Non-Abelian Gauge Fields as Pseudo-Goldstone Vector Bosons

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Abstract

We argue that non-Abelian gauge fields can be treated as the pseudo-Goldstone vector bosons caused by spontaneous Lorentz invariance violation (SLIV). To this end, the SLIV which evolves in a general Yang-Mills type theory with the nonlinear vector field constraint \( Tr(A_\mu A^\mu) = \pm M^2 \) (\( M \) is a proposed SLIV scale) imposed is considered in detail. With an internal symmetry group \( G \) having \( D \) generators not only the pure Lorentz symmetry \( SO(1,3) \), but the larger accidental symmetry \( SO(D,3D) \) of the SLIV constraint in itself appears to be spontaneously broken as well. As a result, while the pure Lorentz violation still generates only one genuine Goldstone vector boson, the accompanying pseudo-Goldstone vector bosons related to the \( SO(D,3D) \) breaking also come into play in the final arrangement of the entire Goldstone vector field multiplet. Remarkably, they remain strictly massless, being protected by gauge invariance of the Yang-Mills theory involved. We show that, although this theory contains a plethora of Lorentz and \( CPT \) violating couplings, they do not lead to physical SLIV effects which turn out to be strictly cancelled in all the lowest order processes considered. However, the physical Lorentz violation could appear if the internal gauge invariance were slightly broken at very small distances influenced by gravity. For the SLIV scale comparable with the Planck one the Lorentz violation could become directly observable at low energies.
1 Introduction

The old idea\cite{1} that spontaneous Lorentz invariance violation (SLIV) may lead to an alternative theory of quantum electrodynamics still remains extremely attractive in numerous theoretical contexts\cite{2} (for some later developments, see the papers\cite{3}). The SLIV could generally cause the appearance of massless vector Nambu-Goldstone modes which are identified with photons and other gauge fields underlying the modern particle physics framework like as Standard Model and Grand Unified Theory. At the same time, the Lorentz violation by itself has attracted a considerable attention in recent years as an interesting phenomenological possibility appearing in various quantum field and string theories\cite{4-9}.

Early models realizing the SLIV conjecture were based on the four fermion (current-current) interaction, where the proposed gauge field may appear as a fermion-antifermion pair composite state\cite{1}, in a complete analogy with a massless composite scalar field in the original Nambu-Jona-Lazinio model\cite{10}. Unfortunately, owing to the lack of a starting gauge invariance in such models and composite nature of Goldstone modes appeared it is hard to explicitly demonstrate that these modes really form together a massless vector boson being a gauge field candidate. Actually, one must make a precise tuning of parameters, including a cancellation between terms of different orders in the $1/N$ expansion (where $N$ is the number of fermion species involved), in order to achieve the massless photon case\cite{11}. Rather, there are in general three separate massless Goldstone modes, two of which may mimic the transverse photons polarizations, while the third one must properly be suppressed.

In this connection, the more instructive laboratory for SLIV consideration proves to be some simple class of the QED type models having from the outset a gauge invariant form, whereas the Lorentz violation is realized through the nonlinear dynamical constraint imposed on the starting vector field $A_\mu$

$$A^2_\mu = n^2_\mu M^2$$ (1)

where $n_\mu$ is an properly oriented unit Lorentz vector, while $M$ is a proposed SLIV scale. This constraint means in essence that the vector field $A_\mu$ develops the vacuum expectation value $\langle A_\mu(x) \rangle = n_\mu M$ and Lorentz symmetry $SO(1,3)$ breaks down to $SO(3)$ or $SO(1,2)$ depending on the time-like ($n^2_\mu = +1$) or space-like ($n^2_\mu = -1$) SLIV. Such QED model was first studied by Nambu a long time ago\cite{12}, but only for the time-like SLIV case and in the tree approximation. For this purpose he applied the technique of nonlinear symmetry realizations which appeared successful in the handling of the spontaneous breakdown of chiral symmetry in the nonlinear $\sigma$ model\cite{13} and beyond\cite{4}.

