THE PSEUDOANALYTIC EXTENSIONS FOR SOME SPACES OF ANALYTIC FUNCTIONS

GUANLONG BAO, HASI WULAN AND FANGQIN YE

ABSTRACT. Using the Cauchy-Riemann operator, we characterize $Q_K$ spaces, Besov spaces and analytic Morrey spaces in terms of pseudoanalytic extensions of primitive functions. Our results are also true on some classical Banach spaces, such as the Bloch space, $BMOA$ and the Dirichlet space.

Keywords: the Cauchy-Riemann operator; pseudoanalytic extension; primitive functions; $Q_K$ spaces; Besov spaces; Morrey spaces.

1. INTRODUCTION

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$. Denote by $H(\mathbb{D})$ the space of functions analytic in $\mathbb{D}$. The Green function in the unit disk with singularity at $a \in \mathbb{D}$ is given by

$$g(a, z) = \log \frac{1}{|\sigma_a(z)|}, \quad z \in \mathbb{D}.$$ 

Here

$$\sigma_a(z) = \frac{a - z}{1 - \overline{a}z},$$

is a Möbius transformation of $\mathbb{D}$.

Throughout this paper, we assume that $K : [0, \infty) \to [0, \infty)$ is a right-continuous and increasing function. A function $f \in H(\mathbb{D})$ belongs to the space $Q_K$ if

$$\|f\|_{Q_K}^2 = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K(g(a, z)) \, dA(z) < \infty,$$

where $dA(z)$ is the Lebesgue measure in $\mathbb{D}$. By [14 Theorem 2.1], $f \in Q_K$ if and only if

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K(1 - |\sigma_a(z)|^2) \, dA(z) < \infty.$$ 

See [14] and [15] for more results of $Q_K$ spaces. If $K(t) = t^p$, $0 \leq p < \infty$, then the space $Q_K$ gives the space $Q_p$ (see [5 29 30]). In particular, $Q_0$ is the Dirichlet space $D$; $Q_1 = BMOA$, the space of bounded mean oscillation (see [6 17]); by [3], for all $p \in (1, \infty)$, the spaces $Q_p$ are the same and equal to the Bloch space $B$ which consists of all functions $f \in H(\mathbb{D})$ with

$$\|f\|_B = \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)| < \infty.$$
For $0 < p < \infty$, the Hardy space $H^p$ consists of all functions $f \in H(\mathbb{D})$ with

$$
\|f\|_{H^p}^p = \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.
$$

It is well-known that if $f \in H^p$, then its nontangential limit $f(e^{i\theta})$ exists almost everywhere (see [9,16]).

In this paper, we need the Cauchy-Riemann operator

$$
\partial = \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \quad z = x + iy.
$$

Dyn’kin [13] said that the so-called pseudoanalytic extension is a method of extension with a given estimate of the Cauchy-Riemann operator. Dyn’kin [13] characterized many classical smoothness spaces applying the pseudoanalytic extension method. Note that

$$
\bigcup_{0 < p < 1} Q_p \subseteq BMOA \subseteq \bigcap_{0 < q < \infty} H^q.
$$

Dyakonov and Girela [12] obtained an interesting characterization of $Q_p$ spaces in terms of pseudoanalytic extension as follows.

**Theorem A.** If $0 < p < 1$ and $f \in \bigcap_{0 < q < \infty} H^q$, then the following conditions are equivalent.

(i) $f \in Q_p$.

(ii) 

$$
\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 \left( \frac{1}{|\sigma_a(z)|^2} - 1 \right)^p dA(z) < \infty.
$$

(iii) There exists a function $\tilde{f} \in C^1(\mathbb{C} \setminus \mathbb{D})$ satisfying

$$
\tilde{f}(z) = O(1), \text{ as } z \to \infty,
$$

$$
\lim_{r \to 1^+} \tilde{f}(re^{i\theta}) = f(e^{i\theta}), \text{ a.e. and in } L^q([-\pi, \pi]) \text{ for all } q \in [1, \infty),
$$

and

$$
\sup_{a \in \mathbb{D}} \int_{\mathbb{C} \setminus \mathbb{D}} |\partial \tilde{f}(z)|^2 \left( |\sigma_a(z)|^2 - 1 \right)^p dA(z) < \infty.
$$

The proof of the above theorem in [12] involved the Calderón-Zygmund operators and the Muckenhoupt weights (see [19]). For $a \in \mathbb{D}$ and $0 < p < 1$, let

$$
U_a(z) = \left| 1 - \frac{1}{|\sigma_a(z)|^2} \right|^p = \frac{(1 - |a|^2)^p|z|^2 - 1|^p}{|z - a|^{2p}}, \quad z \in \mathbb{C}.
$$

Dyakonov and Girela [12] showed that $U_a$ is a Muckenhoupt weight with

$$
\sup_Q \left[ \frac{1}{|Q|} \int_Q U_a(z) dA(z) \right] \left[ \frac{1}{|Q|} \int_Q (U_a(z))^{-1} dA(z) \right] < \infty
$$

for all $a \in \mathbb{D}$. Here $Q$ ranges over the disks in $\mathbb{C}$ and $|Q|$ denotes the area of $Q$. The above estimate of $U_a$ with $a = 0$ was also used to establish the $Q_p$ corona theorem (see [28, Theorem 3.1]). Applying Theorem A, Dyakonov and Girela [12] also gave some nice properties of $Q_p$ spaces.
Motivated by Theorem A, it is nature to ask some questions as follows.

**Question 1.** Can we obtain pseudoanalytic extensions for all $Q_p$, $0 \leq p < \infty$?

**Question 2.** Let $n$ be a positive integer. Does there exist a pseudoanalytic extension characterization of $Q_p$ spaces such that the extension function is $O(z^n)$ as $z \to \infty$?

