Abstract
An update of the ODEtools Maple package, for the analytical solving of 1st and 2nd order ODEs using Lie group symmetry methods, is presented. The set of routines includes an ODE-solver and user-level commands realizing most of the relevant steps of the symmetry scheme. The package also includes commands for testing the returned results, and for classifying 1st and 2nd order ODEs.

(Accepted for publication in Computer Physics Communications)

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Available as http://dft.if.uerj.br/preprint/e7-1.tex
Title of the software package: ODEtools.

Catalogue number: (supplied by Elsevier)

Software obtainable from: CPC Program Library, Queen’s University of Belfast, N. Ireland (see application form in this issue)

Licensing provisions: none

Operating systems under which the program has been tested: UNIX systems, Macintosh, DOS (AT 386, 486 and Pentium based) systems, DEC VMS, IBM CMS.

Programming language used: Maple V Release 3/4.

Memory required to execute with typical data: 16 Megabytes.

No. of lines in distributed program, excluding On-Line Help, etc.: 10159.

Keywords: 1st/2nd order ordinary differential equations, symmetry methods, symbolic computation.

Nature of mathematical problem
Analytical solving of 1st and 2nd order ordinary differential equations using symmetry methods, and the inverse problem; that is: given a set of point and/or dynamical symmetries, to find the most general invariant 1st or 2nd order ODE.

Methods of solution
Computer algebra implementation of Lie group symmetry methods.

Restrictions concerning the complexity of the problem
Besides the inherent restrictions of the method (there is as yet no general scheme for solving the associated PDE for the coefficients of the infinitesimal symmetry generator), the present implementation does not work with systems of ODEs nor with ODEs of differential order higher than two.

Typical running time
This depends strongly on the ODE to be solved. For the case of first order ODEs, it usually takes from a few seconds to 1 or 2 minutes. In the tests we ran with the first 500 1st order ODEs from Kamke’s book \[3\], the average times were: 8 sec. for a solved ODE and 15 sec. for an unsolved one, using a Pentium 200 with 64 Mb. RAM, on a Windows 95 platform. In the case of second order ODEs, the average times for the non-linear 2nd order examples of Kamke’s Book were 35 seconds for a solved ODE and 50 seconds for an unsolved one. The tests were run using the Maple version under development, but almost equivalent results are obtained using the available Maple R4 and R3 (the code presented in this work runs in all these versions).

Unusual features of the program
The ODE-solver here presented is an implementation of all the steps of the symmetry method solving scheme; that is, the command receives an ODE, and when successful it directly returns a closed form solution for the undetermined function. Also, this solver permits the user to optionally participate in the solving process by giving advice concerning the functional form for the coefficients of the infinitesimal symmetry generator (infinitesimals). Many of the intermediate steps of the symmetry scheme are available as user-level commands too. Using the package’s commands, it is then possible to obtain the infinitesimals, the related canonical coordinates, the finite form of the related group transformation equations, etc. Routines for testing the returned results, especially when they come in implicit form, are also provided. Special efforts were put in commands for solving the inverse problem too; that is, commands returning the most general 1st or 2nd order ODE simultaneously invariant under given symmetries. One of the striking new features of the package related to 2nd order ODEs is its ability to deal with dynamical symmetries, both in finding them and in using them in the integration procedures. Finally, the package also includes a command for classifying ODEs, optionally popping up Help pages based on Kamke’s advice for solving them, facilitating the study of a given ODE and the use of the package with pedagogical purposes.
Introduction

In a previous work [2], we presented an implementation of Lie symmetry methods (SM) for solving first order ODEs. The key idea of that work was to prepare routines for finding particular solutions for the PDE determining the coefficients of the infinitesimal symmetry generators (infinitesimals), as well as providing extra routines for the user’s input of functional form ansätze when the default routines fail. This approach is presently solving 85% of Kamke’s examples using only the defaults, apart from being a concrete way to tackle non-classifiable 1st order ODEs, for which the standard computer algebra solvers usually fail.

The same idea can be implemented for 2nd order ODEs too. To understand the motivation, we recall that most implementations of SM for high order ODEs are based on the setup and solving of the so-called determining equations for the infinitesimals -an overdetermined system of PDEs-, which arise when we assume we are interested only in point symmetries [3]. However, this poses a limitation on the 2nd order ODEs that can be dealt with, since only a restricted subset of them have point symmetries. On the other hand, all ODEs have infinite dynamical symmetries [4, 5], which arise as particular solutions for a single linear 2nd order PDE for the infinitesimals.

This paper then presents the implementation of the ideas of our previous work to tackle 2nd order ODEs. One of the tricky things related to the use of SM to solve n-th order ODEs is that the knowledge of n symmetries does not directly reduce the problem to a line integral as in the 1st order case. Moreover, the alternatives found in the literature for constructing the solution departing from dynamical symmetries are few. We then extended some of the standard integration methods for point symmetries, and implemented them as routines for dynamical symmetries too.

As a second issue, we invested in the research design of the package, extending both the number and the capabilities of extra user-lever routines related to the intermediate steps of the symmetry scheme. Worth mentioning are a routine for finding the most general 1st / 2nd order ODE simultaneously invariant under many point or dynamical symmetries, and a routine returning the symmetries of an unknown ODE, given its solution.

As a third issue, we invested in augmenting the pedagogical potential of the package by extending the classification capabilities of the odeadvisor command to work with most of the standard classifications for 2nd order ODEs (see [6]).

The exposition is organized as follows. In sec. 1, the SM scheme for solving 2nd order ODEs is briefly reviewed. In sec. 2 a compact table-summary of all ODEtools routines and a detailed description of the ODE-solver are presented. Sec. 3 briefly illustrates the extension of the methods presented in 2 for finding the infinitesimals. In sec. 4 we comment about the methods implemented for integrating ODEs from the knowledge of their symmetries, focusing on the case in which these symmetries are of dynamical type. Sec. 5 displays the results of testing the package with the non-linear 2nd order examples of Kamke’s book as well as an update of the results obtained for the Kamke’s first 500 1st order ODEs. In sec. 6 the main differences between ODEtools and other existing packages for symmetry analysis of ODEs are highlighted. Finally, the conclusions contain a brief discussion about this work and its possible extensions.

1 Symmetry methods for 2nd order ODEs

Generally speaking, the key point of Lie’s method for solving ODEs is that the knowledge of a (Lie) group of transformations which leaves a given ODE invariant may help in reducing the problem of finding its solution to quadratures [7, 8, 9]. Aside from the subtleties which arise when considering different cases, we can summarize the computational task of using SM for solving a given 2nd order ODE, say,

\[ \frac{d^2y}{dx^2} = \Phi(x, y, \frac{dy}{dx}) \]  

as the finding of the infinitesimals of a one-parameter Lie group which leaves Eq.(1) invariant, i.e., a pair
of functions\footnote{\{\(\xi(y, x, y_1), \eta(y, x, y_1)\}\} satisfying
\[
\left(2y_1 \frac{\partial^2 \eta}{\partial y \partial y_1} + 2 \frac{\partial \eta}{\partial y} - 3y_1 \frac{\partial \xi}{\partial y} - 2 \frac{\partial \xi}{\partial x} - 2y_1 \frac{\partial \xi}{\partial x \partial y_1} + \frac{\partial \eta}{\partial y} - 2y_1 \frac{\partial^2 \xi}{\partial y \partial y_1}\right) \Phi \\
+ \left(-y_1 \frac{\partial \eta}{\partial y} + y_1 \frac{\partial \xi}{\partial x} - y_1 \frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial x}\right) \frac{\partial \Phi}{\partial y_1} + \left(2 \frac{\partial^2 \eta}{\partial y \partial y_1} - y_1 \frac{\partial \xi}{\partial y_1^2} - 2 \frac{\partial \xi}{\partial y_1}\right) \Phi^2 \\
+ \left(\frac{\partial \eta}{\partial y} - y_1 \frac{\partial \xi}{\partial y_1} - \xi\right) \frac{\partial \Phi}{\partial x} - \frac{\partial^2 \xi}{\partial y \partial y_1} y_1^3 - 2 \frac{\partial^2 \xi}{\partial y \partial x} y_1^2 + 2 \frac{\partial^2 \eta}{\partial y \partial x} y_1 - y_1 \frac{\partial^2 \xi}{\partial x^2} = 0
\]
followed by the integration of the ODE by either:

\begin{enumerate}
\item[a)] determining the differential invariants of order 0 and 1 of the symmetry group;
\item[b)] determining first integrals \(\psi(x, y, y_1) = \psi_0\) (from the knowledge of \(\xi\) and \(\eta\));
\item[c)] determining the canonical coordinates, say \(\{r, s(r)\}\), of the associated Lie group.
\end{enumerate}

A first look at the symmetry scheme may lead to the conclusion that finding solutions to Eq.(2) would be much more difficult than solving the original Eq.(1). However, what we are really looking for is a particular solution to Eq.(2), and in many cases this particular solution is the only thing one can actually obtain. As an example, consider

\[
\text{ode} := \frac{d^2 y}{dx^2} = \left(\frac{dy}{dx} x - y\right)\frac{2}{x^3}
\]

