Separation of variables for the Ruijsenaars system

V.B. Kuznetsov, F.W. Nijhoff and E.K. Sklyanin

† Department of Applied Mathematical Studies, University of Leeds, Leeds LS2 9JT, UK
‡ Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606, Japan

Abstract

We construct a separation of variables for the classical $n$-particle Ruijsenaars system (the relativistic analog of the elliptic Calogero-Moser system). The separated coordinates appear as the poles of the properly normalised eigenvector (Baker-Akhiezer function) of the corresponding Lax matrix. Two different normalisations of the BA functions are analysed. The canonicity of the separated variables is verified with the use of $r$-matrix technique. The explicit expressions for the generating function of the separating canonical transform are given in the simplest cases $n = 2$ and $n = 3$. Taking nonrelativistic limit we also construct a separation of variables for the elliptic Calogero-Moser system.
1. Introduction

One of the most powerful methods in studies of Liouville integrable systems is that of Separation of Variables (SoV). Originated with the development of the Hamiltonian mechanics as a method to solve the Hamilton-Jacobi equation for particular Hamiltonians, nowadays it has been applied to many families of finite-dimensional (Liouville) integrable systems (see recent review [31]).

For a very long time a great deal of attention has been given to so-called coordinate separation of variables or to separation in the configuration space (see, for instance, [8, 28, 14, 15, 4, 9, 10, 31] and references therein). In this case the separation variables $u_j$ do not depend on the momenta $p_i$ and are functions of the coordinates $x_i$ only:

$$u_j = u_j(x_1, \ldots, x_N).$$

Such kinds of integrable systems admitting a coordinate (local) separation of variables were studied in detail, although in the same time it was understood that far not every Liouville integrable system can be separated through such a transition to new “coordinates” $u_j$. The class of admissible transformations should be enlarged for a generic integrable system up to a general canonical transformation

$$u_j = u_j(x_1, \ldots, x_N, p_1, \ldots, p_N), \quad v_j = v_j(x_1, \ldots, x_N, p_1, \ldots, p_N).$$

In the context of the Inverse Scattering Method [13, 3, 31] the separation variables $(u, v)$ appear usually as pairs of canonically conjugate variables sitting on the spectral curve of the related $n \times n$ Lax matrix $L(u)$. The coordinates $u_j$ are obtained respectively as the poles of the associated Baker-Akhiezer (BA) function $f(u)$ satisfying the linear problem

$$L(u) f(u) = v f(u), \quad f(u) = (f_1(u), \ldots, f_n(u))^t,$$

with some fixed normalisation $\vec{\alpha}(u)$

$$\vec{\alpha} \cdot f \equiv \sum_{j=1}^n \alpha_j(u) f_j(u) = 1.$$

The method of SoV in such a formulation was successfully applied to many particular integrable systems, here are some of the relevant references [23, 26, 27, 29, 1, 24, 31, 17, 11, 10, 16, 20].

In the present paper we prove the SoV for the classical $n$-particle Ruijsenaars system with the $n \times n$ Lax matrix found in [23] and with the Hamiltonian

$$H_1 = \sum_{j=1}^n e^{p_j} \prod_{k \neq j} \frac{\sigma(x_j - x_k - \lambda)}{\sigma(x_j - x_k)}, \quad \{p_j, x_k\} = \delta_{jk}, \quad (1.1)$$

where $\sigma(x)$ is the Weierstrass $\sigma$-function, $\lambda \in \mathbb{R}$ is a parameter of the model and $(p_j, x_j)$ are canonical Darboux variables. It is shown that the method of SoV applies to this system if we use the standard normalisation vector $\vec{\alpha}$

$$\vec{\alpha} = \vec{\alpha}_0 \equiv (0, 0, \ldots, 0, 1), \quad \text{i.e.} \quad f_n(u) = 1.$$
The structure of the paper is the following. In Section 2 we collect known information about the Ruijsenaars system (Lax matrix, integrals of motion, etc). In Section 3 we give an overview of the method of separation of variables and apply it then, in Section 4, to the system in question. In that key Section we also discuss the possibility of an alternative choice for the normalisation vector $\vec{\alpha}(u)$. The generating functions of the canonical separating transform given in terms of the initial and separation variables are constructed in Section 5 in explicit form for the case of two and three degrees of freedom. We also provide the separation of variables for the nonrelativistic limit $\lambda \to 0$ to the elliptic Calogero-Moser system in Section 6. The Section 7 contains some concluding remarks.

2. The system

Let us first recall some properties of the Weierstrass functions which we will need in the main text. Let $2\omega_{1,2} \in \mathbb{C}$ be a fixed pair of the primitive periods and $\Gamma = 2\omega_1 \mathbb{Z} + 2\omega_2 \mathbb{Z}$ the corresponding period lattice. Let us fix also the primitive domain $D := \{ z = 2\omega_1 x + 2\omega_2 y \mid x, y \in [0, 1) \}$ such that $D \sim \mathbb{C}/\Gamma$. The Weierstrass sigma-function is defined by the infinite product (cf., for instance, [33])

$$\sigma(x) = x \prod_{\gamma \in \Gamma \setminus \{0\}} \left( 1 - \frac{x}{\gamma} \right) \exp \left[ \frac{x}{\gamma} + \frac{1}{2} \left( \frac{x}{\gamma} \right)^2 \right], \tag{2.1}$$

the relations between $\sigma$-, $\zeta$- and $\wp$- functions being given by

$$\zeta(x) = \frac{\sigma'(x)}{\sigma(x)}, \quad \wp(x) = -\zeta'(x), \tag{2.2}$$

where $\sigma(x)$ and $\zeta(x)$ are odd functions and $\wp(x)$ is an even function of its argument. We recall also that the $\sigma(x)$ is an entire function, and $\zeta(x)$ is a meromorphic function having simple poles at $\omega_{kl}$, both being quasi-periodic, obeying

$$\zeta(x + 2\omega_{1,2}) = \zeta(x) + 2\eta_{1,2}, \quad \sigma(x + 2\omega_{1,2}) = -\sigma(x) e^{2\eta_{1,2}(x + \omega_{1,2})},$$

in which $\eta_{1,2}$ satisfy $\eta_1 \omega_2 - \eta_2 \omega_1 = \frac{\pi i}{2}$, whereas $\wp(x)$ is doubly periodic. From an algebraic point of view, the most important property of these functions is the existence of a number of functional relations, the most fundamental being

$$\zeta(\alpha) + \zeta(\beta) + \zeta(\gamma) - \zeta(\alpha + \beta + \gamma) = \frac{\sigma(\alpha + \beta) \sigma(\beta + \gamma) \sigma(\gamma + \alpha)}{\sigma(\alpha) \sigma(\beta) \sigma(\gamma) \sigma(\alpha + \beta + \gamma)} \tag{2.3}$$

which can be cast into the following form

$$\Phi_{\kappa}(x) \Phi_{\kappa}(y) = \Phi_{\kappa}(x + y) \left[ \zeta(\kappa) + \zeta(x) + \zeta(y) - \zeta(\kappa + x + y) \right] \tag{2.4}$$

with the function $\Phi_{\kappa}(x)$ defined as follows:

$$\Phi_{\kappa}(x) := \frac{\sigma(x + \kappa)}{\sigma(x) \sigma(\kappa)}.$$
Two other useful identities have the form
\[ \Phi_{\kappa-\tilde{\kappa}}(a - b) \Phi_{\kappa}(x + b) \Phi_{\tilde{\kappa}}(y + a) - \Phi_{\kappa-\tilde{\kappa}}(x - y) \Phi_{\kappa}(y + a) \Phi_{\tilde{\kappa}}(x + b) = \Phi_{\kappa}(x + a) \Phi_{\tilde{\kappa}}(y + b) \left[ \zeta(a - b) + \zeta(x + b) - \zeta(x - y) - \zeta(y + a) \right], \] (2.5)

\[ \Phi_{\kappa-\tilde{\kappa}}(x - y) \Phi_{\kappa}(y + a) \Phi_{\tilde{\kappa}}(x + a) = \Phi_{\kappa}(x + a) \Phi_{\tilde{\kappa}}(y + a) \left[ \zeta(x - y) - \zeta(\kappa + x + a) + \zeta(\tilde{\kappa} + y + a) + \zeta(\kappa - \tilde{\kappa}) \right]. \] (2.6)

The generalised Cauchy identity has the following form [6]
\[ \det (\Phi_{\kappa}(x_i - y_j)) = \Phi_{\kappa}(\Sigma) \sigma(\Sigma) \prod_{k<l} \sigma(x_k - x_l) \sigma(y_l - y_k) \prod_{k<l} \sigma(x_k - y_l) \] (2.7)

where \( \Sigma \equiv \sum_i (x_i - y_i) \).

Now we can introduce the \( n \)-particle (\( A_{n-1} \) type) Ruijsenaars system [23]. It is an integrable system with the following integrals of motion \( (i = 1, \ldots, n) \)
\[ H_i = \sum_{j \in \{1, \ldots, n\} \backslash \{i\}} \exp \left( \sum_{j \in J} p_j \right) \prod_{j \in J} \sigma(x_j - x_i - \lambda) \sigma(\lambda - x_j), \] (2.8)

The variables \( (p_j, x_j), j = 1, \ldots, n, \) on a 2n-dimensional symplectic manifold form a canonical system, i.e. they possess the Poisson brackets
\[ \{p_j, x_k\} = \{x_j, x_k\} = 0, \quad \{p_j, x_k\} = \delta_{jk}, \quad j, k = 1, \ldots, n, \] (2.9)
or, equivalently, the symplectic form \( \omega \) is expressed as \( \omega = \sum_j dp_j \wedge dx_j = d(\sum_j p_j dx_j) \). The \( \lambda \) is a parameter of the model. This system was proposed by Ruijsenaars as a relativistic analog of the Calogero-Moser system.