**Question 3.** Can we give characterizations for some spaces of analytic functions such as $Q_K$ spaces, Besov spaces and analytic Morrey spaces in terms of pseudoanalytic extensions?

Throughout this article, denote by $F$ the primitive function of $f \in H(D)$; that is

$$F(z) = \int_0^z f(w)dw, \quad z \in \mathbb{D}.$$ 

Related to the above questions, in this paper, we characterize some analytic function spaces on $\mathbb{D}$ by pseudoanalytic extension of primitive functions. Our method, without using Calderón-Zygmund operators and Muckenhoupt weights, can be applied to some spaces, such as $Q_K$ spaces, Besov spaces, analytic Morrey spaces and so on. Our results are also new for $Q_p$ spaces.

In this paper, the symbol $A \approx B$ means that $A \lesssim B \lesssim A$. We say that $A \lesssim B$ if there exists a constant $C$ such that $A \leq CB$.

## 2. $Q_K$ Spaces and Pseudoanalytic Extension

In this section, we need two more constraints on $K$ as follows.

$$\int_0^1 \frac{\varphi_K(s)}{s} ds < \infty \quad (2.1)$$

and

$$\int_1^\infty \frac{\varphi_K(s)}{s^{1+q}} ds < \infty, \quad 0 < q < 2, \quad (2.2)$$

where

$$\varphi_K(s) = \sup_{0 \leq t \leq 1} \frac{K(st)}{K(t)}, \quad 0 < s < \infty.$$ 

Under conditions (2.1) and (2.2), $Q_K$ spaces have been studied extensively (see [24, 25, 26]). From now on we always assume the function $K$ satisfying the double condition, namely $K(2t) \approx K(t)$ for all $t \in (0, 1)$.

A very useful tool in the study of $Q_K$ spaces is $K$-Carleson measure. Let $\ell(I)$ be the length of an arc $I$ of the unit circle $\partial \mathbb{D}$. Define the Carleson box by

$$S_G(I) = \begin{cases} \{r\zeta \in G : 1 - \frac{\ell(I)}{2\pi} < r < 1, \zeta \in I\}, & G = \mathbb{D}, \\ \{r\zeta \in G : 1 < r < 1 + \ell(I), \zeta \in I\}, & G = \mathbb{C} \setminus \overline{\mathbb{D}}. \end{cases}$$

Following [15] and [24], a positive Borel measure $\mu$ on $G = \mathbb{D}$ or $G = \mathbb{C} \setminus \overline{\mathbb{D}}$ is said to be a $K$-Carleson measure if

$$\sup_{I \subset \partial \mathbb{D}} \int_{S_G(I)} K\left(\frac{|1 - |z||}{\ell(I)}\right) d\mu(z) < \infty.$$ 

By [14], we know that all $Q_K$ spaces are subsets of the Bloch space. Then the primitive function $F$ of a $Q_K$ function $f$ must be in the Hardy space $H^2$ since the Bloch functions’ Taylor
coefficients is bounded (see [1]). Therefore, the primitive function $F$ has its nontangential limit $F(e^{i\theta})$ almost everywhere on the unit circle.

The following is the main result of this section.

**Theorem 2.1.** Suppose that $K$ satisfies (2.1) and (2.2). Let $f \in H(D)$ with its primitive function $F \in H^2$ and let $n \geq 2$ be an integer. Then the following conditions are equivalent.

(i) $f \in Q_K$.

(ii) There exists a function $\widetilde{F}_n \in C^1(C \setminus D)$ satisfying

$$\lim_{r \to 1^+} \widetilde{F}_n(re^{i\theta}) = F(e^{i\theta}) \text{ a.e. } \theta \in [0, 2\pi],$$

$$\widetilde{F}_n(z) = O(z^n), \text{ as } z \to \infty,$$

$$\partial \widetilde{F}_n(z) = O(z^{n-2}), \text{ as } z \to \infty,$$

and

$$\sup_{a \in D} \int_{C \setminus D} \frac{|\partial \widetilde{F}_n(z)|^2}{(|z|^n - 1)^2} K \left(1 - \frac{1}{|\sigma_a(z)|^2}\right) dA(z) < \infty. \quad (2.6)$$

(iii) There exists a function $\widetilde{F}_n \in C^1(C \setminus D)$ satisfying (2.3), (2.4), (2.5) and $|\partial \widetilde{F}_n(z)|^2 / (|z|^n - 1)^2$ is a $K$-Carleson measure on $C \setminus D$.

Before embarking into the proof of Theorem 2.1, we state some lemmas below.

**Lemma 2.2.** Suppose that $K$ satisfies (2.1). Let $\mu$ be a positive Borel measure on $D$. Then the following conditions are equivalent.

(i) $\mu$ is a $K$-Carleson measure on $D$.

(ii) $$\sup_{a \in D} \int_D K(1 - |\sigma_a(z)|^2) d\mu(z) < \infty.$$  

(iii) $$\sup_{a \in D} \int_D K \left(\frac{1}{|\sigma_a(z)|^2} - 1\right) d\mu(z) < \infty.$$  

**Proof.** Note that (i) $\Leftrightarrow$ (ii) was proved in [15]. To show the equivalence of (ii) and (iii), let

$$K_1(t) = K \left(\frac{t}{1-t}\right), \text{ for } 0 < t < \frac{1}{2}.$$  

By the monotonicity of $K$, $K_1(s)$ is also increasing for $s \in (0, 1)$. Moreover,

$$K(t) \leq K_1(t) \leq K(2t), \text{ for } 0 < t < \frac{1}{2}.$$  

Since $K(2t) \approx K(t)$ for $t \in (0, 1)$, $K_1(t) \approx K(t)$ for $t \in (0, 1/2)$. Thus, the proof is complete.  

