Critical and Non-critical $W_{2,4}$ Strings

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ABSTRACT

Nilpotent BRST operators for higher-spin $W_{2,s}$ strings, with currents of spins 2 and $s$, have recently been constructed for $s = 4, 5$ and 6. In the case of $W_{2,4}$, this operator can be understood as being the BRST operator for the critical $WB_2$ string. In this paper, we construct a generalised BRST operator that can be associated with a non-critical $W_{2,4}$ string, in which $WB_2$ matter is coupled to the $WB_2$ gravity of the critical case. We also obtain the complete cohomology of the critical $W_{2,4}$ BRST operator, and investigate the physical spectra of the $s = 5$ and $s = 6$ string theories.

* Supported in part by the U.S. Department of Energy, under grant DE-FG05-91ER40633.
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1. Introduction

There has been much progress in understanding $W$-string theories recently. $W$ strings are two-dimensional quantum field theories with non-linear local symmetry algebras, namely $W$ algebras, which are the higher-spin generalisations of the Virasoro algebra. There exist many higher-spin extensions of the Virasoro algebra, and the corresponding string theories are interesting subjects which may help us better to understand the fundamental properties of two-dimensional quantum field theories. There are various approaches to the quantisation of the usual bosonic string since the symmetry algebra, namely the Virasoro algebra, is linear. However, since $W$ algebras are generically non-linear and a physical gauge seems not to exist, BRST methods appear to provide the only viable approach to the quantisation of $W$-string theories. The characteristics of the $W$ string are therefore governed by the structure of the corresponding nilpotent BRST operator. Since $W$ algebras are non-linear, BRST operators can be complicated to construct, and in fact there may exist intrinsically different nilpotent BRST operators for a given $W$ algebra, which give rise to different $W$-string theories.

The simplest example of a $W$ algebra is the $W_3$ algebra of Zamolodchikov [1]. The algebra contains a spin-3 primary current $W$ in addition to the spin-2 energy-momentum tensor $T$. The first requirement of building a $W_3$ string is an anomaly-free theory of $W_3$ gravity. In [2], it was shown that such a theory could be obtained by standard BRST techniques. The result then leads on to the construction of a $W_3$ string [3,4,5]. A BRST operator for $W_3$ was first constructed in [6]; its nilpotency demands that the matter currents $T$ and $W$ generate the $W_3$ algebra with central charge $c = 100$ [6,7]. The detailed study of the $W_3$ string based on this BRST operator can be found in [5,8,9,10,11]. This $W_3$ string is sometimes called the "critical" $W_3$ string, for reasons that will be clarified presently.

In [12,13], a BRST operator for the so-called non-critical $W_3$ string was found. The non-critical $W_3$ string is a theory of $W_3$ matter coupled to $W_3$ gravity; it is a generalisation of two-dimensional matter coupled to gravity. The BRST operator of the non-critical $W_3$ string contains two copies of the $W_3$ algebra. One is the $W$-gravity, or Toda, contribution, which generalises the Liouville gravity sector of the non-critical bosonic, and the other is the matter contribution. That such a tensoring of two copies of the $W_3$ algebra can give rise to a sensible theory is quite non-trivial since the $W_3$ algebra is non-linear. The nilpotency of the BRST operator requires the total central charge of the Toda and matter sectors together to be $c = 100$.

Although we are following the conventional terminology of [12], and using the term "critical" to describe the original BRST operator of [6] involving only one copy of the $W_3$ algebra, whilst using the term "non-critical" to describe the BRST operator of [12,13] that involves two commuting copies of the $W_3$ algebra, it should be emphasised that both BRST operators are in fact critical in the sense that they are nilpotent operators built from a set of fields satisfying free-field OPEs. In the case of the ordinary bosonic string the term "non-critical" tends to be reserved for the situation where one of the fields (the Liouville field)
arises dynamically, with an exponential potential term, as a consequence of the Weyl anomaly associated with the non-nilpotence of the BRST operator built just from the ghosts and the matter fields. Although the “non-critical” $W_3$ BRST operator of [12,13] might provide the appropriate arena for studying the analogous emergence of dynamical Toda fields in $W_3$ gravity coupled to $W_3$ matter, in all of the discussions to date it has been used only as a description of a critical $W_3$ string theory with free-field quantisation rules. Further discussion of this point may be found in [14]. Even in the absence of the exponential potential terms, the “non-critical” $W_3$ BRST operator can be, and indeed is, different from the “critical” one. This can happen because, unlike the usual Virasoro case, the algebra is non-linear and so different kinds of BRST operator can occur.

A natural generalisation of the $W_3$-string is a higher-spin string with local spin-2 and spin-$s$ symmetries on the world-sheet, instead of spin-2 and spin-3. Critical BRST operators for such theories have been constructed for $s = 4, 5, 6$ [15]. Presumably they exist for all $s$, although the complexity rises rapidly with increasing $s$. In fact, it was found in [15] that there are two different BRST operators when $s = 4$, one operator when $s = 5$, and four different ones when $s = 6$. For each value of $s$, it appears that one of the BRST operators is associated with a unitary multi-scalar string theory [15]. It is these BRST operators with which we shall be concerned in this paper. We shall refer to the associated gauge theories as $W_{2,s}$ strings. The physical spectrum of the multi-scalar critical $W_{2,s}$ string is closely related to the lowest unitary $W_{(s-1)}$ minimal model [15,16].

The $W_{2,4}$ BRST operator is in fact a BRST operator for the $WB_2$ algebra. However, it should be remarked that for the higher $W_{2,s}$ BRST operators, there does not necessarily exist a corresponding closed $W_{2,s}$ algebra at the quantum level. For example, a closed algebra of spin-2 and spin-5 currents exists only at certain discrete values of central charge [17,18], which do not include the value that would be needed for criticality. Although a closed algebra of spin-2 and spin-6 currents, namely $WG_2$, exists for all values of central charge [19], the $W_{2,6}$ string that we are considering is not related to this algebra. (In fact two of the other three BRST operators with $s = 6$ constructed in [15] are related to the $WG_2$ algebra.) For all $s > 6$, it is known that closed algebras of spin-2 and spin-$s$ currents could exist for at most a finite number of central-charge values, which would presumably not include the critical values.

In this paper, we begin in section 2 by reviewing the critical $W_{2,4}$ string theory that was obtained in [15]. Then, in section 3, we construct a non-critical BRST operator for the $W_{2,4}$ string, generalising the $W_3$ results of [12,13,20]. We use this in section 4 to obtain some of the physical states, including some of the ghost-number zero ground-ring generators. It would be interesting to determine the complete cohomologies of the non-critical and critical $W_{2,4}$ strings along the lines of the results obtained in [21,11] for the $W_3$ string. For the non-critical case, this is a difficult problem, and in the present paper, in section 5, we only consider the simpler critical case. Here, we may use the methods of [11] to obtain the cohomology by
acting with powers of certain invertible physical operators on a basic set of physical states. A partial analysis for higher-spin $W_{2,s}$ strings in section 6 reveals interesting new features.

2. Review of the Critical $W_{2,s}$ String

It was shown in [10] that the BRST operator for the usual critical $W_3$ string could be brought into a simpler form by performing a non-linear transformation under which the ghosts $(b, c)$ and $(\beta, \gamma)$ for the spin-2 and spin-3 currents, and one of the matter fields, are mixed. In terms of the redefined fields, the BRST operator has a double grading, with respect to the ghost numbers for $(b, c)$ and $(\beta, \gamma)$ respectively. In [15], it was shown that this graded BRST operator can be generalised to one where the matter currents have spins 2 and $s$ rather than 2 and 3. Note that $\beta$ therefore has spin $s$, and $\gamma$ has spin $(1-s)$. The BRST operator for the spin-2 plus spin-$s$ string then takes the form [15]:

\[
Q_B = Q_0 + Q_1, \\
Q_0 = \oint dz c (T_{\varphi_1} + T_{\varphi_2} + T_{\gamma,\beta} + \frac{1}{2} T_{c,b}) , \\
Q_1 = \oint dz \gamma F(\varphi_1, \beta, \gamma) ,
\]

where the energy-momentum tensors are given by

\[
T_{\varphi_1} \equiv -\frac{1}{2} (\partial \varphi_1)^2 - \alpha, \\
T_{\varphi_2} \equiv -\frac{1}{2} (\partial \varphi_2)^2 - a, \\
T_{\gamma,\beta} \equiv -s \beta \partial \gamma - (s - 1) \partial \beta \gamma , \\
T_{c,b} \equiv -2 b \partial c - \partial b c ,
\]

The operator $F(\varphi_1, \beta, \gamma)$ has spin $s$ and ghost number zero. The BRST operator is graded, as discussed above, with $Q_0^2 = Q_1^2 = \{Q_0, Q_1\} = 0$. The first of these conditions is satisfied provided that the total central charge vanishes, i.e.

\[
0 = -26 - 2(6s^2 - 6s + 1) + 2 + 12\alpha^2 + 12a^2 .
\]

The remaining two nilpotency conditions determine the precise form of the operator $F(\varphi_1, \beta, \gamma)$ appearing in (2.3). Solutions for $s = 4, 5$ and 6 were found in [15]. (In fact in [15] it was found that whilst there is just one possible BRST operator for $s = 5$, there are two different BRST operators when $s = 4$, and four different ones when $s = 6$. However, only one BRST operator for each $s$, for which the $(\varphi_1, \beta, \gamma)$ system has the central charge $\frac{2(s-2)}{(s+1)}$ discussed above, seems to be associated with a unitary string theory. It is this choice that we shall be concentrating on in the present paper.)
In this paper, we shall principally be interested in the case \( s = 4 \). The BRST operator is then given by (2.1)–(2.7), with \( \alpha^2 = \frac{243}{20}, \ a^2 = \frac{121}{60} \), and [15]

\[
F(\varphi_1, \beta, \gamma) = (\partial \varphi_1)^4 + 4\alpha \partial^2 \varphi_1 (\partial \varphi_1)^2 + \frac{41}{5} (\partial^2 \varphi_1)^2 + \frac{124}{15} \partial^3 \varphi_1 \partial \varphi_1 + \frac{46}{135} \alpha \partial^4 \varphi_1 + 8(\partial \varphi_1)^2 \beta \partial \gamma - \frac{16}{9} \alpha \partial^2 \varphi_1 \beta \partial \gamma - \frac{32}{9} \alpha \partial \varphi_1 \beta \partial^2 \gamma - \frac{4}{3} \beta \partial^3 \gamma + \frac{16}{3} \partial^2 \beta \partial \gamma.
\]

(2.9)

The results for \( s = 5 \) and \( s = 6 \) may be found in [15].

Finally, we remark that the scalar field \( \varphi_2 \) appears in the BRST operator only via its energy-momentum tensor (2.5), and thus appears only in \( Q_0 \) but not \( Q_1 \). Consequently, there exist more general critical \( W_{2,4} \) strings in which \( T_{\varphi_2} \) is replaced by an arbitrary energy-momentum tensor \( T_{\text{eff}} \) that has the same central charge as \( T_{\varphi_2} \), namely \( c_{\text{eff}} = \frac{126}{5} \), and that commutes with \( \varphi_1 \). In particular, one can take \( T_{\text{eff}} \) to be realised in terms of a set of \( d \) scalar fields \( X^\mu \), with a background charge, giving a multi-scalar critical \( W_{2,4} \) string [15]. This procedure can be applied to the \( W_{2,s} \) strings for higher values of \( s \) also [15].

3. The Non-critical \( W_{2,4} \) String

The non-critical \( W_{2,4} \) string is a theory of \( W_{2,4} \) gravity coupled to a matter system on which the \( W_{2,4} \) algebra is realised. The \( W_{2,4} \) algebra was constructed in [17,18] in terms of Laurent modes by imposing the Jacobi identity. It is sometimes called \( WB_2 \), since it can be obtained by Hamiltonian reduction from the (non-simply-laced) algebra \( B_2 \). The algebra \( WB_2 \) can be re-expressed in terms of the spin-two and spin-four currents \( T \) and \( W \). Since \( W \) is primary under \( T \), the only OPE which we need give is \( W(z)W(w) \) [17,18]:

\[
W(z)W(w) \sim \left\{ \frac{2T}{(z-w)^6} + \frac{\partial T}{(z-w)^5} + \frac{3}{10} \frac{\partial^2 T}{(z-w)^4} + \frac{1}{15} \frac{\partial^3 T}{(z-w)^3} + \frac{1}{84} \frac{\partial^4 T}{(z-w)^2} + \frac{1}{560} \frac{\partial^5 T}{(z-w)} \right\}
+ b_1 \left\{ \frac{U}{(z-w)^4} + \frac{1}{2} \frac{\partial U}{(z-w)^3} + \frac{5}{36} \frac{\partial^2 U}{(z-w)^2} + \frac{1}{36} \frac{\partial^3 U}{(z-w)} \right\}
+ b_2 \left\{ \frac{W}{(z-w)^4} + \frac{1}{2} \frac{\partial W}{(z-w)^3} + \frac{5}{36} \frac{\partial^2 W}{(z-w)^2} + \frac{1}{36} \frac{\partial^3 W}{(z-w)} \right\}
+ b_3 \left\{ \frac{G}{(z-w)^2} + \frac{1}{2} \frac{\partial G}{(z-w)} \right\}
+ b_4 \left\{ \frac{A}{(z-w)^2} + \frac{1}{2} \frac{\partial A}{(z-w)} \right\}
+ b_5 \left\{ \frac{B}{(z-w)^2} + \frac{1}{2} \frac{\partial B}{(z-w)} \right\}
+ \frac{c/4}{(z-w)^8}
\]

(3.1)

where the (quasi-primary) composites \( U \) (spin 4), and \( G, A, B \) (all spin 6), are defined by

\[
U \equiv (TT) - \frac{3}{10} \partial^2 T, \quad G \equiv (\partial^2 TT) - \partial(\partial T T) + \frac{2}{9} \partial^2 (TT) - \frac{1}{42} \partial^4 T, \quad A \equiv (TU) - \frac{1}{6} \partial^2 U, \quad B \equiv (TW) - \frac{1}{6} \partial^2 W.
\]

