Tail estimates for sums of variables sampled by a random walk

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Abstract

We prove tail estimates for variables $\sum_i f(X_i)$, where $(X_i)_i$ is the trajectory of a random walk on an undirected graph (or, equivalently, a reversible Markov chain). The estimates are in terms of the maximum of the function $f$, its variance, and the spectrum of the graph. Our proofs are more elementary than other proofs in the literature, and our results are sharper. We obtain Bernstein and Bennett-type inequalities, as well as an inequality for subgaussian variables.

1 Introduction

One of the basic concerns of sampling theory is economising on the ‘cost’ and quantity of samples required to estimate the average of random variables. Drawing states by conducting a random walk is often considerably ‘cheaper’ than the standard Monte-Carlo procedure of drawing independent random states. Despite the loss of independence when sampling by a random walk, the empirical average may converge to the actual average at a comparable rate to the rate of convergence for independent sampling (depending, of course, on the structure of the specific random walk). This approach plays an important role in statistical physics and in computer science (a concise summary of applications is provided in [WX]). Results concerning the rate of convergence of empirical averages sampled by a random walk have been obtained by several authors in [G, D, L, LP] (for vector and matrix valued functions consult [K] and [WX]). Of these, only [L] and [K] allowed the variance to play a role in their estimates.

This paper is a further step in this direction. We improve known Bernstein-type inequalities, and prove a new Bennett-type inequality and a new inequality for subgaussian variables. Our methods are much more elementary than the ones prevailing in the literature, as we do not apply Kato’s perturbation theory for eigenvalues. Our results are sharper for the case of graphs with large spectral gaps (expanders) and tails which go far beyond the variance (large deviations). This regime often features in applications such as the recent [AM].
2 The results

Let $G$ be a finite undirected, possibly weighted, connected graph with $N$ vertices (random walks on such graphs can represent any finite irreducible reversible Markov chain). Denote by $s$ the stationary distribution of the random walk on the graph. Let $f$ be a function on the vertices of $G$, normalised to have maximum 1 and mean 0 relative to the stationary distribution, namely $\sum_i f(i) s(i) = 0$. Let $V = \sum_i f^2(i) s(i)$ denote the variance of $f$ with respect to the stationary distribution. We will think of functions on $G$ as vectors in $\mathbb{R}^N$ and vice versa, so where $u$ and $v$ are vectors, expressions such as $e^u$ and $uv$ will stand for coordinatewise operations.

Denote by $P$ the transition matrix of the random walk, such that $P_{ij}$ is the probability of moving from node $j$ to node $i$. By the Perron-Frobenius theorem the eigenvalues of this matrix are all real, the top eigenvalue is 1 (with $s$ as the only corresponding eigenvector up to scalar multiplication), and the absolute value of all other eigenvalues is smaller or equal to 1. Let $\alpha < 1$ be the second largest eigenvalue of $P$, and $\beta \leq 1$ the second largest absolute value of an eigenvalue of $P$.

Given a starting distribution $q$, the random variables $X_0, X_1, \ldots$ will denote the trajectory of the random walk. $P_q$ and $E_q$ will stand for the probability and expectation of events related to this walk respectively. Let $S_n = \sum_{i=1}^n f(X_i)$. Our concern in this paper is tail estimates for the distribution of $S_n$.

We will prove inequalities in terms of both $\alpha$ and $\beta$. Note that inequalities in terms of $\alpha$ ‘cost’ an additional multiplicative factor outside the exponent, whereas inequalities in terms of $\beta$ are useless in the case of $\beta = 1$ (i.e. bipartite graphs), and become relatively poor if $\alpha$ is small and $\beta$ is large, which may be the case.

**Theorem 1.** In the above setting,

\[
P_q\left(\frac{1}{n}S_n > \gamma\right) \leq \min_{1 \geq \beta^2 e^{2r} + \beta^2 V(e_r-1)^2} \left\| \frac{q}{\sqrt{s}} \right\|_2 e^{-\frac{n}{2} \left[2e^{2r-V}\left(e^{2r-1-2r+\frac{4\beta^2 e^{2r}(e_r-1)^2}{1-3\beta^2 e^{2r}-\beta^2 V(e_r-1)^2}}\right]\right]},
\]

and

\[
P_q\left(\frac{1}{n}S_n > \gamma\right) \leq \min_{1 \geq \alpha e^{2r} + \alpha V(e_r-1)^2} \left\| \frac{q}{\sqrt{s}} \right\|_2 e^{-2r} e^{-n \left[2e^{2r-V}\left(e^{2r-1-2r+\frac{4\alpha e^{2r}(e_r-1)^2}{1-3\alpha e^{2r}-\alpha V(e_r-1)^2}}\right]\right]}.
\]

**Remark.** Note that the results are the same up to the factor $\frac{1}{2}$ in the exponent, the multiplicative factor $e^{2r}$ and the replacement of $\beta^2$ by $\alpha$.

