Electrodynamics of spin currents in superconductors

J. E. Hirsch

Department of Physics, University of California, San Diego
La Jolla, CA 92093-0319

In recent work we formulated a new set of electrodynamic equations for superconductors as an alternative to the conventional London equations, compatible with the prediction of the theory of hole superconductivity that superconductors expel negative charge from the interior towards the surface. Charge expulsion results in a macroscopically inhomogeneous charge distribution and an electric field in the interior, and because of this a spin current is expected to exist. Furthermore, we have recently shown that a dynamical explanation of the Meissner effect in superconductors leads to the prediction that a spontaneous spin current exists near the surface of superconductors (spin Meissner effect). In this paper we extend the electrodynamic equations proposed earlier for the charge density and charge current to describe also the space and time dependence of the spin density and spin current. This allows us to determine the magnitude of the expelled negative charge and interior electric field as well as of the spin current in terms of other measurable properties of superconductors. We also provide a 'geometric' interpretation of the difference between type I and type II superconductors, discuss the relationship between our model and Slater’s seminal work on superconductivity, and discuss the magnitude of the expected novel effects for elemental and other superconductors.

PACS numbers:

I. INTRODUCTION

The theory of hole superconductivity predicts that superconductors expel negative charge from the interior towards the surface\[1\,][2\,][3\,], resulting in the existence of an electric field in the interior of superconductors. To account for this fact we have proposed a new set of electrodynamic equations for superconductors\[3\,][4\,] in place of the conventional London equations\[3\]. The new equations predict a charge density $\rho(\vec{r})$ in the interior of superconductors that satisfies the differential equation (in the static case)

$$\nabla^2 \rho(\vec{r}) = \frac{1}{\lambda_L^2}(\rho(\vec{r}) - \rho_0)$$

with $\rho_0$ a positive constant. This equation (together with a similar equation for the electric potential) predicts that the charge density in the deep interior of a superconducting body is $\rho_0$, and that an excess of negative charge density $\rho_-$ exists within a London penetration depth ($\lambda_L$) of the surface. It also gives rise to an electric field in the interior of superconductors that points towards the surface. For a cylindrical geometry and cylinder radius $R >> \lambda_L$, charge conservation implies that

$$\rho_0 = -\frac{2\lambda_L}{R}\rho_-$$

and for a sphere of radius $R >> \lambda_L$

$$\rho_0 = -\frac{3\lambda_L}{R}\rho_-.$$

(2a)

In both cases, the electric field increases linearly away from the center, attains its maximum value

$$E_m = -4\pi\lambda_L\rho_-$$

at a distance $\lambda_L$ from the surface, and drops to zero at the surface.

Energetic arguments\[3\] indicate that $\rho_-$ and $E_m$ should be independent of the size of the sample and that $E_m$ should be related to the square root of the superconducting condensation energy per unit volume. However, the precise value of $E_m$ could not be determined from the treatment of refs.\[3\,][4\,]. The determination of the value of $E_m$ is one of the central results of this paper. The interior charge density $\rho_0$ depends on the sample dimensions (Eq. (2)) and becomes smaller as the sample size increases.

We have also pointed out in earlier work that an electric field in the interior of a superconductor should give rise to a macroscopic spin current\[2\,][3\,], and showed that in fact the microscopic Hamiltonian proposed by the theory of hole superconductivity\[3\] favors the existence of such a current. In very recent work we have put forth a more detailed picture of how the charge expulsion proposed to exist in superconductors occurs and how the spin current is generated: namely, that electrons ‘expand’ their wavefunction from being confined to a lattice spacing (as corresponds to a nearly filled band) to a much larger (mesoscopic) extent. As electrons move radially outward, the spin-orbit interaction gives rise to azimuthal velocities resulting in the two members of the Cooper pair circulating in opposite directions in an orbit of radius $2\lambda_L$, which encloses precisely one flux quantum of spin-orbit flux. This picture also provides for a dynamical explanation of the Meissner effect. It predicts that in the presence of an external magnetic field, a charge and a spin current will circulate near the surface of a superconductor. In the absence of an external magnetic field, a pure spin current will circulate, with a universal expression for the magnitude of the spin current...
speed at the surface \( v_s^0 = \frac{\hbar}{(4m_e \lambda_L)}. \)

For the charge current and charge density we derived their space and time dependence in Ref. 6 (the charge current behavior predicted by our equations is the same as predicted by the conventional London equations). In this paper we derive the equations governing the space and time dependence of the spin current, and show how the spontaneous spin current is related to the expelled negative charge. Furthermore we provide a new interpretation of type I and type II regimes in superconductors, and we explain how superconductors manage to conserve angular momentum. We also calculate the magnitude of the effects predicted for various materials. Finally, we discuss the relation of our theory with earlier pre-BCS work.

We should point out at the outset that the aim of this paper is limited to providing a macroscopic description of the electrodynamics of charge and spin currents in superconductors, and is not to provide a full microscopic description of the superconducting state. Just like conventional BCS theory is consistent with conventional London electrodynamics, a full quantum-mechanical description of type I and type II regimes in superconductors, and is not to provide a full microscopic description of the superconducting state. Just like conventional BCS theory is consistent with conventional London electrodynamics, a full quantum-mechanical description of the problem should be compatible with the electrodynamics discussed here and will be the subject of future work.

II. SPIN CURRENT CONSTITUTIVE RELATION

For the charge current

\[ \mathbf{J} = e n_s \mathbf{v}_s \]  

with \( n_s \) the superfluid density and \( \mathbf{v}_s \) the superfluid charge velocity, the governing equation is

\[ \mathbf{J}(\mathbf{r}) = -\frac{e}{4\pi \lambda_L^2} \mathbf{A}(\mathbf{r}) \tag{5} \]

with \( \mathbf{A} \) the magnetic vector potential, related to the magnetic field \( \mathbf{B} \) by

\[ \nabla \times \mathbf{A} = \mathbf{B}. \tag{6} \]

The London penetration depth is given by

\[ \frac{1}{\lambda_L^2} = \frac{4\pi n_s e^2}{m_e c^2} \tag{7} \]

and Eq. (5) is equivalent to

\[ \mathbf{v}_s(\mathbf{r}) = -\frac{e}{m_e c} \mathbf{A}(\mathbf{r}) \tag{8} \]

which can be understood as resulting from the 'rigidity' of the wave function that 'forces' \( \mathbf{p} \) in the relation \( \mathbf{p} = m_e \mathbf{v}_s + (e/c) \mathbf{A} \) to stay zero at all times and locations in a simply connected superconductor.

We have shown in ref. 8 that the Meissner effect can be understood 'dynamically' by assuming that electrons move radially outward a distance \( 2\lambda_L \) in the presence of an unscreened magnetic field \( \mathbf{B} \) that gives rise to a vector potential

\[ \mathbf{A} = \frac{\mathbf{B} \times \mathbf{r}}{2}, \tag{9} \]

and acquire through the action of the Lorentz force an azimuthal velocity given by Eq. (8). In a cylindrical geometry, the magnetic field at the surface equals the applied magnetic field \( \mathbf{B} \), and the vector potential at the surface is given by

\[ \mathbf{A}(R) = \lambda_L \mathbf{B} \times \hat{n} \tag{10} \]

(\( \hat{n} = \hat{r} \) denotes the direction normal to the surface) which coincides with Eq. (9) for \( r = 2\lambda_L \). Eq. (10) results from solving London’s equation for an infinitely long cylinder of radius \( R >> \lambda_L \), or from simply assuming that the magnetic field penetrates a distance \( \lambda_L \) and using that

\[ \oint \mathbf{A} \cdot d\mathbf{l} = \int \mathbf{B} \cdot d\mathbf{a} \]

Similarly, we showed in ref. 8 that an 'effective' unscreened magnetic field \( \mathbf{B}_\sigma \) acts on the superfluid electrons arising from the spin-orbit interaction

\[ \mathbf{B}_\sigma = 2\pi n_s \mu \tag{11} \]

with

\[ \mu = \frac{e\hbar}{2m_e c} \]  

the electron magnetic moment. Electrons acquire a spin current velocity\[ 8\]

\[ \mathbf{v}_\sigma^0 = -\frac{\hbar}{4m_e \lambda_L} \mathbf{\hat{\sigma}} \times \mathbf{r} \equiv -\mathbf{v}_\sigma^0 \mathbf{\hat{\sigma}} \times \mathbf{r} \tag{13} \]

through the action of the 'Lorentz' force from the field Eq. (11) in moving radially outward a distance \( 2\lambda_L \). The spin-orbit vector potential corresponding to 'magnetic field' Eq. (11) at radial distance \( 2\lambda_L \) is, from Eqs. (11) and (9)

\[ \mathbf{A}_\sigma = 2\pi n_s \lambda_L \mu \mathbf{\hat{r}}. \tag{14} \]

Just like in the case of the real magnetic field, we assume that this is the value of the spin-orbit vector potential at the surface of the cylinder. Using Eq. (7), we can rewrite the spin-orbit magnetic field and vector potential at the surface Eqs. (11) and (14) as

\[ \mathbf{B}_\sigma(R) = \frac{\hbar c}{4\pi \lambda_L} \mathbf{\hat{\sigma}} = \frac{m_e c}{e \lambda_L} v_\sigma^0 \mathbf{\hat{\sigma}} \tag{15a} \]

\[ \mathbf{A}_\sigma(R) = \frac{\hbar c}{4e \lambda_L} \mathbf{\hat{\sigma}} \times \hat{n} = -\frac{m_e c}{e} v_\sigma^0 \tag{15b} \]
respectively. Next, we need to deduce their behavior as we move away from the surface towards the interior.

We consider a cylindrical geometry throughout. We expect the spin current to flow within a London penetration depth of the surface, just like the charge current, with the electron spin parallel (antiparallel) to the cylinder axis and antiparallel (parallel) to its orbital angular momentum. The 'spontaneous' current of the spin-\(\sigma\) carriers is

\[
\mathcal{J}_\sigma (\hat{r}) = \frac{en_s}{2} \mathbf{v}_\sigma (\hat{r}).
\]

The total velocity of a carrier of spin \(\hat{\sigma}\) in the presence of an external magnetic field is \(\mathbf{v}_\sigma = \mathbf{v}_\sigma + \mathbf{v}_s\). At the surface \((r = R)\) the spin current velocity \(\mathbf{v}_\sigma(R)\) is given by Eq. (13), and the spin-orbit vector potential by Eq. (14) (or (15b)). The equation relating the spin current to the spin vector potential is, in analogy with Eq. (5)

\[
\mathcal{J}_\sigma (\hat{r}) = - \frac{e}{m c} \mathbf{A}_\sigma(\hat{r})
\]

and from Eqs. (16) and (17)

\[
\mathbf{v}_\sigma(\hat{r}) = - \frac{e}{mc^2} \mathbf{A}_\sigma(\hat{r})
\]

analogous to Eq. (8).

