Structure functions and angular ordering at small $x$\footnote{Research supported in part by the Italian MURST and the EC contract CHRX-CT93-0357}

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Abstract

We compute the gluon distribution in deep inelastic scattering at small $x$ by solving numerically the angular ordering evolution equation. The leading order contribution, obtained by neglecting angular ordering, satisfies the BFKL equation. Our aim is the analysis of the subleading corrections. Although not complete — the exact next-to-leading contribution is not yet available — these corrections are important since they come from the physical property of coherence of QCD radiation. In particular we discuss the subleading correction to the BFKL characteristic function and the gluon distribution’s dependence on the maximum available angle. Conformal invariance of the BFKL equation is lost, however this is not enough to bring the small-$x$ gluon distribution into the perturbative regime: although large momentum regions are enhanced by angular ordering, the small momentum regions are not fully suppressed. As a consequence, the gluon anomalous dimension is finite and tends to the BFKL value $\gamma = 1/2$ for $\alpha_s \to 0$. The main physical differences with respect to the BFKL case are that angular ordering leads to 1) a larger gluon anomalous dimension, 2) less singular behaviour for $x \to 0$ and 3) reduced diffusion in transverse momentum.
1 Introduction

Angular ordering is an important feature of perturbative QCD [1] with a deep theoretical origin and many phenomenological consequences [2]. It is the result of destructive interference: outside angular ordered regions amplitudes involving soft gluons cancel. This property is quite general. It is present in both time-like processes, such as $e^+e^-$ annihilation, and in space-like processes, such as deep inelastic scattering (DIS). Moreover it is valid in the regions both of large and small $x$, in which $x$ is the registered energy fraction in the $e^+e^-$ fragmentation function or the Bjorken variable in the DIS structure function. Due to the universality of angular ordering one has a unified leading order description of all hard processes involving coherent soft gluon emission.

Angular ordering is important in the calculation of multi-parton distributions by resummation of powers of $\ln Q^2$, with $Q$ the hard scale, and of powers of $\ln x$ or $\ln(1-x)$ for small or large $x$. This is due to the fact that angular ordering defines the structure of the collinear singularities and, to leading order, their relation to the infra-red (IR) singularities for $x \to 0$ or $x \to 1$. In particular one finds that collinear singularities in the emitted transverse momenta contribute both to $\ln Q$ and $\ln x$ or $\ln(1-x)$. This is because angular ordering implies ordering in the emitted transverse momenta divided by the energies.

In this paper we start a systematic study of multi-parton emission in DIS at small $x$. The detailed analysis of angular ordering in multi-parton emission at small $x$ and the related virtual corrections has been done in Ref. [3] (see also [4]), where it was shown that to leading order the initial-state gluon emission can be formulated as a branching process in which both angular ordering and virtual corrections are taken into account. In this first paper we study the fully inclusive gluon density which gives the structure function at small $x$. This gluon density is given by an inclusive recurrence equation deduced from the small-$x$ coherent branching (the CCFM equation).

In spite of the universality of angular ordering, the space-like and time-like distributions in the small-$x$ region are profoundly different even to leading order. In $e^+e^-$ annihilation processes the small-$x$ distributions are obtained by resumming the $\ln x$ powers which come both from IR and from collinear singularities in the angular ordered regions. In DIS, angular ordering is essential for describing the structure of the final state, but not for the gluon density at small-$x$. This is because in the resummation of singular terms of the gluon density, there is a cancellation between the real and virtual contributions. The only remaining collinear singularity is the one originating from the first gluon emission. As a result, to leading order the small-$x$ gluon density is obtained by resumming $\ln x$ powers coming only from IR singularities, and angular ordering contributes only to subleading corrections.

The calculation of the gluon density by resummation of $\ln x$ powers without angular ordering was done 20 years ago [5] and leads to the BFKL equation for $F(x, k)$, the gluon density at fixed transverse momentum $k$, related to the small-$x$ part of the gluon structure function $F(x, Q)$ by

$$F(x, Q) = \int dk^2 F(x, k) \Theta(Q - k).$$ (1)
In the moment representation the BFKL equation has solutions of the form

\[ x F(x, k) = \int \frac{d\omega}{2\pi i} \left( \frac{1}{x} \right)^\omega F_\omega(k), \quad F_\omega(k) \sim \frac{1}{k^2} \left( \frac{k^2}{k_0^2} \right)^\gamma, \tag{2} \]

with \( k_0 \) an arbitrary constant and \( \gamma \) given by a solution of the well known BFKL characteristic equation

\[ 1 = \frac{\bar{\alpha}_S}{\omega} \chi(\gamma), \quad \chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1 - \gamma), \tag{3} \]

where \( \bar{\alpha}_S = \frac{G A \alpha_S}{\pi} \) and \( \psi \) is the logarithmic derivative of the gamma function. The QCD coupling \( \alpha_S \) is taken as a fixed parameter. The renormalisation group dependence of \( \alpha_S \) on a scale is an effect which goes beyond this leading order contribution in which one resums the powers \( (\alpha_S/\omega)^n \) for \( \omega \to 0 \). The next-to-leading order contribution, \( \alpha_S(\alpha_S/\omega)^n \), is so far only partially computed \[6\].

In eq. (1) \( \gamma \) plays a rôle analogous to that of the gluon anomalous dimension, however its origin is not the renormalisation group but conformal invariance, which is a consequence of the absence of collinear singularities and which implies that in \( F(x, k) \) all regions of \( k^2 \) large or small are equally important. This is reflected in the symmetry \( \gamma \to 1 - \gamma \) of the characteristic function.

The gluon distribution in the angular ordered equation depends on an additional variable \( p \), which corresponds to the maximum available angle for the initial state radiation. We denote by \( A_\omega(k, p) \) the gluon distribution in this case. The solutions have a form similar to (2)

\[ A_\omega(k, p) \sim \frac{1}{k^2} \left( \frac{k^2}{k_0^2} \right)^{\tilde{\gamma}} G(p/k), \tag{4} \]

where \( G(p/k) \), describing the angular dependence, has a structure typical of a form factor, vanishing for \( p \to 0 \). As in (3), the gluon anomalous dimension \( \tilde{\gamma} \) is given by a modified characteristic function \( \tilde{\chi}(\tilde{\gamma}, \alpha_S) \) which depends also on \( \alpha_S \). Angular ordering breaks the conformal invariance of the gluon density, so that the modified characteristic function is not symmetric for \( \tilde{\gamma} \to 1 - \tilde{\gamma} \).

To leading order, i.e. for the leading powers \( \alpha_S^2/\omega^n \), the two equations are equivalent, so that \( A_\omega(k, p) \to F_\omega(k) \). We have therefore that the angular dependence in \( G(p/k) \) is a subleading correction of order \( \alpha_S \). The gluon anomalous dimension has the expansion

\[ \tilde{\gamma} = \gamma + \alpha_S \gamma_1 + \cdots \tag{5} \]

where the leading order result \( \gamma \), given by the BFKL characteristic function, and the first correction \( \gamma_1 \) are functions of the ratio \( \alpha_S/\omega \). We have that \( \alpha_S \gamma_1 \) is the part of the next-to-leading correction of the gluon anomalous dimension which comes from angular ordering.

