Quantum corrections of work statistics in closed quantum systems

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(Dated: April 3, 2018)

We investigate quantum corrections to classical work characteristic function (CF) as a semi-classical approximation to the full quantum work CF. In addition to explicitly establishing the quantum-classical correspondence of Feynman-Kac formula, we find that these quantum corrections must be in even powers of $\hbar$. Exact formulas of the lowest corrections ($\hbar^2$) are proposed and their physical origins are also clarified. We use a forced harmonic oscillator to illustrate our results.

PACS numbers: 05.70.Ln, 05.30.-d

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I. INTRODUCTION

Recently, statistics of quantum work has attracted considerable attention [1–3]. This issue was initially motivated by theoretical efforts of extending classical fluctuation theorems [4–13] into quantum regime. The practical feasibility of manipulating and/or controlling energy of small quantum systems further boosts these research interests [16–19]. In closed quantum systems, quantum work is defined by the two energy measurement scheme (TEM) [20, 21]. Although this definition has been criticized because of destroying possible initial coherence [22, 23], several works support its justification from the aspect of quantum-classical correspondence principle [24, 26]. Obviously, this classical correspondence is not the ultimate goal of these studies; quantum characteristics of quantum work is really what we are concerned with. A possible improvement to the full classical work statistics is to develop semiclassical approaches. In addition to academic interest, we expect that these semiclassical approaches shall provide practical methods to compute complex quantum work statistics of general quantum systems. In this paper, we present such an approach. We will follow Wigner’s idea [27] to represent an evolution equation for the characteristic function (CF) of quantum systems. In this paper, we present such an approach. This paper is organized as follows. In Sec. (II), we briefly review the CF method of quantum work in closed quantum systems. In Sec. (III), a quantum-classical correspondence of Feynman-Kac formula is established. In Sec. (IV) we present the lowest quantum corrections to the classical CF. A forced harmonic oscillator is used to illustrate our approach.

II. AN OVERVIEW OF THE CF METHOD

Let us start with the quantum work definition of a closed quantum systems with Hamiltonian $\hat{H}(t)$. Throughout this paper, all operators are denoted by hat in order to distinguish them from their classical correspondences in the system’s phase space. Because our aim here is to illustrate the idea, our discussion is limited in the simplest single particle and one-dimensional situation. According to the two-energy-measurement (TEM) scheme [20, 21], given the system’s phase space. By expanding the equation in powers of the Planck constant, $\hbar$, we will follow Wigner’s idea [27] to represent an evolution equation for the characteristic function (CF) of quantum work in system’s phase space. By expanding the equation in powers of the Planck constant, $\hbar$, the classical work statistics and its quantum corrections are clearly revealed.

$P(W) = \sum_{n,m} \delta(W - W_{nm}) |\langle \epsilon_n(t)|U(t)|\epsilon_m(0)\rangle|^2 P_m(0)$

where $U(t)$ is the time evolution operator of the system, and $P_m(0)$ is the probability of finding the system at the eigenvector $|\epsilon_m(0)\rangle$ at time 0. We assume that the initial system is in the thermal equilibrium state, that is, $P_m(0) = \exp[-\beta\epsilon_m(0)]/Z(0)$, where $Z(0)$ is the partition function at time 0 and is equal to $\text{Tr}\{\exp[-\beta\hat{H}(0)]\}$, and $\beta$ is the inverse temperature. Because of the involvement of the Dirac function, Eq. (11) is not the most convenient form if one wants to analyze the statistical properties of the work distribution. An alternative way is to resort to its Fourier transform or CF [1–4, 20], and it can be reexpressed as taking the trace over an operator:

$\Phi(\eta) = \text{Tr} \left[ e^{i\eta\hat{H}(t)} U(t) e^{-i\eta\hat{H}(0)} \rho_0 U^\dagger(t) \right] \equiv \text{Tr}[\hat{K}(t)].$

We called $\hat{K}(t)$ the work characteristic operator (WCO) [3]. It is easy to prove that the operator satisfies the following evolution equation [28]:

$\partial_t \hat{K}(t) = \frac{1}{i\hbar} [\hat{H}(t), \hat{K}(t)] + \left[ \partial_t e^{i\eta\hat{H}(t)} \right] e^{-i\eta\hat{H}(t)} \hat{K}(t)$