This 2\textsuperscript{nd} order ODE is non-linear and it doesn’t match any pattern for which we know the solving method \textit{a priori}; standard classification based ODE-solvers then fail when trying to solve it. However, a polynomial \textit{ansatz} for the infinitesimals (here made by \texttt{symgen}, the routine for determining the symmetries) rapidly leads to 3 particular solutions to Eq.(2);\footnote{In what follows, \(y_1 = \frac{dy}{dx}\).}

\[
\begin{align*}
\text{ode} &:= \frac{d^2 y}{dx^2} = \left(\frac{dy}{dx} x - y\right)\frac{2}{x^3} \\
\text{symgen(\texttt{ode});} &\quad \text{\# input = ODE, output = pairs of infinitesimals} \\
\end{align*}
\]

\[
[\xi = 0, \eta = x], [\xi = x, \eta = y], [\xi = x^2, \eta = xy]
\]

Passing the ODE directly to \texttt{odsolve} (the solver), it will internally call \texttt{symgen} and use the result above to solve the ODE as follows:

\[
\begin{align*}
\text{ode} &:= \frac{d^2 y}{dx^2} = \left(\frac{dy}{dx} x - y\right)\frac{2}{x^3} \\
\text{odsolve(\texttt{ode});} &\quad y = (\ln(x) - \ln(1 + xC_1) + C_2) x
\end{align*}
\]

What is amazing, and characteristic of symmetry methods is that if we change \(\left(\frac{dy}{dx} - y\right)/x\) to \(F\left(\left(\frac{dy}{dx} - y\right)/x\right)\) in Eq.(3), where \(F\) is an arbitrary function of its argument, the first two symmetries of Eq.(4) will remain valid and the solving scheme will succeed as well:

\[
\begin{align*}
\text{ode} &:= \frac{d^2 y}{dx^2} = \frac{1}{x} F\left(\left(\frac{dy}{dx} - y\right)/x\right) \\
\text{odsolve(\texttt{ode});} &\quad y = (\ln(x) - \ln(1 + xC_1) + C_2) x
\end{align*}
\]

\footnote{In what follows, the \textit{input} can be recognized by the Maple prompt \texttt{>}.}
\[ y = \left( \int \text{RootOf} \left( \ln(x) + C_1 + \int_{-\infty}^{x} \frac{1}{(\alpha - F(\alpha))} d\alpha \right) \frac{dx}{x} + C_2 \right) x \]  

(7)

The answer above is expressed using \text{RootOf}, and the inner integral uses the new \text{intat} command\footnote{\text{intat} is a command of the last version of PDEtools \cite{PDEtools}, and represents an \text{integral evaluated at a point} - analogous to a derivative evaluated at a point. \text{intat} displays the evaluation point as an upper limit of integration.}

This kind of general answer can be interpreted as a \text{mapping} in that, given \( F \), it returns the answer after calculating the roots of the resulting expression; for example, the ODE Eq.(3) is a particular case of Eq.(6), and its answer Eq.(5) is what one would obtain introducing \( F : u \rightarrow u^2 \) in Eq.(7).

Also, due to the fact that we are just looking for particular solutions to Eq.(2), symmetry methods can be an alternative in solving linear ODEs too. Consider, for instance:

\[ \text{ode} := \frac{d^2 y}{dx^2} = F(x) \left( \frac{dy}{dx} - \frac{y}{x} \right) \]  

(8)

This ODE does not match a known pattern related to special functions, nor can it be solved using the standard schemes for rational or exponential solutions, due to the presence of an arbitrary function \( F(x) \). However, its invariance under \( [\xi = 0, \eta = x] \) and \( [\xi = 0, \eta = y] \) can be easily determined by a polynomial \textit{ansatz}, from which its solution follows straightforwardly:

\[ > \text{odsolve(ode)} ; \]  

\[ y = x \left( C_1 \int e \left( \int \frac{x F(x) - 2}{x} dx \right) dx + C_2 \right) \]  

(9)

2 The package’s commands

A detailed description of the package’s commands, with examples and explanations concerning their calling sequences, is found in the On-Line Help, and is already present in \cite{ODtools}. Therefore, we have restricted this section to a brief table-summary and a detailed description only of the solver, \texttt{odsolve}. Some \textit{input/output} examples can be seen in sec. 3 and 4.

2.1 Summary

A compact summary of the commands of the package is as follows:

| Command     | Purpose:                        |
|-------------|---------------------------------|
| \texttt{odsolve} | solves ODEs using the symmetry method scheme |
| \texttt{intfactor} | looks for an integrating factor for first order ODEs |
| \texttt{canoni} | looks for a pair of canonical coordinates of a given Lie group |
| \texttt{eta\_k} | returns the k-extended infinitesimal |
| \texttt{transinv} | looks for the finite group transformation |
| \texttt{infgen} | returns the k-extended symmetry generator as an operator |
| \texttt{symgen} | looks for pairs of infinitesimals |
| \texttt{equinv} | looks for the most general ODE invariant under a given set of symmetries |
| \texttt{buildsym} | looks for the infinitesimals given the solution of an ODE |
| \texttt{odepde} | returns the PDE for the infinitesimals |
| \texttt{odetest} | tests explicit/implicit results obtained by ODE-solvers |
| \texttt{symtest} | tests a given symmetry w.r.t a given ODE |
| \texttt{odeadvisor} | classifies 1\textsuperscript{st}/2\textsuperscript{nd} order ODEs and pop up related Help-pages |

Table 1. \textit{Summary of the ODEtools commands}
## 2.2 Description

**Command name:** odsolve  
**Feature:** 1st and 2nd order ODE-solver based on symmetry methods  
**Calling sequence:**

\[
> \text{odsolve}(\text{ode}); \\
> \text{odsolve}(\text{ode}, \text{y(x)}, \text{way}=\text{xxx}, \text{HINT}=[\text{[e1, e2]}, \ldots], \text{int\_scheme});
\]

### Parameters:

- **ode** - a 1st or 2nd order ODE.
- **y(x)** - the dependent variable (required when not obvious).
- **way=xxx** - optional, forces the use of only one (xxx) of the 8 internal algorithms \{abaco1, 2, 3, 4, 5, 6, abaco2, pdsolve\} for determining the coefficients of the infinitesimal symmetry generator (infinitesimals).
- **HINT = [e1, e2]** - optional, e1 and e2, indicate possible functional forms for the infinitesimals.
- **HINT=[e1, e2],[e3, e4],...** - optional, a list of hints for the infinitesimals.
- **int\_scheme** - optional, one of: fat, can, can2, gon, gon2, dif.

Optional parameters can be given alone or in conjunction, and in any order.

### Synopsis:

Given a 1st or 2nd order ODE, odsolve's main goal is to solve it in two steps: first, determine pairs of infinitesimals of 1-parameter symmetry groups which leave the ODE invariant, and then use these infinitesimals to integrate the ODE.

To determine the infinitesimals, odsolve makes calls to syngen, another command of the package. To integrate the ODE using these infinitesimals, odsolve has seven schemes, almost all of them explained in connection with point symmetries in [2, 3]:

1. building an integrating factor (fat, only for 1st order ODEs)
2. reducing the ODE to a quadrature using the canonical coordinates of that group (can)
3. reducing a 2nd order ODE to a quadrature at once, using 2 pairs of infinitesimals forming a 2-D subalgebra (can2)
4. reducing a 2nd order ODE to a quadrature, constructively, using a normal form of the generator in the space of first integrals (gon)
5. reducing a 2nd order ODE to a quadrature at once, using 2 pairs of infinitesimals (a 2-D subalgebra) and normal forms of generators in the space of first integrals (gon2)
6. using differential invariants constructively (dif).
7. solving a 2nd order ODE at once, using 3 pairs of infinitesimals, when no two of them can be used to form a 2-D subalgebra.

The integration schemes \([1, 2]\) are used with 1st order ODEs, while schemes \([3, 4, 5, 6]\) work with 2nd order ODEs. The integration schemes \([7, 8, 9, 10]\) work with dynamical symmetries too. odsolve does not classify the ODE before tackling it and is mainly concerned with non-classifiable ODEs for which the standard Maple dsolve fails. By default, odsolve starts off trying to isolate the derivative in the given ODE, then sequentially uses subsets of the algorithms of syngen to try to determine the infinitesimals, and finally sequentially tries the integration schemes mentioned above. The default order for trying these schemes is:

- 1st order ODEs: can, fat
- 2nd order ODEs: gon, can2, gon2, can, dif

When odsolve succeeds in solving the ODE, it returns, in order of preference:
• an explicit closed form solution;
• an implicit closed form solution;
• a parametric solution (default strategy for 1st order ODEs when the derivative cannot be isolated).

All these defaults can be changed by the user; the main options she/he has are:

• To request the use of only one of the algorithms for determining the infinitesimals (way=xxx option; different algorithms may lead to different symmetries for one and the same ODE, sometimes making the integration step easier).
• To enforce the use, in a specific order, of only one or more of the alternative schemes for integrating a given ODE after finding the infinitesimals (fat, can, can2, gon, gon2, and dif optional arguments, useful to select the best integration strategy for each case).
• To indicate a possible functional form for the infinitesimals (HINT=xxx option). This option is valuable when the solver fails, or to study the possible connection between the algebraic pattern of the given ODE and that of the symmetry generators.

A brief description of how the HINT=xxx option can be used is as follows:

• HINT=[e1,e2], indicates to the solver that it should take e1 and e2 as the infinitesimals and determine the form of (a maximum of two) indeterminate functions possibly contained in e1 and/or e2, such as to solve the problem.
• HINT=[[e1,e2], [e3,e4],...], where [e1,e2] is any of the above.
• HINT=parametric, indicates to the solver that it should only look for a parametric solution for the given (1st order) ODE.

