LOCALITY OF RANDOM DIGRAPHS ON EXPANDERS

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We study random digraphs on sequences of expanders with bounded average degree which converge locally in probability. We prove that the threshold for the existence of a giant strongly connected component, as well as the asymptotic fraction of nodes with giant fan-in or nodes with giant fan-out are local, in the sense that they are the same for two sequences with the same local limit. The digraph has a bow-tie structure, with all but a vanishing fraction of nodes lying either in the unique strongly connected giant and its fan-in and fan-out, or in sets with small fan-in and small fan-out. All local quantities are expressed in terms of percolation on the limiting rooted graph, without any structural assumptions on the limit, allowing, in particular, for non tree-like graphs.

In the course of establishing these results, we generalize previous results on the locality of the size of the giant to expanders of bounded average degree with possibly non-tree like limit. We also show that regardless of local convergence of a sequence, uniqueness of the giant and convergence of its relative size for unoriented percolation imply the bow-tie structure for directed percolation.

An application of our methods shows that the critical threshold for bond percolation and random digraphs on preferential attachment graphs is $p_c = 0$, with an infinite order phase transition at $p_c$.

1. Introduction. Many stochastic processes, from statistical physics models to epidemics or information diffusion, take place on an underlying network. This naturally gives rise to random subgraphs of the original graph, which in the simplest cases is described by unoriented or oriented percolation, see [33] for infections with constant recovery time, and [31] for information diffusion. In both cases, the oriented subgraph stems from the fact the process is inherently directed, with nodes infecting or informing their neighbors independently with probability $p$.

This leads to the question whether the important properties of these processes depend on global properties of the network, like connectivity or bipartiteness (as in the case of antiferromagnetic spin models), or whether it is enough to know just local information, represented by the $k$-neighborhoods of random vertices in the original graph. Specifically, we will look at the relative size of the giant component for unoriented percolation, while for oriented percolation, we will look at the fraction of nodes with large fan-out (corresponding to the probability that a random seed leads to an outbreak / successful campaign) or large fan-in (corresponding to nodes likely to be infected in an outbreak).

As we will see, local information is not quite enough - in addition, we will need what we call large-set expansion, a condition which guarantees that for large sets, the size of the edge boundary of a set grows linearly in its size. Under this condition, we show that the proportion of nodes with large fan-in or fan-out is indeed local. Here locality will be formalized by the notion of local convergence [2, 8], see Section 2.1 below for the precise definition.

\textsuperscript{MSC2020 subject classifications:} Primary 60K37, 05C48, 05C80, 05C20; secondary 90B15.

\textsuperscript{Keywords and phrases:} expanders, local convergence, random graphs, percolation, giant component, oriented percolation, epidemic models, Susceptible-Infected-Removed (SIR) process.
The question of locality of unoriented percolation on expanders has recently received much attention in the probability community. In [3], it was shown that on bounded degree expanders, there exists at most one linear size component (giant). If in addition, one assumes the existence of a local weak limit, one obtains locality of the threshold for the appearance of a giant [6, 41], in the sense that it can be inferred from the limit. Less is known for the relative size of the giant. Indeed, for bounded degree expanders, locality of the size of the giant is only known for high girth regular expanders [32]. In this case, the relative size of the giant is given by the survival probability of a percolated branching process.

To our knowledge, no results are known for oriented percolation on expanders with local limit.

While somewhat tangential to the purpose in this paper, we would be amiss not to mention the vast literature on the percolation threshold and the size of the giant for random graphs and for percolation on random graphs, starting with the work of Erdős and Rényi [21]. Since then, various other random models have been studied, from the random digraph of Karp [30], to so-called configuration models [11, 37, 29] and their directed analogues [17], to percolation on regular random graphs [24, 39] and configuration models [23, 28, 13]. Note that in all these models, the size of the giant is again given in terms of the survival probability of a suitable branching process.

To state our results formally, we need the notion of large-set expanders. Formally, it is defined as follows: Given a graph $G = (V, E)$ and a constant $\epsilon < 1/2$, we define

$$\phi(G, \epsilon) = \min_{A \subset V : e(A, V \setminus A) \leq |A|/2} \frac{e(A, V \setminus A)}{|A|}$$

where $e(A, V \setminus A)$ is the number of edges joining $A$ to its complement. Call a graph $G$ an $(\alpha, \epsilon, \bar{d})$ large-set expander if the average degree of $G$ is at most $\bar{d}$ and $\phi(G, \epsilon) \geq \alpha$. A sequence of possibly random graphs $\{G_n\}_{n \in \mathbb{N}}$ is called a large-set expander sequence with bounded average degree, if there exists $\bar{d} < \infty$ and $\alpha > 0$ such that for all $\epsilon \in (0, 1/2)$, the probability that $G_n$ is an $(\alpha, \epsilon, \bar{d})$ large-set expander goes to $1$ as $n \to \infty$.

To simplify our notation, we will take $G_n$ to be a graph on $n$ vertices. As usual, we use $G(p)$ to denote the random subgraph obtained from a graph $G$ by independently keeping each edge with probability $p$. Given a probability measure $\mu$ on $\mathcal{G}_n$, we then define

$$\zeta(p) = \mathbb{E}_\mu[\mathbb{P}_{G(p)}(|C(o)| = \infty)]$$

where $o$ is the root in $(G, o) \sim \mu$ and $C(o)$ is the connected component of $o$ in $G(p)$, and we define the percolation threshold $p_c(\mu)$ of $\mu$ as

$$p_c(\mu) = \inf_{p \in [0, 1]} \{p : \zeta(p) > 0\}.$$

Finally, we use a quenched notion of local weak convergence, namely that of local convergence in probability, see Section 2.1 for the precise definition.

**Theorem 1.1.** Let $\{G_n\}_{n \in \mathbb{N}}$ be a sequence of (possibly random) large-set expanders with bounded average degree converging locally in probability to $(G, o) \in \mathcal{G}_n$ with non-random distribution $\mu$. Let $C_i$ be the $i^{th}$ largest component of $G_n(p)$. If $p \neq p_c(\mu)$, then

$$\frac{|C_1|}{n} \xrightarrow{\mathbb{P}} \zeta(p),$$

with $\xrightarrow{\mathbb{P}}$ denoting convergence in probability with respect to both $\mu$ and percolation. Moreover, for all $p \in [0, 1]$, $\frac{|C_2|}{n} \xrightarrow{\mathbb{P}} 0$, where the convergence is uniform on any closed interval $I \subset (0, 1)$ in that $\sup_{p \in I} \mathbb{P}(|C_2| \geq \epsilon n) \to 0$ for all $\epsilon > 0$. 

Remark 1.2. The restriction $p \neq p_c(\mu)$ can be removed for models where it is known that $\zeta$ is continuous at $p_c$. This includes many models where $\mu$ is supported on trees, including preferential attachment (where $p_c(\mu) = 0$) and all $\mu$ supported on trees with more than 3 ends, i.e., trees with at least 3 disjoint path to $\infty$.

In fact, when proving the statement about the asymptotic size of the giant we first prove that it holds whenever $\zeta$ is continuous at $p$, and then prove that under the assumption of the theorem, $\zeta$ is continuous except possibly at $p_c$. To this end, we generalize a result of Sarkar [41], and prove that a deterministic measure $\mu$ on $G_n$ is extremal in the set of unimodular measures on $G_n$ when it is the local limit in probability of some (possibly random) sequence of large set expanders with bounded average degree. By a theorem of Aldous and Lyons [1], this in turn implies that $\zeta$ is continuous except possibly at $p_c(\mu)$\(^1\). See Section 2.3 for a more detailed discussion.

Our theorem generalizes previous results [3, 5, 41, 32] in several directions, allowing for applications to graph sequences sharing some of the features of more realistic network models. First, we remove the condition of bounded degrees, and replace it by bounded average degree, a condition which allows for power law graphs which were not included before. As an illustrative example, we consider preferential attachment and show that the critical threshold for a linear sized giant is $p_c = 0$, with $\zeta(p) = e^{-\theta(1/p)}$ as $p \to 0$, corresponding to an infinite order phase transition (Theorem 6.1). Second, we remove the assumption that the graph $G_n$ is locally tree-like, and give an explicit expression for the asymptotic size of the giant regardless of whether or not the limit is given by a birth process. Third, we relax the condition of expansion, to include graphs which are not necessarily connected - as a side benefit, we obtain a condition which in many cases is easier to verify (see Appendix D for the case of preferential attachment).

Despite these generalizations, the proof of Theorem 1.1 relies mainly on extensions of known methods, including those of Alon, Benjamini and Stacey [3] and Krivelevich, Lubetzky and Sudakov [32]. These were a major motivation for our proofs, even though the randomness of the sequence $G_n$ induces some subtleties which need to be taken into account to avoid trivial counter examples. We discuss these in Section 3, and relegate the more standard techniques to an appendix.

As a corollary of Theorem 1.1, one obtains a generalization of the results from [6, 41] on the “locality” of $p_c$; indeed, our theorem implies that for sequences of large-set expanders with bounded average degree converging locally in probability, the critical threshold for the appearance of a giant is equal to $p_c$ defined in (3). See Section 2.2 for the precise definition of a critical threshold, and Corollary 3.6 below for a formal statement of this corollary.

The second (and we believe technically more novel) part of the paper concerns oriented percolation. To state our results, we need some additional notation: First, as usual, we say that a sequence of events, $(E_n)$, holds with high probability if the probability of $E_n$ goes to 1 as $n \to \infty$. Next, given a digraph, let $C^+(v)$ (and $C^-(v)$) be the set of nodes $w$ that can be reached by an oriented path from $v$ to $w$ (from $w$ to $v$). We refer to these sets as the fan-out (and fan-in) of $v$. As usual, the set $SCC(v) = C^+(v) \cap C^-(v)$ is called the strongly connected component of $v$. For a strongly connected component $SCC$, we use the symbol $SCC^+$ for the set of nodes $SSC^+ = \bigcup_{v \in SCC} C^+(v)$ and the symbol $SCC^-$ for the set of nodes $SSC^- = \bigcup_{v \in SCC} C^-(v)$. Finally, we use the symbol $D_G(p)$ to denote the random digraph obtained from a graph $G$ by first replacing each edge $\{u, v\}$ by two oriented edges $uv$ and $vu$ and then keeping each oriented edge independently with probability $p$.

\(^1\)We thank the anonymous referee for pointing out this connection, and suggesting the generalization of Sarkar’s results to our settings.
Our next theorem establishes the structure of $D_G(p)$ for any sequence that has a unique giant whose relative size converges in probability after undirected percolation, i.e., the sequence satisfies the conclusion of Theorem 1.1.

**Theorem 1.3.** Let $p \in (0,1]$ and let $\{G_n\}_{n \in \mathbb{N}}$ be a sequence of (possibly random) graphs such that

(i) there exists $q \in (0,p]$ and a function $\zeta : [p-q,p] \to [0,1]$ that is left-continuous at $p$ such that $\frac{|C_1|}{n} \xrightarrow{P} \zeta(p')$ for all $p' \in [p-q,p]$;

(ii) $\frac{|C_2|}{n} \xrightarrow{P} 0$ uniformly in $[p-q,q]$.

Let $SCC_i$ be the $i$th largest strongly connected component in $D_G(p)$. Then

1. Uniformly for all $p' \in [p-q,q]$,
   $$\frac{|SCC_2|}{n} \xrightarrow{P} 0.$$

2. If $\zeta(p) = 0$ and $v \in V(G_n)$ is chosen uniformly at random, then
   $$\frac{|C^+(v)|}{n} \xrightarrow{P} 0, \quad \frac{|C^-(v)|}{n} \xrightarrow{P} 0 \quad \text{and} \quad \frac{|SCC_1|}{n} \xrightarrow{P} 0.$$

3. If $\zeta(p) > 0$ then
   $$\frac{|SCC^+_1|}{n} \xrightarrow{P} \zeta(p), \quad \frac{|SCC^-_1|}{n} \xrightarrow{P} \zeta(p) \quad \text{and} \quad \frac{|SCC_1|}{n} \xrightarrow{P} 1,$$

   with
   $$\liminf_{n \to \infty} \frac{1}{n} \mathbb{E}[|SCC_1|] \geq \zeta^2(p).$$

Furthermore, if $v \in V(G_n)$ is chosen uniformly at random, then with high probability, the following two statements hold:

* either $v \notin SCC^-_1$ and $|C^+(v)| = o(n)$ or $v \in SCC^-_1$ and $|C^+(v) \Delta SCC^+_1| = o(n)$;

* either $v \notin SCC^+_1$ and $|C^-(v)| = o(n)$ or $v \in SCC^+_1$ and $|C^-(v) \Delta SCC^-_1| = o(n)$.

In particular, if $v \notin SCC^+_1 \cup SCC^-_1$, both $|C^+(v)| = o(n)$ and $|C^-(v)| = o(n)$.

**Remark 1.4.** The theorem implies the following “bow-tie structure” for $D_G(p)$ when $\zeta(p) > 0$: Define the bow-tie as the induced graph on $SCC^-_1 \cup SCC^+_1$ with the left and right wing being given by the vertices in $SCC^-_1 \setminus SCC_1$ and $SCC^+_1 \setminus SCC_1$, respectively. The theorem then implies that (up to $o(n)$ exceptions) with high probability all vertices not in the bow-tie will have fan-in and fan-out of size $o(n)$. In fact, all but at most $o(n)$ vertices fall into one of the following four classes: (i) the giant strongly connected component (the center of the bow-tie), consisting of the vertices with large fan-in and large fan-out; (ii) the left (and (iii) the right wing) consisting of the vertices with large fan-out and small fan-in (large fan-in and small fan-out), and (iv) the remaining “dust”, consisting of vertices which have small fan-in and fan-out. See Figure 1 for a demonstration.

The bow-tie structure was first described on an experimental analysis of the web graph [15]. Later, Cooper and Frieze [17] established the bow-tie structure of directed configuration

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2The theorem is actually slightly stronger, since it says that this structure holds even if we define large fan-in and fan-out by requiring only that these sets contain order $n$ vertices - for almost all vertices, large fan-in or fan-out will then automatically give fan-in / fan-out of at least the size of the giant SCC.
model with maximum degree $o(n^{1/12})$. Later work, weakened the condition on maximum degree to $o(n^{1/4})$ in [25], and $o(\sqrt{n})$ in [16]. To the best of our knowledge, this is the first work showing the bow-tie structure for oriented percolation on general expanders (and more generally, oriented percolation on any model for which the conclusions of Theorem 1.1 hold).

**Remark 1.5.** The bow-tie structure established in Theorem 1.3 says in particular that the size of the giant SCC is asymptotically equal to the number of vertices whose fan-in and fan-out is linear in $n$. Under the assumption of local convergence of $G_n$ in probability, one might therefore conjecture that $\frac{1}{n}|SCC_1|$ converges in probability to

$$
(4) \quad \zeta^{+,-}(p) = \mu \left( \mathbb{P}_{D_G(p)}(|C^+(o)| = \infty \text{ and } |C^-(o)| = \infty) \right).
$$

While our technology is not strong enough to prove this for general limits $\mu$, we can prove that $\zeta^{+,-}(p)$ is an asymptotic upper bound on $\frac{1}{n}|SCC_1|$ whenever $G_n$ converges locally in probability to $\mu$, see Lemma 5.1 below. See also Remark 5.2 for a simple case where we can prove that $\frac{1}{n}|SCC_1|$ converges to $\zeta^{+,-}(p)$ in probability.

Let us mention two applications of our results. The first one is the SIR (Susceptible-Infected-Recovered) infection model with fixed recovery time. In this model, each node can have three states: susceptible, infected or recovered. Each infected node infects each of its neighbors independently according to a Poisson process with rate of $\lambda$, and recovers after a fixed time (say one unit of time). So, an infected vertex has an opportunity to infect any of its neighbors independently with probability $p = \frac{1}{\lambda + 1}$. While in this model, an actual infection starting from a particular node gives an infection tree describing all nodes that get eventually infected, it is often useful to capture the structure of a possible infection independently of the initial node, by defining an infection digraph which in our notation is nothing but the random digraph $D_G(p)$. This gives a coupling of the infections starting at all possible seed vertices $v$, with the fan-out of $v$ being exactly the set of nodes getting sick eventually in an infection starting at $v$. The structure of the bow-tie following from Theorem 1.3 then implies that with high probability

- an infection starting in $SCC_1^-$ will infect all vertices in $SCC_1^+$, plus at most $o(n)$ extra vertices;
with the exception of up to $o(n)$ vertices, an infection starting in the complement of $SCC^{-}_1$ will only infect $o(n)$ other vertices.

Together with the third statement of the theorem, we conclude that if we infect a uniform random vertex in the network, the asymptotic probability and the size of an outbreak is $\zeta(p)$ and $n\zeta(p)$, respectively.

Another application of oriented percolation concerns information cascades. In this model, agents (nodes) are either informed or uninformed. Once an agent is informed they have only one chance to communicate the information to any of their contacts (neighbors in network), and the information will be shared successfully with probability $p$. This is a special case of information cascade model considered in [31] and many follow ups, where the success probability over all edges of the network is equal to $p$. Similar to the infection digraph in SIR model, one can define an information digraph in which a directed edge from $u$ to $v$ represents the event that conditioned on $v$ being the first note to be informed, the information is shared successfully along the edge $uv$. As a result of our theorem on oriented percolation, one can estimate the expected number of nodes that will have been informed at the end of the cascade if the initial seed is chosen uniformly at random, or more generally, if a set of initial seeds are chosen uniformly at random, provided the underlying network is an expander with bounded average degree that converges locally in probability.

We close this introduction with a final remark.

REMARK 1.6. In parallel to our work, Remco van der Hofstad developed a different approach to the locality of the giant in unoriented random graphs [44]. He showed that if the random graph sequences has a local limit, and the property that it is unlikely that two random vertices lie in distinct, large components, the relative size of the giant is given by the probability that the origin in the limit lies in an infinite component. This hints at a possible, alternative approach to proving the first statement in Theorem 1.1; as in our proof, one would first establish local convergence of the percolated sequence in probability, but then use a sprinkling argument to prove that the assumptions of [44] are satisfied at the points of continuity of $\zeta(p)$, rather than directly proving our Proposition 3.4 below. Unfortunately, an application of our sprinkling arguments only gives a condition which is weaker than required for an application of the results of [44]. So at the moment, the two methods seem to be complementary, establishing uniqueness of the giant for different sets of random graphs.

1.1. Overview of the Paper. In Section 2, we set up notations and terminology, including the notion of local convergence in probability (Section 2.1) and the formal definition of the threshold for the appearance of a giant and strongly connected giant (Section 2.2). Finally, in Section 2.3 we discuss the continuity of $\zeta$, and in Section 2.4, we review a concentration bound that follows from a beautiful result of Falik and Samorodnitsky [22] and will be used later in the proof of Theorem 1.3. The reduction of the concentration bound to the results of [22] is given in Appendix B.

Theorems 1.1 is proved in Section 3, where on the way of proving it we also show that the relative size of the second largest component in expanders with bounded average degree converges uniformly to zero (see Lemma 3.5). The proof of this lemma uses the techniques of Alon, Benjamini and Stacey [3], extending their results for expanders with bounded degrees to large set expanders with bounded average degrees and local limit in probability, and is given in Appendix C.

Section 4 is a stand-alone section that explores the relation between oriented and unoriented percolation on general graphs via a natural coupling. Building upon this coupling and our results for unoriented percolation, Theorem 1.3 is then proved in Section 5. One of the main technical difficulties in this section is the proof of concentration of the size of the
strongly connected giant, without having an explicit formula for its expectation. While a proof based on Russo’s lemma and suitable bounds on influences might seem natural to the expert, it turns out be quite tricky for the oriented case, due to the fact that a single edge can join many small strongly connected components which without this edge were just sitting on a directed path, “without a path back”. This makes the size of strongly connected giant much less “local” than the undirected analog.

In Section 6, we apply Theorem 1.1 to preferential attachment models. The details of many of the proofs are deferred to appendices: The proof of expansion for preferential attachment models is proven in Appendix D; and finally, upper and lower bounds for the survival probability of the limiting branching process after percolation are proven in Appendix E.

2. Notations, Definitions, and Preliminaries. For a graph $G$, let $V(G)$ be the set of vertices and $E(G)$ be the set of edges. As usual, a rooted graph is a graph with one particular node $v$ designated as the root; we will use the notation $(G, v)$ to denote a rooted graph with root $v$. A graph isomorphism between two graphs $G_1$ and $G_2$ is a bijection $\phi : V(G_1) \to V(G_2)$ such that $\{v, w\} \in E(G_1)$ if and only if $\{\phi(v), \phi(w)\} \in E(G_2)$. If the two graphs are rooted, we also require that $\phi$ maps the root of $G_1$ to that of $G_2$. We will use $G_*$ to denote the space of equivalence classes of locally finite, rooted graphs under these isomorphisms.

The $k$-neighborhood of a vertex $v$ in $G$ is defined as the induced subgraph on the set of nodes of graph distance at most $k$ from $v$, and will be denoted by $B_k(G, v)$. If $G$ is clear from the context, we just write $B_k(v)$ instead of $B_k(G, v)$.

2.1. Local Convergence in Probability. As usual, local convergence [2, 8] is defined in terms of a metric $d_{loc}$ on $G_*$: given two rooted graphs $(G_1, o_1)$ and $(G_2, o_2)$, their “local distance” is defined as

$$d_{loc}((G_1, o_1), (G_2, o_2)) = \frac{1}{1 + \inf_k \{B_k(G_1, o_1) \not\approx B_k(G_2, o_2)\}}.$$ 

where $\approx$ denotes equivalence under isomorphisms which map the roots $o_1$ and $o_2$ into each other. The function $d_{loc}$ defines a topology on the space of rooted graphs and local convergence of a sequence of graphs is defined with respect to that topology. Since the finite graph $G_n$ we consider is typically not rooted, we choose a root uniformly at random,

$$P_n = \frac{1}{n} \sum_{o_n \in V(G_n)} \delta_{(G_n, o_n)}.$$ 

For non-random sequence $G_n$, local weak convergence to a measure $\mu$ on $G_*$ is defined by the requirement that $\mathbb{E}_{P_n}[f] \to \mathbb{E}_\mu[f]$ for all bounded continuous functions $f$ on $G_*$. If $G_n$ is random, there are three commonly considered notions of local convergence: convergence in distribution, convergence in probability, and almost sure convergence, see Chapter 2 in [45] for an overview. For convergence in distribution (also called annealed), one requires that the expectations of $f$ with respect to both $P_n$ and the randomness of $G_n$ converge, while for the other two, the randomness of $G_n$ is fixed (quenched). In this paper, we will use convergence in probability for the quenched version which we now define formally.

Consider thus a sequence of random graphs $\{G_n\}_{n \in \mathbb{N}}$, and a (non-random) probability measure $\mu$ on $G_*$. We say $G_n$ converges locally in probability to $\mu$ if for any bounded continuous function $f : G_* \to \mathbb{R}$,

$$\mathbb{E}_{P_n}[f|G_n] \overset{P}{\to} \mathbb{E}_\mu[f],$$

where in $\mathbb{E}_{P_n}[f|G_n]$ we only take expectations with respect to the random root in $G_n$. So, $\mathbb{E}_{P_n}[f|G_n]$ can be random variable due to the conditional dependence on graph $G_n$ (in the
case where the sequence $G_n$ is random). While in principle, convergence in probability allows for convergence to a random measure $\mu$, in which case $E_\mu[f]$ would be random, in this paper, we will assume that the limiting measure $\mu$ on rooted graphs is non-random.