The study of the differences between the two equations will be done by analytical and especially by numerical calculations. As we shall discuss, the equation with angular ordering can be diagonalised only partially thus numerical methods are needed. The study of some of
Figure 1: Kinematic diagram for a DIS process at parton level: $k_{in}$ is the incoming gluon, defined to have energy $E$; the $k_i$ are the exchanged gluons ($k_n$ is the gluon which undergoes the hard collision) and the $q_i$ are the gluons emitted in the initial state.

the phenomenological features of the gluon density with angular ordering has been done in Ref. [7].

In Sect. 2 we recall the main elements of the CCFM equation and its relation to the BFKL one. We discuss how a hard scale enters. We deduce some simple analytical properties and the behaviour of the solution. In Sect. 3 we discuss the numerical methods used. In Sect. 4 we present the results. Sect. 5 contains a summary of the main points and some conclusions.

2 Equation for the gluon density

In this section we recall the basic elements for the small-$x$ coherent branching and the inclusive equation for the gluon density.

We start by considering the kinematic diagram for a DIS process at parton level presented in fig. 1. All partons involved are gluons since gluons dominate the small-$x$ region. The last exchanged parton $n$ undergoes the hard collision at the scale $Q$. For the exchanged gluon $i$ we denote by $x_i$ and $k_i$ the energy fraction and transverse momentum with respect to the incoming gluon. Introducing the energy ratio $z_i = x_i/x_{i-1}$, we have that $(1 - z_i)x_{i-1}$ and $q_i$ are the energy fraction and transverse momentum of the emitted gluon $i$. We shall use $k_i$ and $q_i$ also to denote the moduli of the transverse momenta ($k_i = |k_i|$ and $q_i = |q_i|$). We consider the region $z_i \ll 1$, which gives the leading IR singularity.

The emission process takes place in the angular ordered region given by $\theta_i > \theta_{i-1}$ with $\theta_i$ the angle of the emitted gluon $q_i$ with respect to the incoming gluon $k_{in}$. In terms of the emitted transverse momenta $q_i$ this region is given by

$$\theta_i > \theta_{i-1}, \quad \Rightarrow \quad q_i > z_{i-1}q_{i-1}.$$  \hspace{1cm} (6)

The branching process given in [3] is accurate to leading IR order and, at the inclusive level,
does not require any collinear approximation [8]. The distribution for the emission of gluon \(i\) is given by

\[
dP_i = \frac{d^2 q_i}{\pi q_i^2} dz_i \frac{\bar{\alpha}_S}{z_i} \Delta(z_i, q_i, k_i) \Theta(q_i - z_{i-1}q_{i-1}),
\]

where angular ordering (6) is included. The function \(\Delta\) is the form factor which resums important virtual corrections for small \(z_i\)

\[
\ln \Delta(z_i, q_i, k_i) = - \int_{z_i}^{1} \frac{dz'}{z'} \frac{\bar{\alpha}_S}{z'} \int \frac{dq'^2}{q'^2} \Theta(k_i - q') \Theta(q' - z'q_i).
\]

This form factor has a simple probabilistic interpretation. It corresponds to the probability for having no radiation of gluons with energy fraction \(x' = z'x_{i-1}\) in the region \(x_i < x' < x_{i-1}\), and with a transverse momentum \(q'\) smaller than the total emitted transverse momentum \(k_i\) and with an angle \(\theta' > \theta_i\). Angles and momenta are related by \(q_i \simeq x_{i-1}E\theta_i\) and \(q' \simeq x'E\theta'\) so that angular ordering gives \(q' > z'q_i\). The two boundaries in \(q'\) are due to coherence in the exchanged gluon \((k > q')\) and in the emitted one \((\theta' > \theta_i)\).

One has

\[
\Delta(z_i, q_i, k_i) = \exp \left( -\bar{\alpha}_S \ln \frac{1}{z_i} \ln \frac{k_i^2}{z_i q_i^2} \right), \quad k_i > q_i,
\]

so that this form factor has a double logarithmic form, suppressing the radiation both for \(z_i \ll 1\) and for the emitted transverse momentum \(q_i \ll k_i/\sqrt{z_i}\).

From the form in (9) and from the probabilistic interpretation, we have that the function \(\Delta\) plays a rôle similar to that of the Sudakov form factor. However there are important differences. The Sudakov form factor resums virtual corrections with IR singularities due to soft emitted gluons, i.e. powers of \(\ln(1 - z)\), regularising the \(z \to 1\) singularity in the splitting function. This implies that when weighted with the energy fraction \(z_i\), the usual branching with the Sudakov form factor [1] can be normalised to unity, corresponding to the \(\omega = 1\) energy sum rule.

The form factor (8) resums virtual corrections with IR singularities due to soft exchanged gluons, i.e. powers of \(\ln x\). The branching (6) cannot be easily normalised to unity for \(\omega = 1\). However this normalisation is not relevant for our study of small \(x\). Note that \(\Delta(z, q, k)\) depends not only on two transverse momentum scales, as does the Sudakov form factor, but also on the energy fraction. This extra dependence in the form factor is one of the important features of DIS coherence at small \(x\). \(\Delta(z, q, k)\) corresponds in the BFKL equation to the gluon Regge form factor, which depends only on \(k\), \(z\) and a collinear cutoff (see later).

Since the intermediate real and virtual transverse momenta are bounded by angular ordering, no collinear cutoff is needed, except on the emission angle of the first gluon. However for \(z_{i-1} \to 0\) one exposes the collinear singularity of \(q_i \to 0\). Thus by integrating over the real and virtual transverse momenta one generates powers of \(\ln z_{i-1}\). After integrating over the energy fractions \(z_{i-1}\) one finds that the general perturbative term is of the form

\[
\frac{\alpha_S n}{x} \ln^{n+\ell-1} x \Rightarrow \frac{\alpha_S n}{\omega^{n+\ell}}, \quad \ell < n.
\]
Here each energy fraction integration gives a power of $\ln x$ while each transverse momentum integration gives either $\ln x$ or $\ln k$. We have therefore that the contributions with $\ell > 0$ are obtained from collinear singularities. Thus if collinear singularities cancel then the leading $\ln x$ contributions are obtained only from IR singularities, i.e. for $\ell = 0$.

In order to deduce a recurrence relation for the inclusive distribution in the last gluon with fixed $x = x_n$ and $k = k_n$ one has to introduce a transverse momentum $p$ associated with the maximum available angle

$$\theta_n < \bar{\theta} \quad \Rightarrow \quad z_n q_n < p \simeq x E \bar{\theta}, \quad (11)$$

with $xE$ the energy of the last gluon, $k_n$, which undergoes the hard collision at the scale $Q$. Then one defines the distribution for emitting $n$ initial state gluons

$$\mathcal{A}^{(n)}(x, k, p) = \int \prod_{i=1}^{n} dP_i \, \Theta(p - z_n q_n) \, \delta(k^2 - k_n^2) \, \delta(x - x_n). \quad (12)$$

The fully inclusive gluon density

$$\mathcal{A}(x, k, p) = \sum_{n=0}^{\infty} \mathcal{A}^{(n)}(x, k, p), \quad (13)$$

satisfies the following recurrence relation

$$\mathcal{A}(x, k, p) = \mathcal{A}^{(0)}(x, k, p) + \int d^2 q \, \pi q^2 \, \frac{\alpha_s}{z} \Delta(z, q, k) \Theta(p - z q) \, \mathcal{A}(x/z, |k + q|, q), \quad (14)$$

where the inhomogeneous term $\mathcal{A}^{(0)}(x, k, p)$ is the distribution for no gluon emission. This equation can be partially diagonalised by introducing the $\omega$-representation

$$\mathcal{A}_\omega(k, p) = \int_0^1 dx \, x^\omega \mathcal{A}(x, k, p). \quad (15)$$

One finds

$$\mathcal{A}_\omega(k, p) = \mathcal{A}^{(0)}_\omega(k, p) + \int d^2 q \, \pi q^2 \, \frac{\alpha_s}{z} \Delta(z, q, k) \Theta(p - z q) \, \mathcal{A}_\omega(|k + q|, q). \quad (16)$$

This equation cannot be further diagonalised in transverse momentum since the kernel depends both on the total momentum $k$ and on $q$ and $p$. Numerical studies are then necessary.

