Poisson formula for a family of non-commutative Lobachevsky spaces

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Dedicated to the memory of Fridrikh Izrailevich Karpelevich

Abstract

We define an analog of the Poisson integral formula for a family of the non-commutative Lobachevsky spaces. The \( q \)-Fourier transform of the Poisson kernel is expressed through the \( q \)-Bessel-Macdonald function.

1 Introduction

The classical Lobachevsky space \( L^3 \) can be identified with 3d hyperboloid

\[
L^3 = \{ x_0^2 - x_1^2 - x_2^2 - x_3^2 = 1, \ x_0 > 0 \},
\]

equipped with the hyperbolic metric. It can be represented as the set of the second order positive definite Hermitian matrices

\[
x = \begin{pmatrix} x_0 - x_1 & x_2 - ix_3 \\ x_2 + ix_3 & x_0 + x_1 \end{pmatrix}, \quad \det x = 1.
\]

It means that \( L^3 \) is the quotient space \( SU_2 \backslash SL_2(\mathbb{C}) \). In this way the classical Lobachevsky space is a particular example of the symmetric spaces.

We introduce the horospheric coordinates \((H, z, \bar{z})\) on \( L^3 \)

\[
\begin{cases}
x_0 = \frac{1}{2}(|z|^2H + H + H^{-1}), \\
x_1 = \frac{1}{2}(-|z|^2H + H - H^{-1}), \\
x_2 = \frac{i}{2}H(z + \bar{z}), \\
x_3 = -\frac{i}{2}H(z - \bar{z}).
\end{cases}
\]

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Let \( \Omega \) be the Laplace-Beltrami operator on \( \mathbf{L}^3 \) corresponding to the metric that comes from the embedding (1.1). In the horospheric coordinates it has the form

\[
\Omega = \frac{1}{2} H^2 \frac{\partial^2}{\partial H^2} + \frac{3}{2} H \frac{\partial}{\partial H} + 2 H^{-2} \frac{\partial^2}{\partial z \partial \bar{z}}. \tag{1.2}
\]

Consider the equation

\[
\left( \frac{1}{2} \Omega + \frac{1}{4} \right) F_\nu(\bar{z}, H, z) = \frac{\nu^2}{4} F_\nu(\bar{z}, H, z), \quad \nu \geq 0. \tag{1.3}
\]

The non-negative solutions of this equation are described in the following way. Let

\[
\mathbf{C}^3 = \{ x_0^2 - x_1^2 - x_2^2 - x_3^2 = 0, \quad x_0 > 0 \}.
\]

be the upper pole of the cone. It can be identified with the Hermitian matrices with \( \det x = 0 \) and \( x_0 \geq 0 \). The horospheric coordinates \((\alpha, \xi, \bar{\xi})\) on \( \mathbf{C}^3 \) take the form

\[
\begin{cases}
  x_0 = \frac{1}{2} \alpha (|\xi|^2 + 1), \\
  x_1 = \frac{1}{2} \alpha (1 - |\xi|^2), \\
  x_2 = \frac{\alpha}{2} (\xi + \bar{\xi}), \\
  x_3 = -\frac{\alpha}{2} (\xi - \bar{\xi}).
\end{cases}
\]

We identify the absolute \( \Xi \) of \( \mathbf{L}^3 \) with the section \( \alpha = \text{const} \) of \( \mathbf{C}^3 \) completed with the point \((z = \infty)\). Then solutions of (1.3) can be represented as the Poisson integral

\[
F_\nu(\bar{z}, H, z) = \int_{\Xi} \mathcal{P}(\xi - \bar{z}, H, \xi - z) d\mu(\xi, \bar{\xi}), \tag{1.4}
\]

where

\[
\mathcal{P}(\xi - \bar{z}, H, \xi - z) = \left( (H^{-1} + |\xi|^2 H)^{-\nu - 1} \right)
\]

is the Poisson kernel and \( d\mu(\xi, \bar{\xi}) = \phi(\xi, \bar{\xi}) d\xi \bar{\xi} \) is a Borel distribution \( \Xi \). The classical theorem states that:

i. non-negative solutions exist for \( \nu \geq 0 \) and have the representation (1.4);

ii. \( \phi(\xi, \bar{\xi}) \) is uniquely reconstructed from \( F_\nu(\bar{z}, H, z) \).

This theorem was generalized by Karpelevich and Furstenberg on arbitrary symmetric spaces of the non-compact type [1, 2].

We investigate an analog of this representation for a family of non-commutative Lobachevsky spaces constructed in Ref. [3]. We use also another form of (1.4). Let \( \Phi_\nu(\bar{s}, H, s) \) be the Fourier transform of \( F_\nu(\bar{z}, H, z) \) with respect to \((\bar{z}, z)\). It satisfies the ordinary differential equation

\[
\left( \frac{1}{4} H^2 \frac{d^2}{dH^2} + \frac{3}{4} H \frac{d}{dH} - H^{-2} \bar{s}s + \frac{1}{4} \right) \Phi_\nu(\bar{s}, H, s) = \frac{\nu^2}{4} \Phi_\nu(\bar{s}, H, s). \tag{1.6}
\]

The solutions to (1.6) decreasing for \( H \to 0 \) are the functions

\[
\Phi_\nu(\bar{s}, H, s) = \frac{\pi}{\Gamma(\nu + 1)} H^{-1} K_\nu(2 \sqrt{\bar{s}s} H^{-1})(\bar{s}s)^{\frac{\nu}{2}} \psi(\bar{s}, s), \tag{1.7}
\]

where \( K_\nu \) is the Bessel-Macdonald function, and \( \psi_\nu(\bar{s}, s) \) is the Fourier transform of \( \phi(\xi, \bar{\xi}) \).

