Optimal homotopy analysis method with Green’s function for a class of nonlocal elliptic boundary value problems

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Abstract
In this paper, we present the optimal homotopy analysis method (OHAM) with Green’s function technique to acquire accurate numerical solutions for the nonlocal elliptic problems. We first transform the nonlocal boundary value problems into an equivalent integral equation, and then use an OHAM with convergence control parameter $c_0$. To demonstrate convergence and accuracy characteristics of the OHAM method, we compare the OHAM and Adomian decomposition method (ADM) with Green’s function. The numerical experiments confirm the reliability of the approach as it handles such nonlocal elliptic differential equations without imposing limiting assumptions that could change the physical structure of the solution. We also discuss the convergence and error analysis of proposed method. In summary: (i) the present approach does not require any additional computational work for unknown constants unlike ADM and VIM [1] (ii) guarantee of convergence (iii) flexibility on choice of initial guess of solution and (iv) useful analytic tool to investigate a class of nonlocal elliptic boundary value problems.

Keyword: Optimal homotopy analysis method; Nonlinear nonlocal elliptic boundary value problems; Convergence analysis; Adomian decomposition method; Integral equations.

1 Introduction
We first consider a class of linear nonlocal elliptic boundary value problems [1, 2]:

\[
\begin{cases}
-\alpha \left( \int_0^1 y(s) ds \right) y''(x) = h(x), & x \in (0, 1) \\
y(0) = a, & y(1) = b, & a, b \in [0, \infty).
\end{cases}
\]

1 Introduction
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y(0) = a, \quad y(1) = b, & a, b \in [0, \infty).
\end{cases}
\]
This problem (1.1) may be called a class of linear nonlocal boundary value problem since the coefficient of the derivative of the unknown solution \( y \) depends upon the integral of \( y \) itself, which in turn depends on the whole domain \((0, 1)\) rather than on a single point.

We also study a class of nonlinear nonlocal elliptic nonlinear boundary value problems [1, 3, 4]:

\[
\begin{cases}
-\alpha \left( \int_0^1 y(s) \, ds \right) y''(x) + y^{2n+1}(x) = 0, & x \in (0, 1) \\
y(0) = a, & y(1) = b, & a, b \in [0, \infty), & n \in 0 \cup \mathbb{Z}^+
\end{cases}
\]  

(1.2)

The problem (1.2) may be called a class of nonlinear nonlocal boundary value problem. Such nonlocal boundary value problems arise in modeling various physical questions such as the aero-elastic behavior of suspended flexible cables subjected to icing conditions and wind action [4–6] or the dust production and diffusion in the fusion devices [4]. For details on such applications of nonlocal boundary value problems see [1, 7] and the references therein.

In [2], the existence and uniqueness of solution of (1.1) was discussed by using a fixed point theorem and then the numerical solutions were obtained via a finite difference scheme. In [3], Cannon and Galiffa developed a numerical method for (1.2), in which they established a priori estimates and the existence and uniqueness of the solution to the nonlinear auxiliary problem via the Schauder fixed point theorem. They proved the existence and uniqueness to the problem and analyzed a discretization of problem (1.2) and showed that a solution to the nonlinear difference problem exists and is unique and that the numerical procedure converges with error \( O(h) \). In [1], Khuri and Wazwaz applied the variational iteration method to a class of nonlocal, elliptic boundary value problems (1.1) and (1.2) and established uniform convergence of the scheme. In [4], Themistoclakis and Vecchio provided the sufficient conditions for the unique solvability and a more general convergence theorem for (1.2) and suggested different iterative procedures to handle the nonlocal nonlinearity of the discrete problem.

