\textbf{\textit{L}}^p \textit{ Estimates for an Oscillating Dunkl Multiplier}

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\textbf{Abstract.} In this paper, we study the \( L^p \) boundedness of a class of oscillating multiplier operator for the Dunkl transform \( \mathcal{F}_k \), \( T_{m,\alpha}(f) = \mathcal{F}_k^{-1}(m,\alpha \mathcal{F}_k(f)) \) with \( m,\alpha(\xi) = |\xi|^{-\alpha} e^{\pm\imath \phi(\xi)} \). We obtain an \( L^p \)-bound result for the corresponding maximal functions. As a specific applications, we give an extension of the \( L^p \) estimate for the wave equation and of Stein’s theorem for the analytic family of maximal spherical means (Stein, Proc Natl Acad Sci USA 73:2174–2175, 1976).

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1. Introduction and Statement of the Results

For \( \alpha > 0 \) and \( \beta > 0 \), the standard oscillating multiplier \( T_{\alpha,\beta} \) is defined via Fourier transform by \( T_{\alpha,\beta}(f) = m_{\alpha,\beta} \hat{f} \), where \( m_{\alpha,\beta} = |\xi|^{-\alpha} e^{\imath \phi(\xi)} \) and \( \phi \) is a \( C^\infty \) function on \( \mathbb{R}^n \) which vanishes near the origin and is equal to 1 for all sufficiently large \( \xi \). The study of their \( L^p \) properties going back to the works of Hirschman [9] in the case \( n = 1 \) and Wainger [18] in higher dimensions. Later on, they have been extensively studied again by many authors in several different contexts, see [8,10,11,14]. This paper is devoted to the study of \( L^p \) boundedness of oscillating multiplier in the context of Dunkl analysis. We will focus on the case \( \beta = 1 \), because of the close connection to the wave equation associated with the Dunkl Laplacian \( \Delta_k \), and to the spherical maximal function. The latter is already studied by Deleaval [5]. More generally, Dunkl-type multipliers have received a considerable attention in recent times, see for instance [2,3,6]. To describe more precisely the results studied in this paper, we shall start by giving a brief summary of the Dunkl analysis.

1.1. Background

Dunkl theory generalizes classical Fourier analysis on \( \mathbb{R}^n \). It started thirty years ago with Dunkl’s seminal work [7] and was further developed by several
mathematicians. We refer for more details to the articles [4,7,12] and the references cited therein.

Let $\mathbb{R}^n$ be equipped with the canonical inner product $\langle.,.\rangle$ and its induced norm $|.|$. Let $G \subset O(\mathbb{R}^n)$ be a finite reflection group associated to a reduced root system $R$ and $k : R \to [0, +\infty)$ be a $G$-invariant function (called multiplicity function). Let $R^+$ be a positive root subsystem. The Dunkl operators $D^k_\xi$ on $\mathbb{R}^n$ are the following $k$-deformations of directional derivatives $\partial_\xi$ by difference operators:

$$D^k_\xi f(x) = \partial_\xi f(x) + \sum_{\nu \in R^+} k(\nu) \frac{\langle \nu, \xi \rangle}{\langle \nu, x \rangle} \left( f(x) - f(\sigma_v x) \right),$$

where $\sigma_v x = x - \langle v, x \rangle^2 v$ denotes the reflection with respect to the hyperplane orthogonal to $v$. The definition of $D^k_\xi$ is of course independent of the choice of the positive system, since $k$ is $G$-invariant. The Dunkl operators are antisymmetric with respect to the measure $w_k(x) dx$ with density

$$w_k(x) = \prod_{\nu \in R^+} |\langle \nu, x \rangle|^{2k(\nu)}.$$

The operators $\partial_\xi$ and $D^k_\xi$ are intertwined by a Laplace-type operator

$$V_k f(x) = \int_{\mathbb{R}^n} f(y) d\mu_x(y)$$

associated to a family of compactly supported probability measures $\{\mu_x | x \in \mathbb{R}^n\}$. Specifically, $\mu_x$ is supported in the convex hull $\text{co}(Gx)$.

For every $y \in \mathbb{C}^n$, the simultaneous eigenfunction problem

$$D^k_\xi f = \langle y, \xi \rangle f \quad \forall \in \mathbb{R}^n$$

has a unique solution $f(x) = E_k(x,y)$ such that $E_k(0,y) = 1$, called the Dunkl kernel and given by

$$E_k(x,y) = V_k(e^{\langle\cdot, y\rangle})(x) = \int_{\mathbb{R}^n} e^{\langle z, y \rangle} d\mu_x(z) \quad \forall x \in \mathbb{R}^n.$$ 

Furthermore this kernel has a holomorphic extension to $\mathbb{C}^n \times \mathbb{C}^n$ and the following equalities hold: for $x, y \in \mathbb{C}^n$,

(i) $E_k(x,y) = E_k(y,x)$,

(ii) $E_k(\lambda x,y) = E_k(x, \lambda y)$, for $\lambda \in \mathbb{C}$,

(iii) $E_k(g.x,g.y) = E_k(x,y)$, for $g \in G$.

In dimension $n = 1$, these functions can be expressed in terms of Bessel functions. Specifically,

$$E_k(x,y) = J_{k-\frac{1}{2}}(ixy) + \frac{xy}{2k+1} J_{k+\frac{1}{2}}(ixy),$$

where

$$J_\nu(z) = \Gamma(\nu+1) \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!\Gamma(\nu + n + 1)} \left( \frac{z}{2} \right)^{2n}$$

are normalized Bessel functions.
The Dunkl transform is defined on $L^1(\mathbb{R}^n, w_k(x)dx)$ by

$$\mathcal{F}_k f(\xi) = c_k \int_{\mathbb{R}^n} f(x) E_k(x, -i \xi) w_k(x) \, dx,$$

where

$$c_k = \int_{\mathbb{R}^n} e^{-|x|^2/2} w_k(x) \, dx.$$

We list some known properties of this transform:

(i) The Dunkl transform is a topological automorphism of the Schwartz space $S(\mathbb{R}^n)$.

(ii) (Plancherel Theorem) The Dunkl transform extends to an isometric automorphism of $L^2(\mathbb{R}^n, w_k(x)dx)$.

(iii) (Inversion formula) For every $f \in S(\mathbb{R}^n)$, and more generally for every $f \in L^1(\mathbb{R}^n, w_k(x)dx)$ such that $\mathcal{F}_k f \in L^1(\mathbb{R}^n, w_k(\xi)d\xi)$, we have

$$f(x) = \mathcal{F}_k^2 f(-x) \quad \forall x \in \mathbb{R}^n.$$ 

(iv) If $f$ is a radial function in $L^1(\mathbb{R}^n, w_k(\xi)d\xi)$ such that $f(x) = \tilde{f}(|x|)$, $x \in \mathbb{R}^n$. Then, $\mathcal{F}_k(f)$ is also radial and

$$\mathcal{F}_k(f)(x) = d_k \int_0^{\infty} \tilde{f}(s) J_{\gamma_k+n/2-1}(s|x|) s^{2\gamma_k+n} ds; \quad x \in \mathbb{R}^n \quad (1.1)$$

where $d_k = 2^{-(\gamma_k+n/2-1)}/\Gamma(\gamma_k + n/2)$ and

$$\gamma_k = \sum_{v \in \mathbb{R}_+} k(v).$$

Let $x \in \mathbb{R}^n$, the Dunkl translation operator $\tau_x$ is given for $f \in L^2_k(\mathbb{R}^n, w_k(x)dx)$ by

$$\mathcal{F}_k(\tau_x(f))(y) = \mathcal{F}_k f(y) E_k(x, iy), \quad y \in \mathbb{R}^n.$$ 

