Large characteristic subgroups of surface groups
not containing any simple loops

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Abstract

We determine the largest (i.e. smallest index) characteristic subgroup
of surface groups not containing any simple loops.

1 Introduction

For any compact orientable surface $S$ we determine the smallest characteristic
non-geometric quotient of $\pi_1(S)$. Non-geometric means that no non-trivial
element that can be represented by a simple closed curve is mapped to the identity
and characteristic means that its kernel is kept fixed under all automorphisms.
We write $\pi, H$ and $g$ for $\pi_1(S), H_1(S, \mathbb{Z})$ and the genus of $S$ respectively. We
assume that $g$ is at least 2. Consider the following characteristic subgroups of $\pi$:
$\pi^{[1]} := \pi$ and inductively $\pi^{[k+1]} := [\pi, \pi^{[k]}]$. We have a well known isomorphism
$\pi^{[2]} / \pi^{[3]} \to \wedge^2 H / \omega, [x, y] \mapsto x \wedge y$, where $\omega$ is the intersection form on $H$. We
have an intersection product $\wedge^2 H \to \mathbb{Z}$. Let $K$ be the kernel of the composition
of the map $\pi^{[2]} \to \wedge^2 H / \omega$ with the intersection product $\wedge^2 H / \omega \to \mathbb{Z} / g\mathbb{Z}$. Our
result is

Proposition 1.1 If $g$ is odd, then the largest (i.e. smallest index) characteristic
non-geometric subgroup of $\pi$ is given by all $g$th powers and $K$. If $g$ is even, then
the largest characteristic non-geometric subgroup of $\pi$ is given by all $2g$th powers
and $K$. The indices of these are $g^{2g+1}$ and $(2g)^{2g}g$ respectively.

Recently, Livingston proved that for $g = 2$ the smallest non-geometric quotient
of $\pi$ is a group of order $2^5$, and he raised the question whether an easy
generalisation of his result (which yields a group of order $g^{2g+1}$), holds true for
any genus. The theorem above shows that his generalisation for odd genus is in
any case the smallest non-geometric characteristic quotient.

2 Notation and preliminary computations

Before we start, we pick once and for all a set of generators for $\pi$: $x_1, \ldots, x_{2g}$ with
defining relation $\Pi_{i=1}^{g}[x_{2i-1}, x_{2i}]$. We call a pair of integers $(i, j)$ related if and
only if there exists an integer $h$ with $1 \leq h \leq g-1$ such that $(i, j) = (2h-1, 2h)$. 

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We write $[a, b]$ for $a^{-1}b^{-1}ab$, such that $ab = ba[a, b]$. Furthermore, if $\alpha$ and $\beta$ are elements of $\pi$, then $\alpha \beta$ means: perform $\alpha$ first, then $\beta$.

Consider the following normal subgroups of $\pi$:

$$
\begin{align*}
\pi^{[1]} & : = \pi; \\
\pi^{[k+1]} & : = [\pi, \pi^{[k]}]; \\
K & : = Ker(\pi^{[2]} \to \mathbb{Z}/g\mathbb{Z}); \\
\pi^n & : = < x^n | x \in \pi >.
\end{align*}
$$

Notice that $K$ contains $\pi^{[3]}$. All these are clearly characteristic subgroups. For $K$ this holds since $\pi^{[2]}$ is characteristic and all involved maps are natural. The quotient we are interested in, is $\pi/\pi^nK$ respectively $\pi/\pi^{2g}K$. In order to prove the proposition, we first describe $\pi/\pi^{[3]}$.

**Lemma 2.1** $\pi/\pi^{[3]} \cong \{\Pi_{i=1}^{2g} x_i^n, \Pi_1^{m_{i,j}} | n_i, m_{i,j} \in \mathbb{Z}, * meaning : (i, j) \neq (2g - 1, 2g)\} \to \pi/\pi^{[3]} \to \pi/\pi^{[2]} \to \mathbb{Z}$.

where multiplication on the right hand side is defined as follows:

$$
(\Pi_{i=1}^{2g} x_i^n, \Pi_1^{m_{i,j}}) \cdot (\Pi_{i=1}^{2g} x_i^n, \Pi_1^{m_{i,j}}) = \Pi_{i=1}^{2g} x_i^{n_i + m_{i,j}} \Pi_1^{m_{i,j} - k_{i,j} + \delta_{i,j}}.
$$

Here $\delta_{a,b} = 1$ if $a = b$ and $a$ is even, else $\delta_{a,b} = 0$.