Proposition 1 ([23]). The Hamiltonians \( H_j \) Poisson commute
\[ \{H_j, H_k\} = 0, \quad j, k = 1, \ldots, n. \] (2.10)

The Lax matrix for this model has the form
\[ L(u) = \sum_{i,j=1}^n h_i \Phi_u(x_i - x_j + \lambda) E_{ij}, \quad h_i := e^{p_i} \prod_{j \neq i} \sigma(x_i - x_j - \lambda) \sigma(x_i - x_j), \] (2.11)

where the matrix \( E_{ij} \) have the following entries: \( (E_{ij})_{kl} = \delta_{ik} \delta_{jl} \). Notice that Ruijsenaars [23] used another gauge of the momenta such that two are connected by the following canonical transformation:
\[ p_i \to p_i + \log \prod_{j \neq i} \sqrt[\sigma(x_i - x_j + \lambda)]{\sigma(x_i - x_j - \lambda)}, \quad x_i \to x_i. \] (2.12)
Proposition 2 ([23]). The characteristic polynomial of the matrix $L(u)$ (2.11) generates the Hamiltonians (2.8)

$$\det(L(u) - v \cdot 1) = \sum_{j=0}^{n} (-v)^{n-j} \frac{H_j}{\sigma^j(\lambda)} \frac{\sigma(u + j\lambda)}{\sigma(u)}$$

(2.13)

where we assume $H_0 \equiv 1$.

3. The method

Recall, first, the standard definitions of Liouville integrability and SoV in the Hamilton-Jacobi equation [2]. An integrable Hamiltonian system with $N$ degrees of freedom is determined by a $2N$-dimensional symplectic manifold (phase space) and $N$ independent functions (Hamiltonians) $H_j$ commuting with respect to the Poisson bracket

$$\{H_j, H_k\} = 0, \quad j, k = 1, \ldots, N.$$ (3.1)

To find a SoV means then to find a canonical transformation $M : (x, p) \mapsto (u, v)$, $M : H_i(x, p) \mapsto H_i(u, v)$ such that there exist $N$ relations

$$\Phi_j(u_j, v_j; H_1, \ldots, H_N) = 0, \quad j = 1, \ldots, N,$$ (3.2)

separating the variables $u_j$. The most common way to describe a canonical transformation is the one in terms of its generating function $F(u|x)$.

Presently, no algorithm is known of constructing a SoV for any given integrable system. Nevertheless, there exists a fairly effective practical recipe based on the classical inverse scattering method. A detailed description of the procedure with many examples can be found in the review paper [31], see also the works [24, 17, 13, 10, 13]. Here we describe very briefly its main steps.

A Lax matrix for a given integrable system is a matrix $L(u)$ dependent on a “spectral parameter” $u \in \mathbb{C}$ such that its characteristic polynomial obeys two conditions

(i) Poisson involutivity:

$$\{\det(L(u) - v \cdot 1), \det(L(\tilde{u}) - \tilde{v} \cdot 1)\} = 0, \quad \forall u, \tilde{u}, v, \tilde{v} \in \mathbb{C};$$

(ii) $\det(L(u) - v \cdot 1)$ generates all integrals of motion $H_i$.

A Baker-Akhiezer (BA) function is the eigenvector

$$L(u) f(u) = v(u) f(u)$$

(3.3)

of the Lax matrix $L(u)$, provided that a normalisation of the eigenvectors $f(u)$ is fixed

$$\tilde{\alpha} \cdot f \equiv \sum_{i=1}^{n} \alpha_i(u) f_i(u) = 1, \quad (f(u) \equiv (f_1(u), \ldots, f_n(u))^T).$$ (3.4)
The pair \((u, v)\) can be thought of as a point of the spectral curve
\[
\det(L(u) - v \cdot 1) = 0. \tag{3.5}
\]
The BA function \(f(u)\) is then a meromorphic function on the spectral curve.

The recipe for finding an SoV is simple:

The separation variables \(u_j\) are poles of the Baker-Akhiezer function, provided it is properly normalised. The corresponding eigenvalues \(v_j\) of \(L(u_j)\), or some functions of them, serve as the canonically conjugated variables.

It is easy to see that the pairs \((u_j, v_j)\) thus defined satisfy the separation equations (3.2) for \(\Phi_j \equiv \det(L(u_j) - v_j \cdot 1)\). The canonicity of the variables \((u_j, v_j)\) should be verified independently. No general recipe is known how to guess the proper (that is producing canonical variables) normalisation for the BA function. In many cases the simplest standard normalisation,
\[
\vec{\alpha}(u) = \vec{\alpha}_0 \equiv (0, 0, \ldots, 0, 1), \tag{3.6}
\]
works. In other cases the vector \(\vec{\alpha}\) may depend on the spectral parameter \(u\) and the dynamical variables \((x, p)\). We shall refer to such normalisation as a dynamical one.

From the linear problem (3.3) and normalisation (3.4) we derive that
\[
\vec{\alpha} \cdot L^k f = v^k, \quad k = 0, \ldots, n - 1, \quad f = f(u) = \begin{pmatrix} \vec{\alpha} \\ \vec{\alpha} \cdot L(u) \\ \vdots \\ \vec{\alpha} \cdot L^{n-1}(u) \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ v \\ \vdots \\ v^{n-1} \end{pmatrix}, \tag{3.7}
\]
Another useful representation of the eigenvector \(f(u)\), which can be directly verified, is as follows:
\[
f_j(u) = \frac{(L(u) - v \cdot 1)_{j_k}}{(\vec{\alpha} \cdot (L(u) - v \cdot 1)^\wedge)_{k}}, \quad \forall k = 1, \ldots, n, \tag{3.8}
\]
where the wedge denotes the classical adjoint matrix (matrix of cofactors).

To derive equations for the separation variables, let \(f_i^{(j)} = \text{res}_{u=u_j} f_i(u)\) and \(v_j \equiv v(u_j)\). Then from (3.3)–(3.4) we have the overdetermined system of \(n + 1\) linear homogeneous equations for \(n\) components \(f_i^{(j)}\) of the vector \(f^{(j)}\):
\[
\begin{cases}
L(u_j) f^{(j)} = v_j f^{(j)}, \\
\sum_{i=1}^{n} \alpha_i(u_j) f_i^{(j)} = 0.
\end{cases} \tag{3.9}
\]
The pair \((u, v) \equiv (u_j, v_j)\) is thus determined from the condition
\[
\text{rank} \begin{pmatrix} \vec{\alpha}(u) \\ L(u) - v \cdot 1 \end{pmatrix} = n - 1. \tag{3.10}
\]
Finally, the condition (3.10) can be rewritten as the following vector equation:
\[ \vec{\alpha} \cdot (L(u) - v \cdot 1)^\wedge = 0. \] (3.11)

One can eliminate \( v \) from (3.11) to get the equation for \( u_j \)'s in the following way. From the linear system (3.9) it follows that \( \vec{\alpha} \cdot (L(u_j))^k f^{(j)} = 0, \ k = 0, \ldots, n - 1, \) so that (because \( f^{(j)} \) is not a zero vector) the following determinant has to vanish on the separation variables \( u_j \):
\[
B(u) = \det \begin{pmatrix}
\vec{\alpha} \\
\vec{\alpha} \cdot L(u) \\
\vdots \\
\vec{\alpha} \cdot L^{n-1}(u)
\end{pmatrix} = 0. \] (3.12)

The formula (3.12) for the separation variables appeared already in [24] (see also [7]) in the case of standard normalisation: \( \vec{\alpha} = \vec{\alpha}_0 \) (3.6) (see, for instance, formula (22) in [24]).

Notice that the fact, that equations (3.11) and (3.12) are the ones for the poles of the BA function, is already hinted, respectively, by the formulas (3.8) and (3.7).

Also, from equations (3.11) we can get many various formulas for \( v \) in the form
\[ v = A(u) \] (3.13)
with \( A(u) \) being rational functions of the entries of \( L(u) \). Let us describe those formulas for \( A(u) \) explicitly.

Define the matrices \( L^{(p)}, \ p = 1, \ldots, n, \) with the following entries:
\[
L_{ij}^{(p)} := \sum_{i_1=1}^{n} \cdots \sum_{i_{p-1}=1}^{n} \begin{vmatrix}
L_{i,j} & L_{i,i_1} & \cdots & L_{i,i_{p-1}} \\
L_{i_1,j} & L_{i_1,i_1} & \cdots & L_{i_1,i_{p-1}} \\
\vdots & \vdots & \ddots & \vdots \\
L_{i_{p-1},j} & L_{i_{p-1},i_1} & \cdots & L_{i_{p-1},i_{p-1}}
\end{vmatrix}, \ p = 2, 3, \ldots, n, \quad (3.14)
\]
and put \( L^{(1)} \equiv L \). These matrices satisfy the recursion relation of the form
\[ L^{(p)} = L \left( \text{tr} L^{(p-1)} \right) - (p - 1) L^{(p-1)} L. \] (3.15)

Introduce the matrix \( B(u) \) by the formula
\[
B(u) := \begin{pmatrix}
\vec{\alpha} \cdot L^{(1)}(u) L^{-1}(u) \\
\vec{\alpha} \cdot L^{(2)}(u) L^{-1}(u) \\
\frac{1}{2} \vec{\alpha} \cdot L^{(3)}(u) L^{-1}(u) \\
\vdots \\
\frac{1}{(n-1)!} \vec{\alpha} \cdot L^{(n)}(u) L^{-1}(u)
\end{pmatrix}. \] (3.16)

Then we have the following statement.

**Proposition 3.**
\[
\vec{\alpha} \cdot (L(u) - v \cdot 1)^\wedge = ((-v)^{n-1}, (-v)^{n-2}, \ldots, 1) \cdot B(u). \] (3.17)
**Proof.** The characteristic determinant \( \det(L(u) - v \cdot 1) \) has the following representation

\[
\det(L(u) - v \cdot 1) = (-v)^n + \sum_{j=1}^{n} \frac{(-v)^{n-j}}{j!} \text{tr}(L^{(j)}(u)).
\]  
**(3.18)**

The adjoint matrix \((L(u) - v \cdot 1)^\wedge\) is a matricial polynomial in \(v\) of the degree \(n - 1\),

\[
(L(u) - v \cdot 1)^\wedge = (-v)^{n-1} \cdot 1 + \sum_{j=1}^{n-1} (-v)^{n-1-j} A^{(j)}(u).
\]  
**(3.19)**

In order to find the matrices \(A^{(j)}\), substitute (3.18) and (3.19) into the definition of the adjoint matrix,

\[
\det(L(u) - v \cdot 1) \cdot 1 = (L(u) - v \cdot 1) (L(u) - v \cdot 1)^\wedge,
\]

and equate coefficients with the degrees of \(v\). In this way we get the following recursion relation for the \(A^{(j)}\)'s

\[
A^{(j)} = \frac{1}{j!} \text{tr}(L^{(j)}) - L A^{(j-1)}
\]  
**(3.20)**

with the initial data

\[
A^{(0)} = 1, \quad A^{(n-1)} = \frac{1}{n!} \text{tr}(L^{(n)}) L^{-1}(u).
\]  
**(3.21)**

The matrix \(\frac{1}{j!} L^{(j+1)} L^{-1}\) (cf. (3.15)) satisfies the same recursion and the same initial values which means that

\[
A^{(j)}(u) = \frac{1}{j!} L^{(j+1)}(u) L^{-1}(u).
\]

From the system of linear homogeneous equations

\[
\vec{\alpha} \cdot (L(u) - v \cdot 1)^\wedge \equiv ((-v)^{n-1}, (-v)^{n-2}, \ldots, 1) \cdot \mathcal{B}(u) = 0 \tag{3.22}
\]

(cf. (3.17)) we derive that

\[
(-v)^{j-i} = \frac{\mathcal{B}^\wedge(u))_{ki}}{(\mathcal{B}^\wedge(u))_{kj}} , \quad \forall k.
\]  
**(3.23)**

The formula (3.23) gives plenty of different representations for the function \(A(u)\), all of them being compatible on the separation variables since, because of the equality

\[
\mathcal{B}(u) = \det(\mathcal{B}(u)),
\]  
**(3.24)**

the matrix \(\mathcal{B}^\wedge(u_j)\) has rank 1.

