Gravitational time dilation, free fall, and matter waves

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We demonstrate that the de Broglie wave of a particle in a gravitational field turns towards the region of lower gravitational potential, causing the particle to fall. This turning is caused by clocks running slower in the smaller potential. We use the analogy of ocean waves that are slower in shallower water and turn towards beaches. This approach implies that the motion is along a geodesic and explains the free fall qualitatively and quantitatively with only elementary algebra.

I. INTRODUCTION

Bodies fall because matter waves refract due to the gravitational time dilation — this is the new interpretation we present here. That the free fall is caused by the gravitational time dilation has been demonstrated, in a more complicated way, in a beautiful paper by Roy Gould. He points out that although Einstein’s model of gravity predicts distortions of both time and space near massive bodies, ordinary objects travel primarily through time and their motion is mainly influenced by time dilation. In a region of lower gravitational potential, time flows more slowly.

Before continuing, we should make this statement more precise. An observer far away from massive bodies assigns positions and times to all events. From the point of view of this observer, a clock placed near a massive body runs slow, and this effect is more pronounced for clocks in a smaller (more negative) gravitational potential.

For example, two clocks near Earth’s surface, separated by height \( h \), tick at different rates. If the average time measured by them is \( t \), the upper clock measures more time by \( \Delta t \) (see Section II),

\[
\Delta t = \frac{gh}{c^2}t,
\]

where \( g \simeq 9.8 \text{ m/s}^2 \) is the gravitational acceleration and \( c \simeq 3 \cdot 10^8 \text{ m/s} \) is the speed of light.

This phenomenon is so important that it has rightly become the subject of stories for young children. In Ref. 1 it is qualitatively explained with the analogy to air travel along great circles, contrasted with straight lines on maps. For a quantitative description of the trajectory, the Schwarzschild metric is applied to determine the shape of the geodesic. This analysis is further developed in Ref. 3.

We propose a simpler approach using de Broglie matter waves, as illustrated in Figure 1. We shall present our argument in terms of wave packets in Section III. Here we explain its gist.

![Figure 1. Trajectory of a freely falling particle in spacetime (curved line). The de Broglie wave front (tilted solid line) changes direction in spacetime because time flows faster at a higher altitude. The horizontal dashed line, labeled \( c\Delta t \), is the extra displacement of the de Broglie wave at \( z = h \) in the time dimension relative to the displacement in time at \( z = 0 \). The tilted arrow indicates the velocity which, in spacetime, is a direction (characterized by the ratio \( dz : dt \)). This refraction is analogous to ocean waves turning towards a beach: they travel slower in shallower water.](image-url)

Start with a particle initially at rest at a height \( z = h \) above Earth’s surface, with \( h \) small so that the gravitational acceleration can be assumed to be constant in the range \( 0 \leq z \leq h \). Throughout this paper we are interested only in the motion in the vertical \( z \) direction. Also, we assume that the time flow rate differences are tiny and make corresponding approximations of the type \( (1 + \delta)^{-1} \simeq 1 - \delta \).

If there were no gravity, the particle would remain at rest at \( z = h \). In spacetime, the particle would move only through time, with its de Broglie wave oscillating as \( \exp(-imc^2t/\hbar) \), where \( m \) denotes the particle’s mass.

Consider now the effect of gravity on the de Broglie wave (see Figure 1). Since the local time flows faster at larger altitudes, de Broglie wave tilts and the particle starts moving through space towards smaller values of \( z \) (it falls).
The slope of the trajectory plotted in Figure 1 is \(\tan \alpha = -\frac{\Delta h}{c \Delta t} = \frac{v}{c}\) where \(v\) is the vertical speed of the particle. Because the velocity is perpendicular to the wave front, \(\alpha\) is also the angle between the wave front and the vertical, as shown in Fig. 1, and \(\tan \alpha\) is the ratio of \(c\) times the extra time elapsed at \(h\) to the distance \(h\), thus

\[
\frac{v}{c} = \frac{c \Delta t}{h} = \frac{cgh}{c^2h} \Rightarrow v = gt.
\]

(2)

The tilting (refraction) of the de Broglie wave reproduces the free fall kinematics.

The diagram in Figure 1 is inspired by Figure 6 in Ref. 6. That very pedagogical paper does not, however, use de Broglie waves; its reasoning is classical, based on Ref. 1. In that approach, the trajectory follows from the postulate that the particle moves on a geodesic.

