OPENING NODES ON HOROSPHERE PACKINGS

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Abstract. We use Bryant representation to construct constant mean curvature-1 surfaces in hyperbolic space by desingularisation of a horosphere packing.

1. Introduction

In this paper, a horosphere packing means a finite, connected set of distinct horospheres in hyperbolic space $H^3$ such that any two horospheres are either disjoint or tangent. We denote by $n$ the number of horospheres and $m$ the number of tangency points.

Horospheres have constant mean curvature equal to 1 (CMC-1 for short). In [10], Pacard and Pimentel have constructed complete, embedded CMC-1 surfaces in hyperbolic space by desingularization of a horosphere packing. Heuristically, the horospheres are slightly shrunk and a suitable small catenoidal neck is glued at each tangency point (see Figure 1).

On the other hand, CMC-1 surfaces in hyperbolic space admit a Weierstrass-type representation in term of meromorphic data, as discovered by Bryant [1]. In particular, they have a meromorphic Gauss map $G$ and a holomorphic quadratic differential $Q$. Our goal in this paper is to revisit the result of Pacard and Pimentel using Bryant’s representation. We prove:

Theorem 1. Given a packing of $n$ horospheres with $m$ tangency points, there exists a smooth family $(M_s)_{0<s<\varepsilon}$ of complete, embedded CMC-1 surfaces in hyperbolic space $H^3$ such that $M_s$ converges when $s \to 0$ to the given horosphere packing. The surfaces $M_s$ have genus $m - n + 1$ and $n$ catenoid-cousin-type ends.

We prove this theorem using Bryant’s representation. The construction follows the general strategy developed by the author in [16–18] to construct minimal surfaces in euclidean space $\mathbb{R}^3$ by opening nodes, using the Weierstrass representation. The strategy can be described as follows: The underlying Riemann surface $\Sigma$ is defined by opening nodes. The meromorphic differentials $dG$ and $\Omega = Q/dG$ are defined on $\Sigma$ by prescribing their periods. Bryant representation gives us a “recipe” to recover the immersion from this meromorphic data. With respect to the minimal case, the new technicality is that instead of having to solve a period problem, which is homology invariant, we have to solve a monodromy problem, which is only homotopy invariant.

As previously stated, Theorem 1 is already proved in [10] using the P.D.E. gluing machinery, which has been recognized as a very robust and versatile tool to

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construct examples in the theory of surfaces. So let us explain the motivation for giving another proof of this result.

Firstly, Bryant representation has been used by many authors to construct CMC-1 surfaces in hyperbolic space: let us mention Benoît Daniel [3–5], Ricardo Sa Earp and Eric Toubiana [13], Wayne Rossman, Masaaki Umehara and Kotaro Yamada [12], Masaaki Umehara and Kotaro Yamada [20–22]. However, only low genus or highly symmetric examples have been constructed. Moreover, most of these examples are close cousins to known minimal surfaces in euclidean space. It seemed interesting to see if high genus examples of a truly hyperbolic nature, such as the ones proposed in Theorem 1, can be constructed using Bryant representation.

Secondly, the method of opening nodes is in many respects more elementary and economical than the P.D.E. gluing machinery. So far, it has only been applied to minimal surfaces in euclidean space. There are, however, many other problems in surface theory for which a “holomorphic recipe” such as the Weierstrass representation is available. For example, CMC surfaces in space forms can be constructed from holomorphic data using the Dorfmeister, Pedit, Wu (DPW) loop group method [6], owing to the fact that the Gauss of a CMC surface is harmonic. More generally, the DPW recipe applies whenever the surface can be represented in terms of a harmonic map into a symmetric space. It seems interesting to see if opening nodes and the DPW method can be combined to work together. The work in this paper is a first step in this direction. Indeed, Bryant representation and DPW have in common that a matrix-valued O.D.E. must be solved, which leads to a monodromy problem.

Returning to Theorem 1, there are free parameters in the construction; namely, $n$ complex parameters $c_1, \ldots, c_n$, which are the limit points of the horospheres on the ideal boundary of $\mathbb{H}^3$ (identified with the Riemann sphere), and $n$ positive real parameters $\xi_1, \ldots, \xi_n$, which represent the speed at which each horosphere is “deflated” to accommodate the catenoidal necks. We can normalize $\xi_1 = 1$, so together with the parameter $s$ our family depends on $3n$ real parameters, which is the expected dimension for the space of CMC-1 surfaces with $n$ ends.

One may ask what are the possible topologies that one can achieve with this construction. In other words, given the number of ends $n$, what are the possible genera $g = m - n + 1$? If $n = 2$, then clearly $g = 0$. For $n \geq 3$, it is easy to construct...
horosphere packings with up to $3n - 6$ tangency points using the Appolonius 4 sphere theorem (see [10]). These are called appolonian packings.

Here is how to do better: the starting position for the american pool is made up of 15 unit spheres (arranged in a triangle) with 30 tangency points. Stack on top of that a horizontal plane, and you get a packing of 16 horospheres with 45 tangency points in the half-space model. This is better than an appolonian packing which has only 42 tangency points for the same number of horospheres. Following this idea of stacking a horizontal plane on top of a dense packing of unit spheres, it is easy to exhibit horosphere packings whose number of tangency points grows like $4n$ (compared to $3n$ for the appolonian packing).

The paper is organized as follows. In Section 2 we present Bryant representation of CMC-1 surfaces in hyperbolic space and recall standard material about monodromy and opening nodes. In Section 3 we construct a family of candidates for the holomorphic data of the CMC-1 immersions we want to construct. The monodromy problem is solved in Section 4 using the implicit function theorem. In Section 5 we study the geometry of the surfaces we have constructed and prove that they are embedded. Finally, an appendix contains several results of independent interest that we use in this construction.

2. Background material

2.1. Bryant representation.

2.1.1. The Minkowski model of $\mathbb{H}^3$. Let $L^4$ denote the 4-dimensional lorentzian space, namely $\mathbb{R}^4$, with coordinates denoted $x_0, x_1, x_2, x_3$ and metric $-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2$. The hyperboloid

$$\{x \in L^4 : \langle x, x \rangle = -1, \ x_0 > 0\}$$

with the induced metric is the Minkowski model of $\mathbb{H}^3$. Bryant identifies $L^4$ with the space of $2 \times 2$ hermitian matrices by identifying $(x_0, x_1, x_2, x_3)$ with the matrix

$$X = \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix}.$$ 

Then $\mathbb{H}^3$ becomes the set of positive definite hermitian matrices with determinant 1.

2.1.2. Bryant representation. Recall that a holomorphic map $F : \Sigma \to SL(2, \mathbb{C})$ is null if $\det(F^{-1}dF) = 0$.

Theorem 2 (Bryant [1]). Let $\Sigma$ be a simply connected Riemann surface. Let $F : \Sigma \to SL(2, \mathbb{C})$ be a holomorphic null immersion. Then $FF^* : \Sigma \to \mathbb{H}^3$ is a smooth conformal CMC-1 immersion. Conversely, if $f : \Sigma \to \mathbb{H}^3$ is a conformal CMC-1 immersion, there exists a holomorphic null immersion $F : \Sigma \to SL(2, \mathbb{C})$ such that $f = FF^*$. Moreover, $F$ is unique up to right multiplication by a constant matrix $H \in SU(2)$.

If $\Sigma$ is not simply connected, then $F$ is only well defined on the universal cover of $\Sigma$ and will have $SU(2)$-valued monodromy.
2.1.3. **Global meromorphic data.** Assume we are given a CMC-1 immersion \( f : \Sigma \to \mathbb{H}^3 \). By Theorem 2, we can write locally \( f = F F^* \), where \( F \) is a null holomorphic, \( SL(2, \mathbb{C}) \)-valued map. Consider the matrix of holomorphic 1-forms

\[
A(z) = (dF(z))F(z)^{-1} \in \mathfrak{sl}(2, \mathbb{C}),
\]

where \( \mathfrak{sl}(2, \mathbb{C}) \) is the Lie algebra of \( 2 \times 2 \) matrices whose trace is zero. Then \( A \) is well defined on \( \Sigma \): replacing \( F \) by \( F H \) does not change \( A \). Define \( G = A_{11} A_{21} \) and \( \Omega = A_{21} \), where \( A_{ij} \) denote the coefficients of \( A \). Then \( G \) is a meromorphic function and \( \Omega \) is a holomorphic 1-form, both globally defined on \( \Sigma \) (except in the exceptional case where \( A_{21} \equiv 0 \)). Since the trace and determinant of \( A \) are zero, we have

\[
A = \begin{pmatrix} G & -G^2 \\ 1 & -G \end{pmatrix} \Omega.
\]

The function \( G \) is the Gauss map introduced by Bryant, and \( Q = \Omega dG \) is the Hopf quadratic differential. We will see the geometric meaning of the Gauss map \( G \) in the next section. We call \((G, \Omega)\) the meromorphic data for the immersion \( f \).

Conversely, here is a recipe to construct CMC-1 immersions in \( \mathbb{H}^3 \). Start with a Riemann surface \( \Sigma \), a meromorphic function \( G \) and a holomorphic 1-form \( \Omega \) on \( \Sigma \), such that \( \Omega \) has a zero at each pole of \( G \) with twice the multiplicity and has no other zeros. Define the matrix \( A \) by (1). Then \( A \) is a holomorphic, \( \mathfrak{sl}(2, \mathbb{C}) \)-valued 1-form on \( \Sigma \) which does not vanish. Solve the linear differential system

\[
dF(z) = A(z)F(z)
\]

with initial data \( F(z_0) = F_0 \in SL(2, \mathbb{C}) \). The solution is a multi-valued holomorphic null immersion \( F : \Sigma \to SL(2, \mathbb{C}) \). (It is an immersion because \( A(z) \neq 0 \).) If its monodromy happens to be in \( SU(2) \), then the immersion \( f = FF^* : \Sigma \to \mathbb{H}^3 \) is well defined and has CMC-1.

**Remark 1.** Many authors instead consider the matrix \( F^{-1} dF \) and write

\[
F^{-1} dF = \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} \omega,
\]

where \( g \) is a meromorphic function and \( \omega \) is a holomorphic 1-form. It turns out that \( (g, \omega) \) is the Weierstrass data for the corresponding minimal surface in \( \mathbb{R}^3 \) via the Lawson correspondence [8], which is why (3) has become so popular. The problem is that unless \( \Sigma \) is simply connected, \( F^{-1} dF \) is not globally defined on \( \Sigma \). This is not surprising, since the Lawson correspondence is local. So this point of view is not appropriate if we are to construct high genus examples.

2.1.4. **Bryant representation in the half-space model.** A more familiar model of \( \mathbb{H}^3 \) is the half-space \( x_3 > 0 \) in \( \mathbb{R}^3 \), with conformal metric \( x_1^{-2}(dx_1^2 + dx_2^2 + dx_3^2) \). The following results are proved in [13]. An orientation preserving isometry from the Minkowski model to the half-space model is given by

\[
\Phi(x_0, x_1, x_2, x_3) = \left( \frac{x_1}{x_0 - x_3}, \frac{x_2}{x_0 - x_3}, \frac{1}{x_0 - x_3} \right).
\]

The immersion \( \Phi \circ f \) is given in the half-space model by

\[
x_1 + i x_2 = \frac{F_{11} F_{21} + F_{12} F_{22}}{|F_{21}|^2 + |F_{22}|^2}, \quad x_3 = \frac{1}{|F_{21}|^2 + |F_{22}|^2},
\]
where \( F_{ij} \) denote the coefficients of the matrix \( F \). The ideal boundary of \( \mathbb{H}^3 \) in this model is \( C \cup \{ \infty \} \). The Gauss map \( G \) has the following geometric interpretation: the normal geodesic ray originated from \( \Phi \circ f(z) \) (in the direction of the mean curvature vector) hits the ideal boundary at the point \( G(z) \).

