Perfect truncated-metric codes and a union of ternary squares expanding the square lattice

I. Dejter
University of Puerto Rico
Rio Piedras, PR 00936-8377
italo.dejter@gmail.com

L. Fuentes
Universidad Nacional Abierta y a Distancia
Cartagena, Colombia
luis.fuentes@unad.edu.co

and

C. Araujo
Universidad del Atlantico
Barranquilla, Colombia
carlosaraujo@unad.edu.co

Abstract
Perfect truncated-metric codes (PTMC’s) in the n-dimensional grid Λ_n of \( \mathbb{Z}^n \) (0 < n ∈ \( \mathbb{Z} \)) and its quotient toroidal grids were obtained via the truncated distance \( \rho(u, v) \) in \( \mathbb{Z}^n \) given between vertices \( u = (u_1, \ldots, u_n) \in \mathbb{Z}^n \) and \( v = (v_1, \ldots, v_n) \in \mathbb{Z}^n \) as the Hamming distance \( h(u, v) \) in \( \mathbb{Z}^n \) (or graph distance \( h(u, v) \) in \( \Lambda_n \)) if \( |u_i - v_i| \leq 1 \), for all \( i \in \{1, \ldots, n\} \), and as \( n + 1 \), otherwise. While this \( \rho \) is related to the \( \ell_p \) metrics, PTMC’s associated with lattice tilings of \( \mathbb{Z}^n \) were recently worked upon as rainbow perfect dominating sets. Now, PTMC’s are extended to ternary compounds \( \Gamma_n \) obtained by glueing, or locking, ternary \( n \)-cubes along their codimension 1 ternary subcubes. Such compounds may be taken as “alternate reality” graphs, since they offer a third option in each coordinate direction at each vertex. We ascertain the existence of an infinite number of isolated PTMC of radius 2 in \( \Gamma_n \) for \( n = 2 \) and conjecture such existence for \( n > 2 \) with radius \( n \). We ask whether there exists a suitable notion replacing that of quotient toroidal grids of \( \Lambda_n \) for the case of \( \Gamma_n \).

1 Perfect truncated-metric codes

We were motivated by computer architecture problems (cf [1]) associated to signal transmission in a finite field (cf. [10, 11, 14]). Related topics of coding theory and lattice domination are based on the Lee metric arising from the Minkowsky \( \ell_p \) norm [4] with \( p = 1 \). These themes, related to the still unsolved Golomb-Welch Conjecture [13, 15, 16], were investigated via perfect Lee codes [10], diameter perfect Lee codes [14], tilings with generalized Lee spheres [2, 11], perfect dominating sets (PDS’s) [20], perfect-distance dominating sets (PDDS’s) [1] and efficient dominating sets [8].
Let $0 < n \in \mathbb{Z}$. The aforementioned codes or dominating sets are generally realized in lattice tilings of $\mathbb{Z}^n$ whose tiles are translates of a common generalized Lee sphere. A natural follow-up here would be to consider lattice tilings of $\mathbb{Z}^n$ with two different generalized Lee spheres. Focus on such follow-up for $n > 3$ is only possible by modifying the notion of the Lee distance $d$ to that of a truncated distance $\rho$, defined in the next paragraph (not a standard distance, but see Remark 6) and subsequently applied in Subsection 1.3 that restates results of [12]. Furthermore, new graphs $\Gamma_n$ built as compounds of ternary $n$-cubes, presented below in Remark 1 and Sections 2-3 that generalize the lattice graph of $\mathbb{Z}^n$, extend those applications.

Recall that the Hamming distance $h(u,v)$ between vectors $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ in $\mathbb{Z}^n \subset \mathbb{R}^n$ is the number of positions $i \in [n] = \{1, \ldots, n\}$ for which $u_i \neq v_i$. Now, let $\rho : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ be given by:

$$
\rho(u,v) = \begin{cases} 
    h(u,v), & \text{if } |u_i - v_i| \leq 1 \text{ for all } i \in [n]; \\
    n + 1, & \text{otherwise}.
\end{cases}
$$

Given $S \subseteq \mathbb{Z}^n$, we denote by $[S]$ the induced subgraph of $S$ in the $n$-dimensional grid $\Lambda_n$, namely the graph whose vertex set is $\mathbb{Z}^n$ with exactly one edge between each two vertices at Euclidean distance 1. In the above definition of $\rho$, the Hamming distance $h$ may be replaced alternatively by the graph distance in $\Lambda_n$ and is also our choice definition to be used on the graphs $\Gamma_n$ of Sections 2-3. Such graphs may be taken as “alternate reality” graphs, since they offer a third option in each coordinate direction at each vertex.

To each component $H$ of $[S]$, we assign an integer $t_H > 0$ to be understood in the context below as the radius of a truncated sphere with center $H$. For every component $H'$ of $[S]$ in the translation class $\langle H \rangle$ of $H$ in $\mathbb{Z}^n$, we assume $t_{H'} = t_H$. This yields a correspondence $\kappa$ from the set of translation classes $\langle H \rangle$ of components $H$ of $[S]$ into $\mathbb{Z}$ such that $\kappa(\langle H \rangle) = t_H$, for every $H \in \langle H \rangle$.

