Individual-Based Stability in Hedonic Diversity Games

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Abstract
In hedonic diversity games (HDGs), recently introduced by [Bredereck, Elkind, and Igarashi (2019)], each agent belongs to one of two classes (men and women, vegetarians and meat-eaters, junior and senior researchers), and agents’ preferences over coalitions are determined by the fraction of agents from their class in each coalition. Bredereck et al. show that while an HDG may fail to have a Nash stable (NS) or a core stable (CS) outcome, every HDG in which all agents have single-peaked preferences admits an individually stable (IS) outcome, which can be computed in polynomial time. In this work, we extend and strengthen these results in several ways. First, we establish that the problem of deciding if an HDG has an NS outcome is NP-complete, but admits an XP algorithm with respect to the size of the smaller class. Second, we show that, in fact, all HDGs admit IS outcomes that can be computed in polynomial time; our algorithm for finding such outcomes is considerably simpler than that of Bredereck et al. We also consider two ways of generalizing the model of Bredereck et al. to \( k \geq 2 \) classes. We complement our theoretical results by empirical analysis, comparing the IS outcomes found by our algorithm, the algorithm of Bredereck et al. and a natural better-response dynamics.

1 Introduction
A number of French exchange students at a Spanish university have signed up for a game theory class. All students taking the class are advised to form study groups to discuss the material and to work on problem sheets. Now, some French students see this as an excellent opportunity to improve their Spanish and would like to join study groups where no one else speaks French. Other students have less confidence in their ability to communicate in Spanish and therefore want to be in a group where at least a few other students speak French; in fact, some of the students prefer to be in an exclusively French-speaking group. Spanish students, too, have different preferences over the fraction of French students in their groups: some are eager to meet new friends, while others are worried that language issues will affect their learning.

Many important aspects of the setting described in the previous paragraph can be captured by the recently introduced framework of hedonic diversity games (HDG) [Bredereck, Elkind, and Igarashi (2019)]. In these games agents can be split into two classes (say, red and blue), and each agent has preferences over the fraction of red agents in their group. The outcome of a game is a partition of agents into groups. This model is relevant for analyzing a variety of application scenarios, ranging from interdisciplinary collaborations to racial segregation.

In their work, Bredereck et al. aim to understand whether HDGs admit stable outcomes, for several common notions of stability for hedonic games, such as Nash stability, individual stability and core stability; the first two concepts are based on deviations by individual agents, while the third concept captures resilience against group deviations. Bredereck et al. show that an HDG may fail to have a Nash stable outcome or a core stable outcome and that deciding if an HDG has a core stable outcome is NP-complete. For individual stability, they get a positive result under the additional assumption that agents’ preferences are single-peaked, i.e., each agent \( i \) has a preferred ratio of red agents in her group (say, \( \rho_i \)), and for any two ratios \( \rho, \rho' \) such that \( \rho < \rho' \leq \rho_i \) or \( \rho_i \leq \rho' < \rho \) she prefers a group with ratio \( \rho' \) to a group with ratio \( \rho \). Specifically, Bredereck et al. show that every HDG with single-peaked preferences admits an individually stable outcome and describe a polynomial-time algorithm for finding some such outcome. Their work leaves open the question whether HDGs with non-single-peaked preferences always have an individually stable outcome.

Our Contribution In this paper, we answer two open questions from the paper of Bredereck et al., as well as extend their model to an arbitrary number of agent classes.

First (Section 3), we show that deciding if an HDG has a Nash stable outcome is NP-complete. Our hardness result holds even if agents have dichotomous preferences, i.e., approve some ratios and disapprove the remaining ratios; in fact, it remains true if each agent approves at most 4 ratios. On the other hand, we show that the existence of a Nash stable outcome can be decided in polynomial time if the size of one of the classes can be bounded by a constant.

We then turn our attention to individual stability (Section 4). We describe a polynomial-time algorithm that finds an individually stable outcome of any HDG; our algorithm is significantly simpler than that of Bredereck et al. However, we show that neither algorithm Pareto-dominates the

\*A shortened version of this paper appears in the proceedings of AAAI-20
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other: there are settings where the algorithm of Bredereck et al. produces a much better partition than our algorithm, and there are settings where the converse is true.

In Section 5 we consider two different ways of extending HDGs to settings with $k$ agent classes, $k > 2$. First, we consider a very general model, where each agent may have arbitrary preferences over the ratios of different classes in her group. We show that our positive result for individual stability does not extend to this model: we describe a game with $k = 3$ that has no individually stable outcomes and prove that deciding the existence of such outcomes is NP-complete if $k \geq 5$. We then propose a more restrictive model, where an agent only cares about the fraction of agents that belong to her class. We show that this model encompasses both HDGs and another well-known class of hedonic games, namely, anonymous games, i.e., games where agents have preferences over the size of their group.

In Section 6, we empirically compare the outcomes produced (1) by our algorithm for finding individually stable outcomes, (2) by the algorithm of Bredereck et al. and (3) by a natural better-response dynamics, with respect to several measures, such as the average social welfare and the diversity of resulting groups. We summarize our findings in Section 7.

**Related Work** Hedonic games were introduced by Drieze and Greenberg (1980) and have received a lot of attention in the computational social choice literature, as they offer a simple, but powerful formalism to study group formation in strategic settings; see, e.g., the survey by Aziz and Savani (2016). Hedonic diversity games share common features with two other well-studied classes of hedonic games, namely, anonymous games (Bogomolnaia and Jackson 2002) and fractional hedonic games (Aziz et al. 2019).

In anonymous games, agents cannot distinguish among other agents, and therefore the only feature of a group that matters to them is its size. HDGs may appear to be more general than anonymous games, since in HDGs each agent can distinguish between two classes of agents. Indeed, many proof techniques developed for anonymous games turn out to be relevant for HDGs. However, in a technical sense, anonymous games are not a subclass of HDGs, and some of the positive results for HDGs do not hold for anonymous games. In particular, while we prove that every HDG has an individually stable outcome, Bogomolnaia and Jackson (2002) show that this is not the case for anonymous games.

Throughout the paper, we compare our results for HDGs to relevant results for anonymous games, and in Section 5 we propose a succinct representation formalism for hedonic games that captures both HDGs and anonymous games.

In fractional hedonic games, every agent assigns a numerical value to every other agent, and an agent’s value for a group of size $s$ that includes her is equal to the sum of the values she assigns to the group members, divided by $s$. Now, if agents are divided into two classes, so that each agent assigns the same value to all agents in each class, the resulting game is a hedonic diversity game. However, not all hedonic diversity games can be obtained in this fashion; in particular, an HDG where each agent prefers groups that have the same number of agents from each class may not be represented in this way. Conversely, there are fractional hedonic games that cannot be represented as HDGs.

**2 Preliminaries**

For every positive integer $n$, we write $[n]$ to denote the set $\{1, \ldots, n\}$.

A hedonic game is a pair $G = (N, (\succeq_i)_{i \in N})$, where $N = [n]$ is the set of agents, and for each $i \in N$ the relation $\succeq_i$ is a weak order over all subsets of $N$ that contain $i$. The subsets of $N$ are called coalitions; the set of all coalitions containing agent $i$ is denoted by $\mathcal{N}(i)$. We refer to the set $N$ as the grand coalition.

