ODD COLOURINGS OF GRAPH PRODUCTS

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Abstract. The odd colouring number is a new graph parameter introduced by Petruševski and Škrekovski [7]. In this note, we show that graphs with so called product structure have bounded odd-colouring number. By known results on the product structure of \( k \)-planar graphs, this implies that \( k \)-planar graphs have bounded odd-colouring number, which answers a question of Cranston, Lafferty, and Song [2].

1 Introduction

Let \( G \) be a graph. A (not necessarily proper)\(^1\) vertex colouring \( \varphi : V(G) \to \mathbb{N} \) is odd if the neighbourhood of each non-isolated vertex of \( G \) contains a colour that occurs an odd number of times. More precisely, if \( N_G(v) := \{ w \in V(G) : vw \in E(G) \} \) denotes the neighbourhood of a vertex \( v \) in \( G \), then \( \varphi \) is an odd colouring of \( G \) if and only if, for each \( v \in V(G) \) with \( |N_G(v)| > 0 \), there exists a colour \( \alpha \) such that \( |\{ w \in N_G(v) : \varphi(w) = \alpha \}| \) is odd.

Odd colourings were recently introduced by Petruševski and Škrekovski [7], who showed that every planar graph \( G \) has a proper odd colouring using at most 9 colours\(^2\), and conjectured that 5 colours always suffice. Caro, Petruševski, and Škrekovski [1] showed that 5 colours always suffice for outerplanar graphs and showed that 8 colours always suffice for some special cases of planar graphs. Building on the work of Caro et al. [1], Petr and Portier [6] showed that every planar graph has an odd colouring using at most 8 colours.

A minor-closed family of graphs\(^3\) \( \mathcal{G} \) is \( d \)-degenerate if every graph in \( \mathcal{G} \) contains a vertex of degree at most \( d \). Cranston et al. [2] proved that any graph from a \( d \)-degenerate minor-closed family of graphs has a proper odd colouring using at most \( 2d+1 \) colours. This result, which has a short and elegant proof, includes outerplanar graphs and, more generally, partial 2-trees (with \( d = 2 \)); planar graphs (with \( d = 5 \)); graphs embeddable on surfaces of Euler genus \( g \); and graphs of treewidth at most \( t \) (with \( d = t \)).

Cranston et al. [2] also consider 1-planar graphs, which do not form a minor-closed family, and show that any 1-planar graph\(^4\) has a proper odd colouring using at most 31

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\(^1\)\( \varphi \) is a proper colouring of \( G \) if \( vw \in E(G) \) implies that \( \varphi(v) \neq \varphi(w) \).

\(^2\)We say that a colouring \( \varphi \) of \( G \) uses \( c \) colours if \( c = |\{ \varphi(v) : v \in V(G) \}| \).

\(^3\)A graph \( M \) is a minor of a graph \( G \) if a graph isomorphic to \( M \) can be obtained from a subgraph of \( G \) by contracting edges. A class \( \mathcal{G} \) of graphs is minor-closed if for every graph \( G \in \mathcal{G} \), every minor of \( G \) is in \( \mathcal{G} \).

\(^4\)A graph is \( k \)-planar if it has an embedding in the plane in which no edge contains a vertex other than its endpoints and each edge is involved in at most \( k \) crossings with other edges.
colours. They ask if this can be extended to k-planar graphs for general k > 1. Our main result, Theorem 1 below, is a more general results for graphs having product structure and implies that any k-planar graph has a proper odd colouring using O(k^5) colours.

