Slow relaxation in weakly open vertex-splitting rational polygons

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Abstract

The problem of splitting effects by vertex angles is discussed for non-integrable rational polygonal billiards. A statistical analysis of the decay dynamics in weakly open polygons is given through the orbit survival probability. Two distinct channels for the late-time relaxation of type $t^\delta$ are established. The primary channel, associated with the universal relaxation of ”regular” orbits, with $\delta = 1$, is common for both the closed and open, chaotic and nonchaotic billiards. The secondary relaxation channel, with $\delta > 1$, is originated from ”irregular” orbits and is due to the rationality of vertices.

Key words: Dynamics of systems of particles, control of chaos, channels of relaxation.

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I. INTRODUCTION

Polygonal classical billiards is an active subject of research in mathematics and physics (see for review Ref. [1]). In view of the null Lyapunov exponent and the null Kolmogorov metric entropy the rational polygons, formed by the piece-line billiard boundary with the vertex angles are rational multiplies of $\pi$, are known to be nonchaotic systems [1–6]. They are therefore well distinct from the Sinai billiard [7] (SB) and the Bunimovich billiard [8] (BB) where classical chaotic motion regimes are due to, respectively, dispersive effects caused by the circle disk and the squared boundary, and the interplay between boundary segments formed by the circle and the square. Meanwhile, the rational polygons of $m$ equal sides and equal vertices (hereafter, the $m$-gons [1]) revealed [9] positive Lyapunov exponents with increasing of $m$. Furthermore, the polygonal billiards exposed chaoticlike changes in the associated quantum-level spectra [10], which fluctuations are shown [11,12] to be very close to the Gaussian orthogonal-ensemble-type statistics. In view of the splitting effects by the angle vertices, polygons do not satisfy the conditions of integrability [2,13].

These evidences for the chaoticlike features of the nonintegrable rational polygons were recently questioned by Mantica [6] through the orbital complexity analysis. Unlike the case of the integrable billiards [14], a delicate problem of the interplay between the regular (piece-line) and the irregular (vertex-angle) boundary segments in polygons cannot be solved in terms of the first-order averaged polygonal characteristics, such as the average-orbit coding length [6], the mean wall-collision time [14], or the average collision number [15]. As shown through analysis of the orbit-wall-collision statistics [15], the higher order correlation effects induced by vertices play a crucial role in the intrinsic dynamics of $m$-gons. Earlier this was experimentally corroborated [11] through the quantum-level spectra statistics: ”unlike the case of the irrational polygons, the long-range level correlation effects are due to the rationality of vertex angles” [11]. In the case of the curved-by-circle infinite-$m$ rational polygon ($\infty$-gon) the late-time memory effects arise from the sliding orbits [15], which have no analog in the ballistic-type dynamics of its counterpart given by the circle billiard.
(hereafter CB), where the orbit classification is well established [10]. This implies that the quasi-classical approach has no justification for the ”quantized” vertex-splitting $\infty$-gon that is geometrically equivalent to the CB. In other words, the quantum-to-classical dynamics correspondence suggested [3] between a given $m$-gon and its circumscribing counterpart is violated [15] regardless of the fact that the geometric correspondence exists and can be achieved with any precision when $m \to \infty$ (with the help of the aforementioned first-order averaged characteristics). This is in agreement with a conclusion on inapplicability in polygons of the quantum-to-classic correspondence principle. The latter was elaborated [12] within the scope of the conventional Wentsell-Kramer-Brillouin picture that failed to establish a one-to-one correspondence between classical orbits and their quantum counterparts.

The issue of the current paper is an investigation of the vertex-splitting effects revealed in the decay dynamics of finite-$m$ rational polygons. An analysis is given through the survival orbit statistics in the weakly open billiards, which boundaries permit orbits to escape through a small opening. We will see that, similarly to the case of the intrinsic dynamics in the closed $m$-gons [15], the vertex-splitting effects are dual of vertex-ordering and vertex-disordering motion effects, that is manifested through, respectively, the orderlike and chaoticlike behavior observed in the late-time relaxation. The paper is organized as follows. The recent findings for the decay dynamics in nonchaotic [14] and chaotic [17] billiards are given in Sec.II and discussed within concepts of the primary and the secondary relaxation channels. The weakly open rational polygons are analyzed numerically and analytically in Sec.III for the cases of small and large numbers $m$. Discussion and conclusions are summarized in Sec.IV.

