ON CERTAIN QUESTIONS OF THE FREE GROUP AUTOMORPHISMS THEORY

VALERIY G. BARDAKOV AND ROMAN MIKHAILOV

Abstract. Certain subgroups of the groups Aut(Fn) of automorphisms of a free group Fn are considered. Comparing Alexander polynomials of two poly-free groups Cb4 and P4 we prove that these groups are not isomorphic, despite the fact that they have a lot of common properties. This answers the question of Cohen-Pakianathan-Vershinin-Wu from [8]. The questions of linearity of subgroups of Aut(Fn) are considered. As an application of the properties of poison groups in the sense of Formanek and Procesi, we show that the groups of the type Aut(G∗Z) for certain groups G and the subgroup of IA-automorphisms IA(Fn) ⊂ Aut(Fn) are not linear for n ≥ 3. This generalizes the recent result of Pettet that IA(Fn) are not linear for n ≥ 5.

1. Introduction

1.1. The group of basis conjugating automorphisms. Let Fn be a free group of rank n ≥ 2 with a free generator set \( \{x_1, x_2, ..., x_n\} \) and Aut(Fn) the group of automorphisms of Fn. Taking the quotient of Fn by its commutator subgroup F′n, we get a natural homomorphism

\[ \xi : \text{Aut}(F_n) \rightarrow \text{Aut}(F_n/F'_n) = \text{GL}_n(\mathbb{Z}), \]

where GLn(\mathbb{Z}) is the general linear group over the ring of integers. The kernel of this homomorphism consists of automorphisms acting trivially modulo the commutator subgroup F′n. It is called the group of IA-automorphisms and denoted by IA(Fn) (see [2, Chapter 1, § 4]). The group IA(F2) is isomorphic to the group of inner automorphisms Inn(F2), which is isomorphic to the free group F2.

J. Nilsen (for the case n ≤ 3) and W. Magnus (for all n) have shown that (see [2, Chapter 1, § 4]) the group IA(Fn) is generated by the following automorphisms

\[ \varepsilon_{ijk} : \begin{cases} x_i \mapsto x_i[x_j, x_k] & \text{if } k \neq i, j, \\ x_l \mapsto x_l & \text{if } l \neq i, \end{cases} \]

\[ \varepsilon_{ij} : \begin{cases} x_i \mapsto x_{j}^{-1}x_ix_j & \text{if } i \neq j, \\ x_l \mapsto x_l & \text{if } l \neq i. \end{cases} \]

The subgroup of the group IA(Fn) is generated by the automorphisms \( \varepsilon_{ij} \), 1 ≤ i ≠ j ≤ n is called the group of basis conjugating automorphisms. We will denote this group by Cbni. The group Cbn is a subgroup of the group of conjugating automorphisms Cn. (Recall that any automorphism of Cn sends a generator \( x_i \) to an element of the type \( f_i^{-1}x_{\pi(i)}f_i \), where \( f_i \in F_n \), and \( \pi \) is a permutation from the symmetric group S_n.) Clearly, if \( \pi \) is the identity permutation, then the described element lies in the group Cbn.

It is shown by J. McCool [3] that the group of basis conjugating automorphisms Cbni is generated by the automorphisms \( \varepsilon_{ij} \), 1 ≤ i ≠ j ≤ n, defined above has the following relations

\[ \varepsilon_{ij} \varepsilon_{ij} = 1, \quad \varepsilon_{ij} \varepsilon_{ik} = \varepsilon_{jk} \varepsilon_{ik}, \quad \varepsilon_{ij} \varepsilon_{jk} = \varepsilon_{kj} \varepsilon_{ij}. \]

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(we denote different indexes by different symbols):
\[
\varepsilon_{ij}\varepsilon_{kl} = \varepsilon_{kl}\varepsilon_{ij},
\]
\[
\varepsilon_{ij}\varepsilon_{kj} = \varepsilon_{kj}\varepsilon_{ij},
\]
\[
(\varepsilon_{ij}\varepsilon_{kj})\varepsilon_{ik} = \varepsilon_{ik}(\varepsilon_{ij}\varepsilon_{kj}).
\]

It is shown in [6] that the group of conjugating automorphisms \(Cb_n\), \(n \geq 2\) decomposes into a semi-direct product:
\[
Cb_n = D_{n-1} \rtimes (D_{n-2} \rtimes (\ldots \rtimes (D_2 \rtimes D_1)) \ldots),
\]
where the group \(D_i\), \(i = 1, 2, \ldots, n - 1\), is generated by the elements \(\varepsilon_{i+1,1}, \varepsilon_{i+1,2}, \ldots, \varepsilon_{i+1,i}, \varepsilon_{1,i+1}, \varepsilon_{2,i+1}, \ldots, \varepsilon_{i,i+1}\). The elements \(\varepsilon_{i+1,1}, \varepsilon_{i+1,2}, \ldots, \varepsilon_{i+1,i}\) generate a free group of rank \(i\), but the elements \(\varepsilon_{1,i+1}, \varepsilon_{2,i+1}, \ldots, \varepsilon_{i,i+1}\) generate a free abelian group of rank \(i\).