In this paper, we present the OHAM with Green’s function technique to acquire accurate numerical solutions for the nonlocal elliptic problems given in (1.1) and (1.2). We first transform the given nonlocal boundary value problems into an equivalent integral equation, and then use OHAM to obtain accurate numerical solutions. To demonstrate convergence and accuracy characteristics of the OHAM, a number of test examples are included. We compare the present method and ADM with Green’s function which confirms the accuracy and superiority of the OHAM. The numerical experiments confirm the reliability of the approach as it handles such nonlocal elliptic differential equations without imposing limiting assumptions that could change the physical structure of the solution. We also discuss the convergence and error analysis of proposed method. In summary: (i) the present approach does not require any additional computational work for unknown constants unlike ADM and VIM [1] (ii) guarantee of convergence (iii) flexibility on choice of an initial guess of solution and (iv) useful analytic tool to investigate a class of nonlocal elliptic boundary value problems.
2 Homotopy analysis method with Green’s function

We consider a general form of (1.1) and (1.2) nonlocal elliptic boundary value problems as

\[
\begin{aligned}
\begin{cases}
\alpha(p)y''(x) = f(y(x)) & x \in (0, 1) \\
y(0) = a, & y(1) = b,
\end{cases}
\end{aligned}
\tag{2.1}
\]

where \(\alpha(p)\) is a continuous positive function. By setting \(f(y(x)) \equiv -h(x)\) or \(f(y(x)) \equiv y^{2n+1}(x)\), we obtain the original boundary value problems (1.1) or (1.2).

Following Singh et al. \[8, 9\], we transform nonlocal elliptic boundary value problems (2.1) into an equivalent integral equation as

\[
y(x) = a + (b - a)x + \frac{1}{\alpha(p)} \int_0^1 G(x, s)f(y(s))ds,
\tag{2.2}
\]

where \(G(x, s)\) is given by

\[
G(x, \xi) = \begin{cases}
x(s - 1), & x \leq s, \\
s(x - 1), & s \leq x.
\end{cases}
\tag{2.3}
\]

According to homotopy analysis method \[10, 11\], we use \(q \in [0, 1]\) as an embedding parameter, the general zero-order deformation equation is constructed as

\[
(1 - q)[\phi(x; q) - y_0(x)] = q c_0 N[\phi(x; q)],
\tag{2.4}
\]

where \(y_0(x)\) denotes an initial guess, \(c_0 \neq 0\) is convergence-controller parameter, \(\phi(x; q)\) is an unknown function and \(N[\phi(x; q)]\) is given by

\[
N[\phi(x; q)] := \phi(x; q) - [a + (b - a)x] - \frac{1}{\alpha(p[\phi(x; q)])} \int_0^1 G(x, s)f(\phi(s; q))ds = 0.\tag{2.5}
\]

The zero-order deformation (2.4) becomes \(\phi(x; 0) = y_0(x)\) at \(q = 0\) and it becomes \(N[\phi(x; 1)] = 0\) at \(q = 1\) which is exactly the same as the original problem (2.1) provided that \(\phi(x; 1) = y(x)\).

Expanding \(\phi(x; q)\) in a Taylor series with respect to the parameter \(q\), we obtain

\[
\phi(x; q) = y_0(x) + \sum_{k=1}^{\infty} y_k(x)q^k,
\tag{2.6}
\]

where \(y_k(x)\) is given by

\[
y_k(x) = \frac{1}{k!} \frac{\partial^k}{\partial q^k}[\phi(x; q)] \bigg|_{q=0}.
\tag{2.7}
\]
The series (2.6) converges for \( q = 1 \) if \( c_0 \neq 0 \) is chosen properly and it reduces to

\[
\phi(x; 1) \equiv y(x) = \sum_{k=0}^{\infty} y_k(x),
\]

which will be one of solutions of the problem (2.2).

