STUDY OF THE EXISTENCE OF SUPERSOLUTIONS FOR
NONLOCAL EQUATIONS WITH GRADIENT TERMS

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Abstract. We study the existence of positive supersolutions of nonlocal equations

\((-\Delta)^s u + |\nabla u|^q = \lambda f(u)\)

in exterior domains where the datum \(f\) can be compared with \(u^p\) near the origin. We prove that the existence or bounded
supersolutions its depend of the values of \(p, q\) and \(s\).

1. Introduction

Establish nonexistence results for positive solutions of nonlinear equations have
been object of study by many authors in the last decades due to its own interest
and its applications like proving a priori bounds for positive solutions (see for instance [26]) or studying the singularities of such solutions (see [30]). The first
kind of nonexistence results, commonly called Liouville’s type results, was obtained
by Gidas and Spruck for the seminal equation

\[-\Delta u = u^p\]

in \(\mathbb{R}^N\), (see [25]). Later on the question of nonexistence of positive solutions for equations
that involve different differential operators than the Laplacian, and other powers,
was ahead by several authors in, for example, [6, 10, 11, 19, 29]. Recently a very general
result have been obtained by Armstrong and Sirakov in [5] where they proved
a very general Liouville type result for viscosity supersolutions of the equation

\[Mu = f(u)\]

in \(\mathbb{R}^N \setminus B_{R_0}\), where \(R_0 > 0\), \(M\) is a general fully local nonlinear operator and \(f\) is a real positive
function in \((0, \infty)\) satisfying

\[\lim_{t \to 0^+} \frac{f(t)}{t^{p^*}} > 0, \quad p^* := \frac{N}{N - 2}.
\]

This result in particular implies that the equation \(-\Delta u = u^p\) does not admit any
positive supersolutions if \(0 < p \leq p^*\) when it is posed, not only in the Euclidean
space \(\mathbb{R}^N\), but also in an exterior domain. Since the result obtained in [5] it is based
in a Hadamard type property of the solutions which requires that the differential
operator \(M\) is homogeneous, it is natural to ask if this kind of nonexistence result
can be obtained if the operator is not homogeneous. In fact it was the main objective
of the recent work of Alarcón, García-Melián and Quaas [2] where the authors prove
a general nonexistence result for the supersolutions of

\[-\Delta u + |\nabla u|^q = \lambda f(u)\]

in \(\mathbb{R}^N \setminus B_{R_0}\), \(\lambda > 0\),

depending of the range of \(p, q\) and \(\lambda\). We have to mention here that, before [2] other
authors addressed this problem but establishing results related with the existence or
nonexistence of radial supersolutions (see for instance [32, 37]). We also notice that
the presence of the gradient term introduce the possibility to have supersolutions
of (1) that are bounded or those who diverges at infinite because the equation is
posed in an exterior domain. To complete the study of (1), in the more general
framework of fully differential operators, Rossi in [31] proved the nonexistence of supersolutions that does not blow up at infinity of
\begin{equation}
-\Delta u + b(x)|\nabla u| = c(x)u \quad \text{in } \mathbb{R}^N \setminus B_R,
\end{equation}
when $b(x)$ and $c(x)$ are bounded functions. Finally we also show up that in [1] the nonexistence of positive supersolutions of (1) for general unbounded weights $b(x)$ and $c(x)$ was getting using a completely different approach than in [31].

The previous results commented here are, as far as we known the optimal Liouville’s type results for supersolutions obtained until the date for the equation (1). Thus, the situation can be summarized in the following picture where in the red zone the problem (1) does not admit positive supersolutions which do not blow up at infinity and in the blue one a bounded supersolution exists depending on the value of $\lambda$ (see [2, Theorems 1,2,3] and [31]).

Motivated by the previous results our objective in the work at hands is to study the Liouville’s type result for supersolutions of the nonlocal equation
\begin{equation}
(-\Delta)^s u + |\nabla u|^q = \lambda f(u) \quad \text{in } \mathbb{R}^N \setminus B_R, \frac{1}{2} < s < 1, \lambda > 0,
\end{equation}
where $(-\Delta)^s$, $0 < s < 1$, is the well-known fractional Laplacian, defined, on smooth functions as
\begin{equation}
(-\Delta)^s u(x) := C_{N,s} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy,
\end{equation}
where $C_{N,s}$ is a normalization constant that is usually omitted for brevity. The integral in (1) has to be understood in the principal value sense, that is, as the limit as $\epsilon \to 0$ of the same integral taken in $\mathbb{R}^N \setminus B_\epsilon(x)$, i.e, the complementary of the ball of center $x$ and radius $\epsilon$. It is clear that the fractional Laplacian is well defined for functions that belong, for instance, to $\mathcal{L}_{2s} \cap \mathcal{C}^2_{loc}$ where
\begin{equation}
\mathcal{L}_{2s} := \left\{ u : \mathbb{R}^N \to \mathbb{R} : \int_{\mathbb{R}^N} \frac{|u(x)|}{1 + |x|^{N+2s}} < \infty \right\}.
\end{equation}
Alternatively, the fractional Laplacian can be written as
\begin{equation}
(-\Delta)^s u(x) := C_{N,s} \int_{\mathbb{R}^N} \frac{2u(x) - u(x + y) - u(x - y)}{|y|^{N+2s}} dy,
\end{equation}
for any $u \in \mathcal{L}_{2s} \cap \mathcal{C}^2_{loc}$. The above representation is useful because the integral is absolutely convergent. Problems with non local diffusion that involve the fractional Laplacian operator, and other integro-differential operators, have been intensively studied in the last years since they appear when we try to model different physical situations as anomalous diffusion and quasi-geostrophic flows, turbulence and water
Theorem 1.1. Let $N > 2s > 1$. Assume that $f: \mathbb{R} \to \mathbb{R}$ is a continuous function verifying (1) and such that $f(t) > 0$ in $(0,\infty)$. If
\[ 1 < q < \frac{N}{(N+1-2s)} \text{ and } 0 < p < \frac{(2s-1)q}{2s-q}, \]
or
\[ q \geq \frac{N}{(N+1-2s)} \text{ and } 0 < p < \frac{N}{(N-2s)}, \]
then there are no positive supersolutions to (1) which do not blow up at infinity.

Theorem 1.2. Let $N > 2s > 1$. Assume that $f: \mathbb{R} \to \mathbb{R}$ is a continuous nondecreasing function verifying (1) and such that $f(t) > 0$ in $(0,\infty)$. If $p = \frac{N}{(N-2s)}$, and $q > \frac{N}{(N+1-2s)}$ then there are no positive supersolution of (1) which do not blow up at infinity.

That is, we obtain the equivalent result as [2, Theorem 1, Theorem 3] in the nonlocal framework.

Regarding with the existence of supersolutions that does not diverges at infinity, doing a carefully analysis we obtain the next

Theorem 1.3. Let $N > 2s > 1$. Then there exists a positive bounded supersolution of
\[ (-\Delta)^s u + |\nabla u|^q = \lambda u^p, \quad \text{in } \mathbb{R}^N \setminus B_{R_0}, \lambda > 0, \]
if one of the following cases hold
\begin{align*}
(i) & \quad p = \frac{N}{N-2s} \text{ and } 0 < q < 1; \\
(ii) & \quad p = \frac{N}{N-2s}, q = 1 \text{ and } R_0 > \bar{R}_0 \text{ for some } \bar{R}_0 \text{ big enough}; \\
(iii) & \quad p = \frac{N}{N-2s}, 1 < q \leq \frac{N}{N+1-2s} \text{ and } \lambda \in (0,\lambda_0) \text{ for some } \lambda_0 > 0; \\
\end{align*}
(iv) \( \frac{2s-1}{2s-q} < p < \frac{N}{N-2s}, \quad \frac{N+2s}{N+1} < q < \frac{N}{N+1-2s}, \) and \( \lambda \in (0, \lambda_0) \) for some \( \lambda_0 > 0; \)

(v) \( \frac{N+2s}{N+2s-q} < p < \frac{N}{N-2s}, \quad 0 < q < \frac{N+2s}{N+1}, \) \( q \neq 1 \) and \( \lambda \in (0, \lambda_0) \) for some \( \lambda_0 > 0. \) In the case of \( q = 1, R_0 \) should be bigger than \( \tilde{R}_0 \) for some \( \tilde{R}_0 \) big enough.

