Radiative decays of heavy-light quarkonia through $M_1$, and $E_1$ transitions in the framework of Bethe-Salpeter equation

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Abstract

In this work we study the radiative decays of heavy-light quarkonia through $M_1$ and $E_1$ transitions that involve quark-triangle diagrams with two hadron vertices, and are difficult to evaluate in BSE-CIA. We have expressed the transition amplitude $M_{fi}$ as a linear superposition of terms involving all possible combinations of $++, -+$ and $--$ components of Salpeter wave functions of final and initial hadron, with coefficients being related to results of pole integrations over complex $\sigma$-plane. We evaluate the decay widths for $M_1$ transitions ($^3S_1 \to ^1S_0 + \gamma$), and $E_1$ transitions ($^3S_1 \to ^1P_0 + \gamma$ and $^1P_0 \to ^3S_1 + \gamma$). We have used algebraic forms of Salpeter wave functions obtained through analytic solutions of mass spectral equations for ground and excited states of $0^{++}$, $1^{--}$, and $0^{-+}$ heavy-light quarkonia in approximate harmonic oscillator basis to calculate their decay widths. The input parameters used by us were obtained by fitting to their mass spectra. We have compared our results with experimental data and other models, and found reasonable agreements.

Key words: Bethe-Salpeter equation, Heavy-Light Quarkonia, $M_1$ and $E_1$ transitions, Transition amplitudes, Form factors, Radiative decay widths

1. Introduction

The most important goals of hadronic physics is to bridge the gap between the QCD lagrangian, and the observed hadronic properties. One of the challenging areas in hadronic physics presently is probing the inner structure of hadrons. There has been a renewed interest in recent years in spectroscopy of these heavy hadrons in charm and beauty sectors, which was primarily due to experimental facilities the world over such as BABAR, Belle, CLEO, DELPHI, BES etc. $^{[15]}$, which have been providing accurate data on $c\pi$, and $b\bar{b}$ hadrons with respect to their masses and decays. In the process
many new states have been discovered such as $\chi_{b0}(3P)$, $\chi_{b0}(2P)$, $X(3915)$, $X(4260)$, $X(4360)$, $X(4430)$, $X(4660)$ [8], some of which are exotic states, which can not be readily explained through the predictions of the quark model. The radiative transitions of heavy quarkonia are of considerable experimental and theoretical interest, and provide an insight into the dynamics of quarkonium. The radiative transitions between $0^{-+}$ (pseudoscalar), and $1^{-+}$ (vector) mesons (for instance, $J/\Psi(nS) \rightarrow \eta_c(n'S) + \gamma$), which proceeds through the emission of photon is characterized by $\Delta L = 0$, there is change in C-parity between the initial and final hadron states, though the total C-parity is conserved. These are the magnetic dipole transition, $M1$. This transition mode is sensitive to relativistic effects, specially between different spatial multiplets ($n > n'$). The E1 transitions are characterized by $|\Delta L| = 1$. Thus in these transitions, there is change in parity between the initial and final hadronic states, for instance, $\Psi(2S) \rightarrow \chi_c(1P) + \gamma$ or $\chi_c(1P) \rightarrow J/\Psi(1S) + \gamma$. In both M1 and E1 transitions, C-parity is conserved. Electric dipole transitions are much stronger than magnetic dipole transitions, and involve transitions between excited states. These transitions have been recently studied in various models, such as relativistic quark models [6,7], effective field theory [8,9], Light-front quark models [10,11], Lattice QCD [12,13], Bethe-Salpeter equation [14–16].

In this work we focus on the radiative decays of the charmed and bottom vector mesons through the processes, $V \rightarrow P \gamma$, $V \rightarrow S \gamma$, and $S \rightarrow V \gamma$, where, $V,P,S$ refer to vector, pseudoscalar and scalar quarkonia, and calculate the radiative decay widths of $B^*$, and $D^*$ mesons for the above mentioned processes in the framework of $4 \times 4$ Bethe-Salpeter equation. In our recent works [17,18], we had studied the mass spectrum of ground and excited states of heavy-light scalar ($0^{++}$), pseudoscalar ($0^{-+}$), and vector ($1^{-+}$) quarkonia, along with the leptonic decays of ground and excited states of $0^{-+}$, and $1^{-+}$ quarkonia. These studies were used to fit the input parameters of our model as $C_0 = 0.69$, $\omega_0 = 0.22$ GeV, $\Lambda_{QCD} = 0.250$ GeV, and $A_0 = 0.01$, with input quark masses $m_u = 0.300$ GeV, $m_s = 0.430$ GeV, $m_c = 1.490$ GeV, and $m_b = 4.690$ GeV.

In the present work on radiative decays, we use these same input parameters to calculate the single photon decay widths for the above processes, However, due to the highly involved algebra, we take only the leading Dirac structures ($\gamma_5, I,$ and $\gamma \cdot \epsilon$ respectively) in the hadronic Bethe-Salpeter wave functions of these hadrons, which as we have earlier shown using our naive power counting rule [19,20], contributes maximum to the calculation of any meson observable. However the price we have to pay for this simplification is that we are unable to get good agreement of decay widths for M1 transitions ($nS \rightarrow n'S + \gamma$, with $n' \neq n$), with data, which would imply that the incorporation of sub-leading Dirac structures in wave functions of $V$ and $P$ quarkonia are important for studying transitions between different spatial multiplets.

Now, as mentioned in our previous works [17,18,21], we are not only interested in studying the mass spectrum of hadrons, which no doubt is an important element to study dynamics of hadrons, but also the hadronic wave functions that play an important role in the calculation of decay constants, form factors, structure functions etc. for $Q\bar{Q}$, and $Q\bar{q}$ hadrons. These hadronic Bethe-Salpeter wave functions were calculated algebraically by us in [17,18,21]. The plots of these wave functions [18] show that they can provide information not only about the long distance non-perturbative physics, but also act as a bridge between the long distance, and short distance physics, and are provide us information about the contribution of the short ranged coulomb interactions in the mass spectral calculation of heavy-light quarkonia. These wave functions and can also lead to studies on a number of processes involving $Q\bar{Q}$, and $Q\bar{q}$ states, and provide a guide for future experiments.

This paper is organized as follows: In section 2, we introduce the formulation of the $4 \times 4$ Bethe-Salpeter equation under the covariant instantaneous ansatz, and derive the hadron-quark vertex. In sections 3, 4, and 5, we calculate the
single photon decay widths for the processes, $V \to P\gamma$, $V \to S\gamma$, and $S \to V\gamma$, where, P, S, and V are the pseudoscalar, scalar and vector heavy-light quarkonium states. In section 6, we provide the numerical results and discussion.

2. Formulation of the BSE under CIA

Our work is based on QCD motivated BSE in ladder approximation, which is an approximate description, with an effective four-fermion interaction mediated by a gluonic propagator that serves as the kernel of BSE in the lowest order. The precise form of our kernel is taken in analogy with potential models, which includes a confining term along with a one-gluon exchange term. Such effective forms of the BS kernel in ladder BSE have recently been used in \cite{23–27}, and can predict bound states having a purely relativistic origin (as shown recently in \cite{23}). As mentioned above, the BSE is quite general, and provides an effective description of bound quark-antiquark systems through a suitable choice of input kernel for confinement.

The Bethe-Salpeter equation that describes the bound state of two quarks ($QQ$ or $Qq$) of momenta $p_1$ and $p_2$, relative momentum $q$, and meson momentum $P$ is

$$S_F^{-1}(p_1)\Psi(P,q) = i \int \frac{d^4q}{(2\pi)^4} K(q,q')\Psi(P,q'),$$

(1)

where $K(q,q')$ is the interaction kernel, and $S_F^{-1}(\pm p_{1,2}) = \pm i\slashed{p}_{1,2} + m_{1,2}$ are the usual quark and antiquark propagators. The 4D B.S. wave function can be written as

$$\Psi(P,q) = S_1(p_1)\Gamma(\hat{q})S_2(-p_2),$$

(2)

where the hadron-quark vertex is

$$\Gamma(\hat{q}) = \int \frac{d^3q'}{(2\pi)^3} K(\hat{q},q')\psi(q').$$

(3)

The 4D B.S. wave function can be expressed in terms of the projected wave functions as

$$\psi(\hat{q}) = \psi^{++}(\hat{q}) + \psi^{+}(\hat{q}) + \psi^{-+}(\hat{q}) + \psi^{--}(\hat{q}),$$

(4)

where

$$\psi^{\pm \pm}(\hat{q}) = \Lambda^\pm_1(\hat{q}) \frac{\slashed{P}}{M} \psi(\hat{q}) \frac{\slashed{P}}{M} \Lambda^\pm_2(\hat{q})$$

(5)

and the projection operators

$$\Lambda^\pm_j(\hat{q}) = \frac{1}{2\omega_j} \left[ \frac{\slashed{P}}{M} \omega_j \pm J(j)(im_j + \hat{q}) \right], \quad J(j) = (-1)^{j+1}, \quad j = 1, 2$$

(6)

with the relation

$$\omega^2_j = m^2_j + q^2$$

(7)

3. Radiative decays of heavy-light quarkonia through $V \to P\gamma$

The single photon decay of vector ($1^{--}$) quarkonia is described by the direct and exchange Feynman diagrams as in Figure[1]