II. DECAY DYNAMICS OF THE CHAOTIC AND NONCHAOTIC BILLIARDS

The intrinsic dynamics of the closed classical billiards is commonly considered in terms of a temporal decay of the correlation functions for certain dynamical variables (see e.g. Ref. [18]). Pure exponential lost of amount of memory on the initial states is not a unique
channel of relaxation even in the chaotic systems (see e.g. Ref. [19]). By studying the chaotic billiards, such as the SB [19–22], that is dynamically equivalent to the correspondent Lorentz gas (LG) model [18] and the BB [8,23,24] with a stadium geometry, it has been recognized that a crossover from the short-time exponential to the late-time algebraic decay is due to the long-term memory effects on a free regular motion. The algebraic tail of the correlation functions seems to be vanished only in the case of the fully hyperbolic systems correspondent to such geometries as the finite-horizon SB [18,21] (equivalent to the high-density LG) or the diamond [18]. Qualitatively the same can be referred to the decay dynamics in weakly open billiards that describes a crossover from a bounded to unbounded orbit free motion. Such a decay dynamics is initially established by uniformly distributed \( N_0 \) point particles (of unit mass and unit velocity) moving inside of the closed planar billiard table and then allowing to escape through a small opening of width \( \Delta \). A temporal behavior of the dynamic observables can be scaled to the characteristic billiard times, namely

\[
\tau_c = \frac{\pi A}{P} \quad \text{and} \quad \tau_e = \frac{P}{\Delta} \tau_c.
\]

Here the mean collision time [18,25,26] \( \tau_c \) and by the mean escape time [14,18,23,27] \( \tau_e \) are introduced through the accessible area \( A \) and the perimeter \( P (\gg \Delta) \) for a given billiard table. The late-time (\( t \gg \tau_e \)) algebraic-type evolution of non-escaped orbits (or particles) \( N_t \propto N_0(\tau_e/t)^\delta \), which time derivation is related to the survival probability, is definitely characterized by the decay dynamic exponent \( \delta \).

In the integrable nonchaotic billiards, the first observation of the algebraic decay with exponents \( \delta \lesssim 1 \) was given [27] for the case of the square billiard. Recent study of the decay dynamics in the 4–gon revealed [14] two distinct channels of the algebraic-type slow relaxation. The first one is due to the regular-orbit motion with the decay dynamic exponent \( \delta = 1 \), and the second channel is originated from the ”irregular” orbits induced by the singular vertex-spliting effects, which give rise to the subdiffusion motion regime indicated [14] by \( \delta \approx 0.85 \). Such a kind of diverse dynamical behavior was observed through the survival orbit spectra defined by a number for the survived orbits and simulated in the integrable CB
and in the almost-integrable \([1]\) square billiard (see, respectively, Figs. 5,6 and 3,4 in Ref. \([14]\)). In both the cases an irregularlike motion is due to orbit families known as, respectively, the "whispering-gallery" (polygonlike) and the "bouncing-ball" orbits. Meanwhile, in the late-time decay of the CB (see inserts in Figs. 3,4 and 5,6 in Ref. \([14]\)) the short-time living, frequently escaped "whispering-gallery" orbits, unlike the nonintegrable case, did not contribute to the second channel of relaxation. Thus, the solely exponent \(\delta = 1\) was observed in the CB.

In the chaotic closed and weakly open (including Hamiltonian) classical systems, presented by the BB \([23,24,28,29]\), the infinite-horizon SB \([17,19,20,22,30,31]\), and by the low-density LG model \([21,32,33]\), the algebraic-type decay was numerically revealed \([34]\) by the dynamic exponents \(\delta \geq 1\). Similarly to the nonchaotic case, it has been repeatedly recognized that the algebraic tail is caused by the "arbitrary long segments" observed in the evolution of stochastic orbits \([24]\), or by the regularlike orbit motion due to "sticking particles" \([23,33]\). This implies that in both the cases this relaxation is due to a free motion of the corresponding trajectories in the infinite (with respect to the observation time) distinct corridors which are open in the relevant phase space \([30–32]\).

