ON STRONGLY QUASI-HEREDITARY ALGEBRAS

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Abstract. Let $A$ be a finite dimensional algebra over an algebraically closed field $k$. If $A$ is quasi-hereditary and the projective dimensions of all standard modules are at most one, then $A$ is called left strongly quasi-hereditary. In this paper, we construct a special heredity chain for left strongly quasi-hereditary algebras. Moreover, we show the quotient algebra by an ideal which appears in a special heredity chain of left strongly quasi-hereditary algebra is also left strongly quasi-hereditary algebra.

1. Introduction

1.1. Quasi-hereditary algebras. Quasi-hereditary algebras were introduced by Scott [Sco87] to study highest weight categories in the representation theory of semisimple complex Lie algebras and algebraic groups. Cline, Parshall and Scott proved many important results in [CPS88], see also [PS88]. In [DR89c], for a semiprimary ring, Dlab and Ringel gave another definition of quasi-hereditary by using an ideal chain.

Definition 1.1 (Dlab-Ringel [DR89c], Cline-Parshall-Scott [CPS88]). Let $R$ be a semiprimary ring.

(1) A two-sided ideal $H$ of $R$ is called a heredity ideal of $R$ if it satisfies the following conditions:

(a) $H$ is a projective module as left $R$-modules;
(b) $\text{Hom}_R(H,R/H) = 0$;
(c) $HJ(R)H = 0$, where $J(R)$ is the Jacobson radical of $R$.

(2) A chain of two-sided ideals of a semiprimary ring $R$

$$R = H_0 > H_1 > \cdots > H_i > H_{i+1} > \cdots > H_n = 0$$

is called a heredity chain if $H_i/H_{i+1}$ are heredity ideals in $R/H_{i+1}$ for $0 \leq i < n$.

(3) $R$ is called a hereditary ring if there exists a heredity chain.

From this definition, it immediately follows that if $R$ is quasi-hereditary and $H$ appears in a heredity chain of $R$, then $R/H$ is also quasi-hereditary. In [DR89c], they showed that all semiprimary rings of global dimension at most two are quasi-hereditary. In particular, the Auslander algebra of a finite-representation type is also quasi-hereditary. In [DR89a], various kinds of “splitting filtration” and associated heredity chains are studied.
1.2. Left strongly quasi-hereditary algebras. Left strongly quasi-hereditary algebras and right strongly quasi-hereditary algebras were introduced by Ringel [Rin10] as a special class of quasi-hereditary algebras to give a concise proof of Iyama’s finiteness theorem [Iya03a]. This theorem states that every finitely generated module over an artin algebra $\Gamma$ is a direct summand of some modules whose endomorphism ring $\Sigma$ is quasi-hereditary, with a heredity chain of length $n$, and the global dimension of $\Sigma$ is bounded by $n$. One can regard this result as a generalization of [DR89b]. In general, given a quasi-hereditary algebra with a heredity chain of length $n$, it is known that its global dimension is bounded by $2n - 2$. Ringel proved that the ring $\Sigma$ in Iyama’s finiteness theorem is not only quasi-hereditary, but even left strongly quasi-hereditary in [Rin10].

Let $\Lambda_w$ be the finite dimensional quotient algebra of the preprojective algebra associated with an element $w$ in the Coxeter group. We denote by $\text{Sub}_\Lambda_w$ the category of free $\Lambda_w$-modules of finite rank. For a standard cluster tilting object $M \in \text{Sub}_\Lambda_w$, the 2-Auslander algebra $\text{End}_{\Lambda_w}(M)$ is a left strongly quasi-hereditary algebra [IR11]. Geiss, Leclerc and Schröer found this result more previously for adaptable elements $w$ of the Coxeter group [GLS07].

If $Q$ is a finite quiver without loops, then there exists an admissible ideal $I$ such that the algebra $kQ/I$ has the global dimension at most two and is a left strongly quasi-hereditary algebra [DR89d, Theorem 3], see also [HZ13, Theorem 3.1] and [Poe10, Theorem 1. (a)].

1.3. Main results. The aim of this paper is to establish the following results.

**Theorem** (Theorem 3.10). Let $A$ be a finite dimensional algebra over an algebraically closed field $k$. $A$ is a left (resp. right) strongly quasi-hereditary algebra if and only if there exists a heredity chain

$$A = H_0 > H_1 > \cdots > H_i > H_{i+1} > \cdots > H_n = 0$$

such that, for any $0 \leq i < n$, the projective dimension of $H_i/H_{i+1}$ as left (resp. right) $A$-modules is at most one.

From Theorem 1.3, we give the following definition.

**Definition** (Definition 3.11). Let $A$ be a left (resp. right) strongly quasi-hereditary algebra with a heredity chain

$$A = H_0 > H_1 > \cdots > H_i > H_{i+1} > \cdots > H_n = 0$$

such that, for any $0 \leq i < n$, the projective dimension of $H_i/H_{i+1}$ as left (resp. right) $A$-modules is at most one. We call such a chain of two-sided ideals a left (resp. right) strongly heredity chain of $A$.