\footnote{Actually, the simplest possible way to obtain the above supplementary condition (1) could be an inclusion the “standard” quartic vector field potential $V(A) = -\frac{m^2}{2} A^2_\mu + \frac{\lambda}{4} (A^2_\mu)^2$ into the QED type Lagrangian, as can be motivated to some extent\cite{14} from the superstring theory. This potential inevitably causes the spontaneous violation of Lorentz symmetry in a standard way, much as an internal symmetry violation is caused in a linear $\sigma$ model for pions\cite{13}. As a result, one has a
In the present paper, we mainly address ourselves to the Yang-Mills gauge fields as the possible vector Goldstone modes (Sec.3) once some basic ingredients of the Goldstonic QED model are established in a general SLIV case (Sec.2). This problem has been discussed many times in the literature within quite different contexts, such as the Yang-Mills gauge fields as the Goldstone modes for the spontaneous breaking of general covariance in a higher-dimensional space [17] or for the nonlinear realization of some special infinite parameter gauge group [18]. However, all these considerations look rather speculative and optional. Specifically, they do not give a correlation between the SLIV induced photon case, from the one hand, and the Yang-Mills gauge field case, from the other. In contrast, our approach is solely based on the spontaneous Lorentz violation thus properly generalizing the Nambu’s QED model [12] to the non-Abelian internal symmetry case. Just in this approach evolved the interrelation between both of cases appears most transparent. We will see that in the Yang-Mills theory case with an internal symmetry group \( G \) having \( D \) generators not only the pure Lorentz symmetry part \( SO(1,3) \) in the symmetry \( SO(1,3) \times G \) of the Lagrangian, but the larger accidental symmetry \( SO(D,3D) \) of the SLIV constraint \( \text{Tr}(A^\mu A^\mu) = \pm M^2 \) in itself is spontaneously broken as well. Because the starting non-Abelian theory proves to be expanded about the vacuum which violates the much higher accidental symmetry appeared, many extra massless modes, the pseudo-Goldstone vector bosons (PGB), have to arise. Actually, while the spontaneous Lorentz violation on its own still generates only one genuine Goldstone vector boson, the accompanying vector PGBs related to the \( SO(D,3D) \) breaking also come into play in the final arrangement of the entire Goldstone vector field multiplet. Remarkably, in contrast to the familiar scalar PGB case [13], the vector PGBs remain strictly massless being protected by the non-Abelian gauge invariance of the Yang-Mills theory involved. Then in Sec.4 we show by some examples of the lowest order SLIV processes that, while the Goldstonic non-Abelian theory evolved contains a rich variety of Lorentz and CPT violating couplings, it proves to be physically indistinguishable from a conventional Yang-Mills theory. Actually, one of the goals of the present work is to explicitly demonstrate that a conventional Yang-Mills theory (as well as QED) is in fact the spontaneously broken theory. The Lorentz violation, due to the quadratic field constraint of the type (1), renders this theory highly nonlinear in the Goldstone vector modes, while physically equivalent to the usual one. So, as well as in the pure QED case, the SLIV only means the noncovariant gauge choice in the otherwise gauge invariant and Lorentz invariant Yang-Mills theory. However, even a tiny breaking of the starting gauge invariance at massive Higgs mode (with mass \( \sqrt{2m_A} \)) together with a massless Goldstone mode associated with photon. Furthermore, just as in the pion model one can go from the linear model for the SLIV to the non-linear one taking a limit \( \lambda_A \to \infty, m_A^2 \to \infty \) (while keeping the ratio \( m_A^2/\lambda_A \) to be finite). This immediately leads to the constraint (1) for vector potential \( A_\mu \) with \( n_A^2 M^2 = m_A^2/\lambda_A \), as it appears from a validity of its equation of motion. Another motivation for the nonlinear vector field constraint (1) might be an attempt to avoid the infinite self-energy of the electron in classical electrodynamics, as was originally indicated by Dirac [15] and extended later to various vector field theory cases [16].
very small distances influenced by gravity would render the SLIV physically significant. For the SLIV scale comparable with the Planck one the spontaneous Lorentz violation could become directly observable at low energies. We summarize the results obtained in the final Sec.5.

2 Goldstonic quantum electrodynamics

The simplest SLIV model is given by a conventional QED Lagrangian for the charged fermion field $\psi$

$$\mathcal{L}(A, \psi) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi} \left( i \gamma \cdot \partial - m \right) \psi - eA_\mu \overline{\psi} \gamma^\mu \psi$$  \hspace{1cm} (2)

where the nonlinear vector field constraint \[1\] is imposed\[12\]. For the resulting Lorentz violation, one can rewrite the Lagrangian $\mathcal{L}(A, \psi)$ in terms of the standard parametrization for the vector potential $A_\mu$

$$A_\mu = a_\mu + \frac{n_\mu(n \cdot A)}{n^2} \quad (n^2 \equiv n_\mu n^\mu)$$  \hspace{1cm} (3)

where the $a_\mu$ is pure Goldstonic mode

$$n \cdot a = 0$$  \hspace{1cm} (4)

while the effective Higgs mode (or the $A_\mu$ component in the vacuum direction) is given according to the above nonlinear constraint \[1\] by

$$n \cdot A = (M^2 - n^2 a_\mu^2)^{1/2} = M - \frac{n^2 a_\mu^2}{2M} + O(1/M^2)$$  \hspace{1cm} (5)

where, for definiteness, the positive sign for the above square root was taken when expanding it in powers of $a^2_\mu/M^2$. Putting the parametrization \[3\] with the SLIV constraint \[1, 5\] into our basic gauge invariant Lagrangian \[2\] one comes to the truly Goldstonic model for QED. This model might look unacceptable due to the inappropriately large Lorentz violating fermion bilinear $eM \overline{\psi}(\gamma \cdot n)\psi$ stemming from the vector-fermion current interaction $eA_\mu \overline{\psi} \gamma^\mu \psi$ in the Lagrangian $\mathcal{L}$ \[2\] when the expansion \[5\] is taken. However, thanks to a local invariance of the Lagrangian $\mathcal{L}$ this term can be gauged away by a suitable redefinition of the fermion field

$$\psi \to e^{ieM(n \cdot x)} \overline{\psi}$$  \hspace{1cm} (6)

after which the above fermion bilinear is exactly cancelled by an analogous term stemming from the fermion kinetic term. So, one eventually comes to the essentially nonlinear SLIV Lagrangian for the Goldstonic $a_\mu$ field of the type (taken in the first approximation in $a^2_\mu/M^2$)

$$\mathcal{L}(a, \psi) = -\frac{1}{4} f_{\mu\nu} f^{\mu\nu} - \frac{1}{2} \delta(n \cdot a)^2 - \frac{1}{4} f_{\mu\nu} h^{\mu\nu} \frac{n^2 a_\mu^2}{M} +$$

