On a comparative study between dependence scales determined by linear and non-linear measures

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(10th December 2008)

Abstract

In this manuscript we present a comparative study about the determination of the relaxation (i.e., independence) time scales obtained from the correlation function, the mutual information, and a criterion based on the evaluation of a non extensive generalisation of mutual entropy. Our results show that, for systems with a small degree of complexity, standard mutual information and the criterion based on its nonextensive generalisation provide the same scale, whereas for systems with a higher complex dynamics the standard mutual information presents a time scale consistently smaller.

1 Introduction

The description of the degree of dependence between variables is of capital importance, namely in several applications like time series analysis in which it is valuable to define how long there is a relevant relation between its elements. As examples, we mention: i) the determination of time scales from which value on a system is considered to be in a stationary state, i.e.,

\[ \frac{\partial P(z(t), t)}{\partial t} = 0, \]

or, in other words, the time needed for a system to achieve such a state (\(z(t)\) represents an element of a time series \(Z \equiv \{z(t)\}\) at time \(t\)), ii) the existence of ageing phenomena, i.e., the dependence of the correlation function,

\[ C_z(t_w, \tau) = \frac{\langle z(t_w) z(t_w + \tau) \rangle - \langle z(t_w) \rangle \langle z(t_w + \tau) \rangle}{\sqrt{\langle z(t_w)^2 \rangle - \langle z(t_w) \rangle^2} \sqrt{\langle z(t_w + \tau)^2 \rangle - \langle z(t_w + \tau) \rangle^2}}, \]

on the waiting time, \(t_w\), iii) the appraisal of how good the forecasting of future events can be or even how long we can produce reliable predictions based on previous values of the time series, iv) embedding of time series used in state space reconstruction and independent component analysis \([2,3,4,5]\), among many other cases.

The most straightforward way of performing this assessment has been the evaluation of the correlation function. Even though it has widespread applications, in truth, for a large class of processes, specifically complex systems \([2]\), such a procedure is unable to give a proper answer \([2]\). Explicitly, the correlation function is a normalised covariance that is
only effective at determining the dependences which are either linear or can be written in a linear way. Hence, by simply applying \( C_z (t_w, \tau) \), the dependences that do not fit in the linear classification can not be correctly measured. It is worth stressing that non-linear dependences rule a large part of the systems presently studied [6]. In other words, if we aim to characterise this sort of systems we must look at higher-order correlations to check for statistical independence. In order to make it, the correlation function is many times replaced by the computation of the mutual information which is able to detect the existence of non-linearities in the system [5, 6, 7, 8, 9]. In the present work, we carry out a comparative study between the correlation function, the mutual information (based on the Kullback-Leibler entropy [10]) and a generalised measure of mutual information [11] which emerged from a non-additive entropy [12] and that has broadly been applied [13]. The comparisons presented are made in discrete time series that correspond to the large majority of the time series available for analysis. The results show that when the degree of complexity is small the two mutual information measures studied provide the same answer. However, if we augment the complexity of the signal, then we verify that the two non-linear measures give different results.

2 Theoretical preliminaries

Consider Shannon entropy, \( S \), as the average of the surprise, \( s_i \), associated with a system which has a certain probability distribution \( \{ p'_i \} \)

\[
S \equiv \sum_i p'_i \ln \frac{1}{p'_i} = \sum_i p'_i s'_i,
\]  

(3)

Suppose now that the system is modified or new measurements are made giving rise to a new set, \( \{ p_i \} \), of distributions associated with the several states allowed by the system. From this new set, and for each state \( i \), we can define a new value for the surprise, \( s_i = \ln \frac{1}{p_i} \), and its variation,

\[
\Delta s_i = s'_i - s_i.
\]  

(4)

Averaging \( \Delta s_i \) with respect to distribution \( \{ p_i \} \) we have,

\[
I (\{ p_i \} , \{ p'_i \}) \equiv \sum_i p_i \Delta s_i = \sum_i p_i \ln \frac{p_i}{p'_i},
\]  

(5)

which is the Kullback-Leibler entropy. This measure has well-known properties such as positiveness and concaveness among many others [14]. Moreover, contrarily to \( S \), \( I (\{ p \} , \{ p' \}) \) is invariant under a change of variables \( x \rightarrow \bar{x} = f (x) \), and is not symmetric when we swap \( p_i \) and \( p'_i \). The latter property invalidates the possibility that \( I (\{ p \} , \{ p' \}) \) can be considered a metric distance. Nevertheless, we can still use it as a distance measure in probability space.