Note that this restriction rules out certain random graph sequences: a sequence $\{G_n\}_{n \in \mathbb{N}}$ where $G_n$ is a random 3-regular graph on $n$ nodes with probability $1/2$, and a random 4-regular graph on $n$ nodes with probability $1/2$ will not have a deterministic limit $\mu$, while the union of two disconnected graphs of the same size where one is 3-regular and one is 4 regular has a deterministic limit $\mu$ (with $\mu(G,o)$ being $1/2$ if $G$ is a 3 or 4 regular infinite tree, and 0 otherwise). Note that by contrast, both sequences converge to this deterministic measure $\mu$ if we consider convergence in distribution.

### 2.2. Thresholds for the Existence of a Giant for Unoriented and Oriented Percolation

The notion of a threshold for the appearance of various structures in finite, random graphs is a well known concept from random graph theory, with the question of the locality of the threshold for the appearance of a giant for percolation on bounded degree expanders being part of the literature on which this paper is building. For the convenience of the reader, and to define these concepts for the oriented case, we give a precise definition below.

We use $P_{G(p)}$ and $E_{G(p)}$ to denote probabilities and expectations with respect to percolation on a graph $G$, and $P_{D_G(p)}$ and $E_{D_G(p)}$ for the oriented analogues. Expectations with respect to the distribution $\mu$ of rooted random graphs describing the local limit are denoted by $E_\mu$. Finally, we use the standard notation $a \land b$ and $a \lor b$ for the maximum and minimum of two real number $a$ and $b$.

We will say that a sequence $p_n$ is a threshold sequence\(^3\) for the existence of a giant component (in short, the percolation threshold) if for all $\epsilon > 0$ and $c > 0$

$$\mathbb{P}(G_n(0 \lor (p_n - \epsilon))) \text{ contains a component of size at least } cn \to 0,$$

and for all $\epsilon > 0$ there exists some $c > 0$ such that

$$\mathbb{P}(G_n(1 \land (p_n + \epsilon))) \text{ contains a component of size at least } cn \to 1.$$

With a slight abuse of notation, we will write $p_n = p_c(G_n)$ to denote a threshold sequence\(^4\). If $G_n$ is random, the above probabilities are with respect to percolation, $\mathbb{P} = \mathbb{P}_{G_n(0 \lor (p_n - \epsilon))}$ and $\mathbb{P} = \mathbb{P}_{G_n(1 \land (p_n + \epsilon))}$, and convergence becomes convergence in probability (with respect to the randomness of $G_n$). We define the critical threshold for the appearance of a giant SCC in $D_{G_n}(p)$ in the same way, with the only difference being that the word “components” is replaced by strongly connected components; we will use the notation $p_c^{\text{SCC}}(G_n)$ such a threshold sequence.

**Remark 2.1.** These two definitions immediately raise the question whether the two thresholds are related. First, it turns out that asymptotically, the two must be the same if they both exist (see Corollary 4.4 below). But even without the assumption that both exist, we know quite a bit; in fact, without any prior assumptions on the existence of either threshold, we know that if the probability that $G_n(p_n)$ contains a giant of size $cn$ or larger goes to 0, then the probability that $D_{G_n}(p_n)$ contains a giant SCC of this size goes to zero as well (Corollary 4.2 below). In the other direction we know that if the expectation of the giant component in $G_n(p_n)$ is bounded below by $cn$ for some $c > 0$, then the expectation of the giant

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\(^3\)When $p_n > \alpha$ for some $\alpha > 0$ independent of $n$ this coincides with the notion of a sharp threshold, as defined, e.g., in [27].

\(^4\)Strictly speaking, the correct formal notation would be $(p_n) \in p_c(G_n)$ to stress the fact that $(p_n)$ remains a threshold sequence if we add a term $\epsilon_n$ which goes to 0 as $n \to \infty$. 
SCC in $D_{G_n}(p_n)$ is bounded by $c'n$ for some $c' > 0$. While this allows us to conclude that once $G_n(p_n)$ has a giant with high probability, the expectation of the giant SCC in $D_{G_n}(p_n)$ is at least of order $n$, this does not imply existence of a giant SCC with high probability; for arbitrary sequences of graphs, we just don’t have enough control over the variance.

Recall our definition (3) of the percolation threshold for an infinite rooted graph $(G, o)$ with law $\mu$. While it might seem natural to define $p^\text{SCC}_c(\mu)$ for the appearance of a strongly connected component similarly, the naive definition turns out not to be useful for identifying the threshold of a locally convergent sequence. This is because a giant strongly connected component for graphs with large girth might not correspond to a SCC in the limit graph. Instead, we consider the event that both the fan-in and the fan-out of the root is infinite,

$$p^{\pm}_c(\mu) = \inf\{p \in [0, 1]: \zeta^{\pm}(p) > 0\},$$

with $\zeta^{\pm}(p)$ given by (4). With this definition, we will prove $p^{\pm}_c(\mu)$ is the threshold for the appearance of a giant SCC in $D_{G_n}(p)$ if $G_n$ is a sequence of large-set expanders with bounded average degrees that converges to $(G, o)$. In fact, it is not hard to see that for arbitrary rooted random graphs

$$p^{\pm}_c(\mu) = p_c(\mu)$$

(see Lemma 4.5), consistent with the fact that if both the appearance of a giant and the appearance of a giant SCC have a threshold sequence, the two must be asymptotically equal (Corollary 4.4).

2.3. Continuity of $\zeta$. In this section, we elaborate on Remark 1.2. To this end, we recall that when $\mu$ is a local limit of some (possibly random) sequence $G_n$, it obeys a symmetry relation known as unimodularity [8], see, e.g., [3] for the definition of unimodularity.

Next, we point out that our definition of $p_c(\mu)$ in (3) differs from the standard definition of $p_c$ for random rooted graphs, as in, e.g., in [34]. In particular, in these papers, $p_c$ is a function of the random graph $(G, o)$ drawn from $\mu$, i.e.,

$$p_c(G, o) = \inf\{p: \exists x \in V(G) \text{ s.t. } P_{G(p)}(|C(x)| = \infty)\},$$

with $C(x)$ denoting the connected cluster of $x$ in $G(p)$. If the limit is extremal in the set $U$ of unimodular measures then $p_c(G, o)$ is almost surely a constant, and in that case, it will be equal to our definition of $p_c(\mu)$ in (3).

Finally, for extremal measures in $U$, Aldous and Lyons showed that if $p_c(\mu) < p_1 < p_2$, then $\mu$-almost surely, every infinite cluster in $G(p_2)$ contains an infinite cluster in $G(p_1)$ (Theorem 6.7 in [1]). This in turn implies continuity of $\zeta$ for all $p \neq p_c(\mu)$ by the standard arguments (see, e.g., [42]). Furthermore, Theorem 8.11 from [1] gives continuity at $p_c$ for non-amenable extremal $\mu \in U$, which in particular holds for extremal trees with at least three ends, i.e., three disjoint infinite path in the tree under consideration.

Extremality of $\mu$ was proven in [41] when $\mu$ is the local weak limit of a non-random sequence of expanders of bounded degree. In Appendix A we generalize this proof to measures $\mu$ that arises as the local limit in probability of a (possibly random) sequence of large-set expanders of uniformly bounded average degree. As just explained, this immediately gives continuity of $\zeta$ for $p \neq p_c(\mu)$.

**Corollary 2.2.** Let $\{G_n\}$ be a sequence of (possibly random) large-set expanders of bounded average degree that converge locally in probability to $\mu$, and let $\zeta(p)$ and $p_c(\mu)$ be as in (2) and (3). Then $\zeta(p)$ is continuous for all $p \neq p_c(\mu)$. 
2.4. Concentration Bounds. One of the most difficult parts of this paper is the proof of concentration for the size of the giant strongly connected component without explicit control of its expectation. To this end, we will use a concentration inequality going back to the work of Falik and Samorodnitsky [22].

Given a positive integer \( m \), let \( x \) denote vectors \( x = (x_e)_{e \in [m]} \), and for \( 0 < p < 1 \), let \( \mathbb{P}_p \) be the independent product measure on \( \{0,1\}^m \) with marginals \( \mathbb{P}(x_e = 1) = p \) (in our application, \( m \) will be twice the number of edges in \( G \), and \( \mathbb{P}_p \) will be the oriented percolation measure \( \mathbb{P}_{D_x(p)} \)). For \( x \in \{0,1\}^m \) and \( e \in [m] \), we use \( x \oplus e, x \cup \{e\} \) and \( x \setminus \{e\} \) to denote the Boolean vector obtained from \( x \) by flipping the bit \( e \), replacing it by 1, or replacing it by 0, respectively. For an increasing function \( f : \{0,1\}^m \to \mathbb{R} \), define the influence of an edge

\[
\Delta_{e} f(x) = f(x) - \left( p f(x \cup \{e\}) + (1 - p) f(x \setminus \{e\}) \right),
\]

and

\[
\mathcal{E}_2(f) = \sum_{e \in [m]} \mathbb{E}_p[|\Delta_{e} f|^2] \quad \text{and} \quad \mathcal{E}_1(f) = \sum_{e \in [m]} \mathbb{E}_p^2[\Delta_{e} f].
\]

The results of Falik and Samorodnitsky [22] then imply the following bounds.

**Lemma 2.3.** Let \( f, \mathcal{E}_1(f) \) and \( \mathcal{E}_2(f) \) be as above. Then

\[
\frac{1 - 2p}{\log \frac{1}{p}} \frac{\text{var}(f)}{\mathcal{E}_1(f)} \log \left( \frac{\text{var}(f)}{\mathcal{E}_1(f)} \right) \leq \mathcal{E}_2(f).
\]

For \( p = 1/2 \) (with the prefactor replaced by its limit, 1/2), the lemma is essentially equivalent to Lemma 2.1 in [7]. While apparently not realizing that its proof required the uniform measure, Lemma 2.1 from [7] was restated in [5] for general \( p \), rendering several of the technical lemmas in that paper incorrect. However, the needed changes do not invalidate the main results of [5], since they all concern values of \( p \) bounded away from zero, where the difference just amounts to a difference in an overall constant. We give the reduction of Lemma 2.3 to the results of reference [22] in Appendix B.

We close our preliminaries by recalling Russo’s formula. Given an increasing event \( A \subset \{0,1\}^m \) and \( x \in \{0,1\}^m \), we define \( e \in [m] \) to be pivotal if exactly one of \( x \) and \( x \oplus e \) is in \( A \). The Margulis–Russo formula [40] then says that

\[
\frac{d}{dp} \mathbb{P}_p(A) = \sum_{e \in [m]} \mathbb{P}_p(e \text{ is pivotal}).
\]

We will also need the analog for an increasing function \( f : \{0,1\}^m \to \mathbb{R} \), which states that

\[
\sum_{e \in [m]} \mathbb{E}_p[|\Delta_{e} f|] = 2p(1 - p) \frac{d}{dp} \mathbb{E}_p[f].
\]

3. Locality of Unoriented Percolation on Expanders. In this section, we prove Theorem 1.1. To this end, we first establish a lemma stating that local convergence in probability allows for the control of both expectations and concentration of local quantities in the percolated graph \( G_n(p) \) (Lemma 3.1). Note that despite its apparent simplicity, the lemma is slightly subtle, and in particular does not hold if one only assumes local convergence in distribution instead of local convergence in probability.

Using this lemma, it will be straightforward to upper bound the asymptotic size of the giant by \( \zeta(p)n \). For the lower bound, we will use a sprinkling argument, which in its simplest form
goes back to Erdős [21], and is at the core of most previous work on locality in percolation including that of Alon, Benjamini and Stacey [3]. We recall that throughout this paper, we assume that all our sequences \((G_n)\) are growing; in particular, we assumed without loss of generality that \(G_n\) is a graph on \(n\) vertices, which may or may not be random.

**Lemma 3.1.** Let \(\mu\) be a (non-random) probability distribution on \(\mathcal{G}_s\), and \(\{G_n\}_{n \in \mathbb{N}}\) be a sequence of (possibly random) graphs that converge locally in probability to \((G, o) \in \mathcal{G}_s\) with distribution \(\mu\). For \(p \in [0, 1]\) and a positive integer \(k\), let \(f_k : \mathcal{G}_s \to \mathbb{R}\) be a bounded and continuous function defined on the \(k\)-neighborhood of a node. Then

\[
\mathbb{E}_{\mathcal{P}_n}\left[f_k(G_n(p), o_n) \mid G_n\right] \xrightarrow{P} \mathbb{E}_{\mu_o}\left[f_k(G(p), o)\right],
\]

where convergence in probability is over the possible randomness of \(G_n\) and percolation, and \(\mu_p\) is the deterministic measure on \(\mathcal{G}_s\) defined by first choosing \((G, o)\) with respect to \(\mu\), then drawing a graph \(G(p)\) via percolation, and finally replacing \(G\) by the connected component of \(o\) in \(G(p)\).

**Remark 3.2.** The lemma implies that if \(\{G_n\}_{n \in \mathbb{N}}\) is locally convergent in probability to \((G, o) \sim \mu\), then \(\{G_n(p)\}_{n \in \mathbb{N}}\) is locally convergent in probability to \((G(p), o) \sim \mu_p\). To prove this, we need to extend the statement to all bounded, continuous functions, which in turn requires tightness. But tightness is obvious here. Indeed, all one needs to observe is that if \(\mathcal{G}_N\) is the set of all \((G, o) \in \mathcal{G}_s\) with at most \(N\) nodes, then \(\mu(\mathcal{G}_N) \geq 1 - \epsilon\) for some \(N\). Local convergence in probability then implies that the same statement holds (with \(\epsilon\) replaced by \(2\epsilon\)) for the probability distribution of \((G_n, x_n)\) and all large enough \(n\), with probabilities with respect to both the randomness of \(G_n\) and \(x_n\). But this property is inherited by the percolated graphs \(G_n(p)\), which gives the desired tightness.

**Proof.** Recall that \(\mathbb{E}_{G_n(p)}\) denotes expectation with respect to percolation only; if \(G_n\) is random, these expectations are still conditioned on the random graph \(G_n\). To prove lemma we use the second moment method.

As a preliminary, we start with the observation that the distance of two vertices does not decrease after percolation i.e., if \(\text{dist}_{G(p)}(x, y) \leq k\), then \(\text{dist}_{G}(x, y) \leq k\) as well. As a consequence, the values of \(f_k\) evaluated on a percolated graph \((G(p), x)\) only depend on the induced subgraph on the vertices of distance at most \(k\) from \(x\) in \(G\). This has two consequences: (i) the expectation of \(f_k\) with respect to percolation depends only on the \(k\)-neighborhood of the root, and (ii) expectations of products factor if the roots are at least distance \(2k + 1\) apart. Explicitly, if we define \(f_{k,p} : \mathcal{G}_s \to \mathbb{R}\) by \(f_{k,p}(G, o) = \mathbb{E}_{G(p)}[f_k(G(p), o)]\), then (i) \(f_{k,p}(G_1, o_1) = f_{k,p}(G_2, o_2)\) whenever \(d_{\text{loc}}((G_1, o_1), (G_2, o_2)) \leq 1\), and (ii) \(\mathbb{E}_{G(p)}[f_k(G(p), o_1) f_k(G(p), o_2)] = f_{k,p}(G, o_1) f_{k,p}(G, o_2)\) whenever \(\text{dist}_{G}(x, y) \geq 2k + 1\).

With these preparations, the proof now is relatively straightforward once we take into account a corollary from [45] concerning the distance of two random vertices in sequences of locally convergent graphs, see below.

We start with the first moment, i.e.,

\[
\mathbb{E}_{G_n(p)}\left[\mathbb{E}_{\mathcal{P}_n}\left[f_k(G_n(p), o_n) \mid G_n\right] \mid G_n\right] \xrightarrow{P} \mathbb{E}_{\mu_o}\left[f_k(G(p), o)\right],
\]

where convergence in probability is over possible randomness of \(G_n\). By the linearity of expectation,

\[
\mathbb{E}_{G_n(p)}\left[\mathbb{E}_{\mathcal{P}_n}\left[f_k(G_n(p), o_n) \mid G_n\right] \mid G_n\right] = \mathbb{E}_{\mathcal{P}_n}\left[\mathbb{E}_{G_n(p)}\left[f_k(G_n(p), o_n) \mid G_n\right] \mid G_n\right] = \mathbb{E}_{\mathcal{P}_n}\left[f_{k,p} \mid G_n\right].
\]
Our observation (i) above, together with the assumption that $f$ is bounded, shows that $f_{k,p}$ is a bounded continuous function on $G_n$. By the definition of local convergence in probability, we have that

$$\mathbb{E}_{\mu_n}[f_{k,p}|G_n] \xrightarrow{p} \mathbb{E}_{\mu}[f_{k,p}(G,o)] = \mathbb{E}_{\mu}[f_k(G(p),o)].$$

For the second moment, we compute

$$\mathbb{E}_{G_n}(p) \left[ \left( \mathbb{E}_{P_n}[f_k(G_n(p),o_n)] \right)^2 | G_n \right] = \frac{1}{n^2} \sum_{u,v \in [n]} \mathbb{E}_{G_n}(p) \left[ f_k(G_n(p),v)f_k(G_n(p),u) | G_n \right],$$

which we write as

$$\frac{1}{n^2} \sum_{u,v \in [n]} \mathbb{E}_{G_n}(p) \left[ f_k(G_n(p),v)f_k(G_n(p),u) | G_n \right] =$$

$$\frac{1}{n^2} \sum_{u,v \in [n]: \text{dist}_{G_n}(u,v) \leq r} \mathbb{E}_{G_n}(p) \left[ f_k(G_n(p),v)f_k(G_n(p),u) | G_n \right] + \frac{1}{n^2} \sum_{u,v \in [n]: \text{dist}_{G_n}(u,v) > r} \mathbb{E}_{G_n}(p) \left[ f_k(G_n(p),v)f_k(G_n(p),u) | G_n \right]$$

where we choose $r = 2k$. By our observation (ii) from the beginning of this proof, the second term can be rewritten as

$$\sum_{u,v \in [n]: \text{dist}_{G_n}(u,v) > r} \mathbb{E}_{G_n}(p) \left[ f_k(G_n(p),v)f_k(G_n(p),u) | G_n \right] =$$

$$\sum_{u,v \in [n]: \text{dist}_{G_n}(u,v) > r} \mathbb{E}_{G_n}(p) \left[ f_k(G_n(p),v) | G_n \right] \mathbb{E}_{G_n}(p) \left[ f_k(G_n(p),u) | G_n \right],$$

implying that

$$\mathbb{E}_{G_n}(p) \left[ \left( \mathbb{E}_{P_n}[f_k(G_n(p),o_n)] \right)^2 | G_n \right] - \mathbb{E}_{G_n}(p) \left[ \mathbb{E}_{P_n}[f_k(G_n(p),o_n)] | G_n \right]^2 \leq 2\alpha^{(2)}_{\leq r}(G_n) \|f\|_\infty$$

where $\alpha^{(2)}_{\leq r}(G_n)$ is the fraction of pairs $u,v \in V(G_n)$ such that $\text{dist}_{G_n}(u,v) \leq r$. By Corollary 2.20 in [45], for any two vertices $u$ and $v$ chosen independently and uniformly at random from $[n]$, the distance between them grows with $n$, i.e., $\text{dist}_{G_n}(u,v) \xrightarrow{p} \infty$, where the probability is with respect to both the randomness of $G_n$ and the random choice of $u$ and $v$. But this implies that $\alpha^{(2)}_{\leq r}(G_n) \rightarrow 0$ in probability (with respect to the random choice for $G_n$), and thus

$$\mathbb{E}_{G_n}(p) \left[ \mathbb{E}_{P_n}[f_k(G_n)] | G_n \right] \xrightarrow{p} 0,$$

where convergence in probability is on random graphs $G_n$. So, by Chebyshev’s inequality,

$$\frac{\mathbb{E}_{P_n}[f_k(G_n)]}{\mathbb{E}_{G_n}(p) \mathbb{E}_{P_n}[f_k(G_n)] | G_n} \xrightarrow{p} 1,$$
Then by convergence of the first moment,
\[ \mathbb{E}_{\mathcal{P}_n}(f_k | G_n) \xrightarrow{p} \mathbb{E}_{\mu_n}(f_k). \]

Next we state a lemma which will be used in our sprinkling argument. Its statement (and its proof) are similar to those used in the work of Alon Benjamini and Stacey [3] and later in [6, 32], but we avoid the assumption of uniformly bounded maximal degree, and only uses large-set expansion instead of expansion. We recall (1) and the definition of \((\alpha, \epsilon, \bar{d})\)-large-set expanders as graphs \(G\) with average degree at most \(\bar{d}\) and \(\phi(G, \epsilon) \geq \alpha\).

**Lemma 3.3 (Sprinkling Lemma).** Let \(G\) be an \((\alpha, \epsilon, \bar{d})\)-large-set expander on \(n\) vertices, and let \(H\) be an instance of \(G(\beta)\) for some \(\beta \in (0, 1]\). Given \(R > 0\), let \(\mathcal{S}\) be a family of disjoint subsets of \(V(G)\), each of size at least \(R\). For two sets \(A\) and \(B\) define an \(A\)-\(B\) path as a path with one endpoint in \(A\) and another in \(B\). Then
\[
\mathbb{P}(\exists A, B \subset \mathcal{S} : \bigcup_{V_i \in A} |V_i| \geq \epsilon n, |V_i| \geq \epsilon n, \text{no } A\text{-}B \text{ path in } H) \leq e^{\frac{\alpha n}{2} - cn},
\]
where \(c\) is a constant that depends on \(\beta, \epsilon, \alpha,\) and \(\bar{d}\), but it is independent of \(R\) and \(n\).

**Proof.** Let \(A\) and \(B\) be two disjoint subsets of \(\mathcal{S}\) that each contain at least \(\epsilon n\) vertices of \(G\). By the large-set expansion of \(G\), we know that we need to remove at least \(\epsilon cn\) edges to disconnect vertices of \(A\) and \(B\). Therefore, by Menger’s theorem there are \(\epsilon cn\) edge-disjoint paths between \(A\) and \(B\) in \(G\). There are at most \(\bar{d}n/2\) edges in the graph in total. Therefore, at least half of these paths has a length bounded by \(l = \frac{d}{\epsilon \alpha}\). Let \(P\) be the set of paths between \(A\) and \(B\) of length at most \(l\). The probability that none of these paths appear in \(H\) is at most \((1 - \beta^l)^{\epsilon cn/2}\). Hence,
\[
\mathbb{P}(\text{no path between } A \text{ and } B \text{ in } H) \leq e^{-\beta d^{\epsilon cn}/2}.
\]

Given \(\mathcal{S}\), there are at most \(2^{|\mathcal{S}|}\) ways to choose disjoint subsets \(A\) and \(B\). By a union bound over all possible partitions we find that the probability that such a partition exists is at most
\[
2^{|\mathcal{S}|} e^{-\beta d^{\epsilon cn}/2} \leq 2^{|\mathcal{S}|} e^{-\beta d/\epsilon \alpha^{\epsilon cn}/2},
\]
which gives the result for \(c = \beta d/\epsilon \alpha^{\epsilon cn}/2\).

The proof of Theorem 1.1 follows from the following proposition which generalizes a recent result of Krivelevich, Lubetzky and Sudakov [32], and Lemma 3.5 below which is a straightforward generalization of a results of Alon, Bejamini and Stacy [3].

