NON-RELATIVISTIC CONFORMAL STRUCTURES (1)

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Abstract. The “Kaluza-Klein-type” geometric structure appropriate to study the central extension of the Galilei group and non-relativistic physics is reviewed.

1. Bargmann structures.

The fundamental invariance group of non-relativistic physics is the Galilei group, which acts on space-time according to

\[ r^* = Ar + bf + c, \]
\[ t^* = t + e, \]

where \( A \in SO(3), b, c \in \mathbb{R}^3, e \in \mathbb{R}. \) However, unlike in the relativistic case, the Galilei group acts on the wave functions only up to a phase [1],

\[ \psi^*(r, t) = e^{-(im/\hbar)(b.Ar + b^2t/2)} \psi(r^*, t^*). \]

Hence, it is only a central extension of the Galilei group (called the Bargmann group), which is a symmetry group at the quantum level.

The natural geometric setting for realizing the central extension of a Lie group is to add a new, ‘vertical’ variable, \( s, \) and consider the \( \mathbb{R}-\)bundle \( M := (\mathbb{R} \times \mathbb{R}^3) \times \mathbb{R} = \{(t, r, s)\} \) [2]. The Bargmann group acts on \( M \) according to (1a-b), augmented by

\[ s^* = s - b.Ar - \frac{1}{2}b^2 - h, \quad h \in \mathbb{R}. \]

The action (2) on the wave functions can then be recovered by lifting the wavefunctions to the bundle as equivariant functions i.e. by replacing \( \psi(r, t) \) by \( \Psi(r, t, s) := e^{ims/\hbar}\psi(r, t). \)

The idea of Duval et al. [3] is to view the extended manifold \( M \) as the proper arena for classical mechanics. Potentials of the Newtonian type can be incorporated into a Lorentz metric defined on the extended space; the dynamical trajectories in ordinary space then correspond to null-geodesics in the extended space. This has been noted many years ago by Eisenhart [4], but has long been forgotten. Extending to \( M \) has the additional advantage the resulting quantities become invariant [2].

The construction is reminiscent of Kaluza-Klein theory, except for that the extra dimension is null rather than space-like. Notice also that, while Kaluza-Klein theory involves electromagnetism, the present framework is adapted to (non-relativistic) gravitational interactions.

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2. Classical dynamics.

Let us first consider a free, non-relativistic particle with Lagrangian \( L_0 = \frac{1}{2} m (\dot{r}')^2 \), \( ((\cdot)' = \frac{d}{d\tau}) \). A Galilei transformation (1a-b) changes \( L_0 \) as \( L_0 \rightarrow L_0 + \frac{m}{2} (b^2/2 + b \cdot \dot{r}') \). The non-invariance of the Lagrangian can however be compensated by adding a fifth coordinate, \( s \), and by considering rather

\[
L_0 = L_0 + m \frac{ds}{dt},
\]

which is indeed invariant with respect to the action (1a-c) of the Bargmann group.

Remarkably, the new Lagrangian \( L_0 \) is associated with geodesic motion in extended space,

\[
L_0 = \frac{1}{2} m \left( \left( \frac{dr}{dt} \right)^2 + 2 \frac{ds}{dt} \right) = \frac{1}{2} m g_{ab}^{0} \dot{x}^a \dot{x}^b, \quad (a,b = 1,\ldots,5)
\]

for the 5-metric \( g_{ab}^{0} dx^a dx^b = dr^2 + 2 ds dt \).

Potentials can be included at this stage. If \( U(r,t) \) is a potential function, we can consider the modified metric

\[
g_{ab} dx^a dx^b = g_{ab}^{0} dx^a dx^b - 2 U dt^2 = dr^2 + 2 ds dt - 2 U dt^2.
\]

Its extremals (geodesics in 5 dimensions) are conveniently described in a homogeneous framework, i.e. by the Lagrangian

\[
\mathcal{L}(r,t,s,\dot{r},\dot{t},\dot{s}) = \left( \frac{1}{2} m g_{ab}^{0} \dot{x}^a \dot{x}^b \right) \dot{t} = \frac{1}{2} m g_{ij}^{0} \dot{r}^i \dot{r}^j - m U t + m \dot{s},
\]

where the \( r, t, s, \dot{r}, \dot{t}, \dot{s} \) are coordinates on the tangent bundle \( TM \).

The value of the quadratic quantity \( h_0 = g_{ab} \dot{x}^a \dot{x}^b \) is conserved along any geodesic, and is interpreted as (minus) the internal energy of the particle. It is convenient to restrict our attention to those “motions” in external space for which \( h_0 \) vanishes [3, 4], i.e. to consider null geodesics. As it is readily verified, these latter project onto the extremals in ordinary space of the Lagrangian \( L = L_0 - m U \). (The vertical coordinate satisfies \( s(t) = s_0 - \int L dt \)). We describe classical system henceforth by the Lagrangian (6), supplemented with the constraint of having vanishing internal energy \( h_0 = 0 \).

