INTRODUCTION TO UNIVERSAL ALGEBRA AND CLONES

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Abstract. The purpose of this note is to provide a gentle introduction to basic universal algebra and (abstract) clones.

1. Introduction

In almost every field of pure and applied mathematics, algebras (in a broad sense) arise quite naturally in one way or another. An algebra, typically, is a set equipped with a family of operations on it. So for example the symmetric group of degree five $S_5$ and the ring of integers $\mathbb{Z}$ are both algebras. Structural similarities between important algebras have led to the introduction and study of various types of algebras, such as monoids, groups, rings, vector spaces over a field, lattices, Boolean algebras, and Heyting algebras. A type of algebras is normally specified by a family of operations and a family of equational axioms. We shall call such a specification of a type of algebras an algebraic theory.

Subsequently, various authors have set out to develop a background theory, or a metatheory for a certain type of algebraic theories. The most famous classical example is Birkhoff’s universal algebra [Bir35]. By working at this level of generality, one can prove theorems for various types of algebras once and for all; for instance, the homomorphism theorems in universal algebra (see e.g., [BS81, Section II.6]) generalise the homomorphism theorems for groups to monoids, rings, lattices, etc. A metatheory also provides a method to relate different types of algebras, by means of morphisms between algebraic theories.

In this note, we explain the basics of universal algebra. We shall confine ourselves to the most basic definitions; we focus on presentations of equational theories, the type of algebraic theories universal algebra deals with. We then describe a presentation independent version of them, namely (abstract) clones. This note is based on Sections 2.1 and 2.2 of the author’s thesis [Fuj18].

1 Other examples of metatheories include those of non-symmetric and symmetric operads [May72], PROs and PROPs [ML65], generalised operads [Bur71, Kel92, Her00, Lei04], and monads [EM65, Lin66]; cf. [Fuj18, Fuj19].
2. Universal algebra

Universal algebra [Bir35] deals with types of algebras defined by finitary operations and equations between them. As a running example, let us consider groups. A group may be defined as a set $G$ equipped with an element $e^G \in G$ (the unit), and two functions $i^G : G \to G$ (the inverse) and $m^G : G \times G \to G$ (the multiplication), satisfying the following axioms:

- for all $g_1 \in G$, $m^G(g_1, e^G) = g_1$ (the right unit axiom);
- for all $g_1 \in G$, $i^G(m^G(g_1, g_1)) = e^G$ (the right inverse axiom);
- for all $g_1, g_2, g_3 \in G$, $m^G(m^G(g_1, g_2), g_3) = m^G(g_1, m^G(g_2, g_3))$ (the associativity axiom).\(^2\)

This definition of group turns out to be an instance of the notion of presentation of an equational theory, one of the most fundamental notions in universal algebra.

First we introduce the notion of graded set, which provides a convenient language for our exposition.

**Definition 2.1.** (1) An $(\mathbb{N} \text{-})$graded set $\Gamma$ is a family $\Gamma = (\Gamma_n)_{n \in \mathbb{N}}$ of sets indexed by natural numbers $\mathbb{N} = \{0, 1, 2, \ldots \}$. By an element of $\Gamma$ we mean an element of the set $\prod_{n \in \mathbb{N}} \Gamma_n = \{(n, \gamma) \mid n \in \mathbb{N}, \gamma \in \Gamma_n\}$. We write $x \in \Gamma$ iff $x$ is an element of $\Gamma$.

(2) If $\Gamma = (\Gamma_n)_{n \in \mathbb{N}}$ and $\Gamma' = (\Gamma'_n)_{n \in \mathbb{N}}$ are graded sets, then a morphism of graded sets $f : \Gamma \to \Gamma'$ is a family of functions $f = (f_n : \Gamma_n \to \Gamma'_n)_{n \in \mathbb{N}}$. \(\blacksquare\)

We can routinely extend the basic notions of set theory to graded sets. For example, we say that a graded set $\Gamma'$ is a graded subset of a graded set $\Gamma$ (written as $\Gamma' \subseteq \Gamma$) iff for each $n \in \mathbb{N}$, $\Gamma'_n$ is a subset of $\Gamma_n$. Given arbitrary graded sets $\Gamma$ and $\Gamma'$, their cartesian product (written as $\Gamma \times \Gamma'$) is defined by $(\Gamma \times \Gamma')_n = \Gamma_n \times \Gamma'_n$ for each $n \in \mathbb{N}$. An equivalence relation on a graded set $\Gamma$ is a graded subset $R \subseteq \Gamma \times \Gamma$ such that each $R_n \subseteq \Gamma_n \times \Gamma_n$ is an equivalence relation on the set $\Gamma_n$. Given such an equivalence relation $R$ on $\Gamma$, we can form the quotient graded set $\Gamma/R$ by setting $(\Gamma/R)_n = \Gamma_n/R_n$, the quotient set of $\Gamma_n$ with respect to $R_n$. These notions will be used below.

A graded set can be seen as a (functional) signature. That is, we can regard a graded set $\Sigma$ as the signature whose set of $n$-ary functional symbols is given by $\Sigma_n$ for each $n \in \mathbb{N}$. We often use the symbol $\Sigma$ to denote a graded set when we want to emphasise this aspect of graded sets, as in the following definition.

**Definition 2.2.** Let $\Sigma$ be a graded set.

(1) A $\Sigma$-algebra $A$ is a set $A$ equipped with, for each $n \in \mathbb{N}$ and $\sigma \in \Sigma_n$, a function $[\sigma]^A : A^n \to A$ called the interpretation of $\sigma$.\(^3\) We write such a $\Sigma$-algebra $A = (A, ([\sigma]^A)_{n \in \mathbb{N}, \sigma \in \Sigma_n})$ simply as $(A, [-]^A)$. We sometimes omit the superscript in $[-]^A$.

