ON THE PRECANONICAL STRUCTURE OF THE SCHRÖDINGER
WAVE FUNCTIONAL

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Abstract

An expression of the Schrödinger wave functional as the product integral of precanonical wave functions on the space of fields and space-time variables is obtained. The functional derivative Schrödinger equation in the canonical quantization is derived from the partial derivative covariant analogue of the Schrödinger equation, which appears in the precanonical quantization based on the De Donder-Weyl Hamiltonization of field theory. The representation of precanonical quantum operators typically contains a parameter \( \kappa \) of the dimension of the inverse spatial volume. The transition from the precanonical description of quantum fields in terms of Clifford-valued wave functions and partial derivative operators to the standard functional Schrödinger representation obtained from canonical quantization is accomplished if \( \frac{1}{\kappa} \to 0 \) and \( \frac{1}{\kappa} \gamma_0 \) is mapped to the infinitesimal spatial volume element \( dx \). Thus the standard QFT corresponds to the precanonical QFT in the limiting case of the infinitesimal parameter \( \frac{1}{\kappa} \).

1 Introduction

Field theories are commonly thought to be systems with an infinite number of degrees of freedom. This notion originates in the canonical Hamiltonian treatment of field theory and it is transferred to quantum field theory by the procedure of canonical quantization. The resulting version of quantum field theory has evolved into a very successful framework of contemporary theoretical physics whose applications range from the condensed matter physics to quantum cosmology, even with its divergences, viewed as pathologies in earlier days of QFT, turned into the triumph of the concepts of renormalization and effective field theory.

However, there are conceptual tensions between quantum theory and relativity, which we face in the context of discussions of foundational issues, quantum field theory in curved
space-times, quantum gravity and unification, where the standard picture of QFT, or our current understanding of it, is possibly pushed to the limits of its applicability due to the distinguished role of time in the formalism of quantum theory on the one side and the generally covariant, geometric and nonlinear nature of the general relativity theory on the other side.

When trying to tackle those questions it may be useful to pay attention to the fact that the progress of QFT has essentially overlooked the developments in the calculus of variations of multiple integral problems, where an extension of the Hamiltonian formulation from mechanics to field theory is known to be by far not unique \[1\]. Moreover, as opposite to the canonical Hamiltonian formalism, the alternative Hamiltonizations neither need to distinguish the “time variable” in the set of space-time variables (i.e. the independent variables of the variational problem which defines a field theory), nor they necessarily imply the picture of fields as infinite-dimensional systems evolving in time (which would fail on not globally hyperbolic space-times).

Thus the question arises whether a formulation of quantum field theory could be built based on those alternative space-time symmetric Hamiltonizations of field theory, and if the inherent features of the latter, such as the manifest respect of the space-time symmetries and the finite dimensionality of the corresponding analogue of the configuration space (i.e. the bundle of fields variables over the space-time, whose sections are the fields configurations appearing in the standard formulations) would be helpful for the clarification of the fundamental issues of QFT at the leading edge of current research, e.g. in the context of quantum gravity.

Moreover, the existence of the Hamilton-Jacobi formulations of field theories which are associated with each of the alternative Hamiltonizations \[1,4\] naturally leads to the question as to whether the alternative formulations of quantum field theories exist which would reproduce the corresponding Hamilton-Jacobi equations in the classical limit, and what would be their physical significance.

To be more specific, let us recall that in the field theory given by the first order Lagrangian density \( L = L(y^a, y^a_\mu, x^\mu) \), where \( y^a \) denote the fields variables of any nature, \( y^a_\mu \) are their first derivatives, and \( x^\mu \) are the space-time variables, the simplest of the above mentioned alternative Hamiltonizations is the so-called De Donder-Weyl (DW) theory (see e.g. \[1,2\]). It is based on the following Hamiltonian-like covariant reformulation of the Euler-Lagrange field equations:

\[
\partial_\mu p^a_\mu = -\frac{\partial H}{\partial y^a}, \quad \partial_\mu y^a = \frac{\partial H}{\partial p^a_\mu},
\]  

(1.1)

which uses the covariant Legendre transformation to new variables: \( p^a_\mu := \partial L/\partial y^a_\mu \) (polymomenta) and \( H(y^a, p^a_\mu, x) := p^a_\mu \partial_\mu y^a - L \) (DW Hamiltonian function).

