Effective Field Theories for the Exactly Solvable Stabilizer Spin Models

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A series of exactly solvable spin models has enriched our understanding of spin liquids and topological matter in the past few decades. The toric code in two and three dimensions and the X-cube model are among the prototypes of such models. Recently, two of the present authors added a variant of the X-cube model called the F3 model to the list of stabilizer Hamiltonians. In this paper, we provide a coherent prescription for obtaining all these stabilizer Hamiltonians from their respective parent lattice gauge theories, which are rank-1 U(1) lattice gauge theories (LGT) for the toric codes, rank-2 U(1) LGT for the X-cube, and a hybrid of rank-1 and rank-2 U(1) LGTs for the F3 model. We then develop the effective field theory (EFT) description of the quasiparticle excitations and gauge field fluctuations through the consistent application of the gauge principle. Well-known BF actions in (2+1) and (3+1) dimensions as well as the EFT of the X-cube can be derived in this way. The effective action for the F3 model is also obtained. Non-trivial effective Lagrangians of the matter fields such as the e and m quasiparticles in the toric codes, and fractons and lineons in the X-cube model are derived. They are usually of quartic order in the field implying strong interaction (and even confinement) among the quasiparticles and result in conservation laws consistent with the restricted mobilities of fractons and lineons. The effective field theory for the F3 model is also developed by following the gauge principle. A tight-binding description of the quasiparticle dynamics is proposed for all these models, which in some cases resembles the quadrupole model studied in the context of the higher-order topological insulator.

I. INTRODUCTION

The importance of exactly solvable spin models such as toric codes (TC) in two and three dimensions [1,13] and the X-cube (XC) model in three dimensions [5–13] is increasingly clear. They are nice examples of models exhibiting topological order and fractional statistics [14–17] (in two dimensions) and host a variety of topological gauge theories, such as the BF theory [17–21] as well as their low-energy effective field theories (EFT) [22–24]. Realizing these spin models in the laboratory has become within reach in recent years [25].

The novelty of the exactly solvable spin models has rooted in the multi-spin interactions that their Hamiltonians consist of. The four-, six-, and even twelve-spin interactions that appear in the Hamiltonians of these spin models can be understood intuitively in connection to their parent lattice gauge theories (LGT) using ‘Higgsing’ [21,31,32]. The mutually commuting operators in the LGT become, through Higgsing, the stabilizer operators in the spin Hamiltonian. The parent theory of the TC is the rank-1 U(1) LGT, while that of the XC model is the rank-2 U(1) LGT [32]. Recently, the authors have studied in detail the spin model obtained by Higgsing the rank-2 U(1) LGT in two dimensions [33].

The elementary excitations of the toric codes are the electric and magnetic quasiparticles exhibiting mutual anyonic statistics, while those of the X-cube model are the fractons and lineons. The relevant EFTs capturing their dynamics and mutual statistics have been constructed [18,27]. Two of the present authors recently proposed a new kind of exactly solvable spin model [34] exhibiting features akin to those of the three-dimensional toric code (3DTC) as well as the XC model. Three types of quasiparticle excitations dubbed freeons, fractons, and fluxons were identified, and for this reason, we refer to it as the ‘F3 model’. Many properties of the F3 model were analyzed in a prior publication. On the other hand, neither the parent LGT of the F3 model nor its EFT was properly understood. We will show in this paper that the F3 model is the descendant of the hybrid rank-1 and rank-2 U(1) LGTs through Higgsing. We also construct the EFT of the F3 model by following the gauge principle that works well in producing the EFTs of the TC and the XC model.

In Sec. II we describe the parent LGT of the toric codes and its transformation into the toric codes through Higgsing. Gauge transformation properties of the gauge fields are clarified and subsequently exploited to derive the field theories of the matter fields (e and m quasiparticles) and the gauge fields. The gauge field parts are the well-known BF theories, while the matter parts of the action have not been properly constructed in the past. We predict, in particular, that the action for the m particles in three dimensions (but not in two dimensions) are highly interactive and may explain their confinement as already known from analysis of the lattice model. A tight-binding model for the m quasiparticle involving the simultaneous hopping of two particles on a cubic lattice is constructed. In Sec. III we perform analogous analysis for the X-cube model by first delineating the Higgsing procedure that leads to the X-cube model, and then constructing the field theories by application of the gauge principle.
principle. The matter Lagrangians for the fractons and lineons we construct and the conservation laws which follow from them directly imply their restricted mobilities. Tight-binding models for the fracton and lineon hopping are constructed. In Sec. [IV] we carry out a similar analysis for the F3 model. Both the matter and the gauge field Lagrangians are non-trivial. Section [V] gives the summary.

II. RANK-1 U(1) LATTICE GAUGE THEORY, TORIC CODES AND THEIR FIELD THEORIES

We show how the toric codes in two and three dimensions follow “naturally” from the parent U(1) LGT, then proceed to construct relevant field theories.

A. Toric Codes from Rank-1 U(1) Lattice Gauge Theory

The rank-1 U(1) LGT has the vector gauge fields $A_i^a$ residing on the $(i, i+\hat{a})$ links of the lattice, where $a = x, y$ or $a = x, y, z$ in two (three) dimensions. The canonically conjugate electric fields $E_i^a$ satisfy the commutation relation $[A_i^a, E_j^b] = i\delta_{ij}\delta_{ab}$. The gauge transformation of $A_i^a$ is generated as

$$A_i^a \rightarrow U_A^a A_i^a U_A^{-1} = A_i^a + f_{i+\hat{a}} - f_i,$$

$$U_A = \exp \left[ i \sum_j f_j (\nabla \cdot E)_j \right]$$

for arbitrary scalar function $f_i$. The lattice divergence of the electric field is defined by

$$\nabla \cdot E_i \equiv \sum_a (E_i^a - E_{i-\hat{a}}^a),$$

where $a = x, y$ or $a = x, y, z$ for spatial dimensions $D = 2$ or $D = 3$, is the generator of the gauge transformation for $A_i^a$.

The magnetic field operator

$$B_i^c \equiv (\nabla \times A)_i^c = \sum_{bc} \epsilon_{abc} (A_i^b - A_{i+\hat{c}}^b)$$

is given by the directed sum of gauge fields around the elementary plaquette. Only one component $B_i^c$ is meaningful in two dimensions while three magnetic field orientations $(B_i^x, B_i^y, B_i^z)$ exist in three dimensions. Importantly, we have the commutativity of the two operators

$$[\nabla \cdot E_i^a, B_j^b] = 0,$$

following from their definitions. The magnetic field acts as a generator for the transformation of the $E$ field:

$$E_i^a \rightarrow U_E^a E_i^a U_E^{-1} = E_i^a + \sum_{bc} \epsilon_{abc} (g_i^c - g_{i-\hat{b}}^c),$$

$$U_E = \exp \left[ i \sum_j g_j \cdot B_j \right]$$

for arbitrary function $g_i$. In the literature on lattice gauge theory, it is customary to treat $A$ as the gauge field and $E$ as the gauge-invariant, physical field. In developing the effective field theory of the toric codes or indeed of all other exactly solvable spin models, however, it is essential that we view both $A$ and $E$ as gauge fields subject to their own gauge transformations, Eqs. (2.1) and (2.5). A certain combination of one gauge field becomes the generator responsible for the gauge transformation of the other, and vice versa.

Now we introduce the exponentiation procedure known as Higgsing, given by

$$X_{a,i} = \exp(-2\pi i E_i^a/p), \quad Z_{a,i} = \exp(i A_i^a)$$

for $a = 1, 2, 3$ corresponding to $a = x, y, z$, respectively. Only the $x, y$ ($1, 2$) components are meaningful in two dimensions. The generalized Pauli operators $X, Z$ obey the commutation algebra

$$ZX = \omega XZ \quad (\omega = e^{2\pi i/p})$$

for integer $p \geq 2$. The $p$-dimensional Hilbert space on which they act is defined by

$$X|g\rangle = |g+1\rangle, \quad Z|g\rangle = |\omega^g|g\rangle.$$

The eigenstates are $|0\rangle, \cdots, |p-1\rangle$ and the additions are by mod $p$.

One can define the following set of operators

$$a_i \equiv \exp \left( \frac{-2\pi i}{p} (\nabla \cdot E) \right)$$

where $a = x, y$ or $a = x, y, z$ for spatial dimensions $D = 2$ or $D = 3$.

$$c_i^a \equiv \exp(i B_i^a) = Z_{2,i} Z_{3,i+\hat{y}} Z_{2,i+\hat{z}}^{-1} Z_{3,i+\hat{x}}^{-1}$$

$$c_i^b \equiv \exp(i B_i^b) = Z_{3,i} Z_{1,i+\hat{z}} Z_{3,i+\hat{x}}^{-1} Z_{1,i+\hat{y}}^{-1}$$

$$c_i^c \equiv \exp(i B_i^c) = Z_{1,i} Z_{2,i+\hat{y}} Z_{1,i+\hat{y}}^{-1} Z_{2,i+\hat{z}}^{-1},$$

which one recognizes as the Higgsed version of the unitary operators generating gauge transformations of $A$ and $E$ fields in the parent U(1) LGT. Note how the four-spin and six-spin interactions arise naturally as a result of Higgsing. The commutativity

$$[a_i, c_i^a] = 0$$

follows automatically from that of generators as given Eq. (2.4). The mutually commuting Hermitian projectors are
subsequently constructed from these unitary operators as
\[ A_i = \frac{1}{p} \sum_{j=0}^{p-1} (a_i)^j, \quad C_i^z = \frac{1}{p} \sum_{j=0}^{p-1} (c_i^z)^j. \] (2.11)

One can readily verify their projector properties \( A_i^2 = A_i \) and \( (C_i^z)^2 = C_i^z \). The Hamiltonians given by the sum of the projectors,
\[ H_{2\text{DTC}} = -\sum_i (A_i + C_i^z), \]
\[ H_{3\text{DTC}} = -\sum_i (A_i + C_i^x + C_i^y + C_i^z) \] (2.12)

are precisely the \( \mathbb{Z}_p \) toric codes in two and three dimensions, respectively.

**B. Field Theory of Two-dimensional Toric Code**

In this subsection, we will show how the BF theory arises naturally as the effective low-energy theory of the excitations in the 2DTC. The field theory is constructed in terms of the variables of the parent U(1) LGT, namely \( A_i \) and \( E_i \) obeying \([A_i^x, E_j^y] = i\delta_{ij}\delta_{ij} \). Such commutation implies the Lagrangian \( \mathcal{L} = \sum_i E_i \cdot \partial_t A_i \). The task of effective field theory (EFT) construction is to identify the remaining terms in the Lagrangian. The final form of the EFT, namely the BF action, is already well known in the literature. Nevertheless, we present the derivation of the BF action in some detail since the methodology developed to derive it will be persistently applicable to all the other effective actions in this paper.