(3.2)
with normal ordering of products of currents understood. The coefficients \(b_1, b_2, b_3, b_4, b_5\) are given by

\[
\begin{align*}
    b_1 &= \frac{42}{5c + 22}, \\
    b_2 &= \sqrt{\frac{54(c + 24)(c^2 - 172c + 196)}{(5c + 22)(7c + 68)(2c - 1)}}, \\
    b_3 &= \frac{3(19c - 524)}{10(7c + 68)(2c - 1)}, \\
    b_4 &= \frac{24(72c + 13)}{(5c + 22)(7c + 68)(2c - 1)}, \\
    b_5 &= \frac{28}{3(c + 24)}b_2.
\end{align*}
\]  

We are now in a position to present our results for the non-critical \(W_{2,4}\) string. The BRST operator takes the form

\[
\begin{align*}
    Q_B &= Q_0 + Q_1, \\
    Q_0 &= \oint dz c(T_{\varphi_1} + T_{\varphi_2} + T_M + T_{\gamma,\beta} + \frac{1}{2}T_{c,b}), \\
    Q_1 &= \oint dz \gamma F(\varphi_1, \beta, \gamma, T_M, W_M),
\end{align*}
\]  

where the matter currents \(T_M\) and \(W_M\) generate the \(WB_2\) algebra, the ghost energy-momentum tensors are given by (2.6) and (2.7), with \(s = 4\) in (2.6), \(T_{\varphi_1} = -\frac{1}{2}(\partial\varphi_1)^2 - \alpha \partial^2\varphi_1\), and \(T_{\varphi_2} = -\frac{1}{2}(\partial\varphi_2)^2 - a \partial^2\varphi_2\). The \(\varphi_1\) and \(\varphi_2\) fields are the “Liouville fields,” or more properly, Toda fields, of the \(W\)-gravity sector. The BRST operator generalises the one given in [20] for the non-critical \(W_3\) string [12,13]. Again \(Q_B\) is graded, with \(Q_0^2 = Q_1^2 = \{Q_0, Q_1\} = 0\).

Our strategy for obtaining the BRST operator consists of writing down the most general possible form for the operator \(F(\varphi_1, \beta, \gamma, T_M, W_M)\), which has conformal weight 4 and ghost number 0, and imposing the conditions for nilpotence of \(Q_B\). The condition \(Q_0^2 = 0\) implies that the total central charge vanishes, \(i.e\.

\[
12\alpha^2 + 12a^2 + c_M - 170 = 0 ,
\]  

where \(c_M\) is the central charge of the matter. We find that the remaining nilpotence conditions \(\{Q_0, Q_1\} = Q_1^2 = 0\) imply

\[
2(\alpha - 2a)(\alpha - 3a) + 1 = 0 ,
\]
and they determine the coefficients of the terms in \( F(\varphi_1, \beta, \gamma, T_M, W_M) \) to be such that

\[
Q_1 = \int dz \gamma \left[ (\partial \varphi_1)^4 + 4 \alpha (\partial \varphi_1)^2 \partial^2 \varphi_1 + \frac{1}{t} (130 - c_M - 8\alpha^2) \partial^2 \varphi_1 \partial^2 \varphi_1 
\right.
\]

\[
+ \frac{1}{6} (-242 + c_M + 24\alpha^2 \partial^4 \varphi_1 + \frac{1}{180\alpha} (-43092 + 252c_M + 2150\alpha^2 + 5c_M \alpha^2 + 120\alpha^4) \partial^4 \varphi_1
\]

\[
+ \frac{3}{5\alpha} (-171 + c_M + 20\alpha^2) \partial \varphi_1 \partial \beta \partial \gamma + \frac{4}{5\alpha} (513 - 3c_M - 40\alpha^2) \partial^2 \varphi_1 \beta \partial \gamma
\]

\[
+ \frac{1}{2} (-196 + c_M + 16\alpha^2) \beta \partial^3 \gamma + \frac{1}{15} (809 - 4c_M - 60\alpha^2) \partial^2 \beta \partial \gamma - 4 T_M \beta \partial \gamma
\]

\[
- 2 (\partial \varphi_1)^2 T_M + \frac{1}{90} (171 - c_M - 20\alpha^2) \partial \varphi_1 \partial T_M + \frac{2}{30} (-171 + c_M + 10\alpha^2) \partial^2 \varphi_1 T_M
\]

\[
+ \frac{2}{5} \left( \frac{-5051 + 36c_M + 420\alpha^2}{(22 + c_M)} \right) (T_M)^2 + \frac{1}{5} \left( \frac{4091 + 215c_M - c_M^2 - 340\alpha^2 - 20c_M \alpha^2}{(22 + 5c_M)} \right) \partial^2 T_M
\]

\[
+ (240\alpha^2 + 17c_M - 2902) \sqrt{\frac{(7c_M + 68)(4c_M - 2)(c_M + 24)}{75(22 + 5c_M)}(196 - 172c_M + c_M^2)} W_M \right].
\]

(3.9)

It is convenient to express the background charges \( \alpha \) and \( a \), which are related by (3.8), in terms of a parameter \( t \):

\[
\alpha = \frac{2}{t} + \frac{3t}{2}, \quad a = \frac{1}{t} + \frac{t}{2}.
\]

(3.10)

From (3.7), it then follows that

\[
c_M = 86 - \frac{60}{t^2} - 30 t^2.
\]

(3.11)

Note that if we were to set \( c_M = 0 \), in which case \( T_M \) and \( W_M \) would be null fields that could be set to zero, (3.5) and (3.9) would reduce to the critical \( W_{2,4} \) BRST operator that we discussed in section 2. Indeed, (3.11) would then give \( t^2 = \frac{5}{3} \) or \( \frac{6}{5} \). The first of these implies that \( \alpha^2 = \frac{343}{20}, \ a^2 = \frac{121}{60} \), and (3.9) reduces to the BRST operator given by (2.9). The second solution, \( t^2 = \frac{6}{5} \), gives \( \alpha^2 = \frac{364}{30}, \ a^2 = \frac{32}{15} \); this was also found in [15], and appears to be associated with a critical string theory that is non-unitary in the multi-scalar case, in that some of the longitudinal modes of spacetime-excited states have negative norms. As a two-scalar theory, however, there is presumably no unitarity problem. The non-critical BRST operator that we have constructed here has the property that it interpolates between the two critical \( W_{2,4} \) BRST operators found in [15]. The fact that there are two inequivalent solutions of \( c_M = 0 \) is related to the fact that the \( B_2 \) algebra is not simply laced. Both of the critical BRST operators are associated with the \( WB_2 \) algebra.

In order to build a non-critical \( W_{2,4} \) string theory, we need an explicit realisation for the matter currents \( T_M \) and \( W_M \) appearing in (3.5) and (3.9) above. A two-scalar realisation of
the \( WB_2 \) algebra was given in [17], and takes the form

\[
T_M = T_{X_1} + T_{X_2} = -\frac{1}{2} (\partial X_1)^2 + i \left( \frac{2}{t} - \frac{3t}{2} \right) \partial^2 X_1 - \frac{1}{2} (\partial X_2)^2 + i \left( \frac{1}{t} - \frac{t}{2} \right) \partial^2 X_2 ,
\]

\[
W_M = \lambda \left[ g_1 (\partial X_1)^4 + g_2 \partial^2 X_1 (\partial X_1)^2 + g_3 (\partial^2 X_1)^2 + g_4 (\partial^3 X_1) \partial X_1 + g_5 \partial^4 X_1 g_6 (\partial X_1)^2 T_{X_2} + g_7 (\partial X_1) (\partial T_{X_2}) + g_8 (\partial^2 X_1) T_{X_2} + g_9 (T_{X_2})^2 + g_{10} \partial^2 T_{X_2} \right] ,
\]

where the coefficients \( g_i \) are given by

\[
\begin{align*}
&g_1 = \frac{1}{16} t^2(3-t^2)(-32+27 t^2) , \\
g_2 = -\frac{i}{8} t(-3+t^2)(-4+3 t^2)(-32+27 t^2) , \\
g_3 = \frac{1}{16} (-3+t^2)(-152+336 t^2-246 t^4+63 t^6) , \\
g_4 = \frac{1}{4} (-3+t^2)(-60+144 t^2-115 t^4+30 t^6) , \\
g_5 = \frac{i}{8} \left( -3+t^2 \right)(-4+3 t^2)(-60+144 t^2-115 t^4+30 t^6) , \\
g_6 = \frac{9}{4} t^2(-68+113 t^2-41 t^4) , \\
g_7 = -\frac{i}{2} (1+t) t (1+t) (150-226 t^2+75 t^4) , \\
g_8 = -\frac{i}{4} t(-216+464 t^2-305 t^4+69 t^6) , \\
g_9 = \frac{1}{4} t^2(3-t^2)(-32+27 t^2) , \\
g_{10} = \frac{1}{8} (240-724 t^2+865 t^4-465 t^6+90 t^8) ,
\end{align*}
\]

and \( \lambda \) is a normalisation constant given by

\[
\lambda^{-2} = \frac{1}{8 t^2}(3-t^2)(-7+3 t^2)(-5+3 t^2)(-2+3 t^2)(-5+4 t^2)(-8+5 t^2)(-6+5 t^2)(-6+7 t^2)(150-226 t^2+75 t^4) .
\]

The realisation (3.12) generates the \( WB_2 \) algebra with central charge given by (3.11). Substituting (3.12) into (3.9) gives the final result for \( Q_1 \) for the non-critical \( W_{2,4} \) string.

It should be emphasised that in the case of the critical \( W_{2,4} \) string in section 2, the fact that there exists a closed \( WB_2 \) algebra at the quantum level did not appear to play an essential rôle in the construction of the BRST operator. Indeed, as we remarked earlier, the analogous construction of a critical \( W_{2,s} \) BRST operator can be carried out even for values of \( s \) for which no closed algebra (at least not with the correct central charge) exists [15]. For the non-critical BRST operator of this section, on the other hand, it is essential that the matter currents \( T_M \) and \( W_M \) do generate a closed algebra, namely \( WB_2 \), at the quantum level.
4. Physical States in the Non-critical $W_{2,4}$ String

We have already remarked that the $W_{2,4}$ algebra can be obtained from Hamiltonian reduction for the algebra $B_2$. For convenience we shall begin this section by setting up some notation. We take the two simple roots of the $B_2$ algebra to be

$$e_1 = (0, 1) \quad e_2 = (1, -1) .$$

The associated fundamental weights $\lambda_i$, defined by $\frac{2e_i \cdot \lambda_j}{(e_i, e_i)} = \delta_{ij}$ are

$$\lambda_1 = \left(\frac{1}{2}, \frac{1}{2}\right), \quad \lambda_2 = (1, 0) .$$

Since $B_2$ is not simply-laced, we need also to introduce the simple co-roots, defined by $e_i^\vee = \frac{2e_i}{(e_i, e_i)}$, and the fundamental co-weights $\lambda_j^\vee$, defined by $\frac{2e_j^\vee \cdot \lambda_j^\vee}{(e_j^\vee, e_j^\vee)} = \delta_{ij}$:

$$e_1^\vee = (0, 2) , \quad e_2^\vee = (1, -1) ,$$

$$\lambda_1^\vee = (1, 1) , \quad \lambda_2^\vee = (1, 0) .$$

The Weyl vector and the Weyl co-vector are consequently

$$\rho = \left(\frac{3}{2}, \frac{1}{2}\right) , \quad \rho^\vee = (2, 1) .$$

In this language, the $W_{2,4}$ matter energy-momentum tensor in (3.12) becomes

$$T_M = -\frac{1}{2} \partial X \cdot \partial X - i(t \rho - \frac{1}{t} \rho^\vee) \cdot \partial^2 X .$$  

Note that since $\rho$ and $\rho^\vee$ are independent vectors, there is no choice of the parameter $t$ for which the background charge vector in (4.6) vanishes. This contrasts with the situation for the $W_3$ algebra, where, since $A_2$ is simply laced, there is no distinction between $\rho$ and $\rho^\vee$. A related difference is that here one can, for purely real or purely imaginary $t$, only achieve matter central charges within certain ranges. Specifically, as may be seen from (3.11), if $t$ is real then we must have $c_M \leq 86 - 60\sqrt{2} \approx 1.147$, whilst if $t$ is imaginary then $c_M \geq 86 + 60\sqrt{2} \approx 170.853$. We note also that the energy-momentum tensor in the gravity sector can be written as $T_L = T_{\varphi_1} + T_{\varphi_2} = -\frac{1}{2} \partial \varphi \cdot \partial \varphi - (t \rho + \frac{1}{t} \rho^\vee) \cdot \partial^2 \varphi$.