To apply the framework of BSE to study radiative decays, $V \to P\gamma$, we have to remember that there are two Lorentz frames, one the rest frame of the initial meson, and the other, the rest frame of final meson. To calculate further, we first write relationship between the momentum variables of the initial and final meson. Let $P$, and $q$ be the
total momentum and the internal momentum of initial hadron, while $P'$, and $q'$ be the corresponding variables of the final hadron, and let $k$, and $\epsilon^\lambda$ be momentum and polarization vectors of emitted photon, while $\epsilon^\lambda$ be the polarization vector of initial meson. Thus if $p_1, 2$, and $p'_1, 2$ are the momenta of the two quarks in initial and final hadron respectively, then, we have, the momentum relations:

\begin{align*}
P &= p_1 + p_2; p_{1,2} = \hat{m}_{1,2}P + q \\
P' &= p'_1 + p'_2; p'_{1,2} = \hat{m}_{1,2}P' + q'
\end{align*}

for initial and final hadrons respectively. From the Feynman diagrams we see that conservation of momentum demands that, $P = P' + k$, while from the first diagram, $p_1 = p'_1 + k$, and $-p_2 = -p'_2$, where $k = P - P'$ is the momentum of the emitted photon. Making use of the above equations, we can express, the relationship between the internal momenta of the two hadrons in terms of the photon momentum, $k$ as,

\begin{align*}
q' &= q + (\hat{m}_1 - 1)k = q - \hat{m}_2k, \\
\hat{m}_{1,2} &= \frac{1}{2}[1 \pm (\frac{m_1^2 - m_2^2}{M^2})] \text{\ before Eq.(10)}
\end{align*}

with $\hat{m}_{1,2} = \frac{1}{2}[1 \pm (\frac{m_1^2 - m_2^2}{M^2})] \text{\ acting like momentum partitioning functions for the two quarks in a hadron. We now decompose the internal momentum $q$ of the initial hadron into two components, $q = (\vec{q}, iM\sigma)$, where $\vec{q}_\mu = q_\mu - \sigma P_\mu$ is the component of internal momentum transverse to $P$ such that $\vec{q}.P = 0$, while $\sigma = \frac{P.P}{P^2}$ is the longitudinal component in the direction of $P$. Similarly for final meson, we decompose its internal momentum, $q'$ into two components $q' = (\vec{q}', iM'\sigma')$. We now first try to find the relationship between the transverse components of internal momenta of the two hadrons, $\vec{q}$, and $\vec{q}'$. For this, we resolve all momenta in Eq.(9) along the direction transverse to the momentum of the initial meson, $P$. Thus we can express Eq.(9) as

\begin{align*}
\vec{q}' &= \vec{q} + \hat{m}_2(\vec{P}' - \vec{P}) \\
\vec{P} &= 0 \\
\vec{P}' &= P' - \frac{P'.P}{P^2}P
\end{align*}

where, it can be easily checked that $\vec{P}.P = 0$, and thus $\vec{P}'$ is orthogonal to $P$. Now, the kinematics gets simplified in the rest frame of the initial meson, where we have $P = (0, iM)$, while for emitted meson, $P' = (\vec{P}', iE')$, where
\[ E' = \sqrt{\vec{P}'^2 + M'^2}, \] and since the photon momentum can be decomposed as, \( k = (\vec{k}, i|\vec{k}|) \), where \( \vec{k} = -\vec{P}' \), since final meson and photon would be emitted in opposite directions. Hence we get, \( |\vec{P}'| = |\vec{k}| = \frac{M^2 - M'^2}{2M} \). Thus the energy of the emitted meson can be expressed as, \( E' = \frac{M^2 + M'^2}{2M} \).

Further the dot products of momenta of the initial and the emitted meson can be expressed as, \( P'.P = -ME' = -\frac{M^2 + M'^2}{2M} \). Thus, we can write the relation between the transverse components of internal momenta of the two hadrons, and their squares as,

\[ q' = \hat{q} + \hat{m}_2(P' - \frac{(M^2 + M'^2)}{2M^2}P). \]

From above equations, it can be easily checked that \( P.q' = 0 \), and \( P'.q' = -\hat{m}_2\frac{(M^2 - M'^2)}{4M^2} \).

Now, we try to find relationship between the time components, \( \sigma \) and \( \sigma' \) of the two hadrons. Taking dot product of Eq.(9) with \( P \), the momentum of the initial hadron, we obtain,

\[ P.q' = P.q - \hat{m}_2 P\hat{k}. \]

Making use of the above decomposition of internal momenta, we obtain the relation between the longitudinal components of internal momenta of the two hadrons as,

\[ M'.\sigma' = M\sigma + \alpha; \]
\[ \alpha = \hat{m}_2\frac{M^2 - M'^2}{2M}. \]

Thus, up to Eq.(13), the kinematics is the same for all the three processes \((V \to P\gamma, V \to S\gamma, \) and \( S \to V\gamma)\) studied in this work.

It is to be noted that \( 4D \) BS wave functions of the two hadrons (vector and pseudoscalar) involved in the process are:

\[ \Psi_V(P, q) = S_F(p_1)\Gamma_V(\hat{q})S_F(-p_2), \]
\[ \Psi_P(P', q') = S_F(p'_1)\Gamma_P(\hat{q}')S_F(-p'_2), \]
\[ (14) \]

where \( S_F \) are the quark propagators, while \( \Gamma_V(\hat{q}) \), and \( \Gamma_P(\hat{q}') \), are the hadron-quark vertex functions for vector and pseudoscalar mesons respectively, and \( \overline{\Psi}_P(P', q') = \gamma_4\Psi_P(P', q')\gamma_4 \) is the adjoint wave function of the emitted pseudoscalar meson.

The EM transition amplitude of the process is

\[ M_{fi} = -i\int \frac{d^4q}{(2\pi)^4} Tr[\epsilon_q\overline{\Psi}_P(P', q')q^\lambda \Psi_V(P, q)S_P^{-1}(-p_2) + \epsilon_{Q}\overline{\Psi}_P(P', q')\gamma_4S_F^{-1}(p_1)\Psi_V(P, q)q'^\lambda], \]
\[ (15) \]

where the first term corresponds to the first diagram, where the photon is emitted from the quark \((q)\), while the second term corresponds to the second diagram where the photon is emitted from the antiquark \((\overline{Q})\) in vector meson.

In the above expression, \( \Psi_P \) and \( \Psi_V \) are the \( 4D \) BS wave functions of pseudoscalar and vector quarkonia involved in the process, and are expressed above, while \( \epsilon_q \), and \( \epsilon_Q \) are the electric charge of quark, and antiquark respectively, and \( \epsilon^\lambda_{\mu} \) is the polarization vector of the emitted photon.
Using the fact that the contribution of the second term is the same as that of the first term (except that $e_q \neq e_{ar{q}}$), we rewrite above equation in terms of the electronic charge, $e$, as,

$$M_{fi} = -ie \int \frac{d^3 \hat{q}}{(2\pi)^3} \text{Tr}[\bar{\Psi}_F(P', \hat{q}')\psi\Psi_V(P, \hat{q})S_F^{-1}(-p_2)].$$

(16)

This can be expressed as,

$$M_{fi} = -ie \int \frac{d^3 \hat{q}}{(2\pi)^3} \int \frac{iMd\sigma}{(2\pi)} \text{Tr}[\Gamma_P(q')S_F(p_1)\gamma_sS_F(p_1)\Gamma_V(q)S_F(-p_2)].$$

(17)

To calculate $M_{fi}$, we express the propagators $S_F(\pm p_{1,2})$ as,

$$S_F(p_1) = \frac{\Lambda_+^+(\hat{q})}{M \sigma + \hat{m}_1 M - \omega_1} + \frac{\Lambda_-^+(\hat{q})}{M \sigma + \hat{m}_1 M + \omega_1},$$

$$S_F(-p_2) = \frac{\Lambda_+^-(-\hat{q})}{M \sigma + \hat{m}_2 M - \omega_2} + \frac{\Lambda_-^-(-\hat{q})}{M \sigma + \hat{m}_2 M + \omega_1}. \quad (18)$$

Here we wish to mention that in transitions involving single photon decays, such as $V \to P + \gamma$, the process requires calculation of triangle quark-loop diagram, which involves two hadron-quark vertices that we attempt in the $4 \times 4$ representation of BSE. We now put the propagators expressed as Eq. (18) into Eq. (17), and multiplying this equation from the left by the relation, $\frac{P}{M} \frac{P}{M} = -1 = \frac{P}{M} (\Lambda_+^+(q') + \Lambda_-^+(q'))$, and making use of Eq. (13), where $\alpha < 1$, the transition amplitude can be expressed as,

$$M_{fi} = i \int \frac{d^3 \hat{q}}{(2\pi)^3} (\Omega_1 + \Omega_2 + \Omega_3 + \Omega_4);$$

$$\Omega_1 = \int \frac{d\sigma}{(2\pi) M^3} \text{Tr} \left[ \frac{-P \Lambda_+^+(q')\Gamma_P(q')\Lambda_+^+(q')\gamma_s\Lambda_+^+(q')\Gamma_V(q)\Lambda_+^+(q)}{[\sigma - (-\frac{\hat{m}_1 M}{M} + \frac{\omega_1}{2M})][\sigma - (-\hat{m}_1 + \frac{\omega_1}{2M})][\sigma - (\hat{m}_2 + \frac{\omega_1}{2M})]};\right]$$

$$\Omega_2 = \int \frac{d\sigma}{(2\pi) M^3} \text{Tr} \left[ \frac{-P \Lambda_+^+(q')\Gamma_P(q')\Lambda_-^+(q')\gamma_s\Lambda_-^+(q')\Gamma_V(q)\Lambda_+^+(q)}{[\sigma - (-\frac{\hat{m}_1 M}{M} + \frac{\omega_1}{2M})][\sigma - (-\hat{m}_1 + \frac{\omega_1}{2M})][\sigma - (\hat{m}_2 + \frac{\omega_1}{2M})]};\right]$$

$$\Omega_3 = \int \frac{d\sigma}{(2\pi) M^3} \text{Tr} \left[ \frac{-P \Lambda_+^+(q')\Gamma_P(q')\Lambda_+^+(q')\gamma_s\Lambda_-^+(q')\Gamma_V(q)\Lambda_-^+(q)}{[\sigma - (-\frac{\hat{m}_1 M}{M} - \frac{\omega_1}{2M})][\sigma - (-\hat{m}_1 - \frac{\omega_1}{2M})][\sigma - (\hat{m}_2 + \frac{\omega_1}{2M})]};\right]$$

$$\Omega_4 = \int \frac{d\sigma}{(2\pi) M^3} \text{Tr} \left[ \frac{-P \Lambda_+^+(q')\Gamma_P(q')\Lambda_+^+(q')\gamma_s\Lambda_-^+(q')\Gamma_V(q)\Lambda_-^+(q)}{[\sigma - (-\frac{\hat{m}_1 M}{M} - \frac{\omega_1}{2M})][\sigma - (-\hat{m}_1 - \frac{\omega_1}{2M})][\sigma - (\hat{m}_2 + \frac{\omega_1}{2M})]};\right]. \quad (19)$$

where the rest of the terms are anticipated to be zero on account of 3D Salpeter equations. The contour integrations over $M d\sigma$ are performed over each of the four terms taking into account the pole positions in the complex $\sigma$ plane:

$$\sigma_\pm^+ = -\frac{\alpha}{M} - \frac{\hat{m}_1 M'}{M} + \frac{\omega_1}{M} \pm i\epsilon$$

$$\sigma_\pm^- = -\frac{\hat{m}_1}{M} \pm i\epsilon$$

$$\sigma_\pm^0 = \frac{\omega_1}{M} \pm i\epsilon. \quad (20)$$

In Eq. (19), the contour integral over each of the four terms can be performed by closing the contour either above or below the real axis in the complex $\sigma$-plane with pole positions displayed in Fig. 2. This leads to the expression for effective 3D form of transition amplitude, $M_{fi}$, under Covariant Instantaneous Ansatz as,

$$M_{fi} = -ie \int \frac{d^3 \hat{q}}{(2\pi)^3} \frac{1}{M^2} \text{Tr} \left[ \alpha_1 \bar{\psi}_P^{\mp}(q')\gamma_s\psi_V^{\mp}(\hat{q}) + \alpha_2 \bar{\psi}_P^{\mp}(q')\gamma_s\psi_V^{\mp}(\hat{q}) + \alpha_3 \bar{\psi}_P^{\mp}(q')\gamma_s\psi_V^{\mp}(\hat{q}) + \alpha_4 \bar{\psi}_P^{\mp}(q')\gamma_s\psi_V^{\mp}(\hat{q}) \right]. \quad (21)$$
Figure 2: Pole positions in the complex $\sigma$-plane