In our approach the refraction of the de Broglie wave naturally determines the trajectory. Since time flows slower closer to Earth’s surface, de Broglie waves evolve more slowly there and their front turns towards the surface, just like the ocean waves turn towards a beach because they propagate more slowly in shallower water. We do not need to use the notions of a metric tensor or of a geodesic, let alone calculate its shape, creating significant conceptual, technical, and pedagogical simplification.

Eq. (1) is derived in Section II. Section III repeats the above discussion of de Broglie wave refraction slightly more rigorously, using wave packets. Application of Eq. (1) to the twin paradox is described in Section IV. We conclude in Section V. Appendix A summarizes Einstein’s 1907 derivation of Eq. (1). In Appendix B we determine the momentum evolution of a wave packet by examining the non-relativistic limit of the Klein-Gordon equation.

A companion 3-minute film presents the main idea and explains how the gravitational time dilation causes bodies to fall.

II. GRAVITATIONAL TIME DILATION

A. Gravitational time dilation from red shift

In this Section we derive Eq. (1) by considering the energy a photon gains when falling in a gravitational field. Einstein’s original derivation using the Lorentz transformation is summarized in Appendix A.

Consider a model clock consisting of a charged harmonic oscillator with frequency \(\nu_1\), placed in the Earth’s gravitational field, at point 1 where the gravitational potential is \(V_1\). When photons emitted by the oscillating charge arrive at another point 2 with the gravitational potential \(V_2\), conservation of energy requires their frequency to change to \(\nu_2\),

\[
\nu_1 \left(1 + \frac{V_1}{c^2}\right) = \nu_2 \left(1 + \frac{V_2}{c^2}\right).
\]

(3)

All processes occurring at point 1 with time intervals \(\Delta t_1\) are observed from point 2 at intervals \(\Delta t_2\) such that, according to Eq. (3),

\[
\frac{\Delta t_2}{\Delta t_1} = \frac{\nu_1}{\nu_2} = \frac{1 + \frac{V_2}{V_1}}{1 + \frac{V_1}{c^2}}.
\]

(4)

Near Earth’s surface \(V_i = gh_i\). Assuming \(h_1 = 0\), \(h_2 = h\),

\[
\frac{\Delta t_2}{\Delta t_1} = 1 + \frac{gh}{c^2}.
\]

(5)

Denoting \(\Delta t_2 - \Delta t_1 = \Delta t\) and \(\Delta t_1 = t\) we reproduce Eq. (1), \(\Delta t = ght/c^2\).

B. Experimental verification of \(\Delta t\)

Many experiments have demonstrated relativistic effects on clocks, including recently with optical lattice clocks on the Tokyo Skytree tower. Clocks placed on airplanes, rockets, and satellites have also been used.

Especially valuable from a pedagogical point of view is Project GREAT, conducted by Tom Van Baak and his family; it is exceptionally well documented.

Van Baak purchased three surplus portable cesium atomic clocks on eBay and converted his minivan into a mobile time laboratory. Before the clocks were taken to a higher altitude, their readings were compared against reference
atomic clocks. After three days, the portable clocks were transported by car by the Van Baak family 1340 meters up Mount Rainier. Measurements were collected for 40 hours.

After returning, the clocks ran for another three days while being compared with reference clocks that had remained at ground level. The average extra time counted in the three clocks while up on the mountain for two days was 23 nanoseconds (see Figure 2), in good agreement with Eq. (1),

\[ \Delta t = \frac{g \hbar}{c^2} = \frac{9.8 \cdot 1340}{9 \cdot 10^{16}} \cdot 40 \cdot 3600 \text{ s} = 21 \text{ ns.} \]  

(6)

![Figure 2. Average readings of the atomic clock ensemble taken on the trip minus the readings of the reference clocks.](image)

The discontinuity between values before and after the mountain trip confirms gravitational time dilation, Eq. (1).

