Quantum quasi-Lie systems: properties and applications

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Abstract

A Lie system is a non-autonomous system of ordinary differential equations describing the integral curves of a t-dependent vector field taking values in a finite-dimensional Lie algebra of vector fields. Lie systems have been generalised in the literature to deal with t-dependent Schrödinger equations determined by a particular class of t-dependent Hamiltonian operators, the quantum Lie systems, and other differential equations through the so-called quasi-Lie schemes. This work extends quasi-Lie schemes and quantum Lie systems to cope with t-dependent Schrödinger equations associated with the here called quantum quasi-Lie systems.

To illustrate our methods, we propose and study a quantum analogue of the classical nonlinear oscillator searched by Perelomov and we analyse a quantum one-dimensional fluid in a trapping potential along with quantum t-dependent Smorodinsky–Winternitz oscillators.

Keywords: Hilbert space; Lie system; Schrödinger equation; Smorodinsky–Winternitz oscillator; quasi-Lie scheme; quantum non-linear oscillator.

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1 Introduction

A Lie system is a non-autonomous system of first-order ordinary differential equations whose general solution can be written via a (generally nonlinear) autonomous function, referred to as superposition rule, a finite set of particular solutions and constants related to the initial conditions [17, 22, 65]. For instance, non-autonomous inhomogeneous linear systems of first-order ordinary differential equations, Bernoulli equations, Riccati equations, and matrix Riccati equations are examples of Lie systems [51, 65].

Lie systems are mathematically interesting due to their geometric and algebraic properties [13, 17, 45]. The Lie–Scheffers theorem establishes that a Lie system is equivalent to a non-autonomous vector field taking values in a finite-dimensional Lie algebra V of vector fields, known as a Vessiot–Guldberg Lie algebra (VG Lie algebra henceforth) of the Lie system [17, 22]. As a consequence, the study of Lie systems leads to the investigation of superposition rules through projective foliations on an appropriate bundle [17], generalised distributions [17], Lie group actions [45, 65], etc. Furthermore, there has been a recent interest in Lie systems possessing VG Lie algebras of Hamiltonian vector fields regarding different geometric structures [3, 6, 10, 15, 25, 37, 41, 51].
The Lie–Scheffers theorem implies that being a Lie system is a very restrictive condition [13]. Nonetheless, Lie systems play a relevant rôle in mathematics, physics, and control theory, e.g. [22] cites more than 200 works on Lie systems and their applications. In particular, the theory of Lie systems provides methods to determine the integrability of certain systems of first-order differential equations [2, 18, 22, 24]. Furthermore, Lie systems have been proved to be very helpful in the study of geometric phases [34], the solution of important nonlinear oscillators [22], the analysis of Wei–Norman equations [24, 26, 30, 31], and the research on problems occurring in quantum mechanics [10, 24] and biology [3, 6]. Riccati equations and their generalisations are ubiquitous in physics, occurring for instance in cosmology and financial models [2, 3, 22, 50, 61]. Riccati equations appear also in the study of epidemiological models [32].

A generalisation of the concept of Lie systems has recently been proposed in [12, 19, 22], where the theory of quasi-Lie systems was developed. This theory investigates non-autonomous systems of first-order differential equations described by \( t \)-dependent vector fields, \( X \), taking values in a finite-dimensional linear space of vector fields \( V \) containing a VG Lie algebra \( W \subset V \) such that \([W,V] \subset V\). If \( V \) is also a Lie algebra, then \( X \) is a Lie system. The Lie algebra \( W \) can be integrated to give rise to a local Lie group action of its associated Lie group, which can be employed to transform the original system into a Lie system with a VG Lie algebra within \( V \) (cf. [12]). This idea allows one to analyse the integrability of non-harmonic classical oscillators, dissipative Milne–Pinney equations, and other systems that cannot be studied through Lie systems [12, 19, 20, 22, 23].

Besides, the theory of Lie systems has been extended to the quantum realm in [11, 21, 22, 24, 27]. A quantum Lie system is a \( t \)-dependent Hamiltonian operator \( \hat{H}(t) \) such that \( i\hat{H}(t) \) takes values in a finite-dimensional real Lie algebra of skew-Hermitian operators [11, 21, 22, 24]. This allows us to develop powerful techniques to study \( t \)-dependent Schrödinger equations associated with these \( t \)-dependent Hamiltonian operators [10, 11, 21, 22, 24]. For instance, \( t \)-dependent Schrödinger equations associated with quantum Lie systems can be investigated through Lie systems on Lie groups and the corresponding real Lie algebra of skew-Hermitian operators is a clue to solve them [22]. An example treatable with this aid is the case of quantum \( t \)-dependent dissipative harmonic oscillators of Calogero–Moser type [22].

To enlarge the field of applications of quantum Lie systems, we propose as a main result a quantum analogue of the theory of quasi-Lie schemes based on the hereupon denominated quantum quasi-Lie schemes. Every quantum quasi-Lie scheme gives rise to a group of \( t \)-dependent gauge transformations, the group of the quantum quasi-Lie scheme, which enables one to transform certain \( t \)-dependent Schrödinger equations into new ones that can be studied through quantum Lie systems. This stands out a particular family of \( t \)-dependent Schrödinger equations, described by the here defined quantum quasi-Lie systems (QQLS), which cover the \( t \)-dependent Schrödinger equations associated with quantum Lie systems as a particular case. In this manner, quantum quasi-Lie systems can be investigated via the theory of quantum Lie systems.

One of the main uses of quantum quasi-Lie schemes is to map a \( t \)-dependent Hamiltonian operator \(-i\hat{H}(t)\) described through a quantum quasi-Lie scheme into a quantum quasi-Lie system or a simpler \( t \)-dependent Hamiltonian operator \(-i\hat{H}'(t)\) through an element \( U(t) \) of the group of the quantum quasi-Lie scheme. Instead of applying straightforwardly a generic \( U(t) \) to transform \(-i\hat{H}(t)\) into a simpler \(-i\hat{H}'(t)\), which is operationally complicated and tedious but ubiquitous in the literature, we provide a series of results, e.g. Propositions 6.1 and 6.2, giving easy criteria to predict some interesting properties of \(-i\hat{H}'(t)\). This will eventually gives hints on the form of \( U(t) \) to simplify \(-i\hat{H}'(t)\) or to ensure that \(-i\hat{H}(t)\) is a quantum quasi-Lie system. Mathematically,
Propositions 6.1 and 6.2 concern the study of real non semi-simple Lie algebra representations. This is the least studied case in the literature, which mainly focuses upon representations of complex semi-simple Lie algebras.

As applications, we show that quantum quasi-Lie schemes allow us to study the quantum evolution in problems in which the theory of quantum Lie systems does not apply. In particular, we develop a quantum analogue of the classical anharmonic oscillator formerly studied by Perelomov [55]. This solves the problem proposed by Perelomov of finding a quantum analogue of his oscillator. Subsequently, we focus on the application of quantum quasi-Lie systems in $t$-dependent Schrödinger equations describing a one-dimensional quantum fluid in a $t$-dependent trapping potential [64] and a quantum $t$-dependent frequency Winternitz–Smorodinsky oscillator.

The paper is organised as follows. Section 2 reviews fundamental notions of differential geometry on Hilbert spaces. Section 3 presents quantum Lie systems and their properties. Section 4 introduces the theory of quantum quasi-Lie schemes, while the theory of quantum quasi-Lie systems is presented in Section 5. Section 6 addresses the description of new methods to study $t$-dependent Schrödinger equations by means of quantum quasi-Lie schemes. Subsequently, we present an application of the theory of quantum quasi-Lie schemes and systems to quantum non-linear oscillators with a homogeneous potential in Section 7. Section 8 is devoted to the application of the theory of quantum quasi-Lie systems to the particular case of homogeneous potentials of degree minus two. In particular, this entails studying quantum $t$-dependent frequency Calogero–Moser systems, quantum fluids in a $t$-dependent homogeneous trapping potential, and $t$-dependent quantum Smorodinsky–Winternitz systems. To conclude, Section 9 contains a summary of the obtained results and a commentary on future research prospects.

2 Differential geometry in Hilbert spaces.

This section surveys the differential geometry of (possibly infinite-dimensional) Hilbert spaces. For further details on the geometry of infinite-dimensional manifolds and the theory of skew-Hermitian operators, we refer to [36, 38, 47, 49].

Let $H$ be a complex Hilbert space endowed with the topology inherited from the norm $\| \cdot \|$ associated with its scalar product $\langle \cdot | \cdot \rangle$. It is hereupon assumed that $H$ is separable, as it usually happens in quantum mechanics. The separability of $H$ means that there exists a countable orthonormal basis. The orthonormal basis, let us say $\{\psi_n\}_{n \in \mathbb{N}}$, gives rise to a global chart $\varphi : \psi \in H \mapsto \{\Re \langle \psi_n | \psi \rangle, \Im \langle \psi_n | \psi \rangle\}_{n \in \mathbb{N}} \in \ell^2(\mathbb{R})$ onto the real Hilbert space $\ell^2(\mathbb{R})$ of square summable sequences, which allows us to consider $H$ as a manifold $H_\mathbb{R}$ modelled over a Hilbert space [10] [22].

We call derivative at a point $p$ in an open subset $U \subset H$ a linear mapping $D : C^\infty(U) \to \mathbb{R}$ satisfying that $D(fg) = (Df)g(p) + f(p)(Dg)$ for every $f, g \in C^\infty(U)$. Every $\psi \in H$ induces a derivative $\dot{\psi} : C^\infty(U) \to \mathbb{R}$ at the point $\phi$ in an open subset $U$ of $H$ given by

$$\left. \dot{\psi}_\phi f \right|_{t=0} = \left. \frac{d}{dt} \right|_{t=0} f(\phi + t\psi), \quad \forall f \in C^\infty(U). \quad (2.1)$$

A derivative of the above particular form is called a kinematic tangent vector with foot point $\phi$. The term ‘kinematic’ is used in the literature to distinguish kinematic tangent vectors from
operational tangent vectors, which refer to derivatives on $C^\infty(U)$. There is, in general, no one-to-one relation between both types of tangent vectors on infinite-dimensional manifolds (see e.g. [33][41]). The term kinematic in kinematic tangent vectors and related notions will be hereafter skipped to simplify the presentation of our results. The tangent space $T_\phi \mathcal{H}_R$ at $\phi \in \mathcal{H}_R$ is the space of all tangent vectors with foot point $\phi$ [33][47][49]. It is immediate to prove, e.g. by inspection of (2.1) on smooth functions $f_\psi: \phi \in \mathcal{H} \mapsto \Re \langle \psi | \phi \rangle$ for $\psi \in \mathcal{H}$, that $\psi_\phi \neq \psi'_\phi$, for $\psi, \psi', \phi \in \mathcal{H}$, if and only if $\psi \neq \psi'$. Hence, each $T_\phi \mathcal{H}_R$ is isomorphic to $\mathcal{H}_R$ for every $\phi \in \mathcal{H}_R$. The isomorphism associates $\psi \in \mathcal{H}_R$ with the tangent vector $\dot{\psi}_\phi \in T_\phi \mathcal{H}_R$. To simplify the notation, $\dot{\psi}_\phi$ will be denoted by $\psi$ if the foot point is known from the context. The space $T_\phi \mathcal{H}_R$ admits a unique norm turning the isomorphism with $\mathcal{H}_R$ into an isometry. Similarly, its tangent bundle, $T\mathcal{H}_R := \bigsqcup_{t \in \mathbb{R}} T_\phi \mathcal{H}_R$, where $\bigsqcup$ stands for a disjoint union [38][49], is naturally diffeomorphic to $\mathcal{H}_R \oplus \mathcal{H}_R$. The space $T\mathcal{H}_R$ is indeed a vector bundle relative to $\pi: \dot{\psi} \in T\mathcal{H}_R \mapsto \phi \in \mathcal{H}_R$.

A vector field is a mapping $X: D(X) \subset \mathcal{H}_R \rightarrow T\mathcal{H}_R$ from a dense subspace $D(X) \subset \mathcal{H}_R$ satisfying that $\pi \circ X = \text{id}_{\mathcal{H}_R} |_{D(X)}$. For any vector field $X$ on $\mathcal{H}_R$ and $\phi \in D(X)$, we denote $X(\phi) := (\phi, X_\phi) \in T_\phi \mathcal{H}_R \subset T\mathcal{H}_R \simeq \mathcal{H}_R \oplus \mathcal{H}_R$.

A vector field $X$ admits a flow if there exists a continuous map $\text{Fl}^X: (t, \phi) \in \mathcal{U} \subset \mathbb{R} \times \mathcal{H}_R \rightarrow \text{Fl}^X(t, \phi) \in \mathcal{H}_R$ such that $\{0\} \times \mathcal{H}_R \subset \mathcal{U}$, $\text{Fl}^X(0, \phi) = \phi$ for every $\phi \in \mathcal{H}_R$ and

$$\frac{d}{dt} \text{Fl}^X(t, \phi) = \lim_{s \rightarrow 0} \frac{\text{Fl}^X(t + s, \phi) - \text{Fl}^X(t, \phi)}{s} = X(\text{Fl}^X(t, \phi)),$$

where the limit is relative to the norm topology of $\mathcal{H}_R$ [38][50].

