CATEGORIFICATION OF DAHA AND MACDONALD POLYNOMIALS

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Abstract. We describe a categorification of the Double Affine Hecke Algebra \( \mathcal{H} \) associated with an affine Lie algebra \( \hat{\mathfrak{g}} \), a categorification of the polynomial representation and a categorification of Macdonald polynomials.

All categorification results are given in the derived setting. That is, we consider the derived category associated with graded modules over the Lie superalgebra \( \mathcal{I}[\xi] \), where \( \mathcal{I} \subset \hat{\mathfrak{g}} \) is the Iwahori subalgebra of the affine Lie algebra and \( \xi \) is a formal odd variable. The Euler characteristic of graded characters of a complex of \( \mathcal{I}[\xi] \)-modules is considered as an element of a polynomial representation. First, we show that the compositions of induction and restriction functors associated with minimal parabolic subalgebras \( p_i \) categorify Demazure operators \( T_i + 1 \in \mathcal{H} \), meaning that all algebraic relations of \( T_i \) have categorical meanings. Second, we describe a natural collection of complexes \( \mathbb{E}M_\lambda \) of \( \mathcal{I}[\xi] \)-modules whose Euler characteristic is equal to nonsymmetric Macdonald polynomials \( E_\lambda \) for dominant \( \lambda \) and a natural collection of complexes of \( \mathfrak{g}[z, \xi] \)-modules \( \mathbb{P}M_\lambda \) whose Euler characteristic is equal to the symmetric Macdonald polynomial \( P_\lambda \).

We illustrate our theory with the example \( \mathfrak{g} = \mathfrak{sl}_2 \) where we construct the cyclic representations of Lie superalgebra \( \mathcal{I}[\xi] \) such that their supercharacters coincide with renormalizations of nonsymmetric Macdonald polynomials.

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0. Introduction

Let \( \mathfrak{g} \) be a semisimple Lie algebra with weight lattice \( \mathcal{P} \) and Weyl group \( W \). Then the ring of symmetric functions \( \mathbb{Z}[\mathcal{P}]^W \) is the ring of characters of finite-dimensional \( \mathfrak{g} \)-modules. The collection of sums of monomials over orbits \( m_\lambda = \sum_{\alpha \in W\lambda} e^\alpha \) is called the monomial basis of the ring of symmetric functions. Here \( \lambda \) belongs to the set of integral dominant weights \( \mathcal{P}_+ \). I. G. Macdonald \( [MD] \) defined a two-parameter pairing \( \langle \cdot, \cdot \rangle_{q,t} \) on \( \mathbb{Z}_q,t[\mathcal{P}]^W := \mathbb{Q}(q, t) \otimes \mathbb{Z}[\mathcal{P}]^W \) and a family of orthogonal polynomials \( \{ P_\lambda(q, t), \lambda \in \mathcal{P}_+ \} \) resulting from the Gram-Schmidt orthogonalization process applied to the monomial basis with respect to
the standard partial ordering of weights:

\[ \langle P_\lambda, P_\mu \rangle_{q,t} = 0 \text{ if } \lambda \neq \mu \quad \& \quad P_\lambda = m_\lambda + \sum_{\mu < \lambda} c_{\lambda,\mu} m_\mu. \]

Though the ordering is partial:

\[ \lambda \geq \mu \iff \lambda - \mu = \text{sum of positive roots}, \]

even so, the set of all polynomials is still orthogonal. That is, the polynomials \( P_\lambda \) and \( P_\mu \) are orthogonal even when dominant weights \( \lambda \) and \( \mu \) are not comparable.

One of the main purposes of this paper is to describe a possible categorification of Macdonald polynomials. We consider the category of bigraded modules over the Lie superalgebra of currents \( g \otimes \mathbb{C}[z, \xi] \). Here the first \( q \)-grading is assigned to the even variable \( z \) and the second \( t \)-grading is assigned to the odd variable \( \xi \). As shown in [Kh] (see also [FGT],[F]) the Euler characteristic of extension groups categorifies the Macdonald pairing \( \langle \cdot, \cdot \rangle_{q,t} \):

\[
\langle \text{gch}(M), \text{gch}(N) \rangle_{q,t} = \sum_{i \geq 0, k,l \in \mathbb{Z}} (-1)^i q^k t^l \dim \text{Ext}^i g \otimes \mathbb{C}[z,\xi] \text{-gmod}(M \{k,l\}, N')
\]

whenever the right hand side is a well defined power series. Here \( \text{gch}(M) \) denotes the graded supercharacter of a bigraded \( \mathfrak{h} \)-semisimple \( g \otimes \mathbb{C}[z,\xi] \)-module \( M \), \( M \{k,l\} \) denotes the graded shift of the module \( M \), and \( N' \) denotes the restricted dual module twisted by the Cartan antiinvolution.

Unfortunately, it is impossible to find a family of bigraded modules whose graded characters are equal to Macdonald polynomials \( P_\lambda \), because it is known that \( P_\lambda \) are not Schur-positive for \( \lambda \) sufficiently large. Therefore, one can expect a categorification in terms of derived category instead of abelian one. For each dominant \( \lambda \) we construct a complex \( PM_\lambda \) of bigraded \( g \otimes \mathbb{C}[z,\xi] \)-modules whose Euler characteristic of characters is proportional to the Macdonald polynomials \( P_\lambda \) and whose cohomology does not contain \( g \)-submodules with weights not less than or equal to \( \lambda \) (Corollary 4.24). In order to convince the reader that the complexes \( PM_\lambda \) we constructed are natural let us outline several other categorification results invented in this paper. For technical reasons, we assume that \( g \) has no \( G_2 \) components in the below.

First, as suggested by I.Cherednik in [Ch2], we replace the symmetric pairing by the nonsymmetric one and consider the orthogonal basis consisting of nonsymmetric Macdonald polynomials \( E_\lambda \). These polynomials constitute an orthogonal basis of the ring of all (not only symmetric) functions \( \mathbb{Z}[P]_{q,t} \) and appear as a result of the Gram-Schmidt process with respect to the Cherednik partial ordering \( \prec \) of all weights (see (4.1) for a detailed definition of the ordering). On the categorical level, Cherednik’s nonsymmetric pairing corresponds to the \( \text{Ext} \)-pairing of modules over the Lie superalgebra \( \mathcal{I}[\xi] \). Here \( \mathcal{I} \) is the Iwahori subalgebra of the affine Lie algebra \( \hat{g} \) and we have \( \mathcal{I}[\xi] \subset g[z,\xi] \subset \hat{g}[\xi] \).

Second, we explain the following categorification of the (extended) Affine Hecke Algebra \( \mathcal{H} \) which we consider as the main categorification result of this paper. Let \( b \subset g \) be the Borel subalgebra and \( p_i \supset b \) be the minimal parabolic subalgebra associated with a given simple root \( \alpha_i \) from a root system \( \Delta \) associated with \( g \). Then there exists a restriction functor \( \text{Res}_i \) between the categories of graded \( p_i[\xi] \)-modules and \( b[\xi] \)-modules. The corresponding left adjoint functor \( \text{Ind}_i \) is the maximal \( p_i \)-integrable quotient of the induced module. \( \text{Ind}_i \) is not exact and the corresponding derived functor, called the derived induction, is denoted \( \text{LInd}_i \).
The main protagonists of this article are the endofunctors $T_i$ defined from the distinguished triangles of derived endofunctors of the bounded derived category $D^b(\mathcal{B}[\xi]-\text{gmod})$:

$$\text{Id}_{\mathcal{B}[\xi]} \xrightarrow{\partial} \text{Res}_i \circ \text{LInd}_i \to T_i \xrightarrow{\pm 1} \text{Id}_{\mathcal{B}[\xi]}[1].$$

**Theorem (Theorem 4.10).** The following collection of endofunctors of the derived category $D(\mathcal{B}[\xi]-\text{gmod})$ categorifies the affine Hecke algebra $\mathcal{H}$ associated with the root system $\Delta$:

- The $\xi$-degree shift endofunctor $t$ categorifies the Hecke parameter $t$;
- for each $\lambda \in P$ the weight shift endofunctors $X^\lambda$ categorifies $X^\lambda \in \mathcal{H}$;
- for each simple root $\alpha_i$ the endofunctors $T_i$ categorifies the Hecke generator $T_i$.

In other words, with each generating relation in $\mathcal{H}$ we assign an isomorphism of the corresponding derived endofunctors:

- The set of all $T_i$ satisfy braid relations:

$$T_i \circ T_j \circ \ldots \simeq T_j \circ T_i \circ \ldots \text{ with } m_{i,j} = \text{ord}_W(s_is_j).$$

- Each endofunctor $T_i$ is an autoequivalence of $D(\mathcal{B}[\xi]-\text{gmod})$ and the inverse endofunctor is isomorphic to the endofunctor

$$\text{cone}(t \circ \text{Res}_i \circ \text{LInd}_i \to \text{Id}_{\mathcal{B}}[-1]).$$

What means that we categorify the following equality

$$q_i^{-1} = t^{-1}q_i + t^{-1} - 1 \iff (q_i + 1)(q_i - t) = 0$$

- The pairs of functors $T_i$ and $X^\lambda$ do satisfy the following compatibility conditions:

$$X^\lambda \circ T_i \simeq T_i \circ X^\lambda, \text{ if } \langle \alpha_i, \lambda \rangle = 0;$$

$$X^\lambda \circ T_i \simeq t^{-1}T_i \circ X^\lambda-\alpha_i[1], \text{ if } \langle \alpha_i, \lambda \rangle = 1.$$  

The double affine Hecke algebra $\mathcal{H}$ almost coincides with the affine Hecke algebra of the affine Lie algebra $\hat{\mathfrak{g}}$ and its categorification is straightforward. In particular, the operators $\gamma^\mu \in \mathcal{H}$ indexed by elements of coroot lattice $Q'$ admit a well defined categorification (as endofunctors $Y^\mu$ of the derived category of the modules over supercurrents of Ivahori subalgebra $\mathcal{J}[\xi]$) and we can state our third categorification result:

**Theorem (Theorem 4.10).** For each dominant weight $\lambda$ there exists a well defined object $\mathcal{E}M_{\lambda} \in D^{-}(\mathcal{J}[\xi]-\text{gmod})$ such that the components of weight $\mu$ vanish in cohomology whenever $\mu \notin \lambda$, and, in addition,

- The subspace of weight $\lambda$ in $H^s(\mathcal{E}M_{\lambda})$ is equal to zero for $s \neq 0$, and is one-dimensional for $s = 0$;
- For all $\mu \in Q_+$ there exists an isomorphism $Y^\mu(\mathcal{E}M_{\lambda}) \cong q^{-\langle \mu, \lambda \rangle}(\mathcal{E}M_{\lambda})$.

Recall that nonsymmetric Macdonald polynomials are eigen-functions for the operators $\gamma^\mu \in \mathcal{H}$ and, therefore, the graded supercharacter of $\mathcal{E}M_{\lambda}$ is equal to $E_{\lambda}$.

The forth categorification result deals with the Cherednik symmetrization functor:

**Theorem (Theorem 2.20).** The composition of derived induction and restriction functors:

$$D(\mathcal{J}[\xi]-\text{gmod}) \xrightarrow{\text{LInd}_g} D(\mathfrak{g}[z, \xi]-\text{gmod}) \xrightarrow{\text{Res}_g} D(\mathcal{J}[\xi]-\text{gmod})$$
categorifies the Cherednik symmetrization functor \( \mathcal{P} := \sum_{w \in W} \tau_w \in \mathcal{H} \). Here \( \text{Ind}_g \) denotes the left adjoint to the restriction functor \( \text{Res}_b : \mathfrak{g}[z, \xi]-\text{gmod} \to \mathcal{I}[\xi]-\text{gmod} \).

Consequently, for each dominant \( \lambda \) the complex \( \mathbb{P}M_\lambda := \text{LInd}_g(\mathcal{E}M_\lambda) \in \mathcal{D}(\mathfrak{g}[z, \xi]-\text{gmod}) \) categorify the corresponding Macdonald polynomials \( P_\lambda \) (Corollary 4.24).

0.1. **Relation to the geometry of flag manifolds.** Let us discuss the relation between our construction and the geometry of flag manifolds. Let \( G \) be the simple algebraic group such that \( \mathfrak{g} = \text{Lie} \, G \), and let \( B \subset G \) be the Borel subgroup such that \( \mathfrak{b} = \text{Lie} \, B \). We consider the truncation Lie algebra

\[
\mathfrak{b}[\xi] := \frac{\mathfrak{b} \otimes \mathbb{C}[z]}{\mathfrak{b} \otimes \mathbb{C}[z] z^2} \cong \mathfrak{b} \times \mathfrak{b},
\]

where the last Lie algebra contains (the second, degree one copy of) \( \mathfrak{b} \) as an ideal with trivial bracket. We similarly define \( \mathfrak{b} \times \mathfrak{b}^* \) by the action \( \mathfrak{b} \circ \mathfrak{b}^* \). Let \( \mathcal{I} \) be the direct sum of injective objects in \( \mathfrak{b}\text{-gmod} \) prolonged to \( \mathfrak{b}[\xi]\text{-gmod} \) trivially. We have an equivalence of categories

\[
\mathbb{D}^{+}(\mathfrak{b}[\xi]\text{-gmod}) \longrightarrow \mathbb{D}^{-}(\mathfrak{b} \times \mathfrak{b}^*\text{-gmod})^{\text{op}}
\]

sending \( M^* \mapsto \text{Ext}_{\mathfrak{b}[\xi]\text{-gmod}}(M^*, \mathcal{I}) \). The abelian category \( \text{Coh}_{B \times \mathbb{G}_m, \mathfrak{b}} \) of \( (B \times \mathbb{G}_m)\text{-equivariant coherent sheaves on an affine space} \mathfrak{b} \) has a fully faithful embedding

\[
\text{Coh}_{B \times \mathbb{G}_m, \mathfrak{b}} \hookrightarrow \mathfrak{b} \times \mathfrak{b}^*\text{-gmod}
\]

by differentiating the \( B \)-action and regarding the \( \mathbb{G}_m \)-action as a grading. We can rewrite the LHS as

\[
\text{Coh}_{B \times \mathbb{G}_m, \mathfrak{b}} \hookrightarrow \text{Coh}_{G \times \mathbb{G}_m, (G \times B) \mathfrak{b}}
\]

by the \( G \)-action, where \( (G \times B) \mathfrak{b} \) is the \( G \)-equivariant vector bundle on \( G/B \) whose fiber at the point \( B/B \in G/B \) is \( \mathfrak{b} \), and the functor is the restriction to the fiber at \( B/B \).

Note that the fiber of \( T^* (G/B) \) at \( B/B \) is \( (\mathfrak{g}/\mathfrak{b})^* \cong \mathfrak{n} \), that induces an embedding

\[
i : T^* (G/B) \subset G \times B \mathfrak{b}
\]

through a \( (B \times \mathbb{G}_m) \)-equivariant embedding of affine spaces \( \mathfrak{n} \subset \mathfrak{b} \). The categorical affine Hecke algebra action \((\text{[BR]})\) on the LHS of

\[
(0.2) \quad \mathbb{D}^b(\text{Coh}_{G \times \mathbb{G}_m, T^* (G/B)}) \xrightarrow{i^*} \mathbb{D}^b(\text{Coh}_{G \times \mathbb{G}_m, (G \times B) \mathfrak{b}})
\]

consists of two portions: one is the twist by \( (G \times \mathbb{G}_m)\text{-equivariant line bundle on} G/B \) pulled back to \( T^* (G/B) \), while the other (corresponding to our \( \mathfrak{T}_i \)) is induced by the composition

\[
(0.3) \quad (s_i)_* \circ p_i^* \circ (p_i)_* \circ s_i^*
\]

of push-pull functors using the natural \( (G \times \mathbb{G}_m)\text{-equivariant maps}:

\[
T^* (G/B) \xleftarrow{s_i} G \times B \mathfrak{n}_i \xrightarrow{p_i} G \times P_i \mathfrak{n}_i,
\]

where \( P_i \supset B \) is a minimal parabolic subgroup of \( G \) and \( \mathfrak{n}_i \subset \mathfrak{n} \) is the nilradical of the Lie algebra of \( P_i \). These two actions naturally prolong to the RHS of \((0.2)\). We can check that our \( \mathfrak{T}_i \)-action is also related to an incarnation of \((0.3)\), and the line bundle twists on \( T^* (G/B) \) and the character twists on \( \text{Coh}_{B \times \mathbb{G}_m, \mathfrak{n}} (\subset \text{Coh}_{B \times \mathbb{G}_m, \mathfrak{b}}) \) correspond to each other.

The geometric construction in \([\text{BR}]\) itself extend to the case of affine flag variety \( X \) (or more general Kac-Moody flag variety). In the case of affine flag variety, we have a categorical action of \( \mathcal{H} \) on the derived category of the equivariant coherent sheaves on \( T^* X \). This triggers at least two problems in our considerations. The first problem is that one needs
to find some class of category that cannot be the category of coherent sheaves if one wants to obtain some numerical consequence (see e.g. [VV]). In particular, the tie between the algebraic and geometric constructions explained above (that was not an equivalence from the beginning) becomes even less transparent. The second problem is that one needs to be more careful on objects corresponding to the Macdonald polynomials. We have operators corresponding to $X$ and $Y$ in the categorical $\mathcal{H}$-action. The joint eigenvectors of the action of $X$, coming from the character twists, correspond to the torus fixed points of $X$. However, this is only via the localization theorem, and it does not have a straight-forward meaning in $\mathfrak{g}[z, \xi]$-gmod (as its objects correspond to Iwahori equivariant objects in $T^*X$, that cannot be supported on a point). The action of $Y$, coming from the composition of $T$, have some meaning in $\mathfrak{g}[z, \xi]$-gmod, but its geometric meaning in terms of $T^*X$ is rather unclear and not useful to analyze its eigenobjects.

These motivate us to provide an algebraic framework that are morally equivalent to the derived category of $T^*X$ that captures the object corresponding to the Macdonald polynomials with respect to the action of $Y$ (as in the authentic treatment of Macdonald polynomials [MD1]).

0.2. Conjectures verified for $\mathfrak{g} = \mathfrak{sl}_2$. The following list of conjectures should persuade the reader to dwell on the categorifications we suggest:

**Conjecture 0.4.** For each $\lambda \in \mathbb{P}$ with $\lambda = u_\lambda \lambda_-$ (where $\lambda_-$ is antidominant and $u_\lambda \in W$ is the shortest element in the finite Weyl group) there exists a complex of graded $\mathcal{I}[\xi]$-modules $EM_\lambda$ yielding the following properties:

- The weight support of the cohomology of $EM_\lambda$ is less or equal to $\lambda_-$;
- For all $\mu \in \mathbb{Q}^\vee$ there exists an isomorphism $Y^\mu(EM_\lambda) \cong t^{l(\mu)+\langle u_\lambda^{-1}(2\rho), \mu \rangle} q^{-\langle \lambda, \mu \rangle} EM_\lambda$, where $2\rho = \sum_{\beta \in \Delta_+} \beta$ and $l(\mu)$ is the length of the coweight $\mu$ considered as an element of the affine Weyl group $W^{af}$.

As we already mentioned we explain the existance of $EM_\lambda$ for dominant $\lambda$ in §4.5, the similar arguments can be used to construct $EM_\lambda$ for antidominant $\lambda$, however, these arguments does not work for general $\lambda$. The supercharacter of the conjectured complex $EM_\lambda$ should coincide with Macdonald polynomial $E_\lambda$ up to a common factor depending on $q$ and $t$ and this is why we can call them Macdonald modules.

