On the semi-scalar equivalence of polynomial matrices

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Abstract — Polynomial matrices $A(\lambda)$ and $B(\lambda)$ of size $n \times n$ over a field $F$ are semi-scalar equivalent if there exist a nonsingular $n \times n$ matrix $P$ over $F$ and an invertible $n \times n$ matrix $Q(\lambda)$ over $F[\lambda]$ such that $A(\lambda) = PB(\lambda)Q(\lambda)$. The aim of the present report is to present a triangular form of some nonsingular polynomial matrices with respect to semi-scalar equivalence.

Keywords — Polynomial matrix; Equivalence of matrices; Smith normal form.

I. INTRODUCTION

Let $F$ be a field. Denote by $M_{n,n}(F)$ the set of $n \times n$ matrices over $F$ and by $M_{n,n}(F[\lambda])$ the set of $n \times n$ matrices over the polynomial ring $F[\lambda]$. In what follows, $I_n$ is the identity $n \times n$ matrix and $O_n$ is the zero $n \times n$ matrix. A polynomial $a(\lambda) = a_0 + a_1 \lambda + \cdots + a_k \lambda^k \in F[\lambda]$ is said to be monic if the first non-zero term $a_k = 1$.

Let $A(\lambda) \in M_{n,n}(F[\lambda])$ be a nonzero matrix and rank $A(\lambda) = r$. For the matrix $A(\lambda)$ there exist matrices $U(\lambda), V(\lambda) \in GL(n, F[\lambda])$ such that $U(\lambda)A(\lambda)V(\lambda) = S_A(\lambda) = \text{diag}(s_1(\lambda), s_2(\lambda), \ldots, s_r(\lambda), 0, \ldots, 0)$, where $s_i(\lambda)$ are monic polynomials for all $i = 1, 2, \ldots, r$ and $s_1(\lambda)|s_2(\lambda)|\ldots|s_r(\lambda)$ (divides) are the invariant factors of $A(\lambda)$. The diagonal matrix $S_A(\lambda)$ is called the Smith normal form of $A(\lambda)$.

Matrices $A(\lambda), B(\lambda) \in M_{n,n}(F[\lambda])$ are said to be semi-scalar equivalent if there exist matrices $P \in GL(n, F)$ and $Q(\lambda) \in GL(n, F[\lambda])$ such that $A(\lambda) = PB(\lambda)Q(\lambda)$ (see [1], Chapter 4).

Let $A(\lambda) \in M_{n,n}(F[\lambda])$ be nonsingular matrix over an infinite field $F$. Then $A(\lambda)$ is semi-scalar equivalent to the lower triangular matrix $[1]$ with the following properties:

- $s_i(\lambda)$, $i = 1, 2, \ldots, n$; are the invariant factors of $A(\lambda)$;
- $s_j(\lambda)$ divides $s_j(\lambda)$ for all $1 \leq i < j \leq n$.

Let $F = \{0, 1\}$ be a field of two elements. It is easily verified that the polynomial matrix

$$A(\lambda) = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^2 + 1 \end{bmatrix}$$

over the field $F$ is not semi-scalar equivalent to the lower triangular matrix $S_A(\lambda) = \begin{bmatrix} 1 & 0 \\ \lambda(\lambda^2 + 1)(\lambda^2 + \lambda + 1) & 1 \end{bmatrix}$.

Thus, the triangular form $S_A(\lambda)$ for nonsingular matrices over a finite field not always exists. It may be noted that for a singular matrix $A(\lambda)$ the matrix $S_A(\lambda)$ does not always exist.

Example. Let $F = R$ be the field of real numbers. For $2 \times 2$ matrices

$$A(\lambda) = \begin{bmatrix} \lambda & 0 \\ \lambda^3 - 3\lambda^2 - \lambda & \lambda^2 - 1(\lambda^2 - 2\lambda) \end{bmatrix}$$

and

$$B(\lambda) = \begin{bmatrix} \lambda & 0 \\ \lambda^3 - \lambda^2 - \lambda & \lambda^2 - 1(\lambda^2 - 2\lambda) \end{bmatrix}$$

with entries from $R[\lambda]$ there exist matrices

$$Q(\lambda) = \begin{bmatrix} 2\lambda^3 - 6\lambda^2 - 2\lambda + 9 \\ -2\lambda^2 + 4\lambda + 4 \end{bmatrix}$$

and $P = \begin{bmatrix} 1/9 & -2/9 \\ 0 & 1 \end{bmatrix} \in GL(2, R)$ such that $A(\lambda) = PB(\lambda)Q(\lambda)$.

