On the Exact Solution of Models based on Non-Standard Representations

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(revised version)

Abstract

The algebraic Bethe ansatz is a powerful method to diagonalize transfer-matrices of statistical models derived from solutions of (graded) Yang Baxter equations, connected to fundamental representations of Lie (super-)algebras and their quantum deformations respectively. It is, however, very difficult to apply it to models based on higher dimensional representations of these algebras in auxiliary space, which are not of fusion type. A systematic approach to this problem is presented here. It is illustrated by the diagonalization of a transfer-matrix of a model based on the product of two different four-dimensional representations of $U_q(\hat{gl}^′(2,1;C))$.

I. INTRODUCTION

The starting point for the construction of (Bethe ansatz) integrable models is the famous Yang-Baxter equation (YBE) [1,2].
\[ R_{23}^{V'V''}(v, w)R_{13}^{VV''}(u, w)R_{12}^{VV'}(u, v) \]  

(1a)

\( V, V' \) and \( V'' \) are three in general different spaces. The operators \( R_{ij}^{VV'}(u) \) act on the direct product \( V \times V' \rightarrow V \times V' \). Both sides of equation (1a) act on the three-fold product \( V \times V' \times V'' \). The lower indices \( i, j \in 1, 2, 3 \) on the \( R \)-operators denote as usual the two factors in this product on which the corresponding \( R \)-operator acts non-trivially. In general the so-called spectral parameters \( u, v \) and \( w \) are complex variables.

Up to now, there is no general classification of the solutions to (1a). The situation is much better understood, if \( V, V' \) and \( V'' \) are carrier spaces for the representation of a simple Lie-algebra or its quantum-deformation. The corresponding theory is mainly due to Drinfel’d, who also introduced the concept of the universal \( R \)-matrix. The existence of the latter guarantees the existence of \( R \)-operators as matrices acting on direct products of usually, but not always finite dimensional carrier spaces \( V \). A good account of these developments has been given by Chari and Presley. Powerful methods to construct these matrices explicitly were developed by Jimbo and many others, see e.g. the book by Ma. The dependence on only one complex parameter is due to the use of evaluation representations of affine algebras. In this case (1a) takes the more common difference form

\[ R_{12}^{VV'}(u - v)R_{13}^{VV''}(u)R_{23}^{VV'}(v) \]

\[ = R_{23}^{V'V''}(v)R_{13}^{VV''}(u)R_{12}^{VV'}(u - v) \]  

(1b)

The first space \( V \) is called auxiliary space, the second relabeled to \( V^{(n)} \), in general taken out of some countable set \( \{V^{(m)}\}_{m=1}^{N} \), a (local) quantum space. An \( L \)-operator acting on the direct product of these is defined as

\[ \hat{L}^{V}(n|u) := R^{VV^{(n)}}(u, w^{(n)}) . \]  

(2a)

It is assumed to act trivially on all other quantum spaces \( V^{(m)} \) with \( m \neq n \). Assuming, that \( w^{(n)} \) in (2a) just labels \( V^{(n)} \) and that \( u \) is a spectral parameter of difference type as in (1a), it is possible to introduce additional inhomogeneities \( \delta^{(n)} \) into the monodromy-matrix.
\[ \hat{T}^V(N|u) := \hat{L}^V(N|u - \delta^{(N)}) \cdots \hat{L}^V(1|u - \delta^{(1)}) . \] (2b)

Here \( \delta^{(n)} \) and \( w^{(n)} \) will be some complex numbers.

\[ \hat{\tau}^V(N|u) = \text{tr}_V \left\{ \hat{T}^V(N|u) \right\} \] (2c)

can be viewed as row-to-row transfer-matrix of a two dimensional (classical) statistical model, with \( N \) sites per row, acting on the (global) quantum space \( V^{(N)} \times \cdots \times V^{(1)} \). If \( \delta^{(n)} \) vanishes and \( w^{(n)} \) is independent of \( n \), the transfer-matrix (2c) is called homogeneous. In any case integrability of the latter is established via (1), written as

\[ R_{12}^{VV'}(u,v) \hat{L}_1^V(n|u) \hat{L}_2^V(n|v) = \hat{L}_2^V(n|v) \hat{L}_1^V(n|u) R_{12}^{VV'}(u,v) , \] (3a)

From that the fundamental commutation relations (FCR) are obtained immediately:

\[ R_{12}^{VV'}(u,v) \hat{T}^V(N|u) \hat{T}^{V'}(N|v) = \hat{T}^{V'}(N|v) \hat{T}^V(N|u) R_{12}^{VV'}(u,v) \] (3b)

Provided \( R_{12}^{VV'} \) is invertible, which is guaranteed for finite dimensional \( V \) and \( V' \), this yields

\[ \left[ \hat{\tau}^V(N|u), \hat{\tau}^{V'}(N|v) \right] \overset{!}{=} 0 \] (3c)

Expanding \( \tau^{V'}(N|v) \) in \( v \) one obtains an infinite family of operators commuting with \( \hat{\tau}^V(N|u) \). The question, if this family contains the right number of “independent” integrals of motion for every finite \( N \), is difficult to answer and usually taken for granted.

The set of equations (2) and (3) was derived by Baxter and can be found together with the original references in his excellent book [7].

The notation here is due to Faddeev and coworkers, who created a purely algebraic way for diagonalizing \( \hat{\tau}^V(N|u) \), the algebraic Bethe ansatz (ABA). Their quantum inverse scattering method (QISM) [8] provided the background for Drinfel’d’s theory [3], but is more general and to the author’s opinion not fully exploited yet. A good account including original references can be found in the book by Korepin et al. [9] and the reprint collection [10].
ABA is a powerful method to construct eigenvectors and eigenvalues of $\hat{\tau}^V (N | u)$. In some sense it is more systematic than the original coordinate Bethe ansatz \cite{11}. In general this is only true, if the auxiliary space $V$ is the carrier space of the fundamental representation of a Lie (super-)algebra or a deformation of the latter. Especially if the auxiliary space $V$ is a higher dimensional carrier space of another representation of the same algebra, simplicity is lost and ABA becomes cumbersome. Drinfel’d’s theory \cite{8} suggests, that a simple generalization should exist. A systematic approach to this problem will be developed in the following.

II. MODELS

In the case of general graded algebras Drinfel’d’s constructions \cite{8} are still not completely understood. However for simple (affine) Lie superalgebras and their quantum deformations a proper algebraic construction has been given by Yamane recently \cite{12}. Also QISM and ABA are not very sensitive to grading and the graded version of the YBE has been established by Kulish and Sklyanin long ago \cite{27}.

The $R$-matrices, which will be used as concrete examples, are related to the “quantum universal enveloping superalgebra” $U_q(\hat{gl}(2,1|\mathbb{C}))$. No use will be made of any peculiar features of this symmetry. The interested reader is referred to the book by Cornwell \cite{14} on Lie superalgebras, from which the notation is borrowed, the book by Kac \cite{15} for more details on affinization and to the paper \cite{12} for the proper construction of the $q$-deformed universal enveloping superalgebra.

The carrier space $V_3$ of the fundamental representation of $U_q(gl(2,1|\mathbb{C}))$ is complex and three-dimensional. Basis and cobasis will be denoted by

$$|i>, \quad <j|i> = \delta_{ij} \quad \text{for} \quad i, j = 1, 2, 3$$

A basis of the complex carrier space $V_4$ of the four-dimensional representation will be denoted similarly. These representations are $\mathbb{Z}_2$-graded: To each basis-vector $|i\rangle$ a number $p(i) \in \{0, 1\}$ is assigned, i.e.
\begin{align}
p(1) = p(2) = 0 \quad , \quad p(3) = 1 \tag{4b}
\end{align}

for \( V_3 \) and analogously

\begin{align}
p(1) = p(2) = 0 \quad , \quad p(3) = p(4) = 1 \tag{4c}
\end{align}

for \( V_4 \). Local basis-vectors are divided into even (bosonic, \( p = 0 \)) and odd (fermionic, \( p = 1 \)) ones. Local operators acting in \( V_3 \) or \( V_4 \) etc. are expressed in the natural basis

\begin{align}
e_{ij} = |i\rangle \langle j| \tag{4d}
\end{align}

If the corresponding space is a (local) quantum space, it will be denoted with a hat, e.g. \( \hat{e}_{ij} \) for clarity. These operators act trivially on all other (local) quantum spaces. A grading is assigned to this basis according to

\begin{align}
p(e_{ij}) = [p(i) + p(j)] \quad \text{mod} \ 2 \tag{4e}
\end{align}

It is possible to extend these definitions of grading naturally to those vectors \( |\psi\rangle \) and operators \( \hat{a} \), which are homogeneous with respect to the grading.

It is convenient to expand operators as well as vectors in the natural (tensor) product basis, which is ordered according to \( V^{(N)} \times \cdots \times V^{(1)} \), see (2b). Grading imposes signs on products of homogeneous operators, i.e.:

\begin{align}
(\hat{a} \otimes \hat{b})(\hat{c} \otimes \hat{d}) = (-1)^{p(b)p(c)} (\hat{a}\hat{c}) \otimes (\hat{b}\hat{d}) \tag{4f}
\end{align}

or on the action of homogeneous operators on homogeneous vectors, i.e.

\begin{align}
(\hat{a} \otimes \hat{b})(|\psi\rangle \otimes |\varphi\rangle) = (-1)^{p(b)p(|\psi\rangle)} (\hat{a}|\psi\rangle) \otimes (\hat{b}|\varphi\rangle) \tag{4g}
\end{align}

The only other effect of grading is, that \( tr_V \) in (2c) has to be interpreted as supertrace:

\begin{align}
tr_V \left\{ \hat{T}^V(N|u) \right\} = \sum_i (-1)^{p(i)} \langle i|\hat{T}^V(N|u)|i \rangle. \tag{4h}
\end{align}
Kulish and Sklyanin found [27], that additional signs, which appear in an explicit representation of the YBE (1) due to grading can be absorbed into a redefinition of matrix elements, so that every solution of the graded YBE is equivalent to a solution of the conventional one. The four dimensional representation can be characterized by a set of complex parameters, symbolically denoted by

\[ V_4 \approx \{ C, \kappa, \kappa^*, \mu, \mu^* \} . \]  

(5a)

This is a peculiarity of Lie superalgebras [14], which is conserved under quantum deformation; \( \kappa, \kappa^* \) and \( \mu, \mu^* \) are not necessarily complex conjugate to each other, but related to \( C \) by

\[ \kappa \kappa^* = [C]_q , \quad \mu \mu^* = [C + 1]_q , \]  

(5b)

where \( q \) is the deformation parameter,

\[ q := e^{2\eta} , \]  

(5c)

and \( q \)-brackets are defined as usual by

\[ [C]_q := \frac{q^C - q^{-C}}{q - q^{-1}} = \frac{\sinh(2\eta C)}{\sinh(2\eta)} \]  

(5d)

Different choices of \( \kappa, \kappa^*, \mu, \mu^* \) can be related to each other by a similarity transformation of the algebra, which conserves grading, but is not unitary in general. That makes it convenient to keep these parameters. Note that the representation \( V_4 \) can be deformed continuously into \( V'_4 \), which is characterized by a set of primed parameters also connected by (5b).

