Explicit formulae for Chern-Simons invariants of the twist-knot orbifolds and edge polynomials of twist knots

J. Ham and J. Lee

Abstract. We calculate the Chern-Simons invariants of twist-knot orbifolds using the Schl"afli formula for the generalized Chern-Simons function on the family of twist knot cone-manifold structures. Following the general instruction of Hilden, Lozano, and Montesinos-Amilibia, we here present concrete formulae and calculations. We use the Pythagorean Theorem, which was used by Ham, Mednykh and Petrov, to relate the complex length of the longitude and the complex distance between the two axes fixed by two generators. As an application, we calculate the Chern-Simons invariants of cyclic coverings of the hyperbolic twist-knot orbifolds. We also derive some interesting results. The explicit formulae of the $A$-polynomials of twist knots are obtained from the complex distance polynomials. Hence the edge polynomials corresponding to the edges of the Newton polygons of the $A$-polynomials of twist knots can be obtained. In particular, the number of boundary components of every incompressible surface corresponding to slope $-4n+2$ turns out to be 2.

Bibliography: 39 titles.

Keywords: Chern-Simons invariant, twist knot, orbifold, $A$-polynomial, edge polynomial.

§ 1. Introduction

In the early 1970s, Chern and Simons [1] defined an invariant of a compact 3 (mod 4)-dimensional Riemannian manifold, $M$, which is now called the Chern-Simons invariant, $\text{cs}(M)$. In the 1980s, Meyerhoff [2] extended the definition of $\text{cs}(M)$ to cusped manifolds. It is the integral of a certain 3-form and an invariant of the Riemannian connection on the principal tangent bundle of $M$.

Various methods of finding the Chern-Simons invariant using ideal triangulations have been introduced [3]–[8] and implemented [9], [10]. But, for orbifolds, to our knowledge, there does not exist a single convenient program which computes the Chern-Simons invariant. The Chern-Simons invariant can also be obtained from the $\eta$-invariant $\eta(M)$: $\text{cs}(M) = \frac{3}{2} \eta(M) \pmod{\frac{1}{2}}$ [11], [12]. But it is easier to compute Chern-Simons invariants than $\eta$-invariants.

Instead of working on the complicated combinatorics of 3-dimensional ideal tetrahedra to find the Chern-Simons invariants of twist-knot orbifolds, we deal with simple one-dimensional singular loci. To make the computation simpler, we express

AMS 2010 Mathematics Subject Classification. Primary 57M25, 51M10, 57M27, 57M50.

© 2016 Russian Academy of Sciences (DoM), London Mathematical Society, Turpion Ltd.
the complex length of the singular locus in terms of the complex distance between
the two axes fixed by two generators. To find the complex length of the singular
locus, we work in $\text{SL}(2, \mathbb{C})$. The singular locus appears in $\text{SL}(2, \mathbb{C})$ as a series
of matrix multiplications. If we first calculate the complex distance and recover
the complex length of the singular locus from the complex distance using Lemma 3.3,
the multiplication needed in $\text{SL}(2, \mathbb{C})$ can be cut down approximately by half. Sim-
ilar methods for volumes can be found in [13]. We use the Schl"afli formula for
the generalized Chern-Simons function on the family of twist knot cone-manifold
structures [14]. In [15] a method for calculating the Chern-Simons invariants of
two-bridge knot orbifolds was introduced but without explicit formulae. Similar
approaches for $\text{SU}(2)$-connections can be found in [16] and for $\text{SL}(2, \mathbb{C})$-connections
in [17]. Explicit integral formulae for Chern-Simons invariants of the Whitehead link
(the two-component twist link) orbifolds and their cyclic coverings are presented
in [18], [19]. A brief explanation for twist knot cone-manifolds is given in [13]. See
also [20]–[25].

The main purpose of the paper is to find explicit and efficient formulae
for the Chern-Simons invariants of twist-knot orbifolds. For a two-bridge
hyperbolic link, there exists an angle $\alpha_0 \in [2\pi/3, \pi)$ for each link $K$ such
that the cone-manifold $K(\alpha)$ is hyperbolic for $\alpha \in (0, \alpha_0)$, Euclidean for $\alpha = \alpha_0$ and
spherical for $\alpha \in (\alpha_0, \pi]$; see [22]–[25]. We will use the Chern-Simons invariant of
the lens space $L(4n + 1, 2n + 1)$ calculated in [15]. The following theorem gives
formulae for $T_m$ for even integer $m$. For odd integer $m$, we can replace $T_m$ by $T_{-m-1}$ as explained in §2. So, the following theorem actually covers all possible
hyperbolic twist knots. We exclude the nonhyperbolic case $n = 0, -1$.

**Theorem 1.1.** Let $T_{2n}$ be a hyperbolic twist knot. Let $T_{2n}(\alpha)$, $0 \leq \alpha < \alpha_0$ be
a hyperbolic cone-manifold with underlying space $S^3$ and with singular set $T_{2n}$ of
cone-angle $\alpha$. Let $k$ be a positive integer such that $k$-fold cyclic covering of $T_{2n}(2\pi/k)$
is hyperbolic. Then the Chern-Simons invariant of $T_{2n}(2\pi/k)$ (mod $1/k$ if $k$ is even
or mod $1/(2k)$ if $k$ is odd) is given by the following formula:

$$
cs\left( T_{2n} \left( \frac{2\pi}{k} \right) \right) \equiv \frac{1}{2} \cs(L(4n + 1, 2n + 1))
$$

$$
+ \frac{1}{4\pi^2} \int_{\alpha_0}^{2\pi/k} \text{Im} \left( 2\log \left( M^{-2}A+iV \right) \right) d\alpha
$$

$$
+ \frac{1}{4\pi^2} \int_{\alpha_0}^{\pi} \text{Im} \left( \log \left( M^{-2}A+iV_1 \right) + \log \left( M^{-2}A+iV_2 \right) \right) d\alpha,
$$

where for $A = \cot(\alpha/2), V (\text{Im}(V) \leq 0), V_1$ and $V_2$ are zeros of the complex distance
polynomial $P_{2n} = P_{2n}(V, B)$ which is either given recursively by

$$
P_{2n} = \begin{cases} 
((4B^4 - 8B^2 + 4)V^2 - 4B^4 + 8B^2 - 2)P_{2(n-1)} - P_{2(n-2)} & \text{if } n > 1, \\
((4B^4 - 8B^2 + 4)V^2 - 4B^4 + 8B^2 - 2)P_{2(n+1)} - P_{2(n+2)} & \text{if } n < -1
\end{cases}
$$

with the initial conditions

$$
P_{-2}(V, B) = (2B^2 - 2)V + 2B^2 - 1, \quad P_0(V, B) = 1,
$$

$$
P_2(V, B) = (4B^4 - 8B^2 + 4)V^2 + (2 - 2B^2)V - 4B^4 + 6B^2 - 1,
$$
or is given explicitly by

\[
P_{2n} = \begin{cases} 
\sum_{i=0}^{2n} \left( n + \left\lfloor \frac{i}{2} \right\rfloor \right) (2(B^2 - 1)(1 - V))^i (1 + 2(V - 1)^{-1})^{\lfloor (i+1)/2 \rfloor} & \text{if } n \geq 0, \\
\sum_{i=0}^{-2n-1} \left( -n + \left\lfloor \frac{i-1}{2} \right\rfloor \right) (2(B^2 - 1)(V - 1))^i (1 + 2(V - 1)^{-1})^{\lfloor (i+1)/2 \rfloor} & \text{if } n < 0,
\end{cases}
\]

where \( B = \cos(\alpha/2) \) and \( V_1 \) and \( V_2 \) approach the common \( V \) as \( \alpha \) decreases to \( \alpha_0 \) and come from the components of \( V \) and \( \overline{V} \).

