D-Brane Actions in Non-Relativistic String Theory and T-Duality

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Abstract

We study D-brane actions in non-relativistic string theory. We consider single D-brane and analyse its properties under T-duality. We also derive an action for \( N \) D8-branes and also for lower dimensional D-branes through T-duality transformations.
1 Introduction and Summary

Non-relativistic string theory is specific form of two dimensional quantum field theory defined on the string world-sheet \[1, 2\]. The target space geometry of this two-dimensional field theory is stringy Newton-Cartan geometry, see also \[4, 5, 6, 7, 8, 9, 10, 11\].

Stringy Newton-Cartan geometry can be physically defined as a theory where probe of the geometry is two dimensional object (string). As a result target space-time naturally splits into two dimensional longitudinal sector and 8−dimensional transverse sector where we will implicitly work with the parent string theory with critical dimension equal to ten. As is well known from the study of non-relativistic string theory \[1, 2\] in the flat space-time longitudinal sector is Lorentzian while transverse sector is Euclidean.

It is very interesting to analyse open string theory in this background where now boundary conditions imposed at the end of string world-sheet play an important role. Very careful analysis of open string theory in non-relativistic background was performed recently in \[1, 5\]. It was shown there that imposing Dirichlet boundary conditions in longitudinal sector leads to non-relativistic open string theory on D8-brane. It was shown in \[5\] that in the flat space-time low energy effective action of open strings on \(N\)-non-relativistic D8-branes is described by Galilean invariant \(U(N)\)−Yang-Mills theory. T-duality of non-relativistic open string theory was further analysed in \[4\] firstly in the flat space-time and then it was studied in general space-time using sigma model description or using Dirac-Born-Infeld (DBI) description. The goal of this paper is to extend the analysis of T-duality of non-relativistic open string using DBI description and its non-abelian generalization.

We start our paper with the standard construction of non-relativistic Dp-brane. In more details, we consider DBI action for string in the background with light-like isometry and we presume that Dp-brane is extended along this direction. Then we perform dimensional reduction along this direction \(^2\). As a result we obtain non-relativistic D(p-1)-brane which is generalization of the construction presented in \[4\] when some of the direction in transverse sector obey Dirichlet boundary conditions. Note that using this construction we get D(p-1)-brane that is localized at light-like direction. We further study T-duality along \(k\)−spatial directions transverse to the world-volume of D(p-1)-brane and we obtain an action for non-relativistic D(p-1-k)-brane in the background that is related to the original one by Buscher’s rules \[13, 14\] which are generalization of standard Buscher’s prescription \[13, 14\] to the case of T-duality along \(k\)− directions \(^3\). However it is important to stress that the resulting D(p-1-k)-brane action has the same form as non-relativistic D(p-1)-brane on condition when the components of time form \(\tau\) along spatial directions that we dualize, are zero. Note that this is the same condition that was imposed in \[11\] when T-duality of closed string along transverse directions was analysed.

\(^2\)We could presume that light-like direction is compact that can be defined by infinite boosting longitudinal spatial circle.

\(^3\)For earlier review of T-duality, see \[12\].
As the next step we proceed to the construction of an action for $N$ non-relativistic Dp-branes. To do this we closely follow [19]. Explicitly, we start with the space-time filling DBI action for $N$ D9-branes in the background with light-like isometry. Performing T-duality along light-like direction we obtain action for $N$ non-relativistic D8-branes. An action for $N$ D(8-k)-branes is derived by performing T-duality along $k$—spatial dimensions. This action is crucial for the analysis of T-duality along direction transverse to the D-brane world-volume. In more details, in order to study T-duality of Dp-brane along directions transverse to its world-volume we should consider an infinite array of Dp-branes on the covering space when all world-volume fields obey quotient conditions [17, 18, 21]. We firstly apply this approach to the case of non-relativistic D8-brane transverse to light-like direction. We replace it with the action for infinite D8-branes on the covering space and we find that this configuration is equivalent to D9-brane in the background with light-like direction. Then we generalize this approach to the case of T-duality along light-like direction transverse to D(8-k)-brane and we again show that it is equivalent to D(9-k)-brane where this D(9-k)-brane is extended along light-like direction.

Let us outline our results and suggest possible extension of this work. We study an effective actions for non-relativistic Dp-branes. We find their form and we analyse as their transform under T-duality transformations. We then generalize this result to the case of $N$ non-relativistic D8 and D(8-k)-branes. We argue that these non-abelian generalizations are crucial for analysis how non-relativistic D(8-k)-brane transforms under T-duality along transverse light-like directions when we show that non-relativistic D(8-k)-brane maps to relativistic D(9-k)-brane in the background with light-like isometry. We mean that this is nice consistency check of the proposal of the action for $N$ non-relativistic D(8-k)-branes.

The natural extension of this work would be to analyse properties of the Wess-Zumino term which describes how Dp-brane couples to Ramond-Ramond forms. However in order to do this we should know non-relativistic limit of Ramond-Ramond fields which has not been found yet. On the other hand it would be natural to start with the fact that non-relativistic D(p-1)-brane is defined using T-duality of relativistic Dp-brane along light-like directions. Then we should perform the same T-duality transformations in case of Wess-Zumino term and we could find term that expresses coupling of non-relativistic D(p-1)-brane to Ramond-Ramond form. We hope to return to this problem in future.

