VALUE SETS OF BIVARIATE CHEBYSHEV MAPS OVER FINITE FIELDS

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ABSTRACT. We determine the cardinality of the value sets of bivariate Chebyshev maps over finite fields. We achieve this using the dynamical properties of these maps and the algebraic expressions of their fixed points in terms of roots of unity.

INTRODUCTION

The Chebyshev polynomials show remarkable properties and they have applications in many areas of mathematics. There is a generalization of these polynomials to several variables introduced by Lidl and Wells [LW72]. It is a well known fact that the Dickson polynomial, a normalization of one variable Chebyshev polynomial, induces a permutation on $\mathbb{F}_q$ if and only if $\gcd(k, q^s - 1) = 1$ for $s = 1, 2$. It is in perfect analogy with one variable case that the $n$ variable Chebyshev map is a bijection of $\mathbb{F}_q^n$ if and only if $\gcd(k, q^s - 1) = 1$ for $s = 1, 2, \ldots, n + 1$.

Let $f : \mathbb{F}_q^n \to \mathbb{F}_q^n$ be a polynomial map in $n$ variables defined over $\mathbb{F}_q$. Denote its value set by $V(f, \mathbb{F}_q^n) = \{f(c) : c \in \mathbb{F}_q^n\}$. Clearly $f$ is a bijection of $\mathbb{F}_q^n$ if and only if its value set has cardinality $q^n$. If $f$ is not a bijection, then it is natural to ask how far it is away from being a bijection. There are several results in the literature which give bounds on the cardinality of the value set. We refer to the work of Mullen, Wan and Wang [MWW13] for a nice introduction to this problem which include several references and historical remarks.

For an arbitrary polynomial map, there is no easy formula giving the cardinality of the value set. However Chou, Gomez-Calderon and Mullen [CGM88] achieve in finding such a formula for the Dickson polynomials. In our previous work [Ku14], we gave a shorter proof of this formula by using a singular cubic curve and generalized those computations to the elliptic case. More precisely we have found the cardinality of the value sets of Lattès maps, which are induced by isogenies of elliptic curves, over finite fields.

In this paper we study the Chebyshev maps with two variables. The bivariate Chebyshev map $T_k$ is given by the formula

$$T_k(x, y) = (g_k(x, y), g_k(y, x))$$

where $g_k(x, y)$ is the generalized Chebyshev polynomial defined by Lidl and Wells [LW72]. We have $g_{-1}(x, y) = y, g_0(x, y) = 3$ and $g_1(x, y) = x$ and these polynomials satisfy the recurrence relation

$$g_k(x, y) = xg_{k-1}(x, y) - yg_{k-2}(x, y) + g_{k-3}(x, y).$$
The recurrence relation work in both ways and \( g_k(x, y) \) is defined for all integers \( k \in \mathbb{Z} \). Note that \( g_k \) has integral coefficients and one can consider the map induced on finite fields. The main result of this paper is Theorem 4.2 which provides a formula for the cardinality of the value set

\[
V(T_k, F_q^2) = \{T_k(x, y) : (x, y) \in F_q^2\}.
\]

We achieve in finding such a formula by using the dynamical properties of bivariate Chebyshev maps over complex numbers which are studied by Uchimura [Uc09]. Uchimura shows that the set of points with bounded orbits is a certain closed domain \( S \) in \( \mathbb{C}^2 \). This set is enclosed by Steiner’s hypocycloid, see Figure 1. Moreover he shows that the number of periodic points of order \( n \) is equal to \( |k|^{2n} \) if \( |k| \geq 2 \). Another tool for our computations is the nice expression of periodic points of \( T_k \). Algebraically each periodic point is a triple sum of roots of unity, a characterization due Koornwinder [Ko74]. We combine these facts together with the identity \( T_q(x, y) \equiv (x^q, y^q) \pmod{p} \). It follows that \( q^2 \) fixed points of \( T_q \) reduce to distinct elements in \( F_q^2 \) modulo a certain prime ideal of a number field. This is the idea we have used in order to compute the size of the value sets for Lattès maps [Ki14]. After characterizing the elements in \( F_q^2 \) in a compatible fashion under the action of \( T_q \), determining the cardinality of \( V(T_k, F_q^2) \) reduces to a combinatorial argument.

The organization of the paper is as follows: In the first section we give an alternative computation of the value set of Dickson polynomials in order to give the idea of our computations in the bivariate case. In the second section, we review some known facts about the dynamics of bivariate Chebyshev maps and classify the points which have bounded orbits. In the third section we focus on the periodic and preperiodic points of \( T_k \) over complex numbers and their algebraic expressions. In the last section, we find the cardinality of the value set of bivariate Chebyshev maps. We finish our paper by giving an example.

## 1. Single variable case

In this section, we will consider the Dickson polynomials, a normalization of one variable Chebyshev polynomials, and give an alternative computation of the cardinality of their value sets. This alternative computation will be a summary of the ideas that will be used in the rest of the paper.