### 1.2. The braid group as a subgroup of \(\text{Aut}(F_n)\).

The braid group \(B_n\), \(n \geq 2\), on \(n\) strands is defined by the generators \(\sigma_1, \sigma_2, \ldots, \sigma_{n-1}\) with defining relators
\[
\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad \text{if} \quad i = 1, 2, \ldots, n - 2,
\]
\[
\sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{if} \quad |i - j| \geq 2.
\]

There exists a natural homomorphism from the braid group \(B_n\) onto the permutation group \(S_n\) sending the generator \(\sigma_i\) to the transposition \((i, i+1)\), \(i = 1, 2, \ldots, n - 1\). The kernel of this homomorphism is called the pure braid group and denoted by \(P_n\). The group \(P_n\) is generated by the elements \(a_{ij}\), \(1 \leq i < j \leq n\) which can be expressed in terms of generators of \(B_n\) as follows:
\[
a_{i,i+1} = \sigma_i^2,
\]
\[
a_{ij} = \sigma_{j-1}\sigma_{j-2}\ldots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\ldots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}, \quad i + 1 < j \leq n.
\]

The pure braid group \(P_n\) is a semi-direct product of the normal subgroup \(U_n\), which is the free group with free generators \(a_{1n}, a_{2n}, \ldots, a_{n-1,n}\) with the group \(P_{n-1}\). Analogously, \(P_{n-1}\) is a semi-direct product of the free subgroup \(U_{n-1}\) with the free generators \(a_{1n-1}, a_{2n-1}, \ldots, a_{n-2,n-1}\) with the subgroup \(P_{n-2}\) etc. Hence the group \(P_n\) has the following decomposition
\[
P_n = U_n \rtimes (U_{n-1} \rtimes (\ldots \rtimes (U_2 \rtimes U_3) \ldots)), \quad U_i \simeq F_{i-1}, \ i = 2, 3, \ldots, n.
\]

The pure braid group \(P_n\) is defined by relations (for \(\nu = \pm 1\))
\[
a_{ik}^{-\nu}a_{kj}a_{ik}^\nu = (a_{ij}a_{kj})^\nu a_{kj} (a_{ij}a_{kj})^{-\nu},
\]
\[
a_{km}^{-\nu}a_{kj}a_{km}^\nu = (a_{kj}a_{mj})^\nu a_{kj} (a_{kj}a_{mj})^{-\nu}, \quad m < j,
\]
\[
a_{im}^{-\nu}a_{kj}a_{im}^\nu = [a_{ij}^{-\nu}, a_{mj}^{-\nu}] a_{kj} [a_{ij}^\nu, a_{mj}^\nu]^{-\nu}, \quad i < k < m,
\]
\[
a_{km}^{-\nu}a_{kj}a_{km}^\nu = a_{kj}, \quad k < i; \ m < j \text{ or } m < k,
\]
where \([a, b] = a^{-1}b^{-1}ab\) is the commutator of elements \(a\) and \(b\).

The braid group \(B_n\) can be embedded into the automorphism group \(\text{Aut}(F_n)\). In this embedding the generator \(\sigma_i, i = 1, 2, \ldots, n - 1\), defines the following automorphism
\[
\sigma_i : \begin{cases} x_i \mapsto x_ix_{i+1}x_i^{-1}, \\
x_{i+1} \mapsto x_i, \\
x_l \mapsto x_l & \text{if } l \neq i, i + 1.\end{cases}
\]
The generator $a_{rs}$ of the pure braid group $P_n$ defines the following automorphism

$$a_{rs} : \begin{cases} 
    x_i \mapsto x_i & \text{if } s < i \text{ or } i < r, \\
    x_r \mapsto x_r x_s x_r^{-1} x_r^{-1}, \\
    x_i \mapsto [x_r^{-1}, x_s^{-1}] x_i [x_r^{-1}, x_s^{-1}]^{-1} & \text{if } r < i < s, \\
    x_s \mapsto x_r x_s x_r^{-1}.
\end{cases}$$

As was shown by E. Artin, the automorphism $\beta$ from $\text{Aut}(F_n)$ belongs to the braid group $B_n$ if and only if $\beta$ satisfies the following two conditions:

1) $\beta(x_i) = a_i^{-1} x_{\pi(i)} a_i$, $1 \leq i \leq n$,

2) $\beta(x_1 x_2 \ldots x_n) = x_1 x_2 \ldots x_n$,

where $\pi$ is a permutation from $S_n$, and $a_i \in F_n$.

The set of automorphisms from $C_n$, which act trivially on the product $x_1 x_2 \ldots x_n$ is exactly the braid group $B_n$. It was shown by A. Savushkina [4], that the braid group $B_n$ intersects with the subgroup $Cb_n$ by the pure braid group $P_n$. Thus the group $P_n$ is a subgroup of the basis conjugating automorphism group $Cb_n$ for all $n \geq 2$. Furthermore, the concordance of decompositions (1) and (2) of groups $Cb_n$ and $P_n$ respectively, takes place: there are corresponding embeddings $U_{i+1} \leq D_i$, $i = 1, 2, \ldots, n - 1$.