2.1.5. **Isometries.** The Lie group \( SL(2, \mathbb{C}) \) acts isometrically on \( L^4 \) by the representation
\[
H \cdot X = HXH^*,
\]
where \( H \in SL(2, \mathbb{C}) \) and \( X \in L^4 \). The action preserves \( \mathbb{H}^3 \) and its kernel is \( \{ \pm I_2 \} \), so the group of direct isometries of \( \mathbb{H}^3 \) is \( PSL(2, \mathbb{C}) \). The action of \( SL(2, \mathbb{C}) \) on \( \mathbb{H}^3 \) extends to the ideal boundary as homographic transformation of the Riemann sphere, namely
\[
H \cdot z = \frac{H_{11}z + H_{12}}{H_{21}z + H_{22}}.
\]
If \( f : \Sigma \to \mathbb{H}^3 \) is a conformal CMC-1 immersion with Gauss map \( G \) and null holomorphic map \( F \), then \( H \cdot f \) has Gauss map \( H \cdot G \) and null holomorphic map \( HF \).

2.1.6. **Horospheres.** The Gauss map is constant on a horosphere, and that constant is the limit point of the horosphere on the ideal boundary of \( \mathbb{H}^3 \) in the half-space model. (This follows from the geometric interpretation of the Gauss map in the half-space model.) If the horosphere is not a horizontal plane, then its meromorphic data is \( \Sigma = \mathbb{C}, G = c, \Omega = \lambda dz \), where \( c \) and \( \lambda \) are complex constants. The constant \( \lambda \) has no geometrical meaning and depends on the chosen conformal parametrization of the horosphere. The matrix \( A \) is given by
\[
A = \lambda \begin{pmatrix} c & -c^2 \\ 1 & -c \end{pmatrix}.
\]
If the horosphere is a horizontal plane, then \( G \equiv \infty \) and we are in the exceptional case where the meromorphic data \((G, \Omega)\) is not defined. In this case, one has
\[
A = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}
\]
for some complex number \( \lambda \). In any case, since the matrix \( A \) is constant, the solution to (2) is \( F(z) = \exp(zA)F_0 \).

2.2. **Linear differential systems.** In this section, we recall standard facts about linear differential systems on a Riemann surface and set up some notation. A basic reference is [15]. Let \( \Sigma \) be a Riemann surface and \( A \) an \( n \times n \) matrix of holomorphic 1-forms on \( \Sigma \). We consider the first order linear differential system on \( \Sigma \),
\[
dY(z) = A(z)Y(z).
\]
2.2.1. **Local theory: The principal solution.** Assume that \( \Sigma \) is simply connected. Given \( z_0 \in \Sigma \), [3] has a unique solution \( Y : \Sigma \to GL(n, \mathbb{C}) \) such that \( Y(z_0) = I_n \). Following [15], we write \( Y(z) = \Pi(z, z_0) \). The map \( \Pi : \Sigma \times \Sigma \to GL(n, \mathbb{C}) \) is holomorphic in both variables and is called the principal solution. It satisfies
\[
\Pi(z_3, z_2)\Pi(z_2, z_1) = \Pi(z_3, z_1).
\]
Given \( Y_0 \in GL(n, \mathbb{C}) \), the solution \( Y \) such that \( Y(z_0) = Y_0 \) is given by \( Y(z) = \Pi(z, z_0)Y_0 \).
2.2.2. Global theory. Now assume that $\Sigma$ is not simply connected. Then the principal solution $\Pi(z, z_0)$ is not well defined; it depends on the homotopy class of the path from $z_0$ to $z$. If $\gamma : [0, 1] \to \Sigma$ is a path from $z_0$ to $z$, the solution $Y$ of \([5]\) such that $Y(z_0) = I_n$, which exists in a simply connected neighborhood of $\gamma(0)$, can be analytically continued along $\gamma$. Its value at $\gamma(1)$ will be denoted $\Pi(\gamma)$. When the path $\gamma$ is clear from the context, we will still use the notation $\Pi(z, z_0)$.

Let $\gamma_1, \gamma_2 : [0, 1] \to \Sigma$ be two paths such that $\gamma_1(1) = \gamma_2(0)$. We denote by $\gamma_2 \cdot \gamma_1$ the path obtained by composing $\gamma_1$ and $\gamma_2$. (The usual notation is $\gamma_1 \cdot \gamma_2$, but in this context it is more convenient, and customary, to reverse the order.) Then

$$\Pi(\gamma_2 \cdot \gamma_1) = \Pi(\gamma_2)\Pi(\gamma_1).$$

In particular, $\Pi$ is a morphism from the fundamental group $\pi_1(\Sigma, z_0)$ to $GL(n, \mathbb{C})$. (With the usual notation for the product in the fundamental group, it would be an anti-morphism.)

2.2.3. Monodromy. The monodromy of a solution is usually defined as follows. Let $Y$ be a solution of \([5]\) in a simply connected neighborhood $U$ of $z_0$. Let $\gamma \in \pi_1(\Sigma, z_0)$. Analytic continuation of $Y$ along $\gamma$ gives another solution of \([5]\) in $U$, which is denoted $\gamma \cdot Y$. There exists a matrix $M \in GL(n, \mathbb{C})$ such that $\gamma \cdot Y = YM$. The matrix $M$ is called the monodromy of $Y$ along $\gamma$ and is denoted $M_\gamma(Y)$. In terms of the principal solution, one has

$$M_\gamma(Y) = Y(z_0)^{-1}\Pi(\gamma)Y(z_0).$$

2.3. Opening nodes. We recall the standard construction of opening nodes. Consider $n$ copies of the Riemann sphere $\mathbb{C} = \mathbb{C} \cup \{\infty\}$, labelled $\mathbb{C}_1, \ldots, \mathbb{C}_n$. Consider $2m$ distinct points $p_1, \ldots, p_m, q_1, \ldots, q_m$ in the disjoint union $\mathbb{C}_1 \cup \cdots \cup \mathbb{C}_n$. Identify $p_i$ with $q_i$ for $1 \leq i \leq m$. This defines a Riemann surface with nodes which we denote $\Sigma_0$. We assume $\Sigma_0$ is connected.

To open nodes, consider local complex coordinates $v_i : V_i \to D(0, 1)$ in a neighborhood of $p_i$ and $w_i : W_i \to D(0, 1)$ in a neighborhood of $q_i$, with $v_i(p_i) = 0$ and $w_i(q_i) = 0$. We assume that the neighborhoods $V_1, \ldots, V_m, W_1, \ldots, W_m$ are disjoint in $\mathbb{C}_1 \cup \cdots \cup \mathbb{C}_n$. Consider, for each $1 \leq i \leq m$, a complex parameter $t_i$ with $|t_i| < 1$. If $t_i = 0$, identify $p_i$ with $q_i$ as above. If $t_i \neq 0$, remove the disks $|v_i| \leq |t_i|$ and $|w_i| \leq |t_i|$. Identify the point $z \in V_i$ with the point $z' \in W_i$ such that

$$v_i(z)w_i(z') = t_i.$$

This creates a Riemann surface, possibly with nodes, which we denote $\Sigma_t$, where $t = (t_1, \ldots, t_n)$. When all $t_i$ are non-zero, $\Sigma_t$ is a genuine Riemann surface of genus $g = m - n + 1$.

Observe that if $t_i \neq 0$, the circle $|v_i| = 1$ is homologous in $\Sigma$ to the circle $|w_i| = 1$ with the opposite orientation. Consequently, if $\omega$ is a holomorphic 1-form on $\Sigma_t$,

$$\int_{|v_i|=1} \omega = -\int_{|w_i|=1} \omega.$$

This makes the following definition natural.

**Definition 1** (Bers). A regular differential on $\Sigma_t$ is a holomorphic 1-form which is allowed to have simples poles at $p_i$ and $q_i$ if $t_i = 0$, with opposite residues.
By a theorem of Fay [7], the space of regular differentials on $\Sigma_t$ has dimension $g$ and admits a basis which depends holomorphically on $t$ in a neighborhood of 0. For $1 \leq i \leq n$, let $J_i^+$ and $J_i^-$ be the set of indices $j$ such that $p_j \in \mathbb{C}_i$ and $q_j \in \mathbb{C}_i$, respectively. As a consequence of Fay’s theorem, one has:

**Theorem 3.** For $t$ in a neighborhood of 0 and for $a = (a_j)_{1 \leq j \leq m} \in \mathbb{C}^m$ satisfying

$$\sum_{j \in J_i^+} a_j - \sum_{j \in J_i^-} a_j = 0 \quad \text{for } 1 \leq i \leq n,$$

there exists a unique regular differential $\omega = \omega_{t,a}$ on $\Sigma_t$ such that

$$\int_{|v_j|=1} \omega = a_j \quad \text{for } 1 \leq j \leq m.$$

Moreover, $\omega_{t,a}$ depends holomorphically on $t$ (away from the nodes).

**Proof.** From the Cauchy theorem in $\mathbb{C}_i$ and (8), we see that (9) is necessary for $\omega$ to exist. If $t = 0$, the map $\omega \mapsto (\int_{|v_j|=1} \omega)_{1 \leq j \leq m}$ is an isomorphism from the space of regular differentials on $\Sigma_0$ to the space of vectors $a \in \mathbb{C}^m$ satisfying (9). (This follows from the fact that a holomorphic 1-form on the Riemann sphere with simple poles is entirely determined by its residues.) Using Fay’s theorem, this remains true for $t$ in a neighborhood of 0. \hfill \Box

Fay’s proof is rather abstract and non-constructive. For an elementary proof of Theorem 3 based on the contraction mapping principle, see [19]. One has a similar result for meromorphic 1-forms with poles at some points $r_1, \cdots, r_k$ distinct from the nodes:

**Theorem 4.** For $t$ in a neighborhood of 0, one can define a regular meromorphic differential $\omega$ on $\Sigma_t$ with poles at the points $r_1, \cdots, r_k$ by prescribing its principal part at each pole and its periods as in (10), replacing (9) by the restriction coming from the residue theorem, namely

$$\sum_{j \in J_i^+} a_j - \sum_{j \in J_i^-} a_j + 2\pi i \sum_{r_j \in \mathbb{C}_i} \operatorname{Res}_{r_j} \omega = 0 \quad \text{for } 1 \leq i \leq n.$$

Recall that the principal part of a meromorphic 1-form $\omega$ at a pole $r$ is its equivalence class under the relation: $\omega \sim \omega'$ if $\omega - \omega'$ is holomorphic in a neighborhood of $r$. The analogue of Fay’s theorem for meromorphic differentials with simple poles is proved by Masur in [9]. For a proof of Theorem 4 in the case of poles of arbitrary order, see [19]. We will also need the following result to compute the partial derivatives of $\omega$ with respect to $t$.

