D=4, N=1, Type IIB Orientifolds

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**Abstract**

We study different aspects of the construction of \(D=4, N=1\) type IIB orientifolds based on toroidal \(Z_N\) and \(Z_M \times Z_N\), \(D=4\) orbifolds. We find that tadpole cancellation conditions are in general more constraining than in six dimensions and that the standard Gimon-Polchinski orientifold projection leads to the impossibility of tadpole cancellations in a number of \(Z_N\) orientifolds with even \(N\) including \(Z_4, Z_8, Z'_8\) and \(Z'_{12}\). We construct \(D=4, Z_N\) and \(Z_N \times Z_M\) orientifolds with different configurations of 9-branes, 5-branes and 7-branes, most of them chiral. Models including the analogue of discrete torsion are constructed and shown to have features previously conjectured on the basis of F-theory compactified on four-folds. Different properties of the \(D=4, N=1\) models obtained are discussed including their possible heterotic duals and effective low-energy action. These models have in general more than one anomalous \(U(1)\) and the anomalies are cancelled by a \(D=4\) generalized Green-Schwarz mechanism involving dilaton and moduli fields.
1 Introduction

Although all $D=10, N=1$ superstring theories are thought to be equally consistent, only the space of classical $D=4, N=1$ vacua of the $E_8 \times E_8$ heterotic string has been studied in some detail. On the contrary, perturbative $D=4, N=1$ vacua of type I have remained very much unexplored because of different reasons. Compactification of type I theory on a Calabi-Yau threefold with standard embedding in the gauge degrees of freedom gives rise to consistent (order by order in $\alpha'$) $D=4, N=1$ classical vacua but the gauge group $SO(26) \times U(1)$ is non-chiral. The construction of fully-fledged four-dimensional type I strings is relatively recent. Type I strings can be understood as an orbifold (orientifold) of type IIB closed strings with respect to the world-sheet parity operation $\Omega$ \cite{Gimon:1997ev,Polchinski:1995mt}. Type IIB $N=1, D=6$ orientifolds have been constructed in the last few years in refs. \cite{Duff:1996sk,Duff:1996xt,Derendinger:1996zr,Derendinger:1997fz,Hull:1997jv,Shiu:1997bm,Arinkin:1996vq,Matsuura:1996ua}. A crucial ingredient in this construction is the existence of Dp-branes \cite{Douglas:1995bn} on which open type I strings must end. Their presence from this perspective is enforced upon us by the tadpole cancellation conditions which guarantee the cancellation of gauge and gravitational anomalies. Type I vacua are not only worth constructing by themselves but also because they are supposed to be S-dual to strongly coupled $SO(32)$ heterotic vacua and hence there is a hope to get information about non-perturbative heterotic physics.

Our knowledge of the structure of $D=4, N=1$ type IIB orientifolds is much less complete, although some examples have been constructed \cite{Choi:1997mg,Matsuura:1998pt,Choi:1998eg,Arinkin:1998ip,Matsuura:1999ep,Arinkin:1999ug} and general conditions for tadpole cancellation in $Z_N \times Z_M$ type IIB orientifolds have been recently presented \cite{Marchesano:2019ryc,Choi:2020qmo}. In this paper we undertake a systematic study of $D=4, N=1$ type IIB orientifolds and extend previous work in several different directions. We present a detailed study of tadpole cancellation conditions for general $D=4, Z_N$ orientifolds and find the surprising result that the usual Gimon-Polchinski orientifold projection \cite{Gimon:1997ev} leads to the impossibility of tadpole cancellation for most of the even order $Z_N, D=4$ orientifolds. In particular this is the case for the $Z_4, Z_8, Z'_8$ and $Z'_{12}$ orientifolds. This is to be contrasted with the $D=6$ case in which all $Z_N$ actions have a Chan-Paton realization compatible with tadpole cancellation \cite{Douglas:1995bn,Matsuura:1998pt}.

We explicitly construct the massless spectrum of all $D=4$ consistent orientifolds with at most one set of 5-branes sitting at the fixed point at the origin. We find particularly useful a Cartan-Weyl realization of the unitary matrices $\gamma_{\theta,p}$ which induce the orbifold action on the Dp-brane degrees of freedom. In this formulation the massless spectra is found in a straightforward manner reminiscent of the computation of massless spectra in heterotic orbifolds. Another feature shown (which also occurs in $D=6$) is
that tadpole cancellation conditions vary depending on what particular fixed points do host 5-branes. We also discuss the effect of the addition of (quantized) Wilson lines on the 9-brane sector and the T-dual of this which is the distribution of 5-branes on different fixed points. T-duality also maps continuous Wilson lines to the emission of sets of 5-branes from the fixed points to the bulk of the orbifold.

In refs. [21, 8, 9, 10] it was shown that in even order $D=6$ orientifolds there are alternative ways to project the closed string sector with respect to the world-sheet parity $\Omega$. This is an equivalent of the discrete torsion degree of freedom already found in heterotic orbifolds [22, 23]. We construct the first $D=4$, $N=1$ orientifolds with these characteristics. This class of models is interesting since their existence was conjectured on the basis of F-theory compactified on four-folds [24].

We also examine the construction of candidate heterotic duals for type IIB $N=1$ orientifolds. The Cartan-Weyl basis for the gauge embedding of the twists on D-branes mentioned above is specially useful in identifying the candidate heterotic duals. We argue that the heterotic duals of the class of type IIB orientifolds discussed in this paper are in general non-perturbative heterotic orbifolds. These are orbifolds of the class introduced in ref. [25] in which the embedding of the twist in the $Spin(32)/Z_2$ lattice violates the standard modular invariance constraints of perturbative orbifolds. The gauge interactions and charged chiral fields from the type I (99) sector are mapped into the untwisted sector of the heterotic orbifold. The (55) and (95) sectors map into non-perturbative degrees of freedom associated to small instanton effects. We describe a few examples of candidate heterotic duals.

We finally study some aspects of the effective low-energy Lagrangian of IIB orientifolds, focusing on those with both 9-branes and one sector of 5-branes. A truncation to four dimensions of the $D=10$ Lagrangian and symmetry arguments allow us to obtain some generic qualitative information on the form of the Kähler potential, superpotential and gauge kinetic functions. Among the generic features is the presence of several anomalous $U(1)$’s whose anomalies are cancelled by a generalized Green-Schwarz mechanism in which both dilaton and moduli fields are involved. This is to be contrasted to perturbative heterotic vacua which have at most one anomalous $U(1)$.

\section{$D = 4$, $N = 1$, Type IIB Orientifolds}

In this chapter we summarize the basic ingredients [1, 3, 4] and notation needed in the construction of orientifolds and generalize them to the $D=4$ case.
An orientifold is a generalization of an orbifold in which a toroidally compactified theory is divided out by an internal discrete symmetry $G_1$ such as $Z_N$. In type IIB string theory there is symmetry operation $\Omega$ that exchanges left and right worldsheet movers. Gauging away this symmetry produces the orientifold and leads to the emergence of non-oriented surfaces spanned by string propagation. Generically, the $\Omega$ parity transformation can be accompanied by other internal or space time symmetry operations. Examples of this are considered in section 4. The complete orientifold group can thus be written as $G_1 + \Omega G_2$ with $\Omega h \Omega' \in G_1$ for $h, h' \in G_2$ [1].

In most of this article we will be mainly concerned with $G_1 = G_2$ and $G_1 = Z_N$ or $G_1 = Z_N \times Z_M$ actions on $T^6$ in type IIB string theory (in section 4 we will consider cases with $G_2 \neq G_1$). The $Z_N$ orbifold action is realized by powers of the twist generator $\theta$ ($\theta^N = 1$) which can be written in the form

$$\theta = \exp(2i\pi(v_1 J_{15} + v_2 J_{67} + v_3 J_{89}))$$ (2.1)

where $J_{mn}$ are $SO(6)$ Cartan generators. In terms of the complex bosonic coordinates $Y_1 = X_4 + iX_5$, $Y_2 = X_6 + iX_7$ and $Y_3 = X_8 + iX_9$ that parametrize the torus, $\theta$ acts diagonally as

$$\theta^k Y_i = e^{2i\pi v_i} Y_i$$ (2.2)

Similarly, we define complex fermionic fields $\psi^i$ as $\psi^1 = \psi^4 + i\psi^5$, etc.. It is convenient to define a twist vector $v = (v_1, v_2, v_3)$ associated to $\theta$. $N=1$ supersymmetry requires $\pm v_1 \pm v_2 \pm v_3 = 0$ for some choice of signs [20]. $Z_N \times Z_M$ actions are described in a similar way [23]. In this case we have $\theta$ and $\omega$ generators whose associated twist vectors are $v_\theta = \frac{1}{N}(1, -1, 0)$ and $v_\omega = \frac{1}{M}(0, 1, -1)$. In what follows we focus on $Z_N$.

To derive the massless spectra of orientifolds we will work in light-cone gauge. For example, in the closed untwisted sector the NS massless states are $\psi^i_{-\frac{1}{2}} |0\rangle$ which is invariant under $\theta$, and $\psi^i_{-\frac{1}{2}} |0\rangle$ which transforms as

$$\theta^k \psi^i_{\frac{1}{2}} |0\rangle = e^{2i\pi v_i} \psi^i_{\frac{1}{2}} |0\rangle$$ (2.3)

Complex conjugates $\psi^i_{\frac{1}{2}}$ transform with a phase $e^{-2i\pi v_i}$. The untwisted massless Ramond states are of the form $|s_0 s_1 s_2 s_3\rangle$ with $s_0, s_i = \pm \frac{1}{2}$ and odd number of minus signs to implement the GSO projection. These states transform as

$$\theta^k |s_0 s_1 s_2 s_3\rangle = e^{2i\pi v_i} |s_0 s_1 s_2 s_3\rangle$$ (2.4)

The condition $\pm v_1 \pm v_2 \pm v_3 = 0$ ensures that there is a gravitino in both the NS-R and R-NS type IIB untwisted sectors. Projecting under $\Omega$ then leads to $N=1, D=4$ supersymmetry.
Although type IIB is a theory of closed strings, the orientifold projection requires both closed and open string sectors for consistency. The presence of the closed sector is clear. Its content is obtained by retaining only those states which are invariant under the orientifold group action and by including twisted sectors. Details and examples are discussed in the next section.

The need for open string sectors can be justified in different ways. An operative way of identifying them is to compute the partition function (or generically, scattering amplitudes) in the closed sector Klein bottle unoriented surface. Tadpole divergences are found. To cancel these tadpoles and render the theory consistent, new contributions must be included \[27\]. Introducing open strings leads to the required cancellation for a specific structure of Chan-Paton charges. Recall also that open strings are expected since when type IIB string coordinates ends are identified up to the action of \(\Omega\), namely, \(\Omega X(\sigma) = X(\sigma)\), the mode expansion of an open string is obtained.

The modern version of the above picture relies on the identification of the tadpoles as non-cancelled charges. Orientifold fixed planes are sources for \((p + 1)\)-forms originated in the Ramond-Ramond (R-R) sector. Charge cancellation can be generically achieved by including the right number of \(D_p\)-branes, carrying opposite charge with respect to these forms \([12]\). Open strings have one end, labeled by \(a\), on a \(D_p\)-brane and the other end, labeled by \(b\), on a \(D_q\)-brane. They give rise to \(pq\) string sectors. The \(a, b\) labels correspond to the Chan-Paton factors at each end of the string. We will construct models with \(D_9\), \(D_5\) and \(D_7\)-branes.

In some cases no tadpoles are present in the Klein bottle amplitude and therefore there is no need for open string sectors. As we explain in section \(2.3\) and the appendix, there are on the other hand cases cases where even the inclusion of open string sectors is not enough to achieve tadpole cancellation and the orientifolds are thus inconsistent.

### 2.1 Closed string sector

The spectrum in the closed sector of the orientifold is obtained from those type IIB orbifold states invariant under \(\Omega\) parity transformations. Orbifold states are constructed by coupling left and right moving states of equal chirality, invariant under the orbifold group action.

The massless left NS states correspond to \(\psi^{\mu}_{\frac{1}{2}} |0\rangle\) vectors and to matter scalars \(\psi^{\pm i}_{\pm \frac{1}{2}} |0\rangle\) \((i, j = 1, 2, 3)\). Vectors are invariant under the orbifold twist \(\theta\) action \((2.1)\) while scalars acquire a phase \(e^{\pm 2\pi i \nu_v}\). Right movers are obtained by replacing \(\psi \to \bar{\psi}\).

By coupling left and right helicity \(\pm 1\) (under Lorentz little group \(SO(2)\)) vectors,
the graviton, an axion (remnant of the $D=10$ NS-NS antisymmetric tensor) and a
dilaton are found. Since parity projection keeps only symmetric combinations, only
graviton ($\pm 2$) and dilaton multiplets are present in the orientifold.

The number of matter states depends on the type of twist $(v_1, v_2, v_3)$ under con-
sideration. $\theta$ invariance requires $\pm v_i \mp v_j = \text{integer}$. We may rephrase this condition
as

$$(r - \bar{r}) \cdot v = \text{integer} \quad (2.5)$$

where $r, \bar{r} = (0, \pm 1, 0, 0)$ are $SO(8)$ vector weights corresponding to the bosonized
world-sheet fermions. As an example consider $v = \frac{1}{N}(1, 1, -2)$. In this case, there are
ten massless scalars,

$$\psi_{\pm i}^j |0\rangle_L \otimes \tilde{\psi}_{\pm j}^i |0\rangle_R \quad (2.6)$$

coming from $i = j = 1, 2, 3, i = 1, j = 2$ and $i = 2, j = 1$. This completes the NS-NS
sector of type IIB $Z_6$ orbifold for instance while eight extra states $(i = 1, 2; j = 3)$ are
present for $Z_3$ case. They combine into five and nine $\Omega$ invariant states respectively.

Massless Ramond left states, $|s_0s_1s_2s_3\rangle$ ($s_0, s_i = \pm \frac{1}{2}$) carrying $\pm \frac{1}{2}$ helicity, transform
as indicated in $(2.4)$. Right Ramond states transform similarly. When both sector are
coupled, twelve (twenty) massless scalars survive the orbifold projection in the $Z_6$ ($Z_3$)
example above. Antisymmetric combinations lead thus to six (ten) $\Omega$ invariant states.
The twist invariant state with $s_1 = s_2 = s_3$ (and similarly for $\tilde{s}_i$) (which is left from
$D=10$ R-R antisymmetric tensor) will combine with the NS-NS dilaton into a dilaton
chiral multiplet $S$. The residual states combine into five (nine) chiral massless states.

Construction of the R-NS sector proceeds in the same manner. Supersymmetric
partners of the above NS-NS and R-R bosons are found. States invariant under $\Omega$ are
obtained by taking the symmetrized combinations $R-NS + NS-R$

We have explicitly shown how to build up states in the untwisted sector. The
full one-loop torus amplitude can also be easily constructed. For type IIB, we just
couple the right movers sector (given for instance in [28]) with an identical (conjugate)
expression for left movers. In general, the trace over states can be written as

$$\mathcal{Z} = \frac{1}{N} \sum_{n,k=0}^{N-1} \mathcal{Z}_T(\theta^n, \theta^k) \quad (2.7)$$

The sum over $n$ is over twisted sectors whereas the sum over $k$ implements the orbifold
projection. The first terms in a $q, \bar{q}$ expansion, $(q = e^{-2\pi t})$ read

$$\mathcal{Z}_T(\theta^n, \theta^k) \sim \tilde{\chi}(\theta^n, \theta^k) \sum_r e^{2i\pi(r+nv)k \frac{1}{2} - \frac{1}{2} qE_0 (1 + \ldots)}$$

$$\sum_{\tilde{r}} e^{-2i\pi(\tilde{r}+nv)k \frac{1}{2} - \frac{1}{2} \bar{q}E_0 (1 + \ldots)} \quad (2.8)$$
where $E_0 = \sum_i \frac{1}{2} n_i |v_i| (1 - n_i |v_i|)$. Here $r$ and $\tilde{r}$ are $SO(8)$ weights and $\sum_{i=0}^{3} r_i = \text{odd}$ implements the GSO projection (similarly for $\tilde{r}$). $\tilde{\chi}(\theta^n, \theta^k)$ takes into account the fixed point degeneracy \cite{28}. In many cases $\tilde{\chi}(\theta^n, \theta^k)$ is just the number of points left simultaneously fixed by $\theta^n$ and $\theta^k$, also $\tilde{\chi}(1, \theta^k) = 1$.

