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Abstract

It is noted that the internal space-time symmetries of relativistic particles are dictated by Wigner’s little groups. The symmetry of massive particles is like the three-dimensional rotation group, while the symmetry of massless particles is locally isomorphic to the two-dimensional Euclidean group. It is noted also that, while the rotational degree of freedom for a massless particle leads to its helicity, the two translational degrees of freedom correspond to its gauge degrees of freedom. It is shown that the $E(2)$-like symmetry of of massless particles can be obtained as an infinite-momentum and/or zero-mass limit of the $O(3)$-like symmetry of massive particles. This mechanism is illustrated in terms of a sphere elongating into a cylinder. In this way, the helicity degree of freedom remains invariant under the Lorentz boost, but the transverse rotational degrees of freedom become contracted into the gauge degree of freedom.

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1 Introduction

If the momentum of a particle is much smaller than its mass, the energy-momentum relation is \( E = \frac{p^2}{2m} + mc^2 \). If the momentum is much larger than the mass, the relation is \( E = cp \). These two different relations can be combined into one covariant formula \( E = \sqrt{m^2 + p^2} \). This aspect of Einstein’s \( E = mc^2 \) is also well known.

In addition, particles have internal space-time variables. Massive particles have spins while massless particles have their helicities and gauge degrees of freedom. As a “further content” of Einstein’s \( E = mc^2 \), we shall discuss that the internal space-time structures of massive and massless particles can be unified into one covariant package, as \( E = \sqrt{m^2 + p^2} \) does for the energy-momentum relation.

The mathematical framework of this program was developed by Eugene Wigner in 1939 \[1\]. He constructed the maximal subgroups of the Lorentz group whose transformations will leave the four-momentum of a given particle invariant. These groups are known as Wigner’s little groups. Thus, the transformations of the little groups change the internal space-time variables of the particle, while leaving its four-momentum invariant. The little group is a covariant entity and takes different forms for the particles moving with different speeds.

In order to achieve the zero-mass and/or infinite-momentum limit of the \( O(3) \)-like little group to obtain the \( E(2) \)-like little group, we use the group contraction technique introduced by Inonu and Wigner \[2\], who obtained the \( E(2) \) group by taking a flat-surface approximation of a spherical surface at the north pole. In 1987, Kim and Wigner \[3\] observed that it is also possible to make a cylindrical approximation of the spherical surface around the equatorial belt. While the correspondence between \( O(3) \) and the \( O(3) \)-like little group is transparent, the \( E(2) \)-like little group contains both the \( E(2) \) group and the cylindrical group \[4\]. We study this aspect in detail in this report.

The space-time symmetries we are discussing in this report are applicable to all theoretical models of elementary and composite particles. Thus, model builders should be aware that their models should satisfy these basic symmetries. They are not going to build theoretical models which will violate the conservation of energy, nor are they going to come up with models which will violate these basic space-time symmetries.

In Sec. 2 we present a brief history of applications of the little groups to internal space-time symmetries of relativistic particles. It is pointed out in Sec. 3 that the translation-like transformations of the \( E(2) \)-like little group corresponds to gauge transformations.

In Sec. 4 we discuss the contraction of the three-dimensional rotation group to the two-dimensional Euclidean group. In Sec. 5, we discuss the little group for a massless particle as the infinite-momentum and/or zero-mass limit of the little group for a massive particle.

The Lorentz covariance is one of the fundamental issues in modern physics. In this paper, we study in this paper for spin-1 particles as a four-by-four representations of
the Lorentz group. However, there are many other interesting particles. Of immediate interest is whether this formalism can be applied to spin-1/2 particles. Another interesting case is a relativistic extended hadrons. We summarize the current status of these research lines in Sec. 6.

2 Little Groups of the Poincaré Group

The Poincaré group is the group of inhomogeneous Lorentz transformations, namely Lorentz transformations preceded or followed by space-time translations. In order to study this group, we have to understand first the group of Lorentz transformations, the group of translations, and how these two groups are combined to form the Poincaré group. The Poincaré group is a semi-direct product of the Lorentz and translation groups. The two Casimir operators of this group correspond to the (mass)$^2$ and (spin)$^2$ of a given particle. Indeed, the particle mass and its spin magnitude are Lorentz-invariant quantities.