Let us now consider that instead of using the surprise as we have just defined, we have a \( q \)-surprise,

\[
s^{(q)}_i = \ln_q \frac{1}{p_i},
\]  

(6)
where,

\[ \ln_q x \equiv \frac{1 - x^{1-q}}{1-q}, \]  

(when \( q \to 1, \ln_q x = \ln x \)) \[15\]. Therefore, the variation of the \( q \)-surprise is

\[ \Delta s_i^{(q)} \equiv s_i^{(q)'} - s_i^{(q)} = \frac{(1 - [p_i']^{1-q}) - (1 - [p_i]^{1-q})}{1-q}. \]  

Computing the \( q \)-average of \( \Delta s_i^{(q)} \) with respect to the distribution \( \{p_i\} \) \[12, 16\],

\[ E_p \left[ \Delta s_i^{(q)} \right] \equiv \sum_i [p_i]^q \Delta s_i^{(q)} = \sum_i [p_i]^q \frac{[p_i]^{1-q} - [p_i']^{1-q}}{1-q}, \]

and using Eq. (7), we obtain the \( q \)-generalisation of Kullback-Leibler entropy,

\[ K_q (\{p\}, \{p'\}) = - \sum_i p_i \ln_q \frac{p_i'}{p_i}, \]

for which \( K_1 (\{p\}, \{p'\}) = I (\{p\}, \{p'\}) \). Entropy \( K_q (\{p\}, \{p'\}) \) is positive for \( q > 0 \), negative for \( q < 0 \), and null for \( q = 0 \) or \( p_i' = p_i (\forall,i,q) \). It is also provable that \( K_q (\{p\}, \{p'\}) \) is concave for \( q > 0 \) and convex for \( q < 0 \) (other properties can be found in Ref. \[17\]).

\( K_q \) \textbf{as a measure of dependence} \[11\] - We shall now consider a bidimensional variable \( z = (x, y) \) for which we want to quantify the degree of dependence between \( x \) and \( y \). In the application of \( K_q \) to the analysis of the scale of dependence, the most plausible distribution to be considered as the reference distribution is the product of the marginal distributions,

\[ p' (x, y) = p_1 (x) p_2 (y), \]

where

\[ p_1 (x) = \sum_y p (x, y) \]
\[ p_2 (y) = \sum_x p (x, y) \]

and \( p (x, y) \) is the joint probability distribution.

Using Eq. (5) we can verify that the Kullback-Leibler entropy, which for this case is named as mutual information, can be written as,

\[ I (x, y) = S (x) + S (y) - S (x, y), \]
\[ = S (x) - S (x|y), \]
\[ = S (y) - S (y|x). \]

From the first equation it is simple to see that \( I (x, y) \) only becomes equal to zero when the variables \( x \) and \( y \) are independent, \( i.e., p (x, y) = p_1 (x) p_2 (y) \). Both of \( S (x) \) and \( S (y) \) refer to the entropies of the respective marginal distributions and the entropy \( S (x, y) \) renders the entropy of the joint distribution. Entropies like \( S (x|y) \) are computed as

\[ S (x|y) = - \sum_{x,y} p (x, y) \ln p (x|y) \equiv -E_{p(x,y)} [\ln p (x|y)] , \]
in which \( E_\Pi [Y] \) represents the average of \( Y \) associated with distribution \( \Pi \).