Given two coalitions $C, D \in \mathcal{N}(i)$, we write $C \sim_i D$ if $C \succeq_i D$ and $D \succeq_i C$; we write $C \succ_i D$ if $C \succeq_i D$ and $C \not\succeq_i D$. We say that $i$ weakly prefers $C$ to $D$ if $C \succeq_i D$; if $C \succ_i D$, we say that $i$ strictly prefers $C$ to $D$, and if $C \sim_i D$, we say that $i$ is indifferent between $C$ and $D$. For succinctness, when describing agents’ preferences, we often omit coalitions $C$ with $|\{i\}| \geq |C|$.

An outcome of a hedonic game with the set of agents $N$ is a partition $\pi = \{C_1, \ldots, C_k\}$ of $N$; we write $\pi_i$ to denote the coalition in $\pi$ that contains agent $i$. An agent $i$ has an NS-deviation from an outcome $\pi$ if there exists a coalition $C \in \pi \cup \{\emptyset\}$ such that $C \cup \{i\} \succ_i \pi_i$; $i$ has an IS-deviation from an outcome $\pi$ if there exists a coalition $C \in \pi \cup \{\emptyset\}$ such that $C \cup \{i\} \succ_i \pi_i$ and, additionally, $C \cup \{i\} \succeq_j C$ for each $j \in C$. An outcome $\pi$ is Nash stable (NS) (respectively, individually stable (IS)) if no agent has an NS-deviation (respectively, an IS-deviation). An outcome $\pi$ Pareto dominates an outcome $\pi'$ if $\pi'_i \succeq \pi_i$ for each $i \in N$ and $\pi'_j \succ \pi_j$ for some $j \in N$; an outcome is Pareto optimal if it is not Pareto-dominated by another outcome.

Consider a hedonic game $G = (N, (\succeq_i)_{i \in N})$. We say that $G$ is dichotomous if for every agent $i \in N$ there exist disjoint sets $N^+(i)$ and $N^-(i)$ such that $N(i) = N^+(i) \cup N^-(i)$, and for every pair of coalitions $C, D \in \mathcal{N}(i)$, we have $C \sim_i D$ if $C, D \in N^+(i)$ or $C, D \in N^-(i)$ and $C \succ_i D$ if $C \in N^+(i)$, $D \in N^-(i)$; we say that $i$ approves coalitions in $N^+(i)$ and disapproves coalitions in $N^-(i)$. We say that $G$ is anonymous if for every agent $i \in N$ there exists a weak order $\succeq_i^0$ on $[|N|]$ such that for every pair of coalitions $C, D \in N(i)$, we have $C \succeq_i D$ if and only if $|C| \succeq^0_i |D|$.

In a hedonic diversity game (HDG), the set of agents $N$ is partitioned as $N = R \cup B$, and each agent $i$ is indifferent between any two coalitions $C, D \in \mathcal{N}(i)$ that have the same fraction of agents in $R$; we refer to agents in $R$ and $B$ as red and blue agents, respectively. In such games, the preferences of every agent $i$ can be described by a weak order $\succeq_i^0$ over the set $\Theta = \{\frac{j}{k} \mid 0 \leq j \leq |R|, j \leq k \leq |N|\}$; for all $C, D \in \mathcal{N}(i)$ we have $C \succeq_i D$ if and only if $|C| \succeq^0_i |D|$. We say that a coalition $C$ is homogeneous if $C \subseteq R$ or $C \subseteq B$.

An anonymous game $(N, (\succeq_i)_{i \in N})$ is said to be single-peaked if for every agent $i \in N$ there exists a preferred size $s_i \in [n]$ such that for every pair of coalitions $C, D \in \mathcal{N}(i)$ with $|C| < |D| \leq s_i$ or $s_i \leq |D| < |C|$ it holds that $D \succeq_i C$. Similarly, a hedonic diversity game $(R \cup B, (\succeq_i)_{i \in R \cup B})$ is said to be single-peaked if for every agent $i \in R \cup B$ there
exists a preferred value \( p_i \in \Theta \) such that for every \( \rho, \rho' \in \Theta \) such that \( p_i \leq \rho < \rho' \) or \( \rho' < \rho \leq p_i \), it holds that \( \rho \geq \rho' \).

In what follows, we use the fact that the problem the Truthful Hedonic Diversity Game is NP-complete \((\text{Garey and Johnson} 1979)\). An instance of this problem is given by a set \( X = \{1, \ldots, m\} \) and a collection \( C = \{A_1, \ldots, A_k\} \) of 3-element subsets of \( X \). It is a yes-instance if there exists a subset \( C' \subseteq C \) such that \( C' \) is a partition of \( X \); otherwise it is a no-instance. For each \( x \in X \), let \( J^x = \{j^x_1, \ldots, j^x_m\} \) be the set of all indices of sets in \( C \) to which \( x \) belongs, i.e., \( j \in J^x \) if and only if \( x \in A_j \).

### 3 Nash Stability

Bredereck, Elkind, and Igarashi (2019) show that a hedonic diversity game may fail to have a Nash stable outcome, even if agents’ preferences are single-peaked: indeed, it is possible to construct a set of agents such that there exists a preferred value \( \rho \). In this case, we construct the respective hedonic diversity game. For each \( A_j \in \pi' \) of the respective hedonic diversity game. For each \( A_j \in \pi' \), the outcome \( \pi' \) contains a coalition \( P_j = \{b_x | x \in A_j\} \cup \{r_j^p | p \in [2f(j) + 3]\} \). All remaining agents are put into singleton coalitions and are added to \( \pi' \).

We claim that \( \pi' \) is Nash stable: By construction, no blue agent has an NS-deviation in \( \pi' \), as all blue agents are in one of their most preferred coalitions. Moreover, no filling agent \( r_j, j \in [k] \), has an NS-deviation, as \( r_j \) is either in her most preferred coalition or in a singleton coalition. In the latter case, \( r_j \) can only deviate to red singleton coalitions or to coalitions with ratio \( \frac{2f(j)+3}{2f(j)+6} \) for some \( i \in [k] \). However, deviating to a red singleton coalition does not make a difference for her and deviating to a coalition with ratio \( \frac{2f(j)+3}{2f(j)+6} \). For some \( i \in [k] \), this is never individually rational, as for every \( i \in [k] \) we have \( \frac{2f(i)+4}{2f(i)+7} \). Furthermore, applying the same reasoning, no stalking agent \( z_j \) has an NS-deviation, as for every \( i \in [k] \) we have \( \frac{2f(i)+4}{2f(i)+7} \). The proof of correctness holds even for anonymous games with dichotomous preferences.

#### Theorem 3.1

Given an HDG \( G = (R \cup B, (\preceq_i)_{i \in R \cup B}) \), it is NP-complete to decide whether \( G \) has a Nash stable outcome. The hardness result holds even if \( G \) is dichotomous or if each relation \( \preceq_i \) is a strict order over \( \Theta \).

**Proof.** To see that this problem is in NP, note that it is possible to check in polynomial time whether an outcome \( \pi \) is Nash stable by iterating over all agents and all coalitions in \( \pi \) and checking whether the agent prefers joining this coalition to her current coalition.

We start by proving hardness for the case where the agents have unrestricted preferences. To this end, we construct a reduction from X3C. To begin with, we define \( f(j) = 2j - 1 - \lfloor \frac{j}{2} \rfloor \), which is a bijection from the natural numbers to all numbers not divisible by three. Due to reasons that will become apparent in the proof of correctness, it will be convenient to use \( f(j) \) instead of \( j \) in the construction.