As discussed in ref. [3], the spin current exists due to the presence of an electric field \(\mathbf{E}\) in the interior of the superconductor, that satisfies the differential equation [4]

\[
\nabla^2 \mathbf{E} = \frac{1}{\lambda_L} (\mathbf{E} - \mathbf{E}_0),
\]

where \(\mathbf{E}_0\) is the electric field arising from the uniform positive charge distribution \(\rho_0\) \((\nabla \cdot \mathbf{E}_0 = 4\pi \rho_0)\). \(\mathbf{E}\) is maximum at a distance \(\lambda_L\) from the surface and decays to zero at the surface (for a cylindrical or spherical sample). As discussed by Aharonov and Casher [11] and other authors [12], the spin-orbit interaction term in the non-relativistic limit of Dirac’s equation for a particle with magnetic moment \(\mu\) in an electric field can be written in terms of a spin-orbit vector potential proportional to \(\mu \times \mathbf{E}\). We expect our spin-orbit vector potential to be related to the electric field in a similar fashion, and postulate that the vector potential driving the spin current in Eq. (17) is determined by the difference between the electric field near the surface \(\mathbf{E}\) and the electric field in the deep interior \(\mathbf{E}_0\) through the relation:

\[
\mathbf{A}_\sigma(\hat{r}) = - \frac{me \mathbf{c} \mathbf{v}_0}{e E_m} \hat{\sigma} \times (\mathbf{E}(\hat{r}) - \mathbf{E}_0(\hat{r}))
\]

with \(E_m\) related to the charge density near the surface \(\rho_0\) by the charge conservation condition Eq. (3).

Eq. (20) reduces to Eq. (15b) for the spin-orbit vector potential at the surface, as is seen by setting \(\mathbf{E}(R) = 0\) and \(\mathbf{E}_0(\hat{r}) = E_m \hat{n}\). From Eq. (18), the spin current velocity as function of position is given by

\[
\mathbf{v}_\sigma(\hat{r}) = \frac{e}{m} \frac{\mathbf{E}}{E_m} \hat{\sigma} \times (\mathbf{E}(\hat{r}) - \mathbf{E}_0(\hat{r})).
\]

which reduces to the un-screened spin-orbit field Eq. (15a) upon replacing \(\rho_\sigma\) by its value given by Eq. (3). The result Eq. (23) followed from using Gauss’ law and imposing the global charge conservation condition implicit in Eq. (3): that the positive charge expelled from the interior that gives rise to the maximum electric field \(E_m\) at distance \(\lambda_L\) from the surface is precisely the same as the extra negative charge \(\rho_m\) residing in the surface layer of thickness \(\lambda_L\). Thus the coincidence of Eq. (23) with Eq. (15a) was not built in and supports the consistency and validity of our framework.

## III. THE FOURTH COMPONENT

In ref. [4] we showed that the electrodynamics in the charge sector, under the assumption that the magnetic vector potential obeys the Lorentz gauge, could naturally be formulated in a 4-dimensional covariant form in terms of the 4-dimensional charge current and vector potential \(J = (\mathbf{J}, ic\phi)\), \(A = (\mathbf{A}, i\phi)\) (\(\phi\) = electric potential, \(\rho\) = charge density). The four-dimensional divergence of \(J\) and of \(A\) vanish due to charge conservation and the Lorentz gauge condition respectively. Similarly we seek here the fourth component of a 4-dimensional spin vector potential \(A_\sigma = (\mathbf{A}_\sigma, i\phi_\sigma)\) by demanding that the four-dimensional divergence vanishes:

\[
\text{Div} \mathbf{A}_\sigma = \nabla \cdot \mathbf{A}_\sigma + \frac{1}{c} \frac{\partial}{\partial t} \phi_\sigma = 0.
\]

Using the relation

\[
\nabla \cdot (\mathbf{A}_\sigma \times \mathbf{E}) = - \mathbf{A}_\sigma \cdot (\nabla \times \mathbf{E})
\]

and Faraday’s law we obtain from Eqs. (20) and (24)

\[
\phi_\sigma = \frac{mc^2 \mathbf{v}_0}{e E_m} \hat{\sigma} \cdot \mathbf{B}.
\]
and correspondingly we add the fourth component \( \rho_\sigma \) to form the four-current for carriers of spin \( \sigma \), \( J_\sigma = (\hat{J}_\sigma, ic\rho_\sigma) \), satisfying the equation (cf. Eq. (17))

\[
\rho_\sigma = -\frac{me_v\sigma^0}{8\pi eE_m\lambda_L^2} \hat{\sigma} \cdot \hat{B}.
\]  

(26a)

This equation can be rewritten as

\[
\rho_\sigma = -\frac{en_s v_\sigma^0}{2E_m c} \hat{\sigma} \cdot \hat{B}
\]  

(26b)

or as

\[
\rho_\sigma = -\frac{n_s}{4\lambda_L E_m} \hat{\mu} \cdot \hat{B}
\]  

(26c)

and implies that in the presence of an applied magnetic field there is excess negative charge corresponding to spin direction antiparallel to the applied field \( \hat{B} \). This corresponds to the electrons near the surface that increase their velocity when \( \hat{B} \) is applied. Correspondingly, the charge density of the spin component that decreases its velocity when \( \hat{B} \) is applied decreases. Note that \( \hat{J}_\sigma \) and \( \rho_\sigma \) are related by the continuity equation \( \nabla \cdot \hat{J}_\sigma = 0 \):

\[
\nabla \cdot \hat{J}_\sigma + \frac{\partial \rho_\sigma}{\partial t} = 0
\]  

(27)

When an external magnetic field is applied, \( \hat{J}_\sigma \) near the surface acquires a divergence through the induced \( \nabla \times \vec{E} \), which creates the spin imbalance through Eq. (27).

IV. MAGNITUDE OF EXPELLED NEGATIVE CHARGE AND INTERNAL ELECTRIC FIELD

An applied magnetic field generates a charge current near the surface, and according to Eq. (26) it also generates a charge density. It is natural to conclude that this charge density arises due to the changed carrier velocity. From Eqs. (8) and (10), the magnetic field expressed in terms of the superfluid charge velocity at the surface is

\[
\vec{B} = \frac{me_c}{e\lambda_L} \vec{v}_s(R) \times \hat{n}
\]  

(28)

and the induced charge density Eq. (26a) is

\[
\rho_\sigma = -\frac{me_v\sigma^0}{8\pi eE_m} \hat{\sigma} \cdot (\vec{v}_s(R) \times \hat{n})
\]  

(29)

We argue that it is natural to conclude that the expelled charge density \( \rho_- \) will obey the same relation with respect to the spin current velocity \( v_\sigma \) that \( \rho_\sigma \) bears to the charge current velocity \( v_s \) (Eq. (29)), with an extra factor of 2 because of the two spin components contributing to the spin current. Hence we postulate that

\[
\rho_- = -\frac{me_v\sigma^0}{e\lambda_L} \frac{v_\sigma^0}{4\pi E_m} \hat{\sigma} \cdot (\vec{v}_\sigma \times \hat{n})
\]  

(30)

or, substituting for \( \rho_- \) using Eq. (3)

\[
E_m = (\frac{me_v\sigma^0}{e\lambda_L})^2 \frac{1}{E_m}
\]  

(31)

with solution

\[
E_m = -\frac{me_v\sigma^0}{e\lambda_L}
\]  

(32)

or, substituting for the spin current velocity from Eq. (13)

\[
E_m = -\frac{\hbar c}{4e\lambda_L} = \frac{\phi_0}{4\pi \lambda_L} = 2\pi n_s \mu_B
\]  

(33)

with \( \phi_0 = \hbar c/2e \) the flux quantum and \( \mu_B \) the Bohr magneton. Hence the magnitude of the internal electric field near the surface is the same as that of the spin-orbit effective magnetic field (Eq. (11) or (15a)).

The value of the expelled charge density near the surface, \( \rho_- \), follows from Eqs. (3) and (32):

\[
\rho_- = \frac{n_s e\hbar}{4\lambda_L m_e c} = -\frac{n_s}{2\lambda_L} \mu_B
\]  

(35)

or, using Eq. (7), as

\[
\rho_- = \frac{\hbar c}{16\pi e\lambda_L^2}
\]  

(36)

The charge density induced by the external magnetic field can now be written, substituting in Eq. (26a) \( E_m \) by its value Eq. (32)

\[
\rho_\sigma = \frac{1}{8\pi \lambda_L} \hat{\sigma} \cdot \hat{B}
\]  

(37a)

or, substituting for \( E_m \) in Eq. 26(b)

\[
\rho_\sigma = \frac{n_s e\sigma v_\sigma^0}{2e} (\vec{v}_s \times \hat{n})
\]  

(37b)

as one would expect by analogy with Eq. (34). Thus the total charge density for spin \( \sigma \) is simply

\[
\rho_{\sigma,\text{tot}} = \frac{n_s e}{2}\frac{(v_\sigma^0 + \sigma v_s)}{c} = \frac{n_s e}{2} \frac{\sigma v_{\sigma,\text{tot}}}{c}
\]  

(38)

with \( v_{\sigma,\text{tot}} = v_s + \sigma v_\sigma^0, \sigma = +/ - 1 \) for spin parallel/antiparallel to the applied magnetic field, and

\[
\rho_- = \rho_{\sigma=1,\text{tot}} + \rho_{\sigma=-1,\text{tot}}
\]  

(39)

independent of whether or not an external magnetic field is applied.

Equation (5) for the charge current can be written in terms of the expelled charge density \( \rho_- \) using Eq. (3) as

\[
\vec{J} = \rho_- \frac{\vec{B} \times \hat{n}}{E_m}
\]  

(40)
where we have used Eq. (10) for the vector potential. Eq. (40) has the following interpretation: rather than the entire superfluid moving with speed \( v_s \) (Eq. (4)), we may think of the charge current as being carried solely by the *excess* negative charge density \( \rho_- \), moving with velocity

\[
\vec{v}_\rho^- = c\vec{B} \times \hat{n}/E_m
\]  

(41)

(Note that the total charge density \( \rho_- \) is not changed by the applied magnetic field since \( \rho_\sigma = -\rho_{-\sigma} \). The speed Eq. (41) would exceed the speed of light if the applied magnetic field would exceed \( E_m \). However, the value we deduced for \( E_m \) Eq. (33) is just right to prevent this from happening, since it is of the order of the lower critical field for a type II superconductor [13]:

\[
H_{c1} = \frac{\phi_0}{4\pi\lambda_L} \ln \kappa
\]  

(42)

with \( \kappa \) the Ginzburg Landau parameter. Furthermore, we argued in Ref. [3] that superconductivity is destroyed when one of the components of the spin current is stopped by the applied magnetic field, which corresponds to a magnetic field of magnitude \( E_m \). Thus, Eq. (41) can be understood as meaning that superconductivity is destroyed when the speed of the excess charge carriers near the surface \( (\rho_-) \) reaches the speed of light.

An alternative and perhaps even more remarkable interpretation follows from writing the charge current Eq. (4) solely in terms of the charge density induced by the magnetic field Eq. (37), as:

\[
\vec{J} = en_s \vec{v}_s = e(\rho_\sigma = -1 - \rho_{\sigma = 1}) \vec{v}_s.
\]  

(43)

Eq. (43) can be read as meaning that the charge current induced by the magnetic field is carried solely by the induced charge carriers moving at the speed of light. Within this interpretation, superconductivity is destroyed when the induced charge density of the carriers that are slowed down by the applied magnetic field (carriers with spin parallel to the magnetic field) completely depletes the component of \( \rho_- \) with that spin orientation.

