FROM COLORED JONES INVARIANTS TO LOGARITHMIC INVARIANTS

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Abstract. In this work, we give a formula for the logarithmic invariant of knots in terms of certain derivatives of the colored Jones invariant. This invariant is related to the logarithmic conformal field theory, and was defined by using the centers in the radical of the restricted quantum group at root of unity. A relation between logarithmic invariant and the hyperbolic volume of a cone manifold is also investigated.

Introduction

The logarithmic invariants of knots are introduced by Nagatomo and the author [14] by using the centers in the Jacobson radical of the restricted quantum group $U_q(sl_2)$ at root of unity. In this paper, we give a formula for the logarithmic invariant in terms of the colored Jones invariant. Let $N$ be a positive integer greater than 1 and let $\xi$ be the $2N$-th root of unity given by $\xi = \exp(\pi \sqrt{-1}/N)$. The center of $U_\xi(sl_2)$ is $3N - 1$ dimensional, and its good basis

(1) \[ \{ \hat{\rho}_1, \hat{\rho}_2, \cdots, \hat{\rho}_{N-1}, \hat{\varphi}_1, \hat{\varphi}_2, \cdots, \hat{\varphi}_{N-1}, \hat{\kappa}_0, \hat{\kappa}_1, \cdots, \hat{\kappa}_N \} \]

is given by [2, §5.2] which behaves well under certain action of $SL(2,\mathbb{Z})$. For a knot $L$, let $\gamma_s^{(N)}(L)$ be the logarithmic invariant corresponding to $\hat{\kappa}_s$ of the above basis, and let $V_m(L)$ be the colored Jones invariant corresponding to the $m$ dimensional representation of $U_q(sl_2)$ at generic $q$. We get the following two formulas to explain the logarithmic invariant $\gamma_s^{(N)}(L)$ by using derivatives of the colored Jones invariant $V_m(L)$.

Theorem (in Theorem 2.5). The invariant $\gamma_s^{(N)}(L)$ ($1 \leq s \leq N$) is given by

(2) \[ \gamma_s^{(N)}(L) = \frac{\xi}{2N} \frac{d}{dq}(q - q^{-1})(V_m(L) + V_{2N-m}(L)) \bigg|_{q=\xi} \]

\[ = \frac{N}{\pi \sqrt{-1}} (\xi - \xi^{-1}) \frac{d}{dm}V_m(L) \bigg|_{m=s}^{m=2N-s}. \]

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Remark. The first formula in (2) is given by the derivative of $V_m(L)$ with respect to the parameter $q$ and the second formula is given by the derivative of $V_m(L)$ with respect to the parameter $m$ corresponding to the dimension, which is an integer. However, we can differentiate $V_m(L)$ with respect to $m$ by using the following universal expression of $V_m(L)$ given by Habiro [4, Theorem 3.1] (see also [8]).

\[(3) \quad V_m(L) = \sum_{i=0}^{\infty} a_i(L) \frac{\{m + i, 2i + 1\}q}{\{1\}q}.\]

Here $\{n\}q = q^n - q^{-n}$, $\{n, k\}q = \prod_{j=0}^{m-1} \{n - j\}q$ and the coefficient $a_i(L)$ is a Laurent polynomial in $q$ which does not depend on $m$ (see [4, Theorem 2.1]). For $\frac{d}{dm}V_m(L)$ in (2), $V_m(L)$ is given by (3) and is considered to be an infinite sum with the indeterminate $m$. The integer $s$ is substituted to $m$ after the differentiation, and the sum reduces to a finite sum when $q$ is specialized to $\xi$.

The above theorem suggests some relation between the logarithmic invariant and the hyperbolic volume since relations between the colored Jones invariants and the hyperbolic volume are known for various cases by [5], [11], [12], [3], [10] and [13]. Let $L$ be a hyperbolic knot. In [5], Kashaev found a relation between the hyperbolic volume of the knot complement and the series of invariants $J_N(L)$ he constructed. Kashaev’s invariant turned out to be a specialization of the colored Jones invariant by [11], more precisely, $J_N(L) = V_N(L)|_{q=\xi}$. Then Kashaev’s conjecture is generalized as follows.

Conjecture (Complexified volume conjecture [12]). Let $L$ be a hyperbolic knot in $S^3$. Then

\[(4) \quad \lim_{N \to \infty} \frac{2 \pi \log J_N(L)}{N} = \text{Vol} (S^3 \setminus L) + \sqrt{-1} \text{CS} (S^3 \setminus L),\]

where $\text{Vol} (S^3 \setminus L)$ and $\text{CS} (S^3 \setminus L)$ are the hyperbolic volume and the Chern-Simons invariant of $S^3 \setminus L$ respectively.

There are several generalizations of this conjecture. For example, if we deform $\xi$ to $\xi^\alpha = \exp(\pi \sqrt{-1} \alpha/N)$ by a complex number $\alpha$ near 1, a conjecture for the relation between $V_N(L)$ at $q = \xi^\alpha$ and the complex volume of certain deformation of the hyperbolic structure of $S^3 \setminus K$ is proposed by [3] and [10]. For the figure-eight knot, this conjecture is proved partially by Murakami-Yokota [13].

Our invariant $\gamma^{(N)}_s(L)$ can be considered as a deformation of $J_N(L)$ since $J_N(L)$ is equal to $\gamma^{(N)}_s(L)$. Changing the parameter $N$ to $s$ can be considered as a deformation (not continuous but discrete) of the weight parameter $\lambda$ instead of the deformation of the parameter $q$. Comparing with the deformations in [3], [10], [13], we propose the following conjecture.

Conjecture (Volume conjecture for the logarithmic invariant). Let $L$ be a hyperbolic knot and let $M_\alpha$ be the cone manifold along the singularity set $L$ with the cone angle
\( s_N \) be a sequence of integers such that \( \lim_{N \to \infty} s_N = \frac{\alpha}{2\pi} \). If \( M_\alpha \) is a hyperbolic manifold, then
\[
\lim_{N \to \infty} \frac{2\pi \log s_N^{(N)}(L)}{N} = \text{Vol}(M_\alpha) + \sqrt{-1} \text{CS}(M_\alpha).
\]

For the figure-eight knot, we prove this conjecture for \( \alpha \) satisfying \( 0 \leq \alpha < \frac{\pi}{3} \), and check numerically for entire \( \alpha \).

This paper is organized as follows. In the first section, we recall some properties of the quantum groups, their representations and their centers. These materials are explained in [2]. In the second section, we discuss about the logarithmic invariant of knots. For a knot \( L \), there is a tangle \( T_L \) corresponding to \( L \), and by passing through the universal invariant by Lawrence [7] and Ohtsuki [15], we get a center \( z(T_L) \) of \( U_\xi(sl_2) \), which is an invariant of \( L \). We introduce a representation of \( U_q(sl_2) \) for generic \( q \), which coincides with a projective representation of the restricted quantum group \( U_\xi(sl_2) \) when \( q \) is specialized to \( \xi \). Then, by applying this specialization to \( z(T_L) \), we give a formula to explain the logarithmic invariant in terms of the colored Jones invariant. Moreover, the invariant \( z(T_L) \) is a linear combination of the basis (1), and the coefficients of this linear combination are again invariants of \( L \). Then, we give formulas for these coefficients in terms of the colored Jones invariant. In the last section, we investigate the relation between the logarithmic invariant of the figure-eight knot \( K_{4,1} \) and the hyperbolic volume of a cone manifold along \( K_{4,1} \).

