VECTOR BRAIDS

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Abstract. In this paper we define a new family of groups which generalize the classical braid groups on \( \mathbb{C} \). We denote this family by \( \{B_{m}^{n}\}_{n \geq m+1} \) where \( n, m \in \mathbb{N} \). The family \( \{B_{1}^{n}\}_{n \in \mathbb{N}} \) is the set of classical braid groups on \( n \) strings. The group \( B_{m}^{n} \) is the set of motions of \( n \) unordered points in \( \mathbb{C}^{m} \), so that at any time during the motion, each \( m+1 \) of the points span the whole of \( \mathbb{C}^{m} \) as an affine space. There is a map from \( B_{m}^{n} \) to the symmetric group on \( n \) letters. We let \( P_{m}^{n} \) denote the kernel of this map. In this paper we are mainly interested in understanding \( P_{2}^{n} \). We give a presentation of a group \( \mathcal{P}_{m}^{n} \) which maps surjectively onto \( P_{2}^{n} \). We also show the surjection \( \mathcal{P}_{m}^{n} \rightarrow P_{2}^{n} \) induces an isomorphism on first and second integral homology and conjecture that it is an isomorphism. We then find an infinitesimal presentation of the group \( P_{2}^{n} \).

1. Introduction

Let \( \mathbb{A}^{m} \) denote \( m \)-dimensional, complex affine space. Let \( X_{n}^{m} \) be the space of ordered \( n \)-tuples of elements of \( \mathbb{A}^{m} \), with \( n \geq m+1 \) so that each \( m+1 \) of the components of each \( n \)-tuple span the whole of \( \mathbb{A}^{m} \) in the sense of affine geometry. The symmetric group on \( n \) letters, \( \Sigma_{n} \), acts on \( X_{n}^{m} \) via permuting the components of each point. This action is fixed point free. We can therefore form the quotient space \( X_{n}^{m}/\Sigma_{n} \).

Definition 1.1. Let \( B_{m}^{n} = \pi_{1}(X_{n}^{m}/\Sigma_{n}) \) and \( P_{m}^{n} = \pi_{1}(X_{n}^{m}) \). Call these groups the group of \( n \) stringed vector braids on \( \mathbb{A}^{m} \) and the group of \( n \) stringed pure vector braids on \( \mathbb{A}^{m} \), respectively.

The long exact sequence of a fibration gives us the short exact sequence

\[
1 \rightarrow P_{m}^{n} \rightarrow B_{m}^{n} \rightarrow \Sigma_{n} \rightarrow 1. \tag{1}
\]

The space \( X_{n}^{1} \) is the well known configuration space of \( n \) points in \( \mathbb{C} \). In general we can describe the space \( X_{n}^{m} \) as

\[
X_{n}^{m} = \mathbb{A}^{m} \times \cdots \times \mathbb{A}^{m} - \Delta
\]

where

\[
\Delta = \{(x_{1}, \ldots, x_{n}) \mid x_{i} \in \mathbb{A}^{m}, \text{ span}\{x_{i}, \ldots, x_{i+1}\} \neq \mathbb{A}^{m}\}.
\]

In the case \( m = 1 \), the set \( \Delta \) is simply the “fat diagonal”. In general we will call the set \( \Delta \) the dependency locus. Choose coordinates \( (x_{1}, \ldots, x_{n}) \) for \( \mathbb{A}^{m} \) i.e. an

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isomorphism between $\mathbb{A}^m$ and $\mathbb{C}^m$. Let $x_i = (z_{i1}, \ldots, z_{im})$, where $1 \leq i \leq n$ and $z_{ij} \in \mathbb{C}$. Then the defining equations of $\Delta$ are all possible $(m+1) \times (m+1)$ minors of the matrix

\[
\Delta_{i_1 \ldots i_{m+1}} = \begin{vmatrix}
1 & \cdots & 1 \\
z_{i_1} & \cdots & z_{i_{m+1}} \\
\vdots & \ddots & \vdots \\
z_{i_m} & \cdots & z_{im+1}
\end{vmatrix}
\]

of the matrix

\[
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
z_{11} & z_{12} & \cdots & z_{1n} \\
\vdots & \vdots & \ddots & \vdots \\
z_{m1} & z_{m2} & \cdots & z_{mn}
\end{bmatrix}
\]

**Remark 1.2.** We can also consider motions of $n$ points in $\mathbb{P}^m$ instead of $\mathbb{A}^m$ (see Section 2).

The group $P_n^1$ is the classical pure braid group of $n$ strings on $\mathbb{C}$. The pure braid group has a very nice presentation, which may be understood geometrically [1]. The main aim of this paper is to discover a geometrical presentation for the group $P_n^2$ analogous to that of the classical pure braid group. Elements of $P_n^2$ may be thought of as motions of $n$ points in $\mathbb{A}^2$ so that at any time during this motion no three of the $n$ points lie on a line.

We can now informally state the main results of this paper. We define a group $PL_n$ via a presentation, and a surjective homomorphism $\varphi_n : PL_n \to P_n^2$. The presentation of $PL_n$ is given in Definition 7.3. Theorems 7.14 and 7.20 state that the homomorphism $\varphi_n$ induces isomorphisms on the first and second integral homology groups. In light of these facts the following conjecture seems reasonable.

**Conjecture 1.3.** The homomorphism $\varphi_n : PL_n \to P_n^2$ is an isomorphism.

Despite the fact that it appears to be difficult to prove Conjecture 1.3, it is relatively straightforward to find an infinitesimal presentation for $P_n^2$. In Proposition 8.1 we prove that the completion of the group algebra $\mathbb{C}[P_n^2]$ with respect to the augmentation ideal is isomorphic to the non-commutative power series ring in the indeterminates $X_{ijk}$, $1 \leq i < j < k \leq n$, modulo the two sided ideal generated by the relations

\[
[X_{ijk}, X_{jkl} + X_{ikl} + X_{ijl}] \quad i < j < k < l;
\]

\[
[X_{jkl}, X_{ijk} + X_{ikl} + X_{ijl}] \quad i < j < k < l;
\]

\[
[X_{ikl}, X_{ijk} + X_{ijl} + X_{ikl}] \quad i < j < k < l;
\]

\[
[X_{ijk}, X_{12k} + \cdots + X_{k,n-1,n}] \quad 1 \leq k \leq n;
\]

\[
[X_{ijk}, X_{rst}] \quad i, j, k, r, s, t \text{ distinct.}
\]

We now summarize the contents of this paper. First it is useful to recall one method for finding a presentation of the classical pure braid group, $P_n = P_n^1$. Recall that the map

\[
p_n^1 : X_n^1 \to X_{n-1}^1
\]

1Tomohide Terasoma from Tokyo Metropolitan University has recently found relations in the fundamental group of the space of $n$-tuples of points in $\mathbb{P}^m$, such that each $m$ of the points span a hyperplane in $\mathbb{P}^m$ [9].
which forgets the last point in each \( n \)-tuple is a fibration with fiber equal to \( \mathbb{C} \) less \( n-1 \) points \( \mathbb{C} \). For \( n \geq 3 \), the long exact sequence for a fibration provides us with the following sequence:

\[
1 \to \mathbb{L}_{n-1} \to \mathbb{P}_n \to \mathbb{P}_{n-1} \to 1 \tag{2}
\]

where \( \mathbb{L}_{n-1} \) is a free group on \( n-1 \) generators. Let \( a_{in}, 1 \leq i \leq n-1 \) denote the loop \( a_{in} \) in which the \( n \)th point goes around the \( i \)th point in the punctured \( \mathbb{C} \). Then the set of loops \( \{a_{in} | 1 \leq i \leq n-1\} \) generates \( \mathbb{L}_{n-1} \). We picture the loop \( a_{in} \) as an element of \( \mathbb{P}_n \) in Figure 1.

Using sequence (2) and the fact that \( \mathbb{P}_2 = \mathbb{Z} \), we can inductively prove that the group \( \mathbb{P}_n \) admits a presentation with generators

\[ a_{ij}, \ 1 \leq i < j \leq n, \]

and defining relations,

\[
a_{ij}^{-1} a_{rs} a_{ij} = \begin{cases} 
a_{rs} & \text{if } r < i < j < s \text{ or } i < j < r < s; \\
a_{is} a_{rs} a_{is}^{-1} & \text{if } i < r = j < s; \\
a_{is} a_{js} a_{rs} a_{js}^{-1} a_{is}^{-1} & \text{if } r = i < j < s; \\
a_{is} a_{js}^{-1} a_{is}^{-1} a_{js} a_{rs} a_{js} a_{is} a_{is}^{-1} a_{is}^{-1} & \text{if } i < r < j < s. \end{cases}
\]

Since the generators of \( \mathbb{P}_{n-1} \) clearly lift to \( \mathbb{P}_n \), finding a presentation for \( \mathbb{P}_n \) at each stage involves three main operations. First, we have to add the generators from the fiber group \( \mathbb{L}_{n-1} \) to the group \( \mathbb{P}_n \). Second, we have to add relations to \( \mathbb{P}_n \) obtained by conjugating each generator of \( \mathbb{L}_{n-1} \) by the generators of \( \mathbb{P}_{n-1} \). Finally, we have to lift relations from \( \mathbb{P}_{n-1} \) to \( \mathbb{P}_n \).

The way in which we find presentations for the groups \( \mathbb{P}L_n \) will be modelled on this approach, although, as we shall see, there will be significant complications.

In Section 2 we define the analogues of \( \mathbb{P}m_n \) with \( \mathbb{A}^m \) replaced by \( \mathbb{P}^m \), and compare these groups to \( \mathbb{P}^m_n \) and \( \mathbb{B}^m_n \), respectively. In Section 3 we discuss the groups \( \mathbb{P}^m_n \) and \( \mathbb{B}^m_n \) for \( m \leq n+2 \). In Section 4 we show that the map

\[
p_n^2 : X_n^2 \to X_{n-1}^2
\]

obtained by forgetting the last point in each \( n \)-tuple is not a fibration for \( n \geq 5 \). However, we will be able to repair this defect in some sense by using the fact that these maps are fibrations except over a set of complex codimension one. A consequence of this fact is Corollary 4.3, which states that there is an exact sequence

\[
\pi_1(GF_n^2) \to P_n^2 \to P_{n-1}^2 \to 1,
\]

where \( GF_n^2 \) denotes the generic fiber of the map \( p_n^2 \).

Figure 1. The generator \( a_{in} \) of \( P_n \)
We use this sequence to find relations within the group $P_{2n}$. However, there are two major complications which did not occur when we found a presentation for the classical pure braid group. First, the fiber is more intricate than in the braid case; it is the complement of lines in $\mathbb{A}^2$ rather than a punctured copy of $\mathbb{C}$. Second, and more importantly, we do not know if the group $\pi_1(GF_{2n})$ injects into $P_{2n}$. If this were the case, then Conjecture 1.3 would be true.

In Section 5 we give a way of describing loops in the complexification of a real arrangement of lines in $\mathbb{C}^2$. In Section 6 we use some techniques from stratified Morse theory to find nice presentations for the fiber groups, $\pi_1(GF_{2n})$. In Section 7 we define the group $PL_n$ and a surjective homomorphism $\phi_n : PL_n \to P_{2n}$. We then state the main theorems of this paper, Theorems 7.2, 7.14 and 7.20, and prove Theorems 7.14 and 7.20. In Section 8 we find an infinitesimal presentation for the group $P_{2n}$. In Section 9 we describe the consequences of considering motions of points in $P_{2n}$ as opposed to $A^2$.

The remaining sections of this paper are devoted to the proof of Theorem 7.2. In Section 10 we see how to conjugate the generators of $\pi_1(GF_{2n})$ by the generators of $P_{2n-1}$. In Section 11 we describe a move within the group $P_{2n}$, called the reciprocity law, which we use to lift relations from $P_{2n-1}$ to $P_{2n}$. This manoeuvre is justified in Section 13. Finally, in Section 12 we explain how to lift relations from $P_{2n-1}$ into $P_{2n}$.

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2. Affine Versus Projective

Our definition of $X_{m}^n$ involved looking at points in $\mathbb{A}^m$. By thinking of $\mathbb{A}^m$ as being the affine part of $\mathbb{P}^m$, we may extend our definitions to motions of points in $\mathbb{P}^m$.

Let us first consider the classical braid groups. We denote the classical pure braid group of $n$ strings on $\mathbb{P}^1$ by $Q_n$. Since $\mathbb{P}^1$ is homeomorphic to the two sphere we also see that $Q_n$ is the classical pure braid group on the sphere. By considering $\mathbb{C}$ to be the affine part of $\mathbb{P}^1$ we get a surjective map $P_n \to Q_n$. Note that the following relations, which do not hold in $P_n$, hold in $Q_n$:

$$
a_{12} a_{13} \ldots a_{1n} = 1; \quad a_{1k} a_{2k} \ldots a_{k-1,k} a_{k,k+1} \ldots a_{kn} = 1 \text{ for } 2 \leq k \leq n. \quad (3)
$$

(for example see Figure 3). In fact it is not hard to show that these are all of the extra relations required in the presentation of $P_n$ to get a presentation of $Q_n$. It is also interesting to note that the introduction of these relations into the presentation of $P_n$ introduces torsion (e.g. the center of $Q_n$ contains an element of order 2—see Corollary 2.6).

We now generalise these notions to our situation. Let $Y_n^m$ be the space of ordered $n$-tuples in $\mathbb{P}^m$, with $n \geq m + 1$ so that each $m + 1$ of the points of each $n$-tuple span the whole of $\mathbb{P}^m$. As with $X_n^m$, the symmetric group on $n$ letters acts fixed point freely on $Y_n^m$ by permuting the components of each point. Define $C_n^m =$
We have $\pi_1(Y_n^m/\Sigma_n) = \pi_1(Y_n^m)$. Call these groups the group of $n$ stringed vector braids on $\mathbb{P}^m$ and the homing group of $n$ stringed vector braids on $\mathbb{P}^m$ respectively. Note that we have natural maps $P^m_n \to Q^m_n$ and $B^m_n \to C^m_n$.

Lemma 2.1. The natural map $P^m_n \to Q^m_n$ is surjective.

Proof. This follows as $X^m_n$ is a Zariski open subset of the smooth variety $Y^m_n$.

Corollary 2.2. The natural map $B^m_n \to C^m_n$ is surjective.

We now define an action of the affine and projective linear groups on $X^m_n$ and $Y^m_n$.

The affine group $AGL(m+1)(\mathbb{C})$ acts on the space $X^m_n$ via the diagonal action. If $n \geq m+1$, the the isotropy group of a point is trivial. If $n = m+1$ then $AGL(m+1)(\mathbb{C})$ acts transitively. It follows that $X^m_{m+1}$ is diffeomorphic to $AGL(m+1)(\mathbb{C})$. Let $\bar{X}^m_n$ denote the quotient space $X^m_n/AGL(m+1)(\mathbb{C})$. Then $AGL(m+1)(\mathbb{C}) \to X^m_n \to \bar{X}^m_n$ is a principal $AGL(m+1)(\mathbb{C})$ bundle. Moreover, it has a section (cf. [3, page 421]). Hence we have the following result.

Lemma 2.3. The space $X^m_n$ is diffeomorphic to $\bar{X}^m_n \times AGL(m+1)(\mathbb{C})$.

Corollary 2.4. The group $P^m_n$ has a central element of infinite order.

Proof. We have $\pi_1(X^m_n) \cong \pi_1(\bar{X}^m_n) \times \pi_1(AGL(m+1)(\mathbb{C}))$. But $\pi_1(AGL(m+1)(\mathbb{C}))$ is isomorphic to $\mathbb{Z}$.

