The functional equations of the Selberg and Ruelle zeta functions for non-unitary twists

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Abstract. We consider the dynamical zeta functions of Ruelle and Selberg, defined on one complex variable \( s \), on a compact hyperbolic odd dimensional manifold \( X \). Earlier research has proved that these dynamical zeta functions admit a meromorphic continuation to the whole complex plane. In this paper, we provide functional equations for them, relating their values at \( s \) with those at \(-s\). We prove also a determinant representation of the zeta functions, using the regularized determinant of certain twisted differential operators. Further, we investigate the connection between the Ruelle zeta function at the central point \( s = 0 \) with the refined analytic torsion as it is introduced by Braverman and Kappeler.

Keywords: Twisted Dirac operator, eta function, Ruelle zeta function, Selberg zeta function, regularized determinant, refined analytic torsion.

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1 Introduction

The Selberg and Ruelle zeta functions are dynamical zeta functions, which can associated with the geodesic flow of a compact hyperbolic manifold $X$. In particular, they are defined in terms of the lengths $l(\gamma)$ of the prime closed geodesics on $X$. They are defined by the following infinite products.

- Selberg zeta function
  \[ Z(s; \sigma, \chi) := \prod_{[\gamma] \neq e} \prod_{k=0}^{\infty} \det \left( \text{Id} - \chi(\gamma) \otimes \sigma(m_\gamma) \otimes S^k(\text{Ad}(m_\gamma a_\gamma)|_\pi) \right) e^{-(s+|\rho|)l(\gamma)}; \]
  \hspace{1cm} (1.1)

- Ruelle zeta function
  \[ R(s; \sigma, \chi) := \prod_{[\gamma] \neq e} \det \left( \text{Id} - \chi(\gamma) \otimes e^{-s l(\gamma)} \right)^{(-1)^{d-1}}, \]
  \hspace{1cm} (1.2)

for the complex variable $s$ in some half-plane of $\mathbb{C}$ (see Definitions 2.1 and 2.2). We consider locally symmetric hyperbolic manifolds of odd dimension $d$, obtained as $X = \Gamma \backslash \mathbb{H}^d$, where $\Gamma$ is a discrete torsion free cocompact subgroup of $\text{SO}^0(d, 1)$. One can observe from (1.1) and (1.2) that we associate these zeta functions with finite dimensional representations $\chi$ of $\Gamma$.

For unitary representations $\chi$, these zeta functions has been studied by Bunke and Olbrich in [BO95] for all the locally symmetric spaces of real rank 1. They have proved that the zeta functions admit a meromorphic continuation to the whole complex plane and in addition derive functional equations for them.

In [Spi15a] and [Spi15b], the zeta functions of Ruelle and Selberg associated with non-unitary representations of $\Gamma$ are defined. It is proved also that they admit a meromorphic continuation to $\mathbb{C}$, using tools from spectral analysis and namely the Selberg trace formula for non-unitary twists from [Müll].

In this paper, we follow [BO95] and obtain similar results, i.e., functional equations for the Selberg and Ruelle zeta functions, relating their values at $s$ with those at $-s$. Moreover, we prove a determinant formula, which provides a connection between the regularized determinant of certain twisted Dirac and Laplace-type operators with the zeta functions.
One of our goals is to find an analogue to the theorem of Fried in [Fri86]. In
his paper [Fri86], Fried considered the standard representation of \( M = \text{SO}(d - 1) \) on
\( \Lambda_j \mathbb{C}^{d-1} \) and an orthogonal representation \( \rho: \Gamma \to \text{O}(m) \) of \( \Gamma \). Using the Selberg trace
formula for the heat operator \( e^{-t\Delta_j} \), where \( \Delta_j \) is the Hodge Laplacian on \( j \)-forms on
\( X \), he managed to prove the meromorphic continuation of the zeta functions to the
whole complex plane \( \mathbb{C} \), as well as functional equations for the Selberg zeta function
([Fri86, p.531-532]). He proved also that, in the case of \( d = \text{dim}(X) \) being odd and
\( \rho \) acyclic, i.e., the twisted cohomology groups \( H^*(X; \rho) \) vanish for all \( j \), the Ruelle
zeta function
\[
R(s; \rho) := \prod_{[\gamma] \neq e, \, [\gamma] \text{ prime}} \det(\text{Id} - \rho(\gamma)e^{-s\text{sl}(\gamma)}),
\]
which converges for \( \text{Re}(s) > d - 1 \), admits a meromorphic extension to \( \mathbb{C} \) an is
holomorphic at \( s = 0 \). Futher, for \( \varepsilon = (-1)^{d-1} \)
\[
|R(0; \rho)^\varepsilon| = T_X(\rho)^2,
\]
where \( T_X(\rho) \) is the Ray-Singer analytic torsion defined in [RS71]. This result is
of interest, since it connects the Ruelle zeta function evaluated at zero with the
analytic torsion under certain assumptions. In our case, we have to consider the
refined analytic torsion, as it is introduced in [BK08]. Once we have the determinant
formula at hand, we concern ourselves with relating the refined analytic torsion and
the Ruelle zeta function at zero.

We state our main results. We consider the universal covering \( G = \text{Spin}(d, 1) \)
of \( \text{SO}^0(d, 1) \). Let \( G = KAN \) be the standard Iwasawa decomposition of \( G \). Let
\( M := \text{Centr}_K(A) \) be the centralizer of \( A \) in \( K \) and \( M' = \text{Norm}_K(A) \) the normalizer
of \( A \) in \( K \). We define the restricted Weyl group by the quotient \( W_A := M'/M \). Let \( \tilde{M} \)
be the set of equivalent classes of irreducible unitary representations of \( M \). Let \( \sigma \in \tilde{M} \). We will distinguish the following two cases:

- **case (a):** \( \sigma \) is invariant under the action of the restricted Weyl group \( W_A \).
- **case (b):** \( \sigma \) is not invariant under the action of the restricted Weyl group \( W_A \).

In case (b), we define the also the symmetrized \( S(s; \sigma, \chi) \) and super zeta \( Z^s(s; \sigma, \chi) \)
functions and the super Ruelle zeta function \( R^s(s; \sigma, \chi) \) (Definitions 2.3, 2.4 and 2.5).

**Theorem 1.1.** The Selberg zeta function \( Z(s; \sigma, \chi) \) satisfies the functional equation
\[
\frac{Z(s; \sigma, \chi)}{Z(-s; \sigma, \chi)} = \exp \left( -4\pi \text{dim}(V_{\chi}) \text{Vol}(X) \int_0^s P_\sigma(r)dr \right),
\]
where \( P_\sigma \) denotes the Plancherel polynomial associated with \( \sigma \in \tilde{M} \).
Theorem 1.2. The symmetrized zeta function $S(s; \sigma, \chi)$ satisfies the functional equation
\[ S(s; \sigma, \chi) = \exp \left( - 8\pi \dim(V_\chi) \Vol(X) \int_0^s P_\sigma(r) dr \right), \]
where $P_\sigma$ denotes the Plancherel polynomial associated with $\sigma \in \hat{M}$.

Theorem 1.3. The super zeta function $Z^s(s, \sigma, \chi)$ satisfies the functional equation
\[ Z^s(s; \sigma, \chi)Z^s(-s; \sigma, \chi) = e^{2\pi i \eta(0, D^s_\chi(\sigma))}, \]
where $\eta(0, D^s_\chi(\sigma))$ denotes the eta invariant associated with the Dirac operator $D^s_\chi(\sigma)$. Furthermore,
\[ Z^s(0; \sigma, \chi) = e^{\pi i \eta(0, D^s_\chi(\sigma))}. \]

Theorem 1.4. The Ruelle zeta function satisfies the functional equation
\[ R(s; \sigma, \chi)R(-s; \sigma, \chi) = \exp \left( - 4\pi(d + 1) \dim(V_\sigma) \dim(V_\chi) \Vol(X)s \right). \]

Theorem 1.5. The super Ruelle zeta function associated with a non-Weyl invariant representation $\sigma \in \hat{M}$ satisfies the functional equation
\[ R^s(s; \sigma, \chi)R^s(-s; \sigma, \chi) = e^{2\pi i \eta(D^s_{p,\chi}(\sigma))}, \]
where $\eta(D^s_{p,\chi}(\sigma))$ denotes the eta invariant of the twisted Dirac operator $D^s_{p,\chi}(\sigma)$. Moreover, the following equation holds
\[ R(s; \sigma, \chi)R(-s; w_\sigma \chi) = e^{i\pi \eta(D^s_{p,\chi}(\sigma))} \exp \left( - 4\pi(d + 1) \dim(V_\sigma) \dim(V_\chi) \Vol(X)s \right). \]

Theorem 1.6. Let $\det(A^2_\chi(\sigma) + s^2)$ be the regularized determinant associated to the operator $A^2_\chi(\sigma) + s^2$. Then,

1. case(a) the Selberg zeta function has the representation
\[ Z(s; \sigma, \chi) = \det(A^2_\chi(\sigma) + s^2) \exp \left( - 2\pi \dim(V_\chi) \Vol(X) \int_0^s P_\sigma(t) dt \right). \]

2. case(b) the symmetrized zeta function has the representation
\[ S(s; \sigma, \chi) = \det(A^2_\chi(\sigma) + s^2) \exp \left( - 4\pi \dim(V_\chi) \Vol(X) \int_0^s P_\sigma(t) dt \right). \]
Proposition 1.7. The Ruelle zeta function has the representation

- case (a)

\[
R(s; \sigma, \chi) = \prod_{p=0}^{d} \det(A_{\chi}^p(\sigma_p \otimes \sigma) + (s + \rho - \lambda)^2)^{(-1)^p} \exp \left( -2\pi(d + 1) \dim(V_{\chi}) \dim(V_{\sigma}) \Vol(X)s \right).
\]

- case (b)

\[
R(s; \sigma, \chi)R(s; w\sigma, \chi) = \prod_{p=0}^{d} \det(A_{\chi}^p(\sigma_p \otimes \sigma) + (s + \rho - \lambda)^2)^{(-1)^p} \exp \left( -4\pi(d + 1) \dim(V_{\chi}) \dim(V_{\sigma}) \Vol(X)s \right).
\]

2 Preliminaries

We first introduce our geometrical setting. Let \(X\) be a compact hyperbolic locally symmetric manifold with universal covering the real hyperbolic space \(\mathbb{H}^d\), obtained as follows. We consider the universal coverings \(G = \text{Spin}(d,1)\) of \(O(0,d,1)\) and \(K = \text{Spin}(d)\) of \(SO(d)\), respectively. We set \(\tilde{X} := G/K\).

Let \(\mathfrak{g}, \mathfrak{k}\) be the Lie algebras of \(G\) and \(K\), respectively. Let \(\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}\) be the Cartan decomposition of \(\mathfrak{g}\). We denote by \(\Theta\) the Cartan involution of \(G\) and \(\theta\) be the differential of \(\Theta\) at \(e_G = e\), which is the identity element of \(G\). Let \(\mathfrak{a}\) be a Cartan subalgebra of \(\mathfrak{p}\), i.e., a maximal abelian subalgebra of \(\mathfrak{p}\). There exists a canonical isomorphism \(T_{eK} \cong \mathfrak{p}\).

We consider the subgroup \(A\) of \(G\) with Lie algebra \(\mathfrak{a}\). Let \(M := \text{Centr}_K(A)\) be the centralizer of \(A\) in \(K\). Then, \(M = \text{Spin}(d-1)\) or \(M = \text{SO}(d-1)\). Let \(\mathfrak{m}\) be its Lie algebra and \(\mathfrak{b}\) a Cartan subalgebra of \(\mathfrak{m}\). Let \(\mathfrak{h}\) be a Cartan subalgebra of \(\mathfrak{g}\). We consider the complexifications \(\mathfrak{g}_C := \mathfrak{g} \oplus i\mathfrak{g}\), \(\mathfrak{h}_C := \mathfrak{h} \oplus i\mathfrak{h}\) and \(\mathfrak{m}_C := \mathfrak{m} \oplus i\mathfrak{m}\). Let \(G = KAN\) be the standard Iwasawa decomposition of \(G\). Let \(\Delta^+(\mathfrak{g}, \mathfrak{a})\) be the set of positive roots of the system \((\mathfrak{g}, \mathfrak{a})\). Then, \(\Delta^+(\mathfrak{g}, \mathfrak{a}) = \{\alpha\}\).

Let \(B(X, Y)\) be the Killing form on \(\mathfrak{g} \times \mathfrak{g}\) defined by \(B(X, Y) = \text{Tr}(\text{ad}(X) \circ \text{ad}(Y))\). It is a symmetric bilinear form. Let \(\langle Y_1, Y_2 \rangle_0 := \frac{1}{2(d-1)} B(Y_1, Y_2), \quad Y_1, Y_2 \in \mathfrak{g}\) be the inner product on \(\mathfrak{g}\), induced by the Killing form \(B\). The restriction of \(\langle \cdot , \cdot \rangle_0\) to \(\mathfrak{p}\) defines an inner product on \(\mathfrak{p}\) and hence induces a \(G\)-invariant riemannian metric.
on $\tilde{X}$, which has constant curvature $-1$. Then, $\tilde{X}$, equipped with this metric, is isometric to $\mathbb{H}^d$.

Let $\Gamma \subset G$ be a lattice, i.e., a discrete subgroup of $G$ such that $\text{Vol}(\Gamma \backslash G) < \infty$. $\Gamma$ acts properly discontinuously on $\tilde{X}$ and $X := \Gamma \backslash \tilde{X}$ is a locally symmetric space of finite volume. We assume that $\Gamma$ is torsion free. i.e., there exists no $\gamma \in \Gamma$ with $\gamma \neq e$ such that for $k = 2, 3, \ldots$, $\gamma^k = e$. Then, $X$ is a locally symmetric manifold. If in addition $\Gamma$ is cocompact, then $X$ is a locally symmetric compact hyperbolic manifold of odd dimension $d$.

We will give here the definition of the twisted Ruelle and Selberg zeta function associated with the geodesic flow on the sphere vector bundle $S(X)$ of $X$. We consider the identification

$$S(X) = \Gamma \backslash G/M.$$  

The dynamical zeta functions provide information about the lengths of the closed geodesics on our manifold. It is a well known fact ([GKM68]) that there is a 1-1 correspondence between the closed geodesics on a manifold $X$ with negative sectional curvature and the non-trivial conjugacy classes of the fundamental group $\pi_1(X)$ of $X$. Let $\gamma \in \Gamma$, with $\gamma \neq e$ and $\gamma$ hyperbolic. Then, by [Wal76, Lemma 6.5] there exist a $g \in G$, a $m_\gamma \in M$, and an $a_\gamma \in A^+$, such that $g^{-1} \gamma g = m_\gamma a_\gamma$. The element $m_\gamma$ is determined up to conjugacy classes in $M$, and the element $a_\gamma$ depends only on $\gamma$. Since $\Gamma$ is a cocompact subgroup of $G$, we realize every element $\gamma \in \Gamma - \{e\}$ as hyperbolic. We denote by $c_\gamma$ the closed geodesic on $X$, associated with the hyperbolic conjugacy class $[\gamma]$. We denote also by $l(\gamma)$ the length of $c_\gamma$. An element $\gamma \in \Gamma$ is called primitive if there exists no $n \in \mathbb{N}$ with $n > 1$ and $\gamma_0 \in \Gamma$ such that $\gamma = \gamma_0^n$. We associate to a primitive element $\gamma_0 \in \Gamma$ a prime geodesic on $X$. The prime geodesics correspond to the periodic orbits of minimal length.

Let $\hat{K}, \hat{M}$ be the sets of equivalent classes of irreducible unitary representations of $K$ and $M$, respectively. We define the dynamical zeta functions in terms of irreducible unitary representations of $M$ and arbitrary finite dimensional representations of $\Gamma$.

**Definition 2.1.** Let $\chi : \Gamma \to \text{GL}(V_\chi)$ be a finite dimensional representation of $\Gamma$ and $\sigma \in \hat{M}$. The twisted Selberg zeta function $Z(s; \sigma, \chi)$ for $X$ is defined by the infinite product

$$Z(s; \sigma, \chi) := \prod_{\substack{[\gamma] \neq e \\text{prime} \\\text{[\gamma]}}} \prod_{k=0}^{\infty} \det \left( \text{Id} - (\chi(\gamma) \otimes \sigma(m_\gamma)) \otimes S^k(\text{Ad}(m_\gamma a_\gamma|\pi)) \right) e^{- (s \gamma + |\rho|) l(\gamma)},$$

(2.1)
where $s \in \mathbb{C}$, $\overline{\mathfrak{p}} = \theta \mathfrak{n}$ is the sum of the negative root spaces of $\mathfrak{a}$, $S^k(\text{Ad}(m_\gamma a_\gamma))_\pi$ denotes the $k$-th symmetric power of the adjoint map $\text{Ad}(m_\gamma a_\gamma)$ restricted to $\overline{\mathfrak{p}}$, and $\rho$ is defined by
\[
\rho := \frac{1}{2} \sum_{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})} \dim(\mathfrak{g}_\alpha) \alpha.
\]

By [Spi15a, Proposition 3.4], there exists a positive constant $c$, such that the infinite product in (2.1) converges absolutely and uniformly on compact subsets of the half-plane $\text{Re}(s) > c$.