Figure 2: Pole positions in the complex $\sigma$-plane

where

\[ \alpha_1 = \frac{-[M - \omega_1' - \omega_2']M - \omega_1 - \omega_2}{[\frac{M}{M'} + \hat{m}_1(\frac{M'}{M} - 1) - \frac{M}{M'}(\omega_1 + \omega_1')]} \]
\[ \alpha_2 = \frac{-[M - \omega_1' - \omega_2']M + \omega_1 + \omega_2}{[\frac{M}{M'} + \hat{m}_1(\frac{M'}{M} - 1) - \frac{M}{M'}(\omega_1 + \omega_1')]} \]
\[ \alpha_3 = \frac{[M + \omega_1' + \omega_2']M + \omega_1 + \omega_2}{[\frac{M}{M'} + \hat{m}_1(\frac{M'}{M} - 1) + \frac{M}{M'}(\omega_1 + \omega_1')]} \]
\[ \alpha_4 = \frac{-[M + \omega_1' + \omega_2']M + \omega_1 + \omega_2}{[\frac{M}{M'} + \hat{m}_1(\frac{M'}{M} - 1) + \frac{M}{M'}(\omega_1 + \omega_1')]} \]

(22)

and the projected wave functions, $\psi^{\pm\pm}$ being taken from the 3D Salpeter equations [18] derived earlier, which for initial meson in internal variable $\hat{q}$ are:

\[ (M - \omega_1 - \omega_2)\psi^{++}(\hat{q}) = \Lambda_1^+(\hat{q})\Gamma(\hat{q})\Lambda_1^+(\hat{q}) \]
\[ (M + \omega_1 + \omega_2)\psi^{--}(\hat{q}) = -\Lambda_1^-(\hat{q})\Gamma(\hat{q})\Lambda_1^-(\hat{q}) \]
\[ \psi^{+-}(\hat{q}) = 0. \]
\[ \psi^{-+}(\hat{q}) = 0 \]

(23)

Similar Salpeter equations can be written for final mesons. Thus, we have given a generalized method for handling quark-triangle diagrams with two hadron -quark vertices in the framework of $4 \times 4$ BSE, by expressing the transition amplitude, $M_{fi}$ (Eq.(21-22)) as a linear superposition of terms involving all possible combinations of $++$, and $--$ components of Salpeter wave functions of final and initial hadrons through not only the $+++$, and $---$ terms but also terms like $+--$, and $-++$, with each of the four terms being associated with a coefficient, $\alpha_i (i = 1, ..., 4)$, which is the result of pole integration in the complex $\sigma$-plane, with pole positions in Eq.(20) (shown in Fig.2). This superposition of all possible terms in Eq.(21-22) should be a feature of relativistic frameworks.

Now, to calculate the process, we need the 4D BS wave functions for vector and pseudoscalar mesons. We again start with the general 4D decomposition of BS wave functions [29]. Using 3D decomposition under Covariant Instantaneous Ansatz, the wave function of vector mesons of dimensionality, $M$ can be written as [17,21]:

\[ \psi^V(\hat{q}) = iM\phi_1(\hat{q}) + \phi\chi_2(\hat{q}) + [\phi\hat{q} - \hat{q}\phi]\chi_3(\hat{q}) - i[\hat{q}\phi + \phi\hat{q}]\chi_4(\hat{q}) + (\hat{q}\phi)\chi_5(\hat{q}) - i\hat{q}\phi \cdot \chi_6(\hat{q}) \]

(24)

where $\epsilon^\lambda$ is the vector meson polarization vector. Similarly for a pseudoscalar meson, the 3D wave function with
dimensionality $M$ can be written as,

$$
\psi^P(\bar{q}) = N_P[M\phi_1(\bar{q}) - iP\phi_2(\bar{q}) + i\bar{q}\phi_3(\bar{q}) + \frac{p\bar{q}}{M}\phi_4(\bar{q})\gamma_5].
$$

(25)

Now, in accordance with the naive power counting rule \cite{19,20} proposed (in which one of us (SB) was involved), the Dirac structures associated with $\chi_1$, and $\chi_2$ (in case of vector mesons), and $\phi_1$, and $\phi_2$ (in case of pseudoscalar mesons), are leading, and would contribute maximum to the calculation of any meson observable. And among these two leading Dirac structures (for both V and P mesons), $M\phi_1$ and $M\gamma_5$ are the most dominant, and contribute the maximum in any calculation. Hence to simplify algebra, we make use of the most dominant Dirac structures for both vector and pseudoscalar mesons. Thus, the 4D Bethe-Salpeter wave functions of heavy-light pseudoscalar and vector quarkonia are taken as,

$$
\psi_P(\bar{q}) = N_P(M\gamma_5)\phi_P(\bar{q}'),
$$

$$
\psi_V(\bar{q}) = N_V(iM\phi)\phi_V(\bar{q}),
$$

(26)

Such dominant Dirac structures, $\Gamma = \gamma_5$ (for $0^{++}$), $\gamma_\mu$ (for $1^{--}$), and 1 (for $0^{++}$) have also been used recently in lattice calculations of radiative decays in \cite{22} recently.

The 4D Bethe-Salpeter normalizers are

$$
N_P^{-2} = 4\hat{m}_1\hat{m}_2 M^2 \frac{1}{m_1} \int \frac{d^4\bar{q}'}{(2\pi)^3} \phi_P(\bar{q}),
$$

$$
N_V^{-2} = 4\hat{m}_1\hat{m}_2 M^2 \frac{1}{m_1} \int \frac{d^4\bar{q}'}{(2\pi)^3} \phi_V(\bar{q})
$$

(27)

The 3D wave functions of ground and excited states of pseudoscalar $0^{-+}$ and vector $1^{--}$ quarkonia are \cite{18},

$$
\phi_{P,V}(1S,\bar{q}) = \frac{1}{\pi^{3/2}} \frac{1}{\beta_{P,V}} e^{-\frac{q^2}{\beta_{P,V}}}
$$

$$
\phi_{P,V}(2S,\bar{q}) = \sqrt{\frac{3}{2}} \frac{1}{\pi^{3/2}} \frac{1}{\beta_{P,V}^{3/2}} \left(1 - \frac{2q^2}{3\beta_{P,V}}\right) e^{-\frac{q^2}{\beta_{P,V}}}
$$

$$
\phi_V(1D,\bar{q}) = \sqrt{\frac{4}{15}} \frac{1}{\pi^{3/2}} \frac{1}{\beta_{P,V}^{3/2}} q^2 e^{-\frac{q^2}{\beta_{P,V}}}
$$

$$
\phi_{P,V}(3S,\bar{q}) = \sqrt{\frac{15}{8}} \frac{1}{\pi^{3/2}} \frac{1}{\beta_{P,V}^{3/2}} \left(1 - \frac{4q^2}{3\beta_{P,V}} + \frac{4q^4}{15\beta_{P,V}}\right) e^{-\frac{q^2}{\beta_{P,V}}},
$$

(28)

where the inverse range parameters are

$$
\beta_{P,V} = \left(\frac{\frac{1}{2}\omega_3^2(m_1 + m_2)}{\sqrt{1 + 8\hat{m}_1\hat{m}_2A_0(N + \frac{1}{2})}}\right)^{1/4}
$$

(29)

The $^{++}$ and $^{--}$ components of the B.S. wave function for pseudoscalar meson are \cite{21,26}:

$$
\psi_P^{\pm\pm}(q') = \Lambda_\varphi^{\pm\pm}(q') \frac{P}{M} \psi_P(q') \frac{P}{M} \Lambda_\varphi^{\pm\pm}(q')
$$

(30)

Substituting the 4D BS wave function of pseudoscalar meson, the $^{++}$ and $^{--}$ components of the 4D BS wave function of pseudoscalar meson can be obtained using Eq. (30) as given in Eq. (A.57) of Appendix A1. The corresponding adjoint wave functions are given in Eq. (A.58) of Appendix A1.

Whereas, the positive and negative energy components of the vector meson wave function are

$$
\psi_V^{\pm\pm}(q') = \Lambda_\psi^{\pm\pm}(q') \frac{P}{M} \psi_V(q') \frac{P}{M} \Lambda_\psi^{\pm\pm}(q')
$$

(31)
Following the same steps as in Eq.(A.59), we obtain the ++ and −− components of the 4D BS wave function of vector meson as through Eq.(31). These components of vector meson wave function are given in Eq.(A.59), and their corresponding adjoint wave functions are given in Eq.(A.60) of Appendix A1.

We now calculate the individual terms, \( P\overline{\psi}^{++}(q')\overline{\psi}^{++}((q), P\overline{\psi}^{−−}(q')\overline{\psi}^{−−}((q), P\overline{\psi}^{−−}(q')\overline{\psi}^{++}((q) \), and \( P\overline{\psi}^{−−}(q')\overline{\psi}^{−−}((q) \) in the transition amplitude, \( M_{f1} \). These terms are given in Eqs.(A.61-A.64) of Appendix A1.

