A Wong-Zakai theorem for mass critical NLS

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Abstract

We prove a Wong-Zakai theorem for the defocusing mass-critical stochastic nonlinear Schrödinger equation (NLS) on $\mathbb{R}$. The main ingredient are careful mixtures of bootstrapping arguments at both deterministic and stochastic levels. Several subtleties arising from the proof mark the difference between the dispersive case and corresponding situations in SDEs and parabolic stochastic PDEs, as well as the difference between the large-$n$ case and the limiting ($n = \infty$) case.

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1 Introduction

1.1 Statement of the result

We continue our study of the defocusing mass-critical stochastic nonlinear Schrödinger equation on $\mathbb{R}$ with a conservative noise. Consider the model

$$i\partial_t u + \Delta u = |u|^4 u + u \circ \dot{W}, \quad u(0, \cdot) = f \in L^\infty_\omega(\Omega, L^2_x(\mathbb{R})). \quad (1.1)$$

Here, $W$ is a Wiener process on real-valued functions with proper integrability conditions, and $\circ$ denotes the Stratonovich product. Such a choice of noise assures pathwise mass conservation laws.

**Assumption 1.1.** Throughout, we assume the initial data $f \in L^\infty_\omega(\Omega, L^2_x(\mathbb{R}))$, and there exists $\Lambda_{ini} > 0$ such that

$$\|f\|_{L^\infty(\Omega, L^2(\mathbb{R}))} \leq \Lambda_{ini} \lesssim 1. \quad (1.2)$$

The Wiener process $W(x, t)$ has the form

$$W(t, x) := \sum_{k \in \mathbb{N}} B_k(t)V_k(x), \quad (1.3)$$

where $\{B_k\}$ are standard Brownian motions on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $\{V_k\}$ is a sequence of real-valued functions on $\mathbb{R}$. Furthermore, they satisfy

$$\|W\| := \sum_{k \in \mathbb{N}} (\|V_k\|_{L^1} + \|V_k\|_{L^\infty}) =: \sum_{k \in \mathbb{N}} \Lambda_k \leq \Lambda_{noi} \lesssim 1 \quad (1.4)$$

for some $\Lambda_{noi} > 0$.

**Remark 1.2.** From the view point of the assumption (1.4) and the fact that the topic of this article is of well posedness nature, the noise we considered is not very different from $W(x, t) = V(x)B(t)$, $V \in L^1_x \cap L^\infty_x$. However, sometimes, in other types of problems, it will be of interest to require some non-degeneracy of noise, thus we emphasize that our results also works for infinite dimensional noise. It should be remarked we do not require any smoothness of the noise in the space variable. It may also be of interest to generalize our results to more general noise based on the language of $\gamma-$ radonifying operator. We do not handle this technical issue here, though we believe most of our arguments can also work in such a setting. This paper does not aim to handle re-normalization type problems for very singular noise, such as space time white noise.
The equation (1.1), is called mass-critical since its deterministic version (in general dimension $d$) has the following scaling property: suppose $v$ satisfies
\[ i\partial_t v + \Delta v = |v|^4 v, \quad v(0) = v_0 \in L^2_x(\mathbb{R}^d), \] (1.5)
then for every $\lambda > 0$, the rescaled function $v_\lambda(t, x) := \lambda^{-\frac{d}{2}}v(t/\lambda^2, x/\lambda)$ satisfies the same equation, and the $L^2_x$ norm of the initial data is invariant.

The well-posedness for (1.5) is highly nontrivial for general $L^2_x$ initial data. It was proved by Dodson (\cite{Dod13, Dod16b, Dod16a}) that for every initial data in $L^2_x(\mathbb{R}^d)$, $v$ has a global space-time bound in a suitable Strichartz space. In other words, $v$ scatters for every $L^2_x$ initial data.

These results are crucial if one wants to construct global solutions to the stochastic equation. In \cite{FX18a, FX18b, Zha18}, a global solution to (1.1) was constructed. The purpose of this article is to present a Wong-Zakai type result for (1.1).

Let $\pi(n)$ be a sequence of partitions of $[0, 1]$ of the form
\[ \pi(n) := \{0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_m^{(n)} = 1\} \] (1.6)
such that $|\pi(n)| := \max_j |t_{j+1}^{(n)} - t_j^{(n)}| \to 0$ as $n \to +\infty$. For every $n$, let $W^n$ be defined such that $W^n(t_j^{(n)}) = W(t_j^{(n)})$ for every $t_j^{(n)} \in \pi(n)$, and linearly interpolate in between. Let $u^{(n)}$ be the solution to
\[ i\partial_t u^{(n)} + \Delta u^{(n)} = |u^{(n)}|^4 u^{(n)} + u^{(n)} \frac{dW^n}{dt}, \quad u^{(n)}(0, \cdot) = u(0, \cdot), \] (1.7)
Unlike SPDE (1.1) which needs to be formulated via Ito’s stochastic integration, Equation (1.7) is well defined in the classical sense. Note that $\frac{dW^n}{dt}$ is well defined in the classical sense since $W^n$ is piecewise linear. Thus, (1.7) is no longer a stochastic PDE but a classical PDE (with randomness). More precisely, one can rewrite (1.7) as
\[ i\partial_t u^{(n)} + \Delta u^{(n)} = |u^{(n)}|^4 u^{(n)} + u^{(n)} \frac{W(t^{(n)}_{j+1}) - W(t^{(n)}_j)}{t^{(n)}_{j+1} - t^{(n)}_j}, \quad t \in [t_j^{(n)}, t^{(n)}_{j+1}] \] (1.8)

Given the results of Dodson, \cite{Dod13, Dod16b, Dod16a}, for every given realization of $W$, one can see that $u^{(n)}$ is globally well posed. Such global existence of $u^{(n)}$ can be derived from a pure perturbation view point, See Remark 2.5. Our main result is

**Theorem 1.3.** Let $u$ be the solution to (1.1) as constructed in \cite{FX18a, FX18b}. Let $u^{(n)}$ solve (1.7) with same initial data $u(0, \cdot) \in L^\infty_x(\Omega, L^2_x(\mathbb{R}))$, we have that, for every $\rho \geq 1$
\[ E\|u^{(n)} - u\|^\rho_{X(\mathcal{I})} \to 0 \] (1.9)
as $n \to +\infty$.

Here, we have
\[ \mathcal{X}(\mathcal{I}) = L^\infty_t(\mathcal{I}, L^2_x(\mathbb{R})) \cap L^5_t(\mathcal{I}, L^{10}_x(\mathbb{R})). \] (1.10)
Remark 1.4. The results can be easily extended to sub-critical model.

Remark 1.5. Due to the mass critical nature of the problem and the fact we are working on general $L^2$ data, the dynamic in time interval on $[0, 1]$ should not be understood as a short time dynamic, and one indeed has nonlinear dynamic (rather than a perturbation of linear dynamic) within such an interval. For those kinds of problems, global existence is equivalent to existence on any fixed time interval, if one formulate the notion of local solution in such a way.

Remark 1.6. One can also understand this result as another way to construct solution to (1.1). Formula (1.9) has two levels of meaning. First, $u^{(n)}$ converges and limiting process solves (1.1). Second, the limiting process coincides with the process $u$ constructed in [FX18a]. We remark here the construction of solution in [FX18a] is canonical in the sense that one gets a unique limit from natural approximations, see more in Theorem 1.8. Nevertheless, the uniqueness is not in the sense of a prescribed function space. Another motivation of this article is to demonstrate the solution constructed in [FX18a] is the right solution one should consider.

Remark 1.7. One may also want to state and prove a similar result for energy critical problem. But one may probably only get convergence in probability. To get convergence in some space of form $L^\rho_x X[0, 1]$, it seems that it is crucial one consider the mass based model rather than energy based model. And path-wise mass conservation law plays a crucial role. Strictly speaking, one does not need pathwise mass conservation, as long as one can have deterministic pathwise control of growth of mass which are independent of the choice of path. Nevertheless, from the view point of dispersive PDE, we will restrict ourself with a model such that pathwise mass conservation law holds, for the purpose of studying long time behavior/asymptotic later.

1.2 Review of the construction of solution to (1.1)

The Itô form of (1.1) is formally given by

$$i\partial_t u + \Delta u = |u|^4 u + u\dot{W} - \frac{i}{2} V^2 u,$$

where the product between $u$ and $\dot{W}$ is in the Itô sense, and $V^2 = \sum_k V_k^2$ is the Itô-Stratonovich correction.

Let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^+, \mathbb{R}^+)$ such that $\varphi \in [0, 1]$, $\varphi = 1$ on $[0, 1]$ and vanishes outside the interval $[0, 2]$. For every $m > 0$, let $\varphi_m(\cdot) := \varphi(\cdot/m)$. We summarise the construction in [FX18a] in the following.

**Theorem 1.8 (FX18a).** Fix $f \in L^\infty_x L^2_x$, $\forall \rho \geq 1, \forall m > 0$, there exists a unique $u_m \in L^\rho_x X(0, 1)$ adapted to the filtration generated by $W$ such that

$$u_m(t) = e^{it\Delta} f - i \int_0^t e^{i(t-s)\Delta} \left( \varphi_m(||u_m||_{X_2(0,t)})||u_m(s)||^4 u_m(s) \right) ds$$

$$- i \int_0^t e^{i(t-s)\Delta} (u_m(s)dW(s)) - \frac{1}{2} \int_0^t e^{i(t-s)\Delta} (V^2 u_m(s)) ds,$$
where the integral with respect to $W$ is in the Itô sense, and the formula holds in $L^p_\omega X(0, 1)$. Since $u_m \in L^p$ implies $u_m \in L^{p'}$ if $p \geq p'$. We indeed have $u_m \in L^p_\omega X(0, 1)$ for all $p \geq 1$.

Furthermore, $\forall p \geq 1$, $\{u_m\}_m$ is Cauchy in $L^p_\omega X(0, 1)$ as $m \to +\infty$, and the limit $u$ satisfies the Duhamel formula

$$u(t) = e^{it\Delta} f - i \int_0^t e^{i(t-s)\Delta} \left( |u(s)|^4 u(s) \right) ds$$

$$- i \int_0^t e^{i(t-s)\Delta} (u(s)dW(s)) - \frac{1}{2} \int_0^t e^{i(t-s)\Delta} (V^2 u(s)) ds,$$

where the stochastic integral is also in the Itô sense, and the two sides are equal in $L^p_\omega X(0, 1)$.

Remark 1.9. When we wrote [FX18a], the result was stated for smooth noise, but the result in [FX18a] is valid with essentially same proof, with the noise assumption in this article. Furthermore, in some sense, the current article considers all the discretization of the model considered in [FX18a] with stronger estimates, and one can also use the argument in this article to derive the result in [FX18a] with the noise assumption in this article.

Remark 1.10. Even the construction of $u_m$ for a given $m$ does not directly follow from a Picard Iteration regime, and was achieved by applying a sub-critical approximation. But we do have the natural uniqueness in the sense if $u_m$ is in the prescribed space and solves (1.12), then $u_m$ is uniquely determined. We have uniqueness of $u$ in the sense $u_m$ is Cauchy in $m$, thus if one follows our approach, one will have a unique output $u$ for any given initial data. Such a solution $u$ should not be understood as a weak solution derived from compactness argument, which are usually not unique for a given initial data. However, we didn’t prove that if a process $u$ falls in our prescribed space and solves (1.13), then the $u$ must be the same as the one we constructed. One goal of the current article is to argue the solution $u$ we constructed is natural in the sense we have the desired Wong-Zakai type convergence. We finally remark it seem to natural to believe $u = u_m$ for $t \leq \tau_m$, where $\tau_m$ is the stopping time when $\chi_2(0, \tau_m)$ hit $m$. We don’t know whether this is true. If this is true, then we will also have uniqueness for $u$ by only assuming it is in the prescribed space and solves (1.13).

1.3 Background

1.3.1 Mass critical NLS

Mass critical NLS is a typical model for nonlinear dispersive equations. The local well posedness is well known, and can be established following a Picard iteration scheme with Strichartz estimate. See more for Strichartz esitmates in the section of preliminaries. One may refer to [CW89], see also textbook [Caz03, Tao06]. The global well posedness for (defocusing) mass critical NLS with general $L^2$ initial data was a famous open problem and was finally solved by Dodson, [Dod13, Dod16b, Dod16a], see also the reference therein for more background. We summarize his result for the 1D model
**Theorem 1.11 (Dodson).** Let $v$ solves

$$i\partial_t v + \Delta v = |v|^4 v, \quad v(0, \cdot) = v_0 \in L^2(\mathbb{R}).$$

Then $v$ is global and

$$\|v\|_{X^2(\mathbb{R})} \lesssim \|v_0\|_{L^2} 1,$$

It should pointed out that bound (1.15) implies there is $v^+ \in L^2_x$, depending on $v_0$, so that

$$\|v(t) - e^{it\Delta} v^+\|_{L^2_x} \to 0.$$

### 1.3.2 Wellposedness of stochastic NLS

We will focus on the stochastic NLS with a multiplicative noise on the whole space in this part. The study of stochastic NLS with a multilicative noise was initiated in the work of De Bouard and Debussche, [dBD99], [dBD03], where well posedness for sub-critical non-linearity was established. See also refinement and further development in [BRZ14, BRZ16, Hor18]. In [FX18a, FX18b], we extended the result of [dBD99], and proved wellposedness result for 1D mass critical model with $L^2_x$ initial data. Later, Zhang (Zha18) generalized well poseness for mass critical model to all dimensions and proved well posedness for energy critical model, via different method (rescaling method), with different assumption on the noise and notions of solution.

We want to point out if one is interested in $L^p_{\rho\omega}$ bound as in the work of [dBD99], [FX18a, FX18b], it should be expected mass critical model is very different from energy critical model.

We do a short discussion about the difference between the (local) well posedness between stochastic NLS and deterministic NLS. Note that such difference will arise even for very simple noise $W(x,t) = V(x)B(t)$ and sub-critical nonlinearity.

Following the local well posedness of deterministic NLS, one may want to use Duhamel formula, for example, (1.13), and construct solutions via a Picard Iteration in certain space $L^p_{\rho}Y$, where $Y$ is some Banach space. Such an effort will fail for the following simple reason. If $u \in L^p_{\rho}Y$, then the nonlinear term, in our case $|u|^4 u$, can be expected in at most $L^{\rho/5}_{\rho/5} Y'$, no matter which pair of space $(Y, Y')$ one chooses. To overcome this difficulty, one needs to explore the so-called pathwise mass conservation law in the model. Thus, even in the local theory of [dBD99], [FX18a], some non-perturbative information is used.