$= \frac{1}{i\hbar} [\hat{H}(t), \hat{K}(t)] + \hat{\Omega}(t) \hat{K}(t).$

Note that the initial condition is $\hat{K}(0) = \exp[-\beta\hat{H}(0)]/Z(0)$. A collection of Eqs. (22–44) was called the quantum Feynman-Kac (FK) formula, since it is fully consistent with the spirit of Kac’s original paper [29] in establishment of the CF method of evaluating the distributions of classical stochastic functional [4].
III. QUANTUM-CLASSICAL CORRESPONDENCE OF FK FORMULA

Following Wigner’s idea [27], we reformulate Eq. (4) in the phase space representation [27, 30, 31]: let \( K(z, t) \) be the Weyl symbol of \( \hat{\dot{K}}(t) \), where \( z = (x, p) \) is the phase point, and \( x \) and \( p \) are the position and momentum of the particle, respectively, then

\[
\partial_t K = -\frac{2}{\hbar} H \sin \left( \frac{\hbar \Lambda}{2} \right) K + \Omega \exp \left( -\frac{i\hbar}{2} \Lambda \right) K,
\]

(5)

where the simplectic operator is [30]

\[
\Lambda = \vec{\partial}_p \vec{\partial}_x - \vec{\partial}_x \vec{\partial}_p,
\]

(6)

and the arrows indicate on which direction the derivatives act. \( \Omega(z, t) \) is the Weyl symbol of \( \hat{\Omega} \):

\[
\Omega(z, t) = \left[ \partial_t e^{i\eta H(t)} \right]_w \exp \left( -\frac{i\hbar}{2} \Lambda \right) e^{-i\eta H(t)} \]

(7)

Here the subscript "w" was used to indicate the Weyl symbols of these exponential operators. If one solves Eq. (5), the CF of the quantum work is then evaluated directly as

\[
\Phi(\eta) = \int_{-\infty}^{+\infty} dz K(z, t).
\]

(8)

Eq. (7) seems very complex. Importantly, Wigner [27] has obtained the Weyl symbol of the exponential Hamiltonian by expanding it in powers of \( \hbar \) when he investigated the quantum corrections to the classical thermodynamical quantities [32]:

\[
\left[ e^{-i\eta H(t)} \right]_w = e^{-i\eta H(z, t)} \left[ 1 + (i\hbar)^2 f(\eta, z, t) + o(\hbar^2) \right],
\]

(9)

where

\[
f(\eta, z, t) = \frac{(i\eta)^2}{8m} \left[ \partial_t^2 U - \frac{i\eta}{3} \left( \partial_x U \right)^2 - \frac{i\eta}{3m} \rho^2 \partial_x^2 U \right].
\]

(10)

To obtain this equation, the Hamiltonian of the quantum system is assumed to be simple

\[
\hat{H}(t) = \frac{\hat{p}^2}{2m} + U(\hat{x}, t),
\]

(11)

where \( m \) is the mass of the single particle system. Substituting Eqs (7) and (9) into the right-hand side of Eq. (5), and expanding it till the second powers of \( \hbar \), we have

\[
\partial_t K = -H \Lambda K + i\partial_t H K + \frac{i\hbar}{2} \left[ (i\eta)^2 (\partial_t H \Lambda H) - i\eta \partial_t H \Lambda \right] K + (i\hbar)^2 (\cdots K \cdots) + \cdots
\]

(12)

The exact expression of (\( \cdots \)) is presented in Appendix A. We do not include more terms with higher powers of \( \hbar \), though there are no fundamental difficulties. In order to investigate the quantum-classical correspondence and possible quantum corrections, we expand \( \hat{K} \) in powers of \( \hbar \) as follows:

\[
K = K^{(0)} + (i\hbar) K^{(1)} + (i\hbar)^2 K^{(2)} + \cdots
\]

(13)

Substituting it into Eq. (5) and collecting all terms with the same powers of \( \hbar \), we obtain

\[
\partial_t K^{(0)} = -H \Lambda K^{(0)} + i\eta \partial_t H K^{(0)},
\]

(14)

\[
\partial_t K^{(1)} = -H \Lambda K^{(1)} + i\eta \partial_t H K^{(1)} + \frac{1}{2} \left[ (i\eta)^2 (\partial_t H \Lambda H) - i\eta \partial_t H \Lambda \right] K^{(0)},
\]