Considering Eq. (11), the \( q \)-generalisation, \( K_q (x, y) \), which is now called generalised mutual information, can be expressed as,

\[
K_q (x, y) = \sum_{x,y} \left[ \frac{p(x,y)^q}{1-q} \right] \{1 - [p_1(x)p_2(y)]^{1-q}\} - \{1 - [p(x,y)]^{1-q}\}, \tag{15}
\]
or,

\[
K_q (x, y) = -E_{p(x,y)}^q \{ \ln_q p_1(x) + \ln_q p_2(y) + (1-q) \ln_q p_1(x) \ln_q p_2(y) - \ln_q p(x,y) \}. \tag{16}
\]

Writing,

\[
p(x,y) = p_1(x) \tilde{p}(y|x), \tag{17}
\]
and after some algebra, it is then possible to write Eq. (16) as,

\[
K_q (x, y) = -E_{p(x,y)}^q \{ \ln_q p_1(x) - \ln_q \tilde{p}(y|x) - (1-q) (\ln_q p_1(x) \ln_q \tilde{p}(y|x) - \ln_q p_1(x) \ln_q p_2(y)) \}. \tag{18}
\]

From Eqs. (16) and (18), it is possible to determine the maximum and the minimum values of \( K_q (x, y) \). The minimum value of \( K_q (x, y) = 0 \), exactly corresponds to the case in which \( p(x,y) = p_1(x)p_2(y) \). Complementary, the maximum value occurs when there is a bi-univocal dependence between the two variables, \( i.e., \) the maximum distance to independence. In this case, the conditional entropy,

\[
S_{\tilde{p}(x|y)}^{(\tilde{q}(x|y))} = \sum_y [\tilde{p}(x|y)]^q \ln_q \tilde{p}(x|y), \tag{19}
\]
must vanish since the uncertainty of having a value \( x \) given \( y \) is absent. Analytically, this implies,

\[
E_{p(x,y)}^q [\ln_q \tilde{p}(y|x)] = E_{p(x,y)}^q [\ln_q p_1(x) \ln_q p_2(y)] = 0. \tag{20}
\]
This means that the maximum of \( K_q (x, y) \) yields,

\[
K_q^{MAX} (x, y) \equiv -E_{p(x,y)}^q [\ln_q p_1(x) + (1-q) \ln_q p_1(x) \ln_q p_2(y)]. \tag{21}
\]

The existence of upper and lower bounds allows us to define a ratio, \( R_q \),

\[
R_q = \frac{K_q}{K_q^{MAX}} \in [0,1], \tag{22}
\]
that defines the degree of dependence between the two variables \( x \) and \( y \). For every case, there exists an optimal entropic index, \( q^{op} \), which is related to the degree of dependence, such that the gradient of \( R_q \) is more sensitive and therefore more capable of determining small variations in the degree of dependence. In other words, \( q^{op} \) is recognised as the inflexion point of \( R_q \) versus \( q \) curves. Regarding \( q^{op} \) values, it is simple to verify that when \( x \) and \( y \) are independent \( R_q = 0 (\forall q > 0) \) and optimal value is equal to infinity, \( q^{op} = \infty \). In the case of bi-univocal dependence, we have \( R_q = 1 (\forall q > 0) \), which implies in the limit of total dependence that \( q^{op} = 0 \). Thence, for a certain finite and positive value
of $q^{op}$, it is valid to ascribe a given degree of dependence between the variables $x$ and $y$ that we are analysing.

To conclude this part let us briefly discuss a very specific case of correlation in which the system tends to deflect from its past behaviour, anti-correlation. Anti-correlation is easily verified in the space of variables since the covariance provides to this case a negative value yielding $C_z(t_w, \tau) < 0$. In the probability space, i.e., when information measures are used, negative values cannot be obtained (at least when $q > 0$). In this case, it has been observed that value of $q^{op}$ presented by a anti-correlated time series is smaller than the value presented by the same time series after shuffling [18]. Therefore $q^{op} < q^{op}_{(shuffled)}$ can be taken as a signature of anticorrelation.