The family of the non-commutative Lobachevsky spaces depends on the deformation parameter \( 0 < q < 1 \) and \( \delta = 0, 1, 2 \). The classical (commutative) limit corresponds to \( q = 1 \). The discrete parameter \( \delta \) is responsible for the form of the Laplace-Beltrami operator. Our results only partly reproduce the classical situation. We just construct an analog of the both representations (1.4) and (1.7) for a space of functions on the non-commutative absolute and its Fourier dual. In the limit \( q \to 1 \) we come to (1.4) and (1.7). Some kernels on the unit disc were considered in Ref. [4].
2 Non-commutative Lobachevsky spaces (NLS).

1. General definition.

The description of NLS is based on an analog of the horospherical coordinates. Let $L_{\delta,q}$ be an associative $\ast$-algebra over $\mathbb{C}$ with the unit and three generators $(z, H, z^\ast)$, $H^\ast = H$, $(z)^{\ast} = z^\ast$, and the commutation relations depending on two parameters $q \in (0, 1)$ and $\delta = 0, 1, 2$

$$Hz = q^\delta zH, \quad z^\ast H = q^\delta H z^\ast, \quad z^\ast z = q^{2-2\delta} z z^\ast - q^{-\delta} (1 - q^2) H^{-2}. \quad (2.1)$$

To eliminate the ambiguities related to the non-commutativity we consider only the ordered monomials putting $z^\ast$ on the left side, $z$ on the right side and keeping $H$ in the middle of the monomials:

$$w(m, k, n) = (z^\ast)^m H^k z^n.$$

The symbol $\hat{f}(z^\ast, H, z)\dagger$ denotes that all monomials are ordered

$$\hat{f}(z^\ast, H, z)\dagger = \sum_{m, k, n = -\infty}^{\infty} a_{m, k, n} (z^\ast)^m H^k z^n, \quad a_{m, k, n} \in \mathbb{C}.$$

For technical reasons we consider another type of the generators $x = Hz$, $x^\ast = z^\ast H$. They satisfy the commutation relations

$$Hx = q^\delta xH, \quad x^\ast H = q^\delta H x^\ast, \quad x^\ast x = q^2 x x^\ast + q^\delta (1 - q^2). \quad (2.2)$$

In the definition of $L_{\delta,q}$ we assume that (2.3) are the formal series. We also consider the "self-conjugate" monomials such as $H^\nu$ and $(x^\ast x)^\nu$, where $\nu > 0$.

We define the non-commutative cone $C_q$ as the associative $\ast$-algebra with the unit and the three generators $(\zeta, \alpha, \zeta^\ast)$, $\alpha^\ast = \alpha$, $(\zeta)^{\ast} = \zeta^\ast$ that satisfy the commutation relations

$$\alpha\zeta = \zeta\alpha, \quad (\zeta^\ast)^{\ast} = (\zeta^\ast)^\ast, \quad \zeta^\ast\zeta = q^2 \zeta\zeta^\ast. \quad (2.4)$$

Since $\alpha$ commutes with $\zeta^\ast$, $\zeta$ we can define the quantum absolute $\Xi_q$ as the associative $\ast$-algebra generated by $(\zeta^\ast, \zeta, 1)$

$$\hat{f}(\zeta^\ast, \zeta)\dagger = \sum_{m, n = -\infty}^{\infty} b_{m, n} (\zeta^\ast)^m \zeta^n, \quad (b_{m, n} \in \mathbb{C}). \quad (2.5)$$

We formulate the conditions on the coefficients in next Section.

2. Quantum Lorentz groups.
As in the classical case the NLS are related to the quantum Lorentz group. We consider here the quantum deformation of the universal enveloping algebra $U_q(SL(2, \mathbb{C}))$ ($0 < q \leq 1$) and describe a twisted two parameter family $U_q^{(r,s)}(SL(2, \mathbb{C}))$ \cite{3}.

We start with a pair of the standard $U_q(SL(2, \mathbb{R}))$ Hopf algebra. The first one is generated by $A, B, C, D$ and the relations

\begin{align}
AD &= DA = 1, \quad AB = qBA, \quad BD = qDB, \\
AC &= q^{-1}CA, \quad CD = q^{-1}DC, \quad [B, C] = \frac{1}{q - q^{-1}}(A^2 - D^2).
\end{align}

There is a copy of this algebra $U_q^*(SL(2, \mathbb{R}))$ generated by $A^*, B^*, C^*, D^*$ with the relations following from (2.6). The star generators commute with $A, B, C, D$. This algebra is the Hopf algebras with the coproduct

\begin{align}
\Delta(A) &= A \otimes A, \\
\Delta(B) &= (A^*)^{-r}A \otimes B + B \otimes D(A^*)^s, \\
\Delta(C) &= (A^*)^rA \otimes C + C \otimes D(A^*)^{-s},
\end{align}

with the counit

$\varepsilon\left(\begin{array}{cccc} A & B & C & D \\ C & D & A & B \end{array}\right) = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right),$}

and the antipode

$S\left(\begin{array}{cccc} A & B & C & D \\ C & D & A & B \end{array}\right) = \left(\begin{array}{cccc} D & -q^{-1}(A^*)^{r-s}B & -q^{-1}(A^*)^{-s}A \\ -q(A^*)^{-r} & A \end{array}\right).$

There is the Casimir element in $U_q^{(r,s)}(SL(2, \mathbb{C}))$ commuting with any $u \in U_q^{(r,s)}(SL(2, \mathbb{C})).$

\begin{align}
\Omega_q := \frac{q^{-1} + q(A^2 + A^{-2}) - 4}{2(q^{-1} - q)^2} + \frac{1}{2}(BC + CB). \quad (2.8)
\end{align}

