Quantum-Statistical Current Correlations in Multi-Lead Chaotic Cavities

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Quantum mechanics requires that identical particles are treated as indistinguishable. This requirement leads to correlations in the fluctuating properties of a system. Theoretical predictions are made for an experiment on a multi-lead chaotic quantum dot which can identify exchange effects in electronic current-current correlations. Interestingly, we find that the ensemble averaged exchange effects are of the order of the channel number, and are insensitive to dephasing.

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Quantum theory requires that identical particles are treated as indistinguishable. In particular, the wave function has to remain invariant up to a sign under the exchange of any pair of particles. A well known consequence of this symmetry is the exchange hole in the equal time density-density correlation of an electron gas. It is the purpose of this paper to investigate exchange effects in current-current correlations of a mesoscopic chaotic cavity connected to four leads. A four-lead geometry is the simplest structure which allows an unambiguous identification of exchange effects. While a similar experiment has already been proposed and analyzed under conditions where electron motion is along one-dimensional edge channels, the investigation presented here is the first for a non-trivial many-channel conductor. One might expect from the former analysis that exchange effects are washed out in an ensemble of chaotic conductors, and can thus be observed at best in the fluctuations away from the average. The most important result of our discussion is that this is not the case: exchange effects survive ensemble averaging and are even of the same order of magnitude as the direct terms.

The general existence of exchange effects in a scattering process with two detectors and two mutually incoherent sources has been pointed out already by Goldberger et al. following the seminal experiments of Hanbury-Brown and Twiss with a stellar interferometer based on this principle. A clear account of two-particle interference of bosons and fermions has been given by Loudon. In electronic conductors we deal with a Fermi sea, instead of only two particles. Nevertheless, for electrons moving in a fixed Hartree potential the current-current correlations can be expressed via two-particle exchange amplitudes in terms of the single-particle scattering matrix. The resulting correlations are negative, except in the case of normal-superconductor hybrid structures. Our work is closely related to recent theoretical and experimental efforts to understand shot noise in mesoscopic conductors. Most of this effort has concentrated on conductors which are effectively two-terminal, and has focused on the suppression of the shot-noise power below the uncorrelated Poisson limit \( 2e|I| \). The situation is potentially richer in multi-lead conductors, where there is the possibility of investigating correlations between current fluctuations at different leads.

We compute the current correlations for a chaotic quantum dot. During the last few years, there has been an increasing activity of experimentalists in tailoring electronic confinement potentials, and characterizing the conductive properties of such cavities. If the classical dynamics of the dot is fully chaotic, the quantum transport properties are well described by a relatively simple statistical ensemble for the scattering matrix. Using this approach, we find that both the direct and exchange contributions to the correlations are of the order of the channel number \( N \), and that they are insensitive to dephasing. This is remarkable, since in single-channel scattering geometries the exchange terms depend sensitively on phases, in contrast with the direct terms. Furthermore, we find that the sign of the averaged exchange contribution reverses if the cavity is closed up by tunnel barriers. The particular dependence of our results on the barrier transparency offers a possibility to distinguish these correlations from other effects. In the end we compare with a purely classical resistor network.

The quantity that we study, is the zero-frequency spectral density of current correlations,

\[
P_{\alpha\beta} = 2 \int_{-\infty}^{\infty} dt \Delta I_\alpha(t + t_0)\Delta I_\beta(t_0),
\]

where \( \Delta I_\alpha = I_\alpha - \langle I_\alpha \rangle \) is the fluctuation of the current in lead \( \alpha \) away from the time-average. It is shown in Ref. that

\[
P_{\alpha\beta} = 2\frac{e^2}{h} \sum_{\gamma,\delta} \int dE f_\gamma(1 - f_\delta) \text{Tr} \left[ A_\gamma(\alpha)A_\delta(\beta) \right]
\]

where \( f_\alpha(E) \) is the distribution of reservoir \( \alpha \), and

\[
A_\beta(\alpha) = 1_\alpha \delta_{\alpha\beta} \delta_{\alpha\gamma} - s^\dagger_{\alpha\beta}(E)s_{\alpha\gamma}(E).
\]

Here \( s_{\alpha\beta}(E) \) is the sub-block of the scattering matrix \( S \) for scattering from lead \( \beta \) (\( N_\beta \) channels) to lead \( \alpha \) at energy \( E \), and \( 1_\alpha \) is the \( N_\alpha \times N_\alpha \)-unit matrix. If all reservoirs are at zero-temperature equilibrium, \( f_\alpha(E) = \frac{1}{e^{\frac{E - E_\alpha}{k_B T}} + 1} \).
\(\theta(E - eV_\alpha)\), the summation in Eq. (4) is over \(\gamma \neq \delta\), and the trace becomes a noise conductance
\[
C_{\gamma\delta}(\alpha\beta) = \text{Tr}(s_{\gamma\delta}^1 s_{\alpha\beta}^1 s_{\beta\gamma}^1).
\] (4)