We also believe that the complexes $PM_\lambda$ defined above as $L\text{Ind}_\mathfrak{g}(EM_\lambda)$ (for dominant $\lambda$) are eigen objects with respect to the action of the endofunctor $PY^\mu P$, just as the Macdonald polynomials $P_\lambda$ are eigen functions for the operators $PY^\mu P \in \mathcal{H}$. An essential difference to the nonsymmetric Macdonald polynomials us that the corresponding eigen value is not a monomial in $q$ and $t$ and the precise meaning of the categorical eigen function requires a delicate description which we want to omit here.

Moreover, any bounds of the vanishing of hom-spaces between different Macdonald complexes $EM_\lambda$ (whose existance we conjecture) will imply the categorification of the orthogo-

nality of nonsymmetric polynomials as outlined in the following

**Conjecture 0.5.** For all pairs of integral weights $\lambda \neq \mu$ the following vanishing property holds:

$$\text{hom}^*(EM_\lambda, EM_\mu) = 0.$$
For all pairs of integral dominant weights $\lambda \neq \mu$ the following holds:

$$\text{hom}^*(\mathbb{P}M_\lambda, \mathbb{P}M_\mu) = 0$$

The good news is that all aforementioned conjectures are verified for $g = \mathfrak{sl}_2$ and, moreover, everything can be categorified on the level of modules rather than complexes. For each $k \in \mathbb{Z}$ the $\mathfrak{g}[\xi]$-module $EIM_{k\omega}$ is said to be a cyclic module generated by a cyclic vector $w_{k\omega}$ of the weight $k\omega$ subject to the following list of relations:

$$\begin{align*}
e z^a \xi^b w_{k\omega} &= 0, a \geq 0, b = 0, 1; (fz)^k w_{k\omega} = 0; h\xi w_{k\omega} = 0, \text{ if } k \geq 0, \\
fz^a \xi^b w_{k\omega} &= 0, a > 0, b = 0, 1; e^{-k+1} w_{k\omega} = 0; h\xi w_{k\omega} = 0, \text{ if } k < 0.
\end{align*}$$

Let us outline the properties of the modules $EIM_{k\omega}$ we proved:

(Theorem 5.32) The supercharacter of the cyclic $\mathfrak{g}[\xi]$-module $EIM_{k\omega}$ is equal to the integral form of the nonsymmetric Macdonald polynomial $E_{k\omega}$. In particular, we have $\text{dim} EIM_{k\omega} = \text{dim} EIM_{-(k+1)\omega} = 4^k$ for $k \in \mathbb{Z}_{\geq 0}$.

(Theorem 5.34) There are isomorphisms for all $k \in \mathbb{Z}_{\geq 0}$:

$$Y^\omega (EIM_{k\omega}) \simeq q^{-k} EIM_{k\omega}; \quad Y^{-\omega} (EIM_{-k\omega}) \simeq t^{-1} q^k EIM_{k\omega}.$$  

(Corollary 5.40) The Ext-vanishing property $\text{hom}^*(EIM_{k\omega}, EIM_{m\omega}) = 0$ holds for all $k \neq m \in \mathbb{Z}$.  

0.3. Structure of the paper. The paper is organized as follows:

The main triangulated functors: restriction $\text{Res}_i$, derived induction $\text{LInd}_i$, and their superposition $D_i := \text{Res}_i \circ \text{LInd}_i$ are defined in §1.3 in order to formulate the categorification of the affine Hecke algebra in §1.4.

We work out by direct computation the detailed description of the Demazure functors $D_i$ in §2 which has to be done carefully for the case $g = \mathfrak{sl}_2$ (§2.1, §2.2). We compute the iterated superpositions of the corresponding abelian functors for sufficiently dominant weights in §2.3.

The modules we describe are found to be cyclic and may be viewed as the generalization of the Demazure modules and generalized Weyl modules introduced in [FM].

The proof of the categorification theorem for the affine Hecke algebra is contained in §3. We start with an explanation of the common methods in §3.1 and prove the categorification of each equality in a separate subsection. The exception is §3.3 where we prove an intermediate result on duality between induction and coinduction which is used several times later in the text.

Section 4 is devoted to categorification of (nonsymmetric) Macdonald polynomials. After recalling the standard notations in §4.1 we review the convex Cherednik partial order in §4.2. We then go to adapt the categories and triangulated functors we considered in §2.2 to the affine setting in §4.3 and §4.4. We conclude Section 4 by describing the desired categorification of nonsymmetric Macdonald polynomials for dominant weights. Section 4.6 contains the categorification of the Cherednik symmetrization operator and explains the relationship between complexes that categorify symmetric and nonsymmetric Macdonald polynomials.

The final Section 5 contains a detailed description of all objects of §4 worked out for $g = \mathfrak{sl}_2$. Surprisingly, for $\mathfrak{sl}_2$ one can categorify both symmetric and nonsymmetric Macdonald polynomials by means of cyclic modules, whose description is presented in §5.1.
Acknowledgements. We would like to thank I.Anono, A.Bondal, B.Feigin, A.Kuznetsov for stimulating discussions. We thank I.Marshall and E.Feigin for useful comments on the exposition of the text. A.Kh. was carried out within the HSE University Basic Research Program and funded (jointly) by the Russian Academic Excellence Project '5-100'. Ie.M. was partially supported by the grant RSF 19-11-00056 and JSPS KAKENHI Grant Number 18F18014. S.K was partially supported by JSPS KAKENHI Grant Number JP19H01782.

1. Categorification of AHA: statement of results

We work over an algebraically closed field $\k$ of characteristic zero. It is worth mentioning that many our constructions are well defined for positive characteristics and will be discussed elsewhere.

1.1. Notation conventions. It is worth mentioning to say that there are many different objects, that are denoted in the literature by the same letter. For example, the standard notations for the weight lattice, the Macdonald polynomial and the Cherednik symmetrization functor uses the capital letter $P$. We use different fonts in order to separate objects of different nature.

- the mathsf font is used for root and weight lattices $Q \subset P$;
- the font mathpzc is used for elements of Hecke algebra: e.g. $T_i$, $P$, $X$, $Y$;
- the ordinary font is used for Macdonald polynomials: e.g. $P_\lambda$, $E_\lambda$;
- the bold letters are used for the (derived) functors: e.g. $D_i$, $T_i$, $X^\lambda$, $t_i$;
- the mathbold letters are used for modules and complexes. E.g. $PM_\lambda$, $EM_\lambda$ are symmetric and nonsymmetric Macdonald complexes.

1.2. Setup. Suppose that $g$ is a simple Lie algebra that is not of type $G_2$. We denote its set of roots, together with a chosen set of positive roots by $\Delta = \Delta_+ \sqcup \Delta_-$ ($\Delta_- = -\Delta_+$). Let $\Pi := \{\alpha_1, \ldots, \alpha_r\} \subset \Delta_+$ be the set of simple roots and $I := \{1, \ldots, r\}$ be the indexing set of simple roots. These data uniquely defines the Cartan decomposition: $g = n_- \oplus h \oplus n_+ = n_- \oplus b_+$, we will mostly omit the index for the positive Borel subalgebra identifying $b$ and $b_+$. The $sl_2$-triple assigned to a positive root $\alpha_i \in \Delta_+$ is denoted by $\langle e_i = e_{\alpha_i}, h_i, f_i := e_{-\alpha_i} \rangle$, where $e_j$ is a fixed root vector for every $\beta \in \Delta$.

Let $P \subset h^*$ be the integral weight lattice of $g$. Let $Q \subset P$ be the sublattice spanned by $\Delta$ (root lattice). Denote by $P_+$ - the set of dominant weights, i.e. $\lambda \in P_+$ iff $\forall \alpha_i \in \Pi$ we have $\langle \lambda, \alpha_i \rangle \geq 0$. Let $Q^\vee$ be the dual lattice of $P$ with a natural pairing $\langle \bullet, \bullet \rangle : Q^\vee \times P \rightarrow \mathbb{Z}$. We define $\Pi^\vee \subset Q^\vee$ to be the set of positive simple coroots, and let $Q^\vee_+ \subset Q^\vee$ be the set of non-negative integer span of $\Pi^\vee$.

Let $W$ be the Weyl group of $g$. We fix bijections $I \cong \Pi \cong \Pi^\vee$ so that $i \in I$ corresponds to $\alpha_i \in \Pi$, its coroot $\alpha_i^\vee \in \Pi^\vee$, and a simple reflection $s_i \in W$ corresponding to $\alpha_i$. Let $\{\varpi_i\}_{i \in I} \subset P_+$ be the set of fundamental weights (i.e. $\langle \alpha_i^\vee, \varpi_j \rangle = \delta_{ij}$). We denote by $G$ the simply connected algebraic group over $\k$ whose Lie algebra is $g$. For each $J \subset I$, we have a parabolic Lie subalgebra $p_J \subset g$ generated by $b_+$ and the root subspaces $\{f_i\}_{i \in J}$. In case $J = \{i\}$, we might write $p_i$ or $p_{\alpha_i}$ instead of $p_{\{i\}}$. The corresponding algebraic groups are denoted by $B \subset P_J \subset G$.

Let $\k[\xi]$ be the polynomial superalgebra in one odd variable $\xi$. We use the shorter notation $g[\xi]$ for the corresponding current Lie superalgebra $g \otimes \k[\xi]$ and, moreover, for $x \in g$ and
For $p \in k[\xi]$ we write $xp$ instead of $x \otimes p$ and $x$ stands for $x \otimes 1$. The parity of $\xi$ predicts the following decomposition of $\mathfrak{g}$-modules:

$$g[\xi] = g + g\xi.$$  

We denote by $\mathcal{O} = \mathcal{O}(b_+[\xi])$ the category of $(\mathfrak{h}, \xi)$-graded finitely generated $n_+\text{-finite } b_+[\xi]-$modules. That is, a finitely generated $b_+[\xi]$-module $M$ belongs to $\mathcal{O}$ if it is a $\mathfrak{h}$-semisimple $b_+$-module with locally nilpotent actions of elements $e_i$ for all $\alpha \in \Delta_+$. The main object of study for us is the appropriate derived category associated with the aforementioned abelian one. If $\mathfrak{g}$ is finite we consider the ordinary bounded derived category that does not contain all objects we are interested in, however, we may consider the certain subcategory of $D^b(\mathcal{O}(b_+[\xi]))$ for which we can take the character: complexes that are bounded in each $\mathfrak{h}$-graded component and has the total bounds on the region of $\mathfrak{h}$-weights (see §4 for details).

The category $\mathcal{O}(b_+[\xi])$ (as well as all other categories in this paper) has two gradings: the weight grading and the $\xi$-degree grading. With each pair $(\lambda, m) \in P \oplus \mathbb{Z}$ a weight $\lambda$ and an integer $m$ we assign the bigrading shift functor $X^\lambda t^m : \mathcal{O}(b_+[\xi]) \rightarrow \mathcal{O}(b_+[\xi])$. The functor $X^\lambda t^m$ can be realized as a tensor product with the one-dimensional graded $b_+[\xi]$-module denoted by $k_\lambda \{m\}$ of $\xi$-degree $m$ and of $\mathfrak{h}$-weight $\lambda$. The functors $X^\lambda t^m$ defines the action of the free abelian group $P \oplus \mathbb{Z}$ on $\mathcal{O}(b_+[\xi])$:

$$X^\lambda t^m \circ X^\mu t^n = X^{\lambda + \mu} t^{m+n} \quad \lambda, \mu \in P, m, n \in \mathbb{Z}.$$  

The derived categories $D^\pm(\mathcal{O}(b_+[\xi]))$, $D^b(\mathcal{O}(b_+[\xi]))$ admit one more grading: the homological degree. We denote by $M \mapsto M[n]$ the $n$-shift of the homological degree. It is worth mentioning that homological shifts as well as the $\xi$-shift functor $t$ commutes with all other (endo)functors considered in this note.

1.3. Induction-restriction functors. With each parabolic subalgebra $p_i := b_+ \oplus \langle f_i \rangle$ assigned to a simple root $\alpha_i \in \Pi$ we associate a Lie subalgebra of currents:

$$g[\xi] \ni p_i[\xi] \ni b_+[\xi].$$  

The corresponding category of $(\mathfrak{h}, \xi)$-graded, $p_i$-integrable $p_i[\xi]$-modules is denoted by $\mathcal{O}(p_i)$. The embedding $n_+ [\xi] \subset p_i [\xi]$ of Lie superalgebras produces the restriction functor:

$$(1.1) \quad \text{Res}_i : \mathcal{O}(p_i[\xi]) \rightarrow \mathcal{O}(b_+[\xi]).$$  

The restriction functor is always exact since it does not change the underlying vector space. We show in Section 2.2 that $\text{Res}_i$ has both left and right adjoints which we denote by $\text{Ind}_i$ and $\text{CoInd}_i$ following the classical Frobenius reciprocity principle:

$$(1.2) \quad \mathcal{O}(b_+[\xi]) \xleftarrow{\text{Ind}_i} \mathcal{O}(p_i[\xi]) \xrightarrow{\text{Res}_i} \mathcal{O}(p_i[\xi])$$  

Note, however, that integrability affects the story in the following way: the left adjoint functor $\text{Ind}_i$ maps $b_+[\xi]$-module $M$ to the maximal $p_i$-integrable quotient of the induced module:

$$(1.3) \quad \text{Ind}_i : M \mapsto U(p_i[\xi]) \otimes_{U(b_+[\xi])} M/\sim,$$  

where $\sim$ is the equivalence relation generated by

$$(1.4) \quad U(p_i[\xi]) \otimes_{U(b_+[\xi])} \sim U(p_i[\xi]) \otimes_{U(b_+[\xi])} \mathfrak{h}_i,$$  

with $\mathfrak{h}_i$ the Cartan subalgebra of $p_i$. The right adjoint $\text{CoInd}_i$ is defined similarly.
respectively the image of the right adjoint functor \( \text{Coind}_i(N) \) coincides with the maximal \( p_i \)-integrable submodule in the coinduced module \( \text{Hom}_{b_i[\xi]}(U(p_i[\xi]), N) \). For algebraic groups the corresponding induction functor is known under the name Induction (see e.g. [Ya]), one also uses the name Zuckermann functor (see e.g. [KV]). We explain in Section 2 (Corollary 2.9) that the corresponding left (right) derived functors \( \text{LInd}_i \) and \( \text{RCoid}_i \) are well defined in the bounded derived category.

**Notation 1.4.**
- The endofunctors \( \text{Res}_i \circ \text{LInd}_i, \text{Res}_i \circ \text{RCoid}_i \in \text{End}(\mathbb{D}(\mathcal{O}(b_+[\xi]))) \) are denoted by \( \mathcal{D}_i \) (resp. \( \mathcal{D}_i' \)) and are called Demazure (resp. dual Demazure) functors;
- The mapping cone of the unit morphism \( \text{Id} \xrightarrow{\eta} \text{Res}_i \circ \text{LInd}_i \) assigned with adjunction (1.2) between restriction and induction functors is called the Demazure twist and is denoted by \( \mathcal{T}_i \);
- The mapping cocone (=shifted cone) of the counit morphism \( \text{Res}_i \circ \text{RCoid}_i \xrightarrow{\epsilon} \text{Id} \) is called the Demazure cotwist and is denoted by \( \mathcal{T}_i' \).

In other words we have a pair of adjoint exact triangles of endofunctors of the triangulated category \( \mathbb{D}(\mathcal{O}(b_+[\xi])) \):

\[
\text{Id}_{b[\xi]} \xrightarrow{\eta} \mathcal{D}_i \rightarrow \mathcal{T}_i \xrightarrow{+1} \text{Id}_{b[\xi]}[1], \quad \mathcal{T}_i' \rightarrow \mathcal{D}_i' \xrightarrow{\epsilon} \text{Id}_b \xrightarrow{+1} \mathcal{T}_i'[1]
\]

We show that the functor \( \text{Res}_i \) is spherical in the sense of [AL] (Proposition 3.6) and use the standard terminology of twist and cotwist functors assigned to spherical functors.

We want to underline that we use the fat bold font for (triangulated) functors and the ordinary font for the elements of Hecke algebras.

### 1.4. Categorification of Affine Hecke Algebra

Recall that the Affine Hecke Algebra (AHA) associated with the root system \( \Delta \) is the associative algebra over the field of rational functions \( k(t) \) generated by \( \{ \mathcal{T}_i \mid i \in I \} \) and \( \{ x^\lambda \mid \lambda \in \mathcal{P} \} \) that satisfy the following list of relations:

\[
\begin{align*}
(1.5) \quad & x^\lambda \mathcal{T}_i = \mathcal{T}_i x^\lambda, \text{ if } \langle \alpha_i, \lambda \rangle = 0; \\
(1.6) \quad & x^\lambda \mathcal{T}_i = t \mathcal{T}_i^{-1} x^{\lambda-\alpha_i}, \text{ if } \langle \alpha_i, \lambda \rangle = 1; \\
(1.7) \quad & \mathcal{T}_i^{-1} = t^{-1} \mathcal{T}_i + t^{-1} - 1 \Leftrightarrow (\mathcal{T}_i + 1)(\mathcal{T}_i - t) = 0; \\
(1.8) \quad & \underbrace{\mathcal{T}_i \circ \mathcal{T}_j \circ \ldots}_{\text{m}_{i,j} \text{ factors}} \simeq \underbrace{\mathcal{T}_j \circ \mathcal{T}_i \circ \ldots}_{\text{m}_{i,j} \text{ factors}}, \text{ with } m_{i,j} = \text{ord}_W(s_i s_j).
\end{align*}
\]

**Remark 1.9.** There are many different renormalizations of parameters in Hecke algebras (see e.g. [L1], [MD1], [Ch1]). First, in general one uses several parameters \( t_\alpha \) with \( \alpha \in \Delta \) and \( t_\alpha = t_{\sigma(\alpha)} \) for all \( \sigma \in W \), that might depend on the length of a simple root. It is important for our purposes, that all \( t_\alpha \) are equal to a single variable \( t \). Second, in order to have more symmetric presentation of Hecke Relation (1.7) one frequently uses the generators \( \widetilde{T}_i := t^{-1/2} \mathcal{T}_i \) instead of \( \mathcal{T}_i \):

\[
(\widetilde{T}_i + t^{-1/2})(\widetilde{T}_i - t^{1/2}) = 0 \Leftrightarrow (\mathcal{T}_i + 1)(\mathcal{T}_i - t) = 0
\]
The AHA has a standard (basic) polynomial representation $Z[t][P] := Z[P] \otimes Z[t, t^{-1}]$ (see e.g. [MD1 §4.3], [Ch2 p.310]) with the following (normalized) action of generators $T_i:
\[ ts_i + (s_i - 1) \frac{t - 1}{\chi^{\alpha_i} - 1}. \]

One of the main purposes of this paper is to state and to prove the following categorification result:

**Theorem 1.10.** The endofunctors $\{T_i|i \in \mathbb{I}\}$, $\{X^\lambda|\lambda \in P\}$, and $\{t^m|m \in \mathbb{Z}\}$, categorify the action of the affine Hecke algebra in the basic representation $Z[t][P]$ under the following correspondence:

$T_i \leftrightarrow T_i$, $X^\lambda \leftrightarrow X^\lambda$, $t \leftrightarrow t[1]

More precisely, there exist the following list of isomorphisms of triangulated endofunctors

(1.11) $X^\lambda \circ T_i \simeq T_i \circ X^\lambda$, if $\langle \alpha_i, \lambda \rangle = 0$;

(1.12) $X^\lambda \circ T_i \simeq t^{-1}T_i \circ X^{\lambda-\alpha_i}[1]$, if $\langle \alpha_i, \lambda \rangle = 1$.

