On some Leindler’s theorem on application of the class NMCS

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Abstract
We show the results in the class $GM(\beta)$ corresponding to the theorem of L. Leindler [A note on strong approximation of Fourier series, Analysis Mathematica, 29(2003), 195–199] on strong approximation by matrix means of Fourier series constructed by the sequences from the class NMCS.

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1 Introduction
Let $C_{2\pi}$ be the class of all $2\pi$–periodic real–valued functions continuous over $Q = [-\pi, \pi]$ with the norm

$$\|f\| := \sup_{t \in Q} |f(t)|$$

and consider the trigonometric Fourier series of $f \in C_{2\pi}$ with the partial sums $S_k f$.

Let $A := (a_{n,k})$ be an infinite matrix of real nonnegative numbers such that

$$\sum_{k=0}^{\infty} a_{n,k} = 1, \text{ where } n = 0, 1, 2, \ldots,$$

and let the $A$–transformation of $(S_k f)$ be given by

$$T_{n,A} f(x) := \sum_{k=0}^{\infty} a_{n,k} S_k f(x) \quad (n = 0, 1, 2, \ldots).$$
Let us consider the strong mean
\[ T_{n,A}^p f (x) = \sum_{k=0}^\infty a_{n,k} |S_k f (x) - f (x)|^p \]
and as measures of approximation by such quantity we use the best approximation of \( f \) by trigonometric polynomials \( t_k \) of order at most \( k \) and the modulus of continuity of \( f \) defined by the formulas
\[ E_k(f) = \inf_{t_k} \| f - t_k \| \]
and
\[ \omega_f(\delta) = \sup_{|t| \leq \delta} \| f (\cdot + t) - f (\cdot) \| , \]
respectively.

In [7] S. M. Mazhar and V. Totik proved the following theorem:

**Theorem 1** Suppose \( A := (a_{n,k}) \) satisfies (1), \( \lim_{n \to \infty} a_{n,0} = 0 \) and \( a_{n,k} \geq a_{n,k+1} \) \( k = 0, 1, 2, ... \) \( n = 0, 1, 2, ..., \)
then
\[ \| T_{n,A} f - f \| \leq K \sum_{k=0}^\infty a_{n,k} \omega_f \left( \frac{1}{k+1} \right) . \]

Recently, L. Leindler [2] defined a new class of sequences named as sequences of rest bounded variation, briefly denoted by \( RBVS \), i.e.,
\[ RBVS = \left\{ a := (a_n) \in \mathbb{C} : \sum_{k=m}^{\infty} |a_k - a_{k+1}| \leq K(a) |a_m| \text{ for all } m \in \mathbb{N} \right\} , \quad (2) \]
where here and throughout the paper \( K(a) \) always indicates a constant only depending on \( a \).

Denote by \( MS \) the class of monotone decreasing sequences and \( CQMS \) the class of classic quasimonotone decreasing sequences \( (a_n) \in \mathbb{R}_+ \) and there exists an \( \alpha > 0 \) such that \( a_n/n^\alpha \) is decreasing), then it is obvious that
\[ MS \subset RBVS \cap CQMS. \]

L. Leindler [3] proved that the class \( CQMS \) and \( RBVS \) are not comparable. In [3] L. Leindler considered the class of mean rest bounded variation sequences \( MRBV S \), where
\[ MRBV S = \left\{ a := (a_n) \in \mathbb{C} : \sum_{k=m}^{\infty} |a_k - a_{k+1}| \leq K(a) \frac{1}{m} \sum_{k \geq m/2} |a_k| \text{ for all } m \in \mathbb{N} \right\} . \quad (3) \]
It is clear that \( \text{RBV} \subseteq \text{MRBV}. \)

In [9] the second author proved that \( \text{RBV} \not= \text{MRBV}. \) Moreover, the above theorem was generalized for the class \( \text{MRBV} \) in [8].

