Regular black holes in an asymptotically de Sitter universe

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Abstract

A regular solution of the system of coupled equations of the nonlinear electrodynamics and gravity describing static and spherically-symmetric black holes in an asymptotically de Sitter universe is constructed and analyzed. Special emphasis is put on the degenerate configurations (when at least two horizons coincide) and their near horizon geometry. It is explicitly demonstrated that approximating the metric potentials in the region between the horizons by simple functions and making use of a limiting procedure one obtains the solutions constructed from maximally symmetric subspaces with different absolute values of radii. Topologically they are $\text{AdS}_2 \times S^2$ for the cold black hole, $dS_2 \times S^2$ when the event and cosmological horizon coincide, and the Plebański-Hacyan solution for the ultraextremal black hole. A physically interesting solution describing the lukewarm black holes is briefly analyzed.

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1 Introduction

It is a well known fact that the classical general relativity cannot be trusted when curvature of the manifold approaches the Planck regime. It means that understanding the nature of any classical singularity that resides in the black hole interior as well as its closest vicinity requires profound changes of the standard theory. It is therefore natural that a great deal of effort has been concentrated on construction of the singularity-free models. (See for example Refs. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] and the references cited therein).

One of the most intriguing solutions of this type has been constructed by Ayon-Beato and Garcia [9] and by Bronnikov [10] (ABGB). This solution to the system of coupled equations of the nonlinear electrodynamics and gravity describes a regular static and spherically-symmetric black hole characterized by the total mass $M$ and the magnetic charge $Q$. The status of the nonlinear electrodynamics in this model is to provide a matter source. The casual structure of the solution is governed by the null geodesics ("ordinary photons") rather

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than the photons of the nonlinear theory. The latter move along geodesics of the effective
graph [12, 13].

The recent popularity of the models constructed within the framework of the nonlinear
electrodynamics has been stimulated by the fact that the latter appears naturally in the
low-energy limit of certain formulations of the string and M-theory [14, 15].

The Lagrangian of the nonlinear electrodynamics adopted by Ayon-Beato and Garcia and
by Bronnikov has Maxwell asymptotic in a weak field limit, and, consequently, far from the
ABGB black hole as well as for \( Q/M \ll 1 \) the line element resembles the Reissner-Nordström
(RN) solution; noticeable differences appear near the extremality limit. On the other hand,
as \( r \to 0 \) one has the asymptotic behaviour

\[
-g_{tt} = \frac{1}{g_{rr}} \sim 1 - \frac{4M}{r} \exp \left( -\frac{Q^2}{Mr} \right) \tag{1}
\]

and this very information suffices to demonstrate the finiteness of the curvature invariants.

Although more complicated than the RN geometry the ABGB solution allows exact an-
alytical treatment. Indeed, the location of the horizons can be expressed in terms of known
special functions that certainly simplifies investigations of its causal structure. The ABGB
geometry has been studied by a number of authors and from various points of view. See,
for example [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26], where the stability of the ABGB
spacetime, its gravitational energy, generalizations and the stress-energy tensor of the quan-
tized field propagating in such a geometry have been analyzed. Especially interesting are the
thermodynamic considerations presented in Refs. [27, 28].

In this paper we shall generalize the ABGB solution to the cosmological background.
Although the solution is valid for any \( \Lambda \) we shall restrict ourselves to the positive cosmological
constant. Consequently, an interesting group of related solutions describing topological black
holes are not considered here. The paper is organized as follows: In section 2 we construct the
solution describing the ABGB black hole in the de Sitter geometry and give its qualitative
discussion. The quantitative discussion of its main featur es is contained in Sec.3. The
near-horizon geometry of the degenerate configurations is constructed and discussed in Sec.4
whereas the analysis of the lukewarm black holes is presented in Sec.5. Finally, Sec.6 contains
short discussion and suggestions for extending this work. Throughout the paper the geometric
system of units is used and our conventions follow the conventions of MTW.

2 Cosmological solutions of the coupled system of equations of nonlinear electrodynamics and gravity

In the presence of the cosmological constant the coupled system of the nonlinear electrodyn-
amics and gravity is described by the action

\[
S = \frac{1}{16\pi G} \int (R - 2\Lambda) \sqrt{-g} \, d^4x + S_m, \tag{2}
\]

where

\[
S_m = -\frac{1}{16\pi} \int \mathcal{L}(F) \sqrt{-g} \, d^4x. \tag{3}
\]

Here \( \mathcal{L}(F) \) is some functional of \( F = F_{ab} F^{ab} \) (its exact form will be given later) and all
symbols have their usual meaning. The tensor \( F_{ab} \) and its dual \( *F^{ab} \) satisfy

\[
\nabla_a \left( \frac{d\mathcal{L}(F)}{dF} F^{ab} \right) = 0 \tag{4}
\]

and

\[
\nabla_a * F^{ab} = 0. \tag{5}
\]
The stress-energy tensor defined as

\[ T^{ab} = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{ab}} S \]

is given therefore by

\[ T^b_a = \frac{1}{4\pi} \left( \frac{d\mathcal{L}(F)}{dF} F_{ca} F^{ca} - \frac{1}{4} \delta^b_a \mathcal{L}(F) \right). \]