1.3. The group $Cb^+_n$. Denote by $Cb^+_n$ the subgroup of the group $Cb_n$, $n \geq 2$, generated by the elements $\varepsilon_{ij}$, $i > j$. Groups $Cb^+_n$ are poly-free groups: it is shown in [14] that there exists the splitting exact sequence:

$$1 \to F_{n-1} \to Cb^+_n \to Cb^+_{n-1} \to 1,$$

where $F_{n-1}$ is the free subgroup in $Cb^+_n$, generated by the elements

$$\varepsilon_{n1}, \varepsilon_{n2}, \ldots, \varepsilon_{nn-1}.$$

Description of the Lie algebra constructed from lower central filtration of the basis conjugating automorphism group, is a non-trivial problem; the methods used in the description of Lie algebras for pure braid groups, do not work in this case. However, in the case of the group $Cb^+_n$, the situation is different. Lie algebras and cohomology rings of groups $Cb^+_n$ were described in [8]. It was also shown in [8] that the group $Cb^+_n$ is isomorphic to the pure braid group $P_n$ for $n = 2, 3$. Groups $Cb^+_n$ and $P_n$ are quite similar. For instance, the groups $Cb^+_n$ and $P_n$ are stably isomorphic, namely, suspensions over their classifying spaces $\Sigma K(Cb^+_n, 1)$ and $\Sigma K(P_n, 1)$ are homotopically equivalent for all $n \geq 1$ (see [8]). The conjecture that $Cb^+_n$ and $P_n$ are isomorphic for all $n \geq 1$ comes naturally (question 1, [8]). One of the main results of this paper is the proof that the groups $Cb^+_n$ and $P_3$ are non-isomorphic (Theorem 1).

1.4. The questions of linearity. As it was shown above, for any $n \geq 2$, there is the following chain of subgroups of the automorphism group $\text{Aut}(F_n)$:

$$P_n \subset Cb_n \subset IA_n(F_n) \subset \text{Aut}(F_n).$$

It was shown in [15], [16] that the braid groups $B_n$ are linear. Hence the pure braid groups $P_n$ are also linear. The situation with groups $\text{Aut}(F_n)$ is as follows. The group $\text{Aut}(F_n)$ is not linear for all $n \geq 3$ (see [10]), however the group $\text{Aut}(F_2)$ is linear (such a representation was constructed, for example, in [11]). A certain non-linear subgroup (called poison group) of $\text{Aut}(F_n)$ was constructed in [10]. The following question rises naturally (see question 15.17 [12]): are the groups of IA-automorphisms $IA(F_n)$ and the groups of basis conjugating automorphisms...
Cb_n linear for n ≥ 2? Recently A. Pettet showed that the group IA(F_n) is not linear for n ≥ 5. In this paper we show that the groups IA(F_n) are not linear for all n ≥ 3 (Theorem 5). Therefore, the complete answer to the question 15.17 from [12] is given.

2. The group Cb_4^+ is not isomorphic to the pure braid group P_4

2.1. Fox differential calculus. Recall the definition and main properties of Fox derivatives [5], chapter 3, [7], chapter 7.

Let F_n be a free group of rank n with free generators x_1, x_2, . . . , x_n. Let φ be an endomorphism of the group F_n. Denote by F_n^φ the image of F_n under the endomorphism φ. Let Z be the ring of integers, ZG the integral group ring of G.

For every j = 1, 2, . . . , n define a map

$$\frac{\partial}{\partial x_j} : \mathbb{Z}F_n \rightarrow \mathbb{Z}F_n$$

by setting

1) \(\frac{\partial x_i}{\partial x_j} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{otherwise}, \end{cases}\)

2) \(\frac{\partial x_i^{-1}}{\partial x_j} = \begin{cases} -x_i^{-1} & \text{for } i = j, \\ 0 & \text{otherwise}, \end{cases}\)

3) \(\frac{\partial (vw)}{\partial x_j} = \frac{\partial w}{\partial x_j} (v)^\tau + w \frac{\partial v}{\partial x_j}, \quad w, v \in \mathbb{Z}F_n,\)

where \(\tau : \mathbb{Z}F_n \rightarrow \mathbb{Z}\) is the trivialization operation, sending all elements of F_n to the identity,

4) \(\frac{\partial}{\partial x_j} (\sum a_g g) = \sum a_g \frac{\partial g}{\partial x_j}, \quad g \in F_n, \ a_g \in \mathbb{Z}.\)

Denote by \(\Delta_n\) the augmentation ideal of the ring \(\mathbb{Z}F_n\), i.e. the kernel of the homomorphism \(\tau\). It is easy to see that for every \(v \in \mathbb{Z}F_n\), the element \(v - v^\tau\) belongs to \(\Delta_n\). Furthermore, there is the following formula

\(v - v^\tau = \sum_{j=1}^{n} \frac{\partial v}{\partial x_j} (x_j - 1),\)  

which is called the main formula of Fox calculus. Formula (3) implies that the elements \(\{x_1 - 1, x_2 - 1, \ldots, x_n - 1\}\) determine the basis of the augmentation ideal \(\Delta_n\).

2.2. Groups \(Cb_4^+\) and \(P_4\). As it was noted in the introduction, the poly-free groups \(Cb_n^+\) and \(P_n\) have similar properties. Clearly, one has

\(Cb_3^+ \simeq P_2 \simeq \mathbb{Z};\)

\(Cb_4^+ \simeq P_3 \simeq F_2 \rtimes \mathbb{Z}.\)

However, the next result shows that the groups \(Cb_4^+\) and \(P_4\) being semidirect products of \(F_3\) with \(F_2 \rtimes \mathbb{Z}\), are not isomorphic.