The family of Chebyshev polynomials (of the first kind) are defined by the recurrence relation \( T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x) \) where \( T_0(x) = 1 \) and \( T_1(x) = x \). One can normalize the Chebyshev polynomials by the relation \( D_k(x) = 2T(x) \) to obtain the Dickson polynomials (of the first kind). This is done in order to cancel the repeating factors of two and consider the arithmetic over fields of characteristic two. The Dickson polynomials satisfy a similar recurrence relation \( D_{k+1}(x) = xD_k(x) - D_{k-1}(x) \) where \( D_0(x) = 2 \) and \( D_1(x) = x \).
The first few Chebyshev and Dickson polynomials are:

- $T_0(x) = 1$, $D_0(x) = 2$,
- $T_1(x) = x$, $D_1(x) = x$,
- $T_2(x) = 2x^2 - 1$, $D_2(x) = x^2 - 2$,
- $T_3(x) = 4x^3 - 3x$, $D_3(x) = x^3 - 3x$,
- $T_4(x) = 8x^4 - 8x^2 + 1$, $D_4(x) = x^4 - 4x^2 + 2$,
- $T_5(x) = 16x^5 - 20x^3 + 5x$, $D_5(x) = x^5 - 5x^3 + 5x$.

The dynamics of Dickson polynomials $D_k(x)$ over complex numbers is well-understood. We refer to Silverman [Si07] for a nice summary of these results. In contrast with our terminology, Silverman uses the term Chebyshev polynomials understood. We refer to Silverman [Si07] for a nice summary of these results.

In order to emphasize the analogy between the one variable case and the two variable case, let us set $D$ be a fixed integer with $k \geq 2$. It is a well-known fact that the Julia set of $D_k(x)$ is the closed interval $J = [-2, 2]$ in $\mathbb{C}$. Observe that the Julia set can be given as $J = \{\alpha(\sigma) : \sigma \in \mathbb{R}\}$. The (forward) orbit of $x$ under $D_k$ is the set $\mathcal{O}(x) = \{D_k^n(x) : n \geq 0\}$ by definition. Observe that the interval $[-2, 2]$ can also be obtained as the set of complex numbers $x$ whose orbits $\mathcal{O}(x)$ are bounded sets. Note that any preperiodic or periodic point must be in the set $[-2, 2]$ since their orbits have finitely many elements.

The value sets of Dickson polynomials are first computed by Chou, Gomez-Calderon and Mullen [CGM88]. In [Kü14], we gave an alternative computation of their result by using a singular cubic curve. Now we will give another approach which is a summary of ideas that will be used in the rest of the paper.

The map $\alpha : \mathbb{R}/\mathbb{Z} \to [-2, 2]$ is a two to one covering with two exceptional points. Namely the points $-2 = \alpha(1/2)$ and $2 = \alpha(0)$. The fixed points of $D_k$ satisfy the relation $D_k(x) = x$ by definition. Moreover $x = \alpha(\sigma)$ for some $\sigma \in \mathbb{R}$ and we have $D_k(\alpha(\sigma)) = \alpha(k\sigma)$. For real numbers $\sigma$ and $\hat{\sigma}$, we have $\alpha(\sigma) = \alpha(\hat{\sigma})$ if and only if $\{\sigma, 1 - \sigma\}$ and $\{\hat{\sigma}, 1 - \hat{\sigma}\}$ are equal as subsets of $\mathbb{R}/\mathbb{Z}$. It follows from these observations that any fixed point of $D_k$ is of the form $x = \alpha(r)$ for some rational number $r$. Moreover if $r$ is written in its lowest terms, its denominator must be relatively prime to $k$. Using this characterization, we can write

$$\text{Fix}(D_k, \mathbb{C}) = \left\{ \alpha\left(\frac{a}{k-1}\right) : a \in \mathbb{Z}\right\} \cup \left\{ \alpha\left(\frac{a}{k+1}\right) : a \in \mathbb{Z}\right\}.$$ 

Now let us count the elements in $\text{Fix}(D_k, \mathbb{C})$. Both sets in the above union contain the element $2 = \alpha(0)$. If $k$ is odd, then $-2$ is in their intersection as well. Other than these two elements, the above sets are disjoint since $\gcd(k-1, k+1)|2$. It follows easily that there are $k$ distinct elements in $\text{Fix}(D_k, \mathbb{C})$ by the inclusion and exclusion principle.

Let $\mathbb{F}_q$ be a finite field of characteristic $p$. Consider the number field $K = \mathbb{Q}(\text{Fix}(D_q, \mathbb{C}))$ which is obtained by adjoining the fixed points of $D_q$ to the rational numbers. Let $\mathfrak{p}$ be a prime ideal of $K$ lying over $p$. We have $D_q(x) \equiv x^q \pmod{p}$ and therefore the fixed points of $D_q$ reduced modulo $\mathfrak{p}$ are the solutions of $x^q - x = 0$. 

The value sets of bivariate Chebyshev maps over finite fields
Thus each element of $F_q$ is obtained by reducing a fixed point modulo $p$. Since there are $q$ fixed points of $D_q$, we conclude that there is a one-to-one correspondence
\[ \text{Fix}(D_q, \mathbb{C}) \leftrightarrow F_q \]
which is obtained by the reduction modulo $p$.

From this point on finding a formula for the size of the value set is straightforward. One can use the representations $a(a/(q \pm 1))$ of elements in $\text{Fix}(D_q, \mathbb{C})$ in order to analyze the value set of $D_k$. Applying the inclusion and exclusion principle, we find that
\[ |V(D_k, F_q)| = \frac{q - 1}{2 \gcd(k, q - 1)} + \frac{q + 1}{2 \gcd(k, q + 1)} + \eta(k, q). \]
Here $\eta(k, q)$ is a function which takes the values 0 or 1 if and only if $\gcd(k, q - 1) \equiv \gcd(k, q + 1) \pmod{2}$.

