Multirate Numerical Scheme for Large-Scale Vehicle Traffic Simulation

V. V. Kurtc* and I. E. Anufriev

St. Petersburg Polytechnic University, St. Petersburg, 195251 Russia
*e-mail: kurtsvv@gmail.com

Received January 13, 2015

Abstract—In modeling vehicular traffic on large scales in large cities, the number of vehicles could reach tens of thousands, which, for the microscopic approach, corresponds to the solution of ordinary differential equations (ODEs) with large dimensions. The speed of changes in the size of the components of such systems usually lies in a wide range, as the dynamics and behavior of the vehicles can strongly differ. In this paper we introduce a multirate numerical scheme with a self-adjusting time stepping strategy. Instead of using a single step size for the whole system, the step size for each component is determined by estimating its own local variation. The stability analysis for the developed scheme is performed and the stability conditions are obtained. The presented multirate scheme provides a significant speed-up in the CPU times compared to the corresponding single-rate one. The use of multiple time steps allows parallel computing.

Keywords: multirate time stepping, a priori estimates, stability, citywide microscopic traffic simulation, ordinary differential equations (ODEs)

INTRODUCTION

Large-scale vehicular traffic simulation at the moment is quite important. In the case of modeling on a scale of large cities the number of cars can reach tens of thousands. At the same time, it is required to reproduce the dynamics of vehicles with a high degree of realism and simultaneously to carry out calculations in real time. In order to achieve these aims, it is necessary to use a microscopic approach, advanced mathematical models, and fast numerical methods. Modeling in city conditions means the simultaneous presence of different modes of traffic flow dynamics, such as uniform velocity motion, acceleration and braking to a complete stop, and lane changing. As a result the acceleration of vehicles varies widely. The standard methods of numerical integration of systems of ordinary differential equations (ODEs) for all components use the same integration step. The step, corresponds to the “fastest” component, i.e., the smallest, is chosen for the entire system to satisfy the accuracy applicable to the solution that significantly increases the estimated time. Using the algorithms with multiple steps, in which the integration step is determined individually for each component based on its rate of change, is effective for such problems.

In Section 2 of this paper, a brief review of the research on multirate algorithms is carried out. The microscopic model of vehicular traffic is described in Section 3. A new computational method with multiple steps and an algorithm for the automated selection of an integration step are presented in Section 4. The choice of the size of the macrostep is also justified. The detailed derivation of an error estimate of the numerical integration, based on which the rule to select a step for each of the components of the solution’s vector is received, is given for the presented scheme. The stability analysis is conducted in Section 5. Due to the special structure of the solvable ODE system, the stability condition is obtained in the common form. The numerical results are given in Section 6, including a comparison of the presented algorithm with multiple steps and its one-speed analog. The article concludes with a discussion of the results obtained and proposals for further investigations.

1. REVIEW OF THE LITERATURE

The first work on the multirate numerical methods is devoted to linear algorithms [1]. In [2] there is a scheme based on the Runge-Kutta methods with coupling between the active (fast) and latent (slow) com-
ponents by interpolation and extrapolation of the components values of the solution’s vector. The partitioning into active and latent components is carried out \textit{a priori} and based on the additional information about the system to be solved. A similar scheme based on the Rosenbrock method was studied in [3]. Some results on stability analysis of the simplified multirate methods in the case of systems of two linear equations with one fast and one slow component can be found in [4]. The algorithm based on the finite element method is proposed in [5, 6].

The multirate schemes for non-stiff problems and explicit methods are investigated in [7, 8]. The automatic partitioning into fast and slow components, which is carried out during the extrapolation procedure, is implemented in these works. Savcenco et al. have proposed an adaptable strategy for implicit methods, which is applicable for stiff problems [9]. The local error, the interpolation error propagation, and the stability of the multirate scheme based on the $\theta$-method are investigated in [10].

2. CAR-FOLLOWING MODEL

The class of car-following models is most often used in application to the simulation of urban traffic flows on the scale of large cities, where there are roads with several lanes, intersections of various types, crosswalks, etc. For each vehicle the acceleration function, which generally depends on its speed, distance to the vehicle in front (leader), and difference in their speeds, is defined in explicit form. The dynamics of the traffic flow, in general, are described by a system of ODEs.

