Probing Quantum Gravity Through Exactly Soluble Midi-Superspaces I

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Abstract

It is well-known that the Einstein-Rosen solutions to the 3+1 dimensional vacuum Einstein’s equations are in one to one correspondence with solutions of 2+1 dimensional general relativity coupled to axi-symmetric, zero rest mass scalar fields. We first re-examine the quantization of this midi-superspace paying special attention to the asymptotically flat boundary conditions and to certain functional analytic subtleties associated with regularization. We then use the resulting quantum theory to analyze several conceptual and technical issues of quantum gravity.

1 Introduction

Many of the central problems of quantum gravity can be traced back to two main difficulties: i) the absence of a background space-time metric; and, ii) the presence of an infinite number of degrees of freedom.

Let us begin with the first set of issues. The absence of a background geometry implies that the theory has to be diffeomorphism invariant and this feature makes it difficult to construct observables and formulate precisely questions of direct physical interest. It also gives rise to the celebrated “problem of time”: if there is no background metric, what are we to make of the notion of “time evolution”? Indeed, if the diffeomorphisms are to be regarded as gauge, at first sight, dynamics also appears as a part of gauge. Can one disentangle dynamics from gauge unambiguously? These questions are of course not new. (For a detailed discussion, see, e.g. [1].) To gain insight into these issues, a number of mini-superspace models have been discussed in the literature (see, e.g., [2]). In Bianchi models, for example, one restricts attention only to spatially homogeneous solutions of Einstein’s equations and, in the quantum theory, addresses the issue of time via “de-parametrization”. Perhaps a more striking
model is presented by 2+1-dimensional vacuum general relativity which, like the 3+1 theory, is fully diffeomorphism invariant. Quantization of this model \[3, 4\] has shed light on the notion of observables, role of discrete symmetries, etc. These models have also given us considerable insights into the technical problems that arise due to the underlying diffeomorphism invariance. For example, since we have no Poincaré group to help us, the problem of finding the correct inner-product on the space of quantum states requires a new strategy. The 2+1 model has provided a method which, moreover, is free of ambiguities that arise, e.g., in the de-parametrization procedure.

However, these models do not come to grips with the second main difficulty mentioned above: the presence of an infinite number of degrees of freedom. To face this difficulty, we need to consider genuine field theories which do not require a background space-time metric. An obvious strategy would be to again consider symmetry reductions which, however, are mild enough to leave behind local degrees of freedom. To locate convenient choices, let us briefly return to the 2+1 dimensional vacuum general relativity. This theory can be obtained by a symmetry reduction of 3+1-dimensional general relativity with respect to a single space-like Killing field which is hyper-surface orthogonal and whose norm is constant. Therefore, as a next step, it is natural to drop the severe condition on the norm. The symmetry reduced system now has an infinite number of degrees of freedom. In fact it is now equivalent to 2+1-dimensional general relativity coupled to a zero rest mass scalar field (which is given by the logarithm of the norm of the Killing field) \[7, 8\]. Unfortunately, this midi-superspace is a bit too complicated in that the issue of global existence of such solutions is still largely unexplored in the classical theory. However, if we require that there be another hyper-surface orthogonal Killing field in the 3+1 theory which commutes with the first one, the situation simplifies dramatically. For, now one can in effect “decouple” gravity and the scalar field. More precisely, the equation satisfied by the scalar field on the curved 2+1 dimensional space-time is equivalent to the wave equation on a fictitious flat 2+1-dimensional space-time. Therefore, one can first solve the second equation without any reference to the physical metric and then use the solution to obtain the physical metric by simple integration. Classically, one now has complete control on the issue of global existence.

Such space-times were considered by Einstein and Rosen in the thirties for the case when the first Killing field is a translation in the “z-direction” and the second is a “x-y rotation”. Thus, they represent cylindrical gravitational waves (with only one polarization because of the hyper-surface orthogonality requirement.) Their quantization was considered in a remarkable paper by Kuchař \[6\] already in 1971. The problem was considered again from a 2+1-dimensional perspective by Allen \[6\] in 1987 (without, however, realizing that this is precisely a symmetry reduced version of \[6\].) In the present paper, we shall return to this midi-superspace. Our purpose is two-folds: i) to supplement the analyses by Kuchař and Allen with a careful treatment of boundary conditions in the classical theory and of certain functional analytic issues in the quantum theory; and, ii) to use the resulting quantum theory to analyze
several conceptual and technical problems of quantum gravity. Since the model itself is simple enough to be exactly soluble, it provides a concrete arena to examine these vexing issues and to see how they can be resolved in practice.

Specifically, following \[5, 6\], we will use a canonical approach. Since in this approach one begins with a 2+1 decomposition of space-time, apriori it is not clear if quantized space-time geometries can emerge in the final theory. Indeed, one often hears the criticism that, since it is tied to space-like surfaces, the canonical approach may be inadequate to handle “space-time issues” such as “fluctuations of the light cone”. Here, we have a complete quantum theory. It is therefore natural to ask: are there operators on the final Hilbert space corresponding to space-time geometries? If so, is there adequate structure to analyze how the light cones fluctuate? More generally, can we tie the canonically quantized theory to the quantum description that emerges from covariant approaches? Can we compute S-matrices? In the classical theory, there is a positive energy theorem \[8, 9\]. Does it continue to hold in the quantum theory? Is the true ground state “peaked around” Minkowski space-time? Or, does the ground state contain wild quantum fluctuations with Planck energy density as suggested by Wheeler \[10\]? If so, the true ground state would not have much resemblance to Minkowski space, except perhaps on a suitable coarse-graining. Another question which plays an important role in semi-classical considerations is: Are there “coherent states” which are peaked at classical solutions?

There is a non-perturbative approach to full quantum gravity which is based on connections and triads (see, e.g., \[1\]). A basic assumption in that approach is that the “Wilson-loop operators” –which correspond to traces of holonomies of a connection around space-like loops– should be well-defined. Apriori it is not clear if this assumption is a reasonable one since in the definition of these operators, one appears to smear a quantum field along a one dimensional object (rather than three or four). It is natural to ask for the status of this assumption in a completely solved model. Are these Wilson loop operators well-defined on the explicitly known quantum Hilbert space?

Of course, just because such questions are answered in one way in a specific solution to this model, does not imply that they are not answered in another way in another solution and, more importantly, in full 3+1-dimensional quantum gravity. Nonetheless, the ability to answer them in detail in an explicitly solution can contribute substantially to our overall intuition for quantum gravity. Our analysis is primarily motivated by such considerations. We will find that most of these questions can be answered in detail but that the analysis involves several rather subtle points.

The plan of the paper is as follows. In section 2, we consider the classical Hamiltonian formulation and isolate the true degrees of freedom by a gauge fixing procedure. Because we are in an asymptotically flat situation, by treating the boundary conditions carefully, we can distinguish gauge from dynamics. In particular, the true degrees of freedom are naturally subject to non-trivial dynamics (without the need of
any “deparametrization”). In section 3, we calculate the classical Wilson loop functions and express them in terms of the true degrees of freedom. Quantization is taken up in section 4. As in [5, 6] the Hilbert space of states is a Fock space for scalar fields in 2+1 dimensions. Subtleties arise, however, because the geometrical observables—such as the space-time metric and the Wilson loops—are expressed as integrals of quadratic functionals of these elementary excitations. Thus, in a rough terminology, geometric excitations arise as non-local “collective modes” of the primary mathematical entity, the quantum scalar field. Finally, questions raised earlier in this section are analyzed within this solution. Section 5 summarizes the main results and points out directions for further work.

2 Hamiltonian Formulation

2.1 The midi-superspace

Let us begin with a precise specification of our midi-superspace. For definiteness, we will work in the 2+1-dimensional formulation. Thus, we will consider asymptotically flat, axi-symmetric solutions of 2+1-dimensional general relativity coupled to zero rest mass scalar-fields (where the rotational Killing field is hyper-surface orthogonal). The underlying manifold \( M \) will be topologically \( \mathbb{R}^3 \) and the space-time metric will have signature \(-,+,+\). For simplicity, we will assume that all fields under consideration are \( C^\infty \).

Denote by \( \sigma^a \) the rotational Killing field. Hyper-surface orthogonality of \( \sigma^a \) implies that the space-time metric \( g_{ab} \) has the form:

\[
g_{ab} = h_{ab} + R^2 \nabla_a \sigma \nabla_b \sigma
\]

where \( R \) is the norm of the Killing field and \( \sigma \) is the “angular coordinate”; \( \nabla_a \sigma = R^{-2} g_{ab} \sigma^b \). The field \( h_{ab} \) so defined is a metric of signature 

\( -,+,+ \) on the 2-manifolds orthogonal to \( \sigma^a \). Let us introduce a space-like foliation of this 2-manifold by lines \( t = \text{const} \) and a dynamical vector field \( t^a = N n^a + N^r \hat{r}^a \), where \( n^a \) is the unit, time-like normal to the foliation and \( \hat{r}^a \) the unit (outgoing) vector field within each slice. The pair \( N, N^r \) constitutes the lapse and the shift. If we now introduce a radial coordinate \( r \) on any one leaf such that \( r = 0 \) at the axis (i.e., where \( R = 0 \)) and \( r \) tends to infinity at spatial infinity, the 2-metric \( h_{ab} \) can be written as:

\[
h_{ab} = (-N^2 + (N^r)^2) \nabla_a t \nabla_b t + 2N^r \nabla_a t \nabla_b r + e^\gamma \nabla_a r \nabla_b r,
\]

where \( N, N^r \) and \( \gamma \) are functions of \( r \) and \( t \). It is because of axi-symmetry, that the 3-metric \( g_{ab} \) has only four independent components and they are functions only of two variables.

Thus, our midi-superspace consists of five functions, \( (N, N^r, \gamma, R, \psi) \) on the space-time manifold \( M \) where \( \psi \) is the zero rest mass scalar field (which is also Lie-dragged
The five fields are subject to the following field equations:

\[ G_{ab} = T_{ab} \quad \text{and} \quad g^{ab} \nabla_a \nabla_b \psi = 0, \] (3)

where \( G_{ab} \) is the Einstein tensor of \( g_{ab} \) which is determined by the fields \((N, N^r, \gamma, R)\) via (2) and \( T_{ab} \) is the stress-energy tensor of the scalar field \( \psi \):

\[ T_{ab} = \nabla_a \psi \nabla_b \psi - \frac{1}{2} (g^{cd} \nabla_c \psi \nabla_d \psi) g_{ab}. \] (4)

(Here, we have used a normalization that arises naturally in the reduction from the 3+1 theory to the 2+1. From the 2+1 perspective, it is natural to regard \( \phi := \psi/\sqrt{8\pi G} \) as the physical Klein-Gordon field, where \( G \) is Newton’s constant.)