The infimum is hard to compute, so we must optimise separately for different parameter regimes.

First we use the above result to derive a Bennett-type inequality (cf. [3]).

**Corollary 2.** For any $C > 1$ we have

\[
P_q\left(\frac{1}{n}S_n > \gamma\right) \leq \left\| \frac{q}{\sqrt{s}} \right\|_2 e^{-\frac{n}{2} CV\left(1+\frac{1}{C}\right)\log(1+\frac{1}{C})-\frac{\gamma}{2}} \leq \left\| \frac{q}{\sqrt{s}} \right\|_2 e^{-\frac{n}{2} \gamma \log \frac{1}{C+1}}.
\]
as long as \( \gamma \leq \left( \frac{(C-1)(1+\beta^2)}{(C+1)2\beta^2} - 1 \right)CV \).

\[
\mathbb{P}_q\left( \frac{1}{n} S_n > \gamma \right) \leq (1 + \frac{\gamma}{CV}) \left\| \frac{q}{\sqrt{s}} \right\|_2 e^{-nCV\left[ (1+\frac{\gamma}{CV}) \log (1+\frac{\gamma}{CV}) - \frac{\gamma}{CV} \right]}
\leq (1 + \frac{\gamma}{CV}) \left\| \frac{q}{\sqrt{s}} \right\|_2 e^{-n\gamma \log \frac{\gamma}{CV}}
\]

where the same restriction applies as above with \( \beta^2 \) replaced by \( \alpha \).

Our theorem also allows us to reproduce Lezaud’s estimates from [L] with improved constants:

**Corollary 3.**

\[
\mathbb{P}_q\left( \frac{1}{n} S_n > \gamma \right) \leq \left\| \frac{q}{\sqrt{s}} \right\|_2 e^{-n\gamma^2(1-\beta)\frac{1}{4(V+\gamma)}} \quad \text{and} \quad \mathbb{P}_q\left( \frac{1}{n} S_n > \gamma \right) \leq e^{-\gamma\frac{1-\sqrt{\gamma}}{V+\gamma}} \left\| \frac{q}{\sqrt{s}} \right\|_2 e^{-n\gamma^2(1-\sqrt{\gamma})\frac{1}{2(V+\gamma)}}.
\]

**Remark.** These inequalities imply

\[
\mathbb{P}_q\left( \frac{1}{n} S_n > \gamma \right) \leq \left\| \frac{q}{\sqrt{s}} \right\|_2 e^{-n\gamma^2(1-\beta)\frac{1}{8V}}
\]

for \( \gamma \leq V \), and

\[
\mathbb{P}_q\left( \frac{1}{n} S_n > \gamma \right) \leq \left\| \frac{q}{\sqrt{s}} \right\|_2 e^{-n\gamma(1-\beta)\frac{1}{8}}
\]

for \( \gamma \geq V \). If \( \gamma \) is much larger or much smaller than \( V \), the constant 8 can be decreased towards 4.

Similar results apply to the \( \alpha \)-inequalities. Note that in the \( \alpha \)-case we can also prove

\[
\mathbb{P}_q\left( \frac{1}{n} S_n > \gamma \right) \leq e^{\frac{\gamma^2(1-\alpha)}{4(V+\gamma)}} \left\| \frac{q}{\sqrt{s}} \right\|_2 e^{-n\gamma^2(1-\alpha)\frac{1}{4(V+\gamma)}}.
\]

The Bennett-type bound improves upon this Bernstein-type result for \( \gamma >> V \), provided \( \beta \) is small enough. This allows to see how a smaller \( \beta \) reduces the number of required samples.

Finally, our technique can be adapted to situations where we have additional information on the distribution of \( f \), such as subgaussian tails. Let \( s \) denote here, by abuse of notation, the measure on the vertices of the graph which corresponds to the stationary distribution.

**Theorem 4.** In the above setting assume also that \( s(f \geq t) \leq Ce^{-Kt^2} \) for positive \( t \). Then

\[
\mathbb{P}_q\left( \frac{1}{n} S_n > \gamma \right) \leq \left\| \frac{q}{\sqrt{s}} \right\|_2 e^{-\frac{n}{2}\left( \gamma^2K - \log(C\sqrt{\pi K}\gamma + 2) \right)}
\]

as long as \( \gamma \leq \frac{\log(\frac{1}{2} + \frac{1}{2})}{2K} \).
We also have
\( \Pr_q \left( \frac{1}{n} S_n > \gamma \right) \leq \left\| \frac{q}{\sqrt{s}} \right\|_2 e^{2\gamma K} e^{-n(\gamma^2 K - \log (C \sqrt{\pi K} \gamma + 2))} \),

as long as \( \gamma \leq \frac{\log \left( \frac{1}{2} \sqrt{\alpha} + \frac{1}{2} \right)}{2K} \).