Finally, there are three global variables managing the solving process, which are automatically set by internal routines but can also be assigned by the user, as desired. They are dgun, ngun, sgun, for setting, respectively, the maximum degree of polynomials entering some of the ansätze for the infinitesimals, the maximum number of subproblems into which the original ODE should be mapped, and the maximum size permitted for such subproblems. The dgun variable is automatically set each time symgen is called, according to the given ODE, whereas, by default, ngun and sgun have their values assigned to 1. Increasing the value of dgun usually helps, especially in the case of polynomial ODEs; but, although the user-assigning of the ngun or sgun variables might increase the efficacy of the algorithms, each increase of one unit can slow down the solving process geometrically.

3 Finding the infinitesimals

In the context of SM, the infinitesimals we are looking for are solutions of Eq.(2), which is linear in the functions (ξ,η) and their derivatives. The command which looks for these pairs of infinitesimals for a given ODE is symgen, and the version here presented is an extension to 2nd order ODEs of the schemes presented in [2].

3.1 The symgen subroutines

The key idea underlying this extension is that to look for dynamical symmetries for 2nd order ODEs is mainly equivalent to looking for point symmetries for 1st order ODEs; except that the routines are now going to look for solutions involving (x,y,y1) instead of only (x,y). More specifically, we adapted the previous symgen/... subroutines, whose main purpose is to look for particular solutions to linear problems, in order to look for such particular solutions considering y1 as a new variable in equal footing as x and y. An explanation with details about how these symgen/... routines work can be found in [2]; so, we here restricted the discussion to some examples illustrating the type of results which can now be obtained.
The 1st algorithm, \textit{abaco}_1, typically looks for infinitesimals of the form $[\xi = 0, \eta = \mathcal{F}(q)]$ and $[\xi = \mathcal{F}(q), \eta = 0]$, where $q$ is one of $(x, y, y_1)$.

\textbf{Example:} Kamke's 2nd order non-linear ODE 206

\begin{equation}
\text{ode} := (a^2 - x^2) (a^2 - y^2) \frac{d^2 y}{dx^2} + (a^2 - x^2) y \left( \frac{dy}{dx} \right)^2 - x (a^2 - y^2) \frac{dy}{dx} = 0 \tag{10}
\end{equation}

\begin{verbatim}
> symgen(ode, way=abaco1);
\end{verbatim}

\begin{align}
[\xi = 0, \eta = \sqrt{-a^2 + y^2 \left( 1 + \ln(y + \sqrt{-a^2 + y^2}) \right)}], \\
[\xi = \sqrt{x^2 - a^2 \left( 1 + \ln(x + \sqrt{x^2 - a^2}) \right)}, \eta = 0] \tag{11}
\end{align}

It is also not difficult to find the patterns of 2nd order ODEs having the type of symmetries which \textit{abaco}_1 is prepared in principle to look for. These ODE-patterns can be obtained using the \texttt{equinv} routine, programmed to solve the inverse problem (\textit{input}=symmetries, \textit{output}= invariant ODE; see sec. 3.2). For example, in the first of the six cases mentioned above,

\begin{verbatim}
> equinv([F(x),0],y(x),2);
\end{verbatim}

\begin{equation}
\frac{d^2 y}{dx^2} = \frac{1}{F'(x)^2} \left( \mathcal{F}_1 \left( y, F(x) \frac{dy}{dx} \right) - \frac{dF(x)}{dx} F(x) \frac{dy}{dx} \right) \tag{12}
\end{equation}

is the most general 2nd order ODE invariant under $[F(x),0]$, $F$ and $\mathcal{F}_1$ being arbitrary functions of its arguments, and Eq.(11) is a particular case of this ODE family.

The 2nd and 3rd algorithms look for polynomial in $(x, y, y_1)$ solutions to Eq.(2).

\textbf{Example:} Kamke’s 2nd order non-linear ODE 181

\begin{equation}
\text{ode} := x^2 (x + y) \frac{d^2 y}{dx^2} - \left( x \frac{dy}{dx} - y \right)^2 = 0 \tag{13}
\end{equation}

\begin{verbatim}
> symgen(ode, way=3);
\end{verbatim}

\begin{align}
[\xi = x, \eta = y], \ [\xi = -x, \eta = x], \ [\xi = x^2, \eta = xy] \tag{14}
\end{align}

\begin{verbatim}
> odsolve(ode);
\end{verbatim}

\begin{equation}
y = xe^\left( -\frac{c_1}{x} + c_2 \right) - x \tag{15}
\end{equation}

The 4th algorithm looks for rational in $(x, y, y_1)$ solutions to Eq.(2).

\textbf{Example}

\begin{equation}
\text{ode} := \frac{d^2 y}{dx^2} = \frac{1}{x + y} \left( 2 \frac{dy}{dx} + 1 \right) \frac{dy}{dx} \tag{16}
\end{equation}

\begin{verbatim}
> symgen(ode, way=4);
\end{verbatim}

\begin{align}
[\xi = -\frac{1}{x + y}, \eta = 0], \ [\xi = -\frac{y}{x + y}, \eta = 0], \ [\xi = \frac{x(x + 2y)}{x + y}, \eta = 0], \ [\xi = -1, \eta = 1], \ [\xi = x, \eta = y] \tag{17}
\end{align}

\begin{verbatim}
> odsolve(ode, way=4, can2);
\end{verbatim}
\[ y = \frac{x^2 + 2C_1}{2x + 2C_2} \] (18)

The 5th algorithm uses a polynomial of degree two constructed from a basis of functions and algebraic objects, together with their derivatives, taken from the given ODE (see [2]).

**Example:** Kanke’s 2nd order non-linear ODE 238

\[
\text{ode} := 2 \left(1 + x^2\right) \left(\frac{d^2y}{dx^2}\right)^2 - x \left(\int \frac{dy}{dx} + 4 \frac{dy}{dx}\right) \frac{d^2y}{dx^2} + 2 \left(\int \frac{dy}{dx} + \frac{dy}{dx}\right) \frac{dy}{dx} - 2y = 0
\] (19)

\[ \text{symgen(ode, way=5)}; \]

\[ [\xi = 0, \eta = \frac{4y_1 + 2x + x^3 - x\sqrt{x^4 - 8x^3y_1 - 16x y_1 - 16y_1^2 + 16y + 16x^2y_1}}{4 + 4x^2}] \] (20)

In the example above, the square root, which appeared after isolating \( \frac{d^2y}{dx^2} \), was viewed as an algebraic object, and the symmetry found is of dynamical type.

The next algorithm, abaco2, looks for infinitesimals of the form \([\xi = F(q_1), \eta = G(q_2)]\), where \(q_1\) and \(q_2\) are members of \(\{x, y, y_1\}\).

**Example**

\[
\text{ode} := \frac{d^2y}{dx^2} = \left(\int \frac{dy}{dx}\right)^2 - e^y \left(\int \frac{1}{e^y + \ln(x)} + \frac{x}{e^y} \left(\frac{dy}{dx}\right)\right)
\] (21)

\[ \text{symgen(ode, way=abaco2)}; \]

\[ [\xi = x, \eta = e^y] \]

Apart from setting up the \texttt{symgen/...} routines to work with 2nd order ODEs, two new algorithms for 1st order ODEs were implemented. The first one is specific for Riccati-type ODEs and the second one is based on a direct attempt to solve the problem using the \texttt{pdsolve} routine from the \textit{PDEtools} package. The routine for Riccati ODEs makes three sequential attempts to solve Eq.(2), the key idea being to map the problem of finding the infinitesimals for a given Riccati ODE into that of finding particular solutions to a related linear 2nd order PDE for \(\eta(x, y)\). These three attempts consist of tackling this PDE by: using the 5th algorithm explained above, trying separation of variables, and using the \textit{abaco1} scheme.

**Example:** Kanke’s 1st order ODE 16

\[
\text{ode} := \frac{dy}{dx} + y^2 + (xy - 1)f(x) = 0
\] (22)

\[ \text{symgen(ode, way=6)}; \]

\[ [\xi = 0, \eta = e^{-\int \int f(x)x^2 + 2 \ln(x) \frac{x}{x^2}} (y - 1)^2} dx \]

\[ \text{odsolve(ode, way=6)}; \]

\[ y = \frac{1}{x} + e^{-\int \int f(x)x^2 + 2 \ln(x) \frac{x}{x^4} dx} dx - C_1 \] (24)
### 3.2 The inverse problems

For many reasons, it appeared interesting to develop routines to solve the inverse problems too; that is: find the most general 1st/2nd order ODE simultaneously invariant under a given set of symmetries, or given the answer of an unknown ODE, find its symmetries (and then the corresponding invariant ODE). Typical applications for these routines are: we look for a description of a problem for which we only know the symmetries or some particular solutions; or we look for solving methods for ODEs having a given pattern, for which we know the solution in some cases, and we want to find the solution for the general case.

