Electrovacuum solutions of non-local actions

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Abstract

We consider electrovacuum solutions resulting from an action comprising of Einstein-Hilbert in addition to the non-local term $R\Box^{-2}R$ associated with a mass scale $m$. Applying the Kaluza ansatz to the five-dimensional pure gravitational action, we derive the four-dimensional Einstein-Maxwell action with additional non-local terms involving both electromagnetic and gravitational fields. A recursive algorithm is provided to derive the solution to arbitrary order in $m^2$ about any known solution of General Relativity. We then derive the leading order correction to the metric and electromagnetic potential about the Reissner-Nordström background. In contrast with the case where the Maxwell action is minimally coupled, the solutions of the equations of motion from our Kaluza reduced action are a natural generalization of known solutions about the Schwarzschild background. We discuss possible implications of the non-local terms derived in our effective action on the near horizon coupling of fields on black hole backgrounds.

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I. INTRODUCTION

Experimental observations pertaining to the late-time accelerated expansion of our Universe \cite{1, 2} have inspired numerous modified gravity theories aimed at providing a natural explanation. One possible modification involves the inclusion of non-local terms, which either modify Einstein’s equations or the Einstein-Hilbert action\cite{3–9}. These models were in part inspired by the non-local anomaly-induced quantum effective actions which arise in the context of the trace anomaly and have many applications on curved backgrounds \cite{10, 11}. A key property of these actions are the manifestation of infrared effects through their involvement of inverse differential operators. The phenomenologically motivated models aim to use this feature to provide an explanation for the cosmological constant as an infrared effect.

However, the construction of a consistent infrared deformation of General Relativity (GR) provides a considerable challenge. The theory is required to respect causality, and be free of ghosts \cite{12, 13}, at least up to a reasonable UV cutoff. In an attempt to overcome some of these difficulties, there have been proposals of non-local theories which introduce non-localities with respect to a mass scale, chosen to be of the order of the present value of the Hubble parameter. One of the earliest proposals in this regard was considered in \cite{14, 15}, where non-local terms were included in the Einstein field equations as a part of degravitation idea. These equations were later shown to admit a stress tensor which is in general not conserved on curved backgrounds. The first successful resolution to this was provided in \cite{6},

\[
G_{\mu\nu} - \frac{m^2}{3} (g_{\mu\nu} \Box^{-1} R) T = 8\pi G T_{\mu\nu} \tag{1.1}
\]

where the non-local term now explicitly considers the transverse component which ensures that ghosts remain non-radiative. This equation, when \(m \sim H_0\), is consistent with known cosmological observations and unlike massive gravity theories with a \(\text{vDVZ}\) discontinuity, it reproduces the results of GR when \(m \to 0\). The theory when expressed in terms of local variables has also been shown to not involve propagating ghosts. While a covariant action from which this equation follows has not yet been derived, another action where the equations agree at the linearized level was introduced in \cite{16}

\[
S_{MM} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[ R - \frac{m^2}{6} R \frac{1}{\Box^2} R \right] \tag{1.2}
\]
This action, as in the case of Eq. (1.1), has been also shown to be consistent with known cosmological observations\cite{13, 17}. The spherically symmetric corrections to the Schwarzschild and FRW backgrounds were also derived in \cite{16}. These solutions do not involve non-linear instabilities in the region outside the horizon. This is unlike the scenario in massive gravity theories generically, where non-linearities do creep in below a Vainshtein radius thereby causing a breakdown of the theory well outside of the event horizon.

The success of non-local theories such as Eq. (1.2) however cannot be extended to include general theories with tensorial non-localities, involving curvature terms other than the Ricci scalar, as they do not lead to stable cosmological evolution \cite{18}. We will therefore restrict our attention to Eq. (1.2) and consider the coupling of the electromagnetic field to this action only. In the treatment of non-local theories, we are required to work within a local formulation through the introduction of auxiliary fields and the constraints that they must satisfy. These constraints will be generic to a large class of non-local theories. For instance, in the case of any pure gravitational action involving a $f(R)\Box^{-n}R$ correction term to the Einstein-Hilbert action, where $f(R)$ is an arbitrary function of the Ricci scalar and ‘$n’$ is an arbitrary power, we will always be required to at least introduce an auxiliary field $U$ satisfying $\Box U = -R$. This field plays a central role in our consideration of the coupling of the electromagnetic field to Eq. (1.2). Our analysis however should hold for all local formulations of non-local theories containing this field. As such, we believe that the inferences drawn from our analysis will have a broader applicability to general non-local theories.

In the case of local field theories, it is natural to consider the minimal coupling prescription. However, as just discussed, the dynamics of non-local theories requires going to a local formulation which introduces certain constraints which the gravitational fields must satisfy. Thus the consideration of a local matter theory added to a non-local gravitational theory, as prescribed by the minimal coupling prescription, may not be the most appropriate manner to couple matter fields. In this paper, we will consider the modified coupling of the electromagnetic field by applying the Kaluza ansatz to Eq. (1.2) in five-dimensions. This approach has been considered previously in the context of other modified gravity theories \cite{19, 20}, where in general, the resulting action contains non-minimal couplings between the electromagnetic field strength and curvature tensors. Unlike anomaly-induced quantum effective actions resulting from theories with mixed trace anomalies, the case of phenomenologically motivated non-local gravity theories do not involve a clear prescription to include non-local terms which
involve both gravitational and gauge fields. The action resulting from the Kaluza ansatz in
the present case will lead to non-local terms with both the electromagnetic field strength
tensor and the Ricci scalar.