(1.13)

$T_i \circ T'_i \simeq T'_i \circ T_i \simeq \text{Id}_{\mathbb{B}[g]}$;

$T_i \simeq \text{cone}(t \circ D_i) \rightarrow \text{Id}_{\mathbb{B}[g]}[1]) \simeq \text{cone}(t \circ \text{cone}(T_i[1] \rightarrow \text{Id}_{\mathbb{B}[g]}) \rightarrow \text{Id}_{\mathbb{B}[g]}[1])$

(1.14) $T_i \circ T_j \circ \ldots \simeq T_j \circ T_i \circ \ldots$, here $m_{i,j} = \text{ord}_W(s_is_j)$

and $t$ commutes with all other endofunctors.

**Corollary 1.15.**

- The endofunctors $\{T_i|i \in \mathbb{I}\}$, $\{X^\lambda|\lambda \in P\},\{t^m|m \in \mathbb{Z}\}$ categorify the Affine Hecke Algebra $\mathbb{H}$ assigned to the root system $\Delta$;
- Suppose, moreover, that $\Delta$ is the untwisted affine root system with the finite system $\Delta_0$. $W$ denotes the finite Weyl group, $W_{af} := W \times Q^\vee$ is the affine Weyl group and $\tilde{W} := W \times P^\vee$ is the extended affine Weyl group. The endofunctors $\{T_i|i \in \mathbb{I}\}, \{X^\lambda|\lambda \in P\},\{t^m|m \in \mathbb{Z}\}$ together with the group of diagram automorphisms $\Omega$ representing the elements of the quotient $P^\vee/Q^\vee$ categorify the Double Affine Hecke Algebra $\mathbb{H}$.

**Proof.** The proof of Theorem 1.10 is contained in Section 3.

In Section 3.4 we prove that the restriction functor $\text{Res}_i$ is a spherical functor. We compute its dual twist and cotwist based on computations with $g = \mathfrak{sl}_2$, Hecke Relations (1.13) will follow from the general properties of spherical functors known for specialists in triangulated categories.

Relations (1.11), (1.12) are also based on a computation for $g = \mathfrak{sl}_2$ and are explained in Section 3.2.

The most complicated equivalences for us are Braid Relations (1.14). The case-by-case proof (except $G_2$) is contained in Sections 3.3.1. The strategies in all cases are the same: First, we expand all arrows in the iterated composition, representing $T_i$ as a cone of a morphism $\text{Id}_{\mathbb{B}} \rightarrow \mathbb{D}_i$ and get a complex of functors. Second, thanks to the computation of the iterated inductions given in Section 2.3 we know the 0th cohomology of compositions of $\mathbb{D}_i$'s and we can construct an equivalence from the superposition of $T_i$'s to the complex of functors (called $\text{BGG}_{ij}$) whose components are numbered by vertices of the Bruhat graph associated.
with the corresponding rank 2 subsystem. The latter complex $\mathbf{BGG}_{ij}$ is symmetric with respect to the interchanging $\alpha$ and $\beta$ what implies Braid relation \((1.14)\).

\[ \]

\section{2. Detailed descriptions of the derived superinduction}

\subsection{2.1. The derived induction for $\mathfrak{sl}_2$}

Let $\mathfrak{b} := \langle e, h \rangle$ be a Borel subalgebra of the Lie algebra $\mathfrak{sl}_2 = \langle e, h, f \rangle$. The category of $\mathfrak{sl}_2$-integrable modules is equivalent to the category of $\mathfrak{SL}_2$-integrable modules. Therefore, the derived induction $\text{Ind} : \mathcal{O}(\mathfrak{b}) \to \mathcal{O}(\mathfrak{sl}_2)$ from the category of $\mathfrak{b}$-integrable modules to the category of $\mathfrak{sl}_2$-integrable modules has a presentation via the induction for groups (see \cite{Ya}). Each indecomposable $\mathfrak{b}$-integrable module admits a cyclic vector $v_\lambda$ and is uniquely defined by the lowest and the highest weights:

$$V_{\lambda,\mu} := \mathbb{C}[e]/(e^{\mu+\lambda+1})v_\lambda = \langle v_\lambda, e v_\lambda, e^2 v_\lambda, \ldots, e^k v_\lambda \rangle, \text{ here } k = \frac{\mu - \lambda}{2} \in \mathbb{N} \& hv_\lambda = \lambda v_\lambda.$$ 

Each indecomposable $\mathfrak{sl}_2$-integrable module is irreducible and is indexed by its highest weight:

$$L(\lambda) := \langle v_\lambda, f v_\lambda, \ldots, f^\lambda v_\lambda = v_{\lambda -} \rangle = \langle v_{-\lambda}, e v_{-\lambda}, \ldots, e^\lambda v_{-\lambda} \rangle$$

\textbf{Lemma 2.1.} The functor $L^\bullet$ defined on indecomposables $\mathfrak{b}$-modules in the following way:

$$L^0(V_{\lambda,\mu}) = \begin{cases} L(\mu) \oplus L(\mu - 2) \oplus \ldots \oplus L(\mu - |\lambda|), & \text{if } \mu \geq \lambda \geq -\mu; \\ 0, & \text{otherwise}. \end{cases}$$

$$L^{-1}(V_{\lambda,\mu}) = \begin{cases} L(-\lambda - 2) \oplus L(-\lambda - 4) \oplus \ldots \oplus L(|\mu| - 1 + \text{sgn}(\mu)), & \text{if } -\lambda - 2 \geq \mu \geq \lambda; \\ 0, & \text{otherwise}. \end{cases}$$

satisfies the properties of the universal $\delta$-functor assigned to the right exact functor $\text{Ind}$ if it commutes with the tensor product with the standard two-dimensional representation of $\mathfrak{sl}_2$.

\textbf{Proof.} This is an easy direct computation whose details are left to the reader. First, one has to compute the induction $\text{Ind}(V_{\lambda,\mu})$ and, second, verify the exactness of the image of $L^\bullet$ for all nontrivial extensions of $\mathfrak{b}$-modules:

$$0 \to V_{\lambda,\mu} \to V_{\mu+2,\mu} \to V_{\lambda,\mu} \to 0 \text{ with } \nu \geq \mu \geq \lambda.$$ 

This latter exact sequence is determined uniquely if we take into account the $\mathfrak{b}$-module isomorphism $\mathbb{C}^2 \otimes V_{\lambda,\mu} \cong V_{\lambda+1,\mu+1} \oplus V_{\lambda+1,\mu-1}$. \hfill \Box

Recall that an object $M$ of an abelian category $\mathcal{C}$ is called $F$-acyclic if $L^i F(M) = 0$ for all negative $i < 0$. Here $F : \mathcal{C} \to \mathcal{C}'$ is an additive right exact functor between abelian categories. In particular, Lemma \textbf{2.1} explains that modules $V_{\lambda,\mu}$ are $L\text{Ind}$-acyclic whenever $\lambda + \mu \geq 0$. Moreover, the short exact sequence \textbf{(2.2)} for $\nu > -\lambda$ is an $L\text{Ind}$-acyclic resolution of $V_{\lambda,\mu}$.

\subsection{2.2. The derived superinduction $L\text{Ind}_i$}

Let us denote by $p_\uparrow^i$ the intermediate Lie subalgebra spanned by $\mathfrak{b}_+$ and $p_\downarrow^i$:

$$\mathfrak{b}_+[\xi] \subset p_\uparrow^i \subset p_\downarrow^i[\xi]$$

The quotient $\mathfrak{p}_\uparrow^i/[\mathfrak{b}_+[\xi]$ is a one-dimensional odd space spanned by $f_i \xi$. Respectively $p_\downarrow^i[\xi]/p_\uparrow^i$ is even and is spanned by $f_i$. The (co)induction functors admits decomposition into consecutive
(co)inductions:

\[
\begin{array}{ccc}
\mathcal{O}(b_+[\xi]) & \xrightarrow{\operatorname{Ind_i}^1} & \mathcal{O}(p_i[\xi]) \\
\mathcal{O}(p_i[\xi]) & \xrightarrow{\operatorname{Res_i}^1} & \mathcal{O}(b_+[\xi])
\end{array}
\]

(2.3)

The key advantage of this decomposition is that the odd derived induction \(\operatorname{Ind}_i^1\) is an exact functor and coincides with the usual induction. This happens because the universal enveloping algebra of an odd Lie one-dimensional superalgebra \(\langle f, \xi \rangle\) is two-dimensional. Consequently, \(\dim(\operatorname{Ind}_i^1 M) = 2 \dim M\) and, moreover, we have a short exact sequence of \(b[\xi]\)-modules:

\[
0 \to M \to \operatorname{Ind}_i^1 M = U(p_i) \otimes_{U(b_+)} \mathcal{O}(b_+[\xi]) \to (f, \xi) M \to 0
\]

(2.4)

what implies that the following triangle of functors is distinguished:

\[
\operatorname{Id}_{b[\xi]} \xrightarrow{} \operatorname{Res}_i^1 \circ \operatorname{Ind}_i^1 \to X^{-\alpha} t
\]

(2.5)

Similarly, we have the similar distinguished triangle of functors with coinduction:

\[
X^\alpha t^{-1} \to \operatorname{Res}_i^1 \circ \operatorname{Coind}_i^1 \xrightarrow{e_i} \operatorname{Id}_{b[\xi]}
\]

Remark 2.6. The short exact sequence (2.4) may not split in general. For example, consider \(g = sl_2\) and two-dimensional cyclic module \(M = U(b[\xi])/\langle ev, hv, (e\xi) v \rangle\) whose basis is \(\langle v, (h\xi) v \rangle\). The module \(\operatorname{Ind}_i^1 M\) is the cyclic module generated by the same cyclic vector \(v \) subject to the same set of relations \(ev = hv = (e\xi) v = 0\) and has the basis \(\langle v, (f\xi) v, (h\xi) v, (f\xi)(h\xi) v \rangle\). We have \(e(f\xi) v = (h\xi) v \neq 0\) and on the other hand the action of \(e\) on \(M\) is trivial what implies that \(\operatorname{Ind}_i^1 M\) is not isomorphic to the direct sum of \(M\) and \(M\) shifted.

Lemma 2.7. The categories \(\mathcal{O}(p_i^1)\) (resp. \(\mathcal{O}(b[\xi])\)) has enough \(L\operatorname{Ind}_i^0\)-acyclic (resp. \(L\operatorname{Ind}_i\)-acyclic) objects and, moreover, each object admits an acyclic resolution of length at most 2.

Proof. Let us denote the Borel subalgebra \(\langle e_i, h_i \rangle\) of the \(sl_2\)-triple assigned to the simple root \(\alpha\) by \(b_i\). Thanks to Lemma 2.1 we know that a \(p_i^1\)-module \(M\) is \(L\operatorname{Ind}_i^0\)-acyclic if each direct summand \(V_{\lambda, \mu}\) of \(M\) (considered as \(b_i\)-module) satisfies \(\lambda + \mu \geq 0\). Consequently, a \(b[\xi]\)-module \(M\) is \(L\operatorname{Ind}_i\)-acyclic whenever each direct summand \(V_{\lambda, \mu}\) of the \(b_i\)-module \(M\) yields the inequality \(\lambda + \mu \geq 4\). Denote by \(L_i^r\) the following \(p_i^1\)-module generated by a cyclic vector \(v\), subject to the following list of relations

\[
h_i v = (g_\xi) v = e_j v = e_i^{r+1} v = 0, \text{ with } j \neq i.
\]

The module \(L_i^r\) is of dimension \(r + 1\) and has a basis \(\{e_i^j v | 0 \leq i \leq r\}\). The submodule generated by \(e_i v\) is isomorphic to the \(\alpha\)-shifted module \(X^\alpha(L_i^r) = L_i^r \otimes_k k_i\). Thus we have a short exact sequence of \(p_i^1\)-modules:

\[
0 \to X^\alpha(L_i^r) \to L_i^r \to k \to 0
\]

(2.8)
The tensor product over the base field $\mathbb{k}$ of any given finite-dimensional $p^{\xi}_i$-module (resp. $b^{\xi}_i$-module) $M$ with the above short exact sequence (2.8) defines an $L\text{Ind}^{\bar{1}}_i$-acyclic (resp. $L\text{Ind}_i$-acyclic) resolution of $M$ for $r \gg 0$:

$$0 \to M \otimes_\mathbb{k} X^\alpha(L^r_i) \to M \otimes_\mathbb{k} L^r_i \to M \to 0$$

because the tensor product of $b_i$-module $V_{\lambda, \mu}$ and $L^r_i$ is $L\text{Ind}_i$-acyclic whenever $r > 4 \max(0, -\lambda)$.  

One can repeat the same arguments to the derived coinduction functors $\text{RCoind}_i$ and we end up with the following

**Corollary 2.9.** The right exact induction functors $\text{Ind}_i : O(b^{\xi}_i) \to O(p^{\xi}_i)$ as well as the left exact coinduction functor $\text{Coind}_i$ admit derived functors $L\text{Ind}_i, \text{RCoind}_i : D^b(O(b^{\xi}_i)) \to D^b(O(p^{\xi}_i))$, such that the corresponding $\delta$-functors will have only two nonzero cohomology:

$$H^i(L\text{Ind}(-)) = 0, \text{ if } i \neq 0, -1;$$
$$H^i(\text{RCoind}(-)) = 0, \text{ if } i \neq 0, 1.$$

The direct inspection of $\mathfrak{sl}_2$ computations also leads to the following useful Corollary:

**Corollary 2.10.** The (cohomological) (super)induction commutes with the tensor product with integrable representations. Namely, for any $p^{\xi}_i$-integrable module $M$ and an $b^{\xi}_i$-module $N$ one has an isomorphism of $b^{\xi}_i$ modules:

$$\text{Ind}_i(\text{Res}_i(M) \otimes N) \simeq M \otimes \text{Ind}_i(N), \text{ Ind}_i(\text{Res}_i(M)) = M;$$

$$\text{LInd}_i(\text{Res}_i(M) \otimes N) \simeq M \otimes \text{LInd}_i(N).$$

Let us finish this section with the detailed description of $D_i$ and $D'_i$ in particular cases:

**Example 2.13.** Suppose that $\langle \lambda, \alpha^\vee_i \rangle = n \geq 2$ then $D_i(\mathbb{k}_\lambda)$ is a module concentrated in 0'th homological degree:

$$D_i(\mathbb{k}_\lambda) = \begin{pmatrix}
\begin{array}{c}
\cdots \\
\bullet \\
\bullet \\
\bullet \\
\bullet
\end{array}
\end{pmatrix}
$$

We denote by $v_\lambda$ the generator of the one-dimensional module $\mathbb{k}_\lambda$. Vertices correspond to the basis vectors of the module $D_i(\mathbb{k}_\lambda)$, edges are responsible for the action of elements $e_i$ and $e_i \xi$. The horizontal grading is the weight $\mathfrak{sl}_2^{\alpha=1}$-grading and the vertical one corresponds to the $\xi$-grading. The complex $T_i(\mathbb{k}_\lambda) = \text{cone}(\mathbb{k}_\lambda \to D_i(\mathbb{k}_\lambda))$ consists of the quotient module placed in 0'th homological degree:

$$T_i(\mathbb{k}_\lambda) = \begin{pmatrix}
\begin{array}{c}
\cdots \\
\bullet \\
\bullet \\
\bullet \\
\bullet
\end{array}
\end{pmatrix} \times \begin{pmatrix}
\cdots \\
\bullet \\
\bullet \\
\bullet \\
\bullet
\end{pmatrix}.$$
On the other hand the coinduction functors $D'_i(k_\lambda)$ is concentrated in 1’st homological degree, whenever $\langle \lambda, \alpha_i^\vee \rangle = n \geq 2$:

$$D'_i(k_\lambda) = \begin{pmatrix} e_\xi & \cdots & \cdots & \cdots & e_i \\ e_i & \cdots & \cdots & \cdots & e_i \end{pmatrix} [-1].$$

The complex $T'_i$ is also concentrated in one homological degree and consists of the extension of the aforementioned module $D'_i(k_\lambda)[1]$ with a vector $v_\lambda$:

$$T'_i(k_\lambda) = \begin{pmatrix} e_\xi & \cdots & \cdots & \cdots & e_i \\ e_i & \cdots & \cdots & \cdots & e_i \end{pmatrix} [-1].$$

Note that all modules we described above admits a cyclic vector and there is an isomorphism

$$(2.16) \quad tT'_i(k_\lambda)[1] \simeq T_i(k_{\lambda_\alpha_i}) \text{ for } \langle \lambda, \alpha_i^\vee \rangle \geq 2.$$

**Remark 2.17.** It is easy to derive from Example 2.13 that the character of the functor $T_i$ is equal to the action of the generator $T_i$ of the affine Hecke algebra $H$ in the polynomial representation. Compare with [Ch2] Equation (3.2.27) on page 311.

2.3. **Iterating inductions for sufficiently dominant weights.** In order to get better feeling of the induction functor, we describe the family of modules $\mathbb{W}_D_\lambda$ which we call super-Demazure modules and realize them as a composition of induction functors.

**Definition 2.18.** With each weight $\lambda \in \mathbb{P}$ we assign a cyclic $b[\xi]$-module $\mathbb{W}_D\lambda$ generated by a cyclic vector $v_\lambda$ of weight $\lambda$ subject to the integrability relations:

$$(2.19) \quad \forall i \in I \text{ such that } \langle \alpha_i, \lambda \rangle < 0 \quad e_i^{-\langle \alpha_i, \lambda \rangle + 1} v_\lambda = 0,$$

$$\forall i \in I \text{ such that } \langle \alpha_i, \lambda \rangle \geq 0 \quad e_i v_\lambda = 0,$$

$$\forall h \in \mathfrak{h} \quad (h\xi)v_\lambda = 0.$$

**Theorem 2.20.** Suppose $\lambda$ is a dominant integral weight and $s_{i_1} \ldots s_{i_k}$ is a reduced decomposition of a given element $\sigma$ of the Weyl group associated with the root system $\Delta$. Then the iterated induction $(\text{Res}_{\alpha_{i_1}} \circ \text{Ind}_{\alpha_{i_1}}) \circ \ldots \circ (\text{Res}_{\alpha_{i_k}} \circ \text{Ind}_{\alpha_{i_k}})(k_\lambda)$ is isomorphic to the cyclic $b[\xi]$-module $\mathbb{W}_D\lambda$ and, in particular, does not depend on a reduced decomposition of $\sigma$.

**Proof.** The proof is by induction on the length of $\sigma$ and repeats the standard arguments known for the Demazure modules in the classical case. The base of induction corresponds to the case $\sigma = 1$. For dominant $\lambda$ we have $\langle \lambda, \alpha_i \rangle \geq 0$ for all $i \in I$ and, consequently, $\mathbb{W}_D\lambda = k_\lambda$ since the elements $\{e_i | i \in I\} \cup \mathfrak{h}[\xi]$ generates the Lie superalgebra $b[\xi]$.