The question then is how to construct the representations of the Lorentz group which are relevant to physics. For this purpose, Wigner in 1939 studied the subgroups of the Lorentz group whose transformations leave the four-momentum of a given free particle invariant. The maximal subgroup of the Lorentz group which leaves the four-momentum invariant is called the little group. Since the little group leaves the four-momentum invariant, it governs the internal space-time symmetries of relativistic particles. Wigner shows in his paper that the internal space-time symmetries of massive and massless particles are dictated by the $O(3)$-like and $E(2)$-like little groups respectively.

The $O(3)$-like little group is locally isomorphic to the three-dimensional rotation group, which is very familiar to us. For instance, the group $SU(2)$ for the electron spin is an $O(3)$-like little group. The group $E(2)$ is the Euclidean group in a two-dimensional space, consisting of translations and rotations on a flat surface. We are performing these transformations everyday on ourselves when we move from home to school. The mathematics of these Euclidean transformations are also simple. However, the group of these transformations are not well known to us. In Sec. 4, we give a matrix representation of the $E(2)$ group.

The group of Lorentz transformations consists of three boosts and three rotations. The rotations therefore constitute a subgroup of the Lorentz group. If a massive particle is at rest, its four-momentum is invariant under rotations. Thus the little group for a massive particle at rest is the three-dimensional rotation group. Then what is affected by the rotation? The answer to this question is very simple. The particle in general has its spin. The spin orientation is going to be affected by the rotation!

If the rest-particle is boosted along the $z$ direction, it will pick up a non-zero momentum component. The generators of the $O(3)$ group will then be boosted. The boost will take the form of conjugation by the boost operator. This boost will not change
the Lie algebra of the rotation group, and the boosted little group will still leave the
boosted four-momentum invariant. We call this the $O(3)$-like little group. If we use the
four-vector coordinate $(x, y, z, t)$, the four-momentum vector for the particle at rest is
$(0, 0, 0, m)$, and the three-dimensional rotation group leaves this four-momentum invari-
ant. This little group is generated by

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (1)$$

which satisfy the commutation relations:

$$[J_i, J_j] = i\epsilon_{ijk}J_k. \quad (2)$$

It is not possible to bring a massless particle to its rest frame. In his 1939 paper [1],
Wigner observed that the little group for a massless particle moving along the $z$ axis
is generated by the rotation generator around the $z$ axis, namely $J_3$ of Eq.(1), and two
other generators which take the form

$$N_1 = \begin{pmatrix} 0 & 0 & -i & i \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & i \\ 0 & i & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}. \quad (3)$$

If we use $K_i$ for the boost generator along the $i$-th axis, these matrices can be written
as

$$N_1 = K_1 - J_2, \quad N_2 = K_2 + J_1, \quad (4)$$

with

$$K_1 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}. \quad (5)$$

The generators $J_3, N_1$ and $N_2$ satisfy the following set of commutation relations.

$$[N_1, N_2] = 0, \quad [J_3, N_1] = iN_2, \quad [J_3, N_2] = -iN_1. \quad (6)$$

In Sec. 4, we discuss the generators of the $E(2)$ group. They are $J_3$ which generates
rotations around the $z$ axis, and $P_1$ and $P_2$ which generate translations along the $x$
and $y$ directions respectively. If we replace $N_1$ and $N_2$ by $P_1$ and $P_2$, the above set of
commutation relations becomes the set given for the $E(2)$ group given in Eq.(18). This
is the reason why we say the little group for massless particles is $E(2)$-like. Very clearly,
the matrices $N_1$ and $N_2$ generate Lorentz transformations.
It is not difficult to associate the rotation generator $J_3$ with the helicity degree of freedom of the massless particle. Then what physical variable is associated with the $N_1$ and $N_2$ generators? Indeed, Wigner was the one who discovered the existence of these generators, but did not give any physical interpretation to these translation-like generators. For this reason, for many years, only those representations with the zero-eigenvalues of the $N$ operators were thought to be physically meaningful representations \cite{5}. It was not until 1971 when Janner and Janssen reported that the transformations generated by these operators are gauge transformations \cite{6, 7, 9}. The role of this translation-like transformation has also been studied for spin-1/2 particles, and it was concluded that the polarization of neutrinos is due to gauge invariance \cite{8, 10}.