One can easily verify that there are four screening currents, i.e. vertex operators that are total derivatives under the matter currents (3.12):

$$S_j^+ = e^{-i \alpha_+ e_j \cdot X} , \quad S_j^- = e^{-i \alpha_- e_j^\vee \cdot X} ,$$

where

$$\alpha_+ = t , \quad \alpha_- = -\frac{1}{t} .$$
By following standard arguments, we find that vertex operators
\[ V_{r_i; s_i}^{M} = e^{i p^M X} \]
describing primaries of the \( W_{2,4} \) minimal models have momenta given by
\[ p^M = \sum_{i=1}^{2} \left[ (r_i - 1) \alpha_+ \lambda_i + (s_i - 1) \alpha_- \lambda_i^\vee \right]. \] (4.10)

We also define vertex operators
\[ V_{r_i; s_i}^{L} = e^{p^L \varphi} \]
for the Liouville sector, with momenta given by
\[ p^L = \sum_{i=1}^{2} \left[ (r_i - 1) \alpha_+ \lambda_i - (s_i - 1) \alpha_- \lambda_i^\vee \right]. \] (4.12)

We begin our discussion of physical states by looking at level \( \ell = 0 \), i.e. tachyons. The form of these physical operators is
\[ U = c \partial^2 \gamma \partial \gamma \gamma e^{p^L_1 \varphi_1 + p^L_2 \varphi_2 + i p^M_1 X_1 + i p^M_2 X_2}. \] (4.13)

For operators of this kind, where the \((b, c)\) dependence occurs just as an overall factor of \(c\), the grading of \(Q_B\) implies that we may consider the \(Q_0\) and \(Q_1\) parts of the BRST operator separately. Acting with \(Q_0\) gives the mass-shell condition, which can be written as
\[ (\hat{p}^L_1)^2 + (\hat{p}^L_2)^2 = (\hat{p}^M_1)^2 + (\hat{p}^M_2)^2, \] (4.14)
where the hatted momenta are shifted by the background charges:
\[ \hat{p}^L_1 \equiv p^L_1 + \alpha^L, \quad \hat{p}^L_2 \equiv p^L_2 + \alpha^L, \quad \hat{p}^M_1 \equiv p^M_1 + \alpha^M, \quad \hat{p}^M_2 \equiv p^M_2 + \alpha^M. \] (4.15)

Here, \(\alpha^L = \alpha\) and \(a^L = a\) given by (3.10), and the background charges for the matter sector, which can be read off from \(T_M\) in (3.12), are
\[ \alpha^M = \frac{3t}{2} - \frac{2}{t}, \quad a^M = \frac{t}{2} - \frac{1}{t}. \] (4.16)

Acting with \(Q_1\) on the operator (4.13) gives the further physical-state condition
\[ (\hat{p}^L_1 - \hat{p}^M_1)(\hat{p}^L_1 + \hat{p}^M_1)(\hat{p}^L_2 - \hat{p}^M_2)(\hat{p}^L_2 + \hat{p}^M_2) = 0. \] (4.17)

There are eight classes of tachyons, corresponding to the four different roots of (4.17) with, as a consequence of (4.14), \(\hat{p}^L_2 = \pm \hat{p}^M_2\) (for each of the first two roots of (4.17)) or
\[ \hat{p}_2^L = \pm \hat{p}_1^M \] (for each of the remaining two roots) respectively. The physical operators (4.13) can conveniently be written in terms of the vertex operators (4.9) and (4.11) in the eight cases:

\[
\begin{align*}
t_1 &= f V^L_{-r_1,-r_2;s_1,s_2}, & t_2 &= f V^L_{r_1,-r_2;-s_1,2s_1+s_2}, \\
t_3 &= f V^L_{r_1,r_2;-s_1,-s_2}, & t_4 &= f V^L_{-r_1,r_1+r_2;s_1,-2s_1-s_2}, \\
t_5 &= f V^L_{-r_1-2r_2,r_2;s_1+s_2,-s_2}, & t_6 &= f V^L_{r_1+2r_2,-r_1-r_2;-s_1-s_2,2s_1+s_2}, \\
t_7 &= f V^L_{r_1+2r_2,-r_2;-s_1-s_2,s_2}, & t_8 &= f V^L_{-r_1-2r_2,r_1+r_2;s_1+s_2,-2s_1-s_2},
\end{align*}
\] (4.18)

where \( f = c \partial^2 \gamma \partial \gamma V^M_{r_1,r_2;s_1,s_2} \) in each case. Note that there is some redundancy in the parametrisation of the momenta in (4.18), since the physical-state conditions (4.14) and (4.17) place two conditions on the four momentum components in (4.13). Thus there are really just two independent free parameters in the tachyon solutions. Nonetheless, as in [12] we find that the parametrisation (4.18) is a useful one.

Turning now to higher-level physical states, we expect that there should exist a ground-ring structure of physical operators at ghost number zero, analogous to those that exist in the two-scalar Virasoro string [22], two-scalar \( W_3 \) string [23], and non-critical \( W_3 \) string [12]. Indeed, in [15] it was found that there exists a physical operator at level \( \ell = 10 \) and ghost number \( G = 0 \) in the multi-scalar (critical) \( W_{2,4} \) string, which has zero momentum in the effective spacetime. This operator, \( D \), is associated with the very simple screening current \( S \equiv \oint dw b(w)D = \beta e^{\frac{2}{9} r^2 V^M_{r_1,r_2;s_1,s_2}} \) [15], which satisfies \( \{Q_B, S\} = \partial D \). This physical operator and screening current of course also exist in the special case of the two-scalar critical \( W_{2,4} \) string.

In the \( W_3 \) string, ground-ring operators exist in the non-critical case [12] at the same level number, \( \ell = 6 \), as ghost-number-zero discrete operators in the critical \( W_3 \) string [23]. Thus it is natural, by analogy, to begin our search for ground-ring operators in the non-critical \( W_{2,4} \) string at level \( \ell = 10 \). They can be found by writing down the most general structure at this level and ghost number, and then requiring that \( Q_B \) given by (3.4), (3.5), (3.9), (3.12) annihilate it. In view of the complexity of some of the calculations, it can be advantageous in practice to begin by constructing the most general possible form for the discrete state \( D \), then acting with \( \oint dw b(w) \) to obtain the associated screening current \( S \), and then solving the equation \( \{Q_B, S\} = \partial D \). By using this procedure one has only to calculate the OPE of the BRST current with the relatively-simple screening current \( S \), rather than with the more complicated physical operator \( D \). At level \( \ell = 10 \), we find that there are two discrete physical operators in the non-critical \( W_{2,4} \) string. The first of these, which we shall call \( x \), is associated with the screening current \( S_x \) given by

\[ S_x = \beta V^L_{1112} V^M_{1112}. \] (4.19)

This is the generalisation of the \( \ell = 10 \) discrete operator of the critical \( W_{2,4} \) string that we mentioned above. The second \( \ell = 10 \) discrete physical operator of the non-critical \( W_{2,4} \)
string, which we shall call $y$, has a more complicated structure. The associated screening current $S_y$ takes the form

$$S_y = (\beta + \cdots) V_{2111}^L V_{2111}^M ,$$

(4.20)

where \(\ldots\) indicates 34 omitted terms, all of which have spin 4 and ghost number $-1$. The coefficients, which we have calculated, are $t$-dependent. We shall not present them here owing to the complexity of the expression.

By analogy with the non-critical $W_3$ string, where there are four ground-ring operators for generic values of the matter central charge [12], we might expect that there should be more ground-ring generators in our $W_{2,4}$ case, in addition to the two described above. To find them, we note that a feature of the ground-ring generators of the Virasoro and the $W_3$ strings is that they can normal order with tachyonic physical operators to give tachyonic physical operators again. For example, in the two-scalar Virasoro string there are two branches of tachyons, corresponding to the two solutions, $\hat{p}_L = \pm \hat{p}_M$, of the mass-shell condition $\hat{p}_L^2 = \hat{p}_M^2$. Of the two $\ell = 2$ ground-ring generators, one can normal order with one of the tachyon branches to give a tachyon in the same branch, whilst the other ground-ring generator can do the same for the tachyons of the other branch. In our $W_{2,4}$ case we have the eight branches of tachyons $t_i$ listed in (4.18), corresponding to the eight classes of solutions of (4.14) and (4.17). It is easy to see that the $x$ operator associated with (4.19) can map $t_1$ and $t_2$ into themselves, whilst $y$, associated with (4.20), can map $t_3$ and $t_7$ into themselves. By considering what other discrete operators might be able to map tachyons in other branches, we find that the vertex operator $V_{1211}^L V_{1211}^M$ has momentum that could map the tachyons $t_3$ and $t_4$, whilst the vertex operator $V_{1121}^L V_{1121}^M$ has momentum that could map $t_5$. From the spin of these vertex operators, it is easy to see that they would have to be associated with $\ell = 11$ states. In fact we find that the first, which we shall call $\tilde{x}$, does indeed exist, and has a relatively simple structure, with its screening current $S_{\tilde{x}}$ given by

$$S_{\tilde{x}} = \left(\beta \partial\varphi_1 + \left(\frac{1}{2t} + \frac{t}{2}\right) \partial\beta\right) V_{1211}^L V_{1211}^M .$$

(4.21)

The second, which we shall call $\tilde{y}$, is much more complicated, with $S_{\tilde{y}}$ of the form

$$S_{\tilde{y}} = (\beta \partial\varphi_1 + \cdots) V_{1121}^L V_{1121}^M .$$

(4.22)

We have not been able to carry out the computation in this case, but it seems highly plausible that such a ground-ring generator exists.

In the case of the non-critical $W_3$ string, it was shown in [21] that whilst there are just four ground-ring operators for generic values of the matter central charge [12], the number enlarges to six in the special case $c_M = 2$, which corresponds to vanishing background charge in the matter sector. In the case of $W_{2,4}$ on the other hand, we have already noted that there is no choice of the parameter $t$ in (4.6) for which the background charge totally vanishes in the
matter sector. It is not clear therefore whether one should expect any additional ground-ring operators at any special values of $c_M$.

In [12,21] the structure of the higher-level states with non-standard ghost structure in the non-critical $W_3$ string is discussed. In particular, the subset of physical states that exist in the special case of $c_M = 2$ can be grouped together into multiplets whose members are related by transformations under the Weyl group of $SU(3)$. A similar discussion can be given for the $W_{2,4}$ string, where in this case the Weyl group is that of the underlying $B_2$ algebra. It acts on the momenta of tachyonic vertex operators according to the rule

$$V_{r_1; -s_1} V_{r_1; s_1} \rightarrow V_{r_1; w_*(s_1)} V_{r_1; s_1},$$

(4.23)

with the Liouville momentum (4.12) transformed according to the rule

$$\sum_{i=1}^{2} (r_i - 1)\alpha_+ \lambda_i - (s_i - 1)\alpha_- \lambda_i^V \rightarrow \sum_{i=1}^{2} (r_i - 1)\alpha_+ \lambda_i - s_i \alpha_- w*(\lambda_i^V) + \alpha_- \lambda_i^V.$$  (4.24)

Here, the Weyl transformations of the fundamental co-weights are dual to the standard Weyl transformations of the simple co-roots, and are thus generated by $w_1$ and $w_2$ defined by

$$w_1 * (\lambda_i^V) = 2\lambda_i^V - \lambda_1^V, \quad w_1 * (\lambda_i^V) = \lambda_2^V,$$

$$w_2 * (\lambda_i^V) = \lambda_1^V, \quad w_2 * (\lambda_i^V) = \lambda_2^V - \lambda_2^V.$$  (4.25)

From (4.24), it follows that the action of the Weyl group on the $(s_1, s_2)$ indices of $V_{r_1; -s_1}$ is given by

$$w_1 * (s_1, s_2) = (-s_1, 2s_1 + s_2), \quad w_2 * (s_1, s_2) = (s_1 + s_2, -s_2).$$  (4.26)

The elements $\{1, w_1, w_2, w_1 w_2, w_2, w_1, w_2 w_1, w_1 w_2 w_1, w_1 w_2 w_1 w_2\}$ generate the Weyl group of $B_2$. Starting from the tachyons $t_3$ at level $\ell = 0$, the levels of the 8 Weyl-related states under the action (4.23) are $\{0, r_1s_1, r_2s_2, r_1s_1 + r_1s_2 + r_2s_2, r_1s_1 + 2r_2s_1 + r_2s_2, (r_1 + 2r_2)(s_1 + s_2), (r_1 + r_2)(2s_1 + s_2), 2r_1s_1 + 2r_2s_1 + r_1s_2 + 2r_2s_2\}$. The ghost numbers of the operators are $\{4, 3, 3, 2, 2, 1, 1, 0\}$. The last case here corresponds to the ground-ring operators. From (4.26), it follows that the ground-ring operators have the form $V_{r_1; s_1} V_{r_1; s_1}^M$. The basic ground-ring generators correspond to $(r_1, r_2, s_1, s_2) = (2, 1, 1, 1), (1, 2, 1, 1), (1, 1, 2, 1)$ and $(1, 1, 1, 2)$, with levels $\ell = 10, 11, 11, 10$ respectively. In fact, these correspond to the physical operators $y, \tilde{x}, \tilde{y}$ and $x$ that we discussed previously.