2. Bivariate Chebyshev maps

There is a generalization of Chebyshev maps to higher dimensions introduced by Lidl and Wells [LW72]. For any integer $n$, the polynomial equation $z^2 - nz + 1 = 0$ has roots $y$ and $1/y$ in the complex numbers. If $y^k$ and $1/y^k$ are the roots of $z^2 - n'z + 1 = 0$ then $n'$ is also an integer, and it is a well known fact that $D_k(n) = n'$ where $D_k$ is the Dickson polynomial (of the first kind).

Lidl and Wells generalize this construction by considering a polynomial equation of degree $n + 1$ with integral coefficients and with roots $t_1, t_2, \ldots, t_{n+1}$. Then they consider another polynomial equation with roots $t_1^k, t_2^k, \ldots, t_{n+1}^k$. It turns out that there is a system of polynomials with integral coefficients which give the coefficients of the latter equation in terms of the coefficients of the former equation. They are called the generalized Chebyshev polynomials. For details see [LW72], or [LN83].

Now we focus on the bivariate case. Suppose that $x = t_1 + t_2 + t_3$ and $y = t_1t_2 + t_1t_3 + t_2t_3$ with $t_1t_2t_3 = a$. We assume that $a = 1$ for simplicity. According to the construction of Lidl and Wells, there exists a bivariate polynomial $g_k(x, y)$ with integer coefficients which maps $(x, y)$ to $t_1^k + t_2^k + t_3^k$. Moreover it turns out that $g_k(x, y)$ is equal to $t_1^k g_1 + t_2^k g_2 + t_3^k g_3$. It is easy to see that $g_{-1}(x, y) = y, g_0(x, y) = 3$ and $g_1(x, y) = x$. Further this family satisfies the recurrence relation
\[ g_k(x, y) = xg_{k-1}(x, y) - yg_{k-2}(x, y) + yg_{k-3}(x, y). \]

The first few bivariate Chebyshev polynomials are:
\[ g_0(x, y) = 3, \]
\[ g_1(x, y) = x, \]
\[ g_2(x, y) = x^2 - 2y, \]
\[ g_3(x, y) = x^3 - 3xy + 3, \]
\[ g_4(x, y) = x^4 - 4x^2y + 2y^2 + 4x, \]
\[ g_5(x, y) = x^5 - 5x^3y + 5x^2y^2 + 5x^2 - 5y. \]

As we have seen from the first section, the dynamical properties of Dickson polynomials play an important role in the analysis of the map induced over finite fields. Thus we start with reviewing some known facts about the bivariate Chebyshev map which is defined by
\[ T_k(x, y) = (g_k(x, y), g_k(y, x)). \]
Dynamical properties of $T_k$ on complex numbers are studied by Uchimura [Uc09]. Uchimura shows that the map $T_k$ admits an invariant plane $\{x = \bar{y}\} \subseteq \mathbb{C}^2$. The restriction of $T_k$ to this plane is the polynomial considered by Koornwinder [Ko74]. With Koornwinder's notation, we have

$$T_k(x, \bar{x}) = P_{(k,0)}^{-1/2}(x, \bar{x}).$$

The key property we get from Koornwinder’s work is the nice action of Chebyshev maps on certain elements. Define

$$\alpha(\sigma, \tau) = e^{2\pi i \sigma} + e^{2\pi i \tau} + e^{2\pi i (-\sigma - \tau)}, \quad \sigma, \tau \in \mathbb{R}.$$ 

We have

$$T_k(\alpha(\sigma, \tau), \alpha(\sigma, \tau)) = (\alpha(k\sigma, k\tau), \alpha(k\sigma, k\tau)).$$

Let $k$ be a fixed integer with $|k| \geq 2$. Uchimura shows that any point in $\mathbb{C}^2$, whose orbit under $T_k$ is a bounded set, must be in $\{(x, \bar{x}) : x \in S\}$ where

$$S = \{\alpha(\sigma, \tau) : \sigma, \tau \in \mathbb{R}\}.$$

Note that any periodic or preperiodic point must have a bounded orbit and therefore it must be in $\{(x, \bar{x}) : x \in S\}$ too. If we write $x = u + vi$, then the set $S$ is a closed domain enclosed by Steiner’s hypocycloid

$$(u^2 + v^2 + 9)^2 + 8(-u^3 + 3uv^2) - 108 = 0.$$

The Steiner’s hypocycloid is a simple closed curve which can also be parametrized by $\alpha(\sigma, \sigma)$ with $0 \leq \sigma \leq 1$. There is a symmetry under multiplication by a third root of unity because we have $\alpha(\sigma, \tau) \zeta_3 = \alpha(\sigma + 1/3, \tau + 1/3)$.

Periodic points will play an important role in our computations. A periodic point must have a bounded orbit and therefore its coordinates are of the form $\alpha(\sigma, \tau)$ for some $\sigma, \tau \in \mathbb{R}$. The following lemma is the key to count the elements in the value sets of bivariate Chebyshev maps over finite fields.