**Theorem** (Theorem 3.13). Let $A$ be a left (resp. right) strongly quasi-hereditary algebra over an algebraically closed field $k$ with a left (resp. right) strongly heredity chain

$$A = H_0 > H_1 > \cdots > H_i > H_{i+1} > \cdots > H_n = 0.$$ 

Then $A/H_i$ is again a left (resp. right) strongly quasi-hereditary algebra for $0 \leq i \leq n$.

From Theorem 1.3, we give the following definition.
Definition (Definition 3.14). We say that a finite dimensional algebra $A$ is strongly quasi-hereditary if there exists a heredity chain
\[ A = H_0 > H_1 > \cdots > H_i > H_{i+1} > \cdots > H_n = 0 \]
such that, for any $0 \leq i < n$, the projective dimension of $H_i/H_{i+1}$ as left $A$-modules and the projective dimension of $H_i/H_{i+1}$ as right $A$-modules are at most one. We call the above heredity chain a strongly heredity chain of $A$.

In [Rin10], he proved that if $A$ is strongly quasi-hereditary, then the global dimension of $A$ is at most two. However the question of whether every finite dimensional algebra of global dimension at most two is strongly quasi-hereditary is answered in the negative by providing an example of an Auslander algebra which is not strongly quasi-hereditary.

2. Preliminaries

In this section, we recall the definition of quasi-hereditary algebra and prove some lemmas needed later. We shall use the same notation as in [Don98, Appendix].

2.1. Definitions and basic properties. Let $A$ be a finite dimensional algebra over an algebraically closed field $k$. Let $J$ be the Jacobson radical of $A$. We fix a set of isomorphism classes of simple $A$-modules $\{L(\lambda) \mid \lambda \in \Lambda\}$ and fix a partial ordering $\leq$ on the index set $\Lambda$. For $\lambda \in \Lambda$, we denote by $P(\lambda)$ the projective cover of $L(\lambda)$. We write $A$-mod for the category of finitely generated left $A$-modules and, for $X \in A$-mod, we write $[X : L(\lambda)]$ for the composition multiplicity of $L(\lambda)$. Let $\pi$ be a subset of the index set $\Lambda$. For $X \in A$-mod, we denote by $O^\pi(X)$ a unique minimal submodule of $X$ such that all composition factors of $X/O^\pi(X)$ belong to $\{L(\lambda) \mid \lambda \in \pi\}$. Sending an $A$-module $M$ to $O^\pi(M)$ yields the functor $O^\pi : A$-mod $\to A$-mod. Note that $O^\pi$ is a right exact functor. Since $O^\pi(A)$ is a two-sided ideal of $A$, we set $A(\pi) := A/O^\pi(A)$.

Lemma 2.1 (cf. Donkin [Don98, A.1]). Let $\pi$ be a subset of $\Lambda$. For $X \in A$-mod, we have $O^\pi(X) = O^\pi(A).X$. In particular, if all composition factors of $X$ belong to $\{L(\lambda) \mid \lambda \in \pi\}$, then $O^\pi(X) = 0$.

We regard $X/O^\pi(X)$ as $A(\pi)$-modules for $X \in A$-mod. Note that $\{L(\lambda) \mid \lambda \in \pi\}$ is a set of isomorphism classes of simple $A(\pi)$-modules, and $P(\lambda)/O^{\pi}(P(\lambda))$ is the projective cover of $L(\lambda)$ as left $A(\pi)$-modules.

Definition 2.2. For $\lambda \in \Lambda$, we put $\pi(\lambda) := \{\mu \in \Lambda \mid \mu < \lambda\}$. We define $\Delta(\lambda) = P(\lambda)/O^{\pi}(P(\lambda))$ for all $\lambda \in \Lambda$ and call the modules $\Delta(\lambda)$ the standard modules.

Remark 2.3. Let $\Delta := \{\Delta(\lambda) \mid \lambda \in \Lambda\}$ be the set of standard modules. We denote by $F(\Delta)$ the full subcategory of $A$-mod of modules which have a filtration with factors in $\Delta$. Thus $M \in F(\Delta)$ if and only if $M$ has a submodule series $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_l = 0$ such that $M_i/M_{i+1}$ is isomorphic to a module in $\Delta$. 
For $M \in \mathcal{F}(\Delta)$, the element $[M]$ in the Grothendieck group $K_0(A)$ of $A \mod$ corresponding to $M$ can be written as

$$[M] = \sum_{\lambda \in \Lambda} m_\lambda[\Delta(\lambda)] = \sum_{\mu \in \Lambda} (\sum_{\lambda \in \Lambda} m_\lambda[\Delta(\lambda) : L(\mu)])[L(\mu)].$$

If $[\Delta(\lambda) : L(\mu)] \neq 0$, then we have $\mu \leq \lambda$. Thus the coefficients $m_\lambda$ are uniquely determined (cf. [Don98, A.1 (7)]). In other words, the filtration multiplicities and the length of $\Delta$-filtration do not depend on the choice of $\Delta$-filtration. Thus we denote by $(M : \Delta(\lambda))$ for the filtration multiplicity of $\Delta(\lambda)$ and denote by $fl(M)$ the length of $\Delta$-filtration of $M$.

**Definition 2.4** (Cline-Parshall-Scott [CPS88]). We say that a pair $(A \mod, \leq)$ is a highest weight category if for any $\lambda \in \Lambda$ there exists a short exact sequence

$$0 \to K(\lambda) \to P(\lambda) \to \Delta(\lambda) \to 0$$

with the following properties:

(a) $K(\lambda) \in \mathcal{F}(\Delta)$ for any $\lambda \in \Lambda$;

(b) if $(K(\lambda) : \Delta(\mu)) \neq 0$, then we have $\lambda < \mu$.