\(^2\)From these three axioms it follows that for all $g_1 \in G$, $m^G(e^G, g_1) = g_1$ (the left unit axiom) and $m^G(i^G(g_1), g_1) = e^G$ (the left inverse axiom) hold.

\(^3\)Note that we allow the set $A$ to be empty. In traditional universal algebra the underlying set of a $\Sigma$-algebra is usually required to be nonempty.
(2) If \( A = (A, [-]^A) \) and \( B = (B, [-]^B) \) are \( \Sigma \)-algebras, then a \( \Sigma \)-homomorphism from \( A \) to \( B \) is a function \( f: A \to B \) such that for any \( n \in \mathbb{N} \), \( \sigma \in \Sigma_n \) and \( a_1, \ldots, a_n \in A \),

\[
f([\sigma]^A(a_1, \ldots, a_n)) = [\sigma]^B(f(a_1), \ldots, f(a_n))
\]

holds (that is, the diagram

\[
\begin{array}{ccc}
A^n & \xrightarrow{f^n} & B^n \\
[\sigma]^A & \downarrow{f} & [\sigma]^B \\
A & \xrightarrow{f} & B
\end{array}
\]

commutes).

As an example, let us consider the graded set \( \Sigma_{Grp} \) defined as \( \Sigma_{Grp}^0 = \{e\} \), \( \Sigma_{Grp}^1 = \{i\} \), \( \Sigma_{Grp}^2 = \{m\} \) and \( \Sigma_{Grp}^n = \emptyset \) for all \( n \geq 3 \). Then the structure of a group is given by that of a \( \Sigma_{Grp} \)-algebra. Note that to give an element \( e^G \in G \) is equivalent to give a function \( [e]: 1 \to G \) where 1 is a singleton set, and that for any set \( G \), \( G^0 \) is a singleton set. Also, between groups, the notions of group homomorphism and \( \Sigma_{Grp} \)-homomorphism coincide.

However, not all \( \Sigma_{Grp} \)-algebras are groups; for a \( \Sigma_{Grp} \)-algebra to be a group, the interpretations must satisfy the group axioms. Notice that all group axioms are equations between certain expressions built from variables and operations. This is the fundamental feature shared by all types of algebras expressible in universal algebra. The following notion of \( \Sigma \)-term defines “expressions built from variables and operations” relative to arbitrary graded sets \( \Sigma \).

**Definition 2.3.** Let \( \Sigma \) be a graded set. The graded set \( \text{T}(\Sigma) = (\text{T}(\Sigma)_n)_{n \in \mathbb{N}} \) of \( \Sigma \)-terms is defined inductively as follows.

1. For each \( n \in \mathbb{N} \) and \( i \in \{1, \ldots, n\} \),

\[
x_i^{(n)} \in \text{T}(\Sigma)_n.
\]

We sometimes omit the superscript and write \( x_i \) for \( x_i^{(n)} \).

2. For each \( n, k \in \mathbb{N} \), \( \sigma \in \Sigma_k \) and \( t_1, \ldots, t_k \in \text{T}(\Sigma)_n \),

\[
\sigma(t_1, \ldots, t_k) \in \text{T}(\Sigma)_n.
\]

When \( k = 0 \), we usually omit the parentheses in \( \sigma() \) and write instead as \( \sigma \).

An immediate application of the inductive nature of the above definition of \( \Sigma \)-terms is the canonical extension of the interpretation function \([-]\) of a \( \Sigma \)-algebra from \( \Sigma \) to \( \text{T}(\Sigma) \).

**Definition 2.4.** Let \( \Sigma \) be a graded set and \( A = (A, [-]^A) \) be a \( \Sigma \)-algebra. We define the interpretation \( [-]^A \) of \( \Sigma \)-terms recursively as follows.
(1) For each $n \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$, 
\[
[x_i^{(n)}]^A : A^n \to A
\]
is the $i$-th projection $(a_1, \ldots, a_n) \mapsto a_i$.

(2) For each $n, k \in \mathbb{N}$, $\sigma \in \Sigma_k$ and $t_1, \ldots, t_k \in T(\Sigma)_n$, 
\[
[\sigma(t_1, \ldots, t_k)]^A : A^n \to A
\]
maps $(a_1, \ldots, a_n) \in A^n$ to $[\sigma]^A ([t_1]^A(a_1, \ldots, a_n), \ldots, [t_k]^A(a_1, \ldots, a_n))$; that is, the function $[\sigma(t_1, \ldots, t_k)]^A$ is the following composite:
\[
A^n([t_1]^A, \ldots, [t_k]^A) \xrightarrow{\sigma^\ast} A^k \xrightarrow{[\sigma]^A} A.
\]

Note that for any $n \in \mathbb{N}$ and $\sigma \in \Sigma_n$, $[\sigma]^A = [\sigma(x_1^{(n)}, \ldots, x_n^{(n)})]^A$. Henceforth, for any $\Sigma$-term $t$ we simply write $[t]^A$ for $[\sigma]^A$ defined above.

**Definition 2.5.** Let $\Sigma$ be a graded set. An element of the graded set $T(\Sigma) \times T(\Sigma)$ is called a $\Sigma$-equation. We write a $\Sigma$-equation $(n, (t, s)) \in T(\Sigma)^n \times T(\Sigma)^s$ (that is, $n \in \mathbb{N}$ and $t, s \in T(\Sigma)_n$) as $t \approx_n s$ or $t \approx s$.

