Reciprocity sheaves and logarithmic motives

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Abstract

We connect two developments that aim to extend Voevodsky’s theory of motives over a field in such a way as to encompass non-$\mathbf{A}^1$-invariant phenomena. One is theory of reciprocity sheaves introduced by Kahn, Saito and Yamazaki. The other is theory of the triangulated category $\text{logDM}^{\text{eff}}$ of logarithmic motives launched by Binda, Park and Østvær. We prove that the Nisnevich cohomology of reciprocity sheaves is representable in $\text{logDM}^{\text{eff}}$.

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Introduction

We fix once and for all a perfect base field $k$. The main purpose of this paper is to connect two developments that aim to extend Voevodsky’s theory of motives over $k$ in such a way as to encompass non-$\mathbf{A}^1$-invariant phenomena. One is the theory of reciprocity sheaves introduced by Kahn, Saito and Yamazaki [KSY16, KSY22] and developed in [Sai20, BRS22]. Voevodsky’s theory is based on the category $\text{PST}$ of presheaves with transfers, defined as the category of additive presheaves of abelian groups on the category $\text{Cor}$ of finite correspondences: $\text{Cor}$ has the same objects as the category $\text{Sm}$ of separated smooth schemes of finite type over $k$, and morphisms in $\text{Cor}$ are finite correspondences. Let $\text{NST} \subset \text{PST}$ be the full subcategory of Nisnevich sheaves, that is, those objects $F \in \text{PST}$ whose restrictions $F_X$ to the small étale site $X_{\text{ét}}$ over $X$ are Nisnevich sheaves for all $X \in \text{Sm}$. Voevodsky proved that $\text{NST}$ is a Grothendieck abelian...
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category and defined the triangulated category \( \text{DM}^{\text{eff}} \) of effective motives as the localization of the derived category \( D(\text{NST}) \) of complexes in \( \text{NST} \) with respect to an \( \mathbb{A}^1 \)-weak equivalence (see [MVW06, Definition 14.1]). It is equipped with a functor \( M: \text{Sm} \rightarrow \text{DM}^{\text{eff}} \) associating the motive \( M(X) \) of \( X \in \text{Sm} \).

Let \( \text{HI}_{\text{Nis}} \subset \text{NST} \) be the full subcategory consisting of \( \mathbb{A}^1 \)-invariant objects, namely such \( F \in \text{NST} \) that the projection \( \pi_X : X \times \mathbb{A}^1 \rightarrow X \) induces an isomorphism \( \pi_X^* : F(X) \cong F(X \times \mathbb{A}^1) \) for any \( X \in \text{Sm} \). We say that \( F \in \text{HI}_{\text{Nis}} \) is strictly \( \mathbb{A}^1 \)-invariant if \( \pi_X \) induces isomorphisms

\[
\pi_X^* : H^i_{\text{Nis}}(X, F_X) \cong H^i_{\text{Nis}}(X \times \mathbb{A}^1, F_{X \times \mathbb{A}^1}) \quad \text{for all } i \geq 0.
\]

The following theorem plays a fundamental role in Voevodsky’s theory.

**Theorem 0.1** (Voevodsky [Voe00]). Any \( F \in \text{HI}_{\text{Nis}} \) is strictly \( \mathbb{A}^1 \)-invariant and we have a natural isomorphism

\[
H^i_{\text{Nis}}(X, F_X) \cong \text{Hom}_{\text{DM}^{\text{eff}}}(M(X), L^{\mathbb{A}^1} F[i]) \quad \text{for } X \in \text{Sm},
\]

(0.1.1)

where \( L^{\mathbb{A}^1} : D(\text{NST}) \rightarrow \text{DM}^{\text{eff}} \) is the localization functor.

Notice that there are interesting and important objects of \( \text{NST} \) which do not belong to \( \text{HI}_{\text{Nis}} \). Such examples are given by the sheaves \( \Omega^j \) of (absolute or relative) differential forms; the \( p \)-typical de Rham–Witt sheaves \( W_m \Omega^j \) of Bloch, Deligne and Illusie; smooth commutative \( k \)-group schemes with a unipotent part (seen as objects of \( \text{NST} \)); and the complexes \( R\varepsilon_* \mathbb{Z}/p^n(\varepsilon) \) with \( \text{ch}(\varepsilon) = p > 0 \), where \( \mathbb{Z}/p^n(\varepsilon) \) is the étale motivic complex of weight \( n \) with \( \mathbb{Z}/p^n \) coefficients and \( \varepsilon \) is the change of site functor from the étale to the Nisnevich topology. For such examples, (0.1.1) fails to hold since \( \pi_X : X \times \mathbb{A}^1 \rightarrow X \) induces an isomorphism \( M(X \times \mathbb{A}^1) \cong M(X) \) in \( \text{DM}^{\text{eff}} \) but the maps induced on the cohomology of those sheaves are not isomorphisms.

The category \( \text{RSC}_{\text{Nis}} \) of reciprocity sheaves is a full abelian subcategory of \( \text{NST} \) that contains \( \text{HI}_{\text{Nis}} \) as well as the non-\( \mathbb{A}^1 \)-invariant objects mentioned above. Heuristically, its objects satisfy the property that for any \( X \in \text{Sm} \), each section \( a \in F(X) \) ‘has bounded ramification at infinity’ and the objects of \( \text{HI}_{\text{Nis}} \) are special reciprocity sheaves with the property that every section \( a \in F(X) \) has ‘tame’ ramification at infinity.\(^1\) Slightly more exotic examples of reciprocity sheaves are given by the sheaves \( \text{Conn}^1 \) (for \( \text{ch}(\varepsilon) = 0 \)), whose sections over \( X \) are rank 1-connections, or \( \text{Lisse}^1 \) (in case \( \text{ch}(\varepsilon) = p > 0 \)), whose sections on \( X \) are the lisse \( \mathbb{Q}_p \)-sheaves of rank 1. Since \( \text{RSC}_{\text{Nis}} \) is an abelian category equipped with a lax symmetric monoidal structure by [RSY22], many more interesting examples can be manufactured by taking kernels, quotients and tensor products (see [BRS22, §11.1] for more examples).

The main purpose of this paper is to establish formula (0.1.1) for all \( F \in \text{RSC}_{\text{Nis}} \) in a new category which enlarges \( \text{DM}^{\text{eff}} \) (see (0.2)). It is the triangulated category \( \text{logDM}^{\text{eff}} \) of logarithmic motives introduced by Binda, Park and Østvær in [BPO22]. Let \( I\text{Sm} \) be the category of log smooth and separated fs log schemes of finite type over \( k \), and \( I\text{Cor} \) be the category with the same objects as \( I\text{Sm} \) and whose morphisms are log finite correspondences (see [BPO22, Definition 2.1.1]). Let \( I\text{Shv}^{\text{ltr}} \) be the category of additive presheaves of abelian groups on \( I\text{Cor} \) and \( I\text{Shv}^{\text{ltr}}_{\text{dNis}} \subset I\text{Shv}^{\text{ltr}} \) be the full subcategory consisting of those \( F \) whose restrictions to \( I\text{Sm} \) are dividing Nisnevich sheaves (see [BPO22, Definition 3.1.4]). It is shown in [BPO22, Theorem 1.2.1] that \( I\text{Shv}^{\text{ltr}}_{\text{dNis}} \) is a Grothendieck abelian category, and \( \text{logDM}^{\text{eff}} \) is defined as the localization of the derived category \( D(I\text{Shv}^{\text{ltr}}_{\text{dNis}}) \) of complexes in \( I\text{Shv}^{\text{ltr}}_{\text{dNis}} \) with respect to a \( \square \)-weak equivalence, where \( \square \) is \( \mathbb{P}^1 \) with the log structure associated to the effective divisor \( \infty \leftarrow \mathbb{P}^1 \).

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\(^1\) This heuristic viewpoint is manifested in [RS21a, Theorem 2].
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(see [BPØ22, Definition 5.2.1]). It is equipped with a functor $M : lSm \to \logDM_{\text{eff}}$ associating the logarithmic motive $M(X)$ of $X \in lSm$. Thanks to [BM12, Theorem 1.1], the standard $t$-structure on $D(\Shv_{\text{dNis}}^{\text{triv}})$ induces a $t$-structure on $\logDM_{\text{eff}}$ called the homotopy $t$-structure, and its heart is identified with the abelian full subcategory $\CI_{\text{dNis}}^{\text{triv}} \subset \Shv_{\text{dNis}}^{\text{triv}}$ consisting of strictly $\Box$-invariant objects in the sense of [BPØ22, Definition 5.2.2]. We can now state the main result of this paper.

**Theorem 0.2 (Theorems 6.1 and 6.3).** There exists an exact and fully faithful functor

$$\Log : RSC_{\text{Nis}} \to \CI_{\text{dNis}}^{\text{triv}} : F \to F^{\log} = \Log(F).$$

For $X \in Sm$ we have a natural isomorphism

$$H^i_{\text{Nis}}(X, F_X) \simeq \Hom_{\logDM_{\text{eff}}}((M(X, \text{triv}), L^i F^{\log}),)$$

where $L^i : D(\Shv_{\text{dNis}}^{\text{triv}}) \to \logDM_{\text{eff}}$ is the localization functor and $(X, \text{triv})$ is the log scheme with the trivial log structure.

We remark (see Remark 5.6) that, for $F = \Omega^i$, $F^{\log}(X)$ for $X \in lSm$ whose underlying scheme is smooth agrees with the sheaf of logarithmic differential forms of $X$ at least assuming $\text{ch}(k) = 0$.

We now explain the organization of the paper.

In §1 we discuss some preliminaries and fix notation. We recall the definitions and basic properties of modulus (pre)sheaves with transfers from [KMSY21a, KMSY21b, KSY22, Sai20]. These are a generalization of Voevodsky’s (pre)sheaves with transfers to a version with modulus. The category $\MCor$ of modulus correspondences is introduced. Its objects are pairs $X = (\overline{X}, D)$, where $\overline{X}$ is a separated scheme of finite type over $k$ equipped with an effective Cartier divisor $D$ such that the interior $\overline{X} - D = X$ is smooth. The morphisms are finite correspondences on the interiors satisfying admissibility and a properness condition. Let $\MPST$ be the category of additive presheaves of abelian groups on $\MCor$. A full subcategory $\MNST \subset \MPST$ of Nisnevich sheaves is defined and there is a functor (see §1(20))

$$\omega^{\CI} : RSC_{\text{Nis}} \to \MNST.$$

For every $F \in RSC_{\text{Nis}}$ and $X \in Sm$, it provides an exhaustive filtration on the group $F(X)$ of sections over $X$ which measures the depth of ramification along a boundary of a partial compactification of $X$: for $(\overline{X}, D) \in \MCor$ with $\overline{X} - D = X$, we get the subgroups $F(\overline{X}, D) \subset F(X)$ with $F = \omega^{\CI}F$ such that $F(\overline{X}, D_1) \subset F(\overline{X}, D_2)$ if $D_1 \leq D_2$.

In §2 we prove as a key technical input an analogue of the Zariski–Nagata purity theorem [SGA2, X 3.4] for $\tilde{F}(\overline{X}, D)$ as above. This asserts the exactness of the sequence

$$0 \to F(\overline{X}, D) \to F(X) \to \bigoplus_{\xi \in D^{(0)}} \frac{F(\overline{X}^h_{\xi} - \xi)}{\tilde{F}(\overline{X}^h_{\xi}, \xi)},$$

where $\overline{X} \in Sm$ and $D$ is a reduced simple normal crossing divisor, and where $D^{(0)}$ is the set of the irreducible components of $D$ and $\overline{X}^h_{\xi}$ is the henselization of $X$ at $\xi$. In [RS21b] this result is generalized to the case where $D$ may not be reduced under the assumption that $\overline{X}$ admits a smooth compactification.

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2 In fact it is defined in [BPØ22, Definition 5.2.1] as the localization of the homotopy category of complexes in $\Shv_{\text{dNis}}^{\text{triv}}$ with respect to a $\Box$-local descent model structure.

3 It is a logarithmic analogue of Voevodsky’s strict $\Delta^+$-invariance.

4 The assumption is necessary to use [RS21a, Corollary 6.8] proved in the case $\text{ch}(k) = 0$. We expect that it can be dispensed with by using a forthcoming work of K. Riuling extending [RS21a, Corollary 6.8] to the case $\text{ch}(k) > 0$. 
In §3 we review higher local symbols for reciprocity sheaves constructed in [RS21e]. These are an effective tool with which one can decide when a given element of \( F(X) \) with \( F \in \text{RSC}_{\text{Nis}} \) and \( X \in \text{Sm} \) belongs to \( \tilde{F}(X, D) \) as above. The construction of the pairing depends on pushforward maps for the cohomology of reciprocity sheaves constructed in [BRS22] (which means that Theorem 0.2 depends on the result of [BRS22]).

In §4 we prove the following result. Let \( \text{MCor}_{\text{ls}}^\text{fin} \) be the subcategory of \( \text{MCor} \) whose objects are pairs \((X, D)\) such that \( X \in \text{Sm} \) and the reduced divisor \( D_{\text{red}} \) underlying \( D \) is a simple normal crossing divisor on \( X \) and whose morphisms are modulus correspondences satisfying a finiteness conditions instead of the properness condition (see §1(5)). Then, for \( F \in \text{RSC}_{\text{Nis}} \), the association

\[
\tilde{F}^\text{log}: (X, D) \to \omega^{\text{CI}} F(X, D_{\text{red}})
\]

gives a sheaf on \( \text{MCor}_{\text{ls}}^\text{fin} \), which gives rise to a cohomology theory \( H^i_{\text{log}}(-, \tilde{F}^\text{log}) \) on \( \text{MCor}_{\text{ls}}^\text{fin} \), called the \( i \)th logarithmic cohomology with coefficient \( F \) (see Definition 4.4). The higher local symbols for \( F \) play a fundamental role in the proof of the result.

