GLOBAL WELL-POSEDNESS FOR THE 2D BOUSSINESQ EQUATIONS WITH ZERO VISCOSITY

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ABSTRACT. We prove the global well-posedness of the two-dimensional Boussinesq equations with zero viscosity and positive diffusivity in bounded domains for rough initial data $u_0 \in L^2$, $\text{curl} u_0 \in L^\infty$ and $\theta_0 \in B_q^{2/p}$ with $p \in (1, \infty), q \in (2, \infty)$. Our method is based on the maximal regularity for heat equation.

1. Introduction

In the present paper, we investigate the global well-posedness of the 2D Boussinesq equations with zero viscosity and positive diffusivity in a smooth bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary $\partial \Omega$. The corresponding system reads

\begin{align*}
\begin{cases}
\partial_t u + u \cdot \nabla u + \nabla p = \theta e_2, \\
\partial_t \theta + u \cdot \nabla \theta - \kappa \Delta \theta = 0, \\
\nabla \cdot u = 0,
\end{cases}
\end{align*}

(1.1)

where $u$ is the velocity vector field, $p$ is the pressure, $\theta$ is the temperature, $\kappa > 0$ is the thermal diffusivity, and $e_2 = (0, 1)$. We supplement the system (1.1) with the following initial boundary value conditions

\begin{align*}
\begin{cases}
(u, \theta)(x, 0) = (u_0, \theta_0)(x), & x \in \Omega, \\
u(x, t) \cdot n|_{\partial \Omega} = 0, \theta(x, t)|_{\partial \Omega} = \bar{\theta},
\end{cases}
\end{align*}

(1.2)

where $n$ is the outward unit normal vector to $\partial \Omega$, and $\bar{\theta}$ is a constant.

The general 2D Boussinesq equations with viscosity $\nu$ and diffusivity $\kappa$ are

\begin{align*}
\begin{cases}
\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p = \theta, \\
\partial_t \theta + u \cdot \nabla \theta - \kappa \Delta \theta = 0, \\
\nabla \cdot u = 0.
\end{cases}
\end{align*}

The Boussinesq equations are of relevance to study a number of models coming from atmospheric or oceanographic turbulence where rotation and stratification play an important role (see [24], [27]). From the mathematical view, the 2D Boussinesq equations serve as a simplified model of the 3D Euler and Navier-Stokes equations (see [25]). Better understanding of the 2D Boussinesq equations will shed light on the understanding of 3D flows.

Recently, there are many works devoted to the well-posedness of the 2D Boussinesq equations, see [3]-[19], [28]-[31]. In particular, when $\Omega = \mathbb{R}^2$, Chae in [9]
showed that the system (1.1)-(1.2) has a global smooth solution for \((u_0, \theta_0) \in H^3\).

In the bounded domains case, the boundary effect requires a careful mathematical analysis. In this direction, Zhao in [30] was able to generalize the study of [9] to smooth bounded domains. This result was later extended by Huang in [19] to the case of Yudovich’s type data: \(\text{curl } u_0 \in L^\infty\) and \(\theta_0 \in H^2\). We intend here to improve Huang’s result further by lowering the regularity for initial data.

Our main result is stated in the following theorem.

**Theorem 1.1.** Let \(\Omega\) be a bounded domain in \(\mathbb{R}^2\) with \(C^{2+\epsilon}\) boundary for some \(\epsilon > 0\). Suppose that \(u_0 \in L^2\), \(\text{curl } u_0 \in L^\infty\), and \(\theta_0 \in B^{2-2/p}_{q,p}\) with \(p \in (1, \infty)\), \(q \in (2, \infty)\). Then there exists a unique global solution \((u, \theta)\) to the system (1.1)-(1.2), which satisfies that for all \(T > 0\)

\[
\theta \in C([0, T]; B^{2-2/p}_{q,p}) \cap L^p(0, T; W^{2,q}), \quad \partial_t \theta \in L^p(0, T; L^q),
\]

\[
u \in L^\infty(0, T; L^2) \text{ and } \text{curl } u \in L^\infty(0, T; L^q).
\]

**Remark 1.2.** We only require mild regularity for the initial temperature \(\theta_0\), as the “regularity index” \(2 - 2/p\) can be arbitrarily closed to zero. Thus, our result significantly improves the previous results [9, 19, 30].