In a type I superconductor, superconductivity is destroyed when the applied magnetic field reaches the thermodynamic critical field \( H_c \), which is smaller than Eq. (33) \( (H_c \sim H_{c1} \times \lambda_L/\xi_0 \), with \( \xi_0 \) the Pippard/BCS coherence length). However, in that case the response to an applied magnetic field is non-local since the magnetic field varies strongly over a coherence length, so these considerations may need to be modified. We will argue in a later section that for a type I superconductor \( E_m = H_c \) rather than Eq. (33). The expelled charge density \( \rho_- \) is still given by Eq. (3), which implies that the interpretation of Eq. (41) is the same as discussed above for type II superconductors. Fig. 1 shows schematically the spin current and charge distribution in a cross section of a cylindrical sample.

The electrostatic energy cost per unit volume for a surface charge density \( \rho_- \) expelled from a volume \( V \) is

\[
\frac{1}{2} m_e (v_{\sigma=1}^2 + v_{\sigma=-1}^2) n_s V = \frac{E_m^2 + B^2}{8\pi}
\]  

(48)
The charge current screens the external magnetic field so it does not get into the superconductor. The spin current screens the internal electric field so it does not leak out of the superconductor.

V. SPIN CURRENT SPATIAL DISTRIBUTION

Having determined the value of the internal electric field $E_{m}$, we return to the formulation of the equations governing the spin current. We will slightly generalize the equations in Sect. II. As the equation relating spin current to spin vector potential we take instead of Eq. (17)

$$\vec{J}_{s}(\vec{r}) - \vec{J}_{s0} = -\frac{c}{8\pi\lambda_{L}^{2}}(\vec{A}_{s}(\vec{r}) - \vec{A}_{s0}(\vec{r}))$$

(49)

with

$$\vec{A}_{s}(\vec{r}) = \lambda_{L}\vec{\sigma} \times \vec{E}(\vec{r})$$

(50a)

$$\vec{A}_{s0}(\vec{r}) = \lambda_{L}\vec{\sigma} \times \vec{E}_{0}(\vec{r})$$

(50b)

and

$$\vec{J}_{s}(\vec{r}) = \frac{en_{s}}{2}\vec{v}_{s}(\vec{r})$$

(51a)

$$\vec{J}_{s0} = \frac{en_{s}}{2}\vec{v}_{s0}$$

(51b)

The coefficient $\lambda_{L}$ in Eq. (50) results from replacing in Eq. (20) the value found for $E_{m}$, Eq. (32). Note the similarity between Eq. (50a) and Eq. (10), which was not "built in". Aside from an inconsequential redefinition of $\vec{A}_{s}$ ($\vec{A}_{s}$ in Eq. (17) is ($\vec{A}_{s} - \vec{A}_{s0}$) in Eq. (49)). Eqs. (49)-(51) differ from Eqs. (17) and (20) in that we allow for the possibility of a constant current $\vec{J}_{s0}$ deep in the interior of the superconductor, where $\vec{E} = \vec{E}_{0}$. From Eq. (49) we obtain using Eqs. (50) and (51):

$$\vec{v}_{s}(\vec{r}) - \vec{v}_{s0} = -\frac{c}{4\pi en_{s}\lambda_{L}}\vec{\sigma} \times (\vec{E}(\vec{r}) - \vec{E}_{0}(\vec{r}))$$

(52)

and taking the curl

$$\nabla \times (\vec{v}_{s}(\vec{r}) - \vec{v}_{s0}) = -\frac{c}{en_{s}}\frac{\rho(\vec{r}) - \rho_{0}}{\lambda_{L}}\vec{\sigma}$$

(53)

Eq. (53) is approximately satisfied by

$$\vec{v}_{s}(\vec{r}) = -\frac{c}{en_{s}}\rho(\vec{r})\vec{\sigma} \times \hat{r}$$

(54a)

with

$$\vec{v}_{s0} = \vec{v}_{s}(r << R) = -\frac{c}{en_{s}}\rho_{0}\vec{\sigma} \times \hat{r}$$

(54b)

since using these expressions we obtain

$$\nabla \times (\vec{v}_{s}(\vec{r}) - \vec{v}_{s0}) = -\frac{c}{en_{s}}\frac{\partial \rho(\vec{r})}{\partial r}\vec{\sigma} - \frac{c}{en_{s}}\frac{\rho(\vec{r}) - \rho_{0}}{r}\vec{\sigma}$$

(55)

The second term in Eq. (55) is zero in the deep interior and is smaller than the first term by a factor $\lambda_{L}/R$ near the surface, and the first term in Eq. (55) is approximately the same as Eq. (53).

Eq. (54) generalizes the relation found between charge density and velocity of the spin current carriers near the surface, Eq. (37), to the entire volume. The form Eq. (54) is only valid in the absence of applied magnetic field, when only the pure spin current exists ($\vec{v}_{\sigma=1} = -\vec{v}_{\sigma=-1}$). In the presence of both charge and spin current, we write instead of Eq. (54)

$$\vec{v}_{s}(\vec{r}) = -\frac{c}{en_{s}}\rho_{\sigma}(\vec{r})\vec{\sigma} \times \hat{r}$$

(56)

where (modifying the convention used in Sect. IV) we denote by $\rho_{\sigma}$ the total charge density for spin $\sigma$, produced by both the pure spin current and any superimposed charge current induced by an external magnetic field (this was called $\rho_{\sigma,tot}$ in Sect. IV), and $v_{\sigma}$ denotes what was called $v_{\sigma,tot}$ in Sect. IV Eq. (38).

In Sect. II we had argued that the spin current should die down as we move beyond a London penetration depth of the surface towards the interior, by analogy with the behavior of the charge current, and formulated the equations accordingly. They corresponded to taking $\vec{J}_{s0} = 0$ in Eq. (49). However, the finding of a general relation between charge density and current in Sect. IV led us to include here the $\vec{J}_{s0}$ term in the theory, representing a counterflowing spin current in the interior induced by the charge density $\rho_{0}$, smaller than the spin current near the surface by a factor $\lambda_{L}/R$.

It should be noted however that the addition of the constant term $\vec{J}_{s0}$ leads to a singularity in the vorticity in the deep interior: namely, for $r << R$

$$\nabla \times \vec{v}_{s}(\vec{r}) = -\frac{c}{en_{s}}\frac{\rho_{0}}{r}\vec{\sigma}$$

(57)

diverges as $r \rightarrow 0$. We return to this point in Sect. XII.

The spin current Eq. (51) expressed in terms of the carrier’s velocity Eq. (56) takes the form

$$\vec{J}_{s}(\vec{r}) = -\rho_{\sigma}(\vec{r})\vec{\sigma} \times \hat{r}$$

(58)

and denotes the total current carried by the carriers of spin $\sigma$, including the contribution from the spontaneous pure spin current and from the charge current induced by an applied magnetic field, if any. Eq. (58) is consistent with Eq. (43) deduced for the charge current only.

VI. SPILL-OVER

The addition of the constant term $\vec{J}_{s0}$ to the spin current constitutive relation has another interesting consequence. Consider the equation for the spin current

$$\vec{J}_{s}(\vec{r}) - \vec{J}_{s0} = -\frac{c}{8\pi\lambda_{L}}\vec{\sigma} \times [\vec{E}(\vec{r}) - \vec{E}_{0}(\vec{r})]$$

(59)
We argued earlier that at the surface \( \mathbf{E} = 0 \), \( \mathbf{E}_0 = E_m \hat{r} \), and the spin current velocity is given by the universal form Eq. (13). However, these arguments were based on the constitutive relation Eq. (17), that did not include the \( J_{s0} \) term. Eq. (59) no longer satisfies this condition. How is this inconsistency resolved?

Consider a cylinder of radius \( R \). The electric field \( \mathbf{E}_0 \) at the surface is

\[
\mathbf{E}_0(R) = 2\pi R \rho_0 \hat{n}
\]  (60)

Assuming the electric field \( \mathbf{E} \) vanishes at the surface, Eq. (59) yields

\[
\frac{e n_s}{2} \mathbf{E}_s(R) - \frac{e \rho_0}{2} \hat{\sigma} \times \hat{n} = \frac{c}{4\lambda_L} R \rho_0 (\hat{\sigma} \times \hat{n})
\]  (61)

Bringing the second term to the right side, we have

\[
\frac{e n_s}{2} \mathbf{E}_s(R) = \frac{c}{4\lambda_L} \rho_0 (R + 2\lambda_L) (\hat{\sigma} \times \hat{n})
\]  (62)

which is satisfied if we replace \( R \) by \( R + 2\lambda_L \) in Eq. (2a), i.e. if the ‘effective’ radius of the cylinder is assumed to be

\[
R_{eff} = R + 2\lambda_L
\]  (63)

instead of \( R \), which indicates that the negative superfluid ‘spills over’ a distance \( 2\lambda_L \) beyond the radius of the cylinder.

To obtain the magnitude of the charge that spilled over, note that at \( r = R \) the electric field \( E(R) \) will no longer be zero. To satisfy Eq. (56) with the spin current velocity Eq. (13) and \( E_0 = E_m \) (Eq. (33)) requires an electric field at \( R \) pointing outward, of magnitude

\[
E(R) = 4\pi \lambda_L \rho_0
\]  (64)

which corresponds to a sheet of surface charge density

\[
\sigma_{spill} = -\lambda_L \rho_0
\]  (65)

having spilled out beyond the radius \( R \). Being spread out over a radial distance \( 2\lambda_L \) beyond the surface, it corresponds to a spill-over volume charge density \( \sigma_{spill} / (2\lambda_L) \), i.e.

\[
\rho_{spill} = -\frac{\rho_0}{2}.
\]  (66)

Note also that because of this spill-over effect, the actual negative charge density near the surface of the superconductor is not \( \rho_- \), rather it is \( \rho_- + \rho_0 \). Hence upon taking the curl of Eq. (20) we need to use

\[
\nabla \cdot (\mathbf{E} - \mathbf{E}_0)_{r=R} = 4\pi ((\rho_- + \rho_0) - \rho_0) = 4\pi \rho_-
\]  (67)

This resolves the small discrepancy that we had found in Sect. II between the expressions for the unscreened spin-orbit magnetic field Eqs. (22) and Eq. (15a).

Note also that this scenario is consistent with the microscopic picture put forth in ref. [8], that electrons expand their wavefunctions effective radius from a lattice spacing to circular orbits of radius \( 2\lambda_L \) when a metal goes superconducting. For pairs that are right at the surface in the normal state, these orbits will expand to a distance \( 2\lambda_L \) beyond the body’s surface in the superconducting state.