The eight classes of tachyon in (4.18) can be related to one another under an action of the Weyl group that we have just been discussing. First, we define the Weyl group action on the $r_1$ indices of a vertex operator, in a similar fashion to the action on $s_1$ indices described above:

$$w_1 * (r_1, r_2) = (-r_1, 2r_1 + r_2), \quad w_2 * (r_1, r_2) = (r_1 + r_2, -r_2).$$  (4.27)
Defining \((W, W') V^L_{r; s_i} \equiv V^L_{W^*(r_i), W'^*(s_i)}\) as the action of Weyl-group elements \(W = w_j w_2 \cdots\) and \(W' = w_k w_2 \cdots\), acting on the Liouville vertex operators, we find that we may write the tachyons in (4.18) as follows:

\[
\begin{align*}
t_1 &= (w_1 w_2 w_1 w_2, w_1 w_2 w_1 w_2) t_3, \\
t_3 &= (1, 1) t_3, \\
t_5 &= (w_1 w_2 w_1, w_1 w_2 w_1) t_3, \\
t_7 &= (w_2, w_2) t_3,
\end{align*}
\]

\[(4.28)\]

We have looked also at the physical states at level \(\ell = 1\), and find the following physical operators

\[
\begin{align*}
c \partial \gamma \gamma V^L_{-2-r, 1, s, 1} V^M_{r, 1, s, 1} &\sim (1, w_2) t_5 \\
c \partial \gamma \gamma V^L_{r, 1, -1-s, 1} V^M_{r, 1, s, 1} &\sim (1, w_2) t_3 \\
c \partial \gamma \gamma V^L_{2+r, -1-r, -1-s, 1+2s} V^M_{r, 1, s, 1} &\sim (w_1, w_1 w_2) t_5 \\
c \partial \gamma \gamma V^L_{-r, 1+r, 1+s, -1-2s} V^M_{r, 1, s, 1} &\sim (w_1, w_1 w_2) t_3 \hspace{1cm} (4.29)
\end{align*}
\]

\[
\begin{align*}
c \partial \gamma \gamma V^L_{2r-1-r, 1-s, 1} V^M_{1-2r, r, 1-s, s} &\sim (w_2, w_2 w_1) t_6 \\
c \partial \gamma \gamma V^L_{1-2r, r, s-1-2s} V^M_{1-2r, r, 1-s, s} &\sim (w_2, w_2 w_1) t_7 \\
c \partial \gamma \gamma V^L_{2-2r, r, s-1, s} V^M_{2-2r, r, 1-s, s} &\sim (w_2, w_2 w_1) t_7 \\
c \partial \gamma \gamma V^L_{2r-2, r, s-1, s} V^M_{2-2r, r, 1-s, s} &\sim (w_2, w_2 w_1) t_7 \\
c \partial \gamma \gamma V^L_{-3, 2r, -1} V^M_{1, 1, 1, 1} &\sim (w_2, w_2 w_1) t_2 \\
c \partial \gamma \gamma V^L_{3, -1, -2, 3} V^M_{1, 1, 1, 1} &\sim (w_2, w_2 w_1) t_3
\end{align*}
\]

As in the case of the tachyons, there is redundancy in the parametrisation of the momentum (in each case, \(r\) and \(s\) occur in a fixed combination, so there is really only one continuous parameter here), but nonetheless the parametrisation is a convenient one. In addition, there are three more physical operators of the form \((c \partial \gamma \gamma + \lambda \partial^2 \gamma \partial \gamma \gamma) V^L V^M\), with discrete
momenta, where the vertex operators are
\begin{align*}
V_{1,-1-r,1,s}^L V_{1,r,1,s}^M &\sim (1, w_1) t_2 \\
V_{1,r,1,-2-s}^L V_{1,r,1,s}^M &\sim (1, w_1) t_3 \\
V_{1,-1,1,-1}^L V_{1001}^M &\sim (1, w_1) t_6
\end{align*}

In both (4.29) and (4.30), the expression to the right of the \( \sim \) symbol indicates how the momentum of the state may be related to that of a tachyon by a Weyl-group transformation, according to the rule \((W, W') V_{ri;si}^L \equiv V_{W*(ri);W'(si)}^L\).

5. The Cohomology of the Critical \( W_{2,4} \) String.

One would like to have a systematic procedure for determining the spectra of physical states in the various \( W \)-string theories. For the non-critical \( W_{2,4} \) string of section 4, this would be a very complicated problem, which we shall not attempt to solve here. Methods analogous to those used in [24] for the two-scalar Virasoro string, and in [21] for the non-critical \( W_3 \) string, could presumably be used. For the critical \( W_{2,4} \) string on the other hand, we can use a method that was recently discovered for solving the cohomology of the one-scalar Virasoro string and the two-scalar critical \( W_3 \) string [11]. The basic idea is similar to that of the ground-ring construction for the two-scalar Virasoro string [22], in that higher-level physical operators are obtained by acting with powers of certain generators (which are themselves physical operators). However, there are important differences from the case of the two-scalar string. First of all, in our case the generators will be physical operators with non-zero ghost numbers; these are the natural objects to use in a theory where the spectrum of physical states fans out over a wider and wider range of ghost numbers as one goes to higher and higher levels. Secondly, and most crucially, the generating operators will have inverses. A consequence of this is that, when powers of the generators are normal ordered with any physical operator, the result is guaranteed to be a BRST non-trivial physical operator. Since the generators carry momentum, this means that all physical operators of the theory can be mapped into physical operators in a certain fundamental cell in the momentum plane, where an exhaustive construction of all possible physical operators can be carried out. Having done this, acting with all possible powers of the generators will yield the entire cohomology of physical states [11].

We shall begin our discussion with the two-scalar critical \( W_{2,4} \) string described in section 2. Many physical states for this theory were found in [15]; a characteristic feature is that the momenta in the vertex operators \( e^{p_1 \varphi_1 + p_2 \varphi_2} \) of physical states are always quantised:
\begin{equation}
(p_1, p_2) = \left( \frac{1}{27} \alpha k_1, \frac{1}{11} a k_2 \right),
\end{equation}

15
where \( k_1 \) and \( k_2 \) are integers. For this reason, we find it convenient to characterise the momenta of the physical states by \((k_1, k_2)\), and we shall commonly call these the momenta. The mass-shell condition for a physical state at level \( \ell \) therefore implies the relation

\[
(k_1 + 27)^2 + (k_2 + 11)^2 = 10(12\ell + 1) .
\]

(5.2)

A necessary condition for any physical operator is that its momentum should satisfy (5.2) for integer \( k_1, k_2 \) and \( \ell \). Thus candidates for the two invertible physical operators that could generate the cohomology must have momenta \((k_1, k_2)\) for which \((-k_1, -k_2)\) also satisfies (5.2). They must also be able to normal order with all other physical operators. It is not hard to see that the simplest two possibilities are to have operators, which we shall call \( X \) and \( Y \), with momenta \((30, 0)\) and \((0, 30)\) respectively. From (5.2), it follows that \( X \) has level \( \ell = 28 \), and \( Y \) has level \( \ell = 20 \). Their inverses, \( X^{-1} \) and \( Y^{-1} \), have momenta \((-30, 0)\) and \((0, -30)\), and occur at levels \( \ell = 1 \) and \( \ell = 9 \) respectively.

Despite its high level number, the \( X \) operator at \( \ell = 28 \) is easily constructed. Its associated screening current,

\[
S_X = \oint dw b(w)X,
\]

takes the simple form

\[
S_X = \partial^3 \beta \partial^2 \beta \partial \beta e^{\frac{10}{9} \alpha \varphi_1} .
\]

(5.3)

It is quite straightforward now to verify that \([Q_B, S_X] = \partial X\), thus establishing that \( X \) is a physical operator. The form of \( X \) itself is quite complicated, and we shall not present it here. We just remark that, as may be seen from (5.3), \( X \) has ghost number \( G = -3 \). The inverse of the \( X \) operator is the very simple level 1 physical operator, with ghost number \( G = 3 \);

\[
X^{-1} = \left( c \partial \gamma \partial \gamma - \frac{28}{15} \partial^2 \gamma \partial \gamma \right) e^{-\frac{10}{9} \alpha \varphi_1} .
\]

(5.4)

It is a straightforward matter to verify that indeed the normal-ordered product of \( X^{-1} \) and \( X \) gives the identity (up to an overall non-vanishing constant factor, whose precise value is unimportant in the subsequent argument). Another way of obtaining the \( X \) operator is by noting that there exists a \( G = 0 \) physical operator \( x \) at \( \ell = 10 \), with momentum \((6, 0)\), with corresponding screening current \( S_x = \beta e^{\frac{2}{15} \alpha \varphi_1} \) [15]. By acting appropriately on \( x \) with four of these screening currents (and a ghost booster \([Q_B, \varphi_1]\)), one can construct the \( \ell = 28 \) operator \( X \) that we have described above. The detailed calculation would be quite involved, with care being needed to handle the multiple contour integrals properly, and in practice the direct verification that \([Q_B, S_X] = \partial X\) is much simpler.

The \( Y \) operator, and its associated screening current \( S_Y \), is much more complicated, and in fact we have not yet managed to compute it. The reason for this is that its ghost number is considerably higher (in fact \( G = -1 \)) than the lowest possible one at level \( \ell = 20 \), and also, unlike the screening current (5.3) for the \( X \) operator, \( S_Y \) will involve the \((b, c)\) ghosts. Thus there are a very large number of terms in the expression for \( Y \). Its inverse, on the
other hand, at level $\ell = 9$ with $G = 1$, is quite simple and was in fact already constructed in [15]. It takes the form

$$Y^{-1} = c U e^{-\frac{30}{\pi} a \varphi^2},$$

(5.5)

where $U$ is the spin-3 current

$$U = \frac{5}{3} (\partial \varphi_1)^3 + 5\alpha \partial^2 \varphi_1 \partial \varphi_1 + \frac{25}{4} \partial^3 \varphi_1 + 20 \partial \varphi_1 \beta \partial \gamma$$

$$+ 12 \partial \varphi_1 \partial \beta \gamma + 12 \partial^2 \varphi_1 \beta \gamma + 5 \alpha \partial \beta \partial \gamma + 3 \alpha \partial^2 \beta \gamma.$$  

(5.6)

Interestingly enough, as was shown in [16], the spin-3 current $U$ and the energy-momentum tensor $T = T_{\varphi_1} + T_{\gamma,\beta}$ form the spin-3 and spin-2 currents of a realisation of the $W_3$ algebra at $c = \frac{4}{5}$. For now, we shall proceed by assuming that the $\ell = 20$ physical operator exists, and that its normal-ordered product with $Y^{-1}$ is a non-zero constant, which may be taken to be unity by an appropriate rescaling.

It was shown in [11] that if an invertible physical operator is normal ordered with any BRST non-trivial physical operator $V$, then the result is itself a BRST non-trivial physical operator. This is because, modulo BRST-trivial terms that do not affect the argument, if one normal orders, for example, the invertible operator $X$ with $V$, then one can recover $V$ by a further normal ordering with $X^{-1}$: *i.e.* $V \approx (X^{-1} X V)$. Thus $(X V)$ itself must be BRST non-trivial, since if $P$ is BRST-trivial, then so is $(X^{-1} P)$.

As a consequence of the above argument, it follows that we may take any physical operator in the critical two-scalar $W_{2,4}$ string, and map it into a BRST non-trivial physical operator whose momentum $(k_1, k_2)$ lies in a fundamental unit cell, of size 30 by 30, by acting with appropriate powers (positive or negative) of the invertible operators $X$ and $Y$. We find it convenient to choose this fundamental unit cell to be the one defined by

$$-32 \leq k_1 \leq -3, \quad -25 \leq k_2 \leq 4.$$  

(5.7)

If we now solve explicitly for all physical operators whose momenta lie in this unit cell, it follows that the complete cohomology of physical operators for the theory is then obtained by acting on this set of fundamental physical operators with $X^m Y^n$ for all integer $m$ and $n$.

The momentum $(k_1, k_2)$ of any physical operator must satisfy (5.2) for integer $k_1, k_2$ and level number $\ell$. It is an elementary exercise to enumerate all the possible lattice points in the unit cell (5.7) that satisfy this necessary condition for physical operators. It turns out that 36 of the 900 points fulfil this requirement, with level numbers lying in the range $0 \leq \ell \leq 5$. In fact our choice (5.7) for the fundamental unit cell was motivated by the desideratum that all the candidate solutions within it have low level numbers. Of the 36 candidates, 28 correspond to physical operators that were already found in [15]; these generalise to continuous-momentum operators the multi-scalar critical $W_{2,4}$ string. They comprise eight tachyons $t_i$ at ghost number $G = 4$, level $\ell = 0$; ten operators $u_i$ at $G = 3$, with levels $\ell = 1$ and 2; and ten operators $v_i$ at $G = 2$, with levels $\ell = 3, 4$ and 5. In addition, we find that
there are a further four operators $d_i$ at $\ell = 1$, which would generalise to physical operators with discrete spacetime momenta in the multi-scalar case. (We commonly refer to these as discrete operators, even in the two-scalar case.) One of these is the operator $X^{-1}$ itself, given in (5.4). Thus we have a total of 32 physical operators in the fundamental unit cell. In detail, they comprise eight $G = 4$ operators at $\ell = 0$:

\begin{align}
  t_1 &= c \partial^2 \gamma \partial \gamma \gamma e^{-\frac{30}{27} \alpha \varphi_1 - \frac{12}{11} a \varphi_2}, \\
  t_2 &= c \partial^2 \gamma \partial \gamma \gamma e^{-\frac{30}{27} \alpha \varphi_1 - \frac{10}{11} a \varphi_2}, \\
  t_3 &= c \partial^2 \gamma \partial \gamma \gamma e^{-\frac{24}{27} \alpha \varphi_1 - \frac{12}{11} a \varphi_2}, \\
  t_4 &= c \partial^2 \gamma \partial \gamma \gamma e^{-\frac{24}{27} \alpha \varphi_1 - \frac{10}{11} a \varphi_2}, \\
  t_5 &= c \partial^2 \gamma \partial \gamma \gamma e^{-\frac{28}{27} \alpha \varphi_1 - \frac{14}{11} a \varphi_2}, \\
  t_6 &= c \partial^2 \gamma \partial \gamma \gamma e^{-\frac{28}{27} \alpha \varphi_1 - \frac{8}{11} a \varphi_2}, \\
  t_7 &= c \partial^2 \gamma \partial \gamma \gamma e^{-\frac{26}{27} \alpha \varphi_1 - \frac{14}{11} a \varphi_2}, \\
  t_8 &= c \partial^2 \gamma \partial \gamma \gamma e^{-\frac{26}{27} \alpha \varphi_1 - \frac{8}{11} a \varphi_2},
\end{align}

six $G = 3$ operators at $\ell = 1$:

\begin{align}
  u_1 &= c \partial \gamma \gamma e^{-\frac{20}{27} \alpha \varphi_1 - \frac{20}{11} a \varphi_2}, \\
  u_2 &= c \partial \gamma \gamma e^{-\frac{20}{27} \alpha \varphi_1 - \frac{2}{11} a \varphi_2}, \\
  u_3 &= c \partial \gamma \gamma e^{-\frac{18}{27} \alpha \varphi_1 - \frac{18}{11} a \varphi_2}, \\
  u_4 &= c \partial \gamma \gamma e^{-\frac{18}{27} \alpha \varphi_1 - \frac{4}{11} a \varphi_2}, \\
  u_5 &= c \partial \gamma \gamma e^{-\frac{16}{27} \alpha \varphi_1 - \frac{14}{11} a \varphi_2}, \\
  u_6 &= c \partial \gamma \gamma e^{-\frac{16}{27} \alpha \varphi_1 - \frac{8}{11} a \varphi_2},
\end{align}