From eq. (2.8) it follows that massless states in the $\theta^n$-twisted sector are given by $r, \tilde{r}$ such that

$$\frac{1}{2} (r + n v)^2 - \frac{1}{2} + E_0 = \frac{1}{2} (\tilde{r} + n v)^2 - \frac{1}{2} + E_0 = 0$$

(2.9)

In the NS-NS sector both $r$ and $\tilde{r}$ are vector weights, whereas in the R-R sector they are spinorial weights. In NS-R and R-NS we suitably combine weights of both types.

From eq. (2.8) we can read the multiplicity $D(n)$ of massless states in the $\theta^n$ twisted sector, namely

$$D(n) = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{\chi}(\theta^n, \theta^k) e^{2\pi i (r - \tilde{r}) k v}$$

(2.10)

For $n = 0$ we recover eq. (2.5) above.

Only symmetric (antisymmetric) combinations are to be kept in the NS-NS (R-R) sectors in the orientifold. Thus, (2.10) gives the number of chiral massless states in the type IIB orientifold. Including all twisted sectors gives the multiplicities shown in Table [ ]. We have included in the table only the $Z_N$, $Z_N \times Z_M$ orientifolds which are free of tadpoles and have at most one sector of 5-branes (see below).

The generalization to $Z_N \times Z_M$ orbifolds \cite{19} is straightforward. Denote by $(\theta, \omega)$ the corresponding twists with eigenvalues $(v_\theta, v_\omega)$. Then, the contribution in a $\theta^n \omega^m$ twisted sector is obtained from the above just by replacing $k v \rightarrow k v_\theta + l v_\omega$. Recalling that a discrete torsion phase $\epsilon$ is now allowed \cite{23}, the degeneracy factor reads,

$$D(n, m) = \frac{1}{NM} \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} e^{(nk - ml)} \tilde{\chi}(\theta^n \omega^m, \theta^k \omega^l) e^{2\pi i (r - \tilde{r}) (k v_\theta + l v_\omega)}$$

(2.11)

\subsection{2.2 Open string sector}

Let us now move to the open string sector. The type of D-branes present in this sector depends on the content of the orientifold group. This is expected since there are orientifold planes charged under R-R fields twisted by the orbifold action \cite{4}. When the identity is in $G_2$, the orientifold group contains $\Omega$ as an element and there will be D9-branes. Following ref. \cite{6} we refer to these as type A orientifolds. Contrariwise, type B orientifolds are those in which D9-branes are not needed.

D5-branes are present whenever the orientifold group contains an action of the type $\Omega R_i$, where $R_i$ is an order two element acting on the two complex directions transverse
Table 1: Number of chiral multiplets in closed string sectors for some $Z_N$ and $Z_N \times Z_M$, $D=4$, $N=1$ type IIB orientifolds. A dilaton multiplet must be added in the untwisted sector.

| Twist Group | Untwisted moduli | Twisted moduli |
|-------------|-----------------|----------------|
| $Z_3$       | 9               | 27             |
| $Z_3 \times Z_3$ | 3           | 81             |
| $Z_7$       | 3               | 21             |
| $Z_6$       | 5               | 29             |
| $Z'_6$      | 4               | 42             |
| $Z_3 \times Z_6$ | 3           | 71             |
| $Z_{12}$    | 3               | 25             |

The corresponding 5i-branes live in $D=4$ space-time plus the complex torus with coordinate $Y_i$. For instance the $Z_6$ orientifold with $v = \frac{1}{6}(1, 1, -2)$ has 53-branes whereas $Z_2 \times Z_2$ has sets of 51, 52 and 53-branes \[13\]. It follows that $Z_3$, $Z_7$, and $Z_3 \times Z_3$ orientifolds do not contain D5-branes.

From T-duality arguments D7-branes are expected whenever $\Omega$ acts jointly with a reflection $R_i$ on one complex plane and $(-1)^{F_L}$ that changes sign of all Ramond left states \[8, 9\]. We postpone the treatment of this case to section 4.

Open string states are denoted by $|\Psi_{ab}\rangle$, where $\Psi$ refers to world-sheet degrees of freedom while the $a, b$ Chan-Paton indices are associated to the string endpoints on Dp-branes and Dq-branes. These Chan-Paton labels must be contracted with a hermitian matrix $\lambda_{ab}$. The action of a group element $g \in G_1$ is given by

$$g : |\Psi_{ab}\rangle \rightarrow (\gamma_{g,p})_{aa'}|g.\Psi_{a'b'}(\gamma_{g,q})^{-1}_{b'b}$$

(2.12)

where $\gamma_{g,p}$ and $\gamma_{g,q}$ are unitary matrices associated to $g$. The action of $\Omega h$, $h \in G_2$, is instead given by

$$\Omega h : |\Psi_{ab}\rangle \rightarrow (\gamma_{\Omega h,p})_{aa'}|h.\Psi_{a'b'}(\gamma_{\Omega h,q})^{-1}_{b'b}$$

(2.13)

Consistently with group multiplication, $\gamma_{\Omega g,p}$ can be defined as

$$\gamma_{\Omega g,p} \overset{\text{def}}{=} \gamma_{g,p} \gamma_{\Omega,p}$$

(2.14)

The matrices $\gamma_{\Omega,p}$ and hence $\gamma_{\Omega g,p}$ are unitary.

Consistency with the orientifold group multiplication law implies several constraints on the $\gamma$ matrices since they must provide a representation of the group up to a phase
For example, consider the $G_1 = G_2 = Z_N$ case. To a twist $\theta^k$ there corresponds $\gamma_{\theta^k,p} \equiv \gamma_{k,p}$ and $\gamma_{\Omega \theta^k,p} \equiv \gamma_{\Omega k,p}$. It can be shown that necessarily $\gamma_{0,p} = 1$ \[4\]. Also, without loss of generality we can choose

$$\gamma_{k,p} = \gamma_{1,p}^k$$

Since $\theta^N = 1$, eq. (2.12) leads to

$$\gamma_{1,p}^N = \pm 1$$

Similarly, from $\Omega^2 = 1$ it follows that

$$\gamma_{\Omega,p} = \pm \gamma_{\Omega,p}^T$$

Or in general, from $(\Omega \theta^k)^2 = \theta^{2k}$,

$$\gamma_{\Omega k,p} = \pm \gamma_{k,p} \gamma_{\Omega k,p}^T$$

Now, using eqs. (2.14), (2.15), (2.18) and the unitarity of the $\gamma$ matrices we obtain

$$\gamma_{k,p}^* = \pm \gamma_{\Omega,p}^* \gamma_{k,p} \gamma_{\Omega,p}$$

When there are different types of branes it is also necessary to consider the action of $(\Omega \theta^k)^2$ on pq states. In chapter 3 we will mostly consider the same action of $\Omega$ analyzed by Gimon and Polchinski (GP) \[4\]. The GP action is such that $\Omega^2 = (\pm i)^{(9-p)/2}$ on 9p states. In particular, $\Omega^2 = -1$ on 95 states, implying that in eqs. (2.18) and (2.19) there are opposite signs for 9 and 5-branes.

Cancellation of tadpoles imposes further conditions on the $\gamma$ matrices. For instance it requires $\gamma_{\Omega,9}^T = \gamma_{\Omega,9}$. Hence, for 9-branes we must take the plus sign in eqs. (2.18) and (2.19). Since we can choose $\gamma_{\Omega,9}$ real we then have the condition

$$\gamma_{k,9} = \gamma_{\Omega,9} \gamma_{k,9} \gamma_{\Omega,9}$$

Now, for 5-branes the GP action implies $\gamma_{\Omega,5}^T = -\gamma_{\Omega,5}$. Hence, for 5-branes we must take the minus sign in eqs. (2.18) and (2.19). Since we can choose $\gamma_{\Omega,5}$ pure imaginary we then have the condition

$$\gamma_{k,5}^* = \gamma_{\Omega,5} \gamma_{k,5} \gamma_{\Omega,5}$$

The $\gamma_{k,p}$ matrices are determined from cancellation of ‘twisted’ tadpoles as discussed in next section. It turns out that they can always be chosen diagonal.

The open string spectrum can be computed once the $\gamma$ matrices are found. According to the string endpoints there are various pq sectors and moreover, Dp-branes
with $p < 9$ can sit at different fixed or non-fixed points. Here we will concentrate on models containing 9 and 5-branes, with all the latter located on the particular fixed point corresponding to the origin in the compact (transverse) space. This is an important case since in this configuration one gets maximal gauge symmetry. Furthermore, verification of tadpole cancellation is much simpler. Different distributions of 5-branes on the various fixed points have to be analyzed case by case since tadpole cancellation conditions might imply that there must always be a number of residual 5-branes sitting at the origin which cannot move to other fixed points. We will discuss this in some specific examples below.

We now describe the massless bosonic states in each $pq$ sector. For the sake of clarity we restrict here to the orientifold group generated by $\{G, \Omega\}$ with $G = \mathbb{Z}_N$. Generalization to $G = \mathbb{Z}_N \times \mathbb{Z}_M$ is straightforward. Extra operations accompanying the action of $\Omega$ are considered in specific examples.

**99-States**

The massless NS states include gauge bosons $\psi_{0,ab}^{\mu} \lambda_{ab}^{(0)}$ and matter scalars $\psi_{0,ab}^{i} \lambda_{ab}^{(i)}$. The Chan-Paton matrices must be such that the full states are invariant under the action of the orientifold group. Hence,

$$
\lambda^{(0)} = \gamma_{1,9} \lambda_{1,9}^{(0)} \gamma_{1,9}^{-1} ; \quad \lambda^{(0)} = -\gamma_{1,9} \lambda_{1,9}^{(0)T} \gamma_{1,9}^{-1}
$$

$$
\lambda^{(i)} = e^{2\pi i v} \gamma_{1,9} \lambda_{1,9}^{(i)} \gamma_{1,9}^{-1} ; \quad \lambda^{(i)} = -\gamma_{1,9} \lambda_{1,9}^{(i)T} \gamma_{1,9}^{-1} \quad (2.22)
$$

**55-States**

Massless NS states also include gauge bosons $\psi_{0,ab}^{\mu} \lambda_{ab}^{(0)}$ and matter scalars $\psi_{0,ab}^{i} \lambda_{ab}^{(i)}$. For 5$_i$-branes at fixed points the Chan-Paton matrices must satisfy

$$
\lambda^{(0)} = \gamma_{1,5} \lambda_{1,5}^{(0)} \gamma_{1,5}^{-1} ; \quad \lambda^{(0)} = -\gamma_{1,5} \lambda_{1,5}^{(0)T} \gamma_{1,5}^{-1}
$$

$$
\lambda^{(i)} = e^{2\pi i v} \gamma_{1,5} \lambda_{1,5}^{(i)} \gamma_{1,5}^{-1} ; \quad \lambda^{(i)} = -\gamma_{1,5} \lambda_{1,5}^{(i)T} \gamma_{1,5}^{-1} \quad (2.23)
$$

But for $j \neq i$

$$
\lambda^{(j)} = e^{2\pi i v} \gamma_{1,5} \lambda_{1,5}^{(j)} \gamma_{1,5}^{-1} ; \quad \lambda^{(j)} = \gamma_{1,5} \lambda_{1,5}^{(j)T} \gamma_{1,5}^{-1} \quad (2.24)
$$

The sign change in the $\Omega$ projection is due to the DD boundary conditions on the $j \neq i$ directions transverse to the 5$_i$-branes.

**5$_i$ 9-States**
For 5\(_i\)-branes, coordinates orthogonal to \(Y_i\) obey mixed DN boundary conditions and have expansions with half-integer modded creation operators. By world-sheet supersymmetry their fermionic partners in the NS sector are integer modded. Their zero modes span a representation of a Clifford algebra and can be labelled as \(|s_j, s_k\rangle\), \(j, k \neq i\), with \(s_j, s_k = \pm \frac{1}{2}\). The GSO projection further imposes \(s_j = s_k\). Under \(\theta\), \(|s_j, s_k\rangle\) picks up a phase \(e^{2\pi i (v_j s_j + v_k s_k)}\). Hence, the orientifold projection on a state \(|s_j, s_k, ab\rangle\) implies \(\lambda_{ab} = e^{2\pi i (v_j s_j + v_k s_k)}\gamma_{1,5}^{-1}\) (2.25)

Notice that here the index \(a\) (\(b\)) lies on a 5-brane (9-brane). \(\Omega\) relates 59 with 95 sectors and does not impose extra constraints on \(\lambda\).

Interestingly enough, if the \(\lambda\) matrices are recast in a Cartan-Weyl basis, constraints on Chan Paton matrices emerge as restrictions on weight vectors. In this way computing the spectrum becomes greatly simplified. Formally it appears equivalent to the computation of the untwisted spectrum of heterotic orbifolds.

For the sake of clarity let us first discuss how this is achieved in the 99 sector. We know that projecting under \(\Omega\) parity, represented by a symmetric \(\gamma_{1,9}\) matrix in the 99 sector, gives the equation \(\lambda = -\gamma_{1,9}\lambda^T \gamma_{1,9}^{-1}\). The original 32 \(\times\) 32 unrestricted Chan-Paton (Hermitian) matrices are therefore constrained to be \(SO(32)\) generators. They can be organized into charged generators \(\lambda_a = E_a\), \(a = 1, \cdots, 480\), and Cartan generators \(\lambda_I = H_I\), \(I = 1, \cdots, 16\), such that

\[
[H_I, E_a] = \rho_I^a E_a \quad (2.26)
\]

where the 16 dimensional vector with components \(\rho_I^a\) is the root associated to the generator \(E_a\). These vectors are of the form \((\pm 1, \pm 1, 0, \cdots, 0)\), where underlining indicates that all possible permutations must be considered. The matrix \(\gamma_{1,9}\) and its powers represent the action of the \(Z_N\) group on Chan Paton factors, and they correspond to elements of a discrete subgroup of the Abelian group spanned by the Cartan generators. Hence, we can write

\[
\gamma_{1,9} = e^{-2\pi V \cdot H} \quad (2.27)
\]

This equation defines the 16-dimensional ‘shift’ vector \(V_{(99)} = V\). In section 2 we have seen that \(\gamma_{1,9}\) can be chosen diagonal and furthermore \(\gamma_{1,9}^N = \pm 1\). Then, for example, when \(\gamma^N = -1\) we will have a general structure

\[
V = \frac{1}{2N} (1, \cdots, 1, 3, \cdots, 3, 5, \cdots, 5, \cdots) \quad (2.28)
\]

The number of \(\frac{2l+1}{2N}\) entries is determined by tadpole cancellation. Cartan generators are represented by \(2 \times 2\) \(\sigma_3\) submatrices. Explicit examples are given in section 3.
Recalling the formula

\[ e^{-B} A e^B = \sum_{n=0}^{\infty} [A, B]_n \]  

(2.29)

with \([A, B]_{n+1} = [[A, B]_n, B], [A, B]_0 = A\), and using eq. (2.26), it is easy to show that

\[ \gamma_{1,9} E_a \gamma_{1,9}^{-1} = e^{-2i\pi \rho^a \cdot V_a} \]  

(2.30)

Therefore we see from equation (2.22) that gauge bosons correspond to both, Cartan generators which trivially satisfy the \(\lambda^{(0)}\) constraint, plus charged generators belonging to a subset of \(SO(32)\) root vectors selected by

\[ \rho^a \cdot V_{(99)} = 0 \mod \mathbb{Z} \]  

(2.31)

Similarly, from the equation for \(\lambda^{(i)}\) in (2.22) it follows that matter states correspond to charged generators with

\[ \rho^a \cdot V_{(99)} = v_i \mod \mathbb{Z} \]  

(2.32)

Other open string sectors can be treated in a similar way. For 5-branes on the bulk only \(\Omega\) parity constraints are present in eq. (2.24). Since \(\gamma_{\Omega,5}\) is antisymmetric in this sector, symplectic group generators are obtained. Furthermore, each dynamical 5-brane contains \(2N\) D5-branes and the rank of the group is reduced. The generators are associated to the same \(SO\) root vectors given before plus long roots \((\pm 2, 0, \cdots, 0)\). For branes at fixed points we must impose the rest of the constraints in eq. (2.24) with \(\gamma_{1,5}\) written in terms of an equivalent shift \(V_{(55)}\) as in (2.27). There is then a projection like (2.31). Moreover, since the equivalent shifts can be shown to be always of the form (2.28), whenever 5-branes are at fixed points, long roots are projected out. If all 5-branes sit at the same fixed point we can take \(V_{(55)} = V_{(99)}\) and therefore, exactly the same spectrum as in the 99 sector arises, now corresponding to multiplets of the \(SO(32)_{(55)}\) unitary subgroups of the 5-brane theory.