Defining the vector \( \vec{y}_k = \{y_0(x), y_1(x), \ldots, y_k(x)\} \) and differentiating (2.4), \( k \) times with respect to the parameter \( q \), dividing it by \( k! \), setting subsequently \( q = 0 \), the \( k \)th-order deformation equation is obtained

\[
y_k(x) - \chi_k y_{k-1}(x) = c_0 \, R_k(\vec{y}_{k-1}, x),
\]

where

\[
\chi_k = \begin{cases} 
0, & k \leq 1 \\
1, & k > 1
\end{cases}
\]

and

\[
R_k(\vec{y}_{k-1}, x) = \frac{1}{(k - 1)!} \left\{ \frac{\partial^{k-1}}{\partial q^{k-1}} N[\phi(x; q)] \right\}_{q=0} = \frac{1}{(k - 1)!} \left\{ \frac{\partial^{k-1}}{\partial q^{k-1}} N \left( \sum_{m=0}^{\infty} y_m(x) q^m \right) \right\}_{q=0} = y_{k-1}(x) - (1 - \chi_k) [a + (b - a)x] - \frac{1}{\alpha(p_{k-1})} \int_0^1 G(x, s) \mathcal{H}_{k-1} ds.
\]

Thus we have

\[
R_k(\vec{y}_{k-1}, x) = y_{k-1}(x) - (1 - \chi_k) [a + (b - a)x] - \frac{1}{\alpha(p_{k-1})} \int_0^1 G(x, s) \mathcal{H}_{k-1} ds
\]

where \( p_k \) and \( \mathcal{H}_k \), are given by

\[
p_k = \int_0^1 \left( \sum_{m=0}^{k} y_m(x) q^m \right) ds, \quad \mathcal{H}_k = \frac{1}{k!} \frac{\partial^k}{\partial q^k} \left\{ f \left( \sum_{m=0}^{k} y_m(x) q^m \right) \right\}_{q=0}.
\]

Using (2.9) - (2.12), the \( m \)th-order deformation equation is simplified as

\[
y_k(x) - \chi_k y_{k-1}(x) = c_0 \left[ y_{k-1}(x) - (1 - \chi_k) [a + (b - a)x] \right. \left. - \frac{1}{\alpha(p_{k-1})} \int_0^1 G(x, s) \mathcal{H}_{k-1} ds \right].
\]
Choosing an initial guess \( y_0(x) = a + (b-a)x \), the components \( y_k \) are successively obtained. Hence, the \( n \)th-order approximate solution of the problem (2.2) is given by

\[
\phi_n(x, c_0) = \sum_{k=0}^{n} y_k(x, c_0).
\] (2.14)

The optimal value of the parameter \( c_0 \) can be obtained by minimizing the squared residual of governing equation

\[
E_n(c_0) = \int_{0}^{1} \left( N[\phi_n(x, c_0)] \right)^2 dx.
\] (2.15)

However, the exact squared residual error is expensive to calculate when \( n \) is large. So, we approximate \( E_n \) by the discrete averaged residual error defined by

\[
E_n(c_0) \approx \frac{1}{M} \sum_{k=1}^{M} \left( N[\phi_n(x_k, c_0)] \right)^2,
\] (2.16)

where \( x_k = kh, k = 1, 2, \ldots, M \) and the optimal value \( c_0 \) is obtained by solving \( \frac{dE}{dc_0} = 0 \).

**Remark 2.1.** By setting \( c_0 = -1 \), the scheme (2.13) reduces to the ADM with Green’s function [8, 9].

### 3 Convergence analysis

In this section, we establish the convergence of solution defined in (2.14) of integral (2.2). Let \( X = (C[0, 1], \|y\|) \) be a Banach space with

\[
\|y\| = \max_{x \in [0,1]} |y(x)|, \ y \in X.
\]

**Theorem 3.1.** Let \( 0 < \delta < 1 \) and the solution components \( y_0(x), y_1(x), y_2(x), \ldots \) obtained by (2.14) satisfy the condition: \( \exists \ k_0 \in \mathbb{N} \ \forall k \geq k_0 : \|y_{k+1}\| \leq \delta\|y_k\| \), then the series solution \( \sum_{k=0}^{\infty} y_k(x) \) is convergent.