To validate the choice of normalisation \(\vec{\alpha}(u)\) it remains, first, to make sure that the number of \(u_j\)'s is exactly \(N\) (in some degenerate cases one has to supply a couple of extra variables to make a complete set) and, second, to verify (somehow) the canonicity of brackets between the whole set of separation variables, namely: between zeros \(u_j\) of \(\mathcal{B}(u)\) and their conjugated variables \(v_j \equiv v(u_j) = A(u_j)\). To do this final calculation one needs information about Poisson brackets between entries of the Lax matrix \(L(u)\).
4. The separation

We now proceed with applying the general method to the system in question. For the Ruijsenaars model the number $N$ of degrees of freedom coincides with the number $n$ of particles and, respectively, with the dimension $n$ of the Lax matrix (2.11), so we can put $N = n$ in the formulas of the above Section. Let us first prove two useful Lemmas.

**Lemma 1.** Let $c_i \in \mathbb{C}, x_j^{(i)} \in \mathcal{D}, i = 1, \ldots, M, j = 1, \ldots, N$, be arbitrary constants such that

$$\sum_{j=1}^{N} x_j^{(i)} \equiv x \pmod{\Gamma} \quad \forall i.$$

Then there exist $C \in \mathbb{C}, y_j \in \mathcal{D}, j = 1, \ldots, N$, such that

$$p(u) \equiv \sum_{i=1}^{M} c_i \prod_{j=1}^{N} \sigma(u - x_j^{(i)}) = C \prod_{j=1}^{N} \sigma(u - y_j) , \quad \forall u \in \mathbb{C},$$

where

$$\sum_{j=1}^{N} y_j \equiv x \pmod{\Gamma}.$$

The $p(u)$ can be thought of as $\sigma$-function version of the $N$th degree polynomial (in $u$) which is represented in terms of its zeros $y_j$.

**Proof.** Let $z_j \in \mathcal{D}, j = 1, \ldots, N$, be $N$ distinct constants such that $\sum_{j=1}^{N} z_j \equiv x \pmod{\Gamma}$. Consider the elliptic function $\tilde{p}(u)$ of the form

$$\tilde{p}(u) = \sum_{i=1}^{M} c_i \prod_{j=1}^{N} \frac{\sigma(u - x_j^{(i)})}{\sigma(u - z_j)}.$$  \hspace{1cm} (4.1)

Any elliptic function can be represented through the ratio of products of $\sigma$-functions depending on its zeros, $y_j$, and its poles, $z_j$ (cf., for instance, [3]), i.e.

$$\tilde{p}(u) = C \prod_{j=1}^{N} \frac{\sigma(u - y_j)}{\sigma(u - z_j)},$$  \hspace{1cm} (4.2)

where $\sum_{j=1}^{N} y_j \equiv \sum_{j=1}^{N} z_j \equiv x \pmod{\Gamma}$. The statement follows if we equate right hand sides of (4.1) and (4.2).

Consider the Lax matrix $L(u)$ for the Ruijsenaars system

$$L(u) = \sum_{i,j=1}^{n} h_i \Phi_u(x_i - x_j + \lambda) E_{ij}.$$  \hspace{1cm} (4.3)
Lemma 2. For any integer $p = 1, 2, \ldots, n$ we have the identity

$$(L(u))^p + \sum_{j=1}^{p-1} (-1)^j \frac{H_j}{\sigma^j(\lambda)} \frac{\sigma(u + j\lambda)}{\sigma(u)} (L(u))^{p-j} = \sum_{i,j=1}^n h_i C_{ij}^{(p)} \Phi_u(x_i - x_j + p\lambda) E_{ij}$$

(4.4)

where the scalars $C_{ij}^{(p)}$ do not depend on the spectral parameter $u$ and are given by the formula

$$C_{ij}^{(p)} = (-1)^{p-1} \sum_{i_1 < \cdots < i_{p-1}} h_{i_1} \cdots h_{i_{p-1}} \prod_{k<l} \frac{\sigma(x_{i_k} - x_{i_l}) \sigma(x_{i_l} - x_{i_k})}{\sigma(x_{i_l} - x_{i_l} + \lambda)} \times$$

$$\times \frac{\sigma(x_i - x_j + p\lambda)}{\sigma(x_i - x_j + \lambda)} \prod_{k=1}^{p-1} \left[ \frac{\sigma(x_i - x_{i_k} - x_{j_k} + \lambda)}{\sigma(x_i - x_{i_k} + \lambda)} \right]$$

(4.5)

and $C_{ij}^{(1)} = 1$.

This Lemma does actually say that it is possible to arrange for the degree $p$ polynomial in $L(u)$ (the left hand side of (4.4)) such that $u$-dependence of its $(ij)$-entry occurs only through the factor $\Phi_u(x_i - x_j + p\lambda)$. This fact reflects some hidden internal structure of the Lax matrix $L(u)$ and is essential for further proof of the separation of variables. Notice also that the usage of the generalised Cauchy identity is very important for the proof of the Lemma given below.

Proof. Iterating the recursion (3.15) for the matrix $L^{(p)}(u)$, we get the formula

$$L^{(p)} = (-1)^{p-1} (p-1)! L^p + \sum_{j=1}^{p-1} (-1)^{p-1-j} \frac{(p-1)!}{j!} \text{tr} (L^{(j)}) L^{p-j}.$$  

(4.6)

Noticing that the traces of the $L^{(j)}$ matrices are expressed in terms of the integrals of motion (cf. (2.13) and (3.18))

$$\text{tr} L^{(j)} = \frac{H_j}{\sigma^j(\lambda)} \frac{\sigma(u + j\lambda)}{\sigma(u)}$$

(4.7)

we have that

$$h_i C_{ij}^{(p)} \Phi_u(x_i - x_j + p\lambda) = (-1)^{p-1} \frac{(p-1)!}{(p-1)!} L_i^{(p)}.$$ 

The right hand side being evaluated with the help of the generalised Cauchy identity (2.7), we arrive to the statement of the Lemma.

In order to separate variables in the Ruijsenaars system, first of all we have to fix the normalisation vector $\vec{\alpha}(u)$. The crucial observation is that we can use the standard normalisation (3.6). Then we have the following “characteristic equations” for the separation variables $u = u_j$ and $v = v_j$ (cf. (3.11))

$$(L(u) - v \cdot 1)_{nk}^\wedge = 0, \quad k = 1, \ldots, n.$$  

(4.8)
The “σ-polynomial” $B(u)$ (3.12) has now the form

$$B(u) = \det \begin{pmatrix} 0 & \ldots & 1 \\ L_{n1} & \ldots & L_{nn} \\ \vdots & \ddots & \vdots \\ (L^{n-1})_{n1} & \ldots & (L^{n-1})_{nn} \end{pmatrix}. \quad (4.9)$$

Its zeros, $u_j$, are the poles of the BA function $f(u)$ and are the separation variables. Let us first verify that we have got the right number of the $u_j$'s.

**Theorem 1.** σ-polynomial $B(u)$ (4.2) has $n-1$ zeros $u_j \in \mathcal{D}$ and can be represented by the formula

$$B(u) = \tilde{C} \prod_{j=1}^{n-1} \Phi_u(-u_j) \quad (4.10)$$

where $\tilde{C}$ does not depend on the spectral parameter $u$ and has the form

$$\tilde{C} = (-1)^{n-1} \frac{1}{\Pi_{n-1}^{n-1}} \begin{vmatrix} 1 & \ldots & 1 \\ C^{(2)}_{n1} & \ldots & C^{(2)}_{n,n-1} \\ \vdots & \ddots & \vdots \\ C^{(n-1)}_{n1} & \ldots & C^{(n-1)}_{n,n-1} \end{vmatrix}. \quad (4.11)$$

Variables $u_j$ obey the restriction

$$\sum_{j=1}^{n-1} u_j \equiv \sum_{j=1}^{n-1} (x_j - x_n) - \frac{n(n-1)}{2} \lambda \pmod{\Gamma}. \quad (4.12)$$

**Proof.** Using Lemma 2 we can represent $B(u)$ in the form

$$B(u) = (-1)^{n-1} \frac{1}{\Pi_{n-1}^{n-1}} \times \begin{vmatrix} \Phi_u(x_{n1} + \lambda) & \ldots & \Phi_u(x_{n,n-1} + \lambda) \\ C^{(2)}_{n1} \Phi_u(x_{n1} + 2\lambda) & \ldots & C^{(2)}_{n,n-1} \Phi_u(x_{n,n-1} + 2\lambda) \\ \vdots & \ddots & \vdots \\ C^{(n-1)}_{n1} \Phi_u(x_{n1} + (n-1)\lambda) & \ldots & C^{(n-1)}_{n,n-1} \Phi_u(x_{n,n-1} + (n-1)\lambda) \end{vmatrix}. \quad (4.13)$$

Then, using Lemma 1, we conclude that the σ-polynomial $B(u)$ can be rewritten in terms of its zeros in the form (4.10) where $\tilde{C}$ is given by the formula (4.11) and we also have the restriction (4.12).

To avoid discontinuities when discussing the Poisson brackets it is convenient to think of $u_j$'s as lying on the torus $\mathbb{C}/\Gamma$ rather than on $\mathcal{D}$.

