Charm Quark Mass from Inclusive Semileptonic B Decays

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The MS charm quark mass is determined to be $\overline{m}_c (\overline{m}_c) = 1224 \pm 17 \pm 54$ MeV from a global fit to inclusive $B$ meson decay data, where the first error is experimental, and includes the uncertainty in $\alpha_s$, and the second is an estimate of theoretical uncertainties in the computation. We discuss the implications of the pole mass renormalon in the determination of $m_c$.

I. INTRODUCTION

Precise determinations of the bottom and charm quark masses are becoming increasingly important for the measurement of CKM parameters and in the search for new physics from experiments on $B$-meson and Kaon decays. For example, the theoretical uncertainty in the $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ rate is now dominated by the uncertainty in $\overline{m}_s (\overline{m}_s)$ \textsuperscript{3}. At present, the most precise determinations of the charm quark mass have been obtained from sum rule analyses of the $e^+ e^-$ $R$-ratio and from lattice QCD (though all lattice results still use the quenched approximation) \textsuperscript{4}. The results of recent analyses are consistent \textsuperscript{3,4,5,6,7}; nevertheless it is important to consider other independent methods to extract the charm quark mass. Recently, fundamental standard model parameters such as the bottom quark mass and the mixing angle $\theta_{cB}$ have been determined with high precision from inclusive semileptonic $B$-decays \textsuperscript{8,9,10}. In this paper, we extend the inclusive $B$ decay analysis to also obtain the MS charm quark mass, $\overline{m}_c (\overline{m}_c)$. The charm quark mass was extracted previously in Ref. \textsuperscript{8} by the methods discussed here. We redo the analysis including a more careful consideration of renormalon effects in the quark pole masses.

The experimental input consists of inclusive $B$ decay spectra, such as the electron energy and hadronic invariant mass spectra in inclusive semileptonic $B \rightarrow X_c e \nu$ decay, the inclusive photon energy spectrum in $B \rightarrow X_c \gamma$ decay, and the $B$ meson masses and lifetimes. These quantities can be computed theoretically using heavy quark effective theory (HQET), as an expansion in $\alpha_s (m_b)$ and $\Lambda_{\text{QCD}}/m_b$. At present, all the quantities are known to order $\alpha_s^2 \beta_0$, $\Lambda_{\text{QCD}}^3/m_b^3$, and $\alpha_s \Lambda_{\text{QCD}}/m_b$, and fits to the experimental data were performed in Ref. \textsuperscript{8} including all theoretical quantities to this order. We use the results of this fit to determine the charm quark mass.

It is often convenient to do computations using the quark pole masses $m_{b,c}$ since they define the phase space boundaries in partonic perturbation theory. The heavy quark pole masses are known to have a renormalon ambiguity of order $\Lambda_{\text{QCD}}$ \textsuperscript{11,12}. The order $\Lambda_{\text{QCD}}$ renormalon means that the pole masses $m_{b,c}$ cannot be determined accurately (i.e. with an uncertainty much smaller than $\Lambda_{\text{QCD}}$), and the $\alpha_s$ perturbation expansion for the pole masses is poorly behaved, with a factorial growth in the coefficients at high orders. The bad behavior of the pole masses is already manifest in the first two orders of perturbation theory. In Ref. \textsuperscript{8}, fits to $B$ decay spectra in terms of the pole mass were shown to be poorly behaved. Luckily, there are other quark mass definitions which do not suffer from a renormalon ambiguity, and which can be determined accurately. One of these, the 1S-mass for the bottom quark, $m_{b,1S}$, was used as the bottom quark mass definition in the fits of Refs. \textsuperscript{8,9}. It was shown in Ref. \textsuperscript{8} that fits to inclusive $B$ decays using the 1S-mass are well-behaved, with good convergence of the perturbation series. Similarly, for the charm quark, a quark mass definitions without the $\Lambda_{\text{QCD}}$ renormalon is the MS mass $\overline{m}_c (\mu)$, frequently given for $\mu = \overline{m}_c$, $\overline{m}_s (\overline{m}_c)$.

The difference between the bottom and charm quark masses is constrained by the meson mass difference,

\begin{equation}
\overline{m}_B - \overline{m}_D = m_{b,1S}^\text{pole} - m_{c,1S}^\text{pole} - \frac{\lambda_1}{2} \left( \frac{1}{m_b} - \frac{1}{m_c} \right) + \frac{\rho_1 - \tau_1 - \tau_3}{4} \left( \frac{1}{m_b^2} - \frac{1}{m_c^2} \right)
\end{equation}

where $\overline{m}_{B,D}$ are the spin-averaged $B, B^*$ and $D, D^*$ meson masses, and $\lambda_1, \rho_1, \tau_1, \tau_3$ are HQET parameters. This equation was used to determine the bottom-charm pole mass difference $m_b - m_c = 3.41 \pm 0.01$ GeV in Ref. \textsuperscript{8}, where we frequently suppress the superscript “pole” for the pole masses. The order $\Lambda_{\text{QCD}}$ renormalon cancels in bottom-charm pole mass difference, so $m_b - m_c$ has no $\Lambda_{\text{QCD}}$ renormalon, and can be accurately determined.