For the induction step we suppose that a simple root $i \in I$ is chosen in such way that $l(s_i \sigma) = l(\sigma) + 1$. Note that the Zukerman induction functor $\text{Ind}_i$ is the quotient of the usual induction. Therefore, the $b[\xi]$-module $\text{Ind}_i(\mathbb{W}_D(\lambda))$ admits a cyclic vector $v'$ of weight $\sigma(\lambda)$ (that equals the image of the cyclic vector $v'$ of the $b[\xi]$-module $\mathbb{W}_D(\lambda)$ of weight $\sigma(\lambda)$).
However, the condition \( l(s_i \sigma) = l(\sigma) + 1 \) implies that \( d := \langle \sigma(\lambda), \alpha \rangle > 0 \) and the \( \mathfrak{sl}_2 \)-submodule generated by \( v' \) is of dimension \( d + 1 \). Consequently, the following list of relations are satisfied for the action of \( \mathfrak{p}_i[\xi] \) on \( v' \) in the cyclic module \( \text{Ind}_i(\mathbb{V}D_{\sigma(\lambda)}) \):

\[
\begin{align*}
  e_i v' &= 0, \quad f_i^{d+1} v' = 0, \\
  &\forall j \in I \text{ such that } \langle \alpha_j, \sigma(\lambda) \rangle < 0 \quad e_j^{-(\langle \alpha_j, \sigma(\lambda) \rangle + 1)} v_\lambda = 0, \\
  &\forall j \in I \text{ such that } \langle \alpha_j, \sigma(\lambda) \rangle \geq 0 \quad e_j v_\lambda = 0, \\
  &\forall h \in \mathfrak{h} \quad (h \xi) v_\lambda = 0.
\end{align*}
\]

(2.21)

Let us show that this is the full set of relations in the module \( \text{Ind}_i(\mathbb{V}D_{\sigma(\lambda)}) \) and that the restriction functor \( \text{Res}_i \) of this module coincides with \( \mathbb{V}D_{s_i \sigma(\lambda)} \). Indeed, let us look more carefully on the cyclic \( \mathfrak{b}[\xi] \)-module \( \mathbb{V}D_{s_i \sigma(\lambda)} \) generated by the cyclic vector \( v \) of weight \( s_i \sigma(\lambda) \).

We claim that \( \mathbb{V}D_{s_i \sigma(\lambda)} \) is isomorphic to the cyclic \( \mathfrak{p}_i[\xi] \)-module generated by the same cyclic vector \( v \) subject to Relations (2.19) and the additional trivial relations \( f_i v = (f_i \xi) v = 0 \). The actions of \( f_i \) and \( e_i \) are locally nilpotent by inspection. Therefore, \( \mathbb{V}D_{s_i \sigma(\lambda)} \) belongs to the category \( \mathcal{O}(\mathfrak{p}_i[\xi]) \) and Isomorphism (2.21) implies the coincidence

\[
\text{Ind}_i(\mathbb{V}D_{s_i \sigma(\lambda)}) = \mathbb{V}D_{s_i \sigma(\lambda)}.
\]

Finally, let us consider the automorphism \( s_i \) of the parabolic Lie subalgebra \( \mathfrak{p}_i \). The image of the set of positive roots \( \Delta_+ \) is the set \( (\Delta_+ \cup \{-\alpha_i\}) \setminus \{\alpha_i\} \) and, therefore, \( \mathbb{V}D_{s_i \sigma(\lambda)} \) is the cyclic \( s_i(\mathfrak{b})[\xi] \)-module generated by the cyclic vector \( s_i(v) = e_i^d v \) subject to Relations (2.21) that differs from Relations (2.19) by the action of \( s_i \). Therefore, the quotient of \( U(\mathfrak{p}_i)v' \) subject to Relations (2.21) is an \( \mathfrak{sl}_2 \)-integrable module \( \mathbb{V}D_{s_i \sigma(\lambda)} \) and no additional relations has to be applied after induction \( \text{Ind}_i \). Consequently, modules \( \mathbb{V}D_{s_i \sigma(\lambda)} \) and \( \text{Ind}_i(\mathbb{V}D_{\sigma(\lambda)}) \) coincide and we finish the proof of Theorem 2.20. \( \square \)

3. Proof of the main categorification Theorem 1.10

3.1. General strategy. Theorem 1.10 consists of several isomorphism of different endofunctors of the bounded derived category of the abelian category \( \mathcal{O}(\mathfrak{b}[\xi]) \). The strategy of all proofs of these isomorphisms uses the following scheme:

First, we present a natural transformation \( \varepsilon : F \Rightarrow G \) connecting the pair of endofunctors \( F, G \in \text{End}(\mathbb{D}^b(\mathcal{O}(\mathfrak{b}[\xi]))) \) that appear in the left hand and in the right hand sides of the stated isomorphism. This is the most conceptual part of each isomorphism and is explained in details in each particular case. Note, that all functors under consideration interact with the finite subsystem \( \Pi_0 \subset \Pi \) of ranks 1 or 2. Therefore, it is enough to explain the corresponding equivalence \( \varepsilon : F \Rightarrow G \) for the corresponding rank \( \leq 2 \) finite-dimensional Lie algebra \( \mathfrak{g}_0 \).

For the second step, we check that for all sufficiently dominant integral weights \( \lambda \) the map \( \varepsilon_{\lambda} : F(\mathfrak{k}_\lambda) \rightarrow G(\mathfrak{k}_\lambda) \) is a quasiisomorphism. Recall that \( \mathfrak{k}_\lambda \) denotes the irreducible one-dimensional \( \mathfrak{b}[\xi] \) module of weight \( \lambda \in \mathfrak{p} \). Where sufficiently dominant means that \( \langle \lambda, \alpha_i \rangle \gg 0 \) for all \( \alpha_i \in \Pi_0 \). This step is also checked directly in each particular case. The dominance condition implies vanishing of almost all cohomologies of \( F(\mathfrak{k}_\lambda) \) and \( G(\mathfrak{k}_\lambda) \).

The remaining third step is the same in all cases and we will not repeat it many times.

Thanks to Corollary 2.10 we notice that both endofunctors \( F \) and \( G \) commutes with tensor products with \( \mathfrak{p}[\xi] \)-integrable modules for the parabolic subalgebra \( \mathfrak{p} \supset \mathfrak{b} \) associated with the subset \( \Pi_0 \):

\[
\forall M \in \mathcal{O}(\mathfrak{p}[\xi]), \ L \in \mathcal{O}(\mathfrak{b}[\xi]) \quad F(\text{Res}(M) \otimes L) \simeq M \otimes F(L); \quad G(\text{Res}(M) \otimes L) \simeq M \otimes G(L)
\]
The abelian category of finite-dimensional \( b[x] \) modules are generated by the \( k_\lambda \) with \( \lambda \) sufficiently dominant together with their tensor products with \( p[\xi] \)-integrable modules. The category \( \mathcal{O}(b[\xi]) \) is the full subcategory of the abelian category generated by pro-objects constructed from finite-dimensional ones. This implies that \( \epsilon : F \to G \) is an equivalence of triangulated functors.

3.2. Relations \([1.12]\). The commutativity of \( X^\mu \) and \( T_i \) for \( \langle \mu, \alpha_i \rangle = 0 \) is obvious since they work with noninteracting weights. Relation \([1.12]\) is more involved and requires the map of functors.

As we know from \([2.5]\) there is an equivalence:

\[
\text{cone}(\text{Id}_{b[\xi]} \to \text{Res}_i^1 \circ \text{Ind}_i^1) \simeq X^{-\alpha_i} t
\]

Together with the unit morphism \( \text{Id}_{p_i^1} \to \text{Res}_i^0 \circ \text{LInd}_i^0 \), we get a morphism of functors \( \nu \):

\[
\begin{array}{ccc}
X^{-\alpha_i} t & \xrightarrow{\nu} & T_i \\
\text{cone} \left( \text{Id}_{b[\xi]} \to \text{Res}_i^1 \circ \text{Ind}_i^1 \right) & \xrightarrow{\nu} & \text{cone} \left( \text{Id}_{b[\xi]} \to \text{Res}_i^1 \circ (\text{Res}_i^0 \circ \text{LInd}_i^0 \circ \text{Ind}_i^1) \right).
\end{array}
\]

From the definition of \( T_i' \) we know that the functor \( tX^{-\alpha_i/2} \circ T_i'[1] \circ X^{-\alpha_i/2} \) is equivalent to the cone:

\[
\text{cone} \left( tX^{-\alpha_i/2} \circ D_i' \circ X^{-\alpha_i/2} \xrightarrow{\text{1gpg}} X^{-\alpha_i} t \right).
\]

Consequently, \( \nu \) defines a map of functors:

\[
tX^{-\alpha_i/2} \circ T_i'[1] \circ X^{-\alpha_i/2} \xrightarrow{\nu} T_i \iff tX^{-\alpha_i/2} \circ T_i'[1] \xrightarrow{\nu} T_i \circ X^{-\alpha_i/2}.
\]

**Proposition 3.1.** The map \( \nu : tX^{\mu-\alpha_i} T_i'[1] \to T_i X^\mu \) defined for \( \langle \mu, \alpha_i \rangle = 1 \) is an equivalence of functors.

**Proof.** The equivalence can be checked directly for all irreducible one-dimensional modules \( k_\lambda \) with \( \lambda \) sufficiently dominant. Indeed, the module \( T_i X^\mu(k_\lambda) \) is the quotient of the \( sl_2^{\alpha_i} \)-integrable module \( \mathbb{W}D_{\mu+\lambda} \) by the one-dimensional submodule \( k_{\lambda+\mu} \). On the other hand, \( T_i'(k_\lambda) \) is the homologically shifted extension of the one-dimensional module \( k_\lambda \) and the \( sl_2^{\alpha_i} \)-integrable module. Both modules \( T_i(k_{\lambda+\mu}) \) and \( T_i'(k_\lambda)[1] \) are cyclic \( b_i[\xi] \) modules. In order to show that \( \nu \) is an equivalence between \( T_i(k_{\lambda+\mu}) \) and \( tX^{\mu-\alpha_i} T_i'(k_\lambda)[1] \) it remains to show the coincidence of characters and that the image of a cyclic vector is a cyclic vector.

The general strategy of tensoring with \( p_i \)-integrable modules outlined in Section 3.1 finishes the proof. \( \square \)

3.3. Duality between Induction and Coinduction. Let \( p_j \) be a parabolic subalgebra of \( g \) assigned to the subset \( J \subset I \) of the indexing set of simple roots \( \Pi \). Let \( n_J \) be the corresponding nilpotent subalgebra generated by \( e_i \) with \( i \in J \) and \( \rho_J \) denotes the half-sum of all weights in \( n_J \). One can repeat the arguments of Section 2.2 in order to explain the existence of the corresponding collection of restriction and derived (co)induction functors:

\[
\begin{array}{ccc}
\mathbb{D}^b(\mathcal{O}(b[\xi])) & \xrightarrow{\text{LInd}_J} & \mathbb{D}^b(\mathcal{O}(p_J[\xi])) \\
\mathbb{D}^b(\mathcal{O}(p_J[\xi])) & \xleftarrow{\text{RCoind}_J} & \mathbb{D}^b(\mathcal{O}(b[\xi]))
\end{array}
\]
We can also split (co)induction $\mathbf{L}\text{Ind}_j$ into the composition of the even $\mathbf{L}\text{Ind}_j^0$ and the odd $\mathbf{L}\text{Ind}_j^1$ (co) inductions while considering an intermediate Lie subalgebra $\mathfrak{b}[\xi] \subset \mathfrak{b} \oplus \mathfrak{p}_j \xi \subset \mathfrak{p}_j[\xi]$.

**Proposition 3.2.** The derived inductions and coinductions are equivalent up to appropriate shifts:

\[
\text{(3.3)} \quad \text{RCoid}_{j}^i \simeq \text{LInd}_{j}^1 \circ X^{2\rho_j} t^{-\dim n_j} \simeq X^{2\rho_j} t^{-\dim n_j} \circ \text{LInd}_{j}^1
\]

\[
\text{(3.4)} \quad \text{RCoid}_{j}^0 \simeq \text{LInd}_{j}^0 \circ X^{-2\rho_j} [-\dim n_j]
\]

\[
\Rightarrow \text{RCoid}_{j}^i \simeq \text{LInd}_{j} \circ t^{-\dim n_j} [-\dim n_j]
\]

**Proof.** The odd induction $\mathbf{L}\text{Ind}_j^1$ is an exact functors on the level of abelian categories given by the tensor product with the symmetric algebra $\Lambda^\ast (\mathfrak{n}_- \xi)$. Respectively, the odd coinduction $\text{RCoid}_{j}^1$ is given by the tensor product with the dual space $\text{Hom}(\Lambda^\ast (\mathfrak{n}_- \xi), \mathbb{k})$. The Cartan pairing between $\mathfrak{n}_-$ and $\mathfrak{n}_j^+$ leads the isomorphism

$$\text{Hom}(\Lambda^\ast (\mathfrak{n}_j), \mathbb{k}) \simeq (\Lambda^\ast (\mathfrak{n}_j^+), \mathbb{k}) \simeq X^{2\rho_j} \Lambda^\ast (\mathfrak{n}_j^-),$$

what implies the Isomorphism (3.3).

The existence of the even derived(co)induction functors $\mathbf{L}\text{Ind}_j^0$ (resp. $\text{RCoid}_{j}^0$) is a straightforward generalization of the arguments considered for the case $\mathfrak{g} = \mathfrak{sl}_2$ in §2.1. The Isomorphism (3.4) is the Serre duality isomorphism for the flag variety $\text{Fl} := \mathbb{P}_j[\mathbb{R}]$. Here $-2\rho_j$ is the weight of the top exterior power of the cotangent bundle $\mathcal{T}_{\text{Fl}}$ and $\dim \mathfrak{n}_j$ is equal to the dimension of the flag manifold $\text{Fl}$. See e.g. [Ya] for the detailed proof of the Serre duality in this case and its comparison with derived(co)induction functors. \(\square\)

We want the reader to get used to the shorter notations

$$\mathcal{D}_i := \text{Res}_i \circ \mathbf{L}\text{Ind}_i \text{ and } \mathcal{D}'_i := \text{Res}_i \circ \text{RCoid}_i.$$

Here we repeat the main outcome of Proposition 3.2

$$\mathcal{D}'_i t[1] \simeq \mathcal{D}_i.$$

### 3.4. Hecke quadratic relation for $T_i$.

**Proposition 3.6.** The derived functors $\mathbf{L}\text{Ind}_i$ are spherical in the sense of [Kn]. That is, the following two maps are isomorphisms:

\[
\text{(3.7)} \quad \text{RCoid}_i \oplus \text{LInd}_i \quad \xrightarrow{\rho \circ \text{RCoid}_i + \text{LInd}_i \circ \rho} \quad \text{LInd}_i \circ \text{Res}_i \circ \text{RCoid}_i,
\]

\[
\text{(3.8)} \quad \text{RCoid}_i \oplus \text{LInd}_i \quad \xleftarrow{\rho \circ \text{RCoid}_i + \rho \circ \text{LInd}_i} \quad \text{RCoid}_i \circ \text{Res}_i \circ \text{LInd}_i,
\]

**Proof.** Following the computations for $\mathfrak{sl}_2$-modules considered in §2.1, we know that any irreducible $\mathfrak{b}[\xi]$-module is a one-dimensional graded module isomorphic to $\mathbb{k}_\lambda = \mathbb{k}_\lambda = X^\lambda (\mathbb{k})$ and for $\lambda \gg 0$ yields the following list of isomorphisms of graded $\mathfrak{b}_-$-modules:

$$\text{LInd}_i(\mathbb{k}_\lambda) = \text{LInd}_i^0 (\text{Span}(v_\lambda, f_\xi v_\lambda)) = L(\lambda) \oplus L(\lambda - 2)\xi,$$

$$\text{RCoid}_i(\mathbb{k}_\lambda) = \text{LInd}_i(t^{-1}k_\lambda[-1]) = (L(\lambda)\xi^{-1} \oplus L(\lambda - 2))[-1],$$

$$\text{LInd}_i \circ \text{Res}_i \circ \text{RCoid}_i(\mathbb{k}_\lambda) \cong \text{LInd}_i \circ (L(\lambda)\xi^{-1} \oplus L(\lambda - 2))[-1] \cong \text{LInd}_i^0 ((V_{-\lambda+2\lambda-2} V_{-\lambda-2\lambda-2} V_{-\lambda-4\lambda})[-1] \cong (L(\lambda)\xi^{-1} \oplus L(\lambda - 2))[-1] \oplus (L(\lambda) \oplus L(\lambda - 2)\xi)(-1 + 1) \cong \text{RCoid}_i(\mathbb{k}_\lambda) \oplus \text{LInd}_i(\mathbb{k}_\lambda).$$
Both functors $R\text{Coind}_i \circ \varepsilon$ and $\varepsilon \circ L\text{Ind}_i$ are non-trivial, and there images of $R\text{Coind}_i(\langle k, \lambda \rangle)$ and $L\text{Ind}_i(\langle k, \lambda \rangle)$ contain $L(\lambda)\xi^{-1}$ and $L(\lambda)$ by inspection. Consequently, $R\text{Coind}_i \circ \varepsilon + \varepsilon \circ L\text{Ind}_i$ must induce the above isomorphism.

As mentioned in Corollary 2.10 all functors commute with tensoring with $sl_2^{\text{int}}$-integrable modules. From the above, we know the equivalence (3.7) holds for $\langle \lambda, \alpha \rangle \geq N_0$ for some $N_0 \in \mathbb{Z}$. By tensoring with the standard irreducible two-dimensional $sl_2$-module and applying the long exact sequences, we deduce the isomorphism between the lefthand side and the righthand side of (3.7) as a $b$-modules for $\langle \lambda, \alpha \rangle \geq N \in \mathbb{Z}$ provided if we know the isomorphism for $\langle \lambda, \alpha \rangle \geq N + 1$. Thus, (3.7) yields an isomorphism for every $\lambda$. Consequently, the natural transformation of functors (3.7) is an isomorphisms of functors. The proof of the isomorphism (3.8) is completely analogous.

The same computations for irreducibles shows that one has a pair of adjoint distinguish triangles of endofunctors of $D^b(\mathcal{O}(p_i[\xi]))$:

$$
(3.9) \quad \text{Id}_b \to \mathcal{R}\text{Coind}_i \circ \mathcal{R}\text{Res}_i \to t^{-1}[-1] \to t[1], \quad t[1] \to \mathcal{L}\text{Ind}_i \circ \mathcal{R}\text{Res}_i \to \text{Id}_b \to t
$$

Proposition 2.9 of [11] stated for spherical functors gives another definition of spherical functors whose general formalism was worked out carefully in [11].

**Corollary 3.10.** The endofunctors $T_i$ and $T'_i$ defined by distinguished triangles

$$
\text{Id}_b \to \mathcal{R}\text{Res}_i \circ \mathcal{L}\text{Ind}_i \to T_i \to \text{Id}_b[1], \quad T'_i \to \mathcal{R}\text{Res}_i \circ \mathcal{R}\text{Coind}_i \to \text{Id}_b[1]
$$

are mutually inverse autoequivalences of the derived category $D^b(\mathcal{O}(p_i[\xi]))$.

In the general formalism of spherical functors of [11], the functor $T_i$ is called twist and $T'_i$ is called the dual twist.

