Fractional topological phase on spatially encoded photonic qudits

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We discuss the appearance of fractional topological phases on cyclic evolutions of entangled qudits encoded on photonic degrees of freedom. We show how the spatial correlations between photons generated by spontaneous parametric down conversion can be used to evidence the multiple topological phases acquired by entangled qudits and the role played by the Hilbert space dimension. A realistic experimental proposal is presented with numerical predictions of the expected results.

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I. INTRODUCTION

Geometrical phases are robust means for implementing unitary gates useful for quantum information processing \cite{1,2}. The phase evolution of entangled qubits was investigated in refs.\textsuperscript{3,4}, and the role of entanglement on the topological nature of geometric phases has been discussed both theoretically \textsuperscript{5,6} and experimentally \textsuperscript{7,8} in the context of two-qubit systems. More recently, it has been shown that the dimension of the Hilbert space plays a crucial role on the topological phases acquired by entangled qubits \textsuperscript{9}. The appearance of fractional phases is a remarkable property of two-qudit systems, also shared by multiple qubits \textsuperscript{10}. However, experimental demonstrations of these interesting features are still rare. Fractional phases were originally investigated in quantum Hall systems in connection with different homotopy classes in the configuration space of anyons. Moreover, the topological origin of fractional phases in quantum Hall systems open interesting perspectives related to fault tolerant quantum computation \textsuperscript{11}.

In the present work we propose an experimental setup to observe fractional topological phases with entangled qudits encoded on the transverse position of quantum correlated photon pairs generated by spontaneous parametric down conversion (SPDC) \textsuperscript{12,13}. Local unitary operations can be applied in this degree of freedom with the aid of spatial light modulators that introduce a programmable phase profile on the wavefront of the quantum correlated photons. Then, the desired fractional phases can be observed through polarization controlled two-photon interference \textsuperscript{14,15}. The roles played by entanglement and Hilbert space dimension are unraveled by the behavior of the interference fringes as the unitary operations are continuously changed, raising the conjecture that entanglement and dimensionality witnesses are involved in the quantities to be measured. These ideas are supported by analytical and numerical calculations of the expected interference fringes. The paper is organized as follows: in section II we introduce the fractional topological phase concept; the experimental proposal is presented in section III; section IV is devoted to numerical simulations, and in section V conclusions are outlined.

II. FRACTIONAL TOPOLOGICAL PHASES

Let $|\psi\rangle = \sum_{i,j=1}^{d} \alpha_{ij} |ij\rangle$ be the most general two-qudit pure state. We shall represent this state by the $d \times d$ matrix $\alpha$ whose elements are the coefficients $\alpha_{ij}$. With this notation the norm of the state vector becomes $\langle \psi | \psi \rangle = \text{Tr}(\alpha^\dagger \alpha) = 1$ and the scalar product between two states is $\langle \phi | \psi \rangle = \text{Tr}(\beta^\dagger \alpha)$, where $\beta$ is the $d \times d$ matrix containing the coefficients of state $|\phi\rangle$ in a given basis. In order to characterize a general vector in the Hilbert space, note that any invertible matrix admits a polar decomposition $\alpha = e^{i\theta} QS$, where $Q$ is a positive definite Hermitian matrix, and $S \in SU(d)$. Following the usual terminology of spontaneous parametric down conversion, we shall label the two qudits as signal (s) and idler (i). Under local unitary operations $U_s$ and $V_i$ the coefficient matrix is transformed according to $\alpha' = U_s \alpha V_i^T$. It can be readily seen that this kind of transformation preserves the polar decomposition and can be represented separately in each sector of the coefficient matrix: $\alpha' = U'_{s} \alpha V'^T_{i}$, where $e^{i\theta} = e^{i\theta} \sqrt{\text{det} U_s} \cdot \text{det} V_i \cdot Q' = U_s Q U_i^T$, and $S' = U_s S V_i^T$. $U_s$ and $V_i$ are the $SU(d)$ parts of the local unitary operations: $U_s = \sqrt{\text{det} U_s} U_s$ and $V_i = \sqrt{\text{det} V_i} V_i$. Therefore, we identify the transformation in three sectors of the matrix structure: an explicit phase transformation $\phi \mapsto \phi'$, a transformation closed in the space of positive definite Hermitian matrices $Q \mapsto Q'$, and a transformation $S \mapsto S'$ closed in $SU(d)$. Since the roots of a complex number are multivalued functions, some caution is necessary in defining the above quantities. For time varying unitary operations, we shall assume that any one of the possible roots is taken at the initial time, then the subsequent values must form a continuous evolution as a function of time, so that $\phi(t)$ is a smooth, everywhere differentiable function.