The procedure for accurately determining the charm quark mass is simple in principle — one combines the values of $m_b - m_c$ and $m_{b,1S}$ to determine $\overline{m}_c (\mu)$. None of the three quantities suffers from a $\Lambda_{\text{QCD}}$ renormalon, and so $\overline{m}_c (\mu)$ can be determined reliably. However, it turns out there are some subtleties in the actual computation, and one must take care to ensure that the pole mass renormalon cancellation is properly implemented in $m_b - m_c$ at intermediate stages of the calculation. This was not done correctly in Ref. \textsuperscript{8}. Including the renormalon cancellation properly, as done here, leads to significantly different results.
II. PERTURBATIVE MASS RELATIONS

The computations of the moments of the semileptonic decay spectra are carried out in the theory where the top quark is integrated out and where the bottom quark is treated as static. Thus the top and bottom quarks do not contribute in virtual quark loops, and all $\overline{\text{MS}}$ parameters are defined in the theory for $n_f = 4$ running flavors.

The 3-loop perturbative series for the relation between the charm pole and the charm $\overline{\text{MS}}$ mass at the scale $\mu_m$ is

$$m_c^{\text{pole}} = m_c^{(n_f=4)}(\mu) \cdot g\left(\alpha_s(\mu), \mu, \overline{m}_c^{(n_f=4)}(\mu_m)\right),$$

$$g(\alpha_s, \mu, \overline{m}, \mu_m) = 1 + \epsilon \delta^{(1)}(\alpha_s, \mu, \overline{m}, \mu_m) + \epsilon^2 \delta^{(2)}(\alpha_s, \mu, \overline{m}, \mu_m) + \epsilon^3 \delta^{(3)}(\alpha_s, \mu, \overline{m}, \mu_m)$$

where $\alpha_s \equiv \alpha_s^{(n_f=4)}(\mu)$ is the running coupling in the theory with four dynamical flavors, $\overline{m} \equiv \overline{m}_c^{(n_f=4)}(\mu_m)$ is the $\overline{\text{MS}}$ charm mass in the theory with four dynamical flavors, and $\delta^{(n)}(\alpha_s, \mu, \overline{m}, \mu_m)$ are:

$$\delta^{(1)} = \frac{\alpha_s}{\pi} \left(\frac{4}{3} + L_m\right),$$

$$\delta^{(2)} = \frac{\alpha^2_s}{\pi^2} \left(10.3193 + 2.77778 L + \left(2.98611 + 2.08333 L\right) L_m - 0.541667 L_m^2\right),$$

$$\delta^{(3)} = \frac{\alpha^3_s}{\pi} \left(116.504 + 47.2748 L + 5.78704 L^2 + \left(6.30337 + 15.6505 L + 4.34028 L^2\right) L_m + (-6.20023 - 2.25694 L) L_m^2 + 0.571759 L_m^3\right),$$

where

$$L_m \equiv \ln\left(\frac{\mu^2_m}{\overline{m}^2}\right), \quad L \equiv \ln\left(\frac{\mu^2}{\overline{m}^2}\right).$$

For most of the paper, the superscript $(n_f = 4)$ for $\overline{\text{MS}}$ parameters is dropped for simplicity. The powers of the auxiliary parameter $\epsilon = 1$ indicate the respective orders in the loop expansion. Note that the charm quark mass and $\alpha_s$ are renormalized at the scale $\mu_m$ and $\mu$, respectively. Choosing $\mu = \mu_m = \overline{m}_c(\overline{m}_c)$ and $\alpha_s(\overline{m}_c) = 0.39$ gives

$$g = 1 + 0.42\alpha_s(\overline{m}_c) + 1.05\alpha_s(\overline{m}_c)^2 \epsilon^2 + 3.76\alpha_s(\overline{m}_c)^3 \epsilon^3 = 1 + 1.7\epsilon + 0.16\epsilon^2 + 0.22\epsilon^3.$$  

The first few terms of this series show no sign of convergence, and seem to indicate that one cannot determine $\overline{m}_c(\overline{m}_c)$ to an accuracy better than, say, 20%. The poor perturbative behavior of Eq. (7) is a reflection of the $O(\Lambda_{\text{QCD}})$ pole mass renormalon, which implies that the series for the pole mass behaves at high orders as $\sim \mu^{\epsilon} \alpha_s(\mu)^{\varphi} \phi^{1/\epsilon}$, where $\varphi$ is the order in perturbation theory, $\beta_0 = 11 - 2/3n_f$ is the one-loop coefficient of the QCD $\beta$ function, $n_f$ is the number of light quark flavors, and $\mu$ the renormalization scale for $\alpha_s$. As we will see explicitly in the next section, the bad behavior manifest in Eq. (7) does not limit the accuracy with which one can determine $\overline{m}_c(\mu)$ because in the relation between physical observables and short-distance masses, such as $\overline{m}_c(\mu)$ and $m_{1S}^{1S}$, the $O(\Lambda_{\text{QCD}})$ renormalon contributions always completely cancel. To achieve this cancellation in practical numerical analyses a few subtle guidelines for the treatment of truncated perturbative series have to be followed.