Let $\rho(u,S) = \min\{\rho(u,s) | s \in S\}$. Let $(H)^{\kappa(H)} = \{u \in \mathbb{Z}^n | \rho(u,H) \leq \kappa(H)\}$. A $\kappa$-perfect truncated-metric code, or $\kappa$-PTMC, is a set $S \subseteq \mathbb{Z}^n$ such that $\forall u \in \mathbb{Z}^n$ there exists a unique $s \in S$ with $\rho(u,s) = \rho(u,S)$ and such that the collection

$$\{(H)^{\kappa(H)} | H \text{ is a connected component of } S\}$$

forms a partition of $\mathbb{Z}^n$. For each component $H$ of $S$, $(H)^{\kappa(H)}$ is referred to as the truncated sphere centered at $H$ (or around $H$) with radius $\kappa(\langle H \rangle)$). Here, $H$ is said to be the truncated center of $(H)^{\kappa(H)}$ and its vertices are said to be its central vertices. Given $0 < t \in \mathbb{Z}$, a $\kappa$-PTMC is said to be a $t$-PTMC if all $\kappa(\langle H \rangle)$’s are equal to $t$. If $|H| = 1$, then $|(H)^t| = \sum_{i=0}^t 2^i \binom{n}{i}$, for $0 \leq t \leq n$, (which is less than the cardinal of the corresponding $\ell_1$ sphere of radius $t$, if $t > 1$ [4]). We note that 1-PTMC’s coincide with PDS’s [20].

**Remark 1.** A binary (resp. ternary) $n$-cube graph $Q_n^i$, succinctly called $n$-cube (resp. $n$-tercube), is a cartesian product $K_i \square \cdots \square K_i$ of $n$ complete graphs $K_i$ with $V(K_i) = \{0, \ldots, i-1\} = F_i$, the field of order $i = 2$ (resp. 3). In Sections 2-3, we extend the study of PTMC’s to edge-disjoint unions $\Gamma_n \supset \Lambda_n$ of triangles; these are connected unions of ternary $n$-cubes to be called ternary $n$-cube compounds, (denomination that make us think of $\Gamma_n$ as binary $n$-cube compounds) and that cannot have non-isolated PDS’s or non-isolated 1-PTMC’s. (However,
the ternary perfect single-error-correcting codes of length \( \frac{3^t - 1}{2} \) [18, 2] are isolated PDS’s, or efficient dominating sets [8] in the ternary \( \frac{3^t - 1}{2} \)-cubes; these also are edge-disjoint unions of triangles, for \( t > 0 \). We will say for \( G = \Gamma_n \) in Sections 2-3 that: an edge-disjoint union \( G \) of triangles has a non-isolated PDS \( S \) if every vertex of \( G \) not in \( S \) is adjacent to just either one vertex of \( G \) (as in PDS’s) or the two end-vertices of an edge of \( G \). Section 3 shows that \( \Gamma_2 \) has an infinite number of isolated 2-PTMC and conjectures that \( \Gamma_n \) has isolated \( n \)-PTMC’s, \( \forall n > 2 \).

![Figure 1: Example accompanying Remark 2](image)

**Remark 2.** We restate some results of [12] in Theorem 11, (resp. Theorems 12-13) below. They assert that PTMC’s in \( \Lambda_n \) that are lattice as in [14] (or lattice-like [1]) exist having every induced component \( H \) with vertex set (a) whose convex hull is an \( n \)-parallelotope (resp. either a 0-cube or an \( (n - 1) \)-cube) and (b) contained in a truncated sphere centered at \( H \) with radius \( n \) (resp. either \( n - 2 \) or \( 1 \)) forming part of the associated lattice tiling of \( Z^n \). In the center-left of Figure 1, such 0-cubes and \( (n - 1) \)-cubes are suggested in black and dark-gray colors for \( n = 3 \) (Theorem 12), representing \( Z^n \) with dark-gray thin-edges and vertices, but dominating edges in thick black trace and other edges of a specific tile in thick dark-gray trace. This is schematized on the upper-right of the figure that represents the assignment of the last coordinate value mod 3 on the projection of this PTMC onto \( Z^{n-1} \). The case \( n = 4 \) of Theorem 13 (the first PTMC which is neither PDS nor PDDS), can be visualized similarly. For example, the dominating edges, red colored in the copy of \( Q_4^2 \subset Z^4 \) in the lower-right of Figure 1 indicate a truncated sphere of radius 2 (resp. 1) at \( (0, 0, 0, 0) \) (resp. \( (1, 1, 1, 0) \)). Also in the center-left of Figure 1, light-gray 3-parallelotopes represent convex hull parts of truncated 3-spheres whose central vertices form a lattice PTMC, the simplest case of Theorem 11. A coloring-graph viewpoint of these constructions is found in [12].
Remark 3. (I) A \(t\)-PTMC (resp. a truncated \(t\)-sphere) of \(\Lambda_n\) is defined like a P DDS \([1]\) (resp. generalized Lee sphere \([11]\)) with \(\rho\) instead of the Lee distance \(d\). (II) \(1\)-PTMC’s of \(\Lambda_n\) (resp. truncated \(1\)-spheres) coincide with PDS’s \([20]\) (resp. generalized Lee \(1\)-spheres). (III) An example \([1]\), also in Remark 10, illustrated in Figure 2 below, justifies that our definition of PTMC above employs translation classes instead of isomorphism classes, for in Figure 2 the truncated spheres have common radius (namely 1) for both translation classes, but that does not exclude the eventuality of an example with differing radii.

Remark 4. Motivation for Theorems 12-13 on building “lattice” \(\kappa\)-PTMC’s whose induced components are \(r\)-cubes (\(0 \leq r \leq n\)) of different dimensions \(r\) comes from: (A) the perfect covering codes with spheres of two different radii in Chapter 19 \([5]\) and (B) a negative answer \([19]\) to a conjecture \([20]\) claiming that the induced components of every \(1\)-PTMC \(S\) in an \(n\)-cube \(Q^2_n\) are all \(r\)-cubes \(Q^2_r\) (not necessarily in the same translation class \([9]\)), where \(r\) is fixed. In fact, it was found in \([19]\) that a \(1\)-PTMC in \(Q^2_{13}\) whose induced components are \(r\)-cubes \(Q^2_r\) of two different values \(r = r_1\) and \(r = r_2\) exists, specifically for \(r_1 = 4\) and \(r_2 = 0\). This seems to be the only known counterexample to the cited conjecture.