Note that if we give up the assumption that \( f \) is bounded by 1, a simple renormalisation shows that the estimate remains the same, except that now we must assume \( \gamma \leq \frac{\log \left( \frac{1}{2} \beta + \frac{1}{2} \right)}{2K} \| f \|_\infty \) (respectively for \( \sqrt{\alpha} \) instead of \( \beta \)). For some parameter regimes Theorem 4 asymptotically improves upon Theorem 12 from [AM].

3 Proofs of results in terms of \( \beta \)

In this section we will prove the inequalities involving \( \beta \). Sketches of proofs for inequalities involving \( \alpha \) are deferred to the next section. Before we begin proving we introduce some notation. We will denote \( \| u \|_{1/s} = \sum_i \frac{u(i)^2}{s(i)} \) the \( \frac{1}{s} \)-weighted \( \ell_2 \) norm on \( \mathbb{R}^N \). The inner product associated with this norm is \( \langle u, v \rangle = \sum_i \frac{u(i)v(i)}{s(i)} \). When we refer to the standard \( \ell_2 \) norm we will use the notation \( \| \cdot \|_2 \) for norm and \( \langle \cdot, \cdot \rangle_2 \) for inner product.

The transition matrix \( P \) is not necessarily symmetric, and so its eigenvectors need not be orthogonal (this would be the case only if \( G \) were a regular graph). Reversibility, however, promises that \( s_j P_{ij} = s_i P_{ji} \), and so \( P \) is self adjoint and its eigenvectors are mutually orthogonal with respect to the \( \frac{1}{s} \)-weighted Euclidean structure. Therefore the \( \| \cdot \|_{1/s} \) norm of \( P \) restricted to the subspace orthogonal to \( s \) is \( \beta \), the second largest absolute value of the eigenvalues of \( P \).

Proof of Theorem 1. The beginning of our proof is identical to that of Gillman’s and of those which follow its reasoning. By Markov’s inequality
\[
\Pr_q \left( \frac{1}{n} S_n > \gamma \right) \leq e^{-rn\gamma} \mathbb{E}_q e^{\gamma S_n},
\]
where the expectation can be directly expressed and estimated as
\[
\sum_{(x_0, \ldots, x_n) \in G^{n+1}} \left( e^{\gamma S_n} q(x_0) \prod_{i=0}^{n-1} (P^T)_{x_i, x_{i+1}} \right) = \langle s, (e^{r \gamma} P^T)^n q \rangle \leq \| q \|_{1/s} \| P e^{r \gamma} \|_{1/s}^n.
\]
Here \( e^{r \gamma} \) stands for the diagonal matrix with \( e^{r \gamma(i)} \) as diagonal entries, and the inner product is, we recall, the inner product associated with the \( \frac{1}{s} \)-weighted \( \ell_2 \) norm.

At this point Gillman’s proof and its variations symmetrise the operator so that its norm will equal its top eigenvalue, and use Kato’s spectral perturbation theory.
to estimate this eigenvalue. Our proof, on the other hand, will proceed to simply estimate the norm directly. To do that we will use the equality

\[ \|P e^{rf}\|_{1/s}^2 = \max_{\|u\|_{1/s} = 1} \langle P e^{rf} u, P e^{rf} u \rangle. \]

In order to perform the computation we split the vector \( u \) into stationary and orthogonal components, \( u = as + b\rho \), where \( \rho \) is normalised and orthogonal to \( s \) in the weighted Euclidean structure. Applying similar decompositions \( e^{rf}s = xs + z\sigma \) and \( e^{rf}\rho = ys + wt \) we get

\[ \|P e^{rf}\|_{1/s}^2 = \max_{a^2 + b^2 = 1, \rho,\sigma,\tau} (a(xs + zP\sigma) + b(ys + wP\tau), a(xs + zP\sigma) + b(ys + wP\tau)). \]

We open the inner product and obtain

\[ \|P e^{rf}\|_{1/s}^2 = \max_{a^2 + b^2 = 1, \rho,\sigma,\tau} a^2(x^2 + z^2\|P\sigma\|^2) + b^2(y^2 + w^2\|P\tau\|^2) + 2ab(xy + zw(P\sigma, P\tau)). \]