**Remark 2.5.** $(A \mod, \leq)$ is a highest weight category if and only if $A$ is a quasi-hereditary algebra [CPS88, Theorem 3.6]. Thus the highest weight category $(A \mod, \leq)$ and the quasi-hereditary algebra $A$ can be regarded interchangeably.

**Lemma 2.6** (cf. Donkin [Don98, Proposition A. 2.2]). Let $(A \mod, \leq)$ be quasi-hereditary. For $X \in A \mod$ and $\lambda \in \Lambda$, if $\text{Ext}^1_\Lambda(\Delta(\lambda), X) \neq 0$, then $X$ has a composition factor $L(\mu)$ with $\mu > \lambda$.

For a subset $\pi \subseteq \Lambda$, $\pi$ is called a poset ideal if $\lambda \in \Lambda$ whenever $\lambda < \mu$ and $\mu \in \pi$.

**Lemma 2.7** (cf. Donkin [Don98, Lemma A 3.1]). We assume that $(A \mod, \leq)$ is quasi-hereditary. Let $\pi$ be a poset ideal of $\Lambda$. Then for $X \in \mathcal{F}(\Delta)$, we have $O^\pi(X), X/O^\pi(X) \in \mathcal{F}(\Delta)$. Moreover

$$(X/O^\pi(X) : \Delta(\lambda)) = \begin{cases} (X : \Delta(\lambda)), & \lambda \in \pi; \\ 0, & \text{otherwise}. \end{cases}$$

2.2. From the rest of this section, we assume that $A$ is a quasi-hereditary algebra with a heredity chain $A = H_0 > H_1 > \cdots > H_i > H_{i+1} > \cdots > H_n = 0$. For $1 \leq j < n$, we put $F_j := A/H_j \otimes_A -$. Then $F_j$ are the right exact functors from $A \mod$ to $A/H_j \mod$.

**Lemma 2.8.** Let $\lambda, \mu, \nu \in \Lambda$. If $F_j(L(\lambda)) = L(\lambda)$ and $\mu < \lambda$, then we have $F_j(L(\mu)) = L(\mu)$ for $1 \leq j < n$. If $F_j(L(\lambda)) = 0$ and $\lambda < \nu$, then we have $F_j(L(\nu)) = 0$ for $1 \leq j < n$.

**Proof.** For the functor $F_j$, there exists a poset ideal $\pi_j \subseteq \Lambda$ such that $F_j(M) = M/O^{\pi_j}(M)$ for any $M \in A \mod$. Thus the assertion holds. □

**Lemma 2.9.** We fix an element $\lambda \in \Lambda$ such that $F_j(L(\lambda)) = L(\lambda)$. We denote by $\iota : K(\lambda) \hookrightarrow P(\lambda)$ the inclusion map. Then $F_j(\iota) : F_j(K(\lambda)) \hookrightarrow F_j(P(\lambda))$ is injective for $1 \leq j < n$. 

Firstly, we show this statement in the case $i$ for $1 \leq i < n$. We assume that $\Delta(\lambda)$ is isomorphic to $M_j$, and there exists the following short exact sequence:

$$0 \to K(\lambda) \to P(\lambda) \to \Delta(\lambda) \to 0. \tag{2-1}$$

Thus we have

$$0 \to \mathbb{L}_i F_j(\Delta(\lambda)) \to F_j(K(\lambda)) \xrightarrow{F_j(\iota)} F_j(P(\lambda)) \to F_j(\Delta(\lambda)) \to 0$$

for $1 \leq j < n$. We assume that $F_j(L(\lambda)) = 0$. Then we deduce from Lemma 2.8 and the condition (b) that if $(K(\lambda) : \Delta(\mu)) \neq 0$, then $F_j(L(\mu)) = 0$. Thus we have $F_j(\Delta(\mu)) = 0$ and we deduce from induction on the filtration length of $K(\lambda)$ that $F_j(K(\lambda)) = 0$. Therefore we have $\mathbb{L}_i F_j(\Delta(\lambda)) = 0$. Let $F_j(L(\lambda)) = L(\lambda)$. Then it follows from Lemma 2.9 that $F_j(\iota)$ is injective. Thus we deduce $\mathbb{L}_i F_j(\Delta(\lambda)) = 0$.

We assume that $fl(M) > 1$. Then there exists a standard module $\Delta(\lambda)$ and a factor module $Q$ of $M$ such that

$$0 \to \Delta(\lambda) \to M \to Q \to 0 \tag{2-2}$$

is a short exact sequence. Since $\mathbb{L}_i F_j(\Delta(\lambda)) = 0$, we have

$$0 \to \mathbb{L}_i F_j(M) \to \mathbb{L}_i F_j(Q) \to F_j(\Delta(\lambda)) \to F_j(M) \to F_j(Q) \to 0.$$

Therefore we deduce from the induction hypothesis that $\mathbb{L}_i F_j(M) = 0$.