From this example it follows, that the triangular form $S_A(\lambda)$ is not uniquely determined for a nonsingular polynomial matrix $A(\lambda)$ with respect to semi-scalar equivalence.
Dias da Silva J.A. and Laffey T.J. studied polynomial matrices up to PS-equivalence [2]. Matrices $A(\lambda), B(\lambda) \in M_{n,n}(F[\lambda])$ are PS-equivalent if $A(\lambda) = P(\lambda)B(\lambda)Q$ for some $P(\lambda) \in GL(n, F[\lambda])$ and $Q \in GL(n, F)$.

Let $F$ be an infinite field. A nonsingular matrix $A(\lambda) \in M_{n,n}(F[\lambda])$ is PS-equivalent to the upper triangular matrix (see [2], Proposition 2)

$$S_n(\lambda) = \begin{bmatrix} s_1(\lambda) & s_2(\lambda) & s_3(\lambda) & \cdots & s_n(\lambda) \\ 0 & s_2(\lambda) & s_3(\lambda) & \cdots & s_n(\lambda) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

with the following properties:

1. $s_i(\lambda), i = 1, 2, \ldots, n$, are the invariant factors of $A(\lambda)$;
2. $s_1(\lambda)$ divides $s_2(\lambda)$ for all $1 \leq i < j \leq n$;
3. if $i \neq j$ and $s_i(\lambda) \neq 0$ then $s_j(\lambda)$ is a monic polynomial and $\deg s_i(\lambda) < \deg s_j(\lambda) < \deg s_1(\lambda)$.

The matrix $S_n(\lambda)$ is called a near canonical form of the matrix $A(\lambda)$ with respect to PS-equivalence. We note that conditions (1) and (2) for semi-scalar equivalence were proved in [1]. It is evident that matrices $A(\lambda), B(\lambda) \in M_{n,n}(F[\lambda])$ are PS-equivalent if and only if the transpose matrices $A^T(\lambda)$ and $B^T(\lambda)$ are semi-scalar equivalent. It is clear that semi-scalar equivalence and PS-equivalence represent an equivalence relation on $M_{n,n}(F[\lambda])$. On the basis of the semi-scalar equivalence of polynomial matrices in [1] algebraic methods for factorization of matrix polynomials were developed. We note that these equivalences were used in the study of the controllability of linear systems (see [3], [4]).

The semi-scalar equivalence and PS-equivalence of matrices over a field $F$ contain the problem of similarity between two families of matrices ([1], [2], [5–7]). In most cases, these problems are involved with the classic unsolvable problem of a canonical form of a pair of matrices over a field with respect to simultaneous similarity. At present, such problems are called wild [5].

The semi-scalar equivalence of matrices includes the following two tasks: (1) the determination of a complete system of invariants and (2) the construction of a canonical form for a matrix with respect to semi-scalar equivalence. But these tasks have satisfactory solutions only in isolated cases. The canonical and normal forms with respect to semi-scalar equivalence for a matrix pencil $A(\lambda) = A_0 + A_1 \lambda \in M_{n,n}(F[\lambda])$ over arbitrary field $F$, where $A_0$ is nonsingular, were investigated in [8] and [9]. A canonical form with respect to semi-scalar equivalence for a polynomial matrix over a field is unknown in general case.

### II. MAIN RESULTS

In this part we present main results of this report.

**Theorem.** Let $A(\lambda) \in M_{n,n}(F[\lambda])$ be a nonsingular matrix with the Smith normal form $U(\lambda)A(\lambda)V(\lambda) = S_n(\lambda) = \text{diag}(1, s(\lambda), \ldots, s(\lambda))$, where $s(\lambda)$ is a monic polynomial and $\deg s(\lambda) = n$.

