THE 4-CLASS GROUP OF REAL QUADRATIC NUMBER FIELDS

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ABSTRACT. In this paper we give an elementary proof of results on the structure of 4-class groups of real quadratic number fields originally due to A. Scholz. In a second (and independent) section we strengthen C. Maire’s result that the 2-class field tower of a real quadratic number field is infinite if its ideal class group has 4-rank ≥ 4, using a technique due to F. Hajir.

1. Introduction

Let \( d \) and \( d' \) be discriminants of real quadratic number fields, and suppose that they are the product of positive prime discriminants (equivalently, they are sums of two squares). If \((d, d') = 1\), and if \((d/p') = +1\) for all primes \(p' \mid d'\), then we can define a biquadratic Jacobi symbol by \( (d/d')_4 = \prod_{p'\mid d'} (d/p')_4 \). Here \((d/p')_4\) is the rational biquadratic residue symbol (it is useful to put \((d/2)_4 = (−1)^{(d−1)/8}\); note that \(d \equiv 1 \mod 8\) if \(8 \mid d'\)). Observe, however, that this symbol is not multiplicative in the numerator. We also agree to say that a discriminant \(d'\) divides another discriminant \(d\) if there exists a discriminant \(d''\) such that \(d = d'd''\).

The following result due to Rédei is well known:

**Proposition 1.** Let \(d\) be the discriminant of a quadratic number field. The cyclic quartic extensions of \(k\) which are unramified outside \(\infty\) correspond to \(C_4^+\)-factorizations of \(d\), i.e. factorizations \(d = d_1d_2\) into two relatively prime positive discriminants such that \((d_1/p_2) = (d_2/p_1) = +1\) for all \(p_1 \mid d_1\) and all \(p_2 \mid d_2\). If \(d = d_1d_2\) is such a \(C_4^+\)-factorization, then the extension \(k(\sqrt{d'})\) can be constructed by choosing a suitable solution of \(x^2 - d_1y^2 = d_2z^2\) and putting \(\alpha = x + y\sqrt{d_1}\). In this case, every cyclic extension of \(k\) which is unramified outside \(\infty\) and contains \(\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})\) has the form \(k(\sqrt{d''}\alpha)\), where \(d''\) is a discriminant dividing \(d\).

In particular, if \(d\) is divisible by a negative prime discriminant \(d'\), then for every \(C_4^+\)-factorization \(d = d_1d_2\) there is always a cyclic quartic extension of \(k = \mathbb{Q}(\sqrt{d})\) which is unramified everywhere: should \(k(\sqrt{d'})\) be totally negative, simply take \(k(\sqrt{d''}\alpha)\). The problem of the existence of cyclic quartic extension which are unramified everywhere is thus reduced to the case where \(d\) is a product of positive prime discriminants. The next section is devoted to an elementary proof of a result in this case; Scholz sketched a proof using class field theory in his own special way in [10].

2. Ramification at \(\infty\)

Let \(k\) be a real quadratic number field with discriminant \(d = \text{disc} k\), and assume that \(d\) is the product of positive prime discriminants. Let \(d = d_1 \cdot d_2\) be a \(C_4^+\)-factorization, i.e. assume that \((d_1/p_2) = (d_2/p_1) = +1\) for all \(p_1 \mid d_1\) and all \(p_2 \mid d_2\).
Pick a solution of \(x^2 - d_1y^2 = d_2z^2\) such that \(k(\sqrt{d_1}, \sqrt{\alpha})\) (with \(\alpha = x + y\sqrt{d_1}\)) is a cyclic quartic extension of \(k\) which is unramified outside \(\infty\).

**Theorem 2.** The extension \(K = k(\sqrt{d_1}, \sqrt{\alpha})\) defined above is totally real if and only if \((d_1/d_2)_4 = (d_2/d_1)_4\). If there is a cyclic octic extension \(L/k\) containing \(K\) and unramified outside \(\infty\), then \((d_1/d_2)_4 = (d_2/d_1)_4 = +1\).