In the case where $f(x) = \tilde{f}(|x|)$ is a radial function in $S(\mathbb{R}^n)$, the Dunkl translation is represented by the following integral

$$\tau_x(f)(y) = \int_{\mathbb{R}^n} \tilde{f} \left( \sqrt{|y|^2 + |x|^2 + 2\langle y, \eta \rangle} \right) \, d\mu_x(\eta).$$

This formula shows that the Dunkl translation operators can be extended to all radial functions $f$ in $L^p(\mathbb{R}^n, w_k(x)dx)$, $1 \leq p \leq \infty$ and the following holds

$$\|\tau_x(f)\|_{p,k} \leq \|f\|_{p,k} \quad (1.2)$$

where $\| . \|_{p,k}$ denotes the norm of $L^p(\mathbb{R}^n, w_k(x)dx)$, $1 \leq p < \infty$.

We define the Dunkl convolution product for suitable functions $f$ and $g$ by

$$f *_k g(x) = \int_{\mathbb{R}^n} \tau_x(f)(-y) g(y) \, d\mu_k(y), \quad x \in \mathbb{R}^n.$$ 

We note that it is commutative and satisfies the following property:

$$\mathcal{F}_k(f *_k g) = \mathcal{F}_k(f) \mathcal{F}_k(g). \quad (1.3)$$
Moreover, the operator \( f \to f *_{k} g \) is bounded on \( L^{p}(\mathbb{R}^{n}, w_{k}(x)dx) \) provided \( g \) is a bounded radial function in \( L^{1}(\mathbb{R}^{n}, w_{k}(x)dx) \). In particular, we have the following Young’s inequality:

\[
\|f *_{k} g\|_{p,k} \leq \|g\|_{1,k} \|f\|_{p,k}.
\]  

We now come to the subject of this paper.

1.2. Statement of the Results

For \( \phi \) be a \( C^{\infty} \) radial function which vanishes for \( |\xi| \leq 1/2 \) and is equal to 1 for \( |\xi| \geq 1 \). Let \( \alpha > 0 \), we put

\[
m_{\alpha}(\xi) = |\xi|^{-\alpha} e^{\pm i|\xi| \phi(\xi)}.
\]

The oscillating Dunkl multiplier associated to \( m_{\alpha} \) is defined on \( L^{2}(\mathbb{R}^{n}, w_{k}(x)dx) \) by

\[
T_{m_{\alpha}}(f) = \mathcal{F}^{-1}_{k}(m_{\alpha} \mathcal{F}_{k}(f)).
\]

We will establish the following.

**Theorem 1.1.** \( T_{m_{\alpha}} \) is a bounded operator on \( L^{p}(\mathbb{R}^{n}, w_{k}(x)dx) \) for all \( 1 \leq p \leq \infty \) such that

\[
\alpha > (2\gamma_{k} + n - 1) \left| \frac{1}{2} - \frac{1}{p} \right|.
\]

The proof follows closely the proof given by Sjostrand in [14]. This contains \( L^{p} \) estimates of the solution operator for the Cauchy problem associated to Dunkl wave equation

\[
\Delta_{k} u(x, t) = \partial_{t}^{2} u(x, t), \quad u(x, y) = 0, \quad \partial_{t} u(x, 0) = f(x)
\]

where \( \Delta_{k} = \sum_{i=1}^{n}(D_{e_{i}}^{k})^{2} \), which is referred to as the Dunkl-Laplace operator on \( \mathbb{R}^{n} \). Note that the solution to this problem has been already described in [13] and is given by means of Dunkl’s transform,

\[
\mathcal{F}_{k}(u(., t))(\xi) = \frac{\sin t|\xi|}{|\xi|} \mathcal{F}_{k}(f)(\xi).
\]

As a result, we have

**Theorem 1.2.** For fixed \( t \), the linear operator \( f \to u(x, t) \) is bounded on \( L^{p}(\mathbb{R}^{n}, w_{k}(x)dx) \), for \( 1 \leq p \leq \infty \) such that

\[
\left| \frac{1}{2} - \frac{1}{p} \right| < \frac{1}{2\gamma_{k} + n - 1}.
\]

Theorem 1.2 is, therefore, the special case \( \alpha = 1 \) in Theorem 1.1. Indeed, we just write

\[
\frac{\sin(|\xi|)}{|\xi|} = \frac{e^{i|\xi|}}{2i|\xi|} \phi(\xi) - \frac{e^{-i|\xi|}}{2i|\xi|} \phi(\xi) + \frac{\sin(|\xi|)}{|\xi|} (1 - \phi(\xi)) = a_{1}(\xi) + a_{2}(\xi) + a_{3}(\xi).
\]

As \( a_{3} \) a radial \( C^{\infty} \)-function with compact support, the Dunkl multiplier associated to \( a_{3} \) is the convolution operator with kernel \( \mathcal{F}^{-1}_{k}(a_{3}) \) which is a radial Schwartz function and then by Young’s inequality it is a bounded operator on \( L^{p}(\mathbb{R}^{n}, w_{k}(x)dx) \) for all \( 1 \leq p \leq \infty \).
Remark 1.3. In the statements of Theorems 1.1 and 1.2 nothing can be said about the endpoint of the range of $p$. Adaptation of classical arguments as in [8, 10, 11] is actually not available, in the Dunkl setting the theory of Hardy space $H^1$ and BMO space are not yet much elaborated, a recent development can be found in [1]. Also, the $L^p$ theory of Dunkl multiplier is still ambiguous, since the latter is closely related to the generalized translation operators which require more information than its known properties, in particular about their integral representations.

The next main result is the boundedness of the maximal operator

$$A_\alpha(f)(x) = \sup_{t > 0} |A_\alpha(f)(x,t)|,$$

where

$$A_\alpha(f)(x,t) = \int_{\mathbb{R}^n} \frac{e^{\pm it|\xi|}}{|t\xi|^k} \phi(t\xi) \mathcal{F}(f)(\xi) E_k(ix, \xi) w_k(\xi) d\xi.$$

Theorem 1.4. The maximal operator $A_\alpha$ is bounded on $L^p(\mathbb{R}^n, w_k(x)dx)$, provided

$$\alpha > (2\gamma_k + n - 1) \left| \frac{1}{p} - \frac{1}{2} \right| + \frac{1}{p}.$$

Our arguments follow some ideas in [15] for the study of the boundedness of certain Fourier integral operators. As an application, we provide an extension of the result in [16], obtaining boundedness for the maximal function $M_\alpha(f)(x) = \sup_{t > 0} |M_\alpha(f)(x,t)|$ where

$$M_\alpha(f)(x,t) = \int_{\mathbb{R}^n} j_{\alpha+\gamma_k+n/2-1}(t|\xi|) \mathcal{F}(f)(\xi) w_k(\xi) d\xi.$$

In particular, $M_0$ reduces to the analogous of Stein’s spherical maximal function,

$$M_0(f)(x) = \sup_{t > 0} \left| \int_{S^{n-1}} \tau_x(f)(ty) d\sigma(y) \right|,$$

where $S^{n-1}$ denotes the standard unit sphere in $\mathbb{R}^n$ and $d\sigma$ corresponds to the normalized surface measure. We obtain the following

Theorem 1.5. The maximal operator $M_\alpha$ is bounded on $L^p(\mathbb{R}^n, w_k(x)dx)$ under the following conditions

(a) $\alpha > 1 - 2\gamma_k - n + (2\gamma_k + n)/p$, if $2 \leq p < \infty$,
(b) $\alpha > (2 - 2\gamma_k - n)/p$, if $1 < p \leq 2$.