The 59 sector is handled using an auxiliary \(SO(64) \supset SO(32)_{(99)} \otimes SO(32)_{(55)}\) algebra. Since we have generators acting simultaneously on both 9-branes and 5-branes only roots of the form

\[ W_{(95)} = W_{(9)} \otimes W_{(5)} = (\pm 1, 0, \cdots, 0; \pm 1, 0, \cdots, 0) \]  

(2.33)

must be considered. Here the first (second) 16 components transform under \(SO(32)_{(99)}\) (\(SO(32)_{(55)}\)). The shift in this sector is defined to be \(V_{95} = V_{(99)} \otimes V_{(55)}\). Using (2.22) we learn that massless states correspond to \(W_{(95)}\) roots satisfying

\[ W_{(95)} \cdot V_{(95)} = (s_j v_j + s_k v_k) \mod \mathbb{Z} \]  

(2.34)

with \(s_j = s_k = \pm \frac{1}{2}\), plus (minus) sign corresponding to particles (antiparticles).
2.3 Tadpole cancellation

In the orientifold theory the one-loop vacuum amplitudes include the torus, the Klein bottle ($K$), the Möbius strip ($M$) and the cylinder ($C$). The last three have tadpole divergences from exchange of massless states in the closed string channels. By supersymmetry the total divergences vanish but consistency requires separate cancellation of NS-NS and R-R tadpoles [27]. In refs. [1, 3, 4] it was shown how to extract these tadpoles from the amplitudes in a $Z_2$ orientifold in $D=6$. These results have been extended in both $D=6,4$ [6, 7, 17, 19, 20]. However, the general $Z_N, D=4$ orientifold tadpole cancellation conditions have not been explicitly presented in those references. In the appendix we give the general form of the amplitudes for $T^6/\{Z_N, \Omega\}$ orientifolds in $D=4$ and comment briefly on extracting the tadpoles. In this section we just describe and apply the main results.

The $Z_N$ actions that can act crystallographically on a $T^6$ lattice and lead to $N=1$ supersymmetry were classified long time ago [26]. The list, with corresponding twist vectors, is given in table 2. Clearly, for even $N$, all the $Z_N$ have only one order two element $R = \theta^{N/2}$ that reflects $Y_1$ and $Y_2$. The corresponding orientifolds will then have $5_3$-branes.

| $Z_3$ | $\frac{1}{3}(1, 1, -2)$ | $Z_6'$ | $\frac{1}{6}(1, -3, 2)$ | $Z_8'$ | $\frac{1}{8}(1, -3, 2)$ |
|-------|--------------------------|--------|--------------------------|--------|--------------------------|
| $Z_4$ | $\frac{1}{4}(1, 1, -2)$  | $Z_7$  | $\frac{1}{4}(1, 2, -3)$  | $Z_{12}$ | $\frac{1}{12}(1, -5, 4)$ |
| $Z_6$ | $\frac{1}{6}(1, 1, -2)$  | $Z_8$  | $\frac{1}{8}(1, 3, -4)$  | $Z_{12}'$ | $\frac{1}{12}(1, 5, -6)$ |

Table 2: $Z_N$ actions in $D=4$.

The various tadpoles can be classified according to their volume dependence. We denote by $V_i$, $i = 1, 2, 3$, the volumes of the internal tori and by $V_4$ the regulated space-time volume. Also, as explained in the appendix, the $K$ amplitude has contributions of type $K_1(\theta^k)$ from untwisted closed strings, and, for $N$ even, $K_R(\theta^k)$ from $R$-twisted closed strings. The $M$ amplitude receives contributions $M_p(\theta^k)$ from open strings with both ends on $Dp$-branes. The $C$ amplitude has pieces $C_{pq}(\theta^k)$ from open strings with ends on $Dp$ and $Dq$-branes.

In the $K$ amplitude, $K_1(1)$ has tadpoles proportional to $V_1V_2V_3$ that can be cancelled by introducing $n_9$ 9-branes. Taking into account the $M_9(1)$ and $C_{99}(1)$ divergences also proportional to $V_4V_1V_2V_3$, there is factorization and cancellation of tadpoles
provided that
\[
\gamma^T_{\Omega,9} = \gamma_{\Omega,9} \tag{2.35}
\]
and \(n_9 = 32\). Recall that, as explained in section 2.2, this condition implies eq. (2.20). If we assume the GP action of \(\Omega\) we also obtain the eq. (2.21).

For \(N\) odd the remaining divergences are all proportional to \(V_4\) and factorize since we can always choose
\[
\text{Tr} \left( \gamma^{-1}_{\Omega,k,9} \gamma^T_{\Omega,k,9} \right) = \text{Tr} \gamma_{2k,9} \tag{2.36}
\]
Cancellation of twisted tadpoles then implies
\[
\text{Tr} \gamma_{2k,9} = 32 \prod_{j=1}^{3} \cos \pi k v_j \tag{2.37}
\]
Also, we are free to choose \(\gamma^{N}_{1,9} = 1\) or \(\gamma^{N}_{1,9} = -1\).

For \(N\) even, \(K_1(R)\) has tadpoles proportional to \(V_4 V_3/V_1 V_2\) that can be cancelled by introducing \(n_5\) 53-branes. Taking into account the \(\mathcal{M}_5(R)\) and \(C_{55}(1)\) divergences also proportional to \(V_4 V_3/V_1 V_2\) there is factorization and cancellation of tadpoles provided that
\[
\gamma^T_{\Omega R,5} = \gamma_{\Omega R,5} \tag{2.38}
\]
and \(n_5 = 32\). If we assume the GP action of \(\Omega\) we then have \(\gamma^T_{\Omega R,9} = -\gamma_{\Omega R,9}\). In this case, using eqs. (2.14) and (2.20), we immediately find \(\gamma^2_{R,9} = 0\). Since \(R = \theta^{N/2}\) this then implies the condition \(\gamma^{N}_{1,9} = -1\). Similarly, \(\gamma^{N}_{1,5} = -1\).

For \(N\) even the type of other twisted tadpoles depends on the specific form of \(v\). In all cases there are tadpoles proportional to \(V_4 V_3\) that arise from divergences in the amplitudes \(C_{99}(R), C_{55}(R)\) and \(C_{59}(R)\). These divergences factorize into a square and give the condition
\[
\text{Tr} \gamma_{R,9} + 4\text{Tr} \gamma_{R,5,I} = 0 \tag{2.39}
\]
where \(I = 0, \cdots 15\), refers to the fixed points of \(R = \theta^{N/2}\).

Other tadpoles are most easily described case by case. In particular, in \(Z_4, Z_8, Z_8'\) and \(Z_12'\) we find that the Klein bottle amplitude has divergences proportional to \(V_4/V_3\) that cannot be cancelled against any of the Möbius strip or cylinder contributions. Indeed, for instance in \(Z_4\), if one tries to use \(\gamma_{k,p}\) matrices satisfying eq. (2.39) together with \(\text{Tr} \gamma_{k,9} + 2\text{Tr} \gamma_{k,5} = 0\) from cancellation of tadpoles proportional to \(V_4\), one finds a massless spectrum that is not free of gauge anomalies. This is a signal that there are left-over tadpoles. This is a surprising result, since in \(D=6\) one finds consistent solutions, fully cancelling tadpoles, for all allowed \(Z_N\). Notice that in \(D=6\) these \(V_4/V_3\) tadpoles would be absent because \(V_3 \to \infty\).
For $Z_6$ there are also tadpoles proportional to $V_4$ that imply the conditions

\begin{align*}
\text{Tr} \gamma_{k,9} + \text{Tr} \gamma_{k,5,0} &= 0 \quad ; \quad k = 1, 5 \\
\text{Tr} \gamma_{2,9} + 3\text{Tr} \gamma_{2,5,0} &= 16 \\
\text{Tr} \gamma_{2,9} + 3\text{Tr} \gamma_{2,5,J} &= 4 \\
\text{Tr} \gamma_{4,9} + 3\text{Tr} \gamma_{4,5,0} &= -16 \\
\text{Tr} \gamma_{4,9} + 3\text{Tr} \gamma_{4,5,J} &= -4
\end{align*} (2.40)

where $J = 1, \cdots, 8$, refers to the remaining fixed points of $\theta^2$ in the $(Y_1, Y_2)$ planes. In deriving these conditions we have used eqs. (2.36) and also

\[ \text{Tr} (\gamma_{\Omega_{k,5}}^{-1} T) = -\text{Tr} \gamma_{2k,5} \] (2.41)

The minus sign in the right hand side is due to the GP action of $\Omega$. Notice that as explained previously, $\gamma_{1,9}^0 = -1$ and $\gamma_{1,5}^0 = -1$.

In $Z'_6$ the cylinder amplitudes for $\theta, \theta^5$ have tadpoles proportional to $V_4$. Cancellation gives the conditions

\[ \text{Tr} \gamma_{k,9} - 2\text{Tr} \gamma_{k,5,L} = 0 \quad ; \quad k = 1, 5 \] (2.42)

where $L = 0, \cdots, 3$, refers to the fixed points of $\theta$ in the first two tori. In $C_{99}(\theta^k)$, $M_9(\theta^k)$ and $K_1(\theta^k)$, $k = 2, 4$, there are tadpoles proportional to $V_4 V_2$. Using eq. (2.36) we find factorization leading to

\[ \text{Tr} \gamma_{2,9} = -8 \quad ; \quad \text{Tr} \gamma_{4,9} = 8 \] (2.43)

Finally, in $C_{55}(\theta^k)$, $k = 2, 4$, $M_5(\theta^k)$ and $K_1(\theta^k)$, $k = 1, 5$, there are tadpoles proportional to $V_4 V_2$. Using eq. (2.41) we obtain

\begin{align*}
\sum_M (\text{Tr} g_{2,5,M})^2 + 16 \sum_L \text{Tr} g_{2,5,L} &= -64 \\
\sum_M (\text{Tr} g_{4,5,M})^2 - 16 \sum_L \text{Tr} g_{4,5,L} &= -64
\end{align*} (2.44)

where $M = 0, 1, 2$, refers to the fixed sets of $\theta^2$ in the first two tori. Notice that when all 5-branes sit at the origin (2.44) gives $\text{Tr} \gamma_{2,5} = -8$ and $\text{Tr} \gamma_{4,5} = 8$. $Z'_6$ can also be treated as $Z_2 \times Z_3$ with twist vectors $(\frac{1}{2}, \frac{1}{2}, 0)$ and $(-\frac{1}{3}, 0, \frac{1}{3})$. Applying the general results of Zwart \[19\] for $Z_N \times Z_M$ we obtain tadpole cancellation conditions in complete agreement with the above.
In $Z_{12}$ there are extra tadpoles proportional to $V_4 V_3$ since $\theta^3$ and $\theta^9$ do not rotate the third direction. Cancellation of these gives the conditions

$$\text{Tr} \gamma_{k,9} + 2\text{Tr} \gamma_{k,5,L} = 0 \quad ; \quad k = 3, 9$$

(2.45)

where $L$ refers to the fixed points of $\theta^3$ in the first and second complex coordinates. All other tadpoles are proportional to $V_4$ and imply the constraints

$$\begin{align*}
\text{Tr} \gamma_{k,9} - \text{Tr} \gamma_{k,5,0} &= 0 \quad ; \quad k = 1, 2, 5, 7, 10, 11 \\
\text{Tr} \gamma_{4,9} + 3\text{Tr} \gamma_{4,5,0} &= 16 \\
\text{Tr} \gamma_{4,9} + 3\text{Tr} \gamma_{4,5,J} &= 4 \\
\text{Tr} \gamma_{8,9} + 3\text{Tr} \gamma_{8,5,0} &= -16 \\
\text{Tr} \gamma_{8,9} + 3\text{Tr} \gamma_{8,5,J} &= -4
\end{align*}$$

(2.46)

where $J = 1, \cdots 8$, refers to the remaining fixed points of $\theta^4$ in the $(Y_1, Y_2)$ planes.

3 $Z_N$ and $Z_N \times Z_M$ models with GP action

In this section we study type IIB orientifolds based on $T^6/\{\Omega, G\}$ where $G$ denotes generators of a discrete group $Z_N$ or $Z_N \times Z_M$, as discussed in the previous chapter, leading to an unbroken $N=1$ supersymmetry in four dimensions. Since $\Omega$ is one of the generators of the orientifold, there are 9-branes in all of these models. In addition, they contain as many independent sets of 5-branes as different order two generators $G$ has. Thus there will be models with zero, one or three different sets of 5-branes. Models in which $G$ contains a $Z_2 \times Z_2$ subsector will have in general three different sets of 5-branes. The simplest $Z_2 \times Z_2$ case was studied in [13] and has a non-chiral spectra, a property shared by this subclass of models.

According to eq. (2.20), in all models $\gamma_{\Omega,9}$ is determined by requiring that it be a real symmetric matrix that exchanges the eigenvalues of $\gamma_{k,9}$ with their complex conjugates. In this chapter we will assume the GP action of $\Omega$ so that eq. (2.21) also holds. Thus, $\gamma_{\Omega,5}$ is determined by requiring that it be a pure imaginary antisymmetric matrix that exchanges the eigenvalues of $\gamma_{k,5}$ with their complex conjugates. We have also seen that when $G$ contains a reflection $R$, the GP action requires $\gamma_{R,9}^2 = -1$. Similarly, $\gamma_{R,5}^2 = -1$. In next chapter we will discuss some aspects of different $\Omega$ actions.

Odd order orientifolds are particularly simple. Cancellation of untwisted tadpoles requires 32 9-branes and there are no 5-branes since there is no order two generator
in $G$. The $Z_3$ case was studied in refs. [14, 15] while $Z_7$ and $Z_3 \times Z_3$ were studied in [16, 17, 19]. We include them in our discussion for completeness.

We concentrate here on even order orientifolds with a single sector of 5-branes. We will first treat models in which all 5-branes sit on the particular fixed point corresponding to the origin in the compact space. Other distributions of 5-branes are analyzed in section 3.8. As explained in section 2.3 and the appendix, $Z_4$, $Z_8$, $Z_8'$ and $Z_{12}$ orientifolds have tadpoles that cannot be cancelled by simply including 9 and 5-branes. The only other orientifolds with one set of 5-branes are based on the twists $Z_6$, $Z_6'$, $Z_{12}$ and $Z_3 \times Z_6$.

We now consider each orientifold in more detail.

### 3.1 $Z_3$

With the choice $\gamma_3 = 1$, cancellation of twisted tadpoles, as given in eq. (2.37), requires $\text{Tr} \gamma_\theta = -4$. This fixes $\gamma_\theta$ uniquely:

$$
\gamma_\theta = \text{diag} (\alpha I_{12}, I_4, \alpha^2 I_{12}, I_4)
$$

(3.1)

with $\alpha = e^{2i\pi/3}$. Here and in the following $I_r$ stands for the $r \times r$ identity matrix. The open string spectrum can be easily computed by using the auxiliary shift

$$
V = \frac{1}{3} (1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0)
$$

(3.2)

We find a $U(12) \times SO(8)$ group and charged chiral fields as shown in Table 3. It is also possible to choose $\gamma_3 = -1$ leading to $\text{Tr} \gamma_\theta = 4$. However, the resulting $\gamma_\theta$ leads to the same group and spectrum.

### 3.2 $Z_3 \times Z_3$

The orbifold group is generated by twists $\theta, \omega$ whose action on the three complex coordinates are given by $v_\theta = (\frac{1}{3}, -\frac{1}{3}, 0)$ and $v_\omega = (0, \frac{1}{3}, -\frac{1}{3})$. Cancellation of twisted tadpoles requires [17, 19]

$$
\begin{align*}
\text{Tr} \gamma_\theta &= \text{Tr} \gamma_\omega = \text{Tr} \gamma_{\theta \omega} = 8 \\
\text{Tr} \gamma_{\theta \omega^2} &= -4
\end{align*}
$$

(3.3)

Also, $\text{Tr} \gamma_3 = 1$. Hence,

$$
\begin{align*}
\gamma_\theta &= \text{diag} (\alpha I_4, \alpha^2 I_4, I_4, I_4, \alpha^2 I_4, \alpha I_4, I_4, I_4) \\
\gamma_\omega &= \text{diag} (I_4, \alpha I_4, \alpha^2 I_4, I_4, I_4, \alpha^2 I_4, \alpha I_4, I_4)
\end{align*}
$$

(3.4)
Table 3: Gauge group and charged chiral multiplets in some $Z_N$ and $Z_N \times Z_M$, $D=4$, $N=1$ type IIB orientifolds with GP action. Only models with at most one set of 5-branes are shown. All 5-branes sit on the fixed point at the origin so that in models with 5-branes the spectrum is explicitly T-dual.
with $\alpha = e^{2i\pi/3}$. The open string spectrum can be easily computed by using the associated shift vectors:

$$
V_\theta = \frac{1}{3}(1, 1, 1, 1, -1, -1, -1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)
$$
$$
V_\omega = \frac{1}{3}(0, 0, 0, 0, 1, 1, 1, -1, -1, -1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0) \quad (3.5)
$$

The gauge group is $U(4)^3 \times SO(8)$. The charged spectrum is displayed in Table 3.

