Anisotropic universes with conformal motion

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Abstract. By imposing natural geometrical and kinematical conditions on a conformal Killing vector in Bianchi I spacetime, we show that a class of axisymmetric metrics admits a conformal motion. This class contains new exact solutions of Einstein’s equations, including anisotropic radiation universes that isotropise at late times.

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1. Introduction

A conformal Killing vector (ckv) $\xi^\mu$ generates a local $G_1$ of conformal motions of the metric:

$$\begin{align*}
\ ds^2 & \equiv g_{\mu\nu}dx^\mu dx^\nu \ \rightarrow \ e^\Phi ds^2 \ \Leftrightarrow \\
\mathcal{L}_\xi g_{\mu\nu} & \equiv \xi^\alpha \partial_\alpha g_{\mu\nu} + g_{\alpha\nu} \partial_\mu \xi^\alpha + g_{\mu\alpha} \partial_\nu \xi^\alpha = 2\psi g_{\mu\nu} \\
\end{align*}$$

where $\psi$ is the conformal factor, with $\psi = 0$ if $\xi^\mu$ is a Killing vector (kv). Conformally flat spacetimes admit a maximal $G_{15}$ of conformal motions, with up to 10 of the generators being kv. Non-conformally flat spacetimes admit a maximal $G_7$ of conformal motions, with at least 1 of the generators being a proper ckv (i.e. $\psi \neq 0$) [1].

Solving the conformal Killing equation (1) to find whether, and under what conditions, a spacetime admits conformal motions, can lead to new solutions of the field equations (see for example [2], [3], [4], [5], [6]), or to new geometrical and kinematical insights (see for example [7], [8], [9], [10]). As is well-known, the standard Friedmann–Robertson–Walker (FRW) universes are conformally flat, and admit 9 proper ckv, all pointing out of the homogeneous hypersurfaces [8].
Spatially homogeneous but anisotropic universes are not conformally flat, and the existence of proper CKV is subject to severe constraints. These are the integrability conditions of (1), including invariance of the Weyl tensor:

\[ \mathcal{L}_\xi C^{\alpha}_{\beta\mu\nu} = 0 \]  

Some conditions arise directly from (1) via kinematical considerations. Let \( u^\mu \) be the four–velocity field orthogonal to the homogeneous hypersurfaces, so that its acceleration and vorticity are zero: \( \dot{u}_\mu = 0 = \omega_{\mu\nu} \). Then

\[ \psi = (u_\mu \xi^\mu) \]  

\[ \mathcal{L}_\xi u^\mu = -\psi u^\mu - h^{\mu\nu} \partial_\nu (u_\alpha \xi^\alpha) \]  

\[ 0 = \sigma^{\mu\nu} \left[ \sigma_{\mu\nu} + (h_{\mu\alpha} \xi^\alpha)_{,\nu} \right] \]  

where \( h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu \) and \( \sigma_{\mu\nu} \) is the shear. (The first two equations hold in any spacetime without acceleration and vorticity; the last holds in all spacetimes.)

It follows from (3) that there are no proper CKV tangent to the homogeneous hypersurfaces, while (4) shows that if \( \xi^\mu \) is parallel to \( u^\mu \), then the shear vanishes and the spacetime degenerates to FRW. Thus any proper CKV must be ‘tilted’, neither parallel nor orthogonal to \( u^\mu \). It is natural to impose the condition that the conformal motion of the metric should also be a conformal mapping of the four–velocity field \( u^\mu \), i.e. that \( \xi^\mu \) should map \( u^\mu \)–curves into themselves [7]:

\[ \mathcal{L}_\xi u^\mu = -\psi u^\mu \]  

However, as shown in [9], there are no perfect fluid spatially homogeneous anisotropic spacetimes with a proper CKV that satisfies the symmetry inheritance condition (6).

Thus we are led to search for a proper CKV that is tilted and does not satisfy the simple inheriting condition (3). The simplest geometrical generalisation of (3) is that \( \xi^\mu \) be surface–forming with \( u^\mu \). In section 2, we impose this requirement, together with the requirement that \( \xi^\mu \) forms a Lie algebra with the KV of the spacetime. Given the complexity of the problem for general spatially homogeneous spacetimes, we begin by tackling the simplest anisotropic generalisation of FRW spacetimes, the Bianchi I spacetimes. Then the geometrical and kinematical requirements allow us to solve the conformal Killing equation (1). The integrability conditions force the metric to be axisymmetric, and impose a second–order equation on the metric components.

For a perfect fluid, this means that there is no freedom to specify an equation of state, and in section 3 we show that all perfect fluid solutions which satisfy the weak energy condition (\( \rho \geq 0 \)) violate the dominant energy condition (\( \rho + p \geq 0 \)). We consider fluids with anisotropic pressure, and find new radiation solutions that isotropise at late times. This represents a possible early–universe model with conformal motion.