As a direct consequence of Theorem 1.5, we get the following result, already proved in [5].

Corollary 1.6. Stein’s spherical maximal function $M_0$ is bounded on $L^p(\mathbb{R}^n, w_k(x)dx)$, $n \geq 3$, provide

$$p > \frac{2\gamma_k + n}{2\gamma_k + n - 1}.$$
2. Details of the Proofs

We begin by choosing the Littlewood–Paley dyadic decomposition of unity that we shall use it all along this section. The existence of such a partition is standard, it is given by a $C^\infty$-function $\psi$ that is supported in \( \{ t \in \mathbb{R}; 1/2 \leq |t| \leq 2 \} \) and satisfies

\[
\sum_{-\infty}^{\infty} \psi(2^{-\nu}t) = 1, \quad t \neq 0.
\]

We can assume further that $0 \leq \psi \leq 1$.

For convenience, we use the same notation $C$ to denote the different constants in the different place.

2.1. Proof of Theorem 1.1

Without loss of generality, we will prove Theorem 1.1 by taking $\phi$ the function

\[
\phi(\xi) = \widetilde{\phi}(|\xi|) = \sum_{\nu=0}^{\infty} \psi(2^{-\nu}|\xi|),
\]

which is clearly a $C^\infty$-function on $\mathbb{R}^n$, with $\phi(\xi) = 0$ for $|\xi| \leq 1/2$ and $\phi(\xi) = 1$ for $|\xi| \geq 1$. Indeed, let $\phi_1$ be an arbitrary $C^\infty$-function on $\mathbb{R}^n$, with $\phi_1(\xi) = 0$ for $|\xi| \leq 1/2$ and $\phi_1(\xi) = 1$ for $|\xi| \geq 1$. We write

\[
m_{\alpha}(\xi) = |\xi|^{-\alpha} e^{i|\xi|} (\phi_1(\xi) - \phi(\xi)) + |\xi|^{-\alpha} e^{i|\xi|} \phi(\xi) = m_{\alpha}^{(1)} + m_{\alpha}^{(2)}.
\]

Since $m_{\alpha}^{(1)}$ is a $C^\infty$-function with compact support, then the corresponding multiplier is a bounded linear operator on $L^p(\mathbb{R}^n, w_k(x)dx)$, $1 \leq p \leq \infty$.

Assume as a first step that $\alpha \in ]\gamma_k + (n - 1)/2, \gamma_k + (n + 1)/2[$. Here we shall prove that $T_{m_{\alpha}}$ is a convolution operator whose kernel $K_{\alpha}$ belongs to $L^1(\mathbb{R}^n, w_k(x)dx)$.

We begin, by writing via (2.1)

\[
m_{\alpha}(\xi) = \sum_{\nu=0}^{\infty} m_{\alpha}^{\nu}(\xi),
\]

where $m_{\alpha}^{\nu}(\xi) = \psi(2^{-\nu}|\xi|)|\xi|^{-\alpha} e^{i|\xi|}$. Since the function $m_{\alpha}^{\nu}$ is radial, it follows that $K_{\alpha}^{\nu} = \mathcal{F}_k^{-1}(m_{\alpha}^{\nu})$ is radial and from (1.1),

\[
K_{\alpha}^{\nu}(x) = d_k \int_{0}^{\infty} \tilde{\phi}(2^{-\nu}s) s^{-\alpha+2\gamma_k+n-1} e^{is} \mathcal{J}_{\gamma_k+n/2-1}(|x|s)ds.
\]

In the next we claim that the sum $\sum K_{\alpha}^{\nu}(x)$ is convergent for $|x| \neq 0, 1$ and we have

\[
K_{\alpha}(x) = \sum_{\nu=0}^{\infty} K_{\alpha}^{\nu}(x) = d_k \int_{0}^{\infty} \tilde{\phi}(s) s^{-\alpha+2\gamma_k+n-1} e^{is} \mathcal{J}_{\gamma_k+n/2-1}(|x|s)ds.
\]
For this purpose, we recall the appropriate asymptotic expansion of Bessel function (see [19]),

$$\mathcal{J}_{\gamma_k+n/2-1}(|x|s) = e^{i|x|s} \sum_{\ell=0}^{N-1} a_\ell (|x|s)^{-\gamma_k-(n-1)/2-\ell} + e^{-i|x|s} \sum_{\ell=0}^{N-1} a'_\ell (|x|s)^{-\gamma_k-(n-1)/2-\ell} + R_N(|x|s), \quad (2.3)$$

where $a_\ell$ and $a'_\ell$ are constants and the function $R_N$ satisfies the estimate

$$|R_N(t)| \leq c_N |t|^{-N-\gamma_k-(n-1)/2}. \quad (2.4)$$

Thus for a large enough $N$, one can write

$$K^{\nu}_{\alpha}(x) = d_k \sum_{\ell=0}^{N-1} |x|^{-\gamma_k-(n-1)/2-\ell} \left\{ a_\ell f^{\nu,\ell}_{\alpha}(1 + |x|) + a'_\ell f^{\nu,\ell}_{\alpha}(1 - |x|) \right\}$$

$$+ d_k G^{N,\nu}_{\alpha}(x)$$

where we put

$$f^{\nu,\ell}_{\alpha}(t) = \int_0^\infty \psi(2^{-\nu}s)s^{-\alpha+\gamma_k+(n-1)/2-\ell} e^{its} ds,$$

$$G^{N,\nu}_{\alpha}(x) = \int_0^\infty \psi(2^{-\nu}s)s^{-\alpha+2\gamma_k+n-1} R_N(|x|s) e^{is} ds.$$

After possibly interchanging sum and integral, we have

$$\sum_{\nu=0}^\infty G_{N,\nu}(x) = \int_0^\infty \tilde{\phi}(s)s^{-\alpha+2\gamma_k+n-1} R_N(|x|s) e^{is} ds,$$

$$\sum_{\nu=0}^\infty f_{\nu,\ell}(t) = \int_0^\infty \tilde{\phi}(s)s^{-\alpha+\gamma_k+(n-1)/2-\ell} e^{its} ds; \quad \ell \geq 1.$$

When for $\ell = 0$ we shall need to integrate by parts,

$$\int_0^\infty \psi(2^{-\nu}s)s^{-\alpha+\gamma_k+(n-1)/2} e^{its} ds$$

$$= it^{-1} \int_0^\infty 2^{-\nu} \psi'(2^{-\nu}s)s^{-\alpha+\gamma_k+(n-1)/2} e^{its} ds$$

$$+ i(-\alpha + \gamma_k + (n-1)/2)t^{-1} \int_0^\infty \psi(2^{-\nu}s)s^{-\alpha+\gamma_k+(n-1)/2-1} e^{its} ds.$$