In the following, given \( X \) and \( C \), we construct the respective hedonic diversity game. Intuitively, for each element \( x \in X \) we introduce a blue set agent \( b_x \) as well as some red agents so as to map each set \( A_j \) to a unique coalition with ratio \( \frac{2f(j)+3}{2f(j)+6} \) consisting of the three relevant set agents and \( 2f(j)+3 \) designated red agents. The game is constructed so that in every Nash stable outcome each set agent \( b_x \) needs to be in a coalition corresponding to a set \( A_j \) with \( x \in A_j \).

**Construction:** For each element \( x \in X \), we introduce one blue set agent \( b_x \) with the following preference relation:

\[
b_x : \frac{2f(j_x^1)+3}{2f(j_x^1)+6} \sim b_x \cdots \sim b_x \frac{2f(j_x^{m_x})+3}{2f(j_x^{m_x})+6} \sim b_x 0.
\]

Moreover, for each \( j \in [k] \), let us introduce \( 2f(j)+3 \) red filling agents with the following preference relation:

\[
r_j^p : \frac{2f(j)+3}{2f(j)+6} \sim r_j^p 1, \text{ for all } p \in [2f(j)+3].
\]

In addition, for each \( j \in [m] \), we insert a red stalking agent \( z_j \) with the following preference relation:

\[
z_j : \frac{1}{j+1} \sim z_j 1.
\]

**Correctness:** \((\Rightarrow)\) Assume that there exists a partition \( \pi \subseteq C \) of \( X \). We transform \( \pi \) into a Nash stable outcome \( \pi' \) of the respective hedonic diversity game. For each \( A_j \in \pi' \), the outcome \( \pi' \) contains a coalition \( P_j = \{b_x | x \in A_j\} \cup \{r_j^p | p \in [2f(j)+3]\} \). All remaining agents are put into singleton coalitions and are added to \( \pi' \).

We claim that \( \pi' \) is Nash stable: By construction, no blue agent has an NS-deviation in \( \pi' \), as all blue agents are in one of their most preferred coalitions. Moreover, no filling agent \( r_j, j \in [k] \), has an NS-deviation, as \( r_j \) is either in her most preferred coalition or in a singleton coalition. In the latter case, \( r_j \) can only deviate to red singleton coalitions or to coalitions with ratio \( \frac{2f(j)+3}{2f(j)+6} \), for some \( i \in [k] \). However, deviating to a red singleton coalition does not make a difference for her and deviating to a coalition with ratio \( \frac{2f(j)+3}{2f(j)+6} \). For some \( i \in [k] \), this is never individually rational, as for every \( i \in [k] \) we have \( \frac{2f(i)+4}{2f(i)+7} \neq \frac{2f(i)+4}{2f(i)+7} \). Furthermore, applying the same reasoning, no stalking agent \( z_j \) has an NS-deviation, as for every \( i \in [k] \) we have \( \frac{2f(i)+4}{2f(i)+7} \neq \frac{1}{j+T} \).

\((\Leftarrow)\) Let us assume that there exists a Nash stable outcome \( \pi \) of the hedonic diversity game. Then, no coalition of fraction \( \frac{1}{T+\cdot} \) for some \( j \in [m] \) can be part of \( \pi \), as such a coalition is not individually rational for any blue agent. Consequently, all stalking agents need to be in homogeneous coalitions in \( \pi \). Thereby, \( \pi \) contains no homogeneous blue coalitions of size \( j \in [m] \), as \( z_j \) would have an NS-deviation to it. Therefore, all blue agents need to be in one of their most preferred coalitions in \( \pi \).

Let \( \{P_1, \ldots, P_t\} \subseteq \pi \) be the set of all coalitions containing at least one agent. Then, the reasoning above shows that for all \( \ell \in \pi \) we have \( \theta(P_\ell) = \frac{2f(j)+3}{2f(j)+6} \) for some \( i \in [k] \). As \( \gcd(2f(j)+3, 2f(j)+6) = 1 \) for all \( j \in [k] \), at least three blue and \( 2f(j)+3 \) red agents are part of coalition \( P_i \). Since there only exist three blue agents for which this ratio is individually rational, it follows that all three blue set agents who belong to \( A_j \) (and no other blue agent) are part of \( P_i \). Consequently, by removing all red agents from the coalitions in \( \{P_1, ..., P_t\} \), we obtain a cover of \( X \) by sets in \( C \).

Note that the reduction still works if no indifferences in the preferences are allowed. In this case, it is possible to replace the indifferences in the set agents’ preference relations by strict preferences in an arbitrary way without affecting the correctness of the proof: the first direction still holds, as no blue agent can deviate to a different coalition such that the resulting coalition is individually rational for her, since
we can enumerate all possible guesses in time \( x \) problem as follows. We create a source, a sink, a node for each \( i \). Assume without loss of generality that \( C \) has a capacity \( p \) with \( P \) if and only if there is a partition \( \{A_1, \ldots, A_k, D_0\} \) for some \( i \). An edge from \( x \) to \( y \) is weakly prefers being in a coalition of size \( i \) to \( y \). By construction, there is an edge of capacity \( \ell \) with \( \ell \).

In the second stage, given a guess \( \{C_1, \ldots, C_k\}, (n_1, \ldots, n_k) \), we construct an instance of the network flow problem as follows. We create a source, a sink, a node \( x_i \) for each \( i \) and a node \( y_j \) for each \( j = 0, \ldots, k \). The source is connected to all nodes \( x_i, i \) by an edge of capacity 1, and each node \( y_j, j = 0, \ldots, k \), is connected to the sink by an edge of capacity \( n_j - |C_j| \). If there is an edge of capacity 1 from \( x_i \) to \( y_j \), \( j = 0, \ldots, k \), it is weakly prefers being in a coalition of size \( n_j \) with \( C_j \) blue agents to deviating to a coalition of size \( n_j \) with \( C_j \) blue agents, for \( s \neq j \), to a homogeneous red coalition; an edge from \( x_i \) to \( y_0 \) indicates that \( i \) weakly prefers a homogeneous red coalition to other available options. By construction, there is a flow of size \(|R|\) in this network if and only if there is a partition \( \{A_1, \ldots, A_k, D_0\} \) of \( R \) such that \( C_i \subseteq A_i, |A_i| = n_i \) for each \( i \) and \( s \) and no red agent has an NS-deviation from this partition. Thus, if this instance of network flow does not admit a flow of size \(|R|\), we reject the current guess, and otherwise we proceed to the next stage.

In the third stage, we split \( D_0 \) into \( D_1, \ldots, D_\ell \) for some \( \ell \geq 0 \). Note that, no matter how we do this, if no red agent had an NS-deviation in \( \{A_1, \ldots, A_k, D_0\} \), this will also be the case for the new partition. Thus, we can focus on the blue agents. Given a \( t \in [n_0] \), we say that \( t \) is safe for an agent \( j \) if \( j \) weakly prefers her current coalition in \( \{A_1, \ldots, A_k, D_0\} \) to a coalition consisting of herself and \( t \) red agents; we say that \( t \) is safe if it is safe for each \( j \). Let \( T \subseteq [n_0] \) be a collection of all safe integers. Then, we can subdivide \( D_0 \) so that in the resulting partition no blue agent has an NS-deviation if and only if \( T \) can be represented as a sum of integers from \( T \); the latter problem is a variant of \( \text{KNAPSACK} \), and can be solved by dynamic programming in time \( O(n^2) \).

