A note on the existence of traveling-wave solutions to a Boussinesq system

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Abstract. We obtain a one-parameter family

$$(u_\mu(x, t), \eta_\mu(x, t))_{\mu \geq \mu_0} = (\phi_\mu(x - \omega_\mu t), \psi_\mu(x - \omega_\mu t))_{\mu \geq \mu_0}$$

of traveling-wave solutions to the Boussinesq system

$$\begin{cases} u_t + \eta_x + uu_x + c\eta_{xxx} = 0 \quad (x, t) \in \mathbb{R}^2 \\ \eta_t + u_x + (\eta u)_x + au_{xxx} = 0 \end{cases}$$

in the case $a, c < 0$, with non-null speeds $\omega_\mu$ arbitrarily close to 0 ($\omega_\mu \xrightarrow[\mu \to +\infty]{} 0$). We show that the $L^2$-size of such traveling-waves satisfies the uniform (in $\mu$) estimate $\|\phi_\mu\|_2^2 + \|\psi_\mu\|_2^2 \leq C \sqrt{|a| + |c|}$, where $C$ is a positive constant. Furthermore, $\phi_\mu$ and $-\psi_\mu$ are smooth, non-negative, radially decreasing functions which decay exponentially at infinity.

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1. Introduction

In [1], with the purpose of describing the dynamics of small-amplitude long waves propagating on the surface of an ideal fluid in a channel of constant depth, the authors introduced the four-parameter family of Boussinesq systems

$$\begin{cases} u_t + \eta_x + uu_x + c\eta_{xxx} - du_{xxt} = 0 \\ \eta_t + u_x + (\eta u)_x + au_{xxx} - b_{xxt} = 0 \end{cases}$$

(1.1)

These systems are first-order approximations to the Euler equations in the small parameters $\alpha = \frac{A}{h} << 1$ and $\beta = \frac{h^2}{\lambda^2} << 1$, where $h$ is the depth of the channel (subsequently scaled to 1), and $A$ and $\lambda$ represent a typical wave amplitude and a typical wavelenght respectively. Here, $\eta(x, t)$ denotes the

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deviation of free surface with respect to the undisturbed state (i.e. \(1 + \eta(x,t)\) is the total depth of the liquid at time \(t\) and position \(x\)) and \(u(x,t)\) is the horizontal velocity at height \(\theta\), \(0 \leq \theta \leq 1\). The four parameters \(a, b, c\) and \(d\) are given by

\[
a = \frac{1}{2} \left( \theta^2 - \frac{1}{3} \right) \lambda, \quad b = \frac{1}{2} \left( \theta^2 - \frac{1}{3} \right) (1 - \lambda)
\]

\[
c = \frac{1}{2} (1 - \theta^2) \mu, \quad d = \frac{1}{2} (1 - \theta^2) (1 - \mu),
\]

where, as stated in [1], \(\lambda\) and \(\mu\) are modeling parameters that do not possess a direct physical interpretation.

In [2], a correction of the \(c\) parameter is proposed in order to include the contribution of the surface tension:

\[
c = \frac{1}{2} (1 - \theta^2) \mu - \tau.
\]

The Bond number \(\tau\) is given by \(\tau = \frac{\Gamma}{\rho g h^2}\), where \(\Gamma\) is the surface tension coefficient and \(\rho\) the density of water.

In [3], the authors prove the existence (and orbital stability) of traveling-waves to system (1.1) of the form

\[
\begin{align*}
\phi(x - \omega t) &= \psi(x - \omega t)
\end{align*}
\]

in the case \(a, c < 0\), \(b = d > 0\) and \(ac > b^2\). Furthermore, in [4], the existence of such traveling-waves with small propagation speed is obtained in the case \(a, c < 0\) and \(b = d\).