The 2DTC supports two types of elementary excitations called \( e \) and \( m \) particles. The \( e \) (\( m \)) particles are the excitations of the \( A_i \) (\( C_i^z \)) operators defined in Eq. (2.11). The integer-valued charge of \( e \) (\( m \)) quasiparticle is denoted by \( \rho^e \) (\( \rho^m \)), and are the eigenvalues of the operators
\[ A_i(n) = \frac{1}{p} \sum_{j=0}^{p-1} (\omega^{-n} a_i)^j, \quad C_i^z(n) = \frac{1}{p} \sum_{j=0}^{p-1} (\omega^{-n} c_i^z)^j. \] (2.13)

Explicitly, \( A_i(\rho^e)|\psi\rangle = |\psi\rangle \) and \( C_i^z(\rho^m)|\psi\rangle = |\psi\rangle \) are the quasiparticle eigenstates with integer charges mod \( p \). The statistical interaction between \( e \) and \( m \) quasiparticles generates the braiding phase equal to \( \omega^{\rho^e \rho^m} \) as one species of quasiparticle is moved round the other in a counter-clockwise loop. In the flux attachment picture, each species of quasiparticles sees the other as carrying a magnetic flux \( 2\pi \rho^e/p \) or \( 2\pi \rho^m/p \) (mod 2\pi), giving rise to the Aharonov-Bohm (AB) phase of \( e^{2\pi i \rho^e \rho^m/p} = \omega^{\rho^e \rho^m} \).

Braiding of the \( e \) particle with the charge \( \rho^e \) is performed by a product of \( Z \) operators forming a Wegner-Wilson (WW) loop,
\[ W_Z = \prod_{i \in C} (Z_i)^{\rho^e}, \] (2.14)

where \( C \) expresses a closed contour. The action of \( W_Z \) on an excited state consisting of one \( e \) and one \( m \) quasiparticle, denoted \(|\rho^e, \rho^m\rangle\), is the phase factor
\[ W_Z(\rho^e, \rho^m) \equiv |\rho^e, \rho^m\rangle = \omega^{\rho^e \rho^m} |\rho^e, \rho^m\rangle. \] (2.15)

Generalizing, one obtains the phase \( \omega^# \) where \# counts the total charge of \( m \) quasiparticles enclosed inside the loop \( C \) traced by the \( e \) particle. Referring to the Higgsing formula, Eq. (2.6), the WW loop in Eq. (2.14) becomes
\[ W_Z = \exp \left( i \rho^e \sum_{C} A_i \right) \]
\[ = \exp \left( i \rho^e \sum_{A} (\nabla \times A)^z \right) \]
\[ = \exp \left( i \frac{2\pi}{p} \rho^e \sum_{i, 1d} \rho^m_{1d} \right). \] (2.16)

The first line expresses \( W_Z \) as the sum of the gauge fields along the closed contour. (The upper indices in the gauge fields \( A_i^x \) are suppressed.) The second line is the discrete Stokes’s theorem whereby the line sum \( \sum_{i \in C} \) is replaced by the plaquette sum \( \sum_{1d \in A} \) of the discrete curls \( (\nabla \times A)^z \), with each curl centered at the plaquette carrying the dual coordinate \( 1d \). The braiding statistics obtained from the lattice model calculation is reflected in the third line. From the identity of the second and the third expressions above, we deduce the constraint
\[ (\nabla \times A)^z_{1d} = \frac{2\pi}{p} \rho^m_{1d} \] (2.17)

relating the gauge field Hilbert space (occupied by \( A \)) to the matter field Hilbert space (occupied by \( m \) quasiparticles) in a constraint equation. We will shortly introduce the \( e \) quasiparticle operator \( \psi^m \) and write the density as \( \rho^m = (\psi^m)^\dagger \psi^m \).

In the next step of the field theory construction, we assume the \( m \) quasiparticles obey the usual conservation law
\[ \partial_t \rho^m + (\nabla \cdot j^m)_{1d} = 0, \]
where \( \nabla \cdot j \) is the two-dimensional gradient. Combining the conservation law with the time derivative of Eq. (2.17) gives \((\nabla \times \partial_t A)^z_{1d} = -(2\pi/p)(\nabla \cdot j^m)_{1d}\) which is solved by
\[ \partial_t A_i = \frac{2\pi}{p} j^m_i \times \hat{z}. \] (2.18)

The two constraints, Eqs. (2.17) and (2.18), embody the flux attachment scheme and must be faithfully implemented in the field theory construction. Next, we turn to the braiding of the \( m \) particle with charge \( \rho^m \), which is performed by a product of \( X \) operators
\[ W_X = \prod_{i \in C_d} (X_i)^{\rho^m}, \] (2.19)
where \( C_d \) expresses a closed contour in the dual square lattice. Referring to the Higgsing formula, Eq. \( (2.17) \), the WW loop becomes

\[
W_X = \exp \left( -\frac{2\pi i}{p} \rho^m \sum_{i \in \mathcal{A}} E_i \right) \\
= \exp \left( \frac{2\pi i}{p} \rho^m \sum_{i \in \mathcal{A}} (\nabla^z \cdot E)_i \right) \\
= \exp \left( \frac{2\pi i}{p} \rho^m \sum_{i \in \mathcal{A}} \rho^m_i \right). \tag{2.20}
\]

The second line follows from the first by the discrete Stokes’ theorem and the third line reflects the results of the braiding calculation from the lattice model. We can therefore deduce the identity

\[
(\nabla^z \cdot E)_i = -\rho^e_i \tag{2.21}
\]

relating the gauge field Hilbert space of \( E \) with the matter field Hilbert space of \( m \). Later we will introduce \( \psi^e_i \) for the \( e \) quasiparticle operator and write \( \rho^e_i = (\psi^e_i)\dagger \psi^e_i \).

Similar to the \( m \) particles, we assume for the moment that the \( e \) quasiparticles obey the conservation law \( \partial_i \rho^e_i + (\nabla^z \cdot j^e_i) = 0 \) for some current vector \( j^e_i \). Then we get

\[
(\nabla^z \cdot \partial_i E_i) = (\nabla^z \cdot j^e_i) \quad \text{and} \quad \partial_i E_i = j^e_i. \tag{2.22}
\]

All told, the four equations \( (2.11), (2.13), (2.21) \) and \( (2.22) \) formalize the flux attachment constraints in the 2DTC. All of them can be encoded in terms of Lagrange multipliers in the action:

\[
\mathcal{L} = E_i \cdot \partial_i A_i + A_i \cdot j^e_i - \frac{2\pi}{p} E_i \cdot j^m_i \times \hat{z} \\
+ \frac{p}{2\pi} E^0_i \left( (\nabla \times A)_i - \frac{2\pi}{p} \rho^m_i \right) + A^0_i \left( (\nabla^z \cdot E)_i + \rho^e_i \right). \tag{2.23}
\]

Here, \( E^0_i \) and \( A^0_i \) are the Lagrange multipliers implementing the flux attachment constraints. To confirm that this is precisely the well-known \((2+1)\)-dimensional BF action, one just needs to perform some re-definitions

\[
-(2\pi/p)E_i \times \hat{z} \rightarrow E_i, \\
(\rho^e_i, j^e_i) \rightarrow (-\rho^e_i, -j^e_i)
\]

and obtain

\[
\mathcal{L} = \frac{p}{2\pi} \partial_i (A_i \times E)_i - A_i \cdot j^e_i - E_i \cdot j^m_i \\
+ \frac{p}{2\pi} E^0_i \left( (\nabla \times A)_i - \frac{2\pi}{p} \rho^m_i \right) + A^0_i \left( (\nabla^z \cdot E)_i - \frac{2\pi}{p} \rho^e_i \right). \tag{2.24}
\]

The continuum version of the Lagrangian cast in the three-vector notation is the familiar BF Lagrangian,

\[
\mathcal{L}_{BF} = \frac{p}{2\pi} E \cdot (\nabla \times A) - A \cdot j^e - E \cdot j^m, \tag{2.25}
\]

where \( A = (A^0, A) \), \( E = (E^0, E) \), \( j^e = (\rho^e, j^e) \), \( j^m = (\rho^m, j^m) \), \( \nabla = (\partial_t, \nabla^z) \). The accompanying commutation relations in the continuum are

\[
[A^a(r), E^b(r')] = \frac{2\pi i}{p} \epsilon_{ab} \delta^2(r - r'). \tag{2.26}
\]

The gauge transformations of the \( A \) and \( E \) fields can be read off directly from their lattice formulations \( (2.11) \) and \( (2.23) \):

\[
A(r) \rightarrow U_A^\dagger A(r) U_A = A(r) + \nabla^z f(r) \\
E(r) \rightarrow U_E^\dagger E(r) U_E = E(r) - \nabla^z g(r), \tag{2.27}
\]

where the two unitary operators

\[
U_A = \exp \left[ \frac{ip}{2\pi} \int d^2 r f(r)(\nabla \times E(r)) \right] \\
U_E = \exp \left[ \frac{ip}{2\pi} \int d^2 r g(r)(\nabla \times A(r)) \right] \tag{2.28}
\]

generate the gauge transformations of the \( A \) and \( E \) fields, respectively. The \( A \) and \( E \) fields obey the commutation relation as given in Eq. \( (2.29) \). These transformations of the spatial components of the gauge fields are to be supplemented by those of the temporal components,

\[
A^0(r) \rightarrow A^0(r) + \partial_t f(r) \\
E^0(r) \rightarrow E^0(r) - \partial_t g(r). \tag{2.29}
\]

Together, Eqs. \( (2.27) \) and \( (2.29) \) guarantee the conservation laws of \( e \) and \( m \) quasiparticles. If we assume the compactness of the gauge fields \( A \) and \( E \), the BF action

\[
S_{BF} = \int dt \, \mathcal{L}_{BF} \tag{2.30}
\]

is gauge invariant mod \( 2\pi n \ (n \in \mathbb{Z}) \) under Eq. \( (2.29) \). Note that the coupling of the matter fields to the gauge fields is already reflected in the BF construction given in Eq. \( (2.25) \), but the exact expression of the matter currents in terms of the matter field operators are not known. Indeed, the missing thread in the EFT construction so far is the precise definition of the matter current operators \( j^e \) and \( j^m \) and their conservation laws, which we assumed to be of the familiar form \( \partial_t \rho^{(e,m)} + \nabla^z \cdot j^{(e,m)} = 0 \). As the theory unfolds later in this paper, we will learn that this is not the unique conservation law one can write down for the quasiparticles as they can be many other versions that will take its place.

