Note on the Fractional Mittag-Leffler Functions by Applying the Modified Riemann-Liouville Derivatives

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ABSTRACT: In this article, the fractional derivatives in the sense of the modified Riemann-Liouville derivative is employed for constructing some results related to Mittag-Leffler functions and established a number of important relationships between the Mittag-Leffler functions and the Wright function.

Key Words: Fractional calculus, Modified Riemann-Liouville derivative, Mittage-Leffler function.

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1. Introduction

It is well known that with the classical Riemann-Liouville definition of fractional derivative [2,5,15], the fractional derivative of a constant is not zero. The most useful alternative which has been proposed to cope with this feature is known Caputo derivative [6], but in this derivative fractional derivative would be defined for differentiable functions only. A modification of the Riemann-Liouville has been defined to deal with non-differentiable functions [3,4,9,21,16,23] and it is given as:

Definition 1.1. Let $f : \mathbb{R} \to \mathbb{R}$, $x \to f(x)$ denote a continuous function. The modified Riemann-Liouville derivative of order $\alpha$ is defined by the expression

$$D^\alpha f(x) = \begin{cases} 
\frac{1}{\Gamma(-\alpha)} \int_a^x (x - \eta)^{-\alpha-1} f(\eta) d\eta & ; \alpha < 0, \\
\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x - \eta)^{-\alpha} [f(\eta) - f(a)] d\eta & ; 0 < \alpha < 1, \\
(f^{(\alpha-m)}(x))^{(m)} & ; m \leq \alpha < m + 1.
\end{cases}$$

Some important properties for this kind of derivatives were given in [20] as follows:

1. $D^\alpha x^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1-\alpha)} x^{\mu-\alpha}, \mu > 0$,
2. $D^\alpha (f(x)g(x)) = (D^\alpha f(x)) g(x) + f(x) (D^\alpha g(x))$,
3. $D^\alpha f(u(x)) = D^\alpha f(u) (D(u))^\alpha$,
4. $D^\alpha (m) = 0$ where $m$ is constant function.

There are some special functions which are studied their fractional derivative by several researchers (Agarwal [1], Erdelyi [7] and Miller [18]). In this article, we deal with some of these functions such as Mittag-Leffler and Wright functions.

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The Mittag-Leffler function \([7,8]\) of one parameter is denoted by \(E_\alpha(x)\) and defined by:

\[
E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}, \quad \alpha \in \mathbb{C}, \alpha > 0. \tag{1.1}
\]

This function plays a crucial role in classical calculus for \(\alpha = 1\), for \(\alpha = 1\) it becomes the exponential function, that is \(e^x = E_1(x)\)

\[
e^x = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k + 1)}.
\]

The other important function which is a generalization of series is represented by:

\[
E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta \in \mathbb{C}, \alpha > 0. \tag{1.2}
\]

The functions (1.1) and (1.2) play important role in fractional calculus, also we note that when \(\beta = 1\) in (1.2), then (1.1) is obtained which mean that \(E_{\alpha,1}(x) = E_\alpha(x)\).

Another form which is generalization of (1.1) and (1.2) was introduced by Prabhakar [22] such as:

\[
E^\delta_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{(\delta)_k x^k}{\Gamma(\alpha k + \beta)} \Gamma(\alpha k + \beta), \quad \alpha, \beta, \delta \in \mathbb{C}, \alpha > 0, \tag{1.3}
\]

where \((\delta)_k\), the Pochhammer symbol, is defined by

\[(\delta)_k = \delta(\delta + 1)\cdots(\delta + k - 1), \quad \delta \in \mathbb{C}, k \in \mathbb{N},\]

while

\[(\delta)_0 = 1, \delta \neq 0.\]

There are some special cases of (1.3) such as:

1. \(E^1_{\alpha,1}(x) = E_\alpha(x)\),
2. \(E^1_{\alpha,\beta}(x) = E_{\alpha,\beta}(x)\).

The second functions will be discussed is Wright function, which is defined as

\[
W(x; \alpha, \beta) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)k!}.
\]

This function plays an important role in the solution of a linear partial differential equation. Furthermore, there is an interesting link between the Wright function and the Mittag-Leffler function. Hence, some useful relationships between those functions have been obtained in this work.

2. Main Result

Now, we point out some formulas which do not hold for the classical Riemann-Liouville definition, but apply with the modified Riemann-Liouville definition.

