Characteristic polynomials of random Hermitian matrices and Duistermaat-Heckman localisation on non-compact Kähler manifolds

Yan V Fyodorov and Eugene Strahov
Department of Mathematical Sciences, Brunel University
Uxbridge, UB8 3PH, United Kingdom
Yan.Fyodorov@brunel.ac.uk
Eugene.Strahov@brunel.ac.uk

Abstract

We reconsider the problem of calculating a general spectral correlation function containing an arbitrary number of products and ratios of characteristic polynomials for a $N \times N$ random matrix taken from the Gaussian Unitary Ensemble (GUE). Deviating from the standard "supersymmetry" approach, we integrate out Grassmann variables at the early stage and circumvent the use of the Hubbard-Stratonovich transformation in the "bosonic" sector. The method, suggested recently by one of us [19], is shown to be capable of calculation when reinforced with a generalization of the Itzykson-Zuber integral to a non-compact integration manifold. We arrive to such a generalisation by discussing the Duistermaat-Heckman localisation principle for integrals over non-compact homogeneous Kähler manifolds. In the limit of large $N$ the asymptotic expression for the correlation function reproduces the result outlined earlier by Andreev and Simons [14].

1 Introduction

Recently there was an outburst of research activity related to investigating the moments and correlation functions of characteristic polynomials $Z_N(\mu) = \det (\mu 1_N - \hat{H})$ for random $N \times N$ matrices $H$ of various types. Those studies were motivated by hope to relate statistics of zeroes of the Riemann zeta function to that of eigenvalues of large random matrices [1, 2, 3, 4, 12], as well as by numerous applications of spectral determinants in the theory of quantum chaotic and disordered systems [5, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22], quantum chromodynamics [21], and relations to interesting combinatorial problems [23].

There are several analytical techniques for dealing with the integer moments (positive or negative) of characteristic polynomials. Their applicability varies with the nature of the underlying random matrix ensemble. For unitary random matrices (the so-called "unitary circular ensemble") one can either relate evaluation of the moments to the Selberg-type integral [1, 2], or rely upon variants of the character expansion, directly [23] or indirectly [24]. Another well-studied case relates to the ensembles of Hermitian random matrices characterized by unitary invariant probability measures. That invariance allows for positive integer moments of the characteristic polynomials to be evaluated by methods resorting to orthogonal polynomials [4, 11]. The particular case of the Gaussian measure can be studied very efficiently by the following procedure. First, one represents each of the characteristic polynomials as a Gaussian integral over anticommuting (Grassmann) variables. This allows to average the resulting expressions immediately. At the next step one employs the so-called Hubbard-Stratonovich transformation combined with the subsequent exploitation of the Itzykson-Zuber-Harish-Chandra integral [24, 25]:

$$
\int_{g \in U(N)/T} d\mu(g) \exp \left[ i \operatorname{Tr} (X g Y g^\dagger) \right] = \text{const} \frac{\det \left[ \exp (ix^i y^k) \right]}{\Delta(X) \Delta(Y)} |_{1 \leq i, k \leq N} 
$$

(1)
Here $X, Y$ are diagonal matrices with eigenvalues $x^k, y^l$ correspondingly, where indices $k$ and $l$ take values from 1 to $N$. $\Delta(X), \Delta(Y)$ stand for the Vandermonde determinants and $T = U(1) \times \cdots \times U(1)$ is the maximal torus of the group $U(N)$. A detailed outline of the method can be found, e.g. in [2].

The evaluation of the negative integer moments of characteristic polynomials turns out to be more tricky as care must be taken to avoid divergences and account for presence of poles [19, 22]. The standard way goes back to the work by Schäfer and Wegner [27] and relies upon representing the (regularized) inverse determinants as the Gaussian integrals over commuting complex variables. This method then exploits an extension of the Hubbard-Stratonovich transformation to a non-trivial manifold with inherent "hyperbolic" structure. An alternative variant of the method was suggested in a recent paper by one of the present authors [19], referred as [I] henceforth. The latter work contains a detailed discussion of the problem as well as many related references.

A much more general correlation function of characteristic polynomials is one combining presence of both positive and negative integer moments. Such a correlation function contains a very detailed information about spectra of random matrices and thus it is most important for applications in physics. Correlation functions of that type are also interesting for the sake of comparison with more refined conjectures on the behavior of the Riemann zeta-function [24].

The standard technique in that case naturally combines Gaussian integrals over commuting and anticommuting variables and the subsequent Hubbard-Stratonovich transformation of usual and "hyperbolic" nature. The method is known in the literature as the supermatrix (or "supersymmetry") approach pioneered by Efetov [29] in the theory of disordered systems, and taken over to random matrices by Verbaarschot and Zirnbauer [30].

It should work, in principle, for the general case but technically any general calculation beyond the two-point correlation function proved to be extremely difficult. The main problem is related to the so-called "anomalous", or "boundary" terms arising when changing variables in the superintegrals. Those anomalies can be traced back to the admixture of nilpotent terms (those containing even number of grassmannian factors) to the usual commuting variables. All such anomalous terms were classified by Rothstein [31] in a general form. However, to write down their contribution explicitly in a specific parameterization proves to be a very daunting job. The latter fact makes the standard supersymmetric calculations usually impractical beyond a few lower-order correlation functions. A notable exception is the general many-point correlation function of spectral densities (see works by Zirnbauer [32] and Szabo [33]). In the latter case the boundary terms do not contribute and the final result is provided solely by the "bulk" integral which is evaluated by standard methods. At the same time our main object of interest - the correlator of spectral determinants - contains anomalous terms on equal footing with the bulk contribution.

A modification of the supermatrix method was suggested by Guhr [34]. It relied upon a generalization of the Itzykson-Zuber integral to a unitary supergroup and allowed one to go beyond the two-point function for the case of Gaussian Unitary Ensemble (GUE). A few years ago Andreev and Simons presented an asymptotic formula for a correlation function containing both products and ratios of characteristic polynomials for GUE matrices. In their short communication [14] they claimed that Guhr's method equipped with the further extension of the Itzykson-Zuber integral to a pseudounitary supergroup solved the problem. The authors indicated that the arguments behind that generalization were similar in spirit to those by Guhr, but "technically involved". They promised to present details of the method "in a longer paper" which, unfortunately, never appeared. Let us mention that the role of boundary contributions in the Guhr’s method seems to be not clearly discussed in the literature.

In the present paper we show that the method suggested in [I] enables one to calculate the general correlation function of integer moments of GUE characteristic polynomials. In fact, we
integrate out Grassmann variables at a very early stage thus seriously departing from the general spirit of supersymmetry. One of the advantages is that no anomalous terms can ever arise along such a route.

The calculation required, however, the knowledge of an analogue of the Itzykson-Zuber type integral over non-compact manifolds related to pseudo-unitary groups. A pseudo-unitary group $G$ is defined by the following conditions imposed on the group elements:

$$g^\dagger \lambda g = \lambda, \quad \forall g \in G$$  \hspace{1cm} (2)

Here $\lambda$ is a diagonal matrix with the elements $\pm 1$ in a matrix representation.

In the mathematical literature the integrals of the Itzykson-Zuber type over semi-simple Lie groups are interpreted as Fourier transforms over adjoint orbits $gXg^{-1}, g \in G$ and formulas for such integrals are known (see the works of Rossmann [36], Berline and Vergne [37], Prato and Wu [35], Paradan [38]). However the derivation of the Itzykson-Zuber type integrals over non-compact groups in a manner accessible to physicists was not presented before to the best of our knowledge.

A particular case of non-compact extension of the Itzykson-Zuber integral was presented recently in the Appendix C of [I]. The method of calculation, however, relied upon a specific parameterization of the integration manifold and seemed hardly applicable to a general non-compact case.

The standard procedure of the derivation of the Itzykson-Zuber type integrals is to employ the diffusion equation arguments. This method goes back to the original paper by Itzykson and Zuber [24] and was also used by Guhr [34], and, apparently by Andreev and Simons [14]. We provide such derivation for the Itzykson-Zuber type integral over the pseudo-unitary group in our Appendix A.

In this paper we consider in detail a different approach based on the Duistermaat-Heckman localisation principle [26] outlined in the context of random matrices by Zirnbauer [39]. Indeed, it is well known that the Itzykson-Zuber integral is a representative of the family of integrals of the form $\int \Omega^m \exp iH$ going over a $2m$-dimensional phase space. These integrals are semiclassically exact provided the Hamiltonian $H$ is "localisable". In the case of compact phase spaces localisability is equivalent to the condition for the phase flow generated by the Hamiltonian $H$ to preserve the Riemannian metric of the phase space. In particular, considering the manifold $U(N)/T$ as a phase space, the Itzykson-Zuber Hamiltonian

$$H(X,Y,g) = \text{Tr} (XgYg^\dagger)$$  \hspace{1cm} (3)

appears to be localisable in the sense of Duistermaat and Heckman.

From the geometrical point of view the underlying phase space $U(N)/T$ of the Itzykson-Zuber formula belongs to the family of compact flag manifolds. The flag manifolds are Kähler homogeneous spaces of the type $G/H$, where $G$ is called a transformation group and $H$ is the centralizer of a sub-torus of $T$ in the group $G$, i.e. $H = \{g \in G | g^{-1}T_0g = T_0\}$, where $T_0$ is a sub-torus of $T$. The compact flag manifolds were considered in details by Picken [15] in the context of the Duistermaat-Heckman formula. Picken has obtained an explicit expression for the set of localizable Hamiltonians on a compact flag manifold. Then the fact that the Itzykson-Zuber Hamiltonian is localisable follows as a particular case ($G = U(N), H = U(1) \times \ldots \times U(1)$).

The possibility to extend the Duistermaat-Heckman localisation principle to non-compact symplectic spaces was first discussed by Prato and Wu [35]. The case of the non-compact counterpart of the space $CP^N$ was considered by Fujii and Funahashi [55]. However the calculations done in [I] suggest to concentrate on more general homogeneous non-compact symplectic manifolds. We show that an extension of the Duistermaat-Heckman formula is possible when the homogeneous manifold under considerations, $G/H$ is a non-compact Kähler manifold. The key feature of such manifolds enabling one to apply the Duistermaat-Heckman theorem is that their invariant metrics
can be chosen to be sign-definite. In turn, the review paper by Bor demann, Forger and Romer provides conditions under which a homogeneous manifold turns out to be Kählerian. The conditions can be summarized as follows. Assume that the group $G$ is a connected and semisimple and the subgroup $H$ is a compact centralizer of a torus in $G$. Then

1. If $M = G/H$ is compact, $M$ is a Kähler manifold.

2. Let $M = G/H$ be non-compact and $L$ be the maximal compact subgroup of $G$ containing $H$. Then $M = G/H$ is a Kähler manifold if and only if $G/L$ is a Hermitian symmetric space.

In particular, the manifold $U(n_1, n_2)/T$ appears to be Kählerian in this case. Indeed, the maximal compact subgroup of $U(n_1, n_2)$ containing $H = T$ is $L = U(n_1) \times U(n_2)$. Since $U(n_1, n_2)/U(n_1) \times U(n_2)$ is a Hermitian symmetric space of the type $A_{III}$ (see Helgason [41], p.354) the manifold $U(n_1, n_2)/T$ is a Kähler manifold as follows from the criterions presented above.

Our general construction is then exploited to derive the Itzykson-Zuber type integral over a pseudo-unitary group

$$
\int_{g \in U(n_1, n_2)/T} d\mu(g) \exp \left[ i \text{Tr} \left( X g Y g^{-1} \right) \right] =
$$

$$
\text{const} \frac{\det \left[ \exp(ix^iy^k) \right] |_{1 \leq l,k \leq n_1} \det \left[ \exp(ix^iy^k) \right] |_{n_1+1 \leq l,k \leq n_1+n_2}}{\Delta(X)\Delta(Y)}
$$

(4)

It then can be seen that the remarkable formula of Harish-Chandra remains valid also for the non-compact group $U(n_1, n_2)$. Indeed, the righthand part of Eq.(4) can be rewritten as

$$
\text{const} \prod_{\alpha>0} \alpha(X) \prod_{\alpha>0} \alpha(Y) \sum_{w \in W} (-)^{|w|} \exp \left[ i \text{Tr} \left( Xw(Y) \right) \right]
$$

where $W$ is the Weyl group corresponding to $U(n_1, n_2)$, i.e. $W = S_{n_1} \times S_{n_2}$. The $\alpha(X)$ is a root corresponding to a Cartan subalgebra element $X$, $w(Y) = wYw^{-1}$ and $|w|$ denotes the parity of $w$.

The structure of the paper is as follows. In sections 2 and 3 we provide a necessary background information on Kähler geometry and discuss the Duistermaat-Heckman localisation on non-compact Kähler manifolds. Then in section 4 we give an account of (a refined version of) the method suggested in [1] for calculating the general correlation function of characteristic polynomials for the GUE matrices. In section 5 we analyse the derived matrix integral representation. For this purpose we use the formulae Eqs.(3) and (4) and evaluate the remaining integrals by the saddle-point method in the limit $N \to \infty$. The open questions are summarized in the Conclusions. Technical details are presented in the appendices.

## 2 Basic properties of Kähler manifolds

In this section we show how the Duistermaat-Heckman localisation principle can be reformulated for the Kählerian dynamical systems. These are dynamical systems whose phase spaces are (simply connected) homogeneous Kähler manifolds (for definitions and basic properties of homogeneous Kähler manifolds see, for example, Kobayashi and Nomizu [49]).

Throughout the paper we use a complex parameterization on flag manifolds which is introduced following Borel’s method [46]. A detailed exposition of the method can be found in the papers by
Bar-Moshe and Marinov \cite{47, 48}. Below we provide a concise description of the main features of the underlying structures.

Given a semi-simple Lie group $G$ we introduce the canonical Cartan-Weil basis for the corresponding complex semi-simple Lie algebra $g$: $\{\tau_a\} = \{h_j, e_{\pm q}\}$. Here $a = 1, 2, \ldots, n = \text{dim } g$; $j = 1, 2, \ldots, r = \text{rank } g$, and $\{q\} \in \triangle^+_G$ are the positive roots of the Lie algebra $g$. The complex parameters which are introduced in the flag manifold $G/T$ have the following decomposition:

$$g(z, \bar{z}) = u(z)p(z, \bar{z})$$

where we used the bar to denote the complex conjugation.

To find an explicit expression for $y^q$ and $k^j$ as functions of the complex coordinates $z^q, \bar{z}^q = \bar{z}$, one has to exploit the condition $g^I(z, \bar{z}) = g^{-1}(z, \bar{z})$ when the group $G$ is unitary, or the constraint (3) for the case of a pseudo-unitary group. The element $g(z, \bar{z}) = u(z)p(z, \bar{z})$ constructed in this way represents a point with the coordinates $(z^q, \bar{z}^q)$ on the flag manifold $G/H$. To provide a reader with a simple but informative example we show in the Appendix B an application of the general principles outlined in this section for the case of the compact manifold $U(2)/U(1) \times U(1)$ and its non-compact counterpart $U(1,1)/U(1) \times U(1)$.

Let us note that a complex parameterization introduced above can be looked at as a convenient way of describing the action of the transformation group $G$ on its flag manifold $G/H$. Indeed, for any $g \in G$ the (unique) decomposition $gu(z) = u(gz)p(z, g)$ allows one to find $gz$ and thus to determine the (holomorphic) action of the element $g$ of the transformation group $G$ on a point of the flag manifold $G/H$.