We proceed to construct the field theory for the \( e \) and \( m \) particles in the 2DTC. The gauge principle will play a
pivotal role in such construction and indeed all other constructions that we encounter. Invoking the constraints
\[ \nabla \times E = \frac{2\pi}{p} \rho^e, \quad \nabla \times A = \frac{2\pi}{p} \rho^m \]  
(2.31)
one can regard \( U_g \) and \( U_A \) as generators of gauge transformations \( U_e \) and \( U_m \) for the matter fields,
\[ \psi_e \rightarrow U_g^1 \psi_e U_e = e^{i f} \psi_e, \]
\[ \psi_m \rightarrow U_m^1 \psi_m U_m = e^{i g} \psi_m. \]  
(2.32)

We ask what sort of action will be invariant under the simultaneous transformation of the matter fields \[ \text{Eq. (2.28)} \] and the gauge field \[ \text{Eq. (2.27)} \], and the answer is
\[ \mathcal{L}_e = i \psi_e^\dagger \partial_\mu \psi_e - \frac{1}{2m_e} |D_\mu \psi_e|^2, \]
\[ \mathcal{L}_m = i \psi_m^\dagger \partial_\mu \psi_m - \frac{1}{2m_m} |D_\mu \psi_m|^2, \]  
(2.33)
with covariant derivatives given by
\[ D_e = \nabla^\mu - i A, \quad D_m = \nabla^\mu + i E. \]  
(2.34)

This is the standard minimal coupling of the matter fields to the \( U(1) \) gauge fields. The only unique feature is that both \( A \) and \( E \) are gauge fields, coupled to their respective matter fields \( \psi_e \) and \( \psi_m \), due to the fact that both electric and magnetic charges are on an equal footing in the 2DTC while in the real world only the electric charges are seen. It become straightforward to derive the current conservation laws for the quasiparticles from the action in Eq. \[ \text{Eq. (2.33)} \] as \( \partial_\mu \rho^e + \nabla^\mu j^e = 0 \) and \( \partial_\mu \rho^m + \nabla^\mu j^m = 0 \), with
\[ \rho^e = \psi_e^\dagger \psi_e, \quad j^e = \frac{1}{2m_e} [\psi_e^\dagger D_\mu \psi_e - \text{c.c.}], \]
\[ \rho^m = \psi_m^\dagger \psi_m, \quad j^m = \frac{1}{2m_m} [\psi_m^\dagger D_\mu \psi_m - \text{c.c.}]. \]  
(2.35)

The derivation of EFT for the 2DTC is thus complete. The resulting overall EFT is invariant under the gauge transformations of \( A \) and \( E \) given in Eq. \[ \text{Eq. (2.27)} \]. We will learn that the strategy used to construct the EFT of the 2DTC works consistently well for the construction of EFTs for other exactly solvable spin models.

We complete this subsection by writing down the lattice version of the effective matter Lagrangian in Eq. \[ \text{Eq. (2.28)} \]. There are two kinds of particles \( \psi_i^e \) and \( \psi_i^m \) at each site \( i \), and they hop on a lattice while coupled to the respective gauge fields \( A_i \) and \( E_i \). The transformation properties of the matter fields are
\[ [\psi_i^a] \rightarrow e^{i f_i} [\psi_i^a], \quad [\psi_i^b] \rightarrow e^{i g_i} [\psi_i^b], \]
and together with
\[ A_i^a \rightarrow A_i^a + f_{i+\langle \rangle} - f_i, \quad E_i^a \rightarrow E_i^a + g_{i+\langle \rangle} - g_i, \]
the gauge-invariant hopping terms are dictated to be
\[ [\psi_i^a]_1^+ [\psi_i^a]_2 e^{i A_i^a}, \quad [\psi_i^b]_1^+ [\psi_i^b]_2 e^{i E_i^a}. \]  
(2.36)
and their Hermitian conjugates. Taking the continuum limits of these terms leads to the continuum action already written down in Eq. \[ \text{Eq. (2.33)} \].

The emerging picture is that of \( e \) and \( m \) quasiparticles that are free and coupled to their respective gauge fields \( A \) and \( E \). The gauge fields are, however, not independent as they are dual to each other and are governed by the BF Lagrangian. The non-trivial mutual statistics of \( e \) and \( m \) particles arises as a consequence.

\[ \text{C. Field Theory of Three-dimensional Toric Code} \]

As with the 2DTC, construction of the EFT for 3DTC begins with the commutation relation \[ [A_i^a, E_j^b] = i \delta_{ab} \delta_{ij} \] and the Lagrangian \( \mathcal{L} = \sum E_i \partial_\mu A_i \). The \( e \) particles are still the elementary excitations of the Gauss’s law operator \( A_i \) in the 3DTC. There also seem to be three types of \( m \) particles associated with \( C_i^x, C_i^y, C_i^z \) and \( C_i^a \), \( C_i^b \) operators, respectively, at first sight. We will name them \( m_a \) particles where \( a = x,y,z \). In contrast to the free \( m \) quasiparticle in 2DTC, analysis of the 3DTC showed that the \( m_a \) quasiparticles in 3DTC are not free, but are confined as a closed loop excitation. One obvious implication of it is that the EFT of the \( m_a \) quasiparticles cannot contain free particles. By invoking the gauge principle, one can write down such kind of EFT as explained below.

The two sets of commuting operators \( A_i \) and \( C_i^a, C_i^b, C_i^c \) in the 3DTC correspond to exponentials of the underlying field operators \( \nabla \cdot E \) and \( \nabla \times A \) through Higgsing. They also serve as generators of the gauge transformations:
\[ A(r) \rightarrow U_A A(r) U_A^\dagger = A_r = A(r) + \nabla f(r), \]
\[ E(r) \rightarrow U_E E(r) U_E^\dagger = E(r) + \nabla \times g(r). \]  
(2.37)
The two unitary operators are
\[ U_A = \exp \left[ i \int d^3 r (\nabla \cdot E(r)) \right], \]
\[ U_E = \exp \left[ i \int d^3 r (\nabla \times A(r)) \right]. \]  
(2.38)

The transformation rules in Eq. \[ \text{Eq. (2.37)} \] follow readily from the commutation \[ [A_i^a(r), E_j^b(r')] = i \delta_{ab} \delta^3(r-r') \]. They are the continuum limits of the unitary operators in the lattice theory introduced in Sec. \[ \text{IIA} \].

We then introduce a scalar matter field \( \psi_e \) and a vector matter field \( \psi_m = (\psi_m^x, \psi_m^y, \psi_m^z) \) corresponding to the \( e \) and the three kinds of \( m \) quasiparticles in 3DTC. The quasiparticle densities \( \rho_e = \psi_e^\dagger \psi_e \) and \( \rho_m = (\psi_m^x)^\dagger \psi_m^x, (\psi_m^y)^\dagger \psi_m^y, (\psi_m^z)^\dagger \psi_m^z \) are related to the gauge fields through the constraints
\[ \nabla \cdot E = \rho_e, \quad \nabla \times A = \rho_m. \]  
(2.39)
As a result, the unitary operator $U_A$ and $U_E$ in Eq. \((2.38)\) become $U_e$ and $U_m$, responsible for the gauge transformations

$$
\psi_e \rightarrow U_1 U_e \psi_e = e^{i f} \psi_e
$$
$$
\psi_m \rightarrow U_m \psi_m U_m = (e^{i g_x} \psi_{m,x} + e^{i g_y} \psi_{m,y} + e^{i g_z} \psi_{m,z}).
$$

(2.40)

The unusual structure of the gauge transformation for $E$ as well as $\psi_m$ fields involving a vector of phase functions $g$ has profound implication for the EFT of the $m_a$ quasiparticles. First of all, the effective action of the quasiparticles are inherently interacting. This is the action is quartic in the fields and implies that the derivatives and the matter fields when we discuss EFT of the fractons. (By contrast, we will soon encounter which are first order in the gradient but second order in the matter fields. The antisymmetry implies that are inherent in the 3DTC and can be encoded in the Lagrangian.

The accompanying EFT of the $m_a$ quasiparticles is

$$
\mathcal{L}_e = i \psi_e^\dagger \partial_t \psi_e - \frac{1}{2|e|} |D_e \psi_e|^2,
$$

(2.41)

with $D_e \psi_e = (\nabla - iA) \psi_e$. It is invariant under $\psi_e \rightarrow e^{i f} \psi_e$ and $A \rightarrow A + \nabla f$. On the other hand, the covariant derivatives of $\psi_m$ that are consistent with the rules of gauge transformation are (no sum on $a,b$)

$$
D^{ab}_{m \psi_m} \psi^{b} = \psi_{m,a} \partial_a \psi_m - \psi_{m,b} \partial_b \psi_m^a - i \sum_c \epsilon_{abc} E^c_\rho \psi_m \psi_m^b
$$

(2.42)

which are first order in the gradient but second order in the matter fields. (By contrast, we will soon encounter covariant derivatives that are second order in both the derivatives and the matter fields when we discuss EFT of the fractons.) The accompanying EFT of the $m_a$ quasiparticles is

$$
\mathcal{L}_m = i \psi_m^\dagger \cdot \partial_t \psi_m - \frac{1}{2|m_a|} \sum_{a < b} |D^{ab}_{m \psi_m} \psi^{b} |^2.
$$

(2.43)

The action is quartic in the fields and implies that the quasiparticles are inherently interacting. This is the likely source of the confinement of the $m_a$ quasiparticles in the 3DTC, although direct verification of such claims through calculations using the EFT will be formidable.

The conservation law for the $\varepsilon$ quasiparticle is, as expected, $\partial_t \rho^\varepsilon + \nabla \cdot j^\varepsilon = 0$ with $j^\varepsilon = [\psi_e^\dagger D_e \psi_e - h.c.]/(2|m_e|)$. For the magnetic part, we obtain

$$
\partial_t \rho^{m_a} + \sum_b \partial_b [j^{m}]^{ab} = 0,
$$

(2.44)

where

$$
\rho^{m_a} = (\psi^{a}_{m}) \psi^{a}_{m},
$$

$[j^{m}]^{ab} = \frac{1}{2|m_a|} (\psi^{a}_{m}) \psi^{b}_{m} D^{ab}_{m \psi_m} \psi^{b}_{m} - h.c.].
$$

(2.45)

The vector charge $\rho^{m_a}$ calls for a tensorial current operator $[j^{m}]^{ab}$, which is antisymmetric ($[j^{m}]^{ab} = -[j^{m}]^{ba}$) by virtue of the definition in Eq. \((2.43)\). The magnetic current operator is first order in the derivative, but fourth order in the matter fields. The antisymmetry implies $[j^{m}]^{aa} = 0$, which is consistent with the analysis of the microscopic 3DTC that the motion of the $m_a$ quasiparticle takes place entirely in the plane normal to the $\hat{a}$ direction.

Unlike the 2D toric code, there is an identity $\nabla \cdot \rho^m = 0$ following from the relation $\rho^m = \nabla \times A$ in Eq. \((2.39)\). Due to the antisymmetry of the current tensor we can write it in a vector form $j = (j_x, j_y, j_z)$, introduce the vector charge density $\rho^m = (\rho^{m_x}, \rho^{m_y}, \rho^{m_z})$, and simplify the current conservation law of the magnetic particles:

$$
\partial_t \rho^m + \nabla \times j = 0.
$$

(2.46)

This completes the matter part of the EFT construction for the 3DTC. We will move to show how these constraints are included in the Lagrangian.