(vi) \( \frac{N+2s}{N} < p < \frac{N}{N-2s}, \quad q = \frac{N+2s}{N+1}, \) and \( \lambda \in (0, \lambda_0) \) for some \( \lambda_0 > 0. \)

We show up that on the contrary of the local case, radial reductions are not useful to study our equations (see for instance [14] and references therein) and, moreover, the function \( m \) does not satisfies that \( m(R_1, R_2) = \min\{m(R_1), m(R_2)\} \) where \( m(R) = \min_{|x| = R} u(x) \) is, in the local case, a monotone function for \( R > R_1, \) (see [2, Lemma 1]). This, together with the fact that not much is known for the explicit value of general radial functions of the nonlocal operator, forces us to study different cases separately specially when \( p \) is critical. For that, optimal estimates for the fractional Laplacian of some particular radial functions will be needed (see Lemmas 2.1, 2.3, 3.4 and the proof of Theorem 1.2). Due to the complications that the nonlocal operator introduced, we left as an open problem the existence of bounded supersolutions in the range

\[
\frac{2s - 1}{2s - q} < p < \frac{N}{N + 2s - q}, \quad 0 < q \leq \frac{N + 2s}{N + 1},
\]

because in this case, it is not clear to us how can we find a suitable function \( \psi \) such that \(|(-\Delta)^s \psi|\) decays faster than \(|\nabla \psi|^q\) at infinity (see Cases 4 and 5 in the proof of Theorem 1.3)

We also notice that the essential feature that appear in most of the proof is a comparison principle that in the local framework was proved, for example, in [27, Theorem 10.1]. Finally we want also to mention that, in the supercritical case, that is when \( N > 2s \) and \( p > \frac{N}{(N-2s)}, \) Felmer and Quaas in [23, proof of Theorem 1.3] have recently shown that for any \( \beta \in (1/(p-1), (N-2s)/2s) \) there exists \( C > 0 \) such that

\[
v_\beta(x) = C(1 + |x|)^{-2s\beta},
\]

is a bounded supersolution of \((-\Delta)^s u = u^p\) in \( \mathbb{R}^N. \) Therefore, trivially, \( v_\beta \) is also a supersolution of (1) when \( \lambda = 1 \) and \( f(u) = u^p \) for all \( q \geq 0. \) So that we can also formulate the next

**Theorem 1.4.** Let \( N > 2s, \) \( p > \frac{N}{(N-2s)} \) and \( q \geq 0. \) Then there is a classical positive and bounded supersolution of \((-\Delta)^s u + |\nabla u|^q = \lambda u^p\) in \( \mathbb{R}^N \setminus B_{R_0}, \lambda \leq 1. \)

The conclusion obtained in the Theorems 1.1, 1.2, 1.3 and 1.4 for \( f(u) = \lambda u^p, \) \( \lambda \leq 1, \) can be summarized in the following graphic where

\[
q_1 = \frac{N + 2s}{N + 1} \quad \text{and} \quad q_2 = \frac{N}{N + 1 - 2s}.
\]
In the red zone there is not any positive supersolution which do not blow up at infinite. In the pink one (supercritical case \( p > \frac{N}{(N-2)s} \) and critical \( p = \frac{N}{(N-2)s} \), with \( 0 < q < 1 \)) it is not needed to assume any extra restrictions on \( \lambda \) or \( R_0 \) to prove the existence of bounded supersolutions. The blue region corresponds to the cases in which the existence of bounded supersolutions depends on some upper bound on the parameter \( \lambda \) (see cases iii)-vi) of Theorem 1.3). Finally the green color shows the situation obtained when \( q = 1 \) in which is needed that \( R_0 \) is big enough.

2. Some preliminaries

In this section, we gather some preliminary properties and definitions which will be useful in the forthcoming ones.

2.1. Classical solution.

Definition 2.1. Let \( s \in (1/2, 1) \), \( q, p, R_0 > 0 \), \( B_{R_0} \) be the \( N \)-ball with radius \( R_0 \) and center 0, and \( f: \mathbb{R} \rightarrow \mathbb{R} \) be a function. We say that \( u \in C^2(\mathbb{R}^N \setminus B_{R_0}) \cap L^{2s} \) is a classical supersolution (resp. subsolution) of (1) if

\[
(-\Delta)^s u(x) + |\nabla u(x)|^q \geq (\text{resp.} \leq) f(u) \text{ in } \mathbb{R}^N \setminus B_{R_0},
\]

where \( L^{2s} \) was given in (1). We also call \( u \) a classical solution of (1) if it is both a sub- and supersolution.

Throughout this article we will be always dealing with positive classical supersolutions. However, as we commented in the introduction, we will not make any assumption about if these supersolutions are bounded or not. In fact we will distinguish between those that are bounded and others that not.

2.2. Particular subsolutions. We show up now some results regarding with the existence of subsolutions when the datum \( f \) is equal to zero. Before that we give a useful pointwise inequality that will be also the main tool to prove the existence of bounded supersolutions of (1) (see Section 4). The proof of it can be found in [12, Lemma 2.1].

Lemma 2.1. Let \( \varphi \in C^2(\mathbb{R}^N) \) be a positive real function that is radially symmetric and decreasing for every \( |x| > 1 \). Assume also that \( \varphi(x) \leq |x|^{-\sigma}, |\nabla \varphi(x)| \leq c_0 |x|^{-\sigma-1} \) and \( |D^2 \varphi(x)| \leq c_0 |x|^{-\sigma-2} \) for some \( \sigma > 0 \) and \( |x| \) large enough. Then
there exist $\tilde{R} > 1$ and some positive constants $c_1, c_2, c_3$ that depend only on $s, N$, and $\|\varphi\|_{C^2(\mathbb{R}^n)}$, such that
\[
|(-\Delta)^s \varphi(x)| \leq \begin{cases} 
\frac{c_1}{|x|^{\sigma + 2s}} & \text{if } \sigma < N, \\
\frac{c_2 \log(|x|)}{|x|^{N+2s}} & \text{if } \sigma = N, \\
\frac{c_3}{|x|^{N+2s}} & \text{if } \sigma > N,
\end{cases}
\]
for every $|x| > \tilde{R} > 1$. For $\sigma > N$ the reverse estimate hold from below, that is, if $\varphi \geq 0$ then there is a positive constant $c_4$ such that
\[
|(-\Delta)^s \varphi(x)| \geq \frac{c_4}{|x|^{N+2s}},
\]
for all $|x| > \tilde{R} > 1$.

For any $q > 1$ let us consider now
\begin{equation}
(-\Delta)^s u(x) + |\nabla u|^q = 0 \quad \text{in } \mathbb{R}^N \setminus B_R,
\end{equation}
for large enough $R$. Using the previous result we get the next

**Lemma 2.2.** Let $1 < 2s < N$, $\phi \in C^2(\mathbb{R}^N)$ be as in Lemma 2.1. If
\[
1 < q < \frac{N + 2s}{N + 1} \quad \text{and} \quad \sigma \geq \frac{N + 2s}{q} - 1 > N, \quad \text{or}
\]
\[
q = \frac{N + 2s}{N + 1} \quad \text{and} \quad \sigma > N,
\]
then for any $A > 0$ small enough the function $\phi_A(x) = A\phi(x)$ is a classical subsolution of (2.2).

