ON COCYCLE CONJUGACY OF QUASIFREE ENDOMORPHISMS SEMIGROUPS ON THE CAR ALGEBRA

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W. Arveson has described a cocycle conjugacy class \( \mathcal{U}(\alpha) \) of \( E_0 \)-semigroup \( \alpha \) on \( \mathcal{B}(\mathcal{H}) \) which is a factor of type I. Under some conditions on \( \alpha \) there is a \( E_0 \)-semigroup \( \beta \in \mathcal{U}(\alpha) \) being a flow of shifts in the sense of R.T.Powers (see [11]). We study quasifree endomorphisms semigroups \( \alpha \) on the hyperfinite factor \( \mathcal{M} = \pi(\mathcal{A}(\mathcal{K}))'' \) generated by the representation \( \pi \) of the algebra of the canonical anticommutation relations \( \mathcal{A}(\mathcal{K}) \) over a separable Hilbert space \( \mathcal{K} \). The type of \( \mathcal{M} \) can be I, II or III depending on \( \pi \). The cocycle conjugacy class \( \mathcal{U}(\alpha) \) is described in the terms of initial isometrical semigroup in \( \mathcal{K} \) and an analogue of the Arveson result for the hyperfinite factor \( \mathcal{M} \) of type \( II_1 \) and \( III_\lambda \), \( 0 < \lambda < 1 \), is introduced.

1. Introduction. Let \( \mathcal{M} \) be the \( W^* \)-algebra acting in a Hilbert space. One-parameter unital semigroup \( \alpha_t \in \text{End}(\mathcal{M}), \ t \geq 0 \), is called a \( E_0 \)-semigroup if every function \( \eta(\alpha_t(x)) \) is continuous in \( t \) for \( x \in \mathcal{M} \) and \( \eta \in \mathcal{M}^* \). Given a \( E_0 \)-semigroup \( \alpha \) one can define its generator \( \delta(a) = \lim_{t \to 0} \frac{\alpha_t(a) - a}{t} \) for \( a \in \text{dom}\delta \), where \( \text{dom}\delta \) is \( \sigma \)-week dense in \( \mathcal{M} \) (see [14]). Two \( E_0 \)-semigroups \( \alpha \) and \( \beta \) are called to be cocycle conjugate if there is a strong continuous family of unitaries \( U_t \in \mathcal{M}, \ t \geq 0 \), named a cocycle, such that \( \beta_t(\cdot) = U_t \alpha_t(\cdot) U_t^*, \ U_{t+s} = U_t \alpha_t(U_s), \ t, s \geq 0 \) (see [11,14,15]). Notice that if semigroups \( \alpha \) and \( \beta \) have generators differ on a bounded derivation, then \( \alpha \) and \( \beta \) are cocycle conjugate (see [15]). This case is associated with the differentiable cocycle \( (U_t)_{t \geq 0} \) and it is not the general one. The discussion on the cocycle conjugacy of automorphisms semigroups on the \( W^* \)-algebra \( \mathcal{M} = \mathcal{B}(\mathcal{H}) \) one can see in monograph [14]. A notion of the cocycle conjugacy of endomorphisms semigroups on the \( W^* \)-algebra \( \mathcal{M} \) was given by W. Arveson. He studied the case of \( \mathcal{M} = \mathcal{B}(\mathcal{H}) \) in [11].

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Let $\mathcal{A} = \mathcal{A}(\mathcal{K})$ be the $C^*$-algebra of the canonical anticommutation relations (CAR) over a Hilbert space $\mathcal{K}$. It means that there is a map $f \to a(f)$ from $\mathcal{K}$ to the $C^*$-algebra $\mathcal{A}$ (with the unit $1$, the involution $*$ and the norm $||\cdot||$) satisfying the following properties:

1) $a(\lambda f + g) = \lambda^* a(f) + a(g)$ for all $f, g \in \mathcal{K}, \lambda \in \mathbb{C}$,
2) (CAR) $a(f)a(g) + a(g)a(f) = 0$,
3) the polynomials in all $a(f), a^*(g)$ are dense in $\mathcal{A}$ by the norm, $||a(f)|| = ||f||_{\mathcal{K}}$.

Every state on $\mathcal{A}$ that is a positive linear functional $\phi \in \mathcal{A}^*$, $\phi(1) = 1$, is determined by its values on (Wick) normal ordered monomials $a^*(f_1)...a^*(f_m)a(g_1)...a(g_n)$. The operator $R, 0 < R < 1$, in $\mathcal{B}(\mathcal{K})$ determines a state $\omega_R$ satisfying the condition

$$\omega_R(a^*(f_m)...a^*(f_1)a(g_1)...a(g_n)) = \delta_{nm} \det((f_i, Rg_j)).$$

Such $\omega_R$ is called a quasifree state. Let us define a representation $\pi_R$ of the $C^*$-algebra $\mathcal{A}$ in a Hilbert space $\mathcal{H} = \mathcal{F}(\mathcal{K}) \otimes \mathcal{F}(\mathcal{K})$ by the formula

$$\pi_R(a(f)) = a((1 - R)^{1/2}f) \otimes \Gamma + 1 \otimes a^*(JR^{1/2}f), f \in \mathcal{K},$$

$$\pi_R(1) = \text{Id},$$

where $\mathcal{F}(\mathcal{K})$ is the antisymmetric (fermion) Fock space over $K$ with a vacuum vector $\Omega$, $J$ is some antiunitary in $K$, $\Gamma$ is a hermitian unitary operator completely defined by the condition $\Gamma a(f) = -a(f)\Gamma, f \in \mathcal{H}, \Gamma \Omega = \Omega$. In this representation the state $\omega_R$ becomes the vector one, $\omega_R(x) = (\Omega \otimes \Omega, \pi_R(x)\Omega \otimes \Omega), x \in \mathcal{A}$. Therefore the triple $(\pi_R, \mathcal{H}_R, \Omega \otimes \Omega)$, where $\mathcal{H}_R = \pi_R(\mathcal{A})\Omega \otimes \Omega$, is the Gelfand-Neumark-Segal (GNS) representation of the $C^*$-algebra $\mathcal{A}$ associated with the state $\omega_R$. The state $\omega_R$ is pure if and only if $R = P, P^2 = P$ is an orthogonal projection. In this case the GNS representation $\pi_P$ acting in a Hilbert space $\mathcal{H} = \mathcal{F}((I - P)\mathcal{K}) \otimes \mathcal{F}(JPJK)$ yields the $W^*$-algebra $\mathcal{M}_P(\mathcal{K}) = \pi_P(\mathcal{A})'' = \mathcal{B}(\mathcal{H})$. Setting a two points function of the state $\omega$ to be