**Proof.** Define the sequence \( \{\phi_n\}_{n=0}^{\infty} \) as,

\[
\begin{aligned}
\phi_0 &= y_0(x) \\
\phi_1 &= y_0(x) + y_1(x) \\
\phi_2 &= y_0(x) + y_1(x) + y_2(x) \\
& \vdots \\
\phi_n &= y_0(x) + y_1(x) + y_2(x) + \cdots + y_n(x)
\end{aligned}
\] (3.1)

and we show that is a Cauchy sequence in the Banach space \( X \). For this purpose, consider

\[
\|\phi_{n+1} - \phi_n\| = \|y_{n+1}\| \leq \delta\|y_n\| \leq \delta^2\|y_{n-1}\| \leq \cdots \leq \delta^{n-k_0+1}\|y_{k_0}\|.
\]
For every \( n, m \in \mathbb{N}, n \geq m > k_0 \), we have
\[
\| \phi_n - \phi_m \| = \| (\phi_n - \phi_{n-1}) + (\phi_{n-1} - \phi_{n-2}) + \cdots + (\phi_{m+1} - \phi_m) \|
\leq \| \phi_n - \phi_{n-1} \| + \| \phi_{n-1} - \phi_{n-2} \| + \cdots + \| \phi_{m+1} - \phi_m \|
\leq (\delta^{n-k_0} + \delta^{n-k_0-1} + \cdots + \delta^{m-k_0+1}) \| y_{k_0} \|
= \frac{1 - \delta^{n-m}}{1 - \delta} \delta^{m-k_0+1} \| y_{k_0} \| \tag{3.2}
\]
and since \( 0 < \delta < 1 \) so it follows that
\[
\lim_{n,m \to \infty} \| \phi_n - \phi_m \| = 0. \tag{3.3}
\]
Therefore, \( \{ \phi_n \}_{n=0}^\infty \) is a Cauchy sequence in the Banach space \( X \).

**Theorem 3.2.** Assume that the series solution \( \sum_{k=0}^\infty y_k(x) \) converges to \( y(x) \). If the truncated series \( \phi_m(x, c_0) = \sum_{k=0}^m y_k(x, c_0) \) is used as an approximation to the solution \( y(x) \), then the maximum absolute truncated error is estimated as
\[
|y(x) - \phi_m(x, c_0)| \leq \frac{\delta^{m-k_0+1}}{1 - \delta} \| y_{k_0} \|. \tag{3.4}
\]

**Proof.** From Theorem 3.1 following inequality (3.2), we have
\[
\| \phi_n - \phi_m(x, c_0) \| \leq \frac{1 - \delta^{n-m}}{1 - \delta} \delta^{m-k_0+1} \| y_{k_0} \|,
\]
for \( n \geq m \). Now, as \( n \to \infty \) then \( \phi_n \to y \) and \( \delta^{n-m} \to 0 \). So,
\[
\| y(x) - \phi_m(x, c_0) \| \leq \frac{\delta^{m-k_0+1}}{1 - \delta} \| y_{k_0} \|. \tag{3.5}
\]

Theorems 3.1 and 3.2 together confirm that the convergence of series solution (2.14). \( \square \)

**Theorem 3.3.** If the series solution \( \sum_{k=0}^\infty y_k(x) \) converges to \( y(x) \) then it must be a solution of (2.2).

**Proof.** Since the series \( \sum_{k=0}^\infty y_k(x) \) is convergent, then
\[
\lim_{n \to \infty} y_n(x) = 0, \quad \forall \ x \in [0, 1]. \tag{3.6}
\]
By summing up the left hand-side of (2.9), we get
\[
\sum_{k=1}^n [y_k(x) - \chi_k y_{k-1}(x)] = y_1(x) + \cdots + (y_n(x) - y_{n-1}(x)) = y_n(x). \tag{3.7}
\]
Letting \( n \to \infty \) and using (3.6), equation (3.7) reduces to
\[
\sum_{k=1}^\infty (y_k(x) - \chi_k y_{k-1}(x)) = 0. \tag{3.8}
\]
Using (3.8) and right hand-side of the relation (2.9), we obtain
\[ \sum_{k=1}^{\infty} c_0 R_k(\overline{y}_{k-1}, x) = \sum_{k=1}^{\infty} (y_k(x) - \chi_k y_{k-1}(x)) = 0. \] (3.9)