In considering the solutions resulting from the iterative approach described in this work,
we will demonstrate that the usual minimal coupling of the electromagnetic field provides
unsatisfactory solutions. In considering corrections about the Reissner-Nordström (RN)
background, the solutions are found to multivalued and in general complex outside the event
horizon of the black hole. Further, the limit of vanishing charge also does not lead to the
solutions noted for the Schwarzschild background which signals a deeper inconsistency. In
contrast, the solutions resulting from the Kaluza reduced action involve first order corrections
which are a natural generalization of those for the Schwarzschild background. Our results
also demonstrate that just as in the Schwarzschild case, the solutions for the metric (and
electric field) are well behaved in the region outside the event horizon of the RN black hole.
Although our treatment in this paper will be entirely classical, in deriving an action via
the Kaluza ansatz we have in mind the construction of an effective action which could be
relevant in studying quantum effects. This will be discussed in more detail in the Conclusion.

The organization of our paper is as follows. In Sec. II we review the basic properties
of the action introduced in [16] and the correction derived in the case of the Schwarzschild
background. In Sec. III we apply the Kaluza ansatz to the 5-dimensional gravitational action
to derive an effective action non-local in both electromagnetic and gravitational fields. In
Sec. IV the equations of motion of this action are derived and compared with those for the
original non-local theory with a minimally coupled electromagnetic field. Sec. V describes
an iterative procedure which can be used to solve the equations of motion resulting from
all non-local actions considered in this paper. We then conclude with a discussion of our
results and future directions in Sec. VI.

II. THE NON-LOCAL ACTION

We will denote the gravitational action introduced in [16] by $S_{MM}$

\[
S_{MM} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[ R - \mu R \frac{1}{\Box^2} R \right],
\]  
(2.1)
where $G$ is the Newton constant, $g$ is the determinant of the metric $g_{\mu\nu}$, $\mu = \frac{m^2}{6}$ is the mass term associated with the additional non-local contribution in the action and $\Box$ is the D’Alembertian operator. For the rest of the paper, we will set $G = 1$. The non-local action in the presence of a minimally coupled matter field is then given by

$$S_{NL} = S_{MM} + S_M, \quad (2.2)$$

where $S_M$ is the minimally coupled matter action. Two auxiliary fields $U$ and $S$ can be introduced which satisfy the following constraints

$$\Box U = -R, \quad \Box S = -U, \quad (2.3)$$

These fields can now be used in Eq. (2.2) with the help of Lagrange multipliers $\xi_1$ and $\xi_2$ to provide the following local formulation

$$S_{NL} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ R - \mu RS - \xi_1(\Box U + R) - \xi_2(\Box S + U) \right] + S_M. \quad (2.4)$$

The equations of motion for $U$ and $S$ are given by

$$\Box \xi_1 + \xi_2 = 0, \quad \Box \xi_2 + \mu R = 0, \quad (2.5)$$

respectively, from which we can identify $\xi_2 = \mu U$ and $\xi_1 = \mu S$ on comparing with Eq. (2.3). With these expressions for $\xi_1$ and $\xi_2$, we have the following equation of motion for the metric

$$G_{\mu\nu}(1 - 2\mu S) - 8\pi T_{\mu\nu} = \mu K_{\mu\nu}, \quad (2.6)$$

where the stress energy tensor $T_{\mu\nu}$ is defined in the usual way

$$T_{\mu\nu} := -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} \quad (2.7)$$

and $K_{\mu\nu}$ is given by

$$K_{\mu\nu} = g_{\mu\nu} \left( 2\Box S + \nabla_{\alpha} U \nabla^{\alpha} S - \frac{1}{2} U^2 \right) - 2\nabla_{\mu} \nabla_{\nu} S - (\nabla_{\mu} U \nabla_{\nu} S + \nabla_{\nu} U \nabla_{\mu} S). \quad (2.8)$$

It follows from Eq. (2.8) that $\nabla_{\mu} K^{\mu\nu} = 0$. Since the matter theory is minimally coupled to the background, its equations of motion are unaffected in the present case. Thus the modified theory satisfies the usual conservation equations. The trace of the field equations
on the other hand is modified. Denoting the trace of the stress-energy tensor as $T$, the trace of Eq. (2.6) is given by

$$R(1 - 2\mu S) + 8\pi T = -\mu \left(6\Box S - 2U^2 + 2\nabla_\alpha U\nabla^\alpha S\right), \quad (2.9)$$

The corrections resulting from Eq. (2.6) for the Schwarzschild and FRW backgrounds were also considered in [16], based on the analysis carried out for Eq. (1.1) in [21]. We will now briefly review this derivation for the Schwarzschild background. The following spherically symmetric ansatz for the $4d$ metric was assumed