**Theorem 2.1.** Assume that \(\alpha > 0, \beta > 0\) for \(\lambda \in \mathbb{R}\), then the following formula holds

\[
D^\alpha \left[ x^{\beta-1} E_{\alpha,\beta}^\delta(\lambda x^\alpha) \right] = \frac{x^{\beta-\alpha-1}}{\Gamma(\beta - \alpha)} + \lambda x^{\beta-1} E_{\alpha,\beta}^\delta(\lambda x^\alpha) \tag{2.1}
\]
Proof.

\[
D^\alpha \left[ x^{\beta-1} E_{\alpha, \beta}^\delta (\lambda x^\alpha) \right] = D^\alpha \sum_{k=0}^{\infty} \frac{(\delta)_k \lambda^k}{\Gamma(k+\beta)} x^{\alpha k + \beta - 1} = D^\alpha \left[ x^{\beta-1} \frac{1}{\Gamma(\beta)} + \frac{(\delta)_1 \lambda}{\Gamma(\alpha + \beta)} x^{\alpha + \beta - 1} + \frac{(\delta)_2 \lambda^2}{\Gamma(2\alpha + \beta)2!} x^{2\alpha + \beta - 1} + \frac{(\delta)_3 \lambda^3}{\Gamma(3\alpha + \beta)3!} x^{3\alpha + \beta - 1} + \ldots \right]
\]

Then we obtain the following relation

\[
D^\alpha \left[ x^{\beta-1} E_{\alpha, \beta}^\delta (\lambda x^\alpha) \right] = \frac{x^{\beta-1}}{\Gamma(\beta - \alpha)} + \lambda x^{\beta-1} \sum_{k=0}^{\infty} \frac{(\delta)_{k+1} \lambda^k}{(k+1)} W(x^\alpha; \alpha, \beta).
\]

Also, the following formula is given

\[
D^\alpha \left[ x^{\beta-1} E_{\alpha, \beta}^\delta (\lambda x^\alpha) \right] = \frac{x^{\beta-1}}{\Gamma(\beta - \alpha)} + \lambda x^{\beta-1} E_{\alpha, \beta}^\delta (\lambda x^\alpha).
\]

\[\square\]

Remark 2.2. 1. Since

\[
x^{\beta-1} E_{\alpha, \beta}^{-1} (\lambda x^\alpha) = \frac{x^{\beta-1}}{\Gamma(\beta)} - \frac{\lambda x^{\alpha + \beta - 1}}{\Gamma(\alpha + \beta)},
\]

then

\[
D^\alpha \left[ x^{\beta-1} E_{\alpha, \beta}^{-1} (\lambda x^\alpha) \right] = \frac{x^{\beta-1}}{\Gamma(\beta - \alpha)} - \frac{\lambda x^{\beta-1}}{\Gamma(\beta)} = x^{\beta-1} E_{\alpha, \beta-\alpha}^0 (\lambda x^\alpha) - \lambda x^{\beta-1} E_{\alpha, \beta}^0 (\lambda x^\alpha).
\]

2. When \(\delta = 1\) in formula (2.1), then we obtain

\[
D^\alpha \left[ x^{\beta-1} E_{\alpha, \beta} (\lambda x^\alpha) \right] = \frac{x^{\beta-1}}{\Gamma(\beta - \alpha)} + \lambda x^{\beta-1} E_{\alpha, \beta} (\lambda x^\alpha).
\]

(2.2)

3. When \(\delta = 1\) and \(\beta = 1\) in formula (2.1) and \(1 - \alpha \to 0^+\), then we have the following interesting formula

\[
D^\alpha E_{\alpha} (\lambda x^\alpha) = \lambda E_{\alpha} (\lambda x^\alpha).
\]

(2.3)
Also, we can show this formula by another method such as

\[
D^\alpha E_\alpha (\lambda x^\alpha) = D^\alpha \sum_{k=0}^{\infty} \frac{\lambda^k x^{\alpha k}}{\Gamma(\alpha k + 1)}
= \sum_{k=1}^{\infty} \frac{\lambda^{k-1} x^{\alpha k}}{\Gamma(\alpha k + 1)}
= \sum_{k=0}^{\infty} \frac{\lambda^{k+1} x^{\alpha (k+1)-\alpha}}{\Gamma(\alpha k + 1)}
= \lambda \sum_{k=0}^{\infty} \frac{\lambda^k x^{\alpha k}}{\Gamma(\alpha k + 1)}
= \lambda E_\alpha (\lambda x^\alpha).
\]

The following figures show some modified Riemann-Liouville derivative of order close to zero for \(E_\alpha (x^\alpha)\).

Figure 1: \(D^{0.1} E_{0.1}(x^{0.1})\).

Figure 2: \(D^{0.4} E_{0.4}(x^{0.4})\).

Figure 3: \(D^{0.7} E_{0.7}(x^{0.7})\).

Figure 4: \(D^{0.8} E_{0.8}(x^{0.8})\).