1.1 Notation, \(\ell_p\) metrics and periodicity

If no confusion arises, every \((a_1, \ldots, a_n) \in \mathbb{Z}^n\) is expressed as \(a_1 \cdots a_n\). Let \(O = 00 \cdots 0\), \(e_1 = 10 \cdots 0\), \(e_2 = 01 \cdots 0\), \(\ldots\), \(e_{n-1} = 0 \cdots 01\) and \(e_n = 00 \cdots 01\).

Let \(z \in \mathbb{Z}^n\), let \(H = (V, E)\) be a subgraph of \(\Lambda_n\) and let \(H + z\) be the subgraph \(H' = (V', E')\) whose vertex set is \(V' = V + z = \{w \in \mathbb{Z}^n; \exists v \in V\ \text{such that} \ w = v + z\}\) so that \(uv \in E \iff (u + z)(v + z) \in E'\). For example, let \(H\) be induced in \(\Lambda_n\) by the vertices with entries in \(\{0, 1\}\). Then, every translation \(H + z\) of \(H\) in \(\Lambda_n\) (in particular \(H\) itself) is isomorphic to \(Q^2_n\).

Let \(i \in [n]\). Each edge (resp. segment) \(uv\) of \(\Lambda_n\) (resp. \(\mathbb{R}^n \supset \mathbb{Z}^n\)) with \(u \neq v\) is parallel to \(Oe_i \in E(\Lambda_n) \iff u - v \in \{\pm e_i\}\) (resp. \(\iff \exists a \in \mathbb{R} \setminus \{0\}\ \text{with} \ u - v = ae_i\)). A parallelotope \(P\) in \(\mathbb{R}^n\) is a cartesian product of segments parallel to some of the \(Oe_i\’s\ (i \in [n])\). We restrict to \(P\’s\ having their vertices in \(\mathbb{Z}^n\). An edge of \(P\) is a segment of unit length parallel to some \(Oe_i\ (i \in [n])\) separating a pair of vertices of \(\Lambda_n\) in \(P\). Recall that a trivial subgraph of \(\Lambda_n\) is composed by just one vertex. The proof of the following is similar to that of Theorem 1 \([1]\).

Theorem 5. Let \(0 < t \in \mathbb{Z}\). If \(S\) is a \(t\)-PTMC in \(\Lambda_n\), then the convex hull in \(\mathbb{R}^n\) of the vertex set \(V(H)\) of each nontrivial component \(H\) of \([S]\) is a parallelotope in \(\mathbb{R}^n\) whose edges are parallel to some or all of the segments \(Oe_i \in E(\Lambda_n), \ i \in [n]\).

Let \(H\) be a nontrivial induced subgraph of \(\Lambda_n\) with the convex hull of \(V(H)\) in \(\mathbb{R}^n\) as a parallelotope whose edges are parallel to some or all of the segments \(Oe_i \in E(\Lambda_n), \ (i \in [n])\). If exactly \(r\) elements of \([n]\) are such values of \(i\), then we say that \(H\) is an \(r\)-box. Note that \(H\) is a cartesian product \(\Pi_{i=1}^n P^i\), where \(P^i\) is a finite path, \(\forall i \in [n]\), with exactly \(r\) paths \(P^i\) having positive length. Clearly, the convex hull of an \(r\)-box is an \(r\)-parallelotope in \(\mathbb{R}^n\). Also, every \(r\)-cube in \(\Lambda_n\) is an \(r\)-box, for \(r \in \{0\} \cup [n]\).

Let \(W_{n,H,t}\) be the truncated sphere of radius \(t\) around an \(H\) as above, where \(0 \leq t \in \mathbb{Z}\). Then, \(H\) is the truncated center of \(W_{n,H,t}\). A \(t\)-PTMC \(S\) of \(\Lambda_n\) determines a partition of \(\mathbb{Z}^n\) into spheres \(W_{n,H,t}\) with \(H\) running over the components of \([S]\). Such an \(S\) with the
components of \([S]\) obtained by translations from a fixed finite graph \(H\) is said to be a \(t\)-PTMC\([H]\). (This imitates the definition of a \(t\)-PDDS\([H]\) in \([1]\), mentioned in Remark 3(i) above). Let \(S\) be a \(t\)-PTMC\([H]\) and let \(H'\) be a component of \([S]\) obtained from \(H\) by means of a translation. Then \(S\) is said to be a lattice \(t\)-PTMC\([H]\) if and only if there is a lattice \(L \subseteq \mathbb{Z}^n\) such that: \(H''\) is a component of \([S]\) \(\Leftrightarrow H'' = H' + z\), for some \(z \in L\).

**Remark 6.** It is relevant to note the connections of \(\rho\) and the \(\ell_p\) metrics \([4]\): A truncated sphere of truncated radius 1 centered at some vertex \(v\) is a Lee \((p = 1)\) sphere of radius 1, whereas a truncated sphere of truncated radius \(n\) centered at \(v\) is a sphere of radius 1 in the maximum \((p = \infty)\) distance, namely \(W_{n,\{v\},1}\) (which has an \(n\)-dimensional cube for convex hull). Thus, all truncated spheres in this work are spheres in some \(\ell_p\) metric, and the truncated metrics are a convenient way to consider different \(\ell_p\) metrics for the graph components. This observation allows connections with other previous works such as (for just one \(t\)) perfect codes in the maximum metric and in the \(\ell_p\) metrics \([4]\).

**Remark 7.** To finish this subsection, we rephrase graph-coloring facts of \([12]\).

Announced as Theorem 11 below, there is a construction of lattice \(n\)-PTMC’s \(S\) whose induced subgraphs \([S]\) in \(\Lambda_n\) have their components \(H\) with:

(i) vertex sets \(V(H)\) whose convex hulls are \(n\)-boxes in \(\mathbb{R}^n\) with distance 3 to \([S] \setminus H\); and representatives (one per component \(H\)) forming a lattice with generators along the coordinate directions;

(ii) each \(V(H)\) contained in a truncated sphere \((H)^{\kappa(H)}\) whose radius is \(n\).

