LETTER TO THE EDITOR

New low-frequency nonlinear electromagnetic wave in a magnetized plasma

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Abstract

A new nonlinear electromagnetic mode in a magnetized plasma is predicted. Its existence depends on the interaction of an intense, circularly polarized electromagnetic wave with a plasma, where quantum electrodynamical photon–photon scattering is taken into account. This scattering gives rise to a new coupling between the matter and the radiation. Specifically, we consider an electron–positron plasma and show that the propagation of the new mode is admitted. It could be of significance in pulsar magnetospheres, and result in energy transport between the pulsar poles.

Astrophysical environments can be extremely violent and energetic. Physics considered ‘exotic’ in Earth based laboratory applications can be common throughout our Universe, and sometimes even vital for the existence of certain observed phenomena. Pulsars, surrounded by strong magnetic fields, are the most prolific sources of exotic physics. Quantum electrodynamics (QED) is an indispensable explanatory model for much of the observed pulsar phenomena. Scattering of photons off photons is predicted by QED and it can be a prominent component of pulsar physics, since pulsars offer the necessary energy scales for such scattering to occur. Related to the scattering of photons is the concept of photon splitting in strong magnetic fields (Adler 1971). It has been suggested that such effects could be important in explaining the radio silence of magnetars (Kouveliotou et al 1998, Baring and Harding 2001). In this letter we will point out the existence of a new low-frequency nonlinear electromagnetic wave that may exist in a magnetized plasma, due to the interaction of photons with the quantum vacuum. A discussion of the properties of this new electromagnetic wave using parameters relevant to strongly magnetized pulsars will be given.

The weak field theory of photon–photon scattering can be formulated in terms of the effective Lagrangian density

\[ \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_\text{HE}, \]

where \( \mathcal{L}_0 = -\frac{1}{4} \epsilon_0 F_{ab} F^{ab} = \frac{1}{2} \epsilon_0 (E^2 - c^2 B^2) \) is the classical free field Lagrangian, and

\[ \mathcal{L}_\text{HE} = \kappa \epsilon_0^2 \left[ 4 \left( \frac{1}{2} F_{ab} F^{ab} \right)^2 + 7 \left( \frac{1}{2} F_{ab} \tilde{F}^{ab} \right)^2 \right], \]

is the Heisenberg–Euler correction (Heisenberg and Euler 1936, Schwinger 1951), where \( \tilde{F}_{ab} = \frac{1}{2} \epsilon_{cde} F^{cd} \), and \( \frac{1}{2} F_{ab} \tilde{F}^{ab} = -c E \cdot B \). Here \( \kappa \equiv 2 \alpha^2 \hbar^3 / 45 m_e c^5 \approx 1.63 \times 10^{-30} \text{ m}^2 \text{ kg}^{-1}. \)
\( \alpha \) is the fine-structure constant, \( \hbar \) is the Planck constant, \( m_e \) is the electron mass and \( c \) is the speed of light in vacuum. With \( F_{ab} = \partial_a A_b - \partial_b A_a \), \( A^b \) being the four-potential, we obtain, from the Euler–Lagrange equations, the field equations \( \partial_b [\partial^a F^{bc} / \partial F_{ab}] = 0 \), i.e. (see, e.g. Shukla et al (2004))

\[
\partial_b F^{ab} = 2 \epsilon_0 \kappa \partial_b \left[ \left( F_{cd} F^{cd} \right) F^{ab} + \frac{7}{4} \left( F_{cd} \hat{F}^{cd} \right) \hat{F}^{ab} \right] + \mu_0 j^a,
\]

where \( j^a \) is the four current.