**Proof.** Assume first that \(d \equiv 1 \mod 4\); then we can always choose \(x \in \mathbb{Z}, y \in \mathbb{N}\) in such a way that \(x - 1 \equiv y \equiv 1 \mod 2\). Then \(\alpha\) is 2-primary if and only if \(x + y \equiv 1 \mod 4\). We now proceed as in [3] and reduce the equation \(x^2 - d_1y^2 = d_2z^2\) modulo certain primes. From \(x^2 \equiv d_1y^2 \mod d_2\) and \(x^2 \equiv d_2z^2 \mod d_1\) we get

\[
\left( \frac{x}{d_2} \right) = \left( \frac{d_1}{d_2} \right)_4 \left( \frac{y}{d_2} \right), \quad \text{and} \quad \left( \frac{x}{d_1} \right) = \left( \frac{d_2}{d_1} \right)_4 \left( \frac{z}{d_1} \right).
\]

Multiplying these equations gives

\[
\left( \frac{d_1}{d_2} \right)_4 \left( \frac{d_2}{d_1} \right)_4 = \left( \frac{x}{d_1d_2} \right)_4 \left( \frac{y}{d_2} \right) \left( \frac{z}{d_1} \right).
\]

Considering \(x^2 - d_1y^2 = d_2z^2\) modulo \(x\) shows

\[
\left( -\frac{d_1}{x} \right) = \left( \frac{d_2}{x} \right), \quad \text{i.e.} \quad \left( \frac{x}{d_1d_2} \right)_4 = \left( \frac{d_1d_2}{x} \right)_4 = \left( -\frac{1}{x} \right).
\]

Now write \(y = 2^j u\) for some odd \(u \in \mathbb{N}\); then

\[
\left( \frac{y}{d_2} \right) = \left( \frac{2}{d_2} \right)_j \left( \frac{u}{d_2} \right) = \left( \frac{2}{d_2} \right)_j \left( \frac{d_1}{u} \right) = \left( \frac{2}{d_2} \right)_j.
\]

But \(j = 1\) implies \(d_2 \equiv 5 \mod 8\), and for \(j \geq 2\) we have \(d_2 \equiv 1 \mod 8\); this shows that \((2/d_2)^j = (-1)^{y/2}\). Finally we easily see that \((z/d_1) = (d_1/z) = 1\), hence we have shown

\[
\left( \frac{d_1}{d_2} \right)_4 \left( \frac{d_2}{d_1} \right)_4 = \left( -\frac{1}{x} \right) (-1)^{y/2} = (-1)^{(\lfloor x/2 \rfloor^2)}.
\]

The right hand side equals \(+1\) if \(x > 0\) (the assumption that \(\alpha\) is 2-primary says \(x + y \equiv 1 \mod 4\)), and is \(-1\) if \(x < 0\). Therefore \(K\) is real if and only if \((d_1/d_2)_4(d_2/d_1)_4 = +1\).

Now assume that \(L/k\) is a cyclic unramified octic extension containing \(K\). We claim that the prime ideals \(p\) ramified in \(k_1(\sqrt{\alpha'})/k_1\) (where \(k_1 = \mathbb{Q}(\sqrt{d_1})\)) must split in \(k_1(\sqrt{\alpha'})/k_1\), where \(\alpha'\) is the conjugate of \(\alpha\).

We first show that such \(p\) are not ramified in \(k_1(\sqrt{\alpha'})/k_1\): in fact, ramifying primes must divide \(d_1d_2\), since \(L/k\) is unramified. If \(p \mid d_1\), then \(p\) ramifies completely in \(k_1(\sqrt{\alpha'})/\mathbb{Q}\), which contradicts the fact that all ramification indices must divide 2 (again because \(L/k\) is unramified). Assume that an \(p \mid d_2\) ramifies in both \(k_1(\sqrt{\alpha'})/k_1\) and \(k_1(\sqrt{\omega})/k_1\); since primes dividing \(d_2\) split in \(k_1/\mathbb{Q}\) and ramify in \(F/k_1\) (where \(F = k_2k_1\), \(p\) would ramify in all three quadratic extensions of \(k_1\) contained in \(L\), and again its ramification index would have to be \(\geq 4\).

Now suppose that \(p\) is inert in \(k_1(\sqrt{\omega})/k_1\). Then the prime ideal \(\mathfrak{P}\) in \(F\) above \(p\) is inert in \(K/F\), and since \(L/F\) is cyclic, it is inert in \(L/F\). Let \(e, f\) and \(g\) denote the order of the ramification, inertia and decomposition group \(V, T\), and \(Z\) of \(\mathfrak{P}\), respectively; we have seen that \(e = 2, f \geq 4\) and \(g \geq 2\). Since \(efg = (L : \mathbb{Q}) = 16\), we must have equality. Thus \(k_1\) is the splitting field, and we have \(Z = \text{Gal}(L/k_1)\).

