Hybridgen: A Model for the Study of QCD Hybrid States

M.M. Brisudova\textsuperscript{a,\dagger} and T. Goldman\textsuperscript{b,†}

\textsuperscript{a} Nuclear Theory Center, Indiana University, 2401 Milo B. Sampson Lane, Bloomington, IN 47408

\textsuperscript{b} Theoretical Division, MS B283, Los Alamos National Laboratory, Los Alamos, NM 87545

(September 19, 2002)

Abstract

We study the mixing of excited states of a Hydrogen atom in a cavity with de-excited states plus a confined photon as a model for the coupling of quark-antiquark and quark-antiquark-gluon hybrid states in QCD. For an interesting range of parameters, the results are analytic. We find a case for which wavefunctions (and hence decay patterns) may be at odds with mass with respect to identification of a state as hybrid or not.

Key words: hybrids, spectroscopy, potential models

\textsuperscript{*}On leave from Physics Institute, Slovak Acad. Sci., Dúbravská cesta 9, 842 28 Bratislava, Slovakia. Electronic address: brisuda@niobe.iucf.indiana.edu

\textsuperscript{†}Electronic address: tgoldman@lanl.gov
I. INTRODUCTION

Hybrid (quark-gluon) states are expected to occur in QCD but have not yet been unambiguously experimentally observed. Lattice efforts to study hybrid states directly are hampered by two problems: First, these states are generally not the lowest energy ones of a given spin-parity combination, which reduces the statistical power available for studying them, since the underlying states must first be removed from the Monte Carlo signal. In addition, since, unlike photons, individual gluon states are not gauge invariant, it is difficult to unambiguously identify the contribution of the gluonic component to the detailed structure and even the energy of the state. The problems accompanying direct lattice calculations of hybrid masses are avoided when, instead of individual physical states, static potentials such that the glue can carry angular momentum are determined. Masses of physical states can then be found in the leading Born-Oppenheimer approximation, at least for very heavy quarks.

These lattice data provide a standard with which all models may be compared (for a concise review see, e.g.,). At small quark separation, lattice data are reasonably well described by bag models. Flux tube models form the basis of a different physical picture of hybrids. Hybrid states have been studied in the Coulomb gauge Hamiltonian approach, which, owing to a nontrivial BPS vacuum, naturally leads to a mass gap. Physically, the transverse glue in the hybrid constitutes a valence particle. This approach is not applicable at large separation, where flux tube models and string pictures seem to provide a better description in terms of oscillation mode excitations of the string or the tube. Earlier constituent gluon models generally do not agree with the lattice mass predictions.

With a view to elucidating the nature of such states, we examine a modification of the hydrogenic problem. We describe our motivation in choosing this analogy in detail in the next Section, providing only an overview here.

To model the effect of gauge boson confinement, we place the hydrogen atom in a (spher-
ical) cavity within a magnetic superconducting material. This prevents photon propagation away from the atom, thus confining it to the same cavity. Separate sets of electronic and photonic eigenstates are thus defined. By occupying a photon state simultaneously with an electron state, we obtain our analog hybrid state. Through photon absorption, these states couple to the purely electronic states, and so mix. This model also has the advantage of allowing for removal of the degeneracy between the excited atomic state and combined lower atomic state plus photon. This is more closely analogous to the case in hadronic systems, where the phenomenologically-based expectation is that states including gluon excitations have considerably larger energies than corresponding states including only quarks and antiquarks.

The paper is organized as follows. In Sec.2, we present details of the motivation for our model. In Sec.3, we describe the electronic and photonic states and the relevant transition matrix elements for photon emission and absorption. In Sec.4, we discuss the energy eigenvalues and mixing amplitudes in various limits. Finally, we close with concluding remarks in Sec.5.

II. MOTIVATION: U(1) VS. SU(3)

The definition of hybrid states is rarely made explicit unless presented in a non-relativistic or constituent context. From a relativistic point of view, intermediate state projections should be able to demonstrate the contribution of gluonic excitations to the Fock space of states, since gluon exchanges are responsible for the binding of quarks and antiquarks in even those states not viewed as hybrid.

Such excitations are the analog of Coulomb photonic contributions to atomic states. In a hydrogen atom, the spectrum of states is normally viewed as consisting entirely of electron eigenstates in the Coulomb field of the proton (in the limit of neglecting proton recoil effects). From a field theoretic point of view, however, there are cuts of the off-shell electron-proton scattering amplitude which may traverse one or more photons of the Coulomb exchanges.
producing the binding. These are not viewed as contributions of (electron-photon) ‘hybrid’ intermediate states to the wavefunction since neither these photons, nor the electrons, in the virtual three particle (including the proton) intermediate state are in eigenstates of the overall system.

There are, nonetheless, hybrid contributions to the wavefunctions, in principle. These can be seen for excited states, such as the 2P state for example, when it undergoes a transition to the 1S state and emits a photon. Although a free outgoing wave, this photon is in an eigenstate of the overall system to leading order in the electromagnetic perturbations. Since there is a finite, although small, probability for this photon to be reabsorbed and promote the 1S state back to the 2P state, there is, in principle, a hybrid electron-photon state contribution to the precise description of the nominal 2P excited state. Part of the reason that this may generally be ignored is that the contribution is extremely small, as the photon is free to propagate away from the location of the atom, making reabsorption impossible.

