Closed Self-Similar Solutions to Flows by Negative Powers of Curvature

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Abstract
In some warped product manifolds including space forms, we consider closed self-similar solutions to curvature flows whose speeds are negative powers of mean curvature, Gauss curvature, and other curvature functions with suitable properties. We prove such self-similar solutions, not necessarily strictly convex for some cases, must be slices of warped product manifolds. A new auxiliary function is the key of the proofs.

Keywords  Self-similar solution · Rigidity of hypersurface · Warped product manifold · Curvature flow

Mathematics Subject Classification  53C24 · 53C45 · 53E10

1 Introduction
In this paper, a closed, immersed hypersurface \( X: M^n \to \mathbb{R}^{n+1} \) \((n \geq 2)\) satisfying equation

\[
F^\beta(\mathcal{W}(x)) = \langle X(x), \nu(x) \rangle
\]

is called a closed self-similar solution, where \( F \) is a suitable function of the shape operator \( \mathcal{W} \) of \( M^n \), \( \beta \neq 0 \) is a constant, and \( \nu \) denotes the unit outward normal vector. In fact, this hypersurface is corresponding to the self-similar solution to curvature flow which satisfies

\[
\partial_t X = -\text{sign}(\beta) F^\beta \nu.
\]
Flow by negative powers of curvature (also called inverse curvature flow) refers to the flow with \( \beta < 0 \) and \( F(W) = f(\kappa(W)) \) where \( f \) is a 1-homogeneous (homogeneous of degree 1) function of principal curvatures \( \kappa = (\kappa_1, \ldots, \kappa_n) \). This type of flow has been studied by many researchers (see \([14, 16, 18, 21, 26, 27, 29]\) etc.). Urbas \([28]\) also considered noncompact self-similar solutions to flow by negative powers of Gauss curvature.

Self-similar solutions to mean curvature flow and Gauss curvature flow have been widely studied. Although closed self-similar solution is not unique in general (see example of Angenent \([4]\)), it must be a sphere provided suitable conditions. Huisken \([17]\) showed that any closed self-similar solution to mean curvature flow (also known as self-shrinker) must be a sphere if mean curvature is nonnegative. Colding–Minicozzi \([9]\) introduced a variational characterization of self-shrinker and proved spheres are the only closed \( F \)-stable self-shrinkers. For flow by powers of Gauss curvature, Brendle–Choi–Daskalopoulos \([8]\) proved closed and strictly convex self-similar solutions must be spheres when the power \( \alpha > 1/(n + 2) \). For curvature flow with general \( F \) and positive power \( \beta \), uniqueness results of closed self-similar solution were obtained by McCoy \([22, 23]\), Gao–Li–Ma \([12]\) and Gao–Li–Wang \([13]\) etc.

The concept of self-similar solution can be extended to hypersurfaces in warped product manifolds. Let \( \overline{M}^{n+1} = (0, \bar{r}) \times_{\lambda} N^n \) be a warped product manifold with metric

\[
\bar{g} = dr \otimes dr + \lambda^2(r) g_N,
\]

where \((N^n, g_N)\) is a closed Riemannian manifold and \( \lambda(r) \) is a smooth, positive function of \( r \in (0, \bar{r}) \). It is known that \( \lambda(r) \partial_r \) is a conformal vector field on \( \overline{M}^{n+1} \). A hypersurface \( M^n \) in \( \overline{M}^{n+1} \) is also called a self-similar solution if it satisfies

\[
F^\beta(W(x)) = \bar{g}(\lambda(r(x)) \partial_r(x), v(x)).
\]

In fact, we can generate a family of hypersurfaces by \( M^n \) and \( \lambda \partial_r \) which satisfies the equation of corresponding curvature flow (see \([1, 10]\) for details). The ambient space \( \overline{M} \) is actually Euclidean space \( \mathbb{R}^{n+1} \), sphere \( S^{n+1} \) or hyperbolic space \( \mathbb{H}^{n+1} \) if \( N^n = S^n \) and \( \lambda(r) = r, \sin r \) or \( \sinh r \) correspondingly.

Self-shrinkers in warped product manifolds were studied by Wu \([31]\), Alias–de Lira–Rigoli \([1]\) etc. Ma and the author \([10]\) considered closed self-similar solutions to flow with some general \( F \) and positive power \( \beta \) in warped product manifolds. And they proved uniqueness of solutions if the ambient space is a hemisphere. Later, Gao–Li–Wang \([13]\) extended the uniqueness result to more general \( F \) which is an inverse concave function of principal curvatures.

For flow by negative powers of the \( k \)th mean curvature \( (k < n) \), Ma and the author \([11]\) proved uniqueness of closed, strictly convex self-similar solutions in a class of warped product manifolds.

In this paper, we consider closed self-similar solutions which are not necessarily strictly convex. A hypersurface is called \textit{mean convex} if its mean curvature \( H = \frac{1}{n}(\kappa_1 + \cdots + \kappa_n) \) is positive everywhere.
Theorem 1 Suppose that \( \mathcal{M}^{n+1} = [0, \bar{r}) \times_\lambda N^n \) is a warped product manifold satisfying
\[
\text{Ric}_N \geq (n-1)(\lambda'^2 - \lambda''') g_N,
\]
and \( M^n \) be a closed, immersed hypersurface in \( \mathcal{M}^{n+1} \). If \( M^n \) is mean convex and satisfies
\[
H - \alpha = \bar{g} (\lambda \partial_r, \nu),
\]
where \( \alpha > 0 \) is a constant, then \( M^n \) is a slice \( \{ r_0 \} \times N^n \) for some \( r_0 \in (0, \bar{r}) \).

