Webs of $\mathcal{W}$-algebras

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Plan

- $\mathcal{W}_3$, construction of $\mathcal{W}$-algebras
- $\mathcal{W}_\infty$ as Yangian of affine $u(1)$
- Gaiotto-Rapčák gauge theory construction
- Gluing with examples
$\mathcal{W}$-algebras: extensions of the Virasoro algebra by higher spin currents - appear in many different contexts:

- integrable hierarchies of PDE (KdV, KP)
- (old) matrix models
- instanton partition functions and AGT
- algebras of BPS states in 4d gauge theories
- holographic dual description of 3d higher spin gravities
- quantum Hall effect
- topological strings

Recent: Gaiotto&Rapčák: trivalent junction of D5-NS5-(1,1) bc in Kapustin-Witten twist of 4d $\mathcal{N} = 4$ SYM
Zamolodchikov $\mathcal{W}_3$ algebra

As in illustration, the $\mathcal{W}_3$ algebra constructed by Zamolodchikov (1984) has a stress-energy tensor (Virasoro algebra) with OPE

$$T(z) T(w) \sim \frac{c/2}{(z - w)^4} + \frac{2 T(w)}{(z - w)^2} + \frac{\partial T(w)}{z - w} + \text{reg.}.$$ 

together with spin 3 primary field $W(z)$

$$T(z) W(w) \sim \frac{3W(w)}{(z - w)^2} + \frac{\partial W(w)}{z - w} + \text{reg.}.$$ 

We need to find the OPE of $W$ with itself such that the resulting chiral algebra is associative.
The result:

\[ W(z)W(w) \sim \frac{c/3}{(z - w)^6} + \frac{2T(w)}{(z - w)^4} + \frac{\partial T(w)}{(z - w)^3} \]

\[ + \frac{1}{(z - w)^2} \left( \frac{32}{5c + 22} \Lambda(w) + \frac{3}{10} \partial^2 T(w) \right) \]

\[ + \frac{1}{z - w} \left( \frac{16}{5c + 22} \partial \Lambda(w) + \frac{1}{15} \partial^3 T(w) \right) \]

\( \Lambda \) is a quasiprimary ‘composite’ (spin 4) field,

\[ \Lambda(z) = (TT)(z) - \frac{3}{10} \partial^2 T(z). \]

The algebra is non-linear, not a Lie algebra in the usual sense (in fact linearity should not be expected for spins \( \geq 3 \)).
Construction of $\mathcal{W}$-algebras

Many different ways of constructing $\mathcal{W}$-algebras:

- solving the associativity conditions for a given spin content; e.g. Zamolodchikov $\mathcal{W}_3$, Gaberdiel-Gopakumar $\mathcal{W}_\infty$
- affine Lie algebra $\rightsquigarrow$ Casimir subalgebra $\widehat{sl(N)}_1/sl(N)$ or more generally GKO coset
- Hamiltonian reduction via BRST: Drinfeld-Sokolov reduction; e.g. $\mathcal{W}_4^{(2)}$ from $\widehat{su}(4)$

$$\begin{pmatrix}
\ast & 1 & 0 & 0 \\
\ast & \ast & 1 & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & 0 & \ast
\end{pmatrix}$$

- free field constructions - Miura transformation
Construction #1: (Gaberdiel-Gopakumar) solving the associativity conditions (crossing relations)

- assuming the spin content 2, 3, 4, ... a two-parametric family of algebras is found: \( \mathcal{W}_\infty[\lambda, c] \)
- specializing \( \lambda \to N \rightsquigarrow \) truncation to \( \mathcal{W}_N[c] \)
- in this sense \( \mathcal{W}_\infty \) interpolates between all \( \mathcal{W}_N \)
- triality symmetry: for each value of \( c \) three solutions \( \lambda_j \) giving the same algebra \( \mathcal{W}_\infty[\lambda_1, c] \simeq \mathcal{W}_\infty[\lambda_2, c] \simeq \mathcal{W}_\infty[\lambda_3, c] \),

\[
\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = 0 \quad (\lambda_1 - 1)(\lambda_2 - 1)(\lambda_3 - 1) = c
\]

- the vacuum character is the MacMahon function (counting three-dimensional partitions)
Construction #2: (Maulik-Okounkov) R-matrix, spin chain