**Definition 2.6.** A presentation of an equational theory $\langle \Sigma \mid E \rangle$ is a pair consisting of:

- a graded set $\Sigma$ of basic operations, and
- a graded set $E \subseteq T(\Sigma) \times T(\Sigma)$ of equational axioms.

**Definition 2.7.** Let $\langle \Sigma \mid E \rangle$ be a presentation of an equational theory.

1. A model of $\langle \Sigma \mid E \rangle$, or a $\langle \Sigma \mid E \rangle$-model, is a $\Sigma$-algebra $A$ such that for any $t \approx_n s \in E$, $[t]^A = [s]^A$ holds.
2. A homomorphism between models of $\langle \Sigma \mid E \rangle$ is just a $\Sigma$-homomorphism between the corresponding $\Sigma$-algebras.

Consider the presentation of an equational theory $\langle \Sigma_{\text{Grp}} \mid E_{\text{Grp}} \rangle$, where
\[
E_{1_{\text{Grp}}} = \{ m(x_1^{(1)}, e) \approx x_1^{(1)}, \ m(x_1^{(1)}, i(x_1^{(1)})) \approx e \},
\]
\[
E_{3_{\text{Grp}}} = \{ m(m(x_1^{(3)}, x_2^{(3)}), x_3^{(3)}) \approx m(x_1^{(3)}, m(x_2^{(3)}, x_3^{(3)})) \}
\]
and $E_{n_{\text{Grp}}} = \emptyset$ for all $n \in \mathbb{N} \setminus \{1, 3\}$. Clearly, groups are the same as models of $\langle \Sigma_{\text{Grp}} \mid E_{\text{Grp}} \rangle$. Many other types of algebras—indeed all examples we have mentioned in the first paragraph of the introduction—can be written as models of $\langle \Sigma \mid E \rangle$ for a suitable choice of the presentation of an equational theory $\langle \Sigma \mid E \rangle$ (see e.g., [BS81]).

We now describe the machinery of equational logic, which enables us to investigate consequences of equational axioms without referring to their models. We assume that the reader is familiar with the basics of mathematical logic, such as substitution of a term $t$ for a variable $x$ in a term $s$ (written as $s[x \mapsto t]$), simultaneous substitution (written as $s[x_1 \mapsto t_1, \ldots, x_k \mapsto t_k]$), and the notion of proof (tree) and its definition by inference rules.

**Definition 2.8.** Let $\langle \Sigma \mid E \rangle$ be a presentation of an equational theory.
(1) Define the set of $\langle \Sigma \mid E \rangle$-proofs inductively by the following inference rules. Every $\langle \Sigma \mid E \rangle$-proof is a finite rooted tree whose vertices are labelled by $\Sigma$-equations.

$$\begin{align*}
(\text{Ax}) & \quad t \approx_n s \quad \text{(if } t \approx_n s \in E) \\
(\text{Refl}) & \quad t \approx_n t \\
(\text{Sym}) & \quad t \approx_n s \quad s \approx_n t \\
(\text{Trans}) & \quad t \approx_n s \quad s \approx_n u \quad \Rightarrow \quad t \approx_n u \\
(\text{Cong}) & \quad \frac{s \approx_{n} s'}{t_{1} \approx_{n} t_{1}'} \quad \frac{s \approx_{n} s'}{t_{2} \approx_{n} t_{2}'} \\
& \quad \vdots \\
& \quad s[x_{1}^{(k)} \mapsto t_{1}^{(k)}, \ldots, x_{k}^{(k)} \mapsto t_{k}^{(k)}] \approx_{n} s'[x_{1}^{(k)} \mapsto t_{1}'^{(k)}, \ldots, x_{k}^{(k)} \mapsto t_{k}'^{(k)}]
\end{align*}$$

(2) A $\Sigma$-equation $t \approx_n s \in T(\Sigma) \times T(\Sigma)$ is called an equational theorem of $\langle \Sigma \mid E \rangle$ iff there exists a $\langle \Sigma \mid E \rangle$-proof whose root is labelled by $t \approx_n s$.

We write $\langle \Sigma \mid E \rangle \vdash t \approx_n s$ to mean that $t \approx_n s$ is an equational theorem of $\langle \Sigma \mid E \rangle$, and denote by $T \subseteq T(\Sigma) \times T(\Sigma)$ the graded set of all equational theorems of $\langle \Sigma \mid E \rangle$. ■

The assertion $\langle \Sigma \mid E \rangle \vdash t \approx_n s$ says that the $\Sigma$-equation $t \approx s$ is a syntactic consequence of the equational axioms $E$. Its counterpart is the semantic consequence relation $\models$, defined as follows.

**Definition 2.9.** (1) Let $\Sigma$ be a graded set and $A$ be a $\Sigma$-algebra. For any $\Sigma$-equation $t \approx_n s \in T(\Sigma) \times T(\Sigma)$, we write

$$A \models t \approx_n s$$

to mean $[t]^A = [s]^A$.

(2) Let $\langle \Sigma \mid E \rangle$ be a presentation of an equational theory. For any $\Sigma$-equation $t \approx_n s \in T(\Sigma) \times T(\Sigma)$, we write

$$\langle \Sigma \mid E \rangle \models t \approx_n s$$

to mean that for any $\langle \Sigma \mid E \rangle$-model $A$, $A \models t \approx_n s$. ■

Equational logic is known to be both sound and complete, meaning that the two relations $\vdash$ and $\models$ coincide.