The ODEtools routines related to solving the inverse problems are equinv and buildsym. The former is prepared to work, in principle, with an “arbitrary number” of symmetries. As a first example, consider the determination of the most general 2nd order ODEs simultaneously invariant under

\[ X := \begin{bmatrix} [\xi = 1, \eta = 1], [\xi = x, \eta = y], [\xi = x^2, \eta = y^2] \end{bmatrix} \]  

\[ \text{> equinv}(X, y(x), 2); \]

\[ \frac{d^2 y}{dx^2} = \left( 2 \sqrt{\frac{dy}{dx}} + C_1 \left( \frac{dy}{dx} \right)^{3/2} \frac{1}{y-x} \right) \]  

As an example involving dynamical symmetries, consider the determination of the 2nd order ODE simultaneously invariant under

\[ X := [[\xi = y_1, \eta = 0], [\xi = 0, \eta = \frac{1}{y_1}]]; \]

\[ \text{> equinv}(X, y(x), 2); \]

\[ \frac{d^2 y}{dx^2} = -\frac{\left( \frac{dy}{dx} \right)^2}{\left( -2x \frac{dy}{dx} + 2y - F_1 \left( \frac{dy}{dx} \right) \right) \left( \frac{dy}{dx} \right)^2} \]  

where \( F_1 \) is an arbitrary function of its argument. Proceeding as in the examples above, it is possible to determine the general ODE-patterns which can be solved by the abaco1 scheme and some of those which can be solved by abaco2, as well as to build matching-pattern routines to match families of ODEs for which we know the form of the symmetries \emph{a priori}. Of course, since the number of possible ODE families is infinite, it is not realistic to think of producing matching pattern routines for all the cases. On the other hand, for a variety of concrete problems which arise in different applied areas, it may be convenient to build these routines and have the related solving schemes at hand. Remarkably, most of the standard methods actually work in this manner (i.e., by matching a pattern and then applying the corresponding solving scheme), and these patterns actually arose from applied problems. As a benchmark for equinv, the routine succeeded in finding the most general invariant ODEs for all the twenty examples found in sec. 8.3 or Ibragimov’s book [11] on Lie methods.

Concerning the computational scheme used in equinv, the idea is, roughly speaking, to introduce the given symmetries into Eq.\([\text{2}]\), and solve it for the right-hand-side of the ODE, which is now viewed as “the unknown of the problem”. One of the interesting advantages of this simple approach is that we can consider the invariance of an ODE under point and dynamical symmetries in equal footing, and forget about the technicalities involved in defining differential invariants when derivatives are present in the transformation equations.\footnote{This scheme works just as well to determine high order ODEs (differential order \( > 2 \)) invariant under dynamical symmetries.}

When many symmetries are given at once, from the computational point of view, the problem is a bit more complicate. A brief summary of the scheme implemented is as follows:

1. introduce the first pair of infinitesimals into Eq.\([\text{2}]\) setting up a PDE for the right-hand-side (RHS) - here denominated \( \Phi(x, y, y_1) \) - of the most general explicit invariant second order ODE;
2. solve this PDE for \( \Phi \); the solution will involve an arbitrary function of two differential invariants, say \( F(\psi_1(x,y,y_1),\psi_2(x,y,y_1)) \);

3. update the symbolic value of \( \Phi \) introducing the result of the previous step (i.e., introducing \( F \)). This updated \( \Phi(x,y,y_1) \) is an explicit expression of \( \{x, y, y_1\} \) hereafter called \( \phi(x,y,y_1) \). When only one symmetry is given, \( \phi(x,y,y_1) \) is already the RHS of the invariant ODE we are looking for;

4. take Eq. (2) again and set up a new PDE using the second given pair of infinitesimals, and substitute \( \Phi \) found in PDE by its updated value \( \phi \);

5. we now need to solve this new PDE for the arbitrary function \( F \) introduced in step (3), but this is not directly possible since the unknown \( F \) depends on algebraic expressions - \( \psi_1(x,y,y_1) \) and \( \psi_2(x,y,y_1) \); so introduce these differential invariants as new variables to obtain a PDE for \( F(\psi_1, \psi_2) \) where \( (\psi_1, \psi_2) \) are now symbol variables;

6. solve the PDE of the previous step for \( F \) and use the inverse transformation to restore the original variables \( \{x, y, y_1\} \) in the answer;

7. update the symbolic value of \( \Phi \) with respect to the result of the previous step. The resulting ODE \( y'' = \phi(x,y,y_1) \) is now simultaneously invariant under the first two pairs of infinitesimals;

8. if more pairs of infinitesimals are given, repeat this process from step (4);

To solve all the intermediate PDEs we use the \texttt{pdsolve} command of the PDEtools package, with which ODEtools is completely integrated. What requires a bit more of care is the determination and handling of all the changes of variables involved at each step.

Concerning the \texttt{buildsym} command, the idea is to build the symmetries of an unknown ODE departing from its solution. When the command succeeds it is then possible to use the \texttt{equinv} command to build the most general ODE having the symmetries of the given ODE solution. As a simple example, consider again Eq. (8). Passing the answer of this ODE (Eq. (9)) to \texttt{buildsym} it returns two symmetries:

\[
X := \left\{ \xi = 0, \eta = x \right\}, \left\{ \xi = 0, \eta = x \frac{\int e^{\int F_1(x)dx}}{x^2} dx \right\}
\] (29)

In this case, since the ODE is linear and the symmetries are of the form \( \{0, F(x)\} \), the returned symmetries are actually two particular solutions to Eq. (9). A less simpler example is given by:

\[
\frac{d^2 y}{dx^2} = \frac{x^2}{\sqrt{y}} + \frac{dy}{dx} - \frac{1}{2y} \left( \frac{dy}{dx} \right)^2
\] (30)

An implicit solution to this ODE is given by:

\[
\frac{4}{3} y^{3/2} + \frac{2}{3} x^3 + 2 x^2 + 4x - 2 C_1 e^x - C_2 = 0
\] (31)

This ODE solution leads to the following pair of underlying symmetries

\[
X := \left\{ \xi = 0, \eta = \frac{1}{\sqrt{y}} \right\}, \left\{ \xi = 0, \eta = \frac{e^x}{\sqrt{y}} \right\}
\] (32)

In turn this information leads to the general ODE family having a solution with the same two symmetries of Eq. (10):

\[
> \texttt{equinv}(X, y(x));
\]

\texttt{In turn, the solving of intermediate ODEs built by \texttt{pdsolve} to solve PDEs is carried up by ODEtools commands. PDEtools and ODEtools are both compiled in a single binary maple library.}
\[
\frac{d^2 y}{dx^2} = \frac{F_1(x)}{\sqrt{y}} + \frac{dy}{dx} - \frac{1}{2y} \left( \frac{dy}{dx} \right)^2
\]  

(33)

Concerning the computational scheme implemented in \textbf{buildsym}, the idea is, roughly speaking, to use the ODE-solution to build two \textit{first integrals}, from which two pairs of infinitesimals can be obtained. To build these pairs of infinitesimals, we note that since the first integrals, say \( \phi \) and \( \psi \), are solutions of the linear operator

\[
\Lambda := f \to \frac{\partial f}{\partial x} + y_1 \frac{\partial f}{\partial y} + \Phi(x, y, y_1) \frac{\partial f}{\partial y_1}
\]  

(34)

and from the definition of a symmetry \( X \) such that it satisfies \([X, \Lambda] = \lambda \Lambda \) (see [7]), it is always possible to formulate the problem as: to determine two infinitesimal generators \( X_1 \) and \( X_2 \) of the form \([0, \eta]\) satisfying

\[
X_1 \phi_1 = \eta_1 \frac{d\phi}{dy} + \eta_1^{(1)} \frac{d\phi}{dy_1} = 0, \quad X_1 \phi_2 = \eta_1 \frac{d\phi}{dy} + \eta_1^{(1)} \frac{d\phi}{dy_1} = 1, \\
X_2 \phi_1 = \eta_2 \frac{d\phi}{dy} + \eta_2^{(1)} \frac{d\phi}{dy_1} = 1, \quad X_2 \phi_2 = \eta_2 \frac{d\phi}{dy} + \eta_2^{(1)} \frac{d\phi}{dy_1} = 0.
\]

This system of equations can be solved for \( \{\eta_1, \eta_2\} \), arriving at

\[
\eta_1 = \begin{vmatrix}
0 & \frac{d\phi}{dy_1} \\
1 & \frac{d\phi}{dy_1}
\end{vmatrix}, \quad \eta_2 = \begin{vmatrix}
1 & \frac{d\phi}{dy_1} \\
0 & \frac{d\phi}{dy_1}
\end{vmatrix}
\]  

(35)

The determination of \( X_1 \) and \( X_2 \) then amounts to the determination of \( \phi \) and \( \phi \). These first integrals can be determined from the given solution as follows:

1. solve the given ODE solution with respect to one of the integration constants, say \( C_2 \), obtaining \( C_2 = F(x, y(x), C_1) \);
2. differentiate the result of the previous item w.r.t \( x \) and solve it w.r.t \( C_1 \) obtaining the first integral (a solution of Eq. (34)) \( C_1 = \varphi(x, y, y_1) \);
3. repeat the two previous items interchanging \( C_1 \leftrightarrow C_2 \) to obtain the second first integral \( C_2 = \psi(x, y, y_1) \);

4 \textbf{ Integrating the ODE after finding the infinitesimals }

The methods implemented for integrating the ODE once the infinitesimals were found (methods (a), (b) and (c) mentioned in sec. 2.2) are extensions to those found in [7, 8, 9]. The extension consists of making it possible to use them to integrate the given ODE even when the symmetries found are of dynamical type. Since their use in the framework of point symmetries is rather standard, we restricted the discussion here to the extension of these schemes to work with dynamical symmetries.