Theorem 3.2 puts the problem of finding a Nash stable outcome in the complexity class \( \text{XP} \) with respect to the parameter \( k \); however, we do not know if this problem is fixed-parameter tractable (FPT) with respect to this parameter. For HDGs with dichotomous preferences, another natural parameter is the number of ratios approved by each agent. However, our problem turns out to be para-NP-hard with respect to this parameter: the hardness proof in Theorem 3.1 goes through even if each agent only approves at most four ratios in \( \Theta \). Similarly, Peters (2016) shows that finding a Nash stable outcome in dichotomous anonymous games remains NP-hard if each agent approves at most four coalition sizes. Interestingly, we can prove that the latter problem becomes polynomial-time solvable if each agent approves at most one coalition size, but it is not clear how to extend this proof to dichotomous HDGs.

Theorem 3.3. Given a dichotomous anonymous game \( G = (N, (\geq_l)_{l \in N}) \) where for each agent \( i \) in \( N \) the set \( N^+(i) \) is of the form \( \{C \in N(i) : |C| = s_i\} \) for some \( s_i \in \mathbb{N} \), we can decide in polynomial time whether \( G \) admits a Nash stable outcome.

Proof. If there does not exist an agent \( i \) in \( N \) with \( s_i = 1 \), it follows that the grand coalition is Nash stable. Thus, in the following, we assume that there exists at least one agent approving a singleton coalition. For each \( j = 0, \ldots, n \), let \( N_j \) be the set of all agents \( i \) in \( N \) with \( s_i = j \). Moreover, let \( \ell = \max\{i : N_i \neq \emptyset \text{ for all } j\} \).

We claim that for each \( j \in [\ell] \) all agents in \( N_j \) need to be in coalitions of size \( j \) in every Nash stable outcome. This follows easily by induction on \( j \). Indeed, if an agent in \( N_j \) is not in a singleton coalition, she can always deviate to being in a singleton coalition. Further, if \( 1 < j \leq \ell \) and for each \( k \in [j - 1] \) all agents in \( N_k \) are in coalitions of size \( k \), it follows by definition of \( \ell \) that \( j \) needs to exist a coalition of size \( k \) for each \( k \in [j - 1] \). Now, assuming that an agent in \( N_j \) is not in a coalition of size \( j \), she can always deviate to a coalition of size \( j - 1 \).

For each \( j \in [\ell] \), let \( d_j = \ell \times \frac{|N_j|}{|N|} - |N_j| \). Intuitively, agents in \( N_j \) need \( d_j \) additional agents to split into coalitions of size \( j \). Let \( d = \sum_{j=1}^{\ell} d_j \) and set \( N' = N_0 \cup \bigcup_{j>\ell} N_j \).

If \( d \leq |N'| \), we can construct a Nash stable outcome as follows: we put the agents from \( N_0 \) into \( \ell \) pairwise disjoint sets \( D_1, \ldots, D_\ell \subseteq N' \) so that \( |D_j| = d_j \) for each \( j \). Then, by construction, \( |N_j \cup D_j| \) is divisible by \( j \) for each \( j \in [\ell] \). Hence, we can split agents in \( N_j \cup D_j \) into coalitions...
of size \( j \). The remaining agents in \( N' \) are placed in singleton coalitions. This partition is Nash stable, since all agents in \( N \setminus N' \) approve their coalition sizes and none of the agents in \( N' \) approves coalition sizes \( 1, \ldots, \ell \), or \( \ell + 1 \).

On the other hand, if \( d > |N'| \), it follows that there cannot exist a Nash stable outcome: indeed, in this case, there are not enough agents in \( N' \) to construct an outcome in which each agent in \( N \setminus N' \) is in a coalition of her approved size, and we have argued that this is necessary for stability.

\[ \square \]

### 4 Individual Stability

Bredereck, Elkind, and Igarashi [2019] describe an algorithm that, given an HDG with single-peaked preferences, outputs an individually stable outcome in polynomial time. This algorithm is fairly complex: its description and analysis take up almost four pages of the AAMAS’19 paper. It is similar in spirit to the algorithm of Bogomolnaia and Jackson [2002] that finds an IS outcome of an anonymous game with single-peaked preferences in polynomial time. The main contribution of this section is a much simpler polynomial-time algorithm that can find an individually stable outcome of any HDG; this result is particularly surprising, because it is known that not every anonymous game admits an IS outcome [Bogomolnaia and Jackson, 2002].

**Theorem 4.1.** Given an HDG \( G = (R \cup B, (\succ_i')_{i \in R \cup B}) \), we can compute an individually stable outcome of \( G \) in polynomial time.

**Proof.** In what follows, we say that a coalition \( C \subseteq R \cup B \) is balanced if \( |C \cap R| = |C \cap B| \).

We claim that Algorithm 1 outputs an IS outcome of \( G \).

The first phase of this algorithm creates a maximum-size balanced coalition \( C \) such that all agents in \( C' \) prefer \( C \) to being alone; all other agents are placed in singleton coalitions. In the second phase, the algorithm checks if any of the remaining agents has an IS-deviation to \( C' \); if yes, some such agent is invited to join \( C' \). This step is repeated until none of the remaining agents has an IS-deviation to \( C' \). To avoid ambiguity, we use \( C_0 \) and \( C_1 \) to denote the ‘large’ coalition obtained at the end of the first phase and at the end of the second phase, respectively.

Consider the sets \( B^* \) and \( R^* \) defined by our algorithm, and assume without loss of generality that \( |B^*| \geq |R^*| \). By construction, \( C_0 \) contains all red agents who weakly prefer being in a balanced coalition to being in a homogeneous coalition. In particular, this means that no red agent in \( N \setminus C_1 \) would allow a blue agent to join her singleton coalition.

To prove that the partition computed by our algorithm is individually stable, we consider three classes of agents:

**Agents in \( C_0 \):** By deviating, these agents can form a homogeneous coalition or a balanced coalition. They weakly prefer \( C_0 \) to a homogeneous coalition, and, since they approve all subsequent changes to \( C \), they weakly prefer \( C_1 \) to \( C_0 \) (which is balanced). Thus, they have no IS-deviation.

**Agents in \( N \setminus C_1 \):** By construction, these agents do not have an IS-deviation to \( C_1 \), and they are indifferent between being alone and joining another agent of the same color. Further, a deviation that results in a two-agent balanced coalition is not an IS-deviation: all red agents in \( N \setminus C_1 \) strictly prefer being alone to being in a balanced coalition. Thus, agents in \( N \setminus C_1 \) have no IS-deviation.