This paper is organized as follows. In the next section (2) we introduce non-relativistic D(p-1)-brane. Then in section (3) we study its properties under T-duality. In section (4) we perform its non-abelian generalization. Finally in section (5) we study how non-relativistic D-branes transform under T-duality performed along light-like directions which is transverse to their world-volumes.
2 Non-Relativistic D-Brane and T-Duality

In this section we review and extend construction of Dp-branes in non-relativistic background, following recent work [4].

We start with Dp-brane action in the relativistic background with light like isometry along $y-$ direction which means that the metric component $g_{yy} = 0$ and all space-time fields do not depend on $y$. Let us now consider DBI action for Dp-brane in general background

$$S = -T_p \int d^{p+1}\xi e^{-\phi} \sqrt{-\det A_{\alpha\beta}},$$

$$A_{\alpha\beta} = g_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu + b_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu + \lambda F_{\alpha\beta},$$

where $\xi^\alpha, \alpha, \beta = 0, 1, \ldots, p$ label world-volume of p-brane and where $x^\mu(\xi), \mu = 0, 1, \ldots, 9$ are world-volume fields that describe embedding of Dp-brane in the target space-time with the dilaton $\phi$, metric $g_{\mu\nu}$ and NSNS two form field $b_{\mu\nu}$. Further, $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$, where $A_\alpha$ is gauge field propagating on the world-volume of Dp-brane. Finally $\lambda = 2\pi\alpha'$ and $T_p = \lambda^{\frac{3(p+1)}{2}}$ is Dp-brane tension.

Let us presume that Dp-brane is extended along $y \equiv x^9$ direction so that we impose following static gauge

$$y = \xi^p.$$  

Next step is to perform dimensional reduction when we presume that all world-volume fields do not depend on $y$. As a result we obtain following form of the matrix $A_{\alpha\beta}$

$$A_{\alpha\beta}, A_{\beta\alpha} = 0,$$

$$A_{\alpha y} = g_{ij} \partial_i x^\alpha \partial_j x^\beta + b_{ij} \partial_i x^\alpha \partial_j x^\beta - \lambda \partial_\beta A_y,$$

$$A_{\hat{\alpha} y} = \partial_i x^i g_{iy} + \partial_i x^i b_{iy} + \lambda \partial_\beta A_y,$$

$$A_{\hat{\alpha} \hat{\beta}} = g_{ij} \partial_i x^\alpha \partial_j x^\beta + b_{ij} \partial_i x^\alpha \partial_j x^\beta + \lambda F_{\alpha\beta},$$

where $\xi^{\hat{\alpha}}, \hat{\alpha}, \hat{\beta} = 0, 1, \ldots, p-1$ label world-volume coordinates on D(p-1)-brane and where $i, j = 0, 1, \ldots, 8$.

Using (3) we find that the action (1) has the form

$$S = -T_p \int dy \int d^p\xi e^{-\phi} \sqrt{-\det M_{\alpha\beta}} \left( \begin{array}{cc} 0 & (g_{ij} + b_{ij}) \partial_\beta x^j - \lambda \partial_\beta A_y \\ A_{\hat{\alpha}\hat{\beta}} & A_{\hat{\alpha}\hat{\beta}} \end{array} \right).$$

Note that the determinant in the action (1) has the form

$$\det \left( \begin{array}{cc} 0 & \mathcal{M}_{y\hat{\beta}} \\ \mathcal{M}_{\hat{\alpha} y} & \mathcal{M}_{\hat{\alpha}\hat{\beta}} \end{array} \right),$$

that, using properties of the determinant and also the fact that $\mathcal{M}_{yy} = 0$ is equal to

$$\det \left( \begin{array}{cc} 0 & \mathcal{M}_{y\hat{\beta}} \\ \mathcal{M}_{\hat{\alpha} y} & \mathcal{M}_{\hat{\alpha}\hat{\beta}} \end{array} \right) = \det \left( \begin{array}{cc} 0 & \mathcal{M}_{y\hat{\beta}} \\ \mathcal{M}_{\hat{\alpha} y} & \mathcal{M}_{\hat{\alpha}\hat{\beta}} + V_{\hat{\alpha}} \mathcal{M}_{y\hat{\beta}} + \mathcal{M}_{\hat{\alpha} y} W_{\hat{\beta}} \end{array} \right).$$