Further, the class of general monotone coefficients, \( \text{GM} \), is defined as follows (see [10]):

\[
\text{GM} = \left\{ a := (a_n) \in \mathbb{C} : \sum_{k=m}^{2m-1} |a_k - a_{k+1}| \leq K(a) |a_m| \text{ for all } m \in \mathbb{N} \right\}. \tag{4}
\]

It is clear \( \text{RBV} \cup \text{CQMS} \subset \text{GM}. \)

In [5, 10, 11, 12] was defined the class of \( \beta \)-general monotone sequences as follows:

**Definition 2** Let \( \beta := (\beta_n) \) be a nonnegative sequence. The sequence of complex numbers \( a := (a_n) \) is said to be \( \beta \)-general monotone, or \( a \in \text{GM}(\beta) \), if the relation

\[
\sum_{k=m}^{2m-1} |a_k - a_{k+1}| \leq K(a) \beta_m \tag{5}
\]

holds for all \( m \).

In the paper [12] Tikhonov considered the following examples of the sequences \( \beta_n \):

1. \( \beta_1_n = |a_n| \),
2. \( 2\beta_n = \sum_{k=n}^{n+N} |a_k| \) for some integer \( N \),
3. \( 3\beta_n = \sum_{\nu=0}^{N} |a_{c\nu}| \) for some integers \( N \) and \( c > 1 \),
4. \( 4\beta_n = |a_n| + \sum_{k=n+1}^{[cn]} \frac{|a_k|}{k} \) for some \( c > 1 \),
5. \( 5\beta_n = \sum_{k=[n/c]}^{[cn]} \frac{|a_k|}{k} \) for some \( c > 1 \).

It is clear that \( \text{GM}(\beta_1) = \text{GM}. \) Moreover (see [12] Remark 2.1)

\[
\text{GM}(\beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5) \equiv \text{GM}(\beta_5) \).
\]

Consequently, we assume that the sequence \( (K(\alpha_n))_{n=0}^{\infty} \) is bounded, that is, that there exists a constant \( K \) such that

\[
0 \leq K(\alpha_n) \leq K
\]

holds for all \( n \), where \( K(\alpha_n) \) denote the sequence of constants appearing in the inequalities [24-25] for the sequences \( \alpha_n := (a_{nk})_{k=0}^{\infty} \).
Now we can give the conditions to be used later on. We assume that for all \( n \)
\[
\sum_{k=m}^{2m-1} |a_{n,k} - a_{n,k+1}| \leq K \sum_{k=[m/c]}^{[cm]} \frac{a_{n,k}}{k} \tag{6}
\]
holds if \( \alpha_n = (a_{n,k})_{k=0}^\infty \) belongs to \( GM (5\beta) \), for \( n = 1, 2, ... \)

Following by L. Leindler [1] a sequence \( a := (a_n) \) of nonnegative numbers is called a Nearly Monotone Convergent Sequence, or briefly \( a \in NMCS \), if
\[
\sum_{k=1}^{\infty} a_k < \infty \quad \text{and} \quad ka_k \to 0 \text{ as } k \to \infty,
\]
for all positive integer \( r \).

**Remark 1** If \( \sum_{k=1}^{\infty} |a_k| < \infty \) and \( (a_k) \in GM (5\beta) \) then \( (a_k) \in NMCS \).

The deviation \( H_{n,A}f \) was estimated by L. Leindler in [1] as follows:

**Theorem 3** [1] If \( f \in C_{2\pi}, p > 0, (a_{n,k})_{k=0}^\infty \in NMCS \) for all \( n \), and \( \lim_{n \to \infty} a_{n,0} = 0 \) hold, then
\[
\|T_{n,A}^p f\| \leq K \sum_{k=0}^{\infty} a_{n,k} E_k^p(f).
\]

In this note we show that the class \( NMCS \) is not proper for the above estimate.

In our theorem we consider the class \( GM (5\beta) \) instead of \( NMCS \). Thus we essentially extend the result of S. M. Mazhar and V. Totik (see [7]).

We shall write \( I_1 \ll I_2 \) if there exists a positive constant \( K \), sometimes depended on some parameters, such that \( I_1 \leq KI_2 \).

2 Statement of the results

Our main result is the following

**Theorem 4** If \( f \in C_{2\pi}, p > 0, (a_{n,k})_{k=0}^\infty \in GM (5\beta) \) for all \( n \), [1] and \( \lim_{n \to \infty} a_{n,0} = 0 \) hold, then
\[
\|T_{n,A}^p f\| \ll \sum_{k=0}^{\infty} a_{n,k} E_k^p(f) \tag{7}
\]
for some \( c > 1 \).

