Quantization of quadratic Liénard-type equations by preserving Noether symmetries

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Abstract

The classical quantization of a family of a quadratic Liénard-type equation (Liénard II equation) is achieved by a quantization scheme (M. C. Nucci. \textit{Theor. Math. Phys.}, 168:994–1001, 2011) that preserves the Noether point symmetries of the underlying Lagrangian in order to construct the Schrödinger equation. This method straightforwardly yields the Schrödinger equation as given in (A. Ghose Choudhury and Partha Guha. \textit{J. Phys. A: Math. Theor.}, 46:165202, 2013).

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1 Introduction

In \cite{33} it was inferred that Lie symmetries should be preserved if a consistent quantization is desired. In \cite{8} [ex. 18, p. 433] an alternative Hamiltonian for the simple harmonic oscillator was presented. It is obtained by applying a nonlinear canonical transformation to the classical Hamiltonian of the harmonic oscillator. That alternative Hamiltonian was used in \cite{32} to demonstrate what nonsense the usual quantization scheme\textsuperscript{1} produce. In \cite{23} a quantization scheme that preserves the Noether symmetries was proposed and applied to Goldstein’s example in order to derive the correct Schrödinger equation. In \cite{24} the same quantization scheme was applied in order to quantize the second-order Riccati equation while in \cite{25} the quantization of Calogero’s goldfish system was achieved. In \cite{11} it was shown that this method straightforwardly yields the Schrödinger equation in the momentum space of a Liénard-type nonlinear oscillator as given in \cite{4}.

If a system of second-order equations is considered, i.e.

\[ \ddot{x}(t) = F(t, x, \dot{x}), \quad x \in \mathbb{R}^N, \quad (1) \]

that comes from a variational principle with a Lagrangian of first order, then the method that was first proposed in \cite{23} consists of the following steps:

\textsuperscript{1}Such as normal-ordering \cite{3,20} and Weyl quantization \cite{38}.
Step I. Find the Lie symmetries of the Lagrange equations

\[ \Upsilon = W(t, x) \partial_t + \sum_{k=1}^{N} W_k(t, x) \partial_{x_k} \]

Step II. Among them find the Noether symmetries

\[ \Gamma = V(t, x) \partial_t + \sum_{k=1}^{N} V_k(t, x) \partial_{x_k} \]

This may require searching for the Lagrangian yielding the maximum possible number of Noether symmetries \[30, 31, 34\].

Step III. Construct the Schrödinger equation\[2\] admitting these Noether symmetries as Lie symmetries, namely

\[
2i\psi_t + \sum_{k,j=1}^{N} f_{kj}(x) \psi_{x_j}x_{x_k} + \sum_{k=1}^{N} h_k(x) \psi_{x_k} + f_0(x)\psi = 0 \quad (2)
\]

admitting the Lie symmetries

\[
\Omega = V(t, x) \partial_t + \sum_{k=1}^{N} V_k(t, x) \partial_{x_k} + G(t, x, \psi) \partial_\psi
\]

without adding any other symmetries apart from the two symmetries that are present in any linear homogeneous partial differential equation\[3\], namely

\[
\psi \partial_\psi, \quad \alpha(t, x) \partial_\psi,
\]

where \( \alpha = \alpha(t, x) \) is any solution of the Schrödinger equation \((2)\).

If the system \((1)\) is linearizable by a point transformation, and it possesses the maximal number of admissible Lie point symmetries, namely \( N^2 + 4N + 3 \), then in \[11\] it was proven that the maximal-dimension Lie symmetry algebra of a system of \( N \) equations of second order is isomorphic to \( sl(N + 2, \mathbb{R}) \), and that the corresponding Noether symmetries generate a \( (N^2 + 3N + 6)/2 \)-dimensional Lie algebra \( g^V \) whose structure (Levi-Malčev decomposition and realization by means of a matrix algebra) was determined. It was also proven that the corresponding linear system is

\[
y''(s) + 2A_1(s) \cdot y'(s) + A_0(s) \cdot y(s) + b(s) = 0, \quad (3)
\]

with the condition

\[
A_0(s) = A'_1(s) + A_1(s)^2 + a(s)I, \quad (4)
\]

where \( A_0, A_1 \) are \( N \times N \) matrices, and \( a \) is a scalar function.