The DW Hamiltonian equations (1.1) can be compared with the standard Hamilton’s equations in the canonical formalism:

\[
\partial_\mu p^0_a(x) = -\frac{\delta H}{\delta y^a(x)}, \quad \partial_\mu y^a(x) = \frac{\delta H}{\delta p^0_a(x)},
\]  

(1.2)
where the canonical Hamiltonian functional is introduced:

\[
H([y(x), p^0(x)]) := \int d^4x \left( \partial_t y^a(x)p_0^a(x) - L \right),
\]

and a decomposition into the space and time is performed, so that \(x^\mu := (x, t)\). Here and in what follows the capital bold letters denote functionals.

When both formulations are regular the equivalence between (1.1) and (1.2) can be established by noticing that

\[
H = \int d^4x \left( H - \partial_i y^a(x)p_i^a(x) \right).
\]

Then it is easy to check that the canonical Hamilton’s equations can be derived from the (precanonical) DW Hamiltonian equations (1.1).

Whereas the fields quantization based on the canonical Hamiltonization is well elaborated and underlies QFT as we know it, an approach to quantization of fields which is based on the De Donder-Weyl (DW) generalization of Hamiltonian mechanics to field theory was put forward only recently [5–7] (c.f. also discussions of similar ideas by other authors in [8–12]). In the context of quantization of gravity [13–15] the approach was later called precanonical quantization.

While the connection between the canonical and DW Hamiltonization is sufficiently clear on the classical level (see e.g. [16]), the relation between the respective quantizations has been rather problematic for long time (see [5, 17] for earlier discussions). In the recent paper [18] we found a formula connecting the Schrödinger wave functional with the precanonical wave functions in the case of scalar field theory. However, the derivation was based on an ad hoc Ansatz, so that it remained unclear how general the result. In this paper we establish a connection between QFT based on canonical quantization in the functional Schrödinger representation and, respectively, precanonical quantization without any a priori assumptions regarding the form of this relation, except the general assumption that the relation between the Schrödinger wave functional and precanonical wave function may exist (c.f. Eq. (3.6) below).

In Sect. 2 we present a comparative overview of the elements of canonical and precanonical quantization, which are essential for our purposes. In Sect. 3 we derive the functional derivative Schrödinger equation for quantum scalar field theory from the corresponding Dirac-like partial derivative precanonical analogue of the Schrödinger equation. The consideration leads to the relation between the Schrödinger wave functional in the canonical quantization and the Clifford-valued wave function in precanonical quantization. As an example of the application of our result, we construct the vacuum state functional of the free scalar field theory from the precanonical ground state wave functions. The concluding discussion is found in Sect. 4.

2 Canonical and precanonical quantization

Let us present a brief comparative overview of the elements of canonical and precanonical quantization, which are relevant for the following discussion.
Canonical quantization (in the Schrödinger picture \[19\]) is known to lead to the description of quantum fields in terms of the Schrödinger wave functional \(\Psi([y(x)], t)\) on the infinite-dimensional configuration space of fields configurations \(y(x)\) at the moment of time \(t\). Precanonical quantization \[5–7\] leads to the description in terms of Clifford algebra valued wave functions \(\Psi(y, x)\) on the finite dimensional “covariant configuration space” (in the terminology of \[16\]) of fields variables \(y\) and space-time variables \(x\).

While the Schrödinger wave functional \(\Psi\) fulfills the Schrödinger equation
\[
\imath \hbar \partial_t \Psi = \hat{H} \Psi,
\] (2.1)
where \(\hat{H}\) stands for the functional derivative operator of the canonical Hamiltonian, the precanonical wave function \(\Psi(y, x)\) satisfies the following covariant generalization of the Schrödinger equation \[5–7\]
\[
\imath \hbar \gamma^\mu \partial_\mu \Psi = \hat{H} \Psi,
\] (2.2)
where \(\hat{H}\) is the partial derivative operator of the De Donder-Weyl Hamiltonian function, \(\gamma^\mu\) are Dirac matrices of \(n\)-dimensional space-time, and \(\kappa\) is a “very large” constant of the dimension \(\ell^{-(n-1)}\). The latter routinely appears on the dimensional grounds in the representation of precanonical quantum operators, which follows from quantization of the Poisson-Gerstenhaber brackets of differential forms representing the dynamical variables in field theory. These brackets were found in our earlier work on the mathematical structure of the DW Hamiltonian formulation \[20–22\] (for further generalizations see e.g. \[23–28\]), and their geometric prequantization, which can be viewed as an intermediate step toward precanonical quantization, was considered in \[29\].

More specifically, let us consider the theory of interacting scalar fields, which is given by
\[
L = \frac{1}{2} \partial_\mu y^a \partial^\mu y^a - V(y),
\] (2.3)
where the potential term \(V(y)\) also includes the mass terms like \(\frac{1}{2} m^2 y^2\) (henceforth we set \(\hbar = 1\) and use the metric signature \(+\ldots-)\).