Let us consider the spherically symmetric and static configuration described by the line element of the form

\[ ds^2 = -e^{2\psi(r)} f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \]

where \( f(r) \) and \( \psi(r) \) are unknown functions. The spherical symmetry places restrictions on the components of \( F_{ab} \) tensor and its only nonvanishing components compatible with the assumed symmetry are \( F_{01} \) and \( F_{23} \). Simple calculations yield

\[ F_{23} = Q \sin \theta \]

and

\[ r^2 e^{-2\psi} \frac{d\mathcal{L}(F)}{dF} F_{10} = Q e, \]

where \( Q \) and \( Q_e \) are the integration constants interpreted as the magnetic and electric charge, respectively. In the latter we shall assume that the electric charge vanishes, and, consequently, \( F \) is given by

\[ F = \frac{2Q^2}{r^4}. \]

The stress-energy tensor (6) calculated for this configuration is

\[ T^t_t = T^r_r = -\frac{1}{16\pi} \mathcal{L}(F) \]

and

\[ T^\theta_\theta = T^\phi_\phi = \frac{1}{4\pi} \frac{d\mathcal{L}(F)}{dF} \frac{Q^2}{r^4} - \frac{1}{16\pi} \mathcal{L}(F), \]

which reduces to its Maxwell form for \( \mathcal{L}(F) = F \). With the substitution

\[ f(r) = 1 - \frac{2M(r)}{r} \]

the left hand side of the time and radial components of the Einstein field equations

\[ L^b_a = G^b_a + \Lambda \delta^b_a = 8\pi T^b_a \]

assume simple and transparent form

\[ L^t_t = -\frac{2}{r^2} \frac{dM}{dr} + \Lambda, \quad L^r_r = L^t_t + \frac{2}{r} \left( 1 - \frac{2M}{r} \right) \frac{d\psi}{dr} + \Lambda, \]

and the resulting equations can be easily (formally) integrated.

Further considerations require specification of the Lagrangian \( \mathcal{L}(F) \). We demand that it should have proper asymptotic, i.e., in a weak field limit it should approach \( F \). Following Ayón-Beato, García [9] and Bronnikov [10] let us chose it in the form

\[ \mathcal{L}(F) = F \left[ 1 - \tanh^2 \left( \sqrt{s} \frac{Q^2 F}{2} \right) \right], \]

(17)
where
\[ s = \frac{|Q|}{2b} \]  \hspace{1cm} (18)
and the free parameter \( b \) will be adjusted to guarantee regularity at the center. Inserting Eq. (18) into (17) and making use of Eq. (11) one has
\[
8\pi T_i^i = 8\pi T_r^r = -\frac{Q^2}{r^4} \left( 1 - \tanh^2 \frac{Q^2}{2br} \right), \tag{19}
\]
Now the equations can easily be integrated in terms of the elementary functions:
\[
M(r) = C_1 - b \tanh \frac{Q^2}{2br} + \Lambda \frac{r^3}{6}, \hspace{1cm} \psi(r) = C_2 \tag{20}
\]
where \( C_1 \) and \( C_2 \) are the integration constants. Making use of the conditions
\[
M(\infty) = M, \hspace{1cm} \psi(\infty) = 0 \tag{21}
\]
gives \( C_1 = M \) and \( C_2 = 0 \). On the other hand, demanding the regularity of the line element as \( r \to 0 \) yields \( b_1 = M \), and, consequently, the resulting line element has the form (8) with \( \psi(r) = 0 \) and
\[
f(r) = 1 - 2\frac{M}{r} \left( 1 - \tanh \frac{Q^2}{2Mr} \right) - \frac{\Lambda r^2}{3}. \tag{22}
\]
We shall call this solution the Ayón-Beato-García-Bronnikov-de Sitter solution (ABGB-dS).
It could be easily shown that putting \( Q = 0 \) yields the Schwarzschild-de Sitter (Kottler) solution, whereas for \( \Lambda = 0 \) one gets the Ayón-Beato, García line element as reinterpreted by Bronnikov (ABGB).