(15)

\[
\partial_t K^{(2)} = -H \Lambda K^{(2)} + i\eta \partial_t H K^{(2)} + \frac{1}{2} \left[ (i\eta)^2 (\partial_t H \Lambda H) - i\eta \partial_t H \Lambda \right] K^{(1)} + \left( \cdots K^{(0)} \cdots \right).
\]

(16)

Their initial conditions are

\[
K^{(0)}(z, 0) = P_{eq}(\beta, z, 0),
\]

(17)

\[
K^{(1)}(z, 0) = 0,
\]

(18)

\[
K^{(2)}(z, 0) = P_{eq}(\beta, z, 0) \delta f(\beta, z, 0),
\]

(19)
respectively, where the classical canonical distribution is

$$P_{eq}(\beta, z, 0) = \frac{e^{-\beta H(z, 0)}}{Z_C(0)},$$  \hspace{1cm} (20)$$

$Z_C(0)$ is the classical partition function of the system at time 0,

$$\delta f(\beta, z, 0) = f(\beta, z, 0) - (f(0))_{eq},$$  \hspace{1cm} (21)$$

and $(f(0))_{eq}$ is an average of $f(\beta, z, 0)$ with respect to the canonical distribution \((20)\). Here we explicitly mark the parameter $\beta$, since we will replace it by other parameters in the next section. Eq. \((19)\) is due to the quantum correction to the classical distribution \((27)\). We immediately find that Eq. \((14)\) (the zeroth power of $\bar{\hbar}$) is nothing but the celebrated FK formula for the classical work \([8, 33–37]\).

$$W = \int_0^t \partial_s H[z(s), s] ds,$$  \hspace{1cm} (22)$$

and the solution is

$$K^{(0)}(z, t) = \left\langle e^{i\eta \int_0^t \partial_s H(z(s), s) \bullet \delta(z - z(t))} \right\rangle = e^{i\eta [H(z, t) - H(z(0), 0)]} P_{eq}(\beta, \psi^{-1}_0(z, t), 0).$$  \hspace{1cm} (23)$$

The angular brackets indicate an average over all classical phase trajectories started from initial canonical distribution and weighted by the exponential work. The second equation is valid only for closed classical systems, where $\psi^{-1}_0$ is the inverse of the flow map of the classical Hamiltonian system,

$$z = \psi_t(z_0, 0).$$  \hspace{1cm} (24)$$

That is, the phase points $z_0$ at time 0 and $z$ at time $t$ are at the same phase trajectory described by the map $\psi_t$. Therefore, a quantum-classical correspondence of the FK formula is explicitly established. Obviously, the corresponding principle of work statistic is then a natural consequence. This is the first important result obtained in this paper.

Given the flow map \((24)\), we can also easily construct the solution of Eq. \((15)\) (the first power of $\bar{\hbar}$):

$$K^{(1)}(z, t) = \int_0^t dt' e^{i\eta [H(z, t) - H(z', t')]} \left\{ \frac{1}{2} (i\eta)^2 [\partial_{z'} H(z', t') \Lambda' H(z', t')] \right\} - i\eta \partial_{z'} H(z', t') K_0(z', t') \big|_{z' = \psi^{-1}_t(z, t)}.$$  \hspace{1cm} (25)$$

where $z'$ is the phase point at time $t'$ which is connected to the phase point $z$ at time $t$, namely, $z = \psi_t(z', t')$. The reader is reminded that the operator $\Lambda'$ in this equation is defined with respect to $z' = (z', p')$. Very analogous expression for $K^{(2)}(z, t)$ can be obtained as well, and it obviously depends on the initial condition \((14)\) besides the functions $K^{(0)}$ and $K^{(1)}$. Based on these above results, we arrive at the series expansion of the CF \((9)\) in powers of $\bar{\hbar}$ as follows:

$$\Phi(\eta) = \Phi^{(0)}(\eta) + (i\bar{\hbar}) \Phi^{(1)}(\eta) + (i\bar{\hbar})^2 \Phi^{(2)}(\eta) + \cdots ,$$  \hspace{1cm} (26)$$

