The Eta decay into three neutral pions

is mainly electromagnetic

A. Nehme

Lebanese University
Faculty of Sciences at Nabatieh
Mount Amel, Lebanon

(Dated: June 20, 2011)

Abstract

In the framework of Chiral Perturbation Theory including photons, we found that the contribution of the photon exchange between two intermediate charged Kaons to the slope parameter of the decay $\eta \to 3\pi^0$ amounts to $-0.0221 \pm 0.0034$. When compared with the experimental value, $\alpha = -0.0317 \pm 0.0016$, on the one hand, and with the contribution of the up and down quark mass difference, $+0.013 \pm 0.032$, on the other hand, our result leads to the direct conclusion: The $\eta \to 3\pi^0$ decay cannot be used to determine $m_d - m_u$. 

*Electronic address: anehme@ul.edu.lb
I. INTRODUCTION

The decay $\eta \to 3\pi$, being forbidden by isospin symmetry, takes contributions from the up and down quark mass difference, $m_d - m_u$, and from the electric charge, $e$. Owing to Sutherland’s theorem\cite{1, 2}, the latter contribution is suppressed with respect to the former. As a consequence, the decay in question has been considered for a long while as the principal source of information about $m_d - m_u$. Theoretically, the decay was studied in the framework of Chiral Perturbation Theory up to two loops\cite{3, 4}. The predicted value for the slope parameter in the neutral channel, $\alpha_{str} = 0.013 \pm 0.032$ is in complete discrepancy with the observed value as quoted by the Particle Data Group\cite{5}, $\alpha_{exp} = -0.0317 \pm 0.0016$, if we compare central values. The situation did not improve when taking into account the electromagnetic interaction up to one loop\cite{6, 7} and the suppression of the latter has been firmly confirmed. Recently, a (partial) two-loop calculation\cite{8} showed that the correction induced by the diagram with a photon exchange between two intermediate charged pions on the slope parameter amounts to $\alpha_{\gamma} = 0.0029$ and cannot be simply neglected. Motivated by this promising result, we calculate in the present work the correction induced by the exchange of a virtual photon between two intermediate charged kaons on the slope parameter.

II. THE SLOPE PARAMETER

We follow the notation of \cite{8} unless mentioned. The $s$-channel amplitude is written as

$$M(s) = -\frac{\epsilon_{\eta\pi^0}}{3F_\pi^2} (M_\eta^2 - M_\pi^0) \left[ 1 + \delta_{str}(s) + \delta_{em}(s) + \delta_{\gamma}(s) \right] + \tilde{\delta}_{em}(s) + \tilde{\delta}_{\gamma}(s),$$

(1)

with $\epsilon_{\eta\pi^0}$ is the $\eta$-$\pi^0$ mixing angle and reads

$$\epsilon_{\eta\pi^0} = \frac{\sqrt{3}}{4} \frac{m_d - m_u}{m_s - \bar{m}} + \text{higher-order terms}.$$  

(2)

The strong interaction correction $\delta_{str}$ contains both one- and two-loop contributions\cite{3, 4} and is of $\mathcal{O}(p^4)$ in the chiral counting. The one-loop electromagnetic corrections, $\delta_{em}$ and $\tilde{\delta}_{em}$ are of respective chiral orders $\mathcal{O}(e^2)$ and $\mathcal{O}(e^2p^2)$, and have been calculated in \cite{7, 9} and \cite{6, 7, 9}, respectively. The two-loop electromagnetic correction $\delta_{\gamma}$ is of $\mathcal{O}(e^2p^2)$ and has been partially calculated in \cite{8}. Finally, the two-loop electromagnetic correction $\tilde{\delta}_{\gamma}$ is of $\mathcal{O}(e^2p^4)$ and is calculated in the present work. The corresponding Feynman diagrams with a virtual
FIG. 1: Feynman diagrams with photon exchange between two intermediate charged Kaons. The dashed arrow represents the Eta, plain arrows represent neutral pions, plain lines represent charged kaons, and the wavy line represent a photon. The last diagram vanishes.

photon exchange between two intermediate charged Kaons are sketched in Fig. 1. The slope parameter is written as

\[ \alpha = \alpha_{\text{str}} + \alpha_{\text{em}} + \tilde{\alpha}_{\text{em}} + \alpha_{\gamma} + \tilde{\alpha}_{\gamma}, \]  

(3)

where,

\[ \alpha_{\text{str}} = \frac{1}{9} M_\eta^2 (M_\eta - 3M_{\pi^0})^2 \text{Re} \delta''_{\text{str}}(s_0), \]  

(4)

\[ \alpha_{\text{em}} = \frac{1}{9} M_\eta^2 (M_\eta - 3M_{\pi^0})^2 \text{Re} \delta''_{\text{em}}(s_0), \]  