The interchange of summation being permissible because

$$\sum_{\nu=0}^\infty \int_0^\infty |2^{-\nu} \psi'(2^{-\nu}s)s^{-\alpha+\gamma_k+(n-1)/2} e^{its}| ds$$

$$= \sum_{\nu=0}^\infty \int_0^\infty |2^{-\nu} s \psi'(2^{-\nu}s)|s^{-\alpha+\gamma_k+(n-1)/2-1} ds.$$
Thus,
\[ C \sum_{\nu=0}^{\infty} \int_{2^{\nu-1}}^{2^{\nu+1}} s^{-\alpha+\gamma_k+(n-1)/2-1} ds \leq 2C \int_{1/2}^{\infty} s^{-\alpha+\gamma_k+(n-1)/2-1} ds < \infty. \]

Thus,
\[ \sum_{\nu=0}^{\infty} \int_{0}^{\infty} 2^{-\nu} \psi'(2^{-\nu} s)s^{-\alpha+\gamma_k+(n-1)/2} e^{its} ds = \int_{0}^{\infty} \tilde{\phi}'(s)s^{-\alpha+\gamma_k+(n-1)/2} e^{its} ds \]
and
\[ \sum_{\nu=0}^{\infty} f_{\tilde{\alpha}}^{\nu,0}(t) = it^{-1} \int_{0}^{\infty} \tilde{\phi}'(s)s^{-\alpha+\gamma_k+(n-1)/2} e^{its} ds \]
\[ + i(-\alpha + \gamma_k + (n-1)/2)t^{-1} \int_{0}^{\infty} \tilde{\phi}(s)s^{-\alpha+\gamma_k+(n-1)/2-1} e^{its} ds \]
\[ = \int_{0}^{\infty} \tilde{\phi}(s)s^{-\alpha+\gamma_k+(n-1)/2} e^{its} ds. \]

We, therefore, conclude that
\[ K_{\alpha}(x) = d_k \sum_{\ell=0}^{N-1} |x|^{-\gamma_k-(n-1)/2-\ell} \left\{ a_{\ell} \int_{0}^{\infty} \tilde{\phi}(s)s^{-\alpha+\gamma_k+(n-1)/2-\ell} e^{isx} e^{it|x|s} ds \right. \]
\[ + a_{\ell}' \int_{0}^{\infty} \tilde{\phi}(s)s^{-\alpha+\gamma_k+(n-1)/2-\ell} e^{isx} e^{-it|x|s} ds \right\} \]
\[ + d_k \int_{0}^{\infty} \tilde{\phi}(s)s^{-\alpha+2\gamma_k+n-1} R_N(|x|s) e^{is} ds. \] (2.5)

and in view of (2.3)
\[ K_{\alpha}(x) = d_k \int_{0}^{\infty} \tilde{\phi}(s)s^{-\alpha+2\gamma_k+n-1} e^{is} \mathcal{J}_{\gamma_k+n/2-1}(|x|s) ds. \]

To study the behavior of $K_{\alpha}$, we will need the following elementary lemmas.

**Lemma 2.1.** Let $\alpha > 0$. The function given by
\[ h(t) = \int_{0}^{\infty} \phi(s)s^{-\alpha} e^{\pm its} ds, \quad t \neq 0 \]
satisfies $h(t) = o(t^{-N})$ when $t \to \infty$, for any integer $N > 0$.

**Proof.** Using integration by parts and Leibniz rule,
\[ |t^N h(t)| = \left| \int_{0}^{\infty} \left( \frac{d}{ds} \right)^N (\phi(s)s^{-\alpha}) e^{\pm its} ds \right| \]
\[ \leq C \left\{ \int_{1/2}^{1} \sum_{\ell=0}^{N} |\phi^{(\ell)}(s)s^{-\alpha+N+\ell}| ds + \int_{1}^{\infty} s^{-\alpha-N} ds \right\} \]
\[ \leq C. \]
**Lemma 2.2.** The following are continuous and bounded functions on \((0, \infty)\)

\[
G(t) = \int_0^t s^{-\alpha + \gamma_k + (n-1)/2} e^{\pm is} \, ds, \quad H(t) = \int_1^2 \psi(s)s^{-\alpha + \gamma_k + (n-1)/2} e^{\pm its} \, ds.
\]

**Proof.** The continuous of \(G\) and \(H\) is obvious. Since the integral defining \(G\) is convergent then the function \(G\) has a finite limit at \(\infty\), from which and continuity we obtain the boundedness of \(G\). Similarly for the boundedness of \(H\), since by integration by parts we have that

\[
\left| \int_1^2 \psi(s)s^{-\alpha + \gamma_k + (n-1)/2} e^{\pm its} \, ds \right| \leq Ct^{-1},
\]

and then \(\lim_{t \to \infty} H(t) = 0\). \(\square\)

Let us now consider the asymptotic behavior of \(K_\alpha\).

**Lemma 2.3.** The kernel \(K_\alpha\) has the following properties:

(i) \(K_\alpha(x) = O(|x|^{-N})\), as \(|x| \to \infty\), for big \(N\),

(ii) \(K_\alpha(x) = O\left((1 - |x|)^{\alpha - \gamma_k - (n-1)/2 - 1}\right)\), as \(|x| \to 1\),

(iii) \(K_\alpha(x)\) is bounded near the origin,

(iv) \(K_\alpha\) is an integrable function.

**Proof.** Clearly (iv) is a consequence of (i), (ii) and (iii). The proof of (i) follows from (2.5), Lemma 2.1 and (2.4). Consider (2.5) we see that except the term corresponding to \(\ell = 0\) all terms are bounded near \(|x| = 1\). On the other hand, we write

\[
\int_0^\infty \tilde{\phi}(s)s^{-\alpha + \gamma_k + (n-1)/2} e^{i\pm is} \, ds
\]

\[
= \int_0^\infty s^{-\alpha + \gamma_k + (n-1)/2} e^{i\pm is} \, ds
\]

\[
+ \int_0^\infty (\tilde{\phi}(s) - 1)s^{-\alpha + \gamma_k + (n-1)/2} e^{i\pm is} \, ds
\]

\[
= (1 \pm |x|)^{\alpha - \gamma_k - (n-1)/2 - 1} \int_0^\infty s^{-\alpha + \gamma_k + (n-1)/2} e^{is} \, ds
\]

\[
+ \int_0^\infty (\tilde{\phi}(s) - 1)s^{-\alpha + \gamma_k + (n-1)/2} e^{i\pm is} \, ds
\]

\[
= C(1 \pm |x|)^{\alpha - \gamma_k - (n-1)/2 - 1} + H(x),
\]

where \(H\) is bounded. This gives the property (ii). To prove (iii) we can use the well-known integral representation for Bessel function to write
\[ K_\nu^\alpha(x) = d_k \int_0^\infty \psi(2^{-\nu} s) s^{-\alpha+2\gamma_k+n-1} e^{is} J_{\gamma_k+n/2-1}(|x|s) ds \]
\[ = b_k \int_{-1}^1 (1 - t^2)^\gamma_k+(n-3)/2 \]
\[ \times \left\{ \int_0^\infty \psi(2^{-\nu} s) s^{-\alpha+2\gamma_k+n-1} e^{i(1+t|x|)s} ds \right\} dt, \quad (2.6) \]

