FULLY ADAPTIVE DENSITY-BASED CLUSTERING

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The clusters of a distribution are often defined by the connected components of a density level set. However, this definition depends on the user-specified level. We address this issue by proposing a simple, generic algorithm, which uses an almost arbitrary level set estimator to estimate the smallest level at which there are more than one connected components. In the case where this algorithm is fed with histogram-based level set estimates, we provide a finite sample analysis, which is then used to show that the algorithm consistently estimates both the smallest level and the corresponding connected components. We further establish rates of convergence for the two estimation problems, and last but not least, we present a simple, yet adaptive strategy for determining the width-parameter of the involved density estimator in a data-depending way.

1. Introduction. One definition of density-based clusters, which was first proposed by Hartigan [10], assumes i.i.d. data $D = (x_1, \ldots, x_n)$ generated by some unknown distribution $P$ that has a continuous density $h$. For a user-defined threshold $\rho \geq 0$, the clusters of $P$ are then defined to be the connected components of the level set $\{h \geq \rho\}$. This so-called single level approach has been studied by several authors; see, for example, [6, 10, 14, 17, 20] and the references therein. Unfortunately, however, different values of $\rho$ may lead to different (numbers of) clusters (see, e.g., the illustrations in [5, 19]), and there is no generally accepted rule for choosing $\rho$, either. In addition, using a couple of different candidate values creates the problem of deciding which of the resulting clusterings is best. For this reason, Rinaldo and Wasserman [20] note that research on data-dependent, automatic methods for choosing $\rho$ (and the width parameter of the involved density estimator) “would be very useful.”
A second, density-based definition for clustering, which is known as the cluster tree approach, avoids this issue by considering all levels and the corresponding connected components simultaneously. Its focus thus lies on the identification of the hierarchical tree structure of the connected components for different levels; see, for example, [5, 10, 13, 27, 28] for details. For example, Chaudhuri and Dasgupta [5] show, under some assumptions on $h$, that a modified single linkage algorithm recovers this tree in the sense of [11], and Kpotufe and von Luxburg [13] obtain similar results for an underlying $k$-NN density estimator. In addition, Kpotufe and von Luxburg [13] propose a simple pruning strategy that removes connected components that artificially occur because of finite sample variability. However, the notion of recovery taken from [11] only focuses on the correct estimation of the cluster tree structure and not on the estimation of the clusters itself; cf. the discussion in [24].

Defining clusters by the connected components of one or more level sets clearly requires us to estimate level sets in one form or the other. Level set estimation itself is a classical nonparametric problem, which has been considered by various authors; see, for example, [1–3, 7, 12, 15, 16, 18, 21, 22, 26, 29]. In these articles, two different performance measures are considered for assessing the quality of a density level set estimate, namely the mass of the symmetric difference between the estimate and the true level set, and the Hausdorff distance between these two sets. Estimators that are consistent with respect to the Hausdorff metric clearly capture all topological structures eventually, so that these estimators form an almost canonical choice for density-based clustering with fixed level $\rho$. In contrast, level set estimators that are only consistent with respect to the first performance measure are, in general, not suitable for the cluster problem, since even sets that are equal up to measure zero may have completely different topological properties.

Another, very recent density-based cluster definition (see [4]) uses Morse theory to define the clusters of $\mathcal{P}$. The idea of this approach is best illustrated by water flowing on a terrain. Namely, for each mode $x_0$ of $h$, the corresponding modal cluster is the set of points from which water flows, on the steepest descent path, to $x_0$ on the terrain described by $-h$. Under suitable smoothness assumptions on $h$, it turns out that these modal clusters form a partition of the input space modulo a Lebesgue zero set. Unlike in the single level approach, essentially all points of the input domain are thus assigned to a cluster. However, the required smoothness assumptions are somewhat strong, and so far, a consistent estimator has only been found for the one-dimensional case; see [4], Theorem 1.

In this work, we consider none of these approaches. Instead, we follow the approach of [24]; that is, we are interested in estimating (a) the infimum of all $\rho$ at which the level set has more than one component and (b) the corresponding components. In addition, the usual continuity assumption on
Let us therefore briefly describe the approach of [24] here; more details can be found in Section 2.

Its first step consists of defining level sets $M_\rho$ that are independent of the actual choice of the density; see (2.1). Here we note that this independence is crucial for avoiding ambiguities when dealing with discontinuous densities. So far, some approaches have been made to address these difficulties. For example, Cuevas and Fraiman [6] introduced a thickness assumption for sets $C$ that rules out cases in which neighborhoods of $x \in C$ have not sufficient mass. This thickness assumption excludes some topological pathologies such as topologically connecting bridges of zero mass, while others, such as cuts of measure zero, are not addressed. These issues are avoided in [20] by considering level sets of convolutions $k \ast P$ of the underlying distribution $P$ with a continuous kernel $k$ on $\mathbb{R}^d$ having a compact support. Since such convolutions are always continuous, these authors cannot only deal with discontinuous densities, but also with distributions that do not have a Lebesgue density at all. However, different kernels or kernel widths may lead to different level sets, and consequently, their approach introduces new parameters that are hard to control by the user. In this respect, recall that for some other functionals of densities, Donoho [8] could remove these ambiguities, but so far it is unclear whether this is also possible for cluster analysis.

In a second step, the infimum $\rho^*$ over all levels $\rho$ for which $M_\rho$ contains more than one connected component is considered. To reliably estimate $\rho^*$, it is further assumed that there exists some $\rho^{**} > \rho^*$ such that the component structure of $M_\rho$ remains persistent for all $\rho \in (\rho^*, \rho^{**}]$. Note that such persistence is assumed either explicitly or implicitly in basically all density-based clustering approaches (see, e.g., [5, 13]), as it seems intuitively necessary for dealing with vertically uncertainty caused by finite sample effects. Another assumption imposed on $P$, namely that $M_\rho$ has exactly two components between $\rho^*$ and $\rho^{**}$, seems to be more restrictive at first glance. However, the opposite is true: if, for example, $h : [0, 1] \rightarrow (0, \infty)$ is a continuous density with exactly two distinct, strict local minima at say $x_1$ and $x_2$, then we only have more than two connected components in a small range above $\rho^*$ if $h(x_1) = h(x_2)$. Compared to the case $h(x_1) \neq h(x_2)$, the latter seems to be rather singular, in particular, if one considers higher-dimensional analogs. Finally note that we could look for further splits of components above the level $\rho^{**}$ in a similar fashion. This way we would recover the cluster tree approach, and, at least for the one-dimensional case, also the Morse approach by some trivial modifications already discussed in [4]. However, such an iterative approach is clearly out of the scope of this paper.

The first main result of this paper is a generic algorithm, which is based on an arbitrary level set estimator, for estimating both $\rho^*$ and the corresponding clusters. In the case in which the underlying level set estimator enjoys guarantees on its vertical and horizontal uncertainty, we further provide an
error analysis for both estimation problems in terms of these guarantees. A detailed statistical analysis is then conducted for histogram-based level set estimators. Here, our first result is a finite sample bound, which is then used to derive (as in [24]) consistency. We further provide rates of convergence for estimating $\rho^*$ under an assumption on $P$ that describes how fast the connected components of $M_\rho$ move apart for increasing $\rho \in (\rho^*, \rho^{**}]$. The next main result establishes rates of convergence for estimating the clusters. Here we additionally need the well-known flatness condition of Polonik (see [16]) and an assumption that describes the mass of $\delta$-tubes around the boundaries of the $M_\rho$’s. Unlike previous articles, however, we do not need to restrict our considerations to (essentially) rectifiable boundaries. All these rates can only be achieved if the histogram width is chosen in a suitable, distribution-dependent way, and therefore we finally propose a simple data-driven parameter selection strategy. Our last main result shows that this strategy often achieves the above rates without knowing characteristics of $P$.

Since this work strongly builds upon [23, 24], let us briefly describe our main new contributions. First, in [24], only the consistency of the histogram-based algorithm is established; that is, no rate of convergence is presented. While in [23], such rates are established, the situation considered in [23] is different. Indeed, in [23], an algorithm that uses a Parzen window density estimator to estimate the level sets is considered. However, this algorithm requires the density to be $\alpha$-Hölder continuous for known $\alpha$. Second, neither of the papers considers a data-dependent way of choosing the width parameter of the involved density estimator. Besides these new contributions, this paper also adds a substantial amount of extra information regarding the imposed assumptions and, last but not least, polishes many of the results from [24].

The rest of this paper is organized as follows. In Section 2 we recall the cluster definition from [24] and generalize the clustering algorithm from [24]. In Section 3 we provide a finite-sample analysis for the case, in which the generic algorithm is fed with plug-in estimates of a histogram. In Section 4 we then establish consistency and the new learning rates. Section 5 contains the description and the analysis of the new data-driven width selection strategy. Proofs of some of our results that are new, compared to those in [23, 24], can be found in Section 6. The remaining proofs, auxiliary results and an example of a large class of distributions on $\mathbb{R}^2$ with continuous densities that satisfy all the assumptions made in this paper can be found in [25].

2. Preliminaries: Level sets, clusters and a generic algorithm. In this section we recall and refine several notions related to the definition of clusters in [24]. In addition, we present a generic clustering algorithm, which is based on the ideas developed in [24].
Let us begin by fixing some notation and assumptions used throughout this paper: \((X,d)\) is always a compact metric space, and \(\mathcal{B}(X)\) denotes its Borel \(\sigma\)-algebra. Moreover, \(\mu\) is a known \(\sigma\)-finite measure on \(\mathcal{B}(X)\), and \(P\) is an unknown \(\mu\)-absolutely continuous distribution on \(\mathcal{B}(X)\) from which the data \(D = (x_1, \ldots, x_n) \in X^n\) will be drawn in an i.i.d. fashion. In the following, we always assume that \(\mu\) has full support, that is, \(\text{supp}\, \mu = X\). Of course, the example we are most interested in is that of \(X = [0,1]^d\) and \(\mu\) being the Lebesgue measure on \(X\), but alternatives such as the surface measure on a sphere are possible, too.

Given an \(A \subset X\), we write \(\overset{\circ}{A}\) for its interior, \(\overline{A}\) for its closure and \(\partial A := \overline{A} \setminus \overset{\circ}{A}\) for its boundary. Finally, \(1_A\) denotes the indicator function of \(A\) and \(A \triangle B\), the symmetric difference of two sets \(A\) and \(B\).

2.1. **Density-independent density level sets.** Unlike most papers dealing with density-based clustering, we will not assume that the data-generating distribution \(P\) has a continuous density. Unfortunately, this generality makes it more challenging to define density-level-based clusters. Indeed, since the data is generated by \(P\), we actually need to define clusters for distributions and not for densities. Consequently, a well-defined density-based notion of clusters either needs to be independent of the choice of the density, or pick, for each \(P\), a somewhat canonical density. Now, if we assume that each considered \(P\) has a continuous density \(h\), then these \(h\)’s may serve as such canonical choices. In the absence of continuous densities, however, it is no longer clear how a “canonical” choice should look. In addition, the level sets of two different densities of the same \(P\) may have very distinct connected components (see, e.g., Figure 1) so that defining the clusters of \(P\) by the connected components of \(\{h \geq \rho\}\) becomes inconsistent. In other words, neither of the two alternatives above is readily available for general \(P\).

This issue is addressed in [24] by considering “density level sets” that are independent of the choice of the density. To recall this idea from [24], we fix an arbitrary \(\mu\)-density \(h\) of \(P\). Then, for every \(\rho \geq 0\),

\[
\mu_\rho(A) := \mu(A \cap \{h \geq \rho\}), \quad A \in \mathcal{B}(X)
\]

defines a \(\sigma\)-finite measure \(\mu_\rho\) on \(\mathcal{B}(X)\) that is actually independent of our choice of \(h\). As a consequence, the set

\[
M_\rho := \text{supp} \, \mu_\rho,
\]

which in [24] is called the density level set of \(P\) to the level \(\rho\), is independent of this choice, too. It is shown in [24] (see also [25], Lemma A.1.1) that these sets are ordered in the usual way, that is, \(M_{\rho_2} \subset M_{\rho_1}\) whenever \(\rho_1 \leq \rho_2\). Furthermore, for any \(\mu\)-density \(h\) of \(P\), the definition immediately gives

\[
\mu(\{h \geq \rho\} \setminus M_\rho) = \mu(\{h \geq \rho\} \cap (X \setminus M_\rho)) = \mu_\rho(X \setminus M_\rho) = 0;
\]
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Fig. 1. topologically relevant changes on sets of measure zero. Left: The thick solid lines indicate a set consisting of two connected components $A_1$ and $A_2$. If $h = c 1_{A_1 \cup A_2}$ is a density of $P$ for a suitable constant $c$, then $A_1$ and $A_2$ are the connected components of $\{ h \geq \rho \}$ for all $\rho \in [0,c]$. Right: This is a similar situation, but with topologically relevant changes on sets of measure zero. The straight horizontal thin line indicates a line of measure zero connecting the two components, and the dashed lines indicate cuts of measure zero. Clearly, $h' := c 1_{A_1 \cup A_2 \cup A_3 \cup A_4}$ is another density of $P$, but the connected components of $\{ h' \geq \rho \}$ are the four sets $A_1, \ldots, A_4$ for all $\rho \in [0,c]$.

that is, modulo $\mu$-zero sets, the level sets $\{ h \geq \rho \}$ are not larger than $M_\rho$. In fact, $M_\rho$ turns out to be the smallest closed set satisfying (2.2), and it is shown in [24] (see also [25], Lemma A.1.2) that we have both

\begin{equation}
(2.3) \quad \{ h \geq \rho \} \subset M_\rho \subset \{ h \geq \rho \} \quad \text{and} \quad M_\rho \triangle \{ h \geq \rho \} \subset \partial \{ h \geq \rho \}.
\end{equation}

For technical reasons we will not only need (2.2) but also the “converse” as well as a modification of (2.2). The exact requirements are introduced in the following definition, which slightly deviates from [24].

**Definition 2.1.** We say that $P$ is normal at level $\rho \geq 0$ if there exist two $\mu$-densities $h_1$ and $h_2$ of $P$ such that

\[ \mu(M_\rho \setminus \{ h_1 \geq \rho \}) = \mu(\{ h_2 > \rho \} \setminus \overset{\circ}{M_\rho}) = 0. \]

Moreover, we say that $P$ is normal if it is normal at every level.

It is shown in [25], Lemma A.1.3, that $P$ is normal if it has both an upper semi-continuous $\mu$-density $h_1$ and a lower semi-continuous $\mu$-density $h_2$. Moreover, if $P$ has a $\mu$-density $h$ such that $\mu(\partial \{ h \geq \rho \}) = 0$, then $P$ is normal at level $\rho$ by (2.3). Finally, note that if the conditions of normality at level $\rho$ are satisfied for some $\mu$-densities $h_1$ and $h_2$ of $P$, then they are actually satisfied for all $\mu$-densities $h$ of $P$, and we have $\mu(M_\rho \triangle \{ h \geq \rho \}) = 0$.

The remarks made above show that most distributions one would intuitively think of are normal. The next lemma demonstrates that there are also distributions that are not normal at a continuous range of levels.

**Lemma 2.2.** There exists a Lebesgue absolutely continuous distribution $P$ on $[0,1]$ and a $c > 0$ such that $P$ is not normal at $\rho$ for all $\rho \in (0,c]$. 

2.2. Comparison of partitions and some notions of connectivity. Following [24] we will define clusters with the help of connected components over a range of level sets. To prepare this definition, we recall some notions related to connectivity in this subsection. Moreover, we introduce a tool that makes it possible to compare the connected components of two level sets.

To motivate the following definition, which generalizes the ideas from [24], we note that the connected components of a set form a partition.

**Definition 2.3.** Let $A \subset B$ be nonempty sets and $\mathcal{P}(A)$ and $\mathcal{P}(B)$ be partitions of $A$ and $B$, respectively. Then $\mathcal{P}(A)$ is comparable to $\mathcal{P}(B)$, and we write $\mathcal{P}(A) \sqsubseteq \mathcal{P}(B)$ if, for all $A' \in \mathcal{P}(A)$, there is a $B' \in \mathcal{P}(B)$ with $A' \subset B'$.

Informally speaking, $\mathcal{P}(A)$ is comparable to $\mathcal{P}(B)$ if no cell $A' \in \mathcal{P}(A)$ is broken into pieces in $\mathcal{P}(B)$. In particular, if $\mathcal{P}_1$ and $\mathcal{P}_2$ are two partitions of $A$, then $\mathcal{P}_1 \sqsubseteq \mathcal{P}_2$ if and only if $\mathcal{P}_1$ is finer than $\mathcal{P}_2$.

Let us now assume that we have two partitions $\mathcal{P}(A)$ and $\mathcal{P}(B)$ such that $\mathcal{P}(A) \sqsubseteq \mathcal{P}(B)$. Then it is easy to see (cf. [25], Lemma A.2.1) that there exists a unique map $\zeta : \mathcal{P}(A) \to \mathcal{P}(B)$ such that, for all $A' \in \mathcal{P}(A)$, we have $A' \subset \zeta(A')$.

Following [24], we call $\zeta$ the cell relating map (CRM) between $A$ and $B$. Moreover, we write $\zeta_{A,B} := \zeta$ when we want to emphasize the involved pair $(A,B)$. Note that $\zeta$ is injective, if and only if no two distinct cells of $\mathcal{P}(A)$ are contained in the same cell of $\mathcal{P}(B)$. Conversely, $\zeta$ is surjective, if and only if every cell in $\mathcal{P}(B)$ contains a cell of $\mathcal{P}(A)$. Therefore, $\zeta$ is bijective, if and only if there is a structure preserving a one-to-one relation between the cells of the two partitions. In this case, we say that $\mathcal{P}(A)$ is persistent in $\mathcal{P}(B)$ and write $\mathcal{P}(A) \sqsubseteq \mathcal{P}(B)$.

The next lemma establishes a very useful composition formula for CRMs. For a proof, which is again inspired by [24], we refer to [25], Section A.2.

**Lemma 2.4.** Let $A \subset B \subset C$ be nonempty sets with partitions $\mathcal{P}(A)$, $\mathcal{P}(B)$, and $\mathcal{P}(C)$ such that $\mathcal{P}(A) \sqsubseteq \mathcal{P}(B)$ and $\mathcal{P}(B) \sqsubseteq \mathcal{P}(C)$. Then we have $\mathcal{P}(A) \sqsubseteq \mathcal{P}(C)$, and the corresponding CRMs satisfy

$$\zeta_{A,C} = \zeta_{B,C} \circ \zeta_{A,B}.$$
have \( A' = \emptyset \) or \( A'' = \emptyset \). The maximal connected subsets of \( A \) are called the connected components of \( A \). It is well known that these components form a partition of \( A \), which we denote by \( \mathcal{C}(A) \). Moreover, for closed \( A \subset \subset B \) with \(|\mathcal{C}(A)| < \infty \) we have \( \mathcal{C}(A) \subset \subset \mathcal{C}(B) \); see [24] or [25], Lemma A.2.3.

Following [24], we will also consider a discrete version of path-connectivity. To recall the latter, we fix a \( \tau > 0 \) and an \( A \subset \subset X \). Then \( x, x' \in A \) are \( \tau \)-connected in \( A \) if there exist \( x_1, \ldots, x_n \in A \) such that \( x_1 = x \), \( x_n = x' \) and \( d(x_i, x_{i+1}) < \tau \) for all \( i = 1, \ldots, n-1 \). Clearly, being \( \tau \)-connected gives an equivalence relation on \( A \). We write \( \mathcal{C}_\tau(A) \) for the resulting partition and call its cells the \( \tau \)-connected components of \( A \). It is shown in [24] (see also [25], Lemma A.2.7) that \( \mathcal{C}(A) \subset \subset \mathcal{C}_\tau(A) \) for all \( A \subset \subset B \) and \( \tau > 0 \).

For a closed \( A \) and \( \tau > 0 \), we have \( \mathcal{C}(A) \subset \subset \mathcal{C}_\tau(A) \) with a surjective CRM \( \zeta : \mathcal{C}(A) \to \mathcal{C}_\tau(A) \); see [24] or [25], Proposition A.2.10. To characterize, when this CRM is even bijective, let us assume that \( 1 < |\mathcal{C}(A)| < \infty \). Then

\[
\tau^*_A := \min\{d(A', A'') : A', A'' \in \mathcal{C}(A) \text{ with } A' \neq A''\}
\]

denotes the minimal distance between mutually different components of \( \mathcal{C}(A) \). Now it is shown in [24] (or [25], Proposition A.2.10) that

\[
\mathcal{C}(A) = \mathcal{C}_\tau(A) \iff \tau \in (0, \tau^*_A];
\]

see also Figure 2 for an illustration. In other words, \( \tau^*_A \) is the largest (horizontal) granularity \( \tau \) at which the connected components of \( A \) are not glued together. Finally, this threshold is ordered for closed \( A \subset \subset B \) in the sense that \( \tau^*_A \geq \tau^*_B \) whenever \(|\mathcal{C}(A)| < \infty , |\mathcal{C}(B)| < \infty \), and the CRM \( \zeta : \mathcal{C}(A) \to \mathcal{C}(B) \) is injective. We refer to [24] or [25], Lemma A.2.11.

2.3. Clusters. Using the concepts developed in the previous subsections, we can now recall the definition of clusters from [24].
Fig. 3. Definition of clusters. Left: A 1-dimensional mixture of three Gaussians together with the level $\rho^*$ and a possible choice for $\rho^{**}$. The component structure at level $\rho_2 \in (\rho^*, \rho^{**})$ coincides with that at level $\rho^{**}$, while for $\rho_1 < \rho^*$, we only have one connected component. The levels $\rho_3, \rho_4 > \rho^{**}$ are not considered by Definition 2.5, and thus the component structure at these levels is arbitrary. Finally, the clusters of the distribution are the open intervals $(x_1, x_2)$ and $(x_2, x_3)$. Right: Here we have a similar situation for a mixture of three 2-dimensional Gaussians drawn by contour lines. The thick solid lines again indicate the levels $\rho^*$ and $\rho^{**}$, and the thin solid lines show a level $\rho \in (\rho^*, \rho^{**})$. The dashed lines correspond to a level $\rho < \rho^*$ and a level $\rho > \rho^{**}$. This time the clusters are the two connected components of the open set that is surrounded by the outer thick solid line.

**Definition 2.5.** The distribution $P$ can be clustered between $\rho^* \geq 0$ and $\rho^{**} > \rho^*$ if $P$ is normal and for all $\rho \in [0, \rho^{**}]$, the following three conditions are satisfied:

(i) we have either $|\mathcal{C}(M_\rho)| = 1$ or $|\mathcal{C}(M_\rho)| = 2$;
(ii) if we have $|\mathcal{C}(M_\rho)| = 1$, then $\rho \leq \rho^*$;
(iii) if we have $|\mathcal{C}(M_\rho)| = 2$, then $\rho \geq \rho^*$ and $\mathcal{C}(M_{\rho^{**}}) \subseteq \mathcal{C}(M_\rho)$.

Using the CRMs $\zeta_\rho : \mathcal{C}(M_{\rho^{**}}) \to \mathcal{C}(M_\rho)$, we then define the clusters of $P$ by

$$A^*_i := \bigcup_{\rho \in (\rho^*, \rho^{**})} \zeta_\rho(A_i), \quad i \in \{1, 2\},$$

where $A_1$ and $A_2$ are the two topologically connected components of $M_{\rho^{**}}$.

By conditions (iii) and (ii), we find $\rho < \rho^* \Rightarrow |\mathcal{C}(M_\rho)| = 1 \Rightarrow \rho \leq \rho^*$ as well as $\rho > \rho^* \Rightarrow |\mathcal{C}(M_\rho)| = 2 \Rightarrow \rho \geq \rho^*$ for all $\rho \in [0, \rho^{**}]$; see also Figure 3. At each level below $\rho^*$ there is thus only one component, while there are two components at all levels in between $\rho^*$ and $\rho^{**}$. Moreover, in both cases the corresponding partitions are persistent.

Since all $\zeta_\rho$’s are bijective, we find $\zeta_\rho(A_1) \cap \zeta_\rho(A_2) = \emptyset$ for all $\rho \in (\rho^*, \rho^{**})$, and using $\zeta_\rho(A_1) \nrightarrow A^*_i$ for $\rho \searrow \rho^*$, we conclude that $A^*_1 \cap A^*_2 = \emptyset$. In general, the sets $A^*_i$ are neither open nor closed, and we may have $d(A^*_1, A^*_2) = 0$; that is, the clusters may touch each other; see again Figure 3.

2.4. **Cluster persistence under horizontal uncertainty.** In general, we can only expect nonparametric estimates of $M_\rho$ that are both vertically and
horizontally uncertain. To some extent the vertical uncertainty, which is caused by the estimation error, has already been addressed by the persistence assumed in our cluster definition. In this subsection, we complement this by recalling tools from [24] for dealing with horizontal uncertainty, which is usually caused by the approximation error.

To quantify horizontal uncertainty, we need for \( A \subset X, \delta > 0 \), the sets
\[
A^+\delta := \{ x \in X : d(x, A) \leq \delta \}, \\
A^-\delta := X \setminus (X \setminus A)^{+\delta},
\]
where \( d(x, A) := \inf_{x' \in A} d(x, x') \) denotes the distance between \( x \) and \( A \). Simply speaking, adding a \( \delta \)-tube to \( A \) gives \( A^{+\delta} \), while removing a \( \delta \)-tube gives \( A^{-\delta} \). These operations, as well as closely related operations based on the Minkowski addition and difference have already been used in the literature on level set estimation; see, for example, [30]. Some simple properties of these operations can be found in [25], Lemma A.3.1.

Now let \( L_\rho \) be an estimate of \( M_\rho \) having vertical and horizontal uncertainty in the sense of
\[
M^{-\delta}_{\rho+\varepsilon} \subset L_\rho \subset M^{+\delta}_{\rho-\varepsilon},
\]
for some \( \varepsilon, \delta > 0 \). Ideally, we additionally have \( C(M^{-\delta}_{\rho+\varepsilon}) \subseteq C(L_\rho) \subseteq C(M^{+\delta}_{\rho-\varepsilon}) \). To reliably use \( C(L_\rho) \) as an estimate of \( C(M_\rho) \), it then suffices to know \( C(M^{-\delta}_{\rho+\varepsilon}) \subseteq C(M_\rho) \subseteq C(M^{+\delta}_{\rho-\varepsilon}) \). Unfortunately, however, the latter is typically not true. Indeed, even in the absence of horizontal uncertainty, we do not have \( C(M_{\rho+\varepsilon}) \subseteq C(M_{\rho-\varepsilon}) \) if \( \rho + \varepsilon > \rho^* \) and \( \rho - \varepsilon < \rho^* \). Moreover, in the absence of vertical uncertainty, we usually do not have \( C(M^{-\delta}_{\rho}) \subseteq C(M_\rho) \subseteq C(M^{+\delta}_{\rho}) \), either, as components of \( C(M_\rho) \) may be glued together in \( C(M^{+\delta}_{\rho}) \) or cut apart in \( C(M^{-\delta}_{\rho}) \); see Figure 5. To repair such cuts, our algorithm will consider \( \tau \)-connected components instead of connected components. In the rest of this section we thus investigate under which conditions we do have \( C_\tau(M^{-\delta}_{\rho+\varepsilon}) \subseteq C(M_\rho) \subseteq C_\tau(M^{+\delta}_{\rho-\varepsilon}) \). We begin with the following definition taken from [24] that excludes bridges and cusps that are too thin.

**Definition 2.6.** We say that \( P \) has thick level sets of order \( \gamma \in (0, 1] \) up to the level \( \rho^{**} > 0 \), if there exist constants \( c_{\text{thick}} \geq 1 \) and \( \delta_{\text{thick}} \in (0, 1] \) such that, for all \( \delta \in (0, \delta_{\text{thick}}] \) and \( \rho \in [0, \rho^{**}] \), we have
\[
\sup_{x \in M_\rho} d(x, M^{-\delta}_{\rho}) \leq c_{\text{thick}} \delta^\gamma.
\]
In this case, we call \( \psi(\delta) := 3c_{\text{thick}} \delta^\gamma \) the thickness function of \( P \).

Thicknes assumptions have been widely used in the literature on level set estimation (see, e.g., [22]), where the case \( \gamma = 1 \) is considered. To some
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Fig. 4. Thick level sets. Left: The thick solid line indicates a level set $M_\rho$ below or at the level $\rho^*$, and the thin solid lines show the two components $B'$ and $B''$ of $M_\rho - \delta$. Because of the quadratic shape of $M_\rho$ around the thin bridge, the set $M_\rho$ has thickness of order $\gamma = 1/2$. Right: Here we have the same situation for a distribution that has thick level sets of order $\gamma = 1$. Note that the smaller $\gamma$ on the left leads to a significantly wider separation of $B'$ and $B''$ than on the right, which in turn requires larger $\tau$ to glue the parts together.

extent, the latter is a natural choice, as is discussed in detail in [25], Section A.3. In particular, for $d = 1$ we always have $\gamma = 1$, and for $d = 2$ [25], Example B.2.1, provides a rich class of continuous densities with $\gamma = 1$. Figure 4 illustrates how different shapes of level sets lead to different $\gamma$’s.

The following result, which summarizes some findings from [24] (see also [25], Theorems A.4.2 and A.4.4), provides an answer to our persistence question.

**Theorem 2.7.** Assume that $P$ can be clustered between $\rho^*$ and $\rho^{**}$ and that it has thick level sets of order $\gamma$ up to $\rho^{**}$. Let $\psi$ be its thickness function. Using (2.4), we define the function $\tau^* : (0, \rho^{**} - \rho^*) \to (0, \infty)$ by

$$\tau^*(\varepsilon) := \frac{1}{4} \tau_{M_\rho^* + \varepsilon}.$$  

Then $\tau^*$ is increasing, and for all $\varepsilon^* \in (0, \rho^{**} - \rho^*)$, $\delta \in (0, \delta_{\text{thick}})$, $\tau \in (\psi(\delta), \tau^*(\varepsilon^*))$, and all $\rho \in [0, \rho^{**}]$, the following statements hold:

(i) we have $1 \leq |C_{\tau}(M^+_{\rho^*})| \leq 2$ and $1 \leq |C_{\tau}(M^-_{\rho^*})| \leq 2$;

(ii) if $\rho < \rho^*$ or $\rho \geq \rho^* + \varepsilon^*$, then we have

$$C_{\tau}(M^-_{\rho^*}) \subseteq C(M_\rho) = C_{\tau}(M_\rho) \subseteq C_{\tau}(M^+_{\rho^*}).$$

Theorem 2.7 in particular shows that for sufficiently small $\delta$ and $\tau$, the component structure of $M_\rho$ is not changed when $\delta$-tubes are added or removed and $\tau$-connected components are considered instead. Not surprisingly, however, the meaning of “sufficiently small,” which is expressed by the functions $\tau^*$ and $\psi$, changes when we approach the level $\rho^*$ from above. Moreover, note that even for sufficiently small $\delta$ and $\tau$, Theorem 2.7 does not specify the structure of $C_{\tau}(M^-_{\rho^*})$ and $C_{\tau}(M^+_{\rho^*})$ at the levels $\rho \in [\rho^*, \rho^* + \varepsilon^*)$. In fact, for such $\rho$, the components of $M_\rho$ may be accidentally glued together.
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Fig. 5. Difficulties around $\rho^*$. Left: The thick solid line indicates an $M_\rho$ for $\rho < \rho^*$, and the thin solid lines show $M_\rho^\delta$. While $M_\rho$ consists of one connected component, $M_\rho^\delta$ has two such components, $B'$ and $B''$, and hence $C(M_\rho^\delta)$ is not persistent in $C(M_\rho)$. The two types of dotted lines indicate the set of all points that are within $\tau$-distance of $B'$, respectively $B''$ for two values of $\tau$. Only for the larger $\tau$ we have $C_\tau(M_\rho^\delta) \subseteq C(M_\rho)$; that is, in this case $\tau$-connectivity does glue the separated regions together. Right: The thick solid lines indicate an $M_\rho$ for some $\rho \in [\rho^*, \rho^{**}]$ having two connected components, $A'$ and $A''$, and thin solid lines show the two components of $M_\rho^{\delta \rho}$. The two types of dotted lines indicate the set of all points that are within $\tau$-distance of $(A')^{\delta \rho}$, respectively $(A'')^{\delta \rho}$ for the two values of $\tau$ used left. This time, we have $C(M_\rho) \subseteq C_\tau(M_\rho^{\delta \rho})$ only for the smaller value of $\tau$. Together, these graphics thus illustrate that good values for $\delta$ and $\tau$ at one level may be bad at a different level. However, Theorem 2.7 shows that this undesired behavior can be excluded with the help of the functions $\tau^*$ and $\psi$ for all levels $\rho \notin [\rho^*, \rho^{**} + \varepsilon^*]$.

in $C_\tau(M_\rho^{\delta \rho})$; see, for example, Figure 5. This effect complicates our analysis significantly.

Let us now summarize the assumptions that will be used in the following.

Assumption C. We have a compact metric space $(X, d)$, a finite Borel measure $\mu$ on $X$ with $\text{supp} \mu = X$ and a $\mu$-absolutely continuous distribution $P$ that can be clustered between $\rho^*$ and $\rho^{**}$. In addition, $P$ has thick level sets of order $\gamma \in (0, 1]$ up to the level $\rho^{**}$. We denote the corresponding thickness function by $\psi$ and write $\tau^*$ for the function defined in (2.6).

2.5. A generic clustering algorithm and its analysis. In this section, we present and analyze a generic version of the clustering algorithm from [24]. The main difference between our algorithm and the algorithm of [24] is that our generic algorithm can use any level set estimator that has control over both its vertical and horizontal uncertainty.

Our first result, which is a generic version of [24], Theorem 24, relates the component structure of a family of level set estimates to the component structure of certain sets $M_{\rho+\varepsilon}^\delta$. For a proof we refer to [25], Section A.6.

Theorem 2.8. Let Assumption C be satisfied. Furthermore, let $\varepsilon^* \in (0, \rho^{**} - \rho^*], \delta \in [0, \delta_{\text{thick}}], \tau \in [\psi(\delta), \tau^*(\varepsilon^*)]$ and $\varepsilon \in (0, \varepsilon^*]$. In addition, let $(L_\rho)_{\rho \geq 0}$ be a decreasing family of sets $L_\rho \subseteq X$ such that

$$(2.7) \quad M_{\rho+\varepsilon}^\delta \subseteq L_\rho \subseteq M_{\rho-\varepsilon}^\delta$$
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Fig. 6. Illustration of Algorithm 1. Left: A density (thick solid line) has two modes on the left and a flat part on the right. A plug-in approach based on a density estimate (thin solid line with three modes) is used to provide the level set estimator $L_\rho$ (bold horizontal line at level $\rho$), which satisfies $M^{\delta}_\rho \subseteq L_\rho \subseteq M^{\delta+\epsilon}_\rho$. Only the left component of $L_\rho$ does not vanish at $\rho + 2\epsilon$, and thus Algorithm 1 identifies only one component at its line 3.

Right: Here we have the same situation at a higher level. This time both components of $L_\rho$ do not vanish at $\rho + 2\epsilon$, and hence Algorithm 1 identifies two components at its line 3.

holds for all $\rho \geq 0$. Then, for all $\rho \in [0, \rho^{**} - 3\epsilon]$ and the corresponding CRMs $\zeta : C_\tau(M^{-\delta}_{\rho+\epsilon}) \to C_\tau(L_\rho)$, the following disjoint union holds:

$$C_\tau(L_\rho) = \zeta(C_\tau(M^{-\delta}_{\rho+\epsilon})) \cup \{B' \in C_\tau(L_\rho) : B' \cap L_{\rho+2\epsilon} = \emptyset\}.$$ (2.8)

Theorem 2.8 shows that for suitable $\delta$, $\epsilon$, and $\tau$, all $\tau$-connected components $B'$ of $L_\rho$ are either contained in the image $\zeta(C_\tau(M^{-\delta}_{\rho+\epsilon})$ or vanish at level $\rho + 2\epsilon$, that is, $B' \cap L_{\rho+2\epsilon} = \emptyset$. Now assume we can detect the latter components. By Theorem 2.8 we can then identify the $\tau$-connected components $B'$ that are contained in $\zeta(C_\tau(M^{-\delta}_{\rho+\epsilon})$, and if, in addition, $\zeta$ is injective, these identified components have the same structure as $C_\tau(M^{-\delta}_{\rho+\epsilon})$. By Theorem 2.7 we can further hope that $C_\tau(M^{-\delta}_{\rho+\epsilon}) \subseteq C(M_{\rho+\epsilon})$, so that we can relate the identified components to those of $C(M_{\rho+\epsilon})$. Assuming these steps can be carried out precisely, we obtain Algorithm 1; see also Figure 6, which scans through the values of $\rho$ from small to large and stops as soon as it identifies either no component or at least two.

The following theorem provides bounds for the level $\rho^*_D$ and the components $B_i(D)$ returned by Algorithm 1. It extends the analysis from [24].

**Theorem 2.9.** Let Assumption C be satisfied. Furthermore, let $\epsilon^* \leq (\rho^{**} - \rho^*)/9$, $\delta \in (0, \delta_{\text{thick}})$, $\tau \in (\psi(\delta), \tau^*(\epsilon^*)]$ and $\epsilon \in (0, \epsilon^*)$. In addition, let $D$ be a data set and $(L_{D,\rho})_{\rho \geq 0}$ be a decreasing family satisfying (2.7) for all $\rho \geq 0$. Then the following statements are true for Algorithm 1:

(i) the returned level $\rho^*_D$ satisfies both $\rho^*_D \in [\rho^* + 2\epsilon, \rho^* + \epsilon^* + 5\epsilon]$ and

$$\tau - \psi(\delta) < 3\tau^*(\rho^*_D - \rho^* + \epsilon);$$ (2.9)
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(ii) algorithm 1 returns two sets $B_1(D)$ and $B_2(D)$, and these sets can be ordered such that we have

$$
\sum_{i=1}^{2} \mu(B_i(D) \cup A_i^\ast) \leq 2 \sum_{i=1}^{2} \mu((A_i^\ast) - (D_{\rho_D+\varepsilon}))^{-\delta}) + \mu(M_{\rho_D}^\varepsilon \{ h > \rho \})
$$

(2.10)

Here, $A_i^\ast \in C(M_{\rho_D}^\varepsilon)$ are ordered in the sense of $A_i^\ast \subset A^\ast_i$.

3. Finite sample analysis of a histogram-based algorithm. In this section, we consider the case where the level set estimates $L_{D,\rho}$ fed into Algorithm 1 are produced by a histogram. The main result in this section shows that the error estimates of Theorem 2.9 hold with high probability.

To ensure (2.7), we will use, as in [24], partitions that are geometrically well behaved. To this end, recall that the diameter of an $A \subset X$ is

$$
\text{diam} \ A := \sup \{d(x, x') : x, x' \in A \}.
$$

Now, the assumptions made on the used partitions are as follows:

**Assumption A.** For each $\delta \in (0, 1]$, $A_\delta = (A_1, \ldots, A_{m_\delta})$ is a partition of $X$. Moreover, there exist constants $d > 0$ and $c_{\text{part}} \geq 1$ such that, for all $\delta \in (0, 1]$ and $i = 1, \ldots, m_\delta$, we have

$$
\text{diam} \ A_i \leq \delta, \quad m_\delta \leq c_{\text{part}} \delta^{-d} \quad \text{and} \quad \mu(A_i) \geq c_{\text{part}}^{-1} \delta^d.
$$

**Algorithm 1** Clustering with the help of a generic level set estimator

**Require:** Some $\tau > 0$ and $\varepsilon > 0$.

A decreasing family $(L_{D,\rho})_{\rho \geq 0}$ of subsets of $X$.