**Lemma 2.1.** The complex numbers $\alpha(\sigma, \tau)$ and $\alpha(\tilde{\sigma}, \tilde{\tau})$ are equal if and only if $\{\sigma, \tau, -(\sigma + \tau)\}$ and $\{\tilde{\sigma}, \tilde{\tau}, -(\tilde{\sigma} + \tilde{\tau})\}$ are equal as subsets of $\mathbb{R}/\mathbb{Z}$.

**Proof.** To understand the representations of elements in $S$ in terms of $\alpha(\sigma, \tau)$, a useful idea is to consider the tangent lines to the hypocycloid

$$C = \{\alpha(\sigma, \sigma) : \sigma \in [0, 1]\}.$$

Define $\ell_\sigma$ to be the line passing through the point $\alpha(\sigma, \sigma)$ with slope $-\tan(\pi \sigma)$. If $\sigma \in 1/2 + \mathbb{Z}$, then set $\ell_\sigma$ as the vertical line $\Re(z) = -1$. Note that the lines $\ell_\sigma$ are distinct for $\sigma \in [0, 1]$.

**Figure 1.** The domain $S$. 

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The subset \( \{ \sigma, \tau, -(\sigma + \tau) \} \) of \( \mathbb{R}/\mathbb{Z} \) has one element if \( \alpha(\sigma, \tau) \) is one of the three corner points of \( C \). It has two elements if \( \alpha(\sigma, \tau) \) is on \( C \) but not a corner point. The lemma is trivially true in these cases.

We assume that \( \alpha(\sigma, \tau) \) is an interior point of \( S \). In this case the subset \( \{ \sigma, \tau, -(\sigma + \tau) \} \) of \( \mathbb{R}/\mathbb{Z} \) has three elements and the lines \( \ell_\sigma \), \( \ell_\tau \) and \( \ell_{-(\sigma + \tau)} \) are pairwise distinct. Observe that each interior point of \( S \) is realized precisely three times by the lines \( \ell_\sigma \) as \( \sigma \) varies on the interval \([0, 1)\). We will show that the lines \( \ell_\sigma \), \( \ell_\tau \) and \( \ell_{-(\sigma + \tau)} \) intersect at \( \alpha(\sigma, \tau) \). This geometric result will finish the proof because it gives a one-to-one correspondence between the interior points of \( S \) and the subsets of \( \mathbb{R}/\mathbb{Z} \) with three elements.

Consider \( L_\sigma = \{ (\sigma, t) : t \in \mathbb{R} \} \), a vertical line in \( \mathbb{R}^2 \). We claim that \( \alpha \), regarded as a map from \( \mathbb{R}^2 \) to \( \mathbb{C} \), maps \( L_\sigma \) to the line segment \( \ell_\sigma \cap S \). It follows that \( \alpha(\sigma, \tau) \) lies on \( \ell_\sigma \). By symmetry \( \alpha(\sigma, \tau) \) lies on \( \ell_\tau \), too. Therefore the lines \( \ell_\sigma \) and \( \ell_\tau \) intersects at \( \alpha(\sigma, \tau) \). It is clear that \( \ell_{-(\sigma + \tau)} \) passes through the same point because \( \alpha(\sigma, \tau) = \alpha(\sigma, -(\sigma + \tau)) \). To justify the claim \( \alpha(L_\sigma) = \ell_\sigma \cap S \), we start with

\[
\alpha(L_\sigma) = \{ \zeta^{2\pi i \sigma} + \zeta^{2\pi i t} + \zeta^{-2\pi i (\sigma + t)} : t \in \mathbb{R} \}.
\]

The parametric curve \( \alpha(L_\sigma) \) in \( \mathbb{C} \) has the following components:

\[
f(t) = \Re(\alpha(L_\sigma)) = \cos(2\pi \sigma) + \cos(2\pi t) + \cos(2\pi(\sigma + t))
\]

\[
g(t) = \Im(\alpha(L_\sigma)) = \sin(2\pi \sigma) + \sin(2\pi t) - \sin(2\pi(\sigma + t)).
\]

The slope of the tangent line to the curve \( \alpha(L_\sigma) \) at any point \( \alpha(\sigma, t) \) is given by \( m = g'(t)/f'(t) \) provided that \( f'(t) \neq 0 \). We have

\[
f'(t)/(2\pi) = -\sin(2\pi t) - \sin(2\pi(\sigma + t)) = -2 \sin(\pi(\sigma + 2t)) \cos(\pi \sigma)
\]

\[
g'(t)/(2\pi) = \cos(2\pi t) - \cos(2\pi(\sigma + t)) = 2 \sin(\pi(\sigma + 2t)) \sin(\pi \sigma).
\]

Here, the second equalities follow from the sum to product formulas for the trigonometric functions. Thus, \( m = -\tan(\pi \sigma) \). This computation shows that \( L_\sigma \) is mapped under \( \alpha \) to a line segment with slope \(-\tan(\pi \sigma)\). If \( \sigma \in 1/2 + \mathbb{Z} \), then it is mapped to the vertical line \( \Re(z) = -1 \). Moreover, \( \alpha(L_\sigma) \) is a line segment with end points lying on the hypocycloid \( C \). To see this, note that the functions \( f(t) \) and \( g(t) \) have common critical values if and only if \( \sin(\pi(\sigma + 2t)) = 0 \). This is possible only at \( t = 1/2 - \sigma/2 \) and \( t = 1 - \sigma/2 \) for those \( t \in [0, 1) \). The points corresponding to these two \( t \) values are on the hypocycloid \( C \).

