New Turan-type bounds for Johnson graphs

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A new estimate is obtained for the number of edges in induced subgraphs of Johnson graphs.

1. Introduction

In this paper, we consider the graph $G(n, r, s)$, whose vertices are $r$-element subsets of the set $\{1, 2, \ldots, n\}$, and an edge between two vertices is drawn if the size of the intersection of the corresponding subsets is $s$. Another definition of the graph $G(n, r, s)$: the graph vertices are the unit-cube vertices in $n$-dimensional space, which have exactly $r$ ones in coordinate notation, and an edge between two vertices is drawn when the distance between them is $\sqrt{2(r-s)}$. It is clear that these two formulations are equivalent. Graphs $G(n, r, s)$ are called Johnson graphs. They play a huge role in combinatorial geometry problems (see, e.g., [1,3,6,7,12,14,16,19,28,29]), in the coding theory (see, e.g., [2,13,21]), in the Ramsey theory (see, e.g., [5,8,20]) etc.

In this paper, we study the extremal properties of the graph $G(n, r, s)$. Namely, we investigate the number of edges in an arbitrary subgraph of this graph. Notice that independent vertex set of the $G$ graph – such vertex subset, that no two vertices from this subset are connected by an edge. Independence number $\alpha(G)$ is the largest cardinality of an independent set of vertices of the graph.

We denote by $r(W)$ the number of edges of the graph $G = (V, E)$ on the set $W \subseteq V$. In other words,

$$r(W) = |\{(x, y) \in E \mid x \in W, y \in W\}|.$$

We also define

$$r(l) = \min_{|W| = l, W \subseteq V} r(W).$$

The question arises about the study of this value. Classical Turan’s theorem 1941 gives the answer to this question in the general case.

Theorem 1. Let $G$ be an arbitrary graph, let $\alpha$ be its independence number, $l > \alpha$. Then $r(l) \geq \frac{l^2}{2\alpha} - \frac{l}{2}$.

The proof of this theorem does not take into account any special properties of the graph $G$, and, moreover, this theorem is not improveable in general. However, it is reasonable to assume that for graphs with some constraints, the estimate can be improved. We consider distant graphs – graphs, whose vertices are points in $\mathbb{R}^n$ space, and the edge between such vertices is present if and only if the distance between them is equal to some constant. It is clear that defined graph $G(n, r, s)$ is distant graph.

For arbitrary distant graphs the following theorem has been proven (22).

Theorem 2. Let $G_n$ be sequence of the distant graphs, which $V(G_n) \subset \mathbb{R}^n$. Let $\alpha_n = \alpha(G_n)$. Let $W_n$ be a subset of $V(G_n)$. If $|W_n| = l(n)$ and $n\alpha_n = o(l(n))$, then with $n \to \infty$

$$r(l(n)) \geq \frac{l(n)^2}{\alpha_n} (1 + o(1)).$$

Then, we see, that on distant graphs, forming sequences with certain asymptotic properties, Turan assessment has been improved twice. It can be assumed that at an even more narrow class of distant graphs $G(n, r, s)$ the assessment allows for further improvements. And indeed, in the (27) paper the following theorem has been proven (see, e.g., (23, 26)).

Theorem 3. Consider graph $G(n, 3, 1)$. Let the function $l : \mathbb{N} \to \mathbb{N}$ satisfy $n^2 = o(l)$ as $n \to \infty$. Then there is a function $h : \mathbb{N} \to \mathbb{N}$, that $h \sim \frac{n^2}{2m}$ with $n \to \infty$ and $r(l(n)) \geq h(n)$ for any large enough $n \in \mathbb{N}$.

To understand how the results of theorems 2 and 3 correlate, note, that $\alpha(G(n, 3, 1)) \in \{n - 2, n - 1, n\}$ (see (27)). This means, that on its class of graphs theorem 3 is one and a half times stronger than the general theorem 2. Our main result will be a generalization of theorem 3 in case of fixed $r, s$ with the condition, that $r = 2s + 1$ and $r - s$ is the power of prime number. Obviously, the parameters of theorem 3 satisfy these conditions. Note, that it can be in this conditions in the (1) was shown, that $\alpha(G(n, r, s)) \sim n^{r(2r - 2s - 1)}$. This means that there is a function $q(n) = (1 + o(1))$, that $\alpha(G(n, r, s)) = q(n) \cdot n^s (2r - 2s - 1)!$. Also, since the function $q(n)$ is limited, there is the constant $C_0$, that $\alpha(G(n, r, s)) \leq C_0 n^s$. We will sometimes need these formulations. So, we have