The group $PGL(m+1)(\mathbb{C})$ acts on the space $Y^m_n$ via the diagonal action. If $n \geq m+2$ the the isotropy group of a point is trivial. If $n = m+2$ then $PGL(m+1)(\mathbb{C})$ acts transitively. It follows that $Y^m_{m+2}$ is diffeomorphic to $PGL(m+1)(\mathbb{C})$. Let $\bar{Y}^m_n$ denote the quotient space $Y^m_n/PGL(m+1)(\mathbb{C})$. Then $PGL(m+1)(\mathbb{C}) \to Y^m_n \to \bar{Y}^m_n$ is a principle $PGL(m+1)(\mathbb{C})$ bundle. It has a section (cf. [3, page 421]). Hence we have the following result.

Lemma 2.5. The space $Y^m_n$ is diffeomorphic to $\bar{Y}^m_n \times PGL(m+1)(\mathbb{C})$.

Corollary 2.6. The group $Q^m_n$ has a central element of order $m+1$.

Proof. We have $\pi_1(Y^m_n) \cong \pi_1(\bar{Y}^m_n) \times \pi_1(PGL(m+1)(\mathbb{C}))$. But $\pi_1(PGL(m+1)(\mathbb{C}))$ is isomorphic to $\mathbb{Z}/(m+1)\mathbb{Z}$.

We shall see later that the $AGL(m+1)(\mathbb{C})$ action on $X^m_n$ and the $PGL(m+1)(\mathbb{C})$ action on $Y^m_n$ are useful in understanding some of the properties of the groups $P^m_n$ and $Q^m_n$.  

\footnote{The affine group $AGL(m+1)(\mathbb{C})$ is defined to be the ralalizer of the line at infinity in $PGL(m+1)(\mathbb{C})$. It is not difficult to show that $AGL(m+1)(\mathbb{C})$ is the semidirect product of $GL(m)(\mathbb{C})$ by $\mathbb{C}^\times$.}
3. Getting Started

In this section we study the groups \(P_n^m, Q_n^m, B_n^m\) and \(C_n^m\) when \(n \leq m + 2\).

**Proposition 3.1.** The groups \(P_n^m\) and \(Q_n^{m+1}\) are trivial for all \(m \in \mathbb{N}\).

**Proof.** We begin by extending the definition of the space \(X_n^m\) to the case where \(n \leq m\). Let \(X_n^m, 1 \leq n \leq m - 1\), be the space of \(n\)-tuples of points in \(\mathbb{A}^m\) such that the \(m\)-tuple spans an affine subspace of maximal dimension. The map \(X_n^m \to X_{n-1}^m\) obtained by forgetting the last point in each \(n\)-tuple is a fibration. The fiber of this map is equal to \(\mathbb{A}^m\) less an affine subspace of complex codimension \(m - (n - 1)\). Hence, the fiber of each each of these maps has trivial fundamental group when \(n < m\). Since \(X_1^m\) is diffeomorphic to \(\mathbb{A}^m\) for each \(m\), we can use the exact sequence of a fibration to inductively show that, for fixed \(m\), \(X_n^m\) has trivial homotopy groups when \(n < m\).

A similar argument applies to the space \(Y_n^m\), giving us the result for \(Q_n^{m+1}\). \(\square\)

**Corollary 3.2.** The natural homomorphisms
\[
B_n^m \to \Sigma_n^m \quad \text{and} \quad C_n^{m+1} \to \Sigma_{n+1}^m
\]
are isomorphisms. \(\square\)

The fact that the group \(Q_n^{m+1}\) is trivial and \(P_n^{m+1}\) is not is because the space \(\mathbb{A}^m\) less a hyperplane is homotopic to \(S^1\), whereas the space \(\mathbb{P}^m\) less a hyperplane is contractible.

**Proposition 3.3.** The group \(P_n^{m+1}\) is isomorphic to \(\mathbb{Z}\) for all \(m \in \mathbb{N}\).

**Proof.** The map \(X_{n+1}^{m+1} \to X_n^m\) is a fibration, with fiber equal to \(\mathbb{C}^2\) less a line. This is a \(K(\mathbb{Z}, 1)\). Since \(\pi_i(X_n^m)\), is trivial for \(i \geq 1\) (Proposition 3.1), we obtain the result from the long exact sequence of a fibration. \(\square\)

We immediately get a similar result for \(Q_n^{m+2}\) as a consequence of Lemma 2.5.

**Proposition 3.4.** The group \(P_n^{m+2}\) is isomorphic to \(\mathbb{Z} / (m+1)\mathbb{Z}\) for all \(m \in \mathbb{N}\). \(\square\)

We now use the \(PGL_{m+1}(\mathbb{C})\) action on the space \(Y_n^m\) to find a presentation of the group \(C_n^{m+2}\).

**Proposition 3.5.** The group \(C_n^{m+2}\) admits a presentation with generators
\[
\sigma_1, \ldots, \sigma_{m+1}, \tau,
\]
and defining relations
\[
\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i - j| > 2;
\]
\[
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1};
\]
\[
\sigma_i \tau = \tau \sigma_i;
\]
\[
\sigma_i^2 = \tau;
\]
\[
\tau_{m+1} = 1.
\]

**Proof.** In Section 2 we saw that the spaces \(Y_n^m\) and \(PGL_{m+1}(\mathbb{C})\) are diffeomorphic. Let \(e_i, 1 \leq i \leq m + 1\) be the standard basis of \(\mathbb{C}^{m+1}\). Let \(g \in PGL_{m+1}(\mathbb{C})\). Then the map
\[
\theta : PGL_{m+1}(\mathbb{C}) \to Y_n^m
\]
The map \( \theta : g \mapsto (ge_1, \ldots, ge_{m+1}, ge_1 + \cdots + ge_{m+1}) \) is a diffeomorphism. We know that \( \Sigma_{m+2} \) acts on the right of the space \( Y^m_{m+2} \) by permuting coordinates. We now see that \( \Sigma_{m+2} \) also acts on the right of \( PGL_{m+1}(\mathbb{C}) \). We do this by embedding \( \Sigma_{m+2} \) into \( PGL_{m+1}(\mathbb{C}) \).

Let \( s_i, 1 \leq i \leq m+1 \) denote the transpositions \((i, i+1)\). Then \( \Sigma_{m+2} \) has presentation

\[
\langle s_1, \ldots, s_{m+1} \mid s_i s_j = s_j s_i, |i-j| > 2, s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, 1 \leq i \leq m+1 \rangle
\]

Let \( P_i \in PGL_{m+1}(\mathbb{C}) \) denote the coset of the permutation matrix corresponding to the transposition \( s_i \) (i.e. the identity matrix with its \( i \)th and \((i+1)\)st columns swapped). Map the element \( s_i \) to \( P_i \) for \( 1 \leq i \leq m \). Let \( P_{m+1} \in PGL_{m+1}(\mathbb{C}) \) be the coset of the matrix

\[
\begin{bmatrix}
1 & 0 & \cdots & 0 & -1 \\
0 & 1 & \cdots & 0 & -1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & -1 \\
0 & 0 & \cdots & 0 & -1
\end{bmatrix}
\]

Map the element \( s_{m+1} \) to \( P_{m+1} \). Matrix computations show that this map may be extended to an injective homomorphism from \( \Sigma_{m+2} \) to \( PGL_{m+1}(\mathbb{C}) \).

Let \( \Sigma_{m+2} \) act on the right of \( PGL_{m+2}(\mathbb{C}) \) by group multiplication. Note that

\[
\theta(gP_i) = (ge_{s_i(1)}, \ldots, ge_{s_i(m+1)}, ge_{s_i(1)} + \cdots + ge_{s_i(m+1)}) = \theta(g)s_i
\]

when \( 1 \leq i \leq m \). This is because \( P_i e_j \) is equal to the \( j \)th column of \( P_i \). Also

\[
\theta(gP_{m+1}) = (e_1, \ldots, e_m, -g(e_1 + \cdots + e_{m+1}), -g e_{m+1}).
\]

Thus the map \( \theta \) is \( \Sigma_{m+2} \) equivariant and \( PGL_{m+2}(\mathbb{C})/\Sigma_{m+2} \) is diffeomorphic to \( Y^m_{m+2}/\Sigma_{m+2} \).

Note that that \( PGL_{m+2}(\mathbb{C}) \) is isomorphic to \( PSL_{m+1}(\mathbb{C}) \). Consequently \( SL_n(\mathbb{C}) \) is the universal cover of \( PGL_n(\mathbb{C}) \). The natural homomorphism \( SL_{m+1}(\mathbb{C}) \twoheadrightarrow PSL_{m+1}(\mathbb{C}) \) is a \( \mathbb{Z}/(m+1)\mathbb{Z} \) covering whose kernel is generated by the diagonal matrix \( \tau \) whose diagonal entries are all equal to \( \exp(2\pi i/(m+1)) \). Let \( G = \pi^{-1}(\Sigma_{m+2}) \). Then we have the short exact sequence

\[
1 \to \mathbb{Z}/(m+1)\mathbb{Z} \xrightarrow{\pi} G \to \Sigma_{m+2} \to 1.
\]

We use this now to show that \( G \) is given by the same presentation as that stated in the theorem. Let \( \omega = \exp(2\pi i/(2(m+1))) \) and \( \sigma_i := \omega P_i, 1 \leq i \leq m+1 \). Then each of these matrices lies in \( SL_{m+1}(\mathbb{C}) \). Use the matrix \( \sigma_i \) as a lift of \( P_i \) for \( 1 \leq i \leq m+1 \) and the matrix \( \tau \) as the generator of the image of \( \pi \). Simple matrix calculations show that the stated relations between the matrices \( \sigma_i, 1 \leq i \leq m+1 \), and \( \tau \) hold in \( G \).

To complete the proof we have to show that \( G \) is isomorphic to \( C^m_{m+2} \). First note that since \( \mathbb{Z}/(m+1)\mathbb{Z} \) is central in \( G \) we have the isomorphism

\[
SL_{m+1}(\mathbb{C})/G \cong [\mathbb{Z}/(m+1)\mathbb{Z} \setminus SL_{m+1}(\mathbb{C})]/\Sigma_{m+2}.
\]
Figure 3. Parallel problems

But \([\mathbb{Z}/(m+1)\mathbb{Z}] \setminus SL_{m+1}(\mathbb{C})\) is isomorphic to \(PSL_{m+1}(\mathbb{C})\), which is in turn isomorphic to \(Y_{m+2}^m\). Hence \(SL_{m+1}(\mathbb{C})/G\) is isomorphic to \(Y_{m+2}^m/\Sigma_{m+2}\). Since \(SL_{m+1}(\mathbb{C})/G\) is the universal cover of \(SL_{m+1}(\mathbb{C})\), we conclude that

\[ G \cong \pi_1(Y_{m+2}^m/\Sigma_{m+2}) \cong C_{m+2}. \]

4. FORGETTING A POINT

Let \(p_n^m : X_n^m \to X_{n-1}^m\) denote the map which takes \(n\)-tuples in \(X_n^m\) to \((n-1)\)-tuples in \(X_{n-1}^m\) by forgetting the \(n\)th point. In [6] it is shown that the map \(p_n^1\) is a fibration for all \(n \geq 2\). However, in general these maps fail to be fibrations when \(m\) is greater than one. For example, in the case \(m = 2\) we have the following result.

**Proposition 4.1.** The map \(p_n^2\) is not a fibration for any \(n \geq 5\).

**Proof.** First consider the case when \(n = 5\). Let \((x_1, \ldots, x_5) \in X_5^2\) and let \(L_{ij}\) denote the line through the points \(x_i\) and \(x_j\), in the fiber of \(p_{n+1}^2\) over \((x_1, \ldots, x_n)\). Then the fiber over the point \((x_1, \ldots, x_5)\) will be equal to \(\mathbb{A}^2\) less the union of the lines \(L_{ij}\). The homotopy type of the fibers will not be constant since some of the fibers will contain parallel lines, which do not intersect in \(\mathbb{A}^2\) (see Figure 3 for a non-generic fiber of the map \(p_5^2 : X_5^2 \to X_4^2\)). This problem occurs for all \(n \geq 5\).

In the case when \(n \geq 7\) another type of degeneration also occurs. We refer to Figure 4. This is a real picture of the fibers of \(p_n^2\) which shows how one may get non-generic fibers. When lines with disjoint indices intersect only in double points we are in the generic situation. However, we see in Figure 4 that as the line \(L_{ij}\) moves upwards it passes through a double point giving us a triple intersection. The homotopy type of the fiber changes when we obtain triple points; we no longer have a fibration.

Given that the map \(p_n^2\) fails to be a fibration when \(n \geq 5\), it might seem hopeless to use the same method that we used in Section 1 to find a presentation of the pure braid group in finding a presentation for the group \(P_n^2\). However, we are able to partially salvage this situation as follows.

**Proposition 4.2.** The map \(p_n^m\) is topological fibration except over a subset of \(X_n^m\) of complex codimension one.

**Proof.** The map \(p_n^m\) is a surjective algebraic map, being the restriction of a projection map from \((\mathbb{A}^m)^n \to (\mathbb{A}^m)^{n-1}\). The fibers are complements of \(\binom{n}{m}\) hyperplanes in \(\mathbb{A}^m\). The combinatorics of the fibers are constant over a Zariski open subset of \(X_n^m\), and hence the map \(p_n^m\) is a topological fibration over this set (a proof of this fact may be found in [20]).
Using the results contained in [11] we immediately obtain.

**Corollary 4.3.** Let $GF^m_n$ denote the generic fiber of the map $p^m_n$, $m \geq 1$. The sequence

$$\pi_1(GF^m_n) \to \pi_1(X^m_n) \to \pi(X^m_{n-1}) \to 1$$

is exact. \hfill $\square$

In the case $m = 2$ we have the exact sequence

$$\pi_1(GF^2_n) \to P^2_n \to P^2_{n-1} \to 1,$$

for each $n \geq 4$. This sequence is analogous to the short exact sequence

$$1 \to L_{n-1} \to P_n \to P_{n-1} \to 1,$$

where $n \geq 3$, involving the classical pure braid groups. To date, I have not been able to prove that the group $\pi_1(GF^2_n)$ injects into $P^2_n$. If this were the case then Conjecture 1.3 would be true.

We close this section by noting that we also have the map $q^m_n : Y^m_n \to Y^m_{n-1}$, obtained by forgetting the $n$th point. Both Propositions 4.1 and 4.2 are also true for the map $q^m_n$. In fact, we can say slightly more in this case.

**Proposition 4.4.** The map $q^2_n : Y^2_n \to Y^2_{n-1}$ is a fibration for $n = 4, 5$ and 6. However, the map $q^2_n$ is not a fibration for any $n \geq 7$.

**Proof.** First consider the cases when $n$ is equal to 4, 5 and 6. Let $(x_1, \ldots, x_n) \in X^2_n$ and let $L_{ij}$ denote the line through the points $x_i$ and $x_j$, in the fiber of $p^2_{n+1}$ over $(x_1, \ldots, x_n)$. Then the fiber over the point $(x_1, \ldots, x_n)$ will be equal to $P^2$ less the union of the lines $L_{ij}$. When $n = 5$ and $n = 6$, the lines $L_{ij}$ and $L_{kl}$ only intersect in a double point when $\{i, j\} \cap \{k, l\} = \emptyset$. Hence, since we are fibering over a connected manifold and the combinatorics of all of the fibers are the same. The proof now follows from results contained in [20].

In the case when $n \geq 7$ we refer to Figure 4. \hfill $\square$

**Lemma 4.5.** Let $F^2_n$ denote the fiber of the map $q^2_n : Y^2_n \to Y^2_{n-1}$. For $n = 5$ and $n = 6$ the sequence

$$1 \to \pi_1(F^2_n) \to Q^2_n \to Q^2_{n-1} \to 1.$$

is exact.

**Proof.** Since $Y^2_4$ is diffeomorphic to $PGL_{m+1}(\mathbb{C})$ the group $\pi_2(X^2_4)$ vanishes. Thus, the case $n = 5$ is an immediate consequence of Proposition 4.1 and the long exact sequence of fibration.