In the present paper we exhibit a new family of one-parameter traveling-waves in the case \(a, c < 0\) and \(b = d = 0\). Our method has the advantage of providing radially decreasing functions and a uniform bound for the \(L^2\)-size of the solution. More precisely, we prove the following result:

**Theorem 1.1.** Let \(a, c < 0\). There exists a constant \(\mu_0 = \mu_0(a, c)\) and a one-parameter family of nontrivial traveling-wave solutions to the Boussinesq system

\[
\begin{align*}
\phi_t + \eta_x + uu_x + cu_{xxx} &= 0 \\
\eta_t + u_x + (\eta u)_x + au_{xxx} &= 0
\end{align*}
\]

of the form

\[
(u, \eta) = (\phi_\mu(x - \omega_\mu t), \psi_\mu(x - \omega_\mu t)), \quad \mu \geq \mu_0,
\]

with \((\phi_\mu, \psi_\mu) \in H^\infty(\mathbb{R}) \times H^\infty(\mathbb{R})\), \(\phi_\mu\) and \(-\psi_\mu\) non-negative and radially decreasing, with exponential decay at infinity.

Furthermore, the speed \(\omega_\mu\) satisfies the estimate

\[
0 > \omega_\mu > -\frac{1}{C_1} \sqrt{|a| + |c|} \mu^\frac{3}{4}
\]

and the following uniform control of the \(L^2\)-norm of the traveling-wave holds:

\[
\|\phi_\mu\|^2_2 + \|\psi_\mu\|^2_2 \leq C \sqrt{|a| + |c|},
\]

(1.6)
where \( C \) is a positive constant independent of \( a, c \) and \( \mu \).

**Remark 1.2.** We stated the existence of traveling-waves
\[
(\phi(x - \omega t), \psi(x - \omega t)),
\]
with \( \phi \geq 0 \) and \( \psi \leq 0 \), propagating with negative speed \( \omega \). Noticing that
\((-\phi(x + \omega t), \psi(x + \omega t))\) is also a solution to (1.5), it is straightforward to deduce the existence of non-positive traveling-waves propagating with positive speed.

**Remark 1.3.** As stated above, in [4], the authors also establish the existence of traveling-waves with small speed in the particular case \( a, c < 0 \) and \( b = d = 0 \), although it is not clear, with the method used, if the solutions have a sign, are radially-decreasing or if an uniform (in the speed \( \omega \)) estimate such as (1.6) holds. On the other hand, of course, the method used in [4] has the important advantage of covering the case \( b = d \neq 0 \). Either way, it does not seem obvious to prove or to disprove that the traveling-waves found in both papers are the same.

**Remark 1.4.** Note, in view of (1.2) and (1.3), that the case treated in Theorem 1.1 corresponds to \( \lambda = \mu = 1 \), that is
\[
a = \frac{1}{2} \left( \frac{\theta^2}{3} - \frac{1}{3} \right) \quad \text{and} \quad c = \frac{1}{2} (1 - \theta^2) - \tau.
\]
For \( \theta^2 \to \frac{1}{3} \) and \( \tau \to \frac{1}{3} \) we get \( a, c \to 0 \). The estimate (1.6) then suggests that the traveling-wave solutions vanish in this regime. This is consistent with the known fact (see [5]) that in the case \( \frac{1}{3} - \tau = O(\beta) \), \( \beta \to 0 \) it is necessary to introduce higher order terms in the Boussinesq approximation of the Euler equations in order to model solitary waves.

### 2. The minimization problem

In what follows we fix \( a, c < 0 \). In order to prove Theorem 1.1 we introduce a variational problem whose minimizers, up to a rescaling, correspond to a traveling-wave profile. Let us first note that
\[
(u, \eta) = (\phi(x - \omega t), \psi(x - \omega t)), \quad \phi(y), \psi(y) \xrightarrow{y \to \infty} 0
\]
is a solution to the Boussinesq system (1.5) if and only if \( \phi \) and \( \psi \) satisfy the stationary equation
\[
\begin{align*}
\begin{cases}
a\phi'' + \phi - \omega \psi + \phi \psi = 0 \\
c\psi'' + \psi - \omega \phi + \frac{1}{2} \phi^2 = 0.
\end{cases}
\end{align*}
\]
(2.1)