For the series between the bottom quark pole and the bottom 1S masses the higher order corrections each contain a factor of the inverse $\Upsilon$ Bohr radius $\sim m_b^{1S} \alpha_s$ in addition to the powers of the strong coupling. For the systematic cancellation of the $O(\Lambda_{\text{QCD}})$ renormalon contributions in each order of the perturbative series one has to use the upsilon expansion [14, 15], i.e. terms of order $\alpha_s^{n+1}$ in the perturbative series between the bottom quark pole and the bottom 1S masses are formally treated of order $\alpha_s^n$. For this reason, the powers of $\epsilon$ and $\alpha_s$ differ by one in Eq. (8). From now on, we will use $\epsilon$ as the perturbation series expansion parameter, and the order to which we expand will be denoted by $\varphi$.

The third order relation between the bottom pole and 1S masses reads

$$m_b^{\text{pole}} = m_b^{1S} f(m_b^{1S}, \alpha_s(\mu), \mu, \overline{m}(\mu_m), \mu_m)$$

$$f(m_b^{1S}, \alpha_s, \mu, \overline{m}, \mu_m) = 1 + \epsilon \left[\Delta^{(1)}(m_b^{1S}, \alpha_s)\right]$$

$$\quad + \epsilon^2 \left[\Delta^{(2)}(m_b^{1S}, \alpha_s, \mu) + \Delta^{(3)}(m_b^{1S}, \alpha_s, \mu, \overline{m})\right]$$

$$\quad + \epsilon^3 \left[\Delta^{(3)}(m_b^{1S}, \alpha_s, \mu) + \Delta^{(3)}(m_b^{1S}, \alpha_s, \mu, \overline{m}, \mu_m)\right]$$

where

$$\Delta^{(1)} = \frac{2}{9} \alpha_s^2,$$

$$\Delta^{(2)} = \frac{2 \alpha^3_s}{9} \left(11.2778 + 8.33333 L_{1S}\right),$$

$$\Delta^{(3)} = \frac{2 \alpha^4_s}{9} \left(192.693 + 119.083 L_{1S} + 52.0833 L_{1S}^2\right)$$

$$\quad + 2 \Delta^{(1)} \Delta^{(2)} + \Delta^{(1)} \Delta^{(3)} - \frac{\alpha_s \beta_0}{\pi} \left(\Delta^{(1)}\right)^2.$$
\[ L_{1S} \equiv \ln \left( \frac{3\mu}{4\alpha_s M_{b_{1S}}} \right), \]

and

\[ \Delta_n^{(2)} = \Delta^{NLO}_{\text{massive}}(m, m_{1S}, \alpha_s), \]

\[ \Delta_n^{(3)} = \Delta^{NLLO}(m, m_{1S}, \alpha_s) + 2\Delta^{NLLO}(m, m_{1S}, \alpha_s) + \left[ \beta^{(1)} - \Delta^{(1)} \right] \Delta^{NLO}(m, m_{1S}, \alpha_s, \mu). \]

The latter two terms describe the corrections arising from the non-zero charm quark mass. These contributions are included for consistency, since the charm quark is treated as massive in the charm pole-M\( \overline{\text{S}} \) mass relation of Eq. (2). Note that accounting for the charm mass corrections in Eq. (8) is mandatory to ensure the complete cancellation of the pole mass renormalon contributions in \( m_b - m_c \). This is because, in the perturbation series, only quarks that are treated as massless contribute to the bad higher order behavior that leads to the \( \Lambda_{\text{QCD}} \) renormalon ambiguity. To be more specific, only massless quarks contribute to the number of light quarks \( n_l \) that governs the asymptotic large order behavior; light quarks whose masses are not set to zero in the computations, as well as heavy quarks that are integrated out or treated as static, do not contribute to the \( \Lambda_{\text{QCD}} \) renormalon ambiguity in the perturbative series since their masses effectively act as an infrared cutoff for perturbative contributions from small momenta. The charm quark has been treated as massive in Eq. (2), and so must also be treated as massive in Eq. (8). We discuss this issue again in Section IV. The functions \( \Delta^{NLLO, NLLO}_{\text{massive}} \) are somewhat involved and can be found in [16, 18]. Charm mass contributions first enter at order \( \varphi = 2 \) in the \( \epsilon \) expansion. We will denote order \( \varphi = 2, 3 \) results including the charm mass corrections shown in Eq. (8) by \( \varphi = 2, 3c \).

