Ideal Fluids, the Quark Gluon Plasma, and Hadronic Gases

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Abstract. A detailed comparison between two methods to calculate the shear viscosity coefficient of a hot hadronic gas is presented. We choose two systems in this comparison which are massless particles with current algebra cross section and a mixture comprised of pions with rho resonances. The two methods involved are the Green-Kubo method, applied using the Ultrarelativistic Quantum Molecular Dynamics (UrQMD) model to simulate the hadronic medium, and the Chapman-Enskog method. In addition, the effect of the resonance lifetime on the shear viscosity coefficient is investigated.

1. Introduction

Ultrarelativistic heavy ion collisions at the Relativistic Heavy Ion Collider (RHIC) and the Large Hadron Collider (LHC) are thought to have created a Quark Gluon Plasma (QGP) with characteristics of an ideal fluid [1, 8, 11]. As such, the characterization of the shear viscosity coefficient \( \eta \) and the shear viscosity to entropy density ratio \( \eta/s \) of hot, dense QCD matter is of significant interest.

One should note that the shear viscosity, especially in the context of a relativistic heavy ion collision, is a time-dependent quantity. While the partonic phase of such a collision is expected to have a very low value of \( \eta/s \), after hadronization occurs \( \eta/s \) is expected to rapidly increase [6]. In order to quantify the viscosity of the deconfined phase of a relativistic heavy ion collision, it is necessary to separately determine the viscosity of the hadronic phase in order to quantitatively constrain the viscosity of the deconfined phase. Although several calculations for the hadronic shear viscosity exist, in order to validate a method and test the limits to which a calculation can be reliable, it is necessary to perform a systematic comparison between two different methods and see where the differences exist, such as in [13], and that is the goal of this paper.

2. Theory

Two independent methods here are presented which are used to calculate the shear viscosity coefficient \( \eta \). They are the Chapman Enskog Method and the Green Kubo method.

2.1. Calculating Shear Viscosity: The Chapman Enskog Method

The idea of the Chapman Enskog approximation is that the system, which is slightly disturbed from its equilibrium, will have a perturbed distribution function which can be expressed in terms of hydrodynamic variables [5].
For a single component gas, the first approximation to the shear viscosity takes the form of [13]
\[ \eta = \frac{1}{10} k_B T \frac{\gamma_0^2}{c_{00}}, \]  
(1)
where \( \gamma_0 = -10 \hat{h}, \hat{h} = K_3(z)/K_2(z) \) and \( z = mc^2/k_B T \) is called the relativity parameter. The term \( c_{00} \) contains the interaction in the system which is defined as
\[ c_{00} = 16 \left( w_2^{(2)} - \frac{1}{z} w_4^{(2)} + \frac{1}{3z^2} w_6^{(2)} \right). \]  
(2)

The quantity \( w_i^{(s)} \) is the relativistic omega integral given by
\[ w_i^{(s)} = \frac{2\pi z^3 c}{K_2(z)^2} \int_0^\infty d\psi \sinh^7 \psi \cosh^i \psi K_j(2z \cosh \psi) \times \int_0^\pi d\Theta \sin \Theta \sigma(\psi, \Theta) (1 - \cos^s \Theta), \]  
(3)
where \( j = 5/2 + 1/2(-1)^i, \sinh \psi = g/mc \) and \( \cosh \psi = P/2mc \). The quantities \( g \) and \( P \) are the relative and center of mass momenta, respectively. The interaction in the system is represented by the differential cross section, \( \sigma(\psi, \Theta) \).

### 2.2. Simulating the Hadronic Medium

To simulate the hadronic medium we utilize the Ultrarelativistic Quantum Molecular Dynamics (UrQMD) model. UrQMD is based upon the Boltzmann equation. Interactions in our simulation are based only upon scattering, and we neglect any interparticle potential. The criterion for a collision to occur is based upon the geometric interpretation of the cross section:
\[ d_{\text{min}} \leq \sqrt{\frac{\sigma_{\text{tot}}}{\pi}}, \]  
(4)
where \( d_{\text{min}} \) is the transverse distance at closest approach between two particles. UrQMD is described in great detail in [2].

In order to use the Green-Kubo formalism to extract the shear viscosity coefficient, it is necessary to establish that our system reaches a state of thermal equilibrium. To force our system into equilibrium, we confine our hadronic medium to a cubic box with periodic boundary conditions in real space. This technique was also used in [3, 9, 10]. The input parameters of our system are: volume of the box, initial particle species, along with their chemical potentials. We initialize our systems to be in thermal and chemical equilibrium (\( \mu_i = 0 \)). For the test case involving a gas of massless pions with energy dependent cross sections, the initial yields corresponding to a given temperature and chemical potential are calculated using the Boltzmann distribution function. For the test case involving a pion rho mixture, the initial particle yields are calculated using the Statistical Hadronization with Resonances (SHARE) model [12], with the resonance and decay tables modified to represent the degrees of freedom of the system.