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1. Quantum groups \( U_q(sl_2) \) and \( U_\xi(sl_2) \)

1.1. Notations. We will use following notations.

\[
\{n\}_q = q - q^{-1}, \quad \{n, m\}_q = \prod_{k=0}^{m-1} \{n - k\}_q, \quad \{n\}_q! = \{n, n\}_q,
\]

\[
[n]_q = \frac{\{n\}_q}{\{1\}_q}, \quad [n]_q! = \prod_{k=1}^{n} [k]_q, \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}.
\]

For the \( 2N \)-th root of unity \( \xi = e^{\frac{2\pi i}{N}} \), we also use

\[
\{n\} = \{n\}_\xi, \quad \{n\}_! = \{n\}_\xi!, \quad [n] = [n]_\xi, \quad [n]_! = [n]_\xi!, \quad \begin{bmatrix} n \\ k \end{bmatrix}_\xi = \begin{bmatrix} n \\ k \end{bmatrix}, \quad \{n\}_+ = \xi + \xi^{-1}.
\]

1.2. Quantum group \( U_q(sl_2) \). Let \( U_q(sl_2) \) be the quantum group defined by

\[
U_q(sl_2) = \langle K, E, F \mid KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}} \rangle.
\]
The Hopf algebra structure of $U_q(sl_2)$ is given by
\[
\Delta(K) = K \otimes K, \quad \Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1,
\]
\[
\epsilon(K) = 1, \quad \epsilon(E) = \epsilon(F) = 0,
\]
\[
S(K) = K^{-1}, \quad S(E) = -E K^{-1}, \quad S(F) = -K F,
\]
where $\Delta$ is the coproduct, $\epsilon$ is the counit and $S$ is the antipode. The universal $R$-matrix of $U_q(sl_2)$ is given by
\[
R = q^{H \otimes H} \sum_{n=0}^{\infty} \frac{\{1\}_q^{2n}}{(n)!} q^{\frac{n(n-1)}{2}} (E^n \otimes F^n),
\]
where $H$ is an element such that $q^H = K$.

1.3. Restricted quantum group $\overline{U}_\xi$.

**Definition 1.1.** The restricted quantum group $\overline{U}_\xi(sl_2)$ is
\[
\overline{U}_\xi(sl_2) = U_\xi(sl_2)/(E^N, F^N, K^{2N} - 1),
\]
i.e. $\overline{U}_\xi(sl_2)$ is defined from $U_\xi(sl_2)$ by adding new relations $E^N = F^N = 0$ and $K^{2N} = 1$.

The $R$ matrix of $\overline{U}_\xi(sl_2)$ is given by
\[
R = (\xi^{\frac{1}{2}})^{H \otimes H} \sum_{n=0}^{N-1} \frac{\{1\}_q^{2n}}{(n)!} q^{\frac{n(n-1)}{2}} (E^n \otimes F^n).
\]
Here $\xi^{\frac{1}{2}} = \exp\left(\frac{\pi \sqrt{-1}}{2N}\right)$, $H$ is given by $\xi^H = K$ and satisfies
\[
HE - EH = 2E, \quad HF - FH = -2F.
\]
Moreover, for every $\overline{U}_\xi(sl_2)$-module $V$, if $K v = v$ for $v \in V$, then we assume $H v = 0$.
With the above assumptions, the representation of the $R$ matrix on the tensor representation of two projective modules explained in the next subsection is uniquely determined, and coincides with the representation of the universal $R$-matrix of $\overline{U}_\xi(sl_2)$ given by Drinfeld’s quantum double construction. For the universal $R$ matrix, see, for example, [2].

1.4. Projective modules of $\overline{U}_\xi(sl_2)$. We first explain irreducible representations of $\overline{U}_\xi(sl_2)$. Let $U_\xi^\pm s$ be the $s$-dimensional irreducible representations of $\overline{U}_\xi(sl_2)$ labeled by $1 \leq s \leq N$. The module $U_\xi^\pm s$ is spanned by elements $u_\xi^\pm n$ for $0 \leq n \leq s - 1$, where the action of $\overline{U}_\xi(sl_2)$ is given by
\[
K u_\xi^\pm n = \pm \xi^{s-1-2n} u_\xi^\pm n,
\]
\[
E u_\xi^\pm n = \pm [n][s-n] u_\xi^\pm n_{n-1}, \quad 1 \leq n \leq s - 1, \quad E u_\xi^\pm 0 = 0,
\]
\[
F u_\xi^\pm n = u_\xi^\pm n_{n+1}, \quad 0 \leq n \leq s - 2, \quad F u_\xi^\pm s-2 = 0.
Especially, $U^+_1$ is the trivial module for which $k$ acts by 1 and $e$, $f$ act by 0. The weights (eigenvalues of $K$) occurring in $U^+_s$ are $\xi^{s-1}$, $\xi^{s-3}$, \ldots, $\xi^{-s+1}$, and the weights occurring in $U^-_{N-s}$ are $-\xi^{N-s-1}$, $-\xi^{N-s-3}$, \ldots, $-\xi^{-N+s+1}$.

Let $V^+_s$ ($1 \leq s \leq N$) be the $N$ dimensional representation with highest-weight $\xi^{s-1}$ spanned by elements $v^+_n$ for $0 \leq n \leq N - 1$, where the action of $U(\xi)(sl_2)$ is given by

$$K v^+_n = \xi^{s-1-2n} v^+_n, \quad E v^+_n = \pm[n][s-n] v^+_{n-1}, \quad F v^+_n = v^+_{n+1},$$

for $0 \leq n \leq N - 2$.

Note that $V^+_N = U^+_N$. For $1 \leq s \leq N - 1$, $V^+_s$ satisfies the exact sequence

$$0 \longrightarrow U^+_{N-s} \longrightarrow V^+_s \longrightarrow U^+_s \longrightarrow 0,$$

and there are projective modules $P^+_s$ satisfying the following exact sequence.

$$0 \longrightarrow V^+_{N-s} \longrightarrow P^+_s \longrightarrow V^+_s \longrightarrow 0.$$
\[ E x_j^- = -[j][N-s-j] x_{j-1}^-, \quad 0 \leq k \leq N - s - 1, \]
\[ E y_j^- = \begin{cases} -[j][N-s-j] y_{j-1}^- + x_{j-1}^-, & 1 \leq j \leq N - s - 1, \\ a_{s-1}^-, & j = 0, \end{cases} \]
\[ E a_n^- = [n][s-n] a_{n-1}^-, \quad 0 \leq n \leq s - 1, \]
\[ E b_n^- = \begin{cases} [n][s-n] b_{n-1}^-, & 1 \leq n \leq s - 1, \\ x_{N-s-1}^-, & n = 0, \end{cases} \]
\[ F x_j^- = x_{j+1}^-, \quad 0 \leq j \leq N - s - 2, \]
\[ F y_j^- = \begin{cases} y_{j+1}^-, & 0 \leq j \leq N - s - 2, \\ b_0^-, & j = N - s - 1, \end{cases} \]
\[ F a_n^- = \begin{cases} a_{n+1}^-, & 0 \leq n \leq s - 2, \\ x_0^-, & n = s - 1. \end{cases} \]

