Calculation of the strength of water management facilities for the sustainable development of rural areas

N A Gureeva¹, R Z Kiseleva², Yu V Klochkov² and A P Nikolaev²

¹ Federal State Educational Budgetary Institution of Higher Education “Financial University under the Government of the Russian Federation”, 49 Leningradsky Avenue, Moscow, 125993 (City service post office-3), Russia
² Federal State Educational Institution of Higher Education “Volgograd State Agrarian University” 26 University Avenue, Volgograd, 400002, Russia

E-mail: nagureeve@fa.ru

Abstract. The physical relations are presented at the loading step of a bulk body in two variants. In the first version, the theory of plastic flow is used. In the second variant, to determine the increments of plastic deformations, a hypothesis is proposed about the proportionality of the increments of plastic deformations to the components of the deviator of stress increments, which made it more correct and easier to obtain the defining equations at the loading step. Variants of physically nonlinear defining equations in the calculation of three-dimensional elements of water management objects are implemented in the developed tetrahedral finite element with nodal unknowns in the form of displacement increments and stress increments.

1. Introduction

When determining the stress-strain state of engineering objects, the finite element method is widely used [1, 2, 3, 4, 5].

The calculation of the elements of water management facilities for strength, taking into account the physical nonlinearity, is relevant and is carried out using the theory of plasticity – deformation or the theory of plastic flow [6, 7, 8, 9]. In the step-by-step loading process, the flow theory is particularly widely used, in which, at the loading step, the increments of deformations are divided into elastic and plastic parts. Elastic deformations are determined by Hooke’s law. The increments of plastic deformations are determined on the basis of the hypothesis of proportionality of the components of the tensor of the increments of plastic deformations and the components of the deviator of total stresses. The proportionality coefficient in the implementation of this hypothesis turns out to be a function of the increment of the stress intensity, represented by the differential of the components of the stress increment tensor.

By summing the components of the increments of elastic and plastic deformations, the defining equations of the flow theory at the loading step are obtained.

In this paper, a variant of the flow theory for plastic strain increments is developed, based on the proposed hypothesis of the proportionality of the components of the tensor of plastic strain increments and the components of the deviator of stress increments. The defining equations of the proposed version of the flow theory turned out to be more correct. To implement physical nonlinearity, a tetrahedral finite element in a mixed formulation with nodal unknowns in the form of displacement increments and stress...
increments is developed.

2. Materials and methods

2.1. Variants of the defining equations of the theory of plastic flow

In the first variant, when constructing the defining relations of the theory of plastic flow at the loading step, the hypothesis of the separation of the increments of deformations into elastic and plastic parts is used [9]. The increments of elastic deformations \( \Delta e^e_{ij} \) are determined by the equations of Hooke’s law [10]:

\[
\Delta e^e_{11} = \frac{1}{E} (\Delta \sigma_{11} - \nu \Delta \sigma_{22} - \nu \Delta \sigma_{33});
\]
\[
\Delta e^e_{22} = \frac{1}{E} (\Delta \sigma_{22} - \nu \Delta \sigma_{11} - \nu \Delta \sigma_{33});
\]
\[
\Delta e^e_{33} = \frac{1}{E} (\Delta \sigma_{33} - \nu \Delta \sigma_{11} - \nu \Delta \sigma_{22});
\]

\[
\Delta e^e_{12} = \frac{1+\nu}{E} \Delta \sigma_{12}; \quad \Delta e^e_{13} = \frac{1+\nu}{E} \Delta \sigma_{13}; \quad \Delta e^e_{23} = \frac{1+\nu}{E} \Delta \sigma_{23},
\]

where \( \Delta \sigma_{ij} \) is the increments of normal stresses; \( E \) is the elastic modulus of the material; \( \nu \) - coefficient of transverse deformation.

To determine the increments of plastic deformations, the hypothesis of proportionality of the components of the increments of plastic deformations to the components of the total stress deviator is used, on the basis of which the relations are written in the form [9]:

\[
\Delta e^p_{ij} = \frac{3}{2} \frac{\Delta e^p_i}{\sigma_i} \left( \sigma_{ij} - \frac{1}{3} \delta_{ij} \sigma_c \right),
\]

where \( \Delta e^p_i \) is the increment of the intensity of plastic deformations; \( \sigma_{ij} \) - the total stresses achieved during the previous loading steps; \( \sigma_c = \sigma_{11} + \sigma_{22} + \sigma_{33} \) - the average stress; \( \delta_{ij} \) - Kronecker symbol; \( \sigma_i \) - stress intensity.