Consequently if system \((1)\) admits \( sl(N + 2, \mathbb{R}) \) as Lie symmetry algebra then in \[11\] we reformulated the algorithm that yields the Schrödinger equation as follows:

---

2We assume \( \hbar = 1 \) without loss of generality.

3In the following we will refer to those two symmetries as the homogeneity and linearity symmetries.
Step 1. Find the linearizing transformation which does not change the time, as prescribed in non-relativistic quantum mechanics.

Step 2. Derive the Lagrangian by applying the linearizing transformation to the standard Lagrangian of the corresponding linear system [3], namely that which admits the maximum number of Noether symmetries [4].

Step 3. Apply the linearizing transformation to the Schrödinger equation of the corresponding classical linear problem. This yields the Schrödinger equation corresponding to system [11].

This quantization is consistent with the classical properties of the system, namely the Lie symmetries of the obtained Schrödinger equation correspond to the Noether symmetries admitted by the Lagrangian of system [11].

In [5] the quantization of the following quadratic Liénard-type equation (also called Liénard II equation):
\[ \ddot{x} + f(x) \dot{x}^2 + g(x) = 0, \]  
(5)
where \( f(x), g(x) \) are arbitrary smooth functions of \( x \), was tackled, and the method of the Jacobi Last Multiplier [12, 39] was applied in order to find a Lagrangian without acknowledging that it was already known [34] as detailed in the following Remark.

Remark 1: The Liénard II equation (5) is a special case of the following class of second-order ordinary differential equations:
\[ \ddot{x} + \frac{1}{2} \frac{\partial \phi(t, x)}{\partial x} x^2 + \frac{\partial \phi(t, x)}{\partial t} \dot{x} + B(t, x) = 0. \]  
(6)
that was studied by Euler [6] [Sect. I, Ch. VI, §§915 ff.]. In [12], Jacobi found that the last multiplier is:
\[ M = e^{\phi(t,x)}. \]  
(7)
In [31] the corresponding Lagrangian was found to be:
\[ L_{EJ} = \frac{e^{\phi(t,x)}}{2} \dot{x}^2 + \int^x e^{\phi(t,\xi)} B(t, \xi) d\xi. \]  
(8)
Consequently a Lagrangian for the Liénard II equation (5) is given by:
\[ L_{L-II} = \frac{\dot{x}^2}{2} e^2 f^e f(\xi) d\xi + \int^x g(\xi) e^2 f^e f(\eta) d\eta d\xi. \]  
(9)

In [35] the Lie point symmetry classification of Liénard II equation (5) was performed. In particular it was shown that the following family of equations
\[ \ddot{x} + \frac{h''}{h} \dot{x}^2 + \lambda \frac{h}{h'} + \frac{A}{h^2 h'} = 0, \]  
(10)

4In [10] it was shown that any diffeomorphism between two systems of second-order differential equations takes Noether symmetries into Noether symmetries, and therefore the Lagrangian is unique up to a diffeomorphism.

5With the substitutions \( \partial \phi / \partial t = 0, \partial \phi / \partial x = 2 f(x) \) and \( B(t, x) = g(x) \).

6With \( \phi(t, x) \) and \( B(t, x) \) arbitrary functions of \( t \) and \( x \).
where \( \lambda, A \in \mathbb{R} \) are arbitrary constants and \( h(x) \) is an arbitrary smooth function of \( x \), admits a three-dimensional Lie symmetry algebra \( \mathfrak{sl}(2, \mathbb{R}) \) if \( A \neq 0 \), and an eight-dimensional Lie symmetry algebra \( \mathfrak{sl}(3, \mathbb{R}) \) if \( A = 0 \). In [35] it was also proven that if \( \lambda = \omega^2 > 0 \) then (10) becomes:

\[
\ddot{x} + \frac{h''}{h'} x^2 + \omega^2 \frac{h}{h'} \dot{x} + \frac{A}{h'^3} = 0. \tag{11}
\]

that is isochronous\(^7\) since the point transformation:

\[
\xi = h(x) \tag{12}
\]

transforms (11) into the isotonic oscillator

\[
\ddot{\xi} + \omega^2 \dot{\xi} + \frac{A}{\xi^3} = 0. \tag{13}
\]

If \( A = 0 \) then equation (11) is transformed by means of (12) into an harmonic oscillator with frequency \( \omega \).