Here, it is to be mentioned that, the transverse component of internal momentum of the pseudoscalar meson can be expressed as, \( \hat{q} = \hat{q} + m_2[P' - M^2/M'^2 P] \), as in Eq.(11), where \( m_2 = \frac{1}{2}(1 - \frac{(m_1^2 - m_2^2)}{2M'^2}) \) are the Wightman-Garding definitions of masses of the quarks, that act as momentum partitioning parameters. Further, \( \omega_1^2 = m_{1,2}^2 + \hat{q}^2 \), while, \( \omega_1^2 = m_{1,2}^2 + \hat{q}^2 \). Therefore, the relationship between \( \hat{q}^2 \) and \( \hat{q}^2 \) can be expressed as,

\[
\hat{q}^2 = \hat{q}^2 - \hat{m}_2^2(M^2 - M'^2)^2/4M^2,
\]

(32) where, we have made use of the fact that in the rest frame of initial meson, \( |\hat{B}| = |\hat{k}| = \frac{M^2-M'^2}{2M} \)

The transition amplitude, \( M_{f1} \), is expressed as,

\[
M_{f1} = F_{VP} \epsilon_{\mu\nu\alpha\beta} P_{\mu}e_{\nu}^{\lambda}e_{\alpha}^{\lambda}P_{\beta},
\]

(33) where the antisymmetric tensor, \( \epsilon_{\mu\nu\alpha\beta} \) ensures its gauge invariance. We wish to mention that we choose to work in the rest frame of the decaying particle (as in [22]), and since the results are Lorentz-covariant due to use of Covariant Instantaneous Ansatz (CIA), the choice of the frame is irrelevant [22].

Here, \( F_{VP} \) is the transition form factor for \( V \to P\gamma \), with expression,

\[
F_{VP}(k^2) = -\epsilon N_P N_V \frac{M'}{M^3} \int \frac{d^3\hat{q}}{(2\pi)^3} \frac{\phi_P(\hat{q})\phi_V(\hat{q})}{16\omega_1\omega_2\omega'_1\omega'_2} \left[ T_1\hat{m}_2(M^2 - M'^2)^2/4M^2 + T_2 - \frac{\hat{q}^2}{3} T_4 \right];
\]

\[
T_1 = (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)M^2(m_1 - \hat{q}^2 + \hat{q}^2 - \omega_1\omega_2 + \omega'_1\omega'_2 + m_2(\hat{q}^2 - \hat{q}^2 + \omega_1\omega_2 + \omega'_1\omega'_2 + E'(\omega'_1 + \omega'_2))]
\]

\[
+ (\alpha_1 - \alpha_2 - \alpha_3 + \alpha_4)M^2(m_1 - \hat{q}^2 - \omega'_1\omega'_2 + \omega_1\omega_2)
\]

\[
+ (\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4)E'M^2(m_1 - \hat{q}^2 - \omega_1\omega_2 - \omega'_1\omega'_2 + (\alpha_1 - \alpha_2 + \alpha_3 + \alpha_4)E'M^2m_1(\omega'_1 + \omega'_2);
\]

\[
T_2 = - (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)M^2(m_1 - \hat{q}^2 - \omega_1\omega_2 + m_2m_1) + q^2
\]

\[
T_4 = 2(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)M^2(m_1 - \hat{q}^2)\hat{m}_2(m_2m_1 - m_2m_2)
\]

(34)

The above expression corresponds to \( F_{VP}(k^2 = 0) \), that corresponds to emission of a real photon. However, since in this work, we were mainly interested in calculation of decay widths for various transitions, detailed calculations of \( F_{VP}(k^2) \) on lines of [10 15] will be relegated to a separate paper. Now we proceed to calculate the decay widths for the process \( V \to P\gamma \), which corresponds to emission of a real photon, for which we need, \( F_{VP}(k^2 = 0) \) given above. The kinematical relation, \( \hat{q}^2 = \hat{q}^2 - \hat{m}_2^2(M^2 - M'^2)^2/4M^2 \) connecting \( \hat{q}^2 \), with \( \hat{q}^2 \), is given in Eq.(32). To calculate the decay widths, for which we need to calculate the spin averaged amplitude square, \( |\overline{M}_{f1}|^2 \), where \( |\overline{M}_{f1}|^2 = \frac{1}{32\pi} \sum_\lambda \lambda' |\overline{M}_{f1}|^2 \), where we average over the initial polarization states \( \lambda \) of V-meson, and sum over the final polarization \( \lambda' \) of photon. We make use of the normalizations, \( \Sigma_\lambda \epsilon_\mu^\lambda \epsilon_\nu^\lambda = \frac{1}{2}(\delta_{\mu\nu} + \frac{P_\mu P_\nu}{M^2}) \) for vector meson, and \( \Sigma_\lambda \epsilon_\mu^\lambda \epsilon_\nu^\lambda = \delta_{\mu\nu} \), for the emitted photon, with \( M_{f1} \) taken from Eq.(33).

The spin-averaged amplitude square of the process, obtained after dividing by the total spin states \((2j + 1)\) of the
initial vector meson can be obtained as

\[ |M_{fi}|^2 = \frac{2e^2}{3}[M^2M'^2 - (P.P')^2] |F_{VP}(0)|^2 \]  

In the above equation, we evaluate \(P.P' = -ME'\) in the rest frame of initial vector meson, where \(E' = \sqrt{P'^2 + M'^2}\). Thus, \(|M_{fi}|^2\) can be expressed as,

\[ |M_{fi}|^2 = \frac{2}{3} \left( \frac{M^2 - M'^2}{4} \right)^2 |F_{VP}(0)|^2. \]  

The decay width of the process \((V \to P\gamma)\) in the rest frame of the initial vector meson is expressed as

\[ \Gamma_{V \to P\gamma} = \frac{|M_{fi}|^2}{8\pi M^2} [P'], \]  

where we make use of the fact that modulus of the momentum of the emitted pseudoscalar meson can be expressed in terms of masses of particles as, \(|P'| = |k| = \omega_k = \frac{1}{2M}(M^2 - M'^2)\), where, \(\omega_k\) is the kinematically allowed energy of the emitted photon. Thus, \(\Gamma\) in turn can be expressed as:

\[ \Gamma = \frac{\alpha_{e.m.}}{3} |F_{VP}|^2 \omega_k^3. \]  

We now calculate the radiative decay widths for the process, \(V \to S + \gamma\) in the next section.

### 4. Radiative decays of heavy-light quarkonia through \(V \to S\gamma\)

E1 transitions always involve excited states. The scattering amplitude of the decay process \(V \to S\gamma\) can be written as

\[ M_{fi} = -ie \int \frac{d^3\hat{q}}{(2\pi)^3} \frac{1}{M^2} Tr \left[ \alpha_1 \mathcal{P} \psi_{P'}^{+} (\hat{q'}) \bar{\psi}_{\psi}^{++} (\hat{q}) + \alpha_2 \mathcal{P} \psi_{P'}^{+} (\hat{q'}) \bar{\psi}_{\psi}^{+-} (\hat{q}) + \alpha_3 \mathcal{P} \psi_{P'}^{-} (\hat{q'}) \bar{\psi}_{\psi}^{++} (\hat{q}) + \alpha_4 \mathcal{P} \psi_{P'}^{-} (\hat{q'}) \bar{\psi}_{\psi}^{+-} (\hat{q}) \right] \]  

After the 3D reduction of the 4D BS wave function of scalar meson under CIA, we express the 3D BS wave function with dimensionality \(M\) as

\[ \psi_S(\hat{q}) = N_S [M_{f_1}(\hat{q}) + iPf_2(\hat{q}) - i\tilde{q}f_3(\hat{q}) + 2\frac{P\tilde{q}}{M} f_4(\hat{q})]. \]

Making use of the fact that the most leading Dirac structure in scalar meson BS wave function is \(MI\) (I being the unit \(4 \times 4\) unit matrix), and making use of \([17]\), we express the 3D scalar meson BS wave function as,

\[ \psi_S(\hat{q}) = N_S (M') \phi_S(\hat{q}'), \]

where \(\phi_S(\hat{q})\) is the spatial part of this wave function, whose analytic form is obtained by solving the 3D mass
The 4D BS normalizer of scalar meson, $N_S$, can be obtained by solving the current conservation conditions, and is expressed as,

$$N_S^{-2} = 4m_1 m_2 M^2 \frac{1}{m_1} \int \frac{d^3 \hat{q}}{(2\pi)^3} \phi_S^2(\hat{q}).$$

(43)

We now obtain the ++ and -- components of the scalar meson wave function through Eq. (30) as given in Eq. (A.65) with the corresponding adjoint wave functions in Eq. (A.66) of Appendix A1. The expressions for + + + +, + + --, -- ++, and -- -- terms of the scattering amplitude in Eq. (21) is relegated to Appendix A2.

We can now express the transition amplitude, $M_f$, for the process, $V \rightarrow S \gamma$ as,

$$M_{fi} = -ie N_S N_V \frac{1}{M^2} \int \frac{d^3 \hat{q}}{(2\pi)^3} \phi_S(\hat{q}) \phi_V(\hat{q}) \left[ (\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4)iM^3(\omega'_1 \omega'_2 - m_1 m_2 + \hat{q}^2)(\omega_1 m_2 + m_1 \omega_2) + (\alpha_1 + \alpha_2 - \alpha_3 + \alpha_4)iM^3(\omega'_1 m_2 - m_1 \omega'_2)(\omega_1 \omega_2 + m_1 m_2 + \hat{q}^2) + (\alpha_1 + \alpha_2 - \alpha_3 + \alpha_4)iM^3(m_1 - m_2)(\omega'_1 + \omega'_2)(\hat{q}' \hat{q}) - (\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4)iM^3(m_1 + m_2)(\omega_1 + \omega_2)(\hat{q}' \hat{q}) \right] 4(e^\lambda \cdot e^\lambda) - 2 \left( (\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4)iM^3(\omega'_1 m_2 - m_1 \omega'_2) \right) 4(e^\lambda \cdot e^\lambda) + \left( (\alpha_1 + \alpha_2 - \alpha_3 + \alpha_4)iM^3(m_1 + m_2)(\omega'_1 + \omega'_2) - (\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4)iM^3(m_1 + m_2)(\omega_1 - \omega_2) \right) 4(\hat{q}' \cdot e^\lambda)(\hat{q} \cdot e^\lambda) + \left( - (\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4)iM^3(m_1 + m_2)(\omega'_1 + \omega'_2) + (\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4)iM^3(m_1 + m_2)(\omega_1 + \omega_2) \right) 4(\hat{q}' \cdot e^\lambda)(\hat{q} \cdot e^\lambda) \right],$$

(44)

which can be reexpressed as

$$M_{fi} = -ie N_S N_V \frac{1}{M^2} \int \frac{d^3 \hat{q}}{(2\pi)^3} \phi_S(\hat{q}) \phi_V(\hat{q}) [TR],$$

(45)

where

$$[TR] = A \, 4(e^\lambda \cdot e^\lambda) + B \, 4(e^\lambda \cdot \hat{q})(\hat{q} \cdot e^\lambda) + C \, 4(\hat{q}' \cdot e^\lambda)(\hat{q} \cdot e^\lambda) + D \, 4(\hat{q}' \cdot e^\lambda)(\hat{q} \cdot e^\lambda),$$