We present here some derived results. Theorem 1.2 gives the recursive formulae of \( A \)-polynomials of twist knots. In [26], Theorem 1, Hoste and Shanahan presented the recursive formulae of \( A \)-polynomials of twist knots with the opposite orientation. Theorem 1.3 gives the explicit formulae of the \( A \)-polynomials of twist knots. In [27], Theorem 1.1, Mathews presented the explicit formulae of the \( A \)-polynomials of the twist knots with the opposite orientation. In the case \( n \leq 0 \) in [27] there is a typo; \( (\frac{M^2-1}{L+M^2})^i \) has to be changed to \( (\frac{1-M^2}{L+M^2})^i \); see [28]. For each side of slope \( a/b \) of the Newton polygon of \( A_{2n} \), we can obtain the edge polynomial. By substituting \( t \) for \( L^b M^a \) of each term appearing along the side edge, we have the edge polynomial, \( f_{a/b}(t) \). Theorem 1.4 gives the edge polynomials of twist knots. In [29], Corollary 11.5, Cooper and Long showed that the edge polynomials of a two-bridge knot are all of the form \( \pm(t - 1)^k(t + 1)^l \). We pin them down in the case of twist knots. Corollary 1.5 gives the number of boundary components in the case of slope \( -4n + 2 \) of twist knots. From [30], we know that the number of boundary components of two-bridge knots is one or two. We pin this down in the case of slope \( -4n + 2 \) of twist knots. The proofs of derived results are in §3.

**Theorem 1.2.** The \( A \)-polynomial \( A_{2n} = A_{2n}(L, M) \) is given recursively by

\[
A_{2n} = \begin{cases} 
A_u A_{2(n-1)} - M^4(1 + LM^2)^4A_{2(n-2)} & \text{if } n > 1, \\
A_u A_{2(n+1)} - M^4(1 + LM^2)^4A_{2(n+2)} & \text{if } n < -2
\end{cases}
\]

with the initial conditions

\[
A_{-4}(L, M) = 1 - L + 2LM^2 + 2LM^4 + L^2M^4 - L^2M^6 - LM^8 + LM^10 + 2L^2M^{10} + 2L^2M^{12} - L^2M^{14} + L^3M^{14}, \\
A_{-2}(L, M) = 1 + LM^6, \\
A_0(L, M) = -1, \\
A_2(L, M) = 1 + LM^6.
\]

where

\[
A_u = 1 - L + 2LM^2 + M^4 + 2LM^4 + L^2M^4 + 2LM^6 - LM^8 + L^2M^8.
\]
Theorem 1.3. The A-polynomial $A_{2n} = A_{2n}(L, M)$ is given explicitly by

$$A_{2n} = \begin{cases} 
-M^{2n}(1 + LM^2)^{2n} \sum_{i=0}^{2n} \left( n + \left\lceil \frac{i}{2} \right\rceil \right) (1 - M^2)^i \\
\times (1 + LM^2)^{-i} (L - 1)^{[i/2]} (LM^2 - M^{-2})^{\lfloor (1+i)/2 \rfloor} & \text{if } n \geq 0, \\
M^{-2n}(1 + LM^2)^{-2n-1} \sum_{i=0}^{-2n-1} \left( -n + \left\lceil \frac{i-1}{2} \right\rceil \right) (M^2 - 1)^i \\
\times (1 + LM^2)^{-i} (L - 1)^{[i/2]} (LM^2 - M^{-2})^{\lfloor (1+i)/2 \rfloor} & \text{if } n < 0.
\end{cases}$$

Theorem 1.4. When $n > 0$, the edge polynomials of twist knots are

$$\pm (t - 1) \quad \text{if the slope is } -4n,$$

$$- (t - 1)^n \quad \text{if the slope is } 4 \text{ and } n \text{ is even},$$

$$\pm (t - 1)^n \quad \text{if the slope is } 4 \text{ and } n \text{ is odd}.$$

When $n < 0$, the edge polynomials of twist knots are

$$t + 1 \quad \text{if the slope is } -4n + 2,$$

$$\pm (t - 1)^{-n-1} \quad \text{if the slope is } 4 \text{ and } n \text{ is even},$$

$$(t - 1)^{-n-1} \quad \text{if the slope is } 4 \text{ and } n \text{ is odd}.$$

Corollary 1.5. The number of boundary components of every incompressible surface corresponding to slope $-4n + 2$ of twist knots is 2.

§ 2. Twist knots

A knot $K$ is a twist knot if $K$ has a regular two-dimensional projection of the form shown in Figure 1. Such a $K$ has two left-handed vertical crossings and $m$ right-handed horizontal crossings. We will denote it by $T_m$. One can easily check that the slope of $T_m$ is $2/(2m+1)$, which is equivalent to the knot with slope $(m+1)/(2m+1)$ [31]. For example, Figure 2 shows two different regular projections of the knot $6_1$; one with slope $2/9$ (left) and the other with slope $5/9$ (right). Note that $T_m$ and its mirror image have the same fundamental group up to orientation and hence have the same fundamental domain up to isometry in $\mathbb{H}^3$. It follows that $T_m(\alpha)$ and its mirror image have the same fundamental set up to isometry in $\mathbb{H}^3$ and have the same Chern-Simons invariant up to sign. Since the mirror image of $T_m$ is equivalent to $T_{-m-1}$, when $m$ is odd we will use $T_{-m-1}$ for $T_m$. Hence a twist knot can be represented by $T_{2n}$ for some integer $n$ with slope $2/(4n+1)$ or $(2n+1)/(4n+1)$.

Let us denote by $X_{2n}$ the exterior of $T_{2n}$. In [26], the fundamental group of $X_{2n}$ is calculated with two right-handed vertical crossings as positive crossings instead of two left-handed vertical crossings. The following theorem is tailored to our purpose. It can also be obtained by reading off the fundamental group from the Schubert normal form of $T_{2n}$ with slope $(2n+1)/(4n+1)$ [32].
Chern-Simons invariants and edge polynomials

Figure 1. Twist knot (left) and its mirror image (right).

Figure 2. The knot 6_1 with slope 2/9 (left) and with slope 5/9 (right).