### 3.3 $Z_7$

The twist $\theta$ is generated by $v = \frac{1}{7}(1, 2, -3)$. Taking $\gamma_\theta^7 = 1$, the twisted tadpole cancellation condition eq. (2.37) implies $\text{Tr} \gamma_\theta = 32 \cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \frac{3\pi}{7} = 4$. Then,

$$
\gamma_\theta = \text{diag} (\delta I_4, \delta^2 I_4, \delta^3 I_4, I_4, \delta I_4, \delta^2 I_4, \delta^3 I_4, I_4) \quad (3.6)
$$

where $\delta = e^{2i\pi/7}$ and $\bar{\delta} = \delta^*$. The open string spectrum can be computed using the associated shift

$$
V = \frac{1}{7}(1, 1, 1, 1, 2, 2, 2, -3, -3, -3, -3, 0, 0, 0, 0, 0, 0, 0, 0, 0) \quad (3.7)
$$

The gauge group is again $U(4)^3 \times SO(8)$ although the charged spectrum is slightly different from the $Z_3 \times Z_3$ case (see Table 2). It is also possible to choose $\gamma_\theta^7 = -1$ leading to $\text{Tr} \gamma_\theta = -4$. However, the resulting $\gamma_\theta$ leads to the same group and spectrum.

### 3.4 $Z_6$

The twist $\theta$ is generated by $v = \frac{1}{6}(1, 1, -2)$. The twisted tadpole cancellation conditions are given in eqs. (2.39) and (2.40). To simplify we consider the case with maximal gauge symmetry in which all 32 5-branes sit at the origin and we drop the fixed point subscript in $\gamma_{k,5}$. This configuration is T-selfdual (under duality transformations in the first two complex planes) and the gauge group from 9-branes and 5-branes is the same. In this case tadpole cancellation allows equal $\gamma_k$ matrices for 9-branes and 5-branes. Indeed, we find

$$
\text{Tr} \gamma_{k,9} = \text{Tr} \gamma_{k,5} = 0 \quad ; \quad k = 1, 3, 5
$$
$$
\text{Tr} \gamma_{2,9} = \text{Tr} \gamma_{2,5} = 4
$$
$$
\text{Tr} \gamma_{4,9} = \text{Tr} \gamma_{4,5} = -4 \quad (3.8)
$$
Also, condition (2.38) and the GP condition imply $\gamma_{1,9}^6 = -1$ and $\gamma_{1,5}^6 = -1$ as we explained before. The twist matrix is then

$$
\gamma_{1,9} = \gamma_{1,5} = \text{diag} (\beta I_6, \bar{\beta}^5 I_6, \beta^3 I_4, \bar{\beta} I_6, \bar{\beta}^5 I_6, \bar{\beta}^3 I_4)
$$

(3.9)

where $\beta = e^{i\pi/6}$. The matrices $\gamma_{\Omega,9}$ and $\gamma_{\Omega,5}$ are determined as described before. We find

$$
\gamma_{\Omega,9} = \begin{pmatrix} 0 & I_{16} \\ I_{16} & 0 \end{pmatrix} \; ; \; \gamma_{\Omega,5} = \begin{pmatrix} 0 & -i I_{16} \\ i I_{16} & 0 \end{pmatrix}
$$

(3.10)

$\gamma_{\Omega,9}$ and $\gamma_{\Omega,5}$ have this same form in other even orientifolds in this section.

Computing the spectrum is substantially simplified using the shift notation acting on $SO(32)$ roots. The associated shift corresponding to $\gamma_{1,9}$ and $\gamma_{1,5}$ is

$$
V_{(99)} = V_{(55)} = \frac{1}{12} (1, 1, 1, 1, 1, 5, 5, 5, 5, 5, 5, 3, 3, 3, 3, 3, 3, 3, 3)
$$

(3.11)

The gauge group in the (99) sector is given by $SO(32)$ roots $\rho$ verifying $\rho \cdot V_{(99)} = 0$. The (55) sector is identical so that the full group is $(U(6) \times U(6) \times U(4))^2$. Charged chiral matter fields in the (99) sector correspond to roots verifying $\rho \cdot V_{(99)} = \frac{1}{6}, \frac{1}{6}, -\frac{1}{3}$ mod $\mathbb{Z}$ for each of the three compact complex planes. The (55) sector has the same matter content. As explained in section 2.2 to find the massless chiral fields in the (95), (59) sectors we look for weights $W_{(95)}$ verifying $W_{(95)} \cdot V_{(95)} = \frac{1}{6}$ mod $\mathbb{Z}$. In the end we obtain the spectrum displayed in Table 4.

### 3.5 $Z'_6$

The twist $\theta$ has $v = \frac{1}{6} (1, -3, 2)$. Tadpole cancellation conditions were given in section 2.3. With all 5-branes at the origin they imply

$$
\text{Tr} \, \gamma_{k,9} = \text{Tr} \, \gamma_{k,5} = 0 \; ; \; k = 1, 3, 5 \\
\text{Tr} \, \gamma_{2,9} = \text{Tr} \, \gamma_{2,5} = -8 \\
\text{Tr} \, \gamma_{4,9} = \text{Tr} \, \gamma_{4,5} = 8
$$

(3.12)

We also have $\gamma_{1,9}^6 = -1$ and $\gamma_{1,5}^6 = -1$. Hence,

$$
\gamma_{1,9} = \gamma_{1,5} = \text{diag} (\beta I_4, \beta^5 I_4, \beta^3 I_8, \bar{\beta} I_4, \bar{\beta}^5 I_4, \bar{\beta}^3 I_8)
$$

(3.13)

The associated shift acting on $SO(32)$ roots is

$$
V = \frac{1}{12} (1, 1, 1, 1, 5, 5, 5, 5, 3, 3, 3, 3, 3, 3, 3, 3)
$$

(3.14)
both on the 9-brane and 5-brane sectors. The unbroken (99) or (55) gauge group corresponds to $SO(32)$ roots $\rho$ verifying $\rho \cdot V = 0$. This yields $(U(4) \times U(4) \times U(8))^2$.

Charged chiral multiplets in the (99) or (55) sector are given by roots satisfying $\rho \cdot V = \frac{1}{6}, \frac{1}{12}$, $\frac{1}{3}, \frac{1}{18}$ respectively for each of the three compact complex planes. To find the (95), (59) chiral multiplets we look for weights $W_{(95)}$ verifying $W_{(95)} \cdot V_{(95)} = -\frac{1}{6} \mod \mathbb{Z}$.

In this way we obtain the spectrum displayed in Table 3.6

### 3.6 $Z_{12}$

The twist $\theta$ is generated by $v = \frac{1}{12}(1, -5, 4)$. Tadpole cancellation conditions were given in section 2.3. We choose a T-selfdual configuration in which all the 5-branes sit on the fixed point at the origin. We then find

\[
\begin{align*}
\text{Tr} \gamma_{k,9} &= \text{Tr} \gamma_{k,5} = 0 ; \quad k \neq 4, 8 \\
\text{Tr} \gamma_{4,9} &= \text{Tr} \gamma_{4,5} = 4 \\
\text{Tr} \gamma_{8,9} &= \text{Tr} \gamma_{8,5} = -4
\end{align*}
\]

(3.15)

Also, $\gamma_{1,9}^{12} = -1$ and $\gamma_{1,5}^{12} = -1$. The solution is then

\[
\gamma_{1,9} = \gamma_{1,5} = \text{diag} (\zeta I_3, \zeta^5 I_3, \zeta^7 I_3, \zeta^{11} I_3, \zeta^2 I_2, \zeta^9 I_2, \zeta I_3, \zeta^5 I_3, \zeta^7 I_3, \zeta^{11} I_3, \zeta^3 I_2, \zeta^3 I_2) \quad (3.16)
\]

where $\zeta = e^{i\pi/12}$. Computing the massless spectrum with such a matrix is really very cumbersome but becomes straightforward using the shift notation. In this case

\[
V_{(99)} = V_{(55)} = \frac{1}{24}(-1, -1, -1, 5, 5, 5, -7, -7, -7, 11, 11, 11, 3, 3, 9, 9) \quad (3.17)
\]

Proceeding as in the previous examples we obtain gauge group $(U(3)^4 \times U(2)^2)^2$ and charged chiral spectrum as displayed in Table 3. One can check that the (99) and (55) sectors have $SU(3)$ anomalies which are appropriately cancelled by the chiral fields from the (95) sector.

### 3.7 $Z_3 \times Z_6$

The two generators $\theta$ and $\omega$ are realized by $v_{\theta} = (\frac{1}{3}, 0, -\frac{1}{3})$ and $v_{\omega} = (\frac{1}{6}, -\frac{1}{6}, 0)$. In this case $R = \omega^3$ leads to 5$_{\text{g}}$-branes. Again we will treat the T-selfdual configuration with all 5-branes at the origin. Applying the results of Zwart [19] we find that tadpole cancellation requires

\[
\text{Tr} \gamma_{\theta,9} = \text{Tr} \gamma_{\theta,5} = 8
\]

20
\[
\begin{align*}
\text{Tr} \gamma_{\omega^2,9} &= \text{Tr} \gamma_{\omega^2,5} = -8 \\
\text{Tr} \gamma_{\theta,\omega^2,9} &= \text{Tr} \gamma_{\theta,\omega^2,5} = 4 \\
\text{Tr} \gamma_{\omega^3,9} &= \text{Tr} \gamma_{\omega^3,5} = \text{Tr} \gamma_{\omega,9} = \text{Tr} \gamma_{\omega,5} = 0 \\
\text{Tr} \gamma_{\theta,\omega,9} &= \text{Tr} \gamma_{\theta,\omega,5} = 0
\end{align*}
\]

These constraints are fulfilled by the matrices
\[
\begin{align*}
\gamma_{\theta,9} &= \gamma_{\theta,5} = \text{diag} (\alpha^2I_2, I_2, \alpha I_2, I_2, \alpha I_2, I_2, \alpha^2 I_2, I_2, \alpha^2 I_2, I_2), \\
\gamma_{\omega,9} &= \gamma_{\omega,5} = \text{diag} (\beta I_4, \beta^5 I_4, \beta^3 I_8, \beta I_4, \beta^5 I_4, \beta^3 I_8)
\end{align*}
\]

where \( \alpha = e^{2i\pi/3} \) and \( \beta = e^{i\pi/6} \). In this case it is particularly useful the use equivalent shifts to compute the open string spectrum. These are
\[
\begin{align*}
V_{\theta,9} &= V_{\theta,5} = \frac{1}{3} (2, 2, 0, 0, 1, 1, 0, 1, 1, 1, 2, 2, 0, 0, 0, 0) \\
V_{\omega,9} &= V_{\omega,5} = \frac{1}{12} (1, 1, 1, 5, 5, 5, 3, 3, 3, 3, 3, 3, 3, 3, 3)
\end{align*}
\]

The gauge group is \((U(4) \times U(2)^6)^2\). The charged particle spectrum from the different sectors is shown in Table 3. In this case the \(SU(4)\) anomalies are separately cancelled in each of the (99), (55) and (95) sectors.

### 3.8 Wilson lines and non-coincident 5-branes

In all the previous examples we considered the most symmetric situation in which there are no discrete nor continuous Wilson lines and all 5-branes sit at the same fixed point at the origin. New models with different spectra and smaller gauge groups can be obtained in the more general case in which both possibilities (which in fact are T-dual to each other) are present. We now discuss these possibilities and provide some examples. We restrict to \(D=4\), although the analysis applies equally well to Wilson lines in \(D=6\) orientifolds.

The orbifold action underlying the IIB orientifolds is generated by the space group which involves elements \((\theta, 1)\), with \(\theta\) representing \(Z_N\) rotations, and elements \((1, e_m)\), with \(e_m \in \Lambda, m = 1, \ldots, 6\), where \(T^6 = R^6/\Lambda\). The full space group is in general non-Abelian. The element \((\theta, 1)\) is embedded in the open string sector through unitary matrices \(\gamma_{\theta,p}\), according to the \(Dp\)-branes at the endpoints. In addition there can be background Wilson lines which correspond to embeddings of the elements \((1, e_m)\) through matrices \(W_m\) into the 9-brane sector. To a fixed point \(f\) of \(\theta^k\) there corresponds an element \((\theta^k, c_m e_m)\) such that \((1 - \theta^k)f = c_m e_m\), for some integers \(c_m\). The 9-brane
monodromy associated to this fixed point will thus be $(\prod_{m} W_{m}^{c_{m}})_{\gamma,9}$. The structure of the space group imposes constraints on $\gamma_{\theta,9}$ and $W_{m}$. In particular, if $\theta$ rotates the lattice vector $e_{m}$, $(\theta, e_{m})^{N} = (1, 0)$ and this in turn implies $(W_{m} \gamma_{\theta,9})^{N} = 1$ (up to a phase). If $[W_{m}, \gamma_{\theta,9}] = 0$, then this actually implies $W_{m}^{N} = 1$ and we are dealing with a quantized Wilson line. If $[W_{m}, \gamma_{\theta,9}] \neq 0$ the matrix $W_{m}$ is in principle allowed to vary continuously and we are dealing with a continuous Wilson line.

Let us consider first the case of discrete Wilson lines. Now there is not just one $\gamma_{\theta,9}$ matrix that must obey the tadpole cancellation conditions. The different fixed points split into sets feeling different gauge monodromies in the 9-brane sector and tadpole conditions should apply to all different embeddings $(\prod_{m} W_{m}^{c_{m}})_{\gamma,9}$. This turns out to be a very stringent constraint. Once the $W_{m}$ are determined we can compute the massless spectrum. In the (99) sector we have to project with respect to the Wilson lines, for both vector and chiral multiplets, according to

$$\lambda^{(0)} = W_{m} \lambda^{(0)} W_{m}^{-1} \quad ; \quad \lambda^{(i)} = W_{m} \lambda^{(i)} W_{m}^{-1}$$

for all $m = 1, \ldots, 6$. In the (95) sector when the 5-branes sit at a fixed point in addition one has to take into account the precise gauge monodromy corresponding to that specific fixed point. In particular, if the point is fixed with respect to the space group element $(\theta, c_{m} e_{m})$, in eq. (2.25) one should replace $\gamma_{1,9}$ by $(\prod_{m} W_{m}^{c_{m}})_{\gamma_{1,9}}$.

Consider as a first example the $Z_{3}$ orientifold discussed at the beginning of this chapter. One can take for the torus lattice $\Lambda$ the root lattice of $SU(3)^{3}$. Consider the addition of a discrete Wilson line $W_{1}$ along the first lattice vector $e_{1}$. $W_{1}$ must be unitary and verify $W_{1}^{3} = 1$. Since $\theta e_{1} = e_{2}$, there must be also an identical Wilson line $W_{2} = W_{1}$ along $e_{2}$. The three fixed points in the first lattice are the origin, $w_{1}$ with $(1 - \theta)w_{1} = e_{1}$ and $w_{2}$ with $(1 - \theta)w_{2} = e_{1} + e_{2}$. Hence, the 27 fixed points split into three sets of nine fixed points feeling monodromy $\gamma_{\theta,9}$, $W_{1} \gamma_{\theta,9}$ and $W_{1}^{2} \gamma_{\theta,9}$ respectively. Tadpole cancellation conditions will require

$$\text{Tr} \gamma_{\theta,9} = \text{Tr} W_{1} \gamma_{\theta,9} = \text{Tr} W_{1}^{2} \gamma_{\theta,9} = -4$$

We rewrite the twist (3.1) as $\gamma_{\theta,9} = (\alpha I_{4}, \alpha I_{4}, \alpha^{2} I_{4}, I_{4}, \alpha^{2} I_{4}, \alpha^{2} I_{4}, \alpha I_{4}, I_{4})$ in order to simplify calculations. Then, the following choice for $W_{1}$ verifies the constraints:

$$W_{1} = \text{diag} (I_{4}, \alpha I_{4}, \alpha I_{4}, I_{4}, \alpha^{2} I_{4}, \alpha^{2} I_{4}, \alpha^{2} I_{4}, \alpha^{2} I_{4})$$

This Wilson line background breaks the $U(12) \times SO(8)$ gauge symmetry down to $U(4)^{4}$. The charged chiral multiplets transform as

$$3(1, 4, 4, 1) + 3(4, 1, 4, 1) + 3(4, 1, 1, 1) + 3(1, 1, 1, 6)$$
This particular model was discussed in [18] and has the peculiarity that the field theory associated to the first three $SU(4)$ factors is finite. The scalar potential in this model has a flat direction under which the (99) chiral multiplets associated to one of the three complex dimensions get a vev. As it is well known, this corresponds to the addition of a continuous Wilson line. The $SU(4)^3$ gauge symmetry is broken to the diagonal $SU(4)$ and in this particular example one gets a model with $N=4$ global supersymmetry.