A well-known solution of (1b) with \( V = V' = V'' = V_3 \) is

\[ R^{V_3 V_3}(u) = e_{11} \otimes \hat{e}_{11} + e_{22} \otimes \hat{e}_{22} - d(u) e_{33} \otimes \hat{e}_{33} \]  

(6)

\[ + c(u) [e_{11}(\hat{e}_{22} + \hat{e}_{33}) + e_{22}(\hat{e}_{11} + \hat{e}_{33})] \]  

\[ + a(u) e_{21} \otimes \hat{e}_{12} + b(u) e_{12} \otimes \hat{e}_{21} \]  

\[ + a(u) [e_{31} \otimes \hat{e}_{13} + e_{32} \otimes \hat{e}_{23}] \]  

\[ - b(u) [e_{13} \otimes \hat{e}_{31} + e_{23} \otimes \hat{e}_{32}] \]
with coefficients

\[
\begin{align*}
a(u) & := \frac{\sinh(2\eta)}{\sinh(2\eta + u)} \cosh(u) + \sinh(u) \\
b(u) & := \frac{\sinh(2\eta)}{\sinh(2\eta + u)} \cosh(u) - \sinh(u) \\
c(u) & := \frac{\sinh(u)}{\sinh(2\eta + u)} \\
d(u) & := \frac{\sinh(2\eta - u)}{\sinh(2\eta + u)}
\end{align*}
\]

To the authors knowledge it appeared first in a different notation in the work of Perk and Schultz \[16\]. It is the standard \(q\)-deformation of the \(Y(gl(2, 1|C))\)-symmetric \(R\)-matrix given by Kulish and Sklyanin \[27\].

Kulish and Sklyanin wrote down the \(Y(gl(m, n|C))\)-symmetric \(R\)-matrix for arbitrary positive integers \(m\) and \(n\). Its generalization to the \(U_q(\hat{gl}(m, n|C))\)-symmetric case can also be taken from the paper by Perk and Schultz \[16\]. It is a simple generalization of (6).

The \(R\)-matrix (7) is related to the following \(U_q(\hat{gl}(2, 1|C))\)-symmetric \(R\)-matrix

\[
R^{V_3V_4}(u) = \rho(u) \left[ e_{11} \otimes (\hat{e}_{11} + \hat{e}_{33}) + e_{22} \otimes (\hat{e}_{11} + \hat{e}_{44}) \right] + \alpha_0(u) \left[ e_{11} \otimes (\hat{e}_{22} + \hat{e}_{44}) + e_{22} \otimes (\hat{e}_{22} + \hat{e}_{33}) \right] + e_{33} \otimes \left[ \beta_0(u) \hat{e}_{11} - \hat{e}_{22} + \gamma_0(u)(\hat{e}_{33} + \hat{e}_{44}) \right] + \delta_1(u)e_{12} \otimes \hat{e}_{43} + \delta_2(u)e_{21} \otimes \hat{e}_{34} - \varepsilon_1(u) \left[ e_{13} \otimes \hat{e}_{23} + e_{23} \otimes \hat{e}_{24} \right] + \varepsilon_2(u) \left[ e_{31} \otimes \hat{e}_{32} + e_{32} \otimes \hat{e}_{42} \right] + \delta_1(u)e_{12} \otimes \hat{e}_{43} + \delta_2(u)e_{21} \otimes \hat{e}_{34} - \zeta_1(u) \left[ e_{13} \otimes \hat{e}_{41} - q^{-1}e_{23} \otimes \hat{e}_{31} \right] + \zeta_2(u) \left[ e_{31} \otimes \hat{e}_{14} - q e_{32} \otimes \hat{e}_{13} \right]
\]

with coefficients (A1), listed in appendix A, in the sense, that it fulfills the YBE (1b) with \(V = V' = V_3\) and \(V'' = V_4\):
The coefficients are again listed in appendix A. The construction of this $R$-matrix \((7)\), and the proof of \((8)\) is standard (see e.g. [3,17]).

From \((7)\) a transfer-matrix $\tau^V(N|u)$ is defined by \((7)\). It is sufficient to consider the homogeneous case, i.e. $\delta^{(n)}=0$ and $V^{(n)}=V_4$ for all $n$ in \((2a)\). Integrability follows from \((3c)\). It is easily tractable by ABA, which will be demonstrated in the next section.

Another $U_q(\hat{ gl}(2,1|\mathcal{C}))$-symmetric $R$-matrix acting on the direct product of two different four dimensional representations, characterized by the corresponding parameter sets \((5a)\), is given by

\[
R^{V_4V_4'}(u) = f(u)e_{11} \otimes \hat{e}_{11} + g(u)e_{22} \otimes \hat{e}_{22} - e_{33} \otimes \hat{e}_{33} - e_{44} \otimes \hat{e}_{44} \\
+ r_5 e_{22} \otimes \hat{e}_{11} + r'_5 e_{11} \otimes \hat{e}_{22} - r_{10} (e_{33} \otimes \hat{e}_{44} - e_{44} \otimes \hat{e}_{33}) \\
- r_7 (e_{33} + e_{44}) \otimes \hat{e}_{11} - r'_7 e_{11} \otimes (\hat{e}_{33} + \hat{e}_{44}) \\
- r_9 (e_{33} + e_{44}) \otimes \hat{e}_{22} - r'_9 e_{22} \otimes (\hat{e}_{33} + \hat{e}_{44}) \\
+ r_1 e_{21} \otimes \hat{e}_{12} + r'_1 e_{12} \otimes \hat{e}_{21} - r_4 e_{43} \otimes \hat{e}_{34} - r'_4 e_{34} \otimes \hat{e}_{43} \\
+ r_2 (e_{31} \otimes \hat{e}_{13} + e_{41} \otimes \hat{e}_{14}) - r'_2 (e_{13} \otimes \hat{e}_{31} + e_{14} \otimes \hat{e}_{41}) \\
+ r_3 (e_{32} \otimes \hat{e}_{23} + e_{42} \otimes \hat{e}_{24}) - r'_3 (e_{23} \otimes \hat{e}_{32} + e_{24} \otimes \hat{e}_{42}) \\
- r_6 (e_{24} \otimes \hat{e}_{13} - q^{-1} e_{34} \otimes \hat{e}_{14}) \\
+ r'_6 (e_{42} \otimes \hat{e}_{31} - q e_{32} \otimes \hat{e}_{41}) \\
+ r'_6 (e_{13} \otimes \hat{e}_{24} - q e_{14} \otimes \hat{e}_{23}) \\
- r_8 (e_{31} \otimes \hat{e}_{42} - q^{-1} e_{41} \otimes \hat{e}_{32})
\]

The coefficients are again listed in appendix A. The construction of this $R$-matrix, and a proof of \((11)\),

\[
R^{V_4V_4'}(u-v)R^{V_4V_4''}(u)R^{V_4V_4'''}(v) \\
= R^{V_4V_4''}(v)R^{V_4V_4'}(u)R^{V_4V_4''}(u-v)
\]

(10a)
or

\[
R_{12}^{V_3V_4}(u-v)R_{13}^{V_3V_4'}(u)R_{23}^{V_4V_4'}(v)
= R_{23}^{V_4V_4'}(v)R_{13}^{V_3V_4'}(u)R_{12}^{V_3V_4}(u-v) 
\]  

(10b)

with \( R^{V_3V_4} \) from (7) will be given elsewhere [17]. A special case \((V_4' = V_4)\), leading to considerable simplifications, has been constructed explicitly by Gould et al. [18].

One may fix \( u \) and \( v \) in (10a) and regard \( C, C' \) and \( C'' \) instead as spectral parameters in order to satisfy the general form (1a) of the YBE.

¿From (9) the transfer-matrix \( \tau^{V_4}(N|u) \) is defined by (2). It is again sufficient to consider the homogeneous case \( \delta^{(n)} = 0 \) and \( V^{(n)} = V'_n \) for all \( n \) in (2b). Integrability follows from (3c) with the choice between \( \tau^{V_4}(N|v) \) and \( \tau^{V_3}(N|v) \) as generating functionals for “integrals of motion”.

Here ABA is not straight-forward. This model requires a new strategy in order to obtain equations for all eigenvalues \( \tau^{V_4}(N|u) \).

III. ALGEBRAIC BETHE ANSATZ

The original recipe for ABA is simple [8]:

1. Determine a vacuum state, preferably a highest or lowest weight state of the underlying group structure, if available, which tridiagonalizes \( \hat{L}^V(n|u) \) locally, and extend it via the product structure (2b) to a global vacuum, tridiagonalizing \( \hat{T}^V(N|u) \).

2. Take the off-diagonal elements of \( \hat{T}^V(N|u) \), not annihilating the global vacuum, as creation-operators and use the associative algebra defined by the FCR (3b), to generate eigenvectors to all eigenvalues of \( \hat{\tau}^V(N|u) \) (2c). Equations determining the latter are also derived from the algebra.

The first point is more or less a precondition for the applicability of ABA; the second is crucial: Only if \( V \) is a carrier space of the fundamental representation of a possibly deformed
and graded Lie algebra, the choice of \textit{creation-operators} is obvious.

\( \hat{\tau}^{V_3}(N|u) \) and \( \hat{\tau}^{V_4}(N|u) \), as defined in the previous section, are sufficiently complex to illustrate the general situation.