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where $V_\alpha$ and $W_\beta$ are arbitrary functions. To proceed further we define $T$–dual coordinate $\tilde{y}$ as
\[ \tilde{y} = \lambda A_y \] (7)
and, following [4] we define $\tau_\mu^A, A = 0, 1$ as
\[ \tau_i^0 = C g_{y_i}, \quad \tau_i^0 = 0, \quad \tau_i^1 = C b_{y_i}, \quad \tau_y^1 = C \] (8)
so that we can write
\[ \mathcal{M}_{\hat{\alpha}y} = \tau_{\hat{\alpha}} , \quad \mathcal{M}_{y\hat{\beta}} = \bar{\tau}_{\hat{\beta}} , \quad \tau_{\hat{\alpha}} = \tau_{\hat{\alpha}}^0 + \tau_{\hat{\alpha}}^1 , \quad \tau_{\hat{\beta}} = \tau_{\hat{\alpha}}^0 - \tau_{\hat{\alpha}}^1 , \] (9)
where $\tau_{\hat{\alpha}} = \tau_\mu \partial_{\hat{\alpha}} \bar{x}^\mu = \tau_\mu \partial_{\hat{\alpha}} x^i + \tau_y \partial_{\hat{\alpha}} y$. Let us now define $V_\hat{\alpha}$ and $W_\hat{\alpha}$ as
\[
V_\hat{\alpha} = \partial_{\hat{\alpha}} \bar{x}^\mu V_\mu = \partial_{\hat{\alpha}} x^i (C_i^0 + C_i^1) + \partial_{\hat{\alpha}} \bar{y} (C_y^0 + C_y^1), \\
W_\hat{\alpha} = \partial_{\hat{\alpha}} \bar{x}^\mu W_\mu = \partial_{\hat{\alpha}} x^i (C_i^0 - C_i^1) + \partial_{\hat{\alpha}} \bar{y} (C_y^0 - C_y^1).
\] (10)
Now using (10) we obtain
\[ \mathcal{M}_{\hat{\alpha}\hat{\beta}} + V_\hat{\alpha} M_{y\hat{\beta}} + M_{\hat{\alpha}y} W_\hat{\beta} = \]
\[ H_{\hat{\alpha}\hat{\beta}} + \mathcal{B}_{\hat{\alpha}\hat{\beta}} + \lambda F_{\hat{\alpha}\hat{\beta}}, \] (11)
where $H_{\hat{\alpha}\hat{\beta}} = H_{\mu\nu} \partial_{\hat{\alpha}} \bar{x}^\mu \partial_{\hat{\beta}} \bar{x}^\nu$ where $H_{\mu\nu}$ is equal to
\[
H_{ij} = g_{ij} + C_i^0 \tau_j^0 + \tau_i^0 C_{j}^0 - C_i^1 \tau_j^1 - \tau_i^1 C_{j}^1, \\
H_{\bar{y}y} = 2 (C_y^0 \tau_y^0 - C_y^1 \tau_y^1), \\
H_{\bar{y}j} = C_y^0 \tau_j^0 + \tau_y^0 C_j^0 - C_y^1 \tau_j^1 + \tau_y^1 C_j^1, \\
H_{i\bar{y}} = C_i^0 \tau_y^0 + \tau_i^0 C_y^0 - C_i^1 \tau_y^1 - \tau_i^1 C_y^1. \]
(12)
In the same way we find that $\mathcal{B}_{\mu\nu}$ is equal to
\[
\mathcal{B}_{ij} = b_{ij} + C_i^1 \tau_j^0 - C_i^0 \tau_j^1 + \tau_i^1 C_j^0 - C_i^0 \tau_j^1, \\
\mathcal{B}_{i\bar{y}} = C_i^1 \tau_y^0 - C_i^0 \tau_y^1 + \tau_i^1 C_y^0 - \tau_i^0 C_y^1, \\
\mathcal{B}_{\bar{y}j} = C_y^1 \tau_j^0 + \tau_y^1 C_j^0 - C_y^0 \tau_j^1 - \tau_y^0 C_j^1, \]
(13)
so that the action for D(p-1)-brane has the form
\[
S = -T_{p-1} \int d^p \xi e^{-\phi} \sqrt{- \det \begin{pmatrix}
\tau_\mu \partial_{\beta} \bar{x}^\mu \\
\partial_{\alpha} \bar{x}^\mu \bar{\tau}_\nu \\
H_{\hat{\alpha}\hat{\beta}} + \mathcal{B}_{\hat{\alpha}\hat{\beta}} + \lambda F_{\hat{\alpha}\hat{\beta}}
\end{pmatrix}},
\] (14)
where we identified \( T_{p-1} \) as
\[
T_{p-1} = T_p \int dy
\]
and also \( \dot{\phi} \) as
\[
\dot{\phi} = \phi + \ln |C|
\]
which is the same result as was found in [3]. Finally using properties of determinant (6) we see that there is natural redefinition of the background fields given by prescription
\[
\bar{H}_{a\dot{a}} + \bar{B}_{a\dot{a}} = H_{a\dot{a}} + B_{a\dot{a}} + X_{\dot{a}} \tau_{\dot{a}} + \tau_{\bar{a}} Y_{\bar{a}}
\]
that can be written in equivalent form
\[
\bar{H}_{\mu\nu} = H_{\mu\nu} - Z_\mu A_\nu \eta_{AB} - \tau_{\mu} A Z_\nu B \eta_{AB},
\]
\[
\bar{B}_{\mu\nu} = B_{\mu\nu} - Z_\mu A_\nu \epsilon_{AB} - \tau_{\mu} A Z_\nu B \epsilon_{AB},
\]
where we defined \( Z_\mu A, A = 0, 1 \) as
\[
X_\mu = Z_\mu^0 + Z_\mu^1, \quad Y_\mu = Z_\mu^0 - Z_\mu^1,
\]
and where \( \eta_{AB} = \text{diag}(-1, 1) \) and \( \epsilon_{01} = -\epsilon_{10} = 1 \). These transformations are known as Stuckelberg transformations of the non-relativistic fields [5, 4, 6, 11].