The homogeneous Kähler manifold $M$ comes with the Kähler potential $K(z, \bar{z})$, which is a scalar function defined on any open neighborhood of $M$ with local complex coordinates $z^\alpha$, $(\alpha = 1, 2, \ldots, m = \text{dim}_C M)$. When a group $G$ acts holomorphically on $M$,

$$z \to gz, \forall g \in G$$

the Kähler potential $K(z, \bar{z})$ is transformed as

$$K(z, \bar{z}) \to K(gz, \bar{g}z) = K(z, \bar{z}) + \Phi_g(z) + \Phi_g(z), \quad g \in G$$

where $\Phi_g(z)$ is a holomorphic function of $z$. Once the Kähler potential is provided, the exact $(1,1)$ differential form on the Kähler manifold is introduced as follows:

$$\Omega = \omega_{\alpha\bar{\beta}}(z, \bar{z}) dz^\alpha \wedge d\bar{z}^\beta, \quad \omega_{\alpha\bar{\beta}}(z, \bar{z}) = -\frac{1}{2\pi i} \partial_{\alpha} \partial_{\bar{\beta}} K(z, \bar{z})$$

where the factor $-1/(2\pi i)$ is chosen for convenience. Equations Eqs.\cite{3, 4, 5} show that the $(1,1)$ form $\Omega$ is invariant under the holomorphic action of the group $G$ on the homogeneous Kähler manifold $M$. When the phase space (Kähler manifold $M$) and the $(1,1)$ form are specified, the classical mechanics is defined by the Poisson brackets for any two smooth functions $F_1(z, \bar{z})$ and $F_2(z, \bar{z})$:

$$\{F_1(z, \bar{z}), F_2(z, \bar{z})\}_{PB} = w^{\alpha\bar{\beta}}(z, \bar{z}) \left( \partial_\alpha F_1(z, \bar{z}) \partial_{\bar{\beta}} F_2(z, \bar{z}) - \partial_\alpha F_2(z, \bar{z}) \partial_{\bar{\beta}} F_1(z, \bar{z}) \right)$$

\footnote{Throughout the paper we assume tensor notations, i.e. summation over repeating indices.}
where the antisymmetric field \( w^{\alpha\dot{\beta}}(z, \bar{z}) \) is inverse to \( w_{\alpha\dot{\beta}}(z, \bar{z}) \), i.e.

\[
    w^{\alpha\dot{\beta}}(z, \bar{z})w_{\nu\dot{\beta}}(z, \bar{z}) = \delta^\alpha_\nu, \quad w^{\nu\dot{\beta}}(z, \bar{z})w_{\nu\dot{\beta}}(z, \bar{z}) = \delta^{\dot{\beta}}_\beta
\] (10)

Provided that a Hamiltonian \( H(z, \bar{z}) \) is given, the equations of motion on the Kähler homogeneous manifold \( M \) can be written in terms of the Poisson brackets,

\[
dF(z, \bar{z})/dt = \{ F, H \}_{P.B.}
\] (11)

From Eqs.(7)-(11) we then obtain a time evolution of the complex coordinates:

\[
    \dot{z}^\alpha = w^{\alpha\dot{\beta}}(z, \bar{z})\partial_{\dot{\beta}}H, \quad \dot{z}^{\dot{\beta}} = -w^{\alpha\dot{\beta}}(z, \bar{z})\partial_{\alpha}H
\] (12)

In the formulation of the Duistermaat-Heckman localisation principle one uses an important notion of the Hamiltonian phase flow defined on the homogeneous Kähler manifold \( M \) as a one-parameter group \( g(t) \) of diffeomorphisms \( M \to M \) (see, for example, Arnold [50]):

\[
d/dt|_{t=0} \left( g(t) \cdot z^\alpha \right) = w^{\alpha\dot{\beta}}(z, \bar{z})\partial_{\dot{\beta}}H, \quad d/dt|_{t=0} \left( g(t) \cdot z^{\dot{\beta}} \right) = -w^{\alpha\dot{\beta}}(z, \bar{z})\partial_{\alpha}H
\] (13)

From the above definition and Eqs.(7), (12) it follows that the Kähler metric remains invariant under the holomorphic phase flow.

Let us now represent an arbitrary element of the Lie group \( G \) acting holomorphically on the Kähler manifold \( M \) by the set of cartesian coordinates:

\[
g(\xi) = \exp \left( \xi^a \tau_a \right), \quad a = 1, 2, \ldots, n = \dim g
\] (14)

where \( \tau_a \) are basis elements of the Lie algebra \( g \) of the group \( G \) satisfying the commutation relations

\[
    [\tau_a, \tau_b] = f_{ab}^c \tau_c,
\] (15)

with \( f_{ab}^c \) being the structure constants of the Lie algebra \( g \).

The Lie group \( G \) is itself a homogeneous space so the left action of the group on itself induces vector fields:

\[
    \tau_a \to D_a(\xi) = L^b_a(\xi)\partial_b^\xi
\] (16)

where \( \partial_b^\xi \) denotes a partial derivative with respect to the parameter \( \xi^b \).

The holomorphic action of the Lie group \( G \) on the Kähler manifold \( M \) \( z \in M \to gz \in M, \forall g \in G \) induces the vector fields of the form

\[
    \nabla_a(z, \bar{z}) = \kappa_a^\alpha(z)\partial_\alpha + \kappa_\alpha^a(z)\partial_\bar{z}
\] (17)

where the fields \( \kappa_a^\alpha(z) \) are expressed by the induced vector fields on the group:

\[
    \kappa_a^\alpha(z) = D_a(\xi) \left( g(\xi)z \right)^\alpha |_{\xi=0}
\] (18)

The conjugate fields \( \kappa_\alpha^a(z) \) are defined similarly. Once the \( \nabla_a(z, \bar{z}) \) are induced vector fields they also satisfy the commutation relations (15) of the Lie algebra \( g \).

Introduce now the linear operators \( i_a \) acting on the differential form \( \Omega \) as:

\[
i_a\Omega = w_{\alpha\dot{\beta}}(z, \bar{z})\kappa_a^\alpha(z)dz^{\dot{\beta}} - w_{\alpha\dot{\beta}}(z, \bar{z})\kappa_\alpha^a(z)d\bar{z}^{\dot{\beta}}
\] (19)
The linear operators $i_a$ are related to the Lie derivatives $L_{\nabla a}$ corresponding to the vector fields $\nabla_a(z, \bar{z})$:

$$L_{\nabla a} = di_a + i_a d$$

where $d$ stands for the usual external derivative operator acting on forms. The invariance of the $(1,1)$ Kähler form $\Omega$, Eq. (8) under the holomorphic action of the Lie group $G$ can be represented by the condition

$$L_{\nabla a} \Omega = 0, \quad \forall a = 1, 2, \ldots, n = \dim g$$

Since the Kähler form $\Omega$ is closed: $d\Omega = 0$ the condition Eq.(21) can be rewritten as

$$di_a \Omega = 0$$

As the Kähler manifold $M$ is assumed to be simply connected, and in simply connected spaces all closed forms are exact: $d\Omega = 0 \Rightarrow \Omega = d\omega$, the equation above implies the existence of $n = \dim g$ functions $T_a(z, \bar{z})$ (unique up to an additive constant) such that

$$i_a \Omega = (2\pi i)^{-1}dT_a$$

In terms of the local complex coordinates on the Kähler manifold $M$ differential equations (23) acquire the form:

$$\partial_\alpha T_a(z, \bar{z}) = -2\pi i \kappa^\beta_a(z) w_{\alpha\beta}(z, \bar{z}), \quad \partial_\beta T_a(z, \bar{z}) = 2\pi i \kappa^\alpha_a(z) w_{\alpha\beta}(z, \bar{z})$$

An immediate consequence of these equations is

$$\omega^{\alpha\beta}(z, \bar{z}) \partial_\alpha T_a(z, \bar{z}) = -2\pi i \kappa^\beta_a(z), \quad \omega^{\alpha\beta}(z, \bar{z}) \partial_\beta T_a(z, \bar{z}) = 2\pi i \kappa^\alpha_a(z)$$

which, in turn, implies the relations:

$$\partial_\beta (\omega^{\mu\nu}(z, \bar{z}) \partial_\mu T_a(z, \bar{z})) = 0, \quad \partial_\alpha (\omega^{\mu\nu}(z, \bar{z}) \partial_\mu T_a(z, \bar{z})) = 0$$

The functions $T_a(z, \bar{z})$ are called equivariant momentum maps, and we will see that they play an important role in providing the relation between the Kähler geometry and the Duistermaat-Heckman localisation principle. The explicit construction of the equivariant momentum maps $T_a(z, \bar{z})$ for the compact flag manifolds was performed by Bar-Moshe and Marinov\cite{47}\cite{48} and will be discussed later on in the present paper.

### 2.1 Duistermaat-Heckman localisation and equivariant momentum maps

Now we will obtain the conditions under which a Hamiltonian $H(z, \bar{z})$ on the Kähler manifold can be localisable in the sense of the Duistermaat-Heckman theorem. In this section we adopt the method developed previously by Bismut\cite{51}, Witten\cite{52}, Zirnbauer\cite{39} and apply it for the particular case of the Kähler manifolds.

We consider the integral

$$I = \int_M \Omega^m \exp(iH(z, \bar{z}))$$

where $M$ is a Kähler manifold on which the complex parameterization is introduced. When $M$ is a flag manifold it has a complex parameterization as discussed above, and the $(1,1)$ form $\Omega$ is given by Eq.(8), with $m$ being the complex dimension of the manifold $M$, $\dim C M = m$. Let us introduce
2m anticommuting variables \((\xi^\alpha, \bar{\xi}^{\bar{\alpha}})\) that are counterparts to commuting complex coordinates \((z^\alpha, \bar{z}^{\bar{\alpha}})\) of the manifold \(M\). Integral \((27)\) can be rewritten as that with a flat integration measure:

\[
I = \int \prod_{\alpha, \bar{\alpha} = 1}^m dz^\alpha dz^{\bar{\alpha}} \prod_{\beta, \bar{\beta} = 1}^m d\xi^\beta d\bar{\xi}^{\bar{\beta}} \exp S(z, \bar{z}, \xi, \bar{\xi})
\]  

where

\[
S(z, \bar{z}, \xi, \bar{\xi}) = iH(z, \bar{z}) - w_{\alpha\beta}(z, \bar{z})\xi^\alpha \bar{\xi}^{\beta}
\]

We will refer to the expression in the exponent as to the "action" depending on the commuting complex coordinates \((z^\alpha, \bar{z}^{\bar{\alpha}})\) of the manifold \(M\) and on the anticommuting variables \((\xi^\alpha, \bar{\xi}^{\bar{\alpha}})\).

Let us introduce a first-order differential operator \(D\) defined by the formula

\[
D = \xi^\alpha \partial_\alpha + \xi^{\bar{\alpha}} \partial_{\bar{\alpha}} - iw^{\alpha\bar{\beta}}(z, \bar{z}) \left( \partial_\alpha H(z, \bar{z})\partial_{\bar{\beta}} - \partial_{\bar{\beta}} H(z, \bar{z})\partial_\alpha \right)
\]

Using explicit expressions given above one can verify that such a differential operator annihilates the action \(S\), i.e. \(DS = 0\). Next step is to construct a function \(\lambda(z, \bar{z}, \xi, \bar{\xi})\) on the extended space which is annihilated by a repeated action of the operator \(D\), i.e. \(D^2\lambda = 0\). When such a function exists and integral \((27)\) converges it is possible to deform integral \((28)\) as follows,

\[
I \rightarrow I_t = \int \prod_{\alpha, \bar{\alpha} = 1}^m dz^\alpha dz^{\bar{\alpha}} \prod_{\beta, \bar{\beta} = 1}^m d\xi^\beta d\bar{\xi}^{\bar{\beta}} \exp \left( S(z, \bar{z}, \xi, \bar{\xi}) + tD\lambda(z, \bar{z}, \xi, \bar{\xi}) \right)
\]

where \(t\) is an arbitrary parameter. Indeed, one can expand the integrand in a series with respect to the parameter \(t\) and use the integration by parts together with the properties of the differential operator \(D\) \((DS = 0, D^2\lambda = 0)\) to verify that the integral \(I_t\) does not depend on the parameter \(t\), i.e \(I_t = I\). Following the general procedure described by Zirnbauer\((39)\) we make the following choice for the function \(\lambda(z, \bar{z}, \xi, \bar{\xi})\):

\[
\lambda(z, \bar{z}, \xi, \bar{\xi}) = i \left( \partial_\alpha H(z, \bar{z})\xi^\alpha - \partial_{\bar{\alpha}} H(z, \bar{z})\xi^{\bar{\alpha}} \right)
\]

The action of the first-order differential operator \(D\) on this function gives

\[
D\lambda(z, \bar{z}, \xi, \bar{\xi}) = -2 \left( w^{\alpha\bar{\beta}}(z, \bar{z}) \partial_\alpha H(z, \bar{z})\partial_{\bar{\beta}} - i\partial_{\bar{\alpha}} H(z, \bar{z})\partial_\alpha \right)
\]

A repeated action of the operator \(D\) on the function \(\lambda\) gives the following expression:

\[
D^2\lambda(z, \bar{z}, \xi, \bar{\xi}) = -2\xi^\alpha \partial_\alpha \left( w^{\mu\bar{\nu}}(z, \bar{z}) \partial_\mu H(z, \bar{z}) \right) + 2\bar{\xi}^{\bar{\alpha}} \partial_{\bar{\alpha}} \left( w^{\mu\bar{\nu}}(z, \bar{z}) \partial_\mu H(z, \bar{z}) \right)
\]

It immediately follows that the function \(\lambda(z, \bar{z}, \xi, \bar{\xi})\) satisfies \(D^2\lambda = 0\) if and only if the Hamiltonian \(H(z, \bar{z})\) satisfies the following conditions:

\[
\partial_\alpha \left( w^{\mu\bar{\nu}}(z, \bar{z}) \partial_\mu H(z, \bar{z}) \right) = 0, \quad \partial_{\bar{\beta}} \left( w^{\mu\bar{\nu}}(z, \bar{z}) \partial_\mu H(z, \bar{z}) \right) = 0
\]

The property of the Kähler cosets that ensures localisation is that one can always choose on them a sign-definite Riemann metric (see, for example, Kobayashi and Nomizu\((14)\) and a review article by Bordemann, Forger and Romer\((40)\)). When the metric is positive definite the numerical part of the expression for \(D\lambda\) proportional to

\[
(dH)^2 = w^{\alpha\bar{\beta}}(z, \bar{z}) \partial_\alpha H(z, \bar{z})\partial_{\bar{\beta}} H(z, \bar{z})
\]
is positive definite as well. It then follows that the limit \( t \to \infty \) localises the integral \( I_t \) in Eq.(31) on the critical set \( dH = 0 \). In turn, it implies that the original integral \( I \) is localized on the critical set of the Hamiltonian \( H \) as well. For the negative definite case one can just set \( t \to -\infty \) with the same result.

As is seen from the definition of the Hamiltonian phase flow on a Kähler manifold, Eq.(13), conditions Eq.(35) mean that the phase flow generated by the localizable Hamiltonian is holomorphic. Such phase flow preserves the Kähler metric, as follows from definition (13).

In particular, consider a Hamiltonian on the Kähler manifold \( M \) which can be represented as a linear combination of the momentum maps \( T_a(z, \bar{z}) \) satisfying the Eqs.(24),

\[
H(z, \bar{z}) = \sum_{a=1}^{n} c_a T_a(z, \bar{z}) \tag{37}
\]

Then the relation Eq.(25) ensures that such Hamiltonians conform to the conditions (35). Hence they are localisable in the sense of the Duistermaat-Heckman principle provided the integral (27) converges and \((dH)^2\) defined by equation (36) is sign-definite.

2.2 Localisable Hamiltonians on flag manifolds with unitary transformation group

Let us consider first the case of the flag manifold \( G/H \), with \( G \) being a unitary transformation group, \( G^\dagger = G^{-1} \). In any matrix representation there exist projection matrices \( \eta_j \) which correspond to the elements \( h_j \) of the Cartan subalgebra of the Lie algebra \( \mathfrak{g} \). The projection matrices are defined by the following set of equations

\[
\begin{align*}
\eta_j &= \eta_j^\dagger \\
\eta_j^2 &= \eta_j \\
\eta_j \hat{h}_k &= \hat{h}_k \eta_j \\
\eta_j \hat{e}_q \eta_j &= \hat{e}_{-q} \eta_j \\
\eta_j \hat{e}_q \eta_j &= \eta_j \hat{e}_q \\
\eta_j \hat{e}_{-q} \eta_j &= \eta_j \hat{e}_{-q} \\
\end{align*}
\tag{38}
\]

Here the hat stands for the matrix representation. When the Cartan subalgebra elements are represented by \( N \times N \) diagonal matrices, the projection matrices are also diagonal, and any \( N \times N \) diagonal matrix is a linear combination of the projection matrices and the unit \( N \times N \) matrix, \( \mathbf{1}_N \).