For the fully inclusive gluon density $\mathcal{A}(x, k, p)$ there is a cancellation between the collinear singularities which appear in the real and virtual contributions of the kernel. To see this we convert the recurrence relation into an inclusive form. By using the identity

$$\int_0^1 dz \, z^\omega \frac{\alpha_s}{z} \Delta(z, q, k) \Theta(p - z q) = \frac{\alpha_s}{\omega} \left\{ 1 - \int_0^1 dz \, z^\omega \frac{\partial}{\partial z} \Delta(z, q, k) \Theta(p - z q) \right\}, \quad (17)$$

one finds

$$\mathcal{A}_\omega(k, p) = \mathcal{A}^{(0)}_\omega(k, p) + \frac{\alpha_s}{\omega} \int d^2 q \, \pi q^2 \left[ \mathcal{A}_\omega(|k + q|, q) - \Theta(k - q) \, \mathcal{A}_\omega(k, q_1) \right] + \delta_\omega(k, p), \quad (18)$$

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where the inhomogeneous term is
\[
\tilde{A}_\omega^{(0)}(k, p) = A_\omega^{(0)}(k, p) + \frac{\tilde{\alpha}_S}{\omega} \int \frac{dq^2}{q^2} \Theta(k - q) A_\omega^{(0)}(k, q_1),
\]
and \(q_1 = \min(q, p)\). The correction \(\delta_\omega(k, p)\) is given by
\[
\delta_\omega(k, p) = \frac{\tilde{\alpha}_S}{\omega} \int p d_2q \pi q^2 \left[ \frac{|p|}{q} \Delta(|p|, q) - 1 \right],
\]
where the integration range is \(q > p\). In equation (18) the first term in the integral comes from the real emission contribution while the second is due to the virtual correction. One sees explicitly that, as in the BFKL equation, the real and virtual \(q \to 0\) collinear singularities in the kernel cancel. Also \(\delta_\omega(k, p)\) has no collinear singularities since \(q > p \neq 0\).

For \(p \to \infty\) the term \(\delta_\omega(k, p)\) vanishes and the gluon density \(A_\omega(k, p)\) becomes independent of \(p\). In fact neglecting \(\delta_\omega(k, p)\) and the \(p\) dependence in \(A_\omega(k, p)\) one finds that (18) becomes the BFKL equation for the gluon density
\[
F_\omega(k) = \Phi_\omega^{(0)}(k) + \int d_2q \pi q^2 \left[ F_\omega(|k + q|) - \Theta(q - k) F_\omega(k) \right].
\]
Neglecting the \(p\)-dependence in \(A_\omega(k, p)\) corresponds to neglecting angular ordering. To see this we modify the branching distribution in (7) and the virtual corrections (8) by neglecting angular ordering so that the transverse momenta have no lower bound. To avoid singularities we have to set a collinear cutoff \(\mu\), which, at the inclusive level, becomes irrelevant. The modified branching distribution is given by
\[
dP_i^{(0)} = \frac{d_2q}{\pi q^2} dz_i \frac{\tilde{\alpha}_S}{z_i} \Delta^{(0)}(z_i, k_i) \Theta(q_i - \mu),
\]
\[
\ln \Delta^{(0)}(z, k) = - \int_z^1 dz' \frac{\tilde{\alpha}_S}{z'} \int \frac{dq'^2}{q'^2} \Theta(k - q') \Theta(q' - \mu),
\]
obtained from (7) and (8) by the substitution \(\Theta(q_i - z_{i-1}q_{i-1}) \to \Theta(q_i - \mu)\), and \(\Theta(q' - z'q) \to \Theta(q' - \mu)\) respectively. This modification has no effect to leading order since the collinear singularities cancel. Proceeding as for \(A\), the gluon density satisfies the following recurrence relation:
\[
F(x, k) = F^{(0)}(x, k) + \int d_2q \pi q^2 dz \frac{\tilde{\alpha}_S}{z} \Delta^{(0)}(z, k) \Theta(q - \mu) F(x/z, |k + q|),
\]
where the inhomogeneous term is related to the one in (21) as in (18). From this modified branching one obtains the BFKL equation (21) in which the momentum has the cutoff \(\mu\) which can be neglected.

### 2.1 General properties of gluon distributions

In the following we discuss some of the properties of the gluon distribution \(A_\omega(k, p)\) and its comparison with the BFKL distribution \(F_\omega(k)\). As mentioned before, at large \(p\) the leading
order contribution to $A_\omega(k,p)$ tends to $F_\omega(k)$. We are interested in analysing the subleading corrections contained in $A_\omega(k,p)$ which are due to angular ordering.

We start by recalling some well known properties of the BFKL distribution. The solution of (21) is given by

$$F_{\omega}(k) = \frac{1}{k^2} \left( \frac{k^2}{k_0^2} \right)^{\gamma_0} \Theta(p),$$

where $k_0$ and the function $f_0$ are fixed by the inhomogeneous term and $\chi(\gamma)$ is the BFKL characteristic function (3).

Taking the initial condition

$$\tilde{F}_{\omega}^{(0)}(k) = \frac{1}{k^2} \left( \frac{k^2}{k_0^2} \right)^{\gamma_0},$$

with a given $\gamma_0$ one has that the solution $F_\omega(k)$ has the same form, and therefore the small-$x$ behaviour of $F(x,k)$ is given by

$$x F(x,k) \sim x^{-\omega_c} \left( \frac{k^2}{k_0^2} \right)^{\gamma_0}, \quad \omega_c = \bar{\alpha}_S \chi(\gamma_0).$$

For a general initial condition the asymptotic behaviour of $F_\omega(k)$ for $k \gg k_0$ and for $k \ll k_0$ is given by the expression (2) where $\gamma$ is the solution of the characteristic equation (3) in the region $0 < \gamma < \frac{1}{2}$ and $\frac{1}{2} < \gamma < 1$ respectively. The behaviour of $F(x,k)$ at small $x$ is determined by the leading singularity of $\gamma(\alpha_S/\omega)$ in the $\omega$-plane which is at $\gamma_c = \gamma(\alpha_S/\omega_c) = 1/2$ giving $\omega_c = \bar{\alpha}_S \chi(\frac{1}{2}) = 4\bar{\alpha}_S \ln 2$.