where

$$\Phi^{(0)}(\eta) = \left\langle e^{i\eta \int_0^t \partial_s H(z(s), s) ds} \right\rangle ,$$  \hspace{1cm} (27)$$

$$\Phi^{(1)}(\eta) = \frac{(i\eta) (i\eta + \beta)}{2} \int dz e^{i\eta [H(\psi_t(z, 0), 0) - H(z, 0)]} P_{eq}(z, 0) \int_0^t dt' \partial_{z'} H(\psi_t(z, 0), t') \Lambda H(z, 0)$$

$$= \frac{(i\eta) (i\eta + \beta)}{2} \left\langle e^{i\eta \int_0^t \partial_s H(z(s), s) ds} \int_0^t ds \partial_{z'} H(z(s), s) \Lambda H(z(0), 0) \right\rangle .$$  \hspace{1cm} (28)$$

In the derivations of these equations, we have used the Liouville theorem. In addition, we did not write $\Phi^{(2)}(\eta)$ temporarily since its current form is too long to be useful.
IV. LOWEST ORDER QUANTUM CORRECTION

As we pointed out at the beginning, the ultimate goal of studying the quantum-classical correspondence of work statistics is to deepen our understanding of quantum characteristics of work. Hence, we are interested in the quantum corrections with higher orders of $\hbar$ of the CF, e.g., $\Phi^{(1)}(\eta)$ above. However, nonzero Eq. (28) would imply that the different work moments, which are calculated by taking different orders of derivatives of the CF with respect to $i\eta$ are complex. Hence, $\Phi^{(1)}(\eta)$ must be zero. This fact is not very apparent if we simply see Eq. (25). After a carefully revisiting Eqs. (14)-(15), we find that there is a key relation between their solutions:

$$K^{(1)} = -\frac{i\eta}{2}HAK^{(0)}.$$

(29)

Obviously, the above equation ensures that the first order quantum correction $\Phi^{(1)}$ is exactly zero. As a result, if one wants to obtain meaningful quantum corrections, we must expand the quantum CF at least till the second order of $\hbar^2$. In principle, Eq. (17) has provided the answer. However, this equation is too complicated to obtain its solution.

In fact the quantum CF (3) has an alternative expression, $\Phi(\eta) = \text{Tr} \left[ e^{i\eta H(t)} U(t) e^{-i\eta H(0)} \rho_0 U^\dagger(t) \right] \equiv \text{Tr} \left[ e^{i\eta H(t)} \hat{\rho}(t) \right].$ (30)

We called $\hat{\rho}(t)$ the heat characteristic operator (HCO) $[4]$. The operator satisfies the Liouville-von Neumann equation

$$\partial_t \hat{\rho}(t) = \frac{1}{ih} \left[ \hat{H}(t), \hat{\rho}(t) \right],$$

(31)

with a modified initial condition

$$\hat{\rho}(0) = \frac{e^{-i(\eta + \beta)\hat{H}(0)}}{Z(0)}.$$

(32)

We may write Eq. (31) in the phase space representation as well: let the Weyl symbol $[\hat{\rho}(t)]_w = P(z,t)$, then it satisfies the following equation,

$$\partial_t P(z,t) = -H(z,t)\Delta P(z,t) + (i\hbar)^2 \frac{1}{24} \partial_z^2 U \partial_z^2 P(z,t) + \cdots,$$

(33)

and the initial condition is

$$P(z,0) = P_{eq}(i\eta + \beta, z,0) + (i\hbar)^2 P_{eq}(i\eta + \beta, z,0) \delta f(i\eta + \beta, z,0) + \cdots.$$  

(34)

The reader is reminded that Eqs. (33) and (34) contain only terms of even powers of $\hbar$. If one can solve Eq. (33), the CF is evaluated by

$$\Phi(\eta) = \int_{-\infty}^{+\infty} dz \{ \exp[i\eta \hat{H}(t)] \}_w P(z,t).$$

(35)