In §5 we prove the invariance of logarithmic cohomology under blowups. Let \( \Lambda_{\text{ls}}^\text{fin} \) be the subcategory of \( \text{MCor}_{\text{ls}}^\text{fin} \) whose objects are the same as \( \text{MCor}_{\text{ls}}^\text{fin} \) and whose morphisms are those \( \rho: (Y, E) \to (X, D) \) where \( E = \rho^*D \) and \( \rho \) are induced by blowups of \( X \) in smooth centers \( Z \subset D \) which are normal crossing to \( D \) (see the beginning of the section). Then, for \( F \in \text{RSC}_{\text{Nis}} \) and \( \rho: Y \to X \) in \( \Lambda_{\text{ls}}^\text{fin} \), we have

\[
\rho^*: H^i_{\text{log}}(X, F) \cong H^i_{\text{log}}(Y, F) \quad \forall i \geq 0.
\]

In §6 we prove Theorem 0.2, which is a formal consequence of the theorems in §§4 and 5.

1. Preliminaries

We fix once and for all a perfect base field \( k \). In this section we recall the definitions and basic properties of modulus sheaves with transfers from [KMSY21a, Sai20].

1. Denote by \( \text{Sch} \) the category of separated schemes of finite type over \( k \) and by \( \text{Sm} \) the full subcategory of smooth schemes. For \( X, Y \in \text{Sm} \), an integral closed subscheme of \( X \times Y \) that is finite and surjective over a connected component of \( X \) is called a prime correspondence from \( X \) to \( Y \). The category \( \text{Cor} \) of finite correspondences has the same objects as \( \text{Sm} \), and for \( X, Y \in \text{Sm} \), \( \text{Cor}(X, Y) \) is the free abelian group on the set of all prime correspondences from \( X \) to \( Y \) (see [Voe00]). We consider \( \text{Sm} \) as a subcategory of \( \text{Cor} \) by regarding a morphism in \( \text{Sm} \) as its graph in \( \text{Cor} \).

Let \( \text{PST} \) be the category of additive presheaves of abelian groups on \( \text{Cor} \) whose objects are called presheaves with transfers. Let \( \text{NST} \subseteq \text{PST} \) be the category of Nisnevich sheaves with transfers and let

\[
a^\text{V}_{\text{Nis}}: \text{PST} \to \text{NST}
\]

be Voevodsky’s Nisnevich sheafification functor, which is an exact left adjoint to the inclusion \( \text{NST} \to \text{PST} \). Let \( \text{HI} \subseteq \text{PST} \) be the category of \( A^1 \)-invariant presheaves and put \( \text{HI}_{\text{Nis}} = \text{HI} \cap \text{NST} \subseteq \text{NST} \).

2. Let \( \text{Sm}_{\text{pro}} \) be the category of \( k \)-schemes \( X \) which are essentially smooth over \( k \), that is, \( X \) is a limit \( \lim_{i \in I} X_i \) over a filtered set \( I \), where \( X_i \) is smooth over \( k \) and all transition maps are étale. Note that \( \text{Spec} K \in \text{Sm}_{\text{pro}} \) for a function field \( K \) over \( k \) thanks to the assumption that \( k \) is perfect. We define \( \text{Cor}_{\text{pro}} \) whose objects are the same as \( \text{Sm}_{\text{pro}} \) and whose morphisms are defined as [RS21a, Definition 2.2]. We extend \( F \in \text{PST} \) to a presheaf on \( \text{Cor}_{\text{pro}} \) by \( F(X) := \lim_{i \in I} F(X_i) \) for \( X \) as above.

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(3) We recall the definition of the category \( \mathbf{MCor} \) from [KMSY21a, Definition 1.3.1]. A pair \( \mathcal{X} = (X, D) \) consisting of \( X \in \mathbf{Sch} \) and an effective Cartier divisor \( D \) on \( X \) is called a modulus pair if \( X - D \in \mathbf{Sm} \). Let \( \mathcal{X} = (X, D_X) \), \( \mathcal{Y} = (Y, D_Y) \) be modulus pairs and \( \Gamma \in \mathbf{Cor}(X - D_X, Y - D_Y) \) be a prime correspondence. Let \( \Gamma \subseteq X \times Y \) be the closure of \( \Gamma \), and let \( \Gamma^N \rightarrow X \times Y \) be the normalization. We say that \( \Gamma \) is admissible (respectively, left proper) if \( (D_X)^N \geq (D_Y)^N \) (respectively, if \( \Gamma \) is proper over \( X \)). Let \( \mathbf{MCor}(\mathcal{X}, \mathcal{Y}) \) be the subgroup of \( \mathbf{Cor}(X - D_X, Y - D_Y) \) generated by all admissible left proper prime correspondences. The category \( \mathbf{MCor} \) has modulus pairs as objects and \( \mathbf{MCor}(\mathcal{X}, \mathcal{Y}) \) as the group of morphisms from \( \mathcal{X} \) to \( \mathcal{Y} \).

(4) Let \( \mathbf{MCor}_{ls} \subseteq \mathbf{MCor} \) be the full subcategory of \( (X, D) \in \mathbf{MCor} \) with \( X \in \mathbf{Sm} \) and \( |D| \) a normal crossing divisor on \( X \).

(5) Let \( \mathbf{MCor}^{fin} \subseteq \mathbf{MCor} \) be the full subcategory of the same objects such that \( \mathbf{MCor}^{fin}(\mathcal{X}, \mathcal{Y}) \) are generated by all admissible finite prime correspondences, where finite prime correspondences are defined by replacing the left properness in (3) by finiteness. We also define \( \mathbf{MCor}^{fin}_{ls} = \mathbf{MCor}^{fin} \cap \mathbf{MCor}_{ls} \).

(6) There is a canonical pair of adjoint functors \( \lambda \dashv \omega \):

\[
\lambda : \mathbf{Cor} \rightarrow \mathbf{MCor} \quad X \mapsto (X, \emptyset),
\]

\[
\omega : \mathbf{MCor} \rightarrow \mathbf{Cor} \quad (X, D) \mapsto X - D.
\]

(7) There is a full subcategory \( \mathbf{MCor} \subseteq \mathbf{MCor} \) consisting of proper modulus pairs, where a modulus pair \( (X, D) \) is proper if \( X \) is proper. Let \( \tau : \mathbf{MCor} \hookrightarrow \mathbf{MCor} \) be the inclusion functor and \( \omega = \omega\tau \).

(8) Let \( \mathbf{MPST} \) (respectively, \( \mathbf{MPST}^{pro} \)) be the category of additive presheaves of abelian groups on \( \mathbf{MCor} \) (respectively, \( \mathbf{MCor}^{pro} \)) whose objects are called modulus presheaves with transfers. For \( \mathcal{X} \in \mathbf{MCor} \), let \( \mathbb{Z}_{tr}(\mathcal{X}) = \mathbf{MCor}(-, \mathcal{X}) \) be the representable object of \( \mathbf{MPST} \). We sometimes write \( \mathcal{X} \) for \( \mathbb{Z}_{tr}(\mathcal{X}) \) for simplicity.

(9) In the same manner as (2), the category \( \mathbf{MCor}^{pro} \) is defined and \( F \in \mathbf{MPST} \) is extended to a presheaf on \( \mathbf{MCor}^{pro} \) (see [RS21a, §3.7]).

(10) The adjunction \( \lambda \dashv \omega \) induces a string of four adjoint functors \( (\lambda!, \lambda^*, \omega_!, \omega^*) \) (see [KMSY21a, Proposition 2.3.1]):

\[
\begin{array}{ccc}
\mathbf{MPST} & \xrightarrow{\omega^*} & \mathbf{PST} \\
\mathbf{PST} & \xleftarrow{\omega_!} & \mathbf{MPST} \\
\end{array}
\]

where \( \omega_!, \omega_* \) are localizations and \( \omega_! \) and \( \omega^* \) are fully faithful.

(11) The functor \( \tau \) yields a string of three adjoint functors \( (\tau_!, \tau^*, \tau_*): \)

\[
\begin{array}{ccc}
\mathbf{MPST} & \xrightarrow{\tau_*} & \mathbf{MPST} \\
\end{array}
\]

where \( \tau_!, \tau_* \) are fully faithful and \( \tau^* \) is a localization; \( \tau_! \) has a pro-left adjoint \( \tau_! \), hence is exact (see [KMSY21a, Proposition 2.4.1]). We will denote by \( \mathbf{MPST}^\tau \) the essential image of \( \tau_! \) in \( \mathbf{MPST} \).

(12) The modulus pair \( \square := (\mathbb{P}^1, \infty) \) has an interval structure induced by that of \( \mathbb{A}^1 \) (see [KSY22, Lemma 2.1.3]). We say that \( F \in \mathbf{MPST} \) is \( \square \)-invariant if \( p^* : F(\mathcal{X}) \rightarrow F(\mathcal{X} \otimes \square) \) is an isomorphism for any \( \mathcal{X} \in \mathbf{MCor} \), where \( p : \mathcal{X} \otimes \square \rightarrow \mathcal{X} \) is the projection. Let \( \mathbf{CI} \) be
the full subcategory of $\text{MPST}$ consisting of all $\square$-invariant objects and $\text{CI}^\tau \subset \text{MPST}$ be the essential image of $\text{CI}$ under $\tau$.

(13) Recall from [KSY22, Theorem 2.1.8] that $\text{CI}$ is a Serre subcategory of $\text{MPST}$, and that the inclusion functor $\iota^\square : \text{CI} \to \text{MPST}$ has a left adjoint $h_0^\square$ and a right adjoint $h_1^\square$ given for $F \in \text{MPST}$ and $\mathcal{X} \in \text{MCor}$ by

$$h_0^\square(F)(\mathcal{X}) = \text{Coker}(i_0^\square - i_1^\square : F(\mathcal{X} \otimes \square) \to F(\mathcal{X})),$$
$$h_1^\square(F)(\mathcal{X}) = \text{Hom}(h_0^\square(\mathcal{X}), F).$$

For $\mathcal{X} \in \text{MCor}$, we write $h_0^\square(\mathcal{X}) = h_0^\square(\mathcal{Z}_{\text{et}}(\mathcal{X})) \in \text{CI}$, and by abuse of notation we also write $h_0^\square(\mathcal{X})$ for $\tau h_0^\square(\mathcal{X}) \in \text{CI}^\tau$.

(14) For $F \in \text{MPST}$ and $\mathcal{X} = (X, D) \in \text{MCor}$, write $F_{\mathcal{X}}$ for the presheaf on the small étale site $X_{\text{et}}$ over $X$ given by $U \to F(\mathcal{X}_U)$ for $U \to X$ étale, where $\mathcal{X}_U = (U, D|_U) \in \text{MCor}$. We say that $F$ is a Nisnevich sheaf if $F_{\mathcal{X}}$ is also one for all $\mathcal{X} \in \text{MCor}$ (see [KMSY21a, §3]). We write $\text{MNST} \subset \text{MPST}$ for the full subcategory of Nisnevich sheaves and put

$$\text{MNST}^\tau = \text{MNST} \cap \text{MPST}^\tau, \quad \text{CI}^\text{Nis} = \text{CI}^\tau \cap \text{MNST}^\tau.$$ 

By [KMSY21a, Proposition 3.5.3] and [KMSY21b, Theorem 2], the inclusion functor $\iota_{\text{Nis}} : \text{MNST} \to \text{MPST}$ has an exact left adjoint $a_{\text{Nis}}$ such that $a_{\text{Nis}}(\text{MPST}^\tau)$ $\subset \text{MNST}^\tau$. The functor $a_{\text{Nis}}$ has the following description. For $F \in \text{MPST}$ and $\mathcal{Y} \in \text{MCor}$, let $F_{\mathcal{Y}, \text{Nis}}$ be the usual Nisnevich sheafification of $F_{\mathcal{Y}}$. Then, for $(X, D) \in \text{MCor}$, we have

$$a_{\text{Nis}}F(X, D) = \lim_{f : Y \to X \text{ proper}} F(Y, f_* D)_{\text{Nis}}(Y)$$

where the colimit is taken over all proper maps $f : Y \to X$ that induce isomorphisms $Y - |f_* D| \overset{\sim}{\to} X - |D|$.

(15) By [KMSY21b, Proposition 6.2.1], $\omega^*$ and $\omega_!$ from (10) respect $\text{MNST}$ and $\text{NST}$ and induce a pair of adjoint functors (which for simplicity we write $\omega_!$ and $\omega^*$). Moreover, we have

$$\omega_! a_{\text{Nis}} = a_{\text{Nis}}^V \omega_!.$$ 

By [KSY22, Lemma 2.3.1] and [KMSY21b, Proposition 6.2.1a], for $F \in \text{PST}$, we have $F \in \text{HI}$ (respectively, $F \in \text{HI}_{\text{Nis}}$) if and only if $\omega^* F \in \text{CI}^\tau$ (respectively, $\omega^* F \in \text{CI}^\text{Nis}$).

(16) We say that $F \in \text{MPST}$ is semipure if the unit map

$$u : F \to \omega^* \omega_! F$$

is injective. For $F \in \text{MPST}$ (respectively, $F \in \text{MNST}$), let $F_{\text{sp}} \in \text{MPST}$ (respectively, $F_{\text{sp}} \in \text{MNST}$) be the image of $F \to \omega^* \omega_! F$ (called the semipurification of $F$. See [Sai20, Lemma 1.30]). For $F \in \text{MPST}$ we have

$$a_{\text{Nis}}(F_{\text{sp}}) \cong (a_{\text{Nis}}F)^{\text{sp}}.$$ 

This follows from the fact that $a_{\text{Nis}}$ is exact and commutes with $\omega^* \omega_!$. For $F \in \text{MPST}^\tau$ we have $F_{\text{sp}} \in \text{MPST}^\tau$ since $\tau$ is exact and $\omega^* \omega_! \tau = \tau \omega^* \omega_!$.

(17) Let $\text{CI}^{\tau, \text{sp}} \subset \text{CI}^\tau$ be the full subcategory of semipure objects and consider the full subcategory

$$\text{CI}^{\tau, \text{sp}, \text{Nis}} = \text{CI}^{\tau, \text{sp}} \cap \text{MNST}^\tau \subset \text{CI}^{\text{Nis}}.$$ 

By [Sai20, Theorems 0.1 and 0.4], we have $a_{\text{Nis}}(\text{CI}^{\tau, \text{sp}}) \subset \text{CI}^{\tau, \text{sp}, \text{Nis}}$. 

