Complex bifurcations in fast-slow climate dynamics

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Abstract. We consider a class of quadratic systems with slow and fast variables, which exhibit complicated bifurcations when the dynamic parameter (physically interpreted as the load on a system) is changing. We prove that this system with any two different structurally stable dynamical regimes bifurcates from the first regime to the second one as a result of load variation. We apply these ideas to study the Earth’s climate system dynamics evolving on a wide range of time scales. We examine how the limitation on carbon-climate feedback in the proposed mathematical climate model can affect bifurcations in climate dynamics. Then, we consider the dynamic model with random parameters for the climate-biosphere coupling to explain why the climate may stay stable over long time intervals. The model shows that climate stability can be explained by mutual annihilation of many independent factors. One of the important consequences is that if biodiversity decreases then the random evolution of the biosphere can lead to global climate changes.

Key words. quadratic systems, dynamical systems, random systems, vector field realization, bifurcation, climate, feedback, carbon, emission, biosphere

AMS subject classifications. 37H20, 37C10, 86A05

1. Introduction. Modeling of physical processes in the climate system leads to difficult problems, involving complicated systems of partial differential equations for biological and chemical processes [17]. There exist climate models with different levels of realism; they can include thousands and even millions of equations, thousands of parameters to adjust. Usually, one investigates these models by computer simulations [16]. However, it is difficult to estimate the reliability of these computations, since it is connected with a difficult mathematical problem on the structural stability of attractors [18, 36]. The theory of linear response of climate systems to perturbations [23] is based on the Ruelle theory of linear response for dynamical systems that holds on the formal hypothesis that the dynamical system is of the type axiom A one. The last fact implies structural stability. However, S. Smale’s A-axiom systems [36] seldom appear in practical applications. The class of structurally stable systems is very narrow; this mainly includes systems with hyperbolic or almost hyperbolic behavior. One can expect, therefore, that the attractors of climate systems are sensitive and unstable, and therefore, they can exhibit complicated bifurcations under small parameter perturbations. Consequently, it is possible that ‘realistic’ Earth System Models (ESMs), involving many pa...
parameters, are not sufficiently reliable. Possibly, an adequate approach is to take into account random fluctuations and study random dynamical systems.

In this paper, we consider a general approach, which allows proving the existence of complex large time behavior and bifurcations for climate models. Indeed, the climate system as a complex system has a large number of interconnected and interacting subsystems, including the following: the atmosphere, the oceans, the biosphere etc. Determination of how the dynamics of these subsystems change to reach equilibria of the entire system is the main problem of so-called conceptual climate models.

There are different types of conceptual climate models. Many of them may be classified as the energy balance climate models. Typically, this is an ordinary differential equation describing energy conservation in the climate system. The most popular model is a \textit{zero-dimensional model} (e.g., the well-known Budyko-Sellers model) \cite{26} based on the theory of blackbody radiation determining global temperature changes due to the difference in incoming and outgoing solar radiation. This difference may be caused by changing of control parameters: surface albedo, greenhouse gas emission, and even the solar constant. The equilibria and the ideas how to find them by the bifurcation theory tools can be found here \cite{11}. Another type of conceptual climate models is \textit{radiative-convective} models. The thermal/radiative balance can be studied through simplified Navier-Stokes equations for the motion of the planet (oceanic or atmospheric) fluids. The analytical analysis of equilibria in these models is presented in \cite{38}. If dynamical variables in conceptual climate models are average values of physical quantities over some large volumes or boxes, that models are called \textit{box models} \cite{22}.

Here we consider conceptual climate models where physical variables may be decomposed as slow and fast modes. Then for large times fast mode dynamics is captured by the slow dynamics on a stable slow manifold of a slow-fast system. The slow variables determine a long-term climate evolution under external factors (e.g., the insolation (Milankovitch factors)) whereas the fast modes may be associated with rapid factors (e.g., daily changes in atmospheric dynamics; intrinsic changes in the biosphere). The simplest example of such parameterization is introduced for the two-box energy balance model \cite{12, 13, 34, 45}:

\begin{align}
C_s \frac{dT_s}{dt} &= -\beta T_s - \eta \gamma (T_s - T_o) + h(t), \\
C_o \frac{dT_o}{dt} &= \gamma (T_s - T_o)
\end{align}

The first equation (1.1) defines the short-term changes of near-surface global mean temperature $T_s$, which is the fast mode. The second equation (1.2) describes the long-term changes of deep-ocean temperature $T_o$ determining the dynamics of the slow component. Respective heat capacities are $C_s$ and $C_o$. The coefficients $\gamma$ and $\beta$ are the climate feedback parameters; the factor $\eta$ characterizes effects of ocean temperature rise on climate feedback; the quantity $h(t)$ is a radiative forcing proportional to the temperature of the system through the feedback parameter $\beta$.

One can extend the model defined by equations (1.1) and (1.2) by considering a few of boxes \cite{8} or making a coupling between climate and Earth’s biosphere through $h$ or other parameters. For example, the plant albedo affects the climate system but, on the other hand,
the plant populations depend on temperature $T_s$. At desert zones, an increase of $T_s$ can lead to plant extinctions while at temperate zones the same effect can produce growth of the plant populations. Thus we can represent $h$ as a sum of an industrial contribution $h_{ind}$ and a biotic contribution $h_{bio}$:

\begin{equation}
    h = h_{ind}(t) + h_{bio}, \quad h_{bio} = \sum_{i=1}^{m} b_k x_k,
\end{equation}

where $b_k$ are coefficients and $x_k$ is the population of the $k$-th plant species. For $x_k$ we can write down dynamical equations, which involve $T_s$ as a parameter

\begin{equation}
    \frac{dx_k}{dt} = x_k X_k(x, T_s, v), \quad x = (x_1, ..., x_m)
\end{equation}

and $v = (v_1, ..., v_n)$ are resources. The resource dynamics is defined by equations

\begin{equation}
    \frac{dv_j}{dt} = V_j(v, x, T_s).
\end{equation}

(see subsection 5.2 for a specific choice of $X_k$ and $V_j$). Then we obtain a coupling with (1.1) and (1.2).

Note that systems with many temperature boxes can involve polynomial nonlinearities up to the 4-th order because the Stephan-Boltzmann law involves $T^4$, see [8].

The other important slow-fast model is the low order model of K.A. Maasch and B. Saltzman: model [24]

\begin{equation}
    \frac{dx}{dt} = -x - y,
\end{equation}

\begin{equation}
    \frac{dy}{dt} = ry - pz + sz^2 - yz^2,
\end{equation}

\begin{equation}
    \frac{dz}{dt} = -qx - qz,
\end{equation}

where $x, y, z$ are the anomalies (deviations from long-term averages) of the total global ice mass, the atmospheric CO$_2$ concentration, and the volume of the North Atlantic Deep Water (NADW), respectively, and $q, r, p, s$ are positive coefficients. This model, studied recently in detail in [9, 35], well explains climate transition with respect to insolation forcing and show different bifurcations. The model is a slow-fast one if $q >> 1$ [9].