In general, it is a very difficult task. Hence, we have to restore to the $\hbar$ series expansion again. Expanding the solution of Eq. (33) in even powers of $\hbar$ [38],

$$P(z,t) = P^{(0)}(z,t) + (i\hbar)^2 P^{(2)}(z,t) + \cdots$$

(36)

we obtain their solutions as follows:

$$P^{(0)}(z,t) = \rho_{eq}[i\eta + \beta, \psi_0^{-1}(z,t), 0],$$

$$P^{(2)}(z,t) = \rho_{eq}[i\eta + \beta, \psi_0^{-1}(z,t), 0] \delta f[i\eta + \beta, \psi_0^{-1}(z,t), 0] +$$

$$\frac{1}{24} \int_0^t dt' \partial_{z'}^2 U \partial_{z'}^2 P^{(0)}(z', t') \bigg|_{z' = \psi_0^{-1}(z,t)}.$$

(37)

(38)

According to Eq. (35), $P^{(0)}(z,t)$ obviously leads to the zeroth-order CF, Eq. (27). If we substitute $P^{(2)}(z,t)$ and collect all terms of the second power of $\hbar$, we find that the quantum correction of the second order of $\hbar$ is composed by three terms,

$$\Phi^{(2)}(\eta) = \Phi_m^{(2)}(\eta) + \Phi_i^{(2)}(\eta) + \Phi_d^{(2)}(\eta),$$

(39)
respective, where

\[ \Phi_{m}^{(2)}(\eta) = \left\langle e^{i\eta \int_{0}^{t} ds \tilde{p} H[z(s), s]} f[-i\eta, z(t), t] \right\rangle, \quad (40) \]

\[ \Phi_{d}^{(2)}(\eta) = \left\langle e^{i\eta \int_{0}^{t} ds \tilde{p} H[z(s), s]} \delta f[i\eta + \beta, z(0), 0] \right\rangle, \quad (41) \]

\[ \Phi_{d}^{(2)}(\eta) = \left\langle e^{i\eta \int_{0}^{t} ds \tilde{p} H[z(s), s]} \int_{0}^{t} ds Q[z(s), s, i\eta + \beta] \right\rangle, \quad (42) \]

the integrand in the last equation is

\[ \frac{1}{24} \partial_{\eta}^{2} U \left[ -(i\eta + \beta)\partial_{\eta}^{2} \tilde{H} + 3(i\eta + \beta)^{2}(\partial_{\eta}^{2} \tilde{H})(\partial_{\eta} \tilde{H}) - (i\eta + \beta)^{3} \left( \partial_{\eta} \tilde{H} \right)^{3} \right], \quad (43) \]

and

\[ \tilde{H}(z, s) \equiv H[\psi(t)^{-1}(z, s), 0]. \quad (44) \]

Although these terms seem complex in form, particularly Eq. (42), their physical origins are very clear: \( \Phi_{m}^{(2)}(\eta) \) is the quantum effect of the second energy projective measurement, \( \Phi_{d}^{(2)}(\eta) \) is arisen from the quantum correction of the initial condition, and \( \Phi_{d}^{(2)}(\eta) \) is the quantum correction to the classical dynamic equation. Therefore, these quantum effects play their roles independently in the corrections of the second power of \( \hbar \). Before closing this section, we want to present two comments. One is that Eq. (29) has a simple explanation using the dynamics about \( \omega \) and the dot denotes a derivative with respect to time. We used the subscript star to denote that it is the result achieved from the exact quantum work statistics. Expanding Eq. (10) in powers of \( \hbar \), and we have

\[ \Phi_{s}(\eta) = \exp \left[ -\frac{i\eta F(t)^{2}}{2m\omega^{2}} + c(t) \left( e^{i\eta \hbar \omega} - 1 \right) \right]. \quad (46) \]

where

\[ c(t) = \frac{1}{2m\omega^{2}} \left\vert \int_{0}^{t} ds F e^{i\omega s} \right\vert^{2}, \quad (47) \]

and the dot denotes a derivative with respect to time. We used the subscript star to denote that it is the result achieved from the exact quantum work statistics. Expanding Eq. (10) in powers of \( \hbar \), and we have

\[ \Phi_{s}^{(0)}(\eta) = \exp \left[ -\frac{i\eta F(t)^{2}}{2m\omega^{2}} + i\eta c(t) \right], \quad (48) \]

\[ \Phi_{s}^{(2)}(\eta) = \frac{(\beta + i\eta)^{2} \omega^{2} c(t)}{12\beta} \Phi_{s}^{(0)}(\eta). \quad (49) \]

It is also easy to verify that in this expansion there are only even powers of \( \hbar \).

