The Relativistic Particle: Dirac observables and Feynman propagator

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We analyze the algebra of Dirac observables of the relativistic particle in four space-time dimensions. We show that the position observables become non-commutative and the commutation relations lead to a structure very similar to the non-commutative geometry of Deformed Special Relativity (DSR). In this framework, it appears natural to consider the 4d relativistic particle as a five-dimensional massless particle. We study its quantization in terms of wave functions on the 5d light cone. We introduce the corresponding five-dimensional action principle and analyze how it reproduces the physics of the 4d relativistic particle. The formalism is naturally subject to divergences and we show that DSR arises as a natural regularization: the 5d light cone is regularized as the de Sitter space. We interpret the fifth coordinate as the particle’s proper time while the fifth momentum can be understood as the mass. Finally, we show how to formulate the Feynman propagator and the Feynman amplitudes of quantum field theory in this context in terms of Dirac observables. This provides new insights for the construction of observables and scattering amplitudes in DSR.

Introduction

There has recently been an increasing interest in theories of Deformed Special Relativity (DSR). Tentatively introduced as Lorentz invariant theories with modified dispersion relation taking into account a universal length (or mass) scale [1], they are believed to provide grounds for a phenomenology of quantum gravity in the semi-classical regime. In three space-time dimensions, it has been shown that matter degrees of freedom are described after integration over the metric fluctuations by an effective non-commutative quantum field theory which provides an explicit realization of a DSR theory [2]. There also are several heuristic arguments for 4d quantum gravity [3], even though we do not yet have a definitive derivation of a DSR quantum field theory from quantum gravity. It is nevertheless important to understand how to build a consistent quantum field theory based on such Deformed Special Relativity. On one hand, it would provide us modifications of scattering amplitudes which could be tested experimentally in particle accelerators or in cosmological context; on the other hand, it might provide us some insights in the structure of a full quantum gravity theory.

DSR is usually presented as a theory based on a curved momentum space: the momentum does not live in the standard flat 4d Minkowski space but in the de Sitter space (e.g. [4]). This curvature induces by duality the non-commutativity of the space-time coordinates. The goal is to write a quantum field theory on such a background. Our strategy is to re-examine Special Relativity (SR) and the structure of the algebra of observables of the relativistic particle in order to understand its extension to DSR and provide the deformed theory with solid foundations.

Starting with a standard massive relativistic particle, we first construct the set of strong Dirac observables. These are the phase space functions which commute everywhere in phase space with the Hamiltonian constraint. These “constants of motion” correspond to the measurable quantities. In this simple case, they are generated by the 4-momentum $p_\mu$ and the Lorentz generators $j_{\mu\nu}$. Together, they generate the Poincaré Lie algebra. However, if we are interested in probing the structure of space-time, we would like to identify suitable space-time coordinates which are Dirac observables. The coordinates $x_\mu$ obviously are not observables. To construct good coordinate functions, we use the concept of relational observables: we choose one of the degree of freedom of the system as the clock and we describe the evolution of the remaining degrees of freedom in term of that internal time. One usually chooses the time coordinate $x_0$ as the clock. This leads to the Newton-Wigner position operators [5]. They are Dirac observables, but they are not Lorentz covariant. Thus we chose to work with the Lorentz invariant clock $x_\mu p^\mu$. They lead to well-defined Lorentz covariant posi-
tion observables $X_\mu$. Together with the $p^\mu$'s, these new coordinates generate the whole algebra of observables.

Observing that using $X_\mu$ leads to the impossibility to define a time evolution as well as complications in the quantization procedure, we extend the analysis to the Lorentz covariant position weak observables $X_\mu$, that is the position observables that commute with the Hamiltonian constraint only on shell.

In section II, we recall how the choice of scalar product is of fundamental importance to define the quantum observables: dealing with the kinematical or the physical scalar product leads to different result for the self adjoint observables: dealing with the kinematical or the physical scalar product leads to different result for the self adjoint position operator $\hat{X}_\mu$.

In section III, we recall how the $\hat{X}_\mu$ non-commutativity reflects the impossibility of localizing the quantum relativistic particle with an accuracy better than the Compton length. This non-commutativity turns out to be very similar to the one encountered in DSR. This initial observation shows that Special Relativity already contains the seeds of its extension to DSR.

In section IV, we show that the algebra of the $X, p, j$ observables is naturally quantized as operators acting on the space of functions on the five-dimensional light cone: the massive 4d relativistic particle becomes a massless 5d system. We also make explicit the isomorphism between this new space of wave functions on the 5d light cone and the standard wave functions on the flat Minkowski space.

In section V, we further introduce a 5d action principle for the 4d massive relativistic particle. We study the map between the new 5d coordinates and the usual 4d coordinates $(x, p)$. It appears that the fifth moment actually generates the Hamiltonian flow of the relativistic particle: this fifth component of the momentum can be considered as the (rest) mass of the particle.

However, most of the Poisson (and Dirac) brackets induced in 4d become singular on the 5d light cone. Therefore, we introduce a regularization slightly moving away from the 5d light cone: the 5d momentum now lives on the de Sitter space. The resulting modified 5d action has been shown in to generate the DSR theories as different gauge fixing choices. From this point of view, we here show that DSR appears as a natural regularization of SR at the level of the algebra of observables.

In section VI, we exploit the 5d reformulation of the relativistic particle to write a 5d representation of the Feynman propagator (for a massive scalar field). Once again, the expression becomes singular on the 5d light cone. Nevertheless, we show that the Feynman propagator can be written exactly as a integral on the de Sitter space of moments. This analysis leads to interpreting the fifth space-time coordinate as the proper time of the particle.

Going further in the analysis of the Quantum Field Theory amplitudes, we show how the Feynman loop dia-

gram evaluations can be written as expectation values of some (time ordered) Dirac observables of the relativistic particle. We hope to be able to generalize this to DSR. This would be a definite first step towards deriving the Feynman amplitudes in DSR and constructing a consistent S-matrix describing the scattering of particles.

In the last section, we introduce the necessary framework to deal with particles with spin. We perform the canonical analysis and write the corresponding Dirac observables. We insist on the fact that the spin already induces a (Moyal-like) non-commutativity of the space-time coordinates at the classical level. This non-commutativity is distinct from the non-commutativity of the position observables (which is of a $\kappa$-deformed Poincaré type) and leads to further difficulties in the quantization of the algebra of Dirac observables.

**I. DIRAC OBSERVABLES FOR THE RELATIVISTIC PARTICLE**

**A. Position observables**

We start with the phase space of the relativistic particle:

$$\{x_\mu, p_\nu\} = \eta_{\mu \nu},$$ (1)

where we choose the flat metric $\eta_{\mu \nu} = (+ - - -)$. The Hamiltonian constraint for a massive particle is $H = (p^2 - m^2) = 0$ and the action is:

$$S = \int p^\mu dx_\mu - \lambda H,$$ (2)

where $\lambda$ is a Lagrange multiplier. We are interested into the Dirac observables, which are the phase space functions which Poisson-commute with $H$. They are generated by the momenta $p_\mu$, which generate the translations, and the generators of the Lorentz transformation:

$$J_{\mu \nu} = x_\mu p_\nu - x_\nu p_\mu.$$ (3)

Nevertheless, we would like to have some Dirac observables giving the position of the particle. For this purpose, we use relational observables $\tilde{b}$. These are constructed from two arbitrary phase space functions $a, b$ and defined as a function of an arbitrary real parameter $T$:

$$A_b(T) = \int_R d\tau a(\tau) \tilde{b}(\tau) \delta(b(\tau) - T).$$ (4)

We have introduced the notation $f(\tau)$ for the Hamiltonian flow of the phase space function $f$:

$$f(\tau) \equiv e^{-\frac{i}{\hbar} \tau \{H,\}} f.$$
It is straightforward to check that \(A_b(T)\) is a Dirac observable, whatever the value of \(T\). It represents the value of \(a\) when the clock \(b\) indicates \(T\). It is then natural to choose a clock \(b = x \cdot v\) with \(v\) an arbitrary fixed (time-like) vector and the function \(a = x_\mu\) indicating the particle’s position. This way, we define the following Dirac observables:

\[
X^{(v)}_\mu(T) = x_\mu + \frac{p_\mu}{p^2}(T - x \cdot v) = \frac{j_{\mu\nu}v^\nu + Tp_\mu}{p \cdot v}, \tag{5}
\]

which defines the values of the four coordinates at the time \(x^\alpha v_\alpha = T\). When \(v = (1, 0, 0, 0)\), the time is simply the time coordinate \(x_0\). These position observables commute with each other:

\[
\{X^{(v)}_\mu(T), X^{(v)}_\nu(T)\} = 0.
\]

However, the \(X^{(v)}_\mu(T)\) are not Lorentz-covariant, since they are constructed using a fixed vector \(v_\mu\). The simplest solution to address this problem is to take the special \(v\) vector and the function \(a = x_\mu\) which defines the values of the four coordinates at the time \(T\) servable, whatever the value of \(T\).