We know that \(\text{Gal}(L/k_1) \simeq D_4\), and since \(Z/T\) is always cyclic of order \(f, T\) must
fix a cyclic quartic extension of \( k_1 \) in \( L \). But such an extension does not exist; this contradiction shows that \( p \) splits.

Finally, prime ideals \( p \) ramifying in \( k_1(\sqrt{\alpha})/k_1 \) divide \( \alpha \) (otherwise \( p \) would be a prime above 2; but here we assume that \( d \) is odd, i.e. \( K/\mathbb{Q} \) is unramified above 2); they split in \( k_1(\sqrt{\alpha'})/k_1 \) if and only if \( (\alpha'/p) = 1 \). But now

\[
\left( \frac{x - y\sqrt{d_1}}{p} \right) = \left( \frac{2x}{p} \right) = \left( \frac{2x}{p} \right),
\]

where \( p \) is the prime under \( p \); here we have used that \( x - y\sqrt{d_1} \equiv x - y\sqrt{d_1} + (x + y\sqrt{d_1}) \equiv 2x \mod p \). The proof above shows that \( (2x/d_2) = (2/d_2)^i(d_1/d_2)^i \); since \( (2/d_2) = (2/d_2)^i \), we conclude that \( p \) splits in \( k_1(\sqrt{\alpha})/k_1 \) only if \( (d_1/d_2)^i = 1 \).

The proof in the case where one of the \( d_i \) is divisible by 8 is left to the reader. \( \square \)

Theorem 2 contains many results on the solvability of negative Pell equations (due to Dirichlet, Epstein, Kaplan and others) as special cases. In fact, consider the criteria or proving existing ones (e.g. those in [3]) is no problem at all.

3. Maire’s Result

In [9], C. Maire showed that a real quadratic number field \( k \) has infinite 2-class field tower if its class group contains a subgroup of type \((4,4,4,4)\). Here we will show that it suffices to assume that its class group in the strict sense contains a subgroup of type \((4,4,4,4)\). The method employed is taken from F. Hajir’s paper [3].

**Theorem 3.** Let \( k \) be a real quadratic number field. If the strict ideal class group of \( k \) contains a subgroup of type \((4,4,4,4)\), then the 2-class field tower of \( k \) is infinite.

For the proof of this theorem we need a few results. For an extension \( K/k \), let the relative class group be defined by \( \text{Cl}(K/k) = \ker(N_{K/k} : \text{Cl}(K) \to \text{Cl}(k)) \). Moreover, let \( G_p \) denote the \( p \)-Sylow subgroup of a finite abelian group \( G \); we will denote the dimension of \( G/G^p \) as an \( \mathbb{F}_p \)-vector space by \( \text{rank}_p G \). Let \( \text{Ram}(K/k) \) denote the set of all primes in \( k \) (including those at \( \infty \)) which ramify in \( K/k \); we will also need the unit groups \( E_k \) and \( E_K \), as well as the subgroup \( H = E_k \cap N_{K/k}K^\times \) of \( E_k \). Then Jehne [9] has shown

**Proposition 4.** Let \( K/k \) be a cyclic extension of prime degree \( p \). Then

\[
\text{rank}_p \text{Cl}(K/k) \geq \# \text{Ram}(K/k) - \text{rank}_p E_k/H - 1.
\]

Let us apply the inequality of Golod-Shafarevic to the \( p \)-class group of \( K \). We know that the \( p \)-class field tower of \( K \) is infinite if

\[
\text{rank}_p \text{Cl}(K) \geq 2 + 2\sqrt{\text{rank}_p E_K} + 1.
\]
By Prop. 4 we know that
\[
\text{rank}_p \text{Cl}(K) \geq \text{rank}_p \text{Cl}(K/k) \geq \#\text{Ram}(K/k) - \text{rank}_p E_k/H - 1.
\]