$$\omega(a^*(f)a(g)) = \nu(f, g), f, g \in \mathcal{K},$$

and fixing a numerical parameter $\nu \in [0, 1/2]$ one can consider the quasifree state $\omega = \omega_\nu$ associated with the operator $R = \nu 1$ in $\mathcal{K}$. If $\nu \neq 0$, then $\omega$ is exact ($\omega(x^*x) = 0$ implies $x = 0$). In the GNS representation $\pi_\nu$ acting in a Hilbert space
\[ \mathcal{H} = \mathcal{F}(\mathcal{K}) \otimes \mathcal{F}(\mathcal{K}) \] the \( C^* \)-algebra \( \pi_\nu(\mathcal{A}) \) generates the \( W^* \)-algebra \( \mathcal{M}_\nu = \mathcal{M}_\nu(\mathcal{K}) = \pi_\nu(\mathcal{A}(\mathcal{K}))'' \). If \( 0 < \nu < 1/2 \), then \( \mathcal{M}_\nu \) is a hyperfinite factor of type \( III_\lambda \), where \( \lambda = \nu/(1 - \nu) \). In the case of \( \nu = 1/2 \) the state \( \omega \) is a trace and \( \mathcal{M}_\nu \) is a hyperfinite factor of type \( II_1 \). The case of \( \nu = 0 \) is associated with the vacuum state and the Fock representation with \( \mathcal{M}_\emptyset = \mathcal{B}(\mathcal{F}(\mathcal{K})) \) (see [9-10,16-20]).

Let a quasifree endomorphism \( \alpha \) act on the algebra \( \pi_R(\mathcal{A}) \) act on the generating elements by the formula \( \alpha(\pi_R(a(f))) = \pi_R(a(Vf)), f \in \mathcal{K} \), where \( V \) is an isometry in a Hilbert space \( K \) commuting with \( R \). If \( kerR = ker(I - R) = 0 \) the state \( \omega_R \) is exact and the vector \( \Omega \otimes \Omega \) is separating for the \( W^* \)-factor \( \mathcal{M}_R = \pi_R(\mathcal{A})'' \). It allows to show that \( \alpha \) can be extended to a quasifree endomorphism of \( \mathcal{M}_R \) (see [16]). We denote this endomorphism by \( B_R(V) \). The procedure of the passage from an operator in a Hilbert space \( \mathcal{K} \) to a map on the algebra \( \mathcal{M}_R \) is called the (quasifree) lifting. If \( (V_t)_{t \geq 0} \) is a \( C_0 \)-semigroup of isometries in \( \mathcal{K} \) commuting with \( R \), then the semigroup \( (B_R(V_t))_{t \geq 0} \) is a \( E_0 \)-semigroup on the \( W^* \)-factor \( \mathcal{M}_R \). We call these semigroups the quasifree ones (see [9-10,16-19]).

The endomorphism \( \alpha \) of \( W^* \)-algebra \( \mathcal{M} \) is called a shift if \( \cap_{n=1}^{\infty} \alpha^n(\mathcal{M}) = \mathcal{C}1 \). The \( E_0 \)-semigroup \( (\alpha_t)_{t \geq 0} \) is called a flow of shifts if \( \alpha_t \) is a shift for every fix \( t > 0 \). In [12] R.T. Powers introduced a flow of shifts on the \( W^* \)-algebra \( \mathcal{M}_\emptyset = \mathcal{B}(\mathcal{F}(\mathcal{K})) \). It was obtained by an extension of the semigroup \( (\alpha_t)_{t \geq 0} \) acting on the generating elements of the \( C^* \)-algebra \( \pi_0(\mathcal{A}(\mathcal{K})), \mathcal{K} = L_2(0, +\infty) \), by the formula \( \alpha_t(\pi_0(a(f))) = \pi_0(a(S_t f)), t \geq 0, f \in \mathcal{K} \), where \( (S_t)_{t \geq 0} \) is a \( C_0 \)-semigroup of right shifts in \( \mathcal{K} \) defined by the formula \( (S_t f)(x) = f(x - t) \) for \( x > t \) and \( (S_t f)(x) = 0 \) for \( 0 < x < t, f \in \mathcal{K} \). In [13] it is asserted that the quasifree \( E_0 \)-semigroup \( (B_\nu(S_t))_{t \geq 0} \) on the hyperfinite factor \( \mathcal{M}_\nu, 0 < \nu \leq 1/2 \), consists of shifts.

In [11] W. Arveson posed a question: to describe the class of cocycle conjugacy of the flow of shifts on the \( W^* \)-algebra \( \mathcal{M} \) defined by R.T. Powers in [12]. One can see the answer on this question in the case of \( \mathcal{M} = \mathcal{B}(\mathcal{H}) \) in [11]. We investigate cocycle conjugacy of quasifree automorphisms and endomorphisms semigroups on the hyperfinite factors of type \( II_1 \) and \( III_\lambda, 0 < \lambda < 1 \).

In what follows we denote the trace class, the Hilbert Schmidt class, compact operators and the Hilbert-Schmidt norm by symbols \( s_1, s_2, s_\infty \) and \( || \cdot ||_2 \) correspondently.

2. An extension on \( \mathcal{B}(\mathcal{H}) \) of quasifree automorphisms of hyperfinite
factors \( \mathcal{M} \subset \mathcal{B}(\mathcal{H}) \).