The algebraic-type relaxation channel with \(\delta = 1\) (\(\equiv \alpha\)) established in the chaotic \([17,19,23,30,31]\) as well as in nonchaotic \([27,14]\) billiards, seems to be generic for all non-fully hyperbolic systems with smooth convex boundaries. Its independence of the billiard space dimension \([31]\), its insensitiveness to details of the boundary shape \([14]\), including that to a position of the small opening \([23]\), and to the initial conditions \([14]\), suggests that the late-time \(\alpha\)-relaxation arise in classical systems as the universal primary relaxation. The latter is a part of the two-step relaxation scenario discussed in the chaotic \([17]\) and non-chaotic \([14]\) weakly open classical systems. This universal scenario was introduced by the short-time pure exponential and by the first-power-algebraic overall orbit decays given by \(N_t \propto N_0 e^{-(t/\tau_\epsilon)^\gamma}\) with \(\gamma = 1\), and by \(N_t \propto N_0 (\tau_\epsilon/t)^\delta\) with \(\delta = 1\), respectively. Meanwhile the real relaxation does not exclude the existence of the additional intermediate transient regimes approximated by the stretched-exponential form \([17]\) with \(\gamma < 1\) (earlier discussed
for chaotic billiards in analytical [30] and numerical [18] forms, and by other algebraic-decay forms [30,31] with $\delta \leq 1$. The escape mechanism of the primary relaxation in the chaotic and nonchaotic billiards was described in details within a coarse-grained approximation (see, respectively, Eq.(12) in Ref. [17] and Eq.(15) in Ref. [14]). Unlike the case of the primary relaxation, the temporal observation conditions (observation windows) for the secondary relaxation in the chaotic billiards ($\delta = \beta > 1$) are shown to be very sensitive to the billiard geometry [14], to the dimension [31] $d$ of a billiard table, and to the initial conditions [14]. This can be exemplified by the dynamic-exponent constraint $1 < \beta \leq d$ proposed in Ref. [31] and observed in the chaotic BB [23] and SB [19,30,31] billiards.

The algebraic-type decay of correlations in the low-density chaotic LG is due to evolution of trajectories within the infinite principal and/or ”hindered” open corridors [30,33]. In the corresponding disk-dispersing SB (of side $L$ and disk radius $R$) with the infinite-horizon geometry ($R < L/2$) the principal and the ”hindered” corridors, respectively, can be governed by the limiting disk radii [32] $R_\alpha = L/2$ and $R_\beta = \sqrt{2}L/4$ and observed through the established [17] algebraic-tail temporal windows. Thus, the primary relaxation was observed [40] within the domain $R_\beta < R < R_\alpha$, when the ”hindered” corridors are closed [41]. Under the geometric conditions $0 < R \leq R_\beta$ the transient $\beta$-relaxation channel was activated and indicated by the decay dynamic exponents $\beta = 1.2$ and $\beta = 1.1$, respectively, in Refs. [30] and [40]. One can see that the primary and the secondary relaxation channels can be geometrically associated with Bleher’s [32] principal and ”hindered” corridors, respectively. The latter case was additionally characterized [32] as a superdiffusive motion regime, observed earlier in the chaotic [18,20,22,33] and very recently in the polygonal [13] closed billiards.

We see that non-fully hyperbolic billiards, on one hand, are indiscernible within decay dynamics observed through the primary universal $\alpha$-relaxation channel. On the other hand, the chaotic and nonchaotic, the open and closed billiards are well distinguished with respect to the secondary $\beta$-relaxation given by the decay dynamic exponents $\beta > 1$ and $\beta < 1$, respectively. In what follows we give numerical and analytical analysis of the conditions of stabilization of both the primary and secondary relaxation channels in the weakly open
III. ORBIT DECAY IN POLYGONS

We deal with the rational polygons of \( m \) equal sides, denominated as \( m \)-gons, circumscribed below a circle of radius \( R \). The mean collision time \( \tau_{cm} = (\pi R/2) \cos(\pi/m) \) and the mean escape time \( \tau_{em} = (\pi R^2 m/2\Delta) \sin(2\pi/m) \) are given with the help of Eq.(1) through area \( A_m = (mR^2/2) \sin(2\pi/m) \) and perimeter \( P_m = 2mR\sin(\pi/m) \). In the limit \( m \to \infty \) one naturally arrives at the circle geometry of the \( \infty \)-gon with the mean times \( \tau_{c\infty} = \tau^{(CB)}_{eR} = \pi R/2 \) and \( \tau_{e\infty} = \tau^{(CB)}_{eR} = \pi^2 R^2/\Delta \), both characteristic of the CB. This demonstrates how the first-order dynamic characteristics can be introduced through the aforementioned geometrical correspondence that takes place between the \( \infty \)-gon and the CB. Meanwhile the dynamical correspondence does not exist \[15\] because the vertex-memory effects violate commutation between of the temporal \( (t \to \infty) \) and spatial \( (m \to \infty) \) limits.