The case $n = 5$ also yields the exact sequence

$$\pi_2(F^2_5) \to \pi_2(Y^2_5) \to \pi_2(Y^2_4).$$
We now show that $F_5$ vanishes. Consider the pencil of lines in $F_5^2$, through the point $b$ (see Figure 5). This pencil fibers $F_5^2$. The base is $\mathbb{P}^1$ less 3 points and the fiber is $\mathbb{P}^1$ less 4 points. Hence $F_5^2$ is a $K(\pi, 1)$ space.

By applying Proposition 4.1 and using the long exact sequence of a fibration once more we obtain the result for the case $n = 6$.  

5. GETTING AROUND

In this section we find generators for the fundamental group of the complement of a set of complexified real lines in $\mathbb{A}^2$. Some of the material in this section is drawn from [1].

We begin by stating some conventions that we will use from now on. Fix a real structure on $\mathbb{A}^n$. We denote the real points of this structure by $\mathbb{A}^n(\mathbb{R})$. If $V$ is an affine linear subspace of $\mathbb{A}^n$ then let $V(\mathbb{R})$ be equal to $V \cap \mathbb{A}^n(\mathbb{R})$.

Let $\{L_i(\mathbb{R})\}$ be a set of oriented lines in $\mathbb{A}^2(\mathbb{R})$. Denote the union of the $L_i(\mathbb{R})$ by $\mathcal{A}(\mathbb{R})$. Let $(x_1, x_2)$ be coordinates for $\mathbb{A}^2(\mathbb{R})$. Then $(x_1 + iy_1, x_2 + iy_2)$, $y_1, y_2 \in \mathbb{R}$, are coordinates for $\mathbb{A}^2$. Once we have chosen coordinates for $\mathbb{A}^2$ we will abuse notation and write $\mathbb{C}^2$ for $\mathbb{A}^2$ and $\mathbb{R}^2$ for $\mathbb{A}^2(\mathbb{R})$. Let $\epsilon > 0$ be an arbitrary real number. It will be convenient to work in the tubular neighborhood $N_\epsilon = \{ (x_1 + iy_1, x_2 + iy_2) \in \mathbb{C}^2 | y_1^2 + y_2^2 \leq \epsilon \} - \mathcal{A}$ of $\mathbb{R}^2 - \mathcal{A}(\mathbb{R})$. To justify this we require the following proposition.

**Proposition 5.1.** For all $\epsilon > 0$, the set $N_\epsilon$ is homotopy equivalent to $\mathbb{C}^2 - \mathcal{A}$.

*Proof.* We prove this fact using stratified Morse theory [7]. Begin by stratifying the space $\mathbb{C}^2$ using the arrangement $\mathcal{A}$ (for details see [3] page 245). Let $f : \mathbb{C}^2 \to \mathbb{R}$ be the function defined by the formula $f(x_1 + iy_1, x_2 + iy_2) = y_1^2 + y_2^2$. This is a Morse function (in the sense of stratified Morse theory) on the stratified space $\mathbb{C}^2$. Let $X_{>t}$ denote the set of points $x \in \mathbb{C}^2 - \mathcal{A}$ such that $f(x) > t$. Outside the set of points $x \in \mathbb{C}^2$ where $f(x) < \epsilon$ the function $f$ has no critical points. Hence the set $X_{>t}$ is homotopy equivalent to the set $N_\epsilon$ for any $t \geq \epsilon$.  

Let $L(\mathbb{R})$ be a generic, oriented line in $\mathbb{R}^2 - \mathcal{A}(\mathbb{R})$ and let $v$ be a vector which orients $L(\mathbb{R})$. Its complexification has a canonical orientation and the orientation of the frame formed by the vectors $v$ and $iv$ agrees with this orientation. Let $a$ be any point in $L(\mathbb{R})$. Then we let $\tilde{a} \in L$ denote the point which is distance $\epsilon$ from $a$ in the direction $iv$. Let $p \in L(\mathbb{R}) - \mathcal{A}(\mathbb{R})$ and $q \in L(\mathbb{R}) \cap \mathcal{A}(\mathbb{R})$. We now define a loop in $N_\epsilon$ based at the point $p$ (see Figure 6). First, move in the direction $iv$ from point $p$ to point $\tilde{p}$. Then move along the real line in $L$, which joins $\tilde{p}$ and $q$ towards $\tilde{q}$. On reaching the point $\tilde{q}$, pick a loop $l_q$ with center $q$ and of radius $\epsilon$ within $L$. Now go around this loop in the positive direction with respect to the orientation.
Proposition 5.3. The homotopy class of the loop which we have just defined depends only upon the choices of $L$, its orientation and the points $p$ and $q$.

Remark 5.2. The homotopy class of the loop which we have just defined depends only upon the choices of $L$, its orientation and the points $p$ and $q$.

Denote the set of points in $L(R) \cap A(R)$ by $\{q_i\}$. Let $\zeta_i$ be the loop in $L$, based at $p$ which goes around $q_i$. Theorems of Lefschetz and Zariski (cf. Chapter 2 of [7]) immediately imply the following result.

Proposition 5.3. The set of loops $\{\zeta_i\}$ generates $\pi_1(C^2 - A, \sqrt{})$.

Remark 5.4. If we instead considered $A(R)$ as being an arrangement of lines in $P^2(R)$, then the loops $\zeta_i$ would generate $\pi_1(P^2 - A, \sqrt{})$ (cf. Lemma [2]).

We now need a pair of lemmas which will help us manipulate loops in the complement of a set of complexified real lines in $\mathbb{A}^2(R)$. To do this we first define a hop. Let $H(R)$ an arbitrary real line in $\mathbb{A}^2(R)$. Pick a point $p \in \mathbb{A}^2(R) - H(R)$ near to the line $H(R)$. The line $H(R)$ divides $\mathbb{A}^2(R)$ into two regions. Let $q \in \mathbb{A}^2(R)$ be a point in $\mathbb{A}^2(R) - H(R)$ close to $p$ that lies in the connected component of $\mathbb{A}^2(R) - H(R)$ not containing the point $p$. To hop from the point $p$ to the point $q$ in $\mathbb{A}^2 - H$, choose a line $L(R)$ joining the points $p$ and $q$ and follow a loop in $L$ from $p$ to $q$, in the negative direction with respect to the canonical orientation of $L$.

The first lemma is local in nature. Let $p$ be an element of $\mathbb{A}^2(R)$. Suppose that $n$ lines $L_j(R), 1 \leq j \leq n$, pass through the point $p$. Label the lines from 1 through $n$ in anticlockwise order. Choose a line $L(R)$ passing through $p$, which lies between the lines $L_1(R)$ and $L_n(R)$. Let $C$ be a small circle in $\mathbb{A}^2(R)$ centered at $p$. Let $a, b \in L(R)$ denote the two points of intersection of the circle $C$ with $L(R)$. (see Figure 4). Let $\gamma$ be the path in $\mathbb{A}^2 - \cup L_j$, joining the points $a$ and $b$, which is obtained by following the circle $C$ in the anticlockwise direction and hopping over each line $L_j(R)$. Let $\gamma'$ be the path in $\mathbb{A}^2 - \cup L_j$, joining points $a$ and $b$, which is obtained by following the circle $C$ in the clockwise direction and hopping over each line $L_j(R)$.

Lemma 5.5. The paths $\gamma$ and $\gamma'$ are homotopic in $\mathbb{A}^2 - \cup L_j$, relative to their endpoints $a$ and $b$.

Proof. Let $v$ be the vector in $L(R)$ from the point $a$ to the point $p$. Let $P_v$ be the plane $\mathbb{A}^2(R) + iv$. Then the intersection $P_v \cap L_j$ is empty for $1 \leq j \leq n$. Hence, the intersection $P_v \cap \cup L_j$ is empty.

If $p$ is an element of $\mathbb{A}^2(R)$ then let $\tilde{p}$ denote the point $a + iv$ in $P_v$. Let $C + iv$ denote the circle in $P_v$ which lies above $C$. Let $\rho$ be the path obtained by going
in direction \(iv\) from \(a\) to \(\tilde{a}\), going along \(C + iv\) in the clockwise direction and finally by going in direction \(-iv\) to \(b\). The path \(\rho\) is homotopic to \(\gamma\) relative to the points \(a\) and \(b\). Let \(\rho'\) be the path obtained by going in direction \(iv\) from \(a\) to \(\tilde{a}\), going along \(C + iv\) in the clockwise direction and finally by going in direction \(-iv\) to \(b\). The path \(\rho'\) is homotopic to \(\gamma'\) relative to the points \(a\) and \(b\).

To conclude the proof we see that the paths \(\rho\) and \(\rho'\) are homotopic. This is because the paths contained in \(C + iv\) which were used to define \(\rho\) and \(\rho'\) are homotopic in \(P_v\) relative to \(\tilde{a}\) and \(\tilde{b}\). \(\square\)

The second lemma is global in nature. Let \(R\) be a bounded region in contained in \(\mathbb{A}^2(\mathbb{R})\) which is diffeomorphic to a real closed disc. Also assume that the boundary of \(R\) is smooth. Let \(\mathcal{A}(\mathbb{R})\) be an arrangement of real lines contained in \(\mathbb{A}^2(\mathbb{R})\) such that each line in \(\mathcal{A}(\mathbb{R})\) is transverse to the boundary of \(R\) and the number of components of \(R - \mathcal{A}(\mathbb{R})\) is finite. Also assume that no three lines in \(\mathcal{A}(\mathbb{R})\) intersect in a point in \(R\) and that the boundary of \(R\) contains none of the multiple points of \(\mathcal{A}(\mathbb{R})\) (see Figure 8). Let \(M(\mathbb{R})\) denote the set \(R - \mathcal{A}(\mathbb{R})\) and let \(M\) be the set

\[\{u + iv : u \in M(\mathbb{R}), v \in \mathbb{R}^2, \text{ and } ||v|| < \epsilon\}\]

We now define some loops in \(M\). Let \(\{L_\alpha(\mathbb{R}), \alpha \in A\}\), denote the set of line segments in \(M(\mathbb{R})\), obtained by intersecting \(\mathcal{A}(\mathbb{R})\) with the set \(R\). Let \(p\) be a base point of \(M(\mathbb{R})\). For each line segment \(L_\alpha(\mathbb{R})\) we define a loop \(l_\alpha\) in \(M\), based at \(p\), as follows. Pick a point \(q\) on the line segment \(L_\alpha(\mathbb{R})\) which lies in between any two intersection points. Let \(l_\alpha(\mathbb{R})\) be a path in \(R\) joining the point \(p\) and \(q\) which intersects each line segment \(L_\alpha(\mathbb{R})\) transversely only once and avoids all intersection points. We now define the loop \(l_\alpha\). Follow the path \(l_\alpha(\mathbb{R})\), hopping over any line segment, until reaching the line \(L_\alpha(\mathbb{R})\). Then choose a line \(L(\mathbb{R})\) passing through \(q\), which is transverse to \(L_\alpha(\mathbb{R})\). Now follow a small loop in \(L\), which encircles \(L_\alpha\), in the positive direction with respect to the orientation of \(L\). Finally, return to the point \(p\) along the same path which was taken outward from \(p\).
Lemma 5.6. Suppose that there exists a Morse function defined on the closure of the set $M$ for which each line segment has a unique critical point, which is not a double point. Then the group $\pi_1(M,p)$ admits a presentation with generators $l_\alpha$, $\alpha \in A$ and relations

$$[l_\alpha, l_\beta] = 1, \text{ if } L_\alpha(\mathbb{R}) \cap L_\beta(\mathbb{R}) \neq \emptyset.$$ 

Proof. Using stratified Morse theory it is not hard to show that we can find a presentation for the group $\pi_1(M,p)$ with a generator $l_\alpha'$ for each line segment $L_\alpha$. Moreover the generator $l_\alpha'$ can be chosen to be the composition of a path, which hops over each line segment once, from $p$ to $L_\alpha(\mathbb{R})$ and a loop which goes around $L_\alpha(\mathbb{R})$, by following a small loop in a complex line $L$, in the positive direction with respect to the orientation of $L$. Also, using Morse theory again, it can be seen that each pair of the loops $l_\alpha'$ and $l_\alpha$ commute only if $L_\alpha(\mathbb{R}) \cap L_\beta(\mathbb{R}) \neq \emptyset$.

To complete the proof we show that the loops $l_\alpha'$ and $l_\alpha$ are homotopic in $M$. The loops $l_\alpha$ and $l_\alpha'$ were defined by following a path to the line segment $L_\alpha(\mathbb{R})$, which hopped over any line segment once, and then by following a small loop which encircled $L_\alpha(\mathbb{R})$. Note that $L_\alpha(\mathbb{R})$ intersects any other line segment in at most a double point. Thus we can deform the small loop used in the definition of $l_\alpha'$ past any intersection points in $L_\alpha(\mathbb{R})$, into the small loop which was used to define the loop $l_\alpha$. In this way we can deform the loop $l_\alpha'$ into a new loop $l_\alpha''$ which is based at $p$, hops over every line segment on its way to $L_\alpha(\mathbb{R})$, and which follows the same small loop around $L_\alpha(\mathbb{R})$ as the loop $l_\alpha$. Thus we are reduced to showing that we are able to deform the path used to define the loop $l_\alpha''$ into the path which is used to define $l_\alpha$. This can be done by deforming the path which defines $l_\alpha''$, over the double points in $A(\mathbb{R})$, into the path which defines the loop $l_\alpha$. We are able to do this using Lemma 5.5. $\square$

6. The Generic Fiber

In this section we choose a basepoint for the space $X_n^2$. We then find a natural presentation for the fundamental group of the generic fiber, $GF_n^2$, of the map $p_n^2 : X_n^2 \to X_{n-1}^2$.

To give us some flexibility later, we begin by finding a contractible subset $B$ of $X_n^2(\mathbb{R})$ which we will use to “fatten the base point”. Standard homotopy theory implies that the inclusion $(X_n^2,\ast) \hookrightarrow (X_n^2,B)$ induces a canonical isomorphism $\pi_1(X_n^2,\ast) \cong \pi_1(X_n^2,B)$ for all $\ast \in B$. Hence, elements of $\pi_1(X_n^2)$ may be represented by paths whose endpoints lie within $B$. Let $x = (x_1,\ldots,x_n)$ be an element of $X_n^2(\mathbb{R})$ which is mapped to the point $(x_1,\ldots,x_{n-1})$ by the map $p_n^2$. Denote the line in the fiber of $p_n^2$ over the point $x$, which passes through $x_i$ and $x_j$, by $L_{ij}$. Let $A_{ij}$ be equal to the union of the lines $L_{ij}$, $1 \leq i < j \leq n-1$. The fiber of the map $p_n^2$ over the point $(x_1,\ldots,x_{n-1}) \in X_{n-1}^2$ is then equal to $\mathbb{A}^2 - A_{ij}$. Choose coordinates for $\mathbb{A}^2$. Let

$$\psi : \mathbb{C} \to \mathbb{C}^2$$
$$\psi : t \to (t,t^2)$$

be the rational normal curve. Define the set $B$ to be equal to

$$\{(\psi(t_1),\ldots,\psi(t_n)) \mid t_i \in \mathbb{R}, t_1 \leq \cdots \leq t_n\}.$$ 

\[\text{In the sense of stratified Morse theory.}\]
Since the curve $\psi(\mathbb{R})$ is convex, the set $B$ is a subset of $X_n^2(\mathbb{R})$. The set $B$ is clearly contractible.