To be specific we introduce the correlator \(P_{34}^{\text{ex}}\) for three experiments A, B, and C. In experiment A (B) a small voltage \(V\) is applied only to lead 1 (2), whereas in experiment C a voltage \(V\) is applied to both reservoir 1 and 2. It follows from Eq. (3) that the result of experiment C is not identical to the sum of the results from experiments A and B. We identify the difference as the exchange correlation:
\[
P_{34}^{\text{ex}} = P_{34}^C - P_{34}^A - P_{34}^B = -2P_0 \text{Re} C_{12}(34),
\] (5)
where \(P_0 = 2e|V|^2/h\). Although \(P_{\alpha\beta} \leq 0\) in every experiment for \(\alpha \neq \beta\), the difference can still have both signs.

We first consider ideal coupling of the four \(N\)-channel leads to the cavity (Fig. 1). If time-reversal symmetry is (un)broken, the \(4N \times 4N\)-scattering matrix \(S\) is uniformly distributed on the set of unitary (unitary symmetric) matrices. This is the circular unitary (orthogonal) ensemble of random matrix theory. Since our results for both ensembles differ only to order \(N^{-2}\), we will focus on the simpler unitary ensemble, thus assuming that there is a sufficiently large magnetic field in the cavity.

From the general formula of Ref. 15 one finds the exact ensemble average \(\langle \cdot \rangle\) of the noise conductances:
\[
\langle C_{\gamma\delta}(\alpha\beta) \rangle = (\delta_{\alpha\beta} + \delta_{\gamma\delta} - 1/4) \times N^3/(16N^2 - 1).
\] (6)

Fluctuations are of order one. Thus we find
\[
\langle P_{\alpha\beta} \rangle = \frac{e^2}{h} (\delta_{\alpha\beta} - 1/4) - \frac{N^3}{16N^2 - 1} \sum_{i,j=1}^4 |V_i - V_j|. \] (7)

The average exchange correlation \(\langle P_{34}^{\text{ex}} \rangle = \frac{1}{4}P_0 N^3/(16N^2 - 1)\) is positive and, just as the direct terms, of order \(N\). This is remarkable, given the fact that the sign of the noise conductance \(C_{12}(34) = \text{Tr}(s_{34}^1 s_{12}^1 s_{42}^1 s_{41}^1)\) can vary from sample to sample. The large average is essentially due to correlations between scattering matrix elements imposed by unitarity.

The size of the exchange effect, and the important role played by unitarity, makes it more robust than a normal wave-interference effect. We next show that the current correlations are the same even under conditions where phase-coherent transfer through the sample is completely destroyed, but energy is conserved in the dephasing process. We model dephasing by connecting the sample to an additional fictitious reservoir. Attaching an equilibrium reservoir, like a voltage probe, causes inelastic scattering, which reduces shot noise far below the value in phase-coherent transport. Following De Jong and van Hove we rather consider dephasing by a reservoir with a fluctuating non-equilibrium distribution \(f_\phi(E,t)\) such that no current is drawn at every energy and instant of time. Such quasi-elastic scattering does not change the shot noise of a chaotic cavity with wide leads. Below we generalize this result to the current correlations in a multi-terminal geometry.

The total current through lead \(\alpha\) at time \(t\) and energy \(E\) is
\[
I_\alpha(E,t) = \frac{e}{h} \sum_\beta (N_\alpha \delta_{\alpha\beta} - G_{\alpha\beta}) f_\beta + \delta I_\alpha(E,t),
\] (8)
where \(G_{\alpha\beta} = \text{Tr}(s_{\alpha\beta}^1 s_{\alpha\beta}^1)\) and the indices run over all five leads. Apart from the intrinsic fluctuations defined by \(\delta I_\alpha(E,t)\), there is now also a time-dependence due to the fluctuating \(f_\phi(E,t)\). The requirement \(I_\phi(E,t) = 0\) determines \(f_\phi(E,t)\) which, by substitution back into Eq. (8), yields the total fluctuation \(\Delta I_k\) at a real lead \(k\),
\[
\Delta I_k = \delta I_k + G_{k\phi} \delta I_\phi / \sum_{m=1}^4 G_{\phi m}.
\] (9)