The value \( \Delta e^p_i \) is determined by the expression [9]:

\[
\Delta e^p_i = \frac{\Delta e^e_i - \Delta e^e_f}{E_K - E_1} = \Delta \sigma_i \left( \frac{1}{E_K} - \frac{1}{E_1} \right)
\]

where \( \Delta e^e_i \) is the increment of the intensity of the total deformations; \( \Delta e^e_f \) - the increment of the intensity of elastic deformations; \( E_K \) - the tangent modulus of the deformation diagram; \( E_1 \) - the modulus of the initial section of the deformation diagram; \( \Delta \sigma_i \) - the increment of the intensity of total stresses.

The value \( \Delta \sigma_i \) is defined in this general form:

\[
\Delta \sigma_i = \frac{\partial \sigma_i}{\partial \sigma_{11}} \Delta \sigma_{11} + \frac{\partial \sigma_i}{\partial \sigma_{22}} \Delta \sigma_{22} + \frac{\partial \sigma_i}{\partial \sigma_{33}} \Delta \sigma_{33} + \frac{\partial \sigma_i}{\partial \sigma_{12}} \Delta \sigma_{12} +
\]
\[
+ \frac{\partial \sigma_i}{\partial \sigma_{13}} \Delta \sigma_{13} + \frac{\partial \sigma_i}{\partial \sigma_{23}} \Delta \sigma_{23}.
\]

Taking into account (3) and (4), the increments of plastic deformations (2) are written as:
\[
\Delta e_{ij}^p = \frac{3}{2} \left( \frac{\sigma_{ij} - \frac{1}{3} \delta_{ij} \sigma_c}{\sigma_i} \right) \left( \frac{1}{E_K} - \frac{1}{E_1} \right) \frac{\partial \sigma_i}{\partial \sigma_{mn}} \Delta \sigma_{mn}.
\]

By summing (5) and (1), the matrix relation is formed:

\[
\{\Delta e\} = \left[ C_{11}^{\Pi} \right] \{\Delta \sigma\}, \quad \text{6x1} \quad \text{6x1}
\]

where \(\{\Delta e\}^T = \{\Delta \sigma_1 \Delta \sigma_2 \Delta \sigma_3 \Delta \sigma_{12} \Delta \sigma_{13} \Delta \sigma_{23}\}; \quad \text{6x6}

\[
\{\Delta \sigma\}^T = \{\Delta \sigma_1 \Delta \sigma_2 \Delta \sigma_3 \Delta \sigma_{12} \Delta \sigma_{13} \Delta \sigma_{23}\};
\]

where \(C_{11}^{\Pi}\), is the elastic-plasticity matrix of the first variant.

In the second version of the defining equations for determining the increments of plastic deformations, a hypothesis is proposed about the proportionality of the increments of plastic deformations to the components of the deviator of the stress increments:

\[
\Delta e_{ij}^p = \psi_1 \left( \Delta \sigma_{ij} - \frac{1}{3} \delta_{ij} \Delta \sigma_c \right),
\]

where the proportionality coefficient is determined \(\psi_1\) by the expression:

\[
\psi_1 = \frac{3}{2} \frac{\Delta e_i^p}{\Delta \sigma_i}.
\]

Taking into account (8) and (3) the relations (7) will take the form:

\[
\Delta e_{ij}^p = \frac{3}{2} \left( \frac{1}{E_K} - \frac{1}{E_1} \right) \left( \Delta \sigma_{ij} - \frac{1}{3} \delta_{ij} \Delta \sigma_c \right).
\]

By summing (9) and (1), the matrix relation of the second variant of the relationship between strain increments and stress increments is formed:

\[
\{\Delta e\} = \left[ C_{12}^{\Pi} \right] \{\Delta \sigma\}, \quad \text{6x1} \quad \text{6x1}
\]

2.2. Displacements and deformations

Under the action of the load an arbitrary point of the body at the loading step receives an displacement, which is determined by the vector:

\[
\vec{W} = w_1 \vec{i} + w_2 \vec{j} + w_3 \vec{k}.
\]