**Proposition 3.4.** Let \(\{G_n\}_{n \in \mathbb{N}}\) be a sequence of graphs satisfying the assumptions of Theorem 1.1, and let \(p\) be a continuity point of \(\zeta\). Then for any \(\epsilon > 0\)
\[
\mathbb{P}\left(\zeta(p) - \epsilon \leq \frac{|C_1(G_n(p))|}{n} \leq \zeta(p) + \epsilon\right) \rightarrow 1,
\]
For the upper bound, neither the continuity assumption at \(p\), nor the assumption of expansion, nor that of bounded average degrees is needed.

Before giving the proof of the proposition, we remark that an analogue of this statement for the case where the limit is a regular tree was established in [32], using again a sprinkling argument, combined with branching process techniques (which do not apply here).
PROOF. We begin by proving the upper bound on $|C_1|$. Note that this part of the proof will hold for any sequence of locally convergent graphs (without the assumption of expansion or bounded average degrees). Fix $k \geq 1$. Define $f_k(G_n(p), v) = 1(|B_k(G_n(p), v)| \geq k)$, as the indicator that $v$ is in a component of size at least $k$ in $G_n(p)$. Let $Z_{\geq k}(G_n(p)) = \mathbb{E}_{P_{G_n}}[f_k(G_n(p), v)]$ be the fraction of vertices in $G_n(p)$ that are in a component of size at least $k$. Then by Lemma 3.1

\[ Z_{\geq k}(G_n(p)) \xrightarrow{p} \zeta_k(p), \]

where $\zeta_k(p) = \mu_p(|C(o)| \geq k)$. Note that $\zeta(p) = \lim_{k \to \infty} \zeta_k(p)$. Suppose that $k$ is large enough that $|\zeta_k(p) - \zeta(p)| \leq \epsilon/2$. Then the desired upper bound will be proved once we prove the following,

\[ \mathbb{P}\left( \frac{|C_1|}{n} \leq \zeta_k(p) + \epsilon/2 \right) \to 1. \]

We consider two cases: $\zeta_k(p) > 0$ and $\zeta_k(p) = 0$. If $\zeta_k(p) > 0$, then $n \zeta_k(p) \geq k$ for large enough $n$. But then $\frac{|C_1|}{n} > \zeta_k(p) + \epsilon/2$ implies $|C_1| \geq k$ which in turn implies $n Z_{\geq k} \geq |C_1|$. Therefore,

\[ \mathbb{P}\left( \frac{|C_1|}{n} > \zeta_k(p) + \epsilon/2 \right) \leq \mathbb{P}(Z_{\geq k}(G_n(p)) > \zeta_k(p) + \epsilon/2), \]

for $n$ large enough which implies

\[ \mathbb{P}\left( \frac{|C_1|}{n} \geq \zeta_k(p) + \epsilon/2 \right) \to 0, \quad \text{as } n \to \infty. \]

If $\zeta_k(p) = 0$ then $Z_{\geq k} \to 0$ in probability. For the event that $Z_{\geq k} = 0$, we have $|C_1| \leq k < \epsilon n/2$ if $n$ is large enough. Therefore, if $\zeta_k(p) = 0$ we get

\[ \mathbb{P}\left( \frac{|C_1|}{n} \geq \epsilon/2 \right) \leq \mathbb{P}(Z_{\geq k} > 0) \to 0, \quad \text{as } n \to \infty. \]

The two cases $\zeta_k(p) = 0$ and $\zeta_k(p) > 0$ give the desired upper bound on $|C_1|$. Note that this part also implies that if $\zeta(p) = 0$ then $\frac{|C_1|}{n} \to 0$ in probability.

To prove the lower bound, we may assume without loss of generality that $\zeta(p) > 0$, i.e., we may assume that $p_p = \inf_{p} \{\zeta(p) > 0\}$. Recalling the definition of large-set expanders, choose $\alpha > 0$ such that for all $0 < \epsilon < 1/2$ $G_n$ is an $(\alpha, \epsilon/8, d)$-large-set expander with probability tending to 1 as $n \to \infty$. Choose $\epsilon$ small enough to make sure that $\zeta(p) \geq 3\epsilon/2$, and let $\delta > 0$ be such that for $p - \delta < p' < p$, $\zeta(p) - \zeta(p') \leq \epsilon/2$ (implying in particular that $\zeta(p') \geq \epsilon$). Choose $p' \in (p - \delta \lor p_e), p$ and $\epsilon' > 0$ such that $1 - p \geq (1 - p')(1 - \epsilon')$.

We will use Lemma 3.3 to show that, with high probability, after raising $p'$ to $p$, most of the vertices in large components in $G_n(p')$ will merge into one giant component in $G_n(p)$.

Let $S_k$ be the set of components of size greater than $k$ in $G_n(p')$. Since $\zeta_k(p') \geq \zeta(p') \geq \zeta(p) - \epsilon/2$ we may use (8) to concluded that for all $k$, with high probability the total number of vertices in the sets in $S_k$ is at least $\epsilon n$. By our choice of $\epsilon'$, “sprinkling” edges with probability $\epsilon'$ on top of percolation with probability $p'$ will give a percolated graph which is stochastically bounded by $G_n(p)$. Thus, by Lemma 3.3, we see that if $k$ is large enough, with high probability all but $\epsilon n/8$ vertices that are in a component of size at least $k$ in $G_n(p')$ are in $C_1(G_n(p))$. So, there exist $K_1$ such that for all $k > K_1$ and large enough $n$,

\[ \mathbb{P}\left( \frac{|C_1(G_n(p))|}{n} \geq Z_{\geq k}(G_n(p')) - \frac{\epsilon}{8} \right) \to 1. \]
Combined with (8), this shows that there exists a constant $K_2$ such that for $k \geq K_2$,  
$$
P\left(\frac{|C_1(G_n(p))|}{n} \geq \zeta_k(p') - \frac{\epsilon}{4}\right) \to 1.
$$
Since $\lim_{k \to \infty} \zeta_k(p') = \zeta(p')$, we get that there exists a constant $K_3$ such that for $k \geq K_3$,  
$$
P\left(\frac{|C_1(G_n(p))|}{n} \geq \zeta(p') - \frac{\epsilon}{2}\right) \to 1.
$$
By the choice of $\delta$ and $p'$, $\zeta(p') \geq \zeta(p) - \epsilon/2$. Hence, we get desired lower bound on $|C_1|$:  
$$
P\left(\frac{|C_1(G_n(p))|}{n} \geq \zeta(p) - \epsilon\right) \to 1.$$

Our next lemma generalizes Theorem 2.1 in [3], replacing an assumption of bounded degree expanders by the assumption of large set expansion plus a tightness bound on the largest degree, $\Delta_R(G_n, o_n)$, in a ball of radius $R$ around a random root $o_n$ in $G_n$. Specifically, we will assume that  
\begin{equation}
\forall R < \infty \limsup_{n \to \infty} P(\Delta_R(G_n, o_n) \geq \Delta) \to 0 \quad \text{as} \quad \Delta \to \infty,
\end{equation}
where the probability is with respect to a random root $o_n$ and the randomness of $G_n$.

**Lemma 3.5.** Let $\{G_n\}$ be a sequence of bounded average degree large-set expanders obeying the tightness condition (9), let $0 < q < 1/2$ and $c > 0$ be arbitrary. Then for any $\epsilon > 0$ there exists $N_{\epsilon, q, c}$ such that for all $n > N_{\epsilon, q, c}$ and all $p \in [q, 1-q]$ 
$$
P\left(\frac{|C_2|}{n} \geq c\right) \leq \epsilon
$$
where $P$ denotes probabilities with respect to both the randomness of $G_n$ and percolation.

The proof of the lemma closely follows that of [3], and is given in Appendix C.

**Proof of Theorem 1.1.** The theorem follows immediately from Proposition 3.4, Corollary 2.2, and Lemma 3.5 and the fact that local convergence in probability implies tightness, which by Theorem A.16 of [45] implies (9).

Next, we observe that by Theorem 1.1, the critical threshold is local for expanders with bounded average degree.

**Corollary 3.6.** Let $G_n$ be a sequence of expanders with bounded average degree that converges locally in probability to $(G, o) \in \mathcal{G}_*$ with the law $\mu$. Then
$$
p_c(G_n) \xrightarrow{\text{P}} p_c(\mu).
$$

**Proof.** Given any $\epsilon > 0$ and $p > p_c(\mu) + \epsilon$, since $\zeta(p)$ is continuous, we can apply Theorem 1.1 to get that $\frac{|C_1|}{n} > 0$ in $G_n(p)$ for large enough $n$, and hence,  
$$
\lim_{n \to \infty} P(p_c(G_n) \geq p_c(\mu) + \epsilon) = 0,
$$
for all $\epsilon > 0$. 
If \( p = (p_c(\mu) - \epsilon \lor 0) \), we know that \( \zeta(p) = 0 \). So, by Theorem 1.1, \( \frac{\epsilon}{n} \to 0 \). Therefore, if \( p_c(\mu) > 0 \)

\[
\lim_{n \to \infty} \mathbb{P}(p_c(G_n) \leq p_c(\mu) - \epsilon) = 0, \text{ for all } \epsilon > 0.
\]

Note that if \( p_c(\mu) = 0 \), we already know that \( p_c(G_n) \geq p_c(\mu) \) for all \( n \). As a result of the above limits,

\[
\lim_{n \to \infty} \mathbb{P}(|p_c(G_n) - p_c(\mu)| \geq \epsilon) = 0, \text{ for all } \epsilon > 0.
\]

\[
\square
\]

4. Coupling Oriented and Unoriented Percolations. In this section, we relate oriented and unoriented percolation for general graphs. In particular, we will compare the thresholds \( p_c(G_n) \) for the appearance of a giant component in \( G_n(p) \) to that of the appearance of a giant SCC in \( D_{G_n}(p) \) introduced in Section 2, and show that they are asymptotically the same if both exist (Section 4.1). Next, in Section 4.2, we analyze oriented percolation when the giant component in the unoriented case is unique, and show that under this assumption, the linear-sized SCC is unique if it exists.

Throughout Section 4, we make no assumptions on graph expansion or the existence of local limit, and the result carry over to general graphs.

4.1. Comparing unoriented and oriented thresholds. The following lemma introduces a coupling between oriented and unoriented percolation. The second part couples non-overlapping fan-outs of two vertices to the undirected components of those vertices. To state the lemma, we need the following notation: Given a node \( u \) in \( G \) and a subset \( S \subset V(G) \), we define \( C^+_\setminus S(u) \) as the fan-out of \( u \) in the induced digraph on the complement of \( S \). If \( u \in S \), then \( C^+_\setminus S(u) = \emptyset \).

**Lemma 4.1.** Let \( G \) be a graph on \( n \) nodes, and let \( p \in (0, 1) \). Then,

1. For \( v \) in \( G \) and \( \alpha > 0 \),

\[
\mathbb{P}_{D_{G}(p)}(|C^+(v)| \geq \alpha n) = \mathbb{P}_{G(p)}(|C(v)| \geq \alpha n) = \mathbb{P}_{D_{G}(p)}(|C^-(v)| \geq \alpha n).
\]

2. Let \( u \) and \( v \) be vertices in \( G \), and let \( k_1 \) and \( k_2 \) be positive integers,

\[
\mathbb{P}_{D_{G}(p)}(|C^+(v)| \geq k_1, |C^+_{\setminus C^+(v)}(u)| \geq k_2) = \mathbb{P}_{G(p)}(|C(v)| \geq k_1, |C(u)| \geq k_2, C(v) \neq C(u))
\]

\[
= \mathbb{P}_{D_{G}(p)}(|C^+(v)| \geq k_1, |C^+_{\setminus C^+(v)}(u)| \geq k_2).
\]

**Proof.** 1. This part is a special case of part 2. To see this, one can add a dummy isolated node \( u \) and let \( k_2 = 0 \).

We will prove the first equality and during the proof we will point out how it can be extended to prove the second inequality. For any vertex \( w \), define \( T(w) \) and \( T^+(w) \) as the tree rooted at \( w \) obtained by breadth-first exploration of \( C(w) \) and \( C^+(w) \), respectively. Also, define \( T^+(u \setminus v) \) as the breadth-first exploration of \( C^+_{\setminus C^+(v)}(u) \). Finally, given a tree \( T \subset G \) and a root \( v \) in \( T \), define the corresponding oriented graph \( T^+ \) by directing edges away from the root.

Consider now two arbitrary trees \( T_1 \) and \( T_2 \) rooted at \( v \) and \( u \), respectively. If they intersect, the probability that \( T^+(v) = T_1^+ \) and \( T^+(u \setminus v) = T_2^+ \) is zero, and so is the
probability that \( T(v) = T_1, T(u) = T_2 \) and \( C(v) \neq C(u) \). It they are disjoint, we will define a coupling of \( G(p) \) and \( D_{C}(p) \) which shows that

\[
P_{D_{C}(p)}(T^+(v) = T_1^+, T^+(u \setminus v) = T_2^+) = P_{G(p)}(T(v) = T_1, T(u) = T_2).
\]

Define \( l_1(w) \) and \( l_2(w) \) to be the distance of a vertex \( w \) from the root in \( T_1 \) and \( T_2 \), respectively (with the root having level 0 and nodes that are not in the tree having level \( \infty \)). We express the instances of \( D_{C}(p) \) by choosing, for each edge \( \{x, y\} \in E(G) \), two Bernoulli random variables \( X_{x,y} \) and \( Y_{x,y} \), so that \( X_{x,y} = 1 \) if and only if the directed edge from \( x \) to \( y \) exists in \( D_{C}(p) \) and is 0 otherwise. Similarly, let \( Y_{x,y} \) be a Bernoulli random variable corresponding to the existence of an undirected edge between \( x \) and \( y \) in \( G(p) \).

To define the coupling, first consider the case that \( \{x, y\} \in E(G) \) and \( l_1(x) < l_1(y) \): if the edge \( \{x, y\} \) does not exist in \( T \), let \( X_{x,y} = 0, Y_{x,y} = 0 \). If \( x \) is the successor of \( y \) in \( T_1 \) let \( X_{y,x} = 1, Y_{y,x} = 1 \). Note in particular that the events \( T^+(v) = T_1^+ \) and \( T(v) = T_1 \) happen only if we have set \( X_{x,y} = Y_{x,y} = 0 \) whenever \( x \) is a vertex in \( T_1 \) and \( y \) is a vertex in \( T_2 \).

Next we couple the binary random variables for edges \( \{x, y\} \in E(G) \) such that \( l_2(x) < l_2(y) \) and \( x, y \notin V(T_1) \). Since the event \( T^+(u \setminus v) = T_2 \) involves only edges with both endpoints in \( V(G) \setminus V(T_1) \), the edges coupled in the second step determine whether \( T^+(u \setminus v) = T_2 \) or not. On the other hand, the event \( T(u) = T_2 \) does involve edges between the vertices in \( T_1 \) and \( T_2 \), namely, it requires that \( Y_{x,y} = 0 \) if \( \{x, y\} \) is an edge pointing from \( T_1 \) to \( T_2 \). But as remarked before, these edges have already been set in our first coupling step, and have been set in such a way that if \( T(v) = T_1 \) then all these edges are absent in \( G(p) \), as required. Setting finally all remaining edges independently, we obtain a coupling such that the events in \( D_{C}(p) \) happen if and only the corresponding events in \( G(p) \) happen, and with the same probability.

This completes the proof of the first identity in the lemma. The second one is essentially the same, except that in the second step, we orient all edges in \( D_{C}(p) \) in the opposite direction.

\[ \square \]

Lemma 4.1 immediately gives the following corollary, which in particular shows that the existence of a giant SCC in \( D_{C}(p) \) implies the existence of a giant component in \( G(p) \).

**Corollary 4.2.** Given a (possibly random) graph \( G \) on \( n \) nodes, \( p \in [0, 1] \) and \( \alpha > 0 \), we have that

\[
P_{D_{C}(p)}(|SCC_1| \geq \alpha n) \\
\leq P_{D_{C}(p)}(\text{there exists } \geq \alpha n \text{ vertices } v \text{ with } |C^+(v)| \geq \alpha n) \\
\leq \frac{1}{\alpha} P_{G(p)}(|C_1| \geq \alpha n),
\]

where the probability \( \mathbb{P} \) is first over the randomness of \( G \) and then oriented/unoriented percolation.

**2. Proof.** We first prove the result for a non-random graph \( G \); it can then be generalized to random graphs \( G \) by conditioning on the random graph instance, and then taking a weighted average over all possible instances.

Let \( k = \lceil \alpha n \rceil \). In \( D_{C}(p) \), if \( |SCC_1| \geq k \), then there are at least \( k \) vertices \( v \) with \( |SCC(v)| \geq k \), which implies that there are at least \( k \) vertices with fan-out at least \( k \), proving the first inequality. Next define \( Z_{\geq k} \) as the number of vertices such that \( |C(v)| \geq k \), and
define \(Z_{\geq k}^+\) to be the number of vertices such that \(|C^+(v)| \geq k\). Using first Lemma 4.1, and then the fact that either \(Z_k = 0\) or \(k \leq Z_{\geq k} \leq n\), we have,

\[
\mathbb{P}(\text{there exists } \geq \alpha \text{ vertices } v \text{ with } |C^+(v)| \geq \alpha n) = \mathbb{P}(Z_{\geq k}^+ \geq \alpha n) = \frac{1}{k} \mathbb{E}[Z_{\geq k}^+] = \frac{1}{k} \mathbb{E}[Z_{\geq k}1_{Z_{\geq k} \geq k}] \leq \frac{n}{k} \mathbb{P}(Z_{\geq k} \geq k) = \frac{n}{k} \mathbb{P}(|C_1| \geq k).
\]

Since \(n/k \leq 1/\alpha\), this proves the corollary.

The next lemma gives a bound in the opposite direction. Recall that the strongly connected component of a vertex \(v\), \(SCC(v)\), is the intersection of \(C_n^+(v)\) and \(C_n^-(v)\). Also, recall that \(C_i\) and \(SCC_i\) denote the \(i\)th largest component/strongly connected component in \(G(p)\) and \(D_G(p)\), respectively.

**Lemma 4.3.** Given a graph \(G\) and a constant \(p \in [0, 1]\),

\[
\frac{1}{n} \mathbb{E}_{D_G(p)}[|SCC_1|] \geq \frac{1}{n^2} \sum_i \mathbb{E}_{D_G(p)}[|SCC_i|^2] \geq \left(\frac{1}{n} \mathbb{E}_{G(p)}[|C_1|]\right)^4.
\]

**Proof.** First we note that for all vertices \(u, v\) we have

\[
\mathbb{P}_{D_G(p)}(u \in C^+(v) \text{ and } v \in C^+(u)) \geq \mathbb{P}_{D_G(p)}(u \in C^+(v)) \mathbb{P}_{D_G(p)}(v \in C^+(u))
\]

by the standard FKG inequality. Indeed, let \(D\) be the digraph obtained from \(G\) by replacing every edge in \(G\) by two oriented edges, and let \((0, 1)^D\) be the set of subgraphs of \(D\), equipped with the natural partial order (with \(D_1 \leq D_2\) if each edge in \(D_1\) is an edge in \(D_2\)). Then the functions \(1(u \in C^+(v))\) and \(1(v \in C^+(u))\) are both increasing functions on \(\Omega\), so (10) follows from the Harris inequality [26].

For any two vertices \(u\) and \(v\) define \(q_{uv}\) to be the probability that \(v \in C^+(u)\) in \(D_G(p)\). By the coupling from the proof of Lemma 4.1, \(q_{uv}\) is equal to the probability that \(v \in C(u)\) in \(G(p)\), which is also the same probability that \(u \in C(v)\). Therefore, \(q_{uv} = q_{vu}\). Using that \(\mathbb{E}_{D_G(p)}[SCC(v)] = \sum_u \mathbb{P}_{D_G(p)}(u \in C^+(v) \text{ and } v \in C^+(u))\), we therefore get

\[
\frac{1}{n^2} \sum_v \mathbb{E}_{D_G(p)}[SCC(v)] \geq \frac{1}{n^2} \sum_{u,v \in V(G)} q_{uv}^2 \geq \left(\frac{1}{n} \sum_{u,v \in V(G)} q_{uv}\right)^2 = \left(\frac{1}{n} \sum_v \mathbb{E}_{G(p)}[|C(v)|]\right)^2
\]

where the second inequality follows from Cauchy–Schwarz. As a consequence,

\[
\sum_i \mathbb{E}_{D_G(p)}[|SCC_i|^2] = \sum_v \mathbb{E}_{D_G(p)}[|SCC(v)|] \geq \frac{1}{n^2} \left(\sum_v \mathbb{E}_{G(p)}[|C(v)|]\right)^2 \geq \frac{1}{n^2} \left(\mathbb{E}_{G(p)}[|C_1|^2]\right)^2 \geq \left(\frac{1}{n} \mathbb{E}_{G(p)}[|C_1|]\right)^4.
\]

To complete the proof, we note that \(\sum_i |SCC_i|^2 \leq |SCC_1| \sum_i |SCC_i| = n|SCC_1|\).

**Corollary 4.4.** Fix a (possibly random) sequence of graphs \(\{G_n\}_{n \in \mathbb{N}}\). If \(p_n\) and \(p_{n,SCC}\) are threshold sequences for the existence of a giant in \(G_n(p)\) and the existence of a giant SCC in \(D_{G_n}(p)\), respectively, then \(|p_n - p_{n,SCC}| \to 0\) as \(n \to 0\).
Proof. Fix $\epsilon > 0$. By Corollary 4.2 and the definition of a threshold sequence, we know that there exists a $c = c(\epsilon) > 0$ and an $N < \infty$ such that
\[
\mathbb{P}\left(|SCC_1| \geq cn \text{ in } D_{G_n}(1 \land (p_n^{SCC} + \epsilon))\right) \geq \frac{3}{4}
\]
and
\[
\mathbb{P}\left(|SCC_1| \geq cn \text{ in } D_{G_n}(0 \lor (p_n - \epsilon))\right) \leq \frac{1}{c} \mathbb{P}\left(|C_1| \geq cn \text{ in } G_n(0 \lor (p_n - \epsilon))\right) \leq \frac{1}{3}
\]
for all $n \geq N$, where the probabilities are first over the possible randomness of $G_n$ and then percolation. Since the size of $SCC_1$ is increasing in $p$, this immediately implies that
\[
1 \land (p_n^{SCC} + \epsilon) \geq 0 \lor (p_n - \epsilon)
\]
for all $n \geq N$.

To prove a matching bound in the other direction, let $c > 0$ and $\bar{N}$ be such that for $n \geq \bar{N}$
\[
\mathbb{P}\left(|C_1| \geq cn \text{ in } G_n(1 \land (p_n + \epsilon))\right) \geq \frac{3}{4}.
\]
Then $\mathbb{E}[C_1] \geq \frac{3cn}{4}$, where the expectation is over the possible randomness of $G_n$ and percolation. So by using Lemma 4.3 for all possible instances of $G$ and Jensen’s inequality,
\[
\frac{1}{n} \mathbb{E}_G \mathbb{E}_{D_c(1 \land (p_n + \epsilon))}[|SCC_1|] \geq 2C,
\]
for some $C > 0$ that depends on $c$. Since $\mathbb{E}[|SCC_1|] \leq Cn + n\mathbb{P}(|SCC_1| \geq Cn)$, we conclude that
\[
\mathbb{P}_{D_c(1 \land (p_n + \epsilon))}(|SCC_1| \geq Cn) \geq C.
\]
Using this fact, we now can proceed as in the derivation of the lower bound on $p_n^{SCC} - p_n$ to show that for $n$ large enough,
\[
1 \land (p_n + \epsilon) \geq 0 \lor (p_n^{SCC} - \epsilon).
\]
Since $\epsilon > 0$ was arbitrary, this bound together with the matching bound above implies that $|p_n - p_n^{SCC}| \to 0$ as $n \to \infty$. \hfill \Box

Recall the definition of $\zeta^{+-}(p)$ from (4). Similar to proof of Lemma 4.3 we can give bounds on $\zeta^{+-}(p)$.