To study ABGB-dS line element it is convenient to introduce the dimensionless quantities
\[
x = r/M, \hspace{1cm} q = |Q|/M \hspace{1cm} \text{and} \hspace{1cm} \lambda = \Lambda M^{-2}. \]
For \( \lambda > 0 \) the equation
\[
1 - \frac{2}{x} \left( 1 - \tanh \frac{q^2}{2x} \right) - \frac{1}{3}\lambda x^2 = 0 \tag{23}
\]
has, in general, four roots; the case \( \lambda = 0 \) has been treated analytically in Refs [22, 23, 24]. Unfortunately, Eq. (23) cannot be solved in terms of known transcendental functions and one is forced to refer to some approximations or employ numerical methods. Simple analysis indicate that for \( x > 0 \) it can have, depending on the values of the parameters, three, two or one distinct real solutions. The above configurations can, therefore, have three distinct horizons located at zeros of \( f(r) \), a degenerate and a nondegenerate horizon, and, finally, one triply degenerate horizon. Let us consider each of the configuration more closely and denote the inner, the event and the cosmological horizon by \( r_- \), \( r_+ \) and \( r_c \), respectively. The first configuration is characterized by \( r_- < r_+ < r_c \). The second configuration can be realized in two different ways depending on which horizons do merge and is characterized either by \( r_- = r_+ < r_c \) (degenerate horizons of the first type, referred to as the cold black hole) or \( r_- < r_+ = r_c \) (degenerate horizons of the second type sometimes referred to as the charged Nariai black hole \[1\]). Finally, for the third (ultracold) configuration one has \( r_- = r_+ = r_c \). The degenerate horizons are located at simultaneous zeros of \( f(r) \) and \( f'(r) \) for the cold or the Nariai black hole and \( f(r) = f'(r) = f''(r) = 0 \) for the ultracold black hole. Additionally one can single out the lukewarm configuration, for which the Hawking temperature of the black hole equals the temperature of the cosmological horizon.

The Penrose diagrams visualizing two-dimensional sections of the conformally transformed ABGB-dS geometry is precisely of the type considered earlier for the Reissner-Nordström-de Sitter black hole \[24\] with the one notable distinction: the central singularity must be replaced be a regular region.

\[1\]It must not be confused with the charged Nariai solution which will be discussed in section 4.
Although the line element \([8, 22]\) is rather complicated and cannot be studied analytically one can easily analyze its main features simply by referring to its important limits. First, let us observe that for small \(q\) \((q \ll 1)\) as well as at great distances from the center \((x \gg 1)\) the ABGB-dS solution closely resembles that of RN-dS. Indeed, expanding \(f(r)\) one obtains
\[
f = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3} - \frac{Q^6}{12M^2r^4} + \ldots \tag{24}
\]
On the other hand, as \(r \to 0\), one has
\[
f \sim 1 - \frac{4M}{r} \exp\left(\frac{-Q^2}{4Mr}\right) - \frac{\Lambda r^2}{3} \tag{25}
\]
and the metric in the vicinity of the center may be approximated by the de Sitter line element. One concludes, therefore, that the solution is regular at \(r = 0\) and, in a view of the asymptotic behaviour of the line element, to demonstrate this it is unnecessary to calculate the curvature invariants explicitly. For example, at \(r = 0\) the Kretschmann scalar is equal \(8\Lambda^2/3\), as expected. Further, observe that for small \(\lambda\) the structure of the ABGB-dS geometry is to certain extent qualitatively similar to the ABGB spacetime. Simple analysis indicates that there are, at most, three positive roots of the equation \(f(r) = 0\). Two of them are located closely to the inner and event horizons of the ABGB black hole whereas the third one, located approximately at \(x_c \simeq (3/\lambda)^{1/2}\) (26) is to be interpreted as the cosmological horizon.

3 Horizon structure of the ABGB-dS black holes

Having established the main features of the ABGB-dS solution qualitatively let us study it in more detail and consider the approximate solutions for \(\lambda \ll 1\) first. We shall start, however, with a brief discussion of the \(\lambda = 0\) case and present the results that will be needed in the subsequent calculations. In Ref. \([22]\) it has been shown that for \(\lambda = 0\) the location of the inner, \(r_-^{(0)}\), and event horizon, \(r_+^{(0)}\), of the ABGB black holes can be expressed in terms the real branches of the Lambert functions, \(W_+(s)\):
\[
\rho_\pm = r_\pm^{(0)}/M = -\frac{4q^2}{4W_+(s) \left( -\frac{2q^2}{4} \exp\left(\frac{q^2}{4}\right) \right) - q^2}. \tag{27}
\]
Here \(W_+\) (the principal branch) and \(W_-\) are the only real branches of the Lambert function. Simple manipulations shows that \(\rho_+\) and \(\rho_-\) for
\[
q = q_c = 2\sqrt{W_+(1/e)} \equiv 2\sqrt{w_0} \tag{28}
\]
merge at
\[
\rho_c = \frac{4w_0}{1 + w_0} \tag{29}
\]
For \(q^2 > q_c^2\) the degenerate solution bifurcate into a pair of two complex roots.