In the sequel we obtain few statements which are valid for a general Lax matrix $L(u)$. Let us introduce the following matrices:

$$L(u, v) := L(u) - v \cdot 1, \quad L^\wedge(u, v) := (L(u) - v \cdot 1)^\wedge. \quad (4.14)$$

We can express the Poisson brackets of $L^\wedge(u, v)$ with $L(\tilde{u}, \tilde{v})$ in terms of the Poisson brackets of $L(u, v)$ with $L(\tilde{u}, \tilde{v})$. The answer is given by the following Lemma.
Lemma 3.

\[
\{L_1^\wedge(u,v), L_2(\bar{u},\bar{v})\} = \Delta_1^{-1} \left( \text{tr}_1 [ L_1^\wedge(u,v) \{L_1(u,v), L_2(\bar{u},\bar{v})\} ] \\
- L_1^\wedge(u,v) \{L_1(u,v), L_2(\bar{u},\bar{v})\} \right) L_1^\wedge(u,v),
\]

\[
\{L_1(u,v), L_2^\wedge(\bar{u},\bar{v})\} = \Delta_2^{-1} \left( \text{tr}_2 [ L_2^\wedge(\bar{u},\bar{v}) \{L_1(u,v), L_2(\bar{u},\bar{v})\} ] \\
- L_2^\wedge(\bar{u},\bar{v}) \{L_1(u,v), L_2(\bar{u},\bar{v})\} \right) L_2^\wedge(\bar{u},\bar{v}),
\]

where \( L_1(u,v) = L(u,v) \otimes 1, \ L_2(\bar{u},\bar{v}) = 1 \otimes L(\bar{u},\bar{v}), \ L_1^\wedge(u,v) = L^\wedge(u,v) \otimes 1 \) etc, \( \Delta_1 = \det(L(u,v)) \), \( \Delta_2 = \det(L(\bar{u},\bar{v})) \) and \( \text{tr}_{1,2} \) means trace in the first, respectively, the second space of the tensor product of two spaces and is defined by the rule:

\[
\text{tr}_1[A_1 B_2] \equiv \text{tr}_1[A \otimes B] := \text{tr}(A) (1 \otimes B) \equiv \text{tr}(A) B_2, \quad (4.15)
\]

\[
\text{tr}_2[A_1 B_2] \equiv \text{tr}_2[A \otimes B] := \text{tr}(B) (A \otimes 1) \equiv \text{tr}(B) A_1. \quad (4.16)
\]

Proof. The matrix \( L(u,v) \) and its classical adjoint \( L^\wedge(u,v) \) by definition satisfy the relation

\[
L^\wedge(u,v) L(u,v) = L(u,v) L^\wedge(u,v) = \Delta_1 \cdot 1. \quad (4.17)
\]

Differentiating this formula with respect to a parameter \( t \) and using the formula

\[
\frac{d}{dt} (\det L) = \text{tr} \left( L^\wedge \frac{d}{dt} L \right)
\]

one obtains (cf. (1.45)–(1.47) from [1])

\[
\frac{dL^\wedge}{dt} = \frac{L^\wedge \text{tr} \left( L^\wedge \frac{d}{dt} L \right) - L^\wedge \left( \frac{d}{dt} L \right) L^\wedge}{\Delta_1}.
\]

From which we have the following derivatives in the component-wise form:

\[
\frac{\partial L^\wedge_{ij}}{\partial L_{pq}} = \frac{L^\wedge_{pq} L^\wedge_{ij} - L^\wedge_{ip} L^\wedge_{jq}}{\Delta_1}. \quad (4.18)
\]

Now, using the derivation property of the bracket,

\[
\{L^\wedge_{ij}(u,v), L_{kl}(\bar{u},\bar{v})\} = \sum_{pq} \frac{\partial L^\wedge_{ij}(u,v)}{\partial L_{pq}(u,v)} \{L_{pq}(u,v), L_{kl}(\bar{u},\bar{v})\},
\]

\[
\{L_{ij}(u,v), L^\wedge_{kl}(\bar{u},\bar{v})\} = \sum_{pq} \frac{\partial L^\wedge_{kl}(\bar{u},\bar{v})}{\partial L_{pq}(\bar{u},\bar{v})} \{L_{ij}(u,v), L_{pq}(\bar{u},\bar{v})\},
\]

we verify both statements of the Lemma by substitution and straightforward calculation.

From the involutivity of the characteristic polynomials of a Lax matrix \( L \), \( \Delta_1 \) and \( \Delta_2 \), we have the equality:

\[
\begin{align*}
0 & = \{\Delta_1 \cdot 1 \otimes 1, \Delta_2 \cdot 1 \otimes 1\} = \{L_1(u,v) L_1^\wedge(u,v), L_2(\bar{u},\bar{v}) L_2^\wedge(\bar{u},\bar{v})\} \\
& = L_1 L_2 \{L_1^\wedge, L_2^\wedge\} + L_1 \{L_1^\wedge, L_2^\wedge\} L_2^\wedge + L_2 \{L_1^\wedge, L_2^\wedge\} L_1^\wedge + \{L_1, L_2\} L_1^\wedge L_2^\wedge.
\end{align*}
\]

\[12\]
Hence, using Lemma 3, we can get from here an expression for the bracket of $L^\wedge$ with $L^\wedge$ in terms of the brackets of $L$ with $L$.

**Lemma 4.**

\[
\{ L_1^\wedge(u, v), L_2^\wedge(\bar{u}, \bar{v}) \} = \Delta_1^{-1} \Delta_2^{-1} ( L_1^\wedge L_2^\wedge \{ L_1, L_2 \} - \text{tr}_1 [ L_1^\wedge L_2^\wedge \{ L_1, L_2 \} ] - \text{tr}_2 [ L_1^\wedge L_2^\wedge \{ L_1, L_2 \} ] ) L_1^\wedge L_2^\wedge .
\]

Suppose now that a Lax matrix $L(u)$ satisfies the quadratic (dynamical) $(r, s)$-bracket, then we have the following statement.

**Lemma 5.** Let a Lax matrix $L(u)$ satisfy the quadratic $(r, s)$-bracket of the form

\[
\{ L_1(u), L_2(\bar{u}) \} = L_1 L_2 r_+ - r_- L_1 L_2 + L_1 s_+ L_2 - L_2 s_- L_1 \tag{4.19}
\]

where

\[
r_+ - r_- + s_+ - s_- = 0, \quad \mathcal{P} r_{\pm} \mathcal{P} = -r_{\pm}|_{u \to \bar{u}} , \quad \mathcal{P} s_{\pm} \mathcal{P} = s_{\pm}|_{u \to \bar{u}}.
\]

Here $\mathcal{P}$ is the flip in tensor product of two spaces, i.e. $\mathcal{P} (A \otimes B) \mathcal{P} = B \otimes A$. Then the matrix $L^\wedge(u, v) \equiv (L(u) - v \cdot 1)^\wedge$ obeys the bracket of the form

\[
\{ L_1^\wedge, L_2^\wedge \} = (r_+ - \text{tr}_1 r_+ - \text{tr}_2 r_+) L_1^\wedge L_2^\wedge - L_1^\wedge L_2^\wedge (r_- - \text{tr}_1 r_- - \text{tr}_2 r_-) \tag{4.20}
\]

\[
+ L_2^\wedge (s_+ - \text{tr}_1 s_+ - \text{tr}_2 s_+) L_1^\wedge - L_1^\wedge (s_- - \text{tr}_1 s_- - \text{tr}_2 s_-) L_2^\wedge
\]

\[
+ v \Delta_1^{-1} [ (L_1^\wedge (r_+ - s_-) - \text{tr}_1 [ L_1^\wedge (r_+ - s_-) ] ) L_1^\wedge L_2^\wedge
\]

\[
- L_1^\wedge L_2^\wedge ( (r_- - s_+ ) L_1^\wedge - \text{tr}_1 ( [ r_- - s_+ ) L_1^\wedge ] ) ]
\]

\[
+ \bar{v} \Delta_2^{-1} [ (L_2^\wedge (r_+ + s_+) - \text{tr}_2 [ L_2^\wedge (r_+ + s_+) ] ) L_1^\wedge L_2^\wedge
\]

\[
- L_1^\wedge L_2^\wedge ( (r_- + s_- ) L_1^\wedge - \text{tr}_2 ( [ r_- + s_- ) L_2^\wedge ] ) ] .
\]

**Proposition 4 ([21, 32]).** The Lax matrix (2.11) of the Ruijsenaars model satisfies quadratic $(r, s)$-algebra (4.13) where $(r, s)$-matrices can be chosen to be as follows:

\[
r_+ = a - b + c - d , \quad r_- = a + d , \quad s_+ = b + d , \quad s_- = c - d , \tag{4.21}
\]

where

\[
a := \sum_{i \neq j} \Phi_{u \to \bar{u}} (x_i - x_j) E_{ij} \otimes E_{ji} + \zeta(u - \bar{u}) \sum_k E_{kk} \otimes E_{kk} , \tag{4.22}
\]

\[
b := \sum_{i \neq j} \Phi_u (x_i - x_j) E_{ij} \otimes E_{ii} + \zeta(u) \sum_k E_{kk} \otimes E_{kk} , \tag{4.23}
\]

\[
c := \sum_{i \neq j} \Phi_{\bar{u}} (x_i - x_j) E_{ii} \otimes E_{ij} + \zeta(\bar{u}) \sum_k E_{kk} \otimes E_{kk} , \tag{4.24}
\]

\[
d := \sum_{i \neq j} \zeta(x_i - x_j) E_{ii} \otimes E_{jj} . \tag{4.25}
\]
Notice here that one needs to use three algebraic relations (2.5)–(2.6) and (2.4) for the function $\Phi$ to verify this $(r, s)$-structure (cf. [21]).