□
Wulan and Zhu [27] proved that if $K$ satisfies (2.1) and $n$ is a positive integer, then $f \in Q_K$ if and only if

$$
\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f^{(n)}(z)|^2 (1 - |z|^2)^{2n-2} K \left( 1 - |\sigma_a(z)|^2 \right) dA(z) < \infty.
$$

(2.7)

Combining this with Lemma 2.2, we obtain immediately a characterization of $Q_K$ spaces as follows. If $K$ satisfies (2.1) and $n$ is a positive integer, then $f \in Q_K$ if and only if

$$
\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f^{(n)}(z)|^2 (1 - |z|^2)^{2n-2} K \left( \frac{1}{|\sigma_a(z)|^2} - 1 \right) dA(z) < \infty.
$$

For the case of $\mu$ on $\mathbb{C} \setminus \overline{D}$, we give a similar description of $K$-Carleson measure as follows.

**Lemma 2.3.** Suppose that $K$ satisfies (2.1). Let $\mu$ be a positive Borel measure on $\mathbb{C} \setminus \overline{D}$. Then the following conditions are equivalent.

(i) $\mu$ is a $K$-Carleson measure on $\mathbb{C} \setminus \overline{D}$.

(ii) $\sup_{a \in \mathbb{D}} \int_{1<|z|<1+2\pi} K \left( |\sigma_a(z)|^2 - 1 \right) d\mu(z) < \infty$.

(iii) $\sup_{a \in \mathbb{D}} \int_{1<|z|<1+2\pi} K \left( 1 - \frac{1}{|\sigma_a(z)|^2} \right) d\mu(z) < \infty$.

**Proof.** $(i) \Rightarrow (ii)$. Fix $a = re^{i\theta} \in \mathbb{D}$. Let $I$ be the arc of center $e^{i\theta}$ and $\ell(I) = \frac{2\pi(1-|a|)}{2\pi|a|+1}$. Then for any $z \in S_{\mathbb{C} \setminus \overline{D}}(I)$, one gets

$$
|1 - \bar{a}z| \geq 1 - (1 + \ell(I))|a| = \frac{1}{2\pi} \ell(I)
$$

and

$$
|1 - \bar{a}z| \leq |a| \left| e^{i(\theta + \frac{\ell(I)}{2\pi})} - \frac{1}{\bar{a}} \right| = r \left| e^{i\frac{\ell(I)}{2\pi}} - \frac{1}{r} \right| = \left[ (1 - r)^2 + 4r \left( \sin \frac{\ell(I)}{4} \right)^2 \right]^{\frac{1}{2}}
$$

$$
\leq \left[ \left( \frac{2\pi r + 1}{2\pi} \ell(I) \right)^2 + 4r \left( \frac{\ell(I)}{4} \right)^2 \right]^{\frac{1}{2}} \leq 2\ell(I).
$$

Hence

$$
|1 - \bar{a}z| \approx 1 - |a| \approx \ell(I), \ z \in S_{\mathbb{C} \setminus \overline{D}}(I).
$$

Set

$$
S_n = \{ r\zeta \in \mathbb{C} \setminus \overline{D} : 1 < r < 1 + \ell(2^n I), \zeta \in 2^n I \}.
$$
Then for any
\[
\int_{1<|z|<1+2\pi} K \left( |\sigma_a(z)|^2 - 1 \right) d\mu(z)
\]
holds for all
\[
\int_{S_{c\backslash D}(t)} K \left( |\sigma_a(z)|^2 - 1 \right) d\mu(z) + \sum_{n=1}^{\infty} \int_{S_n \setminus S_{n-1}} K \left( |\sigma_a(z)|^2 - 1 \right) d\mu(z)
\]
\[
\leq \int_{S_{c\backslash D}(t)} K \left( \left|\frac{z}{\ell(I)} - 1 \right| \right) d\mu(z) + \sum_{n=1}^{\infty} \int_{S_n \setminus S_{n-1}} K \left( \left|\frac{z}{2^{2n}\ell(I)} - 1 \right| \right) d\mu(z)
\]
\[
\leq \int_{S_{c\backslash D}(t)} K \left( \left|\frac{z}{\ell(I)} - 1 \right| \right) d\mu(z) + \sum_{n=1}^{\infty} \sup_{z \in S_n} K \left( \frac{\left|z\right| - 1}{2^{2n}\ell(I)} \right) \int_{S_n} K \left( \left|\frac{z}{\ell(2^nI)} - 1 \right| \right) d\mu(z)
\]
\[
\leq \int_{S_{c\backslash D}(t)} K \left( \left|\frac{z}{\ell(I)} - 1 \right| \right) d\mu(z) + \sum_{n=1}^{\infty} \varphi_K \left( 2^{-n} \right) \int_{S_n} K \left( \left|\frac{z}{\ell(2^nI)} - 1 \right| \right) d\mu(z).
\]
Since \(d\mu\) is a \(K\)-Carleson measure and
\[
\sum_{n=1}^{\infty} \varphi_K \left( 2^{-n} \right) \approx \int_{0}^{1} \varphi_K(s) \frac{ds}{s} < \infty,
\]
we obtain that
\[
\sup_{a \in \mathbb{D}} \int_{1<|z|<1+2\pi} K \left( |\sigma_a(z)|^2 - 1 \right) d\mu(z) < \infty.
\]

\((ii) \Rightarrow (iii)\). It follows from that
\[
K \left( 1 - \frac{1}{|\sigma_a(z)|^2} \right) \leq K \left( |\sigma_a(z)|^2 - 1 \right)
\]
holds for all \(a \in \mathbb{D}\) and \(1 < |z| < 1 + 2\pi\).