Another important development along this line of research is the application of group contractions to the unifications of the two different little groups for massive and massless particles. We always associate the three-dimensional rotation group with a spherical surface. Let us consider a circular area of radius 1 kilometer centered on the north pole of the earth. Since the radius of the earth is more than 6,450 times longer, the circular region appears flat. Thus, within this region, we use the $E(2)$ symmetry group for this region. The validity of this approximation depends on the ratio of the two radii.

In 1953, Inouu and Wigner formulated this problem as the contraction of $O(3)$ to $E(2)$ \cite{2}. How about then the little groups which are isomorphic to $O(3)$ and $E(2)$? It is reasonable to expect that the $E(2)$-like little group be obtained as a limiting case for of the $O(3)$-like little group for massless particles. In 1981, it was observed by Ferrara and Savoy that this limiting process is the Lorentz boost \cite{11}. In 1983, using the same limiting process as that of Ferrara and Savoy, Han et al showed that transverse rotation generators become the generators of gauge transformations in the limit of infinite momentum and/or zero mass \cite{12}. In 1987, Kim and Wigner showed that the little group for massless particles is the cylindrical group which is isomorphic to the $E(2)$ group \cite{3}. This is illustrated in Fig. 1.

### 3 Translations and Gauge Transformations

It is possible to get the hint that the $N$ operators generate gauge transformations from Weinberg’s 1964 papers \cite{5, 8}. But it was not until 1971 when Janner and Janssen explicitly demonstrated that they generate gauge transformations \cite{6}. In order to fully appreciate their work, let us compute the transformation matrix

$$\exp (-i(uN_1 + vN_2))$$

generated by $N_1$ and $N_2$. Then the four-by-four matrix takes the form

$$\begin{pmatrix}
1 & 0 & -u & u \\
0 & 1 & -v & v \\
u & v & 1 - (u^2 + v^2)/2 & (u^2 + v^2)/2 \\
u & 0 & -(u^2 + v^2)/2 & 1 + (u^2 + v^2)/2
\end{pmatrix}.$$  

(8)
Figure 1: Contraction of O(3) to E(2) and to the cylindrical group, and contraction of the O(3)-like little group to the E(2)-like little group. The correspondence between E(2) and the E(2)-like little group is isomorphic but not identical. The cylindrical group is identical to the E(2)-like little group. The Lorentz boost of the O(3)-like little group for a massive particle is the same as the contraction of O(3) to the cylindrical group.

If we apply this matrix to the four-vector to the four-momentum vector

\[ p = (0, 0, \omega, \omega) \]  

of a massless particle, the momentum remains invariant. It therefore satisfies the condition for the little group. If we apply this matrix to the electromagnetic four-potential

\[ A = (A_1, A_2, A_3, A_0) \exp(i(kz - \omega t)) \]  

with \( A_3 = A_0 \) which is the Lorentz condition, the result is a gauge transformation. This is what Janner and Janssen discovered in their 1971 and 1972 papers [6]. Thus the matrices \( N_1 \) and \( N_2 \) generate gauge transformations.

## 4 Contraction of O(3) to E(2)

In this Appendix, we explain what the \( E(2) \) group is. We then explain how we can obtain this group from the three-dimensional rotation group by making a flat-surface or cylindrical approximation. This contraction procedure will give a clue to obtaining the \( E(2) \)-like symmetry for massless particles from the \( O(3) \)-like symmetry for massive particles by making the infinite-momentum limit.
The \( E(2) \) transformations consist of rotation and two translations on a flat plane. Let us start with the rotation matrix applicable to the column vector \((x, y, 1)\):

\[
R(\theta) = \begin{pmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix}.
\] (11)

Let us then consider the translation matrix:

\[
T(a, b) = \begin{pmatrix}
1 & 0 & a \\
0 & 1 & b \\
0 & 0 & 1
\end{pmatrix}.
\] (12)

If we take the product \( T(a, b)R(\theta) \),

\[
E(a, b, \theta) = T(a, b)R(\theta) = \begin{pmatrix}
\cos \theta & -\sin \theta & a \\
\sin \theta & \cos \theta & b \\
0 & 0 & 1
\end{pmatrix}.
\] (13)

This is the Euclidean transformation matrix applicable to the two-dimensional \( xy \) plane. The matrices \( R(\theta) \) and \( T(a,b) \) represent the rotation and translation subgroups respectively. The above expression is not a direct product because \( R(\theta) \) does not commute with \( T(a,b) \). The translations constitute an Abelian invariant subgroup because two different \( T \) matrices commute with each other, and because

\[
R(\theta)T(a,b)R^{-1}(\theta) = T(a',b').
\] (14)

The rotation subgroup is not invariant because the conjugation

\[
T(a,b)R(\theta)T^{-1}(a,b)
\] (15)

does not lead to another rotation.