Let a road section has $N$ vehicles, moving one after another (Fig. 1). The acceleration of the $i$th vehicle is determined by the function $a^{(i)}(v_i, h_i)$. As a result, we have a system of $2N$ ODEs with initial values \textbf{v} \text{ and } \textbf{h}

\begin{equation}
\begin{aligned}
\dot{v}_1 &= a^{(i)}(v_1, h_1), \\
\dot{h}_1 &= v_L - v_1, \\
\vdots \\
\dot{v}_N &= a^{(N)}(v_N, h_N), \\
\dot{h}_N &= v_{N-1} - v_N, \\
v_i(0) &= v_{i0}, \\
h_i(0) &= h_{i0}, \\
i &= 1, \ldots, N,
\end{aligned}
\end{equation}

where $v_i$ and $h_i$ are the speed and distance to the leader of the $i$th car, respectively.

The most popular microscopic traffic model in the last decade is the Intelligent driver model (IDM) proposed by Treiber et al. in 1999. In [11] some shortcomings in the application to vehicular traffic simulation in a driving simulator and a modified model is proposed, in which the acceleration function is determined as follows

\begin{equation}
\nu(v, h) = wa \left(1 - \frac{v}{v_0} \right)^\delta + (1 - w) a \left(1 - \frac{d^*}{h} \right)^2.
\end{equation}

The model parameters have the same meaning as in the IDM [12]. The first term determines the acceleration of the vehicle in a free flow, while the second one is responsible for interacting with the leader. For the partitioning of these two modes, the continuous weight function $w$, which depends on the distance to the leader $h$ and the parameter $D$, is defined.

\begin{equation}
w = w(h, d^*, D) =
\begin{cases}
0, & h \in (-\infty, d^*), \\
-2t^3 - 3t^2 + 1, & t = (h - d^*)/D - 1, \\
1, & h \in (d^* + D, +\infty),
\end{cases}
\end{equation}
For brevity, we rewrite the system (1) as follows:

\[
\dot{x}(t) = F(x(t)), \quad x(0) = x^{(0)}
\]

with the initial values \(x^{(0)} \in \mathbb{R}^{2N}\). The approximation of the solution at time \(t_n\) is denoted as \(x^{(n)}\).

### 3. LOCAL ERROR OF THE METHOD

3.1. Numerical scheme with automatic step selection. The new multirate numerical scheme, which is proposed in this paper, is based on the estimation of the local error of the numerical integration. Let us consider one macrostep of the method, which connects the solutions at the neighboring macrolevels \(t_{n-1}\) and \(t_n = t_{n-1} + \Delta T\). Determining the value of the macrostep \(\Delta T\) will be described in Section 4.2. Within one macrostep for each component of the solution vector \(x_i\), one carries out \(k_i\) consecutive microsteps \(\Delta t_i = \Delta T/k_i\) according to the basic numerical scheme — the explicit Euler method. The multiplicity indicators \(\{k_i\}_{i=1}^N\) are determined so as to satisfy the accuracy:

\[
I_n = \|x^{(n)} - \tilde{x}(t_n)\|_\infty < \varepsilon.
\]

Here \(\tilde{x}\) is the exact solution of the following system:

\[
\dot{x}(t) = F(x(t)), \quad x(t_{n-1}) = x^{(n-1)}.
\]

In this case, the infinity norm is used, because the accuracy condition must be satisfied for all components.

It is worth noting that within the macrostep, the values of other components, which, however, are not calculated at present, may be necessary to calculate a specific component. In our case, we consider them equal to values at the beginning of the current macrostep and take it into account in deriving error estimate \(I_n\). At the end of each macrostep, the values of the solution vector are updated.

3.2. Macrostep value. The macrostep value \(\Delta T\) is determined \textit{a priori} and remains constant throughout the integration process. On the one hand, the value \(\Delta T\) should be chosen as much as possible to reduce the number of synchronization operations, which can be time-consuming. On the other hand, traffic modeling is assumed urban scenario, where traffic lights and intersections are present. The interaction with them should be implemented realistically, so \(\Delta T\) should not exceed the human reaction time. As a result, the value \(\Delta T = 0.5\) seconds was chosen for the macrostep that corresponds to the average driver’s reaction time.