In what follows we put $r = 0.$

It was proven in Ref. \cite{3} that $L_{\delta,q}$ is a right $U_q^{(0,s)}(SL(2, \mathbb{C}))-module. The right actions of $A, B, C, A^*$ on generators $z^*, H, z$ take the form

\begin{align}
w(m, k, n).A &= q^{-n+\frac{k}{2}}w(m, k, n), \quad w(m, k, n).A^* = q^{\frac{(1-\delta)(-2m+k)}{s}}w(m, k, n), \\
w(m, k, n).B &= q^{-n+\frac{k+1}{2}}\frac{1 - q^{2n}}{1 - q^2}w(m, k, n - 1), \\
w(m, k, n).C &= q^{n-\frac{3(k-1)}{2} + \delta(k-1)}\frac{1 - q^{2m}}{1 - q^2}w(m - 1, k - 2, n) - q^{-n+\frac{k+3}{2}}\frac{1 - q^{2n-2k}}{1 - q^2}w(m, k, n + 1),
\end{align}

and it follows from (2.8) (2.9) that

\begin{align}
w(m, k, n).\Omega_q &= q^{k+1}\frac{(1 - q^{k+1})^2}{(1 - q^2)^2}w(m, k, n) + \\
&+ q^{(\delta-1)(k-1)}\frac{(1 - q^{2m})(1 - q^{2n})}{(1 - q^2)^2}w(m - 1, k - 2, n - 1). \quad (2.10)
\end{align}
or

$$f \cdot \Omega_q = \frac{q^{-1}f(q^{-1}x^*, q^{-1}H, q^{-1}x) + qf(qx^*, qH, qx)}{(1 - q^2)^2} - \left(1 - q^2\right)^2 q^{\delta-1} \partial_x^* \partial_x f(q^{-1}x^*, q^{\delta-1}H, q^{-1}x).$$

Note, that when $q \to 1\ \Omega_q \to \Omega (1.2)$.

It follows from (2.2), (2.9), and (2.10) that the actions of $A, B, C, A^*$ on $\tilde{w}(m, k, n) = (x^*)^m H^k x^n$ have the form

$$\tilde{w}(m, k, n)A = q^{\frac{m+k-n}{2}} \tilde{w}(m, k, n), \quad \tilde{w}(m, k, n)A^* = q^{\frac{(1-\delta)(-m+k+n)}{s}} \tilde{w}(m, k, n),$$

$$\tilde{w}(m, k, n)B = q^{\frac{m+k-n+1}{2}-\delta(n-1)} \frac{1 - q^{2n}}{1 - q^2} \tilde{w}(m, k + 1, n - 1),$$

$$\tilde{w}(m, k, n)C = q^{-\frac{(3m+3k+n-3)}{2}+\delta(k+n)} \frac{1 - q^{2m}}{1 - q^2} \tilde{w}(m - 1, k - 1, n) +$$

$$+ q^{\frac{3m+3k+n-3}{2}+\delta(n)} \frac{1 - q^{2m+2k}}{1 - q^2} \tilde{w}(m - 1, k, n + 1),$$

and

$$\tilde{w}(m, k, n)\Omega_q = q^{-m-k+n}\frac{1 - q^{m+k+n+1}}{(1 - q^2)^2} \tilde{w}(m, k, n) +$$

$$+ q^{-m-k-n+1+\delta(k+1)} \frac{1 - q^{2m}(1 - q^{2n})}{(1 - q^2)^2} \tilde{w}(m - 1, k, n - 1).$$  \hspace{1cm} (2.11)

The non-commutative cone $C_q$ is also the right module. We define the actions of $A, B, C, A^*$ on $\tilde{v}(m, k, n) = (\zeta^*)^m \alpha^k \zeta^n$ that compatible with the coproduct in $U_q^{(0,s)}(SL(2, \mathbb{C}))$

$$v(m, k, n)A = q^{\frac{m+k-n}{2}} v(m, k, n), \quad v(m, k, n)A^* = q^{-\frac{m+k+n}{s}} v(m, k, n),$$

$$v(m, k, n)B = q^{\frac{m+k-n+1}{2}} \frac{1 - q^{2n}}{1 - q^2} v(m, k + 1, n - 1),$$

$$v(m, k, n)C = q^{\frac{3m+3k+n-3}{2}} \frac{1 - q^{2m+2k}}{1 - q^2} v(m, k - 1, n + 1),$$

and

$$v(m, k, n)\Omega_q = q^{-m-k+n}\frac{1 - q^{m+k+n+1}}{(1 - q^2)^2} v(m, k, n) +$$

$$+ q^{-m-k-n+1}(1 - q^{2m})(1 - q^{2n}) v(m - 1, k, n - 1).$$ \hspace{1cm} (2.12)

It follows from (2.12) that $\Omega_q$ acts on $v(m, k, n)$ in the same way as on $w(m, k, n)$ for $\delta = 0$ and $\alpha$ playing the role of $H$. 


3 The q-Fourier transform and the functional spaces

Consider the algebra $\tilde{\Xi}_q$ generated by $(\xi^*, \xi)$ with the commutation relation

$$\xi \xi^* = q^2 \xi^* \xi \quad (3.1)$$

and the formal series

$$\psi(\xi^*, \xi) = \sum_{m=-\infty}^{\infty} a_{m,n}(\xi^*)^m \xi^n. \quad (3.2)$$

Let $L$ be the map of $\tilde{\Xi}_q$ to the space of functions on the two-dimensional lattice $C(\mathbb{Z} \oplus \mathbb{Z})$

$$L : \tilde{\Xi}_q \to C(\mathbb{Z} \oplus \mathbb{Z}) \quad (3.3)$$

$$L(\psi) = \sum_{m=-\infty}^{\infty} a_{m,n} q^{2mk} q^{2nl} = \psi_{k,l}, \quad (3.4)$$

and $K$ be the algebra of the functions (3.2) such that the series (3.4) converge absolutely for any $k, l \in \mathbb{Z}$.

**Definition 3.1** The factor space $\hat{K} = K / \ker L$ is called the skeleton space and $L$ (3.3) is the skeleton map.

In what follows we deal with the skeleton space only. We can define the skeleton space in the one-dimensional case as well.

**Definition 3.2** $\psi \in \hat{K}$ (3.2) is a finite function if $\psi_{k,l} = 0$ for any $k < -K$ or $l < -L$ and $K, L$ are some positive integers.