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(18) We write $\mathbf{RSC} \subseteq \mathbf{PST}$ for the essential image of $\mathbf{CI}$ under $\omega^!$ (which is the same as the essential image of $\mathbf{CI}_{\text{sp}}^\tau$ under $\omega_!$ since $\omega_! \cong \omega_i^!$ and $\omega_i^! F = \omega_i F_{\text{sp}}$). Put $\mathbf{RSC}_{\text{Nis}} = \mathbf{RSC} \cap \mathbf{NST}$. The objects of $\mathbf{RSC}$ (respectively, $\mathbf{RSC}_{\text{Nis}}$) are called reciprocity presheaves (respectively, sheaves). By [Sai20, Theorem 0.1], we have

$$\alpha_{\text{Nis}}^V(\mathbf{RSC}) \subseteq \mathbf{RSC}_{\text{Nis}}. \quad (1.0.1)$$

We have $\mathbf{HI} \subseteq \mathbf{RSC}$ which also contains smooth commutative group schemes (which may have non-trivial unipotent part), the sheaf $\Omega^i$ of Kähler differentials, and the de Rham–Witt sheaves $W_i \Omega^i$ (see [KSY16, KSY22]).

(19) $\mathbf{NST}$ is a Grothendieck abelian category by [Voe00, Lemma 3.1.6] and we can make $\mathbf{RSC}_{\text{Nis}}$ its full subabelian category as follows. We define the kernel (respectively, cokernel) of a map $\phi : F \to G$ in $\mathbf{RSC}_{\text{Nis}}$ to be that of $\phi$ as a map in $\mathbf{NST}$. Here we need (1.0.1) to ensure that the cokernel of $\phi$ in $\mathbf{NST}$ stays in $\mathbf{RSC}_{\text{Nis}}$. By definition, a sequence $0 \to F \to G \to H \to 0$ is exact in $\mathbf{RSC}_{\text{Nis}}$ if and only if it is exact in $\mathbf{NST}$.

(20) By [KSY22, Proposition 2.3.7] we have a pair of adjoint functors

$$\mathbf{CI} \xrightarrow{\omega_{\mathbf{CI}}} \mathbf{RSC}, \quad (1.0.2)$$

where $\omega_{\mathbf{CI}} = h^0 F^* \omega^*$ and is fully faithful. It induces a pair of adjoint functors

$$\mathbf{CI}^\tau \xrightarrow{\omega_{\mathbf{CI}}} \mathbf{RSC}, \quad (1.0.3)$$

where $\omega_{\mathbf{CI}} = \tau h^0 F^* \omega^*$ and is fully faithful. Indeed, let $F = \tau \hat{F}$ for $\hat{F} \in \mathbf{CI}$ and $G \in \mathbf{RSC}$. In view of (13) and the exactness and full faithfulness of $\tau$, we have

$$\text{Hom}_{\mathbf{CI}}(F, \tau h^0 F^* \omega^* G) \simeq \text{Hom}_{\mathbf{CI}}(\hat{F}, h^0 \omega^* G) \simeq \text{Hom}_{\mathbf{MPST}}(\hat{F}, \omega^* G) \simeq \text{Hom}_{\mathbf{RSC}}(\omega_i F, G).$$

In view of (15), (1.0.3) induces a pair of adjoint functors

$$\mathbf{CI}^\tau_{\text{sp}} \xrightarrow{\omega_{\mathbf{CI}}} \mathbf{RSC}_{\text{Nis}}. \quad (1.0.4)$$

2. Purity with reduced modulus

For $F \in \mathbf{MPST}$, we put

$$F_{-1} = \text{Ker} \left( \text{Hom}_{\mathbf{MPST}}((\mathbb{P}^1 - 0, \infty), F) \xrightarrow{i^!} F \right),$$

$$F_{-1}^{(1)} = \text{Ker} \left( \text{Hom}_{\mathbf{MPST}}((\mathbb{P}^1, 0 + \infty), F) \xrightarrow{i^!} F \right)$$

Note that if $F \in \mathbf{CI}^\tau_{\text{sp}}_{\text{Nis}}$, then $F_{-1}, F_{-1}^{(1)} \in \mathbf{CI}^\tau_{\text{sp}}_{\text{Nis}}$ and

$$F_{-1}^{(1)}(\mathcal{X}) = \lim_n \text{Hom}_{\mathbf{MPST}}(h^0_{\text{sp}}(\mathbb{P}^1, n \cdot 0 + \infty)^0, \text{Hom}_{\mathbf{MPST}}(\mathbb{Z}_{\text{tr}}(\mathcal{X}), F)), \quad (2.0.1)$$

for $\mathcal{X} \in \mathbf{MCor}$, where

$$h^0_{\text{sp}}(\mathbb{P}^1, n \cdot 0 + \infty)^0 = \text{Coker} \left( \mathbb{Z} = \mathbb{Z}_{\text{tr}}(\text{Spec } k, \emptyset) \xrightarrow{i^!} h^0_{\text{sp}}(\mathbb{P}^1, n \cdot 0 + \infty) \right).$$
Definition 2.1. For $e_1, \ldots, e_r \in \{0, 1\}$, put
\[
\tau^{(e_1, \ldots, e_r)} F = \tau^{(e_r)} \cdots \tau^{(e_1)} F,
\]
where
\[
\tau^{(0)} F = F_{-1} \quad \text{and} \quad \tau^{(1)} F = F_{-1}/F_{-1}^{(1)},
\]
where the quotient is taken in $\text{MPST}$. The existence of retractions in the following lemma was suggested by A. Merici. It implies
\[
\tau^{(e_1, \ldots, e_r)} F \in \text{CI}_{\text{Nis}}^{r, \text{sp}} \quad \text{if} \quad F \in \text{CI}_{\text{Nis}}^{r, \text{sp}}.
\]

Lemma 2.2. For $F \in \text{CI}_{\text{Nis}}^{r, \text{sp}}$, the inclusion $F^{(1)} \rightarrow F_{-1}$ admits a retraction $s_F : F_{-1} \rightarrow F^{(1)}$ such that for any map $\phi : F \rightarrow G$ in $\text{CI}_{\text{Nis}}^{r, \text{sp}}$, the following diagram is commutative:
\[
\begin{array}{ccc}
F_{-1} & \xrightarrow{s_F} & F^{(1)} \\
\downarrow \phi & & \downarrow \phi \\
G_{-1} & \xrightarrow{s_F} & G^{(1)}
\end{array}
\]
In particular, $\tau^{(1)} F \in \text{CI}_{\text{Nis}}^{r, \text{sp}} \quad \text{if} \quad F \in \text{CI}_{\text{Nis}}^{r, \text{sp}}$.

Proof. In view of (2.0.1), this follows from [BRS22, Lemma 2.4].

Theorem 2.3. Let $F \in \text{CI}_{\text{Nis}}^{r, \text{sp}}$. Let $K \{ t_1, \ldots, t_n \}$ be the henselization of $K[ t_1, \ldots, t_n ]$ at $(t_1, \ldots, t_n)$ and $X = \text{Spec} \ K \{ t_1, \ldots, t_n \}$ and $D = \{ t_1^{e_1} \cdots t_n^{e_n} = 0 \} \subset X$ with $e_1, \ldots, e_n \in \{0, 1\}$. For a subset $I \subset [1, n]$ let $i_I : \mathcal{H} \hookrightarrow X$ be the closed immersion defined by $\{ t_i = 0 \}_{i \in I}$ and $D \mathcal{H} = \{ \prod_{j \in [1, n]} - I t_j^{e_j} = 0 \} \subset \mathcal{H}$. Then
\[
R^\nu i^!_H F_{(X, D)} = 0 \quad \text{for} \quad \nu \neq q := |I|,
\]
and there is an isomorphism
\[
(\tau^{(e_1)} F)_{(\mathcal{H}, D \mathcal{H})} \simeq R^q i^!_H F_{(X, D)} \quad \text{with} \quad e_I = (e_i)_{i \in I} \in \mathbb{Z}^q_{\geq 0}.
\]

Proof. The proof is divided into two steps.

Step 1: we prove (2.3.1) and (2.3.2) for $q = |I| = 1$. For $\nu = 0$, (2.3.1) follows from the semipurity of $F$ and [Sai20, Theorem 3.1]. Thus, it suffices to show (2.3.1) only for $\nu > 1$. Let $J = \{ j \in [1, n] | e_j \neq 0 \}$ and $r = |J|$. If $\dim(X) = 0$, the assertion is trivial. If $r = 0$, the assertion follows from [Sai20, Corollary 8.6(3)]. Assume $r > 0$ and $\dim(X) \geq 1$, and proceed by the double induction on $r$ and $\dim(X)$. Without loss of generality, we may assume
\[ (\rhd) \quad e_1 \neq 0, \quad \text{and} \quad \mathcal{H} = \{ t_1 = 0 \} \quad \text{if} \quad \mathcal{H} \subset |D|.
\]
Let $i : \mathcal{Z} \hookrightarrow X$ be the closed immersion defined by $\{ t_1 = 0 \}$ and $D \mathcal{Z} = \{ t_2^{e_2} \cdots t_r^{e_r} = 0 \} \subset \mathcal{Z}$ and $D' = \{ t_2^{e_2} \cdots t_r^{e_r} = 0 \} \subset X$. By [Sai20, Lemma 7.1], we have an exact sequence sheaves on $X_{\text{Nis}}$
\[
0 \rightarrow F_{(X, D')} \rightarrow F_{(X, D)} \rightarrow i_*(F^{(e_1)}_{-1})_{(Z, D \mathcal{Z})} \rightarrow 0,
\]
which gives rise to a long exact sequence of sheaves on $H_{\text{Nis}}$
\[
\cdots \rightarrow R^\nu i^!_H F_{(X, D')} \rightarrow R^\nu i^!_H F_{(X, D)} \rightarrow R^\nu i^!_H t^*_H (F^{(e_1)}_{-1})_{(Z, D \mathcal{Z})} \rightarrow \cdots
\]

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By the induction hypothesis, $R^\nu i_{\mathcal{H}}^! F_{(X,D')} = 0$ for $\nu > 1$. If $\mathcal{H} \neq \mathcal{Z}$, we have a Cartesian diagram of closed immersions

$$
\begin{array}{c}
\mathcal{H} \cap \mathcal{Z} \\
\downarrow \iota_{\mathcal{H} \cap \mathcal{Z}} \\
\mathcal{Z} \\
\downarrow \iota \\
\mathcal{X}
\end{array}
$$

and we have an isomorphism

$$
R^\nu i_{\mathcal{H} \cap \mathcal{Z}}^! (F^{(e_1)}_{-1})(\mathcal{Z}, D_{\mathcal{Z}}) \simeq \iota^! \iota_{\mathcal{H} \cap \mathcal{Z}}^! (F^{(e_1)}_{-1})(\mathcal{Z}, D_{\mathcal{Z}}).
$$

By the induction hypothesis, $R^\nu i_{\mathcal{H} \cap \mathcal{Z}}^! (F^{(e_1)}_{-1})(\mathcal{Z}, D_{\mathcal{Z}}) = 0$ for $\nu > 1$, noting that $F^{(e_1)}_{-1} \in \text{Cl}^{\text{sp}}_{\text{Nis}}$ by Lemma 2.2. So the desired vanishing follows from (2.3.3). Moreover, the assumptions (♠) and $\mathcal{H} \neq \mathcal{Z}$ imply that $\mathcal{H} \not\subset [D]$. Then (2.3.2) (with $q = 1$) follows from [Sai20, Lemma 7.1(2)].

If $\mathcal{Z} = \mathcal{H}$, we have

$$
R^\nu i_{\mathcal{H}}^! t^* (F^{(e_1)}_{-1})(\mathcal{Z}, D_{\mathcal{Z}}) = R^\nu i_{\mathcal{H}}^! (F^{(e_1)}_{-1})(\mathcal{Z}, D_{\mathcal{Z}}),
$$

which vanishes for $\nu > 0$. Hence, (2.3.3) gives the desired vanishing together with an exact sequence

$$
0 \to (F^{(e_1)}_{-1})(\mathcal{H}, D_{\mathcal{H}}) \to \delta \to R^1 i_{\mathcal{H}}^! F_{(X,D')} \to R^1 i_{\mathcal{H}}^! F_{(X,D)} \to 0.
$$

By [Sai20, Lemma 7.1(2)] we have an isomorphism

$$
(F^{(e_1)}_{-1})(\mathcal{H}, D_{\mathcal{H}}) \simeq R^1 i_{\mathcal{H}}^! F_{(X,D')}
$$

through which $\delta$ is identified with the map induced by the canonical map $F^{(e_1)}_{-1} \to F_{-1}$. This proves the desired isomorphism (2.3.2) for $\mathcal{Z} = \mathcal{H}$ and completes step 1.

Step 2: we prove the theorem by induction on $q$ assuming $q > 0$. Let $I = \{i_1, \ldots, i_q\} \subset [1, n]$ and $\mathcal{Y} \subset \mathcal{X}$ be the closed subscheme defined by $\{i_1 = 0\}$. Let $i_\mathcal{Y} : \mathcal{Y} \hookrightarrow \mathcal{X}$ and $i_\mathcal{H}, i_\mathcal{Y} : \mathcal{H} \hookrightarrow \mathcal{Y}$ be the induced closed immersions. By step 1 we have $R^\nu i_{\mathcal{Y}}^! F_{(X,D)} = 0$ for $\nu \neq 1$ and we have an isomorphism

$$
(\tau^{(e_1)}_{\mathcal{Y}} F)_{(\mathcal{Y}, D_{\mathcal{Y}})} \simeq R^1 i_{\mathcal{Y}}^! F_{(X,D)} \quad \text{with} \quad D_{\mathcal{Y}} = \{t_1^{i_1} \cdots t_n^{i_n} = 0\} \subset \mathcal{Y}.
$$

Note $\tau^{(e_1)}_{\mathcal{Y}} F \in \text{Cl}^{\text{sp}}_{\text{Nis}}$ by Lemma 2.2. Thus, by the induction hypothesis, we have $R^\nu i_{\mathcal{H}}^! \tau^{(e_1)}_{\mathcal{Y}} F_{(\mathcal{Y}, D_{\mathcal{Y}})} = 0$ for $\nu \neq q - 1$. By the spectral sequence

$$
E_2^{a,b} = R^b i_{\mathcal{H}}^! R^a i_{\mathcal{Y}}^! F_{(X,D)} \Rightarrow R^{a+b} i_{\mathcal{H}}^! F_{(X,D)},
$$

we get the desired vanishing (2.3.1) and an isomorphism

$$
R^q i_{\mathcal{H}}^! F_{(X,D)} \simeq R^{q-1} i_{\mathcal{H}}^! \tau^{(e_1)}_{\mathcal{Y}} F_{(X,D)} \simeq R^{q-1} i_{\mathcal{H}}^! (\tau^{(e_1)}_{\mathcal{Y}} F)_{(\mathcal{Y}, D_{\mathcal{Y}})} \simeq (\tau^{(e_{i_2}, \ldots, e_{i_q})}_{\mathcal{Y}} (\tau^{(e_1)}_{\mathcal{Y}} F))_{(\mathcal{H}, D_{\mathcal{H}})} \simeq (\tau^{(e_{i_1}, e_{i_2}, \ldots, e_{i_q})} F)_{(\mathcal{H}, D_{\mathcal{H}})},
$$

where the third isomorphism holds by the induction hypothesis. This completes the proof of the theorem.

