AUTOMORPHISMS AND REAL STRUCTURES FOR A Π-SYMMETRIC SUPER-GRASSMANNIAN

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APPENDIX BY

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Abstract. Any complex-analytic vector bundle $E$ admits naturally defined homotheties $\phi_\alpha$, $\alpha \in \mathbb{C}^*$, i.e., $\phi_\alpha$ is the multiplication of a local section by a complex number $\alpha$. We investigate the question when such automorphisms can be lifted to a non-split supermanifold corresponding to $E$. Further, we compute the automorphism supergroup of a $\Pi$-symmetric super-Grassmannian $\Pi Gr_{n,k}$, and, using Galois cohomology, we classify the real structures on $\Pi Gr_{n,k}$ and compute the corresponding supermanifolds of real points.

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1. Introduction

Let $E$ be a complex-analytic vector bundle over a complex-analytic manifold $M$. There are natural homotheties $\phi_\alpha$, $\alpha \in \mathbb{C}^*$, defined on local sections as the multiplication by a complex number $\alpha \neq 0$. Any automorphism $\phi_\alpha : E \to E$ may be naturally extended to an automorphism $\wedge \phi_\alpha$ of $\wedge E$. Let $E$ be the locally free sheaf corresponding to $E$. Then the ringed space $(M, \wedge E)$ is a split supermanifold equipped with the supermanifold automorphisms $(id, \wedge \phi_\alpha)$, $\alpha \in \mathbb{C}^*$. Let $\mathcal{M}$ be any non-split supermanifold with retract $(M, \wedge E)$. We investigate the question whether the automorphism $\wedge \phi_\alpha$ can be lifted to $\mathcal{M}$. We show that this question is related to the notion of the order of the supermanifold $\mathcal{M}$ introduced in [Roth]; see Section 2.4.

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Let $\mathcal{M} = \PiGr_{n,k}$ be a II-symmetric super-Grassmannian; see Section 3 for the definition. We use obtained results to compute the automorphism group $\text{Aut} \mathcal{M}$ and the automorphism supergroup, given in terms of a super-Harish-Chandra pair.

**Theorem** (Theorem [24]). (1) If $\mathcal{M} = \PiGr_{n,k}$, where $n \neq 2k$, then
$$\text{Aut} \mathcal{M} \simeq \text{PGL}_n(\mathbb{C}) \times \{\text{id}, \Psi_{-1}^a\}.$$  

The automorphism supergroup is given by the Harish-Chandra pair
$$(\text{PGL}_n(\mathbb{C}) \times \{\text{id}, \Psi_{-1}^a\}, \mathfrak{q}_n(\mathbb{C})/(E_{2n})).$$

(2) If $\mathcal{M} = \PiGr_{2k,k}$, where $k \geq 2$, then
$$\text{Aut} \mathcal{M} \simeq \text{PGL}_{2k}(\mathbb{C}) \times \{\text{id}, \Theta, \Psi_{-1}^a, \Psi_{-1}^a \circ \Theta\},$$
where $\Theta^2 = \Psi_{-1}^a$, $\Psi_{-1}^a$ is a central element of $\text{Aut} \mathcal{M}$, and $\Theta \circ g \circ \Theta^{-1} = (g^t)^{-1}$ for $g \in \text{PGL}_{2k}(\mathbb{C})$.

The automorphism supergroup is given by the Harish-Chandra pair
$$(\text{PGL}_{2k}(\mathbb{C}) \times \{\text{id}, \Theta, \Psi_{-1}^a \circ \Theta\}, \mathfrak{q}_{2k}(\mathbb{C})/(E_{4k})).$$

where $\Theta \circ C \circ \Theta^{-1} = -C^{t_i}$ for $C \in \mathfrak{q}_{2k}(\mathbb{C})/(E_{4k})$ and $\Psi_{-1}^a \circ C \circ (\Psi_{-1}^a)^{-1} = (-1)^C C$, where $C \in \mathbb{Z}/2\mathbb{Z}$ is the parity of $C$.

(3) If $\mathcal{M} = \PiGr_{2,1}$, then
$$\text{Aut} \mathcal{M} \simeq \text{PGL}_2(\mathbb{C}) \times \mathbb{C}^*.$$  

The automorphism supergroup is given by the Harish-Chandra pair
$$(\text{PGL}_2(\mathbb{C}) \times \mathbb{C}^*, \mathfrak{g} \times (z)).$$

Here $\mathfrak{g}$ is a $\mathbb{Z}$-graded Lie superalgebra described in Theorem 2. $z$ is the grading operator of $\mathfrak{g}$. The action of $\text{PGL}_2(\mathbb{C}) \times \mathbb{C}^*$ on $z$ is trivial, and $\phi_\alpha \in \mathbb{C}^*$ multiplies $X \in \mathfrak{u}(\PiGr_{2,1})_k$ by $\alpha^k$.

Here $\Psi_{-1}^a = (\text{id}, \psi_{-1}^a) \in \text{Aut} \mathcal{M}$, where $\psi_{-1}^a$ is an automorphism of the structure sheaf $\mathcal{O}$ of $\mathcal{M}$ defined by $\psi_{-1}^a(f) = (-1)^{\frac{1}{2}}f$ for a homogeneous local section $f$ of $\mathcal{O}$, where we denoted by $\frac{1}{2} \in \mathbb{Z}/2\mathbb{Z}$ the parity of $f$. We denote by $C^{t_i}$ the $i$-transposition of the matrix $C$, see [24]. The automorphism $\Theta$ is constructed in Section 3.2.2. We denoted by $g^t$ the transpose of $g$.

We classify the real structures on a II-symmetric super-Grassmannian $\PiGr_{n,k}$ using Galois cohomology.

**Theorem** (Theorem 27). The number of the equivalence classes of real structures $\mu$ on $\mathcal{M}$, and representatives of these classes, are given in the list below:

(i) If $n$ is odd, then there are two equivalence classes with representatives
$$\mu^\circ, \ (1, \Psi_{-1}^a) \circ \mu^\circ.$$  

(ii) If $n$ is even and $n \neq 2k$, then there are four equivalence classes with representatives
$$\mu^\circ, \ (1, \Psi_{-1}^a) \circ \mu^\circ, \ (c, 1) \circ \mu^\circ, \ (c, \Theta, \Psi_{-1}^a) \circ \mu^\circ.$$  

(iii) If $n = 2k \geq 4$, then there are $k + 3$ equivalence classes with representatives
$$\mu^\circ, \ (c, 1) \circ \mu^\circ, \ (c, \Theta) \circ \mu^\circ, \ r = 0, \ldots, k.$$
(iv) If \((n, k) = (2, 1)\), then there are two equivalence classes with representatives 
\[ \mu^0, \quad (c, 1) \circ \mu^0. \]

Here \(\mu^0\) denotes the standard real structure on \(M = \Pi\text{Gr}_{n,k}\), see Section \ref{sec:real-structures}. Moreover, \(c \in \text{PGL}_n(\mathbb{C})\) and \(c_r \in \text{PGL}_{2k}(\mathbb{C})\) for \(r = 0, \ldots, k\) are certain elements constructed in Proposition \ref{prop:splitting} and Subsection \ref{subsec:splitting} respectively.

Further, we describe the corresponding real subsupermanifolds when they exist. Let \(\mu\) be a real structure on \(M = \Pi\text{Gr}_{n,k}\), and assume that the set of fixed points \(M^\mu\) is non-empty. Consider the ringed space \(M^\mu := (M^\mu, \mathcal{O}^\mu)\) where \(\mathcal{O}^\mu\) is the sheaf of fixed points of \(\mu^*\) over \(M^\mu\). Then \(M^\mu\) is a real supermanifold. We describe this supermanifold in Theorem \ref{thm:fixed-points}.

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## 2. Preliminaries

### 2.1. Supermanifolds

This paper is devoted to the study of the automorphism supergroup of a \(H\)-symmetric super-Grassmannian \(\Pi\text{Gr}_{n,k}\), and to a classification of real structures on \(\Pi\text{Gr}_{n,k}\). Details about the theory of supermanifolds can be found in \cite{Ber, L, BLMS}. As usual, the superspace \(\mathbb{C}^{n|m} := \mathbb{C}^n \oplus \mathbb{C}^m\) is a \(\mathbb{Z}_2\)-graded vector space over \(\mathbb{C}\) of dimension \(n|m\). A superdomain in \(\mathbb{C}^{n|m}\) is a ringed space \(U := (U, \mathcal{F}_U \otimes \bigwedge (\mathbb{C}^n)^*),\) where \(U \subset \mathbb{C}^n\) is an open set and \(\mathcal{F}_U\) is the sheaf of holomorphic functions on \(U\). If \((x_a)\) is a system of coordinates in \(U\) and \((\xi_b)\) is a basis in \((\mathbb{C}^n)^*\) we call \((x_a, \xi_b)\) a system of coordinates in \(U\). Here \((x_a)\) are called even coordinates of \(U\), while \((\xi_b)\) are called odd ones. A supermanifold \(M = (M, \mathcal{O})\) of dimension \(n|m\) is a \(\mathbb{Z}_2\)-graded ringed space that is locally isomorphic to a superdomain in \(\mathbb{C}^{n|m}\). Here the underlying space \(M\) is a complex-analytic manifold. A morphism \(F : (M, \mathcal{O}_M) \to (N, \mathcal{O}_N)\) of two supermanifolds is, by definition, a morphism of the corresponding \(\mathbb{Z}_2\)-graded locally ringed spaces. In more details, \(F = (F_0, F^*)\) is a pair, where \(F_0 : M \to N\) is a holomorphic map and \(F^* : \mathcal{O}_N \to (F_0)_*(\mathcal{O}_M)\) is a homomorphism of sheaves of \(\mathbb{Z}_2\)-graded local superalgebras. We see that the morphism \(F\) is even, that is, \(F\) preserves the \(\mathbb{Z}_2\)-gradings of the sheaves. A morphism \(F : M \to M\) is called an automorphism of \(M\) if \(F\) is an automorphism of the corresponding \(\mathbb{Z}_2\)-graded ringed spaces. The automorphisms of \(M\) form a group, which we denote by \(\text{Aut} M\). Note that in this paper we also consider the automorphism supergroup, see a definition below.

A supermanifold \(M = (M, \mathcal{O})\) is called split, if its structure sheaf is isomorphic to \(\bigwedge E\), where \(E\) is a sheaf of sections of a holomorphic vector bundle \(E\) over \(M\). In this case the structure sheaf of \(M\) is \(\mathbb{Z}\)-graded, not only \(\mathbb{Z}_2\)-graded. There is a functor assigning to any supermanifold a split supermanifold. Let us briefly remind this construction. Let \(M = (M, \mathcal{O})\) be a supermanifold. Consider the following filtration in \(\mathcal{O}\)

\[ \mathcal{O} = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \mathcal{F}_2 \supset \cdots \supset \mathcal{F}_p \supset \cdots, \]
where \( J \) is the subsheaf of ideals in \( \mathcal{O} \) locally generated by odd elements of \( \mathcal{O} \). We define
\[
\text{gr}\mathcal{M} := (M, \text{gr}\mathcal{O}), \quad \text{where} \quad \text{gr}\mathcal{O} := \bigoplus_{p \geq 0} \mathcal{J}^p / \mathcal{J}^{p+1}.
\]

The supermanifold \( \text{gr}\mathcal{M} \) is split and it is called the retract of \( \mathcal{M} \). The underlying space of \( \text{gr}\mathcal{M} \) is the complex-analytic manifold \( (M, \mathcal{O})/\mathcal{J} \), which coincides with \( M \).

The structure sheaf \( \text{gr}\mathcal{O} \) is isomorphic to \( \wedge \mathcal{E} \), where \( \mathcal{E} = \mathcal{J} / \mathcal{J}^2 \) is a locally free sheaf of \( \mathcal{O}/\mathcal{J} \)-modules on \( M \).

Further let \( \mathcal{M} = (M, \mathcal{O}_M) \) and \( \mathcal{N} = (N, \mathcal{O}_N) \) be two supermanifolds, \( \mathcal{J}_M \) and \( \mathcal{J}_N \) be the subsheaves of ideals in \( \mathcal{O}_M \) and \( \mathcal{O}_N \), respectively. Any morphism \( F : \mathcal{M} \to \mathcal{N} \) preserves these sheaves of ideals, that is \( F^*(\mathcal{J}_N) \subset (F_0)^*(\mathcal{J}_M) \), and more generally \( F^*(\mathcal{J}_N^p) \subset (F_0)^*(\mathcal{J}_M^p) \) for any \( p \). Therefore \( F \) induces naturally a morphism \( \text{gr}(F) : \text{gr}\mathcal{M} \to \text{gr}\mathcal{N} \). Summing up, the functor \( \text{gr} \) is defined.

### 2.2. A classification theorem for supermanifolds.

Let \( \mathcal{M} = (M, \mathcal{O}) \) be a (non-split) supermanifold. Recall that we denoted by \( \text{Aut}_{\mathcal{M}} \) the group of all (even) automorphisms of \( \mathcal{M} \). Denote by \( \text{Aut}_{\mathcal{O}} \) the sheaf of automorphisms of \( \mathcal{O} \). Consider the following subsheaf of \( \text{Aut}_{\mathcal{O}} \)
\[
\text{Aut}_{(2)} \mathcal{O} := \{ F \in \text{Aut}_{\mathcal{O}} \mid \text{gr}(F) = \text{id} \}.
\]

This sheaf plays an important role in the classification of supermanifolds, see below.

The sheaf \( \text{Aut}_{\mathcal{O}} \) has the following filtration
\[
\text{Aut}_{\mathcal{O}} = \text{Aut}_{(0)} \mathcal{O} \supset \text{Aut}_{(2)} \mathcal{O} \supset \ldots \supset \text{Aut}_{(2p)} \mathcal{O} \supset \ldots,
\]
where
\[
\text{Aut}_{(2p)} \mathcal{O} = \{ a \in \text{Aut}_{\mathcal{O}} \mid a(u) \equiv u \mod \mathcal{J}^{2p} \text{ for any } u \in \mathcal{O} \}.
\]

Recall that \( \mathcal{J} \) is the subsheaf of ideals generated by odd elements in \( \mathcal{O} \). Let \( \mathcal{E} \) be the bundle corresponding to the locally free sheaf \( \mathcal{E} = \mathcal{J} / \mathcal{J}^2 \) and let \( \text{Aut}_{\mathcal{E}} \) be the group of all automorphisms of \( \mathcal{E} \). Clearly, any automorphism of \( \mathcal{E} \) gives rise to an automorphism of \( \text{gr}\mathcal{M} \), and thus we get a natural action of the group \( \text{Aut}_{\mathcal{E}} \) on the sheaf \( \text{Aut}(\text{gr}\mathcal{O}) \) by \( \text{Int} : (a, \delta) \mapsto a \circ \delta \circ a^{-1} \), where \( \delta \in \text{Aut}(\text{gr}\mathcal{O}) \) and \( a \in \text{Aut}_{\mathcal{E}} \). Clearly, the group \( \text{Aut}_{\mathcal{E}} \) leaves invariant the subsheaves \( \text{Aut}_{(2p)} \text{gr}\mathcal{O} \). Hence \( \text{Aut}_{\mathcal{E}} \) acts on the cohomology sets \( H^1(M, \text{Aut}_{(2p)} \text{gr}\mathcal{O}) \). The unit element \( e \in H^1(M, \text{Aut}_{(2p)} \text{gr}\mathcal{O}) \) is fixed under this action. We denote by \( H^1(M, \text{Aut}_{(2p)} \text{gr}\mathcal{O})/\text{Aut}_{\mathcal{E}} \) the set of orbits of the action in \( H^1(M, \text{Aut}_{(2p)} \text{gr}\mathcal{O}) \) induced by \( \text{Int} \).

Denote by \([\mathcal{M}]\) the class of supermanifolds which are isomorphic to \( \mathcal{M} = (M, \mathcal{O}) \). (Here we consider complex-analytic supermanifolds up to isomorphisms inducing the identical isomorphism of the base spaces.) The following theorem was proved in [Gr].

**Theorem 1 (Green).** Let \((M, \wedge \mathcal{E})\) be a fixed split supermanifold. Then
\[
([\mathcal{M}] \mid \text{gr}\mathcal{O} \simeq \wedge \mathcal{E}) \xrightarrow{1:1} H^1(M, \text{Aut}_{(2)} \text{gr}\mathcal{O})/\text{Aut}_{\mathcal{E}}.
\]

The split supermanifold \((M, \wedge \mathcal{E})\) corresponds to the fixed point \( e \).
2.3. Tangent sheaf of $\mathcal{M}$ and $\text{gr}\mathcal{M}$. Let again $\mathcal{M} = (M, \mathcal{O})$ be a (non-split) supermanifold. The tangent sheaf of a supermanifold $\mathcal{M}$ is by definition the sheaf $T = \text{Der}\mathcal{O}$ of derivations of the structure sheaf $\mathcal{O}$. Sections of the sheaf $T$ are called holomorphic vector fields on $\mathcal{M}$. The vector superspace $\mathfrak{v}(\mathcal{M}) = \mathcal{H}^0(\mathcal{M}, T)$ of all holomorphic vector fields is a complex Lie superalgebra with the bracket

$$[X, Y] = X \circ Y - (-1)^{\tilde{Z}X} \tilde{Y} \circ X, \quad X, Y \in \mathfrak{v}(\mathcal{M}),$$

where $\tilde{Z}$ is the parity of an element $Z \in \mathfrak{v}(\mathcal{M})$. The Lie superalgebra $\mathfrak{v}(\mathcal{M})$ is finite dimensional if $\mathcal{M}$ is compact.

Let $\dim \mathcal{M} = n|\text{m}$. The tangent sheaf $T$ possesses the following filtration:

$$T = T_{(-1)} \supset T_{(0)} \supset T_{(1)} \supset \cdots \supset T_{(m)} \supset T_{(m+1)} = 0,$$

where

$$T_{(p)} = \{ v \in T \mid v(\mathcal{O}) \subset J^p, \ v(J) \subset J^{p+1} \}, \quad p \geq 0.$$

Denote by $T_{\text{gr}}$ the tangent sheaf of the retract $\text{gr}\mathcal{M}$. Since the structure sheaf $\text{gr}\mathcal{O}$ of $\text{gr}\mathcal{M}$ is $\mathbb{Z}$-graded, the sheaf $T_{\text{gr}}$ has the following induced $\mathbb{Z}$-grading

$$T_{\text{gr}} = \bigoplus_{p \geq -1} (T_{\text{gr}})_p,$$

where

$$(T_{\text{gr}})_p = \{ v \in T_{\text{gr}} \mid v(\text{gr}\mathcal{O}_q) \subset \text{gr}\mathcal{O}_{q+p} \text{ for any } q \in \mathbb{Z} \}.$$

We have the following exact sequence the sheaves of groups

$$0 \to \text{Aut}_{(2p+2)}\mathcal{O} \to \text{Aut}_{(2p)}\mathcal{O} \to (T_{\text{gr}})_{2p} \to 0$$

for any $p \geq 1$, see [Roth]. More details about this sequence can be also found in [Oni1, Proposition 3.1].

2.4. Order of a supermanifold. Let again $\mathcal{M} = (M, \mathcal{O})$ be a (non-split) supermanifold. According to Theorem [1] a supermanifold corresponds to an element $[\gamma] \in H^1(M, \text{Aut}_{(2)}\mathcal{O})/\text{Aut} E$. Furthermore for any $p \geq 1$ we have the following natural embedding of sheaves

$$\text{Aut}_{(2p)}\mathcal{O} \hookrightarrow \text{Aut}_{(2)}\mathcal{O},$$

that induces the map of 1-cohomology sets

$$H^1(M, \text{Aut}_{(2p)}\mathcal{O}) \to H^1(M, \text{Aut}_{(2)}\mathcal{O}).$$

(Note that our sheaves are not abelian.) Denote by $H_{2p}$ the image of $H^1(M, \text{Aut}_{(2p)}\mathcal{O})$ in $H^1(M, \text{Aut}_{(2)}\mathcal{O})$. We get the following Aut $E$-invariant filtration

$$H^1(M, \text{Aut}_{(2)}\mathcal{O}) = H_2 \supset H_4 \supset H_6 \supset \cdots.$$

Let $\gamma \in [\gamma]$ be any representative. As in [Roth] we define the order $o(\gamma)$ of the cohomology class $\gamma \in H^1(M, \text{Aut}_{(2)}\mathcal{O})$ to be equal to the maximal number between the numbers $2p$ such that $\gamma \in H_{2p}$. The order of the supermanifold $\mathcal{M}$ is by definition the order of the corresponding cohomology class $\gamma$. We put $o(\mathcal{M}) := \infty$, if $\mathcal{M}$ is a split supermanifold.
2.5. The automorphism supergroup of a complex-analytic compact supermanifold. Let us remind a description of a Lie supergroup in terms of a super-Harish-Chandra pair. A supermanifold \( G \) is a group object in the category of supermanifolds, see for example [Vi1], [Vi3] for details. Any Lie supergroup can be described using a super-Harish-Chandra pair, see [Ber] and also [BCC, Vi3], due to the following theorem, see [Vi3] for the complex-analytic case.

**Theorem 2.** The category of complex Lie supergroups is equivalent to the category of complex super Harish-Chandra pairs.

A complex super Harish-Chandra pair is a pair \((G,\mathfrak{g})\) that consists of a complex-analytic Lie group \(G\) and a Lie superalgebra \(\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1\) over \(\mathbb{C}\), where \(\mathfrak{g}_0 = \text{Lie}(G)\), endowed with a representation \(\text{Ad} : G \to \text{Aut} \mathfrak{g}\) of \(G\) in \(\mathfrak{g}\) such that

- \(\text{Ad}\) preserves the parity and induces the adjoint representation of \(G\) in \(\mathfrak{g}_0\),
- the differential \((d\text{Ad})_e\) at the identity \(e \in G\) coincides with the adjoint representation \(\text{ad}\) of \(\mathfrak{g}_0\) in \(\mathfrak{g}\).