We come now to discuss the properties for $A(x,k)$ by taking solutions of the form

$$A_\omega(k,p) = \frac{1}{k^2} \left( \frac{k^2}{k_0^2} \right)^{\tilde{\gamma}} G(p/k),$$

where $\tilde{\gamma}$ has to be specified and the function $G(p/k)$ takes into account angular ordering.

The equation for $G(p/k)$ is obtained by taking the derivative of (16) with respect to $p$:

$$p \partial_p G(p/k) = \bar{\alpha}_S \int \frac{d^2q}{\pi q^2} \left( \frac{p}{q} \right)^\omega \Delta(p/q,k) G \left( \frac{q}{|k + q|^2} \right) \left( \frac{k + q}{k^2} \right)^{\tilde{\gamma} - 1}.$$  

We consider the case of $0 < \tilde{\gamma} < 1$ and as a boundary condition we take $G(\infty) = 1$. This function depends on $\alpha_S$, $\omega$ and $\tilde{\gamma}$.

If one takes the initial condition as for the BFKL case, (see (23))

$$A_\omega^{(0)}(k,p) = \frac{1}{k^2} \left( \frac{k^2}{k_0^2} \right)^{\gamma_0 - 1} \Theta(p - k),$$

with a given $\gamma_0$, then the solution of the angular ordering equation has the form (27) with $\tilde{\gamma} = \gamma_0$. In this case $\tilde{\gamma}$ is a free parameter independent of $\alpha_S$ and $\omega$. 
The expression (27) is a solution of the homogeneous equation, as in the BFKL case, provided that \( \tilde{\gamma} \) is given by the generalised characteristic function which is obtained from (27) and (18) in the limit \( p \to \infty \).

\[
1 = \frac{\bar{\alpha}_S}{\omega} \tilde{\chi}(\tilde{\gamma}, \alpha_S), \quad \tilde{\chi} = \int \frac{d^2q}{\pi q^2} \left\{ \left( \frac{|k + q|^2}{k^2} \right)^{\tilde{\gamma} - 1} G \left( \frac{q}{|k + q|} \right) - \Theta(k - q) G(q/k) \right\} . \tag{30}
\]

There may be various solutions to this equation and we will consider the leading one, i.e. that with the largest \( \omega \). In this case \( \tilde{\gamma} \) is not an independent variable, but is a function of \( \alpha_S \) and \( \omega \).

From these equations one finds that the leading order contribution to \( \mathcal{A}(x, k, p) \) is given by \( \mathcal{F}(x, k) \). First from (28) one has that the \( p \)-dependence of \( G(p/k) \) is a subleading correction proportional to \( \bar{\alpha}_S \) without \( 1/\omega \) enhancement. Moreover for \( G(p/k) \to 1 \), one has \( \tilde{\chi}(\tilde{\gamma}, \alpha_S) \to \chi(\tilde{\gamma}) \), the BFKL characteristic function in (3). We now list some properties of the angular ordering function.

**Behaviour of \( G(p/k) \) for \( p \gg k \).** In the region \( q > p \gg k \) we have \( \Delta(p/q, q, k) = 1 \) and \((k + q)^2 \simeq q^2\). From (28) we have

\[
p \partial_p G(p/k) \simeq \bar{\alpha}_S G(1) \int_p \frac{d^2q}{\pi q^2} \left( \frac{p}{q} \right)^\omega \left( \frac{q^2}{k^2} \right)^{\tilde{\gamma} - 1} . \tag{31}
\]

Since \( \tilde{\gamma} < 1 \), the derivative vanishes at large \( p \) and one finds

\[
G(p/k) \simeq 1 - \frac{\bar{\alpha}_S G(1)}{(1 - \tilde{\gamma})(2 - 2\tilde{\gamma} + \omega)} \left( \frac{p^2}{k^2} \right)^{\tilde{\gamma} - 1} . \tag{32}
\]

**Behaviour of \( G(p/k) \) for \( p \ll k \).** By using

\[
\ln \Delta(p/q, q, k) = -\bar{\alpha}_S \left[ \ln^2(k/p) \Theta(k - p) - \ln^2(k/q) \Theta(k - q) \right] ,
\]

and \((k + q)^2 \simeq k^2\) for \( q \ll k \), from (28) one obtains

\[
p \partial_p G(p/k) \simeq \bar{\alpha}_S e^{-\bar{\alpha}_S \ln^2(k/p)} \left( \frac{p}{k} \right)^\omega \left[ \int_p \frac{d^2q}{\pi q^2} \left( \frac{k}{q} \right)^\omega e^{\bar{\alpha}_S \ln^2(q/k)} G(q/k) + C(\omega, \alpha_S) \right] , \tag{33}
\]

where \( C(\omega, \alpha_S) \) is a constant which is independent of \( p \). We find then

\[
G(p/k) \simeq \frac{\bar{\alpha}_S C(\omega, \alpha_S)}{\omega} \left( \frac{p}{k} \right)^\omega e^{-\bar{\alpha}_S \ln^2(k/p)} . \tag{34}
\]

This behaviour is similar to that of a Sudakov form factor. The distribution is suppressed when the maximum angle \( \tilde{\theta} \) available for the initial state emission vanishes, more precisely for \( p \simeq x E \tilde{\theta} \) much smaller than the total emitted momentum \( k \).

**Behaviour for \( \tilde{\gamma} \to 0 \) and \( \alpha_S \) fixed.** In this limit the angular ordering function assumes the perturbative form, \( G(p/k) \to \Theta(p - k) \). This is as one would expect from (32) and (34), and can be proved directly from (28).
Correction to the characteristic function. The correction $\delta \chi(\tilde{\gamma}, \alpha_S)$ is given by eq. (30) in which one substitutes $G(p/k)$ with $\delta G(p/k) = 1 - G(p/k)$. From (32) one has that $\delta \chi = \chi(\tilde{\gamma}) - \tilde{\chi}(\tilde{\gamma}, \alpha_S)$ is regular for $\tilde{\gamma} \to 0$. From $\delta \chi$ one obtains the subleading corrections to the gluon anomalous dimension due to angular ordering. By taking into account that $\gamma$ is the solution of $1 = (\bar{\alpha}_S/\omega) \chi(\gamma)$ we can write the characteristic equation (30) in the form

$$\tilde{\chi}(\tilde{\gamma}, \alpha_S) = \chi(\gamma).$$

(35)

Expanding $\tilde{\chi}(\gamma, \alpha_S)$ around $\alpha_S = 0$ we find

$$\tilde{\gamma} = \gamma + \alpha_S \gamma_1 + \cdots, \quad \gamma_1 = -\frac{\partial \tilde{\chi}(\gamma, \alpha_S)}{\chi'(\gamma) \partial \alpha_S |_{\alpha_S=0}}.$$  

(36)

From these equations we have that the first correction to $\tilde{\gamma}$ is $\alpha_S \gamma_1 \sim \alpha_S^3/\omega^2$. To prove this, observe that $\chi'(\gamma) \sim \gamma^{-2}$ for small $\gamma$. Moreover, for $\alpha_S \to 0$ and $\gamma$ fixed, we have $\delta \chi(\gamma, \alpha_S) \sim c\alpha_S$ where $c$ tends to a constant as $\gamma \to 0$.