The main objects to be considered hereafter are the vector fields of the form

$$X_A: \psi \in D(A) \subset \mathcal{H}_R \mapsto (A\psi)_\phi \in T_\phi \mathcal{H}_R \subset T\mathcal{H}_R$$

for a skew-Hermitian operator $A: D(A) \subset \mathcal{H} \rightarrow \mathcal{A}$, where $D(A)$ is the domain of the operator $A$. It is worth noting that an operator on an infinite-dimensional Hilbert space only needs to be defined on a dense subspace of the Hilbert space [39]. Even if $A$ is not bounded, which implies in virtue of the Hellinger–Toeplitz Theorem [38] that $X_A$ can only be defined on a dense subset of $\mathcal{H}_R$, the Stone’s Theorem ensures that $A$ gives rise to a strongly continuous one-parameter group $\{e^{tA}\}_{t \in \mathbb{R}}$ of operators [62][63].

Previous results ensure that the integral curves of $X_A$, namely $t \mapsto e^{tA}\phi$ for every $\phi \in \mathcal{H}$, are defined on the whole $\mathcal{H}_R$ and $X_A$ is complete. This fact can be used to consider skew-Hermitian operators as fundamental vector fields relative to the standard action of the unitary group $U(\mathcal{H})$ on the Hilbert space $\mathcal{H}$, relative to the strong topology. If $\hat{A} := -i\hat{H}$ for an autonomous self-Hermitian Hamiltonian $\hat{H}$, then the solutions to the associated $t$-dependent Schrödinger equation describe the integral curves of $X_{-i\hat{H}}$ [24][40].

Assume that $\hat{B}_1$ and $\hat{B}_2$ are two, possibly not bounded, operators acting on $\mathcal{H}$ admitting a common dense domain $D$ that is also invariant under $\hat{B}_1$ and $\hat{B}_2$, namely $\hat{B}_1, \hat{B}_2$ are defined on $D$ and $\hat{B}_1(D), \hat{B}_2(D) \subset D$. The commutator of $\hat{B}_1$ and $\hat{B}_2$ is defined to be $[\hat{B}_1, \hat{B}_2]\phi = (\hat{B}_1\hat{B}_2 - \hat{B}_2\hat{B}_1)\phi$ for every $\phi \in D$.

For two vector fields $X_{\hat{B}_1}, X_{\hat{B}_2} \in X(\mathcal{H}_R)$ satisfying that $\hat{B}_1, \hat{B}_2$ are skew-Hermitian operators having a common invariant dense domain, we define

$$[X_{\hat{B}_1}, X_{\hat{B}_2}] := -X_{[\hat{B}_1, \hat{B}_2]}.$$
The above expression, when the integral curves of $\hat{B}_1, \hat{B}_2$ are smooth enough, arises as a consequence of the definition of the geometric Lie bracket of vector fields on a Banach manifold. Let us analyse this fact in detail. Assume that $X_{\hat{B}_1}, X_{\hat{B}_2}$ are such that $\hat{B}_1, \hat{B}_2$ admit a common dense invariant domain. If the curve $\alpha(t) := \exp(-t\hat{B}_1)\exp(-t\hat{B}_2)\exp(t\hat{B}_1)\exp(t\hat{B}_2)\phi$ is smooth at $\phi \in \mathcal{H}$, it is possible to retrieve the Lie bracket of the vector fields $X_{\hat{B}_1}, X_{\hat{B}_2}$ geometrically [53, Theorem 1.33]:

$$[[X_{\hat{B}_1}, X_{\hat{B}_2}]](\phi) = \frac{1}{2} \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} \left( F^X_{-t} \circ F^X_{-t} \circ F^X_t \circ F^X_t (\phi) \right) = \frac{1}{2} \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} e^{-t\hat{B}_2} e^{-t\hat{B}_1} e^{t\hat{B}_2} e^{t\hat{B}_1} \phi.$$

In particular, the above always holds when $\mathcal{H}$ is finite-dimensional. A t-dependent vector field on $\mathcal{H}_\mathbb{R}$ is a mapping $X : \mathcal{U} \subset \mathbb{R} \times \mathcal{H}_\mathbb{R} \rightarrow T\mathcal{H}_\mathbb{R}$, for a certain open subset $\mathcal{U}$ in $\mathbb{R} \times \mathcal{H}_\mathbb{R}$, such that $\pi \circ X(t, \psi) = \psi$ on $\mathcal{U}$ and the domain of each $X_t : \phi \in \mathcal{U} \cap (\{t\} \times \mathcal{H}_\mathbb{R}) \mapsto X(t, \phi) \in T\mathcal{H}_\mathbb{R}$, with $t \in \mathbb{R}$, is dense in $\mathcal{H}_\mathbb{R}$. The integral curves of $X$ are sections $\gamma : \mathbb{R} \rightarrow \mathbb{R} \times \mathcal{H}_\mathbb{R}$ of the bundle $\pi_\mathcal{H} : \mathcal{H}_\mathbb{R} \rightarrow \mathbb{R}$ that are simultaneously integral curves of the vector field $\partial_t + X(t, \phi)$ on $\mathcal{H}_\mathbb{R}$. If $\pi_2 : \mathbb{R} \times \mathcal{H}_\mathbb{R} \rightarrow \mathcal{H}_\mathbb{R}$ is the projection onto the second manifold, the curves $\gamma$ are given by the solutions to the differential equation

$$\frac{d(\pi_2 \circ \gamma)}{dt}(t) = X \circ \gamma(t).$$

Every t-dependent Hermitian operator $\hat{H}(t)$ on $\mathcal{H}$ gives rise to a t-dependent vector field $X_{-i\hat{H}(t)}$ on $\mathcal{H}_\mathbb{R}$ and an associated t-dependent Schrödinger equation

$$\frac{\partial \psi}{\partial t} = -i\hat{H}(t)\psi, \quad \psi \in \mathcal{H}, \quad t \in \mathbb{R}.$$

3 Quantum Lie systems

Let us survey the quantum version of Lie systems [22, 24]. On a finite-dimensional manifold $N$, every finite-dimensional real Lie algebra $V$ of vector fields on $N$ can be integrated to give rise to a local Lie group action on $N$. In the infinite-dimensional analogue, additional technical conditions on $V$ may be necessary to integrate $V$. In particular, we are interested in the integration to a Lie group action of finite-dimensional real Lie algebras of vector fields induced by skew-Hermitian operators on (possibly infinite-dimensional) manifolds (see [33, 52, 58] for details). In all cases studied in this work, $V$ can be found to be integrable to a (at least local) Lie group action. This Lie group action allows us to study quantum Lie systems via the standard theory of Lie systems [27, 28]. To focus on the applications of our ideas, quantum quasi-Lie schemes, and quantum Lie systems to be studied in next sections, further comments on the integrability of $V$ to a Lie group action will be omitted.

Definition 3.1. A quantum Lie system is a t-dependent operator $\hat{H}(t)$ on $\mathcal{H}$ of the form

$$\hat{H}(t) := \sum_{\alpha=1}^{r} b_\alpha(t) \hat{H}_\alpha,$$

(3.1)

where $b_1(t), \ldots, b_r(t)$ are t-dependent real functions and $\mathfrak{V} := \{ i\hat{H}_1, \ldots, i\hat{H}_r \}$ is a $r$-dimensional real Lie algebra of skew-Hermitian operators. We call $\mathfrak{V}$ a Vessiot–Guldberg (VG) Lie algebra for the quantum Lie system [33, 24, 27, 28].
Each quantum Lie system (3.1) determines a $t$-dependent Schrödinger equation

$$\frac{\partial \psi}{\partial t} = -i\hat{H}(t)\psi = -\sum_{\alpha=1}^{r} b_\alpha(t)i\hat{H}_\alpha \psi, \quad \psi \in \mathcal{H}, \quad t \in \mathbb{R}. \quad (3.2)$$

Its particular solutions are then integral curves of the $t$-dependent vector field $X_{-i\hat{H}(t)}$ (see [24] for details).

Let us illustrate quantum Lie systems through an example. Consider the quantum one-dimensional *Caldirola–Kanai oscillator* [4, 9, 22, 43], which is determined by the $t$-dependent operator

$$\hat{H}_{\text{CK}}(t) := \exp\left(-2\gamma(t)\hat{p}^2/2\right) + \omega^2 \exp\left(2\gamma(t)\hat{x}^2/2\right), \quad \gamma \in \mathbb{R}, \quad \omega \in \mathbb{R}, \quad (3.3)$$

where $\hat{x}$ and $\hat{p}$ refer to the usual position and momentum operators on $\mathbb{R}$. This model is a quantum analogue of the classical harmonic oscillator with a $t$-dependent mass $m(t) := \exp(2\gamma t)$ and a constant frequency $\omega$ [24, 35].

The $t$-dependent operator $\hat{H}_{\text{CK}}(t)$ can be written as a linear combination with $t$-dependent real functions of the Hermitian operators

$$\hat{H}_1 := \hat{x}^2/2, \quad \hat{H}_2 := \hat{p}^2/2, \quad \hat{H}_3 := (\hat{x}\hat{p} + \hat{p}\hat{x})/4.$$ 

The real linear space $\mathfrak{U}_{\text{CK}} := \{i\hat{H}_1, i\hat{H}_2, i\hat{H}_3\}$ is a vector space of skew-Hermitian operators. Additionally,

$$[i\hat{H}_1, i\hat{H}_2] = -2i\hat{H}_3, \quad [i\hat{H}_1, i\hat{H}_3] = -i\hat{H}_2, \quad [i\hat{H}_3, i\hat{H}_2] = -i\hat{H}_1.$$

Hence, $\mathfrak{U}_{\text{CK}}$ is a finite-dimensional Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Since $i\hat{H}_{\text{CK}}(t)$ takes values in $\mathfrak{U}_{\text{CK}}$, the $t$-dependent operator $\hat{H}_{\text{CK}}(t)$ is a quantum Lie system.

The solutions of the $t$-dependent Schrödinger equation associated with (3.3), namely

$$\frac{\partial \psi}{\partial t} = -i\hat{H}_{\text{CK}}(t)\psi = -i \left( \exp\left(-2\gamma(t)\hat{p}^2/2\right) + \omega^2 \exp\left(2\gamma(t)\hat{x}^2/2\right) \right) \psi,$$

are the integral curves of $X_{\text{CK}}(t) := \omega^2 \exp\left(2\gamma(t)X_{-i\hat{H}_3} + \exp\left(-2\gamma(t)X_{-i\hat{H}_3}\right).$

Similarly to standard Lie systems, the associated $t$-dependent Schrödinger equation related to a quantum Lie system can be solved by means of a Lie system on a Lie group [16, 22]. Let us explain this fact (see [22] for applications). Reconsider again the quantum Lie system (3.1), which satisfies that

$$[i\hat{H}_\alpha, i\hat{H}_\beta] = \sum_{\gamma=1}^{r} c_{\alpha\beta\gamma} i\hat{H}_\gamma, \quad c_{\alpha\beta\gamma} \in \mathbb{R}, \quad \alpha, \beta, \gamma = 1, \ldots, r.$$ 

Let $\varphi : \mathfrak{g} \to \mathfrak{U}$ be an isomorphism of Lie algebras, where $\mathfrak{g}$ is an abstract Lie algebra, $G_\mathfrak{g}$ is the connected and simply connected Lie group related to $\mathfrak{g}$, and $\mathfrak{U}$ is the Lie algebra relative to the commutator of skew-Hermitian operators (more precisely defined on a common domain for all its elements given by a dense subspace $D_\mathfrak{U} \subset \mathcal{H}$). We define a continuous unitary action $\Phi : G_\mathfrak{g} \times \mathcal{H} \to \mathcal{H}$ (continuous relative to the norms induced by the one on $\mathcal{H}$ and the one in $G_\mathfrak{g}$), such that

$$\varphi(v)\psi = \left. \frac{d}{dt} \right|_{t=0} \Phi(\exp(-tv), \psi), \quad \forall \psi \in D_\mathfrak{U}, \quad \forall v \in \mathfrak{g}. \quad (3.4)$$
Note that $\Phi$ amounts to a Lie group morphism $\tilde{\Phi} : g \in G \mapsto \Phi^g \in U(H)$ where $\Phi^g(\psi) := \Phi(g, \psi)$ for every $g \in G$ and $\psi \in H$, whilst $U(H)$ amounts to the space of unitary operators on $H$. We define $\Phi_{\psi} : g \in G \mapsto \Phi(g, \psi) \in H_{\mathbb{R}}$ for every $\psi \in H_{\mathbb{R}}$. Note that $\varphi$ is the infinitesimal Lie algebra morphism induced by $\Phi$. Due to the relation $[X_{\tilde{\beta}_1}, X_{\tilde{\beta}_2}] = -X_{[\beta_1, \beta_2]}$, it follows from (3.4) that we can define $X_{-\varphi(v)}$ to be the fundamental vector field on $H_{\mathbb{R}}$ corresponding to the element $v \in \mathfrak{g}$ and there exists a Lie algebra isomorphism $v \mapsto X_{-\varphi(v)}$ between the elements of the Lie algebra $\mathfrak{g}$ and the Lie algebra of vector fields $V$ given by the vector fields on $H_{\mathbb{R}}$ induced by the skew-Hermitian operators $\varphi(v)$ for $v \in \mathfrak{g}$.