**Theorem 2.10.** Let $\langle \Sigma \mid E \rangle$ be a presentation of an equational theory.

(1) (Soundness) Let $t \approx_n s \in T(\Sigma) \times T(\Sigma)$. If $\langle \Sigma \mid E \rangle \vdash t \approx_n s$ then $\langle \Sigma \mid E \rangle \models t \approx_n s$.

(2) (Completeness) Let $t \approx_n s \in T(\Sigma) \times T(\Sigma)$. If $\langle \Sigma \mid E \rangle \models t \approx_n s$ then $\langle \Sigma \mid E \rangle \vdash t \approx_n s$.

**Proof.** The soundness theorem can be shown by a straightforward induction over $\langle \Sigma \mid E \rangle$-proofs.

To prove the completeness theorem, first observe that the graded set $\overline{T} \subseteq T(\Sigma) \times T(\Sigma)$ of all equational theorems of $\langle \Sigma \mid E \rangle$ (Definition 2.8) is an equivalence relation on $T(\Sigma)$, thanks to the rules (Refl), (Sym) and (Trans). Hence we can consider the quotient graded set $T(\Sigma)/\overline{T}$. We claim that for each $n \in \mathbb{N}$, the set $T_n^{(\Sigma \mid E)} = (T(\Sigma)/\overline{T})_n$ has a natural structure of $\langle \Sigma \mid E \rangle$-model.
We start with endowing a Σ-algebra structure on the set $T_n^{(\Sigma \mid E)}$; that is, we define for each $k \in \mathbb{N}$ and each $\sigma \in \Sigma_k$, its interpretation $[\sigma]: (T_n^{(\Sigma \mid E)})^k \rightarrow T_n^{(\Sigma \mid E)}$. This is defined as

$$[\sigma]([t_1]^E, \ldots, [t_k]^E) = [\sigma(t_1, \ldots, t_k)]^E$$

for each $t_1, \ldots, t_k \in T(\Sigma)_n$. To see that it is indeed well-defined, consider the instances of the (CONG) rule where $s = s' = \sigma(x_1^{(k)}, \ldots, x_k^{(k)})$. Observe that in this Σ-algebra, the interpretation of a Σ-term $s \in T(\Sigma)_k$ is given by

$$[s]([t_1]^E, \ldots, [t_k]^E) = [s[x_1^{(k)} \mapsto t_1, \ldots, x_k^{(k)} \mapsto t_k]]^E.$$ 

The Σ-algebra $T_n^{(\Sigma \mid E)} = (T_n^{(\Sigma \mid E)}, [-])$ satisfies all equational axioms of $\langle \Sigma \mid E \rangle$. To see this, notice that if $s \approx_k s' \in E$, then for each $t_1, \ldots, t_k \in T(\Sigma)_n$, the Σ-equation $s[x_1^{(k)} \mapsto t_1, \ldots, x_k^{(k)} \mapsto t_k] \approx_n s'[x_1^{(k)} \mapsto t_1, \ldots, x_k^{(k)} \mapsto t_k]$ is an equational theorem of $\langle \Sigma \mid E \rangle$, by the rules (Ax), (REFL) and (CONG). Hence $[s] = [s']$ holds in $T_n^{(\Sigma \mid E)}$.

Now suppose that for a Σ-equation $t \approx_n s$ we have $\langle \Sigma \mid E \rangle \models t \approx_n s$. Then in particular $T_n^{(\Sigma \mid E)} \models t \approx_n s$, and in particular the images of $[x_1^{(n)}]^E, \ldots, [x_n^{(n)}]^E \in T_n^{(\Sigma \mid E)}$ under the functions $[t]$ and $[s]$ agree. Hence we have

$$[t]^E = [t[x_1^{(n)} \mapsto x_1^{(n)}, \ldots, x_n^{(n)} \mapsto x_n^{(n)}]]^E$$

$$= [t([x_1^{(n)}]^E, \ldots, [x_n^{(n)}]^E)]$$

$$= [s([x_1^{(n)}]^E, \ldots, [x_n^{(n)}]^E)]$$

$$= [s[x_1^{(n)} \mapsto x_1^{(n)}, \ldots, x_n^{(n)} \mapsto x_n^{(n)}]]^E$$

$$= [s]^E,$$

namely $\langle \Sigma \mid E \rangle \models t \approx_n s$. \qed

Before closing this section, we remark that the $\langle \Sigma \mid E \rangle$-model $T_n^{(\Sigma \mid E)}$ used in the above proof is in fact the free $\langle \Sigma \mid E \rangle$-model generated by the $n$-element set $X_n = \{x_1^{(n)}, \ldots, x_n^{(n)}\}$, in the following sense.

**Proposition 2.11.** Let $\langle \Sigma \mid E \rangle$ be a presentation of an equational theory and $n$ be a natural number. Define the function $\eta_{X_n}: X_n \rightarrow T_n^{(\Sigma \mid E)}$ by $\eta_{X_n}(x_i^{(n)}) = [x_i^{(n)}]^E$ for each $i \in \{1, \ldots, n\}$. Given any $\langle \Sigma \mid E \rangle$-model $A = (A, [-])$ and any function $f: X_n \rightarrow A$, there exists a unique homomorphism of $\langle \Sigma \mid E \rangle$-models...
\( g: T_n^{(\Sigma \mid E)} \rightarrow A \) such that \( g \circ \eta_n = f \).

\[
\begin{array}{c}
X_n \xrightarrow{\eta_n} T_n^{(\Sigma \mid E)} \xrightarrow{g} A \\
\uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \\
A \quad A \quad A \quad A
\end{array}
\]

\((\text{sets}) \quad (\langle \Sigma \mid E \rangle \text{-models})\)

**Proof.** The required homomorphism \( g \) can be defined from \( f \) by recursion; the details are omitted. \( \square \)

### 3. Clones

The central notion we have introduced in the previous section is that of **presentation of an equational theory** (Definition 2.6), whose main purpose is to define its **models** (Definition 2.7). It can happen, however, that two different presentations of equational theories define the "same" models, sometimes in a quite superficial manner.