For small \(\lambda\) one expects that the inner and the event horizon lies closely to the \(\rho_-\) and \(\rho_+\), respectively, and the position of the cosmological horizon can always be approximated by Eq. \((26)\). Depending on \(q\) there will be two horizons located at the roots \(x_-\) and \(x_+\), which for \(q^2 = q_c^2\) coalesce into the degenerate horizon \(x_{cr}\). For \(q^2 > q_c^2\) there are no real solutions for \(x_\pm\) and \(x_c\) tends to \((3/\lambda)^{1/2}\) with increasing \(q\).
Now, let us consider the cold black hole, i.e. the one for which \( x_- = x_+ = x_{cr} \). Taking \( \lambda \) to be a small parameter of the expansion one obtains

\[
q_{cr}^2 = 4w_0 + \frac{64w_0^3}{3(1+w_0)^2} \lambda + \frac{1024w_0^5(5+3w_0)}{9(1+w_0)^3} \lambda^2 + \frac{32768}{81(1+w_0)^7}(59+65w_0+18w_0^2)\lambda^3 + O(\lambda^4)
\]  

(30)

and

\[
x_{cr} = \frac{4w_0}{1+w_0} + \frac{64w_0^3(3+w_0)}{3(1+w_0)^4} \lambda + \frac{1024w_0^5(25+18w_0+3w_0^2)}{9(1+w_0)^7} \lambda^2 + \frac{32768w_0^7}{81(1+w_0)^{10}}(413+461w_0+18w_0^2+59w_0^3)\lambda^3 + O(\lambda^4).
\]  

(31)

For \( q^2 < q_{cr}^2 \), following Romans [30], we shall introduce the dimensionless parameter \( \Delta = \sqrt{q_{cr}^2 - q^2} \) and look for solutions of Eq. (23) of the form

\[
x_{\pm} = \rho_{\pm} + x_{1,(\pm)\lambda} + x_{2,(\pm)\lambda^2} + O(\lambda^3).
\]  

(32)

where

\[
\rho_{\pm} = \frac{4(q_{cr}^2 - \Delta^2)}{4W_{\pm}(\eta) - q_{cr}^2 + \Delta^2}
\]  

(33)

and

\[
\eta = \frac{\Delta^2 - q_{cr}^2}{4} \exp\left(\frac{q_{cr}^2 - \Delta^2}{4}\right).
\]  

(34)

Now, solving a chain of equations of ascending complexity, one has

\[
x_{1,(\pm)} = \frac{4\rho_{\pm} \left[ 64w_0^9 - 16w_0^3\rho_{\pm} - (1+w_0)^2\rho_{\pm}^3 \right]}{3(1+w_0)^2 [(4-\rho_{\pm})(4w_0 - \Delta^2) - 4\rho_{\pm}]}
\]  

(35)

and

\[
x_{2,(\pm)} = \frac{4}{(4-\rho_{\pm})[(4-\rho_{\pm})(4w_0 - \Delta^2) - 4\rho_{\pm}]} \left[ 2\left(x_{1,(\pm)}\right)^2 + \frac{1}{9}(\rho_{\pm} - 2)\rho_{\pm}^6 + \frac{2}{3}(\rho_{\pm} - 6)\rho_{\pm}^3x_{1,(\pm)} + \frac{256\rho_{\pm}w_0^9}{9(1+w_0)^7}(5+3w_0)(\rho_{\pm} - 4)^2 \right].
\]  

(36)

It could be easily shown that putting \( \Delta = 0 \) and taking limit of (32) as \( \rho_{\pm} \to 4w_0/(1+w_0) \) one obtains (31).

In the situations when the cosmological constant cannot be regarded as small, the analytical treatment of the horizon structure of the ABGB-dS black holes is impossible. However, although we are unable to calculate the exact location of horizons in the spacetime of ABGB-dS black holes, one can use a simple trick. Indeed, the form of the equation (23) suggests that it can be solved easily with respect to \( q \) yielding

\[
q = \pm \sqrt{x \ln \frac{12 - 3x + \lambda x^2}{x(3 - \lambda x^2)}},
\]  

(37)

This allows to draw curves \( q = q(x) \) for various (constant) \( \lambda \). The extrema of the curves represent either the degenerate horizons of the cold black holes or the charged Nariai black
holes. A closer examination shows that for \( q > 0 \) one has minima for the configurations with \( r_+ = r_c \) and maxima for the cold black hole. Drawing, on the other hand, \( \lambda = \lambda(x) \) curves for constant values of \( q \) one has minima for the configurations with \( r_- = r_+ \) and maxima for the charged Nariai black holes. Except the ultracold black hole the configurations with the cosmological horizon only are not considered in this paper. The results of such a calculation is displayed in Fig.1. Now, rotating the diagram counter-clockwise by ninety degrees and subsequently reflecting the thus obtained curves by the vertical axis one gets the desired result. On the other hand one can employ numerical methods and the results of such calculations are presented in Fig. 2, where, for better clarity, only \( q > 0 \) region has been displayed. To investigate the horizon structure it is useful to focus attention on \( x = x(q) \) curves of constant \( \lambda \), where \( x \) is, depending on its position on the curve, one the three horizons. For each \( \lambda \geq \lambda_0 \) (where \( \lambda_0 \) denotes some critical value of the cosmological constant to be given later) the (rescaled) radii of the inner horizon (the lower branch), the event horizon (the middle branch) and the cosmological horizon (the upper branch) comprise an S-shaped curve and the turning points of each curve represent degenerate horizons. On the other hand, for \( \lambda < \lambda_0 \) there is only one turning point representing cold black hole with \( r_+ = r_- \) and the cosmological horizon branch is separated form the rest of the curve. The degenerate horizon of the second type appears precisely for \( \lambda_0 = 1/9 \).