For a circularly polarized wave \( E_0 = E_0 (\hat{x} \pm i \hat{y}) \exp(ik \cdot x - i \omega t) \) propagating along a constant magnetic field \( B_0 = B_0 \hat{z} \), the invariants satisfy

\[
F_{cd} F^{cd} = -2 E_0^2 \left( 1 - \frac{k^2 c^2}{\omega^2} \right) + 2 c^2 B_0^2 \quad \text{and} \quad F_{cd} \hat{F}^{cd} = 0,
\]

where \( k \) is the wave number and \( \omega \) is the frequency of the circularly polarized electromagnetic wave. Thus, equation (3) can be written as

\[
\Box A^a = -4 \epsilon_0 \kappa \left[ \frac{E_0^2}{\omega^2} \left( 1 - \frac{k^2 c^2}{\omega^2} \right) \right] \Box A^a + \mu_0 j^a,
\]

in the Lorentz gauge, and \( \Box = \partial_a \partial^a \). For circularly polarized electromagnetic waves propagating in a magnetized cold multicomponent plasma, the four current can be ‘absorbed’ in the wave operator on the left-hand side by the replacement

\[
\Box \rightarrow -D(\omega, k),
\]

where \( D \) is the plasma dispersion function, given by (see, e.g. Stenflo (1976) and Stenflo and Tsintsadze (1979))

\[
D(\omega, k) = k^2 c^2 - \omega^2 + \sum_j \frac{\omega \omega_{pj}^2}{\omega_{\gamma j} \pm \omega_{\epsilon j}},
\]

where the sum is over the plasma particle species \( j \),

\[
\omega_{\epsilon j} = \frac{q_{j} B_0}{m_{0j}} \quad \text{and} \quad \omega_{pj} = \left( \frac{n_{0j} q_{j}^2}{\epsilon_0 m_{0j}} \right)^{1/2},
\]

is the gyrofrequency and plasma frequency, respectively, and

\[
\gamma_j = (1 + v_j^2)^{1/2},
\]

is the gamma factor of species \( j \), with \( v_j \) satisfying

\[
v_j^2 = \left( \frac{e E_0}{cm_{0j}} \right)^2 \frac{1 + v_j^2}{\omega (1 + v_j^2)^{1/2} \pm \omega_{\epsilon j}}.
\]

Here \( n_{0j} \) denotes particle density in the laboratory frame and \( m_{0j} \) particle rest mass. It should be emphasized that for the case of circularly polarized waves propagating along an external magnetic field, the effects due to the electron and ion currents can be calculated without linearizing the plasma governing equations (see, e.g. Derby (1978), Goldstein (1978), Stenflo and Tsintsadze (1979) and Stenflo and Shukla (2001)). Thus, in equation (7), the relativistic nonlinearity, the full Lorentz force as well as the other plasma nonlinearities are fully accounted for. The only limiting assumption of the electromagnetic field amplitude is due to the validity of the Euler–Heisenberg Lagrangian. We note that the application of equation (2) requires field strengths smaller than the Schwinger critical field \( E_S = m_e^2 c^3 / e \hbar \sim 10^{18} \text{ V m}^{-1} \) (see, e.g. Schwinger (1951)). The dispersion relation, obtained from equation (5), then reads

\[
D(\omega, k) = \frac{4 \alpha}{45 \pi} (\omega^2 - k^2 c^2) \left[ \left( \frac{E_0}{E_S} \right)^2 \frac{\omega^2 - k^2 c^2}{\omega^2} - \left( \frac{e B_0}{E_S} \right)^2 \right].
\]
We note that as the plasma density goes to zero, the effect due to photon–photon scattering, as given by the right-hand side of equation (11), vanishes, since then $\omega^2 - k^2c^2 = 0$.

It may be instructive to first consider the small amplitude limit of equation (11). In a magnetized pair plasma this reads

$$
\frac{k^2c^2}{\omega^2} = 1 - \frac{2\omega_p^2}{\omega^2 - \omega_e^2},
$$

(12)

which shows the existence of fast and slow electromagnetic waves. In the low-frequency long wavelength limit, and for $\omega_p \ll \omega_e$ (as we typically have in pulsars) we then have $\omega/k \sim c$.

The phase speed of the slow wave will thus be close to the speed of light. However, for large amplitude waves, the situation is drastically different. As we will see below, we indeed have quite new kinds of dispersive electromagnetic waves whose frequencies depend on the wave amplitudes and for which the phase velocities fulfill $\omega/k \ll c$.