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Theorem 4. Let \( r = 2s + 1 \) and \( r - s \) is power of the prime number, and \( l(n) \) is any function with limitations \( l(n) = o(n^{2s+1}) \) and \( n^{2s} = o(l(n)) \). Let \( \alpha_n = o(G(n, r, s)) \). Then there is function \( h : \mathbb{N} \to \mathbb{N} \), satisfy \( h \sim \frac{3(\alpha)^2}{2s_n} \) with \( n \to \infty \), and \( r(l(n)) \geq h(n) \).

To prove theorem 4, we will need an additional lemma.

Lemma. Let the parameters \( r, s \) and function \( l \) satisfy the conditions of the theorem 4. Let \( W \) be a random set of the vertices of the graph \( G(n, r, s) \), whose size is \( l(n) \). Let \( \Gamma \) be a maximal independent set of vertices in a subgraph of a graph \( G(n, r, s) \), based on vertices from \( W \). Let \( w \in W \setminus \Gamma \). Denote by \( n(\Gamma, w) \) number of vertices in \( \Gamma \), related to the \( w \). Let \( U_1 \) and \( U_2 \) be sets of such vertices \( w \in W \setminus \Gamma \), that \( n(\Gamma, w) = 1 \) or \( 2 \) respectively. Then there exists a constant \( C_1 \) such that \( |U_1 \cup U_2| \leq C_1n^{2s} \).

We emphasize that the constant in the lemma will depend only on \( r \) and \( s \), but not on the \( l, W \) or \( \Gamma \).

In the next section, we first give a proof of the lemma, and then, in subsection 2.2, we prove theorem 4. In the proofs, in order to avoid confusion, it will sometimes be convenient for us to distinguish between the notation for one vertex \( u \) or another of the graph \( G(n, r, s) \) and the corresponding \( r \)-element subset. The latter will be denoted by \( \text{supp}(u) \) and will be called the support of the vertex \( u \).

Finally, note that similar results for the case of arbitrary distance graphs in the plane can be found in the paper [15].

2. Proofs

2.1. Proof of the lemma. Firstly, let us prove that there exists \( C_2 \) such that \( |U_1| \leq C_2n^{s+1} \). Let’s choose the vertex \( u \in \Gamma \). Let

\[
U_{1, u} = \{ w : w \in W \setminus \Gamma, n(\Gamma, w) = 1 \text{ and } (w, u) \in E \}.
\]

Denote by \( U_1 \) the union of sets \( U_{1, u} \) by all \( u \in \Gamma \). Let us fix \( u \) and estimate the cardinality of \( U_{1, u} \). Let \( v \in U_{1, u} \). supports of the vertices \( u \) and \( v \) intersect by \( s \) elements, and these elements can be selected in \( C^s_n \) ways. Next, \( (s + 1) \), element of the support \( v \) can be chosen in \( n - r \) ways. And for its \( s + 1 \) elements there is at most one way to select all the others. Let it is wrong, so there are at least two vertices \( v_1 = \{v^1, v^2, \ldots, v^{s+1}, a, \ldots \} \) and \( v_2 = \{v^1, v^2, \ldots, v^{s+1}, b, \ldots \} \). Note that there is no edge between them, because its supports intersect by \( s + 1 \) elements. Also note, that each of the vertices has only one edge with the set \( \Gamma \) and this edge leads to the selected vertex \( u \) (see pic. 1). Then the set \( (\Gamma \setminus \{u\}) \cup \{v_1, v_2\} \) doesn’t have any edges, that is, it is independent, and has cardinality greater than the cardinality of the set \( \Gamma \), which contradicts the assumption of maximality of \( \Gamma \). So,

\[
|U_1| \leq C^s_n \cdot |\Gamma| \cdot n \leq C^s_r \cdot \alpha_n \cdot n \sim n^{s+1}C^s_r \left(\frac{2r - 2s - 1}{r!}\right)^n.
\]