Although the qualitative behaviour of the degenerate horizons as function of the cosmological constant, such as rather weak dependence of the location of the degenerate horizon of the first type, may easily be inferred from the above analysis, we shall discuss it in more detail. The dependence of the location of the degenerate horizons as functions of \( \lambda \) can be calculated from Eq. (37) and the results are displayed in Fig 3. The lower branch represents degenerate horizons of the first type whereas the upper one represents degenerate horizons of the second kind, and, finally, the branch point represents the triply degenerated horizon. Such a configuration occurs at \( x = 1.34657 \) for \( \lambda = 0.246019 \) and \( |q| = 1.1082 \).

4 Extreme configurations

The ABGB-dS solution gives rise to a number of solutions constructed by applying some limiting procedure in the near extreme geometry. For example, it is a well known fact that the near horizon geometry of the Reissner-Nordstrom solution is properly described by the Bertotti-Robinson line element \([31,32,33,34]\) whereas Ginsparg and Perry \([35]\) demonstrated that the extreme Schwarzschild- de Sitter black hole is naturally connected with the Nariai
Figure 2: The positive roots of Eq. (23) as a function of $q$. Bottom to top the curves are drawn for $\lambda = 0.008, 0.02, 0.05, 0.08, 0.1, 0.11, 0.12, 0.13, 0.14, 0.15, 0.16$. The lower branches represent the inner horizon, which, for $q < 1$ is practically insensitive to the changes of the cosmological constant. For $\lambda > 1/9$ there are two additional branches representing the event and the cosmological horizon comprising S-shaped curves. For $\lambda < 1/9$ the upper branch (the cosmological horizon) is disjoint from the rest of the curve (consisting of the lower and middle branches) and, for small $\lambda$, it is located approximately at $\sqrt{3/\lambda}$.

Figure 3: The radii of the extremal horizons as function of $\lambda$. The lower branch represents the extreme horizons of the cold black hole ($x_-=x_+$) whereas the upper branch represents the extreme horizons of the second type ($x_+=x_c$). The branch point represents ultracold black hole ($x_-=x_+=x_c$) and the point $(1/9, 3)$ of the upper branch represents the extreme Schwarzschild-de Sitter solution.
solution. The procedure of Ginsparg and Perry has been subsequently generalized and employed in a number of physically interesting cases, such as C-metrics, D-dimensional black holes and in construction of various instantons, to name a few. In this section we shall construct the exact solutions to the Einstein field equations which are asymptotically congruent to the near horizon geometry of the ABGB-dS black holes.

First let us consider the situation when the inner horizon is close to the event horizon. For \( r^- \leq r \leq r^+ \) the function \( f \) can be approximated by a parabola \( \alpha(x - x_+)(x - x_-) \) and the degenerate horizon, \( x_d \), by \( (x_+ + x_-)/2 \). Putting \( q^2 = q_{d^2}^2 - \varepsilon^2 \Delta^2 \), where \( \varepsilon \) is a small parameter that measures deviation from the extremal configuration and \( \Delta \) should be chosen such a way to guarantee \( x^\pm = x_d \pm \varepsilon \) one can easily determine \( \alpha \). Indeed, it can be shown that for a given \( \lambda \) one has

\[
\alpha = 4 \frac{\partial f}{\partial q_{\pm d}^2} \frac{\Delta^2 \varepsilon^2}{(r^+ - r^-)^2} = \frac{\partial f}{\partial q_{\pm d}^2} \Delta^2 > 0
\]  

Similarly, for \( r^+ \leq r \leq r_c \), one can approximate the function \( f \) by a parabola \( \beta(x - x_+)(x - x_c) \) and the degenerate horizon by \( (x_+ + x_c)/2 \). Putting \( q^2 = q_{d^2}^2 + \varepsilon^2 \tilde{\Delta}^2 \) one obtains

\[
\beta = -\frac{\partial f}{\partial q_{\pm d}^2} \tilde{\Delta}^2 < 0.
\]

We shall illustrate the procedure by the example of the ABGB black hole. First, observe that making use of the expansions of the Lambert functions \( W_+ \) and \( W_- \),

\[
W_{\pm}(z) = -1 + p - \frac{1}{3}p^2 + ...,
\]

where \( p = \sqrt{2(\varepsilon z + 1)} \) for the principal branch and \( p = -\sqrt{2(\varepsilon z + 1)} \) for \( W_-(z) \), one has

\[
x^\pm = \frac{4w_0}{1 + w_0} \pm \frac{\sqrt{8w_0}}{(1 + w_0)^{3/2}} \Delta \varepsilon + ..., \quad (41)
\]

and, consequently,

\[
\Delta = \frac{(1 + w_0)^{3/2}}{\sqrt{8w_0}}. \quad (42)
\]