(5.9a)

together with four more $G = 3$ operators at $\ell = 2$:

\begin{align}
  u_7 &= c \left( \partial \varphi_1 \partial \gamma \gamma - \frac{2}{11} \alpha \partial^2 \gamma \gamma \right) e^{-\frac{18}{27} \alpha \varphi_1 - \frac{24}{11} a \varphi_2}, \\
  u_8 &= c \left( \partial \varphi_1 \partial \gamma \gamma - \frac{2}{11} \alpha \partial^2 \gamma \gamma \right) e^{-\frac{18}{27} \alpha \varphi_1 + \frac{2}{11} a \varphi_2}, \\
  u_9 &= c \left( \partial \varphi_1 \partial \gamma \gamma - \frac{2}{11} \alpha \partial^2 \gamma \gamma \right) e^{-\frac{14}{27} \alpha \varphi_1 - \frac{20}{11} a \varphi_2}, \\
  u_{10} &= c \left( \partial \varphi_1 \partial \gamma \gamma - \frac{2}{11} \alpha \partial^2 \gamma \gamma \right) e^{-\frac{14}{27} \alpha \varphi_1 - \frac{2}{11} a \varphi_2}.
\end{align}

(5.9b)

At $G = 2$ there are four operators with $\ell = 3$:

\begin{align}
  v_1 &= c \gamma e^{-\frac{10}{27} \alpha \varphi_1 - \frac{20}{11} a \varphi_2}, \\
  v_2 &= c \gamma e^{-\frac{10}{27} \alpha \varphi_1 - \frac{2}{11} a \varphi_2}, \\
  v_3 &= c \gamma e^{-\frac{8}{27} \alpha \varphi_1 - \frac{14}{11} a \varphi_2}, \\
  v_4 &= c \gamma e^{-\frac{8}{27} \alpha \varphi_1 - \frac{8}{11} a \varphi_2},
\end{align}

(5.10a)

two more at $\ell = 4$:

\begin{align}
  v_5 &= c \left( \partial \varphi_1 \gamma - \frac{8}{27} \alpha \partial \gamma \right) e^{-\frac{6}{27} \alpha \varphi_1 - \frac{18}{11} a \varphi_2}, \\
  v_6 &= c \left( \partial \varphi_1 \gamma - \frac{8}{27} \alpha \partial \gamma \right) e^{-\frac{6}{27} \alpha \varphi_1 - \frac{4}{11} a \varphi_2}.
\end{align}

(5.10b)
and four further operators at $\ell = 5$:

\[
v_7 = c \left( \beta \partial \gamma \gamma - \frac{35}{108} \alpha \partial \varphi_1 \partial \gamma + \frac{5}{36} \alpha \partial^2 \varphi_1 \gamma + \frac{5}{8} (\partial \varphi_1)^2 \gamma \right) e^{-\frac{6}{27}a \varphi_1 - \frac{24}{11}a \varphi_2},
\]

\[
v_8 = c \left( \beta \partial \gamma \gamma - \frac{35}{108} \alpha \partial \varphi_1 \partial \gamma + \frac{5}{36} \alpha \partial^2 \varphi_1 \gamma + \frac{5}{8} (\partial \varphi_1)^2 \gamma \right) e^{-\frac{6}{27}a \varphi_1 + \frac{2}{11}a \varphi_2},
\]

\[
v_9 = c \left( \beta \partial \gamma \gamma - \frac{4}{9} \alpha \partial \varphi_1 \partial \gamma + \frac{4}{18} \alpha \partial^2 \varphi_1 \gamma + \frac{3}{4} (\varphi_1)^2 \gamma + \frac{3}{10} \partial^2 \gamma \right) e^{-\frac{4}{27}a \varphi_1 - \frac{30}{11}a \varphi_2},
\]

\[
v_{10} = c \left( \beta \partial \gamma \gamma - \frac{4}{9} \alpha \partial \varphi_1 \partial \gamma + \frac{4}{18} \alpha \partial^2 \varphi_1 \gamma + \frac{3}{4} (\varphi_1)^2 \gamma + \frac{3}{10} \partial^2 \gamma \right) e^{-\frac{4}{27}a \varphi_1 - \frac{2}{11}a \varphi_2}.
\]

The four additional operators in the fundamental unit cell, at $\ell = 1$, are as follows:

\[
d_1 = \left( c \partial \gamma \gamma - \frac{28}{15} \partial^2 \gamma \partial \gamma \right) e^{-\frac{30}{27}a \varphi_1},
\]

\[
d_2 = \left( c \partial \gamma \gamma - \frac{4}{15} \partial^2 \gamma \partial \gamma \right) e^{-\frac{24}{27}a \varphi_1},
\]

\[
d_3 = c \left( \partial^4 \gamma \partial^2 \gamma \partial \gamma \gamma - \frac{10}{9} \partial \varphi_1 \partial^3 \gamma \partial \gamma \partial \gamma \right) e^{-\frac{20}{27}a \varphi_1 - \frac{22}{11}a \varphi_2},
\]

\[
d_4 = c \left( \partial^4 \gamma \partial^2 \gamma \partial \gamma \gamma - \frac{8}{3} \partial \varphi_1 \partial^3 \gamma \partial \gamma \partial \gamma \right) e^{-\frac{24}{27}a \varphi_1 - \frac{22}{11}a \varphi_2}.
\]

The physical operators $d_1$ and $d_2$ have $G = 3$, whilst $d_3$ and $d_4$ have $G = 5$. The operator $d_1$ is the same as $X^{-1}$. The remaining four candidate momenta in the fundamental cell (5.7) do not correspond to any physical operators.

The operators that we have listed above are all prime operators, *i.e.* they are the physical operators, for given momentum, with the lowest possible ghost number. As has been discussed extensively in [22,23,9,10,11], each such operator is associated with a quartet of physical operators, whose remaining three members are obtained by normal-ordering the prime operator with $a_{\varphi_1}, a_{\varphi_2}$ or $a_{\varphi_1}a_{\varphi_2}$, where $\partial a_{\varphi_i} \equiv [Q_B, \partial \varphi_i]$. Thus starting from a prime state at ghost number $G$, the quartet of operators has ghost numbers $\{G, G + 1, G + 1, G + 2\}$. We shall just focus our discussion on the prime operators, it being understood that there is a quartet associated with each prime operator.

The complete cohomology of prime operators for the two-scalar critical $W_{2,4}$ string is then given by

\[
X^m Y^n t_i : \quad G = 4 - 3m - n,
\]

\[
X^m Y^n u_i : \quad G = 3 - 3m - n,
\]

\[
X^m Y^n v_i : \quad G = 2 - 3m - n,
\]

\[
X^m Y^n d_1 : \quad G = 3 - 3m - n,
\]

\[
X^m Y^n d_2 : \quad G = 3 - 3m - n,
\]

\[
X^m Y^n d_3 : \quad G = 5 - 3m - n,
\]

\[
X^m Y^n d_4 : \quad G = 5 - 3m - n,
\]
In all cases, \( m \) and \( n \) can be arbitrary integers. The momenta \((k'_1, k'_2)\) of the resulting physical operators are given by \((k'_1, k'_2) = (k_1 + 30m, k_2 + 30n)\), where \((k_1, k_2)\) is the momentum of the original physical operator in the fundamental unit cell. Their level numbers are obtained by substituting \((k'_1, k'_2)\) into (5.2).

It was implicit in the above construction that the normal-ordered products of \( X \) and \( Y \) with any physical operator are well-defined, in other words that the operator products give integer-degree poles. It is easy to check that this is indeed the case. Recalling that the OPE of vertex operators with momenta \( p \) and \( p' \) gives a factor \((z - w)^{-p p'}\), we see that if \( X \) is normal ordered with a physical operator whose momentum is given by \((k_1, k_2)\), then the vertex operators will give a pole of degree \( \frac{1}{2}k_1 \). Similarly, if \( Y \) is normal ordered with the physical operator the vertex operators will give a pole of degree \( \frac{1}{2}k_2 \). One can easily see from the mass-shell condition (5.2), which is a necessary condition that must be satisfied by any physical operator, that \( k_1 \) and \( k_2 \) are always even integers. Thus we see that normal-ordering \( X \) or \( Y \) with any physical operator will always give an integer-degree pole, and so the product is well-defined.

In view of the fact that we have only been able to conjecture the existence of the invertible operator \( Y \), because of the complexity of the calculation necessary for proving its existence, it is worthwhile to carry out some consistency checks in order to verify the plausibility of the conjecture. One way to do this is to consider some of the physical operators that would be generated by acting with \( Y \) on the fundamental states. It is easy to see from (5.2) that if one acts with \( Y \) on a fundamental operator with momentum \((k_1, k_2)\) at level \( \ell \), one obtains a new physical operator with momentum \((k_1 + 30, k_2 + 11)\) at level \( \ell' = \ell + 13 + \frac{1}{2}k_2 \). From this, we see that a simple example is to consider \((Y u_7)\), where \( u_7 \) is the fundamental \( \ell = 2, G = 3 \) operator given in (5.9b). This would yield a physical operator with momentum \((-18, 6)\) at level \( \ell' = 3 \) and ghost number \( G = 2 \). In fact just such a physical operator exists, and was found in [15]:

\[
V = \left( c \gamma + \frac{4}{3} \partial^2 \gamma \gamma - \frac{16}{3} \alpha \partial \varphi_1 \partial \gamma \gamma - \frac{8}{11} a \partial \varphi_2 \partial \gamma \gamma - \frac{8}{9} b c \partial \gamma \gamma \right) e^{-\frac{2}{3} \alpha \varphi_1 + \frac{6}{11} a \varphi_2} . \tag{5.13}
\]

The existence of this physical operator thus provides supporting evidence for the existence of the \( Y \) operator. We have also checked some other examples of a similar kind. In particular we have verified that \((Y d_3)\) at \( \ell = 3 \) and \( G = 4 \); \((Y u_1)\) at \( \ell = 4 \) and \( G = 2 \); and \((Y v_1)\) and \((Y v_7)\) at \( \ell = 6 \) and \( G = 1 \) all indeed correspond to actual physical operators.

We are now in a position to discuss how the Weyl group of \( B_2 \) acts on the physical states of the two-scalar critical \( W_{2,4} \) string. We first define a momentum \((\hat{k}_1, \hat{k}_2)\) that is shifted by the background charges; \((\hat{k}_1, \hat{k}_2) \equiv (k_1 + 27, k_2 + 11)\), in terms of which the mass-shell condition becomes

\[
\hat{k}_1^2 + \hat{k}_2^2 = 10(12\ell + 1) . \tag{5.14}
\]
This is clearly invariant under the $B_2$ Weyl group, generated by

$$
S_1: \quad (\hat{k}_1, \hat{k}_2) \longrightarrow (\hat{k}_1, -\hat{k}_2) ,
$$

$$
S_2: \quad (\hat{k}_1, \hat{k}_2) \longrightarrow (\hat{k}_2, \hat{k}_1) .
$$

Of course equation (5.14) is in fact invariant under $O(2)$, but we are interested in its $B_2$ Weyl subgroup since this also acts covariantly on the other constraints arising from the physical-state conditions. For example, in the case of tachyons $t$ the physical-state condition $[Q_0, t] = 0$ gives the mass-shell condition (5.14) (with $\ell = 0$), and the remaining physical-state condition $[Q_1, t] = 0$ implies the vanishing of the quartic polynomial

$$
W = (\hat{k}_1^2 - 1)(\hat{k}_1^2 - 9) ,
$$

which one can easily see is invariant under the Weyl group (after using the mass-shell condition if necessary). Thus, the set of eight tachyonic physical states (5.8) are mapped into each other under the action of the Weyl group, as may easily be verified using (5.15). For higher-level physical states the action of the Weyl group is a little more complicated, and in fact although it still preserves the level number $\ell$, it now maps between prime physical states at a set of different ghost numbers $G$ [11]. Its action is most easily understood by noting that if one takes any integer solution of the mass-shell equation (5.14) that actually corresponds to a physical state (recall that, for example, in the fundamental cell (5.7), 32 of the 36 integer solutions correspond to physical states), then after acting with any of the Weyl group elements generated by (5.15) one arrives at a momentum that corresponds to another physical state. This can be verified by showing that the resulting momentum is one that can be obtained by acting with some powers of $X$ and $Y$ on one of the fundamental operators given in (5.8)–(5.11). Thus we see that the Weyl group maps prime physical states into other prime physical states at the same level. As discussed in [11], the ghost numbers of these states will be unequal, except in the special case of the tachyons.

So far in this section, we have concentrated on constructing the cohomology of the two-scalar critical $W_{2,4}$ string. We may also extend the discussion to the multi-scalar critical $W_{2,4}$ string. We recall that the multi-scalar critical $W_{2,4}$ string [15] is obtained by replacing the energy-momentum tensor $T_{\varphi_2}$ in (2.2) by

$$
T_{\text{eff}} = -\frac{1}{2} \partial X^\mu \partial X^\nu \eta_{\mu\nu} - i a_\mu \partial^2 X^\mu ,
$$

where the background-charge vector $a_\mu$ is chosen so that $T_{\text{eff}}$ has the same central charge, namely $c_{\text{eff}} = \frac{126}{5}$, as did $T_{\varphi_2}$ given by (2.5).

As described in [11] for the case of $W_3$ strings, one may derive the multi-scalar cohomology by considering the subset of two-scalar physical operators that can be generalised to the multi-scalar case. To be generalisable, two-scalar physical operators must fall into one of the following three categories:
1) If there is a pair of two-scalar prime physical operators with momenta \((k_1, k_2)\) and \((k_1, -22 - k_2)\), both at the same ghost number. This pair, which have the same conformal dimension \(\Delta = -\frac{1}{120}(k_2 + 11)^2 + \frac{121}{120}\) under \(T_{\text{eff}}\), generalise to a continuous-momentum multi-scalar operator where \(e^{ip \cdot X}\) has the same dimension \(\Delta = \frac{1}{2}p^\mu(p_\mu + 2a_\mu)\).

2) If there is a two-scalar prime physical operator with momentum \((k_1, 0)\). This generalises to a discrete multi-scalar operator with \(p_\mu = 0\) in the effective spacetime, and hence \(\Delta = 0\).