This is not true for the analog case in QCD. In a meson composed of a very heavy antiquark and a light quark, for example, a higher excited state may emit a gluon, leaving the quarks in a lower state, except that they now occupy an overall color octet state instead of a singlet. However, the gluon cannot propagate away, but remains confined to the region of the color octet source formed by the quark-antiquark pair.

Thus, to form an analogous atomic system, we must arrange for the transition photon to be confined to the region of the parent atomic state. We implement this by imagining the atom to be centered in a spherical region (bubble) of vacuum contained within an effectively infinite, perfectly conducting medium. We choose to use magnetic superconductor boundary conditions instead of the usual (for QED) electric superconductor boundary conditions in order to emulate the QCD case more closely. To avoid unnecessary additional complications, we suppose also that the proton is fixed at the center of the cavity, obviating concerns about

\[1\] In field theoretic terms, this constitutes a self-energy correction to the 2P state propagator.
both center of mass motion effects and distortions from sphericity.

There are, however, some additional effects introduced by the cavity that must still be accounted for: The truncation of the Coulomb field of the proton (fixed at the cavity center) affects the electronic eigenstates by altering the boundary condition from the usual one at infinity, and the boundary condition produced by the reflecting interior surface defines a set of eigenmodes for the photon, replacing the continuum of outgoing waves previously available. This part simply corresponds to the well-studied case of the bag model [9,10].

Because the quarks in the QCD case are not in a color singlet state after the gluon emission, our analogy is imperfect. To reduce the complications associated with this, we examine an atomic case with a magnetic transition, which does not alter the spatial wave functions, rather than the 2P-1S case referred to above. In the analog QCD case, the quark color magnetic spin flip also does not alter the quark spatial wave function. This allows us to ignore questions regarding the interaction of a quark and antiquark in a color octet state and to proceed with our analysis in the absence of a more analogous, but still calculable, concept.

Finally, we note that, in the free atomic case, the 2P and 1S states have differing energies, but the hybrid 1S plus photon state is necessarily degenerate, in leading order, with the 2P state. In order to match the 2P-1S energy difference in our atomic magnetic spin flip model, we include the hyperfine interaction between the electron and (fixed polarization) proton to split the energies of the two spin orientation states of the electron. The hybrid state of the lower energy electron spin orientation state plus the photon is again necessarily degenerate, in leading order, with the higher energy electron spin orientation state, in the free atomic case.

However, in both the QCD and cavity atom cases, the hybrid and the excited states need not be degenerate: In the cavity atom case, this is due to the energy eigenstates of the photon being determined by the cavity size, while in QCD we simply don’t know which state is more massive. We therefore take advantage of the flexibility of our cavity atom, magnetic spin flip model to examine all three cases: the excited state more massive than
the hybrid, less massive, and approximately degenerate. Of course, when the ‘excited’ state is less massive, it is actually the lower or ground state, and the terminology inherited from the free atomic case no longer applies.

III. HYDROGEN IN A CAVITY

We begin by studying the lowest states of an electron coupled to a (spin-up) proton.

The interaction of an electron with an electromagnetic field is described by the Pauli equation:

\[
i \hbar \frac{\partial}{\partial t} \phi = \left[ \frac{1}{2m} \left( \frac{\mathbf{p} - e \mathbf{A}}{c} \right)^2 + eA^0 - \frac{e}{2mc} \mathbf{\sigma} \cdot (\nabla \times \mathbf{A}) \right] \phi
\]

(This also shows how the electron interacts with the transverse photon.)

In the simplest treatment of the hydrogen atom, the field \(A^0\) is treated as an external field due to an infinitely heavy charge, i.e., the proton. Corrections due to the mass of the proton being finite are included in bound state perturbation theory. The field due to a single infinitely heavy charge is easily found from Gauss’ law. The problem then becomes how to satisfy the boundary condition on the surface of the cavity, namely, a continuous and vanishing normal component of the electric field. Contrary to an ordinary, electric superconductor (in which the role of electric and magnetic fields is reversed compared to the case at hand), a surface charge to neutralize the field of the proton cannot form.\(^2\) Therefore, a single charge in a cavity inside a magnetic superconductor cannot exist. This is the appealing physical picture of confinement behind the bag model \([10]\).