We always have \(a(0)\) are simply the transversal coordinates of the vector \(x\) with respect to the particle trajectory. More precisely \(D = x \cdot p = 0\) corresponds to the 'perihelion' \(P\) of the particle’s trajectory, i.e. the event when the particle is the closest to the origin. Then \(T\) counts the proper time along the particle’s trajectory from \(P\).

Let us work at some fixed time \(T_0\). The coordinates \(x_\mu\) can be reconstructed solely from the variables \(X^{(v)}(T_0), p_\mu\) since they are Dirac observables and \(x_\mu\) is not. To invert the relation between the \(x_\mu\)’s and the \(X(T_0)\)'s, we need the dilatation \(D\), which is not a Dirac observable\(^1\)

Then specifying the four coordinates \(x_\mu\) is equivalent to specifying the 5-vector \((D, X_\mu(T_0))\), which is the parallel and transversal projections of \(x\) with respect to the vector \(p\) (up to the shift \(T_0p_\mu\)).

Let us look at the algebra generated by the \(X^{(v)}_\mu(T)\). The key remark is that the \(j_{\mu\nu}\)'s and \(k_\mu\)'s form a \(so(1, 4)\) algebra:

\[
\{k_\mu, k_\nu\} = -(p^2)j_{\mu\nu}. \tag{8}
\]

This implies that the \(X^{(v)}_\mu\)'s do not commute with each other:

\[
\{X^{(v)}_\mu(T), X^{(v)}_\nu(T)\} = -\frac{j_{\mu\nu}}{p^2}. \tag{9}
\]

We also have a deformation of the canonical Poisson bracket:

\[
\{X^{(v)}_\mu(T), p_\nu\} = \eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}. \tag{10}
\]

This is the projector onto transversal modes (orthogonal to the direction of the motion \(p_\mu\)). Finally, we compute the action of the dilatation:

\[
\{D, X^{(v)}_\mu\} = -X^{(v)}_\mu, \quad \{D, p_\nu\} = +p_\nu, \quad \{D, j_{\mu\nu}\} = 0.
\]

The essential point for our discussion is that the \(j_{\mu\nu}\) and the rescaled\(^2\) positions \(\sqrt{p^2}X^{(v)}_\mu\) form a \(so(4, 1)\) algebra under the Poisson bracket. This is reminiscent of the Snyder algebra for a Lorentz covariant non-commutative geometry \([9]\). Actually this link can be made precise as we will see in the next section \([13]\). The interesting point is that the Snyder algebra is related (through a change of basis) to the \(\kappa\)-deformed Poincaré algebra encountered in theories of Deformed Special Relativity \([6]\). In some sense, we show here that standard Special Relativity itself already contains the seeds of DSR.

We now compute the values of the two Casimir operators of the \(so(4, 1)\) algebra. For the quadratic Casimir, we find:\(^3\)

\[
C_2 = \frac{1}{2}j_{\mu\nu}j^{\mu\nu} - p^2X^{(v)}_\mu(T)X^{(v)}_\mu(T) = -T^2. \tag{11}
\]

The time becomes a Casimir of our algebra of Dirac observables \((j, X(T))\). Then, introducing the Pauli-Lubanski vector \(\omega_\mu \equiv \epsilon_{\mu\alpha\beta\gamma}X^{(v)}_{\alpha\beta}^{\gamma}\), the quartic Casimir turns out to be trivial:

\[
C_4 \equiv \omega_\mu\omega^\mu = 0. \tag{12}
\]

so that \(D/2, p^2/2\) and \(x^2/2\) form a \(sl(2, \mathbb{R})\) algebra. This is the starting point of 2-time physics \([8]\).

\(^1\) Indeed \(\{D, p^2\}\) does not vanish. More precisely, one can check: \(\{D, p^2\} = 2p^2\), \(\{D, x^2\} = -2x^2\), \(\{p^2, x^2\} = -4D\).

\(^2\) Since the \(X^{(v)}_\mu\) commute with the Hamiltonian, we can rescale them by any function of \(p^2\) without complicating their Poisson brackets. Moreover the coordinate choice \(\sqrt{p^2}X^{(v)}_\mu\) is invariant under dilatations generated by \(D\): these are the natural coordinates to consider when using \(D\) as time \([10]\).

\(^3\) We compute the Lorentz invariant:

\[
X^{(v)}_\mu X^{(v)}_\mu = x_\mu x^\mu + \frac{1}{p^2}(T^2 - D^2).
\]
This means that we are dealing with a simple representation of the algebra \( so(4, 1) \), which can be realized as functions on the 5d light cone \( C_0 \), on the one-sheet hyperboloid \( SO(4, 1)/SO(3, 1) \) (which actually is the de Sitter space) or on the two-sheets hyperboloid \( SO(4, 1)/SO(4) \).

At the level of the action, we compute the kinetic term \( p^\mu dx_\mu \) in term of the coordinates \( X_\mu(T) \) and find:

\[
p^\mu dx_\mu = p^\mu dX_\mu + \frac{(T - D)}{2p^2} dH + dD. \tag{13}
\]

Therefore the action can be written up to total derivatives as:

\[
S' = \int p^\mu dX_\mu + H d \left( \frac{D - T}{2p^2} \right) - \lambda H - \mu (p^\mu X_\mu - T). \tag{14}
\]

This leads to two considerations. First, \( D/p^2 \) can be considered as a new fifth coordinate, conjugate to the Hamiltonian constraint \( H = p^2 - m^2 \). It thus seems possible to provide the relativistic particle with a five-dimensional action principle. Second the five dimensions are reduced to the usual four dimensions by an extra constraint. This second constraint do not commute with the Hamiltonian constraint \( H = 0 \) and can therefore be considered as a gauge fixing condition lifting the first class constraint \( H \) to a second class constraint system. This is analyzed in details in section \[IV]\.

### B. Strong observables versus weak observables

Up to now, we have considered strong observables, that commute exactly with the Hamiltonian constraint. We would like to propose to use weak observables instead, that commute with the Hamiltonian constraint only on the mass-shell.

Indeed, first, from the point of view of the quantization, the factors \( 1/p^2 \) and \( \sqrt{p^2} \) occurring when considering the Dirac observables \( X_\mu(T) \) would not be easy to deal with when quantizing the algebra of observables. Then, from the point of the dynamics, the “time” evolution in \( T \) of the coordinates \( X_\mu(T) \) can not be generated from a Hamiltonian. More precisely, there does not exist any phase function \( H_{eff} \) such that:

\[
\forall T, \{H_{eff}, X_\mu(T)\} = \frac{dX_\mu(T)}{dT} = \frac{p_\mu}{p^2}.
\]

This is due to the fact that the coordinates \( X_\mu(T) \) are basically the projection of the space-time coordinates \( x_\mu \) orthogonally to the momentum \( p_\mu \). We can relax this condition by introducing the following weak observables:

\[
X_\mu(T) = \left( \eta_{\mu\nu} - \frac{p_\mu p_\nu}{m^2} \right) x^\nu + T \frac{p_\mu}{m^2} = x_\mu + \frac{p_\mu}{m^2} (T - x^\nu p_\nu).
\]

In contrast to the previous situation, this relation is invertible off shell\(^4\) and we can express \( x_\mu \) in terms of \( X_\mu(T = 0) \):

\[
x_\mu = \left( \eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2 - m^2} \right) X_\mu. \tag{15}
\]

These do not strongly commute with the Hamiltonian constraint anymore:

\[
\{H, X_\mu(T)\} = -2p_\mu \left( 1 - \frac{p^2}{m^2} \right) = \frac{2p_\mu}{m^2} H, \tag{16}
\]

but the Poisson bracket still vanishes on-shell. The \( X_\mu(T) \) still do not commute with each other:

\[
\{X_\mu(T), X_\nu(T)\} = -\frac{j_{\mu\nu}}{m^2}. \tag{17}
\]

The \( (p, X, j) \) algebra is very similar to the previous \( (p, X, j) \) algebra, but the \( 1/p^2 \) factors are replaced by constant \( 1/m^2 \) factors. The observables \( j_{\mu\nu} \) and \( mX_\mu \) form a \( so(4, 1) \) algebra. Moreover, the \( (p, X, j) \) algebra is now exactly the Snyder algebra related to a \( \kappa \)-deformed Poincaré symmetry with deformation parameter \( \kappa \approx m \). This provides an exact link between the algebra of Dirac observables of the relativistic particle and the phase space structure of the deformed relativistic particle.