Since the auxiliary space is graded, it is useful to transform the matrix-elements of \( \hat{L}^{V_3}(n|u) \) (2a) in the \( V_3 \) basis according to

\[
\left[ \hat{L}^{V_3}(n|u) \right]_{ij}^{(V_3)} \rightarrow (-1)^{p(j)+p(i)} \left[ \hat{L}^{V_3}(n|u) \right]_{ij}^{(V_3)}
\]

(11)

This absorbs just a troublesome minus sign from the commutation of \( |3 \rangle^{V_3} \) with \( [\hat{L}^{V_3}(n|u)]_{13} \) and \( [\hat{L}^{V_3}(n|u)]_{23} \). All four local basis vectors of \( V_4^{(n)} \) (13) are suitable as (local) vacuum, preferably

\[
\Omega^{(n)} := |2 \rangle^{(n)}
\]

(12)

\( \Omega^{(n)} \) is a lowest weight state of the representation of \( U_q(gl(2,1|C)) \) on \( V_4^{(n)} \) and its equivalent was used by Kulish and Reshetikhin to treat the non-graded \( Y(gl(3|C)) \)-symmetric case [19]. Their calculation was generalized to the fundamental representation of \( U_q(\hat{gl}'(m,n)) \) by Schultz [20].

\[
\hat{L}^{V_3}(n|u)\Omega^{(n)} = \begin{pmatrix}
\omega_1^{(n)}(u) & 0 & 0 \\
0 & \omega_2^{(n)}(u) & 0 \\
* & * & \omega_3^{(n)}(u)
\end{pmatrix} \Omega^{(n)}
\]

(13)

with * denoting non-zero entries. The vacuum-eigenvalues of the diagonal elements are given by.

\[
\omega_1^{(n)}(u) = \frac{\sinh(\eta C + u)}{\sinh(\eta C + 2) - u}
\]

\[
\omega_2^{(n)}(u) = \omega_1^{(n)}(u)
\]

\[
\omega_3^{(n)}(u) = -1
\]

(14)

The index \( (n) \) will be omitted, due to homogeneity. Immediately from (13), (2b) and the definition
\[ |0\rangle_N = \Omega^{(N)} \otimes \Omega^{(N-1)} \otimes \cdots \otimes \Omega^{(1)} \quad (15) \]

of the (global) vacuum \(|0\rangle_N\) follows

\[
\hat{T}^{V_3}(N|u) \ |0\rangle_N = \begin{pmatrix} [\omega_1(u)]^N & 0 & 0 \\ 0 & [\omega_1(u)]^N & 0 \\ \hat{C}_1(u) & \hat{C}_2(u) & (-1)^N \end{pmatrix} |0\rangle_N \quad (16)
\]

where

\[ \hat{C}_i(u) := [\hat{T}^{V_3}(N|u)]_{3i} \quad \text{for} \quad i = 1, 2 \quad (17) \]

will later serve as \textit{creation-operators}.

ABA step 1 is finished: From (16),(2c) and (4h) follows the vacuum-eigenvalue of \(\tau^{V_3}(N|u)\):

\[
\Lambda^{V_3}_N(u) = 2[\omega_1(u)]^N - (-1)^N \quad (18)
\]

As mentioned before, Kulish and Reshetikhin solved a model built from the fundamental representation of \(Y(gl(3|\mathcal{C}))\), whose \(R\)-matrix differs from the \(\eta \to 0\)-limit of (3) only in minor details. The FCR (3b) derived from (3):

\[
R^{V_3V_3}_{12}(u - v)\hat{T}^{V_3}_1(N|u)\hat{T}^{V_3}_2(N|v) = \hat{T}^{V_3}_2(N|v)\hat{T}^{V_3}_1(N|u)R^{V_3V_3}_{12}(u - v) \quad (19)
\]

are almost identical to the ones in [19]: Trigonometric functions in (3) do not show up, if appropriate abbreviations are used. Apart from a few signs due to grading, which was also realized in [19], the formal algebra defined by (19) becomes exactly the same.

Of course it is possible to write down equations for eigenvectors and eigenvalues immediately, using the result of [19]. Again apart from a few signs, just the vacuum eigenvalues have to be replaced by (14). This is a well-known feature of ABA.

However some more details will be needed, in order to tackle the more complicated problem
of diagonalizing $\hat{\tau}^V(N|u)$ in the following section:

The (nested, see below) algebraic Bethe ansatz for (right) eigenvectors of $\hat{\tau}^V(N|u)$ is

$$|\lambda_1, \ldots, \lambda_M|F> = F^{a_1, \ldots, a_M} \hat{C}_{a_1}(\lambda_1) \cdots \hat{C}_{a_M}(\lambda_M) |0\rangle_N,$$

(20)

where $\{\lambda_1, \ldots, \lambda_M\}$ is some set of yet unknown parameters and $F^{a_1, \ldots, a_M}$ are some coefficients, yet undetermined. Summation over repeated $a_i = 1, 2$ with $i = 1, \ldots, M$ is implied. From (19) follows immediately

$$\hat{T}_{33}(u) \hat{C}_i(v) = \frac{1}{c(u-v)} \hat{C}_i(v) \hat{T}_{33}(u) + \frac{a(v-u)}{c(v-u)} \hat{C}_i(u) \hat{T}_{33}(v)$$

(21a)

$$\hat{T}_{ij}(u) \hat{C}_k(v) = \frac{1}{c(u-v)} \sum_{l,m=1}^2 r_{lm,ij}(u-v) \hat{C}_m(v) \hat{T}_{il}(u) - \frac{b(u-v)}{c(u-v)} \hat{C}_j(u) \hat{T}_{ik}(v)$$

(21b)

$$\hat{C}_i(u) \hat{C}_j(v) = \frac{1}{d(u-v)} \sum_{k,l=1}^2 r_{kl,ij}(u-v) \hat{C}_k(v) \hat{C}_l(u)$$

(21c)

with $i, j, k \in \{1, 2\}$. $a(u), b(u), c(u)$ and $d(u)$ originate from (6). For brevity $[\hat{T}^V(N|u)]_{ij}^V$ has been denoted by $\hat{T}_{ij}(u)$. In the present case $r_{ik,jl}(u)$ are elements of the non-graded $U_q(\hat{g}l'(2|C))$-symmetric $R$-matrix,

$$R^{V_2V_2}(u) = \sum_{i,j,k,l=1}^2 r_{ik,jl} e_{ij} \otimes \hat{e}_{kl}$$

(22)

which acts on the direct product of two two-dimensional, purely even subspaces $V_2$ of $V_3$, spanned by $|1>$ and $|2>$ from (6a). It is crucial to realize the appearance of $R^{V_2V_2}(u)$ as a proper submatrix in $R^{V_3V_3}(u)$ (6), because it defines a simpler BA-solvable model. Nested
algebraic Bethe ansatz (NABA) is typical for models, based on fundamental representations of dimension larger than 2.

It was preceded by the ingenious, but complicated nested coordinate Bethe ansatz, invented by Gaudin \[21\] and Yang \[1\] independently. Their method was applied to the fundamental representation of the $\mathfrak{g}l(m,n|\mathcal{C})$-symmetric problem by Lai \[23\] and Sutherland \[24\]. The formal algebraic formulation of the method is apparently due to Takhtajan \[22\].

The transfer-matrix $\hat{\tau}^{V_3}(N|u)$ applied to the Bethe ansatz eigenvector (20) should yield

$$\hat{\tau}^{V_3}(N|u) |\lambda_1, \ldots, \lambda_M| F >$$

$$= \Lambda^{V_3}(N|u) |\lambda_1, \ldots, \lambda_M| F >$$

Leaving some technical details for appendix B, it turns out, that this is true, iff the coefficients $F$ in (20) fulfill "6-vertex-type" eigenvalue equations \[13\]:

$$\left[ \hat{\tau}^{V_2}(M|\lambda_k) \right]^{a_1,\ldots,a_m}_{b_1,\ldots,b_m} F^{b_1,\ldots,b_m}$$

$$= \frac{1}{[-\omega_1(\lambda_k)]^N} F^{a_1,\ldots,a_m}$$

for $k = 1, \ldots, M$, of course solvable by ABA \[8\]. This is the second nested Bethe ansatz. $\hat{\tau}^{V_2}(M|u)$ is an inhomogeneous transfer-matrix obtained according to (3) with $\delta^{(n)} = \gamma_n$ from (22). The eigenvalue of $\tau^{V_2}(M|u)$ corresponding to the BA-eigenvector $F$ is given by

$$\Lambda^V_M(u; \mu_1, \ldots, \mu_m)$$

$$= \left( \prod_{n=1}^{M} c(u - \lambda_n) \right) \left( \prod_{\alpha=1}^{m} \frac{1}{c(u - \mu_\alpha)} \right)$$

$$+ \left( \prod_{\alpha=1}^{m} \frac{1}{c(\mu_\alpha - u)} \right)$$

with rapidities $\mu_\alpha$ ($\alpha = 1, \ldots, m$), determined by the BA-equations

$$\prod_{n=1}^{M} c(\mu_\alpha - \lambda_n) = \prod_{\beta=1}^{m} \frac{c(\mu_\alpha - \mu_\beta)}{c(\mu_\beta - \mu_\alpha)}$$

for $\alpha = 1, \ldots, m$. These and expressions for the actual BA-vectors $F$ also depending on $\mu_1, \ldots, \mu_m$, may be found in the literature \[8\].

Using (25) the eigenvalue condition (23) reads
\[-\omega_1(\lambda_k)]^N = \prod_{\alpha=1}^m c(\mu_\alpha - \lambda_k) \tag{26b}\]

for \(k = 1, \ldots, M\), which is the second set of BA-equations, determining \(\lambda_1, \ldots, \lambda_M\). Collecting the wanted terms in (24) the eigenvalue of \(\hat{\tau}_V^3(N|u)\) corresponding to the NABA-eigenvector \(\mathbf{20}\) follows immediately:

\[
\Lambda^V_N(u; \lambda_1, \ldots, \lambda_M|\mu_1, \ldots, \mu_m) = \left(\prod_{\lambda_i=1}^M \frac{1}{c(u - \lambda_i)}\right) \times \left\{[\omega_1(u)]^N \Lambda^V_M(u; \mu_1, \ldots, \mu_m) - (-1)^N\right\} \tag{27}\]

According to Baxter [7] BA-equations guarantee analyticity of all eigenvalues in \(u\). Here a \(q\)-deformed, graded version of the \(R\)-matrix \(\mathbf{3}\) has been used and the \(\hat{C}_l\)-operators act on a different quantum space, i.e. \(V_4\) instead of \(V_3\). However not knowing about \(\mathbf{20}\), the whole calculation has been borrowed from \(\mathbf{19}\). A highest weight state, i.e. \(|1\rangle\) instead of \(|2\rangle\) in (12) and (15), could have been used as vacuum, but this leads to a very similar calculation. The result (27) is new, but it differs just by the vacuum eigenvalues (14) and signs from the well-known one in \(\mathbf{19}\). It is also complete. This is not true for the set of eigenvectors \(\mathbf{20}\). However the missing ones may be produced using the lowest weight property of the ABA-vectors with respect to the group action on quantum space, which can be proved by standard-methods \(\mathbf{8}\).