It is interesting to see how this diagonal group spectrum emerges when a continuous Wilson line $W$ is turned on. Consider the following proposal

$$W = \begin{pmatrix} I_4 & 0 & 0 & 0 \\ 0 & W & 0 & 0 \\ 0 & 0 & I_4 & 0 \\ 0 & 0 & 0 & W^* \end{pmatrix}$$

(3.25)

where the $12 \times 12$ matrix $W$ is defined as

$$W = \begin{pmatrix} w & a & a \\ a & w & a \\ a & a & w \end{pmatrix}$$

(3.26)

with $a$ an arbitrary complex number. It is easy to check that constraints (3.22) are verified. Moreover, $(W\gamma_{\theta,9})^3 = \text{diag}(I_{12}, cI_4, I_{12}, c^*I_4)$ where $c = (w^3 - 3a^2w + 2a^2)$. For $a = 0$ and $w = \alpha$ we recover the discrete Wilson line (3.23) discussed above. For $a \neq 0$ (and $\frac{W}{a} \neq 1, -2$) we can choose $a$ such that $c = 1$. Thus, we are left with a Wilson line completely rotated by the group action $\gamma_{\theta,9}$ and depending on a complex continuous parameter $w$. When projections (3.21) are imposed, $U(4)_{\text{diag}} \times U(4)$ gauge group and three chiral multiplets $3(16, 1) + 3(1, 6)$ are found.

It is illustrative to consider now an orientifold obtained from $Z_3$ by a T-duality with respect to the first two complex planes. In the notation of [18] this would correspond to the $D=4$ analogue of the $Z_6^B$ orientifold. Under this duality $\Omega \to \Omega R$ and the orientifold is generated by the order six element $\Omega \theta$, with $\theta$ generated by $v = \frac{1}{2}(1, 1, -2)$. Thus, the orientifold group is $\mathcal{G} = \{1, \theta^2, \theta^4, \Omega R, \Omega \theta, \Omega \theta^5\}$. Since $\Omega$ is not an element of the orientifold group, this model has no 9-branes. Cancellation of untwisted tadpoles requires the presence of 32 5-branes. Cancellation of twisted tadpoles further requires:

$$\text{Tr} \gamma_{k,5,0} = -4 \quad ; \quad \text{Tr} \gamma_{k,5,J} = 0 \quad ; \quad k = 2, 4$$

(3.27)

where $J = 1, \ldots, 8$, refers to fixed points in the first two complex dimensions away from the origin ($J = 0$ corresponds to the origin). If all the 32 5-branes sit at the origin,
the conditions on $\gamma_{2,5,0}$ are analogous to those for 9-branes in the T-dual model so that eq. (3.1) would give a solution for $\gamma_{2,5,0}$. This yields then gauge group $U(12) \times SO(8)$ and chiral fields in $3(12,8) + 3(66,1)$. Now suppose that we send some of the 5-branes to some other fixed point away from the origin. Due to conditions (3.27) we cannot send them all away from the origin, a minimum of eight 5-branes must remain with $\gamma_{2,5,0} = (\alpha I_4, \alpha^2 I_4)$, so that $\text{Tr} \, \gamma_{2,5,0} = -4$. The other 24 5-branes can leave the origin in groups of six 5-branes (so that they form sets invariant under the orientifold action) and reach some of the other fixed points. Take for example the case in which 12 5-branes sit at the same fixed point away from the origin. The other 12 are related to the former by $\Omega R$ so that projection under this generator does not give extra constraints. Then, the tadpole condition on the 12 5-branes has solution

$$\gamma_{2,5,J} = \text{diag}(\alpha I_4, \alpha^2 I_4, I_4)$$

with $\alpha = e^{2i\pi/3}$. The overall spectrum of this 5-brane configuration is as follows. The 8 5-branes at the origin give gauge group $U(4)$ and three copies of chiral fields in the antisymmetric representation. The other 24 5-branes yield $U(4)^3$ and matter fields in $3(1,\bar{4},4) + 3(4,1,\bar{4}) + 3(\bar{4},4,1)$. Thus, we recover exactly the same massless spectrum as the T-dual. Now, if we send all the 24 5-branes to the bulk, they must travel in $Z_3$ and $\Omega R$ invariant configurations. This means that there are only four dynamical 5-branes leading to $SU(4)_{\text{diag}}$ with three chiral multiplets in the adjoint. This corresponds to giving non-vanishing vevs to the bi-fundamental fields present in the configuration with 24 5-branes at the fixed point.

Let us now study Wilson lines in a model with both 9-branes and 5-branes. Consider the $Z'_6$ orientifold generated by the twist $v = \frac{1}{6}(1, -3, 2)$. This can be realized taking $\Lambda$ to be the $SU(3) \times SO(4) \times SU(3)$ root lattice. The properties of the Wilson lines that can be added depend on what complex direction the Wilson line wraps around. Consider Wilson lines $W_3$, $W_4$, wrapping around the second complex plane. The condition $(\theta, e_{3,4})^6 = 1$ implies that $W^2_{3,4} = 1$. Suppose we add a Wilson line $W_3$ around the direction $e_3$. The four $\theta$ fixed points in the first and second complex directions will split into two points with monodromy $\gamma_{\theta,9}$ and another two points with monodromy $W_3 \gamma_{\theta,9}$. The three fixed points under $\theta^2$ will not feel the Wilson line whereas the sixteen fixed points under $R = \theta^3$ will split in two sets of eight fixed points each. In addition to eqs. (3.12), note that tadpole cancellation conditions will require

$$\text{Tr} \, W_3 \gamma_{3,9} = \text{Tr} \, W_3 \gamma_{1,9} = 0$$

(3.29)
Consider the following Wilson line
\[
W_3 = \text{diag} \left( I_8, I_r, -I_8 - r, I_8, I_r, -I_8 - r \right)
\]
(3.30)

This matrix verifies all the constraints. The effect of \( W_3 \) is to break the gauge symmetry down to \( U(4)^2 \times U(r) \times U(8 - r) \). Consider the open string spectrum in the particular \( r = 0 \) simple case. The gauge group in the (99) sector is as in the case without Wilson line, \( U(4)^2 \times U(8) \). The chiral multiplets in this sector transform as
\[
(1, 1, 28) + (1, 1, \overline{28}) + (4, 4, 1) + (\overline{4}, \overline{4}, 1) + (6, 1, 1) + (1, 6, 1) + (4, \overline{4}, 1)
\]
(3.31)

Concerning the chiral multiplets in the (59) sector, if all 5-branes sit at the origin, since the fixed point at the origin does not feel the Wilson line, the spectrum is just that given in Table 3. If all the 5-branes sit at one of the fixed points feeling the Wilson line (those with coordinates \( \frac{1}{2} e_3 \) and \( \frac{1}{2} (e_3 + e_4) \)) the (59) spectrum is still similar except for a flipping \( 8 \leftrightarrow \overline{8} \) in the \( U(8) \) of the 9-branes. This is due to the fact remarked above that in the projection one should replace \( \gamma_{\theta, 9} \) by \( W_3 \gamma_{\theta, 9} \).

Due to T-duality, in this \( Z'_6 \) orientifold there must be an operation on the 5-branes degrees of freedom which gives an analogous spectrum. In other words, certain distributions of 5-branes on different fixed points must produce analogous physics. It is easy to find the configuration of 5-branes that reproduces the spectrum that we found for the 9-brane sector with the Wilson line \( W_3 \). We locate 16 of the the 32 5-branes at the origin in the first two complex planes. Those must fulfill the tadpole conditions in section 2.3 that have solution \( \gamma_{1, 5, 0} = (\beta I_4, \beta I_4, \overline{\beta} I_4, \overline{\beta} I_4) \). These 16 5-branes give rise to \( U(4)^2 \) group with charged fields transforming as \( (4, 4) + (\overline{4}, \overline{4}) + (6, 1) + (1, 6) + (4, \overline{4}) \). Now we locate the remaining 16 5-branes at one of the other three fixed points \( L = 1, 2, 3 \), in the first two complex planes. The choice \( \gamma_{1, 5, L} = (\beta^3 I_8, \overline{\beta}^3 I_8) \) is consistent with tadpole cancellation. It gives rise to gauge group \( U(8) \) and matter fields in \( 28 + \overline{28} \). Notice that the overall spectrum is the same as that obtained in the 9-brane sector with the addition of the Wilson line \( W_3 \). Notice also that there are no fields which transform non-trivially with respect to both the gauge group from the 5-branes at the origin and the group from the 5-branes at a different fixed point.

4 \( D = 4 \) Orientifolds with alternative discrete torsion projections

As we already remarked, in even order type IIB orientifolds tadpole cancellation is consistent with alternative ways of realizing the orientifold projection. This has been
known in \( D=6 \) for some time. Shortly after ref. \( \text{[4]} \) appeared, another way of orientifolding was presented \( \text{[5]} \), in which D9-branes are absent. On the other hand, refs. \( \text{[6, 8]} \) working in an F-theory framework, suggested another consistent way of performing the orientifold projection. Compactifying F-Theory on a \((h_{21}, h_{11}) = (51, 3)\) Calabi-Yau (CY) manifold, and using Sen’s identification of the fiber inversion with \( \Omega(-1)^F \) in IIB theory \( \text{[29]} \), gives an orientifold group \{\( \Omega(-1)^F \mathcal{R}_3, \Omega(-1)^F \mathcal{R}_4, \mathcal{R}_3 \mathcal{R}_4, 1 \}\). Here \( \mathcal{R}_i \) is a \( \mathbb{Z}_2 \) inversion of coordinate \( Y_i \). Tadpole conditions require the presence of two sets, \( 7, 7' \), of 7-branes. Furthermore, the action of \( \Omega^2 \) in the 7-7’ sector changed from -1 (as in the GP model), to +1. Together with this, the new model had extra twisted tensors and symmetric \( \gamma_R \) matrices (compared to antisymmetric in the GP case). In fact the GP model, or rather its T-dual, can also be constructed in terms of these same generators and two sets of 7-branes \( \text{[30]} \). Hence, it became clear that the sign of \( \Omega^2 \) in the mixed 7-7’ sector is just a choice that can be made each time a \( \mathbb{Z}_2 \) projection is realized. This is thus analogous to the discrete torsion degree of freedom already encountered in heterotic orbifolds \( \text{[22, 23]} \). This connection with discrete torsion was in fact suggested in \( \text{[8, 24]} \).

The choice of sign for \( \Omega^2 \) can be shown to be related to the symmetry or antisymmetry of the matrix \( \gamma_R \) that realizes the order two orientifold twist on the Chan-Paton matrices. It was noted in \( \text{[21]} \) that the presence of certain couplings of R-R scalars to open string vectors required the constraints

\[
\gamma_R = -\gamma_{\Omega} \gamma_R^T \gamma_{\Omega}^{-1}
\]

for the standard \( \Omega^2 = -1 \) projection and

\[
\gamma_R = +\gamma_{\Omega'} \gamma_R^T \gamma_{\Omega'}^{-1}
\]

for an alternative \( \Omega^2 = +1 \). In what follows we will construct \( D=4, N=1 \) orientifolds realizing the alternative projection.

### 4.1 \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) orientifolds and discrete torsion

The \( D=4 \) analogue of the GP orientifold is the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) orientifold of \( \text{[13]} \). In this model the standard \( \Omega^2 = -1 \) GP action is consistently assumed in mixed subsectors. We would like now to construct \( D=4 \) models with alternative actions.

In fact, the existence of these new models was conjectured in ref. \( \text{[24]} \) that constructed \( D=4 \) models in terms of F-theory compactified in particularly simple Calabi-Yau four-folds. In a few examples, such as their \((C,C)\) model, the complete spectrum
coming from F-theory computations is available. This is an affordable four-fold calculation in F-theory, as it admits a $T^8/(Z_2)^3$ orbifold point. Indeed, it is the analogue of the six-dimensional F-theory compactification on the (51,3) CY threefold that we mentioned above. The structure of the $Z_2$ fiber degenerations over the base implies $SO(8)^{12}$ gauge group, no charged chiral multiplets and fifty-five moduli.

In $D=6$ it can be shown that the usual blowing up procedure, corresponding to deforming the Kähler class, produces the (51,3) model with $SO(8)^8$ gauge group and 17 tensors [24]. This is the model in [4, 8]. Instead, to get the GP model, with just one tensor, we have to resolve the singularities by complex structure deformations that lead to F-theory on the $(h_{21}, h_{11}) = (3, 243)$ CY. These facts provide new clues for the discrete torsion analogy. In ref. [24] it was conjectured that the model in [13] (which only uses GP-type projections) can be obtained by an F-theory compactification on a CY four-fold resolved by complex structure deformations. Then, it is expected that blowing up the F-theory (C,C) model corresponds to an orientifold with a ‘complementary Ω action’. This is what we now describe.

The orientifold group is generated by $\{1, R_2 R_3, \Omega(-1)^F L R_1, \Omega(-1)^F L R_2\}$. We expect three sets of 7-branes, associated to elements

$$\Omega(-1)^F L R_i \stackrel{\text{def}}{=} \Omega_i \rightarrow 7_i\text{-branes} \quad (4.3)$$

In principle there could also be D3-branes, due to the $\Omega(-1)^F L R_1 R_2 R_3$ element, but its absence in the F-theory formulation of the same model indicates that they are not present. We will make use of what we know from the $D=6$ model in [6, 8]. As in that case we can put eight $7_i$-branes at each of the four orientifold fixed points generated by $\Omega_i$. Then, the $\gamma$ matrices are written as $8 \times 8$ matrices. Twisted tadpole cancellation requires

$$\begin{align*}
\text{Tr} \gamma_{R_1 R_2 7_1} &+ \text{Tr} \gamma_{R_1 R_2 7_2} = 0 \\
\text{Tr} \gamma_{R_2 R_3 7_2} &+ \text{Tr} \gamma_{R_2 R_3 7_3} = 0 \\
\text{Tr} \gamma_{R_1 R_3 7_1} &+ \text{Tr} \gamma_{R_1 R_3 7_3} = 0
\end{align*} \quad (4.4)$$

where the relative + sign is due to $\Omega^2_{7_i 7_j} = +1$ [3, 8]. We can then take

$$
\begin{array}{cccccccc}
\gamma_{\Omega_1} & \gamma_{\Omega_2} & \gamma_{\Omega_3} & \gamma_{R_1 R_2} & \gamma_{R_2 R_3} & \gamma_{R_1 R_3} \\
7_1 & I & I & I & I & I & I \\
7_2 & I & -I & I & -I & -I & I \\
7_3 & -I & I & I & -I & I & -I
\end{array} \quad (4.5)
$$
The signs for $\gamma_{\Omega_i}$ are such that $\gamma_{\Omega_i}\gamma_{\Omega_j} = \gamma_{R_iR_j}$. The closed sector, as in [24], gives fifty-five moduli. In open sectors, we have to project with the $\gamma$’s above. Each of the three $7_i\bar{7}_i$ sectors gives gauge group $SO(8)^4$. For example, in the $7_1$ case, the Chan-Paton matrices satisfy

$$\lambda = -\gamma_{\Omega_1;7_1}\lambda^T\gamma_{\Omega_1;\bar{7}_1}^{-1} = -\lambda^T$$

$$\lambda = \gamma_{R_1R_2;\bar{7}_1}\lambda\gamma_{\bar{R}_1\bar{R}_2;\bar{7}_1}^{-1} = \lambda$$

$$\lambda = \gamma_{\bar{R}_2\bar{R}_3;\bar{7}_1}\lambda\gamma_{\bar{R}_2\bar{R}_3;\bar{7}_1}^{-1} = \lambda$$

so that $\lambda$ is orthogonal and gives $SO(8)$ at each orientifold plane. For chiral multiplets, as in $D=6$, the fact that the monodromy is trivial ($\gamma_{R_iR_j} = \pm 1$) kills all matter. In the same way, considering the phases due to group actions, we find that mixed $7_i\bar{7}_j$ sectors do not add any other massless state.