Here we assume that \( x_{-1}^- = a_{-1}^- = x_{N-s}^- = b_s^- = 0. \)

1.5. Centers of \( \mathcal{U}_\xi(sl_2) \). The center of \( \mathcal{U}_\xi(sl_2) \) is investigated in \( [2] \).

**Proposition 1.2 (\[2\], 4.4.4.).** The dimension of the center \( Z \) of \( \mathcal{U}_\xi(sl_2) \) is \( 3N - 1 \).

Its commutative algebra structure is described as follows. There are two special central idempotents \( e_0 \) and \( e_N \), other central idempotents \( e_s \) \( (1 \leq s \leq N - 1) \), centers in the radical \( w_s^+ \) \( (1 \leq s \leq N - 1) \), and they satisfy the following commutation relation.

\[
\begin{align*}
e_s e_t &= \delta_{s,t} e_s, \quad s, t = 0, 1, \ldots, N, \\
e_s w_t^+ &= \delta_{s,t} w_t^+, \quad 0 \leq s \leq N, \quad 1 \leq t \leq N - 1, \\
w_s^+ w_t^+ &= w_s^+ w_t^+, \quad 0 \leq s, t \leq N - 1.
\end{align*}
\]

The center \( e_N \) acts as identity on \( U_N^+ \) and as 0 on the other modules. \( e_0 \) acts as identity on \( U_N^- \) and as 0 on the other modules. \( e_s \) acts as identity on \( P_{s}^+ \) and \( P_{-s}^- \) and as 0 on the other modules. The center \( w_s^+ \) acts as \( P_{s}^+ \) by \( w_s^+ b_n^+ = a_n^+ \), \( w_s^+ a_n^+ = 0 \), \( w_s^+ x_k^+ = 0 \), \( w_s^+ y_k^+ = 0 \), and acts on the other modules as 0. Similarly, \( w_s^- \) acts on \( P_{-s}^- \) by \( w_s^- y_k^- = x_k^- \), \( w_s^- x_k^- = 0 \), \( w_s^- a_n^- = 0 \), \( w_s^- b_n^- = 0 \), and acts on the other modules as 0.

According to \( [2] \), the basis \([1]\) is expressed by \( e_s \) and \( w_s^\pm \) as follows.

\[
\begin{align*}
\hat{\rho}_s &= (-1)^{N+s} \frac{1}{N (q^s - q^{-s})} \left( e_s - \frac{q^s + q^{-s}}{|s|^2} (w_s^+ + w_s^-) \right) \quad (1 \leq s \leq N - 1), \\
\hat{\varphi}_s &= \frac{1}{|s|^2} \left( \frac{N-s}{N} w_s^+ - \frac{s}{N} w_s^- \right) \quad (1 \leq s \leq N - 1), \\
\hat{\kappa}_0 &= e_0, \quad \hat{\kappa}_s = \frac{1}{|s|^2} \left( w_s^+ + w_s^- \right) \quad (1 \leq s \leq N - 1), \quad \hat{\kappa}_N = -e_N.
\end{align*}
\]

Let \( z \) be a center of \( \mathcal{U}_q(sl_2) \) given by

\[
z = a_o e_0 + a_N e_N + \sum_{s=1}^{N-1} (a_s e_s + b_s^+ w_s^+ + b_s^- w_s^-).
\]
Then \( z \) can be also expressed by the good basis \( \hat{\kappa}_s, \hat{\rho}, \hat{\varphi} \) by

\[
z = \sum_{s=1}^{N-1} \alpha_s^{(N)} \hat{\rho}_s + \sum_{s=1}^{N-1} \beta_s^{(N)} \hat{\varphi}_s + \sum_{s=0}^{N} \gamma_s^{(N)} \hat{\kappa}_s,
\]

where

\[
\alpha_s^{(N)} = (-1)^{N+s} (q^s - q^{-s}) N a_s, \quad \beta_s^{(N)} = [s]^2 (b^+_s - b^-_s), \quad (1 \leq s \leq N - 1)
\]

(10) \( \gamma_s^{(N)} = [s]^2 \left( \frac{s}{N} b^+_s + \frac{N - s}{N} b^-_s \right) + (q^s + q^{-s}) a_s, \quad (1 \leq s \leq N - 1) \)

\( \gamma_0^{(N)} = a_0, \quad \gamma_N^{(N)} = -a_N. \)

2. Logarithmic invariants of knots

2.1. Modified representations of \( \mathcal{U}_q(sl_2) \). Let \( W_m \) be the highest weight representation of the quantum group \( \mathcal{U}_q(sl_2) \) given by the following basis and actions. Let \( f_0, f_1, \ldots, f_{m-1} \) be the weight basis of \( V_m \) and the actions of \( E, F, K \) are given by

(11) \( E f_i = [i]_q f_{i-1}, \quad F f_i = [m - 1 - i]_q f_{i+1}, \quad K f_i = q^{m-1-2i} f_i. \)

For an integer \( m = 1, 2, \ldots, N - 1 \), we introduce a \( 2N \) dimensional representation \( \mathcal{Y}_m^+ \) which is isomorphic to \( W_{2N-m} \oplus W_m \). The basis of \( \mathcal{Y}_m^+ \) is \( \alpha_0^+, \alpha_1^+, \ldots, \alpha_{2N-m-1}^+, \beta_0^+, \beta_1^+, \ldots, \beta_{m-1}^+ \), and the actions of \( E, F, K \in \mathcal{U}_q(sl_2) \) are given by

\[
E \alpha_i^+ = \begin{cases} 
[i]_q \alpha_{i-1}^+ & \text{if } i \leq N - m \text{ or } i \geq N + 1,

[i]_q \alpha_i^+ + \left[ \frac{2N - m - i - 1}{N - i} \right]_q \beta_{m-N+i-1}^+ & \text{if } N - m + 1 \leq i \leq N,
\end{cases}
\]

\[
E \beta_i^+ = [i]_q \beta_{i-1}^+,
\]

\[
F \alpha_i^+ = \begin{cases} 
[2N - m - i - 1]_q \alpha_{i+1}^+ & \text{if } i \neq N - m - 1,

[N] \alpha_{i+1}^+ + \left[ \frac{N - 1}{m - 1} \right]_q \beta_0^+ & \text{if } i = N - m - 1,
\end{cases}
\]

\[
K \alpha_i^+ = q^{2N-m-1-2i} \alpha_i^+, \quad K \beta_i^+ = q^{m-1-2i} \beta_i^+.
\]

Theorem 2.1. If \( q \) is specialized to \( \xi \), then \( \mathcal{Y}_m^+ \) is isomorphic to the projective module \( \mathcal{P}_m^+ \) given by (8).