A further advantage of the \( X \) observables on the \( \mathcal{X} \) coordinates is that the \( p^2 \) factors are replaced by \( m^2 \) constants: we do not have to deal with any \( \sqrt{p^2} \) factor when quantizing (compare \( \sqrt{p^2} X_\mu \) to the simpler \( mX_\mu \)). Of course, other subtleties will arise. First, although the quartic Casimir will still vanish, the quadratic Casimir \( C_2 \) will only be equal to \( -T^2 \) on the mass-shell. Then, as we will discuss in the paragraphs below, although the operators \( \widetilde{X}_\mu \) will of course be Hermitian for the physical scalar product, they will not be Hermitian with respect to the kinematical scalar product.

Moreover, it is possible to generate the time evolution of the observables \( X_\mu(T) \) with an effective Hamiltonian which turns out to be exactly the logarithm of the original Hamiltonian constraint:

\[
\left\{ \frac{1}{2} \ln |H|, X_\mu(T) \right\} = \frac{p_\mu}{m^2} = \frac{dX_\mu(T)}{dT}. \tag{18}
\]

\(^4\) This is due to the fact that \( p.X \) is not a constant anymore but is easily related to \( p.x \):

\[
p.x = p_\mu x^\mu \left( 1 - \frac{p^2}{m^2} \right).
\]
Finally, we can re-write the action in term of these new coordinates (up to a total derivative):

\[ p^\mu dx_\mu = p^\mu dX_\mu - \frac{1}{2} (D - T) dH, \]

which shows explicitly the canonical relation between the Hamiltonian constraint and the dilatation generator.

### C. Gauge fixing and the Dirac bracket

The choice of a time variable can be understood as an explicit gauge fixing that breaks the symmetry of the action under time reparametrization. After gauge fixing, the symplectic form on the reduced phase space is given by the Dirac bracket. Given the constraint \( H \) and a gauge fixing condition \( C \) such that \( \{ C, H \} \neq 0 \), the Dirac bracket is defined as

\[
\{\phi, \psi\}_D = \{\phi, \psi\} - \{\phi, C\} \left(\frac{1}{\{H, C\}}\right) \{H, \psi\} - \{\phi, H\} \left(\frac{-1}{\{H, C\}}\right) \{C, \psi\}. \tag{20}
\]

For the standard time choice \( C = x.v \), where \( v \) is an arbitrary time-like vector, we have \( \{H, x.v\} = -2p.v \) and we obtain:

\[
\{x_\mu, p.v\}_D = \left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) v^\nu,
\]

\[
\{x_\mu, x_\nu\}_D = 0. \tag{21}
\]

\( p.v \) acts as the Hamiltonian on the gauge fixed system: the time \( x.v \) commute with \( p.v \) and the space coordinates (orthogonal to \( v \)) evolve with the usual speed defined by the momentum \( p_\mu \). The important relation linking the relational Dirac observables written in the previous section and the gauge fixing procedure is the equality between the Dirac bracket and the Poisson bracket of the observables:

\[
\{x_\mu, p_\nu\}_D = \{A^{(\nu)}_\mu(T), p_\nu\},
\]

\[
\{x_\mu, x_\nu\}_D = \{X^{(\nu)}_\mu(T), X^{(\nu)}_\mu(T)\}, \tag{22}
\]

for any value of the parameter \( T \).

Choosing the dilatation as gauge fixing condition, \( C = D \), we compute

\[
\{x_\mu, p_\nu\}_D = \eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2},
\]

\[
\{x_\mu, x_\nu\}_D = -\frac{j_{\mu\nu}}{p^2}. \tag{23}
\]

And we similarly obtain the following equalities:

\[
\{x_\mu, p_\nu\}_D = \{X^{(\nu)}_\mu(T), p_\nu\},
\]

\[
\{x_\mu, x_\nu\}_D = \{X^{(\nu)}_\mu(T), X^{(\nu)}_\mu(T)\}. \tag{24}
\]

### II. QUANTIZATION OF THE WEAK OBSERVABLES: HERMITICITY

We are interested in the Hermicity properties of the quantum operator \( \hat{X}_\mu \). As mentioned earlier, since \( X^{(\mu)}_\nu(T) \) are only weak observables, their hermiticity properties will differ depending if we consider the kinematical inner product or the physical inner product.

In the following, we will focus on \( X^{(\mu)}_\nu(T = 0) \) which we will simply denote \( X^{(\mu)}_\nu \). The other term \(+ T p_\mu/m^2 \) can be easily taken into account. We work in the \( p \)-polarisation: wave functions are functions of the momentum \( p_\mu \), \( \hat{p} \) acts by multiplication while \( \hat{x}_\mu = i\partial/\partial p_\mu \) acts as a derivation operator. The kinematical inner product is simply defined as \( \langle \psi|\phi \rangle = \int d^4p \overline{\psi}(p)\phi(p) \), while the physical inner product takes the Hamiltonian constraint into account:

\[
\langle \psi|\phi \rangle_{ph} = \int d^4p \overline{\psi}(p)\phi(p). \tag{25}
\]

Then, if we choose the trivial ordering for \( \hat{X}^{(\mu)}_\nu \):

\[
\hat{X}^{(\mu)}_\nu = i \frac{\partial}{\partial p^\mu} - i \frac{p_\mu p_\nu}{m^2} \frac{\partial}{\partial p_\nu},
\]

a straightforward calculation gives:

\[
\langle \hat{X}^{(\mu)}_\nu \rangle_{kin} = \hat{X}^{(\mu)}_\nu - i(d+1)\frac{p_\mu}{m^2},
\]

\[
\langle \hat{X}^{(\mu)}_\nu \rangle_{ph} = \hat{X}^{(\mu)}_\nu - i(d-1)\frac{p_\mu}{m^2}, \tag{26}
\]

where \( d = 4 \) is the space-time dimension. Therefore, at the quantum level, for the physical inner product, the correct self-adjoint operator representing the observable \( X^{(\mu)}_\nu(T) \) requires a complex shift:

\[
\hat{X}^{(\mu)}_\nu(T) \equiv i \partial_\mu - i \frac{p_\mu p_\nu}{m^2} \partial_\nu + \left( T - i \frac{(d-1)}{2} \right) \frac{p_\mu}{m^2}, \tag{27}
\]

where \( \partial_\mu \) is the partial derivative with respect to the momentum variable \( p_\mu \).

In the following section, we will quantize the relativistic particle using the \( so(4,1) \) structure of the algebra of observables and we will check that we recover the exact same shift from the 5d perspective.

### III. ON THE LOCALIZATION OF THE QUANTUM PARTICLE

Since we know the quantization of the position observables \( \hat{X}_\mu \), we can study the issue of the localization of
the relativistic particle at the quantum level in terms of Dirac observables.

$\hat{X}_\mu(T)$ is quantized in term of the $\mathfrak{so}(4,1)$ generator $J_{\mu\nu}/m$. In the space-like sector $\mu = i = 1, 2, 3$, the spectrum of $\hat{X}_i$ is discrete, $\hbar/m \mathbb{Z}$, and space distances are quantized in units of Compton length $l_C = \hbar/m$. On the other hand, the spectrum of the time coordinate $\hat{X}_0 \propto J_{0i}$ remains continuous.

Furthermore, one can compute the exact spectrum of the distance operator $\hat{X}_\mu \hat{X}^\mu$ [10] and one shows that the negative eigenvalues (corresponding to the space-like sector) are discrete while its positive eigenvalues are continuous (corresponding to the time-like sector). As explained in [10], the discreteness of the distances does not contradict the Lorentz invariance of the theory.