Concerning “why” so many integration strategies were implemented, we would like to recall that the knowledge of 2 symmetries of a 2\textsuperscript{nd} order ODE does not guarantee a successful integration of it since the different methods available do not apply in \textit{all} the cases. Also, two methods which may be “equivalent” from the mathematical point of view, may not be equivalent for the computer algebra system, since the intermediate tasks involved in each integration process are different. We then preferred to implement all these methods, provide a \textit{default} for their use, and make them available to the user by giving extra arguments to \texttt{odsolve} (see sec. 2.2).
4.1 Using differential invariants

As is well known, once a pair of infinitesimals for a given 2nd order ODE is found, it is possible to integrate the ODE if one succeeds in using these infinitesimals to find the differential invariants of order 0 and 1 and in solving an intermediate 1st order ODE (see for instance section 3.3.2 of [8]). However, if the symmetry is of dynamical type, the invariant of order 0 will be a function of $y_1$, the invariant of order 1 will be a function of $y_2$, and so on. Then, the whole scheme breaks down since, to reduce the order, the scheme requires that the differential invariants of order $n$ do not depend on derivatives of order higher than $n$.

The alternative we implemented is based on the observation that, if we transform the symmetry to the evolutionary form \[9\], then $x$ will always be an invariant (of order 0), so it is possible to work as follows:

1. put the symmetry in the evolutionary form;
2. use $u = x$ as the invariant of order 0 and find the invariant of order 1, $v = v(x, y, y_1)$;
3. construct the invariant of order 2: $(\frac{dv}{du})$;
4. change the variables and write the ODE in terms of $u, v$ and $\frac{dv}{du}$ obtaining the expected reduction of order;
5. solve this 1st order ODE, and reintroduce the original variables $(x, y, y_1)$, obtaining another 1st order ODE, which can be integrated straightforwardly since it has the same symmetry of the original 2nd order ODE.

A tricky point in this last step is that, when the symmetry is of dynamical type (depends on $y_1$), to obtain the symmetry which holds for the 1st order ODE one should replace all occurrences of $y_1$ in the infinitesimals of the original ODE by the right-hand-side of this 1st order ODE.

4.2 Using first integrals

Another way of integrating a 2nd order ODE is to use 2 pairs of infinitesimals to build 2 first integrals, from which one can expect to obtain a closed form solution (see section 7.3 of [7]). One of the advantages of this method is that one always work with the original variables avoiding potential problems related to the introduction of changes of variables (e.g., the system fails in finding the inverse transformation etc.). Taking into account the way we use “dynamical symmetries” to solve the 1st order ODE obtained in the reduction process (see sec. 4.1), the extension of this scheme to work with such symmetries was straightforward.

Now, it is also possible to tackle a 2nd order ODE from the knowledge of only 1 pair of infinitesimals (either of point or dynamical type), by looking for first integrals, exploiting similar ideas as those of the previous subsection; that is:

(i) Put the known symmetry in the evolutionary form. :

One can then prove that there exists a first integral $\varphi(x, y, y_1)$ which satisfies:

\[
\begin{align*}
A(\varphi) &= 0 \\
X(\varphi) &= 0
\end{align*}
\]

where $A$ is given by Eq.(34), and $X$ is the infinitesimal symmetry generator

\[
X := f \rightarrow \xi(x, y, y_1) \frac{\partial f}{\partial x} + \eta(x, y, y_1) \frac{\partial f}{\partial y} + \eta_1(x, y, y_1) \frac{\partial f}{\partial y_1}
\]

(ii) Solve the characteristic strip of $X(\varphi) = 0$ to obtain the invariants of order 0 and 1: $c_0(x, y, y_1)$ and $c_1(x, y, y_1)$.

\footnote{Since $[\xi = 1, \eta = y_1]$ is always a trivial symmetry, a general symmetry can always be “gauged”, that is, rewritten in many different forms. The evolutionary form is given by $[\xi = 0, \eta = \Phi(x, y, y_1)]$.}
Since the symmetry is in evolutionary form, it always happens that $c_0 = x$, and the characteristic strip is reduced to a single 1st order ODE. Now, observing that the solution to $X(\varphi) = 0$ can always be written as $\varphi(c_0, c_1)$,

(iii) change the variables in $A(\varphi) = 0$, from $(x, y, y_1)$ to $(c_0, y, c_1)$.

One can then prove that $A(\varphi) = 0$ becomes

$$\left( \frac{\partial}{\partial c_0} + A(c_1) \frac{\partial}{\partial c_1} \right) \varphi(c_0, c_1) = 0$$

that is, a PDE in only two variables; solve it, obtaining $\varphi(c_0, c_1)$ now simultaneously satisfying equations (36). Finally,

(iv) reintroduce the original variables to obtain $\varphi(x, y, y_1)$; that is, a 1st order ODE which can be integrated straightforwardly since it already satisfies $X(\varphi) = 0$.

4.3 Using canonical coordinates

It is also possible to use a pair of infinitesimals to constructively reduce the order of a 2nd order ODE by introducing the canonical variables of the related invariance group. Also, if one knows two or more symmetries at once, one can use this information to integrate the ODE in almost only one step (see sec. 7.4 of [7]). These two methods were implemented too, although their extension to work with dynamical symmetries doesn’t appear to us as advantageous if compared to the extensions outlined in the previous subsections.

4.4 Examples

We present here some examples to illustrate how these integration schemes work and the different types of answers they may return. As the first example, consider Kamke’s non-linear 2nd order ODE 99:

$$ode := x^4 \frac{d^2 y}{dx^2} + \left( \frac{dy}{dx} - y \right)^3 = 0 \quad (37)$$

This ODE has simple polynomial point symmetries:

> symgen(ode);

$$[\xi = 0, \eta = x], \quad [\xi = x, \eta = y] \quad (38)$$

Its integration is rather fast using the method explained in sec. 4.2:

> odsolve(ode, gon);

$$y = \left( - \arctan \left( \frac{1}{\sqrt{-1 + x^2} C_1} \right) + C_2 \right) x \quad (39)$$

Remarkably, if in this case we use the integration scheme which uses the 2 symmetries to build 2 first integrals, the answer will not be so compact due to the manipulations required to transform the two first integrals $\varphi_1(x, y, y_1), \varphi_2(x, y, y_1)$ into the solution $\varphi_3(x, y)$. The opposite happens with Kamke’s ODE 227:

$$ode := \left( x \frac{dy}{dx} - y \right) \frac{d^2 y}{dx^2} + 4 \left( \frac{dy}{dx} \right)^2 = 0 \quad (40)$$

for which symgen finds two point symmetries, $[\xi = x, \eta = 0]$ and $[\xi = 0, \eta = y]$, and the more compact answer is returned, using special functions, by this scheme which builds two first integrals:

> odsolve(ode, gon2);
\[ y = \frac{1}{2} \left( \text{LambertW} \left( \frac{-x}{e^{(x+C_2)}} \right) + 2 \right) e^{-\left( \text{LambertW} \left( \frac{-x}{e^{(x+C_2)}} \right) + 2 \right)} e^{C_1} \]  \hspace{1cm} (41)

As the next example, consider Kamke’s ODE 122:

\[ \text{ode} := \left( \frac{d^2 y}{dx^2} \right) y - \left( \frac{dy}{dx} \right)^2 - f(x) y \left( \frac{dy}{dx} \right) - g(x) y^2 = 0 \]  \hspace{1cm} (42)

This example involves two arbitrary functions of \( x \) and the package defaults only succeed in finding 1 symmetry:

\[ > \ \text{symgen(ode)}; \]
\[ [\xi = 0, \eta = y] \]  \hspace{1cm} (43)

In this situation only the methods prepared to work with just 1 symmetry will work; in this example, all of them lead to the same result:

\[ > \ \text{odsolve(ode,gon)}; \]
\[ y = e^{\int \int f(x) dx + \int g(x) e^{-\int f(x) dx} dx + \int \int C_1 dx + C_2} \]  \hspace{1cm} (44)

The following example, Kamke’s ODE number 20, is rather critical in the sense that all the integration schemes fail in solving the problem, except the least elaborated one (constructive reduction of order introducing canonical coordinates, using only 1 symmetry at a time). Due to the presence of an arbitrary function of \( x \) and \( y \), the answer appears in implicit form:

\[ \text{ode} := \frac{d^2 y}{dx^2} = x^{(-\frac{1}{2})} F \left( \frac{y}{\sqrt{x}} \right) \]  \hspace{1cm} (45)

\[ > \ \text{odsolve(ode,can)}; \]
\[ \frac{1}{2} \ln(x) - \int \sqrt{x} \frac{1}{\sqrt{b}^2 + 8 \int F(b) db + C_1} db - C_2 = 0 \]  \hspace{1cm} (46)

As the last example, consider the case in which only 1 symmetry has been found, and the integration schemes only succeed in reducing the order but not in solving the resulting 1\textsuperscript{st} order ODE; e.g., Kamke’s ODE number 225:

\[ \text{ode} := F(y) \frac{d^2 y}{dx^2} - D(F(y)) \left( \frac{dy}{dx} \right)^2 - (F(y))^2 \Phi \left( x, \frac{1}{F(y)} \left( \frac{dy}{dx} \right) \right) = 0 \]  \hspace{1cm} (47)

A symmetry for this ODE is found by the fifth algorithm,

\[ > \ \text{symgen(ode,way=5)}; \]
\[ [0, F(y)] \]  \hspace{1cm} (48)

In these cases, an answer with the structure of the solution is returned:

\[ > \ \text{odsolve(ode,way=5)}; \]
\[ y = \text{RootOf} \left( \left( C_1 + \int a \left( \frac{dy}{dx} \right) dx - \int \int \frac{1}{F(a)} da \right) \right), \left( \left\{ \left. \frac{d}{dx} a \right| \right. \right) \left( \frac{dy}{dx} \right) = \Phi(x, \int a dx), \left( \left. \frac{d}{dx} a \right| \right) \left( \frac{dy}{dx} \right) = \frac{1}{F(y) \frac{dy}{dx}} \right) \]  \hspace{1cm} (49)
In the structure above, we see the answer for \( y \) up to a change of variables. More specifically, this change of variables was not performed because the system doesn’t know how to solve the displayed intermediate 1st order ODEs. Although in this particular case it is impossible to go ahead - the unsolved 1st order ODE is the most general we can imagine - in many cases the user may be able to solve the reduced first order ODE by other means. Once \( s1(x) \) is obtained, a change of variables using the displayed transformation, will lead to the desired answer for \( y \).