Proof. We compare the actions of \( \mathcal{U}_q(sl_2) \) on \( \mathcal{Y}_m^+ \) and \( \mathcal{P}_m^+ \). Let \( f \) be a linear map defined by

(12)

\[
f(x_k^+) = \frac{(-1)^k}{[N-m-1-k]!} \alpha_k^+, \quad f(y_k^+) = \frac{(-1)^k [N-1]!}{[N-m-1-k]!} \alpha_{N+k}^+ \quad \text{for } 0 \leq k \leq N - m - 1,
\]

\[
f(a_k^+) = \frac{[m-1]!}{[m-1-k]!} \beta_k^+, \quad f(b_k^+) = \frac{[k]!}{[m]} \alpha_{k+N-m}^+ \quad \text{for } 0 \leq k \leq m - 1.
\]
Then a simple computation shows that the actions of $K$, $E$, $F$ on $\mathcal{Y}_m^+$ and $\mathcal{P}_m^+$ are commute with $f$. Therefore, the specialization of $\mathcal{Y}_m^+$ at $q = \xi$ is isomorphic to $\mathcal{P}_m^+$ as an $\overline{U}_\xi(sl_2)$ module.

For an integer $m = 1, 2, \cdots, N - 1$, introduce a $4N$ dimensional representation $\mathcal{Y}_m^-$ which is isomorphic to $W_{2N+1-m} \oplus W_{2N-m}$. The basis of $\mathcal{Y}_m^-$ is $\alpha_0^-, \alpha_1^-, \cdots, \alpha_{2N+m-1}^-, \beta_0^-, \beta_1^-, \cdots, \beta_{2N-m-1}^-$, and the actions of $E$, $F$, $K \in \mathcal{U}_\xi(sl_2)$ are given by

$$E \alpha_i^- = \begin{cases} [i]_q \alpha_{i-1}^- & \text{if } i \leq m \text{ or } i \geq 2N + 1, \\ [i]_q \alpha_{i-1}^- + \frac{2N + m - 1 - i}{2N - i} \beta_{i-m-1}^- & \text{if } m + 1 \leq i \leq 2N, \end{cases}$$

$$E \beta_i^- = [i]_q \beta_{i-1}^-, \quad F \alpha_i^- = \begin{cases} [2N + m - 1 - i]_q \alpha_{i+1}^- & \text{if } i \neq m - 1, \\ [2N]_q \alpha_{i+1}^- + \frac{2N - 1}{2N - m - 1} \beta_0^- & \text{if } i = m - 1, \end{cases}$$

$$F \beta_i^- = [2N - m - 1 - i]_q \beta_{i+1}^-,$$

$$K \alpha_i^- = q^{2N+m-1-2i} \alpha_i^-, \quad K \beta_i^- = q^{2N-m-1-2i} \beta_i^-.$$

As for $\mathcal{Y}_m^+$, we get the following.

**Theorem 2.2.** If $q$ is specialized to $\xi$, then $\mathcal{Y}_m^-$ is isomorphic to the direct sum $\mathcal{P}_{N-m}^- \oplus \mathcal{P}_{N-m}^-$ of the projective module $\mathcal{P}_{N-m}$ given by (1.4).

**Proof.** Let $Y_1$ be the subspace of $\mathcal{Y}_m^-$ spanned by $\alpha_0^-, \alpha_1^-, \cdots, \alpha_{N+m}^-, \beta_0^-, \beta_1^-, \cdots, \beta_{N-m-1}^-$. And let $Y_2$ be the subspace spanned by the remaining basis $\alpha_{N+m}^-, \alpha_{N+m+1}^-, \cdots, \alpha_{2N+m-1}^-, \beta_{N-m}^-, \beta_{N-m+1}^-, \cdots, \beta_{2N-m-1}^-$. Then $Y_1$ is invariant under the action of $\mathcal{U}_\xi(sl_2)$. We prove that $Y_1$ and $\mathcal{Y}_m^-/Y_1$ are both isomorphic to $\mathcal{P}_{N-m}^-$. Let $g$ be a linear map from $\mathcal{P}_{N-m}^-$ to $Y_1$ defined by

$$g(x_k^-) = \frac{(-1)^{m+k}[N-m]!}{[N-m-k]!} \beta_k^-, \quad g(y_k^-) = (-1)^k [k]! \alpha_{m+k}^- \quad \text{for } 0 \leq k \leq N-m-1,$$

$$g(a_k^-) = \frac{[m]}{[m-k]!} \alpha_k^-, \quad g(b_k^-) = \frac{(-1)^{N+m+k} [N-1]!}{[m-k]!} \alpha_{N+k}^- \quad \text{for } 0 \leq k \leq m-1.$$ 

Then, by checking the actions of $K$, $E$, $F$, we see that $g$ gives an isomorphism from $\mathcal{P}_{N-m}^-$ to $Y_1$ as $\mathcal{U}_\xi(sl_2)$ modules.

Next, we define a linear map $h$ from $\mathcal{P}_{N-m}^-$ to $Y_2$ to show that $\mathcal{P}_{N-m}^-$ are isomorphic to $\mathcal{Y}_m^-/Y_1$.

$$h(x_k^-) = \frac{(-1)^k[N-m]!}{[N-m-1-k]!} \beta_{N+k}^-, \quad h(y_k^-) = (-1)^k[k]! \alpha_{N+m+k}^- \quad \text{for } 0 \leq k \leq N-m-1,$$

$$h(a_k^-) = \frac{[k]}{[m-k]!} \beta_{N-m+k}^-, \quad h(b_k^-) = (-1)^{N-m-1} \frac{[N-1]!}{[m-k]!} \alpha_{2N+k}^- \quad \text{for } 0 \leq k \leq m-1.$$
Then $h$ defines an isomorphism from $\mathcal{P}^+_{N-m}$ to $\mathcal{Y}_m^+/Y_1$. This isomorphism induces an inclusion from $\mathcal{P}^-_m$ to $\mathcal{Y}_m^-$ since $\mathcal{P}^+_{N-m}$ is a projective module. Hence $\mathcal{Y}_m^-$ at $q = \xi$ is isomorphic to $\mathcal{P}^+_{N-m} \oplus \mathcal{P}^+_{N-m}$.

2.2. **Specialization of a center of $\mathcal{U}_q(sl_2)$ at $q = \xi$.** Let $z$ be a center of $\mathcal{U}_q(sl_2)$. Let $\rho_m$, $\eta_m^+$ be representations of $\mathcal{U}_q(sl_2)$ on $\text{End}(W_m)$ and $(\mathcal{Y}_m^+)$ given by the above actions. Then there is a scalar $x^+$ satisfying

$$\eta_m^+(z) \alpha^+_{N-m} = \rho_m(z)_{N-m,N-m} \alpha^+_{N-m} + x^+ \beta^+_0.$$ 

Here $\rho_m(z)_{N-m,N-m}$ is the $(N - m)$-th diagonal element of the representation matrix $\rho_m(z)$ given by (11). By applying $F$, we have

$$\eta_m^+(Fz) \alpha^+_{N-m-1} = \rho_{2N-m-1}(z)_{N-m-1,N-m-1} \eta_m^+(F) \alpha^+_{N-m-1} =$$

$$\rho_{2N-m-1}(z)_{N-m-1,N-m-1} \left( [N]_q \frac{N-1}{m-1} q \beta^+_0 \right),$$

and

$$\eta_m^+(z F) \alpha^+_{N-m-1} = \rho_{2N-m-1}(z)_{N-m,N-m} [N]_q \alpha^+_{N-m} + [N]_q x^+ \beta^+_0 + \rho_\lambda(z)_{0,0} \frac{N-1}{m-1} q \beta^+_0.$$