**Ensure:** An estimate of $\rho^\ast$ and the clusters $A_1^\ast$ and $A_2^\ast$.

1: $\rho \leftarrow 0$
2: repeat
3: Identify the $\tau$-connected components $B'_1, \ldots, B'_M$ of $L_{D,\rho}$ satisfying

$$
B'_i \cap L_{D,\rho+\varepsilon} \neq \emptyset.
$$

4: $\rho \leftarrow \rho + \varepsilon$
5: until $M \neq 1$
6: $\rho \leftarrow \rho + 2\varepsilon$
7: Identify the $\tau$-connected components $B'_1, \ldots, B'_M$ of $L_{D,\rho}$ satisfying

$$
B'_i \cap L_{D,\rho+2\varepsilon} \neq \emptyset.
$$

8: return $\rho_D^\ast := \rho$ and the sets $B_i(D) := B'_i$ for $i = 1, \ldots, M$. 

The most important examples of families of partitions satisfying Assumption A are hyper-cube partitions of $X \subset \mathbb{R}^d$ in combination with the Lebesgue measure; see [25], Example A.7.1, for details. Other situations in which partitions satisfying Assumption A can be found include spheres $X := \mathbb{S}^d \subset \mathbb{R}^{d+1}$ together with their surface measures and $d = d - 1$, sufficiently compact metric groups in combination their Haar measure and known, sufficiently smooth $d$-dimensional sub-manifolds equipped their surface measure. For details we refer to [25], Lemma A7.2 and Corollary A.7.3.

Let us now assume that Assumption A is satisfied. Moreover, for a data set $D = (x_1, \ldots, x_n) \in X^n$ we denote, in a slight abuse of notation, the corresponding empirical measure by $\hat{D}$, that is, $\hat{D} := \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i}$, where $\delta_{x}$ is the Dirac measure at $x$. Then the resulting histogram is

$$h_{D,\delta}(x) = \frac{\sum_{j=1}^{m} D(A_j)}{\mu(A_j)} \cdot 1_{A_j}(x), \quad x \in X.$$ (3.1)

The following theorem provides a finite sample analysis for using the plug-in estimates $L_{D,\rho} := \{h_{D,\delta} \geq \rho\}$ in Algorithm 1.

**Theorem 3.1.** Let Assumptions A and C be satisfied. For a fixed $\delta \in (0, \delta_{\text{thick}}], \varsigma \geq 1, n \geq 1$ and $\tau > \psi(\delta)$, we find an $\varepsilon > 0$ satisfying the bound

$$\varepsilon \geq c_{\text{part}} \sqrt{\frac{E_{\varsigma,\delta}}{2\delta^d n}},$$ (3.2)

where $E_{\varsigma,\delta} := \varsigma + \ln(2c_{\text{part}}) - d\ln\delta$, or if $P$ has a bounded $\mu$-density $h$, the bound

$$\varepsilon \geq \sqrt{\frac{2c_{\text{part}}(1 + \|h\|_{\infty})E_{\varsigma,\delta}}{\delta^d n}} + \frac{2c_{\text{part}}E_{\varsigma,\delta}}{3\delta^d n}.$$ (3.3)

We further pick an $\varepsilon^* > 0$ satisfying

$$\varepsilon^* \geq \varepsilon + \inf\{\varepsilon' \in (0, \rho^{**} - \rho^*) : \tau^*(\varepsilon') \geq \tau\}. \quad (3.4)$$

For each data set $D \in X^n$, we now feed Algorithm 1 with the parameters $\tau$ and $\varepsilon$, and with the family $(L_{D,\rho})_{\rho \geq 0}$ given by

$$L_{D,\rho} := \{h_{D,\delta} \geq \rho\}, \quad \rho \geq 0.$$ If $\varepsilon^* \leq (\rho^{**} - \rho^*)/9$, then with probability $P^n$ not less than $1 - e^{-\varsigma}$, we have a $D \in X^n$ satisfying the assumptions and conclusions of Theorem 2.9.

At this point we like to emphasize that a finite sample bound in the form of Theorem 3.1 can be derived from our analysis whenever Algorithm 1 uses a density level set estimator guaranteeing the inclusions $M_{\rho+\varepsilon}^{-\delta} \subset L_{D,\rho} \subset M_{\rho-\varepsilon}^{\delta}$.
with high probability. A possible example of such an alternative level set estimator is a plug-in approach based on a moving window density estimator, since for the latter it is possible to establish a uniform convergence result similar to [25], Theorem A.8.1; see, for example, [9, 23]. Unfortunately, the resulting level sets become \textit{computationally unfeasible} when used na"ively, and hence we have not included this approach here. It is, however, an interesting open question, whether sets $L_{D, \rho}$ that are constructed differently from the moving window estimator can address this issue. So far, the only known result in this direction [23] constructs such sets for $\alpha$-H"older-continuous densities $h$ with \textit{known} $\alpha$, but we conjecture that a similar construction may be possible for general $h$, too. In addition, strategies such as approximating the sets $L_{D, \rho}$ by fine grids may be feasible, at least for small dimensions, too.

4. Consistency and rates. The first goal of this section is to use the finite sample bound of Theorem 3.1 to show that Algorithm 1 estimates both $\rho^*$ and the clusters $A^*_i$ consistently. We then introduce some assumptions on $P$ that lead to convergence rates for both estimation problems.

The following consistency result is a modification of [24], Theorem 26; see also [25], Section A.9, for a corresponding modification of its proof.

**Theorem 4.1.** Let Assumptions A and C be satisfied, and let $(\varepsilon_n)$, $(\delta_n)$ and $(\tau_n)$ be strictly positive sequences converging to zero such that $\psi(\delta_n) < \tau_n$ for all sufficiently large $n$, and

\begin{equation}
\lim_{n \to \infty} \frac{\ln \delta_n^{-1}}{n\delta_n^{2d} \varepsilon_n^2} = 0.
\end{equation}

For $n \geq 1$, consider Algorithm 1 with the input parameters $\varepsilon_n$, $\tau_n$ and the family $(L_{D, \rho})_{\rho \geq 0}$ given by $L_{D, \rho} := \{h_{D, \delta_n} \geq \rho\}$. Then, for all $\epsilon > 0$, we have

$$\lim_{n \to \infty} P^n(\{D \in X^n: 0 < \rho^*_D - \rho^* \leq \epsilon\}) = 1,$$

and if $\mu(A^*_1 \cup A^*_2 \setminus (A^*_1 \cup A^*_2)) = 0$, we also have

$$\lim_{n \to \infty} P^n(\{D \in X^n: \mu(B_1(D) \triangle A^*_1) + \mu(B_2(D) \triangle A^*_2) \leq \epsilon\}) = 1,$$

where, for $B_1(D)$ and $B_2(D)$, we use the same numbering as in (2.10).

Note that the assumption $\mu(A^*_1 \cup A^*_2 \setminus (A^*_1 \cup A^*_2)) = 0$ is satisfied if there exists a $\mu$-density $h$ of $P$ such that $\mu(\{h \leq \rho^*\}) = 0$; see [25], Section A.9.

Theorem 4.1 shows that for suitably chosen parameters and histogram-based level set estimates Algorithm 1 asymptotically recovers both $\rho^*$ and the clusters $A^*_1$ and $A^*_2$, if the distribution $P$ has level sets that are thicker than a user-specified order $\gamma$. To illustrate this, suppose that we choose $\delta_n \sim n^{-\alpha}$ and $\varepsilon_n \sim n^{-\beta}$ for some $\alpha, \beta > 0$. Then it is easy to check that (4.1)
is satisfied if and only if \( 2(\alpha d + \beta) < 1 \). For \( \tau_n \sim n^{-\alpha} \ln n \), we then have \( \psi(\delta_n) < \tau_n \) for all sufficiently large \( n \), and therefore, Algorithm 1 recovers the clusters for all distributions \( P \) that have thick levels of order \( \gamma \). Similarly, the choice \( \tau_n \sim (\ln n)^{-1} \) leads to consistency for all distributions \( P \) that have thick levels of some order \( \gamma > 0 \). Finally note that (4.1) can be replaced by

\[
\frac{\ln \delta_n^{-1}}{n \delta_n^{\beta_2}} \rightarrow 0
\]

if we restrict our consideration to distributions with bounded \( \mu \)-densities. The proof of this is a straightforward modification of the proof of Theorem 4.1.

To give two examples, recall from the discussion in [25], Section A.5, that for the one-dimensional case \( X = [a, b] \), we always have \( \gamma = 1 \). In two dimensions this is, however, no longer true as, for example, Figure 4 illustrates. Nonetheless, there do exist many examples of both discontinuous and continuous densities for which we have thickness \( \gamma = 1 \); see [25], Section B.2. Finally note that the construction used there can be easily generalized to higher dimensions.

For our next goal, which is establishing rates for both \( \mu(B_i(D) \cap A_i) \rightarrow 0 \) and \( \rho_D \rightarrow \rho^* \), we need, as usual, some assumptions on \( P \). Let us begin by introducing an assumption that leads to rates for the estimation of \( \rho^* \).

**Definition 4.2.** Let Assumption C be satisfied. Then the clusters of \( P \) have separation exponent \( \kappa \in (0, \infty] \) if there is a constant \( c_{\text{sep}} > 0 \) such that

\[
\tau^*(\varepsilon) \geq c_{\text{sep}} \varepsilon^{1/\kappa}
\]

for all \( \varepsilon \in (0, \rho^{**} - \rho^*) \). Moreover, the separation exponent \( \kappa \) is exact if there exists another constant \( \overline{c}_{\text{sep}} > 0 \) such that, for all \( \varepsilon \in (0, \rho^{**} - \rho^*) \), we have

\[
\tau^*(\varepsilon) \leq \overline{c}_{\text{sep}} \varepsilon^{1/\kappa}.
\]

The separation exponent describes how fast the connected components of the \( M_\rho \) approach each other for \( \rho \searrow \rho^* \). Note that the separation exponent is monotone, that is, a distribution having separation exponent \( \kappa \) also has separation exponent \( \kappa' \) for all \( \kappa' < \kappa \). In particular, the "best" separation exponent is \( \kappa = \infty \), and this exponent describes distributions, for which we have \( d(A_1^*, A_2^*) \geq \overline{c}_{\text{sep}} \); that is, the clusters \( A_1^* \) and \( A_2^* \) do not touch each other.

To illustrate the separation exponent, let us consider \( X := [-3, 3] \) and, for \( \theta, \beta \in (0, \infty] \) and \( \rho^* \in [0, 1/6) \), the distribution \( P_{\theta, \beta} \) that has the density

\[
h_{\theta, \beta}(x) := \rho^* + c_{\theta, \beta}(1_{[0, 1]}(|x|)|x|^\theta + 1_{[1, 2]}(|x|) + 1_{[2, 3]}(|x|)(3 - |x|)^\beta),
\]

(4.2)
where $c_{\theta, \beta}$ is a constant ensuring that $h_{\theta, \beta}$ is a probability density; see also Figure 7 for two examples. Note that $P_{\theta, \beta}$ can be clustered between $\rho^*$ and $\rho^{**} := \rho^* + c_{\theta, \beta}$. Moreover, $P_{\theta, \beta}$ always has exact separation exponent $\theta$.

The polynomial behavior in the upper vicinity of $\rho^*$ of the distributions (4.2) is somewhat archetypal for smooth densities on $\mathbb{R}$. For example, for $C^2$-densities $h$ whose first derivative $h'$ has exactly one zero $x_0$ in the set $\{h = \rho^*\}$ and whose second derivative satisfies $h''(x_0) > 0$, one can easily show with the help of Taylor’s theorem that their behavior in the upper vicinity of $\rho^*$ is asymptotically identical to that of (4.2) for $\kappa = \theta = 2$ and $\beta = 1$. Moreover, larger values for $\kappa = \theta$ can be achieved by assuming that higher derivatives of $h$ vanish at $x_0$. Analogously, the class of continuous densities on $\mathbb{R}$ from [25], Section B.2, have separation exponent $\kappa = 2$ (see [25], Example B.2.1), as these densities, similar to Morse functions, behave like $x_1^2 - x_2^2$ in the vicinity of the saddle point. Again, the construction can be modified to achieve other exponents.

In the following we show how the separation exponent influences the rate for estimating $\rho^*$. We begin with a finite sample bound.

**Theorem 4.3.** Let Assumptions A and C be satisfied, and assume additionally that $P$ has a bounded $\mu$-density $h$ and that its clusters have separation exponent $\kappa \in [0, \infty]$. For some fixed $\delta \in (0, \delta_{\text{thick}}]$, $\varsigma \geq 1$, $n \geq 1$ and $\tau \geq 2\psi(\delta)$, we pick an $\varepsilon > 0$ satisfying (3.3), that is,

$$
\varepsilon \geq \sqrt{\frac{2c_{\text{part}}(1 + \|h\|\infty) (\varsigma + \ln(2c_{\text{part}}) - d \ln \delta)}{\delta^3 n}} + \frac{2c_{\text{part}}(\varsigma + \ln(2c_{\text{part}}) - d \ln \delta)}{3\delta^3 n}.
$$

Let us assume that $\varepsilon^* := \varepsilon + (\tau/\omega_{\text{sep}})^\kappa$ satisfies $\varepsilon^* \leq (\rho^{**} - \rho^*)/9$. Then if Algorithm 1 receives the input parameters $\varepsilon$, $\tau$ and the family $(L_{D, \rho})_{\rho \geq 0}$ given by $L_{D, \rho} := \{h_{D, \delta} \geq \rho\}$, the probability $P^n$ of a $D \in X^n$ that satisfies

$$
(4.3) \quad \varepsilon < \rho_D^* - \rho^*,
$$

$$
(4.4) \quad \rho_D^* - \rho^* \leq (\tau/\omega_{\text{sep}})^\kappa + 6\varepsilon
$$

is not less than $1 - e^{-\varsigma}$. Moreover, if the separation exponent $\kappa$ is exact and $\kappa < \infty$, then we can replace (4.3) by

$$
(4.5) \quad \frac{1}{4} \left(\frac{\tau}{\omega_{\text{sep}}}\right)^\kappa + \varepsilon < \rho_D^* - \rho^*.
$$

The finite sample guarantees of Theorem 4.3 can be easily used to derive (exact) rates for $\rho_D^* \to \rho^*$. The following corollary presents, modulo (double) logarithmic factors, the best rates we can derive by this approach.
Corollary 4.4. Let Assumptions A and C be satisfied, and assume that $P$ has bounded $\mu$-density and that its clusters have separation exponent $\kappa \in (0, \infty)$. Furthermore, let $(\varepsilon_n)$, $(\delta_n)$ and $(\tau_n)$ be sequences with

$$
\varepsilon_n \sim \left(\frac{\ln n \cdot \ln \ln n}{n}\right)^{\gamma \kappa/(2\gamma \kappa + d)}, \quad \delta_n \sim \left(\frac{\ln n}{n}\right)^{1/(2\gamma \kappa + d)} \quad \text{and} \quad \tau_n \sim \varepsilon_n^{1/\kappa},
$$

and assume that, for $n \geq 1$, Algorithm 1 receives the input parameters $\varepsilon_n$, $\tau_n$ and the family $(L_{D, \rho})_{\rho \geq 0}$ given by $L_{D, \rho} := \{h_{D, \delta_n} \geq \rho\}$. Then there exists a constant $K \geq 1$ such that for all sufficiently large $n$, we have

$$
P^n\left(\{D \in X^n : \rho^*_{\Delta} - \rho^* \leq K\varepsilon_n\}\right) \geq 1 - \frac{1}{n}. \quad (4.6)
$$

Moreover, if the separation exponent $\kappa$ is exact, there exists another constant $K \geq 1$ such that for all sufficiently large $n$, we have

$$
P^n\left(\{D \in X^n : K\varepsilon_n \leq \rho^*_{\Delta} - \rho^* \leq K\varepsilon_n\}\right) \geq 1 - \frac{1}{n}. \quad (4.7)
$$

Finally, if $\kappa = \infty$, then (4.7) holds for all sufficiently large $n$ if

$$
\varepsilon_n \sim \left(\frac{\ln n \cdot \ln \ln n}{n}\right)^{1/2}, \quad \delta_n \sim (\ln \ln n)^{-1/(2d)} \quad \text{and} \quad \tau_n \sim (\ln \ln n)^{-\gamma/(3d)}.
$$

Recall that for the one-dimensional distributions (4.2) we have $\gamma = 1$ and $\kappa = \theta$, so that the exponent in the rates above becomes $\frac{\theta}{2d + 1}$. In particular, for the $C^2$-case discussed there, we have $\theta = 2$, and thus we get a rate with exponent $2/5$, while for $\theta \to \infty$ the exponent converges to $1/2$. Similarly, for the typical, two-dimensional distributions considered in [25], Section B.2, we have $\gamma = 1$, $\kappa = 2$ and $d = 2$, and hence the exponent in the rate is $1/3$.

Our next goal is to establish rates for $\mu(B_i(D) \cap A_i^*) \to 0$. Since this is a modified level set estimation problem, let us recall some assumptions on $P$, which have been used in this context. The first assumption in this direction is a one-sided variant of a well-known condition introduced by Polonik [16].

Definition 4.5. Let $\mu$ be a finite measure on $X$ and $P$ be a distribution on $X$ that has a $\mu$-density $h$. For a given level $\rho \geq 0$, we say that $P$ has flatness exponent $\vartheta \in (0, \infty]$ if there exists a constant $c_{\text{flat}} > 0$ such that

$$
\mu(\{0 < h - \rho < s\}) \leq (c_{\text{flat}} s)^{\vartheta}, \quad s > 0. \quad (4.8)
$$

Clearly, the larger the $\vartheta$, the more steeply $h$ must approach $\rho$ from above. In particular, for $\vartheta = \infty$, the density $h$ is allowed to take the value $\rho$ but is otherwise bounded away from $\rho$. For example, the densities in (4.2) have a flatness exponent $\vartheta = \min\{1/\theta, 1/\beta\}$ if $\theta < \infty$ and $\beta < \infty$ and a flatness
Fig. 7. Separation and flatness. Left: The density $h_{\theta,\beta}$ described in (4.2) for $\theta = 3$ and $\beta = 3/2$. The bold horizontal line indicates the set $\{\rho^* < h < \rho^* + \varepsilon\}$, and $3\tau^*(\varepsilon)$ describes the width of the valley at level $\rho^* + \varepsilon$. Right: Here we have the same situation for $\theta = 2/3$ and $\beta = 3$. The value of $\varepsilon$ is chosen such that $3\tau^*(\varepsilon)$ equals the value on the left. The smaller value of $\theta$ narrows the valley, and hence $\varepsilon$ needs to be chosen larger. As a result, it becomes more difficult to estimate $\rho^*$ and the clusters. Indeed, ignoring logarithmic factors, Corollary 4.4 gives a rate of $n^{-3/7}$ on the left and a rate of $n^{-2/7}$ on the right, while Corollary 4.8 gives a rate of $n^{-1/7}$ on the left and a rate of $n^{-2/21}$ on the right. Finally, in the most typical case $\theta = 2$ and $\beta = 1$ not illustrated here, we obtain the rates $n^{-1/3}$ and $n^{-1/5}$.

exponent $\vartheta = \infty$ if $\theta = \beta = \infty$. Finally, for the two-dimensional distributions of [25], Section B.2, the flatness exponent is not fully determined by their definition, but some calculations show that we have $\vartheta \in (0, 1]$.

Next, we describe the roughness of the boundary of the clusters.

**Definition 4.6.** Let Assumption C be satisfied. Given some $\alpha \in (0, 1]$, the clusters have an $\alpha$-smooth boundary if there exists a constant $c_{\text{bound}} > 0$ such that, for all $\rho \in (\rho^*, \rho^{**}]$, $\delta \in (0, \delta_{\text{thick}}]$ and $i = 1, 2$, we have

$$
\mu((A^i_\rho)^{+\delta} \setminus (A^i_\rho)^{-\delta}) \leq c_{\text{bound}} \delta^\alpha,
$$

where $A^1_\rho$ and $A^2_\rho$ denote the two connected components of the level set $M_\rho$.

In $\mathbb{R}^d$, considering $\alpha > 1$ does not make sense, and for an $A \subset \mathbb{R}^d$ with rectifiable boundary, we always have $\alpha = 1$; see [25], Lemma A.10.4. The $\alpha$-smoothness of the boundary thus enforces a uniform version of this, which, however, is not very restrictive; see, for example, the densities of (4.2), for which we have $\alpha = 1$ and $c_{\text{bound}} = 4$, and [25], Example B.2.2, for which we also have $\alpha = 1$.

The following assumption collects all conditions we need to impose on $P$ to get rates for estimating the clusters.

**Assumption R.** Assumptions A and C are satisfied, and $P$ has a bounded $\mu$-density $h$. Moreover, $P$ has a flatness exponent $\vartheta \in (0, \infty]$ at level $\rho^*$, its clusters have an $\alpha$-smooth boundary for some $\alpha \in (0, 1]$ and its clusters have a separation exponent $\kappa \in (0, 1]$. 
Let us now investigate how well our algorithm estimates the clusters $A_1^*$ and $A_2^*$. As usual, we begin with a finite-sample estimate.

**Theorem 4.7.** Let Assumption $R$ be satisfied, and assume that $\delta, \varepsilon, \tau, \varepsilon^*, \varsigma, n$ and $(L_D, \rho)_{\rho \geq 0}$ are as in Theorem 4.3. Then the probability $P^n$ of having a data set $D \in X^n$ satisfying (4.3), (4.4) and

$$\mu(B_1(D) \triangle A_1^*) + \mu(B_2(D) \triangle A_2^*) \leq 6c_{\text{bound}}\delta^\alpha + \left(c_{\text{flat}}(\tau/\Delta_{\text{sep}})^\kappa + 7c_{\text{flat}}\varepsilon\right)^\delta$$

is not less than $1 - e^{-\varsigma}$, where the sets $B_1(D)$ and $B_2(D)$ are ordered as in (2.10). Moreover, if the separation exponent $\kappa$ is exact and satisfies $\kappa < \infty$, then (4.5) also holds for these data sets $D$.

Note that for finite values of $\vartheta$ and $\kappa$, the bound in Theorem 4.7 behaves like $\delta^{\alpha + \tau^\kappa + \varepsilon^\vartheta}$, and in this case it is thus easy to derive the best convergence rates our analysis yields. The following corollary presents corresponding results and also provides rates for the cases $\vartheta = \infty$ or $\kappa = \infty$.

**Corollary 4.8.** Assume that Assumption $R$ is satisfied, and write $\varrho := \min\{\alpha, \vartheta \gamma \kappa\}$. Furthermore, let $(\varepsilon_n)$, $(\delta_n)$ and $(\tau_n)$ be sequences with

$$\varepsilon_n \sim \left(\frac{\ln n}{n}\right)^{\frac{\vartheta}{2(\vartheta + \vartheta d)}} (\ln \ln n)^{-\frac{\vartheta d}{8(\vartheta + 4 \vartheta d)}},$$

$$\delta_n \sim \left(\frac{\ln n \cdot \ln \ln n}{n}\right)^{\frac{\vartheta}{2(\vartheta + \vartheta d)}}$$

and

$$\tau_n \sim \left(\frac{\ln n \cdot (\ln \ln n)^2}{n}\right)^{\frac{\varrho \gamma}{2(\vartheta + \vartheta d)}}.$$

Assume that, for $n \geq 1$, Algorithm 1 receives the parameters $\varepsilon_n$, $\tau_n$ and the family $(L_D, \rho)_{\rho \geq 0}$ given by $L_D, \rho := \{h_D, \delta_n \geq \rho\}$. Then there is a constant $K \geq 1$ such that, for all $n \geq 1$ and the ordering as in (2.10), we have

$$P^n \left(D : \sum_{i=1}^2 \mu(B_i(D) \triangle A_i^*) \leq K \left(\frac{\ln n \cdot (\ln \ln n)^2}{n}\right)^{\frac{\vartheta \gamma}{2(\vartheta + \vartheta d)}}\right) \geq 1 - \frac{1}{n}.$$
If $\vartheta \gamma \kappa \leq 1$, we can thus achieve the best rates for estimating $\rho^*$ and the clusters simultaneously. Unfortunately, this changes if $\vartheta \gamma \kappa > 1$. Indeed, while the exponent for $(\delta_n)$ in Corollary 4.4 remains the same, it changes from $\frac{1}{2\gamma \kappa + d}$ to $\frac{\vartheta}{2 + \vartheta d}$ in Corollary 4.8, and a similar effect takes place for the sequences $(\varepsilon_n)$ and $(\tau_n)$. The reason for this difference is that in the case $\vartheta \gamma \kappa > 1$ the estimation of $\rho^*$ is easier than the estimation of the level set $M_{\rho^*}$, and since for estimating the clusters we need to do both, the level set estimation rate determines the rate for estimating the clusters.

To illustrate this difference between the estimation of $\rho^*$ and the clusters in more detail, let us consider the toy model (4.2) in the case $\theta = \beta = \infty$, that is, $\kappa = \infty$. Then the clusters are stumps, and the sets $M_{\rho}$ do not change between $\rho^*$ and $\rho^{**}$. Intuitively, the best choice for estimating $\rho^*$ are then sufficiently small but fixed values for $\delta_n$ and $\tau_n$, so that $\varepsilon_n$ converges to 0 as fast as possible. In Corollary 4.4 this is mimicked by choosing very slowly decaying sequences $(\delta_n)$ and $(\tau_n)$. On the other hand, to find $A^*_1$ and $A^*_2$ it suffices to identify one $\rho \in (\rho^*, \rho^{**}]$ and to estimate the connected components of $M_{\rho}$. The best way to achieve this is to use a sufficiently small but fixed value for $\varepsilon_n$ and sequences $(\delta_n)$ and $(\tau_n)$ that converge to zero as fast as possible. In Corollary 4.8 this is mimicked by choosing a very slowly decaying sequence $(\varepsilon_n)$ and quickly decaying sequences $(\delta_n)$ and $(\tau_n)$.

As for estimating the critical level $\rho^*$, we do not know so far, whether our rates for estimating the clusters are minmax optimal, but our conjecture is that they are optimal modulo the logarithmic terms. To motivate our conjecture, let us consider the case $\alpha = \gamma = 1$. Moreover, assume that two-sided versions of [25], (A.10.4) and (A.10.6), hold for all $\rho \in (\rho^*, \rho^{**}]$, respectively, $\rho = \rho^*$. Then we have $\kappa = \theta$ and $\vartheta = 1/\theta$ by [25], Lemmas A.10.1 and A.10.5, and thus we find $\vartheta = 1$. Consequently, the rates in Corollary 4.8 have the exponent $\frac{1}{2\vartheta + d}$. This is exactly the same exponent as the one obtained in [22] for minmax optimal and adaptive Hausdorff estimation of a fixed level set. In addition, it seems that their lower bound, which is based on [29], is, modulo logarithmic factors, the same for assessing the estimator in the way we have done it in Corollary 4.8. While this coincidence indicates that our rates may be (essentially) optimal, it is, of course, not a rigorous argument. A detailed analysis is, however, out of the scope of this paper. Another interesting question, which is also out of the scope, is whether the estimates $B_i(D)$ approximate the true clusters $A^*_i$ in the Hausdorff metric, too, and if so, whether we can achieve the rates reported in [22].

5. Data-dependent parameter selection. In the last section we derived rates of convergence for both the estimation of $\rho^*$ and the clusters. In both cases, our best rates required sequences $(\varepsilon_n)$, $(\delta_n)$ and $(\tau_n)$ that did depend on some properties of $P$, namely $\alpha$, $\kappa$, $\vartheta$. Of course, these parameters are
not available to us in practice, and therefore the obtained rates are of little practical value. The goal of this final section is to address this issue by proposing a simple data-dependent parameter selection strategy that is able to recover the rates of Corollary 4.4 without knowing anything about \( P \). We further show that this selection strategy recovers the rates of Corollary 4.8 in the case of \( \vartheta \gamma \kappa \leq \alpha \).

We begin by presenting the parameter selection strategy. To this end, let \( \Delta \subset (0, 1] \) be finite and \( n \geq 1 \), \( \varsigma \geq 1 \). For \( \delta \in \Delta \), we fix a \( \tau^{\delta, n} > 0 \) and define

\[
\varepsilon^{\delta, n} := C \sqrt{c_{\text{part}}(\varsigma + \ln(2c_{\text{part}}|\Delta|) - d \ln \delta) \ln n \over \delta^{d n}} + 2c_{\text{part}}(\varsigma + \ln(2c_{\text{part}}|\Delta|) - d \ln \delta) \over 3\delta^{d n},
\]

where \( C \geq 1 \) is some user-specified constant. Now assume that, for each \( \delta \in \Delta \), we run Algorithm 1 with the parameters \( \varepsilon^{\delta, n} \) and \( \tau^{\delta, n} \), and the family \( (L_{D, \rho})_{\rho \geq 0} \) given by \( L_{D, \rho} := \{h_{D, \delta} \geq \rho\} \). We write \( \rho^{\delta, \Delta} \) for the corresponding level returned by Algorithm 1. Let us consider a width \( \delta^{\Delta} \in \Delta \) that achieves the smallest returned level, that is,

\[
\delta^{\star, \Delta} \in \arg \min_{\delta \in \Delta} \rho^{\star, \Delta, \delta}.
\]

Note that in general, this width may not be uniquely determined, so that in the following we need to additionally assume that we have a well-defined choice, for example, the smallest \( \delta \in \Delta \) satisfying (5.2). Moreover, we write

\[
\rho^{\star, \Delta, \delta} := \rho^{\delta, \Delta, \delta} = \min_{\delta \in \Delta} \rho^{\delta, \Delta, \delta}
\]

for the smallest returned level. Note that unlike \( \delta^{\star, \Delta} \), the level \( \rho^{\star, \Delta} \) is always unique. Finally, we define \( \varepsilon^{\Delta} := \varepsilon^{\star, \Delta, \delta} \) and \( \tau^{\delta, \Delta, \delta} := \tau^{\delta, \Delta, \delta} \).

Our first goal is to show that \( \rho^{\star, \Delta} \) achieves the rates of Corollary 4.4 for suitably chosen \( \Delta \) and \( \tau^{\delta, n} \). We begin with a finite sample guarantee.

**Theorem 5.1.** Let Assumptions A and C be satisfied, and assume that \( P \) has a bounded \( \mu \)-density \( h \), and that the two clusters of \( P \) have separation exponent \( \kappa \in (0, \infty] \). For a fixed finite \( \Delta \subset (0, \delta_{\text{thick}}] \), and \( n \geq 1 \), \( \varsigma \geq 1 \) and \( C \geq 1 \), we define \( \varepsilon^{\delta, n} \) by (5.1) and choose \( \tau^{\delta, n} \) such that \( \tau^{\delta, n} \geq 2 \psi(\delta) \) for all \( \delta \in \Delta \). Furthermore, assume that \( C^{2} \ln \ln n \geq 2(1 + \|h\|_{\infty}) \) and \( \varepsilon^{\delta} := \varepsilon^{\delta, n} + (\tau^{\delta, n} / \mathcal{L}_{\text{sep}})^{\kappa} \leq (\rho^{\star} - \rho^{\star}) / 9 \) for all \( \delta \in \Delta \). Then we have

\[
P^{n}\left( \left\{ D \in X^{n} : \varepsilon^{\Delta} < \rho^{\star, \Delta} \leq \min_{\delta \in \Delta} (\tau^{\delta, n} / \mathcal{L}_{\text{sep}}^{\kappa} + 6 \varepsilon^{\delta, n}) \right\} \right) \geq 1 - e^{-\varsigma}.
\]

Moreover, if the separation exponent \( \kappa \) is exact and \( \kappa < \infty \), then the assumptions above actually guarantee

\[
P^{n}\left( \left\{ D : \min_{\delta \in \Delta} (c_{1} \tau^{\delta, n} + \varepsilon^{\delta, n}) < \rho^{\star, \Delta} \leq \min_{\delta \in \Delta} (c_{2} \tau^{\delta, n} + 6 \varepsilon^{\delta, n}) \right\} \right) \geq 1 - e^{-\varsigma},
\]
where \( c_1 := \frac{1}{4}(6\text{sep})^{-\kappa} \) and \( c_2 := \text{sep}^{-\kappa} \), and similarly
\[
P^n\left( \{ D \in X^n : c_1\tau_{D,\Delta} + \varepsilon_{D,\Delta} < \rho^*_D - \rho^* \leq c_2\tau_{D,\Delta} + 6\varepsilon_{D,\Delta} \} \right) \geq 1 - e^{-c}.
\]

Theorem 5.1 establishes the same finite sample guarantees for the estimator \( \rho^*_{D,\Delta} \) as Theorem 4.3 did for the simpler estimator \( \rho^*_D \). Therefore, it is not surprising that for suitable choices of \( \Delta \), the rates of Corollary 4.4 can be recovered, too. The next corollary shows that this can actually be achieved for candidate sets \( \Delta \) that are completely independent of \( P \).

**Corollary 5.2.** Assume that Assumptions A and C are satisfied, that \( P \) has a bounded \( \mu \)-density \( h \) and that the two clusters of \( P \) have separation exponent \( \kappa \in (0, \infty] \). For \( n \geq 16 \), we consider the interval
\[
I_n := \left[ \left( \frac{\ln n \cdot (\ln \ln n)^2}{n} \right)^{1/d}, \left( \frac{1}{\ln \ln n} \right)^{1/d} \right]
\]
and fix some \( n^{-1/d} \)-net \( \Delta_n \subset I_n \) with \( |\Delta_n| \leq n \). Furthermore, for some fixed \( C \geq 1 \) and \( n \geq 16 \), we write \( \tau_{\delta,n} := \delta \ln \ln n \) and define \( \varepsilon_{\delta,n} \) by (5.1) for all \( \delta \in \Delta_n \) and \( \varsigma = \ln n \). Then there exists a constant \( K \) such that, for all sufficiently large \( n \), we have
\[
P^n\left( D : \varepsilon_{D,\Delta_n} < \rho^*_{D,\Delta_n} - \rho^* \leq K \left( \frac{\ln n \cdot (\ln \ln n)^2}{n} \right)^{\gamma\kappa/(2\gamma\kappa+d)} \right) \geq 1 - \frac{1}{n}.
\]
(5.4)

If, in addition, the separation exponent \( \kappa \) is exact and \( \kappa < \infty \), then there is another constant \( K \) such that for all sufficiently large \( n \), we have
\[
P^n\left( D : K \left( \frac{\ln n \cdot (\ln \ln n)^2}{n} \right)^{\gamma\kappa/(2\gamma\kappa+d)} \leq \rho^*_{D,\Delta_n} - \rho^* \right) \left( \frac{\ln n \cdot (\ln \ln n)^2}{n} \right)^{\gamma\kappa/(2\gamma\kappa+d)} \geq 1 - \frac{1}{n}.
\]

Finally, we show that our parameter selection strategy partially recovers the rates for estimating the clusters \( A^*_i \) obtained in Corollary 4.8.

**Corollary 5.3.** Assume that Assumption R is satisfied with \( \alpha \geq \vartheta \gamma \kappa \) and exact separation exponent \( \kappa \). Then, for the procedure of Corollary 5.2, there is a \( K \geq 1 \) such that for \( n \geq 1 \) and the ordering as in (2.10), we have
\[
P^n\left( D : \sum_{i=1}^{2} \mu(B_i(D) \Delta A^*_i) \leq K \left( \frac{\ln n \cdot (\ln \ln n)^2}{n} \right)^{\vartheta\gamma\kappa/(2\gamma\kappa+\vartheta d)} \right) \geq 1 - \frac{1}{n}.
\]
Unfortunately, the simple parameter selection strategy (5.2) is not adaptive in the case \( \alpha < \vartheta \gamma \kappa \), that is, in the case in which the estimation of \( \rho^* \) is easier than the estimation of the corresponding clusters. It is unclear to us whether in this case a two-stage procedure that first estimates \( \rho^* \) by \( \rho^*_{D, \Delta_n} \) as above, and then uses a different strategy to estimate the connected components at the level \( \rho^*_{D, \Delta_n} \) can be made adaptive.

6. Selected proofs. In this section we present some selected proofs. All remaining proofs can be found in [25].

Proof of Lemma 2.2. Let \((x_n)\) be an enumeration of \( \mathbb{Q} \cap [0, 1] \) and \( I_n := [x_n - 2^{-n-2}, x_n + 2^{-n-2}] \cap [0, 1] \) for \( n \geq 1 \). For \( x \in [0, 1] \) and \( I_0 := [0, 1] \), we further define

\[
f(x) := \sup_{n \geq 0} n \mathbb{1}_{I_n}(x),
\]

that is, \( f(x) \) equals the largest integer \( n \geq 0 \) (including infinity) such that \( x \in I_n \). For \( c > 0 \) specified below, we now define

\[
h(x) := \begin{cases} 
2c - \frac{c}{f(x)}, & \text{if } f(x) > 0, \\
0, & \text{else.}
\end{cases}
\]

Then \( h \) is measurable, nonnegative and Lebesgue-integrable, and hence we can choose \( c \) such that \( \int_0^1 h(x) \, dx = 1 \). Then \( h \) is a density of a Lebesgue-absolutely continuous distribution \( P \). Moreover, note that \( h(x) \geq 2c - c/n \) for all \( x \in I_n \) and \( n \geq 1 \). For a fixed \( \rho \in (0, 2c) \) we now write \( n_\rho := c/(2c - \rho) \). Then we have \( 2c - c/n \geq \rho \) if and only if \( n \geq n_\rho \). Consequently, the set

\[
A_\rho := \bigcup_{n \geq n_\rho} I_n
\]

satisfies \( A_\rho \subset \{ h \geq \rho \} \). Moreover, since \( A_\rho \) is open, we find \( A_\rho \subset \{ \overset{\circ}{h} \geq \rho \} \), and thus

\[
\overline{A_\rho} \subset \{ \overset{\circ}{h} \geq \rho \} \subset M_\rho
\]

by [25], Lemma A.1.2. In addition, we have \( \{ x_n : n \geq n_\rho \} \subset A_\rho \), and since the former set is dense in \([0, 1] \), we conclude that \( M_\rho = [0, 1] \). On the other hand, the Lebesgue measure \( \lambda \) of \( \{ h \geq \rho \} \) can be estimated by

\[
\lambda(\{ h \geq \rho \}) \leq \lambda(\{ h > 0 \}) = \lambda\left( \bigcup_{n=1}^{\infty} I_n \right) \leq \sum_{n=1}^{\infty} \lambda(I_n) \leq \sum_{n=1}^{\infty} 2^{-n-1} = \frac{1}{2},
\]

and hence we conclude that \( \lambda(M_\rho \setminus \{ h \geq \rho \}) \geq 1/2 \). In other words, \( P \) is not normal at level \( \rho \). \( \square \)
Proof of Theorem 2.7. The monotonicity of $\tau^*$ is shown in [25], Theorem A.4.2, and (i) follows from parts (i) of [25], Theorems A.4.2 and A.4.4.