where \( b_k = d_k \Gamma(\gamma_k + n/2)/\Gamma(\gamma_k + (n+1)/2 - 1) \). Let \(|x| \leq 1/2\) and \( N > -\alpha + 2\gamma_k + n\), sufficiently large. Using the fact that
\[
\left| \frac{d}{ds} (\psi(2^{-\nu} s)) \right| \leq C |s|^{-\ell}, \quad \ell \in \mathbb{N}
\]
and applying integration by parts \( N \) times, we derive that
\[
\int_0^\infty \psi(2^{-\nu} s) s^{-\alpha+2\gamma_k+n-1} e^{i(1+t|x|)s} ds
\]
\[ = (i(1 + t|x|))^{-N} \int_0^\infty \left( \frac{d}{ds} \right)^N \left( \psi(2^{-\nu} s) s^{-\alpha+2\gamma_k+n-1} \right) e^{i(1+t|x|)s} ds
\]
\[ \leq C 2^{-N} \int_{2^{-\nu-1}}^{2^{\nu+1}} s^{-\alpha+2\gamma_k+n-1-N} ds. \]

From which and (2.6) it follows that
\[
|K_\nu^\alpha(x)| \leq C \int_{2^{-\nu-1}}^{2^{\nu+1}} s^{-\alpha+2\gamma_k+n-1-N} ds \quad (2.7)
\]
and then \( |K_\alpha(x)| \leq C \). This concludes (iii). \( \square \)

**Lemma 2.4.** Let \( \alpha \in [\gamma_k + (n-1)/2, \gamma_k + (n+1)/2] \). The operator \( T_{m\alpha} \) can be represented through the kernel \( K_\alpha \) as the integral,
\[
T_{m\alpha}(f)(x) = \int_{\mathbb{R}^n} K_\alpha(y) \tau_x(f)(y) w_k(y) dy, \quad f \in S(\mathbb{R}^n) \quad (2.8)
\]
and satisfies the inequality
\[
\|T_{m\alpha}(f)\|_{1,k} \leq \|K_\alpha\|_{1,k} \|f\|_{1,k}. \quad (2.9)
\]

**Proof.** Let \( f \in S(\mathbb{R}^n) \). In view of (2.2) we can write
\[
T_{m\alpha} = \sum_{\nu=0}^\infty T_{m\nu}^\alpha.
\]
Each of \( T_{m\nu}^\alpha \) will be written in its integral form
\[
T_{m\nu}^\alpha(f)(x) = K_\nu^\alpha \ast_k f(x) = \int_{\mathbb{R}^n} K_\alpha(y) \tau_x(f)(-y) w_k(y) dy.
\]
Hence to establish (2.8) we only need to interchange the order of integration and summation. For this purpose we split the integral

\[ \int_{\mathbb{R}^n} K^\nu_{\alpha}(y) \tau_x(f)(y) w_k(y) \, dy = \int_{|y| \leq 1/2} K^\nu_{\alpha}(y) \tau_x(f)(y) w_k(y) \, dy \]

\[ + \int_{1/2 \leq |y| \leq 2} K^\nu_{\alpha}(y) \tau_x(f)(y) w_k(y) \, dy \]

\[ + \int_{|y| \geq 2} K^\nu_{\alpha}(y) \tau_x(f)(y) w_k(y) \, dy. \]

From the estimate (2.7) we have

\[ \sum_{\nu=0}^{\infty} \int_{|y| \leq 1/2} K^\nu_{\alpha}(y) \tau_x(f)(y) w_k(y) \, dy = \int_{|y| \leq 1/2} K_{\alpha}(y) \tau_x(f)(y) w_k(y) \, dy. \]

Similarly for the integral over $|y| \geq 2$, by writing

\[ K^\nu_{\alpha}(y) = \sum_{\ell=0}^{N-1} |y|^{-\gamma_k-(n-1)/2-\ell} \left\{ c_{\ell} \int_0^{\infty} \psi(2^{-\nu} s) s^{-\alpha+\gamma_k+(n-1)/2-\ell} e^{i(1+|y|) s} \, ds \right. \]

\[ \left. + c_{\ell}^2 \int_0^{\infty} \psi(2^{-\nu} s) s^{-\alpha+\gamma_k+(n-1)/2-\ell} e^{i(1-|y|) s} \, ds \right\} \]

\[ + \int_0^{\infty} \psi(2^{-\nu} s) s^{-\alpha+2\gamma_k+n-1} R_N(|y| s) e^{is} \, ds, \]
for $N$ sufficiently large, we get that

\[ |K^\nu_{\alpha}(y)| \leq C \int_{2^\nu-1}^{2^{\nu+1}} s^{-\alpha+\gamma_k+(n-1)/2-1} \, ds. \]

Notice that we have integrated by parts the integral of first term (for $\ell = 0$). Thus,

\[ \sum_{\nu=0}^{\infty} \int_{|y| \geq 2} K^\nu_{\alpha}(y) \tau_x(f)(y) w_k(y) \, dy = \int_{|y| \geq 2} K_{\alpha}(y) \tau_x(f)(y) w_k(y) \, dy. \]

Now for the integral over $1/2 \leq |y| \leq 2$ we proceed by applying the dominated convergence theorem. Put

\[ \Psi_\nu(s) = \sum_{j=0}^{\nu} \psi(2^{-j} s), \quad S^\nu_\alpha(y) = \sum_{j=0}^{\nu} K^\nu_{\alpha}(y). \]

First, observe that $\Psi_\nu(s) = 1$ if $1 \leq s \leq 2^\nu$ and $\Psi_\nu(s) = 0$ if $s \geq 2^{\nu+1}$ or $s \leq 1/2$. Using this fact, we have

\[ \int_0^{\infty} \Psi_\nu(s) s^{-\alpha+\gamma_k+(n-1)/2} e^{i(1 \pm |y|) s} \, ds \]

\[ = \int_0^1 \psi(s) s^{-\alpha+\gamma_k+(n-1)/2} e^{i(1 \pm |y|) s} \, ds + \int_1^{2^\nu} s^{-\alpha+\gamma_k+(n-1)/2} e^{i(1 \pm |y|) s} \, ds \]

\[ + \int_2^{2^{\nu+1}} \psi(2^{-\nu} s) s^{-\alpha+\gamma_k+(n-1)/2} e^{i(1 \pm |y|) s} \, ds. \]
Clearly
\[ \left| \int_0^1 \psi(s) s^{-\alpha + \gamma_k + (n-1)/2} e^{i(1 + |y|)^s} ds \right| \leq \int_0^1 \psi(s) s^{-\alpha + \gamma_k + (n-1)/2} ds \leq C. \]

However, Lemma 2.2 implies that
\[ \left| \int_1^{2^\nu} s^{-\alpha + \gamma_k + (n-1)/2} e^{i(1 + |y|)^s} ds \right| = |1 \pm |y||^{\alpha - \gamma_k - (n+1)/2} \]
\[ \leq C |1 \pm |y||^{\alpha - \gamma_k - (n+1)/2} \]
and
\[ \left| \int_{2^\nu}^{2^{\nu+1}} \psi(2^{-\nu} s) s^{-\alpha + \gamma_k + (n-1)/2} e^{i(1 + |y|)^s} ds \right| \]
\[ = |1 \pm |y||^{\alpha - \gamma_k - (n+1)/2} H \left( (1 \pm |y|)2^{\nu} \right) \]
\[ \leq C |1 \pm |y||^{\alpha - \gamma_k - (n+1)/2}. \]