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2). \quad (2.10)$$

Eq. (2.6) can now be used to consider the equations for $e^{2(\beta - \alpha)}R_{00} + R_{11}$ and $R_{22}$. These two equations, along with those of Eq. (2.3), provide the following four independent equations in four unknowns

$$(1 - 2\mu S)(\alpha' + \beta') = -\mu r \left[S'' - (\alpha' + \beta' - U')S'\right], \quad (2.11)$$

$$(1 - 2\mu S) \left[1 + e^{-2\beta(r)}(\beta' - \alpha') - 1\right] = \mu r^2 \left[U + \frac{U^2}{2}\right] - 2re^{-2\beta S'}, \quad (2.12)$$

$$r^2U'' + \left[2r + (\alpha' - \beta')r^2\right]U' = -2e^{2\beta} + 2 \left[1 + 2r(\alpha' - \beta') + r^2(\alpha'' + \alpha^2 - \alpha'\beta')\right] \quad (2.13)$$

$$S'' + (\alpha' - \beta' + \frac{2}{r})S' = e^{2\beta}U. \quad (2.14)$$

The primes in Eqs. (2.11), (2.12), (2.13) and (2.14) denote differentiation with respect to $r$, i.e. $f' = \frac{\partial f}{\partial r}$. Due to the non-vanishing $U$ and $S$ fields, the right hand side of Eq. (2.11) does not vanish as it does in GR and leads to $\alpha$ being in general different from $\beta$. These coupled equations were solved about the Schwarzschild background by considering the region far away from the black hole. In the Newtonian limit, where $r_S \ll r$ with $r_S$ denoting the Schwarzschild radius, we can consider perturbations about flat space and $m$ arbitrary. The solution for $U(r)$ resulting from Eq. (2.13) is now given by the Green function for the Helmholtz equation. With this solution for $U$, the following expressions for $A(r)(= e^{2\alpha})$ and $B(r)(= e^{2\beta})$ were derived

$$A(r) = 1 - \frac{r_S}{r} \left[1 + \frac{1}{3} \left(1 - \cos mr\right)\right],$$

$$B(r) = 1 + \frac{r_S}{r} \left[1 - \frac{1}{3} \left(1 - \cos mr - mr \sin mr\right)\right], \quad (2.15)$$
The solutions can also be derived in the small $m$ limit, i.e. $r \ll m^{-1}$. This region has an overlap with the region considered previously about flat space, for which $r_S \ll r$. Thus in relating the solutions given in Eq. (2.15) with those in the region $r_S \ll m^{-1}$, we are collectively considering solutions in the region $r_S \ll r \ll m^{-1}$.

When $mr$ is small, $\alpha \approx \ln(1 - \frac{r_S}{r}) \approx -\beta$ holds and Eq. (2.13) provides the following general solution

$$U(r) = u_0 - u_1 \ln(1 - \frac{r_S}{r}).$$

The constant $u_0$ provides a cosmological constant to the field equations. This can be noted from Eq. (2.6), where the $\mu \frac{U^2}{2} g_{\mu\nu}$ term contained in $K_{\mu\nu}$ would provide such a contribution. However, since we are considering corrections about the Schwarzschild background, this constant should be set to vanish. The analogous constant in the solution for $U(r)$ in the post-Newtonian approximation was also set to vanish on similar grounds. The correction for $A(r)$ which result from Eq. (2.16) were found to agree with the corresponding expression in Eq. (2.15), provided $u_1 = 1$. With these values for the constants, the leading order correction to the metric was found to be

$$A(r) = e^{2\alpha} \approx 1 - \frac{r_S}{r} \left(1 + \mu r^2\right).$$

The solutions were further considered via numerical integration to account for corrections beyond first-order. This verified that the corrections to GR are $O(m^2 r^2)$ and that the theory remains linear up to $r \sim r_S$ (since $m \sim H_0$). Thus one recovers the vacuum solution of Einstein’s equations in the limit of $m \to 0$, demonstrating that the theory contains no vDVZ discontinuity. This is in contrast with the result in the case of the Einstein-Hilbert action with a Fierz-Pauli term, where a vDVZ discontinuity does result and where linear expansions break down below a Vainshtein radius $r_V = (\frac{GM}{m^4})^{1/5}$, which is parametrically larger than the Schwarzschild radius. In Sec. V, we will demonstrate that the analogous calculation about the RN background requires a careful handling of the constants to ensure that solutions exist in the region outside the event horizon of the black hole.

It will also be relevant to discuss the conventions we will use for the Maxwell action. Following [22], we will adopt

$$S_{EM} = -\frac{1}{16\pi} \int d^4x \sqrt{-g} F^{\beta\gamma} F_{\beta\gamma}.$$  (2.18)
The corresponding stress-energy tensor following Eq. (2.7) is
\[ T_{\mu\nu} = \frac{1}{4\pi} \left( F_{\mu\alpha} F_{\nu}^{\alpha} - \frac{1}{4} F^2 g_{\mu\nu} \right). \tag{2.19} \]

Substituting Eq. (2.18) in Eq. (2.2) as \( S_M \) provides the standard minimal coupling of the electromagnetic field to the given theory. In the next section, we will consider an alternative to this prescription of involving the electromagnetic field by applying the Kaluza Ansatz.

