Approximate solutions for robust multiobjective optimization programming in Asplund spaces

Maryam Saadati and Morteza Oveisiha
Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, Iran

ABSTRACT
In this paper, we study a nonsmooth/nonconvex multiobjective optimization problem with uncertain constraints in arbitrary Asplund spaces. We first provide necessary optimality condition in a fuzzy form for approximate weakly robust efficient solutions and then establish necessary optimality theorem for approximate weakly robust quasi-efficient solutions of the problem in the sense of the limiting subdifferential by exploiting a fuzzy optimality condition in terms of the Fréchet subdifferential. Sufficient conditions for approximate (weakly) robust quasi-efficient solutions to such a problem are also driven under the new concept of generalized pseudo convex functions. Finally, we address an approximate Mond-Weir-type dual robust problem to the reference problem and explore weak, strong, and converse duality properties under assumptions of pseudo convexity.

ARTICLE HISTORY
Received 7 April 2022
Accepted 15 July 2022

KEYWORDS
Approximate solutions; optimality conditions; limiting subdifferential; generalized convexity; robust multiobjective optimization

MATHEMATICS SUBJECT CLASSIFICATION (2020)
41A65; 49K99; 65K10; 90C29; 90C46

1. Introduction

Robust optimization investigates the cases in which optimization problems often consider uncertain data due to prediction errors, lack of information, fluctuations, or disturbances [1–3]. In particular, in such cases these problems rely on conflicting goals due to different multiobjective optimization criteria. Hence, the robust multiobjective optimization is highly of interest in optimization theory and substantial in applications.

The first concept of robustness as a kind of sensitivity against perturbations for multiobjective optimization problems was explored by Branke [4] and provided by Deb and Gupta [5]. In addition, various concepts in minimax robustness for multiobjective optimization were introduced by Kuroiwa and Lee [6], Jeyakumar et al. [7], Ehrhoff et al. [8], and Ide and Köbis [9]. Recently, some different concepts of robustness used in multiobjective optimization in the face of data uncertainty have been established in [10–14].
Approximate efficient solutions of multiobjective optimization problems can be viewed as feasible points whose objective values display a prescribed error $\varepsilon$ in the optimal values of the vector objective. This concept has been widely studied in [15–18]. Optimality conditions and duality theories of $\varepsilon$-efficient solutions and $\varepsilon$-quasi-efficient solutions for convex programming problems under uncertainty have been presented in [19,20].

The most significant results have been introduced to approximate robust optimization in the finite-dimensional case. So, an infinite-dimensional framework would be proper to study when involving optimality and duality in approximate robust multiobjective optimization. From this, we are motivated to articulate and analyse problems that consider infinite-dimensional frameworks.

Let $f: X \to Y$ be a locally Lipschitzian vector-valued function between Asplund spaces, and let $\Omega \subset X$ be a nonempty closed set. Suppose that $K \subset Y$ be a pointed (i.e. $K \cap (-K) = \{0\}$) closed convex cone. We consider the following multiobjective optimization problem:

\[
\begin{align*}
\min_K &\quad f(x) \\
\text{s.t.} &\quad g_i(x) \leq 0, \quad i = 1, 2, \ldots, n,
\end{align*}
\]

where the functions $g_i: X \to \mathbb{R}$, $i = 1, 2, \ldots, n$, define the constraints. Problem (P) under data uncertainty in the constraints can be captured by the following uncertain multiobjective optimization problem:

\[
\begin{align*}
\min_K &\quad f(x) \\
\text{s.t.} &\quad g_i(x, v_i) \leq 0, \quad i = 1, 2, \ldots, n,
\end{align*}
\]

where $x \in X$ is the vector of decision variable, $v_i$ is the vector of uncertain parameter and $v_i \in \mathcal{V}_i$ for some sequentially compact topological space $\mathcal{V}_i$, $v := (v_1, v_2, \ldots, v_n) \in \mathcal{V} := \prod_{i=1}^n \mathcal{V}_i$, and $g_i: X \times \mathcal{V}_i \to \mathbb{R}$, $i = 1, 2, \ldots, n$, are given functions.

One of the powerful deterministic structures to study problem (UP) is the robust optimization, which is known as the problem that the uncertain objective and constraint are satisfied for all possible scenarios within a prescribed uncertainty set. We now associate with them:

\[
\begin{align*}
\min_K &\quad f(x) \\
\text{s.t.} &\quad g_i(x, v_i) \leq 0, \quad \forall v_i \in \mathcal{V}_i, \quad i = 1, 2, \ldots, n.
\end{align*}
\]

The feasible set $F$ of problem (RP) is defined by

\[
F := \{ x \in \Omega \mid g_i(x, v_i) \leq 0, \quad \forall v_i \in \mathcal{V}_i, \quad i = 1, 2, \ldots, n \}.
\]

**Definition 1.1:** Let $\vartheta \in K$, one says a vector $\bar{x} \in F$ is

(i) a robust $\vartheta$-efficient solution of problem (UP), denoted by $\bar{x} \in \vartheta \cdot S(RP)$, iff

\[
f(x) - f(\bar{x}) + \vartheta \notin -K \setminus \{0\}, \quad \forall x \in F,
\]
(ii) a weakly robust $\vartheta$-efficient solution of problem (UP), denoted by $\bar{x} \in \vartheta$-S$^w(RP)$, iff

$$f(x) - f(\bar{x}) + \vartheta \notin -\text{int}K, \quad \forall x \in F,$$

(iii) a robust $\vartheta$-quasi-efficient solution of problem (UP), denoted by $\bar{x} \in \vartheta$-quasi-S(RP), iff

$$f(x) - f(\bar{x}) + \|x - \bar{x}\| \vartheta \notin K \setminus \{0\}, \quad \forall x \in F,$$

(iv) a weakly robust $\vartheta$-quasi-efficient solution of problem (UP), denoted by $\bar{x} \in \vartheta$-quasi-S$^w(RP)$, iff

$$f(x) - f(\bar{x}) + \|x - \bar{x}\| \vartheta \notin \text{int}K, \quad \forall x \in F.$$

The organization of this paper is as follows. In Section 2, we recall some preliminary definitions from variational analysis and several auxiliary results. Section 3 provides necessary condition for weakly robust $\vartheta$-efficient solutions and also necessary/sufficient optimality conditions for (weakly) robust $\vartheta$-quasi-efficient solutions of problem (UP) in the sense of the limiting subdifferential. In Section 4, we formulate duality relations for (weakly) robust $\vartheta$-quasi-efficient solutions between the corresponding problems.

2. Preliminaries

Our notation and terminology are basically standard in the area of variational analysis; see, e.g. [21]. Throughout this paper, all the spaces are Asplund, unless otherwise stated, with the norm $\| \cdot \|$ and the canonical pairing $\langle \cdot, \cdot \rangle$ between the space $X$ in question and its dual $X^*$ equipped with the weak* topology $w^*$. By $B_X(x, r)$, we denote the closed ball centred at $x \in X$ with radius $r > 0$, while $B_X$ and $B_{X^*}$ stand for the closed unit ball in $X$ and $X^*$, respectively. Given a nonempty set $\Omega \subset X$, the symbols $co\Omega$, $cl\Omega$, and $int\Omega$ signify the convex hull, topological closure, and topological interior of $\Omega$, respectively, while $cl^*\Omega$ stands for the weak* topological closure of $\Omega \subset X^*$. The dual cone of $\Omega$ is the set

$$\Omega^+ := \{ x^* \in X^* | \langle x^*, x \rangle \geq 0, \forall x \in \Omega \}.$$

Furthermore, $\mathbb{R}^n_+$ indicates the nonnegative orthant of $\mathbb{R}^n$ for $n \in \mathbb{N} := \{1, 2, \ldots \}$.

A given set-valued mapping $H : \Omega \subset X \rightrightarrows X^*$ is called weak* closed at $\bar{x} \in \Omega$ if for any sequence $\{x_k\} \subset \Omega$, $x_k \to \bar{x}$, and any sequence $\{x^*_k\} \subset X^*$, $x^*_k \in H(x_k)$, $x^*_k \rightrightarrows x^*$, one has $x^* \in H(\bar{x})$. 


For a set-valued mapping $H : X \rightrightarrows X^*$, the sequential Painlevé-Kuratowski upper/outer limit of $H$ as $x \to \bar{x}$ is defined by

$$\operatorname{Lim sup}_{x \to \bar{x}} H(x) := \left\{ x^* \in X^* \mid \exists \text{ sequences } x_k \to \bar{x} \text{ and } x_k^* \rightrightarrows x^* \right\},$$

with $x_k^* \in H(x_k)$ for all $k \in \mathbb{N}$.

Let $\Omega \subset X$ be locally closed around $\bar{x} \in \Omega$, i.e. there is a neighbourhood $U$ of $\bar{x}$ for which $\Omega \cap \text{cl } U$ is closed. The Fréchet normal cone $\hat{N}(\bar{x}; \Omega)$ and the Mordukhovich normal cone $N(\bar{x}; \Omega)$ to $\Omega$ at $\bar{x} \in \Omega$ are defined by

$$\hat{N}(\bar{x}; \Omega) := \left\{ x^* \in X^* \mid \limsup_{x \to \bar{x}} \frac{\langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq 0 \right\}, \quad (4)$$

$$N(\bar{x}; \Omega) := \operatorname{Lim sup}_{x \to \bar{x}} \hat{N}(x; \Omega), \quad (5)$$

where $x \to \bar{x}$ stands for $x \to \bar{x}$ with $x \in \Omega$. If $\bar{x} \notin \Omega$, we put $\hat{N}(\bar{x}; \Omega) = N(\bar{x}; \Omega) := \emptyset$.

For an extended real-valued function $\phi : X \to \overline{\mathbb{R}}$, the limiting/Mordukhovich subdifferential and the regular/FrÉchet subdifferential of $\phi$ at $\bar{x} \in \text{dom } \phi$ are given, respectively, by

$$\partial \phi(\bar{x}) := \left\{ x^* \in X^* \mid (x^*, -1) \in N((\bar{x}, \phi(\bar{x})); \text{epi } \phi) \right\}$$

and

$$\hat{\partial} \phi(\bar{x}) := \left\{ x^* \in X^* \mid (x^*, -1) \in \hat{N}((\bar{x}, \phi(\bar{x})); \text{epi } \phi) \right\}.$$ 

If $|\phi(\bar{x})| = \infty$, then one puts $\partial \phi(\bar{x}) := \hat{\partial} \phi(\bar{x}) := \emptyset$.

Assign $(y^*, f)(x) := (y^*, f(x))$, $x \in X$, $y^* \in Y^*$, for a vector-valued map $f : X \to Y$, and denote $\text{gph } f := \{(x, y) \in X \times Y \mid y = f(x)\}$. Next we recall the required results known as the scalarization formulae of the coderivatives.

**Lemma 2.1:** Let $y^* \in Y^*$, and let $f : X \to Y$ be Lipschitz around $\bar{x} \in X$. We have

(i) (See [22, Proposition 3.5]) $x^* \in \hat{\partial} (y^*, f)(\bar{x}) \Leftrightarrow (x^*, -y^*) \in \hat{N}((\bar{x}, f(\bar{x})); \text{gph } f)$.

(ii) (See [21, Theorem 1.90]) $x^* \in \partial (y^*, f)(\bar{x}) \Leftrightarrow (x^*, -y^*) \in N((\bar{x}, f(\bar{x})); \text{gph } f)$.

Another calculus result is the sum rule for the limiting subdifferential.

**Lemma 2.2** (See [21, Theorem 3.36]): Let $\phi_i : X \to \overline{\mathbb{R}}$, ($i \in \{1, 2, \ldots, n\}, n \geq 2$), be lower semicontinuous around $\bar{x}$, and let all but one of these functions be Lipschitz continuous around $\bar{x} \in X$. Then, one has

$$\partial (\phi_1 + \phi_2 + \cdots + \phi_n)(\bar{x}) \subset \partial \phi_1(\bar{x}) + \partial \phi_2(\bar{x}) + \cdots + \partial \phi_n(\bar{x}).$$
The following lemma computes the limiting subdifferential for the maximum functions in Asplund spaces. The interested readers are referred to [14,23,24] for more details and proofs. The notation $\partial_x$ indicates the limiting subdifferential operation with respect to $x$.