For example, consider the following presentation of an equational theory \( \langle \Sigma_{\text{Grp}}' \mid E_{\text{Grp}}' \rangle \):

\[
\Sigma_{\text{Grp}}' = \Sigma_{\text{Grp}},
\]

\[
E_{1,\text{Grp}}' = \{ m(x_1^{(1)}, e) \approx x_1^{(1)}, \ m(e, x_1^{(1)}) \approx x_1^{(1)}, \ m(x_1^{(1)}, i(x_1^{(1)})) \approx e, \ m(i(x_1^{(1)}), x_1^{(1)}) \approx e \},
\]

\[
E_{n,\text{Grp}}' = E_{n,\text{Grp}} \quad \text{for all} \quad n \in \mathbb{N} \setminus \{1\}.
\]

It is a classical fact that a group can be defined either as a model of \( \langle \Sigma_{\text{Grp}} \mid E_{\text{Grp}} \rangle \) or as a model of \( \langle \Sigma_{\text{Grp}}' \mid E_{\text{Grp}}' \rangle \). Indeed, we may add arbitrary equational theorems of \( \langle \Sigma_{\text{Grp}} \mid E_{\text{Grp}} \rangle \), such as \( i(i(x_1)) \approx x_1, \ i(m(x_1, x_2)) \approx m(i(x_2), i(x_1)) \) and \( x_1 \approx x_1 \), as additional equational axioms and still obtain the groups as the models.

As another example, let us consider the presentation of an equational theory \( \langle \Sigma_{\text{Grp}}'' \mid E_{\text{Grp}}'' \rangle \) defined as:

\[
\Sigma_{0,\text{Grp}}'' = \{e, e'\}, \quad \Sigma_{n,\text{Grp}}'' = \Sigma_{\text{Grp}}' \quad \text{for all} \quad n \in \mathbb{N} \setminus \{0\},
\]

\[
E_{0,\text{Grp}}'' = \{e \approx e'\}, \quad E_{n,\text{Grp}}'' = E_{n,\text{Grp}} \quad \text{for all} \quad n \in \mathbb{N} \setminus \{0\}.
\]

To make a set \( A \) into a model of \( \langle \Sigma_{\text{Grp}}'' \mid E_{\text{Grp}}'' \rangle \), formally we have to specify two elements \([e]\) and \([e']\) of \( A \), albeit they are forced to be equal and play the role of unit with respect to the group structure determined by \([m]\). We cannot quite say that models of \( \langle \Sigma_{\text{Grp}}'' \mid E_{\text{Grp}}'' \rangle \) are equal to models of \( \langle \Sigma_{\text{Grp}} \mid E_{\text{Grp}} \rangle \),
since their data differ; however, it should be intuitively clear that there is no point in distinguishing them.\footnote{In precise mathematical terms, our claim of the “sameness” amounts to the existence of an isomorphism of categories between the categories of \(\langle \Sigma \text{Grp} | E \text{Grp} \rangle\)-models and of \(\langle \Sigma^{\prime\prime} \text{Grp} | E^{\prime\prime} \text{Grp} \rangle\)-models preserving the underlying sets of models, i.e., commuting with the forgetful functors into the category \textbf{Set} of sets.}

A presentation of an equational theory has much freedom in choices both of basic operations and of equational axioms. It is really a presentation. In fact, there is a notion which may be thought of as an equational theory itself, something that a presentation of an equational theory presents; it is called an (abstract) \textit{clone} (see e.g., [Tay93]).

\textbf{Definition 3.1.} A clone \(T\) consists of:

(CD1): a graded set \(T = (T_n)_{n \in \mathbb{N}}\):\footnote{In traditional universal algebra, people often omit \(T_0\).}

(CD2): for each \(n \in \mathbb{N}\) and \(i \in \{1, \ldots, n\}\), an element \(p_i^{(n)} \in T_n\);

(CD3): for each \(k, n \in \mathbb{N}\), a function \(\circ_k^{(n)} : T_k \times (T_n)^k \rightarrow T_n\) whose action on an element \((\phi, \theta_1, \ldots, \theta_k) \in T_k \times (T_n)^k\) we write as \(\phi \circ_k^{(n)} (\theta_1, \ldots, \theta_k)\) or simply as \(\phi \circ (\theta_1, \ldots, \theta_k)\);

satisfying the following equations:

(CA1): for each \(k, n \in \mathbb{N}\), \(j \in \{1, \ldots, k\}\) and \(\theta_1, \ldots, \theta_k \in T_n\),
\[
p_j^{(k)} \circ_k^{(n)} (\theta_1, \ldots, \theta_k) = \theta_j;
\]

(CA2): for each \(n \in \mathbb{N}\), \(\theta \in T_n\),
\[
\theta \circ_n^{(n)} (p_1^{(n)}, \ldots, p_k^{(n)}) = \theta;
\]

(CA3): for each \(l, k, n \in \mathbb{N}\), \(\psi \in T_l\), \(\phi_1, \ldots, \phi_l \in T_k\), \(\theta_1, \ldots, \theta_k \in T_n\),
\[
\psi \circ_k^{(k)} (\phi_1 \circ_k^{(n)} (\theta_1, \ldots, \theta_k), \ldots, \phi_l \circ_k^{(n)} (\theta_1, \ldots, \theta_k)) = (\psi \circ_l^{(k)} (\phi_1, \ldots, \phi_l)) \circ_k^{(n)} (\theta_1, \ldots, \theta_k);
\]