**Proof.** If $\sigma > N$, using Lemma 2.1 it follows that
\[
(-\Delta)^s \phi(x) \leq c(N, s, \sigma)|x|^{-N-2s} \quad \text{in } \mathbb{R}^N \setminus B_{\tilde{R}},
\]
with $c(N, s, \sigma) < 0$. Thus
\[
(-\Delta)^s \phi_A(x) + |\nabla \phi_A(x)|^q \leq c(N, s, \sigma)A|x|^{-N-2s} + A^q c_0 |x|^{-(\sigma + 1)q} \quad \text{in } \mathbb{R}^N \setminus B_{\tilde{R}},
\]
Hence, if $1 < q < \frac{N + 2s}{N + 1}$ and $\sigma \geq \frac{N + 2s}{q} - 1$ or if $q = \frac{N + 2s}{N + 1}$ and $\sigma > N$ then
\[
(-\Delta)^s \phi_A(x) + |\nabla \phi_A(x)|^q \leq (c(N, s, \sigma)A + A^q c_0)|x|^{-N-2s} \leq 0 \quad \text{in } \mathbb{R}^N \setminus B_{\tilde{R}},
\]
as long as $A$ is small enough.

Choosing a particular function in the class of those referred in Lemma 2.1, we can also get the existence of a subsolution of the homogeneous equation (2.2) for $q > \frac{(N+2s)}{(N+1)}$ for a suitable range of the positive parameter $\sigma$. Indeed, we have the following

**Lemma 2.3.** Let $1 < 2s < N$. If
\[
\frac{N + 2s}{N + 1} < q < \frac{N}{N + 1 - 2s} \quad \text{and} \quad N > \sigma \geq \frac{2s - q}{q - 1} > 0, \quad \text{or}
\]
\[
q \geq \frac{N}{N + 1 - 2s} \quad \text{and} \quad N > \sigma > N - 2s > 0,
\]
then for all $A, \varepsilon > 0$ small enough
\[
w_{\sigma, \varepsilon}^A(x) := \begin{cases} 
A\varepsilon^{-\sigma} & \text{if } |x| \leq \varepsilon, \\
A|x|^{-\sigma} & \text{if } |x| > \varepsilon,
\end{cases}
\]
is a classical subsolution of (2.2).
Proof. By [21, Lemma 4.1] (see also [9, Remark 4.7 (iii)] and [24]) we get that
\[ v_\sigma(x) := |x|^{-\sigma}, \quad \sigma \in (-2s, N), \]
is a classical solution of
\[ (-\Delta)^s v(x) = \gamma_\sigma |x|^{-2s} v \quad \text{in } \mathbb{R}^N \setminus \{0\}, \]
where
\[ \gamma_\sigma = \frac{2^{2s} \Gamma \left( \frac{N - \sigma}{2} \right) \Gamma \left( \frac{2s + \sigma}{2} \right)}{\Gamma \left( \frac{\sigma}{2} \right) \Gamma \left( \frac{N - 2s - \sigma}{2} \right)}, \]
is a concave function that is negative if \( \sigma \in (N - 2s, N) \) and, by symmetry, also if \( \sigma \in (-2s, 0) \). Taking \( \varepsilon > 0 \) and
\[ w_{\sigma, \varepsilon}(x) := \begin{cases} \varepsilon^{-\sigma} & \text{if } |x| \leq \varepsilon, \\ v_\sigma(x) & \text{if } |x| > \varepsilon, \end{cases} \]
for every \( |x| \gg 1 \) we get
\[ (-\Delta)^s w_{\sigma, \varepsilon}(x) = (-\Delta)^s v_\sigma(x) + \int_{B_\varepsilon} \frac{|y|^{-\sigma} - \varepsilon^{-\sigma}}{|x - y|^{N+2s}} dy \leq \gamma_\sigma |x|^{-\sigma - 2s} + C \varepsilon^{-\sigma+N} e^{-\sigma+N} |x|^{N+2s}. \]
Then, for every \( \sigma \in (N - 2s, N) \), \( \varepsilon \) small enough and \( |x| \gg 1 \) we have
\[ (-\Delta)^s w_{\sigma, \varepsilon}(x) \leq \gamma_{\sigma, \varepsilon} |x|^{-\sigma - 2s}, \]
where \( \gamma_{\sigma, \varepsilon} = \gamma_\sigma + C \varepsilon^{-\sigma+N} < 0 \). We observe now that
\[ \sigma + 2s \leq (1 + \sigma)q \iff \sigma \geq \frac{2s - q}{q - 1} =: \alpha(q), \]
and, moreover, the function \( \alpha(q) \) is decreasing in \((1, \infty)\) and satisfies
\[ \alpha \left( \frac{N + 2s}{N + 1} \right) = N \quad \text{and} \quad \alpha \left( \frac{N}{N + 1 - 2s} \right) = N - 2s. \]
Hence, under our hypothesis,
\[ (-\Delta)^s w_{\sigma, \varepsilon}(x) + |\nabla w_{\sigma, \varepsilon}(x)|^q \leq \gamma_{\sigma, \varepsilon} A |x|^{-\sigma - 2s} + A^q \sigma^q |x|^{-(\sigma+1)q} \leq (\gamma_{\sigma, \varepsilon} A + A^q \sigma^q) |x|^{-\sigma - 2s}, \quad |x| > 1, \]
due to \( w_{\sigma, \varepsilon}^A(x) = Aw_{\sigma, \varepsilon}(x) \). Then the conclusion follows taking \( A \) small enough. \( \square \)

2.3. Comparison Principle and some pointwise inequalities. To conclude this section we now show two different versions of the Comparison Principle for our equations and some useful inequalities. We begin with the next

**Theorem 2.1.** Let \( q > 0 \), \( u \) and \( v \) be classical sup and subsolution of
\[ (-\Delta)^s w + |\nabla w|^q = 0 \quad \text{in } \mathbb{R}^N \setminus B_{R_0}, \]
respectively. If \( u \) is positive in \( \mathbb{R}^N \setminus B_{R_0} \),
\[ v(x) \to 0 \quad \text{as } |x| \to \infty, \]
and \( u(x) \geq v(x) \) in \( B_{R_0} \) then \( u(x) \geq v(x) \) in \( \mathbb{R}^N \).
Proof. Let \( w(x) := v(x) - u(x) \leq 0 \) in \( B_{R_0} \) and let us suppose that there exists \( x_1 \in \mathbb{R}^N \setminus B_{R_0} \) such that \( w(x_1) > 0 \). Since \( u \) is positive in \( \mathbb{R}^N \setminus B_{R_0} \), \( v(x) \to 0 \) as \( |x| \to \infty \), and \( u(x) \geq v(x) \) in \( B_{R_0} \) there is \( x_0 \in \mathbb{R}^N \setminus B_{R_0} \) such that
\[
 w(x_0) \geq u(x), \quad x \in \mathbb{R}^N.
\]
Thus
\[
0 = \nabla w(x_0) = \nabla v(x_0) - \nabla u(x_0),
\]
and
\[
0 \leq (-\Delta)^s w(x_0) = (-\Delta)^s v(x_0) - (-\Delta)^s u(x_0) \leq 0.
\]
Therefore \( w(x) = w(x_0) \) for every \( x \in \mathbb{R}^N \) which is a contradiction, because \( w(x_0) > 0 \) and \( w(x) \leq 0 \) in \( B_{R_0} \). Thus \( w(x) \leq 0 \) in \( \mathbb{R}^N \setminus B_{R_0} \) as wanted. \( \square \)

**Theorem 2.2.** Let \( q > 0 \), \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function and \( u \) and \( v \) be such that
\[
(-\Delta)^s u + |\nabla u|^q \geq f(x) \geq (-\Delta)^s v + |\nabla v|^q \quad \text{in } R_0 < |x| < R,
\]
in classical sense. If \( u \geq v \) in \( B_{R_0} \cup (\mathbb{R}^N \setminus B_R) \), then \( u(x) \geq v(x) \) in \( \mathbb{R}^N \).

We finish this section with the next two lemmas that are simply a generalization of [2, Lemmas 3 and 4] so that, since their proofs are analogous we omit it.

**Lemma 2.4.** Let \( q > 1 \), \( p > 0 \), \( 2s > 1 \) and \( h(R) \) be a positive decreasing function defined for \( R > R_0 \) and verifying
\[
h(2R)^p \leq C \left( \frac{h(R)}{R^{2s}} + \frac{h(R)^q}{R^q} \right),
\]
for \( R > R_0 \) and some positive constant \( C > 0 \). Then.

a) If \( 0 < p \leq 1 \) then for every \( \theta < 0 \) there is a positive constant \( C \) such that
\[
h(R) \leq CR^\theta,
\]
for every \( R > R_0 \).

b) If \( 1 < q < 2s \) and \( 1 < p < \frac{2s}{2s - q} \) or \( q \geq 2s \) and \( p > 1 \), then for every \( \theta \in \left( -\frac{2s}{p-1}, 0 \right) \) there is a positive constant \( C \) such that
\[
h(R) \leq CR^\theta,
\]
for every \( R > R_0 \).