Note that one can prove Corollary 3.10 by direct computations without using the formalism of spherical functors but we are happy to use here a relatively easy language of spherical functors. The Hecke relation (1.13) follows from Corollary 3.10 and Proposition 3.2 substituted for $J = \{i\}$.

We want to underline another equivalent categorical interpretation of the Hecke relation (1.7) which follows immediately from isomorphisms (3.7) and (3.8):

$$
(3.11) \quad \mathcal{D}_i \circ \mathcal{D}_i \simeq \mathcal{D}_i \oplus \mathcal{D}_i t[1].
$$

Note that, thanks to Serre duality isomorphism (3.5), we have the following collection of adjunctions

$$
\mathcal{D}^b(\mathcal{O}(p_i[\xi])) \xleftarrow{\mathcal{R}\text{Res}_i} \mathcal{D}^b(\mathcal{O}(b[\xi])) \xrightarrow{\mathcal{L}\text{Ind}_i} \mathcal{D}^b(\mathcal{O}(p_i[\xi]))
$$

and the functor $\mathcal{R}\text{Res}_i : \mathcal{D}(\mathcal{O}(p_i[\xi])) \to \mathcal{D}(\mathcal{O}(b[\xi]))$ is also spherical because the adjoint to the spherical functor is spherical. Consequently, we have the following isomorphisms:

$$
(3.12) \quad \mathcal{R}\text{Res}_i t^{-1}[-1] \oplus \mathcal{R}\text{Res}_i \simeq \mathcal{R}\text{Res}_i t^{-1}[-1] \circ \mathcal{L}\text{Ind}_i \circ \mathcal{R}\text{Res}_i \Rightarrow \mathcal{R}\text{Res}_i \oplus \mathcal{R}\text{Res}_i t[1] \simeq \mathcal{R}\text{Res}_i \circ \mathcal{L}\text{Ind}_i \circ \mathcal{R}\text{Res}_i
$$

3.5. **Braid relations for $T_i, T_j$.** It is obvious that if $\langle \alpha_i, \alpha_j \rangle = 0$ then $\mathcal{D}_i$ and $\mathcal{D}_j$ commute and consequently $T_i T_j = T_j T_i$. 

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3.5.1. $A_2$ case. Suppose that the rank 2 root subsystem generated by $i, j \in I$ is $A_2$. In particular, we assume that $(\alpha_i, \alpha_j) = -1$ and we denote by $\mathfrak{s}_i^j$ (resp. $p_{ij}[\xi]$) the corresponding simple (resp. parabolic) Lie subalgebras of $\mathfrak{g}$ (resp. $\mathfrak{g}[[\xi]]$) associated to the subset $\{i, j\} \subset I$.

Recall that the derived induction functor $\text{Lind}_i$ as well as the composition $\text{D}_i = \text{Res}_i \circ \text{Lind}_i$ are the left derived functors associated with the right exact functors $\text{Ind}_i$ and $\text{Res}_i \circ \text{Ind}_i$, respectively. First, let us describe a morphism of additive functors which implies a morphism of derived one’s.

Lemma 3.13. There exists a natural equivalence of additive right exact endofunctors of the abelian category $\mathcal{O}(\mathfrak{b}[[\xi]])$:

$$\pi_{ij} : \text{Res}_i \circ \text{Ind}_i \circ \text{Res}_j \circ \text{Ind}_j \circ \text{Res}_i \circ \text{Ind}_i \rightarrow \text{Res}_{ij} \circ \text{Ind}_{ij}$$

Lemma 3.13 is very well known for even inductions and was first noticed by Demazure. See e.g. [Jos] and [AK] where the modern categorical setup of Demazure functors is used. Most of the proofs for the classical (nonsuper) version are based on the geometry of Bott-Samelson varieties. We suggest below the algebraic proof of this statement without referring to geometry.

Proof. Recall that the inductions we are dealing with are the maximal integrable quotients of the appropriate ordinary inductions from subalgebras. Consequently, the embedding of parabolic subalgebras $p_i[\xi] \subset p_{ij}[\xi]$ predicts the natural transformation between functors of corresponding inductions:

$$\pi_i : \text{Res}_i \circ \text{Ind}_i \rightarrow \text{Res}_{ij} \circ \text{Ind}_{ij}.$$  

Moreover, for each $\mathfrak{b}[[\xi]]$-module $M$ the module $\text{Res}_{ij} \circ \text{Ind}_{ij}(M)$ is $\mathfrak{s}_{ij}$-integrable and, in particular, $\mathfrak{sl}_{ij}$-integrable. The commutativity with the tensor product stated in Corollary 2.10 explains the isomorphism of additive (underived) functors

$$(3.15) \quad \text{Res}_j \circ \text{Ind}_j(\text{Res}_{ij} \circ \text{Ind}_{ij}(M)) = (\text{Res}_{ij} \circ \text{Ind}_{ij}(M)) \otimes \text{Res}_j \circ \text{Ind}_j(\mathbb{k}_0) = \text{Res}_{ij} \circ \text{Ind}_{ij}(M)$$

The last equality follows from the observation $\text{Ind}_i(\mathbb{k}_0) = \mathbb{k}_0$ that can be either checked by hands or figured out from Theorem 2.20. The same isomorphism also makes sense for $\beta$. Finally, we define the morphism of functors $\pi_{ij} := (\text{Res}_i \circ \text{Ind}_i \circ \text{Res}_j \circ \text{Ind}_j) \circ \pi_i$. Together with Isomorphisms (3.15) we have

$$(3.16) \quad \pi_{ij} : (\text{Res}_i \circ \text{Ind}_i) \circ (\text{Res}_j \circ \text{Ind}_j) \circ (\text{Res}_i \circ \text{Ind}_i) \xrightarrow{\pi_i} (\text{Res}_i \circ \text{Ind}_i) \circ (\text{Res}_j \circ \text{Ind}_j) \xrightarrow{3.15} \text{Res}_i \circ \text{Ind}_i \circ (\text{Res}_{ij} \circ \text{Ind}_{ij}) \xrightarrow{3.15} \text{Res}_{ij} \circ \text{Ind}_{ij}.$$ 

Thanks to adjunction between restriction and induction functors we know that all induction functors under consideration are right exact and restriction functors are exact and a composition of right exact functors is right exact. We conclude that both functors in (3.14) are right exact.

We explain in Theorem 2.20 that the superposition of consecutive inductions applied to the irreducible module $\mathbb{k}_\lambda$ leads to the super-Demazure module $\text{W}_D_{s_i, s_j, s_k}(\lambda)$ that happens to be the cyclic $p_{ij}$-integrable module generated by the cyclic vector of weight $s_is_j s_k(\lambda)$. On the other hand we know that $\text{Ind}_{ij}(\mathbb{k}_\lambda)$ is the universal cyclic $p_{ij}$-integrable module generated by the cyclic vector of weight $\lambda$. The $p_{ij}$-integrability implies that one can also choose as a generating cyclic vector any other extremal vector in the $\mathfrak{s}_{ij}$ irreducible representation. In
particular the vector of the weight $s_i s_j s_t(\lambda)$. Hence, $\pi_{ij}$ is an isomorphism for irreducible $k_\lambda$ with $\lambda$ sufficiently dominant. On the other hand, all functors commute with tensor products with $\mathfrak{sl}_d$-integrable modules. Therefore, $\pi_{ij}$ is an isomorphism of additive functors. \qedhere

Corollary 2.9 says that $D_i$ has nonvanishing cohomology in the $-1$st and in the $0$th degrees. Consequently, the composition $D_i \circ D_j \circ D_i$ has only nonpositive cohomology and the natural transformation $\pi_{ij}$ of $0$th cohomology described in Lemma 3.13 extends by universal property to a natural transformation of derived functors:

$$L\pi_{ij} : D_i \circ D_j \circ D_i \to D_{ij}.$$ 

This is not an equivalence of derived functors and the next paragraph explains the difference (the cone) between these functors.

Consider the composition of two counit morphisms coming from adjunctions:

$$\text{Id}_{D_i} \Rightarrow R\text{Coind}_i \circ \text{Res}_i \Rightarrow R\text{Coind}_i \circ \text{Res}_i \Rightarrow R\text{Coind}_i \circ (\text{Res}_j \circ L\text{Ind}_j) \circ \text{Res}_i$$

Together with the composition $\text{Res}_i$ from the left and $L\text{Ind}_i$ from the right we get the following morphism of endofunctors:

$$D_i = \text{Res}_i \circ L\text{Ind}_i \to \text{Res}_i \circ R\text{Coind}_i \circ \text{Res}_j \circ L\text{Ind}_j \circ \text{Res}_i \circ L\text{Ind}_i = D'_i \circ D_j \circ D_i$$

Finally, thanks to Duality (3.5) between induction and coinduction functors we end up with the following morphism of endofunctors:

$$(3.17) \quad D_i \to t^{-1}[-1]D_i \circ D_j \circ D_i \iff \eta : D_i t[1] \to D_i \circ D_j \circ D_i$$

**Proposition 3.18.** There is a distinguish triangle of endofunctors of $D^b(O(b_+[\xi]))$:

$$(3.19) \quad D_i t[1] \to D_i \circ D_j \circ D_i \xrightarrow{L\pi_{ij}} D_{ij}$$

\textit{Proof.} First, we check the acyclicity of the images of functors on irreducible (one-dimensional) $b[\xi]$-modules $k_\lambda$ for sufficiently $i, j$-dominant weight $\lambda \gg 0$. We already know the coincidence of $0$th cohomologies of $D_i(D_j(D_i(k_\lambda)))$ and $D_{ij}(k_\lambda)$ with the super-Demazure module $\mathcal{W}_{D_i, s_j s_t(\lambda)}$ thanks to Lemma 3.13 and Theorem 2.20. We claim that the direct inspection of weights shows that $D_i(k_\lambda)$, $D_j(D_i(k_\lambda))$ as well as $D_{ij}(k_\lambda)$ do not have nonzero cohomology. Thus, the only nonzero cohomology in Triangle (3.19) is $-1$st. The coincidence of $D_i^{-1}(\mathcal{W}_{D_j, s_t(\lambda)})$ and $D_j(k_\lambda) = \text{Ind}_j k_\lambda$ can be either checked directly or verified from the equality of characters for cohomology of functors. The compatibility with tensor products of the finite-dimensional $p_{\alpha \beta}$-module $p_{\alpha \beta}$ forces these isomorphisms to be functorial since the $p_{\alpha \beta}$-action is responsible for extensions. Consequently, Triangle (3.19) is distinguished for $k_\lambda$ with $\lambda \gg 0$.

Finally, we again use the fact that all functors under consideration commute with the tensor multiplication with $\mathfrak{sl}_d$-integrable modules (Corollary 2.10). It remains to notice that iterated tensor products of $\mathfrak{sl}_d$ generate the Grothendieck category $K_0(O(b_+[\xi]))$. Therefore, Triangle (3.19) is distinguished. \qed
Consider the commutative diagram of endofunctors:

\[
\begin{array}{cccccc}
\text{Id} & \rightarrow & D_i & \rightarrow & D_i D_j & \rightarrow & D_j D_j \\
\eta_i & & \eta_j & & (L \pi_{ij}) \circ \eta_i & & (L \pi_{ij}) \circ \eta_i \\
D_j & \rightarrow & D_j D_i & \rightarrow & D_i D_j & &
\end{array}
\]

(3.20)

All morphisms are unit maps except the rightmost terms where one takes the composition of unit and the left derived functor \( \pi_{ij} : D_i D_j D_i \rightarrow D_{ij} \). The commutativity of the diagram follows from the commutativity of unit maps.

We denote by \( \text{BGG}_{ij} \) the total complex of the corresponding double complex:

\[
\text{Id}[3] \rightarrow (D_i \oplus D_j)[2] \rightarrow (D_i D_j \oplus D_j D_i)[1] \rightarrow D_{ij}.
\]

**Proposition 3.21.** Assume that \( \langle \alpha_i, \alpha_j \rangle = -1 \). Then there exists a subcomplex \( C_{iji} \) and a pair of equivalences \( \zeta_i \) and \( \xi_i \)

\[
T_i \circ T_j \circ T_i \leftrightarrow C_{iji} \xrightarrow{\zeta_i} \text{BGG}_{ij}.
\]

In particular, we have an isomorphism of triangulated functors \( T_i \circ T_j \circ T_i \) and \( T_j \circ T_i \circ T_j \) because of the symmetry \( \text{BGG}_{ij} = \text{BGG}_{ji} \).

**Proof.** If we rephrase Proposition 3.18 in terms of spherical functors we end up with sufficient properties of the braid relations formulated in [AL]. However, the arguments are not too complicated and we want to have the self-contained proof in order to compare it with the analogous relations for other rank 2 root subsystems that are not covered by [AL].

The endofunctor \( T_i \circ T_j \circ T_i \) is the successive cones that yield a triple complex, given by the commutative cube of units:

\[
\begin{array}{cccccc}
\text{Id} & \rightarrow & D_i & \rightarrow & D_i D_j & \rightarrow & D_j D_j \\
\eta_i & & \eta_j & & \eta_j & & \eta_j \\
D_j & \rightarrow & D_i & \rightarrow & D_i D_j & &
\end{array}
\]

Denote by \( \tilde{D}_i \subset D_i \oplus D_i \) the kernel of the map \( D_i \oplus D_i \rightarrow D_i \circ D_i \) which is clearly isomorphic to \( D_i \). Denote by \( D_i t[1] \) the direct summand of \( D_i \circ D_i \) defined in (3.11). We include these.
summands into a subcomplex called $C_{ij}$:

$$C_{ij} := \text{Id}[3] \oplus \left( \frac{D_j \oplus D_i}{D_i} \right) [2] \oplus \left( \frac{D_i \circ D_j \oplus D_j \circ D_i}{D_j \circ D_i[D_i][1]} \right) [1] \oplus D_i \circ D_j \circ D_i$$

We claim, that it is not difficult to show that the subspace $C_{ij}$ is indeed a subcomplex. Indeed, notice that the restriction of the counit map $\eta_j$ onto the kernel $\tilde{D}_i$ coincides with $\eta_j$. Moreover, all elements in the direct summand $D_i \subset D_i \circ D_i$ belongs to the image of the differential and hence the restriction of the total differential on it is zero by the commutativity of the diagram and the sign convention of the double complex. The cone of the morphism $\zeta_i : C_{ij} \to T_i \circ T_j \circ T_i$ is isomorphic to the acyclic complex $D_i \to D_i$ and, therefore, $\zeta_i$ is a quasiisomorphism.

We claim that there exists a straightforward map of complexes: $\xi_i : C_{ij} \to \text{BGG}_{ij}$ which is the componentwise maps of induction functors. The cone of this morphism is isomorphic to the distinguished triangle (3.19). Consequently, $\xi_i$ is an isomorphism.

By interchanging $\alpha$ and $\beta$ we get another equivalences:

$$T_j \circ T_i \circ T_j \xleftarrow{C_{ij}} \xi_i \text{BGG}_{ji} \xrightarrow{\xi_i} \text{BGG}_{ij}.$$ 

Therefore, complexes $T_j \circ T_i \circ T_j$ and $T_i \circ T_j \circ T_i$ are isomorphic in derived category.

### 3.5.2. Braid relations: $B_2$ case

In this section we omit the superposition sign "$\circ$" in order to make formulas with functors more compact.

Suppose that the root subsystem generated by $i, j \in \Pi$ is $B_2$. In particular, we assume $\langle \alpha_i, \alpha_j \rangle = -2$. The goal of this subsection is to explain the corresponding isomorphism of triangulated functors:

$$T_i T_j T_i T_j \simeq T_j T_i T_j T_i.$$  

The strategy repeats the one suggested for $A_2$ case:

First, we use Decomposition (3.11) and replace $D_i \to D_i D_i$ by the subcomplex $0 \to D_i t[1]$ (that is quasi-isomorphic to the original). Second, we use the properties of the spherical functors and make several more cancellations of this kind. Third, we use the universal properties of derived functors and relate the remaining complex with the BGG-complex whose zero term is the derived induction on the parabolic subalgebra $p_{ij}$. Finally, we notice that the BGG complex is invariant under the switch of $\alpha$ and $\beta$ that implies the equivalence (3.22).

We start from the description of the key ingredient of the BGG-complex the functor $D_{ij} = \text{Res}_{ij} \circ \text{LInd}_{ij}$ that is the derived superinduction to the parabolic subalgebra $p_{ij}[\xi]$:

**Lemma 3.23.** There exists a distinguished triangle of endofunctors:

$$\left( \frac{D_i \circ D_j t[1]}{D_i D_j t[1]} \right) \xrightarrow{\eta_{D_i} + \eta_{D_j}} D_i D_j D_i D_j D_j D_i L \xrightarrow{\pi_{ij}} D_{ij}. $$

$\Box$
Proof. First, we notice that the 0’th cohomology of \( D_{ij} \) coincides with 0’th cohomology of \( D_i D_j D_i D_j \). We illustrated this fact for \( k_\lambda \) with sufficiently dominant \( \lambda \) in Section 2.3 and the standard trick with the tensor product with integrable modules explains the coincidence for all \( \lambda \). Let us denote the corresponding isomorphism \( \pi_{ij} \) and the corresponding derived functor \( L \pi_{ij} : D_i D_j D_i D_j \to D_{ij} \) exists thanks to the universal property of the left derived functor \( D_{ij} \).