(Notation has been slightly modified as compared to Sec. 3). Further observe that

\[
\frac{\partial f}{\partial q_{d^2}} = \frac{1}{4w_0} \quad (43)
\]

and \( f \) may be approximated by

\[
f = \frac{(w_0 + 1)^3}{32w_0^3}(x - x_-)(x - x_+) = A(r - r_-)(r - r_+). \quad (44)
\]

Finally, introducing new coordinates \( \tilde{T} = T/\varepsilon A \), \( r = r_d + \varepsilon \cosh y \), and taking limit \( \varepsilon \to 0 \) one obtains the line element in the form

\[
ds^2 = \frac{1}{A} \left( -\sinh^2 y \, d\tilde{T}^2 + dy^2 \right) + r_d^2 (d\theta^2 + \sin^2 \theta \, d\phi^2). \quad (45)
\]

Since the modulus of curvature radii of the maximally symmetric subspaces are different this solution does not belong to the Bertotti-Robinson class. Topologically it is a product of the round two sphere of a constant radius and the two dimensional anti-de Sitter geometry. We will call this solution a generalized Bertotti-Robinson solution.
Now, let us return to the ABGB-dS geometry and observe that \( \frac{\partial f}{\partial q_{q_d}} \) is always non-negative. Since there are no analytical expressions describing the exact location of the horizons we shall employ the perturbative approach. Starting with the configurations with \( r_- \) close to \( r_+ \), and repeating the steps above, one obtains the line element (45) with

\[
A = \frac{\partial f}{\partial q_{q_d}} \frac{\Delta^2}{2M^2},
\]

where

\[
\Delta^2 = 2q_d^2 x_d - \cosh^2 \frac{q_d^2}{2x_d} - \frac{q_d^4}{2x_d} \tanh \frac{q_d^2}{2x_d}
\]

and

\[
\frac{\partial f}{\partial q_{q_d}^2} = \frac{1}{x_d^2 \cosh^2 \frac{q_d^2}{2x_d}}.
\]

The curvature scalar of the geometry (45) is a sum of curvatures of the maximally symmetric subspaces

\[
R = R_{AdS_2} + R_{S^2},
\]

where \( R_{AdS_2} = -2A \) and \( R_{S^2} = 2/r_d^2 \). We shall call (45) a generalized cosmological Bertotti-Robinson solution. It can easily be checked that putting \( q_d = q_c \) and \( x_d = p_c \) as given by Eqs. (28) and (29), respectively, one obtains (42) and (43).

On the other hand, for the near extreme configurations with \( r_+ \) close to \( r_\epsilon \) we shall put \( q^2 = \tilde{q}_d^2 + \epsilon^2 \Delta^2 \). Repeating calculations one has \( \tilde{\Delta}^2 = -\Delta^2 \). It should be noted however that for \( x_d = x_+ = x_e \), the parameter \( \tilde{\Delta}^2 \) is positive and hence \( \beta \) is negative, as required. Introducing new coordinates \( \tilde{t} = T/(\epsilon B) \) and \( r = r_d + \epsilon \cos y \), in the limit \( \epsilon \to 0 \), one obtains

\[
ds^2 = \frac{1}{B} \left( -\sin^2 y \, dT^2 + dy^2 \right) + r_d^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right),
\]

where \( B = -\beta \). Topologically it is a product of the round two-sphere and the two-dimensional de Sitter spacetime and the curvature scalar is given by

\[
R = R_{ds_2} + R_{S^2},
\]

where \( R_{ds_2} = 2B \). We shall call this solution a generalized Nariai solution.

Differentiating the function \( f \) with respect to \( x \) twice, subtracting \( 2f(x)/x^2 = 0 \) and dividing thus obtained result by 2 one concludes that

\[
A = \frac{1}{2} f''(r_d).
\]

It follows then that \( A \) vanishes at the ultraextremal horizon. Finally observe that putting \( y = \xi A^{1/2} \) in (45) and taking the limit \( A \to 0 \), one obtains

\[
ds^2 = -\xi^2 dT^2 + d\xi^2 + r_d^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right).
\]

Topologically it is a product of the two-dimensional Minkowski space and the round two-sphere of fixed radius and can be identified with the Plebański-Hacyan [45, 46, 47] solution.

Although we have adopted the point of view that the cosmological constant is not a parameter in the solutions space but, rather, the parameter of the space of theories, one can equally well keep \( q \) constant and change \( \lambda \). Indeed, repeating the calculations with \( \lambda = \lambda_d + \epsilon^2 \Delta^2 \) for \( r_- \) close to \( r_+ \) and \( \lambda = \lambda_d - \epsilon^2 \Delta^2 \) for \( r_+ \) close to \( r_\epsilon \) one obtains precisely (13) and (50), respectively. The sign can be deduced form the analysis of the \( \lambda = \lambda(x) \) curves obtained from Eq. (28), and, as before, the subscript \( d \) denotes degenerate configurations.
Figure 4: The radii of horizons of the lukewarm black hole as functions of $\lambda$. The branch point should be excluded as it represents the degenerate configuration of the second type. The almost horizontal line displays the values of $q$.