\(^2\)Interestingly, for an electric superconductor neutralization is achievable even in the presence of a fractional charge, by a coherent superposition of different net-integer-charge surface charge distributions. Violation of the superselection rules between differing charge sectors is a consequence (frequently unnoted) of the formation of the charged Bose-Einstein condensate of Cooper pairs. Indeed, from the point of view of the pairs, a unit electric charge is already a fraction, namely \(1/2\), of the ‘unit’, \(2e\), for the effective degrees of freedom in the medium, that is, the Cooper pairs.
Imagine, however, a proton at the center of cavity surrounded by a unit net charge bound to a spherical surface infinitesimally close to the surface. By Gauss’ law, this would not affect the electric field distribution in the interior of the cavity. We can achieve the same effect by using the electron to neutralize the charge of the proton, provided the boundary condition becomes the constraint that the wavefunction of the atomic electron must vanish at (within a penetration length of) the surface of the vacuum bubble. With both charges now completely confined to the interior of the vacuum bubble, the net total charge presented at the surface is zero and so no questions regarding electric flux tubes need be addressed. Thus, we can still use the usual Coulomb potential due to the proton to describe the atomic electric field, experienced by the electron, to the edge of the cavity.

We ignore the issue of the multipole moments of the atom. For self-consistency in this, the hydrogen atom is a better choice of a system than positronium, because the proton is better localized. For positronium, due to motion of the positron, the Coulomb interaction can be expected to be more affected by the presence of the boundary, requiring higher multipole moments. Hence, we choose the hydrogen atom, even though the hybrid states in QCD are likely to be more interesting (not to mention confusing!) when the quarks are of comparable mass rather than in heavy-light systems. For this reason, and to make our model as simple and intuitive as possible, we choose to consider the ground state doublet of the hydrogen atom. If the mass of the proton is sufficiently large, these two states constitute a simple two level system. One would be hard-pressed to find a simpler model for hybrid-ordinary meson mixing.

A. Electron wave functions

We proceed with the description of the atom in a standard way: First we solve the Schrodinger equation for the electron moving in the Coulomb potential, and then calculate the hyperfine splitting in bound state perturbation theory. For the leading order, we solve

\[
\left( -\frac{1}{2m}\nabla^2 - \frac{\alpha}{r} \right) \psi = E_0 \psi .
\]

7
The general solution for the ground state is the standard, exponentially suppressed confluent hypergeometric function (with the normalization, $N^{-1}$, described below)

$$\psi(r) = N^{-1}e^{-\frac{r}{n_0}}F_1(1-n_0,2,\frac{2r}{n_0})$$

where we have introduced a dimensionless separation, $x$, and eigenvalue, $n_0$:

$$x \equiv \alpha mr,$$

$$n_0 \equiv \sqrt{\frac{(\alpha m)^2}{-2E_0}}.$$  

We use $R$ to denote the radius of the spherical cavity and $x_0$ to denote its dimensionless size,

$$x_0 \equiv \alpha m R.$$  

The eigenvalue is determined by the boundary condition

$$\psi(r = R) = 0$$

appropriate to this nonrelativistic analysis. We solve this transcendental equation for the eigenvalue, $n_0$, numerically. The results are shown in the Table. Note that the numerical solution shows that, for $x_0 > 3$, the difference between the eigenvalue in the cavity and the continuum limit is already $\leq 6\%$. This should not be surprising. The size of the atom is determined dynamically, and once the cavity becomes larger than the typical Bohr radius, the dynamical structure of the atom is not significantly affected.

The normalization of the wavefunction is required to be unity. Hence, from Eq.(3),

$$N^{-1} = \left[4\pi R^3 \int_0^1 dt t^2 e^{-2t/x_0} |_{1} F_{1}(1-n_0,2,\frac{2x_0}{n_0})^2\right]^{-1/2}.$$  

3This should be compared with the relativistic condition found in the MIT bag model; see, e.g., Ref. [10]. Changing to that boundary condition affords an excellent opportunity to study relativistic corrections.
As stated above, we consider the hydrogen ground state doublet. By tuning the ‘proton’ mass, we can arrange for this doublet to be arbitrarily separated from the rest of the spectrum, i.e., $\Delta E_{spin} \ll \Delta E_{radial, orbital}$. The spin-spin splitting is due to the standard magnetic (Breit-Fermi) hyperfine interaction,

$$V_{spin-spin} = \frac{8\pi\alpha}{3mM} \delta^3(r) \mathbf{s}_{\text{proton}} \cdot \mathbf{s}. \quad (9)$$

associated with the ordinary Coulomb potential.

Let the spin of the proton be fixed in the $+z$ direction. Then the changes to the electron energies in the ground state doublet arising from hyperfine interaction are

$$\frac{8\pi\alpha}{3mM} |\psi(0)|^2 \frac{1}{2} \langle s_z \rangle. \quad (10)$$

where the 1/2 in front of the expectation value of $s_z$ comes from the proton spin and $\langle s_z \rangle = \pm 1/2$ is the expectation value of the electron spin. The factor $|\psi(0)|^2$ is just the calculated norm, $N^{-2}$. (See the Table.)

In the limit of large $R$ (in practice $x_0 > 3$), the eigenvalues and the wavefunctions are well approximated by their respective continuum limits. The transition between the large $R$ and small $R$ regimes is relatively sharp, as is evident from the Table.