Fix a positive operator \( R, \ 0 < R < I, \ ker R = ker(I-R) = 0 \), and an antiunitary operator \( J \) in \( K \) and construct a representation \( \pi \) of the algebra \( \mathcal{A}(\mathcal{K} \oplus \mathcal{K}) \) in a Hilbert space \( \mathcal{H} = \mathcal{F}(\mathcal{K}) \otimes \mathcal{F}(\mathcal{K}) \) by the formula
\[
\pi(a(f \oplus 0)) = a((1 - R)^{1/2}f) \otimes \Gamma + 1 \otimes a(R^{1/2}Jf),
\]
\[
\pi(a(0 \oplus f)) = a(R^{1/2}f) \otimes \Gamma - 1 \otimes a((1 - R)^{1/2}Jf),
\]
where \( f \in \mathcal{K} \). Here \( \Gamma \) is a unitary operator completely defined by the relations \( \Gamma a(f) = -a(f)\Gamma \), \( \Gamma \Omega = \Omega \), \( f \in \mathcal{K} \). Define the hyperfinite factors \( M_R = \pi(\mathcal{A}(\mathcal{K} \oplus 0))'' \) and \( M_P = \pi(\mathcal{A}(\mathcal{K} \oplus \mathcal{K}))'' = \mathcal{B}(\mathcal{H}) \) and consider two vector states on its,
\[
\omega_R(x) = < \Omega \otimes \Omega, \pi(x \oplus 0)\Omega \otimes \Omega >, \ x \in \mathcal{A}(\mathcal{K} \oplus 0).
\]
\[
\omega_P(x) = < \Omega \otimes \Omega, \pi(x)\Omega \otimes \Omega >, \ x \in \mathcal{A}(\mathcal{K} \oplus \mathcal{K}).
\]
Note that
\[
\omega_R(\pi(a^*(f)a(g))) = (f, Rg), \ f, g \in \mathcal{K},
\]
\[
\omega_P(\pi(a^*(f)a(g))) = (f, Pg), \ f, g \in \mathcal{K} \oplus \mathcal{K},
\]
where \( P = \begin{pmatrix} R & R^{1/2}(I-R)^{1/2} \\ R^{1/2}(I-R)^{1/2} & I-R \end{pmatrix} \) is an orthogonal projection in a Hilbert space \( \mathcal{K} \oplus \mathcal{K} \). The state \( \omega_R \) is exact and the state \( \omega_P \) is pure and obtained by the purification procedure (see [19]) from \( \omega_R \). Operators
\[
b(f) = \Gamma \otimes \Gamma \pi(a(0 \oplus f)), \ b^*(f) = \pi(a^*(0 \oplus f))\Gamma \otimes \Gamma, \ f \in \mathcal{K},
\]
generate the commutant \( M'_R \) and satisfy the relation \( b(f) = J\pi(a(f \oplus 0))J, \ f \in \mathcal{K}, \) where \( J \) is a modular involution on \( M_R \) associated with \( \omega_R \) (see Appendix). Let \( V \) and \( W \) be isometrical operators in \( K \) commuting with \( R \). Consider quasifree endomorphisms \( \alpha \) of \( M_R \) and \( \beta \) of its commutant \( M'_R \) obtained by the lifting of \( V \) and \( W \): \( \alpha(\pi(a(f \oplus 0))) = \pi(a(Vf \oplus 0)), \ \beta(b(f)) = b(Wf), \ f \in \mathcal{K} \). Consider minimal unitary dilations \( V' \) and \( W' \) of the operators \( V \) and \( W \) acting in the same Hilbert space \( K', \ K \subset K' \) and a positive contraction \( R' \) in \( K' \) such that \( V'R = R'V, \ W'R = R'W \) (on the existence of \( R' \) see in [17]). Determine a quasifree endomorphism \( \theta \) of the \( C^*\)-algebra generated by \( M_R \) and \( M'_R \) such that \( \theta|_{M_R} = \alpha, \ \theta|_{M'_R} = \beta. \)
Theorem 1. The endomorphism \( \theta \) defined by the quasifree lifting of the isometrical operators \( V \) and \( W \), can be extended on \( M_P \) if and only if the following inclusion holds, \( R^{1/2}(I - R')^{1/2}(V' - W') \in s_2 \).

Remark. The condition of the proposition is sufficient for the cocycle conjugacy of endomorphic semigroups on \( M_R \) obtained by the quasifree lifting of the isometrical operators \( V \) and \( W \) included in the semigroups \( V \) and \( W \) in a Hilbert space \( K \) (see below).

Proof.

As it was proved by H. Araki (see [9-10]), any quasifree *-automorphism \( \theta' \) given on the \( C^* \)-algebra \( \pi(A(K' \oplus K')) \) by the formula \( \theta'(\pi(a(f \oplus g))) = \pi(a(V'f \oplus W'g)) \), \( f, g \in K' \), can be extended on the factor \( M_{P'} \), \( P' = \begin{pmatrix} R' & R^{1/2}(I - R')^{1/2} \\ R^{1/2}(I - R')^{1/2} & I - R' \end{pmatrix} \) if and only if \( \left( \begin{array}{cc} V' & 0 \\ 0 & W' \end{array} \right) \left( \begin{array}{cc} V' & 0 \\ 0 & W' \end{array} \right)^{-1} \in s_2 \). The unitary operator \( \left( \begin{array}{cc} V' & 0 \\ 0 & W' \end{array} \right) \) in the space \( K' \oplus K' \) satisfies this condition and correctly define \( \theta' \). The automorphism \( \theta' \) has the property \( \theta'(\Gamma \otimes \Gamma) = \Gamma \otimes \Gamma \), such that \( \theta'(b(f)) = b(W'f) \), \( f \in K' \). The factor \( M_P \subset M_{P'} \) is invariant under the action of \( \theta' \). Considering the restriction we obtain \( \theta'|_{M_P} = \theta \). Note that in the case of \( V = W \), the endomorphism \( \theta \) is a regular extension of \( \alpha \) in the sense of [8]. \( \Delta \)

Now let \( (V_t)_{t \in R_+} \) be a \( C_0 \)-semigroup of isometrical operators in \( K \). Then a family of quasifree endomorphisms on \( M_R \) defined by the formula \( \alpha_t(\pi(a(f \oplus 0))) = \pi(a(V_tf \oplus 0)) \), \( f \in K \), \( t \in R_+ \), is a \( E_0 \)-semigroup. Involve an expanding family of Hilbert spaces \( (\mathcal{H}_t)_{t \in R_+} \) embedded in \( F(K) \) such that \( \mathcal{H}_t \) is generated by all vectors \( a^\#(f_1)a^\#(f_2)\ldots a^\#(f_n))\Omega, f_i \in ker V_t^*, 1 \leq i \leq n \), where \( a^\# = a^* \) or \( a \). The family \( (\mathcal{H}_t)_{t \in R_+} \) is a product-system of Hilbert spaces (see [8]). Consider a product-system \( K_t = \mathcal{H}_t \otimes J\mathcal{H}_t \), \( t \in R_+ \). Let \( (\theta_t)_{t \in R_+} \) be a regular extension \( (\alpha_t)_{t \in R_+} \), then
\[
K_t = \theta_t(|\Omega \otimes \Omega > < \Omega \otimes \Omega|)\mathcal{H}, \ t \geq 0.
\]

By this way, the product-system associated with the regular extension is obtained by the doubling of the product-system associated with the initial semigroup. Note that in the common case (see [22]) \( \theta_t(P) \), where \( P \) is a one-dimensional projection, is not obliged to be a monotonous increasing family of projections. Such situation can characterize the complete compatibility with the exact state (in this case it is
\( \omega_R \).