Let us compute the cohomology of the image of the triangle \( \text{(3.21)} \) for \( k_\lambda \) with sufficiently dominant \( \lambda \). As we mentioned the 0-th cohomology vanishes thanks to Theorem 2.20.

\[
H^0(D_i D_j D_i D_j(k_\lambda)) = H^0(D_{ij}(k_\lambda)) = \text{Ind}_{ij}(k_\lambda).
\]

Moreover, for \( \lambda \) sufficiently dominant \( D_j(k_\lambda) \) has no higher cohomology and all weights in the \( \mathfrak{sl}_2 \)-integrable module \( \text{Ind}_j(k_\lambda) \) remain to be \( \alpha \)-dominant. Therefore, \( D_i D_j(k_\lambda) \) also does not have nonzero cohomology. As in the \( A_2 \)-case the complex \( D_{ij}(k_\lambda) \) also do not have higher cohomology for sufficiently dominant \( \lambda \). Thus, the cohomology of all complexes in the triangle \( \text{(3.21)} \) differs from zero only for \( H^0 \) and \( H^{-1} \). The coincidence for \( H^{-1} \) follows from the coincidence of characters which was computed by I. Cherednik in \[\text{Ch2}]. \]

Next, let us rewrite the complex \( T_1 T_j T_i T_j \) in terms of \( D \)’s:

\[
(\text{Id} \xrightarrow{\eta_i} D_i) \circ (\text{Id} \xrightarrow{\eta_j} D_j) \circ (\text{Id} \xrightarrow{\eta_i} D_i) \circ (\text{Id} \xrightarrow{\eta_j} D_j) = \\
\text{Id} \xrightarrow{} \begin{pmatrix} D_i & D_j \end{pmatrix} \xrightarrow{} \begin{pmatrix} D_i D_j & D_i D_j \end{pmatrix} \xrightarrow{} \begin{pmatrix} D_i D_j D_i D_i & D_i D_j D_i D_i \end{pmatrix} \xrightarrow{} D_i D_j D_i D_j.
\]

Consider the subcomplex \( C_1 \) where in the second column we replace two copies of \( D_i \) by the kernel of the map \( D_i \oplus D_i \to D_i D_i \) and the component \( D_i D_i \) is replaced by the direct summand \( D_i t[1] \). The same cancellation associated with the simple root \( \beta \) leads to the quasiisomorphic subcomplex \( C_2 \). The next subcomplex \( C_3 \) corresponds to the replacement of the counit morphisms

\[
D_i D_j \oplus D_i D_j \oplus D_i D_j \to D_i D_i D_i D_j \oplus D_i D_j D_j
\]

by its kernel and cokernel which is isomorphic to

\[
D_i D_j \xrightarrow{\eta^0} D_i D_j t[1] \oplus D_i D_j t[1]
\]

and the underlying space of the quasiisomorphic subcomplex \( C_3 \) looks as follows:

\[
\text{Id} \xrightarrow{} \begin{pmatrix} D_i \oplus D_j \end{pmatrix} \xrightarrow{} \begin{pmatrix} D_i t[1] \oplus D_j t[1] \oplus D_j t[1] \oplus D_i D_j \end{pmatrix} \xrightarrow{} \begin{pmatrix} D_i D_i D_i D_i \oplus D_i D_j t[1] \oplus D_i D_j t[1] \oplus D_i D_j D_j \end{pmatrix} \xrightarrow{} D_i D_j D_i D_j.
\]
Let us collect certain parts together due to the morphisms described in \((3.17)\):

\[
\text{(3.25)} \quad \text{Id} \to \bigl( \mathbf{D}_i \oplus \mathbf{D}_j \bigr) \to \bigl( \mathbf{D}_j \mathbf{D}_i \oplus \mathbf{D}_i \mathbf{D}_j \bigr) \to \\
\to \biggl( \text{cone}(\mathbf{D}_i t[1] \to \mathbf{D}_j \mathbf{D}_i t[1]) \oplus \text{cone}(\mathbf{D}_j t[1] \to \mathbf{D}_j \mathbf{D}_j t[1]) \biggr) \to \text{cone} \biggl( \mathbf{D}_i \mathbf{D}_j t[1] \oplus \eta \mathbf{D}_i \mathbf{D}_j t[1] \to \mathbf{D}_i \mathbf{D}_j \mathbf{D}_i \mathbf{D}_j \biggr).
\]

By construction, this object also represents \(C_3\) up to quasi-isomorphism. By replacing the cone in the right most part of \((3.25)\) using \((3.24)\), we obtain a complex that we call \(\text{BGG}_{ij}\) with the following presentation:

\[
\text{(3.26)} \quad \text{Id} \to \bigl( \mathbf{D}_i \oplus \mathbf{D}_j \bigr) \to \bigl( \mathbf{D}_j \mathbf{D}_i \oplus \mathbf{D}_i \mathbf{D}_j \bigr) \to \biggl( \text{cone}(\mathbf{D}_i t[1] \to \mathbf{D}_j \mathbf{D}_i t[1]) \oplus \text{cone}(\mathbf{D}_j t[1] \to \mathbf{D}_j \mathbf{D}_j t[1]) \biggr) \to \mathbf{D}_{ij}.
\]

By construction, \(\text{BGG}_{ij}\) is quasi-isomorphic to \(C_3\). The Complex \((3.26)\) is invariant under the switch \(\alpha_i \leftrightarrow \alpha_j\) that implies the desired equivalence \((3.22)\) given by the following collection of quasiisomorphisms:

\[\mathbf{T}_i \mathbf{T}_j \mathbf{T}_i \mathbf{T}_j \leftarrow \mathbf{C}_1^i \leftarrow \mathbf{C}_3^i \to \text{BGG}_{ij} \leftarrow \mathbf{C}_3^j \to \mathbf{C}_1^j \to \mathbf{T}_j \mathbf{T}_i \mathbf{T}_j \mathbf{T}_i.\]

3.5.3. \(G_2\) case. We believe that the analogous proof can be worked out if the rank 2 subsystem generated by \(\alpha_i\) and \(\alpha_j\) is \(G_2\). However, the proof seems to be extremely technical and we want to save time and skip it.

4. Categorification of DAHA and Macdonald polynomials

4.1. Affine Lie algebras. We keep the notations from the previous section.

Denote by \(\hat{\mathfrak{g}}\) the untwisted affine Lie algebra corresponding to \(\mathfrak{g}\). We set \(\mathfrak{g}[z^{\pm 1}] := \mathfrak{g} \otimes \mathbb{K}[z^{\pm 1}]\). This acquires the natural structure of a Lie algebra. The Lie algebra \(\mathfrak{g}[z^{\pm 1}]\) admits a natural surjection from \([\hat{\mathfrak{g}}, \hat{\mathfrak{g}}]\) by taking quotient by the center.

Let \(\Delta_{af} := \Delta \times \mathbb{Z} \delta \cup \{m \delta\}_{m \neq 0}\) be the untwisted affine root system of \(\Delta\) with its positive part \(\Delta_+ \subset \Delta_{af,+}\). We set \(\alpha_0 := -\vartheta + \delta\), \(I_{af} := \Pi \cup \{\alpha_0\}\), and \(I_{af} := \Pi \cup \{0\}\), where \(\vartheta\) is the highest root of \(\Delta_+\). Let \(\mathfrak{J} \subset \hat{\mathfrak{g}}\) be the (image of the) upper-triangular subalgebra of \([\mathfrak{g}, \mathfrak{g}]\) (that contains \(\mathfrak{h}\); the Iwahori subalgebra). We set \(\mathfrak{U} := U(\mathfrak{J})\) (the enveloping algebra of \(\mathfrak{J}\)). For each \(\mathfrak{J} \subset I_{af}\), we have a Lie subalgebra \(\mathfrak{p}_J \subset \mathfrak{g}[z^{\pm 1}]\) generated by \(\mathfrak{J}\) and the root subspaces of \(\{-\alpha_i\}_{i \in \mathbb{Z}}\). In case \(\mathfrak{J} = \{i\}\), we might write \(\mathfrak{p}_i\) or \(\mathfrak{p}_{\alpha_i}\) instead of \(\mathfrak{p}_{\{i\}}\).

We set \(W_{af} := W \ltimes \mathbb{Q}^\vee\) and call it the affine Weyl group. It is a reflection group generated by \(\{s_i \mid i \in I_{af}\}\), where \(s_0\) is the reflection with respect to \(\alpha_0\). We also have a reflection \(s_i \in W_{af}\) corresponding to \(\alpha \in \Delta \times \mathbb{Z} \delta \subset \Delta_{af}\). Let \(\ell : W_{af} \to \mathbb{Z}_{\geq 0}\) be the length function (based on \(I_{af}\)) and let \(w_0 \in W\) be the longest element in the finite Weyl group \(W \subset W_{af}\). Together with the normalization \(t_{-\vartheta} := s_\vartheta s_0\) (for the coroot \(\vartheta^\vee\) of \(\vartheta\)), we introduce the translation element \(t_j \in W_{af}\) for each \(\beta \in \mathbb{Q}^\vee\). We set \(\Omega\) to be the group of diagram automorphism of the affine Dynkin diagram of \(\mathfrak{g}\) such that \(\Omega \ltimes W_{af}\) gives the extended affine Weyl group \(\hat{W}_{af}\). We have

\[\hat{W}_{af} \cong W \ltimes \mathbb{P}^\vee \supset W_{af} = W \ltimes \mathbb{Q}^\vee.\]

Every element of \(\hat{W}_{af}\) is written as \(w \pi\) for \(w \in W_{af}\) and \(\pi \in \Omega\). As \(\Omega\) preserves \(\{s_i \mid i \in I_{af}\}\), we define

\[\ell(w \pi) = \ell(\pi w) := \ell(w).\]
Recall that $\Omega$ permutes the set of level one dominant weights of $\widehat{\mathfrak{g}}$, and their projections to $\mathfrak{h}^*$ induces $\Omega$-permutations of a finite subset $\Lambda_0$ of $P$. We call $\Lambda_0$ the set of miniscule weights. Below we consider the level one action of $\widehat{W}_{af}$ on $P$, that extends the $\Omega$-action on $\Lambda_0$:

$$s_i\lambda := \begin{cases} 
\lambda - \langle \alpha_i^\vee, \lambda \rangle \alpha_i & (i \in I) \\
\lambda + (\langle \vartheta_i^\vee, \lambda \rangle + 1)\alpha_i & (i = 0)
\end{cases}.$$

4.2. Convex orders on $P$. For $\lambda \in P$, we set $\lambda_+ \in P$ to be the unique dominant weight in $W\lambda$ and set $\lambda_- \in P$ to be the unique anti-dominant weight. We define three partial preorders on $P$ as:

$$\lambda \leq \mu \iff \lambda \in \mu - Q_+$$

$$\lambda < \mu \iff \lambda_+ \neq \mu_- \text{ and } \lambda_- \in \mu_- + Q_+$$

$$\lambda \leq \mu \iff \lambda < \mu, \text{ or } \lambda_- = \mu_- \text{ and } \lambda \geq \mu.$$

We call $\leq$ the dominance order, $<$ the Cherednik order, and $<$ the Macdonald order. Note that we have $\lambda < \mu$ if and only if $\lambda < \mu_-$. For each $\lambda \in P$, we consider

$$P[< \lambda] := \{ \mu \in P \mid \mu < \lambda \} \subset P[\leq \lambda] := \{ \mu \in P \mid \mu \leq \lambda \}$$

$$P[< \lambda] := \{ \mu \in P \mid \mu < \lambda \}.$$

By the definition of $<$, the set $P[< \lambda]$ is $W$-stable.

**Theorem 4.2** (Cherednik [Ch1] §1). Let $i \in I_{af}$ and $\lambda, \mu \in P$. We have:

1. if $\mu$ and $\mu - m\alpha_i$ ($m \geq 0$) belongs to $P[< \lambda]$, then we have $\mu - c\alpha_i$ for each $0 \leq c \leq m$;
2. if $\lambda \leq s_i\lambda$, then we have $s_iP[\leq \lambda] \cup P[\leq s_i\lambda] \subset P[\leq \lambda]$;
3. if $\lambda \geq s_i\lambda$, then we have $s_iP[\leq \lambda] \subset P[\leq \lambda]$.

For $\lambda \in P$, let $u_\lambda \in W$ be the minimal length element such that $u_\lambda \lambda \in P_-$. 

4.3. Lie superalgebra and the category of graded modules. Let us adopt the setup given in Section 1.2 to the affine case. Namely we consider the Lie superalgebras $\mathfrak{J}[\xi] \subset p_J[\xi] \subset g[z^\pm][\xi]$ that corresponds to the super currents from the Iwahori, parabolic assigned with $J \subset I_{af}$ and affine Lie algebras correspondingly. These algebras are bigraded with respect to the $(z, \xi)$-gradings. We set

$$Z[P_+] := Z[q, t] \otimes_Z Z[Q_+] \text{ and } Z[P] := Z[q^\pm 1, t^\pm 1] \otimes_{Z[q,t][Q_+]} Z[P_+].$$

Let $\mathcal{O}(\mathfrak{J}[\xi])$ be the category of $(z, \xi)$-graded $\mathfrak{J}[\xi]$-modules with semi-simple $\mathfrak{h}$-action whose eigenvalues belongs to $P$. Moreover we require that for each $M \in \mathcal{O}(\mathfrak{J}[\xi])$ the subspace $M_{a,b}$ of $z$-degree $a$ and $\xi$-degree $b$ is finite-dimensional and the formal power sum

$$gch(M) := \sum_{a,b} q^a(-t)^b ch M_{a,b}$$

has to belong to $Z[P]$ what means that $M_{a,b} = 0$ for $a \ll 0$ or $b \ll 0$.

We denote by $\mathcal{O}(p_J[\xi])$ the corresponding category of bigraded $\mathfrak{h}$-semisimple finitely generated modules for any parabolic subalgebra assigned with $J \subset I_{af}$.

For a pair of formal series $f, g \in Z[P]$ with respect to $q,t$ and $e^\lambda$ ($\lambda \in P$) with integer coefficients, we say that $g \leq f$ if and only if we have the corresponding inequality for all coefficients.
The weight support $\Psi(M)$ of a graded module $M \in \mathcal{O}(\mathfrak{g}[\xi])$ is the set of $\mathfrak{g}$-weights whose weight space is nonzero. To each weight $\lambda \in \mathcal{P}$ we assign a subcategory $\mathcal{O}_{\prec \lambda} \subset \mathcal{O}(\mathfrak{g}[\xi])$ consisting of graded modules whose weight support $\Psi(M) \subset \mathcal{P}[\prec \lambda]$ is bounded from above by $\lambda$ with respect to the Cherednik ordering $\prec$.

4.4. Derived category $\mathbb{D}_c^-(\mathcal{O}(\mathfrak{g}[\xi]))$. The Macdonald polynomials are not polynomials in $q, t$ and in order to work out the corresponding categorification we have to change the derived category we are working with.

**Definition 4.3.** A bounded from above complex of bigraded $\mathfrak{g}$-semisimple $\mathfrak{g}[\xi]$-integrable modules $M^\prime$ is called convergent if for each pair $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ the $(z, \xi)$-graded component $M^\prime_{(a,b)}$ is a bounded complex of finite dimensional $b_+$-modules and for either $a \ll 0$ or $b \ll 0$ the graded component $M^\prime_{(a,b)}$ is empty.

The triangulated category $\mathbb{D}_c^-(\mathcal{O}(\mathfrak{g}[\xi]))$ is the derived category associated with the category of convergent complexes.

Note that each module $M \in \mathcal{O}(\mathfrak{g}[\xi])$ is convergent but might be infinite-dimensional, however, each complex $M^\prime \in \mathbb{D}_c^-(\mathcal{O}(\mathfrak{g}[\xi]))$ admits the Euler characteristics $\text{gch}(M^\prime)$ of graded supercharacters which belongs to $\mathbb{Z}[\mathcal{P}]$.

We denote by $q$ and $t$ the endofunctors of $\mathcal{O}$ (and various categories constructed from $\mathcal{O}$) that raise the $z$-degree and $\xi$-degree by one, respectively.

**Notation 4.4.** From now on in order to underline that we are working with affine root systems we will use the simpler notation for the lower index. For example, we will write $D_i$ instead of $D_{-\alpha_i}$ for $i \in I_{af}$.

**Proposition 4.5.** The derivedfunctors: induction $L\text{Ind}_i$, restriction $R\text{es}_i$ and coinduction $R\text{Coind}_i$ are well defined adjoint functors between convergent catego ry of complexes:

$$
\begin{array}{ccc}
\mathbb{D}_c^-(\mathcal{O}(b[\xi])) & \xleftarrow{L\text{Ind}_i} & \mathbb{D}_c^-(\mathcal{O}(p_i[\xi]));
\end{array}
$$

**Proof.** There exists straightforward generalization of the arguments of Section 2.2 that shows that each module $M \in \mathcal{O}(\mathfrak{g}[\xi])$ admits a resolution of length at most 2 by $L\text{Ind}_i$-acyclic (resp. $R\text{Coind}_i$-acyclic) modules from $\mathcal{O}(\mathfrak{g}[\xi])$. □

Note that the functor $R\text{es}_i$ commutes with limits and colimits. Its right adjoint $R\text{Coind}_i$ commutes with colimits, respectively the left adjoint $L\text{Ind}_i$ commutes with limits, whenever they exist.

**Theorem 4.6.** The category $\mathbb{D}_c^-(\mathcal{O}(\mathfrak{g}[\xi]))$ together with endofunctors $\{T_i | i \in I_{af}\}$, $X^\lambda$, the $\xi$-grading shift functor $t$, the $z$-grading shift functor $q$ and the group of diagram automor-phismin $\Omega := \mathcal{P}/Q$ categorify the action of the Double Affine Hecke Algebra $\mathcal{H}$ on the polynomial representation in the sense of Theorem 1.10.

**Proof.** The endofunctors $D_i, D'_i$ as well as the endofunctors $T_i, T'_i$ has at most two nonzero cohomological components and since they commute with direct sums/products it is enough to prove all the relations for the bounded complexes what was already done in the proof of Theorem 1.10. □
Let $\mathcal{O}[< \lambda]$ and $\mathcal{O}[\leq \lambda]$ denote the categories of graded $\mathfrak{g}$-modules $M$ such that the weight support $\Psi(M) \subset P[< \lambda]$ and $\Psi(M) \subset P[\leq \lambda]$ correspondingly.

We denote by $\mathbb{D}_c^{-}[< \lambda]$ and $\mathbb{D}_c^{-}[\leq \lambda]$ the full subcategories of $\mathbb{D}_c^{-}$ that contain complexes of $\mathfrak{g}$-modules whose homology belongs to $\mathcal{O}[< \lambda]$ and $\mathcal{O}[\leq \lambda]$ correspondingly.

**Theorem 4.7.** Let $\lambda \in P$ and let $i \in I_{af}$.

1. If $s_i \lambda < \lambda$, then the endofunctor $D_i$ preserves the full subcategory $\mathbb{D}_c^{-}[\leq \lambda]$;
2. If $s_i \lambda > \lambda$, then the endofunctor $D_i$ maps the subcategory $\mathbb{D}_c^{-}[\leq \lambda]$ to $\mathbb{D}_c^{-}[\leq s_i \lambda]$.

**Proof.** Since the functor $D_i$ commutes with limits, it is enough to check the statement of the theorem for modules whose weight support consists of one element. Therefore, the question reduces to the detailed observation of the $\mathfrak{sl}_2$-case. We showed in Example 2.13 that the derived induction $D_i(M)$ (as well as $T_i(M)$) has no derived components whenever the weight support of a $\mathfrak{g}$-module $M$ consists of one element $\lambda$ which is $\alpha_i$-dominant: $\langle \lambda, \alpha_i^\vee \rangle > 0$. In other words, $D_i(M)$ is a complex concentrated in zero homological degree and the weight support of the $\mathfrak{g}$-modules $D_i(M)$ is $s_i$-symmetric. In particular, the subspace of weight $s_i(\lambda)$ of the modules $D_i(M)$ as well as $T_i(M)$ is isomorphic to $M$ and, moreover, $T_i(M)$ belongs to $\mathbb{D}_c^{-}[\leq \lambda]$. On the other hand, if $\langle \lambda, \alpha_i^\vee \rangle \leq 0$ and the weight support $\Psi(M)$ consists of one element $\lambda$ the complex $D_i(M)$ is concentrated in $-1$st cohomological degree and consists of a module whose weight support is concentrated in $P[\leq \lambda]$. □

**Corollary 4.8.** The endofunctors $D_i$ preserve $\mathbb{D}_c^{-}[\leq \lambda_-]$ for all $i \in I_{af}$ and all $\lambda \in P$.

4.5. $Y$-eigen objects in $\mathbb{D}_c^{-}(\mathcal{O}(b[\mathfrak{g}]))$. For each fundamental coweight $\omega_i \in \Pi'$ with $i \in I$ we have the translation element $t_\omega_i \in W_{af}$ with a reduced decomposition:

$$
(4.9) \quad t_\omega_i = \pi s_i s_2 \cdots s_{i_\ell} \quad i_1, \ldots, i_\ell \in I_{af}, \pi \in \Omega.
$$

This follows the presentation of the functor $Y^{\omega_i}$ which categorifies the $\gamma'$-elements of $\mathbb{H}$:

$$
Y^{\omega_i} = \pi \circ T_i_1 \circ T_i_2 \circ \cdots \circ T_{i_\ell}.
$$

Thanks to the categorification Theorem 4.6 we know that the endofunctor $Y^{\omega_i}$ does not depend on the choice of a reduced decomposition. Moreover, for each $\mu \in Q'$, the endofunctor $Y^\mu := Y^{\mu_+} \circ (Y^{\mu_-})^{-1}$ is uniquely defined up to an isomorphism. Here $\mu = \mu_+ - \mu_-$ with $\mu_+$ and $\mu_-$ being dominant weight.