The projection matrices were introduced by Bando, Kuratomo, Maskawa and Uehara [53] and Itoh, Kugo and Kunitomo [54] to construct explicit formulae for the Kähler potentials. In particular it was found that in the case of a flag manifold with a unitary transformation group \( G \) the most general Kähler potential is a linear combination of scalar functions (the fundamental Kähler potentials). Each fundamental Kähler potential corresponds to a basis element of the Cartan subalgebra of the Lie algebra of the group \( G \). The explicit expression for the fundamental Kähler potential corresponding to the basis element \( h_i \) of the Cartan subalgebra is given by

\[
K_i(z, \bar{z}) = \ln \det \left( \eta_i u^\dagger(z) u(z) \eta_i + I - \eta_i \right), \quad u(z) = \exp(z^q \hat{e}_q) \tag{39}
\]

This formula was used by Bar-Moshe and Marinov [47, 48] to derive the equivariant momentum maps in terms of the local complex coordinates on a Kähler manifold. The expression obtained by Bar-Moshe and Marinov is

\[
T_a(z, \bar{z}) = -\text{Tr}(\rho(z, \bar{z}) \tau_a), \quad \rho(z, \bar{z}) = \sum_{i=1}^r l_i \rho_i(z, \bar{z}) \tag{40}
\]

where \( l_i \) are arbitrary constant coefficients and the matrices \( \rho_i(z, \bar{z}) \) are given by the following equation:

\[
\rho_i(z, \bar{z}) = u(z) \eta_i \left( \eta_i u^\dagger(z) u(z) \eta_i + I - \eta_i \right)^{-1} \eta_i u^\dagger(z), \quad u(z) = \exp(z^q \hat{e}_q) \tag{41}
\]
The Hamiltonians on Kähler manifolds which can be represented as a linear combination of the momentum maps are localisable. For the corresponding integrals the Duistermaat-Heckman formula is applicable as we have seen in the previous section. Using the expressions Eq. (40) for the momentum maps we thus obtain the following formula for the localisable Hamiltonians:

\[
H(z, \bar{z}) = \sum_{a=1}^{n} \sum_{i=1}^{r} v_{ai} \text{Tr} (\rho_i(z, \bar{z}) \hat{\tau}_a),
\]

where \(v_{ai}\) are arbitrary constant coefficients. The matrices \(\rho_i(z, \bar{z})\) are transformed under the action of the group \(G\) on the manifold \(M\) as

\[
\rho_i(z, \bar{z}) \to \rho_i(gz, g\bar{z}) = \rho_i(z, \bar{z}) g^\dagger, \quad g^\dagger = g^{-1}
\]

The above transformation law for the matrices \(\rho_i(z, \bar{z})\) can be verified using the decomposition

\[
gu(z) = u(z) p(z, g)
\]

which determines the action of the transformation group \(G\) on its flag manifold (see Appendix C for explicit calculation). Note that \(\rho_i(0, 0) = \eta_i\) and the projection matrix \(\rho_i(z, \bar{z})\) at the point with the complex coordinates \((z^a, \bar{z}^{\bar{a}})\) can be written as

\[
\rho_i(z, \bar{z}) = g(z, \bar{z}) \eta_i g^{-1}(z, \bar{z}).
\]

Here the group element \(g(z, \bar{z}) \in G\) represents a point of the coset space \(G/H\) parameterized as described earlier in this section.

A short inspection of expressions Eq. (42) and Eq. (44) for the particular case \(G = U(N)\) and \(H = T\) makes it clear that the localizable Hamiltonian \(H(z, \bar{z})\) is the same as that entering the Itzykson-Zuber integral, see Eq. (3).

The construction described above can be taken over to the case of pseudo-unitary transformation group without much modification.

### 3 Duistermaat-Heckman localization principle for manifolds with pseudo-unitary transformation group

We consider the flag manifolds \(G/T\) where the transformation group \(G\) is a pseudo-unitary Lie group with its elements satisfying the condition (2). Localisable Hamiltonians will be expressed as linear combinations of momentum maps similar to the case of the unitary transformation groups. The only new element is the presence of the matrix \(\lambda\). We begin with the formula for the fundamental Kähler potentials (compare with Eq. (39)):

\[
K_i(z, \bar{z}) = \ln \det \left( \eta_i u^\dagger(z) \lambda u(z) \eta_i + 1 - \eta_i \right), \quad u(z) = \exp(z^a e_a)
\]

Once the fundamental Kähler potentials are known it is possible to find the equivariant momentum maps in terms of the local complex coordinates on the given flag manifold. We have found that the equivariant maps have the same form as in the unitary case (see Eq. (40)), but the projection matrices \(\rho_i(z, \bar{z})\) turned out to be slightly different and are given by

\[
\rho_i(z, \bar{z}) = u(z) \eta_i \left( \eta_i u^\dagger(z) \lambda u(z) \eta_i + 1 - \eta_i \right)^{-1} \eta_i u^\dagger(z) \lambda
\]

We note that in the pseudounitary case the element \(u^\dagger(z)\) in the above expression comes together with the matrix \(\lambda\) in the same way as it does in the formula Eq. (13) for the fundamental Kähler potentials. Respectively, the Hamiltonians given by the formula Eq. (42) with \(\rho_i(z, \bar{z})\) specified by
Eq. (46) are localisable provided that the integral Eq. (45) converges. The transformation law for the matrices $\rho(z, \bar{z})$ turns out to be of the same form as one for the unitary case:

$$\rho(z, \bar{z}) \rightarrow \rho(g z, g \bar{z}) = g \rho(z, \bar{z}) g^{-1}, \quad g^\dagger \lambda g = \lambda$$

Then the same argumentation makes it evident that the natural counterpart of the Itzykson-Zuber Hamiltonian is

$$H(g, X, Y) = \text{Tr} \left( g(z, \bar{z}) X g^{-1}(z, \bar{z}) Y \right), \quad g(z, \bar{z}) \in U(n_1, n_2)/T$$

This Hamiltonian is localisable provided that the diagonal matrices $X, Y$ are chosen properly as to ensure that the integral Eq. (4) converges.

In what follows we apply the Duistermaat-Heckman theorem to the integral Eq. (4) with the Hamiltonian given by the expression Eq. (48) and evaluate it by using the method of stationary phase. We conventionally refer to this procedure as to the "semiclassical approximation", but the Duistermaat-Heckman localisation principle ensures that such an approximation yields the exact result.

The elements $g$ of the coset space $U(n_1, n_2)/T$ satisfy the condition Eq. (2) where $G = U(n_1, n_2)$, and the matrix $\lambda$ is given by

$$\lambda = \begin{pmatrix} I_{n_1} & 0 \\ 0 & -I_{n_2} \end{pmatrix}$$

For our calculation we make use of the same complex parameterization for the manifold $G = U(n_1, n_2)/T$ as that described in section 2. In the integral Eq. (4) the matrices $X, Y$ are diagonal, with the complex variables $x_1, \ldots, x_{n_1+n_2}$ and $y_1, \ldots, y_{n_1+n_2}$.

Let us find now the set of solutions for the equation $dH = 0$ (saddle points). We get:

$$dH = \text{Tr} \left( \delta g \left[ Y, g^{-1} X g \right] \right) = 0, \quad \delta g = g^{-1} dg$$

Note that the expression $\text{Tr}(x, y)$, with $x \in g, y \in g$ is a quadratic form on the Lie algebra $g$. Then from the condition $\text{Tr}(x, y) = 0$ follows that $x = 0$ or $y = 0$. We see that the equation (50) is equivalent to the condition of the Lie algebra elements $Y$ and $g^{-1} X g$ commuting with each other, i.e.

$$[Y, g^{-1} X g] = 0$$

Once the matrices $X, Y$ are diagonal all the saddle points on the homogeneous manifold $U(n_1, n_2)/T$ belong to the permutation group $S_{n_1+n_2}$ (which is the permutation group of the diagonal entries of $X$ and $Y$). Denote these saddle points by $P$. The saddle points $P$ should belong to the coset space $U(n_1, n_2)/H$ so they are elements of the pseudo-unitary group $U(n_1, n_2)$ and satisfy the following constraint:

$$P^\dagger \lambda P = \lambda, \quad P \in S_{n_1+n_2}$$

It is clear that only permutation matrices that do not mix the elements $+1$ and $-1$ of the matrix $\lambda$ could satisfy the condition Eq. (52). Thus the relevant saddle points are all elements of the permutation group $S_{n_1} \times S_{n_2}$, where $S_{n_1}$ stands for the permutation group of the top $n_1$ diagonal elements of the matrix $\lambda$, and $S_{n_2}$ is the permutation group of the rest $n_2$ diagonal entries of $\lambda$. Let us recall that in the case of unitary coset space the relevant saddle points were all possible elements of $S_{n_1+n_2}$. We conclude that the condition ensuring the saddle points to be elements of the pseudo-unitary group reduces the initial symmetry group $S_{n_1+n_2}$ of saddle points down to the symmetry group of the lowest order, $S_{n_1} \times S_{n_2}$.
In order to evaluate the integral Eq.(4) by the stationary phase method we rewrite it as a sum over contributions from neighborhoods of the saddle points

\[ I \equiv \int_{g \in U(n_1,n_2)/T} d\mu(g) \exp(iH(g,X,Y)) = \sum_{P \in S_n \times S_2} \int d\mu(g_P) \exp(iH(g_P,X,Y)) \] (53)

where \( g_P = g \cdot P \). Each integral in the sum above corresponds to one saddle point \( P \) and should be taken over a neighborhood of the point \( P \) on the manifold \( U(n_1,n_2)/T \). Consider the shift \( g_P = g \cdot P \) as a change of coordinates. Then it suffices to take the shift element \( g \) to be close to the unit element of the group. This allows one to explore the neighborhood of the chosen saddle point solution in line with the spirit of the Duistermaat-Heckman principle. Introducing the complex parameterization \( g = g(z,\bar{z}) \) we therefore take the complex parameters of the shift element \( g \) to be close to zero. We note that the coset measure is invariant under the group shifts,

\[ d\mu(g_P) = d\mu(g \cdot P) = d\mu(g) \] (54)

enabling us to use the same flat measure at each integral in the sum in the expression Eq.(53). Correspondingly, the Hamiltonian \( H(g_P,X,Y) \) is expanded in the vicinity of the relevant saddle point \( P \) separately at each of the integrals in the right hand side of Eq.(53).

For each \( g \in U(n_1,n_2)/T \) represented by a point from the coset space we introduce the local complex parameterization in the neighborhood of unity, \( g = I \). We have:

\[ g(0,0) = I, \quad g(z,\bar{z}) = I + g^{(1)}(z,\bar{z}) + g^{(2)}(z,\bar{z}) + O(|z|^3) \] (55)

(Here \( g^{(1)}(z,\bar{z}) = O(|z|) \) and \( g^{(2)}(z,\bar{z}) = O(|z|^2) \)). The condition of pseudounitarity of the transformation group underlying the given coset space yields:

\[ g^{-1}(z,\bar{z}) = I + \lambda \left(g^{(1)}(z,\bar{z})\right)^\dagger + \lambda g^{(2)}(z,\bar{z}) \lambda + O(|z|^3) \] (56)

Then it is straightforward to find the Lie algebra element \( g^{(1)}(z,\bar{z}) \) up to the lowest order in \( z,\bar{z} \) using the Cartan-Weyl basis of the Lie algebra \( g \):

\[ g^{(1)}(z,\bar{z}) = Z + N + Q \]

\[ Z = z^q e_q, \quad N = N^q(z,\bar{z}) e_{-q}, \quad Q = Q^j(z,\bar{z}) h_j \] (57)

The Lie algebra elements \( N \) and \( Q \) turn out to be of the first order in \( |z| \). Insert Eqs.(55), (56) with \( g^{(1)}(z,\bar{z}) \) given by Eq.(57) to the pseudo-unitary condition Eq.(2). We neglect the terms of the second order in \( |z| \) and find the Lie algebra elements \( N \) and \( Q \):

\[ N = -\lambda Z \dagger \lambda, \quad Q = 0 \] (58)

We see that the first-order Lie algebra elements in terms of the local complex coordinates are given by:

\[ g^{(1)}(z,\bar{z}) = Z - \lambda Z \dagger \lambda \equiv Z - Z \dagger \] (59)

\[ Z = \sum_{q=1}^{n_1+n_2} z^q e_q = \sum_{1 \leq i < j \leq n_1+n_2} z_{ij} e_{(ij)} \] (60)
where the standard basis elements \( e_{(ij)} \) of the Lie algebra are defined in the matrix representation by the formula:

\[
\{e_{(ij)}\}_{kl} = \delta_{ik} \delta_{jl} \tag{61}
\]

Now we should expand the Hamiltonian

\[
H(g_P(z, \bar{z})) = H(0) + H^{(1)}(z, \bar{z}) + H^{(2)}(z, \bar{z}) + O(|z|^3)
\]

in the vicinity of the saddle point \( P \). For this purpose we insert Eqs.(55), (56) into the Hamiltonian and obtain

\[
H(g_P(z, \bar{z})) = H(0) + H^{(1)}(z, \bar{z}) + H^{(2)}(z, \bar{z}) + O(|z|^3)
\]

where we have used the following notations:

\[
(g^{(1)}_\lambda)^\dagger \equiv \lambda(g^{(1)})^\dagger \lambda, \quad (g^{(2)}_\lambda)^\dagger \equiv \lambda(g^{(2)})^\dagger \lambda, \quad Y_P \equiv PYP^{-1}
\]

The explicit forms for \( H^{(1)} \) and \( H^{(2)} \) in terms of the local complex coordinates can be found from the pseudo-unitary condition \( \bar{P} \). We find that \( H^{(1)} = 0 \) and \( H^{(2)} \) can be written as

\[
H^{(2)}(z, \bar{z}) = \text{tr} \left( Xg^{(1)}(z, \bar{z})Y_P \left( g^{(1)}_\lambda(z, \bar{z}) \right)^\dagger \right) - \text{tr} \left( XYP \left( (g^{(2)}_\lambda(z, \bar{z}) \right) \right) \tag{65}
\]

We further observe that taking into account terms of first and second order with respect to \( |z| \) results in an expression for the Hamiltonian \( H(g_P(z, \bar{z})) \) which depends only on the element \( g^{(1)}(z, \bar{z}) \). Once the expression for \( g^{(1)}(z, \bar{z}) \) is found (Eq.(55)) we obtain from Eqs.(62) and (57):

\[
H(g_P(z, \bar{z})) = \text{tr} (XY_P) + \left( Z_\lambda^X X [ZY_P] + ZX \left[ Z_\lambda^X Y_P \right] \right) + O(|z|^3)
\]

Inserting \( X = \sum_{i=1}^{n_1+n_2} x^i e_{(ii)} \), \( Y_P = \sum_{i=1}^{n_1+n_2} y^i_P e_{(ii)} \) and \( Z = \sum_{1 \leq i < j \leq n_1+n_2} z_{ij} e_{(ij)} \) to the above equation and calculating the traces yields the Hamiltonian \( H(g_P(z, \bar{z})) \) up to the second order terms in \( |z| \):

\[
H(g_P(z, \bar{z})) = \sum_{i=1}^{n_1+n_2} x^i y^i_P - \sum_{1 \leq i < j \leq n_1} (x^i - x^j)(y^i_P - y^j_P) z_{ij} \bar{z}_{ij}
\]

\[
+ \sum_{1 \leq i < j \leq n_1+n_2} (x^i - x^j)(y^i_P - y^j_P) z_{ij} \bar{z}_{ij} - \sum_{1 \leq i < j \leq n_1+n_2} (x^i - x^j)(y^i_P - y^j_P) z_{ij} \bar{z}_{ij} + O(|z|^3) \tag{66}
\]

We use the above expansion of the Hamiltonian \( H(g_P(z, \bar{z})) \) in the vicinity of the saddle point \( P \) to obtain the semiclassical approximation for the integral Eq.(60):

\[
I = \sum_{P \in S_{n_1} \times S_{n_2}} \exp \left( i \sum_{j=1}^{n_1+n_2} x^j y^j_P \right) \int_{1 \leq i < j \leq n_1} dz_{ij} d\bar{z}_{ij} \exp \left( -i \sum_{1 \leq i < j \leq n_1} (x^i - x^j)(y^i_P - y^j_P) z_{ij} \bar{z}_{ij} \right)
\]

\[
13
\]
resent each of the characteristic polynomials in the denominator as the Gaussian integrals:

\[ \prod_{n_1+1 \leq i < j \leq n_1+n_2} \frac{dz_{ij} d\bar{z}_{ij}}{-2\pi i} \exp \left( -i \sum_{n_1+1 \leq i < j \leq n_1+n_2} (x^i - x^j)(y^i_F - y^j_F) z_{ij} \bar{z}_{ij} \right) \]

\[ \prod_{1 \leq i < j \leq n_1+n_2} \frac{dz_{ij} d\bar{z}_{ij}}{-2\pi i} \exp \left( i \sum_{1 \leq i < j \leq n_1+n_2} (x^i - x^j)(y^j_F - y^i_F) z_{ij} \bar{z}_{ij} \right) \]  \hspace{1cm} (68)

Calculating the Gaussian integrals and noting that for the Vandermonde determinantal factors \( \Delta(Y) = \prod_{i<j}(y_i - y_j) \) one has \( \Delta(Y_F) = (-)^c \Delta(Y) \), with \( c \) being equal to unity (zero) for even (odd) permutations, respectively, after straightforward manipulations we obtain the final formula:

\[ I = \left( -1 \right)^{n_1 n_2} \frac{\det \left( \exp(ix^k y^l) \right) \left|_{1 \leq k,l \leq n_1} \right. \det \left( \exp(ix^k y^l) \right) \left|_{n_1+1 \leq k,l \leq n_1+n_2} \right)}{\Delta(X) \Delta(Y)} \]  \hspace{1cm} (69)

4 Correlation function of characteristic polynomials: general formalism

In the present section we derive an integral representation for the correlation function of characteristic polynomials of the GUE matrices suitable for further investigation in the limit of large matrix dimensions. Our method is a refined version of that introduced in [1].