**Theorem 5.** The partial derivative $\frac{\partial}{\partial t_i} \omega_{t,a}$ at $t = 0$ has two double poles at $p_i$ and $q_i$, with principal parts

$$-\frac{dv_i}{v_i^2} \operatorname{Res}_{q_i} \frac{\omega_{0,a}}{w_i} \quad \text{at } p_i,$$

$$-\frac{dw_i}{w_i^2} \operatorname{Res}_{p_i} \frac{\omega_{0,a}}{v_i} \quad \text{at } q_i,$$

and has vanishing periods on all circles $|v_j| = 1$.

This is proved in [18], Lemma 3. See also [19], Remark 5.6.
3. The meromorphic data $(\Sigma, G, \Omega)$

3.1. Notation. The horospheres of our given horosphere packing are denoted $S_1, \cdots, S_n$. We define the following sets, which we use to index various quantities:

$$I = \{(i, j) \in \mathbb{Z}^2 : i < j \text{ and } S_i, S_j \text{ are tangent}\},$$

$$J_i^+ = \{j : (i, j) \in I\} \quad J_i^- = \{j : (j, i) \in I\} \quad J_i = J_i^+ \cup J_i^-.$$  

Without loss of generality, we may assume (by applying an isometry) that each horosphere $S_i$ is not a horizontal plane in the half-space model. We fix a conformal parametrisation $f_i : \mathbb{C} \to S_i$ and let

$$G = c_i \quad \Omega = \lambda_i dz$$

be its meromorphic data (see Section 2.1.6). For $j \in J_i$, we denote by $p_{ij}$ the point in $\mathbb{C}$ such that $f_i(p_{ij})$ is the point $S_i \cap S_j$. Without loss of generality, we may assume (by changing the parametrisation $f_i$) that for each $i$, the disks $D(p_{ij}, 1)$ for $j \in J_i$ and $D(0, 1)$ are pairwise disjoint.

3.2. The Riemann surface $\Sigma$. We consider $n$ copies of the complex plane, denoted $\mathbb{C}_1, \cdots, \mathbb{C}_n$, and $m$ copies of the Riemann sphere $\mathbb{C} \cup \{\infty\}$, denoted $\mathbb{C}^i_j$ for $(i, j) \in I$. We think of $p_{ij}$ as a point in $\mathbb{C}_i$. The points 0, 1 and $\infty$ in $\mathbb{C}^i_j$ will be denoted respectively $0_{ij}$, $1_{ij}$ and $\infty_{ij}$. Heuristically, the reader should think of $\mathbb{C}_1, \cdots, \mathbb{C}_n$ as the parametrization domain for the horospheres $S_1, \cdots, S_n$, and $\mathbb{C}^i_j \setminus \{0, \infty\}$ as a parameter domain for the catenoidal necks connecting them.

We define a Riemann surface $\Sigma_0$ with $2m$ nodes by identifying $p_{ij}$ with $0_{ij}$ and $p_{ji}$ with $\infty_{ij}$, for $(i, j) \in I$, and call it $\Sigma_0$. To open nodes, we consider the natural coordinates $v_{ij} = z - p_{ij}$ in a neighborhood of $p_{ij}$ in $\mathbb{C}_i$, $w_{ij} = z$ in a neighborhood of $0_{ij}$ in $\mathbb{C}^i_j$, $v_{ji} = z - p_{ji}$ in a neighborhood of $p_{ji}$ in $\mathbb{C}_j$ and $w_{ji} = \frac{1}{z}$ in a neighborhood of $\infty_{ij}$ in $\mathbb{C}^j_i$. We open nodes as explained in Section 2.3 introducing a complex parameter $t_{ij}$ to open the node $p_{ij} \sim 0_{ij}$ and another parameter $t_{ji}$ to open the node $p_{ji} \sim \infty_{ij}$. Let $t = (t_{ij}, t_{ji}) (i, j) \in I$ be the collection of these parameters. This defines a Riemann surface (possibly with nodes) which we denote $\Sigma$, or $\Sigma_t$ when we need to emphasize the dependence on the parameter $t$. We denote by $\Sigma$ the compactification of $\Sigma$ obtained by adding the points $\infty_1, \cdots, \infty_n$, where $\infty_i$ denotes the point at infinity in $\mathbb{C}_i$.

For $(i, j) \in I$, we define $\alpha_{ij}$ as the homology class of the circle $|z - p_{ij}| = 1$ in $\mathbb{C}_i$, with the positive orientation. This is homologous, in $\Sigma$, to the unit circle in $\mathbb{C}^i_j$, with the negative orientation, and to the circle $|z - p_{ji}| = 1$ in $\mathbb{C}_j$, also with the negative orientation.

3.3. The Gauss map $G$. Here are our requirements on the Gauss map $G$. At $\infty_i$, it should take the value $c_i$. It should have a simple pole in each Riemann sphere $\mathbb{C}^i_j$ (because on each catenoidal neck, we expect a point where the mean curvature vector is vertical pointing up, in the half-space model). We choose the identification of the Riemann sphere with $\mathbb{C} \cup \{\infty\}$ so that this pole is $z = 1$. The following proposition tells us that these requirements completely determine $G$. 

Proposition 1. For \( t \) small enough, there exists a unique meromorphic function \( G = G_t \) on \( \Sigma_t \) with the following properties:

- \( G_t \) has \( m \) simple poles at the points \( 1_{ij} \) for \( (i, j) \in I \),
- \( G_t(\infty_i) = c_i \) for \( i = 1, \cdots, n \).

Moreover, \( G_t \) depends holomorphically on \( t \) (away from the nodes and its poles), and at \( t = 0 \), we have

\[
G_0(z) = \begin{cases} 
  c_i & \text{in } \mathbb{C}_i, \\
  c_j + \frac{c_j - c_i}{z - 1} & \text{in } \mathbb{C}_{ij}.
\end{cases}
\]

Proof. We first define the differential \( \mu = dG \) and we recover \( G \) by integration. By Theorem 4, there exists a unique meromorphic differential \( \mu_t \) on \( \Sigma_t \) which has \( m \) double poles at \( 1_{ij} \) for \( (i, j) \in I \) with principal part

\[
r_{ij} \frac{dz}{(z - 1)^2}
\]

and has vanishing period on all cycles \( \alpha_{ij} \). Here the \( m \) complex numbers \( r_{ij} \) are free parameters. At \( t = 0 \), we have

\[
\mu_0 = \begin{cases} 
  0 & \text{in } \mathbb{C}_i, \\
  r_{ij} \frac{dz}{(z - 1)^2} & \text{in } \mathbb{C}_{ij}.
\end{cases}
\]

Lemma 1. For \( t \) in a neighborhood of 0, there exist unique values of the parameters \( r_{ij} \) such that

\[
\int_{\infty_i}^{\infty_j} \mu_t = c_j - c_i \quad \text{for } (i, j) \in I.
\]

Moreover, each \( r_{ij} \) depends holomorphically on \( t \), and when \( t = 0 \), \( r_{ij} = c_i - c_j \).

Proof. First observe that (13) is a linear system of \( m \) linear equations with \( m \) unknowns \( r_{ij}, (i, j) \in I \). Hence it suffices to prove that the system (13) is invertible when \( t = 0 \). In (13), it is understood that the path from \( \infty_i \) to \( \infty_j \) goes through \( \mathbb{C}_{ij} \). There is no canonical way to choose this path, but all choices are homologous modulo \( \alpha_{ij} \). Since \( \mu \) has no period on \( \alpha_{ij} \), \( \int_{\infty_i}^{\infty_j} \mu_t \) is a well-defined holomorphic function of \( t \). Moreover, by Lemma 4 in [18], this function extends holomorphically at \( t = 0 \) with value

\[
\int_{\infty_i}^{\infty_j} \mu_0 = \int_{\infty_i}^{p_{ij}} 0 + \int_{0_{ij}}^{\infty_j} r_{ij} \frac{dz}{(z - 1)^2} + \int_{p_{ji}}^{\infty_j} 0 = -r_{ij}.
\]

The lemma follows. \( \square \)

Returning to the proof of Proposition 1, we define the function \( G_t \) on \( \Sigma_t \) by

\[
G_t(z) = c_1 + \int_{\infty_1}^{z} \mu_t.
\]

By Lemma 1 and the fact that \( \mu_t \) has no residues and no period on the cycles \( \alpha_{ij} \), \( G_t \) is well defined on \( \Sigma_t \) (meaning that the integral does not depend on the path from \( \infty_1 \) to \( z \)) and has all the desired properties. \( \square \)
3.4. The holomorphic differential $\Omega$. Here are our requirements on the holomorphic differential $\Omega$. It should have a double pole at $\infty$, with leading term $\lambda_i dz$, just like the horosphere $S_i$. It also needs a double zero at each pole of $G$. We define $\Omega$ by prescribing poles, principal parts and periods, using Theorem 4. Then we adjust the parameter $t$ so that $\Omega$ has the required zeros.

**Definition 2.** Consider $m$ complex parameters $a_{ij}$, $(i, j) \in I$, and let $a = (a_{ij})_{(i, j) \in I}$. We define $\Omega = \Omega_{t, a}$ as the unique meromorphic 1-form on $\Sigma_t$ with $n$ double poles at $\infty_1, \ldots, \infty_n$, with principal part

$$\lambda_i dz + \sum_{j \in J_i^+} a_{ij} \frac{dz}{z} - \sum_{j \in J_i^-} a_{ji} \frac{dz}{z} \quad \text{at } \infty_i$$

and periods

$$\int_{\alpha_{ij}} \Omega = 2\pi i a_{ij} \quad \text{for } (i, j) \in I.$$

It depends holomorphically (away from its poles and the nodes) on $t$. Moreover, at $t = 0$ we have

$$\Omega_{0, a} = \begin{cases} 
\lambda_i dz + \sum_{j \in J_i^+} a_{ij} \frac{dz}{z} - \sum_{j \in J_i^-} a_{ji} \frac{dz}{z} \quad \text{in } \mathbb{C}_i, \\
- a_{ij} \frac{dz}{z} \quad \text{in } \mathbb{C}_{ij}.
\end{cases}$$

Note that the residue of the prescribed principal part at $\infty_i$ is

$$\text{Res}_{\infty_i} \Omega = - \sum_{j \in J_i^+} a_{ij} + \sum_{j \in J_i^-} a_{ji},$$

so (11) holds.