Denote \( p_\sigma = \|P\sigma\|^2 \), \( p_\tau = \|P\tau\|^2 \) and \( p_{\sigma,\tau} = \langle P\sigma, P\tau \rangle \). Our task is reduced to computing the \( \ell_2 \) norm of the following 2 by 2 symmetric bilinear form:

\[
\begin{pmatrix}
  x^2 + z^2p_\sigma & xy + zw p_{\sigma,\tau} \\
  xy + zw p_{\sigma,\tau} & y^2 + w^2p_\tau
\end{pmatrix}.
\]

The norm of a 2 by 2 bilinear form equals its largest eigenvalue, which can be derived from its trace (sum of eigenvalues) and determinant (their product). We obtain:

\[
\left\| \begin{pmatrix} A & B \\ B & C \end{pmatrix} \right\|_2 = \frac{1}{2} \left[ (A + C) + \sqrt{(A + C)^2 - 4(AC - B^2)} \right].
\]

Substituting the entries of our own bilinear form and rearranging some terms inside the square root, we get

\[
\|P e^{rf}\|_{1/s}^2 = \frac{1}{2} \left[ (x^2 + y^2 + z^2p_\sigma + w^2p_\tau) + \right.
\]

\[
\left. \left[ (x^2 + y^2 - z^2p_\sigma - w^2p_\tau)^2 + 4z^2w^2(p_{\sigma,\tau}^2 - p_\sigma p_\tau) \\
+ 4z^2p_\sigma + 4y^2w^2p_\tau + 8xyzwp_{\sigma,\tau} \right]^{1/2} \right]
\]

\[
\leq \frac{1}{2} \left[ (x^2 + y^2 + z^2p_\sigma + w^2p_\tau) + \\
\left[ (x^2 + y^2 - z^2p_\sigma - w^2p_\tau)^2 + 4(|xz\sqrt{p_\sigma}| + |yw\sqrt{p_\tau}|)^2 \right]^{1/2} \right],
\]

where we used the Cauchy-Schwarz inequality \( p_{\sigma,\tau}^2 \leq p_\sigma p_\tau \).
To estimate the square root we use the inequality $\sqrt{1 + X^2} \leq 1 + \frac{X^2}{2}$. This lead us to
\[
\|Pe^r\|^2_{1/s} \leq x^2 + y^2 + \frac{(xz\sqrt{p_x})^2 + (yw\sqrt{p_y})^2}{x^2 + y^2 - z^2p_x - w^2p_y},
\] (4)

Note that this result depends on assuming that $x^2 + y^2 \geq z^2p_x + w^2p_y$. (For the purposes of the proof of Theorem 4 we require the inequality
\[
\|Pe^r\|^2_{1/s} \leq x^2 + y^2 + |xz\sqrt{p_x}| + |yw\sqrt{p_y}|,
\] (5)

which is obtained by using $\sqrt{1 + X^2} \leq 1 + |X|$, and depends on the same inequality.)

Let us now estimate the components of our formula. We recall that $f$ has mean 0 with respect to the stationary distribution $s$ and maximum 1. We obtain
\[
x = \langle e^r f, s \rangle = 1 + \frac{(f, s)_{2r}}{1!} + \frac{(f^2, s)_{2r^2}}{2!} + \frac{(f^3, s)_{2r^3}}{3!} + \ldots
\]
\[
\leq 1 + V\left(\frac{r^2}{2!} + \frac{r^3}{3!} + \frac{r^4}{4!} + \ldots\right) \leq 1 + V(e^r - 1 - r)
\]

Note also that $|f| \leq 1$ implies that $x \leq e^r$, and that by the arithmetic-geometric mean $x = \sum_i s(i)e^{r(i)} \geq e^r \sum_i s(i)f(i) = 1$.

To estimate $y = \langle e^r f, \rho \rangle = \langle e^r f, s \rangle$ recall that $\rho$ is normalised and orthogonal to $s$, and that $\langle f, \rho \rangle \leq \|fs\|_{1/s} = \sqrt{V}$. We get
\[
|y| = |\langle e^r f, \rho \rangle| = \langle s, \rho \rangle + \frac{(f, \rho)_{2r}}{1!} + \frac{(f^2, \rho)_{2r^2}}{2!} + \ldots
\]
\[
\leq \sqrt{V}(r + \frac{r^2}{2!} + \frac{r^3}{3!} + \ldots) \leq \sqrt{V}(e^r - 1)
\]

Note that $x^2 + y^2 \leq \|e^r f\|^2_{1/s} = \langle e^{2rf}, s \rangle$, which, as in the computation of $x$ above, is bounded by $1 + V(e^{2r} - 1 - 2r)$.