Super Harish-Chandra pairs form a category. (A definition of a morphism is natural, see in [Ber] or in [Vi3].)

A supermanifold \(\mathcal{M} = (M,\mathcal{O})\) is called compact if its base space \(M\) is compact. If \(\mathcal{M}\) is a compact complex-analytic supermanifold, the Lie superalgebra of vector fields \(\sigma(\mathcal{M})\) is finite dimensional. For a compact complex-analytic supermanifold \(\mathcal{M}\) we define the automorphism supergroup as the super-Harish-Chandra pair

\[
\text{(Aut}\, \mathcal{M}, \sigma(\mathcal{M})).
\]

3. Super-Grassmannians and II-symmetric super-Grassmannians

3.1. Complex-analytic super-Grassmannians and complex-analytic II-symmetric super-Grassmannians. A super-Grassmannian \(\text{Gr}_{m|n,k|l}\) is the supermanifold that parameterizes all \(k\times l\)-dimensional linear subsuperspaces in \(\mathbb{C}^{m|n}\).

Here \(k \leq m, l \leq n\) and \(k + l < m + n\). The underlying space of \(\text{Gr}_{m|n,k|l}\) is the product of two usual Grassmannians \(\text{Gr}_{m,k} \times \text{Gr}_{n,l}\). The structure of a supermanifold on \(\text{Gr}_{m|n,k|l}\) can be defined in the following way. Consider the following \((m + n) \times (k + l)\)-matrix

\[
\mathcal{L} = \begin{pmatrix}
A & B \\
C & D
\end{pmatrix}.
\]

Here \(A = (a_{ij})\) is a \((m \times k)\)-matrix, whose entries \(a_{ij}\) can be regarded as (even) coordinates in the domain of all complex \((m \times k)\)-matrices of rank \(k\). Similarly \(D = (d_{uv})\) is a \((n \times l)\)-matrix, whose entries \(d_{uv}\) can be regarded as (even) coordinates in the domain of all complex \((n \times l)\)-matrices of rank \(l\). Further, \(B = (b_{pq})\) and \(C = (c_{uv})\) are \((m \times l)\) and \((n \times k)\)-matrices, respectively, whose entries \(b_{pq}\) and \(c_{uv}\) can be regarded as generators of a Grassmann algebra. The matrix \(\mathcal{L}\) determines the following open subsuperdomain in \(\mathbb{C}^{mk+n|ml+nk}\)

\[
\mathcal{V} = (V, F_V \otimes \bigwedge (b_{pq}, c_{uv})),
\]

where \(V\) is the product of the domain of complex \((m \times k)\)-matrices of rank \(k\) and the domain of complex \((n \times l)\)-matrices of rank \(l\), \(F_V\) is the sheaf of holomorphic functions on \(V\) and \(\bigwedge (b_{pq}, c_{uv})\) is the Grassmann algebra with generators \((b_{pq}, c_{uv})\). Let us define an action \(\mu : \mathcal{V} \times \text{GL}_{k|l}(\mathbb{C}) \to \mathcal{V}\) of the Lie supergroup \(\text{GL}_{k|l}(\mathbb{C})\) on \(\mathcal{V}\) on the right in the natural way, that is by matrix multiplication. The quotient space under this action is called the super-Grassmannian \(\text{Gr}_{m|n,k|l}\). Now consider
the case \( m = n \). A II-symmetric super-Grassmannian \( IIGr_{n,k} \) is a subsupersmanifold in \( Gr_{n[n,k]} \), which is invariant under odd involution \( \Pi : C^{n[n} \to C^{n[n}, see below.

Let us describe \( Gr_{m[n,k]} \) and \( IIGr_{n,k} \) using charts and local coordinates \[ \text{Man} \]. First of all let us recall a construction of an atlas for the usual Grassmannian \( Gr_{m,k} \). Let \( e_1, \ldots, e_m \) be the standard basis in \( C^m \). Consider a complex \((m \times k)\)-matrix \( C = (c_{ij}) \), where \( i = 1, \ldots, m \) and \( j = 1, \ldots, k, \) of rank \( k \). Such a matrix determines a \( k \)-dimensional subspace \( W \) in \( C^m \) with basis \( \sum_{i=1}^{m} c_{ij} e_i, \ldots, \sum_{i=1}^{m} c_{ik} e_i \).

Let \( I \subset \{1, \ldots, m\} \) be a subset of cardinality \( k \) such that the square submatrix \( L = (c_{ij}), i \in I \) and \( j = 1, \ldots, k \), of \( C \) is non-degenerate. (There exists such a subset since \( C \) is of rank \( k \).) Then the matrix \( C' := C \cdot L^{-1} \) determines the same subspace \( W \) and contains the identity submatrix \( E_k \) in the lines with numbers \( i \in I \). Let \( U_I \) denote the set of all \((m \times k)\)-complex matrices \( C' \) with the identity submatrix \( E_k \) in the lines with numbers \( i \in I \). Any point \( x \in U_I \) determines a \( k \)-dimensional subspace \( W_x \) in \( C^m \) as above, moreover if \( x_1, x_2 \in U_I \), \( x_1 \neq x_2 \), then \( W_{x_1} \neq W_{x_2} \). Therefore, the set \( U_I \) is a subset in \( Gr_{m,k} \). We can verify that \( U_I \) is open in a natural topology in \( Gr_{m,k} \) and it is homeomorphic to \( C^{(m-k)k} \). Therefore \( U_I \) can be regarded as a chart on \( Gr_{m,k} \). Further any \( k \)-dimensional vector subspace in \( C^m \) is contained in some \( U_I \) for a subset \( J \subset \{1, \ldots, m\} \) of cardinality \( |J| = k \). Hence the collection \( \{U_I\}_{|I|=k} \) is an atlas on \( Gr_{m,k} \).

Now we are ready to describe an atlas \( A \) on \( Gr_{m[n,k]} \). Let \( I = (I_0, I_1) \) be a pair of sets, where
\[
I_0 \subset \{1, \ldots, m\} \quad \text{and} \quad I_1 \subset \{1, \ldots, n\},
\]
with \( |I_0| = k \) and \( |I_1| = l \). As above to such an \( I \) we can assign a chart \( U_{I_0} \times U_{I_1} \) on \( Gr_{m,k} \times Gr_{n,l} \). Let \( A = \{U_I\} \) be a family of superdomains parametrized by \( I = (I_0, I_1) \), where
\[
U_I := (U_{I_0} \times U_{I_1}, F_{U_{I_0} \times U_{I_1}} \otimes \wedge ((m-k)k + (n-l)k)).
\]
Here \( \wedge (r) \) is a Grassmann algebra with \( r \) generators and \( F_{U_{I_0} \times U_{I_1}} \) is the sheaf of holomorphic function on \( U_{I_0} \times U_{I_1} \). Let us describe the superdomain \( U_I \) in a different way. First of all assume for simplicity that \( I_0 = \{m-k+1, \ldots, m\} \), \( I_1 = \{n-l+1, \ldots, n\} \). Consider the following matrix
\[
Z_I = \begin{pmatrix}
X & \Xi \\
E_k & H \\
0 & Y \\
0 & E_l
\end{pmatrix},
\]
where \( E_s \) is the identity matrix of size \( s \). We assume that the entries of \( X = (x_{ij}) \) and \( Y = (y_{rs}) \) are coordinates in the domain \( U_{I_0} \) and the domain \( U_{I_1} \), respectively. We also assume that the entries of \( \Xi = (\xi_{ab}) \) and of \( H = (\eta_{cd}) \) are generators of the Grassmann algebra \( \wedge ((m-k)k + (n-l)k) \). We see that the matrix \( Z_I \) determines a superdomain
\[
U_I := (U_{I_0} \times U_{I_1}, F_{U_{I_0} \times U_{I_1}} \otimes \wedge (\xi_{ab}, \eta_{cd}))
\]
with even coordinates \( x_{ij} \) and \( y_{rs}, \) and odd coordinates \( \xi_{ab} \) and \( \eta_{cd} \).

Let us describe \( U_I \) for any \( I = (I_0, I_1) \). Consider the following \((m+n) \times (k+l)\)-matrix
\[
Z_I = \begin{pmatrix}
X' & \Xi' \\
H' & Y'
\end{pmatrix},
\]
Here the blocks $X', Y', \Xi'$ and $H'$ are of size $m \times k$, $n \times l$, $m \times l$ and $n \times k$, respectively. We assume that this matrix contains the identity submatrix in the lines with numbers $i \in I_0$ and $i \in \{m + j \mid j \in I_1\}$. Further, non-trivial entries of $X'$ and $Y'$ can be regarded as coordinates in $U_{I_0}$ and $U_{I_1}$, respectively, and non-trivial entries of $\Xi'$ and $H'$ are identified with generators of the Grassmann algebra $\bigwedge((m - k)l + (n - l)k)$, see definition of $U_I$. Summing up, we have obtained another description of $U_I$.

The last step is to define the transition functions in $U_I \cap U_J$. To do this we need the matrices $Z_I$ and $Z_J$. We put $Z_J = Z_I C_{IJ}$, where $C_{IJ}$ is an invertible submatrix in $Z_I$ that consists of the lines with numbers $i \in J_0$ and $m + i$, where $i \in J_1$. This equation gives us a relation between coordinates of $U_I$ and $U_J$, in other words the transition functions in $U_I \cap U_J$. The supermanifold obtained by gluing these charts together is called the super-Grassmannian $Gr_{n|k|\mathbb{R}}$. The supermanifold $\Pi Gr_{n,k}$ is defined as a subsupermanifold in $Gr_{n|k|\mathbb{R}}$ defined in $Z_I$ by the equations $X' = Y'$ and $\Xi' = H'$. We can define the $\Pi Gr_{n,k}$ as all fixed points of an automorphism of $Gr_{n|k|\mathbb{R}}$ induced by an odd linear involution $\Pi : \mathbb{C}^n \to \mathbb{C}^n$, given by

$$
\begin{pmatrix}
0 & E_n \\
E_n & 0
\end{pmatrix}
\begin{pmatrix}
V \\
W
\end{pmatrix} =
\begin{pmatrix}
W \\
V
\end{pmatrix},
$$

where $\begin{pmatrix} V \\ W \end{pmatrix}$ is the column of right coordinates of a vector in $\mathbb{C}^n$. In our charts $\Pi Gr_{n,k}$ is defined by the following equation

$$
\begin{pmatrix}
0 & E_n \\
E_n & 0
\end{pmatrix}
\begin{pmatrix}
X & \Xi \\
H & Y
\end{pmatrix} =
\begin{pmatrix}
0 & E_k \\
E_k & 0
\end{pmatrix}
\begin{pmatrix}
X & \Xi \\
H & Y
\end{pmatrix},
$$

or equivalently,

$$X = Y, \quad H = \Xi.$$

An atlas $\mathcal{A}^\Pi$ on $\Pi Gr_{n,k}$ contains local charts $U_I^\Pi$ parameterized by $I \subset \{1, \ldots, n\}$ with $|I| = k$. The retract $gr\Pi Gr_{n,k}$ of $\Pi Gr_{n,k}$ is isomorphic to $(Gr_{n,k}, \bigwedge \Omega)$, where $\Omega$ is the sheaf of 1-forms on $Gr_{n,k}$. More information about super-Grassmannians and II-symmetric super-Grassmannians can be found in [Man], see also [Oni1] [V12].

3.2. II-symmetric super-Grassmannians over $\mathbb{R}$ and $\mathbb{H}$. We will also consider II-symmetric super-Grassmannians $\Pi Gr_{n,k}(\mathbb{R})$ and $\Pi Gr_{n,k}(\mathbb{H})$ over $\mathbb{R}$ and $\mathbb{H}$. These supermanifolds are defined in a similar way as $\Pi Gr_{n,k}$ assuming that all coordinates are real or quaternion. In more details, to define $\Pi Gr_{n,k}(\mathbb{R})$ we just repeat the construction of local charts and transition functions above assuming that we work over $\mathbb{R}$. The case of $\Pi Gr_{n,k}(\mathbb{H})$ is slightly more complicated. Indeed, we consider charts $Z_I$ as above with even and odd coordinates $X = (x_{ij})$ and $\Xi = (\xi_{ij})$, respectively, where by definition

$$x_{ij} := \begin{pmatrix} x_{11}^{ij} & x_{12}^{ij} \\ -x_{12}^{ij} & x_{11}^{ij} \end{pmatrix}, \quad \xi_{ij} := \begin{pmatrix} \xi_{11}^{ij} & \xi_{12}^{ij} \\ -\xi_{12}^{ij} & \xi_{11}^{ij} \end{pmatrix}.$$

Here $x_{ij}$ are even complex variables and $\bar{x}_{ij}$ is the complex conjugation of $x_{ij}$. Further, any $\xi_{ij}$ is an odd complex variable and $\bar{\xi}_{ij}$ is its complex conjugation. (Recall that a complex conjugation of a complex odd variable $\eta = \eta_1 + i\eta_2$ is $\bar{\eta} := \eta_1 - i\eta_2$, where $\eta_1$ is a real odd variable.) To obtain $\Pi Gr_{n,k}(\mathbb{H})$ we repeat step by step the construction above.
3.3. The order of a II-symmetric super-Grassmannian. We start this subsection with the following theorem proved in [Oni1, Theorem 5.1].

**Theorem 3.** A II-symmetric super-Grassmannian $\Pi Gr_{n,k}$ is split if and only if $(n, k) = (2, 1)$.

From [Oni1] Theorem 4.4 it follows that for the II-symmetric super-Grassmannian $\mathcal{M} = \Pi Gr_{n,k}$ we have $H^1(\mathcal{M}, (\mathcal{T}_p)_{\mathcal{O}}) = \{0\}$, $p \geq 3$. This implies the following statement.

**Proposition 4.** A II-symmetric super-Grassmannian $\Pi Gr_{n,k}$ is a supermanifold of order 2 for $(n, k) \neq (2, 1)$. The order of $\Pi Gr_{2,1}$ is $\infty$, since this supermanifold is split.

**Proof.** To show the statement consider the exact sequence (1) for $\mathcal{M} = \Pi Gr_{n,k}$ and the corresponding exact sequence of cohomology sets

$$\rightarrow H^1(\mathcal{M}, Aut(2p \mathcal{O})) \rightarrow H^1(\mathcal{M}, Aut(2p \mathcal{O})) \rightarrow H^1(\mathcal{M}, (\mathcal{T}_p)_{\mathcal{O}}) \rightarrow .$$

Since $H^1(\mathcal{M}, (\mathcal{T}_p)_{\mathcal{O}}) = \{0\}$ for $p \geq 3$, see [Oni1] Theorem 4.4, and $Aut(2p \mathcal{O}) = \text{id}$ for sufficiently large $p$, we have by induction $H^1(\mathcal{M}, Aut(2p \mathcal{O})) = \{\epsilon\}$, $q \geq 2$. Therefore $H_{2p} = \{\epsilon\}$ for $p \geq 2$. Since by Theorem 3 the II-symmetric super-Grassmannian $\Pi Gr_{n,k}$ is not split for $(n, k) \neq (2, 1)$, the corresponding to $\Pi Gr_{n,k}$, where $(n, k) \neq (2, 1)$, cohomology class $\gamma$ is not trivial. Therefore, $\gamma \in H_2 \setminus H_4 = H_2 \setminus \{\epsilon\}$. This completes the proof. $\square$

### 4. Lifting of Homotheties on a Non-split Supermanifold

#### 4.1. Lifting of an Automorphism in Terms of Green's Cohomology

On any vector bundle $E$ over $M$ we can define a natural automorphism $\phi$, where $\alpha \in C^* = C \setminus \{0\}$. In more details, $\phi$ multiplies any local section by the complex number $\alpha$. Let $r$ be the minimum between positive integers $k$ such that $\alpha^k = 1$. The number $r$ is called the order $\text{ord}(\phi)$ of the automorphism $\phi$. If such a number does not exist we put $\text{ord}(\phi) = \infty$. In this section we study a possibility of lifting of $\phi$ on a non-split supermanifold corresponding to $E$.

A possibility of lifting of an automorphism (or an action of a Lie group) to a non-split supermanifold was studied in [Oni2], see also [HO] Proposition 3.1 for a proof of a particular case. In particular the following result was obtained there. Denote by $\text{Aut}E$ the group of automorphisms of $E$, which are not necessary identical on $M$. Clearly, we have $\text{Aut}E \subset \text{Aut}E$.

**Proposition 5.** Let $\gamma \in H^1(\mathcal{M}, Aut(2\text{gr}\mathcal{O}))$ be a Green cohomology class of $\mathcal{M}$. Then $B \in \text{Aut}E$ lifts to $\mathcal{M}$ if and only if for the induced map in the cohomology group we have $B(\gamma) = \gamma$.

Consider the case $B = \phi$, in details. Let us choose an acyclic covering $\mathcal{U} = \{U_a\}_{a \in I}$ of $M$. Then by the Leray theorem, we have an isomorphism $H^1(\mathcal{M}, Aut(2\text{gr}\mathcal{O})) \simeq H^1(\mathcal{U}, Aut(2\text{gr}\mathcal{O}))$, where $H^1(\mathcal{U}, Aut(2\text{gr}\mathcal{O}))$ is the Čech 1-cohomology set corresponding to $\mathcal{U}$. Let $(\gamma_{ab})$ be a Čech cocycle representing $\gamma$ with respect to this isomorphism. Then

$$\gamma = \phi(\gamma) \iff \gamma_{ab} = u_a \circ \phi(\gamma_{ab}) \circ u_b^{-1} = u_a \circ \phi(\gamma_{ab}) \circ u_b^{-1},$$
4.2. Natural gradings in a superdomain. Let us consider a superdomain \( U := (U, \mathcal{O}) \), where \( \mathcal{O} = \mathcal{F} \otimes \Lambda (\xi_1, \ldots, \xi_m) \) and \( \mathcal{F} \) is the sheaf of holomorphic functions on \( U \), with local coordinates \((x_a, \xi_b)\). For any \( \alpha \in \mathbb{C}^* \) we define an automorphism \( \theta_\alpha : \mathcal{O} \to \mathcal{O} \) of order \( r = \text{ord}(\theta_\alpha) \) given by \( \theta_\alpha(x_a) = x_a \) and \( \theta_\alpha(\xi_b) = \alpha \xi_b \). Clearly \( \theta_\alpha \) defines the following \( \mathbb{Z}_r \)-grading (or \( \mathbb{Z} \)-grading if \( r = \infty \)) in \( \mathcal{O} \):

\[
\mathcal{O} = \bigoplus_{k \in \mathbb{Z}_r} \mathcal{O}^k, \quad \text{where} \quad \mathcal{O}^k = \{ f \in \mathcal{O} \mid \theta_\alpha(f) = \alpha^k f \}.
\]

If \( r = 2 \), the decomposition (4) coincides with the standard decomposition of \( \mathcal{O} = \mathcal{O}_0 \oplus \mathcal{O}_1 \) into even and odd parts

\[
\mathcal{O}_0 = \mathcal{O}^0, \quad \mathcal{O}_1 = \mathcal{O}^1.
\]

4.3. Lifting of an automorphism \( \phi_\alpha \), local picture. Let \( E \) be a vector bundle, \( \mathcal{E} \) be the sheaf of section of \( E \), \( (M, \Lambda \mathcal{E}) \) be the corresponding split supermanifold, and \( \mathcal{M} = (M, \mathcal{O}) \) be a (non-split) supermanifold with the retract \( \text{gr}\mathcal{M} \simeq (M, \Lambda \mathcal{E}) \).

Recall that the automorphism \( \phi_\alpha \) of \( E \) multiplies any local section of \( E \) by the complex number \( \alpha \). We say that \( \psi_\alpha \in H^0(M, \text{Aut}\mathcal{O}) \) is a lift of \( \phi_\alpha \) if \( \text{gr}(\text{id}, \psi_\alpha) = (\text{id}, \Lambda \phi_\alpha) \).

Let \( B = \{ V_a \} \) be any atlas on \( \mathcal{M} \) and let \( V_a \in B \) be a chart with even and odd coordinates \((x_a, \xi_a)\), respectively. In any such \( V_a \in B \) we can define an automorphism \( \theta_\alpha^a = \theta_\alpha^a(V_a) \) as in Section 4.2 depending on \( V_a \). This is \( \theta_\alpha^a(x_a) = x_a \) and \( \theta_\alpha^a(\xi_a) = \alpha \xi_a \).

Proposition 6. Let \( \psi_\alpha \) be a lift of the automorphism \( \phi_\alpha \) of order \( r = \text{ord}(\phi_\alpha) \).

1. If \( r \) is even, then there exists an atlas \( A = \{ U_a \} \) on \( \mathcal{M} \) with local coordinates \((x_1^a, \xi_1^a)\) in \( U_a = (U_a, \mathcal{O}|_{U_a}) \) such that

\[
\theta_\alpha^a(\psi_\alpha(x_a^a)) = \psi_\alpha(x_a^a), \quad \theta_\alpha^a(\psi_\alpha(\xi_a^a)) = \alpha \psi_\alpha(\xi_a^a).
\]

2. If \( r > 1 \) is odd or if \( r = \infty \), then there exists an atlas \( A = \{ U_a \} \) on \( \mathcal{M} \) with local coordinates \((x_1^a, \xi_1^a)\) in \( U_a = (U_a, \mathcal{O}|_{U_a}) \) such that

\[
\psi_\alpha(x_a^a) = x_a^a, \quad \psi_\alpha(\xi_a^a) = \alpha \xi_a^a.
\]

Proof. Let \( A \) be any atlas on \( \mathcal{M} \) and let us fix a chart \( U \in A \) with coordinates \((x_1, \xi_1)\). In local coordinates any lift \( \psi_\alpha \) of \( \phi_\alpha \) can be written in the following form

\[
\psi_\alpha(x_1) = x_1 + F_2 + F_3 + \cdots; \quad \psi_\alpha(\xi_1) = \alpha(\xi_1 + G_3 + G_5 + \cdots),
\]

where \( F_s = F_s(x_1, \xi_1) \) is a homogeneous polynomial in variables \( \{ \xi_j \} \) of degree \( s \), and the same for \( G_q = G_q(x_1, \xi_1) \) for odd \( q \). We note that

\[
\psi_\alpha(F_s) = \alpha^s F_s + \mathcal{J}^{s+1}, \quad \psi_\alpha(G_q) = \alpha^q G_q + \mathcal{J}^{q+1}
\]

for any even \( s \) and odd \( q \). The idea of the proof is to use successively the following coordinate change

\[
(I) \quad x_1' = x_1 + \frac{1}{1 - \alpha^2 p} F_{2p}(x_1, \xi_1), \quad \xi_1' = \xi_1;
\]

\[
(II) \quad x_1'' = x_1', \quad \xi_1'' = \xi_1' + \frac{1}{1 - \alpha^2 p} G_{2p+1}(x_1', \xi_1'),
\]
where \( p = 1, 2, 3 \ldots \) in the following way.