(We do not know whether similar lattice \(t\)-PTMC’s \(S\) exist with \([S]\) having \(r\)-parallelotopes as their components, for \(r\) fixed such that \(0 < r < n\).)

Extending the meaning of the adjective “lattice” in Subsection 1.2 below, as in \([12]\) we have in Theorems 12-13 that a lattice \(\kappa\)-PTMC \(S\) exists in \(\Lambda_n\) whose induced subgraph \([S]\) has its components \(H\) with vertex sets \(V(H)\):

(i') having both \((n - 1)\)-cubes and 0-cubes as their convex hulls in \(\mathbb{R}^n\);

(ii') contained in truncated spheres \((H)^{\kappa(H)}\) whose radii, namely 1 and \((n - 2)\), correspond respectively to the \((n - 1)\)-cubes and 0-cubes in item (i').

A set \(S \subseteq \mathbb{Z}^n = E(\Lambda_n)\) is periodic if and only if there exist \(p_1, \ldots, p_n\) in \(\mathbb{Z}\) such that \(v \in S\) implies \(v \pm p_ie_i \in S, \forall i \in [n]\). Since each lattice \(t\)-PTMC\([H]\) \(S\) is periodic \([1]\), then every canonical projection from \(\Lambda_n\) onto a toroidal grid \(\mathcal{T}\), i.e. a cartesian product \(\mathcal{T} = C_{k_1p_1} \square C_{k_2p_2} \square \cdots \square C_{k_np_n}\) of \(n\) cycles \(C_{k_ip_i}\) \((0 < k_i \in \mathbb{Z}, \forall i \in [n])\), takes \(S\) onto a \(t\)-PTMC\([H]\) in \(\mathcal{T}\). This observation adapts to respective situations that complement the statements of Theorems 11-13 via canonical projections from \(\mathbb{Z}^n\) onto adequate toroidal grids \(\mathcal{T}\).

**Question 8.** Do lattice \(t\)-PTMC’s \(S\) exist with \([S]\) having \(r\)-parallelotopes as their components, for \(r\) fixed such that \(0 < r < n\)?
1.2 Further specifications

Given a lattice $L$ in $\mathbb{Z}^n$, a subset $T \subseteq \mathbb{Z}^n$ that contains exactly one vertex in each class mod $L$ (so that $T$ is a complete system of coset representatives of $L$ in $\Lambda_n$) is said to be an FR (acronym suggesting “fundamental region”) of $L$. A partition of $\mathbb{Z}^n$ into FR’s of $L$ is said to be a tiling of $\mathbb{Z}^n$. Those FR’s are said to be its tiles.

We extend the notion of lattice $\kappa$-PTMC so that the associated fundamental region (see [3], pg 26), or FR (see above) of every new lattice contains a finite number of (in our applications, just two) members $H$ of each $\langle H \rangle$.

In the literature, existing constructions of lattice $t$-PTMC’s in $\Lambda_n$ ($t < n$) concern just $t = 1$ (see [1, 2, 11, 14]) but there are not many known lattice 1-PTMC’s. For example, [7] shows that there is only one lattice 1-PTMC[$Q_2^2$] and no non-lattice 1-PTMC[$Q_2^2$]. In addition, there is a lattice 2-PDDS[$Q_2^2$] in $\Lambda_3$ arising from a tiling of Minkowsky cited in [1].

**Conjecture 9.** There is no $t$-PTMC lattice in $\Lambda_n$, for $1 < t < n$.

This conjecture has an analogous form for perfect codes in the $\ell_p$ metrics in [4], and together with the conjecture mentioned in Remark 4(B), produces a contrast with the constructions in Theorems 12-13, below.

If $S$ is a periodic non-lattice $t$-PTMC[$H$] in $\Lambda_n$, then there exists $0 < m \in \mathbb{Z}$ and a tiling of $\Lambda_n$ with tiles that are disjoint copies of the vertex set $V(H^*)$ of a connected subgraph $H^*$ induced in $\Lambda_n$ by the union of:

(a) the vertex sets of $m$ disjoint copies $H^1, \ldots, H^m$ of $H$ (components of $[S]$);

(b) the sets $(H^j)^{\kappa(H)}$ of vertices $v \in \mathbb{Z}^n$ with $\rho(v, H^j) \leq t$, for $j \in [m]$, where $(H^1)^{\kappa(H)}, \ldots, (H^m)^{\kappa(H)}$ are pairwise disjoint copies of $(H)^{\kappa(H)}$ in $\mathbb{Z}^n$.

By taking such an $m$ as small as possible, we say that $S$ is a $t$-PTMC[$H; m$].

![Figure 2: FR of a lattice $L_S$ for a 1-PTMC[$Q_1^2; 4$] $S$ in $\Lambda_3$.](image-url)
Remark 10. In Section 5 [1], a non-lattice 1-PTMC[\(Q^2_1; 4\)] \(S\) is shown to exist with a lattice \(L_S\) based on it, each of the FR’s of \(L_S\) containing two copies of \(Q^2_2\) parallel to \(Oe_1\) and two more copies of \(Q^2_2\) parallel to \(Oe_2\), these four copies being components of \([S]\). This is represented in Figure 2, where the convex hulls of the truncated 1-spheres of such four components (in thick black trace) are shaded in dark-gray color and the remaining area completing their convex hull is shaded in light-gray color. A fixed vertex \(v_T\) can be taken in each resulting tile \(T\) so that all the resulting vertices \(v_T\) constitute \(L_S\).

Thus, even for a non-lattice \(t\)-PTMC \(S\) in a \(\Lambda_n\), a lattice can be recovered and formed by selected vertices \(v_T\) in the corresponding tiles \(T\) associated with \(S\). We say that such set \(S\) is a \(lattice t\)-PTMC\([H; m]\), indicating the number \(m\) of isomorphic components of \([S]\) in a typical tile \(T\) in which to fix a distinguished vertex \(v_T\).