The gauge part of the EFT is constructed by first examining the various constraints on the gauge fields and the matter fields. In three dimensions it is no longer meaningful to speak of the braiding of one quasiparticle around the other. Instead, we derive the necessary constraints directly from the Higgsing formulas in Eq. \((2.49)\). One can read off

$$
(\nabla \times A)_{id} = \frac{2\pi}{p} (\rho^m)_{id},
$$

(2.47)

$$
(\nabla \cdot E)_i = -\rho^0_i
$$

relating the lattice curl of $A$ to the vector charge density $(\rho^m)_{id} = (\rho^{m_x}, \rho^{m_y}, \rho^{m_z})_{id}$ and the lattice divergence $\nabla \cdot E$ to the scalar charge density $\rho^0$. Taking the time derivatives on both sides of the equations and invoking the quasiparticle conservation laws in Eq. \((2.44)\), we get

$$
(\nabla \times \partial_t A)^a_{id} = -(2\pi/p) \sum_b \partial_b [j^{m}]^{ab}_{id},
$$

and

$$
(\nabla \cdot \partial_t E) = (\nabla \cdot j^m)_i.
$$

(2.48)

The equations \((2.47)\) and \((2.48)\) formalize the constraints that are inherent in the 3DTC and can be encoded in the Lagrangian

$$
\mathcal{L} = E_i \cdot \partial_t A_i = A_i \cdot j^m_i + \frac{p}{2\pi} \sum_{abc} \epsilon_{abc} E^a_i [j^{m}]^{bc}_{id} - \frac{p}{2\pi} E^a_i \left( \nabla \times A - \frac{2\pi}{p} \rho^m \right)_{id} - A^a_i \left( \nabla \cdot E + \rho^0 \right)_i.
$$

(2.49)

Here, $E^a_i = (E^{x0}_i, E^{y0}_i, E^{z0}_i)$ and $A^a_i$ are the Lagrange multipliers. Carrying out the re-definition

$$
-\sum_a \epsilon_{abc} E^a_i \rightarrow E^{bc}_i
$$

$$
A_i \rightarrow -A_i
$$

(2.50)
where \( E_{ij}^{bc} \) is antisymmetric by construction, we obtain a new Lagrangian form
\[
\mathcal{L} = \frac{p}{4\pi} \sum_{abc} \epsilon_{abc} E_{i}^{bc} \partial_t A_i^a - A_i \cdot \dot{j}_i - \frac{1}{2} \sum_{ab} E_{i}^{ab}[j^m]_{ab}^i
\]
\[
- \frac{p}{2\pi} \dot{E}_i^i \cdot \left( \nabla \times A - \frac{2\pi}{p} \rho_m \right)_{i+d}
\]
\[
+ \frac{p}{2\pi} A_i^i \left( \frac{1}{2} \sum_{abc} \epsilon_{abc} \partial_t E_{i}^{bc} - \frac{2\pi}{p} \rho^e \right)
\]
(2.51)
The action can be cast in the four-vector continuum notation (\( \epsilon_{0123} = +1 \))
\[
\mathcal{L}_{BF} = \frac{p}{2\pi} \epsilon_{\alpha\beta\mu\nu} \frac{1}{2} E_{i}^{\alpha\beta} \partial_\mu A^\nu - A_i \cdot j^c - \frac{1}{2} E_{i}^{\alpha\beta}\sum_{\alpha\beta}
\]
(2.52)
where \( A \) and \( j^c \) are
\[
A = (A^0, \mathbf{A}) \quad j^c = (\rho^c, j^c)
\]
and \( E^{\mu\nu} \) and \( [j^m]^{\mu\nu} \) are
\[
E^{\mu\nu} = \begin{pmatrix} 0 & -E^{x0} & -E^{y0} & -E^{z0} \\ -E^{0x} & 0 & E^{0y} & E^{0z} \\ -E^{z0} & -E^{y0} & 0 & E^{z0} \\ E^{z0} & -E^{y0} & -E^{x0} & 0 \end{pmatrix}
\]
\[
[j^m]^{\mu\nu} = \begin{pmatrix} \rho_m^x & -\rho_m^y & -\rho_m^z \\ \rho_m^y & \rho_m^x & 0 \\ \rho_m^z & 0 & \rho_m^x \\ -\rho_m^x & \rho_m^z & \rho_m^y \end{pmatrix}
\]
(2.54)
The commutation relation and the gauge transformation of the \( A^a \) and \( E^{ab} \) in the continuum field theory are
\[
[A^a(r), E^{bc}(r')] = \frac{2\pi i}{p} \epsilon_{abc} \delta^3(r - r')
\]
\[
A(r) \rightarrow U_A(r)A(r)U_A^\dagger = A(r) + \nabla f(r)
\]
\[
E^{ab}(r) \rightarrow U_A^\dagger E_{i}^{ab}(r)U_E = E_{i}^{ab}(r) - \partial_a g_b^i(r) + \partial_b g_a^i(r)
\]
(2.55)
where
\[
U_A = \exp \left[ \frac{ip}{4\pi} \sum_{abc} \epsilon_{abc} \int d^3 r f(r) \partial_t E_{i}^{bc}(r) \right]
\]
\[
U_E = \exp \left[ -\frac{ip}{2\pi} \int d^3 r g(r) \cdot \nabla \times A(r) \right]
\]
(2.56)
The gauge transformations of the temporal components \( A^0 \) and \( E^0 \) are
\[
A^0(r) \rightarrow A^0(r) + \partial_t f(r)
\]
\[
E^{ab}(r) \rightarrow E^{ab}(r) - \partial_a g^b(r) + \partial_b g_a^i(r)
\]
(2.57)
Accordingly, \( A \cdot j^c \) and \( \frac{1}{2} E^{\alpha\beta}[j^m]^{\alpha\beta} \) transform under the gauge transformations as
\[
A \cdot j^c \rightarrow A \cdot j^c + \sum_\mu \partial_\mu f[j^c]^\mu
\]
\[
\frac{1}{2} E^{\alpha\beta}[j^m]^{\alpha\beta} \rightarrow \frac{1}{2} E^{\alpha\beta}[j^m]^{\alpha\beta} + \frac{1}{2} (-\partial_a g^i + \partial_b g^a)[j^m]^{\alpha\beta}
\]
(2.58)
Applying integration by parts to the two equations in Eq. (2.58) leads to the conservation laws of \( e \) and \( m \) quasiparticles as well as the identity \( \nabla \cdot \rho^m = 0 \), which means the Lagrangian we just constructed reflects those constraints well.

This completes the derivation of the EFT for the 3DTC as a (3+1) BF theory plus the minimally coupled theory of matter fields. The BF part of the field theory was understood for quite some time, but the construction of the EFT for the \( m \) quasiparticles is new. The quartic nature of the Lagrangian points to the highly interacting nature of the \( m \) quasiparticles that could explain the confinement of \( m \) quasiparticles in 3DTC, but not in 2DTC where the matter field theory is quadratic and free. It is a testimony to the power of the gauge principle that such highly interacting Lagrangian can be derived with no extra effort beyond the requirement of gauge invariance.

A lattice action of the matter field action can be constructed easily. For the \( e \) part we have the usual hopping model with the Peierls substitution \([\psi_{m}^e]_a^+ [\psi_{m}^e]_a e^{iA^a_m} \) which is gauge-invariant. On the other hand, the gauge transformations of the \( E \) field and \( \psi_m \) are given by
\[
E_{i}^{ab} \rightarrow E_{i}^{ab} - \delta_i g_a^b + \delta_i g_b^a - g_i^a,
\]
\[
[\psi_{m}^e]_a \rightarrow (e^{i\delta_i^a}[\psi_{m}^e]_a e^{i\delta_i^b}[\psi_{m}^e]_b), e^{i\delta_i^b}[\psi_{m}^e]_b),
\]
and the gauge-invariant hopping terms are
\[
\left[\psi_{m}^e]_a \rightarrow (e^{i\delta_i^a}[\psi_{m}^e]_a e^{i\delta_i^b}[\psi_{m}^e]_b), e^{i\delta_i^b}[\psi_{m}^e]_b),
\]
(2.59)
and their Hermitian conjugates. A correlated hopping of several species of \( m \) quasiparticles characterize the tight-binding model characterization of the \( m \) particle dynamics in the 3DTC.

III. RANK-2 U(1) LATTICE GAUGE THEORY, X-CUBE MODEL AND ITS FIELD THEORY

Similar to the treatment of toric codes in the previous section, we first lay the ground for the parent rank-2 U(1) LGT in three dimensions which give rise to the X-cube model through Higgsing. The relevant EFT of the X-cube model is constructed subsequently.

A. Rank-2 U(1) LGT and the X-Cube Model

The XC model follows from Higgsing a specific kind of rank-2 U(1) LGT in three dimensions. Construction of the LGT utilizes symmetric gauge fields \( A_{i}^{ab} = A_{i}^{ba} \) (a, b = x, y, z) obeying the hollowness condition \( A_{i}^{aa} = 0 \) (no sum on a) for the diagonal components. The conjugate fields \( E_{i}^{ab} \) satisfying the commutation \([A_{i}^{ab}, E_{j}^{cd}] = \)
i∂f share the same symmetry and hollowness properties: $E_{i}^{aa} = E_{i}^{ab}$ and $E_{i}^{aa} = 0$.

The transformation rule for the gauge fields are

$$A_{i}^{ab} \rightarrow U_{A}^d A_{i}^{ab} U_{A}^{-1}$$

$$= A_{i}^{ab} - f_{i-\hat{a}-\hat{b}} + f_{i-\hat{a}} - f_{i-\hat{b}} - f_{i}$$

$$\sim A_{i}^{ab} \sim \partial_{\hat{a}} \partial_{\hat{b}} f_{i}$$

$$E_{i}^{ab} \rightarrow U_{E}^d E_{i}^{ab} U_{E}^{-1}$$

$$= E_{i}^{ab} + \sum_{c} \epsilon_{abc} (g_{i-\hat{c}}^{\hat{b}} - g_{i}^{\hat{b}} - g_{i-\hat{a}+\hat{c}}^{\hat{a}} + g_{i}^{\hat{a}})$$

$$\sim E_{i}^{ab} + \sum_{c} \epsilon_{abc} (\partial_{\hat{c}} g_{i}^{\hat{b}} - \partial_{\hat{a}} g_{i}^{\hat{b}}).$$

(3.1)

The unitary operators are defined by

$$U_{A} = \exp \left[ i \sum_{j} f_{j}(DE)_{j} \right]$$

$$U_{E} = \exp \left[ i \sum_{j} g_{j} \cdot B_{j} \right].$$

(3.2)

Generators of the gauge transformations are

$$(DE)_{i} \equiv E_{i+\hat{x}+\hat{y}}^{xy} - E_{i+\hat{y}}^{xy} - E_{i+\hat{y}}^{xy} + E_{i}^{xy}$$

$$+ E_{i+\hat{y}}^{y\hat{z}} - E_{i-\hat{y}}^{y\hat{z}} - E_{i-\hat{y}}^{y\hat{z}} + E_{i}^{y\hat{z}}$$

$$+ E_{i+\hat{z}+\hat{y}}^{z\hat{z}} - E_{i+\hat{z}}^{x\hat{z}} - E_{i+\hat{z}}^{x\hat{z}} + E_{i}^{x\hat{z}}$$

$$\sim \partial_{\hat{y}} \partial_{\hat{z}} E_{i}^{xy} + \partial_{\hat{y}} \partial_{\hat{z}} E_{i}^{y\hat{z}} + \partial_{\hat{z}} \partial_{\hat{y}} E_{i}^{x\hat{z}},$$

$$B_{i}^{a} \equiv \sum_{bc} \epsilon_{abc} (A_{i}^{ab} - A_{i-\hat{a}-\hat{c}}^{ab}).$$

(3.3)

One can easily prove the commutativity $[(DE)_{i}, B_{j}^{a}] = 0$. It also follows from the definition that $B_{i}^{x} + B_{i}^{y} + B_{i}^{z} = 0$.