These are well-known and beautiful features of Bethe ansatz solvable systems. Also the equations for the inhomogeneous model with \(w^{(n)} = C^{(n)}\) in \(\mathbf{28}\) can be written down immediately using an argument due to Baxter [7]:

\[
\Lambda^V_N(u; \lambda_1, \ldots, \lambda_M|\mu_1, \ldots, \mu_m) \tag{28a} \\
= \prod_{n=1}^N \left(\frac{\sinh(\eta C^{(n)} + u - \delta^{(n)})}{\sinh(\eta C^{(n)} + 2) - u + \delta^{(n)}}\right) \times \left\{\prod_{i=1}^M \frac{\sinh(u - \lambda_i + 2\eta)}{\sinh(u - \lambda_i)} \prod_{\alpha=1}^m \frac{\sinh(u - \mu_\alpha - 2\eta)}{\sinh(u - \mu_\alpha)} + \prod_{\alpha=1}^m \frac{\sinh(u - \mu_\alpha + 2\eta)}{\sinh(u - \mu_\alpha)}\right\} \right\}
\]
\[- (1)^N \prod_{i=1}^{M} \frac{\sinh(u - \lambda_i + 2\eta)}{\sinh(u - \lambda_i)}\]

The BA-equations (analyticity conditions) are

\[
\prod_{i=1}^{M} \frac{\sinh(\mu_i - \lambda_i + 2\eta)}{\sinh(\mu_i - \lambda_i)} = \prod_{\beta=1, \beta \neq \alpha}^{m} \frac{\sinh(\mu_\alpha - \mu_\beta + 2\eta)}{\sinh(\mu_\alpha - \mu_\beta - 2\eta)}
\]

(28b)

for \(\alpha = 1, \ldots, m\) and

\[
\prod_{n=1}^{N} \frac{\sinh(\lambda_k - \delta^{(n)} - \eta(C^{(n)} + 2))}{\sinh(\lambda_k - \delta^{(n)} + \eta C^{(n)})} = \prod_{\alpha=1}^{m} \frac{\sinh(\mu_\alpha - \lambda_k + 2\eta)}{\sinh(\mu_\alpha - \lambda_k)}
\]

(28c)

for \(k = 1, \ldots, M\). The situation is different in the case of \(\tau^{V_4}(N|u)\), because the innocent looking change of auxiliary space requires the use of an at first sight completely different algebra. In the next section a systematic approach to this problem will be developed, which makes extensive use of the presented solution.

**IV. DIAGONALIZATION OF \(\hat{\tau}^{V_4'}(N|U)\)**

In order to understand the difficulties in diagonalizing the homogeneous version of \(\tau^{V_4'}(N|u)\) defined in section \(\text{III}\) it is convenient to follow the standard procedure from the previous section as far as possible. So \(V_{4}^{(N)} \times \cdots \times V_{4}^{(1)}\) will be chosen as quantum space, while \(V_{4}'\), characterized by primed parameters (3a) will serve as auxiliary space. The sign change (11) will be applied and the local vacuum will be chosen as lowest weight state in \(V_{4}^{(1)}\) (12). Omitting the local index \((n)\), due to homogeneity, this leads to

\[
\hat{L}^{V_4'}(n|u) \Omega^{(n)} = \begin{pmatrix}
\omega_1(u) & 0 & 0 & 0 \\
* & \omega_2(u) & * & * \\
* & 0 & \omega_3(u) & 0 \\
* & 0 & 0 & \omega_4(u)
\end{pmatrix} \Omega^{(n)}
\]

(29)
with the new (local) vacuum eigenvalues

\[
\begin{align*}
\omega_1(u) &= \frac{\sinh(\eta(C - C') + u) \sinh(\eta(C - C' + 2) + u)}{\sinh(\eta(C + C') - u) \sinh(\eta(C + C' + 2) + u)} \\
\omega_2(u) &= \frac{\sinh(\eta(C + C' + 2) - u)}{\sinh(\eta(C + C' + 2) + u)} \\
\omega_3(u) &= \frac{\sinh(\eta(C' - C) - u)}{\sinh(\eta(C + C' + 2) + u)} \\
\omega_4(u) &= \omega_3(u)
\end{align*}
\]

(30)

There are five non-vanishing entries compared to two in (13). This will be the same for the other three possible local vacua. Using (15), (2b) leads to

\[
\begin{align*}
\hat{T}^{V_i}(N|u) \ |0\rangle_N &= \\
= \left( \begin{array}{cccc}
[\omega_1(u)]^N & 0 & 0 & 0 \\
* & [\omega_2(u)]^N & * & * \\
* & 0 & [\omega_3(u)]^N & 0 \\
* & 0 & 0 & [\omega_3(u)]^N
\end{array} \right) \ |0\rangle_N
\end{align*}
\]

(31)

¿From the integrability condition (3c)

\[
\left[ \hat{\tau}^{V_i}(N|u), \hat{\tau}^{V_3}(N|v) \right] = 0
\]

it is clear, that \( \hat{\tau}^{V_3}(N|v) \) and \( \hat{\tau}^{V_i}(N|u) \) share the same eigenvectors. The eigenvalues (27) are in general degenerate. The lowest weight property of the (global) vacuum (15), which is inherited by the BA-vectors (20) via standard arguments [8], guarantees uniqueness of these special vectors. Note that the same argument would hold also for a highest weight state as (global) vacuum, but not for any other choice. From this and (31), following Baxter [7], it can be concluded immediately, that all eigenvalues of \( \hat{\tau}^{V_i}(N|u) \) can be represented in the form

\[
\Lambda_{N}^{V_i}(u; \lambda_1, \ldots, \lambda_M|\mu_1, \ldots, \mu_m)
= [\omega_1(u)]^N F(u) + [\omega_2(u)]^N G(u)
- [\omega_3(u)]^N \{H(u) + J(u)\}
\]

(32)
where $F(u), G(u), H(u)$ and $J(u)$ are meromorphic functions in $u$, whose residua cancel, if the analyticity conditions (26) hold.

In order to determine these unknown functions, the FCR (3b) with with $V = V_3$ and $V' = V_4'$, namely

$$R_{12}^{V_3 V'_4}(u, v)\hat{T}_1^{V_3}(N|u)\hat{T}_2^{V'_4}(N|v)$$

$$= \hat{T}_2^{V'_4}(N|v)\hat{T}_1^{V_3}(N|u)R_{12}^{V_3 V'_4}(u, v)$$

(33)

with $R^{V_3 V'_4}(u)$ from (1) will be chosen. The reasons are

1. $R^{V_3 V'_4}$ is a $12 \times 12$-matrix while $R^{V_4 V'_4}$ is a $16 \times 16$-matrix. The choice $V = V_4$ would greatly increase the number of equations.

2. In contrast to (16) equation (31) does not offer a natural choice of creation-operators, so the invaluable a priori knowledge of unique eigenvectors (20) with BA-parameters obeying (26) would be lost within the alternative choice.

The $R$-matrices $R^{V_3 V'_4}(u)$ (1) and $R^{V_4 V'_4}(u)$ (3) do not contain $R^{V_3 V_2}(u)$ (22) as a proper submatrix.

In particular unwanted terms turn out to be much more complicated. However it is possible to omit their calculation. As will be shown, the knowledge of unique eigenvectors (20) with (26) as well as some details of the calculation given in section III are sufficient to determine the unknown functions in (32) unambiguously.

For brevity (17) will be used as well as

$$\hat{T}^{V_3}_{ij}(u) = [\hat{T}^{V_3}(N|u)]^{V_3}_{ij}, \quad \hat{T}^{V'_4}_{ij}(u) = [\hat{T}^{V'_4}(N|u)]^{V'_4}_{ij}$$

First it is convenient to list all components from (33), containing an operator $\hat{C}_i(u)$ (17) multiplied with a diagonal element of $\hat{T}^{V'_4}_{jj}(v)$ from the right. From (7) and (A1) with primed parameters (5a) and (33) follows:

$$\zeta_2(u - v)\hat{T}^{V_3}_{11}(u)\hat{T}^{V'_4}_{41}(v)$$
\[ - \zeta_2(u - v)q\hat{T}_{21}^V(u)\hat{T}_{31}^V(v) \]  
\[ + \beta_0(u - v)\hat{C}_1(u)\hat{T}_{11}^V(v) = \rho(u - v)\hat{T}_{11}^V(v)\hat{C}_1(u), \]  
\[ - \hat{C}_1(u)\hat{T}_{22}^V(v) = \alpha_0(u - v)\hat{T}_{22}^V(v)\hat{C}_1(u) \]  
\[ \quad - \varepsilon_2(u - v)\hat{T}_{23}^V(v)\hat{T}_{33}^V(u), \]  
\[ \varepsilon_2(u - v)\hat{T}_{11}^V(u)\hat{T}_{23}^V(v) \]  
\[ + \gamma_0(u - v)\hat{C}_1(u)\hat{T}_{33}^V(v) = \rho(u - v)\hat{T}_{33}^V(v)\hat{C}_1(u), \]  
\[ \varepsilon_2(u - v)\hat{T}_{21}^V(u)\hat{T}_{24}^V(v) \]  
\[ + \gamma_0(u - v)\hat{C}_1(u)\hat{T}_{44}^V(v) = \alpha_0(u - v)\hat{T}_{44}^V(v)\hat{C}_1(u) \]  
\[ \quad + \delta_2(u - v)\hat{T}_{43}^V(v)\hat{C}_2(u) \]  
\[ - \zeta_2(u - v)\hat{T}_{41}^V(v)\hat{T}_{33}^V(u) \]  
\[ \zeta_2(u - v)\hat{T}_{12}^V(u)\hat{T}_{41}^V(v) \]  
\[ - \zeta_2(u - v)q\hat{T}_{22}^V(u)\hat{T}_{31}^V(v) \]  
\[ + \beta_0(u - v)\hat{C}_2(u)\hat{T}_{12}^V(v) = \rho(u - v)\hat{T}_{11}^V(v)\hat{C}_2(u), \]  
\[ - \hat{C}_2(u)\hat{T}_{22}^V(v) = \alpha_0(u - v)\hat{T}_{22}^V(v)\hat{C}_2(u) \]  
\[ \quad - \varepsilon_2(u - v)\hat{T}_{24}^V\hat{T}_{33}^V(u) \]  
\[ \varepsilon_2(u - v)\hat{T}_{12}^V(u)\hat{T}_{23}^V(v) \]  
\[ + \gamma_0(u - v)\hat{C}_2(u)\hat{T}_{33}^V(v) = \delta_1(u - v)\hat{T}_{34}^V(v)\hat{C}_1(u) \]  
\[ \quad + \alpha_0(u - v)\hat{T}_{33}^V(v)\hat{C}_2(u) \]  
\[ + \zeta_2(u - v)q\hat{T}_{31}^V(v)\hat{T}_{33}^V(u) \]  
\[ \varepsilon_2(u - v)\hat{T}_{22}^V(u)\hat{T}_{24}^V(u) \]  
\[ + \gamma_0(u - v)\hat{C}_2(u)\hat{T}_{44}^V(v) = \rho(u - v)\hat{T}_{44}^V(v)\hat{C}_2(u) \]
The idea is to keep only contributions leading to wanted terms, when the eigenvector (20) is applied to \( \hat{T}_{11}^{V} (N|u) \) and neglect all others. The set (34) is not complete. For instance in (34a) a term \( \propto \hat{T}_{11}^{V} (u) \hat{T}_{41}^{V} (u) \) and another \( \propto \hat{T}_{21}^{V} (u) \hat{T}_{31}^{V} (v) \) occur. Both will act non-trivially on \( |0\rangle_N \) from (31). However in the set (33) the relations

\[
\alpha_0 (u - v) \hat{T}_{11}^{V} (u) \hat{T}_{41}^{V} (v) \\
+ \delta_1 (u - v) \hat{T}_{21}^{V} (u) \hat{T}_{31}^{V} (v) \\
+ \zeta_1 (u - v) \hat{C}_1 (u) \hat{T}_{11}^{V} (v) = \hat{T}_{41}^{V} (v) \hat{T}_{11}^{V} (u)
\]