The fiber $p_n^2$ over a point in $B$ is not necessarily generic. When choosing a base point in $X_n^2$ we need to ensure that this is the case and thus we impose two extra conditions on points in $B$. To do this we first define some new lines in the fiber of $p_n^2$. Consider the curve $\psi(\mathbb{R})$ as being contained in $\mathbb{P}^2(\mathbb{R})$. Let $L_k^\infty(\mathbb{R})$, $1 \leq k \leq n$, be equal to the line in $\mathbb{R}^2$ which passes through the point $x_k$ and the point at infinity on $\psi(\mathbb{R})$. Orient the line $L_k^\infty(\mathbb{R})$ in the direction from $x_k$ which points to the point at infinity. We now specify the two extra conditions that we will impose on points in $B$:

- Divide the line $L_k^\infty(\mathbb{R})$, $1 \leq k \leq n - 1$, into three segments as follows. Let the first segment be the portion of $L_k^\infty(\mathbb{R})$ between $x_k$ and the point at infinity on $\psi(\mathbb{R})$. Let the second segment be the portion of the line $L_k^\infty(\mathbb{R})$ between the point $x_k \in L_k^\infty(\mathbb{R})$ and the point $L_{12}(\mathbb{R}) \cap L_k^\infty(\mathbb{R})$. Let the third segment be the remaining portion of $L_k^\infty(\mathbb{R})$. A point $x$ of $B$ satisfies the lexicgon condition if it satisfies the following three properties:
  1. The lines $L_{ij}(\mathbb{R})$, $1 \leq i \leq k - 1$ intersect the first segment of the line $L_k^\infty(\mathbb{R})$ in points which lie above the the lines $L_{i,j-1}(\mathbb{R})$ for each $k + 2 \leq j \leq n$.
  2. The lines $L_{ij}(\mathbb{R})$, $1 \leq i \leq j < k$, intersect the the second segment of the line $L_k^\infty(\mathbb{R})$ in lexicigraphical order with respect to the orientation of $L_k^\infty(\mathbb{R})$.
  3. The lines $L_{ij}(\mathbb{R})$, $k + 1 \leq i < j \leq n$, intersect the third segment of $L_k^\infty(\mathbb{R})$ in reverse lexicigraphical order with respect to the orientation of $L_k^\infty(\mathbb{R})$.

- The line $L_k^\infty(\mathbb{R})$, $1 \leq k \leq n - 1$ and the line $L_{kn}(\mathbb{R})$ divide $\mathbb{A}^2(\mathbb{R})$ into four regions. Two of these regions do not contain the curve $\psi(\mathbb{R})$. A point $x$ of $B$ satisfies the double condition if the regions not containing the curve $\psi(\mathbb{R})$ do not contain any double points of the arrangement $A_3$.

Remark 6.1. The lexicgon condition ensures that we do not get parallel lines in the fiber of $p_n^2$ over $x$ as in Figure 3. The double condition ensures that we only get double points in $A_3$ (away from the points $x_k$). Hence we avoid degenerations in the fiber of $p_n^2$ like the one illustrated in Figure 4.

Define the set $S_n$ to be the set of points contained in $B$ which satisfy the lexicgon and triple conditions. Note that $p_n^2(S_n)$ is equal to $S_{n-1}$.

Lemma 6.2. The map $p_n^2 : S_n \to S_{n-1}$ is a fibration whose fiber is homeomorphic to $\mathbb{R}$.

Proof. We proceed by induction on $n$. When $n = 1$ the set $S_1$ is equal to $\mathbb{R}$. Assume the result up to $n - 1$. Let $(x_1, \ldots, x_{n-1}) \in S_{n-1}$. Let $c$ be the final point to the right of $x_{n-1}$ on $\psi(\mathbb{R})$, for which $(x_1, \ldots, x_{n-1}, c)$ fails to satisfy both the lexicgon and double conditions. The set of points, $F$, to the right of $c$ on $\psi(\mathbb{R})$ is homeomorphic to $\mathbb{R}$. Moreover, the point $(x_1, \ldots, x_{n-1}, f)$ is in $S_n$ for all $f \in F$. The result follows.

Corollary 6.3. The set $S_n$ is contractible for each $n \geq 1$.

Let $p_n^2 : X_n^2 \to X_{n-1}^2$ be the projection which forgets the $k$th point, for $1 \leq k \leq n - 1$. The set $S_n$ was chosen specifically so that the following result would be true.
Proposition 6.4. The fiber of the map $p_k^3$ over any point in $S_{n-1}$ is generic. Moreover, the fiber of the map $p_k^3$, $1 \leq k \leq n$, over any point in $S_{n-1}$ is isomorphic to $GF_n^2$ as an oriented matroid.

Let $F_k$ denote the fiber of the map $p_k^3 : X_n^2 \to X_{n-1}^2$ over the a point in $S_{n-1}$. By Theorem 5.3 we immediately obtain the following result.

Corollary 6.5. The groups $\pi_1(GF_n^2)$ and $\pi_1(F_k)$ have the same presentation for $1 \leq k \leq n - 1$.

We can now inductively choose a base point for $X_n^2$. Choose any point in $S_1$. Let $b$ be a fixed point in the set $S_1$, with $b$ lying over the basepoint previously chosen in $S_{n-1}$. From now on define $b$ to be the base point of $X_n^2$. See Figure 9 for a picture of the generic fiber over the point $b \in X_n^2$.

We shall now find a presentation for $\pi_1(GF_n^2, b_n)$, where $GF_n^2 = \mathbb{C}^2 - \mathcal{A}_1$ is the generic fiber over the basepoint $b = (b_1, \ldots, b_{n-1}) \in X_{n-1}$. We work in the neighborhood $N_0$ defined in Section 5. Note that since $N_0$ is homotopic to $\mathbb{C}^2 - \mathcal{A}_1$, the two groups $\pi_1(GF_n^2, b_n)$ and $\pi_1(N_0, b_n)$ are isomorphic.

We begin by finding generators. Orient all lines $L_{ij}$, in the direction from $b_i$ to $b_j$ where $i < j$. Let $p_{ij} = L_{ij} \cap L_n^\infty$. We define loop $a_{ijn}$, $1 \leq i < j \leq n - 1$, to be the loop in $L_n^\infty$, based at $b_n$, which goes around $p_{ij}$ (see Figure 10). As a consequence of Proposition 5.3 we immediately obtain the following result.

Lemma 6.6. The set $\{a_{ijn} | 1 \leq i < j \leq n - 1\}$, generates the group $\pi_1(GF_n^2, b_n)$.

We now wish to find the defining relations amongst the $a_{ijn}$ for the group $\pi_1(GF_n^2, b_n)$. We will do this using Van Kampen’s Theorem. We begin by dividing $\mathbb{R}^2 \subset \mathbb{C}^2$ into three open sets. Let $\epsilon > 0$ be a small real number. Let $T(\mathbb{R})$ be a real tubular neighborhood of the curve $\psi(\mathbb{R})$, with diameter equal to $2\epsilon$. Choose $\epsilon$ small enough so that $T(\mathbb{R})$ does not contain any double intersections of $\mathcal{A}_1$. Let $D_\epsilon(\mathbb{R}) \subset T(\mathbb{R})$ be the disc of radius $\epsilon$ centered at $b_n$. The curve $\psi(\mathbb{R})$ divides $\mathbb{R}^2$ into two regions. Call the open region containing the tangent lines to $\psi(\mathbb{R})$ the exterior region of $\psi(\mathbb{R})$. The complementary open region will be called the interior region of $\psi(\mathbb{R})$. Let $E(\mathbb{R})$ denote the union of the exterior region of $\psi(\mathbb{R})$ and the disc $D_\epsilon(\mathbb{R})$. Let $I(\mathbb{R})$ denote the union of the interior region of $\psi(\mathbb{R})$ and the disc $D_\epsilon(\mathbb{R})$. Note that $\mathbb{R}^2 = T(\mathbb{R}) \cup E(\mathbb{R}) \cup I(\mathbb{R})$. Let $E, I$ and $T$ denote the complexification of each of these sets respectively, intersected with $N_0$.

Our aim is to find a presentation for $\pi_1(GF_n^2, b_n)$ by applying Van Kampen’s Theorem to the sets $I - \mathcal{A}_1$, $E - \mathcal{A}_1$, and $T - \mathcal{A}_1$. To do this, we begin by finding a presentation of $\pi_1(I - \mathcal{A}_1, \{\gamma\})$. We first need to define a new loop $\tilde{a}_{ijn}$, $1 \leq i <
The loops $a_{ijn}$, $\tilde{a}_{ijn}$ and $a'_{ijn}$ 

Figure 10. The loops $a_{ijn}$, $\tilde{a}_{ijn}$ and $a'_{ijn}$

$1 \leq n - 1$, in $GF_n^2$. Define the loop $\tilde{a}_{ijn}$ as follows. Pick a point $p_{ij} \in I(\mathbb{R})$ on the line $L_{ij}(\mathbb{R})$ which lies within the disc of radius $\epsilon$ about the point $b_j$. Let the line $\tilde{L}(\mathbb{R})$ denote the real line joining $b_n$ and $p_{ij}$. We define the loop $\tilde{a}_{ijn}$ to be the loop in $\tilde{L}$, based at $b_n$, which goes around $p_{ij}$ (see Figure 10).

**Lemma 6.7.** The group $\pi_1(I - A_i, |\hspace{.1cm}|)$ admits a presentation with generators $\tilde{a}_{ijn}$, $1 \leq i < j \leq n - 1$, and defining relations

$$[\tilde{a}_{ijn}, \tilde{a}_{rsn}] = 1, \ 1 \leq i < r < j \leq n - 1.$$

**Proof.** Note that the set $I(\mathbb{R})$ is convex. Hence, the lines $L_{ij}(\mathbb{R})$ intersect $I(\mathbb{R})$ in only one segment. It is not hard to see that one can find a Morse function defined on the closure of the set $I - A_i$. To complete the proof apply Lemma 5.6. 

We now find a presentation for the group $\pi_1(E - A_i, |\hspace{.1cm}|)$. We begin by defining a new loop $a'_{ijn}$, $1 \leq i < j \leq n - 1$, contained in $GF_n^2$. Let $\psi'(\mathbb{R})$ be a curve joining $b_n$ and $b_{i-1}$ which is obtained by deforming the curve $\psi(\mathbb{R})$ as follows. Fix the points $b_n$ and $b_{i-1}$ and push the portion of the curve $\psi(\mathbb{R})$ lying between these two points away from the curve $\psi(\mathbb{R})$ into the region $E(\mathbb{R})$. Let $p_{rs}$ be equal to $L_{rs}(\mathbb{R}) \cap \psi'(\mathbb{R})$, $1 \leq r < s \leq n - 1$. We now define the loop $a'_{ijn}$. Start at the point $b_n$. On reaching a point $p_{rs}$ hop over the line $L_{rs}(\mathbb{R})$. Run all the way down $\psi'(\mathbb{R})$, hopping over each point $p_{rs}$, until reaching the line $L_{ij}(\mathbb{R})$. Then choose a line $L(\mathbb{R})$ passing through $p_{ij}$ which is transverse to $L_{ij}(\mathbb{R})$. Let $l$ be a small loop in $L$, which is oriented in the positive direction with respect to the orientation of $L$, and which goes around $L_{ij}$. Run around the loop $l$ in the positive direction. Finally, return to $b_n$ along the same path taken on the outward journey.

**Lemma 6.8.** The group $\pi_1(E - A_i, |\hspace{.1cm}|)$ admits a presentation with generators $a_{ijn}$ and $a'_{ijn}$, $1 \leq i < j \leq n - 1$, and defining relations

$$[a_{ijn}, a_{rsn}] = 1, \ 1 \leq r < i < j \leq n - 1;$$

$$[a_{ijn}, a'_{rsn}] = 1, \ 1 \leq i < j < r \leq n - 1.$$

---

5 For example, in order to define a Morse function, take a family of real expanding sets, which eventually exhaust $I$, are based at $p$, and which grow in following way. The family first intersects each line segment $L_{ij}(\mathbb{R}) \cap I(\mathbb{R})$ tangentially. It also envelopes each intersection point one at a time.
Proof. Note that the set $E(\mathbb{R})$ is not convex. In fact, the intersection of $L_{ij}(\mathbb{R})$ with $E(\mathbb{R})$ consists of precisely two line segments (thus for each line $L_{ij}(\mathbb{R})$ we have to add the two generators $a_{ijn}$ and $a'_{ijn}$ to the presentation of $\pi_1(E - A_{\lfloor, \lfloor})$). It is not hard to see that one can define a Morse function on closure of the set $E - A_{\lfloor}$.

Now apply Lemma 5.6 to complete the proof.

Now we find a presentation for the group $\pi_1(T - A_{\lfloor, \lfloor})$. Let $[g_1, \ldots, g_n]$ denote the relations

$$g_1 \ldots g_n = g_2 \ldots g_n g_1 = \cdots = g_n g_1 \ldots g_{n-1}.$$  

Lemma 6.9. The group $\pi_1(T - A_{\lfloor, \lfloor})$ admits a presentation with generators

$$a_{ij}, \quad a_{ijn}, \quad 1 \leq i < j \leq n - 1,$$

and defining relations

$$[a_{1jn}, a_{j-1,j,n}, \ldots] = 1, \quad 1 \leq j \leq n - 1. \quad (8)$$

Proof. The generators $a_{ij}$ and $\tilde{a}_{ij}$ can be homotoped into the set $T - A_{\lfloor}$. To see this deform the paths that were used to define the loops $a_{ijn}$ and $\tilde{a}_{ij}$ into the set $T - A_{\lfloor}$. This can be done by deforming these paths over the double points in $A_{\lfloor}(\mathbb{R})$, using Lemma 5.5. Then deform the small loops in the definition of the $a_{ijn}$ and $a'_{ijn}$, past the intersection points in lines $L_{ij}(\mathbb{R})$, until they lie in the set $T - A_{\lfloor}$.

By making $\epsilon$ small enough we see that

$$\pi_1(T - A_{\lfloor, \lfloor}) \cong \bigoplus \pi_\infty(B_\epsilon(b_j) - \bigcup L_{ij}),$$

where $B_\epsilon(b_j)$ is a complex ball or radius $\epsilon$ with center $b_j$. We now refer to Randell’s paper [13]. By considering the ends of the loops $a_{ijk}$ and $\tilde{a}_{ijk}$ locally in each ball $B_\epsilon(b_j)$ we see that the relations arise from a calculation made within the complement of the Hopf link, $\delta(B_\epsilon(b_j)) \cap \bigcup L_{ij}$.

We now want to write the loops $\tilde{a}_{ijn}$ and $a'_{ijn}$ in terms of the generators $a_{ij}$. We refer to Randell’s paper [13]. Using the result contained in this paper concerning the Hopf link of a point, we have the formulas

$$\tilde{a}_{ijn} = A^{-1} a_{ijn} A,$$

where,

$$A = a_{i-1,j,n} \ldots a_{1jn}, \quad (10)$$

and

$$a'_{ijn} = B^{-1} \tilde{a}_{ijn} B,$$

where,

$$B = \tilde{a}_{i,j-1,n} \ldots \tilde{a}_{i,i+1,n} a_{i,i-1,n} \ldots a_{1,i,n} \ldots a_{1,i,n}. \quad (12)$$

We are now able to state the main theorem of this section.

Theorem 6.10. The group $\pi_1(GF_n^2, b_n)$ admits a presentation with generators

$$a_{ij}, \quad 1 \leq i < j \leq n - 1,$$

and defining relations (7) – (10).
Figure 11. The generator $a_{ijk}$ of $P_n^2$

Proof. As we have seen, $N_\epsilon \cong \mathbb{C}^2 - A$ can be divided into three sets, $E - A_1, I - A_1$, and $T - A_1$. We know a presentation for the fundamental group of each of these sets. Hence we need only understand how they fit together. The fundamental group of the intersection $E \cap T$ is a free group with generators $a_{ijn}$ and $a_{ij}$. The fundamental group of the intersection $I \cap T$ is a free group with generators $\tilde{a}_{ijn}$. Also, $E \cap I = D_\epsilon$ which is contractible. Now apply Van Kampen’s Theorem.