Finally, we remark that, in the field of stochasite NLS, there seem to be more than one notion of local solutions. Certain local solutions are easier to construct, but may have more difficulty to be extended globally. And if one uses the notions of solutions as in [dBD99], [FX18a], then those local solutions are very easy to be extended to be global, but indeed long time dynamic do appear in very short time, though with small probability. Those solutions are all very natural, and one should be expected they actually equal to each other. One motivation of our article, by showing a Wong-Zakai convergence, is to argue our solutions is the natural candidate of the solutions in the sense it can be approximated by classical solutions of PDEs.
1.3.3 Wong-Zakai convergence

The classical Wong-Zakai type theorem refers to a series of pioneering results by Wong and Zakai ([WZ65a, WZ65b]) on one dimensional SDEs and by Stroock and Varadhan ([SV72]) on multidimensional SDEs, which roughly assert that if $B^{(n)}$ converges to a (finite dimensional) Brownian motion $B$, then the solution $X^{(n)}$ to the multidimensional SDE

$$\text{d}X^{(n)}_t = \mu(X^{(n)}_t)\text{d}t + \sigma(X^{(n)}_t)\text{d}B^{(n)}_t \quad (1.17)$$

converges to the solution $X$ to the Stratonovich SDE

$$\text{d}X_t = \mu(X_t)\text{d}t + \sigma(X_t) \circ \text{d}B_t \quad (1.18)$$

The convergence statement as well as the rate (in terms of $n$) are directly related to sample path continuity of $X_t$.

As for analogous questions for parabolic stochastic PDEs, it is natural to consider the model

$$\text{d}u(t) = \Delta u \text{d}t + f(u)\text{d}t + g(u) \circ \text{d}W_t \quad (1.19)$$

for some Wiener process $W$. The linear operator $e^{t\Delta}$ has a strong smoothing effect (one immediately turns any initial data into a smooth function). Hence, as long as the noise is not too singular and the nonlinear effect is not too strong, the dissipative system essentially behaves like high dimensional ODEs, and Wong-Zakai convergence are well expected.

We point out here if the noise is singular (for example, $W$ being cylindrical Wiener process on $L^2(T)$), then the problem becomes much subtler as the singularity of the noise is strong enough so that the Stratonovich formulation does not exist. The question for singular parabolic SPDEs has been open for a long time, and was successfully handled in [HP15] with the framework of regularity structures.

The aim of this article is to prove a Wong-Zakai type theorem for the nonlinear dispersive (in contrast to dissipative) PDEs, (1.1). In some sense, dispersive equations behave less like high dimensional ODEs than dissipative ones since the linear propagator $e^{it\Delta}$ does not have such smoothing effects on $L^2_x$ initial data. In particular, it does not mild out high frequencies of the solution flow, in particular when one considers a critical nonlinearity. Indeed, even the solution to the linear deterministic equation does not have any Holder continuity as a flow in $L^2_x$. We point out here the nonlinearity we are considering are $L^2_x$-critical, it is strong enough that all levels of frequencies should be taken into account. We remark here, the noise we consider, though not smooth, is essentially finite dimensional and does not have the subtlety as those in singular parabolic PDEs.

Since typical Wong-Zakai convergence are intimately related to the time-continuity of the solution flow, it is then not apriori clear whether such a statement is true even if the limiting equation is well defined, and hence it is our interest to show that this is indeed true. We should remark that we obtain the convergence but without a rate.
1.4 Sketch of proof of the main theorem

In order to show the convergence of \( u^{(n)} \) to \( u \), we will need an extra process \( u_m^{(n)} \) which solves

\[
i \partial_t u_m^{(n)} + \Delta u_m^{(n)} = \phi_m \left( \| u_m^{(n)} \|_{X(t,0)} \right) |u_m^{(n)}|^4 u_m^{(n)} + u_m^{(n)} dW^{(n)}_s, \quad u_m^{(n)}(0, \cdot) = f. \tag{1.20}
\]

We first make a simple observation. Unlike \( u, u_m \) whose formulation relies on stochastic integral, \( u^{(n)}, u_m^{(n)} \) are path-wisely defined, and one easily verifies \( u^{(n)} = u_m^{(n)} \) when \( t \leq \tau^m_n \), where \( \tau^m_n \) is the stopping time \( \| u^{(n)} \|_{X(t,0,\tau^m_n)} = m \).

The main ingredient to prove Theorem 1.3 is the following uniform bound regarding \( u^{(n)} \) and \( u_m^{(n)} \).

**Theorem 1.12.** Given Assumption 1.1 for the noise and initial data, let \( u^{(n)}, u_m^{(n)} \) solve (1.7), (1.20), one has for all \( \rho \geq 1 \),

\[
\| u_m^{(n)} \|_{L^p X(0,1)} \lesssim_{\Lambda_{\text{ini}}, \Lambda_{\text{noi}}, \rho} 1; \quad \| u^{(n)} \|_{L^p X(0,1)} \lesssim_{\Lambda_{\text{ini}}, \Lambda_{\text{noi}}, \rho} 1 \tag{1.21}
\]

uniformly for all \( n \) and \( m \).

Most the estimates in this article involves constant depending \( \Lambda_{\text{ini}}, \Lambda_{\text{noi}} \). Since they are fixed all the time, we don’t emphasize such dependence in the later of the article.

The uniform bound (1.21) will imply the following stability type results, which will allow us to reduce the problem with noise and initial data which are regular in space.

**Corollary 1.13.** Let \( \tilde{u}_m^{(n)} \) and \( \tilde{u}_m \) denote the solutions to (1.20) and (1.12) with initial data \( \tilde{f} \in L^\infty_{\infty} L^2_X \) and noise

\[
\tilde{W}(t, x) = \sum_{k \in \mathbb{N}} B_k(t) \tilde{V}_k(x) \tag{1.22}
\]

satisfying also Assumption 1.1 with the same \( \Lambda_{\text{noi}}, \Lambda_{\text{ini}} \). Note that \( \tilde{W}^{(n)} \), which is the discretisation of \( \tilde{W} \), will be defined similarly as \( W^n \) with the same partition \( \pi^{(n)} \). Then for every \( \varepsilon > 0 \) and \( \rho \geq 1 \), there exist \( \delta > 0 \) depending on \( \rho, \Lambda_{\text{ini}} \) and \( \Lambda_{\text{noi}} \) such that if

\[
\| f - \tilde{f} \|_{L^p L^2} + \sum_{k \in \mathbb{N}} \| V_k - \tilde{V}_k \|_{L^p} \lesssim \delta, 1 = p < \infty \tag{1.23}
\]

then

\[
\| u_m^{(n)} - \tilde{u}_m^{(n)} \|_{L^p X(0,1)} < \varepsilon, \tag{1.24}
\]

and the same is true for \( \| u^{(n)} - \tilde{u}^{(n)} \|_{L^p X(0,1)} \). All the proportionality constants are uniform in both \( n \) and \( m \).

**Remark 1.14.** We will only need finite choice of \( p \) in (1.23). However, since we assume apriori bound for \( \sum_k \| V_k \|_{L^\infty}, \sum_k \| \tilde{V}_k \|_{L^\infty} \), if we assume (1.23) for some \( p = p_0 \), we already implicitly assume (1.23) for all \( p > p_0 \).
To prove Theorem 1.3, one split $u^{(n)} - u$ into
\[ \|u^{(n)} - u\| \leq \|u^{(n)} - u_m^{(n)}\| + \|u_m^{(n)} - u_m\| + \|u_m - u\|, \] (1.25)
where all the norms are $L^p_{\omega}X[0, 1]$. By construction (Theorem 1.8), $u_m \to u$ as $m \to +\infty$. Hence, it remains to show the convergence of the first two terms, which will be the material of the following two propositions. The convergence $u^{(n)} - u$ follows from the uniform bound (1.21) and the aforementioned observation $u^{(n)} = u^{(n)}$ if the $L^p_{t}L^q_{x}$ norm of the latter is smaller than $m$

**Proposition 1.15.** For every initial data $f \in L^{\infty}_{t}L^{2}_{x}$, we have
\[ \sup_{n} \|u^{(n)} - u^{(n)}\|_{L^p_{\omega}X(0,1)} \to 0 \] (1.26)
as $m \to +\infty$.

**Proof.** Let
\[ \Omega^{(n)}_{K} = \{ \omega \in \Omega : \|u^{(n)}\|_{X(0,1)} \geq K \}. \] (1.27)
Recall the definitions of $u^{(n)}$ and $u^{(m)}$ from (1.7) and (1.20). Note that $u^{(m)} = u^{(n)}$ on $(Q_{m}^{(n)})^{c}$. Hence, we have
\[ \|u^{(m)} - u^{(n)}\|_{L^p_{\omega}X(0,1)} = \|1_{Q_{m}^{(n)}}(u^{(m)} - u^{(n)})\|_{L^p_{\omega}X(0,1)} \]
\[ \leq \left( \Pr(\Omega^{(m)}_{r}) \right)^{\frac{1}{q}} \left( \|u^{(m)}\|_{L^q_{\omega}X(0,1)} + \|u^{(n)}\|_{L^q_{\omega}X(0,1)} \right). \] (1.28)

By Theorem 1.12, we have
\[ \Pr(\Omega^{(m)}_{r}) \leq m^{-\rho}\|u^{(n)}\|_{L^p_{\omega}X(0,1)} \lesssim_{\rho} m^{-\rho} \] (1.29)
for all $\rho$. Hence the claim follows. \qed

The convergence $u^{(n)} - u$ is the step we see the Wong-Zakai convergence.

**Proposition 1.16.** For every initial data $u(0, \cdot) \in L^{\infty}_{t}L^{2}_{x}$ and every $m > 0$, we have
\[ \|u^{(m)} - u_m\|_{L^p_{\omega}X(0,1)} \to 0 \] (1.30)
as $n \to +\infty$.

The proof of the above proposition is another main ingredient of the article. It is in this step we see the Wong-Zakai type convergence. It will be split in later sections.

The basic idea is that uniform bound (1.21) allows us to reduce the study of $u^{(n)}$ to $u^{(m)}$, which essentially linearizes the dynamic. This is in particular important since we are working on stochastic problems, the non-linearity cause extra problems due to loss of integrability in probability space. Wong-Zakai convergence is not trivial even for linear stochastic Schrodinger, since $e^{it\Delta}$ does not have time regularity in $t$. That’s why we need the stability arguments Corollary 1.13 to regularize the initial data and noise.
1.5 Organization of the article

According to the above sketch, the proof of the main result (Theorem 1.3) will be complete if one can prove Theorem 1.12, Corollary 1.13, Proposition 1.16. We will prove Theorem 1.12 in Section 3, Corollary 1.13 in Section 5, and Proposition 1.16 Section 5. We present the preliminaries in Section 2.

Notations

We will write $A \lesssim B$ if there exists $C$, so that $A \leq CB$. When such a $C$ depending some parameter, for example, $m$, we will write $A \lesssim_m B$. Similarly we define $B \lesssim A$. In this article, there are several dependence on parameters we typical don’t keep track of, i.e. the dependence of $\rho$ in $L^p$ type estimate and dependence on $\Lambda_{ini}, \Lambda_{noi}$ is (1.1). For example, we will short $\lesssim \rho$ as $\lesssim$. As usual, the constant $C$ may change line by line.

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2 Preliminaries

2.1 Dispersive estimate and Strichartz estimate

We start with the by-now standard dispersive estimates and Strichartz estimates. Let $e^{it\Delta}$ be the free propagator of linear Schrodinger equation, one has the following dispersive estimate,

$$\|e^{it\Delta}f\|_{L^p_t L^p_x} \lesssim t^{\frac{1}{2} - \frac{1}{p}} \|f\|_{L^{p'}}, \quad p \geq 2.$$

and Strichartz estimate,

$$\|e^{it\Delta}f\|_{L_t^q L_x^r(\mathbb{R})} \leq C \|f\|_{L_t^{2} L_x^{\infty}(\mathbb{R})},$$

$$\|\int_0^t e^{i(t-s\Delta)}\sigma(s)ds\|_{L_t^q L_x^r(I)} \leq C \|\sigma\|_{L_t^{q'} L_x^{r'}(I^*)}, \quad (q,r) \text{ admissible.}$$

Here we use $p'$ to denote the conjugate of $p$, such that $\frac{1}{p} + \frac{1}{p'} = 1$. A pair $(q,r)$ is called admissible iff $\frac{2}{q} + \frac{1}{r} = \frac{1}{2}$. In particular, the pair $(5,10)$ is admissible.

One may refer to [Caz03], [KT98] and [Tao06] and reference therein.

2.2 Standard stability and modified stability

In this section, we present several stability results for deterministic NLS. We focus on $d = 1$ to simplify the numeric, but this part has natural generalization to high dimensions. We remark that all the estimates below does not rely on the choice of time interval $[0, T]$. We start with standard and most frequently used stability results for NLS, one may refer to, for example, Lemmas 3.9, 3.10 in [CKS'08].
**Proposition 2.1.** Let \( \tilde{w} \) solves in \([0, T]\)

\[
i\tilde{w}_t + \Delta \tilde{w} = |\tilde{w}|^4 \tilde{w} + e
\]  

(2.3)

with

1. \[ \|\tilde{w}\|_{L^\infty_t L^2_x} \leq M. \]
2. \[ \|\tilde{w}\|_{L^5_t L^{10}_x} \leq E. \]

Then there exists some \( \epsilon_0 \) depends on \( M, E \), so that if \( w \) solves \((1.5)\) with \[ \|w(0) - \tilde{w}(0)\|_{L^2_x} \leq \epsilon \leq \epsilon_0, \] and \[ \|e\|_{L^1_t L^2_x} \leq \epsilon \leq \epsilon_0. \] then

\[ \|w - \tilde{w}\|_{X[0, T]} \lesssim_{E, M} \epsilon \]  

(2.4)

and in particular,

\[ \|w\|_{X[0, T]} \lesssim_{E, M} 1. \]  

(2.5)

While Proposition 2.1 is purely perturbative, one can combine it with Dodson’s global well posedness result, Theorem 1.11, to improve it to derive

**Proposition 2.2.** Let \( \tilde{w} \) solve

\[
i\tilde{w}_t + \Delta \tilde{w} = |\tilde{w}|^4 \tilde{w} + e
\]  

(2.6)

with \[ \|w(0)\|_{L^2_x} \leq M. \] Then there exists \( \epsilon > 0 \) depending on \( M \), so that if

\[ \|e\|_{L^1_t L^2_x[0, T]} \leq \epsilon, \]  

(2.7)

then

\[ \|\tilde{w}\|_{X[0, T]} \lesssim_M 1. \]  

(2.8)

**Remark 2.3.** In other words, the assumption for \( \tilde{w} \) in Proposition 2.1 can be derived if one assume the perturbation is small enough depending on the size of initial data \( w(0) \). Proposition 2.1 relies on Dodson’s GWP result and in particular is non-perturbative.

Note that Prop 2.2 implies the following a priori bound.

**Corollary 2.4.** Let \( w \) solves

\[
iw_t + \Delta w = |w|^4 w + e
\]  

(2.9)

in \([0, T]\) with \[ \|w\|_{L^\infty_t L^2_x} \leq M, \] and \[ \|e\|_{L^1_t L^2_x} \leq E, \] then

\[ \|w\|_{X[0, T]} \lesssim_M (1 + E) \]  

(2.10)

**Remark 2.5.** This a priori bound is enough for one to establish the almost sure, (or in other words, pathwise) global wellposeness for (1.7).