We have therefore recovered the spectrum of the $(C,C)$ F-theory compactification in [24], a ‘discrete torsion’ version of the orientifold of [13]. The matrices in (4.5) representing the $Z_2$ actions clearly are all symmetric, contrary to the matrices of ref. [13] that are antisymmetric, as in the original GP model. We can nevertheless think of a model in which one $Z_2$ twist is realized in GP way and the other in the way of [4, 8]. This option was also suggested in ref. [24] in terms of a four-fold ambiguity in defining the $Z_2 \times Z_2$ orientifold. The construction is quite similar to the one before, we just substitute one $Z_2$ action from the symmetric form by one of the antisymmetric matrices defined in [13]. More concretely, consider

$$
\begin{array}{ccccccc}
\gamma_{\Omega_1} & \gamma_{\Omega_2} & \gamma_{\Omega_3} & \gamma_{R_1R_2} & \gamma_{R_2R_3} & \gamma_{R_1R_3} \\
7_1 & I & M & I & M & M & I \\
7_2 & M & I & M & M & M & I \\
7_3 & I & M & -I & M & -M & -I \\
\end{array}
$$

(4.7)

where

$$M = \begin{pmatrix}
0 & I_2 & 0 & 0 \\
-I_2 & 0 & 0 & 0 \\
0 & 0 & 0 & -I_2 \\
0 & 0 & I_2 & 0
\end{pmatrix}
$$

(4.8)

It is important that the set of matrices in (4.7) satisfy all conditions imposed by group multiplication. These conditions are the same as in [13], except that when $\Omega^2_{7_1\bar{7}_2}$ enters, it gives an extra -1 due to the new projection. Meanwhile $\Omega^2_{7_1\bar{7}_3}$ remains equal to -1. The choice in (4.7) cancels twisted tadpoles in accordance with

$$\text{Tr} \gamma_{R_1R_2;7_1} - \text{Tr} \gamma_{R_1R_2;\bar{7}_2} = 0$$

28
\[ \begin{align*}
\text{Tr} \gamma R_{23} &- \text{Tr} \gamma R_{23} = 0 \\
\text{Tr} \gamma R_{13} + \text{Tr} \gamma R_{13} = 0 \quad (4.9)
\end{align*} \]

The closed spectrum is still the same. In the open sector, the projection by \( M \) breaks \( SO(8) \rightarrow SU(4)^{12} \). The three \( 7_i \) sectors give together \( SU(4)^{12} \). To compute the matter spectrum we just have to recall the phases corresponding to each group action, with \( \Omega \rightarrow \pm 1 \) for DD, NN boundary conditions and \( R_i \rightarrow -1 \) when acting on \( Y_i \). We get two copies of \((6, 1, 1, 1)\) from each \( 7_i \) sector. In mixed sectors, the \( R_i R_j \) twist kills all states. The total spectrum is anomaly free. It would be interesting to obtain the F-theory version of this model for comparison.

### 4.2 A chiral \( Z_6' \) model without 9-branes

One can also extend to \( D=4 \) the projection presented in ref. [3]. One can for example construct a \( Z_2 \times Z_3 \) orientifold with group \( \{1, R\} \times \{1, \omega, \omega^2\} \times \{1, \Omega S\} \), where \( R \) is generated by \( v_R = (0, \frac{1}{2}, -\frac{1}{2}) \) and \( \omega \) by \( v_\omega = (\frac{1}{3}, -\frac{1}{3}, 0) \) whereas \( S \) is the transformation \( S : (Y_2, Y_3) \rightarrow (-Y_2, -Y_3 + \frac{1}{2}) \). In principle, the group elements \( \Omega S \) and \( \Omega R S \) would give D5-branes and D9-branes respectively. But, due to \( S \), the Klein bottle amplitude associated to D9-branes is free of tadpoles [3]. There are still \( K \) tadpoles proportional to \( V_1 V_2 V_3 \) that can be cancelled by introducing D5\(_1\)-branes. To solve the model, we will consider a configuration in which 16 D5-branes are on top of the origin and the other 16 sit on its image under \( S \). As in ref. [3], we can take \( \gamma_{\Omega S} = 1, \gamma_R = (I_8, -I_8) \). Also,

\[ \gamma_\omega = \text{diag} (\alpha^2 I_2, \alpha I_2, I_4, \alpha I_2, \alpha^2 I_2, I_4) \quad (4.10) \]

The constraints on Chan-Paton factors read:

\[ \lambda^{(0)} = \gamma_R \lambda^{(0)} \gamma_R^{-1} \quad ; \quad \lambda^{(0)} = \gamma_\omega \lambda^{(0)} \gamma_\omega^{-1} \quad (4.11) \]

for gauge vectors while for matter

\[ \begin{align*}
\lambda^{(1)} &= \gamma_R \lambda^{(1)} \gamma_R^{-1} \quad ; \quad \lambda^{(1)} = \alpha \gamma_\omega \lambda^{(1)} \gamma_\omega^{-1} \\
\lambda^{(2)} &= -\gamma_R \lambda^{(2)} \gamma_R^{-1} \quad ; \quad \lambda^{(2)} = \alpha^* \gamma_\omega \lambda^{(2)} \gamma_\omega^{-1} \\
\lambda^{(3)} &= -\gamma_R \lambda^{(3)} \gamma_R^{-1} \quad ; \quad \lambda^{(3)} = \gamma_\omega \lambda^{(3)} \gamma_\omega^{-1} \quad (4.12)
\end{align*} \]

where \( \alpha = e^{2i\pi/3} \). From the (55) sector we obtain gauge group \((U(2)^2 \times U(4))^2\) and charged matter fields transforming as

\[ \psi^1 : (2, 1; 4; 1, 1, 1) + (1, 2; 4; 1, 1, 1) + (2, 2; 1; 1, 1, 1) \]
\[ \psi^2 : (1, 2, 1; 1, 1, \overline{4}) + (1, 1, 4; 2, 1, 1) + (2, 1, 1; 1, 2, 1) + (1, 1, \overline{4}; 1, 2, 1) + (1, 1, \overline{4}; 1, 1, 4) + (1, 2, 1; 2, 1, 1) \]

\[ \psi^3 : (1, 1, 4; 1, 1, \overline{4}) + (1, 1, \overline{4}; 1, 1, 4) + 2(2, 1, 1; 2, 1, 1) + 2(1, 2, 1; 1, 2, 1) \]  

(4.13)

The model is chiral and anomaly free.

5 Heterotic duals of $D = 4$, $N = 1$ orientifolds

The ten-dimensional $SO(32)$ heterotic string is supposed to be S-dual to type I strings. This fact is already suggested by the form of the ten-dimensional effective Lagrangian. By dimensional reduction one can obtain that the mapping between the dilatons of both dual theories in $D$ dimensions is \cite{14, 31}

\[ \Phi_I = 6 - \frac{D}{4} \Phi_H - \frac{D - 2}{16} \log \det G^{(10-D)}_H \]  

(5.1)

where $\Phi_I$ ($\Phi_H$) is the type I (heterotic) dilaton. $G_H$ is the metric of the $(10-D)$ compact dimensions in the heterotic frame. Notice that for $D=10$ indeed a strongly coupled heterotic string maps to a weakly interacting type I string. However, for $D=4$ eq. (5.1) shows that there might be a weak-weak coupling duality for regions of moduli space. This means that if we have a $D=4$ type I vacuum and a heterotic vacuum which are dual to each other, their spectra has to exactly match at weak coupling. It is important to realize however that the weakly coupled dual of a perturbative (type I or heterotic) $D=4$ model need not be a perturbative string construction.

In ten dimensions the gauge group in type I originates in open strings ending on 9-branes whereas in $SO(32)$ heterotic they originate in the left-handed bosonic sector with 16 coordinates compactified on the $Spin(32)/Z_2$ lattice. Thus, in searching for the heterotic duals of given $D=4$, $N=1$ orientifolds the obvious idea is to consider heterotic $Z_N$ or $Z_N \times Z_M$ orbifolds whose gauge degrees of freedom and untwisted chiral states match the orientifold spectrum. It turns out that this is possible in all cases. The identification is particularly obvious if we choose the Cartan-Weyl representation for the twist matrices $\gamma_{\theta,9}$, as we discussed in section 2.2, and associate a shift vector $V_{(99)}$ to these matrices. Thus, the natural mapping between (99) orientifold states and heterotic untwisted sector is

\[ \text{Type I} \leftrightarrow SO(32) \text{ Heterotic} \]
\[ \psi_{\frac{1}{2}} |0, ab\rangle \lambda_{\mu,ab}^{(i)} \leftrightarrow \psi_{\frac{1}{2}} |0\rangle_R \otimes |P^I\rangle_L \left( \partial X^I \right)_L \quad I = 1, \cdots, 16 \]
\[ \psi_{\frac{1}{2}} |0, ab\rangle \lambda_{\mu,ab}^{(i)} \quad i = 1, 2, 3 \leftrightarrow \psi_{\frac{1}{2}} |0\rangle_R \otimes |P^I\rangle_L \left( \partial X^I \right)_L \quad i = 1, 2, 3 \]
\[ \gamma_{1,9} = \exp(-2i\pi V_{(99)} \cdot H) \leftrightarrow V_{het} = V_{(99)}, \quad NV_{het} \in \Gamma \quad (5.2) \]

where \( \Gamma \) is the \( \text{Spin}(32)/Z_2 \) lattice and \( P \in \Gamma \) are the gauge quantized momenta of the heterotic string. On the heterotic side the action of the twist \( \theta \) in the gauge degrees of freedom is embedded through the shift \( V_{het}^2 \) and the massless states are obtained by projection \( P \cdot V_{het} = \text{integer} \) for the gauge group and \( P \cdot V_{het} = v_i \text{ mod integer} \) for the \( i = 1, 2, 3 \) untwisted chiral multiplets. One can trivially check that with the identification \( V_{het} = V_{(99)} \) indeed the untwisted heterotic spectrum precisely matches the \( (99) \) sector of the candidate dual orientifold.

For a heterotic orbifold to be perturbatively consistent certain modular invariant constraints must be fulfilled. In particular level matching imposes for a \( Z_N \) twisted sector the constraint \( N(V_{het}^2 - v^2) = \text{even} \). Now, one can check that only the shifts \( V_{(99)} \) in section 3 corresponding to odd \( N \) \( (Z_3, Z_3 \times Z_3, Z_7) \) obey the modular invariant constraints, while none of the even order twists do. Indeed, perturbative heterotic duals for these three orientifolds were proposed in refs. \cite{14, 13, 16}. However, it is easy to obtain new shifts that produce the same untwisted spectrum and \textit{are} modular invariant also for even \( N \). It is enough to consider any of the even order shifts in section 3 and do the replacement

\[ V_{het} = V_{(99)} \rightarrow V_{het} = V_{(99)} - (0, 0, 0, \cdots, 0, 1) \quad (5.3) \]

This fact was already noticed in the \( D=6 \) case in ref. \cite{24}. Indeed, consider the simplest \( Z_2, D=6 \) GP orientifold. One can check that in this case \( V_{(99)} = \frac{1}{4}(1, 1, \ldots, 1, 1) \), which is not modular invariant, but the twist \( V_{het} = \frac{1}{4}(1, 1, \ldots, 1, -3) \) does obey the perturbative modular invariant constraints. The new \( V_{het} \) gives the heterotic dual of a particular configuration in the \( Z_2 \) orientifold in which there are 2 D5-branes at each of the 16 fixed points of \( Z_2 \) in \( D=6 \). It was argued in ref. \cite{23} that the first (non-modular invariant) shift for the heterotic model gives rise to the heterotic dual of a different configuration of the \( Z_2 \) orientifold in which e.g. all 32 5-branes sit at the same fixed point. The same was found for the other \( D=6, N=1 \) orientifolds, the duals of orientifolds with all 5-branes sitting at the origin are (non-perturbative) heterotic models in which the standard modular invariant constraints are violated. This is just as well, since most orientifold models have extra gauge and matter degrees of freedom coming from the 5-brane sector which can only appear on the heterotic side at the non-perturbative level. In \( D=6 \) these non-perturbative effects may be understood as due to small instantons
either in the bulk or located at fixed points. Non-perturbative heterotic orbifolds of this type were constructed in ref. [25].

The $D=4, N=1$ orientifolds discussed in this paper appear to behave in a similar way concerning type I-heterotic duality. It seems that the heterotic duals of orientifolds with all 5-branes sitting at the origin are non-perturbative $D=4, N=1$ heterotic orbifolds in which the usual modular invariance constraints are violated and non-perturbative gauge groups and fields arise due to small instanton effects. Although, as we mentioned above, for each of the orientifolds of even order one can find a perturbative heterotic candidate dual obeying modular invariance constraints, one cannot match the full massless spectra with the orientifolds since, to start with, these perturbative vacua are missing the extra degrees of freedom associated to the 5-branes. Furthermore, these perturbative heterotic vacua have extra charged matter fields in their twisted sectors which also are missing in their orientifold counterparts. From this point of view the $D=6, Z_2$ GP model is exceptional since there is one configuration of the 5-branes (two in each of the 16 fixed points) which precisely matches the perturbative $Z_2$ heterotic orbifold that we mentioned above. With that configuration there is no gauge group left from the 5-brane sector. In the case of generic $D=4, N=1$ orientifolds of even order such privileged 5-brane configurations seem difficult to find, if they exist at all.

To exemplify the above discussion let us consider a candidate perturbative heterotic dual of the $Z'_6$ orientifold. The mapping in eq. (5.2) suggests constructing a perturbative heterotic orbifold with a shift $V_{het} = V_{(99)}$ as given in eq. (3.14) appropriately shifted as in (5.3). As we said, the untwisted sector precisely matches the $(99)$ sector of the orientifold. On the other hand, the twisted sectors have content

$$\theta, \theta^5 : 12(4,1,1) + 12(1,\overline{4},1)$$
$$\theta^2, \theta^4 : 9(6,1,1) + 9(1,6,1) + 6(\overline{4},4,1) + 3(4,\overline{4},1) + 18(1,1,1)$$
$$\theta^3 : 4[(1,1,8) + (1,1,\overline{8}) + (\overline{4},1,1) + (1,4,1)] + 8[(4,1,1) + (1,\overline{4},1)] \quad (5.4)$$

One can check that the contribution to non-Abelian anomalies coming from $\theta, \theta^5$ sectors is cancelled by that coming from the $\theta^2, \theta^4$ sectors. The contribution of the $\theta^3$ particles exactly cancels against the untwisted sector. Notice that the content of the $\theta^3$ twisted sector is identical to that of the $(59)$ sector of the $Z'_6$ orientifold, except for the obvious fact that multiplicities coming from the number of fixed points in the heterotic orbifolds are representations with respect to the $(55)$ gauge group on the orientifold side. Although this coincidence would suggest that this perturbative heterotic model

$$32$$
could be dual to the $Z_6'$ orientifold, there is nothing in the orientifold resembling the spectrum of $\theta^n, n = 1, 2, 4, 5$ sectors. Rather, one would expect this model to be dual to some non-perturbative type I vacuum which has solitonic states reproducing those sectors (in the same way that $SO(32)$ spinorial representations are expected to appear at the non-perturbative level in type I strings).

Heterotic duals for some of the $D=4, N=1$ orientifolds of section 4 with alternative orientifold projections can however be proposed. Consider first the $Z_2 \times Z_2$ orientifold with gauge group $SO(8)^2$ of section 4.1 and the corresponding heterotic $Z_2 \times Z_2$ orbifold generated by the actions $v_\theta = \frac{1}{2}(1, -1, 0), v_\omega = \frac{1}{2}(0, 1, -1)$ acting on a square $SO(4)^3$ compactification lattice. Add quantized Wilson lines $a_1, a_2$ along, say, the first two compact coordinates. If $\theta$ ($\omega$) is embedded in the $Spin(32)/Z_2$ gauge lattice as a shift vector $A$ ($B$) given by

$$A = a_1 = \frac{1}{2}(1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$$

$$B = a_2 = \frac{1}{2}(0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0)$$

(5.5)

the gauge group is broken to $SO(8)^4$. Notice that these shifts violate the perturbative modular invariance constraints. The presence of the Wilson lines projects out any massless untwisted charged field. Now, consider the $\theta$ twisted sector. Its 16 fixed points are split into four groups of four fixed points each feeling respectively the gauge monodromies $A, A + a_2, A + a_1 + a_2, A + a_1$. The first three shifts here have length 2 and there are in principle no perturbative massless states in those sectors. However, as discussed in refs. [32, 25], small instantons sitting at a $Z_2$ singularity with such monodromy generate a tensor multiplet in $D=6$ which in turn gives rise to a massless chiral singlet when reduced to $D=4$. Thus, there are 12 singlets from those 12 fixed points. The other 4 fixed points under $\theta$ have trivial monodromy since $A = a_1$ and $A$ is of order two. As remarked in ref. [33], 5-branes at a $Z_2$ singularity with trivial monodromy originate an $SO(8)$ vector multiplet plus one tensor multiplet in $D=6$. Projecting down to $D=4$ one thus expects from the four fixed points gauge group $SO(8)^4$ and 4 singlets. Now, from the sector twisted under $\theta \omega$ a similar massless spectrum, an $SO(8)^4$ and 16 singlets, is expected. Finally, the 16 fixed points under $\omega$ do not feel the Wilson lines, all feel monodromy given by $B$. Thus one only expects 16 extra chiral singlets in the massless spectrum. Putting all the pieces together, one obtains the expected $SO(8)^12$ gauge group and the correct number of moduli singlets to reproduce the spectrum of the orientifold.