For a sphere of radius \( R \), the total spilled charge is

\[
4\pi R^2 \sigma_{spill},
\]

which using Eq. (2b) and (34) corresponds to a number of spilled electrons

\[
N_{spill} = \frac{3hc}{16\pi e^2 \lambda_L R}
\]  (68)

For example, for \( \lambda_L = 400 \text{ Å} \) and \( R = 1 \text{ cm} \) this yields \( N_{spill} = 2 \times 10^6 \). We had already predicted earlier such a spill-over of negative charge beyond the surface of superconductors based on quite different arguments [2].

### VII. SPIN CURRENT ELECTRODYNAMICS

We can now simply extend these equations to describe the full space and time dependence of the currents and charge densities. In terms of the four-vector for spin-component \( \sigma \) the total current with or without an applied magnetic field is given by

\[
J_\sigma(\mathbf{r}, t) - J_{\sigma 0} = -\frac{c}{8\pi \lambda_L} (A_\sigma(\mathbf{r}, t) - A_{\sigma 0}(\mathbf{r}))
\]  (69)

with

\[
J_\sigma(\mathbf{r}, t) = (\vec{J}_\sigma(\mathbf{r}, t), ic \rho_\sigma(\mathbf{r}, t))
\]  (70a)

\[
J_{\sigma 0} = (\vec{J}_{\sigma 0}, ic \rho_{\sigma 0})
\]  (70b)

and

\[
A_\sigma(\mathbf{r}, t) = (\vec{A}_\sigma(\mathbf{r}, t), ic \phi_\sigma(\mathbf{r}, t))
\]  (71a)

\[
A_{\sigma 0}(\mathbf{r}) = (\vec{A}_{\sigma 0}(\mathbf{r}), ic \phi_{\sigma 0}(\mathbf{r}))
\]  (71b)

with

\[
\vec{A}_\sigma(\mathbf{r}, t) = \lambda_L \vec{\sigma} \times \vec{E}(\mathbf{r}, t) + \vec{A}(\mathbf{r}, t)
\]  (72a)

\[
\vec{A}_{\sigma 0}(\mathbf{r}) = \lambda_L \vec{\sigma} \times \vec{E}_0(\mathbf{r})
\]  (72b)

and

\[
\phi_\sigma(\mathbf{r}, t) = -\lambda_L \vec{\sigma} \cdot \vec{B}(\mathbf{r}, t) + \phi(\mathbf{r}, t)
\]  (73a)

\[
\phi_{\sigma 0}(\mathbf{r}) = \phi_0(\mathbf{r})
\]  (73b)

Here, \( \vec{A} \) and \( \phi \) are the magnetic vector potential and electric potential. \( \vec{E}_0 \) and \( \phi_0 \) are the electrostatic field.
and potential for a uniform charge density \( \rho_0 \) throughout the material. The spatial component of Eq. (69) is
\[
\vec{J}_\sigma(\vec{r}, t) - \vec{J}_\sigma(0) = -\frac{e}{8\pi \lambda L} (\lambda L \vec{\sigma} \times \vec{E}(\vec{r}, t) + \vec{A}(\vec{r}, t)) \tag{74a}
\]
and the fourth component is
\[
\rho_\sigma(\vec{r}, t) - \rho_\sigma(0) = \frac{1}{8\pi \lambda L} \vec{\sigma} \cdot \vec{B}(\vec{r}, t) - \frac{1}{8\pi \lambda L} (\phi(\vec{r}, t) - \phi(\vec{0})) \tag{74b}
\]
The continuity equation sets the four-dimensional divergence of the four-vector \( \vec{J}_\sigma \) equal to zero
\[
\text{Div} \vec{J}_\sigma = 0 \tag{75}
\]
with the fourth component of the divergence operator given by \( \partial / \partial (ict) \). The Lorentz gauge condition
\[
\vec{\nabla} \cdot \vec{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0 \tag{76}
\]
together with Faraday’s law \( \vec{\nabla} \times \vec{E} = (-1/c) \partial \vec{B} / \partial t \) ensures that the four-divergence of \( A_\sigma \) vanishes, consistent with Eq. (69). That \( A_\sigma \) is a four-vector can be seen from the fact that it can be written as
\[
(A_\sigma)_\alpha = \frac{i \lambda L}{2} \epsilon_{\alpha \beta \gamma \delta} F_{\gamma \delta} + A_\alpha \tag{77}
\]
with \( F_{\gamma \delta} \) the electromagnetic field tensor, \( A_\alpha \) the usual electromagnetic four-vector potential obeying the Lorentz gauge condition, and \( \epsilon_{\alpha \beta \gamma \delta} = +1 \ (-1) \) for even (odd) permutations of 1234 and zero otherwise.

The current 4-vectors are given in terms of the velocity of the superfluid charge density per spin \( en_s/2 \), the velocity for each spin component \( v_\sigma \), and the (excess) charge density \( \rho_\sigma \) as
\[
J_\sigma(\vec{r}, t) = \left( \frac{en_s}{2} \vec{v}_\sigma(\vec{r}, t), ic \rho_\sigma(\vec{r}, t) \right) \tag{78a}
\]
\[
J_{\sigma 0} = \left( \frac{en_s}{2} \vec{v}_{\sigma 0}, ic \rho_{\sigma 0} \right) \tag{78b}
\]
with \( \vec{v}_{\sigma 0} \) given by Eq. (51b) and \( \rho_{\sigma 0} = \rho_0/2 \). Using the relation (56), they can be written in the remarkable form
\[
J_\sigma(\vec{r}, t) = \rho_\sigma(\vec{r}, t)c(-\vec{\sigma} \times \vec{r}, i) \tag{79a}
\]
\[
J_{\sigma 0} = \frac{\rho_0 c}{2} (-\vec{\sigma} \times \vec{r}, i) \tag{79b}
\]
which says that the supercurrent density (charge or spin) at any point in the superconductor can be understood as arising from the excess local charge density moving at the speed of light.

The differential equations determining the behavior of all quantities are
\[
\Box^2 (A_\sigma - A_{\sigma 0}) = \frac{1}{\lambda L} (A_\sigma - A_{\sigma 0}) \tag{80a}
\]
with
\[
\Box^2 = \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \tag{81}
\]
and \( J_\sigma \) is given in terms of \( A_\sigma \) by Eq. (69). The equations for the charge sector only are simply obtained by defining the charge four-current and charge four-potential
\[
J_c = J_{\sigma = +1} + J_{\sigma = -1} \tag{82a}
\]
\[
A_c = (A_{\sigma = +1} + A_{\sigma = -1})/2 \tag{82b}
\]
and similarly for \( J_{\sigma 0} \) and \( A_{\sigma 0} \), and they obey
\[
J_c(\vec{r}, t) - J_{\sigma 0} = -\frac{e}{4\pi \lambda L} (A_c(\vec{r}, t) - A_{\sigma 0}(\vec{r})) \tag{83}
\]
and Eq. (83), which of course coincide with the equations derived in ref. [1] for the charge sector.

**VIII. SPIN CURRENT ELECTROSTATICS**

In the absence of time-dependence, the electric field in the interior of the superconductor is described by the differential equation Eq. (19), and the charge density obeys the differential equation Eq. (1). They are related by the equation
\[
\vec{E}(\vec{r}) - \vec{E}_0(\vec{r}) = 4\pi \lambda^2 \vec{\nabla} \rho(\vec{r}) \tag{84}
\]
For a cylinder, the solution to these equations is given in terms of modified Bessel functions \( I_\nu(z) \) (Bessel functions of imaginary argument) as
\[
\rho(r) = \rho_0 (1 - iR L \frac{I_0(i r/\lambda L)}{2 I_1(i r/\lambda L)})(r < R) \tag{85a}
\]
\[
\vec{E}(r) = 2\pi \rho_0 r [1 - \frac{R}{r} I_1(i r/\lambda L)] \tag{85b}
\]
for a sphere
\[
\rho(r) = \rho_0 (1 - \frac{1}{3} \frac{R^3}{\lambda^2 r^2} f(R/\lambda L)) \tag{86a}
\]
\[
\vec{E}(r) = \frac{4}{3} \pi \rho_0 r^2 [1 - \frac{R^3}{r^3} f(R/\lambda L)] \tag{86b}
\]
with \( f(x) = xcoshx - sinh x \), and for a plane
\[
\rho(r) = \rho_0 (1 - \frac{R}{\lambda L} \frac{cosh(r/\lambda L)}{sinh(R/\lambda L)}) \tag{87a}
\]
The magnitude of the spin current near the surface is given in terms of the electric field applied is larger than the critical field which points in the direction of the spin quantization axis. Then the up and down spin components of the total current are simply obtained by using the z axis as the spin quantization axis and adding the charge and spin current contributions. Similarly, a magnetic field applied parallel to a planar surface will polarize the spin current with the spin quantization axis parallel to the applied field.

Consequently, an external electric field that points in (antiparallel to \( E_0 \)) will enhance the spin current, and one that points out will suppress it. If the magnitude of the electric field applied is larger than the critical field \( E_m \) (eq. 33) it will completely suppress the spin current and hence destroy superconductivity.

If an electric current circulates along the axial direction of the cylinder, one expects that in the absence of applied electric field that above a critical current

\[
 J_{c1} = \frac{c}{4\pi} \frac{H_{c1}}{\lambda_L},
\]

vortices start to penetrate the superconductor [13]. If an electric field normal to the surface is applied, this expression will be modified to

\[
 J_{c1} = \frac{c}{4\pi} \frac{H_{c1}}{\lambda_L} \pm \frac{E_{ext}}{\lambda_L},
\]

and in the deep interior

\[
 J_{\sigma 0} = \frac{2\lambda_L}{R} J_{\sigma}(R) .
\]
magnetic field in direction and in direction to the averaged second derivative of the lattice potential we showed in Ref. [15] that the net force is proportional to an electric dipole [16, 17] magnetic moment equivalent to an electric dipole in ferromagnetic metals. An electron with magnetic moment pointing into the paper deflects in the opposite direction.

An azimuthal spin-orbit force originating in the positive magnetic moment pointing out of the paper, i.e. parallel to \( \vec{B} \), gives rise to the spin current, whether or not an external magnetic field is present.

The fact that a magnetic moment propagating in a periodic lattice experiences a net transverse force was proposed in Ref. [15] as an explanation for the anomalous Hall effect in ferromagnetic metals. An electron with magnetic moment \( \vec{\mu} \) propagating with velocity \( \vec{v} \) is equivalent to an electric dipole [16, 17]

\[
\vec{p} = \gamma \frac{\vec{v}}{c} \times \vec{\mu}
\]  

with \( \gamma = (1 - \nu^2/c^2)^{-1} \sim 1 \) for non-relativistic speeds. An electric dipole in an electric field experiences a force \( \vec{F} = \nabla (\vec{p} \cdot \vec{E}) \) and a torque \( \vec{\tau} = \vec{p} \times \vec{E} \). As shown in Fig. 2(a), for a discrete positive charge distribution the force acting when the electron passes through the line joining two ions is in direction \( \vec{\mu} \times \vec{v} \). Averaging over the unit cell we showed in Ref. [15] that the net force is proportional to the averaged second derivative of the lattice potential and in direction \( \vec{\mu} \times \vec{v} \), corresponding to an 'effective' magnetic field in direction parallel to \( \vec{\mu} \).