For two graphs A and B, the strong product A ⊠ B of A and B is the graph with vertex set V(A ⊠ B) := V(A) × V(B) and that contains an edge with endpoints (x_1, y_1) and (x_2, y_2) if and only if (i) x_1 x_2 ∈ E(A) and y_1 = y_2; (ii) x_1 = x_2 and y_1 y_2 ∈ E(B); or (iii) x_1 x_2 ∈ E(A) and y_1 y_2 ∈ E(B). A t-tree H is a graph that is either a clique on t + 1 vertices or a graph that contains a vertex v of degree t whose neighbours form a clique and such that H − {v} is a t-tree. The following is the main result in this paper:

**Theorem 1.** Let H be a t-tree, let P be a path, and let G be a subgraph of H ⊠ P. Then G has a proper odd colouring of G that uses at most 8t + 4 colours.

A number of graph families are known to exhibit so called product structure like that required of the graph G in Theorem 1. For example, for every planar graph G there exists a 6-tree H and a path P such that G is isomorphic to a subgraph of H ⊠ P [8]. Similar results hold (with constants other than 6) for graphs of bounded Euler genus, apex-minor free graphs, and bounded-degree graphs from proper-minor-closed families [3, 4]. Most relevant for the current discussion is the following theorem of Dujmović, Morin, and Wood [5]:

**Theorem 2** (Dujmović et al. [5]). For every k-planar graph G there exists an O(k^5)-tree H and a path P such that G is isomorphic to a subgraph of H ⊠ P.

Combining this with Theorem 1, we immediately obtain the following corollary, answering the question posed by Cranston et al. [2].

**Corollary 3.** Every k-planar graph G has a proper odd colouring that uses O(k^5) colours.

## 2 Proof of Theorem 1

**Proof of Theorem 1.** Let y_1, ..., y_h be the vertices of P, in order. To avoid a boring edge case, we extend P by one vertex in each direction, so that the vertices y_0 and y_{h+1} are defined.

Let x_1, ..., x_r be the vertices of H ordered so that x_1, ..., x_t is a clique and, for each i ∈ {t + 1, ..., r}, the vertices in C_{x_i} := N_G(x_i) ∩ [x_1, ..., x_{i−1}] form a clique of order t. We make crucial use of the following well-known property of t-trees:

(♦) If 1 ≤ i ≤ j ≤ r and x_j ∈ C_{x_i} then C_{x_i} ∪ {x_i} ⊆ C_{x_j} ∩ [x_1, ..., x_j].

With this setup out of the way, we can proceed with the proof, which is by induction on the number of vertices of G. We will prove the following stronger statement: There exists a proper odd colouring ϕ : V(G) → {1, ..., 8t + 4} of G that satisfies the following additional condition:

(★) For each (x_i, y_j) ∈ V(H ⊠ P), define

\[ C_{(x_i, y_j)} := V(G) ∩ \left( \left( C_{x_i} × \{y_{j−1}, y_j, y_{j+1}\} \right) ∪ \{(x_i, y_i), (x_i, y_{j−1})\} \right). \]

\(^5(♦)\) follows from the fact that S := C_{x_i} ∪ {x_i} separates [x_1, ..., x_i] \ S from x_{i′}. In the language of rooted tree-decompositions, the node whose bag contains S is an ancestor of all nodes whose bags contains x_{i′}.\]
Then, for each \( v \in V(H \boxtimes P) \) the vertices in \( C_v \) receive distinct colours, i.e., \( \varphi(u) \neq \varphi(w) \) for each distinct \( u, w \in C_v \).

Note that, for any edge \( vw \) of \( G \), \( v, w \in C_w \) or \( v, w \in C_v \). Therefore, \( (\bigstar) \) implies that the colouring \( \varphi \) is a proper colouring of \( G \).