Asymptotic flatness and regularity at the axis imply certain boundary conditions on our dynamical fields. We first note that \( g_{ab} \) reduces to a Minkowskian metric when \( N = 1, N^r = 0, \gamma = 0, R = r \) and \( \psi = 0 \). The general asymptotic flatness conditions can be written as:

\[ N = 1 + N_1(r, t), \quad N^r = N^r_1(t) + N^r_1(r, t) \]
\[ \gamma(r, t) = \gamma_\infty(t) + \gamma_1(r, t), \quad R(r, t) = r(1 + R_1(r, t)) \] (5)

where, on any \( t = \text{const} \) surface, \( N_1, N^r_1, \gamma_1, R_1 \) and the scalar fields \( \psi \) are of asymptotic order \( O(1/r) \). (We will say that a function \( f(r) \) is of asymptotic order \( 1/r \) if \( rf(r), r^2 f'(r) \) and \( r^3 f''(r) \) admits limits as \( r \) tends to infinity, where a prime denotes a derivative with respect to \( r \).) While the conditions imposed on \( N, N^r, R \) and \( \psi \) are the obvious ones, the condition on the field \( \gamma \) seems surprising at first. For, even at infinity, \( \gamma \) is not required to approach its Minkowskian value, 0. The reason is that the asymptotic value of \( \gamma \) contains the information about mass: If \( \gamma_\infty \neq 0 \), the spatial metric has a deficit angle at infinity which measures the ADM mass [8, 9]. Thus, there is a striking contrast with asymptotic flatness in 3+1 dimensions; the space-time metrics in our midi-superspace do not approach a fixed Minkowskian metric at infinity.

Note finally that these boundary conditions are somewhat simpler than those used in [4] where general 2+1-dimensional space-times were considered. Here, we can exploit the fact that we are now working in a highly restrictive context of cylindrical waves.

Finally, regularity at the axis is ensured by requiring that \( N^r, \gamma \) and \( R \) vanish there for all \( t \). (Recall also that by assumption, \( N, N^r, \gamma, R^2 \) and \( \psi \) are \( C^\infty \) everywhere and, in particular, at \( r = 0 \).)

### 2.2 Phase Space

Let us begin with the 3-dimensional action with appropriate boundary terms:

\[ S(g, \psi) := \frac{1}{16\pi G} \int_{M'} d^3x \sqrt{|R - g^{ab} \nabla_a \psi \nabla_b \psi|} + \frac{1}{8\pi G} \int_{\partial M'} d^2x [K \sqrt{h} - K_o \sqrt{h_o}], \] (6)
metric on $\partial M'$ induced by $g_{ab}$; and, $K_o$ and $h_o$ are the corresponding fields induced by the Minkowski metric $\tilde{g}_{ab}$ (obtained by setting $N = 1, N^r = 0, \gamma = 0, R = r$ and $\psi = 0$).

To pass to the Hamiltonian formulation, one performs a 2+1-decomposition. Let us substitute in (6) the form of the metric given by Eqs. (1) and (2). Then, the action reduces to the standard form:

$$S = \frac{1}{8G} \int dt \left( dr(p_\gamma \dot{\gamma} + p_R \dot{R} + p_\psi \dot{\psi}) - H[N, N^r] \right)$$

(7)

The Hamiltonian $H$ is given by:

$$H[N, N^r] = \frac{1}{8G} \int dr (NC + N^r C_r) + \frac{1}{4G} (1 - e^{-\gamma_\infty/2})$$

(8)

where $C$ and $C_r$ are functions of the canonical variables,

$$C = e^{-\gamma/2}(2R'' - \gamma' R' - p_\gamma p_R) + \frac{1}{2} R e^{-\gamma/2} \left( \frac{p_\psi^2}{R^2} + \psi'^2 \right),$$

$$C_r = e^{-\gamma}(-2p_\gamma' + \gamma' p_\gamma + R' p_R) + e^{-\gamma} p_\psi \psi',$$

(9)

and $\gamma_\infty$ is the value of $\gamma$ at $r = \infty$. (Here primes denote derivatives with respect to $r$.)

As expected, the lapse and shift functions $N, N^r$ appear as Lagrange multipliers; they are not dynamical variables. Thus, the phase-space $\Gamma$ consists of three canonically-conjugate pairs, $(\gamma, p_\gamma; R, p_R; \psi, p_\psi)$, on a 2-manifold $\Sigma$ which is topologically $R^2$. The boundary conditions on the configuration variables $(\gamma, R, \psi)$ have already been discussed. The conditions on the momenta can be deduced from their definitions in terms of these fields and their time derivatives. At infinity, $p_\gamma$ and $p_R$ fall-off as $O(1/r^2)$ while $p_\psi$ falls off as $O(1/r)$. (Note that these conditions imply that action $\int dr p_\gamma \delta \gamma, \int dr p_R \delta R$ and $\int dr p_\psi \delta \psi$ of the momenta on the tangent vectors $\delta \gamma, \delta R, \delta \psi$ to our configuration space are all finite, so that we have a well-defined (weakly non-degenerate) symplectic structure.) There are two first class constraints, $C = 0$ and $C_r = 0$, obtained by varying the action with respect to the Lagrange multipliers $N$ and $N^r$. The Hamiltonian is given by $H$. (It is because of the underlying axi-symmetry that we have only one diffeomorphism constraint, $C_r$.)

Let us begin by analyzing the canonical transformations generated by constraints. For this, we have to first smear the constraints and obtain well-defined functions on the phase space, say, $C[N_g] := \int dr N_g C$ and $C[N^r_g] = \int dr N^r_g C_r$. Using the boundary conditions on the phase space variables, it is straightforward to verify that these functions are well-defined and differentiable on the phase space if $N_g$ vanishes on the axis and is of asymptotic order $O(1/r)$ and $N^r_g$ admits a limit at infinity. (From now on, the subscript $g$ on smearing fields will indicate that they satisfy these boundary conditions.) Since the constraints are of first class, and since we are in the asymptotically flat context, the canonical transformations generated by these
constraints can be regarded as “gauge” in an appropriate sense. As one might expect, $C[N_g]$ generates “bubble time evolutions” via lapses which go to zero at infinity while $C[N^r_g]$ generates spatial diffeomorphisms which are bounded at infinity. The situation with the Hamiltonian constraint is the same as the one we normally encounter in the 3+1-dimensional theory. For the diffeomorphism constraint, on the other hand, the situation is quite different since the diffeomorphisms generated by $N^r_g r^a$ are not necessarily asymptotically identity. This is, however, the standard situation in 2+1 dimensions (see e.g., [8, 9]): In 2+1 dimensions, there are no asymptotic Killing fields corresponding to spatial translations and the ADM 2-momentum vanishes.

To obtain genuine time translations, we have to allow lapses which tend to 1 at infinity and on the axis. When this is done, the constraint function $C[N]$ continues to exist everywhere on the phase space. However, due to the presence of the first two terms involving derivatives of $\gamma$ and $R$ in the expression of $C$, the function $C[N]$ fails to be differentiable. To make it differentiable, one has to add a surface term. As one might expect, this is precisely the surface term in the expression (8) of the Hamiltonian. Thus, the function which generates the canonical transformation corresponding to (asymptotically unit) time translation is precisely the Hamiltonian $H[N]$ (obtained by setting $N^r = 0$ in Eq. (8)). On physical states—the numerical value of the Hamiltonian is given by the surface term in (8):

$$E = \frac{1}{4G}(1 - e^{-\frac{1}{2}\gamma_\infty}),$$

(10)

As usual, in the space-time picture, the evolution generated by the Hamiltonian on the phase space corresponds to motions along the vector field $t^a$.

Let us summarize the discussion of this sub-section. Because we are in the asymptotically flat context, there is a clean separation between gauge and dynamics. As usual, when it comes to physical interpretation, the “gauge transformations” of general relativity have a somewhat different status from that in Yang-Mills theory. It is not that the diffeomorphisms generated by $C[N_g]$ and $C[N^r_g]$ are “unphysical”. Rather, they are “redundant” when it comes to extracting the physical content of the theory. As we will see below, we can gauge fix these constraints and extract the true degrees of freedom. The Hamiltonian generates “time evolution” among these gauge fixed points. Knowing this evolution, we can reconstruct the entire solution; motions generated by constraints are not needed and are in this sense “redundant”.

### 2.3 Gauge Fixing

Since the canonical transformations generated by $C[N_g]$ and $C[N^r_g]$ are to be regarded as gauge, as in Yang-Mills theory, to gauge fix the system we need to extract one point from each orbit of the corresponding Hamiltonian vector fields. This is achieved by imposing gauge fixing conditions which, together with the constraints, constitute a second class system. As in [8], we will choose these conditions to make the space-time
geometry transparent. Let us demand:

\[ R(r) = r \quad \text{and} \quad p_\gamma (r) = 0. \]  \hspace{1cm} (11)

The first condition is motivated by the fact that, in any solution to the field equations (satisfying our boundary conditions), the gradient \( \nabla_a R \) of the norm of the Killing field \( \partial / \partial \sigma \) is space-like everywhere on \( M \). Since furthermore \( R \sim r \) at the axis and at infinity, it is natural to use \( R \) itself as the radial coordinate. After this condition is imposed, \( R \) will no longer be a dynamical variable. The second gauge fixing condition will remove \( \gamma \) from our list of dynamical variables. Thus, if these conditions are admissible, the true degrees of freedom will all reside in the field \( \psi \), in accordance with our general expectation that in 2+1 dimensions, all the local degrees of freedom are carried by matter fields.