For the 'mesoscopic' scale under consideration \( (2\lambda_L) \) it is more appropriate to consider a uniform positive charge density, as shown in Fig. 2(b). The electric field points radially outward,

\[
\vec{E} = 2\pi \rho \vec{r}
\]  

with \( \rho \) the ionic charge density. The torque by the electric field acting on the equivalent electric dipole changes the angular momentum

\[
\frac{d\vec{L}}{dt} = m_e \frac{d}{dt} (\vec{p} \times \vec{v}) = \vec{\tau} = \left( \frac{\vec{v}}{c} \times \vec{\mu} \right) \times \vec{E}
\]  

hence

\[
m_e \vec{r} \times \frac{d\vec{v}}{dt} = -2\pi \rho \vec{r} \times \left( \frac{\vec{v}}{c} \times \vec{\mu} \right)
\]

which is equivalent to the effect of an 'effective' magnetic field in direction parallel to \( \vec{\mu} \)

\[
\vec{B}_\sigma = -2\pi \frac{\rho}{c} \vec{\mu}
\]

on the electron, also causing it to deflect in direction \( \vec{\mu} \times \vec{v} \), as shown in Fig. 2(b). Eq. (105) is the same as Eq. (11), since the superfluid of negative charge density \( e\sigma \) should respond to the positive ionic charge density \( \rho = -e\sigma \).

The corresponding 'unscreened' vector potential, defined by \( \nabla \times \vec{A}_\sigma = \vec{B}_\sigma \) is given by

\[
\vec{A}_e = \frac{\vec{B}_\sigma \times \vec{r}}{2} = -\pi \frac{\rho}{c} \vec{\mu} \times \vec{r}
\]

which yields

\[
\vec{A}_\sigma = -\frac{\hbar}{4m_e c} \sigma \times \vec{E}.
\]

for \( \vec{E} \) given by Eq. (103). The spin-orbit interaction term in the non-relativistic limit of the Dirac equation for an electron in an electric field \( \vec{E} \) is

\[
H_{s.o.} = -\frac{e\hbar}{4m_e c^2} \sigma \cdot (\vec{E} \times \vec{p}) = \frac{e}{m_e c} \vec{p} \cdot \vec{A}_\sigma
\]

where \( \vec{p} \) is the momentum operator and \( \vec{A}_\sigma \) is given by Eq. (107). Thus, we can understand the relation Eq. (18) between velocity and spin-orbit vector potential as arising from minimization of the energy-momentum relation that arises from the non-relativistic limit of the Dirac equation

\[
E(\vec{p}) = \frac{\vec{p}^2}{2m_e} + \frac{e}{m_e c} \vec{p} \cdot \vec{A}_\sigma
\]

with \( \vec{p} = m_e \vec{v}_\sigma \) and \( \vec{A}_\sigma \) given by Eq. (107).

It should be mentioned however that our conclusion is opposite to that of Chudnovsky [19] concerning the sign of the spin-orbit force acting on an electron moving in a positive background. Ref. [19] defines a spin-orbit vector potential as Eq. (107) with opposite sign, and concludes that the force is in opposite direction to that depicted
in Fig. 2. We argue that it is physically clear that the electron will deflect as shown in Fig. 2, so that the electron orbital angular momentum will point antiparallel to its spin angular momentum to minimize the spin-orbit energy, just like in atoms. The difference between our conclusion and that of ref. [19] can be accounted for by the momentum of the electromagnetic field, which was calculated by Aharonov et al as [20]

\[ P_{\text{field}} = \frac{1}{4\pi c} \int \vec{E} \times \vec{B} = \frac{1}{c} \vec{E} \times \vec{\mu} \quad (109) \]

and points in opposite direction to the mechanical momentum of the electron in Fig. (2) and is twice as large.

As electrons of opposite spin move radially outward a distance 2\(\lambda_L\) they acquire azimuthal velocity

\[ \vec{v}_\sigma = -\frac{e}{m_e c} \hat{A}_\sigma (r = 2\lambda_L) = -\frac{\hbar}{4 m_e \lambda_L^2} \hat{\sigma} \times \hat{r} \quad (13) \]

where we have used \(\rho = |e| n_s \hbar\) and Eq. (7).

X. PHYSICAL INTERPRETATION

There are three key elements underlying the physical picture considered here that relate to the theory of hole superconductivity [21]: (i) The wavefunction in the normal state has to be confined to short distances, which requires an almost full band (hole transport in the normal state); (ii) The carriers forming the superfluid are long wavelength electrons [9], with negative charge [22]; (iii) In quantum mechanics, expansion of the wave function is associated with lowering of kinetic energy, which is the driving force for hole superconductivity [22].

It is interesting to note that an early explanation of superconductivity proposed by J.C. Slater in 1937 [24] had some key elements in common with what is being proposed here. Slater proposed a model where “the wave functions correspond to electrons which can wander for some distance through the metal”, and showed that “to produce superconductivity the orbits must be of order of magnitude of 137 atomic diameters”.

Consider an external magnetic field \(\vec{H}\) applied to an electron orbiting with radius \(r\) in a plane perpendicular to \(\vec{H}\). Classically, in changing the field from \(H\) to \(H + dH\) the electron changes its speed by \(d\vec{v} = (e/m_e c)(r dH/2)\) due to Faraday’s law, and its orbital magnetic moment by \(d\mu = (er/2c)d\vec{v}\) antiparallel to the field, resulting in an energy increase \(dE = H d\mu\). When the field is increased from 0 to \(H\), the total energy increase is

\[ \Delta E = \int_0^H dH' \frac{e^2}{4 m_e c} r^2 H' = \frac{e^2}{8 m_e c^3} r^2 H^2 \quad (110) \]

The same result is obtained quantum-mechanically, with \(r^2\) replaced by the mean square radius in the plane perpendicular to \(\vec{H}\), \(r^2 \rightarrow < x^2 + y^2 >\). If there are \(n_s\) electrons per unit volume in such orbits, the magnetic susceptibility per unit volume is

\[ \chi = -n_s \frac{\partial^2 \Delta E}{\partial H^2} = -\frac{n_s e^2}{4m_e c^2} r^2 \quad (111) \]

and for orbits of radius \(r = 2\lambda_L\), as proposed here,

\[ \chi = -\frac{n_s e^2}{m_e c^2} \lambda_L^2 = -\frac{1}{4\pi} \quad (112) \]

using Eq. (7) for the London penetration depth. Eq. (112) is the condition for perfect diamagnetism. Instead, the Landau diamagnetic susceptibility in the normal state is given by Eq. (111) with \(r = k_F^{-1}\), with \(k_F\) the Fermi wavevector.

Similarly, Eq. (110) yields that the increase in energy per unit volume for \(n_s\) electrons per unit volume in orbits of radius \(r = 2\lambda_L\) is

\[ u \equiv n_s \Delta E = \frac{n_s e^2}{8 m_e c^3} (2\lambda_L)^2 H^2 = \frac{H^2}{8\pi} \quad (113) \]

again using Eq. (7), so that the system will remain in this state as long as the ‘condensation energy’ density of the state is greater than the energy cost \(H^2/8\pi\). This is of course the condition that defines the thermodynamic critical field \(H_{c1}\) [13].

Slater’s arguments [24] proceeded similarly. He pointed out that the radius of the orbit in the susceptibility expression Eq. (111) should be such that \(\chi = -1/4\pi\) for perfect diamagnetism; taking \(n_s = 1/d^3\), with \(d^3\) the volume per superconducting electron, and setting \(d = 2a_0\), with \(a_0 = \hbar^2/m_e c^2\) the Bohr radius, yields

\[ \frac{r^2}{d^2} = \frac{m_e c^2 d}{e^2 \pi} = \frac{2}{\pi} \left( \frac{\hbar c}{e^2} \right)^2 \quad (114) \]

hence Slater concluded that “the orbits must be of order of magnitude of 137 (= \hbar c/e^2) atomic diameters”.

Furthermore, Slater proposed as a criterion for the critical magnetic field that will destroy superconductivity to compare the Landau level spacing of energy levels for fixed \(z\)-component of momentum in an external field \(H\)

\[ \Delta E_{ll} = \frac{\hbar}{m_e c} H \quad (115) \]

to the spacing of discrete energy levels in a ‘box’ of size 137 atomic units. Instead, we argued in ref. [8] that superconductivity will be destroyed when the external magnetic field completely stops the spin current orbital motion of the electron with spin antiparallel to \(H\), i.e. for

\[ H = B_\sigma \quad (116) \]

with \(B_\sigma\) given by Eq. (15a). The Landau level energy spacing for such a magnetic field is

\[ \Delta E_{ll} = \hbar \omega_c = \frac{\hbar}{m_e c} B_\sigma = \frac{\hbar^2}{4m_e \lambda_L^2} \quad (117) \]
which is of the same order of magnitude as the energy level spacing of electrons confined to a region of size \( \sim 2\lambda_L \). Thus our criterion is essentially equivalent to Slater’s criterion. An equivalent form of this criterion is that superconductivity will be destroyed when the energy increase for electrons in a magnetic field \( H \) for orbits of radius \( r = 2\lambda_L \), Eq. (110), becomes comparable to the spacing of energy levels in a box of such size.

Of course Slater’s scenario left several questions unanswered that our present scenario addresses, namely: (i) what happens to those orbits when there is no external magnetic field applied? As we showed in [8], orbits with radius \( 2\lambda_L \) arise from the spin-orbit interaction under the constraint that the orbit should enclose precisely one flux quantum of spin orbit flux, so that the wave function is single-valued. And (ii) how do the orbits arise when a metal is cooled into the superconducting state, and how is the Meissner current generated? It is the process of expansion of the orbits from negligible radius (of order of a lattice spacing) to radius \( 2\lambda_L \), that generates dynamically the azimuthal velocities required.

It is also interesting to note that the physics of superconductivity discussed here provides a natural ‘classical’ explanation for the existence of macroscopic phase coherence in superconductors. The fact that electrons traverse overlapping orbits of radius \( 2\lambda_L \) implies that the orbits of many electrons cross the orbit of any given electron, so the motions needs to be synchronized: to avoid collisions the angular position of the i-th electron \( \theta_i(t) = \omega_i t + \theta_{i0} \) has to bear a definite relation with the angular positions of the electrons in the overlapping orbits, that persists over time. It is also not possible to change one orbit without affecting the synchronization with all the others, which gives an intuitive understanding to the notion of ‘rigidity’ of the superconducting state.[3]. Finally, it is interesting to note that several workers in the pre-BCS era proposed mechanisms of superconductivity based on the notion of ‘spontaneous currents’ that would exist in the superconductor in the absence of applied external fields.[22] However these were charge currents rather than spin currents.