III. DETERMINING \( m_c \)

Inclusive B decay spectra have traditionally been computed in terms of the pole masses \( m_b \) and \( m_c \). The formulae used in Ref. [7] eliminated the pole mass dependence by using Eq. (1) and Eq. (8). The resulting expressions eliminate the \( \Lambda_{\text{QCD}} \) renormalon from the perturbation series, and have better behaved perturbative expansions. The fit in Ref. [7] used all expressions consistently to order \( \alpha_s^2 \beta_0 \), and so used Eq. (8), dropping the \( \epsilon^2 \) and non-BLM parts of the \( \epsilon^2 \) terms (which also includes \( \Delta^{(2)} \)). This paper focuses on the determination of \( m_{c}(\mu) \) and how the results depend on the order of perturbation theory and the methods that are used for the analyses. We use the following two methods:

Method A: Take the results of Ref. [7] for \( m_{1S}^{b_{1S}}, \lambda_1, \rho_1, \tau_1 \) and \( \tau_2 \) as an input, and then determine \( m_{c}(\mu_{c}) \) using Eqs. [15] [18].

\[ m_b - m_c = f(\alpha_s) m_{1S}^{b_{1S}} - g(\alpha_s) m_{c}(\mu_{c}) \],

where \( \alpha_s \) is an input, and then determine \( m_{c}(\mu_{c}) \) by using Eq. (15) for \( m_{c}(\mu_{c}) \) gives

\[ m_{c}(\mu_{c}) = \frac{f(\alpha_s) m_{1S}^{b_{1S}} - g(\alpha_s) m_{c}(\mu_{c})}{g(\alpha_s) m_{c}(\mu_{c})}. \]

The difference between Methods A and B is whether \( m_b - m_c \) or the RHS side of Eq. (11) is held fixed. Note that for each method the entire expressions to a given order in \( \alpha_s \) are kept, except for the finite charm mass corrections in Eq. (8) as explained below.

Note that in determining \( m_{c}(\mu) \) to order \( \epsilon^\varphi, \varphi = 0, 1, 2, 3 \), we will use Eqs. (8) to order \( \epsilon^\varphi \), even though the same expression was only used to order \( \alpha_s^2 \beta_0 \) in the fit in Ref. [7]. Thus we break up the problem into two parts, determining \( m_b \), etc. from inclusive B decays, which has already been studied in detail in Ref. [8], and then using the results of the fit to get \( m_{c}(\mu) \). In both parts, the most precise available theoretical predictions are being used. Since in both parts, the leading \( \mathcal{O}(\Lambda_{\text{QCD}}) \) renormalon contributions are being systematically canceled, correlations of the behavior of perturbation theory in the two parts can be neglected. The errors in each part are determined separately, and both errors are included in our final error estimate. The errors in the first part of the problem (including the dependence on the order in \( \epsilon \) were studied in detail in Refs. [7, 8], and we use the same estimate here. In this work we concentrate on the errors in the second part of the problem.

For method B, one substitutes Eqs. (2, 8) to write \( m_b - m_c \)

\[ m_b - m_c = f(\alpha_s) m_{1S}^{b_{1S}} - g(\alpha_s) m_{c}(\mu_{c}) \],

where we have shown explicitly the dependence of \( m_{1S}^{b_{1S}} \) on \( \alpha_s \), suppressing the other variables. Both \( f(\alpha_s) \) and \( g(\alpha_s) \) contain an \( \mathcal{O}(\Lambda_{\text{QCD}}) \) renormalon contribution and have a badly behaved perturbation series. The \( \mathcal{O}(\Lambda_{\text{QCD}}) \) renormalon cancels between the two series, and so is absent in \( m_b - m_c \). To ensure that the series expansion for \( m_b - m_c \) is well-behaved, it is mandatory to truncate \( f(\alpha_s) \) and \( g(\alpha_s) \) to the same order in \( \epsilon \), when they are both written in terms of \( \alpha_s(\mu) \) at the same scale. Using different scales, e.g. \( \mu = m_{1S}^{b_{1S}} \) for \( f \) and \( \mu = m_{c}(\mu_{c}) \) for \( g \) and then truncating at the same order in \( \epsilon \) does not cancel the renormalon, and still leads to a badly behaved perturbation series which gives unreliable results.

Solving Eq. (15) for \( m_{c}(\mu_{c}) \) gives

\[ m_{c}(\mu_{c}) = \frac{f(\alpha_s) m_{1S}^{b_{1S}} - g(\alpha_s) m_{c}(\mu_{c})}{g(\alpha_s) m_{c}(\mu_{c})}. \]

We use Eq. (16) to determine \( m_{c}(\mu_{c}) \) in several different ways. In each case, we use Eq. (16) with \( f \) and \( g \) expanded to order \( \epsilon^\varphi \), with \( \varphi = 0, 1, 2, 3, 3c \). It is important to note that expanding to order \( \epsilon^\varphi \) means that all terms of higher order must be dropped. Since we are dealing with divergent series, one cannot retain some higher order terms, since they are large. The cancellations that convert the badly behaved perturbation series...
for $f$ and $g$ into a better behaved series for the $\overline{\text{MS}}$ charm mass only take place when all terms up to a given order, and no terms of higher order, are included. This point is discussed in more detail at the end of this section.