(ii) Let us first consider the case $\rho < \rho^*$. Since $P$ can be clustered, we have $|C(M_\rho)| = 1$, and [25], Proposition A.2.10, gives both $\tau^*_{M_\rho} = \infty$ and $C(M_\rho) = C_M(M_\rho)$. By [25], Lemma A.4.1, we further find $C_M(M_\rho) \subseteq C_M(M_{\rho^*})$.

Finally, part (ii) of [25], Lemma A.4.3, yields $1 \leq |C(M_{\rho^*})| \leq |C(M_\rho)| = 1$, and hence its part (iii) gives the persistence $C_M(M_{\rho^*}) \subseteq C(M_\rho)$.

In the case $\rho \geq \rho^* + \varepsilon$, $C_M(M_\rho) \subseteq C_M(M_{\rho^*})$ follows from part (ii) of [25], Theorem A.4.2, and the equality $C(M_\rho) = C(M_\rho^*)$ follows from [25], Proposition A.2.10, in combination with $\tau^* \leq \tau^*(\varepsilon^*) \leq \tau^*(\rho - \rho^*)$. By part (ii) of [25], Theorem A.4.4, we further know $C_M(M_{\rho^*}) \subseteq C_M(M_{\rho^*})$. Using $\rho \geq \rho^* + \varepsilon$ and part (iv) of [25], Theorem A.4.2, we find $|C_M(M_{\rho^*})| = 2$, and hence part (iii) of [25], Theorem A.4.4, gives $C_M(M_{\rho^*}) \subseteq C(M_\rho)$. □

Proof of Theorem 2.9. (i) The first bound on $\rho^*_D$ directly follows from part (i) of [25], Theorem A.6.2.

To show (2.9), we observe that parts (iii) and (iv) of [25], Theorem A.6.2, imply $2 = |C_M(M_{\rho^*}^\delta)| = |C(M_{\rho^*}^\delta)|$. Since we further have $\rho^*_D + \varepsilon \leq \rho^* + \varepsilon^* + 6\varepsilon \leq \rho^{**}$ by the first bound on $\rho^*_D$, part (iii) of [25], Lemma A.4.3, thus shows

$$d(B_1, B_2) \geq \tau - 2\psi^*_{M_{\rho^*_D}^\delta + \varepsilon}(\delta) \geq \tau - 2C_{\text{thick}} \gamma > \tau - \psi(\delta),$$

where $B_1$ and $B_2$ are the two connected components of $M_{\rho^*_D}^\delta$. On the other hand, the definition of $\tau^*_{M_{\rho^*_D}^\delta + \varepsilon}$ in [25], Proposition A.2.10, together with the definition of $\tau^*$ in (2.6) gives

$$3\tau^*(\rho^*_D - \rho^* + \varepsilon) = \tau^*_{M_{\rho^*_D}^\delta + \varepsilon} = d(B_1, B_2).$$

Combining both we find (2.9).

(ii) Part (iii) of [25], Theorem A.6.2, shows that Algorithm 1 returns two sets. Our next goal is to find a suitable ordering of these sets. To this end, we adopt the notation of [25], Theorem A.6.2. Moreover, we denote the two topologically connected components of $M_{\rho^{**}}$ by $A_1$ and $A_2$. We further write

$$V^i_{\rho^*_D + \varepsilon} := \zeta^{\rho^{**}, \rho^*_D + \varepsilon(A_i)}, \quad i = 1, 2,$$

for the two $\tau$-connected components of $M_{\rho^*_D + \varepsilon}$. Note that part (iv) of [25], Theorem A.6.2, ensures that we can actually make this definition, and, in addition, it shows $V^1_{\rho^*_D + \varepsilon} \neq V^2_{\rho^*_D + \varepsilon}$. Moreover, by parts (ii) and (iii) of [25],
To this end, we fix an $i \in \{1, 2\}$ and $\rho := \rho_D^i$. Consequently, $A_{\rho+\varepsilon}^1$ and $A_{\rho+\varepsilon}^2$ are the two connected components of $M_{\rho+\varepsilon} = M_{\rho_D^i+\varepsilon}$, which by Definition 2.5 can be ordered in the sense of $A_{\rho+\varepsilon}^1 \subset A_{\rho+\varepsilon}^2$. Moreover, $V_{\rho+\varepsilon}^1$ and $V_{\rho+\varepsilon}^2$ become the two $\tau$-connected components of $M_{\rho+\varepsilon}$. For $i \in \{1, 2\}$, we further write $W_{\rho+\varepsilon}^i := (A_{\rho+\varepsilon}^i)^- \delta$. Our first goal is to show that

$$W_{\rho+\varepsilon}^i \subset V_{\rho+\varepsilon}^i, \quad i \in \{1, 2\}. \tag{6.2}$$

To this end, we fix an $x \in W_{\rho+\varepsilon}^1$. Since $W_{\rho+\varepsilon}^1 \subset A_{\rho+\varepsilon}^1$ and $W_{\rho+\varepsilon}^1 \subset M_{\rho+\varepsilon}^\delta$, where the latter follows from $(A_{\rho+\varepsilon}^1)^- \delta \subset M_{\rho+\varepsilon}^\delta$, we then have $x \in A_{\rho+\varepsilon}^1$ and $x \in V_{\rho+\varepsilon}^1 \cup V_{\rho+\varepsilon}^2$. Let us assume that $x \in V_{\rho+\varepsilon}^2$. Then we have $V_{\rho+\varepsilon}^2 \setminus A_{\rho+\varepsilon}^1 \neq \emptyset$. Now, the diagram of [25], Theorem A.6.2, shows that $\zeta_{\rho+\varepsilon} : \mathcal{C}_\tau(M_{\rho+\varepsilon}^\delta) \to \mathcal{C}(M_{\rho+\varepsilon})$ satisfies $\zeta_{\rho+\varepsilon}(V_{\rho+\varepsilon}^2) = A_{\rho+\varepsilon}^2$, and hence we have $V_{\rho+\varepsilon}^2 \subset A_{\rho+\varepsilon}^2$. Consequently, $V_{\rho+\varepsilon}^2 \cap A_{\rho+\varepsilon}^1 \neq \emptyset$ implies $A_{\rho+\varepsilon}^2 \cap A_{\rho+\varepsilon}^1 \neq \emptyset$, which is a contradiction. Therefore, we have $x \in V_{\rho+\varepsilon}^1$; that is, we have shown (6.2) for $i = 1$. The case $i = 2$ can be shown analogously.

By (6.2) we find $W_{\rho+\varepsilon}^i \subset V_{\rho+\varepsilon}^i \subset B_i$, and thus $\mu(A_{\rho+\varepsilon}^i \setminus B_i) \leq \mu(A_{\rho+\varepsilon}^i \setminus W_{\rho+\varepsilon}^i)$ for $i = 1, 2$. Conversely, using $\mu(B \setminus A) = \mu(B) - \mu(A \cap B)$ twice, we obtain

$$\mu(B_1 \setminus (A_{\rho+\varepsilon}^1 \cup A_{\rho+\varepsilon}^2)) = \mu(B_1) - \mu(B_1 \cap (A_{\rho+\varepsilon}^1 \cup A_{\rho+\varepsilon}^2))$$

$$\geq \mu(B_1) - \mu(B_1 \cap A_{\rho+\varepsilon}^1) - \mu(B_1 \cap A_{\rho+\varepsilon}^2)$$

$$= \mu(B_1 \setminus A_{\rho+\varepsilon}^1) - \mu(B_1 \cap A_{\rho+\varepsilon}^2).$$

Since $B_1 \cap B_2 = \emptyset$ implies $B_1 \cap A_{\rho+\varepsilon}^2 \subset A_{\rho+\varepsilon}^2 \setminus B_2$, we thus find

$$\mu(B_1 \setminus A_{\rho+\varepsilon}^1) = \mu(B_1 \setminus A_{\rho+\varepsilon}^1) + \mu(A_{\rho+\varepsilon}^1 \setminus B_1)$$

$$\leq \mu(B_1 \setminus (A_{\rho+\varepsilon}^1 \cup A_{\rho+\varepsilon}^2)) + \mu(A_{\rho+\varepsilon}^2 \setminus B_2) + \mu(A_{\rho+\varepsilon}^1 \setminus B_1)$$

$$\leq \mu(B_1 \setminus \{h > \rho^*\}) + \mu(A_{\rho+\varepsilon}^1 \setminus W_{\rho+\varepsilon}^1) + \mu(A_{\rho+\varepsilon}^2 \setminus W_{\rho+\varepsilon}^2),$$

where in the last estimate we also used [25], (A.1.3). Repeating this estimate for $\mu(B_2 \setminus A_{\rho+\varepsilon}^1)$ and using $B_1 \cup B_2 \subset L_{D, \rho} \subset M_{\rho+\varepsilon}^\delta$ yields the assertion. □

**Proof of Theorem 3.1.** Let us fix a $D \in X^n$ with $\|h_{D, \delta} - h_{P, \delta}\| \leq \varepsilon$. By the first estimate of [25], Theorem A.8.1, we see that the probability $P^n$ of such a $D$ is not smaller than $1 - e^{-c}$. In the case of a bounded density
and (3.3), the same holds by the second estimate of [25], Theorem A.8.1, and
\[
\sqrt{\frac{6c_{\text{part}} \| h \|_{\infty} \varsigma + \ln(2c_{\text{part}}) - d \ln \delta}{3 \delta^4 n}} + \left(\frac{2c_{\text{part}} \varsigma}{3 \delta^4 n}\right)^2 + \frac{c_{\text{part}} \varsigma}{3 \delta^4 n}
\]
\leq \sqrt{\frac{6c_{\text{part}} \| h \|_{\infty} \varsigma + \ln(2c_{\text{part}}) - d \ln \delta}{3 \delta^4 n}} + \frac{2c_{\text{part}} \varsigma}{3 \delta^4 n}
\leq \sqrt{\frac{2c_{\text{part}} (1 + \| h \|_{\infty}) (\varsigma + \ln(2c_{\text{part}}) - d \ln \delta)}{\delta^4 n}} + \frac{2c_{\text{part}} (\varsigma + \ln(2c_{\text{part}}) - d \ln \delta)}{3 \delta^4 n},
\]
where we use \(\ln(2c_{\text{part}}) \geq d \ln \delta\). Now, [25], Lemma A.8.2, shows (2.7) for all \(\rho \geq 0\). Let us check that the remaining assumptions of Theorem 2.9 are also satisfied if \(\varepsilon^* \leq (\rho^{**} - \rho^*)/9\). Clearly, we have \(\delta \in (0, \delta_{\text{thick}}], \ v \in (0, \varepsilon^*] \) and \(\psi(\delta) < \tau\). To show \(\tau \leq \tau^*(\varepsilon^*)\) we write
\[
E := \{\varepsilon' \in (0, \rho^{**} - \rho^*) : \tau^*(\varepsilon') \geq \tau\}.
\]
Since we assume \(\varepsilon^* < \infty\), we obtain \(E \neq \emptyset\) by the definition of \(\varepsilon^*\). There thus exists an \(\varepsilon' \in E\) with \(\varepsilon' \leq \inf E + \varepsilon \leq \varepsilon^*\). Using the monotonicity of \(\tau^*\) established in [25], Theorem A.4.2, we then conclude that \(\tau \leq \tau^*(\varepsilon^*) \leq \tau^*(\varepsilon^*)\), and hence all assumptions of Theorem 2.9 are indeed satisfied. \(\square\)

**Proof of Theorem 4.3.** Let us begin by checking the conditions of Theorem 3.1. Obviously, \(\varepsilon\) is chosen this way, and the definition of \(\varepsilon^*\) together with the assumption \(\varepsilon^* \leq (\rho^{**} - \rho^*)/9\) yields
\[
(\tau/\mathcal{L}_{\text{sep}})^{\kappa} \leq \varepsilon^* < \rho^{**} - \rho^*.
\]
By the assumed separation exponent \(\kappa\), we thus find in the case \(\kappa < \infty\) that
\[
\inf\{\bar{\varepsilon} \in (0, \rho^{**} - \rho^*) : \tau^*(\bar{\varepsilon}) \geq \tau\} \leq \inf\{\bar{\varepsilon} \in (0, \rho^{**} - \rho^*) : \mathcal{L}_{\text{sep}}^{\bar{\varepsilon}^{1/\kappa}} \geq \tau\} = (\tau/\mathcal{L}_{\text{sep}})^{\kappa}.
\]
Consequently, (3.4) holds in the case \(\kappa < \infty\). Moreover, in the case \(\kappa = \infty\), (6.3) together with \(\rho^{**} < \infty\) implies \(\tau \leq \mathcal{L}_{\text{sep}}\). In addition, the separation exponent \(\kappa = \infty\) ensures \(\tau^*(\bar{\varepsilon}) \geq \mathcal{L}_{\text{sep}}\) for all \(\bar{\varepsilon} > 0\), and hence we obtain
\[
\varepsilon + \inf\{\bar{\varepsilon} \in (0, \rho^{**} - \rho^*) : \tau^*(\bar{\varepsilon}) \geq \tau\} = \varepsilon \leq \varepsilon^*;
\]
that is, (3.4) is also established in the case \(\kappa = \infty\). Now, applying Theorem 3.1, we see that \(\rho_D^* \in [\rho^* + 2\varepsilon, \rho^* + \varepsilon + 5\varepsilon]\) with probability \(P_n\) not less than \(1 - e^{-\varsigma}\); that is, (4.3) is proved. In addition, the definition of \(\varepsilon^*\) yields
\[
\rho_D^* - \rho^* \leq \varepsilon^* + 5\varepsilon \leq (\tau/\mathcal{L}_{\text{sep}})^{\kappa} + 6\varepsilon,
\]
and (3.3), the same holds by the second estimate of [25], Theorem A.8.1, and
and hence we obtain \((4.4)\). Let us finally show \((4.5)\). To this end, we first observe that Theorem 3.1 ensures
\[
\frac{\tau}{2} \leq \tau - \psi(\delta) < 3\tau^*(\rho_D^* - \rho^* + \varepsilon) \leq 3\tau_{\text{sep}}(\rho_D^* - \rho^* + \varepsilon)^{1/\kappa} < 3\tau_{\text{sep}}^{2^{1/\kappa}}(\rho_D^* - \rho^*)^{1/\kappa},
\]
where in the last step, we use the already established \((4.3)\). By some elementary transformations we conclude that
\[
\frac{1}{2} \left( \frac{\tau}{6\tau_{\text{sep}}} \right)^{\kappa} < \rho_D^* - \rho^*.
\]
and combining this with \(2\varepsilon \leq \rho_D^* - \rho^*\), we obtain the assertion. \(\Box\)

Proof of Corollary 4.4. We first show \((4.7)\) for \(\kappa < \infty\) and sufficiently large \(n\) with the help of Theorem 4.3. To this end, we define \(\varepsilon_n := \varepsilon_n + (\tau_n/\varepsilon_{\text{sep}})^{\kappa}\) for \(n \geq 1\). Since \((\varepsilon_n), (\delta_n)\) and \((\tau_n)\) converge to 0, we then have \(\delta_n \in (0, \delta_{\text{thick}}]\) and \(\varepsilon_n^*(\rho^* - \rho^*)/9\) for all sufficiently large \(n\). Furthermore, our definitions ensure \(\tau_n/\delta_n^* \to \infty\), and hence we have \(\tau_n \geq 6\delta_{\text{thick}}\delta_n^* = 2\varepsilon(\delta_n)\) for all sufficiently large \(n\), too. Before we can apply Theorem 4.3, it thus remains to show \((3.3)\) for sufficiently large \(n\). To this end, we observe that for \(s_n := \ln n\) and \(\xi_n := 2c_{\text{part}}(s_n + \ln(2c_{\text{part}}) - d \ln \delta_n)\), we have
\[
\varepsilon_n' := \sqrt{\frac{1 + \|h\|_{\infty}}{\delta_n^{\kappa n}}} + \frac{\xi_n}{2\delta_n^{\kappa n}} \leq \left( \frac{\ln n}{n} \right)^{\gamma n/(2\gamma n + d)}.
\]
Using \(\varepsilon_n \cdot (\ln n)^{-\gamma n/(2\gamma n + d)} \to \infty\), we then see that \(\varepsilon_n \geq \varepsilon_n'\) for all sufficiently large \(n\). Now, applying Theorem 4.3, namely \((4.4)\), we obtain an \(n_0 \geq 1\) and a constant \(K\) such that \((4.6)\) holds for all \(n \geq n_0\). Moreover, if \(\kappa\) is exact, \((4.5)\) yields a constant \(K\) such that \((4.7)\) holds for all \(n \geq n_0\).

In the case \(\kappa = \infty\), we first observe that \(\varepsilon_n^* := \varepsilon_n + (\tau_n/\varepsilon_{\text{sep}})^{\kappa}\) satisfies \(\varepsilon_n^* = \varepsilon_n\) for all \(n\) with \(\tau_n < \varepsilon_{\text{sep}}\); that is, for all sufficiently large \(n\). Moreover, we have \(\tau_n/\delta_n^* \to \infty\), and, like the case \(\kappa < \infty\), it thus suffices to show \((3.3)\) for sufficiently large \(n\). To this end, we observe that for \(s_n := \ln n\) and \(\varepsilon_n'\) as above, we find that, for all sufficiently large \(n\),
\[
\varepsilon_n' \leq c_2 \left( \frac{\ln n \cdot \sqrt{\ln \ln n}}{n} \right)^{1/2} \leq \varepsilon_n,
\]
where \(c_2\) is a suitable constant independent of \(n\). Consequently, \((4.3)\) and \((4.4)\) yield \((4.7)\) for all sufficiently large \(n\). \(\Box\)

Lemma 6.1. Under the assumptions of Theorem 2.9 we have
\[
\sum_{i=1}^{2} \mu(B_i(D) \Delta A_i^*) \leq 2 \sum_{i=1}^{2} \mu(A_{\rho_D^* + \varepsilon}^{i} \setminus (A_{\rho_D^* + \varepsilon}^{i})^{\delta}) + \mu(M_{\rho_D^* + \varepsilon}^{i} \setminus M_{\rho_D^* - \varepsilon}^{i}) + \mu(\{\rho^* < h < \rho_D^* + \varepsilon\}).
\]
Proof of Lemma 6.1. We will use inequality (2.10) established in Theorem 2.9. To this end, we first observe that [25], (A.1.3), implies
\[ \mu(M_{\rho-\varepsilon}^{+\delta} \setminus \{h > \rho^*\}) = \mu\left(M_{\rho-\varepsilon}^{+\delta} \setminus \bigcup_{\rho' \geq \rho^*} M_{\rho'}\right) \leq \mu(M_{\rho-\varepsilon}^{+\delta} \setminus M_{\rho-\varepsilon}). \]

To bound the remaining terms on the right-hand side of (2.10), we further observe that the disjoint relation \( A \cap B^{+\delta} = (A \cap (B^{+\delta} \setminus B)) \cup (A \cap B) \) applied to \( B := X \setminus A_{\rho+\varepsilon} \) yields
\[
\mu(A_i^s \setminus (A_{\rho+\varepsilon}^i)^{-\delta}) = \mu(A_i^s \cap (X \setminus A_{\rho+\varepsilon}^i)^{+\delta}) \\
= \mu(A_i^s \cap (X \setminus A_{\rho+\varepsilon}^i)^{+\delta} \cap A_{\rho+\varepsilon}^i) + \mu(A_i^s \setminus A_{\rho+\varepsilon}^i) \\
= \mu(A_{\rho+\varepsilon}^i \setminus (A_{\rho+\varepsilon}^i)^{-\delta}) + \mu(A_i^s \setminus A_{\rho+\varepsilon}^i).
\]
Moreover, \( A_{\rho+\varepsilon}^i \subset A_i^s, A_i^s \cap A_2^s = \emptyset \) together with [25], (A.1.2) and (A.1.3), imply
\[
\mu(A_i^1 \setminus A_{\rho+\varepsilon}^1) + \mu(A_2^2 \setminus A_{\rho+\varepsilon}^2) = \mu((A_i^1 \cup A_2^2) \setminus (A_{\rho+\varepsilon}^1 \cup A_{\rho+\varepsilon}^2)) \\
= \mu(\{\rho^* < h < \rho + \varepsilon\}).
\]
Combining all estimates with (2.10), we obtain the assertion. \( \square \)

Proof of Theorem 4.7. Since Assumption R includes the assumptions made in Theorem 4.3, we obtain (4.3) and (4.4). Furthermore, recall that the proofs of Theorems 4.3 and 3.1 show that the probability \( P^n \) of having a dataset \( D \in X^n \) satisfying the assumptions of Theorem 2.9 is not less than \( 1 - e^{-c} \). For such \( D \), Lemma 6.1 is applicable, and hence we obtain
\[
\mu(B_1(D) \triangle A_1^s) + \mu(B_2(D) \triangle A_2^s) \\
\leq \mu(M_{\rho_D-\varepsilon}^{+\delta} \setminus M_{\rho_D-\varepsilon}) + \mu(\{\rho^* < h < \rho_D^* + \varepsilon\}) \\
+ 2\mu(A_1^1 \setminus (A_1^{1+\delta})^{-\delta}) + 2\mu(A_2^2 \setminus (A_2^{2+\delta})^{-\delta}) \\
\leq \mu(M_{\rho_D-\varepsilon}^{+\delta} \setminus M_{\rho_D-\varepsilon}) + \mu(\{0 < h - \rho^* < \rho_D^* - \rho^* + \varepsilon\}) + 4c_{\text{bound}}\delta^\alpha,
\]
where in the second estimate we use that the clusters have an \( \alpha \)-smooth boundary by Assumption R. Moreover, the \( \alpha \)-smooth boundaries also yield
\[
\mu(M_{\rho_D-\varepsilon}^{+\delta} \setminus M_{\rho_D-\varepsilon}) \leq \mu((A_1^1)^{+\delta} \setminus M_{\rho_D-\varepsilon}) + \mu((A_2^2)^{+\delta} \setminus M_{\rho_D-\varepsilon}) \\
\leq \mu((A_1^1)^{+\delta} \setminus A_{\rho_D-\varepsilon}) + \mu((A_2^2)^{+\delta} \setminus A_{\rho_D-\varepsilon}) \\
\leq 2c_{\text{bound}}\delta^\alpha.
\]
Finally, by (4.4) and the flatness exponent $\vartheta$ from Assumption R, we find
\[
\mu(\{0 < h - \rho^* < \rho_D^* - \rho^* + \varepsilon\}) \leq (c_{\text{flat}}(\rho_D^* - \rho^* + \varepsilon))^\vartheta \leq ((\tau/\text{sep})^\kappa + 7\varepsilon)^\vartheta.
\]
Combining these three estimates, we then obtain the assertion. \(\square\)

**Proof of Corollary 4.8.** To apply Theorem 4.7 we check that $\varepsilon_n$, $\delta_n$ and $\tau_n$ satisfy the assumptions of Theorem 4.3 for $\varepsilon_n := \ln n$ and all sufficiently large $n$. To this end, we observe that for $\varepsilon_n := \ln n$ and $\xi_n := 2c_{\text{part}}(\varepsilon_n + \ln(2c_{\text{part}}) - d \ln \delta_n)$, we have
\[
\varepsilon_n' := \frac{(1 + \|h\|\infty)\xi_n}{\delta_n^2} + \frac{\xi_n}{3\delta_n^2} \leq \left(\frac{\ln n}{n}\right)^{\vartheta/(2\vartheta + \vartheta d)} (\ln n)^{-\vartheta d/(4\vartheta + 2\vartheta d)}.
\]
Using $\varepsilon_n \cdot \left(\frac{\ln n}{n}\right)^{-\vartheta/(2\vartheta + \vartheta d)} (\ln n)^{\vartheta d/(4\vartheta + 2\vartheta d)} \to \infty$, we then see that $\varepsilon_n \geq \varepsilon_n'$ for all sufficiently large $n$. Moreover, the remaining conditions on $\varepsilon_n$, $\delta_n$ and $\tau_n$ from Theorem 4.3 are clearly satisfied for all sufficiently large $n$, and hence we can apply Theorem 4.7 for such $n$. This yields
\[
\mu(B_1(D) \triangle A_1^*) + \mu(B_2(D) \triangle A_2^*) \leq 6c_{\text{bound}}\delta_n^\kappa + (c_{\text{flat}}(\tau_n/\text{sep})^\kappa + 7c_{\text{flat}}\varepsilon_n)^\vartheta
\]
with probability $P^n$ not smaller than $1 - 1/n$ for all sufficiently large $n$. Some elementary calculations then show that there is a $K$ with
\[
P^n\left(D : \mu(B_1(D) \triangle A_1^*) + \mu(B_2(D) \triangle A_2^*) \leq K\left(\frac{\ln n \cdot (\ln n)^2}{n}\right)^{\vartheta/(2\vartheta + \vartheta d)}\right)
\geq 1 - \frac{1}{n}
\]
for all sufficiently large $n$. Moreover, since we always have
\[
\mu(B_1(D) \triangle A_1^*) + \mu(B_2(D) \triangle A_2^*) \leq 2\mu(X) < \infty,
\]
it is an easy exercise to suitably increase $K$ such that the desired inequality actually holds for all $n \geq 1$. \(\square\)

**Proof of Theorem 5.1.** First observe that $C^2 \ln(\ln n) \geq 2(1 + \|h\|\infty)$ guarantees that all $\varepsilon_{\delta,n}$ satisfy (3.3) for $\varsigma' := \varsigma + \ln |\Delta|$. Consequently, Theorem 4.3, namely (4.3) and (4.4), yields
\[
P^n(\{D \in X^n : \varepsilon_{\delta,n} < \rho_{D,\delta} - \rho^* \leq (\tau_{\delta,n}/\text{sep})^\kappa + 6\varepsilon_{\delta,n}\}) \geq 1 - e^{-\varsigma - \ln |\Delta|}
\]
for all $\delta \in \Delta$. Applying the union bound, we thus find
\[
P^n(D \in X^n : \varepsilon_{\delta,n} < \rho_{D,\delta} - \rho^* \leq (\tau_{\delta,n}/\text{sep})^\kappa + 6\varepsilon_{\delta,n} \text{ for all } \delta \in \Delta) \geq 1 - e^{-\varsigma}.
\]
Let us now consider a $D \in X^n$ such that $\varepsilon_{\delta,n} < \rho_{D,\delta}^* - \rho^* \leq (\tau_{\delta,n}/L_{\text{sep}})^\kappa + 6\varepsilon_{\delta,n}$ for all $\delta \in \Delta$. Then the definitions of $\rho_{D,\Delta}^*$ and $\varepsilon_{D,\Delta}$ [see (5.3)] imply

$$\rho_{D,\Delta}^* - \rho^* = \min_{\delta \in \Delta} \rho_{D,\delta}^* - \rho^* \in \left( \min_{\delta \in \Delta} \varepsilon_{\delta,n} \right)$$

and $\varepsilon_{D,\Delta} = \varepsilon_{D,\Delta}^* < \rho_{D,\delta}^* - \rho^* = \rho_{D,\Delta}^* - \rho^*$; that is, we have shown the first assertion. To show the remaining assertions, we first observe that a literal repetition of the argument above, in which we only replace the use of (4.3) by that of (4.5), yields

$$P_n(D \in X^n : c_1 \tau_{\delta,n}^\kappa + \varepsilon_{\delta,n} < \rho_{D,\delta}^* - \rho^* \leq c_2 \tau_{\delta,n}^\kappa + 6\varepsilon_{\delta,n} \text{ for all } \delta \in \Delta) \geq 1 - e^{-c}.$$ Using (5.3) we then immediately obtain the second assertion, while considering $\delta = \delta_{\Delta}^*$ gives the third assertion. □

**Proof of Corollary 5.2.** Let us fix an $n \geq 16$. For later use we note that this choice implies $I_n \subset (0, 1]$. Our first goal is to show that we can apply Theorem 5.1 for sufficiently large $n$. To this end, we first observe that $\max_{\delta \in \Delta} \varepsilon_{\delta,n} = (\ln \ln n)^{-d/2}$ for all sufficiently large $n$. Analogously, $\max_{\delta \in \Delta} \ln \ln \ln n \to 0$ implies $\max_{\delta \in \Delta} (\tau_{\delta,n}/L_{\text{sep}})^\kappa \leq (\rho^{**} - \rho^*)/18$ for all sufficiently large $n$, and the definition of $\tau_{\delta,n}$ ensures $\min_{\delta \in \Delta} \tau_{\delta,n} \geq \psi(\delta)$ for all sufficiently large $n$. Let us now show that eventually we also have $\max_{\delta \in \Delta} \varepsilon_{\delta,n} \leq (\rho^{**} - \rho^*)/18$.

To this end, note that the derivative of $g_n : (0, \infty) \to \mathbb{R}$ defined by

$$g_n(\delta) := \frac{\ln(2c_{\text{part}}|\Delta_n|n - d \ln \delta)}{\delta^{d+1}}$$

is given by

$$g'_n(\delta) = -\frac{d(1 + \ln(2c_{\text{part}}|\Delta_n|n - d \ln \delta))}{\delta^{d+1}}$$

and using $c_{\text{part}} \geq 1$, we thus find that $g_n$ is monotonically decreasing on $(0, 1]$ for all $n \geq 1$. In addition, using $|\Delta_n| \leq n$ we obtain

$$g_n(\min I_n) = g_n \left( \left( \frac{\ln n \cdot (\ln \ln n)^2}{n} \right)^{1/d} \right) = \frac{\ln(2c_{\text{part}}|\Delta_n|n + \ln n - \ln \ln n - 2\ln \ln n)}{\ln n \cdot (\ln \ln n)^2} \leq \frac{4 \ln n - \ln \ln n - 2\ln \ln n}{\ln n \cdot (\ln \ln n)^2} \leq \frac{4}{(\ln \ln n)^2}$$

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for all \( n \geq \max\{16, 2c_{\text{part}}\} \), and hence \( g_n(\min I_n) \ln \ln n \to 0 \) for \( n \to \infty \). Since the definition of \( \varepsilon_{\delta,n} \) gives \( \varepsilon_{\delta,n} = C \sqrt{c_{\text{part}} g_n(\delta) \ln \ln n + \varepsilon^2 c_{\text{part}} g_n(\delta)} \), we can thus conclude that

\[
\max_{\delta \in \Delta_n} \varepsilon_{\delta,n} \leq \max_{\delta \in \Delta_n} C \sqrt{c_{\text{part}} g_n(\delta) \ln \ln n + c_{\text{part}} g_n(\min I_n)} \to 0
\]

for \( n \to \infty \). This ensures the desired \( \max_{\delta \in \Delta_n} \varepsilon_{\delta,n} \leq (\rho^{**} - \rho^*)/18 \) for all sufficiently large \( n \). Combining this with our previous estimate, we find

\[
\max_{\delta \in \Delta_n} (\tau_{\delta,n}/L_{\text{sep}})^{\kappa} + \varepsilon_{\delta,n} \leq (\rho^{**} - \rho^*)/9
\]

for all sufficiently large \( n \), and thus we can apply Theorem 5.1 for such \( n \).

Before we proceed, let us now fix an \( n \geq 16 \) and assume that without loss of generality that \( \Delta_n \) is of the form \( \Delta = \{\delta_1, \ldots, \delta_m\} \) with \( \delta_{i-1} < \delta_i \) for all \( i = 2, \ldots, m \). We write \( \delta_0 := \min I_n \) and \( \delta_{m+1} := \max I_n \). Our intermediate goal is to show that

\[
(6.4) \quad \delta_i - \delta_{i-1} \leq 2n^{-1/d}, \quad i = 1, \ldots, m+1.
\]

To this end, we fix an \( i \in \{1, \ldots, m\} \) and write \( \bar{\delta} := (\delta_i + \delta_{i-1})/2 \in I_n \). Since \( \Delta_n \) is an \( n^{-1/d} \)-net of \( I_n \), we then have \( \delta_i - \bar{\delta} \leq n^{-1/d} \) or \( \bar{\delta} - \delta_{i-1} \leq n^{-1/d} \), and from both, (6.4) follows. Moreover, to show (6.4) in the case \( i = m+1 \), we first observe that there exists an \( \delta_i \in \Delta_n \) with \( \delta_i - \delta_m \leq n^{-1/d} \) since \( \Delta_n \) is an \( n^{-1/d} \)-net of \( I_n \). Using our ordering of \( \Delta_n \), we can assume without loss of generality that \( i = m \), which immediately implies (6.4).

We now prove the first assertion in the case \( \kappa < \infty \). To this end, we write

\[
\delta_n^* := \left( \frac{\ln n \cdot \ln \ln n}{n} \right)^{1/(2\gamma \kappa + d)},
\]

where we note that for sufficiently large \( n \) we have \( \delta_n^* \in I_n \). In the following we thus restrict our considerations to such \( n \). Then there exists an index \( i \in \{1, \ldots, m+1\} \) such that \( \delta_{i-1} \leq \delta_n^* \leq \delta_i \), and by (6.4) we conclude that \( \delta_n^* \leq \delta_i \leq \delta_n^* + 2n^{-1/d} \). Clearly, this yields

\[
\min_{\delta \in \Delta_n} \left( c_2 \tau_{\delta,n}^\kappa + 6\varepsilon_{\delta,n} \right) = \min_{\delta \in \Delta_n} \left( c_2 \delta_n^\kappa (\ln \ln n)^\kappa + 6\varepsilon_{\delta,n} \right)
\]

\[
\leq c_2 \delta_i^\kappa (\ln \ln n)^\kappa + 6\varepsilon_{\delta,i,n}
\]

\[
\leq c_2 (\delta_n^* + 2n^{-1/d})^{\gamma \kappa} (\ln \ln n)^\kappa + 6\varepsilon_{\delta_i,n}
\]

\[
\leq 6c_2 \left( \frac{\ln n \cdot (\ln \ln n)^2}{n} \right)^{1/(2\gamma \kappa + d)} + 6\varepsilon_{\delta_i,n}
\]

(6.5)
for all sufficiently large $n$, where $c_2 := \xi_{\text{sep}}^{-\alpha}$ is the constant from Theorem 5.1. Moreover, using $|\Delta_n| \leq n$ and the monotonicity of $g_n$, we further obtain
\[
g_n(\delta_i) \leq g_n(\delta_n^*) = \frac{\ln(2c_{\text{part}}|\Delta_n|n) - d \ln \delta_n^*}{(\delta_n^*)^d} \leq \frac{\ln(2c_{\text{part}}) + 2 \ln n - d \ln \delta_n^*}{(\delta_n^*)^d},
\]
\[
\leq \frac{4 \ln n}{(\delta_n^*)^d},
\]
for all sufficiently large $n$. By the relation between $\varepsilon_{\delta,n}$ and $g_n(\delta)$, we then find
\[
\varepsilon_{\delta_i,n} \leq 2C \sqrt{c_{\text{part}}} \left( \frac{\ln n \cdot \ln \ln n}{n} \right)^{\gamma \kappa/(2\gamma \kappa + \delta)} + 3c_{\text{part}} \left( \frac{\ln n}{n} \right)^{2\gamma \kappa/(2\gamma \kappa + \delta)},
\]
and combining this estimate with (6.5) and Theorem 5.1, we obtain the first assertion in the case $\kappa < \infty$.

Let us now consider the case $\kappa = \infty$. To this end, we fix an $n$ such that
\[
\delta_n^* := \left( \frac{1}{\ln \ln n} \right)^{1/d}
\]
satisfies $(\delta_n^* + 2n^{-1/d})^\gamma \ln \ln n < \xi_{\text{sep}}$, and thus
\[
((\delta_n^* + 2n^{-1/d})^\gamma \ln \ln n)/\xi_{\text{sep}})^\kappa = 0.
\]
Since $\delta_n^* \in I_n$, there also exists an index $i \in \{1, \ldots, m+1\}$ such that $\delta_{i-1} \leq \delta^* \leq \delta_i$, and by (6.4) we again conclude $\delta^* \leq \delta_i \leq \delta^* + 2n^{-1/d}$. Clearly, the latter implies
\[
\min_{\delta \in \Delta_n} ((\tau_{\delta,n}/\xi_{\text{sep}})^\kappa + 6\varepsilon_{\delta,n}) \leq (\delta_i^* \ln \ln n/\xi_{\text{sep}})^\kappa + 6\varepsilon_{\delta_i,n}
\]
\[
\leq ((\delta_n^* + 2n^{-1/d})^\gamma \ln \ln n/\xi_{\text{sep}})^\kappa + 6\varepsilon_{\delta_i,n}
\]
\[
= 6\varepsilon_{\delta_i,n},
\]
by our assumptions on $n$. Analogously to (6.6) we further find, for sufficiently large $n$, that
\[
g_n(\delta_i) \leq g_n(\delta_n^*) \leq \frac{3 \ln n - d \ln \delta_n^*}{(\delta_n^*)^d} \leq \frac{3 \ln n + \ln \ln n}{n(\ln \ln n)^{-1}} \leq \frac{4 \ln n \cdot \ln \ln n}{n},
\]
and by the relation between $\varepsilon_{\delta,n}$ and $g(\delta)$, we then find the assertion with the help of Theorem 5.1.
Let us finally prove the second assertion. To this end we first recall that we have already seen that for sufficiently large \( n \), we can apply Theorem 5.1. Thus it suffices to find a lower bound for the right-hand side of

\[
\min_{\delta \in \Delta_n} (c_1 \tau_{\delta, n}^\kappa + \varepsilon_{\delta, n}) \geq \min\{1, c_1\} \cdot \min_{\delta \in \Delta_n} (\tau_{\delta, n}^\kappa + \varepsilon_{\delta, n}),
\]

where \( c_1 \) is the constant appearing in Theorem 5.1. Now, for \( n \geq 16 \), we have \( I_n \subset (0, 1] \), and thus we find \( \delta \in (0, 1] \) for all \( \delta \in \Delta_n \). For sufficiently large \( n \) this yields

\[
\min_{\delta \in \Delta_n} (\tau_{\delta, n}^\kappa + \varepsilon_{\delta, n}) = \min_{\delta \in \Delta_n} \left( \delta^{\gamma^\kappa} (\ln \ln n)^{\kappa} + C \sqrt{c_{\text{part}} \gamma_n(\delta) \ln \ln n} + 2 \frac{c_{\text{part}} \gamma_n(\delta)}{3} \right) \\
\geq \min_{\delta \in \Delta_n} \left( \delta^{\gamma^\kappa} + C \sqrt{c_{\text{part}} \ln n \cdot \ln \ln n} \right) \\
\geq \min_{\delta \in (0, 1]} \left( \delta^{\gamma^\kappa} + C \sqrt{c_{\text{part}} \ln n \cdot \ln \ln n} \right).
\]

An elementary application of calculus then yields the assertion. □

**Proof of Corollary 5.3.** As in the proof of Corollary 4.8 it suffices to show the assertion for sufficiently large \( n \). Now, we have seen in the proof of Corollary 5.2 that for sufficiently large \( n \), Inequality (5.4) follows from the fact that the procedure satisfies the assumptions of Theorem 5.1 for such \( n \) and \( \varsigma := \ln n \). Consequently, for sufficiently large \( n \), the probability \( P^n \) of having a data set \( D \in X^n \) satisfying both (5.4) and the third inequality of Theorem 5.1 is not less than \( 1 - 1/n \). Let us fix such a \( D \). Then we have

\[
(6.8) \quad c_1 \tau_{D, \Delta}^\kappa + \varepsilon_{D, \Delta} \leq \rho_{D, \Delta}^* - \rho^* \leq K \left( \frac{\ln n \cdot (\ln \ln n)^2}{n} \right)^{\gamma^\kappa/(2\gamma^\kappa+d)}.
\]

Moreover, an elementary estimate yields

\[
c_1 \tau_{D, \Delta}^\kappa + \varepsilon_{D, \Delta} \geq \min\{1/7, c_1 \varsigma_{\text{sep}}^\kappa\} \cdot \left( (\tau_{D, \Delta}/\varsigma_{\text{sep}})^\kappa + 7 \varepsilon_{D, \Delta} \right),
\]

and setting \( c := \min\{1/7, c_1 \varsigma_{\text{sep}}^\kappa\} \), we hence obtain

\[
(6.9) \quad (\tau_{D, \Delta}/\varsigma_{\text{sep}})^\kappa + 7 \varepsilon_{D, \Delta} \leq c^{-1} K \left( \frac{\ln n \cdot (\ln \ln n)^2}{n} \right)^{\gamma^\kappa/(2\gamma^\kappa+d)}.
\]

In addition, for sufficiently large \( n \), inequality (6.8) implies

\[
(6.10) \quad \delta_{D, \Delta}^* \leq \tau_{D, \Delta}^{1/\gamma} \leq (4K)^{1/\gamma^\kappa} (6\varsigma_{\text{sep}})^{1/\gamma} \left( \frac{\ln n \cdot (\ln \ln n)^2}{n} \right)^{1/(2\gamma^\kappa+d)}.
\]
Now we have already seen in the proofs of Theorem 5.1 and Corollary 5.2 that for sufficiently large \( n \), the assumptions on \( \delta, \varepsilon_{\delta,n}, \varepsilon^*_\delta, \tau_\delta, S_\delta := \ln n \) and \( n \) of Theorem 4.3 are satisfied for all \( \delta \in \Delta_n \) simultaneously. We can thus combine (6.9) and (6.10) with Theorem 4.7 to obtain the assertion. \( \square \)

**SUPPLEMENTARY MATERIAL**

Supplement to “Fully adaptive density-based clustering” (DOI: 10.1214/15-AOS1331SUPP; .pdf). We provide two appendices A and B. In Appendix A, several auxiliary results, which are partially taken from [24], are presented, and the assumptions made in the paper are discussed in more detail. In Appendix B, we present a couple of two-dimensional examples that show that the assumptions imposed in the paper are not only met by many discontinuous densities, but also by many continuous densities.