**Corollary 2.3.** Let \(\alpha > 0, \beta > 0\) and for \(\lambda \in \mathbb{R}\), then the following formula holds

\[
D^\alpha E_\alpha (\lambda x) = \lambda \alpha^{-\alpha} x^{1-\alpha} E_\alpha (\lambda x). \tag{2.4}
\]

**Proof.** We can write

\[
D^\alpha E_\alpha (\lambda x) = D^\alpha E_\alpha \left( \left( \frac{1}{\alpha} x^{\frac{1}{\alpha}} \right)^\alpha \right).
\]
Let \( u = \lambda^{\frac{1}{\alpha}} x^{\frac{1}{\alpha}} \) and by applying the fractional derivative properties, we get

\[
D^\alpha E_\alpha (\lambda x) = E_\alpha (u^\alpha) \left[ \lambda^{\frac{1}{\alpha}} \alpha^{-1} x^{\frac{1}{\alpha}-1} \right]^\alpha = \lambda \alpha^{-\alpha} x^{1-\alpha} E_\alpha (\lambda x).
\]

\[\Box\]

In the following figures there are some modified Riemann-Liouville derivative of order closed to zero for \( E_\alpha(x) \).

**Figure 5:** \( D^{0.1} E_{0.1}(x) \).

**Figure 6:** \( D^{0.4} E_{0.4}(x) \).

**Figure 7:** \( D^{0.7} E_{0.7}(x) \).

**Figure 8:** \( D^{0.8} E_{0.8}(x) \).

**Theorem 2.4.** Assume that \( \alpha > 0, \beta > 0 \) for \( \lambda \in \mathbb{R} \), then the following formula holds

\[
D^\gamma \left[ x^{\beta-1} E^\delta_{\alpha, \beta}(\lambda x^\alpha) \right] = \frac{x^{\beta-1-\gamma}}{\Gamma(\beta-\gamma)} + \lambda x^{\alpha+\beta-\gamma-1} E^\delta_{\alpha, \alpha+\beta-\gamma}(\lambda x^\alpha).
\]  

(2.5)

**Proof.**

\[
D^\gamma \left[ x^{\beta-1} E^\delta_{\alpha, \beta}(\lambda x^\alpha) \right] = D^\alpha \sum_{k=0}^{\infty} \frac{(\delta)_k \lambda^k}{\Gamma(\alpha k + \beta k!)} x^{\alpha k + \beta - 1}
\]

\[
= D^\gamma \left[ \frac{x^{\beta-1}}{\Gamma(\beta)} + \frac{(\delta)_1 \lambda}{\Gamma(\alpha + \beta)} x^{\alpha + \beta - 1} + \frac{(\delta)_2 \lambda^2}{\Gamma(2\alpha + \beta) 2!} x^{2\alpha + \beta - 1} + \frac{(\delta)_3 \lambda^3}{\Gamma(3\alpha + \beta) 3!} x^{3\alpha + \beta - 1} + \ldots \right]
\]

\[
x^{\beta-\gamma-1} \sum_{k=0}^{\infty} \frac{(\delta)_k \lambda^k}{\Gamma(\alpha k + \beta - \gamma) k!} x^{\alpha k}.
\]
Hence, the relation with the Wright function is

\[ D^\gamma [x^{\beta-1} E_{\alpha,\beta}^\delta(\lambda x^{\alpha})] = x^{\beta-\gamma-1} \sum_{k=0}^{\infty} (\delta)_k \lambda^k W(x^{\alpha}; \alpha, \beta - \gamma). \]

Also,

\[ D^\gamma [x^{\beta-1} E_{\alpha,\beta}^\delta(\lambda x^{\alpha})] = \frac{x^{\beta-\gamma-1}}{\Gamma(\beta - \gamma)} + \lambda x^{\alpha+\beta-\gamma-1} E_{\alpha,\alpha+\beta-\gamma}(\lambda x^{\alpha}). \]

**Remark 2.5.**

1. If we set \( \gamma = \alpha \) in formula (2.5), then the formula (2.1) is obtained.

2. Let \( \delta = 1 \) in (2.5), then

\[ D^\gamma [x^{\beta-1} E_{\alpha,\beta}(\lambda x^{\alpha})] = \frac{x^{\beta-1-\gamma}}{\Gamma(\beta - \gamma)} + \lambda x^{\alpha+\beta-\gamma-1} E_{\alpha,\alpha+\beta-\gamma}(\lambda x^{\alpha}) \]

Also, if \( \beta = 1 \) and \( 1 - \gamma \to 0^+ \) then

\[ D^\gamma E_\alpha(\lambda x^{\alpha}) = \lambda x^{\alpha-\gamma} E_{\alpha,\alpha-\gamma+1}(\lambda x^{\alpha}). \]

This formula is also true when \( \alpha > 0 \) and \( 0 < \gamma < 1 \) for \( \lambda \in \mathbb{R} \) by the following method:

\[ D^\gamma E_\alpha(\lambda x^{\alpha}) = D^\gamma \sum_{k=0}^{\infty} \frac{\lambda^k x^{\alpha k}}{\Gamma(\alpha k + 1)} = \sum_{k=1}^{\infty} \frac{\lambda^k x^{\alpha k-\gamma}}{\Gamma(\alpha k - \gamma + 1)} = \sum_{k=0}^{\infty} \frac{\lambda^{k+1} x^{\alpha k+\alpha-\gamma}}{\Gamma(\alpha k + \alpha - \gamma + 1)} = \lambda x^{\alpha-\gamma} \sum_{k=0}^{\infty} \frac{\lambda^k x^{\alpha k}}{\Gamma(\alpha k + \alpha - \gamma + 1)} = \lambda x^{\alpha-\gamma} E_{\alpha,\alpha-\gamma+1}(\lambda x^{\alpha}). \]