For \( \mu > 0 \), we set
\[
X_\mu = \{(f, g) \in H^1(\mathbb{R}) \times H^1(\mathbb{R}) : N(f, g) = \|f\|_2^2 + \|g\|_2^2 = \mu\}
\]
and consider the minimization problem \( m(\mu) = \inf \{ \tau(f, g) : (f, g) \in X_\mu \} \), where
\[
\tau(f, g) = -a \int f'^2 - c \int g'^2 + \int f^2 g + 2 \int fg. 
\]

**Proposition 2.1.** For all \( \mu > 0 \), \( m(\mu) > -\infty \).
More precisely, \( m(\mu) \geq -\frac{C}{\sqrt{|a|}} \mu^\frac{m}{2} - 2\mu \), where \( C \) is a positive constant.

**Proof** Let \((f, g) \in X_\mu\). One only has to notice that by the Cauchy-Schwarz and Gagliardo-Nirenberg inequalities,
\[
\left| \int fg \right| \leq \|f\|_2 \|g\|_2 \leq \mu \quad \text{and} \\
\left| \int f^2g \right| \leq \|f\|_4^2 \|g\|_2 \leq \|f\|_2^\frac{1}{2} \|f\|_2^\frac{3}{2} \|g\|_2 \leq \mu^\frac{3}{2} \|f'\|_2^\frac{1}{2}.
\]
Hence,
\[
\tau(f, g) \geq -a \|f'\|_2^2 - \mu^\frac{3}{2} \|f'\|_2^\frac{1}{2} - 2\mu = P(\|f'\|_2),
\]
which is enough to conclude since \( \frac{1}{\sqrt{|a|}} \left( 16 - \frac{1}{3} - 4 - \frac{1}{3} \right) \mu^\frac{m}{2} - 2\mu \) is the minimum of \( P(x) = -ax^4 - \mu^\frac{3}{2} x - 2\mu \).

**Proposition 2.2.** For all \( \mu > 0 \), \( m(\mu) < -\frac{C}{\sqrt{|a|} + |c|} \mu^\frac{3}{2} \), where \( C \) is a positive constant.

**Proof** We fix a non-negative function \( h \in H^1(\mathbb{R}) \) such that \( \|h\|_2 = 1 \) and we put \( h_\lambda(x) = \lambda h(\lambda^2 x) \), where \( \lambda \) will be chosen later. Then, for all \( \mu > 0 \),
\[
(f, g) = \left( \frac{1}{\sqrt{2}} \left( \mu^\frac{3}{2} h_\lambda(\mu^\frac{3}{2} x), -\mu^\frac{3}{2} h_\lambda(\mu^\frac{3}{2} x) \right) \right) \in X_\mu \text{ and}
\]
\[
m(\mu) \leq \tau(f, g) = \frac{|a| + |c|}{2} \mu^\frac{3}{2} \int (h_\lambda')^2 - \frac{1}{2\sqrt{2}} \mu^\frac{3}{2} \int h_\lambda^2 \leq \mu^\frac{3}{2} \left( \frac{|a| + |c|}{2} ||h_\lambda'||_2^2 - \frac{1}{2\sqrt{2}} ||h_\lambda||_3^3 \right)
\]
\[
\leq \mu^\frac{3}{2} \lambda \left( \lambda^3 (|a| + |c|) ||h'||_2^2 - \frac{1}{2\sqrt{2}} ||h||_3^3 \right). 
\]
We conclude the proof by choosing \( \lambda = \frac{\epsilon}{\sqrt{|a| + |c|}} \), with \( \epsilon \) such that
\[-C = \lambda^3 (|a| + |c|) ||h'||_2^2 - \frac{1}{2\sqrt{2}} ||h||_3^3 < 0. \]