As a final observation note that the restricted map \( \alpha : [0, 1) \times [0, 1) \to S \) is a six to one map unless \( \alpha(\sigma, \tau) \) is on the boundary \( C = \{ \alpha(\sigma, \sigma) : \sigma \in \mathbb{R} \} \). A point on \( C \) which is not a corner point can be represented in three different ways and the corner points \( \alpha(0, 0), \alpha(1/3, 1/3) \) and \( \alpha(2/3, 2/3) \) can be represented uniquely. We finish this section with an illustration of the lines \( \ell_\sigma \) within the proof of Lemma 2.1.

**Example 2.2.** The point \(-1 + \sqrt{3} \in S \) can be represented by any of the following six expressions: \( \alpha(1/6, 1/3), \alpha(1/3, 1/6), \alpha(1/6, 1/2), \alpha(1/2, 1/6), \alpha(1/2, 1/3) \) and \( \alpha(1/3, 1/2) \). Note that the lines \( \ell_{1/6}, \ell_{1/3} \) and \( \ell_{1/2} \) intersect at the same point, namely \(-1 + \sqrt{3} \). See Figure 2.
Figure 2. There lines meeting at $-1 + \sqrt{-3}$.

3. Periodic and preperiodic points

The family of Dickson polynomials $D_k(x)$ has very explicit dynamical properties. For example a point with bounded orbit must lie in the interval $[-2, 2]$ in $\mathbb{C}$. Moreover a point $x$ is preperiodic if and only if $x = 2 \cos(2\pi r)$ for some rational number $r$.

We want to classify all periodic and preperiodic points of $T_k$. The cases $k = -1, k = 0$ and $k = 1$ are trivial so we assume that $|k| \geq 2$. All points with bounded orbit lie in $\{(x, \bar{x}) : x \in S\}$ and their coordinates are of the form $\alpha(\sigma, \tau)$ for some real numbers $\sigma$ and $\tau$. Periodic and preperiodic points have bounded orbits since there are finitely many elements in their orbits. Thus their coordinates are given by $\alpha(\sigma, \tau)$. Moreover we have the following

Lemma 3.1. Let $k$ be a fixed integer with $|k| \geq 2$. A point $(x, y) \in \mathbb{C}^2$ is a preperiodic point of $T_k$ if and only if there exist rational numbers $r, s \in \mathbb{Q}$ such that $x = \alpha(r, s)$ and $y = \bar{x} = \alpha(-r, -s)$. Moreover if $r$ and $s$ are written in their lowest terms then $(\alpha(r, s), \alpha(-r, -s))$ is a periodic point of $T_k$ if and only if the denominators of $r$ and $s$ are both relatively prime to $k$.

Proof. Suppose that $x = \alpha(r, s)$ where $r$ and $s$ are rational numbers. Then it is easy to see that $(x, \bar{x})$ is a preperiodic point of $T_k$. For the converse, let $\alpha(\sigma, \tau)$ be a preperiodic point under $T_k$ with $|k| \geq 2$. It follows that

$$\alpha(k^n \sigma, k^n \tau) = \alpha(k^l \sigma, k^l \tau)$$

for some positive integers $n \leq l$. This is possible if and only if

$$\{k^n \sigma, k^n \tau, -k^n(\sigma + \tau)\} = \{k^l \sigma, k^l \tau, -k^l(\sigma + \tau)\}$$

as subsets of $\mathbb{R}/\mathbb{Z}$ by Lemma 2.1. There are six possibilities. We will prove only one case. The proofs for the other cases are similar. Suppose that we have

$$k^n \sigma \equiv k^l \tau \pmod{\mathbb{Z}},$$
$$k^n \tau \equiv -k^l(\sigma + \tau) \pmod{\mathbb{Z}}.$$  

We omit the third equation since it can be obtained from these two. Starting with the former equation and then using the latter equation, we obtain

$$k^n \sigma \equiv k^l \tau \equiv k^{l-n}k^n \tau \equiv k^{l-n}[-k^l(\sigma + \tau)] \pmod{\mathbb{Z}}.$$  

Now we plug in the first equation again and get

$$k^n \sigma \equiv k^{l-n}[-k^l \sigma - k^n \sigma] \pmod{\mathbb{Z}}.$$
Therefore
\[ k^{n}\sigma + k^{l-n}\sigma + k^l\sigma \equiv 0 \pmod{\mathbb{Z}}. \]

From this congruence, we see that \( \sigma \) is a rational number with denominator \( k^{2l-n} + k^l + k^n \). Since \( k^{n}\sigma \equiv k^l\tau \pmod{\mathbb{Z}} \), the number \( \tau \) must be rational too.