We say that $\mathcal{X} = (X, D) \in \textbf{MCor}$ is reduced if so is $D$. The following Corollaries 2.4 and 2.5 are immediate consequences of Theorem 2.3.

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Corollary 2.4. Take $F \in \mathcal{C}^{r,sp}_{Nis}$ and $(X, D) \in \mathbf{MCor}_{ls}$ reduced. Let $x \in X^{(n)}$ with $K = k(x)$ and let $\mathcal{X} = X^h_{x}$ be the henselization of $X$ at $x$. Then

$$H^i_{x}(X_{Nis}, F_{(X,D)}) = 0 \text{ for } i \neq n.$$  

Choosing an isomorphism

$$\varepsilon : \mathcal{X} \simeq \text{Spec } K \{t_1, \ldots, t_n\}$$  

such that $D_{\mathcal{X}} = \{t_1^{e_1} \cdots t_n^{e_n} = 0\} \subset \mathcal{X}$ with $e_1, \ldots, e_n \in \{0, 1\}$, there exists an isomorphism depending on $\varepsilon$:

$$\theta_\varepsilon : \tau^{(e_1, e_2, \ldots, e_n)} F(x) \simeq H^n_x(X_{Nis}, F_{(X,D)}).$$

Corollary 2.5. For $F \in \mathcal{C}^{r,sp}_{Nis}$ and $\mathcal{X} = (X, D) \in \mathbf{MCor}_{ls}$ reduced, the following sequence is exact:

$$0 \to F(X, D) \to F(X - D, \emptyset) \to \bigoplus_{\xi \in D(0)} \frac{F(X^h_{\xi}, \xi, \emptyset)}{F(X^h_{\xi}, \xi)}. $$

The idea of deducing the following corollary from the above is due to A. Merici.

Corollary 2.6. Let $\mathcal{X} = (X, D) \in \mathbf{MCor}_{ls}$ be reduced.

(1) Assume given an exact sequence in $\text{MNST}$,

$$0 \to H \xrightarrow{\phi} G \xrightarrow{\psi} F, \quad (2.6.1)$$  

such that $F, G, H \in \mathcal{C}^{r,sp}_{Nis}$ and that $\omega \psi$ is surjective in $\text{NST}$. If $X$ is henselian local, then

$$0 \to H(\mathcal{X}) \to G(\mathcal{X}) \to F(\mathcal{X}) \to 0$$

is exact.

(2) Let $\gamma : F \to G$ be a map in $\mathcal{C}^{r,sp}_{Nis}$ such that $\omega \gamma$ is an isomorphism. Then $F(\mathcal{X}) \to G(\mathcal{X})$ is an isomorphism.

(3) For $F \in \mathcal{C}^{r,sp}_{Nis}$, the unit map $u : F \to \omega \mathcal{C}_{ls} F$ induces an isomorphism $F(\mathcal{X}) \cong \omega \mathcal{C}_{ls} F(\mathcal{X})$.

Proof. To show (1), it suffices to show the surjectivity of $G(\mathcal{X}) \to F(\mathcal{X})$. Let $\eta \in X$ be the generic point and consider the following commutative diagram of the Cousin complexes:

$$
\begin{array}{cccccc}
0 & \longrightarrow & H(\mathcal{X}) & \longrightarrow & H(\eta) & \longrightarrow & \bigoplus_{x \in X^{(1)}} H^1_x(X, H_{\mathcal{X}}) & \longrightarrow & \bigoplus_{y \in X^{(2)}} H^2_y(X, H_{\mathcal{X}}) \\
\downarrow \phi(\eta) & & \downarrow & & \downarrow & & \downarrow \phi(\phi) & & \downarrow \phi(\phi) \\
0 & \longrightarrow & G(\mathcal{X}) & \longrightarrow & G(\eta) & \longrightarrow & \bigoplus_{x \in X^{(1)}} H^1_x(X, G_{\mathcal{X}}) & \longrightarrow & \bigoplus_{y \in X^{(2)}} H^2_y(X, G_{\mathcal{X}}) \\
\downarrow \psi(\eta) & & \downarrow & & \downarrow & & \downarrow \psi(\psi) & & \downarrow \psi(\psi) \\
0 & \longrightarrow & F(\mathcal{X}) & \longrightarrow & F(\eta) & \longrightarrow & \bigoplus_{x \in X^{(1)}} H^1_x(X, F_{\mathcal{X}}) & \longrightarrow & \bigoplus_{y \in X^{(2)}} H^2_y(X, F_{\mathcal{X}})
\end{array}
$$

By Corollary 2.4, the horizontal sequences are exact. By the assumption, $\psi(\eta)$ is surjective. By a diagram chase we are reduced to showing the following claim.

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Claim 2.6.1.

(i) For \( x \in X^{(1)} \), the sequence
\[
H^1_\ast(X, H_X) \to H^1_\ast(X, G_X) \to H^1_\ast(X, F_X)
\]

is exact.

(ii) For \( y \in X^{(2)} \), \( H^2_\ast(\phi) \) is injective.

To show (i), by Corollary 2.4, it suffices to show the exactness of \( \tau^{(e)} H \to \tau^{(e)} G \to \tau^{(e)} F \) for \( e \in \{0, 1\} \). The case \( e = 0 \) follows from the left exactness of the endofunctor \( \text{Hom}_{\text{MPST}}(\mathcal{X}, -) \) on \( \text{MNST} \) for any \( \mathcal{X} \in \text{MCor} \). We have a commutative diagram
\[
\begin{array}{ccc}
\tau^{(1)} H & \phi & \tau^{(1)} G \\
p_H & s_H & p_G \\
\tau^{(0)} H & \phi & \tau^{(0)} G \\
\end{array}
\]

where \( p_* \) are the projections and \( s_* \) is a right inverse of \( p_* \) coming from the retractions from Lemma 2.2. We have
\[
\phi \circ p_H = p_G \circ \phi, \quad \psi \circ p_G = p_F \circ \psi, \quad \phi \circ s_H = s_G \circ \phi, \quad \psi \circ s_G = s_F \circ \psi.
\]

By a diagram chase, the case \( e = 1 \) follows from the case \( e = 0 \).

To show (ii), by Corollary 2.4, it suffices to show the injectivity of \( \tau^{(e)} H \to \tau^{(e)} G \) for \( e \in \{(0, 0), (0, 1), (1, 0), (1, 1)\} \). The case \( e = (0, 0) \) follows from the same left exactness as above, and the other cases from this case thanks to Lemma 2.2.

To show (2), we may assume \( \mathcal{X} \) is henselian local. Then it follows from (1).

Finally, (3) follows from (2) since \( \omega_{\mathcal{X}/\mathbb{Z}} \) is an isomorphism. This completes the proof of the corollary. \( \square \)

3. Review of higher local symbols

In this section we recall from [RS21c] the higher local symbols for reciprocity sheaves, which are a fundamental tool to prove Theorem 4.2, one of the main theorems of this paper. First we introduce some basic notation. In this section \( X \) is a reduced noetherian separated scheme of dimension \( d < \infty \) such that \( X^{(0)} = X^{(d)} \).

Let \( K \) be a field. For an integer \( r \geq 0 \), let \( K_r^M(K) \) be the Milnor \( K \)-group of \( K \). Let \( A \) be a local domain with the function field \( K \). For an ideal \( I \subset A \), let \( \overline{K}_r^M(A, I) \subset K_r^M(K) \) denote the subgroup generated by symbols
\[
\{1 + a, b_1, \ldots, b_{r-1}\} \quad \text{with} \quad a \in I, \ b_i \in A^\times.
\]

Let \( A \) be a noetherian excellent one-dimensional local domain with function field \( K \) and residue field \( F \). Let \( \tilde{A} \) be the normalization of \( A \) and \( S \) be the set of the maximal ideals of \( \tilde{A} \). For \( \mathfrak{m} \in S \), denote \( \kappa(\mathfrak{m}) = \tilde{A}/\mathfrak{m} \). Then we define
\[
\partial_A := \sum_{\mathfrak{m} \in S} \text{Nm}_{\kappa(\mathfrak{m})/F} \circ \partial_{\mathfrak{m}} : K_r^M(K) \to K_{r-1}^M(F),
\]

where \( \partial_{\mathfrak{m}} : K_r^M(K) \to K_{r-1}^M(\kappa(\mathfrak{m})) \) denotes the tame symbol for the discrete valuation ring \( \tilde{A}_\mathfrak{m} \), the localization of \( \tilde{A} \) at \( \mathfrak{m} \), and \( \text{Nm}_{\kappa(\mathfrak{m})/F} \) is the norm map.
The following properties hold for all \( \{x,x',x''\} \in X \).

A maximal chain with break at \( r \) is a chain (3.0.2) with \( n = d - 1 \) and \( x_i \in X(\tau) \), for \( i < r \), and \( x_i \in X(\tau) \), for \( i \geq r \). We denote 

\[
mc_r(X) = \{\text{maximal chain with break at } r \text{ on } X\}.
\]

For \( \{x_0,\ldots,x_{d-1}\} \in mc_r(X) \), we denote by \( b(x) \) the set of \( y \in X(\tau) \) such that 

\[
b(y) := \{x_0,\ldots,x_{r-1},y,x_r,\ldots,x_{d-1}\} \in mc(X).
\] (3.0.3)

In the rest of this section we fix \( F = \omega^\text{CI}G \in \mathcal{C}^\text{p}\mathcal{S}_\text{Nis} \) with \( G \in \mathcal{R}\mathcal{S}_\text{Nis} \) (cf. (1.0.4)). We also fix a function field \( K \) over the base field \( k \). Let \( X \) be an integral scheme of finite type over \( K \) and assume \( d = \dim(X) \geq 1 \). Recall from [RS21c, §5] that we have a collection of bilinear pairings (cf. the convention from §1(9)) 

\[
\{(-,-)_{1/X,K,z} : F(K(X)) \otimes K^\text{M}(K(X)) \to F(K)\}_{z \in mc(X)}.
\] (3.0.4)

The following properties hold for all \( a \in F(K(X)) \) (see Remark 3.1 below).

(1) Let \( X \hookrightarrow X' \) be an open immersion where \( X' \) is an integral \( K \)-scheme of dimension \( d \). Then, for all \( \beta \in K^\text{M}(K(X)) \), 

\[
(a,\beta)_{X/K,z} = (a,\beta)_{X'/K,z'}.
\]

(2) Let \( \{x_0,\ldots,x_{d-1},x_d\} \in mc(X) \) and \( Z \subset X \) be the closure of \( z = x_d \), and set \( \{x_0,\ldots,x_{d-1}\} \in mc(Z) \). Assume \( a \in F(O_{X,z}) \) and let \( a(z) \in F(K(Z)) \) be the restriction of \( a \). Then 

\[
(a,\beta)_{X/K,z} = (a(z),\partial_z \beta)_{Z/K,z'} \quad \text{for } \beta \in K^\text{M}(K(X)),
\]

where \( \partial_z : K^\text{M}(K(X)) \to K^\text{M}(K(Z)) \) is the map (3.0.1) for \( A = O_{X,z} \).

(3) Let \( D \subset X \) be an effective Cartier divisor with \( I_D \subset \mathcal{O}_X \) its ideal sheaf. Assume that \( X \setminus D \) is regular so that \( (X,D) \in \text{MCor}^\text{pro} \) and that \( a \in F(X,D) \). For \( \{x_0,\ldots,x_{d-1},x_d\} \in mc(X) \), we have 

\[
(a,\beta)_{X/K,z} = 0 \quad \text{for } \beta \in \mathcal{K}^\text{M}(O_{X,x_{d-1}},I_D \mathcal{O}_{X,x_{d-1}}).
\]

(4) Let \( \{x_0,\ldots,x_{d-1},x_d\} \in mc_r(X) \) with \( 0 \leq r \leq d-1 \). For \( \beta \in K^\text{M}(K(X)) \), 

\[
(a,\beta)_{X/K,z}(y) = 0 \quad \text{for almost all } y \in \{x_0,\ldots,x_{d-1}\}.
\]

Assume that either \( r \geq 1 \) or \( r = 0 \), \( X \) is quasi-projective, and the closure of \( x_1 \) in \( X \) is projective over \( K \), where \( \{x_0,\ldots,x_{d-1}\} \). Then 

\[
\sum_{y \in \{x_0,\ldots,x_{d-1}\}} (a,\beta)_{X/K,z}(y) = 0.
\]
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Remark 3.1. Properties (HS1)–(HS4) are slight variants of the (stronger) properties (HS1)–(HS4) in [RS21c, Proposition 5.3], where the Milnor $K$-group $K^M_d(K^{h}_{X,x})$ of the iterated henselization $K^{h}_{X,x}$ of $K(X)$ along the chain $x$ is used instead of $K^M_d(K(X))$. The version stated here follows easily using the natural maps $i_x : K(X) \to K^h_{X,x}$ and the commutative diagram in the situation of (HS2),

$$
K^M_d(K^{h}_{X,x}) \xrightarrow{\partial_x} K^M_{d-1}(K^{h}_{Z,x'})
$$

$$
\uparrow i_x \quad \quad \uparrow i'_{x'}
$$

$$
K^M_d(K(X)) \xrightarrow{\partial_x} K^M_{d-1}(K(Z))
$$

and the commutative diagram in the situation of (HS4),

$$
K^M_{d-1}(K^{h}_{X,x'}) \xrightarrow{i_{x'}} K^M_{d-1}(K^h_{X,x'(y)})
$$

where $\partial_x$ (respectively, $i_y$) is defined in [RS21c, (4.1.1)] (respectively, [RS21c, (3.2.3)]). We also note that $K^M_d(O_{X,x_{d-1}}, I_D O_{X,x_{d-1}})$ in (HS2) coincides with the Zariski stalk at $x_{d-1}$ of the sheaf $\mathcal{V}_{d,X|D}$ defined in [RS21c, 4.4].