3.3. Algorithm of automatic step selection. Let us consider the \(i\)th component of the solution’s vector of system (4). We will define the number of consecutive microsteps \(k_i\) and the corresponding value of the microstep \(\Delta t_i = \Delta T/k_i\), which guarantee the fulfillment of the accuracy conditions (5). In order to move from the layer \(t_m\) to the next \(t_m + \Delta T\), it is necessary to execute \(k_i\) consecutive steps according to the basic numerical scheme:

\[
\begin{align*}
    x_i^{(m+1)} &= x_i^{(m)} + \Delta t_i \cdot f_i(x_1^{(m)}, x_2^{(m)}, \ldots, x_i^{(m)}, \ldots, x_{2N}^{(m)}), \\
    x_i^{(m+2)} &= x_i^{(m+1)} + \Delta t_i \cdot f_i(x_1^{(m)}, x_2^{(m)}, \ldots, x_i^{(m+1)}, \ldots, x_{2N}^{(m)}), \\
    \vdots \\
    x_i^{(m+k_i)} &= x_i^{(m+k_i-1)} + \Delta t_i \cdot f_i(x_1^{(m)}, x_2^{(m)}, \ldots, x_i^{(m+k_i-1)}, \ldots, x_{2N}^{(m)}).
\end{align*}
\]

Here \(f_j = (F)_j\). By simple transformations in (7), we obtain the relationship between solutions \(x_i^{(m+k)}\) and \(x_i^{(m)}\) at the time \(t_m\) and \(t_m + \Delta T\), respectively:

\[
x_i^{(m+k)} = x_i^{(m)} + \Delta T \sum_{j=0}^{k_i-1} f_i(x_1^{(m)}, x_2^{(m)}, \ldots, x_i^{(m+j)}, \ldots, x_{2N}^{(m)}).
\]

We will consistently calculate \(x_i^{(m+j)}, j = 2, \ldots, k_i\), expressing it through the solution \(x^{(m)}\) at the time of the beginning of the current macrostep. For \(x_i^{(m+2)}\) we have
Expanding the third summand in the Taylor expansion in the neighborhood of \( x^{(m)} \), we obtain
\[
 x^{(m+2)}_i = x^{(m)}_i + \Delta t_i f_i \left( x^{(m)} \right) + \Delta t_i^2 f_i \left( x^{(m)} \right) \frac{\partial f_i}{\partial x_i} \left( x^{(m)} \right) + O \left( \Delta t_i^3 \right). 
\] (9)

We will execute the same procedure for \( x^{(m+3)}_i \):
\[
 x^{(m+3)}_i = x^{(m)}_i + 3 \Delta t_i f_i \left( x^{(m)} \right) + 3 \Delta t_i^2 f_i \left( x^{(m)} \right) \frac{\partial f_i}{\partial x_i} \left( x^{(m)} \right) + O \left( \Delta t_i^3 \right). 
\] (10)

After \( k_i \) consecutive steps we obtain
\[
 x^{(m+k)}_i = x^{(m)}_i + k_i \Delta t_i f_i \left( x^{(m)} \right) + C_{k_i} \Delta t_i^2 f_i \left( x^{(m)} \right) \frac{\partial f_i}{\partial x_i} \left( x^{(m)} \right) + O \left( \Delta t_i^3 \right), 
\] (12)
where \( C_{k_i} = \binom{k_i}{2} = \frac{k_i(k_i-1)}{2} \) is the binomial coefficient.

Let us calculate the exact solution \( \tilde{x}_i \) of (6) with the initial data \( \tilde{x}(t_m) = x^{(m)} \) at the end of the current macrostep \( t_m + \Delta T \):
\[
 \tilde{x}_i \left( t_m + T \right) = \tilde{x} (t_m) + \Delta T f_i \left( x^{(m)} \right) + \frac{\Delta T^2}{2} \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} \left( x^{(m)} \right) f_j \left( x^{(m)} \right) + O \left( \Delta T^3 \right). 
\] (13)

Subtracting (12) from (13), we obtain the numerical integration error after one macrostep \( (t_m \rightarrow t_m + \Delta T) \):
\[
e_{i,k_i} = \left\lfloor \Delta T f_i \left( x^{(m)} \right) + \frac{\Delta T^2}{2} \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} \left( x^{(m)} \right) f_j \left( x^{(m)} \right) + O \left( \Delta T^3 \right) \right\rfloor.
\] (14)

Discarding \( O(\Delta T^3) \), we obtain an estimate of the numerical integration error for the \( i \)th component:
\[
est_{i,k_i} = \frac{\Delta T^2}{2} \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} \left( x^{(m)} \right) f_j \left( x^{(m)} \right) + O \left( \Delta T^3 \right).
\] (15)

If the number of microsteps within one macrostep is one, i.e., \( k_i = 1 \), expression (14) transforms to well-known one corresponding to the explicit Euler method [15].