The first result can be outlined as follows. Consider quadratic systems of large dimension,

\begin{equation}
    u_t = Au + B(u) + h,
\end{equation}

where $u \in H = \mathbb{R}^N$ is a phase space, $A$ is a linear operator, $B$ is a quadratic operator, and $h$ is an external forcing (we will also call it the load following the terminology from the ecological literature).
It was shown [42] that the dynamics of this slow system is very diverse. Namely, under a choice of $h$ this dynamics can be reduced to another quadratic system with $n$ variables only, where $n << N$. The form of these resulting shorted systems may be practically arbitrary and they exhibit various dynamical phenomena (chaos, Lorenz dynamics for $n = 3$, saddle-node and Andronov-Hopf bifurcations for $n = 2$). Moreover, there are possible bifurcations between these regimes. Recently it was shown that such systems can generate all structurally stable (for example, hyperbolic) dynamics [42]. We obtain thus that there can appear Ruelle-Takens strange attractors [25], Smale horseshoes and other stable chaotic phenomena.

The new result is as follows. We fix all these parameters except the $h$ assuming that we can vary the load $h$. Then the form of the slow system depends on $h$. For any two different structurally stable dynamical regimes (say, the first regime is a periodical cycle and the second one is a strange attractor) there exists a system \(1.9\) such that under a change of $h$ this system bifurcates from the first regime to the second one. For a more rigorous formulation see Theorem 3.1.

These mathematical results can be interpreted as follows. We assume that the dynamics of climate in a conceptual model can vary in a non-predictable manner depending on the load $h$. This load can be directly connected to the carbon emission magnitude [34]. For example, permafrost ecosystems contain more than twice as much carbon as is currently in the atmosphere and are warming six-times as fast as the global mean. As permafrost thaws, some portion of its organic matter will be decomposed by microorganisms, emitting large amounts of greenhouse gases [3]. This permafrost climate feedback is the largest terrestrial feedback to climate change and one of the most likely to occur [39]. Different unexpected scenario, are discussed in section 4.

In the context of our Theorem 3.1 and results on tipping points [2, 21] here arises a key question: Why does climate stays stable over large time intervals? To answer that question, in the second part of the paper we consider the following model:

\[
\frac{dy_i}{dt} = \kappa g_i(y, x),
\]

\[
\frac{dx_j}{dt} = \sum_{l=1}^{m} A_{jl} x_l + \kappa_1 F_j(y, x),
\]

where $i = 1, \ldots, n$, $j = 1, \ldots, p$, and

\[
F_j(y, x) = \sum_{k=1}^{m} b_{jk} f_{jk}(y, x).
\]

In these equations, the unknown vector-valued function $y(t) = (y_1(t), \ldots, y_n(t))$ consists of slow components, the unknown function $x = (x_1, \ldots, x_m)$ consists of fast components, $\kappa, \kappa_1$ are small positive parameters, $g_i, f_{jk}$ are given smooth and uniformly bounded functions, $b_{jk}$ are random coefficients, and the square matrix $A_{jl}$ defines a linear operator $A$, which has a spectrum $\sigma(A)$ such that

\[
\text{Re} \sigma(A) < -\delta_0 < 0.
\]
Then for sufficiently small $\kappa, \kappa_1 > 0$ the system of equations (1.10) and (1.11) has an attracting smooth invariant manifold defined by

\begin{equation}
    x_j = \Phi_j(y, \kappa, \kappa_1) = \kappa_1 \left( \sum_{k=1}^{m} c_{ik} f(y, 0) \right) + \tilde{X}_j(y, \kappa, \kappa_1),
\end{equation}

where

\begin{equation}
    c_{ik} = -\sum_{j=1}^{m} (A^{-1})_{ij} b_{jk},
\end{equation}

the sufficiently smooth functions $\tilde{X}_j(y, \kappa, \kappa_1)$ define small corrections (here $A^{-1}$ stands for a matrix inverse to $A$). As a result, we obtain the following system for slow variables:

\begin{equation}
    \frac{dy_i}{dt} = \kappa g_i(y, \Phi(y, \kappa, \kappa_1)),
\end{equation}

where $\Phi(y, \kappa, \kappa_1) = (\Phi_1(y, \kappa, \kappa_1), ..., \Phi_p(y, \kappa, \kappa_1))$.

The system defined by equations (1.1), (1.2), (1.3), (1.4) and (1.5) is in the general class of systems (1.10) and (1.11) and under some assumptions that system can be reduced to (1.13).

For systems (1.13) we prove an averaging theorem assuming that $c_{ik}$ are random independent parameters. This theorem asserts that if the hyperbolic attractor of the averaged system (1.13) has a low fractal dimension, then the attractor of the original system is close to the attractor of averaged one with probability $P_m$, which is exponentially close to 1 for large $m$.

So, our main idea in the explanation the relative stability of climate is that a large number of independent factors can mutually annihilate. However, this mechanism is violated if there is a rise in the climate feedback strength.

In the first part of the paper, our main technical tool is the method of the realization of vector fields (RVF) proposed by P. Poláčik [27, 28]. The main idea of the RVF method is to consider, instead of a single semiflow, a family of local semiflows $S_t(P)$ depending on some parameters $P$ involved in the problem (see section 2 for more detail).

For systems with random parameters we use arguments from dynamical system theory and the Hoeffding inequality, one of concentration inequalities.

The paper is organized as follows. In section 2 and section 3, we consider the RVF method and show that under changes of the load $\mathcal{h}$ there are possible bifurcations leading from one structurally stable behaviour to another. Section 4 states some simple scenarios of bifurcations that can appear as a result of changes to $\mathcal{h}$. In the remaining part of the paper we investigate systems with random parameters.

2. Method of realization of vector fields. Let us describe the method of the realization of vector fields (RVF) proposed by P. Poláčik [27, 28]. Consider a family of local semiflows $S_t(P)$ in a Banach space $\mathbf{H}$. Suppose these semiflows depend on a parameter $P \in \mathbf{E}$, where $\mathbf{E}$ is another Banach space. Let $B^n(R)$ be the ball $B^n(R) = \{ q : |q| \leq R \}$ in $\mathbf{R}^n$ of radius $R > 0$ centered at 0, where $q = (q_1, q_2, ..., q_n)$ and $|q|^2 = q_1^2 + ... + q_n^2$. We write simply $B^n$ for $R = 1$. Consider a system of differential equations on the ball $B^n(R)$:

\begin{equation}
    \frac{dq}{dt} = Q(q),
\end{equation}
where
\[(2.2)\quad Q \in C^1(B^n), \quad \sup_{q \in B^n(R)} |\nabla Q(q)| < 1.\]

Suppose the vector field \(Q\) is directed strictly inward to the ball \(B^n(R)\) at its boundary \(\partial B^n(R) = \{q : |q| = R\}\):
\[(2.3)\quad Q(q) \cdot q < 0 \quad \text{for} \quad q \in \partial B^n(R).\]

Then equation (2.1) defines a finite dimensional global semiflow on \(B^n(R)\).