Thus $K$ has infinite $p$-class field tower if
\[
\#\text{Ram}(K/k) \geq 3 + \text{rank}_p E_k/H + 2\sqrt{\text{rank}_p E_K + 1}.
\]

We have proved (compare Schoof [11]):

**Proposition 5.** Let $K/k$ be a cyclic extension of prime degree $p$, and let $\rho$ denote the number of finite and infinite primes ramifying in $K/k$. Then the $p$-class field tower of $K$ is infinite if
\[
\rho \geq 3 + \text{rank}_p E_k/H + 2\sqrt{\text{rank}_p E_K + 1}.
\]

Here $H$ is the subgroup of $E_k$ consisting of units which are norms of elements from $K$, and $\text{rank}_p G$ denotes the $p$-rank of $G/G^p$.

**Proof of Thm.** Since the claim follows directly from the inequality of Golod-Shafarevich if $\text{Cl}_2(k) \geq 6$, we may assume that $d = \text{disc}k$ is the product of at most six positive prime discriminants, or of at most seven if one of them is negative. The idea is to show that a subfield of the genus class field of $k$ satisfies the inequality of Prop. 5; this will clearly prove that $k$ has infinite 2-class field tower.

Assume first that $d = \prod_{j=1}^5 d_j$ is the product of five positive prime discriminants. If $\text{Cl}^+(k)$ contains a subgroup of type $(4, 4, 4, 4)$, then $d = d_1 \cdot d_2d_3d_4d_5$ and $d = d_2 \cdot d_1d_3d_4d_5$ must be $C_4^+$-factorizations. Therefore, the $d_j$ ($j \geq 3$) split completely in $F = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$, hence there are 12 prime ideals ramifying in $K/F$, where $K = F(\sqrt{d_3d_4d_5})$. Since $\text{rank}_2 E_K = 8$, the condition of Prop. 5 is satisfied for $K/F$ if $\text{rank}_2 E_F/H \leq 3$. But since $-1$ is a norm in $K/F$ (only prime ideals of norm $\equiv 1 \mod 4$ ramify), we have $-1 \in H$; this implies that $\text{rank}_2 E_F/H \geq 3$.

Next suppose that $d = \prod_{j=1}^6 d_j$ is the product of six positive prime discriminants. We will show in the next section (see Problem 7) that either there is $C_4^+$-factorizations of type $d_1 \cdot d_2d_3d_4d_5d_6$ (then we can apply Prop. 5 to the quadratic extension $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})/\mathbb{Q}(\sqrt{d_1})$, and we are done), or there exist (possibly after a suitable permutation of the indices) $C_4^+$-factorizations $d_2d_3 \cdot d_1d_4d_5d_6$ and $d_1d_3 \cdot d_2d_4d_5d_6$. In this case consider $F = \mathbb{Q}(\sqrt{d_2d_3}, \sqrt{d_1d_5})$. The prime ideals above $d_4, d_5, d_6$ split completely in $F/\mathbb{Q}$, and applying Prop. 5 to $F(\sqrt{d})/F$ shows exactly as above that $F(\sqrt{d})$ has infinite 2-class field tower.

Now we treat the case where $d$ is divisible by some negative prime discriminants. First assume that $d$ is the product of six prime discriminants. In this case, all possible factorizations of $d$ as a product of two positive discriminants must be a $C_4^+$-factorization; unless $d$ is a product of six negative discriminants, there exists a factorization of type $d_1 \cdot \prod_{j=2}^6 d_j$, where $d_1 > 0$ is a prime discriminant. In this case, we can apply Prop. 5 to $\mathbb{Q}(\sqrt{d}, \sqrt{d_1})/\mathbb{Q}(\sqrt{d_1})$. The case where all six prime discriminants are negative does not occur here: since all factorizations are $C_4^+$, we must have $(d, d_1/q_i) = +1$ for all triples $(i, j, \ell)$ such that $1 \leq i < j < \ell \leq 6$ (here $q_\ell$ is the unique prime dividing $d_\ell$). At least one of these triples consist of odd primes $q \equiv 3 \mod 4$; call them $q_1, q_2$ and $q_3$, respectively. We find $(q_1/q_3) = (q_2/q_3) = -(q_1/q_2) = -(q_1/q_2) = (q_2/q_1) = (q_3/q_1)$, contradicting the quadratic reciprocity law.
Finally, if \( d \) is the product of seven prime discriminants, then (see Problem 8) there exists at least one \( C_4^+ \)-factorization of type \( d = p \cdot \ast \ast \) or \( d = rr' \cdot \ast \ast \), where \( p \equiv 1 \mod 4 \) and \( r, r' \) are either both positive or both negative prime discriminants (see Problem 8). Applying Prop. 4 to \( \mathbb{Q}(\sqrt{d}, \sqrt{p})/\mathbb{Q}(\sqrt{p}) \) or \( \mathbb{Q}(\sqrt{d}, \sqrt{rr'})/\mathbb{Q}(\sqrt{rr'}) \) yields the desired result. \( \square \)