**Definition 2.2.** Let $\chi : \Gamma \rightarrow \text{GL}(V_\chi)$ be a finite dimensional representation of $\Gamma$ and $\sigma \in \hat{M}$. The twisted Ruelle zeta function $R(s; \sigma, \chi)$ for $X$ is defined by the infinite product
\[
R(s; \sigma, \chi) := \prod_{[\gamma] \neq e \text{ prime}} \det \left( \text{Id} - \chi(\gamma) \otimes \sigma(m_\gamma) e^{-sl(\gamma)} \right)^{(-1)^{d-1}}. \tag{2.2}
\]

By [Spi15a, Proposition 3.5], there exists a positive constant $r$, such that the infinite product in (2.2) converges absolutely and uniformly on compact subsets of the half-plane $\text{Re}(s) > r$.

Let $M' = \text{Norm}_K(A)$ be the normalizer of $A$ in $K$. We define the restricted Weyl group by the quotient $W_A := M'/M$. Then, $W_A$ has order 2. Let $w \in W_A$ be a non-trivial element of $W_A$, and $m_w$ a representative of $w$ in $M'$. The action of $W_A$ on $\hat{M}$ is defined by
\[
(w\sigma)(m) := \sigma(m_w^{-1}mm_w), \quad m \in M, \sigma \in \hat{M}.
\]

We have already associated the Selberg and Ruelle zeta functions with irreducible representations $\sigma$ of $M$. These representations are chosen precisely to be the representations arising from restrictions of representations of $K$. Let $i^* : R(K) \rightarrow R(M)$ be the pullback of the embedding $i : M \hookrightarrow K$. We will distinguish the following two cases:

- **case (a):** $\sigma$ is invariant under the action of the restricted Weyl group $W_A$.
- **case (b):** $\sigma$ is not invariant under the action of the restricted Weyl group $W_A$.

In case (b), we define the following twisted zeta functions.

**Definition 2.3.** Let $\chi : \Gamma \rightarrow \text{GL}(V_\chi)$ be a finite dimensional representation of $\Gamma$ and $\sigma \in \hat{M}$. The symmetrized zeta function $Z(s; \sigma, \chi)$ for $X$ is defined by
\[
S(s; \sigma, \chi) := Z(s; \sigma, \chi)Z(ws; \sigma, \chi), \tag{2.3}
\]

where $w$ is a non-trivial element of the restricted Weyl group $W_A$. 

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Definition 2.4. Let $\chi: \Gamma \to \text{GL}(V_\chi)$ be a finite dimensional representation of $\Gamma$ and $\sigma \in \hat{M}$. The super zeta function $Z(s; \sigma, \chi)$ for $X$ is defined by

$$Z^s(s; \sigma, \chi) := \frac{Z(s; \sigma, \chi)}{Z(ws; \sigma, \chi)},$$

(2.4)

where $w$ is a non-trivial element of the restricted Weyl group $W_A$.

Definition 2.5. Let $\chi: \Gamma \to \text{GL}(V_\chi)$ be a finite dimensional representation of $\Gamma$ and $\sigma \in \hat{M}$. The super Ruelle zeta function $R^s(s; \sigma, \chi)$ for $X$ is defined by

$$R^s(s; \sigma, \chi) := \frac{R(s; \sigma, \chi)}{R(s; w\sigma, \chi)},$$

(2.5)

where $w$ is a non-trivial element of the restricted Weyl group $W_A$.

3 The eta function of the twisted Dirac operator

In this section, we will define the eta function associated with the twisted Dirac operator $D^\chi_\sharp(\sigma)$, as it is introduced in [Spi15b]. Since the twisted Dirac operator is an elliptic first order differential operator, but no longer self adjoint, we follow the definitions and the notions from [BK08] about the $\theta$-calculus of elliptic operators.

We recall at first the definition of the twisted Dirac operator from [Spi15b]. Let $\sigma \in \hat{M}$ be an irreducible representation of $M$. Let $s$ be the spin representation of $K$. Since $d - 1$ is an even integer, $s$ splits into two irreducible half-spin representations $(s^+, S^+), (s^-, S^-)$ of $M$. Let $\text{Cl}(p)$ be the Clifford algebra of $p$ with respect to the inner product $\langle \cdot, \cdot \rangle_0$, as it is defined in Section 2, restricted to $p$. Let $\cdot: p \otimes S \to S$ be the Clifford multiplication on $p \otimes S$. We consider the connection $\nabla$ in $\text{Cl}(p)$, induced by the canonical connection in the tangent frame bundle of $X$. Let $L$ be any bundle of left modules over $\text{Cl}(p)$ over $\tilde{X}$, i.e., a spinor bundle over $\tilde{X}$. We lift the connection $\nabla$ in $L$ and obtain a connection also denoted by $\nabla$. The Dirac operator $D: C^\infty(X, L) \to C^\infty(X, L)$ locally is defined as

$$Df \equiv \sum_{i=1}^d e_i \cdot \nabla e_i f,$$

where $(e_1, \ldots, e_d)$ is a local orthonormal frame for $T_x X, x \in X$. The operator $D$ is an elliptic ([LM89, Lemma 5.1]), formally self-adjoint ([LM89, Proposition 5.3]) operator of first order.
We want to define twisted Dirac operators acting on smooth sections of vector bundles associated with the representations $\sigma$ of $M$ and arbitrary representations $\chi$ of $\Gamma$. By [BO95, Proposition 1.1, (3)], there exists an unique element $\tau(\sigma) \in \tilde{K}$ and a splitting $s \otimes \tau(\sigma) = \tau^+(\sigma) \oplus \tau^-(\sigma)$ where $\tau^+(\sigma), \tau^-(\sigma) \in R(\tilde{K})$ such that $\sigma + w\sigma = \iota^*(\tau^+(\sigma) - \tau^-(\sigma))$ We define the representation $\tau_s(\sigma)$ of $K$ by $\tau_s(\sigma) := s \otimes \tau(\sigma)$, with representation space $V_{\tau_s(\sigma)} = S \otimes V_{\tau(\sigma)}$; where $V_{\tau(\sigma)}$ is the representation space of $\tau(\sigma)$. We consider the homogeneous vector bundle $\tilde{E}_{\tau(\sigma)} := G \times_{\tau(\sigma)} V_{\tau(\sigma)}$ over $\tilde{X}$. The vector bundle $\tilde{E}_{\tau_s(\sigma)} := \tilde{E}_{\tau(\sigma)} \otimes S$ over $\tilde{X}$ carries a connection $\nabla_{\tau_s(\sigma)}$, defined by the formula

$$\nabla_{\tau_s(\sigma)} = \nabla_{\tau(\sigma)} \otimes 1 + 1 \otimes \nabla,$$

where $\nabla_{\tau(\sigma)}$ denotes the canonical connection in $\tilde{E}_{\tau(\sigma)}$. We extend the Clifford multiplication by requiring that it acts on $V_{\tau_s(\sigma)} = S \otimes V_{\tau(\sigma)}$ as follows.

$$e \cdot (\phi \otimes \psi) = (e \cdot \phi) \otimes \psi, \quad e \in \text{Cl}(p), \phi \in S, \psi \in V_{\tau(\sigma)}.$$

We define locally the Dirac operator $\tilde{D}(\sigma)$ acting on $C^\infty(\tilde{X}, V_{\tau_s(\sigma)})$ by

$$\tilde{D}(\sigma)f = \sum_{i=1}^d e_i \cdot \nabla_{e_i}\tau_{\sigma}f,$$

where $(e_1, \ldots, e_d)$ is local orthonormal frame for $\tilde{X}$ and $f \in C^\infty(\tilde{X}, V_{\tau_s(\sigma)})$. The space of smooth sections $C^\infty(\tilde{X}, V_{\tau_s(\sigma)})$ can be identified with $C^\infty(G; \tau_s(\sigma))$ as in [Spi15a, equation (5.1)].

Let now $\chi : \Gamma \to \text{GL}(V_{\chi})$ be an arbitrary finite dimensional representation of $\Gamma$. Let $E_{\chi}$ be the associated flat vector bundle over $X$. Let $E_{\tau_s(\sigma)} := \Gamma\tilde{E}_{\tau_s(\sigma)}$ be the locally homogeneous vector bundle over $X$. We consider the product vector bundle $E_{\tau_s(\sigma)} \otimes E_{\chi}$ over $X$ and equip this bundle with the product connection $\nabla^E_{\tau_s(\sigma) \otimes E_{\chi}}$ defined by

$$\nabla^E_{\tau_s(\sigma) \otimes E_{\chi}} = \nabla^{E_{\tau_s(\sigma)}} \otimes 1 + 1 \otimes \nabla^{E_{\chi}}.$$

We consider the Clifford multiplication on $(V_{\tau_s(\sigma)} \otimes V_{\chi})$ by requiring that it acts only on $V_{\tau_s(\sigma)}$. Then the twisted Dirac operator $D^t_{\chi}(\sigma)$ associated with $\nabla^{E_{\tau_s(\sigma)} \otimes E_{\chi}}$ is defined locally by

$$D^t_{\chi}(\sigma)\phi = \sum_{i=1}^d e_i \cdot \nabla_{e_i}^{E_{\tau_s(\sigma)} \otimes E_{\chi}}(\phi),$$

where $\phi \in C^\infty(U, E_{\tau_s(\sigma)} \otimes E_{\chi}|U)$ and $U$ open subset of $X$ such that $E_{\chi}|U$ is trivial. If we consider the pullbacks $\tilde{E}_{\tau_s(\sigma)}$, $\tilde{E}_{\chi}$ to $\tilde{X}$ of $E_{\tau_s(\sigma)}$, $E_{\chi}$, respectively, then, $\tilde{E}_{\chi} \cong$
\( \tilde{X} \times V_\chi \) and \( C(\tilde{X}, \tilde{E}_{r_\chi} \otimes \tilde{E}_\chi) \cong C(\tilde{X}, \tilde{E}_{r_\chi}) \otimes V_\chi \). With respect to this isomorphism, it follows that

\[ \tilde{D}_\chi^2(\sigma) = \tilde{D}(\sigma) \otimes \text{Id}_{V_\chi}. \]

We recall the definition of the operator \( A^\chi_\chi(\sigma) \) acting on \( C^\infty(X, E(\sigma) \otimes E_\chi) \) from [Spi15a, equation 5.26].

\[ A^\chi_\chi(\sigma) := \bigoplus_{m_\tau(\sigma) \neq 0} A^\chi_{r,\chi} + c(\sigma), \]

where

\[ c(\sigma) := -|\rho|^2 - |\rho_m|^2 + |\nu_\tau + \rho_m|^2, \]

\( \rho_m := \frac{1}{2} \sum_{\alpha \in \Delta^+(m_c, b)} \alpha \), and \( \nu_\tau \) denotes the highest weight of \( \sigma \). The operator \( A^\chi_{r,\chi} \) is induced by the twisted Bochner-Laplace operator \( \Delta^\chi_{r,\chi} \) and acts on smooth sections of the twisted vector bundle \( E(\sigma) \otimes E_\chi \) (for further details for the operator \( A^\chi_\chi(\sigma) \) see [Spi15b, p. 27-28] and [Spi15b, p. 13-14]). The locally homogeneous vector bundle \( E(\sigma) \) associated with \( \tau \) is of the form \( E(\sigma) = \bigoplus_{m_\tau(\sigma) \neq 0} \chi_{E_{r_\chi}} \), where \( E_{r_\chi} \) is the locally homogeneous vector bundle associated with \( \tau \in \hat{K} \). The two vector bundles \( E_{r_\chi} \) and \( E(\sigma) \) coincide up to a \( \mathbb{Z}_2 \) grading. By [Spi15b, equation (4.8)], the Parthasarathy formula generalizes as

\[ (D^\chi_{\tau}(\sigma))^2 = A^\chi_\chi(\sigma). \]

The principal symbol \( \sigma_{D^\chi_{\tau}(\sigma)}(x, \xi) \) of the twisted Dirac operator \( D^\chi_{\tau}(\sigma) \) is given by

\[ \sigma_{D^\chi_{\tau}(\sigma)}(x, \xi) = (i\xi) \otimes \text{Id}_{(V_{r_\chi} \otimes V_\chi)_x}, \]

\( x \in X, \xi \in T^*_XX, \xi \neq 0 \) and therefore it has nice spectral properties. Namely, by [Spi15b, Lemma 5.6], its spectrum is discrete and contained in a translate of a positive cone \( C \subset \mathbb{C} \). The square of the twisted Dirac operator \( (D^\chi_{\tau}(\sigma))^2 \) acting on smooth sections of \( E_{r_\chi} \otimes E_\chi \) is a second order elliptic differential operator but no longer self-adjoint. Nevertheless, its principal symbol is given by \( \sigma_{(D^\chi_{\tau}(\sigma))^2}(x, \xi) = \|\xi\|^2 \otimes \text{Id}_{(V_{r_\chi} \otimes V_\chi)_x} \), for \( x \in X, \xi \in T^*_XX, \xi \neq 0 \), and hence has nice spectral properties as well. By [Spi15a, Lemma 5.8], its spectrum is discrete and contained in a translate of a positive cone \( C \subset \mathbb{C} \) (see Figure 1).

The induced operators \( D^\chi_{\tau}(\sigma)e^{-t(D^\chi_{\tau}(\sigma))^2}, e^{-t(D^\chi_{\tau}(\sigma))^2} \) are well defined, integral operators acting on smooth sections of the twisted vector bundle \( E(\sigma) \otimes E_\chi \) ([Spi15b, equations (5.2) and (5.3)]). The kernel of the operator \( D^\chi_{\tau}(\sigma)e^{-t(D^\chi_{\tau}(\sigma))^2} \) is given by

\[ K^\tau_{r_\chi}(\gamma)(x, x') = \sum_{\gamma \in \Gamma} K^\tau_{r_\chi}(g^{-1}\gamma g') \otimes \chi(\gamma), \quad (3.1) \]
where \( x = \Gamma g, x' = \Gamma g' \in G \) (see [Spi15b] equation 5.4]). The kernel function \( K_t^{\tau} \) is the kernel associated with the operator \( \tilde{D}(\sigma)e^{-t(\tilde{D}(\sigma))^2} \). It belongs to the Harish-Chandra \( L^2 \)-Schwartz space \((C^2(G) \otimes \text{End}(V_{\tau(\sigma)}))^{K \times K} \), as it is defined in [BM83] p. 161-162. It is important for the proof of the functional equations of the Selberg zeta function to derive a formula that connects the eta invariant \( \eta(0, D^x_\chi(\sigma)) \) and the trace \( \text{Tr}(D^x_\chi(\sigma)e^{-t(D^x_\chi(\sigma))^2}) \). We will use the following lemmata.

**Lemma 3.1.** The asymptotic expansion of the trace of the kernel \( K_t^{\tau;\chi}(x, y) \) of the operator \( D^x_\chi(\sigma)e^{-t(D^x_\chi(\sigma))^2} \) is given by

\[
\text{tr} \ K_t^{\tau;\chi}(x, y) \sim_{t \to 0^+} \text{dim}(V_\chi)(a_0(x)t^{1/2} + O(t^{3/2}, x)),
\]

where \( a_0(x) \) is a \( C^\infty \)-function on \( X \).

**Proof.** By [BF86] Theorem 2.4 we have that the trace of the kernel \( K_t^{\tau;\chi}(x, y) \in C(\tilde{X}, \tilde{E}_{\tau,\chi} \otimes \tilde{E}^*_{\tau,\chi}) \), associated to the operator \( \tilde{D}(\sigma)e^{-t(\tilde{D}(\sigma))^2} \) has the asymptotic expansion

\[
\text{tr} \ K_t^{\tau;\chi}(x, x) \sim_{t \to 0^+} a_0(x)t^{1/2} + O(t^{3/2}, x),
\]

where \( a_0(x) \) a smooth local invariant determined by the total symbol of \( D(\sigma) \). Locally, the twisted Dirac operator \( D^x_\chi(\sigma) \) is described by

\[
\tilde{D}^x_\chi(\sigma) = \tilde{D}(\sigma) \otimes \text{Id}_{V_\chi},
\]

and the symbol \( \sigma_{\tilde{D}^x_\chi(\sigma)} \) of \( D^x_\chi(\sigma) \) for \( \xi \in T^*X, \xi \neq 0 \) is given by

\[
\sigma_{\tilde{D}^x_\chi(\sigma)}(x, \xi) = (i\xi) \otimes \text{Id}_{(V_{\tau(\sigma)} \otimes V_\chi)_x}. 
\]

Hence, by (3.1)

\[
\text{tr} \ K_t^{\tau;\chi}(x, y) \sim_{t \to 0^+} \text{dim}(V_\chi)(a_0(x)t^{1/2} + O(t^{3/2}, x)).
\]
Lemma 3.2. The asymptotic expansion of the trace of the kernel $H_t^{\tau,\chi}(x,y)$ of the operator $e^{-t\tilde{A}_\chi(\sigma)}$ is given by

$$\text{tr} H_t^{\tau,\chi}(x,x) \sim_{t \to 0^+} \dim(V_\chi) \sum_{j=0}^\infty c_j(x)t^{-\frac{d}{2}}$$

(3.4)

where $c_j(x)$ are $C^\infty$-functions on $X$.

Proof. By [Gil95, Lemma 1.8.2] we have that the trace of the kernel $Q_t(x,y) \in C(\tilde{X}, \tilde{E}_\tau \boxtimes \tilde{E}^*_\tau)$, associated to the operator $e^{-t\tilde{A}_\tau}$ has the asymptotic expansion

$$\text{tr} Q_t(x,x) \sim_{t \to 0^+} \sum_{j=0}^\infty c_j(x)t^{-\frac{d}{2}}.$$

where $c_j(x)$ are smooth local invariants determined by the total symbol of the $\Delta_\tau$.