**Theorem 4.10.** For each module $M \in \mathcal{O}(\mathfrak{g})$ whose weight support $\Psi(M)$ consist of a unique dominant weight $\lambda$ there exists a well defined object $EM_\lambda(M) \in D_c^{-}[\leq \lambda]$ such that

1. $EM_\lambda(M) \in D_c^{-[0]}$;
2. The subspace of weight $\lambda$ in $H^s(EM_\lambda(M))$ is equal to zero for $s < 0$ and coincides with $M$ for $s = 0$;
3. For all $\mu \in Q'$ there exists an isomorphism $Y^\mu(EM_\lambda(M)) = q^{-\langle \mu, \lambda \rangle}(EM_\lambda(M))$.

The proof of Theorem 4.10 is based on the derived version of the Banach fixed-point Theorem applied to the generating set of pairwise commuting endofunctors $Y_{\mu_i}$, $i \in I$ with $\mu_i$ assembling a basis of $Q'$. The following Proposition 4.11 is the key intermediate step in the construction of $EM_\lambda(M)$.

Let us denote by $Y^\mu$ the grading shift $q^{\langle \mu, \lambda \rangle} \circ Y^\mu$ of the endofunctor $Y^\mu$. We suppose that $\mu$ is dominant and the power $q^{\langle \mu, \lambda \rangle}$ is the eigen value of the operator $Y^\mu$ on the nonsymmetric Macdonald polynomial $E_\lambda$. 
Proposition 4.11. Suppose that $\lambda, \mu$ is a pair of dominant weights and $M \in \mathcal{O}(\mathcal{J}[\xi])$ is an $\mathcal{J}[\xi]$-module whose weight support $\Psi(M)$ consists of $\lambda$. Then there exists a map $\mathbb{D}_c [-\leq \lambda]$: 
\[ \gamma : \tilde{Y}^\mu(M) \to M, \text{ for } i \in \mathcal{I} \]
and, moreover, the $z$-graded component of the cohomology of the cone($\gamma$) differs from zero only for positive powers of $z$.

**Proof.** Suppose $i \in \mathcal{I}_{st}$ and $\alpha_i$ is the corresponding simple root. Let $\lambda$ be $\alpha_i$-dominant and the weight support $\Psi(M) = \{ \lambda \}$ for a $\mathcal{J}[\xi]$-module $M$. As we already mentioned the complex $\mathcal{D}_i(M)$ consists of a module concentrated in 0-th homological degree, whose weight support is $s_i$-symmetric. In particular, the weight $s_i(\lambda)$ belongs to the weight support $\Psi(\mathcal{D}_i(M))$ and is maximal with respect to the Cherednik order $\leq$ and minimal with respect to the dominance order $\leq$. Therefore, we have a quotient map $\mathcal{D}_i(M) \to s_i(M)$ such that $M$ belongs to the kernel by weight reasons. Consequently, we have the map from the cone $\gamma_i : \mathcal{T}_i(M) \to s_iM$. While iterating this map following the decomposition \((4.9)\) we obtain a collection of morphisms:

\[ (4.12) \]
\[ \tilde{Y}^\mu(M) := \pi \circ \mathcal{T}_{i_1} \circ \mathcal{T}_{i_2} \circ \cdots \circ \mathcal{T}_{i_k}(M) \xrightarrow{\gamma_{i_1}} \pi \circ \mathcal{T}_{i_1} \circ \mathcal{T}_{i_2} \circ \cdots \circ \mathcal{T}_{i_{k-1}}(s_{i_k}(M)) \xrightarrow{\gamma_{i_{k+1}}} \cdots \]
\[ \to \cdots \xrightarrow{\gamma_{i_1}} \pi s_{i_1} \cdots s_{i_k}(M) = t_\mu(M) \approx q^{-\langle \mu, \lambda \rangle} M. \]

The module $t_\mu(M)$ differs from $M$ only by a shift grading along the loop parameter $z$ what is outlined in the last isomorphism of \((4.12)\). Moreover, for all $k \leq \ell$ the weight support of the complex $N_k := \mathcal{T}_{i_k} \circ \cdots \circ \mathcal{T}_{i_1}(M)$ contains a unique element $\nu$ with $\nu_- = \lambda_-$. One can show (by induction) that $\nu$ equals $s_{i_k} \cdots s_{i_1}(\lambda)$. All other weights of $H(N_k)$ belongs to the convex set $\mathcal{P}[\prec \lambda_+]$. Suppose that $\mathbb{K}_\nu$ is a one-dimensional module of weight $\nu$, then $z$-grading of $t_\mu(\mathbb{K}_\nu)$ is greater or equal to $-\langle \mu, \nu_+ \rangle$ and the same bounds take place for the $z$-grading of $\tilde{Y}^\mu(\mathbb{K}_\nu)$ as well as for the partial compositions $\pi \mathcal{T}_{i_1} \cdots \mathcal{T}_{i_k}(\mathbb{K}_\nu)$. Consequently, the $z$-grading of the cone of a morphism $\tilde{Y}^\mu(M) \to M$ is not less than $2 = \langle \mu, \lambda \rangle - \max_{\nu < \mu^+}(\langle \mu, \nu_+ \rangle)$. \(\square\)

**Proof of Theorem 4.10.** Suppose that $\Psi(M)$ consists of one dominant weight $\lambda$ and dominant weight $\mu_1, \ldots, \mu_r$ constitute a basis of the weight lattice $Q^\vee$. Thanks to Proposition 4.11 we have a collection of maps

\[ (4.13) \]
\[ \cdots \to (\tilde{Y}^\mu_1)^{m+1}(M) \xrightarrow{(\tilde{Y}^\mu_1)^{m}(\gamma_1)} (\tilde{Y}^\mu_2)^{m}(M) \to \cdots \to (\tilde{Y}^\mu_r(\gamma_1) \to (\tilde{Y}^\mu_1)^{m}(M) \to \tilde{Y}^\mu(M) \to M. \]

We claim that the categorical version of the famous Banach fixed point Theorem can be easily formulated in the derived setting. The system \((4.13)\) satisfies the Mittag-Leffler condition because each $(z, \xi)$-bigraded component of a convergent complexes has to be finite-dimensional. Therefore, the homotopy colimit $\text{hocolim}_m(\tilde{Y}^\mu_1)^{m}(M)$ is a well defined object of the category of convergent complexes $\mathbb{D}_c^-$. Moreover, thanks to the reduced decomposition of $\mu_i$ and Theorem 4.7 we know that this colimit belongs to $\mathbb{D}_c^- [\leq \lambda]$. The categorification Theorem 4.6 implies that endofunctors $Y$’s commute and the iterated homotopy colimit

$\mathbb{EM}_\lambda := (\tilde{Y}^\mu_1)^{\infty} \circ \cdots \circ (\tilde{Y}^\mu_r)^{\infty}(M)$

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is a well defined object in $D^b_c[\leq \lambda]$. For each $\mu = \sum c_i \mu_i$ we have $Y^\mu = (Y^{\mu_1})^{c_1} \circ \ldots \circ (Y^{\mu_r})^{c_r}$ and, consequently,

$$Y^\mu(EM_\lambda) = q^{-\sum c_i (\mu_i, \lambda)} EM_\lambda = q^{-\langle \mu, \lambda \rangle} EM_\lambda.$$ 

Consequently, the module $EM_\lambda$ satisfies all required conditions $(i)-(iii)$ of Theorem 4.10. □

**Corollary 4.14.** The constructed above objects $EM_\lambda$ associated with the one dimensional modules $k_\lambda$ categorify the nonsymmetric Macdonald polynomials $E_\lambda$:

$$\text{gch}(EM_\lambda) = E_\lambda.$$ 

**Proof.** First, we know that the nonsymmetric Macdonald polynomials are eigen functions of $Y$-operators and are uniquely defined by this property (up to a rational function on $q$ and $t$). Second, we know that the subspace of weight $\lambda$ of $EM_\lambda$ is one-dimensional. What follows that $\text{gch}(EM_\lambda)$ is equal to $E_\lambda$. □

**Corollary 4.15.** The constructed above objects $EM_\lambda$ associated with the one dimensional modules $k_\lambda$ categorify the nonsymmetric Macdonald polynomials $E_\lambda$ and satisfy the Ext-periodicity property for each $\nu \in P$:

$$(4.16) \quad q^{\langle \nu, \lambda \rangle} \text{hom}^\ast (EM_\lambda, EM_\mu) = q^{\langle \nu, \mu \rangle} \text{hom}^\ast (EM_\lambda, EM_\mu), \text{ if } \lambda \neq \mu.$$ 

**Proof.**

$$q^{\langle \nu, \lambda \rangle} \text{hom}^\ast (EM_\lambda, EM_\mu) = \text{hom}^\ast (q^{-\langle \nu, \lambda \rangle} EM_\lambda, EM_\mu) = \text{hom}^\ast (Y^\nu(EM_\lambda), EM_\mu) =$$

$$= \text{hom}^\ast (\pi T_{i_1} \ldots T_{i_r}(EM_\lambda), EM_\mu) = \text{hom}^\ast ((EM_\lambda), T_t' \ldots T_{t_1}' \pi^{-1} EM_\mu) =$$

$$= \text{hom}^\ast (EM_\lambda, (Y^\nu)^{-1}(EM_\mu)) = q^{\langle \nu, \mu \rangle} \text{hom}^\ast (EM_\lambda, EM_\mu).$$ □

It is worth mentioning that Ext-periodicity property implies vanishing of Ext groups whenever we know any bounds on the complex hom$(EM_\lambda, EM_\mu)$. We will show in the preceding Section 2.1 that for $g = sl_2$ the Macdonald complexes $EM_\lambda$ are modules concentrated in one homological degree. Therefore, the vanishing of Ext$(EM_\lambda, EM_\mu)$ holds for different $\lambda$ and $\mu$ for $g = sl_2$ (Corollary 5.40 below).

### 4.6. Symmetrization functor.

The goal of this subsection is to study the induction functor to the derived category of the representations of bigger parabolic subalgebra, namely the current algebra $g[z, \xi]$. We use it to construct the categorification of symmetric Macdonald polynomials.

We consider the symmetrization operator $P \in HH$ from [Ch2], page 324. In our notations it is equal to:

$$P := \sum_{w \in W} T_w.$$ 

It is shown in [Ch2] that $P$ is divisible by $T_i + 1$ from the left and from the right and consequently, the image of $P$ belongs to the subspace of $W$-symmetric functions. Moreover, the operator $P$ is uniquely characterized up to a rational function on $q, t$ by the following list of properties:

1. $\forall i \in I$ one has $T_i P = PT_i = t P$;
2. the image of the operator $P$ is contained in $W$-symmetric functions;
Lemma 4.17. The derived induction and restriction functors

\[ \text{Res}_i : D_c^- (O(J[\xi])) \Rightarrow D_c^- (O(b[z, \xi])) : \text{LInd}_i \]

satisfy the following properties for each \( i \in I \):

\[
\begin{align*}
D_i \circ \text{Res}_i & \simeq \text{Res}_i \oplus \text{Res}_i \circ t[1]; \\
\text{LInd}_i \circ D_i & \simeq \text{LInd}_i \oplus \text{LInd}_i \circ t[1].
\end{align*}
\]

Proof. Denote by \( \text{Res}_i^L \) the restriction functor from the algebra \( g[z, \xi] \) to the parabolic subalgebra \( p_i[\xi] \) and by \( \text{LInd}_i^L, \text{RCoInd}_i \) the corresponding derived induction and coinduction functors. The big restriction, (co)induction is the composition of the smaller ones:

\[ \text{Res}_i = \text{Res}_i \circ \text{Res}_i^L, \quad \text{LInd}_i = \text{LInd}_i^L \circ \text{LInd}_i, \quad \text{RCoInd}_i^L \circ \text{RCoInd}_i. \]

Consequently, we have

\[ D_i \circ \text{Res}_i = (\text{Res}_i \circ \text{LInd}_i) \circ (\text{Res}_i \circ \text{Res}_i^L) \overset{\text{(3.12)}}{=} (\text{Res}_i \oplus \text{Res}_i t[1]) \circ \text{Res}_i^L = \text{Res}_i \oplus \text{Res}_i t[1]. \]

and the similar isomorphism for the coinduction functors:

\[ \text{RCoInd}_i \circ D_i = \text{RCoInd}_i^L \circ \text{RCoInd}_i \circ \text{Res}_i \circ \text{LInd}_i \overset{\text{(3.13)}}{=} \text{RCoInd}_i^L \circ (\text{RCoInd}_i \oplus \text{LInd}_i) \overset{\text{(3.14)}}{=} \text{RCoInd}_i^L \circ (\text{RCoInd}_i \oplus \text{RCoInd}_i \circ t[1]) = \text{RCoInd}_i \oplus \text{RCoInd}_i \circ t[1]. \]

Let us combine the Isomorphisms (4.19) and (4.18) into a collection of isomorphisms:

\[
\begin{align*}
D_i \circ \text{Res}_i \circ \text{LInd}_i & \simeq \text{Res}_i \circ \text{LInd}_i \oplus \text{Res}_i \circ \text{LInd}_i \circ t[1] \simeq \text{Res}_i \circ \text{LInd}_i \circ D_i.
\end{align*}
\]

Denote by \( D_i \) the composition \( \text{Res}_i \circ \text{LInd}_i \) in order to shorten the presentation:

\[
\begin{align*}
T_i \circ D_i = \text{cone}(\text{Id} \rightarrow D_i) \circ D_i = \text{cone}(D_i \rightarrow D_i \circ D_i) \simeq \\
\overset{\text{(4.20)}}{=} \text{cone}(D_i \rightarrow (D_i \oplus D_i t[1])) = D_i t[1].
\end{align*}
\]

By the same computations one can see that

\[
D_i \circ T_i = D_i t[1].
\]

Theorem 4.23. The composition \( D_i \) of the restriction and the derived induction functors \( O(J[\xi]) \overset{\text{LInd}_i}{\rightarrow} O(g[z, \xi]) \overset{\text{Res}_i}{\rightarrow} O(J[\xi]) \) categorifies the Cherednik symmetrization operator \( P \in \mathcal{H} \).

Proof. Notice that \( \mathcal{H} \) acts fully faithfully on the set of characters \( \mathbb{Z}[P] \). It implies that the transformation rule of the supercharacter of objects in \( D_c^- (J[\xi]) \) with respect to any endofunctor is given by an element of an appropriate completion of \( \mathcal{H} \). Thus, in order to prove that the endofunctor \( D_i \) acts on characters as operator \( P \) it is enough to explain the categorical analogues of Properties (1)-(4):

(3) if the weight \( \mu \) is contained in the convex hall of the \( W \)-orbit of the dominant weight \( \lambda \), then the support of \( Pf^\mu \) is contained in this convex hall;

(4) \( Pf = fP \) whenever \( f \) is a \( W \)-symmetric function.
(1) The categorification of Property (1) is proved in Lemma 4.17 and the preceeding isomorphisms (4.21) and (4.22).
(2) The character of the image of the functor $D_i$ is symmetric with respect to the finite Weyl group because the image of $L_{\text{Ind}}I_b$ is $g$-integrable.
(3) Theorem 4.7 explains that the endofunctor $D_i$ preserves the subcategory of modules whose weights lie in the convex hall of $W \lambda_+$ for each dominant weight $\lambda_+$.
(4) Property (4) follows from the standard observation that endofunctors $D_i$ commutes with taking tensor products with $g$-integrable modules.

Finally we are able to describe modules that categorifes Macdonald polynomials:

**Corollary 4.24.** For each dominant weight $\lambda \in Q^+_+$ the character of the complex

$$PM_{\lambda} := L_{\text{Ind}}(EM_{\lambda}) \in D^c_c(g[z, \xi])$$

is proportional to the Macdonald (symmetric) polynomial $P_{\lambda}$.

**Proof.** Cherednik symmetrization operator $\mathcal{P}$ maps a nonsymmetric Macdonald polynomial $E_{\lambda}$ to a polynomial which is proportional to a symmetric Macdonald polynomial $P_{\lambda_+}$ where $\lambda_+$ is the dominant weight in the $W$ orbit of $\lambda$. □

5. Macdonald modules for $\mathfrak{sl}_2$

Let us denote by $\mathfrak{I} := \mathfrak{b}_+ + z\mathfrak{sl}_2[z]$ the Iwahori subalgebra of $\hat{\mathfrak{sl}}_2$. Define the following cyclic $\mathfrak{I}[\xi]$-modules.

**Definition 5.1.** Take $k \in \mathbb{Z}_{\geq 0}$. The cyclic module $EM_{-k\omega}$ is the graded cyclic module generated by the cyclic vector $w_{-k\omega}$ of $q, t$-degree 0, weight $-k\omega$ subject to the following list of relations:

$$(5.2) \quad f z^{a} \xi^{b} w_{-k\omega} = 0, a > 0, b = 0, 1; \quad e^{k+1} w_{-k\omega} = 0; \quad h \xi w_{-k\omega} = 0.$$

The cyclic module $EM_{k\omega}$ is the graded cyclic module with the cyclic vector $w_{k\omega}$ of $q, t$-degree 0, weight $k\omega$ and the following relations:

$$(5.3) \quad e z^{a} \xi^{b} w_{k\omega} = 0, a \geq 0, b = 0, 1; \quad (f z)^{k} w_{k\omega} = 0; \quad h \xi w_{k\omega} = 0.$$

We call these modules by global nonsymmetric Macdonald modules.

**Remark 5.4.** The module $EM_{-k\omega}$ is the $\pi$-automorphism twist of the module $EM_{(k+1)\omega}$.

We need local versions of these modules.

**Definition 5.5.** The quotient module $EM_{k\omega} = EM_{k\omega}/\langle h \rangle[z]w_{k\omega}$ is called the local nonsymmetric Macdonald module.

**Remark 5.6.** Note that in difference to ordinary nonsuper Weyl modules the $k\omega$-weight space of $EM_{k\omega}$ is not one dimensional because we have a nontrivial action of some elements $hq^a \xi$, $a > 0$ on the generator. One more difference is that $EM_{k\omega}$ is not a restriction of any module over the whole current algebra.
5.1. The module $\mathfrak{EM}_-\omega$ and its deformation. In this subsection we study the properties of the smallest nontrivial nonsymmetric Macdonald module $\mathfrak{EM}_-\omega$.

**Proposition 5.7.** The elements $w_-, h\xi w_-, ew_-, e\xi w_-$ assemble a basis of the module $\mathfrak{EM}_-\omega$ and its supercharacter is equal to:

$$\left(1 - qt\right)x^{-1} + \left(1 - t\right)x = \widetilde{E}_-\omega(x, q, t).$$

**Proof.** Using PBW theorem we have that $\mathfrak{EM}_-\omega$ has a basis of the following form:

$$\prod_{i=1}^{s_e} e z_{c_i}^{c_i} \prod_{i=1}^{s_h} h z_{b_i}^{b_i} \prod_{i=1}^{s_f} f z_{a_i}^{a_i} w_-.$$

Relations from Definition 5.1 imply $s_f = 0$ and $b_i \geq 1$. Relations from Definition 5.5 imply $b_i' = 1$. Further we have for $d > 0$:

$$ez^d w_- = -\frac{1}{2} f z^d \cdot e^2 w_- = 0.$$  

Thus for $d > 0$:

$$hz^{d+1} \xi w_- = f \xi \cdot ez^d w_- = 0.$$  

Thus $-\omega$-weight space of $\mathfrak{EM}_-\omega$ is linearly generated by $w_-$ and $hz\xi w_-$ that are linearly independent by degree reasons.