To see if our gauge fixing conditions are admissible, let us compute their Poisson brackets with the constraints. We have:

\[
\{ R(r) - r, C_r [N^r_g] \} = N^r_g e^{-\gamma} R' \\
\{ p_\gamma, C[N_g] \} = \left[ \frac{N_g}{2} \left( -p_\gamma p_R + \frac{p_\psi^2}{2 R} + \frac{R}{2} \psi'^2 \right) - N'_g R' \right] e^{-\frac{\gamma}{2}},
\]  \hspace{1cm} (12)

where, as before, \( N^r_g \) and \( N_g \) are pure gauge lapses and shifts. If \( N^r_g \neq 0 \) and \( N_g \neq 0 \), the right sides of (12) do not vanish at any point on the intersection of the surfaces defined by constraints and gauge fixing conditions (11). Hence, as needed, the gauge fixed surface intersects the gauge orbits transversely.

The question now is whether we can choose lapse and shift such that the dynamical evolution generated by the Hamiltonian \( H[N, N^r] \) preserves the gauge conditions. More precisely, since the Hamiltonian \( H[N, N^r] \) weakly commutes with the constraints \( C[N_g], C[N^r_g] \), we know that the dynamical evolution it generates maps entire gauge orbits to entire gauge orbits. The question is if we can select \( N, N^r \) such that the image under evolution of any gauge fixed point on the constraint surface is another gauge fixed point. General considerations from symplectic geometry imply that if such a pair exists, it is unique. We will now establish the existence. Let us begin with the Poisson brackets between the gauge conditions and the Hamiltonian:

\[
\{ R(r) - r, H[N, N^r] \} \approx N^r e^{-\gamma} \\
\{ p_\gamma (r), H[N, N^r] \} \approx \left[ \frac{N}{4r} \left( \frac{p_\psi^2}{r^2} + r^2 \psi'^2 \right) - N' \right] e^{-\frac{\gamma}{2}},
\]  \hspace{1cm} (13)

where \( \approx \) stands for equality modulo constraints and gauge conditions. We seek \( N \) and \( N^r \) which satisfy our boundary conditions (namely, \( N = 1 + O(1/r) \) and \( N^r = N^r_0 + O(1/r) \) at infinity) and for which the right hand sides of (13) vanish (modulo constraints and gauge conditions). The only solutions are:

\[
N(r) = \exp \left( -\frac{1}{4} \int_r^\infty dr_1 r_1^2 \left( \frac{(p_\psi)^2}{r_1^2} + (\psi')^2 \right) \right) \quad \text{and} \quad N^r(r) = 0.
\]  \hspace{1cm} (14)
Finally, let us extract the true degrees of freedom of the theory. In order to accomplish this, we need to eliminate redundant variables by solving the set of second class constraints (9) and use gauge conditions (11). By setting $R = r$ and $p_\gamma = 0$ in (9), we can trivially solve for $\gamma$ and $p_R$ in terms of $\psi$ and $p_\psi$ (using the Hamiltonian and the diffeomorphism constraints respectively). The result is:

$$\gamma(R) = \frac{1}{2} \int_0^R dR_1 R_1 \left( \frac{p_\psi^2}{R_1^2} + \psi'^2 \right),$$  
$$p_R = -p_\psi \psi'. \quad (15)$$

Substituting (15) in (14), we can also express the lapse $N$ in terms of $\gamma$. Thus, as expected, the true degrees of freedom reside just in the matter variables. Indeed, the space-time metric is now completely determined by $\psi$ and $p_\psi$:

$$g_{ab} = e^{\gamma(R,t)} \left( -e^{-\frac{\gamma_\infty}{2}} \nabla_a t \nabla_b t + \nabla_a R \nabla_b R \right) + R^2 \nabla_a \sigma \nabla_b \sigma, \quad (17)$$

where, from now on, $\gamma$ will only serve as an abbreviation for the right side of (15).

### 2.4 Reduced Phase Space

It is obvious from the above discussion that the reduced phase space $\tilde{\Gamma}$ can be coordinatized by the pair $(\psi(R), p_\psi(R))$. The (non-degenerate) symplectic structure on the reduced phase space $\tilde{\Gamma}$ is the pull-back of the symplectic structure on $\Gamma$. Thus,

$$\{\psi(R_1), p_\psi(R_2)\} = \delta(R_1, R_2) \quad (18)$$

on $\tilde{\Gamma}$. Next, let us write the reduced action by substituting (11), (15) and (16) in (6),

$$S[\psi, p_\psi] = \frac{1}{8G} \int dt \left[ \int dR(p_\psi \dot{\psi}) - 2(1 - e^{-\frac{1}{2} \gamma_\infty}) \right], \quad (19)$$

where, as before, $\gamma_\infty = \gamma(r = \infty)$. By varying the action (19) with respect to $\psi$ and $p_\psi$ we then obtain equations of motion:

$$\dot{\psi} = e^{-\frac{1}{2} \gamma_\infty} \frac{p_\psi}{R} \quad \text{and} \quad \dot{p}_\psi = e^{-\frac{1}{2} \gamma_\infty} (R \psi)''. \quad (20)$$

Due to the presence of $\exp(-\frac{\gamma_\infty}{2})$ factors, these equations are highly non-linear. However, using (20) it is straightforward to check that $\gamma_\infty(t)$ is a constant of motion. Hence, given any one solution, we can define a new time coordinate $T$ on $M$ via a constant rescaling: $T := (\exp -\frac{1}{2} \gamma_\infty) t$. Then, the field $\psi$ satisfies the following linear second-order equation of motion:

$$- \frac{\partial^2 \psi}{\partial T^2} + \frac{\partial^2 \psi}{\partial R^2} + \frac{1}{R} \frac{\partial \psi}{\partial R} = 0. \quad (21)$$
This is exactly the Klein-Gordon equation for a scalar field propagating on a Minkowskian background $g^o_{ab}$, given by:
\[
g^o_{ab} = -\nabla_a T \nabla_b T + \nabla_a R \nabla_b R + R^2 \nabla_a \sigma \nabla_b \sigma. \tag{22}
\]

Thus, a remarkable simplification has occurred. We can just solve for a free scalar field $\psi$ in Minkowski space $(M, g^o_{ab})$, define a function $\gamma$ through (15), and construct a curved metric $g_{ab}$ through (17). Then the pair $(g_{ab}, \psi)$ satisfies the non-linear Einstein-Klein-Gordon equations.

This decoupling is not surprising from the space-time perspective. Indeed, it has been exploited repeatedly in the literature. However, it is illuminating to see how the decoupling comes about from a phase space perspective especially since the dynamics of the true degrees of freedom is driven only by the boundary term Hamiltonian which, furthermore, seems quite complicated at first sight. Note also that, while from a space-time perspective the passage between $t$ and $T$ involves only a constant rescaling, since the constant varies from solution to solution, from a phase space perspective it is a rather complicated, “q-number” transformation. Thus, in quantum theory, if one variable in the pair $(t, T)$ is taken as a “time-parameter”, the other will be a genuine operator. It is therefore instructive to contrast the two notions of time.

By construction, $t$ can be identified with the affine parameter along the Hamiltonian vector field defined by (8) on the phase space. Given any dynamical trajectory, we obtain a space-time metric $g_{ab}$ and $t$ can then be thought of as a time coordinate on $M$ with the property that $\partial/\partial t$ generates an unit time translation at infinity. The parameter $T$, on the other hand does not have a direct and simple physical interpretation in our phase space framework. Its most direct interpretation comes from the fiducial Minkowskian metric $g^o_{ab}$ on $M$. Even at infinity, the norm of the vector field $\partial/\partial T$ varies from one physical metric $g_{ab}$ to another. For the decoupling procedure, on the other hand, it is natural to fix, once and for all, the Minkowskian metric $g^o_{ab}$ on $M$ and regard $g_{ab}$ simply as a “derived” quantity. Then $T$ does have a natural interpretation of time. Finally, note that this somewhat peculiar situation arose because, in 2+1 dimensions, the physical metrics $g_{ab}$ do not approach a fixed Minkowskian metric even at infinity (or alternatively, because in 3+1 dimensions, cylindrical waves fail to be asymptotically flat in the conventional sense.)

We will conclude this section with a remark. To begin with, one can ignore the broad physical problem of interest and focus just on a free scalar field satisfying the wave equation on the Minkowskian background $(M, g^o_{ab})$. The phase space for this system is the same as our reduced phase space and the Hamiltonian is given by $\gamma_\infty$. However, $\gamma_\infty$ does not have a direct physical interpretation in terms of the original, coupled system; the physical energy of our system is given by (10).
3 Holonomy

As explained in the introduction, there is a non-perturbative approach approach to quantum general relativity in 3+1 dimensions which is based on the assumption that traces of holonomies of a certain connection are well-defined operators in the quantum theory. We would like to investigate the status of this assumption in the context of our midi-superspace. Therefore, in this section, we will make a short detour to compute the holonomy in question in the classical theory. Readers who are not familiar with this approach to quantum gravity may skip this section without loss of continuity.

In the first-order (Palatini) formalism for 2 + 1 general relativity the fundamental variables are triads $e^I_a$ and connection 1-forms which take values in the Lie-algebra of $SU(1, 1)$ [11, 4]. Let us denote the $SO(2, 1)$ connection by $A^I_a$ and its pull-back to the 2-dimensional slice $\Sigma$ by $A^I_a$, where $I, J, \cdots = 0, 1, 2$ are internal indices with respect to a basis $\tau^I$ in the Lie algebra of $SU(1, 1)$. The internal indices are raised and lowered with a Minkowski metric $\eta_{IJ}$ with signature $(-, +, +)$.