Hence we get

$$x^+ = \left[ \frac{N-1}{m-1} \right] q \frac{\rho_{2N-m}(z)_{0,0} - \rho_m(z)_{0,0}}{[N]_q}.$$ 

Let $\eta^-_m : \mathcal{U}_q(sl_2) \to \text{End}(\mathcal{Y}_m^-)$. Then there is a scalar $x^-$ satisfying

$$\eta_m^-(z) \alpha^-_m = \rho_{2N+m}(z)_{m,m} \alpha^-_m + x^- \beta^-_0.$$ 

By applying $F$, we have

$$\eta_m^-(Fz) \alpha^-_{m-1} = \rho_{2N+m}(z)_{m-1,m-1} \eta_m^-(F) \alpha^-_{m-1} =$$

$$\rho_{2N+m}(z)_{m-1,m-1} \left( [2N]_q \frac{2N-1}{2N-m-1} q \beta^-_0 \right),$$

and

$$\eta_m^-(z F) \alpha^-_{m-1} = \rho_{2N+m}(z)_{m,m} [2N]_q \alpha^-_m + [2N]_q x^- \beta^-_0 + \rho_{2N-m}(z)_{0,0} \left[ \frac{2N-1}{2N-m-1} q \beta^-_0 \right].$$

Hence we get

$$x^- = \left[ \frac{2N-1}{m} \right] q \frac{\rho_{2N+m}(z)_{0,0} - \rho_{2N-m}(z)_{0,0}}{[2N]_q}.$$
By specializing \( q \) to the \( 2N \)-th root of unity \( \xi \), we get
\[
\begin{align*}
\lim_{q \to \xi} x^+ &= -\frac{\xi}{2N} \frac{d}{dq} \{1\}_q (\rho_{2N-m}(z)_{0,0} - \{1\}_q \rho_{m}(z)_{0,0}) \\
\lim_{q \to \xi} x^- &= \frac{(-1)^m \xi}{4N} \frac{d}{dq} \{1\}_q (\rho_{2N+m}(z)_{0,0} - \rho_{2N-m}(z)_{0,0})
\end{align*}
\]
by using l’Hopital’s rule.

Now, we compare with the coefficients \( b^+_m \) of \( w^+_s \) introduced in \([14]\). Let \( \tilde{\eta}^+_m, \tilde{\eta}^-_m \) be the representations on \( \mathcal{P}^+_m \) and \( \mathcal{P}^-_{N-m} \) in \([14]\), then
\[
\begin{align*}
\tilde{\eta}^+_m(z) &= \rho_m(z)_{0,0} b^+_0 + b^+_m a^+_0, \\
\tilde{\eta}^-_m(z) &= \rho_m(z)_{0,0} y^-_0 + b^-_m x^-_0.
\end{align*}
\]
By the isomorphisms \( f, g \) in \([12], [13]\),
\[
\begin{align*}
\tilde{f}(b^+_0) &= \frac{1}{[m]} \alpha^+_N - m, f(a^+_0) = \beta^+_m, g(y^-_0) = \alpha^-_m \text{ and } g(x^-_0) = (-1)^m [m] \beta^-_m.
\end{align*}
\]
Hence we get
\[
\begin{align*}
b^+_m &= \frac{\xi}{2N[m]} \frac{d}{dq} \{1\}_q (\rho_m(z)_{0,0} - \rho_{2N-m}(z)_{0,0}) \\
b^-_m &= \frac{\xi}{4N[m]} \frac{d}{dq} \{1\}_q (\rho_{2N+m}(z)_{0,0} - \rho_{2N-m}(z)_{0,0})
\end{align*}
\]  
(15)

2.3. Logarithmic invariants. Let \( L \) be a knot with framing \( 0 \), \( T_L \) be a tangle obtained from \( L \) and \( z(T_L) \) be the center corresponding to the universal invariant constructed by Lawrence and Ohtsuki, where we assign the \( R \) matrix given by \([6]\) and \( K^{\pm 1} \) to the maximal and the minimal points as in Figure \([1]\). In \([14]\), \( K^{N \pm 1} \) is assigned instead of \( K^{\pm 1} \), and so the invariant defined here and that in \([14]\) is different by the sign \((-1)^{\langle m-1 \rangle f}\) where \( f \) is the framing of the knot. So these invariants coincide for an unframed knot.

The scalar corresponding \( \rho_m(z(T_L)) \) is given by \( V_m(L)/[m]_q \) where \( V_m \) is the colored Jones invariant normalized as \( V_m(\phi) = 1 \). Then \( \rho_m(z(T_L))_{0,0} = V_m(L)/[m]_q \), and we have
\[
\begin{align*}
b^+_s(L) &= \frac{\xi}{2N[s]} \frac{d}{dq} \{1\}_q \left( \frac{V_s(L)}{[s]_q} - \frac{V_{2N-s}(L)}{[2N-s]_q} \right) \\
b^-_s(L) &= \frac{\xi}{4N[s]} \frac{d}{dq} \{1\}_q \left( \frac{V_{2N+s}(L)}{[2N+s]_q} - \frac{V_{2N-s}(L)}{[2N-s]_q} \right)
\end{align*}
\]  
(16)
from \([15]\).

Let \( s \) be an integer satisfying \( 1 \leq s \leq N - 1 \) and put
\[
\underline{s} = \min(s, N - s), \quad \overline{s} = \max(s, N - s).
\]
By using Habiro’s universal formula \([3]\), \( b^+_s(L) \) is expressed in terms of \( a_i(L) \) as follows. We put \( a_i(L)_{\xi} = a_i(L)|_{q = \xi} \).
Theorem 2.3. For a knot $L$, we have

$$b_+^s(L) = b_-^s(L) =$$

$$\frac{(1)^2}{s} \left( \sum_{i=0}^{s-1} \frac{a_i(L) \xi \{s+i\}!}{|s| \{s-i-1\}!} \sum_{s-i \leq k \leq s+i} \frac{\{k\}+}{\{k\}} + 2 \sum_{i=\xi}^{s-1} a_i(L) \xi \{s+i, i\}' \{s-1, i\}' \right),$$

where $\{n, j\}'$ is obtained from $\{n, j\} = \prod_{k=0}^{j-1} \{n-k\}$ by replacing the term $\{N\}$ by $-1$ and the term $\{0\}$ by 1.