(5)

\[ \alpha_{\gamma} = \frac{1}{9} M_\eta^2 (M_\eta - 3M_{\pi^0})^2 \text{Re} \delta''(s_0), \]  

(6)

\[ \tilde{\alpha}_{\text{em}} = -\frac{F_\pi^2}{3\epsilon_{\eta\pi^0}} \frac{M_\eta^2 (M_\eta - 3M_{\pi^0})^2}{M_\eta^2 - M_{\pi^0}^2} \text{Re} \tilde{\delta}_{\text{em}}''(s_0), \]  

(7)

\[ \tilde{\alpha}_{\gamma} = -\frac{F_\pi^2}{3\epsilon_{\eta\pi^0}} \frac{M_\eta^2 (M_\eta - 3M_{\pi^0})^2}{M_\eta^2 - M_{\pi^0}^2} \text{Re} \tilde{\delta}_{\gamma}''(s_0). \]  

(8)
Diagrams of Fig. 1 contribute to both $\alpha_\gamma$ and $\tilde{\alpha}_\gamma$. Their contribution to the latter can be written, following [8] in terms of five Master Integrals as

$$
\tilde{\alpha}_\gamma = \frac{1}{1152\sqrt{3}\epsilon\pi^0 F^2_\gamma} \frac{e^2}{M^2_{\eta}(M_{\eta} - 3M_{\pi^0})^2} \frac{1}{(4\pi)^D(D - 3)(D - 4)} \times \frac{1}{s^5_0\sigma^6(s_0)} \text{Re} \left( d_1J_1 + d_2J_2 + d_3JT + d_4J^2 + d_5T^2 \right)_{s=s_0}. \tag{9}
$$

We found for the coefficients the following expressions,

$$
d_1 = (D - 2)(512(2D(2D - 7) + 3) + 9)M^4_K
- 128(D - 2)(D(27D - 187) + 294)sM^6_K
- 32((D(D(81D - 76) - 1468) + 2628)s^2
+ (2D(D(2D - 7) + 3) + 9)(M^2_\pi + 3M^2_\eta)^2)M^6_K
+ 8s(2(D(D(21D - 167) + 412) - 312)(M^2_\pi + 3M^2_\eta)^2)M^4_K
+ 2s^2(12(D - 2)(D(79D - 426) + 560)M^6_K
- (D(D(27D - 272) + 868) - 888)(M^2_\pi + 3M^2_\eta)^2)M^6_K
+ (D - 2)s^3(36(D - 2)Ds^2 - (D - 6)(D - 4)(M^2_\pi + 3M^2_\eta)^2)),
$$

$$
d_2 = -6(D - 2)^2(256(D^2 + D - 9)M^8_K - 64(D(5D - 33) + 48)sM^6_K
- 16((D(47D - 14) - 468)s^2 + (D^2 + D - 9)(M^2_\pi + 3M^2_\eta)^2)M^4_K
+ 4s(12(D - 4)(20D - 43)s^2 - (D(7D - 51) + 84)(M^2_\pi + 3M^2_\eta)^2)M^2_K
+ 36(D - 2)Ds^4 - (D - 6)(D - 4)s^2(M^2_\pi + 3M^2_\eta)^2),
$$

$$
d_3 = -256(D(D(7D - 46) + 115) - 108)M^6_K
- 256(D - 2)(D((D - 21)D + 91) - 108)sM^8_K
+ 16((D(D(D(D + 92) - 751) + 1742) - 1028) - 336)s^2
+ (D(D(7D - 46) + 115) - 108)(M^2_\pi + 3M^2_\eta)^2)M^6_K
- 16s((D(D(D(D(3D + 107) - 1195) + 3646) - 3236) - 528)s^2
- (D(D((D - 14)D + 67) - 149) + 132)(M^2_\pi + 3M^2_\eta)^2)M^4_K
+ s^2(12(D - 2)(D(D(3D + 14) - 397) + 1456) - 1520)s^2
- (D(D(D((D - 4)D - 67) + 566) - 1592) + 1536)(M^2_\pi + 3M^2_\eta)^2)M^2_K
+ (D - 2)^2s^3(36(D - 2)Ds^2 - (D - 6)(D - 4)(M^2_\pi + 3M^2_\eta)^2),
$$
\[ d_4 = -2M_K^2(36(D - 3)^2(D - 2)(3D - 8)s^5 - (D - 3)(2048(D - 2)M_K^{10}
\quad + 512(D - 6)(D - 2)sM_K^6 + 64(D - 2)((D - 6)(D - 5)s^2 - 2(M_\pi^2 + 3M_\eta^2)^2)M_K^6
\quad - 16s((D(D(15D - 23) - 108) + 128)s^2 + 2(D - 6)(D - 2)(M_\pi^2 + 3M_\eta^2)^2)M_K^4
\quad + 4s^2(24(D - 3)(D(3D - 1) - 20)s^2 - (D - 6)(D - 2)(M_\pi^2 + 3M_\eta^2)^2)M_K^2
\quad + (D - 5)(D - 4)(3D - 8)s^3(M_\pi^2 + 3M_\eta^2)^2), \]