Now suppose that \((x, y)\) is a periodic point under \( T_k \). Then there exist rational numbers \( r = a/b \) and \( s = c/d \) for some integers \( a, b, c \) and \( d \) such that \( x = \alpha(r, s) \) and \( y = \alpha(-r, -s) \). Suppose that \( r \) and \( s \) are written in their lowest terms, i.e. \( \gcd(a, b) = 1 \) and \( \gcd(c, d) = 1 \). Without loss of generality we can assume that \((x, y)\) is fixed by \( T_k \). The general result will follow from the identity \( T_k \circ T_m = T_{km} \) which is valid on \( \{(x, \bar{x}) : x \in S\} \). It follows by Lemma 2.1 that the set \( \{r, s,-(r + s)\} \) modulo \( \mathbb{Z} \) is permuted under multiplication by \( k \). Thus \( r \equiv k^b r \pmod{\mathbb{Z}} \) and therefore \( r(k^b - 1) \equiv 0 \pmod{\mathbb{Z}} \). From this we conclude that the denominator of \( r \) is relatively prime to \( k \) since it is a divisor of \( k^b - 1 \). The same result holds for \( s \) as well. To see the converse let \( f \) be order of \( k \) modulo the least common multiple of denominators of \( r \) and \( s \). Then \( T_k^f \) fixes the point \((x, y)\) and therefore it is an \( f \)-periodic point of \( T_k \).

Now we want to describe the set of points in \( C \) which are fixed under \( T_k \). Consider the following sets for \( |k| \geq 2 \):

\[
\begin{align*}
A_k &= \left\{ \alpha \left( \frac{d}{k-1}, \frac{e}{k-1} \right) : d, e \in \mathbb{Z} \right\}, \\
B_k &= \left\{ \alpha \left( \frac{d}{k^2 - 1}, \frac{dk}{k^2 - 1} \right) : d \in \mathbb{Z} \right\}, \\
C_k &= \left\{ \alpha \left( \frac{d}{k^2 + k + 1}, \frac{dk}{k^2 + k + 1} \right) : d \in \mathbb{Z} \right\}.
\end{align*}
\]

It is obvious that \( \text{Fix}(T_k, \mathbb{C}^2) \supseteq \{(x, \bar{x}) : x \in A(k) \cup B(k) \cup C(k)\} \). The converse inclusion is also true.

**Theorem 3.2.** Let \( k \) be a fixed integer with \( |k| \geq 2 \). Then

\[ \text{Fix}(T_k, \mathbb{C}^2) = \{(x, \bar{x}) : x \in A_k \cup B_k \cup C_k\}. \]

**Proof.** It is enough to show that the union \( A_k \cup B_k \cup C_k \) has \( k^2 \) elements. In order to do this we will apply the inclusion and exclusion principle.

We start with counting the elements in \( A_k \). The set \( A_k \) have elements of the form \( \alpha(d/(k-1), e/(k-1)) \). It is enough to consider \( 0 \leq d, e \leq k - 2 \) because of the periodicity. There are \((k - 1)^2\) such pairs of \((d, e)\). These pairs do not result in distinct elements because there are some repetitions. A fixed point will be represented six times among these pairs unless \( d = e \). An element of the form \( \alpha(d/(k-1), d/(k-1)) \) will be represented three times unless \( 3d \equiv 0 \pmod{k-1} \). Moreover three corner points \( \alpha(0, 0), \alpha(1/3, 1/3) \) and \( \alpha(2/3, 2/3) \) are in \( A_k \) if \( 3|k-1 \). If \( 3 \nmid k - 1 \), then only \( \alpha(0, 0) \) is in \( A_k \) among the corner points. Thus

\[ |A_k| = \frac{(k - 1)^2 + 3(k - 1) + 2\gcd(k - 1, 3)}{6}. \]

The set \( B_k \) have elements of the form \( \alpha(d/(k^2 - 1), dk/(k^2 - 1)) \). It is enough to consider \( 0 \leq d \leq k^2 - 2 \) because of the periodicity. A fixed point will be
represented two times among these values unless \( d \) is a multiple of \( k + 1 \). In that case the representation will be unique. Therefore

\[
|\mathcal{B}_k| = \frac{(k^2 - 1) + (k - 1)}{2}
\]

The set \( \mathcal{C}_k \) consists of elements of the form \( \alpha(d/(k^2 + k + 1), dk/(k^2 + k + 1)) \) with \( 0 \leq d \leq k^2 + k \). Every element is represented three times in this case unless \( k - 1 \) is divisible by 3. Thus

\[
|\mathcal{C}_k| = \frac{(k^2 + k + 1) + 2 \gcd(k - 1, 3)}{3}
\]

Now we consider the intersections. We start with \( \mathcal{A}_k \cap \mathcal{B}_k \). An element of the form \( \alpha(d/(k^2 - 1), dk/(k^2 - 1)) \) is in \( \mathcal{A}_k \) if and only if \( d \) is a multiple of \( k+1 \). There are \( k-1 \) such integers among \( \{0, 1, 2, \ldots, k^2-2\} \), each of which is represented uniquely. As a result \( |\mathcal{A}_k \cap \mathcal{B}_k| = k - 1 \). The set \( \mathcal{A}_k \cap \mathcal{C}_k \) may only have elements \( \alpha(0, 0), \alpha(1/3, 1/3) \) or \( \alpha(2/3, 2/3) \) since \( \gcd(k - 1, k^2 + k + 1) \) divides 3. Thus \( |\mathcal{A}_k \cap \mathcal{C}_k| = \gcd(k - 1, 3) \). Similarly \( |\mathcal{B}_k \cap \mathcal{C}_k| = \gcd(k - 1, 3) \), and \( |\mathcal{A}_k \cap \mathcal{B}_k \cap \mathcal{C}_k| = \gcd(k - 1, 3) \). Now it is trivial to verify that \( |\mathcal{A}_k \cup \mathcal{B}_k \cup \mathcal{C}_k| = k^2 \) by applying the inclusion and exclusion principle. \( \square \)