### III. THE KALUZA REDUCED ACTION

We begin by considering a five-dimensional spacetime with the following metric
\[ \hat{g}_{AB} = \begin{pmatrix} g_{\mu\nu} + \alpha^2 A_\mu A_\nu & \alpha A_\mu \\ \alpha A_\nu & 1 \end{pmatrix}, \tag{3.1} \]
where \( \alpha \) is a parameter which will be fixed later. Here and elsewhere in this section, five-dimensional objects will be represented with hats, uppercase Latin indices are five-dimensional, \( A, B, \cdots = 0, \cdots, 3, 5 \), while Greek indices are four-dimensional, \( \mu, \nu, \cdots = 0, \cdots, 3 \). The inverse of Eq. (3.1) is given by
\[ \hat{g}^{AB} = \begin{pmatrix} g^{\mu\nu} & -\alpha A^\mu \\ -\alpha A^\nu & \alpha^2 A_\gamma A_\gamma + \frac{1}{\alpha^2} \end{pmatrix}. \tag{3.2} \]
From Eq. (3.1), it follows that \( \sqrt{-g} = \sqrt{-\hat{g}} \). As we are interested in the coupling of electromagnetism to gravity, we will assume the cylindricity condition \( \frac{\partial \hat{g}_{AB}}{\partial x^5} = 0 \). It also follows from this condition that \( \hat{\Box} = \hat{g}^{AB} \hat{\nabla}_A \hat{\nabla}_B = g^{\mu\nu} \nabla_\mu \nabla_\nu = \Box \). The following Ricci tensor components can also now be calculated
\[ \hat{R}_{\mu\nu} = R_{\mu\nu} + \frac{1}{4} \alpha^4 F^{\beta\gamma} F_{\beta\gamma} A_\mu A_\nu - \frac{1}{2} \alpha^2 (A_\mu \nabla_\beta F_{\mu}^\beta + A_\nu \nabla_\beta F_{\nu}^\beta + F_{\beta\mu} F_{\beta}^\mu) \]
\[ \hat{R}_{\mu5} = \frac{1}{4} \alpha^3 F^{\beta\gamma} F_{\beta\gamma} A_\mu - \frac{1}{2} \alpha (\nabla_\beta F_{\mu}^\beta), \quad \hat{R}_{55} = \frac{1}{4} \alpha^2 F^{\beta\gamma} F_{\beta\gamma}, \tag{3.3} \]
thereby giving the five-dimensional Ricci scalar,
\[ \hat{R} = R - \frac{\alpha^2}{4} F^{\beta\gamma} F_{\beta\gamma}. \tag{3.4} \]
Writing the radius of compactification of the fifth dimension as \( \hat{R} \), and the five-dimensional Newton’s constant as \( \hat{G}_5 \), we find that setting
\[ \frac{2\pi \hat{R}}{\hat{G}_5} = \frac{1}{G} = 1, \quad \alpha^2 = 4G = 4 \tag{3.5} \]
leads to the reduction of the five-dimensional Einstein-Hilbert action as
\[
\frac{1}{16\pi G_5} \int d^5x \sqrt{-\hat{g}} \hat{R} = \frac{1}{16\pi} \int d^4x \sqrt{-g} R - \frac{1}{16\pi} \int d^4x \sqrt{-g} F^{\beta\gamma} F_{\beta\gamma}.
\] (3.6)

We will now consider Eq. (2.1) in five-dimensions and apply the above reduction to it. The action in this case is given by
\[
\hat{S}_{MM} = \frac{1}{16\pi G_5} \int d^5x \sqrt{-\hat{g}} \left[ \hat{R} - \hat{\mu} \hat{R} \frac{1}{\Box^2} \hat{R} \right],
\] (3.7)
The five-dimensional mass scale \(\hat{\mu}\) must be such that \(\hat{\mu} = \frac{\hat{m}^2}{6} = \frac{m^2}{6} = \mu\). The reason for this is that following the reduction, we require the mass scale to match the original non-local theory in the limit of a vanishing electromagnetic field. Likewise, since \(\Box^2 = \Box^2\), their inverses should also agree. For the remaining terms, we substitute Eq. (3.5) and Eq. (3.4) to find the following reduced action
\[
S_{KMM} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ R - F^{\beta\gamma} F_{\beta\gamma} - \mu \left( (R - F^{\beta\gamma} F_{\beta\gamma}) \frac{1}{\Box^2} (R - F^{\beta\gamma} F_{\beta\gamma}) \right) \right]
\] (4.1)

In the next section, we will consider the local formulation of this action and derive its equations of motion. These will then be compared with those resulting from Eq. (2.2) with the minimally coupled electromagnetic field. The difference in the constraints satisfied by the auxiliary fields in the two cases will play an important role in the nature of their solutions.