Let \( \hat{H} \) be \( N \times N \) random Hermitian matrix \( \hat{H} = \hat{H}^\dagger \) which is characterized by the standard (GUE) joint probability density:

\[ \mathcal{P}(\hat{H}) = C_N \exp \left( -\frac{N}{2} \text{Tr} \hat{H}^2 \right), \quad C_N = (2\pi)^{-\frac{N(N+1)}{4}} N^{N^2/2} \]  \hspace{1cm} (70)

with respect to the measure \( d\hat{H} = \prod_{i=1}^N dH_{ii} \prod_{i<j} dH_{ij} d\bar{H}_{ij} \), where as usual \( dzd\bar{z} = 2d\text{Re}z d\text{Im}z \).

Regularizing the characteristic polynomial \( Z_N(\mu) = \det \left( \mu 1_N - \hat{H} \right) \) by considering the spectral parameter \( \mu \) such that \( \text{Im} \mu \neq 0 \) we are interested in calculating the following generating function:

\[ K_N(\bar{\mu}_B, \bar{\mu}_F) = \left( \prod_{k=1}^{F} \frac{Z_N(\mu^{(1)}_1) Z_N(\mu^{(2k)}_F)}{Z_N(\mu^{(1)}_{1B}) Z_N(\mu^{(2k)}_{2B})} \right) \]  \hspace{1cm} (71)

where we denote by \( \langle ... \rangle \) the expectation value with respect to the distribution Eq.(70) and

\[ \bar{\mu}_B = \text{diag}(\mu^{(1)}_{1B}, ..., \mu^{(n_B)}_{1B}), \mu^{(1)}_{2B}, ..., \mu^{(n_B)}_{2B}) \], \[ \bar{\mu}_F = \text{diag}(\mu^{(1)}_{1F}, ..., \mu^{(n_F)}_{1F}, \mu^{(1)}_{2F}, ..., \mu^{(n_F)}_{2F}) \]

and \( \text{Im}(\mu^{(1)}_{1B}, -\mu^{(2)}_{2B}) > 0 \).

The generating function is obviously an analytic one with respect to the complex variables \( (\mu^{(1)}_{1F}, \mu^{(2k)}_{2F}) \). It turns out to be technically convenient to change: \( \mu^{(1)}_{1F} \rightarrow -i \mu^{(1)}_{1F}, \mu^{(2k)}_{2F} \rightarrow -i \mu^{(2k)}_{2F} \) when performing the ensemble averaging, and restore the original generating function by a simple analytical continuation.

To calculate the average we first use the standard “supersymmetrisation” procedure and represent each of the characteristic polynomials in the denominator as the Gaussian integrals:

\[ [Z_N(\mu)]^{-1} = \frac{1}{(4\pi i)^N} \int d^2 S \exp \left\{ \pm \frac{i}{2} \left( \mu S^{-1} S^\dagger \hat{H} S \right) \right\} \]  \hspace{1cm} (72)
where we introduced a complex $N$-dimensional vector $S = (s_1, ..., s_N)^T$ (here $T$ stands for the vector transposition) so that $d^2S = \prod_{i=1}^{N} ds_i d\sigma_i$. The sign $\pm$ in the exponential is coordinated with the sign of $\text{Im} a$ to ensure convergence of the integral.

For the characteristic polynomials in the numerator we use the representation in terms of the Gaussian integrals over anticommuting (Grassmannian) $N$-component vectors $\chi, \chi^\dagger$, see e.g. [30]. Taking the product of all the integrals, the generating function can be written down in the following form:

$$K_N(\bar{\mu}_b, \bar{\mu}_f) \propto \int \prod_{k}^{n_F} d\chi_{k,1} d\chi^\dagger_{k,1} \int \prod_{k}^{n_F} d\chi_{k,2} d\chi^\dagger_{k,2} \exp \left\{ \frac{1}{2} \sum_{k}^{n_F} \left( \mu_{1F}^{(k)} \chi_{k,1} \chi^\dagger_{k,1} + \mu_{2F}^{(k)} \chi_{k,2} \chi^\dagger_{k,2} \right) \right\}$$

$$\times \int d^2 \mathbf{s}_{1,1} d^2 \mathbf{s}_{1,2} \exp \left\{ \frac{i}{2} \sum_{l=1}^{n_B} \left[ \mu_{1B} \mathbf{s}_{l,1} \mathbf{s}_{l,1}^\dagger - \mu_{2B} \mathbf{s}_{l,2} \mathbf{s}_{l,2}^\dagger \right] \right\}$$

$$\times \left\langle \exp \left\{ -\frac{i}{2} \text{Tr} \hat{H} \left\{ \sum_{l=1}^{n_B} \left( \mathbf{s}_{l,1} \otimes \mathbf{s}_{l,1}^\dagger - \mathbf{s}_{l,2} \otimes \mathbf{s}_{l,2}^\dagger \right) + \frac{1}{2} \sum_{k}^{n_F} \left( \chi_{k,1} \otimes \chi^\dagger_{k,1} + \chi_{k,2} \otimes \chi^\dagger_{k,2} \right) \right\} \right\rangle_{\text{GUE}}$$

The ensemble average is then easy to perform via the identity:

$$\left\langle e^{-\frac{i}{4} \text{Tr} [\hat{H} \hat{A}]} \right\rangle_{\text{GUE}} \propto e^{-\frac{1}{16} \text{Tr} [\hat{A}^2]}$$

and after straightforward manipulations one arrives at:

$$K_N(\bar{\mu}_b, \bar{\mu}_f) \propto \int \prod_{k}^{n_F} d\chi_{k,1} d\chi^\dagger_{k,1} \int \prod_{k}^{n_F} d\chi_{k,2} d\chi^\dagger_{k,2} \exp \left\{ \frac{1}{2} \sum_{k}^{n_F} \left( \mu_{1F}^{(k)} \chi_{k,1} \chi^\dagger_{k,1} + \mu_{2F}^{(k)} \chi_{k,2} \chi^\dagger_{k,2} \right) + \frac{1}{8N} \text{Tr} \left\{ \hat{Q}_F^2 \right\} \right\}$$

$$\times \int d^2 \mathbf{s}_{1,1} d^2 \mathbf{s}_{1,2} \exp \left\{ \frac{i}{2} \sum_{l=1}^{n_B} \left[ \mu_{1B} \mathbf{s}_{l,1} \mathbf{s}_{l,1}^\dagger - \mu_{2B} \mathbf{s}_{l,2} \mathbf{s}_{l,2}^\dagger \right] - \frac{1}{8N} \text{Tr} \left\{ \hat{Q}_B \hat{L} \hat{Q}_B^\dagger \hat{L}^\dagger \right\} \right\}$$

$$\times \exp \left\{ -\frac{1}{4N} \sum_{k=1}^{n_F} \sum_{p=1}^{n_B} \chi^\dagger_{k,p} \left( \sum_{l=1}^{n_B} \mathbf{s}_{l,1} \otimes \mathbf{s}_{l,1}^\dagger - \mathbf{s}_{l,2} \otimes \mathbf{s}_{l,2}^\dagger \right) \chi_{k,p} \right\}$$

where we introduced the following $2n_F \times 2n_F$ Hermitian matrices $\hat{Q}_F$ and $2n_B \times 2n_B$ Hermitian matrices $\hat{Q}_B$:

$$\hat{Q}_F = \left( \begin{array}{cc} \hat{Q}_F^{(11)} & \hat{Q}_F^{(12)} \\ \hat{Q}_F^{(12)^\dagger} & \hat{Q}_F^{(11)^\dagger} \end{array} \right), \quad \hat{Q}_B = \left( \begin{array}{cc} \hat{Q}_B^{(11)} & \hat{Q}_B^{(12)} \\ \hat{Q}_B^{(12)^\dagger} & \hat{Q}_B^{(11)^\dagger} \end{array} \right)$$

with entries

$$\left[ \hat{Q}_F^{(\mathbf{p}_1, \mathbf{p}_2)} \right]_{l_1 l_2} = \mathbf{s}_{l_1, \mathbf{p}_1} \mathbf{s}_{l_2, \mathbf{p}_2}^\dagger, \quad \left[ \hat{Q}_F^{(q_1, q_2)} \right]_{k_1 k_2} = \chi^\dagger_{k_1, q_1} \chi_{k_2, q_2}$$

and used the notation $\hat{L} = \text{diag}(1_{n_B}, -1_{n_B})$

Now we employ the Hubbard-Stratonovich identity:

$$\exp \left\{ \frac{1}{8N} \text{Tr} \hat{Q}_F^2 \right\} \propto \int d\hat{Q}_F \exp \left\{ -\frac{N}{2} \text{Tr} \hat{Q}_F^2 - \frac{1}{2} \left( \chi_{k,1} \chi^\dagger_{k,1} \right) \hat{Q}_F^T \left( \chi_{k,2} \chi^\dagger_{k,2} \right) \right\}$$

where the integration goes over the manifold of $2n_F \times 2n_F$ Hermitian matrices $\hat{Q}_F$ with the symmetry structure inherited from that of $\hat{Q}_F$ and we introduced the shorthand notation:

$$\left( \chi_{k,1} \chi^\dagger_{k,2} \right) \equiv \left( \chi_{1,1}^\dagger, ..., \chi_{n_p,1}^\dagger, \chi_{1,2}^\dagger, ..., \chi_{n_p,2}^\dagger \right)$$
Exploiting the above identity allows one to perform the gaussian Grassmannian integral explicitly and to bring the expression to the form:

$$K_N(\hat{\mu}_F, \hat{\mu}_f) \propto \int d\tilde{Q}_F e^{-\frac{i}{2} \text{Tr} \tilde{Q}_F^2} \int \prod_{l=1}^{n_B} d^2 \text{S}_{l,1} d^2 \text{S}_{l,2} \exp \left\{ i \sum_{l=1}^{n_B} \left( \mu^{(l)}_{1,B} \text{S}_{l,1}^\dagger \text{S}_{l,1} - \mu^{(l)}_{2,B} \text{S}_{l,2}^\dagger \text{S}_{l,2} \right) \right\}$$

$$\times e^{-\frac{i}{2} \text{Tr} (\tilde{Q}_B \tilde{L} \tilde{Q}_B^\dagger) \det \hat{A}_B}$$

where

$$\hat{A}_B = \begin{pmatrix}
a_{1,1,1} \text{I}_N - \hat{B} & a_{1,2,1} \text{I}_N & \cdots & a_{1,2,n_F} \text{I}_N \\
a_{2,1,1} \text{I}_N & a_{2,2,1} \text{I}_N - \hat{B} & \cdots & a_{1,2,n_F} \text{I}_N \\
\vdots & \vdots & \ddots & \vdots \\
a_{1,2,n_F} \text{I}_N & a_{2,2,n_F} \text{I}_N & \cdots & a_{2,n_F} \text{I}_N - \hat{B}
\end{pmatrix}$$

We have used notations $a_{k_1,k_2} = \mu_F^{(k)} \delta_{k_1,k_2} - \langle q_F \rangle_{k_1,k_2}$, for $k = 1, 2, \ldots, n_F$ and introduced the $N \times N$ matrix $\hat{B} = \frac{1}{2N} \sum_{l=1}^{n_B} \left[ \text{S}_{l,1} \otimes \text{S}_{l,1}^\dagger - \text{S}_{l,2} \otimes \text{S}_{l,2}^\dagger \right]$.

To bring the determinant of the matrix $\hat{A}_B$ in the above exponent to the form suitable for further manipulations we consider the case $N \geq 2n_B$ and further introduce the $N \times N$ matrix

$$\hat{M}_N = \left( \hat{M}, e_1, e_2, \ldots, e_{N-2n_B} \right)$$

where the $N$-component orthonormal vectors $e_1, e_2, \ldots, e_{N-2n_B}$ are chosen to form a basis of the orthogonal complement to the linear span of vectors $\text{S}_{1,1}, \ldots, \text{S}_{n_B,1}, \text{S}_{1,2}, \ldots, \text{S}_{n_B,2}$ (without restricting generality we can consider the latter vectors to be linear independent). Correspondingly, the matrix $\hat{M}$ is chosen to have $2n_B$ columns which are just the vectors $\text{S}_{1,1}$ and $\text{S}_{l,2}$. Simple calculation shows that

$$\hat{M}^\dagger \hat{M} = \tilde{Q}_B \quad \text{and} \quad \hat{M}^\dagger \hat{B} \hat{M} = \frac{1}{2N} \tilde{Q}_B \tilde{L} \tilde{Q}_B$$

Then we can write:

$$\det \begin{pmatrix}
\hat{M}_N^\dagger & 0 \\
0 & \hat{M}_N^\dagger \\
\vdots & \vdots & \ddots & \vdots \\
0 & \hat{M}_N^\dagger
\end{pmatrix} \times \det (A_B) \times \det \begin{pmatrix}
\hat{M}_N & 0 \\
0 & \hat{M}_N \\
\vdots & \vdots & \ddots & \vdots \\
0 & \hat{M}_N
\end{pmatrix}$$

$$= \det a_{k_1,k_2} \left( \hat{Q}_B 1_{N-2n_B} \right) - \delta_{k_1,k_2} \frac{1}{2N} \left( \tilde{Q}_B \tilde{L} \tilde{Q}_B 1_{N-2n_B} \right)$$

$$= \det \left( a_{k_1,k_2} \hat{Q}_B - \delta_{k_1,k_2} \frac{1}{2N} \hat{Q}_B \hat{L} \hat{Q}_B \right)$$

Now it is easy to see that the determinant factor $\det \hat{A}_B$ is given by

$$\det \hat{A}_B = \det \left[ \hat{\mu}_F - \hat{Q}_F \right] \prod_{k=1}^{2n_B} \det \left[ q^{(p)}_{k,F} 1_{2n_b} - \frac{1}{2N} \hat{Q}_B \hat{L} \right]$$

where we introduced the notation $q^{(p)}_{k,F}$ for (real) eigenvalues of the (Hermitian) matrix $\hat{Q}^{(p)}_F = \hat{\mu}_F - \hat{Q}_F$. 