Separation variables $(u, v) = (u_j, v_j)$, $j = 1, \ldots, n - 1$, for the Ruijsenaars model are implicitly defined by the following system of equations

$$
(L^\wedge(u, v))_{nk} \equiv (L(u) - v \cdot 1)_{nk}^\wedge = 0, \quad k = 1, \ldots, n,
$$

where $L(u)$ is the Lax matrix (2.14). The Poisson brackets for these new variables are generally given by the expression:

$$
\{u_i, u_j\} = \{u_i, v_j\} = \{v_i, v_j\} = 0, \quad i \neq j,
$$

and matrices $M$ are defined as follows:

$$
M_{m,kl} := \left( \begin{array}{cc}
\frac{\partial(L^\wedge(u,v))_{nk}}{\partial u} & \frac{\partial(L^\wedge(u,v))_{nk}}{\partial v} \\
\frac{\partial(L^\wedge(u,v))_{nl}}{\partial u} & \frac{\partial(L^\wedge(u,v))_{nl}}{\partial v}
\end{array} \right)_{\mid(u,v) = (u_m,v_m)}.
$$

**Theorem 2.** The separation variables $(u_j, v_j)$, $j = 1, \ldots, n - 1$, for the Ruijsenaars system, defined by the system of equations (4.20), possess the following Poisson brackets:

$$
(i) \quad \{u_i, u_j\} = \{u_i, v_j\} = \{v_i, v_j\} = 0, \quad i \neq j,
$$

$$
(ii) \quad \{v_j, u_j\} = v_j.
$$

**Proof.** Generically the matrix $M_{m,kl}$ (4.28) for $k \neq l$ is invertible which means that in order to prove the statement $(i)$ we have to show that

$$
\{(L^\wedge(u,v))_{nk}, (L^\wedge(u,\tilde{v}))_{nl}\}_{A_{ij}} = 0, \quad \forall k, l = 1, \ldots, n,
$$

when $i \neq j$. The latter fact follows from the Lemma 5 when we substitute in the right hand side of (4.20) the $(r, s)$-matrices from the Proposition 4 and put in both sides $(u, v) = (u_i, v_i)$, $(\tilde{u}, \tilde{v}) = (u_j, v_j)$, $i \neq j$. Indeed, using the definition (4.20), we get then the expression of the form

$$
\{(L^\wedge(u,v))_{nk}, (L^\wedge(u,\tilde{v}))_{nl}\}_{A_{ij}} \quad (i \neq j)
$$

$$
= \sum_{pq} \frac{(a - b + c - d)_{np,nq}}{det(L(u,v))}_{A_{ij}} (L^\wedge(u_i, v_i))_{pk} (L^\wedge(u_j, v_j))_{ql}
$$

$$
+ \sum_{pqr} \frac{v [L^\wedge(u,v)_{np}(L^\wedge(u,v))_{pk} - (L^\wedge(u,v))_{nk}(L^\wedge(u,v))_{qp}]}{det(L(u,v))} (a - b)_{pq,lr} (L^\wedge(u_j, v_j))_{rl}
$$

$$
+ \sum_{pqr} \frac{\tilde{v} [L^\wedge(\tilde{u},\tilde{v})_{np}(L^\wedge(\tilde{u},\tilde{v}))_{pk} - (L^\wedge(\tilde{u},\tilde{v}))_{nk}(L^\wedge(\tilde{u},\tilde{v}))_{qp}]}{det(L(\tilde{u},\tilde{v}))} (a + c)_{nr,pq} (L^\wedge(u_i, v_i))_{rk}.
$$
Each of the above three terms is equal zero, the first one when simply inspecting the inputs from the matrices \(a, b, c, d\); the latter two because the simple zero in the denominator is cancelled by a double zero in the numerator.

In order to prove the statement \((ii)\) we take \(i = j\) in (4.27) to get

\[
\{u_j, v_j\} \det M_{j,kl} = \{(L^\wedge(u, v))_{nk}, (L^\wedge(u, v))_{nl}\}_{(u,v) = (u_j,v_j)},
\]

(4.30)

where we recall that \(k \neq l\). Hence, we have to show that

\[-v_j \det M_{j,kl} = \{(L^\wedge(u, v))_{nk}, (L^\wedge(u, v))_{nl}\}_{(u,v) = (u_j,v_j)}.\]

(4.31)

To calculate the right hand side of (4.31) we use the Proposition 4 and take the limit \(\tilde{u} \to u\) in the \((r, s)\)-bracket (1.19). Using the derivation property of the bracket and substituting \(u = u_j, v = v_j\) we then conclude that the only non-vanishing term in the right of (4.31) has the following form:

\[
\{(L^\wedge(u, v))_{nk}, (L^\wedge(u, v))_{nl}\}_{(u,v) = (u_j,v_j)}
\]

\[
= v_j \sum_{prs} \left( \frac{\partial(L^\wedge(u, v))_{nk}}{\partial(L(u,v))_{pr}} \frac{\partial(L^\wedge(u, v))_{nl}}{\partial(L(u,v))_{rs}} - \frac{\partial(L^\wedge(u, v))_{nk}}{\partial(L(u,v))_{pr}} \frac{\partial(L^\wedge(u, v))_{nl}}{\partial(L(u,v))_{rs}} \frac{\partial(L(u,v))_{ps}}{\partial u} \right)_{(u,v) = (u_j,v_j)}.
\]

On the other hand the determinant of \(M_{j,kl}\) can be evaluated making use of its definition (1.28) and expressing the derivatives by \(u\) and \(v\) in terms of those by \((L(u,v))_{pq}\). Then we have the following formula for the left hand side of (1.31):

\[
-v_j \det M_{j,kl}
\]

\[
= v_j \sum_{prs} \left( \frac{\partial(L^\wedge(u, v))_{nk}}{\partial(L(u,v))_{pr}} \frac{\partial(L^\wedge(u, v))_{nl}}{\partial(L(u,v))_{rs}} - \frac{\partial(L^\wedge(u, v))_{nk}}{\partial(L(u,v))_{pr}} \frac{\partial(L^\wedge(u, v))_{nl}}{\partial(L(u,v))_{rs}} \frac{\partial(L(u,v))_{ps}}{\partial u} \right)_{(u,v) = (u_j,v_j)}.
\]

Straightforward calculation, using (4.18) and the fact that the matrix \(L^\wedge(u_j, v_j)\) has rank 1 shows that these two expressions are equal to each other (cf. here the proof of the analogous Theorem 1.3 from [1] establishing the Poisson brackets for the \(sl(n)\) Gaudin magnet obeying the simplest linear \(r\)-matrix algebra with the rational \(r\)-matrix).

**Theorem 3.** The variables \((u_j, y_j := \log(v_j)), \ j = 1, \ldots, n - 1,\) together with the variables \((X, P)\) describing the “motion of the center-of-mass”,

\[
X := x_n, \quad P := \log(H_n) = \sum_{j=1}^{n} p_j,
\]

(4.33)

constitute the complete canonical set of new (separation) variables.

**Proof.** The bracket \(\{P, X\} = 1\) is easily seen, so, in addition to the statements of the Theorem 2, it is only left to check that

\[
\{P, u_j\} = \{P, v_j\} = 0, \quad j = 1, \ldots, n - 1,
\]

(4.34)

\[
\{X, u_j\} = \{X, v_j\} = 0, \quad j = 1, \ldots, n - 1.
\]

(4.35)
The equalities (4.34) are trivial since \((u_j, v_j), j = 1, \ldots, n - 1\), are defined by the equations \( (L(u) - v \cdot 1)^{\wedge}_{nk} = 0, k = 1, \ldots, n \), and entries of the matrix \( L(u) \) depend only on differences \( x_i - x_j \), therefore

\[
\{ P, (L(u))_{ij}\} = 0, \quad \forall i, j = 1, \ldots, n.
\]

For the brackets in (4.35) we have the following expression \((k \neq l)\):

\[
\mathcal{M}_{j:kl} \left\{ X, u_j \right\} = - \left\{ \left( X, (L^\wedge(u, v))_{nk} \right) \right\}_{|x = u_j, v_j} \left( \left( X, (L^\wedge(u, v))_{nl} \right) \right)_{|x = u_j, v_j}.
\]  

(4.36)

The vector on the right of (4.36) is equal to zero since \( \forall k = 1, \ldots, n \)

\[
\{ X, (L^\wedge(u, v))_{nk} \}_{|x = u_j, v_j} = - \left[ \sum_{pq} \frac{L^{\wedge}_{np} L^{\wedge}_{qk} - L^{\wedge}_{pq} L^{\wedge}_{nk}}{\det(L(u, v))} \delta_{np} (L_{pq} + v \delta_{pq}) \right]_{|u = u_j, v = v_j} = 0.
\]

The equalities (4.37) follow because the matrix \( \mathcal{M}_{j:kl} \) is nondegenerate. \[ \Box \]

The proved SoV for the \( A_{n-1} \) \((n\text{-particle})\) problem with the standard normalisation vector \( \vec{\alpha}_0 \equiv (0, 0, \ldots, 0, 1) \) actually implies another SoV for the \( A_{n-2} \) problem with the non-standard normalisation vector \( \vec{\alpha}_1 \):

\[
\vec{\alpha}_1 := (\Phi_u(\xi - x_1 + \lambda), \ldots, \Phi_u(\xi - x_{n-1} + \lambda)),
\]  

(4.37)

if we choose \( \xi = x_n \). Let us demonstrate this explicitly.

Let us take the Lax matrix (2.11) for the \( n \)-particle system

\[
L(u) = \sum_{i,j=1}^{n} h_i \Phi_u(x_i - x_j + \lambda) E_{ij}.
\]

If we remove the last \((n\text{-th})\) row and the last column from this Lax matrix then we get the following \((n - 1) \times (n - 1)\) matrix

\[
L_{(n-1)}^{\wedge}(u) := \begin{pmatrix}
h_1 \Phi_u(\lambda) & \cdots & h_1 \Phi_u(x_1 - x_{n-1} + \lambda) \\
\vdots & \ddots & \vdots \\
h_{n-1} \Phi_u(x_{n-1} - x_1 + \lambda) & \cdots & h_{n-1} \Phi_u(\lambda)
\end{pmatrix}
\]  

(4.38)

which is the Lax matrix for the integrable system with \( n - 1 \) particles with the Hamiltonian

\[
H_{(n-1)}^{(x)} = \Phi_u(\lambda) \sum_{i=1}^{n-1} h_i = \Phi_u(\lambda) \sum_{i=1}^{n-1} e^{p_i} \prod_{k \neq i}^{n} \frac{\sigma(x_i - x_k - \lambda)}{\sigma(x_i - x_k)}.
\]

(4.39)

Under the simple canonical transformation,

\[
e^{p_i} \to e^{p_i} \frac{\sigma(x_i - x_n)}{\sigma(x_i - x_n - \lambda)}, \quad x_i \to x_i, \quad i = 1, \ldots, n - 1,
\]

(4.40)
the system (4.39) turns into Ruijsenaars’ system with \( n - 1 \) particles. This 1-degree-of-freedom-less system obviously inherits the non-standard SoV with the dynamical normalization (4.37) from the standard one (with \( \vec{\alpha}_0 \)) for the system with \( n \) degrees of freedom. Indeed, to see this, it is sufficient to note that the separation variables \((u_j, v_j), j = 1, \ldots, n - 1,\) for both systems are defined from the intersection of two spectral curves:

\[
\begin{align*}
\text{det}(L(u) - v \cdot 1) &= 0, \\
\text{det}(L_{xn}(u) - v \cdot 1) &= 0.
\end{align*}
\tag{4.41}
\]

In other words, the condition of the standard SoV for the first problem,

\[
\text{rank}\left(L(u) - v \cdot 1\right) = n - 1,
\tag{4.42}
\]

implies the following condition of SoV for the second problem:

\[
\text{rank}\left(L_{xn}(u) - v \cdot 1\right) = n - 2,
\tag{4.43}
\]

where \( \vec{\alpha}_1(u) \) is given by (4.37).