We can write the above transformation matrix in terms of generators. The rotation is generated by

\[
J_3 = \begin{pmatrix}
0 & -i & 0 \\
i & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\] (16)

The translations are generated by

\[
P_1 = \begin{pmatrix}
0 & 0 & i \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad P_2 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & i \\
0 & 0 & 0
\end{pmatrix}.
\] (17)

These generators satisfy the commutation relations:

\[
[P_1, P_2] = 0, \quad [J_3, P_1] = iP_2, \quad [J_3, P_2] = -iP_1.
\] (18)
This $E(2)$ group is not only convenient for illustrating the groups containing an Abelian invariant subgroup, but also occupies an important place in constructing representations for the little group for massless particles, since the little group for massless particles is locally isomorphic to the above $E(2)$ group.

The contraction of $O(3)$ to $E(2)$ is well known and is often called the Inonu-Wigner contraction [4]. The question is whether the $E(2)$-like little group can be obtained from the $O(3)$-like little group. In order to answer this question, let us closely look at the original form of the Inonu-Wigner contraction. We start with the generators of $O(3)$. The $J_3$ matrix is given in Eq.(1), and

$$J_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (19)$$

The Euclidean group $E(2)$ is generated by $J_3, P_1$ and $P_2$, and their Lie algebra has been discussed in Sec. [4].

![Figure 2](image)

**Figure 2:** North-pole and Equatorial-belt approximations. The north-pole approximation leads to the contraction of $O(3)$ to $E(2)$. The equatorial-belt approximation leads corresponds to the contraction the cylindrical group.

Let us transpose the Lie algebra of the $E(2)$ group. Then $P_1$ and $P_2$ become $Q_1$ and $Q_2$ respectively, where

$$Q_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & i & 0 \end{pmatrix}. \quad (20)$$
Together with $J_3$, these generators satisfy the same set of commutation relations as that for $J_3, P_1$, and $P_2$ given in Eq. (18):

$$[Q_1, Q_2] = 0, \quad [J_3, Q_1] = iQ_2, \quad [J_3, Q_2] = -iQ_1.$$  \hspace{1cm} (21)

These matrices generate transformations of a point on a circular cylinder. Rotations around the cylindrical axis are generated by $J_3$. The matrices $Q_1$ and $Q_2$ generate translations along the direction of $z$ axis. The group generated by these three matrices is called the cylindrical group $[3, 4]$.

We can achieve the contractions to the Euclidean and cylindrical groups by taking the large-radius limits of

$$P_1 = \frac{1}{R}B^{-1}J_2B, \quad P_2 = -\frac{1}{R}B^{-1}J_1B,$$  \hspace{1cm} (22)

and

$$Q_1 = -\frac{1}{R}BJ_2B^{-1}, \quad Q_2 = \frac{1}{R}BJ_1B^{-1},$$  \hspace{1cm} (23)

where

$$B(R) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{pmatrix}.$$  \hspace{1cm} (24)

The vector spaces to which the above generators are applicable are $(x, y, z/R)$ and $(x, y, Rz)$ for the Euclidean and cylindrical groups respectively. They can be regarded as the north-pole and equatorial-belt approximations of the spherical surface respectively $[3]$. Fig. 2 illustrates how the Euclidean and cylindrical contractions are made.

5 Contraction of O(3)-like Little Group to E(2)-like Little Group

Since $P_1(P_2)$ commutes with $Q_2(Q_1)$, we can consider the following combination of generators.

$$F_1 = P_1 + Q_1, \quad F_2 = P_2 + Q_2.$$  \hspace{1cm} (25)

Then these operators also satisfy the commutation relations:

$$[F_1, F_2] = 0, \quad [J_3, F_1] = iF_2, \quad [J_3, F_2] = -iF_1.$$  \hspace{1cm} (26)

However, we cannot make this addition using the three-by-three matrices for $P_i$ and $Q_i$ to construct three-by-three matrices for $F_1$ and $F_2$, because the vector spaces are different for the $P_i$ and $Q_i$ representations. We can accommodate this difference by creating two different $z$ coordinates, one with a contracted $z$ and the other with an
expanded $z$, namely $(x, y, Rz, z/R)$. Then the generators become

\[
P_1 = \begin{pmatrix}
0 & 0 & 0 & i \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad P_2 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & i \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

and

\[
Q_1 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad Q_2 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

Then $F_1$ and $F_2$ will take the form

\[
F_1 = \begin{pmatrix}
0 & 0 & 0 & i \\
0 & 0 & 0 & 0 \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad F_2 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & i \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

The rotation generator $J_3$ takes the form of Eq. (2). These four-by-four matrices satisfy the $E(2)$-like commutation relations of Eq. (26).