**Theorem 1.** Groups \(Cb_4^+\) and \(P_4\) are not isomorphic.

**Proof.** It is well-known (see [3]) that the group \(P_4\) decomposes as a direct product of its center generated by a single element \(a_{12}a_{13}a_{23}a_{14}a_{24}a_{34}\) and a subgroup \(H = U_1 \ltimes U_3, \ U_4 = \langle a_{14}, a_{24}, a_{34} \rangle, \ U_3 = \langle a_{13}, a_{23} \rangle\). It is easy to check that the center of the group \(Cb_4^+\) is the infinite cyclic group generated by the element \(\varepsilon_{21}\varepsilon_{31}\varepsilon_{41}\) and the whole group \(Cb_4^+\) is a direct product of its center and a subgroup \(G = D_3^+ \ltimes D_2^+, \ \text{where} \ D_3^+ = \langle \varepsilon_{41}, \varepsilon_{42}, \varepsilon_{43} \rangle, \ D_2^+ = \langle \varepsilon_{31}, \varepsilon_{32} \rangle\). Since
We present these relations in the following form:

\[ \varepsilon_{31}^{-1}\varepsilon_{41}\varepsilon_{31} = \varepsilon_{41}, \quad \varepsilon_{31}^{-1}\varepsilon_{42}\varepsilon_{31} = \varepsilon_{42}, \quad \varepsilon_{31}^{-1}\varepsilon_{43}\varepsilon_{31} = \varepsilon_{41}\varepsilon_{43}\varepsilon_{41}^{-1}, \]
\[ \varepsilon_{32}^{-1}\varepsilon_{41}\varepsilon_{32} = \varepsilon_{41}, \quad \varepsilon_{32}^{-1}\varepsilon_{42}\varepsilon_{32} = \varepsilon_{42}, \quad \varepsilon_{32}^{-1}\varepsilon_{43}\varepsilon_{32} = \varepsilon_{42}\varepsilon_{43}\varepsilon_{42}^{-1}. \]

We present these relations in the following form:

\[ r_{11} = \varepsilon_{41}^{-1}\varepsilon_{31}^{-1}\varepsilon_{41}\varepsilon_{31}, \quad r_{21} = \varepsilon_{42}^{-1}\varepsilon_{31}^{-1}\varepsilon_{42}\varepsilon_{31}, \quad r_{31} = \varepsilon_{43}^{-1}\varepsilon_{31}^{-1}\varepsilon_{43}\varepsilon_{31}\varepsilon_{41}, \]
\[ r_{12} = \varepsilon_{41}^{-1}\varepsilon_{32}^{-1}\varepsilon_{41}\varepsilon_{32}, \quad r_{22} = \varepsilon_{42}^{-1}\varepsilon_{32}^{-1}\varepsilon_{42}\varepsilon_{32}, \quad r_{32} = \varepsilon_{43}^{-1}\varepsilon_{42}^{-1}\varepsilon_{43}\varepsilon_{32}\varepsilon_{42}. \]

We define the homomorphism \( \varphi : G \rightarrow \langle t \rangle \), by setting

\[ \varphi(\varepsilon_{31}) = \varphi(\varepsilon_{32}) = \varphi(\varepsilon_{41}) = \varphi(\varepsilon_{42}) = \varphi(\varepsilon_{43}) = t \]

and extend it to the homomorphism of integral group rings by linearity

\[ \varphi : \mathbb{Z}G \rightarrow \mathbb{Z}\langle t \rangle. \]

Let us find Fox derivatives of the relators \( r_{ij} \), with respect to the symbols \( \varepsilon_{31}, \varepsilon_{32}, \varepsilon_{41}, \varepsilon_{42}, \varepsilon_{43} \) and their \( \varphi \)-images. We have the following

\[
\left( \frac{\partial r_{11}}{\partial \varepsilon_{31}} \right)^\varphi = t^{-2}(t - 1), \quad \left( \frac{\partial r_{11}}{\partial \varepsilon_{41}} \right)^\varphi = t^{-2}(1 - t), \quad \left( \frac{\partial r_{12}}{\partial \varepsilon_{31}} \right)^\varphi = t^{-2}(1 - t), \quad \left( \frac{\partial r_{12}}{\partial \varepsilon_{41}} \right)^\varphi = t^{-2}(1 - t),
\]
\[
\left( \frac{\partial r_{21}}{\partial \varepsilon_{31}} \right)^\varphi = t^{-2}(1 - t), \quad \left( \frac{\partial r_{21}}{\partial \varepsilon_{41}} \right)^\varphi = t^{-2}(1 - t), \quad \left( \frac{\partial r_{22}}{\partial \varepsilon_{31}} \right)^\varphi = t^{-2}(1 - t), \quad \left( \frac{\partial r_{22}}{\partial \varepsilon_{41}} \right)^\varphi = t^{-2}(1 - t),
\]
\[
\left( \frac{\partial r_{31}}{\partial \varepsilon_{31}} \right)^\varphi = t^{-2}(1 - t), \quad \left( \frac{\partial r_{31}}{\partial \varepsilon_{41}} \right)^\varphi = t^{-2}(1 - t), \quad \left( \frac{\partial r_{32}}{\partial \varepsilon_{31}} \right)^\varphi = t^{-2}(1 - t), \quad \left( \frac{\partial r_{32}}{\partial \varepsilon_{41}} \right)^\varphi = t^{-2}(1 - t),
\]
\[
\left( \frac{\partial r_{32}}{\partial \varepsilon_{43}} \right)^\varphi = t^{-3}(1 - t^2), \quad \left( \frac{\partial r_{32}}{\partial \varepsilon_{43}} \right)^\varphi = t^{-3}(1 - t^2).
\]

Derivatives with respect to other generators are zero.