**Proposition 2.** For $a$ in a neighborhood of 0, there exists a unique value $t(a)$, depending holomorphically on $a$, such that $\Omega_{t(a), a}$ has a double zero at each pole of $G$, and has no other zeros in $\Sigma$ (provided all parameters $a_{ij}$ are non-zero). Moreover, for each $(i, j) \in I$, we have

$$a_{ij} = 0 \implies t_{ij}(a) = t_{ji}(a) = 0,$$

$$\frac{\partial t_{ij}(a)}{\partial a_{ij}}|_{a=0} = -\frac{1}{2\lambda_i},$$

$$\frac{\partial t_{ji}(a)}{\partial a_{ij}}|_{a=0} = \frac{1}{2\lambda_j}.$$
The partial derivatives of $\Omega_{t, a}$ in $\mathbb{C}_{ij}$ with respect to all other parameters $t_{k\ell}$ and $a_{k\ell}$ are zero. Write $\Omega_{t, a} = f_{ij}(t, a, z)dz$ in $\mathbb{C}_{ij}$. We want to solve $f_{ij}(t, a, 1) = f'_{ij}(t, a, 1) = 0$. The Jacobian matrix of $(f_{ij}(t, a, 1), f'_{ij}(t, a, 1))$ with respect to $(t_{ij}, t_{ji})$ is
\[
\begin{pmatrix}
-\lambda_i & \lambda_j \\
2\lambda_i & 0
\end{pmatrix}.
\]
The existence of the solution $(t(a))$ then follows from the implicit function theorem applied to the map $(f_{ij}(t, a, 1), f'_{ij}(t, a, 1))(i,j)\in I$ whose Jacobian has block diagonal form. Next consider some $(i, j) \in I$ and assume that $a_{ij} = 0$. If $t_{ij} = t_{ji} = 0$, then $\Omega_{t, a}$ has two simple poles at $0_{ij}$ and $\infty_{ij}$, with residue $\pm a_{ij} = 0$, and hence is holomorphic in $\mathbb{C}_{ij}$, so $\Omega_{t, a} \equiv 0$ in $\mathbb{C}_{ij}$. So (15) follows from uniqueness in the implicit function theorem. Equations (16) and (17) are obtained by differentiating $f_{ij}(t(a), a, 1) = f'_{ij}(t(a), a) = 0$ with respect to $a_{ij}$, using (18), (19) and (20). Finally, if all parameters $a_{ij}$ are non-zero, then (16) and (17) imply that all parameters $t_{ij}$ and $t_{ji}$ are non-zero, so $\Sigma$ is a genuine compact Riemann surface of genus $g = m - n + 1$. $\Omega$ is a meromorphic 1-form with $m$ poles at $0_{ij}$ in $\mathbb{C}_{ij}$, whose Jacobian has block diagonal form. Integrating, we obtain the first line of (21). The proof of the second line is entirely similar. Regarding (22), we have (again using Theorem 5)
\[
\frac{\partial \mu}{\partial t_{ij}} = -\frac{dz}{(z - p_{ij})^2} \text{Res}_{0_{ij}, \mu_0} = -(c_i - c_j) \frac{dz}{(z - p_{ij})^2} \quad \text{in } \mathbb{C}_i.
\]
Hence, by the chain rule and (16),
\[
\frac{\partial \mu_{t(a)}}{\partial a_{ij}} = \frac{c_i - c_j}{2\lambda_i} \frac{dz}{(z - p_{ij})^2} \quad \text{in } \mathbb{C}_i.
\]
Integrating, we obtain the first line of (22). The proof of the second line is entirely similar. Regarding (22), we have (again using Theorem 5)
\[
\frac{\partial \Omega_{t, a}}{\partial t_{ij}} = 0 \quad \frac{\partial \Omega_{t, a}}{\partial a_{ij}} = \frac{dz}{z - p_{ij}} \quad \text{in } \mathbb{C}_i.
The first line of (22) follows from the chain rule. The proof of the second line is similar. Using (18), (19), (20) and the chain rule, we have

\[
\frac{\partial \Omega_{t(a),a}}{\partial a_{ij}} = -\lambda_i \frac{dz}{z^2} \times \left( -\frac{1}{2\lambda_i} \right) + \lambda_j \frac{dz}{z} \times \left( \frac{1}{2\lambda_j} \right) - \frac{dz}{z} = \frac{(1 - z)^2}{2z^2} \frac{dz}{z} \quad \text{in } \mathbb{C}_{ij}.
\]

4. The monodromy problem

4.1. Formulation of the problem. We consider the matrix \( A = A_a \) defined by (1) with \( G = G_{t(a)} \) and \( \Omega = \Omega_{t(a),a} \). Each coefficient of \( A \) is a holomorphic differential on \( \Sigma = \Sigma_{t(a)} \). Let \( 0_i \) be the point \( z = 0 \) in \( \mathbb{C}_i \). Let \( F : \Sigma \to SL(2, \mathbb{C}) \) be the solution of \( dF = AF \) with initial condition \( F(0_i) = M_1 \), where \( M_1 \in SL(2, \mathbb{C}) \) is a matrix we can prescribe. (Observe that \( F(z) \in SL(2, \mathbb{C}) \) because \( A(z) \in \mathfrak{sl}(2, \mathbb{C}) \).) The solution \( F \) is of course only well defined on the universal cover of \( \Sigma \). We need to adjust the parameters so that \( F \) has \( SU(2) \)-valued monodromy, so \( f = FF^* \) is well defined on \( \Sigma \). Taking \( 0_1 \) as a base point for the fundamental group and using (7), this is equivalent to

\[
\forall \gamma \in \pi_1(\Sigma, 0_1), \quad M_1^{-1} \Pi(\gamma) M_1 \in SU(2),
\]

where \( \Pi \) denotes the principal solution of \( dF = AF \) on \( \Sigma \) (see Section 2.2.1).

Instead of using a set of generators of \( \pi_1(\Sigma, 0_1) \), which would involve in a complicated way the “combinatorics” of our given horosphere packing, we reformulate the monodromy problem in a more “local” way as follows. For \((i,j) \in I\), let \( \gamma_{ij} \in \pi_1(\Sigma, 0_i) \) be a loop in \( \mathbb{C}_i \) with base point \( 0_i \) which goes around \( p_{ij} \) and does not encircle any other node. We also define \( \Gamma_{ji} \) as a path connecting \( 0_i \) to \( 0_j \) through \( \mathbb{C}_{ij} \) (to be defined more precisely later on).

**Proposition 4.** Given \( n \) matrices \( M_1, \ldots, M_n \) in \( SL(2, \mathbb{C}) \), assume that for all \((i,j) \in I\):

\[
M_i^{-1} \Pi(\gamma_{ij}) M_i \in SU(2),
\]

\[
M_j^{-1} \Pi(\Gamma_{ji}) M_j \in SU(2).
\]

Then (24) is satisfied. Moreover, the solution \( F \) of \( dF = AF \) with initial condition \( F(0_i) = M_1 \) satisfies \( F(0_i) \in M_i \times SU(2) \); hence \( f(0_i) = M_i M_i^* \) for \( 1 \leq i \leq n \).

**Proof.** Assume that (25) and (26) hold for all \((i,j) \in I\). Let \( \Gamma_{ij} = \Gamma_{ji}^{-1} \). Using (6),

\[
M_i^{-1} \Pi(\Gamma_{ij}) M_j = \left( M_j^{-1} \Pi(\Gamma_{ji}) M_j \right)^{-1} \in SU(2).
\]

Define \( \gamma_{ji} = \Gamma_{ji} \gamma_{ij}^* \gamma_{ji} \in \pi_1(\Sigma, 0_j) \). Using (1) again,

\[
M_j^{-1} \Pi(\gamma_{ji}) M_j = \left( M_j^{-1} \Pi(\Gamma_{ji}) M_j \right) \left( M_i^{-1} \Pi(\gamma_{ij}) M_i \right) \left( M_i^{-1} \Pi(\Gamma_{ij}) M_j \right) \in SU(2).
\]

In other words, (25) and (26) also hold for \((j,i) \in I\). Let \( \mathbb{C}_i^* \) be \( \mathbb{C}_i \) minus the disks \( D(p_{ij}, 1) \) for \( j \in J_i \). The fundamental group \( \pi_1(\mathbb{C}_i^*, 0_i) \) is the free group with generators \( \gamma_{ij} \) for \( j \in J_i \). Hence (25) implies that

\[
\forall \delta \in \pi_1(\mathbb{C}_i^*, 0_i) \quad M_i^{-1} \Pi(\delta) M_i \in SU(2).
\]

Any element \( \gamma \in \pi_1(\Sigma, 0_1) \) is homotopic to a product of the form

\[
\delta_k \Gamma_{i_k} \delta_k^{-1} \delta_{k-1} \Gamma_{i_{k-1}} \cdots \delta_2 \Gamma_{i_2} \delta_1 \delta_1,
\]
where $k \in \mathbb{N}^*$, $i_1 = i_k = 1$ and $\delta_j \in \pi_1(C^*,_i,0_{i_j})$ for $1 \leq j \leq k$. So (24) follows from (26) and (27).

4.2. **Choice of the matrices $M_1, \ldots, M_n$.** By the last statement of Proposition 4, choosing the matrices $M_i$ amounts to prescribing the image of the points $0_1, \ldots, 0_n$. Recall from the introduction that we want to “deflate” the horosphere $S_i$ at speed $\xi_i$. This suggests the following choice. Consider $n$ fixed, positive numbers $\xi_1, \ldots, \xi_n$ and a real parameter $s$. (These are the same parameters as in the introduction.) Let $O_i = f_i(0) \in S_i$, where $f_i$ is the chosen conformal parametrization of the horosphere $S_i$. Choose $M_i(s) \in SL(2, \mathbb{C})$ so that $s \in [0, \infty) \mapsto M_i(s) M_i(s)^*$. The matrix $M_i(s)$ is unique up to right multiplication by an element in $SU(2)$, which is clearly irrelevant for the monodromy problem.

4.3. **Main result.** To solve the monodromy problem, we need to adjust the complex parameters $p_{ij}$ and $p_{ji}$ for $(i,j) \in I$. We will denote $p_{ij}^0$ and $p_{ji}^0$ as the value of these parameters corresponding to the given horosphere packing (namely, such that $f_i(p_{ij}^0) = f_j(p_{ji}^0) = S_i \cap S_j$). The matrix of holomorphic 1-forms $A$ depends holomorphically on the parameters $a = (a_{ij})_{(i,j) \in I}$ and $p = (p_{ij}, p_{ji})_{(i,j) \in I}$ and will be denoted $A_{a,p}$. The principal solution of $dF = A_{a,p} F$ will be denoted $\Pi_{a,p}$. Our goal is to prove:

**Proposition 5** (Solution of the monodromy problem). For $s > 0$ small enough, there exist unique values $a(s)$ and $p(s)$ such that (25) and (26), with $M_i = M_i(s)$ and $\Pi = \Pi_{a(s),p(s)}$, are satisfied for all $(i,j) \in I$. Moreover, $a(s)$ and $p(s)$ are smooth functions of $s$ for $s \neq 0$, and extend continuously at $s = 0$ with value $a_{ij}(0) = 0$ and $p_{ij}(0) = p_{ij}^0$.

4.4. **Choice of an isometry.** From now on, we fix a couple $(i,j) \in I$. We have in mind to solve (25) and (26). The computations will be simplified by applying a well-chosen isometry. Let $h$ be an orientation preserving isometry of $\mathbb{H}^3$ such that $h(S_i)$ is the horosphere $x_3 = 1$ and $h(S_i \cap S_j) = (0,0,1)$, in the half-space model. (This isometry is unique up to composition by a rotation around the vertical axis.) Since the horospheres $S_i$ and $S_j$ are tangent, $h(S_j)$ is the sphere of radius $\frac{1}{2}$ centered at $(0,0,\frac{1}{2})$. The limit points of $h(S_i)$ and $h(S_j)$ are respectively $\infty$ and $0$.

As explained in Section 2.1.5, the isometry $h$ corresponds to a matrix $H \in SL(2, \mathbb{C})$, unique up to sign. We have $H \cdot c_i = \infty$ and $H \cdot c_j = 0$, where the dot means the action by homography on the Riemann sphere. So $H$ may be written in the form

$$H = \frac{1}{\sqrt{c_j - c_i}} \left( \begin{array}{cc} \rho & -\rho c_j \\ \rho^{-1} & -\rho^{-1} c_i \end{array} \right),$$

where $\rho$ is some complex number.