Since \( c_0 \neq 0 \), then equation (3.9) reduces to
\[ \sum_{k=1}^{\infty} R_k(\overline{y}_{k-1}, x) = 0. \] (3.10)

Using (3.10) and (2.9), we have
\[ 0 = \sum_{k=1}^{\infty} R_k(\overline{y}_{k-1}, x) = \sum_{k=1}^{\infty} \left[ y_{k-1}(x) - (1 - \chi_k) [a + (b - a)x] - \frac{1}{\alpha(p_{k-1})} \int_{0}^{1} G(x, s) \mathcal{H}_{k-1} ds \right] \]
\[ = \sum_{k=1}^{\infty} y_{k-1}(x) - [a + (b - a)x] - \frac{1}{\sum_{k=1}^{\infty} \alpha(p_{k-1})} \int_{0}^{1} G(x, s) \sum_{k=1}^{\infty} \mathcal{H}_{k-1} ds, \]

since \( \sum_{k=0}^{\infty} y_k(x) \) converges to \( y(x) \), then \( \sum_{k=0}^{\infty} \mathcal{H}_k \to f(y(x)) \) and \( \sum_{k=1}^{\infty} \alpha(p_{k-1}) \to \alpha(p) \) [12], we obtain
\[ y(x) = a + (b - a)x + \frac{1}{\alpha(p)} \int_{0}^{1} G(x, s) f(y(s)) ds. \]

Hence, \( y(x) \) is the exact solution of integral equation (2.2).

4 Numerical results

In this section, four examples are discussed and the results are compared with existing exact solutions. We define the absolute errors as
\[ E_n(x) := |y(x) - \phi_n(x)|, \quad e_n(x) := |y(x) - \psi_n(x)| \]
where \( \phi_n(x) \) and \( \psi_n(x) \) are OHAM and ADM solutions, respectively.

Example 4.1. Consider the special case of linear nonlocal elliptic boundary value problem (1.1) with \( \alpha(p) = p^{1/3} \) as
\[ \begin{cases} 
 p^{1/3} y''(x) = \frac{6}{\sqrt{4}} x, & x \in (0, 1) \\
 y(0) = 0, & y(1) = 1, \\
 p = \left( \int_{0}^{1} y(s) ds \right). 
\end{cases} \] (4.1)

Its exact solution is \( y(x) = x^3 \).
Applying the OHAM (2.13) to the example (4.1), we have

\[ y_k(x) - \chi_k y_{k-1}(x) = c_0 \left[ y_{k-1}(x) - (1 - \chi_k)y_0(x) - \frac{1}{\alpha(p_{k-1})} \int_0^1 G(x,s) \frac{6s}{\sqrt{4}} ds \right]. \]  

(4.2)

Using (4.2) with an initial approximation \( y_0 = x \), we obtain \( \phi_2(x, c_0) \). With the help of (2.16), the optimal value of the parameter \( c_0 \) is computed as \( c_0 = -0.505595 \), and hence the OHAM solution is given by

\[ \phi_2(x) = 7.7 \times 10^{-16} x + x^3. \]  

(4.3)

A comparison among the numerical results obtained by OHAM solution \( \phi_2(x) \), ADM solution \( \psi_2(x) \) and the exact solution \( y(x) \) is depicted in Table 1.