### 3 T-Duality of Non-Relativistic D(p-1)-Brane

In this section we study how non-relativistic D(p-1)-brane action (14) transforms under T-duality transformations. We start with situation when we perform T-duality along \( k \) longitudinal spatial dimensions where D(p-1)-brane wraps them. Then it is natural to perform gauge fixing
\[
\xi^m = x^m, m = 9 - k, \ldots, 8,
\]
where now all world-volume fields do not depend on \( \xi^m \). Instead they are functions of remaining world-volume coordinates \( \xi^{\alpha} \) where \( \alpha = 0, \ldots, p - 1 - k \). Further, let us denote remaining coordinates as \( x^{\bar{\alpha}}, \bar{\mu}, \bar{\nu} = 0, 1, \ldots, 9 - k \). Finally we introduce \( E_{\mu\nu} = H_{\mu\nu} + B_{\mu\nu} \). In this case the action (14) has the form
\[
S = -T_{(p-1)} \int d^k \xi \times
\]
\[
\times \int d^{p-k} \xi e^{-\hat{\phi}} \sqrt{-\det \left( \begin{array}{ccc}
0 & \tau_{\mu} \partial_{\bar{\beta}} x^{\bar{\mu}} & \tau_{n} \\
\partial_{\bar{\alpha}} \bar{x}^{\bar{\alpha}} \bar{\tau}_{\bar{\mu}} & E_{\bar{\alpha}\bar{\beta}} + \lambda F_{\bar{\alpha}\bar{\beta}} & E_{\alpha n} + \lambda \partial_{\alpha} A_n \\
\bar{\tau}_{m} & E_{m\beta} - \lambda \partial_{\bar{\beta}} A_m & E_{mn}
\end{array} \right)} =
\]
\[
= -T_{(p-1-k)} \int d^{p-k} \xi e^{-\hat{\phi}} \sqrt{-\det E_{mn}} \times
\]
\[
\sqrt{-\det \left( \begin{array}{ccc}
\tau_{n} E_{mn} \bar{\tau}_{m} & -\tau_{n} \bar{\tau}_{m} & \tau_{\mu} \partial_{\bar{\beta}} x^{\bar{\mu}} - \tau_{m} \bar{E}^{mn} (E_{\bar{\alpha}\bar{\beta}} - \lambda \partial_{\bar{\beta}} A_n) \\
\partial_{\bar{\alpha}} \bar{x}^{\bar{\alpha}} \bar{\tau}_{\bar{\mu}} & -\tau_{\bar{\alpha}} \bar{x}^{\bar{\alpha}} \bar{\tau}_{\bar{\mu}} & \bar{E}^{kn} \bar{\tau}_{k} \bar{E}_{\bar{\alpha}\bar{\beta}} - \lambda \partial_{\bar{\alpha}} A_n \bar{E}^{mn} (E_{mn} - \lambda \partial_{\beta} A_m)
\end{array} \right)},
\]
(21)
where $\tilde{E}^{mn}$ is matrix inverse to $H_{mn} + B_{mn}$. We see that the form of the action for D(p-1-k)-brane does not have the form of non-relativistic action as was introduced in the second question due to the presence of the expression $\tau_m \tilde{E}^{mn} \tilde{\tau}_n$. However when we derived the action (21) we presumed that D(p-1)-brane wraps $x^k$ directions which are pure spatial. Then it is natural to presume that

$$\tau_m = \tilde{\tau}_m = 0 .$$  \hspace{1cm} (22)

Note also that these conditions were imposed when T-duality along pure spatial dimension was analysed in [11].

With the help of conditions (22) we obtain that the action for non-relativistic D(p-1-k)-brane has the form

$$S = - T_{p-1-k} \int d^{p-k} \xi e^{-\tilde{\phi}} \sqrt{\det \left( \begin{array}{cc} 0 & \tau_{\alpha} \partial_{\beta} x^{\bar{\mu}} \\ \partial_{a} \tilde{x}^{a} \bar{\tau}^{\bar{\alpha}} & P[\tilde{E}_{\bar{\alpha} \bar{\beta}}] + \lambda F_{\bar{\alpha} \bar{\beta}} \end{array} \right) } ,$$  \hspace{1cm} (23)

where $\tilde{\phi}$ is transformed dilaton field defined by equation

$$e^{-\tilde{\phi}} = e^{-\tilde{\phi}} \sqrt{\det E_{mn}} ,$$  \hspace{1cm} (24)

and where $P[\tilde{E}_{\bar{\alpha} \bar{\beta}}]$ is pull-back of the T-dual metric to the world-volume of D(p-1-k)-brane defined as

$$P[\tilde{E}_{\bar{\alpha} \bar{\beta}}] = \partial_{a} \bar{x}^{a} \tilde{E}_{\bar{\alpha} \bar{\beta}} \partial_{\beta} \bar{x}^{\bar{\mu}} + \partial_{a} \bar{x}^{a} \tilde{E}_{\bar{\alpha} m} \partial_{\beta} \tilde{x}_m + \partial_{a} \bar{x}^{a} \tilde{E}_{\bar{\alpha} n} \partial_{\beta} \tilde{x}_n \tilde{E}^{mn} \partial_{\beta} \tilde{x}_n ,$$  \hspace{1cm} (25)

where T-dual coordinates $\tilde{x}_m$ are defined as

$$\tilde{x}_m = \lambda A_m .$$  \hspace{1cm} (26)