Such a clone is written as \(T = (T, (p_i^{(n)})_{n \in \mathbb{N}, i \in \{1, \ldots, n\}}, (\circ_k^{(n)})_{k, n \in \mathbb{N}})\) or simply \((T, p, \circ)\). \(\blacksquare\)

To understand the definition of clone, it is helpful to look at some pictures known as \textit{string diagrams} (cf. [Cur12, Lei04]). Given a clone \(T = (T, p, \circ)\), let us draw an element \(\theta\) of \(T_n\) as a triangle with \(n\) “input wires” and a single “output wire”:

\[
\begin{array}{c}
\vdots \\
\theta \\
\end{array}
\]

\[
\begin{array}{c}
\vdots \\
\theta \\
\end{array}
\]
The element $p_i^{(n)}$ in (CD2) may also be denoted by

![Diagram](n_{(i-th)})

and $\phi \circ_k^{(n)} (\theta_1, \ldots, \theta_k)$ in (CD3) by

![Diagram](\phi)

Then the axioms (CA1)–(CA3) simply assert obvious equations between the resulting “circuits”. For instance, (CA2) for $n = 3$ reads:

![Diagram](\theta = \theta)

Next we define models of a clone. We first need a few preliminary definitions.

**Definition 3.2.** Let $A$ be a set. Define the clone $\text{End}(A) = (\langle A, A \rangle, p, \circ)$ as follows:

1. **(CD1):** for each $n \in \mathbb{N}$, let $\langle A, A \rangle_n$ be the set of all functions from $A^n$ to $A$;
2. **(CD2):** for each $n \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$, let $p_i^{(n)}$ be the $i$-th projection $A^n \rightarrow A, (a_1, \ldots, a_n) \mapsto a_i$;
3. **(CD3):** for each $k, n \in \mathbb{N}$, $g: A^k \rightarrow A$ and $f_1, \ldots, f_k: A^n \rightarrow A$, let $g \circ_k^{(n)} (f_1, \ldots, f_k)$ be the function $(a_1, \ldots, a_n) \mapsto g(f_1(a_1, \ldots, a_n), \ldots, f_k(a_1, \ldots, a_n))$, that is, the following composite:

$$A^n \xrightarrow{(f_1, \ldots, f_k)} A^k \xrightarrow{g} A.$$

It is straightforward to check the axioms (CA1)–(CA3).

**Definition 3.3.** Let $T = (T, p, \circ)$ and $T' = (T', p', \circ')$ be clones. A clone **homomorphism from $T$ to $T'$** is a morphism of graded sets (Definition 2.1) $h: T \rightarrow T'$ which preserves the structure of clones; precisely,

- for each $n \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$, $h_n(p_i^{(n)}) = p_i^{(n)}$;
for each \( k, n \in \mathbb{N} \), \( \phi \in T_k \) and \( \theta_1, \ldots, \theta_k \in T_n \),
\[
h_n(\phi \circ_k^{(n)} (\theta_1, \ldots, \theta_k)) = h_k(\phi) \circ_k^{(n)} (h_n(\theta_1), \ldots, h_n(\theta_k)).
\]

**Definition 3.4.** Let \( T \) be a clone. A model of \( T \) is a pair \( \mathbf{A} = (A, \alpha) \) consisting of a set \( A \) and a clone homomorphism \( \alpha : T \to \text{End}(A) \).

Let us then define the notion of homomorphism between models. First we extend the definition of the graded set \( \langle A, A \rangle \) introduced in Definition 3.2.

**Definition 3.5.** (1) Let \( A \) and \( B \) be sets. The graded set \( \langle A, B \rangle \) is defined by setting, for each \( n \in \mathbb{N} \), \( \langle A, B \rangle_n \) be the set of all functions from \( A^n \) to \( B \).

(2) Let \( A, A' \) and \( B \) be sets and \( f : A' \to A \) be a function. The morphism of graded sets \( \langle f, B \rangle : \langle A, B \rangle \to \langle A', B \rangle \) is defined by setting, for each \( n \in \mathbb{N} \), \( \langle f, B \rangle_n : \langle A, B \rangle_n \to \langle A', B \rangle_n \) be the precomposition by \( f^n : (A')^n \to A^n \); that is, \( h \mapsto h \circ f^n \).

(3) Let \( A, B \) and \( B' \) be sets and \( g : B \to B' \) be a function. The morphism of graded sets \( \langle A, g \rangle : \langle A, B \rangle \to \langle A, B' \rangle \) is defined by setting, for each \( n \in \mathbb{N} \), \( \langle A, g \rangle_n : \langle A, B \rangle_n \to \langle A, B' \rangle_n \) be the postcomposition by \( g : B \to B' \); that is, \( h \mapsto g \circ h \).