As we shall see from the numerical analysis, the characteristic function $\tilde{\chi}(\tilde{\gamma}, \alpha_S)$ decreases with $\tilde{\gamma}$, reaches a minimum at $\tilde{\gamma}_c < 1$ (for reasonable $\alpha_S$), and then rises again. As in the BFKL case, we shall denote by $\tilde{\omega}_c$ the leading singularity in $\omega$ which corresponds to the minimum of the characteristic function at $\tilde{\gamma} = \tilde{\gamma}_c$.

3 Numerical methods

3.1 Evolution in rapidity

The angular ordered equation is solved by binning the function $A(x, k, p)$ in all three variables. To allow the coverage of a wide range of transverse and longitudinal momenta it is convenient to store the function on a grid of logarithmic variables $y = \ln(1/x)$, $\ln k$ and $\ln p$, where transverse momenta are in units of $k_0$. This allows us to go to very small $x$, and to cover the wide range of transverse momenta needed to correctly take into account the diffusion in $\ln k$. We solve for $A$ using the integral equation (14). From now on we will refer to the gluon density through the following function:

$$A(y, k, p) = e^{-y} A(e^{-y}, k, p).$$

(37)

The equation satisfied by $A$ is

$$A(y, k, p) = A^{(0)}(y, k, p) + \bar{\alpha}_S \int_0^y dy' \int_{q_{\text{min}}}^{q_{\text{max}}} \frac{d^2 q}{\pi q^2} \Delta(e^{-y'}, q, k) \Theta(\ln p + y' - \ln q) A(y - y', |k + q|, q),$$

(38)

where the limits on the $q$ integration, $q, |k + q| > q_{\text{min}}$ and $q, |k + q| < q_{\text{max}}$, are introduced because of the finite extent of the grid in $\ln p$ and $\ln k$. Our approach is based on the fact
that \(A(0, k, p) = A^{(0)}(0, k, p)\). Then we attempt to determine \(A\) on a grid in \(y\), at points \(y = i\delta y\), where \(i\) runs from 1 to some \(i_{\text{max}} = Y/\delta y\), \(Y\) being the highest rapidity in which we are interested. Assuming that we know \(A\) for all points on the grid up to \(i\delta y\), then the procedure for evaluating the \((i+1)^{\text{th}}\) point is the following: as a first approximation we set \(A((i+1)\delta y, k, p) = A(i\delta y, k, p)\). This is put into the integral equation \(\text{[38]}\) to allow us to calculate a second approximation to \(A((i+1)\delta y, k, p)\), which can itself be fed in to yield a still better approximation. This procedure is repeated until we have a stable value for \(A((i+1)\delta y, k, p)\). Generally convergence is reached after about three or four steps. One can also aid the convergence by making a better first approximation (e.g. by taking into account the first derivative of \(A\) with respect to \(y\)). In certain regions the form factor \(\Delta\) varies very rapidly with \(y\), requiring the use of specially adapted integration weights to obtain the correct answer. In cases where \(A\) varies with \(y\) in a known rapid manner, that information can also be used. The BFKL equation is solved in a similar manner.

Later in this article, for the purpose of obtaining precise information about \(\tilde{\chi}\), it will be necessary to go to extremely large rapidities — \(y \sim 100\), or equivalently \(x \sim 10^{-50}\)! At this point the \(y\)' integrations, because of their large extent, become very slow, and also require one to store information about \(A\) at rapidities all the way from 0 to \(y\). This quickly becomes prohibitive both in terms of computing time (which scales as \(y^2\)) and memory requirements (one should bear in mind that for each \(y\) point, we are storing a large 2-dimensional grid in \(\ln k\) and \(\ln p\), as discussed below). Fortunately the integrand is dominated by small values of \(y\)', because \(A(y - y')\) is a function which decreases exponentially with \(y'\) as does the form factor. This allows us to truncate the \(y'\) integration; a limit of \(y' < 4/\alpha_s\) is generally found to be adequate.

The other component of the problem is the \(d^2q\) integration. Given that we have stored \(A(y, k, p)\) on a grid in \(\ln k\) and \(\ln p\), the task is that of obtaining a discretised kernel \(K(i_k, i_{|k+q|}, i_q)\) such that

\[
\int_{q_{\text{min}}}^{q_{\text{max}}} \frac{d^2q}{\pi q^2} A(y, |k + q|, q) = \sum_{i_q = -i_{q,\text{min}}}^{i_{q,\text{max}}} \sum_{i_{|k+q|}} K(i_k, i_{|k+q|}, i_q) A(y, e^{i_{|k+q|}\delta \ell}, e^{i_q\delta \ell}) \tag{39}
\]

where \(k = e^{i_k \delta \ell}\) and \(\delta \ell\) is the logarithmic spacing between grid points in \(\ln k\) and \(\ln p\). The sum over \(i_{|k+q|}\) is the equivalent of the angular integral. The difficulty that arises is that in the region where \(|k + q|\) involves a significant cancellation, a small change in either \(k\) or \(q\) has a large effect on \(\ln |k + q|\) — so moving slightly away from a grid point defined by \(i_k, i_q\), the result of the angular integral changes radically. The solution is to think of \(i_k\), and \(i_q\) not as grid points, but as extended regions in \(k\) and \(q\) (and analogously for \(i_{|k+q|}\)), and when calculating the discretised kernel one must perform an average over the appropriate region. This is found to drastically reduce the discretisation errors.

The main limits on the method described here are due to memory requirements resulting from storing the gluon density on a three dimensional grid. Generally the grid resolution parameters \(\delta y = \delta \ell \sim 0.1\), together with \(q_{\text{min}} \simeq 10^{-6} k_0\) and \(q_{\text{max}} \simeq 10^6 k_0\). With these parameters one can determine \(A\) to an accuracy of better than 1%.

\(\delta y\) and \(\delta \ell\) are kept equal to simplify the treatment of the \(\Theta\)-function in \(\text{[38]}\).
As done in section 4.2, one can impose a dependence of the form \( k^{2(\gamma - 1)}G^{(0)}(y/p/k) \) on \( A^{(0)}(y/k/p) \). The \( k^{2(\gamma - 1)} \) dependence remains in the solution \( A \), so that it is no longer necessary to store the \( k \) dimension of \( A \). This allows one to go to smaller bin spacings, increasing the accuracy, which is necessary when attempting precision studies of the angular ordered characteristic function \( \tilde{\chi}(\gamma, \alpha_S) \). To perform the calculation at small \( \gamma \), one must take into account that the integral over the region of small \( |k + q| \) needs to go to extremely small \( |k + q| \ll e^{-1/(25)} \) (the integral is of the form \( \int dx x^{25-1} \)). In principle this would require a prohibitively large number of bins — but the problem has been resolved by calculating analytically the contribution from the region below the lowest stored bin.