3) If there is a two-scalar prime physical operator with momentum \((k_1, -22)\). This generalises to a discrete multi-scalar operator with \(p_\mu = -2a_\mu\) in the effective spacetime, where \(a_\mu\) is the constant background-charge vector appearing in (5.14). Again, this has conformal dimension \(\Delta = 0\) as measured by \(T_{\text{eff}}\).

By taking the subset of two-scalar physical states that satisfies the above conditions, and generalising them to the multi-scalar case, we obtain all the multi-scalar states that are purely tachyonic in the effective spacetime. The complete set of physical states, including excitations in the effective spacetime, is then obtained by replacing the spacetime vertex operators \(e^{ip \cdot X}\) with arbitrary highest-weight operators with the same conformal weight under \(T_{\text{eff}}\) given in (5.17).

By looking in detail at all the prime physical operators (5.12) of the two-scalar \(W_{2,4}\) string, we find that the subset of physical operators that can be generalised to the multi-scalar case is given by \(X^m t_i, X^m u_i, X^m v_i, X^m (Y^{-1}d_1), X^m (Y d_3), X^m (Y^{-1}d_2), X^m (Y d_4)\) (which all generalise to continuous-momentum physical operators), and \(X^m d_i\) (which generalise to discrete-momentum physical operators). All the fundamental operators \(t_i, u_i\) and \(v_i\) given in (5.8)–(5.10) themselves occur in pairs with conjugate \(\varphi_2\) momenta (those with index \(i = 2r + 2\) are conjugate to those with index \(i = 2r + 1\)). The operators \((Y^{-1}d_1)\) and \((Y d_3)\) comprise a pair with conjugate \(\varphi_2\) momenta: \((k_1, k_2) = (-30, -30)\) and \((-30, 8)\). Similarly, \((Y^{-1}d_2)\) and \((Y d_4)\) comprise a conjugate pair with momenta \((-24, -30)\) and \((-24, 8)\). These generalise to continuous-momentum multi-scalar operators with \(G = 4\) at level \(\ell = 3\), which we shall call \(\tilde{w}_1\) and \(\tilde{w}_2\) respectively. Thus we may re-express the cohomology of prime operators in the multi-scalar \(W_{2,4}\) string using a fundamental basis comprising the multi-scalar generalisations of (5.8)–(5.11), together with \(\tilde{w}_i\). The set of fundamental physical operators for the multi-scalar \(W_{2,4}\) string are therefore given by

\[
\begin{align*}
\tilde{t}_1 &= c \partial^2 \gamma \partial_\gamma \gamma e^{-\frac{30}{27} \alpha \varphi_1} e^{ip \cdot X}; & \Delta &= 1, \\
\tilde{t}_2 &= c \partial^2 \gamma \partial_\gamma \gamma e^{-\frac{24}{27} \alpha \varphi_1} e^{ip \cdot X}; & \Delta &= 1, \\
\tilde{t}_3 &= c \partial^2 \gamma \partial_\gamma \gamma e^{-\frac{28}{27} \alpha \varphi_1} e^{ip \cdot X}; & \Delta &= \frac{14}{15}, \\
\tilde{t}_4 &= c \partial^2 \gamma \partial_\gamma \gamma e^{-\frac{26}{27} \alpha \varphi_1} e^{ip \cdot X}; & \Delta &= \frac{14}{15},
\end{align*}
\]
\[\begin{align*}
\tilde{u}_1 &= c \partial\gamma \gamma e^{-\frac{30}{27} \alpha \varphi_1} e^{ip \cdot X}; \quad \Delta = \frac{1}{3}, \\
\tilde{u}_2 &= c \partial\gamma \gamma e^{-\frac{18}{27} \alpha \varphi_1} e^{ip \cdot X}; \quad \Delta = \frac{3}{5}, \\
\tilde{u}_3 &= c \partial\gamma \gamma e^{-\frac{16}{27} \alpha \varphi_1} e^{ip \cdot X}; \quad \Delta = \frac{14}{15}, \\
\tilde{u}_4 &= c \left( \partial\varphi_1 \partial\gamma \gamma - \frac{4}{9} \alpha \partial^2 \gamma \gamma \right) e^{-\frac{14}{27} \alpha \varphi_1} e^{ip \cdot X}; \quad \Delta = \frac{1}{3}, \\
\tilde{u}_5 &= c \left( \partial\varphi_1 \partial\gamma \gamma - \frac{2}{15} \alpha \partial^2 \gamma \gamma \right) e^{-\frac{18}{27} \alpha \varphi_1} e^{ip \cdot X}; \quad \Delta = -\frac{2}{5}, \\
\tilde{v}_1 &= c \gamma e^{-\frac{10}{27} \alpha \varphi_1} e^{ip \cdot X}; \quad \Delta = \frac{1}{3}, \\
\tilde{v}_2 &= c \gamma e^{-\frac{8}{27} \alpha \varphi_1} e^{ip \cdot X}; \quad \Delta = \frac{14}{15}, \\
\tilde{v}_3 &= c \left( \partial\varphi_1 \gamma - \frac{8}{27} \alpha \partial\gamma \right) e^{-\frac{6}{27} \alpha \varphi_1} e^{ip \cdot X}; \quad \Delta = \frac{3}{5}, \\
\tilde{v}_4 &= c \left( \beta \partial\gamma \gamma - \frac{4}{9} \alpha \partial\varphi_1 \partial\gamma + \frac{1}{18} \alpha \partial^2 \varphi_1 \gamma + \frac{3}{4} \left( \partial\varphi_1 \right)^2 \gamma + \frac{3}{10} \partial^2 \gamma \right) e^{-\frac{4}{27} \alpha \varphi_1} e^{ip \cdot X}; \quad \Delta = \frac{1}{3}, \\
\tilde{v}_5 &= c \left( \beta \partial\gamma \gamma - \frac{35}{108} \alpha \partial\varphi_1 \partial\gamma + \frac{8}{35} \alpha \partial^2 \varphi_1 \gamma + \frac{3}{5} \left( \partial\varphi_1 \right)^2 \gamma \right) e^{-\frac{6}{27} \alpha \varphi_1} e^{ip \cdot X}; \quad \Delta = -\frac{2}{5}, \\
\tilde{w}_1 &= c \left( 41 \left( \partial\varphi_1 \right)^3 \partial^2 \gamma \partial\gamma \gamma + \frac{120}{10} \partial\varphi_1 \partial^3 \gamma \partial^2 \gamma \gamma + \frac{60}{20} \partial\varphi_1 \partial^4 \gamma \partial\gamma \gamma \right. \\
&\quad \left. - \frac{12}{3} \partial^2 \varphi_1 \partial^3 \gamma \partial\gamma \gamma - \frac{26}{9} \alpha \left( \partial\varphi_1 \right)^2 \partial^3 \gamma \partial\gamma \gamma + \frac{2}{3} \alpha \partial^2 \varphi_1 \partial\varphi_1 \partial^2 \gamma \partial\gamma \gamma \right. \\
&\quad \left. - \frac{53}{45} \alpha \partial^3 \gamma \partial^2 \gamma \partial\gamma \gamma - \frac{11}{15} \alpha \partial^4 \gamma \partial^2 \gamma \gamma - \frac{13}{45} \alpha \partial^5 \gamma \partial\gamma \gamma \right) e^{-\frac{30}{27} \alpha \varphi_1} e^{ip \cdot X}; \quad \Delta = -2, \\
\tilde{w}_2 &= c \left( 68 \left( \partial\varphi_1 \right)^3 \partial^2 \gamma \partial\gamma \gamma + 36 \partial\varphi_1 \partial^3 \gamma \partial^2 \gamma \gamma - 50 \partial\varphi_1 \partial^4 \gamma \partial\gamma \gamma \right. \\
&\quad \left. - 120 \partial^2 \varphi_1 \partial^3 \gamma \partial\gamma \gamma + 15 \partial^3 \varphi_1 \partial^2 \gamma \partial\gamma \gamma + \frac{340}{3} \alpha \partial^2 \varphi_1 \partial\varphi_1 \partial^2 \gamma \partial\gamma \gamma \right. \\
&\quad \left. - \frac{40}{3} \alpha \partial^3 \gamma \partial^2 \gamma \partial\gamma \gamma + \frac{10}{9} \alpha \partial^4 \gamma \partial^2 \gamma \gamma + \frac{20}{9} \alpha \partial^5 \gamma \partial\gamma \gamma \right) e^{-\frac{24}{27} \alpha \varphi_1} e^{ip \cdot X}; \quad \Delta = -2, \\
\tilde{d}_1 &= c \left( \partial\gamma \gamma - \frac{28}{15} \partial^2 \gamma \partial\gamma \gamma \right) e^{-\frac{30}{27} \alpha \varphi_1}; \quad \Delta = 0, \\
\tilde{d}_2 &= c \left( \partial\gamma \gamma - \frac{4}{15} \partial^2 \gamma \partial\gamma \gamma \right) e^{-\frac{24}{27} \alpha \varphi_1}; \quad \Delta = 0, \\
\tilde{d}_3 &= c \left( \partial^4 \gamma \partial^2 \gamma \partial\gamma \gamma - \frac{10}{9} \partial\varphi_1 \partial^3 \gamma \partial^2 \gamma \partial\gamma \gamma \right) e^{-\frac{30}{27} \alpha \varphi_1} e^{-2ia \cdot X}; \quad \Delta = 0, \\
\tilde{d}_4 &= c \left( \partial^4 \gamma \partial^2 \gamma \partial\gamma \gamma - \frac{2}{9} \partial\varphi_1 \partial^3 \gamma \partial^2 \gamma \partial\gamma \gamma \right) e^{-\frac{24}{27} \alpha \varphi_1} e^{-2ia \cdot X}; \quad \Delta = 0.
\end{align*}\]

Here $\Delta = \frac{1}{2}p \cdot (p + 2a)$ is the conformal weight of the spacetime vertex operator $e^{ip \cdot X}$, as measured by $T^{\text{eff}}$. The complete cohomology of prime operators in the multi-scalar $W_{2,4}$ string comprises continuous-momentum operators $X^m \tilde{u}_i$ at ghost number $G = 4 - 3m$, $X^m \tilde{w}_i$ at ghost number $G = 3 - 3m$, $X^m \tilde{v}_i$ at ghost number $G = 2 - 3m$ and $X^m \tilde{w}_i$ at ghost number
\[ G = 4 - 3m, \text{ together with discrete-momentum operators } X^m \tilde{d}_1 \text{ and } X^m \tilde{d}_2 \text{ at } G = 3 - 3m \text{ and } X^m d_3 \text{ and } X^m \tilde{d}_4 \text{ at } G = 5 - 3m. \] The \( \varphi_1 \) momenta, spacetime weights \( \Delta \), and level numbers can be obtained from (5.18), and the mass-shell formula (5.2).

We see from (5.18) that the continuous-momentum fundamental operators \( \tilde{t}_i, \tilde{u}_i, \tilde{v}_i \) and \( \tilde{w} \) have conformal weights \( \Delta = \{1, \frac{14}{15}, \frac{3}{5}, \frac{1}{3}, -\frac{2}{5}, -2\} \), with multiplicities \( \{2, 4, 2, 4, 2, 2\} \) respectively. The reason for the doubling of the multiplicities of the \( \Delta = 1 \) tachyon, given in (5.18). In fact there is a freedom to add a BRST-trivial term that must be included in \( \tilde{t}_i \), whereas we may write the operator \( \tilde{w} \) as a \( \Delta = 1 \) descendant operator. Similarly, we may write the operator \( \tilde{w} \) as a \( \Delta = 1 \) descendant. The reason for the doubling of the multiplicities of the \( \Delta = 1 \) tachyon, given in (5.18).

The \( \Delta = -2 \) physical operators \( \tilde{w}_1 \) and \( \tilde{w}_2 \) in (5.18) are examples that correspond to \( W \)-descendant operators in the \( W_3 \) minimal model. For example we find that \( \tilde{w}_1 \) can be written as

\[
\tilde{w}_1 \propto \left( U_{-3} + \frac{1}{582\alpha} (1775 U_{-1} U_{-2} - 605 U_{-2} U_{-1}) + \frac{75}{18928\alpha^2} (U_{-1})^3 \right) \tilde{t}_1,
\]

where \( U \) is the spin-3 primary current of the \( W_3 \) minimal model, given in (5.6), and \( \tilde{t}_1 \) is a \( \Delta = 1 \) tachyon, given in (5.18). In fact there is a freedom to add a BRST-trivial term to \( \tilde{w}_1 \) in (5.18), whilst preserving its factorised form, and only for one particular choice of the associated free parameter, namely the choice we have adopted, can \( \tilde{w}_1 \) be written as a descendant operator. Similarly, we may write the operator \( \tilde{w}_2 \) in (5.18) as a \( W_3 \) descendant. In this case, the BRST-trivial term that must be included in \( \tilde{w}_2 \) in order to allow it to be expressed as a \( W_3 \) descendant results in rather complicated coefficients, and so we have left \( \tilde{w}_2 \) in a simpler form. Thus up to BRST-trivial terms, we find that may write \( \tilde{w}_2 \) as

\[
\left( U_{-3} - \frac{1}{128\alpha} (211 U_{-1} U_{-2} + 1076 U_{-2} U_{-1}) - \frac{519}{64\alpha^2} (U_{-1})^3 \right) \tilde{t}_2.
\]
6. Critical $W_{2,s}$ Strings

It is of interest to consider how the results of the previous sections might be extended to the case of $W_{2,s}$ strings for general values of $s$. A “critical” BRST operator of the form (2.1)–(2.7) has been explicitly constructed for $s = 3$ (the $W_3$ case [6]), and $s = 4, 5$ and 6 [15]. It is expected that such BRST operators will exist for arbitrary values of $s$.