**Lemma 2.3**: Let $\mathcal{V}$ be a sequentially compact topological space, and let $g : X \times \mathcal{V} \to \mathbb{R}$ be a function such that for each fixed $v \in \mathcal{V}$, $g(\cdot, v)$ is locally Lipschitz on $U \subset X$ and for each fixed $x \in U$, $g(x, \cdot)$ is upper semicontinuous on $\mathcal{V}$. Let $\phi(x) := \max_{v \in \mathcal{V}} g(x, v)$. If the multifunction $(x, v) \in U \times \mathcal{V} \Rightarrow \partial_x g(x, v) \subset X^*$ is weak* closed at $(\bar{x}, \bar{v})$ for each $\bar{v} \in \mathcal{V}(\bar{x})$, then the set $cl^*co(\bigcup \{\partial_x g(\bar{x}, v) \mid v \in \mathcal{V}(\bar{x})\})$ is nonempty and

$$\partial \phi(\bar{x}) \subset cl^*co\left(\bigcup \{\partial_x g(\bar{x}, v) \mid v \in \mathcal{V}(\bar{x})\}\right),$$

where $\mathcal{V}(\bar{x}) = \{v \in \mathcal{V} \mid g(\bar{x}, v) = \phi(\bar{x})\}$.

In what follows, we also use a formula for the limiting subdifferential of maximum of finitely many functions in Asplund spaces.

**Lemma 2.4** (See [21, Theorem 3.46]): Let $\phi_i : X \to \overline{\mathbb{R}}$, $(i \in \{1, 2, \ldots, n\}, n \geq 2)$, be Lipschitz continuous around $\bar{x}$. Put $\phi(x) := \max_{i \in \{1, 2, \ldots, n\}} \phi_i(x)$. Then

$$\partial \phi(\bar{x}) \subset \bigcup \left\{ \partial \left( \sum_{i \in I(\bar{x})} \mu_i \phi_i \right)(\bar{x}) \mid (\mu_1, \mu_2, \ldots, \mu_n) \in \Lambda(\bar{x}) \right\},$$

where

$$I(\bar{x}) := \{i \in \{1, 2, \ldots, n\} \mid \phi_i(\bar{x}) = \phi(\bar{x})\}$$

and

$$\Lambda(\bar{x}) := \left\{ (\mu_1, \mu_2, \ldots, \mu_n) \mid \mu_i \geq 0, \sum_{i=1}^n \mu_i = 1, \mu_i (\phi_i(\bar{x}) - \phi(\bar{x})) = 0 \right\}.$$

**Assumptions** (See [14, p.131]): Suppose $\mathcal{V}$ be a sequentially compact topological space, and let $f : X \to Y$ and $g : X \times \mathcal{V} \to \mathbb{R}^n$ are functions satisfying the following hypotheses:

*(A1)* For a fixed $\bar{x} \in \Omega$, $g$ is locally Lipschitz in the first argument and uniformly on $\mathcal{V}$ in the second argument, i.e. there exist an open neighbourhood $U$ of $\bar{x}$ and a positive constant $\ell$ such that $\|g(z, v) - g(y, v)\| \leq \ell \|z - y\|$ for all $z, y \in U$ and $v \in \mathcal{V}$.

*(A2)* For each $i = 1, 2, \ldots, n$, the function $v_i \in \mathcal{V}_i \mapsto g_i(x, v_i) \in \mathbb{R}$ is upper semicontinuous for each $x \in U$. 
For a fixed $A_5$, we define real-valued functions $\phi_i$ and $\phi$ on $X$ via

$$\phi_i(x) := \max_{v_i \in V_i} g_i(x, v_i) \quad \text{and} \quad \phi(x) := \max_{i \in [1, 2, \ldots, n]} \phi_i(x),$$

and we notice that above assumptions imply that $\phi_i$ is well defined on $V_i$. In addition, $\phi_i$ and $\phi$ follow readily that are locally Lipschitz on $U$, since each $g_i(\cdot, v_i)$ is (see [14, (H1), p.131] and [10, p.290]). Note that the feasible set $F$ can be equivalently characterized by:

$$F = \{ x \in \Omega \mid \phi_i(x) \leq 0, \ i = 1, 2, \ldots, n \} = \{ x \in \Omega \mid \phi(x) \leq 0 \}.$$

For each $i = 1, 2, \ldots, n$, the multifunction $(x, v_i) \in U \times V_i \Rightarrow \partial_x g_i(x, v_i) \subset X^*$ is weak* closed at $(\bar{x}, v_i)$ for each $v_i \in V_i(\bar{x})$, where $V_i(\bar{x}) = \{ v_i \in V_i \mid g_i(\bar{x}, v_i) = \phi_i(\bar{x}) \}$.

For a fixed $\bar{x} \in \Omega$, $\vartheta \in K$, and $y^* \in K^+$, we define a real-valued function $\psi$ on $X$ as follows:

$$\psi(x) := \max \{ \langle y^*, f(x) - f(\bar{x}) + \vartheta, x \rangle, \phi(x) \}.$$

Inspired by the concept of pseudo-quasi generalized convexity by Fakhar [15], we introduce a similar concept of pseudo-quasi convexity type for $f$ and $g$.

**Definition 2.1:** Let $\vartheta \in K$, we say that

(i) $(f, g)$ is $\vartheta$-type I pseudo convex on $\Omega$ at $\bar{x} \in \Omega$ if for any $x \in \Omega$, $y^* \in K^+$, $u^* \in \partial (y^*, f)(\bar{x})$, and $v_i^* \in \partial_x g_i(\bar{x}, v_i)$, $v_i \in V_i(\bar{x}), i = 1, 2, \ldots, n$, there exists $w \in -N(\bar{x}, \Omega)^+$ such that

$$\langle y^*, f \rangle(x) < \langle y^*, f \rangle(\bar{x}) - \|x - \bar{x}\| \langle y^*, \vartheta \rangle \implies \langle u^*, w \rangle + \|x - \bar{x}\| \langle y^*, \vartheta \rangle < 0,$$

$$g_i(x, v_i) \leq g_i(\bar{x}, v_i) \implies \langle v_i^*, w \rangle \leq 0, \quad i = 1, 2, \ldots, n,$$

$$\|w\| \leq \|x - \bar{x}\|.$$

(ii) $(f, g)$ is $\vartheta$-type II pseudo convex on $\Omega$ at $\bar{x} \in \Omega$ if for any $x \in \Omega \setminus \{ \bar{x} \}$, $y^* \in K^+ \setminus \{ 0 \}$, $u^* \in \partial (y^*, f)(\bar{x})$, and $v_i^* \in \partial_x g_i(\bar{x}, v_i)$, $v_i \in V_i(\bar{x}), i = 1, 2, \ldots, n$, there exists $w \in -N(\bar{x}, \Omega)^+$ such that

$$\langle y^*, f \rangle(x) \leq \langle y^*, f \rangle(\bar{x}) - \|x - \bar{x}\| \langle y^*, \vartheta \rangle \implies \langle u^*, w \rangle + \|x - \bar{x}\| \langle y^*, \vartheta \rangle < 0,$$

$$g_i(x, v_i) \leq g_i(\bar{x}, v_i) \implies \langle v_i^*, w \rangle \leq 0, \quad i = 1, 2, \ldots, n,$$

$$\|w\| \leq \|x - \bar{x}\|.$$

**Remark 2.1:** If in Definition 2.1,

(i) we set $\Omega = X$ and $\vartheta = 0$, then this definition reduces to [11, Definition 2.2].

(ii) we set $Y = \mathbb{R}^p$, then this definition reduces to [15, Definition 3.8].
(ii) we set $\Omega = X$, $Y = \mathbb{R}^p$, and $\vartheta = 0$, then this definition reduces to [12, Definition 3.2].

**Remark 2.2:**

(i) It follows from Definition 2.1 that if $(f, g)$ is $\vartheta$-type II pseudo convex on $\Omega$ at $\bar{x} \in \Omega$, then $(f, g)$ is $\vartheta$-type I pseudo convex on $\Omega$ at $\bar{x} \in \Omega$, but converse is not true (see Example 2.3).

(ii) It is noted that the generalized (resp., strictly generalized) convexity (see [15, Definition 3.2]) of $(f, g)$ is reduced to the $\vartheta$-type I (resp., type II) pseudo convexity of $(f, g)$. Furthermore, as the next example demonstrates, the class of $\vartheta$-type I pseudo convex functions is properly wider than the class of generalized convex functions, which is properly broader than convex functions (see [16, Example 3.12]).

**Example 2.3:** Let $X := \mathbb{R}^2$, $Y := \mathbb{R}^3$, $\Omega := \mathbb{R}^2$, $\mathcal{V}_i := [-1, -\frac{1}{4}]$, $i = 1, 2$, $\mathcal{V} := \prod_{i=1}^2 \mathcal{V}_i$, and let $K := \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid y_1 \leq 0 \text{ and } y_i \geq 0 \text{ for } i = 2, 3\}$. Consider $f : X \to Y$ and $g : X \times \mathcal{V} \to \mathbb{R}^2$ defined by $f := (f_1, f_2, f_3)$ and $g := (g_1, g_2)$, respectively, where

$$
\begin{align*}
  f_1(x_1, x_2) &= 5|x_1| - \frac{2}{5}x_2 + \frac{4}{5}, \\
  f_2(x_1, x_2) &= \frac{1}{2}|x_1| + 6, \\
  f_3(x_1, x_2) &= 4|x_1| + \frac{1}{2}x_2 + 1 \\
  g_1(x_1, x_2, v_1) &= \frac{1}{4}v_1^2|x_1| + \frac{1}{2}v_1^2x_2 - v_1^2 + \frac{1}{4}|v_1|, \\
  g_2(x_1, x_2, v_2) &= \frac{1}{8}x_1^2 + |v_2|x_2 - |v_2| + \frac{1}{4},
\end{align*}
$$

$v_i \in \mathcal{V}_i$, $i = 1, 2$. Let $\vartheta := (0, 0, \frac{3}{2}) \in K$ and consider $\bar{x} := (0, 0) \in \Omega$, Hence $N(\bar{x}; \Omega) = \{(0, 0)\}$ and $N(\bar{x}; \Omega)^+ = \mathbb{R}^2$. Obviously, from the definitions,

$$
\partial f_1(\bar{x}) = [-5, 5] \times \left\{ -\frac{2}{5} \right\}, \quad \partial f_2(\bar{x}) = \left[ -\frac{1}{2}, \frac{1}{2} \right] \times \{0\}, \quad \text{and} \quad \partial f_3(\bar{x}) = [-4, 4] \times \left\{ \frac{1}{2} \right\}.
$$

Moreover

$$
\partial x g_1(\bar{x}, v_1) = \left[ -\frac{1}{4}v_1^2, \frac{1}{4}v_1^2 \right] \times \left\{ \frac{1}{2}v_1^2 \right\} \quad \text{and} \quad \partial x g_2(\bar{x}, v_2) = (0, |v_2|)
$$

for all $v_i \in \mathcal{V}_i$, $i = 1, 2$. 

Suppose that for some \( x := (x_1, x_2) \in \Omega \) and \( y^* := (y_1^*, y_2^*, y_3^*) \in K^+ \) the condition \( \langle y^*, f \rangle(x) < \langle y^*, f \rangle(\bar{x}) - \|x - \bar{x}\| \langle y^*, \partial \rangle \) is satisfied. Thus

\[
\left(-\frac{2}{5}y_1^* + \frac{1}{2}y_3^* \right) x_2 < - \left(5y_1^* + \frac{1}{2}y_2^* + 4y_3^* \right) |x_1| - \frac{1}{c} \|x - \bar{x}\| \langle y^*, \partial \rangle.
\]

Dividing both sides of above inequality by \( c := -\frac{2}{5}y_1^* + \frac{1}{2}y_3^* > 0 \), we have

\[
x_2 < -\frac{1}{c} \left(5y_1^* + \frac{1}{2}y_2^* + 4y_3^* \right) |x_1| - \frac{1}{c} \|x - \bar{x}\| \langle y^*, \partial \rangle.
\]

Putting \( (w_1, w_2) := w = x \in -N(\bar{x}; \Omega)^+ \) and employing (6), for any \( u^* := y_1^* u_1^* + y_2^* u_2^* + y_3^* u_3^* \in \partial \langle y^*, f \rangle(\bar{x}) \), where \( u_i^* := (u_{i,x}, u_{i,y}) \in \partial f_i(\bar{x}) \), \( i = 1, 2, 3 \), one has \( \|w\| \leq \|x - \bar{x}\| \) and

\[
\langle u^*, w \rangle + \|x - \bar{x}\| \langle y^*, \partial \rangle = \left(\begin{array}{cc} y_1^* u_{1,x} + y_2^* u_{2,x} + y_3^* u_{3,x} \\ -\frac{2}{5}y_1^* + \frac{1}{2}y_3^* \end{array} \right) \times \left(\begin{array}{c} w_1 \\ w_2 \end{array} \right) + \|x - \bar{x}\| \langle y^*, \partial \rangle < 0,
\]

where the latter inequality is due to \( u_{1,x}^* \in [-5, 5] \), \( u_{2,x}^* \in [-\frac{1}{2}, \frac{1}{2}] \), and \( u_{3,x}^* \in [-4, 4] \). So \( \langle u^*, w \rangle + \|x - \bar{x}\| \langle y^*, \partial \rangle < 0 \).