**B. Constituent photon**

The lowest mode in the cavity with magnetic boundary conditions is [9]

$$\tilde{A} = N^{-1}_\gamma j_1(\omega_0 r) \tilde{L} Y_{lm}(\Omega)a_m + H.C. \quad (11)$$

where $a_m$ is the mode annihilation operator, $j_1$ is the spherical Bessel function, $\tilde{L}$ is the angular momentum operator, the $Y_{lm}$ are the spherical harmonics and $\tilde{r} = r\tilde{\Omega}$. The frequency $\omega_0 = k_0/R$ is fixed by the boundary condition:

$$k_0 j_1'(k_0) = -j_1(k_0) \quad (12)$$
which is determined by the requirement that the photon flux leaving the cavity vanishes. The solution is \( k_0 = 2.7437 \) [9]. The normalization of this wavefunction to correspond to a single photon is given by \( N_\gamma \),

\[
N^{-1}_\gamma = \frac{1}{\sqrt{2\omega_0}} \left[ 2 \int_0^R dr \ r^2 j_1^2(\omega_0 r) \right]^{-1/2} \\
= R^{-1} \left[ 2k_0 \left( \frac{1}{k_0^2} + \frac{\sin(k_0) \cos(k_0)}{k_0^3} - 2\left(\sin^2(k_0)\right) \right) \right]^{-1/2} \tag{13}
\]

The term in parenthesis in the last expression is a number,

\[
c(k_0) = 1.2857,
\]

which is independent of \( R \), i.e., \( N^{-1}_\gamma = c(k_0)R^{-1} \). Note that, since \( x_0 = \alpha mR \),

\[
\omega_0 = \alpha m \frac{k_0}{x_0} \tag{14}
\]

so that the energy of this lowest photon mode is \( k_0/x_0 \) in units of \( \alpha m \).

C. Emission of the constituent photon.

From the Pauli equation, Eq.(1), the operator for the interaction of the electron with the constituent photon is:

\[
(-) \frac{e}{2mc} \vec{\sigma} \cdot (\vec{\nabla} \times \vec{A}). \tag{15}
\]

First, we find the magnetic field associated with the constituent photon from Eq.(11).

\[
\vec{\nabla} \times \vec{A}_m = i \sqrt{\frac{3}{4\pi}} \left[ \hat{e}_r \frac{2 \hat{j}_1}{r} \cos \theta - \hat{e}_\theta \left( \frac{\partial \hat{j}_1}{\partial r} + \frac{\hat{j}_1}{r} \right) \sin \theta \right], \ m = 0
\]

\[
= -i \sqrt{\frac{3}{8\pi}} \left[ \hat{e}_r \frac{2 \hat{j}_1}{r} \sin \theta e^{i\phi} + \left( \frac{\partial \hat{j}_1}{\partial r} + \frac{\hat{j}_1}{r} \right) \left( \hat{e}_\theta \cos \theta e^{i\phi} + \hat{e}_\phi i e^{i\phi} \right) \right], \ m = +1
\]

\[
\vec{\nabla} \times \vec{A}_{-m} = (\vec{\nabla} \times \vec{A}_m)^*. \tag{16}
\]

We need only the spin-flip part of the interaction

\[
\vec{\sigma} \cdot \vec{B} = \sigma_+ B_- + \sigma_- B_+ + \sigma_z B_z \tag{17}
\]
where \( B_\pm = (B_x \pm iB_y) / \sqrt{2} = (\hat{x} \pm i\hat{y}) \cdot (\vec{\nabla} \times \vec{A}) / \sqrt{2} \).

Using the expression for specific \( m \) values, it is straightforward to show that

\[
\begin{align*}
  m = 0 & : \quad \sigma_+ B_+ \propto e^{\pm i\phi} \\
  m = +1 & : \quad \sigma_+ B_+ \propto e^{+i\phi \pm i\phi} \\
  m = -1 & : \quad \sigma_+ B_+ \propto e^{-i\phi \pm i\phi}.
\end{align*}
\]

The ground state wavefunction is spherically symmetric, therefore the \( m = 0 \) mode does not contribute at all. The \( m = 1 \) photon gives a nonzero expectation value for \( \sigma_+ B_- \); \( m = -1 \) similarly gives a value for \( \sigma_- B_+ \) and is the complex conjugate of the \( m = 1 \) case. In what follows, we concentrate on the mixing between a spin up pure electron state (\( |a\rangle \equiv | \uparrow \rangle \)), and a spin down electron with a constituent photon (\( |b\rangle \equiv | \downarrow \gamma \rangle \)). Therefore, we need only the \( m = +1 \) case.