Let $\{a_1, \ldots, a_r\}$ be the basis of $T_eG \simeq \mathfrak{g}$ given by $a_{\alpha} := \varphi^{-1}(i\tilde{H}_{\alpha})$ for $\alpha = 1, \ldots, r$. Then, $a_1, \ldots, a_r$ have the same commutation relations, relative to the natural Lie bracket in $T_eG$ (see (1)), as the operators $i\tilde{H}_1, \ldots, i\tilde{H}_r$, which in turn have the opposite structure constants of the vector fields $X_{i\tilde{H}_{\alpha}}$. Hence, if $[i\tilde{H}_\alpha, i\tilde{H}_\beta] = \sum_{\gamma=1}^rc_{\alpha\beta\gamma}i\tilde{H}_\gamma$, then $[X_{-i\tilde{H}_\alpha}, X_{-i\tilde{H}_\beta}] = \sum_{\gamma=1}^rc_{\alpha\beta\gamma}X_{-i\tilde{H}_\gamma}$, where we assume $\alpha, \beta = 1, \ldots, r$. Hence,

$$[a_{\alpha}, a_{\beta}] = \sum_{\gamma=1}^r c_{\alpha\beta\gamma} a_{\gamma}, \quad \alpha, \beta, \gamma = 1, \ldots, r,$$

and

$$\Phi(\exp(-\lambda a_{\alpha}), \psi) = \exp(-i\lambda\tilde{H}_{\alpha})\psi, \quad \forall \psi \in D_{\mathfrak{g}}, \quad \forall \lambda \in \mathbb{R}, \quad \alpha = 1, \ldots, r.$$

It is quite useful in applications to consider $G$ to be a matrix Lie group. In such a case, the elements of $\mathfrak{g}$ are matrices and the commutator of the elements of $T_eG$ is the commutator of the corresponding associated elements of the Lie algebra. It is also worth noting that $\varphi(Ad_{\psi}(v)) = Ad_{\Phi^g(\psi)}(v)$, for all $g \in G$ and $v \in \mathfrak{g}$, where $Ad_g$ stands for the adjoint action of $g$ on $\mathfrak{g}$ and $Ad_{\Phi^g}$ is the adjoint action of $\Phi^g \in U(H)$ relative to $\varphi(v)$.

In particular, let us prove that solving the $t$-dependent Schrödinger equation associated with a quantum Lie system $\tilde{H}(t) = \sum_{\alpha=1}^rb_{\alpha}(t)H_{\alpha}$ reduces to solving the Lie system in $G$ given by

$$\frac{dg}{dt} = -\sum_{\alpha=1}^rb_{\alpha}(t)X^R_{\alpha}(g),$$

where $X^R_{\alpha}(g) = R_{g^{\ast}a_{\alpha}}$ for $\alpha = 1, \ldots, r$ and $R_g : g' \in G \mapsto g'g \in G$. Indeed, if we define $\psi(t) = \Phi(g(t), \psi(0))$, with $\psi(0) \in H_{\mathbb{R}}$, then for every $t_0 \in \mathbb{R}$, one has that

$$\frac{d}{dt}\bigg|_{t=t_0} \Phi(g(t), \psi(t_0)) = \frac{d}{dt}\Phi(g(t)g(t_0)^{-1}g(t_0), \psi(t_0)) = \varphi\left(\frac{dg}{dt}(t_0)g^{-1}(t_0)\right)\Phi(g(t_0), \psi(0)) = -i\tilde{H}(t_0)\psi(t_0),$$

and $\psi(t)$ is the general solution to (3.2).

In the particular case of $\tilde{H}_{CK}(t)$, which is related to a quantum VG Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$, the associated Lie system should be associated with the universal covering group, $\widetilde{SL}(2, \mathbb{R})$, of the Lie group $SL(2, \mathbb{R})$, which has a Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. In consequence, it is defined on the Lie group $G := \widetilde{SL}(2, \mathbb{R})$ and takes the form

$$\frac{dg}{dt} = -e^{-2\gamma t}X^R_2(g) - \omega^2e^{2\gamma t}X^R_1(g), \quad g \in \widetilde{SL}(2, \mathbb{R}), \quad t \in \mathbb{R}, \quad (3.5)$$
where $X^R_\alpha(g) := R_{g*e}a_\alpha$ for $\alpha = 1, 2, 3$ and $\{a_1, a_2, a_3\}$ is a basis of $T_{id}SL_2 \simeq \mathfrak{s}(2, \mathbb{R})$ satisfying the same commutation relations as the $i\tilde{H}_1, i\tilde{H}_2, i\tilde{H}_3$.

As $SL(2, \mathbb{R})$ has a fundamental group isomorphic to $\mathbb{Z}$, its universal covering, $\tilde{S}L(2, \mathbb{R})$, is not a matrix Lie group, which is complicated to deal with in practical applications (cf. [39, p. 127]).

In applications, one is frequently interested in solutions of (3.5) for the variable $t$ taking values close to zero. By the Ado theorem, every finite-dimensional Lie algebra can be written in a matrix form and, then, it admits a matrix Lie group $[12, 22]$. Since all Lie groups with the same Lie algebra are locally diffeomorphic close to the neutral element, one can define an analogue of (3.5) on the matrix Lie group $\mathfrak{s}(2, \mathbb{R})$ given by

$$\frac{dg}{dt} = -e^{-2\gamma t}X_2^R(g) - \omega^2 e^{2\gamma t}X_1^R(g), \quad g \in SL(2, \mathbb{R}), \quad t \in \mathbb{R}. $$

### 4 Quantum quasi-Lie schemes

This section develops the theory of quantum quasi-Lie schemes as an extension of the theory of quasi-Lie schemes $[12, 22]$. Its main aim is to investigate the integrability properties of a class of $t$-dependent Schrödinger equations containing the $t$-dependent Schrödinger equations studied via quantum Lie systems as a particular case.

The following example illustrates the necessity of quantum quasi-Lie schemes. Consider an $n$-dimensional nonlinear quantum oscillator with a $t$-dependent anharmonic potential described by

$$\tilde{H}_{NH}(t) := \frac{1}{2} \sum_{i=1}^{n} \dot{p}_i^2 + \omega(t) \frac{1}{2} \sum_{i=1}^{n} \dot{x}_i^2 + \tilde{\omega}(t) \sum_{i=1}^{n} x_i^\alpha, \quad \alpha \in \mathbb{Z}^+,$$

where $\alpha$ is a negative integer, the term $NH$ stands for 'non-harmonic', and $\omega(t), \tilde{\omega}(t)$ are real $t$-dependent functions such that the points of the curve $(\omega(t), \tilde{\omega}(t))$ in $\mathbb{R}^2$ span the whole space $\mathbb{R}^2$. Let us prove that $\tilde{H}_{NH}(t)$ is not a quantum Lie system through the following no-go proposition.

**Proposition 4.1. (Quantum Lie systems no-go proposition)** Let $\tilde{H}(t)$ be a $t$-dependent Hermitian operator on $\mathcal{H}$ such that there exist $\tilde{H}_a, \tilde{H}_b \in \mathcal{H}(t)$, $t \in \mathbb{R}$ and a Hermitian operator $\tilde{H}_c$ on $\mathcal{H}$ satisfying

$$[i\tilde{H}_c, i\tilde{H}_a] = c_a i\tilde{H}_a, \quad \quad [i\tilde{H}_c, i\tilde{H}_b] = c_b i\tilde{H}_b,$$

for constants $c_a \in \mathbb{R}$ and $c_b \in \mathbb{R} \setminus \{0\}$. Let $pr : V \subset \text{End}(\mathcal{H}) \rightarrow \text{End}(\mathcal{H})$ be a Lie algebra morphism from a Lie subalgebra $V$ of the complex Lie algebra $\text{End}(\mathcal{H})$ of operators on $\mathcal{H}$ containing $(i\tilde{H}(t))_{t \in \mathbb{R}}$ and $i\tilde{H}_c$. If $\text{ad}_{pr(i\tilde{H}_b)}^n \pi(i\tilde{H}_a) \neq 0$ for every natural number $n > 0$, then $\tilde{H}(t)$ is not a quantum Lie system.

**Proof.** By induction, we obtain that $[i\tilde{H}_c, \text{ad}_{pr(i\tilde{H}_b)}^n i\tilde{H}_a] = (nc_b + c_a) \text{ad}_{pr(i\tilde{H}_b)}^n i\tilde{H}_a$ for every $n \in \mathbb{N}$. Since $\text{ad}_{pr(i\tilde{H}_b)}^n \pi(i\tilde{H}_c) \neq 0$ by assumption, $\pi$ is a Lie algebra morphism, and all the $i\tilde{H}_n := \text{ad}_{pr(i\tilde{H}_b)}^n i\tilde{H}_a$ with $n \in \mathbb{N} \cup \{0\}$ belong to the domain of $\pi$, then $\tilde{H}_n \neq 0$ for all $n \in \mathbb{N}$ and the operators $\{i\tilde{H}_n\}_{n \in \mathbb{N}}$ are eigenvectors with different eigenvalues of the linear morphism $\text{ad}_{pr(i\tilde{H}_c)} : E \rightarrow E$ where $E := \langle i\tilde{H}_n \rangle_{n \in \mathbb{N}}$ is an $\mathbb{R}$-linear space. Hence, the $\{i\tilde{H}_n\}_{n \in \mathbb{N}}$ are linearly independent and $\dim E = +\infty$. In consequence, every Lie algebra of operators containing the $i\tilde{H}(t)$ must be infinite-dimensional since it contains $i\tilde{H}_a, i\tilde{H}_b$ and all the $\{i\tilde{H}_n\}_{n \in \mathbb{N}}$. \[\square\]
Let us apply the results of the above proposition to (4.1). Define the Hermitian operators on \( L^2(\mathbb{R}^n) \) of the form

\[
\hat{H}_a := \sum_{j=1}^{n} \hat{x}_j^a, \quad \hat{H}_b := \sum_{j=1}^{n} \hat{p}_j^2, \quad \hat{H}_c := \sum_{j=1}^{n} \frac{\hat{x}_j \hat{p}_j + \hat{p}_j \hat{x}_j}{2}.
\]

Let us define

\[
\text{pr}(i\hat{H}_{NH}(t)) = \frac{i}{2} \hat{p}_1^2 + \omega(t) \frac{i}{2} \hat{x}_1^2 + i\hat{\omega}(t) \hat{x}_1^a, \quad \text{pr}(i\hat{H}_c) = i\left(\frac{\hat{x}_1 \hat{p}_1 + \hat{p}_1 \hat{x}_1}{2}\right), \quad \text{pr}(i\hat{H}_a) = i\hat{x}_1^a, \quad \text{pr}(i\hat{H}_b) = i\hat{p}_1^2.
\]

Since \([i\hat{H}_c, i\hat{H}_a] = \alpha i\hat{H}_a, \quad [i\hat{H}_c, i\hat{H}_b] = -2i\hat{H}_b\) and

\[
\left[\hat{p}_1^2, i\hat{x}_1^a \hat{p}_1^k \right] = 2\alpha i\hat{x}_1^{a-1} \hat{p}_1^{k+1} + \alpha(\alpha - 1)\hat{x}_1^{a-2} \hat{p}_1^k,
\]

it follows that

\[
\left[\text{pr}(i\hat{H}_b), \text{pr}(i\hat{H}_a)\right] = 2\alpha i\hat{x}_1^{a-1} \hat{p}_1 + \alpha(\alpha - 1)\hat{x}_1^{a-2},
\]

and, by induction and recalling that \(\alpha < 0\), we obtain

\[
\text{ad}^n_{\text{pr}(i\hat{H}_b)}(\pi(i\hat{H}_a)) = \sum_{k=0}^{n} a_k \hat{x}_1^{(a+k)\hat{p}_1^{n-k}}.
\]

for certain complex constants \(a_0, \ldots, a_n\). In particular, it can be proved that \(a_0 = 2^n i\alpha(\alpha - 1) \cdots (\alpha - n + 1)\), while \(a_n = (-i)^{n-1} \alpha(\alpha - 1) \cdots (\alpha - 2n + 1)\), which are different from zero since \(\alpha < 0\). Since \(\text{ad}^n_{\text{pr}(i\hat{H}_b)} \neq 0\), the quantum Lie systems no-go proposition ensures that (4.1) is not a quantum Lie system.

Consider now a \(t\)-dependent Schrödinger equation

\[
\frac{\partial \psi}{\partial t} = -i\hat{H}(t)\psi, \quad \psi \in \mathcal{H}, \quad t \in \mathbb{R}, \quad (4.2)
\]

where \(-i\hat{H}(t)\) is a \(t\)-dependent operator taking values in a finite-dimensional real linear space \(\mathcal{W}\) of skew-Hermitian operators. Hence, \(-i\hat{H}(t)\) can be understood as a curve in \(\mathcal{W}\).

This time we do not impose the skew-Hermitian operators \(\{-i\hat{H}(t)\}_{t \in \mathbb{R}}\) and their successive commutators to span a real finite-dimensional Lie algebra of operators (with respect to the operator commutator as in the case of quantum Lie systems). Instead, suppose that \(-i\hat{H}(t)\) takes values in a finite-dimensional real linear space of skew-Hermitian operators \(\mathcal{W}\) and we also assume that there exists a non-zero real Lie algebra \(\mathcal{W} \subset \mathcal{W}\). Hence, \([\mathcal{W}, \mathcal{W}] \subset \mathcal{W}\).