### 3.1 $SO(2, 1)$ Connection

To obtain the internal connection for the space-time metric (17), we need to fix the internal (i.e., $SU(1, 1)$) gauge. This is accomplished by fixing the triads $e^I_a$. Our choice will be:

$$e^I_a \tau^I = \sqrt{2} e^{\gamma(R,t)-\gamma(\infty)} (\nabla_a t) \tau^I_0 + \sqrt{2} e^{\gamma(R,t)} (\nabla_a R) \tau^I_1 + \sqrt{2} R (\nabla_a \sigma) \tau^I_2$$

(23)

It is straightforward to check that the space-time metric (17) is recovered via $g_{ab} = \eta_{IJ} e^I_a e^J_b$ with the convention $\eta_{IJ} = 2 \text{Tr}(\tau^I \tau^J)$. The triad determines the (Christoffel symbols and the) internal connection $A^I_a$ uniquely. Its pull-back to the spatial slice $\Sigma$ turns out to be:

$$A^I_a = A^I_a \tau^I = \frac{\gamma}{2} e^{\frac{1}{2} \gamma(\infty)} (\nabla_a R) \tau^I_0 + e^{-\frac{1}{2} \gamma} (\nabla_a \sigma) \tau^I_0.$$  

(24)

Note, however, that since $R, \sigma$ fail to be smooth at $R = 0$ our connection also fails to be smooth there. However, our boundary conditions do ensure that all physical fields are smooth at the origin. Thus, this singularity is merely a reflection of a bad choice of gauge (which has in effect introduced a “source” at the origin). We can remedy this situation by a gauge transformation. The general form of gauge transformations is:

$$A^I_a = g A^I_a g^{-1} - (\partial_a g) g^{-1} \quad \text{with} \quad g = e^{\tau^I \Lambda^I(R,\sigma)}.$$  

(25)

By choosing the transformation parameters to be $\Lambda^0 = e^{-\frac{1}{2} \gamma(0)} \sigma$ and $\Lambda^1 = \Lambda^2 = 0$, we obtain a smooth connection as desired:

$$A^I_a = A^I_a \tau^I = \frac{\gamma}{2} e^{\frac{1}{2} \gamma(\infty)} \nabla_a R [\cos \sigma \tau^I_2 - \sin \sigma \tau^I_1] + [e^{-\frac{1}{2} \gamma} - 1] \nabla_a \sigma \tau^I_0.$$  

(26)
3.2 Holonomy computation

The holonomy of $A'^I_a$ along a loop $\eta$ is given by a path ordered exponential of the integral of $A'^I_a$ along $\eta$:

$$U_\eta[A] := \mathcal{P} \exp \left( \oint_\eta A_a dS^a \right). \tag{27}$$

For quantum considerations, it turns out that the most interesting loops are the integral curves of the rotational Killing vector $\sigma^a$. Note that, along these curves, only the second term in the expression (26) of the connection contributes. Since the internal vector in this term is constant, this part of the connection is effectively Abelian. Recall that in the case of an Abelian connection the path ordered exponential reduces to an ordinary exponential. Hence, if $\eta$ is chosen to be the integral curve of the Killing field with radius $r_o$, the holonomy can be easily evaluated. We have:

$$U_\eta[A'] = \cos \left[ \pi \left( 1 - e^{-\frac{1}{2} \gamma(r_o)} \right) \right] - 2\tau_0 \sin \left[ \pi \left( 1 - e^{-\frac{1}{2} \gamma(r_o)} \right) \right] \tag{28}$$

where we have used the fundamental representation of $SU(1,1)$. For our purposes, it will suffice to consider these particular loops.

Of special interest to the quantization program under consideration are the functions $T^0_\eta[A]$ of connections defined by the trace of the holonomy. Taking the trace of (28) yields

$$T^0_\eta[A'] = 2 \cos \left[ \pi \left( 1 - e^{-\frac{1}{2} \gamma(r_o)} \right) \right]. \tag{29}$$

Note, incidentally, that if $\eta$ is chosen to be the loop at infinity, $T^0_\eta[A]$ reduces to a simple function of the total energy of the coupled system. For the reduced system, $\gamma(r_o)$ represents precisely the energy of $\psi$ in a box of radius $r_o$ (where $\psi$ is regarded as a scalar field propagating on the Minkowskian background.) The question of whether the $T^0_\eta$ can be promoted to a well-defined operator will therefore reduce to the question of whether the operator corresponding the energy of a scalar field in a box can be satisfactorily regulated.

4 Quantum Theory

4.1 Quantization

The reduced phase space of section 2.4 serves as the natural point of departure for quantization. Since the constraints have been solved, the algebra $\mathcal{A}$ of observables is easy to construct. The obvious complete set of classical observables is given by the smeared fields and momenta, $\hat{\psi}(f) := \int dr f(r) \hat{\psi}(r)$ and $\hat{p}_\psi(g) := \int dr g(r) \hat{p}_\psi(r)$, where $f, g$ belong to the Schwartz space $\mathcal{S}$ of smooth test functions with rapid decay at infinity. Thus, the quantum algebra $\mathcal{A}$ is generated by operators $\hat{\psi}(f)$ and $\hat{p}_\psi(g)$, subject to the canonical commutation relations:

$$[\hat{\psi}(f), \hat{\psi}(g)] = 0, \ [\hat{p}_\psi(f), \hat{p}_\psi(g)] = 0, \ [\hat{\psi}(f), \hat{p}_\psi(g)] = i \int dr f g \hat{I}. \tag{30}$$
Our task is to find a representation of $A$ which, furthermore, carries a well-defined Hamiltonian operator $\hat{H}$, the quantum analog of $\frac{1}{4\pi l^2}(1 - \exp(-\frac{2\gamma}{\infty}))$.

For technical simplicity, we will regard $\hat{\psi}$ and $\hat{p}_\psi$ as operator-valued distributions in two (space) dimensions and incorporate rotational symmetry by restricting the states to be axi-symmetric at the very end. Our experience from low dimensional, interacting scalar quantum field theories now suggests that we use as our Hilbert space $H = L^2(S', d\mu)$ where $S'$ is the space of all tempered distributions on $R^2$, and $\mu$ a suitable measure thereon. (For details, see, e.g., [12]). Since $\gamma_\infty$ is the Hamiltonian of the free scalar field in Minkowski space, to make the quantum Hamiltonian operator well-defined, it is natural to use for $\mu$ the standard Gaussian measure for a free, massless scalar field with covariance $\frac{1}{2}\Delta - \frac{1}{2}$, where $\Delta$ is the Laplacian on $R^2$ with respect to the flat metric $q_{ab}^{\mu} = \nabla_a R \nabla_b R + r^2 \nabla_a \theta \nabla_b \theta$. Thus, $\mu$ is defined by

$$\int_{S'} d\mu e^{i\int d^2x f(\vec{x}) \tilde{\psi}(\vec{x})} = e^{-\frac{1}{2}\int d^2x f(\vec{x}) \Delta - \frac{1}{2} f(\vec{x})} \quad (31)$$

where $\tilde{\psi} \in S'$. (Heuristically, “$d\mu = [\exp -\frac{1}{2}\int d^2x (\psi \Delta \frac{1}{2} \psi)] D\psi$.”) The action of the basic operators is then given by:

$$\hat{\psi}(f) \cdot \Psi(\psi) = \left( \int d^2x f \psi \right) \Psi(\psi) \quad \text{and} \quad \hat{p}_\psi(g) \cdot \Psi(\psi) = -i \int d^2x \left[ g \frac{\delta}{\delta \psi} + \frac{1}{2} \psi \Delta \frac{1}{2} g \right] \Psi(\psi) \quad (32)$$

where $\Psi$ belongs to the dense sub-space of cylindrical functions in $\mathcal{H}$. The operators $\hat{\psi}(f)$ and $\hat{p}_\psi(f)$ admit self-adjoint extensions to $\mathcal{H}$. We will see below that the Hamiltonian is also represented by a self-adjoint operator and that, like its classical counterpart, it is positive.

This choice of representation is also suggested by the mathematical equivalence between our physical system and a free massless scalar field on Minkowski space defined by $g_{ab}^0$ (see Eq (21)). Thus, although our viewpoint is somewhat different, our final choice of representation is the same as that of Refs [5, 6].

In a more familiar terminology, our representation can be obtained by introducing an operator-valued distribution $\hat{\psi}(\vec{x}, T)$ in the fictitious Minkowskian background $(M, g_{ab}^0)$:

$$\hat{\psi}(\vec{x}, T) = \frac{1}{2\pi} \int \frac{d^2k}{\sqrt{2\omega_k}} \left[ \hat{A}(\vec{k}) e^{i(k \cdot \vec{x} - \omega_k T)} + \hat{A}^\dagger(\vec{k}) e^{-i(k \cdot \vec{x} - \omega_k T)} \right], \quad (33)$$

where $\omega_k = \sqrt{k \cdot k}$, and $\hat{A}(\vec{k})$ and $\hat{A}^\dagger(\vec{k})$ are the standard creation and annihilation operators. The Hilbert space $\mathcal{H}$ can be generated by repeated actions of creation operators on the vacuum. There is a well-defined self-adjoint operator $\hat{L}_\sigma$ on $\mathcal{H}$ which
represents the total angular momentum along the Killing field $\partial/\partial \sigma$. The physical Hilbert space $\mathcal{H}_P$ is the eigenspace of $\hat{L}_\sigma$ with zero eigenvalue. Since zero is a discrete eigenvalue, $\mathcal{H}_P$ is a sub-space of $\mathcal{H}$.

The physical Hilbert space can also be obtained more directly by using, instead of (33), an operator valued distribution in which the zero angular momentum constraint has already been incorporated, namely,

$$
\hat{\psi}(R, T) = \int_0^\infty dk \left[ f_k^+(R, T)\hat{A}(k) + f_k^-(R, T)\hat{A}^\dagger(k) \right].
$$

Here $f_k^+(R, T) = \left[ f_k^-(R, T) \right]^* = \frac{1}{\sqrt{2}} J_0(kR)e^{-i\omega_kT}$, where, from now on, $J_n(kR)$ will denote the $n$-th order Bessel function of the first kind. Note that $f_k^+(R)$ are solutions of the equation of motion (21) and provide an orthonormal basis for the one-particle Hilbert space with respect to the Klein-Gordon inner-product. (Our normalization is such that the creation and annihilation operators satisfy the commutation relations $[\hat{A}(k), \hat{A}^\dagger(k')] = \delta(k, k').$) The physical Hilbert space $\mathcal{H}_P$ can be generated by repeatedly acting on the vacuum by the creation operators $\hat{A}^\dagger(k)$. In what follows, we will use both the two dimensional as well as the one dimensional descriptions given by (33) and (34).

We will conclude this sub-section with three remarks.

1) Since the physical Hilbert space has a Fock structure, it is tempting to refer to the quanta created by $\hat{A}^\dagger(k)$ as (scalar) “particles” and we will often do so. Note, however, that from the point of view of the coupled Einstein-Klein-Gordon system we began with, this description is gauge dependent. The system has one local degree of freedom and we chose to put it in the scalar field. Another gauge choice could put it in the gravitational field and the interpretation of quantum states would then be different. However, the interpretation is unambiguous at null infinity –i.e, for asymptotic states– because one does not need to fix gauge there (see below).