In the expectation value of the operator in Eq.(17), the spin and spatial expectation values factor out:

\[
\langle a| -\frac{e}{2mc} \sigma_+ B_- |b\rangle = (-) \frac{e}{2mc} \langle \sigma_+ \rangle I
\]

where

\[
I \equiv -\frac{i}{4} \sqrt{\frac{3}{\pi}} N^{-2} N_\gamma^{-1} \int d^3 r \ e^{-2\alpha mr \ n_0} |1F_1(1-n_0, 2, \frac{2\alpha mr n_0}{n_0})|^2
\times \left[ 2 \frac{j_1(\omega_0 r)}{r} (1 - \cos^2 \theta) + \left( \frac{\partial j_1(\omega_0 r)}{\partial r} + \frac{j_1(\omega_0 r)}{r} \right) (1 + \cos^2 \theta) \right].
\]

After straightforward algebra, the integral reduces to

\[
I = -i \sqrt{\frac{3}{\pi}} N^{-2} N_\gamma^{-1} \frac{4\pi}{3} \int dr \ r \sin(\omega_0 r) e^{-2\alpha mr \ n_0} |1F_1(1-n_0, 2, \frac{2\alpha mr n_0}{n_0})|^2.
\]

\[\text{(21)}\]

**D. Special case: \( R \) large.**

For the sake of consistency, \( \alpha \) in our calculation has to be small to justify the nonrelativistic treatment of the hydrogen atom. In addition to this restriction, the ratio \( m/M \) has
to be small so that the proton is effectively static and the spherical cavity is a good description. Together, these imply that the energy shifts of the doublet must be small compared to \(\alpha m\).

The energy of the photon, \(\frac{k_0}{x_0}\), as measured in units of \(\alpha m\), is of order one for \(x_0 = 3\). (Recall that the large \(R\) regime starts around \(x_0 \simeq 3\)). This is large, not only compared to the spin-spin interaction energy, but also compared to the radial binding, which is of order \(\alpha\) in these units. Thus, it is sufficient to consider just the large \(R\) limit, i.e., \(x_0 \gg 3\), to obtain results over a range of photon energies from large to small relative to all of the energy scales of pure hydrogen.

In the large \(R\) limit, the radial part of the wavefunction, \(\psi(r)\), and its value at the origin, \(N^{-1}\), are well approximated by their respective continuum values, i.e.

\[
n_0 \doteq 1 \quad \text{(22)}
\]

\[
\psi(r) \doteq N^{-1} e^{-x} \quad \text{(23)}
\]

\[
N^{-1} \doteq \left[\pi(\alpha m)^{-3}\right]^{-1/2}. \quad \text{(24)}
\]

The energies of the two states under consideration, i.e. the spin up electron (spin one hydrogen state) and the spin down electron (including a spin zero hydrogen component) together with a constituent photon (for a total spin of one, again) are

\[
E_a \equiv E(\uparrow) = \alpha m \left(\frac{2}{3} \frac{m}{M} \alpha^3\right) \quad \text{(25)}
\]

\[
E_b \equiv E(\downarrow + \gamma) = \alpha m \left(-\frac{2}{3} \frac{m}{M} \alpha^3 + \frac{k_0}{x_0}\right) \quad \text{(26)}
\]

relative to the centroid of the doublet.

Since the wavefunction reduces to a simple exponential, the integral in Eq.(21) can be evaluated analytically,

\[
I = -i \frac{16}{\sqrt{3\pi}} c(k_0)k_0 \frac{x_0^2}{(4x_0^2 + k_0^2)^2} (\alpha m)^2. \quad \text{(27)}
\]

The full transition energy matrix element between the two states is then:
\[ T_{ab} \equiv \langle a | H_I | b \rangle = i \alpha m \sqrt{\frac{2}{3}} \frac{c(k_0)}{k_0} \frac{(\frac{k_0}{x_0})^2}{[1 + (\frac{k_0}{2x_0})^2]^2} \alpha^{3/2}. \] (28)

The Hamiltonian of this two-level system is therefore:

\[
H = \begin{pmatrix}
E_a & T_{ab} \\
T_{ab}^* & E_b
\end{pmatrix},
\] (29)

and its eigenstates are

\[
N_1 | 1 \rangle = | T_{ab} | | a \rangle + (E_1 - E_a) | b \rangle \] (30)

\[
N_2 | 2 \rangle = | T_{ab} | | a \rangle + (E_2 - E_a) | b \rangle \] (31)

where \( N_{1,2} = [|T_{ab}|^2 + (E_{1,2} - E_a)^2]^{1/2} \) (\( E_a, E_b, \) and \( T_{ab} \) are as given in previous expressions), and where we have absorbed a phase of \( \pm i \) into the definition of (either) one of the states for convenience. The corresponding eigenvalues are

\[
E_{1,2} = \frac{\alpha m}{2} \frac{k_0}{x_0} \pm \sqrt{\Delta^2 + \frac{2}{3} \left( \frac{2c(k_0)}{k_0} \right)^2 \frac{(\frac{k_0}{x_0})^4}{[1 + (\frac{k_0}{2x_0})^2]^4} \alpha^3},
\] (32)

where

\[
\Delta \equiv \frac{4 m}{3 M} \alpha^3 - \frac{k_0}{x_0},
\] (33)

is the splitting between the \((a, b)\) states before mixing, in units of \( \alpha m \). The energy difference between the eigenstates is

\[
\frac{\Delta E}{\alpha m} = \frac{(E_1 - E_2)}{\alpha m} = \sqrt{\Delta^2 + \frac{2}{3} \left( \frac{2c(k_0)}{k_0} \right)^2 \frac{(\frac{k_0}{x_0})^4}{[1 + (\frac{k_0}{2x_0})^2]^4} \alpha^3}
\] (34)