Let $K$ be the subalgebra of finite functions from $\hat{K}$. Define the q-Fourier transform on $K$

$$(\mathcal{F}^{-1}\psi)(\zeta^*, \zeta) = \phi(\zeta^*, \zeta) = \frac{1}{4\Theta_0} \int \int d_q \zeta^* d_q \zeta \psi(\xi^*, \xi), \quad (3.5)$$

where the integral is the Jackson integral (A.8) and $\Theta_0$ is determined by (A.5). The $q^2$-integral is well defined because $\psi(\zeta^*, \zeta) \in K$. The inversion formula has the form

$$(\mathcal{F}\phi)(\xi^*, \xi) = \psi(\xi^*, \xi) = \int \int d_q \zeta^* e(\xi^* \zeta) \phi(\xi^*, \zeta), \quad (3.6)$$

Let $Z$ be the image (3.5) of $K$. It follows from the last relation and (3.1) that $(\zeta^*, \zeta)$ can be identified with the absolute generators (2.4), and therefore $Z$ is a subalgebra of $\Xi_q$.

**Proposition 3.1** The maps

$$\mathcal{F}^{-1} \circ \mathcal{F} : Z \to Z,$$

$$\mathcal{F} \circ \mathcal{F}^{-1} : K \to K$$

are the identity maps on $K$ and $Z$ correspondingly.
Proof. For brevity we consider the one-dimensional case. To prove Proposition we consider the value of the Fourier transform on the $q^2$-lattice and show that

$$F \circ F^{-1}(\psi(q^{2n})) = \psi(q^{2n}), \quad (3.7)$$

$$F^{-1} \circ F(\phi(q^{2n})) = \phi(q^{2n}). \quad (3.8)$$

Consider the first relation

$$F \circ F^{-1}(\psi(\xi)) = \frac{1 - q^2}{2\Theta_0} \int e(\xi \zeta) \int d_q u \psi(u) E(-q^2 u \zeta) d_q \zeta =$$

$$\frac{1 - q^2}{2\Theta_0} q^{2n} \int d_q u \psi(u) \int e(\xi \zeta) E(-q^2 u \zeta) d_q \zeta.$$

It follows from Lemma A.1 that we come to (3.7). (3.8) is proving just in the same way. \[\square\]

Now we construct the Fourier transform on $L_{\delta, q}$ with respect to the "horospheric" generators $(x^*, x)$ in the similar way as above. Let $f(x^*, H, x)$ be an element from $L_{\delta, q}$ (2.3), such that the inverse Fourier integral

$$F^{-1}(f)(y^*, H, y) = \frac{1}{4\Theta_0^2} \int d_q x^* f(x^*, H, x) E(-q^2 y^* x^*) E(-q^2 y x) d_q x \quad (3.9)$$

is well defined. We preserve the notion $L_{\delta, q}$ for the space of these functions and define the algebra

$$\tilde{L}_{\delta, q} = F^{-1}(L_{\delta, q}) \quad (3.10)$$

with the generators $(y^*, H, y)$ and the commutation relations

$$yH = Hy, \quad y^*H = Hy^*, \quad (3.11)$$

$$yy^* = q^{-2} y^* y[1 + q^\delta (q^2 - 1)y^* y]^{-1}.$$

The direct Fourier transform takes the form

$$(Fg)(x^*, H, x) = \int \int e(y^* x^*) d_q y^* g(y^*, H, y) y^* d_q y e(yx). \quad (3.12)$$

Then as before

$$F \circ F^{-1} = Id \text{ on } L_{\delta, q}, \quad F^{-1} \circ F = Id \text{ on } \tilde{L}_{\delta, q}.$$

Let $\nu \geq 0$ and $\tilde{W}_\nu$ be the space of functions with a fixed singularity. It is constructed by means of a pair elements $g_1, g_2 \in \tilde{L}_{\delta, q}$ as follows

$$\tilde{W}_\nu = \left\{ \begin{array}{ll}
g_1(y^*, H, y) + (y^*)^\nu g_2(y^*, H, y)y^* & \text{if } \nu \neq n \in \mathbb{N}, \\
g_1(y^*, H, y) + \ln y^* g_2(y^*, H, y) + g_2(y^*, H, y) \ln y & \text{if } \nu = n \in \mathbb{N}. \end{array} \right\} \quad (3.13)$$

We define $W_\nu$ as the image of $\tilde{W}_\nu$ by the Fourier transform (3.12)

$$W_\nu = \mathcal{F}(\tilde{W}_\nu). \quad (3.14)$$
4 The Poisson kernel

The Poisson kernel is the element of the algebra $L_{q,\delta} \otimes \Xi_q$ determined by the series

$$
P_{\nu}((x^* \otimes 1 - 1 \otimes \zeta^*), H, (1 \otimes x - \zeta \otimes 1)) =
$$

$$
= \sum_{k=0}^{\infty} (-1)^k \frac{(q^{2\nu+2}, q^2)_{k}}{(q^2, q^2)_{k}} q^{(2-\nu \delta - 2\delta)k} (x^* \otimes 1 - 1 \otimes \zeta^*)^k H^{\nu+1} (-\zeta \otimes 1 + 1 \otimes x)^k.
$$

Let

$$
\Omega^{\nu}_q = \Omega_q - q^{-\nu+2} \left( \frac{1 - q^\nu}{1 - q^2} \right) Id.
$$

and $F_{\nu}(x^*, H, x) \in L_{q,\delta}$ be a solution to the equation

$$
F_{\nu}(x^*, H, x) \Omega^{\nu}_q = 0.
$$

We can formulate now our main result.

**Proposition 4.1** For any $\phi \in \mathbb{Z}$ on the absolute $\Xi_q$

- The function

  $$
  F_{\nu}(x^*, H, x) = (P_{\nu} * \phi)(x^*, H, x),
  $$

  is a solution to the equation $(4.2)$;

- $F_{\nu}(x^*, H, x) \in W_{\nu}$ $(3.14)$.