III. WAVE PACKET NEAR EARTH

Suppose that at \( t = 0 \) the wave function of a particle has a Gaussian shape in \( z \), of width \( \sigma \) and center at \( z = 0 \),

\[ \psi(z, t = 0) = \frac{1}{\sqrt{\sigma \sqrt{\pi}}} \exp\left(-\frac{z^2}{2\sigma^2}\right). \]  

(7)

Decompose \( \psi \) into normalized momentum eigenstates,

\[ \phi(k) = \int_{-\infty}^{\infty} dz \psi(z) e^{-ikz} = \sqrt{2\sigma \sqrt{\pi}} e^{-\frac{k^2\sigma^2}{2}}. \]  

(8)

A state with a wave vector \( k \) has the energy \( E_k = \sqrt{m^2c^2 + h^2k^2} \simeq mc^2 + \frac{h^2k^2}{2m} \). The wave function evolves as (see Appendix B)

\[ \psi(z, t') = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \phi(k) e^{ikz} \exp\left(-i\frac{E_k t'}{\hbar}\right). \]  

(9)

So far this has been a standard analysis. Now, notice that \( t' \) is a function of \( z \):

\[ t' = t \left(1 + \frac{gz}{c^2}\right), \]  

(10)

\[ \psi(z, t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \phi(k) e^{ikz} \exp\left[-i\frac{E_k t'}{\hbar} \left(1 + \frac{gz}{c^2}\right)\right] \]  

(11)

\[ = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \phi(k) \exp\left[i z \left(k - \frac{E_k gt'}{\hbar c^2}\right)\right] \exp\left(-i\frac{E_k t}{\hbar}\right). \]  

(12)
We see that the wave packet is centered not around zero momentum but around the time-dependent value

\[ p(t) = \hbar k(t) = \frac{E_k}{c^2} \approx -mg, \tag{13} \]

where we have approximated the energy by its rest value, \( E_k \approx mc^2 \), since the additional, \( k \)-dependent kinetic energy gives a correction suppressed by inverse \( c^2 \) and is negligible for non-relativistic motion.

The value of momentum in Eq. (13) corresponds to the speed

\[ v = gt, \tag{14} \]

as expected in the uniformly accelerated motion.

![Figure 3](image-url)

Figure 3. Two-dimensional spatial trajectory of a freely falling projectile (curved line). Similarly to Fig. 1, the de Broglie wave front (tilted solid line) changes direction, but here it is shown in space rather than in spacetime, in order to make it easier to see the analogy with the refraction of ocean waves.

We derive this result once more, in a manner that is less abstract. Consider now a two-dimensional projectile motion, as shown in Fig. 3. The projectile has a constant horizontal component of velocity \( v_x \). The phase velocity of de Broglie waves corresponding to this motion is \( c^2/v_x \). Thus the horizontal side of the large triangle in Fig. 3 is \( c^2\Delta t/v_x \): an extra distance by which the wave advances during the extra time \( \Delta t \) elapsed at the higher altitude. Using similar triangles, we relate the slope of the trajectory to the sides of the large triangle,

\[ \frac{dz}{dx} = \tan \alpha = \frac{\frac{c^2\Delta t}{v_x}}{h}. \tag{15} \]

On the other hand, \( dx = v_x dt \), so that the vertical component of the projectile’s velocity is

\[ \frac{dz}{dt} = \frac{c^2\Delta t}{h} = gt, \tag{16} \]

as before in Eqs. (2) and (14).

**IV. JUMPING ONTO A TRAIN AND THE TWIN PARADOX**

Here we show how Eq. (1) helps to understand the twin paradox. One twin stays at rest; the other sets out to travel with a large velocity \( v \). Each of them sees the other one moving and thus each deduces that the sibling’s clock is running slow. Yet when the twins reunite, the one who traveled turns out to have aged less. Obviously, the symmetry is broken by the traveling twin having to accelerate to reverse the direction of velocity and return. Yet it may be hard to fathom that the extra aging of the twin at rest happens only during that acceleration event.

It is easier to consider a simpler, more localized situation: a railway car of proper length \( L \) is passing a station with speed \( v \). It is equipped with one clock at the front and one at the rear. The clocks are synchronized in the car frame but from the point of view of a ground observer, the rear clock is ahead by

\[ \Delta t = \frac{Lv}{c^2}; \tag{17} \]

see a lucid discussion on p. 513 in Ref. 16.

Imagine that the ground observer decides to get on the train. After sprinting in the direction of the train’s motion (therefore towards the front clock and away from the rear one), the observer sees both clocks showing the same time. During the acceleration, the front clock must have been running faster than the rear one.
Figure 4. The rear clock on a moving train is ahead of the front clock, when seen by a person standing on the ground. In the frame of an accelerating person jumping onto the train, the front clock runs faster than the rear clock. For a person on the train, the clocks show the same time.

Denote the average acceleration of the sprinting observer by $g$, for consistency with Eq. (1). In order to reach the train’s speed $v$, the duration of the spurt is $t = v/g$.