(All energies in this regime are much less than \( \alpha m \) for sufficiently large values of \( x_0 \).) Since, for the values of \( k_0 \) and \( c(k_0) \) the numerical value of \( \left( \frac{2c(k_0)}{k_0} \right)^2 \sim 1 \), in what follows, we omit this combination in approximate formulas below, for simplicity.
IV. DISCUSSION

Below, we will refer to the pure hydrogenic (proton plus electron only) states, corresponding to quark-antiquark (”pure quark”) states in QCD, as pure electron states. The states including a cavity-trapped photon, corresponding to hybrid quark-antiquark-gluon states in QCD, will be referred to as ‘hybrid’ states. There are three distinct limits defined by the relative size of the unperturbed energies:

A. Photon energy much less than the energy splitting between the pure electron states

In this case, the upper pure electron state corresponds to the QCD case of a quark-antiquark color singlet excitation of overall higher energy, whereas the hybrid state (lower pure electron state plus photon) corresponds to a lower energy quark-antiquark color octet state plus gluon making up a hybrid state of overall energy which is still lower than that of the excited singlet. We have

\[
\frac{k_0}{x_0} \ll \frac{4}{3} \frac{m^3}{M^3} \alpha^3.
\]  

(35)

This occurs when the cavity becomes so large that effectively the energy of the lowest lying photon approaches the continuum limit, i.e. zero.

Of course, as the cavity size increases, the energy of the excited photons decreases also, and eventually some photon state would become comparable with the energy splitting of the hyperfine states. We are not interested in this scenario, since we are not concerned with the dynamics of this system for itself. Rather, we use the size of the cavity to model the magnitude of the confining energy of the gluon in a hybrid state. From this point of view, the case at hand corresponds to an ordinary meson and a hybrid which contains a constituent gluon of energy much smaller than the energy difference of the pure quark-antiquark components. Such a case may occur in QCD if the next higher gluonic state turns out to be very much higher in energy, or if some quantum number constraint prevents mixing
between the pure quark-antiquark components and a gluonic state of energy closer to the energy difference of the pure quark-antiquark components, corresponding in our model to one of the higher cavity photon states.

The energies of the eigenstates are, respectively,

$$E_{1,2}/(\alpha m) = \pm \frac{2}{3} \frac{m}{M} \alpha^3 + \frac{1}{2} \left( \frac{k_0}{x_0} + \frac{k_0}{x_0} \right). \quad (36)$$

The energy difference between the eigenstates is then

$$\frac{\delta E}{\alpha m} = \frac{4}{3} \frac{m}{M} \alpha^3 - \frac{k_0}{x_0} = \Delta. \quad (37)$$

Not surprisingly, the mixing is small, as $E_1 - E_a$ is of the order of the fourth power of the photon energy:

$$|1\rangle \simeq |a\rangle + \mathcal{O} \left[ \sqrt{\frac{3}{2} \frac{M}{4m} \left( \frac{k_0}{x_0} \right)^2} |b\rangle \right] \quad (38)$$

$$|2\rangle \simeq |b\rangle - \mathcal{O} \left[ \sqrt{\frac{3}{2} \frac{M}{4m} \left( \frac{k_0}{x_0} \right)^2} |a\rangle \right]. \quad (39)$$

Despite the inverse powers of $\alpha$ and the ratio $M/m$, the corrections from the mixing are extremely small in this case, in view of Eq.(35). The hybrid state has the lower energy of the two and, again in view of Eq.(35), $\Delta \approx \frac{4}{3} \frac{m}{M} \alpha^3$ so the state has almost the same energy as the lower pure electron state.

**B. Photon energy comparable to the energy splitting of the pure electron states**

This corresponds to the QCD case where the quark-antiquark color singlet excited state is comparable in overall energy to the hybrid state.

With decreasing size of the cavity, the energy of the constituent photon increases and it becomes comparable to the energy splitting between the pure electron states. With $\Delta$ as defined above in Eq.(35), we have

$$|\Delta| < \frac{k_0}{x_0} \sim \frac{4}{3} \frac{m}{M} \alpha^3. \quad (40)$$
As long as $\Delta$ is larger than the magnitude of the off diagonal matrix element of the Hamiltonian, the system remains similar to the situation described in the previous section. However, when the energy splitting $\Delta$ is also much smaller than the mixing energy, that is,

$$\Delta^2 \ll \left( \frac{k_0}{x_0} \right)^4 \alpha^3,$$

the energies of the eigenstates are approximately

$$E_{1,2}/(\alpha m) = \frac{1}{2} \left\{ \frac{k_0}{x_0} \pm \left( \frac{k_0}{x_0} \right)^2 \sqrt{\frac{2}{3}} \alpha^3 \pm \frac{\Delta^2}{2 \left( \frac{k_0}{x_0} \right)^2 \sqrt{\frac{2}{3}} \alpha^3} \right\}.$$