If \( r = 2 \) there is nothing to check. If \( r > 2 \), first of all we apply (5)(I) and (5)(II) successively for \( p = 1 \). After coordinate changes (5)(I) we have

\[
\psi_\alpha(x'_i) = \psi_\alpha(x_i + \frac{1}{1 - \alpha^2} F_2) = x_i + F_2 + \frac{\alpha^2}{1 - \alpha^2} F_2 + \cdots = \\
x_i + \frac{1}{1 - \alpha^2} F_2 + \cdots = x'_i + \cdots \in x'_i + J^3; \quad \psi_\alpha(\xi'_i) \in \alpha \xi'_i + J^3.
\]

After coordinate changes (5)(II) similarly we will have

\[
\psi_\alpha(x''_i) \in x''_i + J^4, \quad \psi_\alpha(\xi''_i) \in \alpha \xi''_i + J^4.
\]

Now we change notations \( x_i := x''_i \) and \( \xi_j := \xi''_j \). Further, since (6) holds, we have

\[
\psi_\alpha(x_i) = x_i + F_r + F_{r+2} + \cdots; \quad \psi_\alpha(\xi_j) = \alpha(\xi_j + G_5 + G_7 + \cdots).
\]

Here we used the same notations for monomials \( F_s \) and \( G_q \) as above, however after the first step these functions may change. Now we continue to change coordinates consequentially in this way. If \( \alpha^{2p} \neq 1 \) for any \( p \in \mathbb{N} \), that is the order \( r = \text{ord}(\phi_\alpha) \) is odd or infinite, we can continue this procedure and obtain the required coordinates. This proves the second statement.

If \( r \) is even we continue our procedure for \( p < r/2 \). Now in our new coordinates \( \psi_\alpha \) has the following form

\[
\psi_\alpha(x_i) = x_i + F_r + F_{r+2} + \cdots; \quad \psi_\alpha(\xi_j) = \alpha \xi_j + \alpha G_{r+1} + \alpha G_{r+3} + \cdots.
\]

For any \( p \) such that \( \alpha^{2p} \neq 1 \), the changes of variables inverse to (5)(I) and (5)(II) have the following form

\[
(I) \quad x_a = x'_a + F'(x'_i, \xi'_j)(2p), \quad \xi_b = \xi'_b; \\
(II) \quad x'_a = x''_a, \quad \xi'_b = \xi''_b + G'(x''_i, \xi''_j)(2p+1),
\]

where \( F'(x'_i, \xi'_j)(2p) \in J^{2p} \) and \( G'(x''_i, \xi''_j)(2p+1) \in J^{2p+1} \).

Now we use again the coordinate change (5)(I) and (5)(II) for \( p = r+2 \), successively. Explicitly after coordinate changes (5)(I) using (7) for \( p = r+2 \) we have

\[
\psi_\alpha(x'_i) = \psi_\alpha(x_i + \frac{1}{1 - \alpha^{r+2}} F_{r+2}(x_i, \xi_j)) = x_i + F_r(x_i, \xi_j) + F_{r+2}(x_i, \xi_j) + \\
\frac{\alpha^{r+2}}{1 - \alpha^{r+2}} F_{r+2}(x_i, \xi_j) + \cdots = x_i + F_r(x_i, \xi_j) + \cdots \in x'_i + F_r(x_i, \xi_j) + J^{r+3}; \\
\psi_\alpha(\xi'_j) \in \alpha \xi'_j + \alpha G_{r+1}(x'_i, \xi'_j) + J^{r+3}.
\]

After the coordinate change (5)(II), we will have

\[
\psi_\alpha(x''_i) \in x''_i + F_r(x''_i, \xi''_j) + J^{r+4}, \quad \psi_\alpha(\xi''_j) \in \alpha \xi''_j + \alpha G_{r+1}(x''_i, \xi''_j) + J^{r+4}.
\]

Repeating this procedure for \( p = r+4, \ldots, 2r-2 \) and so on for \( p \neq kr, k \in \mathbb{N} \) we obtain the result.

4.4. **Lifting of an automorphism \( \phi_\alpha \), global picture.** Now we will show that a supermanifold with an automorphism \( \psi_\alpha \) has very special transition functions in an atlas \( \mathcal{A} = \{ \mathcal{U}_a \} \) from in Proposition 6. Recall that in any \( \mathcal{U}_a \in \mathcal{A} \) with coordinates \( (x_i, \xi_j) \) we can define an automorphism \( \theta^a_\alpha = \theta^a_\alpha(\mathcal{U}_a) \) as in Section 4.2 by \( \theta^a_\alpha(x_i) = x_i \) and \( \theta^a_\alpha(\xi_j) = \xi_j \).
Theorem 7. Let $\mathcal{A} = \{U_a\}$ be an atlas as in Proposition 6 and let there exists a lift $\psi_\alpha$ of the automorphism $\phi_\alpha$ of order $r = \text{ord}(\phi_\alpha)$. Let us take two charts $U_a, U_b \in \mathcal{A}$ such that $U_a \cap U_b \neq \emptyset$ with coordinates $(x^a_\lambda, \xi^a_\lambda)$ and $(x^b_\lambda, \xi^b_\lambda)$, respectively, with the transition functions $\Psi_{ab} : U_b \to U_a$.

(I) If $r$ is even, then we have
\[ \theta^b_\alpha(\Psi^*_b(x^a_\lambda)) = \Psi^*_{ab}(x^a_\lambda), \quad \theta^b_\alpha(\Psi^*_b(\xi^a_\lambda)) = \alpha \Psi^*_b(\xi^a_\lambda). \]

Or more generally,
\[ \theta^b_\alpha \circ \Psi^*_b = \Psi^*_{ab} \circ \theta^a_\alpha, \quad \theta^b_\alpha \circ \Psi^*_b = \Psi^*_b \circ \theta^a_\alpha. \]

(II) If we can find an atlas $\mathcal{A}$ with transition functions satisfying (8), the automorphism $\phi_\alpha$ possesses a lift $\psi_\alpha$.

(III) If $r > 1$ is odd or $r = \infty$, then $\mathcal{M}$ is split.

Proof. (III) Let $\Psi^*_b(x^a_\lambda) := L(x^a_\lambda, \xi^a_\lambda) = \sum_k L_{2k}$, where $L_{2k}$ are homogeneous polynomials of degree $2k$ in variables $\{\xi^b_\lambda\}$. Then if $r > 1$ is odd or $r = \infty$ by Proposition 6 we have
\[ \psi_\alpha \circ \Psi^*_b(x^a_\lambda) = \psi_\alpha \sum_k L_{2k} = L_0 + \alpha^2 L_2 + \alpha^4 L_4 + \cdots; \]
\[ \Psi^*_b \circ \psi_\alpha(x^a_\lambda) = \Psi^*_b(x^a_\lambda) = L_0 + L_2 + L_4 + \cdots. \]

Since $\psi_\alpha$ globally defined on $\mathcal{M}$, we have the following equality
\[ \psi_\alpha \circ \Psi^*_b = \Psi^*_b \circ \psi_\alpha, \]
which implies that $L_{2q} = 0$ for any $q \geq 1$. Similarly, the equality $\psi_\alpha \circ \Psi^*_b(\xi^a_\lambda) = \Psi^*_b \circ \psi_\alpha(\xi^a_\lambda)$ implies that $\Psi^*_b(\xi^a_\lambda)$ is linear in $\{\xi^b_\lambda\}$. In other words, $\mathcal{M}$ is split.

(I) Now assume that $r$ is even. Similarly to above we have
\[ \psi_\alpha \circ \Psi^*_b(x^a_\lambda) = \psi_\alpha \sum_k L_{2k} = L_0 + \alpha^2 L_2 + \cdots + \alpha^{r-2} L_{r-2} + L'; \]
\[ \Psi^*_b \circ \psi_\alpha(x^a_\lambda) = \Psi^*_b(x^a_\lambda + F_r + F_{2r} + \cdots) = L_0 + L_2 + \cdots + L_{r-2} + L'', \]
where $L', L'' \in \mathcal{J}^r$. Again the equality (10) implies that $L_2 = \cdots = L_{r-2} = 0$. Similarly, we can show that
\[ \Psi^*_b(\xi^a_\lambda) = M_1 + M_{r+1} + M_{r+3} + \cdots, \]
where $M_{2k+1}$ are homogeneous polynomials of degree $2k + 1$ in variables $\{\xi^b_\lambda\}$. Now if $T = T_0 + T_1 + T_2 + \cdots$ is a decomposition of a super-function into homogeneous polynomials in $\{\xi^b_\lambda\}$, denote by $[T]_q := T_q$ its $q$’s part. Using that $\psi_\alpha(L_0 r)$, where $s \in \mathbb{N}$, is $\theta^a_\alpha$-invariant, we have
\[ [\psi_\alpha \circ \Psi^*_b(x^a_\lambda)]_{2p} = \alpha^{2p} L_{2p}, \quad 2p = r + 2, \ldots, 2r - 2. \]

Further, using $\Psi^*_b(F_r)$ is $\theta^a_\alpha$-invariant mod $\mathcal{J}^{2r}$, we have
\[ [\Psi^*_b \circ \psi_\alpha(x^a_\lambda)]_{2p} = L_{2p}, \quad 2p = r + 2, \ldots, 2r - 2. \]

This result implies that $L_{r+2} = \cdots = L_{2r-2} = 0$. Similarly we work with $M(\xi^a_\lambda, \xi^b_\lambda)$. In the same way we show that $L_s = 0$ for any $p \neq sr$, where $s = 0, 1, 2, \ldots$.

(II) If $\mathcal{M}$ possesses an atlas $\mathcal{A}$ with transition functions satisfying (9), a lift $\psi_\alpha$ can be defined in the following way for any chart $U_a$
\[ \psi_\alpha(x^a_\lambda) = x^a_\lambda; \quad \psi_\alpha(\xi^a_\lambda) = \alpha \xi^a_\lambda. \]
Corollary 10. Observation made in [Kos] about lifting of graded operators.

Indeed, for

Remark 8. Now we can show that (8) is equivalent to Theorem 7 (I). Let again

Remark 9. In case

Further, since, grΨ∗ab : Uab → Uab of gr M applying the automorphism γab. (Here we identified gr U and U in a natural way.)

In the structure sheaf of Uab (respectively Uab) there is an automorphism θα (respectively θβ) defined as above. Since gr U = U, we get θα = θα|U. Recall that the statement Theorem 7 (I) we can reformulate in the following way

ψ∗ab = γab ⋄ grψ∗ab,

where (γab) is a Čech cocycle corresponding to the covering A = {Ua} representing M, see Theorem 7 and γab is written in coordinates of Uab. In other words this means that the transition functions Ψab may be obtained from the transition functions of grΨab : Uab → Uab of gr M applying the automorphism γab. (Here we identified gr U and U in a natural way.)

If the automorphism φα can be lifted to a supermanifold M, then

Corollary 11. Any supermanifold M possesses a lift of an automorphism φ−1. Indeed, by definition M possesses an atlas satisfying (8). Therefore in (any) local coordinates (xα, ξβ) of M we can define an automorphism ψ+1α by the following formulas

We will call this automorphism standard. We also can define this automorphism in the following coordinate free way

Corollary 12. Let r = ord(φα) > 1 be odd or ∞. Then the automorphism φα can be lifted to a supermanifold M if and only if M is split.

Corollary 13. If the automorphism φα can be lifted to a supermanifold M, then

In particular, if o(M) = 2, the automorphism φα can be listed to M only for α = ±1.

Formulas (8) shows that ψα is well-defined. The proof is complete. □

Corollary 13. If the automorphism φα can be lifted to a supermanifold M, then

o(M) ≥ ord(φα), where o(M) is the order of a supermanifold M, see Section 2.3.
4.5. Lifting of the automorphism $\phi_1$ and consequences. By definition any lift $\psi_1$ of the automorphism $\phi_1 = id$ is a global section of the sheaf $H^0(M, Aut_{(2)}O)$, see Section 2.2. The 0-cohomology group $H^0(M, Aut_{(2)}O)$ can be computed using the following exact sequence
\[ \{ e \} \to Aut_{(2q+2)}O \to Aut_{(2q)}O \to (T_{gr})_{2q} \to 0, \quad p \geq 1, \]
see [1]. Further let we have two lifts $\psi_0$ and $\psi'_0$ of $\phi_0$. Then the composition $\Psi_0 := (\psi_0)^{-1} \circ \psi'_0$ is a lift of $\phi_1$. Therefore any lift $\psi'_0$ is equal to the composition $\psi_0 \circ \Psi_0$ of a fixed lift $\psi_0$ and an element from $\Psi_0 \in H^0(M, Aut_{(2)}O)$. In particular, according Corollary [1] there always exists the standard lift $\psi'^0_1$ of $\phi_{-1}$. Therefore for any lift $\psi'_{-1}$ we have $\psi'_{-1} = \psi'^0_1 \circ \Psi_0$, where $\Psi_0 \in H^0(M, Aut_{(2)}O)$.

5. Automorphisms of the structure sheaf of $\Pi_{Gr_{n,k}}$

Let $\mathcal{M} = \Pi_{Gr_{n,k}}$ be a II-symmetric super-Grassmannian. Recall that the retract $\text{gr}\Pi_{Gr_{n,k}}$ of $\Pi_{Gr_{n,k}}$ is isomorphic to $(Gr_{n,k} \, \text{st}, \, \Omega)$, where $\Omega$ is the sheaf of 1-forms on the usual Grassmannian $Gr_{n,k}$. The sheaf $\Omega$ is the sheaf of sections of the cotangent bundle $T^*(\mathcal{M})$ over $\mathcal{M} = Gr_{n,k}$. In the next subsection we recover a well-known result about the automorphism group $\text{Aut}^*(\mathcal{M})$ of $T^*(\mathcal{M})$.

5.1. Automorphisms of the cotangent bundle over a Grassmannian.

Let $\mathcal{M} = Gr_{n,k}$ be the usual Grassmannian, i.e. the complex manifold that parametrizes all $k$-dimensional linear subspaces in $\mathbb{C}^n$ and let $T^*(\mathcal{M})$ be its cotangent bundle. It is well-known result that $\text{End}^*(\mathcal{M}) \simeq \mathbb{C}$. Therefore, $\text{Aut}^*(\mathcal{M}) \simeq \mathbb{C}^*$. For completeness we will prove this fact using use the Borel-Weil-Bott Theorem, see for example [A] for details.

Let $G = \text{GL}_n(\mathbb{C})$ be the general linear group, $P$ be a parabolic subgroup in $G$, $R$ be the reductive part of $P$ and let $E_\chi \to G/P$ be the homogeneous vector bundle corresponding to a representation $\chi$ of $P$ in the fiber $E = (E_\chi)_P$. Denote by $E_\chi$ the sheaf of holomorphic sections of $E_\chi$. In the Lie algebra $\mathfrak{gl}_n(\mathbb{C}) = \mathfrak{Lie}(G)$ we fix the Cartan subalgebra $t = \{\text{diag}(\mu_1, \ldots, \mu_n)\}$, the following system of positive roots
\[ \Delta^+ = \{\mu_i - \mu_j \mid 1 \leq i < j \leq n\}, \]
and the following system of simple roots $\Phi = \{\alpha_1, \ldots, \alpha_{n-1}\}$, $\alpha_i = \mu_i - \mu_{i+1}$, where $i = 1, \ldots, n-1$. Denote by $t^*(\mathbb{R})$ a real subspace in $t^*$ spanned by $\mu_j$. Consider the scalar product $(,) \text{ in } t^*(\mathbb{R})$ such that the vectors $\mu_j$ form an orthonormal basis.

An element $\gamma \in t^*(\mathbb{R})$ is called \textit{dominant} if $(\gamma, \alpha) \geq 0$ for all $\alpha \in \Delta^+$. We assume that $B^- \subset P$, where $B^-$ is the Borel subgroup corresponding to $\Delta^-$. 

\textbf{Theorem 14} (Borel-Weil-Bott). \textit{Assume that the representation $\chi : P \to \text{GL}(E)$ is completely reducible and $\lambda_1, \ldots, \lambda_n$ are highest weights of $\chi|R$. Then the $G$-module $H^0(G/P, E_\chi)$ is isomorphic to the sum of irreducible $G$-modules with highest weights $\lambda_1, \ldots, \lambda_n$, where $\lambda_i$ are dominant highest weights of $\chi|R$.}

Now we apply this theorem to the case of the usual Grassmannian $Gr_{n,k}$. We have $Gr_{n,k} \simeq G/P$, where $G = \text{GL}_n(\mathbb{C})$ and $P \subset G$ is given by

\[ P = \left\{ \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \right\}, \]

where $A$ is a complex $k \times k$-matrix. We see that $R = \text{GL}_k(\mathbb{C}) \times \text{GL}_{n-k}(\mathbb{C})$. The isotropy representation $\chi$ of $P$ can be computed in a standard way, see for instance [Oni11 Proposition 5.2]. The representation $\chi$ is completely reducible and it is equal to $\rho_1 \otimes \rho_2$, where $\rho_1$ and $\rho_2$ are standard representations of the Lie groups $\text{GL}_k(\mathbb{C})$ and $\text{GL}_{n-k}(\mathbb{C})$, respectively.
**Proposition 15.** For usual Grassmannian $M = \text{Gr}_{n,k}$, where $n - k, k > 0$, we have

$$\text{End } T^*(M) \simeq \mathbb{C}, \quad \text{Aut } T^*(M) \simeq \mathbb{C}^*.$$

**Proof.** The cotangent bundle $T^*(M)$ over $M$ is homogeneous and the corresponding representation is the dual to isotropy representation $\chi$. Let us compute the representation $\omega$ of $P$ corresponding to the homogeneous bundle

$$\text{End } T^*(M) \simeq T(M) \otimes T^*(M).$$

The representation $\omega$ is completely reducible and we have

$$\omega|_R = \rho_1 \otimes \rho_2^* \otimes \rho_1^* \otimes \rho_2 \simeq \rho_1 \otimes \rho_1^* \otimes \rho_2 \otimes \rho_2^*.$$

Therefore, we have

1. $\omega|_R = 1 + \text{ad}_1 + \text{ad}_2 + \text{ad}_1 \otimes \text{ad}_2$ for $k > 1$ and $n - k > 1$;
2. $1 + \text{ad}_2$ for $k = 1$ and $n - k > 1$;
3. $1 + \text{ad}_1$ for $k > 1$ and $n - k = 1$;
4. $1$ for $k = n - k = 1$,

where $1$ is the trivial one dimensional representation, $\text{ad}_1$ and $\text{ad}_2$ are adjoint representations of $\text{GL}_k(\mathbb{C})$ and $\text{GL}_{n-k}(\mathbb{C})$, respectively. Then the heights weights of the representation $\omega|_R$ are

1. $0, \mu_1 - \mu_k, \mu_{k+1} - \mu_n, \mu_1 - \mu_k + \mu_{k+1} - \mu_n$ for $k > 1$ and $n - k > 1$;
2. $0, \mu_2 - \mu_n$ for $k = 1$ and $n - k > 1$;
3. $0, \mu_1 - \mu_{n-1}$ for $k > 1$ and $n - k = 1$;
4. $0$ for $k = n - k = 1$,

respectively. We see that the unique dominant weight is 0 in any case. By Borel-Weil-Bott Theorem we obtain the result. $\square$

5.2. The group $H^0(M, \text{Aut}\mathcal{O})$. Recall that $\mathcal{M} = (M,\mathcal{O}) = \Pi \text{Gr}_{n,k}$ is a $\Pi$-symmetric super-Grassmannian. To compute the automorphisms of $\mathcal{O}$ we use the following exact sequence of sheaves

$$(12) \quad e \to \text{Aut}_{(2)}\mathcal{O} \xrightarrow{\iota} \text{Aut}\mathcal{O} \xrightarrow{\sigma} \text{Aut}(\Omega) \to e,$$

where $\text{Aut}(\Omega)$ is the sheaf of automorphisms of the sheaf of 1-forms $\Omega$. Here the map $\iota$ is the natural inclusion and $\sigma$ maps any $\delta : \mathcal{O} \to \mathcal{O}$ to $\sigma(\delta) : \mathcal{O}/\mathcal{J} \to \mathcal{O}/\mathcal{J}$, where $\mathcal{J}$ is again the sheaf of ideals generated by odd elements in $\mathcal{O}$. Consider the corresponding to (12) exact sequence of 0-cohomology groups

$$(13) \quad \{e\} \to H^0(M, \text{Aut}_{(2)}\mathcal{O}) \to H^0(M, \text{Aut}\mathcal{O}) \to \text{Aut } T^*(M),$$

and the corresponding to (12) exact sequence of 0-cohomology groups

$$(14) \quad \{e\} \to H^0(M, \text{Aut}_{(2p+2)}\mathcal{O}) \to H^0(M, \text{Aut}_{(2p)}\mathcal{O}) \to H^0(M, (\mathcal{T}_{\mathcal{Gr}})_{2p}), \quad p \geq 1.$$

In [Oni1] Theorem 4.4 it has been proven that

$$(15) \quad H^0(M, (\mathcal{T}_{\mathcal{Gr}})_s) = \{0\} \quad \text{for} \quad s \geq 2.$$ (For $\mathcal{M} = \Pi \text{Gr}_{2,1}$ this statement follows from dimensional reason.) Therefore,

$$(16) \quad H^0(M, \text{Aut}_{(2)}\mathcal{O}) = \{e\}.$$
Recall that the automorphism $\psi_{st}^{-1}$ of the structure sheaf was defined in Corollary 11.