Our final results for $m_c$ are presented in Table IV and are obtained using the following procedures (in the order given below):

1. Use $\mu = 4.2$ GeV and $\mu_m = 4.2$ GeV, to get $\overline{m}_c(4.2\text{ GeV})$. Eq. (10) is expanded out analytically to order $\phi^2$. The terms $\overline{m}_c(\mu_m)$ in $f$ and $g$ on the RHS are eliminated by iteration resulting in an analytic (but complicated) expression. [analytic]

2. Use $\mu = 4.2$ GeV and $\mu_m = 4.2$ GeV, to get $\overline{m}_c(4.2\text{ GeV})$. Eq. (10) is solved numerically. In particular, the terms ln $[\overline{m}_c(4.2\text{ GeV})]$ in $f$ and $g$ are kept. [numeric]

3. Use the result from 1 for $\overline{m}_c(\mu_m)$ and determine $\overline{m}_c(4.2\text{ GeV})$ using the ($\phi + 1$)-loop QCD renormalization group equations. [analytic+RGE]

4. Use the result from 1 for $\overline{m}_c(\mu_m)$ and determine $\overline{m}_c(4.2\text{ GeV})$ using the ($\phi + 1$)-loop QCD renormalization group equations. [numeric+RGE]

5. Use $\mu = 4.2$ GeV and $\mu_m = \overline{m}_c(\mu_m)$, to get $\overline{m}_c(\overline{m}_c)$. Eq. (10) is expanded out analytically to order $\phi^2$. The terms $\overline{m}_c(\mu_m)$ in $f$ and $g$ are eliminated iteratively, as in 1. [analytic]

6. Use $\mu = 4.2$ GeV and $\mu_m = \overline{m}_c(\mu_m)$, to get $\overline{m}_c(\overline{m}_c)$. Eq. (10) is solved numerically for $\overline{m}_c(\overline{m}_c)$. In particular, the terms ln $[\overline{m}_c(\overline{m}_c)]$ in $f$ and $g$ are kept. [numeric]

7. Use the result from 1 for $\overline{m}_c(4.2\text{ GeV})$ and determine $\overline{m}_c(\overline{m}_c)$ using the ($\phi + 1$)-loop QCD renormalization group equations. [analytic+RGE]

8. Use the result from 1 for $\overline{m}_c(4.2\text{ GeV})$ and determine $\overline{m}_c(\overline{m}_c)$ using the ($\phi + 1$)-loop QCD renormalization group equations. [numeric+RGE]

For method A the same procedures are followed, except that one uses Eq. (11) as the starting point, and all results are (always) expanded to order $\Lambda^4_{\text{QCD}}/m^3$ in addition to the perturbative expansion in $\epsilon$ for the functions $f$ and $g$. We note that for the higher order $\Lambda_{\text{QCD}}/m$ corrections in Eq. (11), we treat the bottom and charm mass parameter as pole masses and always eliminate the charm mass terms iteratively in the $\Lambda_{\text{QCD}}/m$ expansion. For method B, $m_b - m_c$ is determined from Eq. (11) setting the bottom mass terms in the $\Lambda_{\text{QCD}}/m$ corrections equal to $m_{b_{\text{NS}}}$. Methods A differs from method B through the dependence of $m_b$ and $m_c$ in the $1/m^2$ and $1/m^4$ terms in Eq. (10) on the order $\phi$. In method B, $m_b - m_c$ is held fixed, whereas in method A, it can vary by more than 10 MeV between $\phi = 0$ and $\phi = 5$. 

The reference value $\mu = 4.2$ GeV is used in the above methods, since that is the value used in Ref. 7. The estimate of the theory uncertainty in Ref. 7 includes half the $\alpha_s^2/\beta_0$ term, and so already includes any uncertainties due to a different choice of reference scale.

### A. Renormalon Cancellation

There are alternative ways of writing Eq. (11). For example, one could also write the mass relations Eqs. (28) in the inverse form

$$\overline{m}_c(\mu) = \tilde{g} m_c, \quad m_b = \tilde{f} m_b.$$  \hspace{1cm} (17)

leading to four possible expressions,

$$m_b - m_c = f m_b^{1S} - g \overline{m}_c(\mu),$$ \hspace{0.5cm} (18a)

$$m_b - m_c = \frac{1}{f} m_b^{1S} - g \overline{m}_c(\mu),$$ \hspace{0.5cm} (18b)

$$m_b - m_c = f m_b^{1S} - \frac{1}{\tilde{g}} \overline{m}_c(\mu),$$ \hspace{0.5cm} (18c)

$$m_b - m_c = \frac{1}{f} m_b^{1S} - \frac{1}{\tilde{g}} \overline{m}_c(\mu).$$ \hspace{0.5cm} (18d)