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SUPPLEMENT TO “FULLY ADAPTIVE DENSITY-BASED CLUSTERING”

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In this supplement several auxiliary results, which are partially taken from [9], are presented and the assumptions made in the paper are discussed in more detail. This material is contained in the sections A.1 to A.10. In addition, we present a couple of two-dimensional examples that show that the assumptions imposed in the paper are not only met by many discontinuous densities, but also by many continuous densities. This material is contained in the sections B.1 and B.2.

Appendix A. Remaining Proofs and Additional Material. In this appendix, the auxiliary results from [9] are presented and the assumptions are discussed in more detail than it was possible in the main paper.

A.1. Material Related to Level Sets. In this section we present some additional results from [9] related to the definition of \( M_\rho \).

To begin with, we note that using the definition of the support of a measure it becomes obvious that \( M_\rho \) can be expressed by

\[
M_\rho = \{ x \in X : \mu(\rho(U)) > 0 \text{ for all open neighborhoods } U \text{ of } x \}. \tag{A.1.1}
\]

Furthermore, if \( \text{supp} \mu = X \), we actually have \( M_\rho = X \) for all \( \rho \leq 0 \), but typically we are, of course, interested in the case \( \rho > 0 \), only. The next lemma shows that the sets \( M_\rho \) are ordered in the usual way.

**Lemma A.1.1.** Let \( (X,d) \) be a complete separable metric space, \( \mu \) be a \( \sigma \)-finite measure on \( X \), and \( P \) be a \( \mu \)-absolutely continuous distribution on \( X \). Then, for all \( \rho_1 \leq \rho_2 \), we have

\[ M_{\rho_2} \subset M_{\rho_1}. \]

**Proof of Lemma A.1.1.** We fix an \( x \in M_{\rho_2} \) and an open set \( U \subset X \) with \( x \in U \). Moreover, we fix a \( \mu \)-density \( h \) of \( P \). Then we obtain

\[
\mu_{\rho_1}(U) = \mu(\{ h \geq \rho_1 \} \cap U) \geq \mu(\{ h \geq \rho_2 \} \cap U) = \mu_{\rho_2}(U) > 0,
\]

and hence we obtain \( x \in M_{\rho_1} \) by (A.1.1).

The following lemma describes the relationship between \( M_\rho \) and \( \{ h \geq \rho \} \).
Lemma A.1.2. Let \((X,d)\) be a complete separable metric space, \(\mu\) be a \(\sigma\)-finite measure on \(X\) with \(\text{supp} \mu = X\), and \(P\) be a \(\mu\)-absolutely continuous distribution on \(X\). Then, for all \(\mu\)-densities \(h\) of \(P\) and all \(\rho \in \mathbb{R}\), we have

\[
\{h \geq \rho\} \subset M_\rho \subset \overline{\{h \geq \rho\}}.
\]

If \(h\) is continuous, we even have \(\{h > \rho\} \subset M_\rho \subset \{h \geq \rho\}\) and \(\partial M_\rho \subset \{h = \rho\}\).

Proof of Lemma A.1.2. By definition, \(M_\rho\) is the smallest closed set \(A\) satisfying \(\mu(\{h \geq \rho\} \setminus A) = 0\). Moreover, we obviously have

\[
\mu(\{h \geq \rho\} \setminus \overline{\{h \geq \rho\}}) = 0,
\]

and hence we obtain \(M_\rho \subset \overline{\{h \geq \rho\}}\). To show the other inclusion, we fix an \(x \in \{h \geq \rho\}\) and an open set \(U \subset X\) with \(x \in U\). Then \(\{h \geq \rho\} \cap U\) is open and non-empty, and hence \(\text{supp} \mu = X\) yields

\[
\mu(U) = \mu(\{h \geq \rho\} \cap U) \geq \mu(\{h \geq \rho\} \cap U) > 0.
\]

By (A.1.1) we conclude that \(x \in M_\rho\), that is, we have shown \(\{h \geq \rho\} \subset M_\rho\).

Now assume that \(h\) is continuous. Clearly, we have \(\{h > \rho\} \subset \{h \geq \rho\}\) and since \(\{h > \rho\}\) is open, we conclude that \(\{h > \rho\} \subset \{h \geq \rho\} \subset M_\rho\) by the previously shown inclusion. Moreover, since \(\{h \geq \rho\}\) is closed, we find \(M_\rho \subset \overline{\{h \geq \rho\}} = \{h \geq \rho\}\). Recalling that \(M_\rho\) is closed by definition, we further find \(\partial M_\rho \subset \{h \geq \rho\}\), and thus it remains to show \(\partial M_\rho \subset \{h \leq \rho\}\). Let us assume the converse, i.e., that there exists an \(x \in \partial M_\rho\) such that \(h(x) > \rho\). By the continuity we then find an open neighborhood \(U\) of \(x\) with \(U \subset \{h > \rho\}\). Since \(x \in \partial M_\rho\), we further find an \(y \in U \setminus M_\rho\), while our construction together with the previously shown \(\{h > \rho\} \subset M_\rho\) yields the contradicting statement \(U \setminus M_\rho \subset \{h > \rho\} \setminus M_\rho = \emptyset\).

The next lemma provides some simple sufficient conditions for normality.

Lemma A.1.3. Let \((X,d)\) be a complete separable metric space, \(\mu\) be a \(\sigma\)-finite measure on \(X\) with \(\text{supp} \mu = X\), and \(P\) be a \(\mu\)-absolutely continuous distribution on \(X\). Then the following statements hold:

i) If \(P\) has an upper semi-continuous \(\mu\)-density, then it is upper normal at every level.

ii) If \(P\) has a lower semi-continuous \(\mu\)-density, then it is lower normal at every level.

iii) If, for some \(\rho \geq 0\), \(P\) has a \(\mu\)-density \(h\) such that \(\mu(\partial \{h \geq \rho\}) = 0\), then \(P\) is normal at level \(\rho\).
Proof of Lemma A.1.3. \( i \). Let us fix an upper semi-continuous \( \mu \)-density \( h \) of \( P \). Then \( \{ h \geq \rho \} \) is closed, and hence Lemma A.1.2 shows \( M_\rho \subseteq \{ h \geq \rho \} = \{ h \geq \rho \} \). Thus, \( P \) is upper normal at level \( \rho \).

\( ii \). Let \( h \) be a lower semi-continuous \( \mu \)-density of \( P \). By Lemma A.1.2 we then know \( \{ h > \rho \} = \{ h > \rho \} \subseteq \{ h > \rho \} \subseteq \{ h > \rho \} \). This yields the assertion.

\( iii \). The upper normality follows from (2.3). To see that \( P \) is lower normal, we use the inclusion \( \{ h > \rho \} \setminus \{ h > \rho \} \subseteq \{ h > \rho \} \setminus \{ h > \rho \} = \partial \{ h > \rho \} \) which follows from Lemma A.1.2.

Let us now assume that \( P \) is upper normal at some level \( \rho \). By (2.2) we then immediately see that

\[
(A.1.2) \quad \mu(M_\rho \triangle \{ h \geq \rho \}) = 0
\]

for all \( \mu \)-densities \( h \) of \( P \). In other words, up to \( \mu \)-zero measures, \( M_\rho \) equals the \( \rho \)-level set of all \( \mu \)-densities \( h \) of \( P \). Moreover, if for some \( \rho^* > 0 \) and \( \rho^{**} > \rho^* \), the distribution \( P \) is upper normal at every level \( \rho \in (\rho^*, \rho^{**}] \), then using the monotonicity of the sets \( M_\rho \) and \( \{ h \geq \rho \} \) in \( \rho \) as well as \( (\cup_{i \in I} A_i) \triangle (\cup_{i \in I} B_i) \subseteq \cup_{i \in I} (A_i \triangle B_i) \), we find

\[
(A.1.3) \quad \mu \left( \{ h > \rho^* \} \triangle \bigcup_{\rho > \rho^*} M_\rho \right) \leq \mu \left( \bigcup_{n \in \mathbb{N}} ( \{ h \geq \rho^* + 1/n \} \triangle M_{\rho^* + 1/n} ) \right) = 0
\]

for all \( \mu \)-densities \( h \) of \( P \), and if \( P \) has a continuous density \( h \), we even have \( \bigcup_{\rho > \rho^*} M_\rho = \{ h > \rho^* \} \) by an easy consequence of Lemma A.1.2. Similarly, if \( P \) is lower normal at every level \( \rho \in (\rho^*, \rho^{**}] \), we find

\[
(A.1.4) \quad \mu \left( \{ h > \rho^* \} \setminus \bigcup_{\rho > \rho^*} \tilde{M}_\rho \right) \leq \mu \left( \bigcup_{n \in \mathbb{N}} ( \{ h > \rho^* + 1/n \} \setminus \tilde{M}_{\rho^* + 1/n} ) \right) = 0,
\]

and if in addition, (A.1.3) holds, we obtain \( \mu(\bigcup_{\rho > \rho^*} M_\rho \triangle \bigcup_{\rho > \rho^*} \tilde{M}_\rho) = 0 \).

A.2. Proofs and Material on Connected Components. This section contains the proofs related to Subsection 2.2. In addition, we recall several additional results on connected components from [9].

Lemma A.2.1. Let \( A \subset B \) be two non-empty sets with partitions \( \mathcal{P}(A) \) and \( \mathcal{P}(B) \), respectively. Then the following statements are equivalent:

\( i \) \( \mathcal{P}(A) \) is comparable to \( \mathcal{P}(B) \).

\( ii \) There exists a \( \zeta : \mathcal{P}(A) \rightarrow \mathcal{P}(B) \) such that, for all \( A' \in \mathcal{P}(A) \), we have

\[
(A.2.1) \quad A' \subset \zeta(A').
\]
Moreover, if one these statements are true, the map $\zeta$ is uniquely determined by (A.2.1). We call $\zeta$ the cell relating map (CRM) between $A$ and $B$.

**Proof of Lemma A.2.1.** ii) $\Rightarrow$ i). Trivial.

i) $\Rightarrow$ ii). For $A' \in \mathcal{P}(A)$ we find a $B' \in \mathcal{P}(B)$ such that $A' \subset B'$. Defining $\zeta(A') := B'$ then gives the desired Property (A.2.1).

Finally, assume that ii) is true but $\zeta$ is not unique. Then there exist $A' \in \mathcal{P}(A)$ and $B', B'' \in \mathcal{P}(B)$ with $B' \neq B''$ and both $A' \subset B'$ and $A' \subset B''$. Since $A' \neq \emptyset$, this yields $B' \cap B'' \neq \emptyset$, which in turn implies $B' = B''$ as $\mathcal{P}(B)$ is a partition, i.e. we have found a contradiction. $\square$

**Proof of Lemma 2.4.** Clearly, $\zeta := \zeta_{B,C} \circ \zeta_{A,B}$ maps from $\mathcal{P}(A)$ to $\mathcal{P}(C)$. Moreover, for $A' \in \mathcal{P}(A)$ we have $A' \subset \zeta_{A,B}(A')$ and for $B' := \zeta_{A,B}(A') \in \mathcal{P}(B)$ we have $B' \subset \zeta_{B,C}(B')$. Combining these inclusions we find

$A' \subset \zeta_{A,B}(A') \subset \zeta_{A,B,C}(\zeta_{A,B}(A')) = \zeta_{B,C} \circ \zeta_{A,B}(A') = \zeta(A')$

for all $A' \in \mathcal{P}(A)$. Consequently, $\mathcal{P}(A)$ is comparable to $\mathcal{P}(C)$ and by Lemma A.2.1 we see that $\zeta$ is the CRM $\zeta_{A,C}$, that is $\zeta_{A,C} = \zeta = \zeta_{B,C} \circ \zeta_{A,B}$. $\square$

**Lemma A.2.2.** Let $(X,d)$ be a metric space, $A \subset X$ be a non-empty subset and $\tau > 0$. Then every $\tau$-connected component of $A$ is $\tau$-connected.

**Proof of Lemma A.2.2.** Let $A'$ be a $\tau$-connected component of $A$ and $x, x' \in A'$. Then $x$ and $x'$ are $\tau$-connected in $A$, and hence there exist $x_1, \ldots, x_n \in A$ such that $x_1 = x, x_n = x'$ and $d(x_i, x_{i+1}) < \tau$ for all $i = 1, \ldots, n - 1$. Now, $d(x_1, x_2) < \tau$ shows that $x_1$ and $x_2$ are $\tau$-connected in $A$, and hence they belong to the same $\tau$-connected component, i.e. we have found $x_2 \in A'$. Iterating this argument, we find $x_i \in A'$ for all $i = 1, \ldots, n$. Consequently, $x$ and $x'$ are not only $\tau$-connected in $A$, but also $\tau$-connected in $A'$. This shows that $A'$ is $\tau$-connected. $\square$

**Lemma A.2.3.** Let $(X,d)$ be a metric space and $A \subset B$ be two closed non-empty subsets of $X$ with $|\mathcal{C}(B)| < \infty$. Then $\mathcal{C}(A)$ is comparable to $\mathcal{C}(B)$.

**Proof of Lemma A.2.3.** Let us fix an $A' \in \mathcal{C}(A)$. Since $A \subset B$ and $|\mathcal{C}(B)| < \infty$ there then exist an $m \geq 1$ and mutually distinct $B_1, \ldots, B_m \in \mathcal{C}(B)$ with $A' \subset B_1 \cup \cdots \cup B_m$ and $A' \cap B_i \neq \emptyset$ for all $i = 1, \ldots, m$. Since $A$ and $B$ are closed, $A'$ and the sets $A' \cap B_i$ are also closed. Consequently, the sets $A' \cap B_i$ are also closed in $A'$ with respect to the relative topology of $A'$. Let us now assume that $m > 1$. Then $A' \cap B_1$ and $(A' \cap B_2) \cup \cdots \cup (A' \cap B_m)$ are two disjoint relatively closed non-empty subsets of $A'$ whose union equals...
A'. Consequently A' is not connected, which contradicts $A' \in C(A)$. In other words, we have $m = 1$, that is, $C(A)$ is comparable to $C(B)$. \hfill \qed

**Lemma A.2.4.** Let $(X,d)$ be a metric space, $A \subset X$ be non-empty and $\tau > 0$. Then we have $d(A', A'') \geq \tau$ for all $A', A'' \in C_\tau(A)$ with $A' \neq A''$. Moreover, if $A$ is closed, all $A' \in C_\tau(A)$ are closed, and if $X$ is compact we have $|C_\tau(A)| < \infty$.

**Proof of Lemma A.2.4.** Let $A' \neq A''$ be two $\tau$-connected components of $A$. Then we have $d(x', x'') \geq \tau$ for all $x' \in A'$ and $x'' \in A''$, since otherwise $x'$ and $x''$ would be $\tau$-connected in $A$. Thus, we have $d(A', A'') \geq \tau$, and from the latter and the compactness of $X$, we conclude that $|C_\tau(A)| < \infty$. Finally, let $(x_i) \subset A'$ be a sequence in some component $A' \in C_\tau(A)$ such that $x_i \to x$ for some $x \in X$. Since $A$ is closed, we have $x \in A$, and hence $x \in A''$ for some $A'' \in C_\tau(A)$. By construction we find $d(A', A'') = 0$, and hence we obtain $A' = A''$ by the assertion that has been shown first. \hfill \qed

**Lemma A.2.5.** Let $(X,d)$ be a metric space, $A \subset X$ be a non-empty subset and $\tau > 0$. Then the following statements are equivalent:

i) $A$ is $\tau$-connected.

ii) For all non-empty subsets $A^+$ and $A^-$ of $A$ with $A^+ \cup A^- = A$ and $A^+ \cap A^- = \emptyset$ we have $d(A^+, A^-) < \tau$.

**Proof of Lemma A.2.5.** i) $\Rightarrow$ ii). We fix two subsets $A^+$ and $A^-$ of $A$ with $A^+ \cup A^- = A$ and $A^+ \cap A^- = \emptyset$. Let us further fix two points $x^+ \in A^+$ and $x^- \in A^-$. Since $A$ is $\tau$-connected, there then exist $x_1, \ldots, x_n \in A$ such that $x_1 = x^-$, $x_n = x^+$ and $d(x_i, x_{i+1}) < \tau$ for all $i = 1, \ldots, n - 1$. Then, $x^+ \in A^+$ and $x^- \in A^-$ imply the existence of an $i \in \{1, \ldots, n - 1\}$ with $x_i \in A^-$ and $x_{i+1} \in A^+$. This yields $d(A^+, A^-) \leq d(x_i, x_{i+1}) < \tau$.

ii) $\Rightarrow$ i). Assume that $A$ is not $\tau$-connected, that is $|C_\tau(A)| > 1$. We pick an $A^+ \in C_\tau(A)$ and write $A^- := A \setminus A^+$. Since $|C_\tau(A)| > 1$, both sets are non-empty, and our construction ensures that they are also disjoint and satisfy $A^+ \cup A^- = A$. Moreover, for every $A' \in C_\tau(A)$ with $A' \neq A^+$ we know $d(A^+, A') \geq \tau$ by Lemma A.2.4 and since $A^-$ is the union of such $A'$, we conclude $d(A^+, A^-) \geq \tau$. \hfill \qed

**Corollary A.2.6.** Let $(X,d)$ be a metric space, $A \subset B \subset X$ be non-empty subsets and $\tau > 0$. If $A$ is $\tau$-connected, then there exists exactly one $\tau$-connected component $B'$ of $B$ with $A \cap B' \neq \emptyset$. Moreover, $B'$ is the only $\tau$-connected component $B''$ of $B$ that satisfies $A \subset B''$. \hfill \qed
Proof of Corollary A.2.6. The second assertion is a direct consequence of the first, and hence it suffices to show the first assertion. Let us assume the first is not true. Since \( A \subset B \) there exist \( B', B'' \in \mathcal{C}_\tau(B) \) with \( B' \neq B'' \), \( A \cap B' \neq \emptyset \), and \( A \cap B'' \neq \emptyset \). We write \( A^- := A \cap B' \) and \( A^+ := A \cap (B \setminus B') \). Since \( B'' \subset B \setminus B' \), we obtain \( A^+ \neq \emptyset \), and therefore, Lemma A.2.5 shows \( d(A^-, A^+) < \tau \). Consequently, there exist \( x^- \in A^- \) and \( x^+ \in A^+ \) with \( d(x^+, x^-) < \tau \). Now we obviously have \( x^- \in B' \), and by construction, we also find a \( B''' \in \mathcal{C}_\tau(B) \) with \( x^+ \in B''' \). Our previous inequality then yields \( d(B', B''') < \tau \), while Lemma A.2.4 shows \( d(B', B''') \geq \tau \), that is, we have found a contradiction. \( \square \)

Lemma A.2.7. Let \((X, d)\) be a metric space, \( A \subset B \) be two non-empty subsets of \( X \) and \( \tau > 0 \). Then \( \mathcal{C}_\tau(A) \) is comparable to \( \mathcal{C}_\tau(B) \).

Proof of Lemma A.2.7. For \( A' \in \mathcal{C}_\tau(A) \), Corollary A.2.6 shows that there is exactly \( B' \in \mathcal{C}_\tau(B) \) with \( A' \subset B' \). Thus, \( \mathcal{C}_\tau(A) \) is comparable to \( \mathcal{C}_\tau(B) \). \( \square \)

Lemma A.2.8. Let \((X, d)\) be a metric space, \( A \subset X \) be a non-empty subset and \( \tau > 0 \). Then, for a partition \( A_1, \ldots, A_m \) of \( A \), the following statements are equivalent:

i) \( \mathcal{C}_\tau(A) = \{ A_1, \ldots, A_m \} \).

ii) \( A_i \) is \( \tau \)-connected for all \( i = 1, \ldots, m \), and \( d(A_i, A_j) \geq \tau \) for all \( i \neq j \).

Proof of Lemma A.2.8. i) \( \Rightarrow \) ii). Follows from Lemma A.2.4.

ii) \( \Rightarrow \) i). Let us fix an \( A' \in \mathcal{C}_\tau(A) \) and an \( A_i \) with \( A_i \cap A' \neq \emptyset \). Since \( A_i \) is \( \tau \)-connected and \( A' \in \mathcal{C}_\tau(A) \), Corollary A.2.6 applied to the sets \( A_i \subset A \subset X \) yields \( A_i \subset A' \). Moreover, \( A_1, \ldots, A_m \) is a partition of \( A \), and thus we conclude that

\[
A' = \bigcup_{i \in I} A_i,
\]

where \( I := \{ i : A_i \cap A' \neq \emptyset \} \). Now let us assume that \( |I| \geq 2 \). We fix an \( i_0 \in I \) and write \( A'^+: A_i \) and \( A^- := \bigcup_{i \in I \setminus \{i_0\}} A_i \). Since \( |I| \geq 2 \), we obtain \( A^- \neq \emptyset \), and Lemma A.2.5 thus shows \( d(A^+, A^-) < \tau \). On the other hand, our assumption ensures \( d(A^+, A^-) \geq \tau \), and hence \( |I| \geq 2 \) cannot be true. Consequently, there exists a unique index \( i \) with \( A' = A_i \). \( \square \)

Lemma A.2.9. Let \((X, d)\) be a compact metric space and \( A \subset X \) be a non-empty closed subset. Then the following statements are equivalent:

i) \( A \) is connected.
ii) $A$ is $\tau$-connected for all $\tau > 0$.

Proof of Lemma A.2.9. i) $\Rightarrow$ ii). Assume that $A$ is not $\tau$-connected for some $\tau > 0$. Then, by Lemma A.2.4, there are finitely many $\tau$-connected components $A_1, \ldots, A_m$ of $A$ with $m > 1$. We write $A' := A_1$ and $A'' := A_2 \cup \cdots \cup A_m$. Then $A'$ and $A''$ are non-empty, disjoint and $A' \cup A'' = A$ by construction. Moreover, Lemma A.2.4 shows that $A'$ and $A''$ are closed since $A$ is closed, and hence $A$ cannot be connected.

ii) $\Rightarrow$ i). Let us assume that $A$ is not connected. Then there exist two non-empty closed disjoint subsets of $A$ with $A' \cup A'' = A$. Since $X$ is compact, $A'$ and $A''$ are also compact, and hence $A' \cap A'' = \emptyset$ implies $\tau := d(A', A'') > 0$. Lemma A.2.5 then shows that $A$ is not $\tau$-connected. \hfill $\square$

The next proposition investigates the relation between $C_\tau(A)$ and $C(A)$.

Proposition A.2.10. Let $(X, d)$ be a compact metric space and $A \subset X$ be a non-empty closed subset. Then the following statements hold:

i) For all $\tau > 0$, $C(A)$ is comparable to $C_\tau(A)$ and the CRM $\zeta : C(A) \to C_\tau(A)$ is surjective.

ii) If $|C(A)| < \infty$, we have

$$\tau^*_A := \min \{d(A', A'') : A', A'' \in C(A) \text{ with } A' \neq A''\} > 0,$$

where $\min \emptyset := \infty$. Moreover, for all $\tau \in (0, \tau^*_A] \cap (0, \infty)$, we have $C(A) = C_\tau(A)$ and, for such $\tau$, the CRM $\zeta : C(A) \to C_\tau(A)$ is bijective.

Finally, if $\tau^*_A < \infty$, that is, $|C(A)| > 1$, we have

$$\tau^*_A = \max \{\tau > 0 : C(A) = C_\tau(A)\}.$$

Note that, in general, a closed subset of $A$ may have infinitely many topologically connected components as, e.g., the Cantor set shows. In this case, the second assertion of the lemma above is, in general, no longer true.

Proof of Proposition A.2.10. i). Let $A' \in C(A)$ and $\tau > 0$. Since $A$ is closed, so is $A'$, and hence $A'$ is $\tau$-connected by Lemma A.2.9. Consequently, Corollary A.2.6 shows that there exists an $A'' \in C_\tau(A)$ with $A' \subset A''$, i.e. $C(A)$ is comparable to $C_\tau(A)$. Now we fix an $A'' \in C_\tau(A)$. Then there exists an $x \in A''$, and to this $x$, there exists an $A' \in C(A)$ with $x \in A'$. This yields $A' \cap A'' \neq \emptyset$, and since $A'$ is $\tau$-connected by Lemma A.2.9, Corollary A.2.6 shows $A' \subset A''$, i.e. we obtain $\zeta(A') = A''$.

ii). Let $A_1, \ldots, A_m$ be the topologically connected components of $A$. Then the components are closed, and since $A$ is a closed and thus compact subset of
X, the components are compact, too. This shows \( d(A_i, A_j) > 0 \) for all \( i \neq j \), and consequently we obtain \( \tau^*_A > 0 \). Let us fix a \( \tau \in (0, \tau^*_A] \cap (0, \infty) \). Then, Lemma A.2.9 shows that each \( A_i \) is \( \tau \)-connected, and therefore Lemma A.2.8 together with \( d(A_i, A_j) \geq \tau^*_A \geq \tau \) for all \( i \neq j \) yields \( C_\tau(A) = \{ A_1, \ldots, A_m \} \). Consequently, we have proved \( C(A) = C_\tau(A) \). The bijectivity of \( \zeta \) now follows from its surjectivity. For the proof of the last equation, we define \( \tau^* := \sup \{ \tau > 0 : C(A) = C_\tau(A) \} \). Then we have already seen that \( \tau^*_A \leq \tau^* \). Now suppose that \( \tau^*_A < \tau^* \). Then there exists a \( \tau \in (\tau^*_A, \tau^*) \) with \( C(A) = C_\tau(A) \).

Lemma A.2.11. Let \( (X, d) \) be a compact metric space and \( A \subset B \) be two non-empty closed subsets of \( X \) with \( |C(A)| < \infty \) and \( |C(B)| < \infty \). If the CRM \( \zeta : C(A) \to C(B) \) is injective, then we have \( \tau^*_A \geq \tau^*_B \).

Proof of Lemma A.2.11. Let us fix some \( A', A'' \in C(A) \) with \( A' \neq A'' \). Since \( \zeta \) is injective, we then obtain \( \zeta(A') \neq \zeta(A'') \). Combining this with \( A' \subset \zeta(A') \) and \( A'' \subset \zeta(A'') \), we find

\[
d(A', A'') \geq d(\zeta(A'), \zeta(A'')) \geq \tau^*_B,
\]

where the last inequality follows from the definition of \( \tau^*_B \). Taking the infimum over all \( A' \) and \( A'' \) with \( A' \neq A'' \) yields the assertion.

A.3. Additional Material Related to Tubes around Sets. This section contains additional material on the operations \( A^{+\delta} \) and \( A^{-\delta} \).

Let us begin by noting that in the literature there is another, closely related concept for adding and cutting off \( \delta \)-tubes, which is based on the Minkowski addition. Namely, in generic metric spaces \( (X, d) \), we can define

\[
A^{\oplus \delta} := \{ x \in X : \exists y \in A \text{ with } d(x, y) \leq \delta \}
\]

\[
A^{\ominus \delta} := \{ x \in X : B(x, \delta) \subset A \}
\]

for \( A \subset X \) and \( \delta > 0 \), where \( B(x, \delta) := \{ y \in X : d(x, y) \leq \delta \} \) denotes the closed ball with radius \( \delta \) and center \( x \). Some simple considerations then show \( A^{\ominus (\delta + \epsilon)} \subset A^{-\delta} \subset A^{\ominus \delta} \) and \( A^{\oplus \delta} \subset A^{+\delta} \subset A^{\ominus (\delta + \epsilon)} \) for all \( \epsilon, \delta > 0 \), that is, the operations of both concepts almost coincide. In addition, it is straightforward to check that \( A^{\ominus \delta} = X \setminus (X \setminus A)^{\ominus \delta} \).
Usually, the operations $\oplus \delta$ and $\ominus \delta$ are considered for the special case $X := \mathbb{R}^d$ equipped with the Euclidean norm. In this case, we immediately obtain the more common expressions
\[
A^{\oplus \delta} = \{x + y : x \in A \text{ and } y \in \delta B_{\ell_2^d}\}
\]
\[
A^{\ominus \delta} = \{x \in \mathbb{R}^d : x + \delta B_{\ell_2^d} \subset A\},
\]
where $B_{\ell_2^d}$ denotes the closed unit Euclidean ball at the origin. Note that the latter formulas remain true for sufficiently small $\delta > 0$, if we consider the “relative case” $X \subset \mathbb{R}^d$ and subsets $A \subset X$ satisfying $d(A, \mathbb{R}^d \setminus X) \in (0, \infty)$.

In general, it is cumbersome to determine the exact forms of $A^{+ \delta}$ and $A^{- \delta}$, respectively $A^{\oplus \delta}$ and $A^{\ominus \delta}$ for a given $A$. For a particular class of sets $A \subset \mathbb{R}^2$, Example B.1.1 illustrates this by providing both $A^{\oplus \delta}$ and $A^{\ominus \delta}$.

The next lemma establishes some basic properties of the introduced operations.

**Lemma A.3.1.** Let $(X, d)$ be a metric space and $A, B \subset X$ be two subsets. Then the following statements hold:

1. If $A$ is compact, then $A^{+ \delta} = A^{\oplus \delta}$.
2. We have $d(A, B) \leq d(A^{+ \delta}, B^{+ \delta}) + 2\delta$.
3. We have
   \[
   (A.3.1) \quad \bigcap_{\delta > 0} A^{+ \delta} = \overline{A}.
   \]
4. We have $(A \cup B)^{+ \delta} = A^{+ \delta} \cup B^{+ \delta}$ and $(A \cap B)^{+ \delta} \subset A^{+ \delta} \cap B^{+ \delta}$.
5. We have $A^{- \delta} \cup B^{- \delta} \subset (A \cup B)^{- \delta}$ and, if $d(A, B) > \delta$, we actually have $A^{- \delta} \cup B^{- \delta} = (A \cup B)^{- \delta}$.
6. For $A_1, A_2 \subset X$ with $A_1 \cap A_2 = \emptyset$ and $B_i \subset A_i$ with $d(B_1, B_2) > \delta$, we have
   \[
   (A_1^{- \delta} \setminus B_1^{- \delta}) \cup (A_2^{- \delta} \setminus B_2^{- \delta}) \subset (A_1 \cup A_2)^{- \delta} \setminus (B_1 \cup B_2)^{- \delta},
   \]
   and equality holds, if $d(A_1, A_2) > \delta$.
7. For all $\delta > 0$ and $\epsilon > 0$, we have $A \subset (A^{+ \delta+\epsilon})^{- \delta}$ and $(A^{- \delta-\epsilon})^{+ \delta} \subset A$.
8. For all $\delta > 0$ and $\epsilon > 0$, we have $(\partial A)^{+ \delta} \subset A^{+ \delta+\epsilon} \setminus A^{- \delta-\epsilon}$.

**Proof of Lemma A.3.1.** i). Clearly, it suffices to prove $A^{+ \delta} \subset A^{\oplus \delta}$. To prove this inclusion, we fix an $x \in A^{+ \delta}$. Then there exists a sequence $(x_n) \subset A$ with $d(x, x_n) \leq \delta + 1/n$ for all $n \geq 1$. Since $A$ is compact, we may assume without loss of generality that $(x_n)$ converges to some $x' \in A$. Now we easily obtain the assertion from $d(x, x') \leq d(x, x_n) + d(x_n, x')$. 
Let us fix an \( x \in A^{+\delta} \) and an \( y \in B^{+\delta} \). Then there exist two sequences \( (x_n) \subset A \) and \( (y_n) \subset B \) such that \( d(x, x_n) \leq \delta + 1/n \) and \( d(y, y_n) \leq \delta + 1/n \) for all \( n \geq 1 \). For \( n \geq 1 \), this construction now yields
\[
d(A, B) \leq d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n) \leq d(x, y) + 2\delta + 2/n,
\]
and by first letting \( n \to \infty \) and then taking the infimum over all \( x \in A^{+\delta} \) and \( y \in B^{+\delta} \), we obtain the assertion.

To show the inclusion \( \supset \), we fix an \( n \in A \). Then there exists a sequence \( (x_n) \subset A \) with \( x_n \to x \) for \( n \to \infty \). For \( \delta > 0 \) there then exists an \( n_\delta \) such that \( d(x, x_n) \leq \delta \) for all \( n \geq n_\delta \). This shows \( d(x, A) \leq \delta \), i.e. \( x \in A^{+\delta} \). To show the converse inclusion \( \supset \), we fix an \( x \in X \) that satisfies \( x \in A^{+1/n} \) for all \( n \geq 1 \). Then there exists a sequence \( (x_n) \subset A \) with \( d(x, x_n) \leq 1/n \), and hence we find \( x_n \to x \) for \( n \to \infty \). This shows \( x \in \overline{A} \).

If \( x \in (A \cup B)^{+\delta} \), there exists a sequence \( (x_n) \subset A \cup B \) with \( d(x, x_n) \leq \delta + 1/n \). Without loss of generality we may assume that \( (x_n) \subset A \), which immediately yields \( x \in A^{+\delta} \). The converse inclusion \( A^{+\delta} \cup B^{+\delta} \subset (A \cup B)^{+\delta} \), and the inclusion \( (A \cap B)^{+\delta} \subset A^{+\delta} \cap B^{+\delta} \) are trivial.

The first inclusion follows from part \( iv \) and simple set algebra, namely
\[
A^{-\delta} \cup B^{-\delta} = X \setminus ((X \setminus A)^{+\delta} \cap (X \setminus B)^{+\delta}) \subset X \setminus (((X \setminus A) \cap (X \setminus B))^{+\delta} \cap ((X \setminus B) \cap (X \setminus A))^{+\delta}) \\
= X \setminus (X \setminus (A \cup B))^{+\delta} \\
= (A \cup B)^{-\delta}.
\]
To show the converse inclusion, we fix an \( x \in (A \cup B)^{-\delta} \). Since \( (A \cup B)^{-\delta} \subset A \cup B \), we may assume without loss of generality that \( x \in A \). It then remains to show that \( x \in A^{-\delta} \), that is \( d(x, X \setminus A) > \delta \). Obviously, \( A \cap B = \emptyset \), which follows from \( d(A, B) > \delta \), implies
\[
X \setminus A = ((X \setminus A) \cap (X \setminus B)) \cup ((X \setminus A) \cap B) = (X \setminus (A \cup B)) \cup B,
\]
and hence we obtain \( d(x, X \setminus A) = d(x, X \setminus (A \cup B)) \cup d(x, B) > \delta \wedge \delta = \delta \)
where we used both \( x \in (A \cup B)^{-\delta} \) and \( d(A, B) > \delta \).

\( iv) \). Using the formula \( (A_1 \cup A_2) \setminus (B_1 \cup B_2) = (A_1 \setminus B_1) \cup (A_2 \setminus B_2) \), which easily follows from \( A_i \setminus B_j = A_i \) for \( i \neq j \), we obtain
\[
(A_1^{1-\delta} \setminus B_1^{-\delta}) \cup (A_2^{1-\delta} \setminus B_2^{-\delta}) = (A_1^{1-\delta} \cup A_2^{1-\delta}) \setminus (B_1^{-\delta} \cup B_2^{-\delta}) \\
\subset (A_1 \cup A_2)^{-\delta} \setminus (B_1 \cup B_2)^{-\delta},
\]
where in the last step we used \( iv) \). The second assertion also follows from \( v) \).

\( vii) \). Obviously, \( A \subset (A^{+\delta+\epsilon})^{-\delta} \) is equivalent to \( (X \setminus A)^{+\delta+\epsilon} \cap (X \setminus A)^{+\delta+\epsilon} \subset X \setminus A \). To prove the latter, we fix an \( x \in (X \setminus A)^{+\delta+\epsilon} \). Then there exists a
sequence \((x_n) \subset X \setminus A^{+\delta+\epsilon}\) with \(d(x, x_n) \leq \delta + 1/n\) for all \(n \geq 1\). Moreover, \((x_n) \subset X \setminus A^{+\delta+\epsilon}\) implies \(d(x_n, x') > \delta + \epsilon\) for all \(n \geq 1\) and \(x' \in A\). Now assume that we had \(x \in A\). For an index \(n\) with \(1/n \leq \epsilon\), we would then obtain \(\delta + \epsilon < d(x_n, x) \leq \delta + \epsilon\), and hence \(x \in A\) cannot be true.

To show the second inclusion we fix an \(x \in (A^{-\delta-\epsilon})^+\). Then there exists a sequence \((x_n) \subset A^{-\delta-\epsilon}\) such that \(d(x, x_n) \leq \delta + 1/n\) for all \(n \geq 1\). This time, \(x_n \in A^{-\delta-\epsilon}\) implies \(x_n \notin (X \setminus A)^{+\delta+\epsilon}\), that is \(d(x_n, x') > \delta + \epsilon\) for all \(n \geq 1\) and \(x' \in X \setminus A\). Choosing an \(n\) with \(1/n \leq \epsilon\), we then find \(x \in A\).

\(viii\). We fix an \(x \in (\partial A)^{\oplus \delta}\). By definition, there then exists an \(x' \in \partial A\) with \(d(x, x') \leq \delta\). Moreover, by the definition of the boundary, there exists an \(x'' \in A\) with \(d(x', x'') \leq \epsilon\), and hence we find \(d(x, x'') \leq \delta + \epsilon\), i.e. \(x \in A^{+\delta+\epsilon}\). Since \(\partial A = \partial (X \setminus A)\), the same argument yields \(x \in (X \setminus A)^{+\delta+\epsilon}\), i.e. \(x \notin A^{-\delta-\epsilon}\). Thus, we have shown \((\partial A)^{\oplus \delta} \subset A^{+\delta+\epsilon} \setminus A^{-\delta-\epsilon}\). Using \((\partial A)^{\oplus \delta} \subset (\partial A)^{\oplus (\delta+\epsilon)}\) and a simple change of variables then yields the assertion. \(\square\)

**A.4. Additional Material Related to Persistence.** In this section we recall and prove two results of [9] that extend Theorem 2.7.