**Definition 4.2.** A **quantum quasi-Lie scheme**, \(S(\mathcal{W}, \mathcal{W})\), is a pair of non-zero finite-dimensional \(\mathbb{R}\)-linear spaces \(\mathcal{W}, \mathcal{W}\) of skew-Hermitian operators on a Hilbert space \(\mathcal{H}\) satisfying that

\[
\mathcal{W} \subset \mathcal{Y}, \quad [\mathcal{W}, \mathcal{W}] \subset \mathcal{W}, \quad [\mathcal{W}, \mathcal{W}] \subset \mathcal{Y}. \quad (4.3)
\]

To illustrate quasi-Lie schemes, let us consider the \(\mathbb{R}\)-linear space of skew-Hermitian operators

\[
\mathcal{W}_{NH} := \left\{ i\hat{H}_0 := i \sum_{j=1}^{n} \hat{p}_j^2, \quad i\hat{H}_1 := i \sum_{j=1}^{n} \hat{x}_j^a, \quad i\hat{H}_2 := i \sum_{j=1}^{n} \hat{x}_j^2, \quad i\hat{H}_3 := i \sum_{j=1}^{n} \frac{\hat{x}_j \hat{p}_j + \hat{p}_j \hat{x}_j}{2} \right\},
\]
for a fixed $\alpha \in \mathbb{N}\backslash\{2\}$ and $\mathfrak{W}_{NH} := \langle i\tilde{H}_2, i\tilde{H}_3 \rangle$. Since $[i\tilde{H}_2, i\tilde{H}_3] = -2i\tilde{H}_2$, the linear space $\mathfrak{W}_{NH}$ is a two-dimensional real Lie algebra. We can see that $[\mathfrak{W}_{NH}, \mathfrak{W}_{NH}] \subset \mathfrak{V}_{NH}$ since

$$[i\tilde{H}_2, i\tilde{H}_1] = 0, \quad [i\tilde{H}_2, i\tilde{H}_0] = -i\tilde{H}_3, \quad [i\tilde{H}_3, i\tilde{H}_0] = -2i\tilde{H}_0, \quad [i\tilde{H}_3, i\tilde{H}_1] = \alpha i\tilde{H}_1.$$

It follows that $\mathfrak{W}_{NH}$ and $\mathfrak{V}_{NH}$ satisfy the conditions (4.3) and therefore the pair $\mathfrak{W}_{NH}, \mathfrak{V}_{NH}$ gives rise to a quantum quasi-Lie scheme $S(\mathfrak{W}_{NH}, \mathfrak{V}_{NH})$. Moreover, $-i\tilde{H}_{NH}(t)$ takes values in $\mathfrak{V}_{NH}$. Indeed,

$$\tilde{H}_{NH}(t) = \frac{1}{2}\tilde{H}_0 + \omega(t)\frac{1}{2}\tilde{H}_2 + \tilde{\omega}(t)\tilde{H}_1.$$

The last expression will be a clue to study the $t$-dependent Schrödinger equation related to $\tilde{H}_{NH}(t)$ through quasi-Lie schemes. Indeed, it suggests us, along with other examples to be studied in the forthcoming sections, to propose the following definition.

**Definition 4.3.** The $t$-dependent Schrödinger equation (4.2) admits a compatible quantum quasi-Lie scheme $S(\mathfrak{W}, \mathfrak{V})$ if $-i\tilde{H}(t)$ is a curve taking values in $\mathfrak{V}$.

Let us stress that the idea behind Definition 4.3 is that given a $t$-dependent Schrödinger equation (4.2), we can apply the theory of quasi-Lie schemes to study it only via a quasi quasi-Lie scheme that is compatible with it in the sense given above. If not otherwise stated, we hereafter assume that every quasi-Lie scheme and $t$-dependent skew-Hermitian operator is defined on a generic (possibly infinite-dimensional) Hilbert space $\mathcal{H}$.

## 5 Quantum quasi-Lie systems

Many of the works in the literature try to map a certain $t$-dependent operator into a simpler one by means of a gauge transformation (see e.g. 4.1). Similarly, we introduce quantum quasi-Lie systems in this section as a class of $t$-dependent Schrödinger equations with a compatible quantum quasi-Lie scheme that can be transformed into a $t$-dependent Schrödinger equation described via a quantum-Lie system. To achieve this goal, it becomes necessary to extend some previous results on quantum Lie systems.

**Definition 5.1.** The representation of a quasi-Lie scheme $S(\mathfrak{W}, \mathfrak{V})$ is a Lie algebra morphism

$$\rho : \mathfrak{W} \rightarrow \text{End}(\mathfrak{V}),$$

$$A \rightarrow \rho_A := [A, \cdot],$$

where we recall that $\text{End}(\mathfrak{V})$ is the Lie algebra of endomorphisms on $\mathfrak{V}$ relative to the commutator of endomorphisms. A subrepresentation of $S(\mathfrak{W}, \mathfrak{V})$ is a subspace $\mathfrak{V}_1 \subset \mathfrak{V}$ such that $S(\mathfrak{W}, \mathfrak{V}_1)$ is a quasi-Lie scheme.

As the subspace $\mathfrak{W}$ of a quasi-Lie scheme $S(\mathfrak{W}, \mathfrak{V})$ is a real Lie algebra of skew-Hermitian operators, there exists a Lie group action $\Phi : (g, \psi) \in G \times \mathcal{H} \rightarrow \Phi(g, \psi) \in \mathcal{H}$ such that the mappings $\Phi^g : \psi \in \mathcal{H} \mapsto \Phi(g, \psi) \in \mathcal{H}$, with $g \in G$, are unitary operators on $\mathcal{H}$ and, moreover, one has that

$$\frac{d}{dt} \bigg|_{t=t_0} \Phi(\exp(tv), \psi) = \bar{\rho}(v)\Phi(\exp(t_0v), \psi), \quad \forall \psi \in \mathcal{H},$$

where $g$ is the Lie algebra of $G$, while $\bar{\rho} : g \rightarrow \mathfrak{W}$ is a Lie algebra isomorphism relative to the commutator operator on elements of $\mathfrak{W}$. Note that there may be other Lie group action
Proposition 5.3. The elements of the group \( G' \) of \( G' \) is isomorphic to \( g \) and \( G' \) is connected, satisfying similar properties. Let us study the relations between such Lie group actions.

Let as consider \( \Phi : g \in G \mapsto \Phi g \in U(\mathcal{H}) \) and \( \Phi' : g \in G' \mapsto \Phi' g \in U(\mathcal{H}) \). Now, we aim to show that \( \Phi(G) = \Phi'(G') \). Due to the fact that \( \Phi \) and \( \Phi' \) are Lie group actions and \( \tilde{\rho}, \tilde{\rho}' \) are their tangent maps at 0, respectively, one can consider the following commutative diagram

\[
\begin{array}{ccc}
g & \overset{\tilde{\rho}}{\longrightarrow} & \mathfrak{g}' \\
\downarrow{\exp_G} & & \downarrow{\exp} \\
G & \overset{\Phi}{\longrightarrow} & U(\mathcal{H}) \\
\downarrow{\Phi} & & \downarrow{\Phi'} \\
G' & \overset{\exp_{G'}}{\longleftarrow} & \mathfrak{g}'
\end{array}
\]  

(5.1)

where \( \exp_G, \exp_{G'} \) are the exponential maps from \( g, g' \) into the Lie groups \( G, G' \), respectively, and \( \exp \) stands for the exponential of operators in \( \mathfrak{g} \), which exists because the elements of \( \mathfrak{g} \) are skew-Hermitian operators. Since \( G \) is connected, every element of \( G \) can be written as a product of elements in the image of the \( \exp_G \). Hence, the group generated by the elements of \( \Phi(\exp_G(g)) \) is \( \Phi(G) \). Repeating the same argument concerning the right-hand side of diagram (5.1), using that it is commutative, and the fact that \( \text{Im} \tilde{\rho} = \text{Im} \tilde{\rho}' \), we obtain that \( \Phi(G) = \Phi'(G') \). Note that \( \Phi(G) \) is not a Lie subgroup of \( U(\mathcal{H}) \) even when \( \mathcal{H} \) is finite-dimensional and \( U(\mathcal{H}) \) becomes a standard Lie group (cf. [29]). Moreover, the space \( \mathcal{G}_g \) of curves \( U(t) \) in \( \Phi(G) \) with \( U(0) = \text{Id}_\mathcal{H} \) is a group relative to the multiplication

\[
(U_1 \ast U_2)(t) = U_1(t)U_2(t), \quad U_1(t), U_2(t) \in \mathcal{G}_g, \quad \forall t \in \mathbb{R}.
\]

These ideas justify the following definition.

**Definition 5.2.** If \( S(\mathfrak{M}, \mathfrak{G}) \) is a quasi-Lie scheme and \( \Phi : G_g \times \mathcal{H} \to \mathcal{H} \) is a Lie group action obtained by integrating the Lie algebra \( \mathfrak{G} \), we call group of the quasi-Lie scheme \( S(\mathfrak{M}, \mathfrak{G}) \) the space \( \mathcal{G}_g \) of curves \( U(t) \) in \( \Phi(G) \) with \( U(0) = \text{Id}_\mathcal{H} \).

Recall that, given a \( t \)-dependent Schrödinger equation (4.2), one can define a family of unitary operators \( U(t_2, t_1) \), with \( t_2, t_1 \in \mathbb{R} \), such that, given a certain \( \psi_0 \in \mathcal{H} \), then \( U(t_2, t_1)\psi_0 \) is the value of the particular solution \( \psi(t) \) to (4.2) for \( t = t_2 \) with initial condition \( \psi(t_1) = \psi_0 \). To simplify the notation, we will call \( t \)-dependent evolution operator of a Schrödinger equation (4.2) the \( t \)-dependent family of unitary operators, \( U(t) \), such that \( U(t) = U(t, 0) \).

**Proposition 5.3.** The elements of the group \( \mathcal{G}_g \) of a quasi-Lie scheme \( S(\mathfrak{M}, \mathfrak{G}) \) are the \( t \)-dependent evolution operators of the \( t \)-dependent Schrödinger equations of the form

\[
\frac{\partial \psi}{\partial t} = -i\tilde{\mathcal{H}}_0(t)\psi, \quad \psi \in \mathcal{H},
\]

(5.2)

for a certain \( t \)-dependent operator \(-i\tilde{\mathcal{H}}_0(t)\) taking values in \( \mathfrak{G} \). Conversely, the \( t \)-dependent evolution operator to (5.2) can be described by a \( t \)-dependent evolution operator \( U(t) \) in \( \mathcal{G}_g \).

**Proof.** Let \( \mathcal{U} \) be an open neighbourhood of the neutral element of the Lie group \( G_g \), where canonical coordinates of the second-kind can be defined. Consider a curve \( U(t) \) in \( G_g \). Then, \( U(0) = \text{Id}_\mathcal{H} \) and, if \( t \) is so close enough to 0, let us say \( t \in (-\epsilon, \epsilon) \) for a certain real \( \epsilon > 0 \), we can assume that \( U(t) \) takes values in \( \Phi(\mathcal{U}) \) for \( t \in (-\epsilon, \epsilon) \), where we recall that \( \Phi \) is the morphism of groups \( \Phi : G_g \to U(\mathcal{H}) \) related to the Lie group action \( \Phi : G_g \times \mathcal{H} \to \mathcal{H} \) induced by the integration of
\[ U(t) = \exp \left( i f_1(t) \hat{H}_1 \right) \times \ldots \times \exp \left( i f_r(t) \hat{H}_r \right), \quad (5.3) \]

for certain \( t \)-dependent real functions \( f_1(t), \ldots, f_r(t) \) vanishing at \( t = 0 \).

A short calculation shows that

\[ \frac{dU}{dt}(t)\psi = \sum_{j=1}^{r} \frac{df_j}{dt}(t) \text{Ad}_{\Pi_{k=1}^{j-1} \exp(i f_k(t) \hat{H}_k)}(i \hat{H}_j) U(t)\psi, \quad \forall \psi \in \mathcal{H}, \quad (5.4) \]

where \( \text{Ad}_{\exp(i f_k(t) \hat{H}_k)} \hat{H}_j := \exp(i f_k(t) \hat{H}_k) \hat{H}_j \exp(-i f_k(t) \hat{H}_k) \) for every \( j, k = 1, \ldots, r \) and \( t \in \mathbb{R} \).

Note that the exponentials are arranged in ascending order relative to \( k \). We also assume that \( \text{Ad}_{\Pi_{k=1}^{j} \exp(i f_k(t) \hat{H}_k)} := \text{Id}_\mathcal{W} \). As \( \mathcal{W} \) is a real Lie algebra, each morphism \( \text{Ad}_{\exp(i \lambda \hat{H}_j)} \), with \( \lambda \in \mathbb{R} \) and \( j = 1, \ldots, r \), leaves \( \mathcal{W} \) invariant. Moreover, the right-hand side of (5.4) shows that

\[ \frac{dU}{dt}(t)U(t)^{-1} = \sum_{j=1}^{r} \frac{df_j}{dt}(t) \text{Ad}_{\Pi_{k=1}^{j-1} \exp(i f_k(t) \hat{H}_k)}(i \hat{H}_j) \]

becomes a curve \( -i \hat{H}_0(t) \) taking values in \( \mathcal{W} \). Consequently, \( U(t) \) determines a curve \( -i \hat{H}_0(t) \) in \( \mathcal{W} \) and the general solution to its \( t \)-dependent Schrödinger equation has general solution \( U(t)\psi_0 \) for \( \psi_0 \in \mathcal{H} \). Recall that since \( f_k(0) = 0 \), for \( k = 1, \ldots, r \), the \( \psi_0 \in \mathcal{H} \) is the initial condition for the particular solution \( \psi(t) = U(t)\psi_0 \) and \( U(0) = \text{Id}_\mathcal{H} \).