For a scheme $Z$ over $k$, write $Z_K = Z \otimes_k K$. If $Z_K$ is integral, we denote by $K(Z)$ the function field of $Z_K$. We quote the following result from [RS21c, Proposition 7.3]. It is a key tool in the proof of Theorem 4.2.

Proposition 3.2. Let $X \in \text{Sm}$ and assume $D$ is a reduced simple normal crossing divisor on $X$ with $I_D \subset O_X$ its ideal sheaf. Let $U \subset X$ be an open subset containing all the generic points of $D$. Let $a \in F(X \setminus D)$. Assume that, for all function fields $K/k$ and for all $x = (x_0, \ldots, x_{d-1}, x_d) \in \text{mc}(U_K)$ with $x_{d-1} \in D^{(0)}_K$, we have

$$(a, \beta)_{X_K/k,x} = 0 \quad \text{for all} \quad \beta \in K^M(O_{X,x_{d-1}}, I_D O_{X,x_{d-1}}).$$

Then $a \in F(X, D)$.

4. Logarithmic cohomology of reciprocity sheaves

For $\mathcal{X} = (X, D) \in \text{MCor}_{ls}$, we write $\mathcal{X}_{\text{red}} = (X, D_{\text{red}}) \in \text{MCor}_{ls}$. We say that $\mathcal{X} = (X, D) \in \text{MCor}_{ls}$ is reduced if $\mathcal{X} = \mathcal{X}_{\text{red}}$.

Definition 4.1. Let $F \in \text{MPST}$.

(1) We say that $F$ is log-semipure if for any $\mathcal{X} \in \text{MCor}_{ls}$, the map $F(\mathcal{X}_{\text{red}}) \to F(\mathcal{X})$ is injective. Note that if $F$ is semipure, $F$ is log-semipure (cf. §1(16)).

(2) We say that $F$ is logarithmic if it is log-semipure and satisfies the condition that for $\mathcal{X}, \mathcal{Y} \in \text{MCor}_{ls}$ with $\mathcal{X}$ reduced and $\alpha \in \text{MCor}^{\text{fin}}(\mathcal{Y}, \mathcal{X})$, the image of $\alpha^* : F(\mathcal{X}) \to F(\mathcal{Y})$ is contained in $F(\mathcal{Y}_{\text{red}}) \subset F(\mathcal{Y})$.

Let $\text{MPST}^\text{log}$ be the full subcategory of $\text{MPST}$ consisting of logarithmic objects and put $\text{MNST}^\text{log} = \text{MNST} \cap \text{MPST}^\text{log}$.
Theorem 4.2. Any $F \in \text{CI}_{\text{Nis}}^{r,sp}$ is logarithmic, that is, $\text{CI}_{\text{Nis}}^{r,sp} \subset \text{MNST}_{\text{log}}$.

We need a preliminary lemma for the proof of the theorem.

Lemma 4.3. Let $F \in \text{CI}_{\text{Nis}}^{r,sp}$. Let $\mathbf{A}_K^n = \text{Spec} K[x_1, \ldots, x_n]$ be the affine space over an algebraic field $K$ over $k$ and $V = \text{Spec} K \{x_1, \ldots, x_n\}$ be the henselization of $\mathbf{A}_K^n$ at the origin and $\mathcal{L}_i = \{x_i = 0\} \subset V$ for $i \in [1, n]$. For an integer $0 < r \leq n$, the natural map

$$K\{x_{r+1}, \ldots, x_n\}[x_1, \ldots, x_r] \rightarrow K\{x_1, \ldots, x_n\}$$

induces a map in $\text{MCor}_{\text{pro}}$ (cf. §1(9)):

$$\rho_r : (V, \mathcal{L}_1 + \cdots + \mathcal{L}_r) \rightarrow (\mathbf{A}_S^r, \{x_1 \cdots x_r = 0\}) \simeq (A^1, 0)^{\otimes r} \otimes (S, \emptyset),$$

where $S = \text{Spec} K\{x_{r+1}, \ldots, x_n\}$. It induces

$$\rho_r^* : F(\mathbf{A}_S^r, \{x_1 \cdots x_r = 0\}) \rightarrow F(V, \mathcal{L}_1 + \cdots + \mathcal{L}_r).$$

Then $F(V, \mathcal{L}_1 + \cdots + \mathcal{L}_r)$ is generated by the image of $\rho_r^*$ and

$$F(V, \mathcal{L}_1 + \cdots + \mathcal{L}_i + \cdots \mathcal{L}_r) \quad \text{for} \quad i = 1, \ldots, r.$$

Proof. For $Y \subset \text{MCor}$, let $F^Y \subset \text{MPST}$ be defined by $F^Y(Z) = F(Y \otimes Z)$. Clearly, we have $F^Y \subset \text{CI}_{\text{Nis}}^{r,sp}$ for $F \subset \text{CI}_{\text{Nis}}^{r,sp}$. We prove the lemma by induction on $r$. The case $r = 1$ holds since by [Sai20, Lemmas 7.1 and 5.9], $p_1$ induces an isomorphism

$$F(A^1, 0)(S)/F(A^1, 0)(S) \simeq F(V, \mathcal{L}_1)/F(V).$$

By definition $\mathcal{L}_1 = \text{Spec} K\{x_2, \ldots, x_n\}$ and we have a map in $\text{MCor}_{\text{pro}}$,

$$(V, \mathcal{L}_1 + \cdots + \mathcal{L}_r) \rightarrow (A^1, 0) \otimes (\mathcal{L}_1, \mathcal{L}_1 \cap (\mathcal{L}_2 + \cdots + \mathcal{L}_r)),$$

induced by the natural map $K\{x_2, \ldots, x_n\}[x_1] \rightarrow K\{x_1, \ldots, x_n\}$. By [Sai20, Lemmas 7.1 and 5.9], it induces an isomorphism

$$F(A^1, 0)(\mathcal{L}_1, E)/F(A^1, 0)(\mathcal{L}_1, E) \simeq F(V, \mathcal{L}_1 + \cdots + \mathcal{L}_r)/F(V, \mathcal{L}_2 + \cdots + \mathcal{L}_r)$$

with $E = \mathcal{L}_1 \cap (\mathcal{L}_2 + \cdots + \mathcal{L}_r)$. By the induction hypothesis, $F(A^1, 0)(\mathcal{L}_1, E)$ is generated by $F(A^1, 0)(\mathcal{L}_1, E_j)$ with $E_j = \mathcal{L}_1 \cap (\mathcal{L}_2 + \cdots + \mathcal{L}_j) + \cdots + \mathcal{L}_r$ for $j = 2, \ldots, r$ together with the image of the map

$$F(A^1, 0)^{\otimes r-1}(S) = F(A^1, 0)^{\otimes r}(S) \rightarrow F(A^1, 0)(\mathcal{L}_1, E)$$

induced by

$$\mathcal{L}_1 \rightarrow (A_S^{r-1}, \{x_2 \cdots x_r = 0\}) \simeq (A^1, 0)^{\otimes r-1} \otimes (S, \emptyset)$$

coming from the map $K\{x_{r+1}, \ldots, x_n\}[x_2, \ldots, x_r] \rightarrow K\{x_2, \ldots, x_n\}$. This proves the lemma. \qed

Proof of Theorem 4.2. By Corollary 2.6(3), we may assume $F = \omega_{\text{CI} G}$ for $G \in \text{RSC}_{\text{Nis}}$. Take $X = (X, D)$, $Y = (Y, E) \subset \text{MCor}_{\text{Is}}$ with $X'$ reduced, and let $\alpha \in \text{MCor}_{\text{fin}}(Y, X)$ be an elementary correspondence. We need to show that $\alpha^*(F(X')) \subset F(Y_{\text{red}})$. The question is Nisnevich local over $X$ and $Y$. Hence, we may assume $(X, D) = (V, \mathcal{L}_1 + \cdots + \mathcal{L}_r) \subset \text{MCor}_{\text{pro}}$ in the notation of Lemma 4.3. If $r = 0$, we have $\alpha \in \text{MCor}_{\text{fin}}(Y, X)$ by the assumption $\alpha \in \text{MCor}_{\text{fin}}(Y, X)$ so that

$$\alpha^*(F(X)) = \alpha^*(F(X, \emptyset)) \subset F(Y, \emptyset) \subset F(Y_{\text{red}}).$$

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Assume \( r > 0 \) and proceed by induction on \( r \). By Lemma 4.3, we may then assume
\[
(X, D) = M := (\mathbf{A}^1, 0)^{\otimes r} \otimes (S, \emptyset) \quad \text{for} \ S \in \text{Sm}^{\text{pro}}.
\]

On the other hand, by Corollary 2.5, we have an exact sequence
\[
0 \to F(Y, E_{\text{red}}) \to F(Y - E_{\text{red}}, \emptyset) \to \bigoplus_{\xi \in E^{(0)}} F(Y_{[\xi]}^h - \xi, \emptyset).
\]

Hence, we may replace \( Y \) with its Nisnevich neighborhood of a generic point \( \xi \) of \( E \). Using the assumption that \( k \) is perfect, we may then assume the following condition (\( \spadesuit \)). Recall that \( \alpha \) is by definition an integral closed subscheme of \((Y - E) \times (X - D)\) finite surjective over \( Y - E \), and its closure \( \overline{\alpha} \) in \( Y \times X \) is finite surjective over \( Y \).

(\( \spadesuit \)) Let \( Y' \) be the normalization of \( \overline{\alpha} \) and \( E' := E \times_Y Y' \). Then \( X, Y, E \) and \( E' \) are irreducible, and \( \alpha, Y', E_{\text{red}} \) and \( E'_{\text{red}} \) are essentially smooth over \( k \).

Let \( g : Y' \to Y \) and \( f : Y' \to X \) be the induced maps. We have \( E' = g^*E \geq f^*D \) as Cartier divisors on \( Y' \) by the modulus condition for \( \alpha \). Hence, these maps induce
\[
F(X, D) \xrightarrow{f^*} F(Y', E') \xrightarrow{g_*} F(Y, E).
\]

We claim that \( \alpha^* : F(X, D) \to F(Y, E) \) agrees with this map. Indeed, this follows from the equality
\[
\Gamma_f \circ \Gamma_g = \alpha \in \text{Cor}(Y - E, X - D),
\]

where \( \Gamma_g \in \text{Cor}(Y - E, Y' - E') \) is the transpose of the graph of \( g \) and \( \Gamma_f \in \text{Cor}(Y' - E', X - D) \) is the graph of \( f \). By definition this follows from the equality
\[
\Gamma_g \times_{Y' - E'} \Gamma_f = \alpha \subset (Y - E) \times (X - D)
\]

which one can check easily, noting that \( Y' \to \overline{\alpha} \) is an isomorphism over \( \alpha \) since \( \alpha \) is regular by (\( \spadesuit \)). Then we get a commutative diagram
\[
\begin{array}{ccc}
F(Y', E'_{\text{red}}) & \xrightarrow{\sim} & F(Y', E_{\text{red}}) \times_Y Y' \\
\downarrow & & \downarrow \\
F(Y', E_{\text{red}} \times_Y Y') & \xrightarrow{g_*} & F(Y, E_{\text{red}}) \\
\downarrow & & \downarrow \\
F(X, D) & \xrightarrow{f^*} & F(Y', E') \\
\downarrow & & \downarrow \\
& & F(Y, E)
\end{array}
\]

where the top inclusion comes from the inequality \( E_{\text{red}} \times_Y Y' \geq E'_{\text{red}} \) as Cartier divisors on \( Y' \) thanks to the semipurity of \( F \) (cf. §1(16)). Hence, it suffices to show \( f^*(F(X, D)) \subset F(Y', E'_{\text{red}}) \).

By replacing \((Y, E)\) with \((Y', E')\), we may now assume that \( \alpha \) is induced by a morphism \( f : Y \to X = \mathbf{A}^r \times S \). Then \( \alpha \) factors in \text{MCor} as
\[
(Y, E) \xrightarrow{i} (\mathbf{A}^1, 0)^{\otimes r} \otimes (Y, \emptyset) \to (\mathbf{A}^1, 0)^{\otimes r} \otimes (S, \emptyset),
\]

where the first map is induced by the map
\[
i = (pr_{\mathbf{A}^r} \circ f, id_Y) : Y \to \mathbf{A}^r \times Y,
\]

and the second is induced by
\[
id_{\mathbf{A}^r} \times (pr_S \circ f) : \mathbf{A}^r \times Y \to \mathbf{A}^r \times S.
\]

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Note that $i$ is a section of the projection $A^r \times Y \to Y$. Thus, we are reduced to showing $i^*(F((A^1, 0)^{\otimes r} \otimes (Y, 0))) \subset F(Y, E_{\text{red}})$. By Proposition 3.2 this follows from the following claim.