\((iii) \Rightarrow (i)\). For given a subarc \(I \in \partial \mathbb{D}\), set \(e^{i\theta}\) the midpoint of \(I\) and set
\[
a = \frac{2\pi - \ell(I)}{2\pi(\ell(I) + 1)} e^{i\theta}.
\]
Then for any \(z \in S_{c\backslash D}(I)\), we have
\[
0 < |\sigma_a(z)|^2 - 1 \leq \frac{\left( 2\pi a \right) + 1 \left( 2 + \ell(I) \right) \left( \ell(I) \right)^2}{\left( 1 + \ell(I) \right)^2} \leq 8\pi(2\pi + 1)(\pi + 1).
\]
Let
\[
K_2(t) = K \left( \frac{t}{1 + t} \right), \quad 0 < t < 8\pi(2\pi + 1)(\pi + 1).
\]
By the monotonicity of \(K\), \(K_2(s)\) is also increasing for \(s \in (0, 1)\). Moreover,
\[
K \left( \frac{t}{8\pi(2\pi + 1)(\pi + 1) + 1} \right) \leq K_2(t) \leq K(t), \quad 0 < t < 8\pi(2\pi + 1)(\pi + 1).
\]
Since $K(2t) \approx K(t)$ for $t \in (0, 1)$, $K_2(t) \approx K(t)$ for $t \in (0, 8\pi(2\pi + 1)(\pi + 1))$. Thus,
\[
\int_{S \cap \mathbb{C}(I)} K \left( \frac{|z| - 1}{\ell(I)} \right) d\mu(z) \approx \int_{S \cap \mathbb{C}(I)} K \left( |\sigma_a(z)|^2 - 1 \right) d\mu(z)
\]
\[
\approx \int_{S \cap \mathbb{C}(I)} K \left( \frac{|\sigma_a(z)|^2 - 1}{|\sigma_a(z)|^2} \right) d\mu(z)
\]
\[
\lesssim \int_{1<|z|<1+2\pi} K \left( 1 - \frac{1}{|\sigma_a(z)|^2} \right) d\mu(z).
\]
Hence, (i) follows. \qed

From [15] and [25], if $K$ satisfies (2.1) and (2.2), then there exists a small enough constant $c$, $0 < c < q/2$, such that $K(t)/t^c$ is increasing and $K(t)/t^{q-c}$ is decreasing in $(0, 1)$. To prove Theorem 2.1, we also need the following estimate.

**Lemma 2.4.** Let (2.1) and (2.2) hold for $K$. If $s < \min(1 + c, 2 - q + c)$ and $2s + r - 4 \geq 0$, then
\[
\int_{\mathbb{D}} \frac{K \left( 1 - |\sigma_a(w)|^2 \right)}{(1 - |w|^2)^s |1 - \overline{w}z|^r} dA(w) \lesssim \frac{K \left( 1 - |\sigma_a(z)|^2 \right)}{(1 - |z|^2)^{s+r-2}}
\]
for all $a, z \in \mathbb{D}$. Here $c$ is a small enough positive constant depending only on (2.1) and (2.2).

**Proof.** For fixed $a, z \in \mathbb{D}$, let $\lambda = \sigma_z(a)$. Then
\[
|\sigma_a(w)| = |\sigma_\lambda \circ \sigma_z(w)|.
\]
Note that $2s + r - 4 \geq 0$. Checking the proof of Lemma 2.1 in [7], one gets
\[
\int_{\mathbb{D}} \frac{K \left( 1 - |\sigma_a(w)|^2 \right)}{(1 - |w|^2)^s |1 - \overline{w}z|^r} dA(w)
\]
\[
\lesssim \frac{K \left( 1 - |\lambda|^2 \right)}{(1 - |z|^2)^{s+r-2}} \int_{\mathbb{D}} \varphi_K \left( \frac{1 - |u|^2}{|1 - \lambda u|^2} \right) \frac{1}{(1 - |u|^2)^s} dA(u).
\]
Since $K$ satisfies (2.1) and (2.2), there exists a small enough positive constant $c$ depending only on (2.1) and (2.2) (see [15] [25]), such that
\[
\varphi_K(t) \lesssim t^c, \quad 0 < t \leq 1
\]
and
\[
\varphi_K(t) \lesssim t^{q-c}, \quad t \geq 1.
\]
Thus,
\[
\int_{\mathbb{D}} \varphi_K \left( \frac{1 - |u|^2}{|1 - \lambda u|^2} \right) \frac{1}{(1 - |u|^2)^s} dA(u)
\]
\[
\lesssim \int_{\mathbb{D}} \frac{(1 - |u|^2)^c}{|1 - \lambda u|^{2c}} dA(u) + \int_{\mathbb{D}} \frac{(1 - |u|^2)^{q-c-s}}{|1 - \lambda u|^{2q-2c}} dA(u).
\]
For $s < 1 + c < q + 1 - c$ and $s < 2 - q + c$, using Lemma 3.10 in [35], we get
\[
\int_{\mathbb{D}} \frac{(1 - |u|^2)^c}{|1 - \lambda u|^{2c}} dA(u) + \int_{\mathbb{D}} \frac{(1 - |u|^2)^{q-c-s}}{|1 - \lambda u|^{2q-2c}} dA(u) \lesssim 1.
\]
Thus,
\[
\int_D \frac{K (1 - |\sigma_a(w)|^2)^r}{(1 - |w|^2)^s |1 - \overline{w}z|^r} dA(w) \lesssim \frac{K (1 - |\lambda|^2)^r}{(1 - |z|^2)^{s+r-2}} \approx \frac{K (1 - |\sigma_a(z)|^2)^r}{(1 - |z|^2)^{s+r-2}}.
\]

Now we are ready to finish the proof of Theorem 2.1.