$$+ \overline{\psi} (i \gamma \cdot \partial + m) \psi - e a_\mu \overline{\psi} \gamma^\mu \psi + \frac{e n^2 a_\mu^2}{2M} \overline{\psi}(\gamma \cdot n)\psi$$  \hspace{1cm} (7)
We denoted its strength tensor by \( f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \), while \( h^{\mu\nu} = n^\mu \partial^\nu - n^\nu \partial^\mu \) is a new SLIV oriented differential tensor. This tensor \( h^{\mu\nu} \) acts on the infinite series in \( a_\rho^2 \) coming from the expansion of the effective Higgs mode \( 5 \) from which the first order term \( -n^2 a_\nu^2/2M \) was only taken in this expansion throughout the Lagrangian \( L(a, \psi) \). Also, we explicitly included the orthogonality condition \( n \cdot n = 0 \) into Lagrangian through the term which can be treated as the gauge fixing term (taking the limit \( \delta \to \infty \)) and retained the former notation for the fermion \( \psi \).

The Lagrangian \( 7 \) completes the Goldstonic QED construction for the charged fermion field \( \psi \). The model, as one can see, contains the massless Goldstone modes given by the tree broken generators of the Lorentz group, while keeping the massive Higgs mode frozen. These modes, lumped together, constitute a single Goldstone vector boson associated with photon\(^2\). In the limit \( M \to \infty \) the model is indistinguishable from a conventional QED taken in the general axial (temporal or pure axial) gauge. So, for this part of the Lagrangian \( L(a, \psi) \) given by the zero-order terms in \( 1/M \) the spontaneous Lorentz violation only means the noncovariant gauge choice in otherwise the gauge invariant (and Lorentz invariant) theory. Remarkably, furthermore, also all the other \( ( \text{first and higher order in } 1/M ) \) terms in the \( L(a, \psi) \) \( 7 \), though being by themselves the Lorentz and \( CPT \) violating ones, do not lead to the physical SLIV effects which turn out to be strictly cancelled in all the physical processes involved. So, the nonlinear constraint \( 1 \) imposed on the standard QED Lagrangian \( 2 \) appears, in fact, as a possible gauge choice, while the \( S \)-matrix remains unaltered under such a gauge convention. This conclusion was first reached at tree level\(^{12} \) and recently extended to the one-loop approximation\(^{19} \). All the one-loop contributions to the photon-photon, photon-fermion and fermion-fermion interactions violating the physical Lorentz invariance were shown to be exactly cancelled as well. This means that the vector field constraint \( A_\mu^2 = n_\mu^2 M^2 \) which has been treated as the nonlinear gauge choice at a tree (classical) level, remains just as a pure gauge condition when quantum effects are also taken into account. Remarkably, this conclusion appears to work also for a general Abelian theory case\(^{20} \), particularly, when the internal \( U(1) \) charge symmetry is spontaneously broken hand in hand with the Lorentz one. As a result, the massless photon being first generated by the Lorentz violation become then massive due to the standard Higgs mechanism, while the SLIV condition in itself remains to be a gauge choice\(^3\).

\(^2\)Strictly speaking one can no longer use the standard definition of photon as a state being the spin-1 representation of the (now spontaneously broken) Poincare group. However, due to gauge symmetry of the starting QED Lagrangian \( 2 \) the separate SLIV Goldstone modes appear combined in such a way that a standard photon (taken in an axial gauge \( 4 \)) emerges.

\(^3\)Note in this connection that there was discussed\(^{12} \) a possibility of an explicit construction of the gauge function corresponding to the nonlinear gauge constraint \( 1 \) that would eliminate the need for all the kinds of checks of gauge invariance mentioned above. Remarkably, the equation for this gauge function appears to be mathematically equivalent to the classical Hamilton-Jacobi equation of motion for a charged particle. Thus, this gauge function should in principle exist because there is a solution to the classical problem. However, this formal analogy only works for the time-like SLIV \( (n_\mu^2 = +1) \) in the pure QED leaving aside a general Abelian theory when the gauge invariance can spontaneously be broken. Apart from that, it does not generally extend to
3 Goldstonic Yang-Mills theory

In this section, we extend our discussion to the non-Abelian internal symmetry case given by a general group $G$ with generators $t^i$ ($[t^i, t^j] = i\epsilon^{ijk}t^k$ and $Tr(t^i t^j) = \delta^j_i$) where $\epsilon^{ijk}$ are structure constants and $i, j, k = 0, 1, ..., D - 1$). The corresponding vector fields which transform according to its adjoint representation are given in the proper matrix form $A^\mu = A^i_\mu t^i$, while the matter fields (fermions, for definiteness) are presented in the fundamental representation column $\psi_r$ ($r = 0, 1, ..., d - 1$) of $G$.

By analogy with the above Goldstonic QED case we take for them a conventional Yang-Mills type Lagrangian

$$L(A, \psi) = -\frac{1}{4} Tr(F_{\mu\nu}F^{\mu\nu}) + \bar{\psi}(i\gamma \cdot \partial - m)\psi + g\bar{\psi}A_\mu\gamma^\mu\psi$$

(8)