**Lemma 2.5.** Let \( q > 1 \), \( p > 0 \), \( 2s > 1 \) and \( h(R) \) be a positive decreasing function defined for \( R > R_0 \) and verifying
\[
h(R)^p \leq C \left( \frac{h(R)}{R^{2s}} + \frac{h(R)^q}{R^q} \right),
\]
for \( R > R_0 \) and some positive constant \( C > 0 \).

a) If \( 1 < q < 2s \) and \( p = \frac{2s}{2s - q} \) then there is a positive constant \( C \) such that
\[
h(R) \leq CR^{2s/p-1},
\]
for every \( R > R_0 \).

b) If \( q > \frac{N}{N+1-2s} \) and \( p = \frac{N}{N-2s} \) then there is a positive constant \( C \) such that
\[
h(R) \leq CR^{-N+2s},
\]
for every \( R > R_0 \).
3. Nonexistence results

Throughout all this section, we will assume that \( q > 1 \), \( f: \mathbb{R} \rightarrow \mathbb{R} \) is a continuous function, \( \lambda > 0 \) and \( u \) be a positive supersolution of (1) which does not blow up at infinity.

Given \( 0 < R_1 < R_2 \), we define
\[
m(R_1, R_2) := \min \{ u(x) : x \in A(R_1, R_2) \},
\]
where
\[
A(R_1, R_2) = \{ x : R_1 \leq |x| \leq R_2 \}.
\]

Observe that \( m(R_1, \cdot) \) is a nonincreasing function in \((R_1, \infty)\) and \( m(\cdot, R_2) \) is a nondecreasing function in \([0, R_2)\). As we mentioned in the introduction, unlike in the local case, it cannot be proved that \( m(R_1, R_2) = \min \{ m(R_1), m(R_2) \} \) where \( m(R) = \min_{|x|=R} u(x) \) is, in the local framework, a monotone function for \( R > R_1 \).

3.1. Preliminary results: some bounds for the \( m \) function. In order to prove the nonexistence of positive supersolutions which do not blow up at infinity, we will get some previous auxiliary lemmas regarding with the function \( m(R_1, R_2) \), \( 0 < R_1 < R_2 \), defined in (3). More precisely we obtain upper and lower bounds for \( m(0, R) \) that will be the key steps to obtain the nonexistence results (see Lemmas 3.2, 3.3 and 3.4).

**Lemma 3.1.** Let \( \lambda > 0 \), \( f(t) > 0 \) in \((0, \infty)\) be a continuous function and \( u \) be a positive classical supersolution of (1) which does not blow up at infinity. Then
\[
\lim_{R \to \infty} m(R, 4R) = 0,
\]
with \( m \) given in (3).

**Proof.** Let \( R > R_0 \) fixed but arbitrary and \( \eta \in C^\infty(\mathbb{R}) \) be such that
\[
\eta(t) := \begin{cases} 
1 & \text{if } t \in [2, 3], \\
0 & \text{if } t < 1 \text{ or } t > 4.
\end{cases}
\]

We define
\[
v(x) := u(x) - m(2R, 3R) \eta \left( \frac{|x|}{R} \right), \quad R > 0.
\]

Observe that, there exists \( x_R \in A(R, 4R) \) such that
\[
v(x_R) \leq v(x), \quad x \in \mathbb{R}^N.
\]

Then
\[
0 \geq (-\Delta)^s v(x_R) = (-\Delta)^s u(x_R) - \frac{m(2R, 3R)}{R^{2s}} (-\Delta)^s \eta \left( \frac{|x|}{R} \right),
\]
\[
0 = \nabla v(x_R) = \nabla u(x_R) - \frac{m(2R, 3R)}{R} \nabla \eta \left( \frac{|x|}{R} \right) \frac{x_R}{|x_R|}.
\]

Thus, since \( u \) is a positive supersolution of (1), we have that
\[
\lambda f(u(x_R)) \leq (-\Delta)^s u(x_R) + |\nabla u(x_R)|^q \leq C(\eta) \left( \frac{m(2R, 3R)}{R^{2s}} + \frac{m(2R, 3R)^q}{R^q} \right).
\]

Therefore, since \( m(2R, 3R) \) is bounded we get
\[
f(u(x_R)) \rightarrow 0 \quad \text{as } R \rightarrow \infty.
\]

so, using the fact that \( u(x_R) \leq C \), it follows that
\[
u(x_R) \rightarrow 0 \quad \text{as } R \rightarrow \infty.
\]
Thus, since \( x_R \in A(R, 4R) \) the desired conclusion follows.

**Corollary 1.** Under the same hypothesis as in Lemma 3.1, if \( m \) is given in (3), then for all \( R_1 \geq 0 \)
\[
\lim_{R \to \infty} m(R_1, R) = 0.
\]
Moreover
\[
m(R_1, R) = m(0, R),
\]
for all large enough \( R \).

**Proof.** Let \( R_1 \geq 0 \). For any \( R > 4R_1 \), we have that
\[
m(R_1, R) \leq m(R/4, R).
\]
Then, by Lemma 3.1, we get
\[
\lim_{R \to \infty} m(R/4, R) = 0,
\]
so, therefore
\[
(10) \quad \lim_{R \to \infty} m(R_1, R) = 0.
\]
Finally, by the fact that
\[
m(0, R) = \min\{m(0, R_1), m(R_1, R)\}, \quad R > R_1,
\]
by (3.1), taking \( R \) large enough, we conclude
\[
m(0, R) = \min\{m(0, R_1), m(R_1, R)\} = m(R_1, R).
\]

**Lemma 3.2.** Let \( \lambda > 0, f(t) > 0 \) in \((0, \infty)\) be a continuous function that satisfies

(1), and \( u \) be a positive classical supersolution of (1) which does not blow up at infinity. Then there exists a positive constant \( C \) such that

\[
m(0, 2R)^p \leq C \left( \frac{m(0, R)}{R^{2\alpha}} + \frac{m(0, R)^q}{R^q} \right),
\]
for \( R \) large enough. That is, \( m(0, R) \) satisfies the hypothesis of Lemma 2.4 where \( m \) was defined in (3).

**Proof.** Let \( R_1 > R_0 \) and \( R \) be large enough so that
\[
m(0, R) = m(R_1, R),
\]
and
\[
m(0, R_1) > m(R_1, R).
\]
Given \( \xi \in C^\infty(\mathbb{R}) \) such that
\[
\xi(t) := \begin{cases} 1 & \text{if } |t| < 1, \\ 0 & \text{if } |t| > 2, \end{cases}
\]
we define
\[
w(x) := u(x) - m(R_1, R)\xi \left( \frac{|x|}{R} \right).
\]
Note that \( w(x) > 0 \) in \( B_{R_1} \cup (\mathbb{R}^N \setminus B_{2R}) \). Then there is \( x_R \in A(R_1, 2R) \) such that
\[
w(x_R) \leq w(x), \quad x \in \mathbb{R}^N.
\]
From here, the argument proceeds as in the proof of Lemma 3.1 by using (1).
We will give now a lower bound for the function $m$ for all ranges of $q$, that is, we have the next.

**Lemma 3.3.** Under the same hypothesis as in Lemma 2.2, if

(i) $1 < q < (N + 2s)/(N + 1)$ and $\sigma \geq (N + 2s)/q - 1$ or $q = (N + 2s)/(N + 1)$ and $\sigma > N$,

(ii) $N + 2s/(N + 1) < q < N/(N + 1 - 2s)$, and $N > \sigma > (2s - 2)/(q - 1)$,

(iii) $q \geq N/(N + 1 - 2s)$ and $N > \sigma > N - 2s$,

then there exists a positive constant $A$ such that

$$m(0, R) \geq AR^{-\sigma},$$

for $R$ large enough and $m$ given in (3).