**Definition 2.1.** Let \(\epsilon\) be a positive number. We say that the family of the local semiflows \(S_t(P)\) in \(H\) realizes the field \(Q\) with accuracy \(\epsilon\) (briefly, \(\epsilon\)-realizes) on \(B^n\), if there exists a parameter \(P = P(Q, \epsilon, n)\) such that
(i) the semiflow \(S_t(P)\) has a locally invariant manifold \(M_n(P)\). This manifold is defined by a map \(Z : B^n(R) \to H\)
\[(2.4)\quad z = Z(q), \quad q \in B^n(R), \quad z \in H, \quad Z \in C^{1+r}(B^n),\]
where \(r \geq 0;\)
(ii) the restriction of the semiflow \(S_t(P)|_{M_n(P)}\) on \(M_n(P)\) is defined by the system of differential equations
\[(2.5)\quad \frac{dq}{dt} = Q(q) + \tilde{Q}(q, P), \quad \tilde{Q} \in C^1(B^n(R)),\]
where
\[(2.6)\quad |\tilde{Q}(\cdot, P)|_{C^1(B^n(R))} < \epsilon.\]

In this paper all the locally invariant manifolds \(M_n(P)\) are locally attracting.

**Remark.** These manifolds \(M_P\) can be globally attracting, i.e., inertial, positively invariant and locally attracting, or invariant. The theory of invariant and inertial manifolds is well developed, see [4, 5, 7, 10, 14, 44].

We refer to the families \(S_t(P)\) of the semiflows, which are capable to \(\epsilon\)-realize all finite dimensional fields \(Q\) with any accuracy \(\epsilon\), as **maximally dynamically complex families of semiflows** or, for brevity, simply maximally complex semiflows (MCF’s).

According to the definition of structural stability, all finite dimensional dynamics that are stable under small perturbations can be generated by the MCF (up to orbital topological equivalencies).

Recall the basic concept of structural stability introduced by A. Andronov and S. Pontryagin in 1937 [37]. Consider a smooth vector field \(F\) on compact domain \(\mathbb{D}^n\) of \(\mathbb{R}^n\) with a smooth boundary (or on a compact smooth manifold \(M\) of dimension \(n\)). Assume that \(F \in C^1(\mathbb{D}^n)\) and consider all \(\epsilon\)-small perturbations \(\tilde{F}\) such that
\[(2.7)\quad |\tilde{F}|_{C^1(\mathbb{D}^n)} < \epsilon.\]
Consider systems of differential equations $\frac{dx}{dt} = F(x)$ and $\frac{dx}{dt} = F(x) + \tilde{F}(x)$ and suppose that they define global semiflows $S^t_F$ and $S^t_{F+\tilde{F}}$ on $\mathbb{D}^n$. The system $\frac{dx}{dt} = F(x)$ is called structurally stable if there exists an $\epsilon_0$ such that for all positive $\epsilon < \epsilon_0$ trajectories of semiflows $S^t_F$ and $S^t_{F+\tilde{F}}$ are orbitally topologically conjugated (there exists a homeomorphism, which map trajectories of the first system into trajectories of the second one). Roughly speaking, the original system is structurally stable if any sufficiently small $C^1$ perturbations of that system conserve the topological structure of its trajectories, for example, the equilibrium point stays an equilibrium (maybe, slightly shifted with respect to the equilibrium of non-perturbed system), or the perturbed cycle is again a cycle (maybe slightly deformed and shifted).

Note that structurally stable dynamics may be, in a sense, "chaotic". There is a rather wide variation in different definitions of "chaos". In principle, one can use here any concept of chaos. If this chaos is persistent under small $C^1$-perturbations, this kind of chaos occurs in the dynamics of a MCF. Here, following the classical mathematical tradition [1, 18, 25, 31, 32, 33, 37], we focus our attention on compact invariant sets with hyperbolic chaotic dynamics. We use only the following basic property of hyperbolic sets, the so-called persistence [1, 25, 29, 33]. This means that the hyperbolic sets are, in a sense, robust: if (2.1) generates the hyperbolic set $\Gamma$ and $\epsilon$ is sufficiently small, then the dynamics (2.5) also generates another hyperbolic set $\tilde{\Gamma}$. Dynamics (2.1) and (2.5) restricted to $\Gamma$ and $\tilde{\Gamma}$ respectively, are topologically orbitally equivalent (see [1, 18, 29, 33]).

Thus, all hyperbolic sets can appear in the dynamics of a MCF, for example, hyperbolic rest points, cycles and also chaotic hyperbolic sets: the Smale horseshoes, Anosov flows, the Ruelle-Takens-Newhouse chaos, see [1, 18, 25, 29, 33]. Examples of MCF families can be given by some reaction-diffusion equations [6] (in this case, however, invariant manifolds are not locally attracting) and reaction-diffusion systems [41]. Neural and genetic networks present examples of MCF having fundamental applications.

The RVF method can be applied to systems whose dynamics admits a decomposition in slow and fast components.

3. Quadratic systems. Let us consider a class of quadratic systems with slow and fast variables, which exhibit complicated bifurcations when the load $h$ changes. Consider first the system

\begin{equation}
\frac{dy_i}{dt} = (K_i y, y) + B_i y + h_i, \tag{3.1}
\end{equation}

where $y = (y_1(t), ..., y_N(t))$ is a unknown vector function and we, for brevity, use the notation

\begin{align*}
(K_i x, y) &= \sum_{m=1}^{N} K_{ilm} x_l y_m, \\
B_i y &= \sum_{k=1}^{N} B_{ik} y_k,
\end{align*}

where $i = 1, ..., N$.

We will use the following result obtained in [42, 43]. System (3.1) exhibits a maximally complex dynamics, in the sense of the previous section. Parameters are coefficients $K_{ijl}, B_{ik}, h_i$ and $N$. 
More precisely, let us consider a dynamical system defined by (2.1) satisfying (2.3) and \( \epsilon > 0 \). Let \( A_Q \) be an attractor of that system. To \( \epsilon \)-realize the vector field \( Q \) (the system (2.1)) we can take a sufficiently large \( N \), fix coefficients \( K_{ijl} \) satisfying some conditions, and then adjust the matrix \( B \) and the load vector \( h \).