**Theorem 16.** Let $\mathcal{M} = \PiGr_{n,k}$ be a $\Pi$-symmetric super-Grassmannian and $(n,k) \neq (2,1)$. Then

$$H^0(\Gr_{n,k}, \text{Aut}\mathcal{O}) = \{\text{id}, \psi_{st}^{-1}\}.$$  

For $\mathcal{M} = \PiGr_{2,1}$ we have

$$H^0(\Gr_{2,1}, \text{Aut}\mathcal{O}) \simeq \mathbb{C}^*.$$  

**Proof.** From (13), (16) and Proposition 15, it follows that

$$\{e\} \to H^0(M, \text{Aut}\mathcal{O}) \to \{\phi_\alpha \mid \alpha \in \mathbb{C}^*\} \simeq \mathbb{C}^*.$$  

Now the statement follows from Proposition 4 and Corollary 13. In more details, for $(n,k) \neq (2,1)$, we have $o(M) = 2$, therefore $\phi_\alpha$ can be lifted to $M$ if and only if $\text{ord}(\phi_\alpha) = 1$ or $2$. In other words, $\alpha = \pm 1$.

In the case $\mathcal{M} = \PiGr_{2,1}$, we have $\dim M = (1|1)$. Therefore, $\text{Aut}_{(2)}\mathcal{O} = \text{id}$ and any $\phi_\alpha$ can be lifted to $M$. The proof is complete. $\square$

We finish this section with the following theorem.

**Theorem 17.** Let $\text{gr}\mathcal{M} = (M, \text{gr}\mathcal{O}) = \text{gr}\PiGr_{n,k}$, where $\PiGr_{n,k}$ is a $\Pi$-symmetric super-Grassmannian. Then

$$H^0(\Gr_{n,k}, \text{Aut}(\text{gr}\mathcal{O})) = \text{Aut} \mathfrak{T}(M) \simeq \mathbb{C}^*.$$  

**Proof.** In Sequence 14 we can replace $\mathcal{O}$ by $\text{gr}\mathcal{O}$. (This sequence is exact for any $\mathcal{O}'$ such that $\text{gr}\mathcal{O}' \simeq \text{gr}\mathcal{O}$.) By (15) as above we get

$$H^0(M, \text{Aut}_{(2)}(\text{gr}\mathcal{O})) = \{e\}.$$  

By (13) we have

$$\{e\} \to H^0(M, \text{Aut}(\text{gr}\mathcal{O})) \longrightarrow \text{Aut} \mathfrak{T}(M) \simeq \mathbb{C}^*.$$  

Hence any automorphism from $\text{Aut} \mathfrak{T}(M)$ induces an automorphism of $\text{gr}\mathcal{O}$, we obtain the result. $\square$

6. **The automorphism supergroup Aut $\PiGr_{n,k}$ of a II-symmetric super-Grassmannian**

6.1. **The automorphism group of** $\text{Gr}_{n,k}$. The following theorem can be found for example in [A] Chapter 3.3, Theorem 1, Corollary 2.

**Theorem 18.** The automorphism group $\text{Aut}(\text{Gr}_{n,k})$ is isomorphic to $\text{PGL}_n(\mathbb{C})$ if $n \neq 2k$ and if $(n,k) = (2,1)$; and $\text{PGL}_n(\mathbb{C})$ is a normal subgroup of index 2 in $\text{Aut}(\text{Gr}_{n,k})$ for $n = 2k$, $k \neq 1$.

More precisely in the case $n = 2k \geq 4$ we have

$$\text{Aut}(\text{Gr}_{2k,k}) = \text{PGL}_n(\mathbb{C}) \rtimes \{\text{id}, \Phi\},$$

where $\Phi^2 = \text{id}$ and $\Phi \circ g \circ \Phi^{-1} = (g^t)^{-1}$ for $g \in \text{PGL}_n(\mathbb{C})$. 

An additional automorphism $\Phi$ can be described geometrically. (Note that an additional automorphism is not unique.) It is well-known that $\text{Gr}_{n,k} \simeq \text{Gr}_{n,n-k}$ and this isomorphism is given by $\text{Gr}_{n,k} \ni V \mapsto V^\perp \in \text{Gr}_{n,n-k}$, where $V^\perp$ is the orthogonal complement of $V \subset \mathbb{C}^n$ with respect to a bilinear form $B$. In the case $n = 2k$ we clearly have $\text{Gr}_{n,k} = \text{Gr}_{n,n-k}$, hence the map $V \mapsto V^\perp$ induces an automorphism of $\text{Gr}_{2k,k}$, which we denote by $\Phi_B$. This automorphism is not an element of $\text{PGL}_n(\mathbb{C})$ for $(n,k) \neq (2,1)$.

Assume that $B$ is the symmetric bilinear form, given in the standard basis of $\mathbb{C}^n$ by the identity matrix. Denote the corresponding automorphism by $\Phi$. Let us describe $\Phi$ in the standard coordinates on $\text{Gr}_{2}$. Denote the corresponding automorphism by $\Phi$. Let us define $\Phi$ by the identity matrix. We have

$$
\begin{pmatrix}
X \\
E
\end{pmatrix} \xrightarrow{\Phi} \begin{pmatrix}
E \\
-X^t
\end{pmatrix},
$$

since

$$
\begin{pmatrix}
E \\
-X^t
\end{pmatrix}^t \cdot \begin{pmatrix}
X \\
E
\end{pmatrix} = \begin{pmatrix}
E & -X
\end{pmatrix} \cdot \begin{pmatrix}
X \\
E
\end{pmatrix} = 0.
$$

More general, let $U_I$, where $|I| = k$, be another chart on $\text{Gr}_{2k,k}$ with coordinates $(x_{ij})$, $i,j = 1,\ldots,k$, as described in Section 3.3. Denote $J := \{1,\ldots,k\} \setminus I$. Then $U_J$ is again a chart on $\text{Gr}_{2k,k}$ with coordinates $(y_{ij})$, $i,j = 1,\ldots,k$. Then the automorphism $\Phi$ is given by $y_{ij} = -x_{ji}$.

**Remark 19.** In case $(n,k) = (2,1)$ the automorphism $\Phi$ described above is defined as well, however it coincides with the following automorphism from $\text{PGL}_2(\mathbb{C})$

$$
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} \cdot \begin{pmatrix}
x \\
1
\end{pmatrix} = \begin{pmatrix}
1 & -x
\end{pmatrix} = \begin{pmatrix}
1 \\
-x^t
\end{pmatrix}.
$$

The same in another chart.

Let us discuss properties of $\Phi$ mentioned in Theorem 18. Clearly $\Phi^2 = \text{id}$. Further, for $g \in \text{PGL}_n(\mathbb{C})$ we have

$$
\left( [g^t]^{-1} \cdot \begin{pmatrix}
E \\
-X^t
\end{pmatrix} \right)^t \cdot \begin{pmatrix}
X \\
E
\end{pmatrix} = \begin{pmatrix}
E & -X
\end{pmatrix} \cdot g^{-1} \cdot g \cdot \begin{pmatrix}
X \\
E
\end{pmatrix} = 0.
$$

(In other charts $U_I$ the argument is the same.) In other words, if $V \subset \mathbb{C}^{2k}$ is a linear subspace of dimension $k$, then $(g \cdot V)^\perp = (g^t)^{-1} \cdot V$. Hence,

$$
V \xrightarrow{\Phi^{-1}} V^\perp \xrightarrow{g} g \cdot V^\perp \xrightarrow{\Phi} (g^t)^{-1} \cdot V.
$$

Therefore, $\Phi \circ g \circ \Phi^{-1} = (g^t)^{-1}$.

### 6.2. About lifting of the automorphism $\Phi$.

**6.2.1. Lifting of the automorphism $\Phi$ to $\text{gr} \Pi \text{Gr}_{2k,k}$.** Recall that we have

$$
\text{gr}\Pi \text{Gr}_{n,k} \simeq (\text{Gr}_{n,k}, \bigwedge \Omega),
$$

where $\Omega$ is the sheaf of 1-forms on $\text{Gr}_{n,k}$. Therefore any automorphism of $\text{Gr}_{n,k}$ can be naturally lifted to $\text{gr}M = \text{gr}\Pi \text{Gr}_{n,k}$. Indeed, the lift of an automorphism $F$ of $\text{Gr}_{n,k}$ is the automorphism $(F, \wedge d(F))$ of $(\text{Gr}_{n,k}, \bigwedge \Omega)$. Further, by Theorem 17 we have

$$
\{e\} \to H^0(M, \text{Aut}(\text{gr}\mathcal{O})) \simeq \mathbb{C}^* \longrightarrow \text{Aut}(\text{gr}M) \longrightarrow \text{Aut}(\text{Gr}_{n,k}).
$$
we can describe the automorphism $(\Phi, \eta)$.

First consider the following two coordinate matrices, see Section 3:

$$\begin{pmatrix} X & \Xi \\ E & 0 \\ \Xi & X \\ 0 & E \end{pmatrix}, \quad \begin{pmatrix} E & 0 \\ Y & H \\ 0 & E \\ H & Y \end{pmatrix},$$

where $X = (x_{ij})$, $Y = (y_{ij})$ are $k \times k$-matrices of local even coordinates and $\Xi = (\xi_{kl})$, $H = (\eta_{kl})$ are $k \times k$-matrices of local odd coordinates on $\text{IIGr}_{2k,k}$. Denote by $\mathcal{V}_i \in \mathcal{A}^\Pi$ the corresponding to $Z_i$ superdomain. Then $\text{gr}\mathcal{V}_1$ and $\text{gr}\mathcal{V}_2$ are superdomains in $\text{gr}\mathcal{A}^\Pi$ with coordinates $(\text{gr}(x_{ij}), \text{gr}(\xi_{kl}))$ and $(\text{gr}(y_{ij}), \text{gr}(\eta_{kl}))$, respectively. (Note that we can consider any superfunction $f$ as a morphism between supermanifolds, therefore $\text{gr}f$ is defined.)

We can easily check that the coordinate $\text{gr}(\xi_{ij})$ (or $\text{gr}(\eta_{ij})$) can be identified with the 1-form $d(\text{gr}(x_{ij}))$ (or $d(\text{gr}(y_{ij}))$, respectively) for any $(ij)$. Using this fact we can describe the automorphism $(\Phi, \eta)$ on $\text{IIGr}_{2k,k}$. We get in our local charts

$$\begin{pmatrix} \text{gr}X & \text{gr}\Xi \\ E & 0 \\ \text{gr}\Xi & \text{gr}X \\ 0 & E \end{pmatrix} (\Phi, \eta) \begin{pmatrix} E & 0 \\ -\text{gr}X^t & -\text{gr}\Xi^t \\ 0 & E \\ -\text{gr}\Xi^t & -\text{gr}X^t \end{pmatrix}.$$

We can describe the automorphism $(\Phi, \eta)$ in any other charts of $\text{gr}\mathcal{A}^\Pi$ in a similar way. Clearly, $(\Phi, \eta) = id.$

Hence,

$$\text{Aut}(\text{gr}M) \simeq C^* \times \text{Aut}(\text{Gr}_{n,k}).$$

Now we see that $\text{Aut}(\text{gr}M)$ is isomorphic to the group of all automorphisms $\text{Aut}^\Pi(M)$ of $\text{T}^\Pi(M)$. An automorphism $\phi_\alpha \in C^*$ commutes with any $(F, \iota d(F)) \in \text{Aut}(\text{Gr}_{n,k})$. Hence we obtain the following result.

**Theorem 20.** If $\text{gr}M = \text{gr}\text{IIGr}_{2k,k}$, then

$$\text{Aut}(\text{gr}M) \simeq \text{Aut}^\Pi(M) \simeq \text{Aut}(\text{Gr}_{n,k}) \times C^*.$$

In other words, any automorphism of $\text{gr}M$ is induced by an automorphism of $\text{T}^\Pi(M)$.

More precisely,

1. If $\text{gr}M = \text{gr}\text{IIGr}_{2k,k}$, where $k \geq 2$, then

$$\text{Aut}(\text{gr}M) \simeq (\text{PGL}_{2k}(C) \times \{\text{id}, (\Phi, \iota d(\Phi))\}) \times C^*,$$

   where $(\Phi, \iota d(\Phi)) \circ g \circ (\Phi, \iota d(\Phi))^{-1} = (g^t)^{-1}$ for $g \in \text{PGL}_{2k}(C)$.

2. For other $(n, k)$, we have

$$\text{Aut}(\text{gr}M) \simeq \text{PGL}_n(C) \times C^*.$$

**Corollary 21.** We see, Theorem 20, that any lift of the automorphism $\Phi$ to $\text{gr}\text{IIGr}_{2k,k}$ has the following form

$$\phi_\alpha \circ (\Phi, \iota d(\Phi)), \quad \alpha \in C^*.$$
Theorem 5.2. This statement can be deduced from results obtained in [Oni1]. Indeed, by [Oni1, Proposition 4.10] we have

(21) \( H^1(Gr_{n,k}, (T_{Gr})_2) \simeq H^2(Gr_{n,k}, \bigwedge^2 \Omega) \). Further by Dolbeault-Serre theorem we have

(20) \( H^1(Gr_{n,k}, (T_{Gr})_2) \simeq H^{2,1}(Gr_{n,k}, \Omega) \).

Combining Formulas (18), (19) and (20) we get

(19) \( H^1(Gr_{n,k}, \bigwedge^2 \Omega) = \{0\} \).

Further by Dolbeault-Serre theorem we have

(18) \( H^1(Gr_{n,k}, (T_{Gr})_2) \simeq H^{1,1}(Gr_{n,k}, \Omega) \).

Consider the exact sequence for the sheaf \( gr\mathcal{O} \)

\[ e \to Aut_{(2p+2)Gr\mathcal{O}} \to Aut_{(2p)gr\mathcal{O}} \to (T_{Gr})_{2p} \to 0. \]

Since \( H^1(M, (T_{Gr})_p) = \{0\} \) for \( p \geq 3 \), see [Oni1, Theorem 4.4], we have

\[ H^1(Gr_{n,k}, Aut_{(2p)gr\mathcal{O}}) = \{e\} \text{ for } p \geq 2. \]

Hence we have the following inclusion

(21) \( H^1(Gr_{n,k}, Aut_{(2p)gr\mathcal{O}}) \to H^1(Gr_{n,k}, (T_{Gr})_2) \simeq H^{2,1}(Gr_{n,k}, \Omega) \).

Let \( \gamma \in H^1(Gr_{2k,\mathcal{O}}) \) be the Green cohomology class of the supermanifold \( IIGr_{2k,\mathcal{O}} \), see Theorem [Oni1]. Denote by \( \eta \) the image of \( \gamma \) in \( H^{2,1}(Gr_{2k,\mathcal{O}}) \). (The notation \( \eta \) we borrow in [Oni1].)

**Theorem 22.** The automorphism \( \phi_\alpha \circ (\Phi, \wedge d(\Phi)) \), where \( \alpha \in \mathbb{C}^* \), can be lifted to \( IIGr_{2k,\mathcal{O}} \), where \( k \geq 2 \), if and only if \( \alpha = \pm i \).

The II-symmetric super-Grassmannian \( IIGr_{2,1} \) is split, in other words, \( IIGr_{2,1} \simeq grIIGr_{2,1} \). Therefore any \( \phi_\alpha \circ (\Phi, \wedge d(\Phi)) \) is an automorphism of \( IIGr_{2,1} \).

**Proof.** This statement can be deduced from results of [Oni1]. Indeed, by [Oni1, Theorem 5.2 (1)], \( IIGr_{2k,\mathcal{O}} \), where \( k \geq 2 \), corresponds to the \((2,1)\)-form \( \eta \neq 0 \) defined by [Oni1, Formula (4.19)]. Further, the inclusion \( Aut^{\ast}(M) \) is \( Aut^{\ast}(M) \)-invariant. In [Oni1, Lemma 3.2] it was shown that \( \phi_\alpha(\eta) = \alpha^2 \eta \) and in the proof of [Oni1, Theorem 4.6 (2)] it was shown that \( (\Phi, \wedge d(\Phi))(\eta) = -\eta \in H^{2,1}(Gr_{n,k}, \Omega) \). This implies that

\[ [\phi_\alpha \circ (\Phi, \wedge d(\Phi)))(\eta) = \eta \]

if and only if \( \alpha = \pm i \). Hence,

\[ [\phi_\alpha \circ (\Phi, \wedge d(\Phi)))(\gamma) = \gamma \in H^1(M, Aut_{(2p)gr\mathcal{O}}) \]

if and only if \( \alpha = \pm i \). Hence by Proposition [Oni1] we obtain the result. \( \square \)
6.2.4. A geometric construction of a lift of \( \Phi \) to \( \Pi \text{Gr}_{2k,k} \). Together with the supermanifold \( \Pi \text{Gr}_{n,k} \) we can consider a \( \Pi^L \)-symmetric super-Grassmannian \( \Pi \text{Gr}^L_{n,k} \). A construction of \( \Pi \text{Gr}^L_{n,k} \) is similar to the construction of \( \Pi \text{Gr}_{n,k} \) given in Section 3.4. The difference is that we write even and odd coordinates in rows, not columns. For example a coordinate matrix \( \mathcal{Z}_I \), where \( I = \{1, \ldots, k\} \), of \( \Pi \text{Gr}^L_{n,k} \) has the following form

\[
\begin{pmatrix}
  X & 0 & 0 & \Xi \\
  E & X & 0 & \Xi \\
  0 & 0 & X & E \\
\end{pmatrix}
\]

The supermanifold \( \Pi \text{Gr}^L_{n,k} \) is a super-Grassmannian of \( \Pi^L \)-symmetric \( k \)-dimensional superspaces in \( \mathbb{C}^{m[n]} \), where \( \Pi^L \) is an odd involution, which is linear with respect to left coordinates in \( \mathbb{C}^{m[n]} \). (If we write our involution \( \Pi \) in left coordinates, it will be superlinear, not linear, see [L].)

In \( \mathbb{C}^{2k|2k} \) we can consider a bilinear form given in the standard basis by the identity matrix of size 4. This form is not super-symmetric. The form induces a map \( \Pi \text{Gr}_{2k,k} \to \Pi \text{Gr}^L_{2k,k} \) given in charts by

\[
\begin{pmatrix}
  X & \Xi \\
  E & X \\
  \Xi & E \\
  0 & X \\
\end{pmatrix} \mapsto \begin{pmatrix}
  E & -X & 0 & -\Xi \\
  0 & -\Xi & E & -X \\
\end{pmatrix}
\]

For other charts, let \( \mathcal{U}_I \), where \( |I| = k \), be another chart on \( \Pi \text{Gr}_{2k,k} \) with coordinates \((x_{ij}, \xi_{st})\), \( i, j, s, t = 1, \ldots, k \), as described in Section 3.4. Denote \( J := \{1, \ldots, 2k\} \setminus I \). Then \( \mathcal{U}_J^t \) is a chart on \( \Pi \text{Gr}^L_{2k,k} \) with coordinates \((y_{ij}, \eta_{st})\), \( i, j, s, t = 1, \ldots, k \). Then our map is given by \( y_{ij} = -x_{ij}, \eta_{st} = -\xi_{st} \). Since,

\[
\begin{pmatrix}
  E & -X & 0 & -\Xi \\
  0 & -\Xi & E & -X \\
\end{pmatrix} \begin{pmatrix}
  X & \Xi \\
  E & X \\
  \Xi & E \\
  0 & X \\
\end{pmatrix} = 0.
\]

and the same for coordinate matrices of charts \( \mathcal{U}_I \) and \( \mathcal{U}_J^t \), \( J := \{1, \ldots, 2k\} \setminus I \), we see that the map is independent on the choice of a chart.

Further, for \( \Pi \)-symmetric even matrices an \( i \)-transposition is defined, see [25],

\[
\begin{pmatrix}
  A & B \\
  B & A \\
\end{pmatrix}^{t_i} = \begin{pmatrix}
  A^t & iB^t \\
  iB^t & A^t \\
\end{pmatrix},
\]

which satisfies \((C_1 \cdot C_2)^{t_i} = C_2^{t_i} \cdot C_1^{t_i}\). (Here \( iB^t \) is the matrix \( B^t \), transpose to the matrix \( B \), multiplied by the complex number \( i = \sqrt{-1} \).) Now we can define an automorphism of \( \Pi \text{Gr}_{n,k} \) by

\[
\begin{pmatrix}
  X & \Xi \\
  E & X \\
  0 & E \\
\end{pmatrix} \mapsto \begin{pmatrix}
  E & -X & 0 & -\Xi \\
  0 & -\Xi & E & -X \\
\end{pmatrix} \mapsto \begin{pmatrix}
  E & 0 & -\Xi^t & -X^t \\
  0 & E & -X^t & -\Xi^t \\
\end{pmatrix}.
\]

In charts \( \mathcal{Z}_I \to \mathcal{Z}_J \) on \( \Pi \text{Gr}_{n,k} \), where \( J := \{1, \ldots, 2k\} \setminus I \), this map is given by \( y_{ij} = -x_{ij}, \eta_{st} = -i\xi_{ts} \). In notations of Theorem 22, the obtained automorphism is the (unique) lift of \( \phi_i \circ (\Phi, \wedge d(\Phi)) \) for \((n, k) \neq (2, 1)\) and obtained automorphism is equal to \( \phi_i \circ (\Phi, \wedge d(\Phi)) \) for \((n, k) = (2, 1)\).