Now writing
\[ S_{\nu}(y) = \sum_{p=0}^{N-1} |y|^{-\gamma_k - (n-1)/2 - p} \left\{ c_p \int_0^\infty \Psi_{\nu}(s) s^{-\alpha + \gamma_k + (n-1)/2 - p} e^{i(1 + |y|)^s} ds \right\} \]
\[ + c'_p \int_0^\infty \Psi_{\nu}(s) s^{-\alpha + \gamma_k + (n-1)/2 - p} e^{i(1 - |y|)^s} ds \]
\[ + \int_0^\infty \Psi_{\nu}(s) s^{-\alpha + 2\gamma_k + n - 1} R_N(|y|) e^{is} ds. \]

So, as \( 1/2 \leq |y| \leq 2 \) we get that
\[ |S_{\nu}(y)| \leq C |1 - |y||^{\alpha - \gamma_k - (n+1)/2}. \]

(2.10)

Since the right hand side in (2.10) is an integrable function then by the dominated convergence theorem,
\[ \sum_{\nu=0}^\infty \int_{1/2 \leq |y| \leq 2} K_{\alpha}(y) \tau_x(f)(y) w_k(y) dy = \int_{1/2 \leq |y| \leq 2} K_{\alpha}(y) \tau_x(f)(y) w_k(y) dy. \]

This completes the proof of (2.8).

To prove (2.9), we consider the truncated kernels and operators, for \( 0 < \varepsilon < 1 \)
\[ K_{\alpha,\varepsilon} = K_{\alpha} 1_{\{\varepsilon \leq ||y|-1| \leq \frac{1}{2} \}} \]
and for \( f \in S(\mathbb{R}^n) \),
\[ T_{m,\alpha,\varepsilon} f(x) = \int_{\mathbb{R}^n} K_{\alpha,\varepsilon}(y) \tau_x(f)(y) w_k(y) dy = \int_{\mathbb{R}^n} \tau_x(K_{\alpha,\varepsilon})(y) f(y) w_k(y) dy. \]
Since the kernel $K_\alpha$ is in $L^1(\mathbb{R}^n, w_k(y)dy)$ and bounded a way from $|y| = 1$ which imply that $K_{\alpha, \varepsilon}$ is a bounded radial function in $L^1(\mathbb{R}^n, w_k(y)dy)$, we can then use (1.4) and (1.2) to get the following
\[
\|T_{m, \alpha}(f)\|_{1,k} \leq \|\tau_x(K_{\alpha, \varepsilon})\|_{1,k}\|f\|_{1,k} \leq \|K_{\alpha, \varepsilon}\|_{1,k}\|f\|_{1,k} \leq \|K_{\alpha}\|_{1,k}\|f\|_{1,k}.
\]
As $\lim_{\varepsilon \to \infty} T_{m, \alpha}(f) = T_{m, \alpha}(f)(x)$, then we can apply Fatou’s lemma to obtain (2.9).

Now, we are in a position to prove Theorem 1.1.

**Proof of Theorem 1.1.** We can assume that $1 \leq p \leq 2$, the case $2 \leq p \leq \infty$ follows by a well-known duality argument. Let us choose an arbitrary $\alpha_0 \in [\gamma_k + (n - 1)/2, \gamma_k + (n + 1)/2]$. We first write
\[
m_{\alpha}^\nu(\xi) = \left\{ \psi(2^{-\nu}|\xi|)|2^{-\nu}\xi|^{\alpha_0 - \alpha} \right\}\left\{ e^{i|\xi|}\phi(|\xi|) \right\} = 2^{\nu(\alpha_0 - \alpha)}h(2^{-\nu}\xi)m_{\alpha_0}(\xi)
\]
where $h(\xi) = \psi(|\xi|)|\xi|^{\alpha_0 - \alpha}$. The fact that $h$ is a radial $C^\infty$-function with compact support, the corresponding multiplier operator $T_h$ is the convolution operator with the radial Schwartz function $\mathcal{F}^{-1}_k(h)$ and therefore bounded on $L^r(\mathbb{R}^n, w_k(x)dx)$ for all $1 \leq r \leq \infty$. This follows from (1.4). Also, a simple argument shows that the multiplier $T_{h_\nu}$ of symbol $h_\nu(\xi) = h(2^{-\nu}\xi)$ is bounded on $L^r(\mathbb{R}^n, w_k(x)dx)$ with norm $\|T_{h_\nu}\|_r = \|T_h\|_r$. Now, using Lemma 2.9 and that $T_{m_\alpha}^\nu$ is a composition of two multiplier operators on $L^1(\mathbb{R}^n, w_k(x)dx)$, it follows that $T_{m_\alpha}^\nu$ is bounded on $L^1(\mathbb{R}^n, w_k(x)dx)$ with norm
\[
\|T_{m_\alpha}^\nu\|_1 \leq C2^{\nu(\alpha_0 - \alpha)}.
\]

Similarly, if we write $m_{\alpha}^\nu(\xi) = 2^{-\alpha \nu}\left\{ \psi(2^{-\nu}|\xi|)|2^{-\nu}\xi|^{-\alpha} \right\}\left\{ e^{i|\xi|}\phi(|\xi|) \right\}$ then we get that
\[
\|T_{m_\alpha}^\nu\|_2 \leq C2^{-\alpha \nu}.
\]
Therefore, by the Riesz–Thorin interpolation theorem, $T_{m_\alpha}^\nu$ is bounded on $L^p(\mathbb{R}^n, w_k(x)dx)$ for any $1 \leq p \leq 2$, $1/p = 1 - \theta/2$, $\theta \in [0, 1]$, and we have that
\[
\|T_{m_\alpha}^\nu\|_p \leq C2^{\nu(\alpha_0 - \alpha)(1 - \theta) - \alpha \nu \theta} \leq C2^{\nu(\alpha_0 (1 - \theta) - \alpha)}.
\]
We can then sum and obtain the boundedness of $T_{m_\alpha}^\nu$ on $L^p(\mathbb{R}^n, w_k(x)dx)$ if $\alpha > \alpha_0 (1 - \theta)$. Since $\alpha_0$ is arbitrary in $[\gamma_k + (n - 1)/2, \gamma_k + (n + 1)/2]$, it suffices to take
\[
\alpha > \gamma_k + (n - 1)/2(1 - \theta) = (2\gamma_k + (n - 1)) \left(\frac{1}{p} - \frac{1}{2}\right).
\]
Our theorem is, therefore, proved.
2.2. Proof of Theorem 1.4
We will use the following technical Lemma.