Procedure shown above, on how to connect the standard normalization vector \( \vec{\alpha}_0 \) and the alternative one, \( \vec{\alpha}_1 \), does obviously reflect an embedding, \( gl(n - 1) \subset gl(n) \), of one problem into the other. In other words (and it is true in general, for any integrable system of \( A_n \) type), one always has a free choice, namely: to include or not to include the “center-of-mass variable”, \( X \), and its conjugate one, \( P \), in the complete set of separation variables.

## 5. Generating functions

In this Section we derive the explicit formulas for SoV in the simplest cases: \( n = 3 \) with the standard normalisation (3.6) of \( \vec{\alpha} \), and \( n = 2 \) with the dynamical normalisation (4.37) (we skip the trivial case of the purely coordinate SoV \( x_{1,2} \to x_1 \pm x_2 \) for the 2-particle problem). Since the both cases are treated in very much the same manner as their trigonometric prototypes, see, respectively [18] and [19], we present only the main formulas here, omitting the details of the calculations.

Let us start with the \( n = 3 \) case. Following [18] define two functions \( A_1(u) \) and \( A_2(u) \) by the formulas

\[
(L(u) - A_k)_{3,3-k} = 0, \quad k = 1, 2,
\tag{5.1}
\]

or explicitly,

\[
A_k(u) = L_{kk} - \frac{L_{3k} L_{k,3-k}}{L_{3,3-k}} = e^{\rho_k} a_k(u), \quad k = 1, 2,
\tag{5.2}
\]

\[
a_k(u) = \frac{\sigma(u + 2\lambda + x_3 - x_{3-k}) \sigma(x_k - x_{3-k} - \lambda)}{\sigma(\lambda) \sigma(u + \lambda + x_3 - x_{3-k}) \sigma(x_k - x_{3-k} + \lambda)}, \quad k = 1, 2.
\tag{5.3}
\]
The separated variables \( u_j \) are defined from the equation
\[
A_1(u_j) = A_2(u_j)
\] (5.4)
which is equivalent to the equation \( B(u_j) = 0 \) since
\[
B(u_j) = h_3^2 \Phi_u(x_3 - x_1 + \lambda) \Phi_u(x_3 - x_2 + \lambda) \sigma(\lambda) (A_2 - A_1)
\]
and has two roots \( u_{1,2} \in D \). From the easily verified invariance of the ratio \( a_1(u)/a_2(u) \) under the transformation \( u \mapsto x_1 + x_2 - 2x_3 - 3\lambda - u \) it follows that
\[
u_1 + u_2 \equiv x_1 + x_2 - 2x_3 - 3\lambda \pmod{\Gamma},
\] (5.5)
which agrees with (4.12). The conjugated variables \( v_j \equiv e^{y_j} \) are defined as
\[
v_j = A_1(u_j) = A_2(u_j)
\] (5.6)
or, equivalently, through four equations
\[
v_j = e^{p_k a_k(u_j)}, \quad j, k \in \{1, 2\},
\] (5.7)
for four variables \( u_1, u_2, v_1, v_2 \). By virtue of the Theorem 3 the variables \( (u_1, u_2, X; y_1, y_2, P) \) are canonical. The generating function of the separating canonical transformation \( M \) is most conveniently expressed in terms of another set of canonical variables
\[
x_+ = x_1 + x_2 - 2x_3, \quad x_- = x_1 - x_2, \quad X = x_3,
\] (5.8)
\[
p_\pm = \frac{1}{2}(p_1 \pm p_2), \quad P = p_1 + p_2 + p_3,
\] (5.9)
\[
u_\pm = u_1 \pm u_2, \quad y_\pm = \frac{1}{2}(y_1 \pm y_2).
\] (5.10)

We shall need a \( \sigma \)-generalisation of the Euler dilogarithm function,
\[
\text{Li}_2(z) = \int_0^z \log(\sin(\zeta)) \, d\zeta,
\]
which we define as
\[
S(z) := \int_0^z \log(\sigma(\zeta)) \, d\zeta.
\] (5.11)
Notice that this function was introduced in [22] and has been used to construct the Lagrangian function of the integrable map which is a time-discretisation of the Ruijsenaars system. Using the product expansion for the Weierstrass sigma-function \((q = \exp(i\pi \omega_2/\omega_1))\):
\[
\sigma(z) = \frac{2\omega_1}{\pi} e^{\frac{y_2 z^2}{2\omega_1}} \sin\left(\frac{\pi z}{2\omega_1}\right) \prod_{n=1}^{\infty} \left(1 - 2q^{2n} \cos\left(\frac{\pi z}{\omega_1}\right) + q^{4n}\right) \left(1 - q^{2n}\right)^2,
\] (5.12)
cf. [33], we can express the function \( S \) in terms of the following function
\[
\text{Li}_3(z; q) := \sum_{k=1}^{\infty} \frac{z^k}{(1 - q^k) k^2}, \quad |q| < 1, \quad |z| < 1.
\] (5.13)
Notice that similar, but different from (5.13), $q$-deformations of the Euler (di-)
trilogarithm have been proposed in the review article [11]. In terms of (5.13) we
obtain
\[
S(z) = \frac{\eta_1 z^3}{6 \omega_1} + \left( \log \left( \frac{2 \omega_1}{\pi} \right) + 2 \sum_{k=1}^{\infty} \frac{q^{2k}}{(1 - q^{2k})^k} \right) z
\]
\[+ i \frac{\omega_1}{\pi} \left( \text{Li}_3(q^2 t; q^2) - \text{Li}_3(q^2 t^{-1}; q^2) \right),
\]
where $t = \exp(\pi i z / \omega_1)$. This series representation converges for $|q|^2 \leq |t| \leq |q|^{-2}$.

Let \[\mathcal{L}(\nu; x, y) := S(\nu + x + y) + S(\nu - x + y) + S(\nu + x - y) + S(\nu - x - y). \quad (5.15)\]
The generating function $F(y_+, x_+; u_-, x_-)$ of the canonical transformation from
$(x_\pm, p_\pm)$ to $(u_\pm, y_\pm)$, satisfying the defining relations
\[\frac{\partial F}{\partial x_+} = p_+, \quad \frac{\partial F}{\partial y_+} = u_+, \quad \frac{\partial F}{\partial x_-} = p_-, \quad \frac{\partial F}{\partial u_-} = -y_-, \quad (5.16)\]
is given then by the expression
\[F = y_+ (x_+ - 3 \lambda) + x_+ \log \sigma(\lambda) - \mathcal{L} \left( \frac{\lambda, x_- - 2, u_-}{2} \right) \quad (5.17)\]
\[+ S(\lambda - x_-) + S(\lambda + x_-).\]

The case $n = 2$ with the normalisation
\[\vec{\alpha}_1 = (\Phi_u(\xi - x_1 + \lambda), \Phi_u(\xi - x_2 + \lambda)) \quad \text{(cf. (1.37))}
\]
is treated similarly to its trigonometric prototype [19]. Having introduced the functions $A_1(u)$ and $A_2(u)$ by the formulas
\[\left( \vec{\alpha}_1 \cdot (L(u) - A_k)^\lambda \right)_k = 0, \quad k = 1, 2, \quad (5.18)\]
or explicitly,
\[A_1 = L_{11} - \frac{\Phi_u(\xi - x_1 + \lambda)}{\Phi_u(\xi - x_2 + \lambda)} L_{12} = e^{p_1} a_1(u), \quad (5.19)\]
\[A_2 = L_{22} - \frac{\Phi_u(\xi - x_2 + \lambda)}{\Phi_u(\xi - x_1 + \lambda)} L_{21} = e^{p_2} a_2(u), \quad (5.20)\]
\[a_k(u) = \frac{\sigma(u + \xi + 2 \lambda - x_{3-k}) \sigma(\xi - x_k) \sigma(x_k - x_{3-k} - \lambda)}{\sigma(\lambda) \sigma(u + \xi + 2 \lambda - x_{3-k}) \sigma(\xi - x_k) \sigma(x_k - x_{3-k} - \lambda)}, \quad k = 1, 2, \quad (5.21)\]
one proceeds as above with the only difference that the relation (5.3) is replaced by
\[u_1 + u_2 \equiv x_1 + x_2 - 3 \lambda - 2 \xi \quad \text{(mod $\Gamma$)} \quad (5.22)\]
and the variables $x_\pm$ are defined now as $x_\pm = x_1 \pm x_2$. The resulting expression for $F(y_+, x_+; u_-, x_-)$ is
\[F = y_+ (x_+ - 3 \lambda - 2 \xi) + x_+ \log \sigma(\lambda)
\[\quad - \mathcal{L} \left( \frac{\lambda, x_- - 2, u_-}{2} \right) - \mathcal{L} \left( \frac{\lambda, x_+ - \lambda}{2} - \xi, \frac{x_-}{2} \right)
\[+ S(\lambda - x_-) + S(\lambda + x_-). \quad (5.23)\]
6. Nonrelativistic limit to the Calogero-Moser system

The nonrelativistic limit is obtained by letting \( \lambda \to 0 \) while rescaling the momenta \( p_j := i\lambda p_j/g, \ g \in \mathbb{R} \), and making the canonical transformation \( p_j := p_j - ig \sum_{k \neq j} \zeta(x_j - x_k) \) such that \( h_j \to 1 + i\lambda p_j/g + O(\lambda^2) \) in (2.11). The \((r,s)\)-matrix structure is linear in that limit since the \( L \)-matrix behaves as

\[
L(u) \to (\lambda^{-1} + \zeta(u)) \cdot 1 + \frac{i}{g} \ell(u) + O(\lambda), \tag{6.1}
\]

\[
\ell(u) := \sum_j p_j E_{jj} - ig \sum_{j \neq k} \Phi_u(x_j - x_k) E_{jk}. \tag{6.2}
\]

The \( \ell \)-matrix (6.2) is Krichever’s [12] Lax operator for the elliptic Calogero-Moser system with the Hamiltonian

\[
H = \sum_{j=1}^n p_j^2 + g^2 \sum_{j \neq k} \wp(x_j - x_k). \tag{6.3}
\]

**Proposition 5 ([30]).** The Lax matrix \( \ell(u) \) (6.2) of the elliptic Calogero-Moser system satisfies linear \((r,s)\)-algebra of the form

\[
\{ \ell_1(u), \ell_2(\tilde{u}) \} = [\ell_1, r] + [\ell_2, s] \tag{6.4}
\]

where

\[
r = a + c, \quad s = a - b, \quad s = -P_{r} P_{u \leftrightarrow \tilde{u}}, \tag{6.5}
\]

(see (4.22), (4.23), (4.24)), and \([\ldots]\) means matrix commutator.