We finally present a stability argument which will be useful in the study of stochastic dispersive equations, in particular when it is combined with the so-called Da Prato-Debussche method.
Proposition 2.6. Let \([a, b]\) be an interval and \(u, g \in \mathcal{X}(a, b)\) satisfy \(g(a) = 0\) and 
\[
u(t) = e^{i(t-a)\Delta}u(a) - i \int_0^t e^{i(t-s)\Delta}|u|^4u(s)ds + g(t),
\]
Then for every \(M > 0\), there exists \(\eta_M, B_M > 0\) so that if 
\[
\|u\|_{L^\infty} \leq M, \|g\|_{L_5^2} \leq \eta_M,
\]
then 
\[
\|u\|_{L_5^2} \leq B_M.
\]

This technical proposition has played an important role in [FX18a], and one may refer to Proposition 4.6 in that article. (It was stated in a more complicated way there since we take into account of the truncation, but the estimate in Prop 4.6 is uniform in \(m\), thus also work here.)

Remark 2.7. In the study of stochastic NLS, it turns out one can not quite write the solution in the form (2.3) so that error term \(e\) can be well estimated. One needs to study its stability in the form of its integral version.

We finally point out, in this paper, one also has deal NLS with time dependent truncated non-linearity of form \(\phi_m(\|w\|_{\chi^2(0, t)})|w|^4 w\). We have

Remark 2.8. Prop 2.1, Prop 2.2, Cor 2.4, Prop 2.6 bound hold if one replaces the nonlinearity \(|w|^4 w\) by \(\phi_m(\|w\|_{\chi^2(0, t)})|w|^4 w\), and all the implicit constants involved the statement remains unchanged and in particular uniform in \(m\).

2.3 Burkholder inequality

Rather than state a general version of Burkholder inequality ([BDG72, Bur73]) involving the technical notion of \(\gamma\)-randomizing operator, we state a simpler version which will be enough for our purpose. Recall, due to (1.4), our noise is essentially of form \(V(x)B(t)\) when we apply Burkholder inequality.

Proposition 2.9. Let \(B(t)\) be a standard Brownian motion, \(\sigma\) be a right-continuous adapted process in \(L^p\), \(2 \leq p < \infty\). We have 
\[
\| \sup_{a, b \in T} \int_{[a, b]} \sigma(s)dB_s \|_{L^p_x L^p_t} \lesssim_p \| \int_{[0, T]} \| \sigma(s) \|_{L^2_x}^2 ds \|_{L^{p/2}_t}^{1/2}, 1 < p < \infty.
\]

One may refer to , for example, in [BP99, Theorem 2.1] for a proof. See [Brz97, vNVW07] for more details. We show two typical examples when estimate (2.14) is applied in this article. Let \(W\) be the Wiener process as in our article, with (1.4).

1. A discrete version\(^1\) of (2.14). Let \(0 < t_1 < ... t_n = 1\), let \(f_k\) be a sequence in

\(^1\)It can be checked as an application of (2.14), but strictly speaking, (2.14) is derived from this discrete version.
Proof of Theorem 1.12 uniform boundedness

Let $f_k \in F$, let $1 \frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Then
\[
\| \sum_k f_k (W(t_{k+1}) - W(t_k)) \|_{L_r^p L_r^q} \lesssim \rho \sum_k \| W(t_{k+1}) - W(t_k) \|_{L_r^p}^2 \| f_k \|_{L_r^p}^{1/2} \lesssim_{\rho, \lambda} \sum_k \| t_{k+1} - t_k \| f_k \|_{L_r^p}^{1/2} \rho \quad (2.15)
\]

2. Estimate regarding the Duhamel formula for Schrodinger equation. Let $u(s) \in F_j$ be an adapted process in $L_r^q$, let $p \geq 2$, and $\frac{1}{p'} = \frac{1}{q} + \frac{1}{r}$ (this simply means $p' \leq q$), then for any $t \in \mathbb{R}$,
\[
\| \sup_{a,b \in [0,T]} \int_a^b e^{i(t-s)\Delta} u(s) ds \|_{L_r^p L_r^q} \lesssim \rho \sum_k \| e^{i(t-s)\Delta} V_k u(s) \|_{L_r^p}^{1/2} \| \rho \quad (2.16)
\]
In the last step, we have used dispersive estimate (2.1).

2.4 Kolmogorov’s criterion

Finally, we recall the Kolmogorov’s criterion, which is a classical tool to show the Holder continuity of Brownian motion.

Proposition 2.10. Let $q \geq 2, \beta > 1/q$. Let $X(s)$ be some stochastic process in some Banach space $Y$, $s \in [0,T]$. Assuming
\[
\| X(t) - X(s) \|_{L_r^q Y} \lesssim |t - s|^\beta, \quad (2.17)
\]
then for all $\alpha \in [0, \beta - 1/q)$, one can find $K_\alpha(\omega)$ so that
\[
\| X(t) - X(s) \|_{Y} \leq K_\alpha(\omega), \quad \text{and} \quad \| K_\alpha \|_{L_r^q} \lesssim_{\alpha, q, \beta} 1. \quad (2.18)
\]

In particular, for the noise we consider in this article, applying Burkholder inequality (2.14), (2.15) and Kolomprog criteria, we have for all $1 \leq p \leq \infty$ and $\alpha < 1/2$, so that
\[
\| \sup_{t,s \in [0,1]} \frac{\| W(t) - W(s) \|_{L_r^p}}{|t - s|^{\alpha}} \|_{L_r^q} \lesssim_{\text{noi}, \alpha, \rho} 1 \quad (2.19)
\]
(Directly apply Proposition 2.10 gives (2.19) for $\rho$ large enough, which implies the bounds for small $\rho$ via Holder inequality.)

3 Proof of Theorem 1.12 uniform boundedness

We prove the second bound in (1.21) only, which corresponds to $m = +\infty$. Uniform in $m$ bounds in $u^{(m)}$ follows similarly (almost line by line.)
3.1 Overview of the proof

We start by introducing some new notation for simplicity. We aim to do uniform in n estimate. We will fix n and the partition \( \pi(n) \). We, without loss of generality, only consider \( n \gg 1 \) and \( \| \pi(n) \| \ll 1 \). We will denote \( u^{(n)} \) by \( v \), and denote \( t_j^{(n)} \) by \( t_j \). We also define \( j(s) \) be the index so that \( t_{j(s)} < s \leq t_{j(s)+1} \). Finally, we denote \( t_{j(s)} \) by \( [s] \).

Before we start, observe for any \( 1 \leq p < \infty, 1 \leq p \leq \infty \),

\[
\|W(t) - W(s)\|_{L^p_t L^p_x} \leq \sum_k \|V_k(B_k(t) - B_k(s))\|_{L^p_t L^p_x} \lesssim_{\rho, \Lambda_{ini}} |t - s|^{1/2}. \tag{3.1}
\]

We also, recall, a priori, we have, by mass conservation law, almost surely

\[
\|u^{(n)}\|_{L^\infty_t L^2_x} \leq \Lambda_{ini} \lesssim 1. \tag{3.2}
\]

Expanding \( v \) by Duhamel formula based on time \( a \in [0, 1] \), we have

\[
v(t) = e^{it-a} \Delta v(a) - i \int_a^t e^{it-s} \Delta N(v(s))ds - i \int_a^t e^{it-s} \Delta (v(s) \frac{dW^{(n)}}{ds})ds
\]

\[
= e^{it-a} \Delta v(a) - i \int_a^t e^{it-s} \Delta N(v(s))ds - i \int_a^t e^{it-s} \Delta (v(s) \frac{W(t_{j(s)+1}) - W(t_{j(s)})}{t_{j(s)+1} - t_{j(s)}})ds
\]

We need to explore the martingale structure in [3.3]. To do this, we further expand \( u(s) \) within interval \([s], s\) by Duhamel Formula. Observe that for any \( r \in ([s], s) \), we have \( j(r) = j(s) \). Thus, we derive

\[
v(s) = e^{i(s-[s]) \Delta} v([s]) - i \int_{[s]} e^{i(s-r) \Delta} N(v(r))dr - i \int_{[s]} e^{i(s-r) \Delta} (u(r) \frac{W(t_{j(s)+1}) - W(t_{j(s)})}{t_{j(s)+1} - t_{j(s)}})dr
\]

Summarizing (3.3) and (3.4), we derive

\[
v(t) = e^{it-a} \Delta v(a) - i \int_a^t e^{it-s} \Delta N(v(s))ds
\]

\[
- \int_a^t e^{it-s} \Delta \left( \frac{W(t_{j(s)+1}) - W(t_{j(s)})}{t_{j(s)+1} - t_{j(s)}} \int_{[s]} e^{i(s-r) \Delta} N(v(r))dr \right) ds
\]

\[
+ (-i) \int_a^t e^{it-s} \Delta \left( \frac{W(t_{j(s)+1}) - W(t_{j(s)})}{t_{j(s)+1} - t_{j(s)}} e^{i(s-[s]) \Delta} v([s]))ds
\]

\[
- \int_a^t e^{it-s} \Delta \left( \frac{W(t_{j(s)+1}) - W(t_{j(s)})}{t_{j(s)+1} - t_{j(s)}} \int_{[s]} e^{i(s-r) \Delta} \left[ \frac{W(t_{j(s)+1}) - W(t_{j(s)})}{t_{j(s)+1} - t_{j(s)}} v(r) \right] dr \right) ds
\]

We introduce some extra notations to simplify the above formula.

Let \( S_{qua}(a, t), S_{mar(a,t)}, \tilde{N}(v; s) \) be defined as,

\[
S_{qua}(a, t) = (-i) \int_a^t e^{it-s} \Delta \left( \frac{W(t_{j(s)+1}) - W(t_{j(s)})}{t_{j(s)+1} - t_{j(s)}} e^{i(s-[s]) \Delta} v([s]))ds \tag{3.6}
\]
3.2 Control of source term

\( S_{\text{qua}}(a, t) = - \int_a^t e^{i(t-s)\Delta} \left( \frac{W(t_j(s)+1) - W(t_j(s))}{t_j(s)+1 - t_j(s)} \int_s^t e^{i(s-r)\Delta} \left( \frac{W(t_j(r)+1) - W(t_j(r))}{t_j(r)+1 - t_j(r)} v(r) \right) dr \right) ds \)  

\[ \tilde{N}(u^{(n)}; s) = \left( \frac{W(t_j(s)+1) - W(t_j(s))}{t_j(s)+1 - t_j(s)} \int_s^t e^{i(s-r)\Delta} \mathcal{N}(v(r)) dr \right) \]  

We can now simplify (3.5) to

\[ v(t) = e^{i(t-a)\Delta} v(a) - i \int_a^t e^{i(t-s)\Delta} \tilde{N}(v(s)) ds - \int_a^t e^{i(t-s)\Delta} \tilde{N}(v; s) ds + S_{\text{mar}}(a, t) + S_{\text{qua}}(a, t), \]  

We will view \( S_{\text{mar}}, S_{\text{qua}} \) as source term and we will view \( \tilde{N}(v; s) \) as perturbative term.

Roughly speaking, we want to use some maximal type estimate to get rid of the \( a \) parameter in \( S_{\text{mar}}, S_{\text{qua}} \). For the term \( S_{\text{mar}} \), we will explore the martingale structure in this term. For the term \( S_{\text{qua}} \), we will explore the fact \( \sum_j \|W(t_{j+1}) - W(t_j)\|^2 \sim \sum_j |t_{j+1} - t_j| \lesssim 1 \), for reasonable norm \( \|\| \) which will be detailed later. For the term \( \tilde{N}(v; s) \), observe, at least in the average sense, this term is morally \( \|W(t_{j+1}) - W_t\| \|\mathcal{N}(v_s)\| \sim |t_{j+1} - t_j|^{1/2} \|\mathcal{N}(v_s)\| \), and can be treated perturbatively via a bootstrap argument. Again, we don’t specify about the exact norm we will use at this moment.

There is (very) small probability that \( \|W(t_{j+1}) - W(t_j)\| \) is of large size. In such case we will indeed directly go back to (1.8), and directly view \( v(\frac{W(t_{j+1}) - W(t_j)}{t_{j+1} - t_j}) \) as a perturbative term, and expose the fact that such event is of very small probability.

In the following, we will first prove some maximal type control of \( S_{\text{mar}}, S_{\text{qua}} \), and prove some technical lemmas to handle the case when \( \|W(t_{j+1}) - W(t_j)\| \) is large. Then, we will prove the bound for \( v \) based on (3.9).

We emphasize again the \( n \) is fixed in the rest of this section. So is \( \pi^n \). We omit the parameter \( n \) and denote \( u^{(n)} \) by \( \nu \) for notation convenience, but all the estimate should be independent of \( n \). We only consider the case \( n \) is large and \( \pi^n \) is small.

We will fix a small parameter \( \eta \) in this section.

### 3.2 Control of source term \( S_{\text{mar}}, S_{\text{qua}} \)

Let

\[ S_{\text{mar}}^*(t) := \sup_{0 \leq \tau \leq t} \left\| \int_0^\tau e^{i(t-s)\Delta} \left( \frac{W(t_j(s)+1) - W(t_j(s))}{t_j(s)+1 - t_j(s)} e^{i(s-[s])\Delta} v([s]) \right) ds \right\|_{L^2_x}, \]  

\[ S_{\text{qua}}^*(t) := \int_0^t \int_s^t \left\| e^{i(t-s)\Delta} \left( \frac{W(t_j(s)+1) - W(t_j(s))}{t_j(s)+1 - t_j(s)} e^{i(s-r)\Delta} \left[ \frac{W(t_j(r)+1) - W(t_j(r))}{t_j(r)+1 - t_j(r)} v(r) \right] \right) \right\|_{L^2_x} dr ds \]
Then by triangle inequality
\[ \| S_{mar}(a, t) \|_{L^1_t L^2_x} \lesssim S_{mar}^*(t), \| S_{qua}(a, t) \|_{L^1_t L^2_x} \lesssim S_{qua}^*(t). \]  
(3.12)

In this subsection, we show

Lemma 3.1. For \( \rho \geq 1 \), we have
\[ \| S_{mar}^*(t) \|_{L^\infty_t L^2_x[0, 1]} \lesssim \rho, \| S_{qua}^*(t) \|_{L^\infty_t L^2_x[0, 1]} \lesssim \rho. \]  
(3.13)

We only need to prove for \( \rho \geq 5 \). By Minkowski inequality and recall we only work on a finite time interval, we have the embedding
\[ L^\infty_t L^6_x \hookrightarrow L^5_t L^6_x \hookrightarrow L^\infty_t L^5_x. \]  
(3.14)

To prove (3.13), we need only to prove for every \( t \in [0, 1] \),
\[ \| S_{mar}^*(t) \|_{L^\infty_x} \lesssim 1 \]  
(3.15)
\[ \| S_{qua}^*(t) \|_{L^\infty_x} \lesssim 1 \]  
(3.16)

We fix \( t \) until the end of this section.

Before we start the proof of (3.15), (3.16), we first prove the following technical lemma which handles in fluctuation any small interval \([t_j, t_{j+1}]\).