**Proof 2.2** We have the exact sequence

$$1 \to \pi^{[2]}/\pi^{[3]} \to \pi/\pi^{[3]} \to \pi/\pi^{[2]} \to 1,$$

where the outer factors are finitely generated free abelian groups ([Lk, Th.5.12]) of rank $(2g) - 1$ and $2g$ respectively. Explicitly, we have:

$$
\pi/\pi^{[2]} = (x_i)_{a,b}, i = 1, \ldots, 2g,
\pi^{[2]}/\pi^{[3]} = (\{x_i, x_j\})_{a,b}, 1 \leq i < j \leq 2g, (i, j) \neq (2g - 1, 2g).
$$

Here the subscript $ab$ means the free abelian group generated by these elements. Now recall that if $a \in \pi^{[k]}$ and $b \in \pi^{[l]}$ then $ab = ba[a, b]$ with $[a, b] \in \pi^{[k+l]}$, so that

$$
ab \equiv ba \mod \pi^{[k+l]}.
$$

Furthermore we have the following identities, modulo $\pi^{[3]}$: ([MKS, Th.5.1])

$$
\begin{align*}
[a, b] & = [b, a]^{-1}, \\
[a, bc] & = [a, c] [a, b], \\
[ab, c] & = [a, c] [b, c].
\end{align*}
$$

It is clear now that any element of $\pi/\pi^{[3]}$ can be written uniquely in the form $\Pi_{i=1}^{2g} x_i^n, \Pi_1^{m_{i,j}} | n_i, m_{i,j} \in \mathbb{Z}$ and the meaning of the * is:
(i, j) \neq (2g - 1, 2g). Note that the last \((2^g - 1)\) factors commute by (1). Before we start the computation, notice that modulo \(\pi^{[3]}\) we have

\[ [a^i, b^j] = [a, b]^{ij}, \] (5)

as one proves easily by induction. The following identities hold in the group \(\pi/\pi^{[3]}\):

\[
\begin{align*}
(x_1^{n_1} \ldots x_{2g}^{n_{2g}})(x_1^{k_1} \ldots x_{2g}^{k_{2g}}) &= x_1^{n_1 + k_1} \ldots x_{2g}^{n_{2g} + k_{2g}} [x_1, x_2]^{-k_1} \ldots [x_1, x_{2g}]^{-k_{2g}} \\
&= \ldots \\
&= \prod x_i^{n_i + k_i}, \prod_{1 \leq i < j \leq 2g} [x_i, x_j]^{-k_{ij} + \delta_{i,j+1} k_{2g-1} n_{2g}}
\end{align*}
\]

with \(\delta_{i+1,j} k_{2g-1} n_{2g}\) as above. Combining these proves the lemma. (Cf. [PdJ, Lemma 6.1])

Here the term with \(\delta_{i+1,j} k_{2g-1} n_{2g}\) stems from the defining relation for the group \(\pi: \Pi^{[2]}_1[x_{2i-1}, x_{2i}] = 1\), which we used to get rid of \([x_{2g-1}, x_{2g}]\) as a generator for \(\pi^{[2]}/\pi^{[3]}\).

Clearly the subgroup \(\pi^{[2]}/\pi^{[3]}\) is given in terms of these generators by all expressions of the form \(\Pi^{[2]}_{1 \leq i < j \leq 2g} [x_i, x_j]^{m_{ij}}\).

**Lemma 2.3** \(\pi/K \cong \{ \Pi^{2g}_{i=1} x_i^{n_i} [x_1, x_2]^m | n_i \in \mathbb{Z}, m \in \mathbb{Z}/g\mathbb{Z} \}\), where multiplication on the right hand side is defined as follows:

\[
(\Pi_{i=1}^{2g} x_i^{n_i} [x_1, x_2]^m) (\Pi_{i=1}^{2g} x_i^{k_i} [x_1, x_2]^l) = \Pi_{i=1}^{2g} x_i^{n_i + k_i} [x_1, x_2]^{m + l - \sum_{i=1}^{2g} k_{2i-1} n_{2i} + (g-1) k_{2g-1} n_{2g}}.
\]

**Proof 2.4** Clearly, \(K/\pi^{[3]}\) is generated by the elements \([x_i, x_j]\) for \(i\) and \(j\) not related, by the elements \([x_{2k-1}, x_{2k}][x_{2k-1}, x_{2k}]^{-1}\) and by \(\Pi_{i=1}^{g} [x_{2i-1}, x_{2i}]\).