We denote the automorphism described in this section by \( \Theta \).
6.3. The automorphism supergroup of $\text{IIGr}_{n,k}$. As above we denote by $\varpi(M) := H^0(M, T)$ the Lie superalgebra of vector fields on a supermanifold $M$. In this section $M = \text{IIGr}_{n,k}$. Recall that $q_{\alpha}(C)$ is a strange Lie superalgebra, see [Kac] for details. This Lie superalgebra contains all matrices over $C$ in the following form

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix},$$

where $A, B$ are complex square matrix of size $n$. For instance the identity matrix $E_{2n}$ of size $2n$ is an element of $q_{\alpha}(C)$. The corresponding Lie supergroup we denote by $Q_{\alpha}(C)$. This is a subsupergroup in $\text{GL}_{2n}(C)$ which is invariant with respect to the odd involution $\Pi$. This Lie supergroup can be seen as a subsupergroup of the following form in $\text{GL}_{2n}(C)$.

$$\begin{pmatrix} L & M \\ M & L \end{pmatrix},$$

where $L$ is an $n \times n$-matrix of even coordinates, while $M$ is an $n \times n$-matrix of odd coordinates. The corresponding Lie subalgebra of size $2n$ (GL$_n(C), q_{\alpha}(C)$).

In [Oni1] the following theorem was proved, see [Oni1, Theorem 5.2] and [V22, Section 5] for an explicit description of $\varpi(\text{IIGr}_{2,1})$.

Theorem 23. (1) If $M = \text{IIGr}_{n,k}$, where $(n, k) \neq (2, 1)$, then

$$\varpi(M) \simeq q_{\alpha}(C)/\langle E_{2n} \rangle.$$

(2) If $M = \text{IIGr}_{2,1}$, then

$$\varpi(M) \simeq \mathfrak{g} \ltimes \langle z \rangle \simeq q_{2}(C)/\langle E_{4} \rangle \ltimes \langle z \rangle.$$

Here $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{0} \oplus \mathfrak{g}_{1}$ is a $\mathbb{Z}$-graded Lie superalgebra defined in the following way.

$$\mathfrak{g}_{-1} = V, \quad \mathfrak{g}_{0} = \mathfrak{sl}_{2}(C), \quad \mathfrak{g}_{1} = \langle d \rangle,$$

where $V = \mathfrak{sl}_{2}(C)$ is the adjoint $\mathfrak{sl}_{2}(C)$-module, $[\mathfrak{g}_{0}, \mathfrak{g}_{1}] = \{0\}$, $[d, -]$ maps identically $\mathfrak{g}_{-1}$ to $\mathfrak{g}_{0}$, and $z$ is the grading operator of the $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g}$.

the element $d$ corresponds to the matrix $\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} \in q_{2}(C)$.

Recall that the automorphism $\psi_{z1}^{s}$ of the structure sheaf was defined in Corollary [11]. Denote by $\Psi_{z1}^{s}$ the corresponding automorphism of the supermanifold $M$.

Note that for any supermanifold $(M, O)$ the action of $\Psi_{z1}^{s}$ of $H^0(M, T)$ is given by $v \mapsto \psi_{z1}^{s} \circ v \circ (\psi_{z1}^{s})^{-1} = (-1)^{k}v$. If our supermanifold is split and $v$ is a vector field of degree $k$, we have $\phi_{\alpha} \circ v \circ \phi_{\alpha}^{-1} = \alpha^{k}v$. In particular for the graded operator $z$ from Theorem [23] we have $\phi_{\alpha} \circ z \circ \phi_{\alpha}^{-1} = z$.

Now everything is ready to prove the following theorem. Recall that the automorphism supergroup for a compact complex supermanifold is defined in terms of super-Harish-Chandra pairs by formula (2).

Theorem 24. (1) If $M = \text{IIGr}_{n,k}$, where $n \neq 2k$, then

$$A := \text{Aut}M \simeq \text{PGL}_{n}(C) \times \{\text{id}, \Psi_{z1}^{s}\}.$$

The automorphism supergroup is given by the Harish-Chandra pair

$$(\text{PGL}_{n}(C) \times \{\text{id}, \Psi_{z1}^{s}\}, q_{n}(C)/\langle E_{2n} \rangle).$$
(2) If $\mathcal{M} = \text{IIGr}_{2k,k}$, where $k \geq 2$, then

$$A := \text{Aut} \mathcal{M} \simeq \text{PGL}_{2k}(C) \rtimes \{\text{id}, \Theta, \Psi_{st}^{-1}, \Psi_{st} \circ \Theta\},$$

where $\Theta^2 = \Psi_{st}^{-1}$, $\Psi_{st}^{-1}$ is a central element of $A$, and $\Theta \circ g \circ \Theta^{-1} = (g^i)^{-1}$ for $g \in \text{PGL}_{2k}(C)$.

The automorphism supergroup is given by the Harish-Chandra pair

$$(\text{PGL}_{2k}(C) \rtimes \{\text{id}, \Theta, \Psi_{st}^{-1}, \Psi_{st} \circ \Theta\}, q_{2k}(C)/(E_{4k})),$$

where $\Theta \circ C \circ \Theta^{-1} = -C^i$ for $C \in q_{2k}(C)/(E_{4k})$ and $\Psi_{st}^{-1} \circ C \circ (\Psi_{st}^{-1})^{-1} = (-1)^C C$.

(3) If $\mathcal{M} = \text{IIGr}_{2,1}$, then

$$A := \text{Aut} \mathcal{M} \simeq \text{PGL}_{2}(C) \times C^*.$$

The automorphism supergroup is given by the Harish-Chandra pair

$$(\text{PGL}_{2}(C) \times C^*, g \rtimes (z)).$$

Here $g$ is a $\mathbb{Z}$-graded Lie superalgebra described in Theorem 26, the action of $\text{PGL}_{2}(C) \times C^*$ on $z$ is trivial, and $\phi_\alpha \in C^*$ multiplies $X \in \mathfrak{g}(\text{IIGr}_{2,1})$ by $\alpha^k$.

**Proof.** We use the following exact sequence of groups:

$$e \rightarrow H^0(M, \text{Aut}(\mathcal{O})) \rightarrow \text{Aut} \mathcal{M} \rightarrow \text{Aut}(\text{gr}\mathcal{M}).$$

By (10) we have $H^0(M, \text{Aut}(\mathcal{O})) = e$. Therefore, $\text{Aut} \mathcal{M}$ is a subgroup in $\text{Aut}(\text{gr}\mathcal{M})$ and the group $\text{Aut}(\text{gr}\mathcal{M})$ was computed in Theorem 20. If $(n, k) = (2, 1)$, we have $\text{IIGr}_{2,1} \simeq \text{gr}\text{IIGr}_{2,1}$. Hence, the result about the automorphism group follows from Theorem 20.

In [Man] it was proven that $\text{IIGr}_{n,k}$ possesses an effective action of $\text{PGL}_n(C)$, which is compatible with the natural action of $\text{PGL}_n(C)$ on $\text{gr}\mathcal{M}$ and $M$ for any $(n, k)$.

Assume that $(n, k) \neq (2, 1)$. By Theorem [10] we see that $H^0(M, \text{Aut}\mathcal{O}) = \{\text{id}, \psi_{st}^{-1}\}$ (automorphisms which are identical on the base space $M$). In other words, $\phi_\alpha$ can be lifted to $\text{IIGr}_{n,k}$ if and only if $\alpha = \pm 1$. Note that the automorphism $\Psi_{st}^{-1} = (\text{id}, \psi_{st}^{-1})$ is defined on any supermanifold and it always commutes with any other automorphism. This implies the result for $n \neq 2k$.

For $n = 2k, k \geq 2$, by Theorem 22, the automorphism $\phi_\alpha \circ (\Phi, \wedge d(\Phi))$ can be lifted to $\text{IIGr}_{n,k}$ if and only if $\alpha = \pm 1$. Above we denoted the lift of $\phi_1 \circ (\Phi, \wedge d(\Phi))$ by $\Theta$. We check that $\{\text{id}, \Psi_{st}^{-1}, \Theta, \Psi_{st} \circ \Theta\}$ is a subgroup and $\text{PGL}_{2k}(C)$ is a normal subgroup in $\text{Aut} \mathcal{M}$. Indeed, $\Theta^2 = \Psi_{st}^{-1}$ since

$$\text{gr}(\Theta^2) = \text{gr}(\Theta)^2 = \phi_1 \circ (\Phi, \wedge d(\Phi)) \circ \phi_1 \circ (\Phi, \wedge d(\Phi)) = \phi_{-1} = \text{gr}((\Psi_{st}^{-1}).$$

Further, $(\Psi_{st}^{-1})^2 = \text{id}$ and $\Psi_{st}^{-1}$ is central. Moreover, $\text{PGL}_{2k}(C)$ is normal in $\text{Aut}(\text{gr}\mathcal{M})$, hence it is normal in $\text{Aut} \mathcal{M}$ as well. We also have $\Theta \circ g \circ \Theta^{-1} = (g^i)^{-1}$ for $g \in \text{PGL}_{2k}(C)$, since by Theorem 20 we have

$$\text{gr}(\Theta \circ g \circ \Theta^{-1}) = (\Phi, \wedge d(\Phi)) \circ g \circ (\Phi, \wedge d(\Phi))^{-1} = (g^i)^{-1}.$$
is a one-parameter subgroup in \( \mathbb{Q}_{2k}(C)/(E_{4k}) \). Here the parity of \( t \) is the same as the parity of \( C \). We need to compute \( \Theta \circ (E_{4k} + tC) \circ \Theta^{-1} \). We have

\[
\begin{pmatrix} X & \Xi \\ E & 0 \\ \Xi & X \\ 0 & E \end{pmatrix} \xrightarrow{\Theta^{-1}} \begin{pmatrix} E & 0 & -X^t & i\Xi^t \\ 0 & E & i\Xi^t & -X^t \end{pmatrix} \xrightarrow{(E_{4k} + tC)} \begin{pmatrix} E & 0 & -X^t & i\Xi^t \\ 0 & E & i\Xi^t & -X^t \end{pmatrix} \xrightarrow{\Theta} \begin{pmatrix} X & \Xi \\ E & 0 \\ \Xi & X \\ 0 & E \end{pmatrix}.
\]

Therefore,

\[
\Theta \circ C \circ \Theta^{-1} = \frac{d}{dt}
|_{t=0} (\Theta \circ (E_{4k} + tC) \circ \Theta^{-1}) = \frac{d}{dt}
|_{t=0} ((E_{4k} + tC)^{-1})t_0 = -C^t.
\]

The proof is complete.

\[\square\]

7. Real structures on a supermanifold

7.1. Real structures on commutative superalgebras. Let us fix \( \epsilon_i \in \{\pm 1\} \) for \( i = 1, 2, 3 \). Following Manin [Man] Definition 3.6.2 a real structure of type \( (\epsilon_1, \epsilon_2, \epsilon_3) \) on a \( C \)-superalgebra \( A \) is an \( R \)-linear (even) automorphism \( \rho \) of \( A \) such that

\[
\rho(\rho(a)) = \epsilon_3a, \quad \rho(ab) = \epsilon_3\epsilon_2\epsilon_1^b \rho b \rho a, \quad \rho(\lambda a) = \lambda \rho a,
\]

for \( \lambda \in C \) and for homogeneous elements \( a, b \in A \), where \( \tilde{a} \) and \( \tilde{b} \) denote the parities of \( a \) and \( b \), respectively. To any real structure \( \rho \) of type \( (\epsilon_1, \epsilon_2, \epsilon_3) \) on \( A \), we can assign a real structure \( \rho' \) of type \( (\epsilon_1, \epsilon_2, -\epsilon_3) \) as follows: we put \( \rho' = -\rho \). Moreover, we can assign a real structure \( \rho'' \) of type \( (\epsilon_1, -\epsilon_2, \epsilon_3) \) as follows: we put \( \rho''(a) = \rho(a) \) if \( a \) is even, and \( \rho''(a) = i\rho(a) \) when \( a \) is odd, where \( i = \sqrt{-1} \). See [BLMS] Section 1.11.2. Thus in order to classify real structures on \( A \) of all types \( (\epsilon_1, \epsilon_2, \epsilon_3) \), it suffices to classify real structures of the types \((1, -1, 1)\) and \((-1, -1, 1)\). In this paper we consider real structures of the type \((1, -1, 1)\). Then for a commutative superalgebra \( A \) and for a real structure \( \rho \) on \( A \) we have

\[
\rho(\rho(a)) = a, \quad \rho(ab) = \rho a \rho b, \quad \rho(\lambda a) = \bar{\lambda} \rho a.
\]

For a definition of a real structure on a complex Lie superalgebra see for instance [S].

7.2. Real structures on supermanifolds. Recall that a real structure on a complex-analytic manifold \( M \) is an anti-holomorphic involution \( \mu : M \to M \), see for instance [ACF]. This definition is equivalent to the following one. Let \( (M, \mathcal{F}) \) be a complex-analytic manifold, that is \( \mathcal{F} \) is the sheaf of holomorphic functions on \( M \). A homomorphism \( \rho : \mathcal{F} \to \mathcal{F} \) of sheaves of real local algebras is called a real structure on \( M \) if

\[
\rho^2 = id, \quad \rho(\lambda f) = \bar{\lambda} \rho(f),
\]

where \( \lambda \in C \) and \( f, g \in \mathcal{F} \). These definitions are equivalent. Indeed, if \( \mu \) is a complex involution, we put

\[
\rho(f) = \mu^*(f) \in \mathcal{F}.
\]

Conversely, if \( \rho \) satisfies the second definition of a real structure, we put

\[
\mu^*(f) = \rho(f), \quad \mu^*(\bar{f}) = \rho(f) \quad \text{for} \quad f \in \mathcal{F}.
\]
Let us define a real structure on a supermanifold. A real structure of type $(1, -1,1)$ on a complex-analytic supermanifold $\mathcal{M} = (M, \mathcal{O})$ is a homomorphism $\rho : \mathcal{O} \to \mathcal{O}$ of sheaves of real local superalgebras such that

$$\rho^2 = \text{id}, \quad \rho(\lambda f) = \overline{\rho(f)} \quad \text{for} \quad \lambda \in \mathbb{C}, \ f \in \mathcal{O}.$$ 

Since $\rho$ preserves the parity of elements in $\mathcal{O}$, it preserves the sheaf of ideals $\mathcal{J} \subset \mathcal{O}$ generated by odd elements. Clearly the induced sheaves homomorphism $\rho^* : \mathcal{O}/\mathcal{J} \to \mathcal{O}/\mathcal{J}$ is a real structure on the underlying manifold $M$.

As in the case of complex-analytic manifolds let us give another definition of a real structure. To any complex-analytic supermanifold $\mathcal{M} = (M, \mathcal{O})$ of dimension $(n|m)$, we can assign a real supermanifold $\mathcal{M}^R$ of dimension $(2n|2m)$. This procedure is described for instance in [Kal, Section. From complex to real]. There a definition of a complex-conjugation of a function was given. This definition can be regarded as follows. Any super-function $f \in \mathcal{O}$ is at the same time a morphism from $\mathcal{M}$ to $\mathcal{C}^{1|1} = \mathcal{C}^{1|0} \oplus \mathcal{C}^{0|1}$. More precisely, an even function $f \in \mathcal{O}_0$ can be regarded as a morphism $f : \mathcal{M} \to \mathcal{C}^{1|0}$ and an odd element $f \in \mathcal{O}_1$ can be regarded as a morphism $f : \mathcal{M} \to \mathcal{C}^{0|1}$. Further, let $(z, \eta)$ be the standard complex coordinates in $\mathcal{C}^{1|1}$, where $z = z_1 + iz_2$ and $\eta = \eta_1 + i\eta_2$ and $z_1, \eta_1$ are standard real even and odd coordinates in $\mathbb{R}^{2|2}$, respectively. In $\mathcal{C}^{1|1}$ we define a complex conjugation by the following formula

$$\overline{\tau} = z_1 - iz_2 \quad \text{and} \quad \overline{\eta} = \eta_1 - i\eta_2.$$ 

Then for any $f : \mathcal{M} \to \mathcal{C}^{1|1}$ its complex conjugation $\overline{f}$ can be regarded as a composition of the morphism $f$ and the complex conjugation in $\mathcal{C}^{1|1}$. We also have

$$\overline{f_1 \cdot f_2} = \overline{f_1} \cdot \overline{f_2}, \quad f_1, f_2 \in \mathcal{O}.$$ 

If $f : \mathcal{M} \to \mathcal{C}^{1|1}$ is a morphism (or a super-function), we have for any morphism $F = (F_0, F^*) : \mathcal{M} \to \mathcal{M}$ the following formula

$$(23) \quad F^*(f) = f \circ F.$$ 

Now a real structure $\mu$ on a supermanifold $\mathcal{M}$ is an anti-holomorphic involutive automorphism $\mu = (\mu_0, \mu^*)$ of the supermanifold $\mathcal{M}^R$. That is $\mu^*$ maps $\mathcal{O}$ to $\overline{\mathcal{O}}$ and $\mu^2 = \text{id}$.

**Proposition 25.** Two definitions of a real structure on $\mathcal{M}$ coincide.

**Proof.** Let $\mu = (\mu_0, \mu^*)$ be an anti-holomorphic involutive automorphism of $\mathcal{M}^R$. Then we define

$$\rho(f) := \mu^*(f) \quad \text{for} \quad f \in \mathcal{O}.$$ 

Let us check that $\rho \circ \rho = \text{id}$. Using (23) we have

$$\rho(\rho(f)) = \rho(f \circ \mu) = \mu^*(f \circ \mu) = f \circ \mu \circ \mu = f.$$ 

Other properties of $\rho$ are clear.

On the other hand, let $\rho : \mathcal{O} \to \mathcal{O}$ be a real structure according to the first definition. By definition we put

$$\mu^*(f) = \overline{\rho(f)} \quad \text{for} \quad f \in \mathcal{O}; \quad \mu^*(f) = \overline{\rho(f)} \quad \text{for} \quad f \in \overline{\mathcal{O}}.$$ 

Clearly $\mu$ is anti-holomorphic. Further, we have

$$\mu^*(\mu^*(f)) = \mu^*(\overline{\rho(f)}) = \overline{\rho(\rho(f))} = \rho(\rho(f)) = f \quad \text{for} \quad f \in \mathcal{O};$$

$$\mu^*(\mu^*(f)) = \mu^*(\overline{\rho(f)}) = \overline{\rho(\rho(f))} = \rho(\rho(f)) = f \quad \text{for} \quad f \in \overline{\mathcal{O}}.$$
The proof is complete. □

Two real structures $\mu, \mu'$ on $\mathcal{M}$ are called equivalent if the pairs $(\mathcal{M}, \mu)$ and $(\mathcal{M}, \mu')$ are isomorphic, that is, if there exists a (complex-analytic) isomorphism of supermanifolds $\beta: \mathcal{M} \to \mathcal{M}$ such that $\mu' \circ \beta = \beta \circ \mu$.

7.3. **Isotropic $\Pi$-symmetric Hermitian super-Grassmannians.** Let $A$ be a commutative superalgebra. For even matrices over $A$ an $i$-transposition is defined

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}^t_i = \begin{pmatrix}
A^t & iC^t \\
iB^t & D^t
\end{pmatrix},
\]

which satisfies $(C_1 \cdot C_2)^t_i = C_2^t \cdot C_1^t_i$. (Here $iB^t$ is the matrix $B^t$, transpose to the matrix $B$, multiplied by the complex number $i = \sqrt{-1}$.) The $i$-transposition maps a $\Pi$-symmetric even matrix to a $\Pi$-symmetric even matrix

\[
\begin{pmatrix}
A & B \\
B & A
\end{pmatrix}^t_i = \begin{pmatrix}
A^t & iB^t \\
iB^t & A^t
\end{pmatrix}.
\]

We denote an isotropic $\Pi$-symmetric Hermitian super-Grassmannian by $\Pi IGr^H_{2k,k}$, where $H$ is the super-Hermitian form given by the following matrix in the standard basis of $\mathbb{C}^{2k|2k}$

\[
H = \begin{pmatrix}
b_k & 0 \\
0 & b_k
\end{pmatrix}, \quad b_k = \begin{pmatrix}0 & -iE_k \\
iE_k & 0
\end{pmatrix},
\]

Compare with Section A.23. We see that the restriction of the form $H$ to the vector subspace of even $\Pi$-symmetric vectors is equal to the form $F(-, -)$, see A.23.

The isotropic $\Pi$-symmetric Hermitian super-Grassmannian $\Pi IGr^H_{2k,k}$ is a real subsupermanifold in $\Pi Gr_{2k,k}$ defined by the following isotropy condition

\[
(Z_I)^t_i \cdot H \cdot Z_I = 0,
\]

where $Z_I = \begin{pmatrix}X' & \Xi' \\
\Xi' & X'
\end{pmatrix}$ is a $\Pi$-symmetric coordinate matrix of $\Pi Gr_{2k,k}$, $i$-transposition is as in (25) and $\overline{Z}_I$ is complex conjugation of $Z_I$, see Section 7.2.

Let us write this condition (26) explicitly for the coordinate matrix

\[
Z_I = \begin{pmatrix}X & \Xi \\
E & 0 \\
\Xi & X \\
0 & E
\end{pmatrix}.
\]

We get after a direct computation

\[
X = (X)^t, \quad -i\Xi = (\Xi)^t.
\]

The base space of this supermanifold is described in Corollary A.24.