Suppose the observer jumps on the train a distance $(1 - x)L$ from the front, $xL$ from the rear of the car, as shown in Figure 4. According to Eq. (1), the front clock registers an extra time $\Delta t_F = g(1 - x)Lt/c^2$ and the rear clock lags behind by $\Delta t_R = gxLt/c^2$. The sum of these two effects gives the net advance of the front clock,

$$\Delta t = \Delta t_F + \Delta t_R = \frac{gLt}{c^2} = \frac{Lv}{c^2},$$

(18)

which exactly cancels the previous difference between the clocks, Eq. (17). The same mechanism resolves the twin paradox.

V. SUMMARY

The gravitational time dilation effect on the Global Positioning System (GPS) is often portrayed as the most practically important effect of general relativity. We hope that free fall will now inherit this distinction.

Since the altitude of GPS satellites exceeds 20 000 km, it is not surprising that gravitational effects are noticeably different on their orbit than on the earth’s surface. But how can the variation in the time flow be of any relevance for phenomena near the surface? The effect of the terrestrial gravitational field seems to be really small: one centimeter increase of the altitude, $h_1$ cm, causes a relative change of time flow of the order of $\epsilon = gh_1$ cm/$c^2 = 10^{-18}$, (this is approximately the current best precision of atomic clocks.) Yet not only is it relevant, but it is the main mechanism causing bodies to fall.

The crucial reason is that the smallness of $\epsilon$, caused by the inverse square of $c$, is compensated by the square of $c$ present in the rest energy $mc^2$. This cancellation leads to a $c$-independent non-relativistic limit when $c \to \infty$.

From the point of view of a distant observer mentioned in the Introduction, the de Broglie wave of the particle in a force field evolves as $\exp -i (mc^2 + U)t$, where $U$ is the potential energy. For example, gravitational potential energy near Earth’s surface is $U = mgh$. The position dependence of $U$ causes de Broglie waves to refract. This picture is applicable to all conservative forces. Gravity is unique in that one can make a coordinate transformation, $(mc^2 + U)t = mc^2t'$, see Eq. (10), which is universal because the $m$ dependence cancels. This universality leads to the geometrical interpretation of gravity, as in Ref. 1.

We hope that the interpretation of the free fall in terms of de Broglie wave refraction will help students, especially those who do not have the time to learn differential geometry, to grasp the relevance of general relativity.

NOTE ADDED

After this paper was published, Robert Spekkens kindly informed us about Ref. 20, where the role of the gravitational time dilation in the free fall had been pointed out.
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Appendix A: Gravitational time dilation: Einstein’s derivation

Einstein discovered gravitational time dilation in 1907 (Ref. 21, p. 301), long before he created general relativity. We note that it is common to all theories that incorporate the Equivalence Principle.\textsuperscript{22,23} Here we summarize Einstein’s reasoning in deriving Eq. 1.

He started with the observation that physical laws in a uniformly accelerated frame do not differ from those in a frame at rest in a uniform gravitational field. Since he found an accelerated frame more theoretically accessible, he used it to analyse the running of clocks and then inferred the corresponding result in the gravitational field.

He considered three reference frames: $S$ with spacetime coordinates $x$, $t$ is at rest; $\Sigma$ with coordinates $\xi$, $\sigma$ accelerates along the $x$ axis with a constant acceleration $g$ with respect to an instantaneously comoving inertial frame denoted by $S'$, with coordinates $x'$, $t'$. The notion of a constant acceleration was made precise in a later paper (Ref. 21, p. 316), in response to a letter from Max Planck.

At time $t = 0$, $\Sigma$ is instantaneously at rest with respect to $S$ and clocks everywhere in $\Sigma$ are set to 0. The time they measure is called the local time in $\Sigma$ and is denoted by $\sigma$. Local time at the origin of $\Sigma$, that is at $\xi = 0$, is denoted by $\sigma(\xi = 0)$.

Two events at different points $\xi$ are not in general simultaneous with respect to the comoving frame $S'$ when clocks at those points show the same local time $\sigma$. Simultaneous events in $S'$ have the same value of $t'$, related to coordinates in $S$ by the Lorentz transformation,

$$t' = t - \frac{vx}{c^2}. \quad (A1)$$

Time $\sigma(\xi = 0)$ is considered so small that quadratic effects in $\sigma(\xi = 0)$ and thus also in the velocity of $S'$ and $\Sigma$ with respect to $S$, $v = g\sigma(\xi = 0)$, are neglected, thus the factor $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$ is approximated by 1.