**Proposition 2.5** If \(g\) is odd, then

\[ \pi/\pi^{g} K \cong \{ \Pi_{i=1}^{2g} x_i^{n_i} [x_1, x_2]^m | n_i, m \in \mathbb{Z}/g\mathbb{Z} \}, \]

and if \(g\) is even, say \(g = 2h\), then

\[ \pi/\pi^{g} K \cong \{ \Pi_{i=1}^{2g} x_i^{n_i} [x_1, x_2]^m | n_i \in \mathbb{Z}/g\mathbb{Z}, m \in \mathbb{Z}/h\mathbb{Z} \}, \]

where multiplication on the right hand sides is defined as above.
**Proof 2.6** Consider the short exact sequence

\[ 1 \to \pi / (\pi^2 \cap \pi^g K) \to \pi / \pi^g \to \pi / \pi^g \pi^2 \to 1. \]

Clearly, \( \pi / \pi^g \pi^2 \cong H_1(S; \mathbb{Z}/g\mathbb{Z}) \). Furthermore, if \( g \) is odd, \( \pi / (\pi^2 \cap \pi^g K) \cong (\mathbb{Z}/g\mathbb{Z}) \), whereas in case \( g \) is even, say \( g = 2h \), we have \( \pi^2 / (\pi^2 \cap \pi^g K) \cong (\mathbb{Z}/h\mathbb{Z}) \). This follows directly from [PdJ, Lemma 6.3].

**Corollary 2.7** The subgroup \( \pi^g \) is generated by all \( g \)th powers of geometric (i.e. representable by a simple closed curve) elements modulo \( K \).

**Proof 2.8** Clearly the elements \( x_i^g \) and \( [x_1, x_3]^g \) are \( g \)th powers of geometric elements, settling the statement for odd \( g \). For even \( g \), say \( g = 2h \), we have \( (x_1x_2)^{2h} \cong x_1^{2h} x_2^{2h} [x_1, x_2]^{(2h-1)h} \) modulo \( \pi^3 \), as one proves easily by induction (cf. [PdJ, Lemma 6.3]). Therefore, \( [x_1, x_2]^{-h} \cong x_2^{-2h} x_1^{-2h} (x_1 x_2)^{2h} [x_1, x_2]^{-2h} \), proving the corollary.

We define the following simple loops on \( S \): \( \gamma_1 = x_1 \), \( \gamma_{2h} = x_{2h} \), for \( h = 1, \ldots, g \), \( \gamma_{2h-1} = x_{2h-1} [x_{2h-3}, x_{2h-2}]^{-1} x_{2h-3}^{-1} \), for \( h = 2, \ldots, g \), and finally \( \gamma_{2g+1} = [x_{2g-1}, x_{2g}]^{-1} x_{2g-1}^{-1} \).

We write \( \tau_i \) respectively \( \sigma_i \) for the (right handed) Dehn twist around \( \gamma_i \) and \( x_{2i-1} \) respectively. For later convenience we list the action of these Dehn twists on the generators of \( \pi \) and the action modulo \( \pi^3 \) as above (if the action of some Dehn twist on a generator is not given, it is the trivial action):

\[
\begin{align*}
\tau_1(x_2) &= x_1^{-1} x_2, \\
\tau_{2h}(x_{2h-1}) &= x_2 x_{2h-1}^{-1} x_2 x_{2h-1} x_{2h-1}^{-1} x_2, \\
\tau_{2h-1}(x_{2h-2}) &= x_{2h-2}^{-1} x_{2h-1}^{-1} = x_{2h-2} x_{2h-1} x_{2h-2}^{-1} x_{2h-3}, \\
\tau_{2h-1}(x_{2h-1}) &= x_{2h-1}^{-1} x_{2h-2}^{-1} x_{2h-1}^{-1} = x_{2h-1} x_{2h-2} x_{2h-3} x_{2h-2}^{-1} x_{2h-3}^{-1}, \\
\tau_{2h-1}(x_{2h}) &= x_{2h-1}^{-1} x_{2h}, \\
\tau_{2g+1}(x_{2g}) &= x_{2g}^{-1} x_{2g+1} = x_{2g} x_{2g+1} x_{2g}^{-1} x_{2g+1}^{-1} x_{2g+1}^{-1}. \\
\sigma_i(x_{2i}) &= x_{2i-1} x_{2i}.
\end{align*}
\]