8. **Real structures on a $\Pi$-symmetric super-Grassmannian**

8.1. **The action of the complex conjugation on $\text{Aut } \Pi Gr_{n,k}$.** If $a \in \text{Aut } \mathcal{M}$, we denote by $\gamma a \in \text{Aut } \mathcal{M}$ the element satisfying $(\gamma a)^{*}(f) = a^*(\overline{f})$ for any local function $f$ on $\mathcal{M}$. (Recall that $\overline{f}$ is defined in Section 7.2) Let us compute $\gamma a$ for any $a \in \text{Aut } \Pi Gr_{n,k}$. The group $\text{Aut } \Pi Gr_{n,k}$ was computed in Theorem 24.
Proposition 26. For any \((n,k)\) we have
\[\gamma g = \overline{g}, \quad g \in \text{PGL}_n(C), \quad \gamma (\Psi^x_{11}) = \Psi^{x}_{11}.\]
For \((2k,k), k \geq 2\), we have
\[\gamma \Theta = \Theta^{-1} = \Psi^x_{11} \circ \Theta.\]
For \((2,1)\) we have
\[\gamma \phi_\alpha = \phi_{\overline{\alpha}}.\]

Proof. First of all, \(\Psi^x_{11}\) multiplies an odd local function by \(-1\) and it is identical on an even function. Therefore, for any supermanifold we have \(\gamma (\Psi^x_{11}) = \Psi^x_{11}\), see Section 6.2.

Further, let \(Z_t\) be a coordinate matrix of \(\text{II}Gr_{n,k}\), see Section 6.1 and \(g \in \text{PGL}_n(C)\). Then the action of \(\text{PGL}_n(C)\) on \(\text{II}Gr_{n,k}\) is defined by the matrix multiplication \(g \cdot Z_t\), see \[\text{Man}\]. Therefore, \(g \cdot Z_t = \overline{g} \cdot Z_t\). Hence, \(\gamma g = \overline{g}\).

The automorphism \(\Phi\) of \(Gr_{n,k}\) is described in local coordinates in Section 3.1. Clearly, \(\gamma \Phi = \Phi\). Note that \(\gamma (\Phi, \land d(\Phi))\) is the lift of \(\gamma \Phi\) to \(T^*(Gr_{n,k})\). The result follows.

Recall that \(\phi_\alpha\) is defined for split supermanifolds and it multiplies a local section of the corresponding bundle by \(\alpha\). Hence, \(\gamma \phi_\alpha\) multiplies a local section of the corresponding bundle by \(\overline{\alpha}\).

To compute \(\gamma \Theta\) we note that \(\text{gr}(\overline{f}) = \overline{\text{gr}(f)}\) for any function on \(\text{II}Gr_{n,k}\). Therefore,
\[\text{gr}(\gamma \Theta) = \gamma (\text{gr}\Theta) = \gamma (\phi_1 \circ (\Phi, \land d(\Phi))) = \phi_{-1} \circ (\Phi, \land d(\Phi)) = \text{gr}(\Theta^{-1}).\]
Recall that \(\text{gr \text{Aut} II}Gr_{n,k} \to \text{Aut(\text{grII}Gr_{n,k})}\) is injective, see Proof of Theorem 24.

The result follows. \[\square\]

8.2. Real structures on \(\text{II}Gr_{n,k}\). Let \(\mu^o = (\mu^o_1, (\mu^o)^*)\) denote the standard real structure on \(M = \text{II}Gr_{n,k}\). Namely, the anti-holomorphic involution \(\mu^o\) of \(\text{II}Gr_{n,k}\) is induced by the complex conjugation in \(C^{n|n}\). More precisely, let us describe \(\mu^o\) in our charts. For instance, in the chart \(V_1\), see \[17\], with local coordinates \(x_{ij} = x^{1x}_{ij} + ix^{2x}_{ij}, \quad \xi_{ab} = \xi^1_{ab} + i\xi^2_{ab}\), where \(x^{1x}_{ij}, x^{2x}_{ij}, \xi^1_{ab}, \xi^2_{ab}\) are real even and odd coordinates, the real structure \(\mu^o\) is given by
\[(\mu^o)^* (x_{1x}^{1x} + ix_{1x}^{2x}) = x_{1x}^{1x} - ix_{1x}^{2x}, \quad (\mu^o)^* (\xi_{ab}^1 + i\xi_{ab}^2) = \xi_{ab}^1 - i\xi_{ab}^2.\]
In other charts the idea is similar.

If \(n\) is even, write
\[a_j = \text{diag}(J, \ldots, J) \quad (n/2 \text{ times}), \quad \text{where} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.\]
Set \(e := \pi(a_j) \in \text{PGL}_n(C)\), where \(\pi: \text{GL}_n(C) \to \text{PGL}_n(C)\) is the canonical homomorphism. Due to Theorem 23 the element \(e \in \text{PGL}_n(C)\) can be regarded as an holomorphic automorphism of \(M\).

Theorem 27. The number of the equivalence classes of real structures \(\mu\) on \(M\), and representatives of these classes, are given in the list below:

(i) If \(n\) is odd, then there are two equivalence classes with representatives
\[\mu^o, \quad (1, \Psi^x_{11}) \circ \mu^o.\]
(ii) If $n$ is even and $n \neq 2k$, then there are four equivalence classes with representatives 
\[ \mu^o, \ (1, \Psi_{-1}^\ast) \circ \mu^o, \ (c_J, 1) \circ \mu^o, \ (c_J, \Psi_{-1}^\ast) \circ \mu^o. \]

(iii) If $n = 2k \geq 4$, then there are $k + 3$ equivalence classes with representatives 
\[ \mu^o, \ (c_J, 1) \circ \mu^o, \ (c_r, \Theta) \circ \mu^o, \ r = 0, \ldots, k. \]

(iv) If $(n, k) = (2, 1)$, then there are two equivalence classes with representatives 
\[ \mu^o, \ (c_J, 1) \circ \mu^o. \]

Here $\mu^o$ denotes the standard real structure on $\mathcal{M} = \Pi \text{Gr}_{n,k}$ as above. Moreover, $c_J \in \text{PGL}_n(\mathbb{C})$ and $c_r \in \text{PGL}_{2k}(\mathbb{C})$ for $r = 0, \ldots, k$ are certain elements constructed in Proposition \ref{rem} and Subsection \ref{section} respectively.

**Proof.** By Proposition \ref{prop} in Appendix \ref{appendix} the theorem follows immediately from Theorem \ref{thm}.

\[ \square \]

8.3. Ringed space of real points of $\Pi \text{Gr}_{n,k}$. We wish to describe the ringed space of real points of $\mathcal{M} = (\mathcal{M}, \mathcal{O})$ of the corresponding real structures $\mu = (\mu^o, \mu^c)$ from Theorem \ref{thm}. In more details the **ringed space of real points** is by definition the ringed space $\mathcal{M}^\mu := (\mathcal{M}^{\mu^o}, \mathcal{O}^\mu)$, where $\mathcal{M}^{\mu^o}$ is the set of fixed points of $\mu^o$ and $\mathcal{O}^\mu$ is the sheaf of fixed points of $\mu^c$ over $\mathcal{M}^\mu$. Let us first describe real points corresponding to the real structures $\mu^o$ and $\Psi_{-1}^\ast \circ \mu^o$.

**Proposition 28.** (1) The ringed space of real points of $\Pi \text{Gr}_{n,k}$ corresponding to the real structure $\mu^o$ can be identified with $\Pi \text{Gr}_{n,k}(\mathbb{R})$.

(2) The ringed space of real points of $\Pi \text{Gr}_{n,k}$ corresponding to the real structure $\Psi_{-1}^\ast \circ \mu^o$ can be identified with a real subsupermanifold, which we denote by $\Pi \text{Gr}_{n,k}(\mathbb{R})$.

**Proof.** The first statement is obvious. Let us describe the supermanifold $\Pi \text{Gr}_{n,k}(\mathbb{R})$ using charts and local coordinates. An atlas $\mathcal{A}$ of $\Pi \text{Gr}_{n,k}(\mathbb{R})$ contains charts $\mathcal{U}_I$, where $I \subset \{1, \ldots, n\}$ with $|I| = k$. To any $I$ we assign the following $2n \times 2k$-matrix

\[ \mathcal{Z}_I = \begin{pmatrix} X' & \Xi' \\ \Xi' & X' \end{pmatrix}. \]

We assume that $\mathcal{Z}_I$ contains the identity submatrix $E_{2k}$ of size $2k$ in the lines with numbers $i$ and $n + i$, where $i \in I$. We also assume that non-trivial elements $x_{ij}$ of $X'$ are real numbers, i.e., $\tau_{ij} = x_{ij}$, while non-trivial elements $\xi_{ij}$ of $\Xi'$ are pure imaginary odd variables, i.e., $\bar{\xi}_{ij} = -\xi_{ij}$. Transition functions are defined as above.

\[ \square \]

**Remark 29.** Note that for $n = 2k \geq 4$, compare with Theorem \ref{thm} the real structure $\Psi_{-1}^\ast \circ \mu^o$ is equivalent to the real structure $\mu^o$. Indeed, we have $\Theta \circ (\Psi_{-1}^\ast \circ \mu^o) \circ \Theta^{-1} = \mu^o$.

In Proposition \ref{prop} in Appendix \ref{appendix} a description of real points of the base space $\text{Gr}_{2n,2k}$ corresponding to the real structure $(c_J, 1) \circ \mu^o$ was given. Indeed, we have a natural embedding $\text{GL}_n(\mathbb{H}) \hookrightarrow \text{GL}_{2n}(\mathbb{C})$ defined as follows. If $(a_{ij}) \in \text{GL}_n(\mathbb{H})$, 

\[ a_{ij} = \begin{pmatrix} a_{ij} \\ 0 \end{pmatrix}, \quad b_{ij} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]
we replace any quaternion entry $a_{ij}$ by the complex matrix\[ \begin{pmatrix} a_{11}^{ij} & a_{12}^{ij} \\ -a_{12}^{ij} & a_{11}^{ij} \end{pmatrix}, \] see \[35\]. This inclusion leads to an inclusion of homogeneous spaces

\begin{equation}
Gr_{n,k}(H) \cong \text{GL}_n(H)/P(H) \hookrightarrow \text{GL}_{2n}(C)/P(C) \cong Gr_{2n,2k}, \tag{28}
\end{equation}

where $P(H) \subset \text{GL}_n(H)$ contains all matrices in the form\[ \begin{pmatrix} S & 0 \\ T & U \end{pmatrix} \]over $H$, where $S \in \text{Mat}_{n-k}(H)$, and $P(C) \subset \text{GL}_{2n}(C)$ contains all matrices in the form\[ \begin{pmatrix} S' & 0 \\ T' & U' \end{pmatrix} \]over $C$, where $S' \in \text{Mat}_{2n-2k}(C)$.

Let us describe this embedding in terms of local charts on $Gr_{n,k}(H)$ and $Gr_{n,k}(C)$ as above. Let $U_I$, where $I \subset \{ 1, \ldots, n \}$ with $|I| = k$, be a local chart on $Gr_{n,k}(H)$. Denote by $I' \subset \{ 1, \ldots, 2n \}$ the following set

\begin{equation}
I' := \{ 2i, 2i-1 \mid i \in I \}. \tag{29}
\end{equation}

and by $U_{I'}$, the corresponding to $I'$ chart on $Gr_{n,k}(C)$. Now the embedding in charts $U_I \to U_{I'}$ is given by

\[ H \ni x_{ij} \mapsto \begin{pmatrix} x_{11}^{ij} & x_{12}^{ij} \\ -x_{12}^{ij} & x_{11}^{ij} \end{pmatrix}, \]

where $(x_{ij})$ are standard coordinates in $U_I$. For instance we see that the image of $Gr_{n,k}(H)$ in $Gr_{n,k}(C)$ is covered by chart of the form $U_{I'}$, where $I'$ is as in \[29\].

**Proposition 30.** Let $M = \text{II}Gr_{2n,k}$.

1. If $k$ is odd, there are no real points corresponding to the real structure $(c_J, 1) \circ \mu^o$.

2. If $k = 2k'$, the real points of $M$ corresponding to the real structure $(c_J, 1) \circ \mu^o$ can be identified with $\text{II}Gr_{n,k}(H)$.

**Proof.** The first statement follows from Proposition \[A.1.4\] in Appendix \[A\]. Indeed, in this case there is no real point on the base space. Assume that $k = 2k'$. We have to compute the fixed points of $(c_J, 1) \circ \mu^o$. Let us do that for $n = 2$ and $k' = 1$; the general case in charts $U''_{I'}$, see \[29\], is similar. We apply the standard real structure $\mu^o$ to the coordinates $Z_1$, see \[17\], and now we apply $c_J$\n
\[
\text{diag}(J, J, J) \begin{pmatrix}
\bar{x}_{11} & \bar{x}_{12} & \bar{\xi}_{11} & \bar{\xi}_{12} \\
\bar{x}_{21} & \bar{x}_{22} & \bar{\xi}_{21} & \bar{\xi}_{22} \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\bar{\xi}_{11} & \bar{\xi}_{12} & \bar{x}_{11} & \bar{x}_{12} \\
\bar{\xi}_{21} & \bar{\xi}_{22} & \bar{x}_{21} & \bar{x}_{22} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
\bar{x}_{21} & \bar{x}_{22} & \bar{\xi}_{21} & \bar{\xi}_{22} \\
-\bar{x}_{11} & -\bar{x}_{12} & -\bar{\xi}_{11} & -\bar{\xi}_{12} \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
\bar{\xi}_{21} & \bar{\xi}_{22} & \bar{x}_{21} & \bar{x}_{22} \\
-\bar{\xi}_{11} & -\bar{\xi}_{12} & -\bar{x}_{11} & -\bar{x}_{12} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
Therefore, in the standard coordinates, we get
\[
\begin{pmatrix}
\bar{x}_{21} & \bar{x}_{22} & \xi_{21} & \xi_{22} \\
-\bar{x}_{11} & -\bar{x}_{12} & -\xi_{11} & \xi_{12} \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
\bar{\xi}_{21} & \bar{\xi}_{22} & \bar{x}_{21} & \bar{x}_{22} \\
-\xi_{11} & -\xi_{12} & -\bar{x}_{11} & -\bar{x}_{12} \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}
= 
\begin{pmatrix}
\begin{pmatrix}
\bar{x}_{22} & -\bar{x}_{21} & \bar{\xi}_{22} & -\bar{\xi}_{21} \\
-x_{12} & \bar{x}_{11} & -\xi_{12} & \xi_{11} \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{pmatrix} \\
\begin{pmatrix}
\xi_{11} & \xi_{12} \\
-x_{12} & \bar{x}_{11} \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\end{pmatrix}
\begin{pmatrix}
x_{11} & x_{12} & \xi_{11} & \xi_{12} \\
x_{21} & x_{22} & \xi_{21} & \xi_{22} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Therefore, \(x_{11} = \bar{x}_{22}, x_{12} = -\bar{x}_{21}, \xi_{11} = \bar{\xi}_{22}, \xi_{12} = -\bar{\xi}_{21}\). In the matrix form we have
\[
\begin{pmatrix}
x_{11} & x_{12} & \xi_{11} & \xi_{12} \\
-x_{12} & \bar{x}_{11} & -\xi_{12} & \xi_{11} \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\bar{\xi}_{11} & \bar{\xi}_{12} & \bar{x}_{11} & \bar{x}_{12} \\
-\xi_{12} & \xi_{11} & -\bar{x}_{12} & \bar{x}_{11} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x_{11} & x_{12} & \xi_{11} & \xi_{12} \\
x_{21} & x_{22} & \xi_{21} & \xi_{22} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Now we can regard the matrix \(z := \begin{pmatrix} x_{11} & x_{12} \\ -\bar{x}_{12} & \bar{x}_{11} \end{pmatrix}\) as a new quaternion even variable, while \(\zeta := \begin{pmatrix} \xi_{11} & \xi_{12} \\ -\xi_{12} & \xi_{11} \end{pmatrix}\) is a new quaternion odd variable. In short we have
\[
\begin{pmatrix}
z & \zeta \\
1 & 0 \\
\zeta & z \\
0 & 1
\end{pmatrix}
\]
is a standard chart on a \(\Pi\)-symmetric quaternion super-Grassmannian \(\Pi\text{Gr}_{2,1}(\mathbb{H})\), see above. Clearly our computation does not depend on the choice of a local chart \(U'_n\), and it is similar for other \(n\) and \(k'\).

\[\square\]

Proposition 31. \(\text{(1) If } n \text{ is even but } k \text{ is odd, then there are no real points corresponding to the real structure } (c_1, \Psi_{-1}^x) \circ \mu^x.\)

\(\text{(2) If } n = 2n' \text{ and } k = 2k', \text{ then the real points of } \mathcal{M} \text{ corresponding to the real structure } (c_j, \Psi_{-1}^x) \circ \mu^x \text{ is a real supermanifold, which we denote by } \Pi\text{Gr}_{2n', k'}(\mathbb{H}).\)

Proof. The first statement follows from Proposition A.14 in Appendix A since in this case there is no real point on the base space. If \(k = 2k'\) is even, we repeat the argument similar to Proposition 28. For simplicity we assume that \(k' = 1\), further
we apply \((c_J, \Psi_{st}^{-1}) \circ \mu^o\), we will get for odd coordinates (for even coordinates the equation is the same as above):

\[
\begin{pmatrix}
\xi_{11} & \xi_{12} \\
\xi_{21} & \xi_{22}
\end{pmatrix}
= \begin{pmatrix}
-\xi_{22} & -\xi_{21} \\
\xi_{12} & -\xi_{11}
\end{pmatrix}.
\]

Therefore, \(\xi_{11} = -\xi_{22}\) and \(\xi_{21} = \xi_{12}\). Now \(\zeta := \begin{pmatrix}
\xi_{11} & \xi_{12} \\
\xi_{12} & -\xi_{11}
\end{pmatrix}\) is a new quaternion odd variable. Now local charts on \(\PiGr'_{2n', k'}(H)\) are defined. □

**Remark 32.** Note that for \(n = 2k \geq 4\), compare with Theorem 27, the real structure \((c_J, \Psi_{st}^{-1}) \circ \mu^o\) is equivalent to the real structure \((c_J, 1) \circ \mu^o\). Indeed, we have \(\Theta \circ ((c_J, \Psi_{st}^{-1}) \circ \mu^o) \circ \Theta^{-1} = (c_J, 1) \circ \mu^o\).

Let us prove the following proposition.

**Proposition 33.** Let \(n = 2k\).

(1) There are no real points corresponding to the real structure \((c_r, \Theta) \circ \mu^o, r = 0, \ldots, k - 1\).

(2) Let \(k \geq 2\). The real points of \(M\) corresponding to the real structure \((c_k, \Theta) \circ \mu^o\) can be identified with \(\PiGr_{2k, k}(H)\).

**Proof.** The first statement follows from Corollary A.21. In this case there are no real points on the base space.

Let us prove the second statement. We take a coordinate matrix \(Z_I\) of \(\PiGr_{2k, k}\). The idea is to prove that the isotropy condition (26), see also (27), is equivalent to the condition \((c_k, \Theta) \circ \mu^o(Z_I) = Z_I\). Indeed, by Section A.23 the cocycle \(c_k\) may be represented by the matrix

\[
H = \begin{pmatrix}
b_k & 0 \\
0 & b_k
\end{pmatrix}, \quad b_k = \begin{pmatrix}
0 & -iE_k \\
iE_k & 0
\end{pmatrix}.
\]

We have

\[
(c_k, \Theta) \circ \mu^o(Z_I) = H \cdot \Theta(Z_I) = H \cdot (Z_I^+)^{t_i}.
\]

Here we denoted by \(Z_I^+\) the result of application of the map (22). Further, we have

\[
(H \cdot (Z_I^+)^{t_i})^{t_i} \cdot H \cdot Z_I = (Z_I^+)H^t \cdot H \cdot Z_I = -Z_I^+ \cdot Z_I = 0.
\]

Therefore the fixed point condition \(H \cdot (Z_I^+)^{t_i} = Z_I\) is equivalent to the isotropy condition (26) □.

From Propositions 30, 28, 31, 33 and Theorem 27, we obtain the following result.

**Theorem 34.** Let \(M = \PiGr_{n, k}\). The ringed space of real points \(M^\mu\) corresponding to the real structures \(\mu\) are as follows.

(i) If \(n\) is odd, then

\[M^\mu = \PiGr_{n, k}(R) \quad \text{and} \quad M^{(c_J, \Psi_{st}^{-1}) \circ \mu^o} = \PiGr'_{n, k}(R).
\]

(ii) For \((n, k) = (2, 1)\) we have

\[M^\mu = \PiGr_{2, 1}(R) \quad \text{and} \quad M^{(c_J, 1) \circ \mu^o} = \emptyset.
\]
As in [Ser], Section I.5.1, we define the 1-cohomology set
\[ \mathcal{M}^\mu = \text{IGr}_{n,k}(\mathbb{R}), \quad \mathcal{M}^{(e_j,\Theta)}_\mu = \text{IGr}_{n,2}(\mathbb{R}), \quad \mathcal{M}^{(e_j,1)}_\mu = \emptyset, \quad \mathcal{M}^{(e_j,\Psi)}_\mu = \emptyset. \]

(iii) If \( n \) is even, \( n \neq 2k \) and \( k \) is odd, we have
\[ \mathcal{M}^\mu = \text{IGr}_{n,k}(\mathbb{R}), \quad \mathcal{M}^{(e_j,\Theta)}_\mu = \text{IGr}_{n,2}(\mathbb{R}), \quad \mathcal{M}^{(e_j,1)}_\mu = \emptyset, \quad \mathcal{M}^{(e_j,\Psi)}_\mu = \emptyset. \]

(iv) If \( n, k \) are even and \( n \neq 2k \), we have
\[ \mathcal{M}^\mu = \text{IGr}_{n,k}(\mathbb{R}), \quad \mathcal{M}^{(e_j,\Theta)}_\mu = \text{IGr}_{n,2}(\mathbb{R}), \quad \mathcal{M}^{(e_j,1)}_\mu = \text{IGr}_{n/2,k/2}(\mathbb{H}), \quad \mathcal{M}^{(e_j,\Psi)}_\mu = \text{IGr}_{n/2,k/2}(\mathbb{H}). \]

(v) If \( k \) is even and \( n = 2k \), we have
\[ \mathcal{M}^\mu = \text{IGr}_{2k,k}(\mathbb{R}), \quad \mathcal{M}^{(e_j,1)}_\mu = \emptyset, \quad \mathcal{M}^{(e_j,\Theta)}_\mu = \emptyset, \quad r = 0, \ldots, k - 1, \quad \mathcal{M}^{(e_j,\Psi)}_\mu = \Pi \text{IGr}_{H,2k,k}. \]

(vi) If \( k \geq 3 \) is odd and \( n = 2k \), we have
\[ \mathcal{M}^\mu = \text{IGr}_{2k,k}(\mathbb{R}), \quad \mathcal{M}^{(e_j,1)}_\mu = \emptyset, \quad \mathcal{M}^{(e_j,\Theta)}_\mu = \emptyset, \quad r = 0, \ldots, k - 1, \quad \mathcal{M}^{(e_j,\Theta)}_\mu = \Pi \text{IGr}_{H,2k,k}. \]

Appendix A. Real Galois cohomology

by Mikhail Borovoi

In this appendix we prove Theorem A.12 computing \( H^1(\mathbb{R}, \text{Aut}\mathcal{M}) \) where \( M = \Pi \text{Gr}_{n,k}(\mathbb{C}) \). This result gives us Theorem A.27 classifying real structures on \( \Pi \text{Gr}_{n,k}(\mathbb{C}) \). After that, we state and prove Proposition A.14 and Corollary A.24, which compute the set of real points of certain twisted forms of the real Grassmannian \( \text{Gr}_{n,k,\mathbb{R}} \).