Corollary 2.4. Let $L^f$ be the framed knot with framing $f$ which is isotopic to $L$ as a non-framed knot. The colored Jones invariant $V_m$ is generalized to a framed knot by $V_m(L^f) = q^{m-1} f^m V_m(L)$, and the invariants $b_+^s(L^f)$ and $b_-^s(L^f)$ are generalized as follows.

$$b_+^s(L^f) = q^{2 + f} b_+^s(L) + \frac{(-N + s) f \{1\}}{|s|^2} V_s(L)|_{q=\xi},$$

$$b_-^s(L^f) = q^{2 + f} b_-^s(L) + \frac{s f \{1\}}{|s|^2} V_s(L)|_{q=\xi}.$$

Proof of Theorem 2.3. Let $a'_i(L) = \{1\}_q a_i(L)$ for $a_i(L)$ in (3) and $a'_i(L)\xi = a'_i(L)|_{q=\xi}$; then

$$\frac{d}{dq} \left( \frac{\{1\}_q V_s(L)}{|s|^q} \right)_{q=\xi} = \sum_{i=0}^{s-1} a'_i(L) \xi \{s+i\}! \left( \frac{d}{dq} \frac{a'_i(L)}{a'_i(L)} \right)_{q=\xi} + \sum_{s-i \leq k \leq s+i} \frac{k \{k\}+}{\xi \{k\}},$$

where $\{k\}+ = \xi+\xi^{-1}$. Now we compute (16). We use $\{2N-k\}' = -\{k\}$ and $\frac{d}{dq} F(q)\big|_{q=\xi} = -\frac{2N}{\xi} \lim_{q \to \xi} \frac{F(q)}{(N)}$ for a function $F(q)$ of $q$.

$$\frac{d}{dq} \left( \frac{\{1\}_q V_s(L)}{|s|^q} - \frac{V_{2N-s}(L)}{|2N-s|^q} \right)_{q=\xi} =$$
\[
\sum_{i=0}^{s-1} \frac{a_i'(L) \xi \{s+i\}!}{\{s\} \{s-i\}!} \left( \frac{d}{dq} \frac{a_i'(L)}{a_i'(L)} \right)_{q=\xi} + \sum_{s-1 \leq k \leq s+i, k \neq s} k \{k\} \frac{\xi \{k\}_+}{\xi \{k\}} \\
- \sum_{i=0}^{s-1} \frac{2N}{\xi} \lim_{q \to \xi} \frac{a_i'(L) \{s+i,i\}q \{s-1,i\}q}{\{N\}_q} \\
- \sum_{i=0}^{\pi-1} \frac{a_i'(L) \xi \{2N-s+i\}!}{\{2N-s\} \{2N-s-i-1\}!} \left( \frac{d}{dq} \frac{a_i'(L)}{a_i'(L)} \right)_{q=\xi} + \sum_{2N-s-1 \leq k \leq 2N-s+i, k \neq 2N-s} k \{k\} \frac{\xi \{k\}_+}{\xi \{k\}} \\
+ \sum_{i=0}^{\pi-1} \frac{2N}{\xi} \lim_{q \to \xi} \frac{a_i'(L) \{2N-s+i,i\}q \{2N-s-1,i\}q}{\{N\}_q}.
\]

Let
\[
\{n,j\}' = \prod_{0 \leq k \leq j-1} (-1)^t \prod_{0 \leq k \leq j-1} \{n-k\}_q.
\]

Then we know that
\[
\lim_{q \to \xi} \frac{s+i,i\}q \{s-1,i\}q}{\{N\}_q} = \begin{cases} 0 & \text{if } s \leq \frac{N}{2}, \\ \{s+i,i\}' \{s-1,i\} & \text{if } x > \frac{N}{2}, \end{cases}
\]
for \(s \leq i \leq s-1\) and
\[
\lim_{q \to \xi} \frac{\{2N-s+i,i\}q \{2N-s-1,i\}q}{\{N\}_q} = \begin{cases} 2 \{s+i,i\} \{s-1,i\}' & \text{if } s \leq \frac{N}{2}, \\ \{s+i,i\}' \{s-1,i\} & \text{if } x > \frac{N}{2}. \end{cases}
\]
for \(s \leq i \leq \pi - 1\). We also know that
\[
\{2N-k\} = -\{k\}, \quad \frac{\{2N-s+i\}!}{\{2N-s\} \{2N-s-i\}!} = \frac{\{s+i\}!}{\{s\} \{s-i\}!}.
\]

By using (19), (20), (21), we get
\[
\frac{d}{dq} \{1\}_q \left( \frac{V_s(L)}{[s]_q} - \frac{V_{2N-s}(L)}{[2N-s]_q} \right)_{q=\xi} = \\
\frac{2N}{\xi} \sum_{i=0}^{s-1} \frac{a_i'(L) \xi \{s+i\}!}{\{s\} \{s-i\}!} \sum_{s-1 \leq k \leq s+i, k \neq s} k \{k\} + \frac{4N}{\xi} \sum_{i=0}^{\pi-1} a_i'(L) \xi \{s+i,i\}' \{s-1,i\}'.
\]

Similar computation shows that
\[
\frac{d}{dq} \{1\}_q \left( \frac{V_{2N+s}(K)}{[2N+s]_q} - \frac{V_{2N-s}(L)}{[2N-s]_q} \right)_{q=\xi} = \\
\frac{4N}{\xi} \sum_{i=0}^{s-1} \frac{a_i'(L) \xi \{s+i\}!}{\{s\} \{s-i\}!} \sum_{s-1 \leq k \leq s+i, k \neq s} (2N-k) \{k\} + \frac{8N}{\xi} \sum_{i=0}^{\pi-1} a_i'(L) \xi \{s+i,i\}' \{s-1,i\}'.
\]
Therefore

\[
\begin{align*}
   b^+_s(L) &= b^-_s(L) = \\
   \frac{1}{s} \left( \sum_{i=0}^{s-1} \frac{a_i(L)\xi(s+i)!}{s-i} \sum_{k \neq s-i} \{k\}_+ + 2 \sum_{i=0}^{s-1} a_i(L)\xi(s+i)\{s-i\}_+ \right).
\end{align*}
\]

\[\square\]

**Proof of Corollary 2.4.** By using \(V_s(L^f) = \xi^{2-1f} V_s(L)\), we have

\[
\begin{align*}
   b^+_s(L^f) &= \xi^{2-1f} b^+_s(L) + \frac{1}{2N} \left( \frac{s^2-1}{2} - \frac{(2N-s)^2-1}{2} \right) f \{1\}_s V_s(L) |_{q=\xi} \\
   &= \xi^{2-1f} b^+_s(L) - f \frac{(N-s)\{1\}_s}{[s]_2} V_s(L) |_{q=\xi}, \\
   b^-_s(L^f) &= \xi^{2-1f} b^-_s(L) + f \frac{s\{1\}_s}{[s]_2} V_s(L) |_{q=\xi}.
\end{align*}
\]

\[\square\]

2.4. **Coefficients of centers.** The center of the restricted quantum group \(\overline{U}_\xi(sl_2)\) are spanned by \(e_0, \ldots, e_N, w^+_1, \ldots, w^+_N\), where \(e_s\) is the central idempotent and \(w^+_s\) is the center in the radical of \(P^\pm_s\). For a framed knot \(L^f\), let \(z = z(T_{L^f})\) be the center of \(\overline{U}_\xi(sl_2)\) determined from \(L^f\) by using a tangle \(T_{L^f}\) obtained from \(L^f\). Then \(z\) is expressed as a linear combination of the better basis \([11]\). In the following, \(L\) represents the non-framed knot which is historic to \(L^f\).

\[
z = \sum_{s=1}^{N-1} \alpha^{(N)}_s \hat{\rho}_s + \sum_{s=1}^{N-1} \beta^{(N)}_s \hat{\varphi}_s + \sum_{s=0}^{N} \gamma^{(N)}_s \hat{\kappa}_s.
\]