2) We now have the full Hilbert space of states. So, it is natural to examine if one can generate a picture of space-time –as opposed to just spatial– quantum geometry in spite of our use of the canonical approach. As one might expect from our gauge-fixing procedure, the answer is in the affirmative. In the fixed chart $(T, R, \sigma)$ on $M$, the metric operator can be (heuristically) written as:

$$
\hat{g}_{ab} = :e^{\hat{g}(R, T)} : (-\nabla_a T \nabla_b T + \nabla_a R \nabla_b R) + R^2 \nabla_a \sigma \nabla_b \sigma :,
$$

where, as usual, the double-dots indicate normal ordering. (The reason behind the qualification “heuristic” and the quotes will become clear in section 4.3.)

We can now ask if there are well-defined semi-classical states peaked at classical solutions. The answer is again in the affirmative. Consider, in the Fock space, a coherent state $|\Psi_c >$ which is peaked at a classical solution $c(R, T)$ of (21). In
the configuration representation, these are Gaussians for which the uncertainty
in the field operator and its momentum are “shared equally”, the product of
the two uncertainties being minimum for all times $T$. On these states, the
expectation value of the metric operator \( (35) \) is well-defined
and is given just by

\[
< \Psi_c | \hat{g}_{ab} | \Psi_c > = e^{\gamma[c,p_c]}(-\nabla_a T \nabla_b T + \nabla_a R \nabla_b R) + R^2 \nabla_a \sigma \nabla_b \sigma, \tag{36}
\]

where $\gamma[c,p_c]$ is the right side of \( (14) \), evaluated on the initial data of the
classical solution $c$. Thus, every coherent states in our physical Hilbert space
$\mathcal{H}_0$ remains peaked at a classical scalar field $c$ and a metric $g_{ab}$, satisfying the
coupled Einstein-Klein-Gordon equation. While the result is technically rather
simple, conceptually it is somewhat surprising. For, the coupled system satisfies
highly non-linear equations and the wave packets do not disperse in spite of these
non-linearities.

3) It is well-known that there exist an infinite number of unitarily inequivalent
representations of the algebra $\mathcal{A}$. Our additional requirements are that the
Hamiltonian operator be well-defined and that the physical states be invariant
under the rotational symmetry corresponding to $\partial/\partial \sigma$. Unfortunately, these
requirements by themselves are not strong enough to select a representation
uniquely. To single out the Fock representation in Minkowskian quantum field
theories, one needs additional conditions that refer to the action of the Poincaré
group. In our case, the Minkowski space-time $(M, g_{ab}^0)$ is only a fictitious back-
ground and its Poincaré group has no physical significance in the full, coupled
system.

Nonetheless, it is possible to single out our representation by two methods.
The first involves the imposition of reality conditions as indicated in \[4\]: The
measure $\mu$ on $S'$ is singled out by the condition that the operators $\hat{\psi}(f)$ and
$\hat{p}_\psi(g)$ of \( (32) \) be self-adjoint. The second method invokes the S-matrix theory.
It turns out that the Einstein-Rosen waves are all asymptotically flat at null
infinity in 2+1 dimensions \[13\]. Furthermore, the classical S-matrix is well-
deﬁned: the data on past null inﬁnity determines the solution uniquely which
in turn determines the data on future null inﬁnity. Hence, it is natural to use
the asymptotic quantization scheme \[14\] to quantize the coupled system at past
and future null inﬁnity. It turns out that our Fock representation is naturally
isomorphic to the simplest representation obtained by asymptotic quantization
(either at past or future null inﬁnity). Details will appear elsewhere.

### 4.2 Hamiltonian and Time

Recall that, after reduction, the classical Hamiltonian is given by $H = \frac{1}{\sqrt{\pi}}[1 − \exp −(\frac{1}{2} \gamma_\infty)]$. Since $\gamma_\infty$ is the Hamiltonian of a free scalar field in Minkowski space,
the normal-ordered operator $:\hat{\gamma}_\infty:$ admits the standard self-adjoint extension which,
for simplicity, we will denote also by \( \hat{\gamma}_\infty \). Then, the standard spectral theorems ensure that

\[
\hat{H} := \frac{1}{4G} (1 - e^{-\frac{1}{2} \hat{\gamma}_\infty}) = \frac{1}{4G} (1 - e^{-\int kd k \hat{A}^\dagger(k) \hat{A}(k)})
\]

(37)
is a well-defined, self-adjoint operator. Since \( \hat{\gamma}_\infty \) is a non-negative, unbounded operator and since \( f(\lambda) = (1 - e^{-\frac{\lambda}{2}}) \) takes values in \([0, 1]\) for \( \lambda \in [0, \infty] \), it follows that the spectrum of \( H \) is given by \([0, 1/4G]\). If we consider states in \( \mathcal{H}_P \) with higher and higher frequency, the expectation value of \( \hat{\gamma}_\infty \)—i.e., the energy in the field from the mathematical, Minkowskian perspective—increases unboundedly. However, the expectation value of the physical Hamiltonian \( \hat{H} \) remains bounded and tends to the limit \( 1/4G \). Thus, the situation is completely analogous to that in the classical theory [5].

Let us now examine the ground state. Since \( |0\rangle \) is the unique ground state of \( \hat{\gamma}_\infty \) on \( \mathcal{H}_P \), it follows immediately that it is also the unique ground state of \( \hat{H} \). Since \( |0\rangle \) is, in particular, a coherent state, it is peaked at a classical solution to the coupled system. As one might expect, the solution is: \( \psi = 0 \) and \( g_{ab} = g_{ab}^o \). Thus, the quantum ground state is peaked on Minkowski space-time. The ground state geometry is thus quite tame, there is no evidence of wild fluctuations at the Planck scale.

What is the situation with general coherent states? Given a coherent state \( |\Psi_c\rangle := \exp[\int dk c(k) \hat{A}^\dagger(k)] \cdot |0\rangle \), peaked at a classical solution \( c \), we have:

\[
[\exp -\frac{1}{2} \hat{\gamma}_\infty] \cdot |\Psi_c\rangle = [\exp \int dk e^{k} c(k) \hat{A}^\dagger(k)] \cdot |0\rangle := |\Psi_c\rangle'
\]

(38)
where, \( c'(k) = e^k c(k) \). Thus, the image is again a coherent state but its peak is shifted. Therefore, the expectation value of the Hamiltonian in a coherent state \( |\Psi_c\rangle \) is given by:

\[
<\Psi_c, \hat{H} \cdot \Psi_c> \quad \frac{1}{<\Psi_c, \Psi_c>} = \frac{1}{4G} \left[ 1 - \exp \frac{1}{\hbar} \int dk (e^{-\hbar k} - 1)|c(k)|^2 \right],
\]

(39)
where, to bring out the quantum effects, we have restored the factors of \( \hbar \). (Recall also that, from the perspective of the 2+1-dimensional theory, the scalar field has to be rescaled by factors involving \( \sqrt{G} \). The net effect is to replace \( \hbar G \) which has the physical dimension of length.) By contrast, the classical energy (33) of the solution to the Einstein-Klein-Gordon equation determined by \( c \) is \( E(c) = \frac{1}{4G} (1 - \exp - \int dk |c(k)|^2) \). If we expand out \( \exp \hbar k \) in (33), the leading term yields the classical answer. In general, the classical energy is a good approximation to the expectation value of the quantum Hamiltonian if \( c(k) \) is concentrated on low frequencies. Quantum corrections (of order \( G\hbar \) and higher) become more and more significant if the support of \( c(k) \) is shifted to higher and higher frequencies.

Next, let us consider the issue of time. Recall that, in the classical theory, the Hamiltonian evolution is tied to time \( t \), the affine parameter along the Hamiltonian
vector field in the phase space. Each dynamical trajectory gives rise to a space-time and $t$ can then be interpreted as a time coordinate in that space-time, $\partial / \partial t$ being an unit asymptotic time translation. From the decoupling viewpoint, on the other hand, it is the variable $T$ that arises naturally; it represents time in the fixed Minkowskian background. What is the situation in the quantum theory? Now, our measure $\mu$ on $S'$ which dictates the Hilbert structure is rooted in the flat 2-geometry induced by $g^{ab}$ or, alternatively, in the positive and negative frequency decomposition with respect to the Minkowskian time $T$. Indeed, since the field equation (20) in terms of $t$ is non-linear, positive frequency decomposition with respect to $t$ is not meaningful apriori. Thus, while $t$ and $T$ are on equal footing in the classical theory, our choice of representation breaks this symmetry in the quantum theory.

We can mimic the situation in the classical theory and introduce a dynamical parameter $\lambda$—analogous to the classical $t$—associated with the Hamiltonian:

$$i\hbar \frac{\partial \Psi}{\partial \lambda} = \hat{H} \cdot \Psi.$$ (40)

However, unlike in the classical theory, now a solution to the dynamical equation does not define a hyperbolic space-time and hence we can not interpret $\lambda$ as a time parameter in the familiar sense, i.e., in space-time terms. However, a key simplification occurs if we restrict ourselves to coherent states $\Psi_c$. Since each of these states is peaked at a classical space-time, we can ask if, given any one of these states, we can interpret $\lambda$ as a time parameter in the corresponding classical space-time. The answer is in the affirmative. In fact $\lambda$ can be identified with the time coordinate $t$ of that space-time! Thus, as one might have hoped, the familiar notion of time re-emerges in the semi-classical regime. In the full quantum theory, however, the dynamical parameter defined by the Hamiltonian does not have a simple space-time interpretation.

We will conclude this discussion with a remark. There is an obvious alternative form for the Hamiltonian: We can further normal-order $\hat{H}$ and define a new Hamiltonian $\hat{H}' = :\hat{H}:$. One can verify that $\hat{H}'$ is densely defined and admits a self-adjoint extension. It also annihilates $|0>$ and $|\Psi_c>$ equals the classical energy of $c$. It thus appears to be an attractive alternative. However, its spectrum is the entire real line! This comes about because the overall normal ordering ensures that, while acting on $n$-particle states, only the first $n + 1$ terms in the expansion of the exponential in $\hat{H}'$ have non-vanishing contributions. Thus, for example, on 1-particle states, $\hat{H}'$ has the same action as $\frac{1}{8} :\hat{\gamma}_1^\infty :$ which is unbounded above. Similarly, on two particle states, it is unbounded below. Given that the classically allowed energy values lie in the interval $[0, 1/4G]$, we can not take $\hat{H}'$ as the physically admissible quantum analog of the classical Hamiltonian.
4.3 Metric operator

Since we are dealing with a system with an infinite number of degrees of freedom, operators corresponding to physical observables have to be regulated. For the Hamiltonian, this was achieved via normal ordering. In this section, we will focus on the metric operator.

