LAPLACIAN COMPARISON THEOREM ON RIEMANNIAN MANIFOLDS WITH MODIFIED $m$-BAKRY-ÉMERY RICCI LOWER BOUNDS FOR $m \leq 1$

KAZUHIRO KUWAE* AND TOSHIKI SHUKURI

Abstract. In this paper, we prove a Laplacian comparison theorem for non-symmetric diffusion operator on complete smooth $n$-dimensional Riemannian manifold having a lower bound of modified $m$-Bakry-Émery Ricci tensor under $m \leq 1$ in terms of vector fields. As consequences, we give the optimal conditions for modified $m$-Bakry-Émery Ricci tensor under $m \leq 1$ such that the (weighted) Myers’ theorem, Bishop-Gromov volume comparison theorem, Ambrose-Myers’ theorem, Cheng’s maximal diameter theorem, and the Cheeger-Gromoll type splitting theorem hold. Some of these results were well-studied for $m$-Bakry-Émery Ricci curvature under $m \geq n$ ([19, 21, 27, 33]) or $m = 1$ ([34, 35]) if the vector field is a gradient type. When $m < 1$, our results are new in the literature.

1. Introduction

1.1. Modified Bakry-Émery Ricci curvatures. Let $(M, g)$ be an $n$-dimensional smooth complete Riemannian manifold with its volume measure $m := \text{vol}_g$ and $V$ a $C^1$-vector field. Throughout this paper, we assume that the manifold $M$ has no boundary and is connected. We consider a diffusion operator $\Delta_V := \Delta - \langle V, \nabla \cdot \rangle$. In [31, 32, 34], $\Delta_V$ is called the $V$-Laplacian on $(M, g)$.

For any constant $m \in ]-\infty, +\infty[$, we introduce the symmetric 2-tensor

$$\text{Ric}_{m,n}(\Delta_V)(x) = \text{Ric}(x) + \frac{1}{2} \mathcal{L}_V g(x) - \frac{V^*(x) \otimes V^*(x)}{m - n}, \quad x \in M,$$

and call it the modified $m$-Bakry-Émery Ricci tensor of the diffusion operator $\Delta_V$. Here $\mathcal{L}_V g(X, Y) := \langle \nabla_X V, Y \rangle + \langle \nabla_Y V, X \rangle$ is the Lie derivative of $g$ with respect to $V$ and $V^*$ is the dual 1-form of $V$ coming from $g$.

For any $m \in ]-\infty, +\infty[$ and a continuous function $K : M \to \mathbb{R}$, we call $(M, g, V)$ or $L$ satisfies the CD($K, m$)-condition if

$$\text{Ric}_{m,n}(\Delta_V)(x) \geq K(x) \quad \text{for all} \quad x \in M.$$
When \( m = n \), we always assume that \( V \) vanishes so that \( \text{Ric}_{n,n}(\Delta V) = \text{Ric} \). When \( m \geq n \), \( m \) is regarded as an upper bound for the dimension of the diffusion operator \( \Delta V \). Throughout this paper, we focus on the case \( m \leq 1 \) and assume \( n > 1 \) if \( m = 1 \) and \( V \) does not vanish (i.e., \( V \equiv 0 \) and \( \Delta V = \Delta \) if \( m = n = 1 \)). Consequently, for \( m \leq 1 \), we always assume \( n > m \) provided \( V \) does not vanish. Note that, for \( m \leq 1 \), \( N \in [n, +\infty[ \), and for any \( x \in M \), we have

\[
\text{Ric}_{1,n}(\Delta V)(x) \geq \text{Ric}_{m,n}(\Delta V)(x) \geq \text{Ric}_{\infty,n}(\Delta V)(x) \geq \text{Ric}_{N,n}(\Delta V)(x).
\]

If we only consider the case that the lower bounds of the above Ricci tensor are constant, \( \text{Ric}_{1,n}(\Delta V) \geq \text{const.} \) is the weakest one among them. But if we consider the case that the lower bound of Ricci curvature depends on the parameter \( m \) like (2.11) below, the similar condition is no longer the weakest one.

In the literature, there have been intensive works on the study of geometry and analysis of weighted complete Riemannian manifolds with the \( \text{CD}(K,m) \)-condition for \( m \geq n \) and \( K \in \mathbb{R} \) (or \( K \in C(M, \mathbb{R}) \)) in the case \( V = \nabla \phi \) for \( \phi \in C^2(M) \). See \([2, 3, 4, 5, 6, 7, 8, 12, 13, 19, 20, 17, 18, 21, 27, 33]\), and reference therein. During recent years, there are already several papers on the study of weighted Riemannian manifolds with \( m \)-Bakry-Émery Ricci curvature for \( m < 0 \) or \( m < 1 \) with \( V = \nabla \phi \) for a \( C^2 \)-function \( \phi \). For \( V = \nabla \phi \), we write \( L := \Delta_{\nabla \phi} \) in this introduction. In [25], Ohta and Takatsu proved the \( K \)-displacement convexity of the Rényi type entropy under the \( m \)-Bakry-Émery Ricci tensor condition \( \text{Ric}_{m,n}(L) \geq K \), i.e., the \( \text{CD}(K,m) \)-condition, for \( m \in ]-\infty, 0[ \cup ]n, +\infty[ \) and \( K \in \mathbb{R} \). After that, Ohta [24] and Kolesnikov-Milman [15] simultaneously treated the case \( m < 0 \). Ohta [24] extended the Bochner inequality, eigenvalue estimates, and the Brunn-Minkowski inequality under the lower bound for \( \text{Ric}_{m,n}(L) \) with \( m < 0 \). Kolesnikov-Milman [15] also proved the Poincaré and the Brunn-Minkowski inequalities for manifolds with boundary under the lower bound for \( \text{Ric}_{m,n}(L) \) with \( m < 0 \). In [24, Theorem 4.10], Ohta also proved that the lower bound of \( \text{Ric}_{m,n}(L)(x) \) with \( m < 0 \) is equivalent to the curvature dimension condition in terms of mass transport theory as defined by Lott-Villani [22] and Sturm [29, 30]. In [34], Wylie proved a warped product version of Cheeger-Gromoll splitting theorem under the \( \text{CD}(0,1) \)-condition. He also proved an isometric product version of Cheeger-Gromoll splitting under \( \text{CD}(0,m) \)-condition with \( m < 1 \) and \( (V,1) \)-completeness condition. In [35], W. Wylie and D. Yeroshkin proved a Laplacian comparison theorem, a Bishop-Gromov volume comparison theorem, Myers’ theorem and Cheng’s maximal diameter theorem on manifolds with \( m \)-Bakry-Émery Ricci curvature condition for \( m = 1 \) with \( V = \nabla \phi \) for a \( C^2 \)-function \( \phi \). Recently, Milman [23] extended the Heintze-Karcher Theorem, isoperimetric inequality, and functional inequalities under the lower bound for \( \text{Ric}_{m,n}(L)(x) \) with \( m < 1 \). In [16], the first named author and X.-D. Li established the Laplacian comparison theorem on weighted complete Riemannian manifolds with the \( \text{CD}(K,m) \)-condition with \( m \leq 1 \) for \( V = \nabla \phi \) with
\( \phi \in C^2(M) \), and obtained (weighted) Myers’ theorem, Bishop-Gromov volume comparison theorem, Ambrose-Myers’ theorem, Cheeger-Gromoll type splitting theorem, stochastic completeness and Feller property of \( L \)-diffusion process under optimal conditions on the \( m \)-Bakry-Émery Ricci tensor for \( m \leq 1 \) over the weighted complete Riemannian manifolds.

It is important to know whether one can establish the Laplacian comparison theorem on such Riemannian manifolds with the CD\((K, m)\)-condition for \( m \leq 1 \) and \( K \in \mathbb{R} \) for general \( C^1 \)-vector field \( V \). In this paper, we prove such comparison theorem for \( K \) being a continuous function depending on a re-parametrized distance function on \( M \). As consequences, we give the optimal conditions on the modified \( m \)-Bakry-Émery Ricci tensor for \( m \leq 1 \) so that (weighted) Myers’ theorem, Bishop-Gromov volume comparison theorem, Ambrose-Myers’ theorem, Cheng’s maximal diameter theorem, and the Cheeger-Gromoll type splitting theorem hold on weighted complete Riemannian manifolds. These geometric results are complete extensions of the case for \( V = \nabla \phi \) proved in the first part of [16]. When \( m < 1 \), our results are new in the literature.

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2. Main result

Let \( V \) be a \( C^1 \)-vector field on a Riemannian manifold \((M, g)\). Since there may be no function \( \phi \) satisfying \( V = \nabla \phi \) in general, we can still make sense of bounds by integrating \( V \) along geodesics. Define

\[
V_\gamma(r) := \int_0^r \langle V_{\gamma_s}, \dot{\gamma}_s \rangle ds
\]

for a unit speed geodesic \( \gamma : [0, T] \rightarrow M \), and

\[
\phi_V(x) := \inf \left\{ \int_0^{r_p(x)} \langle V_{\gamma_s}, \dot{\gamma}_s \rangle ds \mid \gamma : \text{unit speed geodesic } \gamma_0 = p, \gamma(r_p(x)) = x \right\}.
\]

Note that \( V_\gamma \) depends on the choice of unit speed geodesic \( \gamma \), and \( \phi_V(x) \) depends on \( p \) with \( \phi_V(p) = 0 \) and it is well-defined for \( x \in M \). It is easy to see that \( \phi_V(x) = \int_0^{r_p(x)} V r_p(\gamma_s) ds \) under \( x \notin \text{Cut}(p) \), where \( \gamma \) is the unique unit speed geodesic with \( \gamma_0 = p \) and \( \gamma(r_p(x)) = x \). Hence \( \phi_V \) is a continuous function on \((\text{Cut}(p) \cup \{p\})^c \). Consequently, \( \phi_V \) is an \( m \)-measurable function. Moreover, for \( x \notin \text{Cut}(p) \), \( \phi_V(x) = V_\gamma(r_p(x)) \) for the unique unit speed geodesic \( \gamma \) with \( \gamma_0 = p \) and \( \gamma(r_p(x)) = x \). Hence \( \phi_V(\gamma_t) = V_\gamma(t) \) for any unit speed geodesic \( \gamma \) with \( \gamma_0 = p \) and \( \gamma \notin \text{Cut}(p) \). When \( V = \nabla \phi \) is a gradient vector field for some
\( \phi \in C^2(M) \), then one can see

\[
V_\gamma(t) = \int_0^t \langle \nabla \phi, \dot{\gamma}_s \rangle ds = \int_0^t \frac{d}{ds} \phi(\gamma_s) ds = \phi(\gamma_t) - \phi(\gamma_0).
\]

Throughout this paper, we fix a point \( p \in M \) and a constant \( C_p > 0 \), which may depend on \( p \). For \( x \in M \), we define

\[
s_p(x) := \inf \left\{ C_p \int_0^{r_p(x)} e^{-\frac{2V_\gamma(t)}{n-m}} dt \middle| \gamma : \text{unit speed geodesic} \right. \\
\left. \right. \gamma_0 = p, \gamma(r_p(x)) = x \right\}.
\]

If \((M, g)\) is complete, then \( s_p(x) \) is finite and well-defined from the basic properties of Riemannian geodesics. Let \( s(p, q) := s_p(q) \) for \( p, q \in M \). If \( q \) is not a cut point of \( p \), then there is a unique minimal geodesic from \( p \) to \( q \) and \( s_p \) is smooth in a neighborhood of \( q \) as can be computed by pulling the function back by the exponential map at \( p \). Note that \( s(p, q) \geq 0 \), it is zero if and only if \( p = q \). But \( s(p, q) = s(q, p) \) does not hold in general. If \( V = \nabla \phi \) for some \( \phi \in C^2(M) \) and set \( C_p = \exp \left( -\frac{2\phi(p)}{n-m} \right) \) for the definition of \( s_p(x) \) with \( p \) being arbitrary, then one can see that \( s(p, q) = s(q, p) \) for \( p, q \in M \). However, \( s(p, q) \) does not necessarily satisfy the triangle inequality.

**Definition 2.1.** Let \((M, g)\) be an \( n \)-dimensional complete Riemannian manifold and \( V \) a \( C^1 \)-vector field. Fix \( p \in M \). Then we say that \((M, g, V)\) is \((V, m)\)-complete at \( p \) if

\[
\lim_{r \to +\infty} \inf_{L(\gamma) = r} \int_0^r e^{-\frac{2V_\gamma(t)}{n-m}} dt = +\infty,
\]

where the infimum is taken over all minimizing unit speed geodesics \( \gamma \) with respect to the metric \( g \) such that \( \gamma_0 = p \). We say that \((M, g, V)\) is \((V, m)\)-complete if it is \((V, m)\)-complete at \( p \) for all \( p \in M \).