Secondly, we also show the assertion in the case $i > 1$ by induction on $fl(M)$. If $fl(M) = 1$, then there exists $\lambda \in \Lambda$ such that $\Delta(\lambda)$ is isomorphic to $M$. Thus we have $\mathbb{L}_{i+1} F_j(\Delta(\lambda)) \cong \mathbb{L}_i F_j(K(\lambda))$ from the short exact sequence (2-1). Therefore the assertion follows from the induction hypothesis.

If $fl(M) > 1$, then we have

$$\cdots \to \mathbb{L}_{i+1} F_j(\Delta(\lambda)) \to \mathbb{L}_i F_j(M) \to \mathbb{L}_i F_j(Q) \to \mathbb{L}_i F_j(\Delta(\lambda)) \to \cdots$$

from the short exact sequence (2-2). Thus $\mathbb{L}_i F_j(M) \cong \mathbb{L}_i F_j(Q)$ for all $i \geq 2$. Therefore the assertion follows from the induction hypothesis.

In the following, We write $\mathbb{L}_i F_j$ for the $i$-th left derived functor of $F_j$. In the case $i = 1$, we write $LF_j$.

**Lemma 2.10.** The functors $F_j$ on $\mathcal{F}(\Delta)$ are exact functors for $1 \leq j < n$.

**Proof.** Let $M \in \mathcal{F}(\Delta)$. We show that $\mathbb{L}_i F_j(M) = 0$ for $1 \leq j < n$ by induction on $i$. Firstly, we show this statement in the case $i = 1$ by induction on $fl(M)$. If $fl(M) = 1$, then there exists $\lambda \in \Lambda$ such that $\Delta(\lambda)$ is isomorphic to $M$, and there exists the following short exact sequence:

$$0 \to K(\lambda) \to P(\lambda) \to \Delta(\lambda) \to 0. \tag{2-1}$$

Thus we have

$$0 \to \mathbb{L}_i F_j(\Delta(\lambda)) \to F_j(K(\lambda)) \xrightarrow{F_j(\iota)} F_j(P(\lambda)) \to F_j(\Delta(\lambda)) \to 0$$

for $1 \leq j < n$. We assume that $F_j(L(\lambda)) = 0$. Then we deduce from Lemma 2.8 and the condition (b) that if $(K(\lambda) : \Delta(\mu)) \neq 0$, then $F_j(L(\mu)) = 0$. Thus we have $F_j(\Delta(\mu)) = 0$ and we deduce from induction on the filtration length of $K(\lambda)$ that $F_j(K(\lambda)) = 0$. Therefore we have $\mathbb{L}_i F_j(\Delta(\lambda)) = 0$. Let $F_j(L(\lambda)) = L(\lambda)$. Then it follows from Lemma 2.9 that $F_j(\iota)$ is injective. Thus we deduce $\mathbb{L}_i F_j(\Delta(\lambda)) = 0$.

We assume that $fl(M) > 1$. Then there exists a standard module $\Delta(\lambda)$ and a factor module $Q$ of $M$ such that

$$0 \to \Delta(\lambda) \to M \to Q \to 0 \tag{2-2}$$

is a short exact sequence. Since $\mathbb{L}_i F_j(\Delta(\lambda)) = 0$, we have

$$0 \to \mathbb{L}_i F_j(M) \to \mathbb{L}_i F_j(Q) \to F_j(\Delta(\lambda)) \to F_j(M) \to F_j(Q) \to 0.$$

Therefore we deduce from the induction hypothesis that $\mathbb{L}_i F_j(M) = 0$.

Secondly, we also show the assertion in the case $i > 1$ by induction on $fl(M)$. If $fl(M) = 1$, then there exists $\lambda \in \Lambda$ such that $\Delta(\lambda)$ is isomorphic to $M$. Thus we have $\mathbb{L}_{i+1} F_j(\Delta(\lambda)) \cong \mathbb{L}_i F_j(K(\lambda))$ from the short exact sequence (2-1). Therefore the assertion follows from the induction hypothesis.

If $fl(M) > 1$, then we have

$$\cdots \to \mathbb{L}_{i+1} F_j(\Delta(\lambda)) \to \mathbb{L}_i F_j(M) \to \mathbb{L}_i F_j(Q) \to \mathbb{L}_i F_j(\Delta(\lambda)) \to \cdots$$

from the short exact sequence (2-2). Thus $\mathbb{L}_i F_j(M) \cong \mathbb{L}_i F_j(Q)$ for all $i \geq 2$. Therefore the assertion follows from the induction hypothesis. \[\Box\]

**Proposition 2.11.** The functors $F_j \mid_{A/H_j \text{mod}} : A/H_j \text{mod} \to A/H_j \text{mod}$ are exact for $1 \leq j < n$. 
Lemma 2.12. Let \( X, Y \in A/H_j \mod \). Then we have
\[
\text{Tor}_i^A(X, Y) \cong \text{Tor}_i^{A/H_j}(X, Y)
\]
for any \( i > 0 \) and \( 1 \leq j < n \).