By using \([11]\), we have

\[
\begin{align*}
   \alpha^{(N)}_s(L^f) &= (-1)^{N+s} N \{1\} V_s(L^f) |_{q=\xi}, & 1 \leq s \leq N-1, \\
   \beta^{(N)}_s(L^f) &= [s]^2 \left( b^+_s(L^f) - b^-_s(L^f) \right) = -N f \{1\} V_s(L^f) |_{q=\xi}, & 1 \leq s \leq N-1, \\
   \gamma^{(N)}_0(L^f) &= \frac{V_{2N} \{1\}_q V_{2N}(L)}{[2N]} |_{q=\xi} = \frac{\xi^{1-\frac{m}{2N}}}{4N} \frac{d}{dq} \left\{ \frac{V_{2N}(L)}{2N} \right\} |_{q=\xi} = \frac{N \xi^{-\frac{m}{2}} \{1\}_q}{2N} \frac{V_{m}(L)}{V_{2N-m}(L)} |_{q=\xi}, \\
   \gamma^{(N)}_s(L^f) &= [s]^2 \left( \frac{s}{N} b^+_s(L^f) + \frac{N-s}{N} b^-_s(L^f) \right) + \{s\}_+ V_s(L^f) |_{q=\xi} \\
   &= \xi^{\frac{m}{2}} \left( \sum_{i=0}^{s-1} a_i(L)\xi(s+i,2i+1) \sum_{k=s-i} \{k\}_+ \right) + 2 \sum_{i=0}^{s-1} a_i(L)\xi(s+i,2i+1) \\
   &= \xi \frac{2N}{e_{2N}} \frac{d}{dq} \left\{ \frac{V_s(L) + V_{2N-s}(L)}{2N} \right\} |_{q=\xi}.
\end{align*}
\]
\[
\frac{\gamma_{N}^{(N)}(L)}{[N]} = - \frac{V_{N}(L)}{[N]} \bigg|_{q=\xi}^{m=s} = \frac{N \{1\} \xi^{2r-1-f} \frac{d}{d m} V_{m}(L) \bigg|_{m=s}^{q=\xi}}{2 \pi \sqrt{1 - \xi}}, \quad 1 \leq s \leq N - 1,
\]

\[
\gamma_{N}^{(N)}(L) = - \frac{V_{N}(L)}{[N]} \bigg|_{q=\xi}^{m=N} = \frac{N \xi^{2r-1-f} \frac{d}{d m} V_{m}(L) \bigg|_{m=N}^{q=\xi}}{2 \pi \sqrt{1 - \xi}}.
\]

In \( \frac{d}{d m} V_{m}(L) \), the colored Jones invariant \( V_{m}(L) \) is expressed by Habiro’s universal formula \( (3) \) and considered to be an infinite sum with the variable \( m \). The integer \( s \) is substituted after obtaining the derivative. The sum reduces to a finite sum when \( q \) is specialized to \( \xi \). Hence we get the following.

**Theorem 2.5.** For framed knot \( L^{f} \) with framing \( f \) and let \( L \) be the same knot without framing. Then we have

\[
\alpha_{s}^{(N)}(L) = (-1)^{N+s} N \{1\} V_{s}(L) \bigg|_{q=\xi}, \quad 1 \leq s \leq N - 1,
\]

\[
\beta_{s}^{(N)}(L) = - N f \{1\} V_{s}(L) \bigg|_{q=\xi}, \quad 1 \leq s \leq N - 1,
\]

\[
\gamma_{0}^{(N)}(L) = \frac{V_{2N}(L)}{[2N]} \bigg|_{q=\xi}^{m=2N} = \frac{N \xi^{2r-1-f} \frac{d}{d m} V_{m}(L) \bigg|_{m=2N}^{q=\xi}}{2 \pi \sqrt{1 - \xi}},
\]

\[
\gamma_{s}^{(N)}(L) = \frac{\xi^{1+2r-1-f} d}{dq} \{1\}(V_{s}(L) + V_{2N-s}(L)) \bigg|_{q=\xi}^{m=s} = \frac{N \{1\} \xi^{2r-1-f} \frac{d}{d m} V_{m}(L) \bigg|_{m=s}^{q=\xi}}{2 \pi \sqrt{1 - \xi}}, \quad 1 \leq s \leq N - 1,
\]

\[
\gamma_{N}^{(N)}(L) = - \frac{V_{N}(L)}{[N]} \bigg|_{q=\xi}^{m=N} = \frac{N \xi^{2r-1-f} \frac{d}{d m} V_{m}(L) \bigg|_{m=N}^{q=\xi}}{2 \pi \sqrt{1 - \xi}}.
\]

Especially, if the framing \( f = 0 \),

\[
\alpha_{s}^{(N)}(L) = (-1)^{N+s} N \{1\} V_{s}(L) \bigg|_{q=\xi}, \quad 1 \leq s \leq N - 1,
\]

\[
\beta_{s}^{(N)}(L) = 0, \quad 1 \leq s \leq N - 1,
\]

\[
\gamma_{0}^{(N)}(L) = \frac{V_{2N}(L)}{[2N]} \bigg|_{q=\xi}^{m=2N} = \frac{N \{1\} \xi^{2r-1-f} \frac{d}{d m} V_{m}(L) \bigg|_{m=2N}^{q=\xi}}{2 \pi \sqrt{1 - \xi}},
\]

\[
\gamma_{s}^{(N)}(L) = \frac{\xi^{1+2r-1-f} d}{dq} \{1\}(V_{s}(L) + V_{2N-s}(L)) \bigg|_{q=\xi}^{m=s} = \frac{N \{1\} \xi^{2r-1-f} \frac{d}{d m} V_{m}(L) \bigg|_{m=s}^{q=\xi}}{2 \pi \sqrt{1 - \xi}}, \quad 1 \leq s \leq N - 1,
\]

\[
\gamma_{N}^{(N)}(L) = - \frac{V_{N}(L)}{[N]} \bigg|_{q=\xi}^{m=N} = \frac{N \xi^{2r-1-f} \frac{d}{d m} V_{m}(L) \bigg|_{m=N}^{q=\xi}}{2 \pi \sqrt{1 - \xi}}.
\]
3. Relation to the hyperbolic volume

In this section, we check the volume conjecture (5) of the logarithmic invariant $\gamma_s(K_{41})$ for the figure-eight knot $K_{41}$.

3.1. Logarithmic invariant of figure-eight knot. The colored Jones invariant $V_\lambda(K_{41})$ is expressed as follows.

$$V_\lambda(K_{41}) = \sum_{i=0}^{\infty} \frac{\{\lambda + i + 1, 2i + 1\}}{\{1\}}.$$  

This means that the coefficients $a_i(K_{41})$ are all equal to 1 in Habiro’s formula (3), and $\gamma_s^{(N)}(K_{41})$ is given by

$$\gamma_s^{(N)}(K_{41}) = \sum_{i=0}^{s-1} \{s + i, 2i + 1\} \sum_{k=s-i}^{s+i} \frac{\{k\}+}{\{k\}} + 2 \sum_{i=\underline{s}}^{s-1} \{s + i, 2i + 1\},$$

where $\underline{s} = \min(s, 2N - s)$, $\overline{s} = \max(s, 2N - s)$ as before.