Now, for \( x \in \Omega \) and \( v_1 \in N_1(\bar{x}) \) the condition \( g_1(x, v_1) \leq g_1(\bar{x}, v_1) \) implies \( \frac{1}{2} v_1^2 x_2 \leq -\frac{1}{4} v_1^2 |x_1| \), and therefore for any \( v_1^* := (v_{1,x}^*, v_{1,y}^*) \in \partial x g_1(\bar{x}, v_1) \), we get

\[
\langle v_1^*, w \rangle \leq v_{1,x}^* x_1 - \frac{1}{4} v_1^2 |x_1| \leq 0
\]

due to \( v_{1,x}^* \in [-\frac{1}{4} v_1^2, \frac{1}{4} v_1^2] \). Similarly for \( x \in \Omega \) and \( v_2 \in N_2(\bar{x}) \) satisfying \( g_2(x, v_2) \leq g_2(\bar{x}, v_2) \), we have \( x_2 \leq -\frac{1}{8 |v_1|} x_1^2 \), and thus for any \( v_2^* \in \partial x g_2(\bar{x}, v_2) \), it holds

\[
\langle v_2^*, w \rangle \leq -\frac{1}{8} x_1^2 \leq 0.
\]

Therefore, \( (f, g) \) is \( \vartheta \)-type I pseudo convex on \( \Omega \) at \( \bar{x} \).

Although, there exist \( x := (0, 1) \in \Omega \setminus \{\bar{x}\} \) and \( y^* := (0, 1, 0) \in K^+ \setminus \{0\} \) such that \( \langle y^*, f \rangle(x) = 6 = \langle y^*, f \rangle(\bar{x}) - \|x - \bar{x}\| \langle y^*, \partial \rangle \), but for \( u_1^* := (0, -\frac{2}{5}) \in \partial f_1(\bar{x}), u_2^* := (0, 0) \in \partial f_2(\bar{x}) \), and \( u_3^* := (0, \frac{1}{2}) \in \partial f_3(\bar{x}) \), one has

\[
u^* := y_1^* u_1^* + y_2^* u_2^* + y_3^* u_3^* = (0, 0)
\]

so \( \langle u^*, w \rangle + \|x - \bar{x}\| \langle y^*, \partial \rangle = 0 \) for any \( w \in -N(\bar{x}; \Omega)^+ \). This signifies that \( (f, g) \) is not \( \vartheta \)-type II pseudo convex on \( \Omega \) at \( \bar{x} \). On the other side, there exist \( x :=
Suppose Example 2.4: Suppose that for some \( x \parallel \leq \| x - \tilde{x} \| \) we have
\[
(y^*, f)(x) - (y^*, f)(\tilde{x}) = 0,
\]
\[
g_1(x, v_1) - g_1(\tilde{x}, v_1) = -\frac{9}{4}|v_1|^2 < 0,
\]
\[
g_2(x, v_2) - g_2(\tilde{x}, v_2) = \frac{1}{8} - 5|v_2| < 0.
\]

Hence, \((f, g)\) is not generalized convex on \( \Omega \) at \( \tilde{x} \).

**Example 2.4:** Suppose \( X, Y, \Omega, \mathcal{V}, i = 1, 2, \mathcal{V} := \prod_{i=1}^{2} \mathcal{V}_i \), and \( K \) be the same as Example 2.3. Let \( f : X \rightarrow Y \) defined by \( f := (f_1, f_2, f_3) \), where
\[
\begin{align*}
  f_1(x_1, x_2) &= -\frac{4}{5}x_1^2 + 5|x_1| - \frac{4}{5}x_2^2 - \frac{2}{5}x_2 + \frac{4}{5}, \\
  f_2(x_1, x_2) &= \frac{1}{2}|x_1| + 6, \\
  f_3(x_1, x_2) &= x_1^2 + 4|x_1| + x_2^2 + \frac{1}{2}x_2 + 1,
\end{align*}
\]

and let \( g : X \times \mathcal{V} \rightarrow \mathbb{R}^2 \) be the same as Example 2.3. Let \( \tilde{x} := (0, 0) \in \Omega \) and \( \vartheta \) be the same as Example 2.3. Then
\[
\partial f_1(\tilde{x}) = [-5, 5] \times \left\{ -\frac{2}{5} \right\}, \quad \partial f_2(\tilde{x}) = \left[ -\frac{1}{2}, \frac{1}{2} \right] \times \{0\}, \quad \text{and}
\]
\[
\partial f_3(\tilde{x}) = [-4, 4] \times \left\{ \frac{1}{2} \right\}.
\]

Suppose that for some \( x := (x_1, x_2) \in \Omega \setminus \{ \tilde{x} \} \) and \( y^* := (y_1^*, y_2^*, y_3^*) \in K^+ \setminus \{0\} \) the condition \((y^*, f)(x) \leq (y^*, f)(\tilde{x}) - \|x - \tilde{x}\|(y^*, \vartheta)\) is satisfied. Therefore
\[
\left( -\frac{2}{5}y_1^* + \frac{1}{2}y_3^* \right)x_2 \leq -\left( -\frac{4}{5}y_1^* + y_3^* \right)x_1^2 - \left( 5y_1^* + \frac{1}{2}y_2^* + 4y_3^* \right)|x_1|
\]
\[
-\left( -\frac{4}{5}y_1^* + y_3^* \right)x_2^2 - \|x - \tilde{x}\|(y^*, \vartheta).
\]

Dividing both sides of above inequality by \( c := \frac{2}{5}y_1^* + \frac{1}{2}y_3^* > 0 \), we have
\[
x_2 \leq -2x_1^2 - \frac{1}{c} \left( 5y_1^* + \frac{1}{2}y_2^* + 4y_3^* \right)|x_1| - 2x_2^2 - \frac{1}{c}\|x - \tilde{x}\|(y^*, \vartheta). \tag{7}
\]
Putting \((w_1, w_2) := w = x \in -N(\bar{x}; \Omega)^+\) and using \((7)\), for any \(u^* := y_1^* u_1^* + y_2^* u_2^* + y_3^* u_3^* \in \partial (y^*, f)(\bar{x})\), where \(u_i^* := (u_{ix}^*, u_{iy}^*) \in \partial f_i(\bar{x}), i = 1, 2, 3\), we get

\[
\langle u^*, w \rangle + \|x - \bar{x}\| \langle y^*, \theta \rangle \\
\leq (y_1^* u_{1x}^* + y_2^* u_{2x}^* + y_3^* u_{3x}^*) x_1 - 2c x_1^2 - \left(5y_1^* + \frac{1}{2} y_2^* + 4y_3^*\right) |x| - 2c x_2^2 < 0,
\]

where the latter strict inequality is due to \(u_{1x}^* \in [-5, 5], u_{2x}^* \in [-\frac{1}{2}, \frac{1}{2}], u_{3x}^* \in [-4, 4]\), and \(x \neq \bar{x}\). So \(\langle u^*, w \rangle + \|x - \bar{x}\| \langle y^*, \theta \rangle < 0\). The complete calculation is similar to that of Example 2.3. Hence, \((f, g)\) is \(\vartheta\)-type II pseudo convex on \(\Omega\) at \(\bar{x}\).

In the rest of this section, we present a suitable constraint qualification in the sense of robust, which is required to get a so-called robust \(\vartheta\)-approximate Karush-Kuhn-Tucker (KKT) condition.

**Definition 2.2 (See [15, Definition 4.5]):** Let \(\bar{x} \in F\). We say that the constraint qualification (CQ) condition is satisfied at \(\bar{x}\) if

\[
0 \notin cl^* co \left( \bigcup \left\{ \partial x g_i(\bar{x}, v_i) \mid v_i \in \mathcal{V}_i(\bar{x}) \right\} \right) + N(\bar{x}; \Omega), \quad i \in I(\bar{x}),
\]

where \(I(\bar{x}) := \{i \in \{1, 2, \ldots, n\} \mid \phi_i(\bar{x}) = \phi(\bar{x})\}\).

It is noteworthy here that this condition (CQ) is reduced to the extended Mangasarian-Fromovitz constraint qualification (EMFCQ) in the smooth setting; see e.g. [21] for more details.

**Definition 2.3:** Let \(\vartheta \in K\) and \(\bar{x} \in F\). One says that \(\bar{x}\) satisfies the robust \(\vartheta\)-approximate (KKT) condition if there exist \(y^* \in K^+ \setminus \{0\}\), \(\mu := (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{R}^n_+\), and \(\tilde{v}_i \in \mathcal{V}_i, i = 1, 2, \ldots, n\), such that

\[
\begin{aligned}
0 &\in \partial (y^*, f)(\bar{x}) + \sum_{i=1}^n \mu_i cl^* co \left( \bigcup \left\{ \partial x g_i(\bar{x}, v_i) \mid v_i \in \mathcal{V}_i(\bar{x}) \right\} \right) \\
&\quad + \langle y^*, \vartheta \rangle B_{X^*} + N(\bar{x}; \Omega), \\
\mu_i \max_{v_i \in \mathcal{V}_i} g_i(\bar{x}, v_i) &= \mu_i g_i(\bar{x}, \tilde{v}_i) = 0, \quad i = 1, 2, \ldots, n.
\end{aligned}
\]

Therefore, the robust \(\vartheta\)-approximate (KKT) condition defined above is guaranteed by the constraint qualification (CQ).
3. Robust necessary and sufficient optimality conditions

This section is devoted to study necessary optimality conditions for weakly robust $\vartheta$-efficient solutions and weakly robust $\vartheta$-quasi-efficient solutions of problem (UP) by exploiting the nonsmooth version of Fermat’s rule, the sum rule for the limiting subdifferential and the scalarization formulae of the coderivatives, and to discuss sufficient optimality conditions for (weakly) robust $\vartheta$-quasi-efficient solutions by imposing the pseudo convexity assumptions.

The first theorem presents a necessary optimality condition in a fuzzy form for weakly robust $\vartheta$-efficient solutions of problem (UP).

**Theorem 3.1**: Suppose that $f$ and $g_i$, $i = 1, 2, \ldots, n$, satisfy Assumptions (A1)–(A5). If $\bar{x} \in \vartheta^{-S^w}(RP)$, then there exist $x_\eta \in \Omega$, with $\|x_\eta - \bar{x}\| \leq \eta$, $y^* \in K^+$, $\lambda := (\lambda_1, \lambda_2) \in \mathbb{R}^2_+$, with $\lambda_2 \neq 0$, and $\mu := (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{R}^n_+$, with $\frac{\lambda_1}{\lambda_2} \|y^*\| + \|\mu\| = 1$, and $v_\eta \in V_i$, $i = 1, 2, \ldots, n$, such that

$$0 \in \frac{\lambda_1}{\lambda_2} \partial (y^*, f)(x_\eta) + (1 - \lambda_1) \sum_{i=1}^n \mu_i cI^* \text{co} \left( \bigcup \left\{ \partial_x g_i(x_\eta, v_i) \mid v_i \in V_i(x_\eta) \right\} \right)$$

$$+ \frac{\langle y^*, \vartheta \rangle}{\lambda_2 \eta} Bx^* + N(x_\eta; \Omega),$$

$$\frac{\lambda_1}{\lambda_2} \left( \langle y^*, f(x_\eta) - f(\bar{x}) + \vartheta \rangle \right) - \psi(x_\eta) = 0,$$

$$(1 - \lambda_1) \mu_i \left( g_i(x_\eta, v_i_\eta) - \psi(x_\eta) \right) = 0, \quad i = 1, 2, \ldots, n.$$

**Proof**: If $\bar{x} \in \vartheta^{-S^w}(RP)$, then we have $f(x) - f(\bar{x}) + \vartheta \notin -\text{int} K^+$ for all $x \in F$. Using the separation theorem, there exists $y^* \in K^+$ such that

$$\langle y^*, f(x) - f(\bar{x}) + \vartheta \rangle \geq 0, \quad \forall x \in F. \quad (8)$$

Let us consider the function $\psi$ and take into account (8), it can be easily obtained that

$$0 \leq \psi(x), \quad \forall x \in \Omega, \quad (9)$$

which implies that $\psi$ is bounded from below on $\Omega$, too.