We begin with the following lemma, which shows that \(C_\tau(A)\) is persistent in \(C_\tau(A^{+\delta})\), if \(\tau > 0\) and \(\delta > 0\) are sufficiently small.

**Lemma A.4.1.** Let \((X, d)\) be a compact metric space, and \(A \subset X\) be non-empty. Then, for all \(\delta > 0\) and \(\tau > \delta\), the following statements hold:

\begin{enumerate}
  \item [i)] The set \((A')^{+\delta}\) is \(\tau\)-connected for all \(A' \in C_\tau(A)\).
  \item [ii)] The CRM \(\zeta : C_\tau(A) \to C_\tau(A^{+\delta})\) is surjective.
  \item [iii)] If \(A\) is closed, \(|C(A)| < \infty\), and \(\tau \leq \tau_A/3\), then the CRM \(\zeta : C_\tau(A) \to C_\tau(A^{+\delta})\) is bijective and satisfies
    \[
    (A.4.1) \quad \zeta(A') = (A')^{+\delta}, \quad A' \in C_\tau(A).
    \]
\end{enumerate}

**Proof of Lemma A.4.1.** i). Since \(\tau > \delta\), there exist an \(\epsilon > 0\) with \(\delta + \epsilon < \tau\). For \(x \in (A')^{+\delta}\), there thus exists an \(x' \in A'\) with \(d(x, x') \leq \delta + \epsilon < \tau\), i.e. \(x\) and \(x'\) are \(\tau\)-connected. Since \(A'\) is \(\tau\)-connected, it is then easy to show that every pair \(x, x'' \in (A')^{+\delta}\) is \(\tau\)-connected.

ii). Let us fix an \(A' \in C_\tau(A^{+\delta})\) and an \(x \in A'\). For \(n \geq 1\) there then exists an \(x_n \in A\) with \(d(x, x_n) \leq \delta + 1/n\) and since by Lemma A.2.4 there only exist finitely many \(\tau\)-connected components of \(A\), we may assume without loss of generality that there exists an \(A'' \subset C_\tau(A)\) with \(x_n \in A''\) for all \(n \geq 1\). This yields \(d(x, x_n) \leq \delta + 1/n\) for all \(n \geq 1\), and hence \(d(x, x'') \leq \delta\). Consequently, we obtain \(x \in (A'')^{+\delta}\), i.e. we have \((A'')^{+\delta} \cap A' \neq \emptyset\). Since \((A'')^{+\delta} \subset A^{+\delta}\), we then conclude that \((A'')^{+\delta} \subset A'\) by Corollary A.2.6 and part i). Furthermore, we clearly have \(A'' \subset (A'')^{+\delta}\), and hence \(\zeta(A'') = A'\).
iii). Let us first consider the case $|\mathcal{C}(A)| = 1$. In this case, part i) of Proposition A.2.10 shows $|\mathcal{C}_\tau(A)| = 1$, and thus $|\mathcal{C}_\tau(A^+\delta)| = 1$ by the already established part ii). This makes the assertion obvious. 

In the case $|\mathcal{C}(A)| > 1$ we write $A_1, \ldots, A_m$ for the $\tau$-connected components of $A$. By part iv) of Lemma A.3.1 we then obtain

\begin{equation}
A^+\delta = \bigcup_{i=1}^m A_i^+\delta.
\end{equation}

Since $|\mathcal{C}(A)| > 1$, we further have $\tau^*_A < \infty$, and hence part ii) of Proposition A.2.10 yields $\mathcal{C}(A) = \mathcal{C}_\tau(A)$. The definition of $\tau^*_A$ thus gives $d(A_i, A_j) \geq \tau^* \geq 3\tau$ for all $i \neq j$. Our first goal is to show that

\begin{equation}
d(A_i^+\delta, A_j^+\delta) \geq \tau, \quad i \neq j.
\end{equation}

To this end, we fix $i \neq j$ and both an $x_i \in A_i^+\delta$ and an $x_j \in A_j^+\delta$. Now, the compactness of $X$ yields the compactness of $A_i$ and $A_j$ by Lemma A.2.4, and hence part i) of Lemma A.3.1 shows that there exist $x'_i \in A_i$ and $x'_j \in A_j$ with $d(x_i, x'_i) \leq \delta$ and $d(x_j, x'_j) \leq \delta$. This yields

$$3\tau \leq d(x'_i, x'_j) \leq d(x'_i, x_i) + d(x_i, x_j) + d(x_j, x'_j) \leq 2\delta + d(x_i, x_j),$$

and the latter together with $\delta < \tau$ implies (A.4.3).

Now i) showed that each $A_i^+\delta$, $i = 1, \ldots, m$, is $\tau$-connected. Combining this with (A.4.2), (A.4.3), and Lemma A.2.8, we see that $A_1^+\delta, \ldots, A_m^+\delta$ are the $\tau$-connected components of $A^+\delta$. The bijectivity of $\zeta$ then follows from the surjectivity and a cardinality argument, and (A.4.1) is obvious.

The following theorem is an extended version of the statements of Theorem 2.7 that deal with $\mathcal{C}_\tau(M^+\delta)$.

**Theorem A.4.2.** Let $(X,d)$ be a compact metric space, $\mu$ be a finite Borel measure on $X$ and $P$ be a $\mu$-absolutely continuous distribution on $X$ that can be clustered between $\rho^*$ and $\rho^{**}$. Then the function $\tau^*$ defined by (2.6) is monotonically increasing. Moreover, for all $\varepsilon^* \in (0, \rho^{**} - \rho^*)$, $\delta > 0$, $\tau \in (\delta, \tau^*(\varepsilon^*))$, and all $\rho \in [0, \rho^{**}]$, the following statements hold:

i) We have $1 \leq |\mathcal{C}_\tau(M^+\delta)| \leq 2$.

ii) If $\rho \geq \rho^* + \varepsilon^*$, then $|\mathcal{C}_\tau(M^+\delta)| = 2$ and $\mathcal{C}(M^+\delta) \subseteq \mathcal{C}_\tau(M^+\delta)$.

iii) If $|\mathcal{C}_\tau(M^+\delta)| = 2$, then $\rho \geq \rho^*$ and $\mathcal{C}_\tau(M^+\delta) \subseteq \mathcal{C}(M^+\delta)$.

iv) If $\mathcal{C}(M^+\delta) \subseteq \mathcal{C}_\tau(M^+\delta)$ and $|\mathcal{C}_\tau(M^+\delta)| = 1$, then $\rho < \rho^* + \varepsilon^*$.  


Proof of Theorem A.4.2. Let us first show the assertions related to the function \( \tau^* \). To this end, we first observe that for \( \varepsilon \in (0, \rho^{**} - \rho^*) \) we have \( |C(M_{\rho^* + \varepsilon})| = |C(M_{\rho^{**}})| = 2 \) by Definition 2.5. This shows \( \tau^*(\varepsilon) < \infty \).

Let us now fix \( \varepsilon_1, \varepsilon_2 \in (0, \rho^{**} - \rho^*) \) with \( \varepsilon_1 \leq \varepsilon_2 \). Then Definition 2.5 guarantees that both \( M_{\rho^* + \varepsilon_1} \) and \( M_{\rho^* + \varepsilon_2} \) have two topologically connected components and that the CRM \( \zeta : C(M_{\rho^* + \varepsilon_2}) \to C(M_{\rho^* + \varepsilon_1}) \) is bijective. From Lemma A.2.11 we thus obtain

\[
\tau^*(\varepsilon_2) = \frac{1}{3} \tau^* M_{\rho^* + \varepsilon_2} \geq \frac{1}{3} \tau^* M_{\rho^* + \varepsilon_1} = \tau^*(\varepsilon_1).
\]

i). Since \( \emptyset \neq M_\rho \subset M_{\rho^* + \delta} \), we find \( |C_{\tau}(M_{\rho^* + \delta})| \geq 1 \). On the other hand, since \( \tau > \delta \), part ii) of Lemma A.4.1 and part i) of Proposition A.2.10 yield

\[
(A.4.4) \quad |C_{\tau}(M_{\rho^* + \delta})| \leq |C_{\tau}(M_\rho)| \leq |C(M_\rho)| \leq 2.
\]

ii). Let us fix a \( \rho \in [\rho^* + \varepsilon^*, \rho^{**}] \). For \( \varepsilon := \rho - \rho^* \), the monotonicity of \( \tau^* \) then gives \( \tau^*(\varepsilon^*) \leq \tau^*(\varepsilon) \), and hence we obtain

\[
\tau \leq \frac{1}{3} \tau^* M_{\rho^* + \varepsilon^*} \leq \frac{1}{3} \tau^* M_\rho < \infty.
\]

Part ii) of Proposition A.2.10 thus shows that the CRM \( \zeta_\rho : C(M_\rho) \to C_{\tau}(M_\rho) \) is bijective. Furthermore, \( \delta < \tau \leq \tau^* M_\rho / 3 \) together with part iii) of Lemma A.4.1 shows that the CRM \( \zeta_\delta : C_{\tau}(M_\rho) \to C_{\tau}(M_{\rho^* + \delta}) \) is bijective. Consequently, the CRM \( \zeta = \zeta_\delta \circ \zeta_\rho : C(M_\rho) \to C_{\tau}(M_{\rho^* + \delta}) \) is bijective, and from the latter we conclude that \( |C_{\tau}(M_{\rho^* + \delta})| = |C(M_\rho)| = 2 \).

iii). Since \( |C_{\tau}(M_{\rho^* + \delta})| = 2 \), the already established (A.4.4) yields \( |C(M_\rho)| = 2 \), and hence Definition 2.5 implies both \( \rho \geq \rho^* \) and the bijectivity of the CRM \( \zeta^{**} : C(M_{\rho^{**}}) \to C(M_\rho) \). Moreover, for \( \rho^{**} \), the already established part ii) shows that the CRM \( \zeta_M : C_{\tau}(M_{\rho^{**}}) \to C_{\tau}(M_{\rho^* + \delta}) \) is bijective, and the proof of ii) further showed \( C(M_{\rho^{**}}) = C_{\tau}(M_{\rho^* + \delta}) \). Consequently, \( \zeta_M \) equals the CRM \( C(M_{\rho^{**}}) \to C_{\tau}(M_{\rho^* + \delta}) \). In addition, \( \delta < \tau \) together with part ii) of Lemma A.4.1 and part i) of Proposition A.2.10 shows that the CRM \( \zeta_\rho : C(M_\rho) \to C_{\tau}(M_{\rho^* + \delta}) \) is surjective. Now, by Lemma 2.4 these maps commute in the sense of the following diagram:

\[
\begin{array}{ccc}
C(M_{\rho^{**}}) & \xrightarrow{\zeta^{**}} & C(M_\rho) \\
\downarrow \zeta_M & & \downarrow \zeta_\rho \\
C_{\tau}(M_{\rho^* + \delta}) & \xrightarrow{\zeta} & C_{\tau}(M_{\rho^* + \delta})
\end{array}
\]
and consequently, the CRM $\zeta$ is surjective. Since $|C_\tau(M^{+\delta}_\rho)| = |C(M^{**}_\rho)| = 2$ and $|C_\tau(M^{+\delta})| = 2$, we then conclude that $\zeta$ is bijective.

iv). We proceed by contraposition. To this end, we fix an $\rho \in [\rho^* + \epsilon^*, \rho^{**}]$. By the already established part ii) we then find $|C_\tau(M^{+\delta}_\rho)| = 2$, and part iii) thus shows that the CRM $\zeta_M : C_\tau(M^{+\delta}_\rho) \to C_\tau(M^{+\delta})$ is bijective. Moreover, Lemma 2.4 yields the following diagram

\[
\begin{array}{ccc}
\tau + \delta & \Rightarrow & \tau \\
\uparrow & & \uparrow \\
\zeta_M & & \zeta_M \\
\downarrow & & \downarrow \\
\tau - \delta & \Rightarrow & \tau
\end{array}
\]

where $\zeta$, $\zeta_V$, and $\zeta_{V,M}$ are the corresponding CRMs. Now our assumption guarantees that $\zeta$ is bijective, and hence the diagram shows that $\zeta_{V,M} \circ \zeta_V$ is bijective. Consequently, $\zeta_V$ is injective, and from the latter we obtain $2 = |C_\tau(M^{+\delta})| = |C_\tau(M^{+\delta}_\rho)| \leq |C_\tau(M^{+\delta})|$. \hfill $\square$

The next lemma investigates situations in which $C_\tau(A^{-\delta})$ is persistent in $C(A)$. In particular, it shows that if $\tau$ is sufficiently large compared to $\delta$ and $|C_\tau(A^{-\delta})| = |C(A)|$, then we obtain persistence. Informally speaking this means that gluing $\delta$-cuts by $\tau$-connectivity may preserve the component structure.

**Lemma A.4.3.** Let $(X,d)$ be a compact metric space, and $A \subset X$ be non-empty and closed with $|C(A)| < \infty$. We define $\psi^*_A : (0, \infty) \to [0, \infty]$ by

$$
\psi^*_A(\delta) := \sup_{x \in A} d(x, A^{-\delta}), \quad \delta > 0.
$$

Then, for all $\delta > 0$ and all $\tau > 2\psi^*_A(\delta)$, the following statements hold:

i) For all $B' \in C(A)$, there is at most one $A' \in C_\tau(A^{-\delta})$ with $A' \cap B' \neq \emptyset$.

ii) We have $|C_\tau(A^{-\delta})| \leq |C(A)|$.

iii) If $|C_\tau(A^{-\delta})| = |C(A)|$, then $C_\tau(A^{-\delta})$ is persistent in $C(A)$. Moreover, for all $B', B'' \in C(A)$ with $B' \neq B''$ we have

$$
(A.4.5) \quad d(B', B'') \geq \tau - 2\psi^*_A(\delta). \tag{A.4.5}
$$

**Proof of Lemma A.4.3.** i). Let us fix a $\psi > 2\psi^*_A(\delta)$ with $\psi < \tau$ and a $\tau' \in (0, \tau_A)$ such that $\psi + \tau' < \tau$, where $\tau_A$ is the constant defined in Proposition A.2.10. Moreover, we fix a $B' \in C(A)$. By Proposition A.2.10
we then see that $\mathcal{C}(A) = \mathcal{C}_\tau(A)$, and hence $B'$ is $\tau'$-connected. Now let $A_1, \ldots, A_m$ be the $\tau$-connected components of $A^{-\delta}$. Clearly, Lemma A.2.4 yields $d(A_i, A_j) \geq \tau$ for all $i \neq j$. Assume that $i$ is not true, that is, there exist indices $i_0, j_0$ with $i_0 \neq j_0$ such that $A_{i_0} \cap B' \neq \emptyset$ and $A_{j_0} \cap B' \neq \emptyset$. Thus, there exist $x' \in A_{i_0} \cap B'$ and $x'' \in A_{j_0} \cap B'$, and since $B'$ is $\tau'$-connected, there further exist $x_0, \ldots, x_{n+1} \in B' \subset A$ with $x_0 = x', x_{n+1} = x''$ and $d(x_i, x_{i+1}) < \tau'$ for all $i = 0, \ldots, n$. Moreover, our assumptions guarantee $d(x_i, A^{-\delta}) < \psi/2$ for all $i = 0, \ldots, n + 1$. For all $i = 0, \ldots, n + 1$, there thus exists an index $\ell_i$ with

$$d(x_i, A_{\ell_i}) < \psi/2.$$

In addition, we have $x_0 \in A_{i_0}$ and $x_{n+1} \in A_{j_0}$ by construction, and hence we may actually choose $\ell_0 = i_0$ and $\ell_{n+1} = j_0$. Since we assumed $\ell_0 \neq \ell_{n+1}$, there then exists an $i \in \{0, \ldots, n\}$ with $\ell_i \neq \ell_{i+1}$. For this index, our construction now yields

$$d(A_{\ell_i}, A_{\ell_{i+1}}) \leq d(x_i, A_{\ell_i}) + d(x_i, x_{i+1}) + d(x_{i+1}, A_{\ell_{i+1}}) < \psi + \tau' < \tau,$$

which contradicts the earlier established $d(A_{\ell_i}, A_{\ell_{i+1}}) \geq \tau$.

ii). Since $A^{-\delta} \subset A$, there exists, for every $A' \in \mathcal{C}_\tau(A^{-\delta})$, a $B' \in \mathcal{C}(A)$ with $A' \cap B' \neq \emptyset$. We pick one such $B'$ and define $\zeta(A') := B'$. Now part i) shows that $\zeta : \mathcal{C}_\tau(A^{-\delta}) \to \mathcal{C}(A)$ is injective, and hence we find $|\mathcal{C}_\tau(A^{-\delta})| \leq |\mathcal{C}(A)|$.

iii). As mentioned in part ii), we have an injective map $\zeta : \mathcal{C}_\tau(A^{-\delta}) \to \mathcal{C}(A)$ that satisfies

(A.4.6) $A' \cap \zeta(A') \neq \emptyset$, $A' \in \mathcal{C}_\tau(A^{-\delta})$.

Now, $|\mathcal{C}_\tau(A^{-\delta})| = |\mathcal{C}(A)|$ together with the assumed $|\mathcal{C}(A)| < \infty$ implies that $\zeta$ is actually bijective. Let us first show that $\zeta$ is the only map that satisfies (A.4.6). To this end, assume the converse, that is, for some $A' \in \mathcal{C}_\tau(A^{-\delta})$, there exists an $B' \in \mathcal{C}(A)$ with $B' \neq \zeta(A')$ and $A' \cap B' \neq \emptyset$. Since $\zeta$ is bijective, there then exists an $A'' \in \mathcal{C}_\tau(A^{-\delta})$ with $\zeta(A'') = B'$, and hence we have $A'' \cap B' \neq \emptyset$ by (A.4.6). By part i), we conclude that $A' = A''$, which in turn yields $\zeta(A') = \zeta(A'') = B'$. In other words, we have found a contradiction, and hence $\zeta$ is indeed the only map that satisfies (A.4.6).

Let us now show that $\mathcal{C}_\tau(A^{-\delta})$ is persistent in $\mathcal{C}(A)$. Since we assumed $|\mathcal{C}_\tau(A^{-\delta})| = |\mathcal{C}(A)|$, it suffices to prove that the injective map $\zeta : \mathcal{C}_\tau(A^{-\delta}) \to \mathcal{C}(A)$ defined by (A.4.6) is a CRM, i.e. it satisfies

(A.4.7) $A' \subset \zeta(A')$, $A' \in \mathcal{C}_\tau(A^{-\delta})$.

To show (A.4.7), we pick an $A' \in \mathcal{C}_\tau(A^{-\delta})$ and write $B_1, \ldots, B_m$ for the topologically connected components of $A$. Since $A^{-\delta} \subset A$, we then have
A' \subset B_1 \cup \cdots \cup B_n$, where the latter union is disjoint. Now, we have just seen that $\zeta(A') \in \{B_1, \ldots, B_n\}$ is the only component satisfying $A' \cap \zeta(A') \neq \emptyset$, and therefore we can conclude $A' \subset \zeta(A')$.

Finally, let us show (A.4.5). To this end, we first prove that, for all $A' \in C_\tau(A^{-\delta})$ and $x \in \zeta(A')$ we have

\begin{equation}
(A.4.8) \quad d(x, A') \leq \psi^*_A(\delta),
\end{equation}

where $\zeta : C_\tau(A^{-\delta}) \to C(A)$ is the bijective CRM considered above. Let us assume that (A.4.8) is not true, that is, there exist an $A' \in C_\tau(A^{-\delta})$ and an $x \in \zeta(A')$ such that $d(x, A') > \psi^*_A(\delta)$. Since $d(x, A^{-\delta}) \leq \psi^*_A(\delta)$, there further exists an $A'' \in C_\tau(A^{-\delta})$ with $d(x, A'') \leq \psi^*_A(\delta)$. Obviously, this yields $A' \neq A''$. Let us fix a $\tau' \in (0, \tau_A^*)$ with $2\psi^*_A(\delta) + \tau' < \tau$, and an $x' \in A'$. For $B' := \zeta(A')$, we then have $x' \in B'$ by (A.4.7), and our construction guarantees $x \in B'$. Now, the rest of the proof is similar to that of i). Namely, since $B'$ is $\tau'$-connected, there exist $x_0, \ldots, x_{n+1} \in B'$ with $x_0 = x$, $x_{n+1} = x'$ and $d(x_i, x_{i+1}) < \tau'$ for all $i = 0, \ldots, n$. Let $A_1, \ldots, A_m$ be the $\tau'$-connected components of $A^{-\delta}$. Then, for all $i = 0, \ldots, n+1$, there exists an index $\ell_i$ with

$$d(x_i, A_{\ell_i}) \leq \psi^*_A(\delta),$$

where we may choose $A_{\ell_0} = A''$ and $A_{\ell_{n+1}} = A'$. Since $\ell_0 \neq \ell_{n+1}$, there then exists an $i \in \{0, \ldots, n\}$ with $\ell_i \neq \ell_{i+1}$, and our construction yields

$$\tau \leq d(A_{\ell_i}, A_{\ell_{i+1}}) \leq d(x_i, A_{\ell_i}) + d(x_i, x_{i+1}) + d(x_{i+1}, A_{\ell_{i+1}}) < 2\psi^*_A(\delta) + \tau' < \tau.$$

To prove (A.4.5), we again assume the converse, that is, that there exist $B', B'' \in C(A)$ with $B' \neq B''$ and $d(B', B'') < \tau - 2\psi^*_A(\delta)$. Then there exist $x' \in B'$ and $x'' \in B''$ such that $d(x', x'') < \tau - 2\psi^*_A(\delta)$. Now, since $\zeta$ is bijective, there exists $A', A'' \in C_\tau(A^{-\delta})$ with $A' \neq A''$, $B' = \zeta(A')$, and $B'' = \zeta(A'')$. Using (A.4.8), we then obtain

$$\tau \leq d(A', A'') \leq d(x', A') + d(x', x'') + d(x'', A'') < 2\psi^*_A(\delta) + \tau - 2\psi^*_A(\delta) = \tau,$$

i.e. we again have found a contradiction. \(\square\)

The following theorem provides an extended version of the statements of Theorem 2.7 that deal with $C_\tau(M^{-\delta}_\rho)$.

**Theorem A.4.4.** Let Assumption C be satisfied and $\varepsilon^* \in (0, \rho^{**} - \rho^*)$, $\delta \in (0, \delta_{\text{thick}})$, $\tau \in (\psi(\delta), \tau^*(\varepsilon^*))$, and $\rho \in [0, \rho^{**}]$. Then, we have:

i) We have $1 \leq |C_\tau(M^{-\delta}_\rho)| \leq 2$. 


ii) We have \( C_\tau(M_\rho^{-\delta}) \subseteq C_\tau(M_\rho^{+\delta}) \).

iii) If \( |C_\tau(M_\rho^{-\delta})| = 2 \), then \( \rho \geq \rho^* \) and \( C_\tau(M_\rho^{-\delta}) \subseteq C_\tau(M_\rho^{-\delta}) \subseteq C(M_\rho) \).

**Proof of Theorem A.4.4.** i). We first observe that \( \delta \leq \delta_{\text{thick}} \) implies

\[
\sup_{x \in M_\rho} d(x, M_\rho^{-\delta}) = \psi_{M_\rho}^*(\delta) \leq c_{\text{thick}} \delta^7 < \infty,
\]

and thus \( M_\rho^{-\delta} \neq \emptyset \), i.e. \( |C_\tau(M_\rho^{-\delta})| \geq 1 \). Conversely, we have \( |C_\tau(M_\rho^{-\delta})| \leq |C(M_\rho)| \leq 2 \), where the first inequality was established in part ii) of Lemma A.4.3 and the second is ensured by Definition 2.5.

ii). The monotonicity of \( \tau^* \) established in Theorem A.4.2 yields \( \delta < \psi(\delta) < \tau \leq \tau^*(\varepsilon^\tau) \leq \tau_{\ast M_\rho}^* / 3 \). By part iii) of Lemma A.4.1 we then conclude that the CRM \( C_\tau(M_\rho^{-\delta}) \rightarrow C_\tau(M_\rho^{+\delta}) \) is bijective, and part ii) of Theorem A.4.2 shows \( |C_\tau(M_\rho^{-\delta})| = |C_\tau(M_\rho^{+\delta})| = 2 \). By Lemma 2.4 it thus suffices to show that the CRM \( \zeta : C_\tau(M_\rho^{+\delta}) \rightarrow C_\tau(M_\rho^{-\delta}) \) is bijective. Furthermore, if \( |C_\tau(M_\rho^{+\delta})| = 1 \), this map is automatically injective, and if \( |C_\tau(M_\rho^{-\delta})| = 2 \), the injectivity follows from the surjectivity and the above proven \( |C_\tau(M_\rho^{-\delta})| = 2 \). Consequently, it actually suffices to show that \( \zeta \) is surjective. To this end, we fix a \( B' \in C_\tau(M_\rho^{+\delta}) \) and an \( x \in B' \). Then our assumption ensures \( d(x, M_\rho^{-\delta}) < \psi(\delta) \), and hence there exists an \( A' \in C_\tau(M_\rho^{-\delta}) \) with \( d(x, A') < \psi(\delta) \). Therefore, \( \psi(\delta) < \tau \) implies that \( x \) and \( A' \) are \( \tau \)-connected, which yields \( x \in A' \). In other words, we have shown \( A' \cap B' \neq \emptyset \). By Lemma A.2.6 and the definition of \( \zeta \), we conclude that \( \zeta(A') = B' \).

iii). We have \( 2 = |C_\tau(M_\rho^{-\delta})| \leq |C(M_\rho)| \leq 2 \), where the first inequality was shown in part ii) of Lemma A.4.3 and the second is guaranteed by Definition 2.5. We conclude that \( |C(M_\rho)| = 2 \), and hence Definition 2.5 ensures both \( \rho \geq \rho^* \) and the bijectivity of the CRM \( \zeta_{\text{top}} : C(M_\rho^{+\delta}) \rightarrow C(M_\rho) \).

Furthermore, \( |C_\tau(M_\rho^{-\delta})| = |C(M_\rho)| \), which has been shown above, together with part iii) of Lemma A.4.3 yields a bijective CRM \( \zeta_\rho : C_\tau(M_\rho^{-\delta}) \rightarrow C(M_\rho) \), i.e. the second persistence \( C_\tau(M_\rho^{-\delta}) \subseteq C(M_\rho) \) is shown. Moreover, part ii) of Theorem A.4.2 shows \( |C_\tau(M_\rho^{+\delta})| = 2 \), and hence the already established bijectivity of \( \zeta^{**} : C_\tau(M_\rho^{+\delta}) \rightarrow C_\tau(M_\rho^{-\delta}) \) gives \( |C_\tau(M_\rho^{-\delta})| = |C_\tau(M_\rho^{+\delta})| = 2 = |C(M_\rho^{+\delta})| \). Consequently, part iii) of Lemma A.4.3 yields a bijective CRM \( \zeta_{\rho^{**}} : C_\tau(M_\rho^{+\delta}) \rightarrow C(M_\rho^{+\delta}) \). Then the CRM \( \zeta : C_\tau(M_\rho^{-\delta}) \rightarrow C(M_\rho^{-\delta}) \) enjoys the following diagram
whose commutativity follows from Lemma 2.4. Then the bijectivity of \( \zeta^{**} \), \( \zeta_{\text{top}} \), and \( \zeta_{\rho} \) yields the bijectivity of \( \zeta \), which completes the proof. \(\square\)

A.5. Additional Material Related to Thickness. In this section we discuss some aspects related to the thickness assumption introduced in Definition 2.6.

To this end, let \((X, d)\) be an arbitrary metric spaces and \(A \subset X\). We then define the function \(\psi^*_A : (0, \infty) \to [0, \infty]\) by

\[
\psi^*_A(\delta) := \sup_{x \in M_\rho} d(x, A^{-\delta}), \quad \delta > 0.
\]

Obviously, \(\psi^*_M\) coincides with the left-hand side of (2.5).

Our first observation is that the definition of \(\psi^*_A\) immediately yields \(A \subset (A^{-\delta} + \psi^*_A(\delta))\) for all \(\delta > 0\) with \(\psi^*_A(\delta) < \infty\), and it is also straightforward to see that \(\psi^*_A(\delta)\) is the smallest \(\psi > 0\), for which this inclusion holds, that is

\[
\psi^*_A(\delta) = \min\{\psi \geq 0 : A \subset (A^{-\delta})^+\}
\]

for all \(\delta > 0\). In other words, \(\psi^*_A(\delta)\) gives the size of the smallest tube needed to recover a superset of \(A\) from \(A^{-\delta}\). In particular, if \(\delta\) is too large, that is \(A^{-\delta} = \emptyset\), we obviously have \(\psi^*_A(\delta) = \infty\) and no recovery is possible.

Intuitively it is not surprising that \(\psi^*_A\) grows at least linearly, that is

\[(A.5.1) \quad \psi^*_A(\delta) \geq \delta\]

for all \(\delta > 0\) provided that \(d(A, X \setminus A) = 0\). Indeed, \(\psi^*_A(\delta) < \delta\) for some \(\delta > 0\) gives us an \(\epsilon > 0\) such that \(d(x, A^{-\delta}) < \delta - \epsilon\) for all \(x \in A\). Since \(d(A, X \setminus A) = 0\) there then exists an \(x \in A\) with \(d(x, X \setminus A) < \epsilon\), and for this \(x\) there exists an \(x' \in A^{-\delta}\) with \(d(x, x') < \delta - \epsilon\). Now the definition of \(A^{-\delta}\) gives \(d(x', X \setminus A) > \delta\), and hence we find a contradiction by

\[
\delta < d(x', X \setminus A) \leq d(x', x) + d(x, X \setminus A) < \delta.
\]

For generic sets \(A\), the function \(\psi^*_A\) is usually hard to bound, but for some classes of sets, \(\psi^*_A\) can be computed precisely. For example, for an interval
$I = [a, b]$, we have $\psi^*_A(\delta) = \delta$ for all $\delta \in (0, (b - a)/2]$, and $\psi^*_A(\delta) = \infty$, otherwise. Clearly, this example can be extended to finite unions of such intervals and for intervals that are not closed, the only difference occurs at $\delta = (b - a)/2$. In higher dimensions, an interesting class of sets $A$ with linear behavior of $\psi^*_A$ is described by Serra’s model, see [7, p. 144], that consist of all compact sets $A \subset \mathbb{R}^d$ for which there is a $\delta_0 > 0$ with

$$A = (A^{\oplus \delta_0})^{\oplus \delta_0} = (A^{\oplus \delta_0})^{\oplus \delta_0}.$$ 

If, in addition, $A$ is path-connected, then [11, Theorem 1] shows that this relation also holds for all $\delta \in (0, \delta_0]$. In this case, we then obtain

$$A = (A^{\oplus (\delta + \epsilon)})^{\oplus (\delta + \epsilon)} \subset (A^{\oplus (\delta + \epsilon)})^{+ \delta + \epsilon} \subset (A^{- \delta})^{+ \delta + \epsilon}$$

for all $\delta \in (0, \delta_0)$ and $0 < \epsilon \leq \delta_0 - \delta$. In other words, we have $\psi^*_A(\delta) \leq \delta + \epsilon$, and letting $\epsilon \to 0$, we thus conclude $\psi^*_A(\delta) = \delta$ for all $\delta \in (0, \delta_0)$. With the help of Lemma A.3.1, it is not hard to see that this result generalizes to finite unions of compact, path-connected sets, which has already been observed in [11]. Finally, note that [11, Theorem 1] also provides some useful characterizations of (path-connected) compact sets belonging to Serra’s model.

In a nutshell, these are the sets whose boundary is a $(d - 1)$-dimensional sub-manifold of $\mathbb{R}^d$ with outward pointing unit normal vectors satisfying a Lipschitz condition.

Fortunately, our analysis does not require the exact form of $\psi^*_A$, but only its asymptotic behavior for $\delta \to 0$. Therefore, it is interesting to note that $\psi^*_A$ is also asymptotically invariant against bi-Lipschitz transformations. To be more precise, let $(X, d)$ and $(Y, e)$ be two metric spaces and $I : X \to Y$ be a bijective map for which there exists a constant $C > 0$ such that

$$C^{-1}e(I(x), I(x')) \leq d(x, x') \leq Ce(I(x), I(x'))$$

for all $x, x' \in X$. For $A \subset X$ and $\delta > 0$, we then have $I(A^{+ \delta/C}) \subset (I(A))^{+ \delta} \subset I(A^{C \delta})$, which in turn implies

$$C^{-1}\psi^*_A(\delta/C) \leq \psi^*_{I(A)}(\delta) \leq C\psi^*_A(C\delta)$$

for all $\delta > 0$. In particular, we have $\psi^*_A(\delta) \leq \delta^\gamma$ for some $\gamma \in (0, 1]$ if and only if $\psi^*_{I(A)}(\delta) \leq \delta^\gamma$.

Last but not least we like to mention that based on the sets $A \subset \mathbb{R}^2$ considered in Example B.1.1, Example B.1.2 estimates $\psi^*_A$. In particular, this example provides various sets $A$ with $\psi^*_A(\delta) \sim \delta$ that do not belong to Serra’s model, and this class of sets can be further expanded by using bi-Lipschitz transformations as discussed above.
Now consider Definition 2.6, which excludes thin cusps and bridges, where the thinness and length of both is controlled by $\gamma$. Such assumptions have been widely used in the literature on level set estimation and density-based clustering. For example, a basically identical assumption has been made in [8] for the exponent $\gamma = 1$, which can be taken, if, e.g., the level sets belong to Serra’s model. Moreover, level sets belonging to Serra’s model have been investigated in [10]. In particular, [10, Theorem 2] shows that most level sets of a $C^1$-density with Lipschitz continuous gradient belong to Serra’s model. Unfortunately, however, levels at which the density has a saddle point are excluded in this theorem, and some other elementary sets such as cubes in $\mathbb{R}^d$ do not belong to Serra’s model, either. For this reason, we allow constants $c_{\text{thick}} > 1$ in Definition 2.6. Moreover, the exponent $\gamma < 1$ is allowed to provide more flexibility in situations, in which very thin bridges are expected. However, based on the discussion on $\psi^*_A$ as well as the examples provided in Section B.2, we strongly believe, that in most cases assuming $\gamma = 1$ is reasonable. With the help of the discussion on $\psi^*_A$ it is also easy to see that we have $M_\rho \subset (M^{-\delta}) + \psi(\delta)/2$ for all $\delta \in (0, \delta_{\text{thick}}]$ and all $\rho \in (0, \rho^{\ast\ast})$. In addition, it becomes clear that exponents $\gamma > 1$ are impossible as soon as $d(M_\rho, X \setminus M_\rho) = 0$ for some $\rho \in (0, \rho^{\ast\ast})$. Finally, recall that a less geometric assumption excluding thin features has been used by various authors, see e.g. [3, 2, 6] and the references therein, and an overview of these and similar assumptions can be found in [1].

Understanding (2.5) in the one-dimensional case is very simple. Indeed, if $X \subset \mathbb{R}$ is an interval and $P$ can be topologically clustered between $\rho^{\ast}$ and $\rho^{\ast\ast}$, then, for all $\rho \in [0, \rho^{\ast\ast}]$, the level set $M_\rho$ consists of either one or two closed intervals. Using this, the discussion on $\psi^*_A$ shows that $P$ actually has thick levels of order $\gamma = 1$ up to the level $\rho^{\ast\ast}$. Moreover, a possible thickness function is $\psi(\delta) = 3\delta$ for all $\delta \in (0, \delta_{\text{thick}}]$, where $\delta_{\text{thick}}$ equals the smaller radius of the two intervals at level $\rho^{\ast\ast}$.

Finally, using the discussion on $\psi^*_A$ it is not hard to construct distributions with discontinuous densities that have thick levels of order, e.g. $\gamma = 1$. For continuous densities, however, this task is significantly harder due to the above mentioned saddle point effects at the critical level $\rho^{\ast}$. Therefore, we have added Example B.2.1, which provides a large class of such densities in the case $X \subset \mathbb{R}^2$.

**A.6. Proofs and Results Related to Algorithm 2.1.** The main goals of this section is to prove Theorem 2.8 and to provide background material from [9] for the proof of Theorem 2.9.
Lemma A.6.1. Let $(X,d)$ be a compact metric space and $\mu$ be a finite Borel measure on $X$ with $\text{supp}\mu = X$. Moreover, let $P$ be a $\mu$-absolutely continuous distribution on $X$, and $(L_\rho)_{\rho \geq 0}$ be a decreasing family of sets $L_\rho \subset X$ such that

$$M_{\rho+\delta}^- \subset L_\rho \subset M_{\rho-\varepsilon}^+$$

for some fixed $\delta > 0$, $\varepsilon > 0$, and all $\rho \geq 0$. For some fixed $\rho \geq 0$ and $\tau > 0$, let $\zeta : C_\tau(M_{\rho+\varepsilon}^-) \to C_\tau(L_\rho)$ be the CRM. Then we have:

1) For all $A' \in C_\tau(M_{\rho+\varepsilon}^-)$ with $A' \cap M_{\rho+3\varepsilon}^- \not= \emptyset$ we have $\zeta(A') \cap L_{\rho+2\varepsilon} \not= \emptyset$.