**Claim 4.3.1.** Take $a \in F((A^1, 0)^{\otimes r} \otimes (Y, 0))$. There exists an open neighborhood $U \subset Y$ of the generic point of $E$ such that for every function field $K$ over $k$ and every $\delta = (\delta_0, \ldots, \delta_{e-1}, \delta_e) \in \mathfrak{m}(U_K)$ with $\xi := \delta_{e-1} \in E^{(0)}_K$ and $e = \dim(Y)$, we have

$$
(i^*(a)_{K, \gamma} Y_{K, \delta}) = 0 \quad \forall \gamma \in \mathcal{K}^M_e(O_{Y_K, \xi}, m_\xi)
$$

for the pairing from (3.0.4):

$$
(-, -)_{Y_K, \delta} : F(K(Y)) \otimes K^M_d(K(Y)) \to F(K).
$$

**Proof.** After replacing $Y$ by an open neighborhood of the generic point of $E$, we may assume that $Y = \text{Spec}(A)$ is affine and $E_{\text{red}} = \text{Spec}(A/(\pi))$ for $\pi \in A$ and, moreover, that writing

$$
A^r \times Y = \text{Spec} A[x_1, \ldots, x_r], \quad (A^1, 0)^{\otimes r} \otimes (Y, 0) = (A^r_Y, \{x_1 \cdots x_r = 0\}),
$$

we have

$$
i(Y) = \bigcap_{1 \leq i \leq r} \{x_i - u_i \pi^m = 0\} \quad \text{with} \quad m_i \in \mathbb{Z}_{\geq 0}, \ u_i \in A^r.
$$

Let $\delta = (\delta_0, \ldots, \delta_e)$ be as in the claim and put $\delta' = (\delta_0, \ldots, \delta_{e-1}) \in \mathfrak{m}(E_{\text{red}})_K$. Put $\tilde{X}_K = A^r \times Y_K$ and $F = \{\pi = 0\} \subset \tilde{X}_K$. Note $d := \dim(\tilde{X}_K) = e + r$. Let $z_j$ for $e \leq j \leq d - 1$ be the generic point of

$$Z_j = \bigcap_{1 \leq i \leq d-j} \{x_i - u_i \pi^m = 0\} \subset \tilde{X}_K
$$

which lies over $\delta_e$, and $w_j$ for $e - 1 \leq j \leq d - 2$ be the generic point of

$$W_j = F \cap Z_{j+1} = \{\pi = x_1 = \cdots = x_{d-j} = 0\}
$$

which is contained in the closure of $z_{j+1}$. Note $\dim(Z_j) = \dim(W_j) = j$ and the section $i$ induces isomorphisms

$$Y_K \simeq Z_e \quad \text{and} \quad (E_{\text{red}})_K \simeq W_{e-1}. \quad (4.3.2)
$$

Let $\sigma = (i(\delta'), w_1, w_2, \ldots, w_{d-2}, \eta_1, \nu) \in \mathfrak{m}(\tilde{X}_K)$, where $\nu$ is the generic point of $\tilde{X}_K$ lying over $\delta_e$, $\eta_1$ is the generic point of $D_1 = \{x_1 = 0\} \subset \tilde{X}_K$ contained in the closure of $\nu$, and $i(\delta') \in \mathfrak{m}(W_{e-1})$ is the image of $\delta'$ under (4.3.2). Take any $\gamma \in \mathcal{K}^M_e(O_{Y_K, \xi}, m_\xi)$ and put

$$
\beta = \left\{i(\gamma), \frac{u_1 \pi^m}{u_1 \pi^m}, \ldots, \frac{u_r \pi^m}{u_r \pi^m} \right\} \in \mathcal{K}^M_d(O_{\tilde{X}_K, \nu}), \quad (4.3.3)
$$

where $\iota : K^M_e(O_{Y_K, \delta}) \to K^M_e(O_{\tilde{X}_K, \nu})$ is induced by the projection $\tilde{X}_K \to Y_K$. For $a \in F((A^1, 0)^{\otimes r} \otimes (Y, 0))$ and its restriction $a_K \in F((A^1, 0)^{\otimes r} \otimes (Y_K, 0))$, we have

$$
0 = (a_K, \beta)_{\tilde{X}_K/K, \sigma} = - \sum_{\tau \in \tilde{X}_K^n \setminus \{\eta_1\} \frac{\tau}{\tau > u_{d-2}}} \langle a_K, \beta \rangle_{\tilde{X}_K/K, (i(\delta'), w_1, \ldots, w_{d-2}, \tau, \nu)}
$$

$$
= - (a_K, \beta)_{\tilde{X}_K/K, (i(\delta'), w_1, \ldots, w_{d-2}, z_{d-1}, \nu)}
$$

$$
= \pm (a_K)_{Z_{d-1}, \beta_1} Z_{d-1}/K, (i(\delta'), w_1, \ldots, w_{d-2}, z_{d-1}, \nu).
$$

---

5 Although $Y$ is assumed to be irreducible, $Y_K$ may not be so and possibly a finite product of schemes essentially smooth over $k$, noting that $k$ is perfect.
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\[ \beta_1 = \left\{ \left( \gamma_1, \frac{u_2 \pi^m - x_2}{u_2 \pi^m}, \ldots, \frac{u_r \pi^m - x_r}{u_r \pi^m} \right) \right\} \in K^M_{d-1}(O_{Z_{d-1}, z_{d-1}}), \]

where \( \iota_1 : K^M_e(O_{Y_{K, \delta}}) \to K^M_e(O_{Z_{d-1}, z_{d-1}}) \) is induced by the dominant map \( Z_{d-1} \to Y_K \) induced by the projection \( \tilde{X}_K \to Y_K \). The first equality follows from \( §3 \) (HS3) applied to \( D_1 \subset \tilde{X}_K \), noting that \( \beta \) lies in \( K^M_d(O_{\tilde{X}_K, n_1}, m_n) \) since \( (u_1 \pi^m - x_1)/u_1 \pi^m \in 1 + x_1 O_{X, n_1} \). The second follows from (HS4). The third equality holds since \( z_{d-1} \) is the unique \( \tau \in \tilde{X}_K^{(1)} \) such that \( \tau > w_{d-2} \) and \( (a_K, \beta) \tilde{X}_K/K, (\iota(\delta'), w_{\ldots, w_{d-2}}, \tau, \nu) \) may not vanish, which follows from (HS2), noting that \( \iota(\gamma))F = 0 \). The final equality follows from (HS2). When \( r = 1 \), the last term in the above formula is equal to \( (a_K)_{Y_K, \gamma})_{Y_K/K, \delta} \) by (4.3.2), so that the proof is complete. When \( r > 1 \), we further get

\[ 0 = ((a_K)_{Z_{d-1}, \beta})_{Z_{d-1}/K, (i(\delta'), w_{\ldots, w_{d-2}}, \tau, z_{d-1})} \]

\[ = - \sum_{\tau \in Z^{(1)}_{d-1} \setminus \{w_{d-2}\}} ((a_K)_{Z_{d-1}, \beta})_{Z_{d-1}/K, (i(\delta'), w_{\ldots, w_{d-3}}, \tau, z_{d-1})} \]

\[ = - ((a_K)_{Z_{d-1}, \beta})_{Z_{d-1}/K, (i(\delta'), w_{\ldots, w_{d-3}}, z_{d-2}, z_{d-1})} \]

\[ = \pm ((a_K)_{Z_{d-2}, \beta})_{Z_{d-2}/K, (i(\delta'), w_{\ldots, w_{d-3}}, z_{d-2})} \]

\[ \beta_2 = \left\{ \left( \gamma_2, \frac{u_3 \pi^m - x_3}{u_3 \pi^m}, \ldots, \frac{u_r \pi^m - x_r}{u_r \pi^m} \right) \right\} \in K^M_{d-1}(O_{Z_{d-2}, z_{d-2}}), \]

where \( \iota_2 : K^M_e(O_{Y_{K, \delta}}) \to K^M_e(O_{Z_{d-2}, z_{d-2}}) \) is induced by the dominant map \( Z_{d-2} \to Y_K \) induced by the projection \( \tilde{X}_K \to Y_K \). The above equalities hold by the same arguments as above, except that for the third equality there are a priori two \( \tau \in Z^{(1)}_{d-1} - \{w_{d-2}\} \) with \( \tau > w_{d-3} \) for which \( ((a_K)_{Z_{d-1}, \beta})_{Z_{d-1}/K, (i(\delta'), w_{\ldots, w_{d-3}}, \tau, z_{d-1})} \) may not vanish. One is \( z_{d-2} \) and the other is the generic point \( \eta_2 \) of \( Z_{d-1} \cap D_2 \) with \( D_2 = \{ x_2 = 0 \} \subset \tilde{X}_K \) which is contained in the closure of \( z_{d-1} \). But \( ((a_K)_{Z_{d-1}, \beta})_{Z_{d-1}/K, (i(\delta'), w_{\ldots, w_{d-3}}, \eta_2, z_{d-1})} = 0 \). Indeed, \( (a_K)_{Z_{d-1}} \in F(\text{Spec}(O_{Z_{d-1}, \eta_2}, \eta_2)) \) since \( Z_{d-1} \) and \( D_2 \) intersect transversely in \( \tilde{X}_K \). Hence, the vanishing follows from (HS3) applied to \( Z_{d-1} \cap D_2 \subset Z_{d-1} \), noting that \( \left( (u_2 \pi^m - x_2)/u_2 \pi^m \right)_{Z_{d-1}} \in 1 + x_2 O_{Z_{d-1}, \eta_2} \) so that \( \beta_1 \in K^M_d(O_{Z_{d-1}, \eta_2}, m_\eta_2) \). Repeating the same arguments, we finally get

\[ 0 = ((a_K)_{Z_e, \iota_2(\gamma)})_{Z_e/K, (\iota(\delta'), z_e) = ((a_K)_{Y_K, \gamma})_{Y_K/K, \delta}, \]

where \( \iota_2 : K^M_e(O_{Y_{K, \delta}}) \to K^M_e(O_{Z_e, z_e}) \) is induced by the isomorphism \( Z_e \to Y_K \) induced by the projection \( \tilde{X}_K \to Y_K \) and the second equality follows from (4.3.2). This completes the proof of the claim and Theorem 4.2.

\[ \square \]

**Definition 4.4.** For \( F \in MNST^\log \) and an integer \( i \geq 0 \), consider the association

\[ H^i_{\log, (-, F)} : MCor^{\text{fin}}_{\text{is}} \to \text{Ab} ; (X, D) \to H^i(X_{\text{Nis}}, F_{(X, D, \text{rat})}). \]

By the definition this gives a presheaf on \( MCor^{\text{fin}}_{\text{is}} \), which we call the \( i \)th logarithmic cohomology with coefficient \( F \).

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5. Invariance of logarithmic cohomology under blowups

Retain the notation of § 4.

**Definition 5.1.** Let \( \Lambda_{\text{fin}}^{\text{ls}} \) be the class of morphisms \( \rho : (Y, E) \to (X, D) \) in \( \text{MCor}_{\text{ls}}^{\text{fin}} \) satisfying the following conditions.

(a) \( \rho \) is induced by a proper morphism \( \rho : Y \to X \) inducing an isomorphism \( Y \setminus E \xrightarrow{\sim} X \setminus D \) and \( E = \rho^* D \).

(b) Zariski locally on \( X \), \( \rho : Y \to X \) is the blowup of \( X \) in a smooth center \( Z \subset D \) which is normal crossing to \( D \).

Here, a smooth \( Z \) contained in \( D \) is normal crossing to \( D \) if, letting \( D_1, \ldots, D_n \) be the irreducible components of \( D \), there exists a subset \( I \subset \{ 1, \ldots, n \} \) such that \( Z \subset \bigcap_{i \in I} D_i \) and \( Z \) is not contained in \( D_j \) for any \( j \notin I \) and intersects \( \sum_{j \notin I} D_j \) transversally. Note that the condition is equivalent to that called strict normal crossing in [BPO22, Definition 7.2.1].

**Theorem 5.2.** For \( F \in \text{CI}_{\text{Nis}}^{r, \text{sp}} \) and \( \rho : Y \to X \) in \( \Lambda_{\text{fin}}^{\text{ls}} \), we have

\[
\rho^* : H^i_{\log}(X, F) \cong H^i_{\log}(Y, F) \quad \forall i \geq 0. \tag{5.2.1}
\]

**Proof.** Write \( Y = (Y, E) \) and \( X = (X, D) \). First we prove the theorem for \( i = 0 \). We may assume that \( D \) is reduced and \( E = \rho^* D \). By [KMSY21a, Proposition 1.9.2 b]), \( \rho \) is invertible in \( \text{MCor} \), so that \( \rho^* : F(X) \cong F(Y) \). Since this factors through \( F(Y, E_{\text{red}}) \) by Theorem 4.2, we get (5.2.1) for \( i = 0 \).

To show (5.2.1) for \( i > 0 \), it suffices to prove \( R^i \rho_* F(Y, E_{\text{red}}) = 0 \). The problem is Nisnevich local, so we may assume that \( \rho \) is induced by a blowup \( \rho : Y \to X \) in a smooth center \( Z \subset D \) normal crossing to \( D \). By [KS21, Corollary 9], Nisnevich locally around a point of \( Z \), \((X, D)\) is isomorphic to

\[
(A^c, L_1 + \cdots + L_r) \otimes W \quad \text{with } W = (W, W^{\infty}) \in \text{MCor}_{\text{ls}},
\]

where \( A^c = \text{Spec } k[t_1, \ldots, t_c] \) with \( c = \text{codim}(Z, X) \) and \( L_i = V(t_i) \) for \( i = 1, \ldots, r \) with \( 1 \leq r \leq c \), and \( Z \) corresponds to \( 0 \times W \). Hence, the theorem follows from the following proposition.

**Proposition 5.3.** Let \( F \in \text{CI}_{\text{Nis}}^{r, \text{sp}} \) and \( W = (W, W^{\infty}) \in \text{MCor}_{\text{ls}} \). Let \( A^n = \text{Spec } k[t_1, \ldots, t_n] \) and put \( L_i = V(t_i) \) for \( 1 \leq i \leq n \). Let \( \rho : Y \to A^n \) be the blowup at the origin \( 0 \in A^n \) and \( L_i \subset Y \) be the strict transforms of \( L_i \) for \( 1 \leq i \leq n \) and \( E = \rho^{-1}(0) \subset Y \). For any \( 1 \leq r \leq n \), we have

\[
R^i \rho_{W*} F_{(Y, L_1 + \cdots + L_r + E) \otimes W} = 0 \quad \text{for } i \geq 1, \tag{5.3.1}
\]

where \( \rho_W := \rho \times \text{id}_W : Y \times W \to A^2 \times W \).