Proposition 2.1.

\[ \pi_1(X_{2n}) = \langle s, t \mid swt^{-1}w^{-1} = 1 \rangle, \]

where \( w = (ts^{-1}t^{-1}s)^n \).

§ 3. The complex distance polynomial and A-polynomial

Given a twist knot \( T_{2n} \) and a set of generators, \( \{s,t\} \), we identify the set of representations, \( R \), of \( \pi_1(X_{2n}) \) into \( \text{SL}(2, \mathbb{C}) \) with \( R(\pi_1(X_{2n})) = \{ (\eta(s), \eta(t)) \mid \eta \in R \} \). Since the defining relation of \( \pi_1(X_{2n}) \) gives the defining equation of \( R(\pi_1(X_{2n})) \) [33], \( R(\pi_1(X_{2n})) \) can be thought of as an affine algebraic set in \( \mathbb{C}^2 \). The set \( R(\pi_1(X_{2n})) \) is well-defined up to isomorphisms which arise from changing the set of generators. We say that elements of \( R \) that differ by conjugations in \( \text{SL}(2, \mathbb{C}) \) are equivalent.

We use two sets of coordinates to give the structure of an affine algebraic set to \( R(\pi_1(X_{2n})) \). In an equivalent way, for some \( O \in \text{SL}(2, \mathbb{C}) \) we will consider both \( \eta \) and \( \eta' = O^{-1}\eta O \).

For the complex distance polynomial, we use for coordinates

\[
\eta(s) = \begin{bmatrix}
\frac{M + 1/M}{2} & e^{\rho/2} \frac{M - 1/M}{2} \\
e^{-\rho/2} \frac{M - 1/M}{2} & \frac{M + 1/M}{2}
\end{bmatrix},
\]

\[
\eta(t) = \begin{bmatrix}
\frac{M + 1/M}{2} & e^{\rho/2} \frac{M - 1/M}{2} \\
e^{-\rho/2} \frac{M - 1/M}{2} & \frac{M + 1/M}{2}
\end{bmatrix},
\]
and for the $A$-polynomial we use

$$\eta'(s) = \begin{bmatrix} M & 1 \\ 0 & M^{-1} \end{bmatrix}, \quad \eta'(t) = \begin{bmatrix} M & 0 \\ t & M^{-1} \end{bmatrix}. $$

In [13], the complex distance polynomial of $T_{2n}$ is presented recursively. Theorem 3.2 gives it explicitly. It is the defining polynomial of the algebraic set $R(\pi_1(X_{2n}))$ with the set of generators given in Proposition 2.1 and with the coordinates $\eta(s)$ and $\eta(t)$ in $\text{SL}(2, \mathbb{C})$. The actual computation in [13] is done after setting $M = e^{i\alpha/2}$ and then the variables are changed to $B = \cos(\alpha/2)$ and $V = \cosh \rho$.

**Theorem 3.1** (see [13], Theorem 3.1). For $B = \cos(\alpha/2)$, $V = \cosh \rho$ is a root of the following complex distance polynomial $P_{2n} = P_{2n}(V, B)$ which is given recursively by

$$P_{2n} = \begin{cases} ((4B^4 - 8B^2 + 4)V^2 - 4B^4 + 8B^2 - 2)P_{2(n-1)} - P_{2(n-2)} & \text{if } n > 1, \\
(4B^4 - 8B^2 + 4)V^2 - 4B^4 + 8B^2 - 2)P_{2(n+1)} - P_{2(n+2)} & \text{if } n < -1
\end{cases}$$

with the initial conditions

$$P_{-2}(V, B) = (2B^2 - 2)V + 2B^2 - 1, \quad P_0(V, B) = 1, \quad P_{2}(V, B) = (4B^4 - 8B^2 + 4)V^2 + (2 - 2B^2)V - 4B^4 + 6B^2 - 1.$$

**Theorem 3.2.** For $B = \cos(\alpha/2)$, $V = \cosh \rho$ is a root of the following complex distance polynomial $P_{2n} = P_{2n}(V, B)$ which is given explicitly by

$$P_{2n} = \begin{cases} \sum_{i=0}^{2n} \left( n + \lfloor \frac{i}{2} \rfloor \right) \left( 2(B^2 - 1)(1 - V) \right)^i (1 + 2(V - 1)^{-1})^{\lfloor (1+i)/2 \rfloor} & \text{if } n \geq 0, \\
\sum_{i=0}^{-2n-1} \left( -n - \lfloor \frac{i - 1}{2} \rfloor \right) \left( 2(B^2 - 1)(V - 1) \right)^i (1 + 2(V - 1)^{-1})^{\lfloor (1+i)/2 \rfloor} & \text{if } n < 0.
\end{cases}$$

**Proof.** We write $f_{2n}$ for the right-hand side of the claimed formula and will show that $f_{2n}$ can be rewritten as

$$f_{2n} = \begin{cases} \sum_{j=0}^{n} \left( n + j \right) \left( 2(B^2 - 1)(V - 1) \right)^{2j}(1 + 2(V - 1)^{-1})^{j} - \sum_{j=0}^{n} \left( n + j \right) \left( 2(B^2 - 1)(V - 1) \right)^{2j+1}(1 + 2(V - 1)^{-1})^{j+1} & \text{if } n \geq 0, \\
\sum_{j=0}^{-n-1} \left( -n - 1 + j \right) \left( 2(B^2 - 1)(V - 1) \right)^{2j}(1 + 2(V - 1)^{-1})^{j} + \sum_{j=0}^{-n-1} \left( -n + j \right) \left( 2(B^2 - 1)(V - 1) \right)^{2j+1}(1 + 2(V - 1)^{-1})^{j+1} & \text{if } n < 0.
\end{cases}$$
Now, the theorem follows by solving the recurrence formula with the initial conditions given in Theorem 3.1:

\[
P_{2n} = \begin{cases} 
[(2(B^2 - 1)(V - 1))^2 - 2(B^2 - 1)(V - 1)(1 + 2(V - 1)^{-1}) + 1] \\
\times h_{n-1} - h_{n-2} \\
(2(B^2 - 1)(V - 1)(1 + 2(V - 1)^{-1}) + 1)h_{n-1} - h_{n-2}
\end{cases}
\]

if \( n > 1 \),

if \( n < -1 \),

where

\[
h_n = \sum_{k=0}^{[n/2]} \binom{n+1}{2k+1} (2(B^2 - 1)^2(V - 1)^2(1 + 2(V - 1)^{-1}) + 1)^{n-2k} \\
\times ((2(B^2 - 1)^2(V - 1)^2(1 + 2(V - 1)^{-1}) + 1)^2 - 1)^k
\]

\[
= \sum_{i=0}^{[n/2]} \binom{n-i}{i} (-1)^i \sum_{j=0}^{n-2i} \binom{n-2i}{j} 2^{n-2i-j} \\
\times ((2(B^2 - 1)(V - 1))^2(1 + 2(V - 1)^{-1}))^j
\]

\[
= \sum_{j=0}^{n} \binom{(n-j)/2}{j} \binom{n-i}{i} \binom{n-2i}{j} (-1)^i 2^{n-2i-j} \\
\times ((2(B^2 - 1)(V - 1))^2(1 + 2(V - 1)^{-1}))^j
\]

\[
= \sum_{j=0}^{n} \binom{n+1+j}{2j+1} ((2(B^2 - 1)(V - 1))^2(1 + 2(V - 1)^{-1}))^j.
\]

Then \( f_{2n} \) can be obtained by simplifying the above formula.