3. Inner \(*\)-automorphisms and the cocycle conjugacy on the hyperfinite factor \( \mathcal{M}_R \).

In [9] it was proved that a quasifree derivation \( \delta \) of \( \mathcal{A}(\mathcal{K}) \) acting on the generating elements by the formula \( \delta(a(f)) = a(df), \ f \in \text{dom}\ d \), where \( d \) is some skew-hermitian operator, is inner iff \( d \in s_1 \). Then the automorphism obtained by the quasifree lifting of an unitary \( e^d, d \in s_1 \), is inner. Notice that \( e^d - I \in s_1 \). On the other side every inner automorphism \( \alpha \) can not be represented in the form \( \alpha = e^{\delta} \), where \( \delta \) is some inner derivation. In the following theorem we give the necessary and sufficient condition of an innerness of \( B_R(W) \) in the terms of \( W, WR = RW \).

**Theorem 2.** The quasifree automorphism \( B_R(W) \) of the hyperfinite factor \( \mathcal{M}_R \) is inner iff \( R^{1/2}(I - R)^{1/2}(W - I) \in s_2 \).

**Remark.** The results of [18] yields a sufficiency and a necessity of the condition of theorem 2 in the case of a pure point spectrum of \( W \). Thus we need to prove a necessity of one.

Proof of theorem 2 (necessity).

The quasifree automorphism \( B_R(W) \) has a form \( B_R(W)(\cdot) = U \cdot U^*, U \in \mathcal{M}_R \) by the condition. In the appendix we show that the generating elements of commutant \( \mathcal{M}_R' \) are \( 1, b(f) = \Gamma \otimes \Gamma \pi(a(0 \oplus f)), b^*(f) = \pi(a^*(0 \oplus f)) \Gamma \otimes \Gamma, f \in \mathcal{K} \). Thus \( U\pi(a(0 \oplus f))U^* = U\Gamma \otimes \Gamma U^* \Gamma \otimes \Gamma \pi(a(0 \oplus f)), f \in \mathcal{K} \). Let us show that \( U\Gamma \otimes \Gamma U^* \Gamma \otimes \Gamma = 1 \). Note that \((\Gamma \otimes \Gamma)^2 = I, (\Gamma \otimes \Gamma)^* = \Gamma \otimes \Gamma \) therefore \( U\Gamma \otimes \Gamma U^* \Gamma \otimes \Gamma \) equals \( 1 \) or \(-1 \). By this way \( U\pi(a(0 \oplus f))U^* = \pi(a(0 \oplus f)) \) or \(-\pi(a(0 \oplus f)), f \in \mathcal{K} \).

In the first case the quasifree automorphism \( B(W \oplus I) \) is unitary implementable, in the second case the quasifree automorphism \( B(W \oplus (-I)) \) is that. Therefore by theorem 1, \( R^{1/2}(I - R)^{1/2}(W - I) \in s_2 \) or \( R^{1/2}(I - R)^{1/2}(W + I) \in s_2 \). In the first case the theorem is proved. Suppose \( R^{1/2}(I - R)^{1/2}(W + I) \in s_2 \). Then \( R^{1/2}(I - R)^{1/2}(-W - I) \in s_2 \) and the automorphism \( B_R(-W) \) is inner. Thus the automorphism \( B_R(-I) = B_R(-W)B_R(W) \) is inner. It is a contradiction by [18]. Thus we proved \( U\pi(a(0 \oplus f))U^* = \pi(a(0 \oplus f)) \). Therefore the automorphism \( B(W \oplus I) \) of the \( C^* \)-algebra \( \pi(\mathcal{A}(\mathcal{K} \oplus \mathcal{K})) \) obtained by the lifting of \( W \oplus I \) is unitary implementable and \( R^{1/2}(I - R)^{1/2}(W - I) \in s_2 \) by theorem 1. \( \triangle \)

Let \( (U_t)_{t \geq 0} \) and \( (V_t)_{t \geq 0} \) be \( \mathcal{C}_0 \)-semigroups of unitaries in a Hilbert space \( \mathcal{K} \) commuting with the operator \( R \).
Theorem 3. The quasifree semigroups $(B_R(U_t))_{t \geq 0}$ and $(B_R(V_t))_{t \geq 0}$ on the hyperfinite factor $\mathcal{M}_R$ are cocycle conjugate iff $R^{1/2}(I - R)^{1/2}(U_t - V_t) \in s_2$, $t \geq 0$.

Proof of theorem 3 is based on theorem 1, theorem 2 and one result of [20] that we formulate in the following lemma:

**Lemma.** Let $\sigma$-week continuous groups of $^*$-automorphisms $\alpha = (\alpha_t)_{t \in \mathbb{R}}$ and $\beta = (\beta_t)_{t \in \mathbb{R}}$ on the $W^*$-factor $\mathcal{M}$ having separable predual $\mathcal{M}_s$ be such that $^*$-automorphisms $\beta_t \alpha_t$ are inner for all $t \in \mathbb{R}$. Then the group $\alpha$ and $\beta$ are cocycle conjugate.

Proof of theorem 3.

Necessity. Let the semigroups $(B_R(U_t))_{t \geq 0}$ and $(B_R(V_t))_{t \geq 0}$ be cocycle conjugate. Then $B_R(U_t)(\cdot) = W_t B_R(V_t)(\cdot) W_t^*$, $W_t \in \mathcal{M}_R$, $t \geq 0$. Fix $t \geq 0$. The automorphism $B(V_t \oplus V_t)$ is unitary implementable by theorem 1. Therefore the automorphism $B(U_t \oplus V_t)(\cdot) = W_t B(V_t \oplus V_t)(\cdot) W_t^*$ is unitary implementable too. The result follows from theorem 1.

Sufficiency. The result follows from theorem 2 and the lemma. $\triangle$

4. The cocycle conjugacy of quasifree endomorphisms semigroups.

Let $U = (U_t)_{t \geq 0}$ and $V = (V_t)_{t \geq 0}$ be $C_0$-semigroups of isometries in a Hilbert space $\mathcal{K}$ commuting with the operator $R$. Then there are $C_0$-semigroups of unitaries $U' = (U'_t)_{t \geq 0}$ and $V' = (V'_t)_{t \geq 0}$ in a Hilbert space $\mathcal{K}'$, $\mathcal{K} \subset \mathcal{K}'$, being minimal unitary dilations of semigroups $U$ and $V$ and the positive contraction $R'$ commuting with $U'$, $V'$ and satisfying the relation $U'_t R = R' U_t$, $V'_t R = R' V_t$, $t \geq 0$.

**Definition.** Two $C_0$-semigroups of isometries $U$ and $V$ are called to be approximating each other if $U'_t - V'_t \in s_2$ and $U'_t V'_t^*|_{\mathcal{K} \oplus \mathcal{K}} = I$, $t \geq 0$.