Let us point out that $X_\mu X^\mu$ is not the usual metric $x_\mu x^\mu$. Nevertheless, it is a Lorentz invariant Dirac observable, which coincides with $x_\mu x^\mu$ when $D$ is fixed i.e when working in fixed eigenspace of the dilatation $\hat{D}$. Indeed we recall that:

$$X_\mu(T)X^\mu(T) = x_\mu x^\mu + \frac{1}{p^2}(T^2 - D^2).$$

This discrete lattice-like structure of the coordinates $X_\mu$ naturally leads to some intrinsic uncertainties in the measurement of these position Dirac observables. Indeed, as shown in section [10], the commutator of the coordinates reads:

$$\{X_\mu, X_\nu\} = \frac{1}{\hbar}(l_C)^2 j_{\mu\nu},$$

(28)

where $l_C = \hbar/m$ is the Compton length of the particle (at rest). This Poisson bracket will get quantized as:

$$[\hat{X}_\mu, \hat{X}_\nu] = i(l_C)^2 j_{\mu\nu}.$$

From this, we expect a position uncertainty $\delta X \sim l_C$, which is characteristic of the quantized relativistic particle. More precisely, we have the uncertainty relation:

$$(\delta X_\mu)(\delta X_\nu) \geq \frac{l_C^2}{2} |\langle j_{\mu\nu}\rangle|.$$ 

Let us look at the space sector and consider the uncertainty in the spatial position, $(\delta \ell)^2 \equiv (\delta X_1)^2 + (\delta X_2)^2 + (\delta X_3)^2$. Following arguments from [11], one shows that $\delta \ell$ is always larger than the Compton length $l_C$ as long as the state is not invariant under SO(4) i.e under the $J_{ij}$’s and the $X_i$’s. However, if the state is invariant under SO(4), it can not be invariant under $X_0$ and the uncertainty $(\delta X_0)$ will be larger than $l_C$. A more explicit analysis would require more details on the action of the $\mathfrak{so}(4,1)$ generators. However, from the study of $\mathfrak{so}(3,1)$ in [11], we expect that this would naturally lead to a position uncertainty always larger than $l_C$.

IV. QUANTIZATION ON THE 5D LIGHT-CONE

We now proceed to the quantization of the algebra of observables for a fixed mass $m > 0$. We will see that the observables $j, X(T), p$ can naturally be represented on the five-dimensional light cone. To this purpose, let us introduce the 5d coordinates $y_A$ and their conjugate momentum variables $\pi_A$, with the symplectic structure $\{y_A, \pi_B\} = \eta_{AB}$ and the metric $\eta_{AB} = (+ - - - -)$. We define the 5d light cone $C_0$ in momentum space as the algebraic manifold:

$$C_0 = \{(\pi_A) | \pi_0^2 - \pi_i \pi_i - \pi_4^2 = 0\}. \quad (29)$$

Choosing a particular value of the time $T_0$, we identify $j$ and $X(T_0)$ to the 5d Lorentz generators,

$$J_{AB} = y_A \pi_B - y_B \pi_A,$$

and we will define the 4-momentum $p_\mu$ as a simple function of the 5d momenta $\pi_A$,

$$p_\mu = m \frac{\pi_\mu}{\pi_4}, \quad (30)$$

so that the mass-shell condition $p^2 = m^2$ becomes the light-cone condition,

$$p_\mu p^\mu - m^2 = \frac{m^2}{\pi_4^2} \pi_A \pi^A. \quad (31)$$

It is straightforward to check that this choice has the right Poisson bracket with $X_\mu(T_0)$ and $j_{\mu\nu}$. This maps 4d massive relativistic particles to 5d massless particles (at the level of Dirac observables). This provides us with a natural 5-dimensional action principle for the 4d relativistic particle. We will analyze this in details in the next section.

At the quantum level, we will work in the $y$-polarisation. We represent the 5-momentum $\pi$ as derivation operators,

$$\pi_A = -i \eta_{AB} \frac{\partial}{\partial y_B},$$

and the observables $X_\mu(T_0)$ becomes the differential operators $\hat{J}_{AB}/m$.

Next we would like to identify the reference time $T_0$ to the quadratic Casimir $J_{AB} J^{AB}$. At the classical level, we have:

$$\frac{1}{2} J_{AB} J^{AB} = (y_A \pi^A)(\pi_B \pi^B) - (y_A \pi^A)^2,$$

(32)

where we introduce the 5d dilatation $D = y_A \pi^A$, $(y_A \pi^A)$ and $(\pi_A \pi^A)$.
Then the time the corresponding eigenvalue of the dilatation operator. We can compute the Casimir operator $\hat{J}^2$ for the algebra \( so(4,1) \) (we are working at \( d = 4 \)):

\[
-\frac{1}{2} \hat{J}_{AB} \hat{J}^{AB} = y^2 \Delta - \hat{D}(\hat{D} + d - 1)
\]

\[
= \Delta y^2 - \hat{D}(\hat{D} + d + 3) - 2(d + 1)
\]

\[
= \frac{1}{2}(y^2 \Delta + \Delta y^2) - \hat{D}(\hat{D} + d + 1) - (d + 1).
\]

The extra term arises from the ordering ambiguity since \( \Delta \) and \( y^2 \) do not commute. This is analogous to the extra factor met for example in the vacuum energy of the harmonic oscillator. Note that \( \hat{D} \) (and its square) is not (anti-)Hermitian, and we actually have:

\[
\hat{D}^\dagger = -\hat{D} - (d + 1).
\]

The shifts in the operator \( \hat{J}_{AB} \hat{J}^{AB} \) ensure that the Casimir operator remains self-adjoint.

We would like to compute the eigenvalues of \( \hat{J}^2 \) on the light cone \( C_0 \) i.e. on the states \( \varphi \) satisfying \( \Delta \varphi = 0 \). We introduce the states \( \varphi_{P,\lambda}(y) = (y P_A)^{\lambda} \). They satisfy \( \Delta \varphi_{P,\lambda} = 0 \) as soon as \( P \) is light-like, \( P_A P_A = 0 \). Moreover they diagonalise the dilatation operator, \( \hat{D} \varphi_{P,\lambda} = \lambda \varphi_{P,\lambda} \). Hence, we have:

\[
-\frac{1}{2} \hat{J}_{AB} \hat{J}^{AB} \varphi_{P,\lambda} = -\lambda(\lambda + d - 1) \varphi_{P,\lambda}.
\]

We therefore identify the time to the eigenvalue, \( T_0^2 = -\lambda(\lambda + d - 1) \). This is possible if and only if \( \lambda \) has a fixed real part:

\[
\lambda = -\left(\frac{d - 1}{2}\right) + i\beta, \quad T_0^2 = \beta^2 + \left(\frac{d - 1}{2}\right)^2.
\]

The time remains continuous at the quantum level, even though we have a minimal time unit \( T_{\text{min}} = 3/2 \) (in Compton unit \( h/mc^2 \)).

To summarize, the space of harmonic functions (having a vanishing laplacian) form a reducible representation of \( so(4,1) \) which decomposes into irreducible representations labeled by the parameter \( \lambda \). These irreducible components are formed by homogeneous functions and \( \lambda \) is the corresponding eigenvalue of the dilatation operator. Then the time \( T_0 \) actually fixes which representation we use.

Finally, we can check that since \( p_\mu, j_\mu, X_\mu \) all commute with the 5d dilatation \( \hat{D} \), it is natural that we can represent them in an irreducible representation corresponding to a single eigenvalue \( T \) of \( \hat{D} \).

Now, we are interested in the precise mapping between the usual 4d wave functions and the states of our 5d quantization. For this purpose, it is easier to use the momentum polarization and work with the Fourier transforms. Let us introduce the following family of morphisms between the usual space of wave functions \( \varphi(p) \) and the space of functions on the light cone:

\[
\Theta_\alpha : \Phi(p) \to \phi(\pi_A) = (\pi_4)^\alpha \Phi \left( m \frac{p_\mu}{\pi_4} \right).
\]

We use the Fourier transform on \( C_0 \):

\[
\varphi(y) = \int d^4 \pi \delta(\pi_A \pi^A) e^{iyA\pi^A} \phi(\pi).
\]

Using this Fourier transform, it is straightforward to check that \( \Theta_\alpha \) gives functions which are eigenvectors of \( \hat{D} \) with eigenvalue:

\[
\lambda = -(d - 1) - \alpha.
\]

Using the isomorphism \( \Theta_\alpha \), we can compute the action of the operator \( \tilde{X}_\mu = J_{\mu\lambda}/m \) on the standard wave functions \( \Phi(p) \). Working in the \( \pi \)-polarisation, \( \hat{y} \) acts as \( +i\partial/\partial\pi \) and \( X_\mu \) becomes:

\[
\tilde{X}_\mu = \frac{i}{m} \left( \pi_\mu \frac{\partial}{\partial \pi_4} + \eta_{\mu
u} \pi_4 \frac{\partial}{\partial \pi_\nu} \right).
\]

Then defining \( \tilde{X}_\mu^{(\alpha)} = \Theta_\alpha^{-1} \tilde{X}_\mu \Theta_\alpha \), we obtain:

\[
\tilde{X}_\mu^{(\alpha)} \phi(p) = +i \eta_{\mu
u} \frac{\partial \phi}{\partial p_\nu} - i \frac{p_\mu p_\nu}{m^2} \frac{\partial \phi}{\partial p_\nu} + i\alpha p_\mu \phi.
\]

Computing the action of this operator on standard plane waves \( \varphi_x(p) = e^{-i p \cdot x} \), we get:

\[
\tilde{X}_\mu^{(\alpha)} \varphi_x = \left( x_\mu + \frac{p_\mu}{m^2} (+i\alpha - p_\nu x^\nu) \right) \varphi_x.
\]

Inserting \( +i\alpha = -i(d - 1) - i\lambda = +\beta - i(d - 1)/2 \) in the previous formulæ, we recognize the exact same equation as in \( (27) \) with the same imaginary shift in time \(-i(d - 1)/2\). This imaginary shift is purely a quantum effect.