Conversely, assume that we are given a curve \( -i \hat{H}_0(t) \) taking values in \( \mathcal{W} \). Let us determine a curve \( U(t) \), defined at least for \( t \) in a neighbourhood around zero, describing the \( t \)-dependent evolution operator for (5.2). Around \( t = 0 \), the curve \( U(t) \) can be written in the form (5.3) because \( U(0) = \text{Id}_\mathcal{H} \). We can define a \( t \)-dependent family of mappings of the form \( T_t : \mathcal{W} \rightarrow \mathcal{W} \), for each \( t \in \mathbb{R} \), such that

\[ -i \sum_{j=1}^{r} \lambda_j \hat{H}_j T_t^\dagger = \sum_{j=1}^{r} \lambda_j \text{Ad}_{\Pi_{k=1}^{j-1} \exp(i f_k(t) \hat{H}_k)}(i \hat{H}_j) \]

for every set \( \lambda_1, \ldots, \lambda_r \in \mathbb{R} \). The functions \( f_1(t), \ldots, f_r(t) \) are undetermined now, but \( f_1(0) = \ldots = f_r(0) = 0 \) and then \( T(0) = \text{Id}_\mathcal{W} \). Since \( T_t \) depends continuously on \( t \), then each particular \( T_{t_0} \) will be invertible for \( t_0 \) in a neighbourhood of zero. As a consequence, the system of differential equations given by

\[ -\sum_{j=1}^{r} \frac{df_j}{dt}(t) \text{Ad}_{\Pi_{k=1}^{j-1} \exp(i f_k(t) \hat{H}_k)}(i \hat{H}_j) = -i \hat{H}_0(t) \]

amounts, after applying \( T_t^{-1} \), to

\[ -i \sum_{k=1}^{r} \frac{df_k}{dt}(t) \hat{H}_k = T_t^{-1}(-i \hat{H}_0(t)), \]

which always has a local solution \( f_1(t), \ldots, f_r(t) \) for \( t \) in a neighbourhood of zero. This allows us to determine the values of \( f_1(t), \ldots, f_r(t) \) close to \( t = 0 \) ensuring that \( U(t) \) is the \( t \)-dependent evolution operator of (5.2).
The main property of a quantum quasi-Lie scheme is given in the following theorem, which will be employed to simplify \( t \)-dependent Schrödinger equations 'compatible' with quantum quasi-Lie schemes.

**Definition 5.4.** Let \( S(\mathfrak{W}, \mathfrak{V}) \) be a quasi-Lie scheme and let \( U(t) \) be the \( t \)-dependent evolution operator related to a \( t \)-dependent skew-Hermitian operator \(-i\hat{H}(t)\) taking values in \( \mathfrak{V} \). Then, we write \( U'(t)(-i\hat{H}(t)) \), where \( U'(t) \) stands for a curve in \( \mathcal{G}_{\mathfrak{W}} \), for the \( t \)-dependent skew-Hermitian operator \(-i\hat{H}'(t)\) associated with the evolution operator \((U' \star U)(t)\). We also write \( \mathfrak{W}_R \) and \( \mathfrak{V}_R \) for the spaces of curves in \( \mathfrak{W} \) and \( \mathfrak{V} \), respectively.

**Theorem 5.5.** *(The main theorem of quantum Lie systems)* Let \( S(\mathfrak{W}, \mathfrak{V}) \) be a quantum quasi-Lie scheme and let \(-i\hat{H}(t)\) be a curve in \( \mathfrak{V} \). If \( U_{\mathfrak{W}}(t) \) is an element of \( \mathcal{G}_{\mathfrak{W}} \), then

\[
U_{\mathfrak{W}}(t) \star (-i\hat{H}(t)) := \hat{A}(t) + \text{Ad}_{U_{\mathfrak{W}}(t)}(-i\hat{H}(t)),
\]  

(5.5)

where \( \hat{A}(t) \) is the \( t \)-dependent skew-Hermitian operator taking values in \( \mathfrak{W} \) whose evolution is determined by \( U_{\mathfrak{W}}(t) \), is an element of \( \mathfrak{W}_R \). If \( \psi(t) \) is a particular solution to the \( t \)-dependent Schrödinger equation related to \(-i\hat{H}(t)\), then the curve in \( \mathcal{H} \) of the form \( \psi'(t) = U_{\mathfrak{W}}(t)\psi(t) \) is a solution to the Schrödinger equation related to \( U_{\mathfrak{W}}(t) \star (-i\hat{H}(t)) \).

**Proof.** As \( U_{\mathfrak{W}}(t) \) is the \( t \)-dependent evolution operator related to the \( t \)-dependent skew-Hermitian operator \( \hat{A}(t) \), one has that \( \hat{A}(t) \) is an element of \( \mathfrak{W}_R \subset \mathfrak{V}_R \). Since \(-i\hat{H}'(t) := U_{\mathfrak{W}}(t) \star (-i\hat{H}(t)) \) is by definition the \( t \)-dependent skew-Hermitian operator admitting a \( t \)-dependent evolution operator \( U'(t) := U_{\mathfrak{W}}(t)U(t) \), where \( U(t) \) is the \( t \)-dependent evolution operator for \( \hat{H}(t) \), then

\[
\frac{dU'}{dt}(t)U'^*(t)\psi_0 = \left[ \frac{dU_{\mathfrak{W}}}{dt}(t)U_{\mathfrak{W}}^*(t) + \text{Ad}_{U_{\mathfrak{W}}(t)} \left( \frac{dU}{dt}(t)U^*(t) \right) \right] \psi_0
\]

\[
= \hat{A}(t)\psi_0 - \text{Ad}_{U_{\mathfrak{W}}(t)}(i\hat{H}(t))\psi_0 = \hat{A}(t)\psi_0 + \sum_{n=0}^{\infty} \frac{1}{n!} \text{ad}_{\hat{A}(t)}^n(i\hat{H}(t))\psi_0,
\]

(5.6)

for every \( \psi_0 \in \mathcal{H} \). As \( S(\mathfrak{W}, \mathfrak{V}) \) is a quantum Lie system, the curve \( -[\hat{B}, i\hat{H}(t)] \subset \mathfrak{V} \), for any \( \hat{B} \in \mathfrak{W} \), is a curve in \( \mathfrak{W} \) and \(-i\hat{A}(t)\) takes values in \( \mathfrak{W} \). By induction, \( \text{ad}_{\hat{A}(t)}^n(i\hat{H}(t)) \) takes values in \( \mathfrak{W} \) for every \( n \in \mathbb{N} \cup \{0\} \) and, in view of (5.6), one has that

\[
\frac{dU'}{dt}(t)U'^*(t) = -i\hat{H}'(t)
\]

takes values in \( \mathfrak{W} \).

Summarising, a quantum quasi-Lie scheme induces a group of \( t \)-dependent transformations allowing us to transform the \( t \)-dependent Schrödinger equation described by a \( t \)-dependent skew-Hermitian operator taking values in the linear space \( \mathfrak{W} \) of a quantum quasi-Lie scheme \( S(\mathfrak{W}, \mathfrak{V}) \) into another \( t \)-dependent Schrödinger equations related to \( t \)-dependent skew-Hermitian operators taking values also in \( \mathfrak{W} \). A particular relevant case occurs when the quantum quasi-Lie scheme enables us to transform, via an element of \( \mathcal{G}_{\mathfrak{W}} \), the initial \( t \)-dependent Schrödinger equation into a final one which can be described by means of the usual theory of quantum Lie systems. This motivates the next definition.
Definition 5.6. The $t$-dependent skew-Hermitian operator $-i\hat{H}(t)$ is a quantum quasi-Lie system with respect to the quantum quasi-Lie scheme $S(\mathfrak{W}, \mathfrak{V})$ if $-i\hat{H}(t)$ takes values in $\mathfrak{W}$ and there exists a curve $U(t)$ in $G_{\mathfrak{W}}$ such that $U(t) \star (-i\hat{H}(t))$ is a quantum Lie system.

To simplify the notation, we will call quantum quasi-Lie system and quantum Lie system the $t$-dependent Schrödinger equations related to a quantum quasi-Lie system or a quasi-Lie system, respectively.

6 Transformation properties of quantum quasi-Lie schemes

The group, $G_{\mathfrak{W}}$, of the quantum quasi-Lie scheme $S(\mathfrak{W}, \mathfrak{V})$ allows us to map every $t$-dependent skew-Hermitian operator $-i\hat{H}(t)$ taking values in $\mathfrak{W}$ into new ones taking values in $\mathfrak{V}$. This fact may be helpful in integrating or, at least, simplifying Proposition 6.1.

Proof. An element $U(t)$ in $G_{\mathfrak{W}}$ acts on $-i\hat{H}(t)$ taking values in $\mathfrak{W}$ in the form (5.5). Applying the projection $P$ in both sides of (5.5) and using that $dU(t)/dt(t)U^\dagger(t)$ takes values in $\mathfrak{W}$ for every $t \in \mathbb{R}$, one obtains

$$P[U(t) \star (-i\hat{H}(t))] = P \left( \frac{dU}{dt}(t)U^\dagger(t) - \text{Ad}_{U(t)}(i\hat{H}(t)) \right) = \frac{dU}{dt}(t)U^\dagger(t) - P(\text{Ad}_{U(t)}(i\hat{H}(t))).$$

Consider the differential equation for $U(t)$ of the form

$$\frac{dU}{dt}(t)U^\dagger(t) = P(\text{Ad}_{U(t)}(i\hat{H}(t))). \tag{6.1}$$

Any particular solution $U(t)$ of this equation with $U(0) = \text{Id}_\mathcal{H}$ will map $-i\hat{H}(t)$ into a new $t$-dependent skew-Hermitian operator $-i\hat{H}(t) := U(t) \star (-i\hat{H}(t))$ such that $P[U(t) \star (-i\hat{H}(t))] = 0$.

Let us consider a linear space, $\mathcal{V}$, isomorphic to $\mathfrak{W}$ via $\Theta : \mathcal{V} \to \mathfrak{W}$. Consider also $\mathfrak{W} \subset \mathcal{V}$ to be the linear subspace $\mathfrak{W} := \Theta^{-1}(\mathfrak{W})$. Then, $\mathfrak{W}$ inherits via $\Theta$ a Lie algebra structure that makes that $\mathfrak{W}$ can be considered as the Lie algebra of a Lie group $G_{\mathfrak{W}}$. The isomorphism $\Theta : \mathcal{V} \to \mathfrak{W}$ allows us to define a projector $\mathcal{P} : \mathcal{V} \to \mathcal{V}$ onto $\mathfrak{W}$ of the form $\mathcal{P} := \Theta^{-1} \circ P \circ \Theta$. Consider the system of differential equations on $G_{\mathfrak{W}}$ of the form

$$R_{g^{-1}} \star g \frac{dg}{dt} = \mathcal{P}(\text{Ad}_g a(t)), \tag{6.2}$$

where $a(t) := \Theta^{-1}(i\hat{H}(t))$. Let us prove that the solution to this system satisfying that $g(0) = e$ is such that the related $U(t)$ is a solution to (6.1). Let $\Phi : G_{\mathfrak{W}} \times \mathcal{H} \to \mathcal{H}$ be the Lie group action induced by the Lie algebra isomorphism $\Theta|_{\mathfrak{W}} : \mathfrak{W} \to \mathfrak{W}$. If we define $U(t) = \Phi^{g(t)}$ and $\psi \in \mathcal{H}$, we have that

$$\left. \frac{d}{dt} \right|_{t=t_0} U(t)\psi = \left. \frac{d}{dt} \right|_{t=t_0} \Phi(g(t)g^{-1}(t_0), \Phi(g(t_0), \psi)) = \Theta(\mathcal{P}(\text{Ad}_{g(t_0)} a(t_0)))U(t_0)\psi.$$
Since $\Theta(\text{Ad}_g a(t)) = \Phi^g i \hat{H}(t) \Phi^{-1}$, we get from the definition of $P$ that

$$P(\Phi^g i \hat{H}(t) \Phi^{-1}) = P(\Theta(\text{Ad}_g a(t))) = \Theta(P(\text{Ad}_g a(t))).$$

Hence,

$$\frac{d}{dt} \bigg|_{t=t_0} U(t)\psi = \Theta(P(\text{Ad}_g a(t_0)))U(t_0)\psi = P(U(t)i \hat{H}(t)U^{-1}(t))U(t)\psi.$$

Since the differential equation [6.2] admits a local solution for every initial condition, the same is true for the differential equation [6.1].

If $\mathfrak{U}$ is a Lie algebra, namely $-i \hat{H}(t)$ is a quantum Lie system, then we can choose $\mathfrak{W} = \mathfrak{U}$ and the previous proposition ensures that there exists $U(t)$ in $\mathcal{G}_{2\mathfrak{U}}$ such that $-U(t)\star i \hat{H}(t) = 0$. Hence, $U^{-1}(t)\psi$, for any $\psi \in \mathcal{H}$ becomes the general solution to the $t$-dependent Schrödinger equation related to $-i \hat{H}(t)$.

Let us use Proposition 6.1 to simplify or to integrate $-i \hat{H}(t)$ taking values in $\mathfrak{U}$ when $\mathfrak{U}$ is not a Lie algebra. In such a case, if $-i \hat{H}(t) = \sum_{\alpha=1}^{r+s} b_\alpha(t)iH_\alpha$ for certain $t$-dependent functions $b_1(t), \ldots, b_{r+s}(t)$ and a basis $\{iH_1, \ldots, iH_{r+s}\}$ of $\mathfrak{U}$ such that the first $r$ elements form a basis for $\mathfrak{W}$, then Proposition 6.1 ensures that there exists an $U(t)$ in $\mathcal{G}_{2\mathfrak{U}}$ such that $U(t)\star i \hat{H}(t) = \sum_{\alpha=r+1}^{r+s} b_\alpha(t)iH_\alpha$. This simplifies the expression of the initial $t$-dependent Schrödinger equation related to $-i \hat{H}(t)$.