Let the user-specified accuracy for numerical solution obtained be \( \varepsilon \). Then, for each component \( x_i \), using estimate (15), we can determine the minimum value \( k_i \), which will guarantee obtaining the numerical solution with the required accuracy \( \varepsilon \):
\[
\min_{k_i \in N} k_i : err_{i,k_i} < \varepsilon.
\] (16)

With regard to system (1), describing the dynamics of traffic flow, the estimate of a local error for the speed components of the \( i \)th vehicle takes the form
\[
est_{i,v}^{j,k} = \frac{\Delta T^2}{2k_i} \left| a^{(i)}_{j,k} + a^{(i)}_{j-1,k} \right|.
\] (17)
Here \( a^{(i)}_{j,k} \) and \( a^{(i)}_{j,k} \) are partial derivatives of the acceleration function \( a^{(i)} \) with respect to speed and distance, respectively. The numerical integration error is controlled by the speed only. The microstep \( \Delta t_i \) for the vehicle \( i \), which guarantees that the value \( \est_{i,v}^{j,k} \) is within the required accuracy \( \varepsilon_{v,i} \), is expressed as follows:
where $\lceil . \rceil$ is rounding to the nearest integer in the direction of positive infinity. In the case of a steady flow, i.e., when the denominator (18) is equal to zero, the microstep $\Delta t$ is set to the value of the macrostep $\Delta T$.

4. STABILITY OF THE NUMERICAL SCHEME WITH MULTIPLE STEPS

The results on the stability of multirate schemes can be found in [2, 4, 13, 14]. In this section the stability analysis of the numerical scheme with multiple steps suggested in this paper in application to the system (1), which describes the dynamics of the traffic flow, is conducted.

We will consider one macrostep according to (7) for the entire system (1)

\[
\begin{aligned}
\dot{x}_1^{(m+1)} &= x_1^{(m)} + \Delta t_1 \cdot f_1(x_1^{(m)}, x_2^{(m)}, \ldots, x_{2N}^{(m)}), \\
\vdots \\
\dot{x}_1^{(m+k_1)} &= x_1^{(m+k_1-1)} + \Delta t_1 \cdot f_1(x_1^{(m+k_1-1)}, x_2^{(m)}, \ldots, x_{2N}^{(m)}), \\
\dot{x}_2^{(m+1)} &= x_2^{(m)} + \Delta t_2 \cdot f_2(x_1^{(m)}, x_2^{(m)}, \ldots, x_{2N}^{(m)}), \\
\vdots \\
\dot{x}_2^{(m+k_2)} &= x_2^{(m+k_2-1)} + \Delta t_2 \cdot f_2(x_1^{(m)}, x_2^{(m+k_2-1)}, \ldots, x_{2N}^{(m)}).
\end{aligned}
\]

(19)

It is required to determine the multiplicity indicators $k_i$ and the corresponding values of microsteps $\Delta t_i = T/k_i$ for each vehicle at the beginning of each macrostep so as to provide the stability of the numerical scheme (19). Linearizing (1) in the neighborhood of point $x^{(m)}$, we obtain

\[
\dot{x}^{(m+k)} = R(J, \Delta T, k_1, \ldots, k_n) x^{(m)},
\]

(20)

where $R(J, \Delta T, k_1, \ldots, k_n)$ is the transition matrix. The Jacobi matrix $J$ for (1) has a block structure

\[
J = \begin{bmatrix}
a^{(1)}_v & a^{(1)}_h & 0 & 0 & 0 & 0 & \cdots \\
-1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & a^{(2)}_v & a^{(2)}_h & 0 & 0 & \cdots \\
+1 & 0 & -1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & a^{(3)}_v & a^{(3)}_h & \cdots & \cdots \\
0 & 0 & +1 & 0 & -1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots
\end{bmatrix}
\]

(21)