**Proof.** (i) Let’s take a positive real function $\phi \in C^2(\mathbb{R}^N)$ such that $\phi$ is radially symmetric and decreasing in $|x| > 1$. Assume also that there exists $\sigma > 0$ such that $\phi(x) < |x|^{-\sigma}$, $|\nabla \phi(x)| < c_0|x|^{-\sigma - 1}$, and $|D^2\phi(x)| < c_0|x|^{\sigma - 2}$, for large enough $|x|$. By Lemma 2.2 we have that if $1 < q < (N + 2s)/(N + 1)$ and $\sigma \geq (N + 2s)/q - 1$ or $q = (N + 2s)/(N + 1)$ and $\sigma > N$ then for any $A > 0$ small enough, the function $\phi_A(x) = A\phi(x)$ satisfies

$$(-\Delta)^s \phi_A(x) + |\nabla \phi_A(x)|^q \leq 0 \quad \text{in } |x| > R,$$

for large enough $R$. Moreover, we can take $A$ small enough so that

$$\phi_A(x) \leq u(x), \quad |x| \leq R.$$

Then, by Theorem 2.1, we have

$$\phi_A(x) \leq u(x), \quad x \in \mathbb{R}^N.$$

Thus the conclusion follows.

(ii) By Lemma 2.3 we can take $\varepsilon$ and $A$ small enough and $R_1 \gg \max\{R_0, 1\}$ where $R_0$ is given in (1), so that

$$w_{\sigma, \varepsilon}^A(x) \leq u(x), \quad |x| \leq R_1,$$

and

$$(-\Delta)^s w_{\sigma, \varepsilon}^A(x) + |\nabla w_{\sigma, \varepsilon}^A(x)|^q \leq 0, \quad \text{in } |x| > R_1,$$

where $\sigma \in (\frac{2s - q}{q - 1}, N)$. Then, doing comparison again, by Theorem 2.1, we have

$$w_{\sigma, \varepsilon}^A(x) \leq u(x), \quad x \in \mathbb{R}^N,$$

so (3.3) follows.

(iii) The proof is similar to the case (ii) considering $\sigma \in (N - 2s, N)$. □

To conclude this section regarding with the estimates of the function $m$, we introduce some lower and upper bounds for $m(0, R)$ that will be necessary in the critical case $p = N/(N - 2s)$.

**Lemma 3.4.** Under the same hypothesis of Lemma 3.1 if $q > N/(N + 1 - 2s)$ then

$$m\left(0, \frac{R}{2}\right) \leq Cm(0, R),$$

for $R$ large enough and $m$ given in (3).

**Proof.** For the proof, we borrow some ideas of [23, Lemma 4.2]. In fact, given $R > R_0$, we take

$$R(\varepsilon) = R := R\left[\frac{\varepsilon}{1 + \varepsilon^{2 - N + 2s}}\right]^{\frac{1}{2}},$$

where $\varepsilon$ will be selected later. We first take $\varepsilon > 0$ small enough such that $R < R/2$.

We define the following radial functions
\[ w(x) := \begin{cases} \frac{R^{N+2s}}{|x|^{N+2s}} & \text{if } 0 < |x| < R, \\ \frac{R^{N+2s}}{|x|^{N-2s}} & \text{if } R \leq t, \end{cases} \]

and

\[ w_R(x) := \begin{cases} w(x) & \text{if } |x| \leq 2R, \\ \frac{(2R)^{-N+2s}}{|x|^{N-2s}} & \text{if } 2R \leq |x|. \end{cases} \]

We set now

\[ \phi(x) := m \left( \frac{R}{2} \right) \frac{w_R(x) - w(2R)}{w(R) - w(2R)} \]

and we claim that, for any \( R/2 < |x| < 2R \),

\[
(-\Delta)^s \phi(x) + |\nabla \phi(x)|^q \leq \frac{m \left( \frac{R}{2} \right)}{w(R) - w(2R)} (-\Delta)^s w_R(x) \]

\[ + \left( \frac{m \left( \frac{R}{2} \right)}{w(R) - w(2R)} \right) \frac{1}{|x|^{N+1-2s}}. \]

To check the previous claim we start by observing that, since \( q > \frac{N}{N+1-2s} \), if \( R > 2 \), then for any \( R/2 < |x| < 2R \) it follows that

\[
(-\Delta)^s \phi(x) + |\nabla \phi(x)|^q < \frac{m \left( \frac{R}{2} \right)}{w(R) - w(2R)} (-\Delta)^s w_R(x) \]

\[ + \left( \frac{m \left( \frac{R}{2} \right)}{w(R) - w(2R)} \right) \frac{1}{|x|^N}. \]

Then to prove (3.1), it is enough to get that there exists a positive constant \( C \) such that

\[ (-\Delta)^s w_R(x) \leq -\frac{C}{|x|^N}, \]

for any \( R/2 < |x| < 2R \). For that we notice that, since \( \eta(x) := |x|^{-N+2s} \) is the fundamental solution of \( (-\Delta)^s \), for every \( R/2 < |x| < 2R \), we have that

\[ 0 = (-\Delta)^s \eta(x) \geq (-\Delta)^s w(x) + I_1(\varepsilon, x) + I_2(\varepsilon, x), \]

where

\[ I_1(\varepsilon, x) := \int_{B_{\varepsilon}(x)} \frac{\frac{R}{\varepsilon^{N+2s}} - |x - y|^{-N+2s}}{|y|^{N+2s}} dy < 0, \]

\[ I_2(\varepsilon, x) := \int_{B_{\varepsilon}(-x)} \frac{\frac{R}{\varepsilon^{N+2s}} - |x + y|^{-N+2s}}{|y|^{N+2s}} dy < 0. \]
We choose now $\varepsilon$ small enough such that for any $|x| > R/2$, and $y \in B\overline{R}(x) \cup B\overline{R}(0)$ we have that $|y| \geq R/3$. Then, for any $R > |x| > R/2$,

$$I_1(\varepsilon, x) \geq - \left( \frac{3}{R} \right)^{N+2s} C(N, s)\overline{R}^{2s} = - \frac{C(N, 2s, \varepsilon)}{R^N} \geq - \frac{\tilde{C}(N, 2s, \varepsilon)}{|x|^N},$$

where $\tilde{C}(N, 2s, \varepsilon)$ is a positive constant that goes to 0 as $\varepsilon \to 0$. Doing a similar computation for $I_2(\varepsilon, x)$, by (3.1) it follows that

$$(-\Delta)^s w(x) \leq \frac{C(N, 2s, \varepsilon)}{|x|^N},$$

for every $R/2 < |x| < 2R$, where $C(N, 2s, \varepsilon)$ is also a positive constant that goes to 0 as $\varepsilon \to 0$.

On the other hand, for any $R/2 < |x| < 2R$ we get

$$(-\Delta)^s w_R(x) = (-\Delta)^s w(x) + E(\varepsilon, x),$$

where

$$E(\varepsilon, x) := \int_{B_{2R}(x) \cup B_{2R}(0)} \frac{-w(x + y) - w_R(x - y) + |x + y|^{-N+2s} + |x - y|^{-N+2s}}{|y|^{n+2s}} dy.$$

We observe that if $R/2 < |x| < 2R$ and $y \in \mathbb{R}^N \setminus B_{5R}(0)$ then

$$\min\{|x + y|, |x - y|\} \geq \frac{3}{5} |y|.$$

Thus, for any $x \in A(R/2, 2R)$, we have

$$E(\varepsilon, x) \leq - \frac{C(N, s)}{R^N} \leq - \frac{C(N, s)}{|x|^N}.$$

Therefore by (3.1)-(3.1) we can select $\varepsilon$ small enough such that (3.1) follows and, consequently, also the claim (3.1).