Following \textsuperscript{16}, we shall define as cyclic those evolutions for which the final state is physically equivalent to the initial one, in other words, the initial and final state vectors are related only by a global phase factor: $\alpha' = e^{i\theta} \alpha$. The geometric phase acquired by a time evolving quan-
th state $\alpha(t) = e^{i\phi(t)} Q(t) S(t)$ is always defined as

$$\phi_g = \arg \langle \psi(0) | \psi(t) \rangle + i \int_0^t dt \langle \psi(t) | \dot{\psi}(t) \rangle$$

(1)

$$= \arg \{ \text{Tr} [\alpha^\dagger(0) \alpha(t)] \} + i \int_0^t dt \text{Tr} [\alpha^\dagger(t) \dot{\alpha}(t)]$$

that corresponds to the total phase minus the dynamical phase. At this point we can profit from the polar decomposition by investigating the contribution coming from each sector of the coefficient matrix. Let us consider a cyclic evolution over the time interval $T$: $\alpha(T) = e^{i\theta} \alpha(0)$. First, we identify the trivial phase evolution $\phi(T) = \phi(0) + \Delta \phi$. In the positive Hermitian sector, if we write $Q(T) = e^{i\theta} Q(0)$, its defining condition imposes $\theta_\alpha = 0$, so that no phase contribution could stem from this sector. Finally, in the $SU(d)$ sector we can have $S(T) = e^{i\theta} S(0)$, however, $\det S(T) = e^{i\theta} \det S(0)$, and since both $S(T)$ and $S(0)$ are $SU(d)$ matrices, we arrive at

$$\theta_s = \frac{2n\pi}{d},$$

(2)

where $n = 0, 1, 2, \ldots, d - 1$. Therefore, only fractional phase values can arise from the $SU(d)$ sector and the global phase acquired in a cyclic evolution is

$$\theta = \Delta \phi + \frac{2n\pi}{d}.$$  

(3)

Although $SU(d)$ is simply connected, the topological nature of these fractional phases relies on the multiply connected manifold that represents the set of $SU(d)$ matrices $S$ with the identification $e^{i\theta} S \equiv S$.

We can also investigate the dynamical phase using the polar decomposition. At this point it will be useful to write:

$$\dot{\alpha} = i\phi \alpha(t) + e^{i\phi(t)} \left[ \dot{Q}(t) S(t) + Q(t) \dot{S}(t) \right].$$

(4)

This expression can be used in the dynamical phase integral appearing in eq. (4). Using the normalization condition $\text{Tr}[\alpha^\dagger(t) \alpha(t)] = 1$, we see that the first term in eq. (3) will cancel out the trivial global phase $\Delta \phi$, which do not contribute to the geometric phase, as would be expected. Then, by making $\alpha^\dagger(t) = e^{-i\phi(t)} S^\dagger(t) Q(t)$ and using the cyclic property of the trace, one easily arrives at

$$\phi_g = \frac{2n\pi}{d} + i \int_0^T dt \text{Tr} \left[ Q(t) \dot{Q}(t) \right]$$

$$+ i \int_0^T dt \text{Tr} \left[ Q^2(t) \dot{S}(t) S^\dagger(t) \right].$$

(5)

The normalization condition implies in $\text{Tr}[Q^2(t)] = 1$. Differentiating both sides with respect to $t$ and using the cyclic property of the trace gives $\text{Tr} \left[ Q(t) \dot{Q}(t) \right] = 0$, so that the first integral in eq. (5) vanishes. Now, it is now important to identify the following invariants under local unitary evolutions: $\text{Tr} [\rho_j^p]$, $p = 1, \ldots, d$, where $\rho_j$ is the reduced density matrix of qudit $j$:

$$\rho_s = (\alpha^\dagger \alpha)^T = (S^T Q^2 S)^T,$$

(6)

and

$$\rho_i = \alpha \alpha^\dagger = Q^2.$$  

(7)

In fact, the invariants are $j$-independent since one easily shows that $\text{Tr} [\rho_j^p] = \text{Tr} [\rho_{j+1}^p] = \text{Tr} [Q^p]$. The first one ($p = 1$) is simply the norm of the state vector, as already stated. The second invariant is related to the $I$-concurrence of a two-qudit pure quantum state [19] $C = \sqrt{2(1 - \text{Tr} [\rho_j^2])}$, so that its invariance expresses the well known fact that entanglement is not affected by local unitary operations. The $p = d$ invariant can be rewritten in terms of the former and $D = |\det \alpha|$. In particular, for qubits we have $C = 2D$. For a given dimension $d$, the $I$-concurrence runs between 0 for product states and $C_m = \sqrt{2(d-1)/d}$ for maximally entangled states.