**3. Existence of Minimizers**

Let \( \mu > 0 \) and \((f_n, g_n)\) a minimizing sequence for \( m(\mu) \). By denoting \( f^* \) the Schwarz symmetrization of \( |f| \), it is well known that
\[
\|f''\|_2 \leq \|f'\|_2, \quad \|f^*\|_2 = \|f\|_2, \quad \int fg \leq \int f^* g^* \quad \text{and} \quad \int f^2 g \leq \int f^2 g^*.
\]
Hence, \( \tau(f^*, -g^*) \leq \tau(f, g) \) and \((f, g) \in X_\mu \) implies that \((f^*, -g^*) \in X_\mu \). Therefore we can choose a minimizing sequence \((f_n, g_n)\) with \(f_n \geq g_n \leq 0\) and \(f_n, -g_n\) radially decreasing.

We will now apply the concentration-compactness method ([6], [7]) to prove the compactness, up to translations, of the sequence \((f_n, g_n)\) in \(L^2(\mathbb{R}) \times L^2(\mathbb{R})\) strong.

Following this method, we set the concentration function \(\rho_n = f_n^2 + g_n^2\) and put \(Q_n(t) = \sup_{y \in \mathbb{R}} \int_{y-t}^{y+t} \rho_n\). Also, we set \(Q(t) = \lim_{n \to +\infty} Q_n(t)\) and \(\Omega = \lim_{t \to +\infty} Q(t)\).

We start by ruling out vanishing:

**Proposition 3.1.** There exists \(\mu_0 = \mu_0(a, c)\) such that for \(\mu \geq \mu_0, \Omega > 0\).

**Proof** Assume that \(\Omega = 0\). Since \(Q(t)\) is non-negative and non-increasing, for all \(t, Q(t) = 0\). Hence,

\[
\lim_{n \to +\infty} \sup_{y \in \mathbb{R}} \int_{y-t}^{y+t} f_n^2 = \lim_{n \to +\infty} \sup_{y \in \mathbb{R}} \int_{y-t}^{y+t} g_n^2 = 0.
\]

From the Proof of 2.1 one can infer that \((f_n)\) (and \(g_n\)) is bounded in \(H^1(\mathbb{R})\). Arguing as in [6] (Lemma I.1), \(\|f_n\|_4 \to 0\), hence \(\int f_n^2 g_n \to 0\) by Cauchy-Schwarz. Furthermore,

\[
m(\mu) = \lim_{n \to +\infty} \left( -a \int f_n^2 - c \int g_n^2 + \int f_n^2 g_n + 2 \int f_n g_n \right) \geq 2 \lim_{n \to +\infty} \int f_n g_n \geq -2\mu,
\]

which contradicts Proposition 2.2 for \(\mu \geq \mu_0 = \frac{2^{\frac{3}{2}}}{c_1^2} \sqrt{|a| + |c|}\). □

Next, we rule out dichotomy, that is \(0 < \Omega < \lim_{n \to +\infty} \int \rho_n\). It is sufficient to prove the following lemma:

**Lemma 3.2.** For all \(\mu \geq \mu_0\) and for all \(\theta > 1\), \(m(\theta \mu) < \theta m(\mu)\).

**Proof** We have \(\tau(\theta^\frac{1}{2} f_n, \theta^\frac{1}{2} g_n) = \theta \tau(f_n, g_n) - (\theta^\frac{1}{2} - \theta) \int |f_n|^2 |g_n|\).

Also, there exists \(\delta > 0\) such that for all \(n\) large enough, \(\int |f_n|^2 |g_n| \geq \delta\).