**Theorem 4.** Let a $C_0$-semigroup of isometries $U$ approximates a $C_0$-semigroup of isometries $V$. Then the quasifree semigroup $(B_R(U_t))_{t \geq 0}$ is cocycle conjugate to the quasifree semigroup $(B_R(V_t))_{t \geq 0}$.

Proof of theorem 4.

Theorem 3 leads to an existence of a cocycle $(\mathcal{W}_t)_{t \geq 0}$ such that $B_R(U'_t)(\cdot) = \mathcal{W}_t B_R(V'_t)(\cdot) \mathcal{W}_t^*$, $\mathcal{W}_t \in \mathcal{M}_R(\mathcal{K'})$, $t \geq 0$. We show that the condition $U'_t V'_t^*|_{\mathcal{K} \oplus \mathcal{K}} = I$ implies $\mathcal{W}_t \in \mathcal{M}_R(\mathcal{K})$. Note that if this condition holds a family of unitaries $W_t = U'_t V'_t^*|_{\mathcal{K}}$, $W_t - I \in s_2$, $t \geq 0$, is correctly defined. The quasifree lifting of $(W_t)_{t \geq 0}$ determines a family of inner automorphisms $B_R(W_t)(x) = \mathcal{W}_t x \mathcal{W}_t^*$, $\mathcal{W}_t, x \in s_2$.
\[ M_R(\mathcal{K}), \ t \geq 0. \] We constructed \((\mathcal{W}_t')_{t \geq 0}\) such that \(\mathcal{W}_t'x\mathcal{W}_t'^* = \mathcal{W}_t x\mathcal{W}_t^*, \ x \in M_R'(\mathcal{K}'), \ t \geq 0.\) Therefore \(\mathcal{W}_t = e^{ic(t)} \mathcal{W}_t', \ c(t) \in \mathbb{R}, \ t \geq 0,\) and \(\mathcal{W}_t \in M_R(\mathcal{K}).\)

5. Continuous semigroups of isometries in a Hilbert space.

It is useful to remind that an isometry \(V\) in a Hilbert space \(K\) is called completely nonunitary if there is no subspace \(K_0 \subset K\) reducing \(V\) to an unitary. Every \(C_0\)-semigroup of completely nonunitary isometries \((V_t)_{t \geq 0}\) is unitary equivalent to its model that is a \(C_0\)-semigroup of shifts \((S_t)_{t \geq 0}\) acting in a Hilbert space \(K' = H \otimes L_2(0, +\infty)\) by the formula \((S_t f)(x) = f(x - t)\) for \(x > t, (S_t f)(x) = 0\) for \(0 < x < t, f \in K'\). Here \(H\) is some Hilbert space of the dimension \(n\) equal to the deficiency index of \((V_t)_{t \geq 0}\) (we call \(n\) by the deficiency index of \((V_t)_{t \geq 0}\) in the following). Let \((V_t)_{t \geq 0}\) be a \(C_0\)-semigroup of isometries in a Hilbert space \(K\) and \(K = K_0 \oplus K_1\) be the Wold decomposition of \(K\), where \(K_0\) reduces \((V_t)_{t \geq 0}\) to a \(C_0\)-semigroup of unitaries and \(K_1\) reduces \((V_t)_{t \geq 0}\) to a \(C_0\)-semigroup of completely nonunitary isometries. We shall say \((V_t)_{t \geq 0}\) satisfies the condition \(N\) if the \(C_0\)-semigroup of unitaries \((V_t|_{K_0})_{t \geq 0}\) is uniformly continuous that is \(||V_t|_{K_0} - V_s|_{K_0}|| \to 0, t \to s, s, t \geq 0.\)

**Theorem 5 ([1-3,5-6]).** Let a \(C_0\)-semigroup of isometries \((V_t)_{t \geq 0}\) in a Hilbert space \(K\) with a deficiency index \(n > 0\) satisfy the condition \(N\).

Then there is a \(C_0\)-semigroup of completely nonunitary isometries \((S_t)_{t \geq 0}\) with a deficiency index \(n\) approximating \((V_t)_{t \geq 0}\) in the sense of the definition of part 4.

In the proof of the theorem we use the complex analysis in the Hardy space (see [21]). Let \((\lambda_k)_{1 \leq k \leq N}, \ N \leq +\infty,\) be a system of complex numbers satisfying the following properties,

\[
\text{Re}\lambda_k < 0, |\text{Im}\lambda_k| < R, 1 \leq k \leq N, \sum_{k=1}^{N} |\text{Re}\lambda_k| < +\infty, \quad (1)
\]

where \(R\) is some positive number. In this case the formula \(B(\lambda) = \prod_{k=1}^{N} \frac{\lambda + \lambda_k}{\lambda - \lambda_k}, \ \lambda \in \mathbb{C},\) defines an analitic function being regular in the semiplane \(\text{Re}\lambda > 0\) and equaling the unit by module on the imaginary axis. The function \(B(\lambda)\) is called the Blaschke product.

**Proposition 1.** Let the numbers \((\lambda_k)_{1 \leq k \leq N}\) satisfy the condition (1). Then the
Blaschke product $B(\lambda)$ constructed of $(\lambda_k)_{1 \leq k \leq N}$ can be estimated as follows:

$$|B(\lambda)| < C_1, \ |\lambda| > C_2, \ B(\lambda) = 1 - \frac{C_3}{\lambda} + o\left(\frac{1}{\lambda}\right), \ |\lambda| \rightarrow +\infty, \ \lambda \in \mathbb{C},$$

where $C_1, C_2$ and $C_3$ are some positive constants.

Proof.

$$\ln B(\lambda) = \sum_{k=1}^{N} \ln \frac{1 + \frac{\lambda}{\lambda_k}}{1 - \frac{\lambda}{\lambda_k}} = -\frac{2s}{\lambda} + o\left(\frac{1}{\lambda}\right), \ |\lambda| \rightarrow +\infty,$$

where $s = -\sum_{k=1}^{N} \text{Re}\lambda_k$, $0 < s < +\infty$ by the condition (1). $\triangle$

Let us consider a Hilbert space $\mathcal{K} = L_2(0, +\infty)$. Let $P_{[t_1, t_2]}$ designate a projection on a subspace of $\mathcal{K}$ consisting of functions $f(x) = 0$ for $0 < x < t_1$, $t_2 < x < +\infty$. Let us define an isometry $\Theta$ acting in $\mathcal{K}$ by the formula $\Theta = \mathcal{F}^{-1} B\mathcal{F}$, where $\mathcal{F}$ and $B$ are the Fourier transformation and an operator of the multiplication by the Blaschke product correspondently.