In this case the operator of canonical conjugate momentum of \(y^a(x)\):
\[
p^a_0(x) := \frac{\partial L}{\partial \partial_t y^a(x)},
\]
in the Schrödinger \(y(x)\)-representation is given by
\[
\hat{p}^a_0(x) = -i \frac{\delta}{\delta y^a(x)}.
\] (2.4)

This representation follows from quantization of the equal-time Poisson bracket
\[
\{\hat{p}^a_0(x), y^b(x')\} = \delta^b_0 \delta(x - x').
\] (2.5)
In precanonical quantization the representation of the operators of polymomenta:

\[ \hat{p}_a^\mu = -i \kappa \gamma^\mu \frac{\partial}{\partial y^a}, \]  

(2.6)

follows from quantization of the Heisenberg subalgebra of the above mentioned Poisson-Gerstenhaber algebra of forms (c.f. [5–7]):

\[ \{ p^\mu_a, y^b \} = \delta^b_a, \]

\[ \{ p^\mu_a, y^b \omega^\nu \} = \delta^b_a \delta^\mu_\nu, \]

\[ \{ p^a_\mu, y^b \omega^\nu \} = \delta^b_a \delta^\mu^\nu, \]  

(2.7)

where \( \omega^\mu := \partial^\mu \omega \) are the contractions of the volume \( n \)-form \( \omega := dx^0 \wedge dx^1 \wedge ... \wedge dx^{n-1} \) on the space-time with the basis vectors of its tangent space \( \partial^\mu \). This quantization also implies the map \( q \) from the co-exterior forms on the classical level to the Clifford algebra elements (Dirac matrices) on the quantum level:

\[ \omega_\mu \mapsto q \frac{1}{\kappa} \gamma_\mu, \]  

(2.8)

which is similar to the “Chevalley map” [30], or “quantization map” [31], known in the theory of Clifford algebras. The constant \( \kappa \) appears here on the dimensional grounds. From the association of \( \omega_0 \), which represents the infinitesimal spatial volume element \( dx \), with \( \gamma_0 \), which is dimensionless, it is evident that \( \frac{1}{\kappa} \) corresponds to the “very small” volume and has the meaning of the physically infinitesimal or elementary volume.

Furthermore, while the canonical Hamiltonian operator of the quantum scalar field theory:

\[ \hat{H} = \int dx \left\{ -\frac{1}{2} \frac{\delta^2}{\delta y(x)^2} + \frac{1}{2} (\nabla y(x))^2 + V(y(x)) \right\}, \]  

(2.9)

is formulated in terms of functional derivative operators, the DW Hamiltonian operator in this case:

\[ \hat{H} = \frac{1}{2} \kappa^2 \partial_{yy} + V(y) \]  

(2.10)

(see [5–7]), involves the partial derivative operators with respect to the fields variables.

The question naturally arises, how those two descriptions, which seem to be so different both physically and mathematically, can be related: how the Schrödinger wave functional is related to the precanonical wave function and how the functional derivative Schrödinger equation is related to the precanonical Schrödinger equation.

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1 We have explained in our earlier papers [21] that the natural multiplication of forms here is given by the co-exterior product: \( \alpha \bullet \beta := \ast^{-1} (\ast \alpha \wedge \ast \beta) \), where \( \ast \) is the Hodge duality operator.
3 Schrödinger wave functional from precanonical wave function

To make the above mentioned relation less mysterious, let us first recall our earlier observation [17] that the functional derivative Hamilton-Jacobi equation of the canonical Hamiltonian formalism:

$$\partial_t S + H \left( y^\alpha(x), p_\alpha^0(x) = \frac{\delta S}{\delta y(x)}, t \right) = 0, \quad \text{(3.1)}$$

can be derived from the partial differential Hamilton-Jacobi equation of precanonical De Donder-Weyl theory:

$$\partial_\mu S^\mu + H \left( y^\alpha, p_\mu^\alpha = \frac{\partial S^\mu}{\partial y^\alpha}, x^\mu \right) = 0, \quad \text{(3.2)}$$

if the canonical HJ functional $S([y(x), t])$ is constructed in terms of the DW-HJ functions $S^\mu(y, x)$ as follows

$$S = \int_\Sigma S^\mu_\omega \mu. \quad \text{(3.3)}$$

Here $\Sigma: (y = y(x), t = \text{const})$ is the subspace in the covariant configuration space, which represents the field configuration $y(x)$ at the moment of time $t$.