(Since $\frac{k_0}{x_0} \ll 1$ here, to satisfy Eq.(41), the term in square brackets in the denominator of Eq.(34) is approximately unity.) The wave functions are, to $O(\Delta)$,

$$|1\rangle = \frac{1}{\sqrt{2}} \left\{ \left( 1 - \frac{\Delta}{2 \left( \frac{k_0}{x_0} \right)^2 \sqrt{\frac{2}{3}} \alpha^3} \right) |a\rangle + \left( 1 + \frac{\Delta}{2 \left( \frac{k_0}{x_0} \right)^2 \sqrt{\frac{2}{3}} \alpha^3} \right) |b\rangle \right\}$$

$$|2\rangle = \frac{1}{\sqrt{2}} \left\{ \left( 1 + \frac{\Delta}{2 \left( \frac{k_0}{x_0} \right)^2 \sqrt{\frac{2}{3}} \alpha^3} \right) |a\rangle + \left( -1 + \frac{\Delta}{2 \left( \frac{k_0}{x_0} \right)^2 \sqrt{\frac{2}{3}} \alpha^3} \right) |b\rangle \right\}.$$ 

The mixing is maximum when $\Delta = 0$, as one would expect. (The presence of $\alpha$ in the denominator may be misleading: Eq.(41) shows that, in this case, $|\Delta| \ll \alpha^2$, as $\frac{k_0}{x_0} \ll 1$ in the large $R$ limit considered in this paper. So, in fact, the term with the inverse power of the coupling constant is nonetheless very small.)

The energy splitting between physical states is, not surprisingly, dominated by the off diagonal matrix element of the Hamiltonian, viz.

$$\frac{\delta E}{\alpha m} = \left( \frac{k_0}{x_0} \right)^2 \sqrt{\frac{2}{3}} \alpha^3 \left[ 1 + \frac{3\Delta^2}{4\alpha^3} \left( \frac{x_0}{k_0} \right)^4 \right].$$

(Again, despite the misleading inverse powers of $\alpha$, the correction term in the square brackets is small due to Eq.(41).)

**C. Photon energy much larger than the energy splitting of the pure electron states**

This corresponds to the QCD case where the hybrid state has much larger overall energy than the quark-antiquark color singlet excited state.
As the cavity size is decreased further, the photon energy increases as \( x_0^{-1} \) and can become much larger than the energy splitting of the pure electron states. We can arrange for both \( \frac{4m}{3M} \alpha^3 \) and \( \frac{k_0}{x_0} \) to be much less than one in the \( R \) large limit, while satisfying

\[
\frac{4m}{3M} \alpha^3 \ll \frac{k_0}{x_0}. \tag{46}
\]

Thus, \( \Delta \approx -\frac{k_0}{x_0} \) and the eigenvalues are

\[
E_1/(\alpha m) = \frac{k_0}{x_0} \left[ 1 + \left( \frac{k_0}{x_0} \right)^2 \frac{\alpha^3}{6} \right] - \frac{2m}{3M} \alpha^3 \tag{47}
\]

\[
E_2/(\alpha m) = \frac{2m}{3M} \alpha^3 - \left( \frac{k_0}{x_0} \right)^3 \frac{\alpha^3}{6}. \tag{48}
\]

Hence, we have that

\[
\frac{\delta E}{\alpha m} \approx -\Delta + \left( \frac{k_0}{x_0} \right)^3 \frac{\alpha^3}{3} \cdot \tag{49}
\]

\[
= \frac{k_0}{x_0} - \frac{4m}{3M} \alpha^3 + \left( \frac{k_0}{x_0} \right)^3 \frac{\alpha^3}{3}. \tag{50}
\]

However, since \( \frac{k_0}{x_0} < 1 \), in this case the difference is almost the same as the energy of the trapped cavity photon, corresponding to the constituent gluon in the QCD hybrid case.

The wave functions in this case are

\[
|1\rangle = \sqrt{\frac{\alpha^3}{6}} \left( \frac{k_0}{x_0} \right) |a\rangle + \left[ 1 - \frac{\alpha^3}{12} \left( \frac{k_0}{x_0} \right)^2 \right] |b\rangle \tag{51}
\]

\[
|2\rangle = \left[ 1 - \frac{\alpha^3}{12} \left( \frac{k_0}{x_0} \right)^2 \right] |a\rangle - \sqrt{\frac{\alpha^3}{6}} \left( \frac{k_0}{x_0} \right) |b\rangle. \tag{52}
\]

Note that the energy of the higher state, \( E_1 \), remains dominated by the diagonal element, but in this case, it is the hybrid state. The correction from mixing is further suppressed by \( \alpha^3 \). Conversely, mixing becomes a crucial factor for the energy of the lower state when \( \frac{k_0}{x_0} \) is comparable to \( \left[ \frac{m}{M} \right]^{1/3} \). The energy of the lower state, \( E_2 \), is affected by the mixing even though the state is dominated by the pure non-hybrid component. In particular, cancellation or near-cancellation between the spin-splitting and mixing energy terms may lead to the appearance that this pure electron (non-hybrid) state is at the wrong energy to be identified as a conventional (meson) state within a framework that ignores constituent bosons, e.g. a quark model.
CONCLUDING REMARKS

We have examined the physical system of hydrogen in a cavity and found that, even in the large cavity, small coupling, heavy nucleus limit, the system has a rich range of available characteristics which may illuminate corresponding cases of QCD hybrids, where none of the limits apply. (All parameter ratios are near unity.)