**Remark 2.2.**

(1) If \( V_\gamma \) is upper bounded for any unit speed geodesic \( \gamma \) with \( \gamma_0 = p \), then \((M, g, V)\) is always \((V, m)\)-complete at \( p \) for all \( m \leq 1 \). In particular, if there exists a non-negative integrable function \( f \) on \([0, +\infty[\) such that \( \langle V, \nabla r_p \rangle_x \leq f(r_p(x)) \), then \( V_\gamma(r) \leq \int_0^r f(t) dt \leq \int_0^\infty f(t) dt < \infty \) so that \((M, g, V)\) is always \((V, m)\)-complete at \( p \) for all \( m \leq 1 \).

(2) If \( M \) is compact, then \((M, g, V)\) is always \((V, m)\)-complete for \( m \leq 1 \). Indeed, if so, the set \( G_r := \{ \gamma \mid \gamma \) is a unit speed minimal geodesic, \( L(\gamma) = r \} \) is an empty set for sufficiently large \( r > 0 \). This implies (2.2).

(3) If there exists a non-negative locally integrable function \( f \) on \([0, +\infty[\) satisfying \( f(t) \leq C/t \) on \([1, +\infty[\) for some \( C \in ]0, (n-m)/2] \) and \( \langle V, \nabla r_p \rangle \leq f(r_p) \), then \((V, m)\)-completeness at \( p \) holds for all \( m \leq 1 \). Here we assume \( n > 1 \) for \( m = 1 \).
In fact, we see for \( r > 1 \)
\[
\inf_{L(\gamma) = r} \int_0^r e^{-\frac{2V(t)}{n-m}} dt \geq \int_1^r e^{-\frac{2f(s)}{n-m}} dt \\
\geq e^{-\frac{2L^2_{\gamma} f(s)}{n-m}} \int_1^r e^{-\frac{2L^2_{\gamma} f(s)}{n-m}} dt \\
\geq e^{-\frac{2L^2_{\gamma} f(s)}{n-m}} \int_1^r e^{-\frac{2C \log t}{n-m}} dt \\
\geq e^{-\frac{2L^2_{\gamma} f(s)}{n-m}} \int_1^r \frac{dt}{t^{\frac{n}{n-m}}} \to +\infty \quad \text{as} \quad r \to \infty,
\]
where the infimum is taken over all minimizing unit speed geodesics \( \gamma \) with \( \gamma_0 = p \).

(4) The \((V, 1)\)-completeness at \( p \) defined as in \[34\] Definition 6.2 implies the \((V, m)\)-completeness at \( p \) for every \( m \leq 1 \) provided \( V_\gamma \geq 0 \) for any unit speed geodesic \( \gamma \) with \( \gamma_0 = p \). The converse also holds under \( V_\gamma \leq 0 \) for any unit speed geodesic \( \gamma \) with \( \gamma_0 = p \).

**Lemma 2.3.** Let \((M, g)\) be an \( n\)-dimensional complete non-compact Riemannian manifold and \( V \) a \( C^1 \)-vector field. Fix \( p \in M \) and suppose that \((M, g, V)\) is \((V, m)\)-complete at \( p \). Then, for any sequence \( \{q_i\} \) in \( M \) such that \( d(p, q_i) \to +\infty \) as \( i \to +\infty \), \( s(p, q_i) \to +\infty \).

**Proof.** The proof is similar to that of \[35\] Proposition 3.4. We omit it. \( \square \)

**Remark 2.4.** Recall that \( \phi_V \) depends on \( p \in M \). For a fixed \( p \in M \), we set \( \phi_V(r) : = \inf_{B_r(p)} \phi_V \) and \( \bar{\phi}_V(r) : = \sup_{B_r(p)} \phi_V \) for \( r \in [0, +\infty] \). Then \( \bar{\phi}_V(r) \leq 0 \leq \phi_V(r) \) for \( r > 0 \) and \( \lim_{r \to 0} \phi_V(r) = \lim_{r \to 0} \bar{\phi}_V(r) = 0 \). If \( x \notin \text{Cut}(p) \), we have \( s_p(x) = C_p \int_0^{r_p(x)} e^{-\frac{2 \phi_V(r_p(x))}{n-m}} dt \) for the unique unit speed geodesic \( \gamma \) with \( \gamma_0 = p \) and \( \gamma(r_p(x)) = x \). So \( \lim_{x \to p} \frac{s_p(x)}{r_p(x)} = C_p \).

In particular,
\[
C_p e^{-\frac{2 \phi_V(r_p(x))}{n-m}} r_p(x) \leq s_p(x) \leq C_p e^{-\frac{2 \phi_V(r_p(x))}{n-m}} r_p(x) \quad \text{for} \quad x \notin \text{Cut}(p).
\]

2.1. **Laplacian Comparison.** Let \( \kappa : [0, +\infty] \to \mathbb{R} \) be a continuous function and \( a_\kappa \) the unique solution defined on the maximal interval \( ]0, \delta_\kappa[ \) for \( \delta_\kappa \in ]0, +\infty[ \) of the following Riccati equation
\[
\frac{da_\kappa}{ds}(s) = \kappa(s) + a_\kappa(s)^2
\]
with the boundary conditions
\[
\lim_{s \to 0^+} sa_\kappa(s) = 1,
\]
and
\[
\lim_{s \to \delta_\kappa^+} (s - \delta_\kappa) a_\kappa(s) = 1
\]
under $\delta_\kappa < \infty$. \[(2.4)\] yields
\[
\lim_{s \downarrow 0} a_\kappa(s) = +\infty.
\]
If $\delta_\kappa < \infty$, from (2.5), $\delta_\kappa$ is the explosion time of $a_\kappa$ in the sense that
\[
(2.7) \quad \lim_{s \uparrow \delta_\kappa} a_\kappa(s) = -\infty.
\]
Actually, $a_\kappa(s) = s_\kappa(s)/s_\kappa(s)$, where $s_\kappa$ is the unique solution of Jacobi equation $s''_\kappa(s) + \kappa(s)s_\kappa(s) = 0$ with $s_\kappa(0) = 0$, $s'_\kappa(0) = 1$, and $\delta_\kappa = \inf\{ s > 0 \mid s_\kappa(s) = 0 \}$. We write $a_\kappa(s) = \cot_\kappa(s)$. Moreover, $\]0, \delta_\kappa[ \ni s \mapsto \cot_\kappa(s)$ is decreasing (resp. strictly decreasing) provided $\kappa(s)$ is non-negative (resp. positive) for all $s \in ]0, \delta_\kappa[$ in view of (2.3). If $\kappa$ is a real constant, then
$$a_\kappa(s) = \begin{cases} \sqrt{\kappa} \cot(\sqrt{\kappa}s) & \kappa > 0, \\ 1/s & \kappa = 0, \\ \sqrt{-\kappa} \coth(\sqrt{-\kappa}s) & \kappa < 0 \end{cases}$$
and $\delta_\kappa = \pi/\sqrt{\kappa^+} \leq +\infty$. Fix $m \in ]-\infty, 1[$ and set $m_\kappa(s) := (n-m)\cot_\kappa(s)$. Then (2.3) is equivalent to
\[
- \frac{dm_\kappa}{ds}(s) = (n-m)\kappa(s) + \frac{m_\kappa(s)^2}{n-m},
\]
and (2.4) (resp. (2.5)) is equivalent to $\lim_{s \downarrow 0} s_\kappa(s) = n-m$ (resp. $\lim_{s \uparrow \delta_\kappa} (s - \delta_\kappa) m_\kappa(s) = n - m$ under $\delta_\kappa < \infty$). In view of the uniqueness of the solution to (2.3) with (2.6), we have the scaling property $a_{\alpha\kappa}(s) = \frac{1}{\alpha} a_{\alpha^2\kappa}(s/\alpha)$ for $\alpha > 0$. Here $\kappa_\alpha(s) := \kappa(s/\alpha)$. In particular, $a_\kappa(s) = \frac{1}{\alpha} a_{\alpha^2\kappa}(s/\alpha)$ for $\alpha > 0$ provided $\kappa$ is a constant.

Our first result is the following Laplacian comparison along unit speed geodesic on weighted complete Riemannian manifolds $(M, g, V)$ under the lower bound of modified $m$-Bakry-Émery Ricci tensor for $m \leq 1$.

**Theorem 2.5 (Laplacian Comparison Theorem).** Suppose that $(M, g)$ is an $n$-dimensional complete smooth Riemannian manifold and $V$ is a $C^1$-vector field. Fix $p \in M$. Take $R \in ]0, +\infty[. \ Let \phi_V$ be the function defined in (2.1). Suppose that
\[
\text{Ric}_{m,n}(\Delta_V)x(\nabla r_p, \nabla r_p) \geq (n-m)\kappa(s_p(x))e^{-\frac{4\phi_V(x)}{n-m}}C_p^2
\]
holds under $r_p(x) < R$ with $x \in (\text{Cut}(p) \cup \{p\})^c$. Then
\[
(\Delta_V r_p)(x) \leq (n-m)\cot_\kappa(s_p(x))e^{-\frac{2\phi_V(x)}{n-m}}C_p.
\]

**Corollary 2.6.** Suppose that $(M, g)$ is an $n$-dimensional complete smooth Riemannian manifold and $V$ is a $C^1$-vector field. Fix $p \in M$ and assume $\delta_\kappa < \infty$. Then
$$\lim_{s \uparrow \delta_\kappa} \Delta_V r_p(x) = -\infty.$$
Remark 2.7. The sufficient condition \((2.9)\) under \(r_p(x) < R\) with \(x \in (\text{Cut}(p) \cup \{p\})^c\) for our Laplacian comparison theorem is weaker than the condition:

\[
(2.11) \quad \text{Ric}_{m,n}(\Delta_V)(x) \geq (n - m) \kappa(s_p(x)) e^{-\frac{4\phi_V(x)}{n-m}} C_p^2 g_x
\]

under \(r_p(x) < R\), because \(\nabla r_p(x)\) is defined only for \(x \notin \text{Cut}(p) \cup \{p\}\). In particular, CD\((K, m)\)-condition for \(K(x) = (n - m) \kappa(s_p(x)) e^{-\frac{4\phi_V(x)}{n-m}} C_p^2\) always implies that \((2.9)\) holds for all \(x \in (\text{Cut}(p) \cup \{p\})^c\).

Remark 2.8. The inequality \((2.10)\) is meaningful at \(p\), because \(m_\kappa(0+) = +\infty\) and \(\Delta_V r_p(p) = \Delta r_p(p) = +\infty\) in view of the classical Laplacian comparison theorem for \(\Delta\) under local upper sectional curvature bound (see \([14, \text{Theorem 3.4.2}]\)). Moreover, the following inequality

\[
(2.12) \quad r_p(x)(\Delta_V r_p)(x) \leq (n - m) r_p(x) \cot_\kappa(s_p(x)) e^{-\frac{2\phi_V(x)}{n-m}} C_p
\]

is also meaningful at \(p\). Indeed, the right hand side of \((2.12)\) has the value \(n - m\) at \(x = p\) by Remark 2.7 and the left hand side has the value \(n - 1\) at \(x = p\) by the classical Laplacian comparison theorem for \(\Delta\) as noted above.

Remark 2.9. Theorem 2.5 generalizes \([35, \text{Theorem 4.4}]\).

2.2. Geometric consequences.

Theorem 2.10 (Weighted Myers’ Theorem). Let \((M, g)\) be an \(n\)-dimensional complete Riemannian manifold and a \(C^1\)-vector field \(V\). Fix \(p \in M\). Assume that \((2.9)\) holds for all \(x \in (\text{Cut}(p) \cup \{p\})^c\) and \(\delta_\kappa < \infty\). Then \(s(p,q) \leq \delta_\kappa\) for all \(q \in M\).

Corollary 2.11. Let \((M, g)\) be an \(n\)-dimensional complete Riemannian manifold and a \(C^1\)-vector field \(V\). Fix \(p \in M\) and \(\delta_\kappa < \infty\). Assume that \((2.9)\) holds for all \(x \in (\text{Cut}(p) \cup \{p\})^c\) and \((M, g, V)\) is \((V, m)\)-complete at \(p\). Then \(M\) is compact.

Remark 2.12. (1) Theorem 2.10 (resp. Corollary 2.11) generalizes \([35, \text{Theorem 2.2}]\) (resp. \([35, \text{Corollary 2.3}]\)).

(2) Since \(V_\gamma \leq 0\) for any unit speed geodesic \(\gamma\) with \(\gamma_0 = p\) implies the \((V, m)\)-completeness at \(p\), Corollary 2.11 implies the compactness of \(M\) provided \(\delta_\kappa < \infty\), \((2.9)\) holds for \(x \notin \text{Cut}(p) \cup \{p\}\) and \(V_\gamma \leq 0\) any unit speed geodesic \(\gamma\) with \(\gamma_0 = p\).