Next, using the same estimate as for $y$, we get $|z| = |\langle e^r f, \sigma \rangle| \leq \sqrt{V}(e^r - 1)$. For $w = \langle e^r f, \tau \rangle$ we use the simple estimate $|w| \leq e^r$.

Finally, since the norm of $P$ restricted to the subspace orthogonal to $s$ is $\beta$, we have $p_x, p_y, p_x, y \leq \beta^2$

Now we plug our estimates into inequality (4), and derive
\[
\|Pe^r\|^2_{1/s} \leq 1 + V(e^{2r} - 1 - 2r) + \frac{e^r\sqrt{V}(e^r - 1)\beta + \sqrt{V}(e^r - 1)\beta e^r}{1 - \beta^2e^{2r} - \beta^2V(e^r - 1)^2}
\]
\[
\leq 1 + V\left(e^{2r} - 1 - 2r + \frac{2\beta e^r(e^r - 1)^2}{1 - \beta^2e^{2r} - \beta^2V(e^r - 1)^2}\right)
\]
\[
\leq e^{(e^{2r} - 1 - 2r + \frac{4\beta^2e^{2r}(e^r - 1)^2}{1 - \beta^2e^{2r} - \beta^2V(e^r - 1)^2})}
\]
as long as $1 \geq \beta^2 e^{2r} - \beta^2 V(e^r - 1)^2$. To conclude, recall that
\[
\P_q(\frac{1}{n} S_n > \gamma) \leq e^{-n \gamma r} \|q\|_{1/s} \|Pe^{r f}\|_{1/s},
\]
so we finally obtain
\[
\P_q(\frac{1}{n} S_n > \gamma) \leq \min_{1 \geq \beta^2 e^{2r} + \beta^2 V(e^r - 1)^2} \left\| \frac{q}{\sqrt{s}} \right\|_2 e^{-n \left[ \gamma r - \frac{4}{2} V \left( e^{2r} - 1 - 2r + \frac{4\beta^2 e^{2r}}{1 - \beta^2 (2e^{2r} - 1)} \right) \right]}.
\]
\[\square\]

To derive the corollaries and Theorem 4, we only need to assign suitable values to $r$. We will restrict to the case $q = s$ in order not to have to carry the $\|\frac{q}{\sqrt{s}}\|_2$ term.

**Proof of Corollary 3** Using the inequalities $(e^r - 1)^2 \leq e^{2r} - 1 - 2r$ and $\beta^2 e^{2r} + \beta^2 V(e^r - 1)^2 \leq \beta^2 (2e^{2r} - 1)$ we bound the expression inside the exponent in inequality (1) by
\[
-\frac{n}{2} \left[ 2\gamma r - V(e^{2r} - 1 - 2r) \left( 1 + \frac{4\beta^2 e^{2r}}{1 - \beta^2 (2e^{2r} - 1)} \right) \right] =
-\frac{n}{2} \left[ 2\gamma r - V(e^{2r} - 1 - 2r) \frac{1 + \beta^2 + 2\beta^2 e^{2r}}{1 + \beta^2 - 2\beta^2 e^{2r}} \right].
\]
Take any $C > 1$. If we assume that $e^{2r} \leq \frac{(C-1)(1+\beta^2)}{(C+1)2\beta^2}$, then $\frac{1+\beta^2+2\beta^2 e^{2r}}{1+\beta^2-2\beta^2 e^{2r}} \leq C$, so the above expression is bounded by
\[
-\frac{n}{2} \left[ 2\gamma r - CV(e^{2r} - 1 - 2r) \right].
\]
We set $2r = \log (1 + \frac{\gamma}{CV})$. Substituting this into the above we get
\[
-\frac{n}{2} \left[ \gamma \log (1 + \frac{\gamma}{CV}) - CV(1 + \frac{\gamma}{CV} - 1 - \log (1 + \frac{\gamma}{CV})) \right] \leq -\frac{n}{2} CV \left[ (1 + \frac{\gamma}{CV}) \log (1 + \frac{\gamma}{CV}) - \frac{\gamma}{CV} \right] \leq -\frac{n}{2} \gamma \log \frac{\gamma}{CeV}.
\]
Recall that we have required that $e^{2r} \leq \frac{(C-1)(1+\beta^2)}{(C+1)2\beta^2}$, which reduces to assuming $\gamma \leq \frac{(C-1)(1+\beta^2)}{(C+1)2\beta^2} - 1)CV$. \[\square\]