Finally components of T-dual metric have the form

$$\tilde{E}_{\bar{\mu} \bar{\nu}} = E_{\mu \nu} - E_{\mu m} \tilde{E}^{mn} E_{n \nu} , \hspace{0.5cm} \tilde{E}^{m}_{\bar{\mu}} = E_{\mu m} \tilde{E}^{mn} , \hspace{0.5cm} \tilde{E}^{n}_{\bar{\nu}} = - \tilde{E}^{mn} E_{m \bar{\nu}} .$$  \hspace{1cm} (27)

Note that (27) are standard T-duality rules as are known in relativistic string theories [15, 16].

As the natural step we should analyse T-duality properties of non-relativistic D(p-1)-brane when we dualize along directions transverse to the world-volume of D(p-1)-brane. However as was shown in [17, 18] this can be done when we consider infinite number of D(p-1)-branes on the covering space whose world-volume fields obey appropriate quotient conditions. In order to do this we should firstly find an action for $N$ Dp-branes in non-relativistic background.

4 Non-Abelian Generalization

In order to find non-abelian action for $N$ Dp-branes in non-relativistic background we will follow [19] when we start with the non-abelian action for $N$-space-time filling D9-branes. This action has the form

$$S = - s \text{Tr} T_9 \int d^{10} \xi e^{-\phi} \sqrt{- \det (E_{\mu \nu} + \lambda F_{\mu \nu})} ,$$  \hspace{1cm} (28)
where
\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] , \] (29)
where \( A_\mu \) are \( N \times N \) Hermitian matrices and where \( s\text{Tr} \) means symmetrized trace. Now as in the second section (2) we presume that the background has light like isometry and we label this dimension by \( \xi^0 \equiv y \). We further presume that all world-volume fields do not depend on them. Then we get
\[ \lambda F_{\hat{\mu}y} = D_{\hat{\mu}}\Phi_{\hat{g}} = -\lambda F_{y\hat{\mu}} , \] (30)
where we introduced \( \Phi_{\hat{g}} \) as \( N \times N \) matrix through the formula
\[ \Phi_{\hat{g}} = \lambda A_y , \] (31)
and where \( \hat{\mu} = 0, 1, \ldots, 8 \). Note that \( \Phi_{\hat{g}} \) describes embedding of D8-branes in transverse \( \hat{g} \)-directions. Using this notation we get
\[ \det(E_{\mu\nu} + \lambda F_{\mu\nu}) = \det \begin{pmatrix} 0 & E_{y\hat{\mu}} - D_{\hat{\mu}}\Phi_{\hat{g}} \\ \bar{E}_{\hat{\mu}y} + D_{\hat{\mu}}\Phi_{\hat{g}} & \bar{E}_{\hat{\mu}\hat{\nu}} + \lambda F_{\hat{\mu}\hat{\nu}} \end{pmatrix} . \] (32)

Then performing the same manipulation as in the second section (2) we obtain an action for \( N \) D8-branes in non-relativistic background in the form
\[ S = -T_8 \int d^8\xi \text{Str} \phi \sqrt{-\det \left( \begin{array}{cc} 0 & \tau_{\hat{\mu}} + \tau_{\hat{g}} D_{\hat{\mu}}\Phi_{\hat{g}} \\ \bar{\tau}_{\hat{\mu}} + D_{\hat{\mu}}\Phi_{\hat{g}} & \bar{\tau}_{\hat{\mu}} + \bar{\tau}_{\hat{g}} D_{\hat{\mu}}\Phi_{\hat{g}} \end{array} \right) } . \] (33)
Let us now perform T-duality along \( k \)-directions which means that all world-volume fields do not depend on \( \xi^m, m = (9-k), \ldots, 8 \). Then \( \hat{\mu}, \hat{\nu} = 0, 1, \ldots, 8-k \) are world-volume coordinates. As a result we obtain
\[ F_{\hat{\mu}n} = D_{\hat{\mu}}\Phi_n , \quad F_{\hat{\mu}\hat{\nu}} = -D_{\hat{\nu}}\Phi_{\hat{m}} , \quad F_{mn} = i\lambda^{-1}[\Phi_m, \Phi_n] \] (34)
and consequently
\[ \det \left( \begin{array}{cc} 0 & \tau_{\hat{\mu}} + \tau_{\hat{g}} D_{\hat{\mu}}\Phi_{\hat{g}} \\ \bar{\tau}_{\hat{\mu}} + D_{\hat{\mu}}\Phi_{\hat{g}} & H_{\hat{\mu}\hat{\nu}} + B_{\hat{\mu}\hat{\nu}} + \lambda F_{\hat{\mu}\hat{\nu}} \end{array} \right) = \det \left( \begin{array}{cccc} 0 & \tau_{\hat{\mu}} + \tau_{\hat{g}} D_{\hat{\mu}}\Phi_{\hat{g}} & \tau_{\hat{\mu}} + \tau_{\hat{g}} i\lambda^{-1}[\Phi_n, \Phi_{\hat{g}}] \\ \bar{\tau}_{\hat{\mu}} + D_{\hat{\mu}}\Phi_{\hat{g}} & H_{\hat{\mu}\hat{\nu}} + B_{\hat{\mu}\hat{\nu}} + \lambda F_{\hat{\mu}\hat{\nu}} & H_{\hat{\mu}\hat{n}} + B_{\hat{\mu}\hat{n}} + D_{\hat{\mu}}\Phi_n \end{array} \right) . \] (35)
For simplicity we introduce notation \( E_{\mu\nu} = H_{\mu\nu} + B_{\mu\nu} \). Further, let us now presume that the matrix \( E_{mn} \) has an inverse matrix \( E_{\mu\nu}^{-1} \). This is certainly always possible to define, for example in the simplest case when \( k = 1 \) so that \( H_{zz} \) is one-dimensional. Then we can write the determinant (35) as
\[ \det \left( \begin{array}{cc} A_{\hat{g}\hat{g}} & A_{\hat{g}\hat{\nu}} \\ A_{\hat{\mu}\hat{g}} & A_{\hat{\mu}\hat{\nu}} \end{array} \right) , \] (36)
where