(46)

where
\[ A = M^2 \left( (\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4)(\omega'_1\omega'_2 - m_1m_2 + \hat{q}'^2)(\omega_1m_2 + m_1\omega_2) \right. \]
\[ + (\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4)(\omega'_1m_2 - m_1\omega'_2)(\omega_1m_2 + m_1\omega_2 + \hat{q}'^2) + [(\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4)(m_1 - M_2)(\omega'_1 + \omega'_2) \]
\[ \left. - (\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4)(m_1 + m_2)(\omega_1 + \omega_2)][(\hat{q}' - m_2(\hat{q}, P')) \right) \]
\[ B = 2M^2 \left( (\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4)[-(\omega'_1m_2 - m_1\omega'_2) + m_2(\omega'_1 + \omega'_2)] + (\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4)\omega_2(m_1 + m_2) \right) \]
\[ C = M^3 \hat{m}_2 \left( -(\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4)(m_1 - m_2)(\omega'_1 + \omega'_2) + (\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4)(m_1 + m_2)(\omega_1 + \omega_2) \right) \]
\[ D = M^3 \hat{m}_2 \left( (\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4)(m_1 + m_2)(\omega'_1 + \omega'_2) - (\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4)(m_1 + m_2)(\omega_1 - \omega_2) \right) \]

To calculate the decay widths, we need to calculate the spin averaged amplitude square, \(|\mathcal{M}_{fi}|^2\), where \(|\mathcal{M}_{fi}|^2 = \sum_{\lambda,\lambda'} |\mathcal{M}_{fi}|^2\), where we average over the initial polarization states \(\lambda\) of V-meson, and sum over the final polarization \(\lambda'\) of photon. We make use of the normalizations, \(\Sigma_\alpha \epsilon^\alpha_{\mu} c^\alpha_{\nu} = \frac{1}{2}(\delta_{\mu\nu} + \frac{P_{\mu} P_{\nu}}{M^2})\) for vector meson, and \(\Sigma_\lambda \epsilon_{\mu}^\lambda c^\lambda_{\nu} = \delta_{\mu\nu}\) for the emitted photon, with \(M_{fi}\) taken from the previous equations.

The spin-averaged amplitude square of the process, obtained after dividing by the total spin states \((2j + 1)\) of the initial vector meson can be written as a double integral over \(d^3\hat{q}\) as, The averaged square of the scattering amplitude can be expressed as
\[ |\mathcal{M}_{fi}|^2 = e^2N_f^2N_f^2 \frac{1}{M^4} \int \frac{d^3\hat{q}''}{(2\pi)^3} \int \frac{d^3\hat{q}'}{(2\pi)^3} \frac{\phi^2_\lambda(\hat{q}'')\phi^2_\lambda(\hat{q}')}{|TR|^2}, \]
where
\[ |TR|^2 = \frac{1}{3} \sum_\lambda \sum_{\lambda'} |TR|^2 = \frac{16}{3} \left( A^2 + B^2 \frac{1}{3}(\hat{q}, \hat{q}'')^2 + C^2 \frac{1}{3}(-M'^2 + \frac{M^4}{M^2})(\hat{q}, \hat{q}'') \]
\[ + D^2 \frac{1}{3}(-M'^2)(\hat{q}, \hat{q}'') + AB \frac{1}{3}(\hat{q}' + \hat{q}'')^2 + A(C + D) \frac{1}{3}[(\hat{q}'', P') + (\hat{q}, P')] \]
\[ + B(C + D) \frac{1}{3}[(\hat{q}'', P') + (\hat{q}, P')][(\hat{q}, \hat{q}'') + CD[(\hat{q}, \hat{q}'') + (\hat{q}'', P')]) \frac{1}{3}(\hat{q}, \hat{q}''), \]}
\[ \text{and} \]
\[ \hat{q}, P' = \hat{m}_2 M'^2 - \hat{m}_2 \frac{(M^2 + M'^2)^2}{4M^2} \]
\[ \hat{P}, P' = -\frac{(M^2 + M'^2)}{2} \]
\[ \hat{q}, \hat{q}' = \hat{q}^2 - \hat{m}_2 \frac{(M^2 - M'^2)^2}{4M^2} \]

where we have made use of the relations, \((P, e^\lambda) = 0\), \((P, e^\lambda') = 0\), and \((P, \hat{q}') = 0\). To obtain spin averaged amplitude square, we sum over the polarizations of the vector meson and the emitted photon by making use of the fact that, \(\Sigma_\lambda \epsilon^\lambda_{\mu} c^\lambda_{\nu} = \frac{1}{2}(\delta_{\mu\nu} + \frac{P_{\mu} P_{\nu}}{M^2})\) for vector meson, and \(\Sigma_\lambda \epsilon^\lambda_{\mu} c^\lambda_{\nu} = \delta_{\mu\nu}\) for the emitted photon. This gives \(\Sigma_\lambda \Sigma_\lambda' \vert e^\lambda, c^\lambda' \vert^2 = 1\).

We can write the decay width,
\[ \Gamma_{V\rightarrow S\gamma} = \frac{|\mathcal{M}_{fi}|^2}{8\pi M^2} |\hat{P}|, \]
where we make use of the fact that modulus of the momentum of the emitted pseudoscalar meson can be expressed in terms of masses of particles as, \(|\hat{P}| = \frac{1}{2\pi} (M^2 - M'^2)\).
5. Radiative decays of heavy-light quarkonia through $S \rightarrow V \gamma$

We proceed to evaluate the process in the same manner as $V \rightarrow S \gamma$, using Fig A where the initial scalar meson decays into a vector meson and a photon. Drawing analogy from $V \rightarrow P \gamma$, and $V \rightarrow S \gamma$, the effective 3D form of transition amplitude, $M_{f_1}$ for $S \rightarrow V \gamma$ under Covariant Instantaneous Ansatz can be expressed as,

$$M_{f_1} = -ie \int \frac{d^3 \hat{q}}{(2\pi)^3} \frac{1}{M^2} \text{Tr} \left[ \alpha_1 P \bar{\psi}_V^+(\hat{q}') \psi_S^+(\hat{q}) + \alpha_2 P \bar{\psi}_V^+(\hat{q}') \psi_S^-(\hat{q}) + \alpha_3 P \bar{\psi}_V^-(\hat{q}') \psi_S^+(\hat{q}) + \alpha_4 P \bar{\psi}_V^-(\hat{q}') \psi_S^-(\hat{q}) \right]. \quad (51)$$

The transition amplitude of the $S \rightarrow V \gamma$ process can be obtained as

$$M_{f_1} = eN_S N_V \frac{M'}{M^3} \int \frac{d^3 \hat{q}}{(2\pi)^3} \frac{\phi_S(\hat{q})\phi_V(\hat{q})}{16\omega_1\omega_2\omega'_1\omega'_2} [TR], \quad (52)$$

where

$$[TR] = \text{Tr}[-(\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4)iM^3(\omega'_1\omega'_2 + m_1m_2)(\omega_1m_2 - \omega_2)m\phi'\phi']$$

$$+ (\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4)iM^3(\omega_1m_2 - \omega_2)m\phi'\phi'$$

$$- (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)iM^2(m_1 + m_2)\psi'\phi'\phi'$$

$$- (\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4)iM^3(\omega_1m_2 + \omega_2m_2)(\omega_1\omega_2 - m_1m_2 + q^2)\phi'\phi'$$

$$- (\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4)iM^3(m_1(\omega_1 + \omega_2)\phi'\phi'\phi' + m_2(\omega_1 + \omega_2)\phi'\phi'\phi')$$

$$+ (\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4)iM^3(m_1 + m_2)(\omega'_1\phi'\phi'\phi' - \omega'_2\phi'\phi'\phi') \quad (53)$$

Evaluating trace over gamma matrices, $M_{f_1}$ can be expressed as,

$$M_{f_1} = ieN_S N_V \frac{M'}{M^3} \int \frac{d^3 \hat{q}}{(2\pi)^3} \frac{\phi_S(\hat{q})\phi_V(\hat{q})}{16\omega_1\omega_2\omega'_1\omega'_2} [A'4(e,e') + B'4(e,e')(e',\hat{q}) + C'4(e,P')(e',\hat{q})], \quad (54)$$

where

$$A' = M^3 \left[ - (\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4)[(\omega'_1\omega'_2 + m_1m_2)(\omega_1m_2 - \omega_2)m + (\omega_1m_2 - \omega_2)m] + (m_1 - m_2)(\omega_1 + \omega_2)(\hat{q}' + \hat{m}_2(P',\hat{q}]) + (\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4)[-(\omega'_1m_2 + \omega'_2m_2)(\omega_1\omega_2 - m_1m_2 + \hat{q}^2) + (m_1 + m_2)(\omega'_1 + \omega'_2)(\hat{q}' + \hat{m}_2(P',\hat{q}))] \right]$$

$$B' = M^3 \left[ (\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4)[(\omega_1m_2 - \omega_2m_2) - m_2(\omega_1 + \omega_2)] + (\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4)(m_1 + m_2)\omega'_2 \right]$$

$$C' = M^3 \hat{m}_2 \left[ (\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4)[2(\omega_1m_2 - \omega_2m_2) + (m_1 - m_2)(\omega_1 + \omega_2)] + (\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4)(m_1 + m_2)(\omega'_1 + \omega'_2) \right]$$

To calculate the decay widths, we again need to calculate the spin averaged amplitude square, $|\mathcal{M}_{f_1}|^2$, where $|\mathcal{M}_{f_1}|^2 = \sum_{\lambda,\lambda'} |M_{f_1}|^2$, where we sum over the final polarization states, $\lambda'$ of photon, and $\lambda$ of V-meson.

The spin averaged amplitude modulus square gives,

$$|\mathcal{M}_{f_1}|^2 = e^2 N_V^2 N_S^2 \frac{M^2}{M^6} \int \frac{d^3 \hat{q}}{(2\pi)^3} \int \frac{d^3 \hat{q}'}{(2\pi)^3} \frac{\phi_S(\hat{q})\phi_V(\hat{q})}{16\omega_1\omega_2\omega'_1\omega'_2} |TR|^2, \quad (55)$$

where $|TR|^2 = \sum_{\lambda} |TR|^2$. Thus,
\[ |TR|^2 = 16 \left[ A'^2 + B'^2 \frac{1}{3}(\tilde{q} \tilde{q}'')^2 + C'^2 \frac{1}{3}(-M'^2)(\tilde{q} \tilde{q}'') + A'B' \frac{1}{3}(\tilde{q}^2 + \tilde{q}'')^2 \right. \\
\left. + A'C' \frac{1}{3}[(\tilde{q}''.P') + (\tilde{q}.P')] + B'C' \frac{1}{3}2(\tilde{q} \tilde{q}'') \right] \tag{56} \]

The decay widths \( \Gamma \) for the process, \( S \to V\gamma \), are given by Eq. (50), with \( P' \), now the momentum of the emitted vector meson.