**Remark 1.3.** By modifying slightly the method in the current paper, we can prove the global well-posedness for initial data \((u_0, \theta_0) \in H^{2+s} \times H^s\) with \(s > 0\) or \((u_0, \theta_0) \in W^{2,q} \times W^{1,q}\) with \(q > 2\).

The proof of Theorem 1.1 consists of two main steps. First, we show the global existence of weak solutions to (1.1)-(1.2). Then we improve the regularity of weak solutions using the maximal regularity for heat equation. Our proof is elementary and can be carried over to the case of \(\mathbb{R}^2\) without difficulty.

The rest of our paper is organized as follows. In Section 2, we recall maximal regularity for heat equations as well as some basic facts. Section 3 is devoted to the proof of our main theorem.

2. Notations and Preliminaries

**Notations:**

1) Let \(\Omega\) be a bounded domain in \(\mathbb{R}^2\). For \(p \geq 1\) and \(k \geq 1\), \(L^p(\Omega)\) and \(W^{k,p}(\Omega)\) \((p = 2, H^k(\Omega))\) denote the standard Lesbegue space and Sobolev space respectively. For \(T > 0\) and a function space \(X\), denote by \(L^p(0, T; X)\) the set of Bochner measurable \(X\)-valued time dependent functions \(f\) such that \(t \rightarrow \|f\|_X\) belongs to \(L^p(0, T)\).

2) Let \(s \in (0, \infty), p \in (1, \infty)\) and \(r \in [1, \infty]\). The Besov space \(B^s_{p,r}(\Omega)\) is defined as the real interpolation space between \(L^p(\Omega)\) and \(W^{m,p}(\Omega)\) \((m > s)\),

\[
B^s_{p,r}(\Omega) = (L^p(\Omega), W^{m,p}(\Omega))_{\frac{s}{m-r}}.
\]

See Adams and Fournier ([1], Chapter 7).

3) Throughout this paper, the same letter \(C\) denotes various generic positive constant which is dependent on initial data \((u_0, \theta_0)\), time \(T\), the thermal diffusivity \(\kappa\), and the domain \(\Omega\).

We need the well-known Sobolev Embeddings and Gagliardo-Nirenberg inequality (see Adams and Fournier [1] and Nirenberg [23]).

**Lemma 2.1.** Let \(\Omega \in \mathbb{R}^2\) be any bounded domain with \(C^2\) boundary. Then the following embeddings and inequalities hold:
Lemma 3.1. By a standard argument, see Zhao \[30\], establish the global existence of weak solution to (3.1)-(3.2) in bounded domains.

We also need the maximal regularity for heat equation (see Amann \[2\]), which is critical to the proof of our main theorem.

Lemma 2.2. Let \( \Omega \) be a bounded domain with a \( C^{2+\epsilon} \) boundary in \( \mathbb{R}^2 \) and \( 1 < p, q < \infty \). Assume that \( u_0 \in B_{q,p}^{2-2/p} \), \( f \in L^p(0,\infty;L^q) \). Then the system

\[
\begin{aligned}
(2.1)
\begin{cases}
\partial_t u - \kappa \Delta u = f, \\
u(x,t)|_{\partial\Omega} = 0, \\
u(x,t)|_{t=0} = u_0,
\end{cases}
\end{aligned}
\]

has a unique solution \( u \) satisfying the following inequality for all \( T > 0 \):

\[
\|u(T)\|_{B_{q,p}^{2-2/p}} + \|u\|_{L^p(0,T;W^{2,q})} + \|\partial_t u\|_{L^p(0,T;L^q)} \\
\leq C \left( \|u_0\|_{B_{q,p}^{2-2/p}} + \|f\|_{L^p(0,T;L^q)} \right),
\]

with \( C = C(p,q,\kappa,\Omega) \).

We complete this section by recalling the following well-known inequality (see Yudovich \[29\]), which will be used several times.

Lemma 2.3. For any \( p \in (1,\infty) \), the following estimate holds

\[
\|\nabla u\|_{L^p(\Omega)} \leq C \frac{p^2}{p-1} \|w\|_{L^p(\Omega)},
\]

where \( C = C(\Omega) \) does not depend on \( p \).