**2. Solutions of the conformal Killing equation**

The diagonal Bianchi I metrics have the form

\[ ds^2 = -dt^2 + A_i(t)^2 \left( dx^i \right)^2 \]  

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where \( x^\mu = (x^0, x^i) = (t, \vec{x}) \), and the canonical four–velocity is \( u^\mu = \delta^\mu_0 \). They admit a \( G_3 \) of motions, generated by the KV

\[
Y_i \equiv Y_i^\mu \partial_\mu = \partial_i \quad \text{with} \quad [Y_i, Y_j] = 0
\]  

(8)

From the discussion in Section 1, we are led to look for a CKV \( \xi^\mu \) that (a) closes with \( \{Y_i\} \) to generate a \( G_4 \), and (b) is surface–forming with \( u^\mu \):

\[
[Y_i, \xi] = a_i \xi + b^j_i Y_j
\]  

(9)

\[
\mathcal{L}_\xi u^\mu = -\psi u^\mu - \lambda h^{\mu}_{\nu} \xi^\nu
\]  

(10)

where \( a_i, b^j_i \) are constants, and \( \lambda \) is some scalar field, by Frobenius’ theorem and (4).

From (8) and (9) we get

\[
\partial_i \xi^0 = a_i \xi^0 \quad \text{and} \quad \partial_i \xi^j = a_i \xi^j + b_i^j
\]  

(11)

The integrability condition of the second of equations (11) is

\[
\partial_j \partial_k \xi^i = 0 \Rightarrow a_k b^i_j = 0 \Rightarrow b^i_j = a_j b^i
\]  

for some constant \( b^i \) (square brackets denote anti–symmetrisation). Thus (11) integrate to give

\[
\xi = e^{\vec{a} \cdot \vec{x}} \left[ B(t) \partial_0 + C_i(t) \partial_i \right]
\]  

(12)

where we have used the freedom to add multiples of the \( Y_i \) to \( \xi \) in order to set \( b^i = 0 \).

Then (12) and (10) imply

\[
C_i(t) = C(t) c^i
\]  

(13)

\[
\dot{\psi} = \dot{B} e^{\vec{a} \cdot \vec{x}}
\]  

(14)

\[
\lambda = \frac{\dot{C}}{C}
\]  

(15)

where \( c^i \) are constants. To avoid degenerate cases, we require \( \dot{B} \dot{C} \neq 0 \). Finally, the geometrically and kinematically defined CKV given by (12) - (15) is subject to the conformal Killing equation (1) for the metric (7). It is straightforward to show that the solution and integrability conditions are:

\[
a_1 B - c^1 (A_1)^2 \dot{C} = 0
\]  

(16)

\[
\frac{\dot{B}}{B} - a_1 c^1 \frac{C}{B} = \frac{\dot{A}_1}{A_1}
\]  

(17)

\[
A_2 = A_3 = B
\]  

(18)

\[
a_2 = a_3 = c^2 = c^3 = 0
\]  

(19)

It follows that such a CKV can only occur in the axisymmetric class of Bianchi spacetimes, and that eliminating \( C \) from (14) and (17) gives a condition involving only the two
independent metric functions. Writing $A \equiv A_1$ and $a \equiv a_1$, we can summarise the results as:

$$ds^2 = -dt^2 + A^2 dx^2 + B^2 (dy^2 + dz^2)$$  \hspace{1cm} (20)

$$\xi = e^{ax} \left[ B \partial_t + a \left( \int \frac{B}{A^2} dt \right) \partial_x \right]$$  \hspace{1cm} (21)

$$\psi = B e^{ax}$$  \hspace{1cm} (22)

$$\frac{a^2}{A^2} = \frac{\ddot{B}}{B} - \frac{\ddot{A}}{A} + \frac{\dot{A}^2 - \dot{A}\dot{B}}{AB}$$  \hspace{1cm} (23)

The integrability condition (23) satisfies the conditions for ‘decomposability’ \cite{11, 12}, and thus may be reduced to

$$A \left[ A \left( \frac{B}{A} \right) \right] = a^2 \left( \frac{B}{A} \right)$$

This can be integrated upon defining a new time parameter:

$$\tau = \int \frac{dt}{A} \hspace{1cm} (24)$$

$$B = \left( be^{ax} + ce^{-ax} \right) A \hspace{1cm} (25)$$

where $b, c$ are constants. Clearly $a \neq 0$ to avoid the isotropic FRW case ($B \propto A$).