(i) \Rightarrow (ii). From now on, we write
\[ w^* = 1/\overline{w}, \ w \in \mathbb{C} \setminus \{0\}. \]

Let \( f \in \mathcal{Q}_K \) and
\[ F(z) = \int_0^z f(\zeta) d\zeta, \ z \in \mathbb{D}. \]

Set
\[ \widetilde{F}_n(z) = \sum_{i=0}^n \frac{(-1)^i}{i!} (z^* - z)^i F(i)(z^*), \ z \in \mathbb{C} \setminus \overline{\mathbb{D}}. \]

Clearly, \( \widetilde{F}_n \in C^1(\mathbb{C} \setminus \overline{\mathbb{D}}) \) satisfying
\[ \lim_{r \to 1^+} \widetilde{F}_n(re^{i\theta}) = F(e^{i\theta}) \ \text{a.e.} \ \theta \in [0, 2\pi], \]

and
\[ \widetilde{F}_n(z) = O(z^n), \ \text{as} \ z \to \infty. \]

Note that
\[ \overline{\partial} \widetilde{F}_n(z) = \frac{(-1)^{n+1}}{n!} (z^* - z)^n (z^*)^2 F^{(n+1)}(z^*). \]

Then
\[ \overline{\partial} \widetilde{F}_n(z) = O(z^{n-2}), \ \text{as} \ z \to \infty. \]

Making the change of variable \( z = \zeta^* \), we deduce that
\[
\sup_{a \in \mathbb{D}} \int_{\mathbb{C} \setminus \overline{\mathbb{D}}} \frac{|\overline{\partial} \widetilde{F}_n(z)|^2}{(|z|^n - 1)^2} K \left( 1 - \frac{1}{|\sigma_a(z)|^2} \right) dA(z) \approx \frac{1}{(n!)^2} \sup_{a \in \mathbb{D}} \int_{\mathbb{C} \setminus \overline{\mathbb{D}}} \frac{|z^* - z|^2 |z^*|^4 |f^{(n)}(z^*)|^2}{(|z|^n - 1)^2} K \left( 1 - \frac{1}{|\sigma_a(z)|^2} \right) dA(z).
\]

Combining this with (2.7), we get the desired result.

(ii) \Rightarrow (i). Assume that (ii) holds. Let \( z \in \mathbb{D} \) and \( R > 1 \). Following a technique in [10] or [12] and in view of (2.3), we employ the Cauchy-Green formula to the function that equals \( F \) in \( \mathbb{D} \) and \( \widetilde{F}_n \) in \( \mathbb{C} \setminus \overline{\mathbb{D}} \). Then
\[
F(z) = \frac{1}{2\pi i} \int_{|\zeta| = 1} \frac{F(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{|\zeta| = R} \frac{\widetilde{F}_n(\zeta)}{\zeta - z} d\zeta - \frac{1}{\pi} \int_{1 < |\zeta| < R} \overline{\partial} \widetilde{F}_n(\zeta) \zeta - z \]

(2.8)
By (2.4),

\[
\int_{|\zeta|=R} \frac{\widetilde{F}_n(\zeta)}{(\zeta - z)^{n+2}} d\zeta \to 0, \quad \text{as } R \to \infty.
\]

By (2.5), we know

\[
\left| \int_{\mathbb{C}\setminus \mathbb{B}} \frac{\partial \widetilde{F}_n(\zeta)}{(\zeta - z)^{n+2}} dA(\zeta) \right| < \infty.
\]

These together with (2.8) give

\[
F^{(n+1)}(z) = -\frac{(n+1)!}{\pi} \int_{\mathbb{C}\setminus \mathbb{B}} \frac{\partial \widetilde{F}_n(\zeta)}{(\zeta - z)^{n+2}} dA(\zeta).
\]

By the Hölder inequality, the change of variable and the well known estimates in Zhu’s book [35, Lemma 3.10], we obtain

\[
|F^{(n+1)}(z)|^2 \lesssim \int_{\mathbb{C}\setminus \mathbb{B}} \frac{1}{|\zeta - z|^4} dA(\zeta) \int_{\mathbb{C}\setminus \mathbb{B}} \frac{|\partial \widetilde{F}_n(\zeta)|^2}{|\zeta - z|^{2n}} dA(\zeta)
\]

\[
\approx \int_D \frac{1}{|1-wz|^4} dA(w) \int_{\mathbb{C}\setminus \mathbb{B}} \frac{|\partial \widetilde{F}_n(\zeta)|^2}{|\zeta - z|^{2n}} dA(\zeta)
\]

\[
\approx \frac{1}{(1 - |z|^2)^2} \int_{\mathbb{C}\setminus \mathbb{B}} \frac{|\partial \widetilde{F}_n(\zeta)|^2}{|\zeta - z|^{2n}} dA(\zeta).
\]

Using Lemma 2.4, \(n \geq 2\) and (2.6), we see that

\[
\sup_{a \in \mathbb{D}} \int_D (1 - |z|^2)^{2n-2} |F^{(n+1)}(z)|^2 K (1 - |\sigma_a(z)|^2) dA(z)
\]

\[
\lesssim \sup_{a \in \mathbb{D}} \int_{\mathbb{C}\setminus \mathbb{B}} \frac{|\partial \widetilde{F}_n(\zeta)|^2}{|\zeta - z|^{2n}} dA(\zeta) (1 - |z|^2)^{2n-4} K (1 - |\sigma_a(z)|^2) dA(z)
\]

\[
\lesssim \int_D K (1 - |\sigma_a(z)|^2) \frac{dA(z)}{|1-wz|^4} |\partial \widetilde{F}_n(w^*)|^2 |w|^{2n-4} dA(w)
\]

\[
\lesssim \int_D \frac{|\partial \widetilde{F}_n(w^*)|^2 |w|^{2n-4}}{(1 - |w|^2)^2} K (1 - |\sigma_a(w)|^2) dA(w)
\]

\[
\approx \frac{1}{(1 - |w|^2)^2} \int_{\mathbb{C}\setminus \mathbb{B}} \frac{|\partial \widetilde{F}_n(\zeta)|^2}{(|\zeta|^n - 1)^2} K \left( 1 - \frac{1}{|\sigma_a(\zeta)|^2} \right) dA(\zeta) < \infty.
\]

Note that \(F'(z) = f(z)\) for \(z \in \mathbb{D}\). By (2.7), we get \(f \in Q_K\).