We recall here that the operator $A_\tau$ is just the Bochner-Laplace operator $\Delta_\tau$ minus the Casimir eigenvalue $\lambda_\tau$, i.e.,

$$\tilde{A}_\tau = \tilde{\Delta}_\tau - \lambda_\tau,$$

acting on $C^\infty(\tilde{X}, \tilde{E}_\tau)$ (see [Spi15a, p. 27-28]). On the other hand, locally, the twisted Bochner-Laplace operator $\tilde{\Delta}^{\tau,\chi}_\tau$ is described as follows.

$$\tilde{\Delta}^{\tau,\chi}_\tau = \tilde{\Delta}_\tau \otimes \text{Id}_{V_\chi}$$

(see [Müll11 p. 19-20]). Hence, its symbol is given by

$$\sigma_{\Delta^{\tau,\chi}_\tau}(x,\xi) = (\|\xi\|^2) \otimes \text{Id}_{(V_\tau \otimes V_\chi)_x},$$

for $\xi \in T^*X, \xi \neq 0$. Moreover, if $j$ is an odd integer, then $c_j = 0$. We mention here that one can use the expansion in power series of the term $e^{-tc(\sigma)}$. Then, the assertion follows from equation (3.3). \qed

Definition 3.3. The angle $\theta \in [0,2\pi)$ is a principal angle for an elliptic operator $D^{\tau,\chi}_\chi(\sigma)$ if

$$\text{spec}(\sigma_{D^{\tau,\chi}_\chi(\sigma)}(x,\xi)) \cap R_\theta = \emptyset, \quad \forall x \in X, \forall \xi \in T^*_xX, \xi \neq 0,$$

where $R_\theta := \{\rho e^{i\theta} : \rho \in [0,\infty]\}$. }

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\textbf{Definition 3.4.} The angle $\theta$ is an Agmon angle for an elliptic operator $D$, if it is a principal angle for $D^s_\chi(\sigma)$ and there exists an $\varepsilon > 0$ such that
\[
\spec(D^s_\chi(\sigma)) \cap L_{[\theta-\varepsilon,\theta+\varepsilon]} = \emptyset,
\]
where $L_I$ is a solid angle defined by
\[
L_I := \{\rho e^{i\theta} : \rho \in (0,\infty), \theta \in I \subset [0,2\pi]\}.
\]

\textbf{Definition 3.5.} Let $\theta$ be an Agmon angle for $D^s_\chi(\sigma)$ and let $\spec(D^s_\chi(\sigma)) = \{\lambda_k : k \in \mathbb{N}\}$ be the spectrum of $D^s_\chi(\sigma)$, contained in some discrete subset of $\mathbb{C}$. Let $m_k = m(\lambda_k)$ be the algebraic multiplicity of the the eigenvalue $\lambda_k$ (see [Sp15a, Definition 4.6]). Then, for $\Re s > 0$, we define the eta function $\eta_\theta(s,D^s_\chi(\sigma))$ of $D^s_\chi(\sigma)$ by the formula
\[
\eta_\theta(s,D^s_\chi(\sigma)) = \sum_{\Re(\lambda_k) > 0} m_k(\lambda_k)^{-s} - \sum_{\Re(\lambda_k) < 0} m_k(-\lambda_k)^{-s}.
\]

It has been shown by [GS95, Theorem 2.7] that $\eta_\theta(s,D^s_\chi(\sigma))$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$ with isolated simple poles and is regular at $s = 0$. Moreover, the number $\eta_\theta(0,D^s_\chi(\sigma))$ is independent of the Agmon angle $\theta$. We call the number $\eta_\theta(0,D^s_\chi(\sigma)) = \eta(0,D^s_\chi(\sigma))$ the eta invariant associated with the operator $D^s_\chi(\sigma)$.

We give here a short description of the proof. Definition 3.5 can be read also as
\[
\eta_\theta(s,D^s_\chi(\sigma)) = \zeta_\theta(s,\Pi_\chi,D^s_\chi(\sigma)) - \zeta_\theta(s,\Pi_\chi,D^s_\chi(\sigma)),
\]
where $\Pi_\chi$ (resp. $\Pi_\chi$) is the pseudo-differential projection whose image contains the span of all generalized eigenvectors of $D^s_\chi(\sigma)$ corresponding to eigenvalues $\lambda$ with $\Re(\lambda) > 0$ (resp. $\Re(\lambda) < 0$) (for more details see [BK07, Definition 6.16]). The zeta function $\zeta_\theta(s,\Pi_\chi,D^s_\chi(\sigma))$ is define for $\Re s > d$,
\[
\zeta_\theta(s,\square,D^s_\chi(\sigma)) := \Tr(\square D^s_\chi(\sigma)^{-s}),
\]
where $\square = \Pi_\chi,\Pi_\chi$ (see [BK07, p.24]). Then, the meromorphic continuation of the eta function arises from the meromorphic continuation of the kernel of the operator $\square D^s_\chi(\sigma)^{-s}$ and in case we consider additional $\Re(\lambda) > 0$, from the meromorphic continuation of the kernel of the operator $\square e^{-t(D^s_\chi(\sigma))^2}$.

To define the operator $D^s_\chi(\sigma)^{-s}$, one has to use the contour $\Gamma_{\alpha,\rho_0}$, described as in [Shu87, p.88]. Let $\alpha$ be an Agmon angle for $D^s_\chi(\sigma)$. We assume that 0 is not an eigenvalue of $D^s_\chi(\sigma)$. Then, there exists a $\rho_0 > 0$ such that
\[
\spec(D) \cap \{z \in \mathbb{C} : |z| \leq 2\rho_0\} = \emptyset.
\]
We consider the contour $\Gamma_{\alpha,\rho_0} \subset \mathbb{C}$, defined as $\Gamma_{\alpha,\rho_0} = \Gamma'_1 \cup \Gamma'_2 \cup \Gamma'_3$, where $\Gamma'_1 = \{ re^{i\alpha} : \infty > r \geq \rho_0 \}$, $\Gamma'_2 = \{ \rho_0 e^{i\beta} : \beta \leq \alpha - 2\pi \}$, $\Gamma'_3 = \{ re^{i(\alpha - 2\pi)} : \rho_0 \leq r < \infty \}$. Then, for $\text{Re}(s) > 0$ we define

$$D^s_\chi(\sigma)^{-s} = \frac{i}{2\pi} \int_{\Gamma_{\alpha,\rho_0}} \lambda^{-s}(D^s_\chi(\sigma) - \lambda \text{Id})^{-1} d\lambda.$$ 

If we integrate by parts the integral above, the operator $(D^s_\chi(\sigma) - \lambda \text{Id})^{-k}$ will occur. By ([GS95, Theorem 2.7.]), for $k < -d$, there exists an asymptotic expansion of the trace of the operator $\square(D^s_\chi(\sigma) - \lambda \text{Id})^{-k}$ as $|\lambda| \to \infty$:

$$\text{Tr}(\square(D^s_\chi(\sigma) - \lambda \text{Id})^{-k}) \sim \sum_{j=1}^{\infty} c_j \lambda^{d-j-k} + \sum_{l=1}^{\infty} (c'_l \log \lambda + c''_l) \lambda^{-k-l},$$

where the coefficients $c_j$ and $c'_l$ are determined from the symbols of $D^s_\chi(\sigma)$ and $\square$, and the coefficients $c''_l$ are in general globally determined.

Let $\Pi_\lambda$ be the projection on the span of the root spaces corresponding to eigenvalues $\lambda$ with $\text{Re}(\lambda^2) > 0$. We consider the Agmon angle $\theta$ fixed and we write $\eta_0(s, D^s_\chi(\sigma))$ instead of $\eta(s, D^s_\chi(\sigma))$. We define the functions

$$\eta_0(s, D^s_\chi(\sigma)) := \sum_{\text{Re}(\lambda) > 0} \lambda^{-s} - \sum_{\text{Re}(\lambda^2) \leq 0} \lambda^{-s} \quad \text{and} \quad \eta_1(s, D^s_\chi(\sigma)) := \sum_{\text{Re}(\lambda) > 0} \lambda^{-s} - \sum_{\text{Re}(\lambda) < 0} \lambda^{-s}.$$ 

By Definition 3.5, the eta function $\eta(s, D^s_\chi(\sigma))$ satisfies the equation

$$\eta(s, D^s_\chi(\sigma)) = \eta_0(s, D^s_\chi(\sigma)) + \eta_1(s, D^s_\chi(\sigma))$$

Since the spectrum of $(D^s_\chi(\sigma))^2$ is discrete and contained in a translate of a positive cone in $\mathbb{C}$ (see Figure 1), there are only finitely many eigenvalues of $(D^s_\chi(\sigma))^2$ with $\text{Re}(\lambda^2) \leq 0$.

**Lemma 3.6.** The eta function $\eta(s, D^s_\chi(\sigma))$ satisfies the equation

$$\eta(s, D^s_\chi(\sigma)) = \eta_0(s, D^s_\chi(\sigma)) + \frac{1}{\Gamma(s+1/2)} \int_0^\infty \text{Tr}(\Pi_+ D^s_\chi(\sigma) e^{-t(D^s_\chi(\sigma))^2}) t^{s+1/2} dt.$$  (3.5)
Figure 1: The discrete spectrum of the operator $(D^2_\Lambda(\sigma))^2$. There are only finitely many eigenvalues of $(D^2_\Lambda(\sigma))^2$ with negative real part.
Proof. Let \( \lambda \) be an eigenvalue of \( D^s_\chi(\sigma) \) such that \( \text{Re}(\lambda^2) > 0 \). The Gamma function is defined by
\[
\Gamma(s) = \int_0^{\infty} t^{s-1}e^{-t}dt, \quad \text{Re}(s) > 0.
\]
We apply now the following change of variables \( t \mapsto t' = \lambda^2t \) to see
\[
(\lambda^2)^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1}e^{-\lambda^2t}dt.
\]
Changing variables and using the Cauchy theorem to deform the contour of integration back to the original one, we get
\[
(\lambda^2)^{-s} = \frac{1}{\Gamma(s+1/2)} \int_0^{\infty} e^{-\lambda^2\frac{t^{s+1}}{2}} dt.
\]
We mention here that we can use the Lidskii’s theorem ([Sim05, Theorem 3.7, p.35]) to express the trace of the operator \( D^s_\chi(\sigma)e^{-t(D^s_\chi(\sigma))^2} \) in terms of its eigenvalues \( \lambda_k \)
\[
\text{Tr}(D^s_\chi(\sigma)e^{-t(D^s_\chi(\sigma))^2}) = \sum_{\lambda_k \neq 0} m_k(\lambda_k)\lambda_k e^{-t\lambda_k^2}.
\]
Taking the sum over the eigenvalues \( \lambda_k \) of \( D^s_\chi(\sigma) \), counting also their algebraic multiplicities, we have
\[
\text{Tr}(\Pi_+D^s_\chi(\sigma)((D^s_\chi(\sigma))^2)^{-\frac{s+1}{2}}) = \frac{1}{\Gamma(s+1/2)} \int_0^{\infty} \text{Tr}(\Pi_+D^s_\chi(\sigma)e^{-t(D^s_\chi(\sigma))^2})t^{-\frac{s+1}{2}} dt. \tag{3.6}
\]
To prove the convergence of the above integral, we first observe that
\[
\text{Tr}(\Pi_+D^s_\chi(\sigma)((D^s_\chi(\sigma))^2)^{-\frac{s+1}{2}}) = \int_0^{1} \text{Tr}(\Pi_+D^s_\chi(\sigma)e^{-t(D^s_\chi(\sigma))^2})t^{-\frac{s+1}{2}} dt + \int_1^{\infty} \text{Tr}(\Pi_+D^s_\chi(\sigma)e^{-t(D^s_\chi(\sigma))^2})t^{-\frac{s+1}{2}} dt. \tag{3.7}
\]
Then, for the first integral in the right-hand side of (3.7), we use the asymptotic expansion of the trace of the operator \( D^s_\chi(\sigma)e^{-t(D^s_\chi(\sigma))^2} \), as it is described in Lemma 3.1. We have
\[
\int_0^{1} \text{Tr}(\Pi_+D^s_\chi(\sigma)e^{-t(D^s_\chi(\sigma))^2})t^{-\frac{s+1}{2}} dt = \int_0^{1} \text{dim} V_\chi(a_0t^{1/2} + O(t^{3/2}))t^{-\frac{s+1}{2}} dt < \text{dim} V_\chi a_0 \frac{4}{s + 3}. \tag{3.8}
\]
which is a holomorphic function for \( \text{Re}(s) > 0 \). Here,

\[
\alpha_0 = \int_X a_0(x) d\mu(x),
\]

where \( a_0(x) \) is a smooth local invariant, and \( \mu(x) \) is the volume measure determined by the riemannian metric on \( X \). For the second integral in the right hand side of (3.7), we set \( c_0 := \frac{1}{2} \min\{\text{Re}(\lambda_k^2) : \text{Re}(\lambda_k^2) > 0, \lambda_k \neq 0\} \). Then,

\[
\left| \sum_{\lambda_k \neq 0} \lambda_k e^{-t\lambda_k^2} \right| \leq c_1 e^{-\frac{c_0}{2} t}.
\]

Therefore,

\[
\int_1^\infty |\text{Tr}(\Pi + D^2_\chi(\sigma) e^{-t(D^2_\chi(\sigma))^2})|t^{-\frac{s+1}{2}}|dt \leq c_1 \int_1^\infty e^{-\frac{c_0}{2} t} t^\text{Re}(s)-1 dt < \infty. \tag{3.9}
\]

By equations (3.7), (3.8), and (3.9), it follows that the integral in the right-hand side of (3.6) is well defined and hence

\[
\eta(s, D^2_\chi(\sigma)) = \eta_0(s, D^2_\chi(\sigma)) + \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty \text{Tr}(\Pi + D^2_\chi(\sigma) e^{-t(D^2_\chi(\sigma))^2})t^{-\frac{s+1}{2}} dt.
\]

\[\square\]

4 Functional equations for the Selberg zeta function

The meromorphic continuation of the Selberg and Ruelle zeta functions associated with non-unitary representations of the subgroup \( \Gamma \) is proved in [Spi15a] and [Spi15b]. We will derive here functional equations for the Selberg zeta function in case (a) and for the symmetrized, super and Selberg zeta function in case (b).

Lemma 4.1. The logarithmic derivative of the Selberg zeta function satisfies the following functional equation

\[
L(s) + L(-s) = -4\pi \dim(V_\chi) \text{Vol}(X) P_{\eta}(s). \tag{4.1}
\]
Proof. We recall equation

\[
\sum_{i=1}^{N} \left( \prod_{j=1 \atop j \neq i}^{N} \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i} L(s_i) = - \sum_{i=1}^{N} \left( \prod_{j=1 \atop j \neq i}^{N} \frac{1}{s_j^2 - s_i^2} \right) \frac{\pi}{s_i} \dim(V_{\chi}) \Vol(X) P_{\sigma}(s_i) \\
+ \sum_{\lambda_k} \sum_{i=1}^{N} \left( \prod_{j=1 \atop j \neq i}^{N} \frac{1}{s_j^2 - s_i^2} \right) \frac{m(t_k)}{t_k + s_i^2},
\]

by [Spi15a, equation 6.15]. We fix again the complex numbers \(s_2, \ldots, s_N \in \mathbb{C}\) and we let \(s_1 = s \in \mathbb{C}\) vary. Then, we substitute \(s \mapsto -s\). The resulting equation will differ from (4.2) in the terms for \(i = 1\). i.e.,

\[
\left( \prod_{j=2 \atop j \neq i}^{N} \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s} L(s) \mapsto \left( \prod_{j=2 \atop j \neq i}^{N} \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{-2s} L(-s).
\]

Also, since the Plancherel polynomial \(P_\sigma(s)\) is an even polynomial of \(s\) ([Mia79, p.264-265]),

\[
- \left( \prod_{j=2 \atop j \neq i}^{N} \frac{1}{s_j^2 - s_i^2} \right) \frac{\pi}{s} \dim(V_{\chi}) \Vol(X) P_\sigma(s) \mapsto \left( \prod_{j=2 \atop j \neq i}^{N} \frac{1}{s_j^2 - s_i^2} \right) \frac{\pi}{s} \dim(V_{\chi}) \Vol(X) P_\sigma(s).
\]

We subtract the resulting equation from (4.2). In an obvious way the sum that includes the term

\[
\left( \prod_{j=2 \atop j \neq i}^{N} \frac{1}{s_j^2 - s_i^2} \right) \frac{m(t_k)}{t_k + s_i^2}
\]

will be canceled out, as well as the terms that include the fixed complex numbers \(s_2, \ldots, s_N\). Hence,

\[
\left( \prod_{j=2 \atop j \neq i}^{N} \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s} (L(s) + L(-s)) = - \left( \prod_{j=2 \atop j \neq i}^{N} \frac{1}{s_j^2 - s_i^2} \right) \frac{2\pi}{s} \dim(V_{\chi}) \Vol(X) P_\sigma(s).
\]
We multiply the above equation by the function

\[ 2s \prod_{j=2}^{N} (s_j^2 - s_i^2). \]

Then we have

\[ L(s) + L(-s) = -4\pi \dim(V_\chi) \Vol(X) P_\sigma(s). \]

**Theorem 4.2.** The Selberg zeta function \( Z(s; \sigma, \chi) \) satisfies the following functional equation

\[
\frac{Z(s; \sigma, \chi)}{Z(-s; \sigma, \chi)} = \exp \left( -4\pi \dim(V_\chi) \Vol(X) \int_0^s P_\sigma(r) dr \right). \tag{4.3}
\]

**Proof.** We integrate once over \( s \) and exponentiate equation (4.1). The assertion follows. \qed

We consider now case (b) and use the same argument as in case (a).