We further specify the developments above as follows. A \(t\)-PTMC \(S\) in \(\Lambda_n\) with the components of \([S]\) obtained by translations from two non-parallel subgraphs \(H_0, H_1\) of \(\Lambda_n\) is said to be a \(t\)-PTMC\([H_0, H_1]\). The case of this in Figure 2 clearly is a \(\kappa\)-PTMC, with two translation classes of components of \([S]\) formed by the copies of \(Q^2_2\) parallel to each of the directions coordinates. Here, \(\kappa\) sends those copies onto \(t = 1\). Each tile of this \(\kappa\)-PTMC contains two copies of \(Q^2_2\) parallel to \(Oe_1\) and two copies of \(Q^2_2\) parallel to \(Oe_2\), accounting for \(m = 4\).

More generally, let \(t_i \in [n]\), for \(i = 0, 1\). We say that a set \(S \subseteq V\) is a \((t_0, t_1)\)-PTMC\([H_0, H_1]\) in \(\Lambda_n\) if for each \(v \in V\) there is:

\(\text{(i') a unique index } i \in \{0, 1\} \text{ and a unique component } H^i_v \text{ of } [S] \text{ obtained by means of a translation from } H_i \text{ such that the truncated distance } \rho(v, H^i_v) \text{ from } v \text{ to } H^i_v \text{ satisfies } \rho(v, H^i_v) \leq t_i \text{ and}
\)

\(\text{(ii') a unique vertex } w \text{ in } H^i_v \text{ such that } \rho(v, w) = \rho(v, H^i_v).\)

Even though such a set \(S\) is not lattice as in [14] (or lattice-like [1]), it may happen that there exists a lattice \(L_S \subseteq \mathbb{Z}^n\) such that for \(0 < m_0, m_1 \in \mathbb{Z}\) there exists an FR of \(L_S\) in \(\Lambda_n\) given by the union of two disjoint subgraphs \(H^*_0, H^*_1\), where \(H^*_i\) (\(i = 0, 1\)) is induced in \(\Lambda_n\) by the disjoint union of:

\(\text{(a') the vertex sets of } m_i \text{ disjoint copies } H^{i_1}_i, \ldots, H^{i_{m_i}}_i \text{ of } H_i, (\text{components of } [S]);\)

\(\text{(b') the sets } (H^{j}_{i})^{\kappa(H)} \text{ of vertices } v \in \mathbb{Z}^n \text{ for which } 0 < \rho(v, H^{j}_{i}) \leq t_i, \text{ for } j \in [m_i], \text{ where } (H^{i_1}_i)^{\kappa(H)}, \ldots, (H^{i_{m_i}}_i)^{\kappa(H)} \text{ are pairwise disjoint copies of } H^\kappa(H)) \text{ in } \mathbb{Z}^n.\)

Subsection 1.3 contains a restatement of results of [12], translating its graph-coloring context into our present one. In fact, in Theorem 12 below, a lattice 1-PTMC\([H_0, H_1; m_0, m_1]\) is obtained; in addition, a family of lattice \((t_0, t_1)\)-PTMC\([H_0, H_1; m_0, m_1]\)'s in the graphs \(\Lambda_n\) that extend the lattice 1-PTMC of Theorem 12 is obtained in Theorem 13.

### 1.3 Restatement of known results

A particular case of lattice \(t\)-PTMC\([H]\) is that in which \(H\) is an \(n\)-box of \(\Lambda_n\). For each such \(H\), Theorem 11 below says that there is a lattice \(n\)-PTMC\([H]\) in \(\Lambda_n\). (In [2], \(n\)-boxes of
unit volume in $\Lambda_n$ are shown to determine 1-PDDS$[H]$’s if and only if either $n = 2^r - 1$ or $n = 3^r - 1$.

How to construct a lattice $n$-PTMC$[H]$ $S$ in $\Lambda_n$? From Subsection 1.1 we know that $S$ determines a partition of $\mathbb{Z}^n$ into spheres $W_{n,H',n}$, where $H'$ runs over the components of $[S]$. These spheres must conform a tiling associated to a lattice $L_S \subset \mathbb{Z}^n$ as follows: In each $W_{n,H',n}$, let $b_1b_2 \cdots b_n$ be the vertex $a_1a_2 \cdots a_n$ for which $a_1 + a_2 + \cdots + a_n$ is minimal. We say that this $b_1b_2 \cdots b_n$ is the anchor of $W_{n,H',n}$. The anchors of the spheres $W_{n,H',n}$ form the lattice $L = L_S$. Without loss of generality we can assume that $O$ is the anchor of a $W_{n,H_0,n}$ whose truncated center $H_0$ is a component of $[S]$. Let $c_1c_2 \cdots c_n$ be the vertex $a_1a_2 \cdots a_n$ in $W_{n,H_0,n}$ for which $a_1 + a_2 + \cdots + a_n$ is maximal. Then $L_S$ has generating set $\{(1 + c_1)e_1, (1 + c_2)e_2, \ldots, (1 + c_n)e_n\}$ and is formed by all linear combinations of those $(1 + c_i)e_i$, $(i \in [n])$. This insures that $S$ exists and is lattice [14] (or lattice-like [1]) via $L_S$.

**Theorem 11.** [12] For each $i \in [n]$, let $P_i$ be a path of length $c_i - 2$, parallel to $Oe_i$, and let $H = \Pi_{i=1}^{n}P_i$ be an $n$-box in $\Lambda_n$. Then, there is a lattice $n$-PTMC$[H]$ $S$ of $\Lambda_n$ with minimum $\ell_1$-distance 3 between the components of $[S]$.