With the obtained values of derivatives we form the Alexander matrix. After elementary transformations of rows and columns and deleting zero rows and columns, we get the diagonal matrix

\[ (1 - t) \text{ diag}(-t^{-2}, -t^{-2}, -t^{-2}, -t^{-2}(1 + t)). \]

Since the Alexander polynomial is defined up to the multiplication with a unit of the ring \( \mathbb{Z}\langle t \rangle \), we get

\[ \Delta_G(t) = (1 - t)^4(1 + t). \]

We now consider the group \( H \). Defining relations of the group \( P_4 \) imply that the group \( H \) can be defined by the following relations

\[
a_{13}^{-1}a_{14}a_{13}^{-1} = a_{34}^{-1}a_{14}a_{34}, \quad a_{13}^{-1}a_{24}a_{13} = [a_{14}^{-1}, a_{34}^{-1}]a_{24}[a_{34}^{-1}, a_{14}^{-1}],
\]
\[
a_{23}^{-1}a_{41}a_{23} = a_{41}, \quad a_{23}a_{24}a_{23}^{-1} = a_{34}^{-1}a_{24}a_{34},
\]
\[
a_{13}^{-1}a_{34}a_{13} = a_{14}a_{34}a_{14}^{-1}, \quad a_{23}^{-1}a_{34}a_{23} = a_{24}a_{34}a_{24}^{-1}.
\]
These relations can be presented in the following form
\[ q_{11} = a_{13}a_{14}a_{13}^{-1}a_{14}^{-1}a_{34}, \quad q_{21} = a_{13}^{-1}a_{24}a_{13}a_{14}a_{34}^{-1}a_{14}^{-1}a_{34}^{-1}a_{24}a_{14}a_{34}^{-1}a_{14}^{-1}, \]
\[ q_{12} = a_{14}^{-1}a_{23}a_{14}a_{23}, \quad q_{22} = a_{23}a_{24}^{-1}a_{23}^{-1}a_{24}^{-1}a_{34}, \]
\[ q_{31} = a_{13}^{-1}a_{34}a_{13}a_{34}^{-1}a_{14}^{-1}, \quad q_{32} = a_{23}^{-1}a_{34}a_{24}a_{34}^{-1}a_{24}^{-1}. \]

As above, we define the homomorphism \( \varphi \) from the group \( H \) to the infinite cyclic group \( \langle t \rangle \), by setting
\[ \varphi(a_{13}) = \varphi(a_{23}) = \varphi(a_{14}) = \varphi(a_{24}) = \varphi(a_{34}) = t, \]
and extend it by linearity to the homomorphism of group rings
\[ \varphi : \mathbb{Z}H \rightarrow \mathbb{Z}\langle t \rangle. \]

We compute the Fox derivatives of the relators \( q_{ij} \), written above, with respect to the variables \( a_{13}, a_{23}, a_{14}, a_{24}, a_{34} \) and find their images under the homomorphism \( \varphi \). We get the following identities (the derivatives of all other relators can be written analogously)
\[
\left( \frac{\partial q_{11}}{\partial a_{13}} \right)^\varphi = 1 - t, \quad \left( \frac{\partial q_{11}}{\partial a_{14}} \right)^\varphi = t^{-1}(t^2 - 1), \quad \left( \frac{\partial q_{11}}{\partial a_{34}} \right)^\varphi = t^{-1}(1 - t),
\]
\[
\left( \frac{\partial q_{21}}{\partial a_{13}} \right)^\varphi = t^{-1}(t - 1), \quad \left( \frac{\partial q_{21}}{\partial a_{14}} \right)^\varphi = -(1 - t)^2, \quad \left( \frac{\partial q_{21}}{\partial a_{24}} \right)^\varphi = t^{-1}(1 - t),
\]
\[
\left( \frac{\partial q_{21}}{\partial a_{34}} \right)^\varphi = (t - 1)^2, \quad \left( \frac{\partial q_{32}}{\partial a_{24}} \right)^\varphi = t^{-1}(t - 1), \quad \left( \frac{\partial q_{32}}{\partial a_{24}} \right)^\varphi = t - 1,
\]
\[
\left( \frac{\partial q_{32}}{\partial a_{34}} \right)^\varphi = t^{-1}(1 - t^2).
\]

We form the matrix from the values of the calculated Fox derivatives. After elementary transformations over rows and columns, we get the diagonal matrix
\[(1 - t) \text{ diag}(1, -t^{-2}, 1, t^{-2} + t^{-1} + 1).\]

The Alexander polynomial of the group \( H \) is equal to
\[ \Delta_H(t) = (1 - t)^4(t^2 + t + 1). \]

Thus the Alexander polynomials of the groups \( G \) and \( H \) are different, and hence, the groups \( G \) and \( H \) are not isomorphic (see, for example [7] for the proof that Alexander invariants, in particular the Alexander polynomial, are invariants of a group and do not depend on a given group presentation).