Equation (11) with \( A = 0 \) admits an eight-dimensional Lie symmetry algebra generated by the following operators\(^8\)

\[
\begin{align*}
\Gamma_1 &= \partial_t, \\
\Gamma_2 &= \cos(2\omega t)\partial_t - \omega \frac{h}{h'} \sin(2\omega t)\partial_x, \\
\Gamma_3 &= \sin(2\omega t)\partial_t + \frac{h}{h'} \cos(2\omega t)\partial_x, \\
\Gamma_4 &= \frac{h}{\omega^2} \cos(\omega t)\partial_t - \frac{h^2}{\omega h'} \sin(\omega t)\partial_x, \\
\Gamma_5 &= \frac{h}{\omega^2} \sin(\omega t)\partial_t + \frac{h^2}{\omega h'} \cos(\omega t)\partial_x, \\
\Gamma_6 &= \frac{h}{h'} \partial_x, \\
\Gamma_7 &= \frac{\omega^2}{h'} \sin(\omega t)\partial_x, \\
\Gamma_8 &= \frac{\omega^2}{h'} \cos(\omega t)\partial_x,
\end{align*}
\tag{14}
\]

while equation (11) with \( A \neq 0 \) admits a three-dimensional Lie symmetry algebra admitted generated by \( \Gamma_1, \Gamma_2, \Gamma_3 \), i.e.:

\[
\begin{align*}
\Gamma_1 &= \partial_t, \\
\Gamma_2 &= \cos(2\omega t)\partial_t - \omega \frac{h}{h'} \sin(2\omega t)\partial_x, \\
\Gamma_3 &= \sin(2\omega t)\partial_t + \frac{h}{h'} \cos(2\omega t)\partial_x, \\
\Gamma_4 &= \frac{h}{\omega^2} \cos(\omega t)\partial_t - \frac{h^2}{\omega h'} \sin(\omega t)\partial_x, \\
\Gamma_5 &= \frac{h}{\omega^2} \sin(\omega t)\partial_t + \frac{h^2}{\omega h'} \cos(\omega t)\partial_x.
\end{align*}
\]

\(^7\) We do not consider equation (11) with \( \lambda \leq 0 \), since the isochronous property of (11) would have been lost.

\(^8\) We remark that in the case \( A = 0 \) the linearizing transformation is obtained by means of the canonical representation of a two-dimensional abelian intransitive subalgebra [19]. One such subalgebra is that generated by \( \Gamma_7 \) and \( \Gamma_8 \) and therefore the transformation

\[
\begin{align*}
\hat{t} &= \tan(\omega t), \\
\hat{x} &= \frac{h}{\cos(\omega t)}
\end{align*}
\]

takes equation (11) into the free particle, while transformation (12) takes equation (11) into the harmonic oscillator with frequency \( \omega \).
\[ \Gamma_2 = \cos(2\omega t) \partial_t - \omega \frac{h}{h'} \sin(2\omega t) \partial_x, \quad (15) \]
\[ \Gamma_3 = \sin(2\omega t) \partial_t + \omega \frac{h}{h'} \cos(2\omega t) \partial_x. \]

The three symmetries \((16)\) are a representation of the complete symmetry group of equation \((11)\), namely a group that completely specifies a given differential equation through its algebraic representation \([13]\). Indeed if we impose to the following general second-order ordinary differential equation
\[ \ddot{x}(t) = F(t, x, \dot{x}) \quad (16) \]
to admit the symmetry algebra with generators \((16)\), then we obtain equation \((11)\), a family of equations characterized by the parameter \(A\).

We now show that equation \((11)\) hides linearity. If we solve equation \((11)\) with respect to \(A\) and derive once with respect to \(t\), then we obtain the following third-order equation
\[ \dddot{x} = -3 \left( \frac{h''}{h} + \frac{h'}{h} \right) \dot{x} \ddot{x} - \left( \frac{h'''}{h'} + 3 \frac{h''}{h} \right) \dot{x}^3 - 4\omega^2 \dot{x}, \quad (17) \]
that admits a seven-dimensional Lie symmetry algebra \([13]\) and therefore is linearizable \([19]\). The linearizing transformation, that is obtained by means of the canonical representation of a two-dimensional abelian intransitive subalgebra \([11][19]\), is
\[ \tilde{t} = \tan(2\omega t), \quad \tilde{x} = \frac{h^2}{2 \cos(2\omega t)}, \quad (18) \]
that yields the following linear equation
\[ \frac{d^3\tilde{x}}{d\tilde{t}^3} = - \frac{3\tilde{t}}{1 + \tilde{t}^2} \frac{d^2\tilde{x}}{d\tilde{t}^2}. \quad (19) \]
In particular the transformation \(u = \frac{h^2}{2}\) yields
\[ \ddot{u} = -4\omega^2 u, \quad (20) \]

namely the once-derived linear harmonic oscillator with frequency \(2\omega\).