A. Small Number of Vertices

Similarly to the closed \( m \)-gons \[15\], let us consider the case of small number of vertices, with \( m < 10 \), within the scope of the deterministic approach. This is given by generalization of the regular-orbit description introduced \[14\] for the particular case of \( m = 4 \) and is straightforwardly based on the fact that the wall-collision angles \( \varphi \) (counted off the normal to the boundary and preserved by elastic reflections) are integrals of motion. This is true for the integrable billiards where dispersing or splitting effects are absent. A dynamic description of the regular-orbit motion can be introduced with accounting of the fact that \( m \) (or \( m/2 \)) sides of a given \( m \)-gon, with odd (or even) number of vertices, are dynamically equivalent \[37\]. A description of the wall-collision statistics can be therefore reduced to the collision-angle domain \( \varphi = [0, \varphi_m] \) with
\[ \varphi_m = \begin{cases} 
\pi/2m & \text{for odd } m, \\
\pi/m & \text{for even } m. 
\end{cases} \]  

(2)

In turn, the \( \varphi \)-family regular–orbit sets can be introduced (for details see Appendix) through the characteristic collision times, namely

\[ t_{cm}(\varphi) = \frac{\pi R \cos \pi/m \sin \varphi_m}{2\varphi_m \cos(\varphi - \psi_m)} , \text{ with} \]

(3)

\[ \psi_m = \begin{cases} 
0 & \text{for odd } m \text{ and } m/2, \\
\pi/m & \text{for even } m/2. 
\end{cases} \]  

(4)

The collision time \( t_{cm}(\varphi) \) is related to the billiard mean collision time \( \tau_{cm} \) through the "mean-collision-time" equation, i.e., \( < t_{cm}(\varphi) >_c = \int_0^{\varphi_m} t_{cm}(\varphi) f_{0m}(\varphi) d\varphi = \tau_{cm} \). The latter is equivalent to the known \[25\] "mean-free-path" equation considered in the uniformly populated two-dimensional (2D) collision subspace \( \Omega_{cm} \) (see also Eqs.(3,4) in Ref. \[14\]) the 3D phase space \( \Omega_m \). The \( \varphi \)-set-orbit distribution function \( f_{0m} \) is defined here as

\[ f_{0m}(\varphi) = \begin{cases} 
\frac{\cos(\varphi - \psi_m)}{\sin \varphi_m}, & \text{for subspace } \Omega_{cm}, \\
\frac{t_{cm}(\varphi)}{\tau_{cm}} \cos(\varphi - \psi_m) \equiv \frac{1}{\varphi_m}, & \text{for space } \Omega_m 
\end{cases} \]  

(5)

by generalization of Eqs.(6,7) in Ref. \[14\].

We discuss the late-time \( (t \gg \tau_{em}) \) survival dynamics in a given \( m \)-gon through the \( \varphi \)-set-orbit decay spectra defined by numbers of the survived orbits \( N_m(t, \varphi) \) and by corresponding overall numbers \( N_{tm} = < N_m(t, \varphi) >_c \). Here a procedure of averaging over collisions, denoted by \( < ... >_c \), is introduced through the aforesaid "mean-collision-time" equation. The universal relaxation channel, associated solely with the regular orbits, is given through numbers of the nonescaped orbits, namely

\[ \frac{N_m(t, \varphi)}{N_{0m}} = C_m(\varphi) \frac{t_{cm}(\varphi)}{\tau_{cm}} f_{0m}(\varphi) \frac{\tau_{cm}}{t} \text{ and } \frac{N_{tm}}{N_{0m}} = D_m \frac{\tau_{em}}{t}, \]  

(6)

which are the late-time solutions of the relevant decay-kinetics master equation (see Eqs. (16,17) in Ref. \[14\]). As seen from Eq.(6), the fundamental characteristics \( t_{cm}(\varphi), f_{0m}(\varphi) \) and \( \tau_{cm} \) are common for both the decay and intrinsic dynamics. The \( \varphi \)-set orbit-partial
weight $C_m(\varphi)$ and the orbit-overall weight $D_m$ of the corresponding algebraic tails can be established in explicit form within a certain coarse-grained scheme and directly observed \[14,17\]. However an interesting analysis that indicates a departure of $m$-gons from the true integrable systems due to vertices can be given through the algebraic-tail weights without detail calculations.

By taking into account the mentioned equation $N_{tm} = \langle N_m(t, \varphi) \rangle_c$, the overall regular-orbit-decay weight $D_{cm} = C_{cm}^{(reg)} = \langle C_m(\varphi) \rangle_c^{(reg)}$ immediately follows from Eq.(6). The validity of this relation can be straightforwardly tested for the integrable CB where the irregular orbits do not survive in the late-time relaxation. Indeed, in this case $C_{cR}^{(exp)} = 0.206$ and $D_{cR}^{(exp)} = 0.214$ that follows from the observation (see Tab.2 in Ref. \[14\]) of the primary relaxation of the collision space $\Omega_{cR}$. One can see that the algebraic-tail parametrization of the regular-orbit decay effects in the CB is self-consistent that is experimentally justified with a high precision, \textit{i.e.}, $C_{cR}^{(exp)} = D_{cR}^{(exp)} = 0.210 \pm 2\%$. In the case of polygons, a violation of the relations $C_{cm}^{(reg)} = C_{cm}^{(exp)}$ and $C_{cm}^{(exp)} = D_{cm}^{(exp)}$ is expected in view of the long-living irregular-orbit motion induced by vertex-splitting effects.