IV. EQUATIONS OF MOTION

Given Eq. (3.8), we can define two variables \(\tilde{U}\) and \(\tilde{S}\) in analogy with the original non-local theory which satisfy,
\[
\Box \tilde{U} = - (R - F^{\beta\gamma} F_{\beta\gamma}) , \quad \Box \tilde{S} = - \tilde{U}.
\] (4.2)
The presence of \(F^{\beta\gamma} F_{\beta\gamma}\) as a source for the \(U\) field will have significant implications on the resulting solutions. These can now be substituted in Eq. (3.8) with the help of two Lagrange multipliers \(\xi_1\) and \(\xi_2\) to give,
\[
S_{KMM} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ (R - F^{\beta\gamma} F_{\beta\gamma})(1 - \mu \tilde{S}) - \xi_1 \left( \Box \tilde{U} + R - F^{\beta\gamma} F_{\beta\gamma} \right) - \xi_2 \left( \Box \tilde{S} + \tilde{U} \right) \right]
\] (4.3)
Varying this action with respect to \(\tilde{U}\) and \(\tilde{S}\) we find,
\[
-\xi_2 - \Box \xi_1 = 0 ,
\]
\[
-\mu (R - F^{\beta\gamma} F_{\beta\gamma}) - \Box \xi_2 = 0 ,
\] (4.3)
respectively. These two equations identify $\xi_1 = \mu \tilde{U}$ and $\xi_2 = \tilde{S}$. This then leads to the equation of motion for the metric being given by

$$ (G_{\mu\nu} - 8\pi T_{\mu\nu}) \left( 1 - 2\mu \tilde{S} \right) = \mu \tilde{K}_{\mu\nu}, $$ \hspace{1cm} (4.4)

where $T_{\mu\nu}$ is as defined in Eq. (2.19) and

$$ \tilde{K}_{\mu\nu} = g_{\mu\nu} \left( 2\Box \tilde{S} + \nabla_\alpha \tilde{U} \nabla^\alpha \tilde{S} - \frac{1}{2} \tilde{U}^2 \right) - 2\nabla_\mu \nabla_\nu \tilde{S} - \left( \nabla_\mu \tilde{U} \nabla_\nu \tilde{S} + \nabla_\nu \tilde{U} \nabla_\mu \tilde{S} \right), $$ \hspace{1cm} (4.5)

The equations of motion for the electromagnetic field $A_\mu$ resulting from Eq. (4.2) is given by

$$ \nabla_\mu \left( \left( 1 - 2\mu \tilde{S} \right) F^{\mu\nu} \right) = 0 $$ \hspace{1cm} (4.6)

Thus the Maxwell and Einstein field equations both involve non-local corrections, unlike the situation in the minimally coupled case, where one recovers Maxwell’s equation in its usual form

$$ \nabla_\mu F^{\mu\nu} = 0, $$ \hspace{1cm} (4.7)

The other equations in the minimally coupled case are given in Sec. II. In comparing Eq. (4.4) and Eq. (2.6) we further note that the stress tensor in the Kaluza reduced case also involves $\mu$ corrections. Finally, while Eq. (4.5) and Eq. (2.8) show that the general correction terms $\tilde{K}_{\mu\nu}$ and $K_{\mu\nu}$ are structurally similar, they provide different contributions due to the difference in the definitions of the auxiliary fields $\tilde{U}$ and $\tilde{S}$ from their non-tilded counterparts. In the following subsection, we will derive the solution of Eq. (4.4) through an iterative approach built on known solutions of GR.

V. ITERATIVE APPROACH TO DERIVING THE SOLUTIONS

To construct the solutions of the equations of motion given in the previous section, let us rewrite Eq. (4.4) and Eq. (4.6) in the following way

$$ G_{\mu\nu} - 8\pi T_{\mu\nu} = \mu \left( \tilde{K}_{\mu\nu} + 2\tilde{S}(G_{\mu\nu} - 8\pi T_{\mu\nu}) \right) $$
$$ \nabla_\mu F^{\mu\nu} = 2\mu \left( \nabla_\mu \tilde{S} F^{\mu\nu} + \tilde{S} \nabla_\mu F^{\mu\nu} \right). $$ \hspace{1cm} (5.1)

The form of these equations suggest that we can consider the fields $g_{\mu\nu}$ and $A_\mu$ in terms of their zeroth and first-order (in $\mu$) contributions

$$ g_{\mu\nu} = g_{\mu\nu}^{(0)} + g_{\mu\nu}^{(1)}, \quad A_\mu = A_\mu^{(0)} + A_\mu^{(1)}, $$ \hspace{1cm} (5.2)
Here \( \{g^{(1)}_{\mu\nu}, A^{(1)}_{\mu}\} \) is linear in \( \mu \) and \( \{g^{(0)}_{\mu\nu}, A^{(0)}_{\mu}\} \) satisfy the the Einstein-Maxwell equations
\[
G^{(0)}_{\mu\nu} - 8\pi T^{(0)}_{\mu\nu} = 0, \quad \nabla^{(0)}_{\mu} F^{(0)\mu\nu} = 0, \quad (5.3)
\]
Thus \( g^{(0)}_{\mu\nu} \) and \( A^{(0)}_{\mu} \) represent any known electrovacuum solution of GR. We will assume the RN solution, for which we have
\[
g^{(0)}_{\mu\nu} = \begin{pmatrix}
-f(r) & 0 & 0 & 0 \\
0 & f(r)^{-1} & 0 & 0 \\
0 & 0 & r^2 & 0 \\
0 & 0 & 0 & r^2 \sin^2 \theta
\end{pmatrix}, \quad F^{(0)}_{\mu\nu} = \begin{pmatrix}
0 & \frac{q}{r^2} & 0 & 0 \\
-\frac{q}{r^2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad (5.4)
\]
where \( f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \), with \( M \) and \( Q \) denoting the mass and charge of the black hole respectively, while \( F^{(0)}_{\mu\nu} = 2\partial_{[\mu} A^{(0)}_{\nu]} \) is the lowest order electromagnetic field strength tensor.