### 3.2 Iterative method

We have solved the recurrence relations derived from (16), which in \( \omega \) space read:

\[
A^{(r+1)}(k,p) = \int \frac{d^2 q}{\pi q^2} \Gamma_\omega(k,p,q)A^{(r)}(|k + q|,q),
\]

where the kernel \( \Gamma \) is given by:

\[
\Gamma_\omega(k,p,q) = \int_0^1 dz z^\omega \frac{\bar{\alpha}_S}{z} \Delta(z,k,q) \theta(p - zq)
= \frac{\bar{\alpha}_S}{\omega} \left\{ \left[ \frac{q_1}{q} \right]^\omega - \left[ \frac{q_2}{q} \right]^\omega \right\} + \frac{\sqrt{\bar{\alpha}_S}}{\sqrt{\omega}} \text{erfc} \left[ \frac{\omega}{\sqrt{\bar{\alpha}_S}} + \sqrt{\bar{\alpha}_S} \log \frac{k}{q_2} \right] \exp \left[ \frac{\omega^2}{\bar{\alpha}_S} + \omega \log \frac{k}{p} + \bar{\alpha}_S \log^2 \frac{k}{q_3} \right],
\]

with \( q_1 = \min(p,q) \), \( q_2 = \min(k,p,q) \) and \( q_3 = \min(k,q) \). The corresponding equation for the BFKL distributions is

\[
F^{(r+1)}(k) = \Gamma_\omega(k) \int \frac{d^2 q}{\pi q^2} \theta(q - \mu)F^{(r)}(|k + q|),
\]

where

\[
\Gamma_\omega(k) = \int_0^1 dz z^\omega \frac{\bar{\alpha}_S}{z} \Delta^{(0)}(z,k) = \frac{\bar{\alpha}_S}{\omega} \frac{1}{1 + \frac{\alpha_s}{\omega} \log \frac{k^2}{\mu^2} \theta(k - \mu)}.
\]

The iterative method is not very efficient (in that it requires more computer memory and CPU time than other methods), but has the advantage that it closely mimics the physical branching process, thus giving a simple way to evaluate final state (exclusive) quantities, which will be examined in a future publication.

Our method is based on the truncated expansion of the various distributions on a suitable basis of Chebyshev polynomials (see also [7]). The expansion of the structure function (14) reads:

\[
A^{(r)}(k,p) \approx w(k,p) \sum_{n=1}^N \sum_{m=1}^M a^{(r)}_{nm} V^{(N)}_n[t(k)]V^{(M)}_m[u(p)],
\]

where \( w \) is a weight function, chosen to ensure convergence, the functions \( t \) and \( u \) map the variables \( k \) and \( p \) (which range from 0 to \( \infty \)) onto the interval \([-1,1]\). The efficiency of the method relies heavily on a suitable choice of the function \( t \) and \( u \). We have used the usual logarithmic mapping:

\[
t(k) = \left( \ln \frac{k_{\max}}{k_{\min}} \right)^{-1} \ln \frac{k}{\sqrt{k_{\min}k_{\max}}}, \quad u(p) = \left( \ln \frac{p_{\max}}{p_{\min}} \right)^{-1} \ln \frac{p}{\sqrt{p_{\min}p_{\max}}},
\]
where \( k_{\text{min}}, p_{\text{min}} \) and \( k_{\text{max}}, p_{\text{max}} \) are respectively the lower and upper cutoffs in the \( k \) and \( p \) variables.

The basis functions are defined by

\[
V_n^{(N)}(t) = \frac{2}{N} \left[ \sum_{i=1}^{N-1} T_i(t_n) T_i(t) + \frac{1}{2} \right],
\]

where \( T_i \) are Chebyshev polynomials, and are an orthonormal complete set on the points \( \{ t_n \} \),

\[
t_n = \cos \left( \frac{2N - 2n + 1}{2N} \pi \right), \quad n = 1, \ldots, N
\]

with the notable property that \( V_n^{(N)}(t_m) = \delta_{n,m} \).

With these definitions, the expansion coefficients are just the (rescaled) function values on the \( N \times M \) rectangular grid \( \{(k_n, p_m)\} \):

\[
a_{nm}^{(r)} = \frac{\mathcal{A}_\omega^{(r)}(k_n, p_m)}{w(k_n, p_m)}
\]

where \( t(k_n) = t_n, \ n = 1, \ldots, N \) and \( u(p_m) = u_m, \ m = 1, \ldots, M \).

Inserting expansion (44), the recursion (40) becomes

\[
a_{nm}^{(r+1)} = \sum_{n'=1}^{N} \sum_{m'=1}^{M} L_{nm,n'm'} a_{n'm'}^{(r)}
\]

where the matrix

\[
L_{nm,n'm'} = \int \frac{d^2 q}{\pi q^2} \frac{w(k_n + q, p_m)}{w(k_n, p_m)} \Gamma_\omega(k_n, p_m, q) V_{n'}^{(N)}[t(k_n + q)] V_{m'}^{(M)}[u(q)]
\]

must be evaluated only once, and then used to iterate the equation as many times as desired.

We have checked the stability and the convergence of this method by varying the functions \( w, t \) and \( u \), and the parameters \( N \) and \( M \). The method is weakly sensitive to the choice of \( w \), as long as the starting condition \( \mathcal{A}^{(0)}(k, p) \) is smooth, and numerical convergence is ensured for the integral appearing in (50). We have used \( w(k, p) = \mathcal{A}^{(0)}(k, p) \) for the \( \mathcal{A}_\omega \) distribution and \( w(k) = \sqrt{\frac{\mu^2}{k^2 + \mu^2}} \) for the \( \mathcal{F}_\omega \) case. The choice (45) allows us to obtain good stability and fast convergence with \( N \) and \( M \) of the order of \( 30 - 40 \).

The number of iterations needed to obtain a stable solution depends drastically on the value of \( \omega \). Far from the \( \omega \)-plane singularity (\( \omega \approx 1 \)), \( 20 - 30 \) iterations are sufficient to obtain an accurate solution over all the \( k \)-range (the iteration converges faster for smaller values of \( k \)). On the other hand, the required number of iterations increases dramatically as \( \omega \) approaches the critical value \( \omega_c \): with \( \omega - \omega_c \approx 0.01 \) we need about 5000 iterations to obtain a reliable result.
4 Numerical results

In this section we report the results obtained by solving (14) and (16) for the gluon distributions $A(x,k,p)$ and $A_\omega(k,p)$ respectively. We compare these results with those obtained from the BFKL equation.

4.1 Behaviour at small $x$

First we study $A(x,k,p)$ for a simple initial condition

$$A^{(0)}(x,k,p) = \delta(1-x) \frac{1}{k} \delta(k-k_0) \Theta(p-q_{\text{min}}),$$

(51)

where $k_0$ sets the momentum scale. As a collinear cutoff for the first emission we take $q_{\text{min}}$. This condition is not quite physical but is suitable for studying the general properties of the solution. We first show that $A(x,k,p)$ becomes independent of $p$ for $p \gg k$. Then we show that its behaviour for $x \to 0$ is less singular than that of the BFKL gluon distribution.

$p$-dependence. In fig. 2 we plot $A(x,k,p)$ as a function of $k$ for increasing values of $p$ and for fixed $x$ and $\alpha_S$. As $p$ increases the gluon distribution becomes independent of $p$. To show this we plot the ratio $A(x,k,p)/A(x,k,\bar{p})$ with $\bar{p}$ in the asymptotic region. As expected from the discussion in sect. 2 the limiting value is obtained first at low values of $k$.