This result of [17] demonstrates that the transition from the object of DW (precanonical) theory, such as $S^\mu(y, x)$, to the object of canonical theory, such as $S([y(x), t])$, involves a restriction of the former to the subspace $\Sigma$ and the subsequent integration over it. In this way the functionals of fields configurations are constructed from the functions on the covariant finite-dimensional configuration space, and the functional derivative equations of the canonical formalism are derived from their precanonical partial derivative counterparts.

Having obtained this result on the classical level, we can expect a similar relation between the wave functional and precanonical wave function on the quantum level, because both the Schrödinger wave functional $\Psi$ and the precanonical wave function $\Psi$ are related to the exponentials of, respectively, the HJ functional and DW-HJ functions, viz.,

$$\Psi \sim e^{i S} \quad \text{and} \quad \Psi \sim e^{\frac{i}{\hbar} S^\mu \gamma_\mu} \quad \text{(3.4)}$$

(see [7], where the second expression is used to argue that in the classical limit the DW-HJ equation follows from the precanonical Schrödinger equation). Using the fact that $S$ is the spatial integral of DW-HJ functions, we can anticipate that $\Psi$ is related to the product integral [32] of precanonical wave functions restricted to the subspace $\Sigma$:

$$\Psi([y(x)]) \sim e^{i S} = e^{i \int_\Sigma S^\mu_\omega \mu} = \prod_{x \in \Sigma} e^{i S^\mu(y = y(x), x, t)} \omega^\mu \sim \prod_{x \in \Sigma} \Psi(y = y(x), x, t) |_{\gamma_\mu \rightarrow \omega_\mu}, \quad \text{(3.5)}$$

where the last step implies the inverse of the "quantization map" in Eq. (2.8). The consideration below will make this idea more precise.
Now, let us assume that the Schrödinger wave functional $\Psi$ can be expressed in terms of the precanonical wave function $\Psi(y, x)$ restricted to the subspace $\Sigma$: $\Psi(y, x)|_{\Sigma} := \Psi_{\Sigma}(y(x), x, t)$, i.e.,

$$\Psi([y(x)], t) = \Psi([\Psi_{\Sigma}(y(x), x, t), [y^a(x)])]. \quad (3.6)$$

Then the time evolution of the Schrödinger wave functional is determined by the time evolution of precanonical wave function. By applying the chain rule to the composite functional (3.6) we obtain:

$$i \partial_t \Psi = \int d\mathbf{x} \ Tr \left\{ \frac{\delta \Psi}{\delta \Psi_{\Sigma}^T(y^a(x), x, t)} i \partial_t \Psi_{\Sigma}(y^a(x), x, t) \right\}. \quad (3.7)$$

Note that the additional dependence of $\Psi$ from $y^a(x)$, which is not incorporated in $\Psi|_{\Sigma}$, is supposed to take into account the fact (of quantum reality) that the amplitudes $\Psi|_{\Sigma}$ in space-like separated points in general are not independent.

The equation of time evolution of $\Psi_{\Sigma}(x)$ is given by the space-time decomposed precanonical Schrödinger equation, Eq. (2.2), restricted to $\Sigma$, viz.,

$$i \partial_t \Psi_{\Sigma}(x) = -i \alpha^i \frac{d}{dx^i} \Psi_{\Sigma}(x) + i \alpha^i \partial_i y(x) \partial_y \Psi_{\Sigma}(x) + \frac{1}{\varepsilon} \beta (\hat{H}\Psi_{\Sigma}(x)), \quad (3.8)$$

where

$$\frac{d}{dx^i} := \partial_i + \partial_i y(x) \partial_y + \partial_{ij} y(x) \partial_{y_j} + \ldots \quad (3.9)$$

is the total derivative along the subspace $\Sigma$, $\beta := \gamma^0$, $\beta^2 = 1$, and $\alpha^i := \beta \gamma^i$. Here we also introduced the shorthand notation $\Psi_{\Sigma}(x) := \Psi_{\Sigma}(y(x), x, t)$.

Hence, the time evolution of the wave functional of the quantum scalar field is given by:

$$i \partial_t \Psi = \int d\mathbf{x} \ Tr \left\{ \frac{\delta \Psi}{\delta \Psi_{\Sigma}^T(x, t)} \left[-i \alpha^i \frac{d}{dx^i} \Psi_{\Sigma}(x) + i \alpha^i \partial_i y(x) \partial_y \Psi_{\Sigma}(x) \right. \right. \right.

$$

$$\left. - \frac{1}{2} \varepsilon \beta \partial_{yy} \Psi_{\Sigma}(x) + \frac{1}{\varepsilon} \beta V(y(x)) \Psi_{\Sigma}(x) \right] \right\}. \quad (3.10)$$

Eq. (3.10) is obtained by inserting the explicit expression of the DW Hamiltonian operator of the nonlinear scalar field given by Eq. (2.10) into Eq. (3.8).