2) For all $B' \in C_\tau(L_\rho)$ with $B' \not\in \zeta(C_\tau(M_{\rho+\varepsilon}^-))$, we have

(A.6.1) $$B' \subset (X \setminus M_{\rho+\varepsilon})^+ \cap M_{\rho-\varepsilon}^+$$

(A.6.2) $$B' \cap L_{\rho+2\varepsilon} \subset (X \setminus M_{\rho+\varepsilon})^+ \cap M_{\rho+\varepsilon}^+.$$

Proof of Lemma A.6.1. 1). Using the CRM property $A' \subset \zeta(A')$ and the inclusion $M_{\rho+3\varepsilon}^- \subset L_{\rho+2\varepsilon}$, we obtain

$$\emptyset \not= A' \cap M_{\rho+3\varepsilon}^- \subset \zeta(A') \cap L_{\rho+2\varepsilon}.$$

2). We fix a $B' \in C_\tau(L_\rho) \setminus \zeta(C_\tau(M_{\rho+\varepsilon}^-))$. For $x \in B'$ we then have

$$x \not\in \bigcup_{A' \in C_\tau(M_{\rho+\varepsilon}^-)} \zeta(A'),$$

and hence the CRM property yields

$$x \not\in \bigcup_{A' \in C_\tau(M_{\rho+\varepsilon}^-)} A' = M_{\rho+\varepsilon}^-.$$

This shows $x \in (X \setminus M_{\rho+\varepsilon})^+$, i.e. we have proved $B' \subset (X \setminus M_{\rho+\varepsilon})^+$. Now, (A.6.1) follows from $B' \subset L_{\rho} \subset M_{\rho-\varepsilon}^+$, and (A.6.2) follows from $B' \cap L_{\rho+2\varepsilon} \subset L_{\rho+2\varepsilon} \subset M_{\rho+\varepsilon}^+$. \qed

Proof of Theorem 2.8. We first establish the following disjoint union:

$$C_\tau(L_\rho) = \zeta(C_\tau(M_{\rho+\varepsilon}^-)) \cup \left\{ B' \in C_\tau(L_\rho) \setminus \zeta(C_\tau(M_{\rho+\varepsilon}^-)) : B' \cap L_{\rho+2\varepsilon} = \emptyset \right\}$$

(A.6.3) $$\cup \left\{ B' \in C_\tau(L_\rho) : B' \cap L_{\rho+2\varepsilon} = \emptyset \right\}.$$

We begin by showing the auxiliary result

(A.6.4) $$A' \cap M_{\rho+3\varepsilon}^- \not= \emptyset, \quad A' \in C_\tau(M_{\rho+\varepsilon}^-).$$
To this end, we observe that $i)$ and $ii)$ of Theorem A.4.2 yield $|C_\tau(M_{\rho+\delta})| = 2$, and hence part $ii)$ of Theorem A.4.4 implies $|C_\tau(M^{-\delta}_{\rho-\delta})| = 2$. Let $W'$ and $W''$ be the two $\tau$-connected components of $M^{-\delta}_{\rho-\delta}$. We first assume that $M^{-\delta}_{\rho+\delta}$ has exactly one $\tau$-connected component $A'$, i.e. $A' = M^{-\delta}_{\rho+\epsilon}$. Then $\rho + 3\epsilon \leq \rho^{**}$ and $\rho + \epsilon \leq \rho + 3\epsilon$ imply
\[ \emptyset \neq M^{-\delta}_{\rho+\delta} \subset M^{-\delta}_{\rho+\epsilon} \cap M^{-\delta}_{\rho+3\epsilon} = A' \cap M^{-\delta}_{\rho+3\epsilon}, \]
i.e. we have shown (A.6.4). Let us now assume that $M^{-\delta}_{\rho+\epsilon}$ has more than one $\tau$-component. Then it has exactly two such components $A'$ and $A''$ by $\rho + \epsilon < \rho^{**}$ and part $i)$ of Theorem A.4.4. By part $iii)$ of Theorem A.4.4 we may then assume without loss of generality that we have $W' \subset A'$ and $W'' \subset A''$. Since $\rho + 3\epsilon \leq \rho^{**}$ implies $M^{-\delta}_{\rho+3\epsilon} \subset M^{-\delta}_{\rho+\epsilon}$, these inclusions yield $\emptyset \neq W' = W' \cap M^{-\delta}_{\rho+\epsilon} \subset A' \cap M^{-\delta}_{\rho+3\epsilon}$ and $\emptyset \neq W'' = W'' \cap M^{-\delta}_{\rho+\epsilon} \subset A'' \cap M^{-\delta}_{\rho+3\epsilon}$. Consequently, we have proved (A.6.4) in this case, too.

Now, from (A.6.4) we conclude by part $i)$ of Lemma A.6.1 that $B' \cap L_{\rho+2\epsilon} \neq \emptyset$ for all $B' \in \zeta(C_\tau(M^{-\delta}_{\rho+\epsilon}))$. This yields
\begin{align*}
\{ B' \in C_\tau(L_\rho) \setminus \zeta(C_\tau(M^{-\delta}_{\rho+\epsilon})) : B' \cap L_{\rho+2\epsilon} = \emptyset \} \\
= \{ B' \in C_\tau(L_\rho) : B' \cap L_{\rho+2\epsilon} = \emptyset \},
\end{align*}
which in turn implies (A.6.3).

Let us now show (2.8). Clearly, by (A.6.3) it remains to show
\[ B' \cap L_{\rho+2\epsilon} = \emptyset, \]
for all $B' \in C_\tau(L_\rho) \setminus \zeta(C_\tau(M^{-\delta}_{\rho+\epsilon}))$. Let us assume the converse, that is, there exists a $B' \in C_\tau(L_\rho) \setminus \zeta(C_\tau(M^{-\delta}_{\rho+\epsilon}))$ with $B' \cap L_{\rho+2\epsilon} \neq \emptyset$. Since $L_{\rho+2\epsilon} \subset M^{-\delta}_{\rho+\epsilon}$, there then exists an $x \in B' \cap M^{-\delta}_{\rho+\epsilon}$. By part $i)$ of Lemma A.3.1 this gives an $x' \in M_{\rho+\epsilon}$ with $d(x, x') \leq \delta$, and hence we obtain
\[ d(x', M^{-\delta}_{\rho+\epsilon}) \leq \psi^{-1}_{M_{\rho+\epsilon}}(\delta) \leq c_{\text{thick}}\delta^\gamma < 2c_{\text{thick}}\delta^\gamma. \]
From this inequality we conclude that there exists an $x'' \in M^{-\delta}_{\rho+\epsilon}$ satisfying $d(x', x'') < 2c_{\text{thick}}\delta^\gamma$. Let $A'' \in C_\tau(M^{-\delta}_{\rho+\epsilon})$ be the unique $\tau$-connected component satisfying $x'' \in A''$. The CRM property then yields $x'' \in A'' \subset \zeta(A'') =: B''$, and thus, using $c \geq 1$, we find
\[ d(B', B'') \leq d(x, x'') \leq d(x, x') + d(x', x'') < \delta + 2c_{\text{thick}}\delta^\gamma \leq 3c_{\text{thick}}\delta^\gamma < \tau. \]
However, since $B' \not\in \zeta(C_\tau(M^{-\delta}_{\rho+\epsilon}))$ and $B'' \in \zeta(C_\tau(M^{-\delta}_{\rho+\epsilon}))$ we obtain $B' \neq B''$, and hence Lemma A.2.4 yields $d(B', B'') \geq \tau$. \qed
**Theorem A.6.2.** Let Assumption C be satisfied. Furthermore, let \( \varepsilon^* \leq (\rho^{**} - \rho^*)/9 \), \( \delta \in (0, \delta_{\text{thick}}] \), \( \tau \in (\psi(\delta), \tau^*(\varepsilon^*)) \), and \( \varepsilon \in (0, \varepsilon^*) \). In addition, let \( D \) be a data set and \((L_{D,\rho})_{\rho \geq 0}\) be a decreasing family satisfying

\[
M_{\rho + \varepsilon}^{-\delta} \subset L_{D,\rho} \subset M_{\rho - \varepsilon}^{+\delta}
\]

for all \( \rho \geq 0 \). Furthermore, assume that Algorithm 2.1 receives the parameters \( \tau, \varepsilon \), and \((L_{D,\rho})_{\rho \geq 0}\). Then, the following statements are true:

i) The returned level \( \rho_D^* \) satisfies \( \rho_D^* \in [\rho^* + 2\varepsilon, \rho^* + \varepsilon^* + 5\varepsilon] \).

ii) We have \( |C_\tau(M_{\rho_D^* + \varepsilon}^{-\delta})| = 2 \) and the CRM \( \zeta : C_\tau(M_{\rho_D^* + \varepsilon}^{-\delta}) \to C_\tau(L_{D,\rho_D^*}) \) is injective.

iii) Algorithm 2.1 returns the two \( \tau \)-connected components of \( \zeta(C_\tau(M_{\rho_D^* + \varepsilon}^{-\delta})) \).

iv) There exist CRMs \( \zeta_{\rho^{***}} : C_\tau(M_{\rho_D^* + \varepsilon}^{-\delta}) \to C(M_{\rho^{***}}) \) and \( \zeta_{\rho_D^* + \varepsilon} : C_\tau(M_{\rho_D^* + \varepsilon}^{-\delta}) \to C(M_{\rho_D^* + \varepsilon}) \) such that we have a commutative diagram of bijective CRMs:

\[
\begin{array}{ccc}
C_\tau(M_{\rho_D^* + \varepsilon}^{-\delta}) & \xrightarrow{\zeta_{\rho^{***}}} & C(M_{\rho^{***}}) \\
\downarrow{\zeta_{\rho^{***},\rho_D^* + \varepsilon}} & & \downarrow{\zeta} \\
C_\tau(M_{\rho_D^* + \varepsilon}^{-\delta}) & \xrightarrow{\zeta_{\rho_D^* + \varepsilon}} & C(M_{\rho_D^* + \varepsilon})
\end{array}
\]

**Proof of Theorem A.6.2.** We begin with some general observations. To this end, let \( \rho \in [0, \rho^{**} - 4\varepsilon] \) be the level that is currently considered in Line 3 of Algorithm 2.1. Then, Theorem 2.8 shows that Algorithm 2.1 identifies exactly the \( \tau \)-connected components of \( L_{D,\rho} \) that belong to the set \( \zeta(C_\tau(M_{\rho_D^* + \varepsilon}^{-\delta})) \), where \( \zeta : C_\tau(M_{\rho_D^* + \varepsilon}^{-\delta}) \to C_\tau(L_{D,\rho}) \) is the CRM. In the following, we thus consider the set \( \zeta(C_\tau(M_{\rho_D^* + \varepsilon}^{-\delta})) \). Moreover, we note that the returned level \( \rho_D^* \) always satisfies \( \rho_D^* \geq \rho + 3\varepsilon \) by Line 4 and Line 6, and equality holds if and only if \( |\zeta(C_\tau(M_{\rho_D^* + \varepsilon}^{-\delta}))| \neq 1 \).

i) Let us first consider the case \( \rho \in [0, \rho^* - \varepsilon] \). Then \( \rho + \varepsilon < \rho^* \) together with part i) and iii) of Theorem A.4.4 shows \( |C_\tau(M_{\rho_D^* + \varepsilon}^{-\delta})| = 1 \), and hence \( |\zeta(C_\tau(M_{\rho_D^* + \varepsilon}^{-\delta}))| = 1 \). Our initial consideration then show, that Algorithm 2.1 does not leave its loop, and thus \( \rho_D^* \geq \rho^* + 2\varepsilon \).

Let us now consider the case \( \rho \in [\rho^* + \varepsilon, \rho^* + \varepsilon^* + 2\varepsilon] \). Here we first note that Algorithm 2.1 actually inspects such an \( \rho \), since it iteratively inspects all \( \rho = i\varepsilon \), \( i = 0, 1, \ldots \), and the width of the interval above is \( \varepsilon \). Moreover, our assumptions on \( \varepsilon^* \) and \( \varepsilon \) guarantee \( \rho^* + \varepsilon^* + 2\varepsilon \leq \rho^{**} - 4\varepsilon \), and hence we have \( \rho \in [\rho^* + \varepsilon^* + \varepsilon, \rho^{**} - 4\varepsilon] \), i.e., we are in the situation
described at the beginning of the proof. We write \( \zeta_V : \mathcal{C}_\tau(M_{\rho^*}) \to \mathcal{C}_\tau(M_{\rho^*+\epsilon}) \), \( \zeta_M : \mathcal{C}_\tau(M_{\rho^*}) \to \mathcal{C}_\tau(M_{\rho^*+\epsilon}) \), and \( \zeta_{V,M} : \mathcal{C}_\tau(M_{\rho^*}) \to \mathcal{C}_\tau(M_{\rho^*+\epsilon}) \) for the CRMs between the involved sets. We then obtain the commutative diagram

\[
\begin{array}{ccc}
\mathcal{C}_\tau(M_{\rho^*}) & \xrightarrow{\zeta_{V,M}} & \mathcal{C}_\tau(M_{\rho^*+\epsilon}) \\
\downarrow & & \downarrow \\
\mathcal{C}_\tau(M_{\rho^*}) & \xrightarrow{\zeta_M} & \mathcal{C}_\tau(M_{\rho^*+\epsilon}) \\
\end{array}
\]

where the CRM \( \zeta_{**} \) is bijective by part ii) of Theorem A.4.4. Moreover, \( \rho - \epsilon \geq \rho^* + \epsilon^* \) together with part ii) of Theorem A.4.2 shows \( |\mathcal{C}_\tau(M_{\rho^*})| = 2 \), and by iii) of Theorem A.4.2 we conclude that \( \zeta_M \) is bijective. Similarly, \( \rho + \epsilon \geq \rho^* + \epsilon^* \) and the bijectivity of \( \zeta_{**} \) show by iv) of Theorem A.4.2 that \( |\mathcal{C}_\tau(M_{\rho^*})| = 2 \), and thus \( \zeta_V \) is bijective by part iii) of Theorem A.4.4. Consequently, \( \zeta' : \mathcal{C}_\tau(L_{D,\rho}) \to \mathcal{C}_\tau(M_{\rho^*+\epsilon}) \). Then Lemma 2.4 yields another diagram:

\[
\begin{array}{ccc}
\mathcal{C}_\tau(M_{\rho^*}) & \xrightarrow{\zeta_{V,M}} & \mathcal{C}_\tau(M_{\rho^*+\epsilon}) \\
\downarrow & & \downarrow \\
\mathcal{C}_\tau(L_{D,\rho}) & \xrightarrow{\zeta'} & \mathcal{C}_\tau(M_{\rho^*+\epsilon}) \\
\end{array}
\]

Since \( \zeta_{V,M} \) is bijective, we then find that \( \zeta \) is injective, and since we have already seen that \( M_{\rho^*+\epsilon} \) has two \( \tau \)-connected components, we conclude that \( |\mathcal{C}_\tau(M_{\rho^*+\epsilon})| \) contains two elements. Consequently, the stopping criterion of Algorithm 2.1 is satisfied, that is, \( \rho_{D^*} = \rho + 3\epsilon \leq \rho^* + \epsilon^* + 5\epsilon \).

\textit{ii).} Theorem 2.8 shows that in its last run through the loop Algorithm 2.1 identifies exactly the \( \tau \)-connected components of \( L_{D,\rho} \) that belong to the set \( \zeta_{-3\epsilon}(<\mathcal{C}_\tau(M_{\rho^*+\epsilon}) \text{ where } \rho := \rho_{D^*} - 3\epsilon \text{ and } \zeta_{-3\epsilon} : \mathcal{C}_\tau(M_{\rho^*+\epsilon}) \to \mathcal{C}_\tau(L_{D,\rho}) \text{ is the CRM. Moreover, since Algorithm 2.1 stops at } \rho_{D^*} - 3\epsilon \), we have \( |\zeta_{-3\epsilon}(\mathcal{C}_\tau(M_{\rho^*+\epsilon}))| \neq 1 \) by our remarks at the beginning of the proof, and thus \( |\mathcal{C}_\tau(M_{\rho^*+\epsilon})| \neq 1 \). From the already proven part i) we further know that \( \rho + \epsilon = \rho_{D^*} - 2\epsilon \leq \rho^* + \epsilon^* + 3\epsilon \leq \rho^* + 4\epsilon^* \leq \rho^{**} \), and part i) of Theorem A.4.4 hence gives \( |\mathcal{C}_\tau(M_{\rho^*+\epsilon})| = 2 \). For later purposes, note that the latter together with \( |\zeta_{-3\epsilon}(\mathcal{C}_\tau(M_{\rho^*+\epsilon}))| \neq 1 \) implies the injectivity of \( \zeta_{-3\epsilon} \). Now, part iii) of Theorem A.4.4 shows that the CRM \( \zeta_{\rho^*,\rho^*+\epsilon} : \mathcal{C}_\tau(M_{\rho^*+\epsilon}) \to \mathcal{C}_\tau(M_{\rho^*+\epsilon}) \) is bijective. Let us consider the following commutative diagram:
where the remaining two maps are the corresponding CRMs, whose existence is guaranteed by
\( \rho_D^* + \varepsilon \leq \rho_D^* + 7\varepsilon^* \leq \rho^* \) and \( \rho + \varepsilon \leq \rho_D^* + \varepsilon \), respectively.
Now the bijectivity of \( \zeta_{\rho^*, \rho + \varepsilon} \) shows that \( \zeta_{\rho^*, \rho_D^* + \varepsilon} \) is injective. Moreover, \( \rho^*_D + \varepsilon \leq \rho^* \) implies \( |C_\tau (M_{\rho^*_D + \varepsilon}^-) - 2 \) by part ii) of Theorem A.4.4, while \( \rho^* \geq \rho^*_D + \varepsilon \) implies \( |C_\tau (M_{\rho^*_D + \varepsilon}^-) - 2 \) by part iv) of Theorem A.4.2 and part ii) of Theorem A.4.4. Therefore, \( \zeta_{\rho^*, \rho_D^* + \varepsilon} \) is actually bijective. This yields both \( |C_\tau (M_{\rho_D^* + \varepsilon}^-) - 2 \), which is the first assertion, and the bijectivity of \( \tilde{\zeta} \).

Let us consider yet another commutative diagram

\[
\begin{array}{ccc}
C_\tau (M_{\rho^*_D + \varepsilon}^-) & \xrightarrow{\zeta} & C_\tau (M_{\rho^*_D + \varepsilon}^-) \\
\zeta_{\rho^*, \rho_D^* + \varepsilon} & \downarrow & \zeta_{\rho^*, \rho_D^* + \varepsilon} \\
C_\tau (M_{\rho_D^* + \varepsilon}^-) & \xrightarrow{\zeta} & C_\tau (M_{\rho_D^* + \varepsilon}^-)
\end{array}
\]

where again, all occurring maps are the CRMs between the respective sets.

Now we have already shown that \( \zeta_{-3\varepsilon} \) is injective and that \( \tilde{\zeta} \) is bijective. Consequently, \( \zeta \) is injective.

\( ii.i) \). This assertion follows from Theorem 2.8 and the inequality \( \rho_D^* \leq \rho^* - 3\varepsilon \), which follows from part i).

\( iv \). We have already seen in the proof of part ii) that \( |C_\tau (M_{\rho^*_D}^-) = 2 \), and consequently part iii) of Lemma A.4.3 shows that there exists a bijective CRM \( \zeta_{\rho^*} : C_\tau (M_{\rho^*_D}^-) \to C(M_{\rho^*}) \). Moreover, part ii) shows \( |C_\tau (M_{\rho_D^* + \varepsilon}^-) = 2 \), thus part iii) of Lemma A.4.3 yields another bijective CRM \( \zeta_{\rho_D^* + \varepsilon} : C_\tau (M_{\rho_D^* + \varepsilon}^-) \to C(M_{\rho_D^* + \varepsilon}) \). Furthermore, in the proof of part ii) we have already seen that CRM \( \zeta_{\rho^*, \rho_D^* + \varepsilon} \) is bijective. This gives the diagram.

\textbf{A.7. Additional Material Related to Assumption A.} In this section we discuss Assumption A, which describes the partitions needed for our histogram approach, in more detail.

We begin with an example of partitions satisfying Assumption A.
**Example A.7.1.** Let $X := [0, 1]^d$ be equipped with the metric defined by the supremum norm $\| \cdot \|_{\ell^\infty}$, and $\lambda^d$ be the Lebesgue measure. For $\delta \in (0, 1]$, there then exists a unique $\ell \in \mathbb{N}$ with $\frac{1}{\ell+1} < \delta \leq \frac{1}{\ell}$. We define $h := \frac{1}{\ell+1}$ and write $A_\delta$ for the usual partition of $[0, 1]^d$ into hypercubes of side-length $h$. We define $h := \frac{1}{\ell+1}$ and write $A_\delta$ for the usual partition of $[0, 1]^d$ into hypercubes of side-length $h$. Then, for each $A_i \in A_\delta$, we have $\text{diam} A_i \leq h = \delta$ and $\lambda^d(A_i) = h^d \geq 2^{-d}\delta^d$. Moreover, we obviously have $|A_\delta| = h^d \leq 2^{-d}\delta^d$, and hence $(A_\delta)_{\delta \in (0, 1]}$ satisfies Assumption A with $c_{\text{part}} := 2^d$.

The next lemma describes a general situation in which there exist partitions satisfying Assumption A. For its formulation, recall that the covering numbers of a compact metric space $(X, d)$ are defined by

$$N(X, d, \delta) := \min \left\{ n \geq 1 : \exists x_1, \ldots, x_n \in X \text{ with } X \subset \bigcup_{i=1}^n B(x_i, \delta) \right\}, \; \delta > 0,$$

where again $B(x, \delta)$ denotes the closed ball with center $x$ and radius $\delta$.

**Lemma A.7.2.** Let $(X, d)$ be a compact metric space for which there exist constants $c > 0$ and $d > 0$ such that

$$N(X, d, \delta) \leq c\delta^{-d}, \quad \delta \in (0, 1/4].$$

Moreover, assume that there exists a finite measure $\mu$ on $X$ such that

$$\mu(B(x, \delta)) \geq c^{-1}\delta^d$$

for all $x \in X$ and $\delta \in (0, 1/4]$. Then Assumption A is satisfied for $d$ and $c_{\text{part}} = 4^d c$.

Note that the unit spheres $S^d \subset \mathbb{R}^{d+1}$ together with their surface measures satisfy the assumptions for $d = d - 1$, see also Corollary A.7.3.

**Proof of Lemma A.7.2.** Let us recall that a $\delta$-packing in $X$ is a family $y_1, \ldots, y_m \in X$ with $d(y_i, y_j) > 2\delta$ for all $i \neq j$. Let us write

$$\mathcal{M}(X, d, \delta) := \max \left\{ m \geq 1 : \exists \delta\text{-packing } y_1, \ldots, y_m \text{ in } X \right\}$$

for the size of the largest possible $\delta$-packing in $X$. Then it is well-known that we have the following inequalities between these packing numbers and the covering numbers:

$$\mathcal{M}(X, d, \delta) \leq N(X, d, \delta) \leq \mathcal{M}(X, d, \delta/2), \; \delta > 0.$$
Let us now fix a \( \delta \in (0, 1] \) and a maximal \( \delta/4 \)-packing \( y_1, \ldots, y_m \) in \( X \). By (A.7.1) we conclude that

\[
m = M(X, d, \delta/4) \leq N(X, d, \delta/4) \leq 4^d c\delta^{-d}.
\]

To construct the partition \( A_\delta \), we consider a Voronoi partition \( A_1, \ldots, A_m \) that corresponds to the points \( y_1, \ldots, y_m \), where the behavior of the cells on their boundary may be arbitrary, i.e. ties may be arbitrarily resolved. Our next goal is to show

(A.7.2) \( B(y_i, \delta/4) \subset A_i \subset B(y_i, \delta/2), \quad i = 1, \ldots, m. \)

To prove the left inclusion, we fix an \( x \in B(y_i, \delta/4). \) For \( j \neq i \), we then find

\[
\delta/2 < d(y_i, y_j) \leq d(y_i, x) + d(x, y_j) \leq \delta/4 + d(x, y_j),
\]

and hence \( d(x, y_j) > \delta/4 \geq d(x, y_i). \) From the latter we conclude that \( x \in A_i. \)

For the proof of the right inclusion, we assume that it does not hold for some index \( i \in \{1, \ldots, m\} \). Then there exists an \( x \in A_i \) such that \( d(x, y_i) > \delta/2. \) On the hand, since \( y_1, \ldots, y_m \) is a maximal \( \delta/4 \)-packing in \( X \), there exists a \( j \in \{1, \ldots, m\} \) with \( d(x, y_j) \leq 2\delta/4 = \delta/2, \) and hence we have \( d(x, y_j) \leq \delta/2 < d(x, x_i). \) This implies \( x \notin A_i, \) i.e. we have found a contradiction.

Now, using (A.7.2), we obtain both \( \mu(A_i) \geq \mu(B(y_i, \delta/4)) \geq 4^{-d} c^{-1} \delta^d \) and \( \text{diam } A_i \leq \text{diam } B(y_i, \delta/2) \leq \delta. \)

The next corollary in particular shows that one of the assumptions made in Lemma A.7.2 can be omitted if the measure behaves regularly on balls.

**Corollary A.7.3.** Let \( (X, d) \) be a compact metric space and \( \mu \) be a finite measure on \( X \) for which there exists a constant \( K \geq 1 \) such that

\[
K^{-1} \leq \frac{\mu(B(y, \delta))}{\mu(B(x, \delta))} \leq K, \quad x, y \in X, \delta \in (0, 1/4].
\]

If there exist constants \( c > 0 \) and \( d > 0 \) such that

\[
N(X, d, \delta) \leq c\delta^{-d}, \quad \delta \in (0, 1/4],
\]

then Assumption A is satisfied for \( d \) and \( c_{\text{part}} = 4^d cK. \) Similarly, if

\[
\mu(B(x, \delta)) \geq c^{-1} \delta^d, \quad \delta \in (0, 1/8],
\]

holds true, then Assumption A is satisfied for \( d \) and \( c_{\text{part}} = 8^d cK. \)
If \((X,d,\cdot)\) is a compact group with invariant metric \(d\) and \(\mu\) is its Haar measure, then we have \(K = 1\). Moreover, if \(X \subset \mathbb{R}^d\) is a sufficiently smooth manifold and \(\mu\) is its surface measure, then the corollary is also applicable.

**Proof of Corollary A.7.3.** To show the first assertion, we fix a \(\delta \in (0,1/4]\) and a minimal \(\delta\)-net \(x_1,\ldots,x_n\) of \(X\). For an \(x \in X\) we then obtain

\[
1 = \mu(X) \leq \sum_{i=1}^{n} \mu(B(x_i,\delta)) \leq nK \mu(B(x,\delta)) \leq cK \delta^{-d} \mu(B(x,\delta)).
\]

By Lemma A.7.2 we thus obtain the first assertion.

To prove the second assertion we fix a \(\delta \in (0,1/4]\) and a maximal \(\delta/2\)-packing \(y_1,\ldots,y_m\) of \(X\). Then \(B(y_i,\delta/2) \cap B(y_j,\delta/2) = \emptyset\) for \(i \neq j\) implies

\[
1 = \mu(X) \geq \sum_{i=1}^{m} \mu(B(y_i,\delta/2)) \geq mk^{-1} \mu(B(x,\delta/2)) \geq m2^{-d}c^{-1}K^{-1}\delta^d,
\]

and hence \(N(X,d,\delta) \leq M(X,d,\delta/2) = m \leq 2^d c K \delta^d\) by (A.7.1). Lemma A.7.2 then yields the second assertion. \(\square\)

**A.8. Material Related to Basic Properties of Histograms.** The goal of this section is to establish the key inclusion (2.7) for our histogram-based approach. The material of this section is taken from [9].

Our first result shows that \(h_{D,\delta}\) uniformly approximates its infinite-sample counterpart

\[
h_{P,\delta}(x) := \sum_{j=1}^{m} \frac{P(A_j)}{\mu(A_j)} \cdot 1_{A_j}(x), \quad x \in X,
\]

with high probability, where \(A_\delta = (A_1,\ldots,A_m)\) for a fixed \(\delta > 0\).

**Theorem A.8.1.** Let Assumption A be satisfied and \(P\) be a distribution on \(X\). Then, for all \(n \geq 1\), \(\varepsilon > 0\), and \(\delta > 0\), we have

\[
P^n\left(\{ D \in X^n : \|h_{D,\delta} - h_{P,\delta}\|_\infty \geq \varepsilon \} \right) \leq 2c_{\text{part}} \exp\left(-d \ln \delta - \frac{2n\varepsilon^2\delta^{2d}}{c_{\text{part}}^2} \right).
\]

In addition, if \(P\) is \(\mu\)-absolutely continuous and there exists a bounded \(\mu\)-density \(h\) of \(P\), then, for all \(n \geq 1\), \(\varepsilon > 0\), and \(\delta > 0\), we have

\[
P^n\left( D \in X^n : \|h_{D,\delta} - h_{P,\delta}\|_\infty \geq \varepsilon \right) \leq 2c_{\text{part}} \exp\left(\ln \delta^{-d} - \frac{3n\varepsilon^2\delta^{2d}}{c_{\text{part}}(6\|h\|_\infty + 2\varepsilon)} \right).
\]
Proof of Theorem A.8.1. We fix an \( A \in \mathcal{A}_d \) and write \( f := \mu(A)^{-1}1_A \). Then \( f \) is non-negative and our assumptions ensure \( \|f\|_\infty \leq c_{\text{part}}\delta^{-d} \). Consequently, Hoeffding’s inequality, see e.g. [4, Theorem 8.1], yields
\[
P^n\left( \left\{ D \in X^n : \frac{1}{n} \sum_{i=1}^{n} f(x_i) - \mathbb{E}_P f < \varepsilon \right\} \right) \geq 1 - 2 \exp\left( -\frac{2n\varepsilon^2 \delta^{2d}}{c^2_{\text{part}}} \right)
\]
for all \( n \geq 1 \) and \( \varepsilon > 0 \), where we assumed \( D = (x_1, \ldots, x_n) \). Furthermore, we have \( \frac{1}{n} \sum_{i=1}^{n} f(x_i) = \mu(A)^{-1}D(A) \) and \( \mathbb{E}_P f = \mu(A)^{-1}P(A) \). By a union bound argument and \( |\mathcal{A}_d| \leq c_{\text{part}}\delta^{-d} \), we thus obtain
\[
P^n\left( \left\{ D \in X^n : \sup_{A \in \mathcal{A}_d} \left| \frac{D(A)}{\mu(A)} - \frac{P(A)}{\mu(A)} \right| < \varepsilon \right\} \right) \geq 1 - 2c_{\text{part}}\delta^{-d} \exp\left( -\frac{2n\varepsilon^2 \delta^{2d}}{c^2_{\text{part}}} \right).
\]
Since, for \( x \in X \) and \( A \in \mathcal{A}_d \) with \( x \in A \), we have \( h_{D,\delta}(x) = \mu(A)^{-1}D(A) \) and \( h_{P,\delta}(x) = \mu(A)^{-1}P(A) \), we then find the first assertion.

To show the second inequality, we write \( f := \mu(A)^{-1}(1_A - P(A)) \) for a fixed \( A \in \mathcal{A}_d \). This yields \( \mathbb{E}_P f = 0 \), \( \|f\|_\infty \leq c_{\text{part}}\delta^{-d} \), and
\[
\mathbb{E}_P f^2 \leq \mu(A)^{-2}P(A) \leq \mu(A)^{-1}\|h\|_\infty \leq c_{\text{part}}\delta^{-d}\|h\|_\infty.
\]
Consequently, Bernstein’s inequality, see e.g. [4, Theorem 8.2], yields
\[
P^n\left( \left\{ D \in X^n : \left| \frac{1}{n} \sum_{i=1}^{n} f(x_i) \right| < \varepsilon \right\} \right) \geq 1 - 2 \exp\left( -\frac{3n\varepsilon^2 \delta^{d}}{c_{\text{part}}(6\|h\|_\infty + 2\varepsilon)} \right).
\]
Using \( \frac{1}{n} \sum_{i=1}^{n} f(x_i) = (D(A) - P(A))\mu(A)^{-1} \), the rest of the proof follows the lines of the proof of the first inequality.

The next result specifies the vertical and horizontal uncertainty of a plug-in level set estimate \( \hat{h} \geq \rho \), if \( \hat{h} \) is a uniform approximation of \( h_{P,\delta} \).

Lemma A.8.2. Let Assumption A be satisfied, \( P \) be a \( \mu \)-absolutely continuous distribution on \( X \), and \( \hat{h} : X \to \mathbb{R} \) be a function with \( \|\hat{h} - h_{P,\delta}\|_\infty \leq \varepsilon \) for some \( \varepsilon \geq 0 \). Then, for all \( \rho \geq 0 \), the following statements hold:

i) If \( P \) is upper normal at the level \( \rho + \varepsilon \), then we have \( M_{\rho+\varepsilon}^{-\delta} \subset \{ \hat{h} \geq \rho \} \).

ii) If \( P \) is upper normal at the level \( \rho - \varepsilon \), then we have \( \{ \hat{h} \geq \rho \} \subset M_{\rho-\varepsilon}^{+\delta} \).

Proof of Lemma A.8.2. i). We will show the equivalent inclusion \( \{ \hat{h} < \rho \} \subset (X \setminus M_{\rho+\varepsilon})^{-\delta} \). To this end, we fix an \( x \in X \) with \( \hat{h}(x) < \rho \). If \( x \in X \setminus M_{\rho+\varepsilon} \), we immediately obtain \( x \in (X \setminus M_{\rho+\varepsilon})^{-\delta} \), and hence we may
restrict our considerations to the case $x \in M_{\rho+\varepsilon}$. Then, $\hat{h}(x) < \rho$ together with $\|h - h_{P,\delta}\|_\infty \leq \varepsilon$ implies $h_{P,\delta}(x) \leq \hat{h}(x) + \varepsilon < \rho + \varepsilon$. Now let $A$ be the unique cell of the partition $A_\delta$ satisfying $x \in A$. The definition of $h_{P,\delta}$ together with the assumed $0 < \mu(A) < \infty$ then yields

\[(A.8.1) \quad \int_A h \, d\mu = P(A) = h_{P,\delta}(x) \mu(A) < (\rho + \varepsilon)\mu(A),\]

where $h : X \to [0,\infty)$ is an arbitrary $\mu$-density of $P$. Our next goal is to show that there exists an $x' \in (X \setminus M_{\rho+\varepsilon}) \cap A$. Suppose the converse, that is $A \subset M_{\rho+\varepsilon}$. Then the upper normality of $P$ at the level $\rho + \varepsilon$ yields

$\mu(A \setminus \{h \geq \rho + \varepsilon\}) \leq \mu(M_{\rho+\varepsilon} \setminus \{h \geq \rho + \varepsilon\}) = 0$, and hence we conclude that $\mu(A \cap \{h \geq \rho + \varepsilon\}) = \mu(A)$. This leads to

\[
\int h \, d\mu = \int_{A \setminus \{h \geq \rho + \varepsilon\}} h \, d\mu + \int_{A \setminus \{h \geq \rho + \varepsilon\}} h \, d\mu = \int_{A \setminus \{h \geq \rho + \varepsilon\}} h \, d\mu \geq (\rho + \varepsilon)\mu(A).
\]

However, this inequality contradicts (A.8.1), and hence there does exist an $x' \in (X \setminus M_{\rho+\varepsilon}) \cap A$. This implies $d(x, X \setminus M_{\rho+\varepsilon}) \leq d(x, x') \leq \text{diam } A \leq \delta$, i.e. we have shown $x \in (X \setminus M_{\rho+\varepsilon})^C$.

\(ii\). Let us fix an $x \in X$ with $\hat{h}(x) \geq \rho$. If $x \in M_{\rho-\varepsilon}$, we immediately obtain $x \in M_{\rho-\varepsilon}^C$, and hence it remains to consider the case $x \in X \setminus M_{\rho-\varepsilon}$. Clearly, if $\rho - \varepsilon \leq 0$, this case is impossible, and hence we may additionally assume $\rho - \varepsilon > 0$. Then, $h(x) \geq \rho$ together with $\|h - h_{P,\delta}\|_\infty \leq \varepsilon$ yields $h_{P,\delta}(x) \geq \hat{h}(x) - \varepsilon \geq \rho - \varepsilon$. Now let $A$ be the unique cell of the partition $A_\delta$ satisfying $x \in A$. By the definition of $h_{P,\delta}$ and $\mu(A) > 0$ we then obtain

\[(A.8.2) \quad \int_A h \, d\mu = P(A) = h_{P,\delta}(x) \mu(A) \geq (\rho - \varepsilon)\mu(A),\]

where $h : X \to [0,\infty)$ is an arbitrary $\mu$-density of $P$. Next we show that there exists an $x' \in M_{\rho-\varepsilon} \cap A$. Suppose the converse holds, that is $A \subset X \setminus M_{\rho-\varepsilon}$. Then the assumed upper normality of $P$ at the level $\rho - \varepsilon$ yields

$\mu(M_{\rho-\varepsilon} \setminus \{h \geq \rho - \varepsilon\}) = 0$,

and thus we find $\mu((X \setminus M_{\rho-\varepsilon}) \setminus \{h < \rho - \varepsilon\}) = 0$ by $A \setminus B = (X \setminus A) \setminus (X \setminus B)$. Combining this with the assumed $A \subset X \setminus M_{\rho-\varepsilon}$, we obtain

$\mu(A \setminus \{h < \rho - \varepsilon\}) \leq \mu((X \setminus M_{\rho-\varepsilon}) \setminus \{h < \rho - \varepsilon\}) = 0$,

and this implies

\[
\int h \, d\mu = \int_{A \setminus \{h < \rho - \varepsilon\}} h \, d\mu + \int_{A \setminus \{h < \rho - \varepsilon\}} h \, d\mu = \int_{A \setminus \{h < \rho - \varepsilon\}} h \, d\mu < (\rho - \varepsilon)\mu(A).
\]
This contradicts (A.8.2), and hence there does exist an \( x' \in M_{\rho-\varepsilon} \cap A \). This yields \( d(x, M_{\rho-\varepsilon}) \leq d(x, x') \leq \text{diam } A \leq \delta \), i.e. we have shown \( x \in M_{\rho-\varepsilon}^+ \).

A.9. Proofs and Additional Material Related to the Consistency. In this section we prove Theorem 4.1. Furthermore, it contains additional material related to the assumptions made in that theorem.

**Lemma A.9.1.** Let \((X,d)\) be a metric space, \( \mu \) be a finite Borel measure on \( X \), and \((A_\rho)_{\rho \in \mathbb{R}}\) be a decreasing family of closed subsets of \( X \). For \( \rho^* \in \mathbb{R} \), we write

\[
\dot{A}_\rho^* := \bigcup_{\rho > \rho^*} A_\rho \quad \text{and} \quad \check{A}_\rho^* := \bigcup_{\rho > \rho^*} A_\rho.
\]

Then we have

\[
\dot{A}_\rho^* = \bigcup_{\rho > \rho^*} \bigcup_{\varepsilon > 0} \bigcup_{\delta > 0} A_{\rho + \varepsilon}^{-\delta}.
\]

Moreover, the following statements are equivalent:

i) \( \mu(\dot{A}_\rho^* \setminus \check{A}_\rho^*) = 0 \).

ii) For all \( \varepsilon > 0 \), there exists a \( \rho_\varepsilon > \rho^* \) such that, for all \( \rho \in (\rho^*, \rho_\varepsilon] \), we have \( \mu(A_\rho \setminus \check{A}_\rho) \leq \varepsilon \).