Furthermore, due to $\bar{x} \in F$, it holds that $\psi(\bar{x}) = \langle y^*, \vartheta \rangle$. Thus, from (9) we get that

$$\psi(\bar{x}) \leq \inf_{x \in \Omega} \psi(x) + \langle y^*, \vartheta \rangle.$$ 

For any $\eta > 0$, using the *Ekeland's variable principle* (see [25, Theorem 1.1]), we arrive at $x_\eta \in \Omega$ such that $\|x_\eta - \bar{x}\| \leq \eta$ and

$$\psi(x_\eta) \leq \psi(x) + \frac{\langle y^*, \vartheta \rangle}{\eta} \|x_\eta - x\|, \quad \forall x \in \Omega.$$
This denotes that $x_\eta$ is a minimizer to the optimization problem

$$\min_{x \in \Omega} \omega(x),$$

where

$$\omega(x) := \psi(x) + \frac{\langle y^*, \vartheta \rangle}{\eta} \|x_\eta - x\|, \quad x \in \Omega. \quad (10)$$

Thus $x_\eta$ is a minimizer to the unconstrained optimization problem

$$\min_{x \in X} \omega(x) + \delta(x; \Omega). \quad (11)$$

Applying the generalized Fermat’s rule (see [21, Proposition 1.114]), we obtain

$$0 \in \partial \left( \omega + \delta(\cdot; \Omega) \right)(x_\eta). \quad (12)$$

Since the function $\omega$ is Lipschitz continuous around $x_\eta$ and the function $\delta(\cdot; \Omega)$ is l.s.c around this point, from the sum rule of Lemma 2.2 applied to (12) and from the relation $\partial \delta(x_\eta; \Omega) = N(x_\eta; \Omega)$ we get that

$$0 \in \partial \omega(x_\eta) + N(x_\eta; \Omega). \quad (13)$$

Also note that (see [26, Example 4])

$$\partial \left( \|x_\eta - x\| \right)(x_\eta) = B_{X^*}.$$

Use the summation rule again to $\omega$ defined in (10) and using (13), we arrive at

$$0 \in \partial \psi(x_\eta) + \frac{\langle y^*, \vartheta \rangle}{\eta} B_{X^*} + N(x_\eta; \Omega). \quad (14)$$

Now, applying the formula for the limiting subdifferential of maximum functions in Lemma 2.4, one has

$$\partial \psi(x_\eta) \subset \bigcup \left\{ \partial \left( \alpha_1 \langle y^*, f(\cdot) - f(\bar{x}) + \vartheta \rangle + \alpha_2 \phi(\cdot) \right)(x_\eta) : \alpha_1, \alpha_2 \geq 0, \right.$$

$$\alpha_1 + \alpha_2 = 1,$$

$$\alpha_1 \left( \langle y^*, f(x_\eta) - f(\bar{x}) + \vartheta \rangle - \psi(x_\eta) \right) = 0, \quad \alpha_2 \left( \phi(x_\eta) - \psi(x_\eta) \right) = 0 \biggr\}. \quad (15)$$

This together with (14) and using the sum rule give us $(\bar{\alpha}_1, \bar{\alpha}_2) \in \mathbb{R}_+^2$ with $\bar{\alpha}_1 + \bar{\alpha}_2 = 1$, such that

$$\bar{\alpha}_1 \left( \langle y^*, f(x_\eta) - f(\bar{x}) + \vartheta \rangle - \psi(x_\eta) \right) = 0, \quad (15)$$

$$\bar{\alpha}_2 \left( \phi(x_\eta) - \psi(x_\eta) \right) = 0, \quad (16)$$

and

$$0 \in \bar{\alpha}_1 \partial \langle y^*, f \rangle(x_\eta) + \bar{\alpha}_2 \partial \phi(x_\eta) + \frac{\langle y^*, \vartheta \rangle}{\eta} B_{X^*} + N(x_\eta; \Omega). \quad (17)$$
Invoking again Lemma 2.4, we have

\[ \partial \phi(x_{\eta}) \subset \bigcup \left\{ \partial \left( \sum_{i \in I(x_{\eta})} \mu_i \Phi_i \right) (x_{\eta}) \mid (\mu_1, \mu_2, \ldots, \mu_n) \in \Lambda(x_{\eta}) \right\}, \quad (18) \]

where \( I(x_{\eta}) = \{ i \in \{1, 2, \ldots, n\} \mid \phi_i(x_{\eta}) = \phi(x_{\eta}) \} \) and

\[ \Lambda(x_{\eta}) = \left\{ (\mu_1, \mu_2, \ldots, \mu_n) \mid \mu_i \geq 0, \sum_{i=1}^{n} \mu_i = 1, \mu_i (\phi_i(x_{\eta}) - \phi(x_{\eta})) = 0 \right\}. \]

Using further Lemma 2.3, we arrive at

\[ \partial \phi_i(x_{\eta}) \subset \text{cl}^* \text{co} \left( \bigcup \left\{ \partial x g_i(x_{\eta}, v_i) \mid v_i \in \mathcal{V}_i(x_{\eta}) \right\} \right), \quad i = 1, 2, \ldots, n, \quad (19) \]

where \( \mathcal{V}_i(x_{\eta}) = \{ v_i \in \mathcal{V}_i \mid g_i(x_{\eta}, v_i) = \phi(x_{\eta}) \} \) and the set \( \text{cl}^* \text{co} \left( \bigcup \{ \partial x g_i(x_{\eta}, v_i) \mid v_i \in \mathcal{V}_i(x_{\eta}) \} \right) \) is nonempty. The sum rule of the limiting subdifferential and the relations (17)–(19) results to

\[ 0 \in \tilde{\alpha}_1 \partial \langle y^*, f \rangle(x_{\eta}) + \tilde{\alpha}_2 \bigg\{ \sum_{i \in I(x_{\eta})} \mu_i \text{cl}^* \text{co} \left( \bigcup \left\{ \partial x g_i(x_{\eta}, v_i) \mid v_i \in \mathcal{V}_i(x_{\eta}) \right\} \right) \bigg\} \]
\[ \left( \mu_1, \mu_2, \ldots, \mu_n \right) \in \Lambda(x_{\eta}) \bigg\} + \frac{\langle y^*, \eta \rangle}{\eta} B_{X^*} + N(x_{\eta}; \Omega). \]

So, there exist \( \bar{\mu} := (\bar{\mu}_1, \bar{\mu}_2, \ldots, \bar{\mu}_n) \in \Lambda(x_{\eta}) \), with \( \sum_{i=1}^{n} \bar{\mu}_i = 1 \) and \( \bar{\mu}_i = 0 \) for all \( i \in \{1, 2, \ldots, n\} \setminus I(x_{\eta}) \), such that

\[ 0 \in \tilde{\alpha}_1 \partial \langle y^*, f \rangle(x_{\eta}) + \tilde{\alpha}_2 \sum_{i=1}^{n} \bar{\mu}_i \text{cl}^* \text{co} \left( \bigcup \left\{ \partial x g_i(x_{\eta}, v_i) \mid v_i \in \mathcal{V}_i(x_{\eta}) \right\} \right) \]
\[ + \frac{\langle y^*, \eta \rangle}{\eta} B_{X^*} + N(x_{\eta}; \Omega). \]

Putting \( \lambda_1 := \tilde{\alpha}_1, \lambda_2 := \tilde{\alpha}_2 \|y^*\| + \|\bar{\mu}\|, \) and \( \mu := \frac{\lambda_1}{\lambda_2} \bar{\mu} \) and dividing the above inclusion by \( \lambda_2 \), we have \( y^* \in K^+, \lambda := (\lambda_1, \lambda_2) \in \mathbb{R}^2_+, \lambda_2 \neq 0, \) and \( \mu \in \mathbb{R}_+^n \), with \( \frac{\lambda_1}{\lambda_2} \|y^*\| + \|\mu\| = 1 \), satisfying the first relation in the theorem.

On the other hand, due to the upper semicontinuity of the function \( v_i \in \mathcal{V}_i \mapsto g_i(x_{\eta}, v_i) \) for each \( i = 1, 2, \ldots, n \) and also, the sequentially compactness of \( \mathcal{V}_i \), we can select \( v_i \in \mathcal{V}_i \) such that \( g_i(x_{\eta}, v_i) = \max_{v_i \in \mathcal{V}_i} g_i(x_{\eta}, v_i) = \phi_i(x_{\eta}) \).

By considering \( \phi_i(x_{\eta}) = \phi(x_{\eta}) \) for all \( i \in I(x_{\eta}) \) and using relations (15) and (16), the proof of the theorem is achieved. \( \blacksquare \)
Remark 3.1: Theorem 3.1 develops [16, Theorem 3.4], where the underlying optimization problem has a finite dimension framework.

Similarly, we establish a necessary optimality condition in the sense of the limiting subdifferential for weakly robust $\vartheta$-quasi-efficient solutions of problem (UP). To prove this theorem, it is required to state a fuzzy necessary optimality condition in terms of the Fréchet subdifferential for weakly robust $\vartheta$-quasi-efficient solutions of problem (UP) as follows.

Theorem 3.2 (See [27, Theorem 3.2]): Let $\tilde{x} \in \vartheta$-quasi-$S^w$ (RP). Then for each $k \in \mathbb{N}$, there exist $x^{1k} \in B_X(\tilde{x}, \frac{1}{k})$, $x^{2k} \in B_X(\tilde{x}, \frac{1}{k})$, $x^{3k} \in \Omega \cap B_X(\tilde{x}, \frac{1}{k})$, $y^*_k \in K^+$ with $\|y^*_k\| = 1$, and $\alpha_k \in \mathbb{R}_+$ such that

$$0 \in \partial \langle y^*_k, f \rangle(x^{1k}) + \alpha_k \partial \varphi(x^{2k}) + \hat{N}(x^{3k}; \Omega) + \left( \langle y^*_k, \vartheta \rangle + \frac{1}{k} \right) B_{X^*},$$

$$|\alpha_k \varphi(x^{2k})| \leq \frac{1}{k}.$$

Theorem 3.3: Suppose that $g_i$, $i = 1, 2, \ldots, n$, satisfy Assumptions (A1)–(A4). If $\tilde{x} \in \vartheta$-quasi-$S^w$ (RP), then there exist $y^* \in K^+$, $\mu := (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{R}_+^n$, with $\|y^*\| = 1$, and $\tilde{v}_i \in \mathcal{V}_i$, $i = 1, 2, \ldots, n$, such that

$$0 \in \partial \langle y^*, f \rangle(\tilde{x}) + \sum_{i=1}^n \mu_i c_i \co \left( \bigcup \left\{ \partial_x g_i(\tilde{x}, v_i) \mid v_i \in \mathcal{V}_i(\tilde{x}) \right\} \right)$$

$$+ \langle y^*, \vartheta \rangle B_{X^*} + N(\tilde{x}; \Omega),$$

$$\mu_i \max_{v_i \in \mathcal{V}_i} g_i(\tilde{x}, v_i) = \mu_i g_i(\tilde{x}, v_i) = 0, \quad i = 1, 2, \ldots, n. \tag{20}$$

Furthermore, if the (CQ) is satisfied at $\tilde{x}$, then (20) holds with $y^* \neq 0$.