Finally since $\phi(x) \leq u(x)$ if $x \in B_{R/2} \cup B_{2R}$, by Theorem 2.1, we get that

$$\phi(x) \leq u(x), \quad x \in \mathbb{R}^N.$$

Therefore, by taking the infimum in $0 < |x| \leq R$, there exists a positive constant $C$ such that

$$m \left( 0, \frac{R}{2} \right) \leq C m(0, R),$$

for large enough $R$. \quad \square

To conclude this section we observe that by Lemmas 3.2 and 3.4, we clearly deduce the following

**Lemma 3.5.** Under the same hypothesis of Lemma 3.2, if $q > N/(N+1-2s)$ then there exists a positive constant $C$ such that

$$m(0, R)^p \leq C \left( \frac{m(0, R)}{R^{2s}} + \frac{m(0, R)^q}{R^q} \right).$$

for $R$ large enough. That is, $m(0, R)$ satisfies the hypothesis of Lemma 2.5.
3.2. Nonexistence result in the subcritical case \((p < \frac{N}{N-2s})\). Using the technical lemmas showed in the previous section we can prove now the main result of the work in the subcritical case, that is, Theorem 1.1.

**Proof of Theorem 1.1.**

We suppose the contrary, that is, we assume that there exists a positive supersolution \(u\) of (1) which does not blow up at infinity.

**Case 1:** \(1 < 2s < N,\ 1 < q \leq \frac{N+2s}{N+1}\) and \(0 < p < \frac{(2s-1)q}{2s-q}\).

By (3.3), Lemmas 2.4, 3.2 and 3.3, we get that

- If \(0 < p \leq 1\), either \(1 < q < \frac{(N+2s)}{(N+1)}\) and \(\sigma \geq \frac{(N+2s)}{(N+1)} - 1\) or \(q = \frac{(N+2s)}{(N+1)}\) and \(\sigma > N\) then there is a positive constant \(C\) such that
  \[ AR^{-\sigma} \leq m(0, R) \leq CR^\theta, \quad \theta < 0, \]
  for \(R\) large enough, which is a contradiction.

- If \(q = \frac{(N+2s)}{(N+1)}\), \(1 < p < \frac{(2s-1)q}{(2s-q)} = \frac{(N+2s)}{N}\) and \(\sigma > N\) then there is a positive constant \(C\) such that
  \[ AR^{-\sigma} \leq m(0, R) \leq CR^\theta, \quad \theta \in \left( -\frac{2s}{p-1}, 0 \right), \]
  if \(R\) is large enough, which implies a contradiction as long as we can choose \(\sigma < -\theta\). This is certainly possible due to the fact that
  \[ -\frac{2s}{p-1} < -N \iff p < \frac{N + 2s}{N}. \]

- If \(1 < q < \frac{(N+2s)}{(N+1)}\), \(1 < p < \frac{(2s-1)q}{(2s-q)}\) and \(\sigma \geq \frac{(N+2s)}{q} - 1\), then there is a positive constant \(C\) such that
  \[ AR^{-\sigma} \leq m(0, R) \leq CR^\theta, \quad \theta \in \left( -\frac{2s}{p-1}, 0 \right), \]
  choosing \(R\) large enough. Since \(q < \frac{N + 2s}{N + 1}\) then
  \[ \frac{2s-1}{2s-q} < 1 + \frac{2s}{N + 2s - q}, \]
  with is equivalent to
  \[ p < 1 + \frac{2s}{N + 2s - q}, \]
  that implies
  \[ -\frac{2s}{p-1} < 1 - \frac{N + 2s}{q}. \]
  Then we can take \(\sigma < -\theta\) so that we get a contradiction with (3.2).

**Case 2:** \(1 < 2s < N, \ (\frac{N+2s}{N+1}) < q < \frac{N}{N+1-2s}\) and \(0 < p < \frac{(2s-1)q}{(2s-q)}\).

By (3.3) and Lemmas 2.4, 3.2 and 3.3 we obtain:

- If \(0 < p \leq 1, \frac{N+2s}{N+1} < q < \frac{N}{N+1-2s}\), and \(N > \sigma > \frac{(2s-q)}{(q-1)}\), then, taking \(R\) big enough, there exists a positive constant \(C\) such that
  \[ AR^{-\sigma} \leq m(0, R) \leq CR^\theta, \quad \theta < 0, \]
  which implies a contradiction.
• If $\frac{N+2s}{N+1} < q < \frac{N}{(N+1-2s)}$, $1 < p < \frac{(2s-1)q}{(2s-q)}$, and $N > \sigma > \frac{(2s-1)}{(q-1)}$, then there exists a positive constant $C$ such that

$$AR^{-\sigma} \leq m(0, R) \leq CR^\theta, \quad \theta \in \left( -\frac{2s}{p-1}, 0 \right),$$

for large enough $R$. Since

$$-\frac{2s}{p-1} < \frac{2s-q}{1-q} \iff p < \frac{2s-1}{2s-q}q,$$

we deduce that we can choose $\sigma < -\theta$ so that the contradiction follows.

Case 3: $1 < 2s < N$, $q \geq \frac{N}{(N+1-2s)}$ and $0 < p < \frac{N}{(N-2s)}$. Using (3.3) and Lemmas 2.4, 3.2 and 3.3 it follows that

• If $0 < p \leq 1$ and $\frac{N}{(N-2s)} \leq q$, if $R$ is large enough, there exists a positive constant $C$ such that

$$AR^{-\sigma} \leq m(0, R) \leq CR^\theta, \quad \theta < 0,$$

which implies a contradiction.

• If $1 < p < \frac{N}{(N-2s)}$ and $q \geq 2s$ then there is a positive constant $C$ such that

$$AR^{-\sigma} \leq m(0, R) \leq CR^\theta, \quad \theta \in \left( -\frac{2s}{p-1}, 0 \right),$$

choosing $R$ big enough which implies a contradiction if we choose $\sigma$ close enough to $N - 2s$.

• Finally we show up that if $1 < p < \frac{N}{(N-2s)}$ and $\frac{N}{(N+1-2s)} \leq q < 2s$ then $1 < p < \frac{2s-1}{2s-q}q$. Therefore once again there exists a positive constant $C$ such that

$$AR^{-\sigma} \leq m(0, R) \leq CR^\theta, \quad \theta \in \left( -\frac{2s}{p-1}, 0 \right),$$

as long as $R$ is large enough. As before this implies a contradiction taking $\sigma$ close enough to $N - 2s$.

3.3. Nonexistence results in the critical case ($p = \frac{N}{(N-2s)}$). Before proving the nonexistence result regarding with the critical case (see Theorem 1.2), we need the following auxiliary lemma.

**Lemma 3.6.** Under the same hypothesis of Lemma 3.2, if $p = \frac{N}{(N-2s)}$, $q > \frac{N}{(N+1-2s)}$ and $f$ is a nondecreasing function, there exists $R_1 > R_0$ and a positive constant $C$ such that

$$u(x) > \frac{C}{|x|^{N-2s}},$$

for any $|x| > R_1$.

**Proof.** Set

$$w(x) := \begin{cases} 1 & \text{if } |x| \leq 1, \\ |x|^{-N+2s} & \text{if } |x| \geq 1, \end{cases}$$

By (3.1), replacing $R$ by 1, we know that

$$(-\Delta)^s w(x) \leq \frac{C}{|x|^{N+2s}}, \quad C > 0,$$
for every $|x| > 1$. For any $R_2 > R_1 \gg 1$, we define

$$\phi(x) := \begin{cases} 0 & \text{if } |x| < 1 \text{ or } |x| > 2R_2, \\ m(0, R_1) \frac{w(x) - w(R_2)}{w(1) - w(R_2)} & \text{if } 1 < |x| \leq 2R_2, \end{cases}$$

that satisfies

$$(-\Delta)^s \phi(x) + |\nabla \phi(x)|^q < C \left( \frac{1}{|x|^{N+2s}} + \frac{1}{|x|^{(N+1-2s)p}} \right) \leq C \frac{1}{|x|^{\sigma p}},$$

for some $C = C(N, s, R_1)$, $\sigma(q, N) \in (N - 2s, N)$ and every $x \in A(R_1, 2R_2)$. Observe that the existence of $\sigma$ in this precise range comes from the hypothesis $q > N/(N-2s+1)$.