\[ d_5 = -(D - 2)(384(3D((D - 4)D + 1) + 20)M_K^6
\quad + 32(D(D(D - 48)D + 327) - 776) + 616)sM_K^6
\quad - 8(2(D(D(6D^2 + 75D - 430) - 322) + 1976)s^2
\quad + 3(3D((D - 4)D + 1) + 20)(M_\pi^2 + 3M_\eta^2)^2)M_K^4
\quad + 2s(36(D(D(D + 18) - 217) + 644) - 584)s^2
\quad - (D(D(D + 12) - 201) + 724) - 776)(M_\pi^2 + 3M_\eta^2)^2)M_K^2
\quad + 108(D - 2)^2Ds^4 - (D - 4)^2(3D - 8)s^2(M_\pi^2 + 3M_\eta^2)^2). \] (10)

The analytic expressions for the integrals can be found in [8] with \( M_\pi \) replaced by \( M_K \).

Concerning the \( J \) integral, it should be replaced by the following expression,

\[ J = i \left\{ 1 - M_\pi^{D-4}\Gamma(1 - D/2) - 2 \left( \frac{4M_K^2}{s} - 1 \right)^{1/2} \arctan \left( \frac{4M_K^2}{s} - 1 \right)^{-1/2} \right\}. \] (11)

III. RESULTS AND CONCLUSIONS

We first expand \( \tilde{\alpha}_\gamma \) around \( D = 4 \) and then use the following numerical values

\[ e^2 = \frac{4\pi}{137.04}, \quad (F_\pi, M_{\pi^0}, M_K, M_\eta) = (92.42, 139.57, 493.68, 547.30) \text{ MeV}. \] (12)

For the mixing angle, we use the value \[10,\]

\[ \epsilon_{\eta\pi^0} = 0.013 \pm 0.032. \] (13)

We find that

\[ \tilde{\alpha}_\gamma = -0.0221 \pm 0.0034. \] (14)

On the other hand, we found that the contribution of the diagrams in Fig. 1 to \( \alpha_\gamma \) is equal to \( -0.0005 \). This, together with the pion contribution calculated in [8], give a total of

\[ \alpha_\gamma = 0.0024. \] (15)

5
Adding now the whole electromagnetic contribution to the slope, we get

\[ \alpha_{e^2} = \alpha_{em} + \alpha_{\gamma} + \tilde{\alpha}_{em} + \tilde{\alpha}_{\gamma} = -0.0203 \pm 0.0034. \]  

(16)

Note that the uncertainty comes from the mixing angle only. The obtained result contradicts all the “classical wisdom” based on the suppression of the electromagnetic contribution to the slope parameter. Finally, the main result of the present work, Eq. (14), concerns only Kaon loops with one virtual photon exchanged. In order to obtain finite and scale independent amplitude, one needs to perform three additional calculations:

1. Two-loop diagrams without photons, but with mass difference between charged and neutral mesons (pions and kaons);

2. One-loop diagrams with counterterms of \( \mathcal{O}(e^2p^2) \) in the vertices;

3. Tree-level diagram with \( \mathcal{O}(e^2p^4) \) counterterms.

In order to make the last calculation, the construction of a chiral Lagrangian is necessary.

[1] D. G. Sutherland, Phys. Lett. 23, 384 (1966).
[2] J. S. Bell and D. G. Sutherland, Nucl. Phys. B 4, 315 (1968).
[3] J. Gasser and H. Leutwyler, Nucl. Phys. B250, 539 (1985).
[4] J. Bijnens and K. Ghorbani, JHEP 11, 030 (2007).
[5] K. Nakamura et al., JP G 37, 075021 (2010).
[6] R. Baur, J. Kambor, and D. Wyler, Nucl. Phys. B460, 127 (1996).
[7] C. Ditsche, B. Kubis, and U.-G. Meissner, Eur. Phys. J. C 60, 83 (2009).
[8] A. Nehme and S. Zein, hep-ph/1106.0915.
[9] A. Deandrea, A. Nehme, and P. Talavera, Phys. Rev. D 78, 034032 (2008).
[10] G. Ecker, G. Muller, H. Neufeld and A. Pich, Phys. Lett. B 477, 88 (2000).