Remark 1 Compared with Corollary 2 in [11], instead of strictly convex, \( M^n \) is mean convex in the above theorem. It is easy to check that \( \mathbb{R}^{n+1}, S^{n+1} \) and \( \mathbb{H}^{n+1} \) satisfy the assumption of \( M \).

Let \( \Gamma \subset \mathbb{R}^n \) be an open, convex, symmetric cone with vertex at the origin, which contains the positive cone \( \Gamma_+ = \{ (\kappa_1, \ldots, \kappa_n) \in \mathbb{R}^n : \kappa_i > 0 \text{ for any } i = 1, \ldots, n \} \).

Condition 2 We assume that \( F(\mathcal{W}) = f(\kappa(\mathcal{W})) \) satisfies the following properties in \( \Gamma \):

(i) \( f \) is a smooth, symmetric function of the eigenvalues \( \kappa \) of \( \mathcal{W} \).
(ii) \( f \) is positive in \( \Gamma \) and normalized such that \( f(1, \ldots, 1) = 1 \).
(iii) \( f \) is strictly increasing in each argument, i.e., \( \frac{\partial f}{\partial \kappa_i} > 0 \) in \( \Gamma \) for any \( i = 1, \ldots, n \).
(iv) \( f \) is 1-homogeneous (homogeneous of degree 1), i.e., \( f(s\kappa) = sf(\kappa) \) for any \( s > 0 \) and \( \kappa \in \Gamma \).
(v) The following inequalities hold in \( \Gamma \):
\[
\sum_{i=1}^n \frac{\partial f}{\partial \kappa_i} \geq 1, \quad \sum_{i=1}^n \frac{\partial f}{\partial \kappa_i} \kappa_i^2 \geq f^2, \quad \text{and} \quad f - \frac{\partial f}{\partial \kappa_i} \kappa_i \geq 0 \text{ for any } i = 1, \ldots, n.
\]

Theorem 3 Suppose that \( \mathcal{M}^{n+1} = [0, \bar{r}) \times_\lambda N^n \) is a warped product manifold, where \( (N, g_N) \) is a closed Riemannian manifold with constant sectional curvature \( c \) and \( \lambda(r) \) satisfies \( \lambda'(r) > 0 \) and
\[
\frac{\lambda(r)''}{\lambda(r)} + \frac{c - \lambda(r)'^2}{\lambda(r)^2} \geq 0. \tag{1}
\]
Let \( M^n \) be a closed, immersed hypersurface in \( \mathcal{M}^{n+1} \) satisfying
\[
F^{-\alpha} = \bar{g}(\lambda \partial_r, \nu),
\]
where \( \alpha > 0 \) is a constant. If principal curvatures \( \kappa \) of \( M^n \) are in a cone \( \Gamma \) such that Condition 2 holds, then \( M^n \) is a slice \( \{ r_0 \} \times N^n \) for some \( r_0 \in (0, \bar{r}) \).
Remark 2 The assumption of the ambient space $\overline{M}$ in the above theorem is stronger than it in Theorem 1. In fact, inequality (1) implies

$$(\lambda'^2 - \lambda'' \lambda) g_N \leq c g_N = \frac{1}{n-1} \text{Ric}_N.$$ 

It can be checked that space forms $\mathbb{R}^{n+1}$, $\mathbb{S}^{n+1}$ and $\mathbb{H}^{n+1}$ still satisfy the assumption and readers may refer to [6] for more spaces satisfying (1).

Let $\sigma_k(\kappa)$ denote the $k$th elementary symmetric polynomial of principal curvatures $\kappa = (\kappa_1, \ldots, \kappa_n)$ of hypersurface $M^n$, i.e.,

$$\sigma_k(\kappa) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \kappa_{i_1} \cdots \kappa_{i_k}.$$ 

The $k$th mean curvature is defined by $H_k = \sigma_k(\kappa)/(n \choose k)$.

The following corollary of Theorem 3 holds under weaker convexity assumption of $M^n$ (compared with Corollary 5 in [11]).

Corollary 4 Suppose $\overline{M}^{n+1}$ is the same in Theorem 3 and $M^n$ is a closed, immersed hypersurface in $\overline{M}^{n+1}$. If $M^n$ satisfies $H_{k+1} > 0$ and

$$H_k^{-\alpha} = \tilde{g}(\lambda \partial_r, \nu),$$ 

where $\alpha > 0$ is a constant and $2 \leq k \leq n - 1$, then $M^n$ is a slice $\{r_0\} \times N^n$ for some $r_0 \in (0, \tilde{r})$.

If $M^n$ is strictly convex, we have the following corollaries from Theorem 3.

Corollary 5 Suppose $\overline{M}^{n+1}$ is under the same assumption of Theorem 3 and $M^n$ is a closed, immersed hypersurface in $\overline{M}^{n+1}$. If $M^n$ is strictly convex and satisfies

$$K^{-\alpha} = \tilde{g}(\lambda \partial_r, \nu),$$ 

where $K$ is Gauss curvature and $\alpha > 0$ is a constant, then $M^n$ is a slice $\{r_0\} \times N^n$ for some $r_0 \in (0, \tilde{r})$.