---

6 If we had not include the \( \delta(\pi^2) \) in the measure of the Fourier transform, we would have found \( \lambda = -(d + 1) - \alpha \). This shift \( (d + 1) \to (d - 1) \) is exactly the same as above in the analysis of the Hermiticity of \( X_\mu \) with respect to the kinematical and physical inner product.
At the end of the day, the shift due to the ordering ambiguities in the 5d quantization fits exactly the shift required for the hermiticity of the observables \( X_\mu(T) \) with respect to the physical inner product: the 5d quantization on the light cone is equivalent to the standard quantization of the 4d relativistic particle.

Finally, we point out that when \( \lambda \) is set to zero, we recover the representation of Snyder’s non-commutative coordinates as differential operators in the momentum \( p \). Notice nevertheless that \( \lambda = 0 \) is actually excluded in our analysis due to the quantum shift in \(-i(d-1)/2\).

V. A 5D ACTION PRINCIPLE AND DSR

A. From 5d to 4d

Since we represent the algebra of observables of the relativistic particle on the 5d light cone, it seems natural to check their Poisson brackets:

We have the 5d mass-shell condition

\[
\mathcal{H}_{5d} = \pi_A \pi^A = 0
\]

The second term involves \( p^\mu X_\mu \), which is our (Lorentz-invariant) clock time \( T \) measuring the proper time along the particle’s trajectory in the usual 4d space-time. This identifies \( \pi_4 \) as the conjugate momentum to the proper

time: it generates the Hamiltonian evolution. This should be compared to the (effective) Hamiltonian \( \ln H \) describing the evolution the (weak) observables \( X \) as written in equation (13). Finally, we can interpret \( y_4/\pi_4 \) as the conjugate coordinate to the 5d mass \( \pi_A \).

Up to now, we have dealt with the position observables \( X_\mu \). It would be interesting to recover the standard commutative 4d space-time coordinates. We first notice that:

\[
p^\mu X_\mu = \pi^A y_A - \frac{y_4}{\pi_4} \pi^A \pi_A.
\]

On the 5d mass-shell, \( \pi^A \pi_A = 0 \), fixing \( p^\mu X_\mu = T \) is thus equivalent to fixing \( \mathcal{D} = \pi^A y_A = T \). Assuming this extra condition, \( \mathcal{D} = T \), the coordinate \( X_\mu \) is actually exactly the Dirac observables for the relativistic particle that we introduced earlier (3). More precisely, we introduce 4d coordinates as:

\[
x_\mu = y_\mu \frac{\pi_4}{m}.
\]  (42)

It is straightforward to check that they satisfy the standard canonical Poisson brackets:

\[
\{x_\mu, p_\nu\} = 0, \quad \{x_\mu, x_\nu\} = \delta_{\mu\nu}.
\]

Moreover we now have:

\[
\mathcal{H}_{5d} = 0 \quad \Rightarrow X_\mu = x_\mu + \frac{p_\mu}{m^2} (T - x.p) = X_\mu(T).
\]

Finally, writing the 5d kinetics in terms of \( x_\mu \), we get (up to a total derivative):

\[
\pi^A dy_A = p^\mu dx_\mu + \ln \pi_4 \, d \left( \pi^A y_A \right).
\]  (43)

In light of these remarks, we propose to reduce the 5d to a 4d one through a gauge fixing of the 5d mass-shell condition \( \mathcal{H}_{5d} \). We choose as gauge fixing condition \( \mathcal{D} = T \). It is straightforward to compute the corresponding Dirac bracket:

\[
\{y_A, \pi_B\}_D = \eta_{AB} - \frac{\pi_A \pi_B}{\pi_C \pi_C},
\]

\[
\{y_A, y_B\}_D = -\frac{J_{AB}}{\pi_C \pi_C}.
\]  (44)

Since \( p_\mu \) and \( X_\mu \) commute with both \( \mathcal{H}_{5d} \) and the 5d dilatation \( \mathcal{D} \), their Dirac bracket with any phase space function is equal to the Poisson bracket. Then we check that \( \ln \pi_4 \) generates the Hamiltonian flow on the 4d variables:

\[
\{\ln \pi_4, X_\mu\}_D = \{\ln \pi_4, p_\mu\}_D = 0,
\]

\[
\{\ln \pi_4, x_\mu\}_D = p_\mu \frac{\pi_4}{m^2 \pi^A \pi_A}.
\]  (45)
The usual 4d Hamiltonian constraint $H_{4d} = p^2 - m^2$ is easily expressed in terms of the 5d variables:

$$H_{4d} = m^2 \pi^A \pi_A \pi_4. $$

Its flow is thus the same as the one generated by $\pi_4$ up to a factor $\pi^A \pi_A$. We check that it of course removes all the $\pi_4$ and $\pi^A \pi_A$ factors and simply we recover the usual relation $\{H_{4d}, x_\mu\}_D = -2p_\mu$.

Therefore, if we want to recover the standard relativistic dynamics after gauge fixing, we have to add an extra-constraint to the 5d action and we write:

$$S_{5d} = \int \pi^A dy_A - \lambda \pi^A \pi_A - \mu(\pi_4 - M), \quad (46)$$

where $M$ is an arbitrary parameter. The path integral for this action is obviously equivalent to a relativistic particle with action $\int \pi^A dy_A - \lambda(\pi^A \pi_A - M^2)$. This shows that it is the fifth moment $\pi_4$ which generates the (rest) mass of the 4d particle.

### B. DSR as a regularization

The main problem with the gauge fixing procedure is the singularity of the Dirac bracket at $\pi^A \pi_A = 0$ on the 5d mass-shell. A natural regularization is to allow a (small) deviation from 0 and modify the 5d constraint to:

$$H_{5d} = \pi^A \pi_A + \epsilon\kappa^2 = 0, \quad (47)$$

where $\kappa \in \mathbb{R}_+^*$ is a mass scale and $\epsilon = \pm$ the sign of the deviation. The full 5d action now reads:

$$S_{5d} = \int \pi^A dy_A - \lambda(\pi^A \pi_A + \epsilon\kappa^2) - \mu(\pi_4 - M), \quad (48)$$

This is actually the 5d action which generates DSR. Indeed, it has been shown in [9] that all the various bases of Deformed Special Relativity can be derived as different gauge fixing of the 5d constraint $H_{5d}$. In that framework, $\epsilon$ is required to be positive and the momentum space $\pi^A \pi_A = -\kappa^2$ is the de Sitter space. $\kappa$ is usually set to the Planck mass and induces a discrete spectrum for certain distance operators [10]. The rest mass of the 4d particle is obtained from the $\pi_4$ constraint and is a function of $M$ (and $\kappa$), the exact function depending of the details of the gauge fixing.

Having $\kappa \neq 0$ regulates all the previous expressions. Moreover it allows to explicitly invert the definition of the 4-momenta $p_\mu = m\pi_\mu/\pi_4$ and express the fifth moment $\pi_4$ in term of $p^2$:

$$\pi_4^2 = \frac{\epsilon\kappa^2}{1 - \frac{\kappa^2}{m^2}}. \quad (49)$$

For $\epsilon = +$, we are constrained to work with a bounded momentum $p^2 \leq m^2$. The other choice $\epsilon = -$ leads to $p^2 \geq m^2$ and we discard it as unphysical. Since $\pi_4$ is a simple function of $p^2$, it is now obvious that it generates the 4d Hamiltonian flow for the relativistic particle. Moreover, the parameter $m$ loses its straightforward interpretation as the rest mass of the particle. More precisely, imposing the constraint $\pi_4 = M$, we obtain:

$$p^2 = m^2 \left(1 - \frac{\kappa^2}{M^2}\right).$$

We recover the standard dispersion relation $p^2 = m^2$ in the limit $\kappa \ll M$ when we remove the regulator $\kappa \to 0$.