**Lemma 2.5.** Suppose that $F$ is $C^1(I)$, for an interval $I$. Then, for each $0 < \lambda \leq |I|$ and $p, p' > 1$ with $1/p + 1/p' = 1$, we have

$$
\sup_{t} |F(t)| \leq \lambda^{-1/p} \left( \int_{I} |F(t)|^p dt \right)^{1/p} + \lambda^{1/p'} \left( \int_{I} |F'(t)|^p dt \right)^{1/p}.
$$

(2.12)

**Proof.** Let $I_0 \subset I$ be an interval of length $\lambda$. For $t, s \in I_0$ we have

$$
F(t) = F(s) + \int_{s}^{t} F'(u) du.
$$

So by Hölder’s inequality

$$
|F(t)| \leq |F(s)| + \lambda^{1/p'} \left( \int_{I} |F'(t)|^p dt \right)^{1/p}.
$$

Integrating both sides with respect to $s$, we get

$$
|F(t)| \leq \lambda^{-1} \int_{I} |F(s)| ds + \lambda^{1/p'} \left( \int_{I} |F'(t)|^p dt \right)^{1/p}
$$

and using Holder’s inequality again, we obtain

$$
|F(t)| \leq \lambda^{-1/p} \left( \int_{I} |F(t)|^p dt \right)^{1/p} + \lambda^{1/p'} \left( \int_{I} |F'(t)|^p dt \right)^{1/p}.
$$

Then, (2.12) is established. \qed

The next lemma is essentially contained in the theorems 6.2 and 6.1 of [17].

**Lemma 2.6.** Let $\Phi \in L^1(\mathbb{R}^n w_k(x)dx)$ be a real valued radial function with $F_k(\Phi) \in L^1(\mathbb{R}^n, w_k(x)dx)$ and which satisfies $|\Phi(x)| \leq C(1 + |x|)^{-2\gamma_k - n-1}$. Then, we have

$$
\| \sup_{t>0} \Phi_t * f(x) \|_{p,k} \leq C \| f \|_{p,k}; \quad 1 < p \leq \infty,
$$

where $\Phi_t(x) = t^{-2\gamma_k - n} \Phi(t^{-1}x)$.

Now we consider our maximal operator

$$
A_\alpha(f)(x) = \sup_{t>0} |A_\alpha(f)(x,t)|
$$

where

$$
A_\alpha(f)(x,t) = \int_{\mathbb{R}^n} e^{\pm it|\xi|} \left| \frac{\phi(t\xi)}{|t\xi|^\alpha} \phi(t\xi) \mathcal{F}(f)(\xi) E_k(ix, \xi) w_k(\xi) \right| d\xi.
$$

We write

$$
A_\alpha(f)(x,t) = \sum_{\nu=0}^{\infty} A_\alpha^{\nu}(f)(x,t)
$$

where

$$
A_\alpha^{\nu}(f)(x,t) = \int_{\mathbb{R}^n} e^{\pm it|\xi|} \left| \frac{\phi(t\xi)}{|t\xi|^\alpha} \phi(t\xi) \psi(2^{-\nu} t|\xi|) \mathcal{F}(f)(\xi) E_k(ix, \xi) w_k(\xi) \right| d\xi.
$$
Let \( t \in [1, 2] \). We can use the same argument that provided (2.11) and duality to get the following estimates

\[
\| A^\nu_\alpha(f)(., t) \|_{p,k} \leq C 2^{\nu[\alpha_0[2/p-1]-\alpha]} \| f \|_{p,k}
\]

\[
\left\| \frac{\partial}{\partial t} A^\nu_\alpha(f)(., t) \right\|_{p,k} \leq C 2^{\nu[\alpha_0[2/p-1]-(\alpha-1)]} \| f \|_{p,k}
\]

for an arbitrary \( \alpha_0 \in ]\gamma_k + (n-1)/2, \gamma_k + (n+1)/2 \) and \( p \geq 1 \). Now by applying Lemma 2.5 with \( \ell = 2^{-\nu} \) it follows that

\[
\left\| \sup_{1 \leq t \leq 2} A^\nu_\alpha(f)(., t) \right\|_{p,k} \leq C 2^{\nu[\alpha_0[2/p-1]-\alpha+1/p]} \| f \|_{p,k}.
\] (2.13)

In addition, (2.13) implies that for \( j \in \mathbb{Z} \),

\[
\left\| \sup_{2^j \leq t \leq 2^{j+1}} A^\nu_\alpha(f)(., t) \right\|_{p,k} \leq C 2^{\nu[\alpha_0[2/p-1]-\alpha+1/p]} \| f \|_{p,k}
\]

which can be seen by writing

\[
\sup_{2^j \leq t \leq 2^{j+1}} |A^\nu_\alpha(f)(x, t)| = \sup_{1 \leq t \leq 2} |A^\nu_\alpha(f(2^j .))(2^{-j}x, t)|.
\] (2.14)

In the next, we claim that for \( p \geq 2 \),

\[
\left\| \sup_{t > 0} |A^\nu_\alpha(f)(., t)| \right\|_{p,k} \leq C 2^{\nu[\alpha_0[2/p-1]-\alpha+1/p]} \| f \|_{p,k}
\] (2.15)

which asserts that

\[
\| \sup_{t > 0} |A_\alpha(f)(., t)| \|_{p,k} \leq C \| f \|_{p,k},
\] (2.16)

for \( p \geq 2 \) and \( \alpha > (2\gamma_k + (n-1)) (1/2 - 1/p) + 1/p \). For the proof, we invoke a variant of the Littlewood–Paley operators associated to Dunkl transform, see Corollary 4.2 of [3]. Let \( t \in [2^j, 2^{j+1}] \), \( j \in \mathbb{Z} \). Define the function \( f_\ell \) by \( \mathcal{F}_k(f_\ell)(\xi) = \mathcal{F}_k(f)(\xi) \psi(2^{-\ell}|\xi|) \), \( \ell \in \mathbb{Z} \). It follows that

\[
A^\nu_\alpha(f)(x, t) = A^\nu_\alpha \left( \sum_{|\ell+j-\nu| \leq 2} f_\ell \right)(x, t).
\]

From this and (2.14) we have for all \( p \geq 2 \)

\[
\int_{\mathbb{R}^n} \sup_{t > 0} |A^\nu_\alpha(f)(x, t)|^p w_k(x) dx
\]

\[
\leq \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} \sup_{2^j \leq t \leq 2^{j+1}} |A^\nu_\alpha(f)(x, t)|^p w_k(x) dx
\]

\[
\leq C \, 2^{\nu p [\alpha_0[2/p-1]-\alpha+1/p]} \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} \sum_{|\ell+j-\nu| \leq 2} |f_\ell(x)|^p w_k(x) dx
\]

\[
\leq C \, 2^{\nu p [\alpha_0[2/p-1]-\alpha+1/p]} \int_{\mathbb{R}^n} \sum_{\ell \in \mathbb{Z}} |f_\ell(x)|^p w_k(x) dx
\]
\[
\leq C \, 2^{\nu p|\alpha_0|2/p-1-\alpha+1/p} \int_{\mathbb{R}^n} \left( \sum_{\ell \in \mathbb{Z}} |f_\ell(x)|^2 \right)^{p/2} w(x) \, dx
\]

This yields the claim (2.15).