We say that $f$ is inverse concave if the function

$$f^*(\kappa_1, \ldots, \kappa_n) := \frac{1}{f(\frac{1}{\kappa_1}, \ldots, \frac{1}{\kappa_n})}$$

is concave. It is known that a class of functions are concave and inverse concave in [2], for example $(\frac{\sigma_k}{\sigma_l})^{\frac{1}{l-k}}$ where $\sigma_k$ is the $k$th elementary polynomials and $0 \leq l < k \leq n$. These convexity conditions of functions appear naturally in the study of curvature flows and other fully nonlinear PDEs (see, for example, [2, 5, 22]).
Corollary 6 Suppose $\mathcal{M}^{n+1}$ is under the same assumption of Theorem 3 and $M^n$ is a closed, immersed hypersurface in $\mathcal{M}^{n+1}$. If $M^n$ is strictly convex and satisfies

$$F^{-\alpha} = \bar{g}(\lambda, \partial_r, \nu),$$

where $F$ is concave, inverse concave and satisfies Condition 2 (i)–(iv), $\alpha > 0$ is a constant, then $M^n$ is a slice $\{r_0\} \times N^n$ for some $r_0 \in (0, \bar{r})$.

There are some connections between self-similar solutions and hypersurfaces of constant curvatures. Brendle [6] proves closed and embedded hypersurfaces with constant mean curvature in a class of warped product manifolds must be umbilic. The case of hypersurfaces with constant $H_k$ is showed by Brendle and Eichmair [7]. In these papers, Heintze–Karcher type inequality and Minkowski type formula in warped product manifolds are established, which can also be used to obtain uniqueness of self-similar solutions (see [11]). Rigidity problems of hypersurfaces in warped product manifolds are also considered in [19, 32] etc.

Now, we briefly recall the methods in [11] and compare them with the proofs in this paper. For example, let us consider the case $\mathcal{M}^{n+1} = \mathbb{R}^{n+1}$ and $M^n$ satisfies

$$\frac{1}{n}(\kappa_1 + \cdots + \kappa_n)$$

is normalized mean curvature.

One method in [11] is based on an integral inequality

$$0 = \int_{M^n} \text{div}(HX^T - \nabla (X, \nu)) \geq -(n - 1) \int_{M^n} \langle X, \nabla H \rangle.$$  \hspace{1cm} (3)

If $M^n$ is strictly convex, using (2), we know

$$\langle X, \nabla H \rangle = - \frac{1}{\alpha} \langle X, \nu \rangle^{-\frac{a+1}{\alpha}} \sum_{i} \kappa_i \langle X, e_i \rangle^2 \leq 0.$$  \hspace{1cm} (4)

Combing (3) and (4), we know $M^n$ is a sphere.

The other method uses the Heintze-Karcher inequality for embedded and mean convex $M^n$

$$\int_{M^n} \langle X, \nu \rangle \leq \int_{M^n} \frac{1}{H}$$

and the Minkowski formula

$$|M^n| = \int_{M^n} H \langle X, \nu \rangle.$$

Combining with (2), we have

$$\int_{M^n} H^{-\alpha} \leq \int_{M^n} H^{-1}$$  \hspace{1cm} (5)
and
\[ |M^n| = \int_{M^n} H^{1-\alpha}. \] 

If \( \alpha > 1 \), using Hölder inequality, we obtain
\[
\int_{M^n} H^{1-\alpha} \int_{M^n} H^{-1} \leq |M^n| \int_{M^n} H^{-\alpha}.
\]

From (5) and (6) we know that equality occurs in the above inequality. This implies \( M^n \) is a sphere.

However, it seems that these integral methods can not be generalized for general curvature function \( F \), except for the case \( F = H_k \). This motivates us to seek a proof via the maximum principle.

In fact, for (2), we introduce an auxiliary quantity
\[
P = \frac{|X|^2}{2} - \frac{\alpha}{\alpha + 1} \langle X, \nu \rangle^{\frac{\alpha+1}{\alpha}},
\]
which is similar to Weinberger’s \( P \)-function [30] in spirit. We notice
\[
\Delta P = \langle X, \nu \rangle^{\frac{\alpha+1}{\alpha}} (|h|^2 - nH^2) + \frac{1}{\alpha} \langle X, \nu \rangle^{\frac{1-\alpha}{\alpha}} \sum_i (nH - \kappa_i) \kappa_i \langle X, \epsilon_i \rangle^2
\geq \frac{1}{\alpha} \langle X, \nu \rangle^{\frac{1-\alpha}{\alpha}} \sum_i (nH - \kappa_i) \kappa_i \langle X, \epsilon_i \rangle^2,
\]
where \( |h|^2 = \sum_i \kappa_i^2 \) and the last step is from the Cauchy–Schwarz inequality. Thus, function \( P \) is subharmonic if \( M^n \) is strictly convex, which implies \( M^n \) is a sphere. Proofs of Theorems 1 and 3 are based on this observation.

The paper is organized as follows. In Sect. 2, we show some examples of functions satisfying Condition 2 and recall some facts of hypersurfaces in warped product manifolds. In Sect. 3, we derive a basic formula of auxiliary function \( P \). In Sect. 4, we present the proof of Theorem 1. In Sect. 5, we prove Theorem 3 and its corollaries.

2 Preliminaries

Throughout this paper, repeated indexes will be added up from 1 to \( n \) unless otherwise stated.

2.1 Function \( F \) and Its Defining Cone \( \Gamma \)

We recall some facts of symmetric functions for later calculations (see [15, 25] for example).
If matrix $W = (h_{ij})$ is diagonal, i.e., $h_{ij} = \kappa_i \delta_{ij}$ for any $1 \leq i, j \leq n$, then the following formula holds for function $F(W) = f(\kappa(W))$:

$$\frac{\partial F}{\partial h_{ij}} b_{ij} = \frac{\partial f}{\partial \kappa_p} b_{pp},$$

for any symmetric matrix $B = (b_{ij})$.