3. Solutions of Einstein’s field equations

The class of metrics given by (20), (24) and (25) is now subjected to Einstein’s field equations

$$R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} = \rho u_\mu u_\nu + p h_{\mu \nu} + \Pi_{\mu \nu}$$  \hspace{1cm} (26)

where $\rho$ is the total energy density, $p$ is the isotropic pressure, and $\Pi_{\mu \nu}$ is the trace–free anisotropic pressure tensor. By symmetry there is no heat flux, and the anisotropic pressure has the form

$$\Pi_{\mu \nu} = \text{diag} \left( 0, \Pi, -\frac{1}{2} \Pi, -\frac{1}{2} \Pi \right)$$  \hspace{1cm} (27)

For a perfect fluid, $\Pi = 0$, but $\Pi \neq 0$ in general, e.g. for a perfect fluid with magnetic field (see \cite{13}). The distribution function for a kinetic gas may be expressed as \cite{14}

$$f(x^\mu, p^\nu) = F(t, E) + F_\mu(t, E) e^\mu + F_{\mu \nu}(t, E) e^\mu e^\nu + \cdots$$

where $F_{\mu \nu}$ are the covariant multipoles that determine the anisotropy of the distribution, and $p^\mu = Eu^\mu + \sqrt{E^2 - m^2} e^\mu$, so that $E$ is the particle energy and $e^\mu$ is a unit vector along the particle 3–momentum. The covariant dipole $F_\mu$ determines the heat flux (which in this case is zero), and the covariant quadrupole $F_{\mu \nu}$ determines the anisotropic pressure \cite{14}:

$$\Pi_{\mu \nu}(t) = \frac{8\pi}{15} \int_m^\infty \left( E^2 - m^2 \right)^{3/2} F_{\mu \nu}(t, E) dE$$

Thus $\Pi$ arises from a quadrupole anisotropy of the distribution function, and by (27):

$$\Pi(t) = \frac{8\pi}{15} \int_m^\infty \left( E^2 - m^2 \right)^{3/2} \Gamma(t, E) dE \quad \text{where} \quad F_{\mu \nu} = \text{diag} \left( 0, \Gamma, -\frac{1}{2} \Gamma, -\frac{1}{2} \Gamma \right)$$

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The distribution cannot be isotropic unless the shear vanishes \([15]\), so in general \(\Pi \neq 0\) (see \([16]\), \([17]\), \([18]\), \([19]\) for examples). For collision–free and relaxational (BGK) distributions, the shear vanishes if the dipole, quadrupole and octopole are zero, or if any 4 consecutive multipoles are zero, or if there are only a finite number of non–zero multipoles \([20]\).

Whatever the physical source of anisotropic pressure is, the field equations \((26)\) with \((27)\) and \((20)\) are of the form

\[
\frac{\dot{B}^2}{B^2} + 2\frac{\dot{A}\dot{B}}{AB} = \rho \tag{28}
\]

\[
-2\frac{\ddot{B}}{B} - \frac{\dot{B}^2}{B^2} = p + \Pi \tag{29}
\]

\[
-\frac{\ddot{A}}{A} - \frac{\dot{B}}{B} - \frac{\dot{A}\dot{B}}{AB} = p - \frac{1}{2}\Pi \tag{30}
\]

The system \((28)\) - \((30)\) is closed by \((24)\), \((25)\) and an equation of state when \(\Pi \neq 0\).

### 3.1 Perfect fluid solutions

With \(\Pi = 0\), there is no freedom to specify an equation of state. Subtracting \((29)\) from \((30)\) and using \((24)\) and \((25)\), we get

\[
A^2 = k\left(\frac{1}{b^2e^{2\alpha_t} - c^2e^{-2\alpha_t}}\right) \tag{31}
\]

\[
B^2 = k\left(\frac{bc^{\alpha_t} + ce^{-\alpha_t}}{bc^{\alpha_t} - ce^{-\alpha_t}}\right) \tag{32}
\]

where \(k\) is a constant. Then \((24)\), \((22)\) and \((28)\), \((29)\) give

\[
\rho = -\frac{4a^2bc}{k}\left(\frac{b^2e^{2\alpha_t} + c^2e^{-2\alpha_t} + bc}{b^2e^{2\alpha_t} - c^2e^{-2\alpha_t}}\right) \tag{33}
\]

\[
p = -\frac{4a^2bc}{k}\left(\frac{b^2e^{2\alpha_t} + c^2e^{-2\alpha_t} + 3bc}{b^2e^{2\alpha_t} - c^2e^{-2\alpha_t}}\right) \tag{34}
\]