Here the convolution is defined as

$$
(P_{\nu} * \phi)(x^*, H, x) = \int \int d\mu_2 \zeta^* \frac{1}{\mu_2} P_{\nu}((x^* \otimes 1 - 1 \otimes \zeta^*)^k, (-\zeta \otimes 1 + 1 \otimes x)) \phi(\zeta, \zeta) d\mu_2 \zeta.
$$

We postpone the proof of this statement to last Section and formulate here some intermediate steps.

Let $\sigma$ be the involution $\sigma f(x^*, H, x) = f(-x^*, H, -x)$.

**Proposition 4.2** The Poisson kernel has the integral representation

$$
P_{\nu}(x^*, H, x) = \sigma F(Q_{\nu}),
$$

where

$$
Q_{\nu}(y^*, H, y) =
$$

$$
= \frac{1 + q H^2 + (2^{\nu^2+\nu})}{26\Theta_0^2} \frac{1}{\Gamma_q^2(\nu + 1)} (\frac{1 - q^\nu}{1 - q^2}) \frac{1}{2} y^2 q^{\delta(1 - q^\nu)} H^{\frac{\delta}{4}} y^{\frac{\delta}{2}} K_{\nu}^{(2)} (2(y^*)^\frac{1}{2} y^{\frac{1}{2}} q^{\delta(1 - q^\nu)} ; q^2) H^{\frac{\delta}{4}} H^{\frac{\delta}{2}} y^{\frac{\delta}{2}},
$$

$K_{\nu}^{(2)}$ is the $q^2$-Bessel-Macdonald function of kind 2 [6, 8], and $\Gamma_{q^2}(\nu + 1)$ is the $q^2 - \Gamma$-function $(A.3)$.

Consider the Fourier transform of the left hand side $(4.2)$

$$
\sigma F^{-1}(f, \Omega_q^{\nu}) = \sigma F^{-1}(f) \tilde{\Omega}_q^{\nu}, \quad \tilde{\Omega}_q^{\nu} = F^{-1} \Omega_q^{\nu}.
$$

For $\sigma F^{-1}(f) = g(y^*, H, y)$ we have the equation $g(y^*, H, y) \tilde{\Omega}_q^{\nu} = 0$, or

$$
q^{-1} g(q^{-1} y^*, qH, q^{-1} y) - (q^\nu + q^{-\nu}) g(y^*, H, y) + qg(qy^*, q^{-1} H, qy) =
$$
\[ (1 - q^2)^2 q^4 y^* g(qy^*, q^\delta H, qy) y. \]

The \( q^2 \)-Fourier transform of the \( q^2 \)-Poisson kernel satisfies (4.7)

\[ Q_\nu(y^*, H, y) \tilde{\Omega}_q^\nu = 0. \]  

The statement is verified directly using the series representation of the \( q^2 \)-Bessel-Macdonald function.

**Proposition 4.3** If \( g(y^*, H, y) \) is a solution of (4.7), then the product \( g(y^*, H, y) \psi(\alpha y^*, \alpha y) \) is a solution to the same equation for any function \( \psi(\alpha y^*, \alpha y) \).

Let

\[ Q_\nu(y^*, H, y; \xi^*, \xi) = Q_\nu(y^*, H, y) \delta(y^* \otimes 1 - 1 \otimes \xi^*, y \otimes 1 - 1 \otimes \xi). \]  

Here delta-function is the kernel of the integral transform \( \tilde{\Xi}_q \to \tilde{\mathcal{L}}_{\delta,q} \)

\[ h(y^*, y) = \int d_{q^2} \xi^* H(\xi^*, \xi) \delta(y^* \otimes 1 - 1 \otimes \xi^*, y \otimes 1 - 1 \otimes \xi) \tilde{\Phi}_{q^2} \xi. \]

Thereby, the multiplication by \( Q_\nu \) carries out this map.

**Proposition 4.4** The operator of multiplication on \( Q_\nu(y^*, H, y) \) transforms the space \( \mathcal{K} \) into the space \( \tilde{\mathcal{W}}_{\nu} \).

We illustrate our construction by the following commutative diagram

\[
\begin{array}{ccc}
(\psi \in \mathcal{K}, \tilde{\Xi}_q) & \xrightarrow{F^{-1}} & (\phi \in Z, \Xi_q) \\
\downarrow Q_\nu \times & & \downarrow \mathcal{P}_\nu^* \\
(\tilde{\mathcal{W}}_\nu, \tilde{\mathcal{L}}_{\delta,q}) & \xrightarrow{\sigma F} & (F_\nu \in \mathcal{W}_\nu, \mathcal{L}_{\delta,q})
\end{array}
\]

It means that one can start with a finite function \( \psi \in \mathcal{K} \) and then come to the solution \( F_\nu \) by one of the two possible ways.

### 5 The \( q^2 \)-Fourier transform of spherical symmetric functions

**Definition 5.1** A function \( f \in \mathcal{L}_{\delta,q} \) is spherical symmetric if it depends on the product of \( x^*x \)

\[ f(x^*, H, x) = \sum_{l,k=-\infty}^{\infty} c_{l,k}(x^*)^l H^k x^l. \]  

Any ordered element from \( \mathcal{L}_{\delta,q} \) can be represented in the form [7]

\[ f(x^*, H, x) = \sum_{r=1}^{\infty} (x^*)^r \phi_r(x^*, H, x) + \phi_0(x^*, H, x) + \sum_{r=1}^{\infty} \phi_r(x^*, H, x)x^r, \]  

where \( \phi_r(x^*, H, x) \) are spherically symmetric.