We have performed numerical experiments \[38\] on decay dynamics in $m$-gons with small number of vertices: $m = 3, 4, ..., 8$. The initial particles ($N_{0m} = 10^6$) have been distributed randomly within the two distinct phase spaces described in Eq.(5) and then allowed to escape through a small opening $\Delta \ll R$. The condition $\tau_{em} (= 300)$ is chosen to be common for all $m$ and that has been provided with the accuracy of $\pm 5\%$ with the help of Eq.(1). In all cases of $m \leq 8$ the typical algebraic decay is observed \[10\] within a certain temporal windows given by, approximately, $10^1 \tau_{em} < t \leq 10^3 \tau_{em}$. The particular cases of the observed decay spectra for pentagon and heptagon are exemplified in Fig.1. In general, the overall-orbit late-time decay in $m$-gons with small number of vertices do not show noticeable deviations from the linear relaxation \[39\] (see the left insert in Fig.1). Thus the partial weights $C_m^{(exp)}(\varphi)$ are derived from the observed numbers $N_{tm}^{(exp)}(t, \varphi)$ through Eq.(3) with accounting of the estimated distribution function $f_{0m}(\varphi)$ and the collision times $t_{cm}(\varphi)$ given in, respectively, Eqs.(2) and (3). These equations were additionally experimentally tested \[10\] (\textit{e.g.} see the
As seen from Fig.1, the observed partial weights $C_m^{(\text{exp})}(\varphi)$ exhibit regular (small) and irregular (large) deviations from the mean magnitude $C_m^{(\text{exp})} = C_m^{(\text{tot})}$ shown by a solid (horizontal) line. The latter and the regular-orbit weights $C_m^{(\text{reg})}$ (evaluated with regardless of the large isolated peaks) are accumulated in Table 1. Through analysis of a difference between the overall and partial weights $\Delta C_m = C_m^{(\text{tot})} - C_m^{(\text{reg})}$, one can see with the help of Table 1 that, similarly to the case of the intrinsic dynamics (see Fig.2 in Ref. [15]), the vertex-splitting effects in the even-gons are more pronounced than those in the odd-gons. In all the $m$-gon cases a deviation of the total weights, treated as $\Delta D_m = D_m^{(\text{tot})} - D_m^{(\text{reg})}$ (with $D_m^{(\text{tot})} = D_m^{(\text{exp})}$ and $D_m^{(\text{reg})} = C_m^{(\text{reg})}$ given in Table 1), exceeds experimental error ($\pm 2\%$) established above for the integrable SB, and we therefore infer that $D_m^{(\text{tot})} > D_m^{(\text{reg})}$. This implies that the irregular-orbit motion is involved into the observed relaxation. On the other hand, no noticeable deviation from the primary relaxation is indicated in the observed decay-orbit dynamics when $m \leq 8$. Similarly to the case of the intrinsic dynamics observed in the $m$-gons with small number of vertices, we deduce that the regular-orbit motion dominates in the late-time decay dynamics.

**B. Large Number of Vertices**

The universal two-step relaxation in open billiards is shared by $m$-gons with arbitrary number of vertices [10]. In Figs. 2 and 3 we analyze our numerical results for the late-time overall-orbit decay dynamics in the $m$-gons with large number ($m = 2^n$ with $n = 3,...6$) of vertices for the cases of relatively small and large opening widths. In general, one can observe that the decay dynamics of the rational polygons with increasing of number of sides moves away from that given by the geometrically corresponding CB: the chaotic effects, manifested by the secondary relaxation channel with $\beta > 1$, become more pronounced with number of vertices. As seen from the observed relaxation of the initially equivalent states (given in phase spaces $\Omega_{\infty}$ and $\Omega_R$) is qualitatively distinct, and the open $\infty$-gon and the CB
are not therefore dynamically equivalent. Moreover, the \(m\)-gons with \(m > 8\) do not expose the algebraic decay with \(\beta < 1\), characteristic for subdiffusive motion regimes in nonchaotic open systems observed \(\text{in 4-gon.}\)