16
Next step is to deal with the integrals over $S_{l1}, S_{l2}$. For this we observe that the integrand depends on those variables only via the matrix $\tilde{Q}_B$ and employ the following

**Theorem I**

Consider a function $F(S_1, \ldots, S_m)$ of $N$-component complex vectors $S_l \ 1 \leq l \leq m$ such that

$$\int_{C^N} d^2 S_1 \ldots \int_{C^N} d^2 S_m |F(S_1, \ldots, S_m)| < \infty \quad (80)$$

Suppose further that the function $F$ depends only on $m^2$ scalar products $S_{l1}^t S_{l2} \ 1 \leq l_1, l_2 \leq m$ so that it can be rewritten as a function $F(Q_m)$ of $m \times m$ Hermitian matrix $Q_m$:

$$\tilde{Q}_m = \left( \begin{array}{ccc} S_{11}^t & S_{12}^t & \ldots & S_{1m}^t \\ S_{21}^t & S_{22} & \ldots & S_{2m}^t \\ \ldots & \ldots & \ldots & \ldots \\ S_{m1}^t & S_{m2} & \ldots & S_{mm} \end{array} \right)$$

Then for $N \geq m$

$$\int_{C^N} d^2 S_1 \ldots \int_{C^N} d^2 S_m F(S_1, \ldots, S_m) = C_{N,m} \int_{Q_m > 0} d\tilde{Q}_m \left( \det\tilde{Q}_m \right)^{N-m} F(Q_m) \quad (81)$$

where

$$C_{N,m} = \frac{(2\pi)^{Nm - \frac{m(m-1)}{2}}}{\prod_{k=1}^{m} (N-k)!}$$

and the integration in the right-hand side of Eq. (81) goes over the manifold of Hermitian positive definite $m \times m$ matrices $Q_m$.

In fact, the formula Eq. (81) was already implicitly used in [I]. In that paper it was justified by heuristic arguments employing the Fourier transform of the function $F$ and subsequent exploitation of a matrix integral close to that considered by Ingham and Siegel [2]. A proof of the theorem is given in the Appendix D of the present paper.

In our particular case $m = 2n_B$ and the role of $\tilde{Q}_m$ is played by $\tilde{Q}_B$. The convergency condition $\int_{C^N} d^2 S_1 \ldots \int_{C^N} d^2 S_m$ is ensured by imaginary parts of the spectral parameters $\mu$. Applying the theorem, we get:

$$\kappa_N(\hat{\mu}_B, \hat{\mu}_F) \propto \int d\tilde{Q}_F \left( \det\tilde{Q}_F^{(\mu)} \right)^{N-2n_B} e^{-\frac{N}{2}\text{Tr}\tilde{Q}_F^2}$$

$$\times \int_{Q_B > 0} d\tilde{Q}_B \det\tilde{Q}_B^{N-2n_B} e^{-\frac{N}{2}\text{Tr}(\tilde{Q}_B L)^2 + \frac{N}{2}\text{Tr}[\hat{\mu}_B \tilde{Q}_B L]} \prod_{k=1}^{2n_F} \det \left[ q_k^{(\mu)} 1_{2n_B} - \frac{1}{2N} \tilde{Q}_B L \right] \quad (82)$$

Let us now replace the Hermitian matrix $\tilde{Q}_F$ by $\tilde{Q}_F^{(\mu)}$ as the integration manifold and further change: $\tilde{Q}_B \to 2N\tilde{Q}_B$. Omitting both tilde and $(\mu)$ symbols henceforth we arrive at:

$$\kappa_N(\hat{\mu}_B, \hat{\mu}_F) \propto \int dQ_F (\text{det}Q_F)^{N-2n_B} e^{-\frac{N}{2}\text{Tr}[\hat{\mu}_F - Q_F]^2}$$

$$\times \int_{Q_B > 0} dQ_B (\text{det}Q_B)^{N-2n_B} e^{-\frac{N}{2}\text{Tr}(Q_B L)^2 + iN\text{Tr}[\hat{\mu}_B Q_B L]} \det \left[ Q_F \otimes 1_{2n_B} - 1_{2n_B} \otimes Q_B L \right] \quad (83)$$

$^2$After completion of our work we learned that an equivalent formula was used earlier by David, Duplantier and Guitter [4] in quite a different context.
The unitary group is exactly that by Itzyon-Zuber-Harish-Chandra, Eq. (1). This yields:

\[
\text{ Standing for the Vandermonde determinant (see e.g. [43]).}
\]

\[
\text{dQ} = \text{integration measure in those variables is known to be written by}
\]

\[
U
\]

\[
\text{therein. The matrices satisfy}
\]

\[
N
\]

\[
\text{of these (non-Hermitian!) matrices are discussed at length in the Appendix B of [I], and references}
\]

\[
\text{relevant variables are real eigenvalues}
\]

\[
\text{where the last factor is the invariant measure on the coset space of}
\]

\[
\text{by us in Section I (cf. Eq.(4)).}
\]

\[
\text{To perform the saddle-point evaluation of the integrals we first have to expose those degrees of freedom which are amenable to such a treatment. It is immediately evident that for the matrix Q}_F \text{ the relevant variables are real eigenvalues } -\infty < q_k < \infty, 1 \leq k \leq 2n_F. \text{ Accordingly, we write } Q_F = U \hat{q}_F U^\dagger, \text{ where } U \in U(2n_F) \text{ is } 2n_F \times 2n_F \text{ unitary matrix, and } \hat{q}_F = \text{diag}(q_1, ..., q_{2n_F}). \text{ The integration measure in those variables is known to be written by } dQ_F \propto \Delta^2(\hat{q}_F) d\mu(U) d\hat{q}_F, \text{ with } d\mu(U) \text{ being the corresponding Haar’s measure on the group } U(2n_F) \text{ and } \Delta(\hat{q}_F) = \prod_{k_1 < k_2} (q_{k_1} - q_{k_2}) \text{ standing for the Vandermonde determinant (see e.g. [43]).}
\]

The only term in the integrand of Eq. (83) which depends on the unitary matrix \(U\) is obviously the exponential \(NTr(\hat{\mu}_F U \hat{q}_F U^\dagger)\). We immediately see that the corresponding integral over the unitary group is exactly that by Itzyon-Zuber-Harish-Chandra, Eq. (I). This yields:

\[
\mathcal{K}_N(\hat{\mu}_B, \hat{\mu}_F) \propto e^{-\frac{1}{2} \text{Tr}(\hat{\mu}_F^2)} \int d\hat{q}_F \Delta(\hat{q}) (\text{det}\hat{q}_F)^{N-2n_B} \det\left[ e^{N\hat{\mu}_B^k q_{k_2}} \right]_{k_1, k_2 = 1}^{2n_F} (84)
\]

\[
\times e^{-\frac{1}{2} \text{Tr}(\hat{q}_F^2)} \int_{Q_B > 0} dQ_B (\text{det}Q_B)^{N-2n_F} e^{-\frac{1}{2} \text{Tr}(Q_B L^2) + i N Tr[\hat{\mu}_B Q_B L]} \prod_{k=1}^{2n_F} \text{det}\left[ q_k 1_{2n_B} - Q_B L \right]
\]

It is of little utility, however, to introduce eigenvalues/eigenvectors of \(Q_B > 0\) as the integration variables. Rather, it is natural to treat \(Q_B^{(L)} = Q_B \hat{L}\) as a new matrix to integrate over. Properties of these (non-Hermitian!) matrices are discussed at length in the Appendix B of [I], and references therein. The matrices satisfy \(Q_B^{(L)\dagger} = \hat{L} Q_B^{(L)} \hat{L}\), have all eigenvalues real and can be diagonalized by a (pseudounitary) similarity transformation: \(Q_B^{(L)} = \hat{T} \hat{\rho}_B \hat{T}^{-1}\), where \(\hat{\rho}_B = \text{diag}(\hat{\rho}_1, \hat{\rho}_2)\), and \(n_B \times n_B\) diagonal matrices \(\hat{\rho}_1, \hat{\rho}_2\) satisfy: \(\hat{\rho}_1 > 0, \hat{\rho}_2 < 0\). Pseudounitary matrices \(\hat{T}\) satisfy: \(\hat{T}^\dagger \hat{L} \hat{T} = \hat{L}\) and form the group \(U(n_B, n_B)\) ("hyperbolic symmetry").

We again introduce the diagonal entries \(\hat{\rho}_1\) and \(\hat{\rho}_2\) along with the matrices \(\hat{T} \in U(n_B, n_B) / U(1) \times ... \times U(1)\) as new integration variables. The integration measure \(dQ_B^{(L)}\) is given in new variables as \(30\):

\[
dQ_B^{(L)} \propto d\hat{\rho}_1 d\hat{\rho}_2 \prod_{l_1 < l_2}^{n_B} \left( p_{1(l_1)} - p_{1(l_2)} \right)^2 \left( p_{2(l_1)} - p_{2(l_2)} \right)^2 \prod_{l_1, l_2} \left( p_{1(l_1)} - p_{2(l_2)} \right)^2 \text{d}\mu(T)
\]

where the last factor is the invariant measure on the coset space of \(T\)-matrices.

Again, the only term in the integrand of Eq. (83) which depends on the pseudounitary matrices \(\hat{T}\) is obviously the exponential \(NTr(\hat{L} \hat{\rho}_B \hat{T} \hat{\rho}_B \hat{T}^{-1})\). We immediately see that the corresponding integral over the non-compact ("hyperbolic") manifold of \(T\)-matrices is exactly that addressed by us in Section I (cf. Eq. (I)).

\[
I(\hat{\rho}_B, \hat{\rho}_1, \hat{\rho}_2) = \int d\mu(\hat{T}) \exp\left\{ i NTr\left( \hat{\rho}_B \hat{T} \right) \hat{T} \right\} \hat{\rho}_1 \hat{\rho}_2 \hat{T}^{-1} \right\} (85)
\]
the characteristic polynomials
and perform similar replacements for the other two determinants in Eq.(86).

\[ \Delta \{ \hat{p}_2 \} = \hat{p}_1 - \hat{p}_2 \prod_{l_1,l_2} (\hat{p}_1^{(l_1)} - \hat{p}_2^{(l_2)}) \]

As a final step we change \( \hat{p}_2 \to -\hat{p}_2 \) and with the integrand depending only on the eigenvalues, we arrive to the following expression:

\[
\mathcal{K}_N(\hat{\mu}_B, \hat{\mu}_F) \propto e^{-\hat{\Psi} \text{Tr}[\hat{\rho}_F]^{2}} \int_{R_{+}} d\hat{\rho}_2 \Delta \{ \hat{\rho}_2 \} (\text{det} \hat{\rho}_F)^{N-2n_B} \det \left[ e^{iN \hat{\mu}_F^{(l_2)} \hat{p}_2^{(l_2)}} \right]_{l_1,l_2=1}^{n_B} \]

\[
\times \text{det} \left[ e^{-iN \hat{\mu}_F^{(l_1)} \hat{p}_1^{(l_1)}} \right]_{l_1,l_2=1}^{n_B} \prod_{k=1}^{2n_F} \det [q_k 1_{n_B} - \hat{p}_1] \det [q_k 1_{n_B} + \hat{p}_2] \tag{86} \]

where we denoted \( R_{+}^{(F)} \) the integration domain: \( R_{+}^{(F)} = (-\infty, \infty), k = 1, \ldots, 2n_F \) and \( R_{+}^{(B)} \) the domain \( 0 \leq p^{(l)} < \infty \) for \( l = 1, 2, \ldots, n_B \). Taking into account presence of the Vandermonde determinant antisymmetric in all \( q \)'s as well as the symmetry of the rest of the integrand with respect to \( (2n_F)! \) permutations of entries of the matrix \( \hat{\rho}_F = \text{diag}(q_1, \ldots, q_{2n_F}) \) we see that we can effectively replace the determinant factor:

\[
\text{det} \left[ e^{N \hat{\mu}_F^{(k_1)} q_{k_2}} \right]_{k_1,k_2=1}^{2n_F} \to (2n_F)!e^{N \sum_{k=1}^{2n_F} \mu^{(k)} q_k} \]

and perform similar replacements for the other two determinants in Eq.(86).

Summing up, we derived the following integral representation for the correlation functions of the characteristic polynomials

\[
\mathcal{K}_N(\hat{\mu}_B, -i\hat{\mu}_F) = \left\langle \prod_{k=1}^{2n_F} Z_N(-i\hat{\mu}_F^{(k)}) Z_N(-i\hat{\mu}_F^{(k)}) \prod_{l=1}^{n_B} Z_N(\hat{\mu}_F^{(l)}) Z_N(\hat{\mu}_F^{(l)}) \right\rangle_{GUE} \]

\[
= \frac{1}{\Delta \{ \hat{\mu}_F \} \Delta \{ \hat{\mu}_B \} \int_{R_{+}} d\hat{\rho}_2 \Delta \{ \hat{\rho}_2 \} (\text{det} \hat{\rho}_F)^{N-2n_B} e^{-\hat{\Psi} \text{Tr}[\hat{\rho}_F^{2}]^{2}} \sum_{k=1}^{2n_F} (\mu^{(k)} - q_k)^2 \tag{87} \]

\[
\times \int_{R_{+}^{(B)}} d\hat{p}_1 \Delta \{ \hat{p}_1 \} \int_{R_{+}^{(B)}} d\hat{p}_2 \Delta \{ \hat{p}_2 \} \prod_{l_1,l_2=1}^{n_B} \left( p_1^{(l_1)} + p_2^{(l_2)} \right) e^{-\hat{\Psi} \text{Tr}[\hat{\rho}_F^{2}]^{2}} e^{N \sum_{l=1}^{n_B} (\mu_{1B}^{(l)} p_1^{(l)} - \mu_{2B}^{(l)} p_2^{(l)})} \]

\[
\times \text{det} \left[ \hat{p}_1 \hat{p}_2 \right]^{N-2n_B} \prod_{k=1}^{2n_F} \det [q_k 1_{n_B} - \hat{p}_1] \det [q_k 1_{n_B} + \hat{p}_2] \]

which is still exact for \( N \geq 2n_B \) and valid for arbitrary values of parameters such that \( \text{Im} \mu_{1B}^{(l)} > 0 \) and \( \text{Im} \mu_{2B}^{(l)} < 0 \).

Before treating the integrals in the limit \( N \to \infty \) by the saddle-point method we can restore the normalisation constant \( \mathcal{C} \) by comparing both sides of the equation in the limit \( N \to \infty \), \( \mu_{1B}^{(k)} \to \infty, \text{Im} \mu_{1B}^{(l)} \to \infty, \text{Im} \mu_{2B}^{(l)} \to -\infty \). Obviously, in this limit the presence of the random matrix \( H \in GUE \) is immaterial and by its very definition the correlation function tends to:

\[
\left\langle \prod_{k=1}^{2n_F} Z_N(-i\hat{\mu}_F^{(k)}) Z_N(-i\hat{\mu}_F^{(k)}) \prod_{l=1}^{n_B} Z_N(\hat{\mu}_F^{(l)}) Z_N(\hat{\mu}_F^{(l)}) \right\rangle_{GUE} \to (-1)^{Nn_F} \left[ \prod_{k=1}^{n_F} \frac{\mu_{1B}^{(k)} \mu_{2B}^{(k)}}{\mu_{1B}^{(k)} \mu_{2B}^{(k)}} \right]^{N} \]
In the right hand side close inspection shows that the integrals over \( q_F^{(k)} \) are dominated by vicinity of \( q_F^{(k)} = \mu_F^{(k)} \). They effectively decouple from the integrals over \( p_{1,2}^{(l)} \) and can be straightforwardly calculated yielding exactly the factor:

\[
\Delta\{\hat{\mu}_F\} \left(\frac{2\pi}{N}\right)^{n_F} \left[ \prod_{k=1}^{n_F} \mu_F^{(k)} \right]^N
\]

On the other hand, performing the remaining integrals in the appropriate limit amounts to evaluating the following expression:

\[
I = (-1)^{\frac{n_B(n_B+1)}{2}} \int_{R_+^n} \int_{R_+^n} \frac{\Delta\{\hat{p}_1, -\hat{p}_2\} e^{iN \sum_{l=1}^{n_B} \left[ \mu_B^{(l)} (\hat{p}_1^{(l)} - \hat{p}_2^{(l)}) \right]} \det \{\hat{p}_1\hat{p}_2\}^{N-2n_B}}{\Pi_{k=1}^{n_B} (-iN \mu^{(l)}_{1B}) (-iN \mu^{(l)}_{2B})} \Delta \left\{ \left(-iN \mu^{(l)}_{1B}\right)^{-1}, \left(-iN \mu^{(l)}_{2B}\right)^{-1} \right\}
\]

(88)

It can be done by expanding the Vandermonde determinant in the sum over all permutations, evaluating the corresponding integrals and resuming the resulting expression back to form another Vandermonde determinant:

\[
I = (-1)^{\frac{n_B(n_B+3)}{2}} \prod_{k=1}^{N-1} k! \left[ \prod_{l=1}^{n_B} (-iN \mu^{(l)}_{1B}) (-iN \mu^{(l)}_{2B}) \right]^{N-2n_B} \Delta \left\{ \left(-iN \mu^{(l)}_{1B}\right)^{-1}, \left(-iN \mu^{(l)}_{2B}\right)^{-1} \right\}
\]

Combining all these facts we restore the normalisation constant as:

\[
\tilde{C} = \frac{(-1)^{N(n_B+n_F)-n_B(n_B/2-1)} N^{2n_B(N-n_B)+n_B+n_F}}{(2\pi)^{n_F} \prod_{k=1}^{2n_B} \Gamma(N-k+1)}
\]

(89)

Coming back to investigating the expression Eq(87) we can already continue analytically: \( \mu_F^{(k)} \to i\mu_F^{(k)} \) for \( k = 1, ..., 2n_F \) and set all imaginary parts of the spectral parameters \( \mu^{(l)}_{1B} \) and \( \mu^{(l)}_{2B} \) to zero. As usual, we are interested in the so-called "scaling limit" when all the spectral parameters \( \mu^{(l)}_{1B} \), \( \mu^{(l)}_{2B} \) as well as \( \mu^{(k)}_F \) are around the same point of the spectrum \( \mu \) such that \( |\mu| < 2 \), their mutual distance being of the order of \( N^{-1} \). Correspondingly, we set:\n
\[
\mu^{(l)}_{1,2B} = \mu + \frac{1}{N} \omega^{(l)}_{B(1,2)}, \quad \mu^{(k)}_F = \mu + \frac{1}{N} \omega^{(k)}_F
\]

and consider all \( \omega_B, \omega_F = O(1) \) when \( N \to \infty \).