Lemma 3.2. For any \( \kappa < 1/10 \), we have
\[ \| \sup_{\tau \in [0, \xi]} \| \int_\tau e^{i(t-s)\Delta} \left( W(t_j(s)) - W(t_j(s)) e^{i(s-a)\Delta} v(s) \right) ds \|_{L^1_t L^6_x} \lesssim \rho, \kappa \| \pi \|^{1/10 - \kappa}. \]  
(3.17)

Proof of Lemma 3.2. Observe \([\tau] = t_j(s)\) and \( \tau \leq t_{j(s)+1} \). We thus have
\[ \| \int_\tau e^{i(t-s)\Delta} \left( W(t_j(s+1)) - W(t_j(s)) e^{i(s-a)\Delta} v(s) \right) ds \|_{L^1_t L^6_x} \lesssim \rho, \kappa \| \pi \|^{1/10 - \kappa}. \]  
(3.18)

(We applied dispersive estimate (2.1) in the third line of (3.18), and also recall we a priori have \( \| e^{i(s-a)\Delta} v(s) \|_{L^2_x} \lesssim 1 \). We have also applied holder \( \| f g \|_{L^{10/9}_x} \lesssim \| f \|_{L^{10/9}_x} \| g \|_{L^2_x} \). We have that (3.17) follows from via estimate (3.18) via estimate (2.19), and the fact \( \| \pi \| \geq |t_{j+1} - t_j|, \forall j \).)
Now we go to the estimate for $S^\ast$.

**Proof of (3.15).** Observe

$$
\int_0^\tau e^{i(t-s)\Delta} \left( \frac{W(t_{j(s)}+1) - W(t_{j(s)})}{t_{j(s)+1} - t_{j(s)}} e^{i(s-[s])\Delta} v([s]))ds
= \int_0^{[\tau]} e^{i(t-s)\Delta} \left( \frac{W(t_{j(s)}+1) - W(t_{j(s)})}{t_{j(s)+1} - t_{j(s)}} e^{i(s-[s])\Delta} v([s]))ds
+ \int_{[\tau]}^\tau e^{i(t-s)\Delta} \left( \frac{W(t_{j(s)}+1) - W(t_{j(s)})}{t_{j(s)+1} - t_{j(s)}} e^{i(s-[s])\Delta} v([s]))ds
$$

(3.19)

and the second term is already well estimated by Lemma 3.2. We need only to prove

$$
\|\sup_\tau \| \int_0^{[\tau]} e^{i(t-s)\Delta} \left( \frac{W(t_{j(s)}+1) - W(t_{j(s)})}{t_{j(s)+1} - t_{j(s)}} e^{i(s-[s])\Delta} v([s]))ds \|_{L^p_x} \|_{L^\rho_t} \lesssim 1
$$

(3.20)

Observe

$$
\int_0^{[\tau]} e^{i(t-s)\Delta} \left( \frac{W(t_{j(s)}+1) - W(t_{j(s)})}{t_{j(s)+1} - t_{j(s)}} e^{i(s-[s])\Delta} v([s]))ds
= \sum_{j \leq \tau} \int_{t_{j-1}}^{t_j} W(t_{j(s)}+1) - W(t_{j(s)}) e^{i(s-[s])\Delta} v([s]))ds
$$

(3.21)

can be viewed indeed a dicrete martingale in $L^1_x$, by Burkholder inequality (2.14), we have

$$
\|\sup_\tau \| \int_0^{[\tau]} e^{i(t-s)\Delta} \left( \frac{W(t_{j(s)}+1) - W(t_{j(s)})}{t_{j(s)+1} - t_{j(s)}} e^{i(s-[s])\Delta} v([s]))ds \|_{L^p_x} \|_{L^\rho_t} \lesssim \sum_j \| \int_{t_j}^{t_{j+1}} e^{i(t-s)\Delta} \left( \frac{W(t_{j+1}+1) - W(t_{j+1})}{t_{j+1} - t_j} e^{i(s-[s])\Delta} v([s]))ds \|_{L^p_x} \|_{L^\rho_t}^{2/2}
$$

(3.22)

Argued similar as (3.18), we have\(^2\)

$$
\int_{t_j}^{t_{j+1}} e^{i(t-s)\Delta} \left( \frac{W(t_{j+1}+1) - W(t_{j+1})}{t_{j+1} - t_j} e^{i(s-[s])\Delta} v([s]))ds \|_{L^p_x} \lesssim \| W_{t_{j+1}-W_{t_{j}}} \|_{L^p_x} \|_{L^\rho_t}^{2/2} \int_{t_j}^{t_{j+1}} (t-s)^{-2/5} ds
$$

(3.23)

Plug (3.23) into (3.22), we will have (3.20) follows if we have

$$
\| \sum_j \| W_{t_{j+1}-W_{t_{j}}} \|_{L^p_x} \|_{L^\rho_t}^{2/2} \int_{t_j}^{t_{j+1}} (t-s)^{-2/5} ||_{L^p_x} \lesssim 1.
$$

(3.24)

\(^2\)In this step, we use dispersive estimates rather than Strichartz estimates.
Using (3.1) and triangle inequality, we have

\[ \| \sum_j \left( \frac{\| W_{t_{j+1}} - W_{t_j} \|_{L^2}^{s/2}}{t_{j+1} - t_j} \int_{t_{j+1}}^{t_j} (t - s)^{-2/5} ds \right) \|_{L^2}^{\rho/2} \]

\[ \leq \rho \sum_j \frac{1}{t_{j+1} - t_j} \int_{t_{j+1}}^{t_{j+1}} (t - s)^{-2/5} ds^2 \]

\[ \leq \sum_j \int_{t_j}^{t_{j+1}} (t - s)^{-4/5} = \int_0^1 (t - s)^{-4/5} \lesssim 1. \] (3.25)

We are done. \[ \square \]

**Remark 3.3.** Observe in (3.25), the last step we use \( \int_0^1 (t - s)^{-4/5} \lesssim 1 \). If we work on an small interval \([a, b]\) rather than the whole interval \([0, 1]\), we will gain a small power of \((b - a)^\alpha\). (Here, of course \( \alpha = 1/5 \), the key point is this term is subcritical in these on can gain a positive power of \((b - a)\).)

Now we turn to the control of \( S_{qua}^* \)

**Proof of (3.16).** Unlike (3.15), which explores the martingale structure and relies on certain cancellation, the proof of (3.16) is more straightforward. Estimate similar as (3.18), we have

\[ \| e^{i(t-s)\Delta} \left( \frac{W(t_j(s)+1) - W(t_j(s))}{t_j(s)+1 - t_j(s)} e^{i(s-r)\Delta} [\frac{W(t_j(s)+1) - W(t_j(s))}{t_j(s)+1 - t_j(s)} \nu(r)] \right) \|_{L^2(x)} \]

\[ \lesssim (t - s)^{-2/5} \| \frac{W(t_j(s)+1) - W(t_j(s))}{t_j(s)+1 - t_j(s)} \|_{L^5(x)} \]

\[ \| S_{qua}^*(t) \|_{L^p(x)} \lesssim \| \sum_j \int_{t_j}^{t_{j+1}} (t - s)^{-2/5} (t_{j+1} - t_j)^{-1} \| \Delta_j W \|_{L^5(x)} \| \Delta_j W \|_{L^\infty(x)} ds \|_{L^p(x)} \]

\[ \lesssim \rho \int_0^1 (t - s)^{-2/5} \lesssim 1. \] (3.27)

Here \( \Delta_j := W(t_{j+1}) - W(t_j) \).

We are done. \[ \square \]

**Remark 3.4.** Similar as in Remark 3.3, one can gain a small power of \((b - a)^\alpha\) if we decide to work on a small interval \([a, b]\) rather than a whole interval \([0, 1]\).

### 3.3 Control the large oscillation

Though of small probability, it is possible in some interval \([t_j, t_{j+1}]\), one has a large oscillation in the relative size, we will use the following lemma to control such case.
Lemma 3.5. One always has the following crude bound for any interval \([t_j, t_{j+1}]\).
\[
\|v(t)\|_{X_{[t_j,t_{j+1}]}^1} \lesssim 1 + \|W(t_{j+1}) - W(t_j)\|_{L_x^\infty} \tag{3.28}
\]

**Proof.** Recall (1.7), also recall (3.2), and use estimate
\[
\|v \frac{W(t_{j+1}) - W(t_j)}{t_{j+1} - t_j}\|_{L_t^1 L_x^2} \lesssim \|W(t_{j+1}) - W(t_j)\|_{L_x^\infty}. \tag{3.29}
\]

Then Lemma 3.5 follows from Corollary 2.4
\[\square\]

**Remark 3.6.** Lemma 3.5 is stated for all size of \(\|W(t_{j+1}) - W(t_j)\|\). However, it is only useful when \(\|W(t_{j+1}) - W(t_j)\|\) is as large as \(\sim 1\), which is of very small probability \((\lesssim e^{-\frac{1}{t_{j+1} - t_j}})\).

**Remark 3.7.** Sometimes, one may want to avoid the use of \(L_x^\infty\) norm. There is no problem. One can replace the \(L_t^1 L_x^2\) norm into any \(L_t^q L_x^{2q'}\) in Proposition 2.1, 2.2 and Cor 2.4. For example, if one use the pair \(5, 10\), then similar proof gives
\[
\|v(t)\|_{X_{[t_j,t_{j+1}]}} \lesssim 1 + \frac{\|W(t_{j+1}) - W(t_j)\|}{(t_{j+1} - t_j)^{1/4}} \|W(t_j)\|_{L_x^{2q'/2}}. \tag{3.30}
\]
which is still good for use, since \(W(t_{j+1}) - W(t_j)\) is centered at scale \(\sim |t_{j+1} - t_j|^1/2\).

**3.4 Derive the desired bound**

We are ready to prove the desire bound for \(v\), i.e.

\[
\|v\|_{L_t^\infty X_{[0,1]}^{1,0}} \lesssim_{\rho} 1 \tag{3.31}
\]

(The \(X_1\) part is trivial since we have mass conservation law, or (3.2)).

We will fix a small constant \(\eta\) during the proof, whose value will be determined later but only depends on the mass of the initial data, \(\Lambda_{\text{ini}}\).

Recall our partition \(0 = t_0 < t_1 < ... < t_n\) is fixed all the time. For every \(\omega\) almost surely, we separate the intervals \(\cup_j[t_j, t_{j+1}]\) into two groups from the perspective of Lemma 3.5.

- We call \([t_j, t_{j+1}]\) a type-A interval if \(\|W(t_{j+1}) - W(t_j)\|_{L_x^{10}} \geq \eta\).
- If \([t_j, t_{j+1}]\) is not of type-A, we call it type-B.

Note that whether certain interval is of type-A is a random event. However, when the mesh of the partition \(\|\pi\|\) is small, it is of very small probability for any interval to be type-A.

We further do a partition of all type-B intervals into a collection of sub-intervals \(\cup_l [a_l, b_l]\), i.e.

\[
\{[t_j, t_{j+1}]) \cap [t_j, t_{j+1}]\} \text{ is type-B} = \cup_l [a_l, b_l], \quad \text{and} \quad (a_l, b_l) \cap (a_{l'}, b_{l'}) = \emptyset, l \neq l'. \tag{3.32}
\]

We require
\[
\|S_{qua}^*\|_{L_t^2[a_l, b_l]} + \|S_{mar}^*\|_{L_t^2[a_l, b_l]} \lesssim \eta \tag{3.33}
\]
Thus, we define a random variable \(\omega \rightarrow J(\omega)\), and \(J = J(\omega)\) is the number of sub-intervals \([a_l, b_l]\).

We now state two lemmas to summarize the properties of those partitions. First, we claim \(J\) can not be too large in average sense.
Lemma 3.8.

\[ \|J\|_{L^p} \lesssim_{\rho, \eta} 1 \]  

(3.34)

Second, when the \( \eta \) is chosen small enough, (such smallness only depends on \( \Lambda_{ini} \) and the fact \( \|\pi\| \) is small enough), we have

Lemma 3.9. If in some interval \([a_l, b_l]\) so that (3.33) holds, then we have (deterministic) estimate

\[ \|v\|_{X_2[a_l, b_l]} \lesssim 1 \]  

(3.35)

Assuming Lemma 3.8, 3.9 at the moment, we conclude the proof of (3.31).

Proof of (3.31) assuming Lemma 3.8, 3.9. If some interval is of type-A, we apply Lemma 3.5. All the type-II intervals of type-B are partitioned into subintervals \([a_l, b_l]\) in which (3.33) hold, and we apply Lemma 3.9. To summarize, we derive

\[ \|v\|_{X_2[a_l, b_l]} \lesssim \rho \]  

(3.36)

Thus, we derive

\[ \|v\|_{L^p} \lesssim \rho \|J\|_{L^p}^{1/5} + \sum_j \|\chi\|_{W(t_{j+1}) - W(t_j)_{L^p}} \lesssim (\omega)(1 + \|W(t_{j+1}) - W(t_j)\|_{L^p}) \]  

(3.37)

The first term is controlled by Lemma 3.8 For the second, using the fact that for each \( j \)

\[ P(\|W(t_{j+1}) - W(t_j)\|_{L^p} \sim \alpha) \lesssim e^{-c|t_{j+1} - t_j|^2}, \]

here \( c \) is some number only depends on \( \Lambda_{ini} \)  

(3.38)

We derive

\[ \sum_j \|\chi\|_{W(t_{j+1}) - W(t_j)_{L^p}} \lesssim (\omega)(1 + \|W(t_{j+1}) - W(t_j)\|_{L^p}) \]  

(3.39)

We are done. (Note though it seems the bound depends on \( \eta \), but \( \eta \) is fixed and only depends on \( \Lambda_{ini} \).)

We are left with the proof of Lemma 3.8 and Lemma 3.9. We first handle Lemma 3.8
Proof of Lemma 3.8. Observe
\[ \|v\|_{L^5_t L^5_v[0,1]} \geq \sum_l \|v\|_{L^5_t L^5_v[a_l, b_l]} \sim J\eta^5 \]  
(3.40)

Plug in Lemma 3.1, and then Lemma 3.8 follows.

Finally, we present the proof of Lemma 3.9. Implicitly, part of the proof is in the same spirit of so-called De Prato-Debussche trick.

Proof. Let us fix \( l \) and denote \([a_l, b_l]\) by \([a, b]\). We first point without loss of generality, we may assume \( a = [a] \), otherwise by our definition of \([a_l, b_l]\), the interval \([t_{j(a)}, t_{j(a)+1}]\) is not of type-A, and we can apply Lemma 3.5 to control the dynamic of \( v \) in interval \([t_{j(a)}, t_{j(a)+1}]\), and we need only to study the dynamic of \( v \) in \([t_{j(a)+1}, b]\), which satisfy all the assumption for \([a_l, b_l]\), which we will use below. Similarly, we also assume \( b = [b] \).

For \( t \in [a, b] \), we know \( v \) solves (3.9). Let
\[ h_1(t) := -\int_a^t e^{i(t-s)\Delta} \tilde{N}(v; s) ds \]
(3.41)
and
\[ h_2(t) := S_{mar}(a, t) + S_{qua}(a, t). \]
Observe
\[ h_2(a) = h_1(a) = 0, \text{ and } \|h_2(a)\|_{L^5_t L^5_v[a, b]} \lesssim \eta. \]
(3.42)
(The above estimate follows from our choice of \([a, b], (3.33)\)).

If there is no \( h_1 \) term, then Lemma 3.9 follows from the modified stability Proposition 2.6.

We will treat \( h_1 \) pertubatively, to do this, we will follow a bootstrap scheme. Let \( M \) in Proposition 2.6 be chosen as ini, let us choose \( B_M, \eta_M \) according to Proposition 2.6.