(where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$ and $g$ stands for the universal coupling constant in the theory) with the nonlinear SLIV constraint

$$Tr(A_\mu A^\mu) = n^2_\mu M^2, \quad n^2_\mu = \pm 1$$

imposed\(^4\). One can easily see that, although we propose only the $SO(1, 3) \times G$ invariance in the theory, the SLIV constraint taken \(^9\) possesses, in fact, the much higher accidental symmetry $SO(D, 3D)$ determined by the dimensionality $D$ of the $G$ group adjoint representation to which the vector fields $A^i_\mu$ are belonged. This symmetry is indeed spontaneously broken at a scale $M$

$$\langle A^i_\mu(x) \rangle = n^i_\mu M$$

(10)

with the vacuum direction given now by the ‘unit’ rectangular matrix $n^1_\mu$ which describes both of the generalized SLIV cases at once, time-like ($SO(D, 3D) \rightarrow SO(D - 1, 3D)$) or space-like ($SO(D, 3D) \rightarrow SO(D, 3D - 1)$), respectively, depending on the sign of the $n^2_\mu \equiv n^i_\mu n^{i,\mu} = \pm 1$. This matrix has only one non-zero element for both of cases determined by the proper $SO(D, 3D)$ rotation. They are, particularly, $n^0_0$ or $n^0_3$ provided that the vacuum expectation value \(^10\) is developed along the $i = 0$ direction in the internal space and along the $\mu = 0$ or $\mu = 3$ direction, respectively, in the Minkowskian space-time. In response to each of these two breakings, side by side with one true vector Goldstone boson and the $D - 1$ scalar Goldstone bosons corresponding to the spontaneous violation of actual $SO(1, 3) \otimes G$ symmetry of the total Lagrangian $L$, the $D - 1$ vector pseudo-Goldstone bosons related to breaking of the accidental $SO(D, 3D)$ symmetry of the SLIV constraint taken \(^9\) are also produced. Remarkably, in contrast to the familiar scalar PGB case\(^13\) the non-Abelian case (see next Section).

\(^4\)As in the Abelian case, the existence of such a constraint could be related with some non-linear $\sigma$ type SLIV model proposed for the vector field multiplet $A^i_\mu$ in the Yang-Mills theory \(^5\). Note in this connection that, due to its generic antisymmetry, the familiar quadrilinear terms $-\frac{1}{2} g^2 Tr([A^i_\mu, A^j_\mu])^2$ in the Lagrangian \(^5\) do not contribute into the SLIV since they identically vanish for any single-valued vacuum configuration $\langle A^i_\mu \rangle$. 

\(^5\)
the vector PGBs remain strictly massless being protected by the non-Abelian gauge invariance of the starting Lagrangian \( \mathcal{L} \). Together with the aforementioned true vector Goldstone boson they complete the entire Goldstonic vector field multiplet of the internal symmetry group \( G \).

As in the Abelian case, upon an explicit use of the corresponding SLIV constraint \( (9) \) being so far the only supplementary condition for vector field multiplet \( A^i_\mu \), one comes to the pure Goldstone field modes \( a^i_\mu \) identified in a similar way

\[
A^i_\mu = a^i_\mu + \frac{n^i_\mu}{n^2}(n \cdot A), \quad n \cdot a = n^i_\mu a^{\mu,i} = 0 \quad (n^2 \equiv n^2_\mu), \tag{11}
\]

At the same time, an effective Higgs mode (i.e., the \( A^i_\mu \) component in the vacuum direction \( n^i_\mu \)) is given by the product \( n \cdot A = n^i_\mu A^{\mu,i} \) determined by the SLIV constraint

\[
n \cdot A = [M^2 - n^2(a^i_\nu)^2]^{1/2} = M - \frac{n^2(a^i_\nu)^2}{2M} + O(1/M^2) \tag{12}
\]

where, as earlier in the Abelian case, we took the positive sign for the square root when expanding it in powers of \( (a^i_\nu)^2/M^2 \). Note that the general Goldstonic modes \( a^i_\mu \), apart from pure vector fields, contain the \( D-1 \) scalar ones, \( a^i_0 \) and \( a^i_3 \) \( (i' = 1...D-1) \), for the time-like \( (n^i_\mu = n^i_0 g_{\mu 0} \delta^{i0} \) and space-like \( (n^i_\mu = n^i_3 g_{\mu 3} \delta^{i0}) \) SLIV, respectively. They can be eliminated from the theory if one puts the proper supplementary conditions on the \( a^i_\mu \) fields which were still the constraint free. Using their overall orthogonality \( (11) \) to the physical vacuum direction \( n^i_\mu \) one can formulate these supplementary conditions in terms of a general axial gauge for the entire \( a^i_\mu \) multiplet

\[
n \cdot a^i \equiv n^i_\mu a^{\mu,i} = 0, \quad i = 0...D - 1 \tag{13}
\]

where \( n_\mu \) is the unit Lorentz vector introduced in the Abelian case which is now oriented in Minkowskian space-time so as to be parallel to the vacuum matrix \( n^i_\mu \). For such a choice the simple equation holds

\[
n^i_\mu = s^i n_\mu \quad (s^i \equiv \frac{n \cdot n^i}{n^2}) \tag{14}
\]

which shows that the rectangular vacuum matrix \( n^i_\mu \) has the factorized ”two-vector” form. As a result, apart from the Higgs mode excluded earlier by the orthogonality condition \( (11) \), all the scalar fields also appear eliminated, and only pure vector fields, \( a^{i_\nu} \) \( (\mu' = 1, 2, 3) \) or \( a^{i_\nu} \) \( (\mu'' = 0, 1, 2) \) for time-like or space-like SLIV, respectively, are only left in the theory.