**Proof of Theorem 1.2 and Theorem 1.3.** Using Theorem 4.4 in [13] we obtain the following Lemma 3.3, which relates the zeros of the complex distance polynomial \( P_{2n} = P_{2n}(V, B) \) and the zeros of the \( A \)-polynomial \( A_{2n} = A_{2n}(L, M) \). Then Theorem 1.2 (respectively, Theorem 1.3) can be obtained from \( P_{2n} \) in Theorem 3.1 (respectively, in Theorem 3.2) by replacing \( V \) with \( ((M^2 + 1)(LM^2 - 1)) / ((M^2 - 1)(LM^2 + 1)) \) (using the equalities in Lemma 3.3) and \( B \) with \( (M + M^{-1})/2 \) and by clearing the denominators.

**Lemma 3.3.**

\[
iV = A \frac{LM^2 - 1}{LM^2 + 1}, \quad L = M^{-2} A + iV, \quad \frac{A}{i} = \frac{\cos(\alpha/2)}{\sin(\alpha/2)} = \frac{\cosh(ia/2)}{\sinh(ia/2)} = \frac{(M + M^{-1})/2}{(M - M^{-1})/2} = \frac{M^2 + 1}{M^2 - 1}.
\]

**Proof.** (1) is Theorem 4.4 in [13]. Since

Since

\[
A \frac{\cos(\alpha/2)}{i} = \frac{\cosh(ia/2)}{\sinh(ia/2)} = \frac{(M + M^{-1})/2}{(M - M^{-1})/2} = \frac{M^2 + 1}{M^2 - 1},
\]
we get the first equality of (2). By solving the first equality for $L$, we get the second equality of (2).

**Proof of Theorem 1.4.** By Lemma 3.4, we have the Newton polygons (NP) associated to $A_{2n}$ in Figure 3. Let us only consider nonzero slopes if $n \neq -1$.

![Figure 3. The Newton polygons of $A_{2n}$.](image)

When $n > 0$, the NP has two slopes, $-4n$ and 4. When the slope is $-4n$ and the edge has the term $-M^{4n}$ on it, there are only two terms of $A_{2n}$ appearing along the edge of the slope. From the explicit formula of $A_{2n}$ in Theorem 1.3, they are $-M^{4n}$ and $L$. The term $-M^{4n}$ occurs when $i = 2n$. We get the term $L$ by adding two terms, $(n-1)L$ (which occurs when $i = 2n-1$) and $-nL$ (which occurs when $i = 2n$). From $L - M^{4n}$, by substituting $Lt$ for $M^{4n}$ and dividing by $L$ we get the edge polynomial $1-t$. Hence, on the right above edge with the same slope $-4n$ we have $t - 1$ for the edge polynomial. When the slope is 4 and the term $-M^{4n}$ is on the edge, from the explicit formula of $A_{2n}$ in Theorem 1.3, $nLM^{4n+4}$ is also on the edge and this term occurs when $i = 2n$. We can get the edge polynomial from the sum of the terms which appear on this edge by substituting $t$ for $LM^4$ and dividing by $M^{4n}$. Hence the coefficient of $LM^{4n+4}$ is the coefficient of $t$ in the edge polynomial. Since we know that the constant term is $-1$ and the coefficient of $t$ is $n$, we obtain the edge polynomial $(t-1)^n$ when $n$ is odd and $-(t-1)^n$ when $n$ is even, because the fact that the edge polynomials of two-bridge knots are, up to sign, the products of some powers of $t-1$ and some powers of $t+1$ [29] and the coefficient conditions force the power of $t+1$ to be zero. Hence, on the lower-right edge with the same slope 4 we have $-(t-1)^n$ for the edge polynomial.
When \( n < 0 \), the NP has two slopes, \(-4n + 2\) and 4. When the slope is \(-4n + 2\) and the edge has the term 1 on it, there are only two terms from \( A_{2n} \) appearing along the edge of this slope. By the explicit formula of \( A_{2n} \) in Theorem 1.3, they are 1 and \( LM^{-4n+2} \). The term 1 occurs when \( i = 2n \). We get the term \( LM^{-4n+2} \) by adding two terms, \((n+1)LM^{-4n+2}\), which occurs when \( i = -2n - 2 \), and \(-nLM^{-4n+2}\), which occurs when \( i = -2n - 1 \). From \( 1 + LM^{-4n+2} \), substituting \( t \) for \( LM^{-4n+2} \) we obtain the edge polynomial \( t + 1 \). Hence, on the upper-right edge with the same slope \(-4n + 2\) we have \( t + 1 \) for the edge polynomial. When the slope is 4 and the term \( LM^{-4n+2} \) is on the edge, by the explicit formula of \( A_{2n} \) in Theorem 1.3, the term \((n+1)L^2M^{-4n+6}\) is on the edge. We obtain the term \((n+1)L^2M^{-4n+6}\) by adding two terms, \(\frac{(n+1)(n+2)}{2}L^2M^{-4n+6} \), which occurs when \( i = -2n - 2 \), and \(-\frac{n(n+1)}{2}L^2M^{-4n+6} \), which occurs when \( i = -2n - 1 \). We can get the edge polynomial from the sum of the terms which appear on this edge by substituting \( t \) for \( LM^4 \) and dividing by \( LM^{-4n+2} \). Hence the coefficient of \( L^2M^{-4n+6} \) is the coefficient of \( t \) in the edge polynomial. Since we know the constant term is 1 and the coefficient of \( t \) is \( n + 1 \), we obtain the edge polynomial \((t - 1)^{-n-1}\) when \( n \) is odd and \(-(t - 1)^{-n-1}\) when \( n \) is even, because the fact that the edge polynomials of two-bridge knots are, up to sign, the product of some powers of \( t - 1 \) and some powers \( t + 1 \) [29] and the coefficient conditions force the power of \( t + 1 \) to be zero. Hence on the lower-right edge with the same slope 4, we have \((t - 1)^{-n-1}\) for the edge polynomial.