In this paper we apply the quantization algorithm that preserves the Noether symmetries to equation \((11)\) in the case \(A = 0\) and in the case \(A \neq 0\), and compare our findings with those in \([5]\). We also determine the eigenvalues and the eigenfunctions of the obtained Schrödinger equation using its Lie symmetries i.e. the method developed in a series of papers \([16,17,28,29]\).

\(^9\)This method is described in \([15]\).

\(^{10}\)The seven generators are \(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_6\) in \([15]\) and
\[ X_1 = \frac{1}{h h'} \partial_x, \quad X_2 = \frac{1}{h h'} \cos(2\omega t) \partial_x, \quad X_3 = \frac{1}{h h'} \sin(2\omega t) \partial_x. \]

\(^{11}\)Namely that generated by \(X_2\) and \(X_3\).
2 Quantization of the Liénard II equation (11)

We quantize equation (11) by considering first the case $A = 0$, i.e.:

$$\ddot{x} + \frac{h''}{h'} \dot{x}^2 + \omega^2 \frac{h}{h'} = 0 \quad (21)$$

and then the case $A \neq 0$.

2.1 Equation (21)

As shown in [35] equation (21) is linearizable, therefore in order to quantize it we follow the three Steps [11] as recalled in the Introduction.

Step 1. The transformation that takes equation (21) into the harmonic oscillator with frequency $\omega$ is [12], [35].

Step 2. The Lagrangian (9) corresponding to equation (21) is

$$L_0 = \frac{1}{2} (\dot{h}')^2 - \frac{1}{2} \omega^2 h^2.$$

(22)

It admits five Noether point symmetries, namely $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_7$ and $\Gamma_8$ in (15).

Remark 2: The Jacobi Last Multiplier (7) in the case of equation (21) becomes

$$M = (h')^2.$$

(23)

Since there is a link between Jacobi Last Multiplier and Lie symmetries [18], [22] then it is interesting to underline that the Jacobi Last Multipliers (23) can be obtained by means of the two symmetries $\Gamma_7$ and $\Gamma_8$, that generate an intransitive two-dimensional abelian subalgebra yielding the linear transformation (12). In fact the Jacobi Last Multiplier (23) is the reciprocal of the determinant:

$$\Delta_{78} = \det \begin{bmatrix} 1 & \dot{x} & -\frac{h''}{h'} \dot{x}^2 - \frac{\omega^2 h}{h'} & \frac{\omega^2 \cos(\omega t) \dot{h}' - \sin(\omega t) \dot{x} h''}{(h')^2} \\ 0 & \frac{\omega^2 \sin(\omega t)}{h'} & \frac{\omega^2 \cos(\omega t) \dot{h}' - \sin(\omega t) \dot{x} h''}{(h')^2} \\ 0 & \frac{\omega^2 \cos(\omega t)}{h'} & -\frac{\omega^2 \sin(\omega t) \dot{h}' - \cos(\omega t) \dot{x} h''}{(h')^2} \end{bmatrix} = -\frac{\omega^5}{(h')^2}.$$

(24)

apart from the unessential constant factor $-\omega^5$.

Step 3. The Schrödinger equation of the linear harmonic oscillator with frequency $\omega$ is:

$$2i \psi_t + \psi_{\xi \xi} - \omega^2 \xi^2 \psi = 0 \quad (25)$$

with $\psi = \psi(t, \xi)$. If we apply the transformation (12), then we obtain the Schrödinger equation of equation (21):

$$2i \psi_t + \frac{\psi_{xx}}{(h')^2} - \frac{h'' \psi_x}{(h')^3} - \omega^2 h^2 \psi = 0.$$

(26)
We now check the classical consistency of the Schrödinger equation (26). Using the REDUCE programs [21] we find that its Lie point symmetries are generated by the following operators:

\[
\begin{align*}
\Xi_1 &= \Gamma_1, \\
\Xi_2 &= \Gamma_2 + \frac{\omega \psi}{2} \left( \sin(2\omega t) - 2 \cos(2\omega t)h^2i\omega \right) \partial_\psi, \\
\Xi_3 &= \Gamma_3 - \frac{\omega \psi}{2} \left( \cos(2\omega t) + 2 \sin(2\omega t)h^2i\omega \right) \partial_\psi, \\
\Xi_4 &= \Gamma_7 + \cos(\omega t)h\omega \psi \partial_\psi, \\
\Xi_5 &= \Gamma_8 - \sin(\omega t)h\omega \psi \partial_\psi.
\end{align*}
\] (27)

and the two homogeneity and linearity symmetries.