6. Results and Discussion

We have studied radiative decays of conventional heavy-light quarkonia through M1 and E1 transitions in the framework of Bethe-Salpeter equation. Such processes involve quark-triangle diagrams, and involve two hardon-quark vertices and are difficult to evaluate in BSE under CIA \([14,16]\). In this work we have given a generalized method of handling quark triangle diagrams with two hadron-quark vertices in the framework of \( 4 \times 4 \) BSE, by expressing the transition amplitude, \( M_{fi} \) (Eq.(21-22)) as a linear superposition of terms involving all possible combinations of ++, and -- components of Salpeter wave functions of final and initial hadrons, through not only the etrms, ++ +++, and -- --, but also the terms like, + + --, and -- ++, with each of the four terms being associated with a coefficient, \( \alpha_i (i = 1,...,4) \), which is the result of the pole integration in the complex \( \sigma \)-plane, with pole positions in Eq.(20) (shown in Fig.2). This superposition of all possible terms in Eq.(21-22) should be a feature of relativistic frameworks.

Using this generalized expression for \( M_{fi} \), in Eq.(21-22), we have evaluated the decay widths for M1 transitions, \( ^3S_1 \to ^1S_0 + \gamma \), involving the decays of the ground and excited states of the heavy-light mesons such as, \( B_s \), \( B_s' \), \( J/\Psi, D_s \), and \( D_s' \). Here, we have studied the processes, \( nS \to n'S \) for both \( n = n' \), and \( n \neq n' \). As regards the E1 transitions, we have studied the processes, \( ^3S_1 \to ^1P_0 + \gamma \), that involve the decays of \( \Psi(2S), B_c \to (2S) \), and \( D \to (2S) \), and the processes, \( ^1P_0 \to ^3S_1 + \gamma \), that involve decays of \( \chi_{c0}(1P), B_c(1P) \) and \( B_c(2P) \).

We used algebraic forms of 3D Salpeter wave functions obtained through analytic solutions of mass spectral equations in approximate harmonic oscillator basis for ground and excited states of \( 0^{++}, 1^{--}, \) and \( 0^{--} \) heavy-light quarkonia for calculation of their decay widths. The input parameters used by us are: \( C_0 = 0.69, \omega_0 = 0.22 \text{ GeV}, \Lambda_{QCD} = 0.25 \text{ GeV}, \) and \( A_0 = 0.01 \), along with the input quark masses \( m_u = 0.30 \text{ GeV}, m_s = 0.43 \text{ GeV}, m_c = 1.49 \text{ GeV}, \) and \( m_b = 4.67 \text{ GeV} \), that were obtained by fitting to their mass spectra \([18]\). We have compared our results with experimental data and other models, and found reasonable agreements.

To simplify the algebra, we take only the leading Dirac structures (\( \gamma_5, I, \) and \( \gamma, \epsilon \) respectively) in the hadronic Bethe-Salpeter wave functions of \( P, S \) and \( V \) quarkonia, which as we have earlier shown using our naive power counting rule \([19,20]\), contribute maximum to the calculation of any meson observable. However the price we have to pay for this simplification is that though we get reasonable agreements of our decay widths for M1 transitions, \( nS \to n'S + \gamma \) (with \( n' = n \)), with data, but for \( n' \neq n \), the agreement is not good. This can be seen from Table 1, that for \( J/\Psi(n'S) \to n_c(nS) + \gamma \), though we get reasonable results for \( 1S \to 1S \), and \( 2S \to 2S \), but for \( 2S \to 1S \), our calculated decay widths are half of central values of data, which would imply that the incorporation of sub-leading Dirac structures in wave functions of \( V \) and \( P \) quarkonia might be important for studying transitions between different spatial multiplets. We wish to study this further.

We wish to mention that \( F_{VP}(0) \) in Eq.(34) are the electromagnetic coupling constants, \( g_{VP\gamma} \). It is seen that our coupling constant, \( |g_{J/\Psi n,\gamma}| = 0.622 \text{ GeV}^{-1} \text{(Expt. } = 0.570 \pm 0.110 \text{ GeV}^{-1} \text{ [33])} \), while the coupling constant, \( g_{D\ast P\gamma} = \)
of variations of coupling constants, $g$, radiative decays of $J/\psi \rightarrow \eta_c \gamma$, and $\Psi(2S) \rightarrow \eta_c(2S) \gamma$ are reasonable, though $\Psi(2S) \rightarrow \eta_c(1S)$ is nearly half of central value of data. However our $\eta_s \rightarrow \eta_s(1S)$ is in good agreement with data, but for $\chi_{c0}$ is higher than data, though again there is a lot of variation in results of other models. However, as mentioned above, incorporation of all Dirac structures is expected to give better agreement with data.

Similarly we again see a wide range of variations in different models for $J/\psi$, and $\Psi(2S)$. Our decay widths for $J/\psi \rightarrow \eta_c \gamma$, and $\Psi(2S) \rightarrow \eta_c(2S) \gamma$ are reasonable, though $\Psi(2S) \rightarrow \eta_c(1S)$ is nearly half of central value of data. However our $nS \rightarrow nS$ transitions show a marked decrease as we go from ground to higher excited states, which is in conformity with data and other models. We have also given our predictions for radiative decays of $B_c* (1S), B_c*(2S), B_s*(2S), B*(2S), D*(2S)$, for which data is not yet available, and for $D_s* (1S)$, where PDG [41] gives only the upper limit of the decay width. As regards $E1$ transitions, our decay width result for $\Psi(2S)$ is in good agreement with data, but for $\chi_{c0}$ is higher than data, though again there is a lot of variation in results of other models. However, as mentioned above, incorporation of all Dirac structures is expected to give better agreement with data.

The aim of doing this study was to mainly test our analytic forms of wave functions in Eqs. (28) and (42) obtained as solutions of mass spectral equations in an approximate harmonic oscillator basis obtained analytically from $4 \times 4$ BSE as a starting point, that has so far given good predictions [17, 18, 21] not only of the mass spectrum of heavy-light quarkonia, but also their leptonic decays, two-photon, and two gluon decays. The present work would in turn lead to the validation of our approach, which provides a much deeper insight than the purely numerical calculations in $4 \times 4$ BSE approach that are prevalent in the literature.

This work was mainly focused on evaluation of decay widths for $M1$, and $E1$ transitions. A more detailed study on not only the transition form factors of both $M1$, and $E1$ transitions, but also the "static" form factors describing meson-photon interactions through the vertex $M \gamma M$ for various mesons will be relegated to a separate paper.

|                | BSE-CIA | Expt.          | LFQM            | PM       | RQM       |
|----------------|---------|----------------|-----------------|----------|-----------|
| $\Gamma_{J/\psi(1S_c)\rightarrow \eta_c(1S_c)\gamma}$ | 1.4202  | 1.5793±0.0112 [32] | 1.69±0.05 [11] | 1.8 [35] | 1.050 [39] |
| $\Gamma_{\psi(2S_c)\rightarrow \eta_c(2S_c)\gamma}$     | 0.1843  | 0.2002±0.008 [53]  |                 |          |          |
| $\Gamma_{\psi(2S_c)\rightarrow \eta_c(1S_c)\gamma}$     | 0.4238  | 0.9724          |                 |          |          |
| $\Gamma_{B_s^*(1S_c)\rightarrow B_c(1S_c)\gamma}$      | 0.0664  | 0.06 [36]       | 0.033 [39]      |          |          |
| $\Gamma_{B_s^*(2S_c)\rightarrow B_c(2S_c)\gamma}$      | 0.0360  | 0.01 [36]       | 0.017 [39]      |          |          |
| $\Gamma_{B_s^*(1S_c)\rightarrow B_c(1S_c)\gamma}$      | 0.0861  | 0.064±0.016 [41] | 0.068±0.017     |          |          |
| $\Gamma_{B_s^*(2S_c)\rightarrow B_c(2S_c)\gamma}$      | 0.0531  |                |                 |          |          |
| $\Gamma_{B_c^*(1S_c)\rightarrow B(1S_c)\gamma}$       | 0.1332  | 0.13±0.01 [41]  | 0.13±0.01 [11]  |          |          |
| $\Gamma_{B_c^*(2S_c)\rightarrow B(2S_c)\gamma}$       | 0.1458  |                |                 |          |          |
| $\Gamma_{D_s^*(1S_c)\rightarrow D_c(1S_c)\gamma}$     | 0.2170  | 0.213 [40]      |                 |          |          |
| $\Gamma_{D_s^*(1S_c)\rightarrow D(1S_c)\gamma}$       | 1.267   | 1.334±0.0072 [41] | 0.90±0.02 [11]  |          |          |

Table 1. Radiative decay widths of heavy-light mesons (in Kev) for $M1$ transitions in BSE, along with experimental
data and results of other models. The corresponding branching fractions calculated in our model (in second column along with their experimental data (in square brackets)) are also presented.

|               | BSE-CIA   | Expt.     | LFQM       | PM       | RQM       |
|---------------|-----------|-----------|------------|----------|-----------|
| \( \Gamma_{\psi(2S_1) \rightarrow \chi_{c0}(1P_0) \gamma} \) | 27.9226   | 28.5714 \pm 0.0432 | 33         | 26.3     | 39        |
| \( \Gamma_{B^+_s(2S_1) \rightarrow B_s(1P_0) \gamma} \) | 9.0119    | 9.6       | 37         | 3.78     | 39        |
| \( \Gamma_{D^+(2S_1) \rightarrow D(1P_0) \gamma} \) | 1.0214    |           |            |          |           |
| \( \Gamma_{\chi_{c0}(1P_0) \rightarrow J/\psi(1S_1) \gamma} \) | 133.7740  | 119.5 \pm 8 | 11         | 161      | 39        |
| \( \Gamma_{B_c(1P_0) \rightarrow B_s(1S_1) \gamma} \) | 76.7047   | 65.3      | 36         | 75.5     | 39        |
| \( \Gamma_{B_s(2P_0) \rightarrow B_s(2S_1) \gamma} \) | 27.0924   | 52.5      | 36         | 34       | 39        |

Table 2. Radiative decay widths of heavy-light mesons (in KeV) for E1 transitions, along with experimental data and results of other models.