If function $f = f(\kappa)$ is 1-homogeneous, differentiating $f(s\kappa) = sf(\kappa)$ with respect to $s$ gives

$$\frac{\partial f}{\partial \kappa_i} \kappa_i = f.$$

Let $\sigma_k(\kappa)$ denote the $k$th elementary symmetric polynomial of $\kappa = (\kappa_1, \ldots, \kappa_n)$, i.e.,

$$\sigma_k(\kappa) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \kappa_{i_1} \cdots \kappa_{i_k}.$$  

And let $\sigma_k(\kappa|i)$ denote $\sigma_k(\kappa)$ with $\kappa_i = 0$ for a fixed $i$. We also set $\sigma_0(\kappa) = 1$ and $\sigma_k(\kappa) = 0$ if $k > n$ or $k < 0$.

The Gårding’s cone is defined by

$$\Gamma_k := \{ \kappa \in \mathbb{R}^n | \sigma_m(\kappa) > 0 \text{ for any } 1 \leq m \leq k \}.$$  

We also consider the following cone:

$$\tilde{\Gamma}_k := \left\{ \kappa \in \mathbb{R}^n \bigg| \begin{array}{l} \sigma_m(\kappa) > 0 \text{ for any } 1 \leq m \leq k - 1, \\ \sigma_k(\kappa|i) > -(k - 1)\sigma_k(\kappa) \text{ for any } 1 \leq i \leq n \end{array} \right\},$$

where $1 \leq k \leq n - 1$.

Combining $\sum_{i=1}^n \sigma_k(\kappa|i) = (n - k)\sigma_k(\kappa)$, inequalities $\sigma_k(\kappa|i) > -(k - 1)\sigma_k(\kappa)$ implies $\sigma_k(\kappa) > 0$ which means $\tilde{\Gamma}_k \subset \Gamma_k$. Furthermore, noticing $\sigma_k(\kappa|i) \geq 0$ in $\Gamma_{k+1}$, we know $\tilde{\Gamma}_k$ has the following relation with Gårding’s cones: $\Gamma_{k+1} \subset \tilde{\Gamma}_k \subset \Gamma_k \subset \tilde{\Gamma}_{k-1}$.

Now we show some examples of $F$ satisfying Condition 2.

**Example 1** Function $H_k^{\frac{1}{k}}(\kappa)$ in $\tilde{\Gamma}_k$, where $H_k(\kappa) = \frac{1}{(k)!} \sigma_k(\kappa)$ and $1 \leq k \leq n - 1$.

We only show Condition 2 (v) since (i)–(iv) are easy to be checked. If $f = H_k^{\frac{1}{k}}$, then

$$\sum_{i=1}^n \frac{\partial f}{\partial \kappa_i} \kappa_i = \frac{1}{k} H_k^{\frac{1-k}{k}} \sum_{i=1}^n \frac{\partial H_k}{\partial \kappa_i} \kappa_i = H_k^{\frac{1-k}{k}} H_{k-1} \geq 1,$$

where the last inequality is from MacLaurin inequality. And

$$\sum_{i=1}^n \frac{\partial f}{\partial \kappa_i} \kappa_i^2 = \frac{1}{k} H_k^{\frac{1-k}{k}} \sum_{i=1}^n \frac{\partial H_k}{\partial \kappa_i} \kappa_i^2 = \frac{n}{k} H_1 H_k^{\frac{1}{k}} - \frac{n-k}{k} H_k^{\frac{1-k}{k}} H_{k+1}.$$
From inequality $H_1H_k \geq H_{k+1}$ and MacLaurin inequality, we know
\[
\sum_{i=1}^{n} \frac{\partial f}{\partial \kappa_i} \kappa_i^2 \geq H_1H_k \frac{1}{k} \geq H_k^2 = f^2.
\]
We notice, for any index $1 \leq i \leq n$ fixed,
\[
f - \frac{\partial f}{\partial \kappa_i} \kappa_i = H_k \left( 1 - \frac{\sigma_{k-1}(\kappa|i)\kappa_i}{k\sigma_k(\kappa)} \right).
\]
From $\sigma_k(\kappa) = \sigma_k(\kappa|i) + \sigma_{k-1}(\kappa|i)\kappa_i$ and the definition of $\tilde{\Gamma}_k$, we know
\[
1 - \frac{\sigma_{k-1}(\kappa|i)\kappa_i}{k\sigma_k(\kappa)} = \frac{(k-1)\sigma_k(\kappa) + \sigma_k(\kappa|i)}{\sigma_k(\kappa)} > 0.
\]
Thus, we know $H_k^{1/2}(\kappa)$ in $\tilde{\Gamma}_k$ satisfies Condition 2.

**Remark 3** For function $H_k^{1/2}$, almost all requirements in Condition 2 hold in $\Gamma_k$ except the inequality $f - \frac{\partial f}{\partial \kappa_i} \kappa_i \geq 0$ for any $1 \leq i \leq n$. In fact, there are examples showing that $\Gamma_k$ is not sufficient. For the case $n = 3$ and $k = 2$, if $\kappa_1 = -\frac{1}{2}$, $\kappa_2 = 1$ and $\kappa_3 = \frac{3}{2}$, we can check such $\kappa \in \Gamma_2$ but $f - \frac{\partial f}{\partial \kappa_3} \kappa_3 < 0$ for $f = H_2^{1/2}$.

