A mathematical model for vertical column farming system design of leafy vegetable crops

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Abstract. We constructed two mathematical non-linear programming models for the design of the leafy vegetable Vertical Column Farming System (VCFS), based on different plantation field’s geographical location divided by the Tropic of Capricorn and Cancer. The objective function of both models is set as the maximum gross output amount of vegetable and the constraints inside these non-linear programming models include the vegetable crop’s minimum sunlight exposure time requirement. An example optimal solution of the non-linear programming model, for plantation fields lying beyond the Tropic of Capricorn and Cancer, is submitted by mathematical software using Genetic Algorithm.

1. Introduction

With the rapid development of urbanisation, plantation fields designated for vegetables are becoming less available in the urban areas than before as it moves outwards gradually to the suburban area due to the shrinkage of farmland in the city center. Vertical Column Farming System (VCFS) is known for its high productivity of vegetable crops and its support of greening in urban area [1], comparing to traditional farming on the ground. Urban farming by VCFS not only helps to solve the problems like agricultural land limitation but contributes to the reduction of carbon dioxide emission which is released by transporting vegetables from suburban fields.

In VCFS, vegetables are cultivated on abreast vertical columns which act as growing support structure for plants and provides a flow passage for essential nutrition supply. Instructions of VCFS construction have been shown in a number of papers with example construction parameter values [2], which also verified the outstanding economic benefit of the VCFS. Our research is focused on formulating a general model, which provides optimal configuration parameter values, to readers who have the intention to design and construct a VCFS.

We start with solar access analysis as it affects the vegetables’ quality positively [2]. In Section 2.1, solar access analysis methods including Stick-Shadow Principle (SSP) and Space-Point Solar Access Analysis (SPSAA) are introduced to calculate the daily solar access time of each column. In Section 2.3 and Section 2.4, the primary and advanced mathematical non-linear programming models have been formulated by setting the maximum gross output amount of vegetables from the VCFS as the objective function. The constraints of minimum vegetable light requirements are expressed differently in these two models based on different research method. Heuristic algorithm (Genetics Algorithm) submits an example result of the VCFS’s configuration parameters in Section 3, involving the height,
radius and distance of each column. Conclusion and other issues relating to this research are summarized in Section 4.

| Nomenclature | Definition |
|--------------|------------|
| \( \phi \)  | latitude   |
| \( \delta \) | solar declination |
| \( \omega \) | solar hour angle |
| \( As \)    | solar azimuth |
| \( Hs \)    | solar zenith |
| \( Ci \)    | column serial number |
| \( Va \)    | observational points |
| \( Tsr \)   | sunrise time |
| \( Tsz \)   | sunset time |
| \( InsTi \) | inspection time intervals |
| \( Ti \)    | inspection time |
| \( Si \)    | column shadow length |
| \( ri \)    | column radius |
| \( di \)    | column distance |
| \( hi \)    | column height |
| \( a \)     | field length |
| \( b \)     | field width |
| \( p \)     | number of plant per square meter |
| \( ci \)    | profit per plant |
| \( cz \)    | cost per column |

2. Model Formulation and Optimisation

2.1. Solar Access Analysis Methods

Considering that sunlight impacts the growing quality of leafy vegetables significantly [2], this paper refers the design of vertical columns’ configuration to the high-density residential flats’ construction [3] which takes solar access analysis requirements into consideration. In this paper, we utilise SSP and SPSAA methods to conduct solar access analysis based on the Earth’s Revolution and solar access analysis parameters (including latitude (\( \phi \)), solar declination (\( \delta \)), hour angle (\( \omega \)), azimuth (\( As \)) and zenith (\( Hs \))). An example diagram of vertical column system’s sunlight shadow analysis has been illustrated in figure 1 by the Autodesk® 3ds MAX.

![Figure 1. Illustration diagram of VCFS’s solar access analysis.](image)

The solar angles of azimuth and zenith can be applied into the calculation through SSP for the direction and length of shadow generated by sunlight on each column \( C_i (i = 1, \ldots, m) \). The data which include each column’s shadow direction and length will be stored in matrixes for further computation. SPSAA helps to count the daily solar access hours of observational points \( V_{ai} \) designated on each column (where the vegetables are cultivated). In SPSAA, daily effective sunlight time (i.e. from sunrise time \( T_{sr} \) to the sunset time \( T_{sz} \)) has been divided into several inspection time intervals \( InsTi = [Ti, Ti + AT] \) and inspection time \( Ti (i = 1, \ldots, \frac{T_{sz} - T_{sr}}{AT}) \) to evaluate the observational points’
solar accessibility. The observational point is regarded as inaccessible to sunlight in the entire time interval \(I_{nsT_i}\) if its location lies within the shadow region generated by other columns at the inspection time \(T_i\). Otherwise, the observational point is marked with sunlight access time \(\Delta T\) in the time interval \(I_{nsT_i}\). After computing all the time intervals, the total daily solar access hours of each observational point can be obtained by summation of the results at each time interval \(I_{nsT_i}\).