Next, we focus on mode propagation in an ultra-relativistic electron–positron plasma ($\gamma_e \gg 1$), where the two species have the same number density $n_0$. Then equation (11) gives

$$
k^2c^2 - \omega^2 \pm \frac{\omega_{pe}^2}{\omega^2} = \frac{4\alpha}{45\pi} \left[ \left( \frac{E_0}{E_S} \right)^2 \frac{\omega^2 - k^2c^2}{\omega^2} - \left( \frac{cB_0}{E_S} \right)^2 \right] (\omega^2 - k^2c^2).
$$

(13)

Following Stenflo and Tsintsadze (1979), we have defined $\omega_{pe} = eE_0/cm_0c$.

Looking for low-frequency modes, we now use the approximation $\omega \ll kc$, at which equation (13) gives

$$
\frac{k^2c^2}{\omega^2} \approx \frac{4\alpha}{45\pi} \left[ \left( \frac{E_0}{E_S} \right)^2 \frac{k^2c^2}{\omega^2} + \left( \frac{cB_0}{E_S} \right)^2 \right] \frac{k^2c^2}{\omega^2} \pm \frac{\omega_{pe}^2}{\omega_{pe}^2}.
$$

(14)

It is sometimes advantageous to use the relation $\omega_{pe} = \omega_e(E_0/E_S)$, where $\omega_e = m_ec^2/h$ is the Compton frequency, to write equation (14) as

$$
\frac{k^2c^2}{\omega^2} \approx \frac{4\alpha}{45\pi} \left[ \left( \frac{E_0}{E_S} \right)^2 \frac{k^2c^2}{\omega^2} + \left( \frac{cB_0}{E_S} \right)^2 \right] \frac{k^2c^2}{\omega^2} \pm \frac{\omega_{pe}^2 E_S}{\omega_{pe}^2 E_0}.
$$

(15)

Using the dispersion relation (14) the group velocity $v_g \equiv d\omega/dk$ is

$$
v_g = \left[ 1 - \frac{4\alpha}{45\pi} \left( \frac{cB_0}{E_S} \right)^2 \pm \frac{2v_p \omega_{pe}^2}{kc^2 \omega_{pe}} \right] \left[ 1 - \frac{4\alpha}{45\pi} \left( \frac{cB_0}{E_S} \right)^2 \pm \frac{3v_p \omega_{pe}^2}{2kc^2 \omega_{pe}} \right]^{-1} v_p,
$$

(16)

where $v_p \equiv \omega/k$ is the phase velocity.

Pulsar magnetospheres exhibit extreme field strengths in a highly energetic pair plasma. Ordinary neutron stars have surface magnetic field strengths of the order of $10^6$–$10^9$ T, while magnetars can reach $10^{10}$–$10^{11}$ T (Kouveliotou et al. 1998), coming close to, or even surpassing, energy densities $\epsilon_0 E_S^2$ corresponding to the Schwinger limit. Such strong fields will make the vacuum fully nonlinear, due to the excitation of virtual pairs. Photon splitting can therefore play a significant role in these extreme systems (Harding 1991, Baring and Harding 2001).

The emission of short wavelength photons due to the acceleration of plasma particles close to the polar caps results in the production of electrons and positrons as the photons propagate through the pulsar intense magnetic field (Beskin et al. 1993). Given the Goldreich–Julian density $n_{GJ} = 7 \times 10^{15} \left(0.1 \text{s/P}(B/10^8 \text{T})\right) \text{m}^{-3}$, where $P$ is the pulsar period and $B$ is the pulsar magnetic field, the pair plasma density is expected to satisfy $n_0 = M n_{GJ}$, where $M$ is the multiplicity (Beskin et al. 1993, Luo et al. 2002). Moderate estimates give $M = 10$ (Luo et al. 2002). Thus, the density in a pulsar pair plasma can be of the order $10^{18} \text{m}^{-3}$. The plasma
experiences a relativistic factor $\sim 10^2$–$10^3$ (Asseo 2003). On the other hand, the primary beam will have $n_0 \sim n_{GJ}$ and $\gamma \sim 10^6$–$10^7$ (Asseo 2003).