**Example 2** Function $\sigma_n^{1/2}(\kappa)$ in $\Gamma_n = \Gamma_+$. Inequalities $\sum_{i=1}^{n} \frac{\partial f}{\partial \kappa_i} \geq 1$ and $\sum_{i=1}^{n} \frac{\partial f}{\partial \kappa_i} \kappa_i^2 \geq f^2$ can be checked similar to Example 1. If the defining cone $\Gamma = \Gamma_+$, $f - \frac{\partial f}{\partial \kappa_i} \kappa_i \geq 0$ follows from
\[
f - \frac{\partial f}{\partial \kappa_i} \kappa_i = \sum_{j=1}^{n} \frac{\partial f}{\partial \kappa_j} \kappa_j - \frac{\partial f}{\partial \kappa_i} \kappa_i = \sum_{j \neq i} \frac{\partial f}{\partial \kappa_j} \kappa_j > 0.
\]

**Example 3** Concave and inverse concave function $F$ in $\Gamma_+$ which satisfies Condition 2 (i)–(iv).

We only need to ensure inequalities $\sum_{i=1}^{n} \frac{\partial f}{\partial \kappa_i} \geq 1$ and $\sum_{i=1}^{n} \frac{\partial f}{\partial \kappa_i} \kappa_i^2 \geq f^2$. The proofs can be found in [3, Lemmas 4 and 5], and we show them for the readers’ convenience. Since $f$ is concave and 1-homogeneous, we have
\[
1 = f(1, \ldots, 1) \leq f(\kappa) + \frac{\partial f}{\partial \kappa_i}(\kappa)(1 - \kappa_i) = \sum_{i} \frac{\partial f}{\partial \kappa_i}(\kappa).
\]
Let $\kappa^* = (\kappa_1^*, \ldots, \kappa_n^*)$ where $\kappa_i^* = \frac{1}{\kappa_i}$ for $1 \leq i \leq n$. Since $f$ is also inverse concave, i.e., $f^*(\kappa_1, \ldots, \kappa_n) = \frac{1}{f(\kappa_1, \ldots, \kappa_n)}$ is concave, we have

$$1 \leq \sum_i \frac{\partial f^*}{\partial \kappa_i}(\kappa^*) = \frac{1}{(f(\kappa))^2} \frac{\partial f}{\partial \kappa_i}(\kappa)^2.$$ 

The following proposition shows that the class of functions satisfying Condition 2 has some convexity properties.

**Proposition 7** If $f_1$ and $f_2$ satisfy Condition 2 in the same defining cone $\Gamma \supset \Gamma_+$, then $\lambda f_1 + (1 - \lambda) f_2$ and $f_1^\lambda f_2^{1-\lambda}$ also satisfy Condition 2 in $\Gamma$ for any $\lambda \in [0, 1]$.

**Proof** Denote $f = \lambda f_1 + (1 - \lambda) f_2$ and $\tilde{f} = f_1^\lambda f_2^{1-\lambda}$.

First, we show $f$ satisfies Condition 2. It is clear for (i)-(iv), so we only check (v). By direct calculations, we know

$$\sum_{i=1}^n \frac{\partial f}{\partial \kappa_i} = \lambda \sum_{i=1}^n \frac{\partial f_1}{\partial \kappa_i} + (1 - \lambda) \sum_{i=1}^n \frac{\partial f_2}{\partial \kappa_i} \geq \lambda + 1 - \lambda = 1$$

and

$$f - \frac{\partial f}{\partial \kappa_i} \kappa_i = \lambda \left( f_1 - \frac{\partial f_1}{\partial \kappa_i} \kappa_i \right) + (1 - \lambda) \left( f_2 - \frac{\partial f_2}{\partial \kappa_i} \kappa_i \right) \geq 0.$$ 

And

$$\sum_{i=1}^n \frac{\partial f}{\partial \kappa_i} \kappa_i^2 = \lambda \sum_{i=1}^n \frac{\partial f_1}{\partial \kappa_i} \kappa_i^2 + (1 - \lambda) \sum_{i=1}^n \frac{\partial f_2}{\partial \kappa_i} \kappa_i^2 \geq \lambda f_1^2 + (1 - \lambda) f_2^2 \geq f^2,$$

where we use Jensen’s inequality in the last step.

Next, we consider $\tilde{f}$. It is obvious for (i), (ii), and (iv). From

$$\frac{\partial \tilde{f}}{\partial \kappa_i} = \tilde{f} \left( \frac{\lambda \frac{\partial f_1}{\partial \kappa_i}}{f_1} + \frac{1 - \lambda \frac{\partial f_2}{\partial \kappa_i}}{f_2} \right),$$

we see

$$\frac{\partial \tilde{f}}{\partial \kappa_i} > 0 \quad \text{and} \quad \tilde{f} - \frac{\partial \tilde{f}}{\partial \kappa_i} = \tilde{f} \left( \frac{1 - \lambda \frac{\partial f_1}{\partial \kappa_i}}{f_1} + \frac{\lambda \frac{\partial f_2}{\partial \kappa_i}}{f_2} \right) > 0$$

for any $i = 1, \ldots, n$. Furthermore,

$$\sum_{i=1}^n \frac{\partial \tilde{f}}{\partial \kappa_i} \geq \lambda \left( \frac{f_2}{f_1} \right)^{1-\lambda} + (1 - \lambda) \left( \frac{f_1}{f_2} \right)^\lambda \geq 1.$$
and
\[ \sum_{i=1}^{n} \frac{\partial \tilde{f}}{\partial \kappa_i} \kappa_i^2 \geq \tilde{f}(\lambda f_1 + (1 - \lambda) f_2) \geq \tilde{f}^2, \]
where Young’s inequality is used in the last steps of above inequalities. \(\square\)

Proposition 7 implies more examples satisfying Condition 2 on suitable cones larger than the positive cone. For instance, \(\lambda H_{k-1} + (1 - \lambda) H_k\) and \(H_k^{\lambda} H_{1-\lambda}^{1-\lambda}\) both satisfy Condition 2 in \(\tilde{\Gamma}_k\) for any \(\lambda \in (0, 1)\).