We now show that the such constrained Goldstone vector fields \( a^i_\mu \) (with the supplementary conditions \( (13) \) taken) appear truly massless when the starting non-Abelian Lagrangian \( \mathcal{L} \) is rewritten in the form determined by the SLIV. Actually, putting the parametrization \( (11) \) with the SLIV constraint \( (12) \) into the Lagrangian \( \mathcal{L} \) one is led to the highly nonlinear Yang-Mills theory in terms of the pure Goldstonic gauge field modes \( a^i_\mu \). However, as in the above Abelian case, one should
first gauge away (using the local invariance of the Lagrangian $\mathcal{L}$) the enormously large, while false, Lorentz violating terms appearing in the theory in the form of the fermion and vector field bilinears. As one can readily see, they stem from the couplings $g\bar{\psi}A_\mu\gamma^\mu\psi$ and $-\frac{1}{2}g^2Tr((A_\mu, A_\nu))^2$, respectively, when the effective Higgs mode expansion (12) is taken in the Lagrangian (8). Making the appropriate redefinitions of the fermion ($\bar{\psi}$) and vector ($a_\mu \equiv a_\mu^t t^i$) field multiplets

$$\bar{\psi} \rightarrow U(\omega)\bar{\psi}, \quad a_\mu \rightarrow U(\omega)a_\mu U(\omega)^\dagger, \quad U(\omega) = e^{igM(n^i \cdot x)t^i}$$

and using the evident equalities for the linear (in coordinate) transformations $U(\omega)$ with the single-valued vacuum matrix $n_\mu^i$ ($n_0^0$ or $n_3^0$ for the particular SLIV cases)

$$\partial_\mu U(\omega) = ign_\mu^i t^i U(\omega) = igU(\omega)n_\mu^i t^i$$

one can confirm that the abovementioned Lorentz violating terms are exactly cancelled with the analogous bilinears stemming from their kinetic terms. So, the final Lagrangian for the Goldstonic Yang-Mills theory takes the form (in the first approximation in $(a_\nu^i)^2/M^2$)

$$\mathcal{L}(a, \bar{\psi}) = -\frac{1}{4}Tr(f_{\mu\nu}f^{\mu\nu}) - \frac{1}{2}\delta(n \cdot a)^2 + \frac{1}{4}Tr(h_{\mu\nu})(n^2a_\mu^i)^2 + n_\mu^i \gamma^\mu\psi - \frac{gn^2(a_\mu^i)^2}{2M}\bar{\psi}(\gamma \cdot n^k)t^k\psi$$

(17)

where the tensor $f_{\mu\nu}$ is, as usual, $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu - ig[a_\mu, a_\nu]$, while $h_{\mu\nu}$ is a new SLIV oriented tensor of the type

$$h_{\mu\nu} = n_\mu^i \partial_\nu - n_\nu^i \partial_\mu + ig([n_\mu^i, a_\nu] - [n_\nu^i, a_\mu])$$

(18)

This tensor $h_{\mu\nu}$ acts on the infinite series in $(a_\nu^i)^2$ coming from the expansion of the effective Higgs mode (12) from which only the first order term $-n^2(a_\nu^i)^2/2M$ was taken throughout the Lagrangian $\mathcal{L}(a, \bar{\psi})$. We also retained the former notations for the fermion and vector field multiplets after transformations (15), and explicitly included the (axial) gauge fixing term into Lagrangian according to the supplementary conditions taken (13).

The theory derived gives a proper generalization of the nonlinear QED model (12) for the non-Abelian case. It contains the massless vector boson multiplet $a_\mu^i$ (consisting of one Goldstone and $D - 1$ pseudo-Goldstone vector states) which gauges the starting internal symmetry $G$. In the limit $M \rightarrow \infty$ it is indistinguishable from a conventional Yang-Mills theory taken in the general axial gauge. So, for this part of the Lagrangian $\mathcal{L}(a, \bar{\psi})$ given by the zero-order in $1/M$ terms the spontaneous Lorentz violation only means the noncovariant gauge choice in the otherwise gauge invariant (and Lorentz invariant) theory. However, one may expect that, just as it appears in the nonlinear QED model, also all the first and higher order in $1/M$ terms in the $\mathcal{L}$ (17), though being by themselves the Lorentz and $CPT$ violating ones, do not lead to the physical SLIV effects due to the mutual cancellation of their contributions into all the physical processes appeared.
4 The lowest order SLIV processes

Let us now show that the simple tree level calculations related to the Lagrangian \( \mathcal{L}(a, \psi) \) confirms in essence this proposition. As an illustration, we consider SLIV processes in the lowest order in \( g \) and \( 1/M \) being the fundamental parameters of the Lagrangian (17). They are, as one can readily see, the vector-fermion and vector-vector elastic scattering going in the order \( g/M \), which we turn to once the Feynman rules in the Goldstonic Yang-Mills theory are established.

4.1 Feynman rules

The corresponding Feynman rules, apart from the ordinary Yang-Mills theory rules for

(i) the vector-fermion vertex

\[ -ig \gamma_\mu t^i \tag{19} \]

(ii) the vector field propagator (taken in a general axial gauge \( n^\mu a^i_\mu = 0 \))

\[
D_{\mu \nu}^{ij}(k) = -\frac{i \delta^{ij}}{k^2}\left(g_{\mu \nu} - \frac{n_\mu k_\nu + k_\mu n_\nu}{n \cdot k} + \frac{n^2 k_\mu k_\nu}{(n \cdot k)^2}\right) \tag{20}
\]

which automatically satisfies the orthogonality condition \( n^\mu D_{\mu \nu}^{ij}(k) = 0 \) and on-shell transversality \( k_\mu D_{\mu \nu}^{ij}(k) = 0 \) \( (k^2 = 0) \); the latter means that free vector fields with polarization vector \( \epsilon^\mu_\mu(k, k^2 = 0) \) are always appeared transverse \( k_\mu \epsilon^\mu_\mu(k) = 0 \);