**Lemma 3.4.** The Newton polygons associated to \( A_{2n} \) are the polygons in Figure 3.

**Proof.** The lemma is true when \( n = -2, -1, 1 \).

Since the Newton polygon of \( A_{2n} \) has ones on the corners up to sign [34], to determine the shape of the Newton polygon we only need to consider \( A_{2n} \) modulo 2. We will use the recursive formula of \( A_{2n} \) in Theorem 1.2. Modulo 2, \( A_n \) has six terms and \( M^4(1 + LM^2)^4 \) has two terms \( M^4 + L^4M^{12} \). The lemma can be proved by induction. You just have to combine six copies of the Newton polygons of \( A_{2n-1} \) (if \( n > 1 \)) or \( A_{2n+1} \) (if \( n < -2 \)) shifted by \( A_2 \) and two copies of the Newton polygons of \( A_{2n-2} \) (if \( n > 1 \)) or \( A_{2n+1} \) (if \( n < -2 \)) shifted by \( M^4 + L^4M^{12} \), removing double points.

**Proof of Corollary 1.5.** The number of boundary components of two-bridge knots is one or two [30]. Hence the number of boundary components of every incompressible surface corresponding to slope \(-4n + 2\) is bounded above by 2.

The orders of roots (roots of unity) of \( f_{a/b} \) divide the number of boundary components of every incompressible surface corresponding to slope \( a/b \) [35]. Hence, when the slope is \(-4n + 2\), since we have a single root of unity of degree 2 by Theorem 1.4, the number of boundary components of every incompressible surface corresponding to slope \(-4n + 2\) is bounded below by 2.

§ 4. Generalized Chern-Simons function

The general references for this section are [12], [14], [15] and [36]. We introduce the generalized Chern-Simons function on the family of twist knot cone-manifold structures. For the oriented knot \( T_{2n} \), we orient a chosen meridian \( s \) so that the orientation of \( s \) followed by the orientation of \( T_{2n} \) coincides with the orientation of \( S^3 \).
We use the definition of a lens space in [15] so that we can have the right orientation when the definition of a lens space in [15] is combined with the following frame field. On the Riemannian manifold $S^3 - T_{2n} - s$ we choose a special frame field $\Gamma = (e_1, e_2, e_3)$. This is an orthonormal frame field such that for each point $x$ near $T_{2n}$, $e_1(x)$ has the knot direction, $e_2(x)$ has the tangent direction of a meridian curve, and $e_3(x)$ has the knot-to-point direction. A special frame field always exists by Proposition 3.1 of [14]. From $\Gamma$ we obtain an orthonormal frame field $\Gamma_\alpha$ on $T_{2n}(\alpha) - s$ by the Schmidt orthonormalization process with respect to the geometric structure of the cone manifold $T_{2n}(\alpha)$. Moreover it can be made special by deforming it in a neighbourhood of the singular set and meridian $s$ if necessary. Let $\Gamma'$ be an extension of $\Gamma$ to $S^3 - T_{2n}$. To each cone-manifold $T_{2n}(\alpha)$ we assign the real number

$$I(T_{2n}(\alpha)) = \frac{1}{2} \int_{\Gamma(S^3 - T_{2n} - s)} Q - \frac{1}{4\pi} \tau(s, \Gamma') - \frac{1}{4\pi} \left( \frac{\beta \alpha}{2\pi} \right),$$

where $-2\pi \leq \beta \leq 2\pi$ and $Q$ is the Chern-Simons form:

$$Q = \frac{1}{4\pi^2} \left( \theta_{12} \wedge \theta_{13} \wedge \theta_{23} + \theta_{12} \wedge \Omega_{12} + \theta_{13} \wedge \Omega_{13} + \theta_{23} \wedge \Omega_{23} \right),$$

$$\tau(s, \Gamma') = -\int_{\Gamma'(s)} \theta_{23}.$$ 

where $(\theta_{ij})$ is the connection 1-form, $(\Omega_{ij})$ is the curvature 2-form of the Riemannian connection on $T_{2n}(\alpha)$ and the integral is over the orthonormalizations of the same frame field. When $\alpha = 2\pi/k$ for some positive integer, $I(T_{2n}(2\pi/k))$ (mod $1/k$ if $k$ is even or mod $1/(2k)$ if $k$ is odd) is independent of the frame field $\Gamma$ and of the representative of the equivalence class $\beta$ and hence is an invariant of the orbifold $T_{2n}(2\pi/k)$.

$I(T_{2n}(2\pi/k))$ (mod $1/k$ if $k$ is even or mod $1/(2k)$ if $k$ is odd) is called the Chern-Simons invariant of the orbifold and is denoted by $cs(T_{2n}(2\pi/k))$.

For the generalized Chern-Simons function on the family of twist knot cone-manifold structures we have the following Schl"afli formula.

**Theorem 4.1** (see [15], Theorem 1.2). *For the family of geometric cone-manifold structures $T_{2n}(\alpha)$ and the differentiable functions $\alpha(t)$ and $\beta(t)$ of $t$ we have

$$dI(T_{2n}(\alpha)) = -\frac{1}{4\pi^2} \beta \, d\alpha.$$*

§ 5. Proof of Theorem 1.1

For $n \geq 1$ and $M = e^{i\alpha/2}$ ($B = \cos(\alpha/2)$), $A_{2n}(M, L)$ and $P_{2n}(V, A)$ have a $2n$-component set of zeros, and for $n < -1$ they have a $-(2n + 1)$-component set of zero. The component which gives the maximal volume is the geometric component [13], [37], [38]; it is identified in [13]. For each $T_{2n}$, there exists an angle $\alpha_0 \in [2\pi/3, \pi]$ such that $T_{2n}$ is hyperbolic for $\alpha \in (0, \alpha_0)$, Euclidean for $\alpha = \alpha_0$, and spherical for $\alpha \in (\alpha_0, \pi)$ [22]–[25].

Denote by $D(T_{2n}(\alpha))$ be the set of common zeros of the discriminant of $A_{2n}(L, e^{i\alpha/2})$ over $L$ and the discriminant of $P_{2n}(V, \cos(\alpha/2))$ over $V$. Then $\alpha_0$ belongs to $D(T_{2n}(\alpha))$. 
On the geometric component we can calculate the Chern-Simons invariant of the orbifold $T_{2n}(2\pi/k)$ (mod $1/k$ if $k$ is even or mod $1/(2k)$ if $k$ is odd), where $k$ is a positive integer such that the $k$-fold cyclic covering of $T_{2n}(2\pi/k)$ is hyperbolic:

$$\text{cs}(T_{2n}(\frac{2\pi}{k})) \equiv I(T_{2n}(\frac{2\pi}{k})) \pmod{\frac{1}{k}} \equiv I(T_{2n}(\pi)) + \frac{1}{4\pi^2} \int_{2\pi/k}^{\pi} \beta d\alpha \pmod{\frac{1}{k}},$$

$$\equiv \frac{1}{2} \text{cs}(L(4n + 1, 2n + 1)) + \frac{1}{4\pi^2} \int_{2\pi/k}^{\alpha_0} \text{Im}(2 \log \left( \frac{M^{-2} A + iV}{A - iV} \right)) d\alpha$$

$$+ \frac{1}{4\pi^2} \int_{\alpha_0}^{\pi} \text{Im}(\log \left( \frac{M^{-2} A + iV_1}{A - iV_1} \right) + \log \left( \frac{M^{-2} A + iV_2}{A - iV_2} \right)) d\alpha$$

$$\pmod{\frac{1}{k}} \text{ if } k \text{ is even or mod } \frac{1}{2k} \text{ if } k \text{ is odd},$$

where the second equivalence comes from Theorem 4.1 and the third equivalence comes from the fact that

$$I(T_{2n}(\pi)) \equiv \frac{1}{2} \text{cs}(L(4n + 1, 2n + 1)) \left( \mod \frac{1}{2} \right),$$

Lemma 3.3 and the geometric interpretations of hyperbolic and spherical holonomy representations.