There are only three independent components in the tensors $A_{i}^{ab}$ and $E_{i}^{ab}$ and they can be mapped to vector fields

$$(A_{i}^{yz}, A_{i}^{xz}, A_{i}^{xy}) \rightarrow (A_{i}^{yz}, A_{i}^{xz}, A_{i}^{xy})$$

$$(E_{i}^{yz}, E_{i}^{xz}, E_{i}^{xy}) \rightarrow (E_{i}^{yz}, E_{i}^{xz}, E_{i}^{xy})$$

(3.4)

on the dual lattice as shown in Fig. 1. The original fields $A_{i}^{ab}$ and $E_{i}^{ab}$ are defined on the faces of a cube while the dual fields $A_{i}^{ab}$ and $E_{i}^{ab}$ are defined on the link $(i_{a}, i_{d} + \hat{a})$ of the dual cubic lattice. The gauge variables $f_{i-\hat{a}+\hat{b}}, f_{i-\hat{a}}, f_{i-\hat{b}}, f_{i}$ in Eq. (3.1) reside on the four corners of the plaquette at which $A_{i}^{ab}$ is defined.

We re-label the dual lattice sites $i_{d}$ as $i$ from now on. Correspondingly, the tensor fields ($A_{i}^{ab}, E_{i}^{ab}$) will be replaced by the vector notations ($A^{a}, E^{a}$). The Higgsing of the rank-2 U(1) LGT is performed by ($a = x, y, z$ or $a = 1, 2, 3$)

$$X_{a,i} = \exp(iA_{i}^{a}), \quad Z_{a,i} = \exp(2\pi i E_{i}^{a}/p).$$

(3.5)

Compared to the Higgsing formula used in Eq. (2.6) for the toric codes, the roles of $X$ and $Z$ operators have been switched in Eq. (3.5). This is by design and will be crucial in constructing the hybrid (F3) model in the next subsection.

A set of mutually commuting unitary operators ($a^{a}_{i}, a^{y}_{i}, a^{z}_{i}, b_{i}$) in the $p$-dimensional Hilbert space of spins is constructed as

$$a^{a}_{i} \equiv \exp(iB_{i}^{a}) = X_{-1,3}X_{-3,1}X_{-1,2}X_{1,2},$$

$$a^{y}_{i} \equiv \exp(iB_{i}^{y}) = X_{1,1}X_{1,-1}X_{-1,3}X_{3,1},$$

$$a^{z}_{i} \equiv \exp(iB_{i}^{z}) = X_{2,2}X_{2,-2}X_{1,1}X_{1,-1},$$

$$b_{i} \equiv \exp \left( \frac{2\pi i}{p} (DE)_{i} \right).$$

$$= Z_{1,1}Z_{1,2}Z_{-1,1}Z_{2,2}$$

$$\times Z_{3,2}Z_{3,1}Z_{-3,2}Z_{-3,1}Z_{3,3}. (3.6)$$

The mutually commuting Hermitian projectors are constructed out of them as

$$A_{a}^{t} = \frac{1}{p} \sum_{j=0}^{p-1} (a^{a}_{i})^{j}, \quad B_{i} = \frac{1}{p} \sum_{j=0}^{p-1} (b_{i})^{j},$$

(3.7)

where $a = x, y, z$. Finally, we arrive at the Hamiltonian

$$H_{XC} = - \sum_{i} (A_{a}^{t} + A_{y}^{t} + A_{z}^{t} + B_{i})$$

(3.8)

which is precisely the $\mathbb{Z}_{p}$ X-cube model.

Both the three-dimensional toric code and the X-cube model are built out of the same $p$-dimensional Hilbert space at the links of the cubic lattice. The difference between the two models arises from their parent LGT being rank-1 or rank-2 U(1) LGT. Different sets of mutually
commuting generators are available in each LGT, resulting in a different set of Higgsed spin operators that make up the 3DTC and the XC model, respectively. In the following section, we ask whether it is possible to combine both LGTs in the same model and then, through Higgsing, create a new kind of exactly solvable spin model. The result is the F3 model proposed earlier by two of the present authors. We first finish up the field-theoretic construction of the X-cube model.

B. Field Theory of X-cube Model

The two elementary excitations in the XC model are denoted \(l\)-quasiparticles (\(a = x, y, z\)) and fractons. The \(l\)-quasiparticles are the excitations of the \(A_i^a\) operators while fractons are those of the \(B_i\) operator, both defined in Eq. (3.7). The integer-valued charges of \(l\)-quasiparticles and fractons which are defined mod \(p\) are denoted by \(\rho^a\) and \(\rho^f\), and are respectively determined as the eigenvalues of the operators

\[
A_i^a(n) = \frac{1}{p} \sum_{j=0}^{p-1} (\omega^{-n a_i^a})^j, \quad B_i(n) = \frac{1}{p} \sum_{j=0}^{p-1} (\omega^{-n b_i})^j.
\]

(3.9)

The eigenstate \(|\psi_{l a}\rangle\) with a \(l\)-quasiparticle is defined as \(A_i^a(\rho^a)|\psi_{l a}\rangle = |\psi_{l a}\rangle\) and \(B_i(\rho^f)|\psi_f\rangle = |\psi_f\rangle\) defines the eigenstate with a fracton excitation. In other words, \(|\psi_{l a}\rangle\) is an eigenstate of \(a_i^a(b_i)\) with eigenvalue \(\omega^{\rho^a}(\omega^{\rho^f})\). Then, since \(a_i^a a_i^a a_i^a = 1\) by Eq. (3.6), we can derive a constraint \(\sum_a \rho^a = 0\) (mod \(p\)).

The term lineon is used to refer to the excitation in the XC model that can move freely only in a particular direction. They can be precisely defined in terms of the \(l\)-quasiparticles just defined, as the eigenstates satisfying

\[
(x): A_i^x(0)|\psi_x\rangle = A_i^x(-\rho^x)|\psi_x\rangle = A_i^x(\rho^x)|\psi_x\rangle = |\psi_x\rangle,
\]

\[
(y): A_i^y(p^y)|\psi_y\rangle = A_i^y(0)|\psi_y\rangle = A_i^y(-\rho^y)|\psi_y\rangle = |\psi_y\rangle,
\]

\[
(z): A_i^z(-\rho^z)|\psi_z\rangle = A_i^z(\rho^z)|\psi_z\rangle = A_i^z(0)|\psi_z\rangle = |\psi_z\rangle.
\]

(3.10)

The \(x, y, z\) lineons denoted by \((x), (y), (z)\) with charges \(\rho^x, \rho^y, \rho^z\) are respectively defined as having the eigenvalues \((0, -\rho^x, \rho^x), (\rho^y, 0, -\rho^y), (-\rho^z, \rho^z, 0)\) of the \(l\)-quasiparticles. In other words, lineons are composite excitations consisting of a pair of \(l\)-quasiparticles. We, therefore, view the \(l\)-quasiparticles as the more fundamental excitations and proceed to construct field theories for them, with the lineon dynamics emerging as a natural by-product.

As in the case of 3DTC and its field theory construction, we find it useful to start by examining the gauge transformation properties of the \(A\) and \(E\) fields, given by

\[
A^a(r) \rightarrow U_A^a A^a(r) U_A = A^a(r) - \frac{1}{2} \sum_{bc} |\epsilon_{abc}| \partial_b \partial_c f(r)
\]

\[
E^a(r) \rightarrow U_E^a E^a(r) U_E = E^a(r) + \sum_{bc} |\epsilon_{abc}| \partial_a g^b(r),
\]

(3.11)

where

\[
U_A = \exp \left[ \frac{i}{2} \sum_{abc} |\epsilon_{abc}| \int d^3r f(r) \partial_a \partial_b E^c(r) \right]
\]

\[
U_E = \exp \left[ i \sum_{abc} |\epsilon_{abc}| \int d^3r g^a(r) \partial_c A^f(r) \right].
\]

(3.12)

The sums \(\sum_{bc}\) and \(\sum_{abc}\) span the \(x, y, z\) indices. The generators in \(U_A\) and \(U_E\) are the continuum expressions of the two mutually commuting operators \((DE)_i\) and \(B_i\) defined in Eqs. (3.3) after the suitable transcription from the tensor to the vector notations.

Introducing the fracton matter field \(\psi_f\) and the \(l\)-quasiparticle matter fields \(\psi_i = (\psi_i^x, \psi_i^y, \psi_i^z)\), we seek their minimal coupling structure to the gauge fields \((A,E)\) that is consistent with the transformation rules of the gauge fields in Eq. (3.11) as well as

\[
\psi_f \rightarrow e^{i l_f} \psi_f
\]

\[
\psi_i \rightarrow (e^{i g^x}_1 \psi_i^x, e^{i g^y}_1 \psi_i^y, e^{i g^z}_1 \psi_i^z)
\]

(3.13)

for the matter fields. This is understood as \(U_A^1 \psi_f U_A\) and \(U_E^1 \psi_i U_E\) under the constraints

\[
\frac{1}{2} \sum_{abc} |\epsilon_{abc}| \partial_a \partial_b E^c = \rho^f = \psi_f^\dagger \psi_f, \quad \sum_{abc} |\epsilon_{abc}| \partial_a \partial_b A^b = \rho^n = (\psi_i^\dagger)^1 \psi_i^a.
\]

(3.14)

Given these ingredients, one can now construct covariant derivatives

\[
D_{f}^{ab} \psi_f = \psi_f \partial_a \partial_b \psi_f - \partial_a \psi_f \partial_b \psi_f + i \sum_{c} |\epsilon_{abc}| A^c \psi_f
\]

\[
D_{f}^{i} \psi_i = \psi_i \partial_z \psi_i^x - \psi_i^x \partial_z \psi_i + i E^z \psi_i^y \psi_i^x
\]

\[
D_{f}^{i} \psi_i = \psi_i \partial_x \psi_i^y - \psi_i^y \partial_x \psi_i + i E^x \psi_i^y \psi_i^x
\]

\[
D_{f}^{i} \psi_i = \psi_i \partial_y \psi_i^z - \psi_i^z \partial_y \psi_i + i E^y \psi_i^z \psi_i^x,
\]

(3.15)

both of which are quadratic in the fields. The fracton covariant derivative contains two powers of the gradients while the \(l\)-quasiparticle has only one power of it. The minimally coupled Lagrangians are

\[
\mathcal{L}_f = i \psi_f^\dagger \partial_t \psi_f - \frac{1}{2m_f} \sum_{a<b} |D_{f}^{ab} \psi_f|^2
\]

\[
\mathcal{L}_l = i \psi_i^\dagger \partial_t \psi_i - \frac{1}{2m_m} \sum_a |D_{f}^{a} \psi_i|^2.
\]

(3.16)
The matter field Lagrangians are quartic in the matter fields and interacting.