| \( x \) | \( y(x) \) | \( \psi_2(x, -1) \) | \( \phi_2(x, -0.505595) \) | \( |y(x) - \psi_2(x)| \) | \( |y(x) - \phi_2(x)| \) |
|-------|---------|-----------------|-----------------|-----------------|-----------------|
| 0.0   | 0.000   | -0.0000000000   | 0.000           | 0.0000000000    | 0.0000000000    |
| 0.1   | 0.001   | -0.062559671    | 0.001           | 0.06359671      | 7.67615E-17     |
| 0.2   | 0.008   | -0.115267240    | 0.008           | 0.123267240     | 1.49186E-16     |
| 0.3   | 0.027   | -0.148270607    | 0.027           | 0.175270607     | 2.11636E-16     |
| 0.4   | 0.064   | -0.151717670    | 0.064           | 0.215717670     | 7.67615E-17     |
| 0.5   | 0.125   | -0.115756328    | 0.125           | 0.240756328     | 2.77556E-16     |
| 0.6   | 0.216   | -0.030534480    | 0.216           | 0.246534480     | 3.05311E-16     |
| 0.7   | 0.343   | 0.1137999760    | 0.343           | 0.229200024     | 2.77556E-16     |
| 0.8   | 0.512   | 0.3270991400    | 0.512           | 0.22045E-16     | 2.22045E-16     |
| 0.9   | 0.729   | 0.6192151140    | 0.729           | 1.11022E-16     | 1.11022E-16     |
| 1.0   | 1.000   | 1.0000000000    | 1.000           | 2.22045E-16     | 0.0000000000    |

Example 4.2. Consider the special case of nonlinear nonlocal elliptic boundary value problem (1.2) with \( \alpha(p) = \frac{1}{p} \) as

\[
\begin{align*}
\begin{cases} \\
-\frac{1}{p}y''(x) + \frac{3}{4(\sqrt{2} - 2)}y^5(x) = 0, & x \in (0, 1) \\
y(0) = 1, & y(1) = \frac{\sqrt{2}}{2}, \\
p = \left( \int_0^1 y(s)ds \right). 
\end{cases}
\end{align*}
\]  

(4.4)

Its exact solution is \( y(x) = \frac{1}{\sqrt{1-x^2}} \).

Applying the OHAM (2.13) to the example (4.2), we have

\[ y_k(x) - \chi_k y_{k-1}(x) = c_0 \left[ y_{k-1}(x) - (1 - \chi_k)y_0(x) - \frac{1}{\alpha(p_{k-1})} \int_0^1 G(x,s) H_{k-1} ds \right]. \]  

(4.5)

Using (4.5) with an initial guess \( y_0 = 1 + (\frac{\sqrt{2}}{2} - 1)x \) and (2.16) with \( c_0 = -0.819014 \), we obtain the homotopy optimal solution as

\[
\begin{align*}
\phi_2(x) &= 1 - 0.4973x + 0.3737x^2 - 0.3060x^3 + 0.2235x^4 - 0.1258x^5 + 0.05148x^6 \\
&\quad - 0.0153x^7 + 0.0033x^8 - 0.00055x^9 + 0.0000646x^{10} + \cdots
\end{align*}
\]
A comparison among the numerical solution obtained by OHAM solution \( \phi_2(x) \), ADM solution \( \psi_2(x) \) and the exact solution is depicted in Table 2.

| \( x \) | \( y(x) \) | \( \psi_2(x, c_0 = -1) \) | \( \phi_2(x, -0.819014) \) | \( |y(x) - \psi_2(x)| \) | \( |y(x) - \phi_2(x)| \) |
|---|---|---|---|---|---|
| 0.0 | 1.000000000 | 1.000000000 | 1.000000000 | 0.000000000 | 0.000000000 |
| 0.1 | 0.953462589 | 0.954516555 | 0.953715758 | 0.001053966 | 0.000253169 |
| 0.2 | 0.912870929 | 0.91348055 | 0.913715758 | 0.002038784 | 0.000477126 |
| 0.3 | 0.877058019 | 0.87819116 | 0.877715758 | 0.002761097 | 0.0006477126 |
| 0.4 | 0.845154255 | 0.848286083 | 0.84586767 | 0.002761097 | 0.0006477126 |
| 0.5 | 0.816496581 | 0.819638873 | 0.81733417 | 0.002761097 | 0.0006477126 |
| 0.6 | 0.790569415 | 0.793406404 | 0.791235825 | 0.002836989 | 0.000666410 |
| 0.7 | 0.766964989 | 0.769254375 | 0.767106781 | 0.002836989 | 0.000666410 |
| 0.8 | 0.745355992 | 0.746938889 | 0.745731089 | 0.001582896 | 0.000375097 |
| 0.9 | 0.725476250 | 0.727235455 | 0.725867948 | 0.000797295 | 0.000191698 |
| 1.0 | 0.707106781 | 0.707106781 | 0.707106781 | 0.000000000 | 2.22045E-16 |