In the particular case of \(\Delta = 0.05R\) shown in Fig. , the universal relaxation remains stable until \(m = 64\), but when \(m \geq m_{\alpha}^{(\exp)} = 128\) the primary relaxation channel turns up to be closed. In other words, the regular-orbit-motion relaxation affected by vertices is transformed into the irregularlike-motion relaxation indicated by dynamic exponent \(\beta = 1.2\). Qualitatively the same follows from Fig. 3, but the upper limit for the \(\alpha\)-channel observation window shows its dependence on \(\Delta\), i.e., \(m_{\alpha}^{(\exp)} = 32\) for \(\Delta = 0.20R\). Thus Bleher’s principal corridors of the regular-motion relaxations are open for \(3 \leq m < m_{\alpha}^{(\exp)}\) (similarly to the case of the SB given by \(R_\beta \leq R < R_\alpha\) that is discussed in Sec.II) when the vertex-splitting (or disk-dispersing effects) are weak \(\text{[13]}\). We see that the \(\alpha\)-relaxation \((3 \leq m < m_{\alpha}^{(\exp)})\) occurs as a precursor of the \(\beta\)-relaxation regime, realized for \(m \geq m_{\alpha}^{(\exp)}\), just as in the chaotic SB case the \(\beta\)-relaxation \((0 < R \leq R_\beta)\) was observed \(\text{[10]}\) before the \(\alpha\)-relaxation \((R_\beta \leq R < R_\alpha)\). Conversely, the observed \(\alpha\)-to-\(\beta\)-relaxation crossover in the \(m\)-gons, induced by the interplay between the piece-line regular and the vertex-angle singular boundaries, is similar to that between the semi-square and semi-circle parts of the stadium boundary. The latter was deduced \(\text{[23]}\) in the case of the BB with small \((\Delta = 0.01)\) and large \((\Delta = 0.25)\) opening widths, with the observed exponents \(1 \lesssim \beta \lesssim 2\) (see Table 1 in Ref. \(\text{[23]}\)).

Qualitatively, the effect of closing of the principal corridors in a given \(m\)-gon can be understood by difficulties, increasing with \(m\), to draw the long segments of free motion, which intersect the polygonal sides in the correspondent LG lattice but avoid the vertex angles (see also discussion in Appendix). By contrast, the \(\beta\)-relaxation revealed in Figs. 2, 3 is associated with stabilization of the ”irregular”-type trajectories, which are effectively deviated by vertex angles. More precisely, the observed order-to-chaoticlike crossover can be understood as a regular-to-irregular orbit transformation of the aforesaid sliding orbits (formed \(\text{[13]}\) by \(\varphi \approx \pi/2\) -sets with characteristic free-path times \(t_{\text{cm}}^{(\text{reg})}(\varphi) = \tau_\infty \cos^{-1} \varphi\) following
from Eq.\(3\)), which survive in the open \(m\)-gons with \(m < m_\alpha\), into those, renormalized substantially by vertex angles, called \[15\] by vortexlike orbits (with the large, but finite mean characteristic time \(\tau_{\text{reg}}^{(irreg)} = \tau_{\text{ext}} m / \pi\), which expected to be stable for \(m \geq m_\alpha \gg 1\). Within this context, the survival conditions of the regular (sliding) and the irregular (vortexlike) orbits driven, respectively, by piece-line and vertex-angle parts of the open polygonal boundary, can be introduced as follows. On one hand, the favorable observation conditions for the \(\alpha\)-relaxation (or the \(\beta\)-relaxation) should ensure to exclude (or to include) the vertex-angle effects under the constraint \(m < m_\alpha\) (or \(m \geq m_\alpha\)). On the other hand, in the weakly open \((\Delta \ll P_m) m\)-gon of a side length \(L_m = P_m / m\), the survival conditions for the regular sliding or irregular vortexlike orbits are satisfied by geometric constraints, respectively, \(\Delta \ll L_m\) or \(\Delta \gg L_m\). Hence, the \(\alpha\)-\(\beta\)-relaxation crossover, observed at \(m = m_\alpha\), is ensured by the condition \(\Delta = L_m\). With taking into account that perimeter in the polygons with large number of sides is well approximated by \(P_m \approx 2\pi R\), one arrives at the desirable criterium

\[
m_\alpha = \frac{2\pi R}{\Delta}.
\]

This finding provides the estimates \(m_\alpha^{(\text{theor})} = 126\) and 31 for the particular cases of \(m_\alpha^{(\text{exp})} = 128\) and 32 realized in Figs. 2 and 3, respectively. We infer that unlike the case of the closed polygons \[14\], the vertex-splitting effects in the open \(m\)-gons with large number of vertices give rise to stabilization of the vortexlike-orbit motion.