**Lemma 4.5.** Let $\mu$ be a probability distribution on $G_s$, and let $p \in [0,1]$. Then
\[
\zeta(p) \geq \zeta^{+-}(p) \geq \zeta^2(p).
\]
As a consequence, $p_c(\mu) = p^{+-}_c(\mu)$.

**Proof.** To prove the lower bound, consider a graph $(G,o)$ drawn from the distribution $\mu$. Since $1(|C^+(o)| = \infty)$ and $1(|C^-(o)| = \infty)$ are increasing functions of edges, by the FKG inequality
\[
\mathbb{P}_{D_c(p)}\left(|C^+(o)| = \infty, |C^-(o)| = \infty\right) \geq \mathbb{P}_{D_c(p)}\left(|C^+(o)| = \infty\right) \mathbb{P}_{D_c(p)}\left(|C^-(o)| = \infty\right).
\]
Similar to the coupling of Lemma 4.1 on the infinite graph $G$, we get
\[
\mathbb{P}_{D_c(p)}\left(|C^+(o)| = \infty\right) = \mathbb{P}_{G(p)}\left(|C(o)| = \infty\right) = \mathbb{P}_{D_c(p)}\left(|C^-(o)| = \infty\right).
\]
As a result,

\[ \mu \left( \mathbb{P}_{D_G(p)} \left( |C^+(o)| = \infty, |C^-(o)| = \infty \right) \right) \geq \mu \left( \mathbb{P}_{G(p)}^2 \left( |C(o)| = \infty \right) \right) \]

\[ \geq \mu \left( \mathbb{P}_{G(p)} \left( |C(o)| = \infty \right) \right)^2, \]

where the second inequality is by Cauchy–Schwarz.

The upper bound immediately follows by applying the coupling in Lemma 4.1 to infinite graphs \( G \). In fact,

\[ \zeta^+(p) = \mu \left( \mathbb{P}_{D_G(p)} \left( |C^+(o)| = \infty, |C^-(o)| = \infty \right) \right) \]

\[ \leq \mu \left( \mathbb{P}_{D_G(p)} \left( |C^+(o)| = \infty \right) \right) = \mu \left( \mathbb{P}_{G(p)} \left( |C(o)| = \infty \right) \right) = \zeta(p). \]

The statement about \( p_c \) follows trivially. \( \square \)

4.2. Graphs with a Unique Giant Component. We proceed by considering graphs with a unique giant for unoriented percolation and we analyze the implications for oriented percolation. We will use the following definition.

**Definition 4.6.** Fix \( p \in [0, 1] \) and \( \epsilon > 0 \). A sequence \( \{G_n\}_{n \in \mathbb{N}} \) of (possibly random) graphs is called a sequence of graphs with an \( \epsilon \)-unique giant component if the probability that \( |C^2| \geq \epsilon n \) in \( G_n(p) \) goes to 0 as \( n \to \infty \). We say that \( G_n \) has a *uniformly* \( \epsilon \)-unique giant component in an interval \( I \subset [0, 1] \) if

\[ \lim_{n \to \infty} \sup_{p \in I} \mathbb{P}_{G_n(p)} (|C^2| \geq \epsilon n) = 0, \]

where the probability \( \mathbb{P}_{G_n(p)} \) is over both the randomness of \( G_n \) and the randomness of oriented percolation.

Note that by Lemma 3.5, a sequence \( G_n \) satisfying the assumptions of Theorem 1.1 has a uniformly \( \epsilon \)-unique giant component in \( I \) for all \( \epsilon \) and all closed intervals \( I \subset (0, 1) \). As a first consequence of this fact and Lemma 4.1, we show that if in \( D_G(p) \) two vertices have large non-overlapping fan-ins/fan-outs, then there must be two giant components in \( G(p) \). As a result, we prove that if a giant SCC exists it must be unique and almost all of the vertices with a large fan-out must reach to the giant SCC before exploring many nodes outside the SCC. The next corollary shows that on graphs with an \( \epsilon \)-unique giant, all but \( \epsilon n \) of the large fan-out must have equal size.

**Corollary 4.7.** Fix \( p \in [0, 1] \) and let \( \{G_n\}_{n \in \mathbb{N}} \) be a sequence of (possibly random) graphs with \( \epsilon \)-unique giant component. Then for any two fixed vertices \( u \) and \( v \),

\[ \mathbb{P}_{D_G(p)} (|C^+(v)| \geq \epsilon n, |C^+_G(v)(u)| \geq \epsilon n) \to 0, \]

where the convergence is uniform in \( u \) and \( v \), and the randomness is over oriented percolation, and the possible randomness of \( G_n \). If the sequence \( \{G_n\} \) has a uniformly \( \epsilon \)-unique giant component in the interval \( I \) then the convergence is uniform in \( p \in I \).
PROOF. By applying Lemma 4.1 Part 2, we see that the statement is equivalent to
\[\mathbb{P}_{G_n(p)}(|C(v)| \geq \epsilon n, |C(u)| \geq \epsilon n \text{ and } C(u) \neq C(v)) \to 0,\]
where the probability now goes over the randomness in \(G_n(p)\) (including the possible randomness of \(G_n\)). But if \(|C(v)| \geq \epsilon n, |C(u)| \geq \epsilon n \text{ and } C(u) \neq C(v)\) there exist at least two clusters of size \(\geq \epsilon n\), implying that \(|C_2| \geq \epsilon n\). Therefore the left hand side is bounded by \(\mathbb{P}_{G_n(p)}(|C_2| \geq \epsilon n)\), which goes to zero by \(\epsilon\)-uniqueness of the giant. Note that the convergence is uniform in \(p\) if the sequence has a uniformly \(\epsilon\)-unique giant in \(I\). 

The corollary clearly implies that the result holds for uniform random choice of \(u\) and/or \(v\). Furthermore, one can also bound the size of the second largest SCC in \(D_G(p)\).

**Lemma 4.8.** Let \(p \in [0,1]\) and let \(\{G_n\}_{n \in \mathbb{N}}\) be a (possibly random) sequence of graphs with an \(\epsilon\)-unique giant component. Then with probability tending to 1 the second largest SCC in \(D_G_n(p)\) contains less than \(\epsilon n\) vertices. If the sequence \(\{G_n\}_{n \in \mathbb{N}}\) has a uniformly \(\epsilon\)-unique giant component in \(I\) then the convergence is uniform for all \(p \in I\).

**Proof.** Assume to the contrary that for some \(\delta\) with probability at least \(\delta\), \(D_G_n(p)\) has two SCCs larger than \(\epsilon n\). In an instance of \(D_G_n(p)\), let \(A\) and \(B\) be two disjoint SCCs. Then without loss of generality assume there is no directed path from any vertex of \(A\) to any vertex of \(B\). So, if we pick a random pair of nodes \((u,v)\) of this instance, with probability at least \(\epsilon^2\) we have that \(|C^+(v)| \geq \epsilon n\) and \(|C^+(v)\backslash C^+(u)| \geq \epsilon n\). Now, by considering all instances of \(D_{G_n}(p)\), we have that
\[\mathbb{P}_{u,v \in [n], D_{G_n}(p)}(|C^+(v)| \geq \epsilon n, |C^+(v)\backslash C^+(u)| \geq \epsilon n) \geq \epsilon^2 \delta.\]
Therefore, by Corollary 4.7 we get a contradiction. As in Corollary 4.7, the convergence is uniform in \(I\) if the giant is uniformly \(\epsilon\)-unique in \(I\).

Now that we know if the giant SCC exists, it is unique, with a very similar argument we can prove that (i) all but \(o(n)\) nodes with a large fan-out reach into the giant SCC (if it exists) and (ii) there are only \(o(n)\) nodes in their fan-out before it reaches the set \(SCC_1^+\).

**Lemma 4.9.** Given \(\epsilon > 0\) and \(p \in [0,1]\), let \(\{G_n\}_{n \in \mathbb{N}}\) be a sequence of (possibly random) graphs with an \(\epsilon\)-unique giant component. Let \(SCC_1\) be the largest SCC of \(D_{G_n}(p)\) and let \(O_\epsilon\) be the set of vertices \(u\) such that \(|C^+(u)\backslash SCC_1^+| \geq \epsilon n\), i.e., the set of vertices that have a large fan-out before reaching \(SCC_1^+\). Then
\[\lim_{n \to \infty} \mathbb{P}_{D_{G_n}(p)}(\text{\#}SCC_1 \geq \epsilon n, |O_\epsilon| \geq \epsilon' n) = 0 \quad \text{for all } \epsilon' > 0,\]
and
\[\lim_{n \to \infty} \mathbb{P}_{D_{G_n}(p)}(\text{\#}SCC_1 \geq \epsilon n, |\tilde{O}_\epsilon| \geq \epsilon' n) = 0 \quad \text{for all } \epsilon' > 0,\]
where \(\tilde{O}_\epsilon\) is the set of nodes \(u\) such that \(|C^+(u)| \geq \epsilon n\) and \(C^+(u) \cap SCC_1 = \emptyset\).

**Proof.** Assume to the contrary that there exists \(\epsilon' > 0\) and \(\delta > 0\) that
\[\mathbb{P}_{D_{G_n}(p)}(\text{\#}SCC_1 \geq \epsilon n, |O_\epsilon| \geq \epsilon' n) \geq \delta\]
for infinitely many \(n\). Then given an instance of \(D_{G_n}(p)\) such that \(|SCC_1| \geq \epsilon n\) and \(|O_\epsilon| \geq \epsilon' n\), if we pick two random nodes \(u\) and \(v\), with probability at least \(\epsilon \epsilon'\), \(u \in O_\epsilon\) and \(v \in SCC_1\), which in turn implies that \(|SCC(v)| \geq \epsilon n\) and \(|C^+( SCC_1\backslash v)| \geq \epsilon n\). But
SSC\((v)^+ = C^+(v)\), and \(|SCC\(v)\| \geq \epsilon n\) implies \(|C^+(v)| \geq \epsilon n\), so we have that with probability at least \(\epsilon'\), \(|C^+(v)| \geq \epsilon n\) and \(|C^+(v)(u)| \geq \epsilon n\), which contradicts Corollary 4.7.

The last statement follows by similar arguments and the observation that \(C^+(u) \cap SCC(v) = \emptyset\) implies that \(C^-(v) \cap C^+(u) = \emptyset\), which in turn gives that \(C^+(v)(u) = C^-(v)\).

Together with Lemma 4.3 Lemma 4.8 also allows us to bound the expected size of the square of the largest SCC from below.

**Lemma 4.10.** Given \(\epsilon > 0\) and \(p \in [0, 1]\), let \(\{G_n\}_{n \in \mathbb{N}}\) be a sequence of (possibly random) graphs with \(\epsilon\)-unique giant component. Then for all \(\epsilon' \in (0, \epsilon)\) there exists \(N\) such that for all \(n \geq N\)

\[
\mathbb{E}_{D_{G_n}}(p) \left( \frac{|SCC|}{n^2} \right) \geq \left( \mathbb{E}_{G_n(p)} \left( \left[ \frac{|C|}{n} \right] \right) \right)^4 - \epsilon'.
\]

**Proof.** Fix an instance \(G_n\). If \(G_n\) has an \(\epsilon\) unique giant, then by Lemma 4.8, \(|SCC_2| \leq \epsilon n\) with probability tending to 1, and therefore,

\[
\sum_{i \geq 2} \mathbb{E}_{D_{G_n}(p)}[|SCC_i|^2] \leq n \mathbb{E}_{D_{G_n}(p)}[|SCC_2|] \leq n^2 \epsilon + o(n^2).
\]

Combined with Lemma 4.3 and Jensen’s inequality, this implies the statement of the lemma.

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5. **From Unoriented to Oriented Percolation.** This section analyzes the structure of the oriented percolation. The main goal is to prove Theorem 1.3. For that purpose in Lemma 5.6, we show that in the supercritical case a linear strongly connected component exists with high probability. But first, we give an upper bound on the size of the largest SCC for any sequence of graphs with a local limit in probability.

**Lemma 5.1.** Let \(\{G_n\}_{n \in \mathbb{N}}\) be a (possibly random) sequence of graphs that converges locally in probability to \((G, o) \sim \mu\). Recall the definition of \(\zeta^+(p)\) in (4). Then for any \(\epsilon > 0\) and \(p \in [0, 1]\),

\[
\mathbb{P}_{D_{G_n}}(p) \left( \frac{|SCC_1|}{n} \geq \zeta^+(p) + \epsilon \right) \to 0,
\]

**Proof.** The proof is similar to the unoriented case in Proposition 3.4. For \(k \geq 1\) and a vertex \(v \in G_n\), define

\[
f_k^+(D_{G_n}(p), v) = 1(|C^+(v) \cap B_k(D_{G_n}(p), v)| \geq k, |C^-(v) \cap B_k(D_{G_n}(p), v)| \geq k),
\]

as the indicator that \(v\) has fan-out and fan-in larger than \(k\). Define the fraction of vertices with fan-in and fan-out larger than \(k\) as \(Z_{\geq k}^+(D_{G_n}(p)) = \mathbb{E}_{D_{G_n}}[f_k^+(D_{G_n}(p), v)]\). It is easy to check that Lemma 3.1 also holds for the percolation on digraphs. Therefore,

\[
Z_{\geq k}^+ \overset{p}{\rightarrow} \zeta_k^+(p),
\]

where \(\zeta_k^+(p) = \mu \left( \mathbb{P}_{D_{G_n}}(|C^+(o)| \geq k, |C^-(o)| \geq k) \right)\).

Note that if \(|SCC_1| \geq k\) then \(Z_{\geq k}^+ \geq |SCC_1|\), and if \(Z_{\geq k}^+ \leq k\) then \(|SCC_1| \leq k\). Then by considering two cases \(\zeta_k^+(p) = 0\) and \(\zeta_k^+(p) > 0\), with a similar argument as in Proposition 3.4 we get that

\[
\mathbb{P}_{D_{G_n}}(p) \left( \frac{|SCC_1|}{n} \geq \zeta_k^+(p) + \epsilon/2 \right) \to 0.
\]
Since $\zeta^+(p) = \lim_{k \to \infty} \zeta^+_k(p)$, one can find $K$ such that for $k > K$ we have $|\zeta_k(p) - \zeta(p)| \leq \epsilon/2$. Therefore,

$$\mathbb{P}_{\mathcal{D}_{G_n}} \left( \frac{|SCC_1|}{n} \geq \zeta^+(p) + \epsilon \right) \to 0.$$

**Remark 5.2.** When the limit $\mu$ is a non-random tree, $|C^+(o)|$ and $|C^-(o)|$ become independent. Thus $\zeta^-(p) = \zeta^2(p)$, which matches the lower bound from Theorem 1.3. In other words, under the assumptions of Theorem 1.1, and the additional assumption that the limit $(G, o)$ is a non-random tree,

$$\frac{1}{n}|SCC_1| \xrightarrow{\mathcal{D}} \zeta^+(p) = \zeta^2(p).$$

A simple example is a sequence of $d$-regular expanders of large girth. In general, the asymptotic size of $\frac{1}{n}|SCC_1|$ will not be given by $\zeta^2(p)$, even if the limit is a random tree.

The next lemma gives tail bounds on the number of nodes with a large fan-in/fan-out.

**Lemma 5.3.** Let $L^+_c$ and $L^-_c$ be the set of vertices with fan-out (fan-in) larger than $cn$. Fix $p \in (0, 1]$ and an interval $I \in [0, 1]$ containing $p$. Assume that $\frac{|C_1|}{n} \xrightarrow{\mathcal{D}} \zeta(p)$ and that for all $\epsilon > 0$, the sequence $(G_n)_{n \in \mathbb{N}}$ has a uniformly $\epsilon$-unique giant component in $I$. Then

1. For all $\epsilon > 0$, $\mathbb{P}_{\mathcal{D}_{G_n}}(\frac{|L^+_c(p) + \epsilon|}{n} \geq \epsilon n) \to 0$.
2. For all $a, c \in (0, 1)$ there exists $\epsilon_0 > 0$ such that for all $\delta > 0$ and $0 < \epsilon < \epsilon_0$ there exists $N$ such that for all $n > N$, and all $p' \in I$,

$$\mathbb{P}_{\mathcal{D}_{G_n}}(\frac{|L^-_c|}{n} \geq \alpha n) - \delta \leq \mathbb{P}_{\mathcal{D}_{G_n}}(\frac{|L^-_c|}{n} \geq \alpha n) - \delta \leq \mathbb{P}_{\mathcal{D}_{G_n}}(\frac{|L^-_c|}{n} \geq \alpha n) - \delta.$$

3. If $\zeta(p) > 0$, and $\alpha \in (0, \zeta(p))$, then

$$\frac{|L^+_c|}{n} \to \zeta(p),$$

in expectation and in probability. If $\zeta(p) = 0$, then for any $\alpha > 0$, $\frac{|L^+_c|}{n} \to 0$ in expectation, and hence in probability.

**Proof.** 1. Assume to the contrary that for infinitely many $n$,

$$\mathbb{P}_{\mathcal{D}_{G_n}}(\frac{|L^+_c(p) + \epsilon|}{n} \geq \epsilon n) \geq \delta.$$

Then $\mathbb{E}[|L^+_c(p)|] \geq \delta \epsilon n$, implying that there exists a node $v$ such that

$$\mathbb{P}_{\mathcal{D}_{G_n}}(\frac{|C^+(v)|}{n} \geq \zeta(p)n + \epsilon n) \geq \delta.$$

On the other hand, by Lemma 4.1, part 1 and the convergence of the giant in unoriented percolation $\frac{|C_1|}{n} \xrightarrow{\mathcal{D}} \zeta(p)$,

$$\mathbb{P}_{\mathcal{D}_{G_n}}(\frac{|C^+(v)|}{n} \geq \zeta(p)n + \epsilon n) = \mathbb{P}_{\mathcal{D}_{G_n}}(\frac{|C(v)|}{n} \geq \zeta(p)n + \epsilon n) \to 0,$$

a contradiction.
2. We prove that with probability at least $1 - \delta$, the following statement holds: for all digraphs $D_{G_n}(p')$ with $|L_c^+| \geq \alpha n$ there are at least $(c - \epsilon)n$ nodes in $D_{G_n}(p')$ such that their fan-in is larger than $(\alpha - \epsilon)n$, i.e., $|L_{c^-}| \geq (c - \epsilon)n$. We prove this in two steps:

1) We show that with probability at least $\delta$ there is a single node $u \in L_c^+$ such that its fan-out $C^+(u)$ covers (almost) all of the fan-outs of the nodes in $L_c^+$. 2) this, we prove that the fan-in of most of the nodes $w \in C^+(u)$ is large, and in fact, $|C^-(w)| \geq |L_c^+| - \epsilon n$.

To formally prove the first step, we need the following definition. Call a pair $(x, y)$ bad if $x \in L_c^+$ and $|C^+(x)(y)| \geq \epsilon^2 n/2$. The choice of $\epsilon^2/2$ may seem arbitrary at first, but it will be useful later on. Next, we will bound the number of bad pairs. Let $\epsilon'$ be a uniform $\alpha$-unique giant in $I$, we conclude that there exists $N = N(\epsilon, \delta)$ such that for all $n \geq N$ and all $p' \in I$,

$$\frac{1}{n^2} \sum_{u, v \in V(G_n)} P_{D_{G_n}(p')}(|C^+(u)| \geq cn, |C^+(v)| \geq \epsilon^2 n/2) \leq \delta^2.$$ 

Hence, the expected number of bad pairs is at most $\delta^2 n^2$, and by Markov inequality with probability at least $1 - \delta$, the number of bad pairs is less than $\delta n^2$. To complete the first step, assume without loss of generality that $\delta < \epsilon^2/2$. Given an instance of $D_{G_n}(p')$ that has at most $\delta n^2$ bad pairs and $|L_c^+| \geq \alpha n$, there exists a vertex $u \in L_c^+$ that appears in at most $\epsilon^2 n/2$ bad pairs; as a consequence, for at least $n(1 - \epsilon^2/2)$ nodes $y$ we have $|C^+(y) \setminus C^+(u)| \leq |C^+(u)| < \epsilon^2 n/2$, completing the first step.

Now, we proceed with the second step: We claim that there are $(c - \epsilon)n$ nodes in $C^+(u)$ such that their fan-in contains at least $(\alpha - \epsilon)n$ nodes of $L_c^+$. Let $X_u$ be the set of nodes in $C^+(u)$ that have less than $|L_c^+| - \epsilon n$ nodes in their fan-in. We will use the fact that $u$ appears in at most $\epsilon^2 n/2$ bad pairs to prove $|X_u| \leq \epsilon n$. Before proceeding with its proof, note that if $|X_u| \leq \epsilon n$, then there are $|C^+(u)| - \epsilon n$ nodes with fan-in of size at least $|L_c^+| - \epsilon n \geq (\alpha - \epsilon)n$. Thus, $|L_{c^-}| \geq (c - \epsilon)n$. As a result, for any $\delta < (\alpha \epsilon)^2/2$ there exists $N$ such that for $n \geq N$ and all $p' \in I$,

$$P_{D_{G_n}(p')}(|L_c^+| \geq \alpha n) - \delta \leq P_{D_{G_n}(p')}(|L_c^+| \geq \alpha n, \text{ and } \exists \text{ at most } \delta n^2 \text{ bad pairs}) \leq P_{D_{G_n}(p')}(|L_{c^-}| \geq (c - \epsilon)n).$$

Thus the proof of part 2 follows once we prove $|X_u| \leq \epsilon n$.

Construct a bipartite graph $B$ on $C^+(u) \times L_c^+$ with an edge between $w \in C^+(u)$ and $v \in L_c^+$ whenever $w \in C^+(v)$. To bound $|X_u|$, we find a lower bound and an upper bound for the number of edges of $B$. First, by definition of $X_u$, the number of edges coming out of the side of $C^+(u)$ in $B$ are at most $|X_u|((|L_c^+| - \epsilon n) + (|C^+(u)| - |X_u|)|L_c^+|$. Now, to find a lower bound on the number of edges, note that $u$ appears in at most $\epsilon^2 n/2$ bad pairs $(u, v)$, and if $(u, v)$ is not bad, then $|C^+(u) \cap C^+(v)| \geq |C^+(u)| - \epsilon^2 n/2$. So, there are at least $|L_c^+| - \epsilon^2 n/2$ nodes of $L_c^+$ that have all but $\epsilon^2 n/2$ nodes of $C^+(u)$ in their fan-out. Combining these two bounds, we get

$$|X_u|(|L_c^+| - \epsilon n) + (|C^+(u)| - |X_u|)|L_c^+| \geq (|C^+(u)| - \epsilon^2 n/2)(|L_c^+| - \epsilon^2 n/2).$$

As a result, $|X_u| \leq \epsilon n$. 