Based on Theorems 2.5 and 2.10, we can deduce several geometric fruitful results. Next we will give two versions of the Bishop-Gromov type volume comparison. The first one is for \(\mu_V(A) = \int_A e^{-\phi_V(x)} m(dx)\) of metric annuli \(A(p, r_0, r_1) := \{x \in M \mid r_0 \leq r_p(x) \leq r_1\}\).
The comparison in this case will be in terms of the quantities

\( \overline{V}_p(\kappa, r_0, r_1) := \int_{r_0}^{r_1} \int_{S^{n-1}} s_{\kappa}^{n-m} \left( \sup_{\eta} s_p(r, \eta) \right) \, dr \, d\theta, \quad \overline{\kappa}_p(\kappa, r_1) := \overline{V}_p(\kappa, 0, r_1), \)

\( \underline{V}_p(\kappa, r_0, r_1) := \int_{r_0}^{r_1} \int_{S^{n-1}} s_{\kappa}^{n-m} \left( \inf_{\eta} s_p(r, \eta) \right) \, dr \, d\theta, \quad \underline{\kappa}_p(\kappa, r_1) := \underline{V}_p(\kappa, 0, r_1), \)

under \( s_p(r_1, \theta) \leq \delta_\kappa \) for all \( \theta \in S^{n-1} \). Here

\[ s_p(r, \theta) := C_p \int_0^r e^{-\frac{2V_\kappa(t)}{n-m}} \, dt \]

with \( \theta = \tilde{\gamma}_0 \), and \( \overline{\phi}_V(r) \) and \( \underline{\phi}_V(r) \) are the functions defined in Remark 2.4. If \( \phi_V \) is rotationally symmetric around \( p \), i.e., if there exists a \( C^2 \)-function \( \Phi_V \) on \([0, +\infty[ \) such that \( \phi_V(x) = \Phi_V(r_p(x)) \), then \( s_p(r, \theta) \) is independent of \( \theta \in S^{n-1} \). The second one is for \( \nu_V(A) := \int_A e^{-\frac{2\phi_V(x)}{n-m}} \mu_V(dx) = \int_0^{\frac{2\phi_V(x)}{n-m}} e^{-\frac{n-m+2}{n-m} \phi_V(x)} m(dx) \) of the sets \( C(p, s_0, s_1) := \{ x \in M \mid s_0 \leq s_p(x) \leq s_1 \} \) and \( C_s(p) := C(p, 0, s) \). The set \( C(p, s_0, s_1) \) also depends on \( s_p \) and is quite different from annuli. The comparison in this case will be in terms of the quantities

\[ v(\kappa, s_0, s_1) := \int_{s_0}^{s_1} \int_{S^{n-1}} s_{\kappa}^{n-m}(s) \, ds \, d\theta \quad \text{and} \quad v(\kappa, s_1) := v(\kappa, 0, s_1) \]

under \( s_1 \leq \delta_\kappa \). When \( m \in ]-\infty, 1[ \) is an integer and \( \kappa \) is a constant, (2.16) is the volume of annuli in the simply connected space form of constant curvature \( \kappa \) and dimension \( n - m + 1 \).

**Theorem 2.13 (Bishop-Gromov Volume Comparison).** Fix \( p \in M \) and \( R \in ]0, +\infty[ \). Suppose that \((M, g)\) is an \( n \)-dimensional complete smooth Riemannian manifold and a \( C^1 \)-vector field \( V \). Let \( \kappa : [0, +\infty[ \to \mathbb{R} \) be a continuous function. Assume that (2.9) holds for \( r_p(x) < R \) with \( x \in ( \text{Cut}(p) \cup \{p\})^c \). Then we have the following:

1. Suppose that \( 0 \leq r_0 < r_a \leq r_1 \) and \( 0 \leq r_0 \leq r_b < r_1 \). Then

\[ \frac{\mu_V(A(p, r_b, r_1))}{\mu_V(A(p, r_0, r_a))} \leq \frac{\overline{V}_p(\kappa, r_b, r_1)}{\underline{V}_p(\kappa, r_0, r_a)} \]

holds for \( r_1 < R \). Assume further that \( \phi_V \) is rotationally symmetric around \( p \). Then

\[ \frac{\mu_V(A(p, r_b, r_1))}{\mu_V(A(p, r_0, r_a))} \leq \frac{\nu_p(\kappa, r_b, r_1)}{\nu_p(\kappa, r_0, r_a)} \]

holds for \( r_1 < R \), in particular, the function

\[ ]0, R[ \ni r \mapsto \frac{\mu_V(B_r(p))}{\nu_p(\kappa, r)} \]

is non-increasing.
(2) Suppose that \( 0 \leq s_0 < s_a \leq s_1 \) and \( 0 \leq s_0 \leq s_b < s_1 \). Then

\[
\frac{\nu_V(C(p, s_b, s_1))}{\nu_V(C(p, s_0, s_a))} \leq \frac{v(\kappa, s_b, s_1)}{v(\kappa, s_0, s_a)}
\]

holds for \( s_1 < S \). In particular, the function

\[
\int_0^S \Theta_V(s) \, ds \geq -R \quad \text{is non-increasing. Here } S = \inf_{\theta \in \mathbb{S}^{n-1}} s_p(R, \theta).
\]

**Remark 2.14.** (2.19) (resp. (2.21)) may not be bounded as \( r \to 0 \) (resp. \( s \to 0 \)) unless \( m = 1 \). Note that the Bishop type inequality holds for \( m = 1 \) (see [35, Corollary 4.6]).

**Corollary 2.15.** Fix \( p \in M \) and \( R \in ]0, +\infty[ \). Suppose that \((M, g)\) is an \( n \)-dimensional complete smooth Riemannian manifold and \( V \) is a \( C^1 \)-vector field. Assume that

\[
\operatorname{Ric}_{m,n}(\Delta_V)_{x}(\nabla r_p, \nabla r_p) \geq 0 \quad \text{for} \quad r_p(x) < R \quad \text{with} \quad x \notin \operatorname{Cut}(p) \cup \{p\}.
\]

Then

\[
\operatorname{Ric}_{m,n}(\Delta_V)(\hat{\gamma}_t, \hat{\gamma}_t) \leq e^{2\nu_V(r_1) - \nu_V(r_2)} \left( \frac{r_2}{r_1} \right)^{n-m+1} \quad \text{for all} \quad 0 < r_1 < r_2 < R
\]

holds.

**Theorem 2.16 (Ambrose-Myers’ Theorem).** Let \((M, g)\) be an \( n \)-dimensional complete Riemannian manifold and \( V \) a \( C^1 \)-vector field. Fix \( p \in M \). Assume that \((M, g, V)\) is \((V, m)\)-complete at \( p \). Suppose that for every unit speed (local minimizing) geodesic \( \gamma \) with \( \gamma_0 = p \), we have

\[
\int_0^{\infty} e^{-\frac{2\nu_V(t)}{n-m}} \operatorname{Ric}_{m,n}(\Delta_V)(\hat{\gamma}_t, \hat{\gamma}_t) dt = +\infty.
\]

Then \( M \) is compact.

**Corollary 2.17.** Let \((M, g)\) be an \( n \)-dimensional complete Riemannian manifold and \( V \) a \( C^1 \)-vector field. Fix \( p \in M \). Assume \( \operatorname{Ric}_{m,n}(\Delta_V) \geq 0 \) on \( M \). Suppose that there exists a non-negative measurable function \( f \) on \([0, +\infty[\) satisfying \( \int_0^{\infty} f(s) ds < +\infty \) and \( \langle V, \nabla r_p \rangle \geq -f(r_p) \), and for every unit speed (local minimizing) geodesic \( \gamma \) with \( \gamma_0 = p \), we have

\[
\int_0^{\infty} \operatorname{Ric}_{m,n}(\Delta_V)(\hat{\gamma}_t, \hat{\gamma}_t) dt = +\infty.
\]

Then \( M \) is compact.

**Corollary 2.18.** Let \((M, g)\) be an \( n \)-dimensional complete Riemannian manifold and \( V \) a \( C^1 \)-vector field. Fix \( p \in M \) and a constant \( \kappa > 0 \). Assume that \((M, g, V)\) is \((V, m)\)-complete at \( p \). Suppose that for every unit speed (local minimizing) geodesic \( \gamma \) with \( \gamma_0 = p \),
we have

\[ \text{Ric}_{m,n}(\Delta_V)(\dot{\gamma}_t, \dot{\gamma}_t) \geq (n - m)k \exp \left( -\frac{4V(1)}{n-m} \right) C_p^2. \]

Then \( M \) is compact.

**Remark 2.19.** (1) Theorem 2.16 is a version of Ambrose’s Theorem ([1]). Here Ambrose’s Theorem states that if for any (local minimizing) geodesic \( \gamma \) emanating from a point \( p \in M \),

\[ \int_0^\infty \text{Ric}(\dot{\gamma}_t, \dot{\gamma}_t) dt = +\infty, \]

then \( M \) is compact. Cavalcante-Oliveira-Santos [10] also proved the following different version of Ambrose’s Theorem (see [10] Theorem 2.1): Suppose that every (local minimizing) geodesic \( \gamma \) emanating from \( p \) satisfies

\[ \int_0^\infty \text{Ric}_{m,n}(\Delta_{\nabla \phi})(\dot{\gamma}_t, \dot{\gamma}_t) dt = +\infty \]

under \( m > n \) for \( \phi \in C^2(M) \). Then \( M \) is compact. Tadano [31] Theorem 14] extends [10] Theorem 2.1 for \( \Delta_V \) with modified \( m \)-Bakry-Émery Ricci tensor \( \text{Ric}_{m,n}(\Delta_V) \) under \( m > n \). Our Theorem 2.16 is different from the above mentioned results. Tadano [32] Theorem 25] also proves a version of Ambrose’s Theorem for \( \Delta_V \) with modified 1-Bakry-Émery Ricci tensor \( \text{Ric}_{1,n}(\Delta_V) \) under the condition \( \text{Ric}_{1,n}(\Delta_V) > 0 \) and \( |V| \leq k \exp(-\ell r) \) for some \( k \geq 0, \ell > 0 \). So our condition in Corollary 2.17 is milder than one in [32] Theorem 25].

In the following theorem and its corollary, we assume \( V = \nabla \phi \) for some \( \phi \in C^2(M) \) and set \( C_p = \exp \left( -\frac{2\phi(p)}{n-m} \right) \) for the definition of \( s_p(x) \) with \( p \) being an arbitrary point.

As noted before, \( s(p, q) \) is symmetric for any \( p, q \in M \). Let \( h = \exp \left( -\frac{4\phi}{n-m} \right) g \) be the conformal change of the metric \( g \). Then \( s(p, q) \) is the smallest length in the \( h \) metric of a minimal geodesic between \( p \) and \( q \) in the \( g \) metric. As such, \( d^h(p, q) \leq s(p, q) \) for any \( q \in M \). So Theorem 2.16 tells us that the diameter of the metric \( h \) is less than or equal to \( \delta_\kappa \). For this conformal diameter estimate we also obtain the following rigidity characterization.

**Theorem 2.20 (Cheng’s Maximal Diameter Theorem).** Suppose that \( (M, g), n > 1 \), is a complete Riemannian manifold and \( \phi \in C^2(M) \). Fix \( p, q \in M \). Assume that \( \delta_\kappa < \infty \), \( \kappa \) is positive on \( [0, \delta_\kappa], \kappa(s) = \kappa(\delta_\kappa - s) \) for all \( s \in [0, \delta_\kappa], \) and (2.9) holds for all \( x \in (\text{Cut}(p) \cup \{ p \})^c \). We further assume that (2.9) by replacing \( p \) with \( q \) holds for all \( x \in (\text{Cut}(q) \cup \{ q \})^c \). If \( d^h(p, q) = \delta_\kappa \), then \( m = 1 \), \( \phi \) is rotationally symmetric around \( p \), i.e., \( \phi \) is a function depending only on radial \( r \), and \( g \) is a warped product metric of the form

\[ g = dr^2 + e^{2\phi(r)+2\phi(0)} s^2_\kappa(s(r)) g_{\mathbb{S}^{n-1}}, \quad 0 \leq r \leq d(p, q), \]

where \( s(r) = \int_0^r e^{-\frac{2\phi(t)}{n-1}} dt \) and \( s(d(p, q)) = \delta_\kappa \).
Corollary 2.21. Suppose that \((M, g), n > 1\), is a complete Riemannian manifold and 
\(\phi \in C^2(M)\). Fix \(p, q \in M\). Assume that \(\kappa\) is a positive constant and \((2.9)\) holds for all 
\(x \in (\text{Cut}(p) \cup \{p\})^c\). We further assume that \((2.9)\) by replacing \(p\) with \(q\) holds for all 
\(x \in (\text{Cut}(q) \cup \{q\})^c\). If \(d^h(p, q) = \pi/\sqrt{\kappa}\), then \(m = 1\), \(\phi\) is rotationally symmetric around 
\(p\), i.e., \(\phi\) is a function depending only on radial \(r\), and \(g\) is a warped product metric of the form

\[
g = dr^2 + e^{2\phi(r) + 2\phi(0)} \cdot \frac{\sin^2(\sqrt{\kappa}(s(r)))}{\kappa} g_{\mathbb{R}^{n-1}}, \quad 0 \leq r \leq d(p, q),
\]

where \(s(r) = \int_0^r e^{-\frac{2\phi(t)}{n-1}} dt\) and \(s(d(p, q)) = \pi/\sqrt{\kappa}\).