From Eq. (3.6) we obtain the expressions for the first and the second total functional.
derivatives of $\Psi$ with respect to $y(x)$, viz.,

$$\frac{\delta \Psi}{\delta y(x)} = \text{Tr} \left\{ \frac{\delta \Psi}{\delta \Psi_S^T(x,t)} \partial_y \Psi_S(x) \right\} + \frac{\delta \Psi}{\delta y(x)}. \quad (3.11)$$

$$\frac{\delta^2 \Psi}{\delta y(x)^2} = \text{Tr} \left\{ \frac{\delta \Psi}{\delta \Psi_S^T(x,t)} \delta(0) \partial_{yy} \Psi_S(x) \right\} + \text{Tr} \text{Tr} \left\{ \frac{\delta^2 \Psi}{\delta \Psi_S^T(x,t) \otimes \delta \Psi_S^T(x,t)} \partial_y \Psi_S(x) \otimes \partial_y \Psi_S(x) \right\} + 2 \text{Tr} \left\{ \frac{\delta \Psi}{\delta \Psi_S^T(x,t)} \frac{\partial \delta \Psi}{\partial \Psi_S(x)} \right\} + \frac{\delta^2 \Psi}{\delta y(x)^2}. \quad (3.12)$$

Here and in what follows $\tilde{\delta}$ denotes the partial functional derivative with respect to $y(x)$, and $\delta(0)$ is the result of functional differentiation of a function with respect to itself at the same spatial point.

Our first observation is that the potential energy term in the functional Schrödinger equation for the scalar field, see Eqs. (2.11) and (2.9), should be obtained from the potential energy term in Eq. (3.10), i.e.,

$$\int dx \text{Tr} \left\{ \frac{\delta \Psi_S}{\delta \Psi_S^T(x)} \frac{1}{\kappa} \beta V(y(x)) \Psi_S(x) \right\} \mapsto \int dx \ V(y(x)) \Psi. \quad (3.13)$$

It can be accomplished if

$$\text{Tr} \left\{ \frac{\delta \Psi_S}{\delta \Psi_S^T(x)} \frac{1}{\kappa} \beta \Psi_S(x) \right\} \mapsto \Psi \quad (3.14)$$

in any point $x$. The precise meaning of the operation $\mapsto$ will be established below.

The second observation is obtained by functionally differentiating both sides of Eq. (3.14) with respect to $\Psi_S^T(x)$:

$$\text{Tr} \left\{ \frac{\delta^2 \Psi_S}{\delta \Psi_S^T(x) \otimes \delta \Psi_S^T(x)} \frac{1}{\kappa} \beta \Psi_S(x) \right\} + \frac{\delta \Psi_S}{\delta \Psi_S^T(x)} \frac{1}{\kappa} \beta \delta(0) \mapsto \frac{\delta \Psi_S}{\delta \Psi_S^T(x)}, \quad (3.15)$$

where $\delta(0) = \delta \Psi_S(x)/\delta \Psi_S^T(x)$. We conclude, that the second term in (3.12), which has no counterparts in the familiar functional Schrödinger equation, vanishes, provided

$$\frac{1}{\kappa} \beta \delta(0) \mapsto 1. \quad (3.16)$$

Similarly, our third observation is that the term $\kappa \beta \partial_{yy} \Psi_S$ in (3.10) reproduces the first term in Eq. (3.12) and therefore, in the functional Schrödinger equation with $\tilde{\text{H}}$ given by Eq. (2.9), if

$$\beta \kappa \mapsto \delta(0). \quad (3.17)$$
We see that this condition is consistent with Eq. (3.16) in the sense that (3.16) is fulfilled provided (3.17) is fulfilled.

Now, if we recall the origin of Dirac matrices in precanonical quantization as the quantum representations of differential forms, we can readily recognize the conditions (3.16) and (3.17) as the inverse quantization map $q$ in Eq. (2.8) in the limit of the infinitesimal elementary volume $\frac{1}{\kappa} \to 0$, i.e. Eq. (3.17) is understood as follows:

$$\beta \kappa \to 0 \implies \delta(0).$$  

(3.18)

Note that one may think the mapping in Eq. (3.18) is the “Wick rotation” in the hyperbolic complex plane $(1, \beta)$ combined with the limit $\kappa \to \infty$.