5 Tests and performance

We tested the set of routines here presented, mainly to confirm the correctness of the returned results. In addition, we ran a “performance test” of \texttt{symgen} determining the infinitesimals for Kamke’s set of 246 non-linear 2nd order examples. Also, although the primary goal of \texttt{dsolve} is to complement the standard Maple \texttt{dsolve} in solving non-classifiable ODEs, we ran a “comparison of performances” in solving these Kamke examples too. The idea underlying both performance tests was to see the ability of the \texttt{symgen} command in determining infinitesimals for a well known set of equations, and to compare the possible efficacy of a SM-based solver with that of a classification-based solver such as \texttt{dsolve}.

5.1 Test of \texttt{symgen}

As explained in sec.1, the SM approach for solving ODEs involves two main steps: the determination of pairs of infinitesimals and their subsequent use in the integration process. The first performance test, thus, concerned the explicit determination, by the \texttt{symgen} command, of infinitesimals for the above mentioned 246 non-linear examples of Kamke’s book. To perform these tests we prepared 3 files containing the input of all these ODE examples. The table below displays the total number of successes, the average time spent with each solved/fail case, and the number of successes of each of \texttt{symgen}’s six algorithms when the whole set of ODEs was tackled using only one of them.

| File | ODEs | Symmetries | Average time | infinitesimals determined by |
|------|------|------------|--------------|-----------------------------|
|      |      | only 1     | many         | ok | fail | abaco1 | 2 | 3 | 4 | 5 | abaco2 |
| 1    | 100  | 49 24 7 sec. 22 sec. 54 62 72 65 55 71 |
| 2    | 100  | 26 53 6 sec. 22 sec. 67 53 74 68 56 70 |
| 3    | 46   | 17 22 26 sec. 54 sec. 29 20 30 21 23 31 |
| Total| 246  | 92 99 ≈ 10 sec. ≈ 29 sec. 150 138 176 169 134 171 |

Table 1. Kamke’s non-linear 2nd order ODEs: Infinitesimals determined by \texttt{symgen}.

We here distinguished the cases in which only one symmetry has been found, since in those cases the integration procedure has greater chances of only obtaining a reduction of order. The Kamke ODE numbers related to the table above are:

| \texttt{symgen} found | Kamke’s book ODE-number |
|-----------------------|------------------------|
| only 1 symmetry for 92 ODEs | 11, 12, 16, 17, 20, 21, 22, 23, 24, 25, 26, 28, 31, 32, 40, 45, 46, 47, 48, 49, 54, 58, 64, 65, 66, 68, 69, 70, 72, 73, 74, 75, 76, 77, 79, 80, 82, 83, 86, 87, 89, 90, 91, 92, 94, 96, 97, 98, 100, 102, 103, 105, 106, 118, 119, 120, 121, 122, 129, 132, 152, 153, 156, 160, 165, 170, 172, 174, 177, 183, 186, 187, 196, 197, 200, 201, 202, 203, 204, 206, 208, 213, 219, 223, 224, 225, 226, 230, 231, 238, 241, 242 |
| many symmetries for 99 ODEs | 1, 2, 4, 7, 10, 14, 30, 42, 43, 50, 56, 57, 60, 61, 62, 63, 67, 71, 78, 81, 84, 88, 93, 99, 104, 107, 108, 109, 110, 111, 113, 117, 124, 125, 126, 127, 128, 130, 133, 134, 135, 136, 137, 138, 140, 141, 143, 146, 150, 151, 154, 155, 157, 158, 159, 162, 163, 164, 166, 168, 169, 173, 175, 176, 178, 179, 180, 181, 182, 184, 185, 188, 189, 190, 191, 192, 193, 205, 209, 210, 214, 218, 220, 221, 222, 227, 228, 229, 232, 233, 234, 236, 237, 239, 240, 243, 244, 245, 246 |

Table 2. Kamke’s ODE numbers for which \texttt{symgen} found symmetries: 77%.

In summary, \texttt{symgen} fails in 23%, finds one symmetry in 37%, and finds more than one symmetry in 40% of the examples. As a curiosity, although \texttt{symgen}’s routines are also prepared to look for symmetries not of polynomial or rational type, it is noticeable that most of Kamke’s 2nd order non-linear examples have polynomial symmetries.

\footnote{The \textit{infinitesimals found by \texttt{symgen}} as well as the answers found by \texttt{dsolve} for Kamke’s non-linear 2nd order ODEs are available at: \url{http://dft.if.uerj.br/odetools.html}}
5.2 Test of odsolve’s integration schemes and comparison of performances

As explained in Sec. 4, for 2nd order ODEs the knowledge of one or more symmetries may or may not lead to the desired answer. For some cases the routines may only obtain a reduction of order. So, to evaluate the solving capabilities of the symmetry scheme here presented we ran a performance test for odsolve too, and compared the results with those obtained by using the standard Maple dsolve in solving these 246 Kamke non-linear examples. The results we found can be summarized as follows:

| Symmetry Count | ODEs | dsolve solved | odsolve solved | odsolve reduced |
|---------------|------|----------------|----------------|-----------------|
| many symmetries for: | 99 ODEs | 57 | 98 | 1 |
| 1 symmetry for: | 92 ODEs | 18 | 36 | 52 |
| 0 symmetry for: | 55 ODEs | 0 | 0 | 0 |
| Totals: | 246 ODEs | 75 | 134 | 53 |

Table 3. Comparative performance

Although Kamke’s set of ODEs is just a particular one, some conclusions can be drawn from the table above. First of all, the standard technique of classifying the ODE according to whether it matches a given pattern (for which we know the solving method) did not perform better in any example. On the other hand, the implemented symmetry scheme, which tackles the ODEs without making any distinction between them, almost doubled dsolve’s performance in obtaining closed form solutions, apart from being able to return a reduction of order in more 53 examples. Also, odsolve succeeded in integrating to the end 36 examples for which only 1 symmetry was found, turning clear the relevance of implementing integration alternatives for these cases. Finally, 98 of 99 examples for which symgen obtained more than 1 symmetry were solved by odsolve to the end, convincing us that the problem is now mainly related to finding these symmetries, not their posterior use in the integration process.

Concerning the cases in which symgen found one or more pair of infinitesimals but odsolve fails in integrating or reducing the order of the ODE, the problem was mainly related to: it was not possible to find the inverse of the transformation from canonical variables \((r, s(r))\) to \((x, y(x))\), or this inverse involved RootOfs and integrals. This was the case in Kamke’s ODEs 203, 204, 206, 238 and 246. As an example, consider Kamke’s ODE number 203,

\[
\text{ode} := a y (y - 1) \frac{d^2 y}{dx^2} - (a - 1)(2y - 1) \left( \frac{dy}{dx} \right)^2 + f(x) y (y - 1) \frac{dy}{dx} y = 0
\]

(50)

for which symgen found one symmetry

\[
> X := \text{symgen}(\text{ode});
\]

\[
X := \{ \xi = 0, \ \eta = (y^2 - y) \text{hypergeom}\left(\left[1, \frac{2}{a}\right], \left[\frac{a + 1}{a}\right], y \right) + \left( y^{\frac{2n-1}{a}} - y^{\frac{n-1}{a}} \right) \right\}
\]

(51)

The corresponding canonical variables are given by:

\[
> \text{canoni}(X, y(x), s(r));
\]

\[
\begin{aligned}
\{r &= x, \\
s(r) &= \int y(x) \left( (x^2 - x) \text{hypergeom}\left(\left[1, \frac{2}{a}\right], \left[\frac{a + 1}{a}\right], x \right) + \frac{\left( x^{-\frac{2n-1}{a}} - x^{-\frac{n-1}{a}} \right)}{(1 - x)^{\frac{1}{a}}} \right) \}
\end{aligned}
\]

(52)

To reduce the order using these canonical variables the system solved them w.r.t \(\{x, y(x)\}\), arriving at expressions involving the RootOf an integral; even when the change of variables using those expressions was successful, due to the complicated resulting algebraic structure, it was impossible to verify the reduction of order, and the subroutine gave up with no result.
5.3 Test of symgen with the 1st order Kamke examples

Although the main purpose of this paper is to present the extension of the package’s commands to work with 2nd order ODEs, some improvements have been done in the abilities for solving 1st order ODEs too. An update of the results obtained by symgen for the 1st order Kamke’s examples is as follows:

| File | ODEs | Successes | Average time | $\xi$ and $\eta$ can be determined by |
|------|------|-----------|--------------|-----------------------------------|
| 1    | 90   | 63        | 14.0 sec.    | $abaco_1$                         |
| 2    | 100  | 88        | 2.5 sec.     | $abaco_1$, 3, 4, 5, 6             |
| 3    | 99   | 86        | 1.2 sec.     | $abaco_2$                         |
| 4    | 89   | 80        | 3.5 sec.     | $abaco_1$, 3, 4, 5, 6             |
| 5    | 89   | 77        | 3.3 sec.     | $abaco_1$, 3, 4, 5, 6             |

Total: 467 successes, $\approx 5$ sec., $\approx 10$ sec.