This shows how classical mechanics in DSR can be considered as a regularization of standard Special Relativity from the five-dimensional point of view when quantizing the algebra of Dirac observables. This is consistent with the hope that DSR quantum field theory regularizes Feynman diagrams of standard QFT. This offers a shift of perspective on the interpretation of the non-commutative space-time coordinates of DSR. Indeed the non-commutativity of Lorentz-covariant position observables is already present in Special Relativity (consider the $X_\mu(T)$ coordinates). These coordinates are not the true space-time coordinates, $x_\mu$, but Dirac observables which are constants of motion. This is supported by the fact that the relativistic Dirac observables $X_\mu, p_\mu$ have the same Poisson brackets as the DSR phase space coordinates in the Snyder basis [9]. The only difference is that we represent the Poisson algebra of the 5d light cone while the DSR coordinates are realized as operators on the de Sitter space [1, 9].

### C. A 5d representation of the Feynman propagator

Since we have provided a 5d representation of the relativistic particle at both the level of the action and of the Dirac observables, it would be interesting to investigate whether this picture can be extended to quantum field theory. More precisely, we focus on the Feynman propagator, from which one can then build the whole perturbative expansion of the scattering amplitudes.

In the proper time representation, the Feynman propagator reads:

$$K_m(x_\mu) = \int_{\mathbb{R}^+} dT \int d^4p e^{ip^\mu x_\mu} e^{-3T(p^2 - m^2 + i\epsilon)}, \quad (50)$$

where $\epsilon > 0$ is a regulator. We would like to express this in terms of the 5d variables $(y_A, \pi_A)$. First, we write the mass-shell constraint in terms of the $\pi$'s:

$$p^2 - m^2 = m^2 \pi^A \pi_A \pi_4. \quad (51)$$
We perform a first change of variable $p_{\mu} = m\pi_{\mu}/\pi_4$. It is then natural to introduce the rescaled coordinates $y_{\mu} = m x_{\mu}/\pi_4$ to preserve the symplectic form. Finally, we do a change of variables from the proper time $T$ to the fifth coordinate $y_4 = T m^2 \pi A \pi^A/\pi_4^3$. We obtain in the end:

$$K_m(x) = m^2 \int_{\mathbb{R}^+} dy_4 \int d^5 \pi_{\mu} e^{i\pi_{\mu} y_A} \pi_4^A. \tag{51}$$

A first remark is that $\pi_4$ is still unspecified. It could possibly play the role of a renormalisation scale or some energy cut-off. Nevertheless, the measure $d^5 \pi_{\mu}/\pi_4$ suggests a lift to a 5d integral such as $d^5 \pi_4 \delta(\pi A \pi^A)$. However, this would conflict with the $1/\pi_4 \pi^A$ term in the integral. This is normal since imposing $\pi A \pi^A = 0$ amounts to enforcing the mass-shell constraint, but the Feynman propagator is an off-shell object.

We propose to resolve this issue by the same DSR regularization as used earlier. We introduce the constraint $\delta(\pi A \pi^A + \kappa^2)$. Then we obtain the following 5d representation of the Feynman propagator:

$$K_m(x) = \frac{m^2}{\kappa^2} \int_{\mathbb{R}^+} dy_4 \int d^5 \pi_4 \delta(\pi A \pi^A + \kappa^2) e^{i\pi_{\mu} y_{\mu}}. \tag{52}$$

We have thus written the Feynman propagator as an on-shell object from the 5d point of view.

The only subtle point is that imposing $\pi A \pi^A + \kappa^2 = 0$ truncates the momentum space to the $p^2 < m^2$ sector. To recover the other half of the momentum space, we should switch the sign of $\kappa^2$ and impose $\pi A \pi^A - \kappa^2 = 0$. Therefore the full Feynman propagator, with an integration over the whole $p$ space, is the difference of the two 5d integrals with $\kappa^2$-shell condition respectively $\delta(\pi A \pi^A + \kappa^2)$ and $\delta(\pi A \pi^A - \kappa^2)$.

This 5d representation of the Feynman propagator allows a clear interpretation of the fifth dimension: $\pi_4$ represents the mass of the particle (or more precisely the possibly off-shell $p^2$) while $y_4$ is the proper time (rescaled by some $m/\pi_4$ factor).

This 5d formulation of the Feynman propagator should allow a 5d DSR-like representation of all scattering amplitudes of quantum field theory. From the reverse point of view, it shows that amplitudes computed in QFT based on DSR could simply be equivalent to standard QFT. To evade such a no-go theorem about DSR, we see two alternatives:

- DSR relaxes the $\kappa$-shell condition, either by allowing $\kappa$ to vary or by allowing $\pi A \pi^A$ not to be fixed at $\pm \kappa^2$. In this case, DSR will truly be a 5d theory based on a physical 5d momentum $\pi_{\mu}$.

- The deformation of the scattering in DSR is not strictly contained in the Feynman propagator but due to a modification of the interaction vertices. In algebraic terms, we deform the co-product dictating the law of addition of the moments $\gamma$.

These two viewpoints do not exclude each other.

### VI. ORDERED OBSERVABLES AND FEYNMAN DIAGRAMS

Other useful relational observables are the ones recording whether the particle went through a given fixed space-time point $z_{\mu}$ along its trajectory. Let us introduce the phase space distribution:

$$O_z \equiv \int_{\mathbb{R}} d\tau \delta^{(4)} \left( x_{\mu} \left( \frac{\tau}{m} \right) - z_{\mu} \right), \tag{53}$$

where $x_{\mu}(\tau) = \exp(-\tau (H, \cdot)/2) x_{\mu} = x_{\mu} + \tau p_{\mu}$. The factor $m$ is here for dimensional purposes. It is straightforward to check that this is a Dirac observable. Carrying out the integration in $x_0$, this observable reads as:

$$O_{z} = \frac{m^2}{p_0^2} \delta^{(3)}(K_{i}(z)), \tag{54}$$

where we have defined the boost vector $K_{i}(z) \equiv p_{0}(x_{i} - z_{i}) - p_{i}(x_{0} - z_{0})$. Note that $K_{i}(z) = j_{i0} - (p_{0}z_{i} - p_{i}z_{0})$.

We can easily compute the Fourier transformed observable:

$$O_{q} \equiv \int d^4 z e^{i q \cdot z} O_{z} = e^{i q \cdot z} \delta \left( \frac{q \cdot p}{m^2} \right). \tag{55}$$

We quantize this operator by splitting the exp($iq$) into two and we define:

$$\hat{O}_{q} |p\rangle = \delta \left( \frac{(p + \frac{q}{2}) \cdot q}{m^2} \right) |p + q\rangle. \tag{56}$$

Due to the chosen ordering, this operator is still a Dirac observable at the quantum level, i.e it commutes with the quantum operator $p^2$ and leaves invariant the Hilbert of physical state (annihilated by $p^2 - m^2$). Indeed, the $\delta$-function,

$$m^2 \delta \left( \left( p + \frac{q}{2} \right) \cdot q \right) = 2m^2 \delta \left( (p + q)^2 - p^2 \right),$$

imposes that $(p + q)$ is on the same mass-shell than $p$. Finally, reversing the Fourier transform, we define the quantum operator:

$$\hat{O}_{z} |p\rangle = \int_{\mathbb{R}} d\tau \int dq e^{-i q \cdot \left( z - \frac{p + \frac{q}{2}}{m^2} \right)} |p + q\rangle. \tag{57}$$

The $\frac{q}{2}$ shift in the exponential is due to the quantum ordering.
An interesting related phase space function is defined by restricting the range of \( \tau \)-integration to \( \mathbb{R}_+ \):

\[
F_z = \int_{\mathbb{R}_+} d\tau \delta^{(4)} \left( x_\mu \left( \frac{\tau}{m} \right) - z_\mu \right).
\]

(58)

Taking the Fourier transform, we define:

\[
\hat{F}_q = \int d^4 z e^{iq \cdot z} F_z = e^{iq \cdot x} \frac{im}{q \cdot p + i\epsilon}.
\]

(59)

where \( \epsilon > 0 \) regularizes the \( \tau \)-integration. The \( F \)'s are not Dirac observables but their Poisson bracket with the Hamiltonian constraint generates the plane waves\(^7\):

\[
\{ H, \hat{F}_q \} = 2me^{iq \cdot x}.
\]

(60)