**IV. DISCUSSION AND CONCLUSIONS**

The mild discontinues caused by vertex angles and relative lengths of the edges is the central problem of the intrinsic dynamics of the ”almost integrable” polygonal billiards commonly discussed \[1\] in terms of the orbit ergodicity, mixing, entropy, coding, complexity \[4\], pseudo-integrability \[3\], spectral level statistics \[10, 11\], and the orbit collision statistics \[15\]. The problem is now addressed to the decay dynamics in the \(m\)-gons and is discussed through the orbit survival probability \(\Psi_{\alpha}(t) = |d(N_{tm}/N_{0m})/dt|\) related to the number of the
survived orbits $N_{tm}$. The decay spectra given by the $\varphi$-set regular-orbit numbers $N_{m}(t, \varphi)$, are also studied for the case of small number of vertices $m$.

A general approach to the decay problem based on a simple decay kinetic equation \cite{17} naturally arrives at the primary slow relaxation of the regular-orbit sets given by $\Psi_{m}^{(\alpha)} \propto t^{-2}$ (see Eq.(6)). We have demonstrated that the universal $\alpha$-channel, attributed for both the chaotic \cite{17} and nonchaotic \cite{14} billiards, is also characteristic of nonintegrable rational polygons. The primary relaxation-motion regime originated from the piece-line parts of the polygonal table is associated with long-living sliding orbits with large collision angles $\varphi$. In the corresponding phase space these orbits are unbounded trajectories (see Fig.4) that move without splitting at angle vertices along Bleher’s principal corridors. Following to the simplest polygonal orbit classification by Gutkin \cite{1}, the regular sliding-orbit sets can be presented by the ”infinite-past-to-infinite-future” trajectories. They ”never” hit vertices, preserve the initial linear momenta, and show a regular behavior in the orbit decay spectra $N_{m}(t, \varphi)$ (see Fig.1). Conversely, the singular orbit sets caused by the ”infinite-past-to-vertex”, the ”vertex-to-infinite-future”, and the diagonal ”vertex-to-vertex” trajectories \cite{42} exhibit pronounced weights $C_{m}(\varphi)$ in the orbit-decay process (shown by high peaks in Fig.1). Eventually, they do not play any significant role in the wall-collision statistics in $m$-gons with small number of vertices limited by, approximately, $3 \leq m \leq 8$, and thus the primary relaxation dominates. This corroborates by our numerical study (analyzed in Table 1 and Fig.1) and, in general, is in accord with studies of the closed polygons by the orbit-wall collision statistics \cite{14} and by the orbit complexity \cite{3}.

When the number of vertices is large, the secondary relaxation with the survival probability $\Psi_{m}^{(\beta)} \propto t^{-\beta-1}$ predominates over the primary relaxation (see Figs. 2,3). The established domain for the decay exponent $1 < \beta < 2$ corresponds to that known \cite{31} for the chaotic SB. Qualitatively, the survival probability function $\Psi_{m}^{(\beta)}$ can be associated with the distribution function for trajectories trapped by the strange attractors, discussed in the theory of the open classical chaotic systems, or with the corresponding waiting-time probability function \cite{43}. With accounting of findings for the decay dynamics on the SB by Fendrik’s group
one can expect that the secondary relaxation is due to the singular trapped orbit sets that move freely along Bleher’s ”hidden corridors”. Similarly to the case of the chaotic billiards, the observation conditions for the secondary relaxation in rational polygons are sensitive to the initial conditions and to the geometrical constrains. Indeed, the $\beta$-relaxation channel turns up to be closed if the initial particle distribution is simulated \[40\] in the 2D collision subspace $\Omega_{cm}$. In the case of the 3D $\Omega_m$ space the secondary relaxation appears to be dynamically stable under the geometrical constraint $m > m_\alpha$, where $m_\alpha = 2\pi R/\Delta$ is given by the $\alpha$-to-$\beta$-relaxation criterium estimated in Eq.(7). As shown, this criterium meets the favorable survival conditions for the regular-motion regime with those induced by rationality of the vortices. The latter are generated by the sliding orbits through the vertex-“ordering” effects and are associated with the vortexlike orbits \[15\]. As follows from Eq.(7), the observation window of such a motion disappears in the limit $\Delta \to 0$, when the vortexlike orbits do not survive in the closed polygons (see Fig.2 in Ref. [15]). Finally, we have demonstrated that the vortexlike orbits become stable in the open rational polygons and visible through the secondary slow relaxation common for the chaotic systems.