On the other hand, by Lemma 3.3 iii), there exist $R > R_1$ and a positive constant $A$ such that

$$u(x) \geq \frac{A}{|x|^\sigma}, \quad x \in B_{R_1}.\]$$

Thus, since $f$ is nondecreasing and verifies (1), there exist $R > R_0$ large enough and a positive constant $A$ such that

$$(-\Delta)^s u(x) + |\nabla u(x)|^q \geq \frac{A}{|x|^{\sigma p}}, \quad x \in B_{R_1} \setminus B_{R_0},$$

Then, since $\phi(x) \leq u(x)$ for any $x \in \mathbb{R}^N \setminus A(R_1, R_2)$, by Theorem 2.2 there exists $K > 0$ such that

$$K \phi(x) \leq u(x), \quad x \in \mathbb{R}^N.$$

Passing to the limit as $R_2 \to \infty$, we get

$$\frac{C}{|x|^{N-2s}} \leq u(x), \quad |x| > R_1.$$

as wanted \[\square\]

Now we can prove the main result of this subsection, that is,

**Proof of Theorem 1.2.**

Let us define the radial function

$$\Gamma(x) := \frac{\log(1 + |x|)}{|x|^{N-2s}},$$

that is decreasing in $(r_0, \infty)$ for some $r_0 > 0$. By [23, Lemma 6.1], there exists a positive constant $C$ such that

$$(-\Delta)^s \Gamma(x) \leq \frac{C}{|x|^N}, \quad x \in \mathbb{R}^N \setminus \{0\}.$$

Given $\min\{r_0, R_0\} < \varepsilon < R_1 < R_2$, we define now

$$\phi(x) := \begin{cases} m(0, R_1) \frac{w(x) - w(R_2)}{w(1) - w(R_2)} & \text{if } |x| \leq 2R_2, \\ 0 & \text{if } |x| > 2R_2, \end{cases}$$

where here

$$w(x) \begin{cases} \Gamma(\varepsilon) & \text{if } |x| \leq \varepsilon, \\ \Gamma(x) & \text{if } |x| > \varepsilon. \end{cases}$$

Since it can be proved (see [23, page 2734]) that $(-\Delta)^s w$ also satisfies the same kind of upper bound than $(-\Delta)^s \Gamma$ for $x \in A(R_1, R_2)$ then there exists $C > 0$, a constant independent of $R_2$, such that

$$(-\Delta)^s \phi(x) \leq \frac{m(0, R_1)}{w(\varepsilon) - w(R_2)} \frac{C}{|x|^N}, \quad x \in A(R_1, R_2).$$
Thus if \( x \in A(R_1, R_2) \), we get
\[
(-\Delta)^s \phi(x) + |\nabla \phi(x)|^q \leq \frac{m(0,R_1)}{w(\varepsilon) - w(R_2)} \frac{C}{|x|^N} + \left( \frac{m(0,R_1)}{w(\varepsilon) - w(R_2)} \frac{1}{|x|^N} \right)^q \left( \frac{|x|^{-2s}}{1 + |x|^2} + N - 2s \right) \log(1 + |x|)^q
\]
where \( C \) is a positive constant independent of \( R_2 \). Moreover, since
\[
\frac{m(0,R_1)}{w(\varepsilon) - w(R_2)} \to \frac{m(0,R_1)}{w(\varepsilon)}, \quad R_2 \to \infty,
\]
taking \( R_2 \) big enough it follows that
\[
(-\Delta)^s \phi(x) + |\nabla \phi(x)|^q \leq C \left[ \frac{1}{|x|^N} + \frac{\log(1 + |x|)^q}{|x|^{(N+1-2s)q}} \right],
\]
for every \( x \in A(R_1, R_2) \). We show up that, since \( q > \frac{N}{N+1-2s} \), it is clear that
\[
\beta = \frac{q(N-2s+1) - N}{q} > 0.
\]
Using that fact, and the hypothesis on \( f \), from (3.3) we get that
\[
(-\Delta)^s \phi(x) + |\nabla \phi(x)|^q \leq f \left( \frac{K}{|x|^N} \right), \quad x \in A(R_1, R_2),
\]
with \( K \) independent of \( R_2 \) as long as \( R_1 \) is large enough. Using now the Lemma 3.6 by comparison we conclude the existence of \( C > 0 \) such that
\[
C \phi(x) \leq u(x),
\]
for every \( x \in A(R_1, R_2) \).

On the other hand, by Lemmas 3.5 and 2.5(b), for large enough \( R \) there is a positive constant \( K \) such that
\[
u(x) \leq \frac{K}{|x|^{N-2s}}, \quad x \in B_R.
\]
Finally, by (3.3) and (3.3), we get
\[
C \log(1 + |x|)^q \leq a(x) \leq \frac{K}{|x|^{N-2s}}, \quad x \in A(R_1, R_2),
\]
for large enough \( R_1 \), with \( C \) and \( K \) positive constants independent of \( R_2 \). Therefore
\[
C \log(1 + |x|)^q \leq u(x) \leq \frac{K}{|x|^{N-2s}}, \quad x \in \mathbb{R}^N \setminus B_{R_1},
\]
that clearly implies a contradiction. \( \square \)

4. Existence of Supersolutions

Unlike to the previous section, positive supersolutions can be constructed when we consider \( f(u) = \lambda u^p \) in (1) for some values of \( \lambda \). In fact we give the

Proof of Theorem 1.3.

Let \( \varphi_\sigma = \varphi \in C^2(\mathbb{R}^N) \) be a positive real function that is radially symmetric and decreasing for every \( |x| > 1 \), such that \( \varphi(x) \leq |x|^{-\sigma}, \ |D \varphi(x)| \geq c_0 |x|^{-\sigma-1} \) and
We define
\[ \psi(x, R, \sigma) := \varphi \left( \frac{R}{R_0} x \right). \]

Then, by Lemma 2.1, there exists \( \tilde{R} > 0 \) such that
\[
(\Delta) \psi(x, R, \sigma) = \left( \frac{R}{R_0} \right)^{2s} (\Delta) \psi \left( \frac{R}{R_0} x \right)
\]
for all \( R > \tilde{R} \) and \( |x| > R_0 \).

We now split the rest of the proof in six different cases.

Case 1. \( p = \frac{N}{N-2s}, 0 < q < 1 \).

Taking \( \sigma = N-2s \), by (4), for any \( R > \tilde{R} \) and \( |x| > R_0 \), we get
\[
(\Delta) \psi(x, R, N-2s) + |\nabla \psi(x, R, N-2s)|^q \geq \left( \frac{R}{R_0} \right)^{2q(N-2s)} \frac{1}{|x|^N} \left[ - \left( \frac{R}{R_0} \right)^{2q(N-2s)} \frac{1}{|x|^N} \right] \psi^p(x, R, N-2s) \left(c_1 + \frac{c_0^q |x|^{N-(N+1-2s)q}}{2} \right)
\]
for all \( 1-q > 0 \) it is possible to take \( R > \tilde{R} \) big enough such that
\[
(\Delta) \psi(x, R, N-2s) + |\nabla \psi(x, R, N-2s)|^q \geq \left( \frac{R}{R_0} \right)^{2q(N-2s)} \frac{1}{|x|^N} \left[ - \left( \frac{R}{R_0} \right)^{2q(N-2s)} \frac{1}{|x|^N} \right] \psi^p(x, R, N-2s) \left(c_1 + \frac{c_0^q |x|^{N-(N+1-2s)q}}{2} \right)
\]
for every \( \lambda > 0 \), and \( |x| > R_0 \).

Case 2. \( p = \frac{N}{N-2s}, \) and \( q = 1 \).

Doing the same as Case 1 for \( |x| > R_0 \) we get
\[
(\Delta) \psi(x, R, N-2s) + |\nabla \psi(x, R, N-2s)| \geq \left( \frac{R}{R_0} \right)^{2(N-2s)} \frac{1}{|x|^N} \left[ -c_1 + c_0^q |x|^{2s-1} \right]
\]
As long as \( R_0 > \tilde{R}_0 \) such that
\[-c_1 + c_0 R_0^{2s-1} > 1,\]
then it is possible to consider \( R > \tilde{R} \) big enough in order to have that and
\[
(\Delta) \psi(x, R, N-2s) + |\nabla \psi(x, R, N-2s)| \geq \lambda \psi^p(x, R, N-2s),
\]
for every \( \lambda > 0 \) and \( |x| > R_0 \).
Cases 3. $p = \frac{N}{N - 2s}$, and $1 < q \leq \frac{N}{N + 1 - 2s}$.