Proof. For \( X \in \mod A/H_j \), we define the functors \( G_j := X \otimes_{A/H_j} - \). Then \( G_j \) are right exact functors from \( A/H_j \mod \) to \( k \mod \). For \( 1 \leq j < n \), \( F_j(P) \) is a \( G_j \)-acyclic for all projective objects \( P \) in \( A \mod \). Thus there exists a Grothendieck spectral sequence (for example, see [CE56]) \( E = (E_{p,q}^r, E_n) \) of \( A \mod \) such that for each \( Y \in A \mod \), the following holds:
\[
E_{2}^{p,q} \cong \text{Tor}_{p+q}^{A/H_j}(X, \mathbb{L}F_j(Y)), E_n = \text{Tor}_n^A(X, Y).
\]
Moreover it follows from Lemma 2.11 that \( \mathbb{L}_i F_j(Y) = 0 \) for \( i > 0 \) and \( 1 \leq j < n \). Therefore the assertion holds.

Lemma 2.13. Let \( X \in A/H_j \mod \), \( Y \in A \mod \). Then we have
\[
\text{Ext}_A(X, Y) \cong \text{Ext}_{A/H_j}(X, F_j(Y))
\]
for any \( i \geq 0 \).

Proof. For \( X \in A/H_j \mod \),
\[
\text{Hom}_{A/H_j}(X, -) : A/H_j \mod \to k \mod
\]
is a left exact functor and \( F_j(I) \) is \( \text{Hom}_{A/H_j}(X, -) \)-acyclic for any injective object \( I \) in \( A \mod \). Moreover we have \( \text{Hom}_{A/H_j}(X, -) \circ F_j = \text{Hom}_A(X, -) \). Thus there exists a
Grothendieck spectral sequence \( E = (E_p^q, E^m) \) of \( A \text{mod} \) such that for each \( Y \in A \text{mod} \), the following holds:

\[
E_2^{p,q} \cong \text{Ext}_{A/H_j}^p(X, R^qF_j(Y)), E^n = \text{Ext}_A^{p+q}(X, Y).
\]

Therefore the proof is done. \(\square\)

3. Main results

3.1. Ideal chains for left strongly quasi-hereditary algebras.

**Definition 3.1** (Ringel [Rin10, Definition in §4]). We say that a pair \((A \text{mod}, \leq)\) is **left strongly quasi-hereditary** if for any \(\lambda \in \Lambda\) there exists a short exact sequence

\[
0 \to K(\lambda) \to P(\lambda) \to \Delta(\lambda) \to 0
\]

with following properties:

(a) \(K(\lambda) \in F(\Delta)\) for all \(\lambda \in \Lambda\);
(b) if \((K(\lambda) : \Delta(\mu)) \neq 0\), then we have \(\lambda < \mu\);
(c) \(K(\lambda)\) is a projective left \(A\)-module.

**Remark 3.2** (Ringel [Rin10, Proposition in §4]). If \((A \text{mod}, \leq)\) is left (resp. right) strongly quasi-hereditary, then \((A \text{mod}, \leq)\) is quasi-hereditary.

**Example 3.3.** We assume that a natural number \(e\) is greater than or equal to two. Let \(A_e\) be the algebra over an algebraically closed field defined by the following quiver

\[
\begin{array}{cccccccc}
1 & \overset{\alpha_1}{\cdots} & \overset{\alpha_{i-1}}{\cdots} & \overset{\alpha_i}{\cdots} & \overset{\alpha_{i+1}}{\cdots} & \cdots & \overset{\alpha_{e-1}}{\cdots} & e \\
\beta_1 & \cdots & \beta_i & \cdots & \beta_{i+1} & \cdots & \beta_{e-1}
\end{array}
\]

with relations \(\alpha_i\alpha_{i-1}, \beta_i\beta_{i-1}, \alpha_{i-1}\beta_{i-1} - \beta_i\alpha_i\) for \(2 \leq i \leq e-1\) and \(\alpha_{e-1}\beta_{e-1}\).

If \(e = 2\), then \(A_2\) is a left strongly quasi-hereditary algebra with respect to \(\{1 < 2\}\). If \(e > 2\), then \(A_e\) is a quasi-hereditary algebra with respect to \(\{1 < 2 < \cdots < e\}\). However \(A_e\) cannot be a left strongly quasi-hereditary algebra.

**Remark 3.4.** In general, \(A\) is quasi-hereditary if and only if \(A^{\text{op}}\) is quasi-hereditary [CPS88, Lemma 3.4]. However opposite algebras of left strongly quasi-hereditary algebras are not always left strongly quasi-hereditary.

**Definition 3.5** (Ringel [Rin10]). \(A\) is called **right strongly strongly quasi-hereditary** if the conditions of Definition 3.1 hold for \(A^{\text{op}} \text{mod}\).

**Example 3.6.** Let \(B\) be the algebra over a field defined by the following quiver

\[
\begin{array}{ccc}
\alpha & 1 & \gamma \\
\beta & 2 & 3
\end{array}
\]

with relations \(\gamma\alpha, \alpha\beta\). Then \(B\) is left strongly quasi-hereditary with respect to \(\{1 < 2 < 3\}\), but \(B^{\text{op}}\) is not left strongly quasi-hereditary with respect to \(\{1 < 2 < 3\}\). However \(B^{\text{op}}\)
is left strongly quasi-hereditary with respect to \( \{2 < 1 < 3\} \). In fact, \( B \) and \( B^{\text{op}} \) are not left strongly quasi-hereditary with respect to the same ordering. Thus \( B \) is not strongly quasi-hereditary.