Theorem 2.22 (Cheeger-Gromoll Splitting Theorem). Let \((M, g)\) be an \(n\)-dimensional 
non-compact complete Riemannian manifold and \(V\) a \(C^1\)-vector field. Suppose that \((M, g, V)\) is \((V, m)\)-complete and \(M\) contains a line. Then under \(\text{CD}(0, m)\)-condition with \(m < 1\), 
\(M\) is isometric to \(\mathbb{R} \times N\) and \(V\) depends only on \(N\).

Corollary 2.23. Let \((M, g)\) be an \(n\)-dimensional non-compact complete Riemannian 
manifold and \(C^1\)-vector field \(V\). Suppose that \(V_\gamma \leq 0\) for any unit speed geodesic \(\gamma\) and \(M\) contains a line. Then under \(\text{CD}(0, m)\)-condition with \(m < 1\), we have that \(M\) is isometric to \(\mathbb{R} \times N\) and \(V\) depends only on \(N\).

Proof. If \(V_\gamma \leq 0\) for any unit speed geodesic \(\gamma\), then \((M, g, V)\) is \((V, m)\)-complete for all \(m \leq 1\). So the assertion easily follows Theorem 2.22.

Remark 2.24. Theorem 2.22 partially extends [31] Corollary 6.7] for a restricted case, 
where \(\text{CD}(0, m)\)-condition for \(m < 1\) and \((V, 1)\)-completeness of \((M, g, V)\) are assumed for 
the isometric splitting \(M = \mathbb{R} \times N\). Note that the \((V, m)\)-completeness does not necessarily 
mean the \((V, 1)\)-completeness, and it is weaker than \((V, 1)\)-completeness if \(V_\gamma \geq 0\) for any 
unit speed geodesic \(\gamma\).

3. Proof of Theorem 2.5

Recall the \(V\)-Laplacian \(\Delta_V u := \Delta u - \langle V, \nabla u \rangle\). Letting \(\lambda(r, \theta) = C_p^{-1} e^{\frac{2V_\gamma(r)}{n-m}} \Delta_V r_p(r, \theta)\), 
we find that \(\lambda\) satisfies the Riccati differential inequality in terms of the parameter \(s\).

Lemma 3.1. Let \(\gamma\) be a unit speed minimal geodesic with \(\gamma_0 = p\) and \(\gamma_0 = \theta\). Let \(s\) be 
the parameter \(ds = C_p e^{-\frac{2V_\gamma(r)}{n-m}} dr\). Then

\[
\frac{d\lambda}{ds} \leq -\frac{\lambda^2}{n-m} - C_p^{-1} e^{\frac{4V_\gamma(r)}{n-m}} \text{Ric}_{m,n}(\Delta_V)(\dot{\gamma}_r, \dot{\gamma}_r)
\]

in particular,

\[
\frac{d\lambda}{dr} \leq -C_p e^{-\frac{2V_\gamma(r)}{n-m}} \frac{\lambda^2}{n-m} - C_p^{-1} e^{\frac{2V_\gamma(r)}{n-m}} \text{Ric}_{m,n}(\Delta_V)(\dot{\gamma}_r, \dot{\gamma}_r)
\]

holds for \(x = (r, \theta) \notin \text{Cut}(p) \cup \{p\}\). Moreover, if equality is achieved at a point, then \(m = 1\) 
and at that point \(\nabla_{\nabla r_p}\) has at most one non-zero eigenvalue which is of multiplicity \(n-1\).
We modify the proof of the Laplacian comparison theorem on weighted complete Riemannian manifolds with the CD$(\kappa, 1)$-condition by Wylie and Yeroshkin [35].

The usual Bochner-Weitzenböck formula for functions says that for any $u \in C^3(M)$,

$$\frac{1}{2} \Delta |\nabla u|^2 = |\nabla^2 u|^2 + \text{Ric}(\nabla u, \nabla u) + \langle \nabla \Delta u, \nabla u \rangle.$$ 

The Bochner-Weitzenböck formula for the $V$-Laplacian and the $m$-Bakry-Émery Ricci curvature is given by

$$\frac{1}{2} \Delta_V |\nabla u|^2 = |\nabla^2 u|^2 + \text{Ric}_{\infty, n}(\Delta_V) (\nabla u, \nabla u) - \frac{V^* \otimes V^*}{n - m} (\nabla u, \nabla u) + \langle \nabla \Delta_V u, \nabla u \rangle.$$ 

Consider this equation with $u = r_p$ at an interior point of a minimizing geodesic (so that $r_p$ is smooth in a neighborhood). Then $|\nabla r_p| = 1$ in this neighborhood, so that the left hand side is zero. Now we claim $\nabla \nabla r_p = 0$, i.e., $\nabla r_p$ is a null vector for $\nabla \nabla r_p$. For this, it suffices to show that for any smooth vector field $X$ on $M \setminus \{p\}$

$$\langle \nabla \nabla r_p, X \rangle = 0.$$

This is true if $X$ is parallel to $\nabla r_p$, because for $f \in C^\infty(M \setminus \{p\})$

$$\langle \nabla \nabla r_p, f \nabla r_p \rangle = f \langle \nabla \nabla r_p, \nabla r_p \rangle = f \frac{1}{2} (\nabla r_p |\nabla r_p|^2) = 0.$$

Moreover, (3.3) holds if $X$ is vertical to $\nabla r_p$, because

$$\langle \nabla \nabla r_p, X \rangle = \frac{1}{2} (\nabla r_p |\nabla r_p|^2) = \frac{1}{2} (\nabla r_p) 0 = 0.$$

Hence $\nabla \nabla r_p$ has at most $n - 1$ non-zero eigenvalues and by the Cauchy-Schwarz inequality, it holds on $(\text{Cut}(p) \cup \{p\})^c$ that (see [35])

$$|\text{Hess } r_p|^2 = ||\nabla \nabla r_p||^2 \geq \frac{(\Delta r_p)^2}{n - 1}.$$

Now $m \leq 1$. Hence

$$0 \geq \frac{(\Delta r_p)^2}{n - m} + \text{Ric}_{m, n}(\Delta_V) (\nabla r_p, \nabla r_p) - \frac{1}{n - m} |\langle V, \nabla r_p \rangle|^2 + \langle \nabla \Delta_V r_p, \nabla r_p \rangle.$$

This gives us the following inequality along $\gamma$,

$$\frac{d}{dr} (\Delta_V r_p)(r, \theta) \leq - \frac{(\Delta r_p(r, \theta))^2}{n - m} - \text{Ric}_{m, n}(\Delta_V) (\gamma_r, \gamma_r) + \frac{1}{n - m} |\langle V, \nabla r_p \rangle(r, \theta)|^2.$$
From this, we have
\[
\frac{d\lambda}{ds} = C_p^{-1} e^{\frac{2V_p(r)}{n-m}} \frac{d\lambda}{dr} \left( \frac{d}{dr} e^{\frac{2V_p(r)}{n-m}} \Delta V_{p} r_p(r, \theta) + e^{\frac{2V_p(r)}{n-m}} \frac{d\lambda}{dr} \Delta V_{p} r_p(r, \theta) \right)
\]
\[
= C_p^{-2} e^{\frac{2V_p(r)}{n-m}} \left\{ \left( \frac{d}{dr} e^{\frac{2V_p(r)}{n-m}} \Delta V_{p} r_p(r, \theta) + e^{\frac{2V_p(r)}{n-m}} \frac{d\lambda}{dr} \Delta V_{p} r_p(r, \theta) \right) \right\}
\]
\[
= C_p^{-2} e^{\frac{2V_p(r)}{n-m}} \left\{ \frac{2}{n-m} \frac{\partial V_{\gamma}(r)}{\partial r} \cdot \Delta V_{p} r_p(r, \theta) + \frac{d}{dr} \Delta V_{p} r_p(r, \theta) \right\}
\]
\[
\leq C_p^{-2} e^{\frac{4V_p(r)}{n-m}} \left\{ \frac{2}{n-m} \frac{\partial V_{\gamma}(r)}{\partial r} \cdot \Delta V_{p} r_p(r, \theta) - (\Delta r_p(r, \theta))^2 + |\nabla V_{p} r_p(r, \theta)|^2 \right\}
\]
\[
- C_p^{-2} e^{\frac{4V_p(r)}{n-m}} \text{Ric}_{m,n}(\Delta V_{\gamma}) (\dot{\gamma}_r, \dot{\gamma}_r)
\]
\[
= - \frac{C_p^{-2}}{n-m} e^{\frac{4V_p(r)}{n-m}} (\Delta V_{p} r_p(r, \theta))^2 - C_p^{-2} e^{\frac{4V_p(r)}{n-m}} \text{Ric}_{m,n}(\Delta V_{\gamma}) (\dot{\gamma}_r, \dot{\gamma}_r)
\]
\[
= - \frac{1}{n-m} \left( C_p^{-1} e^{\frac{2V_p(r)}{n-m}} \Delta V_{p} r_p(r, \theta) \right)^2 - C_p^{-2} e^{\frac{4V_p(r)}{n-m}} \text{Ric}_{m,n}(\Delta V_{\gamma}) (\dot{\gamma}_r, \dot{\gamma}_r)
\]
\[
= - \frac{\lambda^2}{n-m} - C_p^{-2} e^{\frac{4V_p(r)}{n-m}} \text{Ric}_{m,n}(\Delta V_{\gamma}) (\dot{\gamma}_r, \dot{\gamma}_r).
\]
Here we use (3.6) at the inequality above and use \( \Delta V_{p} r_p = \Delta r_p - \langle V, \nabla V_{p} r_p \rangle \) in the next equality. If the equality holds for (3.1) at some \( x = (r_0, \theta) \notin \text{Cut}(p) \cup \{p\} \), then the equality for (3.6) equivalently the equality for (3.5) at \( x \notin \text{Cut}(p) \cup \{p\} \) holds, i.e.,
\[
0 = \frac{(\Delta r_p(r, \theta))^2}{n-m} + \text{Ric}_{m,n}(\Delta V_{\gamma}) (\nabla r_p, \nabla r_p) - \frac{1}{n-m} |\langle V, \nabla r_p \rangle|^2 + |\nabla \Delta V_{p} r_p, \nabla r_p|
\]
\[
\geq \frac{(\Delta r_p(r, \theta))^2}{n-1} + \text{Ric}_{m,n}(\Delta V_{\gamma}) (\nabla r_p, \nabla r_p) - \frac{1}{n-m} |\langle V, \nabla r_p \rangle|^2 + |\nabla \Delta V_{p} r_p, \nabla r_p|
\]
holds at \( x \notin \text{Cut}(p) \cup \{p\} \). This and \( m \leq 1 \) yield
\[
\frac{m - 1}{(n-m)(n-1)} (\Delta r_p)^2(x) = 0.
\]
Thus \( m = 1 \) or \( \Delta r_p(x) = 0 \). Since \( M \) has an upper bound \( k_\varepsilon > 0 \) of the sectional curvature on some \( B_\varepsilon(p) \subset \text{Cut}(p) \), the usual Laplacian comparison theorem tells us that \( \Delta r_p(x) \geq (n-1)\sqrt{k_\varepsilon} \cot(\sqrt{k_\varepsilon} r_p(x)) > 0 \) for \( 0 < r_p(x) < \varepsilon \). Therefore we obtain \( m = 1 \), in particular, the equality for (3.4) holds at \( x \). This implies that \( \nabla r_p \) at \( x \) has at most one non-zero eigenvalue of multiplicity \( n - 1 \). \( \square \)

Let \( \kappa \) be a continuous function on \([0, +\infty[\) with respect to the parameter \( s \). Assuming the curvature bound \( \text{Ric}_{m,n}(\Delta V_{\gamma})(\nabla r_p, \nabla r_p) \geq (n-m)\kappa(s_p(x)) e^{-\frac{4V_p(r(x))}{n-m}} C_p^2 \) for \( s_p(x) < S \) with \( x \notin \text{Cut}(p) \cup \{p\} \), we see \( \text{Ric}_{m,n}(\Delta V_{\gamma})(\dot{\gamma}_r, \dot{\gamma}_r) \geq (n-m)\kappa(s) e^{-\frac{4V_p(r(x))}{n-m}} C_p^2 \) for \( s = \)
\( s(r, \theta) < S \) with \( 0 < r < d(p, \text{Cut}(p)) \). From (3.1) we have the usual Riccati inequality

\[
(3.7) \quad \frac{d\lambda}{ds}(s) \geq (n - m)\kappa(s) + \frac{\lambda(s)^2}{n - m} \quad \text{for} \quad s \in ]0, S[ 
\]

with the caveat that it is in terms of the parameter \( s \) instead of \( r \). This gives us the following comparison estimate.