The fundamental set of two-bridge link orbifolds is constructed in [39]. The following theorem gives the Chern-Simons invariant of the lens space $L(4n+1,2n+1)$.

**Theorem 5.1** (see [15], Theorem 1.3).

$$\text{cs}(L(4n + 1, 2n + 1)) \equiv \frac{6n + 4}{8n + 2} \pmod{1}.$$ 

§ 6. Chern-Simons invariants of the hyperbolic twist-knot orbifolds and of their cyclic coverings

Table 1 (respectively, Table 2) gives the approximate Chern-Simons invariants of the hyperbolic twist-knot orbifolds cs($T_{2n}(2\pi/k)$) for $2 \leq n \leq 9$ (respectively, for $-9 \leq n \leq -2$) and for $3 \leq k \leq 10$, and of their cyclic coverings cs($M_k(T_{2n})$). We used Simpson’s rule for the approximation with $10^4$ intervals ($5 \times 10^3$ in Simpson’s rule) from $2\pi/k$ to $\alpha_0$ and $10^4$ intervals ($5 \times 10^3$ in Simpson’s rule) from $\alpha_0$ to $\pi$. Table 3 gives the approximate Chern-Simons invariant of $T_{2n}$ for each $n$ between $-9$ and $9$ except the unknot, the torus knot and the amphicheiral knot. We again used Simpson’s rule for the approximation with $10^4$ intervals ($5 \times 10^3$ in Simpson’s rule) from $0$ to $\alpha_0$ and $10^4$ intervals ($5 \times 10^3$ in Simpson’s rule) from $\alpha_0$ to $\pi$. We used the *Mathematica* software for the calculations. We record here that our data in Table 3 and those obtained using the *SnapPy* software match up to six decimal points.

The authors would like to thank Alexander Mednykh and Hyuk Kim for their assistance, Nathan Dunfield and Daniel Mathews for their prompt help, and also anonymous referees.
Table 1. The Chern-Simons invariant of the hyperbolic twist-knot orbifolds $cs(T_{2n}(2\pi/k))$ and of their cyclic coverings $cs(M_k(T_{2n}))$, for $2 \leq n \leq 9$ and $3 \leq k \leq 10$.

|   | $cs(T_4(2\pi/k))$ | $cs(M_k(T_4))$ |   | $cs(T_6(2\pi/k))$ | $cs(M_k(T_6))$ |
|---|-------------------|----------------|---|-------------------|----------------|
| 3 | 0.0875301        | 0.262590       | 3 | 0.0449535        | 0.134860       |
| 4 | 0.144925         | 0.579699       | 4 | 0.0876043        | 0.350417       |
| 5 | 0.0784576        | 0.392288       | 5 | 0.0165337        | 0.0826684      |
| 6 | 0.0351571        | 0.210943       | 6 | 0.138167         | 0.829004       |
| 7 | 0.00506505       | 0.0354553      | 7 | 0.0120078        | 0.080545       |
| 8 | 0.0218112        | 0.196301       | 8 | 0.0430876        | 0.344700       |
| 9 | 0.0530574        | 0.530574       | 9 | 0.0121250        | 0.109125       |
| 10| 0.0351571        | 0.530574       | 10| 0.0876213        | 0.876213       |

|   | $cs(T_8(2\pi/k))$ | $cs(M_k(T_8))$ |   | $cs(T_{10}(2\pi/k))$ | $cs(M_k(T_{10}))$ |
|---|-------------------|----------------|---|-----------------------|-------------------|
| 3 | 0.0161266         | 0.0483799      | 3 | 0.162697             | 0.488091         |
| 4 | 0.0536832         | 0.214733       | 4 | 0.0320099            | 0.128040         |
| 5 | 0.0817026         | 0.408513       | 5 | 0.0597580            | 0.298790         |
| 6 | 0.103012          | 0.618074       | 6 | 0.0809665            | 0.485799         |
| 7 | 0.0481239         | 0.336867       | 7 | 0.0260276            | 0.182193         |
| 8 | 0.00768503        | 0.0614802      | 8 | 0.110559            | 0.884475         |
| 9 | 0.0322210         | 0.289989       | 9 | 0.0100766            | 0.0906893        |
| 10| 0.0521232         | 0.521232       | 10| 0.0299660            | 0.299660         |

|   | $cs(T_{12}(2\pi/k))$ | $cs(M_k(T_{12}))$ |   | $cs(T_{14}(2\pi/k))$ | $cs(M_k(T_{14}))$ |
|---|----------------------|-------------------|---|----------------------|-------------------|
| 3 | 0.148360             | 0.445081         | 3 | 0.137750             | 0.413249         |
| 4 | 0.0170833            | 0.0683334        | 4 | 0.00620422           | 0.0248169        |
| 5 | 0.0447221            | 0.223610         | 5 | 0.0337900            | 0.168950         |
| 6 | 0.0658884            | 0.395330         | 6 | 0.0549355            | 0.329613         |
| 7 | 0.0109281            | 0.0764969        | 7 | 0.0713932            | 0.499752         |
| 8 | 0.0954474            | 0.763579         | 8 | 0.0844779            | 0.675823         |
| 9 | 0.0505121            | 0.454609         | 9 | 0.0395376            | 0.355839         |
| 10| 0.0148406            | 0.148406         | 10| 0.005386414          | 0.0386414        |

|   | $cs(T_{16}(2\pi/k))$ | $cs(M_k(T_{16}))$ |   | $cs(T_{18}(2\pi/k))$ | $cs(M_k(T_{18}))$ |
|---|----------------------|-------------------|---|----------------------|-------------------|
| 3 | 0.129617             | 0.388850         | 3 | 0.123198             | 0.369593         |
| 4 | 0.247931             | 0.991725         | 4 | 0.241431             | 0.965725         |
| 5 | 0.0254880            | 0.127440         | 5 | 0.0189709            | 0.0948543        |
| 6 | 0.0466221            | 0.279733         | 6 | 0.0400981            | 0.240588         |
| 7 | 0.0630741            | 0.441518         | 7 | 0.0565432            | 0.395803         |
| 8 | 0.0761526            | 0.609221         | 8 | 0.0696194            | 0.556955         |
| 9 | 0.0312096            | 0.280887         | 9 | 0.0246862            | 0.222176         |
| 10| 0.0955375            | 0.955375         | 10| 0.0890057            | 0.890057         |
Table 2. The Chern-Simons invariants of the hyperbolic twist-knot orbifolds $cs(T_{2n}(2\pi/k))$ and of their cyclic coverings $cs(M_k(T_{2n}))$, for $-9 \leq n \leq -2$ and $3 \leq k \leq 10$.