In the same way one can find the candidate heterotic dual for the $Z_2 \times Z_3$ orientifold.
in section 4.2. Here \( v_\theta = \frac{1}{2}(0, 1, -1) \) and \( v_\omega = \frac{1}{3}(1, -1, 0) \). We add a Wilson line \( a_6 \) around the sixth coordinate and embed \( \theta \) and \( \omega \) through shifts \( A, B \) as follows:

\[
A = \frac{1}{2}(1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)
\]
\[
B = \frac{1}{3}(0, 0, 0, 0, 1, 1, -1, -1, 1, -1, 0, 0, 0, 0, 0)
\]
\[
a_6 = \frac{1}{4}(1, 1, 1, \cdots, 1, 1, 1)
\] (5.6)

Again these shifts do not verify the perturbative modular invariance constraint. One can check that the untwisted sector reproduces the (55) sector of the orientifold in section 4.2. Unlike in the previous example, there is no twisted sector with trivial monodromy that could cause extra non-perturbative gauge factors. As discussed in ref.\cite{25} the twisted subsectors of order two and three have associated shifts \( A, B \) which only lead to extra singlets from twisted sectors upon reduction to \( D = 4 \).

6 Effective action and anomalous \( U(1) \)'s in \( D = 4 \) orientifolds

In this section we wish to discuss some general features of the effective low-energy supergravity action of \( D=4, N=1 \) orientifolds. We will concentrate on the class of type IIB orientifolds discussed in chapter 3. Let us first describe a few general properties of the massless closed string sector of these theories. As discussed in chapter 2, besides the supergravity multiplet, closed strings give rise to a number of untwisted chiral moduli singlets \( T_i \) and twisted singlets \( M_\alpha \). Here \( i = 1, 2, 3 \) except for the \( Z_3 \) and \( Z_6 \) orientifolds that have extra off-diagonal moduli. In addition only \( Z_6' \) has one complex structure field \( U_2 \). However, we will consider only the diagonal untwisted moduli \( T_i \) and the dilaton chiral singlet \( S \). The dependence of these four complex scalars on the radii \( R_i \) of the three compact dimensions can be extracted from the \( N=2 \) results of \cite{34, 31}

\[
S = e^{-\phi} R_1 R_2 R_3 + i \theta; \quad T_i = e^{-\phi} \frac{R_i}{R_j R_k} + i \eta_i \quad (i \neq j \neq k) \] (6.1)

where \( \phi \) is the four-dimensional dilaton and \( \theta, \eta_i \) are R-R scalars. Let us concentrate now on orientifolds which contain 9-branes and one sector of 5-branes located at the origin (all even order orientifolds in Table 3). The gauge group in these models has the structure \( G_9 \times G_5 \), where \( G_9 \) (\( G_5 \)) originates in the \( 9(5) \)-brane sector of the theory. There are charged matter fields of three types, fields \( C_i^9 \), \( i = 1, 2, 3 \), charged under \( G_9 \).
only; fields $C^5_i, i = 1, 2, 3,$ charged under $G_5$ only; and fields $C^{95}$ charged under both $G_9$ and $G_5$. It is interesting to see how the different massless chiral fields transform under T-duality. Suppose that the worldvolume of the 5-branes sweep the four non-compact dimensions and the third complex plane. We already remarked that the configuration with all 5-branes at the origin is invariant under T-duality transformations in the first two complex planes. With the above definitions for $S$ and $T_i$ one thus finds

$$
R_1 \leftrightarrow \frac{\alpha'}{R_1} ; \quad R_2 \leftrightarrow \frac{\alpha'}{R_2} \\
S \leftrightarrow T_3 \\
T_{1,2} \leftrightarrow T_{2,1} \\
C^9_i \leftrightarrow C^5_i \\
C^{95} \leftrightarrow C^{95}
$$

(6.2)

Then, we observe that under T-duality the roles of $S$ and $T_3$ are exchanged. The gauge kinetic functions $f_9, f_5$ dependence on the $S, T_i$ fields can also be extracted from the $N=2$ case

$$
f_9 = S \quad ; \quad f_5 = T_3
$$

(6.3)

This is consistent with the fact that under T-duality in the first two complex planes $S$ goes to $T_3$ and the rôles of 5-branes and 9-branes are exchanged. Notice that from the heterotic point of view the gauge interactions from 5-branes are non-perturbative and this tells us that their strengths are governed (in the dual heterotic) by the moduli rather than the dilaton.

In fact there are reasons to argue that both $f_9$ and $f_5$ also depend linearly on closed string twisted singlets $M_\alpha$. One can easily check that all orientifolds in Table 4 have anomalous $U(1)_X$’s in their spectra. We know that in $D=4$, $N=1$ heterotic vacua those anomalies are cancelled by a four-dimensional version of the Green-Schwarz mechanism in which $\text{Im} S$ transforms as $\text{Im} S \rightarrow \text{Im} S - \delta_{GS} \Lambda(x)$ under a $U(1)_X$ gauge transformation with gauge parameter $\Lambda(x)$. Since in the perturbative heterotic $\text{Im} S$ couples to $F \tilde{F}$ in a universal manner to all groups, the $U(1)_X$ anomaly can cancel as long as the mixed anomaly of the $U(1)_X$ with all gauge interactions are in the ratio of the Kac-Moody levels of the gauge algebras. Furthermore, since there is only one dilaton field $S$ to do the trick, in perturbative heterotic vacua there is at most only one anomalous $U(1)_X$. Equation (6.3) already tells us that in the case of type IIB, $D=4$ orientifolds there will be in general more than one anomalous $U(1)_X$ since not only $S$ but moduli fields like $T_3$ have suitable couplings for a Green-Schwarz mechanism to work.
More precisely, one finds that in the absence of Wilson lines the orientifolds \( Z_3 \) and \( Z_3 \times Z_3 \) have only one anomalous \( U(1)_X \), but several if Wilson lines are added. The \( Z_7 \) orientifold has three anomalous \( U(1)_X \)'s when there are no Wilson lines. For models with both 5-branes and 9-branes one finds several anomalous \( U(1)_X \)'s. For example, one finds three anomalous \( U(1)_X \)'s in the \( Z_6 \) orientifold of chapter 3. In order for all the anomalies to cancel, non-linear transformations of the \( S \) and \( T_3 \) fields are not enough. This is particularly obvious in odd models like \( Z_3 \) in which there is no 5-brane sector and the \( S \) field couples universally to the \( SU(12) \) and \( SO(8) \) factors. One can easily show that the mixed anomaly of the \( U(1)_X \) with those two factors is different and could not possibly be cancelled by a shift in \( \text{Im} S \). What happens is that a linear combination of 27 twisted moduli fields \( M_\alpha, \alpha = 1, \ldots, 27 \), do also get shifted under a \( U(1)_X \) transformation. In addition, direct couplings of the \( M_\alpha \) closed string states with \( F_9 \tilde{F}_9 \) must exist for the mechanism to work. That this is the case can also be confirmed by studying the heterotic dual of this model. Something analogous is expected in the case of models with 5-branes. Thus, \( U(1)_X \) anomaly cancellation in this class of orientifolds requires gauge kinetic functions of the form

\[
f_9 = S + \sum_\alpha c^0_\alpha M_\alpha ; \quad f_5 = T_3 + \sum_\alpha c^5_\alpha M_\alpha \quad (6.4)
\]

where \( c^0,5 \) are constant coefficients. Multiple \( U(1)_X \) anomaly cancellation is achieved by non-trivial transformations of \( S, T_3 \) and \( M_\alpha \) under the \( U(1)_X \)'s. Notice that this is analogous, though not identical, to the generalized Green-Schwarz mechanism in \( D=6 \) theories suggested in ref. [36].

Using symmetry arguments one can also extract some of the relevant terms for the Kähler potential in orientifolds with 9-branes and one set of 5-branes. The equivalent terms for the \( N=2 \) case were obtained in ref. [34]. Considering the truncation from the \( D=10 \) type I action and imposing invariance under T-duality with respect to the transformations in eq. (6.2) one gets

\[
K = -\log(S + S^*) - \log(T_3 + T_3^* + |C^3_3|^2) \\
- \log(T_1 + T_1^* + |C^9_1|^2 + |C^5_2|^2) - \log(T_2 + T_2^* + |C^9_2|^2 + |C^5_1|^2) \\
+ \frac{|C^{95}|^2}{(T_1 + T_1^*)^{1/2}(T_2 + T_2^*)^{1/2}} \quad (6.5)
\]

Indeed the above formula is invariant under T-duality with respect to the first two complex planes and with respect to the Peccei-Quinn symmetries corresponding to shifts of the imaginary parts of \( S \) and \( T_i \). The form of the metric of the charged \( C^{95} \) fields is suggested by T-duality invariance and the fact that the (95) sector of these
theories behave as a sort of $Z_2$ twisted sector. The perturbative trilinear superpotential $W$ has the structure

$$W = C_1 C_2 C_3 + C_1 C_2 C_3 + C_1 C_2 C_3 + C_1 C_2 C_3$$

(6.6)

It is also explicitly invariant under T-duality in the first two complex planes. One can also test in specific examples that gauge quantum numbers are consistent with the existence of these couplings.

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7 Appendix

1-loop amplitudes and tadpoles

We want to compute the tadpoles for $T^6/\{Z_N, \Omega\}$ type IIB orientifolds. We start by writing down the Klein bottle amplitude given by

$$\mathcal{K} = \frac{V_4}{8N} \sum_{n,k=0}^{N-1} \int_0^{\infty} \frac{dt}{t} (4\pi^2 \alpha')^{-2} \mathcal{Z}_K(\theta^n, \theta^k)$$

where

$$\mathcal{Z}_K(\theta^n, \theta^k) = \text{Tr} \{ (1 + (-1)^F) \Theta^k e^{-2\pi t (L_0(\theta^n) + L_0(\theta^k))} \}$$

The contribution of the uncompactified momenta is already extracted in (7.1). Recall that $V_4$ denotes the regularized space-time volume. Since $\Omega$ exchanges $\theta^n$ with $\theta^{N-n}$, only $n = 0$ and $n = \frac{N}{2}$, if $N$ is even, do survive the trace. $\mathcal{Z}_K(1, \theta^k)$ and $\mathcal{Z}_K(R, \theta^k)$ lead to pieces $\mathcal{K}_1(\theta^k)$ and $\mathcal{K}_R(\theta^k)$ whose definition is obvious from (7.1).

The trace in $\mathcal{Z}_K$ can be evaluated in a standard way using $\tilde{\vartheta}$ functions to write the contributions of complex bosons and fermions. Also, the GSO projection is implemented by summing over spin structures. Then, taking into account the insertion of $\Omega$ we find

$$\mathcal{Z}_K(1, \theta^k) = \sum_{\alpha, \beta=0, \frac{1}{2}} \eta_{\alpha, \beta} \tilde{\vartheta}[\alpha] \tilde{\vartheta}[\beta] \tilde{\vartheta}[\alpha + \beta, 2k_0] \frac{-2 \sin 2\pi k_0}{\eta^3}$$

where $\eta_{0,0} = -\eta_{0, \frac{1}{2}} = -\eta_{\frac{1}{2}, 0} = 1$. The $\vartheta$ and $\eta$ functions are defined in the compendium at the end of this appendix. The tilde indicates that the argument is $\tilde{q} = q^2 = e^{-4\pi t}$. This result is strictly valid only if $2k_0 \neq \text{integer}$. If $k_0 = \text{integer}$, (7.3) has a well defined limit but we must also include a sum over quantized momenta in the $Y_i$ direction. If $k_0 = \text{half-integer}$, (7.3) has again a well defined limit but we must also include a sum over windings in the $Y_i$ direction. Upon taking the limit $t \to 0$, using the Poisson resummation formula, a sum over quantized momenta in $Y_i$ gives an internal volume factor $V_i$, whereas a sum over windings in $Y_i$ gives a factor $1/V_i$. For example, in $Z_6'$ with $v = \frac{1}{6}(1, -3, 2)$ we have the following overall volume dependence in $\mathcal{K}_1(\theta^k)$: $V_4V_1V_2V_3$ for $k = 0$, $V_4/V_2$ for $k = 1, 5$, $V_4V_2$ for $k = 2, 4$, and $V_4V_3/V_1V_2$ for $k = 3$.

Let us now write $\mathcal{Z}_K(R, \theta^k)$ that appears when $N$ is even. We assume that $v$, as in Table 4, has been chosen so that to $R = \theta^{N/2}$ there corresponds $\frac{N}{2}v = (\frac{1}{2}, \frac{1}{2}, 0)$ mod $\mathbb{Z}$. Then,

$$\mathcal{Z}_K(R, \theta^k) = \tilde{\chi}(\theta^{N/2}, \theta^k) \sum_{\alpha, \beta=0, \frac{1}{2}} \eta_{\alpha, \beta} \tilde{\vartheta}[\alpha] \tilde{\vartheta}[\beta] \tilde{\vartheta}[\alpha + \beta, 2k_0] \frac{-2 \sin 2\pi k_0}{\eta^3}$$

where

$$\tilde{\vartheta}[\alpha] = \sum_{n,k=0}^{N-1} \vartheta[n, k] e^{-2\pi t (L_0(\theta^n) + L_0(\theta^k))}$$

and

$$\tilde{\vartheta}[\alpha + \beta, 2k_0] = \tilde{\vartheta}[\alpha] \tilde{\vartheta}[\beta, 2k_0]$$

(7.4)
$\tilde{\chi}(\theta^{N/2}, \theta^k)$ is a factor that takes into account the fixed point degeneracy [28]. Eq. (7.4) is strictly valid only if $2kv_3 \neq \text{integer}$. If $kv_3 = \text{integer}$, there is a well defined limit but we must also include a sum over quantized momenta in $Y_3$. If $kv_3 = \text{half-integer}$, again there is a well defined limit but we must also include a sum over windings in $Y_3$.

Both (7.3) and (7.4) vanish by virtue of supersymmetry. Indeed, choosing, as in Table 3, $v_1 + v_2 + v_3 = 0$, and using the identities (7.31), we find

$$Z_K(1, \theta^k) = (1 - 1) \frac{\tilde{\chi}[0]}{\tilde{\eta}^2} \prod_{i=1}^3 (-2 \sin 2\pi kv_i) \frac{\tilde{\chi}[0 + 2kv_i]}{\tilde{\chi}[0 + 2kv_i]}$$

$$Z_K(R, \theta^k) = (1 - 1) \tilde{\chi}(\theta^{N/2}, \theta^k) \frac{\tilde{\chi}[0 + 2kv_i]}{\tilde{\chi}[0 + 2kv_i]}$$

(7.5)

Notice that in $Z_K(R, \theta^k)$ the expression multiplying $(1 - 1)$ vanishes identically when $k = 0, N/2$ and when $2kv_2 = \text{integer}$ as in $Z_6'$.

The next step is to extract the divergences as $t \to 0$. To this end we use the identities (7.30) given in the compendium. Roughly speaking, we find $Z_K \to (1 - 1)(2t) \times$ factors. More precisely, take for example $Z_K(1, \theta^k)$ and for simplicity assume $2kv_i \neq \text{integer}$.

The limit $t \to 0$ gives

$$Z_K(1, \theta^k) \to (1 - 1)(2t) \prod_{i=1}^3 |2 \sin 2\pi kv_i|$$

(7.6)

Also, taking into account sums over momenta/windings and using the Poisson resummation formula [4], we find

$$Z_K(1, 1) \to (1 - 1)(16t) \frac{V_1V_2V_3}{(4\pi^2\alpha')^3}$$

$$Z_K(1, R) \to (1 - 1)(16t) \frac{4\pi^2\alpha'V_3}{V_1V_2}$$

(7.7)

After the change of variables $t = \frac{1}{4\ell}$ [4], the $K$ amplitude schematically reduces to $(1 - 1) \int_0^\infty d\ell \sum_k \tilde{Q}_k^2$. The $(1 - 1)$ shows that, as expected, the NS-NS and R-R are equal and cancel in the full amplitude. However, consistency of field equations [27] requires that each divergence vanishes separately but in general the orientifold plane charges $\tilde{Q}_k$ are not zero as seen from the above limits for $Z_K(1, \theta^k)$.