In [16,17,28,29] and more recently in [11,32] it was shown how to find the eigenfunctions and the eigenvalues of the Schrödinger equation by means of its admitted Lie symmetries.

We apply this method to the Schrödinger equation (26).

Let us rewrite the Lie point symmetries (28) of equation (26) in complex form, i.e.:

\[
\begin{align*}
\Omega_1 &= i\partial_t, \\
\Omega_{2\pm} &= e^{\pm 2i\omega t} \left[ \partial_t \pm i\frac{h}{\sqrt{\hbar}} \partial_x - i \left( \omega^2 \hbar^2 \pm \frac{i}{2} \omega \right) \psi \partial_\psi \right], \\
\Omega_{3\pm} &= e^{\pm i\omega t} \left( \frac{1}{\hbar} \partial_x \mp \hbar \omega \psi \partial_\psi \right).
\end{align*}
\] (28)

The operators \(\Omega_{3\pm}\) act as creation and annihilation operators. In fact if we consider the invariant surfaces associated with these two operators:

\[
\begin{align*}
\Omega_{3\pm} F(t,x,\psi) &= 0, \\
F(t,x,\psi) &= f(t,\psi e^{\pm \frac{i}{2} \hbar^2})
\end{align*}
\] (29)

one gets

\[
\begin{align*}
\psi_{\pm} &= T_{\pm}(t)e^{\pm \frac{i}{2} \hbar^2}.
\end{align*}
\] (30)

namely the following similarity solutions of (26):

\[
\psi_0 = e^{-\frac{i}{2} \omega t - \frac{\hbar^2}{2}}.
\] (32)

Since we want a solution that goes to zero at infinity, then we have to choose the invariant surface relative to \(\Omega_{3+}\). Hence the operator \(\Omega_{3-}\) acts as a creation operator, while \(\Omega_{3+}\) as an annihilation one. Substituting \(\psi_+\) into the Schrödinger equation (26) yields \(T_+ = e^{-\frac{i}{2} \omega t}\), thus the ground state [12] is:

\[
psi_0 = e^{-\frac{i}{2} \omega t - \frac{\hbar^2}{2}}.
\] (33)

The operator \(\Omega_1\) acts as an eigenvalue operator:

\[
\begin{align*}
\Omega_1 \psi_0 &= \frac{\omega}{2} \psi_0.
\end{align*}
\] (33)

since it yields the ground state energy \(E_0 = \omega/2\), that corresponds to the quantum harmonic oscillator.

\[12\text{Apart from an unessential normalization constant.}\]
We use the creation operator $\Omega_3^-$ and the linearity operator $\Omega \psi_0$ in order to construct the higher states. Since the commutator:

$$[\Omega_3^-, \Omega \psi_0] = -2\omega \hbar e^{-\frac{1}{2} \omega t - \frac{i}{2} \hbar^2}. \quad (34)$$

then:

$$\psi_1 = -2\omega \hbar e^{-\frac{1}{2} \omega t - \frac{i}{2} \hbar^2}. \quad (35)$$

is another solution of (26) which satisfies the proper boundary conditions and have eigenvalue:

$$\Omega_1 \psi_1 = \frac{3\omega}{2} \psi_0, \quad (36)$$

i.e. $E_1 = 3\omega/2 > E_0$.

If we evaluate the commutator:

$$[\Omega_3^+, \Omega \psi_1] = -2\omega \Omega \psi_0, \quad (37)$$

we have a multiple of the ground state (32).