We use hats to denote the action of the isometry $h$ on various objects: $\hat{F} = HF$ solves $d\hat{F} = \hat{A} \hat{F}$, where $\hat{A} = HAH^{-1}$. An elementary computation gives

$$\hat{A} = \frac{1}{c_j - c_i} \left( \begin{array}{cc} (G - c_j)(G - c_i) & -\rho^2(G - c_j)^2 \\ \rho^{-2}(G - c_i)^2 & -(G - c_i)(G - c_j) \end{array} \right) \Omega.$$
The principal solution of $Y' = \hat{A}Y$ is $\hat{\Pi} = H\Pi H^{-1}$. Equations (25) and (26) are equivalent to

$$\hat{M}_i^{-1}\hat{\Pi}(\gamma_{ij})\hat{M}_i \in SU(2),$$

(29)

$$\hat{M}_j^{-1}\hat{\Pi}(\Gamma_{ji})\hat{M}_i \in SU(2),$$

(30)

where $\hat{M}_i = HM_i$ and $\hat{M}_j = HM_j$.

**Remark 2.** All of these quantities, $H$, $\rho$, $\hat{A}$, $\hat{\Pi}$, $\hat{M}_i$, $\hat{M}_j$, actually depend on both indices $i$ and $j$ because the chosen isometry $h$ does. However, since $i$ and $j$ are fixed until the very end of Section 4.9 this dependence will not be written to make notation lighter.

### 4.5. Computation of the matrices $\hat{M}_i(s)$ and $\hat{M}_j(s)$.

Consider the matrix

$$\Xi(s) = \left( \begin{array}{cc} e^{s/2} & 0 \\ 0 & e^{-s/2} \end{array} \right) \in SL(2, \mathbb{C}).$$

Then $s \mapsto \Phi(\Xi(s)\Xi(s)^*)$ is the parametrization at unit speed of the positive vertical axis (oriented upwards) in the half-space model. (Here $\Phi$ is the isometry from the Minkowski model to the half-space model given in Section 2.1.4.) The horosphere $\hat{S}_i = h(S_i)$ is parametrized by $\hat{f}_i = h \circ f_i$. We need to compute the corresponding null holomorphic map $\hat{F}_i$. By substitution of $G = c_i$ and $\Omega = \lambda_i dz$ in (28), we obtain

$$\hat{A}_i = \left( \begin{array}{cc} 0 & \hat{\lambda}_i \\ 0 & 0 \end{array} \right), \quad \hat{\lambda}_i = \rho^2 \lambda_i (c_i - c_j).$$

(31)

By our choice of the isometry $h$, we have $\hat{F}_i(p^0_{ij}) = I_2$. Hence

$$\hat{F}_i(z) = \exp((z - p^0_{ij})\hat{A}_i).$$

(32)

Then by our choice of the matrix $M_i(s)$ in Section 4.2

$$\hat{M}_i(s) = \exp(-p^0_{ij}\hat{A}_i)\Xi(\xi_is).$$

(33)

In the same way, the horosphere $\hat{S}_j$ has the null holomorphic map $\hat{F}_j$ given by

$$\hat{A}_j = \left( \begin{array}{cc} 0 & 0 \\ \hat{\lambda}_j & 0 \end{array} \right), \quad \hat{\lambda}_j = \rho^{-2} \lambda_j (c_j - c_i),$$

(35)

and recalling that the mean curvature vector of $\hat{S}_j$ at $(0,0,1)$ points down,

$$\hat{M}_j(s) = \exp(-p^0_{ji}\hat{A}_j)\Xi(-\xi_js).$$

(36)
4.6. Partial derivatives of the matrix $\hat{A}$. We return to the matrix $\hat{A}_{\mathbf{a}, \mathbf{p}}$ given by (28) with $G = G_{t(a)}$ and $\Omega = \Omega_{t(a), \mathbf{a}}$. Taking the derivative of (28) and using (12), (14), (21) and (22), we obtain the following result after simplification:

**Proposition 6.** At $\mathbf{a} = 0$, we have:

\[
\hat{A}_{0, \mathbf{p}} = \hat{A}_i \, dz \quad \frac{\partial \hat{A}_{\mathbf{a}, \mathbf{p}}}{\partial a_{ij}} = \frac{c_i - c_j}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{dz}{z - p_{ij}} \quad \text{in } C_i,
\]

\[
\hat{A}_{0, \mathbf{p}} = \hat{A}_j \, dz \quad \frac{\partial \hat{A}_{\mathbf{a}, \mathbf{p}}}{\partial a_{ij}} = \frac{c_j - c_i}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{dz}{z - p_{ji}} \quad \text{in } C_j,
\]

\[
\hat{A}_{0, \mathbf{p}} = 0 \quad \frac{\partial \hat{A}_{\mathbf{a}, \mathbf{p}}}{\partial a_{ij}} = \frac{c_j - c_i}{2} \begin{pmatrix} z & -z^2 \\ \rho^2 z^2 & -z \end{pmatrix} \frac{dz}{z^2} \quad \text{in } C_{ij},
\]

where the matrices $\hat{A}_i$ and $\hat{A}_j$ are given by (31) and (35).

4.7. Expansion of $\hat{\Pi}(\gamma_{ij})$.

**Proposition 7.** We have the following expansion:

\[
\hat{\Pi}_{\mathbf{a}, \mathbf{p}}(\gamma_{ij}) = I_2 + a_{ij} \pi i (c_i - c_j) \exp(-p_{ij}\hat{A}_i) \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \exp(p_{ij}\hat{A}_i) + O(||\mathbf{a}||^2).
\]

**Proof.** When $\mathbf{a} = 0$, the principal solution in $C_i$ is given by $\hat{\Pi}_{0, \mathbf{p}}(z, 0_i) = \exp(z\hat{A}_i)$, which is well defined, so $\hat{\Pi}_{0, \mathbf{p}}(\gamma_{ij}) = I_2$. By Proposition 9 in Appendix A (37) and the residue theorem,

\[
\frac{\partial \hat{\Pi}_{\mathbf{a}, \mathbf{p}}(\gamma_{ij})}{\partial a_{ij}} = \frac{c_i - c_j}{2} \int_{\gamma_{ij}} \exp(-z\hat{A}_i) \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \exp(z\hat{A}_i) \frac{dz}{z - p_{ij}}
\]

\[
= \pi i (c_i - c_j) \exp(-p_{ij}\hat{A}_i) \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \exp(p_{ij}\hat{A}_i).
\]

For $(k, l) \neq (i, j)$, the partial derivative of $\hat{A}$ with respect to $a_{k\ell}$ is holomorphic at $p_{ij}$. Hence the partial derivative of $\hat{\Pi}(\gamma_{ij})$ with respect to $a_{k\ell}$ is zero. Equation (40) follows.

4.8. Expansion of $\hat{\Pi}(\Gamma_{ji})$. Recall that $\exp$ is a local diffeomorphism from a neighborhood of 0 in $M_2(\mathbb{C})$ to a neighborhood of $I_2$ in $GL(2, \mathbb{C})$. We denote the inverse diffeomorphism by log. Also, for a matrix $M$ and a complex number $\lambda$, we define $M^\lambda = \exp(\lambda \log M)$.

**Proposition 8.** We have

\[
\hat{\Pi}_{\mathbf{a}, \mathbf{p}}(\Gamma_{ji}) \times \hat{\Pi}_{\mathbf{a}, \mathbf{p}}(\gamma_{ij}) = \exp(-p_{ji}\hat{A}_j) \exp(p_{ij}\hat{A}_i) + \mathcal{O}(\mathbf{a}),
\]

where $\mathcal{O}(\mathbf{a})$ is a well-defined holomorphic function of $(\mathbf{a}, \mathbf{p})$ which vanishes at $\mathbf{a} = 0$.

**Proof.** Let us fix the value of all parameters except $a_{ij}$. We must first see that the left-hand side of (41) is a well-defined function of $a_{ij}$ for $a_{ij} \neq 0$, which of course means that $\hat{\Pi}(\Gamma_{ji})$ itself is not. To see this, we have to define precisely the path
\( \Gamma_{ji} \) from 0\(_i\) to 0\(_j\). For \( t_{ij} \) and \( t_{ji} \) non-zero, we define \( \Gamma_{ji} \) as the composition of the following two paths:

- The following path from 0\(_i\) to \( p_{ij} + t_{ij} \) in \( \mathbb{C}_i \): a path from 0\(_i\) to \( p_{ij} + 1 \) in \( \mathbb{C}_i \) minus all unit disks around the nodes (depending continuously on the parameter \( p_{ij} \) in a neighborhood of \( p_{ij}^0 \)), composed with the spiral parametrized by \( x \mapsto p_{ij} + (t_{ij})^x \) for \( x \in [0, 1] \).
- The following path from \( p_{ji} + t_{ji} \) to 0\(_j\) in \( \mathbb{C}_j \): the spiral parametrized by \( x \mapsto p_{ji} + (t_{ji})^{1-x} \) for \( x \in [0, 1] \), composed with a path from \( p_{ji} + 1 \) to 0\(_j\).

We can compose these two paths because the points \( p_{ij} + t_{ij} \) and \( p_{ji} + t_{ji} \) are both identified with 1\(_{ij}\) when opening nodes. Observe that we need a determination of the arguments of \( t_{ij} \) and \( t_{ji} \) to define the spirals. In other words, if we take \( t_{ij} \) and \( t_{ji} \) to live in the universal cover of the punctured unit disk (so \( \arg t_{ij} \) and \( \arg t_{ji} \) are well defined), \( \Gamma_{ji} \) depends continuously on \( t_{ij} \) and \( t_{ji} \). Now if the argument of \( a_{ij} \) is increased by \( 2\pi \), then the arguments of \( t_{ij} \) and \( t_{ji} \) are increased by the same amount. Hence the homotopy class of \( \Gamma_{ji} \) is multiplied on the right by \( (\gamma_{ij})^2 \) and \( \hat{\Pi}(\Gamma_{ji}) \) is multiplied on the right by \( \hat{\Pi}(\gamma_{ij})^2 \). Consequently, the left-hand side of (11) is unchanged, so it is a well-defined holomorphic function of \( a_{ij} \) for \( a_{ij} \neq 0 \).

Next we prove that the left-hand side of (11) is uniformly bounded. Because it is a well-defined function of \( a_{ij} \), we can assume that \( \arg a_{ij} \in [-2\pi, 2\pi] \). Using (6), we write

\[
\hat{\Pi}(\Gamma_{ji}) = \hat{\Pi}(0, p_{ij} + 1) \hat{\Pi}(p_{ij} + 1, p_{ji} + t_{ji}) \hat{\Pi}(p_{ij} + t_{ij}, p_{ij} + 1) \hat{\Pi}(p_{ij} + 1, 0_i).
\]

Since the path from 0\(_i\) to \( p_{ij} + 1 \) stays in a fixed compact set of \( \mathbb{C}_i \) minus the nodes, where \( \hat{\mathcal{A}} \) is uniformly bounded, the fourth factor in (42) is uniformly bounded. We estimate the third factor using Proposition 10 from Appendix B. For this, we need an integral estimate of \( ||\hat{\mathcal{A}}|| \) on the circle of center \( p_{ij} \) and radius \( |t_{ij}|/2 \).