In this way we reduce the expression under investigation to the form convenient for starting the saddle-point analysis:

\[
\mathcal{K}_N(\hat{\mu}_B, \hat{\mu}_F) = \frac{\tilde{C}_1 e^{\frac{N}{2} \Gamma[\hat{\mu}_B^2]} \Delta\{\omega_B\} \Delta\{\omega_F\}}{\int_R \int_{R_+^n} \Delta\{\hat{q}_F\} \left[ \prod_{k=1}^{n_F} q_k \right]^{N-2n_B} e^{\sum_{k=1}^{2n_F} \omega^{(k)}_F q_k}} \times
\]

\[
\times e^{-N} \left[ \sum_{k=1}^{2n_F} \mathcal{L}_F(q_k) \right] \times \prod_{l=1}^{n_B} \left[ \prod_{l=1}^{2n_B} q_k \left( p_1^{(l)} + p_2^{(l)} \right) \right] \times \left[ \prod_{l=1}^{n_B} p_1^{(l)} p_2^{(l)} \right]^{-2n_F} e^{\sum_{l=1}^{n_B} \left( -\omega^{(l)}_B p_1^{(l)} - \omega^{(l)}_B p_2^{(l)} \right)} \prod_{l_1, l_2=1}^{n_F} \left[ q_k - p_1^{(l_1)} \right] \left[ q_k + p_2^{(l_2)} \right] \times e^{-N} \sum_{l=1}^{n_B} \mathcal{L}_B(q_k^{(l)}) + \mathcal{L}_F(q_k^{(l)}) \right]
\]

\[
= \frac{\tilde{C}_1 e^{\frac{N}{2} \Gamma[\hat{\mu}_B^2]} \Delta\{\omega_B\} \Delta\{\omega_F\}}{\int_R \int_{R_+^n} \Delta\{\hat{q}_F\} \left[ \prod_{k=1}^{n_F} q_k \right]^{N-2n_B} e^{\sum_{k=1}^{2n_F} \omega^{(k)}_F q_k}} \times \prod_{l=1}^{n_B} \left[ \prod_{l=1}^{2n_B} q_k \right]^{N-2n_B} e^{\sum_{l=1}^{n_B} \left( -\omega^{(l)}_B p_1^{(l)} - \omega^{(l)}_B p_2^{(l)} \right)} \prod_{l_1, l_2=1}^{n_F} \left[ q_k - p_1^{(l_1)} \right] \left[ q_k + p_2^{(l_2)} \right] \times e^{-N} \sum_{l=1}^{n_B} \mathcal{L}_B(q_k^{(l)}) + \mathcal{L}_F(q_k^{(l)}) \right]
\]

\[
= \frac{\tilde{C}_1 e^{\frac{N}{2} \Gamma[\hat{\mu}_B^2]} \Delta\{\omega_B\} \Delta\{\omega_F\}}{\int_R \int_{R_+^n} \Delta\{\hat{q}_F\} \left[ \prod_{k=1}^{n_F} q_k \right]^{N-2n_B} e^{\sum_{k=1}^{2n_F} \omega^{(k)}_F q_k}} \times \prod_{l=1}^{n_B} \left[ \prod_{l=1}^{2n_B} q_k \right]^{N-2n_B} e^{\sum_{l=1}^{n_B} \left( -\omega^{(l)}_B p_1^{(l)} - \omega^{(l)}_B p_2^{(l)} \right)} \prod_{l_1, l_2=1}^{n_F} \left[ q_k - p_1^{(l_1)} \right] \left[ q_k + p_2^{(l_2)} \right] \times e^{-N} \sum_{l=1}^{n_B} \mathcal{L}_B(q_k^{(l)}) + \mathcal{L}_F(q_k^{(l)}) \right]
\]
where
\[ \tilde{C}_1 = (-1)^{n_F(n_F-1)/2} N^{n_F(2n_F-1)+n_B(2n_B-1)} \tilde{C} \]
and
\[ \mathcal{L}_F(q) = \frac{1}{2} q^2 - i\mu q - \ln q \quad , \]
\[ \mathcal{L}_{1B}(p) = \frac{1}{2} p^2 - i\mu p - \ln p \quad , \]
\[ \mathcal{L}_{1B}(p) = \frac{1}{2} p^2 + i\mu p - \ln p \]

Now it is evident that in the limit \( N \to \infty \) the contributions to integrals come from the stationary points of the "actions" \( \mathcal{L}_F(q) \), \( \mathcal{L}_{1B}(p) \) and \( \mathcal{L}_{1B}(p) \) given by solutions of the equation \( q - i\mu - q^{-1} = 0 \):

\[ q_k = \frac{1}{2} \left[ i\mu \pm \sqrt{4 - \mu^2} \right] \equiv q^\pm \quad , \quad k = 1, ..., 2n_F \]
\[ p_1^{(l)} = \frac{1}{2} \left[ i\mu + \sqrt{4 - \mu^2} \right] \equiv q^+ \quad , \quad p_2^{(l)} = \frac{1}{2} \left[ -i\mu + \sqrt{4 - \mu^2} \right] \equiv -q^- \quad l = 1, ..., n_B \]

Here we took into account the restrictions of the original integration domain: \( \text{Rep}^{(1,2)}_{(1,2)} \geq 0 \).

Presence of the Vandermonde determinants as well as the factor \( \prod_{k=1}^{2n_F} \prod_{l=1}^{n_B} [q_k - p_1^{(l)}] [q_k + p_2^{(l)}] \)
makes the integrand vanish at the saddle-point sets and thus care should be taken when calculating the saddle point contribution to the integral. First of all, the totality of \( 2^{2n_F} \) saddle-points \( q_{F}^{\pm} = (q_{1}^{\pm}, ..., q_{2n_F}^{\pm}) \) can be further subdivided into classes giving contributions of different orders of magnitude in powers of the small parameter \( N^{-1} \). A little inspection reveals that the leading contribution comes from the choice of half of saddle-points to be \( q^+ \), the rest being \( q^- \), with total number of such sets \( \binom{2n_F}{n_F} \) (compare \[ \text{[13]} \]). Indeed, for such a choice the number of vanishing brackets inside the Vandermonde determinant \( \prod_{1 \leq k_1 < k_2 < 2n_F} (q_{k_1} - q_{k_2}) \) is minimal.

To find the contribution from each of the relevant saddle-point sets explicitly let us subdivide the index set \( 1, 2, ..., 2n_F \) into the set \( \{K_+\} = \{k_1 < k_2 < ... < k_{n_F}\} \) of those indices \( 1 \leq k_m \leq 2n_F \) for which \( q_{m} = q^+ \) and the rest denoted as \( \{K_-\} \). Let us also present the integration variables \( q_k \) as: \( q_k \in \{K_+\} = q^+ + \alpha_k^{+} \), with two set of variables \( \alpha^{\pm} = (\alpha_k^{\pm} \in \{K_k\}) \) serving to describe deviations from the saddle-point values. Then:

\[ \Delta\{\hat{q}_F\} = \prod_{1 \leq k_1 < k_2 < 2n_F} (q_{k_1} - q_{k_2}) \]
\[ = \prod_{k_1 \in \{K_+\}} (q_{k_1} - q_{k_2}) \prod_{k_1 \in \{K_-\}} (q_{k_1} - q_{k_2}) \prod_{k_1 \in \{K_+\}} (q_{k_1} - q_{k_2}) \]
\[ = (-1)^{\epsilon_{K_+K_-}} (4 - \mu^2)^{n_F} \Delta\{\hat{\alpha}^+\} \Delta\{\hat{\alpha}^-\} + \text{h. o. t.} \]

where \( \epsilon_{K_+K_-} \) is odd or even integer serving to take into account the sign factor arising in the process of rearranging indices in the last of three products in the above equation. The abbreviation h.o.t. stands for higher order terms in \( \alpha \)'s. Further we expand in the exponentials up to terms quadratic with respect to \( \alpha \)'s and have:
where we made use of

where

Similarly, we set 

Collecting all the factors we now can represent the leading order contribution to the correlation 

where we made use of 

Similarly, we set 

In this way we obtain: 

Let us now introduce four diagonal matrices of size 

where 

Collecting all the factors we now can represent the leading order contribution to the correlation function as 

where 

The last relation allows us to write down the final result of the calculation in the form:

\[ \times e^{i q^+ \left[ \sum_{k \in (K_+)} \omega_F^{(k)} + \sum_{l=1}^{n_B} \omega_B^{(l)} \right] + i q^- \left[ \sum_{k \in (K_-)} \omega_F^{(k)} + \sum_{l=1}^{n_B} \omega_B^{(l)} \right]} \]

Here we used the integral formula

\[
\int_{R^m} d\Theta \Delta \left\{ \Theta \right\} \exp \left\{ -\frac{t}{2} \text{Tr} \left[ \Theta \right]^2 + i \text{Tr} \left[ \Theta \hat{\Omega} \right] \right\} = \exp \left\{ -\frac{1}{2t} \text{Tr} \left[ \hat{\Omega} \right]^2 \right\}
\]

and denoted: \( \hat{\epsilon} = \epsilon_{K_+K_-} + Nn_F + n_Bn_F + n_B/2 \).

A close inspection of the quotient of the Vandermonde determinants occurring when substituting Eq.(103) into Eq.(101) reveals that:

\[
\Delta \left\{ \hat{\Omega}^- \right\} \Delta \left\{ \hat{\Omega}^+ \right\} = (-1)^{K_+K_-} F_{n_B,n_F}^{K_+,K_-} (\hat{\omega}_B, \hat{\omega}_F)
\]

where we introduced the notation:

\[
F_{n_B,n_F}^{K_+,K_-} (\hat{\omega}_B, \hat{\omega}_F) = \frac{\prod_{l=1}^{n_B} \left[ \prod_{k \in (K_+)} \left( \omega_F^{(k)} - \omega_B^{(l)} \right) \prod_{k \in (K_-)} \left( \omega_F^{(k)} - \omega_B^{(l)} \right) \right]}{\prod_{k_1 < k_2} \left( \omega_B^{(k_1)} - \omega_B^{(k_2)} \right) \prod_{k_1 \in (K_-)} \left( \omega_B^{(k_1)} - \omega_B^{(k_2)} \right)}
\]

and \((-1)^{K_+K_-}\) is exactly the same factor that appeared in our calculation earlier due to rearranging variables inside the brackets in the product of Vandermonde determinants.

We also observe that:

\[
\mu \text{Tr} \hat{\omega}_F + iq^+ \sum_{k \in (K_+)} \omega_F^{(k)} + iq^- \sum_{k \in (K_-)} \omega_F^{(k)} = -iq^+ \sum_{k \in (K_-)} \omega_F^{(k)} - iq^- \sum_{k \in (K_+)} \omega_F^{(k)}
\]

The last relation allows us to write down the final result of the calculation in the form:

\[
K_{N \to \infty}(\hat{\mu}_B, \hat{\mu}_F) = C_{N,n_B,n_F} e^{\frac{\pi}{2}(n_B-n_f)\mu^2}
\]

and the summation goes over all possibilities of subdividing the index set 1, 2, ..., 2n_F into two index sets \( \{K_+\} \) and \( \{K_-\} \). Here \( C_{N,n_B,n_F} \) stands for the overall normalisation constant:

\[
C_{N,n_B,n_F} = (2\pi)^{n_B} (-1)^{Nn_B+n_B^2-n_F+n_F^2+n_B+n_Bn_F}
\]

and we neglected all the terms of the order of \( O(N^{-1}) \) in the exponential to be consistent with the leading order approximation.

Remembering that the mean spectral density of the GUE eigenvalues in the limit of large \( N \) is given by the Wigner semicircular law: \( \rho(\mu) = \frac{1}{2\pi} \sqrt{4 - \mu^2} \) so that \( q^\pm = \frac{\mu}{2} \pm i\rho(\mu) \) we satisfy ourselves that for \( \mu = 0 \) the derived expression coincides with one announced in [14] and obtained by a rather different method.
6 Conclusions

In the present paper we have demonstrated that the method suggested in [I] allows one to analyse the correlation function containing both positive and negative moments of characteristic polynomials. This technique combines simultaneous exploitation of the standard Hubbard-Stratonovich transformation with an integration theorem (see formula (81)). The latter is a new element as compared with [I] which replaces the Ingham-Siegel integration used there. The method leads to a compact integral representation (83) for the correlation function. To study the asymptotic limit of large GUE matrices we needed to expose variables amenable to the saddle point treatment. For this purpose we had to derive the integration formula extending the Itzykson-Zuber-Harish-Chandra integral to the non-compact Kähler manifold $U(n_1, n_2)/T$ (see expression (4)). Our derivation is based on the Duistermaat-Heckman localization principle.

Our preliminary considerations show that the outlined procedure works well for other ensembles of random matrices, in particular for the chiral GUE and non-Hermitian ensembles. In the latter cases it requires a non-compact analogue of the integral formula found by Guhr, Wettig [27] and by Jackson, Sener and Verbaarschot [28]. The corresponding calculation will be published elsewhere.

7 Acknowledgements

E Strahov is grateful to R Picken for useful communications. Y V Fyodorov would like to thank B Duplantier for pointing out the reference [56] in relation to the formula Eq.(139), to DW Farmer for informing him on the paper [28], to G Akemann for bringing references [21, 33] and to P.-E. Paradan for bringing references [36, 37, 38] to the authors attention.

This research was supported by EPSRC grant GR/13838/01 "Random matrices close to unitary or hermitian."