Their current conservation laws follow from the effective Lagrangians (3.16) as
\[
\partial_t \rho^f - \sum_{a \leq b} \partial_a \partial_b [j^f]^{ab} = 0,
\]
\[
\partial_t \rho^a + \sum_{bc} \epsilon_{abc} \partial_c [j^f]^c = 0,
\]
and we can find an identity \( \sum_a \rho^a = 0 \) derived from the definition of \( \rho^f \) in Eq. (3.14). Those The quasiparticle density and the current operators are given by
\[
\rho^f = \psi_1^\dagger \psi_f,
\]
\[
[j^f]^{ab} = \frac{1}{2m_f^2} \left[ \psi_1^\dagger \psi_f \right] D_f^{ab} \psi_f - h.c. ,
\]
\[
\rho^a = (\psi_1^a)^\dagger \psi_1^a, 
\]
\[
[j^f]^a = -\frac{1}{4m_f^2} \left[ \sum_{bc} \epsilon_{abc} (\psi_1^b)^\dagger (\psi_1^c)^\dagger D_f^{bc} \psi_1 - h.c. \right]. 
\]

The fracton current density tensor is symmetric, \( [j^f]^{ab} = [j^f]^{ba} \). In particular, the fracton conservation law involving two spatial derivatives implies the conservation of the fracton dipole moment \( \int d^3r \, \rho^f (\mathbf{r}) \) over time and the immobility of a single fracton is guaranteed. We can also understand the one-dimensional movement of lineons using the current conservation laws of \( l_a \) quasiparticles. In the case of \( x \)-lineon with charge \( \rho^x \), the conditions \( \rho^x = -\rho^y = \rho^z \) and \( \rho^x = 0 \) renders the current conservation laws for the \( l_a \) quasiparticles,
\[
\partial_t \rho^x + \partial_x [j^x]^x = \partial_y [j^y]^y = \partial_z [j^z]^z. 
\]

Movement of the \( x \)-lineon along the \( x \) direction without extra energy cost corresponds to the case \( \partial [j^y]^y = \partial [j^z]^z = 0 \), whereby Eq. (3.19) becomes \( \partial_t \rho^x + \partial_x [j^x]^x = 0 \) - a simple conservation law of free quasiparticles in one dimension. Another movement allowed for the \( x \)-lineon of charge \( \rho^x \) is that it can be changed to a charge \( -\rho^y \) \( y \)-lineon moving along the \( +y \) direction and a charge \( -\rho^z \) \( z \)-lineon moving along the \(+z\) direction. Such metamorphosis of the \( x \)-lineon into a pair of \( y \) and \( z \)-lineons can be shown to take place in the XC model. In fact, a possible solution of Eq. (3.19) is
\[
\rho^x (\mathbf{r}) = \rho^\ast \delta (x) \delta (y) \delta (z) \theta (-t)
\]
\[
[j^x]^x (\mathbf{r}) = 0
\]
\[
[j^y]^y (\mathbf{r}) = -\rho^\ast \delta (x) \theta (y) \delta (z) \delta (t)
\]
\[
[j^z]^z (\mathbf{r}) = -\rho^\ast \delta (x) \delta (y) \theta (z) \delta (t),
\]
where \( \theta (x) \) is the step function. According to the solution, the \( x \)-lineon charge vanishes as \( t \) becomes positive. In its place, a pair of \( y \)- and \( z \)-lineon currents appear for \( t > 0 \). Overall, the effective theory of the matter fields captures the lineon dynamics of the XC lattice model rather well. This completes the discussion of the matter part of the EFT for the XC model.

The gauge part of the EFT construction begins with a pair of gauge fields \( A_i \) and \( E_i \), obeying \( [A^a_i, E^b_i] = i \delta_{ab} \delta_{ij} \). The corresponding Lagrangian is \( L = \sum_i E_i \partial_t A_i \). Similar to the derivation of EFT of 3DTC, we identify the constraints directly from the Higgsing formula in Eq. (3.17).

The first of the constraints is
\[
\frac{1}{2} \sum_{abc} \epsilon_{abc} (\partial_a \partial_b E^c) = \rho^f. 
\]

Taking the time derivative on both sides and invoking the fracton conservation given by Eq. (3.17), we get
\[
(1/2) \sum_{abc} \epsilon_{abc} (\partial_a \partial_b E^c)_i = \sum_a \epsilon_{abc} (\partial_a \partial_b [j^f]^{ab})_i 
\]
and its solution
\[
E^a_i = \frac{1}{2} \sum_{bc} \epsilon_{abc} (j^f)^{bc}_i. 
\]

The second constraint is
\[
\sum_{bc} \epsilon_{abc} (\partial_c A^c)_i = \frac{2\pi}{p} l_a. 
\]

Invoking the conservation law of the \( l_a \)-quasiparticles given in Eq. (3.17) we get \( \sum_{bc} \epsilon_{abc} (\partial_c A^c)_i = - (2\pi/p) \sum_{bc} \epsilon_{abc} (\partial_c [j^f]^{bc})_i \), and the solution is
\[
A_i = - \frac{2\pi}{p} j_i. 
\]

Equations (3.21)-(3.24) formalize the constraints in the XC model, which are embodied in the Lagrangian
\[
L = E_i \partial_t A_i + \frac{1}{2} \sum_{abc} \epsilon_{abc} A_i^a [j^f]^{bc}_i + \frac{2\pi}{p} E_i \cdot j_i 
\]
\[
+ \frac{p}{2\pi} F_i^{a0} \left( \sum_{bc} \epsilon_{abc} \partial_c A^c - \frac{2\pi}{p} \rho^b \right)_i 
\]
\[
- A_i^0 \left( \sum_{abc} \epsilon_{abc} \frac{1}{2} \partial_a \partial_b E^c \right)_i. 
\]

where \( F_i^{a0} \) and \( A_i^0 \) are the Langrange multipliers. After re-defining
\[
\sum_a \epsilon_{abc} A_i^a \rightarrow A_i^{bc} 
\]
\[
- (2\pi/p) E_i^a \rightarrow E_i^a 
\]
\[
(p_i^f, j_i^f) \rightarrow (-p_i^f, -j_i^f), 
\]
the continuum Lagrangian takes the form
\[
L = \frac{p}{2\pi} \sum_{abc} \epsilon_{abc} \left( \frac{1}{2} A_i^{ab} \partial_c E^c - \frac{1}{2} A_i^{ab} [j^f]^{ab} - E_i \cdot j_i 
\]
\[
+ \frac{p}{2\pi} E_i^{a0} \left( \sum_{bc} \epsilon_{abc} \partial_c A_i^{ab} - \frac{2\pi}{p} \rho^b \right)_i 
\]
\[
+ A_i^0 \left( \frac{p}{2\pi} \sum_{abc} \epsilon_{abc} \frac{1}{2} \partial_a \partial_b E^c - \rho^f \right). 
\]
The commutation relation and the gauge transformation properties of the $A^{ab}$ and $E^a$ in Eq. (3.27) are

$$[E^a(r), A^{bc}(r')] = \frac{2\pi i}{p} \delta^3(r - r')$$

$$A^{ab}(r) \rightarrow A^{ab}(r) + \partial_a \partial_b f$$

$$E^a(r) \rightarrow E^a(r) - \epsilon_{abc} \partial_c g^b$$, \hspace{1cm} (3.28)

where $U_A$ and $U_E$ are

$$U_A = \exp \left[ \frac{ip}{4\pi} \sum_{abc} \epsilon_{abc} \int d^3r f(r) \partial_a \partial_b E^c(r) \right]$$

$$U_E = \exp \left[ \frac{i p}{2\pi} \sum_{abc} \epsilon_{abc} \int d^3r g^a(r) \partial_c A^{ab}(r) \right].$$ \hspace{1cm} (3.29)

In order to properly embed the conservation laws of $l_a$-quasiparticles and fractons as well as the identity $\sum_a \rho^a = 0$ in the Lagrangian, the gauge transformations of $A^0$ and $E^{ab}$ are chosen to be

$$A^0(r) \rightarrow A^0(r) + \partial_i f_i(r)$$

$$E^{ab}(r) \rightarrow E^{ab}(r) + \partial_\rho g^a(r) + g^0(r).$$ \hspace{1cm} (3.30)

An arbitrary function $g^0$ included in the gauge transformation of $E^{ab}(r)$ is essential to derive the constraint $\sum_a \rho^a = 0$. Conservation laws of $l_a$-quasiparticles and fractons follow from requiring gauge invariance of the Lagrangian.

As in the case of the EFT construction of the 3DTC, we managed to reproduce the gauge part of the action and, more significantly, derive the minimally coupled field theory of the matter fields from consistent application of the gauge principle. This Lagrangian agrees with the one derived in Ref. 23 except the gauge transformation of $E^{ab}(r)$.

The matter Lagrangians can be formulated in the lattice hopping model of the fracton fields $\psi^f_i$. The fact that the fracton fields couple to the $A$ field and that its transformation properties are as given in Eq. (3.27): $A^{ab} \rightarrow A^{ab} - f_{i-a-b} + f_{i-a} + f_{i-b} - f_i$, we may write down the gauge-invariant hopping of the fractons as

$$(\psi^f_i)^\dagger \psi^f_{i-a-b} (\psi^f_{i-a})^\dagger \psi^f_{i-a-b} \psi^f_i e^{iA^{ab}_{i-a-b}}$$ \hspace{1cm} (3.31)

and its Hermitian conjugate. Note that this is a three-dimensional extension of the quadrupole model studied in depth by Hughes and collaborators in the context of the theory of multipole topological insulators 27. The $l_a$-quasiparticles are coupled to the $E^{ab}$ fields that transform as in Eq. (3.27). Appropriate hopping terms are

$$(\psi^{f*}_i)^\dagger \psi^{f*}_{i+y} \psi^{f*}_{i+z} \psi^{f*}_{i+x} e^{iE^{cz}_i}$$,

$$(\psi^{f*}_i)^\dagger \psi^{f*}_{i+y} \psi^{f*}_{i+z} \psi^{f*}_{i+x} e^{iE^{cz}_i}$$,

$$(\psi^{f*}_i)^\dagger \psi^{f*}_{i+y} \psi^{f*}_{i+z} \psi^{f*}_{i+x} e^{iE^{cz}_i}$$, \hspace{1cm} (3.32)

and their Hermitian conjugates.

IV. HYBRID RANK-1 AND RANK-2 LATTICE GAUGE THEORY, F3 MODEL AND ITS FIELD THEORY

We finally come to the discussion of the F3 model that we proposed in an earlier publication. Proceeding in analogy with the previous two sections, we first lay out the underlying LGT of the F3 model and its Higgsing, and then construct the relevant field theory.

A. F3 Model from Higgsing the Hybrid Rank-1 and Rank-2 LGT

There are two types of terms in the XC model, and we ask if any of them can be added onto the 3DTC without violating the exact solvability. One sees readily that the addition of the magnetic field terms $a^2_i$ from the XC model to the 3DTC poses a problem because they in general fail to commute with the magnetic flux operators $c^a_i$ in the 3DTC: $[a^a_i, c^b_j] \neq 0$. (Take $a^x_i$ and $c^y_j$ for example, and one finds there is only one link on which the X operator from $a^x_i$ and the Z operator from $c^y_j$ act. This results in the phase factor $\omega$ under the exchange of X and Z operators, which is nonzero and spoils the commutativity.)