The previous score (see Table 1. in [2]) was 75%, and the improvement is mainly due to adjustments in ‘symgen/2’, which is now obtaining the infinitesimals for many rational ODEs, and to the introduction of a new subroutine, ‘symgen/6’, for tackling ODEs of Riccati type. The number and class of Kamke’s 1st order ODEs for which symgen fails in determining a pair of infinitesimals are now given by:

| Class          | Kamke’s numbering (1st order ODE examples) |
|----------------|--------------------------------------------|
| rational       | 203, 205, 234, 257, 432, 480, 482          |
| d’Alembert     | 388, 390, 408, 409, 410, 430, 479          |
| Riccati        | 22, 25, 110                                |
| Abel           | 36, 37, 38, 40, 42, 43, 45, 46, 47, 48, 49, 50, 51, 111, 145, 146, 147, 151, 169, 185, 219, 237, 250, 253, 265, 269 |
| NONE           | 53, 54, 55, 56, 74, 79, 80, 81, 82, 83, 87, 121, 128, 129, 197, 202, 206, 331, 340, 345, 351, 367, 370, 395, 456, 460, 461, 493 |

Table 5. Kamke’s 1st order ODEs for which symgen fails: 14%

6 Computer algebra implementations of Lie symmetry methods

There are various computer algebra packages for performing Lie symmetry analysis of differential equations [3, 12]. It is then appropriate to situate ODEtools in the framework of the packages already existing in Maple and other computer algebra systems (CAS).

The range of applicability of ODEtools

To start with, ODEtools is a package for tackling only first and second order ODEs, while some of the other symmetry packages, as for example CRACK and PDELIE [12], also handle systems of ODEs and PDEs of, in principle, arbitrary order. The project is to extend ODEtools’ commands to handle high order ODEs soon, and concerning PDEs, ODEtools is presently being developed as an ODE-package project, but it is fully integrated with the PDEtools package [10].

Dynamical symmetries and the solving scheme: input=ODE, output=closed-form-solution

Concerning first and second order ODEs, the goals in ODEtools are in some sense a bit different from those of other symmetry packages. In ODEtools the focus is mainly in returning closed form solutions to ODEs instead of in performing a complete symmetry analysis. Thus, the idea implemented is to look for enough symmetries of point or dynamical type as to be able to integrate the ODE - instead of looking for all possible point symmetries. Concretely, the difference is that most of the other symmetry packages restrict the search to point symmetries and are not prepared to build a closed form solution to the ODE from the knowledge of these symmetries (see [3]). It is worth mentioning here that the search for point symmetries is more systematic due to the existence of an overdetermined system of PDEs as departure point, but most of second order ODEs don’t have point symmetries. On the other hand all

---

9 The information present in this section was gathered from [3, 4, 6, 7, 13].
10 Exception made by CRACK; restricted capabilities to determine dynamical symmetries are also found in some other packages [4].
ODEs have infinite symmetries of dynamical type. Also, though the determination of the dimension of the symmetry group (done by other packages) before searching for the symmetries may be useful, this is of no concrete help when searching for dynamical symmetries since the dimension of the group is infinite.

Related to these points, the ODEtools command for determining the symmetries (symgen) does not restrict the type of the symmetries it looks for, and the ODE-solver (odsolve) has built-in routines for using both types of symmetries to directly return a closed form solution for the problem (see sec. 22), thus allowing the scheme input=ODE, output=answer in a single step as shown in the examples throughout this paper. Furthermore, the program is able to proceed with the integration process of a second order ODE even when only one symmetry (of any type) was found, by using it to reduce the ODE’s order and restarting the solving process with this reduced ODE as input. For the case in which the reduced ODE cannot be solved, an appropriate scheme for returning reductions of order was implemented. As an example of all this, consider

\[
\frac{d^2 y}{dx^2} - \frac{1}{y} \left( \frac{dy}{dx} \right)^2 - \sin(x) \left( \frac{dy}{dx} \right) y \cos(x) y^2
\]  

This ODE does not have point symmetries [14], and the routines here presented only succeed in finding a single dynamical symmetry:

> symgen(ode,way=5);

\[
[\xi = 0, \eta = \frac{\sin(x)y^2y_1 + \cos(x)y^3 - y^2 - y_1^2}{y}]
\]

The differential invariants corresponding to the infinitesimals above are given by:

\[
[I_0 = x, I_1 = \sin(x)y + \frac{y_1}{y}]
\]  (54)

and their introduction as variables lead to the following first integral

\[
C_1 = -\frac{1}{y} \left( \frac{dy}{dx} + \sin(x) y^2 \right)
\]  (55)

with known symmetry (see sec. 12). Finding the solution of the above equation is then straightforward. In ODEtools, all these steps are run by giving a single input:

> odsolve(ode);

\[
y = \frac{1 + C_1^2}{-\cos(x) + \sin(x)C_1 + e^{-C_1} x C_2 + e^{-C_1} x C_2} C_1^2
\]

As a second example, let’s consider the ODE shown in [13] illustrating the use of CRACK in solving ODEs using symmetry analysis:

\[
\text{ode} := \frac{d}{d\rho} \left( h \frac{d^2 h}{d\rho^2} - 5 \rho^2 \left( \frac{dh}{d\rho} \right)^2 + (5 \rho h - 20 \rho^2) \frac{dh}{d\rho} - 20 h^4 + 16 h^6 + 4 h^2 \right) = 0
\]  (56)

This ODE resulted from an attempt to generalize Weyl’s class of solutions of Einstein field equations [15]. According to [13], to solve this ODE in the framework of CRACK and related set of packages (LIEPDE, APPLYSYM, etc.) one should:

1. call LIEPDE to determine the ODE symmetries;

2. call APPLYSYM with the ODE and its symmetries and run an interactive process to obtain a new ODE of reduced order;

3. use the second symmetry to integrate by hand the reduced ODE of the previous step, to arrive at a parametric solution.
In the framework of ODEtools, all these steps are compacted into a single call to odsolve. Also, many integration strategies using the same symmetries in different manners are optionally available. Using differential invariants for instance (see sec. 4.1), odsolve directly arrives at an explicit solution for Eq. (56) in terms of the roots of a quartic polynomial:

\[ h = 3 \frac{C_1 \text{RootOf}(3\,Z^4\,\rho^2 - C_1^2\,(1 + 2\,\rho^2\,C_2)\,Z^2 + 9\,\rho^2)^2 - 18\,\rho^2)}{C_1^2\,(1 + 2\,\rho^2\,C_2)\,(\text{RootOf}(3\,Z^4\,\rho^2 - C_1^2\,(1 + 2\,\rho^2\,C_2)\,Z^2 + 9\,\rho^2))^2} \]  

(57)

These roots can be made explicit as well, resulting in an explicit answer without RootOf. In this example, odsolve spent 5.7 sec calling symgen to determine the symmetries, and 7 sec more to find the differential invariants, change variables to reduce Eq.(56) to a first order ODE with known symmetries, integrate the latter, and return to the original variables. Moreover, in this case the two symmetries mentioned in [13] are already in normal form (i.e., they form a closed subalgebra), but in other cases, the step (3) mentioned some paragraphs above would not be possible in a direct manner since the reduced ODE will not have the same symmetries as the original one. In such a case odsolve will automatically either rewrite the symmetries so that they form a closed subalgebra, or determine a third symmetry when that is not possible.

These differences do not mean that one package is better than the other - and CRACK is one of the most powerful and complete symmetry packages available - but emphasize that the goals and strategy of each of the symmetry packages available are not completely equivalent. In CRACK or the Macsyma package PDElie, for instance, there are routines for handling problems completely out of the scope of ODEtools as to search for the symmetries of systems of differential equations.

The symmetries as particular solutions to the determining PDE Eq. (2)

The computational strategy used in ODEtools for finding the symmetries is also different from the strategy implemented in most of the other symmetry packages. The difference is that the ODEtools routines directly look for particular solutions for the determining PDE Eq. (2), instead of building a set of overdetermined PDEs and trying to solve it exactly [3]. Part of the motivation for this different approach is that an overdetermined set of PDEs for the infinitesimals can only be built when one restricts the search to point symmetries and this restriction is not present in ODEtools. Another reason is that even restricting the search to point symmetries, when a solution exists, this solution usually comes up only after completely solving the whole system of PDEs. Of course this requires the appropriate routines for “solving systems of linear PDEs” but this is a major problem on its own, and there is as yet no general algorithm to integrate such a system [3]. Moreover, most of the schemes for tackling systems of PDEs will work only if the system of PDEs satisfies a priori some restrictive conditions - e.g., the functional dependence on the dependent variables should be of polynomial type or the routines only work with rational function coefficients.

On the other hand, the search for particular solutions to Eq. (2) permits searching for point and dynamical symmetries in equal footing, by reducing the problem to one of algebraic type or to auxiliary first and second order ODEs as explained in our previous work [3], and without requiring routines for solving coupled systems of PDEs [4]. Moreover, there are no restrictions (e.g., coefficient fields, functional dependence etc.) to the type of ODE which can be tackled, and as a matter of fact, at least with Kamke’s non-linear examples, the average time consumed to solve an ODE is very acceptable (see Sec. 6), including in that both the determination and the use of the symmetries for building a closed form solution.