\[
\begin{align*}
A_{\bar{g}g} &= -(\bar{\tau}_m + i\lambda^{-1}[\Phi_m, \Phi_{\bar{g}}]\bar{\tau}_{\bar{g}})(Q^{-1})^{mn}(\bar{\tau}_n + i\lambda^{-1}[\Phi_n, \Phi_{\bar{g}}]\bar{\tau}_{\bar{g}}), \\
A_{\bar{g}\bar{v}} &= \bar{\tau}_\bar{v} + \bar{\tau}_{\bar{g}}D_{\bar{v}}\Phi_{\bar{g}} - (\tau_n + i\lambda^{-1}\tau_{\bar{g}}[\Phi_n, \Phi_{\bar{g}}])(Q^{-1})^{mn}(E_{m\bar{v}} - D_{\bar{v}}\Phi_m), \\
A_{\mu\bar{g}} &= \bar{\tau}_\mu + D_{\mu}\Phi_{\bar{g}}\bar{\tau}_{\bar{g}} - (E_{\mu\bar{m}} + D_{\mu}\Phi_m)(Q^{-1})^{mn}(\tau_n + i\lambda^{-1}[\Phi_n, \Phi_{\bar{g}}]\bar{\tau}_{\bar{g}}), \\
A_{\mu\bar{v}} &= E_{\mu\bar{v}} - (E_{\mu\bar{m}} + D_{\mu}\Phi_m)(Q^{-1})^{mn}(E_{n\bar{v}} - D_{\bar{v}}\Phi_n), \end{align*}
\]

(37)

and where \((Q^{-1})^{mn}\) is its inverse \(Q_{mk}(Q^{-1})^{kn} = \delta^m_n\).

We showed in previous section that T-dual of D(p-1)-brane is again non-relativistic D(p-1-k)-brane on condition when components of \(\tau\) along directions we dualize are equal to zero. Let us then impose the same condition. Further, let us express final form of (37) using T-dual form of the metric given in (27) and we obtain

\[
\begin{align*}
A_{\bar{g}g} &= -\lambda^{-2}\tau_{\bar{g}}[\Phi_{\bar{g}}, \Phi_m](Q^{-1})^{mn}[\Phi_n, \Phi_{\bar{g}}]\bar{\tau}_{\bar{g}}, \\
A_{\bar{g}\bar{v}} &= \tau_{\bar{v}} + \tau_{\bar{g}}D_{\bar{v}}\Phi_{\bar{g}} - i\lambda^{-1}\tau_{\bar{g}}[\Phi_{\bar{g}}, \Phi_n](Q^{-1})_m^m(\bar{E}_{\bar{m}\bar{v}} + \bar{E}_{\bar{m}k}D_{\bar{v}}\Phi_k), \\
A_{\mu\bar{g}} &= \bar{\tau}_\mu + D_{\mu}\Phi_{\bar{g}}\bar{\tau}_{\bar{g}} - i\lambda^{-1}(\bar{E}_{\bar{m}}^m + D_{\mu}\Phi_k\bar{E}_{km})(Q^{-1})_m^n[\Phi_n, \Phi_{\bar{g}}]\bar{\tau}_{\bar{g}}, \\
A_{\mu\bar{v}} &= P[\bar{E}_{\mu\bar{v}}] + P[\bar{E}_{\mu\bar{m}}\bar{E}_{rs}((Q^{-1})_s^t - \delta^t_s)\bar{E}_{t\bar{v}}], \end{align*}
\]

(39)

where \(P[\bar{E}_{\mu\bar{v}}]\) is pull-back of the T-dual metric defined as

\[
P[\bar{E}_{\mu\bar{v}}] = \bar{E}_{\mu\bar{v}} + D_{\mu}\Phi_m\bar{E}_{m\bar{v}} + \bar{E}_{\mu}^nD_{\bar{v}}\Phi_n + D_{\mu}\Phi_m\bar{E}_{mn}D_{\bar{v}}\Phi_m,
\]