In the following figures show \( D^\gamma E_\alpha(x^{\alpha}), \lambda = 1 \).
Moreover, we note that \( \frac{x^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)} \to 0 \) when \( \beta - \gamma \to 0^+ \), then
\[
D^\beta \left[ x^{\beta-1} E_{\alpha,\beta}(\lambda x^\alpha) \right] = \lambda x^{\alpha-1} E_{\alpha,\alpha}(\lambda x^\alpha).
\]

3. Assume that \( \alpha = \gamma \), \( \beta = 1 \), \( \delta = 1 \) and \( 1 - \gamma \to 0^+ \) in (2.5), then the formula (2.3) is given.

**Corollary 2.6.** We can write
\[
D^\beta E_\alpha(\lambda x) = D^\beta E_\alpha \left( \left( \lambda^{\frac{1}{\alpha}} x^{\frac{1}{\beta}} \right)^\alpha \right).
\]

Let \( u = \lambda^{\frac{1}{\alpha}} x^{\frac{1}{\beta}} \) and by applying the fractional derivative properties, we get
\[
D^\beta E_\alpha(\lambda x) = u^{\alpha-\beta} E_{\alpha,\alpha-\beta+1}(u^\alpha) \left[ \lambda^{\frac{1}{\alpha}} x^{\frac{1}{\beta}} \right]^{\alpha-1} = \lambda^{\alpha-\beta} x^{1-\beta} E_{\alpha,\alpha-\beta+1}(\lambda x).
\]

Then
\[
D^\beta E_\alpha(\lambda x) = \lambda^{\alpha-\beta} x^{1-\beta} E_{\alpha,\alpha-\beta+1}(\lambda x) \tag{2.8}
\]

Let \( \beta = \alpha \) in the above formula, then
\[
D^\alpha E_\alpha(\lambda x) = \lambda^{\alpha-\alpha} x^{1-\alpha} E_\alpha(\lambda x).
\]

In the following figures show \( D^\beta E_\alpha(x), \lambda = 1 \).
**Theorem 2.7.** Assume that $\alpha, \beta > 0$ for $\lambda \in \mathbb{R}$, then the following formula holds

$$D^\beta E^\delta_{\alpha, \beta} (\lambda x^{-\alpha}) = (-1)^\beta \lambda x^{-\alpha - \beta} E^\delta_{\alpha, \alpha} (\lambda x^{-\alpha})$$  \hspace{1cm} (2.9)

**Proof.**

$$D^\beta E^\delta_{\alpha, \beta} (\lambda x^{-\alpha}) = D^\beta \sum_{k=0}^{\infty} \frac{(\delta)_k \lambda^k}{\Gamma(ak + k + \beta)k!} x^{-ak}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^\beta (\delta)_{k+1} \lambda^{k+1}}{(ak + \alpha)(k + 1)!} x^{-ak - \alpha - \beta}.$$ 

Then

$$D^\beta E^\delta_{\alpha, \beta} (\lambda x^{-\alpha}) = (-1)^\beta \sum_{k=0}^{\infty} \frac{(\delta)_{k+1} \lambda^{k+1} x^{-\alpha - \beta}}{k + 1} W(x^{-\alpha}; \alpha, \alpha).$$

Moreover,

$$D^\beta E^\delta_{\alpha, \beta} (\lambda x^{-\alpha}) = (-1)^\beta \lambda x^{-\alpha - \beta} E^\delta_{\alpha, \alpha} (\lambda x^{-\alpha}).$$
Here when \( \delta = 1 \), then
\[
D^\beta E_{\alpha,\beta}(\lambda x^{-1}) = (-1)^\beta \lambda^\beta x^{-\alpha} x^{-\alpha-\beta} E_{\alpha,\alpha}(\lambda x^{-1}).
\]

**Theorem 2.9.** Assume that \( \alpha > 0, \beta > 0 \) and \( \lambda \in \mathbb{R} \), then the following formula holds
\[
D^{\alpha n} E_\alpha(\lambda x^\alpha) = \lambda^n E_\alpha(\lambda x^\alpha)
\]  
(2.11)

where \( n = 1, 2, 3, \ldots \)

**Proof.**

\[
D^{\alpha n} E_\alpha(\lambda x^\alpha) = D^{\alpha n} \sum_{k=0}^{\infty} \frac{\lambda^k x^{\alpha k}}{\Gamma(\alpha k + 1)}
\]
\[
= \sum_{k=n}^{\infty} \frac{\lambda^{k+n} x^{\alpha k}}{\Gamma(\alpha k + 1 + n)}
\]
\[
= \sum_{k=0}^{\infty} \frac{\lambda^{k+n} x^{\alpha k}}{\Gamma(\alpha k + 1)}
\]
\[
= \lambda^n \sum_{k=0}^{\infty} \frac{\lambda^{k+n} x^{\alpha k}}{\Gamma(\alpha k + 1)}
\]
\[
= \lambda^n E_\alpha(\lambda x^\alpha).
\]

\( \square \)

Note that, if \( n = 1 \), then we obtain formula (2.3).