Observe that the same remarks as in Hajir’s paper apply: our proof yields more than we claimed. If, for example, \( d \) is a product of five positive prime discriminants and admits two \( C_4^+ \)-factorizations of type \( d = d_1 \cdot d_2 d_3 d_4 d_5 \) and \( d = d_2 \cdot d_1 d_3 d_4 d_5 \), then \( k = \mathbb{Q}(\sqrt{d}) \) has infinite 2-class field tower even if \( \text{Cl}(k) \) has 4-rank equal to 2.

We also remark that it is not known whether these results are best possible: we do not know any imaginary quadratic number field with 2-class group of type \( (4, 4, 4) \) and abelian 2-class field tower (cf. \([2]\)).

4. SOME RAMSEY-TYPE PROBLEMS

Let the discriminant \( d = d_1 \ldots d_t \) be the product of \( t \) positive prime discriminants. Let \( X \) be the subspace of \( \mathbb{F}_2^t \) consisting of 0 and \((1, \ldots, 1)\), and put \( V = \mathbb{F}_2^t / X \). Then the set of possible factorizations into a product of two discriminants corresponds bijectively to an element of \( V \): in fact, a factorization of \( d \) is a product \( d = \prod_j d_j^e_j \cdot \prod_j d_j^{f_j} \) with \( e_j, f_j \in \mathbb{F}_2 \) and \( e_j + f_j = 1 \), and it corresponds to the image of \((e_1, \ldots, e_t)\) in the factor space \( V = \mathbb{F}_2^t / X \) (exchanging the two factors corresponds to adding \((1, \ldots, 1)\)). We will always choose representatives with a minimal number of nonzero coordinates. Moreover, the product defined on the set of factorizations of \( d \) corresponds to the addition of the vectors in \( V \): the bijection constructed above is a group homomorphism.

Define a map \( \mathbb{F}_2^t \to \mathbb{N} \) by mapping \((e_1, \ldots, e_t)\) to \( \min(\sum_{j=1}^t e_j, t - \sum_{j=1}^t e_j) \); observe that the sum is formed in \( \mathbb{N} \) (not in \( \mathbb{F}_2 \)). Since \((1, \ldots, 1)\to 0\), this induces a map \( V \to \mathbb{N} \) which we will denote by \( S \). Let \( V_\nu \) denote the fibers of \( V \) over \( \nu \), i.e. put \( V_\nu = \{ u \in V : S(u) = \nu \} \). It is easy to determine their cardinality:

**Lemma 6.** If \( t = 2s \) is even, then \( \#V_\nu = \binom{t}{s} \) if \( \nu < s \); if \( t = 2s + 1 \) is odd, then \( \#V_\nu = \binom{t}{s} \) for all \( \nu \leq s \).

Now let us formulate our first problem:

**Problem 7.** Let \( t = 6 \), and suppose that \( U \subseteq V \) is a subspace of dimension 4. If \( U \cap V_1 \) is empty, then the equation \( a + b + c = 0 \) has solutions in \( U \cap V_2 \).