(iii) the 3-vector vertex (with vector field 4-momenta \( k_1, k_2 \) and \( k_3 \); all 4-momenta in vertexes are taken ingoing throughout)

\[
ge^{ijk}[(k_1 - k_2)_\gamma g_{\alpha \beta} + (k_2 - k_3)_\alpha g_{\beta \gamma} + (k_3 - k_1)_\beta g_{\alpha \gamma}] \tag{21}
\]

include the new ones, violating Lorentz and CPT invariance, for

(iv) the contact 2-vector-fermion vertex

\[
\frac{ig n^2}{M}(\gamma \cdot n^k) \tau^k g_{\mu \nu} \delta^{ij} \tag{22}
\]

(v) another 3-vector vertex

\[
-\frac{in^2}{M}\left[(k_1 \cdot n^i)k_{1, \alpha} g_{\beta \gamma} \delta^{ik} + (k_2 \cdot n^i)k_{2, \beta} g_{\alpha \gamma} \delta^{ki} + (k_3 \cdot n^i)k_{3, \gamma} g_{\alpha \beta} \delta^{ij}\right] \tag{23}
\]

where the second index in the vector field 4-momenta \( k_1, k_2 \) and \( k_3 \) denotes their Lorentz components;

(vi) the extra 4-vector vertex (with the vector field 4-momenta \( k_{1,2,3,4} \) and their proper differences \( k_{12} \equiv k_1 - k_2 \) etc.)

\[
\frac{n^2 g}{M}[\epsilon^{ijkp} \delta^{kl} g_{\alpha \beta} g_{\gamma \delta}(n^p \cdot k_{12}) + c^{klp} \delta^{ij} g_{\alpha \beta} g_{\gamma \delta}(n^p \cdot k_{34}) +
+ c^{jp} \delta^{ikl} g_{\alpha \beta} g_{\gamma \delta}(n^p \cdot k_{13}) + c^{lp} \delta^{ijk} g_{\alpha \gamma} g_{\beta \delta}(n^p \cdot k_{24}) +
+ c^{ip} \delta^{jk} g_{\alpha \delta} g_{\beta \gamma}(n^p \cdot k_{14}) + c^{jp} \delta^{ik} g_{\alpha \delta} g_{\beta \gamma}(n^p \cdot k_{23})] \tag{24}
\]
where only the terms which can not lead to contractions of the rectangular vacuum matrix \( n_i^\mu \) with vector field polarization vectors \( \epsilon^i_\mu(k) \) are presented. These contractions are in fact vanished due to the gauge taken \( n_i^\mu \cdot \epsilon^i = s^p(n \cdot \epsilon^i) = 0 \) (with a factorized two-vector form for the matrix \( n_i^\mu \) used).

Just the rules (i-vi) are needed to calculate the lowest order amplitudes of the processes we have mentioned in the above.

### 4.2 Vector boson scattering on fermion

This process is directly related to two SLIV diagrams one of which is given by the contact \( a^2 \)-fermion vertex \eqref{eq:22}, while another corresponds to the pole diagram with the longitudinal \( a \)-boson exchange between Lorentz violating \( a^3 \) vertex \eqref{eq:23} and ordinary \( a \)-boson-fermion one \eqref{eq:19}. Since ingoing and outgoing \( a \)-bosons appear transverse \( k_1 \cdot \epsilon^i(k_1) = 0, k_2 \cdot \epsilon^j(k_2) = 0 \) only the third term in this \( a^3 \) coupling \eqref{eq:23} contributes to the pole diagram so that one comes to a simple matrix element

\[
iM = \frac{i g a^2}{\sqrt{\lambda}} \bar{u}(p_2) \tau^i \left[ (\gamma \cdot n^i) + i(k \cdot n^i)\gamma^\mu k^\nu D_{\mu\nu}(k) \right] u(p_1) [\epsilon(k_1) \cdot \epsilon(k_2)]
\]

where the spinors \( u(p_{1,2}) \) and polarization vectors \( \epsilon^i_\mu(k_1) \) and \( \epsilon^j_\mu(k_2) \) stand for the ingoing and outgoing fermions and \( a \)-bosons, respectively, while \( k \) is the 4-momentum transfer \( k = p_2 - p_1 = k_1 - k_2 \). Upon the further simplifications in the square bracket related to the explicit form of the \( a \)-boson propagator \( D_{\mu\nu}(k) \) \eqref{eq:20} and matrix \( n_i^\mu \) \eqref{eq:14}, and using the fermion current conservation \( \bar{u}(p_2)(\hat{p}_2 - \hat{p}_1)u(p_1) = 0 \), one is finally led to the total cancellation of the Lorentz violating contributions to the \( a \)-boson-fermion scattering in the \( g/M \) approximation.

Note, however, that such a result may be in some sense expected since from the SLIV point of view the lowest order \( a \)-boson-fermion scattering discussed here is hardly distinct from the photon-fermion scattering considered in the nonlinear QED case\[12\]. Actually, the fermion current conservation which happens to be crucial for the above cancellation works in both of cases, whereas the couplings being peculiar to the Yang-Mills theory have not yet touched on. In this connection the next example seems to be more instructive.