Let \( \epsilon > 0 \) and \( A_{\bar{Q}} \) be an attractor of the system (3.2). To \( \epsilon \)-realize the vector field \( \bar{Q} \) we can take a sufficiently large \( N \), take the same coefficients \( K_{ijl} \) as that above and adjust another matrix \( \bar{B} \) and another load vector \( \bar{h} \).

This construction can be used to show that a quadratic system can exhibit complicated bifurcations between structurally stable dynamics under changes of the load \( h \) and for fixed \( K_{ijl}, N, B \). Let us consider a second dynamical system

\[
\frac{dq}{dt} = \bar{Q}(q),
\]

where \( \bar{Q} \) satisfies the same conditions (2.2) and (2.3) that \( Q \) satisfies.

By results [42] one can obtain the following:

**Theorem 3.1.** Let (2.1) and (3.2) be two systems of differential equations satisfying conditions (2.2) and (2.3) and generating global semiflows with local attractors \( A \) and \( \bar{A} \), respectively. Then for each \( \epsilon > 0 \) there exists a quadratic system (3.1), which \( \epsilon \)-realizes both systems (2.1) and (3.2) by a variation of \( h \) for fixed \( N, K \) and \( B \). If the systems generate structurally stable semiflows, then there exists a quadratic system (3.1), which has a local attractor topologically equivalent to \( A \) for some choice of the load \( h \) and another local attractor \( \bar{A} \) for another choice of \( h \).

**Proof.** Let us take a small \( \epsilon > 0 \) and by [42] construct \( \epsilon \)-realization of those systems by a quadratic systems

\[
\frac{dy_i}{dt} = (K_i y, y) + B^{(l)}_i y + \bar{h}^{(l)}_i,
\]

where for \( l = 1 \) that system \( \epsilon/2 \)-realizes (3.2) and for \( l = 2 \) it \( \epsilon/2 \)-realizes (2.1). Let us consider the system

\[
\frac{dy_i}{dt} = (K_i^{(2)} y, x) + (K_i y, y) + C^{(l)}_i x + \bar{h}_i,
\]

\[
\frac{dx_j}{dt} = A_j x + \bar{h}_j,
\]

where \( i, j = 1, \ldots, N, A, C \) are square matrices and we use notation introduced above,

\[
(K_i^{(p)} x, y) = \sum_{m=1}^{N} K_{ilm}^{(p)} x_l y_m, \quad p = 1, 2, 3.
\]

\[
A_i x = \sum_{k=1}^{N} A_{ik} x_k.
\]
Suppose that the spectrum $\text{Spec } A$ of the matrix $A$ lies in a negative half-plane:

$$\sigma(A) < -\delta_0 < 0.$$ 

Then

$$|x - x^*| < c \exp(-\delta_0 t), \quad x^* = A^{-1}\bar{h}.$$ 

Therefore, for large times we can replace the system (3.4) and (3.5) by system (3.1), where

$$B_{ij}(x^*) = \sum_{l=1}^{N} K_{ilj} x^*_l + C^{(1)}_{ij},$$

and

$$h_i = \bar{h}_i + C_i x^*.$$ 

For each $i$ consider the linear maps $g_i: x^* \rightarrow z$ defined on $\mathbb{R}^N$ by relations

$$z_j = \sum_{l=1}^{N} K^{(2)}_{ilj} x^*_l.$$ 

Suppose that the ranges of all these maps have dimension $N$. It is clear that for generic coefficients $K_{ilj}$ this property is fulfilled. Then we see that for an appropriate choice $x^* = x^{(1)}$ and $x^* = x^{(2)}$ we obtain systems (3.3) with $l = 1$ and $l = 2$, respectively; this proves the first assertion of Theorem 3.1. The second one follows from the first and the definition of structural stability.

In the next section, we will consider a simple bifurcation scenario of such kind.

4. Simple bifurcations scenarios. In this section, we return to conceptual climate models and consider how the limitation on climate feedbacks (for example, restrictions on greenhouse gas emission) can affect climate bifurcations. We will show that there exist different scenarios, for example:

A. Restrictions on greenhouse gas emission postpones an inevitable catastrophe; a bifurcation occurs but later;

B. Restrictions on greenhouse gas emission allows us to avoid a climate bifurcation;

C. Restrictions on greenhouse gas emission lead us to a climate bifurcation.

To show the possibility of these effects, we use a simple dynamic model involving a fast variable $x \in \mathbb{R}^m$ (for example, a near surface temperature anomaly, as in [34]) and slow variables $q \in \mathbb{R}^n$, respectively:

$$\frac{dx}{dt} = -B(q)x + f(x,q),$$

$$\frac{dq}{dt} = \epsilon(g(x,q) + eh),$$

where $h, \epsilon$ are parameters, where $0 < \epsilon << 1$. The parameter $h$ defines the intensity of different greenhouse gases emission and $e \in \mathbb{R}^m$ is a non-zero vector with non-negative components,
that defines the direction of that emission (for example, \( e_1 \) can correspond to emission of \( \text{CO}_2 \), \( e_2 \) can correspond to emission of black carbon, and \( e_3 \) to emission of methane \( \text{CH}_4 \)). We assume that \( f, g \) are smooth functions such that \( f = O(x^2) \) as \( x \to 0 \). The bifurcations in the \( x \) dynamics can arise at \( q \) such that \( \text{Det} \ B(q) = 0 \). Suppose that the set of root of the equation (4.1) \( \text{Det} \ B(q) = 0 \) consists of two connected components, \( M_1 \) and \( M_2 \). Outside bifurcation points, equation (4.1) implies that for large times \( t \) the fast variables are defined by \( x = 0 \) and for slow variables we then have

\[
\frac{dq}{dt} = \epsilon(g(0, q) + eh). 
\]

For \( |g| << h \) scenarios A-C depend on the locations of \( M_1 \) and \( M_2 \) and the direction vector \( e \). These different scenarios A, B, C are illustrated in Figure 1.

![Figure 1. Scenario of different bifurcations depending on intensity and direction of the human impact \( h \).](image)

To see the mechanisms of scenarios A, B, and C, let us consider the simplest case \( g = 0 \). Scenario A is possible if the human impact coefficient \( h \) is diminished but the emission direction vector \( e \) does not change (for example, emission of all greenhouse gases diminishes equally). Then the right line 1 attains the manifold \( M_1 \) later (see Figure 1). The scenario B arises when \( h \) is constant but \( e \) changes (for example, the carbon emission is restricted but the methane emission increases). Then instead of the right line 1 we obtain the right line 2. The scenario C appears by the same mechanism but when the right line 2 transfers to the right line 3 and the right line 1 transfers to the right line 3, respectively.

In the next section we consider systems with random parameters to explain why the climate may stay stable over long periods of time.
5. Systems with random parameters. We consider systems (1.13), which arise, in a natural way, from systems decomposed in slow and fast variables. We will use the following notation. We denote by $E X$ the expectation of a random quantity $X$, and by $Var X$ its variance. Moreover, $Pr[A]$ denotes the probability of a random event $A$. In this section, we formulate general theorems on averaging with respect to the parameters that are applicable to fast-slow climate models.