Let \( \Gamma = \text{Gal}(\mathbb{C}/\mathbb{R}) \) denote the Galois group of \( \mathbb{C} \) over \( \mathbb{R} \). Then \( \Gamma = \{ 1, \gamma \} \), where \( \gamma \) is the complex conjugation.

Let \( (A, \sigma) \) be a pair where \( A \) is a group (not necessarily abelian) and \( \sigma : A \to A \) is an automorphism such that \( \sigma^2 = 1_A \). We define a left action of \( \Gamma \) on \( A \) by
\[ (\gamma, a) \mapsto \gamma a := \sigma(a) \text{ for } a \in A. \]
We say that \( (A, \sigma) \) is a \( \Gamma \)-group.

We consider the set of 1-cocycles
\[ Z^1(A, \sigma) = \{ c \in A \mid c \cdot \gamma c = 1 \}, \]
where \( \gamma c = \sigma(c) \). The group \( A \) acts on the left on \( Z^1(A, \sigma) \) by
\[ a \ast c = a \cdot c \cdot \gamma a^{-1} \text{ for } a \in A, \quad c \in Z^1(A, \sigma). \]
As in [Ser], Section I.5.1, we define the 1-cohomology set
\[ H^1(A, \sigma) = Z^1(A, \sigma)/A, \]
to be the set of orbits of \( A \) in \( Z^1(A, \sigma) \). We shall write \( Z^1A \) for \( Z^1(A, \sigma) \) and \( H^1A \) for \( H^1(A, \sigma) \). We denote by \( [c] \in H^1A \) the cohomology class of a 1-cocycle \( c \in Z^1A \). Note that \( 1 \in Z^1A \); we denote its class in \( H^1A \) by \([1]\) or just by 1. In general (when \( A \) is nonabelian) the cohomology set \( H^1A \) has no natural group structure, but it has a canonical neutral element \([1]\).

Remark A.2. (1) If \( (A, \sigma) = (A', \sigma') \times (A'', \sigma'') \), then
\[ Z^1(A, \sigma) = Z^1(A', \sigma') \times Z^1(A'', \sigma'') \quad \text{and} \quad H^1(A, \sigma) = H^1(A', \sigma') \times H^1(A'', \sigma''). \]
(2) \(H^1(\{\pm 1\}, \text{id}) = Z^1(\{\pm 1\}, \text{id}) = \{\pm 1\}\).

(3) \(H^1(C^*, (z \mapsto \bar{z})) = \{1\}\) (exercise); this is a special case of Hilbert’s Theorem 90.

**A.3.** Let \(G\) be a linear algebraic group defined over \(R\) (we write “an \(R\)-group”). We denote by \(G(R)\) and \(G(C)\) the groups of \(R\)-points and of \(C\)-points of \(G\), respectively. The nontrivial element \(\gamma \in \Gamma\) acts on \(G(C)\) by the complex conjugation \(\sigma: g \mapsto \bar{g}\). We write \(H^1(R, G) = H^1(G(C), \sigma)\). For simplicity, we write \(H^1 G\) or \(H^1 G(C)\) for \(H^1(R, G)\).

**A.4.** Consider the short exact sequence of \(\Gamma\)-groups

\[
1 \rightarrow C^* \rightarrow \text{GL}(n, C) \xrightarrow{\tau} \text{PGL}_n(C) \rightarrow 1,
\]

where \(\gamma \in \Gamma = \text{Gal}(C/R)\) acts on \(g \in \text{GL}(n, C)\) by the complex conjugation \(g \mapsto \bar{g}\). This exact sequence induces a cohomology exact sequence

\[
1 = H^1 \text{GL}(n, C) \rightarrow H^1 \text{PGL}_n(C) \xrightarrow{\Delta} H^2 C^*;
\]

see Serre’s book [Ser], Section I.2.2 for the definition of \(H^2\), and Proposition 43 in Section I.5.7 for the exact sequence. See also [BT2], Section 1.1 and Construction 4.4.

**Proposition A.5** (well-known).

\[
\#H^1 \text{PGL}_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \end{cases}
\]

If \(n\) is even, write

\[
a_J = \text{diag}(J, \ldots, J) \text{ (}n/2\text{ times)} \quad \text{where} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

Then \(c := \pi(a_J) \in \text{PGL}_n(C)\) is a cocycle representing the nontrivial cohomology class in \(H^1 \text{PGL}_n\).

**Proof.** The cardinality \(\#H^1 \text{PGL}_n\) can be computed, for instance, using [BT1] Corollary 13.6. If \(n\) is even, then we have \(a_J \cdot \gamma a_J = a_J^2 = -1\). It follows that \(c \cdot \gamma c = \pi(-1) = 1\). These formulas mean that \(c\) is a 1-cocycle and that \(\Delta[c] = [-1] \neq [1]\), whence \([c] \neq [1]\), as required. \(\square\)

**A.6.** Consider the short exact sequence of real algebraic groups

\[
1 \rightarrow U_1 \rightarrow U_n \xrightarrow{\pi} \text{PU}_n \rightarrow 1,
\]

where \(U_n\) is the unitary group, that is, \(\text{GL}(n, C)\) with complex conjugation \(\sigma\) acting by \(g \mapsto (g^t)^{-1}\) where \(g^t\) denotes the transposed matrix to the complex conjugate matrix \(\bar{g}\). This exact sequence is the exact sequence (30) of complex algebraic groups, but with another complex conjugation. Consider the cocycles \(a_p = \text{diag}(-1, \ldots, -1, 1, \ldots, 1) \in Z^1 U_n\) where \(-1\) appears \(p\) times and \(1\) appears \(n - p\) times, \(0 \leq p \leq n\). Set \(c_p = \pi(a_p) \in Z^1 \text{PU}_n\).

**Proposition A.7** (well-known).

\[
H^1 \text{PU}_n = \{[c_p] \mid 0 \leq p \leq n/2\}.
\]

In particular, when \(n = 2k\), we have \(\#H^1 \text{PU}_n = k + 1\).
We see from the definition of $H$.

**Lemma A.9.** Consider the short exact sequence

\[ 1 \to A_0 \to A \xrightarrow{\pi} C_4 \to 1 \]

where $A_0 = \text{PGL}(n, \mathbb{C})$, and consider the induced cohomology exact sequence

\[ H^1 A_0 \to H^1 A \xrightarrow{\pi_*} H^1 C_4. \]

We see from the definition of $H^1$ that $H^1 C_4 = C_4/\langle \Theta^2 \rangle \cong C_2$ with cocycles $1, \Theta$. Since the exact sequence (31) splits, we see that the map $\pi_*$ in (32) is surjective. Thus

\[ H^1 A_0 \cong \pi_*^{-1}[1] \cup \pi_*^{-1}[\Theta]. \]

We compute $\pi_*^{-1}[1] = \ker \pi_*$. By Serre [Ser, Section I.5.5, Corollary 1 of Proposition 39], the inclusion map

\[ A_0 \hookrightarrow A, \quad a \mapsto (a, 1) \quad \text{for} \quad a \in A_0 = \text{PGL}(n, \mathbb{C}) \]

induces a bijection

\[ (H^1 A_0)/C_4^\Gamma \cong \ker \pi_*. \]

Here $C_4^\Gamma = \{1, \Theta^2\}$. It acts on the right on $H^1 A_0$ as follows. Let $a \in Z^1 A_0$; then

\[ [a, 1] * \Theta^2 = [(1, \Theta^2)^{-1} \cdot (a, 1) \cdot \gamma(1, \Theta^2)]. \]
Since \((1, \Theta^2)\) is of order 2, central in \(A\), and \(\Gamma\)-fixed, we have
\[
(1, \Theta^2)^{-1} \cdot (a, 1) \cdot \gamma(1, \Theta^2) = (a, 1).
\]
Thus \(C_4^\Gamma\) acts on \(H^1 A_0\) trivially, and \(\ker \pi_* \cong H^1 A_0\). By Proposition A.5 we obtain that
\[
\# \ker \pi_* = 2 \quad \text{with cocycles} \ (1, 1), \ (c_1, 1).
\]

We compute \(\pi_*^{-1}[\Theta]\). By Serre [Ser, Section I.5.5, Corollary 2 of Proposition 39], the inclusion map (not a homomorphism)
\[
A_0 \hookrightarrow A, \quad a \mapsto (a, \Theta) \quad \text{for} \quad a \in A_0
\]
induces a bijection
\[
(H^1_{\Theta}A_0) / (\Theta C_4)^\Gamma \cong \pi_*^{-1}[\Theta].
\]
Since \(C_4\) is an abelian group, we have \(\Theta C_4 = C_4\) and \((\Theta C_4)^\Gamma = \{1, \Theta^2\}\). It acts on the right on \(H^1_{\Theta}A_0\) as follows. Let \(a \in \tilde{Z}^1_{\Theta}A_0\); then
\[
[a, \Theta] \ast \Theta^2 = [(1, \Theta^2)^{-1} \cdot (a, \Theta) \cdot \gamma(1, \Theta^2)].
\]
We calculate:
\[
(1, \Theta^2)^{-1} \cdot (a, \Theta) \cdot \gamma(1, \Theta^2) = (1, \Theta^2)^{-1} \cdot (a, \Theta) \cdot (1, \Theta^2) = (\Theta^{-2}(a), \Theta) = (a, \Theta).
\]

Thus \((\Theta C_4)^\Gamma\) acts on \(H^1_{\Theta}A_0\) trivially, and \(\pi_*^{-1}[\Theta] \cong H^1_{\Theta}A_0\). By Proposition A.7 we obtain that
\[
\# \pi_*^{-1}[\Theta] = k + 1 \quad \text{with cocycles} \ (c_0, \Theta), \ldots, (c_k, \Theta).
\]
which completes the proof of the lemma.

**A.10.** Let \(\mathcal{M}\) be a complex supermanifold, and \(\mu\) be a fixed real structure on \(\mathcal{M}\) of the type \((1, -1, 1)\). Then \(\mu^2 = \text{id}_{\mathcal{M}}\). Write \(A = \text{Aut} \mathcal{M}\), the group of even holomorphic automorphisms of \(\mathcal{M}\). For \(a \in A\), we write
\[
\sigma(a) = \mu a \mu^{-1} = \mu a.
\]
Then
\[
\sigma(\sigma(a)) = a \quad \text{for all} \quad a \in A.
\]
We obtain a \(\Gamma\)-group \((A, \sigma)\), where \(\Gamma = \{1, \gamma\}\) acts on \(A\) by
\[
\gamma a = \sigma(a) = \mu a \mu^{-1} \quad \text{for} \quad a \in A.
\]

Let \(\mu'\) be any real structure on \(\mathcal{M}\) of the type \((1, -1, 1)\). Write \(\mu' = c \circ \mu\); then \(c \in \text{Aut} \mathcal{M}\) is an (even) holomorphic automorphism. We have
\[
1 = \mu'^2 = (c \mu)^2 = c \mu c \mu = c \cdot \mu c \mu^{-1} \cdot \mu^2 = c \cdot \gamma c.
\]
We see that the condition \(\mu'^2 = 1\) implies the cocycle condition
\[
(33) \quad c \cdot \gamma c = 1.
\]
Conversely, if an automorphism \(c \in \text{Aut} \mathcal{M}\) satisfies (33), and \(\mu' = c \mu\), then
\[
\mu'^2 = c \mu c \mu = c \cdot \mu c \mu^{-1} \cdot \mu^2 = (c \cdot \gamma c) \cdot \mu^2 = 1,
\]
and hence \(\mu'\) is a real structure on \(\mathcal{M}\).

Now let \(\mu_1 = c_1 \circ \mu\) and \(\mu_2 = c_2 \circ \mu\) be two real structures on \(\mathcal{M}\). Assume that the real structures \(\mu_1\) and \(\mu_2\) are equivalent, that is, \((\mathcal{M}, \mu_1) \cong (\mathcal{M}, \mu_2)\). This means that there exists an automorphism \(a: \mathcal{M} \to \mathcal{M}\) such that
\[
(34) \quad \mu_2 = a \circ \mu_1 \circ a^{-1}.
\]
Then
\[
c_2 \mu = a \cdot c_1 \mu \cdot a^{-1} = a \cdot c_1 \cdot \sigma(a)^{-1} \cdot \mu,
\]
whence
\[ c_2 = a \cdot c_1 \cdot \sigma(a)^{-1} = a \cdot c_1 \cdot \gamma a^{-1}. \]

Conversely, if (35) holds for some \( a \in \text{Aut} \mathcal{M} \), then (24) holds. This means that \( a \) is an isomorphism \((\mathcal{M}, \mu_1) \to (\mathcal{M}, \mu_2)\), and therefore the real structures \( \mu_1 \) and \( \mu_2 \) are equivalent.

We can state the results of this subsection as follows:

**Proposition A.11.** For \( \mathcal{M}, \mu, \) and the \( \Gamma \)-group \((A, \sigma)\) as in Subsection A.10, define an action of the group \( \Gamma = \{1, \gamma\} \) on \( A \) by
\[ \gamma a = \sigma(a) := \mu \cdot a \cdot \mu^{-1} \] for \( a \in A \).

Then:

(i) The map
\( c \mapsto c \circ \mu \) for \( c \in Z^1(A, \sigma) \)

is a bijection between the set of 1-cocycles \( Z^1(A, \sigma) \) and the set of real structures on \( \mathcal{M} \).

(ii) This map induces a bijection between \( H^1(A, \sigma) \) and the set of equivalence classes of real structures on \( \mathcal{M} \), sending \([1] \in H^1(A, \sigma)\) to the equivalence class of \( \mu \).

This proposition is similar to Proposition 5 in Section III.1.3 of Serre’s book [Ser].

Now let \( \mathcal{M} \) be the II-symmetric Grassmannian \( \text{IIGr}_{n,k} \). Theorem 24 describes the automorphism group \( \text{Aut} \mathcal{M} = \text{Aut}(\text{IIGr}_{n,k}) \). We wish to compute \( H^1 \text{Aut}(\text{IIGr}_{n,k}) \) and to classify the real structures on \( \text{IIGr}_{n,k} \). We use the notation of Propositions A.5 and A.7.

**Theorem A.12.** The list below gives the cardinality \( \# H^1 \text{Aut}(\text{IIGr}_{n,k}) \) and a set of representing cocycles for all cohomology classes in \( H^1 \text{Aut}(\text{IIGr}_{n,k}) \):

(i) Case \( n \) is odd: \( \# H^1 = 2 \) with representatives:
\[ (1,1), (1, \Psi_{-1}^*) \in \text{PGL}_n(\mathbb{C}) \times \{\text{id}, \Psi_{-1}^*\}. \]

(ii) Case \( n \) is even, \( n \neq 2k \): \( \# H^1 = 4 \) with representatives:
\[ (1,1), (1, \Psi_{-1}^*), (c_j, 1), (c_j, \Psi_{-1}^*) \in \text{PGL}_n(\mathbb{C}) \times \{\text{id}, \Psi_{-1}^*\}. \]

(iii) Case \( n = 2k \geq 4 \): \( \# H^1 = k + 3 \) with representatives:
\[ (1,1), (c_j, 1), (c_0, \Theta), (c_1, \Theta), \ldots, (c_k, \Theta) \in \text{PGL}_{2k}(\mathbb{C}) \times \{\text{id}, \Theta, \Psi_{-1}^*, \Psi_{-1}^* \circ \Theta\}. \]

(iv) Case \( n = 2, k = 1 \): \( \# H^1 = 2 \) with representatives:
\[ (1,1), (c_j, 1) \in \text{PGL}_2(\mathbb{C}) \times \mathbb{C}^*. \]

**Proof.** (i) and (ii) follow from Theorem 24(1), Remark A.2(1), and Proposition A.5.

(iii) follows from Theorem 24(2) and Lemma A.9.

(iv) for \( n = 2, k = 1 \) by Theorem 24(3) we have \( \text{Aut}(\text{IIGr}_{n,k}) \cong \text{G}_m \times \text{PGL}_{2,R} \). By Hilbert’s Theorem 90, we have \( H^1(R, \text{G}_m) = 1 \). It is well known that \( \# H^1(R, \text{PGL}_{2,R}) = 2 \) with cocycles \( 1, c_j \); see, for instance, [Bor22], Theorem 3.1. \( \square \)
We denote by $GL_{n,R}$, $GL_{n',H}$, etc. algebraic $R$-groups, and by $GL(n,R)$, $GL(n',H)$, etc. their groups of $R$-points. Here $H$ is the division algebra of Hamilton’s quaternions.

Lemma A.13 (well-known). Let $G = GL_{n,R}$ and assume that $n$ is even, $n = 2n'$. Let $c = \pi(a_j) \in Z^1(R, PGL_n(C))$ be as in Proposition A.12. Then the twisted group $cG$ is isomorphic to $GL_{n',H}$. In other words,

$$\{ g \in GL(n,C) \mid c \cdot g \cdot c^{-1} = g \} \cong GL(n',H).$$

Proof. Let $M_n(R)$ denote the algebra of $n \times n$-matrices over $R$. Then $G(R) = GL(n,R) = M_n(R)^\mu$. In order to compute $cG$, it suffices to compute the twisted algebra $cM_n$. Let $X \in M_n(C)$. We write $X$ as a block matrix

$$X = (X_{ij})_{1 \leq i,j \leq n'}, \quad \text{where } X_{ij} \in M_2(C).$$

Then $X \in cM_n(R)$ if and only if

$$a_j \cdot \overline{X} \cdot a_j^{-1} = X,$$

that is

$$J \cdot \overline{X}_{ij} \cdot J^{-1} = X_{ij} \quad \text{for all } i,j.$$

We write $Y$ for $X_{ij}$. Then condition (36) is equivalent to

$$J \cdot \overline{Y} = Y \cdot J.$$

An easy calculation shows that condition (37) on $Y$ means that

$$Y = \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix}, \quad \text{where } u,v \in C.$$

In other words,

$$Y = \lambda_1 + \lambda_2 i + \lambda_3 j + \lambda_4 k,$$

where $\lambda_1, \ldots, \lambda_4 \in R$ and

$$i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad j = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad k = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

One checks that

$$i^2 = -1 = j^2, \quad ij = k = -ji.$$

Thus

$$(cM_2)(R) \cong H, \quad (cM_n)(R) \cong M_{n'}(H), \quad (cGL_n)(R) \cong GL_{n'}(H),$$

and hence $cGL_n \cong GL_{n',H}$, as required. 

Proposition A.14. Let $n = 2n'$ be an even natural number. Consider the Grassmann variety $X = Gr_{n,k,R}$ over $R$ and its twisted form $cX := (Gr_{n,k}, c \circ \mu)$, where $c = \pi(a_j) \in Z^1 PGL_n(C)$ is as in Proposition A.12 and $\mu$ is the standard complex conjugation in $Gr_{n,k,C}$.

(i) if $k$ is even, $k = 2k'$, then the set of real points $\langle cX \rangle(R)$ is in a canonical bijection with the set $Gr_{n',k'}(H)$ of quaternionic $k'$-dimensional subspaces in the quaternionic $n'$-dimensional space $H^{n'}$. In other words,

$$\{ x \in Gr_{n,k}(C) \mid c \cdot \overline{x} = x \} \cong Gr_{n',k'}(H).$$

(ii) If $k$ is odd, then the set $\langle cX \rangle(R)$ is empty.
Proposition A.14. □
Lemma A.15. Set which proves assertion (i) of Proposition A.14.

Proof. Write \( V = \mathbb{R}^n \) with canonical basis \( e_1, \ldots, e_n \). Write \( G = \text{GL}_{n,\mathbb{R}} \), \( X = \text{Gr}_{n,k,\mathbb{R}} \). To any complex point \( x \in X(\mathbb{C}) \) we assign its stabilizer \( P_x \subset G_{\mathbb{C}} \). Let \( x_0 \in X(\mathbb{R}) \) denote the point corresponding to the \( k \)-dimensional subspace \( W_0 = \langle e_1, \ldots, e_k \rangle \subset V \), and write \( P_0 = \text{Stab}_{G_{\mathbb{C}}}(x_0) \). Then
\[
P_0 = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \middle| A \in M_k(\mathbb{C}) \right\}.
\]

We have a canonical \( \Gamma \)-equivariant bijection
\[
x \mapsto P_x, \quad x \in X(\mathbb{C}), \quad P_x \subset G_{\mathbb{C}}, \quad P_x \text{ is conjugate to } P_0.
\]

We twist \( G \) and \( X \) with the same cocycle \( c = \pi(a_1) \in Z^1 \text{PGL}_n(\mathbb{C}) \); then the map \( x \mapsto P_x \) is \( \Gamma \)-equivariant also with respect to the twisted real structures corresponding to the twists \( \cdot X \) and \( \cdot G \).