**Proof.** Clearly \( \#U = 16 \) and \( \#(U \cap V_0) = 1 \); if \( U \cap V_1 = \varnothing \), then \( \#(U \cap V_3) \leq \#V_3 = 10 \) implies that \( \#(U \cap V_2) \geq 4 \). But among four vectors in \( V_2 \) there must exist a pair \( a, b \in U \cap V_2 \) with a common coordinate 1 by Dirichlet’s box principle; after permuting the indices if necessary we may assume that \( a = (1, 1, 0, 0, 0, 0) \) and \( b = (1, 0, 1, 0, 0, 0) \). Clearly \( a + b = (0, 1, 1, 0, 0, 0) \in U \cap V_2 \), and our claim is proved. \( \square \)

What has this got to do with our \( C_4^+ \)-factorizations? Well, consider the case where \( d \) is the product of six positive prime discriminants. The possible \( C_4^+ \)-factorizations correspond to \( V \); since rank \( \text{Cl}_4^+(k) \geq 4 \), we must have at least four
independent \(C_4^+\)-factorizations, generating a subspace \(U \subset V\) of dimension 4. Problem 4 shows that among these there is one \(C_4^+\)-factorization with one factor a prime discriminant, or there are two \(C_4^+\)-factorizations \(d_1d_2 \cdot d_3d_4d_5d_6\) and \(d_1d_3 \cdot d_2d_4d_5d_6\).

Now consider \(F_2^t / X\) and define 'incomplete traces'

\[
T_{2k} : F_2^t / X \rightarrow F_2 : (u_1, \ldots, u_t) \mapsto \sum_{j=1}^{2k} u_j
\]

(Here the sum is formed in \(F_2\)). This is a well defined linear map, hence its kernel \(V^{(2k)} = \ker T_{2k}\) is an \(F_2\)-vector space of dimension \(t - 2\). The elements of \(V^{(2k)}\) correspond to the set of factorizations of a discriminant \(d\) into a product of two positive discriminants when \(2k\) of the \(t\) prime discriminants dividing \(d\) are negative.

Defining \(S : V^{(2k)} \rightarrow \mathbb{N}\) as above and denoting the fibers by \(V_{v_j}^{(2k)}\), we find, for example, that \(#V_0^{(2k)} = 1\) and \(V_1^{(2k)} = t - 2k\). In the special case \(t = 7\) we get, in addition, \(#V_2^{(2)} = 1 + (\frac{5}{2}) = 11\), \(#V_3^{(2)} = 5 + (\frac{5}{2}) = 15\), \(#V_4^{(2)} = 3 + (\frac{1}{2}) = 9\), \(#V_5^{(4)} = 1 + 3(\frac{3}{2}) = 19\), \(#V_6^{(6)} = 6(\frac{6}{2}) = 15\), and \(#V_7^{(6)} = 6(\frac{6}{2}) = 15\).

**Problem 8.** Let \(t = 7, 1 \leq k \leq 3\), and consider a 4-dimensional subspace \(U\) of \(V^{(2k)}\). Then at least one of \(U \cap V_1^{(2k)}\) or \(U \cap V_2^{(2k)}\) is not empty.

**Proof.** Assume that \(U \cap V_1^{(2k)} = U \cap V_2^{(2k)} = \emptyset\). Then \(U \subseteq (V_0^{(2k)} \cup V_3^{(2k)})\). If \(k = 2\) or \(k = 6\), this leads at once to a contradiction, because then \(#V_{3}^{(2k)} = 15 = #U - 1\), and \(V_3^{(2k)} \subseteq V_0^{(2k)}\) is not a subspace in these cases (for example, \((0, 0, 1, 0, 0, 1, 0) + (0, 0, 1, 0, 0, 1, 0) = (0, 0, 0, 1, 1, 1, 0) \in V_2^{(2k)}\) for \(k = 2\) and \(k = 6\)).

Consider the case \(k = 4\); since \(U \cap V_3^{(2k)}\) contains 15 elements, there exists \(u \in U \cap V_3^{(2k)} \setminus \{(0, 0, 0, 1, 1, 1)\}\). Permuting the indices if necessary (of course we are not allowed to exchange one of the first 2k indices with one from the last \(2k\)) we may assume that \(u = (1, 1, 0, 0, 0, 1, 1)\). It is easy to see that \(V_3^{(2k)}\) contains exactly six elements \(v\) such that \(u + v \in V_2^{(2k)}\); since \(#V_{3}^{(4)} = 19\) and \(#U \cap V_3^{(2k)} \setminus \{u, (0, 0, 0, 1, 1, 1)\} \geq 14\), one of these \(v\) must be contained in \(U \cap V_3^{(2k)}\). This shows that \(V_3^{(2k)} \cup V_0^{(2k)}\) does not contain a subspace of dimension 4, and the proof is complete. \(\square\)

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