It follows from \((33)\) and \((34)\) that if \(\rho \geq 0\), then \(\rho + p \leq 0\), so that the dominant energy condition is violated, and the energy density grows with expansion, by the energy conservation equation \(\dot{\rho} + (\rho + p)u^{\mu}_{\;\mu} = 0\). The limiting case \(\rho + p = 0\) occurs if and only if \(b\) or \(c\) vanishes, which in turn implies \(\rho = 0 = p\) by \((33)\), \((24)\), and \(\psi = 0\) by \((32)\), \((22)\). In this case \(\xi^\mu\) degenerates to a null KV of a vacuum Bianchi I solution. Taking \(c = 0\), we have

\[
ds^2 = -dt^2 + a(t - t_*)^2dx^2 + \frac{k}{b^2}(dy^2 + dz^2) \tag{35}
\]

\[
\xi = \sqrt{\frac{k}{b}} e^{\alpha x}\left[\partial_t + \frac{1}{a(t_* - t)}\partial_x\right] \tag{36}
\]

where \(t_*\) is constant.

In summary: although perfect fluid solutions exist for CKV that are surface–forming with \(u^\mu\), and do not exist at all for ‘inheriting’ CKV \((6)\) that map \(u^\mu\) conformally \((6)\), these solutions are unphysical.
3.2 Anisotropic pressure solutions

When \( \Pi \neq 0 \), we impose the linear barotropic equation of state on the isotropic pressure:

\[
p = (\gamma - 1)\rho
\]

where the constant \( \gamma \) satisfies \( 1 \leq \gamma \leq \frac{4}{3} \) for ‘normal’ fluids. By (28) - (30) and (37), and using (24), (25), we get an equation for \( A(\tau) \):

\[
3 \frac{A''}{A} + \frac{3}{2}(3\gamma - 4) \frac{A'^2}{A^2} + 2a(3\gamma - 1) \left[ \frac{b e^{a\tau} - c e^{-a\tau}}{b e^{a\tau} + c e^{-a\tau}} \right] \frac{A'}{A} + \frac{1}{2} a^2 (3\gamma - 2) \left[ \frac{b e^{a\tau} - c e^{-a\tau}}{b e^{a\tau} + c e^{-a\tau}} \right]^2 + 2a^2 = 0
\]

(38)

Thus we have reduced the problem to solving a single second order (but non–autonomous) ODE. We will not consider this equation in general, but only treat the autonomous case \( c = 0 \), when (38) becomes

\[
3 \frac{A''}{A} + \frac{3}{2}(3\gamma - 4) \frac{A'^2}{A^2} + 2a(3\gamma - 1) \frac{A'}{A} + \frac{1}{2} a^2 (3\gamma + 2) = 0
\]

(39)

Although the exact solution for \( 1 \leq \gamma < \frac{4}{3} \) has been found [21], we will only consider the more physically interesting case of \( \gamma = \frac{4}{3} \). This is a radiation universe with anisotropy in the distribution of radiation, and (39) and (24), (25) give

\[
A = k (\tau - \tau_*) e^{-a\tau} \quad B = kb (\tau - \tau_*)
\]

(40)

and

\[
t - t_* = \frac{k}{a^2} \left[ e^{-a\tau_*} - e^{-a\tau} \{ a(\tau - \tau_*) + 1 \} \right]
\]

(41)

where \( k, t_*, \tau_* \) are constants. Then (10) with (28), (29) implies

\[
\rho = \frac{4 e^{2a\tau} [3 - 2a(\tau - \tau_*)]}{3k^2 (\tau - \tau_*)^4}
\]

(42)

\[
\Pi = \left[ \frac{10a(\tau - \tau_*) + 3}{2a(\tau - \tau_*) - 3} \right] \frac{\rho}{12}
\]

(43)

4. Concluding remarks

The exact solution (10) - (13) is an axisymmetric Bianchi I radiation universe with anisotropy in the radiation distribution, that admits a surface–forming conformal motion. There is a big bang at \( \tau = \tau_* \), equivalently \( t = t_* \), and for \( a < 0 \) the universe expands as \( \tau \) and \( t \) increase. The energy density and the anisotropic pressure decay with expansion. The shear is given by

\[
\sigma \propto \frac{1}{A} \left| \frac{A'}{A} - \frac{B'}{B} \right| \propto e^{a\tau} \frac{\rho}{\tau - \tau_*}
\]

(44)
and decays with expansion, so that the solution isotropises at late times. The anisotropic pressure starts from $-\infty$ at the big bang, increases through zero, at time $\tau = \tau_* - (3/10a)$, to a maximum, and then decays like $\frac{5}{12}\rho$ at late times.

Thus the solution has some reasonable physical properties so that it might in principle be used as a model of the early universe. However, we are not putting it forward as such a model. Rather, we have shown how the imposition of natural geometrical and kinematical assumptions on a conformal motion lead to solutions in Bianchi I spacetimes, including solutions which are not immediately ruled out physically - as is often the case with conformal motions.

The methods introduced here could be applied to other spacetimes, including other Bianchi types.

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