We will consider now $u(x) = A\psi(x, R, N - 2s)$ where $A$ is a positive constant that will be chosen in a suitable way later. Repeating the computations done in (4) we get that

$$(-\Delta)^s u(x) + |\nabla u(x)|^q \geq \left(\frac{R}{R_0}\right)^{2s-N} A \frac{A^{q-1}c_0^q}{|x|^{(q+1)s}} \left[ -c_1 + A^{q-1}c_0^q \left(\frac{R}{R_0}\right)^{(2s-N)(q-1)} \right]$$

$$\geq \left(\frac{R}{R_0}\right)^{2s-A^{1-p}u^p(x)} \left[ -c_1 + A^{q-1}c_0^q \left(\frac{R}{R_0}\right)^{(2s-N)(q-1)} \right],$$

as long as $R > \tilde{R}$ and $|x| > R_0$. Now we choose the positive constant $A$ so that, for instance,

$$-c_1 + A^{q-1}c_0^q \left(\frac{R}{R_0}\right)^{(2s-N)(q-1)} = 1.$$ 

Therefore, taking

$$\lambda_0 = \left(\frac{R}{R_0}\right)^{2s} A^{1-p},$$

we have that for any $0 < \lambda < \lambda_0$, and $|x| > R_0$, $u(x)$ is the suitable supersolution we were looking for.

Case 4. $\frac{2s-1}{2s-q} < p < \frac{N}{N+1}$, and $1 < \frac{N+2s}{N+1} < q < \frac{N}{N + 1 - 2s}$.

We take $0 < \sigma = \frac{2s-1}{q-1}$ and $u(x) = A\psi(x, R, \sigma)$ where $A$ is a positive constant that will be chosen later. We observe that since $\frac{N+2s}{N+1} < q$, then $\sigma < N$ so that, since $\sigma + 2s = q(\sigma + 1)$ by (4) we get

$$(-\Delta)^s u(x) + |\nabla u(x)|^q \geq \frac{R}{R_0}^{-\sigma} \frac{1}{|x|^{(\sigma+1)q}} A \left[ -c_1 + c_0 q A^{q-1} \left(\frac{R}{R_0}\right)^{2s-q} \right]$$

$$\geq A^{1-p} \left(\frac{R}{R_0}\right)^{2s} u^{\frac{\sigma+1}{\sigma}q}(x) \left[ -c_1 + c_0 q A^{q-1} \left(\frac{R}{R_0}\right)^{2s-q} \right].$$

for every $R > \tilde{R}$ and $|x| > R_0$. Choosing now $A$ so that

$$-c_1 + c_0 q A^{q-1} \left(\frac{R}{R_0}\right)^{2s-q} = 1,$$

we obtain

$$\lambda_0 = \left(\frac{R}{R_0}\right)^{2s} A^{1-p} \simeq \left(\frac{R}{R_0}\right)^{\frac{2s-q}{q}} (1-p)^{\frac{2s-q}{q}},$$

Therefore, for every $0 < \lambda < \lambda_0$ and $|x| > R_0$, we get

$$(-\Delta)^s u(x) + |\nabla u(x)|^q \geq \lambda u^p(x),$$

as wanted

Case 5. $\frac{N+2s}{N+1-2s} < q < \frac{N}{N+1}$, and $0 < q < \frac{N+2s}{N+1}$, $q \neq 1$.  

We consider now 
\[ u(x) = A\psi(x, R, \sigma) \] 
with \( \sigma = \frac{N+2s-q}{q} \) and \( A > 0 \) to be chosen. Since \( 0 < q < \frac{N+2s}{N+1} \) then \( \sigma > N \) so that using again (4) it follows that

\[
(-\Delta)^s u(x) + |\nabla u(x)|^q 
\geq \left( \frac{R}{R_0} \right)^{-N} \frac{A}{|x|^{(\sigma+1)q}} \left[ -c_3 + c_0^q A^{q-1} \left( \frac{R_0}{R} \right)^{2s-q} \right] 
\geq \left( \frac{R}{R_0} \right)^{2s} \psi^{\frac{N+2s}{N+1}}(x, R, \sigma) \left[ -c_3 + c_0^q A^{q-1} \left( \frac{R_0}{R} \right)^{2s-q} \right].
\]

for \( R > \bar{R} \) and \( |x| > R_0 \) by using the fact that \( (\sigma+1)q = N + 2s \). Choosing the positive constant \( A \) so that

\[
-c_3 + c_0^q A^{q-1} \left( \frac{R_0}{R} \right)^{2s-q} = 1,
\]

and taking

\[
\lambda_0 = \left( \frac{R}{R_0} \right)^{2s} A^{1-p},
\]

we have

\[
(-\Delta)^s u(x) + |\nabla u(x)|^q \geq \lambda u^p(x),
\]

for every \( 0 < \lambda < \lambda_0 \), and \( |x| > R_0 \).

In the case of \( q = 1 \) we will take \( \sigma = N + 2s - 1 \) obtaining

\[
(-\Delta)^s \psi(x, R, \sigma) + |\nabla \psi(x, R, \sigma)| 
\geq \left( \frac{R}{R_0} \right)^{-N} \frac{1}{|x|^{N+2s}} \left[ -c_3 + c_0 \left( \frac{R_0}{R} \right)^{2s-1} \right] 
\geq \left( \frac{R}{R_0} \right)^{2s} \psi^{\frac{N+2s}{N+1}}(x, R) \left[ -c_3 + c_0 \left( \frac{R_0}{R} \right)^{2s-1} \right],
\]

for \( R > \bar{R} \) and \( |x| > R_0 \). Taking \( R > \bar{R} \) big enough such that

\[
-c_3 + c_0 \left( \frac{R_0}{R} \right)^{2s-1} \geq 1,
\]

and

\[
\lambda_0 = \left( \frac{R}{R_0} \right)^{2s},
\]

we conclude that \( u(x) \) is a supersolution of (1.3) for every \( 0 < \lambda < \lambda_0 \).

**Case 6.** \( \frac{N+2s}{N} < p < \frac{N}{N-2s} \), \( q = \frac{N+2s}{N+1} \).

In this last case we observe that

\[
\sigma + 2s > (\sigma + 1)q \text{ if and only if } \sigma < N,
\]

and

\[
\frac{\sigma + 1}{\sigma} q \to \frac{N+2s}{N} \text{ as } \sigma \to N.
\]
Then given $p > \frac{N+2s}{N}$ there exists $\sigma_p \in (0,N)$ such that $\frac{\sigma_p+1}{\sigma_p}q < p$. Considering $u(x) = \nabla \psi(x,R,\sigma_p)$, with $A$ chosen later, for every $R > R_0$ and $|x| > R_0$, we get

\[
(-\Delta)^s u(x) + |\nabla u(x)|^q \geq \left(\frac{R}{R_0}\right)^{-\sigma_p} A x |x|^{\sigma_p+2s-(\sigma_p+1)q} + c_0 \left(\frac{R_0}{R}\right)^{-\sigma_p(q-1)} A^{q-1}
\]

\[
\geq \left(\frac{R}{R_0}\right)^{(\sigma_p+1)q-\sigma_p} x |x|^{\sigma_p+2s-(\sigma_p+1)q} + c_0 \left(\frac{R_0}{R}\right)^{-\sigma_p(q-1)} A^{q-1} \right]
\]

Taking $A > 0$ in order to guarantee that

\[
-R \left(\frac{R}{R_0}\right)^{(\sigma_p+1)q-\sigma_p} A^{1-p} = 1,
\]

and

\[
\lambda_0 = \left(\frac{R}{R_0}\right)^{(\sigma_p+1)q-\sigma_p} A^{1-p},
\]

the conclusion follows.

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