In order to exploit the role played by these invariants in the geometrical phase, we shall make them explicit in the expression of $Q^2$ in terms of the identity and the generators $T_n$ $(n = 1, 2, \ldots, d^2 - 1)$ of $SU(d)$. As usual, the generators are traceless matrices normalized according to $\text{Tr} [T_n T_m] = \delta_{nm}/2$. A general $Q^2$ matrix with $\text{Tr} [Q^2] = 1$ and $\text{Tr} [Q^4] = 1 - C^2/2$ can be written as

$$Q^2(t) = \frac{1}{d} + \sqrt{C_m^2 - C^2} \hat{q}(t) \cdot T,$$

(8)

where $\hat{q}$ is a unit vector in $\mathbb{R}^{d^2 - 1}$. For maximally entangled states ($C = C_m$), the reduced density matrices become $\rho_s = \rho_i = 1/d$.

The expression (5) can be used in the last term of eq. (3) to establish a connection between the geometrical phase and the concurrence of the two-qudit state undergoing a cyclic evolution. However, it will be useful to work out a more convenient parametrization of $S S^\dagger$.

First we recall that for a general invertible matrix $A$ we have [20]:

$$\frac{d (\det A)}{dt} = \det A \text{Tr} \left[ A^{-1} \frac{d A}{dt} \right].$$

(9)

Since the evolution of $S(t)$ is closed in $SU(d)$ ($\det S(t) = 1$), we easily arrive at $\text{Tr} \left[ S S^\dagger \right] = 0$. Besides, note that $S(t) S(t) = -S(t) S(t)$, which allows us to define the velocity vector $s \in \mathbb{R}^{d^2 - 1} \text{ such that } S S^\dagger = i s \cdot T$. Then, using the normalization condition for the generators and the fact that they are traceless matrices, we arrive at a compact expression for the geometrical phase:

$$\phi_g = \frac{2n\pi}{d} - \frac{1}{2} \sqrt{C_m^2 - C^2} \int \hat{q} \cdot \mathbf{d} x,$$

(10)

where we defined $\mathbf{d} x = s dt$. This key result shows that the geometrical phase for a cyclic evolution is composed
by a fractional contribution of topological nature and an integral contribution which depends on the history of the quantum state evolution, parametrized in the Hermitian and SU(d) sectors of the polar decomposition. This integral contribution is weighted by entanglement and vanishes completely for maximally entangled states, for which only the fractional values are allowed. We next turn to a possible physical implementation of the fractional phases using entangled qudits encoded on spatial degrees of freedom of photon pairs generated by spontaneous parametric down conversion (SPDC).

III. POLARIZATION CONTROLLED TWO-PHOTON INTERFERENCE

Let us consider the setup sketched in Fig.(1). Using multiple slit masks, spatial qudits are encoded on the transversal paths of the photon pairs generated by SPDC. Beyond the slits, the wave field can be written as a superposition of the field transmitted through different slits. We shall refer to these contributions as the slit modes $\eta_m(r)$, where $m$ is the slit index. The slit mode functions satisfy the orthonormality condition:

$$\int d^2r \eta_m^*(r) \eta_n(r) = \delta_{mn}. \tag{11}$$

The positive frequency components of signal and idler vector field operators are

$$E_i^+ = E_{sH}^+ \hat{e}_H + E_{sV}^+ \hat{e}_V, \quad E_i^- = E_{iH}^+ \hat{e}_H + E_{iV}^+ \hat{e}_V, \tag{12}$$

where $\hat{e}_\mu (\mu = H, V)$ is the horizontal $(H)$ or vertical $(V)$ polarization unit vector. Each polarization component is expanded in terms of the slit modes as

$$E_i^{+\mu} = \sum_p a_{ip} \eta_p(r_s), \quad E_i^{+\nu} = \sum_q b_{iq} \eta_q(r_i), \tag{13}$$