Appendix A. Diffusion derivation of the Itzykson-Zuber type integral on the pseudo-unitary group $U(n_1, n_2)$

Let us consider a diffusion on matrices that are elements of the Lie algebra $u(n_1, n_2)$. Any such $(n_1 + n_2) \times (n_1 + n_2)$ matrix $A$ satisfies the equation:

$$A^\dagger = \lambda AA$$  \hspace{1cm} (108)

as it follows from the pseudo-unitarity of the group $U(n_1, n_2)$. The corresponding Laplace operator invariant under pseudo-unitary transformations acquires the following form:

$$D_A = \sum_{i=1}^{n_1+n_2} \partial^2/\partial A_{ii}^2 + 1/2 \sum_{1 \leq i < j \leq n_1+n_2} (-)^{\sigma_{ij}} \left[ \partial^2/(\partial \text{Re } A_{ij})^2 + \partial^2/(\partial \text{Im } A_{ij})^2 \right]$$  \hspace{1cm} (109)

where the symbol $\sigma_{ij}$ takes the values 0 when $1 \leq i < j \leq n_1$ or $n_1 < i < j \leq n_2$ and 1 when $1 \leq i \leq n_1 < j \leq n_2$. Once the Laplacian $D_A$ is given, it generates the diffusion on matrices satisfying Eq.(108). Such a diffusion is described by the heat equation

$$\frac{1}{2} D_A \psi(A, t) = \partial_t \psi(A, t)$$  \hspace{1cm} (110)

with the initial condition

$$\psi(A, t = 0) = \phi(A)$$  \hspace{1cm} (111)
The solution of the diffusion problem defined above is represented by an integral over the matrices $B$ satisfying the condition Eq. (108):

$$
\psi(A, t) = \frac{1}{(2\pi t)^{(n_1 + n_2)/2}} \int dB \exp \left( -\frac{1}{2t} \text{Tr}(A - B)^2 \right) \phi(B)
$$

(112)

Let us note that the expression $\text{Tr}(A - B)^2$ is not positive definite for the matrices from the Lie algebra $u(n_1, n_2)$. However one can always choose the initial distribution $\phi(B)$ in a way that ensures the existence of the integral Eq. (112). Let us further assume that the initial distribution $\phi(B)$ is invariant under pseudo-unitary transformations:

$$
\phi(gBg^{-1}) = \phi(B), \; g \in U(n_1, n_2)
$$

(113)

This condition implies that the function $\phi(B)$ depends only on the set eigenvalues of the matrix $B$ and is symmetric under separate permutations of the first $n_1$ and the rest of $n_2$ eigenvalues. Indeed, the corresponding Weyl group is $S_{n_1} \times S_{n_2}$ and not $S_{n_1 + n_2}$ as in the case of unitary transformations.

In what follows we adopt the argumentation used in the original work by Itzykson and Zuber to the present case. Let us take a diagonal matrix $A$ in Eq. (112) and diagonalize the matrix $B$ by a pseudo-unitary transformation:

$$
A = \text{diag} (\alpha_1, \alpha_2, \cdots, \alpha_{n_1 + n_2}), \; B = v \text{diag} (\beta_1, \beta_2, \cdots, \beta_{n_1 + n_2}) v^{-1}, \; v \in U(n_1, n_2)
$$

(114)

Then the integral expressing the solution of the diffusion problem (Eq. 112) can be rewritten as follows:

$$
\psi(\alpha, t) = \frac{1}{(2\pi t)^{(n_1 + n_2)/2}} \int dv \int d\beta \Delta^2(\beta) \exp \left( -1/2t \text{Tr}[(\alpha - \beta v^{-1})^2] \right) \phi(\beta)
$$

(115)

where $\alpha, \beta$ stand for the diagonal matrices $\text{diag} (\alpha_1, \alpha_2, \cdots, \alpha_{n_1 + n_2})$ and $\text{diag} (\beta_1, \beta_2, \cdots, \beta_{n_1 + n_2})$, respectively. We introduce the function $\zeta(\alpha, t)$ antisymmetric with respect to separate permutations inside the sets $(\alpha_1, \alpha_2, \cdots, \alpha_{n_1})$ and $(\alpha_{n_1 + 1}, \alpha_{n_1 + 2}, \cdots, \alpha_{n_1 + n_2})$:

$$
\zeta(\alpha, t) = \Delta(\alpha) \psi(\alpha, t), \; \xi(\alpha, t = 0) = \Delta(\alpha) \phi(\alpha)
$$

(116)

Act on this function by the Laplacian operator $D_A$. For the function $\zeta(\alpha, t)$ depending only on the eigenvalues of the matrix $A$ and $\psi(\alpha, t)$ being a solution of the diffusion problem Eq. (110), the procedure yields the following differential equation:

$$
\partial_t \xi(\alpha, t) = 1/2 \sum_{i=1}^{n_1+n_2} \frac{\partial^2}{\partial \alpha_i^2} \xi(\alpha, t)
$$

(117)

The only diffusion kernel $K(\alpha, \beta, t)$ corresponding to the above equation which is antisymmetric with respect to separate permutations inside the sets $(\beta_1, \beta_2, \cdots, \beta_{n_1})$ and $(\beta_{n_1+1}, \beta_{n_1+2}, \cdots, \beta_{n_1+n_2})$ is given by

$$
K(\alpha, \beta, t) = \frac{\text{const}}{(2\pi t)^{(n_1 + n_2)/2}} \sum_{P \in S_{n_1} \times S_{n_2}} (-)^P \exp \left[ -\frac{1}{2t} \sum_i (\alpha_i - \beta_i^P)^2 \right]
$$

(118)

Comparison of the equations Eqs. (113, 116) and Eq. (118) yields the desired formula Eq. (4) after a simple manipulation.
Appendix B. Complex parameterization of $U(2)/U(1) \times U(2)$ and $U(1,1)/U(1) \times U(1)$

First we note that the manifold $U(2)/U(1) \times U(1)$ is equivalent to $SU(2)/U(1)$. The complex Lie algebra corresponding to the group $SU(2)$ has three elements $e_q, e_{-q}, h$ in its Cartan-Weil basis. In the two-dimensional fundamental representation these basis elements are expressed as follows:

$$h = 1/2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad e_q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad e_{-q} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

(119)

Following the general method of constructing the complex parameterization we decompose the representative $g(z, \bar{z})$ of the coset space $SU(2)/U(1)$ in the same way as it is done in the formula Eq.(5). The factors $u(z)$ and $p(z, \bar{z})$ are given by:

$$u(z) = \exp (ze_q), \quad p(z, \bar{z}) = \exp (y(z, \bar{z})e_q) \cdot \exp (k(z, \bar{z})h)$$

(120)

Any element of a coset of a unitary group must satisfy the unitarity condition. For our case the unitarity condition $g^\dagger(z, \bar{z}) = g^{-1}(z, \bar{z}), \forall g(z, \bar{z}) \in SU(2)/U(1)$ is equivalent to the following algebraic relation:

$$p(z, \bar{z})p^\dagger(z, \bar{z}) = u^{-1}(z) (u^\dagger(z))^{-1}$$

(121)

This relation enables one to derive explicitly the functions $y(z, \bar{z})$ and $k(z, \bar{z})$ entering the formula Eq.(120). We chose to perform the calculations below in the fundamental matrix representation since the obtained expressions for $y(z, \bar{z})$ and $k(z, \bar{z})$ are the same in any representation. In the two-dimensional matrix representation we have:

$$u(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, \quad \exp (y(z, \bar{z})e_q) = \begin{pmatrix} 1 & 0 \\ -y(z, \bar{z}) & 1 \end{pmatrix}$$

$$\exp (k(z, \bar{z})h) = \begin{pmatrix} \exp (1/2k(z, \bar{z})) & 0 \\ 0 & \exp (-1/2k(z, \bar{z})) \end{pmatrix}$$

(122)

Inserting the above matrix expressions to the formula (121) we find:

$$Re \ k(z, \bar{z}) = \ln(1 + z\bar{z}), \quad y(z, \bar{z}) = \bar{z}/(1 + z\bar{z})$$

(123)

Let us note that the function $k(z, \bar{z})$ is specified up to an arbitrary complex part. This means that the corresponding element of the coset space $g(z, \bar{z})$ is determined up to a multiplication by a torus element from the right, as it must be the case.

The parameterization of the coset space $U(1,1)/U(1) \times U(1)$ is obtained by following similar steps. The difference is that the corresponding representative of the coset space should be an element of the pseudo-unitary group $U(1,1)$ rather than of $U(2)$. Therefore the representative $g(z, \bar{z})$ of the coset space $U(1,1)/U(1) \times U(1)$ must satisfy the pseudo-unitary condition Eq.(3). The latter leads to the following algebraic relation (which replaces Eq.(121) above):

$$p(z, \bar{z})p^\dagger(z, \bar{z}) = u^{-1}(z)\lambda (u^\dagger)^{-1}$$

(124)

Using this formula we obtain the expressions for the real part of the function $k(z, \bar{z})$ and for the function $y(z, \bar{z})$:

$$Re \ k(z, \bar{z}) = \ln(1 - z\bar{z}), \quad y(z, \bar{z}) = -\bar{z}/(1 - z\bar{z})$$

(125)

A remarkable feature of the described parameterization is that the expression for $Re \ k(z, \bar{z})$ can be considered as fundamental Kähler potentials. This is a quite general property common to any
homogeneous Kähler manifold with a unitary or pseudo-unitary transformation group as was shown by Itoh, Kugo and Kunitomo \[54\].

An alternative way to find the Kähler potentials in terms of the local complex parameters \(z, \bar{z}\) is to exploit the relation Eq.(33) (or its analogue Eq.(37) for a pseudo-unitary coset). The number of fundamental Kähler potentials is equal to the number of projection matrices or to the rank of the Lie algebra under consideration. The Lie algebra \(su(2)\) has one basis element \(h\) in its Cartan subalgebra, so the rank of \(su(2)\) is equal to unity. The projection matrix corresponding to the basis element \(h\) is determined from the equations (38):

\[
\eta = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\]  

(126)

Now insert the projection matrix \(\eta\) and the matrix \(u(z)\) given by Eq.(122) to the formula Eq.(33) for the unitary coset space \(U(2)/U(1) \times U(1)\) (or to its counterpart Eq.(13) for the pseudo-unitary coset \(U(1,1)/U(1) \times U(1)\)). A simple calculation yields the corresponding Kähler potentials:

\[
K_{U(2)/U(1) \times U(1)}(z, \bar{z}) = \ln(1 + z\bar{z}), \quad K_{U(1,1)/U(1) \times U(1)}(z, \bar{z}) = \ln(1 - z\bar{z})
\]  

(127)

Once the Kähler potentials are known, the (1,1) forms on the manifolds \(U(2)/U(1) \times U(1)\) and \(U(1,1)/U(1) \times U(1)\) can be immediately obtained from the relation Eq.(8):

\[
\Omega_{U(2)/U(1) \times U(1)} = \frac{1}{2\pi i} \frac{dz \wedge d\bar{z}}{(1 + z\bar{z})^2}, \quad \Omega_{U(1,1)/U(1) \times U(1)} = \frac{1}{2\pi i} \frac{dz \wedge d\bar{z}}{(1 - z\bar{z})^2}
\]  

(128)

Finally let us determine the momentum maps on the manifolds \(U(2)/U(1) \times U(1)\) and \(U(1,1)/U(1) \times U(1)\). As it can be seen from Eq.(41) the momentum maps are completely determined by the matrix \(\rho(z, \bar{z})\). In order to find that matrix we insert the projection matrix \(\eta\) and the matrix \(u(z)\) to the formula Eq.(13) for the compact coset \(U(2)/U(1) \times U(1)\) and to the formula Eq.(16) for its non-compact counterpart \(U(1,1)/U(1) \times U(1)\). We obtain:

\[
\rho_{U(2)/U(1) \times U(1)}(z, \bar{z}) = \frac{1}{1 + z\bar{z}} \begin{pmatrix} \bar{z} & z \\ z & 1 \end{pmatrix}, \quad \rho_{U(1,1)/U(1) \times U(1)}(z, \bar{z}) = \frac{-1}{1 - z\bar{z}} \begin{pmatrix} z\bar{z} & z \\ z & -1 \end{pmatrix}
\]  

(129)

Now the momentum maps \(T_q(z, \bar{z}), T_{-q}(z, \bar{z}), T_h(z, \bar{z})\) corresponding to the basis elements \(e_q, e_q\) and \(h\) can be easily constructed. For the space \(U(2)/U(1) \times U(1)\) they are given by

\[
T_q(z, \bar{z}) = -\frac{\bar{z}}{1 + z\bar{z}}, \quad T_{-q}(z, \bar{z}) = \frac{z}{1 + z\bar{z}}, \quad T_h(z, \bar{z}) = \frac{1 - z\bar{z}}{1 + z\bar{z}}
\]  

(130)

The corresponding momentum maps for the non-compact coset \(U(1,1)/U(1) \times U(1)\) are

\[
T_q(z, \bar{z}) = \frac{\bar{z}}{1 + z\bar{z}}, \quad T_{-q}(z, \bar{z}) = -\frac{z}{1 + z\bar{z}}, \quad T_h(z, \bar{z}) = \frac{1 + z\bar{z}}{1 - z\bar{z}}
\]  

(131)

**Appendix C. Transformation of projection matrices \(\rho_i(z, \bar{z})\)**

In order to prove Eq.(13) we note that the decomposition \(gu(z) = u(gz)p(z, g)\) that defines the group action on the flag manifold under consideration leads to the following expressions for \(u(gz)\) and \(u^t(gz)\):

\[
u(gz) = gu(z)p^{-1}(z, g), \quad u^t(gz) = (p^t(z, g))^{-1} u^t(z)g^t
\]  

(132)
Rewrite $\rho_i(gz,\overline{g})$ explicitly using formula Eq. (11):

\[
\rho_i(gz,\overline{g}) = u(gz)\eta_i (\eta_i u^\dagger(gz)u(gz)\eta_i + I - \eta_i)^{-1} \eta_i u^\dagger(gz)
\]  

(133)

Insert $u(gz)$ and $u^\dagger(gz)$ given by Eq. (132) to the above formula and use the properties of the projection matrices $\eta_j$ (equation (38)). Taking into account relations

\[
\eta_i (I - \eta_i) (I - \eta_i) = (I - \eta_i)
\]

we obtain:

\[
\rho_i(gz,\overline{g}) = gu(z)p^{-1}(z,g)\eta_i (\eta_i p^{-1}(z,g)\eta_i + I - \eta_i)^{-1} \times
\]

\[
(\eta_i u^\dagger(z)u(z)\eta_i + I - \eta_i)^{-1} (\eta_i (p^\dagger(z,g))^{-1} \eta_i + I - \eta_i)^{-1} \eta_i (p^\dagger(z,g))^{-1} u^\dagger(z)g^\dagger
\]

(135)

The transformation law for the matrices $\rho_i(z,\overline{z})$ (see Eq. (43)) follows immediately when we simplify the above expression using

\[
p^{-1}(z,g)\eta_i (\eta_i p^{-1}(z,g)\eta_i + I - \eta_i)^{-1} = \eta_i,
\]

\[
(\eta_i (p^\dagger(z,g))^{-1} \eta_i + I - \eta_i)^{-1} \eta_i (p^\dagger(z,g))^{-1} = \eta_i
\]

(136)

**Appendix D. Proof of the Theorem I**

In this Appendix we give a proof of the statement of the Theorem I. In fact, we demonstrate the validity of the closely related

**Theorem Ia**

Consider a function $F(S_1,\ldots,S_m)$ of $N$-component real vectors $S_i$, $1 \leq l \leq m$ such that

\[
\int_{R^N} dS_1\ldots dS_m |F(S_1,\ldots,S_m)| < \infty
\]

(137)

Denoting $S^T$ the transposition suppose further that the function $F$ depends only on $m(m+1)/2$ scalar products $S^*_l S_{l_2}$, $1 \leq l_1, l_2 \leq m$ so that it can be rewritten as a function $F(\hat{Q}_m)$ of $m \times m$ real symmetric matrix $\hat{Q}_m$:

\[
(Q_m)_{k,l} = (S^T_k S_l) |_{1 \leq k, l \leq m}
\]

Then for $N > m$ the integral defined as

\[
I^{(F)}_{N,M} = \int_{R^N} dS_1\ldots dS_m F(S_1,\ldots,S_m)
\]

(138)

is equal to

\[
I^{(F)}_{N,M} = C^{(o)}_{N,m} \int_{\hat{Q}_m > 0} d\hat{Q}_m \left(\det\hat{Q}_m\right)^{(N-m-1)/2} F(\hat{Q}_m)
\]

(139)

where the proportionality constant is given by

\[
C^{(o)}_{N,m} = \frac{\pi^m (N - \frac{m + 1}{2})}{\prod_{k=0}^{m-1} \Gamma \left(\frac{N - k - 1}{2}\right)}
\]
and the integration in Eq. \[(139)\] goes over the manifold of real symmetric positive definite $m \times m$ matrices $\tilde{Q}_m$.