For $N$ even we also have to consider $Z_K(R, \theta^k)$. An interesting example to analyze is $Z_4$. In this case $v_3 = -\frac{1}{2}$ and we must include a sum over windings in $Y_3$ in both $Z_K(1, \theta^k)$ and $Z_K(R, \theta^k)$, $k = 1, 3$. We obtain

$$Z_K(1, \theta^k) + Z_K(R, \theta^k) \to (1 - 1) \frac{16\pi^2\alpha' t}{V_3} \left[ (2 \sin \frac{\pi k}{2})^2 + \tilde{\chi}(\theta^2, \theta^k) \right] ; \quad k = 1, 3 \quad (7.8)$$
Here $\tilde{\chi}(\theta^2, \theta^k) = 4$ is the number of simultaneous fixed points of $\theta^2$ and $\theta$ (or $\theta^3$). Thus, $K$ has a non-vanishing tadpole proportional to $V_4/V_3$. In fact, whenever $\theta^k$ reflects the third coordinate we find that $Z_K(1, \theta^k) + Z_K(R, \theta^k)$ leads to a non-zero tadpole proportional to $V_4/V_3$. In $Z_8$, $Z_{12}'$ this happens for $k = \frac{N}{4}$ mod 2, and in $Z_8'$, for $k = 2, 6$. The $Z_2 \times Z_4$ and $Z_4 \times Z_4$ orientifolds also have non-vanishing Klein-bottle tadpoles proportional to $V_4/V_3$ that arise from the elements corresponding to $v = (\frac{1}{4}, \frac{1}{4}, -\frac{1}{2})$ and $3v$.

To cancel the divergence proportional to $V_4V_1V_2V_3$ in $K_1(1)$ we must introduce 9-branes and to cancel the divergence proportional to $V_4V_3/V_1V_2$ in $K_1(R)$ we must introduce 5-branes. We then have to compute Möbius strip and cylinder amplitudes. These amplitudes will also have divergences of the form $(1 - 1)^{\sum_{\alpha} Q_{\alpha}}$ and $(1 - 1)^{\sum_{\alpha} Q_{\alpha}^2}$. The various divergences from all amplitudes can be organized according to volume dependence. For each type of dependence we must find a sum of squares, each of which has to be zero.

The cylinder amplitudes are given by

$$C_{pq} = \frac{V_4}{8N} \sum_{k=0}^{N-1} \int_0^\infty dt \frac{(8\pi^2\alpha')^2}{t} Z_{pq}(\theta^k)$$

where

$$Z_{pq}(\theta^k) = Tr_{pq} \left\{ (1 + (-1)^F) \theta^k e^{-2\pi t L_0} \right\}$$

The trace is over open string states with boundary conditions according to the $D_p$ and $D_q$-branes at the endpoints. $Z_{pq}(\theta^k)$ gives rise to $C_{pq}(\theta^k)$ with obvious definition from (7.9).

In $Z_{99}$, boundary conditions are NN in all directions. Hence,

$$Z_{99}(\theta^k) = \sum_{\alpha, \beta=0, 1} \eta_{\alpha, \beta} \frac{\vartheta[\alpha_{\beta}]}{\eta^3} \prod_{i=1}^{3} \vartheta[\alpha_{\beta+kv_i}] \frac{-2\sin(\pi kv_i \eta)}{\vartheta[\frac{1}{2} + kv_i]} (\text{Tr} \gamma_{\alpha, 9})^2$$

This vanishes by supersymmetry. Indeed, using the first identity in (7.31) gives

$$Z_{99}(\theta^k) = (1 - 1)^{\sum_{\alpha} \vartheta[\alpha]} \prod_{i=1}^{3} -2\sin(\pi kv_i \eta) \frac{\vartheta[\alpha]}{\vartheta[\frac{1}{2} + kv_i]} (\text{Tr} \gamma_{\alpha, 9})^2$$

When $kv_i = \text{integer}$ there is a well defined limit and we must include a sum over quantized momenta in $Y_i$. In the limit $t \to 0$ we find

$$Z_{99}(1) \to (1 - 1)^{\sum_{\alpha} \vartheta[\alpha]} (\text{Tr} \gamma_{0, 9})^2$$

This can be organized according to the volume dependence. For each type of dependence we must find a sum of squares, each of which has to be zero. The cylinder amplitudes are given by

$$C_{pq} = \frac{V_4}{8N} \sum_{k=0}^{N-1} \int_0^\infty dt \frac{(8\pi^2\alpha')^2}{t} Z_{pq}(\theta^k)$$

where

$$Z_{pq}(\theta^k) = Tr_{pq} \left\{ (1 + (-1)^F) \theta^k e^{-2\pi t L_0} \right\}$$

The trace is over open string states with boundary conditions according to the $D_p$ and $D_q$-branes at the endpoints. $Z_{pq}(\theta^k)$ gives rise to $C_{pq}(\theta^k)$ with obvious definition from (7.9).

In $Z_{99}$, boundary conditions are NN in all directions. Hence,

$$Z_{99}(\theta^k) = \sum_{\alpha, \beta=0, 1} \eta_{\alpha, \beta} \frac{\vartheta[\alpha_{\beta}]}{\eta^3} \prod_{i=1}^{3} \vartheta[\alpha_{\beta+kv_i}] \frac{-2\sin(\pi kv_i \eta)}{\vartheta[\frac{1}{2} + kv_i]} (\text{Tr} \gamma_{\alpha, 9})^2$$

This vanishes by supersymmetry. Indeed, using the first identity in (7.31) gives

$$Z_{99}(\theta^k) = (1 - 1)^{\sum_{\alpha} \vartheta[\alpha]} \prod_{i=1}^{3} -2\sin(\pi kv_i \eta) \frac{\vartheta[\alpha]}{\vartheta[\frac{1}{2} + kv_i]} (\text{Tr} \gamma_{\alpha, 9})^2$$

When $kv_i = \text{integer}$ there is a well defined limit and we must include a sum over quantized momenta in $Y_i$. In the limit $t \to 0$ we find

$$Z_{99}(1) \to (1 - 1)^{\sum_{\alpha} \vartheta[\alpha]} (\text{Tr} \gamma_{0, 9})^2$$

This can be organized according to the volume dependence. For each type of dependence we must find a sum of squares, each of which has to be zero.
Also, if $kv_i \neq \text{integer}$,

$$Z_{99}(\theta^k) \rightarrow (1 - 1) t \prod_{i=1}^{3} |2 \sin \pi k v_i| (\text{Tr} \gamma_{k,9})^2$$  \hspace{1cm} (7.14)

However, if for instance $k v_3 = \text{integer}$,

$$Z_{99}(\theta^k) \rightarrow (1 - 1) t \frac{V_3}{8 \pi^2 \alpha'} \prod_{i=1}^{2} |2 \sin \pi k v_i| (\text{Tr} \gamma_{k,9})^2$$  \hspace{1cm} (7.15)

To extract the divergences in all $C$ amplitudes we have to make the change of variables $t = \frac{1}{2e}$ [4].

In the (55) sector there are DD boundary conditions in directions $Y_1,Y_2$ transverse to the 5-branes. Oscillator expansions with DD boundary conditions have integer modes but include windings instead of momenta. Then, $Z_{55}$ has a form similar to (7.11). More precisely, after using the first identity in (7.31),

$$Z_{55}(\theta^k) = (1 - 1) \frac{\vartheta[\alpha]}{\eta^3} \prod_{i=1}^{3} \frac{-2 \sin \pi k v_i \vartheta[\frac{\alpha}{2+k v_i}]}{\vartheta[\frac{\alpha}{2+k v_i}]} \sum_{I} (\text{Tr} \gamma_{k,5,I})^2$$  \hspace{1cm} (7.16)

where $I$ refers to the fixed points of $\theta^k$. This is valid if $kv_i \neq \text{integer}$. Otherwise we must include a sum over windings in $Y_i, i = 1,2$, or over quantized momenta in $Y_3$. For example, for $k = 0$ we find the $t \rightarrow 0$ limit

$$Z_{55}(1) \rightarrow (1 - 1) \frac{t}{16} \frac{8 \pi^2 \alpha' V_3}{V_1 V_2} (\text{Tr} \gamma_{0,5})^2$$  \hspace{1cm} (7.17)

In the (59) sector there are DN boundary conditions in coordinates $Y_1,Y_2$. Hence, their oscillator expansions include half-integer modes. For fermions, world-sheet supersymmetry requires that in Neveu-Schwarz (Ramond) moddings be opposite (same) to that of the corresponding bosons. Hence,

$$Z_{59}(\theta^k) = \sum_{\alpha,\beta=0,\frac{1}{2}} \frac{\vartheta[\alpha]}{\eta^3} \frac{-2 \sin \pi k v_3 \vartheta[\frac{\alpha}{2+k v_3}]}{\vartheta[\frac{\alpha}{2+k v_3}]} \prod_{i=1}^{2} \frac{\vartheta[\frac{\alpha}{2+k v_i}]}{\eta} \sum_{I} \text{Tr} \gamma_{k,5,I} \text{Tr} \gamma_{k,9}$$  \hspace{1cm} (7.18)

Using the second identity in (7.31) shows that, as expected, $Z_{59}$ vanishes and can be written as

$$Z_{59}(\theta^k) = (1 - 1) \frac{\vartheta[\alpha]}{\eta^3} \frac{-2 \sin \pi k v_3 \eta \vartheta[\frac{\alpha}{2+k v_3}]}{\vartheta[\frac{\alpha}{2+k v_3}]} \prod_{i=1}^{2} \frac{\vartheta[\frac{\alpha}{2+k v_i}]}{\eta} \sum_{I} \text{Tr} \gamma_{k,5,I} \text{Tr} \gamma_{k,9}$$  \hspace{1cm} (7.19)

Notice that for $k = 0$, or for $kv_2 = \text{integer}$ as in $Z_6'$, the expression multiplying $(1 - 1)$ always vanishes. For $k = \frac{N}{2}$ there is a well defined limit and we must include a sum over quantized momenta in $Y_3$. 

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The Möbius strip amplitudes are given by

\[ \mathcal{M}_p = \frac{V_4}{8N} \sum_{k=0}^{N-1} \int_0^\infty \frac{dt}{t} (8\pi^2 \alpha' t)^{-2} Z_p(\theta^k) \]  

(7.20)

where

\[ Z_p(\theta^k) = \text{Tr}_p\{(1 + (-1)^F) \Omega \theta^k e^{-2\pi L_0} \} \]  

(7.21)

\( Z_p(\theta^k) \) gives rise to \( \mathcal{M}_p(\theta^k) \) with obvious definition from (7.20). Here the trace is over open string states with boundary conditions according to the Dp-branes at both endpoints. The main difference between \( Z_p \) and \( Z_{pp} \) is the insertion of \( \Omega \) that acts on the various bosonic and fermionic oscillators thereby introducing extra phases in the expansions in \( q \). More precisely, \( \Omega \) acts on oscillators as [4]

\[ \alpha_r \to \pm e^{i\pi f} \; ; \; \psi_r \to \pm e^{i\pi f} \]  

(7.22)

The upper (lower) sign is for NN (DD) boundary conditions. Furthermore, \( \Omega \) acts as \( e^{-i\pi/2} \) on the NS vacuum. This ensures that \( \Omega(\psi^\mu(0)|_{NS}) = -\psi^\mu(0)|_{NS} \) as needed for the orientifold projection on gauge vectors.

To derive the Möbius trace we can use (7.22) and the results for (99) cylinders. After using \( \vartheta \) identities we obtain

\[ Z_9(\theta^k) = -(1 - 1) \frac{\vartheta[\frac{1}{3}] \vartheta[\frac{2}{3}]}{\vartheta^3 \vartheta[0]} \prod_{i=1}^3 \frac{-2\sin \pi k v_i \vartheta[\frac{1}{k v_i}] \vartheta[\frac{0}{k v_i + 1}] \vartheta[\frac{1}{k v_i + 1}]}{\vartheta[\frac{1}{k v_i + 1}] \vartheta[\frac{0}{k v_i}]} \text{Tr} (\gamma^{-1}_{\Omega_{k,9}} \cdot \gamma^T_{\Omega_{k,9}}) \]  

(7.23)

where again the tilde means variable \( \vartheta = e^{-4\pi t} \). When \( k v_i = \text{integer} \) we must include a sum over quantized momenta in \( Y_i \). Notice that \( \vartheta[\frac{1}{k v_i}] \) vanishes identically when \( k = \frac{N}{2} \) and whenever \( k v_i = \text{half-integer} \). For 5-d-branes we instead find

\[ Z_5(\theta^k) = (1 - 1) \frac{\vartheta[\frac{1}{3}] \vartheta[\frac{2}{3}]}{\vartheta^3 \vartheta[0]} \frac{-2\sin \pi k v_3 \vartheta[\frac{1}{k v_3}] \vartheta[\frac{0}{k v_3}]}{\vartheta[\frac{1}{k v_3}] \vartheta[\frac{0}{k v_3}]} \prod_{i=1}^2 \frac{2\cos \pi k v_i \vartheta[\frac{1}{k v_i + 1}] \vartheta[\frac{0}{k v_i}]}{\vartheta[\frac{1}{k v_i}] \vartheta[\frac{0}{k v_i}]} \sum_T \text{Tr} (\gamma^{-1}_{\Omega_{k,5,T}} \cdot \gamma^T_{\Omega_{k,5,T}}) \]  

(7.24)

For \( k = 0 \) there is a vanishing contribution to tadpoles. For \( k v_i = \text{half-integer} \), \( i = 1, 2 \), we must include a sum over windings in \( Y_i \). For \( k v_3 = \text{integer} \) we must include a sum over momenta in \( Y_3 \). In particular, for \( k = \frac{N}{2} \) we obtain the following \( t \to 0 \) limit

\[ Z_5(R) \to -(1 - 1) t \frac{8\pi^2 \alpha' V_3}{V_1 V_2} \text{Tr} \gamma_{0,5} \]  

(7.25)
To extract tadpoles in $\mathcal{M}_p$ we have to make the change of variables $t = \frac{1}{8\ell} [4].$

To arrive at the tadpole cancellation conditions written in section 2.3, we must take the limit $t \to 0$ in the various traces and next change variable to $\ell$ appropriately to find the large $\ell$ behavior of the amplitudes. The final step is to collect all terms with a given volume dependence.

To finish this appendix we wish to stress that in $Z_4, Z_8, Z'_8$ and $Z'_{12}$ there are leftover tadpoles even after introducing 9-branes and 5$^3$-branes. Indeed, we have seen that in these cases the $K$ amplitude has divergences proportional to $V_4/V_3$. This type of volume dependence cannot arise from any of the $\mathcal{M}$ or $\mathcal{C}$ amplitudes because it would require a sum over windings in $Y_3$ that is not possible for 5$^3$-branes.

### Compendium of $\vartheta$ properties

The $\vartheta$ function of rational characteristics $\delta$ and $\phi$ is given by

$$\vartheta[\delta\phi](t) = \sum_n q^{\frac{1}{2}(n+\delta)^2} e^{2i\pi(n+\delta)\phi}$$

Here the variable $q$ is $q = e^{-2\pi t}$. The $\vartheta$ function also has the product form

$$\frac{\vartheta[\delta]}{\eta} = e^{2\pi i\delta\phi} q^{\frac{1}{2}\delta^2 - \frac{1}{24}} \prod_{n=1}^{\infty} (1 + q^{n+\delta - \frac{1}{2}} e^{2\pi i\phi})(1 + q^{n-\delta - \frac{1}{2}} e^{-2\pi i\phi})$$

where the Dedekind $\eta$ function is

$$\eta = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

Notice that

$$\lim_{\phi \to 0} \frac{-2\sin \pi\phi}{\vartheta[\frac{1}{2} + \phi]} = \frac{1}{\eta^3}$$

The $\vartheta$ and $\eta$ functions have the modular transformation properties

$$\vartheta[\delta\phi](t) = e^{2\pi i\delta\phi} t^{-\frac{1}{2}} \vartheta[-\phi](1/t)$$

$$\eta(t) = t^{-\frac{1}{2}} \eta(1/t)$$

The $\vartheta$'s satisfy several Riemann identities [37]. In particular,

$$\sum_{\alpha, \beta} \eta_{\alpha, \beta} \vartheta[\beta^\alpha] \prod_{i=1}^{3} \vartheta[\alpha_{\beta + u_i}] = 0$$

$$\sum_{\alpha, \beta} \eta_{\alpha, \beta} \vartheta[\beta^\alpha] \vartheta[\alpha_{\beta + u_3}] \prod_{i=1}^{2} \vartheta[\alpha_{\beta + u_i}] = 0$$

provided that $u_1 + u_2 + u_3 = 0.$
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