5.1. A class of system. We consider the following systems of differential equations:

\begin{equation}
\frac{dy_i}{dt} = F_i(y, \Phi(y)),
\end{equation}

where $i = 1, \ldots, n$, $y(t) = (y_1(t), \ldots, y_n(t))$ is an unknown vector-function, and $\Phi = (\Phi_1, \ldots, \Phi_p)$, $\Phi_l(y)$ are functions, which will be defined below. Let $\mathbb{B}^n$ be a compact subdomain of $\mathbb{R}^n$ with a smooth boundary $\partial \mathbb{B}^n$. We suppose that $F_i(y, \Phi)$ are smooth functions uniformly bounded as are the first and second derivatives with respect to all variables $y, \Phi$:

\begin{equation}
|F_i|_{C^2(\mathbb{B}^n \times \mathbb{R}^p)} < C_F,
\end{equation}

here $C_F$ is a positive constant.

We assume, moreover, that the functions $\Phi_l(y)$ are linear combinations of other functions with random coefficients $c_{ij}$:

\begin{equation}
\Phi_l(y) = m^{-1} \sum_{j=1}^{m} c_{ij} f_{ij}(y),
\end{equation}

We suppose that the $f_{ij}$ have uniformly bounded derivatives

\begin{equation}
|f_{ij}|_{C^2(\mathbb{B}^n)} < C_f,
\end{equation}

where a positive constant $C_f$ is uniform in $i, j$.

For (5.1) we set the initial data

\begin{equation}
y(0) = y^{(0)}.\end{equation}

Let the following assumptions hold:

Assumption 5.1. Let $c_{ij}$ be independent random quantities such that $Ec_{ij} = \bar{c}$, and, moreover, almost surely

\[c_{ij} \in (-R_0, R_0),\]

where $R_0 > 0$ is a constant uniform in $m$.

Together with system (5.1) we consider the corresponding averaged system:

\begin{equation}
\frac{d\bar{y}_i}{dt} = \bar{F}_i(y),
\end{equation}
where
\[
(5.7) \quad \bar{F}_i(y) = F_i(\bar{y}, \bar{\Phi}_1(y), \ldots, \bar{\Phi}_p(y)),
\]
and where \(i = 1, \ldots, n\), \(y(t) = (y_1(t), \ldots, y_n(t))\) is an unknown vector-function, and \(\bar{\Phi}_i(y)\) are averages of functions \(\Phi_i(y)\) over random parameters:
\[
(5.8) \quad \bar{\Phi}_i(y) = \bar{c}m^{-1} \sum_{j=1}^m f_{ij}(y)
\]

We assume that there holds the following condition:
\[
(5.9) \quad \bar{F}(y) \cdot e(y) < 0 \quad \forall y \in \partial \mathbb{B}^n,
\]
where \(e(y)\) is a normal vector to the boundary \(\partial \mathbb{B}^n\) at the point \(y\) directed inward on the domain \(\mathbb{B}^n\). For the system (5.6) we set the same initial data (5.5). Condition (5.9) implies that the Cauchy problem (5.5) and (5.6) defines a global semiflow on the domain \(\mathbb{B}^n\).

5.2. Example of a model. Let us consider an example.

A simple climate model is Budyko–Sellers energy balance system. It is defined by the one-dimensional system [11]:
\[
(5.10) \quad \frac{dT}{dt} = \lambda^{-1}(-e\sigma T^4 + \frac{\mu_0 I_0}{4}(1 - A)),
\]
where \(\lambda\) is thermal inertia, \(T\) is the averaged surface temperature, \(t\) is time, and \(A\) is the albedo of the surface. The left term characterizes the time-dependent behavior of the climate system. On the right hand side, the first term is the outgoing emission and the second term represents the incoming solar radiation. Generally, the incoming solar radiation to the Earth’s surface should depend on the total solar radiation incident on the earth \(I_0\), and the solar constant \(\mu_0\), as well as surface albedo. On the other side, the outgoing emission depends on the fourth power of temperature, the effective emissivity \(e\) and a Stefan-Boltzmann constant \(\sigma\).

This system can be coupled with a global ecosystem model as follows. The complete averaged albedo \(A\) can depend on a biosphere state. We restrict ourselves, for simplicity, to only plant species. Suppose there are \(m >> 1\) plant species which share a few of resources \(v_1, \ldots, v_n\) and \(x_1(t), \ldots, x_m(t)\) are corresponding plant species populations. Consider a typical model [20]. We consider the following system of equations:
\[
(5.11) \quad \frac{dx_i}{dt} = x_i(-\mu_i + \phi_i(v) - \gamma_i x_i), \quad i = 1, \ldots, m,
\]
\[
(5.12) \quad \frac{dv_k}{dt} = D_k(S_k - v_k) - \sum_{i=1}^M b_{ki} x_i \phi_i(v), \quad k = 1, \ldots, n.
\]
Here \(x = (x_1, x_2, \ldots, x_m)\) are unknown species abundances, \(v = (v_1, \ldots, v_n)\) is a vector of unknown resource amounts, where \(v_k\) is the resource of the \(k\)-th type consumed by all ecosystem
species, $\mu_i$ are the species mortalities, $D_k > 0$ are resource turnover rates, $S_k$ is the supply of the resource $v_k$, and $b_{ik} > 0$ is the content of $k$-th resource in the $i$-th species. The coefficients $\gamma_i > 0$ define self-limitation effects [30].

We consider general $\phi_j$ which are bounded, non-negative and Lipshitz continuous

(5.13) \[ 0 \leq \phi_j(v) \leq C_+, \quad |\phi_j(v) - \phi_j(\tilde{v})| \leq L_j|v - \tilde{v}|, \]
and

(5.14) \[ \phi_k(v) = 0, \quad \text{for all } k, \quad v \in \partial \mathbb{R}_+^m \]
where $\partial \mathbb{R}_+^m$ denotes the boundary of the hyperoctant $\mathbb{R}_+^m = \{v : v_j \geq 0, \forall j\}$. Moreover, we suppose that

(5.15) \[ \frac{\partial \phi_k(v)}{\partial v_j} \geq 0, \quad \text{for all } k, j, \quad v \in \partial \mathbb{R}_+^m. \]
This assumption means that as the amount of the $j$-th resource increases all the functions $\phi_l$ also increase.

Conditions (5.14) and (5.13) can be interpreted as a generalization of the well known von Liebig law, where

(5.16) \[ \phi_k(v) = r_k \min \left\{ \frac{v_1}{K_{k1} + v_1}, \ldots, \frac{v_m}{K_{km} + v_m} \right\} \]
where $r_k$ and $K_{kj}$ are positive coefficients, and $k = 1, \ldots, M$. The coefficient $r_k$ is the maximal level of the resource consumption rate by the $k$-th species and coefficients $K_{ki}, i = 1, \ldots, M$ define the sharpness of the consumption curve $\phi_k(v)$.