Iterating this process yields all the eigenfunctions, i.e.:

$$[\Omega_3^-, \Omega \psi_{n-1}] = \psi_n \partial_\psi = \Omega \psi_n. \quad (38)$$

The generic eigenvalue and eigenfunction can be derived in the following manner. We evaluate the commutator between $\Omega_3^-$ and $\Omega \chi$, where $\chi$ is a generic solution of (26) i.e.

$$[\Omega_3^-, \Omega \chi] = e^{-i\omega t} \left( \frac{1}{\hbar^2} \partial_x - \omega \hbar \right) \chi \partial_\psi, \quad (39)$$

and then we generate the $n$th eigenfunction by using the iteration procedure (38), i.e.: 

$$\psi_n = e^{-i\omega t} \left( \frac{1}{\hbar^2} \partial_x - \omega \hbar \right) \psi_{n-1}$$

$$= e^{-2i\omega t} \left( \frac{1}{\hbar^2} \partial_x - \omega \hbar \right)^2 \psi_{n-2}$$

$$= \cdots$$

$$= e^{-ni\omega t} \left( \frac{1}{\hbar^2} \partial_x - \omega \hbar \right)^n \psi_0$$

$$= e^{-\left(n+\frac{1}{2}\right)i\omega t} \left( \frac{1}{\hbar^2} \partial_x - \omega \hbar \right)^n \left( e^{-\frac{1}{2} \omega \hbar^2} \right)$$

$$= (-1)^n \omega \frac{n}{2} e^{-\left(n+\frac{1}{2}\right)i\omega t} H_n(\sqrt{\omega} \hbar) e^{-\frac{1}{2} \omega \hbar^2}.$$

where $H_n$ is the $n$-th Hermite polynomial [1]. This trivially gives the $n$-th eigenvalue:

$$\Omega_1 \psi_n = \omega \left( n + \frac{1}{2} \right) \psi_n. \quad (41)$$

Thus we have proven that the spectrum of the Schrödinger equation (26) is equal to that of the quantum harmonic oscillator of frequency $\omega$.

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13Namely $\Omega \chi = \chi \partial_\psi$ with $\chi$ any solution of (26).
2.2 Equation (11)

Equation (11) is not linearizable by means of a point transformation. In fact it admits just three symmetries (16), that generates an algebra \( sl(2, \mathbb{R}) \), therefore we cannot use the three Steps 1,2,3 as in the case \( A = 0 \). Instead we have to use the more general method proposed in [23] and recalled in the Introduction, namely Steps I,II,III, that is to find a Lagrangian admitting those three symmetries as Noether symmetries, and then use them to quantize equation (11).

**Step I.** The three-dimensional Lie symmetry algebra of equation (11) is generated by the operators (16).

**Step II.** The Lagrangian (9) corresponding to equation (11) is:

\[
L_A = \frac{(h')^2}{2} x^2 + \frac{A}{2h^2} - \frac{\omega^2}{2} h^2. \tag{42}
\]

It admits the three symmetries (16) as Noether symmetries.

**Step III.** We now construct the Schrödinger equation admitting the Noether symmetries (16) as Lie symmetries. We begin with equation (2), i.e.:

\[
2i\psi_t + f_{11}(x)\psi_{xx} + h_1(x)\psi_x + f_0(x)\psi = 0, \tag{43}
\]

and thus we obtain that the Schrödinger equation corresponding to the equation (11) is\(^{14}\)

\[
2i\psi_t + \frac{\psi_{xx}}{(h')^2} - \frac{h''\psi_x}{(h')^3} + \left( \frac{A}{h^2} - \frac{\omega^2}{h^2} \right) \psi = 0, \tag{44}
\]

and its Lie symmetries are generated by the operators \( \Xi_1, \Xi_2, \Xi_3 \) in (28) and the usual homogeneity and linearity symmetries.

As before, the eigenfunctions and the eigenvalues of the Schrödinger equation (44) can be derived by means of its admitted Lie symmetries. However here the operators \( \Omega_{2\pm} \) in (29) act as creation and annihilation operators. Since the procedure is the same we do not write down all the details. We have obtained that the eigenfunctions are:

\[
\psi_n = h^{\frac{k+1}{2}} e^{-\frac{i\omega t}{2}} e^{-\frac{\omega t}{2} h^2} L_n^{k/2} \left( \omega h^2 \right), \tag{45}
\]

with \( k = \sqrt{1-4A} \), and \( L_n^{k/2} \) the associated Laguerre polynomials, while the energy eigenvalues are:

\[
E_n = 2\omega \left( n + 1 + \frac{k}{4} \right). \tag{46}
\]

It is not a surprise that the energy eigenvalues (46) are exactly those of the quantum isotonic oscillator [7]. We also underline that they correspond to those of the quantum harmonic oscillator with frequency \( 2\omega \).