The increments of deformations at the point under consideration are determined by the Cauchy relations:

\[
\Delta e_{11} = \frac{\partial w_1}{\partial x}, \quad \Delta e_{22} = \frac{\partial w_2}{\partial y}, \quad \Delta e_{33} = \frac{\partial w_3}{\partial z};
\]

\[
\Delta e_{12} = \frac{1}{2} \left( \frac{\partial w_1}{\partial y} + \frac{\partial w_2}{\partial x} \right), \quad \Delta e_{13} = \frac{1}{2} \left( \frac{\partial w_1}{\partial z} + \frac{\partial w_3}{\partial x} \right), \quad \Delta e_{23} = \frac{1}{2} \left( \frac{\partial w_2}{\partial z} + \frac{\partial w_3}{\partial y} \right),
\]

which are represented in matrix form:

\[
\{\Delta e\} = \left[ L \right] \{v\}, \quad \text{6x1} \quad \text{6x3}
\]

where \(\{v\}^T = \{w_1 \ w_2 \ w_3\}; \quad \left[ L \right] - \text{matrix of differential operators.}
3. Results and discussion

An arbitrary tetrahedron with nodes i, j, k, l is accepted as a finite element in the cartesian coordinate system. To perform numerical integration, an arbitrary tetrahedron is mapped to a local rectangular tetrahedron, the local coordinates of which vary within \(0 \leq \xi, \eta, \zeta \leq 1\).

The cartesian coordinates of the inner point of the tetrahedron are determined through their nodal values using linear relations:

\[
\lambda = \xi \lambda^i + \eta \lambda^j + \zeta \lambda^k + \left(1 - \xi - \eta - \zeta\right) \lambda^l = \phi(\xi, \eta, \zeta) \vec{v},
\]

(14)

where the symbol \(\lambda\) refers to the cartesian coordinates \(x, y,\) and \(z\), and the symbol \(\{\lambda_y\}\) - refers to the column of values of the coordinates of the nodes of the tetrahedron.

By differentiating (14) the derivatives of cartesian coordinates \(x, \xi, x, \eta, x, \zeta, y, \eta, \zeta, z, \eta, \zeta\) in the local coordinate system and the derivatives of local coordinates in cartesian coordinates are determined \(\xi_x, \xi_y, \xi_z, \eta_x, \eta_y, \eta_z, \zeta_x, \zeta_y, \zeta_z\). The components of the displacement vector of the inner point of the finite element \(\{w_i\}\) are approximated by their nodal values also by expressions (14):

\[
w_1 = \phi^T \{w_1\}, \quad w_2 = \phi^T \{w_2\}, \quad w_3 = \phi^T \{w_3\},
\]

(15)

On the basis of (15), a matrix relation is formed:

\[
\{w\} = [A] \{\mu_y\},
\]

(16)

where \(\{\mu_y\}^T = \{w_1\}^T \{w_2\}^T \{w_3\}^T\) - a string of nodal unknown displacements.

Taking into account (16), the relation (13) for the increments of deformations gets the expression:

\[
\{\Delta \varepsilon\} = [L] [A] \{\mu_y\} = [B] \{\mu_y\},
\]

(17)

The approximation of the individual components of the stress tensor was also performed by the relations (14). The string of nodal unknown stresses is taken as:

\[
\{\Delta \sigma_y\}^T = \{\Delta \sigma_{xx} \Delta \sigma_{yx} \Delta \sigma_{yy} \Delta \sigma_{yy} \Delta \sigma_{yz} \Delta \sigma_{yz} \Delta \sigma_{xz} \Delta \sigma_{xz}\}.\]

(18)

For the increments of the stresses of the inner point of the finite element, a matrix expression is formed:

\[
\{\Delta \sigma\} = [S] \{\Delta \sigma_y\},
\]

(19)

where \(\{\Delta \sigma\}^T = \{\Delta \sigma_{xx} \Delta \sigma_{yy} \Delta \sigma_{yy} \Delta \sigma_{xy} \Delta \sigma_{xz} \Delta \sigma_{yz}\}.\)

To form the matrix of elastic-plastic deformation of the tetrahedron at the loading step, a mixed functional in the form of [5]:

\[
\phi = \int_{V} \left[ \{\sigma\}^T \{\Delta \sigma\} \right] \{\Delta \varepsilon\} dV - \frac{1}{2} \int_{V} \left[ \{\Delta \sigma\}^T \{C\} \{\Delta \sigma\} \right] dV - \int_{w} \left[ \{q\}^T \{\mu_y\} \right] dW + \frac{1}{2} \int_{V} \{\Delta \varepsilon\} \{\Delta \sigma\} dV,
\]

(19)

where \(\{\sigma\}\) - is the stress column for the previous loading steps; \(\rho = 1,2\) - for the plasticity matrix by (6) or by (10); \(\{q\}\) - the load for the previous loading steps; \(\{\Delta \sigma\}\) - the load at the considered loading step; \(w\) - the area of application of a given load.