**Lemma 3.2.** Suppose that \((M, g)\) be an \( n \)-dimensional complete Riemannian manifold and \( V \) a \( C^1 \)-vector field. Fix \( R \in ]0, +\infty[ \) and \( x, p \in M \). Assume that (2.6) holds for \( r_p(x) < R \) with \( x \notin (\text{Cut}(p) \cup \{p\}) \). Let \( \gamma, s, \) and \( \lambda \) be defined to be as in Lemma 3.1. Then

\[
(3.8) \quad \lambda(r, \theta) \leq m_\kappa(s) 
\]

holds for \( r < R \), \( s < \delta_\kappa \) and \( x = (r, \theta) \notin \text{Cut}(p) \cup \{p\} \). Here

\[
s = s_p(r) = C_p \int_0^r \exp \left( -\frac{2\phi_V(\gamma_t)}{n - m} \right) dt. 
\]

Suppose further that the equality in (3.8) holds for some \( r_0 < R \) with \( s_0 := s(r_0) < \delta_\kappa \). We choose an orthonormal basis \( \{e_i\}_{i=1}^n \) of \( T_pM \) with \( e_n = \dot{\gamma}_0 \). Let \( \{Y_i\}_{i=1}^{n-1} \) be the Jacobi fields along \( \gamma \) with \( Y_i(0) = 0 \) and \( Y_i'(0) = e_i \). Then we have \( m = 1 \), and at \( x = (r, \theta) \) with \( r \leq r_0 \), \( \nabla y_{r_p} \) has at most one non-zero eigenvalue which is of multiplicity \( n - 1 \), and for all \( r \in ]0, r_0[ \) we have

\[
(3.9) \quad \text{Ric}_{1,n}(\Delta_V)(\dot{\gamma}_r, \dot{\gamma}_r) = (n - 1)\kappa(s_p(\gamma_r))e^{-\frac{2\phi_V(r)}{n - m}}C_p^2. 
\]

Moreover, for all \( i \) we have \( Y_i(r) = C_p^{-1}F_\kappa(r)E_i(r) \) for \( r \in [0, r_0] \), where

\[
(3.10) \quad F_\kappa(r) := \exp \left( \frac{V_\gamma(r)}{n - 1} \right) s_\kappa(s_p(\gamma_r)), 
\]

and \( \{E_i(r)\}_{i=1}^{n-1} \) are the parallel vector fields with \( E_i(0) = e_i \). Consequently,

\[
(3.11) \quad g_{\gamma_r} = dr^2 + C_p^{-2}e^{\frac{2\phi_V(r)}{n - m}}s_\kappa^2(s_p(\gamma_r))g_{\mathbb{S}^{n-1}}. 
\]

Here \( g_{\mathbb{S}^{n-1}} \) is the standard metric on the sphere \( \mathbb{S}^{n-1} \).

**Proof.** Set \( S := s_p(R) \). Then \( r < R \) implies \( s < S \). Since \( \Delta r_p(r, \theta) \to +\infty \) as \( r \to 0 \), we see \( \lambda(r, \theta) \to +\infty \) as \( r \to 0 \) or \( s \to 0 \). We set \( \beta(s) := s_\kappa^2(s)(\lambda - m_\kappa(s)) \). Then, by
We can conclude that \((2.8)\) and \((3.7)\), for \(s < S\)

\[
\beta'(s) = 2s^2(\lambda - m_\kappa(s)) + s^2(s)\left(\frac{d\lambda}{ds} - m'_\kappa(s)\right)
\]

\[
= 2s^2(\lambda - m_\kappa(s)) + s^2(s)\left(\frac{d\lambda}{ds} + (n-m)\kappa(s) + \frac{m^2_\kappa(s)}{n-m}\right)
\]

\[
\leq \frac{s^2(s)}{n-m} (2m_\kappa(s)\lambda - 2m^2_\kappa(s)) + \frac{s^2(s)}{n-m} (m^2_\kappa(s) - \lambda^2)
\]

\[
= -\frac{s^2(s)}{n-m} (\lambda - m_\kappa(s))^2 \leq 0.
\]

We note here that \((3.7)\) is derived from \((2.9)\). If we show \(\beta(0) = 0\), then \(\beta(s) \leq \beta(0) = 0\). For this, it suffices to prove that \(s(\lambda - m_\kappa(s))\) is upper bounded as \(s \to 0\). We already know that \(\lim_{s \to 0} sm_\kappa(s) = n - m\) and the ratio \(s/r = s_p(r)/r\) converges to \(C_p\) as \(r \to 0\). So it suffices to prove \(\lim r \lambda(r, \theta) = C_p^{-1}(n - 1)\) as \(r \to 0\), equivalently \(\lim r \Delta r_p(r, \theta) = n - 1\), because \(\lim r \Delta r_p(r, \theta) = 0\). In view of the usual Laplacian comparison theorem for the Laplace-Bertrami operator \(\Delta\) under the upper (resp. lower) bound \(K_\kappa\) (resp. \(\kappa_\kappa\)) of sectional curvature on \(B_\varepsilon(p)\), we see \((n-1) \cot_\kappa(r) \leq \Delta r_p(r, \theta) \leq (n-1) \cot_\kappa(r)\) on \(B_\varepsilon(p)\). This implies the desired assertion. Next we assume that the equality in \((3.8)\) holds for some \(r_0 < R\), i.e., \(\lambda(r_0, \theta) = (n - m) \cot_\kappa(s_0)\) for \(r_0 < R\) with \(s_0 = s(r_0)\). This implies \(0 = \beta(s_0) \leq \beta(s) \leq \beta(0) = 0\), hence \(\lambda(r) = m_\kappa(s)\) for all \(s \in [0, s_0]\). From this,

\[
\frac{d\lambda}{ds}(s_0) = \frac{dm_\kappa}{ds}(s_0).
\]

In particular, we have at \(r_0\)

\[
\frac{d\lambda}{ds} \leq -\frac{\lambda(r)^2}{n-m} - C_p^{-2} e^{\frac{2\kappa_\kappa(s)}{n-m}} \text{Ric}_{m,n}(\Delta V)(\gamma_r, \gamma_r)\]

\[
\leq -\frac{\lambda(r)^2}{n-m} - (n-m)\kappa(s) = -\frac{m_\kappa(s)^2}{n-m} - (n-m)\kappa(s) = \frac{d\lambda}{ds}.
\]

Then the equality holds in \((3.1)\) at \(x = (r_0, \theta)\). So we have \(m = 1\) by Lemma \(3.3\). We can conclude \(\beta(s) \equiv 0\) on \([0, s_0]\) from \(\beta(0) = \beta(s_0) = 0\) and \(\beta'(s) \leq 0\) so that \(\lambda(r, \theta) = (n - 1) \cot_\kappa(s)\) for \(s \in [0, s_0]\). We then see the equality \((3.12)\) at any \(r \in [0, r_0]\), hence \((3.9)\) holds at any \(r \in [0, r_0]\).

Finally we prove \((3.11)\) at any \(r \in [0, r_0]\) under \(\lambda(r_0) = (n - m) \cot_\kappa(s_0)\). Hereafter, we assume \(r \in [0, r_0]\). By Lemma \(3.3\) at \(x = (r, \theta)\), \(\nabla_{\nabla r_p}\) has a non-zero eigenvalue \(A(r)\) which is of \(n - 1\) multiplicity. Then we have

\[
\lambda(r, \theta) = C_p^{-1} e^{\frac{2\kappa_\kappa(r)}{n-1}} (\Delta r_p(r, \theta) - \langle V, \nabla r_p \rangle (r, \theta))
\]

\[
= C_p^{-1} e^{\frac{2\kappa_\kappa(r)}{n-1}} ((n - 1) A(r) - \langle V, \nabla r_p \rangle (r, \theta)) = (n - 1) \cot_\kappa(s),
\]
where we use the equality \( \frac{\partial}{\partial r} \frac{\partial}{\partial \theta} \) at any \( r \in [0, r_0] \). So we have \( A(r) = C_p e^{-\frac{2\gamma_r(r)}{n-1}} \cot \kappa(s) + \frac{2\gamma_r}{n-1} \cot \kappa(s) + \frac{(V_n V_r + V_{rr})}{n-1} = \langle n - 1 \rangle \Delta r_p(\gamma_r) \). The radial curvature equation (see Theorem 2 in pp. 44) tells us that

\[(3.13) \quad R(E_i, \gamma_r) \gamma_r = -(A'(r) + A(r)^2) E_i.\]

Combining Bochner-Weitzenböck formula with (3.9), we have

\[(3.14) \quad A'(r) + A(r)^2 = \frac{V''_\gamma(r)}{n-1} + \left( \frac{V'_\gamma(r)}{n-1} \right)^2 - \kappa(s) \gamma_r) e^{-\frac{4\gamma_r(r)}{n-1}} C_p^2 = \frac{F''_p(r)}{F_p(r)}.\]

Since \( F_\kappa(0) = 0 \) and \( F'_\kappa(0) = C_p \), we obtain

\[Y_i(r) = C_p^{-1} F_\kappa(r) E_i(r) = C_p^{-1} e^{-\frac{2\gamma_r(r)}{n-1}} s_\kappa(s) \gamma_r) E_i(r).\]

This proves the desired conclusion. \( \square \)

**Corollary 3.3.** Let \((M, g)\) be an \( n \)-dimensional complete Riemannian manifold and \( V \) a \( C^1 \)-vector field. Fix \( p \in M \) and \( R \in [0, +\infty[ \). Assume that (2.9) holds for \( r_p(x) < R \) with \( x \notin \text{Cut}(p) \cup \{p\} \). Then \( s_p(x) < \delta_\kappa \).

**Proof.** We may assume \( \delta_\kappa < \infty \). Take \( x \in B_R(p) \) with \( x \notin \text{Cut}(p) \cup \{p\} \). Let \( x = (r, \theta) \) be the polar coordinate expression around \( p \) and set \( s := s_p(r) = C_p \int_0^r \exp \left( -\frac{2\gamma_t(t)}{n-m} \right) dt \)

and \( S = s_p(R) \), where \( \gamma \) is a unit speed geodesic with \( \gamma_0 = p \) and \( \dot{\gamma}_0 = \theta \). We see \( s_p(x) < S \). Assume \( S > \delta_\kappa \). Then there exists \( r_0 \in ]0, R[ \) such that \( \delta_\kappa = C_p \int_0^{r_0} \exp \left( -\frac{2\gamma_t(t)}{n-m} \right) dt \). By (3.8), \( \lambda(r, \theta) \leq (n - m) \cot \kappa(s) \) holds for \( s < \delta_\kappa \). Since \( r \uparrow r_0 \) is equivalent to \( s = s(r) \uparrow \delta_\kappa \), we have

\[\lambda(r_0, \theta) = \lim_{r \uparrow r_0} \lambda(r, \theta) \leq \lim_{r \uparrow r_0} (n - m) \cot \kappa(s(r)) = -\infty.\]

This contradicts the well-definedness of \( \lambda(r, \theta) = C_p^{-1} \left( e^{2\gamma_t(t)} \Delta V_p \right) (r, \theta) \) for \( r \in ]0, R[ \). Therefore \( S \leq \delta_\kappa \) under \( \delta_\kappa < \infty \) and we obtain the conclusion \( s_p(x) < S \leq \delta_\kappa \). \( \square \)

Let \( p \in M \) and let \((r, \theta), r > 0, \theta \in S^{n-1}\) be exponential polar coordinates (for the metric \( g \)) around \( p \) which are defined on a maximal star shaped domain in \( T_p M \) called the segment domain. Write the volume element \( d\mathbf{m} = J(r, \theta) dr \wedge d\theta \).

Let \( s_p(\cdot, \cdot) \) be the re-parametrized distance function defined above. Inside the segment domain, \( s_p \) has the simple formula

\[s_p(r, \theta) = C_p \int_0^r e^{-\frac{2\gamma_t(t, \theta)}{n-m}} dt.\]

Therefore, \( s_p \) is a smooth function in the segment domain with the property that \( \frac{\partial s_p}{\partial r} = C_p e^{-\frac{2\gamma_t(r, \theta)}{n-m}} \). We can then also take \((s, \theta)\) to be coordinates which are also valid for the entire segment theorem. We can not control the derivative of \( s \) in directions tangent to
the sphere, so the new \((s, \theta)\) coordinates are not orthogonal as in the case for geodesic polar coordinates. However, this is not the issue when we computing volumes as

\[
(3.15) \\
e^{-\frac{2\phi}{n-m}} d\mu_V = e^{-\frac{n-m+2}{n-m}\phi_V} J(r, \theta) dr \wedge d\theta = C_p^{-1} e^{-\frac{2\phi_V(r, \theta)}{n-m}} \frac{\partial}{\partial r}.
\]

Here \(d\mu_V = e^{-\phi} dm\). We denote the derivative in the radial direction in terms of this parameter by \(\frac{d}{ds}\). In geodesic polar coordinates \(\frac{d}{ds}\) has the expression \(\frac{d}{ds} = C_p^{-1} e^{-\frac{2\phi_V(r, \theta)}{n-m}} \frac{\partial}{\partial r}\). Note that it is not the same as \(\frac{\partial}{\partial s}\) in \((s, \theta)\) coordinates.