**XI. ANGULAR MOMENTUM**

Within our model each electron in the superfluid moves in a circular orbit of radius \( 2\lambda_L \), with electron spin perpendicular to the plane of the orbit in direction antiparallel to the orbital angular momentum, and velocity given by Eq. (13). Hence the mechanical orbital angular momentum of the i-th electron is

\[
\vec{l}_{i,orb} = m_e (2\lambda_L) \hat{r} \times \vec{v}_\sigma = -\frac{\hbar}{2} \hat{\sigma} \tag{118}
\]

where the last equality follows from Eq. (13). The electron spin angular momentum is

\[
l_{i,spin} = \frac{\hbar}{2} \hat{\sigma} \tag{119}
\]

and consequently its total angular momentum (orbital plus spin) is exactly zero. Thus, the transition to superconductivity can be thought of as a “quenching” of the electron’s spin angular momentum by the development of an opposite-pointing orbital angular momentum of the same magnitude.

Let us consider the total mechanical angular momentum for electrons of spin \( \sigma \) in a superconducting cylinder of radius \( R \) and height \( h \), choosing as usual the spin quantization axis parallel to the cylinder axis. The total number of electrons with spin \( \sigma \) for superfluid density \( n_s \) in the entire volume \( V = \pi R^2 h \) is

\[
N_{vol,\sigma} = \frac{n_s}{2} V \tag{120}
\]

Hence the total mechanical angular momentum carried by electrons with spin \( \sigma \) is the product of Eq. (120) and (118):

\[
\vec{L}_{vol,\sigma} = N_{vol,\sigma} \vec{l}_{i,orb} = -n_s V m_e \lambda_L V \vec{\sigma} = -\frac{n_s}{2} V \frac{\hbar}{2} \vec{\sigma} \tag{121}
\]

On the other hand, the orbital angular momentum for an electron of spin \( \sigma \) in the surface layer of thickness \( \lambda_L \) is

\[
\vec{l}_{surf} = m_e R \hat{r} \times \vec{v}_\sigma = \frac{R}{2\lambda_L} \vec{l}_{i,orb} \tag{122}
\]

and the total number of electrons of spin \( \sigma \) in the surface layer of thickness \( \lambda_L \) is

\[
N_{surf,\sigma} = 2\pi R \lambda_L \frac{n_s}{2} V = \frac{2\lambda_L}{R} \frac{n_s}{2} V = \frac{2\lambda_L}{R} N_{vol,\sigma} \tag{123}
\]

so that the total mechanical angular momentum carried by the electrons of spin \( \sigma \) in the surface layer is

\[
\vec{L}_{surf,\sigma} = N_{surf,\sigma} \vec{l}_{surf} = N_{vol,\sigma} \vec{L}_{i,orb} = \vec{L}_{vol,\sigma}, \tag{124}
\]

the same as Eq. (121). So we may think of the total mechanical angular momentum in two equivalent ways: (i) each superfluid electron in the bulk carries mechanical angular momentum \( \hbar/2 \), or (ii) the superfluid electrons in the surface layer of thickness \( \lambda_L \) each carry a mechanical angular momentum \( R \hbar/4 \lambda_L \) (resulting from their orbital motion with speed \( v_\sigma \) and radius \( R \)), and the electrons in the interior carry no mechanical angular momentum. The total mechanical angular momentum of the super-fluid electrons of each spin component in the surface layer equals in magnitude the total spin angular momentum of the superfluid electrons of that spin component in the entire volume.

If we cool the system into the superconducting state in the presence of an applied magnetic field \( \vec{B} \), the electrons of both spin orientations acquire an extra contribution to their orbital angular momentum as they expand to their orbits of radius \( 2\lambda_L \). The azimuthal velocity acquired is

\[
\vec{v}_\phi = \frac{e\lambda_L}{m_e c} \hat{r} \times \vec{B} \tag{125}
\]
due to the action of the Lorentz force as the orbits expand to radius $2\lambda_L$. $\tilde{v}_\phi$ increases the speed of the electrons with $\tilde{\mu}$ parallel to $\vec{B}$, and decreases the speed of the electrons with $\tilde{\mu}$ antiparallel to $\vec{B}$. The extra angular momentum per electron due to the magnetic field is

$$\vec{I}_B = m_e(2\lambda_L)\vec{r} \times \vec{v}_\phi = -\frac{2e\lambda_L^2}{c}\vec{B}$$  \hspace{1cm} (126)

and the total extra angular momentum acquired by the electrons is obtained by multiplying $\vec{I}_B$ by the total number of electrons:

$$\vec{L}_e = -\frac{2eN_e\lambda^2}{c}\vec{B} = -\frac{m_e e R^2 h}{2e} \vec{B}$$  \hspace{1cm} (127)

as expected, since

$$\vec{L}_e = \frac{2m_e c}{e} \vec{m}$$  \hspace{1cm} (128)

for electrons, with $\vec{m} = \pi R^2 h \vec{M}$ the induced magnetic moment and $\vec{M} = -\vec{B}/4\pi$ the required magnetization to cancel the magnetic field in the interior. Hence, just like for the angular momentum in the spin current, the angular momentum in the Meissner current in the surface layer can be interpreted as arising from the orbits of radius $2\lambda_L$ of each electron in the bulk.

How is this electronic angular momentum compensated? We raised this puzzle in ref. [20]: in the conventional theory of superconductivity the charge distribution is homogeneous, so there is no angular momentum in the electromagnetic field. The electrons in the Meissner current carry mechanical angular momentum $\vec{L}_e$, but there is no mechanism in the conventional theory to generate a compensating angular momentum of the ions.

This conundrum is naturally resolved in the present scenario. Recall that in the absence of magnetic field the $2\lambda_L$ circular orbits arise from the interaction of the magnetic moment moving outward with the positive ionic lattice. The physics is illustrated schematically in Fig. 3. The ions exert a torque on the electron

$$\vec{\tau}_{ie} = \vec{p} \times \vec{E}_i$$  \hspace{1cm} (129)

where $\vec{E}_i$ is the ionic electric field and $\vec{p}$ is the equivalent electric dipole Eq. (102). As the ions exert the torque on the electrons the ionic lattice is subject, by Newton’s third law, to an equal and opposite torque:

$$\vec{\tau}_{ei} = -\vec{\tau}_{ie}$$  \hspace{1cm} (130)

In the absence of applied magnetic field, $\vec{\tau}_{ei}$ is equal and opposite from electrons with spin up and spin down, hence there is no net torque on the ions. However, in the presence of $\vec{B}$ the velocity of both spin electrons is modified according to Eq. (125), and hence the net $\vec{\tau}_{ei}$ is no longer zero: as the electrons acquire the extra angular momentum Eq. (126), the extra torque on the ions points opposite to the direction of $\vec{B}$ and generates an equal and opposite angular momentum for the lattice, $\vec{L}_{ions} = -\vec{I}_B$ per electron, and a total ionic angular momentum

$$\vec{L}_i = -\vec{L}_e$$  \hspace{1cm} (131)

so it would appear that the total angular momentum is conserved.

However that is not quite the whole story. Because the resulting charge distribution is inhomogeneous, there will also be some angular momentum stored in the electromagnetic field:

$$\vec{L}_{field} = \frac{1}{4\pi c} \int d^3 r' \vec{r} \times (\vec{E} \times \vec{B})$$  \hspace{1cm} (132)

which we can estimate as

$$\vec{L}_{field} = -\frac{V\lambda_L}{2\pi c} E_m \vec{B}$$  \hspace{1cm} (133)

since the region where both electric and magnetic fields are nonzero is only the surface layer of thickness $\sim \lambda_L$. Using Eqs. (130), (124), (33) and (7)

$$L_{field} = \frac{\hbar}{8m_e c \lambda_L} L_e = \frac{1}{8\pi} \left(\frac{\lambda}{\lambda_L}\right) L_e \sim 10^{-6} L_e$$  \hspace{1cm} (134)
with \( \lambda_c = h/m_e c \) the Compton wavelength. Hence we conclude that the spin-orbit interaction can account for 99.9999% of the angular momentum conservation puzzle[20] through Eq. (131), but we are still missing 0.0001%!

This 'missing' angular momentum is of course the tiny bit of extra electronic angular momentum that is acquired by the electrons in \( \rho_\rightarrow \) that moved outward cutting magnetic field lines near the surface in the process. The change in azimuthal velocity of an electron near the surface moving outward a distance \( \lambda_L \) in a magnetic field \( B \) due to the Lorentz force is

\[
\Delta v = \frac{e}{m_e c} \Delta A = \frac{e}{m_e c} \frac{\Delta \phi}{2 \pi R} = \frac{e\lambda_L B}{m_e c} \tag{135}
\]

and the acquired angular momentum is

\[
\vec{l} = -\frac{e}{c} R \lambda_L \vec{B} \tag{136}
\]

parallel to the magnetic field. The number of electrons that moved out to the region within \( \lambda_L \) of the surface and in the process cut through magnetic field lines can be estimated as

\[
N_{\rho_\rightarrow, \text{out}} = \frac{\rho_\rightarrow}{e} 2 \pi R \lambda_L h \tag{137}
\]

hence the mechanical angular momentum gained by the outflowing \( \rho_\rightarrow \) charge is

\[
\vec{l}'_e = N_{\rho_\rightarrow, \text{out}} \vec{v} = -\frac{2 \rho_\rightarrow}{e} \frac{V \lambda_L^2}{c} \vec{B} = \frac{V \lambda_L}{2 \pi c} E_m \vec{B} \tag{138}
\]

which is equal and opposite to the angular momentum of the field Eq. (133). (Note that Eq. (2a) was used in deriving Eq. (138)). Consequently, angular momentum is conserved when a metal is cooled into the superconducting state in the presence of a magnetic field, since the total angular momentum above \( T_c \) is zero, and from Eqs. (131), (133) and (138)

\[
\vec{l}_e + \vec{l}'_e + \vec{l}_{\text{field}} + \vec{l}_i = 0. \tag{139}
\]

\[ \text{FIG. 4: Type I versus type II materials.} \]

\[ \xi = 2\lambda_L \]

XII. TYPE I VERSUS TYPE II MATERIALS

In a type II superconductor, magnetic flux penetrates the body for \( H_c > H_{c1} \) and divides itself into filaments, each carrying a flux quantum \( \phi_0 \). In the vortex core of size \( \xi \), the 'coherence length', the system is normal. Instead, in a type I superconductor the flux is excluded until the applied field reaches the value \( H = H_c \), the thermodynamic critical field, and for \( H > H_c \) the entire material becomes normal. The thermodynamic critical field is given by

\[
H_c = \sqrt{\frac{\beta}{2 \pi}} \frac{h c}{\lambda_L} \sim H_{c1} \frac{\lambda_L}{\xi_0} \tag{140}
\]

with \( \xi_0 \) the Pippard-BCS coherence length[13]. A similar relation holds at finite temperatures between the Ginzburg Landau coherence length \( \xi \) and the penetration depth. We will not distinguish here between \( \xi \) and \( \xi_0 \). For type II superconductors, \( \xi < \lambda_L \), hence \( H_{c1} < H_c \).

We propose the following interpretation of type I versus type II behavior, depicted in Fig. 4. Up and down spin electrons have orbits of radius \( 2\lambda_L \), but the orbits coincide only in the extreme type II limit where \( \lambda_L >> \xi \). \( \xi \) represents the (average) distance between the centers of the orbits of spin up and spin down components of the Cooper pair.