A simple way to couple climate subsystem (5.10) and the ecosystem model defined by (5.11) and (5.12) is to suppose that the resource supply parameters $S_k$ depends on the surface temperature $T$. Moreover, we can suppose the albedo is a linear function of $x_i$:

(5.17) \[ A = A(x) = A_0 + m^{-1} \sum_{j=1}^{m} c_j x_j. \]

Suppose that species populations $x_i$ are fast, while $T$ and resources $v_k$ are slow. Such a situation arises if, for example, $\gamma_i >> 1$. Then one can show that for large times there exists a slow locally attracting invariant manifold and $x_i(t) \approx X_i(T, v)$, where $X_i(T, v)$ are time averaged species populations for fixed $T, v$.

Then we obtain the following system:

(5.18) \[ \frac{dT}{dt} = \lambda^{-1}(-e \sigma T^4 + \frac{\mu_0 I_0}{4}(1 - A_0 - m^{-1} \sum_{j=1}^{m} c_j X_j(T, v))), \]

(5.19) \[ \frac{dv_k}{dt} = D_k(S_k(T) - v_k) - \sum_{i=1}^{m} b_{ki} X_i(T, v) \phi_i(v), \quad k = 1, \ldots, n. \]
This system lies in our class defined by (1.10) and (1.11). We suppose that coefficients $c_k$ are random quantities.
5.3. Main result for systems with random parameters. The main result describes a connection between the attractor of the original system and its averaged analogue.

Theorem 5.2. Suppose condition (5.9) holds and that averaged system (5.6) defines a global dissipative semiflow on $\mathbb{B}^n$. Moreover, let us assume that averaged system (5.6) has an hyperbolic attractor $\bar{A}$. Then with probability $Pr_A$ the original system (5.1) also defines a global dissipative semiflow on $\mathbb{B}^n$, which has an attractor $A$ topologically equivalent to $\bar{A}$. The probability $Pr_A$ satisfies the inequality

$$Pr_A > 1 - C_1 n \exp \left( - C_2 m \epsilon^2 - n \ln \epsilon \right),$$

where $C_1, C_2$ are positive constants uniform in $m$ and the value $\epsilon > 0$ does not depend on $m$.

The structurally stable system are seldom found in real applications (if we exclude the cases $n = 1$ and $n = 2$, where they are generic). According to basic result of S. Smale, for dimensions $n > 2$ structurally stable systems are not generic. To overcome this difficulty, we consider an approach, which allows us to show that solutions of the original system stay in a small neighborhood of a local attractor of the corresponding averaged system.

Namely, we state the second theorem concerning Lyapunov functions. The stability of many dynamical regimes can be proved by using such functions. Recall that $L(y)$ is a Lyapunov function of a system $dy/dt = F(y)$ in a domain $V \subset \mathbb{R}^n$ if $L$ is at least $C^1$ smooth and $L(y(t))$ does not increases along trajectories $y(t)$ of the system:

(5.20)
$$\nabla L(y) \cdot F(y) \leq 0, \quad y \in V.$$

For example, if $y^*$ is a stable rest point of the system, then often one can construct a $L(y)$ close to a quadratic form, which is Lyapunov function in a small neighborhood $V$ of $y^*$ and

(5.21)
$$\nabla L(y) \cdot F(y) \leq c|y - y^*|^2, \quad y \in V$$

for some $c > 0$.

Let us formulate a theorem.

Theorem 5.3. Suppose condition (5.9) holds and that the averaged system (5.6) has a Lyapunov function such that

(5.22)
$$\nabla L(y) \cdot F(y) \leq -\epsilon, \quad y \in V$$

where $V$ is an open subdomain of $\mathbb{R}^n$ with a compact closure, and moreover,

$$|L|_{C^2(V)} < C_L$$

for a positive constant $C_L$. Then with the probability $Pr_{L, \epsilon}$ the original system (5.1) also has a Lyapunov function such that

(5.23)
$$\nabla L(y) \cdot F(y) \leq -\epsilon/2, \quad y \in V.$$  

The probability $Pr_{L, \epsilon}$ satisfies the inequality

$$Pr_{L, \epsilon} > 1 - \bar{C}_1 \exp \left( - \bar{C}_2 m \epsilon^2 - \ln \epsilon \right),$$

where $\bar{C}_1, \bar{C}_2$ are positive constants uniform in $m$. 

This theorem can be applied as follows. Let \( \tilde{A} \) be a local attractor of the original system, \( \mathcal{V}(A) \) be an open set, which is a subset of the attraction basin of \( \tilde{A} \) and such that \( \text{dist}(A, \mathcal{V}(A)) > \delta \), where \( \delta > 0 \). Assume there exists a Lyapunov function \( L_A(y) \) such that

\[
H_A(y) = \nabla L(y) \cdot \bar{F}(y) \leq -\epsilon(\delta),
\]

for all \( y \in \mathcal{V}(A) \) and some \( \epsilon(\delta) > 0 \). Then with probability \( P_{rL, \epsilon} \) all the trajectories \( y(t, y(0)) \) of the original system such that its starting point \( y(0) \) lies in \( \mathcal{V}(A) \) converge to a \( \delta \)-small neighborhood of \( \tilde{A} \):

\[
\text{dist}(y(t, y(0)), \tilde{A}) < \delta, \quad \forall \ t > T_0(\epsilon).
\]

However, it is impossible to prove that the original system with a probability close to 1 has an attractor topologically equivalent to \( \tilde{A} \).

5.4. Applications. Let us apply Theorem 5.2 and Theorem 5.3 to a system defined by (5.18) and (5.19). In the general case this system is complicated. We suppose that the \( c_i \) are random independent quantities such that \( Ec_i = \bar{c} \). Moreover, we apply the approximation studied in [20, 40]. We assume that the turnovers satisfy \( D_k > 1 \). Then

\[
v = S_k - \tilde{S}_k, \quad 0 < \tilde{S}_k < \text{const}D^{-1},
\]

and

\[
X_j(T, v) = \gamma_i^{-1}(\phi_i(S(T)) - \mu_i)_+ + O(D^{-1}),
\]

where \( S = (S_1, ..., S_n) \) and \( f_+ = \max(f, 0) \). Suppose that all species \( X_j \) survive and have positive abundances. Then

\[
X_j = \gamma_i^{-1}(\phi_i(S(T)) - \mu_i)_+ := U_j(T) > 0,
\]

and (5.18) reduces to

\[
\frac{dT}{dt} = \lambda^{-1}(-c\sigma T^4 + \frac{\mu_0 I_0}{4}(1 - A_0 - m^{-1} \sum_{j=1}^{m} c_j U_j(T))).
\]