If the functions $f$, $\tilde{f}$, $g$ and $\tilde{g}$ each are truncated to order $\phi$, only the first form represents a well-behaved series in practice. For the remaining three cases, it is necessary to reexpand the final expression in $\epsilon$ and truncate to $\epsilon^2$, leading to the first form. For the analytic procedures described above, all four forms lead then to the same answer, since the final results are always reexpanded and consistently truncated to $\epsilon^2$. However, for the numeric procedures, where $f$ and $g$ are individually truncated to $\epsilon^2$, and then $m_c$ is computed numerically by requiring that the relations analogous to Eq. (16) are satisfied, only the first form is correct. Solving Eq. (16) numerically with any of the replacements $f \rightarrow 1/\tilde{f}$ or $g \rightarrow 1/\tilde{g}$ can lead to unreliable results due to badly behaved higher order contributions that remain uncanceled. This was missed in Ref. 8; the other problem was that $\alpha_s$ was not used at the same renormalization scale $\mu$ for both $f$ and $g$.

### IV. RESULTS AND CONCLUSIONS

The results of our analysis are given in Table IV. The input parameters $m_{b_{\text{NS}}}$, $\lambda_1$, $p_1$ and $\tau_{1,3}$ and their covariance matrix needed for our analysis have been obtained using the 1S-mass fit procedure described in Ref. 9, but with the replacement $\alpha_s(4.2\text{ GeV}) = 0.22 \rightarrow 0.2245$ as the central value, to be consistent with $\alpha_s(M_Z) = 0.118$. This makes tiny changes to the input parameters from the results published in Ref. 9, e.g. $m_{b_{\text{NS}}}$ changes by 6 MeV, which is much smaller than its $\sim 30$ MeV total error. The results for the 6-quark 1S mass (rounded to 10 MeV precision) remains unchanged and reads $m_{b_{\text{NS}}} = 4.68 \pm 0.03$ GeV in full agreement with the T sum rule analyses carried out in Refs. 10-12.
For the input parameters used in our analysis, we have distinguished between experimental uncertainties coming from the data on inclusive $B$ decay spectra, and the theory uncertainties coming from the corresponding theoretical predictions at order $\alpha_s^2 \beta_0$ by carrying out the fits with theory uncertainties (as described in Ref. 5) switched on and switched off. For our analysis we have also accounted for the uncertainty in $\alpha_s(4.2\text{ GeV})$ as an experimental uncertainty, including its correlation with $m_b\beta_1, \lambda_1, \rho_1$ and $\tau_{1.3}$. The corresponding $\alpha_s$ entries of the covariance matrix have been estimated from the dependence of the best fit for $m_b\beta_1, \lambda_1, \rho_1$ and $\tau_{1.3}$ on $\alpha_s(4.2\text{ GeV})$ and assuming that the error on $\alpha_s(4.2\text{ GeV})$ is $\pm0.0114$ which corresponds to an error of $\pm0.003$ for the strong coupling at $\mu = M_Z$. This error has been conservatively chosen to be larger than the PDG estimate $\pm0.002$ for the error on $\alpha_s(M_Z)$. We find that the uncertainty in $\alpha_s(4.2\text{ GeV})$ has virtually no impact on the total experimental uncertainty for $m_c$ ($m_t$), while the experimental error for $m_c(4.2\text{ GeV})$ increases by 15 to 20 MeV.

The various results shown in Table I are stable, and reflect a very good perturbative behavior in each case. For each entry in the table the first error is statistical accounting for the uncertainties and correlations of the input parameters $m_b\beta_1, \lambda_1, \rho_1, \tau_{1,3}$ and $\alpha_s(4.2\text{ GeV})$ due to uncertainties in the experimental data used in the analysis of Ref. 5, and the second error accounts for the theory uncertainties in the predictions used in Ref. 5. The results show some interesting features that are worthy of comment: It is conspicuous that the results for $m_c$ ($m_t$) obtained directly in a single step appear more stable than those obtained by scaling $m_c(4.2\text{ GeV})$ down to the charm mass. For example for method A, the two results for $m_c$ ($m_t$) based on the analytic determination read $(1279 - 66e - 18c^2 + 6c^3)$ MeV [analytic] and $(1521 - 203e - 72c^2 - 15c^3)$ MeV [analytic+RGE]. Both series show very good convergence properties. On the other hand, the differences between the central values for the two series at order $(1, e, e^2, e^3)$ are $(242, 105, 51, 30)$ MeV. These values are much larger than one might guess by using a naive error estimate based on the good behavior of each individual series. This is a feature that appears common to the behavior of perturbation theory at the (relatively low) charm mass scale $\cal M$. Since there is no a-priori reason to consider any single one of the series to be more reliable than the other, we use as an estimate of the theoretical uncertainty in our analysis the spread of central values for the different methods.