The SoV for the elliptic Calogero-Moser system follows, in principle, by taking limit \( \lambda \to 0 \) in the corresponding formulas describing SoV for the Ruijsenaars system. Although, because this limit is not so simple and straightforward, we prefer to do it independently, repeating the steps for proving main statements for the Ruijsenaars system in Section 4.

The normalisation vector is the same:

\[
\vec{\alpha}(u) = \vec{\alpha}_0 \equiv (0, 0, \ldots, 0, 1).\]

We have now the following characteristic equations for the separation variables \( u = u_j \) and \( v = v_j \)

\[
(\ell(u) - v \cdot 1)_{nk}^\wedge = 0, \quad k = 1, \ldots, n. \tag{6.6}
\]

The zeros of the \( \sigma \)-polynomial \( b(u) \)

\[
b(u) := \det \begin{pmatrix}
0 & \cdots & 1 \\
\ell_{n1} & \cdots & \ell_{nn} \\
\vdots & \ddots & \vdots \\
(\ell^{n-1})_{n1} & \cdots & (\ell^{n-1})_{nn}
\end{pmatrix}. \tag{6.7}
\]
give us separation variables $u_j$.

**Theorem 4.** $\sigma$-polynomial $b(u)$ \(b(u)\) has $n-1$ zeros $u_j \in \mathcal{D}$ and can be represented by the formula

$$b(u) = \tilde{C} \prod_{j=1}^{n-1} \Phi_u(-u_j)$$  \hspace{1cm} (6.8)

where $\tilde{C}$ does not depend on the spectral parameter $u$. Variables $u_j$ obey the restriction

$$\sum_{j=1}^{n-1} u_j \equiv \sum_{j=1}^{n-1} (x_j - x_n) \pmod{\Gamma}.$$  \hspace{1cm} (6.9)

**Proof.** From the limit (6.1) and the definitions of $B(u)$ and $b(u)$ we conclude that

$$B(u) = b(u) + O(\lambda).$$

Both $B(u)$ and $b(u)$ are $\sigma$-polynomials in $u$ and, since the degree of such a polynomial must not change with the analytical continuation of the parameter $\lambda$, $b(u)$ has the same degree as $B(u)$ does. Moreover, now the separation variables have to obey the restriction (6.9), the one being the limit of the corresponding relation (6.12).

Let us introduce the following notations:

$$\ell(u, v) := \ell(u) - v \cdot 1, \quad \ell^\wedge(u, v) := (\ell(u) - v \cdot 1)^\wedge,$$  \hspace{1cm} (6.10)

and also $\Delta_1 = \det(\ell(u, v))$ and $\Delta_2 = \det(\ell(u, \tilde{u}))$. Suppose now that a Lax matrix $\ell(u)$ satisfies the linear (dynamical) $(r, s)$-bracket (6.4), then we have the following statement.

**Lemma 6.** Let a Lax matrix $\ell(u)$ satisfy the linear $(r, s)$-bracket of the form

$$\{\ell_1(u), \ell_2(\tilde{u})\} = [\ell_1, r] + [\ell_2, s], \quad s = -\mathcal{P} r \mathcal{P}_{\text{on} u}. \hspace{1cm} (6.11)$$

Then the matrix $\ell^\wedge(u, v) \equiv (\ell(u) - v \cdot 1)^\wedge$ obeys the bracket of the form

$$\{\ell_1^\wedge, \ell_2^\wedge\} = \Delta_1^{-1} \left[ (\ell_1^\wedge s - \text{tr}_1 [\ell_1^\wedge s]) \ell_1^\wedge \ell_2^\wedge - \ell_1^\wedge \ell_2^\wedge (s \ell_1^\wedge - \text{tr}_1 [s \ell_1^\wedge]) \right] + \Delta_2^{-1} \left[ (\ell_2^\wedge r - \text{tr}_2 [\ell_2^\wedge r]) \ell_1^\wedge \ell_2^\wedge - \ell_1^\wedge \ell_2^\wedge (r \ell_2^\wedge - \text{tr}_2 [r \ell_2^\wedge]) \right]. \hspace{1cm} (6.12)$$

**Theorem 5.** The separation variables $(u_j, v_j)$, $j = 1, \ldots, n-1$, for the elliptic Calogero-Moser system, defined by the system of equations (6.4), possess the following Poisson brackets:

\begin{align*}
(i) & \quad \{u_i, u_j\} = \{u_i, v_j\} = \{v_i, v_j\} = 0, \quad i \neq j, \\
(ii) & \quad \{v_j, u_j\} = 1.
\end{align*}
Proof. In analogy with the proof of the Theorem 2 we have to show first that

\[ \{(\ell^\wedge(u,v))_{nk}, (\ell^\wedge(\tilde{u}, \tilde{v}))_{nl}\}_{A_{ij}} = 0, \quad \forall k, l = 1, \ldots, n, \]

when \( i \neq j \). We have from Lemma 6 and Proposition 5 that

\[
\{(\ell^\wedge(u,v))_{nk}, (\ell^\wedge(\tilde{u}, \tilde{v}))_{nl}\}_{A_{ij}} (i \neq j) \quad (6.13)
\]

\[
= \left[ \Delta_{i}^{-1} (\ell^\wedge_{1} s - tr_{1}[\ell^\wedge_{1} s]) \ell^\wedge_{2} + \Delta_{2}^{-1} (\ell^\wedge_{2} r - tr_{2}[\ell^\wedge_{2} r]) \ell^\wedge_{1}\right]_{A_{ij}}
\]

\[
= \sum_{pqr} \left[ \frac{(\ell^\wedge(u,v))_{np}(\ell^\wedge(u,v))_{qk} - (\ell^\wedge(u,v))_{nk}(\ell^\wedge(u,v))_{qp}}{\det(\ell(u,v))} \right]_{A_{ij}} (a-b)_{pq, nr} \right)_{A_{ij}} (\ell^\wedge(u, v))_{rl}
\]

\[
+ \sum_{pqr} \left[ \frac{(\ell^\wedge(\tilde{u}, \tilde{v}))_{np}(\ell^\wedge(\tilde{u}, \tilde{v}))_{qk} - (\ell^\wedge(\tilde{u}, \tilde{v}))_{nk}(\ell^\wedge(\tilde{u}, \tilde{v}))_{qp}}{\det(\ell(u,v))} \right]_{A_{ij}} (a+c)_{nr, pq} \right)_{A_{ij}} (\ell^\wedge(u, v))_{r, k}.
\]

These two terms in the right hand side have the same form as latter two in (4.29) and, again, they are equal to zero since in both expressions the simple zero in the denominator is cancelled by a double zero in the numerator.

The matrix of derivatives \( M \) instead of (4.28) has now the form

\[
M_{m,kl} := \left( \frac{\partial(\ell^\wedge(u,v))_{nk}}{\partial u} \frac{\partial(\ell^\wedge(u,v))_{nk}}{\partial v} \right)_{(u,v)=(u_m, v_m)}.
\]

In order to prove the statement (ii) we have to show that

\[
- \det M_{j,kl} = \{(\ell^\wedge(u,v))_{nk}, (\ell^\wedge(u,v))_{nl}\}_{(u,v)=(u_j, v_j)}, \quad (6.15)
\]

where \( k \neq l \). Again, the right hand side of (6.15) can be evaluated by first taking the limit \( \tilde{u} \to u \) in the \((r, s)\)-bracket of the Proposition 5 and then using the derivation property of the bracket. We derive the following expression

\[
\{(\ell^\wedge(u,v))_{nk}, (\ell^\wedge(u,v))_{nl}\}_{(u,v)=(u_j, v_j)}
\]

\[
= \sum_{pqr} \left( \frac{\partial(\ell^\wedge(u,v))_{nk}}{\partial (\ell(u,v))_{pr}} \frac{\partial(\ell^\wedge(u,v))_{nl}}{\partial (\ell(u,v))_{rs}} - \frac{\partial(\ell^\wedge(u,v))_{nl}}{\partial (\ell(u,v))_{pr}} \frac{\partial(\ell^\wedge(u,v))_{nk}}{\partial (\ell(u,v))_{rs}} \right)_{(u,v)=(u_j, v_j)}.
\]

On the other hand the determinant of \( M_{j,kl} \) (cf. (4.32)) has the form

\[
- \det M_{j,kl} \quad (6.16)
\]

\[
= \sum_{pqr} \left( \frac{\partial(\ell^\wedge(u,v))_{nk}}{\partial (\ell(u,v))_{pr}} \frac{\partial(\ell^\wedge(u,v))_{nl}}{\partial (\ell(u,v))_{rs}} - \frac{\partial(\ell^\wedge(u,v))_{nl}}{\partial (\ell(u,v))_{pr}} \frac{\partial(\ell^\wedge(u,v))_{nk}}{\partial (\ell(u,v))_{rs}} \right)_{(u,v)=(u_j, v_j)}.
\]

Such two expressions are equal to each other by the reasons pointed out in the end of the proof of Theorem 2.

\[\blacksquare\]
Theorem 6. The variables \((u_j, v_j \equiv y_j), j = 1, \ldots, n-1,\) together with the variables \((X, P)\) describing the motion of the center-of-mass,
\[
X := x_n, \quad P := \text{tr } \ell(u) = \sum_{j=1}^{n} p_j,
\]
constitute the complete canonical set of new (separation) variables. 

Proof repeats the proof of the Theorem 3. 