Forth, if Eq. (3.10) is supposed to lead to the description in terms of the wave functional $\Psi$ alone, then the third term in (3.12), which is proportional to $\partial_y \Psi(x)$, should cancel the second term in (3.10), which is also proportional to $\partial_y \Psi(x)$. This requirement leads to the condition which further restricts the dependence of $\Psi$ on $\Psi(x)$ and $y(x)$, viz.,

$$\frac{\delta \Psi}{\delta \Psi(x)} \gamma^i \partial_i y(x) \mapsto - \frac{\delta \delta \Psi}{\delta \Psi(x) \partial y(x)}.$$  

(3.19)

By introducing the notation

$$\Phi(x) := \frac{\delta \Psi}{\delta \Psi(x)},$$  

(3.20)

and taking into account that $\partial y(0) = 0$, the solution of Eq. (3.19) can be found in the form

$$\Phi(x) = \Xi(\Psi(x); \tilde{x}) e^{-iy(x) \gamma^i \partial_i y(x)/\kappa},$$  

(3.21)

where $\Xi(\Psi(x); \tilde{x})$ denotes a functional of $\Psi(x')$ at $x' \neq x$. Consequently,

$$\frac{\delta \Phi(x)}{\delta \Psi(x)} = 0$$  

(3.22)

or equivalently,

$$\frac{\delta^2 \Psi}{\delta \Psi(x) \partial \Psi(x)} = 0.$$  

(3.23)

We note that the latter equality is consistent with Eqs. (3.15) and (3.16).

Now, Eqs. (3.20) and (3.21) lead to the following solution:

$$\Psi = \text{Tr} \left\{ \Xi(\Psi(x); \tilde{x}) e^{-iy(x) \gamma^i \partial_i y(x)/\kappa} \frac{1}{\kappa} \beta \Psi(x) \right\} \bigg|_{\beta \kappa \to \delta(0)},$$  

(3.24)

which is valid for any $x$ and in combination with the inverse quantization map (3.18). It is easy to check that Eq. (3.24) is consistent with Eq. (3.14).
The fifth observation is that the last term in (3.12), evaluated on the solution (3.24), yields:

\[ \frac{\bar{\delta}^2 \Psi}{\delta y(x)^2} \mapsto (\nabla y(x))^2 \Psi. \] (3.25)

Hence, it correctly reproduces the \( \frac{1}{2} (\nabla y(x))^2 \) term in the functional Schrödinger equation (2.1) with \( \hat{H} \) given by Eq. (2.9). This calculation thus indicates that those are the twisting factors \( e^{-iy(x)\gamma^i \partial_i y(x)/\kappa} \) in front of the precanonical wave functions in (3.24) which account for the non-ultralocality (in Klauder’s terminology [33]) of relativistic scalar field theory.

Our sixth observation concerns the first term in the right hand side of Eq. (3.10), which contains the total derivative. Namely, by integration by parts it takes the form

\[ \int d\mathbf{x} \, \text{Tr} \left\{ \left( \frac{d}{dx^i} \Phi(x) \right) \gamma^i \Psi_\Sigma(x) \right\}. \] (3.26)

Then, by taking the total derivative \( \frac{d}{dx^i} \) of the explicit expression of \( \Phi \) in Eq. (3.21):

\[ \frac{d}{dx^i} \Phi(x) = -i \Xi([\Psi_\Sigma]; \mathbf{x}) e^{-iy(x)\gamma^i \partial_i y(x)/\kappa} (\gamma^k \partial_k y(x) \partial_i y(x) + y(x) \gamma^k \partial_k y(x)), \] (3.27)

and using the expression of \( \Psi \) in Eq. (3.24), we transform Eq. (3.26) to the following form

\[ -i \Psi \int d\mathbf{x} \, (\gamma^k \partial_k y(x) \partial_i y(x) + y(x) \gamma^k \partial_k y(x)) \gamma^i, \] (3.28)

which obviously vanishes upon integration by parts. Hence, the first (total derivative) term in the right hand side of (3.10) does not contribute to the functional derivative equation describing the time evolution of \( \Psi \).

Finally, the functional \( \Xi([\Psi_\Sigma]) \) in (3.24) is specified by combining all the above observations together and noticing that the formula Eq. (3.24) is valid for any \( x \). It can be accomplished only if the functional \( \Psi \) has the structure of the continuous product of identical terms at all points \( x \), viz.,

\[ \Psi = \text{Tr} \left\{ \prod_x e^{-iy(x)\gamma^i \partial_i y(x)/\kappa} \Psi_\Sigma(y(x), x, t) \right\} \bigg|_{\beta, \kappa \rightarrow \kappa(0)}^{-1}. \] (3.29)

Thus we have obtained the expression of the Schrödinger wave functional in terms of precanonical wave functions. The equality in the above expression implies the inverse of the Clifford algebraic quantization map \( q \) and the limit of the infinitesimal elementary volume element \( \frac{1}{\kappa} \rightarrow 0 \). Moreover, the preceding consideration also derives term by term the functional Schrödinger equation for the wave functional \( \Psi \) from the precanonical Schrödinger equation for the wave function \( \Psi \) restricted to the subspace \( \Sigma \).