We quantize \( F_q \) using the same ordering as for \( O_q \) splitting the translation \( e^{iq \cdot x} \) into two halves:

\[
\hat{F}_q |p\rangle = \frac{im}{q \cdot (p + \frac{q}{2}) + i\epsilon} |p + q\rangle,
\]

\[
= \frac{2im}{(p + q)^2 - p^2 + i\epsilon} |p + q\rangle.
\]

(61)

(62)

Its commutator with the Hamiltonian generates simple translations in the momentum space:

\[
\left[ \hat{H}, \hat{F}_q \right] |p\rangle = 2im |p + q\rangle.
\]

(63)

It is straightforward to generalize to observables recording whether the particle went through a certain number of space-time points, \( z^i \), ordered in time:

\[
\int \prod_{i=1}^n d\tau_i \delta^{(4)} (x_\mu (\frac{\tau_i}{m}) - z^i_\mu).
\]

We define \( O_q^{(n)} \) for a range \(-\infty < \tau_1 < \ldots < \tau_n < +\infty \) and \( F_q^{(n)} \) for the restricted range \( 0 < \tau_1 < \ldots < \tau_n < +\infty \). We compute their Fourier transform, now depending on \( n \) momenta \( q^i \):

\[
O_q^{(n)} = \frac{m^n \delta(p.Q_1) e^{iq_1 \cdot Q_1}}{\prod_{j=2}^n (p,Q_j + i\epsilon)} \cdot \hat{F}_q^{(n)} = \frac{(im)^n e^{iq_1 \cdot Q_1}}{\prod_{j=1}^n (p,Q_j + i\epsilon)},
\]

\[
O_q^{(n)} = \frac{m^n \delta(p.Q_1) e^{iq_1 \cdot Q_1}}{\prod_{j=1}^n (p,Q_j + i\epsilon)}.
\]

(64)

where we have defined the momenta \( Q_j = \sum_{i=j}^n q^i \). \( O_q^{(n)} \) is obviously once again a Dirac observable. As for the \( F \)'s, it is straightforward to compute:

\[
\{ H, \hat{F}_q^{(n)} \} = 2me^{iq \cdot x} \hat{F}_q^{(n-1)}.
\]

At the quantum level, we similarly define the operators:

\[
\hat{O}_q^{(n)} |p\rangle = \frac{2im \delta((p + Q_1)^2 - p^2)}{\prod_{j=2}^n (Q_j, (p + Q_j) + i\epsilon)} |p + Q_1\rangle,
\]

(65)

\[
\hat{F}_q^{(n)} |p\rangle = \frac{(im)^n}{\prod_{j=1}^n (Q_j, (p + Q_j + i\epsilon)) |p + Q_1\rangle}.
\]

(66)

This leads to the same tower of operators with the following commutators:

\[
\left[ \hat{H}, \hat{F}_q^{(n)} \right] = 2m T_q \hat{F}_q^{(n-1)},
\]

where \( T_q \) is the momentum translation by \( q \).

The one-loop Feynman diagram with two external legs can be extracted from:

\[
\int d^4 s \langle \hat{O}_q^{(4)} | \hat{F}_q^{(3)}_{s,r,-s} | p \rangle,
\]

(67)

where \( p \) is a reference vector on the mass-shell. Indeed up to the \( \epsilon \) regulator, this is equal to:

\[
(2m)^4 \delta^{(4)} (q + r) \frac{\delta((p + q + r)^2 - p^2)}{(p + r)^2 - p^2} \mathcal{I}_2(r),
\]

\[
\mathcal{I}_2(r) = \int d^4 s \frac{1}{((r + s)^2 - p^2)(s^2 - p^2)}.
\]

(68)

Setting \( p^2 = m^2 \), we obtain the one-loop Feynman amplitude with two legs for a massive scalar field, up to a normalization factor \( 1/((p + r)^2 - p^2) \). We can further get rid of the \( \delta((p + q + r)^2 - p^2) \) which leads to a singular result by equivalently considering the operator \( \exp(i\epsilon \cdot q) \mathcal{F}_{s,r,-s}^{(3)} \):

\[
\int d^4 s \langle p | \hat{O}_q^{(4)} \mathcal{F}_{s,r,-s}^{(3)} | p \rangle = \delta^{(4)} (q + r) \frac{(2im)^3}{(p + r)^2 - p^2} \mathcal{I}_2.
\]

Note that despite the fact that we are using \( F \) that operator still defines a Dirac observable since \( O_q^{(n)} \sim \exp(i\epsilon \cdot q) \mathcal{F}_{s,r,-s}^{(n-1)} \).

The physical interpretation of \( O_q^{(4)}_{s,r,-s} \) is as follows. Since we are dealing with \( O_q^{(4)} \), we are constraining the particle to go through four fixed space-time points \( ordered \ in \ time \ along \ the \ particle \ trajectory \). However, due to the
identification of the moments \( s \leftrightarrow -s \), the two last points are actually identified to the two first points, so that the particle does a time-like loop, going through a point \( z_1 \) then through a point \( z_2 \) then point \( z_1 \) again and point \( z_2 \) again. This graph exactly draws the one-loop Feynman diagram with two external legs.

This can be generalized to all Feynman diagrams. This shows that the quantum field theory scattering amplitudes can be expressed as expectation values of the \( \mathcal{O}^{(n)} \) Dirac observables (for a single relativistic particle).

Let us conclude this section by the following remark. If we were to define the \( \mathcal{O} \) observables using the Dirac observables \( X_\mu(\tau) \) instead of the coordinates \( x_\mu(\tau) \),

\[
\hat{O}_2 \equiv \int d\pi \delta^{(4)}(X_\mu \left( \frac{\pi}{m} \right) - z_\mu),
\]

its Fourier transform \( \hat{O}_q \) would not change at all,

\[
\hat{O}_q \equiv \int d^4z e^{q\cdot z} \hat{O}_2 = e^{iq\cdot x} \delta \left( \frac{q\cdot p}{m^2} \right) = \mathcal{O}_q,
\]

and the whole following construction would be identical. It is thus possible to construct the Feynman diagram evaluations from expectation values of the \( X_\mu(T) \) Dirac observables. We believe this can be applied in order to define the Feynman amplitudes for DSR. Following \([2]\), the main ingredient would be a modified Fourier transform reflecting that the momentum space is not the flat Minkowski space but is not curved.

\[\text{VII. THE PARTICLE WITH SPIN}\]

\[\text{A. Dirac observables}\]

In this final section, we investigate if including spin will change the analysis of the algebra of Dirac observables of the relativistic particle. A massive spinning particle is defined by the phase space \((x_\mu, p_\mu, s_{\mu\nu})\). The Poisson bracket still defines \( x \) and \( p \) as canonical variable. The \( s_{\mu\nu} \) commute with \( x \) and \( p \) and form a Lorentz algebra, i.e. have similar brackets as \( J_{\mu\nu} \). The constraints on the phase space are now:

\[
H = p^2 - m^2,
\]

\[
N = \frac{1}{2} s_{\mu\nu} s^{\mu\nu} + \lambda^2,
\]

\[
O^\nu = p_\mu s^{\mu\nu},
\]

where \( \lambda \) is the norm of the spin. This formalism can be entirely derived from a Lagrangian\(^8\).

Dirac observables now need to commute with all 6 constraints. The algebra of observables is now generated by the momentum \( p_\mu \) and the Lorentz generators \( L_{\mu\nu} = J_{\mu\nu} + s_{\mu\nu} \). We actually recover the Poincaré algebra. Let us notice that the spin \( s_{\mu\nu} \) is not an observable although it is a constant of the motion (since it commutes with the Hamiltonian constraint \( H \) generating the trajectories and doesn’t with \( O_\mu \)).

It is still straightforward to construct the coordinate Dirac observables. And we define the position observables in term of a time \( x \cdot v \):

\[
X_\mu(v) = \frac{L_{\mu\nu} v^\nu + T p_\mu}{p \cdot v} = x_\mu + \frac{p_\mu}{p \cdot v} (T - x \cdot v) + \frac{s_{\mu\nu} v^\nu}{p \cdot v},
\]

and in term of the time \( D \):

\[
X_\mu = \frac{L_{\mu\nu} p^\nu + T p_\mu}{p^2} = x_\mu + \frac{p_\mu}{p^2} (T - D) + \frac{O_\mu}{p^2},
\]

which is weakly equal to the spinless case. Similarly to the spinless case, we obtain:

\[
\{ X_\mu, X_\nu \} = -\frac{L_{\mu\nu}}{p^2} - \frac{j_{\mu\nu}}{p^2} - \frac{s_{\mu\nu}}{p^2}.
\]

The spin \( s \) creates an additional non-commutativity.