and

\[
\delta_2 (u - v) \hat{T}_{11}^{V} (u) \hat{T}_{41}^{V} (v) \\
+ \alpha_0 (u - v) \hat{T}_{21}^{V} (u) \hat{T}_{31}^{V} (v) \\
- \zeta_1 (u - v) q^{-1} \hat{C}_1 (u) \hat{T}_{11}^{V} (v) = \rho (u - v) \hat{T}_{31}^{V} (v) \hat{T}_{21}^{V} (u)
\]

can be found and used to eliminate these terms leading to

\[
\left( \beta_0 - \frac{\zeta_1 \zeta_2 [2 \alpha_0 + q^{-1} \delta_1 + q \delta_2]}{\alpha_0^2 - \delta_1 \delta_2} \right) (u - v) \hat{C}_1 (u) \hat{T}_{11}^{V} (v) \\
= \rho (u - v) \hat{T}_{11}^{V} (v) \hat{C}_1 (u) \\
- \left( \frac{\rho \zeta_2}{\alpha_0} \left[ 1 + \frac{\delta_2 [\alpha_0 q + \delta_1]}{\alpha_0^2 - \delta_1 \delta_2} \right] \right) (u - v) \hat{T}_{21}^{V} (v) \hat{T}_{31}^{V} (u) \\
+ \left( \frac{\rho \zeta_2 [\alpha_0 q + \delta_1]}{\alpha_0^2 - \delta_1 \delta_2} \right) (u - v) \hat{T}_{31}^{V} \hat{T}_{21}^{V} (u)
\]

where the dependence on difference variables has been denoted symbolically for brevity. The last two terms on the right hand side will not lead to a contribution proportional to any BA-eigenvector (20). It has been checked – and this is crucial, that these terms are not related to a proper combination of \( \hat{C}_1 \)-operators by unused relations from the set (33). In conclusion they can be identified as leading to unwanted terms.

In the same way two other relations from (33) may be used to eliminate from (34a) terms \( \propto \hat{T}_{12}^{V} (u) \hat{T}_{41}^{V} (v) \) and \( \propto \hat{T}_{22}^{V} (u) \hat{T}_{31}^{V} (v) \), which after omitting contributions leading to unwanted terms yield the same result with \( \hat{C}_1 (u) \) replaced by \( \hat{C}_2 (u) \), i.e.
\[
\hat{T}_{11}^{V_i}(u)\hat{C}_i(v) = \left( \frac{\beta_0 - \zeta_1\zeta_2[2\alpha_0 + q^{-1}\delta_1 + q\delta_2]}{\rho[\alpha_0^2 - \delta_1\delta_2]} \right) (v - u) \\
\times \hat{C}_i(v)\hat{T}_{11}^{V_i}(u) \pm \ldots
\] (35a)

for \( i = 1, 2 \).

In (34b) and (34f) terms \( \propto \hat{T}_{11}^{V_i'}(v)\hat{T}_{33}^{V_i}(u) \) and \( \propto \hat{T}_{24}^{V_i'}\hat{T}_{33}^{V_i}(v) \) can be identified as leading to unwanted terms in the sense explained above and therefore be neglected:

\[
\hat{T}_{22}^{V_i}(u)\hat{C}_i(v) = \frac{-1}{\alpha_0(v - u)} \hat{C}_i(v)\hat{T}_{22}^{V_i}(u) \pm \ldots
\] (35b)

for \( i = 1, 2 \). The other relations from (33) can be treated similarly, leading to

\[
\hat{T}_{33}^{V_i}(u)\hat{C}_1(v) = \left( \frac{\alpha_0\gamma_0 - \varepsilon_1\varepsilon_2}{\alpha_0\rho} \right) (v - u) \hat{C}_1(v)\hat{T}_{33}^{V_i}(u) \\
\pm \ldots
\] (35c)

\[
\hat{T}_{44}^{V_i}(u)\hat{C}_1(v) = \left( \frac{\alpha_0\gamma_0 - \varepsilon_1\varepsilon_2}{\alpha_0^2 - \delta_1\delta_2} \right) (v - u) \hat{C}_1(v)\hat{T}_{44}^{V_i}(u) \\
- \left( \frac{\delta_2\alpha_0\gamma_0 - \varepsilon_1\varepsilon_2}{\alpha_0^2 - \delta_1\delta_2} \right) (v - u) \hat{C}_2(v)\hat{T}_{43}^{V_i}(u) \\
\pm \ldots
\] (35d)

\[
\hat{T}_{44}^{V_i}(u)\hat{C}_1(v) = \left( \frac{\alpha_0\gamma_0 - \varepsilon_1\varepsilon_2}{\alpha_0^2 - \delta_1\delta_2} \right) (v - u) \hat{C}_1(v)\hat{T}_{43}^{V_i}(u) \\
\pm \ldots
\] (35e)

Some details of the calculations are given in appendix C. They are tedious, but straightforward: It is trivial to identify terms proportional to a simple \( M = 1 \) eigenvector, (20), if it is applied. The remaining terms are divided into those, which possibly lead to a contribution proportional to an eigenvector via the algebra (34), and others which cannot be transformed this way. The former terms have been eliminated by using convenient relations from (34) and evaluated again, till this procedure terminated, leaving only terms of the latter type,
Equations (35a) and (35b) contain non-trivial terms $\propto \hat{C}_2(v)\hat{T}_{43}^{V_i}(u)$ and $\propto \hat{C}_1(v)\hat{T}_{43}^{V_i}(u)$. Next it is natural to add to (34) the relations involving terms $\propto \hat{T}_{34}^{V_i}(u)\hat{C}_i(v)$ and $\propto \hat{T}_{43}^{V_i}(u)\hat{C}_i(v)$ with $i = 1, 2$, i.e.

\[
\varepsilon_2(u-v)\hat{T}_{11}^{V_3}(u)\hat{T}_{24}^{V_4}(v) + \gamma_0(u-v)\hat{C}_1(u)\hat{T}_{34}^{V_4}(v) = \alpha_0(u-v)\hat{T}_{34}^{V_4}(v)\hat{C}_1(u) + \delta_2(u-v)\hat{T}_{33}^{V_4}(v)\hat{C}_2(u) + \zeta_2(u-v)\hat{T}_{31}^{V_4}(v)\hat{T}_{33}^{V_3}(u)
\]

\[
\varepsilon_2(u-v)\hat{T}_{21}^{V_3}(u)\hat{T}_{23}^{V_4}(v) + \gamma_0(u-v)\hat{C}_1(u)\hat{T}_{43}^{V_4}(v) = \rho(u-v)\hat{T}_{43}^{V_4}(v)\hat{C}_1(u)
\]

\[
\varepsilon_2(u-v)\hat{T}_{22}^{V_3}(u)\hat{T}_{24}^{V_4}(v) + \gamma_0(u-v)\hat{C}_2(u)\hat{T}_{34}^{V_4}(v) = \rho(u-v)\hat{T}_{34}^{V_4}(v)\hat{C}_2(u)
\]

\[
\varepsilon_2(u-v)\hat{T}_{22}^{V_3}(u)\hat{T}_{23}^{V_4}(v) + \gamma_0(u-v)\hat{C}_2(u)\hat{T}_{43}^{V_4}(v) = \delta_1(u-v)\hat{T}_{44}^{V_4}(v)\hat{C}_1(u) + \alpha_0(u-v)\hat{T}_{43}^{V_4}(v)\hat{C}_2(u) + \zeta_2(u-v)\hat{T}_{41}^{V_4}(v)\hat{T}_{33}^{V_3}(u)
\]

Proceeding as above, leads to

\[
\hat{T}_{34}^{V_i}(u)\hat{C}_1(v) = \left(\frac{\alpha_0\gamma_0 - \varepsilon_1\varepsilon_2}{\alpha_0^2 - \delta_1\delta_2}\right) (v-u) \hat{C}_1(v)\hat{T}_{34}^{V_i}(u) - \left(\frac{\delta_2\alpha_0\gamma_0 - \varepsilon_1\varepsilon_2}{\alpha_0} \right) (v-u) \hat{C}_2(v)\hat{T}_{33}^{V_i}(u) \pm \ldots
\]

\[
\hat{T}_{43}^{V_i}(u)\hat{C}_2(v) = \left(\frac{\alpha_0\gamma_0 - \varepsilon_1\varepsilon_2}{\alpha_0^2 - \delta_1\delta_2}\right) (v-u) \hat{C}_2(v)\hat{T}_{43}^{V_i}(u) - \left(\frac{\delta_1\alpha_0\gamma_0 - \varepsilon_1\varepsilon_2}{\alpha_0} \right) (v-u) \hat{C}_1(v)\hat{T}_{44}^{V_i}(u)
\]
This idea is strongly supported by a comparison of (35) and (37) with (21), used in the algebraic diagonalization of $\hat{\tau}_V(N|u)$, suggesting, that the submatrix $\{\hat{T}_{ij}^{V'}\}$ with $i, j = 3, 4$ will play the same role as the submatrix $\{\hat{T}_{ij}^{V}\}$ with $i, j = 1, 2$ in the previous section. Indeed using the definitions (A1) with primed parameters (5a), (35a) and (35b) can be written

\begin{align}
\hat{T}_{11}^{V'}(u)\hat{C}_i(v) &= \frac{\sinh(u - v + \eta(C' + 2))}{\sinh(u - v - \eta(C' - 2))}\hat{C}_i(v)\hat{T}_{11}^{V'}(u) \\
&\quad \pm \ldots 
\end{align}

(38a)

\begin{align}
\hat{T}_{22}^{V'}(u)\hat{C}_i(v) &= \frac{\sinh(u - v + \eta(C' + 2))}{\sinh(u - v - \eta C')}\hat{C}_i(v)\hat{T}_{22}^{V'}(u) \\
&\quad \pm \ldots 
\end{align}

(38b)

for $i = 1, 2$, while the remaining equations from (35) and (37) may be noted as

\begin{align}
\hat{t}_{ij}(u)\hat{C}_k(v) &= \frac{\sinh(u - v + \eta(C' + 2))}{\sinh(u - v - \eta C')} \\
&\quad \times \sum_{l,m=1}^2 r_{lm,jk}(u - v - \eta C') \hat{C}_m(v)\hat{t}_{il}(u) \\
&\quad \pm \ldots 
\end{align}

(38c)

for $i, j, k = 1, 2$, where the elements $r_{ik,jl}(u)$ of the $R$-matrix (22) and the convenient definition

\begin{align}
\begin{pmatrix}
\hat{t}_{11}(u) & \hat{t}_{12}(u) \\
\hat{t}_{21}(u) & \hat{t}_{22}(u)
\end{pmatrix}
\begin{pmatrix}
\hat{T}_{33}^{V'}(u) & \hat{T}_{43}^{V'}(u) \\
\hat{T}_{34}^{V'}(u) & \hat{T}_{44}^{V'}(u)
\end{pmatrix}
\end{align}

(38d)

have been used. The similarity of (38) to (21) is striking and allows to calculate the eigenvalues of $\hat{T}_{ij}^{V'}(N|u)$ easily.