The numerical scheme (19) is stable if and only if all eigenvalues of the transition matrix $R$ are less than unity in modulus

\[
R = \begin{bmatrix}
r_1^{k_1} & r_1^{k_1-1} a^{(1)}_h \frac{\Delta T}{k_1} & 0 & 0 & \cdots \\
-\Delta T & 1 & 0 & 0 & \cdots \\
0 & 0 & r_3^{k_3} & r_3^{k_3-1} a^{(2)}_h \frac{\Delta T}{k_3} & \cdots \\
0 & 0 & +\Delta T & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots
\end{bmatrix}
\]

(22)
where \( r_{2i-1} = \left(1 + a_{(i)}^{(i)} \frac{\Delta T}{k_{(i)}} \right) \). Due to the block structure of the matrix \( R \), it is possible to find all the eigenvalues and obtain the stability conditions for the scheme with multiple steps (19)

\[
1 + r_{2i-1}^{k_{2i-1}} \pm \sqrt{(1 + r_{2i-1}^{k_{2i-1}})^2 - 4 \frac{r_{2i-1}^{k_{2i-1}} - 1}{r_{2i-1}^{k_{2i-1}} - 1} \frac{a_{(i)}^{(i)} \Delta T^2}{k_{(i)}}} < 2, \quad i = 1, \ldots, n.
\]  

(23)

The numerical experiments show that if a condition (18) is satisfied, inequality (23) can be violated for some components. It is necessary to set values \( \Delta t_i \), \( i = 1, \ldots, n \) so that conditions (18) and (23) are fulfilled.

5. RESULTS

Let us consider a vehicle which is following its leader. As the initial conditions for (1), we will put \( v_1 = 0 \text{ m/s} \), \( h_t = 0 \text{ m} \), and the accuracy required for the speed components is \( \varepsilon_v = 0.1 \). We will define the leader’s trajectory (dashed line in Fig. 2a) and run the calculation. As shown in Fig. 2b, the step selection rule (18) operates correctly; in other words, the values of the microsteps guarantee the required accuracy \( \varepsilon_v = 0.1 \).

The explicit Euler method with the step \( \Delta t = 0.005 \) seconds is used to obtain an exact solution of the system, which is necessary to calculate the integration error. A series of experiments with various trajectories of the leader, which show the correctness of condition (18), are carried out.

Figure 3 compares two numerical integration schemes—the explicit Euler method with a variable step and the multirate solver proposed in this paper. The road network section on which the vehicular traffic is simulated is maximally close to a real urban area and contains intersections and roads with several lanes. The number of simultaneously simulated vehicles is 1000 and the value of a macrostep is 0.5 seconds. The time interval during which the simulation is carried out is 100 seconds. The algorithm of the first method contains the following steps for one macrostep:

1. Computation of steps \( \{\Delta t_i\}_{i=1}^n \) on the basis of the classical a priori estimate [15] and solution accuracy \( \varepsilon_v \).

2. Determination of the general step for the whole system \( \Delta t = \min \{\Delta t_i\}_{i=1}^n \).

3. Performing one step for system (1) according to the explicit Euler method.

The results demonstrate that the multirate solver is faster than the standard method. Moreover, the value of the method’s relative speed-up is proportional to the required accuracy \( \varepsilon_v \). It is worth noting that in the case of the first experiment (Fig. 3a), during the first 30 seconds, the one-speed method is faster.

Fig. 2. (a) Speed profiles for the leader (dashed line) and the following vehicle (solid line). (b) Error of the numerical integration at each macrostep.
than the algorithm with multiple steps. This can be explained by the fact that at the initial stage of simulation, the traffic flow is relatively free, as a result, the integration steps for both methods are comparable with the value of the macrostep $\Delta T$, and the multirate solver is slower because of the calculation of a step according to rule (18). In most cases, the situation when the traffic flow has a low density everywhere is quite rare and therefore the use of an algorithm with multiple steps is in beneficial.