Using these solutions, we can find the first-order corrections in \( \mu \) from Eq. (5.1)
\[
G^{(1)}_{\mu\nu} - 8\pi T^{(1)}_{\mu\nu} = \mu \left( \nabla^{(1)}_{\mu} F^{(0)\mu\nu} + \nabla^{(0)}_{\mu} F^{(1)\mu\nu} \right) = 2\mu (\nabla^{(0)}_{\mu} \tilde{S}^{(0)}) F^{(0)\mu\nu}. \quad (5.5)
\]

In Eq. (5.5), \( \nabla^{(0)}_{\mu} \) and \( \nabla^{(1)}_{\mu} \) imply that the connection in the covariant derivative is expanded to the respective order. Likewise, \( G^{(1)}_{\mu\nu} \) and \( T^{(1)}_{\mu\nu} \) are the first-order contributions of \( G_{\mu\nu} \) and \( T_{\mu\nu} \), while \( \tilde{K}^{(0)}_{\mu\nu} \) and \( \tilde{S}^{(0)} \) are in terms of the zeroth-order fields with the following expressions
\[
\tilde{K}^{(0)}_{\mu\nu} = g^{(0)}_{\mu\nu} \left( 2\Box^{(0)} \tilde{S}^{(0)} + \nabla^{(0)}_{\alpha} \tilde{U}^{(0)} \nabla^{(0)\alpha} \tilde{S}^{(0)} - \frac{1}{2} \tilde{U}^{(0)2} \right) - 2\nabla^{(0)}_{\mu} \nabla^{(0)}_{\nu} \tilde{S}^{(0)} \\
\Box^{(0)} \tilde{S}^{(0)} = -\tilde{U}^{(0)}, \quad (5.6)
\]
In addition to the equations considered in Eq. (5.1), we have the constraint equations given in Eq. (4.1). These in the lowest order satisfy
\[
\Box^{(0)} \tilde{U}^{(0)} = F^{(0)\alpha\beta} F^{(0)\alpha\beta} = -\frac{2q^2}{r^2}, \quad \Box^{(0)} \tilde{S}^{(0)} = -\tilde{U}^{(0)}, \quad (5.7)
\]
To simplify the notation, we will henceforth label \( \tilde{U}^{(0)} \) and \( \tilde{S}^{(0)} \) as \( \tilde{u}(r) \) and \( \tilde{s}(r) \) respectively. To make further progress, we assume the following spherically symmetric ansatz for the
metric and electromagnetic field strength

\[ g_{\mu\nu} = \begin{pmatrix} -(f(r) + \mu A(r)) & 0 & 0 & 0 \\ 0 & (f(r) + \mu (A(r) - B(r)))^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}, \]

\[ F_{\mu\nu} = \begin{pmatrix} 0 & \frac{Q}{r^2} + \mu D(r) & 0 & 0 \\ \frac{Q}{r^2} - \mu D(r) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]  

(5.8)

where \( A(r), B(r) \) and \( D(r) \) represent the first-order correction terms. Following the procedure just described, we can consider the following first-order set of equations

\[ \frac{2Q^2}{r^4} = -f(r)\tilde{u}(r) + f(r) \left( \frac{2\tilde{u}(r)'}{r} + \tilde{u}(r)'' \right) \]  

(5.9)

\[ \tilde{u}(r) = f(r)\tilde{s}(r) + f(r) \left( \frac{2\tilde{s}(r)'}{r} + \tilde{s}(r)'' \right) \]  

(5.10)

\[ f(r)B(r)' - B(r)f(r)' = -2rf(r)^2(\tilde{u}(r)\tilde{s}(r)' + \tilde{s}(r)'') \]  

(5.11)

\[ Q(B(r)f(r)' - B(r)'f(r)) = -2f(r)^2(-Q\tilde{s}(r)' + 2rD(r) + r^2D(r)') \]  

(5.12)

\[ 4rf(r)\tilde{s}(r)' = 2 \left( A(r) + 2QD(r) + r^2 \left( \tilde{u}(r) + \frac{\tilde{u}(r)^2}{2} \right) + r \left( A(r)' - \frac{B(r)'}{2} \right) \right) - B(r) \left( 2 + \frac{2Q^2}{f(r)r^2} + \frac{rf(r)'}{f(r)} \right) \]  

(5.13)

where primes denote differentiation with respect to coordinate ‘\( r \)’. Eq. (5.9) and Eq. (5.10) are those provided in Eq. (5.7), which is solved entirely in the lowest order. Eq. (5.11) is the first-order correction of \((1 - 2\mu S)(g^{00}R_{00} - g^{11}R_{11})\), Eq. (5.12) is the first-order correction to Maxwell’s equation and Eq. (5.13) is the first-order correction of \( R_{22} \). One can note that by beginning with Eq. (5.9), one can solve the next equation provided in Eq. (5.10), with which one can further solve Eq. (5.11), and so on. Thus Eq. (5.9) - Eq. (5.13) provides a sequence of equations which can be used to solve for the corrected Einstein-Maxwell equations at this order. An analogous construction can be carried out for every order in \( \mu \), thereby allowing one to solve the theory to any desired accuracy. However, the prescription given above implicitly uses the fact that \( \mu \), i.e. \( m^2 \), is small. The results one will find should therefore correspond to \( mr \ll 1 \), which covers the region of spacetime which is
cosmologically observable. Further, as the leading order corrections will provide the most significant contribution, our analysis will be restricted to this order for the cases considered below.