We claim when \( \eta \) is small enough, one can do the following bootstrap estimate

Lemma 3.10. If in \([a, T] \subset [a, b]\) one has the bootstrap assumption
\[ \|v\|_{L^5_t L^5_v[a, T]} \leq 2B_M \]
(3.43)
then one has the bootstrap estimate
\[ \|v\|_{L^5_t L^5_v[a, T]} \leq B_M \]
(3.44)

Lemma 3.9 follows from Lemma 3.10 by standard continuity argument.

We are left with the proof of Lemma 3.10. We will first choose \( \eta \) small enough so that \( \|h_2(a)\|_{L^5_t L^5_v[a, b]} \leq \eta_M/10 \). Lemma 3.10 follows from Proposition 2.6 if we can show that (3.43) implies
\[ \|h_1(t)\|_{L^5_t L^5_v} \leq \eta_M/10. \]
(3.45)
Proof of Corollary 1.13

(We will make \( \eta \) even smaller if necessary.)

By Strichartz estimate \([2,2]\), we have

\[
\|h_1(t)\|_{L^1_tL^2_x[a,t]} \lesssim \|W(t_{j(s)+1}) - W(t_{j(s)})\|_{t_{j(s)+1} - t_{j(s)}} \int_{[s]}^t e^{i(s-r)\Delta} \mathcal{N}(v(r))dr\|_{L^3_tL^2_x[a,t]} \tag{3.46}
\]

Note that since we assume \( a = [a], b = [b] \), thus the \( [t_{j(s)}, t_{j(s)+1}] \) in the integrand is always in \([a, b]\), also note that \([t_{j(s)}, t_{j(s)+1}] \) must be of Type B.

We use the pointwise estimate

\[
\left\| \frac{W(t_{j(s)+1}) - W(t_{j(s)})}{t_{j(s)+1} - t_{j(s)}} \int_{[s]}^t e^{i(s-r)\Delta} \mathcal{N}(v(r))dr \right\|_{L^2_t} \lesssim \frac{1}{t_{j(s)+1} - t_{j(s)}} \|W(t_{j(s)+1}) - W(t_{j(s)})\|_{L^1_t[0,1]} \mathcal{N}(v)\|_{L^1_tL^2_x[s,t]} \tag{3.47}
\]

We conclude, by (3.46) and (3.47), with

\[
\|h_1(t)\|_{L^1_tL^2_x} \lesssim \sum_{[t_j,t_{j+1}] \subset [a,b]} \eta \|v\|_{L^5_tL^{10}_x[t_j,t_{j+1}]}^5 \lesssim \eta B_M^5 \tag{3.48}
\]

We have (3.45) when \( \eta \) is small enough. We are done.

Before we end this subsection, we remark sometimes one may wants to avoid the use of \( L^\infty_x \) in stability. There is no problem. We sketch the associated modification here. As already mentioned in Remark 3.7, one can replace the \( \|W(t_{j+1}) - W_t\|_{L^\infty_x} \) in Lemma 3.5 by \( \frac{\|W_{t_{j+1}} - W(t_j)\|_{L^{5/2}_t}}{t_{j+1} - t_j} \). Later, one may define an interval is of type A iff \( \frac{\|W_{t_{j+1}} - W(t_j)\|_{L^{5/2}_t}}{t_{j+1} - t_j} \geq \eta \). And finally, in the proof of Lemma 3.10, one replace the \( L^1_tL^2_x \) into \( L^{5/4}_tL^{10}_x \), by observing as as in Remark 3.7 that (5.10) is also a Strichartz admissible pair.

4 Proof of Corollary 1.13

4.1 Overview of the proof

We first point out, it is very natural that if one can prove Theorem 1.12 then one can prove a stability result as in Corollary 1.13. It may be of some concern since in Corollary 1.13, we only require closeness between \( V_k \) and \( \tilde{V}_k \) in \( L^p_x \) for \( p < \infty \) (and an a priori control of \( L^\infty_x \)), while the proof of Theorem 1.12 do uses \( L^\infty_x \) bound in several places. However, we use \( L^\infty_x \) bound of \( W \) only for the convenience of numerics, it has already been explained how to modify the use of \( L^\infty_x \) into \( L^p \) in the end of previous section. There is one more place we didn’t explain in the end of previous section, i.e. in (3.27), however, the \( \|\Delta_j W\|_{L^\infty_x} \) was multiplied by \( \|\Delta_j W\|_{L^{5/2}_x} \), thus it will be enough for stability arguments as in Corollary 1.13 only requires closedness in \( L^{5/2}_x \) and a priori control in \( L^\infty_x \).
Due to the above discussion and for simplicity and conciseness of numeric, we will only present the proof of Corollary 1.13 replacing (1.23) by the following stronger assumption

\[ \| f - \tilde{f} \|_{L^p_\rho L^2_x} + \sum_{k \in \mathbb{N}} \| V_k - \tilde{V}_k \|_{L^1_x \cap L^\infty_x} < \delta. \] (4.1)

Fix \( \epsilon, \rho, \) and argue similarly as Proposition 1.15, one can find \( m \gg 1, \) so that for any \( n, \)

\[ \| u^{(n)}_m - u^{(n)} \|_{L^p_\rho X([0,1])} \leq \epsilon/10. \] (4.2)

Thus, we only need to prove Corollary 1.13 for \( u^{(n)}_m \) for some fixed \( m. \) Also recall the bound in Theorem 1.12 holds also for \( u^{(n)}_m, \) uniform in \( m \) and \( n. \)

We also note we can further freely assume \( \| \pi^{(n)} \| \) is small enough and such smallness can depend on this \( m. \) Indeed, fix \( \epsilon, \) we can choose \( m \) so that (4.2) holds. For any given number \( c_m, \) we can reduce the proof of Corollary 1.13 into the case \( \| \pi^{(n)} \| \geq c_m \) and \( \pi^{(n)} \leq c. \) If \( \| \pi^{(n)} \| \geq c_m. \) We directly prove Corollary 1.13 without reducing the \( u^{(n)}_m. \) And what we are left is the case for stability of \( u^{(n)}_m \) with \( \pi^{(n)} \leq c. \)

We may only consider \( m \) large, the largeness of \( m \) may depend on \( \epsilon. \)

Again for notation simplicity, we will fix \( \epsilon = \epsilon_0, \) and \( \rho. \) We fix \( n, m, \) denote \( u^{(n)}_m \) by \( v, \) \( \tilde{u}^{(n)}_m \) by \( \tilde{v}, \) and denote \( t_j \) by \( t_j. \)

We let \( w := v - \tilde{v}, \) and we let \( U = W - \tilde{W}. \)

We will need a small constant \( \eta \) similarly as in the previous section. Also recall \( \Lambda_{ini}, \Lambda_{noi} \) are fixed throughout this article.

Note that since \( m \) is fixed, the nonlinearity is essentially, from viewpoint of dispersive equation, linearized.

We finally recall, since we a priori have \( L^p_\rho X \) bound for all \( \rho, \) we are free to drop small probability sets.

We claim

**Lemma 4.1.** There is \( h > 0, \) such that when \( \delta_0 \) is chosen small enough, one can always find \( \delta, \) in (4.1), small enough (depending on \( \delta_0, \rho, m, h, \)) such that if

\[ \| w \|_{L^p_\rho X([0,1])} \leq \delta_0 \] (4.3)

and \( d - c \sim h, \) then one has,

\[ \| w \|_{X([0,d])} \lesssim \delta_0 \] (4.4)

Iterate this lemma \( \sim 1/h \) times and the desired stability follows. It should be expected \( \delta \ll \delta_0, \) and \( -\ln \delta_0 \sim -\frac{1}{h} \ln \epsilon. \)

We now turn to the proof of Lemma 4.1.

For technical convenience, we will only consider the case \( c = t_j, d = t_{j'}, \) for some \( j < j'. \) (Note this is OK since we are allowed to assume \( \| \pi \| \) is small enough, and such smallness could depend on \( m).\) Lemma 4.1 will be reduced to the following bootstrap lemma 3.10.
Lemma 4.2. Given the assumption of Lemma 4.1, one can find $C > 0$, so that assuming bootstrap assumption for $T \in [c, d]$,
\[
\|w(t)\|_{L^\infty[0, T]} \leq 2C\delta_0,
\] (4.5)
then one has
\[
\|w(t)\|_{L^\infty[0, T]} \leq C\delta_0.
\] (4.6)
(This $C$ can be dependent on $m$, which is fixed all the time.)

4.2 Equation for $w$ and a collection of estimates

We first write down the equation for $w$. For any $[a, t]$ in $[c, d]$,(one may recall (3.5)), we have
\[
w(t) = e^{i(t-a)\Delta}w(a) - i \int_a^t e^{i(t-s)\Delta} [\phi_m(\|v\|_{X_2(0,s)}) (\mathcal{N}(v) - \mathcal{N}(\tilde{v}))] ds
\]  
\[\pm \int_a^t i(\phi_m(\|v\|_{X_2(0,s)}) - \phi_m(\|\tilde{v}\|_{X_2(0,s)}))\mathcal{N}(\tilde{v}(s)) ds
\]  
\[\pm \int_a^t e^{i(t-s)\Delta} \left( \frac{W(t_{j(s)+1}) - W(t_{j(s)})}{t_{j(s)+1} - t_{j(s)}} \right) \int_{[s]} e^{i(s-r)\Delta} \left\{ \phi_m(\|v\|_{X_2(0,r)}) \mathcal{N}(v) - \phi_m(\|\tilde{v}\|_{X_2(0,r)}) \mathcal{N}(\tilde{v}) \right\} ds
\]  
\[\pm \int_a^t e^{i(t-s)\Delta} \left( \frac{U(t_{j(s)+1}) - U(t_{j(s)})}{t_{j(s)+1} - t_{j(s)}} \right) \int_{[s]} e^{i(s-r)\Delta} \phi_m(\|\tilde{v}\|_{X_2(0,r)}) \mathcal{N}(\tilde{v}) ds
\]  
\[\pm \int_a^t e^{i(t-s)\Delta} \left( \frac{W(t_{j(s)+1}) - W(t_{j(s)})}{t_{j(s)+1} - t_{j(s)}} \right) \int_{[s]} e^{i(s-r)\Delta} \left\{ \phi_m(\|v\|_{X_2(0,r)}) \mathcal{N}(v) - \phi_m(\|\tilde{v}\|_{X_2(0,r)}) \mathcal{N}(\tilde{v}) \right\} ds
\] (4.7)

We will treat all the term with $\pm$ sign before them perturbatively, so the exact choice of $\pm$ does not matter in the analysis. (Indeed, as far as one chooses the $\pm$ sign consistently, the exact choice of such sign does not matter for any term.)

We introduce some notation to simplify the above equation. Let
- \( S_1(a, t) := -i \int_a^t e^{i(t-s)\Delta} \frac{W(t_{j(s)+1}) - W(t_{j(s)})}{t_{j(s)+1} - t_{j(s)}} e^{i(s-r)\Delta} \phi_m(\|v\|_{X_2(0,r)}) \mathcal{N}(v) - \phi_m(\|\tilde{v}\|_{X_2(0,r)}) \mathcal{N}(\tilde{v}) ds \)
- \( S_2(a, t) := - \int_a^t e^{i(t-s)\Delta} \left( \frac{W(t_{j(s)+1}) - W(t_{j(s)})}{t_{j(s)+1} - t_{j(s)}} \right) \int_{[s]} e^{i(s-r)\Delta} \left\{ \phi_m(\|v\|_{X_2(0,r)}) \mathcal{N}(v) - \phi_m(\|\tilde{v}\|_{X_2(0,r)}) \mathcal{N}(\tilde{v}) \right\} ds \)
- \( e_1(a, t) := \int_a^t e^{i(t-s)\Delta} \left( \frac{W(t_{j(s)+1}) - W(t_{j(s)})}{t_{j(s)+1} - t_{j(s)}} \right) \int_{[s]} e^{i(s-r)\Delta} \left\{ \phi_m(\|v\|_{X_2(0,r)}) \mathcal{N}(v) - \phi_m(\|\tilde{v}\|_{X_2(0,r)}) \mathcal{N}(\tilde{v}) \right\} ds \)
- \( e_2(a, t) := \int_a^t i(\phi_m(\|v\|_{X_2(0,s)}) - \phi_m(\|\tilde{v}\|_{X_2(0,s)})) \mathcal{N}(\tilde{v}(s)) ds \)
- \( e_3(a, t) := \int_a^t e^{i(t-s)\Delta} \left( \frac{U(t_{j(s)+1}) - U(t_{j(s)})}{t_{j(s)+1} - t_{j(s)}} \right) \int_{[s]} e^{i(s-r)\Delta} \phi_m(\|\tilde{v}\|_{X_2(0,r)}) \mathcal{N}(\tilde{v}) ds \)

Proof of Corollary 1.13

\[ e_4(a, t) := \pm \int_a^t e^{i(t-s)} \left( \frac{U(t_{j+1}) - U(t_{j})}{t_{j+1} - t_{j}} \right) e^{i(s-r)} \Delta v(s) ds. \]

\[ e_5(a, t) := \pm \int_a^t e^{i(t-s)} \left( \frac{U(t_{j+1}) - U(t_{j})}{t_{j+1} - t_{j}} \right) \int_{t_{j}}^{t_{j+1}} e^{i(s-r)} \Delta \left( \frac{W(t_{j+1}) - W(t_{j})}{t_{j+1} - t_{j}} \right) v(r) dr ds. \]

\[ e_6(a, t) := \pm \int_a^t e^{i(t-s)} \Delta \left( \frac{W(t_{j+1}) - W(t_{j})}{t_{j+1} - t_{j}} \right) \int_{t_{j}}^{t_{j+1}} e^{i(s-r)} \Delta \left( \frac{U(t_{j+1}) - U(t_{j})}{t_{j+1} - t_{j}} \right) v(r) dr ds. \]

Now, we may write (4.7) as

\[ w(t) = e^{i(t-a)} \Delta w(a) - i \int_a^t e^{i(t-s)} \Delta [\phi_m(v) || v ||_{X(0,s)}] [\mathcal{N}(v) - \mathcal{N}(\tilde{v})] ds + S_1(a, t) + S_2(a, t) + \sum_i e_i(a, t). \]

(4.8)

We first present all the estimates for \( S_i \), and \( e_j \). Since the estimate is of same nature as what we did in the proof of Theorem 1.12, we will do a sketch for the similar part and highlight the difference. We will work on time interval \([a, b] \subset [c, T] \subset [c, d]\).