### A Appendix

#### A1. Radiative decays through \( V \rightarrow P \gamma \)

Substituting the 4D BS wave function of pseudoscalar meson in Eq. (26), we obtain the ++ and −− components as

\[
\psi_{P}^{++}(q^\prime) = \frac{N_P}{4\omega_1^2} \frac{M^\prime}{M} \phi_p(q^\prime)[M(\omega_1^2 + m_1^2)|q^\prime|^2 - i(\omega_1^2m_2 + m_1\omega_2^2)|q^\prime|^2 + iM(m_1 - m_2)|q^\prime|^2 - (\omega_1^2P - \omega_2^2P)]\gamma_5
\]

\[
\psi_{P}^{--}(q^\prime) = \frac{N_P}{4\omega_1^2} \frac{M^\prime}{M} \phi_p(q^\prime)[M(\omega_1^2 + m_1^2)|q^\prime|^2 + i(\omega_1^2m_2 + m_1\omega_2^2)|q^\prime|^2 + iM(m_1 - m_2)|q^\prime|^2 - (\omega_1^2P - \omega_2^2P)]\gamma_5
\]

(A.57)

The adjoint Bethe-Salpeter wave function of pseudoscalar meson can be obtained by evaluating \( \bar{\psi}_P^{\pm}(q^\prime) = \gamma_4(\psi_P^{\pm}(q^\prime))^\dagger \) as

\[
\bar{\psi}_P^{++}(q^\prime) = \frac{-iN_V}{4\omega_1} \phi_v(q^\prime)[M(\omega_1^2 + m_1^2)|\psi|^2 - M(\omega_1^2 + m_1^2)|\psi|^2 + iM(m_1\gamma^\dagger + m_2\gamma^\dagger) + (\omega_1\gamma^\dagger P - \omega_2\gamma^\dagger P)]
\]

(A.58)

Following the same steps as in Eq. (A.59), we obtain the ++ and −− components of vector meson wave function in Eq. (26) as

\[
\psi_V^{++}(q^\prime) = \frac{iN_V}{4\omega_1^2} \phi_v(q^\prime)[M(\omega_1^2 + m_1^2)|\psi|^2 - M(\omega_1^2 + m_1^2)|\psi|^2 + iM(m_1\gamma^\dagger + m_2\gamma^\dagger) + (\omega_1\gamma^\dagger P - \omega_2\gamma^\dagger P)]
\]

(A.59)

where as the adjoint wave functions are

\[
\bar{\psi}_V^{++}(q^\prime) = \frac{-iN_V}{4\omega_1^2} \phi_v(q^\prime)[-M(\omega_1^2 + m_1^2)|\psi|^2 + M(\omega_1^2 + m_1^2)|\psi|^2 + iM(m_1\gamma^\dagger + m_2\gamma^\dagger) - (\omega_1\gamma^\dagger P - \omega_2\gamma^\dagger P)]
\]

(A.60)
The $\mathcal{P}\overline{\psi}^{++}_P(q')\phi^{+}_V(q)$, $\mathcal{P}\overline{\psi}^{++}_P(q')\phi^{+}_V(q)$, $\mathcal{P}\overline{\psi}^{--}_P(q')\phi^{+}_V(q)$, and $\mathcal{P}\overline{\psi}^{--}_P(q')\phi^{+}_V(q)$ in the calculation of the transition amplitude, $M_{f_1}$ for $V \rightarrow P \gamma$ is done by using Eqs. (A.59) and (A.60) as:

$$\mathcal{P}\overline{\psi}^{++}_P(q')\phi^{+}_V(q) = \frac{-iN_PN_V}{16\omega_1\omega_2\omega_1'\omega_2'} \frac{M'}{M} \phi_P(q')\phi_V(q) \left[ -iM(\omega'_1\omega'_2 + m_1m_2 + \hat{q}^2)(\omega_1m_2 + m_1\omega_2)\mathcal{P}\phi\mathcal{P}_\gamma \right. $$

$$+ iM^2(\omega'_1\omega'_2 + m_1m_2 + \hat{q}^2)(m_1\mathcal{P}\phi\mathcal{P}_\gamma + m_2\mathcal{P}\phi\mathcal{P}_\gamma) $$

$$- iM(\omega'_1m_2 + m_1\omega'_2)(\omega_1\omega_2 + m_1m_2)\mathcal{P}\phi\mathcal{P}_\gamma $$

$$+ iM^3(\omega'_1m_2 + m_1\omega'_2)\phi\mathcal{P}_\gamma $$

$$- iM^2(\omega'_1m_2 + m_1\omega'_2)(\omega_1\phi\mathcal{P}_\gamma - \omega_2\phi\mathcal{P}_\gamma) $$

$$- iM^2(m_1 - m_2)(\omega_1\omega_2 + m_1m_2)\mathcal{P}\phi\mathcal{P}_\gamma $$

$$+ iM^2(m_1 - m_2)\mathcal{P}\phi\mathcal{P}_\gamma $$

$$- iM(m_1 - m_2)(\omega_1\mathcal{P}\phi\mathcal{P}_\gamma - \omega_2\mathcal{P}\phi\mathcal{P}_\gamma) $$

$$- iM^2(\omega_1m_2 + m_1\omega_2)(\omega'_1 + \omega'_2)\phi\mathcal{P}_\gamma $$

$$+ iM^3(m_1(\omega'_1 + \omega'_2)\phi\mathcal{P}_\gamma + m_2(\omega'_1 + \omega'_2))\phi\mathcal{P}_\gamma \right], \quad (A.61)$$

$$\mathcal{P}\overline{\psi}^{--}_P(q')\phi^{+}_V(q) = \frac{-iN_PN_V}{16\omega_1\omega_2\omega_1'\omega_2'} \frac{M'}{M} \phi_P(q')\phi_V(q) \left[ iM(\omega'_1\omega'_2 + m_1m_2 + \hat{q}^2)(\omega_1m_2 + m_1\omega_2)\mathcal{P}\phi\mathcal{P}_\gamma \right. $$

$$+ iM^2(\omega'_1\omega'_2 + m_1m_2 + \hat{q}^2)(m_1\mathcal{P}\phi\mathcal{P}_\gamma + m_2\mathcal{P}\phi\mathcal{P}_\gamma) $$

$$- iM(\omega'_1m_2 + m_1\omega'_2)(\omega_1\omega_2 + m_1m_2)\mathcal{P}\phi\mathcal{P}_\gamma $$

$$- iM^3(\omega'_1m_2 + m_1\omega'_2)\phi\mathcal{P}_\gamma $$

$$- iM^2(\omega'_1m_2 + m_1\omega'_2)(\omega_1\phi\mathcal{P}_\gamma - \omega_2\phi\mathcal{P}_\gamma) $$

$$- iM^2(m_1 - m_2)(\omega_1\omega_2 + m_1m_2)\mathcal{P}\phi\mathcal{P}_\gamma $$

$$+ iM^2(m_1 - m_2)\mathcal{P}\phi\mathcal{P}_\gamma $$

$$+ iM(m_1 - m_2)(\omega_1\mathcal{P}\phi\mathcal{P}_\gamma - \omega_2\mathcal{P}\phi\mathcal{P}_\gamma) $$

$$- iM^2(\omega_1m_2 + m_1\omega_2)(\omega'_1 + \omega'_2)\phi\mathcal{P}_\gamma $$

$$+ iM^3(m_1(\omega'_1 + \omega'_2)\phi\mathcal{P}_\gamma + m_2(\omega'_1 + \omega'_2))\phi\mathcal{P}_\gamma \right], \quad (A.62)$$
\[
\mathcal{P}\bar{\psi}_P^{++}(\bar{q}^\prime)\gamma^\dagger\Psi_{\nu}^{-}(\bar{q}) = -iN_P N_V \left( \frac{M^\prime}{16\omega_1\omega_2\omega_1^\prime\omega_2^\prime} \cdot M' \phi_P(\bar{q}^\prime)\phi_V(\bar{q}) \right) - iM(\omega^\prime_{1}\omega^\prime_{2} + m_1m_2 + q^2)(\omega_2 m_2 + m_1\omega_2)\mathcal{P}\gamma^\dagger\mathcal{P}\gamma^5
\]
\[
\quad + iM^2(\omega^\prime_{1}\omega^\prime_{2} + m_1m_2 + q^2)(m_1\mathcal{P}\gamma^\dagger\mathcal{P}\gamma^5 + m_2\mathcal{P}\gamma^\dagger\mathcal{P}\gamma^5)
\]
\[
\quad + iM(\omega^\prime_{1}m_2 + m_1\omega^\prime_{2})(\omega_1\omega_2 + m_1m_2)\mathcal{P}\gamma^\dagger\mathcal{P}\gamma^5
\]
\[
\quad + iM^2(m_1 - m_2)\mathcal{P}\gamma^\dagger\mathcal{P}\gamma^5
\]
\[
\quad - iM^2(m_1 - m_2)(\omega^\prime_1 + m_1\omega^\prime_2)\mathcal{P}\gamma^\dagger\mathcal{P}\gamma^5
\]
\[
\quad + iM^2(m_1 - m_2)\mathcal{P}\gamma^\dagger\mathcal{P}\gamma^5
\]
\[
\quad - iM(\omega^\prime_1 - m_2)(\omega_1\omega_2 + m_1m_2)\mathcal{P}\gamma^\dagger\mathcal{P}\gamma^5
\]
\[
\quad + iM^2(m_1 - m_2)\mathcal{P}\gamma^\dagger\mathcal{P}\gamma^5
\]
\[
\quad - iM^2(m_1 - m_2)(\omega^\prime_1 - m_2)\mathcal{P}\gamma^\dagger\mathcal{P}\gamma^5
\]
\[
\quad + iM^2(m_1 - m_2)\mathcal{P}\gamma^\dagger\mathcal{P}\gamma^5
\]
\[
\quad + iM^2((m_1 + m_2)\mathcal{P}\gamma^\dagger\mathcal{P}\gamma^5 + m_2(\omega^\prime_1 + m_1\omega^\prime_2)\mathcal{P}\gamma^\dagger\mathcal{P}\gamma^5), \quad (A.63)
\]

and

\[
\mathcal{P}\bar{\psi}_P^{--}(\bar{q}^\prime)\gamma^\dagger\Psi_{\nu}^{+}(\bar{q}) = -iN_P N_V \left( \frac{M^\prime}{16\omega_1\omega_2\omega_1^\prime\omega_2^\prime} \cdot M' \phi_P(\bar{q}^\prime)\phi_V(\bar{q}) \right) - iM(\omega^\prime_{1}\omega^\prime_{2} + m_1m_2 + q^2)(\omega_2 m_2 + m_1\omega_2)\mathcal{P}\gamma^\dagger\mathcal{P}\gamma^5
\]
\[
\quad + iM^2(\omega^\prime_{1}\omega^\prime_{2} + m_1m_2 + q^2)(m_1\mathcal{P}\gamma^\dagger\mathcal{P}\gamma^5 + m_2\mathcal{P}\gamma^\dagger\mathcal{P}\gamma^5)
\]
\[
\quad - iM(\omega^\prime_{1}m_2 + m_1\omega^\prime_{2})(\omega_1\omega_2 + m_1m_2)\mathcal{P}\gamma^\dagger\mathcal{P}\gamma^5
\]
\[
\quad - iM^2(m_1 - m_2)\mathcal{P}\gamma^\dagger\mathcal{P}\gamma^5
\]
\[
\quad - iM^2(m_1 - m_2)(\omega^\prime_1 + m_1\omega^\prime_2)\mathcal{P}\gamma^\dagger\mathcal{P}\gamma^5
\]
\[
\quad - iM^2((m_1 + m_2)\mathcal{P}\gamma^\dagger\mathcal{P}\gamma^5 + m_2(\omega^\prime_1 + m_1\omega^\prime_2)\mathcal{P}\gamma^\dagger\mathcal{P}\gamma^5), \quad (A.64)
\]