It is instructive to compare the results at order 2, 3 and $2_c, 3_c$. While accounting for the charm mass corrections in the relation of the bottom pole and $1S$ masses, Eq. 5 leads only to a small difference of at most 5 MeV at order...
TABLE II: Linear formulæ for the $\overline{\text{MS}}$ charm mass in MeV at order 3, in terms of the input parameters for the different procedures used in our analysis. The input parameters are defined by $\Delta \alpha_0 = (\alpha_s(4.2 \text{ GeV}) - 0.2245)$, $\Delta m_b = (m_b^{\text{MS}} \text{ GeV}^{-1} - 4.681)$, $\Delta \lambda = (\lambda_1 \text{ GeV}^{-2} + 0.2426)$, $\Delta \rho = ((\rho_1 - \tau_1 - \tau_3) \text{ GeV}^{-1} + 0.02981)$, $\delta = ((m_b - m_c) \text{ GeV}^{-1} - 3.401)$. The last column shows the maximal deviation in MeV of the linear approximation from our results over the parameter ranges $|\Delta \alpha_0| < 0.015$, $|\Delta m_b| < 0.1$ GeV, $|\Delta \lambda| < 0.1$, $|\Delta \rho| < 0.2$, $|\delta| < 0.05$.

| Method | $3_c$ | max. dev. |
|--------|-------|-----------|
| $\overline{\text{Me}}(4.2 \text{ GeV})$ | analytic | 880.7 - 1761 $\Delta \alpha_0 + 159 \Delta \lambda + 865 \Delta m_b - 59 \Delta \rho$ | 7 |
| $\overline{\text{Me}}(4.2 \text{ GeV})$ | numeric | 895.4 - 1551 $\Delta \alpha_0 + 171 \Delta \lambda + 858 \Delta m_b - 71 \Delta \rho$ | 7 |
| $\overline{\text{Me}}(4.2 \text{ GeV})$ | analytic+RGE | 849.7 - 2473 $\Delta \alpha_0 + 164 \Delta \lambda + 900 \Delta m_b - 61 \Delta \rho$ | 7 |
| $\overline{\text{Me}}(4.2 \text{ GeV})$ | numeric+RGE | 850.2 - 2528 $\Delta \alpha_0 + 182 \Delta \lambda + 913 \Delta m_b - 76 \Delta \rho$ | 6 |
| $\overline{\text{Me}}(m_c)$ | analytic | 1202.0 + 272 $\Delta \alpha_0 + 162 \Delta \lambda + 886 \Delta m_b - 60 \Delta \rho$ | 8 |
| $\overline{\text{Me}}(m_c)$ | numeric | 1202.6 + 217 $\Delta \alpha_0 + 179 \Delta \lambda + 899 \Delta m_b - 75 \Delta \rho$ | 7 |
| $\overline{\text{Me}}(m_c)$ | analytic+RGE | 1232.5 + 937 $\Delta \alpha_0 + 156 \Delta \lambda + 850 \Delta m_b - 58 \Delta \rho$ | 9 |
| $\overline{\text{Me}}(m_c)$ | numeric+RGE | 1247.5 + 1126 $\Delta \alpha_0 + 168 \Delta \lambda + 842 \Delta m_b - 70 \Delta \rho$ | 8 |
| $\overline{\text{Me}}(4.2 \text{ GeV})$ | analytic | 868.0 - 1920 $\Delta \alpha_0 + 836 \Delta m_b - 808 \delta$ | 4 |
| $\overline{\text{Me}}(4.2 \text{ GeV})$ | numeric | 886.6 - 1635 $\Delta \alpha_0 + 823 \Delta m_b - 798 \delta$ | 5 |
| $\overline{\text{Me}}(4.2 \text{ GeV})$ | analytic+RGE | 836.6 - 2650 $\Delta \alpha_0 + 869 \Delta m_b - 840 \delta$ | 2 |
| $\overline{\text{Me}}(4.2 \text{ GeV})$ | numeric+RGE | 840.5 - 2632 $\Delta \alpha_0 + 877 \Delta m_b - 850 \delta$ | 2 |
| $\overline{\text{Me}}(m_c)$ | analytic | 1188.9 + 110 $\Delta \alpha_0 + 858 \Delta m_b - 829 \delta$ | 3 |
| $\overline{\text{Me}}(m_c)$ | numeric | 1192.7 + 124 $\Delta \alpha_0 + 865 \Delta m_b - 838 \delta$ | 3 |
| $\overline{\text{Me}}(m_c)$ | analytic+RGE | 1219.8 + 793 $\Delta \alpha_0 + 822 \Delta m_b - 795 \delta$ | 7 |
| $\overline{\text{Me}}(m_c)$ | numeric+RGE | 1238.1 + 1052 $\Delta \alpha_0 + 809 \Delta m_b - 784 \delta$ | 8 |

2 and 3, for $\overline{\text{Me}}(m_c)$, the difference is already as large as the experimental uncertainties of around 15 MeV at order 3 and 3. This behavior arises because treating the charm as massless in the function $f$ of Eq. [5] leads to a different $O(\Lambda_{\text{QCD}})$-renormalon behavior in the bottom pole-1S mass series than in the charm pole-$\overline{\text{MS}}$ mass series where the charm mass effects are automatically included. Thus, as explained in Section [11] at order 2 and 3 the $O(\Lambda_{\text{QCD}})$ renormalon contributions contained in $f$ and $g$ do not fully cancel. Therefore, to avoid this source of perturbative ambiguity it is mandatory to account for the charm mass effects in the bottom pole-1S mass relation, and only the orders 2 and 3 should be considered for the analysis.