A formal expression for the metric operator was already given in (35), where regularization again consisted of normal ordering. Consider the sub-space of \( H_P \) which is spanned by finite linear combinations of coherent states. It is easy to show that the sub-space is dense and that the matrix elements of the metric operator \( \hat{g}_{ab} \) are well-defined on it. Thus, the formal expression (35) does lead to a well-defined quadratic form; in a field theory terminology, \( \hat{g}_{ab} \) exists in the LSZ sense. However, this does not imply that \( \hat{g}_{ab} \) is well-defined as an operator on this sub-space. Note that this is not a peculiarity of quantum field theory; one encounters such situations already in non-relativistic quantum mechanics. Consider, for example, a 1-dimensional harmonic oscillator. The operator \( \exp(\alpha a^\dagger a^\dagger) \) has finite matrix elements on the basis \( |n> \) for all complex numbers \( \alpha \). However, if \( |\alpha| > 1 \), the norm \( ||e^{\alpha a^\dagger a^\dagger}|n> || \) diverges for any \( |n> \), whence the operator fails to be defined on the sub-space spanned by these basis vectors.

It turns out that the situation with the metric operator is quite analogous (which is the reason behind the quotes in (35)). To see this, let us begin with the first non-trivial term in the expansion of \( \hat{g}_{RR} \) or \( \hat{g}_{TT} \). Setting for simplicity \( T = 0 \) in (34), we have:

\[
: \hat{\gamma}(R) : = \frac{1}{2} \int dk_1 \int dk_2 \left[ 2F_+(R, k_1, k_2) \left( \hat{A}^\dagger(k_1) \hat{A}(k_2) \right) + F_-(R, k_1, k_2) \left( \hat{A}(k_1) \hat{A}(k_2) + \hat{A}^\dagger(k_1) \hat{A}^\dagger(k_2) \right) \right],
\]

(41)

where

\[
F_\pm(R, k_1, k_2) = \pm k_1 k_2 \int_0^R rdr \left( J_0(k_1 r) J_0(k_2 r) \pm J_1(k_1 r) J_1(k_2 r) \right).
\]

(42)

For any fixed \( R \), one can regard the coefficient \( F_-(R, k_1, k_2) \) of \( \hat{A}^\dagger(k_1) \hat{A}^\dagger(k_2) \) as a “potential 2-particle state” in the Fock space. However, a direct calculation shows that its norm is ultra-violet divergent. This immediately implies that the norm \( || : \hat{\gamma}(R) : |0> || \) also diverges, whence the operator fails to be well-defined on the vacuum state. Further calculations show that the same result holds for any coherent state.

What is the origin of this divergence? Recall that : \( \hat{\gamma}(R, T) : \), obtained by promoting (14) to an operator, has the same functional form as the restriction of the Hamiltonian of a scalar field to a box of size \( R \). That is,

\[
: \hat{\gamma}(R) : = \frac{1}{2} \int_0^\infty dr \theta(R - r) \left( \frac{\hat{p}_\psi^2}{r} + r(\dot{\psi})^2 \right),
\]

(43)
where $\theta(R - r)$ denotes the Heaviside step-function, which equals 1 if $r < R$ and 0 otherwise. Normal ordering softens the singularity that arises from the fact that fields are being multiplied at the same point. However, this turns out to be insufficient because of two simultaneous pathologies: the operator contains products of derivatives of the field $\hat{\psi}(R, T)$ and these are integrated on a region with sharp boundary.

Now, a natural strategy to obtain a well-defined metric operator in such circumstance is to soften the sharp boundary of the box. This can be achieved by replacing the Heaviside function $\theta(R - r)$ in (33) with a smooth function $f_R(r)$ which equals 1 for $r \leq R - \epsilon$, then it smoothly decreases to zero and equals zero for $r \geq R + \epsilon$, where $\epsilon$ is a small parameter. An example of such a regulator is:

$$f_R(r) = \begin{cases} 1, & \text{if } r \leq R - \epsilon, \\ \exp \left(-\frac{4\epsilon^2}{(r-R+\epsilon)^2}\right) + 1, & \text{if } R - \epsilon \leq r \leq R + \epsilon, \\ 0, & \text{if } r \geq R + \epsilon. \end{cases}$$

Now, in Minkowskian field theories, while one can begin with such a regulator, after suitable renormalization, one has to take the regulator away to ensure Poincaré invariance. In the present case, however, we need only respect the rotational symmetry and hence there is no apriori need to take the limit $\epsilon \to 0$. Indeed, the Planck length is now a natural candidate for $\epsilon$.

Let us therefore fix a regulator $f_R$ and consider the smeared version of (11):

$$\hat{\gamma}(f_R, T) := \frac{1}{2} \int dk_1 \int dk_2 \left[ 2 F_+(f_R, k_1, k_2) \left( \hat{A}^{\dagger}(k_1) \hat{A}(k_2) e^{i(k_1-k_2)T} \right) + F_-(f_R, k_1, k_2) \left( \hat{A}(k_1) \hat{A}^{\dagger}(k_2) e^{-i(k_1+k_2)T} \right) + \hat{A}^{\dagger}(k_1) \hat{A}^{\dagger}(k_2) e^{i(k_1+k_2)T} \right]$$

where,

$$F_\pm(f_R, k_1, k_2) = \pm k_1 k_2 \int_0^\infty f_R(r) r (J_0(k_1 r)) J_0(k_2 r) \pm J_1(k_1 r) J_1(k_2 r)).$$

The rest of this section is devoted to showing that this operator is well-defined so long as the smearing function $f_R$ belongs to the Schwartz space $S$.

The proof is technically simpler if we adopt the 2-dimensional version of the Fock space introduced before (see (33)). For, we can then mimic the proofs of analogous statements from (13). Now, we can take as our smearing fields, elements $f_R(\vec{x})$ of the Schwartz space on $R^2$. (Thus, the results will in fact be slightly more general than what is need; $f_R(r)$ above is a special case of $f_R(\vec{x})$.)

Let us then write the smeared version of the operator (13) expressed in terms of the creation and annihilation operators given by (33). We have:

$$\hat{\gamma}(f_R, T) := \frac{1}{8\pi} \int d^2k_1 \int d^2k_2 \left[ 2 G_+(f_R, \vec{k}_1, \vec{k}_2) \hat{A}^{\dagger}(\vec{k}_1) \hat{A}(\vec{k}_2) e^{i(\omega_{k_1} - \omega_{k_2})T} - G_-(f_R, \vec{k}_1, \vec{k}_2) \left( \hat{A}(\vec{k}_1) \hat{A}^{\dagger}(\vec{k}_2) e^{-i(\omega_{k_1} + \omega_{k_2})T} + \hat{A}^{\dagger}(\vec{k}_1) \hat{A}^{\dagger}(\vec{k}_2) e^{i(\omega_{k_1} + \omega_{k_2})T} \right) \right]$$

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where
\[ G_\pm(f_R, \vec{k}_1, \vec{k}_2) = \pm \left( \frac{\omega_{k_1}\omega_{k_2} + \vec{k}_1 \cdot \vec{k}_2}{\sqrt{\omega_{k_1}\omega_{k_2}}} \right) f(\vec{k}_1 \mp \vec{k}_2), \] (47)

and \( f(\vec{k}_1 \pm \vec{k}_2) \) is the fourier transform of the smearing function,

\[ f(\vec{k}_1 \pm \vec{k}_2) = \frac{1}{2\pi} \int d^2x f_R(\vec{x}) e^{i(\vec{k}_1 \pm \vec{k}_2) \cdot \vec{x}}. \] (48)

Let us begin by showing that the action of the operator \((16)\) is well-defined on the vacuum state. Since \( \hat{A}(\vec{k}) \) annihilates the vacuum state, we have:

\[ || : \gamma(f_R) : |0> ||^2 = \int d^2k_1 \int d^2k_2 |G_-(f_R, \vec{k}_1, \vec{k}_2)|^2. \] (49)

It follows immediately from \((17)\) that this integral has no infra-red divergences. Therefore, from now on, let us concentrate only on the ultra-violet behavior of the integrand. The factor in the round brackets is ultra-violet divergent. The multiplicative factor \( f \) provides a damping, but only for large \(|k_1 + k_2|\). However, using simple algebra one can bound \( G_-(f_R, \vec{k}_1, \vec{k}_2) \) of Eq \((17)\) by

\[ |G_-(f_R, \vec{k}_1, \vec{k}_2)| \leq \frac{|\vec{k}_1 + \vec{k}_2|^2 |f(\vec{k}_1 + \vec{k}_2)|}{\sqrt{\omega_{k_1}\omega_{k_2}}}. \] (50)

Now, because the smearing function \( f_R(\vec{x}) \) belongs to the Schwartz space, its Fourier transform \( f(\vec{k}_1 + \vec{k}_2) \) falls faster than any polynomial in \(|k_1 + k_2|\). This in turn implies that \( G_-(f_R, \vec{k}_1, \vec{k}_2) \) is square integrable. Note that the smearing function plays a crucial role in this argument. Had we replaced \( f_R(\vec{x}) \) by the Heaviside function \( \theta \) the corresponding Fourier transformed function \( f(\vec{k}_1 + \vec{k}_2) \) would behave as \( 1/|\vec{k}_1 + \vec{k}_2| \) which would not be sufficient to ensure square-integrability of \( G_-(f_R, \vec{k}_1, \vec{k}_2) \) (see \((20)\)). Finally, as a side remark, note that the procedure followed above to prove that \( G_-(f_R, \vec{k}_1, \vec{k}_2) \) is square integrable does not go through for \( G_+ \) because of the minus sign in the argument of the function \( f(\vec{k}_1 - \vec{k}_2) \) (see \((17)\)).