Another useful Dirac observable is the Pauli vector:

\[
W_\mu = \epsilon_{\mu\nu\lambda\rho} p^\nu L^{\lambda\rho} = \epsilon_{\mu\nu\lambda\rho} p^\nu s^{\lambda\rho}.
\]

\( W_\mu \) behaves as a vector under Lorentz transformations. Its norm \( W_\mu W^\mu \) is the second Casimir of the Poincaré algebra: \( W_\mu W^\mu = m^2 \lambda^2 \). We have the following Poisson brackets:

\[
\{ W_\mu, p_\nu \} = 0,
\]

\[
\{ W_\mu, X_\nu \} = \frac{p_\mu W_\nu}{p^2}.
\]

\( \text{---}\)

\( ^8 \) We consider a matrix \( \Lambda \in SO_3^+(1,3) \) and define the momentum and the spin as:

\[
p_\mu = m \Lambda_\mu 0, \quad \frac{1}{2} s_{\mu\nu} J^{\mu\nu} = -i s = \Lambda^{-1} J_{12} \Lambda,
\]

where \( J_{\mu\nu} \) are the Lorentz generators in the fundamental representation in term of \( 4 \times 4 \) matrices. It is easy to check that \( s \) explicitly reads as:

\[
s_{\mu\nu} = \lambda (\Lambda_{\mu1} \Lambda_{\nu2} - \Lambda_{\mu2} \Lambda_{\nu1}).
\]

Then we introduce the action:

\[
S = \int d\tau p^\mu \dot{x}_\mu - \frac{\lambda}{2} \text{Tr}(J_{12} \Lambda^{-1} \dot{\Lambda}).
\]

It is straightforward to check that the spin term in the Lagrangian is \( \frac{1}{2} (\dot{\Lambda}_{12} \Lambda_{12} - \Lambda_{12} \dot{\Lambda}_{12}) \), so that the resulting symplectic structure is simply \( \{ \Lambda_{12}, \dot{\Lambda}_{12} \} = 1/\lambda \). From there, it is obvious that the \( s \)'s form a Lorentz algebra under the Poisson bracket.
Let us also point out the identity:

\[ O_\mu W^\mu = \frac{1}{4} p^2 \epsilon^{\alpha \beta \gamma \delta} s_{\alpha \beta} s_{\gamma \delta}. \]

Next, we can introduce Dirac observables for the spin. As \( s_{\mu \nu} \) commutes with \( H \) and \( N \), we only need to take care of the constraint \( O_\mu \) and gauge fix it. Thus we introduce the following Dirac observables labelled by an arbitrary fixed vector \( a_\mu \):

\[
S_{\mu \nu}^{(a)} = L_{\mu \nu} - (a_\mu p_\nu - a_\nu p_\mu) = s_{\mu \nu} + (x_\mu - a_\mu) p_\nu - (x_\nu - a_\nu) p_\mu, \tag{75}
\]

which gives \( s_{\mu \nu} \) when \( x_\mu = a_\mu \). These observables \( S_{\mu \nu}^{(a)} \) satisfy the same brackets as \( s_{\mu \nu} \) and also form a Lorentz algebra. Moreover \( S_{\mu \nu}^{(a)} \) act as the Lorentz generators on the momentum vector \( p_\mu \).

We can remove the dependence on an arbitrary fixed vector \( a_\mu \) and render the expression covariant by using the Dirac observable \( X_\mu \). Thus we introduce our spin Dirac observable:

\[
S_{\mu \nu} = L_{\mu \nu} - (X_\mu p_\nu - X_\nu p_\mu) = s_{\mu \nu} + (x_\mu - X_\mu) p_\nu - (x_\nu - X_\nu) p_\mu = s_{\mu \nu} + \frac{1}{p^2} (O_\mu p_\nu - O_\nu p_\mu), \tag{76}
\]

which is actually weakly equal to the spin \( s_{\mu \nu} \) itself. Moreover \( \{ S, p \} = 0 \) and the Poisson brackets of the \( S \)'s form the Lorentz algebra on-shell (off-shell, we get a few \( O \wedge p \) terms).

### B. Spin-induced non-commutativity

One can compute the algebra of constraints of the relativistic spinning particle and we found:

\[
\{ H, O_\mu \} = \{ N, O_\mu \} = \{ H, N \} = 0,
\]

\[
\{ O_\mu, O_\nu \} = p^2 s_{\mu \nu} + (O_\alpha p_\nu - O_\nu p_\alpha) = p^2 S_{\mu \nu}. \tag{77}
\]

So \( H \) and \( N \) are first class constraints, while the \( O_\mu \)'s are second class constraints. One can thus introduce the Dirac bracket taking the constraints \( O_\mu = 0 \) into account. Noting \( \Delta_{\mu \nu} \equiv \{ O_\mu, O_\nu \} \) the Dirac matrix, we can compute its inverse:

\[
(\Delta^{-1})^{\mu \nu} = \frac{-\epsilon^{\alpha \beta \gamma \delta} \Delta_{\alpha \beta}}{\epsilon^{\alpha \beta \gamma \delta} \Delta_{\alpha \beta} \Delta_{\gamma \delta}}, \tag{78}
\]

with \( \frac{1}{2} \epsilon \Delta \Delta = \frac{1}{2} p^4 \epsilon SS = \frac{1}{2} p^4 \epsilon ss \). We define the Dirac bracket as:

\[
\{ f, g \}_D = \{ f, g \} - \{ f, O_\mu \}(\Delta^{-1})^{\mu \nu} \{ O_\nu, g \}.
\]

Since \( \{ x_\mu, O_\nu \} = s_{\mu \nu} \) and \( \{ p_\mu, O_\nu \} = 0 \), it is straightforward to compute\(^9\):

\[
\{ x_\mu, p_\nu \}_D = \eta_{\mu \nu}, \quad \{ p_\mu, p_\nu \}_D = 0,
\]

\[
\{ x_\mu, x_\nu \}_D = \frac{1}{2p^2} s_{\mu \nu} + \phi_{\mu \nu}(O_\alpha). \tag{79}
\]

We see that the spin of the particle induces a non-commutativity of the particle position algebra at the classical level.

### C. About the quantification of the Dirac observables

The Dirac observables \( (L, X) \) form as in the spinless case a \( so(4,1) \) algebra under the Poisson bracket. The quadratic Casimir \( C_2 = \frac{1}{2} LL - p^2 XX \) can be computed exactly on-shell and we obtain \( C_2 = -T^2 - \lambda^2 \). The quartic Casimir \( C_4 = w_\mu w^\mu \) is defined in term of the vector:

\[
w_\mu = \epsilon_{\mu \alpha \beta \gamma} X^\alpha(T)L^{\beta \gamma}.
\]

One can check that \( C_4 \neq 0 \) unlike the spinless case. This means that we are not restricted to simple representations anymore as in the spinless case: we need to consider all functions \( L^2(so(1,4)) \) and we cannot restrict ourselves to the 5d light cone. Thus, in the case of the spinning particle, we are necessarily led to work with a 5d representation of the Dirac observables.

### Conclusion

We have looked at the relativistic particle from the perspective of its algebra of (Dirac) observables. We have identified a set of Lorentz covariant position observables, which turn out to be non-commutative. This non-commutativity reflects the fact that one can not localize a massive quantum particle with a precision better than its Compton length. We then showed that the particle admits in this context a natural representation in five dimensions and could be quantized in terms of wave functions on the 5d light cone. This allows a direct comparison with the free particle in DSR (e.g. [6]), which

\[\phi_{\mu \nu}(O) = \frac{4}{p^4 \epsilon ss} \epsilon^{\alpha \beta \gamma \delta} s_{\mu \alpha} s_{\beta \nu} O_{\gamma \delta} p_5.\]
evolves in a non-commutative space-time and whose momentum lives in the curved de Sitter space. Moreover, it turns out that the 5d light cone formulation is subject to divergencies, which are naturally regulated by DSR. This allows a clear understanding of how DSR arises as an extension of Special Relativity. In particular, it lead us to an interpretation of the fifth coordinate as proper time and of the fifth moment as generating the Hamiltonian flow of the particle.

The case of the spinning particle deserves more attention. We have showed that the spin induces an extra non-commutativity of the space-time coordinates. This complicates the quantization of the algebra of observables and a full analysis of the 5d representation of the spinning particle is still under investigation.

Finally, we also described a new representation of the Feynman diagram evaluations in term of Dirac observables. We now hope to extend this approach to DSR and use these new tools in order to build the scattering amplitudes and S-matrix of a Quantum Field Theory in DSR in a consistent way.

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