**Agents in \( C_1 \setminus C_0 \):** Joining \( C \), these agents strictly prefer \( C \) to being in a homogeneous coalition, and they have approved all changes to \( C \) since then. Thus, they strictly prefer being in \( C_1 \) to being in a homogeneous coalition. Further, a blue agent in \( C_1 \setminus C_0 \) cannot join a singleton red coalition, since the red agent strictly prefers to be left alone. On the other hand, every red agent in \( C_1 \setminus C_0 \) strictly prefers being alone to being in a balanced coalition, so by transitivity she strictly prefers staying in \( C_1 \) to joining a singleton blue coalition.

\[ \square \]

**Example 4.2.** Consider an HDG with \( B = \{1, 2, 3\} \), \( R = \{4, 5\} \). The agents’ preferences over \( \Theta \) are given by

\[
\begin{align*}
\frac{2}{3} & \succ_2' \frac{1}{2} \succ_3' 0, \quad \frac{2}{3} \succ_2' \frac{1}{2} \succ_3' 0, \\
\frac{1}{4} & \succ_2' 0, \quad \frac{2}{3} \succ_3' \frac{1}{2} \succ_4' 1, \quad \frac{2}{3} \succ_3' 1.
\end{align*}
\]

The algorithm sets \( B^* = \{1, 2\} \), \( R^* = \{4\} \), and \( C_0 = \{4\} \). Then, in the second phase agent 5 has an IS-deviation to \( C_0 \); we have \( \{1, 4, 5\} \succ_5 \{5\} \), \( \{1, 4, 5\} \succ_4 \{1, 4\} \), and \( \{1, 4, 5\} \succ_4 \{1, 4\} \). Since agents 2 and 3 do not have an IS-deviation to \( \{1, 4, 5\} \), the algorithm stops and outputs \( \{1, 4, 5\}, \{2\}, \{3\} \).

In Example 3.2 our algorithm outputs a Pareto optimal outcome; however, our next example shows that this is not always the case.

**Example 4.3.** Consider an HDG with \( B = \{1, 2, 3, 4\} \), \( R = \{5\} \), where \( \frac{1}{4} \succ_i' 0 \) for each \( i \in B \), and the red agent’s most preferred ratio is \( \frac{1}{4} \). Then, for every agent the formation of the grand coalition is the most preferred outcome. However, as \( B^* = \emptyset \), the algorithm sets \( C_0 = \emptyset \) and hence outputs the partition consisting of five singletons.

---

**Algorithm 1 Computing an individually stable outcome**

**Require:** HDG \( G = (R \cup B, (\succ_i')_{i \in R \cup B}) \)

**Ensure:** Individually stable outcome \( \pi \) of game \( G 

1: Let \( B^* = \{b_1', \ldots, b_k'\} = \{b \in B : \frac{1}{2} \succ_i' 0\}; 
2: Let \( R^* = \{r_1', \ldots, r_k'\} = \{r \in R : \frac{1}{2} \succ_i' 1\}; 
3: Let \( C_0 = \{r_1', \ldots, r_{\min(k,\ell)}\} \cup \{b_1', \ldots, b_{\min(k,\ell)}\}; 
4: Let \( C = C_0; 
5: \textbf{repeat} 
6: \textbf{for} \ i \in N \setminus C \ \textbf{do} 
7: \textbf{if} \ i \text{ has an IS-deviation from} \ \{i\} \text{ to} C \ 
8: \textbf{then} \ C = C \cup \{i\}; 
9: \textbf{until} \ C \text{ has not changed in the previous iteration} 
10: \textbf{Let} \ C_1 = C; 
11: \textbf{return} \ \pi = \{\{i\} \mid i \in N \setminus C_1 \cup \{C_1\}; 

---
Interestingly, the algorithm of Bredereck et al. would output the grand coalition on the instance from Example 4.2. However, it is not the case that on any single-peaked instance the output of Bredereck et al.’s algorithm Pareto-dominates the output of Algorithm 4.1: there exists an example where the converse is true. This means, in particular, that neither of these algorithms is guaranteed to output a Pareto-optimal outcome on single-peaked instances; this is in contrast with the algorithm of Bogomolnaia and Jackson (2002), which, given a single-peaked anonymous game, always outputs an IS outcome that is also Pareto-optimal.

**Better-Response Dynamics** We note that one can also attempt to reach an IS outcome by a sequence of IS deviations: starting from an arbitrary partition, we check if the current partition is individually stable, and if not, we pick an agent who has an IS-deviation and allow her to perform it. We will refer to this procedure as the better-response dynamics (IS-BRD). While IS-BRD is a very general algorithm that can be used for arbitrary hedonic games, it may fail to converge even if an IS outcome exists, and it may need a super-polynomial number of iterations to converge (Gairing and Svarndal 2010); however, no results concerning its convergence are known for hedonic diversity games or for anonymous games.

The following proposition, which applies to arbitrary dichotomous hedonic games, makes partial progress towards the understanding of the performance of IS-BRD: it shows that for such games IS-BRD always converges as long as in the initial partition all agents belong to the grand coalition. We note that existence of IS outcomes in dichotomous games has been established by Peters (2016): his proof provides a polynomial-time algorithm for finding an IS outcome, but this algorithm differs from IS-BRD.

**Proposition 4.4.** For every dichotomous hedonic game with a set of agents $N$, any sequence of IS deviations starting from the grand coalition converges to an IS outcome after at most $|N|$ iterations.

**Proof.** If the grand coalition is individually stable, we are done. Otherwise, let $\pi(k)$ be the partition that forms after a sequence of $k$ deviations, $k \geq 1$, and let $N(k)$ be the set of all agents that have not deviated during this process.

We claim that the agents in $N(k)$ form a coalition in $\pi(k)$, and all agents in $N \setminus N(k)$ approve their coalitions in $\pi(k)$ (and hence only agents in $N(k)$ can perform IS deviations).

This follows by induction on $k$. Indeed, if $k = 1$, then the first agent who deviates forms a singleton coalition she approves, and all remaining agents stay together in one coalition. Now, suppose our statement is true for $k$; we will argue that it is true for $k+1$. Consider the agent $j$ who performs the $(k+1)$-st deviation. By the inductive hypothesis, we have $j \in N(k)$, so $N(k+1) = N(k) \setminus \{j\}$, i.e., agents in $N(k+1)$ form a coalition in $\pi(k+1)$. Now, we consider a coalition $C$ in $\pi(k)$ formed by agents in $N \setminus N(k)$. By the inductive hypothesis, all agents in $C$ approve $C$. If $C$ is unaffected by the deviation, this remains to be the case. On the other hand, if $j$ joins $C$, $j$ approves $C \cup \{j\}$; since all agents in $C$ do not object to $j$ joining them, it follows that they, too, approve $C \cup \{j\}$. This establishes our claim.

It follows that every agent can deviate at most once: after her first deviation, she approves the coalition she is in and does not want to move again. Hence, there can be at most $|N|$ deviations. \hfill $\square$

We will revisit IS-BRD in Section 6 where we compare different approaches to finding IS outcomes in HDGs.

### 5 Diversity Games With $k$ Classes

Bredereck, Elkind, and Igarashi (2019) define hedonic diversity games for two agent classes. However, diversity-related considerations remain relevant in the presence of three or more classes: for instance, in the example discussed in the beginning of the paper, the visiting students may come from several different countries. To capture such settings, we need to reason about games with $k$ agent classes for $k > 2$: e.g., for $k = 3$ we may have red, blue, and green agents. A direct generalization of the model of Bredereck et al. is to allow a red agent to base her preferences on both the ratio of red and blue agents and the ratio of red and green agents; a more restrictive approach is to assume that a red agent only cares about the fraction of red agents in her coalition. We will now explore both of these approaches; as we feel that the latter approach is closer in spirit to the original HDG model, we reserve the term $k$-HDG to refer to the more restrictive model and refer to games where agents can have arbitrary preferences over ratios as $k$-tuple HDGs.