We apply Theorem 5.2 and Theorem 5.3, with \( p = 1 \) and

\[
\Phi_1 = m^{-1} \sum_{j=1}^{m} c_j U_j.
\]

The averaged system has the form

\[
\frac{dT}{dt} = \lambda^{-1}(-c\sigma T^4 + \frac{\mu_0 I_0}{4}(1 - A_0 - \bar{c}w(T))),
\]

where

\[
w(T) = \sum_{j=1}^{m} U_j(T).
\]
Let all $\phi_i(S)$ be uniformly bounded, $\phi_i < a$ for all $i = 1, \ldots, m, W$. Then we find that, with a probability exponentially close to 1, there exists a Lyapunov function defined by

$$L(T) = -\frac{c\sigma T^5}{5} + \frac{\mu_0 I_0}{4}((1 - A_0)T - W(T)).$$

where

$$W(t) = \int_0^t w(s)ds.$$

Local minima of this function correspond to steady points (local attractors) of the averaged system, and local maxima to saddle points of that system. If $\bar{c}$ is small enough, we have only a single local attractor $T = \bar{T}_e$. Our theorems assert that the original system then also has (with a probability close to 1) a single local attractor $T = T_e(m)$ and $|T_e(m) - \bar{T}_e| \to 0$ as $m \to \infty$.

The situation dramatically changes if the condition $\phi_i < a$ is violated, say, one species dominates or if $m$ is small. Then it is impossible to guarantee that $|T_e(m) - \bar{T}_e| \to 0$. This means that if biodiversity decreases then random evolution of the biosphere can lead to climate changes.

6. Conclusions. Many complex systems, in particular, climate models, include slow and fast components. According to classical results [5], large time dynamics of fast modes are captured by a dynamics of slow modes on a slow invariant manifold. It is well known that even low dimensional systems exhibit complex bifurcations [9, 19, 35]. Moreover, such models exhibit multistationarity, i.e., existence of many stationary states that, according to [8], provides the climate stability under variations of insolation.

It is clear, however, that realistic climate models should include thousands of variables that evolve with different rates. What can be observed in the dynamics of such systems?

In the first part of the paper, it is shown that systems with a large number of fast components can exhibit practically arbitrary bifurcations (even if these systems are quadratic). This fact allows us to show that different scenarios of climate catastrophes under human impact are possible, even if we restrict greenhouse gas emission.

Why, however, was the climate system stable over long periods of time in the past? To answer this question, we assumed that parameters of fast subsystems are random and mutually independent. Under such assumptions, we prove a general theorem on connection between attractors of averaged and original systems. If the attractor $\bar{A}$ of the averaged system has a low fractal dimension then, with a probability close to 1, the attractor of the original system is close to $\bar{A}$. We think that this result may have applications for many different fields such as global network systems with unknown parameters, foodwebs, gene networks etc.

So, the climate stability can be explained by the mutual annihilation of many independent factors. For example, if biota has a large diversity, then one can expect that climate-biota interaction does not lead to a catastrophe. However, it is obvious that now we are dealing with an entirely different situation.

7. Appendix. In this section, constants $c$ and $C_i$ can depend on system parameters but are uniform in $m$ for large $m$. Note that we sometimes denote different constants by the same index if it does not lead to confusion.
**Probabilistic estimates.** Let us fix some points \( y^{(k)} \in \mathbb{R}^n \), where \( k = 1, 2, ..., M \) and \( M \) is an positive integer, which will be adjusted later. Let us define the events \( A_{\epsilon, i}(k) \) by

\[
A_{\text{out}, \epsilon, i}(k) = \{|\tilde{F}_i(y^{(k)}) - F_i(y^{(k)}, \Phi(y^{(k)})| > \epsilon/4\},
\]

\[
A_{\epsilon, i}(k) = \neg A_{\text{out}, \epsilon, i}(k),
\]

where \( \neg B \) denotes the negation of \( B \).

The next auxiliary lemma is elementary but useful.

**Lemma 7.1.** One has

\[
\Pr \left[ \prod_{k=1}^{M} \prod_{i=1}^{n} A_{\epsilon, i}(k) \right] \geq 1 - \sum_{k=1}^{M} \sum_{i=1}^{n} \Pr [A_{\text{out}, \epsilon, i}(k)].
\]

**Proof.** That lemma can be proved by de Morgan’s rule.

Furthermore, we use Chernoff bounds to estimate \( \Pr [A_{\text{out}, \epsilon, i}(k)] \).

**Lemma 7.2.** For sufficiently small positive \( \epsilon < \epsilon_0 \) and sufficiently large \( m > m_0(\epsilon) \) one has

\[
\Pr [A_{\text{out}, \epsilon, i}(k)] < c_1 \exp(-c_2 m \epsilon^2), \quad \forall i = 1, ..., n, \quad k = 1, ..., M
\]

where \( c_1, c_2 \) are positive constants uniform in \( M \) and \( m \).

**Proof.** Let us estimate differences \( \Phi_i - E\Phi_i \). To this end, let us fix an index \( i \) and index \( k \) and introduce \( X_j \) by

\[
X_j = c_{ij} f_{ij}(y^{(k)}), \quad X = \sum_{j=1}^{m} X_j.
\]

Due to our **Assumption 5.1** on \( c_{ij} \), we have that \( X_j \) are independent random variables. Moreover, taking into account that \( C^1 \) - norms of \( f_{ij} \) are uniformly bounded we have

\[
|X_j| < C,
\]

where

\[
C = R \max_{i,j,k}(|f_{ij}(y^{(k)})| + |\nabla_y f_{ij}(y^{(k)})|).
\]

Therefore, according to Hoeffding’s inequality (see [15])

\[
\Pr(|X - EX| > \epsilon) < 2 \exp(-m \epsilon^2 / C).
\]

Consequently, for each \( \epsilon > 0 \) we obtain

\[
\Pr[|\Phi_i - E\Phi_i| > \epsilon] < c_1 \exp(-c_2 m \epsilon^2),
\]

where \( l = 1, ..., p \) and \( c_1, c_2 > 0 \) can be taken uniform in \( M \) and \( \epsilon > 0 \). Consider the events

\[
E_{\epsilon,l} = \{|\Phi_l - E\Phi_l| < \epsilon\}.\]
Then, due to Lemma 7.1 and equation (7.6),

\[
\Pr \left[ \prod_{l=1}^{p} B_{\varepsilon,l} \right] \geq 1 - p c_{8} \exp(-c_{2} m \varepsilon^{2}).
\]

Now we use conditions (5.2) to find

\[
(7.7) \quad \Pr[|F_i(y^{(k)}) - F_i(y^{(k)}) - E\Phi(y^{(k)})| > \varepsilon] \geq 1 - p c_{3} \exp(-c_{2} m \varepsilon^{2}).
\]

That estimate, uniform in \( i, k \), proves the lemma.