Now we are in position to check whether Eqs (27) and (39) can reobtain Eqs (48) and (49), respectively. Using Eq. (29) and the exact flow map of the classical harmonic oscillator (see Appendix C), we can straightforwardly calculate the zeroth-order CF, \( \Phi_{s}^{(0)}(\eta) \), and the result agrees with Eq. (48). The calculation of \( \Phi_{s}^{(2)}(\eta) \) is relatively complicated. Because the potential of the harmonic oscillator is

\[ U(x, t) = \frac{m\omega^{2}x^{2}}{2} + F(t)x, \quad (50) \]
FIG. 1. The real (a) and imaginary (b) parts of the CFs of the harmonic oscillator at $\beta = 2$. The black solid and dashed lines are the results of the exactly quantum and fully classical CFs, Eqs (46) and (48), respectively. The dotted lines are the results of the CF having the $\hbar^2$-quantum correction. (c) and (d) are the second and third work moments versus the inverse temperature $\beta$. They evaluated by these CFs, respectively. We do not show the first moment work (or the mean work), since they are the same in the special harmonic oscillator. For convenience, we have let $m = \hbar = \omega = 1$ in the computations.

The dynamic correction term, $\Phi_d^{(2)}(\eta)$, is of course zero. In addition, in this specific system,

$$\langle f(0) \rangle_{eq} = \frac{\omega^2 \beta^2}{24}. \quad (51)$$

Substituting all relevant quantities into Eqs. (40) and (41), doing straightforward algebraic calculations, we obtain these two corrections as follows,

$$\Phi_i^{(2)}(\eta) = \left[ -\frac{\omega^2(i\eta + \beta)^3(2\beta - 2\eta^2c(t))}{24\beta^2} + \frac{\omega^2(i\eta + \beta)^2}{8} - \frac{\omega^2 \beta^2}{24} \right] \Phi_0^{(0)}(\eta). \quad (52)$$

$$\Phi_m^{(2)}(\eta) = \left[ \frac{\omega^2(i\eta)^3(2\beta - 2\eta^2c(t))}{24\beta^2} + \frac{(i\eta)^4\omega^2c(t)}{6\beta} + \frac{\omega^2(i\eta)^3c(t)}{12} - \frac{\omega^2 \eta^2}{8} \right] \Phi_0^{(0)}(\eta). \quad (53)$$

Their sum is just $\Phi^{(2)}(\eta)$ given in Eq. (49). Some useful formulas in the deriving process are presented in Appendix C.

In order to show the importance of the $\hbar^2$-quantum correction, we show the CFs in the panels (a) and (b) of Fig. (V), where includes the exactly quantum CF, the fully classical CF, and the CF with the quantum correction, respectively. For the sake of simplicity, we apply a linear force, $F(t) = t$ ($0 \leq t \leq 1$) therein. Although the semi-classical CF cannot completely recover the exact one, particularly at the large $\eta$-values (lower temperature case), it indeed improves the classical CF. If we check their work moments, this improvement is very significant; see the panels (c) and (d) in the same figure.
VI. CONCLUSION

In this paper, we studied the quantum corrections of the work statistics in the closed quantum systems by expanding the quantum CF of work in powers of $\hbar$. The forced harmonic oscillator clearly shows the satisfactory precision of our formulas, particularly in the range of moderate and high temperature. We think that our results shall be useful when studying complicated quantum systems. Because the phase trajectory of classical system and thermal equilibrium state can be efficiently simulated by molecular dynamics and/or Monte-Carlo methods nowadays, $\hbar^2$-corrections provides a rigor alternative to the full quantum work statistics.

There are several possible theoretical extensions of the current work. For instance, if we take account for higher powers of $\hbar$, it should be interesting to see whether these additional quantum corrections can lead into significant improvements of the quantum CF of work. In addition, if there are many particles present in quantum systems, quantum statistics has to be taken into account. Finally, for open quantum systems, we have established the quantum FK formula as well [4, 40, 41]. The exact meaning of a quantum-classical correspondence in these situations shall be worth investigating in detail.

ACKNOWLEDGMENTS

This work was supported by the National Science Foundation of China under Grant Nos. 11174025 and 11575016. We also appreciate the support of the CAS Interdisciplinary Innovation Team, No. 2060299.