Consider the inverse Fourier transform of the spherical symmetric function (5.1)

\[ F^{-1} f = g(y^*, H, y) = \frac{1}{4\Theta^2} \int \int d_{q^2} x^* x^* E(q^2 y^* x^*) f(x^*, H, x) E(q^2 y x) x d_{q^2} x. \]
Using (A.2) we obtain

\[ g(y^*, H, y) = \frac{1}{4\Theta_0^2}(1 - q^2) \sum_{n=-\infty}^{\infty} q^{2n} \sum_{m=0}^{\infty} \frac{(-1)^m (1 - q^2)^{2m} q^{2m(m+1)} q^{2nm}}{(q^2, q^2)^m} (y^*)^m \sum_{l,k} c_{lk} q^{2nl} H^k y^m = \]

\[ = \frac{1}{4\Theta_0^2}(1 - q^2) \sum_{n=-\infty}^{\infty} q^{2n} J_0^{(2)}(2(y^*)^\frac{1}{2} y^\frac{1}{2} q(1 - q^2)) \sum_{l,k} c_{lk} q^{2nl} H^k y^m, \]

where \( J_0^{(2)} \) is the \( q^2 \)-Bessel function of kind 2 [9]. Using (A.10) the last expression can be rewritten as the integral

\[ (F^{-1}f)(y^*, H, y) = \frac{1 + q}{4\Theta_0^2} \int_0^\infty \frac{1}{\sqrt{2\pi}} J_0^{(2)}(2(y^*)^\frac{1}{2} y^\frac{1}{2} q(1 - q^2)) \sum_{l,k} c_{lk} \rho^{2l} H^{k-\frac{2l}{2}} \rho \frac{\rho}{\rho_d} \rho \]

\[ = \frac{1 + q}{4\Theta_0^2} \int_0^\infty \frac{1}{\sqrt{2\pi}} J_0^{(2)}(2(y^*)^\frac{1}{2} y^\frac{1}{2} q(1 - q^2)) \sum_{l,k} c_{lk} \rho^{2l} H^{k-\frac{2l}{2}} \rho \frac{\rho}{\rho_d} \rho \]

The inversion formula has the form

\[ F(g) = f(x^*, H, x) = \]

\[ = (1 + q) \int_0^\infty \frac{1}{\sqrt{2\pi}} J_0^{(1)}(2(x^*)^\frac{1}{2} x^\frac{1}{2} (1 - q^2)) \sum_{l,k} b_{lk} r^{2l} H^{k-\frac{2l}{2}} \rho \frac{\rho}{\rho_d} \rho \]

\[ = (1 + q) \int_0^\infty \frac{1}{\sqrt{2\pi}} J_0^{(1)}(2(x^*)^\frac{1}{2} x^\frac{1}{2} (1 - q^2)) \sum_{l,k} b_{lk} r^{2l} H^{k-\frac{2l}{2}} \rho \frac{\rho}{\rho_d} \rho \]

Here \( J_0^{(1)} \) is the \( q^2 \)-Bessel function of kind 1 [9].

6 Proofs

1. Proof of 4.2.

Consider the power series

\[ \mathcal{P}_\nu(x^*, H, x) = \sum_{k=0}^{\infty} (-1)^k q^\nu q^{2+1} \frac{q^2}{q^2} \frac{q^{2(2-\nu-\delta-2\delta)k}}{(q^2, q^2)_k} (x^*)^k H^{\nu+1} x^k. \]

and find its Fourier transform \( Q_\nu(y^*, H, y) \) as the function from \( \mathcal{L}_{\delta, q} \). Since \( \mathcal{P}_\nu(x^*, H, x) \) is the spherically symmetric we have, following (5.3)

\[ Q_\nu(y^*, H, y) = \frac{1 + q}{4\Theta_0^2} \int_0^\infty \frac{1}{\sqrt{2\pi}} J_0^{(2)}(2(y^*)^\frac{1}{2} y^\frac{1}{2} q(1 - q^2)) \sum_{l,k} a_{lk} q^{2(2-\nu-\delta-2\delta)k} \rho^{2k} H^{\nu+1+\frac{2l}{2}} \rho \frac{\rho}{\rho_d} \rho \]

where \( a_{lk} = \frac{(-1)^k q^{2+1} q^{2}}{(q^2, q^2)_k} \). On the other hand one can find that

\[ \sum_{k=0}^{\infty} a_{lk} q^{2(2-\nu-\delta-2\delta)k} \rho^{2k} = \frac{(-q^{2+1}(2-\delta) \rho^2, q^2)_\infty}{(-q^{2(2+\nu)\delta} \rho^2, q^2)_\infty}. \]
Hence,
\[ Q_\nu(y^*, H, y) = \frac{1 + q}{4\Theta_0^2} \times \]
\[ \times \int_0^\infty \rho (-q^{2+\nu}(2-\delta) \rho^2, q^2)_\infty^{-1} H^{\nu+1} J_0^{(2)}(2(y^*)^2 q^2 (1-q^2)\rho; q^2) \rho d\rho. \]

It follows from Ref. [5, 6] that the last expression is the integral representation of the $q^2$-Bessel-Macdonald function, i.e.
\[ Q_\nu(y^*, H, y) = B(y^*)^2 H^\frac{\nu}{2} K_0^{(2)}(2(y^*)^2 q y^2 (1-q^2); q^2) H^\frac{\nu}{2} y_2 \frac{\nu}{2}. \]  \hspace{1cm} (6.3)

To calculate $B$ we restrict the integral to the common kernel $y^* = 0$, $y = 0$. Then we obtain [6, 8]
\[ \frac{1 + q}{4\Theta_0^2} \int d_\rho (\rho^2 (-q^{2+\nu}(2-\delta) \rho^2, q^2)_\infty^{-1} = B \frac{1}{2} q^{-\nu^2+\delta(\frac{\nu}{2}+\nu)} \Gamma_{q^2}(\nu). \]

Now we calculate the integral in the right side. Note that
\[ \partial_{\rho^2} E_{q^2}((a \rho^2) = \frac{a}{1-q^2} E_{q^2}(a q^2 \rho^2), \quad \partial_{\rho^2} E_{q^2}(b \rho^2) = \frac{b}{1-q^2} E_{q^2}(b \rho^2). \]