We claim that \( \hat{\mathcal{A}} = \mathcal{O}(a_{ij}) \) in compact subsets of \( \overline{\mathbb{C}}_{ij} \setminus \{0, \infty\} \) (even if the other parameters \( a_{kt} \) are non-zero). Indeed, \( \hat{\mathcal{A}} \) depends holomorphically on \( a_{ij} \), and if \( a_{ij} = 0 \), then by (15), \( t_{ij} = t_{ji} = 0 \), so \( \Omega = 0 \) and \( \hat{\mathcal{A}} = 0 \) in \( \overline{\mathbb{C}}_{ij} \). Also, by (16), \( a_{ij} = \mathcal{O}(t_{ij}) \). Consequently, since \( \hat{\mathcal{A}} \) is a matrix-valued 1-form,

\[
\int_{|z-p_{ij}|=\frac{|t_{ij}|}{2}} ||\hat{\mathcal{A}}|| = \int_{|w_{ij}|=\frac{|t_{ij}|}{2}} ||\hat{\mathcal{A}}|| = \int_{|v_{ij}|=2} ||\hat{\mathcal{A}}|| \leq C|t_{ij}|
\]

for some uniform constant \( C \). By Proposition 10 in Appendix B \( \hat{\Pi}(p_{ij} + t_{ij}, p_{ij} + 1) \) is uniformly bounded. The first and second factors in (42) are estimated in the exact same way. We conclude that \( \hat{\Pi}(\Gamma_{ji}) \) is uniformly bounded (although not well defined – but we assumed that \( \arg a_{ij} \in [-2\pi, 2\pi] \)). The left-hand side of (11) is now a bounded, well-defined holomorphic function of \( (a, p) \) on the set \( a_{ij} \neq 0 \). By the Riemann extension theorem (in several variables), it extends holomorphically at \( a_{ij} = 0 \).

To compute its value at \( a = 0 \), assume that all parameters \( a_{kt} \) for \( (k, \ell) \neq (i, j) \) are zero. By (37), \( \hat{\mathcal{A}}_{a, p} - \hat{\mathcal{A}}_i = \mathcal{O}(a_{ij}) \) in compact subsets of \( \mathbb{C}_i \) minus the nodes. By point (2) of Proposition 10 (with \( A = \hat{\mathcal{A}}_i \)), we obtain

\[
||\hat{\Pi}_{a, p}(p_{ij} + t_{ij}, 0_i) - \hat{\Pi}_i(p_{ij} + t_{ij}, 0_i)|| \leq C |a_{ij} \log |a_{ij}||
\]
where $\hat{\Pi}_i$ is the principal solution of $Y' = \hat{A}_i Y$ in $\mathbb{C}_i$, namely $\hat{\Pi}_i(z, 0_i) = \exp(z\hat{A}_i)$. This gives
\[
\lim_{a_{ij}\to 0} \hat{\Pi}_a(p_{ij} + t_{ij}, 0_i) = \exp(p_{ij}\hat{A}_i).
\]
Arguing in the same way, we obtain
\[
\lim_{a_{ij}\to 0} \hat{\Pi}_a(0_j, p_{ji} + t_{ji}) = \exp(-p_{ji}\hat{A}_j).
\]
Proposition 8 follows.

4.9. Solution of the monodromy problem. We are now ready to prove Proposition 8. The unitary group $SU(2)$ is not a complex manifold, so we have to leave the realm of holomorphic functions. We introduce a small positive real number $\tau$ and have in mind to apply the implicit function theorem at $\tau = 0$. We write
\[
a_{ij} = \tau \frac{b_{ij}}{c_i - c_j},
\]
where $b_{ij}$ is a complex number in a neighborhood of a non-zero central value $b_{ij}^0$. The computation will be simplified by knowing a priori the order of each parameter as a function of $\tau$. The correct orders are
\[
s = -\tau \log \tau, \quad p_{ij} = p_{ij}^0 + sq_{ij},
\]
where $q_{ij}$ is a complex parameter in a neighborhood of 0. One issue here is that the function $\tau \mapsto \tau \log \tau$ does not extend as a differentiable function at $\tau = 0$. We solve this problem by writing $\tau = e^{-1/t^2}$, where $t$ is a real parameter in a neighborhood of 0. Both $\tau$ and $\tau \log \tau$ extend smoothly at $t = 0$, and all parameters are smooth functions of $t$. Let $b = (b_{ij})_{(i,j) \in I}$ and $q = (q_{ij}, q_{ji})_{(i,j) \in I}$.

Recall that $\exp$ maps the Lie algebras $\mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{su}(2, \mathbb{C})$ to the Lie groups $SL(2, \mathbb{C})$ and $SU(2, \mathbb{C})$, respectively. We define
\[
P_{ij} = P_{ij}(t, b, q) = \log \left( \hat{M}_i(s)^{-1}\hat{\Pi}_a(p_{ij})\hat{M}_i(s) \right) \in \mathfrak{sl}(2, \mathbb{C}),
\]
\[
Q_{ij} = Q_{ij}(t, b, q) = \log \left( \hat{M}_j(s)^{-1}\hat{\Pi}_a(p_{ij})\hat{M}_j(s) \right) \in \mathfrak{sl}(2, \mathbb{C}).
\]
We want to solve $P_{ij} \in \mathfrak{su}(2, \mathbb{C})$ and $Q_{ij} \in \mathfrak{su}(2, \mathbb{C})$. We compute $P_{ij}$ using Proposition 7
\[
(43) \quad \hat{\Pi}(\gamma_{ij}) = I_2 + \pi i t b_{ij} \exp(-p_{ij}\hat{A}_i) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \exp(p_{ij}\hat{A}_i) + O(\tau^2) \quad \text{using (40)},
\]
\[
P_{ij} = \pi i t b_{ij} \Xi(-\xi_i s) \exp(-sq_{ij}\hat{A}_i) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \exp(sq_{ij}\hat{A}_i)\Xi(\xi_i s) + O(\tau^2) \quad \text{using (33)},
\]
\[
(44) \quad P_{ij} = \pi i t b_{ij} \begin{pmatrix} 1 & 2\lambda_i s q_{ij} \\ 0 & -1 \end{pmatrix} + O(\tau^2).
\]
We compute $Q_{ij}$ using Proposition 8
\[
\hat{\Pi}(\Gamma_{ij}) = \exp(-p_{ji}\hat{A}_j) \exp(p_{ij}\hat{A}_i)\hat{\Pi}(\gamma_{ij}) \hat{\Pi}(\gamma_{ij})^\pm \log a_{ij} + O(\tau) \quad \text{using (41)},
\]
\[
\hat{\Pi}(\gamma_{ij})^\pm \log a_{ij} = I_2 - sb_{ij} \exp(-p_{ij}\hat{A}_i) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \exp(p_{ij}\hat{A}_i) + O(\tau) \quad \text{using (43)},
\]
The geometry of the image of space model, so on. Equations (28) and (31) hold true, with
\[ \rho, c \]

We consider the following disjoint domains in \( \Sigma \):

\[ \text{Definition of function} \]

For a matrix \( M = (M^{k\ell})_{1 \leq k, \ell \leq 2} \in \mathfrak{sl}(2, \mathbb{C}) \), we have

\[ M \in \mathfrak{su}(2, \mathbb{C}) \iff \text{Re}(M^{11}) = 0 \quad \text{and} \quad M^{12} + \overline{M^{21}} = 0. \]

Define the function \( F = \left( F_{ij}(t, b, q) \right)_{i,j} \) for \( t \neq 0 \) by

\[ \left( \begin{array}{l} 1 \Re(P_{ij}^{11}) \, \frac{1}{s} \Re(Q_{ij}^{11}) \, \frac{1}{s} \Re(Q_{ij}^{12}) \, \frac{1}{s} \Re(Q_{ij}^{21}) \end{array} \right). \]

We want to solve \( F(t, b, q) = 0 \). By (44) and (45), \( F \) extends smoothly at \( t = 0 \), with

\[ F_{ij}(0, b, q) = \left( \begin{array}{c} -\pi \Im(b_{ij}) \, 2\pi i b_{ij} \hat{\lambda}_{ij} q_{ij} \, \xi_{ij} - \Re(b_{ij}) \, \hat{\lambda}_{ij} q_{ij} - \lambda_{ij} q_{ij} \end{array} \right). \]

Taking \( b_{ij}^0 = \frac{\xi_{ij} + \xi_{ij}}{2} > 0 \), we have \( F(0, b^0, 0) = 0 \). It is straightforward that the partial differential of \( F \) with respect to the variables \( (b, q) \) at \( (0, b^0, 0) \) is an isomorphism. By the implicit function theorem, \( t \) in a neighborhood of 0, there exists \( (b(t), q(t)) \) depending smoothly on \( t \) such that \( F(t, b(t), q(t)) = 0 \). Proposition 5 is proved, and the monodromy problem is solved. \( \square \)

5. Embeddedness

Here is what we have achieved so far. For each small enough value of the parameter \( t > 0 \), we have constructed a null holomorphic map \( F \) which has \( SU(2) \)-valued monodromy. All parameters are now smooth functions of \( t > 0 \). To ease notation, the dependence on \( t \) will not be written. Let \( f : \Sigma \to \mathbb{H}^3 \) be the CMC-1 immersion associated to \( F \). It remains to prove that \( f(\Sigma) \) is embedded. We work in the half-space model, so \( f(z) = \Phi(F(z)F(z)^*) \) is given by (44). Fix a small number \( \varepsilon > 0 \). We consider the following disjoint domains in \( \Sigma \):

\[ C_i^\varepsilon = \{ z \in C_i : \forall j \in J_i, |z - p_{ij}^0| > \varepsilon \}, \]

\[ C_{ij}^\varepsilon = \{ z \in C_{ij} : \varepsilon < |z| < \frac{1}{\varepsilon} \}. \]

The complement of these domains in \( \Sigma \) are annuli, which we call transition regions. We also fix some large number \( R \) and define \( C_i^{\varepsilon, R} = C_i \cap D(0, R) \).

5.1. Geometry of the image of \( C_i^{\varepsilon, R} \). Fix some \( i, 1 \leq i \leq n \), and consider an isometry \( h \) such that \( h(S_i) \) is the horosphere \( x_3 = 1 \) and \( h \) maps \( f_i(0) \) to the point \( (0, 0, 1) \). The isometry \( h \) is represented by a matrix \( H \in SL(2, \mathbb{C}) \) which has the form

\[ H = \begin{pmatrix} \rho & -\rho c \\ \rho^{-1} & -\rho^{-1} c_i \end{pmatrix}, \]

where \( \rho, c \) are some complex numbers (\( \rho \) not the same as in Section 4.1). As in Section 4.3, we use hats to denote the action of \( h \), so \( \hat{f} = h \circ f, \hat{F} = HF \), and so on. Equations (28) and (31) hold true, with \( c \) in place of \( c_j \). By construction, \( \hat{f}(0_i) \) parametrizes the vertical axis at speed \( \xi_i \) as \( s \) varies, so \( \hat{F}(0_i) = \Xi(\xi_i s) \), up to right
multiplication by $SU(2)$. We have $\widehat{A} = \widehat{A}_i + O(\tau)$. Since $\mathbb{C}_i^{e,R}$ is a fixed compact domain,

$$\widehat{F}(z) = \exp(z\widehat{A}_i)\Xi(\xi,s) + O(\tau) \quad \text{for } z \in \mathbb{C}_i^{e,R}.$$ 

From this, we conclude that $\widehat{F}(\mathbb{C}_i^{e,R})$ converges smoothly to (a subdomain of) the horosphere $x_3 = 1$ as $t \to 0$. Moreover, from (4), we get

$$x_3(z) = e^{\xi_i s} + O(\tau) \quad \text{for } z \in \mathbb{C}_i^{e,R},$$

so for $t$ small enough, the image of $\mathbb{C}_i^{e,R}$ lies above the horosphere $x_3 = 1$.