**Proposition 2.** Let the conditions of proposition 1 be hold.

Then $\Delta_{t, \delta} = P_{[t, t+\delta]} \Theta P_{[t, t+\delta]} - P_{[t, t+\delta]} \in s_2$, $0 < t, \delta < +\infty$, $||\Delta_{t, \delta}||_2 = O(\delta^{1/2})$, $\delta \rightarrow 0$.

Proof.

Fix $t \geq 0$. Let $\mu_{k, \delta} = -\frac{1}{2|k|} + i\frac{2\pi k}{\delta}$, $k \in \mathbb{Z}$, $\delta > 0$. Let us consider a family of functions $f_{k, \delta}(x) = \frac{(-2\text{Re}\mu_{k, \delta})^{1/2}}{e^{2\text{Re}\mu_{k, \delta}x} - e^{2\text{Re}\mu_{k, \delta}(t+\delta)x}} e^{\mu_{k, \delta}x}$, $t < x < t + \delta$, $f_{k, \delta}(x) = 0$, $0 < x < t$, $t + \delta < x < +\infty$, $k \in \mathbb{Z}$. The family $(f_{k, \delta})_{k \in \mathbb{Z}}$ is the Riesz basis of a Hilbert space $H = P_{[t, t+\delta]} \mathcal{K}$ (see [21]). So there is a bounded operator $V$ having bounded reversion in $\mathcal{K}$ such that the family $(V f_{k, \delta})_{k \in \mathbb{Z}}$ is an orthogonal basis of $H$. Therefore to prove proposition 2 it is sufficient to prove a convergence of the series $\sum_{k \in \mathbb{Z}} ||\Delta_{t, \delta} f_{k, \delta}||^2$ and to investigate its dependence on $\delta$. Let $f_{k, \delta}(1) = P_{[t, t+\delta]} f_{k, \delta}(1)$, $t < x < +\infty$, $f_{k, \delta}(1) = f_{k, \delta}(2) = 0$, $0 < x < t$, $k \in \mathbb{Z}$. Then $f_{k, \delta} = f_{k, \delta}(1) - f_{k, \delta}(2)$, $k \in \mathbb{Z}$ and $||\Delta_{t, \delta} f_{k, \delta}||^2 \leq 2(||\Delta_{t, \delta} f_{k, \delta}(1)||^2 + ||\Delta_{t, \delta} f_{k, \delta}(2)||^2)$, $k \in \mathbb{Z}$. By this way,

$$||\Delta_{t, \delta} f_{k, \delta}(1)||^2 = 2(||f_{k, \delta}(1) - \text{Re}(\Theta f_{k, \delta}(1), f_{k, \delta}(1))||, k \in \mathbb{Z}, \ i = 1, 2,$$

$$||f_{k, \delta}(1)||^2 = \frac{e^{2\text{Re}\mu_{k, \delta}t}}{(e^{2\text{Re}\mu_{k, \delta}t} - e^{2\text{Re}\mu_{k, \delta}(t+\delta)})} = -\frac{1}{2\text{Re}\mu_{k, \delta} \delta} + o(1), |k| \rightarrow +\infty,$$
\[ \|f^{(2)}_{k,\delta}\|^2 = \frac{e^{Re\mu_k(t+\delta)}}{e^{Re\mu_k t} - e^{Re\mu_k(t+\delta)}} = -\frac{1}{2Re\mu_k \delta} + o(1), \ |k| \to +\infty. \]

Using the Laplace transformation technics one can obtain

\[ (\Theta f^{(i)}_{k,\delta}, f^{(i)}_{k,\delta}) = B(\mu_k,\delta)\|f^{(i)}_{k,\delta}\|^2, \ i = 1,2. \]

It follows from proposition 1 that

\[ \|\Delta_{t,\delta} f^{(i)}_{k,\delta}\|^2 = \|f^{(i)}_{k,\delta}\|^2(1 - ReB(\mu_k,\delta)) = \frac{1}{2\delta|\mu_k,\delta|^2} + o\left(\frac{1}{2\delta|\mu_k,\delta|^2}\right) = \]

\[ \frac{\delta}{8\pi^2k^2} + o\left(\frac{\delta}{k^2}\right), \ |k| \to +\infty, \ \delta \to 0, \ i = 1,2, \]

and

\[ \|\Delta_{t,\delta}\|^2 \leq C \sum_{k \in \mathbb{Z}} \|\Delta_{t,\delta} f_{k,\delta}\|^2 \leq 2C \sum_{k \in \mathbb{Z}} (\|\Delta_{t,\delta} f^{(1)}_{k,\delta}\|^2 + \|\Delta_{t,\delta} f^{(2)}_{k,\delta}\|^2) = O(\delta), \ \delta \to 0, \]

where \( C \) is some positive constant, that implies \( \|\Delta_{t,\delta}\|_2 = O(\delta^{1/2}), \ \delta \to 0. \triangle \)

Proof of theorem 5.

Let the operator \( d \) be a generator of uniformly continuous semigroup of unitaries \((U_t)_{t \geq 0}\) being unitary part of some semigroup of isometries. In accordance with the Von Neummann theorem, given a skewhermitian operator \( d \) there is a bounded skewhermitian operator \( D \in \sigma_2 \) such that the skewhermitian operator \( d + D \) has a purely point spectrum. So the operator \( d + D \) is a generator of some uniformly continuous semigroup of unitaries \((V_t)_{t \geq 0}\) having a purely point spectrum. The semigroups \((U_t)_{t \geq 0}\) and \((V_t)_{t \geq 0}\) are known to be bonded by the relation \( V_t - U_t = \int_0^t U_{t-s}DV_s ds, \ t \geq 0, \) therefore \( V_t - U_t \in \sigma_2, \ t \geq 0 \) and \( \|V_{t+\delta} - U_{t+\delta} - V_t + U_t\|_2 = O(\delta), \ \delta \to 0, \ t \geq 0. \)

For the semigroup \((V_t)_{t \geq 0}\) is uniformly continuous its spectrum lies in a circle of radius \( R > 0 \) in the complex plane. Let \((i\mu_k)_{1 \leq k \leq N}\) be eigenvalues of the generator of \((V_t)_{t \geq 0}\) numbering in order of decreasing of its modules. By this way, \( |\mu_k| < R, \ 1 \leq k \leq N \leq +\infty. \) Let complex numbers \((\lambda_k)_{1 \leq k \leq N}\) be such that \( Im\lambda_k = -\mu_k, \ 1 \leq k \leq N, \) and the real parts collected in the accordance of the condition (1). The condition (1) allows to define the Blaschke product associated with \((\lambda_k)_{1 \leq k \leq N}\) (see above).
In what follows we suppose the deficiency index to be 1. It will prove the theorem
because every $C_0$-semigroup of isometries having nonzero deficiency index decom-
poses in an orthogonal sum of a $C_0$-semigroup of isometries having a deficiency index
1 and, probably, a $C_0$-semigroup of completely nonunitary isometries. We shall prove
the existence of a $C_0$-semigroup of isometries in a Hilbert space $\mathcal{K} = L_2(0, +\infty)$ that
is unitary equivalent to given a semigroup of isometries with a deficiency index 1 and a purely point spectrum of its unitary part consisting of numbers $(i\mu_k)_{1 \leq k \leq N}$.