The matrix $A(\lambda)$ is semi-scalar to the matrix

$$S_n(\lambda) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \lambda & s(\lambda) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda^{n-2} & 0 & \cdots & 0 & s(\lambda) \\ \lambda^{n-1} & 0 & \cdots & 0 & 0 \end{bmatrix}$$

if and only if the matrix $A(\lambda)$ admits the representation $B(\lambda) = W(\lambda)A(\lambda)C(\lambda)$,

where $W(\lambda) \in GL(n, F[\lambda])$ and $B(\lambda) = I_n + B_1 \lambda^{-1} + \cdots + B_{n-1} \lambda^{n-1}$, is a monic polynomial matrix of degree $n - 1$. The matrix $S_n(\lambda)$ is uniquely defined for the matrix $A(\lambda)$.

Let $B(\lambda) \in M_{n,n}(F[\lambda])$. The matrix $B(\lambda)$ we write in the form $B(\lambda) = B_0 \lambda^{r_0} + B_1 \lambda^{r_1} + \cdots + B_r \lambda^{r_r}$, where $B_i \in M_{n,n}(F)$, $i = 1, 2, \ldots, n$. It is well known that a matrix polynomial equation $X^{r_0}B_0 + X^{r_1}B_1 + \cdots + XB_r + B_0 = O_n$ is solvable if and only if the matrix $B(\lambda)$ admits the representation $B(\lambda) = (I_n - D\lambda)C(\lambda)$, where $D \in M_{n,n}(F)$ [10]. The problem of solvability of matrix polynomial equations was investigated by many authors (see [1], [11–14] and references therein).

Following propositions gives a complete answer to the question of solvability of a matrix polynomial equation of second order over an infinite field (see also [14]).

Let $A(\lambda) = \sum_{i=-1}^{s} A_i \lambda^{r_i} \in M_{2,2}(F[\lambda])$ be a nonsingular matrix. Further, let

$$S_1(\lambda) = \begin{bmatrix} s_1(\lambda) & 0 \\ s_2(\lambda) & s_2(\lambda) \end{bmatrix}$$

be a near canonical form of the matrix $A(\lambda)$ with respect to semi-scalar equivalence. By [9] and based on the above, we get the following statements.

**Proposition 1.** Let $s_1(\lambda) = (\lambda - \alpha_1)c_1(\lambda)$ and $s_2(\lambda) = (\lambda - \alpha_2)c_2(\lambda)$, where $\alpha_i \in F$. A matrix...
polynomial equation

\[ X^r A_0 + X^{r-1} A_1 + \ldots + X A_{r-1} + A_r = O_2 \] is solvable over a field \( F \) if and only if there exists \( \beta \in F \) such that the matrix

\[
D_\beta (\lambda) = \begin{bmatrix}
\lambda - \alpha_1 & 0 \\
\beta & \lambda - \alpha_2
\end{bmatrix}
\]
is a left divisor of \( S_\delta (\lambda) \), i.e., \( S_\delta (\lambda) = D_\beta (\lambda) C(\lambda) \).

**Proposition 2.** Let \( S_\delta (\lambda) = (\lambda^2 + \lambda \alpha_1 + \alpha_2)c_2(\lambda) \), where \( \lambda^2 + \lambda \alpha_1 + \alpha_2 \in F[\lambda] \). A matrix polynomial equation

\[ X^r A_0 + X^{r-1} A_1 + \ldots + X A_{r-1} + A_r = O_2 \]
is solvable over a field \( F \) if and only if there exists \( \delta_\delta, \delta_1, \delta_2 \in F \) and \( \delta_0 \neq 0 \) such that the matrix

\[
D_{\delta_\delta}(\lambda) = \begin{bmatrix}
1 & 0 \\
\delta_0 \lambda + \delta_1 & \lambda^2 + \lambda \alpha_1 + \alpha_2
\end{bmatrix}
\]
is a left divisor of \( S_\delta (\lambda) \), i.e., \( S_\delta (\lambda) = D_{\delta_\delta}(\lambda) C(\lambda) \).

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