Proof: Let $\tilde{x} \in \vartheta$-quasi-$S^w$ (RP). By using Theorem 3.2, we obtain sequences $x^{1k} \to \tilde{x}$, $x^{2k} \to \tilde{x}$, $x^{3k} \to \tilde{x}$, $y^*_k \in K^+$ with $\|y^*_k\| = 1$, $\alpha_k \in \mathbb{R}_+$, $x^{1k}_1 \in \partial \langle y^*_k, f \rangle(x^{1k})$, $x^{2k}_2 \in \alpha_k \partial \varphi(x^{2k})$, and $x^{3k}_3 \in \hat{N}(x^{3k}; \Omega)$ satisfying

$$0 \in x^{1k}_1 + x^{2k}_2 + x^{3k}_3 + \left( \langle y^*_k, \vartheta \rangle + \frac{1}{k} \right) B_{X^*},$$

$$\alpha_k \varphi(x^{2k}) \to 0 \text{ as } k \to \infty. \tag{21}$$

Now we can consider two possibilities for the sequence $\{\alpha_k\}$:

Case 1: Suppose that $\{\alpha_k\}$ is bounded, therefore without loss of generality we can assume that $\alpha_k \to \alpha \in \mathbb{R}_+$ as $k \to \infty$. In addition, since the sequence $\{y^*_k\} \subset K^+$ is bounded, by applying the weak* sequential compactness of bounded sets in duals to Asplund spaces, there is no loss of generality in assuming that $y^*_k \overset{w^*}{\to} \bar{y}^* \in K^+$ with $\|\bar{y}^*\| = 1$ as $k \to \infty$. Let $\ell_1 > 0$ be a Lipschitz modulus of $f$ around $\tilde{x}$. 


It is obvious that \( \|x_{1k}^*\| \leq \ell_1 \|y_k^*\| \leq \ell_1 \) for all \( k \in \mathbb{N} \) (see, [21, Proposition 1.85]).

As above, by taking a subsequence, if necessary, that \( x_{1k}^* \xrightarrow{w} x_1^* \in X^* \) as \( k \to \infty \). Due to the boundedness of \( \{\alpha_k\} \) and the Lipschitz continuity of \( \phi \) around \( \bar{x} \), the sequence \( \{x_{2k}^*\} \) is also bounded. In this regard, we can have \( x_2^* \in X^* \) such that \( x_{2k}^* \xrightarrow{w} x_2^* \in X^* \) as \( k \to \infty \). Using the part (i) of Lemma 2.1 to the inclusion \( x_{1k}^* \in \partial(y_k^*, f)(x^{1k}) \) gives us,

\[
(x_{1k}^*, -y_k^*) \in N((x^{1k}, f(x^{1k})); gph f), \quad k \in \mathbb{N}.
\]

Passing the limit as \( k \to \infty \) and applying the definitions of normal cones (4) and (5), we obtain \((x_1^*, -\bar{y}^*) \in N((\bar{x}, f(\bar{x})); gph f)\), which equals to

\[
x_1^* \in \partial(\bar{y}^*, f)(\bar{x}), \quad (22)
\]
due to the part (ii) of Lemma 2.1. Similarly, we get

\[
x_2^* \in \alpha \partial \phi(\bar{x}). \quad (23)
\]

From (21), there exists \( b_k^* \in B_{X^*} \) such that

\[
-x_{1k}^* - x_{2k}^* - \left(\langle y_k^*, \vartheta \rangle + \frac{1}{k}\right) b_k^* = x_{2k}^* \in \hat{N}(x^{3k}; \Omega), \quad k \in \mathbb{N}. \quad (24)
\]

Supposing \( b_k^* \xrightarrow{k} b^* \in B_{X^*} \) as \( k \to \infty \) and passing (24) to the limit as \( k \to \infty \), as well as considering (4) and (5), we arrive at

\[
-x_1^* - x_2^* - \langle \bar{y}^*, \vartheta \rangle b^* \in N(\bar{x}; \Omega).
\]

Combining the latter with (22) and (23) gives us

\[
0 \in \partial(\bar{y}^*, f)(\bar{x}) + \alpha \partial \phi(\bar{x}) + \langle \bar{y}^*, \vartheta \rangle B_{X^*} + N(\bar{x}; \Omega).
\]

Now, similar to the proof of Theorem 3.1, there exists \( \tilde{\mu} := (\tilde{\mu}_1, \tilde{\mu}_2, \ldots, \tilde{\mu}_n) \in \Lambda(\bar{x}) \), with \( \sum_{i=1}^n \tilde{\mu}_i = 1 \) and \( \tilde{\mu}_i = 0 \) for all \( i \in \{1, 2, \ldots, n\} \setminus I(\bar{x}) \), such that

\[
0 \in \partial(\bar{y}^*, f)(\bar{x}) + \alpha \sum_{i=1}^n \tilde{\mu}_i \text{cl}^{*}\co\left( \bigcup_{i=1}^n \partial_{x} g_i(\bar{x}, v_i) \mid v_i \in V_i(\bar{x}) \right)
\]

\[
+ \langle \bar{y}^*, \vartheta \rangle B_{X^*} + N(\bar{x}; \Omega).
\]

Dividing the above inclusion by \( \beta := \|\bar{y}^*\| + \alpha \|\tilde{\mu}\| \), and then setting \( y^* := \frac{\bar{y}^*}{\beta} \) and \( \mu := \frac{\alpha}{\beta} \tilde{\mu} \), we have some \( y^* \in K^+ \) and \( \mu := (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{R}^n_+ \), with
∥y∗∥ + ∥μ∥ = 1, such that

\[
0 \in \partial (y^∗, f)(\bar{x}) + \sum_{i=1}^{n} \mu_i \text{cl}^{*} \text{co} \left( \bigcup \{ \partial_{x_i} g_i(\bar{x}, v_i) \mid v_i \in \mathcal{V}_i(\bar{x}) \} \right)
+ \langle y^∗, \vartheta \rangle B_{X^∗} + N(\bar{x}; \Omega). \tag{25}
\]

In addition to that, we can obtain \( \bar{v}_i \in \mathcal{V}_i \) such that \( g_i(\bar{x}, \bar{v}_i) = \max_{v_i \in \mathcal{V}_i} g_i(\bar{x}, v_i) = \phi_i(\bar{x}) \) for each \( i = 1, 2, \ldots, n \). Furthermore, \( \alpha \phi(\bar{x}) = 0 \) since \( \alpha_k \phi(x^{2k}) \to 0 \) as \( k \to \infty \). By considering that \( \phi_i(\bar{x}) = \phi(\bar{x}) \) for all \( i \in I(\bar{x}) \), we arrive at

\[
\mu_i g_i(\bar{x}, \bar{v}_i) = \frac{\alpha}{\beta} \bar{\mu}_i \phi_i(\bar{x}) = \frac{\bar{\mu}_i}{\beta} [\alpha \phi(\bar{x})] = 0,
\]

i.e. \( \mu_i g_i(\bar{x}, \bar{v}_i) = \mu_i \max_{v_i \in \mathcal{V}_i} g_i(\bar{x}, v_i) = 0 \) for all \( i \in \{1, 2, \ldots, n\} \). This together with (25) yields (20).

**Case 2:** Next we assume that \( \{\alpha_k\} \) is unbounded. If \( \ell_2 > 0 \) be a Lipschitz constant of \( \phi \) around \( \bar{x} \), then we have

\[
\|x_{2k}^∗\| \leq \ell_2 \alpha_k \quad \text{for all } k \in \mathbb{N}.
\]

Applying the latter inequality and considering the weak* sequential compactness of bounded sets in duals to Asplund spaces, we may suppose that \( \frac{x_{2k}^*}{\alpha_k} \overset{w^*}{\to} x_2^* \in X^* \) as \( k \to \infty \). From (21), there exists \( b^*_k \in B_{X^*} \) such that

\[
- \frac{x_{1k}^*}{\alpha_k} - \frac{x_{2k}^*}{\alpha_k} - \left( \frac{\langle y_k^*, \vartheta \rangle}{\alpha_k} + \frac{1}{k} \right) b_k^* = \frac{x_{3k}^*}{\alpha_k} \in \hat{N}(x^{3k}; \Omega), \quad k \in \mathbb{N}. \tag{26}
\]

Passing (26) to the limit as \( k \to \infty \) and taking (4) and (5) into account, we obtain

\[
-x_2^* \in N(\bar{x}; \Omega). \tag{27}
\]

Similar to the Case 1, we get from the inclusion \( x_{2k}^* \in \alpha_k \hat{\partial}(x^{2k}) \) that

\[
(x_{2k}^*, -\alpha_k) \in \hat{N}((x^{2k}, \phi(x^{2k})); \text{gph } \phi)
\]

for each \( k \in \mathbb{N} \). So

\[
\left( \frac{x_{2k}^*}{\alpha_k}, -1 \right) \in \hat{N}((x^{2k}, \phi(x^{2k})); \text{gph } \phi), \quad k \in \mathbb{N}.
\]

Assuming \( k \to \infty \) and considering (5) again, we have \( (x_{2}^*, -1) \in N((\bar{x}, \phi(\bar{x})); \text{gph } \phi) \), which is equivalent to

\[
x_2^* \in \partial \phi(\bar{x}).
\]

The latter inclusion with (27) indicate that

\[
0 \in \partial \phi(\bar{x}) + N(\bar{x}; \Omega).
\]
We can arrive at \( 0 = \sum_{i=1}^{n} \mu_i \partial^* f_i(\bar{x}, v_i) + N(\bar{x}; \Omega) \).

Moreover, due to the unboundedness of \( \alpha_k \phi(x^{2k}) \to 0 \) as \( k \to \infty \), we can choose \( \bar{v}_i \in \mathcal{V}_i \) such that \( \mu_i g_i(\bar{x}, \bar{v}_i) = \mu_i \phi_i(\bar{x}) = \mu_i \phi(\bar{x}) = 0 \) for each \( i = 1, 2, \ldots, n \). So, (20) holds by taking \( y^* := 0 \in K^+ \).

Finally, assuming that \( \bar{x} \) satisfies the (CQ) in the Case 1, directly from (20) we can arrive at \( y^* \neq 0 \), which supports the last statement of the theorem and completes the proof.

**Remark 3.2:** Theorem 3.3 reduces to [11, Theorem 3.2] with \( \vartheta = 0 \), [15, Theorem 4.3] with \( Y = \mathbb{R}^p \), and [14, Theorem 3.3] and [16, Theorem 3.7] in the case of finite-dimensional optimization. Note further that our approach here, which involves the fuzzy necessary optimality condition in the sense of the Fréchet subdifferential and the inclusion formula for the limiting subdifferential of maximum functions in the setting of Asplund spaces, is totally different from those ones presented in the aforementioned papers.

We then return to an example to illustrate Theorem 3.3 for an uncertain multiobjective optimization problem.

**Example 3.3:** Suppose that \( X := \mathbb{R}^2 \), \( Y := \mathbb{R}^3 \), \( \Omega := \mathbb{R}^2 \), \( \mathcal{V}_i := [-1, 1] \), \( i = 1, 2 \), \( \mathcal{V} := \prod_{i=1}^{2} \mathcal{V}_i \), and \( K := \mathbb{R}^3_+ \). Consider the following uncertain optimization problem:

\[
\min_{K} \left\{ f(x) \mid g_i(x, v_i) \leq 0, \forall v_i \in \mathcal{V}_i, i = 1, 2 \right\}, \quad (UP)
\]

where \( f : X \to Y, f := (f_1, f_2, f_3) \) are given by

\[
\begin{align*}
f_1(x_1, x_2) &= -2x_1 + |x_2|, \\
f_2(x_1, x_2) &= \frac{1}{|x_1| + 1} - 3x_2 + 2, \\
f_3(x_1, x_2) &= \frac{1}{\sqrt{|x_1| + 1}} - |x_2 - 1| - 1,
\end{align*}
\]

and \( g : X \times \mathcal{V} \to \mathbb{R}^2, g := (g_1, g_2) \) are defined by

\[
\begin{align*}
g_1(x_1, x_2, v_1) &= v_1^2 |x_2| + \max \left\{ x_1, 2x_1 \right\} - 3|v_1|, \\
g_2(x_1, x_2, v_2) &= -3|x_1| + v_2 x_2 - 2,
\end{align*}
\]

where \( v_i \in \mathcal{V}_i, i = 1, 2 \). It is obvious that

\[
\left\{ v_1^2 |x_2| + \max \left\{ x_1, 2x_1 \right\} - 3|v_1| \leq 0 \forall v_i \in \mathcal{V}_1 \right\} = \{(x_1, x_2) \in X \mid x_1 \leq 0 \text{ and } |x_2| \leq -x_1 + 3\},
\]
and, due to $x_1 \leq 0$, it can be verified that

$$\{ -3|x_1| + v_2 x_2 - 2 \leq 0 \ \forall \ v_2 \in V_2 \} = \{(x_1, x_2) \in X \mid x_1 \leq 0 \text{ and } |x_2| \leq -3x_1 + 2 \}.$$ 

Therefore, the robust feasible set is

$$F = \left\{ (x_1, x_2) \in X \mid -\frac{1}{2} \leq x_1 \leq 0 \text{ and } |x_2| \leq -3x_1 + 2 \right\} \bigcup \left\{ (x_1, x_2) \in X \mid x_1 \leq -\frac{1}{2} \text{ and } |x_2| \leq -x_1 + 3 \right\},$$

which is represented in Figure 1.