Now let us prove that there is such a constant \( C_3 \), that \( |U_2| \leq C_3n^{2s} \). In other words, it is necessary to estimate the number of such vertices \( w \in W \setminus \Gamma \), that \( n(\Gamma, w) = 2 \). Let \( w \) have the edges with \( u_1, u_2 \in \Gamma \). supports of the \( u_1, u_2 \) can intersect in \( 0, 1, \ldots, s - 1, s + 1, \ldots, r - 1 = 2s \) elements. Since the vertex \( w \) has an edge with any vertex \( u_1, u_2 \), support of \( w \) and union of supports of \( u_1, u_2 \) can intersect in \( s, s + 1, \ldots, r - 1 = 2s \) elements. Let us define checkmark. We will call checkmark three vertices with such properties: \( u_1, u_2 \in \Gamma \) and \( w \in W \setminus \Gamma, n(\Gamma, w) = 2 \), moreover \((u_1, w) \in E, (u_2, w) \in E \) (see pic. 2). Vertex \( w \) will be called the center of the checkmark, other checkmark vertices will be called sides. We divide the proof into two cases. In the first case support of the center of the checkmark intersects with union of the sides’ supports more, than by \( s \) elements. In the second case checkmark’s center’s support intersects with union sides’ supports by \( s \) elements. It is possible...
if supports of the checkmark’s sides have \( s + 1 \) common element (not \( s \) elements, because \( u_1, u_2 \) don’t have an edge).

\[
\gamma = \gamma \cup \{w_1, w_2, w_3\} \quad \text{and} \quad \Gamma = \Gamma \setminus \{u_1, u_2\} \cup \{w_1, w_2, w_3\}.
\]

**Case 1.** Let \(|\text{supp}(u_1) \cup \text{supp}(u_2)| = k, k \geq s + 1\). Choose pair \( u_1, u_2 \in \Gamma \) and count number of the checkmarks with sides in this pair and centers in certain \( w \). As long as \(|\text{supp}(u_1) \cap \text{supp}(w)| = s\), there are \( C^s_r \) ways choose \( s \) elements of the support \( w \), by which it will intersect with the support \( u_1 \). Among not more than \( r \) elements in the difference of the support \( u_2 \) and support of \( u_1 \), we have to choose \( k - s \geq 1 \) elements (see pic. 3). So, \( s + 1 \)-th element is chosen in at most \( r \) ways in the support \( u_2 \). And there are no more than two ways to select the remaining elements, and it doesn’t matter where to choose these elements — in the support or in the whole set of \( n \) numbers. Suppose the contrary, then there are at least three vertices \( w_1, w_2, w_3 \), whose supports have \( s + 1 \) common element, and therefore, there are no edges between them. Therefore, since from each vertex \( w_1, w_2, w_3 \) there are exactly 2 edges with the set \( \Gamma \), and all the edges go to the vertices \( u_1, u_2 \), the set \((\Gamma \setminus \{u_1, u_2\}) \cup \{w_1, w_2, w_3\}\) is independent and has a larger cardinality, than \( \Gamma \), which contradicts the maximality of \( \Gamma \). So, for any two vertices \( u_1, u_2 \) there are no more \( 2r \cdot C^s_r \) checkmarks. A pair of vertices can be selected at most \( \alpha_n^2 \sim n^{2s} \left( \frac{(2r-2s-1)!}{r!(r-s-1)!} \right)^2 \) ways, and therefore, the number of such vertices \( w \) does not exceed

\[
\alpha_n^2 \cdot 2r \cdot C^s_r < C_4 n^{2s}.
\]
Case 2. Here we are interested in checkmarks that have \(|\text{supp}(u_1) \cup \text{supp}(u_2) \cap \text{supp}(w)| = s\). Choose one vertex \( u_1 \in \Gamma \), fix \( s \) elements in its support. Let there be at least one such checkmark that the support of its center \( w_0 \) intersects with its sides supports \( u_1, u_2 \) precisely along these \( s \) elements (we will call them fixed), and its sides supports, respectively, intersect by fixed \( s \) and at least one more element (see pic. 4). Now let’s calculate how many more sides of the checkmarks with the side at the vertex \( u_1 \) and the same \( s \) elements (we want to estimate exactly the number of sides; we will estimate the number of centers for a given pair of sides later). Since all supports of the sides of the checkmarks have common fixed elements with the support \( u_1 \) and there are \( s \) such elements, each of the side supports must have some \((s+1)\)-th common element with support of the \( u_1 \), otherwise, in an independent set of vertices \( \Gamma \) an edge is formed. Moreover, all checkmark sides, except \( u_1 \) and the second checkmark side with the center in \( w_0 \), do not have edges with \( w_0 \), because the center of the checkmark has exactly two edges with the set \( \Gamma \). However, all the supports of these sides and the support \( w_0 \) have the same fixed \( s \) elements in the intersection. This means each of these sides supports must have at least one additional common element with the support \( w_0 \). Thus, the required number of sides is at most \((r-s)^2 C_{n-s}^r \leq C_{n}n^{r-s-2} = C_{n}n^{s-1}\). Now, for each pair of sides of the checkmark, we will count how many centers \( w \) can exist. First \( s \) elements of the support are fixed — \( w \) intersects with supports of \( u_1 \) and \( u_2 \) by this fixed elements. Next element we choose not more that \( n \) ways besides the supports \( u_1 \) and \( u_2 \). And there are no more than two ways to select all other elements. The proof is similar to the proof in the case 1 — otherwise we will have three vertices \( u_1, u_2, w \) without an edges, and the set \((\Gamma \cup \{u_1, u_2\}) \cup \{w_1, w_2, w_3\}\) will have cardinality more than \( \Gamma \), and still be independent. Hence, for the vertex \( u_1 \) and some \( s \) elements from its support checkmarks centered at some \( w \) at most \( 2n \), whence we get that the number of checkmarks with the given \( u_1 \) and given \( s \) fixed elements no more than \( 2n \cdot C_{n}n^{s-1} = C_{n}n^{s}\). Ways to choose the \( u_1 \) vertex and \( s \) elements in it respectively