Let $(S_t)_{t \geq 0}$ be a $C_0$-semigroup of shifts in a Hilbert space $\mathcal{K} = L_2(0, +\infty)$. Let
us consider a family of functions $f_n(x) = (-2Re\lambda_n)^{1/2}e^{\lambda_n x}$, $1 \leq n \leq N$. The
condition (1) implies an uncompleteness of the system $(f_n)_{1 \leq n \leq N}$ in $\mathcal{K}$ ( the condi-
tion of the convergence of the Blaschke product ). Thus a subspace $\mathcal{K}_1$ being a linear
envelope of $(f_n)_{1 \leq n \leq N}$ does not coincide with $\mathcal{K}$ and defines a subspace $\mathcal{K}_0$ being
invariant under an action of $(S_t)_{t \geq 0}$ and completely describing by the condition of
an orthogonality to all functions $f_n$ such that $\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1$ and the isometry
$\Theta : \mathcal{K} \to \mathcal{K}$, $\Theta = \mathcal{F}^{-1}B\mathcal{F}$, $\Theta S_t = S_t \Theta$, $t \geq 0$, where $\mathcal{F}$ and $B$ are the Fourier
transformation and an operator multiplying by the Blaschke product, defines $\mathcal{K}_0$ and
$\mathcal{K}_1$ by the formula $\mathcal{K}_0 = \Theta \mathcal{K}$. The semigroup $(S_t)_{t \geq 0}$ is intertwined by the operator
\(\Theta\) with its restriction on the subspace $\mathcal{K}_0$: $S_t|_{\mathcal{K}_0} \Theta = \Theta S_t$, $t \geq 0$. The isometric
operator $\Theta : \mathcal{K} \to \mathcal{K}$ sets an unitary map $\mathcal{K} \to \text{Ran} \Theta = \mathcal{K}_0$. Hence the semigroups
$(S_t)_{t \geq 0}$ and $(S_t\mid_{\mathcal{K}_0})_{t \geq 0}$ are unitarily equivalent such that the deficiency index of the
semigroup $(S_t\mid_{\mathcal{K}_0})_{t \geq 0}$ consisting of completely nonunitary isometries in $\mathcal{K}_0$ coincides
with the deficiency index of the semigroup $(S_t)_{t \geq 0}$ and equals 1.

Let the system of functions $(g_n)_{n \in \mathbb{N}}$ be obtained by a successive orthogonalization
of the system $(f_n)_{n \in \mathbb{N}}$. Let us determine a $C_0$-semigroup $(V_t)_{t \geq 0}$ of isometries in $\mathcal{K}$
as follows

$$V_t|_{\mathcal{K}_0} = S_t|_{\mathcal{K}_0}, \quad V_t g_n = e^{i\mu_n t}g_n, \quad t \geq 0, n \in \mathbb{N}. \quad (2)$$

We shall show that for isometries $V_t$, $t \geq 0$, describing in (2) the following
conditions hold, $V_t S^*_t - P_{[t, +\infty]} \in \sigma_2$, $\|V_t - S_t\|_2 \leq \|V_t S^*_t - P_{[t, +\infty]}\|_2 = O(t^{1/2})$, $t \to 0$.

Fix $t > 0$. We need to prove a convergence of the series $\sum_{n=1}^{+\infty} \|(V_t S^*_t - P_{[t, +\infty]}f_n\|^2$
for some orthogonal basis $(f_n)_{n \in \mathbb{N}}$ of $\mathcal{K}$. Choose for this purpose an arbitrary addition
of the system $(S_t g_n)_{n \in \mathbb{N}}$ up to an orthogonal basis of $\mathcal{K}$.

Notice that $V_t S^*_t - P_{[t, +\infty]} = P_{[0, t]} V_t S^*_t + (P_{[t, +\infty]} V_t S^*_t - P_{[t, +\infty]})$ and $(V_t S^*_t -$
\[ P_{[t, +\infty)}|_{\mathcal{K}_t} = 0, \] where \( \mathcal{K}_t \) designates an orthogonal addition of a linear envelope of vectors \((S_t g_n)_{n \in \mathbb{N}}\). An element \( S_t^* g_n \) belongs to a linear envelope of elements \( g_i, \ i = 1, n \), such that \((S_t^* g_n, g_n) = e^{\lambda_n t}, 1 \leq n \leq N\). By this way,

\[
\| (P_{[t, +\infty)} V_t S_t^* - P_{[t, +\infty)}) S_t g_n \|^2 = 1 + \| P_{[t, +\infty)} g_n \|^2 - 2\text{Re}(V_t g_n, S_t g_n) < 2(1 - \text{Re}(V_t g_n, S_t g_n)) = 2(1 - e^{-Re\lambda_n}), \ n \in \mathbb{N}.
\]

For the operator \( P_{[0,t]} V_t S_t^* \) we have the estimate using the Bessel inequality:

\[
\| P_{[0,t]} V_t S_t^* S_t g_n \|^2 = \| P_{[0,t]} g_n \|^2 = 1 - (P_{[t, +\infty)} g_n, g_n) < 1 - |(S_t^* g_n, g_n)|^2 = 1 - e^{-2Re\lambda_n t}, \ n \in \mathbb{N}.
\]

It follows from (3), (4) and (1) that

\[
\sum_{i=1}^{\infty} \| (V_t S_t^* - P_{[t, +\infty)} f_i \| = \sum_{n=1}^{\infty} (\| (P_{[t, +\infty)} V_t S_t^* - P_{[t, +\infty)}) S_t g_n \|^2 + \| (P_{[0,t]} V_t S_t^* S_t g_n \|^2)
\]

\[
= \sum_{n=1}^{\infty} (-4Re\lambda_n t + o(Re\lambda_n)) = O(t), \ \ t \to 0. \text{ Hence } \| V_t S_t^* - P_{[t, +\infty)} \|_2 = O(t^{1/2}), \ \ t \to 0.
\]