Consider now nonrelativistic limit of the generating functions \(F\) (5.17) and (5.23) in the two simplest cases. In analogy with calculations in the previous Section, for the case \(n = 3\) let us define two functions \(A_1(u)\) and \(A_2(u)\) by the formulas (5.1), or explicitly,
\[
A_k(u) = L_{kk} - \frac{L_{3k} L_{3-k}}{L_{33} - k} = p_k + ig a_k(u), \quad k = 1, 2, \tag{6.18}
\]
\[
a_k(u) = \zeta(u) + \zeta(x_3 - x_k) + \zeta(x_k - x_3-k) - \zeta(u + x_3 - x_3-k), \quad k = 1, 2.
\]
The \pm\text{-variables are defined by (5.8)–(5.10) and we have the restriction}
\[
u_+ \equiv x_+ \pmod{\Gamma}. \tag{6.19}
\]
The generating function \(F(y_+, x_+; u_-, x_-)\) (cf. formula (7.12) in [31]) is then given by the expression
\[
F = y_+ x_+ + ig \log \left[ \frac{\sigma \left( \frac{x_++u_2}{2} \right) \sigma \left( \frac{x_+-u_2}{2} \right) \sigma \left( \frac{x_+2}{2} \right) \sigma \left( \frac{x_-2}{2} \right)}{\sigma \left( \frac{x_++u_2}{2} \right) \sigma \left( \frac{x_+-u_2}{2} \right) \sigma \left( \frac{x_+2}{2} \right) \sigma \left( \frac{x_-2}{2} \right)} \right]. \tag{6.20}
\]

Similarly, in the case \(n = 2\), the normalisation vector is taken as follows:
\[
\tilde{a}_1 = (\Phi_u(x_1), \Phi_u(x_2)).
\]
Introduce the functions \(A_1(u)\) and \(A_2(u)\) by the formulas (5.18), or explicitly,
\[
A_1 = L_{11} - \frac{\Phi_u(x_1)\Phi_u(x_1)}{\Phi_u(x_2)\Phi_u(x_2)} L_{12} = p_1 + ig a_1(u), \tag{6.21}
\]
\[
A_2 = L_{22} - \frac{\Phi_u(x_1)\Phi_u(x_1)}{\Phi_u(x_2)\Phi_u(x_2)} L_{21} = p_2 + ig a_2(u), \tag{6.22}
\]
\[
a_k(u) = \zeta(u) + \zeta(x_k - x_k) + \zeta(x_k - x_3-k) - \zeta(u + x_3 - x_3-k), \quad k = 1, 2.
\]
The variables \(x_\pm\) are defined in this case as \(x_\pm = x_1 \pm x_2\) and we have the restriction
\[
u_+ \equiv x_+ - 2\xi \pmod{\Gamma}. \tag{6.23}
\]
The generating function \(F(y_+, x_+; u_-, x_-)\) has the following form
\[
F = y_+ x_+ + ig \log \left[ \frac{\sigma \left( \frac{x_++u_2}{2} \right) \sigma \left( \frac{x_+-u_2}{2} \right) \sigma \left( \frac{x_+2-2\xi}{2} \right) \sigma \left( \frac{x_-2}{2} \right) \sigma \left( \frac{x_+2-2\xi}{2} \right) \sigma \left( \frac{x_-2}{2} \right)}{\sigma \left( \frac{x_++u_2}{2} \right) \sigma \left( \frac{x_+-u_2}{2} \right) \sigma \left( \frac{x_+2-2\xi}{2} \right) \sigma \left( \frac{x_-2}{2} \right) \sigma \left( \frac{x_+2-2\xi}{2} \right) \sigma \left( \frac{x_-2}{2} \right)} \right]. \tag{6.24}
\]
7. Concluding remarks

We have performed the separation of variables for the classical $n$-particle Ruijsenaars system. If we replace the $\sigma$-function $\sigma(x)$ in all the above formulas by $\sin(x)$ ($\sinh(x)$) or by the identity function: $x \to x$, then we get all the above statements valid for the cases of trigonometric (hyperbolic) or rational Ruijsenaars system, respectively.

We have found the explicit generating function $F(u|x)$ of the separating canonical transform in the cases of two and three particles. It is a challenging problem to obtain such a function for $n > 3$ in any explicit form. What is also a problem for possible further studies of this integrable system is to produce a quantum SoV, i.e. to find the corresponding kernel $M_h(u|x)$ of the quantum separating integral operator $M_h$ and related integral representation for eigenfunctions of the quantum integrals of motion $H_j$ (cf. [31, 17, 18, 19, 20]).

Acknowledgments

VBK and FWN wish to acknowledge the support of EPSRC.

References

[1] M. R. Adams, J. Harnad and J. Hurtubise, Darboux coordinates and Liouville-Arnold integration in loop algebras, Commun. Math. Phys. 155 (1993), 385–413.

[2] V. I. Arnol’d, Mathematical methods of classical mechanics, Springer, New-York Heidelberg Berlin, 1974.

[3] B. A. Dubrovin, Theta functions and nonlinear equations, Russ. Math. Surv. 36 (1981), 11–92.

[4] J. C. Eilbeck, V. Z. Enol’skii, V. B. Kuznetsov and A. V. Tsiganov, Linear $r$-matrix algebra for classical separable systems, J. Phys. A: Math. Gen. 27 (1994), 567–578.

[5] A. Erdelyi et al, Higher transcendental functions, volume 3, McGraw Hill: New York, 1953.

[6] G. Frobenius, Über die elliptischen Funktionen zweiter Art, J. Reine Angew. Math. 93 (1882), 53–68.

[7] M. I. Gekhtman, Separation of variables in the classical $SL(N)$ magnetic chain, Commun. Math. Phys. 167 (1995), 593–605.

[8] E. G. Kalnins, Separation of variables for Riemannian spaces of constant curvature, Pitman Monographs and Surveys in Pure and Applied Mathematics 28, Longman Scientific and Technical, Essex, England, 1986.
[9] E. G. Kalnins, V. B. Kuznetsov and W. Miller Jr., Quadrics on complex Riemannian spaces of constant curvature, separation of variables and the Gaudin magnet, J. Math. Phys. 35 (1994), 1710–1731.

[10] E. G. Kalnins, V. B. Kuznetsov and W. Miller Jr., Separation of variables and XXZ Gaudin magnet, Rendiconti del Seminario Matematico dell’Universita e del Politecnico di Torino 53 (1995), 109–120.

[11] A. N. Kirillov, Dilogarithm identities, Progr. Theor. Phys. Suppl. 118 (1995), 61–142.

[12] I. M. Krichever, Elliptic solutions of the Kadomtsev-Petviashvili equation and integrable systems of particles, Func. Anal. Appl. 14 (1980), 282–290.

[13] I. M. Krichever and S. P. Novikov, Holomorphic bundles over algebraic curves and nonlinear equations, Russ. Math. Surv. 32 (1980), 53–79.

[14] V. B. Kuznetsov, Quadrics on real Riemannian spaces of constant curvature: separation of variables and connection with Gaudin magnet, J. Math. Phys. 33 (1992), 3240–3254.

[15] V. B. Kuznetsov, Equivalence of two graphical calculi, J. Phys. A: Math. Gen. 25 (1992), 6005–6026.

[16] V. B. Kuznetsov, Separation of variables for the $D_n$ type periodic Toda lattice, October 1996, submitted.

[17] V. B. Kuznetsov and E. K. Sklyanin, Separation of variables in $A_2$ type Jack polynomials, RIMS Kokyuroku 919 (1995), 27–34.

[18] V. B. Kuznetsov and E. K. Sklyanin, Separation of variables for the $A_2$ Ruijsenaars model and a new integral representation for the $A_2$ Macdonald polynomials, J. Phys. A: Math. Gen. 29 (1996), 2779–2804.

[19] V. B. Kuznetsov and E. K. Sklyanin, Separation of variables and integral relations for special functions, October 1996, submitted.

[20] V. B. Kuznetsov and E. K. Sklyanin, Factorisation of Macdonald polynomials, In: Proceedings of the Second Workshop on Symmetries and Integrability of Difference Equations (SIDEII), July 1996, Canterbury, UK, to appear.

[21] F. W. Nijhoff, V. B. Kuznetsov, E. K. Sklyanin and O. Ragnisco, Dynamical $r$-matrix for the elliptic Ruijsenaars-Schneider system, J. Phys. A: Math. Gen. 29 (1996), L333–L340.

[22] F. W. Nijhoff, O. Ragnisco and V. B. Kuznetsov, Integrable time-discretization of the Ruijsenaars-Schneider model, Commun. Math. Phys. 176 (1996), 681–700.
[23] S. N. M. Ruijsenaars, *Complete integrability of relativistic Calogero-Moser systems and elliptic function identities*, Commun. Math. Phys. **110** (1987), 191–213.

[24] D. R. D. Scott, *Classical functional Bethe ansatz for SL(N): separation of variables for the magnetic chain*, J. Math. Phys. **35** (1994), 5831–5843.

[25] E. K. Sklyanin, *Goryachev-Chaplygin top and the inverse scattering method*, J. Soviet Math. **31** (1985), 3417–3431.

[26] E. K. Sklyanin, *The quantum Toda chain*, In: Non-linear equations in classical and quantum field theory. Ed. by N. Sanchez (Lecture Notes in Physics **226**), N.Y.: Springer (1985), 196–233.

[27] E. K. Sklyanin, *Poisson structure of a periodic classical XYZ chain*, J. Soviet Math. **46** (1989), 1664–1683.

[28] E. K. Sklyanin, *Separation of variables in the Gaudin model*, J. Sov. Math. **47** (1989), 2473–2488.

[29] E. K. Sklyanin, *Separation of variables in the classical integrable SL(3) magnetic chain*, Commun. Math. Phys. **150** (1992), 181–191.

[30] E. K. Sklyanin, *Dynamical r-matrices for the elliptic Calogero-Moser model*, St. Petersburg Math. J. **6** (1994), 397–406.

[31] E. K. Sklyanin, *Separation of variables. New trends*, Progr. Theor. Phys. Suppl. **118** (1995), 35–60.

[32] Yu. B. Suris, *Elliptic Ruijsenaars-Schneider and Calogero-Moser hierarchies are governed by the same r-matrix*, preprint, [solv-int/9603011](http://arxiv.org/abs/solv-int/9603011).

[33] E. T. Whittaker and G. N. Watson, *A course in modern analysis*, Cambridge University Press, 4th ed., 1988.