Otherwise, up to a subsequence, \(\int f_n^2 g_n \to 0\), which is absurd for \(\mu \geq \mu_0\), as seen in the previous proof of Proposition 3.1. Finally,

\[
m(\theta \mu) \leq \tau(\theta^\frac{1}{2} f_n, \theta^\frac{1}{2} g_n) \leq \theta \tau(f_n, g_n) - \delta (\theta^\frac{1}{2} - \theta),
\]

which yields the result:

\[
m(\theta \mu) \leq \lim_{n \to +\infty} \theta \tau(f_n, g_n) - \delta (\theta^\frac{1}{2} - \theta) = \theta m(\mu) - \delta (\theta^\frac{1}{2} - \theta) < \theta m(\mu).\] □

It is standard, from Lemma 3.2, to prove the strict subadditivity of \(m\), that is

\[
\forall \mu \geq \Omega, \quad m(\mu) < m(\Omega) + m(\mu - \Omega),
\]
(see for instance Lemma 2.3 in [8]) which is well-known to rule out dichotomy. Hence, by Lions’ Theorem, we are in the compactness situation. There exists a sequence \((y_n)\) such that, up to a subsequence, \((f_n(\cdot - y_n), g_n(\cdot - y_n))\) converges strongly in \(L^2(\mathbb{R}) \times L^2(\mathbb{R})\) to some \((\tilde{f}, \tilde{g})\) in \(X_\mu\). Since \((f_n, g_n)\) is bounded in \(H^1(\mathbb{R})\), using the compact embedding \(H^1_{rad}(\mathbb{R}) \hookrightarrow L^4(\mathbb{R})\), up to a subsequence, \(f_n\) (respectively \(-g_n\)) converges strongly in \(L^4(\mathbb{R})\) to some radial non-negative function \(f\) (respectively \(-g\)).

Furthermore, \((f_n, g_n) \rightharpoonup (f, g)\) in \(H^1(\mathbb{R}) \times H^1(\mathbb{R})\) weak. Since
\[
\|f_n\|_2^2 + \|g_n\|_2^2 = \|f_n(\cdot - y_n)\|_2^2 + \|g_n(\cdot - y_n)\|_2^2 \xrightarrow{n \to +\infty} \|\tilde{f}\|_2^2 + \|\tilde{g}\|_2^2 = \mu,
\]
we have in fact that \((f_n, g_n) \to (f, g)\) in \(L^2(\mathbb{R}) \times L^2(\mathbb{R})\) strong and, in particular, \((f, g)\) is a minimizer for \(m(\mu)\).

4. End of the Proof of Theorem 1.1

There exists a Lagrange multiplier \(\lambda \in \mathbb{R}\) such that \(\nabla \tau(f, g) = \lambda \nabla N(f, g)\), that is
\[
\begin{cases}
af'' + fg + g &= \lambda f \\
 cg'' + \frac{1}{2} f^2 + f &= \lambda g.
\end{cases}
\] (4.1)

Multiplying these equations by \(f\) and \(g\) respectively and integrating by parts leads to
\[
\begin{cases}
-a \int (f')^2 + \int f^2 g + \int fg &= \lambda \int f^2 \\
-c \int (g')^2 + \frac{1}{2} \int f^2 g + \int fg &= \lambda \int g^2,
\end{cases}
\]
and, adding the equalities,
\[
-\lambda \mu = -\tau(f, g) - \frac{1}{2} \int f^2 g \geq -\tau(f, g) = -m(\mu).
\] (4.2)

Note that, in particular, \(\lambda < 0\). Setting
\[
\phi(x) = -\frac{1}{\lambda} f \left( \frac{x}{\sqrt{-\lambda}} \right) \quad \text{and} \quad \psi(x) = -\frac{1}{\lambda} g \left( \frac{x}{\sqrt{-\lambda}} \right),
\]

we obtain
\[
\begin{cases}
   a \phi'' + \phi + \phi \psi - \frac{1}{\lambda} \psi = 0 \\
   c \psi'' + \psi + \frac{1}{2} \phi^2 - \frac{1}{\lambda} \phi = 0,
\end{cases}
\]
that is, \((\phi, \psi)\) is a solution to (2.1) with speed \(\omega = \frac{1}{\lambda}\).