**Corollary 4.3.** The logarithmic derivative \( L_S(s) \) of the symmetrized zeta function \( S(s; \sigma, \chi) \) satisfies the following functional equation

\[ L_S(s) + L_S(-s) = -8\pi \dim(V_\chi) \Vol(X) P_\sigma(s). \tag{4.4} \]

**Proof.** We recall the following equation from [Spi15b, Equation 7.2].

\[
\Tr \prod_{i=1}^{N} (A^2_\chi(\sigma) + s_i^2)^{-1} = \sum_{i=1}^{N} \left( \prod_{j=1}^{N} \frac{1}{s_j^2 - s_i^2} \right) \frac{\pi}{s_i} 2 \dim(V_\chi) \Vol(X) P_\sigma(s_i)
\]

\[ + \sum_{i=1}^{N} \left( \prod_{j=1}^{N} \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i} L_S(s_i). \tag{4.5} \]

We consider (4.5) at \( s \mapsto -s \). We subtract the resulting equation from (4.5). Using the same argument as in Corollary 4.1, we have

\[
\left( \prod_{j=2}^{N} \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s} \{L(s) + L(-s)\} = - \left( \prod_{j=2}^{N} \frac{1}{s_j^2 - s_i^2} \right) \frac{4\pi}{s} \dim(V_\chi) \Vol(X) P_\sigma(s)
\]
We multiply the above equation by the function
\[ 2s \prod_{j=2}^{N} (s_j^2 - s^2), \]
and get equation (4.3).

**Theorem 4.4.** The symmetrized zeta function satisfies the following functional equation
\[
S(s; \sigma, \chi) = \exp \left( -8\pi \dim(V_{\chi}) \Vol(X) \int_0^s P_s(r) dr \right). \tag{4.6}
\]

*Proof.* We integrate over \( s \) and exponentiate equation (4.4). The assertion follows.

**Theorem 4.5.** The super zeta function satisfies the functional equation
\[
Z(s; \sigma, \chi) Z(-s; \sigma, \chi) = e^{2\pi i \eta(0, D_{\chi}^2(\sigma))}, \tag{4.7}
\]
where the \( \eta(0, D_{\chi}^2(\sigma)) \) is the eta invariant associated to the Dirac operator \( D_{\chi}^2(\sigma) \). Furthermore,
\[
Z(0; \sigma, \chi) = e^{\pi i \eta(0, D_{\chi}^2(\sigma))}. \tag{4.8}
\]

*Proof.* By [Spi15b, Proposition 6.1], we have
\[
\Tr(D_{\chi}^2(\sigma) \prod_{i=1}^{N} (D_{\chi}^2(\sigma)^2 + s_i^2)^{-1}) = \frac{-i}{2} \sum_{i=1}^{N} \left( \prod_{j=1}^{N} \frac{1}{s_j^2 - s_i^2} \right) L^s(s_i).
\]

Equivalently,
\[
\sum_{i=1}^{N} \left( \prod_{j=1}^{N} \frac{1}{s_j^2 - s_i^2} \right) L^s(s_i) = 2i \Tr(D_{\chi}^2(\sigma) \prod_{i=1}^{N} (D_{\chi}^2(\sigma)^2 + s_i^2)^{-1})
\]
\[
= 2i \Tr \int_0^\infty \sum_{i=1}^{N} \left( \prod_{j=1}^{N} \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} D_{\chi}^2(\sigma) e^{-t(D_{\chi}^2(\sigma))^2} dt,
\]
where we have employed the equation
\[
(D_{\chi}^2(\sigma)^2 + s^2)^{-1} = \int_0^\infty e^{-ts^2} e^{-t(D_{\chi}^2(\sigma))^2} dt.
\]
Using the same argument as in the proof of Proposition 6.1 in [Spi15b], we obtain

\[
\sum_{i=1}^{N} \left( \prod_{j=1 \atop j \neq i}^{N} \frac{1}{s_j^2 - s_i^2} \right) L^s(s_i) = 2i \int_{0}^{\infty} \sum_{i=1}^{N} \left( \prod_{j=1 \atop j \neq i}^{N} \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} \text{Tr}(D^s(\sigma)e^{-t(D^s(\sigma))^2})dt.
\]  

(4.9)

We fix now \(s_2, \ldots, s_N \in \mathbb{C}\) and let \(s_1 = s \in \mathbb{C}\) vary. Then, as a function of \(s\), the sum

\[
\sum_{i=1}^{N} \left( \prod_{j=1 \atop j \neq i}^{N} \frac{1}{s_j^2 - s_i^2} \right) L^s(s_i)
\]
determines \(L^s(s)\), up to an even polynomial of \(s\). This polynomial arises from the finite product

\[
\left( \prod_{j=2}^{N} \frac{1}{s_j^2 - s_i^2} \right).
\]

We choose \(s_i\) such that \(\text{Re}(s_i) \to \infty\). Then, in the left-hand side of (4.9), \(L^s(s_i)\) decreases exponentially. Similarly, in the right-hand side of (4.9), the integrals that include the exponentials \(e^{-ts_i^2}\) are

\[
\int_{0}^{\infty} e^{-ts_i^2} \text{Tr}(D^s(\sigma)e^{-t(D^s(\sigma))^2})dt.
\]

Note that each of these integrals is well defined, since as \(t \to \infty\), \(\text{Tr}(D^s(\sigma)e^{-t(D^s(\sigma))^2})\) and \(e^{-ts_i^2}\) decay exponentially, and as \(t \to 0^+\), we use the asymptotic expansion of the trace of the operator \(D^s(\sigma)e^{-t(D^s(\sigma))^2}\). By Lemma 3.1,

\[
\text{Tr}(D^s(\sigma)e^{-t(D^s(\sigma))^2}) \sim_{t \to 0} \dim(V_\chi)(\alpha_0 t^{1/2} + O(t^{3/2})).
\]

Consequently, we can write the right-hand side of (4.9) as the finite sum of the integrals

\[
\int_{0}^{\infty} e^{-ts_i^2} \text{Tr}(D^s(\sigma)e^{-t(D^s(\sigma))^2})dt,
\]

multiplied by the finite product

\[
\prod_{j=1 \atop j \neq i}^{N} \frac{1}{s_j^2 - s_i^2}, \quad i = 1, \ldots, N.
\]
If we choose now $s_i \in \mathbb{C}$ such that $\operatorname{Re}(s_i^2) \to \infty$, then the integrals that contain the term $e^{-ts_i^2}$ decay exponentially. Hence, we can remove the $\sum_{i=1}^{N} \left( \prod_{j=1}^{N} \frac{1}{s_j - s_i} \right)$-structure and get

$$L^s(s) = 2i \int_0^\infty e^{-ts^2} \operatorname{Tr}(D_{\chi}^2(\sigma)e^{-t(D_{\chi}^2(\sigma))^2})dt.$$

Let $\Pi_+$ (resp. $\Pi_-$) be the projection on the span of the root spaces corresponding to eigenvalues $\lambda$ of $D_{\chi}^2(\sigma)$ with $\operatorname{Re}(\lambda^2) > 0$ (resp. $\operatorname{Re}(\lambda^2) \leq 0$). Recall that there are only finitely many eigenvalues of $D_{\chi}^2(\sigma)$ such that $\operatorname{Re}(\lambda^2) \leq 0$ (see Figure 1). We write

$$L^s(s) = 2i \int_0^\infty e^{-ts^2} \operatorname{Tr}(\Pi_+ D_{\chi}^2(\sigma)e^{-t(D_{\chi}^2(\sigma))^2})dt + 2i \int_0^\infty e^{-ts^2} \operatorname{Tr}(\Pi_- D_{\chi}^2(\sigma)e^{-t(D_{\chi}^2(\sigma))^2})dt. \quad (4.10)$$

We set

$$I_+ := \int_0^\infty e^{-ts^2} \operatorname{Tr}(\Pi_+ D_{\chi}^2(\sigma)e^{-t(D_{\chi}^2(\sigma))^2})dt$$

$$I_- := \int_0^\infty e^{-ts^2} \operatorname{Tr}(\Pi_- D_{\chi}^2(\sigma)e^{-t(D_{\chi}^2(\sigma))^2})dt.$$

Then, for the integral $I_+$ we have

$$\int_s^\infty I_+ = \int_s^\infty \int_0^\infty e^{-tw^2} \operatorname{Tr}(\Pi_+ D_{\chi}^2(\sigma)e^{-t(D_{\chi}^2(\sigma))^2})dtdw$$

$$= \int_0^\infty \int_s^\infty e^{-tu^2} \operatorname{Tr}(\Pi_+ D_{\chi}^2(\sigma)e^{-t(D_{\chi}^2(\sigma))^2})dwdt.$$

If we make the change if variables $w \mapsto \frac{1}{\sqrt{t}} u$, we get

$$\int_s^\infty I_+ = \int_0^\infty \int_0^\infty \frac{1}{\sqrt{t}} e^{-u^2} \operatorname{Tr}(\Pi_+ D_{\chi}^2(\sigma)e^{-t(D_{\chi}^2(\sigma))^2})dudt.$$

We use now the error function

$$\Phi(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du.$$
It holds
\[ \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} du = 1 - \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-u^2} du = 1 - \Phi(x). \]

Hence,
\[ \int_{s}^{\infty} I_{+} = \int_{0}^{\infty} \frac{\sqrt{\pi}}{2\sqrt{t}} \left( 1 - \frac{2}{\sqrt{\pi}} \int_{0}^{\sqrt{ts}} e^{-u^2} du \right) \text{Tr}(\Pi_{+} D_{x}^{2}(\sigma)e^{-t(D_{x}^{2}(\sigma))^2}) dt, \tag{4.11} \]
and
\[ \int_{-s}^{\infty} I_{+} = \int_{0}^{\infty} \frac{\sqrt{\pi}}{2\sqrt{t}} \left( 1 - \frac{2}{\sqrt{\pi}} \int_{0}^{\sqrt{ts}} e^{-u^2} du \right) \text{Tr}(\Pi_{+} D_{x}^{2}(\sigma)e^{-t(D_{x}^{2}(\sigma))^2}) dt. \tag{4.12} \]

We add together (4.11) and (4.12) to get
\[ \int_{s}^{\infty} I_{+} + \int_{-s}^{\infty} I_{+} = \int_{0}^{\infty} \frac{\sqrt{\pi}}{\sqrt{t}} \text{Tr}(\Pi_{+} D_{x}^{2}(\sigma)e^{-t(D_{x}^{2}(\sigma))^2}) dt. \tag{4.13} \]

We treat now the integral \( I_{-} \). Since there are only finitely many eigenvalues of \( D_{x}^{2}(\sigma) \) with \( \text{Re}(\lambda^2) \leq 0 \), we can interchange the order of integration and write
\[
\int_{s}^{\infty} I_{-} = \int_{s}^{\infty} \int_{0}^{\infty} e^{-tw^2} \text{Tr}(\Pi_{-} D_{x}^{2}(\sigma)e^{-t(D_{x}^{2}(\sigma))^2}) dt dw
\]
\[ = \sum_{\lambda \text{ Re}(\lambda^2) \leq 0} \lambda \int_{s}^{\infty} \int_{0}^{\infty} e^{-tw^2} e^{-t\lambda^2} dt dw
\]
\[ = \sum_{\lambda \text{ Re}(\lambda) \leq 0} \lambda \int_{s}^{\infty} \frac{1}{w^2 + \lambda^2} dw. \tag{4.14} \]

Substituting at (4.14) \( s \mapsto -s \) and adding the resulting equation to (4.14), we obtain
\[ \int_{s}^{\infty} I_{+} + \int_{-s}^{\infty} I_{-} = \int_{\mathbb{R}} I_{-} = \sum_{\lambda \text{ Re}(\lambda^2) \leq 0} \lambda^{-s} \int_{\mathbb{R}} \frac{1}{w^2 + \lambda^2} dw. \]

By change of variables \( w \mapsto w' = w/\lambda \) we have
\[ \int_{s}^{\infty} I_{-} + \int_{-s}^{\infty} I_{-} = \int_{\mathbb{R}} I_{-} = \sum_{\text{Re}(\lambda) > 0} \pi - \sum_{\text{Re}(\lambda) < 0} \frac{\pi}{\text{Re}(\lambda^2) \leq 0}. \]
The sums over $\lambda$ in the equation above are finite, because we sum over $\lambda$ with $\Re(\lambda^2) \leq 0$ and there are only finitely many eigenvalues such that $\Re(\lambda^2) \leq 0$.

We use now the definition of the function $\eta_0(0, D^\sharp_\chi(\sigma))$ from Section 3:

$$
\eta_0(s, D^\sharp_\chi(\sigma)) := \sum_{\Re(\lambda) > 0} \lambda^{-s} - \sum_{\Re(\lambda) < 0} \sum_{\Re(\lambda^2) \leq 0} \lambda^{-s}.
$$

Then,

$$
\int_{s}^{\infty} I_- + \int_{-s}^{\infty} I_- = \pi \eta_0(0, D^\sharp_\chi(\sigma)). \tag{4.15}
$$

We recall here equation [Spi15b, equation (6.10)]

$$
\log Z^s(s; \sigma, \chi) = \int_{-\infty}^{\infty} L^s(w)dw,
$$

and equation (3.5)

$$
\eta(s, D^\sharp_\chi(\sigma)) = \eta_0(s, D^\sharp_\chi(\sigma)) + \frac{1}{\Gamma(s + \frac{3}{2})} \int_{0}^{\infty} \text{Tr}(\Pi_+ D^\sharp_\chi(\sigma) e^{-t(D^\sharp_\chi(\sigma))^2}) t^{s + \frac{1}{2}} dt.
$$

Hence, by (4.10), (4.13) and (4.15) we get

$$
\log Z^s(s; \sigma, \chi) + \log Z^s(-s; \sigma, \chi) = \int_{s}^{\infty} L^s(w)dw + \int_{-s}^{\infty} L^s(w)dw
$$

$$
= 2i \int_{s}^{\infty} I_- dw + 2i \int_{-s}^{\infty} I_+ dw
$$

$$
+ 2i \int_{s}^{\infty} I_- dw + 2i \int_{-s}^{\infty} I_- dw
$$

$$
= 2\pi i (\eta_1(0, D^\sharp_\chi(\sigma)) + \eta_0(0, D^\sharp_\chi(\sigma)))
$$

$$
= 2\pi i \eta(0, D^\sharp_\chi(\sigma)).
$$

Equation (4.7) follows by exponentiation the equation above. We get equation (4.8) by substituting $s = 0$ in the equation above.

We prove now the functional equation for the Selberg zeta function in case (b).

**Theorem 4.6.** The Selberg zeta function satisfies the following functional equation

$$
\frac{Z(s; \sigma, \chi)}{Z(-s; w\sigma, \chi)} = e^{\pi i \eta(0, D^\sharp_\chi(\sigma))} \exp \left( -4 \pi \dim(V_\chi) \text{Vol}(X) \int_{0}^{s} P_\sigma(r)dr \right). \tag{4.16}
$$
Proof. By Definition 2.3 and 2.4 of the symmetrized and super zeta function, respectively, we have
\[
\frac{Z(s; \sigma, \chi)}{Z(-s; w\sigma, \chi)} = \frac{\sqrt{S(s; \sigma, \chi)Z(s; \sigma, \chi)}}{\sqrt{S(-s; w\sigma, \chi)Z^*(s; \sigma, \chi)}}
= \frac{\sqrt{S(s; \sigma, \chi)Z(s; \sigma, \chi)}}{\sqrt{S(-s; w\sigma, \chi)Z^*(s; \sigma, \chi)}}
= e^{\pi i \eta(0, D^\sharp \chi(\sigma))} \exp \left(-4\pi \dim(V_\lambda) \Vol(X) \int_0^s P_\sigma(r)dr \right),
\]
where in the last equation we have employed Theorem 4.4 and Theorem 4.5. 