Let $\vartheta := (\vartheta_1, \vartheta_2, \vartheta_3) = (0, 1, 0) \in K$ and consider $\bar{x} := (0, 0) \in F$, hence $N(\bar{x}, \Omega) = \{(0, 0)\}$. Suppose $x := (x_1, x_2) \in F$ and take $x_1 \leq 0$, we get $f_1(x) - f_1(\bar{x}) + \|x - \bar{x}\| \vartheta_1 \geq 0$. Therefore

$$f(x) - f(\bar{x}) + \|x - \bar{x}\| \vartheta \notin -\text{int } K$$

**Figure 1.** Robust feasible set of problem (UP) in Example 3.3.
for all \( x \in F \), i.e. \( \bar{x} \) is a weakly robust \( \vartheta \)-quasi-efficient solution of problem \( (\text{UP}) \).

Note further that

\[
\phi_1(\bar{x}) = \max_{v_1 \in V_1} g_1(\bar{x}, v_1) = \max_{v_1 \in V_1} (-3|v_1|) = 0,
\]

\[
\phi_2(\bar{x}) = \max_{v_2 \in V_2} g_2(\bar{x}, v_2) = \max_{v_2 \in V_2} (v_2 - 2) = -1.
\]

So \( \phi(\bar{x}) = \max\{\phi_1(\bar{x}), \phi_2(\bar{x})\} = 0, V_1(\bar{x}) = \{0\}, \) and \( V_2(\bar{x}) = \{1\} \). After calculations, we get

\[
\partial f_1(\bar{x}) = \{-2\} \times [-1, 1], \quad \partial f_2(\bar{x}) = [-1, 1] \times \{-3\},
\]

\[
\partial f_3(\bar{x}) = [-\frac{1}{2}, \frac{1}{2}] \times \{-1, 1\},
\]

and also

\[
\begin{align*}
\text{cl}^*\co\left(\partial_x g_1(\bar{x}, v_1 = 0)\right) &= [1, 2] \times \{0\}, \\
\text{cl}^*\co\left(\partial_x g_2(\bar{x}, v_2 = 1)\right) &= [-3, 3] \times \{1\}.
\end{align*}
\]

On the other hand, since \( I(\bar{x}) = \{i \in \{1, 2\} \mid \phi_i(\bar{x}) = \phi(\bar{x})\} = \{1\} \), it easily follows from \( (29) \) that the (CQ) is satisfied at \( \bar{x} \).

Finally, there exist \( y^* = (\frac{\sqrt{2}}{4}, 0, \frac{\sqrt{2}}{4}) \in K^+ \) and \( \mu = (\frac{1}{2}, 0) \in \mathbb{R}^2_+ \), with \( \|y^*\| + \|\mu\| = 1 \), \( b^* = (0, 0) \in B_{\mathbb{R}^2} \), and \( a^* = (0, 0) \in N(\bar{x}, \Omega) \) such that

\[
0 = \left(\frac{\sqrt{2}}{4}, 0 \right) \left(\begin{array}{ccc} -2 & 0 & 0 \\ 1 & -3 & -1 \end{array}\right) \left(\begin{array}{c} \frac{\sqrt{2}}{2} \\ 0 \\ 0 \end{array}\right) + \left(\begin{array}{c} \frac{1}{2} \\ 0 \end{array}\right) \left(\begin{array}{c} \sqrt{2} \\ 0 \end{array}\right) \\
+ \left(\frac{\sqrt{2}}{4}, 0 \right) \left(\begin{array}{ccc} \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \left(\begin{array}{c} 0 \\ 0 \end{array}\right) + \left(0, 0\right),
\]

and \( \mu_i \max_{v_i \in V_i} g_i(\bar{x}, v_i) = 0 \) for \( i = 1, 2 \).

The next theorem establishes a robust \( \vartheta \)-approximate (KKT) sufficient optimality condition for (weakly) robust \( \vartheta \)-quasi-efficient solutions of problem \( (\text{UP}) \).

**Theorem 3.4:** Assume that \( \bar{x} \in F \) satisfies the robust \( \vartheta \)-approximate (KKT) condition.

(i) If \( (f, g) \) is \( \vartheta \)-type I pseudo convex on \( \Omega \) at \( \bar{x} \), then \( \bar{x} \in \vartheta \)-quasi-\( S^\mu(w)(\text{RP}). \)

(ii) If \( (f, g) \) is \( \vartheta \)-type II pseudo convex on \( \Omega \) at \( \bar{x} \), then \( \bar{x} \in \vartheta \)-quasi-\( S(\text{RP}) \).

**Proof:** Let \( \bar{x} \in F \) satisfy the robust \( \vartheta \)-approximate (KKT) condition. Therefore, there exist \( \gamma^* \in K^+ \setminus \{0\} \), \( u^* \in \partial \langle \gamma^*, f \rangle(\bar{x}) \), \( \mu_i \geq 0 \), \( v_i^* \in \text{cl}^*\co\left(\bigcup\{\partial_x g_i(\bar{x}, v_i) \mid \\
\}ight)\).
\( v_i \in V_i(\bar{x}) \), \( i = 1, 2, \ldots, n \), and \( b^* \in B_{X^*} \) such that
\[
- \left( u^* + \sum_{i=1}^{n} \mu_i v_i^* + \langle y^*, \vartheta \rangle b^* \right) \in N(\bar{x}; \Omega), \tag{30}
\]
\[
\mu_i \max_{v_i \in V_i} g_i(\bar{x}, v_i) = 0, \quad i = 1, 2, \ldots, n. \tag{31}
\]
Firstly, we justify (i). Argue by contradiction that \( \bar{x} \notin \vartheta\text{-quasi-}S^w(\text{RP}) \). Hence, there is \( \hat{x} \in F \) such that \( f(\hat{x}) - f(\bar{x}) + \|\hat{x} - \bar{x}\| \vartheta \in -\text{int} K \). The latter gives us \( \langle y^*, f(\hat{x}) - f(\bar{x}) + \|\hat{x} - \bar{x}\| \vartheta \rangle < 0 \) (see [28, Lemma 3.21]). Since \( (f, g) \) is \( \vartheta \text{-type I pseudo convex on } \Omega \) at \( \bar{x} \), we deduce from this inequality that there exists \( w \in -N(\bar{x}; \Omega)^+ \) such that
\[
\begin{aligned}
\langle u^*, w \rangle + \langle y^*, \vartheta \rangle \|\hat{x} - \bar{x}\| < 0, \\
\|w\| \leq \|\hat{x} - \bar{x}\|.
\end{aligned} \tag{32}
\]
On the other side, it follows from (30) for \( w \) above that
\[
\langle u^*, w \rangle + \sum_{i=1}^{n} \mu_i \langle v_i^*, w \rangle + \langle y^*, \vartheta \rangle \langle b^*, w \rangle \geq 0. \tag{33}
\]
The relations (32) and (33) entail that
\[
\sum_{i=1}^{n} \mu_i \langle v_i^*, w \rangle > 0.
\]
To proceed, we assume that there is \( i_0 \in \{1, 2, \ldots, n\} \) such that \( \mu_{i_0} \langle v_{i_0}^*, w \rangle > 0 \). Taking into account that \( v_{i_0}^* \in \text{cl}^* \text{co} \left( \bigcup \{ \partial_x g_{i_0}(\bar{x}, v_{i_0}) \mid v_{i_0} \in V_{i_0}(\bar{x}) \} \right) \), we get sequence \( \{v_{i_0k}^*\} \subset \text{co}(\bigcup \{ \partial_x g_{i_0}(\bar{x}, v_{i_0}) \mid v_{i_0} \in V_{i_0}(\bar{x}) \}) \) such that \( v_{i_0k}^* \rightharpoonup v_{i_0}^* \). Hence, due to \( \mu_{i_0} > 0 \), there is \( k_0 \in \mathbb{N} \) such that
\[
\langle v_{i_0k_0}^*, w \rangle > 0. \tag{34}
\]
In addition, since \( v_{i_0k_0}^* \in \text{co}(\bigcup \{ \partial_x g_{i_0}(\bar{x}, v_{i_0}) \mid v_{i_0} \in V_{i_0}(\bar{x}) \}) \), there exist \( v_p^* \in \bigcup \{ \partial_x g_{i_0}(\bar{x}, v_{i_0}) \mid v_{i_0} \in V_{i_0}(\bar{x}) \} \) and \( \mu_p \geq 0 \) with \( \sum_{p=1}^{s} \mu_p = 1, p = 1, 2, \ldots, s, s \in \mathbb{N} \), such that \( v_{i_0k_0}^* = \sum_{p=1}^{s} \mu_p v_p^* \). Combining the latter together (34), we arrive at
\[
\sum_{p=1}^{s} \mu_p \langle v_p^*, w \rangle > 0. \quad \text{Thus, we can take } p_0 \in \{1, 2, \ldots, s\} \text{ such that } \langle v_{p_0}^*, w \rangle > 0, \tag{35}
\]
and choose \( v_{p_0} \in V_{i_0}(\bar{x}) \) satisfying \( v_{p_0}^* \in \partial_x g_{i_0}(\bar{x}, v_{p_0}) \) due to \( v_{p_0}^* \in \bigcup \{ \partial_x g_{i_0}(\bar{x}, v_{i_0}) \mid v_{i_0} \in V_{i_0}(\bar{x}) \} \). Invoking now definition of type I pseudo convexity of \( (f, g) \) on \( \Omega \)
at \( \bar{x} \), we get from (35) that
\[
g_i(\bar{x}, \bar{v}_0) > g_i(\bar{x}, \bar{v}_0).
\] (36)

Note that \( \bar{v}_0 \in \mathcal{V}_i(\bar{x}) \), thus we have \( g_i(\bar{x}, \bar{v}_0) = \max_{v_i \in \mathcal{V}_i} g_i(\bar{x}, v_i) \) which together with (31) yields \( \mu_{i_0} g_i(\bar{x}, \bar{v}_0) = 0 \). This implies by (36) that \( \mu_{i_0} g_i(\bar{x}, \bar{v}_0) > 0 \), and hence \( g_i(\bar{x}, \bar{v}_0) > 0 \), which contradicts with the fact that \( \bar{x} \in F \) and completes the proof of (i).

Assertion (ii) is proved similarly to the part (i). If \( \bar{x} \notin \bar{\vartheta} \)-quasi-\( \mathcal{S}(\mathcal{R}P) \), then there exists \( \hat{x} \in F \) such that \( f(\hat{x}) - f(\bar{x}) + \|\hat{x} - \bar{x}\| \vartheta \notin - K \setminus [0] \). Therefore \( \hat{x} \neq \bar{x} \) and \( \langle \mu^*, f(\hat{x}) - f(\bar{x}) + \|\hat{x} - \bar{x}\| \vartheta \rangle \leq 0 \). Now by using the definition of type II pseudo convexity of \( (f, g) \) on \( \Omega \) at \( \bar{x} \), we arrive at the result. \( \blacksquare \)

We immediately get the following robust \( \vartheta \)-approximate (KKT) sufficient optimality condition from Remark 2.2(i) and Theorem 3.4.

**Corollary 3.5:** Let \( \bar{x} \in F \) satisfy the robust \( \vartheta \)-approximate (KKT) condition and \( (f, g) \) is \( \vartheta \)-type I pseudo convex on \( \Omega \) at \( \bar{x} \), then \( \bar{x} \in \vartheta \)-quasi-\( \mathcal{S}(\mathcal{R}P) \).

**Remark 3.4:** Theorem 3.4 reduces to [11, Theorem 3.4] with \( \vartheta = 0 \) and [15, Theorem 4.7] with \( Y = \mathbb{R}^p \), and improves [14, Theorem 3.11], [7, Theorem 3.2], and [16, Theorem 3.13] under pseudo convexity assumptions.

Let us present an example to show the viability of our new concept of pseudo convexity for an uncertain multiobjective optimization problem.