where greek indices refer to polarization modes and roman ones refer to slit modes. The annihilation operators $a_{ip}$ and $b_{iq}$ act on Fock states as usual:

$$a_{ip} b_{iq} |m\sigma, n\epsilon\rangle = \delta_{pm}\delta_{i\mu}\delta_{q\nu}\delta_{\sigma\epsilon}|vac\rangle, \tag{14}$$

where $|vac\rangle$ is the vacuum state and $|m\sigma, n\epsilon\rangle$ is a two-photon Fock state corresponding to one signal photon passing through slit $m$ with polarization $\sigma$ and one idler photon passing through slit $n$ with polarization $\epsilon$.

For type I phase matching and a vertically polarized pump beam, SPDC generates horizontally polarized photon pairs. Meanwhile, the spatial correlations between signal and idler can be tailored by playing with the pump angular spectrum $[13]$ using suitable lenses (omitted in the setup for simplicity) on the pump laser. These spatial correlations determine the two-photon path correlation through the slits. Therefore, the two-photon state after the slits can be written as

$$|\psi\rangle = \sum_{m,n} \alpha_{mn} |mH, nH\rangle, \tag{15}$$

where $\alpha_{mn}$ is the probability amplitude of having one signal photon passing through slit $m$ and one idler photon passing through $n$. Right after the slits, two half wave plates ($\lambda/2$) rotate the photons polarization by $45^\circ$, and the two-photon quantum state becomes

$$|\psi_1\rangle = \sum_{m,n} \frac{\alpha_{mn}}{2} \left( |mH, nH\rangle + |mH, nV\rangle \right) + |mV, nH\rangle + |mV, nV\rangle \right). \tag{16}$$

Then, each photon passes through a polarization controlled unitary gate composed by a Mach-Zehnder interferometer with input and output polarizing beam splitters (PBS), and a spatial phase modulator (SLM) inserted on the vertical polarization arm. The signal and idler evolutions $U_s$ and $V_i$ are then implemented on the vertical polarization component of the spatial qudits encoded on signal and idler photons, respectively. Of course, the efficient implementation of the unitary operations require that the slits are imaged on the SLM’s. For the sake of simplicity, we shall omit the lenses required for this imaging scheme. After the controlled gates, the two-photon quantum state becomes

$$|\psi_2\rangle = \sum_{m,n} \frac{\alpha_{mn}}{2} \left( e^{i(\theta_s + \theta_i)} |mH, nH\rangle + e^{i\theta_s} V_i |mH, nV\rangle \right) + e^{i\theta_s} U_s |mV, nH\rangle + U_s V_i |mV, nV\rangle \right). \tag{17}$$

where $\theta_s$ and $\theta_i$ are longitudinal phases added to signal and idler when they cross the Mach-Zehnder interferometers. The displacement of a piezoelectric ceramic (PZT) coupled to one of the Mach-Zehnder mirrors can do this task.
Finally, signal and idler are sent to different input ports of the same PBS so that only the components $|nH, mH\rangle$ and $|nV, mV\rangle$ will contribute to coincidences. Two polarizers (POL) oriented at 45° are placed before detectors in order to *erase* the polarization information. Then, the detected field operators are:

$$\begin{align*}
E_{1}^+ &= \frac{1}{\sqrt{2}} (iE_{sV}^+ + E_{iH}^+) \\
E_{2}^+ &= \frac{1}{\sqrt{2}} (E_{sH}^+ + iE_{iV}^+) ,
\end{align*}$$

and the coincidence count is proportional to:

$$\langle \psi_2 | E_{1}^+ E_{2}^- E_{j}^+ E_{j}^+ | \psi_2 \rangle = \| E_{2}^+ E_{1}^+ | \psi_2 \rangle \|^2 ,$$

where $E_{j}^+ = (E_{j}^+)^\dagger$ and $j = 1, 2$. From eq. (14), one easily sees that the $|mH, nV\rangle$ and $|mV, nH\rangle$ contributions vanish and the normalized coincidence function reduces to

$$C(r_1, r_2) = \frac{1}{2} \left\| \sum_{m,n} e^{i\theta} \alpha_{mn} E_{iH}^+ E_{sH}^+ |mH, nH\rangle + \alpha_{mn}' E_{iV}^+ E_{sV}^+ |mV, nV\rangle \right\|^2 .$$

where $\theta = \theta_s + \theta_i - \pi$. The coefficients $\alpha_{mn}'$ result from the local unitary transformations on the two-qudit state:

$$U_s V_i \sum_{m,n} \alpha_{mn} |mV, nV\rangle = \sum_{m,n} \alpha_{mn}' |mV, nV\rangle .$$