**Corollary 2.10.** Assume that \( \alpha > 0, \beta > 0 \) and \( \lambda \in \mathbb{R} \), then the following formula holds
\[
D^{\alpha n} E_\alpha(\lambda x) = \lambda^\alpha x^{(1-\alpha)n} E_\alpha(\lambda x)
\]  
(2.12)

where \( n = 1, 2, 3, \ldots \)

**Proof.** Let
\[
D^{\alpha n} E_\alpha(\lambda x) = D^{\alpha n} E_\alpha \left( \left( \frac{\lambda^\frac{\alpha}{n} x^{\frac{1}{n}}} \right)^\alpha \right)
\]
and put \( u = \lambda^\frac{\alpha}{n} x^{\frac{1}{n}} \), then by applying the fractional derivative properties, we get
\[
D^{\alpha n} E_\alpha(\lambda x) = E_\alpha(u^\alpha) \left[ \lambda^{\frac{\alpha}{n} \alpha^{-1} \frac{1}{n} - 1} \right]^{\alpha n}
\]
\[
= \lambda^{\alpha n} x^{(1-\alpha)n} E_\alpha(\lambda x).
\]

Let \( n = 1 \), then formula (2.4) is obtained . 

\( \square \)

**Theorem 2.11.** Assume that \( \alpha > 0, \beta > 0 \) and \( \lambda \in \mathbb{R} \), then the following formula holds
\[
D^{\beta n} E_\alpha(\lambda x^\alpha) = \lambda^{\beta n} x^{(\alpha-\beta)n} E_{\alpha,\alpha n-\beta n+1}(\lambda x^\alpha)
\]  
(2.13)

where \( n = 1, 2, 3, \ldots \)
Proof.

\[ D^{\beta n} E_\alpha (\lambda x^\alpha) = D^{\beta n} \sum_{k=0}^{\infty} \frac{\lambda^k x^{\alpha k}}{\Gamma(\alpha k + 1)} \]

\[ = \sum_{k=n}^{\infty} \frac{\lambda^k x^{\alpha k - \beta n}}{\Gamma(\alpha k - n \beta + 1)} \]

\[ = \sum_{k=0}^{\infty} \frac{\lambda^{k+n} x^{\alpha k + \alpha n - \beta n}}{\Gamma(\alpha k + \alpha n - \beta n + 1)} \]

\[ = \lambda^n x^{\alpha n - \beta n} \sum_{k=0}^{\infty} \frac{\lambda^k x^{\alpha k}}{\Gamma(\alpha k + \alpha n - \beta n + 1)} \]

\[ = \lambda^n x^{\alpha n - \beta n} E_{\alpha, \alpha n - \beta n + 1}(\lambda x^\alpha). \]

Here, when \( n = 1 \), then formula (2.7) is obtained. \( \square \)

**Corollary 2.12.** Assume that \( \alpha > 0, \beta > 0 \) and \( \lambda \in \mathbb{R} \), then the following formula holds

\[ D^{\beta n} E_\alpha (\lambda x) = \lambda^n \alpha^{-\beta n} x^{(1-\beta)n} E_{\alpha, \alpha n - \beta n + 1}(\lambda x) \tag{2.14} \]

where \( n = 1, 2, 3, \ldots \)

**Proof.** Assume that

\[ D^{\beta n} E_\alpha (\lambda x) = D^{\beta n} E_\alpha \left( \left( \lambda^{\frac{1}{\alpha^*}} x^{\frac{1}{\alpha^*}} \right)^\alpha \right) \]

and let \( u = \lambda^{\frac{1}{\alpha^*}} x^{\frac{1}{\alpha^*}} \), then by applying the fractional derivative properties, we get

\[ D^{\beta n} E_\alpha (u^\alpha) = u^{\alpha n - \beta n} E_{\alpha, \alpha n - \beta n + 1}(u^\alpha) \left[ \lambda^{\frac{1}{\alpha^*} - 1} x^{\frac{1}{\alpha^*} - 1} \right]^{\beta n} \]

\[ = \lambda^n \alpha^{-\beta n} x^{(1-\beta)n} E_{\alpha, \alpha n - \beta n + 1}(\lambda x). \]

\( \square \)

Also, when \( n = 1 \), then formula (2.8) is obtained.

Kiryakova introduced and studied the multi-index Mittag-Leffler function as their typical representatives, including many interesting special cases that have already proven their usefulness in FC and its applications [12].