To provide further techniques to integrate $t$-dependent Schrödinger equations determined by a $t$-dependent skew-Hermitian operator $-i \hat{H}(t)$ in $\mathfrak{U}_R$ when $\mathfrak{U}$ is not a Lie algebra, let us extend to the quantum case some structures defined for standard quasi-Lie schemes in [23].

The result of the following theorem allows us to determine when a $t$-dependent skew-Hermitian operator, $-i \hat{H}(t)$, taking values in the space $\mathfrak{U}$ of a quantum quasi-Lie scheme $S(\mathfrak{W}, \mathfrak{U})$ cannot be mapped into zero via an element $U_{2\mathfrak{U}}(t)$ of the group $\mathcal{G}_{2\mathfrak{U}}$ of the quantum quasi-Lie scheme. In this case, we will use a quantum quasi-Lie scheme to study the form of $-U(t)\star i \hat{H}(t)$ for each $U(t)$ in $\mathcal{G}_{2\mathfrak{U}}$.

**Theorem 6.2.** Let $S(\mathfrak{W}, \mathfrak{U})$ be a quantum quasi-Lie scheme with a subrepresentation $\mathfrak{U}_1$ of codimension one, namely $\dim \mathfrak{W}/\mathfrak{U}_1 = 1$, and let $\tau_{\mathfrak{U}_1} : \mathfrak{U} \to \mathfrak{W}/\mathfrak{U}_1$ be the canonical projection onto $\mathfrak{W}/\mathfrak{U}_1$. If $-i \hat{H}(t)$ belongs to $\mathfrak{U}_R$ and it is such that $\tau_{\mathfrak{U}_1}(-i \hat{H}(t)) \neq 0$ and $\tau_{\mathfrak{U}_1}(\mathfrak{W}) = 0$, then $\tau_{\mathfrak{U}_1}(-U(t)\star i \hat{H}(t)) \neq 0$ for every $U(t)$ in $\mathcal{G}_{2\mathfrak{U}}$.

**Proof.** To make the notation clearer, we will simply denote elements of $\mathfrak{W}$ and $\mathfrak{U}$ by lower-case letters, e.g. $w \in \mathfrak{W}$, $v_1 \in \mathfrak{U}$ and the elements of $\mathcal{G}_{2\mathfrak{U}}$ by curves of the form $g(t)$. Since $[\mathfrak{W}, \mathfrak{U}_1] \subset \mathfrak{U}_1$, each Lie algebra morphism $\rho_w = [w, -]$, with $w \in \mathfrak{W}$, induced by the representation of $S(\mathfrak{W}, \mathfrak{U})$ leads to a well-defined morphism $\tilde{\rho}_w : \tau_{\mathfrak{U}_1}(v) \in \mathfrak{W}/\mathfrak{U}_1 \mapsto \tau_{\mathfrak{U}_1}(\rho_w(v)) \in \mathfrak{W}/\mathfrak{U}_1$. In fact, if $v_1, v_2 \in \mathfrak{U}$ and $\tau_{\mathfrak{U}_1}(v_1) = \tau_{\mathfrak{U}_1}(v_2)$, then $[w, v_1 - v_2] \in \mathfrak{U}_1$ for every $w \in \mathfrak{W}$, $\tau_{\mathfrak{U}_1}([w, v_1 - v_2]) = 0$, and

$$\tilde{\rho}_w(\tau_{\mathfrak{U}_1}(v_1)) = \tau_{\mathfrak{U}_1}([w, v_1]) = \tau_{\mathfrak{U}_1}([w, v_2]) = \tilde{\rho}_w(\tau_{\mathfrak{U}_1}(v_2)).$$

This allows us to define a Lie algebra morphism $\tilde{\rho} : w \in \mathfrak{W} \mapsto \tilde{\rho}_w \in \text{End}(\mathfrak{W}/\mathfrak{U}_1)$. Indeed,

$$[\tilde{\rho}_w_1, \tilde{\rho}_w_2](\tau_{\mathfrak{U}_1}(v)) := \tilde{\rho}_w_1(\tilde{\rho}_w_2(\tau_{\mathfrak{U}_1}(v))) - \tilde{\rho}_w_2(\tilde{\rho}_w_1(\tau_{\mathfrak{U}_1}(v))) = \tilde{\rho}_w(\tau_{\mathfrak{U}_1}([w_2, v])) - \tilde{\rho}_w(\tau_{\mathfrak{U}_1}([w_1, v])),$$

and

$$\tilde{\rho}_w(\tau_{\mathfrak{U}_1}(v)) = \tau_{\mathfrak{U}_1}([w, v_1]) = \tau_{\mathfrak{U}_1}([w, v_2]) = \tilde{\rho}_w(\tau_{\mathfrak{U}_1}(v_2)),$$
for all $w_1, w_2 \in \mathfrak{W}$ and every $v \in \mathfrak{V}$. Using the Jacobi identity and the definition of $\tau_{\mathfrak{g}_1}$,

$$\lbrack \hat{\rho}_{w_1}, \hat{\rho}_{w_2} \rbrack (\tau_{\mathfrak{g}_1}(v)) := \tau_{\mathfrak{g}_1}([w_1, [w_2, v]] - [w_2, [w_1, v]]) = \hat{\rho}_{[w_1, w_2]}(\tau_{\mathfrak{g}_1}(v)), \quad \forall w_1, w_2 \in \mathfrak{W}, \forall v \in \mathfrak{V}.$$ 

Hence, $[\hat{\rho}_{w_1}, \hat{\rho}_{w_2}] = \hat{\rho}_{[w_1, w_2]}$ and $\hat{\rho}$ is a Lie algebra morphism. Moreover, $\hat{\rho}$ and $\rho : w \in \mathfrak{W} \mapsto [w, \cdot] \in \text{End}(\mathfrak{V})$ are equivariant relative to $\tau_{\mathfrak{g}_1}$, namely $\hat{\rho}_w \circ \tau_{\mathfrak{g}_1} = \tau_{\mathfrak{g}_1} \circ \rho_w$, for all $w \in \mathfrak{W}$. In fact,

$$\hat{\rho}_w \circ \tau_{\mathfrak{g}_1}(v) = \tau_{\mathfrak{g}_1}([w, v]) = \tau_{\mathfrak{g}_1} \circ \rho_w(v), \quad \forall w \in \mathfrak{W}, \forall v \in \mathfrak{V} \implies \hat{\rho}_w \circ \tau_{\mathfrak{g}_1} = \tau_{\mathfrak{g}_1} \circ \rho_w, \quad \forall w \in \mathfrak{W}.$$ 

Using that $\rho$ and $\hat{\rho}$ are equivariant relative to $\tau_{\mathfrak{g}_1}$, we are going to finally prove that if $v(t)$ takes values in $\mathfrak{V}$ and $\tau_{\mathfrak{g}_1}(v(t)) \neq 0$, then $\tau_{\mathfrak{g}_1}(g(t) \star v(t)) \neq 0$ for every $g(t)$ in $\mathcal{G}_{\mathfrak{m}}$. Since dim $\mathfrak{V}/\mathfrak{V}_1 = 1$, there exists a unique function $\theta \in \mathfrak{V}^*$ such that $\hat{\rho}_w(\tau_{\mathfrak{g}_1}(v)) = \theta(w)\tau_{\mathfrak{g}_1}(v)$.

Consider the case of $g(t) = \exp(f(t)w)$ for some $w \in \mathfrak{W}$, a $t$-dependent function $f(t)$, and an element $v(t)$ in $\mathfrak{V}_R$. It is clear that

$$g(t) \star v(t) = \frac{dg}{dt}(t)g(t)^{-1} + \text{Ad}_{g(t)}v(t) = \frac{df}{dt}(t)w + v(t) + f(t)[w, v(t)] + \frac{f^2(t)}{2}[w, [w, v(t)]] + \ldots$$

Applying $\tau_{\mathfrak{g}_1}$ on both sides, using the assumptions on $\mathfrak{W}$ and $\mathfrak{V}$, and recalling the $\hat{\rho}$ is equivariant relative to $\tau_{\mathfrak{g}_1}$, we obtain

$$\tau_{\mathfrak{g}_1}(g(t) \star v(t)) = \tau_{\mathfrak{g}_1} \left( \frac{dg}{dt}(t)g(t)^{-1} + \tau_{\mathfrak{g}_1}(v(t)) + \tau_{\mathfrak{g}_1} \left( [f(t)w, v(t)] + \frac{f^2(t)}{2}[w, [w, v(t)]] + \ldots \right) \right)$$

$$= (1 + f(t)\theta(w) + \frac{f^2(t)}{2}\theta^2(w) + \ldots)\tau_{\mathfrak{g}_1}(v(t))$$

$$= e^{f(t)\theta(w)}\tau_{\mathfrak{g}_1}(v(t)).$$

Therefore, if $\tau_{\mathfrak{g}_1}(v(t)) \neq 0$, then $\tau_{\mathfrak{g}_1}(g(t) \star v(t)) \neq 0$. Since every $g(t)$ in $\mathcal{G}_{\mathfrak{m}}$ can be written as a product of elements of the form $g(t) = \prod_{i=1}^s \exp(f_i(t)w_i)$ for some elements $w_1, \ldots, w_s$ of $\mathfrak{W}$ and functions $f_1(t), \ldots, f_s(t)$, it follows from the above result that if $\tau_{\mathfrak{g}_1}(v(t)) \neq 0$, then $\tau_{\mathfrak{g}_1}(g(t) \star v(t)) \neq 0$ for every $g(t)$ in $\mathcal{G}_{\mathfrak{m}}$. 

As an immediate consequence of the proof of Proposition 6.2, we obtain the following corollary.

**Corollary 6.3.** Let us assume the same conditions of Theorem 6.2 and let $i\hat{H}_1, \ldots, i\hat{H}_r$ be a basis of $\mathfrak{V}$ such that $i\hat{H}_1, \ldots, i\hat{H}_{r-1}$ is a basis of $\mathfrak{V}_1$. If $\theta$ is the element of $\mathfrak{V}^*$ associated with the induced representation $\hat{\rho} : \mathfrak{W} \to \text{End}(\mathfrak{V}/\mathfrak{V}_1)$, namely if $\tau_{\mathfrak{g}_1} : \mathfrak{V} \to \mathfrak{V}/\mathfrak{V}_1 \simeq \langle i\hat{H}_r \rangle$, the $\theta$ is the only element in $\mathfrak{V}^*$ such that $\tau_{\mathfrak{g}_1}(\theta(a) i\hat{H}_r) = \theta(a)\tau_{\mathfrak{g}_1}(i\hat{H}_r)$ for every $a \in \mathfrak{W}$, then, for any functions $a_1(t), \ldots, a_r(t)$, we have

$$U(t) = \prod_{i=1}^r \exp(a_i(t)i\hat{H}_i) \implies \tau_{\mathfrak{g}_1}(U(t) \star i\hat{H}_r(t)) = \left[ \prod_{i=1}^r \exp(a_i(t)\theta(i\hat{H}_i)) \right] i\hat{H}_r.$$ 

### 7 Applications of quantum quasi-Lie schemes

This section concerns the application of the theory of quantum quasi-Lie schemes to the quantum anharmonic oscillator model given by $n$ interacting particles via a $t$-dependent Hermitian operator of the form

$$\hat{H}_{nH}(t) := \frac{1}{2} \sum_{i=1}^n \left( \hat{p}_i^2 + \omega^2(t)\hat{x}_i^2 \right) + c(t)U_{nH}(\hat{x}_1, \ldots, \hat{x}_n),$$

(7.1)
where \( c(t) \) is a non-vanishing real function, \( \omega(t) \) is any real \( t \)-dependent function describing a sort of \( t \)-dependent frequency, and \( U_{nH}(\vec{x}_1, \ldots, \vec{x}_n) \) is a quantum potential determined by an homogeneous polynomial of order \( k \) depending on the position operators \( \vec{x}_1, \ldots, \vec{x}_n \), i.e. \( U_{nH}(\lambda \vec{x}_1, \ldots, \lambda \vec{x}_n) = \lambda^k U_{nH}(\vec{x}_1, \ldots, \vec{x}_n) \) for every \( \lambda \in \mathbb{R} \). As a particular instance, (7.1) covers many types of anharmonic quantum oscillators, which have been extensively studied in the literature [46, 59, 60].

Perelomov found some conditions ensuring the existence of a \( t \)-dependent change of variables mapping a classical analogue of (7.1) onto an autonomous Hamiltonian system, up to a trivial time-reparametrisation, related to the Hamiltonian

\[
H_{nH} = \frac{1}{2} \sum_{i=1}^{n} \dot{p}_i^2 + U_{nH}(x_1, \ldots, x_n),
\]

where \( U_{nH} \) is now understood as a real homogeneous polynomial of order \( k \) on the position variables \( x_1, \ldots, x_n \) (see [55] for details). As many Hamiltonians of the form (6.2) are known to be explicitly integrable, the \( t \)-dependent change of variables found by Perelomov relates the solutions of such Hamiltonians with the solutions of an associated non-autonomous one \( H_{nH} \).

Perelomov left as an open problem to look for a quantum analogue of his results (cf. [55]). Additionally, the classical anharmonic oscillator related to the \( t \)-dependent Hamiltonian (7.1) was briefly analysed via quasi-Lie schemes in [12]. Subsequently, these results are extended to the quantum realm by means of a quantum quasi-Lie scheme, resulting in a solution to Perelomov’s open problem.