3. ISOMORPHISM PROBLEM FOR A CERTAIN CLASS OF LIE ALGEBRAS

3.1. A class of Lie algebras. The last section gives a motivation for finding methods for proving that two given subgroup of automorphism groups are not isomorphic. In this section we will consider a similar question.

Consider a homomorphism
\[ \phi : F_n \rightarrow IA(F_k), \]
where \( n, k \geq 2 \). We can form a natural semi-direct product
\[ G_\phi := F_k \rtimes F_n, \]
and ask whether \( G_{\phi_1} \) is isomorphic to \( G_{\phi_2} \) for two different homomorphisms \( \phi_1, \phi_2 : F_n \rightarrow IA(F_k) \).
For a given group $G$, denote by $L(G)$ the Lie algebra constructed from the lower central series filtration:

$$L(G) := \bigoplus_{i \geq 1} \gamma_i(G)/\gamma_{i+1}(G),$$

$$\gamma_1(G) = G, \quad \gamma_{i+1}(G) = [\gamma_i(G), G].$$

Here we will consider the class of Lie algebras $L(G_\phi)$ for different homomorphisms $\phi$ and prove that some Lie algebras from this class are not isomorphic.

Since $F_\text{in}$ acts trivially on $H_1(F_k)$, there is the following exact splitting sequence of Lie algebras:

$$0 \rightarrow L(F_k) \rightarrow L(G_\phi) \rightarrow L(F_\text{in}) \rightarrow 0$$

by [17].

3.2. Scheuneman’s invariants. Let $k$ be a field and $L$ a finitely-generated Lie algebra of nilpotence class two. Let us choice a basis of $L$:

$$\{x_1, \ldots, x_n, y_1, \ldots, y_r\}$$

such that $\{y_1, \ldots, y_r\}$ is a basis of $[L, L]$. Let $U(L)$ be a universal enveloping algebra of $L$. Then $U(L)$ can be identified with a polynomial algebra over $k$ with non-commutative variables $x_1, \ldots, x_n, y_1, \ldots, y_r$ and relations

$$x_i y_j = y_j x_i, \text{ for all } i, j;$$

$$y_i y_j = y_j y_i, \text{ for all } i, j;$$

$$x_i x_j - x_j x_i = \text{linear combination of } y_k \text{'s;}$$

Consider an alternative sum

$$I(x_1, \ldots, x_n) = \sum_{\sigma=(i_1, \ldots, i_n) \in S_n} (-1)^{|\sigma|} x_{i_1} \ldots x_{i_n},$$

where $|\sigma|$ denotes the sign of the permutation $\sigma$.

Two polynomials $f(z_1, \ldots, z_m)$ and $g(z_1, \ldots, z_m)$ are $k$-equivalent if

$$f(z_1, \ldots, z_m) = a g(z'_1, \ldots, z'_n),$$

where $a \in k, a \neq 0$, and

$$z'_j = \sum b_{ij} z_j, \quad b_{ij} \in k, \quad \det(b_{ij}) \neq 0.$$

The following theorem is due to Scheuneman:

**Theorem 2.** [13] The $k$-equivalence class of $I(x_1, \ldots, x_n)$ is an invariant of $k$-isomorphism of the Lie algebra $L$.

For $k$-equivalent polynomials $f(z_1, \ldots, z_m)$ and $g(z_1, \ldots, z_m)$ and $z'_i = \sum b_{ij} z_j, \quad b_{ij} \in k, \quad \det(b_{ij}) \neq 0$, such that $f(z_1, \ldots, z_m) = a g(z'_1, \ldots, z'_n)$,

$$\text{Hes}(f)(z_1, \ldots, z_m) = a^n (\det(b_{ij}))^2 \text{Hes}(g)(z'_1, \ldots, z'_m),$$

where $\text{Hes}(f)(z_1, \ldots, z_m)$ is the Hessian $\det\left(\frac{\partial^2 f}{\partial z_i \partial z_j}\right)$ (see, for example, Lemma 8 [13]).
3.3. Non-isomorphic Lie algebras. Consider the following homomorphism $\phi_\alpha$ of a free group $F_3$ with basis $\{u_1, u_2, u_3\}$ to the automorphism group of a free group $F_3$ with basis $\{t_1, t_2, t_3\}$:

$$
\phi_\alpha : u_1 \mapsto \begin{cases} 
t_1 & \text{for } t = 2, 3, \\
t_i & \text{for } t = 1, 2,
\end{cases}
$$

$$
\phi_\alpha : u_2 \mapsto \begin{cases} 
t_2 & \text{for } t = 1, 3, \\
t_i & \text{for } t = 1, 2,
\end{cases}
$$

$$
\phi_\alpha : u_3 \mapsto \begin{cases} 
t_3 & \text{for } t = 1, 3, \\
t_i & \text{for } t = 1, 2,
\end{cases}
$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$,

$$
\alpha_1 \in \{[t_1, t_2], [t_2, t_3], [t_1, t_3]\},
$$

$$
\alpha_2 \in \{[t_2, t_1], [t_2, t_3], [t_1, t_3]\},
$$

$$
\alpha_3 \in \{[t_1, t_2], [t_3, t_2], [t_3, t_1]\}.
$$

Lie algebras $L(G_\alpha)$ which correspond to the groups $G_\alpha$ can be described as Lie algebras with generators $t_1, t_2, t_3, u_1, u_2, u_3$ and defining relations:

$$
[t_1, u_1] = \alpha_1, \ [t_2, u_2] = \alpha_2, \ [t_3, u_3] = \alpha_3,
$$

$$
[t_2, u_1] = [t_3, u_1] = [t_1, u_2] = [t_3, u_2] = [t_1, u_3] = [t_2, u_3] = 0.
$$