**Proof of Lemma A.9.1.** To show the first equality, we observe that (A.3.1) implies

\[
\bigcap_{\rho > \rho^*} \bigcap_{\varepsilon > 0} \bigcap_{\delta > 0} (X \setminus A_{\rho + \varepsilon})^{+\delta} = \bigcap_{\varepsilon > 0} \bigcap_{\rho > \rho^*} X \setminus A_{\rho + \varepsilon} = \bigcap_{\rho > \rho^*} X \setminus A_\rho.
\]

Moreover, every set \( A \subset X \) satisfies \( \overline{X \setminus A} = X \setminus \check{A} \), and hence we obtain

\[
\bigcap_{\rho > \rho^*} \bigcap_{\varepsilon > 0} \bigcap_{\delta > 0} (X \setminus A_{\rho + \varepsilon})^{+\delta} = \bigcap_{\rho > \rho^*} X \setminus A_\rho = \bigcap_{\rho > \rho^*} (X \setminus \dot{A}_\rho) = X \setminus \bigcup_{\rho > \rho^*} \check{A}_\rho.
\]

Therefore, by taking the complement we find

\[
\bigcup_{\rho > \rho^*} \check{A}_\rho = X \setminus \left( \bigcap_{\rho > \rho^*} \bigcap_{\varepsilon > 0} \bigcap_{\delta > 0} (X \setminus A_{\rho + \varepsilon})^{+\delta} \right) = \bigcup_{\rho > \rho^*} \bigcup_{\varepsilon > 0} \bigcup_{\delta > 0} (X \setminus (X \setminus A_{\rho + \varepsilon})^{+\delta}) = \bigcup_{\rho > \rho^*} \bigcup_{\varepsilon > 0} \bigcup_{\delta > 0} A_{\rho + \varepsilon}^{-\delta}.
\]
i) ⇒ ii). Let us fix an \( \varepsilon > 0 \). Since \( \hat{A}_\rho = \bigcup_{\rho' \geq \rho} \hat{A}_{\rho'} \) for \( \rho > \rho^* \), the \( \sigma \)-continuity of finite measures yields a \( \rho_\varepsilon > \rho^* \) such that \( \mu(\hat{A}_{\rho^*} \setminus \hat{A}_{\rho}) \leq \varepsilon \) for all \( \rho \in (\rho^*, \rho_\varepsilon] \). Using \( A_\rho \subset \hat{A}_{\rho^*} \) for \( \rho > \rho^* \), we then obtain the assertion 
\[
\mu(A_\rho \setminus \hat{A}_{\rho}) \leq \mu(\hat{A}_{\rho^*} \setminus A_\rho) \leq \varepsilon.
\]

ii) ⇒ i). Let us fix an \( \varepsilon > 0 \). For \( \rho \in (\rho^*, \rho_\varepsilon] \), we then have \( \hat{A}_\rho \subset \hat{A}_{\rho^*} \), and hence our assumption yields \( \mu(A_\rho \setminus \hat{A}_{\rho^*}) \leq \varepsilon \). In other words, we have \( \lim_{\rho \nearrow \rho^*} \mu(A_\rho \setminus \hat{A}_{\rho^*}) = 0 \). Moreover, we have \( A_\rho \not\supset \hat{A}_{\rho^*} \) for \( \rho \searrow \rho^* \), and hence the \( \sigma \)-continuity of \( \mu \) yields \( \lim_{\rho \searrow \rho^*} \mu(A_\rho \setminus \hat{A}_{\rho^*}) = 0 \).

**Lemma A.9.2.** Let \( f : (0, 1) \to (0, \infty) \) be a monotonously increasing function and \( g : (0, f(1)) \to [0, 1] \) be its generalized inverse, that is 
\[
g(y) := \inf \{ x \in (0, 1] : f(x) \geq y \}, \quad y \in (0, 1].
\]

Then we have \( \lim_{y \to 0^+} g(y) = 0 \).

**Proof of Lemma A.9.2.** Let \( (y_n) \subset (0, f(1)) \) be a sequence with \( y_n \to 0 \). For \( n \geq 1 \), we write \( E_n := \{ x \in (0, 1] : f(x) \geq y_n \} \). Let us fix an \( \varepsilon \in (0, 1] \). Since \( f \) is strictly positive, we then find \( f(\varepsilon) > 0 \), and hence there exists an \( n_0 \geq 1 \) such that \( f(\varepsilon) \geq y_n \) for all \( n \geq n_0 \). Thus, we have \( \varepsilon \in E_n \) for all \( n \geq n_0 \), and from the latter we obtain \( g(y_n) = \inf E_n \leq \varepsilon \) for such \( n \).

Before we prove Theorem 4.1, let us briefly illustrate the additional assumption \( \mu(A_1^* \cup A_2^* \setminus (A_1^* \cup A_2^*)) = 0 \). To this end, we fix a \( \mu \)-density \( h \) of \( P \). Then Lemma A.1.2 tells us that 
\[
A_1^* \cup A_2^* = \bigcup_{\rho > \rho^*} M_\rho \subset \bigcup_{\rho > \rho^*} \{ h \geq \rho \} \subset \bigcup_{\rho > \rho^*} \{ h > \rho \} = \{ h > \rho^* \}.
\]

Using the normality in Assumption C, which implies (A.1.3), we then obtain 
\[
\mu(A_1^* \cup A_2^* \setminus (A_1^* \cup A_2^*)) \leq \mu(\{ h > \rho^* \} \setminus \{ h > \rho^* \}) \leq \mu(\partial \{ h > \rho^* \}) = \mu(\partial \{ h \geq \rho \}).
\]

Consequently, the additional assumption is satisfied, if there exists a \( \mu \)-density \( h \) of \( P \) such that \( \mu(\partial \{ h \leq \rho^* \}) = 0 \). In this respect recall, that Lemma A.1.3 showed that \( P \) is normal, if, for all \( \rho \in \mathbb{R} \), we have a \( \mu \)-density \( h \) of \( P \) with \( \mu(\partial \{ h \geq \rho \}) = 0 \).

**Proof of Theorem 4.1.** We fix an \( \epsilon > 0 \). For \( n \geq 1 \), \( \tau := \tau_n \), and \( \varepsilon := \varepsilon_n \), we define \( \varepsilon_n^* \) by the right hand-side of (3.4). Then, Lemma A.9.2 shows \( 0 < \varepsilon_n^* \leq \epsilon \wedge (\rho^{**} - \rho^*)/9 \) for sufficiently large \( n \). In addition, \( \delta_n \) and
\( \varepsilon_n \) satisfy (3.2) for sufficiently large \( n \) by (4.1), and we also have \( \delta_n \leq \delta_{\text{thick}} \) for sufficiently large \( n \). Thus, there is an \( n_0 \geq 1 \) such that, for all \( n \geq n_0 \), the values \( \varepsilon_n, \delta_n, \tau_n \) and \( \varepsilon_n^* \) satisfy the assumptions of Theorem 3.1 and \( \varepsilon_n^* \leq \varepsilon \).

Let us now consider an \( n \geq n_0 \) and a data set \( D \in X^n \) satisfying both the assertions (i) - (v) of Theorem A.6.2 and (2.10). By Theorem 3.1 and our previous considerations we then know that the probability \( P^n \) of \( D \) is not less than \( 1 - e^{-\varepsilon} \). Now, part i) of Theorem A.6.2 yields \( \rho_D^* - \rho^* \geq 2\varepsilon_n > 0 \) and
\[
\rho_D^* - \rho^* \leq \varepsilon_n^* + 5\varepsilon_n \leq 6\varepsilon_n \leq 6\varepsilon,
\]
i.e. we have shown the first convergence.

To prove the second convergence, we write \( A_i, i = 1, 2 \), for the two topologically connected components of \( M_{\rho^*} \). For \( \rho \in (\rho^*, \rho^{**}] \), we further define \( A^i_{\rho} := \zeta_{\rho}(A_i) \), where \( \zeta_{\rho} : C(M_{\rho^*}) \to C(M_{\rho}) \) is the CRM. In addition, we write \( A^i_{\rho^*} := \emptyset \) for \( \rho > \rho^{**} \) and \( A^i_{\rho} := X \) for \( \rho \leq \rho^* \). Let us first show
\[
(\text{A.9.1}) \quad \mu(\hat{A}^i_{\rho^*} \setminus \hat{A}^i_{\rho^*}) = 0
\]
for \( i = 1, 2 \), where we used the notation of Lemma A.9.1. To this end, we fix an \( \varepsilon > 0 \). Since \( P \) is lower and upper normal at every level \( \rho \in [\rho^*, \rho^{**}] \), we find, for an arbitrary \( \mu \)-density \( h \) of \( P \),
\[
\mu(\hat{M}_{\rho^*} \setminus \hat{M}_{\rho^*}) = \mu(\{h > \rho^*\} \setminus \hat{M}_{\rho^*}) = 0,
\]
where we used (A.1.3), (A.1.4), and the notation of Lemma A.9.1. Lemma A.9.1 then shows that there exists a \( \rho_e > \rho^* \) such that
\[
(\text{A.9.2}) \quad \mu(M_{\rho} \setminus \hat{M}_{\rho}) \leq \varepsilon
\]
for all \( \rho \in (\rho^*, \rho_e] \), where we may assume without loss of generality that \( \rho_e \leq \rho^{**} \). Let us now fix a \( \rho \in (\rho^*, \rho_e] \). Then we obviously have \( \hat{A}^1_{\rho} \cup \hat{A}^2_{\rho} \subset M_{\rho} \).

To prove that the converse inclusion also holds, we pick an \( x \in M_{\rho} \). Without loss of generality we may assume that \( x \in A^1_{\rho} \). Since \( A^2_{\rho} \) is closed and thus compact, we then have \( \varepsilon := d(x, \hat{A}^2_{\rho}) > 0 \). Moreover, since \( \hat{M}_{\rho} \) is open, there exists a \( \delta \in (0, \varepsilon) \) such that \( B(x, \delta) \subset \hat{M}_{\rho} \). This yields \( B(x, \delta) \subset A^1_{\rho} \cup A^2_{\rho} \), and by \( d(x, \hat{A}^2_{\rho}) > \delta \), we conclude that \( B(x, \delta) \subset \hat{A}^1_{\rho} \). This shows \( x \in \hat{A}^1_{\rho} \), and hence we indeed have \( M_{\rho} = A^1_{\rho} \cup \hat{A}^2_{\rho} \) Now we use this equality to obtain
\[
M_{\rho} \setminus \hat{M}_{\rho} = (A^1_{\rho} \setminus (\hat{A}^1_{\rho} \cup \hat{A}^2_{\rho})) \cup (A^2_{\rho} \setminus (\hat{A}^1_{\rho} \cup \hat{A}^2_{\rho})) = (A^1_{\rho} \setminus \hat{A}^1_{\rho}) \cup (A^2_{\rho} \setminus \hat{A}^2_{\rho})
\]
By (A.9.2), this implies \( \mu(A^1_{\rho} \setminus \hat{A}^1_{\rho}) \leq \varepsilon \), and thus Lemma A.9.1 shows (A.9.1).
Let us now fix an $\epsilon > 0$ and a $\zeta \geq 1$. By the equality of Lemma A.9.1 and the $\sigma$-continuity of finite measures there then exist $\delta_\epsilon > 0$, $\varepsilon_\epsilon > 0$, and $\rho_\epsilon \in (\rho^*, \rho^{**}]$ such that, for all $\varepsilon \in (0, \varepsilon_\epsilon]$, $\delta \in (0, \delta_\epsilon]$, $\rho \in (\rho^*, \rho_\epsilon]$, and $i = 1, 2$, we have $\mu(A^i_{\rho^*} \setminus (A^i_{\rho^*+\varepsilon} \setminus \delta]) \leq \epsilon$. Combining this with $A^1_{\rho^*} = A^1_{\rho^*}$, which holds by the definition of the clusters $A^1_{\rho^*}$, and Equation (A.9.1) we then obtain

\[ \mu(A^1_{\rho^*} \setminus (A^1_{\rho^*+\varepsilon} \setminus \delta]) = \mu(\tilde{A}^1_{\rho^*} \setminus (A^1_{\rho^*+\varepsilon} \setminus \delta]) \leq \epsilon. \]

Moreover, our assumption $\mu(A^1_{\rho^*} \cup A^2_{\rho^*} \setminus (A^1_{\rho^*} \cup A^2_{\rho^*})) = 0$ means $\mu(M_{\rho^*} \setminus \tilde{M}_{\rho^*}) = 0$, and since by part iii) of Lemma A.3.1 we know that

\[ \bigcap_{\delta > 0} \left( \bigcup_{\rho > \rho^*} M_{\rho} \right)^{+\delta} = \bigcup_{\rho > \rho^*} M_{\rho} = \tilde{M}_{\rho^*} \]

we find

\[ \mu\left( \left( \bigcup_{\rho > \rho^*} M_{\rho} \right)^{+\delta} \setminus \tilde{M}_{\rho^*} \right) \leq \epsilon \]

for all sufficiently small $\delta > 0$. From this it is easy to conclude that

\[ \mu(M^{+\delta}_{\rho^*} \setminus \tilde{M}_{\rho^*}) \leq \epsilon \]

for all sufficiently small $\epsilon > 0$, $\delta > 0$ and all $\rho > \rho^* + \varepsilon$. Without loss of generality, we may thus assume that (A.9.4) also holds for all $\varepsilon \in (0, \varepsilon_\epsilon]$, $\delta \in (0, \delta_\epsilon]$ and all $\rho > \rho^* + \varepsilon$.

For given $T := T_n$ and $\varepsilon := \varepsilon_n$ we now define $\varepsilon_n^* \in (0, \varepsilon_\epsilon]$ by the right hand-side of (3.4). Then, Lemma A.9.2 shows $\varepsilon_n^* \to 0$, and hence we obtain $\varepsilon_n^* \leq \min\{\varepsilon_n, \varepsilon_\epsilon\}$ for all sufficiently large $n$. In addition, $\delta_n$ and $\varepsilon_n$ satisfy (3.2) for sufficiently large $n$ by (4.1), and we also have $\varepsilon_n \leq \epsilon \land \varepsilon_\epsilon$ and $\delta_n \leq \delta_\epsilon \land \delta_{\text{thick}}$ for sufficiently large $n$. Consequently, there exists an $n_0 \geq 1$ such that, for all $n \geq n_0$, the values $\varepsilon_n$, $\delta_n$, $T_n$ and $\varepsilon_n^*$ satisfy the assumptions of Theorem 3.1 as well as $\varepsilon_n \leq \epsilon \land \varepsilon_\epsilon$ and $\delta_n \leq \delta_\epsilon$.

Let us now consider an $n \geq n_0$ and a data set $D \in X^n$ satisfying both the assertions i) - v) of Theorem A.6.2 and (2.10). By Theorem 3.1 and our previous considerations we then know that the probability $P_n^u$ of $D$ is not less than $1 - e^{-\zeta}$. Now, part i) of Theorem A.6.2 gives both $\rho_D^* \geq \rho^* + 2\varepsilon_n > \rho^* + \varepsilon_n$ and $\rho_D^* \leq \rho^* + \varepsilon_n + 5\varepsilon_n \leq \rho^* + 6\varepsilon_n^* \leq \rho_\epsilon$, and hence (A.9.3) and (A.9.4) hold for $\varepsilon := \varepsilon_n$, $\delta := \delta_n$, and $\rho := \rho_D^*$. Consequently, (2.10) shows

\[
\mu(B_1(D) \setminus A^1_{\rho^*}) + \mu(B_2(D) \setminus A^2_{\rho^*}) \leq 2\mu(A^1_{\rho^*} \setminus (A^1_{\rho^*+\varepsilon} \setminus \delta]) + 2\mu(A^1_{\rho^*} \setminus (A^1_{\rho^*+\varepsilon} \setminus \delta])
\]

\[
\leq 4\varepsilon + \mu(M^{+\delta}_{\rho^*} \setminus \tilde{M}_{\rho^*}) \leq 5\varepsilon,
\]

\[
\mu(M^{+\delta}_{\rho^*} \setminus \tilde{M}_{\rho^*}) \leq \epsilon.
\]
where in the second to last step we also used (A.1.4).

\[ \square \]

**A.10. Additional Material Related to Rates.** In this section, the assumption made in Section 4 are discussed in some more detail.

Let us begin with the following lemma, which gives a sufficient condition for a non-trivial separation exponent.

**Lemma A.10.1.** Let \( X \subset \mathbb{R}^d \) be compact and convex, \( \| \cdot \| \) be some norm on \( \mathbb{R}^d \), and \( P \) be a Lebesgue absolutely continuous distribution on \( X \) that can be clustered between the levels \( \rho^* \) and \( \rho^{**} \). Assume that \( P \) has a continuous density \( h \) and that there exist constants \( c > 0 \) and \( \theta \in (0, \infty) \) such that

\[
\tag{A.10.1} |h(x) - h(x')| \leq c \|x - x'|^\theta
\]

for all \( x \in \{h \leq \rho^*\} \), \( \rho \in (\rho^*, \rho^{**}] \), and \( x' \in \partial_X M_\rho \), where \( \partial_X M_\rho \) denotes the boundary of \( M_\rho \) in \( X \). Then the clusters of \( P \) have separation exponent \( \theta \).

**Proof of Lemma A.10.1.** Let \( \varepsilon \in (0, \rho^{**} - \rho^*] \) and \( A_1 \) and \( A_2 \) be the connected components of \( M_{\rho^*+\varepsilon} \). Since \( A_1 \) and \( A_2 \) are compact, and hence there exist \( x_1 \in A_1 \) and \( x_2 \in A_2 \) with

\[
\tag{A.10.2} a := \|x_1 - x_2\| = d(A_1, A_2),
\]

where we note that \( A_1 \cap A_2 = \emptyset \) implies \( a > 0 \). For \( t \in [0, 1] \), we now consider

\[ x(t) := tx_1 + (1 - t)x_2. \]

Since \( X \) is convex, we note that \( x(t) \in X \) for all \( t \in [0, 1] \). Our first goal is to show that \( x_i \in \partial_X M_{\rho^*+\varepsilon} \) for \( i = 1, 2 \). To this end, we assume the converse, e.g., \( x_2 \in M_{\rho^*+\varepsilon} \). Then there exists an \( \epsilon \in (0, a) \) with \( B_X(x_2, \epsilon) \subset A_2 \), where \( B_X(x_2, \epsilon) := \{x \in X : \|x - x_2\| \leq \epsilon\} \). Now \( \|x(\epsilon/a) - x_2\| = \epsilon \) implies \( x(\epsilon/a) \in A_2 \), while \( \|x(\epsilon/a) - x_1\| = a - \epsilon \) shows \( \|x(\epsilon/a) - x_1\| < d(A_1, A_2) \). Together this contradicts (A.10.2).

For what follows, let us now observe that \( t \mapsto x(t) \) is a continuous map on \( [0, 1] \), and since \( h \) is continuous, there exists a \( t^* \in [0, 1] \) with \( h(x(t^*)) = \min_{t \in [0,1]} h(x(t)) \). Our next goal is to show that

\[
\tag{A.10.3} h(x(t^*)) \leq \rho^*.
\]

To this end, we assume the converse, that is \( h(x(t^*)) > \rho^* \). Then there exists a \( \delta \in (0, \varepsilon] \) such that \( h(x(t)) > \rho^* + \delta \) for all \( t \in [0, 1] \), and therefore an application of Lemma A.1.2 using the continuity of \( h \) yields \( x(t) \in M_{\rho^*+\delta} \) for all \( t \in [0, 1] \). In other words, \( x_1 \) and \( x_2 \) are path-connected in \( M_{\rho^*+\delta} \),
and since the connecting path is a straight line, it is easy to see that $x_1$ and $x_2$ are $\tau$-connected for all $\tau > 0$. Let us pick a $\tau \leq 3\tau^*(\delta) = \tau^*_{M_{\rho^*+\delta}}$. Since $|C(M_{\rho^*+\delta})| = 2$, part ii) of Proposition A.2.10 then shows $C(M_{\rho^*+\delta}) = C_\tau(M_{\rho^*+\delta})$. Let $A_1$ and $A_2$ be the two topologically connected components of $M_{\rho^*+\delta}$. Our previous considerations then showed that $A_1$ and $A_2$ are also the two $\tau$-connected components of $M_{\rho^*+\delta}$. Now, $\delta \leq \varepsilon$ gives a CRM $\zeta : C(M_{\rho^*+\varepsilon}) \rightarrow C(M_{\rho^*+\delta})$, which is bijective, since $P$ can be clustered between $\rho^*$ and $\rho^{**}$. Without loss of generality we may thus assume that $\zeta(A_i) = \tilde{A}_i$ for $i = 1, 2$. This yields $x_i \in A_i \subset \tilde{A}_i$, i.e. $x_1$ and $x_2$ do not belong to the same $\tau$-connected component of $M_{\rho^*+\delta}$. Clearly, this contradicts our observation that $x_1$ and $x_2$ are $\tau$-connected, and hence (A.10.3) is proven.

Now assume without loss of generality that $t^* \in [1/2, 1)$. Since we have already seen that $x_1 \in \partial_X M_{\rho^*+\varepsilon}$, our assumption (A.10.1) and (A.10.3) yield

$$|h(x(t^*)) - h(x_1)| \leq c ||x(t^*) - x_1||^\theta.$$

In addition, Lemma A.1.2 shows $x_1 \in M_{\rho^*+\varepsilon} \subset \{ h \geq \rho^* + \varepsilon \}$. Combining these estimates with (A.10.2) and $d(A_1, A_2) = \tau^*_{M_{\rho^*+\varepsilon}} = 3\tau^*(\varepsilon)$, we find

$$\rho^* + \varepsilon \leq h(x_1) \leq h(x(t^*)) + c ||x(t^*) - x_1||^\theta \leq \rho^* + c ||x(t^*) - x_1||^\theta \leq \rho^* + c 2^{-\theta} d^\theta(A_1, A_2) \leq \rho^* + c (3/2)^{-\theta} \tau^*(\varepsilon)^\theta,$$

and from the latter the assertion easily follows. \hfill $\square$

Note that (A.10.1) holds, if the density $h$ in Lemma A.10.1 is actually $\theta$-Hölder-continuous, and it is easy to see that the converse is, in general, not true. Moreover, using the inclusion $\partial_X M_{\rho} \subset \{ h = \rho \}$ established in Lemma A.1.2, it is easy to check that (A.10.1) is equivalent to

(A.10.4) $$|h(x) - \rho| \leq c d(x, \partial_X M_{\rho})^\theta$$

for all $x \in \{ h \leq \rho^* \}$ and $\rho \in (\rho^*, \rho^{**}]$. Note that a localized but two-sided version of this condition has been used in [8] for a level set estimator that is adaptive with respect to the Hausdorff metric.

Our next goal is to discuss the assumptions made in Theorem 4.7 in more detail. To this end, we need a couple of technical lemmata.

**Lemma A.10.2.** Let $X \subset \mathbb{R}^d$ be compact and convex and $d$ be a metric on $X$ that is defined by a norm on $\mathbb{R}^d$. Then, we have

$$d(x, \partial_X A) \leq d(x, X \setminus A)$$

for all $A \subset X$ and $x \in \overline{A}$, where $\partial_X A$ denotes the boundary of $A$ in $X$.
Proof of Lemma A.10.2. Before we begin with the proof we note that \( B^X = \overline{B}^d \) for all \( B \subset X \) since \( X \) is closed, i.e., taking the closure with respect to \( X \) or \( \mathbb{R}^d \) is the same. Like in the statement of the lemma, we will thus omit the superscript. Let us now write \( \delta := d(x, X \setminus A) \). Then there exists a sequence \( (x_n) \subset X \setminus A \) such that \( d(x, x_n) \rightarrow \delta \). Since \( X \) is assumed to be compact, so is \( X \setminus A \), and thus there exists an \( x_\infty \in X \setminus A \) such that \( d(x, x_\infty) \leq \delta \). Obviously, it suffices to show \( x_\infty \in \partial X A \). Let us assume the converse. Since \( \partial X A = \overline{A} \cap X \setminus \overline{A} \), we then have \( x_\infty \not\in \overline{A} \), that is \( x_\infty \in X \setminus \overline{A} \). Now, the latter set is open in \( X \), and hence there exists an \( \varepsilon > 0 \) such that \( B_X(x_\infty, \varepsilon) \subset X \setminus \overline{A} \), where \( B_X(x_\infty, \varepsilon) \) denotes the closed ball in \( X \) that has center \( x_\infty \) and radius \( \varepsilon \). This \( \varepsilon \) must satisfy \( \varepsilon < \delta \), since otherwise we would find a contradiction to \( x_\infty \in \overline{A} \) by \( x \in B_X(x_\infty, \delta) \subset B_X(x_\infty, \varepsilon) \subset X \setminus \overline{A} \). For \( t := \varepsilon/\delta \in (0, 1) \) we now define \( x' := tx + (1 - t)x_\infty \). The convexity of \( X \) implies \( x' \in X \), and since \( d \) is defined by a norm, we have \( d(x_\infty, x') = td(x, x_\infty) \leq \varepsilon \). Together, this yields \( x' \in B_X(x_\infty, \varepsilon) \subset X \setminus \overline{A} \subset X \setminus \overline{A} \). Consequently, \( d(x, x') = (1 - t)d(x, x_\infty) \leq (1 - t)\delta < \delta \) implies \( d(x, X \setminus A) < \delta \), which contradicts the definition of \( \delta \). \( \square \)

Lemma A.10.3. Let \( X \subset \mathbb{R}^d \) be compact and convex and \( d \) be a metric on \( X \) that is defined by a norm on \( \mathbb{R}^d \). Then, for all \( A \subset X \) and \( \delta > 0 \), we have

\[
A^+ - \delta \setminus A^{-\delta} \subset (\partial X A)^{+\delta},
\]

where the operations \( A^{+\delta} \) and \( A^{-\delta} \) as well as the boundary \( \partial X A \) are with respect to the metric space \((X, d)\).

Proof of Lemma A.10.3. Let us fix an \( x \in A^{+\delta} \setminus A^{-\delta} = A^{+\delta} \cap (X \setminus A)^{+\delta} \). If \( x \in \overline{A} \), then Lemma A.10.2 immediately yields \( d(x, \partial X A) \leq d(x, X \setminus A) \leq \delta \), that is \( x \in (\partial X A)^{+\delta} \). It thus suffices to consider the case \( x \not\in \overline{A} \). Then we find \( x \in X \setminus \overline{A} \subset X \setminus A \subset X \setminus \overline{A} \), and hence another application of Lemma A.10.2 yields \( d(x, \partial X (X \setminus A)) \leq d(x, A) \leq \delta \). Now the assertion easily follows from \( \partial X (X \setminus A) = \overline{X} \setminus \overline{A} \cap (X \setminus A) = \overline{X} \setminus A \cap \overline{A} = \partial X A \). \( \square \)

The next lemma shows that assuming an \( \alpha \)-smooth boundary with \( \alpha > 1 \) does not make sense. It further shows that, for each level set with rectifiable boundary in the sense of [5, 3.2.14], the bound (4.9) holds with \( \alpha = 1 \).

Lemma A.10.4. Let \( \lambda^d \) be the \( d \)-dimensional Lebesgue measure, \( \mathcal{H}^{d-1} \) be the \((d - 1)\)-dimensional Hausdorff measure on \( \mathbb{R}^d \), and \( \sigma_d \) be the volume of the \( d \)-dimensional unit Euclidean ball in \( \mathbb{R}^d \). Then, for every non-empty, bounded, and measurable subset \( A \subset \mathbb{R}^d \) the following statements hold:
i) There exists a $\delta_A > 0$, such that for $\zeta_A := d\sigma_d^{1/d} \lambda^d(\overline{A})^{1-1/d}/2$ and all $\delta \in (0, \delta_A]$, we have

$$\lambda^d(A^+ \setminus A^{-\delta}) \geq \zeta_A \cdot \delta.$$  

ii) If $\partial A$ is $(d-1)$-rectifiable and $\mathcal{H}^{d-1}(\partial A) > 0$, there exists a $\delta_A > 0$, such that, for all $\delta \in (0, \delta_A]$, we have

$$\lambda^d(A^+ \setminus A^{-\delta}) \leq 4\mathcal{H}^{d-1}(\partial A) \cdot \delta.$$  

Proof of Lemma A.10.4. Let us first recall that, for an integer $0 \leq m \leq d$, the upper and lower Minkowski content of a $B \subset \mathbb{R}^d$ is defined by

$$\mathcal{M}_m^{*}(B) := \limsup_{\delta \to 0^+} \frac{\lambda^d(B^+ \delta)}{\sigma_d^m \delta^{d-m}},$$

$$\mathcal{M}_m(B) := \liminf_{\delta \to 0^+} \frac{\lambda^d(B^+ \delta)}{\sigma_d^m \delta^{d-m}},$$

where $\sigma_d^m$ denotes the $\lambda_d^{d-m}$-volume of the unit Euclidean ball in $\mathbb{R}^{d-m}$. It is easy to check that these definitions coincide with those in [5, 3.2.37].

i). Since in the case $\lambda^d(\overline{A}) = 0$ there is nothing to prove, we restrict our considerations to the case $\lambda^d(\overline{A}) > 0$. Now, $A$ is bounded, and hence we have $\lambda^d(\overline{A}) < \infty$. The isoperimetric inequality [5, 3.2.43] thus yields

$$d\sigma_d^{1/d} \lambda^d(\overline{A})^{1-1/d} \leq \mathcal{M}_d^{*}(\partial A),$$

and hence, there exists a $\delta_A > 0$, such that, for all $\delta \in (0, \delta_A]$, we have

$$\frac{d\sigma_d^{1/d} \lambda^d(\overline{A})^{1-1/d}}{2} \leq \frac{\lambda^d((\partial A)^+ \delta)}{\sigma_1 \delta} \leq \frac{\lambda^d(A^{+2\delta} \setminus A^{-2\delta})}{2\delta},$$

where in the last estimate we used part viii) of Lemma A.3.1 and $\sigma_1 = 2$.

ii). Since $\partial A$ is closed and $(d-1)$-rectifiable in the sense of [5, 3.2.14], we find

$$\mathcal{M}^{*(d-1)}(\partial A) = \mathcal{H}^{d-1}(\partial A)$$

by [5, 3.2.39]. Moreover, since $\partial A$ is bounded, the boundary is contained in a compact set $X \subset \mathbb{R}^d$ such that the relative boundary $\partial_X A$ of $A$ in $X$ equals $\partial A$ and the sets $A^+ \delta$ and $A^{-\delta}$ considered in $X$ equal the sets $A^+ \delta$ and $A^{-\delta}$ when considered in $\mathbb{R}^d$ for all $\delta \in (0, 1]$. By Lemma A.10.3 there thus exists a $\delta_A > 0$ such that

$$\frac{\lambda^d(A^{+\delta} \setminus A^{-\delta})}{2\delta} \leq \frac{\lambda^d((\partial A)^+ \delta)}{\sigma_1 \delta} \leq 2\mathcal{H}^{d-1}(\partial A)$$

for all $\delta \in (0, \delta_A].$
The next lemma shows that a bound (4.9) together with a regular behavior of $h$ around the level of interest ensures a non-trivial flatness exponent.

**Lemma A.10.5.** Let $(X,d)$ be a complete, separable metric space, $\mu$ be a finite Borel measure on $X$ with $\text{supp} \mu = X$, and $P$ be a $\mu$-absolutely continuous distribution on $X$. Furthermore, let $\rho \geq 0$ be a level and $h$ be a $\mu$-density of $P$ for which there exist constants $c > 0$, $\alpha \in (0,1]$, $\delta_0 > 0$, and $\theta \in (0,\infty)$ such that

\[
\mu(M^+_{\rho+\delta} \setminus M^-_{\rho-\delta}) \leq c \delta^\alpha
\]

for all $\delta \in (0,\delta_0)$ and

\[
d(x,\partial M_\rho)^\theta \leq c|h(x) - \rho|
\]

for all $x \in \{h > \rho\}$. Then $P$ has flatness exponent $\alpha/\theta$ at level $\rho$.

**Proof of Lemma A.10.5.** Let us fix an $s > 0$. For $x \in \{0 < h - \rho < s\}$ we then find $d(x,\partial M_\rho)^\theta \leq cs$ by (A.10.6), that is $x \in (\partial M_\rho)^{+\delta}$ for $\delta := (cs)^{1/\theta}$. Using part viii) of Lemma A.3.1, we conclude that $x \in M^+_{\rho+2\delta} \setminus M^-_{\rho-2\delta}$. In the case $2\delta \leq \delta_0$, we thus obtain

\[
\mu(\{0 < h - \rho < s\}) \leq \mu(M^+_{\rho+2\delta} \setminus M^-_{\rho-2\delta}) \leq 2^{\alpha}c\delta^\alpha = 2^{\alpha}c^{1+\alpha/\theta}\delta^{\alpha/\theta},
\]

and since $\mu$ is a finite measure, it is then easy to see that we can increase the constant on the right-hand side so that it holds for all $s > 0$.

**Appendix B. Continuous Densities in two Dimensions.** In this appendix, we present a couple of two-dimensional examples that show that the assumptions imposed in the paper are not only met by many discontinuous densities, but also by many continuous densities.

**B.1. Single Two-Dimensional Sets.** In this section we consider the operations $\oplus\delta$ and $\ominus\delta$ for a specific class of sets $A \subset \mathbb{R}^2$.

We begin with an example of a set $A \subset \mathbb{R}^2$, for which we can compute $A \oplus\delta$ and $A \ominus\delta$ explicitly. This example will be the base of all further examples.

**Example B.1.1.** Let $X := [-1,1] \times [-2,2]$ be equipped with the metric defined by the supremums norm. Furthermore, for $x^+_{\pm} \in (-0.6,0.4)$ and $x^-_{\pm} \in (0.4,0.6)$ we fix two continuous functions $f^-, f^+: [-1,1] \to [-1,1]$ such that $f^+$ is increasing on $[-1,x^+_\pm] \cup [0,x^-_\pm]$ and decreasing on $[x^-_\pm,0] \cup$
Finally, we have \( x_+^1, 1 \), while \( f^- \) is decreasing on \([-1, x_-^-] \cup [0, x_-^+] \) and increasing on \([x_-^-, 0] \cup [x_-^+, 1] \). In addition, assume that \( \{ f^- < 0 \} = \{ f^+ > 0 \} \) and \( \{ f^- = 0 \} = \{ f^+ = 0 \} \) as well as \( f^-(\pm 0.5) < 0 \) and \( f^+(\pm 0.5) > 0 \). Now consider the (non-empty) set \( A \) enveloped by \( f^\pm \), that is
\[
A := \{ (x, y) \in [-1, 1] \text{ and } f^-(x) \leq y \leq f^+(x) \}.
\]

To describe \( A^\oplus \delta \) for \( \delta \in (0, 0.1] \), we define \( f^\pm_\delta : [-1, 1] \to [-1, 1] \) by
\[
f^\pm_\delta(x) := \begin{cases} 
\pm(1) & \text{if } x \in [-1, -1 + \delta] \\
\pm(0) & \text{if } x \in [-\delta, \delta] \\
\pm(1) & \text{if } x \in [1 - \delta, 1]
\end{cases}
\]
and \( f^-_\delta(x) := f^-(x - \delta) \lor f^-((x + \delta), \text{ respectively } f^+_\delta(x) := f^+(x - \delta) \land f^+(x + \delta) \) for the remaining \( x \in [-1, 1] \). Then we have
\[
A^\oplus \delta = \{ (x, y) \in [-1, 1] \text{ and } f^-_\delta(x) + \delta \leq y \leq f^+_\delta(x) - \delta \}.
\]

Moreover, to describe \( A^\otimes \delta \), we define
\[
x_{0,-1} := \min \{ x \in [-1, -0.5] : f^+(x) - f^-(x) \geq 0 \} \]
x_{0,-0} := \max \{ x \in [-0.5, 0] : f^+(x) - f^-(x) \geq 0 \} \]
x_{0,+0} := \min \{ x \in [0, 0.5] : f^+(x) - f^-(x) \geq 0 \} \]
x_{0,+1} := \max \{ x \in [0.5, 1] : f^+(x) - f^-(x) \geq 0 \},
\]
where the minima are attained by the continuity of \( f^\pm \) and the fact that all sets are non-empty. Furthermore, we define \( f^\pm_\otimes : [-1, 1] \to [-1, 1] \) by
\[
f^\pm_\otimes(x) := \begin{cases} 
\pm(x + \delta) & \text{if } x \in [-1 \lor (x_{0,-1} - \delta), x_-^+ - \delta] \\
\pm(x_-^+) & \text{if } x \in [x_-^+ - \delta, x_-^+ + \delta] \\
\pm(x_-^+) & \text{if } x \in [x_-^+ - \delta, x_-^+ + \delta] \\
\pm(x^-) & \text{if } x \in [x_-^+ + \delta, (x_{0,+1} + \delta) \land 1]
\end{cases}
\]
as well as \( f^-_\otimes(x) := f^-(x - \delta) \land f^-(x + \delta) \) and \( f^+_\otimes(x) := f^+(x - \delta) \lor f^+(x + \delta) \) for \( x \in [x_-^+ + \delta, x_-^+ - \delta] \setminus (x_{0,-0} + \delta, x_{0,+0} - \delta) \) and \( f^+_\otimes(x) := -2\delta \) for the remaining \( x \in [-1, 1] \). Then we have
\[
A^\otimes \delta = \{ (x, y) \in [-1, 1] \text{ and } f^-_\otimes(x) - \delta \leq y \leq f^+_\otimes(x) + \delta \}.
\]
Finally, we have \( |C(A)| \leq 2 \) with \( |C(A)| = 2 \) if and only if \( x_{0,-0} < x_{0,+0} \), and in the latter case we further have \( \tau^A_{x_{0,+0}} = x_{0,+0} - x_{0,-0} \).
Proof of Example B.1.1. Let us fix a $\delta \in (0, 1/10]$. To simplify notations, we further write $g^- := f^- + \delta$ and $g^+ := f^+ - \delta$.

Proof of “$A_{\oplus \delta} \subset \ldots$”. By $A_{\oplus \delta} = X \setminus (X \setminus A)_{\oplus \delta}$ it suffices to show that

$$\{(x, y) \in X : x \in [-1, 1] \text{ and } (y < g^-(x) \text{ or } y > g^+(x))\} \subset (X \setminus A)_{\oplus \delta}.$$

By symmetry, it further suffices to consider the case $x \geq 0$ and $y > g^+(x)$. Moreover, to show the inclusion above, it finally suffices to find $x' \in [-1, 1]$ and $y' \in [-2, 2]$ with $|x - x'| \leq \delta$, $|y - y'| \leq \delta$ and $y' > f^+(x')$. However, this task is straightforward. Indeed, we can always set $y' := (y + \delta) \wedge 2$, and if $x \in [0, \delta]$ then $x' := 0$ works, since $y' = (y + \delta) \wedge 2 > g^+(x) + \delta = f^+(0) = f^+(x')$,
while for $x \in [1 - \delta, 1]$, the choice $x' := 1$ does by an analogous argument. Finally, if $x \in (\delta, 1 - \delta)$, we set $x' := x - \delta$ if $g^+(x) = f^+(x - \delta) - \delta$ and $x' := x + \delta$ if $g^+(x) = f^+(x + \delta) - \delta$.

Proof of “$A_{\oplus \delta} \supset \ldots$”. Again, it suffices to consider $x \geq 0$. Let us fix a $y$ with $g^-(x) \leq y \leq g^+(x)$. Then, our goal is to show $(x, y) \notin (X \setminus A)_{\oplus \delta}$, i.e.,

\begin{equation}
(B.1.1) \quad \| (x, y) - (x', y') \|_\infty > \delta
\end{equation}

for all $(x', y') \in X \setminus A$. In the following, we thus fix a pair $(x', y') \in X \setminus A$ for which (B.1.1) is not true and show that this leads to a contradiction. We begin by considering the case $x \in [0, \delta]$. Since (B.1.1) is not true, we find $|x - x'| \leq \delta$, and hence $x^+ \leq x' \leq x^+_\oplus$. Then, if $y' > f^+(x')$, this leads to

$$y \leq g^+(x) = f^+(0) - \delta \leq f^+(x') - \delta < y' - \delta,$$

which contradicts the assumed $|y - y'| \leq \delta$. The case $y' < f^-(x')$ analogously leads to a contradiction. Now consider the case $x \in [1 - \delta, 1]$. Then $|x - x'| \leq \delta$ implies $x' \geq x^+_\oplus$. Thus, $y' > f^+(x')$ leads to another contradiction by

$$y \leq g^+(x) = f^+(1) - \delta \leq f^+(x') - \delta < y' - \delta,$$

and the case $y' < f^-(x')$ can be treated analogously. It thus remains to consider the case $x \in [\delta, 1 - \delta]$. Then $|x - x'| \leq \delta$ implies $x - \delta \leq x' \leq x + \delta$. For $x' \leq x^+\oplus$ we thus find $f^+(x - \delta) \leq f^+(x')$, while for $x' \geq x^+_\oplus$ we find $f^+(x + \delta) \leq f^+(x')$. For $y' > f^+(x')$ we hence obtain a contradiction by

$$y \leq g^+(x) = (f^+(x - \delta) \wedge f^+(x + \delta)) - \delta \leq f^+(x') - \delta < y' - \delta,$$

and, again, the case $y' < f^-(x')$ can be shown similarly.