Let us build up a quantum quasi-Lie scheme \( S(\mathfrak{M}_{nH}, \mathfrak{V}_{nH}) \) to deal with the \( t \)-dependent Hamiltonian operator \( -i\hat{H}_{nH}(t) \). This demands to determine subspaces \( \mathfrak{M}_{nH}, \mathfrak{V}_{nH} \) of skew-Hermitian operators satisfying (4.3) and such that \( -i\hat{H}_{c}(t) \) takes values in \( \mathfrak{V}_{nH} \). The construction of \( S(\mathfrak{M}_{nH}, \mathfrak{V}_{nH}) \) is accomplished in the following lemma.

**Lemma 7.1.** The spaces \( \mathfrak{V}_{nH} := \langle i\hat{H}_1, i\hat{H}_2, i\hat{H}_3, i\hat{H}_4 \rangle \) and \( \mathfrak{M}_{nH} := \langle i\hat{H}_2, i\hat{H}_3 \rangle \), with

\[
\begin{align*}
\hat{H}_1 := & \sum_{i=1}^{n} \frac{\dot{p}_i^2}{2}, \quad \hat{H}_2 := \frac{1}{4} \sum_{i=1}^{n} (\tilde{x}_i \ddot{p}_i + \tilde{p}_i \ddot{\tilde{x}}_i), \quad \hat{H}_3 := \frac{1}{2} \sum_{i=1}^{n} \tilde{x}_i^2, \quad \hat{H}_4 := U_{nH}(\tilde{x}_1, \ldots, \tilde{x}_n),
\end{align*}
\]

give rise to quantum quasi-Lie scheme \( S(\mathfrak{M}_{nH}, \mathfrak{V}_{nH}) \) such that \( -i\hat{H}_{nH}(t) \) takes values in \( \mathfrak{V}_{nH} \).

**Proof.** Let us prove first that \( S(\mathfrak{M}_{nH}, \mathfrak{V}_{nH}) \) is a quantum quasi-Lie scheme. It is immediate that \( \mathfrak{V}_{nH} \) and \( \mathfrak{M}_{nH} \) are finite-dimensional linear spaces of skew-Hermitian operators.

Since \( [i\hat{H}_2, i\hat{H}_3] = i\hat{H}_3 \), the space \( \mathfrak{M}_{nH} \) is a real Lie algebra of skew-Hermitian operators as demanded by the definition of a quantum quasi-Lie scheme. Let us verify the only left property of quantum quasi-Lie schemes, namely \( [\mathfrak{M}_{nH}, \mathfrak{V}_{nH}] \subset \mathfrak{V}_{nH} \). It is immediate that

\[
[i\hat{H}_2, i\hat{H}_1] = -i\hat{H}_1, \quad [i\hat{H}_3, i\hat{H}_1] = -2i\hat{H}_2, \quad [i\hat{H}_3, i\hat{H}_4] = 0.
\]

The only non-IMMEDIATE part to corroborate the condition \( [\mathfrak{M}_{nH}, \mathfrak{V}_{nH}] \subset \mathfrak{V}_{nH} \) is to show that \( [i\hat{H}_2, i\hat{H}_4] \subset \mathfrak{V}_{nH} \). As \( [\tilde{x}_j, \tilde{p}_j] = iI \) for \( j = 1, \ldots, n \), then \( \hat{p}_j \tilde{x}_j = \tilde{x}_j \hat{p}_j - i\hat{I} \) and

\[
\sum_{j=1}^{n} (\tilde{x}_j \hat{p}_j + \hat{p}_j \tilde{x}_j) = 2 \sum_{j=1}^{n} \tilde{x}_j \hat{p}_j - i\hat{I},
\]

(7.3)
where $\hat{1}$ is the identity operator on $\mathcal{H}$. Consequently,

$$[i\hat{H}_2, i\hat{H}_4] = \frac{1}{2} \sum_{j=1}^{n} x_j \frac{\partial n_{\hat{H}}(\hat{x}_1, \ldots, \hat{x}_n)}{\partial x_j} = \frac{i}{2} \sum_{j=1}^{n} x_j \frac{\partial n_{\hat{H}}}{\partial x_j}(\hat{x}_1, \ldots, \hat{x}_n).$$

In view of the Euler’s homogeneous function theorem and recalling that $U_{nH}$ is a homogeneous function of degree $k$, it follows that

$$\sum_{j=1}^{n} x_j \frac{\partial n_{\hat{H}}}{\partial x_j} = k n_{\hat{H}} \implies [i\hat{H}_2, i\hat{H}_4] = \frac{i}{2} k n_{\hat{H}}(\hat{x}_1, \ldots, \hat{x}_n).$$

This yields $[i\hat{H}_2, i\hat{H}_4] = \frac{i}{2} k \hat{H}_4$ and $[\mathfrak{W}_{nH}, \mathfrak{V}_{nH}] \subseteq \mathfrak{V}_{nH}$. Thus, $\mathfrak{W}_{nH}, \mathfrak{V}_{nH}$ give rise to a quantum quasi-Lie scheme $S(\mathfrak{W}_{nH}, \mathfrak{V}_{nH})$.

Finally, the form of $\mathfrak{V}_{nH}$ and $-i\hat{H}_{nH}(t)$, which is given by (7.1), ensures that $-i\hat{H}_{nH}(t)$ takes values in $\mathfrak{V}_{nH}$, which finishes the proof. 

Once it is been stated that $S(\mathfrak{W}_{nH}, \mathfrak{V}_{nH})$ is a quantum quasi-Lie scheme and it can be used to describe the $t$-dependent Hamiltonian operators in (7.1), it is time to apply the group of $S(\mathfrak{W}_{nH}, \mathfrak{V}_{nH})$ to analyse $-i\hat{H}_{nH}(t)$. In particular, a relevant case occurs when an element $U(t)$ of $G_{2nH}$ allows us to map $-i\hat{H}_{nH}(t)$ into a $t$-dependent skew-Hermitian operator $-i\hat{H}'_{U}(t) := -U(t)\star i\hat{H}_{nH}(t)$ taking values in a one-dimensional subspace of $\mathfrak{V}_{nH}$. In such a case, a time-reparametrisation allows us to solve the transformed $t$-dependent Schrödinger equation related to $-i\hat{H}'_{U}(t)$ and, applying $U^{-1}(t)$ to the general solution of the latter, we obtain the general solution of the initial $t$-dependent Schrödinger equation.

**Lemma 7.2.** The $t$-dependent operator $-U(t)\star i\hat{H}_{nH}(t)$ takes values in a one-dimensional subspace of $\mathfrak{V}_{nH}$ if and only if $U(t)$, which is assumed without loss of generality to take the form

$$U(t) = \exp\left(i\alpha(t)\hat{H}_2\right) \exp\left(i\beta(t)\hat{H}_3\right),$$

is such that $c(t)e^{\frac{\alpha(t)(k+2)}{2}}$ is a non-zero constant.

**Proof.** In view of the Lie algebra representation of $S(\mathfrak{W}_{nH}, \mathfrak{V}_{nH})$, namely $\rho : w \in \mathfrak{W}_{nH} \mapsto [w, \cdot] \in \text{End}(\mathfrak{V}_{nH})$, the quantum quasi-Lie scheme $S(\mathfrak{W}_{nH}, \mathfrak{V}_{nH})$ admits two subrepresentations given by $\mathfrak{V}_1 := \langle i\hat{H}_1, i\hat{H}_2, i\hat{H}_3 \rangle$ and $\mathfrak{V}_2 := \langle i\hat{H}_2, i\hat{H}_3, i\hat{H}_4 \rangle$.

Since $c(t) \neq 0$, the $t$-dependent operator $-i\hat{H}_{nH}(t)$ has no zero projection neither onto $\mathfrak{V}_1$ nor onto $\mathfrak{V}_2$. Then, Theorem 6.2 yields that the $t$-dependent skew-Hermitian operator of the form $-U(t)\star i\hat{H}_{nH}(t)$ will never take values neither in $\mathfrak{V}_1$ nor in $\mathfrak{V}_2$ for any $U(t)$ in $G_{2nH}$. In other words, if we write $-U(t)\star i\hat{H}_{nH}(t)$ in the basis $iH_1, \ldots, iH_4$ of $\mathfrak{V}_{nH}$, then the $t$-dependent coefficients relative to $iH_1$ or $iH_4$ will never vanish. Moreover, as we want $-U(t)\star i\hat{H}_{nH}(t)$ to take values in a one-dimensional subspace of $\mathfrak{V}_{nH}$, the $t$-dependent coefficients relative to $iH_4$ and $iH_1$ must be proportional. Let us determine their values exactly without determining the exact form of $-U(t)\star i\hat{H}_{nH}(t)$.

As in the proof of Theorem 6.2, the representation of $S(\mathfrak{W}_{nH}, \mathfrak{V}_{nH})$ induces a representation $\hat{\rho}_{\mathfrak{V}_2} : \mathfrak{W}_{nH} \rightarrow \text{End}(\mathfrak{V}_2)$ such that $(\hat{\rho}_{\mathfrak{V}_2})_{i\hat{H}_2}(i\hat{H}_1) = 0$ and $(\hat{\rho}_{\mathfrak{V}_2})_{i\hat{H}_2}(i\hat{H}_1) = -i\hat{H}_1$. Hence, $\theta \in \mathfrak{W}_{nH}$ given by $\theta(i\hat{H}_3) = 0$ and $\theta(i\hat{H}_2) = -1$ is the unique element of $\mathfrak{W}_{nH}$ such that $\tau_{\mathfrak{V}_2}([A, \hat{B}]) = \mathfrak{V}_2$. In particular, $\theta$ induces a representation $\hat{\rho}_{\mathfrak{V}_2}$, and thus $\mathfrak{V}_2$ is a one-dimensional subspace of $\mathfrak{V}_{nH}$.
\( \theta(\hat{A})\tau_{\mathcal{V}_2}(\hat{B}) \) for every \( \hat{A} \in \mathcal{W}_{nH} \) and \( \hat{B} \in \mathcal{V}_{nH} \). If \( U(t) \) has the form (7.4), then Corollary 6.3 gives that \( \tau_{\mathcal{V}_2}(U(t)\star i\hat{H}(t)) = e^{-\alpha(t)}i\hat{H}_1 \).

Meanwhile, the induced representation \( \tilde{\rho}_{\mathcal{V}_1} : \mathcal{W}_{nH} \to \text{End}(\mathcal{V}/\mathcal{V}_1) \) by the second subrepresentation \( \mathcal{V}_1 \) is such that \( (\tilde{\rho}_{\mathcal{V}_1})_{i\hat{H}_3}(i\hat{H}_1) = 0 \) i.e., \( (\tilde{\rho}_{\mathcal{V}_1})_{i\hat{H}_2}(i\hat{H}_4) = ik/2\hat{H}_4 \). Hence, \( \theta \) is determined by \( \theta(iH_3) = 0 \) and \( \theta(i\hat{H}_2) = k/2 \). Thus, if \( U(t) \) has the form (7.4), then Corollary 6.3 gives that \( \tau_{\mathcal{V}_1}(U(t)\star i\hat{H}(t)) = e^{\alpha(t)/2}i\hat{H}_4 \).

Since the \( t \)-dependent coefficients of \( \tau_{\mathcal{V}_1}(U(t)\star i\hat{H}(t)) \) and \( \tau_{\mathcal{V}_2}(U(t)\star i\hat{H}(t)) \) must be proportional, it follows that \( c(t)\exp(\frac{\alpha(t)(k+2)}{2}) \) must be a constant. \( \square \)

Lemma 7.2 provides an easily derivable necessary condition to map \(-i\hat{H}_{nH}(t)\) into a new \( t \)-dependent operator \(-i\hat{H}_{nH}(t)\) taking values in a one-dimensional subspace of \( \mathcal{V}_{nH} \) via an \( U(t) \) belonging to \( \mathcal{G}_{\mathcal{W}_{nH}} \). Nevertheless, the determination of sufficient conditions to obtain such a \(-i\hat{H}_{nH}(t)\) demands to determine the action of a generic \( U(t) \) on \(-i\hat{H}_{nH}(t)\). If \( U(t) \) is in \( \mathcal{G}_{\mathcal{W}_{nH}} \), then

\[
-i\hat{H}_{nH}(t) = U(t)\star (-i\hat{H}_{nH}(t)) = \frac{dU}{dt}(t)^{-1}U(t)(-i\hat{H}_{nH}(t)),
\]

and Theorem 5.5 ensures that \( U(t)\star (-i\hat{H}_{nH}(t)) \) takes values in \( \mathcal{V}_{nH} \). Every element \( U(t) \) in \( \mathcal{G}_{\mathcal{W}_{nH}} \) can be written in the form

\[
U(t) := \exp\left(i\alpha(t)\hat{H}_2\right)\exp\left(i\beta(t)\hat{H}_3\right).
\]

To obtain \(-i\hat{H}_{nH}' = U(t)\star (-i\hat{H}_{nH}(t))\), it is necessary to use that

\[
\text{Ad}\left[\exp\left(i\lambda\hat{H}_2\right)\right]\hat{H}_1 = e^{-\lambda}\hat{H}_1, \quad \text{Ad}\left[\exp\left(i\lambda\hat{H}_3\right)\right]\hat{H}_1 = \hat{H}_1 - 2\lambda\hat{H}_2 + \lambda^2\hat{H}_3,
\]

\[
\text{Ad}\left[\exp\left(i\lambda\hat{H}_2\right)\right]\hat{H}_3 = e^{\lambda}\hat{H}_3, \quad \text{Ad}\left[\exp\left(i\lambda\hat{H}_3\right)\right]\hat{H}_2 = \hat{H}_2 - \lambda\hat{H}_3,
\]

\[
\text{Ad}\left[\exp\left(i\lambda\hat{H}_2\right)\right]\hat{H}_4 = e^{k\lambda}\hat{H}_4, \quad \text{Ad}\left[\exp(i\lambda\hat{H}_3)\right]\hat{H}_4 = \hat{H}_4.
\]

Then,

\[
-i\hat{H}_{nH}'(t) = -e^{-\alpha(t)}\hat{H}_1 + \left(\frac{d\alpha}{dt}(t) + 2\beta(t)\right)\hat{H}_2 + e^{\alpha(t)}\left(\frac{d\beta}{dt}(t) - \beta(t) - \omega^2(t)\right)\hat{H}_3 - e^{\frac{k\alpha(t)}{2}}c(t)i\hat{H}_4,
\]

where we recall that \( \omega(t) \) is the \( t \)-dependent frequency of our initial \( t \)-dependent Hermitian operator (7.1).