**Definition 5.1.** A $k$-tuple hedonic diversity game is a hedonic game $(N, (\succ_i)_{i \in N})$ where $N$ can be partitioned into $k$ pairwise disjoint sets $R_1, \ldots, R_k$ so that for each $j \in [k]$, each agent $i \in R_j$, and every pair of coalitions $C, D \in \mathcal{N}(i)$ with $|C \cap R_j| = |D \cap R_j|$ for each $s \in [k]$ we have $C \sim_i D$.

Given an $n$-agent $k$-tuple HDG and an agent $i \in R_j$, we can map a coalition $C \in \mathcal{N}(i)$ to a $k$-tuple of fractions $\left(\frac{|C \cap R_1|}{|C \cap R_i|}, \ldots, \frac{|C \cap R_k|}{|C \cap R_i|}\right)_{i \in [k]}$; they are indifferent between two coalitions that map to the same tuple. Hence, $i$’s preferences can be described by a partial order over such tuples. As the number of tuples of this form is bounded by $n^{2k}$, if $k$ is bounded by a constant, the size of this representation is polynomial in $n$. On the other hand, every hedonic game with $n$ agents is an $n$-tuple HDG, as we can simply place each agent in a separate class.

Our XP result for Nash stability extends to this more general setting: if the total size of the smallest $k-1$ classes can be bounded by a constant, a Nash stable outcome can be found in time polynomial in the input representation size, which, in turn, can be bounded as $O(n^3)$ in this case. The proof (omitted) is a simple generalization of the proof of Theorem 5.1.

**Theorem 5.2.** Given a $k$-tuple HDG $G = (N, (\succ_i)_{i \in N})$ with classes $R_1, \ldots, R_k$ where $|N| = n$ and $\min_{j \in [k]} |N \setminus R_j| = p$, we can decide whether $G$ has a Nash stable outcome in time $(np)^p \cdot \text{poly}(n)$.
In contrast, the following example shows that Theorem 4.1 does not extend to \( k > 2 \), i.e., \( k \)-tuple HDGs may fail to have an individually stable outcome if \( k > 2 \).

**Example 5.3.** Consider a 3-tuple HDG \( G = (N, \{r \}_{i \in N}) \) with \( k \geq 5 \), it is NP-complete to decide whether \( G \) has an individually stable outcome.

**Theorem 5.4.** Given a \( k \)-tuple HDG \( G = (N, \{r_i\}_{i \in N}) \) with \( k \geq 5 \), it is NP-complete to decide whether \( G \) has an individually stable outcome.

**Proof.** To prove membership, note that it is possible to check in polynomial time whether an outcome is individually stable by iterating over all agent-coalition pairs and checking whether the agent has an IS-deviation to the coalition.

To prove hardness, we construct a reduction from X3C. We map a pair \((X, C)\) to a diversity game with 5 classes of agents (red, blue, green, yellow, white).

The general idea of the construction is to map each element \( x \in X \) to a blue set agent \( b_x \) such that in an individually stable outcome \( b_x \) needs to be in a coalition with ratio \((\frac{j^i + 2}{j^i + 3}, 0, 0, 0)\) for some \( j \in [k] \) such that \( x \in A_j \). This is ensured by an IS-penalizing component ‘destabilizing’ every outcome for which this is not the case. This enables us to map each set \( A_j \subset C \) to a unique coalition with ratio \((\frac{j^i + 2}{j^i + 3}, 0, 0, 0)\).

**Construction:** For each element \( x \in X \), we introduce one blue set agent \( b_x \) with the following preference relation \( \succeq_{b_x} \) over tuples of fractions:

\[
b_x : \left( \frac{1}{j^i + 3}, \frac{j^i + 2}{j^i + 3}, 0, 0, 0 \right) \sim_{b_x} \ldots
\]

\[
\sim_{b_x} \left( \frac{1}{j^i + 3}, \frac{j^i + 2}{j^i + 3}, 0, 0, 0 \right) \succ_{b_x} \left( 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \left( 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right)
\]

\[
\succ_{b_x} \left( 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0 \right) \succ_{b_x} \left( 0, 1, 0, 0, 0 \right).
\]

Moreover, for each \( j \in [k] \), we introduce a red fraction agent \( r_j \) with the following preference relation \( \succeq_{r_j} \):

\[
r_j : \left( \frac{1}{j + 3}, \frac{j + 2}{j + 3}, 0, 0, 0 \right) \succ_{r_j} \left( 1, 0, 0, 0, 0 \right).
\]

Additionally, for each \( j \in [k] \), we introduce \( j - 1 \) redundant blue agents with the following preference relations:

\[
b_j^p : \left( \frac{1}{j + 3}, \frac{j + 2}{j + 3}, 0, 0, 0 \right) \succ_{p} \left( 0, 1, 0, 0, 0 \right), \forall p \in [j - 1].
\]

Finally, we insert an IS-penalizing component consisting of one green, one yellow, and one white agent:

\[
g : \left( 0, \frac{1}{2}, \frac{1}{2}, 0 \right) \succ \left( 0, 0, 0, 1, 0 \right),
\]

\[
y : \left( 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0 \right) \succ \left( 0, 0, 0, 1, 0 \right),
\]

\[
w : \left( 0, \frac{1}{2}, \frac{1}{2}, 0 \right) \succ \left( 0, 0, 0, 0, 1 \right).
\]

**Correctness:** \((\Rightarrow)\) Assume that there exists a partition \( \pi \subset C \) of \( X \). We will transform \( \pi \) into an individually stable outcome \( \pi' \) of the respective 3-tuple hedonic diversity game. For each \( A_j \subset \pi \), the partition \( \pi' \) contains a coalition \( B_p = \{b_x \mid x \in A_j\} \cup \{r_j\} \cup \{b_j^p \mid p \in [j - 1]\} \). We refer to these coalitions as the first part of \( \pi' \). Further, we put all remaining red and blue agents in singleton coalitions, add them to \( \pi' \), and refer to them as the second part of \( \pi' \).

Finally, we add the coalitions \( \{g\} \) and \( \{y, w\} \) to \( \pi' \).

In the following, we prove that no agent has an IS-deviation in \( \pi' \) by iterating over all coalitions and arguing that no agent has an IS-deviation to this coalition. First, every coalition in the first part of \( \pi' \) is the unique most preferred coalition of its fraction agent. Therefore, no fraction agent accepts a deviation to her coalition. Second, no agent has an IS-deviation to a coaliation from the second part, as for no agent in the second part being in a coalition where players of her class account for \( \frac{1}{2} \) is individually rational. Third, no agent has an IS-deviation to \( \{y, w\} \), as this is \( w \)’s most preferred coalition. Fourth, no agent wants to deviate to \( \{g\} \), as all set agents and \( w \) are in one of their most preferred coalitions. Fifth, no agent has an IS-deviation to being in a singleton coalition, as \( \pi' \) is individually rational. Consequently, \( \pi' \) is individually stable.