**Proof**

We prove the statement by induction in $m$ for any $N > m$.

First, for $m = 1$ we parameterize $S = r(O_N e)$, where $r = ||S|| \geq 0$, the matrix $O_N \in O(N)$ is a real orthogonal $N \times N$ satisfying: $O_N^T O_N = 1_N$ and $N - 1$ first components of the vector $e$ are chosen to be zero, the last component being unity: $e^T = (0, ..., 0, 1)$. The integration measure can be written as $dS = r^{N-1} dr d\mu (O)$, the last factor standing for the Haar’s measure on the group $O(N)$, such that:

$$\int_{O(N)} d\mu (O) = 2 \frac{\pi^{N/2}}{\Gamma(N/2)} \equiv \Omega_N$$

Now for $N \geq 2$ we have:

$$\int_{R^N} dSF(S^2) = \Omega_N \int_{R^+} drr^{N-1} F(r^2) = \frac{\Omega_N}{2} \int_{R^+} dq_{11} (q_{11}^{N-2}/2) F(q_{11})$$

which proves the statement and gives the value $C^{(o)}_{N,1} = \frac{1}{2} \Omega_N$ as required.

Suppose now that the statement is true for $(m - 1)$ vectors, each with $(N - 1)$ real components, that means:

$$I_{N-1,m-1}^{(F)} = I_{N-1,m-1}^{(F)} = C^{(o)}_{N-1,m-1} \int_{m-1} d\tilde{Q}_m \left( (\det \tilde{Q}_m)^{N-m-1}/2 \right) F(\tilde{Q}_m) \quad (140)$$

To consider the case of $m$ vectors, each with $N$ components we represent the $m \times m$ matrix $\tilde{Q}_m$ as:

$$\tilde{Q}_m = \begin{pmatrix} (S_1^T S_m) \\ (S_2^T S_m) \\ \vdots \\ (S_m^T S_1) \\ (S_m^T S_2) \\ \vdots \\ (S_m^T S_m) \end{pmatrix}$$

Now parameterize $S_m = r_m (O_N e)$ as before, and for $k = 1, 2, ..., m - 1$ introduce new vectors $\tilde{S}_k = O_N S_k$ as integration variables. Obviously, the entries of the matrix $\tilde{Q}_{m-1}$ do not change, whereas:

$$\begin{pmatrix} S_k^T S_m \end{pmatrix} = r_m \left( \tilde{S}_k^T \tilde{S}_m \right) = r_m \left( \tilde{S}_k^T e \right) = r_m \tilde{S}_{N,k}$$

where $\tilde{S}_{N,k}$ stands for the last $(N - \text{th})$ component of the vector $\tilde{S}_k$. Further, let us consider first $N - 1$ components of the vector $\tilde{S}_k$ as forming the vector $\zeta_k$, for the last $N - \text{th}$ component of the vector $\tilde{S}_k$ using the notation: $\bar{q}_{k,m}$, $k = 1, ..., m - 1$. Obviously:

$$dS_k = d\tilde{S}_k = d\tilde{q}_{k,m}$$

$$\left( \tilde{Q}_{m-1}^c \right)_{k,l} = \left( \tilde{Q}_{m-1} \right)_{k,l} + \bar{q}_{k,m} \bar{q}_{l,m} \quad , \quad 1 \leq k, l \leq m - 1 \quad (141)$$

where $\left( \tilde{Q}_{m-1}^c \right)_{k,l} = (\zeta_k^T \zeta_l)$

After all those preparations we change the order of integrations (which is legitimate in view of the condition Eq. \[(137)\] and the Fubini theorem) and represent the integral in the right-hand side
of Eq. (138) as:

\[ I_{N,M} = \Omega_N \int_{R^+} drr^{N-1} \int_{R^{m-1}} d\vec{q}_{m-1,m} \int_{R^{N-1}} d\xi_1 \cdots \int_{R^{N-1}} d\xi_m F_1 [\vec{Q}] \]

where

\[ F_1 [\vec{Q}] = \mathcal{F} \left[ \begin{pmatrix} r_m q_{1,m} & \hat{\xi}_{k,m} \xi_{m,m} & \cdots \\ \hat{\xi}_{k,m} & \xi_{m,m} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix} \right] \]

Now we can apply the equation Eq. (140) to replace the integration over the vectors \( \zeta_k \) to that over the corresponding positive definite matrices. Further introducing quantities \( q_{k,m} = r_m \hat{\xi}_{k,m}, \ k = 1, \ldots, m-1 \) as integration variables and denoting \( r_m^2 = q_{m,m} \) we immediately see that the above integral can be written as:

\[ C_{N,1}^{(o)} C_{N-1,m-1}^{(o)} \int_{R^+} dq_{m,m} \int_R dq_{1,m} \cdots \int_R dq_{m-1,m} \quad (143) \]

\[ \times \int_{Q_{m-1} > 0} d\hat{Q}_{m-1} q^{(N-n-1)/2} \left[ \det \hat{Q}_{m-1} \right]^{(N-n-1)/2} \mathcal{F} [\hat{Q}] \]

where we denoted

\[ \hat{Q}_m = \begin{pmatrix} \xi_{m-1,k,l} + \frac{q_{m-1,m} q_{k,m}}{q_{m,m}} & q_{1,m} \\ q_{m,1} & q_{m,2} \end{pmatrix} \]

Using the determinant identity

\[ \det \begin{pmatrix} \hat{Q}_{m-1} & q^T \\ q & q \end{pmatrix} = q \times \det \left( \hat{Q}_{m-1} - \frac{q \otimes q^T}{q} \right) \]

we see that:

\[ \det [\hat{Q}_m] = q_{m,m} \times \det [\hat{Q}_{m-1}] \]

A little more thinking shows that the conditions \( q_{m,m} > 0 \) and \( \hat{Q}_{m-1} > 0 \) ensure that \( \hat{Q}_m \) is a general symmetric positive definite of dimension \( m \). For, all minors of the matrix \( \hat{Q}_m \) either just coincide with the minors of the matrix \( \hat{Q}_{m-1} \) or with those of the matrix with elements \( \xi_{m-1,k,l} + \frac{q_{m-1,m} q_{k,m}}{q_{m,m}}, \) both being positive definite.

Combining all this knowledge and the fact that \( C_{N,1}^{(o)} C_{N-1,m-1}^{(o)} = C_{N,m}^{(o)} \) we see that the formula Eq. (143) can be written exactly as the right-hand side of the equation Eq. (139) thus completing the proof.

Theorem I then follows by trivially repeating, mutatis mutandis, all the steps of the proof given above for the case of complex vectors and Hermitian matrices, replacing the orthogonal matrices with unitary one as appropriate.
References

[1] JP Keating, NC Snaith "Random Matrix Theory and $\zeta(1/2 + it)$", Comm. Math. Phys., 214 (2000), 57;

[2] JP Keating, NC Snaith "Random matrix theory and $L$-functions at $s=1/2$" Comm. Math. Phys. 214, (2000) 91

[3] CP Hughes, JP Keating, N O'Connell "Random matrix theory and the derivative of the Riemann zeta function" P Roy Soc Lond A Mat 456 (2000) 2611

[4] E. Brezin and S. Hikami, "Characteristic Polynomials of Random Matrices", Comm. Math. Phys., 214, (2000), 111-135 and "Characteristic Polynomials of Random Real Symmetric Matrices" Comm. Math. Phys., 223, (2001), 363-382

[5] Workshop "L-functions and Random Matrix Theory", The American Institute of Mathematics, www.aimath.org/PWN/Irmt/index.html

[6] DM Gangardt, "Second Quantization approach to characteristic polynomials in RMT", J. Phys.A: Math.Gen. 34 (2001) 3553

[7] DM Gangardt and A Kamenev, "Replica treatment of the Calogero-Sutherland model", Nucl. Phys. B, 610 (2001), 578 (e-preprint arXiv:cond-mat/0102405)

[8] F.Haake, M.Kus, H.-J.Sommers, H.Schomerus, K.Zyczkowski, "Secular determinants of random unitary matrices", J.Phys.A: Math.Gen., 29 (1996), 3641

[9] S Ketteman, D.Klakow and U. Smilansky "Characterization of quantum chaos by the autocorrelation function of spectral determinants" J.Phys.A: Math.Gen., 30 (1997), 3643

[10] Y V Fyodorov "Spectra of Random Matrices Close to Unitary and Scattering Theory for Discrete-Time systems", in: "Disordered and Complex Systems", edited by P.Sollich et al., AIP Conference Proceedings v.553, Melville NY, 2001

[11] ML Mehta, J-M Normand, "Moments of the characteristic polynomial in the three ensembles of random matrices" J.Phys A.Math.Gen: 34 (2001), 4627

[12] A Kamenev and M Mezard, "Wigner-Dyson statistics from the Replica Method", J Phys.A, 32 (1999) 4373

[13] I.V.Yurkevich and I.V.Lerner, "Nonperturbative results for level correlations from the replica nonlinear sigma model," Phys.Rev.B 60, 3955

[14] A V Andreev and BD Simons, "Correlator of the Spectral Determinants in Quantum Chaos", Phys.Rev.Lett. 75 (1995), 2304

[15] Y V Fyodorov and B A Khoruzhenko, "Systematic Analytical Approach to Correlation Functions of Resonances in Quantum Chaotic Scattering", Phys. Rev. Lett. 83 (1999), 66

[16] A Cavagna, J Garrahan, I Giardina, " Index Distribution of random matrices with an Application to Disordered Systems", Phys. Rev. B 61 (2000), 3960

[17] T Shirai, " A Factorization of Determinant Related to Some Random Matrices", J. Stat. Phys. 90 (1998), 1449

31
[18] S Kettemann "Exploring level statistics from quantum chaos to localization with the autocorrelation function of spectral determinants" Phys. Rev. B 59 (1999), 4799 and S Kettemann and A Tsvelik "Information about the integer quantum Hall transition extracted from the autocorrelation function of spectral determinants", Phys. Rev. Lett. 82 (1999) 3689

[19] YV Fyodorov "Negative moments of characteristic polynomials of random matrices: Ingham-Siegel integral as an alternative to Hubbard-Stratonovich transformation", Nucl. Phys. B B[PM] (2001) 621 (e-preprint arXiv:math-ph/0106006).

[20] S Nonnenmacher an MR Zirnbauer, "Det-Det Correlations for quantum maps, dual pair and saddle-point analysis", e-preprint arXiv:math-ph/0109025.

[21] PH Damgaard, SM Nishigaki "Universal spectral correlators and massive Dirac operators", Phys.Rev.B, 57 (1998), 5299

[22] MV Berry and JP Keating, "Clusters of near-degenerate levels dominate negative moments of sectral determinants", J. Phys.A: Math.Gen. 35 (2002) L1-L6

[23] E Strahov "Moments of characteristic polynomials enumerate lexicographic arrays", e-preprint arXiv:math-ph/0112043

[24] C.Itzykson and J.B.Zuber, "The planar approximation. II", J.Math.Phys. 21 (1980), 411

[25] Harish-Chandra , "Differential operators on a semisimple Lie algebra", Proc.Nat.Acad.Sci. 42, (1956) 252

[26] JJ Duistermaat, GJ Heckman "On the variation in the co-homology of the symplectic form of the reduced phase space " "Invent.Math. 69 (1982), 259 and ibid 1983 Invent. Math. 72 153

[27] L Schäfer and F Wegner, "Disordered System with n Orbitals per Site: Lagrange formulation, Hyperbolic Symmetry, and Goldstone modes", Z. Physik B-Condensed Matter, 38 (1980), 113

[28] D W Farmer "Long mollifiers of the Riemann zeta-function", Mathematika, 40 (1993), 71

[29] K.B. Efetov, "Supersymmetry in Disorder and Chaos" (Cambridge University Press, Cambridge 1997).

[30] JJM Verbaarschot and MR Zirnbauer, "Critique of the Replica Trick", J Phys. A:Math.Phys., 17 (1985), 1093

[31] MJ Rothstein "Integration on noncompact supermanifolds" Trans. Amer. Math. Soc. 299 (1987), 387

[32] M R Zirnbauer, "Supersymmetry for systems with unitary disorder: circular ensembles" J Phys. A:Math.Phys., 29 (1996), 7113

[33] RJ Szabo "Microscopic Spectrum of the QCD Dirac Operator in Three Dimensions", Nucl. Phys.B, 598 (2001), 309

[34] T Guhr " Dyson Correlation functions and Graded Symmetry", J.Math.Phys. 32, 336 (1991)

[35] E Prato and S Wu, "Duistermaat-Heckman measures in a non-compact setting", arXiv:alg-geom/9307005, Compos. Math. 94 (1994) 113

[36] W. Rossmann, "Kirilov’s character formula for reductive group", Invent. Math. (1978) 48 207
[37] N. Berline and M. Vergne, ”Fourier transform of orbits of the coadjoint representation” in Representation theory of reductive groups, (1983) pp.53-57, Birkhäuser, Basel

[38] P.-E. Paradan, ”The Fourier Transform of Semi-Simple Coadjoint Orbits” J Funct Anal 163 (1999) 152

[39] M R Zirnbauer, ”Another Critique of the Replica Trick”, e-preprint arXiv:cond-mat/9903338

[40] M Bordemann, M Forger, and H Römer ”Homogeneous Kähler Manifolds: Paving the Way Towards New Supersymmetric Sigma Models” Commun. Math. Phys. (1986) 102 605

[41] S Helgason 2000 Differential Geometry and Symmetric Spaces (AMS Chelsea Publishing)

[42] A E Ingham, ”An integral which Occurs in Statistics”, Proc.Camb.Phil.Soc., 29(1933), 271; C L Siegel, ”Über der analytische Theorie der quadratischen Formen”, Ann. Math. 36(1935), 527

[43] L.K. Hua, Harmonic Analysys of Functions of Several Complex Variables in the Classical Domains (AMS, Providence, 1963).

[44] R J Szabo ”Equivariant localization of path integrals”, e-preprint arXiv:hep-th/9608068

[45] R F Picken, ”The Duistermaat-Heckman integration formula on flag manifolds”, J. Math. Phys. (1990) 31(3) 616

[46] A Borel ” Kählerian coset spaces of semisimple Lie groups” Proc. Natl. Acad. Sci. USA (1954)40, 1147

[47] D Bar-Moshe and M S Marinov ”Realization of compact Lie algebras in Kähler manifolds” J. Phys. A 27 (1994), 6287

[48] D Bar-Moshe and M S Marinov, ”Berezin quantization and unitary representations of Lie groups,” in Berezin Memorial, edited by R. Dobrushin, M. Shubin, and A. Vershik (American Mathematical Society, Providence, RI, 1995)

[49] S Kobayashi and K Nomizu Foundations of Differential Geometry (Interscience, New-York, 1969) Vol. 2, Chap. 9

[50] V I Arnold Mathematical Methods of Classical Mechanics (Springer, 1978)

[51] J M Bismut ”Localisation formulas, Superconnections, and the Index theorem for families” Commun. Math. Phys. 103 (1986) 127

[52] E Witten ”Two Dimensional Gauge Theories Revisited” [ArXiv:hep-th/9204083] J. Geom. Phys. 9 (1992), 303

[53] M Bando , T Kuramoto, T Maskawa and S Uehara ”Structure of non-linear realization in supersymmetric theories” Phys. Lett. B 138 (1984), 94

[54] K Itoh , T Kugo and H Kunitomo, ”Supersymmetric nonlinear realization for arbitrary Kählerian coset space G/H” Nucl. Phys. B 263(1986) , 295

[55] K Fujii and K Funahashi ”Multi-Periodic Coherent States and WKB-Exactness II” J. Math. Phys 38(1997), 2812
[56] F David, B Duplantier and E Guitter, "Renormalization Theory for Interacting Crumpled Manifold", Nucl.Phys.B, 394 (1993), 555

[57] T. Guhr and T. Wettig "An Itzykson-Zuber-like integral and diffusion for complex ordinary and supermatrices", J Math Phys 37(12) (1996) 6395

[58] A. D. Jackson, M. K. Sener and J. J. M. Verbaarschot, "Finite volume partition functions and Itzykson-Zuber integrals", Phys. Lett. B 387 (1996) 355