The correlations of the intrinsic fluctuations \(\delta I_\alpha = \int dE \delta I_\alpha(E,t)\) satisfy Eq. (6) with the time-averaged distribution
\[
f_\phi(E) = \sum_{k=1}^4 G_{\phi k} f(E - eV_k) / \sum_{k=1}^4 G_{\phi k}.
\] (10)

We assume homogeneous and complete dephasing, which implies that the extra lead has \(N_\phi \gg N\) channels, and is coupled ideally to the dot. In this limit, the only non-vanishing transport coefficients are
\[
G_{k\phi} = G_{\phi k} = N, \quad G_{\phi\phi} = N_\phi - 4N, \quad C_{kk}(\phi\phi) = C_{\phi\phi}(\phi\phi) = N_\phi - 4N,
\] (11a)

independent of the magnetic field, for all \(N\), and without fluctuations. Thus
\[
P_{kl} = 2e^2/h(\delta_{kl} - 1/4)N \int dE f_\phi(E)(1 - f_\phi(E)).
\] (12)

For \(N \gg 1\) Eq. (12) coincides exactly with Eq. (6). It shows a striking similarity with a semi-classical expression for the shot-noise power first obtained by Nagaev.
However, it should be noted that for geometries with non-ideal leads, discussed below, the correlations are not determined by the distribution function of the dephasing reservoir only.

We have now shown that exchange effects do survive changes in the elastic scattering potential as well as phase-breaking scattering. This finding sheds new light on the actual issue of shot-noise suppression in two-terminal many-channel geometries. Universal suppression factors were found both quantum mechanically and semi-classically approaches. In fact, the two-terminal shot noise contains direct and exchange terms corresponding to pairs of particles coming from different channels. The reduction below the Poisson noise is partly due to such channel-exchange terms.

The analysis of the cavity connected to open leads is not sufficient for an unambiguous identification of exchange correlations in practice. Current correlations could be established by any other process, and there is no general reason why the results of experiments A, B, and C should be additive. Indeed, in the end we will briefly discuss a classical network where currents are correlated by fluctuations of the self-consistent electrostatic potential inside the “dot”, and where experiment C is not the sum of experiment A and B. To facilitate identification of the exchange effect, we now make specific predictions for the correlations in the case of non-ideal leads, as a function of the probability Γ of transmission through the contact region. We focus on the many-channel limit \( N \gg 1 \), where one cannot distinguish isolated resonances of the system, except if the coupling is so weak that \( \Gamma \ll 1/N \). Below, we discuss this weak coupling limit before turning to the main result of the paper: the correlations in the regime \( 1/N \ll \Gamma \leq 1 \).

We first assume that the contacts are so poorly transmitting that transport is dominated by a single state at the Fermi energy \( E_F \). The scattering amplitude from a point \( b \) to a point \( a \) takes the form

\[
S_{ab} = \delta_{ab} - i\alpha \Delta / \pi \frac{\psi_\alpha^*(a) \psi_\beta(b)}{E_F - E_\nu + i\gamma_\nu/2},
\]

where \( \psi_\nu \) is the resonant eigenstate with energy \( E_\nu \), \( \Delta \) is the mean level spacing, \( \alpha \ll 1 \) is a dimensionless coupling parameter, and \( \gamma_\nu = \alpha \Delta / \pi \sum_r |\psi_\nu(r)|^2 \) is the width of the resonant level. The sum is over positions in the four coupling regions. We assume that each lead contributes equally to the width, which is automatically satisfied for a chaotic cavity with leads of the same width, due to self-averaging of the overlaps with the eigenstate. Then, at small bias \( V \ll \gamma_\nu \), the current correlations are

\[
P_{\alpha\beta}^{A,B} = P_0 (4\delta_{\alpha\beta} - 1)/16,
\]

\[
P_{\alpha\alpha}^{C} = -P_{14}^{C} = -P_{23}^{C} = P_0/4,
\]

\[
P_{13}^{C} = P_{14}^{C} = P_{23}^{C} = P_{24}^{C} = 0.
\]

The result of experiment C is easily understood. For resonant tunneling through a two-terminal symmetric barrier, the shot noise vanishes. Indeed, in experiment C the total current from lead 1 and 2 to lead 3 and 4 is noiseless. All fluctuations and correlations are due to ‘unification’ of the currents from 1 and 2, and ‘partition’ of the total current into 3 and 4. The vanishing of the correlations between incoming and outgoing currents shows that these two choices are made independently. The exchange correlation \( P_{34}^{C} = -P_0/8 \) has a sign opposite to that of the ideally coupled cavity.