3 The proofs

**Proposition 3.1** The quotient \( \pi / \pi^g K \) is characteristic and it is non-geometric if and only if \( g \) is odd. If \( g \) is even, the quotient \( \pi / \pi^{2g} K \) is characteristic and non-geometric.
\begin{proof}
\(\pi^g K\) is characteristic, while \(\pi^g\) and \(K\) are. For \(g\) odd, we have that \(\pi^g K\) is non-geometric since it is exactly the group described by Livingston in [4, Section 5]. For \(g\) even, say \(g = 2h\), it follows from Proposition 2.9 that \(\Pi_i = [x_{2i-1}, x_i]\) is contained in \(\pi^g K\), so \(\pi^g K\) is geometric. On the other hand, for \(g\) even, Livingston’s group is a quotient of \(\pi^{2g} K\), namely the one generated by the elements \(x_i^g\) for \(i = 1, \ldots, 2g\). Thus \(\pi^{2g} K\) is non-geometric.
\end{proof}

**Theorem 3.3** If \(g\) is odd, the largest characteristic non-geometric quotient of \(\pi\) is \(\pi/\pi^g K\) and if \(g\) is even, the largest characteristic non-geometric quotient of \(\pi\) is \(\pi/\pi^{2g} K\). The indices of these groups are \(g^{g+1}\) respectively \((2g)^{2g+1}/2\).

\begin{proof}
The indices follow directly from Proposition 2.3. Let \(M\) be any characteristic finite-index non-geometric subgroup of \(\pi\). Let \(k\) be the smallest positive integer such that for some simple closed not separating curve \(\delta\) we have \(\delta^k \in M\). Since \(M\) is characteristic and all simple closed not separating curves can be mapped one onto the other, we have that \(k\) is the smallest positive integer with \(\delta^k \in M\) for all simple closed not separating curves \(\delta\). Notice that \(k \geq 3\).

Namely, if \(k = 2\), then \([x_1, x_2] = x_1^{-2}(x_1 x_2)^2 x_2^{-2} \in M\), so \(M\) is geometric, contradiction. We have \(\pi/M\pi^2 \cong (\mathbb{Z}/k\mathbb{Z})^{2g}\) for some positive integer \(k\).

Let \(P\) be the subgroup of \(\pi^2\) generated by all \([x_i, x_j]\) for \((i, j)\) not related (recall that \(i < j\)). By an analogous argument we have that \((\pi^2 \cap M)/(\pi^3 \cap M)P\) is a quotient of \((\mathbb{Z}/m\mathbb{Z})^{g-1}\), generated by the elements \([x_{2i-1}, x_{2i}]\) for \(i = 1, \ldots, g\). Again by the classification of surfaces and by the fact that \(M\) is characteristic, we get a uniform power \(l\) such that \([x_{2i-1}, x_{2i}]^l \in M\) if and only if \(l\) divides \(t\). We have that

\[ [x_{2i-1}, x_{2i}] \cong [x_{2i-1}, x_{2i}]^k \text{ modulo } \pi^3 \]

and thus \(l\) divides \(k\) if \(k\) odd and \(2l\) divides \(k\) if \(k\) even, by Proposition 2.7.

Similarly, we obtain a uniform power \(m\) such that for \((i, j)\) not related, the elements \([x_i, x_j]^m\) are in \(M\) if and only if \(m\) divides \(t\). Furthermore we have that \(m\) divides \(l\), since \(\tau_3([x_1, x_2]^m) \equiv [x_1, x_2]^m [x_1, x_3]^m \text{ modulo } \pi^3\).

Now suppose \([x_1, x_2] \notin [x_3, x_4]\) modulo \(M\). Since \(M\) is characteristic and all classes \([x_{2i-1}, x_{2i}]\) can be transformed one into the other by an automorphism of \(\pi\), it follows that they all have different classes modulo \(M\), thus \([x_{2i-1}, x_{2i}][x_{2j-1}, x_{2j}]^{-1}\) is not contained in \(M\). Consider the short exact sequence

\[1 \to M\pi^3/\pi^3 \to \pi/\pi^3 \to M\pi^3/\pi^3 \to 1.\]

We claim that \(M\pi^3/\pi^3\) does not contain any element of the form \([x_i, x_j]\) for \((i, j)\) not related (equivalently, \(m > 1\)). Namely, suppose that there is a pair \((i, j)\), with \((i, j)\) not related, such that \([x_i, x_j] \in M\). Then \([x_i, x_j]\) is in \(M\) for all \((i, j)\) not related, again by the classification of surfaces and the fact that
$M$ is characteristic. We compute $\tau_{2h-1}([x_{2h-2}, x_{2h}])$. We have the following identities modulo $\pi^{[3]}$, where $2 \leq h \leq g$:

\[
\begin{align*}
\tau_{2h-1}([x_{2h-2}, x_{2h}]) & \equiv [x_{2h-3}, x_{2h-2}]^{-1} [x_{2h-2}, x_{2h}]^{-1} [x_{2h-3}, x_{2h}] [x_{2h-3}, x_{2h-1}]^{-1} \frac{1}{[x_{2h-2}, x_{2h-1}, x_{2h}]} [x_{2h-3}, x_{2h-1}]^{-1} [x_{2h-2}, x_{2h}] \equiv [x_{2h-3}, x_{2h-2}]^{-1} [x_{2h-2}, x_{2h}]^{-1} [x_{2h-3}, x_{2h}] [x_{2h-3}, x_{2h-1}]^{-1} \frac{1}{[x_{2h-2}, x_{2h-1}, x_{2h}]} [x_{2h-3}, x_{2h-1}]^{-1} [x_{2h-2}, x_{2h}].
\end{align*}
\]

The product $[x_{2h-3}, x_{2h-2}]^{-1} [x_{2h-1}, x_{2h}]$ is contained in $M$ since the first three commutators are in $M$. This leads to a contradiction.

Now we claim that the abelian $m$-torsion subgroup of $\pi/M\pi^{[3]}$ generated by all $[x_i, x_j]$ for $(i, j)$ not related, has rank $(\frac{2g}{3}) - g = 2g^2 - 2g$. Namely, suppose there is an element $z = \sum_{(i,j)\text{not rel.}} n_{i,j} [x_i, x_j]$ contained in $M$ with all $n_{i,j} \in \{0, \ldots, m-1\}$ (in additive notation). We compute a number of elements of the form $\pi(z) - z$ to show that all these $n_{i,j}$ are actually zero. We use repeatedly that $n_{i,j} [x_i, x_j]$ is in $M$ if and only if $m$ divides $n_{i,j}$

\[
\begin{align*}
2g_2 & := \tau_2(z) - z = \sum_{3 \leq j \leq 2g} n_{1,j} [x_1, x_j] \\
2g_{2-1} & := \sigma_g (2g_2) - z_{2g} = n_{1,2g} [x_1, x_{2g-1}] \Rightarrow n_{1,2g} = 0 \\
2g_{2-2} & := \tau_2(z_{2g-1}) - z_{2g-1} = n_{1,2g-1} [x_1, x_{2g}] \Rightarrow n_{1,2g-1} = 0 \\
2g_{2-3} & := \sigma_g (2g_{2-2}) - z_{2g-2} = n_{1,2g-2} [x_1, x_{2g-3}] \Rightarrow n_{1,2g-2} = 0 \\
2g_{2-4} & := \tau_2(z_{2g-3}) - z_{2g-3} = n_{1,2g-3} [x_1, x_{2g-2}] \Rightarrow n_{1,2g-3} = 0 \\
\vdots
\end{align*}
\]

we get $z = \sum_{2 \leq j \leq 2g} n_{1,j} [x_1, x_j]$.

It is clear that continuing in this way we show that all coefficients $n_{i,j}$ are zero. Since there are $(\frac{2g}{3}) - 2g$ elements of this form, this proves the claim.

It remains to show that the index of this quotient is larger than the index of $\pi^g K$ respectively $\pi^{2g} K$. We have that the index of $M$ is at least $\#(\pi/M\pi^{[2]}) \#(M \cap \pi^{[3]})/\#(M \cap \pi^{[3]})$, which in turn is at least $k^2 m 2g^2 - 2g$. If $m$ is even, this leaves $m = 2$ as smallest possibility. Since $m$ divides $l$, $l$ is also even and therefore $2l$ divides $k$. So the smallest possibility we get is $4^g 2g^2 - 2g = (2g)^2 + 2$. This is larger than $(2g)^2 + 1/2$ for all $g$.

If $m$ is odd, the smallest possibility becomes $3^{2g} 3^{2g^2 - 2g} = (3g)^{2g}$, which is again larger than $g^{2g + 1}$ for all $g$.

On the other hand, suppose $[x_1, x_2] \equiv [x_3, x_4] \text{ modulo } M$. By the same argument, we get that all the classes $[x_{2i-1}, x_{2i}]$ are equal modulo $M$. Since $\Pi_{i=1}^h [x_{2i-1}, x_{2i}] \notin M$ for all $h = 1, \ldots, g - 1$, the smallest possibility for $(\pi^{[2]} \cap M)/(\pi^{[3]} \cap M)P$ is $(Z/gZ)$. This implies that $g$ divides $k$ and $2g$ divides $k$ if $g$ even. Thus, $M$ is contained in $\pi^g K$ or $\pi^{2g} K$ if $g$ is even.
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