We start with term \( S_1, S_2 \), similarly as we did in (3.12), Lemma 3.1, and see also Remark 3.3, 3.4, we can find \( S_1^*, S_2^* \), so that

Lemma 4.3.

\[ \| S_i(a, t) \|_{L^p_c} \lesssim S^*_i(t), \| S^*_i(t) \|_{L^p_c [c, T]} \lesssim \| w \|_{X(0,T)} (T-c)^\alpha \lesssim h^\alpha \| w \|_{L^p_c X(0,T)}. \]

(4.9)

We will not track the exact value of \( \alpha \), and it may change (smaller) line by line. We will need a similar control for the \( L^p_c \) norm for \( S_i \). This is indeed easier by observe

\[ \| \int_a^t e^{i(t-s)} \Delta \left( \frac{W(t_{j+1}) - W(t_{j})}{t_{j+1} - t_{j}} \right) e^{i(s-r)} \Delta w(s) ds \|_{L^2_c} \]

\[ \| \int_a^t e^{i(t-s)} \Delta \left( \frac{W(t_{j+1}) - W(t_{j})}{t_{j+1} - t_{j}} \right) e^{i(s-r)} \Delta w(s) ds \|_{L^2} \]

\[ \lesssim \sup_{\tau \in [c, d]} \int_c^\tau e^{i(s-r)} \Delta \left( \frac{W(t_{j+1}) - W(t_{j})}{t_{j+1} - t_{j}} \right) e^{i(s-r)} \Delta w(s) ds \|_{L^2_c} \]

(4.10)

And the last term does not depend on \( t \). Similar observation works for \( S_2 \). Then, one may derive the following analogue of Lemma 4.3.

Lemma 4.4. There exists \( \tilde{S}^*_i \), \( i = 1, 2 \), so that

\[ \forall t \in [0, T], \| S_i(a, t) \|_{L^2_c} \lesssim \tilde{S}^*_i, \| \tilde{S}^*_i \|_{L^2} \lesssim h^\alpha \| w \|_{L^p_c X(0,T)}. \]

(4.11)

We finally collect the estimates for all the \( e_j \). Recall we let \( \Delta_j W := W(t_{j+1}) - W(t_j) \). We also let \( \Delta_j U := U(t_{j+1}) - U(t_{j}) \). We will use \( C_m \) to denote a constant may depend on \( m \).

Lemma 4.5. Let \([a, b] \subset [a, T] \subset [c, d] \), we have

- Estimate for \( e_1 \).

\[ \| e_1 \|_{X(a,b)} \lesssim C_m (\sup_j \| \Delta_j W \|_{L^\infty_c}) \| w \|_{X(0,b)}. \]

(4.12)

\[ \| E_1 \|_{X(a,b)} \lesssim C_m \| \Delta_j W \|_{L^\infty_c}. \]

(4.13)
- **Estimate for e_2.**

\[
\|e_2(a, t)\|_{X_{a,b}} \lesssim \frac{1}{m} \|w\|_{X_{a,b}} \|\tilde{v}\|_{X_{2[a,b]}}^5. \tag{4.14}
\]

- **Estimate for e_3.**

\[
\|\|e_3(a, t)\|_{X_{a,b}} \lesssim C_m \sup_j \|\Delta_j U\|_{L_x^\infty} \tag{4.15}
\]

- **Estimate for e_4, e_5, e_6.** There exists \(e^*(t), \tilde{e}^*\)

\[
\sum_{i=4}^6 \|e_i(a, t)\|_{L_{x}^{10}} \leq e^*(t), \|e^*(t)\|_{L_{x}^{L_0[0,1]}} \lesssim \delta \tag{4.16}
\]

\[
\sum_{i=4}^6 \|e_i(a, t)\|_{L_x^2} \leq \tilde{e}^*, \|\tilde{e}^*\|_{L_x^L} \lesssim \delta \tag{4.17}
\]

**Proof of Lemma 4.5, a sketch.** For \(e_1(a, t)\), the estimate is similar to (3.47). By Strichartz estimate, (2.2), we have

\[
\|e_1(a, t)\|_{X_{a,b}} \lesssim \left\| \left( \frac{W(t_{j(s)+1}) - W(t_{j(s)})}{t_{j(s)+1} - t_{j(s)}} \int_{[s]} e^{i(s-r)\Delta} \phi_m(\|v\|_{X_{2(0,r)}})\mathcal{N}(v) - \phi_m(\|\tilde{v}\|_{X_{2(0,r)}})\mathcal{N}(\tilde{v}) \right) \right\|_{L_x^1L_{x}^2(a,b)}.
\]

Due to the cut-off \(\phi_m\), one always have

\[
\|\phi_m(\|v\|_{X_{2(0,r)}})\mathcal{N}(v)\|_{L_x^1L_{x}^2} + \|\phi_m(\|\tilde{v}\|_{X_{2(0,r)}})\mathcal{N}(\tilde{v})\|_{L_x^1L_{x}^2} \leq C_m \tag{4.19}
\]

We already obtained (4.14). (Indeed, (4.14) will be enough for this section, we record (4.12) for potential later use.)

Go back to (4.12), If \(\|w\|_{X_{[0,T]}^1} \geq 1\), the desired estimate follows (4.14). Otherwise, if \(\|w\|_{X_{[0,T]}^1} \leq 1\), we further derive

\[
\|\phi_m(\|v\|_{X_{2(0,r)}})\mathcal{N}(v)\|_{L_x^1L_{x}^{10}[0,T]} + \|\phi_m(\|\tilde{v}\|_{X_{2(0,r)}})\mathcal{N}(\tilde{v})\|_{L_x^1L_{x}^{10}[0,T]} \leq C_m \tag{4.20}
\]

(That means, the cut off for \(v\) and \(\tilde{v}\) are essentially same.) we split as

\[
\|\phi_m(\|v\|_{X_{2(0,r)}})\mathcal{N}(v) - \phi_m(\|\tilde{v}\|_{X_{2(0,r)}})\mathcal{N}(\tilde{v})\|_{L_x^1L_{x}^{10}[0,T)} \leq\|\phi_m(\|v\|_{X_{2(0,r)}})\mathcal{N}(v) - \mathcal{N}(\tilde{v})\|_{L_x^1L_{x}^{10}[0,T)} + \|\phi_m(\|v\|_{X_{2(0,r)}}) - \phi_m(\|\tilde{v}\|_{X_{2(0,r)}})\|_{L_x^1L_{x}^{10}[0,T)} \tag{4.21}
\]

and observe

\[
\|\phi_m(\|v\|_{X_{2(0,s)}}) - \phi_m(\|\tilde{v}\|_{X_{2(0,s)}})\| \lesssim \frac{1}{m} \|w\|_{X_{[0,s]}},
\]

then estimate (4.12) follows.
Estimate for $e_2$ follows from Strichartz estimate and the observation
\[
|\phi_m(\|v\|_{X_2(a,s)}) - \phi_m(\|\tilde{v}\|_{X_2(0,s)})| \lesssim \frac{1}{m} \|w\|_{X_2(0,s)}.
\] (4.22)

Estimate for $e_3$ is similar to estimate for $e_1$ except we replace $\Delta_j W$ by $\Delta_j U$, and we use estimate
\[
\|\phi_m(\|v\|_{X_2(0,s)}) \mathcal{W}(\tilde{v}(s))\|_{L^1_t L^2_x} \lesssim C_m.
\] (4.23)

For term $e_4, e_5, e_6$, we will handle similarly as Lemma 4.2 and 4.3, but here we don’t explore the smallness by constrain the analysis in small interval\(^3\). However, we do observe in all those term, there is one $W$ has been replaced by $U$, which gives a $\delta$ in the left hand due to (4.1).

\[ \square \]

4.3 Proof of Lemma 4.2

Now we are ready to prove Lemma 4.2, which will conclude the proof of Corollary 1.13. It should be pointed out, once $h$ is fixed, we are allowed to take $\delta_0$ as small as we want (by choosing $\delta$ even smaller), we will never use and we should never use estimate which is of form $\delta_0 \lesssim h^\alpha$.

Recall we will need a small universal $\eta$. For a.s. every $\omega$, we will do a (random) partition of the interval $[c,T]$ into $c = a_0 < a_1 < ... a_J = T$, such that for every $[a_i, a_{i+1}]$

- For $v$, either $\|v\|_{X_2(a_i, a_{i+1})} \leq \eta$ or $\phi_m(\|v\|_{X_2(0,s)}) = 0, s \geq a_i$.
- For $v$, either $\|\tilde{v}\|_{X_2(a_i, a_{i+1})} \leq \eta$ or $\phi_m(\|v\|_{X_2(0,s)}) = 0, s \geq a_i$

Thus, there can be most $J \sim m^5/\eta^5 + 1 \leq C_m$ such intervals.

In every interval $[a_i, a_{i+1}]$.

We estimated as, via Strichartz and Simple triangle inequality,
\[
\|w\|_{X_2(a_i, a_{i+1})} \lesssim \|w(a_i)\|_{L^2_x} + \eta^4 \|w\|_{X_2(a_i, a_{i+1})} + \sum_i \|S_t\|_{X[a,b]} \] (4.24)

Plug in the estimate for $e_i, i = 1, ..., 6$ and use the definition of $S^*_i(t), \tilde{S}^*_i$, (also note $e_2$ appears means $\|\tilde{v}\|_{X_2(a_i, a_{i+1})} \lesssim \eta$) we derive
\[
\|w\|_{X_2(0, a_{i+1})} \lesssim \|w\|_{X_2(0, a_i)} + \eta^4 \|w\|_{X_2(0, a_{i+1})} + \sum_i \tilde{S}^*_i + \sum_i \|S^*_i\|_{L^\infty_t [c,T]} + C_m \sup_j \|\Delta_j W\|_{L^\infty_x} + \frac{1}{m} \|w\|_{X_2(a_{i+1})} \eta^5 + C_m \sup_j \|\Delta_j U\|_{L^\infty_x} + \sum_{i=4}^6 \|e_s(0)\|_{L^1_t [0,1]} + \bar{e}_i^* \] (4.25)

Note that we $\eta$ is small enough, the term $\eta^4 \|w\|_{X_2(0, a_{i+1})}, \frac{1}{m} \|w\|_{X_2(a_{i+1})} \eta^5$ will be absorbed into the left side. Iterate the above formula $\sim C_m$ times, we derive

\[ \square \]

\(^3\)Such smallness is not useful here. In some sense, the extra smallness $h^\alpha$ is not small enough.
Proof of Proposition 1.16

5 Proof of Proposition 1.16

5.1 Some reduction by Corollary 1.13

By Corollary 1.13, we only need prove Proposition 1.16 for smooth initial data \( f \in L_x^\infty H^1 \) in Assumption 1.1, and we only need to study noise which are finite dimensional and smooth. We, without loss of generality, enhance Assumption 1.1 into

\[
f \in L_x^\infty H^1, W(x,t) = V_1(x)B_1(t)+V_2B_2, V_1, V_2(x) \text{ is some Schawarz class function.}
\]  

(5.1)

(Note that finite dimensional smooth noise is no different as the simple noise above, we consider a dimension 2 noise rather than a simple noise \( V(x)B(t) \) since we want to keep track of the cancellation of non-diagonal term.)

This will give some Hölder regularity in time of the flow \( u^{(n)}_m, u^m \), which is essential to establish Wong-Zakai type convergence.

Note that we deal with truncated equation (with \( \phi_m \) in front of the nonlinearity) only and \( m \) is fixed, hence all the bounds are allowed to depend on \( m \). Again, we use \( C_m \) to denote constant may depend on \( m \), and we allow \( C_m \) to change line by line.

Let \( w = u^{(n)}_m - u_m \). Similar to the previous section, we reduce the proof of Proposition 1.16 with the enhanced assumption (5.1), to the following claim.

Lemma 5.1. There is \( h > 0 \), such that when \( \delta_0 \) is chosen small enough, one can always choose \( \delta \), small enough depending on \( (\delta_0, \rho, m, h, V_1, V_2, f) \), so that if \( \| \pi^n \| \leq \delta \), then if one has

\[
\| w \|_{L_x^\infty X(0,c)} < \delta_0
\]  

(5.2)

and \( d - c \sim h \), then one has,

\[
\| w \|_{X(0,d)} \lesssim \delta_0
\]  

(5.3)
Proof of Proposition 1.16

Note that \( w(0) = 0 \), thus Proposition 5.1 follows by iterating the above Lemma. Lemma 5.1 follows from the following bootstrap type Lemma.

**Lemma 5.2.** Given the assumptions of Lemma 5.1 one can find \( C > 0 \), so that assuming bootstrap assumption that on \( T \in [c, d] \),

\[
\|w(t)\|_{L^\infty_t X([0, T])} \leq 2C\delta_0,
\]
then one has bootstrap estimate

\[
\|w(t)\|_{L^\infty_t X([0, T])} \leq C\delta_0.
\]

(This \( C \) can be dependent on \( m \), which is fixed all the time.)

The rest of the section will be devoted to the proof of Lemma 5.2

### 5.2 Well-posedness results in \( H^1 \)

Before we go to proof of Lemma 5.2 we need some further wellposedness for \( u_m \), \( u_m^{(n)} \) themselves.

We start with he following two propositions on the persistence of regularity.

**Proposition 5.3.** With enhanced assumption (5.1), there exists \( C = C(m, \rho, \Lambda_{\text{ini}}, \Lambda'_{\text{ini}}, \|f\|_{L^\infty_t H^1}) \) such that

\[
\sup_n \|u_m^{(n)}\|_{L^\infty_t L^\infty_t H^1} \leq C, \quad \|u_m\|_{L^\infty_t L^\infty_t H^1} \leq C.
\]

**Proof.** The bound for \( u_m \) is proved in [FX18a Proposition 3.2]. The uniform-in-\( n \) bound for \( \{u_m^{(n)}\} \) can be proved by following exactly the same procedure as that for \( u_m \) and by further local Duhamel expansion as in Section 3. We omit the details here.

**Proposition 5.4.** Let \( \alpha \in (0, 1) \). For every \( m \) and \( \rho \), there exists \( C = C(m, \rho, \Lambda_{\text{ini}}, \Lambda'_{\text{ini}}) \) such that

\[
\|u_m^{(n)}(t) - u_m^{(n)}(s)\|_{L^\infty_t H^\beta} \leq C|t - s|^{\frac{\beta}{2}} \left( \|u_m^{(n)}\|_{L^\infty_t L^\infty_t H^1} + \|u_m^{(n)}\|_{L^\infty_t L^\infty_t H^1}^5 \right).
\]

As a consequence, we have

\[
\|u_m^{(n)}\|_{L^\infty_t L^\infty_t H^\beta} \leq C\left( \|u_m^{(n)}\|_{L^\infty_t L^\infty_t H^1} + \|u_m^{(n)}\|_{L^\infty_t L^\infty_t H^1}^5 \right) \leq C
\]

for every \( \beta < \frac{1-\alpha}{2} \). In the second claim, the constant \( C \) also depends on \( \beta \). The same bounds are true for \( u_m \).

**Proof.** Again, for simplicity, we prove the bounds for \( u_m \) only. For every \( s < t \), we have

\[
u_m(t) - u_m(s) = (e^{\Delta(t-s)} u_m(s) - u_m(s)) - i \int_s^t e^{i(s-r)\Delta} \left( \varphi_m(\|u_m\|_{L^\infty_t X}) \mathcal{N}(u_m(r)) \right) dr
\]

\[\quad - i \int_s^t e^{i(t-r)\Delta}(Vu_m)(r)dB(r) - \frac{1}{2} \sum_k \int_s^t e^{i(t-r)\Delta}(V_k^2 u_m(r))dr.
\]

(5.9)
The last three terms above can be controlled directly via dispersive estimates and in the stochastic integral case, also with Burkholder inequality. To bound the first term, one really uses the flow being in $H^1$, so that

$$\|e^{i(t-s)\Delta}u_m(s) - u_n(s)\|_{H^2_t} \leq C|t-s|^{1/2}\|u_m(s)\|_{H^1_t}. \quad (5.10)$$

Hence, one get the desired bound on $\|u_m(t) - u_m(s)\|_{L_t^0 C_s^3 H^2}$. The bound for $\|u_m\|_{L_t^\infty C_s^3 H^2}$ follows from Kolmogorov criteria.