Furthermore, we note that the pulsar magnetosphere is very far from thermodynamical equilibrium, and consequently there exist numerous possible excitation mechanisms. In addition to Cherenkov excitation and cyclotron excitation that may excite Alfvén wave like modes (e.g. Lyutikov (1999)), free energy sources in the form of temperature anisotropies and/or electron beams can drive Weibel instabilities and beam type instabilities, respectively. Several of these initial excitation mechanisms can lead to the existence of arbitrarily large amplitude waves.

The two QED terms in the squared bracket of equation (15) correspond to the wave field and the static background field, respectively. The former dominates if the oscillating (wave) part of the magnetic field is larger than the static part, and the latter dominates if the opposite is true. Due to the dipole nature of the static field, the former case is more likely to hold at a comparatively large distance from the pulsar surface, whereas close to the surface the static part is likely to dominate. From now on we concentrate on comparatively weak background field strengths (thus excluding regions in the immediate vicinity of the pulsar surface). Accordingly we have $cB_0 \ll E_S$ and we therefore drop the term proportional to $B_0^2$ in equation (15). Next, using the normalized quantities $\Omega = \omega_0 / \omega_{pe}, K = (4\alpha / 45\pi)^{-1/2} k c \omega / \omega_{pe}$ and $\mathcal{E} = (4\alpha / 45\pi) E_0 / E_S$, the dispersion relation (15) reads

$$\Omega^2 = \varepsilon^2 K^2 \mp \frac{\Omega^3}{\varepsilon K^2}. \quad (17)$$

This dispersion relation describe three different modes, two with + polarization and one with $-$ polarization. The normalized frequency as a function of $K$ and $\mathcal{E}$ is shown in figure 1. We note that for $K \ll 1$, the dispersion relation (17) agrees with that of Stenflo and Tsintsadze (1979), whereas in the opposite limit $K \gg 1$, the QED term in (17) is dominating. For the given density, the latter regime applies, except for extremely long wavelengths ($>10^8$ m), and thus we note that QED effects are highly relevant for the propagation of these modes in the pulsar environment. For small $K$ there is only one mode, but as seen from the second and third panels of figure 1, two new modes appear for $K \gtrsim 2.6$. Thus for large $K$, applicable in the pulsar environment, there are three low-frequency modes ($\omega \ll kc$) that depend on nonlinear QED effects for their existence. Using $cB_0 \ll E_S$, the expression (16) for the group velocity

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**Figure 1.** Dispersion surfaces $\Omega = \Omega (K, \mathcal{E})$ as given by equation (17). The first panel corresponds to the $-$ sign in equation (17), and exists for all $K$ and $\mathcal{E}$. The second panel shows the fast $+$ polarized mode, which exists for $K \gtrsim 2.6$. The third panel depicts the slow $+$ polarized mode, also for $K \gtrsim 2.6$. 

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becomes
\[
\frac{d\Omega}{dK} = \frac{\Omega \pm 2\Omega^2 / \varepsilon K^2}{\Omega \pm 3\Omega^2 / 2\varepsilon K^2} \Omega
\]  
(18)

and thus we see that the propagation speed depends nonlinearly on the plasma parameters. Furthermore, from (18), and recalling the normalizations used, we see that the group velocity is typically well below the speed of light. We emphasize that the low group velocity by itself makes our mode more long-living, since the energy supplied to the mode in order to balance the convective energy loss is proportional to the group velocity. We thus suggest that the three new modes presented above can contribute to an understanding of the very complicated energy transport phenomena taking place in the accretion discs of pulsars.

In summary, we have reported the existence of a new low-frequency nonlinear electromagnetic wave in a magnetized plasma. The dispersion relation of the wave has been presented and analysed using relevant astrophysical parameters. Applications to pulsar magnetoplasmas have been pointed out.

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