3. Proof of Main Theorem

We first reformulate the initial-boundary value problem (1.1)-(1.2). Let \( \bar{p} = p - \theta y \) and \( \Theta = \theta - \bar{\theta} \), then we get from the original system

\[
\begin{aligned}
(3.1)
\begin{cases}
\partial_t u + u \cdot \nabla u + \nabla \bar{p} = \Theta e_2, \\
\partial_t \Theta + u \cdot \nabla \Theta - \kappa \Delta \Theta = 0, \\
\nabla \cdot u = 0,
\end{cases}
\end{aligned}
\]

The initial and boundary conditions become

\[
\begin{aligned}
(3.2)
\begin{cases}
(u,\Theta)(x,0) = (u_0,\Theta_0)(x), \ x \in \Omega, \\
u(x,t) \cdot n|_{\partial\Omega} = 0, \Theta(x,t)|_{\partial\Omega} = 0,
\end{cases}
\end{aligned}
\]

where \( \Theta_0 = \theta_0 - \bar{\theta} \). It is clear that (3.1)-(3.2) are equivalent to (1.1)-(1.2). Hence, for the rest of this paper, we shall work on the reformulated problem (3.1)-(3.2).

3.1. Existence. First, we show the existence of weak solutions. Then, we improve the regularity using the maximal regularity for heat equation. In fact, one can establish the global existence of weak solution to (3.1)-(3.2) in bounded domains by a standard argument, see Zhao \[30\].

Lemma 3.1. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^2 \). Assume that \( (u_0,\Theta_0) \in L^2 \times L^2 \). Then there exists one solution \((u,\Theta)\) to (3.1)-(3.2) such that for any \( T > 0 \)

\[
1 \ u \in L^\infty(0,T;L^2), \ \Theta \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1).
\]
4 DAOGUO ZHOU AND ZILAI LI

\[ \int_\Omega u_0 \phi(x,0) \, dx + \int_0^T \int_\Omega (u \cdot \partial_t \phi + u \cdot (u \cdot \nabla \phi) + \Theta e_2 \phi) \, dx \, dt = 0, \text{ for any vector function } \phi \in C_0^\infty (\Omega \times [0,T]), \]

satisfying \( \nabla \cdot \psi = 0 \).

\[ \int_\Omega \Theta_0 \psi(x,0) \, dx + \int_0^T \int_\Omega (\Theta \cdot \partial_t \psi + \Theta u \cdot \nabla \psi - \nabla \Theta \cdot \nabla \psi) \, dx \, dt = 0, \text{ for any scalar function } \psi \in C_0^\infty (\Omega \times [0,T]). \]

It remains to establish the global regularity of solutions obtained in Lemma 3.1. The proof is divided into several lemmas.

**Lemma 3.2.** Let the assumptions in Theorem 1.1 hold. Then the solution obtained in Lemma 3.1 satisfies

\[ \Theta \in L^\infty (0,T; L^2) \cap L^2 (0,T; H^1), \]

\[ u \in L^\infty (0,T; L^2). \]

**Proof.** Multiplying (3.1) \(^2\) by \( T \) and integrating it over \( \Omega \) by parts, we find

\[ \| \Theta \|_{L^2}^2 + 2 \kappa \int_0^T \| \nabla \Theta \|_{L^2}^2 \, dt \leq \| \Theta_0 \|_{L^2}^2. \]

For second estimate, taking \( L^2 \) inner product of (3.1) \(^1\) with \( u \), and using Hölder’s inequality, we get

\[ \frac{1}{2} \frac{d}{dt} \| u \|_{L^2}^2 = \int_\Omega \Theta e_2 \cdot u \, dx \leq \| \Theta \|_{L^2} \| u \|_{L^2}, \]

which, by the Cauchy-Schwarz inequality, gives that

\[ \| u \|_{L^2} \leq \| u_0 \|_{L^2} + \int_0^T \| \Theta \|_{L^2} \, ds \leq \| u_0 \|_{L^2} + T \| \Theta_0 \|_{L^2}. \]

Then the proof of Lemma 3.2 is finished. \( \square \)

**Lemma 3.3.** Let the assumptions in Theorem 1.1 hold. Then the solution obtained in Lemma 3.1 satisfies

\[ w \in L^\infty (0,T; L^2). \]