Since the $AdS_2 \times S^2$ and $dS_2 \times S^2$ (with arbitrary radii of the maximally symmetric subspaces) appear to describe universally the geometry of the vicinity of the (doubly) degenerate horizons, one can use this information in construction of the coefficients $A$ and $B$. Indeed, observe that $f''(r_d) > 0$ at the degenerate horizon of the cold black hole, whereas $f''(r_d) < 0$ at the degenerate horizon of the second type. At the degenerate horizons the Einstein field equations reduce to

$$-\frac{1}{r_d^2} + \Lambda = 8\pi T_t^t$$

and

$$\frac{1}{2} f''(r_d) + \Lambda = 8\pi T_\theta^\theta$$

Consequently, $A = f''(r_d)/2$ and $B = -f''(r_d)/2$ and at the degenerate horizon of the ultracold configuration one has $f''(r_d) = 0$.

5 Lukewarm configuration

Finally, let us consider the important case of the lukewarm black holes, i.e. the black holes with the Hawking temperature of the event horizon equal to that associated with the cosmological horizons. From the point of view of the quantum field theory in curved background the lukewarm black holes are special. It has been shown that for the two-dimensional models it is possible to construct a regular thermal state [48]. Moreover, recent calculations of the vacuum polarization indicate that it is regular on both the event and cosmological horizons of the D=4 lukewarm RN-dS black holes [49].

As the lukewarm black holes are characterized by the condition $T_H = T_C$ the radii of the horizons with this property satisfy the system of equations

$$f(r_+) = f(r_c) = 0, \quad f'(r_+) + f'(r_c) = 0.$$  

(56)

Since one expects that the structure of the horizons of the ABGB-dS black hole is qualitatively similar to its maxwellian counterpart it is worthwhile to analyze briefly lukewarm Reissner-Nordström black holes. Simple calculations indicate that such configurations are possible for $q = 1$ and the locations of the event and the cosmological horizons are given by

$$x_+ = \frac{l}{2} \left(1 - \sqrt{1 - \frac{4M}{l}}\right)$$

(57)
and
\[ x_c = \frac{l}{2} \left( 1 + \sqrt{1 - \frac{4M}{l}} \right), \tag{58} \]

where \( l = \sqrt{3/\lambda} \). For \( \lambda = 3/16 \ r_c \) and \( r_c \) coalesce into the degenerate horizon of the second type. One expects, therefore, that for the lukewarm ABGB-dS black holes \( q \) should always be close to 1. Results of our numerical calculations are presented in Fig.4., where the (rescaled) radii of the event horizon (the lower branch) and the cosmological horizon (the upper branch) are drawn. The function \( q = q(\lambda) \) is almost horizontal. The branch point should be excluded as it refers to the ultracold black holes.

6 Final Remarks

In this paper we have constructed the regular solution to the system of coupled equations of the nonlinear electrodynamics and gravity in the presence of the cosmological constant. We have restricted to the positive \( \Lambda \) and concentrated on the classical issues, such as location of the horizons, degenerate configurations and various solutions constructed form the two dimensional maximally symmetric subspaces. The discussion of the solutions with \( \Lambda < 0 \), both spherically symmetric and topological, have been intentionally omitted. Outside the event horizon the ABGB-dS solution closely resembles RNdS; important differences appear, as usual, for the near extreme configurations. At large distances as well as in the closest vicinity of the center the line element may be approximated by the de Sitter solution.

We indicate a few possible directions of investigations. First, it would be interesting to examine the vacuum polarization effects in the ABGB-dS geometries and compare them with the analogous results calculated in the RNdS spacetime. It should be noted in this regard that the geometries naturally connected with the ABGB-dS geometry, namely the generalized Bertotti-Robinson and the cosmological charged Nariai metric are the exact solutions of the semiclassical Einstein field equations \[51, 52, 53, 54, 55, 56\]. Moreover, the interior of the ultraextremal ABGB-dS black hole provides a natural background for studies initiated in Ref. \[57\]. This group of problems is actively investigated and the results will be published elsewhere.