As we have already mentioned in previous paper [18], the inverse quantization map in the limit of the infinitesimal \( \frac{1}{\kappa} \) means that

\[ \frac{1}{\kappa} \beta \xrightarrow{i \rightarrow i(0)} d\mathbf{x}. \] (3.30)
Therefore the expression of the wave functional in Eq. (3.29) can be written in the form of the multidimensional product integral (c.f. [32])

\[ \Psi = \text{Tr} \left\{ \prod_x e^{-iy(x)\alpha^i \partial_i y(x) dx} \Psi_\Sigma(y(x), x, t) \right\}, \]

(3.31)

which may be more practical to use than Eq. (3.29).

Further, let us recall that \( \Psi_\Sigma \) obeys Eq. (3.8). According to Eqs. (3.26) - (3.28) the total derivative term does not contribute to the functional derivative equation on \( \Psi \). In the case of scalar field theory Eq. (3.8) without the total derivative term can be cast in the form

\[ i \partial_t \Psi_\Sigma = \frac{1}{2} \kappa \beta \left( i \kappa \partial_y + \gamma^i \partial_i y(x) \right)^2 \Psi_\Sigma + \frac{1}{2} \kappa \beta \left( V(y(x)) + \frac{1}{2} (\nabla y(x))^2 \right) \Psi_\Sigma =: \beta \mathcal{E} \Psi_\Sigma. \]

(3.32)

The structure of the operator \( \mathcal{E} \) in the right hand side of Eq. (3.32):

\[ \mathcal{E} = \frac{1}{2 \kappa} \left( i \kappa \partial_y + \gamma^i \partial_i y(x) \right)^2 + \frac{1}{2 \kappa} \left( V(y(x)) + \frac{1}{2} (\nabla y(x))^2 \right), \]

(3.33)

resembles the structure of magnetic Schrödinger operator in \( y \)-space with the “matrix magnetic potential” \( \gamma^i \partial_i y(x) \) and the “electric potential” \( V(y(x)) + \frac{1}{2} (\nabla y(x))^2 \).

The magnetic term in Eq. (3.33) is pure gauge (in \( y \)-space), so that it does not change the eigenvalues of \( \mathcal{E} \) in comparison with \( \hat{H} \). Its influence reduces to the phase shift of the eigenstates of \( \hat{H} \) by the factor \( e^{i \gamma^i \partial_i y(x) / \kappa} \). Note that Eq. (3.32) is valid in the fibers of fields variables and their first jets over each point \( x \), with \( x \)-s here just enumerating the fibers in which Eq. (3.32) is written.

The additive \( \frac{1}{2} (\nabla y(x))^2 \) to the “electric potential” term in Eq. (3.33) modifies the mass term in the potential term of the DW Hamiltonian operator. Namely, by substituting it into Eq. (3.7) and integrating by parts using the property (3.14), we conclude that under the restriction to \( \Sigma \) the mass term \( \frac{1}{2} m^2 y^2 \) in \( V(y) \) is replaced by

\[ \frac{1}{2} y(x)(m^2 - \nabla^2)y(x). \]

(3.34)

Correspondingly, the parameter \( m \) in the expressions of precanonical wave functions is formally replaced by \( \sqrt{m^2 - \nabla^2} \), when they are restricted to \( \Sigma \).

For example, in the case of free massive scalar field theory the ground state of DW Hamiltonian operator, \( \hat{H} = -\frac{1}{2} \kappa^2 \partial_y y + \frac{1}{2} m^2 y^2 \), is given, up to the normalization factor, by \( \Psi_0 \sim e^{-\frac{m}{\kappa} y^2} \), and its eigenvalue is \( \frac{1}{2} m \kappa \) [5-7]. Then the eigenstates of \( \beta \hat{H} \) corresponding to the positive eigenvalues are given by \( \sim (1 + \beta) e^{-\frac{m}{\kappa} y^2} \). Therefore, the corresponding ground state wave function restricted to \( \Sigma \): \( \Psi_{0\Sigma} \), will take the form

\[ \Psi_{0\Sigma} \sim e^{iy(x)\gamma^i \partial_i y(x) / \kappa} (1 + \beta) e^{-\frac{1}{\kappa} y(x) \sqrt{m^2 - \nabla^2} y(x)}. \]