(40)

and where

\[
\begin{align*}
P[\bar{E}_{\mu\bar{v}}\bar{E}_{rs}((Q^{-1})_s^t - \delta^t_s)\bar{E}_{t\bar{v}}] &= \bar{E}_{\mu}^m\bar{E}_{nk}((Q^{-1})_r^k - \delta^k_r)\bar{E}_{r\bar{v}} + \bar{E}_{\mu}^n\bar{E}_{nl}((Q^{-1})_l^t - \delta^t_l)\bar{E}_{k\bar{v}}D_{\bar{v}}\Phi_n + D_{\mu}\Phi_n\bar{E}_{nk}\bar{E}_{r\bar{v}}((Q^{-1})_s^k - \delta^k_s)\bar{E}_{r\bar{v}} + D_{\mu}\Phi_k\bar{E}_{kr}\bar{E}_{rs}((Q^{-1})_t^s - \delta^s_t)\bar{E}_{t\bar{v}}D_{\bar{v}}\Phi_n.\end{align*}
\]

(41)

Finally, \(\tilde{\phi}\) is given in (24). Collecting all these terms together we obtain an action for \(N\) non-relativistic D(8-k)-branes in the form

\[
S = -T_{8-k}\text{Str} \int d^{9-k}\xi e^{-\tilde{\phi}} \sqrt{-\det \begin{pmatrix} A_{\bar{g}g} & A_{\bar{g}\bar{v}} \\ A_{\mu\bar{g}} & A_{\mu\bar{v}} \end{pmatrix}} \det Q_k^l. \]

(42)

This is the final form of the action for \(N\) D(8-k)-branes. Observe that there is general non-zero \(A_{\bar{g}g}\) as opposite to the case of single D(8-k)-brane. However note that for collection \(N\) D(8-k)-brane that are localized at single point \(\bar{y}_0\) we have that \(\Phi_{\bar{g}} = \bar{y}_0I_{N \times N}\) where \(I_{N \times N}\) is unit matrix. As a result \(\Phi_{\bar{g}}\) commutes with all matrices and hence \(A_{\bar{g}g} = 0\). Then the action (42) has similar form as the action for single D(8-k)-brane in non-relativistic background which is nice consistency check.
5 T-Duality Along \( \tilde{y} \)--Direction

Now we are ready to study how non-relativistic D(8-k)-brane transforms under T-duality along directions transverse to its world-volume. Since T-duality along spatial directions that are transverse to its world-volume is the same as in case of relativistic Dp-brane we skip this analysis and recommend [20] for more details. Instead we focus on T-duality along \( \tilde{y} \)--direction.

We begin with the simpler case which is D8-brane transverse to \( \tilde{y} \)-direction and perform T-duality along it. It is well known that in order to perform T-duality along transverse direction we should consider configuration of infinite number of D8-branes on covering space. To do this we should presume that \( \tilde{y} \) coordinate is compact so that the covering space is real line. Then the the world-volume matrix valued fields obey following quotient condition [21]

\[
U \Phi \tilde{y} U^{-1} = \lambda^{1/2} + \Phi \tilde{y}, \\
U A_{\tilde{\mu}} U^{-1} = A_{\tilde{\mu}}. 
\]

(43)

In order to solve quotient equation it is natural to introduce an auxiliary Hilbert space of functions \( f(y) \) on which \( \Phi \tilde{y} \) and \( U \) act. Then \( U \) is generator of the functions on this covering space with coordinate \( y \) in the form

\[
U = e^{i \frac{y}{\sqrt{\lambda}}}, 
\]

(44)

where \( y \) is coordinate of the space on which functions \( f(y) \) are defined. Then \( \Phi \tilde{y} \) has to be equal to

\[
\Phi \tilde{y} = i \lambda \partial_y - A_y(y), 
\]

(45)

where now \( A_y(y) \) is ordinary function that acts on Hilbert space by ordinary multiplication. Using (45) we obtain

\[
D_{\tilde{\mu}} \Phi \tilde{y} = -\lambda F_{\tilde{\mu} \tilde{y}}, \quad F_{\tilde{\mu} \tilde{y}} = \partial_{\tilde{\mu}} A_y - \partial_y A_{\tilde{\mu}} 
\]

(46)

so that the action for \( N \) D8-branes can be written as

\[
S = -\frac{T_8}{\sqrt{\lambda}} \int d^9\xi dye^{-\tilde{\phi}} \sqrt{-\det \begin{pmatrix}
0 & \tau_\nu + \lambda \tau \tilde{y} F_{\tilde{y} \nu} \\
-\tau_{\tilde{\mu}} - \lambda F_{\tilde{\mu} \tilde{y}} \tau_\tilde{y} & H_{\tilde{\mu} \tilde{\nu}} + B_{\tilde{\mu} \tilde{\nu}} + \lambda F_{\tilde{\mu} \tilde{\nu}}
\end{pmatrix}} = \\
= -T_9 \int d^9\xi dye^{-\tilde{\phi}} \sqrt{-\tau \tilde{y} \tau \tilde{y}} \sqrt{-\det \begin{pmatrix}
0 & \tau_\nu \tau \tilde{y}^{-1} + \lambda F_{\tilde{y} \nu} \\
-\tau_{\tilde{\mu}} \tau \tilde{y}^{-1} & E_{\tilde{\mu} \tilde{\nu}} + \lambda F_{\tilde{\mu} \tilde{\nu}}
\end{pmatrix}} 
\]