For comparison, we also provide the set of equations one would find in carrying out the analysis of this section to the equations of Sec. II when the Maxwell field is minimally coupled to the theory. Denoting the corresponding \( U(0) \) and \( S(0) \) fields as \( u \) and \( s \) and performing the same decomposition for the metric and electromagnetic field strength tensor as in Eq. (5.8), we find the following first-order equations

\[
0 = -f(r)'u(r)' + f(r) \left( \frac{2u(r)'}{r} + u(r)'' \right) \tag{5.14}
\]

\[
u(r) = f(r)'s(r)' + f(r) \left( \frac{2s(r)'}{r} + s(r)'' \right) \tag{5.15}
\]

\[
f(r)B(r)' - B(r)f(r)' = -2rf(r)^2(u(r)'s(r)' + s(r)''), \tag{5.16}
\]

\[
Q(B(r)f(r)' - B(r)'f(r)) = -2f(r)^2 \left( 2rD(r) + r^2D(r)' \right) \tag{5.17}
\]

\[
4 \left( rf(r)s(r)' - \frac{Q^2}{r^2} s(r) \right) = 2 \left( A(r) + 2QD(r) + r^2 \left( u(r) + \frac{u(r)^2}{2} \right) \right) + r \left( A(r)' - \frac{B(r)'}{2} \right)
\]

\[- B(r) \left( 2 + \frac{2Q^2}{f(r)r^2} + \frac{rf(r)'}{f(r)} \right) \tag{5.18}
\]

It can be noted that apart from the last two equations, all other equations have a similar form as those of Eqs. (5.9) - (5.13). However, there is a much larger problem in considering \( u \) in generating the solutions as opposed to \( \tilde{u} \) which we will describe in the course of providing the solutions below.

**A. Electrovacuum solutions**

Before proceeding to solve Eqs. (5.9) - (5.13), lets first consider Eq. (5.14) for the minimally coupled case. Given the RN background, the general solution of \( \Box(0)u = 0 \) is

\[
u = c_1 + \frac{c_2}{\sqrt{M^2 - Q^2}} \text{ArcTanh} \left( \frac{r - M}{\sqrt{M^2 - Q^2}} \right) \tag{5.19}
\]

This solution is non-ideal for a variety of reasons. We first note that apart from being a multivalued function, \( \text{ArcTanh}(x) \) is real only when its argument lies between \( x \in [-1, 1] \). In the context of its argument in Eq. (5.19), this region lies between \( M - \sqrt{M^2 - Q^2} \) and \( M + \sqrt{M^2 - Q^2} \). Thus for real coefficients \( c_1 \) and \( c_2 \), it only admits real solutions in the
region between the inner Cauchy horizon and the event horizon of the RN black hole. We can however consider a real solution outside the black hole by choosing a unique complex constant \( c_1 = \frac{ic_2}{\sqrt{M^2 - Q^2}} \). As mentioned in Sec. II, the inclusion of such a term introduces an effective cosmological constant and thus does not provide an appropriate correction to the asymptotically flat RN background. As a final point, we also note that in the limit of \( Q \to 0 \), it would be desirable to recover the \( u \) function of the Schwarzschild background (Eq. (2.16)), which cannot happen in the case of Eq. (5.19).

In considering the \( \tilde{u} \) function on the other hand, the general solution of Eq. (5.9) on the RN background is given by

\[
\tilde{u} = c_1 + \frac{c_2}{\sqrt{M^2 - Q^2}} \text{ArcTanh} \left( \frac{r - M}{\sqrt{M^2 - Q^2}} \right) - \ln \left( 1 - \frac{2M}{r} + \frac{Q}{r^2} \right). \tag{5.20}
\]

By fixing \( c_1 = c_2 = 0 \), we can now eliminate the homogeneous solution for the reasons mentioned above. The solution we are left with is

\[
\tilde{u} = -\ln \left( 1 - \frac{2M}{r} + \frac{Q}{r^2} \right), \tag{5.21}
\]

which agrees with Eq. (5.9) in the limit of vanishing charge. Using this solution, we can now solve the remaining first-order correction equations. The general solution for \( \tilde{s} \) comprises of the homogenous ArcTanh solution, which can be ignored. As in the Schwarzschild case, it also involves logarithmic terms, products of logarithmic terms and polylog functions [21]. Further, unlike the Schwarzschild case, there exist powers of \( \text{ArcTanh} \left( \frac{r - M}{\sqrt{M^2 - Q^2}} \right) \) whose contributions need to be addressed through an appropriate choice of the integration constant.