$\omega$-plane singularity. In fig. 3 we plot the two distributions $A_\omega(k,\bar{p})$ and $F_\omega(k)$ as a function of $\omega$ at fixed $k = k_0$ and $\alpha_S$. The value of $\bar{p}$ in $A_\omega(k,\bar{p})$ is in the asymptotic region. From (24) one has that $F_\omega(k)$ diverges at the singular point $\omega_c = \bar{\alpha}_S \chi(1/2) = 4 \ln 2 \bar{\alpha}_S$, the minimum of the
Figure 3: $A_\omega(k, \bar{p})$ and $F_\omega(k)$ for $\alpha_s = 0.2$ and $\bar{p} = e^{15}k_0$. For the BFKL case the dotted line corresponds to the exact singularity at $\omega = \omega_c = 0.5295$. For the CCFM case the dotted line corresponds to a singularity at $\omega = \bar{\omega}_c = 0.4301$, as determined in section 4.2.

BFKL characteristic function at $\gamma = 1/2$. The gluon distribution $A_\omega(k, \bar{p})$ has a singularity at a value $\bar{\omega}_c \approx 0.4301$ smaller than $\omega_c \approx 0.5295$, the BFKL value.

For the BFKL case the numerical distribution actually tends to diverge at a value of $\omega$ which is $2 - 3\%$ smaller than $\omega_c$. This is due to the presence of lower and upper limits, $q_{\min}$ and $q_{\max}$, in the $q$ and $k$ grid used for the numerical calculation. It is known [9] that if the transverse momentum range is finite, the $\omega$-singularity is shifted to a lower value of the order $\omega_c \rightarrow \omega_c(1 - \pi^2/\ln^2 q_{\min}/q_{\max})$. Similar behaviour has also been noted [10] in the context of the dipole approach to small-$x$ physics [11]. For the values of $q_{\min}$ and $q_{\max}$ considered here one finds $\pi^2/\ln^2 q_{\min}/q_{\max} \approx 0.01$. The reason for this shift is that BFKL diffusion from the edges of the grid modifies the shape of the $k$-distribution which in turn leads to a reduction in the observed power. With angular ordering, the diffusion is reduced (this will be discussed also in section 4.2) and therefore the edges of the grid have a smaller effect.

Small-$x$ behaviour. Another way to obtain the position of the $\omega$-plane singularity consists in studying the small-$x$ behaviour of $A(x, k, \bar{p})$ and $F(x, k)$. In particular in fig. 4 we plot in the small-$x$ region

$$\frac{\partial}{\partial y} \ln x F(x, k) \approx \omega_c - \frac{1}{2y}, \quad y = \ln \frac{1}{x},$$

where $-1/2y$ is the first subasymptotic contribution. One expects that there might be a similar kind of behaviour in the case of $A(x, k, \bar{p})$, with $\omega_c$ replaced by $\bar{\omega}_c$. This is indeed seen and as before we find $\bar{\omega}_c < \omega_c$. The values for $\omega_c$ and $\bar{\omega}_c$ agree with those found in the previous analysis in fig. 3. In the BFKL case we have plotted the analytical result obtained for $q_{\min} \rightarrow 0$.

\[\text{The value quoted here is actually the one determined in section 4.2, which has a higher precision than that obtained by examining the position of the singularity of } A_\omega.\]
Figure 4: The effective power as a function of $1/y$; $\alpha_S = 0.2$ and $p = \bar{p} = 3 \times 10^6$.

and $q_{\text{max}} \to \infty$. With finite values of $q_{\text{min}}$ and $q_{\text{max}}$, increasing $y$, the width in $k$ of the solution increases until it is comparable to the extent of the finite grid, at which point the numerical curve flattens off. This is the same phenomenon that was noted previously for the shift of the singularity of $F_{\omega}$. For the angular ordering case, we have plotted an analytical line analogous to the BFKL one, with the asymptotic power $\tilde{\omega}_c = 0.43$ fitted to give agreement with the numerical results. The tailing off of the numerical curve at large $y$ is also observed in the angular ordering case, but it sets in later than for the BFKL curve — consistent with the idea that diffusion is reduced by angular ordering so that the width of the solution approaches that of the grid only at larger $y$.

We have also studied the $k$-dependence for large $y$. In the BFKL case the numerical result fits well the expected behaviour $\sim 1/kk_0$, corresponding to $\gamma = \gamma_c = 1/2$. With angular ordering we find $A(x, k, \bar{p}) \sim k^{2(\tilde{\gamma} - 1)}$ with $\tilde{\gamma} = \tilde{\gamma}_c \simeq 0.61$. We shall analyse this behaviour in more detail later.

### 4.2 Characteristic function

We now study the generalised characteristic function $\tilde{\chi}$ and the corresponding angular ordering function $G(p/k)$. Recall that for small $x$ the gluon distribution $A(x, k, p)$ has the asymptotic form

$$x A(x, k, p) \sim \frac{x^{\tilde{\omega}_c}}{k^2} \left( \frac{k^2}{k_0^2} \right)^{\tilde{\gamma} - 1} G(p/k), \quad \tilde{\omega}_c = \tilde{\alpha}_S \tilde{\chi}(\tilde{\gamma}, \tilde{\alpha}_S). \quad (53)$$
To obtain \( \tilde{\chi} \) and the corresponding angular ordering function \( G(p/k) \) as a function of \( \tilde{\gamma} \) and \( \alpha_S \) we use the following method. We solve equation (14) by using a trial initial condition

\[
A^{(0)}(x, k, p) = \frac{1}{k^2} \left( \frac{k^2}{k_S^2} \right)^{\tilde{\gamma}} \delta(1 - x) \Theta(p - k),
\]

with a fixed value of \( \tilde{\gamma} \) and \( \alpha_S \). From the discussion in section 2 one has that asymptotically for \( x \to 0 \), \( A(x, k, p) \) has the form (53) with the same \( \tilde{\gamma} \) as the inhomogeneous term (54). The \( x \)- and \( p \)-dependence of the initial condition is not important. Since the solution has the form of (53) we only need to deal with \( G \), which depends just on \( p/k \). This means that one doesn’t need to store the \( k \) dependence of the solution and the finite extent of the grid no longer has a significant effect, drastically reducing the errors.

**Characteristic function \( \tilde{\chi} \).** To obtain \( \tilde{\chi}(\tilde{\gamma}, \alpha_S) \) by using the initial condition (54) we compute \( A(x, k, p) \) with \( p \) in the asymptotic region. By taking the small-\( x \) limit we determine the intercept \( \tilde{\omega}_c \) with a high accuracy — the relative error is of the order of \( 10^{-4} \) for much of the \( \tilde{\gamma} \) region. From \( \tilde{\omega}_c \) we obtain the characteristic function

\[
\tilde{\chi}(\tilde{\gamma}, \alpha_S) = \frac{\tilde{\omega}_c}{\tilde{\alpha}_S},
\]

as a function of \( \tilde{\gamma} \) and \( \alpha_S \). In fig. 5a we plot \( \tilde{\chi} \) as a function of \( \tilde{\gamma} \) for various \( \alpha_S \). We plot for comparison the BFKL characteristic function \( \chi \). We see that \( \delta \chi = \chi - \tilde{\chi} \) is positive, increases with \( \tilde{\gamma} \), and increases with \( \alpha_S \). Moreover we find \( \delta \chi \sim \tilde{\gamma} \) for \( \tilde{\gamma} \to 0 \) (\( \tilde{\alpha}_S \) small and fixed) and \( \delta \chi \sim \tilde{\alpha}_S \) for \( \tilde{\alpha}_S \to 0 \) (\( \tilde{\gamma} \) small and fixed). This agrees with our earlier observation in section 2 and implies that the next-to-leading correction to the gluon anomalous dimension coming from angular ordering is of order \( \alpha_S^2/\omega^2 \). In fig. 5b we plot \( \delta \chi/\tilde{\alpha}_S \) which shows that there are notable next-to-next-to leading corrections especially at large \( \tilde{\gamma} \).