The Gauss’s law $B_i$ of the XC obviously commute with the magnetic fields $c^a_i$ of the 3DTC as they are built out of the same Z operators. On the other hand, the $B_i$’s commute with the Gauss’s law $A_i$ of the 3DTC, only if the local Hilbert space dimension is $p = 3$. The restriction has a geometric origin. The Gauss’s law of the 3DTC is defined on the six links connected to a vertex while that of the X-cube is defined on the twelve links of a cube. When the two operators share a common support, they do so over the three links. Commuting the two Gauss’s law operators, therefore, results in the phase factor $\omega^3$ (since the non-commuting operator pairs occur three times), which equals $\omega^3 = 1$ only for $p = 3$. We call this the compatibility condition as it addresses the compatibility of the Gauss’s laws from rank-1 and rank-2 U(1) LGTs in a given lattice realization. When the compatibility condition is fulfilled, we obtain a new, exactly solvable spin Hamiltonian

$$H_{F3} = -\sum_i A_i - \sum_i B_i - \sum_{i,a} c^a_i,$$ \hspace{1cm} (4.1)

studied previously 34. This model is the sum of the 3DTC ($A_i$ and $C^a_i$) and the Gauss’s law $B_i$ of the XC model. In a previous publication, this model as well as the simplified version consisting of only the sum of the two Gauss’s law operators $H_{F2} = -\sum_i (A_i + B_i)$ had been studied extensively in regard to their ground states and elementary excitations. The quasiparticle excitations associated with $A_i$, $B_i$ and $C^a_i$ were respectively dubbed the freon, fracton, and the fluxon, hence the name F3 model. One can think of F3 as the 3DTC modified by the pres-
ence of the cube term \(-\sum B_i\). This view will permeate the field-theoretic discussion that we will construct shortly.

We can derive the F3 model by Higgsing the gauge theory of two vector gauge fields \(A_i\) and \(E_i\) which satisfy the commutation relation \([A^a_i, E^b_j] = i\delta_{ij}\delta_{ab}\). One can identify the three “mutually commuting” generators, \((\nabla \cdot A)_i\), \((\nabla \times E)_i\), and \((\text{DE})_i\), given by

\[
(\nabla \cdot A)_i = \sum_a (A^a_i - A^a_{i-a}) \\
(\nabla \times E)_i = \sum_{\alpha \beta \gamma} e_{\alpha \beta \gamma} (E^\alpha_i - E^\alpha_{i+\hat{\beta}}) \\
(\text{DE})_i = E^x_{i+\hat{x}+\hat{y}} - E^x_{i+\hat{y}} - E^x_{i+\hat{z}} + E^x_i + \cdots
\]

One can recognize the first two generators as those of the rank-1 LGT in Eqs. (2.2) and (2.3), except for the reversal of the roles \(A \leftrightarrow E\). The rank-2 \(U(1)\) generator \((\text{DE})_i\) is taken from Eq. (4.3). In fact, we do not have strict commutativity of all three operators since the explicit calculation shows

\[
[[(\nabla \cdot A)_i, (\text{DE})_j]] = 3i(\delta_{ij} + \delta_{i,j+\hat{x}+\hat{y}} + \delta_{i,j+\hat{z}+\hat{x}} + \delta_{i,j+\hat{y}+\hat{z}} + \delta_{i,j+\hat{y}} - \delta_{i,j+\hat{z}} + \delta_{i,j+\hat{z}+\hat{y}}).
\]

Nevertheless, we can show that the commutativity is recovered after the Higgsing, justifying the use of “mutually commuting” operators in parenthesis.

The Higgsing formula is given by

\[
X_{a,i} = \exp(iA^a_i), \quad Z_{a,i} = \exp(2\pi i E^a_i/p).
\]

One can easily identify

\[
\exp((\nabla \cdot A)_i) = a_i \\
\exp(2\pi i (\nabla \times E^a_i)/p) = b_i \\
\exp(2\pi i (\text{DE})^a_i)/p) = c_i
\]

as the \(a_i\), \(b_i\), \(c_i\) operators of the F3 model. By virtue of Eq. (4.3), we see that the commutator \([[\nabla \cdot A)_i, (\text{DE})_j]] \) equals \(+3i\delta_{ij}\delta_{ab}\), and therefore \([a_i, b_j] = 0\) provided \(p = 3\). That is, although the generators themselves do not commute, their Higgsed versions do commute in the special case of \(p = 3\).

The gauge transformation rules of \(A_i\) and \(E_i\) are

\[
A^a_i \rightarrow U^a_j A^a_j U^a_A^{-1} \\
E^a_i \rightarrow U^a_j E^a_j U^a_A^{-1}
\]

The two unitary operators responsible for the gauge transformation are

\[
U_A = \exp \left[ i \sum_j \left( f^0_j (\text{DE})_j + f^0_j \cdot (\nabla \times E)_j \right) \right] \\
U_E = \exp \left[ i \sum_j g_j (\nabla \cdot A)_j \right].
\]

for \(f = (f^x, f^y, f^z)\). (When \(f^0_j = 0\), these unitary operators are those of the 3DTC with the roles of \(A\) and \(E\) interchanged.) Since the generators do not commute, \([[\nabla \cdot A)_i, (\text{DE})_j]] \neq 0\), neither do the unitary operators \(U_A U_E \neq U_E U_A\). An explicit calculation shows

\[
\sum_j f^0_j (\text{DE})_j, \sum_i g_i (\nabla \cdot A)_i] = -3i \sum_i g_i [f^0_i + f^0_{i-\hat{x}}, f^0_{i-\hat{x} - \hat{y}}, f^0_{i-\hat{x} - \hat{z}} - f^0_{i-\hat{x} - \hat{y}}, f^0_{i-\hat{x} - \hat{z}} - f^0_{i-\hat{x} - \hat{y}}, f^0_{i-\hat{x} - \hat{y}} - f^0_{i-\hat{x} - \hat{z}}] = -3i \sum_i f^0_i [g_i + g_{i+\hat{x}+\hat{y}} + g_{i+\hat{x}+\hat{z}} - g_{i+\hat{x}+\hat{y}} - g_{i+\hat{x}+\hat{z}}]
\]

In the continuum limit one can show that the commutator reduces to \(-3\sum_i g_i \partial_i \partial_{i+\hat{x}} \partial_{i+\hat{y}} \partial_{i+\hat{z}} f^0_i = +3\sum_i f^0_i \partial_i \partial_{i+\hat{x}} \partial_{i+\hat{y}} \partial_{i+\hat{z}} g_i\). By choosing \(f^0_i\) or \(g_i\) to be functions of a special type, i.e.

\[
f^0_i = f^0_1(i_x, i_y) + f^0_2(i_y, i_z) + f^0_3(i_z, i_x)
\]

(or similarly for \(g_i\), one can prove the commutator vanishes \(\sum_j f^0_j (\text{DE})_j, \sum_i g_i (\nabla \cdot A)_i] = 0\). We thus obtain valid definitions of unitary operators \(U_A\) and \(U_E\) with \([U_A, U_E] = 0\) under the (mild) restriction on the gauge functions \(f^0_i\) or \(g_i\). Such a restriction is lifted when we
turn off $f^0 = 0$ and the model becomes identical to 3DTC. These transformation rules (and the constraint on the gauge functions) will play a vital role in identifying the field theory description of the F3 model.

**B. Field Theory of F3 model**

We have reviewed several well-known exactly solvable spin models and developed relevant EFTs for them. Each spin model had a corresponding parent LGT, i.e. rank-1 U(1) gauge theory in two and three dimensions for the toric codes, and rank-2 U(1) gauge theory for the XC model. Consistent application of the gauge principle rendered effective field theories of both the matter and the gauge fields as well as their interaction through appropriate minimal coupling. The F3 model, as reviewed in Sec. II, was derived from the hybrid of the rank-1 and rank-2 U(1) LGTs. We now apply the gauge principle to derive the appropriate EFT of the F3 model. The construction of the EFT for F3 model begins with the commutation $[A^a, E^b] = i\delta_{ab}\delta_{ij}$ as with other spin models we dealt before. There are three types of quasiparticle excitations in the F3 model called the freeons, fractons, and fluxons, which are the excitations of the $A_i, B_j, C_k$ operators. We will use $e, f, m$ to express the quantities related to freeons, fractons, fluxons, respectively.

First of all, the gauge transformation of $A$ and $E$ in the continuum field theory follows directly from the lattice transformation rules given in Eq. (4.10)

$$A^a(r) \rightarrow U_A^+ A^a(r) U_A$$

$$E(r) \rightarrow U_E^+ E(r) U_E = E(r) - \nabla g(r).$$

The two unitary operators

$$U_A = \exp \left[ i \int d^3 r \left( f^0(r) \mathbf{DE}(r) + f(r) \cdot \nabla \times \mathbf{E}(r) \right) \right]$$

$$U_E = \exp \left[ i \int d^3 r g(r) \nabla \cdot \mathbf{A}(r) \right],$$

$f = (f^x, f^y, f^z)$, are the continuum descendants of the lattice operators given in Eq. (4.7). In particular, we have $\mathbf{DE} = \partial_x \partial_y E^z + \partial_y \partial_z E^x + \partial_z \partial_x E^y$. As in the lattice consideration, the two unitary operators commute provided we choose either $\partial_x \partial_y \partial_z f^0(r) = 0$ or $\partial_y \partial_z \partial_x g(r) = 0$. Such condition is fulfilled by choosing $f^0$ or $g$ to be the sum of functions that depend on at most two of the coordinates but not on all three.

The freeon field $\psi_e$ transforms as $U_E^+ \psi_e U_E = e^{ig} \psi_e$ under the constraint

$$\nabla \cdot \mathbf{A} = \rho_c = \psi_e^\dagger \psi_e.$$  

It behaves in the same way as the $e$ quasiparticle in the 3DTC with the continuity equation $\partial_i \rho^e + \nabla \cdot j^e = 0$.