\((ii) \iff (iii)\). If \(\partial \widetilde{F}_n(z) = O(z^{-2}), \text{ as } z \to \infty,\)

then

\[
\sup_{a \in \mathbb{D}} \int_{|z| \geq n+2\pi} \frac{|\partial \widetilde{F}_n(z)|^2}{(|z|^n - 1)^2} K \left( 1 - \frac{1}{|\sigma_a(z)|^2} \right) dA(z) < \infty.
\]

This together with Lemma 2.3, we see that condition \((ii)\) is equivalent to condition \((iii)\). We finish the proof. \(\square\)
Remark. In Theorem 2.1, by the particular choice of \( K \), we obtain the corresponding characterizations of \( Q_p \) for \( 0 < p < 1 \), \( BMOA \) and the Bloch space.

3. BESOV SPACES, MORREY SPACES AND PSEUDOANALYTIC EXTENSION

In this section, we describe Besov spaces and Morrey spaces by pseudoanalytic extension of primitive functions.

For \( 1 < p < \infty \), the Besov space \( B_p \) is the space of analytic functions \( f \) in \( \mathbb{D} \) such that

\[
\| f \|_{B_p}^p = \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-2} dA(z) < \infty. \tag{3.1}
\]

In particular, if \( p = 2 \), then \( B_2 \) is the Dirichlet space \( D \). Note that (3.1) does not hold for \( p = 1 \).

The Besov space \( B_1 \) consists of all analytic functions \( f \) on \( \mathbb{D} \) which have a representation as

\[
f(z) = \sum_{k=1}^{\infty} c_k \sigma_{a_k}(z), \quad a_k \in \mathbb{D} \quad \text{and} \quad \sum_{k=1}^{\infty} |c_k| < \infty.
\]

For \([2]\), \( f \in B_1 \) if and only if

\[
\int_{\mathbb{D}} |f''(z)| dA(z) < \infty.
\]

Let \( 1 \leq p < \infty \) and \( n \geq 2 \) be an integer. It is well known that \( f \in B_p \) if and only if

\[
\int_{\mathbb{D}} |f^{(n)}(z)|^p (1 - |z|^2)^{np-2} dA(z) < \infty. \tag{3.2}
\]

See \([11, 34, 35]\) for more results of Besov spaces.

Note that all Besov spaces are subsets of the Bloch space. Thus, if \( f \in B_p \), then its primitive function \( F \) belongs to the Hardy space \( H^2 \).

Now we give a pseudoanalytic extension characterization of Besov spaces as follows.

**Theorem 3.1.** Let \( f \in H(\mathbb{D}) \) with its primitive function \( F \in H^2 \) and let \( n \geq 2 \) be an integer. If \( 1 \leq p < \infty \), then the following conditions are equivalent.

(i) \( f \in B_p \).

(ii) There exists a function \( \tilde{F}_n \in C^1(\mathbb{C} \setminus \mathbb{D}) \) satisfying

\[
\lim_{r \to 1^+} \tilde{F}_n(re^{i\theta}) = F(e^{i\theta}) \quad \text{a.e.} \quad \theta \in [0, 2\pi], \tag{3.3}
\]

\[
\widetilde{F}_n(z) = O(z^n), \quad \text{as} \quad z \to \infty, \tag{3.4}
\]

\[
\overline{\partial} \tilde{F}_n(z) = O(z^{n-2}), \quad \text{as} \quad z \to \infty, \tag{3.5}
\]

and

\[
\int_{\mathbb{C} \setminus \mathbb{D}} \frac{|\overline{\partial} \tilde{F}_n(z)|^p}{(|z|^{np/2} - p^2 - 1)^2} dA(z) < \infty. \tag{3.6}
\]
Proof. (i) \Rightarrow (ii). Let \( f \in B_p \). Set

\[
\tilde{F}_n(z) = \sum_{i=0}^{n} \frac{(-1)^i}{i!} (z^* - z)^i F^{(i)}(z^*), \quad z \in \mathbb{C} \setminus \mathbb{D}.
\]

It is easy to check that \( \tilde{F}_n \in C^1(\mathbb{C} \setminus \mathbb{D}) \) and \( \tilde{F}_n \) satisfies (3.3), (3.4) and (3.5). Furthermore, by the following estimates

\[
\int_{\mathbb{C} \setminus \mathbb{D}} \frac{|\partial \tilde{F}_n(z)|^p}{(|z|^{\frac{np}{2}-p+2} - 1)^2} dA(z)
= \frac{1}{(n!)^p} \int_{\mathbb{C} \setminus \mathbb{D}} \frac{|z^* - z|^{np} |z^*|^{2p} |f^{(n)}(z^*)|^p}{(|z|^{\frac{np}{2}-p+2} - 1)^2} dA(z)
\approx \int_{\mathbb{D}} (1 - |w|)^{np/2} |f^{(n)}(w)|^p dA(w)
\approx \int_{\mathbb{D}} (1 - |w|)^{np-2} |f^{(n)}(w)|^p dA(w)
\]

and (3.2), we know that \( \tilde{F}_n \) also satisfies (3.6).