**Proof of Corollary 4** First, we will apply the inequalities $e^{2r} - 1 - 2r \leq 2r^2 e^{2r}$, $e^r - 1 \leq re^r$ and $e^{2r} + V(e^r - 1)^2 \leq 2e^{2r} - 1$ to the exponent in inequality (1). The exponent then turns into
\[
-\frac{n}{2} \left[ 2\gamma r - V r^2 e^{2r} \left( 2 + \frac{4\beta^2 e^{2r}}{1 - \beta^2 (2e^{2r} - 1)} \right) \right] = -n \left[ \gamma r - V r^2 \frac{1 + \beta^2}{(1 + \beta^2)e^{-2r} - 2\beta^2} \right].
\]
Now we set \( r = C\gamma(1-\beta) \). We wish to find \( C \) such that

\[
\gamma r - \frac{V r^2 (1 + \beta^2)}{(1 + \beta^2) e^{-2r} - 2\beta^2} \geq \frac{\gamma r}{2}.
\]

This inequality reduces to verifying that

\[
2C(1 - \beta)(1 + \beta^2) \leq (1 + \beta^2)e^{-2r} - 2\beta^2,
\]

which also guarantees that the denominator is positive, as required by Theorem \( \Box \)

Substituting for \( r \) and using the inequality \( e^{-2r} \geq 1 - 2r \), the above inequality reduces to

\[
(1 - 2C(1 + \frac{\gamma}{V})) + 2C(1 + \frac{\gamma}{V})\beta + (-1 - 2C(1 + \frac{\gamma}{V}))\beta^2 + 2C(1 + \frac{\gamma}{V})\beta^3 \geq 0. \quad (6)
\]

Setting \( C = \frac{V}{2(V + \gamma)} \) guarantees that the above holds for all \( 0 \leq \beta \leq 1 \).

Having verified that \( \gamma r - \frac{V r^2 (1 + \beta^2)}{(1 + \beta^2) e^{-2r} - 2\beta^2} \geq \frac{\gamma r}{2} \)

given our choice of \( C \), we can plug \(-n\gamma r^2\) into the exponent in inequality (5), and obtain the promised bound \( P\left(\frac{1}{n}S_n > \gamma\right) \leq e^{-n\frac{\gamma^2}{2(V + \gamma)}} \).

**Proof of Theorem 4.** To prove this theorem we offer a different analysis of the bound \( x^2 + y^2 \leq \sum_i e^{2r f(i) s(i)} \). This is simply the expectation of \( e^{2rf} \) according to the measure \( s \). We can now evaluate this quantity using the subgaussian information. We get

\[
x^2 + y^2 \leq \int_{-\infty}^{\infty} e^{2rt}d(-s(f \geq t)) = \int_{-\infty}^{\infty} 2re^{2rt}s(f \geq t)dt \leq 1 + \int_0^{\infty} 2e^{2rt}Ce^{-Kt^2}dt = 1 + C\sqrt{\frac{\pi}{K}}r e^{r^2/K}.
\]

Plugging this estimate into inequality (5) together with the simple estimates \( x, w \leq e^r \) and \( y, z \leq e^r - 1 \) we obtain

\[
\|Pe^{rf}\|_{1/s}^2 \leq 1 + C\sqrt{\frac{\pi}{K}}r e^{r^2/K} + 2\beta(e^{2r} - 1).
\]

As noted, inequality (5) depends on taking \( x^2 + y^2 \geq z^2 \beta^2 + w^2 \beta^2 \), which is guaranteed as long as \( \beta^2(2e^{2r} - 1) \leq 1 \). We will make the stronger assumption \( \beta(2e^{2r} - 1) \leq 1 \), and obtain the bound

\[
\|Pe^{rf}\|_{1/s}^2 \leq (C\sqrt{\frac{\pi}{K}}r + 2)e^{r^2/K}.
\]
Recalling that
\[ \mathbb{P}_q \left( \frac{1}{n} S_n > \gamma \right) \leq e^{-n \gamma r} \| P e^{rf} \|^{n}_{1/s}, \]
and setting \( r = \gamma K \), we conclude the required
\[ \mathbb{P}_q \left( \frac{1}{n} S_n > \gamma \right) \leq \| q \|_{1/s} e^{-n \gamma^2 K/2} (C \sqrt{\pi K} \gamma + 2)^{n/2} \]
\[ = \left\| \frac{q}{\sqrt{s}} \right\|_2 e^{-n \gamma^2 K/2} (C \sqrt{\pi K} \gamma + 2)^{n/2}. \]

The condition \( \beta (2e^{2r} - 1) \leq 1 \) now reduces to \( \gamma \leq \frac{\log \left( \frac{1}{2} \beta + \frac{1}{2} \right)}{2K} \).