Moreover we have for $d > 0$:

$$ez^d \xi w_- = \frac{1}{2} h \xi \cdot ez^d w_- = 0.$$  

Moreover by the analogous computations we obtain for any $d > 0, d' = 0, 1$:

$$ez^d \xi^{d'} h z \xi w_- = 0.$$  

What remains consists of monomials $ew_-$ and $e\xi w_-$ that are linearly independent by degree reasons and span the $\omega$-weight space of $\mathfrak{EM}_-\omega$.

Finally, we have:

$$e \xi \cdot ew_- = \frac{1}{4} h \xi \cdot e^2 w_- = 0.$$  

This completes the proof. \qed

**Corollary 5.11.** $\mathfrak{EM}_{2\omega}$ has character equal to $\widetilde{E}_{2\omega}(x, q, t)$.

**Proof.** It follows from Remark 5.4. \qed

The action of the free commutative algebra $k[h z, h z^2, \ldots] = U(h[z])$ on the cyclic vector $w_- \mapsto h z^k w_-$ extends to the right action on the global nonsymmetric Macdonald module $\mathfrak{EM}_-\omega$ by automorphisms. The essential image of this action is isomorphic to $k[h z]$ thanks to the equality:

$$hz^k w_- = (hz)^k w_-$$

that is well known in the nonsuper case [CP].

Thus the local module $\mathfrak{EM}_-\omega$ is the quotient of $\mathfrak{EM}_-\omega$ by the ideal generated by $hz w_-$ that leads to the following deformation $\mathfrak{EM}_-\omega^\alpha$ of the module $\mathfrak{EM}_-\omega$:

**Definition 5.12.**

$$\mathfrak{EM}_-\omega^\alpha = \mathfrak{EM}_-\omega / h(z - \alpha)w_-.$$  


The module $\mathfrak{EM}^{a}_{-\omega}$ is not $z$-graded, but still inherits the $\xi$-degree and the weight decomposition.

**Lemma 5.13.** $\mathfrak{EM}^{a}_{-\omega}$ has a basis $w_{-\omega}$, $hz\xi w_{-\omega}$, $ew_{-\omega}$, $e\xi w_{-\omega}$.

**Proof.** As we know from Proposition 5.7, the elements $w_{-\omega}$, $hz\xi w_{-\omega}$, $ew_{-\omega}$, $e\xi w_{-\omega}$ are linearly independent for $a = 0$, hence they are linearly independent for generic $\alpha$. Moreover, from the theory of Weyl modules for nonsuper Lie algebra $\mathfrak{sl}_2$ we know that for $d > 0$:

$$ez - \alpha^d w_{-\omega} = 0.$$ 

Therefore, we have:

$$h(z - \alpha)^{d+1}\xi w_{-\omega} = -fz \cdot e(z - \alpha)^d w_{-\omega} = 0.$$ 

Thus $-\omega$-weight space has a basis $\{w_{-\omega}, hz\xi w_{-\omega}\}$.

Also we have:

$$e(z - \alpha)^d \xi w_{-\omega} = \frac{1}{2} h\xi \cdot e(z - \alpha)^d w_{-\omega} = 0.$$ 

The following relations

$$e \cdot h(z - \alpha)\xi w_{-\omega} = 0 \quad \text{and} \quad e \cdot e\xi w_{-\omega} = 0$$

are the straightforward generalizations of the one explained in the nondeformed case. $\square$

As one can see from the proof of Lemma 5.13, the following relations are satisfied in $\mathfrak{EM}^{a}_{-\omega}$:

$$ez^k w_{-\omega} = \alpha^k ew_{-\omega}; \quad hz^k \xi w_{-\omega} = \alpha^k h\xi w_{-\omega}; \quad ez^k \xi w_{-\omega} = \alpha^k e\xi w_{-\omega}.$$ 

**Corollary 5.15.** The algebra $k[h\xi]$ acts freely on $\mathfrak{EM}_{-\omega}$.

5.2. **Fusion product.** Take $k$ different numbers $\alpha_1, \ldots, \alpha_k$. Consider the following module:

$$\mathfrak{EM}^{\alpha_1, \ldots, \alpha_k}_{-k\omega} := \mathfrak{EM}^{\alpha_1}_{-\omega} \otimes \cdots \otimes \mathfrak{EM}^{\alpha_k}_{-\omega}.$$ 

**Proposition 5.17.** $\mathfrak{EM}^{\alpha_1, \ldots, \alpha_k}_{-k\omega}$ is cyclic.

**Proof.** There is a standard trick introduced in [FL] showing that Relations 5.14 implies that the product of $k$ cyclic vectors $w := w_{-\omega} \otimes \cdots \otimes w_{-\omega}$ is a cyclic for this module. $\square$

Next we prove that the module $\mathfrak{EM}^{\alpha_1, \ldots, \alpha_k}_{-k\omega}$ satisfies the defining relations of $\mathfrak{EM}_{-k\omega}$. More precisely:

**Lemma 5.18.**

$$fz^a \xi^b w = 0; h\xi w = 0; e^{k+1} w = 0.$$ 

**Proof.** The first two relations hold because elements $fz^a \xi^b$ and $h\xi$ annihilate the cyclic vectors in all tensor factors. The last equality holds because the element $e^{k+1}$ annihilates these vectors. $\square$

The universal enveloping algebra $U(\mathfrak{sl}_2[z, \xi])$ is bigraded with respect to current parameters $z, \xi$. Denote by $F_m$ the sum of subspaces of this algebra whose $z$-degree is $\leq m$, by $F$, the corresponding filtration of $U(\mathfrak{sl}_2[z, \xi])$ and we denote by $F_m(\mathfrak{EM}^{\alpha_1, \ldots, \alpha_k}_{-k\omega})$ the subspace $F_m w$ given by filtration on the cyclic vector $w$. We put $F_{-1} := \{0\}$. Define:

$$\text{gr}^F \mathfrak{EM}^{\alpha_1, \ldots, \alpha_k}_{-k\omega} := \bigoplus_{m \geq 0} F_m/F_{m-1}. $$

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The module \( \text{gr}^{F}\mathfrak{EIM}^{\alpha_1, \ldots, \alpha_k}_{-k\omega} \) is graded by \( z \)-degree, \( \xi \)-degree and weight. By construction it is generated by the image of the element \( w \).

**Proposition 5.21.** The assignment \( w_{-k\omega} \mapsto w \) extends to the surjective homomorphism of \( \mathfrak{sl}_2[z, \xi] \)-modules:

\[
\mathfrak{EIM}_{-k\omega} \twoheadrightarrow \text{gr}^{F}\mathfrak{EIM}^{\alpha_1, \ldots, \alpha_k}_{-k\omega}.
\]

In particular, \( \dim \mathfrak{EIM}_{-k\omega} \geq 4^k \).

**Proof.** It is sufficient to show that the cyclic vector of \( \text{gr}^{F}\mathfrak{EIM}^{\alpha_1, \ldots, \alpha_k}_{-k\omega} \) satisfies the defining relations of \( \mathfrak{EIM}_{-k\omega} \). More precisely, we know that elements \( f z^a \xi^b, h \xi \) and \( e^{k+1} \) annihilate the cyclic vectors in all tensor factors. Consequently, we have

\[
f z^a \xi^b w = 0; \quad h \xi w = 0; \quad e^{k+1} w = 0.
\]

The subspace of \( \mathfrak{EIM}^{\alpha_1, \ldots, \alpha_k}_{-k\omega} \) spanned by vectors of \( \xi \)-degree zero and weight \( -k\omega \) is one-dimensional. Therefore, the element \( h z^k \) annihilates the cyclic vector for any \( k > 0 \) in the graded module \( \text{gr}^{F}\mathfrak{EIM}^{\alpha_1, \ldots, \alpha_k}_{-k\omega} \).

**Corollary 5.24.** \( \dim \mathfrak{EIM}_{(k+1)\omega} \geq 4^k \).

**Proof.** This follows from the action of the diagram automorphism \( \pi \).

### 5.3. Decomposition procedure.

In this subsection we construct four component filtration on the module \( \mathfrak{EIM}_{-k\omega} \) such that the corresponding subquotients are isomorphic to modules \( \mathfrak{EIM}_{(k+1)\omega} \) and \( \mathfrak{EIM}_{k\omega} \) with shifted degree. We use this result to compute the supercharacters of these modules.

Take the following four elements of the module \( \mathfrak{EIM}_{(k+1)\omega} \):

\[
(5.25) \quad w_1 := w; \quad w_2 := e^{k+1} w_1; \quad w_3 := h \xi w_2; \quad w_4 := (f z)^{k+1} w_3.
\]

We define the submodules generated by the aforementioned vectors:

\[
(5.26) \quad \mathfrak{DIM}_i := U(\mathfrak{g}[\xi]) w_i \subset \mathfrak{EIM}_{(k+1)\omega}.
\]

We obviously have the following sequence of inclusions of modules:

\[
\mathfrak{DIM}_1 \supset \mathfrak{DIM}_2 \supset \mathfrak{DIM}_3 \supset \mathfrak{DIM}_4
\]

and the subsequent Lemma explains the inductive description of the successive quotients

**Proposition 5.27.** The assignments \( w_\lambda \mapsto w_i \) of the cyclic vectors extend to the following list of epimorphisms

\[
(5.28) \quad X^{-\omega} \mathfrak{EIM}_{-k\omega} \twoheadrightarrow \mathfrak{DIM}_1/\mathfrak{DIM}_2;
\]

\[
(5.29) \quad X^{-\omega} \mathfrak{EIM}_{(k+1)\omega} \twoheadrightarrow \mathfrak{DIM}_2/\mathfrak{DIM}_3;
\]

\[
(5.30) \quad X^{-\omega} \mathfrak{EIM}_{(k+1)\omega} \twoheadrightarrow \mathfrak{DIM}_3/\mathfrak{DIM}_4;
\]

\[
(5.31) \quad X^{-\omega} q^{k+1} \mathfrak{EIM}_{-k\omega} \twoheadrightarrow \mathfrak{DIM}_4;
\]

**Proof.** Note that all successive quotients \( \mathfrak{DIM}_i/\mathfrak{DIM}_{i+1} \) are cyclic modules by construction and the grading shifts are adapted in order to have a cyclic vector of the same weight and \( (z, \xi) \)-degree in the corresponding nonsymmetric Macdonald module. So in order to have well defined morphisms (5.28)-(5.31) one has to check the defining relations of the Macdonald module.
• The relations for \( w_1 \) follows directly from the defining relations of \( \mathfrak{IM}_{(-k)\omega} \).
• From the theory of Weyl modules we have:
\[
hz^l w_2 = 0, \ l > 0; \ (fz)^{k+1} w_2 = 0; \ ez^l w_2 = 0, \ l \geq 0.
\]
Moreover from the theory of Weyl modules for \( l > 0 \) we have: \( e^k ez^l w = 0 \). Now we get:
\[
e\xi w_2 = e\xi e^{k+1} w = \frac{1}{2(k+2)} h\xi e^{k+2} w = 0
\]
and for \( l > 0 \)
\[
ez^l \xi w_2 = ez^l \xi e^{k+1} w = \frac{1}{2}(hz^l e\xi e^{k+1} - 2(k+1)e\xi ez^l) w = 0.
\]
Moreover, by definition we have: \( h\xi w_2 \in \mathfrak{DIM}_3 \) what implies the map \( (5.29) \).
• We have \([h^l, h\xi] = [ez^l \xi, h\xi] = 0\). Therefore we get for \( l > 0 \):
\[
hz^l w_3 = hz^l h\xi w_2 = h\xi hz^l w_2 = 0;
\]
\[
ez^l \xi w_3 = ez^l \xi h\xi w_2 = h\xi ez^l \xi w_2 = 0.
\]
\[
h\xi w_3 = h\xi h\xi w_2 = 0,
\]
because \( h\xi \) is odd.

Note that \( w_3 = \frac{1}{2(k+1)} e^k e\xi w \). Hence:
\[
ez^l w_3 = 0.
\]
Finally by definition we have:
\[
(fz)^{k+1} w_3 \in \mathfrak{DIM}_4.
\]
• PBW theorem implies that \( w_4 \in U(\langle h \rangle[z, \xi])w \). Therefore we obtain that the elements \( ez^l \xi^l, \ l \geq 0, hz^l, \ l > 0 \) and \( h\xi \) annihilate this element.

Finally we have:
\[
e^{k+1} w_4 = e^{k+1} (fz)^{k+1} w_3 \in k[hz, hz^2, \ldots] w_3.
\]
Thus \( e^{k+1} w_4 = 0 \) because of relations satisfied by the element \( w_3 \).

The diagrammatic automorphism \( \pi \) allows one to define the 4-term filtration of the non-symmetric local Macdonald module \( \mathfrak{IM}_{(k+2)\omega} \) with surjections of smaller Macdonald modules on the successive quotients:
\[
\mathfrak{IM}_{(k+1)\omega} \twoheadrightarrow \mathfrak{DIM}_1 / \mathfrak{DIM}_2;
\]
\[
X^{-\omega} q^{k+1} e\mathfrak{IM}_{-k\omega} \twoheadrightarrow \mathfrak{DIM}_2 / \mathfrak{DIM}_3;
\]
\[
X^{-\omega} q^{k+1} t\mathfrak{IM}_{-k\omega} \twoheadrightarrow \mathfrak{DIM}_3 / \mathfrak{DIM}_4;
\]
\[
X^{-\omega} q^{k+1} t\mathfrak{IM}_{(k+1)\omega} \twoheadrightarrow \mathfrak{DIM}_4.
\]
Lemma 5.37. With the generator \( w \) the cyclic module generated by the element \( w \).

Theorem 5.34. The inductive description of \( \text{Ind}_1(\mathbf{EIM}_{\omega m}) = H^0(\text{LInd}_1(\mathbf{EIM}_{\omega})) \) is the cyclic module generated by the element \( w_{-\omega} \) satisfying the following relations:

\[
\begin{align*}
\varepsilon^a \xi^b w_{-\omega} &= 0, a \geq 0, b = 0, 1; \\
(e)^{m+1} w_{-\omega} &= 0; \\
h \xi w_{-\omega} &= 0.
\end{align*}
\]

Proof. By definition we have that the \( \mathfrak{sl}_2[\zeta, \xi] \) induced module \( U(\mathfrak{sl}_2[\zeta, \xi] \otimes U(\mathfrak{sl}_2[\zeta, \xi])) \mathbf{EIM}_{\omega m} \) is the cyclic module generated by the element \( w_{\omega} \) satisfying the following relations:

\[
\begin{align*}
e^a \xi^b w_{\omega} &= 0, a \geq 0, b = 0, 1; \\
(f)^{m+1} w_{\omega} &= 0; \\
h \xi w_{\omega} &= 0.
\end{align*}
\]

The straightforward check shows that the \( \mathfrak{sl}_2[\zeta, \xi] \)-module, yielding Relations (5.39) is finite-dimensional and, therefore, is integrable. Thus, the module defined by Relations (5.39) is isomorphic to \( \text{Ind}_1(\mathbf{EIM}_{\omega m}) \). The Weyl group automorphism \( s_1 \) keeps the module \( \text{Ind}_1(\mathbf{EIM}_{\omega m}) \), \( s_1(w_{\omega}) = w_{-\omega} \) and \( s_1 \) maps Relations (5.39) to Relations (5.38).

Proof of Theorem 5.37. Suppose that \( m > 0 \). The presentation (5.38) implies the isomorphism of \( \mathcal{J}[\xi] \)-modules:

\[
\text{Res}_1 \circ \text{Ind}_1(\mathbf{EIM}_m) \simeq \text{Ind}_3(\mathbf{EIM}_{\omega m}) \simeq X^{-\omega} q^m \mathbf{EIM}_{(-m+1)\omega}
\]

Consequently, we have:

\[
H^0(\mathbf{T}_1 \mathbf{EIM}_{\omega m}) = H^0(\text{cone}(\mathbf{EIM}_{\omega m} \to \mathbf{D}_1(\mathbf{EIM}_{\omega m})))
\]

and is isomorphic to the quotient \( X^{-\omega} \mathbf{EIM}_{(-m+1)\omega} \). The direct comparison of relations explains the following isomorphism:

\[
\pi X^{-\omega} \mathbf{EIM}_{(-m+1)\omega} \simeq q^m \mathbf{EIM}_{\omega m}.
\]

5.4. Eigenvalues. The goal of this subsection is to prove that the modules \( \mathbf{EIM}_{\omega m} \) are eigenobjects for the endofunctor \( \mathbf{Y}^\omega := \pi \mathbf{T}_1 \) and its inverse \( \mathbf{Y}^{-\omega} := \mathbf{T}_1 \pi \) in the categorical sense:

Theorem 5.34. There are isomorphisms

\[
\begin{align*}
\mathbf{Y}^\omega(\mathbf{EIM}_{\omega m}) &\simeq q^{-m} \mathbf{EIM}_{\omega m}; \text{ if } m > 0 \\
\mathbf{Y}^{-\omega}(\mathbf{EIM}_{\omega m}) &\simeq t^{-1} q^m \mathbf{EIM}_m; \text{ if } m \leq 0.
\end{align*}
\]
The functor $\mathbf{T}^{-1}$ has only 0 and $-1$'st nontrivial cohomology and it remains to show that $-1$'st cohomologies of $\mathbf{T}^{-1}\mathbf{EM}_{m\omega}$ vanishes. The latter follows from the equality of characters that makes sense due to the main categorification Theorem 1.10:

$$gch(H^0(\pi\mathbf{T}^{-1}(\mathbf{EM}_{m\omega}))) = gch(q^{-m}\mathbf{EM}_{m\omega}) = q^{-m}\tilde{E}_{m\omega} =$$

$$= Y^\omega(\tilde{E}_{m\omega}) = gch(Y^\omega\mathbf{EM}_{m\omega}) = gch(\pi\mathbf{T}^{-1}\mathbf{EM}_{m\omega})$$

Here, as before, $\tilde{E}_{m\omega} = \tilde{E}_{m\omega}(x, q, t)$ denotes the integral form of the nonsymmetric Macdonald polynomial $E_m$. $\tilde{E}_m$ differs from $E_m$ by a certain polynomial depending on $q, t$ such that $\tilde{E}_m$ is a polynomial in $q, t$ with integral coefficient and, therefore, is an eigen vector of the endofunctor $Y^\omega$.

The proof for $m \leq 0$ is completely analogous and is based on the isomorphism:

$$\text{Res}_1 \circ \text{Ind}_1(\mathbf{EM}_{-k\omega}) = \text{DIM}_3 t[1].$$

\[ \square \]

**Corollary 5.40.** The modules $\mathbf{EM}_\lambda$ constitute an orthogonal basis of the derived category $\mathbf{D}^{-c}_c(\mathcal{O}(\mathfrak{g}[\xi]))$.

**Proof.** We have $\text{Ext}(\mathbf{EM}_\lambda, \mathbf{EM}_\nu) = 0$ whenever $\lambda \neq \nu$ thanks to Isomorphisms (1.16). \[ \square \]

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