(47)

using also the fact that

\[
\text{Tr} = \frac{1}{\sqrt{\lambda}} \int dy. 
\]

(48)

Remember definition of \( \tau \tilde{y} \), \( \tau \tilde{y} \) as was given in section (2) we obtain the action in the form

\[
S = -T_9 \int d^{10}\xi e^{-\tilde{\phi}} \sqrt{-\det \begin{pmatrix}
0 & E_{\tilde{y} \nu} + \lambda F_{\tilde{y} \nu} \\
E_{\tilde{\mu} \nu} + \lambda F_{\tilde{\mu} \nu} & E_{\tilde{\mu} \tilde{\nu}} + \lambda F_{\tilde{\mu} \tilde{\nu}}
\end{pmatrix}} 
\]

(49)
which is the original form of DBI action for D9-brane in the the relativistic background with light-like isometry.

5.1 T-duality in Case of D(8-k)-brane

Now we proceed to the most interesting problem which is T-duality along $\tilde{y}$ direction that is transverse to the world-volume of non-relativistic D(8-k)-brane. As is previous section we consider an array of infinite number of D(8-k)-branes whose world-volume fields obey quotient conditions (43) and (46). Note that there is additional condition on the matrix $\Phi_m$ in the form

$$U \Phi_m U^{-1} = \Phi_m.$$  \hspace{1cm} (50)

Now $\Phi_m(\xi)$ and $A_{\mu}(\xi)$ should be considered as ordinary functions $\phi_m(\xi, y)$ and $A_{\mu}(\xi, y)$ defined on the space labelled by $y$ that act on test function $f(y)$ by ordinary multiplications.

Then we again find that

$$D_{\mu} \Phi_{\tilde{y}} = -\lambda F_{\tilde{y} \mu}$$  \hspace{1cm} (51)

together with

$$i\lambda^{-1}[\Phi_{\tilde{y}}, \Phi_m] = -\partial_y \phi_m$$  \hspace{1cm} (52)

and also $[\Phi_m, \Phi_n] = [\phi_m(y), \phi_n(y)] = 0$ since this is commutator of ordinary functions. As a result we get that $Q_{mn} = E_{mn}$ and $(Q^{-1})^{mn} = \tilde{E}^{mn}$. Using these results we obtain that components of the matrix $A$ have the form

$$A_{\tilde{y} \tilde{y}} = -\tau_{\tilde{y}} \xi \partial_y \phi_m \tilde{E}^{mn} \partial_y \phi_n,$$

$$A_{\overline{\nu} \tilde{y}} = \tau_{\tilde{y}} + \lambda \tau_{\tilde{y}} F_{\overline{\nu} y} + \partial_y \phi_n (\tilde{E}_{\overline{\nu}} + \tilde{E}^{mk} \partial_y \phi_k) \tau_{\tilde{y}},$$

$$A_{\mu \overline{\nu}} = \tilde{\tau}_{\mu} - \lambda \tilde{F}_{\mu \overline{\nu}} \tilde{\tau}_{\tilde{y}} - (\tilde{E}_{\mu} + \tilde{E}^{km} \partial_y \phi_k) \partial_y \phi_n \tilde{\tau}_{\tilde{y}},$$

$$A_{\mu \overline{\nu}} = P[\tilde{E}_{\mu \overline{\nu}}].$$  \hspace{1cm} (53)

Inserting these results into the action (42) we obtain that it has the form

$$S = -\frac{T_{8-k}}{\lambda^{1/2}} \int dy d^{9-k} \xi e^{-\phi} \sqrt{-\tau_{\tilde{y}} \tau_{\tilde{y}} \det Q_{y}} \times$$

$$\sqrt{-\det \left( \begin{array}{cc} \partial_y \phi_m \tilde{E}^{mn} \partial_y \phi_n & \tau_{\tilde{y}}^{-1} \tau_{\mu} + \partial_y \phi_n (\tilde{E}^{n}_{\overline{\nu}} + \tilde{E}^{mk} \partial_y \phi_k) + \lambda F_{\overline{\nu} y} \\ -\tilde{\tau}_{\tilde{y}}^{-1} \tilde{\tau}_{\mu} + (\tilde{E}_{\mu} + \tilde{E}_{\mu}^{km} \partial_y \phi_k) \partial_y \phi_m + \lambda \tilde{F}_{\mu \overline{\nu}} & P[\tilde{E}_{\mu \overline{\nu}}] + \lambda \tilde{F}_{\mu \overline{\nu}} \end{array} \right)}.$$  \hspace{1cm} (54)

Finally using $\tau_{\tilde{y}} = C$, $\tilde{\tau}_{\tilde{y}} = -C$ we get that the action takes standard form of the relativistic D(9-k)-brane action in the background with the light-like isometry which is nice consistency check. Note also that the background fields are given by T-duality rules (27).
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