The metric in Eq. (5) is a Lorentz metric on \( M \) with signature \((+,+,+,+,−)\), which admits a covariantly constant Killing vector, namely \( \xi = \partial/\partial s \). Following Ref. 3, \((M,g,\xi)\) is called a Bargmann manifold.

At the quantum level, the Schrödinger equation

\[
\left[ -\hbar^2 \frac{\Delta}{2m} + mU \right] \psi = i\hbar \partial_t \psi
\]

where \( \Delta \) is the Laplacian on ordinary 3-space can be written as

\[
\Delta_g \Psi = 0,
\]

\( \Delta_g \) being the Laplacian on \((M,g)\), and \( \Psi = e^{i ms/\hbar} \psi \).
3. Symmetries.

The five-dimensional framework is particularly convenient to describe the symmetries of the problem. Firstly, massless geodesics are permuted by conformal transformations. We should, however, insist on that the mass be conserved and hence we only consider transformations which preserve the vertical vector $\xi$. Thus, we consider those vectorfields $Y$ on $M$ which satisfy

$$L_Yg = \lambda g, \quad [Y, \xi] = 0.$$ \hspace{1cm} (8)

(Killing vectors correspond to $\lambda = 0$). Transformations as in (8) preserve the geodesic Lagrangian, $L_{\tilde{Y}}L = \lambda L$, where $\tilde{Y}$ is the canonical lift of $Y$ to the tangent bundle $TM$. The associated Noether quantities,

$$C = \frac{\partial L}{\partial \dot{x}^a}Y^a,$$ \hspace{1cm} (9)

are constants of the motion. In conventional terms, a vectorfield $Y$ as in Eq. (8) projects onto a vectorfield, $X$, on ordinary spacetime denoted by $Q \times \mathbb{R}$, and also to $\mathbb{R}$, the time axis. Let $\tilde{X}$ denote the canonical lift of $X$ to $TQ \times \mathbb{R}$. The condition $L_{\tilde{Y}}L = 0$ means then that $L_{\tilde{X}}L = m\frac{d}{dt}K$ for some real function $K$. Thus, the usual Lagrangian $L$ changes by a total derivative, which is the definition of a symmetry. In fact, $Y = (X, -K)$. For (6), the Noether quantity (9) reduces to

$$C = \left( \frac{\partial L}{\partial x^i} \right) X^i - \left( \frac{\partial L}{\partial x^a} x'^a - L \right) X^t - mK,$$ \hspace{1cm} (10)

which is the standard conserved quantity associated to the symmetry $X$.

The Killing vector $\xi$ is always a symmetry for a Bargmann system; the associated Noether quantity (10) is, by (6), just $m$, the mass.

For a free particle, for example, Eq. (8) yields a 13-dimensional algebra, whose action $Y = (X^i, X^t, Y^q)$ on extended spacetime is

$$\left( \omega \times r + \left( \frac{1}{2} \delta + \kappa t \right) r + \beta t + \gamma, \kappa t^2 + \delta t + \epsilon, -\left( \frac{1}{2} \kappa r^2 + \beta r + \eta \right) \right).$$ \hspace{1cm} (11)

Here the 11 parameters $\omega \in so(3)$, $\beta, \gamma \in \mathbb{R}^3$, $\epsilon, \eta \in \mathbb{R}$ generate the isometries, with $\omega$ representing rotations, $\beta$ Galilei boosts, $\gamma$ space-translations, $\epsilon$ time-translations, and $\eta$ translations in the vertical direction. They are are readily recognized as the generators of the Bargmann group. The two additional parameters $\delta, \kappa \in \mathbb{R}$ generate the dilatations and the expansions. Eq. (11) is the (extended) Schrödinger algebra [5]. Eq. (10) yields the associated conserved quantities,

$$\begin{align*}
L &= r \times p \quad &\text{angular momentum} \\
g &= m(r - vt) \quad &\text{center of mass} \\
p &= mr^t \quad &\text{momentum} \\
-E &= -\frac{p^2}{2m} \quad &\text{energy} \\
m &= &\text{mass} \\
D &= \frac{1}{2}p \cdot r - tE \quad &\text{dilatation} \\
K &= t^2E + 2tD - \frac{1}{2}mr^2 \quad &\text{expansion}
\end{align*}$$ \hspace{1cm} (12)
as expected.

Let us notice finally, that, due to the conformal invariance of the Laplacian, the classical symmetries (9) are symmetries of the Schrödinger equation also. (Historically, the Schrödinger group was discovered as the maximal invariance group of the Schrödinger equation of a free, non-relativistic particle [5]).

Other examples and the extension to spin are described in References [6] and [7].

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