Note also that Proposition 6.1 ensures that the group of \( S(\mathcal{W}_{nH}, \mathcal{V}_{nH}) \) allows us to map \(-i\hat{H}_{nH}(t)\) into a new \(-i\hat{H}_{nH}(t)\) taking values in \( \langle i\hat{H}_1, i\hat{H}_4 \rangle \) for some \( U(t) \) in \( \mathcal{G}_{\mathcal{W}_{nH}} \). For instance, suppose that \( \beta(t) \) and \( \alpha(t) \) are such that

\[
\frac{d\beta}{dt}(t) = \beta^2(t) + \omega^2(t), \quad \frac{d\alpha}{dt}(t) + 2\beta(t) = 0,
\]

and we recall that \( \alpha(0) = \beta(0) = 0 \). Under such conditions, \( \hat{H}_n(t) = e^{-\alpha(t)}\hat{H}_1 + e^{\frac{\alpha(t)k}{2}}c(t)i\hat{H}_4. \)

To ensure that \(-i\hat{H}_{nH}'(t)\) takes values in a one-dimensional subspace of \( \mathcal{V}_{nH} \), we have to recall the condition in (7.2), i.e.

\[
c(t)e^{\frac{\alpha(t)k}{2}} = e^{-\alpha(t)}l \implies c(t)e^{\frac{\alpha(t)(k+2)}{2}} = l,
\]
for a non-zero constant \( l \in \mathbb{R} \). Note that we can assume without loss of generality that \( l = 1 \) by redefining \( \tilde{H}_t \).

Using (7.8) in (7.7), we obtain that if \( k \neq 2 \) and there exist \( \alpha(t) \) and \( \beta(t) \) satisfying the previous conditions, then
\[
(k + 2) \frac{d^2c}{dt^2}(t) = (k + 3) \frac{(dc/dt)(t)}{c(t)} + \omega^2(t)(k + 2)^2c(t).
\] (7.9)

Conversely, if (7.9) holds, then there exist \( \alpha(t) \) and \( \beta(t) \) satisfying (7.7) and (7.8).

It is worth noting that the definition of a new variable \( v_c := dv/dt \) allows us to transform (7.9) into a Lie system related to a VG Lie algebra isomorphic to \( \text{so}(2, \mathbb{R}) \). In fact, if \( k = 0 \), this is indeed a type of Kummer–Schwarz equation of second-order studied in [14].

Assume that (7.9) is satisfied. Then, the \( t \)-dependent Schrödinger equation related to \( \tilde{H}_{nH}'(t) \) reads
\[
i \frac{\partial \psi}{\partial t} = \tilde{H}_{nH}'(t)\psi = e^{-\alpha(t)}(\tilde{H}_1 + \tilde{H}_4)\psi.
\]
and by means of the time-reparametrization
\[
\tau(t) := \int_t^t e^{-\alpha(t')}dt' = \int_t^t [c(t')t^{-1}]^{1/2} dt',
\]
the previous \( t \)-dependent Schrödinger equation can be mapped into
\[
\frac{\partial \psi'}{\partial \tau} = -i(\tilde{H}_1 + \tilde{H}_4)\psi',
\]
whose solution is
\[
\psi'(t) = \exp \left( -i\tau(t)(\tilde{H}_1 + \tilde{H}_4) \right) \psi'(0),
\]
and \( \psi' \) is the transformed solution and \( \psi'(0) \) is an arbitrary element of \( \mathcal{L}^2(\mathbb{R}^n) \). Hence, the solution of the initial Schrödinger equation determined by the \( t \)-dependent Hamiltonian operator \( \tilde{H}_{nH}(t) \) can be obtained by inverting the unitary transformation (7.5), namely
\[
\psi(t) = \exp \left( -i\beta(t)\tilde{H}_3 \right) \exp \left( -i\alpha(t)\tilde{H}_2 \right) \exp \left( -i\tau(t)(\tilde{H}_1 + \tilde{H}_4) \right) \psi'(0).
\] (7.10)

Therefore, we have mapped a non-autonomous Schrödinger equation into a new one determined by a \( t \)-independent Hermitian Hamiltonian operator, similarly to what it was done by Perelomov in [55], but in a quantum mechanical way. Our result is summarised in the proposition below.

**Proposition 7.3.** Every \( t \)-dependent Schrödinger equation related to a \( t \)-dependent Hermitian Hamiltonian operator \( \tilde{H}_{nH}(t) \) of the form (7.1), with a homogeneous potential of order \( k \neq -2 \) and a non-vanishing function \( c(t) \) that is a particular solution of (7.9), has a general solution (7.10), where
\[
\alpha(t) = -\frac{2}{k + 2} \log c(t), \quad \beta(t) = \frac{dc/dt}{c(t)(k + 2)}, \quad \tau(t) = \int_t^t c(t')^{1/2} dt'.
\]

**Note 7.4.** It was already noted that to transform \(-i\tilde{H}_{nH}(t)\) into an autonomous system up to a time-reparametrization via the group of \( S(\mathfrak{W}_{nH}, \mathfrak{g}_{nH}) \), the condition (7.8) is necessary. Despite that, the condition (7.7) is not mandatory and other alternative ones could be developed. This would lead to new integration methods for (7.4).
8 Homogeneous potentials of degree minus two

Proposition 7.3 cannot be applied to $t$-dependent Hamiltonian operators of the form (7.1) with a homogeneous potential of degree minus two. These potentials include the relevant potential

$$U_{nH}(\tilde{x}_1, \ldots, \tilde{x}_n) := \sum_{j<k} \frac{g}{(\tilde{x}_j - \tilde{x}_k)^2}, \quad g \in \mathbb{R} \setminus \{0\},$$

of a fluid in a $t$-dependent homogeneous trapping potential [64], which is also a type of Calogero-Moser potential, or the celebrated Smorodinsky-Winternitz potentials

$$U_{nH}(\tilde{x}_1, \ldots, \tilde{x}_n) = \sum_{j=1}^{n} \frac{g_j}{\tilde{x}_j^2}, \quad g_j \in \mathbb{R}, \quad \sum_{j=1}^{n} g_j^2 \neq 0.$$

Nevertheless, the procedure to prove Proposition 7.3 can be modified to deal with these pathological potentials with $k = -2$. This is the main aim of this section.

Lemma 7.1, Lemma 7.2, and the expression for $-i\hat{H}_{nH}(t) = U(t) \star (-i\hat{H}_{nH}(t))$ given by (7.6) are still valid when the potential under study is homogeneous of degree minus two. Hence, to map $-i\hat{H}_{nH}(t)$ into a new $t$-dependent skew-Hermitian operator $-i\hat{H}_{nH}'(t) = -U(t) \star i\hat{H}_{nH}(t)$ taking values in a one-dimensional subspace of $\mathfrak{g}_n$, it is mandatory to apply the condition

$$c(t)e^{\alpha(t)(k+2)/2} = \lambda,$$

for a certain non-zero constant $\lambda \in \mathbb{R}$. Since $k = -2$, the function $c(t)$ becomes a constant $c$. If we assume that $\alpha(t)$ and $\beta(t)$ are particular solutions to (7.7), the transformed $t$-dependent Hamiltonian operator (7.6) takes the form $\hat{H}_{nH}'(t) = e^{-\alpha(t)}(\hat{H}_1 + c\hat{H}_4)$. Hence, the general solution to the $t$-dependent Schrödinger equation related to $\hat{H}_{nH}'(t)$, whose explicit $t$-dependence can be removed after a $t$-reparametrisation, reads

$$\psi'(x,t) = \exp \left( \int_0^t e^{-\alpha(t')}dt' \right) \psi(x),$$

where $\psi'(x)$ is an arbitrary element of $L^2(\mathbb{R}^n)$ and $\psi'(x,0) = \psi(x)$. Hence, the general solution to (7.1) for a homogeneous potential of degree minus two reads as (7.10) for some solutions $\alpha(t)$, $\beta(t)$ to equations to (7.7), (7.8) and (7.9). This result is summarised in the following proposition.

**Proposition 8.1.** Every $t$-dependent Hamiltonian operator $\hat{H}(t)$ of the form (7.1) with a homogeneous potential of order minus two and $c(t) = c \in \mathbb{R}$ has a general solution (7.10) where $\beta(t)$ is a particular solution to the Riccati differential equation $d\beta/dt = \beta^2 + \omega^2(t)$ and

$$\alpha(t) = -2 \int_0^t \beta(t')dt', \quad c_0 \in \mathbb{R} \setminus \{0\}, \quad \tau(t) = \int_0^t e^{-\alpha(t')}dt'.$$

Let us apply Proposition 8.1 to the $t$-dependent Hamiltonian

$$\hat{H}_{nH}(t) = \frac{1}{2} \sum_{j=1}^{n} (\tilde{p}_j^2 + K(t)\tilde{x}_j^2) + \sum_{j<k} \lambda(\lambda-1) \frac{1}{(\tilde{x}_j - \tilde{x}_k)^2},$$

whose potential is homogeneous of degree minus two. This $t$-dependent Hamiltonian operator was analysed in [64] to study a quantum one-dimensional fluid in a Paul trap [54]. Sutherland provided
an Ansatz [64] to get a particular solution. Here, we recover some of his results from an algorithmic point of view and show that more useful solutions can be derived to study the properties of the quantum system.

Suppose that we take a $t$-dependent transformation of $G_W$ with functions $\beta(t)$ and $\alpha(t)$ satisfying the differential equations

$$\frac{d\beta}{dt}(t) = \beta^2(t) + K(t), \quad \frac{d\alpha}{dt}(t) + 2\beta(t) = 0. \tag{8.1}$$

In view of Proposition 8.1, the $t$-dependent Schrödinger equation related to $\hat{H}_{nH}(t)$ reads

$$\psi(x,t) = \exp(-\beta(t)i\hat{H}_3) \exp(-\alpha(t)i\hat{H}_2) \exp\left(\int_0^t e^{-\alpha(t')dt'}(\hat{H}_1 + \hat{H}_4)\right) \psi'(x,0).$$

In particular, if we assume

$$\psi(x,0) = \prod_{j<k}^n (x_j - x_k)^\lambda,$$

we obtain

$$(\hat{H}_1 + \hat{H}_4) \prod_{j<k}^n (x_j - x_k)^\lambda = 0$$

and hence

$$\psi(x,t) = \exp(-\beta(t)i\hat{H}_3) \exp(-\alpha(t)i\hat{H}_2) \prod_{j<k}^n (x_j - x_k)^\lambda.$$

Since $\prod_{j<k}(x_j - x_k)^\lambda$ is a homogeneous function with degree $\lambda n(n-1)/2$ and in view of (7.3), it follows that

$$i\hat{H}_2 \prod_{j<k}^n (x_j - x_k)^\lambda = \left(\frac{n}{4} + \frac{\lambda n(n-1)}{4}\right) \prod_{j<k}^n (x_j - x_k)^\lambda,$$

and

$$\psi(x,t) = \exp(-\beta(t)i\hat{H}_3) \exp\left(-\alpha(t)n\left(1 + \lambda(n-1)\right)\right) \prod_{j<k}^n (x_j - x_k)^\lambda.$$ 

In view of the expression of $\hat{H}_3$, it follows that

$$\psi(x,t) = \exp\left(-i \beta(t) \sum_{j=1}^n x_j^2 \right) \exp\left(-\alpha(t)n\left(1 + \lambda(n-1)\right)\right) \prod_{j<k}^n (x_j - x_k)^\lambda.$$ 

From equation (8.1), it turns out that $d\phi/dt(t) = \beta(t)\phi(t)$, with $\phi(t) := e^{-\alpha(t)/2}$. Hence,

$$\psi(x,t) = \exp\left(-i \beta(t) \sum_{j=1}^n x_j^2 \right) \phi(t)^{-n\left(1 + \lambda(n-1)\right)/2} \prod_{j<k}^n (x_k - x_j)^\lambda,$$

recovering, in this way, the particular solution given in [64]. Nevertheless, our approach is more general and our solution is not retrieved through a particular Ansatz as in previous literature.
9 Conclusions and outlook

We have proposed the theory of quantum quasi-Lie schemes as a way to transform an initial $t$-dependent Schrödinger equation into another one, which can be described by means of the usual theory of quantum Lie systems. This enables us to investigate a larger set of $t$-dependent Schrödinger equations than just those related to quantum Lie systems. We have also shown that the theory of quasi-Lie systems admits an equivalent generalisation to that of the theory of Lie systems to the quantum framework.

As a particular instance, we have applied our theory to answer a question made in a paper by Perelomov [55] about the possibility of relating a quantum $t$-dependent Hamiltonian for a nonlinear oscillator to a quantum $t$-independent one of the same type. We have also obtained the solution of a quantum one-dimensional fluid in a Paul trap.

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