3 Application to time series analysis

In what follows, we are going to apply the mutual information measures described here-in-above to time series obtained from mathematical models and a heuristic time series as well. Our goal is to determine the time scale, $T_1$, at which each method considers the elements of a time series $x(t)$ and $y(t) = x(t + \tau)$ as independent from each other. Since we are dealing with finite time series some of the analytical results we have previously presented are no longer valid. For example, the value of total independence that is measured from a finite time series is not $q^{op} = \infty$, but some finite value of $q^{op}$ instead. However, for a specific time series, the level of independence can be assessed by shuffling its elements in such a way that the existent dependencies are wiped out. The scale of interest, $T_K$, is achieved when $q^{op}(\tau)$ reaches the value of $q^{op}$ of a independent shuffled series [19]. The same shuffling procedure allows us to determine the noise level of the correlation function and the minimum value of the mutual information $I$. The minimum concurs (within error margins) to the mutual information of a shuffled series and this match give us the respective time scale of interest $T_I$. The linear correlation scale of independence, $T_C$, is obtained from the intersection of the correlation function with the noise level, similarly to what is currently made in the recurrence plot analysis technique [20]. For sake of simplicity, we are going to consider processes in stationary state whose results are independent of the waiting time.

3.1 Logistic map in the fully chaotic regime

Consider the following non-linear dissipative map,

$$x_{t+1} = 1 - 2x_t^2, \quad x \in [-1, 1],$$  \hspace{1cm} (23)

which corresponds to the logistic map in the fully chaotic regime. Equation (23) is probably the most studied non-linear dynamical system [21]. Elements of a time series obtained from iterating Eq. (23) are associated with the probability distribution,

$$P(x) = \frac{1}{2B(\frac{1}{2}, \frac{1}{2})} \left( \frac{1 - x}{2} \right)^{-\frac{1}{2}} \left( \frac{1 + x}{2} \right)^{-\frac{1}{2}},$$  \hspace{1cm} (24)
with \( B \left( \frac{1}{2}, \frac{1}{2} \right) \) being the Beta function. Furthermore, it can be shown that the summation, 
\[ \xi_N = \sum_{i=1}^{N} x_i \]
approaches the Gaussian distribution as \( N \to \infty \) [22].

As expected, when we have analysed the autocorrelation function, we have verified that \( C_x(\tau) \) promptly attains the noise level. As a matter of fact, we can write \( C_x(\tau) = 0 \) (\( \forall \tau \geq 1 \)), with higher-order correlations different from zero as shown by Beck in [23]. Computing mutual information \( I \) between time series elements \( x(t) \) and \( x(t+\tau) \), we have verified that the noise level is obtained for a lag \( T_I = 15 \). From the normalisation of the generalised mutual information measure, \( R_q(\tau) \), and for each value of \( \tau \), we have computed the optimal values \( q^{op} \). Comparing the values that were obtained from the logistic map time series and the values obtained from the same time series after shuffling their elements we have verified that the characteristic time scale, for which the condition of independence between variables prevails, is \( T_K = 15 \). This scale is exactly the same time scale indicated by the standard mutual information procedure. In Fig. 1 we show typical curves of \( R_q(\tau) \) for several values of \( \tau \). Each curve has been obtained from averages over different runs (for specific values see caption in Fig. 2). From the maximum of every curve \( \frac{dR_q}{dq} \) (right panel of Fig. 1) we have computed \( q^{op}(\tau) \) exhibited in Fig. 2.

Figure 1: Left: Normalised generalisation of mutual information \( R_q \) of \( (x_t, x_{t+\tau}) \) vs. \( q \) for series obtained from Eq. (23) for several values of \( \tau \) and for series obtained after shuffling the elements from logistic map sequences. Right: Derivative of the curves in the left panel with respect to \( q \) vs. \( q \). The maxima correspond to the inflexion points of \( R(q) \), \( q^{op} \), which are represented in Fig. 2.