2.2. VCFS Coordinate System

![Figure 2. Illustration of coordinate system.](image)

Before formulating the VCFS model, we create a coordinate system for the convenience of calculation in later subsections. As shown in figure 2, the number of rows \((N_r)\) and the number of columns \((N_c)\) of the VCFS can be computed by the following equation:

\[
N_r = \frac{a - d}{2r + d} \\
N_c = \frac{b - d}{2r + d}
\] (1)

For each column \(C_i\) in VCFS, a serial number is designated, which is increasing from west to east and south to north as shown in figure 2. For each column \(C_i\), its designated coordinate is calculated by the following equation:

\[
(x_i, y_i) = \begin{cases} \left( (N_r - 1) \cdot (2r + d), k_1 \cdot (2r + d) \right), & i = k_1 \cdot N_r \\ \left( (i - (k_0 \cdot N_c + 1)) \cdot (2r + d), y_{k_0 \cdot N_c + 1} \right), & k_0 \cdot N_c + 1 < i < k_1 \cdot N_r \\ \left( 0, k_0 \cdot (2r + d) \right), & i = k_0 \cdot N_c + 1 \end{cases}
\] (2)

where \(k_0 = 0, 1, ..., N_c - 1; k_1 = 1, ..., N_r; i \in \mathbb{N}\)

In equation (1), \(a, b, r, d\) stand for the length of field, the width of field, the radius of column and the distance of column correspondingly.

2.3. Primary Model for Fields Situating between the Tropic of Capricorn and the Tropic of Cancer

When the plantation field locates between the Tropic of Capricorn and the Tropic of Cancer, it is possible that the plantation field lies directly under the sunlight at noon (i.e. the Solar Declination equals to the location latitude and the angle of zenith reaches its maximum of the whole year) on some days during the year. For example, at twelve o’clock on March 20\(^{th}\), 2018, the Solar Declination of Sanya City \(\left(18.25^\circ\text{N}, 109.51^\circ\text{E}\right)\) is equal to \(18.25^\circ\text{N}\), and the length of each column’s shadow is zero. Therefore, a primary model can be established for these fields locating between the Tropic of Capricorn and Cancer. Figure 2 shows the possible pathway of the sun in one day in these locations.
In this primary model, the lowest observational points (i.e. point A and B as shown in figure 3) on the outermost columns (i.e. the shaded columns as shown in figure 3) are the points with the lowest solar access time in the whole system. As leafy vegetables require adequate exposure time under sunlight to meet its growing conditions [4], each observational point’s daily sunlight access time must be longer than the plant’s minimum light requirements [5]. In order to meet this requirement, the lowest observational points on the outermost columns must satisfy the plant’s minimum solar access time. Based on the conditions mentioned above, a primary mathematical non-linear programming model has been stated as follow by setting the maximum gross output amount of vegetables as the objective function:

\[
\max f(r,d,h) = (2\pi rh) \left( \frac{(a-d)(b-d)}{(2r+d)^2} \right) c_1 p - \frac{(a-d)(b-d)}{(2r+d)^2} c_2
\]

\[s.t. \ \begin{align*}
& \theta_{\text{min}} \leq \theta < \pi, \ \theta = \pi - 2 \arctan \left( \frac{h}{d} \right) \\
& 0 < h \\
& d_{\text{min}} \leq d \leq \max(a,b) \\
& 0 < r < \frac{\max(a,b)}{2}
\end{align*}\]

The first restriction is in accordance with the vegetable crop’s minimum sunlight exposure time requirement, where the correlation between the minimum sunlight exposure time \(T_{\text{sun,min}}\) and \(\theta_{\text{min}}\) is shown below:

\[
\theta_{\text{min}} = \frac{180^\circ}{[T_{\text{max}} - T_{\text{min}}]} T_{\text{sun,min}}
\]

In the third restriction, \(d_{\text{min}}\) stands for the minimum distance of each column due to the real-life maintenance issues. Other restrictions are set based on the limitation of the variables themselves and the limitation of the field size.