5 Functional equations for the Ruelle zeta function

Let \(\nu_p\) be the representation of \(MA\) in \(\Lambda^n n_C\), given by the \(p\)-th exterior power of the adjoint representation
\[
\nu_p := \Lambda^p \text{Ad}_{n_C} : MA \to \text{GL}(\Lambda^n n_C), \quad p = 0, 1, \ldots, d - 1.
\]
For \(p = 0, 1, \ldots, d - 1\), we consider \(J_p \subset \{(\psi_p, \lambda) : \psi_p \in \hat{M}, \lambda \in \mathbb{C}\}\) as the subset consisting of all pairs of unitary irreducible representations of \(M\) and one dimensional representations of \(A\) such that, as \(MA\)-modules, the representations \(\nu_p\) decompose as
\[
\Lambda^n n_C = \bigoplus_{(\psi_p, \lambda) \in J_p} V_{\psi_p} \otimes \mathbb{C}_\lambda,
\]
where \(\mathbb{C}_\lambda \cong \mathbb{C}\) denotes the representation space of \(\lambda\). By Poincaré duality (see \cite[p. 122]{BO95},) we have for \(p < \frac{d - 1}{2}\),
\[
J_{d-1-p} \subset \{(\psi_p, 2\rho - \lambda) : \psi_p \in \hat{M}, \lambda \in \mathbb{C}\}.
\]
(5.1)

We consider now the compact real forms \(G_d\), and \(A_d\) of \(G_C\) and \(A_C\) respectively (see \cite[p.114]{Kna86}). Then \(L := G_d/MA_d\) is Kähler manifold of dimension \(\dim(L) = r\) (\cite[p. 123]{BO95}). For \(\lambda \in \mathbb{Z}\), we extend the one dimensional representation of \(A\) to a representation of \(A_d\). If \(\lambda \in \mathbb{R}\) such that \(\rho + \lambda \in \mathbb{Z}\), then the representation \(\psi_p \otimes \lambda\) exists as a representation of \(MA_d\). Let \(E_{(\psi_p, \lambda)}\) be the holomorphic vector bundle over \(L\), defined by
\[
E_{(\psi_p, \lambda)} := G_d \times_{\psi_p \otimes \lambda} (V_{\psi_p} \otimes \mathbb{C}_\lambda) \to L.
\]
Lemma 5.1. Let $(\sigma, V_\sigma) \in \hat{M}$. Let $P_{\psi_p \otimes \sigma}(s), s \in \mathbb{C}$ be the Plancherel measure associated with the representation $\psi_p \otimes \sigma \in \hat{M}$, $p = 0, \ldots, d - 1$. Let $f(s)$ be the polynomial of $s$ given by

$$f(s) := \frac{d}{ds} F(s) = (-1)^{\frac{d-1}{2}} P_{\psi_p \otimes \sigma}(s) + \sum_{p=0}^{d-1} \sum_{(\psi_p, \lambda) \in J_p} (-1)^p [P_{\psi_p \otimes \sigma}(s + \rho + \lambda) + P_{\psi_p \otimes \sigma}(s - \rho + \lambda)]$$

$$= \sum_{p=0}^{d-1} (-1)^p P_{\psi_p \otimes \sigma}(s; \rho, \lambda). \quad (5.2)$$

Then,

$$f(s) = (d + 1) \dim(V_\sigma). \quad (5.3)$$

Proof. Let $\Lambda_M$ the highest weight of the $\psi_p$. Then, by the Borel-Weil-Bott Theorem (see [War72, Theorem 3.1.2.2]), we have that the representation space of the representation of $G_d$ with highest weight $\Lambda_M$, can be realized as the space of the zero-Dolbeaux cohomology group $H^0(L, E_{(\psi_p, \lambda)})$ of $E_{(\psi_p, \lambda)}$. Moreover, all the higher cohomology groups $H^i(L, E_{(\psi_p, \lambda)})$ of $E_{(\psi_p, \lambda)}$ vanish for $i = 1, \ldots, r$. Hence

$$\dim(H^0(L, E_{(\psi_p, \lambda)})) = \dim(V_{\psi_p} \otimes \mathbb{C}_\lambda). \quad (5.4)$$

On the other hand, by the Weyl’s dimension formula (see [BO95, p. 47]) we have

$$\dim(V_{\psi_p} \otimes \mathbb{C}_\lambda) = P_{\psi_p}(\lambda + \rho). \quad (5.5)$$

By equations (5.4) and (5.5) we have

$$\chi(L, E_{(\psi_p, \lambda)}) := \sum_{q=0}^{r} (-1)^q \dim(H^q(L, E_{(\psi_p, \lambda)}))$$

$$= \dim(H^0(L, E_{(\psi_p, \lambda)}))$$

$$= P_{\psi_p}(\lambda + \rho).$$

Therefore, for $\sigma \in \hat{M}$,

$$P_{\psi_p \otimes \sigma}(\lambda + \rho) = \chi(L, E_{(\psi_p \otimes \sigma, \lambda)}). \quad (5.6)$$
Let $p < \frac{d-1}{2}$. Then, as $MA_d$-modules, the spaces $\{V_{\psi_p} \otimes V_{\sigma} \otimes \mathbb{C}_{s-\lambda} : (\psi_p, \lambda) \in J_p\}$ in the direct sum below decompose as

$$\bigoplus_{(\psi, \lambda) \in J_p} V_{\psi_p} \otimes V_{\sigma} \otimes \mathbb{C}_{s-\lambda} = \bigoplus_{(\psi, \lambda) \in J_p} (V_{\psi_p} \otimes \mathbb{C}_{-\lambda}) \otimes (V_{\sigma} \otimes \mathbb{C}_s) = \Lambda^p \cdot \mathbb{C}_s = \Lambda^p \cdot T^* L \otimes (V_{\sigma} \otimes \mathbb{C}_s).$$ (5.7)

Let $p > \frac{d-1}{2}$. Then, as $MA_d$-modules, the spaces $\{V_{\psi_p} \otimes V_{\sigma} \otimes \mathbb{C}_{s-2\rho+\lambda} : (\psi_p, \lambda) \in J_{p-1+d}\}$ decompose as

$$\bigoplus_{(\psi, \lambda) \in J_{p-1+d}} V_{\psi_p} \otimes V_{\sigma} \otimes \mathbb{C}_{s-2\rho+\lambda} = \bigoplus_{(\psi, \lambda) \in J_{p-1+d}} (V_{\psi_p} \otimes \mathbb{C}_{2\rho-\lambda}) \otimes (V_{\sigma} \otimes \mathbb{C}_s) = \Lambda^{p,0} \cdot T^* L \otimes (V_{\sigma} \otimes \mathbb{C}_s).$$ (5.8)

Therefore, by (5.2), (5.6), (5.7), and (5.8) we get

$$f(s) = \chi(L, \Lambda^{p,0} \cdot T^* L \otimes E_{\sigma,\lambda}).$$ (5.9)

We denote by $\mathcal{A}^{0,q}(L, E_{(\psi_p, \lambda)})$ the vector-valued $(0,q)$-differential forms on $L$. Let the $\overline{\partial}$-operator acting on $\mathcal{A}^{0,q}(L, E_{(\psi_p, \lambda)})$ and the Dirac-type operator

$$D_q := \overline{\partial} + \overline{\partial}^* : \bigoplus_{q=0}^{[r/2]} \mathcal{A}^{0,2q}(L, E_{(\psi_p, \lambda)}) \rightarrow \bigoplus_{q=0}^{[r/2]} \mathcal{A}^{0,2q+1}(L, E_{(\psi_p, \lambda)}).$$ (5.10)

Let $\Box_q$ be the complex Laplace operator, defined as follows.

$$\Box_q := \overline{\partial} \overline{\partial}^* + \overline{\partial} \overline{\partial} + \mathcal{A}^{0,q}(L, E_{(\psi_p, \lambda)}).$$ (5.11)

Then, by Hodge theory applied to $\Box_q$, we have that there is an isomorphism of vector spaces $\mathcal{H}^{0,q}(L, E_{(\psi_p, \lambda)}) \cong H^q(L, E_{(\psi_p, \lambda)})$, where $\mathcal{H}^{0,q}(L, E_{(\psi_p, \lambda)}) := \ker(\Box_q)$. Recall also the definition of the index of the Dirac-type operator $\overline{\partial} + \overline{\partial}^*$:

$$\text{ind}(\overline{\partial} + \overline{\partial}^*) := \dim \ker(\overline{\partial} + \overline{\partial}^*) - \dim \coker(\overline{\partial} + \overline{\partial}^*).$$
We observe that

$$\chi(L, \Lambda^p T^* L \otimes E_{\sigma, \lambda}) = \sum_{q=0}^{r} (-1)^q \dim(H^q(L, E_{(\psi, \lambda)}))$$

$$= \sum_{q=0}^{r} (-1)^q \dim(H^0, q(L, E_{(\psi, \lambda)}))$$

$$= \sum_{q_{\text{even}}} \dim(H^0, q(L, E_{(\psi, \lambda)})) - \sum_{q_{\text{odd}}} \dim(H^0, q(L, E_{(\psi, \lambda)}))$$

$$= \dim \ker(\partial + \partial^*) - \dim \ker(\partial + \partial^*)^*$$

$$= \dim \ker(\partial + \partial^*) - \dim \text{coker}(\partial + \partial^*)$$

$$= \text{ind} (\partial + \partial^*).$$

(5.12)

We will use the index theorem for the operator $(\partial + \partial^*)$. By [GV92, Theorem 4.8], we have

$$\text{ind} (\partial + \partial^*) = \int_L \chi(TL) \wedge ch(E_{\sigma, \lambda}),$$

(5.13)

where $\chi(TL)$ denotes the Euler class of the tangent bundle of $L$, and $ch(E_{\sigma, \lambda})$ is the Chern character associated to $E_{\sigma, \lambda}$. Since $\chi(TL)$ is of top degree, then by the splitting principle for $E_{\sigma, \lambda}$ into line bundles, we have that $ch(E_{\sigma, \lambda})$ is a zero-form, and that

$$ch(E_{\sigma, \lambda}) \equiv ch_0(E_{\sigma, \lambda}) = \dim(E_{\sigma, \lambda}).$$

(5.14)

By [BT82, Proposition 11.24] we have that the Euler number for the Kähler manifold $L$ equals its Euler characteristic

$$\int_L \chi(TL) = \chi(L).$$

(5.15)

By [Bot65, Theorem A], the Euler characteristic of $L$ is equal to the order of the Weyl group $W(G_d, T)$, where $T$ is a maximal torus subgroup of $G_d$. Then, if we consider the principle $MA_d$-fiber bundle $G_d$ over $L$

$$G_d \rightarrow L = G_d/MA_d,$$

we get

$$\chi(G_d) = \chi(\text{MA}_d) \chi(L).$$
Hence, since $\chi(\text{MA}_d) = \text{order}(W(\text{MA}_d), T)$, we have

$$
\chi(L) = \frac{\chi(G_d)}{\chi(\text{MA}_d)} = \frac{\text{order}(W(G_d, T))}{\text{order}(W(\text{MA}_d), T)}.
$$

One can compute (see e.g. [Hei90, p.60]) that

$$
\text{order}(W(SO(m))) = 2^{m-1}m!,
$$

where $m \in \mathbb{N}$ is even. Therefore, we obtain

$$
\chi(L) = \frac{2^{d-1/2}(d + 1/2)!}{2^{d-3/2}(d - 1/2)!} = d + 1. \quad (5.16)
$$

By equations (5.13), (5.14), (5.15) and (5.16) we have

$$
\text{ind } (\mathcal{F} + \mathcal{F}^*) = (d + 1) \text{dim}(V_\sigma). \quad (5.17)
$$

Hence, by (5.9), (5.12) and (5.17) we get

$$
f(s) = (d + 1) \text{dim}(V_\sigma). \quad (5.18)
$$

**Theorem 5.2.** The Ruelle zeta function satisfies the following functional equation

$$
\frac{R(s; \sigma, \chi)}{R(-s; \sigma, \chi)} = \exp \left( -4\pi (d + 1) \text{dim}(V_\sigma) \text{dim}(V_\chi) \text{Vol}(X)s \right). \quad (5.19)
$$

**Proof.** By [Spi15a, Theorem 6.6], we have the following representation of the Ruelle zeta function.

$$
R(s; \sigma, \chi) = \prod_{p=0}^{d-1} \left( \prod_{(\psi_p, \lambda) \in J_p} Z(s + \rho - \lambda; \psi_p \otimes \sigma, \chi) \right)^{(-1)^p}. \quad (5.20)
$$

Then, equation (5.20) becomes by (5.1)

$$
R(s; \sigma, \chi) = Z(s; \psi_{d+1} \otimes \sigma, \chi)^{(-1)^{d+1}} \prod_{p=0}^{d-3} \prod_{(\psi_p, \lambda) \in J_p} Z(s + \rho - \lambda; \psi_p \otimes \sigma, \chi)Z(s - \rho + \lambda; \psi_p \otimes \sigma, \chi)^{(-1)^p}. \quad (5.20)
$$
Hence,

\[
\frac{R(s; \sigma, \chi)}{R(-s; \sigma, \chi)} = \left( \frac{Z(s; \psi_{d-1} \otimes \sigma, \chi)}{Z(-s; \psi_{d-1} \otimes \sigma, \chi)} \right)^{(1)^{d-1}} \prod_{p=0}^{d-3} \prod_{(\psi_p, \lambda) \in J_p} \left( \frac{Z(s + \rho - \lambda; \psi_p \otimes \sigma, \chi)Z(s - \rho + \lambda; \psi_p \otimes \sigma, \chi)}{Z(-s + \rho - \lambda; \psi_p \otimes \sigma, \chi)Z(-s - \rho + \lambda; \psi_p \otimes \sigma, \chi)} \right)^{(-1)^p}.
\]

By Theorem 4.2, we get

\[
\frac{R(s; \sigma, \chi)}{R(-s; \sigma, \chi)} = \exp \left\{ -4\pi \dim(V_{\chi}) \Vol(X) \left( (1)^{d-1} \int_0^s P_{\psi_{d-1} \otimes \sigma}(rdr) + \sum_{p=0}^{d-3} \sum_{(\psi_p, \lambda) \in J_p} (-1)^p \left( \int_0^{s+\rho-\lambda} P_{\psi_p \otimes \sigma}(rdr) + \int_0^{s-\rho+\lambda} P_{\psi_p \otimes \sigma}(rdr) \right) \right) \right\}.
\]

(5.21)

We set

\[
F(s) = (1)^{d-1} \int_0^s P_{\psi_{d-1} \otimes \sigma}(rdr) + \sum_{p=0}^{d-3} \sum_{(\psi_p, \lambda) \in J_p} (-1)^p \left( \int_0^{s+\rho-\lambda} P_{\psi_p \otimes \sigma}(rdr) + \int_0^{s-\rho+\lambda} P_{\psi_p \otimes \sigma}(rdr) \right).
\]

Then,

\[
\frac{d}{ds} F(s) = f(s),
\]

where \( f(s) \) as in (5.2). We can easily write from Lemma 5.1

\[
\frac{R(s; \sigma, \chi)}{R(-s; \sigma, \chi)} = \exp \left( -4\pi \dim(V_{\chi}) \Vol(X)[(d + 1) \dim(V_{\sigma})s + C] \right),
\]

(5.22)

where \( C \in \mathbb{R} \) is a real constant. On the other hand, if we set \( s = 0 \) in (5.22), we get

\[
1 = \exp(-4\pi \dim(V_{\chi}) \Vol(X)C), \quad \text{and hence} \quad C = 0.
\]

The assertion follows.

We exam now case (b). Let \( \tau_p \) be the standard representation of \( K \) on \( \Lambda^p \mathbb{R}^d \otimes \mathbb{C} \).

Let \( (\sigma_p, V_{\sigma_p}) \) be the standard representation of \( M \) in \( \Lambda^p \mathbb{R}^{d-1} \otimes \mathbb{C} \). Let \( \alpha > 0 \) be the
unique positive root of the system \((g, a)\). Let \(\lambda_p : A \to \mathbb{C}^\times\) be the character, defined by \(\lambda_p(a) = e^{pa(\log a)}\). Then as a representation of \(MA\) one has \(\nu_p = \sigma_p \otimes \lambda_p\).

We denote by \(C_p \cong \mathbb{C}\) the representation space of \(\lambda_p\). Then, in the sense of \(MA\)-modules, we have

\[\Lambda^p n_C = \Lambda^p \mathbb{R}^{d-1} \otimes C_p.\]  

(5.23)

Let \(D_p^c (\sigma)\) be the twisted Dirac operator acting on \(C^\infty(X, E_{\tau_1(\sigma)} \otimes E_\lambda)\). For our proposal, we define the twist \(D_{p, \chi}^c (\sigma)\) of the Dirac operator \(D_p^c (\sigma)\) acting on

\[\bigoplus_{p=0}^{d-1} C^\infty(X, E_{\tau_1(\sigma)} \otimes E_\chi \otimes (d-p)\Lambda^p T^*X).\]

The twisted Dirac operator \(D_{p, \chi}^c (\sigma)\) is defined in a similar way as the Dirac operator \(D_p^c (\sigma)\) in Section 3. We equip the bundle \(\Lambda^p T^*X\) with the Levi-Civita connection of \(X\), and we proceed as in [Spi15b, Section 4].

**Theorem 5.3.** The super Ruelle zeta function associated with a non-Weyl invariant representation \(\sigma \in \hat{M}\) satisfies the functional equation

\[R^s(s; \sigma, \chi) R^*(-s; \sigma, \chi) = e^{2i\pi \eta(D_{p, \chi}^c (\sigma))},\]  

where \(\eta(D_{p, \chi}^c (\sigma))\) denotes the eta invariant of the twisted Dirac operator \(D_{p, \chi}^c (\sigma)\).