3.2 Autoregressive conditional heteroskedastic process

Many time series obtained from measurements in complex systems have shown the peculiar feature of having (long-lasting) correlations in the magnitude of its elements albeit their autocorrelation points to a white noise like behaviour. Thus, the characterisation and modelling of the evolution of instantaneous variance, \( \sigma_t \), is of capital importance when we aim to study that type of dynamics. To mimic this kind of time series, it has been introduced by Engle the autoregressive conditional heteroskedastic process (ARCH) [24].
Here, we present an extreme case of a generalisation that can be enclosed within the FIARCH class \[25, 26\]. Our variable is defined as,

\[ x_t = \sigma_t \omega_t, \tag{25} \]

where \( \omega_t \) is a stochastic variable usually associated with a Gaussian distribution with null mean and unitary variance. The variable \( \sigma_t \), also named as volatility for historical reasons, is defined as

\[ \sigma_t^2 = a + b \sum_{i=t_0}^{t-1} K(i - t + 1) x_i^2, \tag{26} \]

where

\[ K(t') = \frac{1}{Z(t')} \exp \left[ \frac{t'}{\zeta} \right], \quad (t' \leq 0, T > 0) \tag{27} \]

with \( Z(t') \) being the normalisation. This process originates non-Gaussian \( x_t \) uncorrelated variables. Despite the latter property, the autocorrelation function of \( x_t^2 \) (so as \( \sigma_t \) or \( |x_t| \)) presents an exponential decay. From numerical implementation of Eqs. (25)-(27) with \( a = 0.5, b = 0.99635 \) (obtained for the case of price fluctuations studied in Ref. [26]), and \( \zeta = 10 \), we have obtained a set of time series from which our results have been derived. To assure that the elements of the analysed time series are in the stationary state, we have left each numerical implementation run unrecorded for \( 10^5 \) steps. As awaited, the correlation function of \( \sigma_t \) presents an exponential decay which intersects the noise level at \( \tau = T_C = 772 \). From our measurements of the standard mutual information we have found a larger value of the independence time which corresponds to a minimum at \( T_I = 1093 \). Regarding the application of the \( q_{op} \) criterion, we have obtained an even larger time to bear out independence between variables, \( T_K \sim 1500 \). This value is clearly apart from \( T_I \). Looking at the dashed (green) line in the lower panel of Fig. 3 we see that the value of \( q_{op}(T_I) \) is below the noise level (even considering error margins). According to this criterion, this discrepancy points that at time scale \( T_K \) there is still a certain degree of dependence between variables \( \sigma_t \).

3.3 Fluctuations of atmospheric temperature

Fluctuations of atmospheric temperature have been intensively studied and a paradigmatic case of time series analysis. In the next case, we analyse fluctuations of the daily temperature with respect to the regularised temperature in Rio de Janeiro (Brazil) between the 1st of January 1995 and the of 13th of January 2008 in a total of 4635 observations [27]. Specifically, from the original time series we have obtained the regularised temperature according to a standard procedure used in climatology. The fluctuations have then been computed by finding the difference between the measured temperature and the regularised temperature (see Fig. 4 left panel). Computing the PDF of these fluctuations we have found that they are very well described by a Gaussian as shown in Fig. 4 right panel. In defiance of such a Gaussian behaviour, when we have estimated

3FIARCH stands for Fractionally Integrated ARCH
the independence scale, we have verified that the dynamics is in fact governed by long-memory effects. Numerically, we have noticed that the correlation function comes into the noise level for $T_C = 35$ days. Using the standard mutual information we have obtained a minimum value for $T_I = 61$ days which indicates the existence of non-linearities governing the dynamics of temperature fluctuations. Nevertheless, the scale given by $T_I$ appears to be an intermediate one, like it has happened in the previous example, since from the application of the $q^{op}$ we have obtained a larger upper bound for dependence $T_K \approx 91$ days. We plot these results in Fig. 5. Again, we verify a hierarchical structure of independence scales furnished by the correlation function, mutual information, and generalised mutual information. This level of dependence might be related to the fact that Rio de Janeiro is a onshore city, thus it is affected by the stability provided at larger scales from the absorption or release of heat by the sea.