Next, one can show that the action of this operator is in fact well-defined on a generic n-particle state on the Fock space,

\[ |\Psi_n> = \int d^2k_1 \cdots d^2k_n g^{(n)}(\vec{k}_1, \cdots, \vec{k}_n) \hat{A}^\dagger(\vec{k}_1) \cdots \hat{A}^\dagger(\vec{k}_n) |0>, \] (51)

where \( g^{(n)}(\vec{k}_1, \cdots, \vec{k}_n) = \langle \vec{k}_1, \cdots, \vec{k}_n | \Psi_n > \), and \( \int d^2k |g^{(n)}(\vec{k}, \cdots, \vec{k})|^2 < \infty \). Now the terms involving annihilation operators will also contribute. The final result is that \(|| : \gamma(f_R) : |\Psi_n > || \) is finite provided that \(|\Psi_n > \) is a state such that \( \int d^2k |\vec{k}|^2 |g^{(n)}(\vec{k}, \cdots, \vec{k}, \cdots)|^2 < \infty \). (This restriction comes from the “particle number preserving term” in \((16)\).) Since finite linear combinations of these states form a
dense subset of the Hilbert space, we have now established that the operator $\hat{\gamma}(f_R)$ is densely defined on $\mathcal{H}_P$.

By inspection, it also symmetric on this space. We will now show that it admits a self-adjoint extension to $\mathcal{H}_P$. For this, by a theorem due to Von-Neumann \cite{16}, it is sufficient to exhibit on $\mathcal{H}_P$ an anti-linear operator $\hat{C}$ with $\hat{C}^2 = 1$ which leaves the domain of $\hat{\gamma}(f_R)$ invariant and commutes with it. We can take $\hat{C}$ to be the complex-conjugation operator on $\mathcal{H}_P = L^2(S', d\mu)$. It is straightforward to show that $\hat{C}$ commutes with $\hat{\gamma}(\vec{x}, T)$ whence $\hat{C}\hat{A}(\vec{k}) = \hat{A}(-\vec{k})\hat{C}$, and, $\hat{C}\hat{A}^\dagger(\vec{k}) = \hat{A}^\dagger(-\vec{k})\hat{C}$. Finally, since $G_\pm(f_R, \vec{k}_1, \vec{k}_2)$ is real and equals $G_\pm(f_R, -\vec{k}_1, -\vec{k}_2)$, it follows that $\hat{C}$ satisfies the conditions of Von-Neumann’s theorem. Again, for notational simplicity, we will denote the self-adjoint extension also by $\hat{\gamma}(f_R)$.

We can now return to the metric. Since $\hat{\gamma}(f_R)$ is a self-adjoint operator on $\mathcal{H}_P$, it follows that $\exp \hat{\gamma}(f_R)$ is also self-adjoint. Thus, we can now give meaning to the formal expression (35) and define a regulated operator for the full space-time metric:

$$\hat{g}_{ab}(f) = e^{\hat{\gamma}(f_R T)}(-\nabla_a T \nabla_b T + \nabla_a R \nabla_b R) + R^2 \nabla_a \sigma \nabla_b \sigma,$$

(52)

within canonical quantization. In the classical theory, the existence theorems ensure that a space-time metric can be recovered from the canonical framework. There is, however, no such general result in the quantum theory. Our success can be traced back to the use of a well-suited gauge fixing procedure. (Whether a different choice of gauge will give equivalent results is far from being clear.)

At first, it is somewhat confusing that while we do not need a smearing function to obtain a well-defined quadratic form, we need one to obtain a well-defined operator. Note however, that the situation is rather similar even in the classical theory! The metric component $\exp \gamma(R)$ is a well-defined functional on (a dense sub-space of) the reduced phase space. However, precisely because of the sharpness of the boundary, this functional fails to give rise to a well-defined Hamiltonian vector field. To obtain a Hamiltonian vector field, one again needs to soften the boundaries using a smearing function. The fact that the unsmeared functional is well-defined is analogous to the fact that, in the quantum theory, the quadratic form is well-defined without smearing. The smeared quantum operator is the analog of the smeared classical observable with a well-defined Hamiltonian vector field. From this perspective, in fact it would have been surprising if a self-adjoint metric operator had existed without smearing; it would then have defined a 1-parameter group of motions on the Hilbert space which would have no classical counterpart.

### 4.4 Quantum geometry

We will now briefly investigate three consequences of the results obtained in the last three sub-sections. The discussion will be rather general and we will only indicate the directions along which more detailed work could be done.
The first concerns the issue of vacuum fluctuations of geometry. To compute these, we need a well-defined operator; quadratic forms do not suffice. Let us therefore consider the regulated metric operator (52). Since it is completely determined by $\hat{\gamma}(f_R, T)$, let us focus on this latter operator. The vacuum expectation value of this operator is zero. However, because of the vacuum fluctuations, there is a non-zero probability of finding other values as well. A qualitative measure of these probabilities is given by the uncertainty:

$$[\delta : \hat{\gamma}(f_R, T) :]^2 := \langle 0 | (\hat{\gamma}(f_R, T)) | 0 \rangle - \langle 0 | \hat{\gamma}(f_R, T) | 0 \rangle^2 = \int dk_1 dk_2 |F_-(f_R, k_1, k_2)|^2.$$

The right side is a measure of the fluctuation of the metric coefficients around the mean. An immediate consequence of the above result is the existence of the fluctuations of the light cone. To see this, consider a vector $k^a$ in the tangent space of a point $(T, R, \sigma)$ which is null with respect to $g_{ab}$. Now, due to the vacuum fluctuations of the metric operator, the value of the norm of $k^a$ is uncertain and, since the fluctuation can have either sign, there is in general a non-zero probability for $k^a$ to be space-like or time-like. The exception occurs if the vector $k^a$ is radial, i.e., orthogonal to $\partial/\partial \sigma$. Then, because of the specific form (52) of the metric operator, $k^a$ continues to be null. (Similar considerations obviously apply to time-like and space-like vectors.) This simple example illustrates that, contrary to an oft-expressed view, the canonical framework is capable of addressing space-time issues such as the fluctuations of the causal structure.

The second feature we wish to discuss concerns the commutator of the metric operators at the same value of $T$. Again, in this calculation, quadratic forms do not suffice and we must use the regulated operator (52). A straightforward calculation yields:

$$[\hat{\gamma}(f_R), \hat{\gamma}(g_{R'})] = \frac{i}{2} \int d^2x \left( (f(\vec{x})\nabla^a g(\vec{x}) - g(\vec{x})\nabla^a f(\vec{x})) \times (\hat{p}_\psi(\vec{x})\nabla_a \hat{\psi}(\vec{x}) + \nabla_a \hat{\psi}(\vec{x})\hat{p}_\psi(\vec{x})) \right).$$

Thus, the commutator does not vanish; the non-vanishing contribution comes from the smeared boundary at the smaller of $R$ and $R'$. At first the result seems surprising since $\hat{\gamma}(f_R)$ and $\hat{\gamma}(g_{R'})$ dictate the “value” of the metric operator at points $R$ and $R'$ which can be widely separated (and have the same value of $T$). However, the result does not contradict any physical principle. For, although the basic field operators $\hat{\psi}$ and $\hat{p}_\psi$ associated with such points do commute, the metric operator is a non-local functional of these.

Indeed, the result has a classical analog. As we pointed out at the end of the last sub-section, the unsmeared metric $g_{ab}$ does not define a Hamiltonian vector field on the reduced phase space. Hence, to evaluate Poisson brackets, we are forced to use the smeared metric. Then, it is easy to verify that the Poisson brackets between the
functionals $\gamma(f_R)$ and $\gamma(g_{R'})$ fail to vanish even when $R$ and $R'$ are widely separated. In fact these Poisson bracket just mirror the commutators given above.

The last point we wish to discuss concerns the holonomies computed in section 3. We found that the expression of the holonomy involves the exponential of the integral of the connection along a loop on $\Sigma$. Now, as we indicated in the Introduction, there is a canonical quantization program which is based on the assumption that the quantum analogs of these holonomies are well-defined operators. The present model provides a good testing ground for the validity of this assumption.

To see that the issue is non-trivial, let us first recall the situation in the well-understood Maxwell theory, say in 2+1 dimensions. There, the connection is generally promoted to an operator-valued distribution and the holonomies (of real connections) fail to be well-defined in the standard Fock representation. For, in a 2+1-dimensional theory, the operator-valued connection has to be smeared with 2-dimensional test fields while loops have only 1-dimensional support. In the present case, we are also using a Fock representation. A natural question therefore arises: Is the situation then analogous to the Maxwell theory? If so, the basic assumption mentioned above would fail to hold in our solution.

Now, because of axi-symmetry, smearing along a path in the radial direction in effect corresponds to a 2-dimensional smearing. Hence, the acid test is provided by loops $R = \text{const}$ where one can not take advantage of axi-symmetry. Can the classical expression (29) of the trace of the holonomy along such a loop, $\eta$, be promoted to a well-defined, regulated operator? Following the procedure we used in section 4.3, we find that the answer is in the affirmative. The quantum operator is given by:

$$\hat{T}^0_\eta = 2 \cos \left[ \pi \left( 1 - e^{-\frac{1}{2} \gamma(f_R)} \right) \right] , \quad (55)$$

The standard spectral theorems ensure that the operator on the right is well-defined, self-adjoint and has spectrum bounded between $-1$ and $+1$. Thus, the situation is very different from that in the Maxwell case. Indeed, in the present case, it is the scalar field that is subject to Fock quantization. The connection –like the metric– is a non-local functional of the elementary scalar field; its expression involves 2-dimensional integrals of the basic fields. It is because of this that the trace of the holonomy can be promoted to a well-defined operator on $\mathcal{H}_P$. As in the case of the metric, if we were interested only in quadratic forms, there would be no need to use any smearing fields; they are needed only if one wishes to obtain genuine operators.

5 Discussion

The mathematical structure of the classical Einstein-Rosen waves has been well-known for a long time. In light of those results, it is not at all surprising that the true degrees of freedom can be coded in a scalar field satisfying the wave equation with respect to a fictitious Minkowski space and quantization of this field in itself is
trivial. Thus, the underlying structure of our final theory is the expected one. The main purpose of the analysis was, rather, to apply the standard canonical quantization method—which is applicable in the more general context—to arrive at this final picture systematically. That is, since the model is technically sufficiently simple to be exactly soluble, we used it to better understand the standard quantization techniques and to probe conceptual and technical issues of quantum general relativity.