| $k$ | $cs(T_{-4}(2\pi/k))$ | $cs(M_k(T_{-4}))$ | $k$ | $cs(T_{-6}(2\pi/k))$ | $cs(M_k(T_{-6}))$ |
|-----|---------------------|-------------------|-----|---------------------|-------------------|
| 3   | 0.0200144           | 0.0600431         | 3   | 0.0749433           | 0.224830          |
| 4   | 0.186811            | 0.747246          | 4   | 0.0126376           | 0.0505506         |
| 5   | 0.00166667          | 0.00833333        | 5   | 0.0873477           | 0.436738          |
| 6   | 0.0504622           | 0.302773          | 6   | 0.140792            | 0.844753          |
| 7   | 0.0163442           | 0.114410          | 7   | 0.0376998           | 0.263898          |
| 8   | 0.116990            | 0.935921          | 8   | 0.0862114           | 0.689691          |
| 9   | 0.0292902           | 0.263612          | 9   | 0.0133130           | 0.119817          |
| 10  | 0.0595432           | 0.595432          | 10  | 0.0552937           | 0.552937          |

| $k$ | $cs(T_{-8}(2\pi/k))$ | $cs(M_k(T_{-8}))$ | $k$ | $cs(T_{-10}(2\pi/k))$ | $cs(M_k(T_{-10}))$ |
|-----|---------------------|-------------------|-----|---------------------|-------------------|
| 3   | 0.109659            | 0.328978          | 3   | 0.133696            | 0.401087          |
| 4   | 0.0564153           | 0.225661          | 4   | 0.0832469           | 0.332988          |
| 5   | 0.0330919           | 0.165460          | 5   | 0.0603968           | 0.301984          |
| 6   | 0.0205610           | 0.123366          | 6   | 0.0480386           | 0.288232          |
| 7   | 0.0130371           | 0.0912595         | 7   | 0.0405999           | 0.284200          |
| 8   | 0.00816423          | 0.0653138         | 8   | 0.0357763           | 0.286210          |
| 9   | 0.00482762          | 0.0434486         | 9   | 0.0324710           | 0.292239          |
| 10  | 0.00244289          | 0.0244289         | 10  | 0.0301075           | 0.301075          |

| $k$ | $cs(T_{-12}(2\pi/k))$ | $cs(M_k(T_{-12}))$ | $k$ | $cs(T_{-14}(2\pi/k))$ | $cs(M_k(T_{-14}))$ |
|-----|---------------------|-------------------|-----|---------------------|-------------------|
| 3   | 0.150597            | 0.451792          | 3   | 0.162874            | 0.488621          |
| 4   | 0.101087            | 0.404347          | 4   | 0.113753            | 0.455011          |
| 5   | 0.0784041           | 0.392020          | 5   | 0.0911448           | 0.455724          |
| 6   | 0.0661095           | 0.396657          | 6   | 0.0788794           | 0.473276          |
| 7   | 0.0587029           | 0.410920          | 7   | 0.0000589596        | 0.000412717       |
| 8   | 0.0538979           | 0.431183          | 8   | 0.0666914           | 0.533532          |
| 9   | 0.0506045           | 0.455441          | 9   | 0.00784780          | 0.0706302         |
| 10  | 0.0482492           | 0.482492          | 10  | 0.0610519           | 0.610519          |

| $k$ | $cs(T_{-16}(2\pi/k))$ | $cs(M_k(T_{-16}))$ | $k$ | $cs(T_{-18}(2\pi/k))$ | $cs(M_k(T_{-18}))$ |
|-----|---------------------|-------------------|-----|---------------------|-------------------|
| 3   | 0.00545933          | 0.0163780         | 3   | 0.0126607           | 0.0379822         |
| 4   | 0.123196            | 0.492785          | 4   | 0.130503            | 0.522012          |
| 5   | 0.000626850         | 0.00313425        | 5   | 0.00795573          | 0.0397786         |
| 6   | 0.0883767           | 0.530260          | 6   | 0.0957144           | 0.574286          |
| 7   | 0.00956397          | 0.0669478         | 7   | 0.0169085           | 0.118359          |
| 8   | 0.0762009           | 0.609607          | 8   | 0.0835496           | 0.668397          |
| 9   | 0.0173633           | 0.156270          | 9   | 0.0247059           | 0.222353          |
| 10  | 0.0705667           | 0.705667          | 10  | 0.0779144           | 0.779144          |
Table 3. Chern-Simons invariant of $T_{2n}$ for $2 \leq n \leq 9$ and $-9 \leq n \leq -2$.

| $2n$ | $\alpha_0$ | $\text{cs}(T_{2n})$ | $2n$ | $\alpha_0$ | $\text{cs}(T_{2n})$ |
|------|-------------|---------------------|------|-------------|---------------------|
| 4    | 2.57414     | 0.344023            | -4   | 2.40717     | 0.346796            |
| 6    | 2.75069     | 0.277867            | -6   | 2.67879     | 0.444846            |
| 8    | 2.84321     | 0.242222            | -8   | 2.80318     | 0.492293            |
| 10   | 2.90026     | 0.220016            | -10  | 2.87475     | 0.0200385           |
| 12   | 2.93897     | 0.204869            | -12  | 2.92130     | 0.0382117           |
| 14   | 2.96697     | 0.193882            | -14  | 2.95401     | 0.0510293           |
| 16   | 2.98817     | 0.185550            | -16  | 2.97825     | 0.0605519           |
| 18   | 3.00477     | 0.179014            | -18  | 2.99694     | 0.0679043           |

Bibliography

[1] Shiing-Shen Chern and J. Simons, “Some cohomology classes in principal fiber bundles and their application to Riemannian geometry”, *Proc. Nat. Acad. Sci. U.S.A.* **68**:4 (1971), 791–794.

[2] R. Meyerhoff, “Hyperbolic 3-manifolds with equal volumes but different Chern-Simons invariants”, *Low-dimensional topology and Kleinian groups* (Coventry/Durham 1984), London Math. Soc. Lecture Note Ser., vol. 112, Cambridge Univ. Press, Cambridge 1986, pp. 209–215.

[3] W.D. Neumann, “Combinatorics of triangulations and the Chern-Simons invariant for hyperbolic 3-manifolds”, *Topology* ’90 (Columbus, OH 1990), Ohio State Univ. Math. Res. Inst. Publ., vol. 1, de Gruyter, Berlin 1992, pp. 243–271.

[4] W.D. Neumann, “Extended Bloch group and the Cheeger-Chern-Simons class”, *Geom. Topol.* **8** (2004), 413–474.

[5] Ch. K. Zickert, “The volume and Chern-Simons invariant of a representation”, *Duke Math. J.* **150**:3 (2009), 489–532.