A2. Radiative decays through $V \rightarrow S\gamma$

The $++$ and $--$ components of scalar meson wave function in Eq. (41) can be obtained through Eq. (36) as

\[
\psi^{++}_S(\bar{q}^\prime) = -\frac{N_S}{4\omega^\prime_{1}\omega^\prime_{2}} \phi_S(\bar{q}^\prime)[-M(\omega^\prime_{1}\omega^\prime_{2} - m_1m_2 + q^2) - i(\omega^\prime_1m_2 - m_1\omega^\prime_2)\mathcal{P} - (\omega^\prime_1\mathcal{P}\gamma^\dagger - \omega^\prime_2\mathcal{P}\gamma^5) - iM(m_1 + m_2)\mathcal{P}]
\]
\[
\psi^{--}_S(\bar{q}^\prime) = -\frac{N_S}{4\omega^\prime_{1}\omega^\prime_{2}} \phi_S(\bar{q}^\prime)[-M(\omega^\prime_{1}\omega^\prime_{2} - m_1m_2 + q^2) + i(\omega^\prime_1m_2 - m_1\omega^\prime_2)\mathcal{P} + (\omega^\prime_1\mathcal{P}\gamma^\dagger - \omega^\prime_2\mathcal{P}\gamma^5) - iM(m_1 + m_2)\mathcal{P}], \quad (A.65)
\]

The corresponding adjoint wave functions are obtained by evaluating $\bar{\psi}^{\pm\pm}_S(\bar{q}^\prime) = \gamma_4(\psi^{\pm\pm}_S(\bar{q}^\prime))^+\gamma_4$ as

\[
\bar{\psi}^{++}_S(\bar{q}^\prime) = -\frac{N_S}{4\omega^\prime_{1}\omega^\prime_{2}} \phi_S(\bar{q}^\prime)[-M(\omega^\prime_{1}\omega^\prime_{2} - m_1m_2 + q^2) - i(\omega^\prime_1m_2 - m_1\omega^\prime_2)\mathcal{P} - (\omega^\prime_1\mathcal{P}\gamma^\dagger - \omega^\prime_2\mathcal{P}\gamma^5) - iM(m_1 + m_2)\mathcal{P}]
\]
\[
\bar{\psi}^{--}_S(\bar{q}^\prime) = -\frac{N_S}{4\omega^\prime_{1}\omega^\prime_{2}} \phi_S(\bar{q}^\prime)[-M(\omega^\prime_{1}\omega^\prime_{2} - m_1m_2 + q^2) + i(\omega^\prime_1m_2 - m_1\omega^\prime_2)\mathcal{P} + (\omega^\prime_1\mathcal{P}\gamma^\dagger - \omega^\prime_2\mathcal{P}\gamma^5) - iM(m_1 + m_2)\mathcal{P}]. \quad (A.66)
\]
The individual terms, $P_{\psi_S^{++}(q')\psi_{\psiV}^{++}(q)}$, $P_{\psi_S^{+-}(q')\psi_{\psiV}^{-+}(q)}$, $P_{\psi_S^{+-}(q')\psi_{\psiV}^{+-}(q)}$, and $P_{\psi_S^{--}(q')\psi_{\psiV}^{--}(q)}$ in the transition amplitude, $M_{fi}$ in Eq. (21), can be obtained as follows:

$$P_{\psi_S^{++}(q')\psi_{\psiV}^{++}(q)} = \frac{-i N_S N_V}{16 \omega_1 \omega_2 \omega_1' \omega_2'} \phi_S(q') \phi_{\psiV}(q) \left[ i M^3(\omega' \omega_2' - m_1 m_2 + \hat{q}^2)(\omega_1 m_2 + \omega_1' m_2)\phi' \phi \right. $$

$$+ i M^2(\omega' \omega_2' - m_1 m_2 + \hat{q}^2)(m_1 P \phi' \phi - m_2 \phi' \phi') + i M^2(\omega_1 m_2 + \omega_1' m_2)(\omega'_1 \phi' \phi' P + \omega_2 \phi' \phi' P)$$

$$+ i M^3(m_1(\omega'_1 + \omega_2 + \omega_2')(\omega_1 \phi' \phi' P - \omega_2 \phi' \phi' P)$

$$- i M^2(1 + m_2)(\omega_1 \phi' \phi' P - \omega_2 \phi' \phi' P), \quad (A.67)$$

$$P_{\psi_S^{+-}(q')\psi_{\psiV}^{-+}(q)} = \frac{-i N_S N_V}{16 \omega_1 \omega_2 \omega_1' \omega_2'} \phi_S(q') \phi_{\psiV}(q) \left[ i M^3(\omega' \omega_2' - m_1 m_2 + \hat{q}^2)(\omega_1 m_2 + \omega_1' m_2)\phi' \phi \right. $$

$$+ i M^2(\omega' \omega_2' - m_1 m_2 + \hat{q}^2)(m_1 P \phi' \phi - m_2 \phi' \phi') + i M^2(\omega_1 m_2 + \omega_1' m_2)(\omega'_1 \phi' \phi' P + \omega_2 \phi' \phi' P)$$

$$+ i M^3(m_1(\omega'_1 + \omega_2 + \omega_2')(\omega_1 \phi' \phi' P - \omega_2 \phi' \phi' P)$

$$- i M^2(1 + m_2)(\omega_1 \phi' \phi' P - \omega_2 \phi' \phi' P), \quad (A.68)$$
\[
\mathcal{P}\bar{\psi}^{++}_S (q') \phi^V_- \psi_V^- (q) = \frac{-iN_g N_V}{16\omega_1 \omega_2 \omega'_1 \omega'_2} \phi_S(q') \phi_V(q) \left[ -iM^3(\omega'_1 \omega'_2 - m_1 m_2 + \hat{q}^2)(\omega_1 m_2 + m_1 \omega_2) \phi^l \phi^l \\
+ iM^2(\omega'_1 \omega'_2 - m_1 m_2 + \hat{q}^2)(m_1 \mathcal{P} \phi^l \phi^l \hat{q} + m_2 \mathcal{P} \phi^l \hat{q}^l \phi^l) \\
+ iM^3(\omega'_1 m_2 - m_1 \omega'_2)(\omega_1 \omega_2 + m_1 m_2) \phi^l \phi^l \\
- iM^3(\omega'_1 m_2 - m_1 \omega'_2) \phi^l \phi^l \hat{q} \hat{q} \\
- iM^2(\omega'_1 m_2 - m_1 \omega'_2)(\omega_1 \phi^l \phi^l \hat{q} \hat{q} - \omega_2 \phi^l \phi^l \hat{q} \hat{q}) \\
+ iM^2(\omega_1 m_2 + m_1 \omega_2)(\omega'_1 \phi^l \phi^l \hat{q} \hat{q} + \omega'_2 \phi^l \phi^l \hat{q} \hat{q}) \\
+ iM^3(m_1(\omega'_1 + \omega'_2) \phi^l \phi^l \hat{q} \hat{q} + m_2(\omega'_1 + \omega'_2) \phi^l \phi^l \hat{q} \hat{q}) \\
- iM^2(m_1 + m_2)(\omega_1 \phi^l \phi^l \hat{q} \hat{q} - \omega_2 \phi^l \phi^l \hat{q} \hat{q}) \right]
\]

and,
\[
\mathcal{P}\bar{\psi}^{--}_S (q') \phi^V_+ \psi_V^+ (q) = \frac{-iN_g N_V}{16\omega_1 \omega_2 \omega'_1 \omega'_2} \phi_S(q') \phi_V(q) \left[ iM^3(\omega'_1 \omega'_2 - m_1 m_2 + \hat{q}^2)(\omega_1 m_2 + m_1 \omega_2) \phi^l \phi^l \\
+ iM^2(\omega'_1 \omega'_2 - m_1 m_2 + \hat{q}^2)(m_1 \mathcal{P} \phi^l \phi^l \hat{q} + m_2 \mathcal{P} \phi^l \hat{q}^l \phi^l) \\
- iM^3(\omega'_1 m_2 - m_1 \omega'_2)(\omega_1 \omega_2 + m_1 m_2) \phi^l \phi^l \\
+ iM^3(\omega'_1 m_2 - m_1 \omega'_2) \phi^l \phi^l \hat{q} \hat{q} \\
- iM^2(\omega'_1 m_2 - m_1 \omega'_2)(\omega_1 \phi^l \phi^l \hat{q} \hat{q} - \omega_2 \phi^l \phi^l \hat{q} \hat{q}) \\
+ iM^2(\omega_1 m_2 + m_1 \omega_2)(\omega'_1 \phi^l \phi^l \hat{q} \hat{q} + \omega'_2 \phi^l \phi^l \hat{q} \hat{q}) \\
- iM^3(m_1(\omega'_1 + \omega'_2) \phi^l \phi^l \hat{q} \hat{q} + m_2(\omega'_1 + \omega'_2) \phi^l \phi^l \hat{q} \hat{q}) \\
- iM^2(m_1 + m_2)(\omega_1 \phi^l \phi^l \hat{q} \hat{q} - \omega_2 \phi^l \phi^l \hat{q} \hat{q}) \right]
\]

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