The two-photon interference pattern is

$$C(r_1, r_2) = \frac{1}{2} \left| \sum_{m,n} \eta_m(r_1) \eta_n(r_2) (\alpha_{mn} e^{i\theta} + \alpha_{mn}') \right|^2 .$$

We are interested only in the phase acquired by the qudits when they evolve under the action of the unitary gates $U_s$ and $V_i$. Since the slit modes are orthonormal, the integrated coincidence count eliminates the spatial interference between different slits, so that only the Mach-Zehnder interference shows up in the two-photon correlations. This corresponds to large aperture detection, insensitive to the detailed spatial structure of the two-photon quantum correlations. However, the Hilbert space dimension remains manifested through the two-qudit coefficients in the integrated coincidence function:

$$C = \int d^2 r_1 \int d^2 r_2 C(r_1, r_2) = \frac{1}{2} \sum_{m,n} |\alpha_{mn} e^{i\theta} + \alpha_{mn}'|^2 .$$

Note that polarization has been used as a subsidiary degree of freedom to provide two paths for the evolution of the spatial qudits. In fact, the interference described by eq. (23) can be thought of as resulting from the superposition between the initial two-qudit state $|\varphi_0\rangle = \sum_{mn} \alpha_{mn} |mn\rangle$ and the evolved state $|\varphi\rangle = U_s V_i |\varphi_0\rangle$ such that

$$C = \left\| |\varphi_0\rangle + |\varphi\rangle \right\|^2 = 1 + |\langle\varphi_0 | \varphi \rangle| \cos (\theta - \gamma) ,$$

where $\gamma = \arg(\varphi_0 | \varphi \rangle)$. The visibility of the interference fringes is the absolute value of the state overlap $\langle\varphi_0 | \varphi \rangle$, while the interference phase is the overlap argument. For a cyclic evolution, $|\varphi\rangle = e^{i\gamma} |\varphi_0\rangle$, the interference recovers maximal visibility with the fringes shifted by $\gamma$. Finally, the topological phases can be implemented by continuously varying the unitary gates according to local time dependent parameters, adjusted by the signal and idler SLM’s respectively. Next, we give some numerical examples of the possible operations and the expected interference patterns that can reveal both the fractional topological phases and the role of entanglement.

**IV. NUMERICAL RESULTS**

We can illustrate the fractional phases with the simplest example of two qutrits ($d = 3$) under the action of diagonal unitary operations $U_s = \text{diag}[e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3}]$ and $V_i = \text{diag}[e^{i\chi_1}, e^{i\chi_2}, e^{i\chi_3}]$, where $\phi_n$ and $\chi_n$ are phase factors imposed by the spatial light modulators (SLM) to the photon path states. In order to guarantee that no trivial dynamical phase is added by the operations, we assume that $\sum_n \phi_n = \sum_n \chi_n = 0$, in other words, we restrict the two-qutrit evolutions to $SU(3) \otimes SU(3)$ operations.

The coefficient matrix evolves as $\alpha' = U_s \alpha V_i^T$, so that

$$\alpha_{mn}' = e^{i(\phi_m + \chi_n)} \alpha_{mn} ,$$

and the coincidence count becomes

$$C = \frac{1}{4} \sum_{m,n} |\alpha_{mn}|^2 \left| e^{i\theta} + e^{i(\phi_m + \chi_n)} \right|^2 = \sum_{m,n} |\alpha_{mn}|^2 \cos^2 \left[ \frac{\phi_m + \chi_n - \theta}{2} \right] .$$

In principle, the two SLMs can be operated independently and implement totally different local evolutions. However, for simplicity, we shall first consider an identical operation capable to evidence both the fractional phases and the signatures of entanglement. Our cyclic operations will be decribed by a real parameter $t$ in the
can see, the behaviour is considerably different for product and entangled states. For the entangled state, the interference fringes completely disappear when \( t = 1/2 \), and the two-qudit entangled state becomes orthogonal to the initial state. Then, the fringes reappear with the fractional phase shift \( 2\pi/3 \), reaching maximal visibility again at \( t = 1 \). On the other hand, the product state exhibits a remarkable different behaviour, the fringes never disappear, and maximal visibility is not recovered when \( t = 1 \).