3. Using Lemma 4.1 part 1 and the convergence of $\frac{|C_1|}{n} \to \zeta(p)$, one can compute the first moment,

$$\mathbb{E}\left[\frac{|L^+|}{n}\right] = \mathbb{E}\left[\frac{1}{n} \sum_{v \in [n]} 1_{|C(v)| \geq \alpha n}\right] = \frac{1}{n} \sum_{v \in [n]} \mathbb{P}_{D_{G_n}(p)}(|C^+(v)| \geq \alpha n)$$

proving convergence in expectation. Note that the same argument also gives that $\mathbb{E}\left[\frac{|L^+|}{n}\right] \to 0$ for all $\alpha > 0$ if $\zeta(p) = 0$. Next, given an arbitrary small $\epsilon > 0$, assume that there exists some $\delta > 0$,

$$\mathbb{P}_{D_{G_n}(p)}(|L^+_\alpha| \geq \zeta(p)n + 2\epsilon n) \geq \delta.$$

Then by Part 2 and the symmetry of changing the directions of all edges we have for large enough $n$

$$P_{D_{G_n}(p)}\left(|L^+_{\zeta(p)+\epsilon}| \geq (\alpha - \epsilon)n\right) = P_{D_{G_n}(p)}\left(|L^-_{\zeta(p)+\epsilon}| \geq (\alpha - \epsilon)n\right) \geq \frac{\delta}{2}.$$

This is a contradiction with Part 1. As a result, for $\epsilon > 0$ small enough

$$\text{var}_{D_{G_n}(p)}\left(\frac{|L^+|}{n}\right) = \mathbb{E}\left[\left(\frac{|L^+|}{n}\right)^2\right] - \zeta(p)^2 + o(1) \leq (\epsilon + \zeta(p))^2 - \zeta(p)^2 + o(1).$$

Since $\epsilon$ was arbitrary, we get that the variance goes to 0, which proves the result.

In Lemma 4.10 we saw that in the supercritical case $\mathbb{E}|SCC_1|^2 \geq \alpha n^2$ for some $\alpha > 0$. To prove $SCC_1$ is linear-sized with high probability, i.e., to prove that $p_c(\mu)$ is a threshold for the existence of a giant SCC, we will want to show that $\text{var}\left(\frac{|SCC_1|}{n}\right) \to 0$. We will do this by invoking Lemma 2.3 from Section 2.4, a bound on how much adding an edge $e$ to $D_{G_n}(p)$ will change the size of $SCC_1$, Lemma 5.5 below, and Russo’s formula in Equation (7) for the expectation of this influence. Recall the definition of $\Delta_e f$ for a Boolean function $f$ from Section 2.4. Russo’s formula then immediately gives the following lemma.

**Lemma 5.4.** For any graph $G$, and $p \in (0, 1)$

$$\sum_e \mathbb{E}_{S \sim D_G(p)}|\Delta_e SCC_1(S)| = p(1-p) \frac{d}{dp} \mathbb{E}_{S \sim D_G(p)}|SCC_1(S)|,$$

where $SCC_1$ is equal to the size of the largest SCC with edges in $S$ and the sum goes over all oriented edges in $E(G)$.

**Proof.** This follows from (7).

The next result bounds the influence of an edge to later bound the variance of $|SCC_1|$.

**Lemma 5.5.** Let $I = [q, p]$ with $0 \leq q \leq p \leq 1$, and assume that in $G_n(q)$, $\frac{|C_1|}{n} \to \zeta(q) > 0$. Furthermore, assume that for all $\epsilon > 0$, $\{G_n\}_{n \in \mathbb{N}}$ has a uniformly $\epsilon$-unique giant component in $I$. Given $\epsilon > 0$ there then exists $N < \infty$ such that for $n \geq N$,

$$\sup_{p' \in I} \mathbb{E}_{D_{G_n}(p')}|\Delta_e SCC_1| \leq \epsilon n$$

for all directed edges $e$ in $E(G_n)$.
PROOF. Consider a digraph $D_{G_n}(p')$ that if one adds the directed edge $e = (y, x)$ to $D_{G_n}(p')$, then the size of $SCC_1$ increases by $\epsilon n$. Let $H_e$ be the corresponding event. We will prove that $\sup_{p' \in (p-q, p]} \mathbb{P}(H_e) \leq \epsilon$.

For the proof we need the following notation. Let $D'$ be a subgraph of $D_{G_n}(p')$ such that each vertex in $D'$ appears in a path from $x$ to $y$. Let $S$ be the set of of maximal strongly connected components in $D'$, and let DAG be the directed acyclic graph obtained by contracting all SCCs in $S$. Since adding $(y, x)$ changes the size of $SCC_1$, we know that $SCC(x) \neq SCC(y)$. Choose a vertex $v_1$ for each strongly connected component in $S$, giving a set of vertices $\{v_0, v_1, \ldots, v_k\}$ (where we choose $v_0 = x$ and $v_k = y$). Order the vertices consistent with the partial order given by DAG. Note that $SCC(v_1), SCC(v_2), \ldots, SCC(v_k)$ do not necessarily form a path, however, for all $i > j$, there are no edges from $SCC(v_i)$ to $SCC(v_j)$ in the DAG.

Define $S_1^k = \bigcup_{j=0}^k SCC(v_j)$ and $S_2^k = \bigcup_{j=i+1}^k SCC(v_j)$. We claim that there exist some index $s$ such that both $|S_1^s| \geq \epsilon n/2$ and $|S_2^s| \geq \epsilon n/2$. We know $|S_1^s| \geq \epsilon n$, so let $s$ be the smallest index such that $|S_1^s| \geq \epsilon n/2$ (note that in particular $|S_1^{s-1}| < \epsilon n/2$ if $s \geq 1$). Then, we show that $|S_2^s| \geq \epsilon n/2$. The reason is that adding the edge $e$ changed the size of $SCC_1$ and any $SCC(v_i)$ by at least $\epsilon n$, and as a result,

$$|SCC(v_s)| + \epsilon n \leq \sum_{\ell=0}^k |SCC(v_{\ell})| = |S_1^{s-1}| + |SCC(v_s)| + |S_2^s| \leq \epsilon n/2 + |SCC(v_s)| + |S_2^s|$$

for $s \geq 1$. If $s = 0$, then

$$|SCC(v_s)| + \epsilon n \leq \sum_{\ell=0}^k |SCC(v_{\ell})| = |SCC(v_s)| + |S_2^s|.$$

In both cases, $|S_2^s| \geq \epsilon n/2$, which proves our claim.

The rest of the proof follows similar ideas as in the proof of Lemma 5.3, part 2. We will show that there are at least $\epsilon n/4$ nodes in $S_1^s$ with large fan-ins such that their fan-ins does not contain $\epsilon n/2$ nodes in the fan-in of $y$. In fact, all the nodes of $S_2^s$ are in the fan-in of $y$, while none of them appear in the fan-ins of any node in $S_1^s$. We will bound the probability of this event (\(\mathbb{P}(H_e)\)) by using Corollary 4.4.

To formalize the proof, let $e \in (0, \zeta(q))$. By part 3 of Lemma 5.3, there exists some $N_1$ such that for all $n > N_1$, with probability at least $\epsilon^3/2$, $|C^+_e| \geq \epsilon n$ in $D_{G_n}(q)$. Since $|C^+_e|$ is increasing in $p$ and $p' \geq q$ for all $p' \in I$, we conclude that the lower bound on $|C^+_e|$ holds in $D_{G_n}(p')$ for all $p' \in I$. Similar to the proof of part 2 in Lemma 5.3, we will prove that there exists some $N_2$ such that for all $p' \in I$ and all $n > N_2$, all but $\epsilon n$ nodes of $L_c^+$ appear in the fan-ins of at least half of the nodes in $S_1^s$.

Recall that $x$ and $y$ are the two endpoints of the edge $e$. Call $(x, v)$ an $x$-bad pair if $|C^+(x)| \geq \epsilon n$ and $|C^+_{C^+(x)}(v)| \geq \epsilon^3 n/2$, and call $(u, y)$ a $y$-bad pair if $|C^-(u)| \geq (c - \epsilon)n$ and $|C^-_{C^-(u)}(y)| \geq \epsilon n/2$. We may use Corollary 4.7 for uniformly $\epsilon$-unique giants, to conclude that for any $\delta > 0$ and all large enough $n$ the expected number of $x$-bad pairs is at most $\delta^2 n$ for all $p' \in I$. By Markov inequality with probability at least $1 - \delta$ the number of $x$-bad pairs is less than $\delta n$. Choosing $\delta = \epsilon^3/2$, we therefore get

$$\mathbb{P}_{D_{G_n}(p')}(H_e) \leq \mathbb{P}_{D_{G_n}(p')}(H_e, |L_c^+| \geq \epsilon n \text{ and } \exists \text{ at most } \frac{\epsilon^3}{2} n \text{ x-bad pairs}) + \epsilon^3.$$

Consider now the event that $H_e$ holds, that there are at most $\frac{\epsilon^3}{2} n$ x-bad pairs and that $|L_c^+| \geq \epsilon n$. Let $B$ be the bipartite graph on $C^+(x) \times L^+_c$ where there is an edge between $u \in C^+(x)$
and \(v \in L^+_i\) if \(u \in C^+(v)\). Let \(X\) be the number of nodes in \(C^+(x)\) that have at most \(|L^+_i| - \epsilon n\) nodes in their fan-ins. We then can proceed as in part 2 of Lemma 5.3 to conclude that

\[
|X|(|L^+_i| - \epsilon n) + (|C^+(u)| - |X|)|L^+_i| \geq (|C^+(u)| - \frac{3n}{2})(|L^+_i| - \frac{3n}{2}).
\]

Therefore, \(|X| \leq \epsilon^2 n\). Since \(S^+_1 \subset C^+(x)\) and \(|S^+_1| \geq \epsilon n/2\), there are at least \(\epsilon n/4\) nodes \(u \in S^+_1\) such that \(|C^-(u)| \geq |L^+_i| - \epsilon n \geq (c - \epsilon)n\). But since the fan-in of a node \(u \in S^+_1\) does not contain any node from \(S^+_2 \subset C^-(y)\), we have that \(|C \setminus C^-(u)| \geq |S^+_2| \geq \epsilon n/2\). Thus, for all these nodes \(u\), the pair \((u, v)\) is a \(y\)-bad pair. As a result,

\[
\mathbb{P}_{D_{G_n}(p')} (H_e) \leq \mathbb{P}_{D_{G_n}(p')} (H_e, |L^+_i| \geq \epsilon n \text{ and } \exists \text{ at most } \frac{\epsilon^3}{2} n \text{ }x\text{-bad pairs}) + \epsilon^3
\]

\[
\leq \mathbb{P}_{D_{G_n}(p')} (\exists \text{ at least } \frac{\epsilon}{4} n \text{ }y\text{-bad pairs}) + \epsilon^3 \leq \epsilon
\]

where the last inequality is obtained again by Corollary 4.7 for large enough \(n\).

In the above arguments, the choice of \(n\) is independent of \(x\) and \(y\) due to the fact that the convergence in Corollary 4.7 is uniform in the fixed vertex \(v\). Thus, the event \(H_e\) takes place with probability at most \(\epsilon\) for any edge \(e\). As a result, for all \(\epsilon > 0\), all large enough \(n\), all \(e \in E(G_n)\), and all \(p' \in I\)

\[
\mathbb{E} |\Delta_\epsilon SCC_1| \leq \epsilon n + n \mathbb{P}_{D_{G_n}(p')} (H_e) \leq 2\epsilon n,
\]

as desired. \(\square\)

The following is the key lemma used in the proof of Theorem 1.3. It shows that the strongly connected component exists with high probability in the super critical regime. The main ingredients of the proof are the concentration bounds given in Section 2.4 and the bounds on the influence of an edge on the size of largest SCC. Combined with Corollary 4.4 and since \(p_c(\mu)\) is a threshold for the giant in \(G_n(\mu)\), the lemma establishes that for a graph sequence satisfying the assumptions of Theorem 1.1, \(p^{SCC}_c(G_n) \to p_c(\mu)\).

**Lemma 5.6.** Let \(\{G_n\}_{n \in \mathbb{N}}\) be a sequence of graphs satisfying the assumptions of Theorem 1.3 in an interval \([p - \epsilon, p]\). If \(\zeta(p) > 0\) and \(\alpha \in (0, \zeta^2(p))\), then

\[
\mathbb{P}_{D_{G}(p)} (|SCC_1| \geq \alpha n) \to 1.
\]

**Proof.** Fix \(\epsilon' > 0\) such that \(0 < \epsilon' < \zeta(p)\). To prove the lemma, we need to show that

\[
\mathbb{P}_{D_{G}(p)} (|SCC_1| \geq (\zeta(p) - \epsilon')^2 n) \to 1.
\]

By the continuity of \(\zeta\) at \(p\) one can find \(q' < q\) such that \(|\zeta(p - q') - \zeta(p)| \leq \frac{\epsilon'}{2}\), implying in particular that \(\zeta(p - q') > 0\).

Let \(I = [p - q', p]\), and let \(m\) be twice the number of edges in \(G_n\), i.e., let \(m\) be the number of possible, oriented edges in \(D_{G_n}(p)\). We first use Lemma 2.3 to prove that given \(\epsilon > 0\), there exists \(N > 0\) such that the following holds for all \(n > N\) and all \(p' \in I\)

\[
\text{var}_{D_{G_n}(p')}(|SCC_1|) \leq \epsilon n \frac{d}{dp'} \mathbb{P}_{D_{G_n}(p')} |SCC_1|.
\]

To prove (12), we consider two cases based on whether \(\frac{\zeta_2(SCC_1)}{\zeta_1(SCC_1)}\) is larger or smaller than \(M = \epsilon^{1/\epsilon}\).
Case 1: For $M$ defined as above $\mathcal{E}_2(SCC_1) \geq M\mathcal{E}_1(SCC_1)$. We claim that this assumption implies that

$$\text{var}_{D_G(p')}(|SCC_1|) \leq \frac{\log \frac{1-p'}{p} \mathcal{E}_2(SCC_1)}{(1-2p') \log(\frac{M}{\log M})}. \tag{13}$$

To see this, assume that $\text{var}_{D_G(p')}(|SCC_1|) \geq \frac{\mathcal{E}_2(SCC_1)}{\log(M)}$ (otherwise (13) holds by the fact that $1 \leq 2 \leq \frac{\log \frac{1-p'}{p}}{1-2p'}$ and $\frac{1}{\log(M/\log M)}$). Then

$$\frac{\text{var}(|SCC_1|)}{\mathcal{E}_1(SCC_1)} \geq \frac{M \text{var}(|SCC_1|)}{\mathcal{E}_2(SCC_1)} \geq \frac{M}{\log M}$$

and (13) follows by Lemma 2.3.

Next note that $|SCC_1| \leq n$ implying that $\mathcal{E}_2(f) \leq n \sum_e E|\Delta_e SCC_1|$. Combined with (13) and Lemma 5.4 we conclude that

$$\text{var}_{D_G(p')}(|SCC_1|) \leq \frac{p'(1-p') \log \frac{1-p'}{p}}{(1-2p')} \frac{n}{\log(\frac{M}{\log M})} \frac{d}{dp'} E_{D_G(p')}|SCC_1|.$$

It is easy to see that the first quotient is bounded by $1/2$ (e.g., expanding both the numerator and denominator around $p' = 1/2$ and comparing the derivatives). Therefore

$$\frac{p'(1-p') \log \frac{1-p'}{p}}{(1-2p') \log(\frac{M}{\log M})} \leq \frac{1}{2 \log(\frac{M}{\log M})} \leq \epsilon,$$

proving (12) for Case 1.

Case 2: For $M$ defined as above, $\mathcal{E}_2(SCC_1) \leq M\mathcal{E}_1(SCC_1)$. In this case, we will use the the Efron-Stein inequality to bound the variance by

$$\text{var}_{D_G(p')}(|SCC_1|) \leq \frac{1}{2} \mathcal{E}_2(SCC_1) \leq \frac{M}{2} \mathcal{E}_1(SCC_1).$$

But in this case, the bound $\mathcal{E}_2(f) \leq n \sum_e E|\Delta_e SCC_1|$ is not strong enough to complete the proof of (12). To overcome this, we use Lemma 5.5, which implies that for any constant $\epsilon > 0$, and large enough $n$, $E|\Delta_e SCC_1| \leq \epsilon n/M$ for all edges $e$ and all $p' \in I$. As a result,

$$\mathcal{E}_1(SCC_1) = \sum_{e \in [m]} (E_{D_G(p')}[\Delta_e SCC_1])^2 \leq \frac{\epsilon n}{M} \sum_{e \in [m]} E_{D_G(p')}[\Delta_e SCC_1] = \frac{\epsilon}{M} p'(1-p')n \frac{d}{dp'} E_{D_G(p')}|SCC_1|,$$

resulting in

$$\text{var}_{D_G(p')}(|SCC_1|) \leq \frac{\epsilon p'(1-p')}{2} \frac{n}{M} \frac{d}{dp'} E_{D_G(p')}|SCC_1|,$$

Since $p'(1-p') \leq 2$, we get (12) in the second case as well.

We are now ready to prove (11). Given $\epsilon > 0$, let $N$ be such that for $n \geq N$ the bound (12) holds for all $p' \in [p-q', p]$. We claim that given $n \geq N$ there exists $p' = p'(n) \in [p-q', p]$

$$\frac{d}{dp'} E_{D_G(p')}|SCC_1| \leq 2n/q'.$$

Indeed, assume this is not the case, then by the fundamental theorem of calculus,

$$E_{D_G(p)}|SCC_1| \geq 2n + E_{D_G(p-q')}|SCC_1|,$$
which is a contradiction, since $|SCC_1| \leq n$ with probability 1. Then by (12) for $n \geq N$ and $p' = p'(n) \in [p - q', p]$

$$\text{var}_{D_{G_n}}(p')(|SCC_1|) \leq \frac{2\epsilon}{q} n^2.$$  

Next we use Lemma 4.10 together with convergence of $|C_1|/n$ to conclude that for all $\epsilon'' > 0$ with $\epsilon'' < \zeta(p - q)$ there exists an $N' < \infty$ such that for $n \geq N'$,

$$\mathbb{E}_{D_{G_n}}(p')(|SCC_1|^2) \geq \mathbb{E}_{D_{G_n}(p-q')}(|SCC_1|^2) \geq \left(\zeta(p - q') - \epsilon'' \right)^4 n^2.$$ 

Choosing $\epsilon'' = \frac{\epsilon'}{4}$ and using that $|\zeta(p) - \zeta(p - q')| \leq \epsilon'/4$, we thus have

$$\mathbb{E}_{D_{G_n}}(p')(|SCC_1|^2) \geq \left(\zeta(p) - \frac{\epsilon'}{2}\right)^4 n^2,$$

for all $n \geq N'$. With this lower bound on the expectation of $|SCC_1|^2$ and the variance bound, (14)

$$\left(\mathbb{E}_{D_{G_n}}(p')(|SCC_1|)\right)^2 \geq \left(\zeta(p) - \frac{\epsilon'}{2}\right)^4 n^2 - \frac{2\epsilon}{q} n^2$$

and, by Chebyshev’s inequality,

$$\mathbb{P}_{D_{G_n}}(p')\left(\left|\frac{|SCC_1|}{n} - \mathbb{E}\left[\frac{|SCC_1|}{n}\right]\right| \geq \frac{1}{4} \epsilon'\right) \leq \frac{32\epsilon}{q \epsilon'^2}.$$ 

We therefore have shown that given $\epsilon > 0$ small enough (depending on $q'$, $\epsilon'$ and $\zeta(p)$) there exists $\hat{N} < \infty$ such that for all $n \geq \hat{N}$ there exists $p' = p'(n) \in [p - q', p]$ such that

$$\mathbb{P}_{D_{G_n}}(p')\left(\frac{|SCC_1|}{n} \geq \left(\zeta(p) - \epsilon'\right)^2\right) \leq \sqrt{\epsilon}.$$ 

Since $|SCC_1|$ is increasing in $p$ this implies that for all sufficiently small $\epsilon > 0$ there exists an $\hat{N}' < \infty$ such that

$$\mathbb{P}_{D_{G_n}}(p)\left(\frac{|SCC_1|}{n} \geq \left(\zeta(p) - \epsilon'\right)^2\right) \leq \sqrt{\epsilon},$$

for all $n \geq \hat{N}'$. This proves (11). \hfill \Box

Now, we are ready to proceed with the proof of Theorem 1.3.

**Proof of Theorem 1.3.** Part 1 follows by the assumption of the theorem on uniqueness of the second largest component: we know that for all $\epsilon > 0$, $G_n$ has a uniformly $\epsilon$-unique giant, and Part 1 follows by Lemma 4.8.

Next, to prove Part 2, Corollary 4.2 implies that for all $\epsilon > 0$, 

$$\mathbb{P}(\text{there exists } \geq \epsilon n \text{ vertices } v \text{ with } |C^+(v)| \geq \epsilon n) \to 0.$$ 

As a result for a uniform random vertex $v$, $\frac{|C^+(v)|}{n} \to 0$. A similar argument implies the statement for fan-ins. Furthermore, in any instance of $D_{G_n}(p)$ with $|SCC_1| \geq \epsilon n$ the probability that a uniform random node has fan-out larger than $\epsilon n$ is at least $\epsilon$. So, $\frac{|SCC_1|}{n}$ must also converge to 0 in probability.

Next we prove Part 3. The statement

$$\liminf_{n \to \infty} \frac{1}{n} \mathbb{E}\left[|SCC_1|^2\right] \geq \zeta^2(p)$$

is a result of Theorem 1.3.
follows from Equation (14). Further, in Lemma 5.6 we showed that for any fixed $\epsilon > 0$,
\[
\limsup_{n \to \infty} \mathbb{P}_{D_G_n(p)} \left( \left| \frac{|SCC_1^+|}{n} - \mathbb{E}[\frac{|SCC_1|}{n}] \right| \geq \epsilon \right) = 0,
\]
which implies $\frac{|SCC_1|}{n} \xrightarrow{p} 1$.

Next, choose $v \in V(G_n)$ uniformly at random. We will prove that either $v \notin SCC_1^-$ and $|C^+(v)| = o(n)$ or $v \in SCC_1^-$ and $|C^+(v)\Delta SCC_1^+| = o(n)$. Recalling the definition of the sets $O_\epsilon$ and $O_\epsilon$ from Lemma 4.9, the statements of the lemma then imply that
\[
\frac{|O_\epsilon|}{n} \to 0 \quad \text{and} \quad \frac{|\tilde{O}_\epsilon|}{n} \to 0
\]
in probability. The second statement implies that if a random vertex $v$ does not fall into $SCC_1^-$ (which is equivalent to $C^+(v) \cap SCC_1^+ = \emptyset$), with high probability its fan-out has $o(n)$ vertices, proving $|C^+(v)| = o(n)$ for this case. If a random vertex $v$ falls into $SCC_1^-$, by the first statement of Lemma 4.9, we know that when considering the induced subgraph on the complement of $SCC_1^+$, the fan-out of $v$ is of size at most $o(n)$. But the fan-out of $v$ in this induced subgraph is nothing but $C^+(v) \setminus SCC_1^+ = C^+(v)\Delta SCC_1^+$, proving $|C^+(v)\Delta SCC_1^+| = o(n)$ for the fan-out of $v$. The same argument works by symmetry for fan-ins.