Let us define now the events \( A_{\epsilon,i,j}(k) \) by

\[
(7.8) \quad A_{\epsilon,i,j}(k) = \{|F_{ij}(y^{(k)}) - F_{ij}(y^{(k)})| > \epsilon/4n\},
\]

where

\[
\bar{F}_{ij}(y) = \frac{\partial F_{i}(y)}{\partial y_j}, \quad F_{ij}(y) = \frac{\partial F_{i}(y, \Phi(y))}{\partial y_j},
\]

and

\[
(7.9) \quad A_{\epsilon,i,j}(k) = \text{Not } A_{\epsilon,i,j}(k).
\]

There holds the following Lemma:

**Lemma 7.3.** For sufficiently small \( \varepsilon < \varepsilon_{0} \) and sufficiently large \( m > m_{0}(\varepsilon) \) one has

\[
(7.10) \quad \Pr [A_{\epsilon,i,j}(k)] < c_{3} \exp(-c_{4} m \varepsilon^{2}), \quad \forall i,j = 1, \ldots, n, \quad k = 1, \ldots, M
\]

where \( c_{3}, c_{4} \) are positive constants uniform in \( M \) and \( m \).

The proof of Lemma 7.3 repeats the same arguments used in the proof of Lemma 7.2 so do not present it.

**Demonstrations of Theorem 5.2 and Theorem 5.3.** First we prove Theorem 5.2.

**Proof.** We use Lemma 7.1, Lemma 7.2 and Lemma 7.3 and the following auxiliary construction. The domain \( \mathbb{B}^{n} \) has the dimension \( n \) therefore we can cover it by \( N(\varepsilon) \sim (\varepsilon)^{-n} \)
balls \( \Omega_{\varepsilon,k} \) of the radius \( \varepsilon \) centered at some points \( y^{(k)} \in \varepsilon \). Here \( r \) is a positive constant uniform in \( \varepsilon \). We denote the union of all those balls by \( U_{\varepsilon} \), it is an open neighborhood of \( \mathbb{B}^{n} \).

Let us consider the perturbation \( \tilde{F}(y) = F(y, \Phi(y)) - F(y) \) and estimate the \( C^{1} \) norm of \( \tilde{F} \) on \( U_{\varepsilon} \). Suppose that all events defined by (7.1) and (7.8) take place. Then

\[
(7.11) \quad \tilde{F}(y^{(k)}) + |\nabla_{y} \tilde{F}(y^{(k)})| < \epsilon/2, \quad k = 1, \ldots, N(\varepsilon).
\]

Then, due to conditions (5.2) on \( F \), and definition of \( \tilde{F} \) we have

\[
|\tilde{F}|_{C^{2}(\mathbb{B}^{n})} < C_{3},
\]

where \( C_{3} \) is independent of \( m \). That inequality and (7.11) imply

\[
(7.12) \quad |\tilde{F}(y) + |\nabla_{y} \tilde{F}(y)| < \epsilon/2 + C_{4} \varepsilon, \quad y \in U_{\varepsilon},
\]
where \( C_4 \) is constant. We set \( r = 1/2C_4 \). Note that

\[
\max_{y \in \partial B_n} \tilde{F}(y) \cdot e(y) < -\delta_0 < 0. \tag{7.13}
\]

Then by Lemma 7.2, Lemma 7.3, the estimates (7.12) and (7.13) we see that for sufficiently small \( \epsilon > 0 \)

\[
\max_{y \in \partial B_n} F(y, \Phi(y)) \cdot e(y) < -\delta_0/2 < 0. \tag{7.14}
\]

Let us apply now the definition of structural stability and Theorem on Persistence of Hyperbolic sets [33]. Then for sufficiently small \( \epsilon > 0 \) the attractor of the original system is topologically equivalent to the attractor of the averaged systems. Note that \( \epsilon \) does not depend on \( m \) and it is defined by averaged system only.

Furthermore, we compute the probability that all the events defined by (7.11) take place by Lemma 7.1, Lemma 7.2 and Lemma 7.3. This finishes the proof.

**Proof of Theorem 5.3**

**Proof.** We apply the same idea used in the previous proof. The domain \( V \) can be covered by \( N(re) \sim (re)^{-n} \) balls \( \Omega_{\epsilon, k} \) of the radius \( \epsilon \) centered at some points \( y^{(k)} \in \). Here \( r \) is a positive constant uniform in \( \epsilon \).

Let us introduce the functions

\[
\tilde{H}(y) = \nabla_y L(y) \cdot \tilde{F}(y), \quad H(y) = \nabla_y L(y) \cdot F(y).
\]

Consider the events

\[
\mathcal{H}_{\text{out}, \epsilon}(k) = \{|H(y^{(k)}) - \tilde{H}(y^{(k)})| > \epsilon/4\}, \tag{7.15}
\]

\[
\mathcal{H}_\epsilon(k) = \neg \mathcal{H}_{\text{out}, \epsilon}(k). \tag{7.16}
\]

Suppose that all events defined by (7.16) take place. Then

\[
|H(y^{(k)}) - \tilde{H}(y^{(k)})| < \epsilon/4, \quad \forall \ k = 1, ..., N(\epsilon).
\]  

\[
|H(y^{(k)}) - H(y)| < \text{Lip}_H \ |y^{(k)} - y|, \tag{7.17}
\]

Now we use the estimate

where \( \text{Lip}_H \) is a Lipshitz constant of \( H \). Let us estimate that constant. By definition of \( H \) one has

\[
\frac{\partial H}{\partial y_k} = m^{-1} \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} \frac{\partial (L_{f_{ij}})}{\partial y_k}.
\]

Due to Assumption 5.1

\[
\left| \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} \frac{\partial (L_{f_{ij}})}{\partial y_k} \right| < mnc_1 R_0,
\]

where \( c_1 \) is a constant.
where
\[
c_1 = \max_{i,j,y \in V} (|f_{ij}(y)||\nabla L(y)| + |\nabla f_{ij}(y)||L(y)|).
\]
The same estimate holds for the Lipshitz constant of $\bar{H}$. Therefore, (7.17) and (7.18) give
\[
(7.19) \quad \sup_{y \in V} |H(y) - \bar{H}(y)| < \epsilon/4 + rC_3\epsilon,
\]
where $C_3 > 0$ is a constant uniform in $m$. Let us set $r = 1/4C_3$. Then condition (5.22) of Theorem 5.3 and equation (7.19) show that equation (5.23) is satisfied. Furthermore, to complete the proof, we compute the probability that all the events defined by equation (7.11) take place by estimates analogous to Lemma 7.1, Lemma 7.2 and Lemma 7.3.

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