Figure 3: Light-cone coordinates. When the system is Lorentz-boosted, one of the axes expands while the other becomes contracted. Both the expansion and contraction are needed for the contraction of the $O(3)$-like little group to $E(2)$-like little group.
Now the $B$ matrix of Eq.(24), can be expanded to

$$B(R) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & R & 0 \\
0 & 0 & 0 & 1/R
\end{pmatrix}. \quad (30)$$

This matrix includes both the contraction and expansion in the light-cone coordinate system, as illustrated in Fig. 3. If we make a similarity transformation on the above form using the matrix

$$\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}, \quad (31)$$

which performs a 45-degree rotation of the third and fourth coordinates, then this matrix becomes

$$\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \cosh \eta & \sinh \eta \\
0 & 0 & \sinh \eta & \cosh \eta
\end{pmatrix}, \quad (32)$$

with $R = e^\eta$. This form is the Lorentz boost matrix along the $z$ direction. If we start with the set of expanded rotation generators $J_3$ of Eq.(1), and perform the same operation as the original Inonu-Wigner contraction given in Eq.(22), the result is

$$N_1 = \frac{1}{R} B^{-1} J_2 B, \quad N_2 = -\frac{1}{R} B^{-1} J_1 B, \quad (33)$$

where $N_1$ and $N_2$ are given in Eq.(3). The generators $N_1$ and $N_2$ are the contracted $J_2$ and $J_1$ respectively in the infinite-momentum and/or zero-mass limit.

It was noted in Sec. 3 that $N_1$ and $N_2$ generate gauge transformations on massless particles. Thus the contraction of the transverse rotations leads to gauge transformations.

6 Further Considerations

We have seen in this report that Wigner’s $O(3)$-like little group can be contracted into the $E(2)$-like little group for massless particles. Here, we worked out explicitly for the spin-1 case, but this mechanism should be applicable to all other spins. Of particular interest is spin-1/2 particles. This has been studied by Han, Kim and Son [8]. They noted that there are also gauge transformations for spin-1/2 particles, and the polarization of neutrinos is a consequence of gauge invariance. It has also been shown that the gauge dependence of spin-1 particles can be traced to the gauge variable of the spin-1/2
particle [13]. It would be very interesting to see how the present formalism is applicable to higher-spin particles.

Another case of interest is the space-time symmetry of relativistic extended particles. In 1973 [14], Kim and Noz constructed a ground-state harmonic oscillator wave function which can be Lorentz boosted. It was later found that this oscillator formalism can be extended to represent the $O(3)$-like little group [15, 16]. This oscillator formalism has a stormy history because it ultimately plays a pivotal role in combining quantum mechanics and special relativity [17, 18].

With these wave functions, we propose to solve the following problem in high-energy physics. The quark model works well when hadrons are at rest or move slowly. However, when they move with speed close to that of light, they appear as a collection of infinite-number of partons [19]. The question then is whether the parton model is a Lorentz-boosted quark model. This question has been addressed before [20, 21], but it can generate more interesting problems [22]. The present situation is presented in the following table.

Table 1: Massive and massless particles in one package. Wigner’s little group unifies the internal space-time symmetries for massive and massless particles. It is a great challenge for us to find another unification: the unification of the quark and parton pictures in high-energy physics.

| Massive, Slow | COVARIANCE | Massless, Fast |
|---------------|-------------|----------------|
| Energy-Momentum | $E = p^2/2m$ | $E = [p^2 + m^2]^{1/2}$ | $E = cp$ |
| Internal Space-time Symmetry | $S_3$ | Wigner’s Little Group | $S_3$ |
| | $S_1, S_2$ | Gauge Trans. | |
| Relativistic Extended Particles | Quark Model | One Covariant Theory | Parton Model |
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