**Example 3.5:** Let \( X, Y, \Omega, \mathcal{V}_i, i = 1, 2, \mathcal{V} := \prod_{i=1}^2 \mathcal{V}_i, K, f, \) and \( g \) be the same as Example 2.3. Take the following uncertain optimization problem:
\[
\min_K \{ f(x) \mid g_i(x, v_i) \leq 0, \forall v_i \in \mathcal{V}_i, i = 1, 2 \}. \tag{UP}
\]

From these constraints of inequality and equality for all \( v_i \in \mathcal{V}_i, i = 1, 2 \), we get
\[
\left\{ \frac{1}{4} v_1^2 x_1 + \frac{1}{2} v_1^2 x_2 - v_1^2 + \frac{1}{4} |v_1| \leq 0 \forall v_1 \in \mathcal{V}_1 \right\} = \left\{ x \in \mathbb{R}^2 \mid x_2 \leq -\frac{1}{2} |x_1| \right\},
\]
\[
\left\{ \frac{1}{8} x_1^2 + |v_2| x_2 - |v_2| + \frac{1}{4} \forall v_2 \in \mathcal{V}_2 \right\} = \left\{ x \in \mathbb{R}^2 \mid x_2 \leq -\frac{1}{2} x_1^2 \right\}.
\]

So, obviously we can verify that
\[
F = \left\{ x \in \mathbb{R}^2 \mid |x_1| \leq 1 \text{ and } x_2 \leq -\frac{1}{2} |x_1| \right\} \cup \left\{ x \in \mathbb{R}^2 \mid |x_1| > 1 \text{ and } x_2 \leq -\frac{1}{2} x_1^2 \right\},
\]
as depicted in Figure 2. Let \( \tilde{x} := (0, 0) \in F \) and \( \vartheta := (\vartheta_1, \vartheta_2, \vartheta_3) \in K \) be the same as Example 2.3. Note that

\[
\phi_1(\tilde{x}) = \max_{v_1 \in V_1} g_1(\tilde{x}, v_1) = \max_{v_1 \in V_1} \left(-v_1^2 + \frac{1}{4}|v_1|\right) = 0,
\]

\[
\phi_2(\tilde{x}) = \max_{v_2 \in V_2} g_2(\tilde{x}, v_2) = \max_{v_2 \in V_2} \left(-|v_2| + \frac{1}{4}\right) = 0.
\]

Then \( \phi(\tilde{x}) = \max\{\phi_1(\tilde{x}), \phi_2(\tilde{x})\} = 0 \), \( V_1(\tilde{x}) = \{-\frac{1}{2}\} \), and \( V_2(\tilde{x}) = \{-\frac{1}{4}\} \). It follows from Example 2.3 that \( \partial f_1(\tilde{x}) = [-5, 5] \times [-\frac{2}{3}] \), \( \partial f_2(\tilde{x}) = [-\frac{1}{2}, \frac{1}{2}] \times \{0\} \), \( \partial f_3(\tilde{x}) = [-4, 4] \times \{\frac{1}{4}\} \), and

\[
\text{cl co} \left( \partial_{x} g_1 \left( \tilde{x}, v_1 = -\frac{1}{4} \right) \right) = \left[-\frac{1}{64}, \frac{1}{64}\right] \times \left\{\frac{1}{32}\right\},
\]

\[
\text{cl co} \left( \partial_{x} g_2 \left( \tilde{x}, v_2 = -\frac{1}{4} \right) \right) = \left(0, \frac{1}{4}\right),
\]

and further that \( (f, g) \) is \( \vartheta \)-type I pseudo convex on \( \Omega \) at \( \tilde{x} \). Note that there exist \( y^* = (-\frac{5}{8}, 0, \frac{1}{2}) \in K^+ \setminus \{0\}, \mu = (0, 1) \in \mathbb{R}^2_+ \), and \( b^* = (0, -1) \in B_{x^*} \). By taking

\[
0 = \begin{pmatrix} -\frac{5}{8} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{2}{5} \end{pmatrix} \begin{pmatrix} 8 \\ 5 \\ -5 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{4} \\ \frac{1}{32} & 4 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{3}{2} \end{pmatrix} (0 \ 1)
\]

and \( \mu_i \max_{v_i \in V_i} g_i(\tilde{x}, v_i) = 0 \) for \( i = 1, 2 \). Therefore, the robust \( \vartheta \)-approximate (KKT) condition is satisfied at \( \tilde{x} \). We have

\[
f_2(x) - f_2(\tilde{x}) + \|x - \tilde{x}\|\vartheta_2 = \frac{1}{2}|x_1| + \|x - \tilde{x}\|\vartheta_1 \geq 0
\]

so \( f(x) - f(\tilde{x}) + \|x - \tilde{x}\|\vartheta \notin -\text{int} K \) for all \( x \in F \), i.e. \( \tilde{x} \) is a weakly robust \( \vartheta \)-quasi-efficient solution of problem (UP).

On the other hand, if \( f \) is the same as Example 2.4, then \( (f, g) \) is \( \vartheta \)-type II pseudo convex on \( \Omega \) at \( \tilde{x} \). We get \( f_2(x) - f_2(\tilde{x}) + \|x - \tilde{x}\|\vartheta_2 > 0 \) for all \( x \in F \setminus \{\tilde{x}\} \), thus \( f(x) - f(\tilde{x}) + \|x - \tilde{x}\|\vartheta \notin -K \setminus \{0\} \) for all \( x \in F \), i.e. \( \tilde{x} \) is a robust \( \vartheta \)-quasi-efficient solution of problem (UP).

4. Robust duality

In this section, we formulate the \( \vartheta \)-Mond-Weir-type dual robust problem (RD\textsubscript{MW}) for (RP), and explore the weak, strong, and converse duality relations between the corresponding problems under pseudo convexity assumptions.
Given $\vartheta \in K$, in connection with the problem (RP), we introduce a dual robust multiobjective optimization problem in the sense of Mond-Weir as follows:

$$\max_K \{ \bar{f}(z, y^*, \mu) := f(z) \mid (z, y^*, \mu) \in F_{MW} \}, \quad \text{(RD)}$$

where $F_{MW}$ is the feasible set defined by

$$F_{MW} := \left\{ (z, y^*, \mu) \in \Omega \times K^+ \setminus \{0\} \times \mathbb{R}^n_+ \mid 0 \in \partial(y^*, f)(z) + \sum_{i=1}^{n} \mu_i v_i^* + \langle y^*, \vartheta \rangle B_{X^*} + N(z; \Omega), \right.$$

$$\left. v_i^* \in \text{cl}^* \text{co} \left( \bigcup \left\{ \partial v_i(z, v_i) \mid v_i \in V_i(z) \right\} \right), \mu_i g_i(z, v_i) \geq 0, i = 1, 2, \ldots, n \right\}.$$ 

**Definition 4.1:** Let $\vartheta \in K$, one says a vector $(\bar{z}, \bar{y}^*, \bar{\mu}) \in F_{MW}$ is
(i) a robust $\vartheta$-quasi-efficient solution of problem (RD$_{MW}$), denoted by $(\check{z}, \check{y}^*, \check{\mu}) \in \vartheta$-quasi-$S$(RD$_{MW}$), iff
\[ \check{f}(\check{z}, \check{y}^*, \check{\mu}) - \check{f}(z, y^*, \mu) < \|z - \check{z}\| \vartheta, \quad \forall (z, y^*, \mu) \in F_{MW}, \]

(ii) a weakly robust $\vartheta$-quasi-efficient solution of problem (RD$_{MW}$), denoted by $(\check{z}, \check{y}^*, \check{\mu}) \in \vartheta$-quasi-$S^w$(RD$_{MW}$), iff
\[ \check{f}(\check{z}, \check{y}^*, \check{\mu}) - \check{f}(z, y^*, \mu) < \|z - \check{z}\| \vartheta, \quad \forall (z, y^*, \mu) \in F_{MW}. \]

In what follows, we use the following notations for convenience:
\[ u < v \iff u - v \in -\text{int} K, \quad u \not< v \text{ is the negation of } u < v, \]
\[ u \leq v \iff u - v \in -K \setminus \{0\}, \quad u \not< v \text{ is the negation of } u \leq v. \]

Weak duality relations between the primal problem (RP) and the dual problem (RD$_{MW}$) is declared in the following theorem.

**Theorem 4.1 (Weak Duality):** Let $x \in F$, and let $(z, y^*, \mu) \in F_{MW}$.

1. If $(f, g)$ is $\vartheta$-type I pseudo convex on $\Omega$ at $z$, then $f(x) \not< \check{f}(z, y^*, \mu) - \|x - z\| \vartheta$.
2. If $(f, g)$ is $\vartheta$-type II pseudo convex on $\Omega$ at $z$, then $f(x) \not< \check{f}(z, y^*, \mu) - \|x - z\| \vartheta$.

**Proof:** By $(z, y^*, \mu) \in F_{MW}$, there exist $u^* \in \partial (y^*, f)(z), \mu_i \geq 0, \nu_i^* \in \text{cl}^* \text{co} (\bigcup \{\partial g_i(z, v_i) | v_i \in V_i(z)\}), i = 1, 2, \ldots, n$, and $b^* \in B_X^*$ such that
\[ -\left( u^* + \sum_{i=1}^n \mu_i v_i^* + \langle y^*, \vartheta \rangle b^* \right) \in N(z; \Omega), \]
\[ \mu_i g_i(z, v_i) \geq 0, \quad i = 1, 2, \ldots, n. \quad (39) \]

To prove (i), suppose that $f(x) < \check{f}(z, y^*, \mu) - \|x - z\| \vartheta$. Hence $\langle y^*, f(x) - \check{f}(z, y^*, \mu) + \|x - z\| \vartheta \rangle < 0$ due to $y^* \neq 0$. This is nothing else but $\langle y^*, f(x) - \check{f}(z) + \|x - z\| \vartheta \rangle < 0$. Since $(f, g)$ is $\vartheta$-type I pseudo convex on $\Omega$ at $z$, we infer from the last inequality that there exists $w \in -N(z; \Omega)^+$ such that
\[ \langle u^*, w \rangle + \langle y^*, \vartheta \rangle \|x - z\| < 0, \]
\[ \|w\| \leq \|x - z\|. \]

Besides, it follows from (39) for $w$ above that
\[ \langle u^*, w \rangle + \sum_{i=1}^n \mu_i \langle v_i^*, w \rangle + \langle y^*, \vartheta \rangle \langle b^*, w \rangle \geq 0. \]
Combining the latter relations, we get that

$$\sum_{i=1}^{n} \mu_i (v_i^*, w) > 0.$$  

Now suppose that there is $i_0 \in \{1, 2, \ldots, n\}$ such that $\mu_{i_0} (v_{i_0}^*, w) > 0$. Proceeding similarly to the proof of Theorem 3.4(i) and replacing $\hat{x} \rightarrow \check{x}$ with $x - z$ give us $g_{i_0}(x, \check{v}_{i_0}) > 0$, which contradicts with $x \in F$.

Next to justify (ii), we proceed similarly to the part (i) by employing $\vartheta$-type II pseudo convexity of $(f, g)$ on $\Omega$ at $z$, if $f(x) \preceq \tilde{f}(z, y^*, \mu) - \|x - z\| \vartheta$, then $x \neq z$ and we infer that there exists $w \in -N(z; \Omega)^+$ such that $(u^*, w) + (y^*, \vartheta)\|x - z\| < 0$ and $\|w\| \leq \|x - z\|$.

We now establish a strong duality theorem which holds between (RP) and $(\text{RD}_{MW})$.

**Theorem 4.2 (Strong Duality):** Let $\check{x} \in S^w(\text{RP})$ be such that the (CQ) is satisfied at this point. Then, there exists $(\check{y}^*, \check{\mu}) \in K^+ \setminus \{0\} \times \mathbb{R}^n_+$ such that $(\check{x}, \check{y}^*, \check{\mu}) \in F_{MW}$. Furthermore,

(i) If $(f, g)$ is $\vartheta$-type I pseudo convex on $\Omega$ at $z$ for all $z \in \Omega$, then $(\check{x}, \check{y}^*, \check{\mu}) \in S^w(\text{RD}_{MW})$.

(ii) If $(f, g)$ is $\vartheta$-type II pseudo convex on $\Omega$ at $z$ for all $z \in \Omega$, then $(\check{x}, \check{y}^*, \check{\mu}) \in S(\text{RD}_{MW})$.