[6] J. Cho, J. Murakami and Y. Yokota, “The complex volumes of twist knots”, *Proc. Amer. Math. Soc.* **137**:10 (2009), 3533–3541.

[7] J. Cho and J. Murakami, “The complex volumes of twist knots via colored Jones polynomials”, *J. Knot Theory Ramifications*, 2010, no. 11, 1401–1421.

[8] J. Cho, H. Kim and S. Kim, “Optimistic limits of Kashaev invariants and complex volumes of hyperbolic links”, *J. Knot Theory Ramifications** **23**:9 (2014), 1450049, 32 pp.

[9] M. Culler, N. Dunfield, M. Goerner and J. Weeks, *SnapPy*, http://www.math.uic.edu/t3m/SnapPy.

[10] O. Goodman, *Snap for hyperbolic 3-manifolds*, http://sourceforge.net/projects/snap-pari.

[11] D. Coulson, O. A. Goodman, C. D. Hodgson and W. D. Neumann, “Computing arithmetic invariants of 3-manifolds”, *Experiment. Math.* **9**:1 (2000), 127–152.

[12] T. Yoshida, “The $\eta$-invariant of hyperbolic 3-manifolds”, *Invent. Math.* **81**:3 (1985), 473–514.

[13] Ji-Young Ham, A. Mednykh and V. Petrov, “Trigonometric identities and volumes of the hyperbolic twist knot cone-manifolds”, *J. Knot Theory Ramifications** **23**:12 (2014), 1450064, 16 pp.
[14] H. M. Hilden, M. T. Lozano and J. M. Montesinos-Amilibia, “On volumes and Chern-Simons invariants of geometric 3-manifolds”, *J. Math. Sci. Univ. Tokyo* **3**:3 (1996), 723–744.

[15] H. M. Hilden, M. T. Lozano and J. M. Montesinos-Amilibia, “Volumes and Chern-Simons invariants of cyclic coverings over rational knots”, *Topology and Teichmüller spaces* (Katinkulta 1995), World Sci. Publ., River Edge, NJ 1996, pp. 31–55.

[16] P. A. Kirk and E. P. Klassen, “Chern-Simons invariants of 3-manifolds and representation spaces of knot groups”, *Math. Ann.* **287**:2 (1990), 343–367.

[17] P. Kirk and E. Klassen, “Chern-Simons invariants of 3-manifolds decomposed along tori and the circle bundle over the representation space of $T^2$”, *Comm. Math. Phys.* **153**:3 (1993), 521–557.

[18] N. V. Abrosimov, “On Chern-Simons invariants of geometric 3-manifolds”, *Sib. Elektron. Mat. Izv.* **3** (2006), 67–70.

[19] N. V. Abrosimov, “The Chern-Simons invariants of cone-manifolds with Whitehead link singular set”, *Mat. Tr.* **10**:1 (2007), 3–15; English transl. in *Siberian Adv. Math.* **18**:2 (2008), 77–85.

[20] D. Cooper, C. D. Hodgson and S. P. Kerckhoff, *Three-dimensional orbifolds and cone-manifolds*, MSJ Memoirs, vol. 5, Math. Soc. Japan, Tokyo 2000, x+170 pp.

[21] W. Thurston, *Geometry and topology of three-manifolds*, Lecture notes, Princeton Univ., 1980, http://library.msri.org/books/gt3m.

[22] S. Kojima, “Deformations of hyperbolic 3-cone-manifolds”, *J. Differential Geom.* **49**:3 (1998), 469–516.

[23] J. Porti, “Spherical cone structures on 2-bridge knots and links”, *Kobe J. Math.* **21**:1–2 (2004), 61–70.

[24] H. Hilden, M. T. Lozano and J. M. Montesinos-Amilibia, “On a remarkable polyhedron geometrizing the figure eight knot cone manifolds”, *J. Math. Sci. Univ. Tokyo* **2**:3 (1995), 501–561.

[25] J. Porti and H. Weiss, “Deforming Euclidean cone 3-manifolds”, *Geom. Topol.* **11** (2007), 1507–1538.

[26] J. Hoste and P. D. Shanahan, “A formula for the A-polynomial of twist knots”, *J. Knot Theory Ramifications* **13**:2 (2004), 193–209.

[27] D. V. Mathews, “An explicit formula for the A-polynomial of twist knots”, *J. Knot Theory Ramifications* **23**:9 (2014), 1450044, 5 pp.

[28] D. V. Mathews, “Erratum: “An explicit formula for the A-polynomial of twist knots””, *J. Knot Theory Ramifications* **23**:11 (2014), 1492001, 1 p.

[29] D. Cooper and D. D. Long, “Remarks on the A-polynomial of a knot”, *J. Knot Theory Ramifications* **5**:5 (1996), 609–628.

[30] A. Hatcher and W. Thurston, “Incompressible surfaces in 2-bridge knot complements”, *Invent. Math.* **79**:2 (1985), 225–246.

[31] H. Schubert, “Knoten mit zwei Brücken”, *Math. Z.* **65** (1956), 133–170.

[32] R. Riley, “Parabolic representations of knot groups. I”, *Proc. London Math. Soc.* (3) **24**:2 (1972), 217–242.

[33] R. Riley, “Nonabelian representations of 2-bridge knot groups”, *Quart. J. Math. Oxford Ser.* (2) **35**:138 (1984), 191–208.

[34] D. Cooper and D. D. Long, “The A-polynomial has ones in the corners”, *Bull. London Math. Soc.* **29**:2 (1997), 231–238.

[35] D. Cooper, M. Culler, H. Gillet, D. D. Long and P. B. Shalen, “Plane curves associated to character varieties of 3-manifolds”, *Invent. Math.* **118**:1 (1994), 47–84.
[36] R. Meyerhoff and D. Ruberman, “Mutation and the $\eta$-invariant”, *J. Differential Geom.* **31**:1 (1990), 101–130.

[37] N. M. Dunfield, “Cyclic surgery, degrees of maps of character curves, and volume rigidity for hyperbolic manifolds”, *Invent. Math.* **136**:3 (1999), 623–657.

[38] S. Francaviglia and B. Klaff, “Maximal volume representations are Fuchsian”, *Geom. Dedicata* **117** (2006), 111–124.

[39] A. Mednykh and A. Rasskazov, *On the structure of the canonical fundamental set for the 2-bridge link orbifolds*, Sonderforschungbereich 343, “Diskrete Strukturen in der Mathematik”, Preprint, 98-062, Universität Bielefeld 1998, 32 pp., http://www.uni-bielefeld.de/fb19/32.htm.

Ji-Young Ham
Seoul National University,
Republic of Korea (South);
Hongik University, Seoul,
Republic of Korea (South)
E-mail: jiyoungham1@gmail.com

Joongul Lee
Hongik University, Seoul,
Republic of Korea (South)
E-mail: jglee@hongik.ac.kr

Received 3/OCT/15 and 28/JAN/16