2.2 Hypersurface in Warped Product Manifold

Let \(M^n\) be a hypersurface in \(\overline{M}^{n+1}\). We will calculate under an orthonormal frame \(\{e_1, \ldots, e_n\}\) on \(M^n\) in this paper. Let \((h_{ij})\) denotes the second fundamental form. Under the orthonormal frame, \((h_{ij})\) equals to the matrix of shape operator \(\mathcal{W}\). Then
\[ \nabla_i \nu = h_{ij} e_j. \]

By Codazzi equation,
\[ \nabla_i h_{jl} = \nabla_l h_{ij} + \left[ \bar{R}_{ijkl} \right], \]
where \(\bar{R}_{ijkl} = \bar{R}(v, e_i, e_l, e_j)\) and \(\bar{R}(\cdot, \cdot, \cdot, \cdot)\) is the \((0, 4)\)-Riemannian curvature tensor of \(\overline{M}^{n+1}\).

Now we assume \(\overline{M}^{n+1} = [0, \bar{r}) \times \lambda N^n\) is a warped product manifold. Let \(\bar{\nabla}\) denote the Levi–Civita connection of \(\overline{M}^{n+1}\). The vector field \(\lambda(r) \partial_r\) satisfies
\[ \bar{\nabla}_\xi \lambda(r) \partial_r = \lambda'(r) \xi \]
for any vector field \(\xi\) on \(\overline{M}^{n+1}\).

Define \(\Phi(r) := \int_0^r \lambda(s) \mathrm{d}s\). Then \(\bar{\nabla} \Phi = \lambda \partial_r\).

Lemma 8 If \(\overline{M}^{n+1} = [0, \bar{r}) \times \lambda N^n\) satisfies
\[ \text{Ric}_N \geq (n - 1)(\lambda'^2 - \lambda'' \lambda^2) g_N, \]
then \(\bar{\text{Ric}}(\nu, \lambda \partial_r \nu) \leq 0\), where \(\partial_r^T = \partial_r - \bar{g}(\partial_r, \nu) \nu\).

Proof A proof can be found in [11, p. 699]. \(\square\)

If we further assume \((N, g_N)\) has constant sectional curvature \(c\). Then
\[ \bar{R}_{ijkl} = - \left( \frac{\lambda''}{\lambda} + \frac{c - \lambda'^2}{\lambda^2} \right) \left( \delta_{ij} r_l - \delta_{jl} r_i \right) r_v, \]
where \(r_i = \bar{g}(\partial_r, e_i)\) and \(r_v = \bar{g}(\partial_r, \nu)\) (see [24], or [11] for convenience).
3 Auxiliary Function P

In this section, we assume that hypersurface $M^n$ in $\overline{M}^{n+1} = [0, \bar{r}) \times \lambda \, N^n$ satisfies

$$F^{-\alpha} = \bar{g}(\lambda \partial_r, \nu).$$

Denote $u := \bar{g}(\lambda \partial_r, \nu)$. And we still use $\Phi$ to denote its pull-back on $M^n$ by the immersion $M^n \to \overline{M}^{n+1}$.

By direct calculation,

$$\nabla_j u = h_{jl} \bar{g}(\lambda \partial_r, e_l), \quad (9)$$

$$\nabla_i \nabla_j u = \nabla_i h_{jl} \bar{g}(\lambda \partial_r, e_l) + \lambda' h_{ij} - h_{il} h_{jl} \bar{g}(\lambda \partial_r, \nu) = \nabla_j h_{ij} \bar{g}(\lambda \partial_r, e_l) + \bar{R}_{vijli} \bar{g}(\lambda \partial_r, e_l) + \lambda' h_{ij} - u h_{il} h_{jl}. \quad (10)$$

It is also easy to check that

$$\nabla_j \Phi = \bar{g}(\lambda \partial_r, e_j), \quad (11)$$

$$\nabla_i \nabla_j \Phi = \lambda' \delta_{ij} - h_{ij} \bar{g}(\lambda \partial_r, \nu) = \lambda' \delta_{ij} - u h_{ij}, \quad (12)$$

where $\delta_{ij}$ is the Kronecker symbol.

Define operator $\mathcal{L} := F^{ij} \nabla_i \nabla_j$, where $F^{ij} = \frac{\partial F}{\partial h_{ij}}$. We consider the following auxiliary function

$$P := \Phi - \frac{\alpha}{\alpha + 1} u^{\frac{\alpha + 1}{\alpha}}. \quad (13)$$

Here $u > 0$ is confirmed by $F^{-\alpha} = u$ and assumption of $F > 0$ in Theorems 1 and 3.