**Remark 3.7.** In [Rin10], Ringel showed that if \( A \) is left strongly quasi-hereditary and right strongly quasi-hereditary with respect to the same ordering, then \( \text{gldim} \ A \leq 2 \).

From now on, we construct a special heredity chain for left strongly quasi-hereditary algebras. This is an analogue of the construction of a heredity chain for quasi-hereditary algebras (cf. [Don98 Appendix]).

**Proposition 3.8.** Let \( A \) be a finite dimensional algebra over an algebraically closed field \( k \). Let \( (A \text{mod}, \leq) \) be left (resp. right) strongly quasi-hereditary. We replace \( \Lambda \) by a totally ordered refinement, that is, \( i < j \) if \( \lambda_i < \lambda_j \). We put \( n := |\Lambda| \) and \( \pi(i) := \{\lambda_1, \ldots, \lambda_i\} \) for \( 1 \leq i \leq n \). Then

\[
A > O^{\pi(1)}(A) > \cdots > O^{\pi(i)}(A) > \cdots > O^{\pi(n)}(A) = 0
\]

is a heredity chain of \( A \) and the projective dimensions of \( O^{\pi(i)}(A)/O^{\pi(i+1)}(A) \) as left (resp. right) \( A \)-modules are at most one for \( 1 \leq i < n \).

**Proof.** Let \( \mu \) be a maximal element of \( \Lambda \). We set \( \sigma := \Lambda \setminus \{\mu\} \). Firstly, we show that \( O^\sigma(A) \) is a heredity ideal in \( A \). Since \( A \in \mathcal{F}(\Delta) \), it follows from Lemma 2.7 that \( O^\sigma(A) \) also belongs to \( \mathcal{F}(\Delta) \). Thus all filtration factors of \( O^\sigma(A) \) are isomorphic to \( \Delta(\mu) \). We assume that there exists a submodule \( X \) of \( O^\sigma(A) \) such that \( \text{Ext}^1_A(\Delta(\mu), X) \neq 0 \). Then there exists a composition factor \( L(\nu) \) with \( \nu > \mu \) because of Lemma 2.6, a contradiction. Thus we have \( O^\sigma(A) = \oplus \Delta(\mu) \) inductively. Since \( \mu \) is a maximal element, \( \Delta(\mu) \) is a projective module, and also for \( O^\sigma(A) \). We have \( \text{Hom}_A(O^\sigma(A), A/O^\sigma(A)) = 0 \) because \( \dim \text{Hom}_A(P(\mu), A/O^\sigma(A)) = [A/O^\sigma(A) : L(\mu)] = 0 \). It follows from the condition (a) that all indices of composition factors of \( J\Delta(\mu) \) belong to \( \sigma \). We have \( O^\sigma(A)J\sigma(\sigma(\sigma)) = 0 \) since it deduces from Lemma 2.1 that \( O^\sigma(A)J\Delta(\mu) = O^\sigma(J\Delta(\mu)) = 0 \).

Secondly, we show that \( O^{\pi(i)}(A)/O^{\pi(i+1)}(A) \) is a heredity ideal in \( A(\pi(i+1)) \). Since \( A(\pi(i+1)) \) is quasi-hereditary with respect to the induced ordering on \( \pi(i+1) \), we can also show that \( O^{\pi(i)}(A)/O^{\pi(i+1)}(A) = O^{\pi(i)}(A(\pi(I+1))) \) is a heredity ideal in \( A(\pi(i+1)) \) like the above discussion.

Finally, we show that the projective dimensions of \( O^{\pi(i)}(A)/O^{\pi(i+1)}(A) \) as left \( A \)-modules are at most one. From Lemma 2.7 we have \( O^{\pi(i)}(A(\pi(i+1))) \in \mathcal{F}(\Delta) \). Thus we deduce from the condition (b) that the projective dimensions of \( O^{\pi(i)}(A(\pi(i+1))) \) as left \( A \)-modules are at most one. Similarly, we obtain the assertion of right strongly quasi-hereditary algebras. \( \Box \)

**Proposition 3.9.** Let \( A \) be a finite dimensional algebra over an algebraically closed field \( k \). If there exists a heredity chain of \( A \)

\[
A = H_0 > H_1 > \cdots > H_i > H_{i+1} > \cdots > H_n = 0
\]

such that, for any \( 0 \leq i < n \), the projective dimension of \( H_i/H_{i+1} \) as left (resp. right) \( A \)-modules is at most one. Then \( A \) is left (resp. right) strongly quasi-hereditary with respect
to the following order \( \preceq \): We put \( \Lambda(i) := \{ \lambda \in \Lambda | [S/H_i : L(\lambda)] \neq 0 \} \), for \( 1 \leq i \leq n \), and \( \Lambda(0) := \emptyset \). For \( \lambda \in \Lambda \), we define a positive integer \( r(\lambda) \) to satisfy \( \lambda \in \Lambda(r(\lambda)) \setminus \Lambda(r(\lambda) - 1) \). Then we define \( \lambda \leq \mu \) if \( r(\lambda) \leq r(\mu) \).