Consider two events simultaneous in $S'$ ($t'_1 = t'_2 \equiv t'$): one with coordinates $(x_1, t_1)$, $(x'_1, t')$, and $(\xi_1, \sigma_1)$ respectively in $S$, $S'$, and $\Sigma$, and the other with subscripts 2 instead of 1. The difference

$$x_2 - x_1 = (x'_2 - vt') - (x'_1 - vt') = x'_2 - x'_1, \quad (A2)$$

is the same as $\xi_2 - \xi_1$, since $\Sigma$ is at rest with respect to $S'$. Further, $t_1 = \sigma_1$ and $t_2 = \sigma_2$ because the duration of motion $\sigma(\xi = 0)$ has been too short to destroy the synchronization of the $\Sigma$ and $S$ clocks. Thus

$$\sigma_2 - \sigma_1 = t_2 - t_1 \quad (A3)$$

$$= \left(t' + \frac{vx_2}{c^2}\right) - \left(t' + \frac{vx_1}{c^2}\right) \quad (A4)$$

$$= \frac{v}{c^2} (x_2 - x_1) = \frac{v}{c^2} (\xi_2 - \xi_1). \quad (A5)$$

Now suppose that event 1 takes place at the origin of $\Sigma$, $\xi_1 = 0$, $\sigma_1 = \sigma(\xi = 0)$. Drop the subscript 2 since the coordinates of event 2 now refer to any event in $\Sigma$:

$$\sigma - \sigma(\xi = 0) = \frac{v}{c^2} \xi = \frac{g\sigma(\xi = 0)}{c^2} \xi. \quad (A6)$$

Finally,

$$\sigma = \sigma(\xi = 0) \left(1 + \frac{gc}{c^2}\right), \quad (A7)$$

equivalent to Eq. 1.
Appendix B: Wave equation in a uniform gravitational field

Here we show that the non-relativistic limit of the Klein-Gordon equation in the freely falling particle reference frame becomes a Schrödinger equation with the gravitational potential \( U(z) = mgz \) in the reference frame of a distant observer. We then apply the Ehrenfest theorem to reproduce the momentum evolution we found in Eq. (13). For a detailed study of the Klein-Gordon equation and matter waves in a gravitational field see Refs. 25 and 26.

In the particle reference frame, the Klein-Gordon equation is

\[
-\partial^2_t \Psi(z, t') + c^2 \nabla^2 \Psi = \left( \frac{mc^2}{\hbar} \right)^2 \Psi. \tag{B1}
\]

Change the time variable to the time of the distant observer, see Eq. (10), \( t' = (1 + \frac{gz}{c^2}) t \), so that \( \partial^2_t \simeq (1 - \frac{2gz}{c^2}) \partial^2_t \). Factor out the leading time dependence, \( \Psi(z, t) = e^{-imc^2t/\hbar^2} \psi(z, t) \). Then

\[
\partial^2_t \Psi(z, t) = e^{-imc^2t/\hbar^2} \left[ \partial^2_z \psi - 2 \frac{imc^2}{\hbar} \partial_z \psi - \left( \frac{mc^2}{\hbar} \right)^2 \psi \right], \tag{B2}
\]

and the Klein-Gordon equation (B1) becomes

\[
\left[ - \left( 1 - \frac{2gz}{c^2} \right) \partial^2_z \psi + 2 \left( 1 - \frac{2gz}{c^2} \right) \frac{imc^2}{\hbar} \partial_z \psi - \frac{2gz}{c^2} \left( \frac{mc^2}{\hbar} \right)^2 \psi \right] + c^2 \nabla^2 \psi = 0. \tag{B3}
\]

Neglecting terms not enhanced by \( c^2 \) we obtain the Schrödinger equation with the gravitational potential \( U(z) = mgz \),

\[
i\hbar \partial_t \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi + mgz \psi. \tag{B4}
\]

The ground state solution of this equation and low-lying excitations have been tested in a series of beautiful experiments with neutrons. Here we are interested in a rather high-energy behavior, where the gravitational potential varies little over the size of a wave packet. In this limit the Ehrenfest theorem is applicable. The evolution of the expectation value of the momentum is given by

\[
\frac{d}{dt} \langle \psi \rangle = -mg, \tag{B5}
\]

in agreement with Eq. (13).
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