**Remark 2** If we suppose that \( (a_{n,k})_{k=0}^\infty \in MS \) then from [7] we deduce
\[
\|T_{n,A}^p f\| \ll \sum_{k=0}^{\infty} a_{n,k} E_k^p(f).
\]
Using the Jackson Theorem [14, Theorem 13.6] we can obtain the following remark.

Remark 3 Under the assumptions of Theorem 3
\[
\left\| T_{n,A}^p f \right\| \ll \sum_{k=0}^{\infty} a_{n,k} \omega^p f \left( \frac{\pi}{k+1} \right). \tag{8}
\]

Remark 4 We can observe that taking \( a_{nn} = 1 \) and \( a_{n,k} = 0 \) for \( k \neq n \) we have \((a_{n,k})_{k=0}^n \in NMCS \) but thus, by Theorem 2, we obtain the estimate
\[
\| S_n f - f \| \ll E_n(f)
\]
which is not true in general.

Remark 5 By the considerations similar to these in [7] we can obtain the estimates
\[
\left\| T_{n,A}^\varphi f \right\| \ll \sum_{k=0}^{\infty} a_{n,k} \varphi \left( E_{\left[ \frac{k}{2^m} \right]}(f) \right)
\]
and
\[
\left\| T_{n,A}^\varphi f \right\| \ll \sum_{k=0}^{\infty} a_{n,k} \varphi \left( \omega f \left( \frac{\pi}{k+1} \right) \right)
\]
instead of (7) and (8) respectively, where
\[
T_{n,A}^\varphi f (x) = \sum_{k=0}^{\infty} a_{n,k} \varphi (\left| S_k f (x) - f (x) \right|)
\]
with a nonnegative monotone increasing continuous function \( \varphi(t) \) \( (t \in [0, \infty)) \) satisfying the conditions
\[
\varphi(0) = 0, \varphi(t) \leq e^{At}, \quad t \in (0, \infty)
\]
and
\[
\varphi(2t) \leq A \varphi(t), \quad t \in (0, 1),
\]
with some constant \( A \).

3 Auxiliary result

We shall use the following

Lemma 1 (see [8], Theorem 1.11). Suppose that \( n = O(\lambda_n) \). Then, for any continuous function \( f \) and for any number \( p > 0 \), we have
\[
\left\| \left\{ \frac{1}{\lambda_n} \sum_{k=n-\lambda_n}^{n-1} |S_k f - f|^p \right\}^{1/p} \right\| \ll E_{n-\lambda_n}(f).
\]
4 Proofs of the results

4.1 Proof of Theorem 3

Let

$$
\left\| T_{n,A}^p f \right\| = \left\| \sum_{k=0}^{2[c]-1} a_{n,k} |S_k f - f|^p + \sum_{k=2^c}^\infty a_{n,k} |S_k f - f|^p \right\|
$$

$$
\leq \left\| \sum_{k=0}^{2[c]-1} a_{n,k} |S_k f - f|^p \right\| + \left\| \sum_{m=[c]}^\infty \sum_{k=2^m}^{2^{m+1}-1} a_{n,k} |S_k f - f|^p \right\| = I_1 + I_2.
$$

for some \( c > 1 \). Using Lemma we get

$$
I_1 \leq \left\| \sum_{k=0}^{2[c]-1} a_{n,k} \frac{k - \lceil k/2^c \rceil + 1}{k} \sum_{l=\lceil k/2^c \rceil}^{k} |S_l f - f|^p \right\|
$$

$$
\leq 2^c \left\| \sum_{k=0}^{2[c]-1} a_{n,k} \frac{1}{k - \lceil k/2^c \rceil + 1} \sum_{l=\lceil k/2^c \rceil}^{k} |S_l f - f|^p \right\|
$$

$$
\ll \sum_{k=0}^{2[c]-1} a_{n,k} E_{\lceil k/2^c \rceil}^p(f).
$$

By partial summation, our Lemma gives

$$
I_2 = \left\| \sum_{m=[c]}^\infty \left[ \sum_{k=2^m}^{2^{m+1}-2} (a_{n,k} - a_{n,k+1}) \sum_{l=2^m}^{k} |S_l f - f|^p 
+ a_{n,2^{m+1}-1} \sum_{l=2^m}^{2^{m+1}-1} |S_l f - f|^p \right] \right\|
$$

$$
\ll \sum_{m=[c]}^\infty \left[ 2^m \sum_{k=2^m}^{2^{m+1}-2} |a_{n,k} - a_{n,k+1}| E_{2^m}^p(f) 
+ 2^m a_{n,2^{m+1}-1} E_{2^m}^p(f) \right]
$$

$$
\ll \sum_{m=[c]}^\infty 2^m E_{2^m}^p(f) \left[ 2^{m+1} \sum_{k=2^m}^{2^{m+1}-2} |a_{n,k} - a_{n,k+1}| + a_{n,2^{m+1}-1} \right].
$$