5.2. Geometry of the end at $\infty_i$. Next we prove that the image of $|z| > R$ in $\mathbb{C}_i$ is embedded. We claim that for $t > 0$ small enough, the Gauss map $G$ has multiplicity 1 at $\infty_i$. This is delicate because $G$ is constant when $t = 0$. We work in the local coordinate $w = \frac{1}{z}$ in a neighborhood of $\infty_i$ and write $\tilde{G}(w) = G(1/w)$. From (21), we obtain

$$\frac{\partial G'(w)}{\partial a_{ij}} = \frac{c_j - c_i}{2\lambda_i(1 - p_{ij}w)^2} \quad \text{for } j \in J_i^+,$$

$$\frac{\partial \tilde{G}'(w)}{\partial a_{ji}} = \frac{c_i - c_j}{2\lambda_i(1 - p_{ji}w)^2} \quad \text{for } j \in J_i^-,$$

$$\tilde{G}'(0) = \sum_{j \in J_i^+} \frac{\partial G'(0)}{\partial a_{ij}} a_{ij} + \sum_{j \in J_i^-} \frac{\partial G'(0)}{\partial a_{ji}} a_{ji} + O(||a||^2)$$

$$= \sum_{j \in J_i^+} \frac{c_j - c_i}{2\lambda_i} \frac{\tau b_{ij}}{c_i - c_j} + \sum_{j \in J_i^-} \frac{c_i - c_j}{2\lambda_i} \frac{\tau b_{ji}}{c_j - c_i} + O(\tau^2)$$

$$= -\frac{\tau \zeta_i}{2\lambda_i} + o(\tau), \quad \text{where} \quad \zeta_i = \frac{1}{2} \sum_{j \in J_i} (\xi_i + \xi_j) > 0.$$ 

Hence for $t > 0$ small enough, $\tilde{G}'(0) \neq 0$, so the Gauss map has multiplicity one at the end. To study the geometry of the end, we again consider the isometry $h$ introduced in Section 5.1. Then $\tilde{G} = H \cdot \tilde{G}$ has a simple pole at $w = 0$ with residue

$$\text{Res}_{w=0} \tilde{G} = \text{Res}_{w=0} \rho^2 \frac{\tilde{G} - c}{\tilde{G}'(0)} = \rho^2 \frac{c_i - c}{\tilde{G}'(0)} = \frac{-2\hat{\lambda}_i}{\tau \zeta_i},$$

where $\hat{\lambda}_i = \rho^2 \lambda_i(c_i - c)$. From (28) we obtain

$$\tilde{G}^2 \Omega = -\widehat{A}_{21} \simeq -\hat{\lambda}_i dz = \hat{\lambda}_i \frac{dw}{w^2}.$$ 

By Theorem 6 in Appendix C (with $\alpha = \frac{-2\hat{\lambda}_i}{\tau \zeta_i}$ and $\alpha^2 \beta = \hat{\lambda}_i$), there exists a uniform positive $\epsilon$ (independent of $t$) such that the image of $0 < |w| < \epsilon$ is the vertical graph $x_3 = u(x_1, x_2)$ of a function $u$. Moreover, at infinity we have

$$\log u(x_1, x_2) \simeq (\tau \zeta_i + o(\tau)) \log \sqrt{x_1^2 + x_2^2},$$

so $x_3 > 1$ on the end. Replacing $R$ by $\epsilon^{-1}$ if necessary, we obtain that $\widehat{F}(\mathbb{C}_i) = \hat{\lambda}_i dz = \hat{\lambda}_i \frac{dw}{w^2}$. This guarantees that the images $f(\mathbb{C}_i)$ for $1 \leq i \leq n$ are disjoint.
5.3. **Geometry of the catenoidal necks.** Fix a couple \((i,j) \in I\). Again consider the isometry \(h\) introduced in Section 4.3 which maps the horosphere \(S_i\) to the horosphere \(x_3 = 1\) and the horosphere \(S_j\) to the sphere of radius \(\frac{1}{2}\) centered at \((0,0,\frac{1}{2})\). In this section, we prove that after a blowup of ratio \(1/\tau\), the image \(\tilde{f}(C_{ij})\) converges to a vertical catenoid.

By computations similar to the computation of \(Q_{ij}\) in Section 4.9 we have

\[ \tilde{F}(1_{ij}) = \tilde{\Pi}(p_{ij} + t_{ij}, O_i)\tilde{F}(0) = I_2 + \frac{s}{2} \begin{pmatrix} \xi_i - b_{ij} & 2\lambda_i q_{ij} \\ 0 & b_{ij} - \xi_i \end{pmatrix} + O(\tau). \]

By (39), we have in \(S^2_{ij}\)

\[ \tilde{F}(z) = \frac{1}{\tau}(\tilde{F}(z) - \tilde{F}(1_{ij})). \]

Then using (46) and (47),

\[ d\tilde{F}(z) = \frac{1}{\tau}A(z)\tilde{F}(z) = (\tilde{A}(z) + O(\tau))(I_2 + O(s)) = \tilde{A}(z) + O(s). \]

Since \(S^2_{ij}\) is a fixed compact set, we obtain by integration

\[ \tilde{F}(z) = -\frac{b_{ij}}{2} \begin{pmatrix} \log z & \rho^2(z^{-1} - 1) \\ \rho^{-2}(z - 1) & -\log z \end{pmatrix} + O(s). \]

Write \(\tilde{x}_k(z) = \tilde{x}_k(1_{ij}) + \tau\tilde{x}_k(z)\) for \(1 \leq k \leq 3\). Using (44) and \(\tilde{F}(z) = I_2 + O(s)\), we obtain

\[ (\tilde{x}_1(z) + i\tilde{x}_2(z), \tilde{x}_3(z)) = \left( \tilde{F}_{12}(z) + \tilde{F}_{21}(z), -2\text{Re}\left(\tilde{F}_{22}(z)\right) \right) + O(s), \]

\[ \lim_{t \to 0} (\tilde{x}_1(z) + i\tilde{x}_2(z), \tilde{x}_3(z)) = -\frac{\xi_i + \xi_j}{4} \begin{pmatrix} \rho^2(z^{-1} - 1) + \rho^{-2}(z - 1), 2\log |z| \end{pmatrix}. \]

This is the parametrization of a vertical catenoid of necksize \(\frac{\xi_i + \xi_j}{2}\). This means that after a blowup of ratio \(\frac{1}{\tau}\) at \(\tilde{f}(1_{ij})\), the image of \(S^2_{ij}\) converges smoothly to a catenoid. Also observe that the image of the circle \(|z| = \epsilon\) lies above the image of \(|z| = \frac{1}{\tau}\). Finally, (46) gives

\[ \tilde{x}_3(1_{ij}) = 1 + s\frac{\xi_i - \xi_j}{2} + O(\tau). \]

Hence the catenoidal neck lies below the image of \(C^\epsilon_{ij}\).

5.4. **Geometry of the transition regions.** Fix \((i,j) \in I\) and let \(U_{ij}\) be the annulus in \(\Sigma\) bounded by the circles \(|z - p_{ij}| = \varepsilon\) in \(C_i\) and \(|z| = \varepsilon\) in \(S^2_{ij}\). We again consider the isometry \(h\) introduced in Section 4.4. Let us prove that the mean curvature vector of \(\tilde{f}\) is almost vertical in \(U_{ij}\). Given the geometric interpretation of the Gauss map given in Section 2.1.4, an elementary computation shows that the angle \(\theta(z)\) between the mean curvature vector at \(f(z)\) and the vertical axis is related to the Gauss map \(G(z)\) by

\[ \frac{\sin \theta(z)}{1 + \cos \theta(z)} = \frac{x_3(z)}{|G(z) - x_1(z) - ix_2(z)|}. \]
The function \( \hat{G}^{-1} \) is holomorphic in the annulus \( U_{ij} \) and is bounded by \( C\varepsilon \) on the boundary circles for some uniform constant \( C \). By the maximum principle, \( \hat{G}^{-1} \) is bounded by \( C\varepsilon \) in \( U_{ij} \). The norm of the holomorphic map \( F(z) - I_2 \) is bounded by \( C\varepsilon \) on the boundary of \( U_{ij} \), and so is bounded by \( C\varepsilon \) in \( U_{ij} \) by the maximum principle. Hence the function \( \hat{x}_1 + i\hat{x}_2 \) is uniformly bounded in \( U_{ij} \), and the height \( \hat{x}_3 \) satisfies \(|\hat{x}_3 - 1| \leq C\varepsilon \) in \( U_{ij} \). Using (18), we obtain

\[
\frac{\sin \hat{\theta}(z)}{1 + \cos \hat{\theta}(z)} \leq C\varepsilon \quad \text{in } U_{ij}.
\]

Hence by choosing \( \varepsilon \) small enough, we can ensure that \( \hat{\theta}(z) < \frac{\pi}{2} \). This implies that \( \hat{f}(U_{ij}) \) is locally a vertical graph. Since we have already seen that it is a graph on the boundary circles, it is globally a graph. Moreover, by the maximum principle, it lies above the lowest point of the top boundary component of the catenoidal neck \( \hat{f}(\hat{C}_{ij}) \). The image of the annulus bounded by the circles \(|z - p_{ji}| = \varepsilon \) in \( \mathbb{C}_j \) and \(|z| = \frac{1}{\varepsilon} \) in \( \hat{C}_{ij} \) is studied in the same way, using an isometry which maps the horosphere \( S_j \) to the horosphere \( x_3 = 1 \). This proves that \( f(\Sigma) \) is embedded and concludes the proof of Theorem 1. □

**Appendix A. Derivative of the monodromy**

Consider a domain \( \Omega \subset \mathbb{C} \) and a point \( z_0 \in \Omega \). Let \( A_\lambda(z) \in GL(n, \mathbb{C}) \) be a family of matrices depending holomorphically on \( (\lambda, z) \) for \( z \in \Omega \) and \( \lambda \) in a neighborhood of 0. Let \( \Pi_\lambda \) denote the principal solution of \( Y' = A_\lambda Y \) in \( \Omega \).