## 4. Value sets over finite fields

It is a well known fact that the Dickson polynomial \( D_k(x) \) induces a permutation on \( \mathbb{F}_q \) if and only if \( \gcd(k, q^s - 1) = 1 \) for \( s = 1, 2 \). It is in perfect analogy with one variable case that the \( n \) variable Chebyshev map is a bijection of \( \mathbb{F}_q^n \) if and only if \( \gcd(k, q^n - 1) = 1 \) for \( s = 1, 2, \ldots, n+1 \). In this section we compute the cardinality of \( V(T_k, F^2_2) \). As a corollary, we recover the result of Lidl and Wells in the case \( n = 2 \).

The coefficients of the Dickson polynomials \( D_k(x) \) can be computed using the following formula:

\[
D_k(x) = \sum_{i=0}^{\lfloor k/2 \rfloor} \frac{k(-1)^i}{k - i} \binom{k}{i} x^{k-2i}.
\]

Let \( q \) be a power of a prime \( p \). It is easily verified using this formula that \( D_k(x) \equiv x^q \) (mod \( p \)). Lidl and Wells provide a similar formula for the bivariate Chebyshev polynomials [LW72, p. 110]. We have

\[
g_k(x, y) = \sum_{i=0}^{\lfloor k/2 \rfloor} \sum_{j=0}^{\lfloor k/3 \rfloor} \frac{k(-1)^i}{k - i - 2j} \binom{k - i - 2j}{i + j} x^{k-2i-3j} y^i,
\]

where only those terms occur for which \( k \geq 2i + 3j \). Recall that \( T_k(x, y) = (g_k(x, y), g_k(y, x)) \). It is clear by this formula that

\[
T_q(x, y) \equiv (x^q, y^q) \pmod{p}.
\]

This congruence enables us to observe that the elements in \( \mathbb{F}_q^2 \) can be obtained by reducing the elements of \( \text{Fix}(T_q, \mathbb{C}^2) \) modulo a certain prime ideal. Because there are precisely \( q^2 \) fixed points of \( T_q \), we obtain the following lemma.

**Lemma 4.1.** Let \( \mathbb{F}_q \) be a finite field of characteristic \( p \). Consider the number field \( K = \mathbb{Q}(\text{Fix}(T_q, \mathbb{C}^2)) \) which is obtained by adjoining the coordinates of fixed points
of \( \mathcal{T}_q \) to the rational numbers. Let \( \mathfrak{P} \) be a prime ideal of \( K \) lying over \( p \). Then there exists a one-to-one correspondence

\[
\text{Fix}(\mathcal{T}_q, C^2) \leftrightarrow \mathbf{F}_q^2
\]

which is given by the reduction modulo \( \mathfrak{P} \).

After characterizing the elements in \( \mathbf{F}_q^2 \) in a compatible fashion under the action of \( \mathcal{T}_q \), determining the cardinality of \( V(\mathcal{T}_k, \mathbf{F}_q^2) \) reduces to a combinatoric argument. This is the idea we have used in order to compute the size of the value sets for Lattès maps [Ku14].

**Theorem 4.2.** Let \( k \) be a nonzero integer and let \( \mathbf{F}_q \) be a finite field of characteristic \( p \). Set

\[
a = \frac{q-1}{\gcd(k, q-1)}, \quad b = \frac{q^2-1}{\gcd(k, q^2-1)} \quad \text{and} \quad c = \frac{q^2+q+1}{\gcd(k, q^2+q+1)}.
\]

Then the cardinality of the value set is

\[
|V(\mathcal{T}_k, \mathbf{F}_q^2)| = \frac{a^2}{6} + \frac{b}{2} + \frac{c}{3} + \eta(k, q)
\]

where \( \eta(k, q) \) is given by

\[
\begin{array}{|c|c|c|}
\hline
\eta(k, q) & 3 \nmid k \text{ or } 3 \nmid a & 3 \mid k \text{ and } 3 \nmid a \\
\hline
2 \nmid k \text{ or } 2 \nmid b & 0 & 2/3 \\
\hline
2 \mid k \text{ and } 2 \mid b & a/2 & a/2 + 2/3 \\
\hline
\end{array}
\]

In particular if \( \gcd(k, 6) = 1 \), then \( \eta(k, q) = 0 \).