References

[1] A. D. Sakharov, Sov. Phys. JETP 22, 21 (1966).
[2] E. B. Gliner, Sov. Phys. JETP 22, 378 (1966).
[3] J. M. Bardeen, GR 5 Proceedings ( Tbilisi, 1968).
[4] V. P. Frolov, M. A. Markov, and V. F. Mukhanov, Phys. Rev. D41, 383 (1990).
[5] V. P. Frolov, M. A. Markov, and V. F. Mukhanov, Phys. Lett. B216, 272 (1989).
[6] I. Dymnikova, Gen. Rel. Grav. 24, 235 (1992).
[7] A. Borde, Phys. Rev. D55, 7615 (1997).
[8] M. Mars, M. M. Martín-Prats, and J. M. M. Senovilla, Classical Quantum Gravity 13, L51 (1996).
[9] E. Ayon-Beato and A. Garcia, Phys. Lett. B464, 25 (1999).
[10] K. A. Bronnikov, Phys. Rev. D63, 044005 (2001).
[11] I. Dymnikova, Class. Quant. Grav. **21**, 4417 (2004).
[12] M. Novello, V. A. De Lorenci, J. M. Salim, and R. Klippert, Phys. Rev. **D61**, 045001 (2000).
[13] M. Novello, S. E. Perez Bergliaffa, and J. M. Salim, Class. Quant. Grav. **17**, 3821 (2000).
[14] N. Seiberg and E. Witten, JHEP **09**, 032 (1999).
[15] E. S. Fradkin and A. A. Tseytlin, Phys. Lett. **B163**, 123 (1985).
[16] N. Breton, Phys. Rev. **D72**, 044015 (2005).
[17] I. Radinschi, Mod. Phys. Lett. **A16**, 673 (2001).
[18] I.-C. Yang and I. Radinschi, Chin. J. Phys. **42**, 40 (2004).
[19] J. Matyjasek, Mod. Phys. Lett. **A23**, 591 (2008).
[20] A. Burinskii and S. R. Hildebrandt, Phys. Rev. **D65**, 104017 (2002).
[21] E. Elizalde and S. R. Hildebrandt, Phys. Rev. **D65**, 124024 (2002).
[22] J. Matyjasek, Phys. Rev. **D63**, 084004 (2001).
[23] W. Berej and J. Matyjasek, Phys. Rev. **D66**, 024022 (2002).
[24] J. Matyjasek, Phys. Rev. **D70**, 047504 (2004).
[25] J. Matyjasek, Phys. Rev. **D76**, 084003 (2007).
[26] W. Berej, J. Matyjasek, D. Tryniecki, and M. Woronowicz, Gen. Rel. Grav. **38**, 885 (2006).
[27] Y. S. Myung, Y.-W. Kim, and Y.-J. Park, Phys. Lett. **B659**, 832 (2008).
[28] J. Matyjasek, Acta Phys. Polon. **B39**, 3 (2008).
[29] F. Mellor and I. Moss, Phys. Rev. **D41**, 403 (1990).
[30] L. J. Romans, Nucl. Phys. **B383**, 395 (1992).
[31] B. Carter, *Black holes*, (Gordon and Breach, New York, 1973), pp. 57–214.
[32] B. Bertotti, Phys. Rev. **116**, 1331 (1959).
[33] I. Robinson, Bull. Acad. Pol. Sci. Ser. Sci. Math. Astron. Phys. **7**, 351 (1959).
[34] P. R. Anderson, W. A. Hiscock, and D. J. Loranz, Phys. Rev. Lett. **74**, 4365 (1995).
[35] P. H. Ginsparg and M. J. Perry, Nucl. Phys. **B222**, 245 (1983).
[36] H. Nariai, Sci. Rep. Töhoku Univ. Ser. I. **34**, 160 (1950).
[37] H. Nariai, Sci. Rep. Töhoku Univ. Ser. I. **35**, 62 (1951).
[38] O. J. C. Dias and J. P. S. Lemos, Phys. Rev. **D68**, 104010 (2003).
[39] M. Caldarelli, L. Vanzo, and S. Zerbini, [hep-th/0008136](https://arxiv.org/abs/hep-th/0008136).
[40] V. Cardoso, O. J. C. Dias, and J. P. S. Lemos, Phys. Rev. **D70**, 024002 (2004).
[41] R. B. Mann and S. F. Ross, Phys. Rev. **D52**, 2254 (1995).
[42] S. W. Hawking and S. F. Ross, Phys. Rev. D52, 5865 (1995).
[43] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth, Adv. Comput. Math. 5, 329 (1996).
[44] D. J. Jeffrey, D. E. G. Hare, and R. M. Corless, Math. Sci. 21, 1 (1996).
[45] J. F. Plebański and S. Hacyan, J. Math. Phys. 20, 1004 (1979).
[46] M. Ortaggio, Phys. Rev. D65, 084046 (2002).
[47] M. Ortaggio and J. Podolsky, Class. Quant. Grav. 19, 5221 (2002).
[48] D. Morgan, S. Thom, E. Winstanley, and P. M. Young, Gen. Rel. Grav. 39, 1719 (2007).
[49] E. Winstanley and P. M. Young, Phys. Rev. D77, 024008 (2008).
[50] L. A. Kofman and V. Sahni, Phys. Lett. B127, 197 (1983).
[51] L. A. Kofman, V. Sahni, and A. A. Starobinsky, Sov. Phys. JETP 58, 1090 (1983).
[52] V. Sahni and L. A. Kofman, Phys. Lett. A117, 275 (1986).
[53] O. B. Zaslavskii, Class. Quant. Grav. 17, 497 (2000).
[54] S. N. Solodukhin, Phys. Lett. B448, 209 (1999).
[55] J. Matyjasek, Phys. Rev. D61, 124019 (2000).
[56] J. Matyjasek and O. B. Zaslavskii, Phys. Rev. D64, 104018 (2001).
[57] J. Matyjasek and O. B. Zaslavskii, Phys. Rev. D71, 087501 (2005).