### 4.3 Vector-vector scattering

The matrix element for this process in the lowest order \( g/M \) is given by the contact SLIV \( a^4 \) vertex \eqref{eq:24} and the pole diagrams with the longitudinal \( a \)-boson exchange between the ordinary \( a^3 \) vertex \eqref{eq:21} and Lorentz violating \( a^3 \) one \eqref{eq:23}, and vice versa. There are six pole diagrams in total describing the elastic \( a - a \) scattering in the \( s \)- and \( t \)-channels, respectively, including also those with an interchange of identical \( a \)-bosons. Remarkably, the contribution of each of them is exactly canceled with one of six terms appeared in the contact vertex \eqref{eq:24}. Actually, writing down the matrix element for one of the pole diagrams with ingoing \( a \)-bosons (with momenta \( k_1 \)
and $k_2$) interacting through the vertex $\{21\}$ and outgoing $a$-bosons (with momenta $k_3$ and $k_4$) interacting through the vertex $\{23\}$ one has

$$iM_{pole}^{(1)} = -i\frac{gm^2}{M} \epsilon^{ijkl}[(k_1 - k_2)_{\mu} g_{\alpha\beta} + (k_2 - k)_{\alpha} g_{\beta\mu} + (k - k_1)_{\beta} g_{\alpha\mu}] \cdot$$

$$\cdot D_{\mu\nu}(k) g_{\gamma\delta} k_{\nu} (n^\gamma \cdot k) [\epsilon^{i,\alpha}(k_1) \epsilon^{j,\beta}(k_2) \epsilon^{k,\gamma}(k_3) \epsilon^{l,\delta}(k_4)] \quad (26)$$

where polarization vectors $\epsilon^{i,\alpha}(k_1), \epsilon^{j,\beta}(k_2), \epsilon^{k,\gamma}(k_3)$ and $\epsilon^{l,\delta}(k_4)$ belong, respectively, to ingoing and outgoing $a$-bosons, while $k = -(k_1 + k_2) = k_3 + k_4$ according to the momentum running in the diagrams taken above. Again, as in the previous case of vector-fermion scattering, due to the fact that outgoing $a$-bosons appear transverse ($k_3 \cdot \epsilon^k(k_3) = 0$ and $k_4 \cdot \epsilon^l(k_4) = 0$), only the third term in the Lorentz violating $a^3$ coupling $\{23\}$ contributes to this pole diagram. Upon evident simplifications related to the $a$-boson propagator $D_{\mu\nu}(k) \{21\}$ and matrix $n^i_{\mu} \{14\}$ one comes to the expression which is exactly cancelled with the first term in the contact SLIV vertex $\{24\}$ when it is properly contracted with $a$-boson polarization vectors. Likewise, other terms in this vertex provide the further one-to-one cancellation with the remaining pole matrix elements $iM_{pole}^{(2-6)}$. So, again, the Lorentz violating contribution to the vector-vector scattering is absent in Goldstonic Yang-Mills theory in the lowest $g/M$ approximation.

4.4 Other processes

Other tree level Lorentz violating processes, related to $a$ bosons and fermions, appear in higher orders in the basic SLIV parameter $1/M$. They come from the subsequent expansion of the effective Higgs mode $\{12\}$ in the Lagrangian $\{17\}$. Again, their amplitudes are essentially determined by an interrelation between the longitudinal $a$-boson exchange diagrams and the corresponding contact $a$-boson interaction diagrams which appear to cancel each other thus eliminating physical Lorentz violation in theory.

Most likely, the same conclusion can be derived for SLIV loop contributions as well. Actually, as in the massless QED case considered earlier $\{19\}$, the corresponding one-loop matrix elements in Goldstonic Yang-Mills theory either vanish by themselves or amount to the differences between pairs of the similar integrals whose integration variables are shifted relative to each other by some constants (being in general arbitrary functions of external four-momenta of the particles involved) that in the framework of dimensional regularization leads to their total cancellation.

So, the Goldstonic vector field theory $\{17\}$ for a non-Abelian charge-carrying matter is likely to be physically indistinguishable from a conventional Yang-Mills theory.

5 Conclusion

The spontaneous Lorentz violation in 4-dimensional flat Minkowskian space-time was shown to generate vector Goldstone bosons both in Abelian and non-Abelian theo-
ries with the corresponding nonlinear vector field constraint (1) or (9) imposed. In the Abelian case such a massless vector boson is naturally associated with photon. In non-Abelian case, although the pure Lorentz violation still generates only one genuine Goldstone vector boson, the accompanying vector PGs related to a violation of the larger accidental symmetry $SO(D, 3D)$ of the SLIV constraint (9) in itself come also into play in the final arrangement of the entire Goldstone vector field multiplet of the internal symmetry group $G$. Remarkably, they remain strictly massless being protected by the gauge invariance of the Yang-Mills theory involved. These theories, both Abelian and non-Abelian, while being essentially nonlinear in the Goldstone vector modes, are physically indistinguishable from conventional QED and Yang-Mills theory. One could actually see that just the gauge invariance not only provides these theories to be free from the unreasonably large Lorentz violation stemming from the fermion and vector field bilinears (see Sections 2 and 3), but also render all the other physical SLIV effects (including those which are suppressed by the Lorentz violation scale $M$) non-observable (Section 4). As a result, Abelian and non-Abelian SLIV theory appear, respectively, as standard QED and Yang-Mills theory taken in the nonlinear gauge (to which the vector field constraints (1) and (9) are virtually reduced), while the $S$-matrix remains unaltered under such a gauge convention.