The range \( p \leq 2 \) follows by interpolation, we proceed as in the proof of Theorem 1.2. Let us first observe that when \( \alpha > \gamma_k + (n + 1)/2 \) the kernel \( K_\alpha \) of the operator \( T_{m_\alpha} \) satisfies the following decay estimates

\[
|K_\alpha(x)| \leq \frac{C_N}{(1 + |x|)^N}; \quad n \in \mathbb{N}, \quad N \in \mathbb{N}
\]

which is immediate by making use of (1.1), (2.3) and integration by parts. Hence using Lemma 2.6 we get the boundedness of \( A_\alpha \) on \( L^p(\mathbb{R}^n, w(x) \, dx) \) for \( 1 < p \leq \infty \). The key tool is the following. For an arbitrary \( \alpha_0 > \gamma_k + (n + 1)/2 \), one can write

\[
m'_\alpha(t \xi) = \left\{ \psi(2^{-\nu} |t \xi|) |2^{-\nu} t \xi|^{\alpha_0-\alpha} \right\} \left\{ e^{it|\xi| |t \xi|^{-\alpha_0} \phi(t \xi)} \right\}
\]

where \( \phi(t \xi) = \psi(|\xi|) |\xi|^{\alpha_0-\alpha} \). Put \( H = F_k^{-1}(h) \), which is a radial Schwartz function. Thus, we can write

\[
A_\alpha^n(f)(x,t) = 2^{\nu(\alpha_0-\alpha)} H_{2^{\nu t^{-1}} \ast_k A_\alpha^n(f)(.,t)}(x)
\]

and we have that

\[
|A_\alpha^n(f)(x,t)| \leq 2^{\nu(\alpha_0-\alpha)} |H_{2^{\nu t^{-1}} \ast_k A_\alpha^n(f)(.,t)}(x)|
\]

\[
\leq 2^{\nu(\alpha_0-\alpha)} |H_{2^{\nu t^{-1}} \ast_k A_\alpha^n(f)}(x)|.
\]

Then, from Lemma 2.6 and the boundedness of \( A_\alpha^n \) we get

\[
\left\| \sup_{t > 0} |A_\alpha^n(f)(.,t)| \right\|_{q,k} \leq C 2^{\nu(\alpha_0-\alpha)} \| f \|_{q,k}; \quad 1 < q \leq \infty.
\]

On the other hand, the boundedness on \( L^2(\mathbb{R}, w_k(x) \, dx) \) in (2.15) gives

\[
\left\| \sup_{t > 0} |A_\alpha^n(f)(.,t)| \right\|_{2,k} \leq C 2^{\nu(1/2-\alpha)} \| f \|_{2,k}.
\]

So, using the Riesz–Thorin interpolation Theorem,

\[
\left\| \sup_{t > 0} |A_\alpha^n(f)(.,t)| \right\|_p \leq C 2^{\nu(\theta(\alpha_0-\alpha) + (1-\theta)(1/2-\alpha))} = C 2^{\nu(\theta(\alpha_0-1/2)+1/2-\alpha)}
\]

(2.17)

for all \( 1 < q \leq p \leq 2 \), where \( 1/p = 1/2 + \theta(1/q - 1/2) \). It is now easy to check from this that the sum \( \sum \nu \| \sup_{t > 0} |A_\alpha^n(f)(.,t)| \|_p \) is finite when

\[
\alpha > \left( \frac{1}{p} - \frac{1}{2} \right) (2\gamma_k + n - 1) + \frac{1}{p}.
\]

In fact, as \( \theta = (1/p - 1/2)(1/q - 1/2)^{-1} \) we see that \( \theta(\alpha_0-1/2)+1/2 \) tends to \( (1/p-1/2)(2\gamma_k + n - 1) + 1/p \) when letting \( \alpha_0 \) go to \( \gamma_k + (n+1)2 \) and \( q \) go to 1.
which guarantees the choice of $\alpha_0$ and $q$ such that $\alpha > \theta(\alpha_0 - 1/2) + 1/2$. This conclude the case $p \leq 2$ and thus we have completely proved our theorem.

### 2.3. Proof of Theorem 1.5

Let $\alpha > 0$. As a first step, we write

$$J_{\alpha + \gamma_k + n/2 - 1}(t|\xi|) = J_{\alpha + \gamma_k + n/2 - 1}(t|\xi|)\phi(t\xi) + J_{\alpha + \gamma_k + n/2 - 1}(t|\xi|)(1 - \phi(t\xi))$$

$$= a_{\alpha}^{(1)}(t\xi) + a_{\alpha}^{(2)}(t\xi)$$

and so, $M_{\alpha}(f) = M_{\alpha}^{(1)} + M_{\alpha}^{(2)}$ with

$$M_{\alpha}^{(i)}(f) = \sup_{t > 0} \left| \int_{\mathbb{R}^n} a_{\alpha}^{(i)}(t|\xi|) F_k(f)(\xi) w_k(\xi) d\xi \right| = \sup_{t > 0} |M_{\alpha}^{(i)}(f)(x, t)|, \quad i = 1, 2.$$ 

Since the function $a_{\alpha}^{(2)}$ is $C^\infty$ with compact support, in view of Lemma 2.6 we get the $L^p$-boundedness of $M_{\alpha}^{(2)}$ for all $1 < p < \infty$. To treat the boundedness of $M_{\alpha}^{(1)}$ we invoke the asymptotic form (2.3) for the Bessel function $J_{\alpha + \gamma_k + n/2 - 1}$. For $N$ big enough,

$$J_{\alpha + \gamma_k + n/2 - 1}(t|\xi|) = e^{it|\xi|} \sum_{\ell = 0}^{N-1} c_\ell(t|\xi|)^{-\alpha - \gamma_k - (n - 1)/2 - \ell}\right.$$

$$+ e^{-it|\xi|} \sum_{\ell = 0}^{N-1} c_\ell'(t|\xi|)^{-\alpha - \gamma_k - (n - 1)/2 - \ell} + R_N(t|\xi|),$$

where $c_\ell$ are constants. The function $R_N$ satisfies for all $j \in \mathbb{N}$

$$|\partial^j_x R_N(s)| \leq C|s|^{-\alpha + N - \gamma_k - (n - 1)/2}. \quad (2.18)$$

It is not hard to show that this requirement is satisfied. Set $V = F_k^{-1}(R_N(|.|) \phi(|.|))$, which can be represented by

$$V(x) = \frac{2^{-\gamma_k - n/2 + 1}}{\Gamma(\gamma_k + n/2)} \int_0^\infty R_N(s) \tilde{\phi}(s)s^{2\gamma_k + n - 1} J_{\gamma_k + n/2 - 1}(|x|s)ds.$$ 

So, with the use of asymptotic form (2.3), integration by part and (2.18) we get that for some larger $N$,

$$|V(x)| \leq C(1 + |x|)^{-N}$$

and because of Lemma 2.6

$$\left\| \sup_{t > 0} V_t \ast_k f \right\|_{p, k} \leq C_p \|f\|_{p, k}; \quad 1 < p \leq \infty.$$ 

It, therefore, only remains to check the needed conditions for the boundedness of the multipliers $A_{\alpha + \gamma_k + (n - 1)/2 + \ell}$, for $\ell = 0, 1 \ldots N - 1$. But, this follows from Theorem 1.4 and can be reduced to the $L^p$-boundedness of $A_{\alpha + \gamma_k + (n - 1)/2}$ that is

$$\alpha + \gamma_k + (n - 1)/2 > (2\gamma_k + n - 1) \left| \frac{1}{p} - \frac{1}{2} \right| + \frac{1}{p}$$

or equivalently
(a) \( \alpha > 1 - 2\gamma_k - n + (2\gamma_k + n)/p \), if \( 1 < p \leq 2 \)
(b) \( \alpha > (2 - 2\gamma_k - n)/p \), if \( 2 \leq p < \infty \),
which is the desired statement of Theorem 1.5.

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