**Proof of Theorem 2.10**. The implication \((2.9) \implies (2.10)\) for \(R < \infty\) follows from Lemma 3.2 because \(r_p\) is smooth on \(M \setminus (\text{Cut}(p) \cup \{p\})\). The implication \((2.9) \implies (2.10)\) for \(R = +\infty\) follows from it.

\[
\square
\]

4. **Proofs of Theorem 2.10 and Corollary 2.11**

**Proof Theorem 2.10**. Suppose that there exist points \(p, q \in M\) such that \(s(p, q) > \delta_\kappa\). Since \(\text{Cut}(p)\) is closed and measure zero, we may assume \(q \notin \text{Cut}(p)\). By Lemma 3.2 along minimal geodesic from \(p\) to \(q\), \(\lambda(r, \theta) \leq m_\kappa(s)\). However, as \(s \to \delta_\kappa\), \(m_\kappa(s) \to -\infty\). This implies \(\Delta r_p(x) \to -\infty\) as \(s(p, x) \to \delta_\kappa\). This contradicts that \(r_p\) is smooth in a neighborhood of \(q\). The final assertion follows Remark 2.4.

**Proof of Corollary 2.11**. Suppose that \(\sup_{q \in M} d(p, q) = +\infty\). Then there exists a sequence \(\{q_i\}\) in \(M\) such that \(d(p, q_i) \to +\infty\) as \(i \to +\infty\). By Lemma 2.3 \(s(p, q_i) \to +\infty\) as \(k \to +\infty\), which contradicts \(\sup_{q \in M} s(p, q) \leq \delta_\kappa\). Therefore, \(\sup_{q \in M} d(p, q) < \infty\), hence \(M\) is compact.

\[
\square
\]

5. **Proof of Theorem 2.13**

Recall that for a Riemannian manifold \(\frac{d}{ds} \log J(r, \theta) = \Delta r_p(r, \theta)\), where \(\Delta r_p\) is the standard Laplacian acting on the distance function \(r_p\) from the point \(p\). (3.15) indicates we should consider the quantity

\[
(5.1) \\
\frac{d}{ds} \log(e^{-V_r(r)} J(r, \theta)) = C_p^{-1} e^{-\frac{2V_r(r)}{n-m}} (\Delta r_p(r, \theta) - \langle V_{r_p}, \dot{\gamma}_r \rangle) = C_p^{-1} e^{-\frac{2V_r(r)}{n-m}} \Delta V r_p(r, \theta).
\]

**Lemma 5.1 (Volume Element Comparison)**. Let \((M, g)\) be an \(n\)-dimensional complete Riemannian manifold and \(V\) a \(C^1\)-vector field. Fix \(p \in M\) and \(R \in ]0, +\infty]\). Assume that \((2.9)\) holds for \(r_p(x) < R\) with \(x \notin \text{Cut}(p) \cup \{p\}\). Let \(J\) be the volume element in geodesic polar coordinates around \(p \in M\) and set \(J_V(r, \theta) := e^{-V_r(r)} J(r, \theta)\). Then for \(r_0 < r_1 < R\) with \(r_1 < \text{cut}(\theta)\),

\[
(5.2) \\
\frac{J_V(r_1, \theta)}{J_V(r_0, \theta)} \leq \frac{g_k(s_p(r_1, \theta))^{n-m}}{g_k(s_p(r_0, \theta))^{n-m}}.
\]

Here \(\text{cut}(\theta)\) is the distance from \(p\) to the cut point along the geodesic with \(\gamma(0) = p\) and \(\dot{\gamma}(0) = \theta\).
Proof. Recall $s = s_p(r) = s_p(r, \theta) = C_p \int_0^r \exp \left( -\frac{2\gamma(s,t)}{n-m} \right) dt$ and $\gamma$ is the unit speed geodesic from $p$ with $\gamma_0 = \theta$. First note that the right hand side of (5.2) is meaningful for $r_0 < r_1 < R$. Indeed, if $R + \infty$, $s_p(r_0, \theta) < s_p(r_1, \theta) < \delta_\kappa$ by Corollary 3.3. If $R = +\infty$, we can take $R_0 \in]r_1, +\infty[$ so that (2.9) holds for $r_p(x) < R_0$, hence $s_p(r_0, \theta) < s_p(r_1, \theta) < \delta_\kappa$ by Corollary 3.3. From Lemma 3.2 and (5.1) we have that
\[
(5.3) \quad \frac{d}{ds} \log J_V(r, \theta) = C_p^{-1} e^{\frac{2\gamma(s,t)}{n-m}} \Delta_V r_p(r, \theta) \leq (n - m) \cot_\kappa(s) = \frac{d}{ds} \log(g_\kappa(s)^{n-m})
\]
for $r \in ]0, R \wedge \text{cut}(\theta)[$. Integrating (5.3) between any $s_0 < s_1$ with $s_i = s_p(r_i, \theta)$ and $r_i \in ]0, R \wedge \text{cut}(\theta)[ \ (i = 0, 1)$ gives
\[
\log \left( \frac{J_V(r_1, \theta)}{J_V(r_0, \theta)} \right) \leq \log \left( \frac{g_\kappa(s_1)^{n-m}}{g_\kappa(s_0)^{n-m}} \right) \Rightarrow \frac{J_V(r_1, \theta)}{J_V(r_0, \theta)} \leq \frac{g_\kappa(s_1)^{n-m}}{g_\kappa(s_0)^{n-m}}
\]
for all $r_0 < r_1 < R \wedge \text{cut}(\theta)$. Note that since $ds$ is an orientation preserving change of variables along the geodesic $\gamma$, the quantity is also non-increasing in terms of the parameter $r \in ]0, R \wedge \text{cut}(\theta)[$. \hfill $\square$

Proof of Theorem 2.13. By Lemma 5.1 for all $r_1, r_2 > 0$ with $r_1 < r_2 < R$ and $r_2 < \text{cut}(\theta)$
\[
\frac{J_V(r_2, \theta)}{J_V(r_1, \theta)} \leq \frac{g_\kappa^{n-m}(s_p(r_2, \theta))}{g_\kappa^{n-m}(s_p(r_1, \theta))} \leq \frac{g_\kappa^{n-m}(\sup_{\eta \in S^{n-1}} s_p(r_2, \eta))}{g_\kappa^{n-m}(\inf_{\eta \in S^{n-1}} s_p(r_1, \eta))}.
\]
So for $0 \leq r_a < r_b \leq r_d$, $0 \leq r_a < r_c < r_d$ and $r_d < R$, we have following inequality
\[
\frac{\int_{\text{cut}(\theta) \wedge r_d} \int_{\text{cut}(\theta) \wedge r_a} J_V(r_2, \theta) \text{d}r_2 \text{d}r_1}{\int_{S^{n-1}} \int_{\text{cut}(\theta) \wedge r_a} J_V(r_1, \theta) \text{d}r_1 \text{d}r_1} \leq \frac{\int_{\text{cut}(\theta) \wedge r_d} \int_{\text{cut}(\theta) \wedge r_a} g_\kappa^{n-m}(s_p(r_2, \theta)) \text{d}r_2 \text{d}r_1}{\int_{S^{n-1}} \int_{\text{cut}(\theta) \wedge r_a} g_\kappa^{n-m}(s_p(r_1, \theta)) \text{d}r_1 \text{d}r_1} \leq \frac{\int_{r_a}^{r_d} g_\kappa^{n-m}(s_p(r_2, \eta)) \text{d}r_2}{\int_{r_a}^{r_c} g_\kappa^{n-m}(s_p(r_1, \eta)) \text{d}r_1}
\]
under $r_a = r_c$ or $r_b = r_d$ by use of [30, Lemma 3.1] (cf. [30, Proof of Theorem 3.2]). From this, we can deduce that
\[
\frac{\int_{S^{n-1}} \int_{\text{cut}(\theta) \wedge r_d} J_V(r_2, \theta) \text{d}r_2 \text{d} \theta}{\int_{S^{n-1}} \int_{\text{cut}(\theta) \wedge r_a} J_V(r_1, \theta) \text{d}r_1 \text{d} \theta} \leq \frac{\int_{S^{n-1}} \int_{r_a}^{r_d} g_\kappa^{n-m}(s_p(r_2, \eta)) \text{d}r_2 \text{d} \theta}{\int_{S^{n-1}} \int_{r_a}^{r_c} g_\kappa^{n-m}(s_p(r_1, \eta)) \text{d}r_1 \text{d} \theta}
\]
holds for general $0 \leq r_a < r_b \leq r_d$, $0 \leq r_a \leq r_c < r_d$ and $r_d < R$. This implies that (2.17) holds for $r_1 < R$. If $\phi$ is rotationally symmetric around $p$, $s_p(r, \theta)$ can be written as $s_p(r)$ and one can derive
\[
\frac{\int_{S^{n-1}} \int_{\text{cut}(\theta) \wedge r_d} J_V(r_2, \theta) \text{d}r_2 \text{d} \theta}{\int_{S^{n-1}} \int_{\text{cut}(\theta) \wedge r_a} J_V(r_1, \theta) \text{d}r_1 \text{d} \theta} \leq \frac{\int_{S^{n-1}} \int_{r_a}^{r_d} g_\kappa^{n-m}(s_p(r_2)) \text{d}r_2 \text{d} \theta}{\int_{S^{n-1}} \int_{r_a}^{r_b} g_\kappa^{n-m}(s_p(r_1)) \text{d}r_1 \text{d} \theta}.
\]
This implies that (2.18) holds for \( r_1 < R \). Similarly, in the modified coordinates \((s, \theta)\), we set

\[
\text{cut}_s(\theta) := \int_0^{\text{cut}(\theta)} e^{-\frac{2V_\gamma(t)}{n-m}} dt,
\]

where \( \gamma \) is the unit speed geodesic with \( \gamma_0 = p \) and \( \gamma_0 = \theta \). Then we have

\[
\nu(V(C(p, s_0, s_1))) = \int_{S^{n-1}} \int_{\text{cut}_s(\theta) \wedge s_1} J_V(r(s, \theta), \theta) ds d\theta,
\]

and

\[
v(\kappa, s_0, s_1) = \int_{S^{n-1}} \int_{s_0}^{s_1} s_{n-m}^n(s) ds d\theta = \omega_{n-1} \int_{s_0}^{s_1} s_{n-m}^n(s) ds.
\]

Therefore, (2) follows. Here \( r(s, \theta) := C_p^{-1} \int_0^s \exp \left( \frac{2V_\gamma(f^{-1}(u))}{n-m} \right) du \) with \( f(r) := s_p(r, \theta) \).

Note that \( s_1 < \delta_\kappa \) always holds under the condition. Indeed, \( s_1 < S \) implies \( s_1 < \delta_\kappa \) under \( R < +\infty \) by Corollary 3.3. When \( R = +\infty \), for any \( \theta \in S^{n-1} \) there exists \( R_0 \in [0, +\infty) \) depending on \( \theta \) such that \( s_1 < s(R_0, \theta) \). Then applying Corollary 3.3 for \( R_0 < \infty \),

\( r_1 := r(s_1, \theta) < r(s(R_0, \theta), \theta) = R_0 \) implies \( s_1 = s(r(s_1, \theta), \theta) < \delta_\kappa \), where we use (2.9) holds for \( r_p(x) < R_0 \).

\[ \Box \]

**Proof of Corollary 2.17.** By Theorem 2.13(1), for \( 0 < r_1 < r_2 < R \)

\[
\frac{\mu_V(B_{r_2}(p))}{\mu_V(B_{r_1}(p))} \leq \frac{\int_0^{r_2} \left( C_p e^{-\frac{2\phi_V(r)}{n-m}} \right)^{n-m} dr}{\int_0^{r_1} \left( C_p e^{-\frac{2\phi_V(r)}{n-m}} \right)^{n-m} dr}
\]

\[
\leq e^{2(\phi_V(r_1) - \phi_V(r_2))} \frac{\int_0^{r_2} r^{n-m} dr}{\int_0^{r_1} r^{n-m} dr} = e^{2(\phi_V(r_1) - \phi_V(r_2))} \left( \frac{r_2}{r_1} \right)^{n-m+1}.
\]

\[ \Box \]

**6. Proofs of Theorem 2.16, Corollaries 2.17 and 2.18**

**Proof of Theorem 2.16.** Suppose that \( M \) is non-compact. Then there exists a unit speed geodesic \( \gamma \) with \( \gamma_0 = p \) satisfying (2.23). Note that the function \( \lambda(t) \) is smooth for all \( t > 0 \) along \( \gamma \). By (2.2), we have

\[
\lambda(t) - \lambda(1) + \frac{C_p}{n-m} \int_1^t e^{-\frac{2V_\gamma(r)}{n-m}} \lambda(r)^2 dr \leq -C_p^{-1} \int_1^t e^{-\frac{2V_\gamma(r)}{n-m}} \text{Ric}_{m,n}(\Delta_V) (\gamma_r, \gamma_r) dr.
\]