As an example of this, consider the determination of the symmetries for Liouville’s ODE

\[ \frac{d^2 y}{dx^2} + g(y) \left( \frac{dy}{dx} \right)^2 + f(x) \frac{dy}{dx} = 0 \]  

(58)

11ODEtools includes user-level commands to run each one of these steps as well, see sec. 4.
12In these cases, for second order ODEs, a third symmetry can always be determined [3].
13For the case of point symmetries, and provided that symmetries of polynomial/rational type exist, there is an interesting work due to G. Reid and McKinnon for finding particular solutions to the determining system of PDEs by solving auxiliary ODEs [4].
where \( f(x) \) and \( g(y) \) are arbitrary functions - i.e., the ODE has arbitrary dependence in both the independent and dependent variables. Such a problem is out of the scope of CRACK (the ODE is not polynomial in the dependent variable), and it is unsolvable using the reduce package SPDE, which doesn’t work with arbitrary functions \( 3 \). It takes 4.8 sec to \texttt{symgen}'s subroutines to determine the following two particular solutions (in this case two point symmetries) to the determining PDE Eq.\textup{(5)}:

\[
\begin{align*}
\xi &= 0, \quad \eta = \left(1 + \int e^{\int g(y) dy} \right) e^{-\int g(y) dy}, & \xi &= \left(1 - \int e^{-\int f(x) dx} \right) e^{\int f(x) dx}, \quad \eta = 0
\end{align*}
\]

and, in this case, the first of the symmetries above is enough to, in two extra seconds, integrate the ODE constructively using canonical coordinates (see sec. \textup{2.2}) via

\[
\texttt{> odsolve(ode, can);}
\]

\[
\ln \left( \int e^{\int g(y) dz} dz + 1 \right) - \ln \left( \int e^{-\int f(x) dx} + C_1 \right) - C_2 = 0
\]

First order ODEs

In the framework of other symmetry packages, Lie methods are usually discarded in the case of first order ODEs. The reason is that the determining PDE for the infinitesimals does not split into an overdetermined system of PDEs, and most of the available symmetry packages are based on the splitting of this PDE. On the other hand in the framework of ODEtools, the approach presented in \( 4 \) put Lie methods as a concrete alternative for tackling non-classifiable first order ODEs, and showed a performance in solving Kamke’s first order examples of \( \approx 85\% \) (see sec. \textup{5.3}), representing the highest performance we are aware of in solving these examples using symmetries or any other methods. Of course the ODEtools scheme will fail with many other examples as well, but until a general solution to the problem is discovered, this performance is convincing us that the approach is worthwhile and a computer algebra implementation of Lie methods can be a powerful alternative also for first order ODEs.

Second order linear ODEs

Another feature of the approach here presented is that it is possible to tackle linear second order ODEs using the Lie scheme, while in symmetry packages these ODEs are excluded from the symmetry analysis process (e.g., SPDE), or tackled in a manner which does not represent a concrete advantage. The point is that when looking for point symmetries by trying to solve an overdetermined system of PDEs, other symmetry packages would look for symmetries of the form \( [\xi = 0, \eta = F(x)] \), where \( F(x) \) is in turn a particular solution to the homogeneous ODE. To find this type of symmetry is then so complicate as solving the original ODE, and consequently many authors tend to disregard the use of Lie methods for linear ODEs. On the other hand, when looking for particular solutions to Eq.\textup{(5)}, these symmetries \( [\xi = 0, \eta = F(x)] \) may arise naturally, as in Eq.\textup{(8)}, and symmetries which are not particular solutions of the original ODE can be found as well. For example, consider Kamke’s second order linear ODE number 142:

\[
\texttt{ode := 5 (a x + b) dy/dx^2 + 8 a dy/dx + c \sqrt{ax + b} y = 0}
\]

In this example, a point symmetry which is not of the form \( [\xi = 0, \eta = F(x)] \) is given by\( 14 \):

\[
\begin{align*}
\xi &= 0, \quad \eta = F(x, y, y_1) \\
[0, F(x, y, y_1)]
\end{align*}
\]

In this example, a point symmetry which is not of the form \( [\xi = 0, \eta = F(x)] \) is given by\( 14 \):

\[
\begin{align*}
\xi &= 0, \quad \eta = \frac{(5a x + 5 b) y_1 + 3 ay}{5(a x + b)^{3/5} a}
\end{align*}
\]

In the implementation here presented, the additional conversion of dynamical symmetries into point symmetries, when possible, is automatic.
Now all linear homogeneous ODEs have the symmetry \([\xi = 0, \eta = y]\), so, in the case of second order ODEs, only one more symmetry as in above is enough to completely solve the problem. Although for linear ODEs the standard schemes of rational, exponential, or special function solutions seem to be the most appropriate methods, an implementation of Lie’s method by looking for particular solutions to Eq. \(\xi \neq 0\) appears as an interesting complement since it can handle ODEs with arbitrary functions or solve the problem when tricky changes of variables are involved.

**Self-contained environment for first and second order ODEs**

As the last difference mentioned in this summary, the ODEtools package has the intention of being a self contained environment for solving and studying first and second order ODEs. For this reason, the project includes a compact on-line manual of solving methods for ODEs, implemented here as the `odeadvisor` command, and commands and options playing the role of `tools` for tackling ODEs which cannot be solved using the defaults.

Among the most relevant of such tools we can mention the `dchange` program, for performing general changes of variables in mathematical expressions; the `buildsym` program for determining the symmetries of an ODE departing from its solution, and the `equinv` program, for determining the most general ODE simultaneously invariant under a given set of point and/or dynamical symmetries. In a typical non-directly solvable ODE, one would first try some changes of variables using `dchange`, and to determine the symmetries using the HINT option of `symgen` as explained in sec. 2.2. When nothing works, one can still consider a similar or simpler problem for which the solution is known, try to build the corresponding symmetries using the `buildsym` command, and then try to understand what would be the symmetry pattern of the unsolvable ODE. In connection with this, using the `equinv` command and departing from the symmetries returned by `buildsym` one may be able determine a family of ODEs containing the unsolvable ODE. Finally, if the ODE is classifiable, the `odeadvisor` command may also be of help, giving either bibliographical references or some computational hints (e.g., possibly useful changes of variables etc.) to study the problem.

7 Conclusions

This paper presented a computer algebra implementation of symmetry methods for solving 1st and 2nd order ODEs. This implementation proved to be a valuable tool for tackling ODEs as shown in sec. 5, resulting in the extension to Maple’s `dsolve` we were looking for. Actually, although the implementation of the method involves much more complicate operations than the traditional matching-pattern schemes, the generality of the families of ODEs which can be concretely solved with it is rather impressive.\(^5\)

Moreover, the solver can be used in an interactive manner, that is, one can participate in the solving process (the HINT and `way=xxx` options, the `gun` variables and the integration schemes; see sec. 2.2), achieving in this manner a significant extension of the solving capabilities of the scheme. This possible user participation plays a fundamental role, because the functional form for the infinitesimals may be out of the scope of `symgen`’s defaults, and because `dsolve`’s defaults, even for the integration schemes (see sec. 4), may not be the best choice for all cases.

Also, two routines, `equinv` and `buildsym`, were designed to tackle the inverse problems, that is to find the most general ODE simultaneously invariant under a given set of symmetries, and to find the symmetries of an unknown ODE when its solution is given. These commands, together with the HINT option of `dsolve`, can be useful in many situations, and for investigating new solving methods. Finally, the set of `ODEtools` commands can be used with educational purposes too, since almost all the relevant

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\(^{15}\)This possibility of participating in the search for the symmetries when desired is a handy and useful feature, almost absent in most of the other symmetry packages.

\(^{16}\)The ODE-solver of ODEtools is now the core of the new ODE-solver of the incomming Maple V Release 5.
steps of the symmetry cycle are available as user-level commands, and classification routines for most of
the 1st and 2nd order ODEs are available through the odeadvisor command.

On the other hand, we also perceived that the complexity of the operations involved in the symme-
try scheme resulted in a slower solving method if compared with the speed of the standard methods
based on matching pattern routines. This difference exists when solving 1st order ODEs (but is almost
insignificant), becomes more noticeable when tackling 2nd order ODEs, and turns out to be relevant when
considering high order ODEs. We then think that an ODE-solver should use both methods, combined
in an appropriate manner in order to achieve the best performance. Concerning what would be the best
combination of methods, it is clear that a testing arena should be defined first, and the results will depend
on this choice. A collection of testing examples most related to real problems seems to be the appropriate
choice, and in this sense Kamke’s book appears to be one of the best candidates available. Then, although
the natural extension of this work would be to implement symmetry methods for higher order ODEs, we
think that it is more convenient to invest in fine tuning the merging of symmetry and standard methods
for the 1st /2nd order cases first. This will open the way for more efficient implementations for the high
order case, as well as permitting us to start more concrete work to tackle PDEs. We are presently working
in some prototypes in these directions and expect to succeed in obtaining reportable results in the near
future.

Acknowledgments

This work was supported by the State University of Rio de Janeiro (UERJ), Brazil, and by the University
of Waterloo, Ontario, Canada. The authors would like to thank K. von Bülow and T. Kolokolnikov for a careful reading of this paper, A.D. Roche for preparing a first version of help pages for 2nd order
ODEs; and Prof. M.A.H. MacCallum for kindly sending us valuable references.

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17 See \url{http://dft.if.uerj.br/odetools.html}
18 Symbolic Computation Group of the Theoretical Physics Department at UERJ - Brazil.
19 Symbolic Computation Group, Faculty of Mathematics, University of Waterloo - Canada.
20 Queen Mary and Westfield College, University of London - U.K.
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