(ii) \Rightarrow (i). Checking the proof of Theorem 2.1, one gets

\[
F^{(n+1)}(z) = -\frac{(n+1)!}{\pi} \int_{\mathbb{C} \setminus \mathbb{D}} \frac{\partial \tilde{F}_n(\zeta)}{(\zeta - z)^{n+2}} dA(\zeta), \quad z \in \mathbb{D}.
\]

Applying the Hölder inequality and [35] Lemma 3.10, we obtain

\[
|F^{(n+1)}(z)|^p \lesssim \left( \int_{\mathbb{C} \setminus \mathbb{D}} \frac{1}{|\zeta - z|^4} dA(\zeta) \right)^{p-1} \int_{\mathbb{C} \setminus \mathbb{D}} \frac{|\partial \tilde{F}_n(\zeta)|^p}{|\zeta - z|^{np-2p+4}} dA(\zeta)
\lesssim \left( \int_{\mathbb{D}} \frac{1}{|1 - \bar{w}z|^4} dA(w) \right)^{p-1} \int_{\mathbb{C} \setminus \mathbb{D}} \frac{|\partial \tilde{F}_n(\zeta)|^p}{|\zeta - z|^{np-2p+4}} dA(\zeta)
\approx \frac{1}{(1 - |z|^{2p-2}} \int_{\mathbb{C} \setminus \mathbb{D}} \frac{|\partial \tilde{F}_n(\zeta)|^p}{|\zeta - z|^{np-2p+4}} dA(\zeta).
\]
Note that \( n \geq 2 \). Using [35, Lemma 3.10] again, we deduce that

\[
\int_{\mathbb{D}} (1 - |z|^2)^{np-2} |F^{(n+1)}(z)|^p \, dA(z)
\]

\[
\leq \int_{\mathbb{D}} \int_{C \setminus B} \frac{1}{|\zeta - z|^{np-2p+4}} dA(\zeta) (1 - |z|^2)^{np-2p} dA(z)
\]

\[
\approx \int_{\mathbb{D}} \int_{C \setminus B} \frac{1}{1 - w|z|^{np-2p+4}} dA(\zeta) (1 - |z|^2)^{np-2p} dA(z)
\]

\[
\approx \int_{\mathbb{D}} (1 - |z|^2)^{np-2p} \frac{1}{1 - w|z|^{np-2p+4}} dA(\zeta) (1 - |z|^2)^{np-2p} dA(z)
\]

\[
\leq \int_{\mathbb{D}} (1 - |w|^2)^{-2} \frac{1}{1 - w|z|^{np-2p+4}} dA(\zeta)
\]

Since \( F'(z) = f(z) \) for \( z \in \mathbb{D} \), by (3.2) and (3.6), we obtain \( f \in B_p \). This finishes the proof. \( \square \)

For \( 0 < \lambda \leq 1 \), the analytic Morrey space \( \mathcal{L}^{2,\lambda} \) is the set of all functions \( f \in H^2 \) satisfying

\[
\|f\|_{L^{2,\lambda}} = \sup_{I \subseteq \partial \mathbb{D}} \left( \frac{1}{\ell(I)^\lambda} \int_I |f(\zeta) - f_I|^2 d\zeta \right)^{1/2} < \infty,
\]

where \( \ell(I) \) denotes the length of \( I \) and

\[
f_I = \frac{1}{\ell(I)} \int_I f(\zeta) d\zeta.
\]

Clearly, \( \mathcal{L}^{2,1} \) is the space \( BMOA \). By [30, p. 54], \( f \in \mathcal{L}^{2,\lambda} \) if and only if

\[
\sup_{a \in \mathbb{D}} (1 - |a|^2)^{1-\lambda} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\sigma_a(z)|^2) dA(z) < \infty.
\]

Let \( n \geq 1 \) be an integer. Using a similar statement in [4], we see that \( f \in \mathcal{L}^{2,\lambda} \) if and only if

\[
\sup_{a \in \mathbb{D}} (1 - |a|^2)^{1-\lambda} \int_{\mathbb{D}} |f^{(n)}(z)|^2 (1 - |\sigma_a(z)|^2)^n dA(z) < \infty. \tag{3.7}
\]

Recently, the interest in analytic Morrey space has grown rapidly. See [8, 18, 20, 21, 23, 31, 32] for more results of \( \mathcal{L}^{2,\lambda} \) spaces.

Note that if \( f \in \mathcal{L}^{2,\lambda} \), then its primitive function \( F \) belongs to \( H^2 \). Applying (3.7), we get the following characterization of analytic Morrey space. The proof is similar to Theorem 2.1, we omit it.

**Theorem 3.2.** Let \( f \in H(\mathbb{D}) \) with its primitive function \( F \in H^2 \) and let \( n \geq 2 \) be an integer. If \( 0 < \lambda \leq 1 \), then the following conditions are equivalent.

(i) \( f \in \mathcal{L}^{2,\lambda} \).
(ii) There exists a function \( \tilde{F}_n \in C^1(\mathbb{C} \setminus \mathbb{D}) \) satisfying
\[
\lim_{r \to 1^+} \tilde{F}_n(re^{i\theta}) = F(e^{i\theta}) \quad \text{a.e.} \quad \theta \in [0, 2\pi],
\]
\[
\tilde{F}_n(z) = O(z^n), \quad \text{as} \quad z \to \infty,
\]
\[
\partial \tilde{F}_n(z) = O(z^{n-2}), \quad \text{as} \quad z \to \infty,
\]
and
\[
\sup_{a \in \mathbb{D}} (1 - |a|^2)^{1-\lambda} \int_{\mathbb{C} \setminus \mathbb{D}} \frac{|\nabla \tilde{F}_n(z)|^2}{(|z|^n - 1)^2} \left(1 - \frac{1}{|\sigma_a(z)|^2}\right) dA(z) < \infty.
\]

**Remark.** When the paper is complete, we found that Wei and Shen obtained the special case of our Theorem 2.1 for \( K(t) = t \) and \( n = 2 \). However, our proof in this paper and those of [22] are different.