\[ \alpha_n \cdot C_t^s \leq n^s C_t, \]

hence, in the current case, the number of vertices \( w \) is at most

\[ (C_0n^s) \cdot (C_7n^s) = C_8n^{2s}. \]

So, in each of the cases, we have an estimate of the form \( C \cdot n^{2s} \). Adding all the constants, we obtain the value stated in the lemma \( C_1n^{2s} \).

2.2. Proof of theorem 4. Let \( W \) — some subset in the set of vertices of the graph \( G(n, r, s) \), having cardinality \( l = l(n) \). Consider the largest independent set in terms of cardinality \( \Gamma_1 \) in the subgraph of \( G(n, r, s) \), generated by the set of vertices \( W \). Let its cardinality equal \( \beta_1 \leq \alpha_n \). Let \( F_1 \) — a subset of such vertices in the set \( W \setminus \Gamma_1 \), that for any vertex \( w \in F_1, (\Gamma_1, w) \leq 2 \). Let \( f_1 = |F_1| \). It follows from the lemma that \( f_1 \leq C_1n^{2s} \). Note that any vertex \( u \in W \setminus (\Gamma_1 \cup F_1) \) has at least three edges with \( \Gamma_1 \). Then found at least \( 3(l(n) - f_1 - \beta_1) + f_1 \geq 3(l(n) - \alpha_n) - 2C_1n^{2s} \) edges. Let us remove from \( W \) the independent set \( \Gamma_1 \) and in the resulting set \( W \setminus \Gamma_1 \) choose a new largest independent set \( \Gamma_2 \) with cardinality equal \( \beta_2 \leq \alpha_n \). Let \( F_2 \) — such subset of the set \( W \setminus (\Gamma_1 \cup F_1) \), that for any vertex \( w \in F_2 \) we have \( |\Gamma_2, w) \leq 2 \). Let \( f_2 = |F_2| \). From the lemma we have the estimate \( f_2 \leq C_1n^{2s} \). We found again at least \( 3(l(n) - 2\alpha_n) - 2C_1n^{2s} \) edges. Repeat this operation \( \left\lceil \frac{l(n)}{\alpha_n} \right\rceil \) times, we get an estimate

\[ r(l(n)) \geq \sum_{i=1}^{\left\lceil \frac{l(n)}{\alpha_n} \right\rceil} (3(l(n) - i\alpha_n) - 2C_1n^{2s}) \sim 3l(n) \cdot \frac{l(n)}{\alpha_n} - \frac{3}{2} \alpha_n \cdot \frac{l(n)}{\alpha_n} (\frac{l(n)}{\alpha_n} + 1) - \frac{l(n)}{\alpha_n} \cdot 2C_1n^{2s} \sim \]

\[ \sim 3\frac{l^2(n)}{\alpha_n} - \frac{3}{2} \frac{l^2(n)}{\alpha_n} - 2C_1n^{2s} \frac{l(n)}{\alpha_n} \sim \frac{3l^2(n)}{2} \frac{1}{\alpha_n} \]

with \( n \to \infty \), since under the assumption \( n^{2s} = o(l(n)) \). Theorem 4 is proven.

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