- Miura transform (free field representation, oper...) for \( \mathcal{W}_N \)

\[
(\alpha_0 \partial + J_1(z)) \cdots (\alpha_0 \partial + J_N(z)) = \sum_{k=0}^{N} U_k(z)(\alpha_0 \partial)^{N-k}
\]

where \( J_j(z) \) are free bosons \( J_j(z)J_k(w) \sim \delta_{jk}(z - w)^{-2} \)

- Intertwiner between different embeddings

\[
R_{12}(\alpha_0 \partial + J_1(z))(\alpha_0 \partial + J_2(z)) = (\alpha_0 \partial + J_2(z))(\alpha_0 \partial + J_1(z))R_{12}
\]

satisfies the Yang-Baxter equation

\[
R_{12}(u_1-u_2)R_{13}(u_1-u_3)R_{23}(u_2-u_3) = R_{23}(u_2-u_3)R_{13}(u_1-u_3)R_{12}(u_1-u_2)
\]

\( \leadsto \) algebraic Bethe ansatz
Construction #3 (Yangian): (Schiffmann-Vasserot, Tsymbaliuk)

an associative algebra with generators $\psi_j, e_j, f_j, j = 0, 1, \ldots$ and

relations of the form

\[
0 = [\psi_j, \psi_k] \\
\psi_{j+k} = [e_j, f_k] \\
0 = [e_{j+3}, e_k] - 3[e_{j+2}, e_{k+1}] + 3[e_{j+1}, e_{k+2}] - [e_j, e_{k+3}] \\
+ \sigma_2 [e_{j+1}, e_k] - \sigma_2 [e_j, e_{k+1}] - \sigma_3 \{e_j, e_k\} \\
0 = \text{Sym}_{(j_1,j_2,j_3)} [e_{j_1}, [e_{j_2}, e_{j_3+1}]]
\]

\[
\ldots
\]

- integrability - an infinite set of commuting charges $\psi_j$
- representations on plane partitions: $e_j/f_j$ are box addition/removal operators
- non-locally related to previous constructions
characters of maximally degenerate $\mathcal{W}_{1+\infty}$ reps are counting plane partitions with given Young diagram asymptotics

$L_0$ level ↔ number of boxes

this is exactly what the topological vertex of topological strings is counting

$\rightsquigarrow$ characters, conformal dimensions, ... can be determined combinatorially

truncation to integer $\lambda$ by restricting the height
for special values of parameters the algebra can be truncated

truncations labeled by triple of non-negative integers

\[ \frac{N_1}{\lambda_1} + \frac{N_2}{\lambda_2} + \frac{N_3}{\lambda_3} = 1 \]

the first singular vector corresponds to a box at \((N_1 + 1, N_2 + 1, N_3 + 1)\)

all the \(\mathcal{W}_N\) minimal models correspond to intersections of two curves of this type, i.e. Ising \(c=1/2\): \((0, 0, 2) \cap (2, 1, 0)\)
Lee-Yang \(c = -22/5\): \((2, 0, 0) \cap (0, 3, 0)\)...
Brane construction (Gaiotto-Rapčák)

trivalent junction of codim 1 interfaces in the Kapustin-Witten twist of $\mathcal{N} = 4$ SYM $\rightsquigarrow$ vertex operator algebra at the corner

- Mikhaylov-Witten: degrees of freedom associated to $(k, 1)$ interface: $U(N|L)_{\psi+k}$ Chern-Simons theory
- Chern-Simons theory at 2d boundary produces vertex operator algebra
• 5-branes meeting at 2 dimensional subspace $\leadsto$ gluing of $U(N|L)_\psi \times U(M|L)_{\psi+1}$ VOA; bc: Gaiotto-Witten

• Gaiotto-Rapčák proposal (for $N > M$):

$$Y_{LMN}[\psi] = \frac{W_{N-M}^{DS}[U(N|L)\psi]}{U(M|L)_{\psi-1}}$$

1. Start with $U(N|L)_\psi \times U(M|L)_{1-\psi}$
2. Drinfeld-Sokolov reduction in $U(N - M) \subset U(N)$ (BRST)
3. Perform BRST coset, gluing $U(M|L) \subset U(N|L)$ with $U(M|L)$