**Proof.** We recall that the vorticity \( w = \text{curl } u \) satisfies the equation

\[ \partial_t w + u \cdot \nabla w = \partial_1 \Theta. \]

Multiplying (3.6) by \( w \), integrating the resulting equations over \( \Omega \) by parts, and using Hölder’s inequality, we have

\[ \frac{1}{2} \frac{d}{dt} \| w \|_{L^2}^2 = \int_\Omega \partial_1 \Theta w \, dx \leq \| \nabla \Theta \|_{L^2} \| w \|_{L^2}, \]

which implies that

\[ \| w \|_{L^2} \leq \| w_0 \|_{L^2} + \int_0^T \| \nabla \Theta \|_{L^2} \, ds \]

\[ \leq \| w_0 \|_{L^2} + T^{1/2} \left( \int_0^T \| \nabla \Theta \|_{L^2}^2 \, ds \right)^{1/2}, \]

\[ \leq \| w_0 \|_{L^2} + \frac{T^{1/2}}{\sqrt{2} \kappa} \| \Theta_0 \|_{L^2}. \]

Then the proof of Lemma 3.3 is finished. \( \square \)
Lemma 3.4. Let the assumptions in Theorem 1.1 hold. Then the solution obtained in Lemma 3.3 satisfies
\[ \Theta \in C^\infty([0,T]; B^{2-2/p}_{\infty,p} \cap L^p(0,T; W^{2,q})) \cap L^p(0,T; L^q), \]
\[ u \in L^\infty(0,T; L^2) \text{ and } \text{curl} u \in L^\infty(0,T; L^\infty). \]

Proof. First, we obtain from (3.5) and Lemma 2.3 that
\[ \nabla u \in L^\infty(0,T; L^2), \]
which implies that for \( 2 \leq q < \infty, \)
\[ (3.7) \quad u \in L^\infty(0,T; L^q). \]

Considering the equation for the temperature, by the maximal regularity for heat equation and Hölder’s inequality, we obtain that for \( 1 < p < \infty, 2 < q < \infty, \)
\[ \|\Theta\|_{L^\infty(0,T; B^{2-2/p}_{\infty,p})} + \|\Theta\|_{L^p(0,T; W^{2,q})} + \|\partial_t \Theta\|_{L^p(0,T; L^q)} \leq C\|\Theta_0\|_{B^{2-2/p}_{\infty,p}} + C\|u \cdot \nabla \Theta\|_{L^p(0,T; L^q)} \]
\[ \leq C\|\Theta_0\|_{B^{2-2/p}_{\infty,p}} + C\|u\|_{L^\infty(0,T; L^q)} \|\nabla \Theta\|_{L^p(0,T; L^\infty)} \]
\[ \leq C\|\Theta_0\|_{B^{2-2/p}_{\infty,p}} + C\|\nabla \Theta\|_{L^p(0,T; L^\infty)}. \]

Using the interpolation inequality in Lemma 2.1, Hölder’s inequality and Young’s inequality, we have for arbitrary \( \epsilon > 0, q > 2, \)
\[ \|\nabla \Theta\|_{L^p(0,T; L^\infty)} \leq \epsilon \|\nabla^2 \Theta\|_{L^p(0,T; L^2)} + C(\epsilon)\|\Theta\|_{L^p(0,T; L^2)}. \]

Plugging the above inequality into (3.8), absorbing the small \( \epsilon \) term, we get
\[ \|\Theta\|_{L^\infty(0,T; B^{2-2/p}_{\infty,p})} + \|\Theta\|_{L^p(0,T; W^{2,q})} + \|\partial_t \Theta\|_{L^p(0,T; L^q)} \leq C(p,q,\kappa, T, \Omega, u_0, \Theta_0), \]
which yields that
\[ \nabla \Theta \in L^1(0,T; L^\infty). \]

Coming back to the vorticity equation (3.6), we derive that
\[ w \in L^\infty(0,T; L^\infty). \]

Then the proof of Lemma 3.4 is finished. \( \square \)