**Proposition 9.** For any \( \gamma \in \pi_1(\Omega, z_0) \),

\[
\left. \frac{\partial \Pi_\lambda(\gamma)}{\partial \lambda} \right|_{\lambda=0} = \Pi_0(\gamma) \int_{\gamma} \Pi_0(z, z_0)^{-1} \frac{\partial A_\lambda(z)}{\partial \lambda} \Pi_0(z, z_0) \, dz.
\]

**Proof.** Let \( Y_\lambda(z) = \Pi_\lambda(z, z_0) \) and \( W = \partial Y_\lambda/\partial \lambda \). Differentiating \( Y'_\lambda = A_\lambda Y_\lambda \) and \( Y_\lambda(z_0) = I_n \) with respect to \( \lambda \) at \( \lambda = 0 \), we get \( W(z_0) = 0 \) and

\[
W' = A_0 W + \frac{\partial A_\lambda}{\partial \lambda} Y_0.
\]

By the variation of constants formula (Theorem 3.12 in [15]),

\[
W(z) = Y_0(z) \int_{z_0}^{z} Y_0(w)^{-1} \frac{\partial A_\lambda(w)}{\partial \lambda} Y_0(w) \, dw.
\]

Taking \( z = \gamma(1) \), the result follows. □

**Appendix B. Uniform estimates of the solution of \( Y' = AY \) in an annulus**

In this section, we consider the annulus \( \Omega \subset \mathbb{C} \) defined by \( \rho^{-1} t < |z| < \rho \), where \( \rho > 1 \) is some fixed number and \( t \) is a small positive parameter. We are aiming for estimates which are uniform with respect to \( t \). Let \( A : \Omega \to sl(n, \mathbb{C}) \) be a holomorphic map. Let \( Y(z) \in SL(n, \mathbb{C}) \) be the solution of \( Y' = AY \) in \( \Omega \), with initial condition \( Y(1) = I_n \). (Of course, \( Y(z) \) is only well defined in the universal cover of \( \Omega \): the value of \( Y(z) \) depends on the determination of \( \arg z \).)
Proposition 10. (1) Assume that for some constant \( c \),
\[
\int_{|z| = \rho} ||A|| \leq c \quad \text{and} \quad \int_{|z| = \rho^{-1}} ||A|| \leq ct.
\]
Then for \( t \leq |z| \leq 1 \) and \( |\arg z| \leq c' \), \( ||Y(z)|| \) is bounded by a constant depending only on \( c, c' \) and \( \rho \).

(2) Let \( \bar{A} \) be another matrix-valued map satisfying the same hypotheses as \( A \) and let \( \bar{Y}(z) \) be the solution of \( \bar{Y}' = \bar{A}\bar{Y} \) with initial condition \( \bar{Y}(1) = I_n \). Assume moreover that
\[
\int_{|z| = \rho} ||A - \bar{A}|| \leq ct.
\]
Then for \( t \leq |z| \leq 1 \) and \( |\arg(z)| \leq c' \),
\[
||Y(z) - \bar{Y}(z)|| \leq Ct|\log t|
\]
for some constant \( C \) depending only on \( c, c' \) and \( \rho \).

Proof. We use the letter \( C \) for uniform constants, depending only on \( c \) and \( \rho \) but not on \( t \). By the Cauchy theorem,
\[
A(z) = \frac{1}{2\pi i} \int_{|w| = \rho} \frac{A(w)}{w - z} dw - \frac{1}{2\pi i} \int_{|w| = \rho^{-1}t} \frac{A(w)}{w - z} dw.
\]
Hence for \( t \leq |z| \leq 1 \),
\[
||A(z)|| \leq \frac{1}{2\pi} \left( \frac{c}{\rho - 1} + \frac{ct}{t(1 - \rho^{-1})} \right) \leq C.
\]
We can connect 1 and \( z \) (in the universal cover of \( \Omega \)) by a path of length less than \( 1 + c' \). The first point of Proposition 10 follows from Gromwall inequality (Lemma 2.7 in [15]). Using the Cauchy formula in the same way, we obtain for \( t \leq |z| \leq 1 \)
\[
||A(z) - \bar{A}(z)|| \leq \frac{1}{2\pi} \left( \frac{ct}{\rho - 1} + \frac{ct}{|z| - \rho^{-1}t} \right) \leq Ct \frac{1}{|z|}.
\]
By the variation of constants formula (Theorem 3.12 in [15]),
\[
\bar{Y}(z) = Y(z) + Y(z) \int_1^z Y(w)^{-1}(\bar{A}(w) - A(w))\bar{Y}(w) dw.
\]
This gives
\[
||\bar{Y}(z) - Y(z)|| \leq Ct \int_1^z \left| \frac{dw}{|w|} \right| \leq Ct (|\arg z| + |\log t|).
\]

Appendix C. Embedded CMC-1 ends

Theorem 6. Let \( f : D^*(0,1) \rightarrow \mathbb{H}^3 \) be a conformal, CMC-1 immersion of the punctured closed unit disk. Assume that the Gauss map \( G \) has a simple pole at 0, with residue \( \alpha \), and the holomorphic differential \( \Omega \) is holomorphic at 0, with \( \Omega(0) = \beta dz \). Assume that \( 0 < |\alpha \beta| \leq {\frac{1}{4}} \). Then there exists \( \varepsilon > 0 \) such that in the half-space model, \( f(D^*(0, \varepsilon)) \) is a vertical graph \( x_3 = u(x_1, x_2) \) of a (positive) function \( u \) over an exterior domain in the plane. The number \( \varepsilon \) only depends on
a bound on $|\alpha^2 \beta|^{\frac{1}{2}}$ and $|F(z)|$ on the unit circle. Moreover, $\alpha \beta$ is real, and at infinity, the function $u$ has the following asymptotic behavior:

$$\log u(x_1, x_2) \simeq (1 - \sqrt{1 + 4 \alpha \beta}) \log |x_1 + ix_2|.$$ 

**Remark 3.** In this paper, we are interested in the case where $\alpha \beta \to 0$ and we need a uniform positive $\varepsilon$. The conclusions of Theorem 6 remain true without the hypothesis $|\alpha \beta| \leq \frac{1}{4}$ but the proof is more involved, as the fuchsian system can be resonant. In particular, one can prove that $\alpha \beta$ is always a real number in $(-\frac{1}{4}, \infty)$.

**Proof.** We use the theory of fuchsian systems to compute $F(z)$ such that $f = FF^*$ in the punctured disk. The system $F' = AF$ is fuchsian provided the matrix $A(z)$ has a simple pole at 0, which is not the case here (it has a double pole). To circumvent this problem, we introduce the matrix

$$N(z) = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}$$

and make the change of unknown $F(z) = N(z)\tilde{F}(z)$. By a straightforward computation, (2) is equivalent to

$$\tilde{F}'(z) = \tilde{A}(z)\tilde{F}(z),$$

where

$$\tilde{A} = \begin{pmatrix} G \omega & -zG^2 \omega \\ z^{-1} \omega & -G \omega - z^{-1} \end{pmatrix}. $$

Now the matrix $\tilde{A}$ has a simple pole at 0, with residue

$$A_0 = \text{Res}_0 \tilde{A} = \begin{pmatrix} \alpha \beta & -\alpha^2 \beta \\ \beta & -\alpha \beta - 1 \end{pmatrix}$$

and the system (50) is fuchsian. The eigenvalues of $A_0$ are

$$\lambda_1 = \frac{-1 + \sqrt{\Delta}}{2}, \quad \lambda_2 = \frac{-1 - \sqrt{\Delta}}{2}, \quad \text{where } \Delta = 1 + 4\alpha \beta.$$ 

The system (50) is called resonant if $\lambda_1 - \lambda_2 = \sqrt{\Delta}$ is a non-zero integer. It follows from our hypothesis that $\frac{1}{2} \leq |\Delta| \leq \frac{3}{2}$ and $\Delta \neq 1$, so the system is non-resonant. By the standard theory of fuchsian systems (Proposition 11.2 in [14]), the solution of (50) has the form

$$\tilde{F}(z) = U(z)z^{A_0}Y_0,$$

where $U(z) \in GL(2, \mathbb{C})$ is well defined, holomorphic in $D(0, 1)$ and satisfies $U(0) = I_2$, and $Y_0 \in GL(2, \mathbb{C})$ is a constant matrix. The monodromy of $\tilde{F}$ on the unit circle $\gamma$ is

$$M_\gamma(\tilde{F}) = M_\gamma(F) = Y_0^{-1}e^{2\pi i A_0} Y_0.$$ 

Since $f$ is well defined, its monodromy $M_\gamma(F)$ belongs to $SU(2)$ so its eigenvalues are a complex number of modulus 1. This implies that the eigenvalues $\lambda_1, \lambda_2$ of $A_0$ are real numbers, so $\alpha \beta$ is real. To compute $z^{A_0}$, we write $A_0 = PDP^{-1}$ with

$$P = \frac{1}{\Delta^{1/4}} \begin{pmatrix} \alpha^{-1} \lambda_1 & -\lambda_2 \\ \alpha \lambda_1 & -\lambda_2 \end{pmatrix} \in SL(2, \mathbb{C}), \quad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. $$

By standard matrix theory we can write $P^{-1}Y_0 = TH$, where $T$ is upper triangular and $H \in SU(2)$. Then (51) gives

$$F(z) = N(z)U(z)Pz^DP^{-1}Y_0 = N(z)U(z)Pz^DTH.$$
Assume that $\frac{1}{c} \leq |\alpha^2\beta| \leq c$ and $||F(z)|| \leq c$ on the unit circle, for some real number $c$. We need a uniform bound (depending only on $c$) of $U(z)$ in the unit disk. The theory of fuchsian systems gives us a bound of $U(z)$ by construction, but this bound is not uniform as $\alpha \beta \to 0$ (because we are approaching the resonant case). To obtain a uniform bound, we must use the fact that the monodromy of $F$ is in $SU(2)$.

First of all, our hypotheses imply the following bounds:

$$\frac{1}{2} \leq \Delta \leq \frac{3}{2}, \quad -\frac{3}{2} \leq \lambda_2 \leq -\frac{1}{2}, \quad |\alpha^{-1}\lambda_1| \leq c \quad \text{and} \quad \frac{1}{2c} \leq |\alpha\lambda_1| \leq \frac{3c}{2}.$$  

Hence the matrix $P$ is uniformly bounded. From (52), we obtain

$$1 = \det F(z) = z \det(U(z))z^{-1} \det T.$$  

Then $T$ is constant, and since $U(0) = I_2$, we obtain $\det U(z) = \det T = 1$. The monodromy of $F$ is given by

$$M_\gamma(F) = H^{-1}T^{-1}e^{2\pi iD}TH \quad \text{where} \quad T^{12}(O(z)) = e^{2\pi i\lambda_1}O(z).$$

Since $\lambda_1 - \lambda_2$ is not an integer, this implies that $T_{12} = 0$, so the matrix $T$ is diagonal. Then $T$ commutes. Equation (52) implies that $U(z)PT$ is uniformly bounded on the unit circle. Since $U(z)PT$ is holomorphic, it is uniformly bounded in the unit disk by the maximum principle. Taking $z = 0$, we obtain that $T$ is uniformly bounded, and hence $U(z)$ is uniformly bounded in the unit disk. Expanding the product in (52), we obtain

$$F(z) = \frac{1}{\Delta^{1/4}} \begin{pmatrix} z^{\lambda_1}(-T_{11}\lambda_2 + O(z)) & z^{\lambda_1}(T_{22}\alpha\lambda_1 + O(z)) \\ z^{1+\lambda_1}(T_{11}\alpha^{-1}\lambda_2 + O(z)) & z^{1+\lambda_2}(-T_{22}\lambda_2 + O(z)) \end{pmatrix} H,$$

where $O(z)$ is holomorphic and uniformly bounded. Using (4) and the bounds (53), we obtain

$$(x_1 + ix_2)(z) = -\frac{\alpha\lambda_1}{z}(1 + O(z) + |z|^\Delta O(1)), \quad x_3(z) = \frac{\sqrt{\Delta}}{|z|^{2+2\lambda_2}}\frac{1}{|T_{22}\lambda_2|^2}(1 + O(z) + |z|^\Delta O(1)),$$

where $O(z)$ and $O(1)$ are real analytic functions that have uniformly bounded derivatives and $O(z)$ vanishes at the origin. The conclusions of Theorem 4 follow.

\[\square\]

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