### 5.3 Equation for $u_m$, $w = u^n_m - u^m$ and a collection of estimates

We rewrite the equation for $u_m$ so that it will be suitable to do comparison of to $u^n_m$. Since we will not compare two different $u^n_m$ and $u^m$, we will still short $t_j^m$ as $t_j$. We still define $j(s)$ be the index so that $t_j(s) < s \leq t_{j(s)+1}$, and denote $t_{j(s)}$ by $[s]$. We denote $u_m^n$ by $v$ and denote $u_m$ by $\tilde{v}$. We will make the argument and the notation similar to the previous section.

Note that $v$ solves $\{3.5\}$, except that we need to use $\phi_m(\|v\|, \chi_{[0,T]}(N(v(t))))$ to replace $N(v(t))$.

We recall $\{1.12\}$, and rewrite the equation of $\tilde{v} = um$ as a perturbation of the equation, which $v = u_m^{(n)}$ satisfies. We have

$$\tilde{v}(t) = e^{it\Delta} \tilde{v}(a) - i \int_a^t e^{i(t-s)\Delta} (\phi_m(\|v\|, \chi_{[0,s)}(N(\tilde{v}(s)))) ds$$

$$- (i) \int_a^t e^{i(t-s)\Delta} \frac{\Delta_j(s)W}{t_{j(s)+1} - t_{j(s)}} e^{i(s-t)\Delta} \tilde{v}([s]) ds$$

$$- \int_a^t e^{i(t-s)\Delta} \left[ \frac{\Delta_j(s)W}{t_{j(s)+1} - t_{j(s)}} \int_s^t e^{i(t-r)\Delta} \frac{\Delta_j(r)W}{t_{j(s)+1} - t_{j(s)}} \tilde{v}(r) dr \right] ds$$

$$\pm i \int_{t_{j(s)+1}}^{[t]} e^{i(t-s)\Delta} (\tilde{v}(s) - \tilde{v}([s])) dW_s$$

$$\pm \frac{1}{2} \int_{t_{j(s)+1}}^{[t]} e^{i(t-s)\Delta} \left\{ V(x)^2 \tilde{v}(s) - 2 \frac{\Delta_j(s)W}{t_{j(s)+1} - t_{j(s)}} \int_s^t e^{i(t-r)\Delta} \frac{\Delta_j(r)W}{t_{j(s)+1} - t_{j(s)}} \tilde{v}(r) dr \right\} ds$$

$$\pm i \int_a^{t_{j(s)+1}} e^{i(t-s)\Delta} \tilde{v} dW_s + \pm i \int_{[t]}^t e^{i(t-s)\Delta} \tilde{v} dW_s$$

$$\pm \frac{1}{2} \int_a^{t_{j(s)+1}} e^{i(t-s)\Delta} V^2 \tilde{v} ds + \pm \frac{1}{2} \int_{[t]}^t e^{i(t-s)\Delta} V^2 \tilde{v} ds$$

(5.11)

The term with $\pm$ sign will be treated in a purely perturbative way, and the choice of $\pm$ sign will not matter in our proof. We use $V^2$ to denote $V_1^2 + V_2^2$. 
Recall $w := v - \tilde{v} = u_m^n - u_m$, we derive

$$w(t) = e^{it\Delta} w(a) - i \int_a^t e^{i(t-s)\Delta} \langle \phi_m(\|v\|, x_{2|0, s}) \mathcal{N}(v(s)) - \phi_m(\|\tilde{v}\|, x_{2|0, s}) \mathcal{N}(\tilde{v}(s)) \rangle ds$$

$$ - (i) \int_a^t e^{it\Delta} \left( \frac{\Delta_{j(s)}W}{t_{j(s)+1} - t_{j(s)}} e^{i(s-[s])\Delta w([s])} \right) ds$$

$$- \int_a^t e^{i(t-s)\Delta} \left( \frac{\Delta_{j(s)}W}{t_{j(s)+1} - t_{j(s)}} \int_s^t e^{i(s-r)\Delta} \left[ \frac{\Delta_{j(s)}W}{t_{j(s)+1} - t_{j(s)}} w(r) \right] dr \right) ds$$

$$\pm i \int_{[t]} e^{it\Delta} \left( \tilde{v}(s) - \tilde{v}([s]) \right) dW_s$$

$$\pm \frac{1}{2} \int_{[t]} e^{it\Delta} \left\{ V(x)^2 \tilde{v}(s) - 2 \frac{\Delta_{j(s)}W}{t_{j(s)+1} - t_{j(s)}} \int_s^t e^{i(s-r)\Delta} \left[ \frac{\Delta_{j(s)}W}{t_{j(s)+1} - t_{j(s)}} \tilde{v}(r) \right] dr \right\} ds$$

$$\pm i \int_a^t e^{i(t-s)\Delta} wdW_s + \pm \frac{1}{2} \int_{[t]} e^{it\Delta} \left( \tilde{v}(s) - \tilde{v}([s]) \right) dW_s$$

$$\pm \frac{1}{2} \int_{a}^{t_{j(a)+1}} e^{i(t-s)\Delta} V^2 \tilde{v} ds + \pm \frac{1}{2} \int_{[t]} e^{it\Delta} \left( \tilde{v}(s) - \tilde{v}([s]) \right) dW_s$$

$$\pm \frac{1}{2} \int_{[t]} e^{it\Delta} \left\{ V(x)^2 \tilde{v}(s) - 2 \frac{\Delta_{j(s)}W}{t_{j(s)+1} - t_{j(s)}} \int_s^t e^{i(s-r)\Delta} \left[ \frac{\Delta_{j(s)}W}{t_{j(s)+1} - t_{j(s)}} \tilde{v}(r) \right] dr \right\} ds$$

$$\pm i \int_a^t e^{i(t-s)\Delta} \left( \frac{W(t_{j(s)+1}) - W(t_{j(s)})}{t_{j(s)+1} - t_{j(s)}} \int_s^t e^{i(s-r)\Delta} \phi_m(\|v\|, x_{2|0, s}) \mathcal{N}(v(r)) dr \right) ds$$

(5.12)

We again introduce some notation to simplify the equation, let

- $M_1(a, t) = (i) \int_a^t e^{i(t-s)\Delta} \left( \frac{\Delta_{j(s)}W}{t_{j(s)+1} - t_{j(s)}} e^{i(s-[s])\Delta w([s])} \right) ds$

- $M_2(a, t) = (i) \int_a^t e^{i(t-s)\Delta} \left( \frac{\Delta_{j(s)}W}{t_{j(s)+1} - t_{j(s)}} e^{i(s-[s])\Delta w([s])} \right) ds$

- $M_3(a, t) = \pm i \int_{a}^{t_{j(a)+1}} e^{i(t-s)\Delta} \left( \tilde{v}(s) - \tilde{v}([s]) \right) ds$

- $g_1(a, t) = \pm \frac{1}{2} \int_{a}^{t_{j(a)+1}} e^{i(t-s)\Delta} \left\{ V(x)^2 \tilde{v}(s) - 2 \frac{\Delta_{j(s)}W}{t_{j(s)+1} - t_{j(s)}} \int_s^t e^{i(s-r)\Delta} \left[ \frac{\Delta_{j(s)}W}{t_{j(s)+1} - t_{j(s)}} \tilde{v}(r) \right] dr \right\} ds$

- $g_2(a, t) = \pm i \int_{a}^{t_{j(a)+1}} e^{i(t-s)\Delta} \left( \frac{W(t_{j(s)+1}) - W(t_{j(s)})}{t_{j(s)+1} - t_{j(s)}} \int_s^t e^{i(s-r)\Delta} \phi_m(\|v\|, x_{2|0, s}) \mathcal{N}(v(r)) dr \right) ds$

Now, we may rewrite the equation for $w$ as

$$w(t) = e^{it\Delta} w(a) - i \int_a^t e^{i(t-s)\Delta} \langle \phi_m(\|v\|, x_{2|0, s}) \mathcal{N}(v(s)) - \phi_m(\|\tilde{v}\|, x_{2|0, s}) \mathcal{N}(\tilde{v}(s)) \rangle ds$$

$$+ \sum_i M_i(a, t) + \sum_i g_i(a, t)$$

(5.13)

We will summarize the estimate for $M_i, i = 1, 2, 3$ and $g_i, i = 2, 3, 4$ in the following Lemma, which will be analogues of Lemma [13], Lemma [14] and Lemma [15].

**Lemma 5.5.** Let $[a, b] \subset [a, T] \subset [c, d], d - c \sim h$, we have the following estimates
Proof of Proposition 1.16

- **Estimate for** \( M_1(a,t), M_2(a,t), M_3(a,t) \). There exist \( \theta > 0 \) and \( M_\ast(t), \tilde{M}_\ast \), so that

\[
\forall t \in [a,T], i = 1,2,3, \| S_i(a,t) \|_{L^p} \lesssim M_\ast(t), \| S_i(a,t) \|_{L^2} \lesssim \tilde{M}_\ast, \tag{5.14}
\]

and

\[
\| M_i(\tau) \|_{L^\infty, \tau \in [c,T]} \lesssim h^\theta \| w \|_{L^\infty, \tau \in (0,T)}, i = 1,2. \tag{5.15}
\]

- **Estimate for** \( g_1(a,t), g_2(a,t) \). There exists \( g_\ast(t), \tilde{g}_\ast, i = 1,2, \) so that

\[
\| g_i(a,t) \|_{L^p} \lesssim g_\ast(t), \| g_i(a,t) \|_{L^2} \lesssim \tilde{g}_\ast, t \in [c,T]. \tag{5.16}
\]

and

\[
\| g_i(\tau) \|_{L^p} + \| \tilde{g}_i(\tau) \|_{L^2} \lesssim C_m \| \pi^{(n)} \|^\theta. \tag{5.17}
\]

- **Estimate for** \( g_3(a,t) \).

\[
\| g_3(a,t) \|_{L^2, \tau \in [c,T]} \lesssim \| \pi^{(n)} \|^\theta, t \in [c,T]. \tag{5.18}
\]

- **Estimate for** \( g_4(a,t) \)

\[
\| g_4(a,t) \|_{X_{[a,b]}} \lesssim C_m \| \sup_j W \|_{L^\infty} \tag{5.19}
\]

**Remark 5.6.** The exact value of \( \theta \) may change line by line, but we only need such \( \theta \) be positive, (so that we can gain a small power). The point is when \( h \), and \( \| \pi^{(n)} \| \) are small enough (depending on \( m \)), the term \( h^\theta, \| \pi^{(n)} \|^\theta \) could always over come loss of any large constant \( C_m \).

**Proof of Lemma 5.5.** Since the structure of the proof is very similar to Lemma 4.5, we will sketch for the parts which are similar and only focus on the difference. , the main difference will be in the estimate of \( M_3(a,t), g_1(a,t), g_2(a,t), \) which relies on the Holder continuity of the flow. The term \( g_4(a,t) \) seem to be the most technically complicated term.

The Estimate for \( M_1(a,t), M_2(a,t) \) are completely same, (though the \( w \) has different meaning), as the estimate for \( S_1(a,t), S_2(a,t) \). For \( M_3(a,t) \), one may take \( M_3^\ast(t) := \sup_{c \leq \tau \leq T} \| \int_c^\tau e^{i(t-s)}(\tilde{v})(s) - \tilde{v}(s) \|_{L^1}, \) \( M_3 := \sup_{c \leq \tau \leq T} \| \int_c^\tau e^{i(t-s)}(\tilde{v})(s) - \tilde{v}(s) \|_{L^2}, \) argue similar as the estimate of \( S_1(a,t) \), one can derive

\[
\| M_3^\ast(t) \|_{L^\infty, L^1} + \| M_3 \|_{L^p} \lesssim \| \tilde{v}(s) - \tilde{v}(s) \|_{L^p} \tag{5.20}
\]

(We could also get a gain of \( h^\theta \) in the estimate, but this is not useful here, and we just put it as \( h^\theta \leq 1 \)).

Plug in the Holder continuity estimate, Propostion 5.4 and Observe \( |s-[s]| \leq \| \pi^n \| \), we derive

\[
\| M_3^\ast(t) \|_{L^\infty, L^1} + \| M_3 \|_{L^p} \lesssim C_m \| \pi^{(n)} \|^\theta, \text{ for some } \theta > 0. \tag{5.21}
\]
Now we turn to the estimate of \( g_1(a, t) \), we will split \( g_1(a, t) \) as
\[
g_1(a, t) := g_{11}(a, t) + g_{12}(a, t)
\]
where
\[
g_{11}(a, t) = \pm \frac{1}{2} \int_{t_j(a)+1}^{t} e^{it-s} \{ V^2(\tilde{v}(s)) - \frac{2\Delta_j W}{t_{j(j)+1} - t_j(s)} \int_{[s]}^s \Delta_j W \tilde{v}(\{s\}) \} ds
\]
\[
g_{12}(a, t) = \pm \frac{1}{2} \int_{t_j(a)+1}^{t} e^{it-s} \{ V^2(\tilde{v}(s) - \tilde{v}(\{s\})) - \frac{2\Delta_j W}{t_{j(j)+1} - t_j(s)} \int_{[s]}^s \Delta_j W \tilde{v}(\{s\}) \} ds
\]

We only need to find the associated \( g_{11}^i(t), \tilde{g}_{11}^i \) for \( g_{11}(a, t) \), \( i = 1, 2 \).

Recall we are working on simple noise \( W(x, t) = V(x)B(t) \) and \( V \) is some nice Schwarz function.

We first give the estimate for \( g_{12} \). Recall that \( \tilde{v} \) has some Holder continuity, Proposition 5.4, i.e we have estimate
\[
\|\tilde{v}\|_{L^2_{\theta}C^\theta L^2_2([0, 1])} \leq C_m,
\]
which of course implies \( \|\tilde{v}\|_{L^2_{\theta}C^\theta L^2_2([0, 1])} \leq C_m \).

Observe \( \|\tilde{v}(s) - tv((s))\|_{L^2_2} \leq \|\pi(n)\|^\theta \|\tilde{v}\|_{C^\theta L^2_2} \), and \( \|\tilde{v}(s) - tv((s))\|_{L^2_2} \leq |t_{j(s)+1} - t_j(s)| \|\pi(n)\|^\theta \|\tilde{v}\|_{C^\theta L^2_2}, \forall r \in ([s], s) \).