\((\Leftarrow)\) Let us assume that there exists an individually stable outcome \( \pi \) of the hedonic diversity game. We claim that every blue set agent \( b_x \) in one of her most preferred coalitions in \( \pi \). For the sake of contradiction, assume that there exists some set agent \( p \) who is not in one of her most preferred coalitions. In the following, we iterate over all possible individually rational coalitions that \( g \) can be part of, and show that \( \pi \) cannot be individually stable if it contains this coalition.

If \( \{g\} \in \pi \), \( p \) needs to be in a homogeneous coalition, as this is her only remaining individually rational coalition. Thereby, \( p \) has an IS-deviation to \( \{g\} \). If \( \{g, p\} \in \pi \), agent \( y \) has an IS-deviation to this coalition. If \( \{g, y, p\} \in \pi \) then \( w \) needs to be in a singleton coalition, as this is her only remaining individually rational coalition. Thereby, \( g \) has an IS-deviation to \( \{w\} \). If \( \{g, w\} \in \pi \), \( w \) has an IS-deviation to...
the coalition \( \{y\} \), as \( \{y\} \) is the only remaining individually rational coalition for \( y \).

Consequently, there is no individually stable outcome where a set agent is not in one of her most preferred coalitions. Let \( \{P_1, \ldots, P_t\} \subseteq \pi \) be the set of all coalitions containing at least one set agent. Then, for all \( \ell \in [t] \) it holds that \( \theta(P_\ell) = \left( \frac{1}{t+1}, \frac{j}{t+1}, 0, 0, 0 \right) \) for some \( j \in [k] \). As \( \left( \frac{1}{t+1}, \frac{j}{t+1}, 0, 0, 0 \right) \) is only individually rational for exactly one red agent, \( P_\ell \) needs to consist of one red and \( j + 2 \) blue agents. Now, recall that this tuple is individually rational for exactly \( j + 2 \) blue agents including the three corresponding set agents. It follows that all three blue set agents who belong to \( A_j \) and no other set agents are part of \( P_\ell \). Consequently, by removing all non-blue agents from the coalitions in \( \{P_1, \ldots, P_t\} \), we obtain a cover of \( X \) by sets in \( C \).

Note that the proof also goes through if no indifferences in the agents’ preferences are allowed, as it is possible to replace the indifferences in the set agents’ preferences by strict preferences in an arbitrary way without affecting the validity of the proof.

Now, in \( k \)-tuple HDGs, an agent may have very complex preferences over ratios of agents from different classes. In our second model, an agent does not distinguish among classes other than her own and hence only cares about the fraction of members of her class in her coalition; we will refer to these games as \( k \)-HDGs. We note that a similar approach has been used recently to provide a game-theoretic model of Schelling segregation with \( k \geq 2 \) agent classes (Elkind et al., 2019; Echzell et al., 2019); this line of work, while similar in spirit to hedonic diversity games, is, however, very different from a technical perspective.

**Definition 5.5.** A hedonic diversity game with \( k \)-classes (\( k \)-HDG) is a hedonic game \((N, (\succeq_i)_{i \in N})\) where \( N \) can be partitioned into \( k \) pairwise disjoint sets \( R_1, \ldots, R_k \) so that for each \( j \in [k] \), each agent \( i \in R_j \), and every pair of coalitions \( C, D \in \mathcal{N}(i) \) we have \( C \sim_i D \) as long as \( \frac{|C \cap R_j|}{|C|} = \frac{|D \cap R_j|}{|D|} \).

By definition, every \( k \)-HDG is a \( k \)-tuple HDG, but the converse is not true, e.g., the game in Example 5.4 is not a \( k \)-HDG for any value of \( k \). Further, a \( 2 \)-HDG is simply an HDG as defined by Bredereck et al. Unlike \( k \)-tuple HDGs, \( k \)-HDGs admit a succinct representation even if \( k \) is not bounded by a constant: for each agent \( i \in R_j \), her preferences over coalitions in \( \mathcal{N}(i) \) can be described by her preferences over fractions of the form \( \frac{r}{\ell} \), where \( r \in [n] \), \( \ell \in \min\{|R_j|, r\} \).

Remarkably, this formalism captures anonymous games.

**Proposition 5.6.** Every anonymous game can be represented as a \( k \)-HDG.

**Proof.** Recall that an anonymous game \( G = (N, (\succeq_i)_{i \in N}) \) with \( |N| = n \) can be equivalently represented by a collection \( (\succeq_i^* \subseteq \mathcal{P}(N))_{i \in N} \) of weak orders over \( [n] \): \( C \succeq_i^* D \) if and only if \( |C \cap \{i\}| \leq |D \cap \{i\}| \). It follows that \( G \) can be viewed as an \( n \)-HDG with partition \( N = R_1 \cup \ldots \cup R_n \), so that \( R_i = \{i\} \) for each \( i \in N \). Indeed, for each \( i \in N \) and each coalition \( C \in \mathcal{N}(i) \), the fraction of agents from class \( R_i \) in \( C \) is exactly \( \frac{|C|}{|\{i\}|} \), so two coalitions \( C, D \in \mathcal{N}(i) \) have the same fraction of agents from \( R_i \) if and only if they have the same size.

It follows that \( k \)-HDGs inherit negative results for anonymous games, such as non-existence of IS outcomes (Bogomolnaia and Jackson, 2002) and hardness of deciding whether a given game has an IS outcome (Ballester, 2004).

**Corollary 5.7.** There exists a \( k \)-HDG that has no individually stable outcome. Moreover, deciding if a given \( k \)-HDG has an individually stable outcome is NP-complete.

Now, Bogomolnaia and Jackson (2002) show that every single-peaked anonymous game has an IS outcome. The definition of single-peaked preferences extends naturally to \( k \)-HDGs, e.g., we can use essentially the same definition as for HDGs. However, it is not clear if the algorithm of Bogomolnaia and Jackson (2002) for finding an IS outcome in single-peaked anonymous games can be extended to single-peaked \( k \)-HDGs: this is an interesting question for future work. Note also that the hardness result of Corollary 5.7 only holds for \( k = n \); it is not clear if the problem of finding an IS outcome in \( k \)-HDGs remains hard for small values of \( k \) (e.g., \( k = 3 \)). Further, the anonymous game with no IS outcomes constructed by Bogomolnaia and Jackson (2002) has 63 agents and therefore translates into a 63-HDG; it remains an open problem whether \( k \)-HDGs with \( k < 63 \) are guaranteed to have an IS-outcome.

## 6 Empirical Analysis

The algorithm for computing individually stable outcomes described in Section 4 has two substantial advantages over the algorithm of Bredereck et al.: first, it works for general preferences, and second, it is much simpler. However, as illustrated by Example 4.3, Bredereck et al.’s algorithm may result in higher agents’ satisfaction. The IS-BRD algorithm is even simpler, but we do not know if it always converges to an IS outcome. To better understand the performance of these three algorithms, in this section, we empirically compare them with respect to three measures: the average social welfare, the average coalition size and the average diversity.

**Preference Models** As Bredereck et al.’s algorithm is only defined for single-peaked HDGs, we only use single-peaked instances in our analysis. We consider three intuitively appealing ways of sampling preferences over ratios that are single-peaked on \( \Theta \).

**Uniform single-peaked preferences** (\( uSP \)). For each agent \( i \), we sample \( \succ_i^* \) uniformly at random among all single-peaked strict orders on \( \Theta \), using the algorithm of Walsh (2015) (see also Lackner and Lackner (2017)).