Applying the (right) eigenvector (20) to $\hat{T}_{11}^{V'}(u)$ and $\hat{T}_{22}^{V'}(u)$ using (38) and (31) yields
\[
\hat{T}_{11}^{V'}(u)|\lambda_1, \ldots, \lambda_M|F > \\
= [\omega_1(u)]^N \prod_{i=1}^M \frac{\sinh(u - \lambda_i + \eta(C' + 2))}{\sinh(u - \lambda_i - \eta(C' - 2))} \\
\times |\lambda_1, \ldots, \lambda_M|F > \pm \ldots
\]

and

\[
\hat{T}_{22}^{V'}(u)|\lambda_1, \ldots, \lambda_M|F > \\
= [\omega_2(u)]^N \prod_{i=1}^M \frac{\sinh(u - \lambda_i + \eta(C' + 2))}{\sinh(u - \lambda_i - \eta C')} \\
\times |\lambda_1, \ldots, \lambda_M|F > \pm \ldots
\]

where unwanted terms have been omitted. Applying it to \([\hat{T}_{33}^{V'}(u) + \hat{T}_{44}^{V'}(u)]\) yields

\[
[\hat{T}_{33}^{V'}(u) + \hat{T}_{44}^{V'}(u)]|\lambda_1, \ldots, \lambda_M|F > \\
= [\omega_3(u)]^N \prod_{i=1}^M \frac{\sinh(u - \lambda_i + \eta(C' + 2))}{\sinh(u - \lambda_i - \eta C')} \\
\times [\hat{\tau}_V^2(M|u - \eta C')] b_1 \ldots b_M F_{a_1 \ldots a_M} \\
\times \hat{C}_{b_1}(\lambda_1) \cdots \hat{C}_{b_M}(\lambda_M) |0\rangle_N \pm \ldots
\]

where \(\hat{\tau}_V^2(M|u)\) is defined by \(\text{(22)}\) via \(\text{(2)}\) with \(\delta^{(n)} = \lambda_n\) as in section \([\text{II}]\). But \(F\) is a (right) eigenvector to \(\hat{\tau}_V^2(M|u)\) corresponding to the eigenvalue from \(\text{(25)}\). The neglected unwanted terms vanish per construction if the supertrace \(\text{(III)}\) is performed according to \(\text{(2c)}\). Therefore the eigenvalue of \(\hat{\tau}_V^4(M|u)\) corresponding to the (right) eigenvector \(\text{(20)}\) is given by

\[
\Lambda_N^{V'}(u; \lambda_1, \ldots, \lambda_M|\mu_1, \ldots, \mu_m) \\
= [\omega_1(u)]^N \prod_{i=1}^M \frac{\sinh(u - \lambda_i + \eta(C' + 2))}{\sinh(u - \lambda_i - \eta(C' - 2))} \\
+ [\omega_2(u)]^N \prod_{i=1}^M \frac{\sinh(u - \lambda_i + \eta(C' + 2))}{\sinh(u - \lambda_i - \eta C')} \\
- [\omega_3(u)]^N \prod_{i=1}^M \frac{\sinh(u - \lambda_i + \eta(C' + 2))}{\sinh(u - \lambda_i - \eta C')} \\
\times \Lambda_M^{V'}(u - \eta C', \mu_1, \ldots, \mu_m)
\]
with vacuum eigenvalues $\omega_i(u) \,(i = 1, 2, 3)$ from (30) and $\Lambda_{M}^{V_i}(u; \ldots)$ from (25).

The BA-parameters $\lambda_1, \ldots, \lambda_M$ and $\mu_1, \ldots, \mu_m$ are to be determined by the BA-equations (26). Note, that these are necessary and sufficient conditions [7] for analyticity of the eigenvalues (39) in $u$. Since up to now no explicit use has been made of these, this is a valuable consistency check on the validity of (39).

(39) is clearly of the expected form (32). It is further obvious, that the eigenvalues for every transfer-matrix based on auxiliary space $V'_4$ can be represented by the same formula (30), provided the (global) quantum space is a lowest weight space. Of course the vacuum eigenvalues have to be replaced by new ones, which are obviously restricted by the BA-equations (26), as discussed in section III.

For completeness the trivial generalization [7] of (39) to the inhomogeneous case with $w^{(n)} = C^{(n)}$ in (2a) and $\delta^{(n)} \neq 0$ in (2b) shall be given explicitly:

$$
\Lambda_{N}^{V_i}(u; \lambda_1, \ldots, \lambda_M|\mu_1, \ldots, \mu_m) = \prod_{n=1}^{N} \frac{\sinh(\eta(C^{(n)} - C') + u - \delta^{(n)})}{\sinh(\eta(C^{(n)} + C') - u + \delta^{(n)})} \\
\times \left( \prod_{i=1}^{M} \frac{\sinh(u - \lambda_i + \eta(C' + 2))}{\sinh(u - \lambda_i - \eta(C' - 2))} \right) \\
+ \prod_{n=1}^{N} \frac{\sinh(\eta(C^{(n)} + C' + 2) - u + \delta^{(n)})}{\sinh(\eta(C^{(n)} + C' + 2) + u - \delta^{(n)})} \\
\times \left( \prod_{i=1}^{M} \frac{\sinh(u - \lambda_i + \eta(C' + 2))}{\sinh(u - \lambda_i - \eta(C' - 2))} \right) \\
- \prod_{n=1}^{N} \frac{\sinh(\eta(C' - C^{(n)}) - u + \delta^{(n)})}{\sinh(\eta(C^{(n)} + C' + 2) + u - \delta^{(n)})} \\
\times \left\{ \left( \prod_{i=1}^{M} \frac{\sinh(u - \lambda_i + \eta(C' + 2))}{\sinh(u - \lambda_i - \eta(C')}) \right) \\
\times \left( \prod_{\alpha=1}^{m} \frac{\sinh(u - \mu_\alpha - \eta(C' + 2))}{\sinh(u - \mu_\alpha - \eta(C'))} \right) \\
+ \left( \prod_{i=1}^{M} \frac{\sinh(u - \lambda_i + \eta(C' + 2))}{\sinh(u - \lambda_i - \eta(C' - 2))} \right) \right\}
$$

(40)
Here the BA-parameters $\lambda_1, \ldots, \lambda_M$ and $\mu_1, \ldots, \mu_m$ are determined by (28b) and (28c). (40) describes all eigenvalues. As mentioned above, additional eigenvectors to the same eigenvalue (40) are obtained by applying shift operators, corresponding to the representation of the group-symmetry on the (global) quantum space, to the eigenvectors (20). Completeness may be assured by the usual arguments [8].

V. CONCLUSION

In the previous section $\hat{\tau}^{V'_4}(N|u)$ has been diagonalized by NABA, combined with analyticity arguments. Obviously the method can be applied to any BA-integrable model, defined by (2), based on an arbitrary, but finite dimensional representation $V'$ of a possibly $q$-deformed Lie (super-)algebra as auxiliary space.

Let the model based on the direct product of a fundamental representation $V$ with itself, here defined by $R^{V_3V_3}(u)$ and (2), be solved by (N)ABA. In order to solve the model under consideration the following scheme may be applied.

1. An auxiliary model based on $V$ as auxiliary and the non-standard representation $V'$ as quantum space, may be constructed by standard methods and its transfer-matrix, i.e. $\hat{\tau}^{V_3}(N|u)$ from (3) via (4), may be diagonalized, using a (global) lowest or highest weight state, e.g. $|0\rangle_N$ (13), as (pseudo-)vacuum.

2. Vacuum eigenvalues may be calculated trivially, see (30). The transfer-matrix of the relevant model and the one of the auxiliary model commute (3c) and share all BA-eigenvectors, which dictates the form of the eigenvalue equations (32).

3. Mixed FCR (34), between creation-operators from auxiliary model (17), should be used as follows:
(a) FCRs (34) between diagonal elements of $\hat{T}^{V'}(N|u)$ and creation-operators multiplied from the right on these (35) should be collected. The remaining terms in these equations are classified as wanted (leading to terms proportional to the known BA-vectors), unwanted (not related to wanted ones by FCRs) and others.

(b) Terms of the last category have to be eliminated by use of other convenient FCRs. **Unwanted terms** may be neglected in final equations, i.e. (33).

(c) Generically the final equations in step (b) involve some off-diagonal elements of $\hat{T}^{V'}(N|u)$ (35). They have to be complemented by all FCRs containing these off-diagonal elements, multiplied from the right with creation-operators (36), to which the same procedure has to be applied (37).

4. The relations obtained in step 3 allow the calculation of the eigenvalue equations (39), if they are written down conveniently, i.e. like (38).

Step one and two are trivial here. Step three is crucial. An unusually large number of FCRs (34) has to be used, because the mixed $R$-matrix (7), does not contain any smaller $R$-matrix like (22) as a proper submatrix, which was true e.g. for (3). The approach is systematic and avoids a complicated discussion of unwanted terms. The author has checked in a number of cases, that these indeed vanish in the present application, but analyticity of the final result (39) is a very strong and usually sufficient test. Step four is simple. Some knowledge of the preceding calculations is a sufficient guideline.

A group theoretical background is not necessary, but helpful. Definitely needed is a commuting (auxiliary) model, algebraically solvable [8], and a unique identification of joint eigenvectors. The theory of quantum groups [3,4,12] provides both. In addition it is implicitly assumed, that the algebra defined by the FCRs is complete, i.e. if two operators are identical, this information should be encoded within the FCRs. This is guaranteed if $R$ has the intertwining property [3].

The more complicated problem of handling the full set of commutation relations of comparable complexity directly, has been tackled more ore less exactly a number of times. The
algebraic solution of a statistical covering model for the one-dimensional Hubbard model, where no commuting transfer-matrix is known, by Ramos and Martins [25]. Also a diagonalization of an $Y(sp(2,1))$-symmetrical model by the same authors should be mentioned [26]. To the authors knowledge no systematic scheme is known and although the eigenvalues are presumably correct, the discussion of unwanted terms is not complete in these works.

It is an interesting, but still unsolved question, if solvability of some statistical model by $n$-fold NABA implies the existence of a commuting transfer-matrix with minimal, that is $(n + 1)$-dimensional, auxiliary space?

In the non-graded $U_q(\hat{gl}'(N|C))$-symmetric case, the quantum-determinant, introduced by Izergin and Korepin [28] and recognized by Drinfel’d [3] to complete the center of this algebra, provides the possibility to construct functional relations [27] for the eigenvalues, extended to an “analytical Bethe ansatz” by Reshetikhin [29]. This is more elegant than the present approach, but does not generalize to the graded case, because no one-dimensional subspace can be separated from a product of transfer-matrices.