In Section 9 we will need to know a presentation for the group $\pi_1(P_n^2 - A_1, \Delta)$, which arises as the fundamental group of the generic fiber of the projection $q^2_{n} : Y_n^2 \to Y_n^2 - 1$.

Lemma 6.11. The group $\pi_1(P_n^2 - A_1, \Delta)$ admits a presentation with the same generators and relations as $\pi(GF_n^2, b_n)$, and with the additional relation

$$a_{12n} \cdot a_{13n} \cdots a_{1,n-1,n} \cdot a_{23n} \cdots a_{2,n-1,n} \cdot a_{34n} \cdots a_{n-2,n-1,n} = 1.$$  (13)

Proof. Use Van Kampen’s theorem to “glue” the line at infinity into $GF_n^2$. Note that this line is homeomorphic to $P^1$; hence the relation (13).

7. Main Theorems

We begin this section by finding generators for the group $P_n^2$. Let $F_k$ be equal to the fiber over the base point $b \in S_n - 1$, of the projection $X_n^2 \to X_{n-1}^2$, obtained by forgetting the $k$th point, for $1 \leq k \leq n$. Let $L(R)$ be the tangent line to $\psi(R)$ passing through the point $b_k$. Define loop $a_{ijk}$, $1 \leq i < j < k \leq n$, to be the loop in $L$, based at $b_k$, which goes around $L(R) \cap L_{ij}(R)$ (see Figure 11). Define loops $a'_{ijk}$ and $a_{ijk}$ in $P_n^2$ using formulas (9)–(12) with $n = k$.

Lemma 7.1. The group $P_n^2$, $n \geq 3$, is generated by the set

$$\{a_{ijk} \mid 1 \leq i < j < k \leq n\}.$$

Proof. We proceed by induction. The group $P_3^2$ isomorphic to $\mathbb{Z}$ by Proposition 7, thus when $n = 3$ the result is clear. Assume the result up to $n - 1$. Now use sequence (10). The generators $a_{ijk}$, where $1 \leq i < j < k \leq n - 1$, generate $P_{n-1}^2$ by induction, and clearly lift from $P_{n-1}^2$ to $P_n^2$. By adding the the generators $a_{ij}$ we obtain the result.

We now wish to discover relations amongst the $a_{ijk}$ in order to find a presentation for the group $P_n^2$. We denote the lexicographical ordering on the set of two element subsets of $\{1, \ldots, n\}$, by $\prec$.

\footnote{Note that when $k$ is equal to $n$ then $F_k$ is equal to $GF_n^2$.}
Theorem 7.2. The following relations hold in $P_n^2$, for $3 \leq k \leq n$.

\[
[a_{ijk}, a_{rsk}] = 1, \quad 1 \leq r < i < j < s \leq k;
\]
\[
[a_{ijk}, a'_{rsk}] = 1, \quad 1 \leq i < j < r < s \leq k;
\]
\[
[\tilde{a}_{ijk}, \tilde{a}_{rsk}] = 1, \quad 1 \leq i < r < j < s \leq k;
\]
\[
[a_{1jk}, \ldots, a_{j-1jk}, \tilde{a}_{j,j+1,k}, \ldots, \tilde{a}_{j,k-1,k}, a_{jk,k+1}, \ldots, a_{jk,n-1}] = 1, \quad 1 \leq j \leq k - 1;
\]
\[
a_{ijk}^{-1} a_{rst} a_{ijk} =
\]
\[
\begin{cases}
  a_{rst} & rs < ij \text{ or } jk < st; \\
  a_{ijt} a_{ikt} a_{rst} a_{ikt}^{-1} a_{ijt}^{-1} & jk = rs; \\
  a_{ijt} a_{ikt} a_{jkt} a_{ikt} a_{ijt}^{-1} a_{ik} a_{ikt}^{-1} a_{ijt}^{-1} & ik < rs < jk; \\
  a_{ijt} a_{ikt} a_{jkt} a_{ikt} a_{ijt}^{-1} a_{ik} a_{ikt}^{-1} & ik = rs \text{ or } ij = rs; \\
  a_{ijt} a_{ikt} a_{jkt} a_{ikt} a_{ijt}^{-1} a_{ikt}^{-1} a_{ijt}^{-1} & ij < rs < ik; \\
  a_{ijt} a_{ikt} a_{jkt} a_{ikt} a_{ijt}^{-1} a_{ik} a_{ikt}^{-1} a_{ijt}^{-1} & \text{where } k < t \leq n.
\end{cases}
\]

Proof. This proof proceeds inductively using sequence (4). When $n = 3$ the relations degenerate, which is correct since $P_3^2$ is isomorphic to $\mathbb{Z}$.

At the nth stage we split the proof up into three main parts and deal with each separately in the following sections.

1. When $k = n$ relations (14) – (17) are simply those coming from the fiber group. These were established in Section 3.
2. When $t = n$ relations (18) come from conjugation of generators in the fiber group by the generators of $P_n^2$. These will be established in Section 10.
3. Relations not dealt with in (1) or (2) come from lifting relations from $P_{n-1}^2$. These will be established in Section 12.

If sequence (4) were short exact, then Theorem 7.2 would give us precisely the relations required for a presentation of $P_n^2$. This motivates the following definition.

Definition 7.3. Let $PL_n$ be the group whose presentation has generators $a_{ijk}$, $1 \leq i < j < k \leq n$, and relations (14) – (17) of Theorem 7.2.

Let $\varphi_n : PL_n \to P_n^2$ be the tautological homomorphism which takes generators to generators. Then, $\varphi_n$ is clearly well defined and onto.

We now want to see how close the group $PL_n$ is to being isomorphic to $P_n^2$. We will do this by studying the relationship between the integral homology of each of these groups. Specifically, we will show that $\varphi_n$ induces an isomorphism on the first two integral homology groups. First, we will need some preliminary results concerning the homology of the space $X_n^2$.

Lemma 7.4. The restriction map

$$H^k(X_n^2, \mathbb{Z}) \to H^k(GF_n^2, \mathbb{Z})$$

is surjective.
Proof. The group $H^1(X^2_n, \mathbb{Z})$ contains the classes of the forms $\frac{1}{2\pi i} d\Delta_{ijk}$, where $\Delta_{ijk}$ are the minors defined in the introduction. The forms $\frac{1}{2\pi i} d\Delta_{ijk}$ restrict to the forms $\frac{1}{2\pi i} d\Delta_{ij}$ on $GF^2_n$ whose $k$-fold wedge products generate $H^k(GF^2_n, \mathbb{Z})$. \hfill \Box

Corollary 7.5. The group $H_k(GF^2_n, \mathbb{Z})$ injects into $H_k(X^2_n, \mathbb{Z})$ for all $k \geq 0$. \hfill \Box

Proposition 7.6. The Leray spectral sequence in the homology for the map $p_n^2 : X^2_n \rightarrow X^2_{n-1}$ has the following properties (here all homology groups have $\mathbb{Z}$ coefficients, unless otherwise stated):

1. For $p + q \leq 2$ the $E^2_{p,q}$ terms are isomorphic to

\[
\begin{array}{ccc}
H_2(GF^2_n) & * & * \\
H_1(GF^2_n) & H_1(GF^2_n) \otimes H_1(X^2_{n-1}) & * \\
\mathbb{Z} & H_1(X^2_{n-1}) & H_2(X^2_{n-1}) \\
\end{array}
\]

2. Let $p + q \leq 2$. Then the $d_2$ differentials whose images lie in $E^2_{p,q}$ all vanish, and thus $E^3_{p,q} = E^{\infty}_{p,q}$.

3. The terms $E^2_{p,q}$ are torsion free for $p + q \leq 2$.

Proof. We begin by proving statement (1). We will work with the cohomology spectral sequence first, and justify the statement in homology later. Let $\pi$ denote the projection map $p^2_n : X^2_n \rightarrow X^2_{n-1}$. Let $U$ be an open cover of $X^2_{n-1}$. Let $\mathcal{F}_U$ be the sheaf defined by $\mathcal{F}_U(U) = H^1(U, \mathbb{Z})$, where $U \in U$. Then the $E_2$ term of the Leray spectral sequence in cohomology has

\[E^2_{p,q} = H^p(X^2_n, \mathcal{F}_U)\]

and converges to $H^{p+q}(X^2_n, \mathbb{Z})$. If the map $\pi$ were a fibration then $\mathcal{F}_U$ would be locally constant for each $q$ and we would obtain the result, since the cohomology groups of the fiber of $\pi$ are torsion free abelian groups of finite rank. However this is not the case. To obtain the result we show that $\mathcal{F}_U$ is locally constant for $q = 1, 2$.

Let $x$ be any point in $X^2_{n-1}$. We will show that we can choose an open ball $B_x \subset X^2_{n-1}$, containing $x$, so that the group $H^q(\pi^{-1}(B_x), \mathbb{Z})$ is isomorphic to $H^q(GF^2_n, \mathbb{Z})$ for $q = 1, 2$.

Denote the complex dimension of the space $X^2_{n-1}$ by $m$. The fiber over the point $x$ in $X^2_{n-1}$ is equal to $\mathbb{C}^2$ less a union of lines $L_{ij}$, $1 \leq i < j \leq n - 1$. Let $D_{ij} \subset (\mathbb{C}^2)^n$ denote the divisor defined by the minor $\Delta_{ij}$ defined in Section 1. Then we choose an open ball $B_x$ containing $x$ so that $\pi^{-1}(B_x)$ is homeomorphic to $L_{ij} \times B_x$. Denote the set $L_{ij} \times B_x$ by $F_{ij}$. Then $\pi^{-1}(B_x)$ is homeomorphic to a complex ball $M$ of dimension $m + 2$ minus the union of the $F_{ij}$.

Since the combinatorics the space $M - \cup F_{ij}$ and the $GF^2_n$ agree in complex codimensions 1 and 2 the groups $H^q(\pi^{-1}(B_x), \mathbb{Z})$ and $H^q(GF^2_n, \mathbb{Z})$ are isomorphic for $q = 1, 2$.

We prove (2) and (3) together, using induction. To begin the induction, note that $H_0(X^2_n) = \mathbb{Z}$, $H_1(X^2_n) = \mathbb{Z}$, and $H_2(X^2_n) = 0$. Since $GF^2_n$ is a complement of lines in $\mathbb{C}^2$, we know that $H_k(GF^2_n)$ (and $H^k(GF^2_n)$) is torsion free for all $k \geq 0$. We work first with the Leray spectral sequence in cohomology with rational coefficients. Assume inductively that $H^k(X^2_{n-1})$ is torsion free for $k = 0, 1$ and 2.

From Lemma 7.4 we deduce that all of the differentials $d_2 : E^2_{0,q} \rightarrow E^2_{2,q-1}$ vanish.
By multiplicativity of the spectral sequence in cohomology, this implies that the differential $d_2 : E^{2,0}_2 \to E^{1,1}_2$ also vanishes.

Now dualize to consider the homology spectral sequence. Note that all of the required differentials vanish when the terms of the spectral sequence have $\mathbb{Q}$ coefficients. However, since the fiber homology is torsion free all of the differentials whose image lie in the fiber homology groups vanish when the terms have $\mathbb{Z}$ coefficients. The only other differential which could be non-zero is $d^2 : E^{2,0}_3 \to E^{1,1}_2$. However, this has to vanish since $H_1(GF^2_n) \otimes H_1(X^2_{n-1})$ is torsion free by induction.

\begin{corollary}
The first and second integral homology groups of $X^2_n$ are torsion free.
\end{corollary}

\begin{proof}
Since the Leray spectral sequence in homology degenerates at $E^{2,2}_{\infty}$, when $p + q \leq 2$, the graded quotients of the corresponding filtration of $H_1(X^2_n, \mathbb{Z})$ and $H_2(X^2_n, \mathbb{Z})$ are torsion free, from which the result follows.
\end{proof}

\begin{remark}
The $E_{\infty}$ term of the Leray spectral sequence in cohomology induces a decreasing filtration $L^\bullet$ on $H^\bullet(X^2_n, \mathbb{Z})$ which we shall call the Leray filtration.

We will also need a technical corollary.

\begin{corollary}
The cup product $\Lambda^2 H^1(X^2_n, \mathbb{Z}) \to H^2(X^2_n, \mathbb{Z})$ is surjective.
\end{corollary}

\begin{proof}
As a graded ring, $E^{p,q}_{\infty}$ is isomorphic to $Gr^p_L H^{p+q}(X^2_n)$, where $Gr^p_L$ denotes the $p$th graded quotient of the filtration $L^\bullet$. From the proof of Proposition 7.4, we deduce that $\Lambda^2 Gr^\bullet_L H^1(X^2_n, \mathbb{Z})$ surjects onto $Gr^\bullet_L H^2(X^2_n, \mathbb{Z})$, which implies the result by induction on $n$.
\end{proof}

Another immediate consequence of Proposition 7.4 is the following result.

\begin{corollary}
The sequence
\begin{align*}
0 \to H_1(GF^2_n, \mathbb{Z}) \to H_1(X^2_n, \mathbb{Z}) \to H_1(X^2_{n-1}, \mathbb{Z}) \to 0
\end{align*}

is short exact.
\end{corollary}

\begin{proof}
This sequence arises from the filtration of $H_1(X^2_n, \mathbb{Z})$ given by the $E_{\infty}$ term of the Leray spectral sequence, plus the fact that $H_1(GF^2_n, \mathbb{Z})$ injects into $H_1(X^2_n, \mathbb{Z})$ by Corollary 7.4.
\end{proof}

This corollary immediately gives us the following lemma.

\begin{lemma}
The group $H_1(X^2_n, \mathbb{Z})$ is free abelian group of rank $\binom{n}{3}$.
\end{lemma}

\begin{proof}
Proceed by induction. To begin the induction note that the group $H_1(X^3_3, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$, since $\pi_1(X^3_3)$ equal to $\mathbb{Z}$ (Lemma 3.3).

Since $GF^2_n$ is the complement in $\mathbb{A}^2$ of $\binom{n-1}{2}$ lines, the group $H_1(GF^3_n, \mathbb{Z})$ is free abelian of rank $\binom{n-1}{2}$. By induction the group $H_1(X^2_{n-1}, \mathbb{Z})$ is free abelian of rank $\binom{n-1}{3}$. The result follows from Corollary 7.10 and the fact that
\[ \binom{n-1}{3} + \binom{n-1}{2} = \binom{n}{3}. \]
Since we do not know if $X_2^n$ is an Eilenberg-MacLane space, it is necessary to prove that the first and second integral homology groups of $X_2^n$ are equal to those of $P_2^n$. To do this we will use some homotopy theory. Denote the Eilenberg-MacLane space $K(G,1)$ associated to $G$ by $BG$. The following result may be deduced from obstruction theory and covering space theory [16].

**Lemma 7.12.** If $(X, \ast)$ is a connected, pointed topological space with the homotopy type of a CW complex and fundamental group $G$, then:

1. there is a natural map $(X, \ast) \to (BG, \ast)$ which is unique up to homotopy;
2. the homotopy fiber $U$ of the map in (1) is weakly homotopy equivalent to the universal cover of $X$.

**Proposition 7.13.** When $k = 1, 2$ the natural map

$$H_k(X_2^n, \mathbb{Z}) \to H_k(P_2^n, \mathbb{Z}),$$

is an isomorphism.