Lemma 9 Function $P$ satisfies the following equality:

$$\mathcal{L} P = \lambda' \left( \sum_i F^{ii} - 1 \right) + u^{\frac{\alpha + 1}{\alpha}} \left( F^{ij} h_{ij} h_{jl} - F^2 \right) - u^{\frac{1}{\alpha}} F^{ij} \bar{R}_{vijli} \bar{g}(\lambda \partial_r, e_l)$$

$$+ \frac{1}{\alpha} \bar{g}(\lambda \partial_r, \nabla \log u) - \frac{1}{\alpha} u^{\frac{\alpha + 1}{\alpha}} F^{ij} \nabla_i \log u \nabla_j \log u. \quad (14)$$

Proof By equality (10), we obtain

$$\mathcal{L} u = \bar{g}(\lambda \partial_r, \nabla F) + F^{ij} \bar{R}_{vijli} \bar{g}(\lambda \partial_r, e_l) + \lambda' F - u F^{ij} h_{ij} h_{jl}. \quad (15)$$
Moreover, using equation $F^{-\alpha} = u$,

\[
\frac{\alpha}{\alpha + 1} \mathcal{L} u^{\alpha+1} = \frac{1}{\alpha} \mathcal{L} u + \frac{1}{\alpha} u^{1-\alpha} F^{ij} \nabla_i u \nabla_j u
\]

\[
= \frac{1}{\alpha} \bar{g}(\lambda \partial_r, \nabla F) + u^{1-\alpha} \bar{R}_{ijkl} \bar{g}(\lambda \partial_r, e_l) + \lambda'
\]

\[
- u^{\alpha+1} F^{ij} h_{ij} + \frac{1}{\alpha} u^{1-\alpha} F^{ij} \nabla_i u \nabla_j u.
\]

By equality (12) and equation $F^{-\alpha} = u$, we know

\[
\mathcal{L} \Phi = F^{ij} (\lambda' \delta_{ij} - u h_{ij}) = \lambda' \sum_i F^{ii} - u F = \lambda' \sum_i F^{ii} - u^{\alpha+1} F^2.
\] (14)

Combining (13) and (14), we obtain

\[
\mathcal{L} P = \mathcal{L} \Phi - \frac{\alpha}{\alpha + 1} \mathcal{L} u^{\alpha+1} = \lambda' \left( \sum_i F^{ii} - 1 \right) + u^{\alpha+1} \left( F^{ij} h_{ij} - F^2 \right)
\]

\[
- u^{1-\alpha} F^{ij} \bar{R}_{ijkl} \bar{g}(\lambda \partial_r, e_l) + \frac{1}{\alpha} u^{1-\alpha} F^{ij} \nabla_i u \nabla_j u.
\]

\[
= \lambda' \sum_i F^{ii} - u^{\alpha+1} F^2.
\] (15)

4 Proof of Theorem 1

For case $F = H$, tensor $F^{ij} = \frac{1}{n} \delta_{ij}$ and operator $\mathcal{L} = \frac{1}{n} \Delta$. Then

\[
\sum_i F^{ii} = 1,
\]

\[
F^{ij} h_{ij} - F^2 = \frac{1}{n}|h|^2 - H^2
\]

and

\[
F^{ij} \bar{R}_{ijkl} \bar{g}(\lambda \partial_r, e_l) = \frac{1}{n} \bar{\text{Ric}}(\nu, \lambda \partial_r^T),
\]

where $\bar{\text{Ric}}$ denotes the Ricci curvature tensor of $\bar{M}$ and $\partial_r^T$ is the tangent part of $\partial_r$. Thus Lemma 9 gives

\[
\frac{1}{n} \Delta P = u^{\alpha+1} \left( \frac{1}{n}|h|^2 - H^2 \right) - \frac{1}{n} u^{\frac{1}{\alpha}} \bar{\text{Ric}}(\nu, \lambda \partial_r^T) + \frac{1}{\alpha} \bar{g}(\lambda \partial_r, \nabla \log u)
\]

\[
- \frac{1}{n\alpha} u^{\alpha+1} |\nabla \log u|^2.
\] (15)
Notice
\[ \nabla P = \lambda \partial^T_r - u^\frac{1}{\alpha} \nabla u. \]

Then
\[ \bar{g}(\nabla P, \nabla \log u) = \bar{g}(\lambda \partial_r, \nabla \log u) - u^\frac{\alpha + 1}{\alpha} |\nabla \log u|^2. \quad (16) \]

From (15) and (16), we obtain
\[ \Delta P - \frac{n}{\alpha} \bar{g}(\nabla P, \nabla \log u) = u^\frac{\alpha + 1}{\alpha} (|h|^2 - nH^2) - u^\frac{1}{\alpha} \bar{\text{Ric}}(v, \lambda \partial^T_r) \\
+ \frac{n - 1}{\alpha} u^\frac{\alpha + 1}{\alpha} |\nabla \log u|^2. \]

Lemma 8 shows \( u^\frac{1}{\alpha} \bar{\text{Ric}}(v, \lambda \partial^T_r) \leq 0 \). Then inequality \(|h|^2 - nH^2 \geq 0\) indicates
\[ \Delta P - \frac{n}{\alpha} \bar{g}(\nabla P, \nabla \log u) \geq \frac{n - 1}{\alpha} u^\frac{\alpha + 1}{\alpha} |\nabla \log u|^2 \geq 0. \]

By the strong maximum principle, we know \( P \) is constant. Then the above inequality shows \( \nabla u = 0 \) for \( n > 1 \). Consequently, \( \nabla r = \partial^T_r = \frac{1}{\lambda} (\nabla P + u^\frac{1}{\alpha} \nabla u) = 0 \) everywhere in \( M^n \). This implies \( r \) is constant in \( M^n \) which means \( M^n \) is a slice.