Consider the quotient

$$
L_\alpha := L(G_\alpha) / [L(G_\alpha), L(G_\alpha), L(G_\alpha)].
$$

Clearly, $L_\alpha$ is 12-dimensional algebra with basis

$$
\{t_1, t_2, t_3, u_1, u_2, u_3, y_1, y_2, y_3, y_4, y_5, y_6\},
$$

where $y_1 = [t_1, t_2], y_2 = [t_1, t_3], y_3 = [t_2, t_3], y_4 = [u_1, u_2], y_5 = [u_1, u_3], y_6 = [u_2, u_3]$. Consider the polynomial $I(t_1, t_2, t_3, u_1, u_2, u_3)$. The direct computation gives the following expression

$$
I(t_1, t_2, t_3, u_1, u_2, u_3) = [t_1, t_2][t_3, u_3][u_1, u_2] - [t_1, t_3][t_2, u_2][u_1, u_3] + [t_1, u_1][t_2, t_3][u_2, u_3] - [t_1, u_1][t_2, u_2][t_3, u_3]
$$

$$
= y_1 y_4 \alpha_3 - y_2 y_5 \alpha_2 + y_3 y_6 \alpha_1 - \alpha_1 \alpha_2 \alpha_3 = P_{(\alpha_1, \alpha_2, \alpha_3)}(y_1, \ldots, y_6).
$$

By Theorem 2 we have

$$
\{k - \text{equivalence classes of cubic forms } P_{(\alpha_1, \alpha_2, \alpha_3)} \} \subseteq \{k - \text{isomorphism classes of } G_\alpha \}
$$

Consider the following cases:

1) Let $a_1 = (\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_1 = [t_2, t_3], \alpha_2 = [t_1, t_3], \alpha_3 = [t_1, t_2]$. Then

$$
P_{a_1}(y_1, \ldots, y_6) = y_1^2 y_4 - y_2^2 y_5 + y_3^2 y_6 - y_1 y_2 y_3,
$$

$$
Hes(P_{a_1})(y_1, \ldots, y_6) = 64y_1^2 y_2^2 y_3^2.
$$

2) Let $a_2 = (\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_1 = [t_2, t_3], \alpha_2 = [t_2, t_3], \alpha_3 = [t_1, t_2]$. Then

$$
P_{a_2}(y_1, \ldots, y_6) = y_1^2 y_4 - y_2 y_5 y_3 + y_3^2 y_6 - y_1 y_3^2,
$$

$$
Hes(P_{a_2}) = 16y_1^2 y_3^4.
$$

3) Let $a_3 = (\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_1 = [t_2, t_3], \alpha_2 = [t_2, t_3], \alpha_3 = [t_1, t_2]$. Then

$$
P_{a_3}(y_1, \ldots, y_6) = y_1 y_4 y_3 - y_2 y_5 y_3 + y_3^2 y_5 - y_3^3,
$$

$$
Hes(P_{a_3}) = 4y_3^6.$$

Clearly, all these three cases define non-$k$-equivalent polynomials. Hence, the corresponding groups and Lie algebras are not isomorphic.

**Proposition 1.** The Lie algebras $L(G_{a_1}), L(G_{a_2}), L(G_{a_3})$ are pairwise non-isomorphic.

**Remark.** The invariant polynomials for the two-step nilpotent quotients of Lie algebras of the groups $Cb^4_1/Z(Cb^4_1)$ and $P_1/Z(P_1)$ define cubic forms which lie on Hilbert’s null-cone, hence they do not differ from each other by an invariant of a Hessian type.$^1$

4. ON NON-LINEARITY OF CERTAIN AUTOMORPHISM GROUPS

4.1. Poison group. Let $G$ be a group. Consider the group $\mathcal{H}(G)$, defined as an HNN-extension:

$$\mathcal{H}(G) = \langle G \times G, t \mid t(g, g)t^{-1} = (g, 1), g \in G \rangle.$$  

It is easy to see that the group $\mathcal{H}(G)$ can be presented as a semi-direct product

$$\mathcal{H}(G) \simeq (G \ast Z) \times G,$$

where the action of $G$ on $G \ast Z = G \ast \langle t \rangle$ is defined by

$$g : g_1 t^{e_1} g_2 t^{e_2} \cdots g_k t^{e_k} \mapsto g_1 (tg_1^{-1})^{e_1} g_2 (tg_1^{-1})^{e_2} \cdots g_k (tg_1^{-1})^{e_n}, \quad g, \ g_i \in G, \ e_i \in Z.$$  

For example, if $G = Z$, we get the following group

$$\mathcal{H}(G) = \langle a, b \mid [a, [b, a]] = 1 \rangle.$$  

It follows from elementary Tietze moves that

$$H(Z) = \langle a, a', t \mid [a, a'] = 1, \ taa't^{-1} = a \rangle = \langle a, a', t \mid [a, a'] = 1, \ a' = a^{-1}t^{-1}a = [a, t] \rangle = \langle a, t \mid [a, [a, t]] = 1 \rangle,$$

where $a = (a, 1), \ a' = (1, a) \in G \times G$.