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\(^{14}\)Introducing \( \hbar \) into (44), i.e.

\[
2i\hbar\psi_t + h^2 \frac{\psi_{xx}}{(h')^2} - h^2 \frac{h''\psi_x}{(h')^3} + \left( \frac{A}{h^2} - \frac{\omega^2}{h^2} \right) \psi = 0,
\]

and performing the classical limit \([14]\) one indeed obtains equation (11).
Comparison with the quantization in [5]

In [5] the von Roos’ ordering method [36,37] for position dependent masses was applied to (5) and the following Schrödinger equation was obtained:

\[ 2i\dot{\phi} + e^2\int f(\xi) d\xi \left\{ \phi_{xx} - 2f\phi_x + \left[ (\beta + 1)(2f^2 - f') + 4\alpha(\alpha + \beta + 1)f^2 \right] \phi \right\} - V\phi, \quad (47) \]

with

\[ V = \int^x e^2 \int f(\eta) d\eta g(\xi) d\xi, \quad (48) \]

and such that the parameters \( \alpha \) and \( \beta \) are related to another parameter \( \gamma \) such that \( \alpha + \beta + \gamma = -1 \).

Then the authors factorized the wave function \( \phi \), i.e.:

\[ \phi(t, x) = e^{-iEt} w(x) G(u(x)), \quad (49) \]

with the function \( G \) being the solution of a generic second-order linear differential equation:

\[ \frac{d^2 G}{du^2} + Q(u) \frac{dG}{du} + R(u) G(u) = 0. \quad (50) \]

In [5], different differential equations were taken into consideration, in particular the Hermite and the associated Laguerre equation.

If equation (50) is the Hermite differential equation, then in [5] it was derived that \( \alpha = \gamma - 1/4, \beta = -1/2 \), and the potential \( V \) had to be:

\[ V = \frac{1}{2} \left( \int^x e^2 \int f(\eta) d\eta d\xi \right)^2. \quad (51) \]

We substitute \( f = h''/h' \) into equation (47) and find to yield the Schrödinger equation (26) that we have derived with \( \omega = 1 \) and \( \phi = \sqrt{h'}\psi \).

We underline that the Schrödinger equation (17) with the potential \( V \) given in (51) admits the symmetries \( \Xi_i, i = 1, \ldots, 5 \) in (28) if and only if \( \alpha = \gamma - 1/4, \beta = -1/2 \).

If equation (50) is the associated Laguerre equation, then in [5] it was derived that \( \alpha = \gamma - 1/4, \beta = -1/2 \), and the potential \( V \) had to be:

\[ V = \frac{1}{8} \left( \int^x e^2 \int f(\eta) d\eta d\xi \right)^2 + \frac{1}{2} \left[ \frac{3}{4} - (1 + \mu)(1 - \mu) \right] \left( \int^x e^2 \int f(\eta) d\eta d\xi \right)^2. \quad (52) \]

We substitute \( f = h''/h' \) into equation (17) and find to yield the Schrödinger equation (44) that we have derived with \( \omega = 1/2 \) and \( \phi = \sqrt{h'}\psi \).

\[ ^{15} \text{In [5] the time-independent Schrödinger equation was derived.} \]
4 Final remarks

A new algorithm for quantization that requires the preservation of Noether symmetries in the passage from classical to quantum mechanics\(^\text{16}\) has been recently introduced and applied to both one-dimensional and two-dimensional Lagrangian equations \([11, 23–27]\).

In this paper we have applied this new method to the quadratic Liénard-type equation \((\text{11})\), both in the linearizable and non-linearizable case, and compared our results with that determined in \([5]\). We have found that the Schrödinger equations obtained in \([5]\) can be determined by means of the quantization that preserves the Noether symmetries.

Even in quantum mechanics whenever differential equations are involved, Lie and Noether symmetries have a fundamental role: Noether symmetries yield the correct Schrödinger equation and its Lie symmetries can be algorithmically used to find the eigenvalues and eigenfunctions.

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\(^{16}\)Namely, the derived Schrödinger equation is such that the independent-variables part of its admitted Lie symmetries corresponds to the Noether symmetries of the classical equations.
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