The rest of this proof is dedicated to convergence of the relative size of $SCC_1^+$ and $SCC_1^-$. Since $\zeta(p) > 0$, there exists some $\alpha > 0$ such that with high probability $|SCC_1| \geq \alpha n$. Since any node in $SCC_1$ has $SCC_1^+$ in their fan-out, for small enough $\epsilon > 0$,
\[
\mathbb{P}\left( \frac{|SCC_1^+|}{n} \geq \zeta(p) + \epsilon \right) \leq \mathbb{P}(\frac{|L_{\zeta(p)+\epsilon}^+|}{n} \geq \alpha n).
\]
Then by part 1 of Lemma 5.3,
\[
\mathbb{P}\left( \frac{|SCC_1^+|}{n} \geq \zeta(p) + \epsilon \right) \to 0.
\]

To prove the lower bound assume to the contrary that there exists $\delta > 0$ such that for infinitely many $n$,
\[
\mathbb{P}\left( \frac{|SCC_1^+|}{n} \leq \zeta(p) - \epsilon \right) \geq \delta.
\]
For any vertices $u, w \in SCC_1$ note that $C^+(w) = C^+(u) = SCC_1^+$. Since $|SCC_1| \geq \alpha n$, with probability greater than $\alpha$ a random node lies in $SCC_1$. Therefore for a random node $v$
\[
\mathbb{P}_{D_G_n(p)}(\alpha \leq \frac{|C^+(v)|}{n} \leq \zeta(p) - \epsilon) \geq \alpha \delta.
\]
By Lemma 4.1,
\[
\mathbb{P}_{G_n(p)}(\alpha \leq \frac{|C(v)|}{n} \leq \zeta(p) - \epsilon) \geq \alpha \delta,
\]
contradicting the bounds in Proposition 3.4. Therefore, we must have
\[
\mathbb{P}\left( \frac{|SCC_1^+|}{n} \leq \zeta(p) - \epsilon \right) \to 0.
\]
Since $\epsilon > 0$ was arbitrary we get the result. By symmetry the same holds for $SCC_1^-$. \hfill \square
6. Applications to Preferential Attachment Graphs. As an application of our method to power law graphs, we consider percolation on preferential attachment graphs. Here we consider the following version of preferential attachment, which closely follows the original formulation by Barabási and Albert [4]. The model has a parameter \( m \in \mathbb{N} \), and is defined as follows. Starting from a connected graph \( G_t \) on at least \( m \) vertices, a random graph \( G_t \) is defined inductively: given \( G_{t-1} \) and its degree sequence \( d_s(t-1) \), we form a new graph by adding one more vertex, \( v_t \), and connect it to \( m \) distinct vertices \( w_1, \ldots, w_m \in V(G_{t-1}) \) by first choosing \( w_1, \ldots, w_m \in V(G_{t-1}) \) i.i.d with distribution \( \mathbb{P}(w_s = i) = \frac{d_s(t-1)}{2d_s(t-1)} \), \( s = 1, \ldots, m \), and then conditioning on all vertices being distinct (thus avoiding multiple edges).

While all our results hold for arbitrary connected starting graphs on at least \( m \) vertices, it will be notationally convenient to choose \( G_{t_0} \) is such a way that at time \( t \geq t_0 \), the graph has \( t \) vertices and \( mt \) edges. For concreteness, we choose \( G_{t_0} \) to be the graph \( K_{2m+1} \), the complete graph on \( t_0 = 2m + 1 \) vertices. We denote the resulting random graph sequence by \( (PA_{m,n})_{n \geq 2m+1} \), and following [9], we call the version of preferential attachment we defined above the conditional model, while the model where the conditioning step is left off will be called the independent model.

There are several papers establishing that the percolation threshold is 0 for variants of this problem, see, e.g., [12] for site percolation on a different preferential attachment model that allows multiple edges and self-loops, and [19, 18] for bond percolation on what is called Bernoulli preferential attachment\(^5\) in [43]. Note that the results of [12] gives an easy proof that \( p_c = 0 \) for bond percolation on the Bollobás-Riordan version of preferential attachment models as well. All one needs to observe is that bond percolation with probability \( p \) gives a stochastic upper bound on site percolation with probability \( p' = p^m \) (take a bond percolation configuration, and delete all vertices for which at least one of the \( m \) initial edges is absent). This does not quite give a proof for the conditional model \( (PA_{m,n})_{n \geq 2m+1} \) considered here since the two models differ in minor technical details, but more importantly, we (a) want to demonstrate the power of the methods developed in this paper, establishing this result from scratch, and (b) we will be able to obtain sharper bound on the relative size of the largest cluster.

Before stating the theorem, we point out that the sequence \( \{PA_{m,n}\}_{n \geq m} \) converges locally in probability to a Pólya-point process [9, 45]. As we will see, the robustness of \( PA_{m,n} \) then reduces to the robustness of Pólya-point processes to bond-percolation. Specifically, the relative size \( \zeta(p) \) appearing in the next theorem is the survival probability of the Pólya-point processes after bond-percolation, and the statement that \( p_c = 0 \) for preferential attachment reduces to the statement that \( \zeta(p) > 0 \) for all \( p > 0 \).

**Theorem 6.1.** Let \( m \geq 2 \), for a positive integer \( n \geq 2m + 1 \) let \( PA_{m,n} \) be the conditional preferential attachment graph defined above, let \( p \in [0,1] \), and let \( C_1 \) and \( C_2 \) be the the largest and second largest connected component in \( PA_{m,n}(p) \), respectively. Then the following limits exist

\[
\frac{|C_1|}{n} \xrightarrow{p} \zeta(p) \quad \text{and} \quad \frac{|C_2|}{n} \xrightarrow{p} 0,
\]

where \( \zeta : [0,1] \rightarrow [0,1] \) is a continuous function with \( \zeta(p) = e^{-\Theta(1/p)} \) as \( p \rightarrow 0 \) and \( \zeta(1) = 1 \). So in particular, the largest component in \( PA_{m,n}(p) \) has linear size for all \( p > 0 \), showing that \( p_c = 0 \).

\(^5\)In this model, the number of new edges is not specified, but instead is a random variable which is sum of \( n \) Bernoulli random variables skewed towards higher degrees.
The theorem will follow from Theorem 1.1 once we establish (1) large-set expansion of $PA_{m,n}$, (2) continuity of the survival probability $\zeta(p)$ of the Pólya-point processes after bond-percolation, and (3) the bounds $\zeta(p) \to 1$ as $p \to 1$ and $\zeta(p) = e^{-\Theta(1/p)}$ as $p \to 0$.

Theorem 1 in [36], shows positive edge expansion for a different version of preferential attachment. Following a similar argument, we prove in Appendix D that $PA_{m,n}$ has positive large set expansion, which is weaker than the expansion established in [36], but sufficient for our purpose.

**Lemma 6.2.** Let $m \geq 2$ and $n \geq m$, and let $PA_{m,n}$ be defined as above. Then there exists some $\alpha > 0$ such that for any $\epsilon \in (0,1/2)$ and all large enough $n$, $\{PA_{m,n}\}_{n \geq m}$ is an $(\alpha, \epsilon, 2m)$ large-set expander with probability $1 - \epsilon$.

To continue, we will use the explicit construction of the local limit of preferential attachment in [9], which gives what the authors call a Pólya-point graph or process. This graph is a random rooted tree, where vertices have types $(S, x) \in \{\emptyset, R, L\} \times [0,1]$, where the discrete label is $S = \emptyset$ for the root, and right (R) or left (L) for all other vertices. We will refer to the continuous label $x$ as the “position” of a point in the Pólya-point graph. The root, with type $(\emptyset, x)$, has a random position $x = \sqrt{\gamma}$ where $\gamma$ is drawn uniformly at random from $[0,1]$. For a vertex of type $(S, x)$ define

$$m(S) = \begin{cases} m, & \text{if } S = L \text{ or } S = \emptyset \\ m - 1, & \text{if } S = R. \end{cases}$$

For a vertex of type $(S, x)$ the off-springs are generated as follows.

- Each such vertex has a deterministic number $m(S)$ of children of type $(L, x_i), i = 1, \ldots, m(S)$, where $x_1, \ldots, x_m(S)$ are chosen i.i.d uniformly at random from $[0, x]$.
- In addition, it has $N \sim \text{Poi}(\frac{1}{x})$ right children, where $\gamma \sim \Gamma(m + 1, 1)$ if $S = L$, and $\gamma \sim \Gamma(m, 1)$ if $S \in \{R, \emptyset\}$. Given $N$, the right children have type $(R, y_1), \ldots, (R, y_N)$, where $y_1, y_2, \ldots, y_N$ are chosen i.i.d uniformly at random from $[x, 1]$.

Next we discuss how to compute $\zeta(p)$ for the Pólya-point graph. To this end, we derive the implicit formula for the survival probability $\rho(S, x)$ of the tree under a node of type $(S, x)$ after percolation, with $(S, x) \in \{L, R\} \times [0,1]$. Let $d_R$ and $d_L$ be the random number of right and left children of such a node after percolation. Intuitively, the extinction probability $1 - \rho(S, x)$ is equal to the probability that all of its children do not appear in an infinite cluster. Taking first the expectation over the positions of these children and then over the number of left and right children after percolation will give an implicit equation for $\rho(S, x)$, showing that $\rho(S, x)$ is a solution of

\begin{equation}
(\Phi f)(S, x) = f(S, x),
\end{equation}

where $S \in \{L, R\}$, and

\begin{equation}
(\Phi f)(S, x) = 1 - \frac{x \left( x - p \int_0^x f(L, y)dy \right)^{m(S)}}{x + p \int_x^1 f(R, y)dy}^{m(S)+1},
\end{equation}

see Appendix E for the derivation of (15). As is typical for implicit equations for survival probabilities in branching processes, the above equation has a trivial solution $f(x, S) \equiv 0$, raising the question of whether there exist other solutions, and if so, which one is the survival probability $\rho(S, x)$. As also typical, $\rho(S, x)$ will be the maximal solution, which here means
the point-wise maximum over all solutions. The exact statement is given in Proposition 6.3 below, whose proof is also given in Appendix E.

Before stating the proposition, we note that once we know \( \rho(S, x) \) for all vertices of discrete type \( L \) or \( R \), we can calculate the survival probability for the root in exactly the same way, except that we now also need to integrate over the position \( x \) of the root, which we recall is equal to \( \sqrt{y} \) where \( y \) is uniform in \([0, 1]\). This leads to the equation

\[
\zeta(p) = \int_0^1 (\Phi p)(\emptyset, \sqrt{y})dy, \quad \text{where} \quad (\Phi f)(\emptyset, x) = 1 - \left( \frac{x - p \int_0^x f(L, z)dz}{x + p \int_x^1 f(R, z)dz} \right)^m, \tag{17}
\]

see again Appendix E for the proof. Note that we extended the domain of \( \Phi \) in (16) to \( (S, x) \in \{\emptyset, L, R\} \times [0, 1] \). To formulate Proposition 6.3, we introduce one more quantity, the probability that a node of type \((S, x)\) reaches level \( k \) after percolation with probability \( p \), a quantity we denote by \( \rho_k(S, x) \).

**Proposition 6.3.** Let \( p > 0 \) and let \( S \in \{L, R\} \). Then the following holds.

1. Let \( \rho_k(S, x) \) be the probability that a node of type \((S, x)\) reaches level \( k \) after percolation with probability \( p \). Then \( \rho_k(S, x) = (\Phi^k 1)(S, x) \) for all \( k \geq 0 \) and all \( x \in [0, 1] \).
2. The survival probability \( \rho(S, x) \) is the maximum solution of (15), i.e., for any other solution \( f \) we have that \( \rho(S, x) \geq f(S, x) \) for all \( x \in [0, 1] \).

We will use this proposition together with (15), (16) and (17) to establish the following bounds on the survival probabilities \( \rho(S, x) \) and \( \zeta(p) \).

**Proposition 6.4.** Let \( p > 0 \). Then

\[
e^{-\frac{1}{2m+1}} \leq \zeta(p) \leq 2me^{-\frac{1-2p}{(m+1)p}}. \tag{18}
\]

Note that by part 1 of Proposition 6.3, \( \rho_{k+1} = \Phi \rho_k \). The main idea of the proof is to establish upper and lower bounds of the form

\[
f_k^-(x) \leq \rho_k(S, x) \leq f_k^+(x), \quad \text{where} \quad f_k^\pm(x) = 1 - \left( \frac{1}{1 + \epsilon_k^\pm} \right)^{m \pm 1},
\]

and where \( \epsilon_k^\pm \) are defined recursively. Then we get the result by showing that the limit \( \lim_{k \to \infty} \epsilon_k^\pm \) exists and is of order \( e^{\Theta(1/\sqrt{p})} \). See Appendix E for the complete proof.

To finish the Theorem of 6.1, we note that the continuity of \( \zeta \) is already known from Corollary 2.2.

After these preparations, the proof of Theorem 6.1 is now almost obvious.

**Proof of Theorem 6.1.** First, we note that by the lower bound in Proposition 6.4 \( p_c = 0 \). So by large-set expansion of PA the continuity of \( \zeta \) for \( p > 0 \) follows from Corollary 2.2. So, we need to show \( \zeta(p) \to 1 \) as \( p \to 1 \). This follows from the fact that in the Pólya-point graph, the root has \( m \) left children, each of these left children have again \( m \) left children, etc., to bound \( \zeta(p) \) from below by the survival probability for percolation on a tree where the root has degree \( m \), and all other vertices have degree \( m + 1 \).

Further, by Lemma 6.2, the sequence \( \{PA_{n,m}\}_{n \geq m} \) are large-set expanders. Also, by [9] and Theorem 5.8 in [45], this sequence converges locally in probability to the Pólya-point graph. Thus we can use Theorem 6.1 to get that \( |C_2|/n \) converges to zero in probability for all \( p \in [0, 1] \), and that \( |C_1|/n \) converges to \( \zeta(p) \) for all continuity points of \( \zeta \), which is all \( p \in [0, 1] \) as well. \( \square \)
Remark 6.5. Using the results from the previous sections, it is easy to see that the relative size of $SCC_1$ for directed percolation on preferential attachment graphs is of order $e^{-O(p^{-1})}$ as well. To see this, we first note that by Lemma 5.1 and Lemma 4.5,
\[
\mathbb{P}_{D_{G_n}(p)} \left( \frac{|SCC_1|}{n} \geq \zeta(p) + \epsilon \right) \leq \mathbb{P}_{D_{G_n}(p)} \left( \frac{|SCC_1^+|}{n} \geq \zeta^+(p) + \epsilon \right) \to 0.
\]
Applying the upper bound in Proposition 6.4, this gives
\[
\mathbb{P}_{D_{G_n}(p)} \left( \frac{|SCC_1|}{n} \geq (2m + 1)e^{-\frac{1-2\rho}{(m+1)p}} + \epsilon \right) \to 0.
\]
For a lower bound, we use Lemma 5.6 and again Proposition 6.4 to get
\[
\mathbb{P}_{D_{G_n}(p)} \left( \frac{|SCC_1|}{n} \geq (1 - p^m)e^{-\frac{1}{\rho} - \epsilon} \right) \to 1.
\]

Acknowledgements. The authors thank Remco van der Hofstad, for insightful communications on local limits for random graph sequences, Jennifer Chayes for discussions concerning percolation, and Persi Diaconis for feedback on an earlier version of this paper. Finally, we would like to thank our anonymous reviewers for their insightful comments and suggestions which greatly improved our paper.

Yeganeh Alimohammadi and Amin Saberi are supported by NSF grant CCF1812919.

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APPENDIX A: CONTINUITY OF $\zeta$ FOR RANDOM SEQUENCE OF EXPANDERS

In this Section, we will prove Corollary 2.2 and show that the percolation function $\zeta$ is continuous for large-set expanders that converge locally in probability. As discussed in Section 2.3, it is enough to prove that the limit of a sequence of large-set expanders is an ergodic (extremal) unimodular random graph. Very recently, Sarkar [41] proved ergodicity of the limit for deterministic expanders with bounded degree. We will show that their proof is extendable to possibly random sequence of large-set expanders with bounded average degree.

To state the lemma we need the following definition. A measurable function $f : \mathcal{G}_x \to \mathbb{R}$ is rerooting-invariant if its value stays invariant under changes in the position of the root.

**Lemma A.1.** Let $\{G_n\}$ be a sequence of possibly random $(\alpha, \bar{d})$ large-set expanders obeying the assumptions of Theorem 1.1, and let $\mu$ be the limit. Then $(G, o) \sim \mu$ is ergodic. That is, if $f$ is any rerooting-invariant function, then $f(G)$ is constant almost surely.

**Proof.** Following the notation of Sarkar [41], for two rational numbers $0 \leq a < b \leq 1$ let

$$
\Gamma_1 = \{(G_i, o) \in \mathcal{G}_x : f(G) \leq a\}, \quad \Gamma_2 = \{(G_i, o) \in \mathcal{G}_x : f(G) \geq b\}.
$$

It is enough to show that $\mu(\Gamma_1)$ and $\mu(\Gamma_2)$ cannot both be positive, which is sufficient to prove the statement of the lemma. To prove this by contradiction assume that there exists $p_0 > 0$ such that $\mu(\Gamma_1) > p_0$ and $\mu(\Gamma_2) > p_0$.

Now, by Theorem A.7 of [45] we know that $\mathcal{G}_x$ is a Polish metric space. So $\mu$ is tight and regular since it is a probability measure on a Polish space (see e.g., Theorem 1.3. [10], and Chapter II, Theorem 1.2 of [38]). So, there exists compact sets $H_i \subseteq \Gamma_i$ such that $\mu(H_i) \geq p_0/2$.

Fix $\epsilon \leq p_0/8$. Recall that $\bar{d}$ is the average degree and $\alpha$ is the expansion. Fix $K = \frac{\bar{d}}{\alpha}$. Then following the proof of Sarkar [41], there exists $R < \infty$ such that for all $(G_i, o_i) \in H_i$,

$$
B_R(G_1, o_1') \neq B_R(G_2, o_2') \quad \forall o_i' \in B_K(G_i, o_i) \text{ for } i = 1, 2.
$$

Further, by tightness of $\mu$ and compactness of $H_i$, Theorem A.16 of [45] implies that there exists some $\Delta R, K$ such that the maximum degree of $R + K$-neighborhoods of any rooted graphs in $H_i$ is bounded by $\Delta = \Delta_{R, K}$, i.e.,

$$
\sup_{(G, o) \sim \mu} \max\{\deg(o') : \forall o' \in B_{K+R}(G_i, o_i) \forall (G_i, o_i) \in H_i\} \leq \Delta.
$$

Let $\mathcal{H}_{R+K, \Delta}$ be the set of connected rooted graphs of radius $K + R$ whose vertices all have degree at most $\Delta$. Note that the number of such graphs is by $1 + \Delta + \cdots + \Delta^{R+K} \leq \Delta^{R+K+1}$,

$$
|\mathcal{H}_{R+K, \Delta}| \leq \Delta^{R+K+1}.
$$

Define a local function $h_{1, R+K}$ as follows:

$$
h_{1, R+K}(G, o) = \max_{(G', o') \in \mathcal{H}_{R+K, \Delta} \cap H_i} \mathbb{1}(B_{R+K}(G, o) \simeq B_{R+K}(G', o')).
$$

Then by local convergence in probability,

$$
\mathbb{E}_{\mathbb{P}_n}[h_{1, R+K}(G_n)] \xrightarrow{\mathbb{P}} \mathbb{E}_\mu[h_{1, R+K}].
$$

Note that $h_{1, R+K}(G, o) = 1$ for all $(G, o) \in H_1$ and hence, $\mathbb{E}_\mu[h_{1, R+K}] \geq p_0/2$. Given $G_n$, let $A_{i, n} = \{v \in V(G_n) : h_{i, R+K}(G_n, v) = 1\}$. Then by local convergence in probability, for large enough $n$,

$$
\mathbb{P}(|A_{i, n}| \geq p_0 n/4) \geq 1 - \epsilon.
$$

Now, combining this with large-set expansion, for large enough $n$,

$$
\mathbb{P}(G_n \text{ is } (\alpha, \epsilon, \bar{d}) \text{ large-set expander, and } |A_{i, n}| \geq p_0 n/4) \geq 1 - 2\epsilon.
$$
Let $G_n$ be an instance such that the above conditions hold. Then by Menger’s Theorem applied to bounded average degree large set expanders (as in the proof of Lemma 3.3), there exists a path of length at most $K = \frac{d}{\alpha}$ between $A_{1,n}$ and $A_{2,n}$ in $G_n$. Let $v_1$ and $v_2$ be the two ends of this path, and $v_1 \in A_{1,n}$, $v_2 \in A_{2,n}$. Then

$$B_R(G_n, v_2) \subseteq B_{R+K}(G_n, v_1) \simeq B_{R+K}(G_1, r_1),$$

for some $(G_1, r_1) \in H_1$. Also, $B_R(G_n, v_2) \simeq B_{R+K}(G_2, r_2)$ for some $(G_2, r_2) \in H_2$. Therefore,

$$\mathbb{P}\left(B_R(G_1, o'_1) \simeq B_R(G_2, o'_2) \quad \text{for some } o'_i \in B_K(G_1, o_i) \text{ for } i = 1, 2\right) \geq 1 - 2\epsilon,$$

which is a contradiction with (19). So, $\mu$ must be extremal.

\textbf{APPENDIX B: PROOF OF LEMMA 2.3}

We start by stating the relevant result of [22] in the general setting considered there. Given an arbitrary probability measure $\mu$ on $\{0, 1\}^m$, we denote expectations with respect to $\mu$ by $E$. We use $x$ to denote elements of $\{0, 1\}^m$, $x_1, \ldots, x_m$ for the coordinates of $x$, and the notation $x \sim y$ to denote elements of $\{0, 1\}^m$ that differ in exactly one coordinate. For $i = 0, 1, \ldots, m$, define $f_i = \mathbb{E}[f|x_1, \ldots, x_i]$ (so in particular $f_m = f$ and $f_0 = \mathbb{E}[f]$), and for $i = 1, \ldots, m$, define $d_i = f_i - f_{i-1}$. Finally, we use $c(\mu)$ to denote the log-Sobolev constant,

$$c(\mu) = \sup_f \frac{E_{x \sim \mu}[\sum_{y \sim x} (f(x) - f(y))^2]}{\text{Ent}(f^2)}$$

where the sup goes over all boolean functions $f$. The theorem we use to prove Lemma 2.3 is Theorem 2.2 in [22], which states that

$$c(\mathbb{P}_p) \text{var}(f) \log \left(\frac{\text{var}(f)}{\sum_{e=1}^m E^2|d_e|}\right) \leq E_x \left[\sum_{y \sim x} (f(x) - f(y))^2\right].$$

If $\mu$ is the product measure for independent $\text{Ber}(p)$ variables, $\mu = \mathbb{P}_p$, the log-Sobolev constant is explicitly known, and is equal to $c(\mathbb{P}_p) = \frac{1 - 2p}{p(1-p)\log \frac{1}{p}}$, see, e.g., Theorem 5.2 in [14]. Lemma 2.3 then follows once we establish that

$$p(1-p)E_x[\sum_{y \sim x} (f(x) - f(y))^2] = E_2(f),$$

and

$$\sum_e E^2|d_e| \leq E_1(f).$$

We start with the observation that $\Delta_e f(x) = (1-p)(f(x) - f(x \oplus e))$ if $x_e = 1$ and $\Delta_e f(x) = p(f(x) - f(x \oplus e))$ if $x_e = 0$. Using the fact that the first event happens with probability $p$, and the second with probability $1 - p$, one easily sees that

$$E_x[|\Delta_e f|] = 2p(1-p)E_x[|f(x) - f(x \oplus e)|]$$

and

$$E_x[|\Delta_e f|^2] = (p(1-p))^2 + (1-p)p^2 E_x[(f(x) - f(x \oplus e))^2].$$

The first identity, together with Lemma 3.1 in [22], which states that

$$E|d_e| \leq 2p(1-p)E_x[f(x) - f(x \oplus e)],$$
then implies (22).