### 2.3. Shear Viscosity Calculation Via Green Kubo Method

The second method to calculate the shear viscosity is the Green Kubo method, which amounts to finding the time integral of the correlations of the shear component of the energy-momentum tensor about the equilibrium state [7]. The energy-momentum tensor is related to the phase-space density of the system through the following equation
\[ \pi^{\mu\nu} = \int d^3p \frac{p^{\mu'} p^{\nu'}}{p^0} f(x, p). \]  
(5)
The phase space density for a system of particles uniformly distributed in phase space is given by

\[ f(x, p) = \frac{1}{V} \sum_{j=1}^{N} \delta(\vec{p} - \vec{p}_j). \]  

(6)

This enables one to calculate \( \pi^{xy} \) directly

\[ \pi^{xy} = \frac{1}{V} \sum_{j=1}^{N} \frac{p^x(j)p^y(j)}{p^0(j)}. \]  

(7)

Note that since we assumed the particles were uniformly distributed in real space, the Green-Kubo formula for the shear viscosity coefficient becomes

\[ \eta = \frac{V}{T} \int dt \langle \pi^{xy}(0)\pi^{xy}(t) \rangle. \]  

(8)

In order to simplify evaluating this integral, we assume the correlation function \( \langle \pi^{xy}(0)\pi^{xy}(t) \rangle \) has an exponential ansatz:

\[ \langle \pi^{xy}(0)\pi^{xy}(t) \rangle \propto \exp\left(-\frac{t}{\tau_{\pi}}\right). \]  

(9)

This ansatz has also been used in [9]. If we assume the exponential dependence of the stress tensor correlation function on post-equilibration time \( t \), the Kubo formula reduces to

\[ \eta = \frac{V}{T} \tau_{\pi} \langle \pi^{xy}(0)^2 \rangle. \]  

(10)

The volume \( V \) of the system is an input parameter, the temperature \( T \) is extracted from fitting momenta distributions to a Boltzmann distribution after checking that the momenta distributions in the system are isotropic, the correlator relaxation time \( \tau_{\pi} \) is obtained from fitting the momentum correlation function to an exponential, and \( \langle \pi^{xy}(0)^2 \rangle \) is obtained from the intercept of the momentum correlation function. Hence, the shear viscosity coefficient can be calculated in this way, and in the next section we present the results of the shear viscosity coefficient for two systems.

3. Results
In this section we present the results for the calculation of the shear viscosity via both the Chapman-Enskog and Green-Kubo method for two systems. One system involves a pure chiral pion gas with an energy dependent cross section given by current algebra. The other system is a \( \pi\rho \) mixture, where massive pions could scatter elastically or inelastically via an intermediate \( \rho \) resonance state. We also scale the resonance lifetime to see if there is any effect on the shear viscosity coefficient.

3.1. Massless Pions with Energy Dependent Cross Section (with Current Algebra)
The total (energy-dependent) cross section for the first system is given by

\[ \sigma(\sqrt{s}) = \frac{5s}{72\pi f_{\pi}^2}, \]  

(11)

where \( s \) is the Mandelstam variable representing the total energy in the center of mass frame and \( f_{\pi} = 93 \text{ MeV} \) is the pion decay constant. The results for the shear viscosity coefficient are shown as a function of temperature in Figure 1, and we find that there is very good agreement between the calculations from the Green-Kubo method and the Chapman-Enskog method.
3.2. Pion Rho Mixture
The $\pi\rho$ mixture involves the two following processes:

- (a) $\sigma_{\pi\pi\rightarrow\pi\pi,\text{elastic}} = 17.778$ mb.
- (b) $\pi\pi\rightarrow\rho\rightarrow\pi\pi$ (inelastic)

The value of the cross section in (a) is based upon the additive quark model, which dictates that meson-meson and baryon-baryon cross sections are related via

$$\sigma_{MM} = \left(\frac{2}{3}\right)^2 \sigma_{BB},$$

where M/B stands for meson/baryon, respectively. Since the elastic nucleon-nucleon cross section is known to be approximately 40 mb, this implies that $\sigma_{\pi\pi,\text{elastic}} = 17.778$ mb. The results for the shear viscosity coefficient are shown as a function of temperature in Figure 2. Also shown in Figure 2 are the results for the shear viscosity coefficient when the resonance lifetime is scaled by a factor of 5.

4. Discussion
As seen in Figure 1 there is good general agreement between the Chapman Enskog and Green Kubo methods for the case of chiral pions with energy dependent cross sections. In Figure 2, we find that, although there is good agreement between the Green-Kubo and Chapman-Enskog methods in the temperature range ($100 < T < 150$) MeV, the results from the Green-Kubo calculation are systematically higher than those for the Chapman-Enskog method in the temperature range ($150 < T < 200$) MeV. This is a noteworthy result, particularly when one realizes that many more inelastic scattering events through the intermediate resonance channel are expected in the temperature range $T > 150$ MeV than $T < 150$ MeV. In order to fully understand why the Green-Kubo method yields a higher result for the shear viscosity than the Chapman-Eskog method does, it is necessary to understand the effect of resonance lifetimes in UrQMD on the shear viscosity coefficient. This is a subtle effect and the effect of timescales...
on transport coefficients in general has been investigated in the past [4]. Scaling the resonance lifetime by a factor of 5 yields no significant effect except in the higher temperature range. This interesting preliminary result should be continued by scaling the resonance lifetime by a factor of 0.2 to see if the trend is witnessed in the higher temperature range ($T > 175$) MeV in the opposite direction.

5. Conclusions
We have compared two methods to calculate the shear viscosity coefficient of hot hadronic matter, which is important to quantify when addressing the general question of transport coefficient behavior in the context of a relativistic heavy ion collision. The two methods used to calculate the shear viscosity were the Chapman-Enskog method and the Green-Kubo method using the UrQMD model to simulate the hadronic medium in equilibrium. When the results of the calculation for the two different methods were compared, excellent agreement was found between the Chapman Enskog and Green Kubo methods for the system of pure massless pions with an energy dependent cross section, whereas a systematic deviation occurred between the Chapman Enskog and Green Kubo methods in the temperature range $150 < T < 200$ MeV, where significant inelastic scattering is expected to occur. Preliminary results indicate changing the resonance lifetime does impact the shear viscosity coefficient at temperatures $T > 175$ MeV, and this result should be investigated further.

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