**Proof:** Thanks to Theorem 3.3, we find $y^* \in K^+ \setminus \{0\}$, $u^* \in \partial (y^*, f)(\check{x})$, $\mu_i \geq 0$, $v_i^* \in \text{cl}^\ast \text{co}(\bigcup \{\partial g_i(\check{x}, v_i) | v_i \in \mathcal{V}_i(\check{x})\}), i = 1, 2, \ldots, n$, and $b^* \in B_{X^*}^\ast$ such that

$$-\left( u^* + \sum_{i=1}^{n} \mu_i v_i^* + (y^*, \vartheta) b^* \right) \in N(\check{x}; \Omega),$$

$$\mu_i \max_{v_i \in \mathcal{V}_i} g_i(\check{x}, v_i) = 0, \quad i = 1, 2, \ldots, n. \quad (40)$$

Putting $\check{y}^* := y^*$ and $\check{\mu} := (\mu_1, \mu_2, \ldots, \mu_n)$, we get $(\check{y}^*, \check{\mu}) \in K^+ \setminus \{0\} \times \mathbb{R}^n_+$. Furthermore, the inclusion $v_i \in \mathcal{V}_i(\check{x})$ means that $g_i(\check{x}, v_i) = \max_{u_i \in \mathcal{V}_i} g_i(\check{x}, u_i)$ for all $i \in \{1, 2, \ldots, n\}$. Thus, it stems from (40) that $\mu_i g_i(\check{x}, v_i) = 0, i = 1, 2, \ldots, n$.

So $(\check{x}, \check{y}^*, \check{\mu}) \in F_{MW}$.

(i) As $(f, g)$ is $\vartheta$-type I pseudo convex on $\Omega$ at $z$ for all $z \in \Omega$, applying (i) of Theorem 4.1 gives us

$$\check{f}(\check{x}, \check{y}^*, \check{\mu}) = f(\check{x}) \not\preceq \check{f}(z, y^*, \mu) - \|\check{x} - z\| \vartheta$$

for each $(z, y^*, \mu) \in F_{MW}$. Therefore $(\check{x}, \check{y}^*, \check{\mu}) \in S^w(\text{RD}_{MW})$. 

\[\Box\]
(ii) As \((f, g)\) is \(\vartheta\)-type II pseudo convex on \(\Omega\) at \(z\) for all \(z \in \Omega\), applying (ii) of Theorem 4.1 allows us

\[
\tilde{f}(\tilde{x}, \tilde{y}^*, \tilde{\mu}) \leq \tilde{f}(z, y^*, \mu) - \|\tilde{x} - z\|\vartheta
\]

for each \((z, y^*, \mu) \in F_{MW}\). Therefore \((\tilde{x}, \tilde{y}^*, \tilde{\mu}) \in S(RDMW)\). ■

**Remark 4.1:** If in Theorems 4.1 and 4.2, we set \(\vartheta = 0\) then these theorems reduce to [11, Theorem 4.1] and [11, Theorem 4.2].

**Theorem 4.3 (Strong Duality):** Let \(\tilde{x} \in F\) be such that the robust \(\vartheta\)-approximate (KKT) condition is satisfied at this point. Then, there exists \((\tilde{y}^*, \tilde{\mu}) \in K^+ \setminus \{0\} \times \mathbb{R}^n_+\) such that \((\tilde{x}, \tilde{y}^*, \tilde{\mu}) \in F_{MW}\). Moreover,

(i) If \((f, g)\) is \(\vartheta\)-type I pseudo convex on \(\Omega\) at \(z\) for all \(z \in \Omega\), then \((\tilde{x}, \tilde{y}^*, \tilde{\mu}) \in S^w(RD_{MW})\) and \(\tilde{x} \in S^w(RP)\).

(ii) If \((f, g)\) is \(\vartheta\)-type II pseudo convex on \(\Omega\) at \(z\) for all \(z \in \Omega\), then \((\tilde{x}, \tilde{y}^*, \tilde{\mu}) \in S(RD_{MW})\) and \(\tilde{x} \in S(RP)\).

**Proof:** Since \(\tilde{x} \in F\) satisfies the robust \(\vartheta\)-approximate (KKT) condition, we find \(y^* \in K^+ \setminus \{0\}, u^* \in \partial (y^*, f)(\tilde{x}), \mu_i \geq 0, v^*_i \in \text{cl}^\vartheta \text{co} \left( \bigcup \{\partial x g_i(\tilde{x}, \nu_i) \mid \nu_i \in V_i(\tilde{x}) \} \right), \ i = 1, 2, \ldots, n, \) and \(b^* \in B_{X^*}\) such that

\[
-\left( u^* + \sum_{i=1}^{n} \mu_i v^*_i + \langle y^*, \vartheta \rangle b^* \right) \in N(\tilde{x} ; \Omega),
\]

\[
\mu_i \max_{\nu_i \in V_i} g_i(\tilde{x}, \nu_i) = 0, \quad i = 1, 2, \ldots, n.
\]

Now similar to the proof of Theorem 4.2, we can arrive at the result. ■

**Remark 4.2:**

(i) Theorems 4.1 and 4.3 develop [15, Theorem 4.18] and [15, Theorem 4.19] with \(Y = \mathbb{R}^p\).

(ii) Theorems 4.1 and 4.3 develop [12, Theorem 5.2] and [12, Theorem 5.3] with \(\Omega = X\) and \(\vartheta = 0\).

Note further that our approach here is totally different from those ones presented in the aforementioned papers.

We conclude this section by presenting converse duality relations between (RP) and (RD_{MW}).

**Theorem 4.4 (Converse Duality):** Let \((\tilde{x}, \tilde{y}^*, \tilde{\mu}) \in F_{MW}\) be such that \(\tilde{x} \in F\).

(i) If \((f, g)\) is \(\vartheta\)-type I pseudo convex on \(\Omega\) at \(\tilde{x}\), then \(\tilde{x} \in S^w(RP)\).

(ii) If \((f, g)\) is \(\vartheta\)-type II pseudo convex on \(\Omega\) at \(\tilde{x}\), then \(\tilde{x} \in S(RP)\).
Proof: Since $(\bar{x}, \bar{y}^*, \bar{\mu}) \in F_{MW}$, there exist $u^* \in \partial \langle \bar{y}^*, f \rangle(\bar{x})$, $\bar{\mu}_i \geq 0$, $v_i^* \in \text{cl}^* \text{co} \left( \bigcup (\partial_x g_i(\bar{x}, v_i) \mid v_i \in \mathcal{V}_i(\bar{x})) \right)$, $i = 1, 2, \ldots, n$, and $b^* \in B_{X^*}$ such that

$$-\left( u^* + \sum_{i=1}^{n} \bar{\mu}_i v_i^* + \langle y^*, \vartheta \rangle b^* \right) \in N(\bar{x}; \Omega),$$

$$\bar{\mu}_i g_i(\bar{x}, v_i) \geq 0, \quad i = 1, 2, \ldots, n.$$ (41)

Let us prove (i) by contradiction. Suppose that $\bar{x} \notin S^w(\text{RP})$. Therefore, there is $\hat{x} \in F$ such that $f(\hat{x}) - f(\bar{x}) + \|\hat{x} - \bar{x}\| \vartheta \in -\text{int} K$. The latter inclusion provides $\langle \bar{y}^*, f(\hat{x}) - f(\bar{x}) + \|\hat{x} - \bar{x}\| \vartheta \rangle < 0$. By the $\vartheta$-type I pseudo convex on $\Omega$ at $\bar{x}$, we infer from this inequality that there exists $w \in -N(\bar{x}; \Omega)^+$ such that

$$\langle u^*, w \rangle + \langle y^*, \vartheta \rangle \|\hat{x} - \bar{x}\| < 0,$$

$$\|w\| \leq \|\hat{x} - \bar{x}\|.$$

Moreover, from (41) we have for $w$

$$\langle u^*, w \rangle + \sum_{i=1}^{n} \bar{\mu}_i \langle v_i^*, w \rangle + \langle y^*, \vartheta \rangle \langle b^*, w \rangle \geq 0.$$

So, the above relationships entail that

$$\sum_{i=1}^{n} \bar{\mu}_i \langle v_i^*, w \rangle > 0.$$

Now argue as in Theorem 3.4(i)’s proof, one can arrive at the result.

The proof of (ii) is similar to that of (i), so we omit the corresponding details.

Remark 4.3: If in Theorem 4.4, we set $\vartheta = 0$ then this theorem reduces to [11, Theorem 4.4].

Disclosure statement

No potential conflict of interest was reported by the author(s).

References

[1] Ben-Tal A, Ghaoui L E, Nemirovski A. Robust optimization. Princeton: Princeton University Press; 2009.
[2] Bertsimas D, Brown D, Caramanis C. Theory and applications of robust optimization. SIAM Rev. 2011;53(3):464--501.
[3] Ben-Tal A, Nemirovski A. Selected topics in robust convex optimization. Math Program. 2008;112:125–158.
Branke J. Creating robust solutions by means of evolutionary algorithms. In: Parallel problem solving from nature – PPSNV; Berlin, Heidelberg: 1998. p. 119–128.

[5] Deb K, Gupta H. Introducing robustness in multiobjective optimization. Evol Comput. 2006;14(4):463–494.

[6] Kuroiwa D, Lee G. On robust multiobjective optimization. Vietnam J Math. 2012;40(2–3):305–317.

[7] Jeyakumar V, Li G, Lee G. Robust duality for generalized convex programming problems under data uncertainty. Nonlinear Anal. 2012;75(3):1362–1373.

[8] Ehrgott M, Ide J, Schöbel A. Minmax robustness for multi-objective optimization problems. Eur J Oper Res. 2014;239(1):17–31.

[9] Ide J, Köbis E. Concepts of efficiency for uncertain multiobjective problems based on set order relations. Math Meth Oper Res. 2014;80(1):99–127.

[10] Lee G, Son P. On nonsmooth optimality theorems for robust optimization problems. Bull Korean Math Soc. 2014;51(1):287–301.

[11] Saadati M, Oveisigha M. Optimality conditions for robust nonsmooth multiobjective optimization problems in Asplund spaces. Bull Belg Math Soc Simon Stevin. 2022;28(4):579–601.

[12] Fakhar M, Mahyarinia M, Zafarani J. On nonsmooth robust multiobjective optimization under generalized convexity with applications to portfolio optimization. Eur J Oper Res. 2018;265(1):39–48.

[13] Chuong T. Robust optimality and duality in multiobjective optimization problems under data uncertainty. SIAM J Optim. 2020;30(2):1501–1526.

[14] Chuong T. Optimality and duality for robust multiobjective optimization problems. Nonlinear Anal. 2016;134:127–143.

[15] Fakhar M, Mahyarinia M, Zafarani J. On approximate solutions for nonsmooth robust multiobjective optimization problems. Optimization. 2019;68(9):1653–1683.

[16] Chuong T, Kim D. Approximate solutions of multiobjective optimization problems. Positivity. 2016;20(1):187–207.

[17] Gutiérrez C, López R, Novo V. Generalized \( \varepsilon \)-quasi-solutions in multiobjective optimization problems: existence results and optimality conditions. Nonlinear Anal. 2010;72(11):4331–4346.

[18] Liu J. \( \varepsilon \)-duality theorem of nondifferentiable nonconvex multiobjective programming. J Optim Theory Appl. 1991;69(1):153–167.

[19] Lee J, Lee G. On \( \varepsilon \)-solutions for convex optimization problems with uncertainty data. Positivity. 2012;16:509–526.

[20] Lee J, Jiao L. On quasi \( \varepsilon \)-solution for robust convex optimization problems. Optim Lett. 2017;11:1609–1622.

[21] Mordukhovich B. Variational analysis and generalized differentiation I: basic theory. Grundlehren der mathematischen Wissenschaften. Berlin, Heidelberg: Springer-Verlag; 2006.

[22] Mordukhovich B, Nam N, Yen N. Fréchet subdifferential calculus and optimality conditions in nondifferentiable programming. Optimization. 2006;55(5–6):685–708.

[23] Mordukhovich B, Nghia T. Nonsmooth cone-constrained optimization with applications to semi-infinite programming. Math Oper Res. 2013;39(2):301–324.

[24] Mordukhovich B, Nghia T. Subdifferentials of nonconvex supremum functions and their applications to semi-infinite and infinite programs with lipschitzian data. SIAM J Optim. 2013;23(1):406–431.

[25] Ekeland I. On the variational principle. J Math Anal Appl. 1974;47(2):324–353.

[26] Ioffe A, Tihomirov V. Theory of extremal problems. Studies in mathematics and its applications. Elsevier; 1979.
[27] Chuong T. Approximate solutions in nonsmooth and nonconvex cone constrained vector optimization. Ann Oper Res. 2022;311(2):997–1015.
[28] Jahn J. Vector optimization: theory, applications, and extensions. Berlin, Heidelberg: Springer-Verlag; 2004.