**Remark.** Note that our method allows to increase \( \gamma \) as far as \( \frac{\log \left( \frac{1}{2} \beta + \frac{1}{2} \right)}{2K} \), where the estimate blows up.

### 4 Proofs of results in terms of \( \alpha \)

The differences between the proofs of results in terms of \( \beta \) and \( \alpha \) are mostly computational, so I will only sketch the relevant differences.

**Proof of Theorem 1.** As above, our task is to estimate \( \| (Pe^{rf})^n \|_{1/s} \). We will use the simple identity \( Pe^{rf} = e^{-\frac{1}{2}rf} e^{\frac{1}{2}rf} Pe^{\frac{1}{2}rf} e^{\frac{1}{2}rf} \) to obtain
\[ \| (e^{rf} P)^n \|_{1/s} \leq e^n \| (e^{\frac{1}{2}rf} Pe^{\frac{1}{2}rf})^n \|_{1/s} \leq e^n \| e^{\frac{1}{2}rf} Pe^{\frac{1}{2}rf} \|^{n}_{1/s}. \]

Since the operator \( e^{\frac{1}{2}rf} Pe^{\frac{1}{2}rf} \) is self-adjoint with respect to the weighted Euclidean structure, we have
\[ \| e^{\frac{1}{2}rf} Pe^{\frac{1}{2}rf} \|^{1/s}_{1/s} = \max_{\| u \|_{1/s} = 1} \langle e^{\frac{1}{2}rf} Pe^{\frac{1}{2}rf} u, u \rangle = \max_{\| u \|_{1/s} = 1} \langle Pe^{\frac{1}{2}rf} u, e^{\frac{1}{2}rf} u \rangle. \]

Decomposing the vectors as in the \( \beta \)-case (with \( \frac{1}{2} r \) replacing \( r \)) we get
\[ \| e^{\frac{1}{2}rf} Pe^{\frac{1}{2}rf} \|^{1/s}_{1/s} = \max_{a^2 + b^2 = 1, \rho, \sigma, \tau} \langle a(xs + zP\sigma) + b(ys + wP\tau), a(xs + z\sigma) + b(ys + w\tau) \rangle. \]

We open the inner product and obtain
\[ \| e^{\frac{1}{2}rf} Pe^{\frac{1}{2}rf} \|^{1/s}_{1/s} = \max_{a^2 + b^2 = 1, \rho, \sigma, \tau} a^2 (x^2 + z^2 \langle P\sigma, \sigma \rangle) + b^2 (y^2 + w^2 \langle P\tau, \tau \rangle) + 2ab (xy + zw \langle P\sigma, \tau \rangle). \]

Our task is reduced to computing the \( \ell_2 \) norm of the same 2 by 2 symmetric bilinear form as in the \( \beta \)-case, except that \( r \) is replaced by \( \frac{1}{2} r \), and the definitions of the \( p \)'s are now \( p_\sigma = \langle P\sigma, \sigma \rangle, p_\tau = \langle P\tau, \tau \rangle \) and \( p_{\sigma, \tau} = \langle P\sigma, \tau \rangle \).
The following identity still holds:

\[ \|e^{2tJ}Pe^{2tJ}\|_{1/s} = \frac{1}{2} \left[ (x^2 + y^2 + z^2 p_\sigma + w^2 p_\tau) + \left[ (x^2 + y^2 - z^2 p_\sigma - w^2 p_\tau)^2 + 4z^2 w^2 (p_{\sigma,\tau}^2 - p_\sigma p_\tau) + 4x^2 z^2 p_\sigma + 4y^2 w^2 p_\tau + 8xyzwp_{\sigma,\tau} \right]^{1/2} \right]. \]

This time, however, the treatment of the terms inside the square root is slightly more delicate. Let \( \lambda_i \) be the eigenvalues of \( P \) in descending order, and let \((\sigma^i)_i\) and \((\tau^i)_i\) be the coordinates of \( \sigma \) and \( \tau \) respectively in terms of the associated orthonormal system. Define

\[ p_\sigma^+ = \sum_{1 > \lambda_i > 0} \lambda_i (\sigma^i)^2 \quad \text{and} \quad p_\sigma^- = -\sum_{\lambda_i < 0} \lambda_i (\sigma^i)^2, \]

and decompose \( p_\tau \) and \( p_{\sigma,\tau} \) analogously. By Cauchy-Schwarz \( |p_{\sigma,\tau}^+| \leq \sqrt{p_\sigma^+ p_\tau^+} \), and the same goes for the \( p^- \)’s.