**Proof.** We have \( \text{Fix}(\mathcal{T}_q, C^2) = \{(x, \bar{x}) : x \in A_q \cup B_q \cup C_q\} \) and there is a one-to-one correspondence between \( \text{Fix}(\mathcal{T}_q, C^2) \) and \( \mathbf{F}_q^2 \). There will be three types of elements \( \mathcal{T}_k(x, y) \) in the value set \( V(\mathcal{T}_k, \mathbf{F}_q^2) \) depending on \( x \) being in \( A_q, B_q \) and \( C_q \). We will refer to these elements as Type-I, Type-II and Type-III, respectively. The proof of the case \( k = 1 \) is similar to the proof of Theorem 3.2. Other cases require a more detailed investigation. For each type we give the form of \( x \) and the number of elements in that type by the following table:

\[
\begin{array}{c|c|c|c|}
\text{Type} & \alpha \left( \frac{d}{a}, \frac{e}{c} \right) & a/3 + 2 \gcd(a, 3)/3 & a/3 + 2 \gcd(a, 3)/3 + b/2 \\
\hline
\text{Type-I} & \alpha \left( \frac{d}{a}, \frac{e}{c} \right) & a/3 + 2 \gcd(a, 3)/3 & a/3 + 2 \gcd(a, 3)/3 + b/2 \\
\hline
\text{Type-II} & \alpha \left( \frac{d}{a}, \frac{e}{c} \right) & a/3 + 2 \gcd(a, 3)/3 & a/3 + 2 \gcd(a, 3)/3 + b/2 \\
\hline
\text{Type-III} & \alpha \left( \frac{d}{a}, \frac{e}{c} \right) & a/3 + 2 \gcd(a, 3)/3 & a/3 + 2 \gcd(a, 3)/3 + b/2 \\
\hline
\end{array}
\]

The number of elements which fit into different types are given by the following table:

\[
\begin{array}{c|c|c|c|c|}
\text{Type-I&II} & \text{Type-I&III} & \text{Type-II&III} & \text{Type-I&II&III} \\
\hline
\gcd(a, b)/2 & \gcd(a, c)/2 & \gcd(b, c)/2 & \gcd(a, b, c)/2 \\
\hline
\end{array}
\]

Applying the inclusion and exclusion principle we see that the cardinality of the value set \( V(\mathcal{T}_k, \mathbf{F}_q^2) \) can be written as

\[
|V(\mathcal{T}_k, \mathbf{F}_q^2)| = \left( \frac{a^2}{6} + \frac{b}{2} + \frac{c}{3} \right) + \left( \frac{a}{2} + \frac{\gcd(b, q-1)}{2} - \gcd(a, b) \right) + \left( \frac{\gcd(a, 3) + 2 \gcd(c, 3)}{3} - \gcd(a, c) - \gcd(b, c) + \gcd(a, b, c) \right).
\]
The second term in the above sum is 0 unless \(2 \mid b\) and \(2 \nmid k\). To see this note that if \(2 \mid b\) and \(2 \mid k\), then \(\frac{\gcd(k,q-1)}{2} = \frac{a}{2}\) and \(\gcd(a,b) = \frac{a}{2}\). If \(2 \nmid b\) or \(2 \nmid k\) then \(\gcd(a,b)\) becomes \(a\). A case by case investigation shows that the third term is 0 unless \(3 \mid k\) and \(3 \mid a\). If \(3 \mid k\) and \(3 \mid a\), then each greatest common divisor appearing in the third term is equal to 1 except \(\gcd(a,3) = 3\). Thus the third term of the sum becomes \(\frac{2}{3}\).

We recover the result of Lidl and Wells for bivariate Chebyshev maps by Theorem 4.2. More precisely we have a sufficient and necessary condition for bivariate Chebyshev maps for being a permutation of \(\mathbb{F}_q^2\).

**Corollary 4.3.** The bivariate Chebyshev map \(T_k(x,y)\) induces a permutation of \(\mathbb{F}_q^2\) if and only if \(\gcd(k,q^s-1) = 1\) for \(s = 1, 2, 3\).

We finish our paper by giving an example to illustrate the invariants introduced in Theorem 4.2.

**Example 4.4.** Let \(k = 6^i\) with \(i = 0, 1, 2, \ldots\) and consider the bivariate Chebyshev map \(T_{6^i}\) on \(\mathbb{F}_{73}^2\). We have \(T_{6^i}(x,y) = (g_{6^i}(x,y), g_{6^i}(y,x))\) where
\[
g_{6^i}(x,y) = x^{6^i} - 6xy^{4^i} + 9y^{2^i}x^{2^i} + 6x^{3^{i}} - 2y^{3^{i}} - 12yx + 3.
\]
The maps \(T_{6^i}\) are not bijections of \(\mathbb{F}_{73}^2\) for \(i = 1, 2, 3, \ldots\) since 6 is not relatively prime to \(73 - 1, 73^2 - 1\) and \(73^3 - 1\). We find the cardinality of the value set by using Theorem 4.2 and obtain the following table.

| \(k\) | \(6^0\) | \(6^1\) | \(6^2\) | \(6^3\) | \(6^4\) | \(6^5\) | \ldots |
|-------|-------|-------|-------|-------|-------|-------|-------|
| \(a\) | 72    | 12    | 2     | 1     | 1     | 1     | \ldots |
| \(b\) | 5328  | 888   | 148   | 74    | 37    | 37    | \ldots |
| \(c\) | 5403  | 1801  | 1801  | 1801  | 1801  | 1801  | \ldots |
| \(\eta(k,73)\) | 0     | 12/2 + 2/3 | 2/2 | 1/2 | 0 | 0 | \ldots |
| \(|V(T_{6^i}, \mathbb{F}_{73}^2)|\) | 5329  | 1075  | 676   | 638   | 619   | 619   | \ldots |

Note that the size of the value set of \(T_{6^i}\) will be 619 for \(i \geq 4\) since \(a, b, c\) are relatively prime to 6 from that point on.

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