In a type I superconductor (Fig. 4(a)), \( \xi > \lambda_L \) and the orbits of \( \uparrow \) and \( \downarrow \) electrons are disjoint. Because orbits are time-reversed partners, a magnetic field cannot thread one of the orbits and not the other. Instead, in a type II material with \( \xi << \lambda_L \) (Fig. 4(b)) magnetic flux can be enclosed by both orbits simultaneously. When
FIG. 5: Type I versus type II materials in a magnetic field (schematic). (a) The wavy lines connect the centers of the two members of a Cooper pair. In the intermediate state, normal regions exist with laminar or other shapes (shaded regions) of size larger than \( \xi \) where Cooper pairs are destroyed and the magnetic field is \( H_c \). (b) \( \xi \ll \lambda_L \) and the orbits of the two members of a Cooper pair almost overlap. The vortex cores (shaded regions) are within the orbits of the two members of a Cooper pair and have magnetic field \( H_{c1} \) when vortices don’t overlap.

the magnetic field first enters a type II superconductor, each orbit will enclose a single flux quantum \( \phi_0 \) and the magnetic field at the center of the vortex is \( H_{c1} \). In the mixed state, at the core of radius \( \xi \) = distance between the centers of the orbits, the magnetic field can have any value between \( H_{c1} \) and \( H_{c2} \) and there will be several vortex cores within the orbits of a Cooper pair with the total flux threaded by each orbit an integer multiple of \( \phi_0 \). The cross-over between type I and type II regimes occurs for \( \xi = 2\lambda_L \) (Fig. (4c)), where the overlapping part of the orbits can just enclose the vortex core. When the centers of the orbits are at distance \( \xi > 2\lambda_L \) (type I), the magnetic flux will destroy the Cooper pairs altogether, but depending on the sample shape there can be intertwined regions of normal and superconducting phases (intermediate state).

Fig. 5 shows schematically type I and type II superconductors in a magnetic field. In type I superconductors with non-zero demagnetizing factor there will be normal regions usually of laminar shape intertwined with the superconducting regions. The uncompensated orbits in the neighborhood of the normal regions and vortices will give rise to spin current and excess negative charge similar to the behavior near the surface.

This interpretation also provides a rationale for the form of the critical field Eq. (140). For \( \xi > \lambda_L \), the ‘confinement’ region for an electron in the Cooper pair is \( \xi \) rather than \( 2\lambda_L \). Following Slater’s reasoning\(^\text{[24]}\), we equate the diamagnetic energy cost for the electron in an orbit of radius \( 2\lambda_L \) in the presence of an external field \( H \) to the spacing of energy levels in a box of length \( \xi \):

\[
\Delta E_H = \frac{e^2}{8m_ec^2} (2\lambda_L)^2 H^2 \sim \frac{\hbar^2}{2m_e \xi^2} \quad (141)
\]

and obtain

\[
H \sim \frac{\hbar c}{e\lambda_L \xi} \quad (142)
\]

which is essentially the thermodynamic critical field Eq. (140). Thus, in a type I material superconductivity is not destroyed when the magnetic field stops the spin current\(^\text{[8]}\) (since \( H_{c1} > H_c \) here) but rather when the diamagnetic energy cost is large enough that the perturbation significantly mixes the unperturbed energy levels, i.e. disturbs the wave function so that it is no longer ‘rigid’.

What is the expelled charge density \( \rho_- \) in type I materials? The conclusion that \( E_m \) is given by Eq. (33) (\( E_m \sim H_{c1} \)) is not valid because the electrodynamic response is non-local. Furthermore, the electrostatic energy cost Eq. (44) would be much larger than the condensation energy in this case. The amount of \( \rho_- \) ‘needed’ to sustain the Meissner current up to \( H = H_c \) is, according to Eq. (41), simply \( \rho_- = -H_c/4\pi \), so we conclude that for type I materials

\[
E_m = H_c \quad (143a)
\]

\[
\rho_- = -\frac{H_c}{4\pi\lambda_L} = \sqrt{\frac{3}{2}} \frac{1}{4\pi^2} \frac{\hbar c}{e\lambda_L^2 \xi_0} \quad (143b)
\]

rather than Eqs. (33) and (34) applicable to type II materials. Because of the non-locality we expect the spin current density near the surface in type I materials will no longer be given by Eq. (16), but rather

\[
\vec{J}_\sigma = -\frac{\rho_- c}{2} \vec{\sigma} \times \hat{r} \quad (144)
\]

In type I materials Eq. (144) is smaller than Eq. (16) by a factor \( \lambda_L/\xi \).

Next we address the fact that disorder is known to increase the value of \( \lambda_L \) and turn a type I into a type II material. This can be easily understood in the picture discussed here. In the presence of defects that weaken or destroy superconductivity locally, the superconductor will expel negative charge towards those regions, thus depleting the magnitude of \( \rho_- \) near the outer surface and decreasing the ability of the superconductor to shield external magnetic fields. These internal weak regions (grain boundaries, defects, etc) will have excess negative charge and accompanying spin current orbiting around them, as shown schematically in Fig. 6. If we cool this superconductor in the presence of a pre-existent magnetic field, it will be unable to expel the magnetic field from those regions, thus resulting in an incomplete Meissner effect.

This picture also suggests that in extreme type II materials there will be an inhomogeneous charge distribution and spin current distribution arising from inhomogeneities and disorder also in the absence of applied magnetic field. Such inhomogeneities are observed in the underdoped regime of high \( T_c \) materials\(^\text{[27]}\), which are
expected to be in the extreme type II limit (largest \( \lambda_L \), smallest \( \xi \))\(^{23} \). The existence of spin currents associated with regions of charge inhomogeneity predicted by our model has not yet been experimentally tested.

Finally, recall that we argued in Sect. V that the equations for the spin current spatial dependence should include the constant term \( J_{\sigma \alpha} \), giving rise to a singularity in the deep interior. Presumably this implies that even in the absence of disorder at least one vortex, i.e. a normal region of size \( \xi \) surrounded by spin current, has to exist in the deep interior of any type II superconductor. This would imply that any type II superconductor is topologically a torus of genus larger or equal to 1 even in the absence of applied magnetic field, hence that a Meissner effect with 100% flux expulsion can never be attained in a type II superconductor no matter how small the applied external field: at the minimum one trapped magnetic flux quantum will always remain in the interior.

FIG. 6: Schematic depiction of a superconductor with strong disorder in the absence of applied magnetic field. Defects, grain boundaries, vacancies, etc. will result in patches of normal regions (hatched areas) surrounded by spin currents (dashed lines, with arrows pointing in the direction of flow of electrons with magnetic moment pointing out of the paper) and excess negative charge density (gray areas). The figure also shows the excess negative charge and spin current near the surface and the spill-over beyond the surface (denoted by the full line). The equivalent electric dipoles (Eq. (102)) point outward near the outer surface and towards the normal regions in the interior. If the system is cooled in the presence of a magnetic field, magnetic flux will be trapped in the hatched regions and a charge current will flow around those regions together with the depicted spin currents. The smallest normal regions have diameter of a coherence length and enclose one flux quantum of spin-orbit flux in the absence of applied magnetic field. .

XIII. SUPERCONDUCTING MATERIAL PARAMETERS

From experimentally measured values of the penetration depth and critical fields we can infer the magnitude of the maximum internal electric field \( E_m \), surface charge density \( \rho_- \), velocity of the spin current carriers at the surface, and the spin current density flowing near the surface. We use the expressions:

\[
H_{c1} = \frac{\hbar c}{4e\lambda_L^2} = 1.64 \times 10^8 \frac{Gauss}{\lambda_L(\AA)^2} \]  \((145)\)

\[
E_m = H_{c1}(G) \times 300 V/cm \]  \((146a)\)

for type II materials (\( H_{c1} < H_c \)) and

\[
E_m = H_c(G) \times 300 V/cm \]  \((146b)\)

for type I materials.

\[
n_s = \frac{m_e c^2}{4\pi e^2 \lambda_L} = \frac{2824 \text{ electrons}}{\lambda_L(\AA)^2 A^3} \]  \((147a)\)

or

\[
n_s(\text{electrons/ion}) = n_s \frac{\bar{A}}{0.602 \rho(\text{gr/cm}^3)} \]  \((147b)\)

with \( \bar{A} \) the average mass number of the compound,

\[
\rho_- = \frac{E_m}{4\pi \lambda_L e} = 5.524 \times 10^{-11} \frac{E_m(\text{V/cm}) \text{electrons}}{\lambda_L(\AA)} \times \frac{A^3}{\text{cm}^3} \]  \((148)\)

\[
v_\sigma = \frac{\hbar}{4m_e \lambda_L} = 2.896 \times 10^7 \frac{\text{cm/s}}{\lambda_L(\AA)} \]  \((149)\)

\[
J_{\text{spin}} = J_{\sigma=+1} - J_{\sigma=-1} = \rho_- c \]

\[
= \frac{\rho_- (el)}{A^3} \times 4.802 \times 10^{15} \text{Amps/cm}^2 \]  \((150)\)

We list the values for a variety of materials including type I and type II superconductors in Table I. From this table we learn that materials with the largest values of \( \rho_- \), \( E_m \) and \( J_{\text{spin}} \) are \( Nb \), \( Pb \) and \( Hg \). They are close to the cross-over between type I and type II regimes and have relatively high \( T_c \). Hence these materials should be favored in experiments aimed at detecting this unconventional physics. Instead, the higher \( T_c \) materials known to date (last entries in Table I and similar) are strongly type II and the magnitude of the spin current and associated quantities is substantially smaller.
TABLE I: Properties of superconducting elements and compounds. The superfluid carrier density \( n_s \) is extracted from the measured penetration depth \( \lambda_L \) using expression Eq. (7), so it corresponds to what is usually interpreted as \( n_s m_e/m^* \), with \( m^* \) the effective mass. The spin current density \( J_{\text{spin}} \) is defined by Eq. (144) and includes the contribution from both spin components.

| Material     | \( T_c (K) \) | \( \lambda_L (\AA) \) | \( H_{c1} (G) \) | \( H_{c2} (G) \) | \( n_s (e/\AA^3) \) | \( n_e (e/\AA^3) \) | \( 10^6 \rho_{\sigma}/cn \) | \( E_m (V/cm) \) | \( v_s (cm/s) \) | \( J_{\text{spin}} (10^6 A/cm^2) \) |
|--------------|----------------|------------------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| Cd          | 0.56           | 1300                   | 30             | 97             | 0.00167        | 0.0361         | 0.23           | 9.000         | 22.275         | 1.83           |
| Zn          | 0.875          | 290                    | 53             | 1955           | 0.0336         | 0.510          | 0.909          | 15.900        | 99.855         | 14.6           |
| Al          | 1.14           | 500                    | 105            | 658            | 0.0113         | 0.188          | 0.31           | 31.500        | 57.916         | 16.7           |
| In          | 3.40           | 640                    | 293            | 401            | 0.0069         | 0.180          | 1.10           | 87.900        | 45.247         | 36.4           |
| Sn          | 3.72           | 510                    | 309            | 632            | 0.0109         | 0.293          | 0.92           | 92.700        | 56.780         | 48.0           |
| Hg          | 4.15           | 410                    | 412            | 978            | 0.0168         | 0.412          | 0.99           | 123.600       | 70.620         | 80.2           |
| Pb          | 7.19           | 390                    | 803            | 1080           | 0.0186         | 0.561          | 1.83           | 240.900       | 74.251         | 163.7          |
| Nb          | 9.50           | 400                    | 1980           | 1028           | 0.0176         | 0.324          | 2.41           | 308.400       | 72.395         | 205            |
| MgB\(_2\)   | 39.2           | 1800                   | 2600           | 51             | 0.00087        | 0.0084         | 0.54           | 15.300        | 16.088         | 2.26           |
| La\(_{0.85}\)Sr\(_{1.15}\)Cu\(_2\)O\(_4\) | 37.3           | 2500                   | 2370           | 26             | 0.00045        | 0.0071         | 0.38           | 7.800         | 11.583         | 0.826          |
| YBa\(_2\)Cu\(_3\)O\(_7\) | 90             | 1500                   | 6450           | 73             | 0.0013         | 0.017          | 0.62           | 21.900        | 19.305         | 3.88           |