**Proof.** An argument for this precedes indeed the statement, and a similar proof is in [12] from a graph-coloring point of view. The claim can be proved alternatively by an additive-group epimorphism technique [1, 14] modified in [12] and containing the tool used to prove Theorems 12-13. □

As in Theorem 11 and commented in Remark 2, Figure 1 displays a representation of the convex hulls of two induced components, in light-gray color, of a lattice $t$-PTMC$[H]$ in $\Lambda_n$, where $t = n = 3$ and $H = Q_0^3$. The components of this PTMC are just 0-cubes, the simplest case of Theorem 11 whose proof, discussed below, is contained in [12] from a graph-coloring perspective.

**Theorem 12.** [12] There exists a lattice 1-PTMC$[Q_2^3, Q_0^2, 2, 2]$ (in particular a PDS) $S$ in $\Lambda_3$ with minimum $\ell_1$-distance 3 between the components of $[S]$.

As also commented in Remark 2, Theorem 12 is illustrated in Figure 1, showing in black color the components of $[S]$ in a typical FR of $L_S$, namely: two copies of $Q_2^3$ and two copies of $Q_0^2$, with the edges in the corresponding truncated 1-spheres shown in thick black trace; the other edges induced in the union of these 4 components are shown in thick dark-gray trace. As also commented in Remark 2, truncated 3-spheres of the 3-PTMC$[Q_0^6]$ resulting from Theorem 11 are shaded in light-gray color. Also, dark-gray color was used to indicate two other copies of $Q_2^3$ appearing in the figure that are components of $[S]$. Notice that the vertices $O, e_1, e_2, e_3$ are indicated in the figure.

To state Theorem 13, let $H_0 = Q_{n-1}^2$, $H_1 = Q_0^2$, $m_0 = m_1 = 2$, $t_0 = 1$, $t_1 = n - 2$.

**Theorem 13.** There exists a lattice $(1, n-2)$-PTMC$[Q_{n-1}^2, Q_0^2, 2, 2]$ $S$ in $\Lambda_n$.

**Remark 14.** From the last observation in Remark 7, it can be deduced that the respective PTMC $S$ covers (via canonical projections) in Theorem

- 11, an $n$-PTMC$[H]$ in toroidal grid $T = C_{c_1k_1} \Box C_{c_2k_2} \Box \cdots \Box C_{c_nk_n}$, $(1 < k_i \in \mathbb{Z}, \forall i \in [n])$;
2 Ternary cube compounds

Let the 2-tercube (or \(\text{tersquare}\)) \(Q^3_2\) be denoted \([\emptyset]\), with vertices given by the 2-tuples \(xy\), \((x, y) \in F_3 = \{0, 1, 2\}\). As a graph, \([\emptyset] = (00,10,20)\Box (00,01,02)\), the cartesian product of two triangles whose vertex sets are both \(F_3\). The 1-sub-tercubes \(Q^3_1\) of the 2-tercube \([\emptyset]\) have vertex sets \(\{0y\}_{y \in F_3}, \{1y\}_{y \in F_3}, \{2y\}_{y \in F_3}, \{x0\}_{x \in F_3}, \{x1\}_{x \in F_3}, \{x2\}_{x \in F_3}\). They will be indicated \(x^0, x^1, x^2, y^0, y^1, y^2\), respectively. To each of these triangles \(t^s\) \((t \in \{x,y\}; s \in F_3)\) of \([\emptyset]\) we glue, or lock, a corresponding tersquare \([s]\) intersecting \([\emptyset]\) exactly in \(t^s\).

This way, six tersquares \([0], \ldots, [2]\) are obtained (that we call \textit{subcentral tersquares}) that intersect \([\emptyset]\) respectively in the triangles \(x^0, \ldots, y^2\). Next, we glue, or lock, to the remaining new triangles (i.e., other than those already in \([\emptyset]\)), a set of nine new tersquares that we denote \([xy], \ldots, [yy]\) and call \textit{corner tersquares}. These nine tersquares share merely a vertex
with $[\emptyset]$. The resulting graph $[[\emptyset]]$, that will be referred to as a 2-hive, contains a total of $1 + 6 + 9 = 16$ tersquares, where the central tersquare $[\emptyset]$ of $[[\emptyset]]$ shares just one triangle with each of the six subcentral tercubes and just one vertex with each of the nine corner tersquares.

The tersquare $[\emptyset]$ is given as a subgraph at the center of Figure 3, and also results in the representation at the lower-right quarter of Table I by identifying the four vertices labelled 22, and also the two vertices labelled 20, resp. 21, resp. 02, resp. 12.

This construction goes ahead with the further iterative gluing of successive tersquares. In the end (or in the limit), a total compound $\Gamma_2$ of glued tersquares results that we call a ternary square compound. Three subgraphs of $\Gamma_2$ are represented in Table I, containing just one 4-cycle (of the 9 in $Q_3^2$) per participating (but not totally shown) glued tersquare. This yields 9 sub-lattice types in $\Gamma_2$ (one per 4-cycle of $Q_3^2$, three partially shown in Table I), each isomorphic to $\Lambda_2$. Thus, $\Gamma_2$ is a superset of each of the copies of $\Lambda_2$ generated by some 4-cycle of $\Gamma_2$. We further specify $\Gamma_2$ after presenting a non-isolated PDS in $[[\emptyset]]$ (Theorem 15).

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Figure 3: Representing a PDS of $[[\emptyset]]$ in $\Gamma_2$
The large connected graph in Figure 3 represents the 2-hive $[[\emptyset]]$, with $[\emptyset]$, $[\emptyset]$ and $[\emptyset]$ in light-blue background, $[\emptyset]$, $[\emptyset]$ and $[\emptyset]$ in yellow background, and the nine tersquares $[\emptyset]$ in green background if $y \neq 1$ and light-gray background if $y = 1$, where the apparent colored areas spatial disposition results in the partial hiding of some of these areas. The thick red edges induce a subgraph of $[[\emptyset]]$ whose vertex set is a non-isolated PDS $S$ (as defined in Remark 1) of $[[\emptyset]]$. The dominating edges are drawn in thick black trace, allowing the reader to verify that all vertices of $[[\emptyset]]$ not in $S$ are dominated as indicated in Remark 1. (It is elementary to verify that no isolated PDS in $[[\emptyset]]$ exists). The remaining edges of $[[\emptyset]]$, not in thick trace, are in dashed trace if they are “exterior” edges of $[[\emptyset]]$ (i.e., those shared by 2-hives other than $[[\emptyset]]$ in $\Gamma_2$) and in thin trace otherwise (i.e., if they are “interior” edges of $[[\emptyset]]$). Motivation for pursuing such a non-isolated PDS arose as a palliative remedy for the impossibility of having here a PDS like the Livingston-Stout PDS in the grid $P_4 \square P_4$ [17], which happens to be the only isolated PDS in grid graphs $P_m \square P_n$, for $2 < \min\{m, n\}$ [6].