So, while at present the Goldstonic nature of gauge fields, both Abelian and non-Abelian, seems to be highly plausible, the most fundamental question of physical Lorentz violation in itself, that only could uniquely point toward such a possibility, is still an open question. Note, that here we are not dealing with direct (and quite arbitrary in essence) Lorentz non-invariant extensions of QED or Standard Model which were intensively discussed on their own in recent years [6-8]. Rather, the case in point is a construction of genuine SLIV models which would generate gauge fields as the proper vector Goldstone bosons, from one hand, and could lead to observed Lorentz violating effects, from the other. In this connection, somewhat natural framework for physical Lorentz violation to occur would be a model where the internal gauge invariance were slightly broken at very small distances through some high-order operators stemming from the gravity-influenced area. Such physical SLIV effects would be seen in terms of powers of ratio $\frac{M}{M_{Pl}}$ (where $M_{Pl}$ is the Planck mass). So, for the SLIV scale comparable with the Planck one they would become directly observable. Remarkably enough, if one has such internal gauge symmetry breaking in an ordinary Lorentz invariant theory this breaking appears vanishingly small at laboratory being properly suppressed by the Planck scale. However, the spontaneous Lorentz violation would render it physically significant: the higher Lorentz scale, the greater SLIV effects observed. If true, it would be of particular interest to have a better understanding of the internal gauge symmetry breaking mechanism that brings out the spontaneous Lorentz violation at low energies. We return to this basic question elsewhere.
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References

[1] W. Heisenberg, Rev. Mod. Phys. 29 (1957) 269;
   J.D. Bjorken, Ann. Phys. (N.Y.) 24 (1963) 174;
   I. Bialynicki-Birula, Phys. Rev. 130 (1963) 465;
   G. Guralnik, Phys. Rev. 136 (1964) B1404;
   T. Eguchi, Phys. Rev. D 14 (1976) 2755;
   H. Terazawa, Y. Chikashige and K. Akama, Phys. Rev. D 15 (1977) 480.

[2] C.D. Froggatt and H.B. Nielsen, Origin of Symmetries (World Scientific, Singapore, 1991).

[3] J.L. Chkareuli, C.D. Froggatt and H.B. Nielsen, Phys. Rev. Lett. 87 (2001) 091601;
   J.L. Chkareuli, C.D. Froggatt and H.B. Nielsen Nucl. Phys. B 609 (2001) 46;
   J.D. Bjorken, [hep-th/0111196];
   Per Kraus and E.T. Tomboulis, Phys. Rev. D 66 (2002) 045015;
   A. Jenkins, Phys. Rev. D 69 (2004) 105007;
   J.L. Chkareuli, C.D. Froggatt, R.N. Mohapatra and H.B. Nielsen, [hep-th/0412225];
   J.L. Chkareuli, C.D. Froggatt and H.B. Nielsen, [hep-th/0610186].

[4] D. Colladay and V.A. Kostelecky, Phys. Rev. D58 (1998) 116002;
   V.A. Kostelecky, Phys. Rev. D69 (2004) 105009;
   R. Bluhm and V.A. Kostelecky, Phys. Rev. D 71(2005) 065008;
   CPT and Lorentz Symmetry, ed. A. Kostelecky (World Scientific, Singapore, 1999, 2002, 2005).

[5] S.M. Carroll, G.B. Field and R. Jackiw, Phys. Rev. D 41 (1990) 1231;
   R. Jackiw and V.A. Kostelecky, Phys. Rev. Lett. 82 (1999) 3572.

[6] S. Coleman and S.L. Glashow, Phys. Rev. D 59 (1999) 116008.
[7] J. W. Moffat, Int. J. Mod. Phys. D2 (1993) 351; J.W. Moffat, Int. J. Mod. Phys. D12 (2003) 1279.

[8] O. Bertolami and D.F. Mota, Phys. Lett. B 455 (1999) 96.

[9] T. Jacobson, S. Liberati and D. Mattingly, Ann. Phys. (N.Y.) 321 (2006) 150.

[10] Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122 (1961) 345.

[11] M. Suzuki, Phys. Rev. D 37 (1988) 210.

[12] Y. Nambu, Progr. Theor. Phys. Suppl. Extra 190 (1968).

[13] S. Weinberg, The Quantum Theory of Fields, v.2, Cambridge University Press, 2000.

[14] V.A. Kostelecky and S. Samuel, Phys. Rev. D 39 (1989) 683; V.A. Kostelecky and R. Potting, Nucl. Phys. B 359 (1991) 545.

[15] P.A.M. Dirac, Proc. Roy. Soc. 209A (1951) 292; P.A.M. Dirac, Proc. Roy. Soc. 212A (1952) 330.

[16] R. Righi and G. Venturi, Lett. Nuovo Cim. 19 (1977) 633; R. Righi, G. Venturi and V. Zamiralov, Nuovo Cim. A47 (1978) 518.

[17] Y.M. Cho and P.G.O. Freund, Phys. Rev. D 12 (1975) 1711.

[18] E. A. Ivanov and V.I. Ogievetsky, Lett. Math. Phys. 1 (1976) 309.

[19] A.T. Azatov and J.L. Chkareuli, Phys. Rev. D 73 (2006) 065026.

[20] J.L. Chkareuli and Z.R. Kepuladze, Phys. Lett. B 644 (2007) 212; J.L. Chkareuli and Z.R. Kepuladze, Proc. of XIV Int. Seminar “Quarks-2006”, eds. S.V. Demidov at al (Moscow, INR, 2006); [hep-th/0610227]