### 2.4. Advanced Model for Fields Situating beyond the Tropic of Capricorn and the Tropic of Cancer

When the plantation field locates beyond the Tropic of Capricorn and the Tropic of Cancer, where it is impossible to lie under direct sunlight in the whole year, it is imperative to optimize the primary model to an advanced. In the advanced model, each column’s shadow length and direction at inspection time and each observational point’s daily solar access time will be calculated separately and will be stored in the designated matrix for SPSAA. SSP provides a formula to compute the shadow length as shown below [6]:

\[
S_i = |\vec{S}_i| = h \cdot \cot H \cdot s = h \cdot \cot \left( \arcsin(\sin \phi \sin \delta + \cos \phi \cos \omega) \right)
\]

This article defines a shadow vector \(\vec{v}_i = (|\vec{v}_i|, \alpha_i)\) for each column (as shown in figure 5) in order to determine the existence of shadow blockage between two columns, where \(|\vec{v}_i|\) and \(\alpha_i\) stand for the...
shadow length and the shadow vector’s direction of column $C_i$ correspondingly. As shown in figure 5, $\alpha_i$ is equal to $\Delta s' = \pi - \Delta s$, where $\Delta s$ is the solar azimuth at inspection time $T_i$.

![Figure 4. Vertical column shadow length calculation.](image)

![Figure 5. Definition of vertical column’s shadow vector.](image)

Each shadow vector at inspection time $T_i$ is stored in a matrix $V$ including the shadow length and direction of the corresponding column:

$$V = \begin{bmatrix} T_1 & T_2 & \cdots & T_n \\ \vec{V}_{11} & \vec{V}_{12} & \cdots & \vec{V}_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \vec{V}_{m1} & \vec{V}_{m2} & \cdots & \vec{V}_{mn} \end{bmatrix}$$  \hspace{1cm} (6)

The comparison results between the distance of two columns and the column shadow length can be utilized to determine the existence of blockage of the observational points on the column. Thus, a distance matrix which includes the distance of two columns has been created as follows:

$$D = \begin{bmatrix} D_{11} & D_{12} & \cdots & D_{1m} \\ D_{21} & D_{22} & \cdots & D_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ D_{m1} & D_{m2} & \cdots & D_{mm} \end{bmatrix}$$  \hspace{1cm} (7)

In matrix $D$, the distance between column $C_i$ and $C_j$ is defined as $D_{ij}$:

$$D_{ij} = |C_i - C_j| = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}, \hspace{0.5cm} i,j = 1, \ldots, m, \hspace{0.5cm} i \neq j$$  \hspace{1cm} (8)

where $(x_i,y_i)$ and $(x_j,y_j)$ are the coordinates of column $C_i$ and $C_j$ as designated in section 2.2.

As shown in figure 4, column $C_j$ is partly blocked by column $C_i$. The shadow area generated by column $C_i$ involves the part lying on the ground and the part lying on column $C_j$. As shown in figure 5, we can tell that column $C_j$ is partly blocked by column $C_i$ because $D_{ij} \leq |\vec{V}|$. The height of shadow on column $C_j$ equals to $(|\vec{V}| - D_{ij}) \cdot \tan \Delta s$. Therefore, in this situation, for any observational point $V_{jk}$
with coordinate \((x_i, y_i, z_{ik})\) on column \(C_j\), we can determine that observational point \(V_{ik}\) has no access to direct sunlight if \(z_{ik} \leq \left(\frac{Y_i}{D_j}\right) \cdot \tan Hs\) at the inspection time \(T_r\).

**Figure 6.** Different stages of blocking by shadow between vertical columns.

The size of the intersection angle (i.e. angle \(\theta_{\text{crit},c}\), as shown in figure 6) between the direction of shadow vector and the connecting line of the center of two columns can be used to evaluate the blockage state of two columns. At the critical moment when the shadow outline is tangent to the column outline (circle), the critical intersection angle (i.e. angle \(\theta_{\text{critical}}\) as shown in figure 6(2)) can be computed by the following equation:

\[
\theta_{\text{critical}} = \arcsin \frac{2r}{D_j}
\]  

Hence, we can use the following equation to determine whether two columns \(C_i\) and \(C_j\) block each other partly or not:

\[
J_y = |\theta_{\text{crit},c}| - |\theta_{\text{critical}}| < 0, \text{ there is partly blockage between two columns}
\]

\[
= 0, \text{ critical moment of blockage (i.e. no blockage)}
\]

\[
> 0, \text{ there is blockage between two columns}
\]  

We apply this method to all columns and store the result in a matrix \(J\):

\[
J = \begin{bmatrix}
J_1 \\
J_2 \\
\vdots \\
J_m
\end{bmatrix} = \begin{bmatrix}
J_{11} & J_{12} & \cdots & J_{1m} \\
J_{21} & J_{22} & \cdots & J_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
J_{m1} & J_{m2} & \cdots & J_{mm}
\end{bmatrix}, \text{ where } J_r = 0
\]  