We may use point wise estimate,
\[
\|e^{it-s}\Delta(V(x)\tilde{v}(s) - \tilde{v}(\{s\}))\|_{L^2_2 \cap L^2_2} \lesssim (t-s)^{-2/5} \|\pi(n)\|^\theta \|\tilde{v}\|_{C^\theta L^2_2}
\]
\[
\|e^{it-s}\Delta\{ \frac{2\Delta_j W}{t_{j(j)+1} - t_j(s)} \int_{[s]}^s e^{i(s-r)\Delta} \Delta_j W \tilde{v}(r) - \frac{\Delta_j W}{t_{j(j)+1} - t_j(s)} \tilde{v}(\{s\}) \} \|_{L^2_2 \cap L^2_2} \lesssim (t-s)^{-5/2} \|\pi(n)\|^\theta \|\tilde{v}\|_{C^\theta L^2_2}
\]
\[
\lesssim (t-s)^{-5/2} \|\pi(n)\|^\theta \sup_j \frac{|B(t_{j+1}) - B(t_j)|^2}{(t_{j+1} - t_j)^{1-\theta/2}} \|\tilde{v}\|_{C^\theta L^2_2} \]
(5.26)

(We have also applied dispersive estimate we have applied dispersive estimate \( \|e^{it-s}\Delta\|_{L^3_{2/1} \rightarrow L^3_2} \lesssim (t-s)^{-2/5}, \) and \( e^{it\Delta} \) is unitary in \( L^2_2 \), and the estimate the difference between \( e^{it-s}\Delta(V(x)\tilde{v}(s)) \) and \( V(x)\tilde{v}(\{s\}) \), one may apply Lemma A.1[A.2]

Thus, we derive
\[
\|g_{12}(a, t)\|_{L^2_2 \cap L^2_2} \lesssim \left( \int_0^1 (t-s)^{-5/2} ds \right)\{ \|\pi(n)\|^\theta \|\tilde{v}\|_{C^\theta L^2_2} + \|\pi(n)\|^\theta \|\tilde{v}\|_{C^\theta L^2_2} \}
\]
(5.27)
Simply take $g_{12}(t), \tilde{g}_{12}$ both be right hand side of (5.27), the desired estimate follows from (5.25) and the observation $\| \sup_{j} \frac{|B_i(t_{j+1})-B_i(t_j)|^2}{(t_{j+1}-t_j)^{\frac{3}{2}}} \|_{L^2} \lesssim 1$.

We now give the estimate for $g_{11}$ Observe

$$g_{11}(a, t) = \sum_{[t_j, t_{j+1}] \subseteq [a, t]} \int_{t_j}^{t_{j+1}} e^{i(t-s)\Delta} \left\{ \mathbf{V}^2 \tilde{v}(t_j) \right\} ds$$

$$- \sum_{i, i' = 1, 2} \frac{(B_i(t_{j+1}) - B_i(t_j))(B_{i'}(t_{j+1}) - B_{i'}(t_j))}{(t_{j+1} - t_j)^2} V_i V_{i'}(x) \int_{t_j}^{t_{j+1}} \tilde{v}([s]) dr ds$$

(There is no typo, there is no $e^{i(s-r)\Delta}$ in this integral).

We further split it as

$$g_{11}(a, t) = g_{111}(a, t) + g_{112}(a, t)$$

where

$$g_{111}(a, t) = \sum_{[t_j, t_{j+1}] \subseteq [a, t]} \int_{t_j}^{t_{j+1}} e^{i(t-t_j)\Delta} \left\{ \mathbf{V}^2 \tilde{v}(t_j) \right\} ds$$

$$- \sum_{i, i' = 1, 2} \frac{(B_i(t_{j+1}) - B_i(t_j))(B_{i'}(t_{j+1}) - B_{i'}(t_j))}{(t_{j+1} - t_j)^2} V_i V_{i'}(x) \int_{t_j}^{t_{j+1}} \tilde{v}([s]) dr ds$$

and $g_{112} = g_{11} - g_{111}$.

The idea is because of Lemma A.1 we can treat some regularity to replace the $e^{i(t-s)\Delta}$ in $[t_j, t_{j+1}]$ by $e^{i(t-t_j)\Delta}$. The estimate of $g_{112}$ is similar to $g_{12}$, we skip the details. We focus on the estimate of $g_{111}(a, t)$, and construct the associated $g^*_{111}(t), \tilde{g}^*_{111}$. Observe we have $g_{111}(a, t)$ equals

$$\pm \frac{1}{2} \sum_{[t_j, t_{j+1}] \subseteq [a, t]} (t_{j+1} - t_j) e^{i(t-t_j)\Delta} \sum_{i, i' = 1, 2} V_i V_{i'}(\delta_{it} - \frac{(B_i(t_{j+1}) - B_i(t_j))(B_{i'}(t_{j+1}) - B_{i'}(t_j))}{(t_{j+1} - t_j)})$$

(5.31)

For notation convenience, we short $(\delta_{it} - \frac{(B_i(t_{j+1}) - B_i(t_j))(B_{i'}(t_{j+1}) - B_{i'}(t_j))}{(t_{j+1} - t_j)})$ as $c^j_{it'}$. We will let

$$g^*_{111}(t) := \sup_{j_0} \left\| \sum_{j \leq j_0} (t_{j+1} - t_j) e^{i(t-t_j)\Delta} \sum_{i, i'} V_i V_{i'}(c^j_{it'}) \right\|_{L^2_{x-t}}$$

$$g^*_{111} := \sup_{j_0} \left\| \sum_{j \leq j_0} (t_{j+1} - t_j) e^{i(t-t_j)\Delta} \sum_{i, i'} V_i V_{i'}(c^j_{it'}) \right\|_{L^2_{x}}$$

(5.32)

The key observation, which should also be expected in any Wong-Zakai convergence result, is that $c^j_{it'}$ is of mean 0. Thus, fix $t$, $\sum_{j \leq j_0} (t_{j+1} - t_j) e^{i(t-t_j)\Delta} V^2 \sum_{i, i'} V_i V_{i'}(c^j_{it'})$ is a martingale in $L^1_{x-t}$, and $\sum_{j \leq j_0} (t_{j+1} - t_j) e^{i(t-t_j)\Delta} \sum_{i, i'} V_i V_{i'}(c^j_{it'})$ is itself a martingale in $L^2_{x}$. What remains is similar to the estimate of $S^*_{mar}$ in Lemma 3.1 We sketch the estimate for $g^*_{111}(t)$ for the convenience of the reader and skip the estimate for $\tilde{g}^*_{111}$.
Again, by (3.14), we may fix $t$, and just estimate $\|g_{111}'(t)\|_{L^5_t L^{10}_x}$, and it follows from Burkholder inequality; (2.14) that

$$
\|g_{111}'(t)\|^p_{L^5_t L^{10}_x} \lesssim \left( \sum_j (t - t_j)^{-4/5} (t_{j+1} - t_j)^2 \left( \sum \omega_i e_i \right)^2 \right)^{p/2} \quad (5.33)
$$

Also note $e_i^{(j)}$ are i.i.d with respect to $j$. Observe

$$
\| \left( \sum_j (t - t_j)^{-4/5} (t_{j+1} - t_j)^2 \right) \|_{L^{10}_x} \lesssim \sum_j (t - t_j)^{-4/5} (t_{j+1} - t_j)^2 \lesssim \| \pi^{(n)} \|
$$

The estimate for $g_{111}'$ now follows. The estimate for $g_1(a, t)$ is now finished.

The estimate for $g_2(a, t)$ is exactly as the estimate for $M_3$. Somehow, we don’t get extra smallness for this term, though it looks like the marginal term of $M_3$.

We now go to the estimate for $g_3(a, t)$. This term one can directly applies Triangle inequality and dispersive estimate (2.1) to derive

$$
\|g_3(a, t)\|_{L^2_t L^5_x} \lesssim \int_{[t_{j_0+1}, a] \cap [t, t_d]} |t - s|^{-2/5} ds \lesssim \| \pi^{(n)} \|^{1/5}.
$$

(In last step, just observe $t_{j_0+1} - a \leq \| \pi^{(n)} \|$, $t - [t] \leq \| \pi^{(n)} \|$.)

We finally go to the estimate for $g_4(a, t)$. The estimate is similar to (3.46), (3.47), also similar to the estimate $e_3$ in Lemma 4.3. Just observe

$$
\| \phi_m(\tilde{v}) N(v) \|_{L^1_t L^2_x} \leq C_m,
$$

and the desired estimate follows.

\[ \Box \]

### 5.4 Proof of Lemma 5.2

We present the proof of Lemma 5.2 here, which will conclude this section. The proof of Lemma 5.2 is very similar to the proof of Lemma 4.2.

We will need a small universal $\eta$. For a.s. every $\omega$, we will do a (random) partition of the interval $[c, T]$ into $c = a_0 < a_1 < \ldots a_J = T$, such that for every $[a_l, a_{l+1}]$

- For $v$, either $\|v\|_{X_2(a_l, a_{l+1})} \leq \eta$ or $\phi_m(\|v\|_{X_2(a_l, s)}) = 0$, $s \geq a_l$.
- For $\tilde{v}$, either $\|\tilde{v}\|_{X_2(a_l, a_{l+1})} \leq \eta$ or $\phi_m(\|\tilde{v}\|_{X_2(a_l, s)}) = 0$, $s \geq a_l$.

Thus, there can be most $J \sim m^5/\eta^5 + 1 \leq C_m$ such intervals. Observe in every $[a_l, a_{l+1}]$, one has

$$
\| \phi_m(\|v\|_{X_2(a_l, s)} N(v) - \phi_m(\|\tilde{v}\|_{X_2(a_l, s)}) n N(\tilde{v}) \|_{L^1_t L^2_x} \lesssim \eta^4 \quad (5.37)
$$

Indeed, one needs to consider the following cases

- Both $\phi_m(\|v\|_{X_2(a_l, s)}), \phi_m(\|\tilde{v}\|_{X_2(a_l, s)})$ are 0 for $s \geq a_l$. There is nothing to prove.
• Both $\phi_m(||v||_{X_2(0,s)}), \phi_m(||\tilde{v}||_{X_2(0,s)})$ are not 0, then one has $||v||_{X_2(a_t,a_{t+1})} \leq \eta, ||\tilde{v}||_{X_2(a_t,a_{t+1})} \leq \eta$, just split as $\phi_m(||v||_{X_2(0,s)})N(v) - \phi_m(||\tilde{v}||_{X_2(0,s)})nN(\tilde{v}) = -\phi_m(||\tilde{v}||_{X_2(0,s)})nN(\tilde{v}) = (\phi_m(||v||_{X_2(0,s)}) - \phi_m(||\tilde{v}||_{X_2(0,s)})nN(\tilde{v})$, and estimate separately with observation $||\phi_m(||v||_{X_2(0,s)}) - \phi_m(||\tilde{v}||_{X_2(0,s)})|| \leq ||w||_{\mathcal{A}(0, a_{t+1})}$.

• Without loss of generality, we only consider $\phi_m(||v||_{X_2(0,s)})$ is 0 for $s \geq a_t$ and $||\tilde{v}||_{X_2(a_t,a_{t+1})} \leq \eta$, we simply observe $\phi_m(||v||_{X_2(0,s)})N(v) - \phi_m(||\tilde{v}||_{X_2(0,s)})nN(\tilde{v}) = -\phi_m(||\tilde{v}||_{X_2(0,s)})nN(\tilde{v}) = (\phi_m(||v||_{X_2(0,s)}) - \phi_m(||\tilde{v}||_{X_2(0,s)})nN(\tilde{v})$, and observe again $||\phi_m(||v||_{X_2(0,s)}) - \phi_m(||\tilde{v}||_{X_2(0,s)})|| \leq ||w||_{\mathcal{A}(0, a_{t+1})}$.

Now, in every interval $[a_t, a_{t+1}]$, we estimated as, via Strichartz estimate and Simple triangle inequality,

$$||w||_{X_2[a_t,a_{t+1}]} \leq ||w(a_t)||_{L^2_x} + \eta^4 ||w||_{X_2[0,a_{t+1}]} + \sum_i ||S_i ||_{X_2[a_t,a_{t+1}]}$$

Absorbing $\eta^4 ||w||_{X_2[0,a_{t+1}]}$ into $||w||_{X_2[0,a_{t+1}]}$, and plug in the estimate in Lemma 5.5, we derive

$$||w||_{X_2[0,a_{t+1}]} \leq ||w||_{X_2[a_t,a_{t+1}]} + \sum_i ||M_i^*(t)||_{L^2_{[c,T]}} + ||\tilde{M}_i^*|| + ||\tilde{g_i}^*||_{L^2_{[c,T]}} + ||\pi^0||^6 + C_m \sup_j \Delta_j W \|_{L^\infty_x} \tag{5.39}$$

Iterate $\sim m^5/\eta^5 + 1 \leq C_m$ times, we derive

$$||w||_{X_2[0,T]} \leq C_m \left\{ ||w||_{X_2[0,c]} + \sum_i ||M_i^*(t)||_{L^2_{[c,T]}} + ||\tilde{M}_i^*|| \right. \tag{5.40}$$

$$\left. + \sum_i ||g_i^*(t)||_{L^2_{[c,T]}} + \tilde{g_i}^* \right. \tag{5.40}$$

$$\left. + ||\pi^0||^6 + C_m \sup_j \Delta_j W \|_{L^\infty_x} \right\}$$

Take $L^\infty_c$ on both sides, we derive

$$||w||_{L^\infty_cX_2[0,T]} \leq C_m \left\{ ||w||_{X_2[0,c]} + \eta^6 ||w||_{L^2_x} + ||\pi^{(m)}||^6 \right\} \tag{5.41}$$

when the $h$ is chosen small enough according to $C_m$, and $||\pi^{(m)}||$ is chosen small enough according to $C_m$ and $\delta_0$, the desired estimate follows.

Appendix A Some useful inequality

We will need following regularity estimate for the linear Schrödinger estimate, it is natural in the sense the natural scaling of Schrödinger will trade two space derivative into one time derivative.
Lemma A.1. For every $0 < \alpha < 1$, we have
\[
\|e^{it\Delta} \psi - \psi\|_{L^2_x(\mathbb{R}^d)} \lesssim_{\alpha, d} t^{\frac{\alpha}{2}} \|\psi\|_{H^{\alpha}(\mathbb{R}^d)},
\]  
uniformly over $t \in (0, 1)$.

Proof. Apply Plancherel Theorem, we have
\[
\|e^{it\Delta} \psi - \psi\|_{L^2_x} = \| (1 - e^{it|x|^2}) \hat{\psi}(\xi) \|_{L^2_{\xi}}.
\]

Split the $\mathbb{R}^d_\xi$ into the $|\xi| \leq \frac{1}{t^{1/2}}$ and $\xi \geq \frac{1}{t^{1/2}}$, plug in $|e^{it|x|^2} - 1| \lesssim t|\xi|^2$ in the first region, and $|e^{it|x|^2} - 1| \leq 2$ in the second region, then the desired estimate follows.

This has an immediate consequence which will be useful in our analysis, we summarize it as the following lemma

Lemma A.2. Let $V$ be some nice Schwarz function, let $\tau_1 \leq \tau_2 \leq \tau_3$, $\tau_3 - \tau_1 l \leq 1$, then we have
\[
\|e^{i(\tau_3 - \tau_2)\Delta}(V(x)e^{i(\tau_2 - \tau_1)\Delta})\psi - \psi\|_{L^2_x} \lesssim (\tau_3 - \tau_1)^{\alpha/2} \|\psi\|_{H^{\alpha}}
\]  
(The bounds, do depend on $V$.)

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