We now compute the ensemble averaged correlations for the cavity with tunnel barriers in the regime \( N \gg 1 \), \( \Delta \gg 1 \). The scattering matrix of the combined system

\[
S = R + T'(1 - UR')^{-1}UT,
\]

where we assume without loss of generality that the reflection and transmission matrices of the barriers are proportional to the \( 4N \times 4N \) unit matrix \( I: R = R' = (1 - \Gamma)I^{1/2}I, T = -T' = I^1/2I \). The distribution of the scattering matrix \( U \) of the cavity is the circular unitary ensemble, as before. The average noise conductances are computed by series expansion of the four fractions \( (1 - UR')^{-1} \), collecting the order \( N \) contributions to each term with the diagrammatic technique of Ref. 24 and resumming. Thus one finds

\[
\langle C_{\gamma\beta}(\alpha\beta) \rangle = [\Gamma(2 - 3\Gamma) + 4\Gamma^2(\delta_{\alpha\beta} + \delta_{\gamma\delta}) - 4\Gamma^2(1 - \Gamma)(\delta_{\alpha\gamma} + \delta_{\alpha\delta} + \delta_{\gamma\delta} + \delta_{\beta\delta}) + 16\Gamma(1 - \Gamma)(\delta_{\alpha\beta}\delta_{\gamma\beta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\beta\gamma}\delta_{\beta\delta}) + 64(1 - \Gamma)^2\delta_{\alpha\beta}\delta_{\gamma\beta}\delta_{\beta\delta}] \times N/64 + O(1),
\]

which for \( \Gamma = 1 \) reduces to the large-N limit of Eq. [9]. The correlators for the experiments A and C are

\[
\langle P_{\alpha\beta}^{C} \rangle = P_0 N\Gamma(2 - \Gamma)(4\delta_{\alpha\beta} - 1)/16, \quad \langle P_{\alpha\alpha}^{A} \rangle = P_0 N\Gamma(10 - 7\Gamma)(4\delta_{\alpha\beta} - 1)/64, \quad \langle P_{\alpha\alpha}^{A} \rangle = P_0 N\Gamma(14 - 5\Gamma)/64 (\alpha \neq 1), \quad \langle P_{\alpha\beta}^{A} \rangle = -P_0 N\Gamma(\Gamma + 2)/64 (\alpha, \beta \neq 1, \alpha \neq \beta). \]

The functional dependence of these results provides a fingerprint for an experimental identification of the correlations. The exchange correlation \( \langle P_{34}^{C} \rangle = P_0 N\Gamma(3\Gamma - 2)/32 \) reverses sign at \( \Gamma = 2/3 \). An observation of this sign change would be a clear indication that measured correlations are due to exchange. One easily checks that in experiment C the shot noise of the total current from lead 1 and 2 to lead 3 and 4 crosses over from one quarter times the Poisson noise at \( \Gamma = 1 \) to one half of the Poisson noise if \( \Gamma \) is small, both in correspondence with the literature.

We now compare these results with a classical circuit, where current conservation induces correlations, which are also non-additive for the experiments A, B, and C. Four resistors \( R \) are connected to four voltage sources as in Fig. 2. Parallel to the resistors there are independent sources of Poisson noise. Temporal fluctuations of the central potential around \( U = \sum_k V_k/4 \) are necessary to
The outgoing currents are correlated, due to fluctuations of the “dot” potential preventing temporal accumulations of charge.

These voltage fluctuations yield the following current correlations:

\[ P_{kl} = e \sum_{m=1}^{4} \left[ 1 + 4(2\delta_{kl} - 1)(\delta_{mk} + \delta_{ml}) \right] \frac{|V_m - U|}{8R}. \]  \[ (18) \]

For experiment A, B and C, Eq. (18) gives the same result as Eq. (17) with \( \Gamma \ll 1 \) and \( \overline{N\Gamma c^2}/h = 1/R \). For other values of the voltages and \( \Gamma \) the correlations differ. This example demonstrates the need to investigate the correlations as a function of a parameter like the barrier strength \( \Gamma \) in order to understand the source of the correlations.

In conclusion, we have evaluated the exchange contributions to current-current correlations in a multilead chaotic cavity. We found that these correlations, instead of being a small interference effect, are of the order of the channel number \( N \), and that they persist in the presence of dephasing. We have made specific predictions for the dependence of the correlations on the transparency of non-ideal leads. Finding such a direct signature of exchange in experiments is likely a challenging task, but would clearly be a fundamental contribution to our understanding of noise in electrical conductors.

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1 L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Pergamon, Oxford, 1959), p. 354.