**Proof.** In this proof all homology groups have integer coefficients. We denote $\pi_1(X_2^n)$ by $G$. By Lemma 7.12 we know that there is a natural map

$$(X_2^n, \ast) \to (BG, \ast)$$

whose homotopy fiber, $U_2^n$, is weakly homotopy equivalent the universal cover of $X_2^n$. The $E^2$ term Leray-Serre spectral sequence of this fibration is

$$
\begin{array}{ccc}
H_2(U_2^n) & H_2(U_2^n, H_1(BG)) & H_2(U_2^n, H_2(BG)) \\
\mathbb{Z} & H_1(BG) & H_2(BG)
\end{array}
$$

It follows that $H_2(BG)$ injects into $H_2(X_2^n)$. Since $H_2(X_2^n)$ is torsion free, so is the group $H_2(BG)$. Since $H_2(BG)$ is isomorphic to $H_2(G)$ we have the commutative diagram

$$
\begin{array}{ccc}
\Lambda^2 H^1(G) & \Lambda^2 H^1(X_2^n) \\
\downarrow & \downarrow \\
H^2(G) & \to & H^2(X_2^n)
\end{array}
$$

By Lemma 7.9 the right hand vertical map is surjective. It follows that the map $i^*$ is surjective. Since $H_2(G)$ is torsion free we can dualize to get the required result.

We can now prove the main theorems.

**Theorem 7.14.** The homomorphism $\varphi_n$ induces an isomorphism between the first integral homology groups of $PL_n$ and $P_2^n$.

**Proof.** First consider the group $H_1(PL_n, \mathbb{Z})$. Since all of the relations in $PL_n$ are commutators $H_1(PL_n, \mathbb{Z})$ is a free group of rank $\binom{n}{3}$ generated by the homology classes of the elements $a_{ijk}$, $i \leq i < j < k \leq n$. The map $PL_n \to PL_{n-1}$ induces a
map from $H_1(PL_n, \mathbb{Z}) \to H_1(PL_{n-1}, \mathbb{Z})$. By considering the abelianisation of the relevant groups we obtain the short exact sequence of torsion free abelian groups

$$0 \to K \to H_1(PL_n, \mathbb{Z}) \to H_1(PL_{n-1}, \mathbb{Z}) \to 0$$

where $K$ is defined to be the kernel. The group $K$ is generated by the homology classes of the elements $a_{ij}$, $1 \leq i < j \leq n - 1$.

The map $i$ induces the following commutative diagram.

$$
\begin{array}{cccccc}
0 & \longrightarrow & K & \longrightarrow & H_1(PL_n, \mathbb{Z}) & \longrightarrow & H_1(PL_{n-1}, \mathbb{Z}) & \longrightarrow & 0 \\
\downarrow & & \downarrow \phi_* & & \downarrow \phi_{n-1} & & \downarrow & & \downarrow \\
0 & \longrightarrow & H_1(GF_n^2, \mathbb{Z}) & \longrightarrow & H_1(P_n^2, \mathbb{Z}) & \longrightarrow & H_1(P_{n-1}^2, \mathbb{Z}) & \longrightarrow & 0
\end{array}
$$

The map $i$ is clearly well defined and the bottom row of the diagram is exact by Lemma 7.10. The map $i$ is surjective since $H_1(GF_n^2, \mathbb{Z})$ is generated by the homology classes of the loops $a_{ij}$, $1 \leq i < j \leq n - 1$. Since the groups $K$ and $H_1(GF_n^2, \mathbb{Z})$ are both torsion free abelian groups the map $i$ is an isomorphism.

We now proceed by induction. Note that $H_1(PL_3, \mathbb{Z})$ and $H_1(P_2, \mathbb{Z})$ are both isomorphic to $\mathbb{Z}$. Assume by induction that the map $\phi_{n-1}$ is an isomorphism. The map $i$ is an isomorphism and so we complete the proof by applying the Five Lemma to the above commutative diagram.

To prove that $H_2(P_n^2, \mathbb{Z})$ is isomorphic to $H_2(PL_n, \mathbb{Z})$ we will use some rational homotopy theory. Let $D : H_2(X, \mathbb{Q}) \to \Lambda^2 H_1(X, \mathbb{Q})$ be the map induced by the diagonal inclusion $X \to X \times X$. Given a group $G$, let $\Gamma^n G$ denote the $n$th term in the lower central series of $G$. The proof of the following result is elementary and may be found in [7].

**Lemma 7.15.** If $G$ is any group then the commutator map

$$\Lambda^2 H_1(G, \mathbb{Z}) \xrightarrow{\ominus} \Gamma^2 G/\Gamma^3 G$$

defined by

$$x \land y \mapsto xyx^{-1}y^{-1}$$

is a surjection.

The following result can be proved using results in either [4, §2.1] or [8, §8].

**Lemma 7.16.** If $X$ is a topological space and the dimension of $H_1(X, \mathbb{Q})$ is finite for $k = 1, 2$, then the sequence

$$H_2(X, \mathbb{Q}) \xrightarrow{D} \Lambda^2 H_1(X, \mathbb{Q}) \xrightarrow{\ominus} [\Gamma^2 \pi_1(X)/\Gamma^3 \pi_1(X)] \otimes \mathbb{Q} \to 0$$

is exact and natural in $X$.

**Corollary 7.17.** The sequence

$$0 \to H_2(X_n^2, \mathbb{Q}) \to \Lambda^2 H_1(X_n^2, \mathbb{Q}) \to [\Gamma^2 \pi_1(X_n^2)/\Gamma^3 \pi_1(X_n^2)] \otimes \mathbb{Q} \to 0$$

is short exact.

**Proof.** Note that the map $H_2(X, \mathbb{Q}) \to \Lambda^2 H_1(X, \mathbb{Q})$ in Lemma 7.16 is injective if and only if $\Lambda^2 H_1(X, \mathbb{Q}) \to H^2(X, \mathbb{Q})$ is surjective. Now apply Lemma 7.9.

We need to compute the map $D : H_2(PL_n, \mathbb{Q}) \to \Lambda^2 H_1(PL_n, \mathbb{Q})$. The following result is proved in [7].
Lemma 7.18. If $F$ is a free group then
\[ \Lambda^2 H_1(F, \mathbb{Z}) \xrightarrow{\sim} \Gamma^2 F/\Gamma^3 F \]
is an isomorphism of $\mathbb{Z}$-modules. \hfill \Box

Let $G$ be the group
\[ G = \langle x_1, \ldots, x_n \mid r \rangle. \]
where $r \in \Gamma^2(x_1, \ldots, x_n)$. The relation $r$ gives an element of $H_2(G, \mathbb{Z})$. We will call $D(r)$ the linearization of $r$. As a consequence of Lemma 7.18 and Lemma 7.16 we have the following result.

Lemma 7.19. If $r$ is as above then;
1. the group $H_2(G, \mathbb{Z})$ is isomorphic to $\mathbb{Z}[r]$;
2. if $r \equiv \prod [x_i, x_j]^{m_{ij}} \mod \Gamma^3 G$, then the map $D$ is given by the formula
\[ D(r) = \sum m_{ij}(x_i \wedge x_j). \] \hfill \Box

Theorem 7.20. The homomorphism $\varphi_n$ induces an isomorphism between the second integral homology groups of $PL_n$ and $P^2_n$.

Proof. Let $N$ be the number of relations in $PL_n$. The rank of $H_2(PL_n, \mathbb{Z})$ is less than or equal to $N$. When equality occurs $H_2(PL_n, \mathbb{Z})$ is torsion free. So if we can show that the rank of $H_2(PL_n, \mathbb{Z})$ is equal to the dimension of $H_2(PL_n, \mathbb{Q})$, then $H_2(PL_n, \mathbb{Z})$ is torsion free. Thus we first use rational coefficients.

We begin by defining filtrations on $H_1(PL_n), \Lambda^2 H_1(PL_n)$ and $H_2(PL_n)$. Let $G_0$ be equal to the set of relations.

To define a filtration on $H_2(PL_n)$ we begin by filtering the relations of $PL_n$. Let $R_0$ be the set of relations (14)–(17) with $k = n$. These relations arise from the fiber group. Let $R_1$ be the set of relations (18) with $t = n$ together with the set relations $R_0$. The set of relations $R_1 - R_0$ come from conjugating generators in the fiber group by generators in $P^2_{n-1}$. Finally, let $R_2$ be the set of all relations for $PL_n$.

We now define a filtration $\tilde{L}$ on $H_2(PL_n)$. Let $\tilde{L}_i$ be the span of the elements of $H_2(PL_n)$ coming from the elements of $R_i$.

We now proceed with the proof using induction. First, note that both $H_2(PL_3, \mathbb{Z})$ and $H_2(P^2_2, \mathbb{Z})$ are trivial. Assume the result up to $n - 1$. As a consequence of Lemma 7.16 we have the following commutative diagram.

First we show that the map $D : H_2(PL_n) \to \Lambda^2 H_1(PL_n)$ is injective. The map $D$ is filtration preserving. Thus the map $\gamma_i : \tilde{L}_i/\tilde{L}_{i-1} \to \tilde{F}_i/\tilde{F}_{i-1}$ induced by $D$ is well defined. To show that $D$ is injective it is sufficient to show that $\gamma_i$ is injective for $0 \leq i \leq 2$. We begin with $\gamma_0$. The image under $D$ of the fiber relations are
\[ a_{ij} \wedge a_{rs}, \text{ if } ij \neq rs; \] (19)
\[ a_{ij} \wedge (a_{1n} + a_{2n} + \cdots + a_{ij} + \cdots + a_{i,n-1,n}), \text{ for } 1 \leq i \leq n - 1. \] (20)

All of these are linearly independent in \( F_0 \). Hence, \( \tilde{L}_0 \) injects into \( F_0 \).

Now consider the map \( \gamma_1 \). The linearized conjugation relations are
\[ a_{ijk} \wedge a_{rsn}, \text{ if } rs \neq ij, ik \text{ or } jk; \] (21)
\[ a_{rsn} \wedge (a_{ijk} + a_{ij} + a_{ik} + a_{ikn}), \text{ if } rs = ij, ik \text{ or } jk. \] (22)

We now quotient out by \( F_0 \). Relations (21) – (22) modulo \( F_0 \) become
\[ a_{rsn} \wedge a_{ijk}, \text{ where } k \neq n. \] (23)

These are linearly independent. Hence \( \gamma_1 \) is injective. The map \( \gamma_2 \) is injective by induction. Hence the map \( D \) is injective.

We now show that the map \( \varphi_n \) induces an isomorphism between \( H_2(PL_n) \) and \( H_2(X^2_n) \). Let \( L \) be the filtration on \( H_*(X^2_n) \) induced by the Leray sequence. Since the map \( D \) is injective for both \( H_2(PL_n) \) and \( H_2(P^2_n) \) we will not differentiate between \( H_2 \) and its image in \( \Lambda^2 H_1 \); for example \( D(L_0) = L_0 \). We show that the map \( \beta : L_i/L_{i-1} \to L_i/L_{i-1} \) induced by \( \varphi_n \) is an isomorphism for \( 0 \leq i \leq 2 \).

Begin with the map \( \beta_0 \). Note that \( L_0 = H_2(GF^2_n) \). Thus we have to analyse the group \( H_2(GF^2_n) \). Recall that \( GF^2_n = \mathbb{C}^2 - A_1 \). Let \( p \) be an intersection point in \( A_1 \) and \( A_\sqrt{2} \) be the set of lines in \( A \) passing through the point \( p \). Let \( M_p = \mathbb{C}^2 - A_\sqrt{2} \). Then there exist inclusion maps \( \iota_p : M_p \to GF^2_n \). By a result of Brieskorn \( \lbrack 3 \rbrack \) the maps \( \iota_p \) induce an isomorphism
\[ \bigoplus_p H_2(M_p) \cong H_2(GF^2_n). \]

Note that each relation in \( \pi_1(GF^2_n) \) arises from an intersection point \( p \) of \( A_1 \). We are thus reduced to the complement of \( n \) lines through the origin in \( \mathbb{C}^2 \). Denote this space by \( M \). The dimension of \( H_2(M) \) is equal to \( n-1 \) \( \lbrack 12 \rbrack \). Now consider the dimension of the space \( \tilde{L} \subset \Lambda^2 \tilde{H}_\infty(M) \) spanned by the linearised relations for \( \pi_1(M) \). The linearizations take the form
\[ a_i \wedge (a_1 + \cdots + a_n), \text{ for } 1 \leq i \leq n \]
where \( a_i \) generates \( H_1(M) \). However, the sum of all of these is equal to zero. Thus the dimension of \( \tilde{L} \) is equal to \( n-1 \), as required.

Now consider the map \( \beta_1 \). The quotient \( L_1/L_0 \) is isomorphic to \( H_1(GF^2_n) \otimes H_1(X^2_n) \). But this is clearly isomorphic to \( L_1/L_0 \) via \( \beta_1 \) by equation (23). The map \( \beta_2 \) is an isomorphism by induction. We conclude that the map \( \varphi_n \) is also an isomorphism.

We now complete the proof by considering integer coefficients. First, since the map \( H_2(PL_n) \to \Lambda^2 H_1(PL_n) \) is injective, the dimension of \( H_2(PL_n) \) is equal to \( N \). Thus \( H_2(PL_n, \mathbb{Z}) \) is torsion free and the map \( D : H_2(PL_n, \mathbb{Z}) \to \Lambda^2 H_1(PL_n, \mathbb{Z}) \) is well defined. Hence we have the following commutative diagram.
\[
\begin{array}{ccc}
H_2(PL_n, \mathbb{Z}) & \longrightarrow & \Lambda^2 H_1(PL_n, \mathbb{Z}) \\
\downarrow \varphi_n & & \downarrow \cong \\
H_2(X^2_n, \mathbb{Z}) & \longrightarrow & \Lambda^2 H_1(X^2_n, \mathbb{Z})
\end{array}
\]
Let $I_1 = \text{Im}\{D : H_2(PL_n, \mathbb{Z}) \to \Lambda^2 H_1(PL_n, \mathbb{Z})\}$, and $I_2 = \text{Im}\{D : H_2(P^2_n, \mathbb{Z}) \to \Lambda^2 H_1(X^2_n, \mathbb{Z})\}$. To complete the proof it suffices to show that $I_1 = I_2$. Note that $I_1 \subset I_2$. Thus it suffices to show that $I_1$ is primitive in $I_2$, i.e.

$$I_1 = (I_1 \otimes \mathbb{Q}) \cap I_2.$$ 

This can be shown easily using the isomorphism $Gr L \cdot I_1 \cong (Gr L \cdot I_1 \otimes \mathbb{Q}) \cap I_2$. 

8. **Infinitesimal Vector Braid Relations**

We begin by recalling the infinitesimal presentation of the classical pure braid group. Given a group $G$ we denote its group algebra over $\mathbb{C}$ by $\mathbb{C}[G]$. Let $\epsilon : \mathbb{C}[G] \to \mathbb{C}$ be the augmentation homomorphism and $J$ be equal to the kernel of $\epsilon$. The powers of $J$ define a topology on $\mathbb{C}[G]$ which is called the $J$-adic topology. In what follows we let $\mathbb{C}\langle Y_i \rangle$ denote the free associative, non-commutative algebra in the indeterminants $Y_i$ and $\mathbb{C}\langle Y_i \rangle$ denote the non-commutative formal power series ring in the indeterminants $Y_i$. In [10] Kohno proves that the $J$-adic completion of the group ring $\mathbb{C}[P_n]$ is isomorphic to $\mathbb{C}\langle\langle X_{ij} \rangle\rangle$ modulo the two-sided ideal generated by the relations

$$[X_{ij}, X_{ik} + X_{jk}] \quad i < j < k;$$

$$[X_{jk}, X_{ij} + X_{ik}] \quad i < j < k;$$

$$[X_{ij}, X_{rs}] \quad i, j, r, s \text{ distinct.}$$

We now find the corresponding infinitesimal presentation for the group $P^2_n$.