**Lemma 5.4.** Proposition 5.3 holds for \( n = 2 \).

**Proof.** The case \( r = 1 \) is proved in [BRS22, Lemma 2.13] and we show the case \( r = 2 \).[6] Put \( D = L_1 + L_2 \). By the case \( i = 0 \) of Theorem 5.2, we get

\[
F_{(A^2, D) \otimes W} \cong \rho_{W*} F_{(Y, L_1 + L_2 + E) \otimes W}. \tag{5.4.1}
\]

Set

\[
F := F_{(Y, L_1 + L_2 + E) \otimes W}
\]

[6] The following argument is adopted from [BRS22, Lemma 2.13], but the present case is easier.
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and $A^2_W = \mathbb{A}^2 \times W$ with the projection $p : A^2_W \to W$. Since $R^i \rho_{W*} \mathcal{F}$ for $i \geq 1$ is supported in $0 \times W$, we have

$$R^i \rho_{W*} \mathcal{F} = 0 \iff p_* R^i \rho_{W*} \mathcal{F} = 0$$

$$\iff (p_* R^i \rho_{W*} \mathcal{F})_w = 0 \quad \forall w \in W$$

$$\iff H^0(\mathbb{A}^2_W, R^i \rho_{W*} \mathcal{F}) = 0 \quad \forall w \in W,$$

where $W_w$ is the henselization of $W$ at $w$. Hence, it suffices to show $H^0(\mathbb{A}^2_W, R^i \rho_{W*} \mathcal{F}) = 0$, assuming $W$ is henselian local. Then we have

$$H^j(\mathbb{A}^2_W, R^i \rho_{W*} \mathcal{F}) = 0, \quad \forall i, j \geq 1.$$ 

By (5.4.1) and [BRS22, Lemma 2.10],

$$H^i(\mathbb{A}^2_W, \rho_{W*} \mathcal{F}) = H^i(\mathbb{A}^2_W, F_{(\mathbb{A}^2, \mathcal{D}) \otimes W}) = 0.$$ 

Thus, the Leray spectral sequence yields

$$H^0(\mathbb{A}^2_W, R^i \rho_{W*} \mathcal{F}) = H^i(Y \times W, \mathcal{F}), \quad i \geq 0,$$

and we have to show that this group vanishes for $i \geq 1$. We can write

$$\mathbb{A}^2 = \text{Spec } k[x, y] \quad \text{and} \quad L_1 = V(x), \ L_2 = V(y) \subset \mathbb{A}^2.$$ 

Then we have

$$Y = \text{Proj } k[x, y][S, T]/(xT - yS) \subset \mathbb{A}^2 \times \mathbb{P}^1.$$ 

Denote by

$$\pi_0 : Y \hookrightarrow \mathbb{A}^2 \times \mathbb{P}^1 \to \mathbb{P}^1 = \text{Proj } k[S, T]$$

the morphism induced by projection, and let $\pi : Y \times W \to \mathbb{P}^1_W$ be its base change. Then $\pi_0$ induces an isomorphism $E \simeq \mathbb{P}^1$, and we have

$$\tilde{L}_1 = \pi_0^{-1}(0), \quad \tilde{L}_2 = \pi_0^{-1}(\infty). \quad (5.4.2)$$

Set $s = S/T = x/y$ and write

$$\mathbb{P}^1 \setminus \{\infty\} = A^1_s := \text{Spec } k[s], \quad \mathbb{P}^1 \setminus \{0\} = \text{Spec } k[\frac{1}{s}].$$ 

Set $U := A^1_s \times W$, $V := (\mathbb{P}^1 \setminus \{0\}) \times W$ and

$$U := (A^1_s, 0) \otimes W, \quad V := (\mathbb{P}^1 \setminus \{0\}, \infty) \otimes W.$$ 

We have

$$\pi^{-1}(U) = A^1_y \times U, \quad \pi^{-1}(V) = A^1_x \times V,$$

and the restriction of $\pi$ to these open subsets is given by projection. Furthermore, $E \times W \subset Y$ is defined by $y = 0$ on $\pi^{-1}(U)$ and by $x = 0$ on $\pi^{-1}(V)$. In view of (5.4.2), we have

$$\mathcal{F}|_{\pi^{-1}(U)} = F_{(A^1_y, 0) \otimes U}, \quad \mathcal{F}|_{\pi^{-1}(V)} = F_{(A^1_x, 0) \otimes V}. \quad (5.4.3)$$

Thus, [BRS22, Lemma 2.10] yields

$$R^j \pi_* \mathcal{F} = 0 \quad \text{for } j \geq 1,$$

and it remains to show

$$H^i(\mathbb{P}^1_W, \pi_* \mathcal{F}) = 0 \quad \text{for } i \geq 1, \quad (5.4.4)$$

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where $P^1_W = P^1 \times W$. For this consider the map
\[
a_0 : Y \to A^1_x \times P^1
\]
which is the closed immersion $Y \hookrightarrow A^2 \times P^1$ followed by the projection $A^2 \to A^1_x$. Let $a : Y \times W \to A^1_x \times P^1 \times W$ be its base change. In view of (5.4.2), the map $a$ induces a morphism in $MCor$,
\[
\alpha : (Y, \tilde{L}_1 + \tilde{L}_2 + E) \otimes W \to (A^1_x, 0) \otimes (P^1, \infty) \otimes W,
\]
which is an isomorphism over $(A^1_x, 0) \otimes (P^1 \setminus \{0\}, \infty) \otimes W$. Setting
\[
F_1 := \text{Hom}(\mathbb{Z}[t](A^1_x, 0), F) \in \text{CT}^{\text{sp}}_{\text{Nis}},
\]
it induces a map of Nisnevich sheaves on $P^1_W$,
\[
\pi_*(\alpha^*) : F_1, (P^1, \infty) \otimes W \to \pi_* F,
\]
which becomes an isomorphism over $(P^1 \setminus \{0\}) \times W$. Hence, (5.4.4) follows from
\[
H^i(P^1_W, F_1, (P^1, \infty) \otimes W) = 0 \quad \text{for } i \geq 1,
\]
which follows from [Sai20, Theorem 0.6].

**Lemma 5.5.** Let $N > 2$ be an integer and assume that Proposition 5.3 holds for $n < N$. Let $(X, D) \in MCor_{\text{ls}}$ and $Z \subset X$ be a smooth integral closed subscheme with $2 \leq \text{codim}(Z, X) =: c < N$. Assume
\[
D = D_1 + \cdots + D_r + D' \quad \text{with } r \leq c,
\]
where $D_1, \ldots, D_r$ are distinct and reduced irreducible components of $D$ containing $Z$, and $D'$ is an effective divisor on $X$ such that none of the component of $D'$ contains $Z$ and $Z$ is transversal to $|D'|$. Let $\rho : Y \to X$ be the blowup of $X$ in $Z$, let $\tilde{D}_i$ and $\tilde{D}' \subset Y$ be the strict transforms of $D_i$ and $D'$ respectively, and let $E_Z = \rho^{-1}(Z)$. Then, for all $W = (W, W^\infty) \in MCor_{\text{ls}}$,
\[
R^i \rho_{W*} F_1, (Y, \tilde{D}_1 + \cdots + \tilde{D}_r + E_Z + \tilde{D}') \otimes W = 0 \quad \text{for } i \geq 1,
\]
where $\rho_W : Y \times W \to X \times W$ denotes the base change of $\rho$.

**Proof.** This proof is adapted from [BRS22, Lemma 2.14]. The question is Nisnevich local around the points in $Z \times W$. Let $z \in Z \times W$ be a point and set $A := \mathcal{O}_{X \times W, z}^h$. For $V \subset Y \times W$, we denote $V(z) := V \times_{X \times W} \text{Spec } A$. By assumption we find a regular system of local parameters $t_1, \ldots, t_m$ of $A$, such that
\[
(D_i \times W)(z) = V(t_i) \quad \text{for } 1 \leq i \leq r, \quad (Z \times W)(z) = V(t_1, \ldots, t_c),
\]
\[
(D' \times W)(z) = V(t^{e_{c+1}}_{c+1} \cdots t^{e_{m_0}}_{m_0}) \quad \text{with } c + 1 \leq m_0 \leq m,
\]
\[
(X \times W^\infty)(z) = V(t^{e_{m_0+1}}_{m_0+1} \cdots t^{e_{m_1}}_{m_1}) \quad \text{with } m_0 \leq m_1 \leq m.
\]
Letting $K$ be the residue field of $A$, we can choose a ring homomorphism $K \to A$ which is a section of $A \to K$. Then we obtain an isomorphism
\[
K\{t_1, \ldots, t_m\} \xrightarrow{\sim} A.
\]
Let $\rho_1 : \tilde{A}^c \to A^c$ be the blowup in 0. By the above,
\[
\rho_W : (Y, \tilde{D}_1 + \cdots + \tilde{D}_r + E_Z + \tilde{D}') \otimes W \to (X, D) \otimes W
\]
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is Nisnevich locally around $z$ isomorphic over $k$ to the morphism

$$(\mathbb{A}^c, \tilde{L}_1 + \cdots + \tilde{L}_r + E) \otimes W' \to (\mathbb{A}^c, L_1 + \cdots + L_r) \otimes W',$$

induced by a map $$(\mathbb{A}^c, \tilde{L}_1 + \cdots + \tilde{L}_r + E) \to (\mathbb{A}^c, L_1 + \cdots + L_r)$$
as in Proposition 5.3. Hence, the statement follows from the proposition for $n = c < N$.

**Proof of Proposition 5.3.** The proof is by induction on $n \geq 2$. The case $n = 2$ follows from Lemma 5.4. Assume that $n > 2$ and that the proposition is proven for $\mathbb{A}^m$ with $m < n$. For $r = 1$, Proposition 5.3 is proved in [BRS22, Theorem 2.12]. Assume that $r \geq 2$. Let $Z := L_1 \cap L_2 \subset \mathbb{A}^n$ and $\hat{Z} \subset Y$ be the strict transform of $Z$. Denote by $\rho : Y' \to Y$ the blowup of $Y$ in $\hat{Z}$, let $\hat{L}_i, E' \subset Y'$ be the strict transforms of $\hat{L}, E$ respectively, and let $E'' = (\rho')^{-1}(\hat{Z})$. Note that $\hat{Z} = \hat{L}_1 \cap \hat{L}_2$ intersecting transversally with $\hat{L}_3 + \cdots + \hat{L}_r + E$ and codim$(\hat{Z}, Y) = 2$. Hence, by Lemma 5.5,

$$R^i\rho_{W*}F(Y', \hat{L}_1 + \cdots + \hat{L}_r + E') \otimes W = 0 \quad \text{for } i \geq 1.$$

Since Theorem 5.2 has been proved for $i = 0$, we have

$$\rho_{W*}F(Y', \hat{L}_1 + \cdots + \hat{L}_r + E') \otimes W = F(Y, \hat{L}_1 + \cdots + \hat{L}_r + E) \otimes W.$$

Hence, we obtain

$$R^i\rho_{W*}F(Y, \hat{L}_1 + \cdots + \hat{L}_r + E) \otimes W = R^i(\rho\rho')_{W*}F(Y', \hat{L}_1 + \cdots + \hat{L}_r + E') \otimes W.$$

Denote by $\sigma : \hat{Y} \to \mathbb{A}^n$ the blowup in $\hat{Z}$, let $\hat{L}_i \subset \hat{Y}$ be the strict transform of $L_i$, and let $\Sigma = \sigma^{-1}(\hat{Z})$. By Lemma 5.5 we get

$$R^i\sigma_{W*}F(\hat{Y}, \hat{L}_1 + \cdots + \hat{L}_r + \Sigma) \otimes W = 0 \quad \text{for } i \geq 1.$$

Denote by $\sigma' : \hat{Y}' \to \hat{Y}$ the blowup in $\hat{Z} = \sigma^{-1}(0) \subset \Sigma$, let $\hat{L}_i, \Sigma' \subset \hat{Y}'$ be the strict transforms of $\hat{L}_i, \Sigma$ respectively, and let $\Sigma'' = \sigma'^{-1}(\hat{Z})$. Note that $\hat{Z} \subset \hat{L}_3 \cap \cdots \cap \hat{L}_n \cap \Sigma$ and codim$(\hat{Z}, \hat{Y}) = n - 1$ and $\hat{Z}$ intersects transversally with $\hat{L}_1 + \hat{L}_2$. Thus, by Lemma 5.5 and the case $i = 0$ of Theorem 5.2, we obtain

$$R\sigma_{W*}F(\hat{Y}', \hat{L}_1 + \cdots + \hat{L}_r + \Sigma') \otimes W = F(\hat{Y}', \hat{L}_1', \ldots, \hat{L}_r', E', \Sigma'') \otimes W.$$

Finally, by [BRS22, Lemma 2.15], there is an isomorphism of $\mathbb{A}^n \times W$-schemes

$$(\hat{Y}', \hat{L}_1', \ldots, \hat{L}_r', \Sigma', \Sigma'') \cong (Y', \hat{L}_1', \ldots, \hat{L}_r', E', \Sigma'').$$

Altogether we obtain, for $i \geq 1$,

$$R^i\rho_{W*}F(Y, \hat{L}_1 + \cdots + \hat{L}_r + E) \otimes W = R^i(\rho\rho')_{W*}F(Y', \hat{L}_1 + \cdots + \hat{L}_r + E') \otimes W,$$

$$= R^i(\sigma\sigma')_{W*}F(\hat{Y}', \hat{L}_1 + \cdots + \hat{L}_r + \Sigma'') \otimes W,$$

$$= R^i\sigma_{W*}F(\hat{Y}, \hat{L}_1 + \cdots + \hat{L}_r + \Sigma) \otimes W,$$

$$= 0,$$

This completes the proof of the proposition. 

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Remark 5.6. For simplicity, we write

$$H^i_{\log}(-, F) = H^i_{\log}(-, \omega^C)$$

for $$F \in \text{RSC}_{\text{Nis}}$$.

By [RS21a, Corollary 6.8], if $$\text{ch}(k) = 0$$ and $$F = \Omega^l$$, we have

$$H^i_{\log}(-, \Omega^l) = H^i(X, \Omega^l(\log |D|))$$

for $$(X, D) \in \underline{\text{MCor}}_{\text{Is}}$$.