The base case, in which \( G \) has no vertices, is trivial. Therefore, we may assume that \( G \) has at least one vertex. Let \( (i, j) \) such that \( G \) satisfies \( (\bigstar) \). We will now extend \( \varphi \) to a colouring of \( G \) by listing colours that we may not choose for \( \varphi(v) \):

- To guarantee that \( \varphi \) satisfies \( (\bigstar) \), observe that assigning a colour to \( v \) can only violate \( (\bigstar) \) if it does so for some vertex \( (x_i, y_j) \in V(H \boxtimes P) \) with \( v \in C_{(x_i, y_j)} \). By \( (\bigstar) \) and the maximality of \( (i, j) \), if \( v \in C_{(x_i, y_j)} \), then \( |j - j'| \leq 1 \) and \( C_{x_i} \cup \{x_i\} \subseteq C_{x_i} \). Therefore,

\[
C_{(x_i, y_j)} \setminus \{v\} \subseteq \left(C_{x_i} \times \{y_j - 2, y_j - 1, y_j, y_j + 1, y_j + 2\}\right) \cup \{(x_i, y_j - 2), (x_i, y_j - 1)\} =: R.
\]

Therefore, in order to satisfy \( (\bigstar) \) it is sufficient to choose \( \varphi(v) \) so that \( \varphi(v) \neq \varphi(w) \) for each \( w \in R \). Let \( X := \{\varphi(w) : w \in R\} \) and observe that \( |X| \leq |R| \leq 5t + 2 \).

Furthermore, if \( v \) is not an isolated vertex of \( G \) then \( (\bigstar) \) ensures that some colour occurs exactly once in \( N_G(v) \). Therefore, to ensure that \( \varphi \) is an odd colouring of \( G \), we need only choose some \( \varphi(v) \notin X \) in such a way that some colour appears an odd number of times in \( N_G(w) \) for each \( w \in N_G(v) \), which is what we do next.

- To guarantee that \( \varphi \) is an odd colouring of \( G \), consider each vertex \( w \in N_G(v) \). If there is exactly one colour \( \alpha \in \{1, \ldots, 8t + 1\} \) that occurs an odd number of times in \( N_G(w) \), then define \( Y_w := \{\alpha\} \) and otherwise define \( Y_w := \emptyset \). Now let \( Y := \bigcup_{w \in N_G(v)} Y_w \) and observe that \( |Y| \leq |N_G(v)| \leq 3t + 1 \). If we choose \( \varphi(v) \notin Y \) then, for each \( w \in N_G(v) \) the following holds:

\begin{itemize}
  \item If \( Y_w = \{\alpha\} \) then \( \varphi(v) \neq \alpha \). Therefore, the colour \( \alpha \) appears an odd number of times in \( N_G(w) \) since it appears an odd number of times in \( N_G(w) \setminus \{v\} \).
  \item If \( Y_w = \emptyset \) then either:
    \begin{itemize}
      \item No colour appears an odd number of times in \( N_G(w) \). Therefore the colour \( \varphi(v) \) appears an even number of times in \( N_G(w) \), so \( \varphi(v) \) appears an odd number of times in \( N_G(w) = N_G(w) \cup \{v\} \).
      \item At least two colours \( \alpha \) and \( \beta \) each appear an odd number of times in \( N_G(w) \). Therefore each colour in \( \{\alpha, \beta\} \setminus \{\varphi(v)\} \) appears an odd number of times in \( N_G(w) \). In particular, at least one of \( \alpha \) or \( \beta \) appears an odd number of times in \( N_G(w) \).
    \end{itemize}
\end{itemize}

Therefore, by choosing \( \varphi(v) \notin X \cup Y \) we obtain an odd colouring of \( G \) that satisfies \( (\bigstar) \). Since \( |X \cup Y| \leq |X| + |Y| \leq 8t + 3 \), there exists some \( \varphi(v) \in \{1, \ldots, 8t + 4\} \setminus (X \cup Y) \) that completes the colouring of \( G \).
3 Remarks

Our proof of Theorem 1 is inspired by the proof of the result on \(d\)-degenerate minor-closed families of Cranston et al. [2], which is an inductive proof that involves contracting an edge \(vw\) incident to a vertex \(v\) of degree at most \(d\). The contraction of this edge (as opposed to the deletion of \(v\)) is crucial to ensuring that \(N_G(v)\) has a colour (namely \(q(w)\)) that appears an odd number of times. However, since the class of graphs with product structure is not minor-closed we can not use edge contractions. Instead, we use vertex deletion along with condition (\(\Box\)) to achieve a similar effect.