Let us consider first the possible generalisation to arbitrary $s$ of the construction of the non-critical $W_{2,4}$ string in section 3. In this construction, the critical $W_{2,s}$ BRST operator is extended by introducing additional matter currents that satisfy the $W_{2,s}$ algebra. As we discussed in section 3, such an algebra, the $WB_2$ algebra, exists at arbitrary central charge in the $s = 4$ case. A similar situation obtains for $s = 6$, in which case the algebra in question is $WG_2$ [17,18]. Thus we expect that a construction of a non-critical $W_{2,6}$ BRST operator along the lines of section 3 should be possible in principle. In practice, the complexity of the system would make an explicit construction quite difficult. For the case of $s = 5$, a $W_{2,5}$ algebra exists only at certain discrete values of the central charge. In principle, one could imagine that a non-critical $W_{2,5}$ BRST operator could be built, in which the matter system realises the $W_{2,5}$ algebra at one or another of these discrete values of central charge. Again, the system is a complicated one, and it would be quite difficult to carry out the construction explicitly. For $s \geq 7$, it is again known that $W_{2,s}$ algebras could exist at most for discrete values of central charge. The complexities of the systems would be even greater in these cases.

Turning now to the critical $W_{2,s}$ BRST operators, it is of interest to consider generalising the $W_{2,4}$ cohomology discussion of section 5. As we shall see, it turns out that there are new features that arise when $s \geq 5$ that complicate the discussion considerably. It was already conjectured in [15] that the physical states of the multi-scalar $W_{2,s}$ string all have the form of effective Virasoro physical states tensored with primary or descendant operators of the lowest unitary $W_{s-1}$ minimal model, which has central charge $c = \frac{2(s-1)}{(s+2)}$. This was investigated further in [16], where strong supporting evidence for the conjecture was obtained. Since the minimal model has $c \geq 1$ when $s \geq 5$, it follows that if one decomposes the operators under the Virasoro subalgebra, an infinite number of primary Virasoro operators will arise. This has the consequence for the multi-scalar $W_{2,s}$ string theory that there will be an infinite number of effective-spacetime physical sectors, with weights $\Delta$ of the form $\Delta_i = 1 - h_{W} - N_i$, where $h_{W}$ takes values in the (finite) list of conformal weights of the primary operators of the lowest $W_{s-1}$ minimal model, and the integers $N_i$ take infinite sets of values, corresponding to the weights $h_{W} + N_i$ of operators that are $W_{s-1}$ descendants but that are still Virasoro primaries [16]. Thus if one looks for the analogue of the set of “basic” physical operators (5.18) for the higher $W_{2,s}$ strings with $s \geq 5$, the set will itself be infinite. (This did not happen for $s = 3$ or $s = 4$, since the relevant minimal models had central charges $c = \frac{1}{2}$ and $c = \frac{4}{5}$ respectively, and thus the lists of Virasoro primaries were finite.)
A related complication when \( s \geq 5 \) is that it is no longer the case that the \( \varphi_2 \) momenta of all physical states in the two-scalar \( W_{2,s} \) string are rational multiples of the background charge \( a \). Let us focus in particular on the subset of two-scalar states that can be generalised to continuous-momentum states in the multi-scalar string. We have looked at the \( W_{2,5} \) string, for which the background charges are given by \( \alpha^2 = \frac{121}{6} \) and \( \alpha^2 = 2 \) [15], in some detail, and we find that although the \( \varphi_1 \) momentum seems to be quantised in units of \( \frac{1}{2} \alpha \), the \( \varphi_2 \) momenta of physical states can have one of two possible forms; \( p_2 = \frac{1}{2} a \) \( k_2 \) or \( p_2 = -a + \frac{a}{2\sqrt{3}} k_2 \), where \( k_2 \) is an integer. The reason for this can be seen by looking at the relation between the conformal weights \( \Delta = -\frac{1}{2} p_2(p_2 + 2a) \) of physical states of the \( W_{2,s} \) string under \( T_{\varphi_2} \), and the conformal weights \( h = 1 - \Delta \) of the primary operators of the associated \( W_{s-1} \) minimal model. Writing

\[
 p_2 = q_2 \ a \tag{6.1}
\]

in each case, we find that the relations are as follows:

\[
\begin{align*}
W_{2,3} : & \quad (q_2 + 1)^2 = \frac{1}{49} (48 \ h + 1) , \tag{6.2a} \\
W_{2,4} : & \quad (q_2 + 1)^2 = \frac{1}{121} (120 \ h + 1) , \tag{6.2b} \\
W_{2,5} : & \quad (q_2 + 1)^2 = h , \tag{6.2c} \\
W_{2,6} : & \quad (q_2 + 1)^2 = \frac{1}{167} (168 \ h - 1) . \tag{6.2d}
\end{align*}
\]

The weights \( h \) for each case are as follows [15,16]:

\[
\begin{align*}
W_{2,3} : & \quad h = \{ 0, \ \frac{1}{16}, \ \frac{1}{2} \} , \tag{6.3a} \\
W_{2,4} : & \quad h = \{ 0, \ \frac{1}{15}, \ \frac{2}{5}, \ \frac{2}{7}, \ \frac{7}{5}, \ 3 \} , \tag{6.3b} \\
W_{2,5} : & \quad h = \{ 0, \ \frac{1}{16}, \ \frac{1}{12}, \ \frac{1}{3}, \ \frac{9}{16}, \ \frac{13}{4}, \ 1; \ \frac{4}{3}, \ \frac{25}{16}, \ \frac{25}{12}, \ \frac{49}{16}, \ \ldots \} \tag{6.3c} \\
W_{2,6} : & \quad h = \{ 0, \ \frac{2}{35}, \ \frac{3}{35}, \ \frac{2}{7}, \ \frac{17}{35}, \ \frac{23}{35}, \ \frac{4}{5}, \ \frac{6}{7}, \ \frac{6}{5}, \ \frac{9}{7}, \ \frac{52}{35}, \ \frac{58}{35}, \ \frac{13}{7}, \ \frac{73}{35}, \ \frac{87}{35}, \ \frac{20}{7}, \ \frac{16}{5}, \ \ldots \} \tag{6.3d}
\end{align*}
\]

In each case, the weights to the left of the semicolon are those of primary operators of the \( W_{s-1} \) minimal model, whilst those after the semicolon (if any) are the weights of \( W_{4-1} \) descendants that are Virasoro primaries. For the \( W_{2,3} = W_3 \) string, we see that for each value of \( h \) in (6.3a), the right-hand side of (6.2a) has a rational square root. The same is true for (6.2b) when one substitutes the \( h \) values (6.3b) of the \( W_{2,4} \) string. However, for the \( W_{2,5} \) string we see that only some of the \( h \) values (6.3c) give rational roots for \( q_2 \),

\[
q_2 = \frac{1}{4} k_2 , \tag{6.4}
\]

whilst the remaining \( h \) values give \( q_2 \) of the form

\[
q_2 = -1 + \frac{1}{2\sqrt{3}} k_2 , \tag{6.5}
\]
where $k_2$ is an integer in each case. Finally, for the $W_{2,6}$ string we see from (6.3d) there are many irrationally-related values for $q_2$ (the occurrence of the prime number 167 in the denominator of (6.2d), unlike the perfect square integers of the previous cases, makes irrationality much more probable).

One consequence of the occurrence of physical states with irrationally-related $\varphi_2$ momenta is that there cannot exist universal $Y$-type operators that can normal order with any physical state to give other physical states. On the other hand, $X$-type operators (which have zero momentum in the $\varphi_2$ direction), can still be expected to exist, and they can still normal order with all physical operators.

Let us consider first the $W_{2,5}$ string. By analogy to these previous cases, we expect that the $X$ operator should have as its associated screening current the simple expression

$$S_X = \partial^4 \beta \partial^3 \beta \partial^2 \beta \partial \beta \beta e^{\frac{12}{\Pi} \alpha \varphi_1}.$$  

Thus the $X$ operator itself will have ghost number $G = -4$. Checking that $[Q_B, S_X] = \partial X$ where $Q_B$ is the BRST operator given in [15] is a somewhat non-trivial calculation, which we have not been able to complete.* However, the inverse of $X$, at level $\ell = 1$ with $G = 4$, is easily computed:

$$X^{-1} = \left( c \partial^2 \gamma \partial \gamma \gamma - \frac{5}{22} \alpha \partial^3 \gamma \partial^2 \gamma \partial \gamma \gamma \right) e^{-\frac{12}{\Pi} \alpha \varphi_1}.$$  

As far as a $Y$-type operator is concerned, one might hope that since, for the $W_{2,5}$ string, there appear to be just the two possible sequences of $q_2$ momenta given by (6.4) and (6.5), there might at least exist an invertible $Y$ operator that could normal order with physical operators whose $q_2$ momenta are given by (6.4). For physical operators of this kind, which we shall call "rational" operators, the level number is related to the momentum $(p_1, p_2) = (\frac{1}{22} \alpha k_1, \frac{1}{4} a k_2)$ by the mass-shell condition

$$(k_1 + 22)^2 + 3(k_2 + 4)^2 = 4(12\ell + 1).$$  

Necessary conditions that should be satisfied by a candidate $Y$ operator are the following. Firstly its momentum should satisfy (6.8) for integer $k_1, k_2$ and $\ell$. Secondly $(-k_1, -k_2)$, which would be the momentum of the $Y^{-1}$ operator, should also satisfy (6.8) for some other integer $\ell'$. Thirdly, the $Y$ operator should give integer-degree poles in its operator product with any other rational physical operator. These conditions imply that $(k_1, k_2)$ should have the form $(k_1, k_2) = (12m, 4n)$, where the integers $m$ and $n$ are either both odd, or both even.

* We do, however, know that there exists a $G = 0$ physical operator $x$ at level $\ell = 15$, whose screening current is given by $S_x = \beta e^{\frac{12}{\Pi} \alpha \varphi_1}$ [15]. By acting with five of these screening currents (and a ghost booster) on the physical operator $x$, one should, in principle, be able to obtain the required $X$ operator. Thus the known existence of $x$ provides supportive evidence for the existence of $X$. 

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Of course these are only necessary conditions for the existence of an invertible $Y$ operator, and unfortunately the physical-state conditions are too complicated for us to be able to determine whether any such operator actually exists. It is also conceivable that the fact that there are also “irrational” physical operators whose $\varphi_2$ momenta satisfy (6.5), which certainly could not normal order with any $Y$, is an indication that no invertible $Y$ operator can exist.

It was shown in [11] for the $W_3$ string, and in section 5 of this paper for the $W_{2,4}$ string, that in the multi-scalar case the entire cohomology can be constructed by acting with arbitrary powers of the relevant $X$ operator on a set of basic multi-scalar states. We may expect therefore that the same should be true for the $W_{2,5}$ string, and thus the lack of a $Y$ operator need not be an obstacle to constructing the cohomology in this multi-scalar case. Of course, as we discussed earlier, there are an infinite number of different matter sectors in the $W_{2,5}$ string, corresponding to an infinite number of different effective-spacetime weights $\Delta$. Consequently, the set of basic operators $u_i$, from which all prime physical operators will be obtained as $X^m u_i$, will itself be infinite. However, as was shown in [16], this infinite set can be understood as arising from decomposing the primary operators of the $c = 1$ $W_4$ algebra into $W$ descendants that are still Virasoro primaries, and so by this means the infinite set $\{u_i\}$ can be generated in principle from just a finite set of physical operators with effective-spacetime weights $\Delta = 1 - h_W$, where $h_W = \{0, \frac{1}{16}, \frac{1}{12}, \frac{1}{3}, \frac{9}{16}, \frac{3}{4}, 1\}$ is the set of conformal weights of the primary fields of the $c = 1$ $W_4$ minimal model. Furthermore, we expect that the multiplicities for the basic set of operators for the $h_W$ values listed above should be $\{2, 4, 2, 2, 4, 4, 2\}$. The reason for the doubling of those with conformal weights $h_W = \{\frac{1}{10}, \frac{9}{16}, \frac{3}{4}\}$ is the same as we discussed in section 5 for certain of the $W_{2,4}$ basic states, namely that the corresponding primary operators of the $W_4$ minimal model have non-zero weights under the spin-3 current, and thus occur in ± pairs. The 20 basic continuous-momentum physical operators, together with their effective-spacetime weights $\Delta = 1 - h_W$, are:

\[
\begin{align*}
  t_1 &= c \partial^3 \gamma \partial^2 \gamma \partial \gamma \gamma e^{-\frac{24}{22} \alpha \varphi_1} e^{ipX}, \quad \Delta = 1 \\
  t_2 &= c \partial^3 \gamma \partial^2 \gamma \partial \gamma \gamma e^{-\frac{20}{22} \alpha \varphi_1} e^{ipX}, \quad \Delta = 1 \\
  t_3 &= c \partial^3 \gamma \partial^2 \gamma \partial \gamma \gamma e^{-\frac{25}{22} \alpha \varphi_1} e^{ipX}, \quad \Delta = 15/16 \\
  t_4 &= c \partial^3 \gamma \partial^2 \gamma \partial \gamma \gamma e^{-\frac{21}{22} \alpha \varphi_1} e^{ipX}, \quad \Delta = 15/16 \\
  u_1 &= c \partial^2 \gamma \partial \gamma \gamma e^{-\frac{15}{22} \alpha \varphi_1} e^{ipX}, \quad w_1 = c \gamma e^{-\frac{5}{22} \alpha \varphi_1} e^{ipX}, \quad \Delta = 15/16 \\
  u_2 &= c \partial^2 \gamma \partial \gamma \gamma e^{-\frac{16}{22} \alpha \varphi_1} e^{ipX}, \quad w_2 = c \partial \gamma \gamma e^{-\frac{10}{22} \alpha \varphi_1} e^{ipX}, \quad \Delta = 11/12 \\
  u_3 &= c \partial^2 \gamma \partial \gamma \gamma e^{-\frac{17}{22} \alpha \varphi_1} e^{ipX}, \quad w_3 = c \partial \gamma \gamma e^{-\frac{11}{22} \alpha \varphi_1} e^{ipX}, \quad \Delta = 7/16 \\
  v_1 &= c (\partial^2 \gamma - \frac{24}{55} \alpha \partial \varphi_1 \partial \gamma \gamma) e^{-\frac{2}{22} \alpha \varphi_1} e^{ipX}, \quad \Delta = 7/16 \\
  v_2 &= c (\partial^2 \gamma - \frac{12}{55} \alpha \partial \varphi_1 \partial \gamma \gamma) e^{-\frac{9}{22} \alpha \varphi_1} e^{ipX}, \quad \Delta = 7/16 \\
  w_3 &= c (\partial^2 \gamma - \frac{12}{55} \alpha \partial \varphi_1 \partial \gamma + \frac{12}{5} (\partial \varphi_1)^2 \gamma + \frac{6}{55} \alpha \partial^2 \varphi_1 \gamma + 3 \beta \partial \gamma \gamma) e^{-\frac{3}{22} \alpha \varphi_1} e^{ipX}, \quad \Delta = 7/16
\end{align*}
\]
\[ u_4 = c \partial^2 \gamma \partial \gamma \gamma e^{-\frac{18}{55} \alpha \varphi_1} e^{i p \cdot X}; \quad w_4 = c \gamma e^{-\frac{6}{55} \alpha \varphi_1} e^{i p \cdot X}, \quad \Delta = \frac{1}{4} \]