Let us now consider a different kind of evolution controlled by independent parameters \( t_s \) and \( t_i \):

\[
\begin{align*}
\phi_1 &= \phi_2 = \frac{\pi}{3} t_s , \\
\phi_3 &= -\frac{2\pi}{3} t_i , \\
\chi_1 &= \chi_2 = \frac{\pi}{3} t_i , \\
\chi_3 &= -\frac{2\pi}{3} t_i . 
\end{align*}
\]

In this case, the coincidence is

\[
C_c = \frac{2}{3} \cos^2 \left( \frac{t\pi}{3} - \frac{\theta}{2} \right) + \frac{1}{3} \cos^2 \left( \frac{2t\pi}{3} + \frac{\theta}{2} \right),
\]

for the entangled state of eq. \( \text{(28)} \), and

\[
C_p = \frac{4}{9} \cos^2 \left( \frac{t\pi}{3} - \frac{\theta}{2} \right) + \frac{1}{9} \cos^2 \left( \frac{2t\pi}{3} + \frac{\theta}{2} \right) + \frac{2}{9} \left[ 1 + \cos(\pi\tau) \cos \left( \frac{t\pi}{3} + \frac{\theta}{2} \right) \right],
\]

for the product state of eq. \( \text{(29)} \), where we defined \( t = (t_s + t_i)/2 \) and \( \tau = (t_s - t_i)/2 \). As we can see, the coincidence interference is completely different in each case. It does not depend on the relative parameter \( \tau \) for the entangled input, while it does for the product one. The behaviour of the fringe visibility is also considerably different, even for \( t_s = t_i = t \) (\( \tau = 0 \)). In Fig. 3 the coincidence interference is plotted for \( \tau = 0 \) and different values of \( t \). In this case, the evolved entangled state never becomes orthogonal to the initial one, but the fringes recover maximum visibility for \( t = 1 \), with the fractional phase shift \( 2\pi/3 \). However, the same operation does not take the product state through a cyclic evolution and the associated interference never recovers maximum visibility.

Finally, in order to observe the effect of the Hilbert space dimension, it is instructive to consider a situation in which only two components of the qutrits are operated, and compare their evolution with a genuine pair of qubits. For example, let us make:

\[
\begin{align*}
\phi_1 &= -\phi_2 = \frac{\pi}{2} t_s , \\
\phi_3 &= 0 , \\
\chi_1 &= -\chi_2 = \frac{\pi}{2} t_i , \\
\chi_3 &= 0 .
\end{align*}
\]
In this case, the coincidence is

\[ \mathcal{C}_e = \frac{1}{2} + \cos \theta \left[ 1 + \frac{2}{3} \cos(\pi t) \right], \quad (34) \]

for the entangled state (36), and

\[ \mathcal{C}_p = \frac{1}{2} + \cos \theta \left[ \frac{1}{2} \cos(\pi t) + \cos(\pi \tau) \right] + 2 \cos \left( \frac{\pi}{2} t \right) \cos \left( \frac{\pi}{2} \tau \right), \quad (35) \]

for the product state (37). As before, the coincidence count associated to an entangled input is independent of the relative control parameter \( \tau \), while the same does not hold true for the product input. However, as can be readily seen in Figs. 3(a) and 3(b), neither the entangled nor the product inputs are driven through a cyclic evolution in the full range of the total control parameter \( t \). The interference fringes never recover maximal visibility.

We can compare these results with those expected for a pair of qubits \( (d = 2) \) prepared either in the entangled state

\[ |\varphi_e\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle), \quad (36) \]

\( (\alpha_{mn} = \delta_{mn}/\sqrt{2}) \) or in the product state having the same single qubit population distribution:

\[ |\varphi_p\rangle = \frac{1}{2} (|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle), \quad (37) \]

\( (\alpha_{mn} = 1/2) \), evolving under

\[ \phi_1 = -\phi_2 = \frac{\pi}{2} t_s, \]
\[ \chi_1 = -\chi_2 = \frac{\pi}{2} t_i. \quad (38) \]