At each inspection time \(T_r\), the sunlight accessibility of each observational point can be evaluated by the method mentioned above with the assistance of shadow vector. The actual solar access time of all observational points \(V_{ik}\) during each time interval \(\text{Ins}T_i = [T_r, T_r + \Delta T]\) is computed by SPSAA with the following formula [3]:

\[
T_{ik} = \begin{cases} 
\Delta T, & \text{if observational point } V_{ik} \text{ is not blocked by any column} \\
0, & \text{if observational point } V_{ik} \text{ is blocked by a column}
\end{cases}
\]

Hence, we have the total length of the daily solar access time of all observational points \(V_{ik}\):

\[
T_{ik} = \sum_{r=1}^{n_r} T_{ikr}, \text{ where } n_r = \frac{T_{ss} - T_{sr}}{\Delta T} \in \mathbb{Z}^*
\]  

Base on the leafy vegetables’ growing requirements as stated in 2.2, an advanced mathematical non-linear programming model has been optimized as follow by setting the maximum gross output amount of vegetables as the objective function:
The first restriction means that the total length of the daily solar access time of each observational point must be longer than the plant’s minimum sunlight exposure time in order that the mathematical programming model submits a global optimal solution. In the third restriction, \( d_{\text{min}} \leq d \leq \max(a,b) \) stands for the minimum distance of each column due to the real-life maintenance issues. Other restrictions are set based on the limitation of the variables themselves and the limitation of the field size.

3. Model Application

This section provides a construction example of the advanced model as stated in 2.3, where the construction field locates in Hangzhou City (30.28° N, 120.15° E) with the intention to plant Lettuce (\textit{Lactuca sativa L.}). This example sets the plant’s minimum sunlight exposure time as four hours (which is adjustable according to different vegetable crops and plantation quality expectation). The day of the summer solstice (Jun. 21st, 2018) has been chosen for solar access analysis by SPSAA. Other values of the pre-set parameters can be found in Table 1. Finally, Genetic Algorithm [7] helps to find a solution of the example as shown in Table 1.

| Table 1. Values of the pre-set and solution parameters. |
|---------------------------------------------------------|
| Pre-set parameters                                      | Symbol | Value | Solution parameters | Symbol | Value |
| Sunrise time                                           | \( T_{sr} \) | 04:59 | Column height (m)   | \( h \) | 1.05  |
| Sunset time                                            | \( T_{ss} \) | 19:06 | Column distance (m) | \( d \) | 0.61  |
| Field length (m)                                       | \( a \) | 50    | Column radius (m)   | \( r \) | 0.31  |
| Field width (m)                                        | \( b \) | 50    | Number of rows      | \( N_c \) | 40    |
| Number of plant per square metre                       | \( p \) | 200   | Number of columns   | \( N_c \) | 40    |
| Profit per plant (yuan)                                | \( c_1 \) | 0.5   |                        |        |      |
| Cost per column                                        | \( c_2 \) | 50    |                        |        |      |

The result (solution parameters) presented in Table 1 is the global optimal solution as all the observational points have met the minimum sunlight exposure time requirement. The estimated economic profit gained from this example field will be 1,966,756 yuan (Chinese Yuan, CNY) excluding any other cost of human resources or first-stage construction expense. Comparing to traditional farming and vertical column farming on the same size field, the economic benefit of the latter is 7.86 times higher than the former as calculated in this example.

4. Conclusion and Discussion

This paper has formulated a vertical column farming model based on solar access analysis. With taking the leafy vegetable crops’ growing requirement of sunlight into consideration, a primary model and an advanced model with example solution have been presented separately according to different locations where plantation field lies in. The example verified that vertical column farming performs much better than traditional plantation (i.e. on the 2-D field) in economic benefit with the same product quality as it utilizes the space effectively. In real-life design and construction, these models mentioned above can be selected in line with the construction requirement considering the adjustability of the pre-set parameters.
Nevertheless, there are limitations in both models during the formulation process. For instance, the variation of weather has not been evaluated in both models due to the complexity of integration of the weather forecasting system. Both models are created by assuming that sunny weather lasts for the whole plantation period. Thus, it is possible to improve the accuracy of the models by linking the actual weather condition in the plantation field with the mathematical programming model. Also, the value $\Delta T$ defined in the time intervals $\Delta t$ in SPSAA determines the precision of solar access analysis. An appropriate value plays a vital role in balancing the calculation complexity (i.e. enumeration times) against the model’s accuracy.

5. References

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