Denote by $N_{AF}$ the class of nilpotent-by-abelian-by-finite groups. The following result due to Formanek and Procesi presents a way of constructing non-linear groups.

**Theorem 3.**$^1$ If $G \notin N_{AF}$, then the group $\mathcal{H}(G)$ is non-linear.

The simplest example of a group which does not lie in the class $N_{AF}$, clearly, is a free non-cyclic group. Thus, for $G = F_2$, the group $\mathcal{H}(F_2)$, called a poison group, is non-linear. This fact plays an important role in the proof of non-linearity of groups $Aut(F_n)$ for $n \geq 3$; the poison group is a subgroup in $Aut(F_n), \ n \geq 3$. This statement can be generalized.

**Theorem 4.** Let $G \notin N_{AF}$, then $Aut(G \ast Z)$ is non-linear.

**Proof.** We will realize the group $\mathcal{H}(G)$ as a subgroup in $Aut(G \ast Z)$ and the statement will follow from Theorem 3. Elements of the subgroup $G \times G$ in $\mathcal{H}(G)$ we will denote as $(g, g')$, i.e. we put dash for elements from the second copy of $G$. The group $G \ast Z$ we will describe in terms of generators $g \in G$ and a free generator $t$. Consider the homomorphism

$$f : \mathcal{H}(G) \to Aut(G \ast Z),$$

given by setting

$$f : g \mapsto i_g, \ g \in G, \ t \mapsto i_t, \ g' \mapsto s_{g'}, \ g' \in G.$$  

where $i_g$ is the conjugation by $g$, $i_t$ is the conjugation by $t$, $s_{g'}$ ($g' \in G$) is the automorphism of $G \ast Z$ acting trivially on $G$ and sending the element $t$ to the element $tg^{-1}$. It can be checked that $f$ is a group homomorphism. Every element of the group $\mathcal{H}(G)$ can be written without

$^1$Authors thank V.L. Popov for helping analyzing these cubic forms and for this remark.
'dash' elements, since \( tgg't^{-1} = g \) and, therefore, \( g' = [g, t] \). Hence, in the case of the existence of a non-trivial kernel of \( f \), there is an element of \( G \ast \mathbb{Z} \), acting trivially by conjugation, i.e. lying in the center of \( G \ast \mathbb{Z} \). However, any non-trivial free product has a trivial center, therefore, \( f \) is a monomorphism. Thus \( \mathcal{H}(G) \) is a subgroup of \( \text{Aut}(G \ast \mathbb{Z}) \). Theorem 3 implies that the group \( \text{Aut}(G \ast \mathbb{Z}) \) is non-linear.

Clearly, one can consider different embeddings of groups \( \mathcal{H}(G) \) in corresponding automorphism groups. Consider the case \( G = F_2 \).

Let \( F_3 \) be a free group with basis \( x_1, x_2, x_3 \) and \( a_1, a_2, a_3 \) some elements of \( F_3 \), such that the subgroup \( \langle a_1, a_2, a_3 \rangle \) is free of rank 3. Define automorphisms \( \alpha_i, i = 1, 2, 3 \) of the group \( F_3 \) as conjugation with \( \alpha_i \).

Theorem 5. The group \( IA(F_3) \) contains a subgroup isomorphic to \( \mathcal{H}(F_2) \), and hence, \( IA(F_3) \) is not linear.

Observe also that the poison group \( \mathcal{H}(F_2) \) is residually finite. It follows from Baumslag's theorem which states that every finitely generated subgroup of an automorphism group of a residually finite group is itself residually finite. Also observe that the non-linearity of the poison group can be used for construction of other non-linear groups given by commutator relations. For example, the group \( \mathcal{H}(F_2) \) contains the following normal subgroup of index 2:

\[
H = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_3] = [x_2, x_4] = [x_5, x_3] = 1, [x_1 x_3, x_2] = [x_2 x_4, x_1] = [x_1 x_5, x_3, x_4] = [x_2 x_5, x_4, x_3] = 1 \rangle,
\]

which is of course non-linear.

5. Questions

(1) Describe the Lie algebra of the group \( Cb_n, n \geq 2 \).
(2) Let \( L_n \) be a free Lie algebra with \( n \) generators, \( n \geq 3 \). Does the group \( \text{Aut}(L_n) \) contain the poison group as a subgroup?
(3) Are the groups \( Cb^+_n \) linear for \( n \geq 3 \)?
(4) Define the chain of subgroups
\[ \text{Aut}(F_n) = IA^1_n \geq IA^2_n \geq IA^3_n \geq \ldots, \]
where \( IA^k_n, \ k \geq 1 \) is the subgroup of \( \text{Aut}(F_n) \), which consists of automorphisms acting trivially modulo the \( k \)-th term of the lower central series of \( F_n \). This chain was introduced in [1]. For which \( k \geq 3, n \geq 3 \) the groups \( IA^k_n \) are non-linear?

(5) Do the groups \( Cb_n \) contain the poison group as a subgroup for \( n \geq 3 \)?

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