Moreover, the following equation holds

\[\frac{R(s; \sigma, \chi)}{R(-s; w\sigma, \chi)} = e^{i\pi \eta(D_{p, \chi}^c (\sigma))} \exp \left(-4\pi (d+1) \dim(V_\sigma) \dim(V_\chi) \text{Vol}(X) s \right).\]  

(5.25)

**Proof.** By [BO95, p. 23], we have

\[\sigma_p = i^*(((-1)^0 \tau_p + (-1)^1 \tau_{p-1} + \ldots + (-1)^{p-1}(\tau_1 - \text{Id})), \quad p = 1, 2, \ldots d-1\]

\[s^+ + s^- = i^*(s), \quad \text{otherwise.}\]

If we take the alternating sum of \(\sigma_p\) over \(p\) we get

\[\sum_{p=0}^{d-1} (-1)^p \sigma_p = i^* \left( \sum_{p=0}^{d-1} (-1)^p (d-p) \tau_p \right).\]  

(5.26)

We write

\[R^s(s; \sigma, \chi) R^*(-s; \sigma, \chi) = \frac{R(s; \sigma, \chi)}{R(s; w\sigma, \chi)} \frac{R(-s; w\sigma, \chi)}{R(s; w\sigma, \chi)} \frac{R(-s; \sigma, \chi)}{R(-s; \sigma, \chi)}.\]  

(5.27)
We will use now the representation (5.20) of the Ruelle zeta function. By the Poincaré duality we obtain

\[
R(s; \sigma, \chi) = Z(s; \sigma_{d-1} \otimes \sigma, \chi)^{(-1)^{d-1}} \\
= \prod_{\nu=0}^{d-3} Z(s + \rho - \lambda; \sigma_\nu \otimes \sigma, \chi)Z(s - \rho + \lambda; \sigma_\nu \otimes \sigma, \chi)^{(-1)^\rho}.
\]

If we substitute the expression above in (5.27), we have

\[
R^s(s; \sigma, \chi)R^s(-s; \sigma, \chi) = \frac{Z(s; \sigma_{d-1} \otimes \sigma, \chi)}{Z(-s; \sigma_{d-1} \otimes w\sigma, \chi)} \left( \prod_{\nu=0}^{d-3} \frac{Z(s + \rho - \lambda; \sigma_\nu \otimes \sigma, \chi)Z(s - \rho + \lambda; \sigma_\nu \otimes \sigma, \chi)^{(-1)^\rho}}{Z(-s; \sigma_{d-1} \otimes w\sigma, \chi)} \right) \prod_{\nu=0}^{d-3} \frac{Z(s + \rho - \lambda; \sigma_\nu \otimes \sigma, \chi)Z(s - \rho + \lambda; \sigma_\nu \otimes \sigma, \chi)^{(-1)^\rho}}{Z(-s; \sigma_{d-1} \otimes w\sigma, \chi)}.
\]

By Theorem 4.5, we get

\[
R^s(s; \sigma, \chi)R^s(-s; \sigma, \chi) = \left( e^{2i\pi\eta(0, D^E_\chi(\sigma \otimes \sigma_{d-1}))} \right)^{(-1)^{d+1}} \prod_{\nu=0}^{d-3} \left( e^{2i\pi\eta(0, D^E_\chi(\sigma \otimes \sigma_\nu))} \right)^{(-1)^\rho},
\]

where we used the fact that the Plancherel polynomial is an even function. Here, \( D^E_\chi(\sigma \otimes \sigma_\nu) \) denotes the Dirac operator acting on the space

\[
\bigoplus_{i=0}^p C^\infty(X, E_{rs}(\sigma) \otimes E_{\chi} \otimes (p - i) \Lambda^i T^* X), \quad p = 0, 1, \ldots, d - 1. \tag{5.28}
\]

Finally, we have

\[
R^s(s; \sigma, \chi)R^s(-s; \sigma, \chi) = e^{2i\pi \sum_{p=0}^{d-3} (-1)^p \eta(0, D^E_\chi(\sigma \otimes \sigma_\nu))} = e^{2i\pi \eta(D^E_{p, \chi}(\sigma))},
\]

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where \( \eta(D^\sharp_{p,\chi}(\sigma)) \) denotes the eta invariant of the operator \( D^\sharp_{p,\chi}(\sigma) \), which by definition of \( D^\sharp_{p,\chi}(\sigma) \) and equation (5.26), is given by

\[
\eta(D^\sharp_{p,\chi}(\sigma)) = \sum_{p=0}^{d-1} (-1)^p \eta(0, D^\sharp_{\chi}(\sigma \otimes \sigma_p)).
\]

For the functional equations (5.25) we have

\[
\frac{R(s; \sigma, \chi)^2}{R(-s; w\sigma, \chi)^2} = \frac{R(s; \sigma, \chi)}{R(-s; w\sigma, \chi)} \frac{R(-s; \sigma, \chi)}{R(s; \sigma, \chi)} \frac{R(s; w\sigma, \chi)}{R(-s; \sigma, \chi)} \frac{R(-s; w\sigma, \chi)}{R(s; w\sigma, \chi)} = e^{2i\pi \eta(D^\sharp_{p,\chi}(\sigma))} \frac{R(s; \sigma, \chi) R(s; w\sigma, \chi)}{R(-s; \sigma, \chi) R(-s; w\sigma, \chi)},
\]

where we have employed the functional equation for the super Ruelle zeta function (5.24). One can easily compute as in the proof of Theorem 5.2 (equation (5.22)) that

\[
\frac{R(s; w\sigma, \chi)}{R(-s; w\sigma, \chi)} = \exp \left( -4\pi (d + 1) \dim(V_\sigma) \dim(V_\chi) \Vol(X) s \right)
\]

Hence,

\[
\frac{R(s; \sigma, \chi)^2}{R(-s; w\sigma, \chi)^2} = e^{2i\pi \eta(D^\sharp_{p,\chi}(\sigma))} \exp 2 \left( -4\pi (d + 1) \dim(V_\sigma) \dim(V_\chi) \Vol(X) s \right).
\]

The assertion follows.

\[\Box\]

6 The determinant formula

By Lemma 3.2, we have

\[
\text{Tr}(e^{-tA^k_\chi(\sigma)}) \sim_{t \to 0^+} \dim(V_\chi) \sum_{j=0}^{\infty} a_j t^{j-\frac{d}{2}}.
\]

where

\[
a_j(x) = \int_X a_j(x) d\mu(x),
\]

\( a_j(x) \) is a smooth local invariant, and \( \mu(x) \) is the volume measure determined by the riemannian metric on \( X \).
Definition 6.1. The xi function associated to the operator $A_\chi^\sharp(\sigma)$ is defined by

$$\xi(z, s; \sigma) := \int_0^\infty e^{-t\lambda^2} \text{Tr}(e^{-tA_\chi^\sharp(\sigma)}) t^{z-1} dt,$$

for $\text{Re}(s^2) > C$, where $C \in \mathbb{R}$ and $\text{Re}(\lambda_i) > 0$, where $\lambda_i \in \text{spec}(A_\chi^\sharp(\sigma))$.

Definition 6.2. We define the generalized zeta function $\zeta(z, s; \sigma)$ by

$$\zeta(z, s; \sigma) = \frac{1}{\Gamma(z)} \int_0^\infty e^{-t\lambda^2} \text{Tr}(e^{-tA_\chi^\sharp(\sigma)}) t^{z-1} dt,$$

for $\text{Re}(s^2) > C$, where $C \in \mathbb{R}$ and $\text{Re}(\lambda_i) > 0$, where $\lambda_i \in \text{spec}(A_\chi^\sharp(\sigma))$.

The two functions converge absolutely and uniformly on compact subsets of the half-plane $\text{Re}(z) > \frac{d}{2}$. Furthermore, they are differentiable in $s \in \mathbb{C}$.

Lemma 6.3. The xi function $\xi(\cdot, s; \sigma)$ admits a meromorphic continuation to the whole complex plane $\mathbb{C}$. Furthermore, it has simple poles at $k_j = \frac{d}{2} - j$ with $\text{res}(k_j, \xi(\cdot, s; \sigma)) = a_j$.

Proof. We define the theta function $\theta(t)$ associated with the operator $e^{-tA_\chi^\sharp(\sigma)}$ by

$$\theta(t) := \text{Tr}(e^{-tA_\chi^\sharp(\sigma)}) = \sum_{\lambda_j \in \text{spec}(A_\chi^\sharp(\sigma))} m(\lambda_j) e^{-t\lambda_j},$$

where $m(\lambda_j)$ denotes the algebraic multiplicity of the eigenvalue $\lambda_j$. Then, $\xi(z, s; \sigma)$ defined by (6.2) is just the Mellin-Laplace transform of $\theta(t)$.

Since the spectrum of $A_\chi^\sharp(\sigma)$ is discrete and contained in a translate of a positive cone in $\mathbb{C}$, there are only finitely many eigenvalues $\lambda_j$ with $\text{Re}(\lambda_j) \leq 0$.

For $N \in \mathbb{N}$ with $N > 1$, we have

$$\left| \sum_{j=1}^\infty m(\lambda_j)e^{-t\lambda_j} - \sum_{j=1}^N m(\lambda_j)e^{-t\lambda_j} \right| = \left| \sum_{j=N+1}^\infty m(\lambda_j)e^{-t\lambda_j} \right| \leq \sum_{j=N+1}^\infty m(\lambda_j)e^{-t\text{Re}(\lambda_j)}.$$ (6.4)

We observe now that there are only finitely many eigenvalues $\lambda_j$ such that $|\lambda_j| \leq c$, where $c$ is positive constant. On the other hand, for every positive constant $c$ there exists a positive integer $N$ such that $\text{Re}(\lambda_j) \geq c$, for every $j \geq N$. We consider an
ordering $\Re(\lambda_{j_1}) \leq \Re(\lambda_{j_2}) \leq \Re(\lambda_{j_3}) \leq \ldots$ of the real parts of the eigenvalues with $\Re(\lambda_j) \geq c$. Then for $t \geq 1$,

$$\sum_{j=N+1}^{\infty} m(\lambda_j)e^{-t\Re(\lambda_j)} \leq \sum_{j=N+1}^{\infty} m(\lambda_j)e^{-t\Re(\lambda_j)/2}e^{-t\Re(\lambda_j)/2} \leq e^{-tc/2}\sum_{j=N+1}^{\infty} m(\lambda_j)e^{-\Re(\lambda_j)/2}. \quad (6.5)$$

To estimate the last sum, we will use the Weyl’s law for the non self-adjoint operator $A^\sharp_\chi(\sigma)$. Given a positive constant $c$, we define the counting function $N(c)$ by

$$N(c) := \sum_{\lambda_j \in \text{spec}(A^\sharp_\chi(\sigma))} m(\lambda_j).$$

In [Mül11], the generalization of the Weyl’s law for the non self-adjoint case is proved. By [Mül11, Lemma 2.2], we have

$$N(c) = \frac{\text{rank}(E(\sigma) \otimes E_\chi) \text{Vol}(X)}{(4\pi)^{d/2}\Gamma(d/2+1)} c^{d/2} + o(c^{d/2}), \quad d \to \infty, \quad (6.6)$$

where $\text{rank}(E(\sigma) \otimes E_\chi)$ denotes the rank of the product vector bundle $E(\sigma) \otimes E_\chi$. To use the Weyl’s law (6.6), we observe that for a real number $a > 1$ (the slope of the straight line of the cone, which all the eigenvalues $\lambda_j$ of $A^\sharp_\chi(\sigma)$ are contained in), we have

$$\sharp\{j: |\Re(\lambda_j)| \leq \lambda\} \leq \sharp\{j: |\lambda_j| \leq a\lambda\} \leq N(a\lambda).$$

By (6.6), we get

$$\sum_{j=N+1}^{\infty} m(\lambda_j)e^{-\Re(\lambda_j)/2} \leq \sum_{k=N+1}^{\infty} \sum_{k \leq \Re(\lambda_j) \leq k+1} m(\lambda_j)e^{-\Re(\lambda_j)/2} \leq \sum_{k=N+1}^{\infty} N(k+1)e^{-k/2} \leq \sum_{k=N+1}^{\infty} C_1(k+1)^{d/2}e^{-k/2} < \infty, \quad (6.7)$$

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where $C_1$ is a positive constant.

Hence, by (6.4), (6.5), (6.7) and the definition of the theta function, we have that given a positive number $C > 0$, there exist a positive integer $N$ and $K > 0$ such that

$$
|\theta(t) - \sum_{j=1}^{N} m(\lambda_j)e^{-t\lambda_j}| \leq Ke^{-Ct}, \quad t \geq 1. \tag{6.8}
$$

Furthermore, by the asymptotic expansion of the trace of the operator $e^{-tA_\chi^\sharp(\sigma)}$ (6.1), we have that for every positive integer $N$,

$$
\theta(t) - \sum_{j=0}^{N} a_j t^{\frac{d}{2} - j} = O(t^{N-\frac{d}{2}}), \quad t \to 0.
$$

All in all, we have proved that $\theta(t)$ satisfies the assumptions as in [JL93, AS 1, AS 2, p. 16]. Hence, we can apply [JL93, Theorem 1.5] for $p = j - \frac{d}{2}$ and obtain the meromorphic continuation of the xi function. The simple poles are located at $k_j = \frac{d}{2} - j$ with $\text{res}(k_j, \xi(\cdot, s; \sigma)) = a_j$.

Let $N(0) \subset \mathbb{C}$ be a neighborhood of zero in $\mathbb{C}$.

**Theorem 6.4.** For every $s \in N(0)$, the xi function $\xi(z, s; \sigma)$ is holomorphic at $z = 0$.

**Proof.** See [JL93, Theorem 1.6].

The generalized zeta function is by definition the xi function divided by $\Gamma(z)$:

$$
\zeta(z, s; \sigma) = \frac{1}{\Gamma(z)} \xi(z, s; \sigma). \tag{6.9}
$$

Consequently, it is also holomorphic at $z = 0$. It holds

$$
\left. \frac{d}{dz} \zeta(z, s; \sigma) \right|_{z=0} = \xi(0, s; \sigma). \tag{6.10}
$$

**Definition 6.5.** The regularized determinant of the operator $A_\chi^\sharp(\sigma) + s^2$ is defined by

$$
\det(A_\chi^\sharp(\sigma) + s^2) := \exp \left( - \left. \frac{d}{dz} \zeta(z, s; \sigma) \right|_{z=0} \right). \tag{6.11}
$$
By (6.10) and (6.11) we get
\[ \det(A_\chi^2(\sigma) + s^2) = \exp(-\xi(0, s; \sigma)). \]
Equivalently,
\[ \log(\det(A_\chi^2(\sigma) + s^2)) = -\xi(0, s; \sigma). \quad (6.12) \]

**Theorem 6.6.** Let \( \det(A_\chi^2(\sigma) + s^2) \) be the regularized determinant associated to the operator \( A_\chi^2(\sigma) + s^2 \). Then,

1. **case (a)** the Selberg zeta function has the representation
   \[ Z(s; \sigma, \chi) = \det(A_\chi^2(\sigma) + s^2) \exp\left(-2\pi \dim(V_\chi) \Vol(X) \int_0^s P_\sigma(t) \, dt\right). \quad (6.13) \]

2. **case (b)** the symmetrized zeta function has the representation
   \[ S(s; \sigma, \chi) = \det(A_\chi^2(\sigma) + s^2) \exp\left(-4\pi \dim(V_\chi) \Vol(X) \int_0^s P_\sigma(t) \, dt\right). \quad (6.14) \]

**Proof.** Proceeding as in the proof of Proposition 6.1 in [Spi15b], we get
\[ \text{Tr} \prod_{i=1}^N (A_\chi^2(\sigma) + s_i^2)^{-1} = \int_0^\infty \sum_{i=1}^N \left( \prod_{j=1}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} \text{Tr}(e^{-tA_\chi^2(\sigma)}) \, dt. \]
We have
\[
\int_0^\infty \sum_{i=1}^N \left( \prod_{j=1}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} \text{Tr}(e^{-tA_\chi^2(\sigma)}) \, dt
= \int_0^\infty \sum_{i=1}^N \left( \prod_{j=1}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i} \frac{d}{ds_i} e^{-ts_i^2} \text{Tr}(e^{-tA_\chi^2(\sigma)}) \, dt.
\]
(6.15)
For $\text{Re}(z) > d/2$, we consider the limit as $z \to 0$

\[
\lim_{z \to 0} \int_0^\infty \sum_{i=1}^N \left( \prod_{j=1, j \neq i}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i} \left( - \frac{d}{ds_i} e^{-ts_i^2} \right) t^{z-1} \text{Tr}(e^{-tA\chi(\sigma)}) dt
\]

\[
= \sum_{i=1}^N \left( \prod_{j=1, j \neq i}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i} \frac{d}{ds_i} \int_0^\infty -e^{-ts_i^2} t^{z-1} \text{Tr}(e^{-tA\chi(\sigma)}) dt.
\]

Hence, the right hand side of (6.15) gives

\[
\int_0^\infty \sum_{i=1}^N \left( \prod_{j=1, j \neq i}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i} \left( - \frac{d}{ds_i} e^{-ts_i^2} \right) \text{Tr}(e^{-tA\chi(\sigma)}) dt
\]

\[
= \lim_{z \to 0} \int_0^\infty \sum_{i=1}^N \left( \prod_{j=1, j \neq i}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i} \left( - \frac{d}{ds_i} e^{-ts_i^2} \right) t^{z-1} \text{Tr}(e^{-tA\chi(\sigma)}) dt
\]

\[
= \sum_{i=1}^N \left( \prod_{j=1, j \neq i}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i} \frac{d}{ds_i} \int_0^\infty -e^{-ts_i^2} t^{z-1} \text{Tr}(e^{-tA\chi(\sigma)}) dt
\]

\[
= \sum_{i=1}^N \left( \prod_{j=1, j \neq i}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i} \frac{d}{ds_i} \left( \log \left( \det(A\chi(\sigma) + s_i^2) \right) \right),
\]

where in the last equation we used (6.12). Therefore, (6.15) becomes

\[
\int_0^\infty \sum_{i=1}^N \left( \prod_{j=1, j \neq i}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} \text{Tr}(e^{-tA\chi(\sigma)}) dt
\]

\[
= \sum_{i=1}^N \left( \prod_{j=1, j \neq i}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i} \frac{d}{ds_i} \log \det(A\chi(\sigma) + s_i^2) \tag{6.16}
\]