To determine our final result for $\overline{\text{Me}}(m_c)$ we use the order $c_1^3$ results. The results obtained using method A and B are very similar. For the central value and the error estimate we therefore only use the numbers from method A. As the central value we use the average of the largest and smallest numbers. For the experimental uncertainty we take the largest experimental error of 17 MeV. For the theory uncertainty, we linearly add the error originating from the theory error of the input parameters $m_b^{\text{MS}}$, $\lambda_1$, $\rho_1$ and $\tau_1,\tau_3$ of 31 MeV, and the perturbative uncertainty of our analysis. For the latter we take half the range of the central values we have obtained, which is 23 MeV. This gives

$$\overline{\text{Me}}(m_c) = 1224 \pm 17 \pm 31_B \pm 23_{m_c} \text{ MeV},$$

with the subscripts denoting the sources of the various uncertainties. We add the two theoretical uncertainties linearly, to be conservative. Moreover, this more conservative treatment should also account for the uncertainty arising from the unknown order $\alpha_s(\Lambda_{\text{QCD}}^2)~m$ corrections in Eq. (11). The latter uncertainty is reflected by the variation of $m_b - m_c$ with the order $\varphi$, and is about 10 MeV. Thus our final result is

$$\overline{\text{Me}}(m_c) = 1224 \pm 17 \pm 54 \text{ MeV}.$$ (20)

The first error is the experimental 1-$\sigma$ error and the second is the theoretical error. Note that the theoretical error should not be interpreted as a 1-$\sigma$ error, since it is not statistical. Rather it represents the range in which we believe that the true charm mass is located with high probability (much greater than the 67% probability of a 1-$\sigma$ interval). Our result is consistent with the most recent result for $\overline{\text{Me}}(m_c)$ from low-$n$ $e^+e^-$ moment sum rules [12]. However, our result is somewhat lower than the most recent results obtained from lattice QCD [13, 14], which were obtained in the quenched approximation.

Linear approximation formulæ showing the dependence of our results on the input parameters $m_b^{\text{MS}}$, $\lambda_1$, $\rho_1$, $\tau_1,\tau_3$, $\alpha_s(4.2 \text{ GeV})$ and $m_b - m_c$ for methods A and B at order $c_2^3$ are displayed in Table II

In our analysis we have also determined $\overline{\text{Me}}(4.2 \text{ GeV})$ using the same procedure as used for $\overline{\text{Me}}(m_c)$,

$$\overline{\text{Me}}(4.2 \text{ GeV}) = 873 \pm 41_{\text{exp}} \pm 34_B \pm 23_{m_c} \text{ MeV}.$$ (21)
Our final result is
\[ \bar{m}_c(4.2\text{ GeV}) = 873 \pm 41 \pm 57 \text{ MeV}. \] (22)

While the theoretical uncertainties are comparable to the corresponding uncertainties for \( m_c(\bar{m}_c) \), we find that the experimental errors are 15 to 20 MeV larger. As mentioned before, this increase originates from the uncertainties in \( \alpha_s \) which have a rather large impact on the uncertainty in \( m_c(\bar{m}_c) \). For our final result is
\[ m_c(\bar{m}_c) = \frac{44 \pm 13}{1.09 \pm 0.13}. \]

The origin of this behavior can be clearly seen from the term for the corresponding uncertainties for \( m_c(\bar{m}_c) \), but almost none for \( m_c(\bar{m}_c) \). Our final result is
\[ m_c(\bar{m}_c) = \frac{44 \pm 13}{1.09 \pm 0.13}. \]

Example QCD where theory uncertainties dominate. A more precise input value for \( \alpha_s \) will not affect the uncertainties of \( m_c(\bar{m}_c) \), but can reduce substantially the experimental error of \( m_c(\bar{m}_c) \). A full \( \mathcal{O}(\alpha_s) \) analysis of the semileptonic inclusive \( B \) decay spectra would reduce the uncertainty of the input parameters \( m_b, \rho_1 \) and \( \tau_{1,3} \) of our charm mass analysis. Moreover, a determination of \( \mathcal{O}(\alpha_s \Lambda_{QCD}^2/m^2) \) corrections would reduce the scheme dependence of the \( \Lambda_{QCD}^2/m^2 \) contributions in the OPE for the \( B-D \) meson mass difference. We believe that these computations are achievable with present technology and could reduce the theory errors shown in Table I by at least a factor of two. Finally, a computation of the full order \( \Lambda^2 \) corrections to the bottom-charm pole mass difference as a function of the \( m_c^* \) charm mass and the bottom IS mass could further reduce the theoretical uncertainty in Eq. (16). Such work would allow one to reduce the uncertainty of \( m_c(\bar{m}_c) \), with a conservative estimation of theory errors, to well below the 50 MeV level.

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