**Definition 2.13.** Assume that \( n > 1 \) is an integer, \( \eta_1, \ldots, \eta_n > 0 \) and \( \beta_1, \ldots, \beta_n \) are arbitrary real numbers. The multi-index Mittag-Leffler function is given as

\[ E_{(\frac{1}{\eta_1}), (\frac{1}{\eta_2}), \ldots, (\frac{1}{\eta_n})}(x) = E_{(\frac{1}{\eta_1}), (\frac{1}{\eta_2}), \ldots, (\frac{1}{\eta_n})}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma\left( \frac{k}{\eta_1} + \beta_1 \right) \cdots \Gamma\left( \frac{k}{\eta_n} + \beta_n \right)}. \]

The same function was given by Luchko [17], called by him Mittag-Leffler function of vector index.

Furthermore, the Wright generalized hypergeometric function \( mW_n \) is defined as

\[ mW_n \left[ \begin{array}{c} (a_i, A_i)_{i=1}^m \\ (b_j, B_j)_{j=1}^n \end{array} \right] (x) = \sum_{k=0}^{\infty} \frac{\Gamma(a_1 k + A_1) \cdots \Gamma(a_m k + A_m) x^k}{\Gamma(b_1 k + B_1) \cdots \Gamma(b_n k + B_n) k!}. \]
The \( m \bar{W}_n \) function is a special case of the Fox \( H \)-function

\[
H^{p,q}_{m,n} \left[ x \begin{pmatrix} (a_i, A_i) \choose (b_j, B_j) \end{pmatrix}^m \right].
\]

In particular, when \( A_i = B_j = 1, \forall i, j \), then Meijer’s G-function is obtained

\[
H^{p,q}_{m,n} \left[ x \begin{pmatrix} (a_i, 1) \choose (b_j, 1) \end{pmatrix}^m \right] = G^{p,q}_{m,n} \left[ x \begin{pmatrix} (a_i) \choose (b_j) \end{pmatrix}^n \right].
\]

For more details see [10,11,13,14].

There are some interesting properties related to multi- Mittag-Leffler function which were proven in [12]:

1. \( E_\alpha = 1 \bar{W}_1 \left[ \begin{pmatrix} 1, 1 \choose (\alpha, 1) \end{pmatrix}^m \right]. \)
2. \( E_{\alpha,\beta} = 1 W_1 \left[ \begin{pmatrix} 1, 1 \choose (\alpha, \beta) \end{pmatrix} \right]. \)
3. \( E^{(\frac{1}{m})}_{\alpha,\beta} = 1 W_n \left[ \begin{pmatrix} 1, 1 \choose (\frac{1}{m}, \beta) \end{pmatrix} \right]. \)
4. \( E^{(\frac{1}{m})}_{\alpha,\beta} = \frac{1}{\Gamma(\delta)} 1 W_1 \left[ \begin{pmatrix} 1, \delta \choose (\alpha, \beta) \end{pmatrix} \right]. \)

In the same paper, the author showed Wright function as a case of multi- Mittag-Leffler function with \( n = 2 \):

\[
W(x; \alpha, \beta) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)k!}
\]

\[
= 0 \bar{W}_1 \left[ \begin{pmatrix} \alpha, \beta \choose x \end{pmatrix} \right]
\]

\[
= E^{(2)}_{(\alpha,1),(\beta,1)}(x).
\]

Indeed, the multi-index Mittag-Leffler function when \( \beta_i = 1, \forall i \) can be written as

\[
E^{(\frac{1}{m})}_{\alpha} (x^{\eta_i}) = \sum_{k=0}^{\infty} \frac{n \prod_{i=1}^{n} \Gamma \left( \frac{k}{\eta_i} + 1 \right) x^{\eta_i k}}{\Gamma \left( \frac{k}{\eta_i} + 1 \right) \eta_i}
\]

Then

\[
D^{\frac{1}{m}} E^{(\frac{1}{m})}_{\alpha} (x^{\frac{1}{m}}) = D^{\eta_i} \sum_{k=0}^{\infty} \frac{x^{\eta_i k}}{\Gamma \left( \frac{k}{\eta_i} + 1 \right) \eta_i}
\]

\[
= \sum_{k=0}^{\infty} \frac{\Gamma \left( \frac{k}{\eta_i} + \frac{1}{\eta_i} + 1 \right)}{\Gamma \left( \frac{k}{\eta_i} + 1 \right) \prod_{i=1}^{n} \Gamma \left( \frac{k}{\eta_i} + 1 \right)} x^{\frac{k}{\eta_i}}
\]

\[
= \sum_{k=0}^{\infty} \frac{\Gamma \left( \frac{k}{\eta_i} + 1 \right) \prod_{i=1}^{n} \Gamma \left( \frac{k}{\eta_i} + 1 \right)}{\Gamma \left( \frac{k}{\eta_i} + 1 \right) \eta_i} 1 \bar{W}_n \left[ \begin{pmatrix} 1, 1 \choose \left( \frac{1}{\eta_i}, \frac{1}{\eta_i} + 1 \right) \right] \right].
\]