To prove (21), we use the second identity and the fact that $p(1-p)^2 + (1-p)p^2 = p(1-p)$ to get

$$\mathcal{E}_2(f) = \sum_{x} \mathbb{E}_x[(\Delta x f(x))^2] = p(1-p) \sum_{x} \mathbb{E}_x[(f(x) - f(x + e))^2].$$

On the other hand,

$$\mathbb{E}_x \sum_{y \sim x} (f(x) - f(y))^2 = \sum_{x} \sum_{y \sim x} \mathbb{P}_p(x)(f(x) - f(y))^2$$

$$= \sum_{x} \sum_{y \sim x} \mathbb{P}_p(x)(f(x) - f(x + e))^2 = \sum_{x} \mathbb{E}_x[(f(x) - f(x + e))^2],$$

which completes the proof of (21) and hence of the lemma.

APPENDIX C: UNIFORM BOUNDS ON THE SIZE OF $C_2$

In this appendix we prove Lemma 3.5. We follow the strategy of [3] where a similar result for expanders with bounded maximum degree is proved. Given a graph $G_n$ on $n$ vertices, a positive number $c > 0$ and an edge $e \in E(G_n)$, let $S(e, c, n)$ be the event that $e$ connects two components of size larger than $cn$ in $G_n(p)$. Let $S(e, n)$ be the event that $S(e, c, n)$ occurs for an edge $e$ chosen uniformly at random from all edges in $E(G_n)$. The following bounds the probability that the event $S(e, c, n)$ holds.

**Lemma C.1.** Given $q > 0$ there exist a constant $\beta$ such that for all $p \in [q, 1-q]$ and all finite graphs $G_n$,

$$\mathbb{P}_{G_n}(S(e, n)) \leq \left(\frac{1}{e} - 1\right) \frac{\beta}{\sqrt{|E(G_n)|}}.$$

**Proof.** This follows from Lemma 2.3 and equation (6) in [3]; note that while equation (6) in [3] appears in the proof of a corollary which assumes expansion and bounded degrees in its statement, neither of these assumptions enter their proof of the bound (6). In fact, $\beta$ is nothing but the constant from Lemma 2.3 in [3] (where it is called $\alpha$), and it just depends on $q$.

To state the next lemma, we use the notation $B_r(A, G)$ for the $r$-neighborhood of a set of vertices $A$ in a graph $G$.

**Proposition C.2.** Let $\alpha > 0 < \bar{d} < \infty$, and $0 < c < 1$, and set

$$r = \left[2\bar{d}/\alpha\right] + \left[\bar{d} / (2c\alpha)\right].$$

Let $\epsilon \leq \min(c, 1-c)$, and let $G$ be a graph with $n$ vertices and average degree at most $\bar{d}$ such that $\phi(G, \epsilon) \geq \alpha$. Then $|B_r(G, A)| \geq \frac{4}{\epsilon} n$ for all $A \subseteq V(G)$ with $|A| \geq cn$.

**Proof.** The proof is adapted from Lemma 2.6 in [3]. Assume by contradiction that $|B_r(G, A)| < 3n/4$. Setting $C = V(G) \setminus B_r(G, A)$ we then have $|C| > n/4$. Let $E(W)$ be the set of edges joining two points in $W$. By the expansion property, if $|B_k(G, C)| \leq n/2$ then

$$|E(B_{k+1}(G, C))| \geq |E(B_k(G, C))| + \alpha n/4,$$

and by induction $|E(B_{k+1}(G, C))| \geq \alpha (k+1) n/4$. Since the total number of edges is at most $dn/2$, we conclude that $|B_k(G, C)| > n/2$ if $k \geq 2\bar{d}/\alpha$, and similarly, $|B_k(G, A)| > n/2$ if $k' \geq \bar{d} / (2c\alpha)$. Therefore, $B_{k'}(G, C) \cap B_k(G, A) \neq \emptyset$, showing that the distance between $C$ and $A$ is at most $r$, which is a contradiction. 

\[\square\]
The following is adapted from the proof of Lemma 2.7. in [3]. The main difference is that we will replace the bounded degree condition used there by the tightness condition (9).

**Proof of Lemma 3.5.** By large-set expansion, there exist $\alpha > 0$ and $\bar{d}$ be such that for all $\epsilon > 0$, with probability tending to 1, $\{G_n\}$ is an $(\alpha, \epsilon, \bar{d})$-large-set expander. Let $r$ be as in (23). Define $V_{k, \Delta}$ as the set of set of vertices such that all vertices in their $k$ neighborhood have degree at most $\Delta$. Given the tightness condition (9), for all $\epsilon > 0$, there exists $\Delta < \infty$ and $N_\epsilon < \infty$ such that for $n \geq N_\epsilon$, with probability $1 - \frac{1}{4}$ we have $\bar{\Delta} \geq 1 - \frac{1}{4}$. Let $A_\epsilon$ be the event that the following conditions hold: $|V_{k, \Delta}| \geq 1 - \frac{1}{4}$, $\phi(G_n, \epsilon) \geq \alpha$, and $G_n$ has average degree at most $\bar{d}$. Increasing $N_\epsilon$ if needed, then for $n \geq N_\epsilon$, $A_\epsilon$ has probability at least $1 - \epsilon/2$.

Fix a $G_n$ such that $\phi(G_n, \epsilon) \geq \alpha$ and $A_\epsilon$ holds. For a vertex $v \in V(G_n)$ let $S'(v, c, n, r)$ be the event that there exists an edge $e$ in the ball $B_r(v)$ such that $S(e, c, n)$ holds. Let $D(v, r)$ be the event that $B_r(v)$ intersects with at least two different connected components of size greater than $cn$. We will use the bound (10) in [3], which states that

$$\mathbb{P}_{G_n(p)} \left( S'(v, c, n, r) \right) \geq q^{2r} \Delta^{-2r^2} \mathbb{P}_{G_n(p)} \left( D(v, r) \right),$$

holds as long as the degree of every vertex in $B_r(v)$ has degree at most $\Delta$, i.e., as long as $v \in V_{r, \Delta}$. On the other hand, for graphs $G_n$ whose average degree is bounded by $\bar{d}$,

$$\frac{1}{n} \sum_{v \in V_{r, \Delta}} \mathbb{P}_{G_n(p)} \left( S'(v, c, n, r) \right) \leq \Delta \frac{1}{n} \sum_{e \in E(G_n)} \mathbb{P}_{G_n(p)} \left( S(e, c, n) \right) \leq 2\bar{d} \Delta r \mathbb{P}_{G_n(p)} \left( S(c, n) \right) \leq 2\bar{d} \Delta r \left( \frac{1}{c} - 1 \right) \frac{\beta}{\sqrt{|E(G_n)|}},$$

where the first inequality follows by a union bound on the edges and the observation that each edge can appear in $r$-neighborhood of at most $\Delta^r$ vertices of $V_{r, \Delta}$. Combining this inequality with (24) we have

$$\frac{1}{n} \sum_{v \in V_{r, \Delta}} \mathbb{P}_{G_n(p)} \left( D(v, r) \right) \leq 2\bar{d} \Delta^{r+2r^2} \left( \frac{1}{c} - 1 \right) \frac{\beta}{q^{2r} \sqrt{|E(G_n)|}}.$$

Given an instance of $G_n(p)$ with two or more components of size larger than $cn$, by $A_\epsilon$, the choice of $r$, and Proposition C.2, the $r$-neighborhood of each of them contains at least $3n/4$ vertices, implying that there are at least $n/2$ vertices with distance $r$ or less from two large components. Thus the event $D(v, r)$ takes place for at least $n/2 - \epsilon n/4 \geq n/4$ nodes $v \in V_{r, \Delta}$. By Markov’s inequality applied to the sum of the indicator functions of $D(v, r)$ over $v \in V_{r, \Delta}$, we therefore get that

$$\mathbb{P}_{G_n(p)} \left( \frac{|C_2|}{n} \geq cn \right) \leq \frac{4}{n} \sum_{v \in V_{2r, \Delta}} \mathbb{P}_{G_n(p)} \left( D(v, r) \right) \leq \frac{C}{\sqrt{|E(G_n)|}},$$

where $C$ is a constant which depends on $\alpha$, $\bar{d}$, $q$ and $\Delta$, but not on $n$ or $p$ (as long as $q \leq p \leq 1 - q$).

Choose $n$ large enough that $A_\epsilon$ holds with probability $\epsilon/2$ and $C / \sqrt{|E(G_n)|} \leq \epsilon/2$. Then we get the lemma by conditioning over all instances of $G_n$ and applying (25) for any instance satisfying $A_\epsilon$ and $\phi(G_n, \epsilon) \geq \alpha$. 

\[\square\]
APPENDIX D: EXPANSION OF PREFERENTIAL ATTACHMENT MODELS

In this appendix, we extend the proof of Theorem 1 in [36] to show that conditional preferential attachment models are good expanders (Lemma 6.2).

The first step is to bound the maximum degree, a step which was needed for the model considered in [36] for reasons that will become clear in the course of our proof. There are stronger bounds on the maximum degree of a vertex in other variations of preferential attachment (see for example Section 4.3 in [20] and Theorem 4.18 in [43]). But Proposition D.2 below is sufficient for our purposes. Before stating the proposition, we state and prove a simple lemma, which will be used in its proof.

**Lemma D.1.** For any sequences \(0 < d_1 \leq d_2 \leq \ldots \leq d_n\) and \(w_1 \geq w_2 \geq \ldots \geq w_n > 0\), then
\[
\sum_{i=1}^{n} \frac{w_i d_i^2}{w_i d_i} \leq \sum_{i=1}^{n} \frac{d_i^2}{d_i}.
\]

**Proof.** The statement is equivalent to
\[
\sum_{i=1}^{n} \sum_{j=1}^{n} w_i d_i^2 d_j \leq \sum_{i=1}^{n} \sum_{j=1}^{n} w_j d_j^2 d_i.
\]
By reordering terms this is equivalent to
\[
0 \leq \sum_{i,j} d_i d_j \left( w_j d_i - w_i d_i + w_i d_j - w_j d_j \right),
\]
which holds since \(w_j - w_i\) and \(d_i - d_j\) have the same sign and
\[
(w_j d_i - w_i d_i + w_i d_j - w_j d_j) = (w_j - w_i)(d_i - d_j) \geq 0.
\]

**Proposition D.2.** Given \(m \geq 2\) and \(n \geq 2m+1\) let \(PA_{m,n}\) be the conditional preferential attachment model defined in Section 6. Let \(M\) be the maximum degree in \(PA_{m,n}\). Then there exists some \(C > 0\) such that
\[
P(M \geq n^{7/8}) \leq e^{-C\sqrt{n}}.
\]

**Proof.** Recall that we start the sequence \((PA_{m,n})_{n \geq t_0}\) with \(K_{2m+1}\) at time \(t_0 = 2m+1\). It has \(mn\) edges at time \(n \geq 2m+1\), giving \(\sum_i d_i(n) = 2mn\). Let \(S_q(n)\) denote the set of sequences \((k_1, \ldots, k_q)\) of pairwise different integers in \([n]\). The probability that the first, second, \ldots, of the \(m\) edges created at time \(n + 1\) attaches to \(m\) different nodes \(k_1, \ldots, k_m \in [n]\) is
\[
P_n^\text{cond}(k_1, \ldots, k_m) = \frac{\prod_{i=1}^{m} d_{k_i}(n)}{\sum_{(k'_1, \ldots, k'_m) \in S_m(n)} \prod_{i=1}^{m} d_{k'_i}(n)}.
\]

Define
\[
w_i(n) = \sum_{(k_2, \ldots, k_m) \in S_{m-1}(n) \atop k_j \neq i, j} \prod_{j=2}^{m} d_{k_j}(n).
\]

For \(s = 1, \ldots, m\), the marginal the \(s\)th new edge connecting to vertex \(i\) is then equal to
\[
p_n(i) = \frac{d_i(n) w_i(n)}{\sum_{j=1}^{m} d_j(n) w_j(n)}.
\]
Let $Q_n = \sum_i d_i^2(n)$. Note that $d_{\max}^2(n) \leq Q_n$. So, it is enough to bound $Q_n$. For that purpose, we use concentration for super-martingales. First, note that

$$E[Q_{n+1} | PA_{m,n}] = Q_n + 2 \sum_{k_1 \ldots k_m} P_{c\text{ond}}(k_1, \ldots, k_m)(d_{k_1}(n) + \cdots + d_{k_m}(n)) + m + m^2$$

$$= Q_n + 2m \sum_i w_i(n) d_i^2(n) + m(m + 1).$$

Note that if $d_i(n) \geq d_j(n)$ then $w_i(n) \leq w_j(n)$. Then by Lemma D.1,

$$E[Q_{n+1} | PA_{m,n}] \leq Q_n + 2m \sum_i d_i^2(n) + m(m + 1) = Q_n(1 + \frac{1}{n}) + m(m + 1).$$

Let $S_n = \frac{Q_n}{n} - m(m + 1)(\sum_{i=2}^n \frac{1}{i})$, then $S_n$ is a supermartingale. Also, since all $d_i$ start at $2m$ and can grow by at most one in each step, we have $d_i(n) \leq n - 1$. Using this fact, we easily see that $Q_{n+1} \leq Q_n + 2m(n - 1) + m(m + 1)$ and hence

$$0 \leq \frac{Q_{n+1} - Q_n - m(m + 1)}{n + 1} \leq S_{n+1} - S_n = \frac{Q_{n+1}}{n + 1} - \frac{Q_n}{n} - \frac{m(m + 1)}{n + 1} \leq 2m,$$

showing that $|S_{n+1} - S_n| \leq 2m$. Finally,

$$E[S_n] \leq E[S_{2m+1}] = 4m^2.$$

So, we can use the Azuma-Hoeffding inequality to get that

$$\mathbb{P}(S_n \geq \lambda + 4m^2) \leq e^{-\frac{\lambda^2}{32m^2}}.$$

Then

$$\mathbb{P}(Q_n \geq \lambda n^{3/2} + 4mn^2 + m(m + 1)n \log \frac{n}{2m}) \leq e^{-\frac{\lambda^2}{32m^2}}.$$

Since $M^2 \leq Q_n$ we get that there exists a constant $C > 0$ such that

$$\mathbb{P}(M \geq n^{7/8}) \leq \mathbb{P}(Q_n \geq n^{7/4}) \leq e^{-C\sqrt{n}}.$$

\[\square\]

**Lemma D.3.** Consider the conditional preferential attachment model specified in Proposition D.2, let $M_t$ be the maximal degree at time $t$, and let $P_{c\text{ond}}(k_1, \ldots, k_m)$ be the conditional probability from (26). Then

$$P_{c\text{ond}}(k_1, \ldots, k_m) \leq 2^{(m-1) \frac{M_t}{2}} \prod_{i=1}^m \frac{d_{k_i}(t)}{2mt}.$$ 

**Proof.** With the definitions from the previous proof, let

$$Z = \sum_{(k_1, \ldots, k_m) \in S_m(t)} \prod_{i=1}^m d_{k_i}(t).$$

Then

$$Z = \sum_{(k'_2, \ldots, k'_m) \in S_{m-1}(n)} \left( \prod_{i=2}^m d_{k'_i}(t) \right) \sum_{k'_1 \notin \{k'_2, \ldots, k'_m\}} d_{k'_1}(t) \geq \left(2mt - (m - 1)M_t\right) \sum_{(k'_2, \ldots, k'_m) \in S_{m-1}(t)} \left( \prod_{i=2}^m d_{k'_i}(t) \right).$$
Since \( M_t \leq t - 1 \), we have that

\[
2mt - (m - 1)M_t \geq \left(1 - \frac{M_t}{2t}\right) 2mt \geq 4^{-\frac{Mt}{2t}} 2mt = 2^{-\frac{Mt}{2t}} 2mt.
\]

Repeating this process, we get that

\[
Z \geq 2^{-(m-1)\frac{Mt}{t}} (2mt)^m.
\]

The lemma follows. \(\square\)

Now, we are ready to prove Lemma 6.2.

**Proof of Lemma 6.2.** We will prove the lemma for \( \alpha = \frac{m-1}{20} \). Our proof will follow the general strategy of the expansion bound in [36], but requires several modifications - the main one stemming from the fact that the conditioning in the conditional model considered here will result in a factor growing exponentially in the largest degree, see (27) below, which differs from Lemma 2 in [36] by the factor \( C^{n^{7/5} \log n} \). We will offset this factor by an extra exponential decay stemming from the fact we only consider large set expanders, whereas [36] proved expansion for sets which can be arbitrary small.

Let \( V = V(PA_{m,n}) \) and \( E = E(PA_{m,n}) \) be the set of vertices and edges of the graph, respectively. Vertices are indexed based on their arrival time, with the first \( 2m + 1 \) vertices ordered in an arbitrary way. Recall that when the vertex \( t + 1 \) arrives, it attaches \( m \) edges to \( m \) distinct old vertices according to (26); we assign indices \( mt + 1, \ldots, m(t + 1) \) to these edges and call them their *arrival index*; for the edges in the original graph \( K_{2m+1} \), we choose the indices arbitrary between 1 and \( m(2m + 1) \), subject to the constraint that the \( m \) edges between a vertex \( t \) and a vertex of lower index lie between \( (t-1)m + 1 \) and \( tm \). We use \( E' = E \setminus E(PA_{m,2m+1}) \) to denote the set of edges with index larger than \( m(2m + 1) \).

Consider a set \( S \) of size \(|S| = k \) with \( \epsilon n \leq k \leq \frac{7}{8} \), and call an edge good if it lies in \( e(S, V \setminus S) \), and bad otherwise. We need to show that for each such \( S \) there are at least \( \lceil \alpha k \rceil \) good edges. Indeed, we will show something slightly stronger, namely that for each such \( S \), there are at least \( \lceil \alpha k \rceil \) good edges in \( E' \). To do so, we will show that with high probability, for any set \( S \), and any set of edge-indices \( A \subset E' \) of size \(|A| \leq k_{\alpha} = \lfloor \alpha k \rfloor - 1 \), there must be at least one good edge in \( E' \setminus A \), i.e., we will show that for \( n \geq n_0 \)

\[
\mathbb{P}(\text{all } e \in E' \setminus A \text{ are bad}) \leq C^{n^{7/5} \log n} \frac{(mn)}{(mk-k_{\alpha})},
\]

where \( n_0 < \infty \) and \( C > 0 \) are constants which depends on \( m, \alpha \), and \( \epsilon \). The proof of this bound is adapted from that of Lemma 2 in [36], see above.

Assuming (27), we first prove the lemma. Given a set \( S \) of size \( k \) with \( 0 \leq k' \leq k_{\alpha} \) good edges in \( E' \) there are \( k_{\alpha} \) choices for \( k' \), and at most \( \binom{mn}{k_{\alpha}} \leq \binom{mn}{k_{\alpha}} \) choices for the set of good edges \( A \). Noting that there are \( \binom{n}{k_{\alpha}} \) choices for sets \( S \) of size \( k \), we get that

\[
\mathbb{P}(\phi(G, \epsilon) < \alpha) \leq C^{n^{7/5} \log n} \sum_{k = \lceil \epsilon n \rceil}^{n/2} (k_{\alpha} + 1) \frac{n}{k} \binom{mn}{k_{\alpha}} \frac{(mk)}{(mn-k_{\alpha})} \frac{(k_{\alpha})}{(k_{\alpha}-k_{\alpha})}
\]

\[
\leq C^{n^{7/5} \log n} \sum_{k = \lceil \epsilon n \rceil}^{n/2} (k_{\alpha} + 1) \binom{km}{k_{\alpha}} \frac{2k_{\alpha}}{k_{\alpha}} \frac{\binom{kn}{k_{\alpha}}}{\binom{mn-k_{\alpha}}{k_{\alpha}}}
\]

\[
\leq C^{n^{7/5} \log n} \sum_{k = \lceil \epsilon n \rceil}^{n/2} (2k_{\alpha})^{8} \frac{2k_{\alpha}}{\binom{mn}{k_{\alpha}}} \frac{1}{2} \binom{(m-1-2\alpha)k}{k_{\alpha}}
\]
\[ \leq 2\alpha n^2 C^{n^{7/8}\log n} \left( \left( \frac{en}{\alpha} \right)^{2\alpha} \left( \frac{1}{2} \right)^{(m-1-2\alpha)} \right)^{en} \]

provided \( n \) is large enough to guarantee that \( \alpha en \geq 2 \). Here the second step follows from the fact that \( \binom{\binom{m-1}{k}}{\binom{m-k}{k}} \leq \binom{m-k}{mk-k} \) combined with standard bounds on binomial coefficients, the third step follows from \( \left( \frac{k}{k_\alpha} \right) \leq \frac{1}{\alpha} \left( 1 - 1/(k_\alpha) \right)^{-1} \leq \frac{1}{\alpha} \frac{1}{1/\left( k_\alpha \right)} \) if \( k \geq en \geq 2/\alpha \), and the third follows from \( k \leq n/2 \). By the choice of \( \alpha \), this bound is of the form

\[ \Pr (\phi(G, e) < \alpha) \leq 2\alpha n^2 C^{n^{7/8}\log n} \beta^en, \]

for some constant \( \beta < 1 \) and drops exponentially fast as \( n \) grows. This reduces the proof of the lemma to the bound (27).

To prove this bound, we first note that the left is a monotone function of \( A \) with respect to inclusion, showing that it is enough to prove the bound for \( |A| = k_\alpha \). Let \( S = V \setminus S \), and let \( B \) be the event that all edges in \( E' \setminus A \) are bad. The event \( B \) is then the intersection of the events \( B_t, t = 2m + 2, \ldots, n \), where \( B_t \) is the event that all edges in \( E' \setminus A \) whose arrival index lies between \( (t-1)m+1 \) and \( tm \) are bad (corresponding to the edges in \( E' \setminus A \) whose younger endpoint is the vertex \( t \)). We will want to bound the probability of the event \( B_t \), conditioned on the graph at time \( t - 1 \). In fact, using the identity

\[ \Pr (B) = \left( \prod_{t=2m+3}^{n} \Pr (B_t | B_{t-1} \cap \cdots \cap B_{2m+2}) \right) \Pr (B_{2m+2}), \]

we will further assume that the graph at time \( t - 1 \) is such that all edges with arrival index between \( m(2m+1)+1 \) and \( m(t-1) \) are bad.