Since the solutions for \( A(r), B(r) \) and \( D(r) \) are significantly more involved in these terms, we will now provide only their leading order contribution. We find that

\[
A(r) = -2Mr - 2(M^2 - Q^2) + c_1 + \mathcal{O}(r^{-1})
\]
\[
B(r) = c_1 + \mathcal{O}(r^{-1})
\]
\[
D(r) = -\frac{2MQ}{r} + \mathcal{O}(r^{-2}) \tag{5.22}
\]

Substituting Eq. (5.22) in Eq. (5.8) and taking \( c_1 = 0 \), we find the following first-order in \( \mu \) corrected metric component and electric field, to leading order in \( r \)

\[
g_{00} \approx \left( 1 - \frac{2M}{r} \right) (1 + \mu(r^2 + Mr)) + \frac{Q}{r^2} (1 + \mu r^2) \tag{5.23}
\]
\[
F_{01} \approx \frac{Q}{r^2} (1 - 2\mu Mr)
\]
FIG. 1: Left: The first-order correction functions $A(r)$ (red), $B(r)$ (blue) and $D(r)$ (green) near the horizon of the black hole ($r=8$). Right: The same functions now considered up to $r=1000$. The plots involve the complete solutions for $A(r)$, $B(r)$ and $D(r)$ and the correction terms are considered with $\mu = 10^{-6}$. The horizon is located at $r = 8$ following the choice of $M = 5$ and $Q = 4$.

FIG. 2: Left: Comparative plots of the corrected $g_{00}$ and $g^{11}$ metric components with that of the uncorrected RN $g_{00}^{(0)}$. Right: Comparative plots of the RN Electric field with that of the first-order corrected Electric field. These plots indicate the excellent agreement of the first-order correction with the RN solution up to $mr \ll 1$.

As can be seen from comparing Eq. (2.17) with Eq. (5.23), the corrected metric is an appropriate extension of the result for the Schwarzschild background. We also note that due to the non-minimal coupling involved in the Kaluza reduced action, the electric field involves charge corrections which involve the mass $M$ of the black hole. There are no $r^{-1}$ corrections for the electric field in the minimally coupled case, making the above result specific to the action of Eq. (3.8).
VI. DISCUSSION

In this work, we have considered the modified coupling of the electromagnetic field by performing the Kaluza reduction on the non-local action of Eq. (3.7). This was proposed as an interesting alternative to the usual minimal coupling prescription, as non-localities result in the Kaluza reduced action in both gauge and gravitational fields thereby constraining both in the resulting local formulation. The difference between the actions and their equations of motion in the two cases could have interesting implications in considering quantum effects, as we will describe in more detail below. In order to derive the classical solutions, we provided an iterative approach in Sec. V catered to the five coupled differential equations one needs to solve. As a result, we found a way of decoupling these equations to sequentially solve for the metric perturbations described by \( B(r) \) and \( A(r) \), in addition to the perturbation of the electromagnetic field strength tensor \( D(r) \).

The standard minimally coupled case was shown to lead to a solution for the auxiliary field \( u \) which is multivalued and in general complex outside the event horizon of the black hole. In considering real coefficients, the solutions in this case exist from the event horizon of the RN black hole to the inner Cauchy horizon (i.e. from \( M + \sqrt{M^2 - Q^2} \) to \( M - \sqrt{M^2 - Q^2} \) in the RN case). For the region outside the black hole, the addition of an imaginary constant term is necessary to derive any real corrections to the Einstein field equations. In the Kaluza reduced action, the analogous solution for the \( \tilde{u} \) field is the logarithm of the lapse function of the background, which is the exact generalization of the known Schwarzschild correction. The solutions outside the BH background exists in this case even in the absence of defining a constant contribution in the solution for \( \tilde{u} \). The plots of the corrected solutions given above demonstrate that insofar as cosmological observations are concerned, the theories are closely related and extend the consistency noted in the previous literature to the case with the electromagnetic field.

However, it is clear that the equations of motion of the Kaluza reduced action could have a greater significance in the context of near horizon physics. Part of this may be anticipated from the solution of \( U \) itself, which in all cases where they are real in the exterior region become imaginary from the event horizon inwards. Since we have built our solution perturbatively about a known GR background, there should in principle be no obstruction in constructing solutions that go just past the horizon of known spherically...
symmetric electrovaccum solution of GR (since $\mu$ in the present theory continues to be small there). Local effective field theory calculations do not break down at the horizon of such black holes, at least in the case when they are large, since the curvature scalars and invariants built from them remain finite and continuous at the horizon. Thus in the context of local effective field theories, drastic changes do not arise and one may expect that the same should extend to certain non-local theories as well. The problem in the non-local case appears to be related to the inability to find a solution of $U(r)$ which continues to be real both outside and inside the black hole horizon. One can thus consider the separate construction of real solutions in both regions, which in either case requires a careful choice of integration constants. These solutions can then be matched across the horizon, which sets the stage to investigate classical non-local effects near the horizon.

A more direct way in which one can investigate quantum effects resulting from non-local theories is known in relation to the anomaly-induced quantum effective action resulting from background gravitational and gauge fields. In particular, interesting contributions to graviton-photon amplitudes result when there exist non-local terms involving both $R$ and $F_{\alpha\beta}F^{\alpha\beta}$, which would otherwise be absent [23]. One can expect these results to extend to the non-local action of Eq. (1.2), where non-local contributions in both gravitational and electromagnetic fields do exist.

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