![Figure 4: Planar projection of a portion of $\Gamma_2$ larger than in Figure 3](image)

**Theorem 15.** There exists a non-isolated PDS, or 1-PTMC $[Q_1^2, Q_2^2, Q_3^3 \square Q_4^2; 4, 1, 1]$, in the 2-hive $[[\emptyset]] \subset \Gamma_2$.

**Proof.** The claimed PDS in $[[\emptyset]]$ is formed by the vertex sets of the following components (of its induced subgraphs):

(a) the edge $(00, 01)$ of triangle $(00, 01, 02)$ shared by tersquares $[\emptyset]$ and $[\emptyset]$;

(b) the edge $(10, 20)$ of triangle $(00, 10, 20)$ shared by tersquares $[\emptyset]$ and $[\emptyset]$.
(c) the edge \((21, 22)\) of triangle \((20, 21, 22)\) shared by tersquares \([00]\) and \([20]\);

(d) the edge \((00, 02)\) of triangle \((00, 02, 01)\) shared by tersquares \([01]\) and \([01]\);

(e) the 4-cycle \((02, 12, 11, 01)\) of tersquare \([02]\), where its edge \((02, 12)\) (of triangle \(\Delta = (02, 12, 22)\)) is shared (with \(\Delta\)) with tersquare \([22]\);

(f) the triangular prism in \([01]\) formed by triangles \((00, 01, 02)\) and \((20, 21, 22)\) and edges \((00, 20)\), \((02, 22)\) (of triangle \((02, 22, 12)\), shared by tersquare \([22]_0\)) and \((01, 21)\) (of triangle \((01, 21, 11)\), shared by tersquare \([11]_0\)).

As can be verified in Figure 3, the edges departing from these red-edge components cover all other vertices of \([\emptyset]_2\), proving the theorem.

![Figure 5: Components of a 2-PTMC\([Q_2]^{\emptyset}\) of \([\emptyset]_2\) in \(\Gamma_2\)](image)

Figure 3 may be augmented by adding the tersquares \([xy]_{ij}\), with \(i \neq j\) in \(F_3\), etc. One may continue by adding tersquares \([xy]_{ij\ldots kl}\) with \(i \neq j \neq \ldots \neq k \neq l\) (resp. \(m \neq l\))...
\( n \neq \ldots \neq p \neq q \), i.e. not two contiguous equal values under the \( x \)'s (resp. \( y \)'s). Taking \( i, j, \ldots, k, l, m, n, \ldots p, q \) in \( F_3 \) to be contiguously different with just two values, (e.g. 0,1; resp. 0,2; resp. 1,2, in the upper-left; resp. upper-right; resp. lower-left quarter in Table I) is a way of obtaining 3 sub-lattices of \( \Gamma_2 \), each isomorphic to \( \Lambda_2 \), as in Table I. But there is an infinite family of “parallel” sub-lattices in \( \Gamma_2 \) for each of the 9 sublattice types, one per each 4-cycle of a typical tersquare as in the lower-left quarter of Table I. For example, the type with values 0,2 represents “parallel” sub-lattices having “shortest” tersquares \([0], [1], [y_1], [z_1], [z_1]_y, [z_1]_y, [y_1]_x, [y_1]_x, [y_1]_x \), etc., including every \([x_1]_x \ldots [x_1]_x \), \([y_1]_y \ldots [y_1]_y \), and \([x_1]_x \ldots [x_1]_x \).

These sub-lattices can be refined by replacing each edge \( e \) in them by the 2-path closing a triangle with \( e \) and adding more vertices and additional edges out of the new vertices obtained by the said replacement, in order to distinguish the glued or locked 2-tercubes in \( \Gamma_2 \). In particular, Figure 3 is obtained from such a procedure by starting from a plane containing the upper-right quarter of Table I. Figure 4 shows a projection of a partial extension of Figure 3 in such a plane, where tersquare colors are kept as in Figure 3, including if they are projected into a 2-path. Colors here still are light-blue, blue, green and light-gray. Each tersquare in Figure 4, represented by four squares with the common central vertex 11, is just recognizable by its upper-left vertex denomination presence. Of the four such squares, the lower-right one has the corresponding tersquare denomination. The other tersquare denominations in the three remaining squares correspond to the counterpart denominations in the representation of \([0]\).

Similarly, a graph \( \Gamma_n \) may be defined as a compound of glued \( n \)-tercubes along their codimension 1 ternary cubes, for any \( 2 < n \in \mathbb{Z} \). Such a compound may be referred to as a ternary \( n \)-cube compound. The \( n \)-tuples representing the vertices of such \( \Gamma_n \) are given in terms of the \( n \)-tercubes \([ \ldots ] \) that contain such vertices; these \( n \)-tuples are assigned locally by reflection on the \((n-1)\)-subtercubes, as done in Figure 3 for \( n = 2 \), which represents the 2-hive \([0]\) with \([0]\) at its center, and its neighboring tersquares \([0], [1], [2] \) in light-blue background, \([3], [4], [5] \) in yellow background, and the tersquares neighboring those tersquares, namely \([y_1], [z_1], (i, j \in F_3) \), in green and light-gray backgrounds. It can be seen that \( \Gamma_n \) may be considered as a superset of \( \Lambda_n \) in three different ways, as was commented above in relation to Table I for \( n = 2 \). We pose the following.