6. DISCUSSIONS AND PLAN OF FURTHER RESEARCH STUDIES

This paper presents a new algorithm with multiple steps to solve ODE systems, which includes automatic identification of an integration step for each component separately. The value of the step (microstep) is determined on the basis of the required accuracy and the estimate of the numerical integration error, which is also obtained in this paper. Due to the multiplicity conditions several consecutive microsteps for each equation of the ODE system are carried out within one macrostep. This scheme is used to solve the ODE system which describes the dynamics of the traffic flows. A modified model of the IDM is used as a microscopic traffic model. The numerical experiments show that the local error of the integration method at the end of each macrostep does not exceed the required accuracy. The results confirm that the speed of the method can be significantly increased while preserving the requirements for the accuracy of the numerical solution if, at the same time, the speed of each component value is considered. In other words, the large steps are used for “slow” components and the small ones are used for the fast components, what is implemented in the proposed algorithm.

For the algorithm with multiple steps presented in this paper, the stability investigation is conducted. Due to the specific structure of the ODE system, the stability conditions are analytically obtained. Their implicit form does not allow us to calculate the value of a microstep directly. However, the algorithm of the method implies that at first the step is determined on the basis of the required accuracy, and then the stability conditions of the method are checked.

It should be noted that the rule of the step selection and the stability conditions are obtained in general, i.e., for any microscopic model, whose acceleration function depends on the speed of the vehicle and the distance to its leader.

As the basic numerical method, i.e., for calculating the solutions in one microstep, the explicit Euler method is used. Throughout a macrostep the interpolation and/or extrapolation of the components values are not applied. The use of higher order methods and interpolation formulas will be the following stages of our research.

In conclusion, it should be added that the multirate solver suggested in this paper can be easily parallelized as follows. The microstep is multiple of a macrostep and is bounded. As a result, there is a finite set of values for the microsteps. Groups of equations are formed, each of which corresponds to a certain
microstep value. The calculations for each group can be executed independently and therefore in parallel. This direction will be the next stage of our research and will further speed-up the calculations.

REFERENCES
1. C. Gear and D. Wells, “Multirate linear multistep methods,” BIT 24, 484–502 (1984).
2. M. Gunther, A. Kvønø, and P. Rentrop, “Multirate partitioned Runge-Kutta methods,” BIT 41, 504–514 (2001).
3. A. Bartel and M. Gunther, “A multirate W-method for electrical networks in state space formulation,” J. Comput. Appl. Math. 147, 411–425 (2002).
4. A. Kvønø, “Stability of multirate Runge-Kutta schemes,” Int. J. Differ. Equ. Appl. 1 (A), 97–105 (2000).
5. A. Logg, “Multi-adaptive Galerkin methods for ODEs I,” SIAM J. Sci. Comput. 24, 1879–1902 (2003).
6. A. Logg, “Multi-adaptive Galerkin methods for ODEs II. Implementation and applications,” SIAM J. Sci. Comput. 25, 1119–1141 (2003).
7. C. Engstler and C. Lubich, “Multirate extrapolation methods for differential equations with different time scales,” Computing 58, 173–185 (1997).
8. C. Engstler and C. Lubich, “MUR8: a multirate extension of the eight-order Dormand–Prince method,” Appl. Numer. Math. 25, 185–192 (1997).
9. V. Savcenco, W. Hundsdorfer, and J. G. Verwer, “A multirate time stepping strategy for stiff ordinary differential equations,” BIT 47, 137–155 (2007).
10. W. Hundsdorfer and V. Savcenco, “Analysis of a multirate theta-method for stiff ODEs,” Appl. Numer. Math. 59, 693–706 (2009).
11. V. Kurtc and I. Anufriev, “Local stability conditions and calibrating procedure for new car-following models used in driving simulators,” in Proceedings of the 10th Conference on Traffic and Granular Flow 2013, pp. 453–461. doi 10.1007/978-3-319-10629-8_50
12. M. Treiber, A. Henneke, and D. Helbing, “Congested traffic states in empirical observations and microscopic simulations,” Phys. Rev. E 62, 1805–1824 (2000).
13. S. Skelboe, “Stability properties of backward differentiation multirate formulas,” Appl. Numer. Math. 5, 151–160 (1989).
14. A. Verhoeven, E. J. W. ter Maten, R. M. M. Mattheij, and B. Tasic, “Stability analysis of the BDF slowest first multirate methods,” CASA-Report 0704 (Tech. Univ. Eindhoven, Eindhoven, 2007).
15. L. F. Shampine, I. Gladwell, and S. Thompson, Solving ODEs with Matlab (Cambridge Univ. Press, New York, 2003).

Translated by A. Sapronova