**References**

[1] J. Anderson, J. Clunie and Ch. Pommerenke, On Bloch functions and normal functions, *J. Reine Angew. Math.*, 270 (1974), 12-37.
[2] J. Arazy, S. Fisher and J. Peetre, Möbius invariant function spaces, *J. Reine Angew. Math.*, 363 (1985), 110-145.
[3] R. Aulaskari and P. Lappan, Criteria for an analytic function to be Bloch and a harmonic or meromorphic function to be normal, *Complex analysis and its applications, Pitman Res. Notes in Math.*, 305, Longman Sci. Tec., Harlow, 1994, 136-146.
[4] R. Aulaskari, M. Nowak and R. Zhao, The \( n \)th derivative characterisation of Möbius invariant Dirichlet space, *Bull. Austral. Math. Soc.*, 58 (1998), 43-56.
[5] R. Aulaskari, J. Xiao and R. Zhao, On subspaces and subsets of \( BMOA \) and \( UBC \), *Analysis.*, 15 (1995), 101-121.
[6] A. Baernstein, Analytic functions of Bounded Mean Oscillation. In: Aspects of Contemporary Complex Analysis, Editors: D. A. Brannan and J. G. Clunie., Academic Press, London, New York (1980), pp. 3-36.
[7] G. Bao, Z. Lou, R. Qian and H. Wulan, Improving multipliers and zero sets in \( Q_K \) spaces, *Collect. Math.*, doi: 10.1007/s13348-014-0113-z.
[8] C. Cascante, J. Fabrega and J. Ortega, The corona theorem in weighted Hardy and Morrey spaces, *Ann. Scuola Norm. Super. Pisa Cl. Sci.*, doi: 10.2422/2036-2145.201202-006.
[9] P. Duren, *Theory of \( H^p \) Spaces*, Academic Press, New York, 1970.
[10] K. Dyakonov, Equivalent norms on Lipschitz-type spaces of holomorphic functions, *Acta Math.*, 178 (1997), 143-167.
[11] K. Dyakonov, Besov spaces and outer functions, *Michigan Math. J.*, 45 (1998), 143-157.
[12] K. Dyakonov and D. Girela, On \( Q_p \) spaces and pseudoanalytic extension, *Ann. Acad. Sci. Fenn. Math.*, 25 (2000), 477-486.
[13] E. Dyn’kin, The pseudoanalytic extension, *J. Anal. Math.*, 60 (1993), 45-70.
[14] M. Essén and H. Wulan, On analytic and meromorphic function and spaces of \( Q_K \)-type, *Illinois J. Math.*, 46 (2002), 1233-1258.
[15] M. Essén, H. Wulan and J. Xiao, Several function-theoretic characterizations of Möbius invariant \( Q_K \) spaces, *J. Funct. Anal.*, 230 (2006), 78-115.
[16] J. Garnett, *Bounded Analytic Functions*, Springer, New York, 2007.
[17] D. Girela, Analytic functions of bounded mean oscillation. In: Complex Function Spaces, Mekrijärvi 1999 Editor: R. Aulaskari. Univ. Joensuu Dept. Math. Rep. Ser. 4, Univ. Joensuu, Joensuu (2001) pp. 61-170.
[18] P. Li, J. Liu and Z. Lou, Integral operators on analytic Morrey spaces, *Sci. China Math.*, 57 (2014), 1961-1974.
[19] E. Stein, Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals, Princeton University Press, Princeton, NJ, 1993.

[20] J. Wang and J. Xiao, Two predualities and three operators over analytic campanato spaces, arXiv:1402.4377.

[21] J. Wang and J. Xiao, Holomorphic campanato spaces on the unit ball, arXiv:1405.6192.

[22] H. Wei and Y. Shen, On the tangent space to the BMO-Teichmüller space, J. Math. Anal. Appl., 419(2014), 715-726.

[23] Z. Wu and C. Xie, Q spaces and Morrey spaces, J. Funct. Anal., 201(2003), 282-297.

[24] H. Wulan and F. Ye, Universal Teichmüller space and QK spaces, Ann. Acad. Sci. Fenn. Math., 39(2014), 691-709.

[25] H. Wulan and J. Zhou, Decomposition theorems for QK spaces and applications, Forum Math., 26(2014), 467-495.

[26] H. Wulan and J. Zhou, QK and Morrey type spaces, Ann. Acad. Sci. Fenn. Math., 38(2013), 193-207.

[27] H. Wulan, K. Zhu, QK spaces via higher order derivatives, Rocky Mountain J. Math., 38(2008), 329-350.

[28] J. Xiao, The Qp corona theorem, Pacific J. Math., 194(2000), 491-509.

[29] J. Xiao, Holomorphic Q Classes, Springer, LNM 1767, Berlin, 2001.

[30] J. Xiao, Geometric Qp functions, Birkhäuser Verlag, Basel-Boston-Berlin, 2006.

[31] J. Xiao and W. Xu, Composition operators between analytic Campanato spaces, J. Geom. Anal., 24(2014), 649-666.

[32] J. Xiao and C. Yuan, Analytic Campanato spaces and their compositions, arXiv:1303.5032.

[33] R. Zhao, Distances from Bloch functions to some Möbius invariant spaces, Ann. Acad. Sci. Fenn. Math., 33(2008), 303-313.

[34] K. Zhu, Analytic Besov spaces, J. Math. Anal. Appl., 157 (1991), 318-336.

[35] K. Zhu, Operator Theory in Function Spaces, American Mathematical Society, Providence, RI, 2007.

DEPARTMENT OF MATHEMATICS, SHANTOU UNIVERSITY, SHANTOU, GUANGDONG, 515063, CHINA

E-mail address: qlbaoah@163.com (G. Bao)
E-mail address: wulan@stu.edu.cn (H. Wulan)
E-mail address: yefqah@163.com (F. Ye)