(2.15)
Here, if we set $\alpha = \frac{1}{\eta_i}$ and $n = 1$, then we obtain formula (2.3), $\lambda = 1$.

$$D^{\gamma}(\frac{1}{\eta_i}) (x) = \eta_i^{-\gamma} x^{1 - \frac{1}{\eta_i}} \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{k}{\eta_i} + \frac{1}{\eta_i} + 1\right)}{\Gamma\left(\frac{k}{\eta_i} + 1\right)} \, i\bar{W}_n \left[ \frac{(1,1)}{\left(\frac{1}{\eta_i}, \frac{1}{\eta_i} + 1\right)_1^n} \right].$$

The above formula can be obtained by putting $u = (x^{\eta_i})^{\frac{1}{\eta_i}}$ and then applying formula (2.15). Especially, if $\alpha = \frac{1}{\eta_i}$ and $n = 1$, formula (2.16) yields to the formula (2.4) when $\lambda = 1$.

**Theorem 2.14.** Assume that $\eta_i > 0$ are arbitrary real numbers and $0 < \gamma < 1$, then the following formula holds

$$D^{\gamma}E\left(\frac{1}{\eta_i}\right) \left( x^{\frac{1}{\eta_i}} \right) = x^{\frac{1}{\eta_i} - \gamma} \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{k}{\eta_i} + \frac{1}{\eta_i} + 1\right)}{\Gamma\left(\frac{k}{\eta_i} + \frac{1}{\eta_i} - \gamma + 1\right)} \, i\bar{W}_n \left[ \frac{(1,1)}{\left(\frac{1}{\eta_i}, \frac{1}{\eta_i} + 1\right)_1^n} \right].$$

Proof.

$$D^{\gamma}E\left(\frac{1}{\eta_i}\right) \left( x^{\frac{1}{\eta_i}} \right) = D^{\gamma} \sum_{k=0}^{\infty} \frac{x^{\frac{k}{\eta_i} - \gamma}}{\prod_{i=1}^{n} \Gamma\left(\frac{k}{\eta_i} + \frac{1}{\eta_i} + 1\right)} = \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{k}{\eta_i} + \frac{1}{\eta_i} + 1\right)}{\Gamma\left(\frac{k}{\eta_i} + \frac{1}{\eta_i} - \gamma + 1\right)} \, i\bar{W}_n \left[ \frac{(1,1)}{\left(\frac{1}{\eta_i}, \frac{1}{\eta_i} + 1\right)_1^n} \right].$$

which is the result. ~

As expected when $\alpha = \frac{1}{\eta_i}$ and $n = 1$, the last formula turns to be the formula (2.7) when $\lambda = 1$.

Since $D^{\gamma}E\left(\frac{1}{\eta_i}\right) (x)$ can be written as $D^{\gamma}E\left(\frac{1}{\eta_i}\right) \left( (x^{\eta_i})^{\frac{1}{\eta_i}} \right)$ and applying (2.17), the following formula is given:

$$D^{\gamma}E\left(\frac{1}{\eta_i}\right) (x) = \eta^{-\gamma} x^{1 - \gamma} \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{k}{\eta_i} + \frac{1}{\eta_i} + 1\right)}{\Gamma\left(\frac{k}{\eta_i} + \frac{1}{\eta_i} - \gamma + 1\right)} \, i\bar{W}_n \left[ \frac{(1,1)}{\left(\frac{1}{\eta_i}, \frac{1}{\eta_i} + 1\right)_1^n} \right].$$

We would like to mention that if $\alpha = \frac{1}{\eta_i}$ and $n = 1$ in formula (2.18), then (2.8) is obtained.

**Corollary 2.15.** For arbitrary $n \geq 2$, let $\forall \eta_i = \infty$ and $\forall \beta_i = 1, i = 1, \ldots, n$. Then

$$D^{\gamma}E^{(n)}_{(0,0,\ldots,0), (1,1,\ldots,1)} (x) = x^{-\gamma} \sum_{k=0}^{\infty} \Gamma(k + 1) \, i\bar{W}_1 \left[ \frac{(1,1)}{1 - (1,1 - \gamma)} \right].$$

Now, we study modified Riemann-Liouville derivative of fractional Sine and Cosine function.

Since

$$\cos_{\alpha}(t^{\alpha}) = \frac{1}{2} [E_{\alpha}(it^{\alpha}) + E_{\alpha}(-it^{\alpha})],$$

then

$$D^{\alpha} \cos_{\alpha}(t^{\alpha}) = \frac{1}{2} [iE_{\alpha}(it^{\alpha}) - iE_{\alpha}(-it^{\alpha})] = - \sin_{\alpha}(t^{\alpha}).$$

Hence, we get a very useful relation

$$D^{\alpha} \cos_{\alpha}(t^{\alpha}) = - \sin_{\alpha}(t^{\alpha}).$$
By using the same technique we can write

\[ D^\alpha \sin_\alpha(t^\alpha) = \cos_\alpha(t^\alpha). \]