If \( k \) is even, \( k = 2k' \), we consider the subgroup
\[
P_{\mathbb{R},0} = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \middle| A \in M_{k'}(\mathbb{R}) \right\} \subset \text{GL}_{n',\mathbb{R}}.
\]

We embed \( \mathbb{H} \) into \( \mathcal{M}_2(\mathbb{C}) \) using the formulas \([35]\). In this way we obtain a canonical isomorphism
\[
\text{GL}_{n',\mathbb{R}} \times_{\mathbb{R}} \mathbb{C} \cong \text{GL}_{n,\mathbb{C}}
\]

sending \( P_{\mathbb{R},0} \times_{\mathbb{R}} \mathbb{C} \) to \( P_0 \). Note that \( H^1 P_{\mathbb{R},0} = 1 \). Since the subgroup \( P_{\mathbb{R},0} \subset cG \) is defined over \( \mathbb{R} \), we see that the corresponding point \( x_0 \in X(\mathbb{C}) = (\cdot X)(\mathbb{C}) \) is defined over \( \mathbb{R} \). We conclude that \( (\cdot X)(\mathbb{R}) \) is nonempty.

To \( x_0 \) we assign the subspace
\[
W_{\mathbb{R},0} = \langle e_1, \ldots, e_{k'} \rangle \subset \mathbb{H}^{n'}.
\]

For any \( k' \)-dimensional quaternionic subspace \( W_{\mathbb{R}} \subset \mathbb{H}^{n'} \) there exists an invertible matrix \( g \in \text{GL}(n',\mathbb{H}) \) such that \( W_{\mathbb{H}} = g \cdot W_{\mathbb{R},0} \), and we obtain a subgroup \( P := g P_0 g^{-1} \subset (\cdot G) \) that is defined over \( \mathbb{R} \) and is conjugate over \( \mathbb{R} \) to \( P_0 \). To this subgroup we assign the real point \( x = g \cdot x_0 \in (\cdot X)(\mathbb{R}) \) with stabilizer \( P \).

Conversely, if \( x \in (\cdot X)(\mathbb{R}) \), we set \( P_x = \text{Stab}_{\cdot G}(x) \subset \cdot G \). Then \( P_x \subset \cdot G \) is a subgroup defined over \( \mathbb{R} \) and conjugate to \( P_0 = P_{\mathbb{R},0} \) over \( \mathbb{C} \). Set
\[
T_x = \{ g \in (\cdot G)(\mathbb{C}) \mid g P_0 g^{-1} = P_x \}.
\]

This variety is clearly defined over \( \mathbb{R} \) and nontrivial over \( \mathbb{C} \). Moreover, it is a torsor (principal homogeneous space) under the normalizer \( N \) of \( P_{\mathbb{R},0} \) in \( cG \). Since \( N = P_{\mathbb{R}} \) and \( H^1 P_{\mathbb{R},0} = 1 \), we conclude that \( T_x \) has an \( \mathbb{R} \)-point \( g_x \); see Serre [Ser, Section I.5.2]. Thus there exists \( g_x \in (\cdot G)(\mathbb{R}) \) such that \( P_x = g_x \cdot P_{\mathbb{R},0} \cdot g_x^{-1} \). To \( x \) we assign the subspace \( W_x = g_x \cdot W_{\mathbb{R},0} \subset \mathbb{H}^{n'} \) (which does not depend on the choice of \( g_x \in T_x(\mathbb{R}) \)). Thus we obtain a bijection
\[
(\cdot X)(\mathbb{R}) \to \text{Gr}_{n',k'}(\mathbb{H}), \quad x \mapsto (P_x, g_x) \mapsto g_x \cdot W_{\mathbb{R},0},
\]
which proves assertion (i) of Proposition [AL4].

If \( k \) is odd, let \( x \in \text{Gr}_{2n',k}(\mathbb{C}) \) be a \( \mathbb{C} \)-point. The stabilizer \( P_x \) of \( x \) is a parabolic subgroup of odd codimension \( k(2n'-k) \). On the other hand, by Lemma [A.15] below, any defined over \( \mathbb{R} \) parabolic subgroup of \( \cdot G \) has even codimension. It follows the parabolic subgroup \( P_x \) is not defined over \( \mathbb{R} \), and hence \( x \) is not an \( \mathbb{R} \)-point. We conclude that when \( k \) is odd, \( \cdot X \) has no real points, which proves assertion (ii) of Proposition [AL4]. □

Lemma A.15. Set \( Q \subseteq cG \cong \text{GL}_{n',\mathbb{H}} \) be a parabolic \( \mathbb{R} \)-subgroup. Then the codimension of \( Q \) in \( cG \) is divisible by 4.
We write the cocycle condition

\[ c(x, y, w) = c(x, y, w') \cdot c(x', y', w') \quad \text{for all } x, x', y, y', w, w' \in V. \]

**Proof.**

\[ \text{Idea of proof.} \]

Let \( S \subset \mathcal{G} \) be the \( \mathbb{R} \)-subtorus such that

\[ S(\mathbb{R}) = \{ \text{diag}(\lambda_1, \ldots, \lambda_n) \mid \lambda_i \in \mathbb{R}^\ast \}. \]

Then \( S \) is a maximal split \( \mathbb{R} \)-torus in \( \mathcal{G} \). Consider the relative root system \( \mathcal{R}(\mathcal{G}, S) \). It is easy to see that the root subspace \( \mathfrak{g}_\beta \) for any root \( \beta \in \mathcal{R}(\mathcal{G}, S) \) has dimension 4. Let \( P \subset \mathcal{G} \) be the \( \mathbb{R} \)-parabolic (parabolic \( \mathbb{R} \)-subgroup) such that \( P(\mathbb{R}) \) is the group of upper triangular quaternionic matrices in \( \text{GL}(n', \mathbb{H}) \); then \( P \) is a minimal \( \mathbb{R} \)-parabolic in \( \mathcal{G} \). Let \( P' \supseteq P \) be any standard \( \mathbb{R} \)-parabolic in \( \mathcal{G} \) with respect to \( S \) and \( P \) in the sense of Borel and Tits [Brl-T, Section 5.12]; then

\[ \text{Lie}(\mathcal{G}) = \text{Lie}(P') \oplus \bigoplus_{\beta \in \Xi} \mathfrak{g}_\beta, \]

where \( \Xi \subseteq \mathcal{R}(\mathcal{G}, S) \) is some subset. It follows that the codimension of \( P' \) is divisible by 4. By [Brl-T Proposition 5.14], any \( \mathbb{R} \)-parabolic \( Q \) of \( \mathcal{G} \) is conjugate (over \( \mathbb{R} \)) to a standard \( \mathbb{R} \)-parabolic, and the lemma follows. \( \square \)

**Lemma A.16.** If \( V \) is a complex vector space, \( W \subset V \) a subspace, and \( B \) is a bilinear form (or a Hermitian form) on \( V \), then we write

\[ W^\perp_B = \{ x \in V \mid B(x, y) = 0 \text{ for all } y \in W \} \]

for the annihilator of \( W \) in \( V \) with respect to \( B \). Let \( n = 2k \). The group \( \mathbb{A} = \text{PGL}_{n, \mathbb{R}} \rtimes \mathbb{C}_4 \) of Subsection A.8 acts by conjugation on its identity component \( G = \text{PGL}_{n, \mathbb{R}} \). Therefore, for any cocycle \( c \in Z^1 \mathbb{A} \) we obtain a twisted group \( \mathcal{G} \).

Moreover, the group \( \mathbb{A} \) naturally acts on \( X := \text{Gr}_{n, k}(\mathbb{C}) \). Namely, for a \( k \)-dimensional subspace \( W \) of \( V = \mathbb{C}^n \), and \( [g] \in \text{PGL}(n, \mathbb{C}) \), the class of \( g \in \text{GL}(n, \mathbb{C}) \), we have \([g] \ast W = g(W)\). Furthermore, the generator \( \Theta \) of \( \mathbb{C}_4 \) sends \( W \) to \( W^\perp_{B_0} \), the annihilator of \( W \) in \( V = \mathbb{C}^n \) with respect to the symmetric bilinear form \( B_0 \) with matrix \( \text{diag}(1, \ldots, 1) \).

We show that this action of \( \mathbb{A} \) on \( X(\mathbb{C}) \) is well defined. For \( g \in \text{PGL}(n, \mathbb{C}) \) we compute

\[ \Theta(g(W)) = g(W)^\perp_{B_0} \]

\[ = \{ x \in V \mid B_0(x, gW) = 0 \} \]

\[ = \{ x \in V \mid B_0(g^t x, W) = 0 \} \]

\[ = \{ g^{-t} y \mid B_0(y, W) = 0 \} \quad y = g^t x, \ x = g^{-t} y \]

\[ = g^{-t} \cdot W^\perp_{B_0} \]

\[ = \Theta(g) \cdot \Theta(W). \]

Thus

\[ \Theta(g(W)) = \Theta(g) \cdot \Theta(W) \]

as required.

For any cocycle \( c \in Z^1 \mathbb{A} \) we obtain a twisted variety \( \mathcal{X} \).

**Lemma A.17.** Consider an element \( c = (F, \Theta) \) where \( F \in \text{PGL}(n, \mathbb{C}) \).

(i) \( c \) is a cocycle if and only if \( F^t = F \).

(ii) If \( c = (F, \Theta) \) is a cocycle and \( c' = (F', \Theta) \) with \( F' = gFg^t \), then \( c' \sim c \).

**Proof.** (i) We write the cocycle condition \( c \overline{c} = 1 \):

\[ (F, \Theta) \cdot (\overline{F}, \Theta^{-1}) = (1, 1), \]
that is,
\[ F \cdot \mathcal{T}^{-t} = 1. \]
Thus
\[ \mathcal{T}^t = F, \]
as required.

(ii) We consider the equivalent cocycle
\[
(g, 1) \cdot (F, \Theta) \cdot (g, 1)^{-1} = (g, 1) \cdot (F \cdot (\bar{g})^{-t}, \Theta)
= (g \cdot F \cdot \bar{g}^t, \Theta) = (F', \Theta),
\]
as required. □

Let \( c = (F, \Theta) \) be a cocycle. We compute the twisted group \( c \mathcal{G} \) for \( \mathcal{G} = \text{PGL}_n, \mathbb{R} \) and the twisted manifold \( c \mathcal{X} \) for \( \mathcal{X} = \text{Gr}_{n,k}, \mathbb{R} \).

Lemma A.18. For a cocycle \( c = (F, \Theta) \) consider the twisted group \( c \mathcal{G} \) with \( \mathcal{G} = \text{PGL}_n, \mathbb{R} \). Then
\[
(c \mathcal{G})(\mathcal{R}) = \{ g \in \text{PGL}(n, \mathbb{C}) \mid g F \bar{g}^t = F \}.
\]

Proof. First, note that \( c^{-1} = (F^t, \Theta^{-1}) \). Indeed,
\[
(F, \Theta) \cdot (F^t, \Theta^{-1}) = (F \cdot (F^t)^{-t}, 1) = (F \cdot F^{-1}, 1) = (1, 1).
\]

Now,
\[
(c \mathcal{G})(\mathcal{R}) = \{ g \mid c(\bar{g}, 1)c^{-1} = (g, 1) \}
= \{ g \mid (F, \Theta) \cdot (g, 1) \cdot (F^t, \Theta^{-1}) = (g, 1) \}.
\]
We calculate:
\[
(F, \Theta) \cdot (\bar{g}, 1) \cdot (F^t, \Theta^{-1}) = (F, \Theta) \cdot (\bar{g} F^t, \Theta^{-1})
= (F \cdot (\bar{g} F^t)^{-t}, 1)
= (F \bar{g}^{-t} F^{-1}, 1).
\]
We obtain
\[
(c \mathcal{G})(\mathcal{R}) = \{ g \in \text{PGL}(n, \mathbb{C}) \mid g = F \bar{g}^{-t} F^{-1} \}
= \{ g \in \text{PGL}(n, \mathbb{C}) \mid g F \bar{g}^t = F \},
\]
as required. □

Lemma A.19. Let \( c = (F, \Theta) \), \( \overline{\mathcal{T}}^t = F \) (then \( F, \Theta \) is a 1-cocycle). Then
\[ (c \mathcal{X})(\mathcal{R}) = \{ W \subset V \mid \dim W = k \text{ and } \bar{x}^t F^{-1} y = 0 \text{ for all } x, y \in W \}. \]

Proof. We write
\[ (c \mathcal{X})(\mathcal{R}) = \{ W \in \text{Gr}_k(V) \mid (F, \Theta) \cdot \overline{W} = W \}. \]
We have
\[ (F, \Theta) \cdot \overline{W} = F \cdot \overline{W}^{\perp_{B_0}}. \]
Thus the condition for \( W \) to lie in \( (c \mathcal{X})(\mathcal{R}) \) is
\[
F \cdot \overline{W}^{\perp_{B_0}} = W,
\]
or, equivalently,
\[
\overline{W}^{\perp_{B_0}} = F^{-1} \cdot W.
\]
This implies
\[ B_0(\bar{x}, F^{-1} \cdot y) = 0 \quad \text{for all } x, y \in W, \]
that is,
\[ \bar{x}^t F^{-1} y = 0 \quad \text{for all } x, y \in W. \]
Conversely, if (41) holds, then
\[ \mathfrak{P}^1 B_0 \supseteq F^{-1} \cdot W, \]
and comparing the dimensions, we obtain in turn the equalities (40) and (39), which gives \( W \in (\cdot, X)(R) \), as required. \( \square \)

**Corollary A.20.** Let \( G = \text{PGL}_{n,R} \). For \( 0 \leq p \leq n \), let the cocycle \( \bar{c}_p = (\pi(a_p), \Theta) \in Z^1 A \) be as in Lemma A.18. Write \( p G = \bar{c}_p G \). Then \( p G \cong \text{PU}(p, n-p); \) namely,
\[ (\bar{c}_p G)(R) \cong \{ g \in \text{PGL}(n, C) \mid g^t \cdot c_p \cdot g = c_p \}. \]

**Proof.** The corollary follows from Lemma A.18 with \( F = c_p := \pi(a_p) \). \( \square \)

**Corollary A.21.** Let \( X = \text{Gr}_{n,k,R} \) where \( n = 2k \geq 4 \). For \( 0 \leq p \leq k \), let the cocycle \( \bar{c}_p = (\pi(a_p), \Theta) \in Z^1 A \) be as in Lemma A.18. Write \( p X = \bar{c}_p X \).

(i) The set of \( R \)-points \( (p X)(R) \) is in a canonical \( (\bar{c}_p G)(R) \)-equivariant bijection with the isotropic Grassmannian \( \text{IGr}(n, k, p) \), that is, the set of \( k \)-dimensional subspaces \( W \subset V = C^n \) that are totally isotropic with respect to the Hermitian form
\[ \mathcal{H}_p(x, y) = x^t \cdot c_p \cdot y \]
where \( x, y \in C^n \) are \( k \)-dimensional column vectors.

(ii) This set \( (p X)(R) \) is non-empty if and only if \( p = k \).

Here we say that \( W \) is totally isotropic with respect to \( \mathcal{H}_p \) if \( \mathcal{H}_p(x, y) = 0 \) for all \( x, y \in W \).

**Proof.** Assertion (i) follows from Lemma A.19 with \( F = c_p \). Note that \( (c_p)^{-1} = c_p \).

If \( p < k \), then the Hermitian form \( \mathcal{H}_p \) admits no \( k \)-dimensional isotropic subspaces. If \( p = k \), then \( \mathcal{H}_k \) admits a \( k \)-dimensional isotropic subspace with basis \( e^1 - e_{k+1}, e_2 - e_{k+2}, \ldots, e_k - e_{2k} \). This proves (ii). \( \square \)

**Remark A.22.** By the Witt theorem for Hermitian forms, the group \( (\cdot c G)(R) = \text{PU}(k, k) \) acts on the isotropic Grassmannian \( (\cdot c X)(R) = \text{IGr}(k, k) \) transitively; see, for instance, Dieudonné [Die71, Assertion (3) in Section I.11].

**A.23.** Set
\[ b_k = i \left( \begin{array}{cc} 0 & -E_k \\ E_k & 0 \end{array} \right) \in \text{GL}(2k, C) \quad \text{where } i^2 = -1. \]
Then
\[ \tilde{b}_k^t = a_k, \quad \tilde{b}_k^{-1} = b_k. \]
Set
\[ d_k = \pi(b_k) \in \text{PGL}(2k, C), \quad \tilde{d}_k = (d_k, \Theta) \in Z^1(\Gamma, A). \]
Consider the Hermitian form \( \mathcal{F} \) on \( C^{2k} \) given by
\[ \mathcal{F}(x, y) = x^t b_k y \quad \text{for } x, y \in C^{2k}. \]

**Corollary A.24.** (i) \( (d_k G)(R) = \{ g \in \text{PGL}(2k, C) \mid g d_k \tilde{g}^t = d_k \}. \)
(ii) \( (\partial_{\ell_{k}}X)(\mathbb{R}) = \{ W \in \text{Gr}_{2k,k}(\mathbb{C}) | F|_{W} = 0 \} \).

The set \( (\partial_{\ell_{k}}X)(\mathbb{R}) \) is nonempty: it contains the \( k \)-dimensional subspace \( W_{0} \) with basis \( \{ e_{1}, \ldots, e_{k} \} \).

Proof. Assertion (i) follows from Lemma A.8 and assertion (ii) follows from Lemma A.19. \( \square \)

References

[A] Akhiezer D. N. Lie Group Actions in Complex Analysis. Aspects of Mathematics, E27. Friedr. Vieweg & Sohn, Braunschweig, 1995.

[ACF] Akhiezer D. N., Cupit-Foutou S. On the canonical real structure on wonderful varieties, Journal für die reine und angewandte Mathematik (Crelles Journal), vol. 2014, no. 693, 2014, pp. 231-244.

[BCC] Baldruzz L., Carmeli C., Cassinelli G. Super G-spaces. Symmetry in mathematics and physics, 159176, Contemp. Math., 490, AMS Providence, RI, 2009.

[BLMS] J. Bernstein, D. Leites, V. Molotkov, V. Shander. Seminar of Supersymmetries: volume 1 (edited by D. Leites) Moscow, Russia: Moscow Center of Continuous Mathematical Education (MCCME).

[BO] Bunegina, V. A., Onishchik, A. L. Homogeneous supermanifolds associated with the complex projective line, preprint no. 33, Inst. Math. Univ. Oslo, Oslo 1993

[Ber] Deligne P., Morgan J.W. Notes on supersymmetry (following Joseph Bernstein), Quantum Fields and Strings: A Course for Mathematicians, Vols. 1,2 (Princeton, NJ, 1996/1997), 41-97. American Mathematical Society. Providence, R.I. 1999.

[Brl-T] Borel A., Tits J. Groupes réductifs. Inst. Hautes Études Sci. Publ. Math. No. 27 (1965), 55–150.

[Bor22] Borovoi, M. Galois cohomology of reductive algebraic groups over the field of real numbers, Commun. Math. 30 (2022), no. 3, 191–201.

[BT1] Borovoi M., Timashev D.A. Galois cohomology of real semisimple groups via Kac labelings. Transform. Groups 26 (2021), 433–477.

[BT2] Borovoi M., Timashev D.A. Galois cohomology and component group of a real reductive group. To appear in Israel J. Math., arXiv:2110.13062 [math.GR].

[Die71] Dieudonné, J.A. La géométrie des groupes classiques. Troisième édition. Ergebnisse der Mathematik und ihrer Grenzgebiete , Band 5. Springer-Verlag, Berlin-New York, 1971.

[Gr] Green P. On holomorphic graded manifolds. Proc. Amer. Math. Soc. 85 (1982), no. 4, 587-590.

[Kac] Kac V.G. Lie superalgebras. Advances in Mathematics Volume 26, Issue 1, October 1977, Pages 8-96.

[Kal] Kaluz M. Almost complex structures on real Lie supergroups, Canad. Math. Bull. 58 (2015), no. 2, 281 - 284.

[Kos] Koszul, J.-L. Connections and splittings of supermanifolds. Differential Geom. Appl. 4 (2) (1994), 151-61.

[L] Leites D.A. Introduction to the theory of supermanifolds. Uspekhi Mat. Nauk, 1980, Volume 35, Issue 1(211), 3-57

[Man] Manin Yu.I. Gauge Field Theory and Complex Geometry, Springer-Verlag, Berlin e.a., 1988.

[Oni1] Onishchik A.L. Non-split supermanifolds associated with the cotangent bundle. Université de Poitiers, Département de Math., N 109. Poitiers, 1997.

[Oni2] Onishchik A.L. Lifting of Holomorphic Actions on Complex Supermanifolds, Advanced Studies in Pure Mathematics, 2002: 317-335 (2002).

[Roth] Rothstein M.J. Deformations of complex supermanifolds, Proc. Amer. Math. Soc. 95 (1985), 255-260.

[S] Serganova V. V. Classification of real simple Lie superalgebras and symmetric super-spaces, Funct Anal Its Appl 17, 200-207 (1983).

[Ser] Serre J.-P. Galois Cohomology, Springer-Verlag, Berlin, 1997.

[V1] Vishnyakova E.G. On holomorphic functions on a compact complex homogeneous supermanifold. Journal of Algebra 350 (1), 2012, 174-196.

[V2] Vishnyakova E.G. Vector fields on II-symmetric flag supermanifolds. Sao Paulo Journal of Mathematical Sciences 10 (1), 20-35.

[V3] Vishnyakova E.G. On complex Lie supergroups and split homogeneous supermanifolds. Transformation groups, Vol. 16, Issue 1, 2011. P. 265 - 285.
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