XIV. SUMMARY AND DISCUSSION

In this paper we have extended the electrodynamic equations for the charge sector of superconductors proposed in Ref. [4] to describe the electrodynamics of the spin sector, based on the proposal of ref. [8] that a spontaneous macroscopic spin current flows within a London penetration depth of the surface of superconductors, in the absence of applied external fields, with carrier speed given by the universal form Eq. (13). We have shown here that the space and time dependence of the spin current can be described within the same four-dimensional framework developed earlier to describe the charge electrodynamics within the theory of hole superconductivity, which predicts that negative charge is expelled from the interior to the surface when a metal goes superconducting. The extension of the theory to describe the spin sector is necessitated by the fact that an interior electric field exists in superconductors within this theory. Instead, in the conventional London theory there is no electric field in the interior of superconductors, hence no separate electrodynamic description of the spin sector is required.

The formalism led us to the determination of the magnitude of the expelled negative charge, which had been left undetermined in our earlier work. Remarkably, the maximum electric field in the superconductor resulting from the charge expulsion was found to have identical magnitude as the spin-orbit field that gives rise to the spin current, and to have similar magnitude, i.e. a spin-orbit interaction, as the wavefunction expands from a small orbit to a mesoscopic orbital in opposite direction with orbital angular momentum antiparallel to their spin angular momentum and of the same magnitude, \( h/2 \). The state arises from the interaction of the electron with the positive ionic background, i.e. a spin-orbit interaction, as the wavefunction expands in the transition to the superconducting state to lower the kinetic energy of confinement of the antibonding electrons at the top of a nearly filled band. As shown in ref. [8], the expansion from a small orbit to a mesoscopic orbit of radius \( 2\lambda_L \) provides a dynamical explanation for the Meissner effect as well as for the dynamical origin of the spin current.

Furthermore, the total mechanical angular momentum carried by each component of the spin current in the surface layer was found to be identical to the aggregate sum of the spin angular momenta of that spin component of the superfluid electrons in the entire volume. This follows from the fact that the orbital angular momentum for each electron in its orbit of radius \( 2\lambda_L \) is \( h/2 \). It implies that the entire superconductor is a single giant Cooper pair, with its giant spin current components reflecting as well as quenching the aggregate sum of the superfluid electron spins.
We also found that our equations predict that a significant amount of negative charge spills out of the superconductor. This effect was anticipated in our earlier work, and we believe it may play an important role in the proximity effect. It also clearly illustrates the tendency of superconductors to get rid of their excess negative charge (antibonding electrons at the top of the Fermi distribution) predicted by the theory of hole superconductivity, which is also reflected in the predicted tunneling asymmetry of universal sign (larger tunneling current for negatively biased superconductor).

The fact that in the proposed scenario spin-orbit coupling is an essential ingredient of superconductivity also provides a natural explanation for the mechanism by which the ionic lattice picks up the ‘missing’ mechanical angular momentum when a metal is cooled into the superconducting state in the presence of a magnetic field. Conventional BCS-London theory neither provides an explanation for how the electrons in the Meissner current acquire their mechanical angular momentum, nor for how the ions acquire a compensating angular momentum in the opposite direction. In our earlier attempt to explain the angular momentum conservation puzzle we could not account for a complete Meissner effect precisely because the spin-orbit interaction mechanism to transfer angular momentum to the lattice was not included. For a 99% Meissner effect, we had to assume a charge expulsion at least 3 orders of magnitude larger than the values discussed here (Eq. (27) of ref. [20]).

Finally, we discussed a new ‘geometric’ interpretation of type II superconducting regimes and of the effect of disorder in superconductors based on the present model, and we proposed that simply connected type II superconductors cannot exist according to this theory.

Note however that we have assumed at various points in this paper that the size of the superconductor is much larger than $\lambda_L$ (e.g. Eqs. (2) and (3)). For superconducting samples of dimensions comparable or smaller than the penetration depth we expect the fundamental equations of Sect. VII to remain valid, but several features will change: in particular, the spin current at the surface will be smaller than the universal form Eq. (13), and the maximum electric field $E_m$ will be smaller than the value given by Eq. (33) [3]. Note also that our treatment assumed local electrodynamics and non-local corrections will be important in strongly type I superconductors.

The hypothesis that electrons in superconductors reside in orbits of diameter hundreds of lattice spacings was originally proposed by Slater [24]. We have shown that the radius of these ‘Slater orbits’ has to be precisely $2\lambda_L$ to give rise to perfect diamagnetism. Indeed, as argued by Slater, large orbits provide a simple and compelling explanation for why superconductors cannot tolerate the presence of a magnetic field in their interior: it simply costs too much energy, proportional to the area subtended by these mesoscopic orbits. Type I superconductors have only two ways to deal with this: either expel the magnetic field from their interior, paying the associated electromagnetic energy price, or if the price is too high, become normal, collapsing the mesoscopic orbits to microscopic Landau orbits of radius $k_F^{-1}$ ~ lattice spacing and pay the associated kinetic energy price. Type II superconductors have a third way: enclose within the mesoscopic electron orbits tubes of magnetic flux where the system is normal. Type II superconductors are able to do this because the two members of a Cooper pair have the centers of their $2\lambda_L$ orbits sufficiently close to each other that they can enclose within them the same flux tube.

Experiments should be able to detect the existence of the predicted spin current near the surface of superconductors. We have given numerical estimates for the magnitude of the spin current and other associated quantities for ‘conventional’ and other superconductors. The spin current density can be as large as $2 \times 10^8$ Amps/cm$^2$, and should be detectable in inelastic neutron scattering experiments with very cold neutrons [30]. The excess charge near the surface as large as 1 electron per $10^6$ ions and the internal electric field as large as 300,000V/cm should be experimentally detectable. It should be pointed out that the calculation of electric fields in the neighborhood of non-spherical samples discussed earlier [31] needs to be modified in light of the results in this paper; this will be discussed elsewhere. Experimental verification of these predictions will support the basic tenets of the theory of hole superconductivity: that superconductivity originates in the fundamental charge asymmetry of condensed matter, that it occurs when a metal has “too many electrons” (an almost full band, small de Broglie wavelength for the carriers at the Fermi energy, and with higher $T_c$ for anions [32]), and that it is driven by kinetic energy lowering and involves dressed holes (antibonding electrons) turning into undressed electrons. Further development of the microscopic theory [7, 21] will be discussed in forthcoming work.

Acknowledgments

A helpful discussion with B. Grinstein is gratefully acknowledged.

[1] J.E. Hirsch, Phys.Rev.B 62, 14498 (2000); Phys.Lett.A 281, 44 (2001).
[2] J.E. Hirsch, Phys.Lett. A 309, 457 (2003).
[3] J.E. Hirsch, Phys.Rev.B 68, 184502 (2003).
[4] J.E. Hirsch, Phys.Rev.B 69, 214515 (2004).
[5] F. London, ‘Superfluids’, Dover, New York, 1961.
[6] J.E. Hirsch, Phys.Rev.B 71, 184521 (2005).
[7] J.E. Hirsch and F. Marsiglio, Phys.Rev.B 39, 11515.
[8] J.E. Hirsch, arXiv:0710.0876 (2007), Europhys. Lett. 81, 67003 (2008).

[9] J.E. Hirsch, Phys.Rev. B71, 104522 (2005), Sect. IX.

[10] M. von Laue, “Theory of Superconductivity”, Chpt. 10, Academic Press, New York, 1952.

[11] Y. Aharonov and A. Casher, Phys.Rev.Lett. 53, 319 (1984).

[12] J. Anandan, Phys.Lett. A 138, 347 (1989); C.R. Hagen, Phys.Rev.Lett. 64, 2347 (1990); S.Oh, C.M. Ryu and S.H.S. Salk, Phys.Rev.A50, 5320 (1994).

[13] M. Tinkham, “Introduction to Superconductivity”, 2nd ed, McGraw Hill, New York, 1996.

[14] J.E. Hirsch, Phys. Rev. Lett. 94, 187001 (2005).

[15] J.E. Hirsch, Phys.Rev.B60, 14787 (1999); see also arXiv:0709.1280 (2007).

[16] T.H. Boyer, Phys.Rev. A 36, 5083 (1987).

[17] J.E. Hirsch, Phys.Rev.B 42, 4774 (1990).

[18] J.D. Bjorken and S.D. Drell, “Relativistic Quantum Mechanics”, Mc Graw Hill, New York, 1964, Chpt. 4.

[19] E.M. Chudnovsky, arXiv: 0709.0725 (2007).

[20] Y. Aharonov, P. Pearle and L. Vaidman, Phys. Rev. A37, 4052 (1988).

[21] J.E. Hirsch, Jour. Phys. Chem. Solids 67, 21 (2006) and references therein.

[22] J.E. Hirsch, Int.J.Mod.Phys. B17, 3236 (2003).

[23] J.E. Hirsch and F. Marsiglio, Phys.Rev.B 62, 15131 (2000); J.E. Hirsch, Physica C 199, 305 (1992); Physica C 201, 347 (1992).

[24] J.C. Slater, Phys.Rev. 52, 214 (1937).

[25] L.D. Landau, Sow.Phys. 4, 43 (1933); W. Heisenberg, Zs. f. Naturf. 2a, 185 (1947); M. Born and K.C. Cheng, Nature 161, 968 (1948); see also references in W. L. Ginsburg, Fortschrritte der Physik 1, 88 (1953).

[26] J.E. Hirsch, Phys.Lett.A 366, 615 (2007).

[27] S.H. Pan et al, Nature 413, 282 (2001).

[28] J.E. Hirsch and F. Marsiglio, Phys.Rev.B 45, 4807 (1992).

[29] F. Marsiglio and J.E. Hirsch, Physica C 159, 157 (1989).

[30] We would like to thank S. Sinha and T. Egami for discussions on this point.

[31] J.E. Hirsch, Phys.Rev.Lett. 92, 016402 (2004).

[32] J.E. Hirsch, in "Studies of High Temperature Superconductors", ed. by A. Narlikar, Nova Sci. Pub., New York, Vol. 38, p. 49 (2002).