Indeed, the analysis shed light on a number of these. At the classical level, we saw that one can effectively exploit asymptotic flatness to disentangle gauge from dynamics. Gauge conditions can be imposed to handle constraints and to extract the true degrees of freedom. In the final picture, we are still left with a non-trivial Hamiltonian. Consequently, the issue of deparametrization never enters our analysis. Similarly, we did not find it necessary to introduce “clock degrees of freedom” \[17\] at infinity to extract dynamics. In the quantum theory, we saw that there exist semi-classical states which are peaked at classical solutions of the coupled Einstein-scalar field system. The positive energy theorem goes over to the quantum theory and the quantum Hamiltonian has the same upper bound as the classical one. The solution also confirms the general expectation about the issue of time in quantum theory in the asymptotically flat context. The parameter \(t\) arises as the affine parameter along the Hamiltonian vector field on the classical phase space and has the space-time interpretation of time in the 3-metric defined by any dynamical trajectory in the phase space. (This is also the situation in full general relativity.) In the quantum theory, an analogous parameter enters the Schrödinger equation \(40\). However, since general quantum states do not correspond to classical space-times, this parameter does not have the standard interpretation of time. This interpretation emerges only in the semi-classical regime: in any coherent state, the parameter can be identified with the classical \(t\). Finally, we saw that the regulated metric and holonomy operators can be constructed by a careful smearing procedure which smoothenes the sharp boundaries that enter the definition of their classical analogs. The associated functional analysis subtleties are non-trivial even from the mathematical perspective of a free field in Minkowski space.

In the technical discussion, we made a liberal use of the fictitious Minkowskian background \(g_{ab}^0\) and the associated time parameter \(T\). However, this was done primarily for pedagogical reasons, i.e., to bring out the relation between the final quantum theory and the expected one. We could have arrived at our Hilbert space of states directly from the reduced phase space either by making use of the “reality condition” strategy \(11\) or by making an appeal to null infinity and the S-matrix theory, without having to explicitly introduce \(g_{ab}^0\).

How do these results compare with those available in the literature? Our analysis is closely related to that of Refs \(3, 4\). In the classical theory, the main difference lies in our systematic handling of the asymptotically flat boundary conditions. In particular, in our treatment, the true Hamiltonian arose directly from the boundary term in the action. This point could not have been realized in the early analyses.
because the relation between 3+1 and 2+1-dimensional theories was not well-known and, more importantly, because a clear understanding of asymptotic flatness in 2+1 dimensions has emerged only recently. (Indeed, given what was known in the early seventies, the treatment of Ref [5] seems to be surprisingly ahead of its time!) In the quantum theory, the difference lies in the treatment of certain functional analytical subtleties. That it is necessary to regularize the metric operator was realized in [6]. However, the suggestion there that the softening of the sharp boundaries can be brought about by a simple ultra-violet cut off in the momentum space is incorrect; one needs suitable smearing fields in space-time. Thus, our regularization differs from that in [6]. Finally, our isolation of true degrees of freedom was carried out in 2+1 dimensions. When translated to a 3+1 dimensional perspective, our result is equivalent to the definition of true observables given in [18].

Since the model has been solved exactly within the standard canonical framework, it opens doors for further analysis in a number of directions. We will conclude by mentioning a few of these.

First, we can now explore quantum field theory on a quantum geometry. Part of the motivation here is similar to that of quantum field theory in curved space-times; one wishes to investigate the effects of a non-trivial background geometry on quantum fields. Furthermore, this analysis can also shed light on the nature of quantum geometry itself. For instance, we may choose as our background, a coherent state. The geometry influences the dynamical evolution of the quantum field because the metric appears in the expression of the Hamiltonian of the test field. Now, in the quantum theory, we have two alternatives. First, we can consider just the quadratic form that is determined by the (normal-ordered, unsmeared) metric (35) and substitute its value in a coherent state in the expression of the Hamiltonian. Since this value is just the classical metric, this would lead us just to the standard quantum field theory in curved space-times. To probe the effects of the quantum nature of geometry, we would have to look beyond just the expectation values. This can not be handled by a quadratic form alone; we need a genuine operator. Thus, the second – and much more interesting – possibility is to use the smeared metric operator in the expression of the Hamiltonian of the test field. Then, one would see the effects of the quantum geometry on the evolution of the matter, even in the case when the geometry is assumed to be in the vacuum state (initially). This analysis would be interesting because much of the standard apparatus of quantum field theory in curved space-times uses the fixed causal structure of the classical geometry which is now absent. Using the canonical framework, one would be able to do quantum theory of test fields even when the causal structure is subject to quantum fluctuations of its own.

Recall that, in the regularization of the metric operator, we needed a smearing function $f_R$. There is, however, no “canonical” choice; while we know what the qualitative behavior of $f_R$ should be, there is considerable freedom in its detailed form. Thus, we do not have a “canonical” regularized metric operator. All choices
provide the required ultra-violet cut-offs but the precise damping depends on the specific form of $f_R$. The differences will show up, for example, in the evolution of test fields. It would be interesting to investigate these differences and see if one can restrict the choice of the smearing functions through thought experiments. If one can not, there would a genuine quantization ambiguity. The situation would be similar to that in non-relativistic quantum mechanics where, in general, the factor ordering ambiguities can not be resolved purely on theoretical grounds.

We saw that the regularized operators corresponding to the traces of holonomies of connections are well-defined on the quantum Hilbert space. Now, in the approach to quantum gravity based on these holonomies [4], a striking picture of quantum geometry has emerged in which geometrical operators such as areas and volumes have a discrete spectrum. It is therefore natural to ask if the same is true in the present case. The question is now manageable, thanks to the regularized metric operators. Since the basic operator $\hat{\gamma}$ is the regularized version of the restriction of the Hamiltonian in a box, it is quite likely that its spectrum is discrete. If so, the lengths in the radial directions and areas will be quantized. This would be a striking result coming from a Fock-like representation.

Another direction for further investigation is provided by the Gowdy models. Since these are spatially compact and have initial curvature singularities, new issues arise. These will be discussed in the sequel to this paper. While both these problems deal only with the “one polarization” case –the two Killing fields are hyper-surface orthogonal in the 3+1-dimensional picture– one can also investigate the two polarization case [19]. In the case when the translational Killing field is time-like, this case was analyzed in detail by Korotkin and Nicolai [20] recently. Their quantization is mathematically complete but somewhat unconventional in the sense that the relation between their Hamiltonian description and the standard Poisson-brackets of classical general relativity is unclear. It would be interesting to compare the results obtained here with the reduction of their model to the one polarization case. More recently, infinite number of conserved quantities have been constructed in the classical theory with two polarizations [21]. Using these, one may be able to extract the true degrees of freedom in this more general case and quantize the model along the lines of this paper.

Finally, the present model itself offers an attractive setting to explore the idea of “fuzzing” of space-time points using techniques involving null infinity [22]. As mentioned in section 4.1, the 2+1-dimensional space-times considered here are asymptotically flat at null infinity [13]. Furthermore, since the form of the metric is sufficiently simple, it should be possible to integrate the null geodesics and express the “light cone cuts of null infinity” explicitly in terms of the initial data for the scalar field at null infinity. These cuts label space-time points. The asymptotic quantization of the scalar field [14] –which is equivalent to the quantization presented here– would then lead to fuzzy points. So far, in this approach, detailed calculations have been carried out only in the linearized approximation [22]. The underlying simplicity of cylindrical
waves provides an interesting arena where these results can be extended beyond the linear context.

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References

[1] Conceptual Problems of Quantum Gravity, eds A. Ashtekar and J. Stachel (Birkhäuser, Boston, 1991); C.J. Isham, in The Proceedings of the 1992 NATO advanced study institute on Recent Problems in Mathematical Physics; K. Kuchař, in The Proceedings of the 4th Canadian conference on General Relativity and Relativistic Astrophysics, eds G. Kunstatter, D. Vincent and J. Williams (World Scientific, Singapore, 1992).

[2] C. W. Misner, In Magic Without Magic: John Archibald Wheeler, edited by J. Klauder (Freeman, San Francisco, 1972); M. Ryan, Hamiltonian Cosmology, (Springer-Verlag, Berlin, 1972); A. Ashtekar, R.S. Tate and C. Uggla, Int. J. Theo. Physics, D2, 15 (1993).

[3] Carlip, In: Knots and Quantum Gravity, ed J. Baez (Clarendon Press, Oxford, 1994); J. Korean Phys. Soc. (Proc. Suppl.), 28, S447 (1995).

[4] A. Ashtekar, Lectures on Non-perturbative canonical gravity, notes prepared in collaboration with R.S. Tate (World Scientific, Singapre, 1991).

[5] K. Kuchař, Phys. Rev. D 4, 955 (1971).

[6] M. Allen, Class. Quantum Grav. 4, 149 (1987).

[7] D. Krammer, H. Stephani, E. Herlt and M. MacCallum, Exact solutions of Einstein’s field equations, (Cambridge University Press, Cambridge, 1980), p 326.

[8] A. Ashtekar and M. Varadarajan, Phys. Rev. D 40, 4944 (1994).

[9] M. Varadarajan, Phys. Rev. D 52, 2020 (1995).

[10] J. A. Wheeler, in Relativity, Groups and Topology, eds C. DeWitt and B.S. DeWitt (Gordon and Breach, New York, 1964).
[11] E. Witten, Nucl. Phys. B311, 46 (1988); A. Ashtekar, V. Husain, C. Rovelli, J. Samuel and L. Smolin, Class. & Grav. 6, L185 (1989).

[12] J. Glimm and A. Jaffe, *Quantum Physics* (Springer-Verlag, New York, 1987).

[13] A. Ashtekar, J. Bicak and B.S. Schmidt, preprint CGPC-96/5-1.

[14] A. Ashtekar, *Asymptotic Quantization* (Bibliopolis, Naples, 1987).

[15] J. T. Cannon and A. M. Gaffe, Common. Math. Pays. 17, 261 (1970).

[16] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Vol II*, theorem X.3 (Academic Press, New York, 1975).

[17] J. Romano and C. Torre, Internal time formalism for space-times with two Killing vectors, Utah preprint (1995).

[18] C. Torre, Class. Quantum Grav. 8, 1895 (1991).

[19] B.K. Berger, P.T. Chruściel, V. Moncrief, Ann. Phys. 237, 322 (1995).

[20] D. Korotkin and H. Nicolai, pre-print hep-th/9605144.

[21] A. Ashtekar and V. Husain (in preparation).

[22] S. Frettelli, C. Kozameh, E.T. Newman, C. Rovelli and R.S. Tate, pre-print gr-qc/9603061.