**Proof.** Since \( A \) has a heredity chain, it is enough to show that the projective dimensions of standard modules are at most one. We show that \( H_i = O^{\Lambda(i)}(A) \) by decreasing induction on \( i \). If \( i = n - 1 \), then we have \( H_{n-1} = \bigoplus_{\lambda \in \Lambda} P(\lambda)^{d_{\lambda}} \). Since \( \dim \operatorname{Hom}_A(H_{n-1}, A/H_{n-1}) = 0 \), we deduce

\[
H_{n-1} = \bigoplus_{\lambda \in \Lambda \setminus \Lambda(n-1)} P(\lambda)^{d_{\lambda}} = O^{\Lambda(n-1)}(A).
\]

We assume that \( H_j = O^{\Lambda(j)}(A) \) for all \( j \geq i + 1 \). Then we deduce

\[
H_i/H_{i+1} = O^{\Lambda(i)}(A/H_{i+1}) = O^{\Lambda(i)}(A/O^{\Lambda(i+1)}(A)).
\]

Since \( H_i/O^{\Lambda(i+1)}(A) = O^{\Lambda(i)}(A)/O^{\Lambda(i+1)}(A) \), we have \( H_i = O^{\Lambda(i)}(A) \). It follows from Lemma 2.7 that

\[
(O^{\Lambda(i)}(A)/O^{\Lambda(i+1)}(A) : \Delta(\lambda)) = (O^{\Lambda(i)}(A)/O^{\Lambda(i+1)}(O^{\Lambda(i)}(A)) : \Delta(\lambda))
\]

\[
= \begin{cases} 
(O^{\Lambda(i)}(A) : \Delta(\lambda)), & \lambda \in \Lambda(i+1); \\
0, & \text{otherwise.}
\end{cases}
\]

On the other hand, if \( (O^{\Lambda(i)}(A) : \Delta(\lambda)) \neq 0 \), then \( \lambda \notin \Lambda(i) \). Thus it follows that \( r(\lambda) = i + 1 \) whenever \( (O^{\Lambda(i)}(A)/O^{\Lambda(i+1)}(A) : \Delta(\lambda)) \neq 0 \). Similarly, we deduce that the projective dimensions of standard modules as right \( A \)-modules are at most one. \( \square \)

Combining Proposition 3.8 and Proposition 3.9, we have the following theorem.

**Theorem 3.10.** Let \( A \) be a finite dimensional algebra over an algebraically closed field \( k \). \( A \) is a left (resp. right) strongly quasi-hereditary algebra if and only if there exists a heredity chain

\[
A = H_0 > H_1 > \cdots > H_i > H_{i+1} > \cdots > H_n = 0
\]

such that, for any \( 0 \leq i < n \), the projective dimension of \( H_i/H_{i+1} \) as left (resp. right) \( A \)-modules is at most one.

From Theorem 3.10, we give the following definition.

**Definition 3.11.** Let \( A \) be a finite dimensional algebra over an algebraically closed field \( k \)

(1) If \( A \) has a heredity chain of \( A \)

\[
A = H_0 > H_1 > \cdots > H_i > H_{i+1} > \cdots > H_n = 0
\]

such that, for any \( 0 \leq i < n \), the projective dimension of \( H_i/H_{i+1} \) as left (resp. right) \( A \)-modules is at most one. Such a heredity chain is called a left (resp. right) strongly heredity chain of \( A \).
(2) We say that $A$ is a left (resp. right) strongly quasi-hereditary algebra if there exits a left (resp. right) strongly heredity chain of $A$.

Remark 3.12. If $A$ has a left (resp. right) strongly heredity chain $A = H_0 > H_1 > \cdots > H_i > H_{i+1} > \cdots > H_n = 0$, then we deduce that $H_i$ are projective modules as left (resp. right) $A$-modules for $0 \leq i \leq n$ inductively. We assume that $A$ has a heredity chain $A = H_0 > H_1 > \cdots > H_i > H_{i+1} > \cdots > H_n = 0$ such that, for any $0 \leq i \leq n$, $H_i$ is a projective module as left (resp. right) $A$-modules. Then the projective dimension of $H_i/H_{i+1}$ are at most one for any $0 \leq i < n$.

Thus we say that $A$ is a left (resp. right) strongly quasi-hereditary algebra if $A$ has a heredity chain $A = H_0 > H_1 > \cdots > H_i > H_{i+1} > \cdots > H_n = 0$ such that, for any $0 \leq i \leq n$, $H_i$ is a projective module as left (resp. right) $A$-modules.

Theorem 3.13. Let $A$ be a left (resp. right) strongly quasi-hereditary algebra over an algebraically closed field $k$ with a left (resp. right) strongly heredity chain

$$A = H_0 > H_1 > \cdots > H_i > H_{i+1} > \cdots > H_n = 0.$$ 

Then $A/H_i$ is again a left (resp. right) strongly quasi-hereditary algebra for $0 \leq i < n$.

Proof. From Theorem 3.10 we write

$$A = H_0 > H_1 > \cdots > H_i > H_{i+1} > \cdots > H_n = 0$$

for a left strongly heredity chain of $A$. Then

$$A/H_i = H_0/H_i > H_1/H_i > \cdots > H_i/H_i = 0$$

is a heredity chain of $A/H_i$.