3.2. Limit of the logarithmic invariant. For this $\gamma_s^{(N)}(K_{41})$, the following theorem holds.

**Theorem 3.1.** Let $\alpha$ be a real number with $0 \leq \alpha < \frac{\pi}{3}$, and let $s_N = \left\lfloor \frac{N\alpha}{2\pi} \right\rfloor$ where $\lfloor x \rfloor$ is the largest integer satisfying $\lfloor x \rfloor \leq x$. Then

$$\lim_{N \to \infty} 2\pi \frac{\left| \gamma_s^{(N)}(K_{41}) \right|}{N} = \text{Vol}(M_\alpha)$$

where $M_\alpha$ is the cone manifold along singular set $K_{41}$ with cone angle $\alpha$.

**Remark.** Numerical computation suggest that

$$\lim_{N \to \infty} 2\pi \frac{\left| \gamma_s^{(N)}(K_{41}) \right|}{N} = \begin{cases} \text{Vol}(M_\alpha), & 0 \leq \alpha < \frac{2\pi}{3}, \\ 0, & \frac{2\pi}{3} \leq \alpha < \frac{4\pi}{3}, \\ \text{Vol}(M_\alpha), & \frac{4\pi}{3} \leq \alpha \leq 2\pi, \end{cases}$$

for the sequence $s_N = \left\lfloor \frac{N\alpha}{2\pi} \right\rfloor$. The values of $\frac{2\pi \left| \gamma_s^{(N)}(K_{41}) \right|}{N}$ for $N = 200$ and $N = 400$ are shown by graphs in Figure 2.

**Proof of Theorem 3.1.** To prove (23), we first estimate the sum

$$\sum_{i=0}^{s_N-1} \{s_N + i, 2i + 1\} \sum_{k=s_N-i}^{s_N+i} \frac{\{k\}+}{\{k\}}.$$
We know \( |\{k\}+| \leq 2N \) since \( 1 \leq k \leq N-1 \), and so \( \sum_{k=s_N-i}^{s_N+i} |\{k\}| \leq 2N^2 \). We also know \( \{s_N+i\}\{s_N-i\} \leq 1 \) since \( 0 \leq s_N, a \leq \frac{N}{6} \) and \( 0 \leq i \leq s_N \). Therefore, we have

\[
\left| \sum_{i=0}^{s_N-1} \{s_N + i, 2i + 1\} \sum_{k=s_N-i}^{s_N+i} |\{k\}| \right| \leq 2N^2.
\]

Next we estimate \( \sum_{i=s_N}^{N-s_N-1} \{s_N + i, 2i + 1\} \). The argument of \( \{s_N + i, 2i + 1\} \) is equal to that of \((-1)^{s_N-1}\) and is not depend on \(i\). Let \( a_i = (-1)^{s_N-1}\{s_N + i, 2i + 1\} \). Then \( a_i \geq 0 \) and we have

\[
a_i \leq \sum_{i=s_N+1}^{N-s_N} a_k \leq N a_{\max}(N)
\]

where \( a_{\max}(N) = \max_{s_N \leq i \leq N-s_N-1} a_i \). Therefore,

\[
(24) \quad -2N^2 + N a_{\max}(N) \leq |\gamma_{s_N}(K_{4_1})| \leq 2N^2 + N a_{\max}(N)
\]

The index \( \hat{i}_{\max}(N) \) for the maximum \( a_i \) must be equal to \( \hat{i}_1(N) = s_N \) or \( \hat{i}_2(N) \) satisfying \( \{s_N + i_1^{(N)}\}\{s_N - i_2^{(N)}\} \geq 1 \) and \( \{s_N + i_2^{(N)}\}\{s_N - i_1^{(N)} - 1\} \leq 1 \) since \( \hat{i}_1^{(N)}, \hat{i}_2^{(N)} \) correspond to the local maximal. The index \( \hat{i}_2^{(N)} \) satisfies

\[
(25) \quad \cos \frac{2\pi (i_2^{(N)} + 1)}{N} \leq \cos \frac{2\pi s_N}{N} - \frac{1}{2} \leq \cos \frac{2\pi i_2^{(N)}}{N}.
\]

If \( N \) is not small, such \( i_2^{(N)} \) exists uniquely between \( \frac{N}{2} + s_N \) and \( N - s_N - 1 \) because of inequalities \( \{2s_N + 1 - N\}\{N - 1\} < 1 \), \( \{\frac{N}{2} + 2s_N\}\{-\frac{N}{2}\} > 1 \) and the shape of the graph of the cosine function. If \( N \) is large enough,

\[
\log a_i = \sum_{k=s_N-i}^{s_N+i} \log |\{k\}|
\]
is almost equal to

\[ N \int_{s_{N-1}^N}^{s_{N+1}^N} \log |2 \sin \pi t| \, dt = -\frac{N}{\pi} \left( \Lambda \left( \frac{2s_N + 2i\pi}{N} \right) - \Lambda \left( \frac{2s_N - 2i\pi}{N} \right) \right), \]

where \( \Lambda(x) = -\frac{1}{2} \int_0^x \log |2 \sin \frac{t}{2}| \, dt \) is the Lobachevski function. Therefore

\[
\lim_{N \to \infty} \frac{2 \pi \log \alpha_{i_1^{(N)}}}{N} = -2 \Lambda(2\alpha), \quad \lim_{N \to \infty} \frac{2 \pi \log \alpha_{i_2^{(N)}}}{N} = -2 \left( \Lambda(\alpha + \theta) - \Lambda(\alpha - \theta) \right)
\]

where \( \theta = \lim_{N \to \infty} \frac{2\pi i_1^{(N)}}{N} \). Then \( \theta > \pi \) since \( s_N^2 + s_N^N < i_2^{(N)} < N - s_N^N - 1 \), and this implies that \( \Lambda(\alpha - \theta) > \Lambda(\alpha + \theta) \). We also know that \( \Lambda(2\alpha) > 0 \). Therefore, \( \lim_{N \to \infty} \alpha_{i_1^{(N)}} < 1 \), \( \lim_{N \to \infty} \alpha_{i_2^{(N)}} > 1 \), and we have \( i_{\text{max}}^{(N)} = i_2^{(N)} \) for sufficient large \( N \). By using the fact that \( \lim_{N \to \infty} \frac{\log s_{N+1}^N - \log s_{N-1}^N}{N} = 0 \), we get

\[
\lim_{N \to \infty} \frac{2 \pi \log |\gamma(s)_N(K_{4t})|}{N} = -2 \left( \Lambda(\alpha + \theta) - \Lambda(\alpha - \theta) \right),
\]

where \( \theta \) satisfies \( \cos \theta = \cos \alpha - \frac{1}{2} \) by (25). The right hand side of this formula is equal to the hyperbolic volume of the cone manifold \( M_\alpha \) given by Mednykh \([9]\) since

\[
\frac{d}{d\alpha} \left( -2 \left( \Lambda(\alpha + \theta) - \Lambda(\alpha - \theta) \right) \right) = \log \left| t - \sqrt{t^2 - 1} \right| = -\arccosh t,
\]

where \( t = 1 + \cos \alpha - \cos 2\alpha \).

\[\square\]

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