The other quasiparticles, i.e. fracton and three species of fluxons, can be grouped as $\psi^\mu$ ($\mu = 0$ for fracton, $\mu = x, y, z$ for fluxons) and transformed as

$$\psi^\mu \rightarrow U_A^+ \psi^\mu U_A = e^{if^\mu} \psi^\mu$$

under the constraints

$$\mathbf{DE}(r) = \rho_f, \ \nabla \times \mathbf{E} = \mathbf{rho}$$

where $\mathbf{rho} = (\mathbf{rho}_x, \mathbf{rho}_y, \mathbf{rho}_z)$. A gauge-invariant Lagrangian can be constructed as

$$\mathcal{L}_f = i \sum_{\mu} (\psi^\mu)^\dagger \partial_t \psi^\mu - \frac{1}{2m} \sum_{a < b} |D^{ab}\psi|^2$$

with the covariant derivatives given by

$$D^{xy}\psi = \psi^x \psi^y \partial_x \partial_y \psi^0 - \psi^x \psi^y \partial_y \partial_x \psi^0 + (\psi^0)^2 [\psi^x \partial_x \psi^y - \psi^y \partial_x \psi^x + iA^z \psi^x \psi^y]$$

$$D^{zx}\psi = \psi^x \psi^z \partial_x \partial_z \psi^0 - \psi^x \psi^z \partial_z \partial_x \psi^0 + (\psi^0)^2 [\psi^x \partial_x \psi^z - \psi^z \partial_x \psi^x + iA^y \psi^x \psi^z]$$

$$D^{yz}\psi = \psi^y \psi^z \partial_y \partial_z \psi^0 - \psi^y \psi^z \partial_z \partial_y \psi^0 + (\psi^0)^2 [\psi^y \partial_y \psi^z - \psi^z \partial_y \psi^y + iA^x \psi^y \psi^z].$$

They are quartic in the matter fields and involve the interaction between fractons and fluxons. (The Lagrangian is octic in the matter fields.) Once the fracton field is turned off by taking $\psi^0 = 1$, we recover the covariant derivative of the 3DTC given in Eq. (2.32), which again reveals the 3DTC root of the F3 model.

The current conservation laws following from the Lagrangian in Eq. (4.15) take the form,

$$\partial_t \rho_f = \partial_x \partial_y j^x + \partial_x \partial_z j^y + \partial_y \partial_z j^z,$nabla \times j.$$ (4.17)

The quasiparticle density and the current operators are given by

$$\rho_f = (\psi^0)^\dagger \psi^0$$

$$\rho^m = (\psi^x)^\dagger \psi^x, (\psi^y)^\dagger \psi^y, (\psi^z)^\dagger \psi^z$$

$$j^x = \frac{1}{2mi} \left[ (\psi^y)^\dagger (\psi^z) + (\psi^z)^\dagger (\psi^y) \right] D^{yz} \psi - h.c.$$  

$$j^y = \frac{1}{2mi} \left[ (\psi^z)^\dagger (\psi^x) + (\psi^x)^\dagger (\psi^z) \right] D^{zx} \psi - h.c.$$  

$$j^z = \frac{1}{2mi} \left[ (\psi^x)^\dagger (\psi^y) + (\psi^y)^\dagger (\psi^x) \right] D^{xy} \psi - h.c.$$  

First of all, note that $\partial_t \rho^m = \nabla \times j$ is identical to the current conservation of magnetic charges in the 3DTC. From the association $\mathbf{rho} = \nabla \times \mathbf{E}$ given in Eq. (4.14), we have $\nabla \cdot \mathbf{rho} = 0$. Note that both time derivatives of the fracton ($\rho_f$) and the fluxon ($\rho^m$) are related to the same current vector $j$ in Eq. (4.17). This goes to show that fractons and fluxons are not independent entities in
the F3 model, but are intertwined objects through the definitions of the current operators \( j \) given above and the continuity equation given in Eq. (4.17). The fracton conservation law in the first line of Eq. (4.17) implies the conservation of the dipole moment \( \int d^3 \mathbf{r} \, \mathbf{r} \rho' \).

Having established the EFT of the freeon, fracton, and fluxon matter fields, we proceed to construct the gauge field Lagrangian. The starting point, as usual, is the constraints relating the matter and the gauge degrees of freedom:

\[
\left( \nabla \cdot \mathbf{A} \right)_i = \frac{2\pi}{p} \rho'_i, \quad \left( \nabla \times \mathbf{E} \right)_i = \rho''_i, \quad \left( \mathbf{DE} \right)_i = \rho'_i.
\]

(4.19)

Taking the time derivatives on both sides of equations and invoking the continuity equations in Eqs. (4.14) and (4.17), we obtain three relations

\[
\left( \nabla \cdot \partial_t \mathbf{A} \right)_i = -(2\pi/p) \left( \nabla \cdot j^c \right)_i \quad \left( \nabla \times \partial_t \mathbf{E} \right)_i = (\nabla \times j)_i \quad \partial_t \left( \mathbf{DE} \right)_i = (\partial_x \partial_y j^z + \partial_y \partial_z j^x + \partial_z \partial_x j^y)_i.
\]

(4.20)

that are solved by

\[
\partial_t A_i = -\frac{2\pi}{p} j^c_i, \quad \partial_t E_i = j_i.
\]

(4.21)

Luckily the second and the third equations in Eq. (4.20) share the same solution, providing consistency to the whole approach we take. The Lagrangian that embodies both Eq. (4.19) and (4.21) is constructed:

\[
\mathcal{L} = E_i \cdot \partial_t A_i + A_i \cdot j_i + \frac{2\pi}{p} E_i \cdot j^c_i + \frac{p}{2\pi} E^0_i \left( \nabla \cdot A - \frac{2\pi}{p} \rho^c \right)_i + A^0_i \cdot \left( \nabla \times \mathbf{E} - \rho'' \right)_i + A^0_i \left( \mathbf{DE} - \rho' \right)_i.
\]

(4.22)

Several Lagrange multipliers, \( (E^0, A^0, A^0) \), are introduced to implement the constraints. The last line is indicating that the obtained Lagrangian is nothing but the Lagrangian of 3DTC supplemented by an additional term coming from the constraint \( \nabla \times \mathbf{E} = \rho'' \). By redefining \( \mathbf{A} \rightarrow -\mathbf{A} \) and \(- (2\pi/p) \mathbf{E} \rightarrow \mathbf{E} \), the continuum Lagrangian would be

\[
\mathcal{L} = \frac{p}{2\pi} \mathbf{E} \cdot \partial_t \mathbf{A} - \mathbf{A} \cdot j - \mathbf{E} \cdot j^c - \frac{p}{2\pi} E^0 \left( \nabla \cdot \mathbf{A} + \frac{2\pi}{p} \rho^c \right) - \mathbf{A}^0 \cdot \left( \frac{p}{2\pi} \nabla \times \mathbf{E} + \rho'' \right) - A^0 \left( \frac{p}{2\pi} \mathbf{DE} + \rho' \right) = \mathcal{L}_{3DTC} - A^0 \left( \frac{p}{2\pi} \mathbf{DE} + \rho' \right)_i.
\]

(4.23)

Again the first two lines are identical to those of the 3DTC. The commutation relation and the gauge transformation of the \( \mathbf{A} \) and \( \mathbf{E} \) in Eq. (3.27) are

\[
\left[ A^a_i (r), E^b_j (r') \right] = \frac{2\pi i}{p} \delta_{ab} \delta^3 (r - r')
\]

\[
A^a_i (r) \rightarrow U_A^i A^a_i (r) U_A^{-i} = A^a_i (r) + \frac{1}{2} \sum_{bc} \left[ \epsilon_{abc} \partial_b \partial_c f^0 (r) + (\nabla \times f (r))^a \right]
\]

\[
E_i (r) \rightarrow U_A^i A^a_i (r) U_A = E_i (r) + \nabla g (r)
\]

(4.24)

where \( U_A \) and \( U_E \) are

\[
U_A = \exp \left[ -i \frac{p}{2\pi} \int d^3 r \left( f^0 (r) \mathbf{DE} (r) + f (r) \cdot \nabla \times \mathbf{E} (r) \right) \right]
\]

\[
U_E = \exp \left[ -i \frac{p}{2\pi} \int d^3 r \frac{\rho \mathbf{E}}{m} \nabla \cdot \frac{\mathbf{A} (r)}{m} \right].
\]

(4.25)

The gauge transformations of \( A^0 \), \( A^0 \), and \( E^0 \) are

\[
A^0_i (r) \rightarrow A^0_i (r) + \partial_i f^0 (r) + \nabla f^0 (r)
\]

\[
A^0_i (r) \rightarrow A^0_i (r) + \partial_i f^0 (r)
\]

\[
E^0_i (r) \rightarrow E^0_i (r) + \partial_i g^0 (r).
\]

(4.26)

All constraints of the model follow from requiring the gauge invariance of the Lagrangian in Eq. (4.23) under the transformations in Eqs. (4.24) and (4.25). This completes the derivation of the EFT for the F3 model.

We note, before concluding the section, that the compatibility condition dictating \( p = 3 \) in the F3 model is no longer playing a role in the field theory construction. A hint of the compatibility is found in the mild restriction imposed on the gauge functions which are, as mentioned earlier, \( \partial_x \partial_y \partial_z f^0 = 0 \) or \( \partial_x \partial_y \partial_z g = 0 \).

V. SUMMARY

We have shown that all the well-known exactly solvable spin models follow from one form of parent lattice gauge theory or the others through Higgsing. The F3 model proposed by two of the present authors is shown to follow from the hybrid of rank-1 and rank-2 U(1) LGTs in three dimensions. A consistent application of the gauge principle leads to effective field theories of the quasiparticles in each spin model which, in turn, capture the known dynamics and statistical interactions at the level of field theories.

The general strategy can be summed up in a few steps. The first one identifies a lattice gauge theory (or a mix of several LGTs if necessary) and obtains the exactly solvable spin model through Higgsing. Secondly, identify the gauge transformation structures of the field variables in the parent LGT. Then invoking the gauge principle, the covariant coupling of the matter fields with the gauge fields can be uniquely identified. It is important to treat both of the conjugate fields \( \mathbf{A} \) and \( \mathbf{E} \) in the parent LGT as gauge fields, with minimal couplings to respective matter fields. In many cases, highly non-trivial Lagrangians
for the matter fields are dictated by the gauge principle. Examples covered in this paper are the quartic action for m quasiparticles in three-dimensional toric codes, fractons and lineons in the X-cube model, and the octic action for fractons and fluxons in the F3 model. One gets the impression that quadratic actions are an exception rather than a norm among the rich variety of gauge structures permitted theoretically. The higher-order actions imply strong interaction among the quasiparticles which is presumably responsible for their confining nature that we see in the lattice model. Explicit demonstration of the confinement from the higher-order action is challenging but could suggest a new way to understand confinement phenomena at large. The continuity equations that follow from the effective action of the matter fields are also unusual and can directly embody the restricted mobility of the quasiparticles as in the case of fractons and lineons.

Finally, identify various constraints relating the gauge fields to the matter fields. A familiar instance of the constraint is the well-known flux attachment mechanism, also responsible for the anyonic braiding statistics in two dimensions. In this paper, we have witnessed an interesting variety of constraints relating to gauge fields and matter fields. Eventually, the constraints find their way into the field theory description through Lagrange multiplier techniques. The resulting field theories of the gauge fields are BF theories in two and three dimensions (for toric codes) and some exotic kinds of field theories for X-cube and F3 models. The X-cube field theory was analyzed in Ref. [25].

The parent lattice gauge theory results in the stabilizer-type spin model through Higgsing. Elementary excitations in the spin model can be analyzed in terms of the effective field theories which obey the gauge principle. In turn, the gauge principle originates from the structure of the parent LGT. Along the way, highly nontrivial effective Lagrangians of the matter and the gauge fields emerge and enrich our view of the field theory at large. The spin model serves as a bridge connecting the parent LGT on one side and the EFT on the other.

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