We find that the state with the constituent boson always mixes with the state of pure constituent fermions, but if the energy of the constituent boson itself is much smaller than the energy difference of the pure fermionic part, the mixing becomes negligibly small. Similarly, when the energy of the constituent boson is much larger than that energy difference, the more energetic state is dominated by the component with the extra boson. However, the energy of the lower lying state may or may not be given by the the state nearly purely composed of fermions, depending on the relative size of the boson and fermion state splitting energies.

When the energies of states prior to mixing are comparable, the mixing is maximum when the energy of the boson coincides with the size of the energy difference of the fermionic part.

Perhaps the most interesting and most relevant case for our understanding of the role of hybrid states in QCD is that when the energy of the constituent boson is larger than the energy difference of the fermionic part. Our results suggest that, if the mixing of the hybrid with the pure fermion states is strong enough in this case, then the energy of the nominally pure quark-antiquark color singlet state is significantly altered even though the wavefunction of the state is essentially unaffected. Note that this may well occur in QCD, where \( \frac{k_0}{x_0} \sim \alpha_S \sim \frac{m}{M} \sim 1 \). This may have important implications for analyses of decay patterns that can influence the interpretation of states as quark-antiquark or hybrid. In particular, the state may lack evidence of hybrid decay patterns even though its mass suggests that it does not belong in a representation of pure quark-antiquark states, and thus superficially requires an alternate interpretation.

Continuation to other parameter regimes of interest (\( \alpha \sim 1, \frac{m}{M} \sim 1 \)) may appear to be straightforward using numerical methods. Unfortunately, the approximations on which the
model is based become invalid, in general, which would require more involved analysis to remedy. However, it should be fairly straightforward to study this problem with a cavity of arbitrary size or relativistic boundary conditions.

As lagniappe, we note that, in addition to the theoretical certainty of the calculations of such a well understood system presented here, it may be possible to realize such systems experimentally using magneto-optical traps (MOTs) for alkali atoms. By adjusting the magnetic field and choice of states, it may be possible to realize a spin flip system with a long enough wavelength that an exterior conducting sphere could be added and still allow for the entry of laser beams required for the MOT. This would allow for the experimental study of electron-photon hybrid states as a model for QCD hybrids.

**ACKNOWLEDGMENTS**

We would like to thank Charles Thorn and Philip Page for discussions. This research is supported by the Department of Energy under contracts W-7405-ENG-36 and DE-FG02-87ER40365.
REFERENCES

[1] C. McNeile, [hep-lat/0207001].

[2] K.J. Juge, J. Kuti, C.J. Morningstar, Phys. Rev. Lett. 82 (1999) 4400; C.J. Morningstar, nucl-th/0110074.

[3] P.R. Page, Nucl. Phys. A663&664 (2000) 585c.

[4] K.J. Juge, J. Kuti, C.J. Morningstar, Nucl. Phys. Proc. Suppl. 63 (1998) 543; G. Karl, J. Paton, Phys. Rev. D 60 (1999) 034015.

[5] T. Barnes, F.E. Close, E.S. Swanson, Phys. Rev. D 52 (1995) 5242.

[6] E.S. Swanson, A.P. Szczepaniak, Phys. Rev. D 59 (1999) 014035.

[7] T.J. Allen, M.G. Olsson, S. Veseli, Phys. Lett. B 434 (1998) 110.

[8] Yu. S. Kalashnikova, Z. Phys. C 62 (1994) 323.

[9] C.B. Thorn, Phys. Rev. D 59 (1999) 116011.

[10] T. Barnes, The Bag Model and Hybrid Mesons, SIN Spring School on Strong Interactions, Zuos (Engadin), Switzerland, 9-17 April, 1985.
TABLES

TABLE I. Numerical results for the ground state hydrogen atom and ground state photon.

| $x_0$ | $n_0$  | $N^{-2}$ in units ($\alpha m)^3$ | $k_0/x_0$ |
|-------|--------|---------------------------------|-----------|
| 1.8456 | 7.38261 | 0.8653                          | 1.48662   |
| 1.8768 | 3.75359 | 0.840668                        | 1.4619    |
| 2.0    | 2.0    | 0.755681                        | 1.37185   |
| 2.03412| 1.84   | 0.733805                        | 1.34884   |
| 2.200  | 1.467  | 0.6505                          | 1.24714   |
| 2.4712 | 1.23562| 0.5519                          | 1.11027   |
| 3.1876 | 1.0625 | 0.4161                          | 0.86842   |
| 4.06228| 1.01557| 0.352562                        | 0.675409  |
| 5.0175 | 1.0035 | 0.3285                          | 0.546826  |
| 6.004  | 1.00072| 0.32094                         | 0.456979  |
| $\infty$ | 1.0 | 0.31831                         | 0         |