The transfer-matrix $\hat{\tau}^{V_4}(N|u)$ has been used mainly for pedagogical reasons. Minus signs due to grading, even in the non-graded version [27] prevent a statistical interpretation. Nevertheless the Hamiltonian limit in the non-difference type spectral parameter (1a), as mentioned above, leads to an additional, unusual Hamiltonian, which will be discussed elsewhere [30].

Note that neither $\hat{\tau}^{V_3}(N|u)$ nor $\hat{\tau}^{V_4}(N|u)$ are hermitian, except if further restrictions are imposed on (5a). The diagonalization of $\hat{\tau}^{V_4}(N|u)$, especially the result (40), may serve as starting point for calculations on the thermodynamics of these models in the non-linear integral equation approach, pioneered by Klümper [31]. For a recent application of this technique see also [32].

The eigenvalue-equation for the transfer-matrix of some other $U_q(\hat{gl}'(2,1|C))$-symmetric models with $V_4'$ as auxiliary and some lowest weight representation as quantum space may be written down by replacing the $\omega_i(u)$ (30) in (39) by new ones.

De Vega and Gonzáles Ruiz [33] and Foerster and Karowski [34] generalized the ABA calculations of Schultz [20] partially to non-periodic, integrable boundary conditions. There
should be no principal problem to combine their techniques with the method presented here. The perhaps most important open question is concerned with the applicability of the method to models with infinite dimensional auxiliary space, which was cautiously excluded here.

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APPENDIX A: COEFFICIENTS OF R-MATRICES

The elements of the $R$-matrix ([4]) are explicitly given by

\[
\rho(u) := \frac{\sinh(\eta(C + 2) + u)}{\sinh(\eta(C + 2) - u)}
\]  
(A1a)

\[
\alpha_0(u) := \frac{1}{[C + 2]_q} \{[C + 1]_q \rho(u) - 1\}
\]  
(A1b)

\[
\beta_0(u) := \frac{1}{[C + 2]_q} \{[2]_q \rho(u) - [C]_q\}
\]  
(A1c)

\[
\gamma_0(u) := \frac{1}{[C + 2]_q} \{\rho(u) - [C + 1]_q\}
\]  
(A1d)

\[
\delta_1(u) := \frac{1}{[C + 2]_q} \{\rho(u)q^{-C+1} + q\}
\]  
(A1e)

\[
\delta_2(u) := \frac{1}{[C + 2]_q} \{\rho(u)q^{C+1} + q^{-1}\}
\]  
(A1f)

\[
\varepsilon_1(u) := \frac{\mu^*}{[C + 2]_q} \{\rho(u)q^{-\frac{C}{2}-1} + q^{\frac{C}{2}+1}\}
\]  
(A1g)

\[
\varepsilon_2(u) := \frac{\mu}{[C + 2]_q} \{\rho(u)q^{\frac{C}{2}+1} + q^{-\frac{C}{2}-1}\}
\]  
(A1h)

\[
\zeta_1(u) := \frac{\kappa^*}{[C + 2]_q} \{\rho(u)q^{-\frac{C+1}{2}} + q^{\frac{C+1}{2}}\}
\]  
(A1i)
\[ \zeta_2(u) := \frac{\kappa}{[C + 2]_q} \{ \rho(u) q^{\frac{C+1}{2}} + q^{-\frac{C+1}{2}} \} \]  

(A1j)

\[ f(u) \text{ and } g(u) \text{ in (9) are defined by} \]

\[ f(u) = \frac{\sinh(\eta(C + C') + u)}{\sinh(\eta(C + C') - u)} \]  

(A2a)

\[ g(u) = \frac{\sinh(\eta(C + C' + 2) - u)}{\sinh(\eta(C + C' + 2) + u)} \]  

(A2b)

Using (5d) and the abbreviations

\[ \alpha = [C + C']_q \]

\[ \beta = [C + C' + 1]_q \]

\[ \gamma = [C + C' + 2]_q \]

\[ \varepsilon = [C']_q q^{\frac{C+C'}{2}+1} - [C]_q q^{\frac{C+C'}{2}-1} \]

\[ \eta = [C]_q q^{\frac{C+C'}{2}+1} - [C']_q q^{\frac{C+C'}{2}-1} \]

the remaining coefficients of (9) can be written as:

\[ r_1 = \frac{\kappa^* \mu^* \kappa' \mu'}{\alpha \beta \gamma} (\gamma q^{C+C'} f(u) + [2]_q \beta + \alpha q^{-C-C'-2} g(u)) \]

\[ r'_1 = \frac{\kappa^* \mu^* \kappa' \mu'}{\alpha \beta \gamma} (\gamma q^{C-C'} f(u) + [2]_q \beta + \alpha q^{C+C'+2} g(u)) \]

\[ r_2 = \frac{\kappa^* \kappa'}{\alpha} \left( q^{\frac{C+C'}{2}} f(u) + q^{\frac{C+C'}{2}} \right) \]

\[ r'_2 = \frac{\kappa^* \kappa'}{\alpha} \left( q^{\frac{C+C'}{2}} f(u) + q^{\frac{C+C'}{2}} \right) \]

\[ r_3 = \frac{\mu \mu'}{\gamma} \left( q^{\frac{C+C'+2}{2}} + q^{\frac{C+C'+2}{2}} g(u) \right) \]

\[ r'_3 = \frac{\mu \mu'}{\gamma} \left( q^{\frac{C+C'+2}{2}} + q^{\frac{C+C'+2}{2}} g(u) \right) \]

\[ r_4 = 1 + \frac{q^{-1}}{\alpha \beta \gamma} \{ [C]_q ([C']_q \gamma f(u) - [C + 1]_q \beta) \]

\[ + [C' + 1]_q ([C + 1]_q \alpha g(u) - [C']_q \beta) \} \]

\[ r'_4 = 1 + \frac{q}{\alpha \beta \gamma} \{ [C]_q ([C']_q \gamma f(u) - [C + 1]_q \beta) \]

\[ + [C' + 1]_q ([C + 1]_q \alpha g(u) - [C']_q \beta) \} \]
\[
\begin{align*}
\frac{r_5}{\alpha \beta \gamma} & = \frac{1}{[C]_q [C + 1]_q \gamma f(u)} - [2]_q [C'q][C + 1]_q \beta \\
& + [C'][C' + 1]_q \alpha g(u) \\
\frac{r_5'}{\alpha \beta \gamma} & = \frac{1}{[C'q][C' + 1]_q \gamma f(u)} - [2]_q [C]_q [C' + 1]_q \beta \\
& + [C][C + 1]_q \alpha g(u) \\
\frac{r_6}{\alpha \beta \gamma} & = \frac{\mu^*}{[C]_q [C + 1]_q \gamma} \left( \frac{C + C'}{2} f(u) - \beta \varepsilon q^{\frac{1}{2}} - [C' + 1]_q \alpha \frac{C + C'}{2} g(u) \right) \\
\frac{r'_6}{\alpha \beta \gamma} & = \frac{\kappa \mu'^*}{[C']_q [C' + 1]_q \gamma} \left( \frac{C + C'}{2} f(u) + \beta \varepsilon q^{-\frac{1}{2}} - [C + 1]_q \alpha \frac{C + C'}{2} g(u) \right) \\
\frac{r_7}{\alpha} & = \frac{1}{[C']_q - [C]_q f(u)} \\
\frac{r'_7}{\alpha} & = \frac{1}{[C]_q - [C']_q f(u)} \\
\frac{r_8}{\alpha \beta \gamma} & = \frac{\kappa^*}{[C]_q [C + 1]_q \gamma} \left( \frac{C + C'}{2} f(u) - \beta \eta q^{\frac{1}{2}} - [C' + 1]_q \alpha \frac{C + C'}{2} g(u) \right) \\
\frac{r'_8}{\alpha \beta \gamma} & = \frac{\mu^*}{[C']_q [C' + 1]_q \gamma} \left( \frac{C + C'}{2} f(u) + \beta \eta q^{-\frac{1}{2}} - [C + 1]_q \alpha \frac{C + C'}{2} g(u) \right) \\
\frac{r_9}{\gamma} & = \frac{1}{[C' + 1]_q - [C + 1]_q g(u)} \\
\frac{r'_9}{\gamma} & = \frac{1}{[C + 1]_q - [C' + 1]_q g(u)} \\
\frac{r_{10}}{\alpha \beta \gamma} & = \frac{1}{[C]_q ([C + 1]_q \beta - [C']_q \gamma f(u)) + [C' + 1]_q ([C']_q \beta - [C + 1]_q \alpha g(u))} \\
\end{align*}
\]

**APPENDIX B: SOME DETAILS ON ABA**

Applying the ansatz (20) to the diagonal elements of \( \hat{T}_{ij}(u) \) using (19) and (16) yields

\[
[\hat{T}_{11}(u) + \hat{T}_{22}(u)] |\lambda_1, \ldots, \lambda_M \rangle F >
\]  

(B1a)
\[
= [\omega_1 (u)]^N \prod_{i=1}^M \frac{1}{c(u - \lambda_i)} \left[ \hat{\tau}^{V_2} (M|u) \right]^{b_1,\ldots,b_M}_{a_1,\ldots,a_M} F^{a_1,\ldots,a_M} \\
\times \hat{C}_{b_1} (\lambda_1) \cdots \hat{C}_{b_M} (\lambda_M) |0\rangle_N \\
+ \sum_{k=1}^M \left[ \hat{\Lambda}_k^{(1,2)} (u; \lambda_1, \ldots, \lambda_M) \right]^{b_1,\ldots,b_M}_{a_1,\ldots,a_M} F^{a_1,\ldots,a_M} \\
\times \hat{C}_{b_k} (u) \prod_{i=1, i \neq k}^M \hat{C}_{b_i} (\lambda_i) |0\rangle_N ,
\]

where \( \hat{\tau}^{V_2} (M|u) \) is an inhomogeneous transfer-matrix obtained according to (2) with \( \delta^{(n)} = \gamma_n \) from (22), and

\[
\hat{T}_{33} (u)|\lambda_1, \ldots, \lambda_M|F > = (-1)^N \prod_{i=1}^M \frac{1}{c(u - \lambda_i)} |\lambda_1, \ldots, \lambda_M|F > \\
+ \sum_{k=1}^M \left[ \hat{\Lambda}_k^{(3)} (u; \lambda_1, \ldots, \lambda_M) \right]^{b_1,\ldots,b_M}_{a_1,\ldots,a_M} F^{a_1,\ldots,a_M} \\
\times \hat{C}_{b_k} (u) \prod_{i=1, i \neq k}^M \hat{C}_{b_i} (\lambda_i) |0\rangle_N ,
\]