To obtain the \(L^2\) size of this solution, a simple computation shows that
\[
\|\phi\|^2 + \|\psi\|^2 = \frac{\mu}{|\lambda|^2},
\]
Also, from (4.2) and Proposition 2.2, \(|\lambda| \mu \geq -m(\mu) \geq C \frac{|a|}{\sqrt{|a| + |c|}} \mu^{-\frac{5}{3}}\), from where we conclude that
\[
\|\phi\|^2 + \|\psi\|^2 \leq C \sqrt{|a| + |c|},
\]
where \(C\) is yet another positive constant independent of \(a\), \(c\) and \(\mu\). Also, note that
\[
|\omega| = \frac{1}{|\lambda|} \leq \frac{1}{C_1} \frac{3}{\sqrt{|a| + |c|}} \mu^{-\frac{2}{3}}.
\]
The regularity of \((\phi, \psi)\) can be obtained by a standard bootstrapping argument (see for instance [4], Proposition 3.2).

In order to prove the exponential decay of \(\phi\) and \(\psi\), following the ideas of Theorem 8.1.1 in [2], we consider, for \(\epsilon, \eta > 0\), \(h(x) = e^{\frac{\epsilon}{1+\eta|x|}} \in L^\infty(\mathbb{R})\). Multiplying equations in (2.1) by \(h\phi\) and \(h\psi\) respectively and integrating, we get
\[
a \int h \phi \phi'' + c \int h \psi \psi'' + \int h (\phi^2 + \psi^2) + \frac{3}{2} \int h \phi^2 \psi - 2 \omega \int h \phi \psi = 0
\]
Integrating by parts and using the fact that \(h' \leq \epsilon h\), we obtain
\[
\int h (\phi^2 + \psi^2 - 2 \omega \phi \psi) \leq a \int h \phi'^2 + c \int h \psi'^2 + \epsilon \int h (|\phi'|^2 + |\psi'|^2) + \frac{3}{2} \int h \phi^2 |\psi|,
\]
and
\[
\int h \left( \left(1 - \frac{\epsilon}{2}\right) \frac{\phi^2}{2} + \left(1 - \frac{\epsilon}{2}\right) \frac{\psi^2}{2} - 2 \omega \phi \psi \right) \leq
\left( a + \frac{\epsilon}{2}\right) \int h \phi'^2 + \left( c + \frac{\epsilon}{2}\right) \int h \psi'^2 + \frac{3}{2} \int h \phi^2 |\psi| \leq \frac{3}{2} \int h \phi^2 |\psi|
\]
for \(\epsilon\) small enough. Since \(\psi \in H^1(\mathbb{R})\), \(\lim_{|x| \to +\infty} \psi(x) = 0\). For \(\epsilon' > 0\) to be chosen later, we set \(r > 0\) such that \(|\psi(x)| \leq \epsilon'\) for \(|x| > r\). We then get
\[
\int h \left( \left(1 - \frac{\epsilon}{2} - \frac{3 \epsilon'}{2}\right) \phi^2 + \left(1 - \frac{\epsilon}{2}\right) \frac{\psi^2}{2} - 2 \omega \phi \psi \right) \leq \frac{3}{2} \int_{|x| \leq r} h \phi^2 |\psi|
\]
and
\[
\int h (C_1 \phi^2 + C_2 \psi^2) \leq \frac{3}{2} \int_{|x| \leq r} h \phi^2 |\psi|
\]
where $C_1 = 1 - \frac{\epsilon}{2} - \frac{3\epsilon_1}{2} - |\omega| > 0$ and $C_2 = 1 - \frac{\epsilon}{2} - |\omega| > 0$ for $\epsilon, \epsilon'$ small enough (and for $|\omega|$ small).

Finally, taking $\eta \to 0$, by Fatou’s Lemma and Lebesgue’s Theorem, we obtain

$$
\int e^{\epsilon |x|} \phi^2 < +\infty \quad \text{and} \quad \int e^{\epsilon |x|} \psi^2 < +\infty.
$$

In view of Theorem 8.1.7 of [2], this is enough to conclude that

$$
e^{\alpha |x|} \phi, e^{\alpha |x|} \psi \in L^\infty(\mathbb{R})
$$

for some $0 < \alpha \leq \epsilon$.

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