(3.35)
By substituting the last expression into (3.31) we see that the magnetic phase factors in Eq. (3.35) and Eq. (3.29) will cancel each other and finally we obtain

$$\Psi \sim \text{Tr} \prod_x (1 + \beta) e^{-\frac{1}{2} \beta y(x) \sqrt{m^2 - \nabla^2} y(x) dx} \sim e^{-\frac{1}{2} \int y(x) \sqrt{m^2 - \nabla^2} y(x) dx},$$  (3.36)

where $\beta(1 + \beta) = (1 + \beta)$ is used (c.f. our earlier treatment in [17]).

The right hand side of Eq. (3.36) reproduces the vacuum state solution of the functional derivative Schrödinger equation for the free scalar field (see e.g. [19]). Usually it corresponds to the picture of the vacuum as the continuum of harmonic oscillators with the zero-point energy $\frac{1}{2} \sqrt{m^2 + k^2}$ at every point of $k$-space. Here the vacuum state of free quantum scalar field is obtained as the product of the ground state wave functions of the DW Hamiltonian operator (which in this case corresponds to the harmonic oscillator in $y$-space) over all points $x$ of the space.

4 Conclusions

Precanonical quantization, which is based on the space-time symmetric generalization of Hamiltonian formalism in field theory (De Donder-Weyl theory), leads to the description of quantum fields in terms of Clifford-valued wave functions on the bundle of fields variables over the space-time. These wave functions obey a Dirac-like generalization of the Schrödinger equation. The formulation includes a small parameter $\frac{1}{\kappa}$ of the dimension of spatial volume, which appears on the dimensional grounds in the representation of precanonical quantum operators.

A proper understanding of the connection between precanonical quantization and the standard methods of quantization in field theory is important for the physical interpretation of the results of precanonical quantization. In this paper we discuss how the results of canonical quantization in the functional Schrödinger representation are related to the precanonical quantization and improve the arguments of the previous discussions in [17, 18].

Summarizing the considerations in Sect. 3 and those in the preceding paper [18], we have proven that

**Proposition:** If $\Psi(y, x)$ is a precanonical wave function obeying the precanonical covariant analogue of the Schrödinger equation (2.2), and $\Psi_\Sigma(y(x), x, t)$ is its restriction to the subspace $\Sigma$ representing a field configuration $y(x)$ at the moment of time $t$, then in the limiting case $\beta x \rightarrow \delta(0)$ there exists a unique composite functional $\Psi$ of $\Psi_\Sigma$ and $y(x)$, whose time evolution is governed by the standard functional derivative Schrödinger equation obtained from canonical quantization. The time evolution of $\Psi$ is completely determined by the time evolution of $\Psi_\Sigma(y(x), x, t)$ given by the precanonical Schrödinger equation restricted to $\Sigma$.

The expression of the Schrödinger wave functional $\Psi$ in terms of the precanonical wave functions $\Psi$ is given by the product integral formula, Eq. (3.37), which is the nec-
necessary and sufficient condition for the functional $\Psi$ to satisfy the canonical Schrödinger equation if $\Psi$ satisfies the precanonical Schrödinger equation.

This result leads us to the conclusion that the canonical QFT in the functional Schrödinger representation is the limiting case $\beta \kappa \mapsto \delta(0)$, or $\beta / \kappa \mapsto d$, of the theory obtained from precanonical quantization.

It is interesting that the introduction of the ultra-violet scale $\kappa$ in precanonical quantization does not modify the relativistic space-time at small distances. It rather defines the scale of “very small” distances for the specific field theory under consideration. It is tempting, however, to interpret $\kappa$ as the universal fundamental ultra-violet scale similar to the Planck scale, where the idea of space-time continuum is supposed to break due to the quantum gravity effects. In this case precanonical quantization might provide new insights into the Planck scale physics.

Note in conclusion that the manifest respect of the space-time symmetry on the level of quantization and the nonperturbative nature of the construction of interacting quantum field theories within the precanonical quantization approach potentially make it a suitable framework in the context of quantum gravity and quantum gauge theories (c.f. [34] for the recent discussions).

Acknowledgements: I am grateful to the University of Wuppertal, University of Göttingen and Erwin Schrödinger Institute in Vienna for their financial support helping me to participate in the Workshops they have hosted in 2013. I am thankful to J. Akram whose remarks on the earlier version of the manuscript have helped me to improve the presentation. Thanks are due to H. Kleinert, A. Pelster and J. Dietel for their kind hospitality at the Research Center of Einstein Physics.

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