Proof of “$A_{\oplus \delta} \subset \ldots$”. Let us fix a pair $(x, y) \in A_{\oplus \delta}$. Without loss of generality we restrict our considerations to the case $y \geq 0$ and $x \in [-1, 0]$. To show that $y \leq f^+_{\oplus \delta}(x) + \delta$ we assume the converse, that is $y > f^+_{\oplus \delta}(x) + \delta$. 
Since \((x, y) \in A^{\geq \delta}\) we then find \((x', y') \in A\) with \(\|(x, y) - (x', y')\|_{\infty} \leq \delta\). From the latter we infer that both \(x - \delta \leq x' \leq x + \delta\) and

\[
y' \geq y - \delta > f_{+\delta}^+(x).
\]

If \(x \in [-1, -1 \lor (x_{0,-1} - \delta))\) we get a contradiction, since \((x', y') \in A\) implies \(x \geq x' - \delta \geq x_{0,-1} - \delta\). Moreover, for \(x \in [-1 \lor (x_{0,-1} - \delta), x_+ - \delta]\), we obtain

\[
f_{+\delta}^+(x) = f^+(x + \delta) \geq f^+(x') \geq y',
\]

which contradicts (B.1.2). If \(x \in [x_+ - \delta, x_+ + \delta]\) we get a contradiction from

\[
f_{+\delta}^+(x) = f^+(x + \delta) \geq f^+(x') \geq y',
\]

and if \(x \in [x_+ + \delta, 0 \land (x_{0,-0} + \delta)]\) we have

\[
f_{+\delta}^+(x) = f^+(x - \delta) \lor f^+(x + \delta) \geq f^+(x - \delta) \geq f^+(x') \geq y'
\]

which again contradicts (B.1.2). Finally, if \(x \in (0 \lor x_{0,-0} + \delta, 0]\) we obtain a contradiction from \(x > x_{0,-0} + \delta \geq x' + \delta\).

Proof of “\(A^{\geq \delta} \supset \ldots\)” Let us fix a pair \((x, y) \in X\) with \(f_{+\delta}^+(x) - \delta \leq y \leq f_{+\delta}^+(x) + \delta\). Without loss of generality we again consider the case \(y \geq 0\) and \(x \in [-1, 0]\), only. To show \((x, y) \in A^{\geq \delta}\) we need to find a pair \((x', y') \in A\) with \(\|(x, y) - (x', y')\|_{\infty} \leq \delta\). Let us assume that we have found an \(x'\) with \(|x - x'| \leq \delta\) and \(f(x') \geq y - \delta\). For \(y'\) defined by

\[
y' := f(x') \land (y + \delta)
\]

we then immediately obtain \(y' \leq y + \delta\). Moreover, if we actually have \(y' = y + \delta\), then we obtain \(|y - y'| \leq \delta\), while in the case \(y' < y + \delta\) we find \(y' = f(x') \geq y - \delta\), that is again \(|y - y'| \leq \delta\). Thus, it suffices to find an \(x'\) with the properties above. To this end, we first observe that we can exclude the case \(x \in [-1, -1 \lor (x_{0,-1} - \delta))\), since for such \(x\) we have \(0 \leq y \leq f_{+\delta}^+(x) + \delta = -\delta\). Analogously, we can exclude the case \(x \in (0 \lor (x_{0,-0} + \delta), 0]\). Now consider the case \(x \in [-1 \lor (x_{0,-1} - \delta), x_+ - \delta]\). For \(x' := x + \delta\) we then have

\[
f(x') = f(x + \delta) = f_{+\delta}^+(x) \geq y - \delta,
\]

and hence \(x'\) satisfies the desired properties. Moreover, for \(x \in [x_+ - \delta, x_+ + \delta]\) we define \(x' := x_+\), which gives \(|x - x'| \leq \delta\). In addition, we again have \(f(x') = f(x_+) = f_{+\delta}^+(x) \geq y - \delta\). Finally, let us consider the case \(x \in [x_+ + \delta, 0 \land (x_{0,-0} + \delta)]\). Let us first assume that \(f(x - \delta) \geq f(x + \delta)\). For \(x' := x - \delta\) we then obtain \(f(x') = f(x - \delta) = f_{+\delta}^+(x) \geq y - \delta\). Analogously, if \(f(x - \delta) \leq f(x + \delta)\), then \(x' := x + \delta\) has the desired properties.

Finally, \(\|C(A)\| \leq 2\) is obvious, and so is the equivalence between \(\|C(A)\| = 2\) and \(x_{0,-0} < x_{0,+0}\). In the latter case, \(A_1 := \{(x, y) \in A : x \leq x_{0,-0}\}\) and \(A_2 := \{(x, y) \in A : x \geq x_{0,+0}\}\) are the two components of \(A\), and from this it is easy to conclude that \(\tau_\lambda^+ = x_{0,+0} - x_{0,-0}\).
Our next example shows how to estimate the function \( \psi_A^* \) for the sets considered in Example B.1.1

**Example B.1.2.** Let us consider the situation of Example B.1.1. To simplify the presentation, let us additionally assume that the monotonicity of \( f^+ \) and \( f^- \) is actually strict and that \( A \) has sufficiently thick parts on both sides of the \( y \)-axis in the sense of

\[
(B.1.3) \quad [-0.8, -0.2] \cup [0.2, 0.8] \subset \{ f^- \leq -0.2 \} \cap \{ f^+ \geq 0.2 \}.
\]

Note that, for all \( \delta \in (0, 0.1] \), this condition in particular ensures that \( A^\delta \) contains open neighborhoods around the points \((-0.5, 0)\) and \((0, 0.5)\). Moreover, for \( \delta \in [0, 0.1] \) we define

\[
\begin{align*}
x_{\delta,-1} &:= \min \{ x \in [-1, -0.8] : f^+(x) - f^-(x) \geq 2\delta \} \\
x_{\delta,0} &:= \max \{ x \in [-0.2, 0] : f^+(x) - f^-(x) \geq 2\delta \} \\
x_{\delta,+0} &:= \min \{ x \in [0, 0.2] : f^+(x) - f^-(x) \geq 2\delta \} \\
x_{\delta,+1} &:= \max \{ x \in [0.8, 1] : f^+(x) - f^-(x) \geq 2\delta \},
\end{align*}
\]

where we note that the minima and maxima are attained by (B.1.3) and the continuity of \( f^\pm \). For the same reason we further have \( x_{\delta,-1} < -0.8 \), \( x_{\delta,0} > -0.2 \), \( x_{\delta,+0} < 0.2 \), and \( x_{\delta,+1} > 0.8 \). Then, \( f^\pm_\delta \) has exactly two local maxima \( x^\pm_{\delta,-} \) and \( x^\pm_{\delta,+} \), satisfying \( x^\pm_{\delta,-} \in [-1, 0] \) and \( x^\pm_{\delta,+} \in [0, 1] \), and \( f^-_\delta \) has exactly two local minima \( x^-_{\delta,-} \) and \( x^-_{\delta,+} \), satisfying \( x^-_{\delta,-} \in [-1, 0] \) and \( x^-_{\delta,+} \in [0, 1] \). Moreover, for all \( \delta \in (0, 0.1] \) we have

\[
\psi_A^*(\delta) \leq \delta + \left( \max \{ |x_{\delta,i} - x_{0,i}| : i \in \{-1, -0, +0, +1\} \} \right. \\
\left. \vee \max \{ |f^i(x^j) - f^j_\delta(x^i_{\delta,j})| : i, j \in \{-, +\} \} \right).
\]

The right hand-side of this inequality can be further estimated under some regularity assumptions. Indeed, if there exist \( c > 0 \) and \( \gamma \in (0, 1] \) such that

\[
(B.1.4) \quad |f^\pm(x^+_\pm) - f^\pm(x)| \leq c|x^+_\pm - x|^{\gamma}, \quad x \in [x^+_\pm - 0.1, x^+_\pm + 0.1],
\]

then, for all \( \delta \in (0, 0.1] \), we can bound the second maximum by

\[
\max \{ |f^i(x^j) - f^j_\delta(x^i_{\delta,j})| : i, j \in \{-, +\} \} \leq c\delta^{\gamma}.
\]

In addition, if, for some \( i \in \{-1, -0, +0, +1\} \), we write \( 2\delta_0 := f^+(x_{0,i}) - f^-(x_{0,i}) \), then \( x_{\delta,i} - x_{0,i} = 0 \) for all \( \delta \in (0, \delta_0] \), i.e. the corresponding term
in the first maximum disappears for these \( \delta \). If \( \delta_0 < 0.1 \), and we additionally assume, for example, that

\[(B.1.5) \quad |f^\pm(x)| \geq c^{-1/\gamma}|x_{0,-1} - x|^{1/\gamma}\]

for all \( x \in [x_{0,-1}, -0.8] \), then we have \( |x_{\delta,-1} - x_{0,-1}| \leq c\delta^\gamma \) for all \( \delta \in (\delta_0, 0.1] \). Combining these assumptions we obtain a variety of sets satisfying \( \psi^*_A(\delta) \leq (c + 1)\delta^\gamma \) for all \( \delta \in (0, 0.1] \), and these examples of sets can be even further extended by considering bi-Lipschitz transformations of \( X \).

Before we can prove the assertions made in the example above, we need to establish the following technical lemma.

**Lemma B.1.3.** Let \( x^* \in [2/5, 3/5] \) and \( f : [0,1] \to \mathbb{R} \) be a continuous function that is strictly increasing on \([0, x^*]\) and strictly decreasing on \([x^*, 1]\). For \( \delta \in (0, 1/\delta] \) we define \( f_{-\delta} : [0,1] \to \mathbb{R} \) by

\[
f_{-\delta}(x) := \begin{cases} f(0) & \text{if } x \in [0, \delta] \\ f(x - \delta) \land f(x + \delta) & \text{if } x \in [\delta, 1 - \delta] \\ f(1) & \text{if } x \in [1 - \delta, 1]. \end{cases}
\]

Then there exists exactly one \( x^*_\delta \in [0,1] \) such that \( f_{-\delta}(x^*_\delta) \geq f_{-\delta}(x) \) for all \( x \in [0,1] \). Moreover, we have \( x^*_\delta \in (x^* - \delta, x^* + \delta) \) and \( x^*_\delta \) is the only element \( x \in [\delta, 1 - \delta] \) that satisfies \( f(x - \delta) = f(x + \delta) \). Finally, we have

\[
f_{-\delta}(x) = \begin{cases} f(x - \delta) & \text{if } x \in [\delta, x^*_\delta] \\ f(x + \delta) & \text{if } x \in [x^*_\delta, 1 - \delta]. \end{cases}
\]

**Proof of Lemma B.1.3.** We first show that there is an \( x_0 \in (x^* - \delta, x^* + \delta) \) such that \( f(x_0 - \delta) = f(x_0 + \delta) \). To this end, we observe \( g : [x^* - \delta, x^* + \delta] \to \mathbb{R} \) defined by \( g := f(\cdot - \delta) - f(\cdot + \delta) \) is continuous, and since \( g(x^* - \delta) = f(x^* - 2\delta) - f(x^*) < 0 \) and \( g(x^* + \delta) = f(x^* + \delta) - f(x^* + 2\delta) > 0 \), we find an \( x_0 \in (x^* - \delta, x^* + \delta) \) such that \( g(x_0) = 0 \) by the intermediate value theorem.

Let us now show that \( f(x - \delta) < f(x + \delta) \) for all \( x \in [\delta, x_0] \) and \( f(x - \delta) > f(x + \delta) \) for all \( x \in [x_0, 1 - \delta] \). Clearly, for \( x \in [\delta, x^* - \delta] \), the strict monotonicity of \( f \) on \([0, x^*]\) yields \( f(x - \delta) < f(x + \delta) \). Moreover, for \( x \in (x^* - \delta, x_0) \), we have \( f(x - \delta) < f(x_0 - \delta) = f(x_0 + \delta) < f(x + \delta) \) since \( f(\cdot - \delta) : [x^* - \delta, x^* + \delta] \to \mathbb{R} \) is strictly increasing, while \( f(\cdot + \delta) : [x^* - \delta, x^* + \delta] \to \mathbb{R} \) is strictly decreasing. This shows the assertion for \( x \in [\delta, x_0] \), and the assertion for \( x \in [x_0, 1 - \delta] \) can be shown analogously.
Combining the two results above, we find that there exists exactly one $x_0 \in [\delta, 1 - \delta]$ satisfying \(f(x_0 - \delta) = f(x_0 + \delta)\), and for this $x_0$ we further know $x_0 \in (x^* - \delta, x^* + \delta)$. In addition, these results show
\[
  f_{-\delta}(x) = \begin{cases} 
    f(x - \delta) & \text{if } x \in [\delta, x_0] \\
    f(x + \delta) & \text{if } x \in [x_0, 1 - \delta].
  \end{cases}
\]

Let us now return to global maximizers of $f_{-\delta}$. To this end, we first observe that the existence of a global maximum of $f_{-\delta}$ follows from the continuity of $f_{-\delta}$ and the compactness of $[0, 1]$. Let us now fix an $x_\delta \in [0, 1]$ at which this global maximum is attained by $f_{-\delta}$. We first observe that $x_\delta \in (\delta, 1 - \delta)$. Indeed, if, e.g., we had $x_\delta \geq 1 - \delta$, we would obtain $f(1) = f_{-\delta}(x_\delta) \geq f_{-\delta}(1 - 2\delta) = f(1 - 3\delta) \wedge f(1 - \delta) = f(1 - \delta) > f(1)$ using $1 - 3\delta > x^*$, and $x_\delta \leq \delta$ would similarly lead to a contradiction. We next show that we actually have $x_\delta \in [x^* - \delta, x^* + \delta]$. To this end, it suffices to show
\[
  (B.1.6) \quad x_\delta \geq x^* - \delta \iff x_\delta \leq x^* + \delta.
\]

To show one implication, assume that $x_\delta \geq x^* - \delta$. Since $f_{-\delta}$ attains its maximum at $x_\delta$, we then obtain
\[
  f(x_\delta + \delta) \geq f(x_\delta - \delta) \wedge f(x_\delta + \delta) = f_{-\delta}(x_\delta) \geq f_{-\delta}(x^* + \delta) = f(x^* + 2\delta).
\]

Now $x_\delta + \delta \leq x^* + 2\delta$ follows from the assumed $x_\delta + \delta \geq x^*$ and the strict monotonicity of $f$ on $[x^*, 1]$. Analogously, $x_\delta \leq x^* + \delta \Rightarrow x_\delta \geq x^* - \delta$ can be shown, and hence (B.1.6) is indeed true.

Finally, we can prove the remaining assertion. To this end, we pick again an $x_\delta$ at which $f_{-\delta}$ attains its maximum. Then we have already seen that $x_\delta \in [x^* - \delta, x^* + \delta]$. Now observe that assuming $x_\delta < x_0$ leads to $f(x_\delta - \delta) < f(x_0 - \delta) = f(x_0 + \delta) < f(x_\delta + \delta)$ using $x_0, x_\delta \in [x^* - \delta, x^* + \delta]$, which in turn yields the contradiction
\[
  f_{-\delta}(x_\delta) = f(x_\delta - \delta) \wedge f(x_\delta + \delta) = f(x_\delta - \delta) < f(x_0 - \delta) \wedge f(x_0 + \delta) = f_{-\delta}(x_0).
\]

Analogously, we find a contradiction assuming $x_\delta > x_0$, and hence we have $x_\delta = x_0$. Consequently, $x_\delta$ is unique and solves $f(x - \delta) = f(x + \delta)$.

**Proof of Example B.1.2.** We first note that the existence and uniqueness of the local extrema is guaranteed by Lemma B.1.3. In addition, this lemma actually shows $x_{\delta,-}^+ \in (x_{\delta,-}^+ - \delta, x_{\delta,-}^+ + \delta)$, $x_{\delta,-}^- \in (x_{\delta,-}^- - \delta, x_{\delta,-}^- + \delta)$, $x_{\delta,+}^+ \in (x_{\delta,+}^+ - \delta, x_{\delta,+}^+ + \delta)$, and $x_{\delta,+}^- \in (x_{\delta,+}^- - \delta, x_{\delta,+}^- + \delta)$. Moreover, we have
\[
  \psi_A^*(\delta) = \sup_{z \in A} d(z, A^{-\delta}) \leq \sup_{z \in A} d(z, A^{\delta})
\]
by $A^{-\delta} \subset A^{\ominus \delta}$. We will thus estimate $d(z, A^{\ominus \delta})$ for $z := (x, y) \in A$.

We begin with the case $x \in [-1, x_{\delta,-1}]$. For later purposes, note that the definition of $A$ yields $x \geq x_{0,-1}$. By the monotonicity of $f^\pm$ on $[-1, -0.8 + \delta]$ we further know $f^\pm_\delta(x + \delta) = f^\pm_\delta(x)$. We write $x' := x_{\delta,-1} + \delta$ and

$$y' := \begin{cases} f^-(x_{\delta,-1}) + \delta & \text{if } y \leq f^-(x_{\delta,-1}) + \delta \\ y & \text{if } y \in [f^-(x_{\delta,-1}) + \delta, f^+(x_{\delta,-1}) - \delta] \\ f^+(x_{\delta,-1}) - \delta & \text{if } y \geq f^+(x_{\delta,-1}) - \delta. \end{cases}$$

If $y \leq f^-(x_{\delta,-1}) + \delta$, we then obtain $y' \leq y'$ and $y' = f^-(x_{\delta,-1}) + \delta \leq f^-(x) + \delta \leq y + \delta$, that is $|y - y'| \leq \delta$, and it is easy to check that the same is true in the two other cases. Consequently, we have $\| (x, y) - (x', y') \|_{\infty} = x_{\delta,-1} + \delta - x$, and our construction further ensures

$$y' \in [f^-(x_{\delta,-1}) + \delta, f^+(x_{\delta,-1}) - \delta] = [f^-_\delta(x') + \delta, f^+_\delta(x') - \delta].$$

By Example B.1.1 we conclude $(x', y') \in A^{\ominus \delta}$, and from this we easily find

(B.1.7) \[ d(z, A^{\ominus \delta}) \leq \delta + x_{\delta,-1} - x \leq \delta + x_{\delta,-1} - x_{0,-1}. \]

To show that (B.1.7) is also true in the case $x \in [x_{\delta,-1}, -0.8 + \delta]$, we first observe that the monotonicity of $f^\pm$ on $[-1, -0.8 + 2\delta]$ yields

$$f^+(x) - f^-(x) \geq f^+(x_{\delta,-1}) - f^-(x_{\delta,-1}) \geq 2\delta,$$

and consequently, we can define

$$y' := \begin{cases} f^-(x) + \delta & \text{if } y \leq f^-(x) + \delta \\ y & \text{if } y \in [f^-(x) + \delta, f^+(x) - \delta] \\ f^+(x) - \delta & \text{if } y \geq f^+(x) - \delta. \end{cases}$$

If $y \leq f^-(x) + \delta$ we then obtain $y \leq y'$ and $y' = f^-(x) + \delta \leq y + \delta$, that is $|y - y'| \leq \delta$, and again it is easy to check that the same is true in the two other cases. Writing $x' := x + \delta$, we thus have $\| (x, y) - (x', y') \|_{\infty} = \delta$. Moreover, the construction together with $f^\pm_\delta(x + \delta) = f^\pm_\delta(x)$ ensures

$$y' \in [f^-(x) + \delta, f^+(x) - \delta] = [f^-_\delta(x') + \delta, f^+_\delta(x') - \delta],$$

and hence we find $(x', y') \in A^{\ominus \delta}$ by Example B.1.1. Thus, we have shown $d(z, A^{\ominus \delta}) \leq \delta \leq \delta + x_{\delta,-1} - x_{0,-1}$, i.e. (B.1.7) is true for all $x \in [-1, -0.8 + \delta]$.

Now consider the case $x \in [-0.8 + \delta, -0.2 - \delta]$. Here, we will focus on the sub-case $y \geq 0$, since the subcase $y \leq 0$ can be treated analogously. For later
purposes, note that we have $f^-(x \pm \delta) \leq -2\delta$. Now, if $x \in [-0.8 + \delta, x_{\delta,-}^+]$, we set $x' := x + \delta$ and $y' := y \land (f^+(x) - \delta)$. This gives $y' \leq y$ and $y - \delta \leq f^+(x) - \delta \leq y'$, and hence we again have $\|(x, y) - (x', y')\|_{\infty} = \delta$. Moreover, our constructions together with Lemma B.1.3 ensures

$$y' \in [-\delta, f^+(x) - \delta] = [-\delta, f_{-\delta}^+(x') - \delta] \subset [f_{-\delta}^-(x') + \delta, f_{+\delta}^+(x') - \delta],$$

that is $(x', y') \in A^{\subseteq\delta}$, and hence (B.1.7) is true in this case, too. The next case, we consider, is $x \in [x_{\delta,-}^+ - \delta, x_{\delta,-}^+ + \delta]$. In this case we set $x' := x_{\delta,-}^+$ and $y' := y \land (f_{+\delta}^+(x_{\delta,-}^+) - \delta)$. This implies

$$y' \in [-\delta, f_{+\delta}^+(x_{\delta,-}^+) - \delta] \subset [f_{-\delta}^-(x') + \delta, f_{+\delta}^+(x') - \delta],$$

and hence $(x', y') \in A^{\subseteq\delta}$. We further have $|x - x'| \leq \delta$ and, if $y \leq f_{+\delta}^+(x_{\delta,-}^+ - \delta$, we also have $|y - y'| = 0$. Conversely, if $y \geq f_{+\delta}^+(x_{\delta,-}^+) - \delta$, we find

$$y \leq f^+(x) \leq f^+(x^+) = f^+(x^+) - (f_{+\delta}^+(x_{\delta,-}^+) - \delta) + y',$$

that is $|y - y'| \leq \delta + f^+(x^+) - f_{+\delta}^+(x_{\delta,-}^+)$. Combining the latter two cases, we therefore obtain $\|(x, y) - (x', y')\|_{\infty} \leq \delta + f^+(x^+) - f_{+\delta}^+(x_{\delta,-}^+)$, that is $d(z, A^{\subseteq\delta}) \leq \delta + f^+(x^+) - f_{+\delta}^+(x_{\delta,-}^+)$. Since all remaining cases can be treated analogously, the proof of the general estimate of $\psi_A^\ast(\delta)$ is finished.

Now consider the additional assumptions of $f^\pm$. For example, assume

$$|f^+(x^+) - f^+(x)| \leq c|x^+ - x|$$

for all $x \in [x^+ - 0.1, x^+ + 0.1]$. Lemma B.1.3 shows $x_{\delta,-}^+ \in (x^+ - \delta, x^+ + \delta)$. Without loss of generality, we assume $x_{\delta,-}^+ \in [x^+, x^+ + \delta)$. Using Lemma B.1.3 and $x_{\delta,-}^+ - \delta \in [x^+, x^+ + \delta)$, we then obtain

$$|f^+(x^+) - f_{+\delta}^+(x_{\delta,-}^+)| = |f^+(x^+) - f^+(x_{\delta,-}^+) - \delta| \leq c|x^+ - x_{\delta,-}^+ + \delta| \leq c\delta^\gamma.$$

Now assume that, for e.g. $i := -1$, we have $\delta_0 > 0$. For $\delta \in (0, \delta_0]$ we then find $f^+(x_{0,-1}) - f^-(x_{0,-1}) \geq 2\delta$, and thus $x_{0,-1} = x_{\delta,-1} = -1$. Conversely, let $\delta \in (\delta_0, 0.1]$. Then we have $f^+(x_{0,-1}) - f^-(x_{0,-1}) < 2\delta$ and a simple application of the intermediate value theorem thus yields $f^+(x_{\delta,-1}) - f^-(x_{\delta,-1}) = 2\delta$. Using the additional assumption on $f^\pm$ around the point $x_{0,-1}$, we then find

$$2c^{-1/\gamma}|x_{\delta,-1} - x_{0,-1}|^{1/\gamma} \leq |f^-(x_{\delta,-1})| + |f^+(x_{\delta,-1})| = f^+(x_{\delta,-1}) - f^-(x_{\delta,-1}) = 2\delta,$$

that is $|x_{\delta,-1} - x_{0,-1}| \leq c\delta^\gamma$. □
B.2. Continuous Densities. In this section we present a class of continuous densities on $\mathbb{R}^2$ that meet the assumptions made in the paper. The first example, which represents the main result of this supplement, shows that many continuous distributions satisfy our thickness assumption.

**Example B.2.1.** Let $X := [-1,1] \times [-2,2]$ be equipped with the metric defined by the supremums norm. Moreover, let $P$ be a Lebesgue absolutely continuous distribution that has a continuous density $h$. Furthermore, assume that there exists a $\rho^{**} > 0$, such that, for all $\rho \in (0, \rho^{**}]$, the level set $M_\rho$ is of the form considered in Example B.1.2. In addition, we assume that there is a constant $K \in (0,1)$ such that

\[(B.2.1) \quad |h(x,y) - \rho^* - x^2 + y^2| \leq K(x^2 + y^2)\]

for some $\rho^* \in [0, \rho^{**})$ and all $(x,y) \in \{h > 0\} \cap \{[-0.2, 0.2] \times (-1.1,1.1)\}$. Moreover, assume that $h$ is continuously differentiable on the sets

\[
A_1 := \{h > 0\} \cap \left(([-0.7, -0.3) \cup [0.3, 0.7)) \times ((-1.1, -0.2) \cup (0.2, 1.1))\right)
\]

\[
A_2 := \{h > 0\} \cap \left(([-1, -0.8) \cup (0.8, 1)) \times ((-1.1, 0) \cup (0.2, 1.1))\right)
\]

\[
A_3 := \{h > 0\} \cap \left\{(x,y) \in X : x \in (-0.2, 0) \cup (0, 0.2) \text{ and } |y| < \sqrt{\frac{1+K}{1-K}|x|}\right\}
\]

with $h_y := \frac{\partial h}{\partial y} \neq 0$ on $A_1$ and $h_x := \frac{\partial h}{\partial x} \neq 0$ on $A_2 \cup A_3$. Finally, assume that there is a constant $C > 0$ such that $|h_x| \leq C|h_y|$ on $A_1$ and $|h_y| \leq C|h_x|$ on $A_2 \cup A_3$. Then $P$ has thick levels of order $\gamma = 1$ with $\delta_{\text{thick}} = 0.1$ and

\[c_{\text{thick}} = 1 + \max\left\{C, \sqrt{\frac{1+K}{1-K}}\right\} .\]

Moreover, $P$ can be clustered between $\rho^*$ and $\rho^{**}$ and we have

\[(B.2.2) \quad \frac{2}{\sqrt{1-K}} \sqrt{\varepsilon} \leq \tau_{\rho^*}^* \leq \frac{2}{\sqrt{1+K}} \sqrt{\varepsilon}, \quad \varepsilon \in (0, \rho^{**} - \rho] .\]

**Proof of Example B.2.1.** Since we consider the Lebesgue measure on $X$, we have $M_0 = X$. Moreover, we have $X^{-\delta} = X$ since we consider the operation in $X$, and from this, we immediately see $\psi^*_X(\delta) = 0$ for all $\delta > 0$. Consequently, there is nothing to prove for $\rho = 0$.

Let us now fix some $\rho \in (0, \rho^{**}]$. Moreover, let $f^\pm : [-1,1] \to [-1,1]$ be the two functions satisfying the assumptions of Example B.1.2 and

\[M_\rho = \{(x,y) \in X : x \in [-1,1] \text{ and } f^-(x) \leq y \leq f^+(x)\} .\]
We pick an \((x, y) \in M_\rho\) with \(y = f^+(x)\) or \(y = f^-(x)\). Then we find \((x, y) \in \partial M_\rho\), and thus we have \(h(x, y) = \rho\) by Lemma A.1.2, that is \(h(x, f^+(x)) = \rho\).

Our first goal is to verify (B.1.4). To this end, we solely focus without loss of generality to the case \(x^+_1\) and \(f^+\), since the other cases can be treated analogously. Let us fix an \(x \in [x^+_1 - 0.1, x^+_1 + 0.1]\). Then we have \(x \in (0.3, 0.7)\) and thus \(f^+(x) \in (0.2, 1.1)\) by (B.1.3). Consequently, \(h\) is continuously differentiable in \((x, f^+(x))\). By the implicit function theorem and the previously shown \(h(x', f^+(x')) = \rho\) for all \(x' \in (0.3, 0.7)\) we then conclude that \(f^+\) is continuously differentiable at \(x\) and

\[
(B.2.3) \quad (f^+(x))' = - \left(\frac{\partial h}{\partial y}(x, f^+(x))\right)^{-1} \cdot \frac{\partial h}{\partial x}(x, f^+(x)) = \frac{h_x(x, f^+(x))}{h_y(x, f^+(x))}.
\]

Using \(|h_x| \leq C|h_y|\) on \(A_1\), we thus find \(|(f^+(x))'| \leq C\), and hence \(f^+\) is Lipschitz continuous on \((0.3, 0.7)\) with Lipschitz constant smaller than or equal to \(C\). This implies (B.1.4) with constant \(C\) and exponent \(\gamma = 1\).

Now consider the endpoints \(x_{0, \pm 1}\), where again it suffices to consider one case, say \(x_{0, -1}\), due to symmetry. Let us write \(2\delta_0 := f^+(x_{0, -1}) - f^-(x_{0, -1})\). Then, if \(\delta_0 \geq 0.1\), we have \(|x_{\delta, -1} - x_{0, -1}| = 0\) for all \(\delta \in (0, 0.1]\) by Example B.1.2, and hence it suffice to show (B.1.5) in the case \(\delta_0 < 0.1\). Observing that it actually suffices to show (B.1.5) for all \(x \in (x_{0, -1}, -0.8)\) by continuity, we begin by fixing such an \(x\). By monotonicity we then have \(0 < f^+(x) < f^+(0.8) < 1.1\), and hence \(h\) is continuously differentiable at \((x, f^+(x))\). The implicit function theorem and the previously shown \(h(x', f^+(x')) = \rho\) for all \(x' \in (x_{0, -1}, -0.8)\), then shows that \(f^+\) is continuously differentiable at \(x\) and (B.2.3) holds. Using \(|h_x| \leq C|h_y|\) on \(A_2\), we then find \(|(f^+(x))'| \geq 1/C\), and the fundamental theorem of calculus thus yields

\[
|f^+(x') - f^+(x)| = \left| \int_x^{x'} (f^+(t))' \, dt \right| \geq C^{-1} |x' - x|
\]

for all \(x, x' \in (x_{0, -1}, -0.8)\). Now, letting \(x' \to x_{0, -1}\), we obtain

\[
|f^+(x)| \geq f^+(x) - f^+(x_{0, -1}) = |f^+(x) - f^+(x_{0, -1})| \geq C^{-1} |x_{0, -1} - x|
\]

for all \(x \in (x_{0, -1}, -0.8)\), i.e. (B.1.5) holds with constant \(C\) and \(\gamma = 1\).

Finally, let us consider the points \(x_{0, \pm 0}\), where yet another time, we only focus on one case, say \(x_{0, +0}\). For \(x \in [x_{0, +0}, 0.2]\), we then have

\[
(B.2.4) \quad \rho = h(x, f^+(x)) \leq \rho^* + (1 + K)x^2 + (K - 1)(f^+(x))^2,
\]

that is \((f^+(x))^2 \leq \frac{\rho^* - \rho}{1 + K} + \frac{1 + K}{1 - K} x^2\). Analogously, we can find a lower bound on \((f^+(x))^2\), so that we end up having

\[
(B.2.5) \quad (f^+(x))^2 \in \left[\frac{\rho^* - \rho}{1 + K} + \frac{1 - K}{1 + K} x^2, \frac{\rho^* - \rho}{1 - K} + \frac{1 + K}{1 - K} x^2\right],
\]
and an analogue result holds for \((f^-(x))^2\). Again, our goal is to show an analogue of (B.1.5). To this end, we first consider the case \(\rho \in (0, \rho^*]\). By (B.2.1), we then know that \(h(0,0) = \rho^* \geq \rho\), and hence \(f^+(0) \geq 0\). Analogously, we find \(f^-(0) \leq 0\), which together implies \(x_{0,+0} = 0\). Furthermore, for \(x \in [x_{0,+0}, 0.2]\), (B.2.5) gives
\[
f^+(x) \geq \sqrt{\frac{\rho^* - \rho}{1 + K} + \frac{1 - K}{1 + K} x^2} \geq \sqrt{\frac{1 - K}{1 + K}} |x_{0,+0} - x|,
\]
that is (B.1.5) holds with constant \(\sqrt{\frac{1+K}{1-K}}\) and exponent \(\gamma = 1\). Let us now consider the case \(\rho \in (\rho^*, \rho^{**}]\). For \(x \in (x_{0,+0}, 0.2)\), (B.2.5) then yields
\[
f^+(x) \leq \sqrt{\frac{\rho^* - \rho}{1 - K} + \frac{1 + K}{1 - K} x^2} < \sqrt{\frac{1 + K}{1 - K}} |x|,
\]
and thus we find \((x, f^+(x)) \in A_3\). Consequently, \(h\) is continuously differentiable at \((x, f^+(x))\), and (B.2.3) holds. As for \(x_{0,-1}\), we can then show that (B.1.5) holds with constant \(C\) and exponent \(\gamma = 1\).

In order to show that \(P\) can be clustered between the levels \(\rho^*\) and \(\rho^{**}\), we first note that the assumed continuity of \(h\) guarantees that \(P\) is normal by Lemma A.1.3. Let us now fix a \(\rho \in (\rho^*, \rho^{**}]\). Since from (B.2.1) we infer that \(h(0,0) = \rho^*\), we then obtain \((0,0) \not\in M_\rho\). The latter implies \(x_{0,-0} < 0 < x_{0,+0}\), where \(x_{0,-0}\) and \(x_{0,+0}\) are the points defined in Example B.1.2 for the set \(M_\rho\). By Example B.1.1 we then see that \(|C(M_\rho)| = 2\). Analogously, for \(\rho \in [0, \rho^*]\), the equality \(h(0,0) = \rho^*\) implies \(x_{0,-0} = 0 = x_{0,+0}\), which shows \(|C(M_\rho)| = 1\). Finally, the bijectivity of \(\zeta : C(M_\rho^{**}) \to C(M_\rho)\) follows from the form of the connected components described in Example B.1.1.

Let us finally prove (B.2.2). To this end, we fix an \(\varepsilon \in (0, \rho^{**} - \rho]\) and define \(\rho := \rho^* + \varepsilon\). Then we have already observed that \(x_{0,-0} < 0 < x_{0,+0}\), and hence \(f^\pm(x_{0,+0}) = 0\). For \(x := x_{0,+0}\) we then obtain
\[
\rho = h(x, f^+(x)) \leq \rho^* + (1 + K)x^2
\]
by (B.2.4), and applying some simple transformations we thus find \(x_{0,+0} = x \geq \sqrt{\frac{\rho - \rho^*}{1 + K}} = \sqrt{\frac{\varepsilon}{1 + K}}\). For \(x := x_{0,+0}\) we further have
\[
\rho = h(x, f^+(x)) \geq \rho^* + (1 - K)x^2,
\]
and thus \(x_{0,+0} \leq \sqrt{\frac{\varepsilon}{1 - K}}\). Since analogous estimates can be derived for \(x_{0,-0}\), the formula \(\tau_{M_\rho^{**}+\varepsilon} = x_{0,+0} - x_{0,-0}\) found in Example B.1.1 gives (B.2.2). \(\Box\)
The last example of this appendix shows that the distributions from the previous example have a smooth boundary.

Example B.2.2. Let $X$ and $P$ be as in Example B.2.1. Then the clusters have an $\alpha$-smooth boundary for $\alpha = 1$ and

$$c_{\text{bound}} = 8\left(10 + C + \sqrt{\frac{1 + K}{1 - K}}\right).$$

Proof of Example B.2.2. Let us first consider the case $0 < \delta \leq 0.1$. To this end, we fix a $\rho \in (\rho^*, \rho^{**}]$. Without loss of generality, we only consider the connected component $A$ with $x < 0$ for all $(x, y) \in A$. We know that $A^{+\delta/2} \backslash A^{-\delta/2} \subset A^{\oplus\delta} \backslash A^{\ominus\delta}$ and the latter two sets have been calculated in Example B.1.1. In the following, we will only estimate $\lambda^2(\{(x, y) : y \geq 0\} \cap A^{\oplus\delta} \backslash A^{\ominus\delta})$, the case $y \leq 0$ can be treated analogously. Our first intermediate result towards the desired estimate is

$$\lambda^2([-1 \lor (x_{0,-1} - \delta), x_{\delta,-1}] \times [0, 2] \cap A^{\oplus\delta} \backslash A^{\ominus\delta}) \leq 2|2(x_{0,-1} - \delta) - x_{\delta,-1}| \leq 2\delta + 2|x_{0,-1} - x_{\delta,-1}| \leq 2(1 + C)\delta,$$

where in the last step we used that the proof of Example B.2.1 showed (B.1.5) for $c = C$ and $\gamma = 1$. Moreover, we have

$$\lambda^2([x_{\delta,-1}, x_{\delta}^+ - \delta] \times [0, 2] \cap A^{\oplus\delta} \backslash A^{\ominus\delta}) = \int_{x_{\delta,-1}}^{x_{\delta}^+ - \delta} f^+(x + \delta) - f^+(x - \delta) + 2\delta \, dx$$

$$\leq 2\delta + \int_{x_{\delta}^+ - \delta}^{x_{\delta}^+ + \delta} f(x) \, dx$$

$$\leq 4\delta$$

and analogously we obtain $\lambda^2([x_{\delta}^+ + \delta, x_{\delta,-0}] \times [0, 2] \cap A^{\oplus\delta} \backslash A^{\ominus\delta}) \leq 4\delta$. In addition, we easily find $\lambda^2([x_{\delta}^- - \delta, x_{\delta}^+ + \delta] \times [0, 2] \cap A^{\oplus\delta} \backslash A^{\ominus\delta}) \leq 4\delta$ and finally, we have

$$\lambda^2([x_{\delta}^- - 0 \land (x_{0,-0} + \delta)] \times [0, 2] \cap A^{\oplus\delta} \backslash A^{\ominus\delta}) \leq 2|x_{\delta}^- - 0 - x_{0,-0} - \delta|$$

$$\leq 2\delta + 2\sqrt{\frac{1 + K}{1 - K}}\delta,$$
where we used that the proof of Example B.2.1 showed (B.1.5) for \( c = \sqrt{1+K \over 1-K} \) and \( \gamma = 1 \). Combining all these estimates we obtain

\[
\lambda^2([-1,0] \times [0,2] \cap A \ominus \delta \setminus A \ominus \delta) \leq 4 \left( 6 + C + \sqrt{1+K \over 1-K} \right) \delta
\]

for all \( \delta \in (0,0.05] \). Moreover, for \( \delta \in [0.05,1] \) we easily obtain

\[
\lambda^2([-1,0] \times [0,2] \cap A \ominus \delta \setminus A \ominus \delta) \leq 2 \leq 40 \delta.
\]

Combining both estimates and adding the case \( y \leq 0 \), we then obtain the assertion.

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