Taking into account (16), (17) and (18), the functional (19) can be represented as:
\[
\Phi = \int \{\sigma\}^T [B] dV \{\mu_y\} + \frac{1}{2} \{\Delta \sigma_y\}^T \int [S]^T [B] dV \{\mu_y\} - 
\frac{1}{2} \{\Delta \sigma_y\}^T \int [S] \int [C^H_{\rho} [S] dV \{\mu_y\}^T - \frac{1}{2} \{\mu_y\}^T \int [A]^T \{\varphi\} dV - \frac{1}{2} \{\mu_y\}^T \int [A]^T \{\varphi\} dV; 
\]

\[(20)\]

After minimizing the functional (20) with respect to the nodal unknowns, \(\{\Delta \sigma_y\}^T\) a \(\{\mu_y\}^T\) system of equations is obtained:

\[
\frac{\partial \Phi}{\partial \{\Delta \sigma_y\}} = \left[ Q \right] \{\mu_y\} - \left[ H \right] \{\Delta \sigma_y\} = 0; 
\]

\[(21)\]

\[
\frac{\partial \Phi}{\partial \{\mu_y\}} = \left[ Q \right]^T \{\Delta \sigma_y\} - \left\{\Delta f_{\sigma}\right\} + \left\{ f_{\sigma}\right\} = 0, 
\]

\[(22)\]

where \(\left[ Q \right] = \int [S]^T [B] dV; \quad \left[ H \right] = - \int [S]^T \left[ C^H_{\rho} [S] dV; \quad \left\{ f_{\sigma}\right\} = \int [A]^T \{\varphi\} dV; \quad \left\{ f_{\Delta \sigma}\right\} = \int [A]^T \{\varphi\} dV. 
\]

\[12 \times 1 \]

\[6 \times 1 \]

\[6 \times 1 \]

Systems (21) and (22) are represented in the finite element form:

\[
\begin{bmatrix} \left[ R \right] \end{bmatrix} \begin{bmatrix} \left[ Z_y\right] \end{bmatrix} = \begin{bmatrix} \left[ F_y\right] \end{bmatrix}, 
\]

\[(23)\]

where \(\begin{bmatrix} \left[ R \right] \end{bmatrix} = \begin{bmatrix} \left[ H \right] & \left[ Q \right] & \left[ 0 \right] \end{bmatrix} \begin{bmatrix} \left[ Q \right] & 0 \end{bmatrix} \begin{bmatrix} H \end{bmatrix} - is the matrix of elastic-plastic deformation of the tetrahedral element;

\[
\begin{bmatrix} \{ \Delta \sigma_y\} \end{bmatrix}^T \begin{bmatrix} \{ \mu_y\} \end{bmatrix}^T \begin{bmatrix} \{ \mu_y\} \end{bmatrix}^T - a string of nodal unknowns of the finite element;
\]

\[
\begin{bmatrix} \{ F_y\} \end{bmatrix}^T = \begin{bmatrix} \{ \Delta \sigma_y\} \end{bmatrix}^T - \begin{bmatrix} \{ f_{\sigma}\} \end{bmatrix}^T = \begin{bmatrix} \{ f_{\Delta \sigma}\} \end{bmatrix}^T - \begin{bmatrix} \{ R\} \end{bmatrix} \begin{bmatrix} \{ f_{\sigma}\} \end{bmatrix}^T - the Newton Raphson discrepancy.
\]

4. Conclusion

In the first version of the theory of plastic flow, the stress increments appeared on the basis of the differential representation of the stress intensity increment.

The developed variant of obtaining the defining equations (10) differs in that in it the stress increments appear directly on the basis of the proposed hypothesis.

The defining equations of the second option are more correct.

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