The operators \( \hat{C}_i (\lambda_1) \) under the products in (B1) are ordered with the index increasing from left to right factors. Note that only the first terms in equation (B1) will contribute to the eigenvalue, while the following terms are unwanted. Their coefficients \( \hat{\Lambda}_k \) are

\[
\left[ \hat{\Lambda}_k^{(1,2)} (u; \lambda_1, \ldots, \lambda_M) \right]^{b_1,\ldots,b_M}_{a_1,\ldots,a_M} = -[\omega_1 (\lambda_k)]^N \frac{b(u - \lambda_k)}{c(u - \lambda_k)} \prod_{i=1}^M \frac{1}{c(\lambda_k - \lambda_i)} \prod_{j=1}^{k-1} \frac{1}{d(\lambda_j - \lambda_k)} \\
\times \left[ \hat{L}_{cMcM-1} (\lambda_k - \lambda_M) \right]^{b_M}_{a_M} \left[ \hat{L}_{cM-1cM-2} (\lambda_k - \lambda_{M-1}) \right]^{b_{M-1}}_{a_{M-1}} \\
\times \cdots \times \left[ \hat{L}_{c_{k+1}c_k} (\lambda_k - \lambda_{k+1}) \right]^{b_{k+1}}_{a_{k+1}} \\
\times \left( \prod_{l=1}^{k-1} \delta_{b_l}^{a_l} \right) \delta_{a_k}^{c_k} \left[ \delta_{b_k}^{c_1} \delta_{c_1}^{c_M} + \delta_{b_k}^{c_2} \delta_{c_2}^{c_M} \right]
\]

where \( \hat{L}_{ij} (u) \) is an abbreviation for \( \hat{L}^{V_2} (n|u) \) of (22), derived from (22) via (2a), and

\[
\left[ \hat{\Lambda}_k^{(3)} (u; \lambda_1, \ldots, \lambda_M) \right]^{b_1,\ldots,b_M}_{a_1,\ldots,a_M} (B2b)
\]

31
\[
\begin{align*}
&= a(\lambda_k - u) \frac{(-1)^{N+M}}{c(\lambda_k - u)} \prod_{i=1}^{M} \frac{1}{c(\lambda_i - \lambda_k)} \prod_{j=k+1}^{M} d(\lambda_j - \lambda_k) \\
&\times \left[ \hat{S}_k(\lambda_1, \ldots, \lambda_k) \right]^{b_1, \ldots, b_k}_{a_1, \ldots, a_k} \left( \prod_{l=k+1}^{M} \delta_{b_l}^{a_l} \right).
\end{align*}
\]

Here a \( k \)-particle \( S \)-matrix has been defined via \( [19] \)

\[
\left[ \hat{S}_k(\lambda_1, \ldots, \lambda_k) \right]^{b_1, \ldots, b_k}_{a_1, \ldots, a_k} = \delta_{b_k}^{c_k} \delta_{a_k}^{c_k} \prod_{i=1}^{k-1} r_{b_i, c_i, c_i+1} (\lambda_i - \lambda_k)
\]

In \( (B2) \) summation over repeated indices \( c_i = 1, 2 \) is implicit. Applying the ansatz \( (20) \) to \( (23) \) forces the unwanted terms in \( (B1) \) to vanish. These equations can be transformed into \( 6 \)-vertex-type eigenvalue equations \( (24) \) in section III \( [19] \).

\section*{APPENDIX C: DERIVATION OF COMMUTATION RELATIONS}

A few more details on the derivation of \( (33) \) are given: In \( (34c) \) the term \( \propto \hat{T}_{11}^{V_2}(u) \hat{T}_{23}^{V'_2}(v) \) acts non-trivially according to \( (14) \) and \( (31) \). It has to be eliminated by use of

\[
\alpha_0 (u - v) \hat{T}_{11}^{V_2}(u) \hat{T}_{23}^{V'_2}(v) + \epsilon_1 (u - v) \hat{C}_1(u) \hat{T}_{33}^{V'_2}(v) = \rho (u - v) \hat{T}_{23}^{V'_2}(v) \hat{T}_{11}^{V_2}(u)
\]

from \( (33) \), which results into \( (35c) \). Similarly \( (34d) \) can be handled, leading to \( (35c) \). In \( (34d) \) the term \( \propto \hat{T}_{21}^{V_3}\hat{T}_{24}^{V'_3} \) has to be eliminated via the relation

\[
\alpha_0 (u - v) \hat{T}_{21}^{V_3}(u) \hat{T}_{24}^{V'_3}(v) + \epsilon_1 (u - v) \hat{C}_1(u) \hat{T}_{44}^{V'_3}(v) = \alpha_0 (u - v) \hat{T}_{24}^{V'_3}(v) \hat{T}_{21}^{V_3}(u) + \delta_2 (u - v) \hat{T}_{23}^{V'_3}(v) \hat{T}_{22}^{V_2}(u) + \zeta_2 (u - v) \hat{T}_{21}^{V'_3}(v) \hat{T}_{32}^{V_3}(u)
\]

from \( (33) \). According to \( (16) \) and \( (31) \) the term \( \propto \hat{T}_{43}^{V'_3}(v) \hat{C}_2(u) \) also acts non-trivially on \( |0\rangle_N \). It has to be eliminated, using the relations following relations from the set \( (33) \),
\begin{align*}
\varepsilon_2(u - v)\hat{T}^{V_3}_{22}(u)\hat{T}^{V'_4}_{23}(v) \\
+ \gamma_0(u - v)\hat{C}_2(u)\hat{T}^{V'_4}_{33}(v) &= \delta_1(u - v)\hat{T}^{V'_4}_{44}(v)\hat{C}_1(u) \\
&+ \alpha_0(u - v)\hat{T}^{V'_4}_{43}(v)\hat{C}_2(u) \\
&+ \zeta_2(u - v)q\hat{T}^{V'_4}_{41}(v)\hat{T}^{V'}_{33}(u)
\end{align*}

and

\begin{align*}
\alpha_0(u - v)\hat{T}^{V_3}_{22}(u)\hat{T}^{V'_4}_{23}(v) \\
+ \varepsilon_1(u - v)\hat{C}_2(u)\hat{T}^{V'_4}_{43}(v) &= \delta_1(u - v)\hat{T}^{V'_4}_{24}(v)\hat{T}^{V'}_{23}(u) \\
&+ \alpha_0(u - v)\hat{T}^{V'_4}_{23}(v)\hat{T}^{V'}_{22}(u) \\
&+ \zeta_2(u - v)q\hat{T}^{V'_4}_{21}(v)\hat{T}^{V'}_{23}(u)
\end{align*}

Both relations have to be used in order to prevent the appearance of \(\hat{T}^{V_3}_{22}(u)\hat{T}^{V'_4}_{23}(v)\), also acting non-trivially on \(|0\rangle_N\), in the result (35e). Applying the same procedure to (34g) leads to (35f).
REFERENCES

[1] C.N. Yang, Phys. Rev. Lett. 19, 1312-5 (1967).
   C.N. Yang, Phys. Rev. 168, 1920-3 (1968).

[2] R.J.Baxter, Ann. Phys., 70, 323-37 (1972).

[3] V.G. Drinfel’d: “Quantum Groups” in “Proceedings of the International Congress of Mathematicians, Berkeley, 1986.

[4] V. Chari and A. Presley: “A Guide to Quantum Groups”, Cambridge University Press, New York, 1994.

[5] M. Jimbo, Comm. Math. Phys. 102, 537-47 (1986).

[6] Z.-Q. Ma: “Yang-Baxter Equation and Quantum Enveloping Algebras”, World Scientific, Singapore, 1993.

[7] R.J. Baxter: “Exactly solved Models in Statistical Mechanics”, Academic Press, London, 1982.

[8] E.K. Sklyanin, L.A. Takhtajan and L.D. Faddeev, Theoret. Math. Phys. 40, 688-706 (1980).
   L.A. Takhtajan and L.D. Faddeev, Russian Math. Surveys 34, 11-68 (1979).
   L.D. Faddeev, Soviet Scientific Reviews C, 1, 107-55 (1980).
   L.A. Takhtajan: “Introduction to Algebraic Bethe Ansatz”, 175-220 in B.S. Shastry, S.S. Jha and V. Singh (eds.): “Exactly Solvable Problems in Condensed Matter and Field Theory”, Lecture Notes in Physics 242, Springer, Berlin Heidelberg, 1985.

[9] V.E. Korepin, N.N. Bogoliubov and A.G. Izergin: “Quantum Inverse Scattering Method and Correlation Functions”, Cambridge University Press, New York, 1993.

[10] M. Jimbo (ed.): “Yang-Baxter Equation in Integrable System”, World Scientific, Singapore, 1989.
[11] H.A. Bethe, Z. Physik 71, 205-26 (1931).

[12] H. Yamane, preprint q-alg/9603015 (1996).

[13] P.P. Kulish and E.K. Sklyanin, J. Soviet. Math. 19, 1596-620 (1982).

[14] J.F. Cornwell: “Group Theory in Physics, Vol. 3 – Supersymmetry and infinite dimensional Algebras”, Academic Press, London, 1989.

[15] V.G. Kac: “Infinite dimensional Lie Algebras”, 3rd ed., Cambridge University Press, New York, 1990.

[16] J.H.H. Perk and C.L. Schultz, Phys. Lett. 84 A, 407-10 (1981).

[17] J. Gruneberg to be published.

[18] M.D. Gould, K.E. Hibberd, J.R. Links and Y.-Z. Zhang, Phys. Lett. A 212, 156-60 (1995).

[19] P.P. Kulish and N.Y. Reshetikhin, JETP, 80 158-83 (1981)

[20] C.L. Schultz, Physica A, 122, 71-88 (1983).

[21] M. Gaudin, Phys. Lett. A 24, 55-6 (1967).

[22] L.A. Takhtajan, LOMI-Proceedings, 1980, 101, 158-83 (1980).

[23] C.K. Lai, J. Math. Phys., 15, 1675-76 (1974).

[24] B. Sutherland, Phys. Rev. B, 12, 3795-805 (1975).

[25] Ramos and Martins, J. Phys. A, 30, L195 (1997).

[26] Ramos and Martins, Nucl. Phys. B, 474, 678-714 (1996).

[27] P.P. Kulish and E.K. Sklyanin: “Quantum Spectral Transform Method – Recent Developments”, 61-119 in J. Hietarina and C. Montonen (eds.): “Integrable Quantum Field Theories” Lecture Notes in Physics 151, Springer, Berlin Heidelberg, 1981.
[28] A.G. Izergin and V.E. Korepin, Sov. Phys. Dokl. 26, 653-4 (1981).

[29] N.Y. Reshetikhin, Sov. Phys. JETP 57, 691-6 (1983).

[30] J. Gruneberg, to be published.

[31] A. Klümper, Ann. Physik, 1, 540 (1992).
   A. Klümper, Z. Phys. B, 91, 507 (1993).

[32] G. Jüttner, A. Klümper and J. Suzuki, Nucl. Phys., 487, 471-502 (1998).

[33] H.J. De Vega and A. Gonzáles-Ruiz, Nucl. Phys. B, 417, 553-578 (1994).
   A. Gonzáles-Ruiz, Nucl. Phys. B, 424, 468-486.

[34] A. Foerster and M. Karowski, Nucl. Phys. B, 408, 512-534 (1993).