**Definition 3.6.** Let \( T \) be a clone, and \( \mathbf{A} = (A, \alpha) \) and \( \mathbf{B} = (B, \beta) \) be models of \( T \). A homomorphism from \( \mathbf{A} \) to \( \mathbf{B} \) is a function \( f : A \to B \) making the following diagram of morphisms of graded sets commute:

\[
\begin{array}{ccc}
T & \xrightarrow{\alpha} & \langle A, A \rangle \\
\downarrow{} \beta & & \downarrow{} \langle A, f \rangle \\
\langle B, B \rangle & \xrightarrow{\langle f, B \rangle} & \langle A, B \rangle \\
\end{array}
\]

Now let us turn to the relation between presentations of equational theories (Definition 2.6) and clones. We start with the observation that the graded set \( T(\Sigma) \) of \( \Sigma \)-terms (Definition 2.3) has a canonical clone structure, given as follows:

**CD2:** for each \( n \in \mathbb{N} \) and \( i \in \{1, \ldots, n\} \), let \( p_i^{(n)} \) be \( x_i^{(n)} \in T(\Sigma)_n \);

**CD3:** for each \( k, n \in \mathbb{N} \), \( s \in T(\Sigma)_k \) and \( t_1, \ldots, t_k \in T(\Sigma)_n \), let \( s \circ_k^{(n)} (t_1, \ldots, t_k) \) be \( s[x_1^{(k)} \mapsto t_1, \ldots, x_k^{(k)} \mapsto t_k] \in T(\Sigma)_n \).

We denote the resulting clone by \( T(\Sigma) \). In fact, this clone is characterised as the free clone generated by \( \Sigma \), in the following sense.

**Proposition 3.7.** Let \( \Sigma \) be a graded set, and let \( \eta_\Sigma : \Sigma \to T(\Sigma) \) be the morphism of graded sets defined by \( (\eta_\Sigma)_n(\sigma) = \sigma(x_1^{(n)}, \ldots, x_n^{(n)}) \) for each \( n \in \mathbb{N} \) and \( \sigma \in \Sigma_n \). Given any clone \( S = (S, p, \circ) \) and any morphism of graded sets \( f : \Sigma \to S \), there exists a unique clone homomorphism \( g : T(\Sigma) \to S \) such that
Proof. The clone homomorphism $g$ may be defined by recursion (recall that $T(\Sigma)$ was defined inductively) as follows:

1. For each $n \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$, let
   
   $g_n(x_i^{(n)}) = p_i^{(n)}$;

2. For each $k, n \in \mathbb{N}$, $\sigma \in \Sigma_k$ and $t_1, \ldots, t_k \in T(\Sigma)_n$, let
   
   $g_n(\sigma(t_1, \ldots, t_k)) = f_k(\sigma) \circ_k^{(n)} (g_n(t_1), \ldots, g_n(t_k))$.

To check that $g$ is indeed a clone homomorphism, it suffices to show for each $s \in T(\Sigma)_k$ and $t_1, \ldots, t_k \in T(\Sigma)_n$,

$$g_n(s[x_1^{(k)} \mapsto t_1, \ldots, x_k^{(k)} \mapsto t_k]) = g_k(s) \circ_k^{(n)} (g_n(t_1), \ldots, g_n(t_k));$$

this can be shown by induction on $s$. The uniqueness of $g$ is clear. \qed

The construction given in Definition 2.4 is a special case of the above; let $S$ be $\text{End}(A)$.

Recall from Definition 2.8 the graded set $\overline{E} \subseteq T(\Sigma) \times T(\Sigma)$ of equational theorems of a presentation of an equational theory $\langle \Sigma \mid E \rangle$. The graded set $\overline{E}$ is an equivalence relation on $T(\Sigma)$, and hence we may consider the quotient graded set $T(\Sigma)/\overline{E}$ (as we did in the proof of Theorem 2.10). By the rule (Cong), the clone operations on $T(\Sigma)$ induce well-defined operations on $T(\Sigma)/\overline{E}$: that is, $\overline{E}$ is not only an equivalence relation on the graded set $T(\Sigma)$, but it is also a congruence relation on the clone $T(\Sigma)$. In particular, we can define $\circ_k^{(n)}$ on $T(\Sigma)/\overline{E}$ by

$$[\phi]_{\overline{E}} \circ_k^{(n)} ([\theta_1]_{\overline{E}}, \ldots, [\theta_k]_{\overline{E}}) = [\phi(\theta_1, \ldots, \theta_k)]_{\overline{E}}.$$

This makes the graded set $T(\Sigma)/\overline{E}$ into a clone; the clone axioms for $T(\Sigma)/\overline{E}$ may be immediately checked from the existence of a surjective morphism of graded sets $q: T(\Sigma) \rightarrow T(\Sigma)/\overline{E}$ (given by $\theta \mapsto [\theta]_{\overline{E}}$) preserving the clone operations. The resulting clone is denoted by $T(\langle \Sigma \mid E \rangle)$; in words, it is the clone consisting of $\Sigma$-terms modulo equational theorems of $\langle \Sigma \mid E \rangle$. It is also characterised by a universal property.

**Proposition 3.8.** Let $\langle \Sigma \mid E \rangle$ be a presentation of an equational theory, and let

$q: T(\Sigma) \rightarrow T(\langle \Sigma \mid E \rangle)$ be the clone homomorphism defined by $q_n(\theta) = [\theta]_{\overline{E}}$ for each $n \in \mathbb{N}$ and $\theta \in T(\Sigma)_n$. Given any clone $S = (S, p, \circ)$ and a clone homomorphism
$g: T(\Sigma) \rightarrow S$ such that for any $t \approx_n s \in E$, $g_n(t) = g_n(s)$ holds, there exists a unique clone homomorphism $h: T(\Sigma | E) \rightarrow S$ such that $h \circ q = g$.

\[
\begin{array}{c}
T(\Sigma) \\
\downarrow^g \quad \downarrow^h \\
T(\Sigma | E) \\
\downarrow^g \\
S
\end{array}
\]

**Proof.** The clone homomorphism $h$ is given by $h_n((\theta|E)) = g_n(\theta)$; this is shown to be well-defined by induction on $\langle \Sigma | E \rangle$-proofs (see Definition 2.8). The uniqueness of $h$ is immediate from the surjectivity of $q$. □

We can now show that for any presentation of an equational theory $\langle \Sigma | E \rangle$, to give a model of $\langle \Sigma | E \rangle$ is equivalent to give a model of the clone $T(\Sigma | E)$. A model of the clone $T(\Sigma | E)$ (Definition 3.4) can be—by Proposition 3.8—equivalently given as a suitable clone homomorphism out of $T(\Sigma)$; this in turn is—by Proposition 3.7—equivalently given as a suitable morphism of graded sets out of $\Sigma$, which is nothing but a model of the presentation of an equational theory $\langle \Sigma | E \rangle$ (Definition 2.7).