• comparing vacuum characters, special choices of $N$ as well as large $N$ limits, one concludes that these $Y_{LMN}$ can be identified with the truncations of $\mathcal{W}_{1+\infty}$ discussed above
We can now consider more complicated interfaces and their associated VOAs:

- **vertices**: degrees of freedom at junction described by $Y_{LMN}$ algebras
- **internal edges**: degrees of freedom associated to line operators along the interfaces - bi-modules

The resulting VOA is obtained by conformally extending $Y_{LMN}$ algebras (vertices) by bi-modules associated to internal edges $\rightsquigarrow$ the total central charge is a sum of individual central charges (no contribution from the line ops) and can be read off directly from the diagram.
Vertex and edge

- five-brane charge conservation at vertex $\sum_j (p_j, q_j) = 0$
- parameters of the vertex algebra are given by $(\Psi \equiv -\epsilon_2/\epsilon_1)$

$$\lambda_j = \frac{\sum_k (N_k p_k \epsilon_1 + N_k q_k \epsilon_2)}{p_j \epsilon_1 + q_j \epsilon_2}$$

- the conformal dimension of $\Box$ is (half-)integral (independent of $\Psi$) and the statistics of the gluing matter (boson/fermion) depends on the relative orientation of the two vertices
Example - $\mathcal{N} = 2$ superconformal algebra

An extension of Virasoro algebra by spin 1 current $J$ and two charge $\pm 1$ fermionic spin $\frac{3}{2}$ supercurrents $G^\pm$

$$J(z)J(w) \sim \frac{c}{3(z-w)^2}$$

$$G^+(z)G^-(w) \sim \frac{2c}{3(z-3)^3} + \frac{2J(w)}{(z-w)^2} + \frac{2T(w) + \partial J(w)}{z-w}$$

$$G^\pm(z)G^\pm(w) \sim \text{reg.}$$

The vacuum character (generic $c$)

$$\chi = \prod_{n=0}^{\infty} \frac{(1 - zq^{3/2+n})(1 - z^{-1}q^{3/2+n})}{(1 - q^{1+n})(1 - q^{2+n})}$$

is up to an $U(1)$ factor exactly what one gets from the gluing
Example - Bershadsky-Polyakov $\mathcal{W}_3^{(2)}$

- An extension of Virasoro algebra by spin 1 current $J$ and two charge $\pm 1$ bosonic spin $\frac{3}{2}$ currents $G^\pm$
- The same spin content as $\mathcal{N} = 2$ SCA but different statistics of the gluing fields
- Can be obtained from $U(3)$ Kac-Moody by DS reduction using the non-principal embedding $3 = 2 + 1$
\( \mathcal{N} = 2 \mathcal{W}_\infty \)

- \( \mathcal{N} = 2 \mathcal{W}_\infty \) symmetry algebra of Kazama-Suzuki coset models is glued from two copies of bosonic \( \mathcal{W}_\infty \)

- the central charge can be decomposed as

\[
c = \frac{c(\mu + 1)(c + 6\mu - 3)}{3(c + 3\mu)^2} - \frac{(c - 3\mu)(c(\mu - 2) - 3\mu)}{3(c + 3\mu^2)} + 1
\]

  corresponding to central charges of the two \( \mathcal{W}_\infty \) vertices

- the symmetry of the diagram is \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) which is also the duality symmetry of the algebra

- the four basic minimal representations correspond to four external legs
Other examples and questions

\[ \hat{U}(3) \]

\[ \hat{U}(4|2) \]

more general DS reduction
Further progress

- free field representations of $Y_{N_1, N_2, N_3}$ by modification of Miura transformation $\rightsquigarrow$ free field representations of a very large class of algebras by gluing
- even spin $\mathcal{W}_\infty$ (orthosymplectic) a subalgebra of $\mathcal{W}_{1+\infty}$
- $W$-algebras associated to $A_n, B_n, C_n, D_n, E_6, E_7, E_8, G_2$ subalgebras of $u(1) \times \mathcal{W}_{n-1}$
- $\mathcal{W}[F_4]$? Grassmannian cosets? gluing of $N = 3$ and $N = 4$ SCA? higher spin square?
- Yangian description of even $\mathcal{W}_\infty$?
Thank you!