In this case, the coincidence is

\[ \mathcal{C}_e (d = 2) = \frac{1}{2} \left[ 1 + \cos(\pi t) \right] \cos \theta, \quad (39) \]

for the entangled state (36), and

\[ \mathcal{C}_p (d = 2) = \frac{1}{2} + \frac{\cos \theta}{4} \left[ \cos(\pi t) + \cos(\pi \tau) \right], \quad (40) \]

for the product state (37). These coincidence counts are shown in Figs. 5(a) and 5(b) for \( \tau = 0 \) and the full range of \( t \). Now, the entangled state is taken through a cyclic evolution and recovers maximal visibility with the topological phase \( \pi \), as expected for maximally entangled qubits [5–10]. On the other hand, the product state does not complete a cyclic evolution, it ends up in a state orthogonal
FIG. 5: Coincidence interference for a (a) maximally entangled state and (b) product state, corresponding to the unitary operations given by eqs. (35). $t = 0$ (continuous line), $t = 0.5$ (dashed line) and $t = 1$ (blue line).

V. CONCLUSIONS

As is well-known, qudit gates based on topological phases are a potentially robust means for implementing quantum algorithms [21–23]. In this work, we proposed a physical implementation of fractional topological phases acquired by entangled qudits operated by local unitary transformations. Spatial qudits encoded on quantum correlated photon pairs are a suitable framework for this investigation, since the qudit components may be efficiently addressed by spatial light modulators. The expected coincidence interferences can reveal both, the role played by entanglement as well as the Hilbert space dimension. Then, a natural conjecture arises as to whether the measured quantities are related to entanglement and dimensionality witnesses, what motivates further investigation.

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[1] J. A. Jones, V. Vedral, A. Ekert, and G. Castagnoli, Nature (London) 403, 869 (2000).
[2] L.-M. Duan, J. I. Cirac, and P. Zoller, Science 292, 1695 (2001).
[3] E. Sjöqvist, Phys. Rev. A 62, 022109 (2000).
[4] B. Hessmo and E. Sjöqvist, Phys. Rev. A 62, 062301 (2000).
[5] R. Mosseri, and R. Dandoloff, J. Phys. A 34, 10243 (2003).
[6] P. Milman, and R. Mosseri, Phys. Rev. Lett. 90, 230403 (2003).
[7] P. Milman, Phys. Rev. A 73, 062118 (2006).
[8] C. E. R. Souza, J. A. O. Huguenin, P. Milman, and A. Z. Khoury, Phys. Rev. Lett. 99, 160401 (2007).
[9] Jiafeng Du, Jing Zhu, Mingjun Shi, Xinhu Peng, and Dieter Suter, Phys. Rev. A 76, 042121 (2007).
[10] L. E. Oxman and A. Z. Khoury, Phys. Rev. Lett. 106, 240503 (2011).
[11] Markus Johansson, Marie Ericsson, Kuldip Singh, Erik Sjöqvist, and Mark S. Williamson, Phys. Rev. A 85, 032112 (2012).
[12] A. Kitaev, Ann. Phys. (N.Y.) 303, 2 (2003).
[13] L. Neves, G. Lima, E. J. S. Fonseca, L. Davidovich, and S. Pádua, Phys. Rev. A 76, 032314 (2007).
[14] I. Rodrigues, O. Cosme, and S. Pádua, J. Phys. B: At. Mol. Opt. Phys. 43, 125505 (2010).
[15] S. P. Walborn, C.H. Monken, S. Pádua, and P.H. Souto Ribeiro, Physics Reports 495, 87 (2010).
[16] M. Franca Santos, P. Milman, A. Z. Khoury, and P. H. Souto Ribeiro, Phys. Rev. A 64, 023804 (2001).
[17] D. P. Caetano, P. H. Souto Ribeiro, J. T. C. Pardal, and A. Z. Khoury, Phys. Rev. A 68, 023805 (2003).
[18] M. Mukunda and R. Simon, Ann. Phys. 228, 205 (1993).
[19] P. Rungta, V. Buzek, C. M. Caves, M. Hillery and G. J. Milburn, Phys. Rev. A 64, 042315 (2001).
[20] K. B. Petersen and M. S. Pedersen, http://matrixcookbook.com, Version Nov. 15, 2012 [c.f. eq.(46) on page 8].
[21] S. S. Bullock, D. P. O’Leary, and G. K. Brennen, Phys. Rev. Lett. 94, 230502 (2005).
[22] A. Muthukrishnan, and C. R. Stroud, Jr., Phys. Rev. A 62, 052309 (2000).
[23] R. Ionicioiu, T. P. Spiller, and W. J. Munro, Phys. Rev. A 80, 012312 (2009).