Example 4.3. Consider the special case of nonlinear nonlocal elliptic boundary value problem (1.2) with \( \alpha(p) = p \) as

\[
\begin{align*}
-p y''(x) + \frac{3(2\sqrt{2} - 2)}{4} y^5(x) &= 0, \quad x \in (0, 1) \\
y(0) &= 1, \quad y(1) = \frac{\sqrt{2}}{2}, \quad p = \left( \int_0^1 y(s) ds \right).
\end{align*}
\]

Its exact solution is \( y(x) = \frac{1}{\sqrt{1+x}} \).

Applying the OHAM (2.13) with \( y_0 = 1 + (\frac{\sqrt{2}}{2} - 1)x \), we have

\[
y_k(x) - \chi_k y_{k-1}(x) = c_0 \left[ y_{k-1}(x) - (1 - \chi_k)y_0(x) - \frac{1}{\alpha(p_{k-1})} \int_0^1 G(x, s) \mathcal{H}_{k-1} ds \right].
\]

Using (4.7) and (2.16) with \( c_0 = -0.933697 \), we obtain the homotopy optimal solution as

\[
\phi_2(x) = 1 - 0.495211 x + 0.362361 x^2 - 0.280911 x^3 + 0.195035 x^4 - 0.107115 x^5 + 0.043453 x^6 - 0.01294 x^7 + 0.002854 x^8 - 0.000464 x^9 + 0.000054 x^{10} - \cdots
\]

A comparison among the numerical solution obtained by OHAM solution \( \phi_2(x) \), ADM \( \psi_2(x) \) and the exact solution is depicted in Table 3.

Example 4.4. Consider the special case of nonlinear nonlocal elliptic boundary value problem (1.2) with \( \alpha(p) = \left( \frac{1}{p} \right)^2 \) as

\[
\begin{align*}
-\left( \frac{1}{p} \right)^2 y''(x) + \frac{2}{\left( \ln 2 \right)^2} y^3(x) &= 0, \quad x \in (0, 1) \\
y(0) &= 1, \quad y(1) = \frac{1}{2}, \quad p = \left( \int_0^1 y(s) ds \right).
\end{align*}
\]

Its exact solution is \( y(x) = \frac{1}{1+x} \).
Table 3 Numerical results of example 4.3

| x   | y(x)          | ψ_2(x, -0.933697) | φ_2(x, -0.933697) | | y(x) - ψ_2(x) | | y(x) - φ_2(x) |
|-----|---------------|------------------|-------------------|---|----------------|---|----------------|
| 0.0 | 1.000000000  | 1.000000000     | 1.000000000       | 0.000000000 | 0.000000000 |
| 0.1 | 0.953462589  | 0.954139200     | 0.953840089       | 0.000676611 | 0.000377500 |
| 0.2 | 0.912870929  | 0.914044009     | 0.913485384       | 0.001173080 | 0.000614454 |
| 0.3 | 0.877058019  | 0.878527575     | 0.877813154       | 0.001494738 | 0.000755135 |
| 0.4 | 0.845154255  | 0.846978784     | 0.845696674       | 0.001643619 | 0.000815419 |
| 0.5 | 0.816496581  | 0.818128146     | 0.817301384       | 0.001631566 | 0.000815419 |
| 0.6 | 0.790569415  | 0.792049712     | 0.791302438       | 0.001480297 | 0.000733023 |
| 0.7 | 0.766964989  | 0.768181846     | 0.767571912       | 0.001216857 | 0.000733023 |
| 0.8 | 0.745355992  | 0.746224331     | 0.745800843       | 0.000868338 | 0.000444851 |
| 0.9 | 0.725476250  | 0.725933776     | 0.725717917       | 0.000457526 | 0.000241667 |
| 1.0 | 0.707106781  | 0.707106781     | 0.707106781       | 0.000000000 | 2.22045E-16  |