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We treat here the case (b). One can proceed similarly for the case (a). The left-hand side of (6.16) can be developed more, if we insert the right-hand side of the trace formula \[\text{Spi15b, Theorem 5.10}\] for the operator \(e^{-tA^\sharp(\sigma)}\). We have

\[
\int_0^\infty \sum_{i=1}^N \left( \prod_{j=1 \atop j \neq i}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} \text{Tr}(e^{-tA^\sharp(\sigma)}) dt = \int_0^\infty \sum_{i=1}^N \left( \prod_{j=1 \atop j \neq i}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} \left( 2 \dim(V_\chi) \text{Vol}(X) \int_\mathbb{R} e^{-t\lambda^2} P_\sigma(i\lambda) d\lambda \right.
\]

\[
+ \sum_{[\gamma] \neq e} \frac{l(\gamma) \text{tr}(\chi(\gamma))}{n_\Gamma(\gamma)} L_{sym}(\gamma; \sigma + w\sigma) e^{-t(\gamma)^2/4t} \left(4\pi t\right)^{1/2} dt.
\]

By \[\text{Spi15a, Lemma 6.4}\], we can interchange the order of integration for the double integral

\[
\int_0^\infty \int_\mathbb{R} \sum_{i=1}^N \left( \prod_{j=1 \atop j \neq i}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} e^{-t\lambda^2} P_\sigma(i\lambda) d\lambda dt.
\]

We use the Cauchy integral formula to calculate this integral. For the calculation of the integral that corresponds to the hyperbolic contribution, we make again use of the identity (see \[\text{EMOT54, p. 146, (27)}\])

\[
\int_0^\infty e^{-ts^2} e^{-t(\gamma)^2/4t} dt = \frac{1}{2s} e^{-s l(\gamma)}.
\]

Hence,

\[
\int_0^\infty \sum_{i=1}^N \left( \prod_{j=1 \atop j \neq i}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} \left( 2 \dim(V_\chi) \text{Vol}(X) \int_\mathbb{R} e^{-t\lambda^2} P_\sigma(i\lambda) d\lambda \right.
\]

\[
+ \sum_{[\gamma] \neq e} \frac{l(\gamma) \text{tr}(\chi(\gamma))}{n_\Gamma(\gamma)} L_{sym}(\gamma; \sigma + w\sigma) e^{-t(\gamma)^2/4t} \left(4\pi t\right)^{1/2} dt
\]

\[
= \sum_{i=1}^N \left( \prod_{j=1 \atop j \neq i}^N \frac{1}{s_j^2 - s_i^2} \right) 2\pi \dim(V_\chi) \text{Vol}(X) P_\sigma(s_i)
\]

\[
+ \sum_{i=1}^N \left( \prod_{j=1 \atop j \neq i}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i} \sum_{[\gamma] \neq e} \frac{l(\gamma) \text{tr}(\chi(\gamma))}{n_\Gamma(\gamma)} L_{sym}(\gamma; \sigma + w\sigma) e^{-s_i l(\gamma)}.
\]
By (6.16), we get

\[
\sum_{i=1}^{N} \left( \prod_{j=1, j \neq i}^{N} \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i} \frac{d}{ds_i} \log \det(A_\chi^2(\sigma) + s_i^2)
\]

\[
= \sum_{i=1}^{N} \left( \prod_{j=1, j \neq i}^{N} \frac{1}{s_j^2 - s_i^2} \right) \frac{2\pi}{s_i} \dim(V_\chi) \Vol(X)P_\sigma(s_i)
\]

\[
+ \sum_{i=1}^{N} \left( \prod_{j=1, j \neq i}^{N} \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i} \sum_{[\gamma] \neq e} \frac{l(\gamma) \tr(\chi(\gamma))}{n_\Gamma(\gamma)} L_{sym}(\gamma; \sigma + w\sigma)e^{-s_i l(\gamma)}.
\]

(6.17)

We fix now the variables \(s_2, \ldots, s_N \in \mathbb{C}\) and let the variable \(s_1 = s \in \mathbb{C}\) vary. Then, we can remove the structure

\[
\sum_{i=1}^{N} \left( \prod_{j=1, j \neq i}^{N} \frac{1}{s_j^2 - s_i^2} \right),
\]

and get

\[
\frac{d}{ds} \log \det(A_\chi^2(\sigma) + s^2) = 4\pi \dim(V_\chi) \Vol(X)P_\sigma(s)
\]

\[
+ \sum_{[\gamma] \neq e} \frac{l(\gamma) \tr(\chi(\gamma))}{n_\Gamma(\gamma)} L_{sym}(\gamma; \sigma)e^{-sl(\gamma)}
\]

\[
+ K'(s),
\]

(6.18)

where \(K'(s)\) is a certain odd polynomial, which is of the form

\[
K'(s) = \prod_{j=2}^{N} (s_j^2 - s^2)2sQ(s_2, \ldots, s_N).
\]

The quantity \(Q(s_2, \ldots, s_N)\) comes form the terms that correspond to the summands over \(i = 2, \ldots, N\) and hence it has a fixed value in \(\mathbb{C}\), since \(s_2, \ldots, s_N\) are fixed.

Next, we can substitute the term that comes from the hyperbolic distribution of the trace formula with the logarithmic derivative of the symmetrized zeta function. By
[Spi15b, equation (3.7)] we have

\[
\frac{d}{ds} \log\det(A^\sharp_\chi(\sigma) + s^2) = 4\pi \dim(V_\chi) \Vol(X) P_\sigma(s) \\
+ L_s(s) + K'(s).
\]

We integrate with respect to \( s \) and get

\[
\log\det(A^\sharp_\chi(\sigma) + s^2) = 4\pi \dim(V_\chi) \Vol(X) \int_0^s P_\sigma(t) dt \\
+ \log S(z, s) + K(s).
\]

Hence,

\[
\log S(z, s) = \log\det(A^\sharp_\chi(\sigma) + s^2) - K(s) \\
- 4\pi \dim(V_\chi) \Vol(X) \int_0^s P_\sigma(t) dt.
\]

(6.19)

We want to show that \( K(s) = 0 \). For that reason, we study the asymptotic behavior of all terms in equation (8.19), as \( s \to \infty \). By [Spi15b, equation (3.7)], \( \log S(z, s) \) decreases exponentially as \( s \to \infty \).

We use now the asymptotic expansion of \( \log\det(A^\sharp_\chi(\sigma) + s^2) \) as \( s \to \infty \), as it is described in [QHS93, p. 219-220]. We write the short time asymptotic expansion (6.1) of the trace of the operator \( e^{-tA^\sharp_\chi(\sigma)} \) as

\[
\Tr(e^{-tA^\sharp_\chi(\sigma)}) \sim_{t \to 0} \sum_{\nu=0}^\infty c_{j_\nu} t^{j_\nu},
\]

where \( j_\nu = j - \frac{d}{2} \), and we use the formula from [QHS93, equation (13)]. In our case, there are no coefficients \( c_{j_\nu} \) that corresponds to integers \( j_\nu' \), because \( d \) is odd and hence \( j_\nu' = 0 \). We have

\[
\log\det(A^\sharp_\chi(\sigma) + s^2) \sim_{s \to \infty} \sum_{k=0}^\infty c_{(2k-d)/2} \Gamma((2k - d)/2) s^{d-2k}.
\]

(6.20)

The right hand side of (6.20) contains only odd powers of \( s \). On the other hand, the Plancherel polynomial is an even polynomial of \( s \). Regarding (6.19), as \( s \to \infty \), we have that an odd polynomial equals an even one. Therefore, the coefficients \( c_{(2k-d)/2} \)
vanish, as well as the coefficients of the even polynomial $K(s)$.
Finally, exponentiating equation (6.19) for $K(s) = 0$, we obtain
\[ S(z, s) = \det(A^2_\chi(\sigma) + s^2) \exp \left( -4\pi \dim(V_\chi) \Vol(X) \int_0^s P_\sigma(t)dt \right). \quad (6.21) \]

We prove now a determinant formula for the Ruelle zeta function. We define the operator
\[ A^2_\chi(\sigma_p \otimes \sigma) := \bigoplus_{\sigma' \in \hat{M}} \bigoplus_{i=1}^{[(\sigma_p \otimes \sigma):\sigma']} A_\chi(\sigma') \quad (6.22) \]
acting on the space $C^\infty(X, E(\sigma') \otimes E_\chi)$, where $\sigma \in \hat{M}$, $E(\sigma')$ is the vector bundle over $X$, and $\sigma_p$ denotes the $p$-th exterior power of the standard representation of $M$.
We distinguish again two cases for $\sigma' \in \hat{M}$.

- **case (a):** $\sigma'$ is invariant under the action of the restricted Weyl group $W_A$.
  Then, $i^*(\tau) = \sigma'$, where $\tau \in R(K)$.

- **case (b):** $\sigma'$ is not invariant under the action of the restricted Weyl group $W_A$.
  Then, $i^*(\tau) = \sigma' + w\sigma'$, where $\tau \in R(K)$.

The vector bundle $E(\sigma')$ is constructed as in [Spi15a, p. 27] for case (a) and [Spi15b, p. 14] for case (b).

**Proposition 6.7.** The Ruelle zeta function has the representation

- **case (a)**
  \[ R(s; \sigma, \chi) = \prod_{p=0}^d \det(A^2_\chi(\sigma_p \otimes \sigma) + (s + \rho - \lambda)^2)^{(-1)^p} \exp \left( -2\pi (d + 1) \dim(V_\chi) \dim(V_\sigma) \Vol(X)s \right). \quad (6.23) \]

- **case (b)**
  \[ R(s; \sigma, \chi)R(s; w\sigma, \chi) = \prod_{p=0}^d \det(A^2_\chi(\sigma_p \otimes \sigma) + (s + \rho - \lambda)^2)^{(-1)^p} \exp \left( -4\pi (d + 1) \dim(V_\chi) \dim(V_\sigma) \Vol(X)s \right). \quad (6.24) \]
Proof. We prove the assertion for case (b). One can proceed similarly for case (a). By [Spi15a, Proposition 6.6], we have the expression of the Ruelle zeta function as a product of Selberg zeta functions. Then, we see

\[
R(s; \sigma, \chi)R(s; w\sigma, \chi) = \prod_{p=0}^{d-1} Z(s + \rho - \lambda; \sigma_p \otimes \sigma, \chi)^{(-1)^p} \prod_{p=0}^{d-1} Z(s + \rho - \lambda; \sigma_p \otimes w\sigma, \chi)^{(-1)^p}
\]

\[
= \prod_{p=0}^{d-1} S(s + \rho - \lambda; \sigma_p \otimes \sigma, \chi)^{(-1)^p}.
\]

Hence, if we equip the determinant formula for the symmetrized zeta function (Theorem 6.6.2), we have

\[
R(s; \sigma, \chi)R(s; w\sigma, \chi) = \prod_{p=0}^{d-1} \det(A^2_{\chi}(\sigma_p \otimes \sigma) + (s + \rho - \lambda)^2)^{(-1)^p} \exp \left( \sum_{p=0}^{d-1} (-1)^p (-4\pi \dim(V_{\chi}) \dim(V_{\sigma}) \Vol(X)) \int_0^{s + \rho - \lambda} P_{\sigma_p \otimes \sigma}(t) dt \right).
\]

On the other hand,

\[
\sum_{p=0}^{d-1} (-1)^p \int_0^{s + \rho - \lambda} P_{\sigma_p \otimes \sigma}(t) dt = \int_0^{s} f(t) dt,
\]

where \(f(t)\) is defined as in (5.2). Therefore, by Lemma 5.1,

\[
\sum_{p=0}^{d-1} (-1)^p \int_0^{s + \rho - \lambda} P_{\sigma_p \otimes \sigma}(t) dt = (d + 1) \dim(V_{\sigma}) s. \quad (6.26)
\]

We substitute equation (6.26) in (6.25) and we get

\[
R(s; \sigma, \chi)R(s; w\sigma, \chi) = \prod_{p=0}^{d-1} \det(A^2_{\chi}(\sigma_p \otimes \sigma) + (s + \rho - \lambda)^2)^{(-1)^p} \exp \left( -4\pi (d + 1) \dim(V_{\chi}) \dim(V_{\sigma}) \Vol(X) s \right).
\]

\[\Box\]
7 The refined analytic torsion

We will recall first the definition of the analytic torsion $T^{RS}_X(\chi; E_\chi)$. Let $X$ be an oriented compact odd dimensional riemannian manifold. We let $(\chi, V_\chi)$ be a finite dimensional representation of $\Gamma$. Let $E_\chi$ be the associated flat vector bundle over $X$. We choose a hermitian metric $h$ in $E_\chi$. Let $g$ be the riemannian metric on $X$.

In general, the analytic torsion does depend on the riemannian metric $g$ and the hermitian metric $h$. However, by [Müll93, Corollary 2.7], if $\dim X$ is an odd integer and $\chi$ is considered to be acyclic (see Assumption 7.3), then, the analytic torsion does not depend on $g$ and $h$. Hence, instead of $T^{RS}_X(\chi; E_\chi)(g, h)$ we simply write $T^{RS}_X(\chi; E_\chi)$.

We follow [RS71, p.148-151] and [MM63, p.370-372]. Let $\Lambda^p(X, E_\chi)$ be the space of smooth differential $p$-forms on $X$ with values in $E_\chi$. We define the space $C^\infty(G, \Lambda^p \Lambda^{p*} \otimes V_\chi)$ by

$$C^\infty(G, \Lambda^p \Lambda^{p*} \otimes V_\chi) := \{ f \in C^\infty(G) : f(kg) = \mu_p^{-1}(k)g, \forall g \in G, \forall k \in K, \quad f(\gamma g) = f(g), \forall g \in G, \forall \gamma \in \Gamma \},$$

where $\mu_p$ denotes the $p$-th exterior power of the $Ad^*$-representation of $K$ in $p$, i.e.,

$$\mu_p := \Lambda^p Ad^*: K \to GL(\Lambda^p \Lambda^{p*}).$$

Then, there exists an isomorphism

$$\Lambda^p(X, E_\chi) \cong C^\infty(G, \Lambda^p \Lambda^{p*} \otimes V_\chi)$$

(see [MP13, p. 12, and p. 15].

Let $\phi$ be a $p$-differential form on the universal covering $\tilde{X}$ with values in $E_\chi$. Then, $\gamma \in \Gamma$ acts on $\phi$ by $\gamma^* \phi = \chi(\gamma)\phi$, where the element $\gamma$ acts on $\tilde{X}$ by deck transformations. Let $d_\chi : \Lambda^p(X, E_\chi) \to \Lambda^{p+1}(X, E_\chi)$ be the exterior derivative operator. Since $E_\chi$ is flat, $d_\chi \circ d_\chi = 0$. Hence, we obtain a de Rham complex

$$\Lambda^0(X, E_\chi) \xrightarrow{d_\chi} \Lambda^1(X, E_\chi) \xrightarrow{d_\chi} \cdots \xrightarrow{d_\chi} \Lambda^d(X, E_\chi).$$

We denote by $H^p(X; E_\chi)$ the $p$-th cohomology group of this complex.

We want to describe $\phi \in \Lambda^p(X, E_\chi)$ locally. Let $(E_\chi)_x$ be the fiber over $x \in X$ and $(E_\chi)_x^*$ its dual vector space. We denote by $(\cdot, \cdot)$ the canonical bilinear pairing on $E_\chi \times E_\chi^*$. For $x \in X$, let $(dx^1, dx^2, \ldots, dx^d)$ be the canonical basis of $T^*_x X$. 