4 Final Remarks

In this manuscript we have performed a comparative study between correlation and dependence measures, namely the mutual information measure and a generalised mutual information, defined within the context of non-additive entropy $S_q$, aiming to obtain the respective independence scale between elements. Our analysis has been performed on discrete time dynamical systems with different levels of non-linearity and memory. Explicitly, we have analysed the logistic map, a heteroskedastic process with long-lasting memory and a natural time series namely the fluctuations of atmospheric temperature. In the overall, our results have conveyed the well-known capability of mutual information for determining the presence of non-linearities. In addition, by means of increasing the memory of the system, i.e., soaring the level of complexity, the differences between the scale provided by mutual information, $I$, and by the criterion based on $q^{op}$ come out with $T_I$ being consistently smaller than $T_K$. Hence, the comparison of the results given by different information measures can be a helpful tool in order to opt for the most appropriate way to model the dynamics related to the measurements of a certain observable or to have an estimative about how further a forecast procedure can go maintaining a sufficient level of reliability. We would also like to refer that the work we have detailed points out the relevance of generalised information measures like as it has been shown with the application of the generalised-escort Tsallis entropy [29] on the distinction of pre-ictal, ictal, and post-ictal stages of epileptic signals [30]. Last of all, we refer that this criterion to set down the independence scale can also be used as an alternative method in the determination of the independence scale and subsequent evaluation of the embedding dimension of recurrence maps [18, 20].

Acknowledgements

SMDQ acknowledges J. de Souza for performing the temperature regularisation used in Sec. 3.3 and R. Zillmer and G. Savill for the comments made on the work presented.

4The manifestation of a Gaussian behaviour is by no means incompatible with the existence of long-range memory as it can be understood from Refs. [26, 28]
here above. This work benefited from financial support from European Union through BRIDGET Project (mktd-cd 200502961).

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Figure 2: *Upper left:* Correlation function of the logistic map *versus* lag, $x_{t+1} = 1 - 2x_t^2$. The correlation function is at the noise level for $\tau \geq 1$. *Upper Right:* Mutual information of the logistic map (points) and mutual information of logistic map shuffled series (line). The matching occurs at the scale $T_I = 15$. *Lower:* Optimal index *versus* lag. The points have been obtained from logistic map time series, the line represents noise level of $q^{op}$, and the grey lines represent the upper and lower bounds of error margins. Once more, the match happens at the scale $T_K = 15$. For every case, averages over time series of $10^6$ elements are made.
Figure 3: *Upper left:* Correlation function of a long-ranged heteroskedastic process, Eq. 26 with $q_m = 1$ [in log-linear scale]. The correlation function is at the noise level for $\tau \geq T_C = 772$ (dotted green vertical line). The dashed blue line has a slope $400^{-1}$. *Upper right:* Mutual information of the same process (black line) and mutual information of logistic map shuffled series (red line). The matching occurs at $T_I = 1093$. *Lower:* Optimal index versus lag. The points have been obtained from logistic map time series and the red line represents noise level of $q^{op}$. The matching happens at $T_K \sim 1500$ clearly different from $T_I$. For every case, averages over time series of $10^6$ elements are made after letting the process evolve for $10^5$ time steps to guarantee stationarity.
Figure 4: Left: Evolution of the atmospheric temperature (black line), the regularised temperature (red line) and the fluctuation, $f$, between measured and regularised temperatures (green line) at Rio de Janeiro between the 1$^{st}$ of January 1995 and the 13$^{th}$ of January 2008 (temperatures in Fahrenheit degrees). Right: Probability density function $P(x)$ vs. $x$ where $x$ represents the detrended and normalised (by its standard deviation) temperature fluctuations, $\langle f \rangle = 0.15$ and $\sigma_f = 3.6$. As can be seen, $P(x)$ is very well fitted by a Normal distribution with the error of adjustment being $\chi^2 = 2.4 \times 10^{-4}$ and $R^2 = 0.989$. 
Figure 5: Upper left: Correlation function of the temperature fluctuations presented in Fig. 4. The correlation function attains the noise level at $\tau = T_C = 35$ days. Upper right: Mutual information of the same series (black line) and mutual information of shuffled series (red line). The matching occurs at $T_I = 61$ days. Lower: Optimal index versus lag. The points have been obtained from the time series and red line represents the noise level of $q^{op}$. The equalisation happens at $T_K \sim 90$ days again plainly away from $T_I$. 