**Proposition 8.1.** The completed group ring of $P^2_n$ is isomorphic to $\mathbb{C}\langle\langle X_{ijk} \rangle\rangle$, $1 \leq i < j < k \leq n$, modulo the two-sided ideal generated by the relations

$$[X_{ijk}, X_{jkl} + X_{ikl} + X_{ijl}] \quad i < j < k < l; \quad (24)$$

$$[X_{jkl}, X_{ijk} + X_{ikl} + X_{ijl}] \quad i < j < k < l; \quad (25)$$

$$[X_{ikl}, X_{ijk} + X_{ijl} + X_{jkl}] \quad i < j < k < l; \quad (26)$$

$$[X_{ijk}, X_{12k} + \cdots + X_{k,n-1,n}] \quad 1 \leq k \leq n; \quad (27)$$

$$[X_{ijk}, X_{rs}] \quad i, j, k, r, s, t \text{ distinct.} \quad (28)$$

**Proof.** This is a special case of a result due to K.-T. Chen [3]. We apply the version of this result which appears in [8] pages 28-29. Observe that the tensor algebra $\bigoplus_{n=0}^{\infty} H_1(X^2_n, \mathbb{C}) \otimes \mathbb{C}$ on $H_1(X^2_n, \mathbb{C})$ is isomorphic to $\mathbb{C}\langle\langle X_{ijk} \rangle\rangle$, where the indeterminant $X_{ijk}$ denotes the homology class of the generator $a_{ijk}$ of the group $P^2_n$. This is a direct consequence of Lemma 7.11 and Theorem 7.14. Let

$$\delta : H_2(X^2_n, \mathbb{C}) \to H_1(X^2_n, \mathbb{C}) \otimes \mathbb{C} \subset \mathbb{C}\langle\langle X_{ijk} \rangle\rangle$$

be the dual of the cup product. According to [8] pages 28-29, the completed group ring of $P^2_n$ is isomorphic to $\mathbb{C}\langle\langle X_{ijk} \rangle\rangle/\text{im} \delta$. Using the results contained in the proof of Theorem 7.20 the ideal (im $\delta$) is simply the two-sided ideal generated by the relations (24)–(28).
9. AFFINE VERSUS PROJECTIVE REVISITED

As we saw in Section 2, we can consider motions of points in $\mathbb{A}^m$ as being motions of points in $\mathbb{P}^m$. This has some interesting consequences for the groups $P^2_x$ and $Q^2_x$. First, note that we have the natural surjective map from $P^2_x \to Q^2_x$ (see Lemma 2.1). Thus, we immediately see that $Q^2_x$ is generated by $a_{ijk}$, for $1 \leq i < j < k \leq n$. However, since we are now looking at points in $\mathbb{P}^2$, we get some extra relations amongst these generators, which are analogous to relations (3) of $Q_n$.

**Lemma 9.1.** For $1 \leq k \leq n$, the following relations hold in $Q^2_n^x$:

\[
\begin{align*}
    a_{123} a_{124} \cdots a_{12n} a_{134} \cdots a_{13n} a_{145} \cdots a_{1,n-1,n} &= 1; \\
    a_{123} a_{124} \cdots a_{12n} a_{234} \cdots a_{23n} a_{245} \cdots a_{2,n-1,n} &= 1; \\
    a_{12k} a_{13k} \cdots a_{k-2,k-1,k} a_{1k,k+1} \cdots a_{k-1,k,k+1} \cdots a_{1kn} a_{k-1,kn} &= 1. (29)
\end{align*}
\]

**Proof.** Each of these relations arises from a product relation which occurs in the fiber of the projection, $Y^2_n \to Y^2_{n-1}$, obtained by forgetting the $k$th point. These can then be written in the required form using the reciprocity law. See Section 12.4 for more details.

We define a group $QL^2_n$ by adjoining the extra relations (29) to the presentation of $PL_n$, and conjecture that this group is isomorphic to $Q^2_n$. An argument similar to that in the proof of Theorem 2.14 can be used to show that the first homology groups of $QL_n$ and $Q^2_n$ are the same, and are free abelian of rank $\binom{n+3}{3} - n$.

We can exploit the action of $PGL_3(\mathbb{C})$ on $Y^2_n$ to better understand the group $Q^2_n$. As we have already seen, the action can be used to show that the group $Q^2_n$ has a central element of order three (see Lemma 2.6). Denote this element by $\tau$. In fact, by analysing the action of $PGL_3(\mathbb{C})$ on $Y^2_n$, we can find an explicit formula for this element in terms of the generators $a_{ijk}$.

**Lemma 9.2.** The element $\tau$ is given by the formula

\[
\tau = \tau_1 \cdots \tau_n
\]

where

\[
\tau_i = a_{12i} a_{13i} \cdots a_{i-2,i-1,i}.
\]

**Proof.** We analyse where the map $PGL_3(\mathbb{C}) \to Y^2_n$ sends the generator $\rho : S^1 \to PGL_3(\mathbb{C})$ of $\pi_1(PGL_3(\mathbb{C}))$, given by the formula

\[
\rho : \theta \longmapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\theta} \end{bmatrix}
\]

where $0 \leq \theta \leq 2\pi$.

Choose coordinates $(x, y)$, $x, y \in \mathbb{R}$ for the affine part of $\mathbb{P}^2(\mathbb{R})$. Let the points $b_1$ and $b_2$ be equal to $(0, 0)$ and $(0, 1)$ respectively. Denote the coordinate of the point $b_i$ by $(x_i, y_i)$, for $i \geq 3$. The line $L^\infty_i(\mathbb{R})$ is equal to the vertical line passing through $(x_i, y_i)$. We “stretch out” the points $b_i$ on the curve $\psi(\mathbb{R})$ so that they satisfy the following condition. We require that the line $L^\infty_{i+1}(\mathbb{R})$ intersects the line $L^{\infty}_{i-1}(\mathbb{R})$ at a point whose $y$ coordinate is less than $-y_{i-1}$.

\[\text{Note we may do this whilst remaining within the basepoint set } B \text{ which was defined in Section 2.}\]
Figure 12. Let’s do the twist!

The loop $\rho$ will be sent to the loop $\tau : S^1 \to Y_n^2$ given by the formula

$$\tau : \theta \to ((0,0), (0,1), (x_3, e^{i\theta} y_3), \ldots, (x_n, e^{i\theta} y_n)).$$

On “squeezing” the points back to their original position, we see that the loop which each point $b_i$ follows in the loop $\tau$ is homotopic to the loop

$$\tau_i = a_{1i} a_{13i} \cdots a_{i-2,i-1,i}.$$

We now show that the loops $\tau_i$ commute with each other. Fix $i < j$. Note that the loop $\tau_j$ encircles the lines $L_{rs}$ for $1 \leq r < s \leq j - 1$. As the point $b_i$ follows the loop $\tau_i$ the lines $L_{rs}$ always remain within the loop $\tau_j$. Thus the motion of the point $b_i$ along $\tau_i$ is independent of the loop $\tau_j$.

Finally note that since the $\tau_i$ commute we have the expression

$$\tau = \tau_3 \cdots \tau_n,$$

as required.

The element $\tau$ of order three in $Q_n^2$ is analogous to the full-twist $\Delta$ of the classical braid group of the sphere. We recall some results about the classical braid groups [1]. In the classical case $P_n$ is a group with center $\mathbb{Z}$ generated by the full-twist

$$\Delta = \delta_2 \cdots \delta_n,$$

where

$$\delta_i = a_{1i} a_{2i} \cdots a_{i-1,i}.$$

The elements $\delta_i$ all commute with each other (see Figure 13). Moreover, $\Delta$ is a central element of $Q_n$ which has order 2.

Using the classical braid groups as a model, it is natural to conjecture that $\tau$ generates the center of the group $P_n^2$.

10. Conjugation

In this section we describe a method for conjugating generators of $\pi_1(GF_n^2)$ by generators of the group $P_n^2$.

It will be helpful to first discuss the classical pure braid case. Recall the short exact sequence [9]. To find a presentation for $P_n$ from one for $P_{n-1}$, we have to be able to write $a_{ij}^{-1} a_r a_{ij}$ as a word in the group $L_{n-1}$, where $1 \leq i < j \leq n - 1$, and $1 \leq r \leq n - 1$. We can do this by picturing the loops $a_{r}a_{n}$ and $a_{ij}$ in $\mathbb{C}$ (Figure 13). As we “unwind” the braid $a_{ij}^{-1} a_r a_{ij}$, the point $j$ moves around point $i$ and “pushes” the loop $a_{r}a_{n}$ with it. At the end of the unwinding process, we are left with a loop in $L_{n-1}$, which will be the required conjugate of $a_{r}a_{n}$. We can describe this
loop in terms of the fiber generators by recording where this loop crosses vertical half lines below the points \(\{1, \ldots, n-1\}\). For example the relation
\[
a_{ij}^{-1} a_{rs} a_{ij} = a_{is} a_{rs} a_{is}^{-1}, \quad 1 \leq i < r < j < s \leq n,
\]
is shown in Figure 14.

We now describe a similar process for conjugating in \(P^2_n\). Let \(L(\mathbb{R})\) denote the line \(L_n^\infty(\mathbb{R})\) which we used to define the loop \(a_{rsn}\), \(1 \leq r < s \leq n - 1\). Let \(p_{rs}\) denote the intersection of line \(L_{rs}(\mathbb{R})\) with \(L(\mathbb{R})\). We picture the generator \(a_{rsn}\) contained in \(L\) in Figure 15.

Note that if we move the point \(b_k\) for \(1 \leq k \leq n - 1\), then we will induce a motion of the points \(p_{rs}\) within \(L\). Hence, when point \(b_k\) follows the loop \(a_{ijk}\), we can picture the induced motions of the points \(p_{rs}\) in \(L\). We can use this picture to determine how generator \(a_{ijk}\) conjugates generator \(a_{rsn}\) by recording how the movements of the \(p_{rs}\) within \(L\) deform the loop \(a_{rsn}\).

At this point we need to make some observations which will simplify the calculation. First, note that the motion of the point \(b_k\) only induces a motion of the points \(p_{rk}\). Moreover, by choosing the loop \(l\) which the point \(b_k\) follows on \(a_{ijk}\) whilst going around \(L_{ij}(\mathbb{R})\) to be small enough\(^8\), we can ensure that only the points \(p_{ik}\) and \(p_{jk}\) go around \(p_{ij}\) as the point \(b_k\) goes around \(l\). Hence, we need only consider the motion of the points \(p_{ik}\) and \(p_{jk}\) within \(L\). We picture the motion of these points induced by the motion of \(b_k\) in Figure 16.

\(^8\)See the definition of \(a_{ijk}\).
In Figure 17 we picture how to obtain the relation
\[ a_{ijk}^{-1} a_{rsn} a_{ijk} = a_{ijn} a_{ikn} a_{rsn} a_{ikn}^{-1} a_{ijn}^{-1}, \quad jk = rs. \]
The other conjugation relations are found in a similar way.

11. The Reciprocity Law.

In this section we define a move within the group \( P_n^2 \), which we will call the reciprocity law. This will be used in Section 12 to lift certain relations from \( P_{n-1}^2 \) to \( P_n^2 \).

To understand the reciprocity law, it is helpful to understand the analogous move within the classical pure braid group. We may consider the generator \( a_{ij} \in P_n \) to be the braid whose \( j \)th string passes around the \( i \)th string. However, by “pulling the \( j \)th string tight” we can also consider this braid to be the one whose \( i \)th string passes around the \( j \)th string (see Figure 18). We label this new loop \( \alpha_{ij} \). The reciprocity law for \( P_n \) is simply the statement \( a_{ij} = \alpha_{ij} \).

We describe the reciprocity law for \( P_n^2 \). First we need to define a new loop \( \alpha_{ijk} \) in \( X_n^2 \). Let \( F_j \) be the fiber defined in Section 6. Choose a point \( q \in I(\mathbb{R}) \) on the line \( L_{ik}(\mathbb{R}) \) within the disc of radius \( \epsilon \) about the point \( b_k \). Let the line \( L'(\mathbb{R}) \) denote the real line joining \( b_j \) and \( q \). Then we define the loop \( \alpha_{ijk} \) to be the loop in \( L' \), based at \( b_j \), which goes around \( q \). To prove the reciprocity law for \( P_n^2 \) we will use the following key lemma.
Lemma 11.1. The two loops $a_{i,k-1,k}$ and $\alpha_{i,k-1,k}$ are homotopic in $X^2_n$ relative to the basepoint $b$, for $2 \leq i < k \leq n$ (see Figure 19).

This proof is computational, and is given in Section 13.

Proposition 11.2. (The reciprocity law) The loops $a_{ijk}$ and $\alpha_{ijk}$ are homotopic in $X^2_n$ relative to the basepoint $b \in B$, for $1 \leq i < j < k \leq n$ (see Figure 20).

Proof. We find an explicit homotopy. Commence the homotopy by shrinking the loop $a_{ijk}$. Let $\epsilon > 0$ be a small real number. Let $L(R)$ denote the line $L^\infty_k$ which was used to define the loop $a_{ijk}$. Let $p$ be a point on $\psi(R)$ lying between the points $b_k - 1$ and $b_k$. Let $v$ be a vector which orients the line $\tilde{L}(R)$ in the direction from $b_j$ to $p$. Move the point $b_j$ up distance $\epsilon$ into $\tilde{L}$ in the direction $iv$. Then move $b_j$ along the line above $\tilde{L}(R)$ until it reaches the point in $\tilde{L}(R)$ in the direction $iv$ above $p$. Finally move $b_j$ back down to $p$. During the motion of $b_j$ the points $p_{ij} = L \cap L_{ij}$ will move within the line $L$. Shrink the loop $a_{ijk}$, so that it follows the motions of the $p_{ij}$.

By sliding points along $\psi(R)$ whilst remaining in the base point set $B$, if necessary, we will now be in the same situation as Lemma 11.1 with $j = k$. The shrunken version of $a_{ijk}$ is thus homotopic to the loop $a_{i,k-1,k}$. By sliding points again, if necessary, we now move the point $b_j$ back to its starting position along the same path which it originally took. During this process the loop $\alpha_{i,k-1,k}$ will get “stretched out”. The resulting loop is homotopic to $\alpha_{ijk}$.

12. Lifting relations

In this section we show how to lift relations from the group $P^2_{n-1}$ to $P^2_n$ in order to complete the proof of Theorem [7.2]. We also show how to lift relation (29) from $Q^2_{n-1}$ to $Q^2_n$. Before we begin lifting relations, we describe the main idea that we shall use.

First, consider the classical pure braid groups. We may lift a relation from $P_{n-1}$ to $P_n$, if the $n$th string does not obstruct the homotopy describing this relation in
$P_{n-1}$. Since the $n$th string does not obstruct any of the homotopies describing the relations in $P_{n-1}$, they all lift.

A similar idea applies in lifting relations from $P_{n-1}^2$ to $P_n^2$, although in this case we get some obstructions. The point $b_n$ introduces the lines $L_{in}(C)$ into $GF_n^2$. If it is possible to describe a relation in $P_{n-1}^2$ by a homotopy within $F_k$, $1 \leq k \leq n - 1$, which does not intersect any of the $L_{in}$, then we can lift this relation. This is because we have the sequence

$$\pi_1(F_k) \to P_n^2 \to P_{n-1}^2 \to 1.$$

However, if it is impossible to describe a relation in $P_{n-1}^2$ by a homotopy which does not intersect the $L_{in}$, then we have to use the reciprocity law to lift the relation.