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consisting of \( \mathbb{C} \)-valued 1-differential forms on \( X \), associated with a local coordinate system \( (x^1, x^2, \ldots, x^d) \). Then, \( \phi \) can be written as

\[
\phi = \sum_{i_1 < \ldots < i_p} u_{i_1 \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p},
\]

where \( u_{i_1 \ldots i_p} \) are smooth sections of \((E_\chi)_x\). Let \( \omega \) be a \( q \)-differential form with values in the dual vector bundle \( E_\chi^* \). This means that \( \omega \) can be written as

\[
\omega = \sum_{j_1 < \ldots < j_q} v_{j_1 \ldots j_q} dx^{j_1} \wedge \ldots \wedge dx^{j_q},
\]

where \( v_{j_1 \ldots j_q} \) are smooth sections of \((E_\chi^*)_x\). We define the wedge product by the \((p + q)\)-differential form, given by

\[
\phi \wedge \omega = \sum_{i_1 < \ldots < i_p} \sum_{j_1 < \ldots < j_q} \langle u_{i_1 \ldots i_p}, v_{j_1 \ldots j_q} \rangle dx^{i_1} \wedge \ldots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_q}.
\]

We define the isomorphism \( \sharp : E_\chi \to E_\chi^* \), induced by the metric \( h \), as

\[
\sharp(v_x)(u_x) := \langle \sharp(v_x), u_x \rangle = h(v_x, u_x),
\]

where \( v_x \in E_\chi^*, u_x \in (E_\chi)^*_x \). We extend this isomorphism to

\[
\sharp : \Lambda^p(X, E_\chi) \to \Lambda^p(X, E_\chi^*).
\]

The riemannian metric on \( X \) defines an operator

\[
*: \Lambda^p(X, E_\chi) \to \Lambda^{d-p}(X, E_\chi),
\]

acting as

\[
*\phi = \sum_{i_1 < \ldots < i_p} u_{i_1 \ldots i_p} *(dx^{i_1} \wedge \ldots \wedge dx^{i_p}),
\]

where \( * \) in the right hand side of the equation above acts as the usual \( * \)-operator on \( \mathbb{C} \)-valued differential forms on \( X \). For every \( p = 0, \ldots, d \), we have \( ** = (-1)^{p(d-p)} \text{Id} \) on \( \Lambda^p(X, E_\chi) \). We consider the following composition

\[
* \circ \sharp := (\ast \otimes \text{Id}) \circ (\text{Id} \otimes \sharp) : \Lambda^p T^* X \otimes E_\chi \to \Lambda^{d-p} T^* X \otimes E_\chi^*.
\]

We define the inner product on \( \Lambda^p(X, E_\chi) \) by

\[
(\theta, \phi) := \int_X \theta \wedge * \circ \sharp \phi. \tag{7.1}
\]
Then, the formal adjoint of $d\chi$ with respect to the inner product (7.1) is the operator $\delta$ on $\Lambda^p(X, E^\chi)$ given by

$$\delta\chi = (-1)^{d(p+1)+1} \circ \delta^p \circ d\chi \circ \delta^p \circ \ast.$$  

We define the Hodge-Laplace operator $\Delta_{\chi,p} : \Lambda^p(X, E^\chi)$ by

$$\Delta_{\chi,p} := d\chi \delta\chi + \delta\chi d\chi.$$  

(7.2)

The operator $\Delta_{\chi,p}$ is an elliptic positive essentially self-adjoint operator. Let $H^p(X, E^\chi) := \ker(\Delta_{\chi,p})$ be the subspaces of the $p$-harmonic forms. By Hodge theory, we have the isomorphism

$$H^p(X; E^\chi) \cong H^p(X, E^\chi).$$

Let $\Delta'_{\chi,p}$ be the restriction of the Hodge-Laplacian $\Delta_{\chi,p}$ to the orthogonal complement of $H^p(X, E^\chi)$ with respect to the inner product (7.1). We define the zeta function $\zeta_{\Delta'_{\chi,p}}(z)$ of $\Delta'_{\chi,p}$ by

$$\zeta_{\Delta'_{\chi,p}}(z) := \text{Tr}(\Delta'_{\chi,p})^{-z},$$

for $\text{Re}(z) > d/2$. It is a well-known fact (see [Gil95, Lemma 1.10.1]) that $\zeta_{\Delta'_{\chi,p}}(z)$ admits a meromorphic continuation to the whole complex plane $\mathbb{C}$ and is regular at $s = 0$. Similarly to the Definition 6.5, we define the regularized determinant of $\Delta'_{\chi,p}$ by

$$\det(\Delta'_{\chi,p}) := \exp \left( -\frac{d}{dz} \zeta_{\Delta'_{\chi,p}}(z) \bigg|_{z=0} \right).$$  

(7.3)

**Definition 7.1.** We define the Ray-Singer analytic torsion associated with a finite dimensional unitary complex representation of $\Gamma$ by the formula

$$\log T^{RS}_X (\chi; E^\chi) := \frac{1}{2} \sum_{p=0}^{d} (-1)^p p \zeta'_{\Delta'_{\chi,p}}(0).$$  

(7.4)

Equivalently, by (7.3) the analytic torsion $T^{RS}_X (\chi; E^\chi)$ can be expressed by the regularized determinants of the Laplacians $\Delta'_{\chi,p}$:

$$T^{RS}_X (\chi; E^\chi) = \prod_{p=0}^{d} (\det(\Delta'_{\chi,p}))^{-\frac{1}{2} p (-1)^p}.$$  

(7.5)

We mention here that since $\Delta'_{\chi,p}$ is a positive essentially self-adjoint operator, the analytic torsion is a positive real number, i.e., $T^{RS}_X (\chi; E^\chi) \in \mathbb{R}^+$.  

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We will give here a short description of the refined analytic torsion $T^C_X(\chi; E_\chi)$ associated with the representation $\chi$ of $\Gamma$. As its name declares, it is a refinement of the Ray-Singer analytic torsion $T^{RS}_X(\chi; E_\chi)$. Whereas the Ray-Singer analytic torsion is a positive real number, the refined analytic torsion is, in general, a complex number.

If $\chi$ is unitary, then the refined analytic torsion can be expressed as the product of the Ray-Singer analytic torsion and a phase factor, which involves the eta invariant of the odd signature operator (see Definition 7.2). Hence, in this case the absolute value of the refined analytic torsion is equal to the Ray-Singer analytic torsion (see Remark 7.8).

We begin with the following definitions as they are given in [BK05] and [BK08]. Let $d = 2n + 1, n \in \mathbb{N}$ be the dimension of $X$. Let $\ast : \Lambda^k(X, E_\chi) \rightarrow \Lambda^{d-k}(X, E_\chi)$ be the $\ast$-operator with respect to the riemannian metric $g$. Let $\Gamma$ be the operator acting on $\Lambda^k(X, E_\chi)$, defined by $\Gamma := i^n(-1)^{(k+1)/2}\ast$.

**Definition 7.2.** We define the odd signature operator $B = B(\nabla, g) : \Lambda^k(X, E_\chi) \rightarrow \Lambda^k(X, E_\chi)$ by

$$B = \Gamma \nabla + \nabla \Gamma.$$  

Explicitly, for a $\omega \in \Lambda^p(X, E_\chi)$ one has

$$B\omega = i^n(-1)^{p+1/2}(\ast \nabla - \nabla \ast)\omega \in \Lambda^{d-p-1}(X, E_\chi) \oplus \Lambda^{d-p+1}(X, E_\chi).$$  

(7.6)

The odd signature operator is an elliptic operator of order 1, but no longer self-adjoint. This is because we assumed that $\chi$ is non-unitary and hence the corresponding flat connection $\nabla$ is not hermitian, i.e., there is no hermitian metric $h$ in $E_\chi$, which is compatible with $\nabla$. Nevertheless, it has a self-adjoint principal symbol, and hence it has nice spectral properties. Namely, its spectrum is discrete and contained in a translate of a cone in $\mathbb{C}$. In addition, we can define the eta function and the eta invariant of the operator $B$. We set

$$\Lambda^{even}(X, E_\chi) := \bigoplus_{p=0}^{n-1} \Lambda^{2p}(X, E_\chi),$$

$$B_{even} := \bigoplus_{p=0}^{n-1} B_{2p} : \Lambda^{even}(X, E_\chi) \rightarrow \Lambda^{even}(X, E_\chi),$$

where $B_{2p}$ denotes the operator $B$ acting on $2p$-differential forms. We call the operator $B_{even}$ the even part of $B$. Since it acts in differential forms of even degree, it can be slightly simplified. We have for a $\omega \in \Lambda^{2p}(X, E_\chi)$,

$$B^{even}\omega = i^n(-1)^{p+1}(\ast \nabla - \nabla \ast)\omega \in \Lambda^{d-2p-1}(X, E_\chi) \oplus \Lambda^{d-2p+1}(X, E_\chi).$$  

(7.7)
In order to define the refined analytic torsion we need the following two assumptions.

**Assumption 7.3.** The representation \( \chi \) of \( \Gamma \) is acyclic, i.e., \( H^p(X;E_\chi) = 0 \), for every \( p = 0, \ldots, d \).

**Assumption 7.4.** The even part \( B^{\text{even}} \) of the odd signature operator is bijective.

We define \( \Lambda^k_+(X, E_\chi) := \ker(\Delta \Gamma) \cap \Lambda^k(X, E_\chi) \) and \( \Lambda^k_-(X, E_\chi) := \ker(\Delta \Gamma) \cap \Lambda^k(X, E_\chi) \), where \( k = 0, \ldots, d \). By Assumption 7.4, we have that \( \Lambda^k(X, E_\chi) = \Lambda^k_+(X, E_\chi) \oplus \Lambda^k_-(X, E_\chi) \), and hence we obtain a grading on \( \Lambda^{2p}(X, E_\chi) \). We put

\[
\Lambda^{2p}_+(X, E_\chi) := \ker(\Delta \Gamma) \cap \Lambda^{2p}_+(X, E_\chi)
\]

\[
\Lambda^{2p}_-(X, E_\chi) := \ker(\Delta \Gamma) \cap \Lambda^{2p}_-(X, E_\chi).
\]

We define

\[
\Lambda^{\text{even}}_\pm(X, E_\chi) := \bigoplus_{p=0}^d \Lambda^{2p}_\pm(X, E_\chi),
\]

and let \( B^{\text{even}}_\pm \) be the restriction of \( B^{\text{even}} \) to \( \Lambda^{\text{even}}_\pm(X, E_\chi) \). Then, \( B^{\text{even}} \) leaves the subspaces \( \Lambda^{\text{even}}_\pm(X, E_\chi) \) invariant. It follows from Assumption 7.4 that the operators

\[
B^{\text{even}}_\pm : \Lambda^{\text{even}}_\pm(X, E_\chi) \to \Lambda^{\text{even}}_\pm(X, E_\chi)
\]

are bijective.

Let \( \theta \in (-\pi, 0) \) be an Agmon angle for \( B^{\text{even}} \). Then, \( \theta \) is an Agmon angle for \( B^{\text{even}}_\pm \) as well. The graded determinant of the operator \( B^{\text{even}} \) is a non-zero complex number defined by

\[
det_{\text{gr}, \theta}(B^{\text{even}}) := \frac{\det_\theta(B^{\text{even}}_+)}{\det_\theta(B^{\text{even}}_-)}.
\]

We set

\[
\xi_\chi = \xi_\chi(\nabla, g, \theta) := \frac{1}{2} \sum_{k=0}^d (-1)^k \zeta'_{2\theta}(0, (\Gamma \nabla)^2 |_{\Lambda^k_+(X, E_\chi)}) ,
\]

where \( \zeta'_{2\theta}(0, (\Gamma \nabla)^2 |_{\Lambda^k_+(X, E_\chi)}) \) is the derivative with respect to \( z \) of the zeta function, of the operator \( (\Gamma \nabla)^2 |_{\Lambda^k_+(X, E_\chi)} \) corresponding to the spectral cut along the ray \( R_\theta \).

**Theorem 7.5.** Let \( \theta \in (-\pi/2, 0) \) be an Agmon angle for \( B^{\text{even}} \) such that there are no eigenvalues of \( B^{\text{even}} \) in the solid angles \( L_{(-\pi/2,\theta]} \) and \( L_{(\pi/2,\theta+\pi]} \). Then

\[
det_{\text{gr}, \theta}(B^{\text{even}}) = e^{\xi_\chi} e^{-i\pi \eta(B^{\text{even}})}.
\]
Proof. This is proved in \cite[Theorem 7.2]{BK08}.

Definition 7.6. Let $\chi : \Gamma \to \GL(V_\chi)$ be a finite dimensional complex representation of $\Gamma$, such that Assumptions 7.3 and 7.4 are satisfied. Let the operator $B^{\text{even}} : \omega \in \Lambda^{2p}(X, E_\chi) \to \Lambda^{d-2p-1}(X, E_\chi) \oplus \Lambda^{d-2p+1}(X, E_\chi)$ be as in (7.7), and $\theta \in (-\pi, 0)$ be an Agmon angle for $B^{\text{even}}$. Then, we define the refined analytic torsion $T^\mathbb{C}_\chi(X; E_\chi) \in \mathbb{C} - \{0\}$ by the formula

$$T^\mathbb{C}_\chi(X; E_\chi) := \det_{\text{gr}, \theta}(B^{\text{even}}) e^{i\pi \text{rank}(E_\chi) \eta^{\text{triv}}(B^{\text{even}})},$$

(7.9)

where $\eta^{\text{triv}}(B^{\text{triv}})$ denotes the eta invariant of the even part of the odd signature operator $B^{\text{triv}}$, associated with the trivial connection on the trivial line bundle $E_{\chi, \text{triv}}$ over $X$.

By \cite{APS75}, if $\dim X \equiv 1 \mod 4$, then $\eta^{\text{triv}}(B^{\text{triv}}) = 0$. Then, we obtain

$$T^\mathbb{C}_\chi(X; E_\chi) := \det_{\text{gr}, \theta}(B^{\text{even}}).$$

(7.10)

In general, the refined analytic torsion does depend on the riemannian metric $g$, as well as on $\nabla$ and $\theta$. Nevertheless, if we consider the set $\mathcal{M}(\nabla)$ of riemannian metrics $g$ that they are admissible for $\nabla$, i.e., the operator $B^{\text{even}}$ satisfies Assumption 7.4, then for an acyclic representation $\chi$ of $\Gamma$ (that is Assumption 7.3 is satisfied), $\mathcal{M}(\nabla)$ is non-empty.

Theorem 7.7. Let $E_\chi \to X$ be a flat vector bundle associated with an acyclic finite dimensional representation $\chi$ of $\Gamma$ over a closed oriented odd dimensional manifold $X$. Let $\nabla$ be the flat connection on $E_\chi$. For each $g \in \mathcal{M}(\nabla)$, the refined analytic torsion $T^\mathbb{C}_\chi(X; E_\chi)$ is independent of the riemannian metric $g$ and the Agmon angle $\theta$.

Proof. See \cite[Theorem 9.3]{BK08}.

Hence, we write $T^\mathbb{C}_\chi(X; E_\chi) \equiv T^\mathbb{C}_\chi(X; E_\chi)(g, \nabla, \theta)$.

Remark 7.8. If the representation $\chi$ of $\Gamma$ is unitary then the expression of $\xi_\chi$ in (7.8) coincides with the expression of the logarithm of the analytic torsion $T^{\text{RS}}_\chi(X; E_\chi)$ in (7.4), i.e.

$$\xi_\chi = \log T^{\text{RS}}_\chi(X; E_\chi),$$

Hence, by Theorem 7.5,

- if $\dim X \equiv 1 \mod 4$,

$$T^\mathbb{C}_\chi(X; E_\chi) = T^{\text{RS}}_\chi(X; E_\chi)e^{-i\pi \eta(B^{\text{even}})};$$
• if \( \dim X \equiv 3 \mod 4 \),
\[
T^C_\chi(X; E_\chi) = T^{RS}_\chi(X; E_\chi) e^{-i\pi \eta(B^{even})} e^{i\pi \text{rank}(E_\chi) \eta_{triv}(B^{even})}.
\]

If the representation \( \chi \) of \( \Gamma \) is not unitary, then by [BK08, Theorem 8.2] we have
\[
\text{Re}(\xi_\chi) = \log T^{RS}_\chi(X; E_\chi).
\]

By Theorem 7.5, we get
\[
|\det_{gr, \theta}(B_{even})| = T^{RS}_\chi(X; E_\chi) e^{\text{Im}(\pi \eta(B_{even}))}.
\]

Since \( B_{triv} \) is self-adjoint, its eta invariant \( \eta(B_{triv}) \) is a real number (see [BK08, Section 9.1]). Hence, for every odd integer \( d = \dim X \), we have
\[
|T^C_\chi(X; E_\chi)| = T^{RS}_\chi(X; E_\chi) e^{\text{Im}(\pi \eta(B^{even}))}.
\]

Proposition 6.7 gives an interpretation of the Ruelle zeta function in terms of the determinants of the operators \( A^p_{\tau, \chi}(\sigma \otimes \sigma) + (s + \rho - \lambda)^2, \quad p = 0, \ldots, d \). By definition, the operators \( A^p_{\tau, \chi}(\sigma \otimes \sigma) \) act on the smooth sections of the vector bundle \( E(\sigma') \otimes E_\chi \).

We consider an acyclic representation \( \chi \) of \( \Gamma \) for defining the refined analytic torsion. If one could apply the well-known Hodge theory, then
\[ H^p(X; E_\chi) \cong \mathcal{H}^p(X, E_\chi), \]
where \( \mathcal{H}^p(X, E_\chi) := \ker(A^p_{\tau, \chi}(\sigma \otimes \sigma) + (\rho - \lambda)^2) \). In this case, we could easily get that \( \ker(A^p_{\tau, \chi}(\sigma \otimes \sigma) + (\rho - \lambda)^2) = 0 \). The situation now is that one can not conclude the triviality of the kernels of the operators. Hence, the regularity of the Ruelle zeta function at zero is not trivial.

**Conjectural Equality**

• case (a)
\[
R(0; \sigma, \chi) = \prod_{p=0}^{d} \det(A^p_{\tau, \chi}(\sigma \otimes \sigma) + (\rho - \lambda)^2)^{(-1)^p}.
\]

• case (b)
\[
R(0; \sigma, \chi) = e^{i \pi \eta(D_{p, \chi}^{even})} \prod_{p=0}^{d-1} \det(A^p_{\tau, \chi}(\sigma \otimes \sigma) + (\rho - \lambda)^2)^{(-1)^p/2}. \tag{7.11}
\]
By Proposition 6.7, we can easily see that once we have the regularity of the Ruelle zeta function at zero, then, we can take the limit as $s \to 0$ of the right hand side of (6.23) and (6.24). In case (b), one has to make use the fact that the Ruelle zeta function can be written as

$$R(s; \sigma, \chi) = \sqrt{R(s; \sigma, \chi)R(s; w\sigma, \chi)R^*(s; \sigma, \chi)},$$

and then use the fact that by the functional equation (5.24) for the super Ruelle zeta function, we have

$$R^*(0; \sigma, \chi) = e^{i\pi\eta(D^\sharp_{p,\chi}(\sigma))}.$$

We recall here that from Theorem 7.5 we have

$$\det_{gr,\theta}(B_{even}) = e^{\xi\chi} e^{i\pi\eta(B_{even})}. \tag{7.12}$$

If we compare equations (7.11) and (7.12), by the definition of the refined analytic torsion $T^C_X(X; E_\chi)$, we are motivated to consider the Ruelle zeta function at zero as a candidate for $T^C_X(X; E_\chi)$.

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