**Question 16.** Does there exist a non-isolated PDS \( S \) in \( \Gamma_n \), for \( n \geq 2 \)? If so, could \( S \) behave like a lattice, for example by restricting itself to a lattice PDS over any sub-lattice of \( \Gamma_n \), as exemplified in Table I?

### 3 Isolated 2-PTMC’s in the ternary square compound

In this section, we keep working in \( \Gamma_2 \), but slightly modifying the definition of a \( \kappa \)-PTMC by replacing the used Hamming distance \( h \) by the graph distance of \( \Gamma_2 \). It is clear that the Hamming distance in our graph-theoretical context is just the graph distance. Then, we have the following result.

**Theorem 17.** There exists 262144 isolated 2-PTMC’s in the 2-hive \([0]\) \( \subset \Gamma_2 \).

**Proof.** In Figure 5, the nine copies \([y_1], (i, j \in F_3) \), of the tersquare are shown with its edges in thick black trace, against the thin black trace of the remaining edges of the 2-hive \([0]\).
These nine copies happen to be the truncated 2-spheres centered at the vertices of an isolated PTMC of \([\emptyset]\) constituted as a selection of one vertex per yellow-faced 4-cycle in the figure. We may say that these yellow-faced 4-cycles are the "external" 4-cycles of \([\emptyset]\). So, there are \(4^9 = 262144\) 2-PTMC’s in \(\Gamma_2\).

Corollary 18. There exists a 2-PTMC\([Q^2_2]\) in the 2-hive \([\emptyset]\).

Proof. The yellow-faced 4-cycles are copies of \(Q^2_2\) and induce the components of the claimed 2-PTMC-\([Q^2_2]\). □

Figure 6: Analysis of the square (00, 01, 11, 10) in tersquare \([x^2y^2]_{22}\).

Theorem 19. There exists an infinite number of isolated 2-PTMC in \(\Gamma_2\).

Proof. Let us analyze one of the yellow squares in Figure 5, say square (00, 01, 11, 10) in tersquare \([x^2y^2]_{22}\), as represented in Figure 6; the remaining eight yellow squares have similar properties to those to be described, once the notations of local vertices and neighboring tercubes are updated in each case in place of those in the figure. Besides the "external" 4-cycle (00, 01, 11, 10) in Figure 6, whose edges are in black color, we set in red and blue, (green and hazel) colors those additional edges defining a pair of 4-cycles in the respective tercubes \([x^{xy}y]_{202}\) and \([x^{xy}y]_{212}\), \(([x^{xy}y]_{220}\) and \([x^{xy}y]_{221}\))}, namely \{(00, 02, 12, 10), (01, 02, 12, 11)\},
((00, 20, 21, 01), (10, 20, 21, 11)), equally denoted but pertaining to different tersquares, as shown. Since it is seen that the maximum truncated distance in Figure 6 is 2, we conclude that any one of the four vertices of the black 4-cycle dominates the remaining vertices in the other 4-cycles in the figure, in positions in their tersquares similar to those of the yellow cycles in Figure 5, namely the four just mentioned tersquares \([xy20], [xy21], [x2y20], [x2y21]\), where these involved tersquare denominations are set in Figure 6 straddling edges common to two of their 4-cycles, for better reference, twice for the last four mentioned tersquares. This particular case is easily generalized to conclude that each vertex in a yellow square dominates via \(\rho\) in the neighboring tersquares in a form similar to that of \([0]\).

With the same structure as the 2-hive \([0]\), we have the 2-hives of its neighboring 2-hive pair \([[xy],[xy]]\), etc., sharing four tersquares with \([0]\], tersquares denoted \((10, 11, 12)\), both in \([xy]\) and in \([xy]\) and \([xy]\) and \([xy]\). These four tersquares belong both in \([[xy]]\) and \([[xy]]\) to the four respective tercubes \([xy],[xy],[xy],[xy]\). Moreover, the union of these four tercubes constitutes the intersection \([[xy]] \cap [[xy]]\). By considering in each of these two 2-hives the equivalent of the yellow squares of \([0]\) in Figure 5, and selecting in each such yellow square a vertex contributing to the construction of an isolated 2-PTMC of \(\Gamma\), iteration of such a procedure eventually allows the completion of such dominating set. Observe that the argument exposed from the 2-hive pair \([[xy],[xy]]\) is one of twelve cases arising from the 2-hive \([0]\). Iteration of these twelve cases departing from any 2-hive of \(\Gamma\) that is reached in the continuing procedure allows to advance a further step in the construction of such isolated 2-PTMC. □

**Remark 20.** The argument of Theorem 19 can be modified to ascertain the existence of 2-PTMC\([H]\), where \(H\) is a 4-cycle, or the union of yellow squares with a common vertex in different 2-hives, like the eight squares, in Figure 6, at vertex 10 in the tersquares \([xy],[xy],[xy],[xy]\), \([xy],[xy],[xy],[xy]\), \([xy],[xy],[xy],[xy]\), \([xy],[xy],[xy],[xy]\), \([xy],[xy],[xy],[xy]\), \([xy],[xy],[xy],[xy]\), \([xy],[xy],[xy],[xy]\), \([xy],[xy],[xy],[xy]\).\]

**Conjecture 21.** There exists an isolated \(n\)-PTMC in \(\Gamma\), \(\forall n > 2\).

**Question 22.** Does there exist a suitable way of declaring a subset \(S \subset V(\Gamma)\) to be *periodic* or *periodic-like*, that extends the notion of periodicity of \(\Lambda_n\) at the end of Remark 7 to one in \(\Gamma\)? so that a replacement of the notion of “quotient” or “toroidal” graphs of \(\Lambda_n\) can be found that way for \(\Gamma\)?

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