3.2. Uniqueness. The method adapted here is essentially due to Yudovich [29], see also Dauchin [10].

Let \((u_1, \Theta_1, p_1)\) and \((u_2, \Theta_2, p_2)\) be two solutions of the system (3.1) - (3.2). Denote \( \delta u = u_1 - u_2, \delta \Theta = \Theta_1 - \Theta_2, \) and \( \delta p = p_1 - p_2. \) Then \( (\delta u, \delta \Theta, \delta p) \) satisfy
\[ \begin{align*}
\partial_t \delta u + u_2 \cdot \nabla \delta u + \nabla \delta p &= -\delta u \cdot \nabla u_1 + \delta \Theta e_2, \\
\partial_t \delta \Theta - \kappa \Delta \delta \Theta &= -u_2 \cdot \nabla \delta \Theta - \delta u \cdot \nabla \Theta_1, \\
\nabla \cdot \delta u &= 0, \\
\delta u(x,t) \cdot n|_{\partial \Omega} &= 0, \delta \Theta(x,t)|_{\partial \Omega} = 0, \\
\delta u(x,0) &= 0, \delta \Theta(x,0) = 0.
\end{align*} \]

By standard energy method and Hölder’s inequality, we have for all \( r \in [2, \infty) \)
\[ \frac{1}{2} \frac{d}{dt} \|\delta u\|^2_{L^2} \leq \|\nabla u_1\|_{L^r} \|\delta u\|^2_{L^2} + \|\delta \Theta\|_{L^2} \|\delta u\|_{L^2} \]
\[ \leq \|\nabla u_1\|_{L^r} \|\delta u\|^2_{L^\infty} \|\delta u\|^2_{L^2} + \|\delta \Theta\|_{L^2} \|\delta u\|_{L^2}, \]
and
\[ \frac{1}{2} \frac{d}{dt} \| \delta \Theta \|_{L^2}^2 \leq \| \nabla \Theta_1 \|_{L^\infty} \| \delta \Theta \|_{L^2} \| \delta u \|_{L^2}, \]
where \( 1/r + 1/r' = 1 \).

Denoting \( X(t) = \| \delta u \|_{L^2}^2 + \| \delta \Theta \|_{L^2}^2 \), we find that
\[ \frac{1}{2} \frac{d}{dt} X \leq \| \nabla u_1 \|_{L^r} \| \delta u \|_{L^r}^{2/r} \| X^{1/r'} + \frac{1}{2} (1 + \| \nabla \Theta_1 \|_{L^\infty}) X. \]

Setting \( Y = e^{-\int_0^t (1 + \| \nabla \Theta_1 \|_{L^\infty}) ds} X \), we deduce that
\[ \frac{1}{r} Y - \frac{1}{r'} \frac{d}{dt} Y \leq 2e^{-\int_0^t (1 + \| \nabla \Theta_1 \|_{L^\infty}) ds} \| \nabla u_1 \|_{L^r} \| \delta u \|_{L^\infty}^{2/r} \]
\[ \leq 2 \frac{r}{r'} \| \nabla u_1 \|_{L^r} \| \delta u \|_{L^\infty}^{2/r}. \]

Integrating in time on \([0, t]\) gives us that
\[ Y(t) \leq \left( 2 \int_0^t \frac{\| \nabla u_1 \|_{L^r}}{r} \| \delta u \|_{L^\infty}^{2/r} ds \right)^r. \]

To proceed, we make two simple observations. First, combing Lemma 2.3 and the bound \( w_1 \in L^\infty(0, T; L^\infty) \), we deduce that
\[ \sup_{1 \leq r < \infty} \frac{\| \nabla u_1(t) \|_{L^r}}{r} \leq C(\Omega). \]
Second, from the fact that \( u_i \in L^\infty(0, T; L^2) \) and \( w_i \in L^\infty(0, T; L^\infty) \) for \( i = 1, 2 \), we have
\[ \delta u \in L^\infty(0, T; L^\infty). \]

Next, choosing \( T^* \) such that \( \int_0^{T^*} \frac{\| \nabla u_1 \|_{L^r}}{r} \leq 1/4 \), together with (3.12), we can rewrite (3.10) as
\[ Y(t) \leq C \left( \frac{1}{2} \right)^r. \]

Sending \( r \) to \( \infty \), we get \( Y(t) \equiv 0 \) on \([0, T^*]\). By standard induction argument, we conclude that \( Y(t) \equiv 0 \) for all \( t \in [0, T] \), which means that \( (\delta u, \delta T, \delta p) \equiv 0 \) for all \( t \in [0, T] \). Thus we obtain the uniqueness of solutions.

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