Hence

\[
(6.1) \lim_{t \to +\infty} \left( \lambda(t) + \frac{C_p}{n-m} \int_1^t e^{-\frac{2V_\gamma(r)}{n-m}} \lambda(r)^2 dr \right) = -\infty.
\]
In particular, \( \lim_{t \to +\infty} \lambda(t) = -\infty \). Next we prove that there exists a finite number \( T > 0 \) such that \( \lim_{t \to T-} \lambda(t) = -\infty \), which contradicts the smoothness of \( \lambda(t) \). By (6.1), given \( C > n - m \) there exists \( t_0 > 1 \) such that
\[
-\lambda(t_0) - \frac{C_p}{n - m} \int_{1}^{t_0} e^{-\frac{2V_{\gamma}(r)}{n-m}} \lambda(r)^2 \, dr \geq \frac{C}{n - m}.
\]
Since
\[
\lim_{t \to +\infty} \int_{1}^{t} e^{-\frac{2V_{\gamma}(r)}{n-m}} \lambda(r)^2 \, dr = +\infty,
\]
there exists \( t_1 \in ]t_0, +\infty[ \) such that \( \int_{t_0}^{t_1} e^{-\frac{2V_{\gamma}(r)}{n-m}} \lambda(r)^2 \, dr \geq 0 \) for all \( t \geq t_1 \). Let \( \psi(t) \) be the function defined by
\[
(6.2)
\psi(t) := -\lambda(t) - \frac{C_p}{n - m} \int_{1}^{t} e^{-\frac{2V_{\gamma}(r)}{n-m}} \lambda(r)^2 \, dr - C_{p-1} \int_{1}^{t_1} e^{-\frac{2V_{\gamma}(r)}{n-m}} \lambda(r)^2 \, dr.
\]
Then we see \( \psi'(t) \geq 0 \) by (3.2). Hence \( \psi(t) \geq \psi(t_0) \) for \( t \geq t_1 > t_0 \). This implies that
\[
(6.3)
-\lambda(t) - \frac{C_p}{n - m} \int_{1}^{t} e^{-\frac{2V_{\gamma}(r)}{n-m}} \lambda(r)^2 \, dr \geq \frac{C}{n - m} > 1
\]
holds for all \( t \geq t_1 \). Let us consider the sequence \( \{t_\ell\} \) defined inductively by
\[
C_p \int_{t_{\ell-1}}^{t_{\ell+1}} e^{-\frac{2V_{\gamma}(r)}{n-m}} \, dr = (n - m) \left( \frac{n - m}{C} \right)^{\ell-1} \quad \text{for} \quad \ell \geq 1.
\]
The existence of such sequence is guaranteed by the \((V, m)\)-completeness of \((M, g, V)\) at \( p \). Let \( T \) be the increasing limit of \( \{t_\ell\} \). Then we see
\[
C_p \int_{t_1}^{T} e^{-\frac{2V_{\gamma}(r)}{n-m}} \, dr = \frac{C(n - m)}{C - n + m}.
\]
In view of the \((V, m)\)-completeness of \((M, g, V)\) at \( p \), we have
\[
\int_{1}^{\infty} e^{-\frac{2V_{\gamma}(r)}{n-m}} \, dr = +\infty.
\]
Thus we obtain \( T < \infty \). Finally we claim that for given \( \ell \in \mathbb{N} \), \( -\lambda(t) \geq \left( \frac{C}{n - m} \right)^{\ell} \) for all \( t \geq t_\ell \). This is true for \( \ell = 1 \) by (6.3). Suppose that \( -\lambda(r) \geq \left( \frac{C}{n - m} \right)^{\ell} \) for all \( r \geq t_\ell \) and fix \( t \geq t_{\ell+1} \). Then using inequality (6.3) again,
\[
-\lambda(t) \geq \frac{C}{n - m} + \frac{C_p}{n - m} \int_{1}^{t_\ell} e^{-\frac{2V_{\gamma}(r)}{n-m}} \lambda(r)^2 \, dr + \frac{C_p}{n - m} \int_{t_\ell}^{t_{\ell+1}} e^{-\frac{2V_{\gamma}(r)}{n-m}} \lambda(r)^2 \, dr
\]
\[
\geq \frac{C_p}{n - m} \int_{t_\ell}^{t_{\ell+1}} e^{-\frac{2V_{\gamma}(r)}{n-m}} \lambda(r)^2 \, dr
\]
\[
\geq \frac{C^{2\ell}}{(n - m)^2} \cdot \frac{(n - m)^{\ell-1}}{C^{\ell-1}} = \left( \frac{C}{n - m} \right)^{\ell+1}.
\]
Therefore we prove the claim. In particular, \( \lim_{t \to T^-} \lambda(t) = -\infty \) which is the desired contradiction.

**Proof of Corollary 2.17.** Suppose that there exists a non-negative integrable function \( f \) on \([0, +\infty[\) satisfying \( \langle V, \nabla r_p \rangle \geq -f(r_p) \). Then \( V_\gamma(r) \geq -\int_0^r f(s)ds \geq -\int_0^{\infty} f(s)ds > -\infty \) and \( \text{Ric}_{m,n}(\Delta V) \geq 0 \) imply

\[
\int_0^\infty e^{\frac{2V_\gamma(t)}{n-m}} \text{Ric}_{m,n}(\Delta V)(\dot{\gamma}_t, \dot{\gamma}_t)dt \\
\geq \exp \left( -\frac{2}{n-m} \int_0^\infty f(s)ds \right) \int_0^\infty \text{Ric}_{m,n}(\Delta V)(\dot{\gamma}_t, \dot{\gamma}_t)dt = +\infty.
\]

This yields the conclusion by Theorem 2.16.

**Proof of Corollary 2.18.** Suppose that \((2.25)\) holds for every unit speed geodesic \( \gamma \) emanating from \( p \). The \((V, m)\)-completeness of \((M, g, V)\) at \( p \) implies

\[
\int_0^\infty e^{-\frac{2V_\gamma(t)}{n-m}} dt = +\infty.
\]

Then we have

\[
\int_0^\infty e^{\frac{2V_\gamma(t)}{n-m}} \text{Ric}_{m,n}(\Delta V)(\dot{\gamma}_t, \dot{\gamma}_t)dt \geq (n-m)\kappa C_p^2 \int_0^\infty e^{-\frac{2V_\gamma(t)}{n-m}} dt = +\infty.
\]

This yields the conclusion by Theorem 2.16.

7. **Proof of Theorem 2.20**

For the proof of Theorem 2.20, we need the following lemma on the solution of Jacobi equation.

**Lemma 7.1.** Let \( \kappa : [0, \infty[ \to \mathbb{R} \) be a continuous function and \( s_\kappa \) the unique solution of the Jacobi equation \( s_\kappa''(s) + \kappa(s)s_\kappa'(s) = 0 \) with \( s_\kappa(0) = 0 \) and \( s_\kappa'(0) = 1 \), and \( \delta_\kappa := \inf\{ s > 0 \mid s_\kappa(s) = 0 \} \) the first zero point of \( s_\kappa \). Assume that \( \delta_\kappa < \infty \) and \( \kappa(s) = \kappa(\delta_\kappa - s) \) holds for all \( s \in [0, \delta_\kappa) \). Then \( s_\kappa'(\delta_\kappa) = -1 \), \( s_\kappa'(\delta_\kappa)/2 = 0 \) and \( s_\kappa(s) = s_\kappa(\delta_\kappa - s) \) for all \( s \in [0, \delta_\kappa] \).

**Proof.** Set \( \mathcal{F}_\kappa(s) := s_\kappa(\delta_\kappa - s) \) for \( s \in [0, \delta_\kappa] \). Then this satisfies \( \mathcal{F}_\kappa''(s) + \kappa(s)\mathcal{F}_\kappa'(s) = 0 \) and \( \mathcal{F}_\kappa(0) = 0 \) and \( \mathcal{F}_\kappa'(0) = -s_\kappa'(\delta_\kappa) \). If we prove \( \mathcal{F}_\kappa'(0) = 1 \), i.e., \( s_\kappa'(\delta_\kappa) = -1 \), then the uniqueness of the solution implies the assertion.

Note that \( s_\kappa(s) := \mathcal{F}_\kappa(s) / \mathcal{F}_\kappa'(0) = -s_\kappa(\delta_\kappa - s) / s_\kappa'(\delta_\kappa) \) also satisfies the Jacobi equation with \( s_\kappa(0) = 0 \) and \( s_\kappa'(0) = 1 \). Then the uniqueness implies \( s_\kappa(s) = s_\kappa(s) \), that is, \( s_\kappa(\delta_\kappa - s) = -s_\kappa'(\delta_\kappa) s_\kappa(s) \) for \( s \in [0, \delta_\kappa] \), in particular, \( s_\kappa(\delta_\kappa)/2 = -s_\kappa'(\delta_\kappa) s_\kappa(\delta_\kappa/2) \). Therefore, \( s_\kappa'(\delta_\kappa) = -1 \) by \( s_\kappa(\delta_\kappa)/2 > 0 \). The proof of \( s_\kappa'(\delta_\kappa)/2 = 0 \) is easy from \( s_\kappa'(s) = -s_\kappa'(\delta_\kappa - s) \) for \( s \in [0, \delta_\kappa] \).
Hereafter, we assume $V = \nabla \phi$ for some $\phi \in C^2(M)$ and set $C_p := \exp \left( -\frac{2\phi(p)}{n-m} \right)$ for the definition of $s_p(x)$ with $p$ being an arbitrary point. We now consider the conformal metric $h = e^{-\frac{4\phi}{n-m}}g$.

**Lemma 7.2.** Fix $p \in M$. Suppose that there exists a point $q \in M$ such that $s(p,q) = d^h(p,q)$ and let $\gamma$ be the minimal unit speed $g$-geodesic from $p$ and $q$ such that $s(p,q) = \int_0^{d(p,q)} e^{-2\frac{\phi(t)}{n-m}} dt$. Then $\nabla \phi$ is parallel to $\dot{\gamma}$ (not parallel along $\gamma$). Moreover if $s(p,x) = d^h(p,x)$ holds for any $x \in M$, then $\phi$ is rotationally symmetric around $p$.

**Proof.** Since $t < d(p,q)$ implies $\gamma_t \notin \text{Cut}(p)$, we have $s(p,q) = \int_0^{d(p,q)} e^{-2\frac{\phi(t)}{n-m}} dt = L^h(\gamma)$. Combining this with $s(p,q) = d^h(p,q)$ we get $d^h(p,q) = L^h(\gamma)$. Then $\gamma$ is a minimal geodesic in the $h$ metric. In particular, $\nabla^h \phi \frac{d\gamma}{ds} = 0$. Applying the formula for connection of $h$ in terms of $g$, we have

$$0 = \nabla^h \phi \frac{d\gamma}{ds}$$

$$= \nabla^g \phi \frac{d\gamma}{ds} - \frac{4}{n-m} \left(\nabla^g \phi, \nabla \phi \right) \frac{d\gamma}{ds} + \frac{2}{n-m} \left(\nabla^g \phi, \frac{d\gamma}{ds} \right) \nabla \phi$$

$$= \frac{2e^{\frac{4\phi}{n-m}}}{n-m} \left(-\left(\gamma_t, \nabla \phi \right) \gamma_r + \nabla \phi \right).$$

Then we obtain that $\nabla \phi = \langle \nabla \phi, \gamma_r \rangle \gamma_r$, i.e., $\nabla \phi$ is parallel to $\dot{\gamma}$. Suppose further that $s(p,x) = d^h(p,x)$ for any $x \in M$. Let $x_1, x_2 \in M$ be the points in the sphere $\partial B_r(p)$ for $r > 0$ and $c : [0, 1] \rightarrow \partial B_r(p)$ a curve on $\partial B_r(p)$ joining $c(0) = x_1$ and $c(1) = x_2$. Then we see $\langle \nabla \phi, \dot{c}_t \rangle = 0$, because $\nabla \phi$ is parallel to $\dot{\gamma}$, where $\gamma$ is the $g$-geodesic from $p$ to a point in $\text{Im}(c)$. Hence $\phi(x_2) - \phi(x_1) = \int_0^1 \langle \nabla \phi, \dot{c}_t \rangle dt = 0$. \hfill \Box

Here we encounter that $s$ does not necessarily satisfy the triangle inequality. To get around this difficulty we utilize again the conformal metric $h$.