Applying the OHAM (2.13) with \( y_0 = 1 + \left( \frac{1}{2} - 1 \right)x \), we have

\[
y_k(x) - \chi_k y_{k-1}(x) = c_0 \left[ y_{k-1}(x) - (1 - \chi_k)y_0(x) - \frac{1}{\alpha(p_{k-1})} \int_0^1 G(x, s) H_{k-1} ds \right]. \tag{4.9}
\]

Using (4.9) and (2.16) with \( c_0 = -0.612671 \), we obtain the homotopy optimal solution as

\[
\phi_2(x) = 1 - 1.00399x + 0.995127x^2 - 0.90086x^3 + 0.655261x^4 - 0.33896x^5 + 0.11522x^6 - 0.0246916x^7 + 0.00308645x^8 - 0.00017147x^9 + \cdots
\]

A comparison among the numerical solution obtained by OHAM solution \( \phi_2(x) \), ADM solution \( \psi_2(x) \) and the exact solution is depicted in Table 4.

Table 4 Numerical results of example 4.4

| x   | y(x)          | ψ_2(x, -0.612671) | φ_2(x, -0.612671) | | y(x) - ψ_2(x) | | y(x) - φ_2(x) |
|-----|---------------|------------------|------------------|---|----------------|---|----------------|
| 0.0 | 1.000000000  | 1.000000000     | 1.000000000       | 0.000000000 | 0.000000000 |
| 0.1 | 0.909909099  | 0.914550054     | 0.908713383       | 0.00549145  | 0.000377526 |
| 0.2 | 0.833333333  | 0.844849352     | 0.832746652       | 0.01151602  | 0.000586882 |
| 0.3 | 0.769230769  | 0.785573674     | 0.768603045       | 0.01634290  | 0.000627724 |
| 0.4 | 0.714285714  | 0.733317575     | 0.713705107       | 0.01903186  | 0.000580607 |
| 0.5 | 0.666666667  | 0.686029523     | 0.666157571       | 0.01936285  | 0.000500906 |
| 0.6 | 0.625000000  | 0.642573190     | 0.624561470       | 0.01753190  | 0.000438530 |
| 0.7 | 0.588235294  | 0.602939944     | 0.587870965       | 0.01415869  | 0.000364329 |
| 0.8 | 0.555555556  | 0.565272431     | 0.555285414       | 0.00971687  | 0.000270141 |
| 0.9 | 0.526315789  | 0.531448042     | 0.526170109       | 0.00483225  | 0.000145681 |
| 1.0 | 0.500000000  | 0.500000000     | 0.500000000       | 6.66134E-16 | 1.66533E-16  |

5 Conclusion

We presented the OHAM with Green’s function technique for obtaining numerical solutions to a class of nonlinear, nonlocal, elliptic boundary value problems. We first trans-
formed the given nonlocal boundary value problems into an equivalent integral equation, and then used the OHAM to obtain accurate solutions. Unlike the ADM, the OHAM always gives better convergent series solution as shown in Tables. The numerical experiments confirm the reliability of the approach as it handles such nonlocal elliptic differential equations without imposing limiting assumptions that could change the physical structure of the solution. In addition, the convergence and error analysis was presented. The proposed scheme will be used further in studying identical applications.

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