The symmetry for \( \gamma \to 1 - \gamma \) of the BFKL characteristic function is not valid for \( \tilde{\chi} \). Recall that this symmetry is based on the fact that the regions of small and large \( k \) are equally important. In the CCFM case, angular ordering favours instead the region of larger \( k \). However, small values are still accessible. Therefore the function \( \tilde{\chi} \) decreases faster than \( \chi \) for increasing \( \tilde{\gamma} \), but, after a minimum at a point \( \tilde{\gamma}_c \) larger than the BFKL value 1/2, \( \tilde{\chi} \) increases again. In fig. 6a and 6b we plot as a function of \( \alpha_S \) the values \( \tilde{\chi}_c \) and \( \tilde{\gamma}_c \) with \( \tilde{\chi}_c \) the minimum of \( \tilde{\chi} \) and \( \tilde{\gamma}_c \) its position. As expected the differences compared to the BFKL values \( \chi_c = 4 \ln 2 \) and \( \gamma_c = 1/2 \) are of order \( \tilde{\alpha}_S \). These results are consistent with the asymptotic solution in fig. 4. From our determination here, we obtain a very accurate estimate of the position of the singularity in \( \omega \): for \( \alpha_S = 0.2 \), we find \( \tilde{\omega}_c \approx 0.4301 \) (the corresponding BFKL value is \( \omega_c \approx 0.5295 \)) and \( \tilde{\gamma}_c \approx 0.6106 \).

Figure 6c shows the second derivative, \( \tilde{\chi}''_c \), of the characteristic function at its minimum; this quantity is important phenomenologically because the diffusion in \( \ln k \) is inversely proportional to the square root of \( \tilde{\chi}''_c \). From the figure, one can see therefore that the inclusion of angular ordering significantly reduces the diffusion compared to the BFKL case.

**Angular ordering function.** By using the trial initial condition (54) and taking the small-\( x \) limit for \( A(x, k, p) \) we compute the angular ordering function \( G(p/k) \) at the given value of \( \tilde{\gamma} \) and \( \tilde{\alpha}_S \) (see eq. (53)). In fig. 7 we plot \( G(p/k) \). The behaviours of \( G(p/k) \) for small and large \( p \) (see (32) and (34)) are as expected. In particular, we note that as \( \alpha_S \to 0 \) (fig. 7a) \( G(p/k) \) tends slowly to become 1 everywhere; as \( \tilde{\gamma} \to 0 \), \( G(p/k) \) tends towards the function \( \Theta(p - k) \).
Figure 5: (a) The characteristic functions with and without angular ordering; $\tilde{\chi}(\tilde{\gamma}, \alpha_s)$ and $\chi(\gamma)$ are plotted as functions of $\tilde{\gamma}$ and $\gamma$ respectively. (b) The difference, $\delta \chi = \chi(\tilde{\gamma}) - \tilde{\chi}(\tilde{\gamma}, \alpha_s)$, between the BFKL and angular ordered characteristic functions, divided by $\tilde{\alpha}_s$. 
Figure 6: (a) The value of the minimum of the characteristic function, $\tilde{\chi}_c$, as a function of $\alpha_S$. (b) The position of the minimum of the characteristic function, $\tilde{\gamma}_c$, as a function of $\alpha_S$. (c) The second derivative of the characteristic function, $\tilde{\chi}_c''$, at its minimum, as a function of $\alpha_S$. 
Figure 7: The angular ordering function $G(p/k)$: (a) for a range of $\alpha_s$, (b) for a range of $\tilde{\gamma}$. 
5 Conclusions

In this paper we have studied the contributions to the subleading corrections of the small-$x$ gluon density which are due to angular ordering. Since they are based on the physical property of QCD coherence, one expects that they are among the important corrections. Another important subleading contribution, that which fixes the scale of the running coupling \[6\], is not included in our study. The calculation has been done mostly by numerical methods, which prove to be quite reliable. In future papers they will be extended to the study of associated distributions \[12, 13, 14\] for which angular ordering is relevant already to leading level.

Our main results are summarised in figs. 5-7 in which we plot the generalised characteristic function $\tilde{\chi}(\tilde{\gamma}, \alpha_S)$ and the angular ordering function $G(p/k)$. From these plots we have studied the subleading corrections $\delta \chi(\tilde{\gamma}, \alpha_S) = \chi(\tilde{\gamma}) - \tilde{\chi}(\tilde{\gamma}, \alpha_S)$ and $\delta G(p/k) = G(\infty) - G(p/k)$. We find that $\tilde{\chi}(\tilde{\gamma}, \alpha_S)$ decreases with $\tilde{\gamma}$ faster than the BFKL characteristic function, it has a minimum at $\tilde{\gamma}_c$ which is larger than $\gamma_c = 1/2$, the BFKL critical point, and it rises again at larger $\tilde{\gamma}$.

The angular ordering function $G(p/k)$ has the structure of a typical form factor: it vanishes when the maximum available angle $\bar{\theta}$ vanishes, i.e. for $p \approx x E \bar{\theta}$ much smaller than $k$, the total emitted momentum.

The BFKL symmetry $\gamma \rightarrow 1 - \gamma$ is lost since conformal invariance is broken by angular ordering. The physical basis of conformal invariance is that in the BFKL equation the regions of small and large momentum are equally important. The coherent branching instead tends to evolve toward large momenta. However, at each intermediate branching, the region of vanishing momentum is still reachable for $x \rightarrow 0$. Within the angular ordering formulation, this effect has been discussed also in \[12\].

The fact that during the branching the intermediate momentum could vanish implies that the evolution contains non-perturbative components in an intrinsic way, not only in an initial boundary condition. It should be noted that in this non-perturbative region the distribution is non-singular (collinear singularities cancel), so that for any small but finite $x$ non-perturbative effects of the small-$k$ region are not too important. However they become very important for the asymptotic limit, $x \rightarrow 0$. As in the BFKL case the small-$k$ region generates a singularity in $\omega$ at $\tilde{\omega}_c > 0$ and an anomalous dimension $\tilde{\gamma}_c$ which is non-vanishing with $\alpha_S$ (see fig. 6b).

By expanding $\tilde{\chi}(\tilde{\gamma}, \alpha_S)$ in powers of $\alpha_S$ at fixed $\alpha_S/\omega$ one obtains the part due to angular ordering of the next-to-leading correction $\alpha_S \gamma_1(\alpha_S/\omega)$ (see \[3\]). We have not obtained its analytical form but we have shown that the angular ordering correction in the small-$x$ limit starts with a power not smaller than $\alpha_S^3/\omega^2$.

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