From $d^h(p,x) \leq s(p,x)$ and the triangle inequality for the $h$-metric we have

$$s(p,x) + s(q,x) \geq d^h(p,x) + d^h(q,x) \geq d^h(p,q).$$

**Proof of Theorem 2.20.** First note that $s_\kappa(s) = s_\kappa(\delta_\kappa - s)$ holds for $s \in [0, \delta_\kappa]$ by Lemma 7.1. In particular, we have $\cot_\kappa(s) = -\cot_\kappa(\delta_\kappa - s)$ for all $s \in [0, \delta_\kappa]$. Let $r_p$ and $r_q$ be the distance functions to $p$ and $q$ respectively. Then by Theorem 2.5 we have

$$\Delta \nabla_\phi (r_p + r_q)(x) \leq (n-m)e^{-\frac{2\phi(x)}{n-m}} \left(\cot_\kappa(s_p(x)) + \cot_\kappa(s_q(x))\right)$$

holds in the barrier sense. We also have $s_p(x) + s_q(x) \geq d^h(p,q) = \delta_\kappa$, so that

$$\cot_\kappa(s_q(x)) \leq \cot_\kappa(\delta_\kappa - s_p(x)) = -\cot_\kappa(s_p(x)).$$

Thus, $\Delta \nabla_\phi (r_p + r_q) \leq 0$ holds in the barrier sense. Note that $\inf_M (r_p + r_q)$ attains its minimum at a point of minimal geodesic joining $p$ and $q$. Then one can apply the strong minimum principle for superharmonic functions in the barrier sense (see [9, 11].
for the strong maximum principle for subharmonic functions in the barrier sense) so that 
\( r_p(x) + r_q(x) = d(p, q) \) for all \( x \in M \) and all geodesics starting point at \( p \) in \( M \) are minimizing and end at \( q \). In particular, we have \( \Delta \nabla_\phi (r_p + r_q) = 0 \) in the classical sense. 
Therefore, we have 
\[
\cot_k(s_p(x)) = \cot_k(\delta_k - s_q(x)) \quad \text{for all} \quad x \in M.
\]
Since \( s \mapsto \cot_k(s) \) is strictly decreasing, we have \( s_p(x) + s_q(x) = \delta_k \). Hence \( s_p(x) + s_q(x) = d^h(p, q) = s(p, q) = \delta_k \) by \( d^h(p, q) \leq s(p, q) \leq \delta_k \) (see Theorem 2.10). We can apply the similar argument so that \( d^h_p(x) + d^h_q(x) = d^h(p, q) = s(p, q) = \delta_k \). Hence \( 0 \leq s_p(x) - d^h_p(x) = d^h_q(x) - s_q(x) \leq 0 \) implies \( s_p(x) = d^h_p(x) \). Taking \( x \notin \text{Cut}(p) \), we see that there exists a unique minimal unit speed geodesic \( \gamma \) with \( \gamma_0 = p \) and \( \gamma_{r_p(x)} = x \) satisfying \( s_p(x) = \int_0^{r_p(x)} e^{-\frac{2d(\gamma_t)}{n-m}} dt \). Applying this with Lemma 7.2, \( \phi \) is rotationally symmetric around \( p \). Secondly, we can deduce that
\[
\begin{align*}
(7.1) \quad & \Delta \nabla_\phi r_p(x) = (n - m)e^{-\frac{2d(x)}{n-m}} \cot_k(s_p(x)), \\
(7.2) \quad & \Delta \nabla_\phi r_q(x) = (n - m)e^{-\frac{2d(x)}{n-m}} \cot_k(s_q(x))
\end{align*}
\]
hold in the barrier sense respectively. Consequently, (7.1) (resp. (7.2)) holds for \( x \in (\text{Cut}(p) \cup \{p\})^c \) (resp. \( x \in (\text{Cut}(q) \cup \{q\})^c \)). Let \( \eta \) be a minimal unit speed geodesic from \( p \) to \( q \) with \( \eta_0 = \theta \). Applying Lemma 3.2 to (7.1), we obtain \( m = 1 \) and the expression of a metric of the form 
\[
g_{\eta_t} = d^2 + e^{\frac{2d(x) + \phi(0)}{n-1}} s^2_k(s(r))g_{\mathbb{R}^{n-1}}, \quad 0 \leq r \leq d(p, q)
\]
with \( s(r) = \int_0^r e^{-\frac{2d(t)}{n-1}} dt \) and \( s(d(p, q)) = \delta_k \). This implies the conclusion.  
\[\square\]

8. Proof of Theorem 2.22

Let \( \gamma \) be a ray in \( M \), i.e. a unit speed geodesic defined on \([0, +\infty[\) such that \( d(\gamma_s, \gamma_t) = |s - t| \) for any \( s, t \geq 0 \). The Busemann function \( b_\gamma : M \to \mathbb{R} \) for a ray \( \gamma \) is defined by 
\[b_\gamma(x) := \lim_{t \to +\infty} (t - d(x, \gamma_t)), \quad x \in M.\]
It follows from the triangle inequality that \( t \mapsto d(x, \gamma_t) \) is monotonically non-decreasing in \( t \), so that the above limit exists. Moreover, it is well-known that \( b_\gamma \) is a 1-Lipschitz function. See e.g. [28].

**Lemma 8.1.** Let \((M, g)\) be an \( n \)-dimensional complete Riemannian manifold and \( V \) a \( C^1 \)-vector field. Fix a point \( p \in M \). Suppose that \( (2.9) \) holds for any \( x \in M \) with \( \kappa \equiv 0 \). Let \( q \in M \) be a point such that \( r_p \) is smooth at \( q \), and let \( \gamma \) be the unique unit speed minimal geodesic from \( p \) to \( q \). Then we have 
\[
(8.1) \quad (\Delta_V r_p)(q) \leq \frac{n - m}{\exp \left( \frac{2V_\gamma(r_p(q))}{n-m} \right)} \int_0^{r_p(q)} \exp \left( -\frac{2V_\gamma(s)}{n-m} \right) ds.
\]
Proof. Applying the Ricatti inequality (3.2) along $\gamma$ under (2.9) with $\kappa \equiv 0$, we see
$$\frac{1}{\lambda(r)^2} \frac{d\lambda}{dr} (r) \leq - \frac{C_p}{n-m} e^{\frac{2\lambda(q(r))}{n-m}}. $$
Integrating this from $\varepsilon > 0$ to $\varepsilon\lambda(q) + \lambda(\varepsilon)$ and letting $\varepsilon \to 0$, we have from $\lim_{\varepsilon \to 0} \lambda(\varepsilon) = +\infty$ that
$$\lambda(r_p(q)) = C_p^{-1} e^{\frac{2\lambda(q(r_p(q)))}{n-m}} (\Delta_V r_p)(q) \leq \frac{n-m}{C_p \int_0^{r_p(q)} e^{-\frac{2\lambda(r)}{n-m}} dr.}$$
This implies the conclusion. \[\square\]

Lemma 8.2. Let $(M, g)$ be an $n$-dimensional complete Riemannian manifold and $V$ a $C^1$-vector field. Suppose that $(M, g, V)$ is $(V, m)$-complete. Suppose that (2.11) holds for any $p, x \in M$ with $\kappa = 0$. Then the Busemann function $b_\gamma$ for any ray $\gamma$ in $M$ is an $\Delta_V$-subharmonic function in the barrier sense, i.e., for each $p \in M$ and any $\varepsilon > 0$, there exists a smooth function $b_{p, \varepsilon}$ defined on a neighborhood $U_{\varepsilon}(p)$ at $p$ such that $b_{p, \varepsilon}(p) = b_\gamma(p)$, $b_{p, \varepsilon} \leq b_\gamma$ on $U_{\varepsilon}(p)$, and $\Delta_V b_{p, \varepsilon}(p) \geq -\varepsilon$.

Proof. Fix $p \in M$ and a ray $\gamma$ in $M$. Take any sequence $\{t_k\}$ satisfying $\lim_{k \to \infty} t_k = +\infty$. Let $\eta_{t_k}$ be a minimal $g$-geodesic joining $p$ and $\gamma_{t_k}$. As stated in [11], there exists a subsequence of $t_k$ such that the initial vector $\eta_{t_k}(0)$ converges to some unit vector $u$ in $T_pM$. Let $\eta$ be the ray emanating from $p$ and generated by $u$. Then $p$ does not belong to the cut-locus of $\eta(r)$, hence $\eta(r) \notin \text{Cut}(p)$ for any $r > 0$. So $b_\gamma^r(x) := r - d(x, \eta(r)) + b_\gamma(p)$ is smooth around $p$ and satisfies $b_\gamma^r \leq b_\gamma$ with $b_\gamma^r(p) = b_\gamma(p)$. By (8.1), we see that for the unique unit speed geodesic $\gamma$ from $\eta(r)$ to $p$

$$\Delta_V b_\gamma^r(p) = -\Delta_V r_\eta(r)(p) \geq \frac{n-m}{\exp \left( \frac{2\lambda(q(r))}{n-m} \right) \int_0^{d(\eta(r), p)} \exp \left( \frac{2\lambda(t)}{n-m} \right) dt.}$$

Note that $\eta(r) = \gamma_{d(p, \eta(r) - r)}$ for $r \in [0, d(p, \eta(r))]$. Then (8.2) becomes

$$\Delta_V b_\gamma^r(p) = -\Delta_V r_\eta(p)(p) \geq \frac{n-m}{\int_0^{d(p, \eta(r))} \exp \left( \frac{-2\lambda(t)}{n-m} \right) du.}$$

Since $(M, g, V)$ is $(V, m)$-complete, we can construct the desired support function. \[\square\]

Proof of Theorem 2.22. Let $\gamma : -\infty, +\infty \to M$ be a line (i.e., $d(\gamma_t, \gamma_s) = |s - t|$ for $s, t \in \mathbb{R}$) and $\gamma^+, \gamma^-$ rays defined by $\gamma^+_t := \gamma_t$ and $\gamma^-_t := \gamma_{-t}$ ($t \geq 0$). Let $b^+$, $b^-$ be the Busemann function associated to $\gamma^+$, $\gamma^-$, respectively. Then, under the $(V, m)$-completeness of $(M, g, V)$, $b^+$ and $b^-$ are continuous $\Delta_V$-subharmonic functions on $M$ in the barrier sense by Lemma 8.2. Since $\gamma$ is a line, for each $x \in M$, we have

$$b^+(x) + b^-(x) = \lim_{t \to +\infty} (2t - d(x, \gamma_t) - d(x, \gamma_{-t})) \leq 0$$

and $b^+ + b^- = 0$ on $\gamma$. In view of the strong maximum principle for $\Delta_V$-subharmonic functions in the barrier sense (see [9, 11] and [12, Lemma 2.4]), we have $b^+ + b^- = 0$ on
M. In particular, $b^+$ and $b^-$ are continuous $\Delta_V$-harmonic functions in the barrier sense. Since $|\nabla r_p| = 1$ on $(\text{Cut}(p) \cup \{p\})^c$, we have $|\nabla b^+| = |\nabla b^-| = 1$ on $M$. Moreover, let $h^\pm$ be the smooth $\Delta_V$-harmonic function on an open ball $B$ such that $b^\pm = h^\pm$ on $\partial B$. Applying the weak maximum principle to the $\Delta_V$-harmonic function $b^+ - h^+ \equiv 0$ on $B$, we can deduce $b^\pm \leq h^\pm$ on $B$, hence $0 = b^+ + b^- \leq h^+ + h^-$. Applying the strong maximum principle again to the smooth $\Delta_V$-harmonic function $h^+ + h^-$ on $B$, we have $h^+ + h^- \equiv 0$ on $B$. Thus, we can get $0 \geq b^+ - h^- = (b^- - h^-) \geq 0$ on $B$, hence $b^\pm = h^\pm$ on $B$. Therefore, $b^\pm$ is smooth on any ball $B$, hence on $M$. Applying [34, Lemma 6.5] to the smooth $\Delta_V$-harmonic function $b_{\gamma \pm}$ and $|\nabla b_{\gamma \pm}| = 1$ on $M$, we can deduce that $\text{Ric}_{1,n}(\Delta_V)(\nabla b_{\gamma \pm}, \nabla b_{\gamma \pm}) = 0$ and $n - 1$ non-zero eigenvalues of Hess $b^\pm |_p$ are all equal, because Hess $b^\pm |_p$ has $n - 1$ non-zero eigenvalues. Applying [34, Lemma 6.6] to the smooth $\Delta_V$-harmonic function $b^\pm$ satisfying $|\nabla b^\pm| = 1$ together with the fact that $\text{CD}(0,m)$-condition implies $\text{CD}(0,1)$-condition for $m < 1$, we have that $g$ has a twisted product of the form $g = dr^2 + e^{\frac{2m}{n-1}} g_N$, where $g_N$ is a metric on $N$ and $\phi : M \to \mathbb{R}$ is a smooth function, $\text{Ric}_{1,n}(\Delta_V)(\nabla b^\pm, \nabla b^\pm) = 0$, and $V = \frac{\partial \phi}{\partial r} \cdot \frac{\partial}{\partial r} + U$ with $U \perp \frac{\partial}{\partial r}$. In the same way of the proof of [34, Corollary 6.7], we can deduce that $\frac{\partial \phi}{\partial r} = 0$, because [34, Proposition 2.1] yields $\text{Ric}_{1,n}(\Delta_V)\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = 0$ and

$$0 \leq \text{Ric}_{m,n}(\Delta_V)\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = \text{Ric}_{1,n}(\Delta_V)\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) + \left(\frac{m - 1}{(n - 1)(n - m)}\right) \left(\frac{d \phi}{dr}\right)^2 = \left(\frac{m - 1}{(n - 1)(n - m)}\right) \left(\frac{d \phi}{dr}\right)^2 \leq 0.$$  

This means that $g$ has the form of product metric $g = dr^2 + e^{\frac{2m(0)}{n-1}} g_N = dr^2 + h_N$ on $\mathbb{R} \times N$. Moreover, we can see that $V$ is a vector field on $N$ by using the fact that $\text{Ric}_{m,n}(\Delta_V)\left(\frac{\partial}{\partial r}, U\right) = 0$ for all $U \perp \frac{\partial}{\partial r}$. \hfill \square

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Department of Applied Mathematics
Fukuoka University
Fukuoka 814-0180
Japan
Email address: kuwae@fukukoa-u.ac.jp

Oita City Takio Junior High School
Oita, 874-0942
Japan
Email address: lo.5.hawks61@docomo.ne.jp