APPENDIX A: THE SECOND POWER OF $\hbar$ IN EQ. (12)

The term $(\cdots K \cdots)$ is complex since it includes the $\hbar^2$ contributions from both the commutator $[,]$ and $\hat{\Omega}$:

$$(\cdots K \cdots) = \frac{1}{24} \frac{\partial^3 U}{\partial z^3} \frac{\partial^3 K}{\partial p^3} + \left\{ \frac{8}{27} \frac{\partial H}{\partial t} e^{i\eta H} \right\} \Lambda^2 \left( e^{-i\eta H} \right) +$$

$$i\eta \frac{\partial H}{\partial t} \left[ f(z,t,i\eta) + f(z,t,-i\eta) \right] + \frac{\partial}{\partial t} f(z,t,-i\eta) - \frac{(i\eta)^2}{4} \left( \frac{\partial H}{\partial t} \Lambda K \right) + \frac{i\eta}{8} \frac{\partial H}{\partial t} \Lambda^2 \right\} K, \quad (54)$$

where

$$\Lambda^2 = \frac{\partial^2}{\partial x^2} - 2 \frac{\partial}{\partial x} \frac{\partial}{\partial p} - \frac{\partial^2}{\partial p^2}. \quad (55)$$

APPENDIX B: AN ALTERNATIVE UNDERSTANDING OF EQ. (29)

According to the definitions of $\hat{K}$ and $\hat{\rho}$, we have

$$\hat{\rho} = e^{iH(t)} \hat{K}. \quad (56)$$

Expressing them in the phase space representation and expanding them till the second powers of $\hbar$, we have

$$P^{(0)}(z,t) + (i\hbar)^2 P^{(2)}(z,t) + \cdots$$

$$= e^{-iH(z,t)} K^{(0)}(z,t) + (i\hbar) e^{-iH(z,t)} \left[ K^{(1)} + \frac{i\eta}{2} H(z,t) \Lambda K^{(0)}(z,t) \right] + \cdots \quad (57)$$

Because $\hbar$-term on the left hand side is zero, we immediately reobtain Eq. (29). Of course, this is imposed by the Liouville-von Neumann equation and the specific initial condition.

APPENDIX C: SEVERAL USEFUL FORMULAS IN THE FORCED HARMONIC OSCILLATOR

The flow map $\psi$ of the classical harmonic oscillator with the Hamiltonian,

$$H(z,t) = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} + F(t)x, \quad (58)$$

...
has an analytical expression:

\[ x(t) = x_0 \cos(\omega t) + \frac{p_0}{m\omega} \sin(\omega t) - l(t), \]  
\[ p(t) = -m\omega x_0 \sin(\omega t) + p_0 \cos(\omega t) - \dot{l}(t). \]

where the function \( l(t) \) is

\[ l(t) = \frac{1}{m\omega} \int_0^t F(s) \sin(\omega(t - s)) ds. \]

Hence, the difference of the Hamiltonian at two phase points along the same phase trajectory is

\[ H(z(t), t) - H(z_0, 0) = a(t)x_0 + b(t)p_0 + c(t) - \frac{F(t)^2}{2m\omega^2}, \]

where

\[ a(t) = m\omega \sin(\omega t) \dot{l}(t) - m\omega^2 \cos(\omega t) l(t) + F(t) \cos(\omega t), \]
\[ b(t) = -\cos(\omega t) \dot{l}(t) - \omega \sin(\omega t) l(t) + \frac{F(t)}{m\omega} \sin(\omega t). \]

To arrive these results, we have used the following relation,

\[ c(t) = \frac{F(t) - m\omega^2 l(t)^2}{2m\omega^2} + \frac{m\dot{l}(t)^2}{2} = \frac{mb(t)^2}{2} + \frac{a(t)^2}{2m\omega^2}. \]

The second equation is applied in deriving the concrete expressions of \( \Phi_i^{(2)}(\eta) \) and \( \Phi_{\nu i}^{(2)}(\eta) \).
To be different from our case, Wigner studied the expansion of $\exp(-\beta H)$ in powers of $\hbar$. However, his formulas remain valid and we only simply replace his results with $\beta$ by $i\eta$. 

The reason is that the evolution equation and initial condition contain only the even powers of $\hbar$.

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