6
Since (6) holds, we have
\[ a_{n,s+1} - a_{n,r} \]
\[ \leq |a_{n,r} - a_{n,s+1}| \leq \sum_{k=r}^{s} |a_{n,k} - a_{n,k+1}| \]
\[ \leq \sum_{k=2^m}^{2^{m+1}-2} |a_{n,k} - a_{n,k+1}| \ll \sum_{k=\lfloor 2^m/c \rfloor}^{\lfloor 2^m \rfloor} \frac{a_{n,k}}{k} \quad (2 \leq 2^m \leq r \leq s \leq 2^{m+1} - 2), \]
whence
\[ a_{n,s+1} \ll a_{n,r} + \sum_{k=\lfloor 2^m/c \rfloor}^{\lfloor 2^m \rfloor} \frac{a_{n,k}}{k} \quad (2 \leq 2^m \leq r \leq s \leq 2^{m+1} - 2) \]
and
\[ 2^m a_{n,2^{m+1}-1} = \frac{2^m}{2^m - 1} \sum_{r=2^m}^{2^{m+1}-2} a_{n,2^{m+1}-1} \ll \sum_{r=2^m}^{2^{m+1}-2} \left( a_{n,r} + \sum_{k=\lfloor 2^m/c \rfloor}^{\lfloor 2^m \rfloor} \frac{a_{n,k}}{k} \right) \ll \sum_{r=2^m}^{2^{m+1}-1} a_{n,r} + 2^m \sum_{k=\lfloor 2^m/c \rfloor}^{\lfloor 2^m \rfloor} \frac{a_{n,k}}{k} \]
whence
\[ I_2 \ll \sum_{m=[c]}^{\infty} \left\{ 2^m E_{2^m}^p (f) \sum_{k=\lfloor 2^m/c \rfloor}^{\lfloor 2^m \rfloor} \frac{a_{n,k}}{k} + E_{2^m}^p (f) \sum_{k=2^m}^{2^m+1} a_{n,k} \right\} \cdot \]

Finally, by elementary calculations we get
\[ I_2 \ll \sum_{m=[c]}^{\infty} \left\{ 2^m E_{2^m}^p (f) \sum_{k=\lfloor 2^m/c \rfloor}^{\lfloor 2^m \rfloor} \frac{a_{n,k}}{k} + E_{2^m}^p (f) \sum_{k=2^m}^{2^m+1} a_{n,k} \right\} \ll \sum_{m=[c]}^{\infty} \sum_{k=\lfloor 2^m/c \rfloor}^{\lfloor 2^m \rfloor} a_{n,k} E_{2^m}^p (f) + \sum_{m=[c]}^{\infty} \sum_{k=2^m}^{2^m+1} a_{n,k} E_{2^m}^p (f). \]
\[ \sum_{m=|c|}^{\infty} \sum_{k=2^m-|c|}^{2^m-1} a_{n,k} E_k^p(f) + \sum_{m=|c|}^{\infty} \sum_{k=2^m}^{2^{m+|c|-1}-1} a_{n,k} E_k^p(\frac{1}{2^m}) (f) + \sum_{m=|c|}^{\infty} E_{2^m}(f) a_{n,2^{m+|c|}} \]

\[ = \sum_{m=|c|}^{\infty} \sum_{r=1}^{[|c|]} \sum_{k=2^{m-r}}^{2^{m-r+1}-1} a_{n,k} E_k^p(f) + \sum_{m=|c|}^{\infty} \sum_{r=0}^{[|c|-1]} \sum_{k=2^{m+r}}^{2^{m+r+1}-1} a_{n,k} E_k^p(\frac{1}{2^{m+r}}) (f) + \sum_{m=|c|}^{\infty} E_{2^m}(f) a_{n,2^{m+|c|}} \]

\[ \leq \sum_{r=1}^{[|c|]} \sum_{k=2^{m-r}}^{2^{m-r+1}-1} a_{n,k} E_k^p(f) + \sum_{r=0}^{[|c|-1]} \sum_{k=2^{m+r}}^{2^{m+r+1}-1} a_{n,k} E_k^p(\frac{1}{2^{m+r}}) (f) + \sum_{m=|c|}^{\infty} a_{n,k} E_k^p(\frac{1}{2^{2|m|}}) (f) \]