We need some notations. Assume the number of indices in \( A \) corresponding to vertices in \( S(S) \) is \( k_1, k_2 \). So, \( |A| = k_1 + k_2 \). Let \( x_1 < x_2 < \ldots < x_{mk-k_1} \) be the arrival indices of edges in \( E' \setminus A \), such that their younger endpoints is in \( S \), and let \( \bar{x}_1 < \bar{x}_2 < \ldots < \bar{x}_{mn-mk-k_2} \) be those whose younger endpoint lies in \( \bar{S} \). If \( t \in S \), let \( x_{i_1+1}, \ldots, x_{i_1+b_i} \) be the arrival indices \( x_i \) such that \( (t-1)m+1 \leq x_i \leq tm \), and similarly for \( \bar{x}_{i_1+1}, \ldots, \bar{x}_{i_1+b_i} \) if \( t \in S \). Then \( B_t \) is the event that for all these indices, the second endpoint lies in \( S \) if \( t \in S \), and in \( S \) if \( t \in \bar{S} \). Let \( d_S(t) \) be the total degree of nodes in \( S \) at the time \( t \), and consider the case \( t \in S \). Then by Lemma D.3,

\[ \Pr (B_t | PA_{m,t-1}) \leq \frac{d_S(t-1)^{b_i}}{2m(t-1)^{b_i}} 2^{(m-1)\frac{b_i}{t}}. \]

Let \( g_t \) be the number of good edges before node \( t \), let \( z_i \) be the number of edges in \( E' \setminus A \) with an index less than \( x_i \), and let \( \bar{z}_i \) be the of edges in \( E' \setminus A \) with an index less than \( \bar{x}_i \). Note that \( z_i \) is bounded from below by the number of edges in \( E' \setminus A \) with an index less than \( x_i \) such that their younger endpoint is in \( S \), i.e., \( z_i \geq (i-1) \), and similarly, \( \bar{z}_i \geq i - 1 \).

By definition of the indices \( i, j, j \in [b_i] \), the number of edges \( E \setminus A \) with both endpoints in \( S \) that appeared before time \( t \) is \( i_1 \). Since \( d_S(t-1) \) is equal to the number of good edges plus twice the number of bad edges with at least one endpoint in \( S \) that have appeared so far, we get that \( d_S(t) \leq g_t + 2i_1 \). To bound the denominator \( m(t-1) \), we note that the total degree of the graph at time \( t - 1 \) is equal to twice the number of good edges plus twice the number of bad edges seen before node \( t \). Recall that we assumed that the events \( B_s \) hold for \( s = 2m+2, \ldots, t-1 \). Therefore, all edges in \( E' \setminus A \) with arrival index between \( m(2m+1)+1 \) and \( m(t-1) \) are bad. Since \( x_{i_1+j} \leq tm \) for \( j \in [d], \) we know that \( z_{i_1+j} - m \) is a lower bound on the number of edges in \( E' \setminus A \) arriving before \( t \). Therefore \( m(t-1) \geq 2g_t + 2z_{i_1+j} - m - 2m(2m+1) \). Hence, for any \( j \in [b_t] \),

\[ \frac{d_S(t-1)}{2m(t-1)} \leq \frac{2i_1 + g_t}{2z_{i_1+j} + 2g_t - 2m(2m+2)} \leq \frac{2i_1 + 2j + g_t}{2z_{i_1+j} + g_t - 2m(2m+2)}. \]
Note that the number of good edges in the graph is at most $|A|$ plus the edges in $PA_{m,2m+1}$, $g_t \leq |A| + m(2m+1)$, we conclude that

$$d_S(t) \leq \frac{i_1 + j + |A| + m(2m+1)}{2m(t-1)}$$

where the second inequality follows from the fact that $|A| = k_\alpha \geq \alpha \epsilon n - 1 \geq m(2m+1)$ for large enough $n$. As a consequence,

$$\Pr(B_t \mid PA_{m,t-1}) \leq 2^{(m-1)\frac{\alpha t}{2}} \prod_{i=1}^{b_t} \frac{i_1 + j + |A|}{z_i + |A| - m(2m+2)}.$$

We can get a similar bound if the vertex $t$ is in $S$.

We will want to use these bounds starting with $t = 2m+3$. Defining $i_0 = \min\{i : x_i \geq 1 + (2m+2)m\}$ and $\tilde{i}_0 = \min\{i : \bar{x}_i \geq 1 + (2m+2)m\}$, bounding $\Pr(B_{2m+2})$ by 1 and $M_t$ by $t^{7/\alpha} \leq n^{7/\alpha}$, we thus get that

$$\Pr(E \setminus A \text{ is bad}) = \Pr(B_{2m+2} \cap \ldots \cap B_n) \leq 2^{n^{7/\alpha} \log n} \prod_{i=i_0}^{mn-\frac{m-1}{k_\alpha} - 1} \frac{i + |A|}{z_i + |A| - m(2m+2)} \prod_{i=\tilde{i}_0}^{mn-\frac{m-1}{k_\alpha} - 1} \frac{i + |A|}{\bar{z}_i + |A| - m(2m+2)},$$

(28)

where in the last step we used that $\frac{|A| + i}{z_i + |A| - m(2m+2)} \geq \frac{|A|}{|A| + mn} \geq \frac{1}{mn}$. Note that $x_i \geq i$ and $\bar{x}_i \geq i$, which implies that $i_0 + \tilde{i}_0 \geq 2 + 2m(2m+2)$, and hence $(mn)^{i_0 + \tilde{i}_0} \leq 2^c \log n$ for some constants $c$ depending on $m$.

Recall that $z_i$ (and $\bar{z}_i$) is the number of edges in $E \setminus A$ that appear before $x_i$ (or $\bar{x}_i$). So $z_1 < z_2 < \ldots < z_{mn-\frac{m-1}{k_\alpha}}$, $\bar{z}_1 < \bar{z}_2 < \ldots < \bar{z}_{mn-\frac{m-1}{k_\alpha}}$, and $z_i \neq \bar{z}_j$ for all $i \leq \frac{m-1}{k_\alpha}$ and $j \leq \frac{m-1}{k_\alpha} - 1$. As a result,

$$\{z_1, \ldots, z_{\frac{m-1}{k_\alpha}}\} \cup \{\bar{z}_1, \ldots, \bar{z}_{mn-\frac{m-1}{k_\alpha}}\} = \{0, \ldots, mn - k_\alpha - 1\}.$$

Using this and the fact that $|A| = k_\alpha = k_1 + k_2$, we get

$$\Pr(E \setminus A \text{ is bad}) \leq 2^{(c+n^{7/\alpha}) \log n} \frac{(mn-\frac{m-1}{k_\alpha} + k_2)!((mn-\frac{m-1}{k_\alpha} + k_1)!(mn)!}{|A|!((mn)!(\prod_{i=1}^{mn-\frac{m-1}{k_\alpha} - 1} |A| - i)!}$$

$$= 2^{(c+n^{7/\alpha}) \log n} \frac{(mn-\frac{m-1}{k_\alpha} + k_2)!((mn-\frac{m-1}{k_\alpha} + k_1)!}{|A|!((mn)!(\prod_{i=1}^{mn-\frac{m-1}{k_\alpha} - 1} |A| - i)!}$$

$$\leq 2^{(c+n^{7/\alpha}) \log n} \frac{(mn-\frac{m-1}{k_\alpha} + k_2)!((mn-\frac{m-1}{k_\alpha} + k_1)!}{|A|!((mn-\frac{m-1}{k_\alpha} + k_2)!((mn-\frac{m-1}{k_\alpha} + k_2)! |A| - i)!}$$

$$\leq 2^{(c+n^{7/\alpha}) \log n} \frac{(mn-\frac{m-1}{k_\alpha} + k_2)!((mn-\frac{m-1}{k_\alpha} + k_1)!}{|A|!((mn-\frac{m-1}{k_\alpha} + k_2)!((mn-\frac{m-1}{k_\alpha} + k_2)! |A| - i)!}$$

provided $n \geq \frac{1+2m(2m+2)}{\alpha \epsilon}$. Here the last bound follows from $|A| = \lceil \alpha k \rceil - 1 \geq \alpha n - 1$. Thus, we have (27) for $C = 4 \left(\frac{2m}{\alpha \epsilon}\right)^{m(2m+2)}$.  

APPENDIX E: PROPERTIES OF THE PÓLYA-POINT GRAPH

In this appendix, we will prove Proposition 6.3, as well as the representation (17) of $\zeta(p)$ for the Pólya-point process. As discussed in the paragraphs preceding Proposition 6.3, this requires us to understand the distribution of the numbers of left and right children of a vertex with a given label $(S, x)$ after percolation, $d^p_L$ and $d^p_R$, respectively. While the distribution of the first kind is just Bin$(m(S), p)$, the second one requires integration out the degree distribution in the Pólya-point process. This leads to the following lemma.

**PROPOSITION E.1.** Given a node of type $(S, x)$ from the Pólya-point graph, where $S \in \{R, L\}$, the degree distribution of off-springs of type $R$ after percolation is

$$\P(d^p_R = k | (S, x)) = q^{m(S)+1}(1 - q)^k \binom{k + m(S)}{k},$$

where $q = \frac{x}{x + p - xp}$. Similarly, for the root conditioned on its position $x$,

$$\P(d^p_R = k | (\emptyset, x)) = q^m(1 - q)^k \binom{k + m - 1}{k}.$$

**PROOF.** Let $d_R$ be the random variable giving the number of off-springs of type $R$ before percolation. Conditioned on $(S, x)$, the distribution of $d_R$ is a mixed Poisson with parameter $\gamma \lambda(x)$, where $\lambda(x) = \frac{1 - x}{x}$ and $\gamma \sim \Gamma(\tilde{m}, 1)$, with $m = m(S) + 1$ if $S \in \{S, R\}$ and $\tilde{m} = m$ if $S = \emptyset$. If we integrate over $\gamma$, we get

$$\P(d_R = k | x) = \int_0^{\infty} e^{-y\lambda(x)} \frac{(y\lambda(x))^k}{k!} \frac{y^{\tilde{m}-1}}{(\tilde{m} - 1)!} e^{-y} dy$$

$$= \frac{\lambda(x)^k}{(1 + \lambda(x))^{k+\tilde{m}}} \frac{(k + \tilde{m} - 1)!}{k!(\tilde{m} - 1)!} = \binom{k + \tilde{m} - 1}{k} x^{\tilde{m}} (1 - x)^k,$$

which is a negative binomial distribution with parameters $x$ and $\tilde{m}$. See also Lemma 5.2 [9], where this distribution was derived as well.

Now, we are ready to find the distribution of $d^p_R$:

$$\P(d^p_R = k | x) = \sum_{d \geq k} \binom{d}{k} p^k (1 - p)^{d-k} \P(d_R = d | x)$$

$$= x^{\tilde{m}} (p(1 - x))^k \sum_{d \geq k} \binom{d}{k} ((1 - p)(1 - x))^{d-k} \binom{d + \tilde{m} - 1}{d}$$

$$= x^{\tilde{m}} (p(1 - x))^k \binom{k + \tilde{m} - 1}{k} \sum_{d \geq k} ((1 - p)(1 - x))^{d-k} \binom{d + \tilde{m} - 1}{d - k}$$

$$= x^{\tilde{m}} (p(1 - x))^k \binom{k + \tilde{m} - 1}{k} \frac{1}{(1 - (1 - x)(1 - p))^{k+\tilde{m}}}$$

$$= \binom{x}{x + p - xp}^{\tilde{m}} (1 - \frac{x}{x + p - xp})^k \binom{k + \tilde{m} - 1}{k},$$

which is again a negative binomial distribution with parameters $\tilde{m}$ and $\frac{x}{x + p - xp}$. \hfill \Box

To compute $\zeta(p)$ for the Pólya-point graph, we derive the implicit formula for the survival probability of a node of type $(S, x)$ given in Proposition 6.3. Given a node of type $(S, x)$,
uniformly at random from \( \{0 \} \). First, on left neighbors similarly. Since the left and right neighbors given \( x \) and their children, and the degrees \( d_R^p \) and \( d_L^p \), and thus will lead us to consider expectations of the form given in the next lemma.

**Lemma E.2.** Fix \( p \in [0,1] \), a measurable function \( g \) on \( \{0,R,L\} \times [0,1] \), and a type \((S,x)\). Let \( d_R^p \) and \( d_L^p \) be defined as above, and let \( E[\cdot | (S,x)] \) denote expectations with respect to the forward degrees \( d_R^p \) and \( d_L^p \) of \((S,x)\), and the positions \( x_i \) of its children. If \( S \in \{R,L\} \), then

\[
E[\prod_{i=1}^{d_L^p} (1-g(L,x_i)) \prod_{j=1}^{d_R^p} (1-g(R,y_j)) | (S,x)] = \frac{x^{m(S)+1}(1-p(T_Lg)(x))^{\frac{m(S)}{N}}}{x+p(1-x)(T_Rg)(x)^{m(S)+1}},
\]

and if \( S = \emptyset \), then

\[
E[\prod_{i=1}^{d_L^p} (1-g(L,x_i)) \prod_{j=1}^{d_R^p} (1-g(R,y_j)) | (\emptyset,x)] = \left( \frac{x(1-p(T_Lg)(x))}{x+p(1-x)(T_Rg)(x)} \right)^m,
\]

where \((T_Rg)(x) = \frac{1}{n-x} \int_x^n g(R,y)dy \) and \((T_Lg)(x) = \frac{1}{x} \int_0^x g(L,y)dy \).

Note that the right hand side of the first identity is just \( 1 - \Phi(g) \), with \( \Phi \) as defined in (16).

**Proof.** We give the proof for the case that \( S \in \{R,L\} \), and the other case can be derived similarly. Since the left and right neighbors given \( x \) are independent, we calculate the expectation independently. First, on left neighbors conditioning on \( d_L^p, x_1, \ldots, x_{d_L^p} \) are generated uniformly at random from \([0, x]\). Then

\[
E[\prod_{i=1}^{d_L^p} (1-g(x_i)) | d_L^p = N, (S,x)] = \prod_{i=1}^{N} (1 - \mathbb{E}g(x_i)) = (1 - (T_Lg)(y))^N
\]

Therefore,

\[
E[\prod_{i=1}^{d_L^p} (1-g(x_i)) | (S,x)] = \sum_{N=0}^{m(S)} (1 - (T_Lg)(x))^N \binom{m(S)}{N} p^N (1-p)^{m(S)-N} = (1 - (T_Lg)(x)p)^{m(S)}.
\]

The position of right off-springs conditioned on \( d_R^p \) is i.i.d. uniformly at random from \([x, 1]\). Hence, similarly

\[
E[\prod_{i=1}^{d_R^p} (1-g(x_i)) | (S,x)] = \sum_{N \geq 0} (1 - (T_Rg)(x))^N \mathbb{P}(d_R^p = N | (S,x))
\]

\[
= \sum_{N \geq 0} (1 - (T_Rg)(x))^N \binom{N + m(S)}{N} q^{m(S)+1}(1-q)^N
\]

\[
= q^{m(S)+1} \frac{1}{(1 - (1 - (T_Rg)(x))(1-q))^{m(S)+1}} = \left( \frac{x}{x+p(1-x)(T_Rg)(x)} \right)^{m(S)+1},
\]

where in the second step we used Proposition E.1, and in the last step we used that \( q = \frac{x}{x+p-xp} \). The product of right and left off-springs gives the statement. \( \square \)
PROOF OF PROPOSITION 6.3.

1. For a vertex of type \((S, x)\), the probability that it does not reach level \(k\) is equal to the probability that none of its children reach level \(k - 1\). Combined with Lemma E.2, we therefore have that

\[
\rho_k(S, x) = 1 - \mathbb{E}[\prod_{i=1}^d (1 - \rho_{k-1}(L, x_i)) \prod_{j=1}^d (1 - \rho_{k-1}(R, y_j))|S, x)] = \Phi \rho_{k-1}(S, x).
\]

Given that all nodes reach level 0, \(\rho_0 = 1\), this implies the first statement of the proposition.

2. By monotone convergence, \(\rho_k(S, x) \downarrow \rho(S, x)\) for all \((S, x) \in \{L, R\} \times [0, 1]\), so \(\lim_{k \to \infty} (\Phi^k)1(S, x) = \rho(S, x)\). Then by dominated convergence we get that \(\rho\) is a fixed point of (15). Let \(f\) be another solution of (15). For any solution we have that \(f(S, x) = \Phi f(S, x) \leq 1\). Therefore, for any \(k\),

\[
f(S, x) = \Phi^k f(S, x) \leq \Phi^k 1(S, x) = \rho_k(S, x).
\]

As a result, \(f(S, x) \leq \rho(S, x)\).

Note that Lemma E.2, together with the fact that the position of the root is \(\sqrt{x}\) for \(x\) chosen uniformly at random from \([0, 1]\), immediately implies (17). We close this appendix with the proof of Proposition 6.4.

PROOF OF PROPOSITION 6.4. We start with some simple observations which we will use throughout the proof. First, we note that if \(0 \leq g \leq g' \leq 1\) point-wise, then

\[
0 = \Phi(0) \leq (\Phi g)(S, x) \leq (\Phi g')(S, x) \leq (\Phi 1)(S, x) \leq 1.
\]

Next, by dividing both the numerator and the denominator in the expressions for \(1 - (\Phi f)(S, x)\) in (16) and in (17) by \(x^{m(S)+1}\) and \(x^{m}\), respectively, we see that

\[
1 - \left(1 - \frac{\int_0^x f(L, z)dz}{1 + \int_0^x f(R, z)dz}\right)^{m-1} \leq (\Phi f)(R, x) \leq (\Phi f)(0, x)
\]

(29)

\[
\leq (\Phi f)(L, x) \leq 1 - \left(1 - \frac{\int_0^x f(L, z)dz}{1 + \int_0^x f(R, z)dz}\right)^{m+1}
\]

To prove upper and lower bounds on \(\rho(S, x)\), we then use that \(\rho(S, x)\) is the pointwise monotone limit, \(\rho_k(S, x) \downarrow \rho(S, x)\), where \(\rho_k = \Phi^k 1\), see Proposition 6.3 and its proof.

For the upper bound, we will inductively bound \(\rho_k\) from above by functions \(f_k\) that don’t depend on the discrete variable \(S\). Assume thus that \(f\) is of this form. Then

\[
(\Phi f)(S, x) \leq 1 - \left(1 - \frac{\int_0^x f(y)dy}{1 + \frac{p}{x} \int_0^1 f(y)dy}\right)^{m+1}
\]

\[
= 1 - \left(1 - \frac{\int_0^x f(y)dy}{1 - \frac{p}{x} \int_0^1 f(y)dy + \frac{p}{x} \int_0^1 f(y)dy}\right)^{m+1}
\]

\[
\leq 1 - \left(1 - \frac{1 - p}{1 - p + \frac{p}{x} \int_0^1 f(y)dy}\right)^{m+1} = 1 - \left(\frac{1}{1 + \frac{p}{x(1-p)} \int_0^1 f(y)dy}\right)^{m+1}.
\]

where in the second bound we used that \(\frac{1}{x} \int_0^x f(y)dy \leq 1\). As a consequence,

\[
\rho(S, x) \leq \rho_k(S, x) \leq f_k(x) \quad \text{where} \quad f_k(x) = 1 - \left(\frac{1}{1 + \frac{p}{x}}\right)^{m+1}
\]
and \( \epsilon_k \) is inductively defined by 
\[
\epsilon_1 = \frac{p}{1-p} \int_0^1 1 = \frac{p}{1-p}
\] 
and
\[
\epsilon_{k+1} = F(\epsilon_k) \quad \text{where} \quad F(\epsilon) = \frac{p}{1-p} \int_0^1 \left(1 - \frac{1}{1 + \frac{1}{x}}\right)^{m+1} dx.
\]

The function \( F : [0, \frac{p}{1-p}] \to [0, \frac{p}{1-p}] \) is monotone increasing and concave, with \( F(0) = 0 \) and \( F\left(\frac{p}{1-p}\right) < \frac{p}{1-p} \), showing that it has two fix-points, the trivial fix-point 0 and another fix-point \( \epsilon^+ > 0 \), with the latter giving the limit \( \epsilon^+ = \lim_{k \to \infty} \epsilon_k \) and the upper bound
\[
\rho(S, x) \leq f_+(x) = 1 - \left(\frac{1}{1 + \frac{\epsilon^+}{x}}\right)^m.
\]

To convert this into an upper bound on \( \zeta(p) \), we first bound
\[
(\Phi p)(\emptyset, x) \leq (\Phi f_+)(\emptyset, x) \leq 1 - \left(\frac{1}{1 + \frac{p}{x(1-p)} \int_0^1 f_+(y) dy}\right)^m = 1 - \left(\frac{1}{1 + \frac{\epsilon^+}{x}}\right)^m,
\]
which gives
\[
\zeta(p) \leq \int_0^1 \left(1 - \left(\frac{1}{1 + \frac{\epsilon^+}{x}}\right)^m\right) 2x dx \leq \int_0^1 \left(\frac{m \epsilon^+}{x}\right) 2x dx = 2m \epsilon^+.
\]

Next we use that
\[
F(\epsilon) \leq \frac{p}{1-p} \int_0^\epsilon dx + \frac{p}{1-p} \int_\epsilon^1 \left((m+1)\frac{\epsilon}{x}\right) dx = \epsilon \frac{p}{1-p} \left(1 + (m+1) \log\left(\frac{1}{\epsilon}\right)\right),
\]
implicating that the non-trivial fix-point of \( F \) obeys the bound \( 1 \leq \frac{p}{1-p}(1 + (m+1) \log(1/\epsilon^+)) \).

This in turn implies that \( \epsilon^+ \leq e^{-\frac{1}{m+1}} \), giving the desired upper bound on \( \zeta(p) \).

In a similar way, one can obtain lower bounds on \( \rho_k \), and thus on \( \rho = \lim_{k \to \infty} \rho_k \). All that changes is that the power \( m+1 \) in our upper bound now becomes a power \( m-1 \), and the upper bound \( \frac{1}{x} \int_0^x f(y) dy \leq 1 \) gets replaced by the lower bound \( \frac{1}{x} \int_0^x f(y) dy \geq 0 \). The resulting lower bound is of the form
\[
\rho(S, x) \geq f_-(x) = 1 - \left(\frac{1}{1 + \frac{\epsilon^-}{x}}\right)^{m-1},
\]
where \( \epsilon^- > 0 \) is the non-trivial fix-point of the function \( \tilde{F} : [0, p] \to [0, p] \) defined by
\[
\tilde{F}(\epsilon) = p \int_0^1 \left(1 - \left(\frac{1}{1 + \frac{\epsilon^-}{x}}\right)^{m-1}\right) dx.
\]

Next, we use the fact that \( (1 + \frac{\epsilon^-}{x})^{m-1} \geq 1 + (m-1)\frac{\epsilon^-}{x} \) to bound
\[
\rho(S, x) \geq 1 - \frac{1}{1 + (m-1)\frac{\epsilon^-}{x}} = \frac{(m-1)\epsilon^-}{(m-1)\epsilon^- + x} \geq \frac{(m-1)\epsilon^-}{(m-1)\epsilon^- + 1}.
\]

By the same reasoning, we may bound \( \tilde{F} \) from below by
\[
\tilde{F}(\epsilon) \geq p \int_0^1 \frac{(m-1)\epsilon^-}{(m-1)\epsilon^- + x} dx = p(m-1)\epsilon \log \frac{1 + (m-1)\epsilon}{(m-1)\epsilon}.
\]

showing that \( \epsilon^- \) is bounded from below by the solution of \( 1 = p(m-1) \log \frac{1 + (m-1)\epsilon}{(m-1)\epsilon} \), which inserted into (30) gives \( \rho(S, x) \geq e^{-\frac{1}{m-1}} \), as claimed. Inserted into (17), this also gives the lower bound on \( \zeta(p) \).