We also remark that every clone is isomorphic to a clone of the form $T(\Sigma | E)$ for some presentation of an equational theory $\langle \Sigma | E \rangle$. Indeed, given any clone $S = (S, p, \circ)$ we can consider its underlying graded set $S$ as a graded set of basic operations, and obtain the surjective clone homomorphism $\varepsilon_S: T(S) \rightarrow S$ extending the identity morphism on $S$ by Proposition 3.7. Define $E_S \subseteq T(S) \times T(S)$ to be the kernel of $\varepsilon_S$, i.e., the graded set of all pairs of elements of $T(S)$ whose images under $\varepsilon_S$ agree. Then we have $S \cong T(S | E_S)$.

The inference rules of equational logic we have given in Definition 2.8 can be understood as the inductive definition of the congruence relation $\mathcal{E} \subseteq T(\Sigma) \times T(\Sigma)$ on the clone $T(\Sigma)$ generated by $E \subseteq T(\Sigma) \times T(\Sigma)$. The notion of clone therefore provides conceptual understanding of equational logic.

We can also shed new light on the soundness and completeness theorem (Theorem 2.10) for equational logic. First we define a variant of the semantical consequence relation $\models$ (Definition 2.9) via the “clone-valued semantics”.

**Definition 3.9.** (1) Let $\Sigma$ be a graded set, $S = (S, p, \circ)$ be a clone and $f: \Sigma \rightarrow S$ be a morphism of graded set. For any $\Sigma$-equation $t \approx_n s \in T(\Sigma) \times T(\Sigma)$, we write

$(S, f) \models_{\rm Clo} t \approx_n s$

iff $g(t) = g(s)$, where $g: T(\Sigma) \rightarrow S$ is the clone homomorphism extending $f$ via Proposition 3.7.

(2) Let $\langle \Sigma | E \rangle$ be a presentation of an equational theory. For any $\Sigma$-equation $t \approx_n s \in T(\Sigma) \times T(\Sigma)$, we write

$\langle \Sigma | E \rangle \models_{\rm Clo} t \approx_n s$

iff for any clone $S = (S, p, \circ)$ and a morphism of graded set $f: \Sigma \rightarrow S$ such that $(S, f) \models_{\rm Clo} t' \approx_{n'} s'$ for all $t' \approx_{n'} s' \in E$, $(S, f) \models_{\rm Clo} t \approx_n s$. ■
Theorem 3.10 (cf. Theorem 2.10). Let \( \langle \Sigma \mid E \rangle \) be a presentation of an equational theory.

1. **(Soundness with respect to the clone-valued semantics)** Let \( t \approx_n s \in T(\Sigma) \times T(\Sigma) \). If \( \langle \Sigma \mid E \rangle \vdash t \approx_n s \) then \( \langle \Sigma \mid E \rangle \models_{\text{Clo}} t \approx_n s \).

2. **(Completeness with respect to the clone-valued semantics)** Let \( t \approx_n s \in T(\Sigma) \times T(\Sigma) \). If \( \langle \Sigma \mid E \rangle \models_{\text{Clo}} t \approx_n s \) then \( \langle \Sigma \mid E \rangle \vdash t \approx_n s \).

**Proof.** The soundness theorem with respect to the clone-valued semantics follows from Proposition 3.8. For the completeness theorem with respect to the clone-valued semantics, consider the clone \( T(\langle \Sigma \mid E \rangle) \) and the morphism of graded set

\[
\Sigma \xrightarrow{\eta} T(\Sigma) \xrightarrow{q} T(\Sigma)/E
\]

(see Proposition 3.7 for the definition of \( \eta_\Sigma \) and Proposition 3.8 for \( q \)); then \( (T(\langle \Sigma \mid E \rangle), q \circ \eta_\Sigma) \models_{\text{Clo}} t \approx_n s \) iff \( \langle \Sigma \mid E \rangle \vdash t \approx_n s \). \( \square \)

Clearly, \( \langle \Sigma \mid E \rangle \models_{\text{Clo}} t \approx_n s \) implies \( \langle \Sigma \mid E \rangle \models t \approx_n s \); the latter amounts to restricting the clone \( S \) in Definition 3.9 to those of the form \( \text{End}(A) \) for some set \( A \). Hence the (original) soundness theorem follows from the soundness theorem with respect to the clone-valued semantics, but observe that the completeness theorem is not an immediate consequence of the completeness theorem with respect to the clone-valued semantics.\(^6\)

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\(^6\)However, one can combine the completeness theorem with respect to the clone-valued semantics with an *embedding theorem for clones*, which claims that every clone \( S \) can be embedded into a product of clones of the form \( \text{End}(A) \), to obtain the completeness theorem. Such an embedding may be obtained, for example, by canonically mapping an arbitrary clone \( S = (S, p, \circ) \) into \( \prod_{n=0}^\infty \text{End}(S_n) \), whose injectivity can be checked by an argument similar to our proof of the completeness theorem (Theorem 2.10).
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