Hence, $$H^i_{\log}(-, F)$$ for $$F \in \text{RSC}_{\text{Nis}}$$ is a generalization of cohomology of sheaves of logarithmic differentials.

6. Relation with logarithmic sheaves with transfers

In this section we use the same notation as [BPØ22].

Let $$\text{lSm}$$ be the category of log smooth and separated fs log schemes of finite type over the base field $$k$$ and $$\text{SmlSm} \subset \text{lSm}$$ be the full subcategory consisting of objects whose underlying schemes are smooth over $$k$$. Let $$\text{lCor}$$ be the category with the same objects as $$\text{lSm}$$ and whose morphisms are log correspondences defined in [BPØ22, Definition 2.1.1]. Let $$\text{lCor}_{\text{SmlSm}} \subset \text{lCor}$$ be the full subcategory consisting of all objects in $$\text{SmlSm}$$.

Let $$\text{PSh}^{\text{ltr}}$$ be the category of additive presheaves of abelian groups on $$\text{lCor}$$ and $$\text{Shv}^{\text{ltr}}_{d\text{Nis}} \subset \text{PSh}^{\text{ltr}}$$ be the full subcategory consisting of those $$F$$ whose restrictions to $$\text{SmlSm}$$ are dividing Nisnevich sheaves (see [BPØ22, Definition 3.1.4]). It is shown in [BPØ22, Theorem 1.2.1 and Proposition 4.7.5] that $$\text{Shv}^{\text{ltr}}_{d\text{Nis}}$$ is a Grothendieck abelian category and there is an equivalence of categories

$$\text{Shv}^{\text{ltr}}_{d\text{Nis}} \simeq \text{Shv}^{\text{ltr}}_{d\text{Nis}}(\text{SmlSm}),$$

(6.0.1)

where the right-hand side denotes the full subcategory of the category $$\text{PSh}^{\text{ltr}}(\text{SmlSm})$$ of additive presheaves of abelian groups on $$\text{lCor}_{\text{SmlSm}}$$ consisting of those $$F$$ whose restrictions to $$\text{SmlSm}$$ are dividing Nisnevich sheaves.

We now construct a functor

$$\text{Log} : \underline{\text{MNST}}_{\text{log}} \to \text{Shv}^{\text{ltr}}_{d\text{Nis}}.$$

(6.0.2)

For $$\mathfrak{X} = (X, \mathcal{M}) \in \text{SmlSm}$$, we put $$\mathfrak{X}^{\text{MP}} = (X, \partial \mathfrak{X})$$, where $$\partial \mathfrak{X} \subset X$$ is the closed subscheme consisting of the points where the log structure $$\mathcal{M}$$ is not trivial. By [BPØ22, Lemma A.5.10], $$\partial \mathfrak{X}$$ with reduced structure is a normal crossing divisor on $$X$$, so that we can view $$\mathfrak{X}^{\text{MP}}$$ as an object of $$\underline{\text{MCor}}_{\text{Is}}$$. For $$F \in \underline{\text{MPST}}_{\text{log}}$$ and $$\mathfrak{X} \in \text{SmlSm}$$, we put

$$F^{\text{log}}(\mathfrak{X}) = F(\mathfrak{X}^{\text{MP}}).$$

(6.0.3)

Take $$\mathfrak{q} \in \text{SmlSm}$$ and $$\alpha \in \text{lCor}((\mathfrak{q}, \mathfrak{X})$$. By [BPØ22, Definition 2.1.1 and Remark 2.1.2(iii)], we have

$$\alpha \in \underline{\text{MCor}}^{\text{fin}}(\mathfrak{q}, \mathfrak{X})$$

for some $$n > 0$$, where $$n \cdot \partial \mathfrak{q} \hookrightarrow Y$$ is the $$n$$th thickening of $$\partial \mathfrak{q} \hookrightarrow Y$$. By the assumption $$F \in \underline{\text{MPST}}_{\text{log}}$$, the induced map

$$F^{\text{log}}(\mathfrak{X}) = F(\mathfrak{X}^{\text{MP}}) \xrightarrow{\alpha^*} F(Y, n \cdot \partial \mathfrak{q})$$

factors through $$F^{\text{log}}(\mathfrak{q}) = F(Y, \partial \mathfrak{q}) \subset F(Y, n \cdot \partial \mathfrak{q})$$ and we get a map

$$\alpha^{\text{log}} : F^{\text{log}}(\mathfrak{X}) \to F^{\text{log}}(\mathfrak{q}).$$
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Moreover, for a map \( \gamma : F \to G \) in \( \textbf{MPST}_{\text{log}} \), the diagram

\[
\begin{array}{ccc}
F^\text{log}(X) & \xrightarrow{\gamma} & G^\text{log}(X) \\
\downarrow \alpha^\text{log} & & \downarrow \alpha^\text{log} \\
F^\text{log}(\mathcal{Y}) & \xrightarrow{\gamma} & G^\text{log}(\mathcal{Y})
\end{array}
\]

is obviously commutative. Hence, the assignment \( X \to F^\text{log}(X) \) gives an object \( F^\text{log} \) of \( \text{PSh}^{\text{ltr}}(\text{Sm}_{\text{Sml}}) \) and we get a functor

\[
\text{Log} : \text{MPST}_{\text{log}} \to \text{PSh}^{\text{ltr}}(\text{Sm}_{\text{Sml}}), \quad F \mapsto F^\text{log}.
\]

(6.0.4)

By the definitions of sheaves ([KMSY21a, Definition 1], [BPØ22, Definition 3.1.4] and [KMSY21a, Proposition 1.9.2]), this induces a functor

\[
\text{MNST}_{\text{log}}^{\text{tr}} \to \text{Shv}^{\text{ltr}}_{\text{dNis}}(\text{Sm}_{\text{Sml}})
\]

which induces the desired functor (6.0.2) using (6.0.1). By the construction, for \( F \in \text{MNST}_{\text{log}}^{\text{tr}} \) and \( X \in \text{Sm}_{\text{Sml}} \) with \( X = X^{\text{MP}} \in \text{MC} \text{or}_{\text{ls}} \), we have

\[
H^i_{\text{Nis}}(X, F_X) = H^i_{\text{Nis}}(X, F^\text{log}) \quad (F^\text{log} = \text{Log}(F)),
\]

(6.0.5)

where the right-hand side is the cohomology for the strict Nisnevich topology (see [BPØ22, Definition 4.3.1]).

**Theorem 6.1.** For \( F \in \text{CI}_{\text{Nis}}^{r, \text{sp}}, \ F^\text{log} = \text{Log}(F) \in \text{Shv}^{\text{ltr}}_{\text{dNis}} \) is strictly \( \Box \)-invariant in the sense of [BPØ22, Definition 5.2.2]. For \( X \in \text{Sm}_{\text{Sml}} \) with \( X = X^{\text{MP}} \in \text{MC} \text{or}_{\text{ls}} \), we have a natural isomorphism

\[
H^i_{\text{Nis}}(X, F_X) \simeq \text{Hom}_{\text{logDM}^{\text{eff}}}(M(X), F^\text{log}[i]),
\]

(6.1.1)

where \( \text{logDM}^{\text{eff}} \) is the triangulated category of logarithmic motives defined in [BPØ22, Definition 5.2.1].

**Proof.** Let \( X_{\text{div}}^{\text{sm}} \) be the category of log modifications \( \mathcal{Y} \to X \) such that \( \mathcal{Y} \in \text{Sm}_{\text{Sml}} \) (see [BPØ22, Definition A.11.12]) and \( X_{\text{div}}^{\text{sm}} \subset X_{\text{div}}^{\text{sm}} \) be the full subcategory given by those maps \( \mathcal{Y} \to X \) that are isomorphic to compositions of log modifications along smooth centers (see [BPØ22, Definitions 4.4.4 and A.14.10]). We have isomorphisms

\[
H^i_{\text{Nis}}(X, F_X) \overset{(6.0.5)}{=} H^i_{\text{Nis}}(X, F^\text{log}) \overset{(\ast 1)}{=} \lim_{\mathcal{Y} \in X_{\text{div}}^{\text{sm}}} H^i_{\text{Nis}}(\mathcal{Y}, F^\text{log})
\]

\[
\overset{(\ast 2)}{=} \lim_{\mathcal{Y} \in X_{\text{div}}^{\text{sm}}} H^i_{\text{Nis}}(\mathcal{Y}, F^\text{log}) \overset{(\ast 3)}{=} H^i_{\text{dNis}}(X, F^\text{log}),
\]

where \((\ast 2)\) follows from [BPØ22, Corollary 4.4.5] and \((\ast 3)\) from [BPØ22, Theorem 5.1.8], and \((\ast 1)\) is a consequence of Theorem 5.2 in view of (6.0.5) and the fact that a log modification of \( X = (X, M) \in \text{Sm}_{\text{Sml}} \) along smooth center is induced Zariski locally by a blowup of \( X \) in an intersection of irreducible components of \( \partial X \) so that it corresponds to a morphism in \( \Lambda_{\text{fin}}^{\text{ls}} \) from Definition 5.1.

Hence, the strict \( \Box \)-invariance of \( F^\text{log} \) follows from [Sai20, Theorem 0.6]. Finally, (6.1.1) follows from [BPØ22, Proposition 5.2.3].
We now consider the composite functor

$$\mathcal{L}og' : \text{RSC}_{\text{Nis}} \xrightarrow{\mathfrak{Cl}_{\text{Nis}}} \text{CI}^{\text{sp}}_{\text{Nis}} \xrightarrow{\mathcal{L}og} \text{CI}_{\text{Nis}}^{\text{tr}},$$

where $\text{CI}_{\text{Nis}}^{\text{tr}} \subset \text{Shv}_{\text{Nis}}^{\text{tr}}$ is the full subcategory consisting of strictly $\square$-invariant objects. By [BM12, Theorem 5.7], $\text{CI}_{\text{Nis}}^{\text{tr}}$ is a Grothendieck abelian category.

**Lemma 6.2.** Log and Log’ have the same essential image.

**Proof.** This follows directly from the construction and Corollary 2.6(3). \qed

In what follows, we let

$$\mathcal{L}og : \text{RSC}_{\text{Nis}} \rightarrow \text{CI}_{\text{Nis}}^{\text{tr}} : F \mapsto F^{\mathcal{L}og}$$

denote $\mathcal{L}og'$ defined as above. By (6.0.3), we have

$$F^{\mathcal{L}og}(X, \text{triv}) = F(X) \quad \text{for } F \in \text{RSC}_{\text{Nis}}, X \in \text{Sm},$$

where $(X, \text{triv})$ denotes the log scheme with the trivial log structure.

**Theorem 6.3.** Log is exact and fully faithful.

**Proof.** First we prove the full faithfulness. Faithfulness follows from (6.2.2). Let $F, G \in \text{RSC}_{\text{Nis}}$ and $\gamma : F^{\mathcal{L}og} \rightarrow G^{\mathcal{L}og}$ be a map in $\text{Shv}_{\text{Nis}}^{\text{tr}}$. By (6.2.2) it induces maps $\gamma_X : F(X) \rightarrow G(X)$ for all $X \in \text{Sm}$. They are compatible with the action of Cor since by [BPØ22, Example 2.1.3(3)],

$$\text{Cor}(Y, X) = \text{lCor}(Y, \text{triv}, (X, \text{triv})) \quad \text{for } X, Y \in \text{Sm}.$$ 

Thus, $\gamma_X$ for $X \in \text{Sm}$ give a map $\gamma_{\text{RSC}_{\text{Nis}}} : F \rightarrow G$ in $\text{RSC}_{\text{Nis}}$. To see $\mathcal{L}og(\gamma_{\text{RSC}_{\text{Nis}}}) = \gamma$, it suffices by (6.0.1) to show that $\mathcal{L}og(\gamma_{\text{RSC}_{\text{Nis}}})$ and $\gamma$ induce the same map $F^{\mathcal{L}og}(X) \rightarrow G^{\mathcal{L}og}(X)$ for $X \in \text{Sm}$. If $X$ has the trivial log structure, this follows immediately from the construction of $\gamma_{\text{RSC}_{\text{Nis}}}$. The general case follows from this in view of the commutative diagram

$$\begin{array}{ccc}
F^{\mathcal{L}og}(X) & \xrightarrow{\gamma} & G^{\mathcal{L}og}(X) \\
\downarrow j^* & & \downarrow j^* \\
F^{\mathcal{L}og}(X \setminus \partial X, \text{triv}) & \xrightarrow{\gamma} & G^{\mathcal{L}og}(X \setminus \partial X, \text{triv})
\end{array}$$

where $j^*$ are induced by the natural map $(X \setminus \partial X, \text{triv}) \rightarrow X$ of log schemes and are injective by the construction and the semipurity of $\omega_{\text{CI}}F$. This completes the proof of the full faithfulness.

Next we show the exactness of $\mathcal{L}og$. It suffices to show the following claim.

**Claim 6.3.1.** Given an exact sequence $0 \rightarrow F \rightarrow G \rightarrow H \rightarrow 0$ in $\text{RSC}_{\text{Nis}}$, the induced sequence

$$0 \rightarrow F^{\mathcal{L}og}(X) \rightarrow G^{\mathcal{L}og}(X) \rightarrow H^{\mathcal{L}og}(X) \rightarrow 0$$

is exact for every $X \in \text{Sm}$. If $X$ has the trivial log structure, this follows immediately from the construction of $\gamma_{\text{RSC}_{\text{Nis}}}$. The general case follows from this in view of the commutative diagram

$$\begin{array}{ccc}
F^{\mathcal{L}og}(X) & \xrightarrow{\gamma} & G^{\mathcal{L}og}(X) \\
\downarrow j^* & & \downarrow j^* \\
F^{\mathcal{L}og}(X \setminus \partial X, \text{triv}) & \xrightarrow{\gamma} & G^{\mathcal{L}og}(X \setminus \partial X, \text{triv})
\end{array}$$

where $j^*$ are induced by the natural map $(X \setminus \partial X, \text{triv}) \rightarrow X$ of log schemes and are injective by the construction and the semipurity of $\omega_{\text{CI}}F$. This completes the proof of the full faithfulness. \qed

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