Hence
\[ \int_0^\infty d_\rho (\rho^2 (-q^{2+\nu}(2-\delta) \rho^2, q^2)_\infty^{-1} E_{q^2}(q^{2+\nu}(2-\delta) \rho^2) = \]
\[ = (1 - q^2) q^{2+\nu}(2-\delta) \int_0^\infty d_\rho \partial_{\rho^2} E_{q^2}(q^{2+\nu}(2-\delta) \rho^2) e_{q^2}(-q^{2+\nu}(2-\delta) \rho^2) = \]
\[ = (1 - q^2) \lim_{m \to \infty} \frac{-q^{2+\nu}(2-\delta)-2m}{-q^{2+\nu}(2-\delta)-2m, q^2}_\infty^{-1} - 1 = \]
\[ = -(1 - q^2) q^{2+\nu}(2-\delta) \int_0^\infty d_\rho \partial_{\rho^2} E_{q^2}(q^{2+\nu}(2-\delta) \rho^2) e_{q^2}(-q^{2+\nu}(2-\delta) \rho^2). \]
\[ \int_0^\infty d_\rho \partial_{\rho^2} E_{q^2}(q^{2+\nu}(2-\delta) \rho^2) e_{q^2}(-q^{2+\nu}(2-\delta) \rho^2). \]

Then we have
\[ \int_0^\infty d_\rho (\rho^2 (-q^{2+\nu}(2-\delta) \rho^2, q^2)_\infty^{-1} E_{q^2}(q^{2+\nu}(2-\delta) \rho^2) = -\frac{1 - q^2}{1-q^{-2\nu}}. \]

It implies that
\[ B = \frac{1 + q}{26\Theta_0^2 \Gamma_{q^2}(\nu + 1)} q^{\nu^2+\delta(\frac{\nu}{2}+\nu)} \]
and the representation (4.6).\hfill \Box

2. Proof of 4.3

Consider the equation (4.7) for the product $g(y^*, H, y)^2 \psi(\alpha y^*, \alpha y)$. Note that $\alpha$ plays the role of $H$ and the action of the generators is the same as for $\delta = 0$ (compare (2.11) and (2.12)). In this way we have
\[ q^{-1} g(q^{-1} y^*, qH, q^{-1} y) \psi(qoq^{-1} y^*, qoq^{-1} y) - (q^\nu + q^{-\nu}) g(y^*, H, y)^2 \psi(\alpha y^*, \alpha y) + \]
\[ qq(qy^*, q^{-1} H, qy) \psi(q^{-1} \alpha y^*, q^{-1} \alpha y) = \]

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(1 - q^2)^2 q^d y^g q g(q y, q y) q^δ q^e H, q y) q^{-1} q^e q^{-1} q^e q y .

3. Proof of 4.4
We can extract the series expansion for $K_n^{(2)}$ from the series expansion of $I_{1}^{(2)}$ and $I_{-1}^{(2)}$ for $\nu \neq n \in \mathbb{N}$ [8]. It gives us the following expansion

$$Q_\nu = \frac{(1 + q)(1 - q^2)}{4\Theta^2 (1 - q^{2\nu})} q^{2\nu} \times$$

$$\times \sum_{l=0}^{\infty} \frac{q^{2l(1-\nu)} (1 - q^2)^{2l}}{(q^2, q^{2l+2})^2 (q^{2\nu+2}, q^2)^2} q^{2l(\nu+2)} (y^*)^l H^{\nu+1} y^l -$$

$$- \frac{\Gamma \nu^2 q^2 (1 - \nu)}{\Gamma q^2 (1 + \nu)} \sum_{l=0}^{\infty} \frac{q^{2l(1+\nu)} (1 - q^2)^{2l}}{(q^2, q^{2l+2})^2 (q^{2\nu+2}, q^2)^2} q^{2l(\nu+2)} (y^*)^{\nu+1} H^{\nu+1} y^{\nu+1} =$$

$$= \Psi_1 + (y^*)^\nu \Psi_2 y^\nu .$$

The functions $\Psi_1$ and $\Psi_2$ are multipliers in $\mathcal{K}$ and we come to the statement of Proposition for $\nu \neq \mathbb{N}$.

Using the expression of $K_n^{(2)}$ [8] it can be proved similarly that $Q_n$ has the logarithmic singularity (3.13). ■

4. Proof of 4.1
Let $\mathcal{F} \phi = \psi$. Remind that $\phi$ and $\psi$ depend on $\alpha$ (see n.4). Taking into account (4.1)

$$\mathcal{P}_\nu = \mathcal{F}[e(\zeta^* y^*) Q_\nu (y^*, H, y) e(y \zeta)] ,$$

we have

$$\sigma \mathcal{F}^{-1}(\mathcal{P}_\nu * \phi) = \sigma \mathcal{F}^{-1}(\mathcal{F}[e(\zeta^* y^*) Q_\nu (y^*, H, y) e(y \zeta)] * \mathcal{F}^{-1} \psi) =$$

$$\frac{1}{4\Theta^2} \int \int d_\nu u^* Q_\nu (u^*, H, u) d_\nu u \int \int d_\nu v^* \psi (v^*, \alpha, v) d_\nu v \int \int d_\nu x^* \mathbf{E} (q^2 x^* y^*) e(-x^* u^*) \times$$

$$\times \mathbf{E} (u x) (q^2 y x) d_\nu x \times$$

$$\times \int \int d_\nu \zeta \mathbf{e} (1 \otimes \zeta^*) u^*) e(-q^2 v^* (\zeta^* \otimes 1)) \mathbf{E} (-q^2 (1 \otimes \zeta^*) v) e(u (\zeta \otimes 1)) d_\nu \zeta .$$

It follows from Lemma A.1 that

$$\int \mathbf{e} (-u x) (q^2 y x) d_\nu x = \begin{cases} \frac{2\Theta_{\nu}}{1 - q^2} \quad \text{for} \ u = y \\ 0 \quad \text{for} \ u \neq y \end{cases}$$

and

$$\int \mathbf{E} (-q^2 (1 \otimes \zeta^*) v) e(u (\zeta \otimes 1)) d_\nu \zeta =$$

$$\begin{cases} \frac{2\Theta_{\nu}}{1 - q^2} \quad \text{for} \ u \otimes 1 = 1 \otimes v \\ 0 \quad \text{for} \ u \otimes 1 \neq 1 \otimes v \end{cases}$$