\[ u_5 = c \left( \partial^3 \gamma \partial \gamma \gamma - \frac{36}{55} \alpha \partial \varphi_1 \partial^2 \gamma \partial \gamma \gamma \right) e^{-\frac{14}{55} \alpha \varphi_1} e^{i p \cdot X}, \quad \Delta = \frac{1}{4} \]

\[ w_5 = c \left( \partial^3 \gamma + 36 (\partial \varphi_1)^2 \partial \gamma - \frac{36}{55} \alpha \partial^2 \varphi_1 \partial \gamma \gamma - \frac{36}{55} \alpha \partial \varphi_1 \partial^2 \gamma \right. \]

\[ \left. - \frac{144}{55} \alpha (\partial \varphi_1)^3 \gamma + \frac{18}{11} \alpha \partial^2 \varphi_1 \partial \gamma - \frac{6}{55} \alpha \partial^3 \varphi_1 \gamma + 12 \beta \partial^2 \gamma \gamma - 21 \beta \partial \beta \partial \gamma \gamma - \frac{108}{11} \alpha \partial \varphi_1 \beta \partial \gamma \gamma \right) e^{-\frac{2}{55} \alpha \varphi_1} e^{i p \cdot X}, \quad \Delta = \frac{1}{4} \]

\[ v_4 = c \alpha \gamma e^{-\frac{12}{22} \alpha \varphi_1} e^{i p \cdot X}, \quad \Delta = 0 \]

\[ v_5 = c \left( \partial^3 \gamma \gamma + \frac{9}{2} \partial^2 \gamma \partial \gamma \gamma - \frac{18}{11} \alpha \partial \varphi_1 \partial^2 \gamma \gamma + \frac{36}{55} (\partial \varphi_1)^2 \partial \gamma \gamma + \frac{18}{55} \alpha \partial \gamma \gamma \right) e^{-\frac{8}{55} \alpha \varphi_1} e^{i p \cdot X}, \quad \Delta = 0 \]

The cohomology of continuous-momentum prime physical operators for the \( W_{2,5} \) string, with \( \Delta = \{1, \frac{15}{9}, \frac{11}{12}, \frac{2}{3}, \frac{7}{10}, \frac{1}{4}, 0\} \), is then given by \( X^m t_i \) at \( G = 5 - 4m \), \( X^m u_i \) at \( G = 4 - 4m \), \( X^m v_i \) at \( G = 3 - 4m \), and \( X^m w_i \) at \( G = 2 - 4m \). The rest of the continuous-momentum prime physical operators, with \( \Delta \leq 0 \), are then obtainable as \( W_4 \) descendents [16]. Some of the higher-level physical operators found in [15] provide non-trivial consistency checks of the existence of the invertible \( X \) operator.

There will in addition be physical operators with discrete momentum in the effective spacetime. These will come from four basic discrete operators at \( \ell = 1 \):

\[ d_1 = \left( c \partial^2 \gamma \partial \gamma \gamma - \frac{5}{55} \alpha \partial^3 \gamma \partial^2 \gamma \partial \gamma \gamma \right) e^{-\frac{12}{11} \alpha \varphi_1}, \quad \Delta = 0 \]

\[ d_2 = \left( c \partial^2 \gamma \partial \gamma \gamma - \frac{35}{117} \alpha \partial^3 \gamma \partial^2 \gamma \partial \gamma \gamma \right) e^{-\frac{10}{11} \alpha \varphi_1}, \quad \Delta = 0 \]

\[ d_3 = c \left( \partial^3 \gamma \partial^3 \gamma \partial^2 \gamma \partial \gamma \gamma - \frac{12}{11} \alpha \partial \varphi_1 \partial^4 \gamma \partial^3 \gamma \partial^2 \gamma \partial \gamma \gamma \right) e^{-\frac{12}{11} \alpha \varphi_1} e^{-2 i a \cdot X}, \quad \Delta = 0 \]

\[ d_4 = c \left( \partial^3 \gamma \partial^3 \gamma \partial^2 \gamma \partial \gamma \gamma - \frac{10}{11} \alpha \partial \varphi_1 \partial^4 \gamma \partial^3 \gamma \partial^2 \gamma \partial \gamma \gamma \right) e^{-\frac{10}{11} \alpha \varphi_1} e^{-2 i a \cdot X}, \quad \Delta = 0 \]

Thus there will be discrete prime operators \( X^m d_1 \) and \( X^m d_2 \) at \( G = 4 - 4m \), and \( X^m d_3 \) and \( X^m d_4 \) at \( G = 6 - 4m \).

It is worth noting that the set of 20 basic continuous-momentum operators given in (6.9), comprises five each at ghost numbers \( G = 5, 4, 3, 2 \) and 2, namely \( \{t_i \}, \{u_i \}, \{v_i \} \) and \( \{w_i \} \). Similarly, if we consider the basic continuous-momentum operators for the \( W_{2,4} \) string, given in (5.18), and restrict attention to the ones associated with the primary fields of the \( c = \frac{4}{3} \) \( W_3 \) minimal model (i.e. exclude the \( \Delta = -\frac{2}{3} \) and \( \Delta = -2 \) operators in (5.18), as well as the \( \Delta = 0 \) discrete operators), then there are a total of twelve, comprising four at each of the ghost numbers \( G = 4, 3, 2 \). In the \( W_3 \) string case, discussed in [11], there are a total of six basic continuous-momentum operators, comprising three at each of the ghost numbers \( G = 3 \) and 2. For the \( W_{2,s} \) string, we can expect that there should be \( s(s-1) \) basic
continuous-momentum operators associated with the $W_{s-1}$ primary fields, with $s$ of them at each of the ghost numbers $G = s, s - 1, \ldots, 2$.

The above considerations presumably give the complete cohomology of prime physical operators in the multi-scalar critical $W_{2,5}$ string. Although we have not been able to construct the enlarged cohomology of the two-scalar case, it is certain that there are then additional physical operators that arise, which cannot generalise to the multi-scalar case. One example that we have found, at level $\ell = 6$ and ghost number $G = 2$, is

$$V = \left( c\gamma + \frac{35}{8} \partial\varphi_1 \partial^2 \gamma \gamma - \frac{1055}{48} \partial^2 \varphi_1 \partial\gamma \gamma - \frac{35}{22} \alpha (\partial\varphi_1)^2 \partial\gamma \gamma - \frac{35}{88} \alpha (\partial\varphi_2)^2 \partial\gamma \gamma 
- \frac{105}{88} \alpha \partial bc \partial\gamma \gamma - \frac{455}{176} \alpha \partial^2 \gamma \partial\gamma + \frac{35}{48} \alpha \partial^3 \gamma \gamma - \frac{35}{16} a \partial\varphi_1 \partial\varphi_2 \partial\gamma \gamma - \frac{35}{44} a \alpha \partial\varphi_2 b c \partial\gamma \gamma 
+ \frac{35}{176} a \alpha \partial\varphi_2 \partial^2 \gamma \gamma - \frac{35}{176} a \alpha \partial^2 \varphi_2 \partial\gamma \gamma \right) e^{-\frac{6}{11} \alpha \varphi_1 + a \varphi_2} .$$

(6.11)

The cohomology of the multi-scalar $W_{2,6}$ string should also be obtainable by using these methods. We expect that there should be an $X$ operator at $\ell = 66$, corresponding to the screening current

$$S_X = \partial^5 \beta \partial^4 \beta \partial^3 \beta \partial^2 \beta \partial \beta \beta e^{\frac{14}{17} \alpha \varphi_1} ,$$

(6.12)

where $\alpha^2 = \frac{845}{28}$ in this case. Checking that $[Q_B, S_X] = \partial X$, with $Q_B$ given in [15] for the $W_{2,6}$ string, would be extremely complicated.* The $X$ operator has ghost number $G = -5$. The inverse $X^{-1}$ is, as usual, a simple level $\ell = 1$ physical operator, with $G = 5$ in this case, given by

$$X^{-1} = \left( c \partial^3 \gamma \partial^2 \gamma \partial\gamma \gamma - \frac{396}{1225} \alpha \partial^4 \gamma \partial^3 \gamma \partial^2 \gamma \partial\gamma \gamma \right) e^{-\frac{14}{17} \alpha \varphi_1} .$$

(6.13)

In accordance with the general observations made above, we expect that the continuous-momentum physical operators should be given by $X^m$ acting on a total of 30 basic operators, comprising six at each of the ghost numbers $G = 6, 5, 4, 3$ and 2. This will give the full cohomology of operators with $\Delta$ weights corresponding to the $W_5$ primary conformal weights $h = 1 - \Delta$ in (6.3d). The rest of the continuous-momentum physical operators, with spacetime weights $\Delta = 1 - h$ corresponding to $h$ values to the right of the semicolon in (6.3d), will arise as $W_5$ descendants [16].

---

* As in the case of the $W_{2,5}$ string discussed earlier, we may use the known existence of a $G = 0$ physical operator $x$ at level $\ell = 21$ in the $W_{2,6}$ string, with associated screening current $S_x = \beta e^{\frac{14}{17} \alpha \varphi_1}$ [15], to argue that the $X$ operator should arise at $\ell = 66$ by acting with six screening currents $S_x$ and a ghost booster on the operator $x$. 

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7. Conclusion

In this paper, we have studied in detail aspects of some higher-spin string theories which have local spin-2 and spin-$s$ symmetries on the two-dimensional worldsheet. Such theories, which we call $W_{2,s}$ strings, apparently exist for all $s \geq 3$, although owing to the increasing complexity at higher values of $s$, they have been explicitly constructed only for $s = 3, 4, 5$ and 6 [15]. The multi-scalar $W_{2,s}$ string has a physical spectrum that is related to a tensor product of effective Virasoro string states times the primary fields of the lowest unitary $W_{s-1}$ minimal model [15,16].

For most of the paper, we have concentrated on the case of the $W_{2,4}$ string. This is the simplest generalisation of the $W_{2,3}$ case, which is just the $W_3$ string. After a brief review of the construction of the critical $W_{2,s}$ string in section 2, we then presented new results in section 3 in which we obtained the “non-critical” $W_{2,4}$ string. This is a generalisation of the non-critical $W_3$ string constructed in [12,13], in which a nilpotent BRST operator is obtained that describes the coupling of $WB_2$ matter to the pure $WB_2$ gravity of the critical $W_{2,4}$ string. In section 4, we studied the spectrum of the theory and obtained some of the physical states, including ground-ring generators with ghost number $G = 0$ at levels $\ell = 10$ and 11. Obtaining a complete solution to the cohomology of physical states would be very complicated in the non-critical $W_{2,4}$ string, but in section 5 we were able to solve the problem for the simpler case of the critical $W_{2,4}$ string, using the methods developed in [11]. Similar considerations for the higher $W_{2,s}$ strings reveal the emergence of new features, seemingly related to the fact that for $s \geq 5$ the multi-scalar states are associated with $W_{s-1}$ minimal models whose primary fields decompose into infinite numbers of Virasoro primaries associated with non-unitary Virasoro models.

As well as the higher-spin BRST operators that we have been considering in this paper, there are others that were found in [15] in the case of $s = 4$ and $s = 6$. These other BRST operators appear to be associated with non-unitary string theories, in the sense that in the multi-scalar case the conformal weights of the effective-spacetime physical operators can exceed 1, and thus the longitudinal modes of excited states can have negative norms. However, in the two-scalar case, there are no longitudinal modes, and the norms of all physical states can be expected to be positive. It may be that there are some interesting new features in these theories.

Finally, we remark that in an interesting recent paper, it was proposed that one can consider hierarchies of string theories in which, for example, the $N = 0$ bosonic string is viewed as a special vacuum of the $N = 1$ superstring, which in turn is viewed as a special vacuum of the $N = 2$ string [25]. Another possible route is to view the bosonic string as being embedded in a hierarchy of $W$-string theories [25,26]. Indeed, the realisation of the critical $W_3$ algebra, with $c = 100$, in terms of an energy-momentum tensor $T^{eff}$ with $c = \frac{51}{2}$ and a scalar field $\varphi_1$, is very reminiscent of the procedure in [25] in which the critical $N = 1$ super-Virasoro algebra, with $c = 15$, is realised in terms of an energy-momentum tensor.
$T_m$ with $c = 26$ and a $(b_1, c_1)$ anticommuting ghost system with spins $(\frac{3}{2}, -\frac{3}{2})$. In both cases, the physical states of the corresponding $W_3$ string or $N = 1$ superstring turn out to be rather trivial, in the sense that they are really just states of a bosonic string. However, we know that there are other realisations of the $N = 1$ super-Virasoro algebra that give a completely different physical spectrum, namely that of the usual $N = 1$ string. It may be that there correspondingly exist other realisations of $W$ algebras that can give rise to a more interesting spectrum of physical states. In view of these points, it is of interest to explore the possible higher-spin string theories, and their hierarchical structures, in more detail.

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