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V. APPENDIX. ORBIT-SET COLLISION TIMES

In a given $m$-gon a number of geometrically equivalent walls $k$ is bounded above by

$$q_m = \begin{cases} m & \text{for odd } m; \\ m/2 & \text{for even } m. \end{cases} \tag{A1}$$

The current collision angle $\varphi_{km}$ with a wall $k (= 1, 2...q_m)$ of a $\varphi$-orbit with $\varphi = [0, \pi/2q_m]$ is reduced through the relation $\varphi_{km} = \varphi - \Theta_{km}$ with the help of $\Theta_{km} = [-\pi/2, \pi/2]$ introduced as the lowest angle between the $k$-wall and axis $x$, namely
\[ \Theta_{km} = \frac{\pi}{2q_m} \begin{cases} 
 q_m - 2k + 1 & \text{for odd } q_m; \\
 q_m - 2k & \text{for even } q_m. 
\end{cases} \] (A2)

As shown in Fig.4 for the particular case \( m = 3 \), the estimates for the wall-collision times \( t_{cm}(\varphi) \) are found through summation of numbers of intersections \( n(t, \varphi_{km}) \) for a trajectory, induced by a given \( \varphi \)-set orbit, considered in the correspondent infinite LG lattice, namely

\[ n_{cm}(t, \varphi) \equiv \frac{t}{t_{cm}(\varphi)} = \sum_{k=1}^{q_m} n(t, \varphi_{km}) \] (A3)

The estimation procedure can be exemplified by a relation \( t \cos(\varphi_{13}) = n(t, \varphi_{13})3a_3. \) The latter employes the fact that a distance between the equivalent walls is \( 3a_3 \), where \( a_m = R \cos(\pi/m) \) stands for the apothem in a given 3-gon. This yields

\[ t_{cm}(\varphi) = a_m q_m \left[ \sum_{k=1}^{q_m} \cos(\varphi - \Theta_{km}) \right]^{-1} \] (A4)

where \( q_m \) and \( \Theta_{km} \) are given in Eqs.(A1) and (A2), respectively. Straightforward estimation of Eq.(A4) [44] results in the collision times \( t_{cm}(\varphi) \) given in Eq.(3).
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FIGURE CAPTIONS

Fig. 1. Analysis of the algebraic decay simulated in pentagon (5-gon) and heptagon (7-gon) within the collision $\Omega_{cm}$ space. Symbols - numerical data on the partial weights $C_5^{(exp)}(\varphi)$ and $C_7^{(exp)}(\varphi)$ deduced from the observed spectra of the survived orbits $N_{t_5}^{(exp)}(\varphi)$ and $N_{t_7}^{(exp)}(\varphi)$ with the help of Eq.(6) and simulated for $\varphi$-set orbits with $0 \leq \varphi \leq \varphi_m$ at distinct times $t = 20, 30\tau_{em}$. Line: the overall-collision-anlle weight $C_{cm}^{(exp)}$.

Insert left: Points - data $N_{t_5}$ for the overall survived orbits at late times and their regular-orbit analysis with the help of Eq.(6).

Insert right: Points - data for $\varphi$-set collision time $t_{c5}(\varphi)$ simulated within the reduced domain $0 \leq \varphi \leq \pi/10$. Line - the same predicted in Eq.(6).

Fig. 2. Temporal evolution of the survived orbits in rational polygons with small opening width ($\Delta = 0.05R$) against the reduced time in log-log coordinates. Reduction is given by the help of Eq.(1) for the escape time $\tau_e = 300$, chosen common for all cases. Points: numerical data for decay of the $\Omega_m$ space phase simulated by $N_0 = 10^6$ particles in the $m$-gons (squares) and the correspondent CB (circles).

Fig. 3. Temporal evolution of the survived orbits in rational polygons with large opening width ($\Delta = 0.20R$) against the reduced time in log-log coordinates. Notations are the same as in Fig.2.

Fig.4. Estimation of the $\varphi$-orbit collision time $t_{cm}(\varphi)$ on the bases of Eq.(A3) for the case of $m = 3$. The regular piece-line orbit $a, b, c, d, e...$ is represented by the infinite straight-line trajectory in the triangle LG lattice with the intersection-point sequences $1, 2, 3, 4...n(t, \varphi_{km})$. The equivalent walls $k$, the unreduced collision angles $\varphi_{km}$, and the axillar angles $\Theta_{km}$ are shown.
Table 1. Fitting parameters of the temporal algebraic decay of the collision space $\Omega_{cm}$ simulated in the weakly open $m$-gons with $\Delta = 0.05R$. Notations: $C_{cm}^{(tot)}$ and $C_{cm}^{(reg)}$ correspond to the data on $\varphi$-sets observed in the decay spectra $C_{cm}^{(exp)}(\varphi)$ (see Fig. 1) and averaged over, respectively, all collision angles and with excluding singular-orbit angles manifested by the high peaks; $D_{cm}^{(tot)} = D_{cm}^{(exp)}$ - the overall-set weights of the algebraic tail given in Eq.(3) and derived within the primary relaxation window (see the left insert in Fig.1).
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