Hence

$$\sigma \mathcal{F}(\mathcal{P}_\nu * \phi) = Q_\nu (y^*, H, y) \psi (y^*, \alpha, y) .$$

■
Appendix A. \(q\)-relations

We assume that \(|q| < 1\). Let us recall some notations \([9]\). We consider the \(q^2\)-exponentials

\[
e_{q^2}(z) = \sum_{n=0}^{\infty} \frac{z^n}{(q^2, q^2)_n} = \frac{1}{(z, q^2)_{\infty}}, \quad |z| < 1,
\]

(A.1)

\[
e_{q^2}(z) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)}z^n}{(q^2, q^2)_n} = (-z, q^2)_{\infty},
\]

(A.2)

and \(q^2 - \Gamma\)-function

\[
\Gamma_{q^2}(\nu) = \frac{(q^2, q^2)_{\infty}}{(q^{2\nu}, q^2)_{\infty}} (1 - q^2)^{1-\nu}.
\]

(A.3)

Introduce the notion of the \(q^2\)-exponentials defined on a tensor product

\[
e(y\zeta) = e_{q^2}(i(1 - q^2)(y \otimes \zeta)), \quad E(y\zeta) = E_{q^2}(i(1 - q^2)(y \otimes \zeta)).
\]

(A.4)

Consider

\[
Q(z, q) = (1 - q^2) \sum_{m=-\infty}^{\infty} \frac{1}{zq^{2m} + z^{-1}q^{-2m}}
\]

and let

\[
\Theta_0 = Q(1 - q^2, q).
\]

(A.5)

Let \(\mathcal{A} = C[z, z^{-1}]\) be the algebra of formal Laurent series. The \(q^2\)-derivative of a function \(f(z) \in \mathcal{A}\) is defined as follows

\[
\partial_z f(z) = (f(z) - f(q^2 z)) \frac{z^{-1}}{1 - q^2}.
\]

(A.6)

The functions (A.4) satisfy conditions:

\[
\partial_y E(y\zeta) = (y \otimes 1) E(q^2 y\zeta),
\]

\[
\partial_q E(y\zeta) = (1 \otimes \zeta) E(q^2 y\zeta),
\]

\[
\partial_y e(y\zeta) = (y \otimes 1) E(y\zeta),
\]

\[
\partial_q e(y\zeta) = (1 \otimes \zeta) e(y\zeta).
\]

(A.7)

The \(q^2\)-integral (Jackson integral \([9]\)) is defined to be the following map \(I_{q^2}\) of the algebra \(\mathcal{A}\) into the space of formal numerical series:

\[
I_{q^2} f = \int dz \, q^2 f(z) = (1 - q^2) \sum_{m=-\infty}^{\infty} q^{2m}[f(q^{2m}) + f(-q^{2m})].
\]

(A.8)

It follows from this definition that

\[
I_{q^2} \partial_z f(z) = 0.
\]

(A.9)

If the series in right side of (A.8) is nonconvergent (A.9) is a regularization of nonconvergent \(q^2\)-integral.
We need also another type of the Jackson integral
\[ \int_0^\infty d_q x f(x) = (1 - q^2) \sum_{m=-\infty}^\infty q^m f(q^m). \]  
(A.10)

The function \( f(z) \) is absolutely \( q^2 \)-integrable if the series
\[ \sum_{m=-\infty}^\infty q^{2m} [ |f(q^{2m})| + |f(-q^{2m})| ] \]
converges.

**Lemma A.1**
\[ \int E(q^2 y \zeta) e(-u \zeta) d_q \zeta = \begin{cases} 2 \frac{\Theta_0 u^{-1}}{1 - q^2} & \text{if } y = u \\ 0 & \text{if } y \neq u \end{cases}, \]
(A.11)

\[ \int d_q y E(q^2 y \xi) e(-y \zeta) = \begin{cases} 2 \frac{\Theta_0 \zeta^{-1}}{1 - q^2} & \text{if } \zeta = \xi \\ 0 & \text{if } \zeta \neq \xi \end{cases}, \]
(A.12)

where \( \Theta_0 \) determined by (A.5).

**Proof.** It follows from (A.7) that
\[ \partial_\zeta [E(y \zeta) e(-u \zeta)] = i((y - u) \otimes 1) E(q^2 y \zeta) e(-u \zeta). \]

Hence, if \( y \neq u \), then
\[ (y - u) \int E(q^2 y \zeta) e(-u \zeta) d_q \zeta = \int \partial_\zeta [E(y \zeta) e(-u \zeta)] d_q \zeta = 0 \]
in accordance with the definition of the \( q^2 \)-integral.

If \( y = u \), then
\[ \int E(q^2 u \zeta) e(-u \zeta) d_q \zeta = u^{-1} \int E(q^2 \zeta) e(-\zeta) d_q \zeta = \]
\[ = u^{-1}(1 - q^2) \sum_{m=-\infty}^\infty q^{2m} \left[ \frac{(i(1 - q^2)q^{2m+2}, q^2)_\infty}{(1 - i(1 - q^2)q^{2m}, q^2)_\infty} + \frac{(i(1 - q^2)q^{2m+2}, q^2)_\infty}{(1 - i(1 - q^2)q^{2m}, q^2)_\infty} \right] = \]
\[ = u^{-1}(1 - q^2) \sum_{m=-\infty}^\infty q^{2m} \left( \frac{1}{1 + i(1 - q^2)q^{2m}} + \frac{1}{1 - i(1 - q^2)q^{2m}} \right) = \]
\[ = u^{-1}2 \sum_{m=-\infty}^\infty \frac{1}{(1 - q^2)^{m-1}q^{2m} + (1 - q^2)q^{2m}} = u^{-1}2 \frac{1}{1 - q^2} \Theta_0 \]
(see (A.5)).

(A.12) is proving in just the same way.  

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