5 Proofs of Theorem 3 and Its Corollaries

Proof of Theorem 3 From equality (8),
\[ F^{ij} \bar{R}_{vji} \bar{g}(\lambda \partial_r, e_i) = -\lambda \left( \frac{\lambda''}{\lambda} + \frac{c - \lambda'^2}{\lambda^2} \right) F^{ij} (\delta_{ij} r_l - \delta_{ji} r_i) r_l r_i \\
= -u \left( \frac{\lambda''}{\lambda} + \frac{c - \lambda'^2}{\lambda^2} \right) \left( (\sum r_i^2) \left( \sum F^{ii} \right) - F^{ij} r_i r_j \right). \]

At any fixed point, we notice
\[ (\sum r_i^2) \left( \sum F^{ii} \right) - F^{ij} r_i r_j = (\sum r_i^2) \left( \sum f^i \right) - f^i r_i^2 \geq 0, \quad (17) \]

where \( f^i := \frac{\partial f}{\partial x_i} > 0 \). And the equality occurs if and only if \( \partial^T_r = 0 \).

Combining with the assumption
\[ \frac{\lambda''}{\lambda} + \frac{c - \lambda'^2}{\lambda^2} \geq 0, \]
we obtain

\[ F_{ij} \tilde{R}_{jl} \tilde{g}(\lambda \partial_r, e_i) \leq 0. \]

Thus, from Lemma 9, we have the following inequality:

\[
\mathcal{L} P \geq \lambda'(\sum_i F^{ii} - 1) + u^{\frac{a+1}{\alpha}} (F^{ij} h_{ij} h_{jl} - F^2) + \frac{1}{\alpha} \tilde{g}(\lambda \partial_r, \nabla \log u)
\]

\[ - \frac{1}{\alpha} u^{\frac{a+1}{\alpha}} F^{ij} \nabla_i \log u \nabla_j \log u. \]

Since

\[ \nabla P = \lambda \partial_r T - u^{\frac{a+1}{\alpha}} \nabla \log u, \]

we have

\[ u^{-\frac{a+1}{\alpha}} \tilde{g}(\lambda \partial_r, \nabla P) = u^{-\frac{a+1}{\alpha}} \lambda^2 \sum_i r_i^2 - \tilde{g}(\lambda \partial_r, \nabla \log u)\]

and

\[ F^{ij} \nabla_i \log u \nabla_j P = \lambda F^{ij} r_j \nabla_i \log u - u^{\frac{a+1}{\alpha}} F^{ij} \nabla_i \log u \nabla_j \log u. \]

Then we know

\[
\mathcal{L} P + \frac{1}{\alpha} u^{-\frac{a+1}{\alpha}} \tilde{g}(\lambda \partial_r, \nabla P) - \frac{1}{\alpha} F^{ij} \nabla_i \log u \nabla_j P
\]

\[ \geq \lambda'(\sum_i F^{ii} - 1) + u^{\frac{a+1}{\alpha}} (F^{ij} h_{ij} h_{jl} - F^2)
\]

\[ + \frac{1}{\alpha u} (\lambda^2 F \sum_i r_i^2 - \lambda F^{ij} r_j \nabla_i u). \]

At any fixed point, choosing a frame such that \( h_{ij} = \kappa_i \delta_{ij} \) and using equality (9), we have

\[ \lambda^2 F \sum_i r_i^2 - \lambda F^{ij} r_j \nabla_i u = \lambda^2 \sum_i (f - f^i \kappa_i) r_i^2. \]

We also know

\[ \sum_i F^{ii} = \sum_i f^i \]

and

\[ F^{ij} h_{ij} h_{jl} = f^i \kappa_i^2. \]
Thus, the following inequality holds
\[
\mathcal{L}P + \frac{1}{\alpha} u^{-\frac{\alpha+1}{\alpha}} \tilde{g}(\lambda \partial_r, \nabla P) - \frac{1}{\alpha} F^{ij} \nabla_i \log u \nabla_j P \\
\geq \lambda' \left( \sum_i f^i - 1 \right) + u^{\frac{\alpha+1}{\alpha}} (f^i \kappa_i^2 - f^2) + \frac{\lambda^2}{\alpha u} \sum_i (f - f^i \kappa_i) r_i^2.
\]

Assumption \( \lambda' > 0 \) and Condition 2 (v) imply
\[
\mathcal{L}P + \frac{1}{\alpha} u^{-\frac{\alpha+1}{\alpha}} \tilde{g}(\lambda \partial_r, \nabla P) - \frac{1}{\alpha} F^{ij} \nabla_i \log u \nabla_j P \geq 0.
\]

From \( \frac{\partial f}{\partial \kappa_i} > 0 \) in \( \Gamma \), we know \( F^{ij} \) is positive definite. By the strong maximum principle, we know \( P \) is constant. It indicates inequality (17) is actually equality. Then \( \partial^2 T = 0 \) implies \( M^n \) is a slice.

**Proof of Corollary 4** By Lemma 2.3 in [20], we know that principal curvatures \( \kappa \in \Gamma_{k+1} \subset \tilde{\Gamma}_k \) from \( H_{k+1} > 0 \). From Example 1 in Sect. 2, we finish the proof by letting \( F = H_k \) and \( \Gamma = \tilde{\Gamma}_k \) in Theorem 3.

**Proofs of Corollary 5 and 6** See Examples 2 and 3 in Sect. 2 and use Theorem 3.

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