Thus we obtain the desired result. \(\square\)

### 4.2 Proof of Remark 1

For \(j = k + 2, k + 3, \ldots, 2k\) we get

\[ \sum_{l=k}^{j-2} |a_l - a_{l+1}| \geq \sum_{l=k}^{j-2} |a_l| - |a_{l+1}| \geq \sum_{l=k}^{j-2} |a_l| - \sum_{l=k+1}^{j-1} |a_l| = |a_k| - |a_{j-1}| \]

Therefore, if \((a_k) \in GM (5\beta)\) then

\[ |a_k| \leq \sum_{l=k}^{j-2} |a_l - a_{l+1}| + |a_{j-1}| \]

\[ \leq 2^{\lfloor j/2 \rfloor - 1} \sum_{l=k}^{\lfloor j/2 \rfloor} |a_l - a_{l+1}| + |a_{j-1}| \]

\[ \ll \sum_{l=\lfloor j/2 \rfloor}^{\lfloor j/2 \rfloor} \frac{|a_l|}{l} + |a_{j-1}| \ll \frac{1}{k} \sum_{l=\lfloor k/2 \rfloor}^{\lfloor k \rfloor} |a_l| + |a_{j-1}|. \]

Summing up on \(j\) and using the assumption \(\sum_{k=1}^{\infty} |a_k| < \infty\) we get for \(k > 1\) that

\[ k |a_k| = \frac{k}{k-1} \sum_{j=k+2}^{2k} |a_k| \ll \sum_{j=k+2}^{2k} \left( \frac{1}{k} \sum_{l=\lfloor k/2 \rfloor}^{\lfloor k \rfloor} |a_l| + |a_{j-1}| \right) \]
\[ \leq \sum_{l=[k/2c]}^{[ck]} |a_l| + 2k \sum_{j=k+2}^{[ck]} |a_{j-1}| = \sum_{l=[k/2c]}^{[ck]} |a_l| + 2k \sum_{j=k+1}^{[ck]+2k} |a_j| \]
\[ \leq 2 \sum_{l=[k/2c]}^{[ck]+2k} |a_l| \to 0, \]
whence \((a_k) \in NMCS. \square\)

4.3 Proof of Remark 2

If \((a_{n,k})_{k=0}^{\infty} \in MS\) then \((a_{n,k})_{k=0}^{\infty} \in GM\ (5\beta)\) and using Theorem 3 we obtain
\[
\left\| T_{n,A}^p f \right\| \leq \sum_{k=0}^{\infty} a_{n,k} E_{(2k+1)}^p (f) = \sum_{k=0}^{\infty} \sum_{m=k2^{[\epsilon]}}^{(k+1)2^{[\epsilon]}-1} a_{n,m} E_{(2m+1)}^p (f) = \sum_{k=0}^{\infty} E_k^p (f) \sum_{m=k2^{[\epsilon]}}^{(k+1)2^{[\epsilon]}-1} a_{n,m} \leq \sum_{k=0}^{\infty} 2^{[\epsilon]} E_k^p (f) a_{n,k2^{[\epsilon]}} \]
\[
\leq \left\{ 2^{[\epsilon]} \right\}^{1/p} \sum_{k=0}^{\infty} E_k^p (f) a_{n,k}. \]

This ends our proof. \(\square\)

4.4 Proof of Remark 5

The proof is similar to the proof of Theorem 3. The difference is such that we use the following Totik estimate (see [13])
\[
\frac{1}{n} \sum_{k=n+1}^{2n} \varphi (|S_k f (x) - f (x)|) \leq K \varphi (E_n (f))
\]
instead of the inequality from Lemma. \(\square\)

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