Notice that \( V_t - S_t = V_t|_{\mathcal{K}_1} - P_{\mathcal{K}_1} S_t|_{\mathcal{K}_1}, \ t \geq 0 \), and the \( C_0 \)-semigroup \((V_t|_{\mathcal{K}_1})_{t \geq 0}\) is uniformly continuous by the condition. The condition \( V_t - S_t \in s_2, \ t \geq 0, \ \| V_t - S_t \|_2 = O(t^{1/2}), \ t \to 0, \) and the uniform continuity of \((V_t|_{\mathcal{K}_1})_{t \geq 0}\) implies that the family \((V_t - S_t)_{t \geq 0}\) is continuous in \( \| \cdot \|_2 \). In fact, \( \| (V_{t+\delta} - S_{t+\delta} - V_t + S_t) \|_2 = \| P_{\mathcal{K}_1} (V_{t+\delta} - S_{t+\delta} - V_t + S_t) \|_2 \leq \| P_{\mathcal{K}_1} (\delta - I) (V_t - S_t) P_{\mathcal{K}_1} \|_2 + \| (V_t - S_t) (\delta - S_t) \|_2 \leq \| P_{\mathcal{K}_1} (\delta - I) P_{\mathcal{K}_1} \|_2 \| V_t - S_t \|_2 + \| V_t - S_t \|_2 (1 + \| V_t - S_t \|_2) \to 0, \ \delta \to 0.
\]

Now we get for operators \( \Delta_t = V_t - S_t, \ t \geq 0 \) the following estimations:

\[
\Delta_t \in s_2, \ \| \Delta_t + \delta - \Delta_t \|_2 \to 0, \ \| \Delta_t \|_2 = O(\delta^{1/2}), \ \delta \to 0, \ t \geq 0.
\]

To complete the proof we need to show that there are \( C_0 \)-semigroups \((V'_t)_{t \geq 0}\) and \((S'_t)_{t \geq 0}\) being unitary dilations of \((V_t)_{t \geq 0}\) and \((S_t)_{t \geq 0}\) which satisfy the conditions \( V'_t - S'_t \in s_2, \ V'_t S'_t|_{\mathcal{K} \oplus \mathcal{K}} = I, \ t \geq 0 \). Let us define unitary dilations in a Hilbert space \( \mathcal{K}' = \mathcal{K} \oplus \mathcal{K} \) by the formula

\[
S'_t(f \oplus g)(x) = ((S_t f)(x) + (P_{[0,t]} g)(t - x)) \oplus (S_t^* g)(x),
\]

\[
V'_t(f \oplus g)(x) = ((V_t f)(x) + (\Theta(P_{[0,t]} g)(t - \cdot))(x)) \oplus (S_t^* g)(x),
\]

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Now the result follows from (5) and proposition 2. △

6. The class of the cocycle conjugacy of the quasifree flow of shifts on the hyperfinite factor $\mathcal{M}_\nu$.

Let $V = (V_t)_{t \geq 0}$ be a $C_0$-semigroup of isometries having uniformly continuous unitary part (see part 5). Then theorem 5 implies the existence of a $C_0$-semigroup of completely nonunitary isometries $S = (S_t)_{t \geq 0}$ in $\mathcal{K}$ approximating $V$ in the sense of part 4. Semigroups $V$ and $S$ have same deficiency indeces. If $(S_t)_{t \geq 0}$ is a $C_0$-semigroup of completely nonunitary isometries, then the quasifree semigroup $(B_\nu(S_t))_{t \geq 0}$ consists of shifts. It follows from theorem 4 that the following assertion holds.

**Theorem 6 ([2-5]).** Let a $C_0$-semigroup of isometries $(V_t)_{t \geq 0}$ with the deficiency index $n > 0$ have uniformly continuous unitary part.

Then there is a $C_0$-semigroup of completely nonunitary isometries $(S_t)_{t \geq 0}$ with the deficiency index $n$ such that the quasifree semigroup $(B_\nu(V_t))_{t \geq 0}$ is cocycle conjugate to the flow of shifts $(B_\nu(S_t))_{t \geq 0}$.

Theorem 6 describes the class of the cocycle conjugacy of the quasifree flow of shifts on the hyperfinite factor $\mathcal{M}_\nu$, so one can reformulate it as follows,

**Theorem 6’.** Let $V = (V_t)_{t \geq 0}$ be a $C_0$-semigroup of isometries in a Hilbert space $\mathcal{K}$ having uniformly continuous unitary part and $S = (S_t)_{t \geq 0}$ be a $C_0$-semigroup of completely nonunitary isometries in $\mathcal{K}$ with the same deficiency index as in $V$.

Then there is an unitary $U$ in $\mathcal{K}$ such that the quasifree semigroup $(B_\nu(UV_tU^*))_{t \geq 0}$ is cocycle conjugate to the flow of shifts $(B_\nu(S_t))_{t \geq 0}$.

Notice that there is an analogue of the result of theorem 6 for discrete quasifree semigroups, see [5,7].

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**Appendix. The commutant of $\mathcal{M}_R$.**

Note that the operators

$$b(f) = \Gamma \otimes \Gamma \pi(a(0 \oplus f)), b^*(f) = \Gamma \otimes \Gamma \pi(a^*(0 \oplus f)), \; f \in \mathcal{K},$$
belong to a commutant of the hyperfinite factor $\mathcal{M}_R$. Therefore its generate the $W^*$-algebra $\mathcal{N} \subset \mathcal{M}'_R$. The formula $Sx\Omega \otimes \Omega = x^*\Omega \otimes \Omega$, $x \in \mathcal{M}_\nu$ correctly defines an antilinear operator $S$ in $\mathcal{H}$ (see [4]). Let $S = J \Delta^{1/2}$ be a polar decomposition of $S$. Then an antiisometrical part $J$ of the operator $S$ is antiunitary and is called a modular involution of $\mathcal{M}_R$. The linear (unbounded) positive operator $\Delta$ is called a modular operator (see [14]). Simple calculation gives the following formula for $J$,

$$Jf_1 \Lambda \ldots A f_n \otimes g_1 \Lambda \ldots A g_m = Jg_m \Lambda \ldots A Jg_1 \otimes J f_n \Lambda \ldots A J f_1, \quad J \Omega \otimes \Omega = \Omega \otimes \Omega.$$ 

By this way,

$$J \pi(a(f \oplus 0))J = b^*(f), \quad f \in \mathcal{K},$$

therefore $J\mathcal{M}_RJ = \mathcal{M}'_R$ and $\mathcal{N} = \mathcal{M}'_R$.

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