12.1. Relations (14) – (16).

**Lemma 12.1.** For $1 \leq k \leq n - 1$ the relations (14) – (16) hold from $P_{n-1}^2$ to $P_n^2$.

**Proof.** For $1 \leq k \leq n - 1$ the relations (14) – (16) hold in $\pi_1(F_k)$. \hfill \Box

12.2. Relation (18).

**Lemma 12.2.** For $1 \leq k \leq n - 1$ the relation (18) lifts from $P_{n-1}^2$ to $P_n^2$.

**Proof.** Relation (18) is described by a homotopy $H : [0, 1] \times [0, 1] \to X_n^2$. Let $R_l \subset L_{in}^\infty$ be equal to $(\text{im } H) \cap L_{in}^\infty$. Then the homotopy $H$ can be chosen so that the lines $L_{in}$ within $F_k$ do not intersect the region $R_l$ for $l = j, k$. This is because the points $b_1, \ldots, b_n$ satisfy the the lexicon condition. Hence $H$ lifts to $X_n^2$. \hfill \Box

12.3. Relation (17). This case is not the same as the previous two, since the lines $L_{in}$ in $F_k$ obstruct the homotopy describing relation (17) within $P_{n-1}^2$. This is because the lines $L_{in}$ always pass through the point $b_i$. First note that for fixed $k$ and $j$ the relation

$$[a_{ijk}, \ldots, a_{j-1, jk}, \tilde{a}_{j, j+1, k}, \ldots, \tilde{a}_{j, k-1, k}, a_{jk, k+1, j}, \ldots, a_{jk, n-1}] = 1 \quad (30)$$

holds in $\pi_1(F_k)$ (and thus in $P_n^2$). To understand why this is the case see Figure 21. Now using the reciprocity law relation (21) can be rewritten as follows

$$[a_{ijk}, \ldots, a_{j-1, jk}, \tilde{a}_{j, j+1, k}, \ldots, \tilde{a}_{j, k-1, k}, a_{jk, k+1, j}, \ldots, a_{jk, n-1}] = 1.$$

Thus relation (17) holds $P_n^2$. Now we simply note that

$$(p_n^2)^*(\{a_{ijk}, \ldots, a_{j-1, jk}, \tilde{a}_{j, j+1, k}, \ldots, \tilde{a}_{j, k-1, k}, a_{jk, k+1, j}, \ldots, a_{jk, n-1}\}) =$$

$$[a_{ijk}, \ldots, a_{j-1, jk}, \tilde{a}_{j, j+1, k}, \ldots, \tilde{a}_{j, k-1, k}, a_{jk, k+1, j}, \ldots, a_{jk, n-2}],$$

for $1 \leq k \leq n - 1$. Hence we have lifted relation (17).

---

9 See Section 4 for the definition of $F_k$.
10 This relation arises from the Hopf link of the point $b_j$. 
12.4. **Relation (29).** We conclude this section by showing how to lift the product relations (29) from $Q_{n-1}^2$ to $Q_n^2$. We can use the same method that we used to lift relation (17). Let $G_k$ denote the fiber of the projection, $Y^2 \to Y_{n-1}^2$, obtained by forgetting the $k$th point. Then for each $k$ the relation (29) holds in $\pi_1(G_k)$ as a consequence of the reciprocity law. Thus relation (29) holds in $Q_n^2$. Now note that,

\[ (p_n^2)(a_{12k}a_{13k}\ldots a_{k-2,k-1,k}a_{1k,k+1}\ldots a_{1kn}\ldots a_{k-1,k,n}) = a_{12k}a_{13k}\ldots a_{k-2,k-1,k}a_{1k,k+1}\ldots a_{1k,n-1}\ldots a_{k-1,k,n-1}, \]

for $1 \leq k \leq n-1$. Hence we have lifted relation (29).

It is informative to understand how this relation lifts. First, let us understand the analogous situation in the pure braid group on $P^1$. In Figure 22 we picture how to lift the relation

\[ a_{1,n-1}a_{2,n-1}\ldots a_{n-2,n-1} = 1, \]

(31)

which holds in $Q_{n-1}$, to the relation

\[ a_{1n}a_{2n}\ldots a_{n-1,n} = 1, \]

which holds in $Q_n$. Note that the $n$th string will always obstruct any homotopy which describes relation (31) in the group $Q_{n-1}$.

We now show how to lift relation (29). First, consider this relation in $Q_{n-1}^2$. This may be rewritten as the product

\[ a_{12k}\ldots a_{1,k,n-1}a_{23k}\ldots a_{k-2,k-1,k}a_{k,k+1}\ldots a_{k,n-1,n-2} \]

using the reciprocity law. We see that a homotopy describing relation (29) can be chosen to lie in the line $L_k^\infty$ (Figure 23). However, the lines $L_{in}$ intersect $L_k^\infty$, and so the product (32) will be homotopic to a loop which encircles the points $L(\mathbb{R}) \cap L_{in}(\mathbb{R})$, within $L_k^\infty$. This is homotopic to the product

\[ \alpha_{k,n-1,n}^{-1}\ldots \alpha_{k,k+1,n}^{-1}\alpha_{k-1,k,n}^{-1}\ldots \alpha_{1,k,n}^{-1}, \]

The reciprocity law allows us to rewrite this expression in terms of the $a_{ijn}$. Thus, we have lifted relation (29).
13.1. We begin by choosing the base point $x$ in $B \subseteq X^2_n$. Note that $X^2_n \subseteq (\mathbb{C}^2)^m$. The curve $\psi$ is then given by the equation $\psi(t) = t^2$. Let

\[(x_m, y_m) = (-m/2n, \psi(-m/2n)) = (-m/2n, (m/2n)^2), \quad 1 \leq m \leq n-2.\]

Choose $\epsilon \in \mathbb{R}$ to be a sufficiently small positive number. Let $u = ((1 - \epsilon), (1 - \epsilon)^2)$ and $v = ((1 + \epsilon), (1 + \epsilon)^2)$. Then, we define $x \in X^2_n$ to be the point

\[(x_1, y_1), \ldots, (x_{n-2}, y_{n-2}), u, v).\]

We now define two loops $\xi_m$ and $\tilde{\xi}_m$ in $X^2_n$, and show that they are homotopic relative to $x$.

Let $p_m$ be the $y$ co-ordinate of the intersection of the line $x = (1 + \epsilon)$ and the line joining points $u$ and $(x_m, y_m)$. Then we have

\[p_m = 2\epsilon((1 - \epsilon) - (m/n)) + (1 - \epsilon)^2.\]

Let $q_m$ be the $y$ co-ordinate of the intersection of the line $x = (1 - \epsilon)$ and the line joining points $v$ and $(x_m, y_m)$. Then we have

\[q_m = -2\epsilon((1 + \epsilon) - (m/n)) + (1 + \epsilon)^2.\]

Let

\[r_m = (1 + \epsilon)^2 - (p_m + p_{m+1})/2,\]

and

\[s_m = (q_m + q_{m+1})/2 - (1 - \epsilon)^2.\]

These will be the “radii” of the loops $\xi_m$ and $\tilde{\xi}_m$ respectively (see Figure 24).
Define
\[ \xi_m(t) = ((1 + e), (1 + e)^2 + r_m(e^{it} - 1)/2) \text{ for } 0 \leq t \leq 2\pi, \]
and
\[ \tilde{\xi}_m(t) = ((1 - e), (1 - e)^2 + s_m(1 - e^{it})/2) \text{ for } 0 \leq t \leq 2\pi. \]
A homotopy, \( H_m(s, t) \), where \( 0 \leq s \leq 1 \) and \( 0 \leq t \leq 2\pi \), between the loops \( \xi_m \) and \( \tilde{\xi}_m \) is now given explicitly by
\[ H_m(s, t) = ((x_1, y_1), \ldots, (x_{n-2}, y_{n-2}), (u_1(s, t), u_2(s, t)), (v_1(s, t), v_2(s, t))), \]
where
\[ (u_1(s, t), u_2(s, t)) = ((1 - e), (1 - e)^2 + (1 - s)s_m(1 - e^{it})/2), \]
and
\[ (v_1(s, t), v_2(s, t)) = ((1 + e), (1 + e)^2 + s_{r_m}(e^{it} - 1)/2). \]
Note that \( H_m(0, t) = \xi_m(t) \) and \( H_m(1, t) = \tilde{\xi}_m(t) \) for \( 0 \leq t \leq 2\pi \). Also, note that \( H_m(s, 0) = x \) and \( H_m(s, 2\pi) = x \) for \( 0 \leq s \leq 1 \). We are thus reduced to showing that the homotopy lies in \( X_r \).

Let \( (x_1, y_1), \ldots, (x_n, y_n) \) be \( n \) points in \( \mathbb{C}^2 \). We will need to check whether three of these points lie on a line. The following condition will be convenient for our calculations. No three of these points will lie on a line if and only if no \( 3 \times 3 \) minor of the matrix
\[
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
x_1 & x_2 & \cdots & x_n \\
y_1 & y_2 & \cdots & y_n
\end{bmatrix}
\]
vanishes. Hence, to prove that the loops \( \xi_m \) and \( \tilde{\xi}_m \) are homotopic, it will be sufficient to show that no \( 3 \times 3 \) minor of the matrix
\[
H(s, t) = \begin{bmatrix}
1 & \cdots & 1 & 1 & 1 \\
x_1 & \cdots & x_{n-2} & u_1(s, t) & v_1(s, t) \\
y_1 & \cdots & y_{n-2} & u_2(s, t) & v_2(s, t)
\end{bmatrix}
\]
vanzishes for any \( 0 \leq t \leq 2\pi \) and \( 0 \leq s \leq 1 \).

We split this calculation up into three cases. Let \( C_i \) denote the \( i \)th column of \( H \), for \( 1 \leq i \leq n \). Since the points corresponding to columns \( C_1, \ldots, C_{n-2} \) lie on \( \psi(\mathbb{R}) \) and are fixed for all \( s \) and \( t \), it suffices to check that the determinants of the following matrices do not vanish, for all \( 1 \leq k < l \leq n - 2 \):
\[
\begin{align*}
[C_k, C_l, C_{n-1}] & ; \\
[C_k, C_l, C_n] & ; \\
[C_k, C_{n-1}, C_n] & .
\end{align*}
\]

13.1.1. **Determinant of** \([33] \). It suffices to check this only when \( t = \pi \), since otherwise the determinant of matrix \([33] \) has non-zero complex part and is thus non-zero. Let \( t \) be equal to \( \pi \). Then matrix \([33] \) is equal to
\[
\begin{bmatrix}
1 & 1 & 1 \\
-(k/2n) & -(l/2n) & (1 - e) \\
(k/2n)^2 & (l/2n)^2 & (1 - e)^2 + (1 - s)s_m
\end{bmatrix}
\]
Thus we need only show that the determinant of this matrix does not vanish for any $0 \leq s \leq 1$. Expanding out the determinant of this matrix gives us the following expression:

$$(k/2n - l/2n)((kl/4n^2) + (1 - \epsilon)^2 + (1 - \epsilon)(l/2n + k/2n) - (1 - s)s_m).$$

Suppose that this expression is equal to zero for some $s$. Then, by choosing $\epsilon$ sufficiently small we can force the inequality

$$s > (1 + 1/2n)^2.$$ 

But $0 \leq s \leq 1$, and we are lead to a contradiction.

### 13.1.2. Determinant of (34).

It suffices to check this only when $t = \pi$, since otherwise the determinant of matrix (34) has non-zero complex part and is thus non-zero.

Let $t$ be equal to $\pi$. Then matrix (34) is equal to

$$
\begin{pmatrix}
1 & 1 & 1 \\
-(k/2n) & -l/2n & (1 + \epsilon) \\
(k/2n)^2 & (l/2n)^2 & (1 + \epsilon)^2 + sr_m
\end{pmatrix}
$$

Thus we need only show that the determinant of this matrix does not vanish for any $0 \leq s \leq 0$. Expanding out the determinant of this matrix gives us the following expression:

$$(k/2n - l/2n)((kl/4n^2) + (1 + \epsilon)^2 + (1 + \epsilon)(l/2n + k/2n) - sr_m).$$

Suppose that this expression is equal to zero for some $s$. Then, by choosing $\epsilon$ sufficiently small, we can force the inequality

$$s > (1 + 1/2n)^2.$$ 

But $0 \leq s \leq 1$, and we are lead to a contradiction.

### 13.1.3. Determinant of (35).

This case is more complicated since the imaginary part of the determinant of matrix (35) may vanish. By simplifying the determinant, we obtain the following expression for the imaginary part of the determinant of matrix (35):

$$-\sin t \left((s_m(1 + l/2n + \epsilon) - s((1 + l/2n - \epsilon)r_m - (1 + l/2n + \epsilon)s_m))/2\right).$$

Note that $\sin t$ can only vanish when $t = 0$ or when $t = \pi$. We explain what happens in the case when $t = \pi$ later. Suppose that $t$ is not equal to $0$ or $\pi$ and that expression (36) vanishes. In this case $s$ would be equal to

$$-s_m(1 + l/2n + \epsilon)/((1 + l/2n - \epsilon)r_m - (1 + l/2n + \epsilon)s_m).$$

A computation shows that this quantity must be greater than one. Hence, we are lead to a contradiction.

The final case we are left with is when $t = \pi$. In this case, by supposing that the determinant vanishes, we obtain:

$$sK = 1 - 2\epsilon(1 + l/n - \epsilon)/s_m,$$

where,

$$K = (((1 + l/2n - \epsilon)r_m)/((1 + l/2n + \epsilon)s_m) - 1).$$

A computation shows that $K$ cannot be equal to zero, and hence we may divide equation (37) by $K$. Again, we are left with $s$ equal to a quantity which, by allowing
$\epsilon$ to be sufficiently small, may be shown to be greater than one. Thus, the image of the map $H_k$ lies in $X^2_n$ and the loops $\xi_m$ and $\bar{\xi}_m$ are homotopic.

13.2. We now generalize the result of Section [13.1]. To prove Lemma [11.1], we will need to place points on the curve $\psi(\mathbb{R})$ to the right of the points $u$ and $v$, without obstructing the homotopy between the loops $\xi_m$ and $\bar{\xi}_m$. We do this by placing our extra points on $\psi(\mathbb{R})$ so that they are “far away” from $u$ and $v$. In this way, we can ensure that the lines introduced by adding in the new points do not intersect the complexified lines $x = 1 + \epsilon$ and $x = 1 - \epsilon$ within the circles of radius $r_m$ and $s_m$ respectively. Now it is clear that the lines introduced by adding the new points will not obstruct the homotopy between the loops $\xi_m$ and $\bar{\xi}_m$.

13.3. We now complete the proof of Lemma [11.1]. Recall that we are trying to show that the loops $\alpha_{i,k-1,k}$ and $\alpha_{i,k-1,k}$ are homotopic relative to $b$. We can slide points up and down $\psi(\mathbb{R})$ as long as we remain in the contractible set $B$ defined in Section [3]. Slide the points $\{b_{k+1}, \ldots, b_k\}$ along the curve $\psi(\mathbb{R})$ until they are “far to the right” of the points $b_{k-1}$ and $b_k$. Now slide points $b_{k-1}$ and $b_k$ into positions $u$ and $v$ of Section [13.1]. Finally, slide points $\{b_1, \ldots, b_{k-2}\}$ so that we are in the situation of Section [13.2]. During the sliding process the loops $\alpha_{i,k-1,k}$ and $\alpha_{i,k-1,k}$ will be deformed; we denote the resulting loops by the same symbol.

The loop $\xi_m$ lies in the complexification of the line $x = 1 + \epsilon$. The loop $\alpha_{i,k-1,k}$ can also be homotoped into this line. Similarly, the loop $\alpha_{i,k-1,k}$ can be homotoped into the complexification of the line $x = 1 - \epsilon$, in which the loop $\bar{\xi}_m$ lies. Hence, we are reduced to showing that certain loops are homotopic in punctured complex lines. In this situation we have

$$a_{i,k-1,k} = \xi_i^{-1},$$

and

$$\alpha_{i,k-1,k} = \xi_i^{-1}.$$

We know from Section [13.2] that the loops $\xi_m$ and $\bar{\xi}_m$ are homotopic for $1 \leq m \leq k - 2$. Thus the loop $\alpha_{i,k-1,k}$ is homotopic to the loop $\alpha_{i,k-1,k}$. By sliding all points back to the basepoint $b$, whilst remaining in the set $B$, we complete the proof of Lemma [11.1].

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