Moreover, since \( \cos_\alpha(t) = \frac{1}{2} [E_\alpha(it) + E_\alpha(-it)] \), then

\[
D^\alpha \cos_\alpha(t) = \frac{1}{2} \left[ i\alpha^{-\alpha} t^{1-\alpha} E_\alpha(it) - i\alpha^{-\alpha} t^{1-\alpha} E_\alpha(-it) \right] \\
= -\alpha^{-\alpha} t^{1-\alpha} \sin_\alpha(t).
\]

The following figures show \( D^\alpha \cos_\alpha(x) \) when \( \alpha = 0.3, 0.5 \) and 0.75:

Figure 15: \( D^{0.3} \cos_{0.3}(x) \)

Figure 16: \( D^{0.5} \cos_{0.5}(x) \)

Figure 17: \( D^{0.75} \cos_{0.75}(x) \)

Also, we can write

\[ D^\alpha \sin_\alpha(t) = \alpha^{-\alpha} t^{1-\alpha} \cos_\alpha(t). \]

The next figures show \( D^\alpha \sin_\alpha(x) \) when \( \alpha = 0.3, 0.5 \) and 0.75:
The next step we study $D^\beta \cos_\alpha(t^\alpha)$ and $D^\beta \cos(t^\alpha)$.

\[
D^\beta \cos_\alpha(t^\alpha) = \frac{1}{2} \left[ it^{\alpha-\beta} E_{\alpha,\alpha-\beta+1}(it^\alpha) - it^{\alpha-\beta} E_{\alpha,\alpha-\beta+1}(-it^\alpha) \right] \\
= -t^{\alpha-\beta} \sin_\alpha,\alpha-\beta+1(t^\alpha),
\]

where

\[
\sin_\alpha,\alpha-\beta+1(t^\alpha) = \frac{t^{\alpha}}{\Gamma(2\alpha - \beta + 1)} - \frac{t^{3\alpha}}{\Gamma(4\alpha - \beta + 1)} + \frac{t^{5\alpha}}{\Gamma(6\alpha - \beta + 1)} - \cdots.
\]

Similarly we can show that

\[
D^\beta \sin_\alpha(t^\alpha) = t^{\alpha-\beta} \cos_\alpha,\alpha-\beta+1(t^\alpha),
\]

where

\[
\cos_\alpha,\alpha-\beta+1(t^\alpha) = \frac{1}{\Gamma(\beta)} - \frac{t^{2\alpha}}{\Gamma(3\alpha - \beta + 1)} + \frac{t^{4\alpha}}{\Gamma(5\alpha - \beta + 1)} - \cdots.
\]

\[
D^\beta \cos(t^\alpha) = \frac{1}{2} \left[ i\alpha^{-\beta} t^{1-\beta} E_{\alpha,\alpha-\beta+1}(it) - i\alpha^{-\beta} t^{1-\beta} E_{\alpha,\alpha-\beta+1}(-it) \right] \\
= -\alpha^{-\beta} t^{1-\beta} \sin_\alpha,\alpha-\beta+1(t).
\]

Similarly

\[
D^\beta \sin(t^\alpha) = \alpha^{-\beta} t^{1-\beta} \cos_\alpha,\alpha-\beta+1(t).
\]

**Theorem 2.16.** The fractional derivative of hyperbolic function of order $m$ is given as

\[
D^\alpha [h_v(x, m)] = \frac{x^{v-\alpha-1}}{\Gamma(v - \alpha)} + x^{v+m-\alpha-1} E_{m,v+m-\alpha}(x^m), v = 1, 2, \cdots.
\]

when $v - \alpha \to 0^+$, then

\[
D^\alpha [h_v(x, m)] = x^{m-1} E_{m,m}(x^m).
\]
Proof. Since hyperbolic function of order $m$ is defined as
\[ h_v(x, m) = \sum_{k=0}^{\infty} x^{mk+v-1} \Gamma(mk + v) = x^{v-1} E_{m, v}(x^m), \quad v = 1, 2, \cdots, \]
then by using formula (2.6) we get the result. \hfill \Box

**Theorem 2.17.** The fractional derivative of Mellin-Ross function,

\[ R_{\alpha}(\beta, x) = x^\alpha \sum_{k=0}^{\infty} \frac{\beta^k x^{(\alpha+1)}}{\Gamma((1 + \alpha)(k + 1))} = x^\alpha E_{\alpha+1, \alpha+1}(\beta x^{\alpha+1}), \]

is given by

\[ D^\alpha \left[ x^\alpha E_{\alpha+1, \alpha+1}(\beta x^{\alpha+1}) \right] = \lambda x^\alpha E_{\alpha+1, \alpha+1}(x^{\alpha+1}) \]

The proof is directed by using formula (2.2).

3. Conclusion

In this note, some useful formulas have been established by using modified Riemann-Liouville definition of fractional derivative. These formulas can be used to solve some linear fractional differential equations which are useful in several physical problems.

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