**Uniform-peak single-peaked preferences** (\( uPSP \)). To generate \( \succ_i^* \), we first select \( i \)'s most preferred ratio by choosing a point in \( \Theta \) uniformly at random. We then continue to place elements of \( \Theta \) in positions \( 2, \ldots, |\Theta| \) of \( \succ_i^* \) one by one; when \( k < |\Theta| \) elements have been ranked, there are at most two elements of \( \Theta \) that can be placed in position \( k + 1 \) so that the resulting ranking is single-peaked on \( \Theta \), and we choose between them with equal probability. This approach to sampling single-peaked preferences was popularized by Conitzer (2009).
Symmetric single-peaked preferences (symSP). For each agent $i$, we choose her preferred point $\theta_i$ from the uniform distribution on $[0,1]$ and define the relation $\succ'_i$ so that $\theta \succ'_i \theta'$ if and only if $|\theta - \theta_i| \leq |\theta' - \theta_i|$. While theoretically the resulting relation may have ties, in our experiments this approach always generated strict orders.

The first two distributions are quite different from each other, e.g., in the uSP model a ranking where 0 appears first is exponentially less likely than a ranking where $\frac{1}{2}$ appears first, while in the upSP model we are equally likely to see 0 and $\frac{1}{2}$ ranked first. On the other hand, upSP and symSP appear to be fairly similar, but our experiments show that our algorithms behave differently on them.

Performance measures The primary measure we are interested in is the social welfare, i.e., the sum of agents’ utilities. However, in general, this measure is difficult to define, since in HDGs agents’ preferences over coalitions are given by weak orders rather than numerical values. For symmetric single-peaked preferences, we can circumvent this difficulty by defining an agent’s disutility as the difference between the fraction of red agents in her coalition and her ideal ratio, so, given a partition $\pi$, for each $i \in N$ we set

$$\omega(i) = 1 - \frac{|\pi_i \cap R|}{|\pi_i|} - \theta_i.$$

For uniform and uniform-peak single-peaked preferences, we identify the utility of agent $i$ in partition $\pi$ with the Borda score of $\theta(i) = \frac{|\pi_i \cap R|}{|\pi_i|}$ in $\succ'_i$: for each $\theta \in \Theta$, we set

$$\beta(i,\theta) = |\{\theta' \in \Theta : \theta \succ'_i \theta'\}| \text{ and } \omega(i) = \frac{\beta(i,\theta(i))}{|\Theta| - 1}.$$

In both cases, we define the average welfare as

$$\omega(\pi) = \frac{1}{n} \sum_{i \in N} \omega(i).$$

To gain additional insight into the behavior of our algorithms, we also consider two other measures, namely, the average coalition size and the average diversity. We measure the diversity of a coalition $C$ as $\delta(C) = 1 - 2 \cdot \frac{|C \cap R|}{|C|}$, note that $\delta(C) = 0$ if $C$ is homogeneous and $\delta(C) = 1$ if $C$ is balanced. Both for the average size and for the average diversity, we average over all agents in the partition rather than all coalitions, as we focus on the experience of an individual agent. That is, the average coalition size and the average diversity are defined, respectively, as

$$\mu(\pi) = \frac{1}{n} \sum_{i \in N} |\pi_i| \text{ and } \delta(\pi) = \frac{1}{n} \sum_{i \in N} \delta(\pi_i).$$

Results The results of our experiments are shown in Figures 1 and 2. In all graphs, BEI19 refers to the algorithm of Bredereck et al., Alg1 refers to our Algorithm I and IS-BRD refers to the IS-BRD algorithm that starts with a uniformly random partition of agents and at each iteration chooses the deviation uniformly at random from among all available deviations. Remarkably, on every instance we have generated, IS-BRD converges in $O(n^2)$ iterations.

We first consider HDGs with the same number of red and blue agents. For each $s = 2,\ldots,50$, we generate 1000 HDGs with $s$ red and $s$ blue agents; thus, $n = 2s$ takes even values from 4 to 100.

Our results for social welfare are shown in Figure 1. For uSP, all algorithms perform similarly and provide fairly high social welfare. Intuitively, this is because under this preference model the agents are likely to rank the ratio $\frac{1}{2}$ highly, and therefore they are quite happy to be in a coalition with an approximately equal number of red and blue agents. Fortunately, all of our algorithms are good at partitioning the agents into such coalitions. In particular, under this distribution it is likely that each agent weakly prefers a balanced coalition to a homogeneous coalition, so Algorithm I will be able to stop after setting $C_0 = N$, and the grand coalition will be well-liked by most agents. However, the other two distributions tell a different story: under these distributions the algorithm of Bredereck et al. substantially outperforms the other two algorithms, with IS-BRD being consistently better than Algorithm I. Thus, if the distribution of agents’ top choices is close to uniform and the numbers of red and blue agents are equal, the more complex algorithm of Bredereck et al. has an advantage over our approach.

Figure 2 shows that the three algorithms are very different in terms of the sizes of coalitions they produce: Algorithm I tends to produce the largest coalitions, and the algorithm of Bredereck et al. tends to produce the smallest coalitions; this holds irrespective of how we sample the agents’ preferences.
Figure 3 shows that all three algorithms achieve almost perfect diversity under the uSP distribution; we have suggested possible reasons for this when discussing the social welfare. For the other two distributions, Algorithm 1 and the algorithm of Bredereck et al. perform similarly, while IS-BRD results in somewhat less diverse partitions.

We also investigate what happens if the two agent classes have different sizes. In Figure 4, we graph the average social welfare of the individually stable outcomes obtained by the three algorithms we consider, as the number of agents is fixed to \( n = 50 \) and the fraction of red agents in the game (denoted by \( \omega(N) \)) changes from 0 to 0.5. In this setting, too, the algorithm of Bredereck et al. outperforms the other two algorithms with respect to the social welfare, for almost all values of \( \theta(N) = \frac{|R|}{|N|+|R|} \); the difference is the most pronounced for the upSP distribution. However, for the uSP distribution, there is a range of values of \( \theta(N) \) where Algorithm 1 has better performance than the algorithm of Bredereck et al.; this can be attributed to the fact that the latter algorithm only produces coalitions that are very unbalanced or perfectly balanced, while Algorithm 1 can output coalitions with more complex ratios.

### 7 Conclusions

Our work contributes to the study of hedonic diversity games—an interesting class of coalition formation games introduced by Bredereck et al. We have focused on stability concepts that deal with deviations by individual agents, namely, Nash stability and individual stability. Remarkably, while our algorithm for finding IS outcomes is theoretically more appealing than the algorithm of Bredereck et al., in that it is simpler and more general, empirically the latter algorithm has better performance for the domain on which it is defined. Perhaps the most interesting open question concerning individual stability is whether IS-BRD is guaranteed to converge to an IS outcome, and if yes, whether convergence always happens after polynomially many iterations; we note that, in our experiments, this is always the case. The same question can be asked in the context of anonymous games. We also feel that the \( k \)-HDG model deserves further attention: in particular, we would like to understand the complexity of finding IS outcomes for small values of \( k \) and/or for single-peaked preferences.

### Acknowledgements

This work was supported by a DFG project “MaMu”, NI 369/19 (Boehmer) and by an ERC Starting Grant ACCORD under Grant Agreement 639945 (Elkind).

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