One might hope that Theorem 1 could be generalized to the setting in which \(H\) belongs to some \(t\)-degenerate minor-closed family of graphs. However, bounding the size of the set \(X\) required to maintain (\(\Box\)) when choosing \(q(v)\) relies critically on (\(\varnothing\)), which is a property of graphs of treewidth \(t\) that is not true for all \(t\)-degenerate minor-closed graph families.

Subgraphs of \(H \boxtimes P \boxtimes K_\ell\). A number of product structure theorems characterize graphs as subgraphs of \(H \boxtimes P \boxtimes K_\ell\) where \(H\) is a \(t\)-tree, \(P\) is a path, and \(K_\ell\) is a complete graph of order \(\ell\). Since \(H \boxtimes P \boxtimes K_\ell\) is isomorphic to \(H \boxtimes K_\ell \boxtimes P\) and \(H \boxtimes K_\ell\) is a \((\ell(t + 1) - 1)\)-tree, Theorem 1 immediately implies that these graphs have odd colourings using \(8\ell t + 8\ell - 4\) colours.

It is possible to improve this slightly by redoing the proof Theorem 1. In this case, the vertex \(v := (x_i, y_j, z_k)\) that is removed is also chose to maximize \((i, j)\), with ties broken arbitrarily. Then one finds that the sizes of the colour sets \(X\) and \(Y\) that must be avoided when choosing \(q(v)\) are bounded by \(|X| \leq 5\ell t + 3\ell - 1\) and \(|Y| \leq 3\ell t + 2\ell - 1\) so that \(|X \cup Y| \leq 8\ell t + 5\ell - 2\). This gives the following variant of Theorem 1:

Theorem 4. Let \(H\) be a \(t\)-tree, let \(P\) be a path, let \(K_\ell\) be a clique on \(\ell\) vertices, and let \(G\) be a subgraph of \(H \boxtimes P \boxtimes K_\ell\). Then \(G\) has an odd colouring using at most \(8\ell t + 5\ell - 1\) colours.

Subgraphs of \(H \boxtimes I\). Perhaps a more interesting generalization comes by replacing \(P\) by some graph \(I\) of maximum-degree \(\Delta\). Again, one can follow the same general strategy used in the proof of Theorem 1, with the following changes.

- The vertices \(y_1, \ldots, y_h\) are the vertices of \(I\) in no particular order.
- The set \(C(x_i, y_j)\) is defined as
  \[C(x_i, y_j) := ([x_i] \cup C_{x_i}) \boxtimes ([y_j] \cup N_{I}(y_j)).\]
- \(|X \cup Y|\) is bounded as follows: \(|Y| \leq |N_G(v)| \leq |C_{v} \setminus \{v\}| \leq (t + 1)(\Delta + 1) - 1\). The set \(R\) used to define \(X\) is given by \(R := ([x_i] \cup C_{x_i}) \times N_{I}^{2}(y_j)\), where \(N_{I}^{2}(y_j)\) denotes the set of at most \(\Delta^2 + 1\) vertices in \(I\) of distance at most 2 from \(y_j\). Then \(|X| \leq |R \setminus \{v\}| \leq (t + 1)(\Delta^2 + 1) - 1\). Therefore \(|X \cup Y| \leq |X| + |Y| \leq (\Delta^2 + \Delta)(t + 1) + 2t\).

These changes prove the following variant of Theorem 1:

Theorem 5. Let \(H\) be a \(t\)-tree, let \(I\) be a graph of maximum-degree \(\Delta\), and let \(G\) be a subgraph of \(H \boxtimes I\). The \(G\) has an odd colouring using at most \((\Delta^2 + \Delta)(t + 1) + 2t + 1\) colours.
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