All this yields

\[ p_{\sigma,\tau}^2 - p_\sigma p_\tau = (p_{\sigma,\tau}^+ - p_{\sigma,\tau}^-)^2 - (p_\tau^+ - p_\tau^-)(p_{\sigma,\tau}^+ - p_{\sigma,\tau}^-) = ((p_{\sigma,\tau}^+)^2 - p_\sigma^+ p_{\sigma}^-) + ((p_{\sigma,\tau}^-)^2 - p_\sigma^- p_{\sigma}^+) + (p_\sigma^+ p_{\sigma}^- + p_\sigma^- p_{\sigma}^+ - 2p_{\sigma,\tau}^+ p_{\sigma,\tau}^-) \leq (\sqrt{p_\sigma^+ p_\tau^+} + \sqrt{p_\sigma^- p_\tau^-})^2 \]

and

\[ x^2 z^2 p_\sigma + y^2 w^2 p_\tau + 2xyzwp_{\sigma,\tau} = (x^2 z^2 p_\sigma^+ + y^2 w^2 p_\tau^+ + 2xyzwp_{\sigma,\tau}^+) - (x^2 z^2 p_\sigma^- + y^2 w^2 p_\tau^- + 2xyzwp_{\sigma,\tau}^-) \leq (|xz\sqrt{p_\sigma^+}| + |yw\sqrt{p_\tau^+}|)^2. \]

We now combine the two estimates to get

\[ 4z^2 w^2 (p_{\sigma,\tau}^2 - p_\sigma p_\tau) + 4x^2 z^2 p_\sigma + 4y^2 w^2 p_\tau + 8xyzwp_{\sigma,\tau} \leq 4 \max(|xz|^2, |yw|^2, |zw|^2) \left( (\sqrt{p_\sigma^+ p_\tau^+} + \sqrt{p_\sigma^- p_\tau^-})^2 + (\sqrt{p_\sigma^+} + \sqrt{p_\tau^+})^2 \right). \]

Since \( \lambda_2 = \alpha \), all \( p^+ \)'s are absolutely bounded by \( \alpha \). Note also that \( p_\sigma^+ + \alpha p_\sigma^- \leq \alpha \|\sigma\|_{1/s} = \alpha \), and the same goes for \( \tau \). So, in fact, the above is bounded by the expression

\[ 4 \max(|xz|^2, |yw|^2, |zw|^2) \left( (\sqrt{p_\sigma^+ (1 - p_\tau^+ / \alpha)} + \sqrt{p_\tau^+ (1 - p_\sigma^+ / \alpha)})^2 + (\sqrt{p_\sigma^+} + \sqrt{p_\tau^+})^2 \right). \]

Rearranging terms and using Cauchy-Schwarz we get

\[ \left( \sqrt{p_\sigma^+ (1 - p_\tau^+ / \alpha)} + \sqrt{p_\tau^+ (1 - p_\sigma^+ / \alpha)} \right)^2 + \left( \sqrt{p_\sigma^+} + \sqrt{p_\tau^+} \right)^2 \leq 2 \left( p_\sigma^+ (1 - p_\tau^+ / \alpha) + p_\tau^+ + \sqrt{(p_\sigma^+ + p_\tau^+ (1 - p_\sigma^+ / \alpha))(p_\sigma^+ + p_\tau^+ (1 - p_\tau^+ / \alpha))} \right) \leq 4 \alpha. \]
So we finally obtain
\[
\|e^{\frac{1}{2}rf} Pe^{\frac{1}{2}rf}\|_{1/s} = \frac{1}{2} \left[ (x^2 + y^2 + z^2 p_\sigma + w^2 p_r) \\
+ [(x^2 + y^2 - z^2 p_\sigma - w^2 p_r)^2 + 16\alpha \max(|xz|^2, |yw|^2, |zw|^2)]^{1/2} \right].
\]

Using the same estimates as in the \(\beta\)-case, replacing \(r\) by \(\frac{1}{2} r\) in the estimates of \(x, y, z\) and \(w\), recalling that \(p_\sigma, p_r \leq \alpha\), and finally changing the bound variable \(r\) into \(2r\) we obtain the desired results. \(\square\)

The other proofs derive from the remark following Theorem 4 which applies also to the proof of Theorem 4. Note that if in the proof of Corollary 3 we set \(r = C \gamma (1 - \sqrt{\alpha}) V\) instead of \(r = C \gamma (1 - \alpha) \sqrt{V}\), we need to satisfy the inequality
\[
(1 - 2C(1 + \gamma) V) - \alpha + 2C(1 + \gamma) V\alpha^2 \geq 0
\]
instead of inequality \(\mathcal{A}\). This holds for \(C = \frac{V}{\sqrt{V} + \gamma}\), leading to inequality \(\mathcal{B}\) in the remark following Corollary 3.

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