Thus it is enough to show that the projective dimensions of $H_j/H_{j+1}$ as left $A/H_i$-modules are at most one for $1 \leq j \leq i - 1$. Since the projective dimension of $H_j/H_{j+1}$ as left $A$-modules is at most one, we write

$$0 \to P_1 \to P_0 \to H_j/H_{j+1} \to 0$$

for a projective resolution of $H_j/H_{j+1}$ as left $A$-modules. It follows from Lemma 2.12 that $A/H_i \otimes_A -$ is an exact functor. Therefore

$$0 \to A/H_i \otimes_A P_1 \to A/H_i \otimes_A P_0 \to A/H_i \otimes_A H_j/H_{j+1} \to 0$$

is a projective resolution of $H_j/H_{j+1}$ as left $A/H_i$-modules. Thus we have

$$\text{proj.dim}_{A/H_i} H_j/H_{j+1} \leq \text{proj.dim}_A H_j/H_{j+1} \leq 1.$$ 

Similarly, we obtain the assertion of right strongly quasi-hereditary algebras. \qed

3.2. Strongly quasi-hereditary algebras. From Theorem 3.10 we give the following definition:

Definition 3.14. Let $A$ be a finite dimensional algebra over an algebraically closed field $k$.

(1) If $A$ has a heredity chain of $A$

$$A = H_0 > H_1 > \cdots > H_i > H_{i+1} > \cdots > H_n = 0$$
such that, for any $0 \leq i < n$, the projective dimension of $H_i/H_{i+1}$ as left $A$-modules and the projective dimension of $H_i/H_{i+1}$ as right $A$-modules are at most one. Such a heredity chain is called a strongly heredity chain of $A$.

(2) We say that $A$ is a strongly quasi-hereditary algebra if there exists a strongly heredity chain of $A$.

**Remark 3.15.** Let $A$ be a finite dimensional algebra with a strongly heredity chain $A = H_0 > H_1 > \cdots > H_i > H_{i+1} > \cdots > H_n = 0$. Then it follows from Theorem 3.13 that $A/H_i$ is again strongly quasi-hereditary, similarly to the definition of quasi-hereditary algebra by Dlab and Ringel [DR89c].

**Remark 3.16.** It deduces from [DR89c, Theorem 1] that $A$ is hereditary if and only if any chain of idempotent ideals of $A$ is a strongly heredity chain of $A$.

Most of the proof follows from [DR89c, Theorem 1]. Let $A$ be hereditary. Then any chain of idempotent ideals of $A$ becomes a heredity chain of $A$ by [DR89c, Theorem 1]. Moreover it follows from the assumption that the projective dimensions of the quotients of ideals are at most one. Since a strongly heredity chain is a special case of a heredity chain, the converse is also true.

**Example 3.17.** Let $A$ be the Auslander algebra of the truncated polynomial algebra $k[x]/(x^3)$. Then $A$ is the algebra over an algebraically closed field $k$ defined by the following quiver

$$
\begin{array}{ccc}
1 & \overset{\alpha_1}{\leftarrow} & 2 \\
& \overset{\beta_1}{\searrow} & \overset{\alpha_2}{\nearrow} \\
& & 3 \\
\end{array}
$$

with relations $\alpha_1\beta_1 - \beta_2\alpha_2$, $\alpha_2\beta_2$. Then $A$ has the following strongly heredity chain:

$$A > A(e_2 + e_3)A > Ae_3A > 0.$$

If $A$ is strongly quasi-hereditary, then the global dimension of $A$ is at most two [Rin10]. However the question of whether every finite-dimensional algebra of global dimension at most two is strongly quasi-hereditary is answered in the negative by providing the following example.

**Example 3.18.** Let $Q$ be the quiver $1 \leftarrow 2 \rightarrow 3$ whose underlying graph is the Dynkin graph $A_3$ and let $A$ be the Auslander algebra of $kQ$. Then $A$ is the algebra over an algebraically closed field $k$ defined by the following quiver

$$
\begin{array}{ccc}
1 & \overset{\alpha}{\leftarrow} & 2 \\
& \overset{\beta}{\searrow} & 3 \\
& & 5 \\
& \overset{\gamma}{\nearrow} & 4 \\
& & 6 \\
\end{array}
$$

with relations $\beta\alpha$, $\delta\gamma$ and $\epsilon\beta - \varphi\delta$. The global dimension of $A$ is two. However we can not construct a strongly heredity chain of $A$. Thus the converse of Ringel’s Theorem [Rin10] in not true.
Remark 3.19. If the global dimension of \( A \) is at most two, then \( A \) is a left (resp. right) strongly quasi-hereditary algebra. The most of the proof follows from [Iya03b, Theorem 3.6]. He showed that if \( A \) has global dimension at most two, then the category of finitely generated projective \( A \)-modules has a complete total left (resp. right) rejective chain. It can be shown that the category of finitely generated projective \( A \)-modules has a complete total left (resp. right) rejective chain if and only if \( A \) is a left (resp. right) strongly quasi-hereditary algebra.

Remark 3.20. It is known that a Ringel dual of a left strongly quasi-hereditary algebra is right strongly quasi-hereditary [Rin10]. Thus the Ringel self dual algebras which are left strongly quasi-hereditary are strongly quasi-hereditary and they have global dimension at most two. Consequently, from Remark 3.19 it follows that for a Ringel self dual algebra \( A \), \( A \) has global dimension at most two if and only if \( A \) is a (left) strongly quasi-hereditary algebra.

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