SUPPORTS, REGULARITY, AND $\boxplus$-INFINITE DIVISIBILITY FOR MEASURES OF THE FORM $(\mu^{\boxplus p})^{\boxplus q}$

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Abstract. Let $\mathcal{M}$ be the set of Borel probability measures on $\mathbb{R}$. We denote by $\mu^{ac}$ the absolutely continuous part of $\mu \in \mathcal{M}$. The purpose of this paper is to investigate the supports and regularity for measures of the form $(\mu^{\boxplus p})^{\boxplus q}$, $\mu \in \mathcal{M}$, where $\boxplus$ and $\boxplus$ are the operations of free additive and Boolean convolution on $\mathcal{M}$, respectively, and $p \geq 1$, $q > 0$. We show that for any $q$ the supports of $(\mu^{\boxplus p})^{\boxplus q}$ and $(\mu^{\boxplus p})^{\boxplus q}$ contain the same number of components and this number is a decreasing function of $p$. Explicit formulas for the densities of $(\mu^{\boxplus p})^{\boxplus q}$ and criteria for determining the atoms of $(\mu^{\boxplus p})^{\boxplus q}$ are given. Based on the subordination functions of free convolution powers, we give another point of view to analyze the set of $\boxplus$-infinitely divisible measures and provide explicit expressions for their Voiculescu transforms in terms of free and Boolean convolutions.

1. Introduction

For measures $\mu$ and $\nu$ in $\mathcal{M}$, the measure $\mu \boxplus \nu$ is the free (additive) convolution of $\mu$ and $\nu$. Thus, $\mu \boxplus \nu$ is the distribution of $X + Y$, where $X$ and $Y$ are free random variables with distributions $\mu$ and $\nu$, respectively. Denote by $\phi_\mu$ the Voiculescu transform of $\mu$ which satisfies the identity $\phi_{\mu \boxplus \nu} = \phi_\mu + \phi_\nu$ in some truncated cone in the upper half-plane $\mathbb{C}^+$. For $n \in \mathbb{N}$, the $n$-fold free convolution $\mu \boxplus \cdots \boxplus \mu$ is denoted by $\mu^{\boxplus n}$. It was shown in [21] that the discrete semigroup $\{\mu^{\boxplus n} : n \in \mathbb{N}\}$ can be embedded in a continuous family $\{\mu^{\boxplus p} : p \geq 1\}$ which satisfies $\mu^{\boxplus p_1} \boxplus \mu^{\boxplus p_2} = \mu^{\boxplus (p_1 + p_2)}$, $p_1, p_2 \geq 1$. Any measure in this family satisfies $\phi_{\mu^{\boxplus p}} = p\phi_\mu$ in some truncated cone in $\mathbb{C}^+$. We refer the reader to [31] and [10] for complete developments on the existence of this continuous family. In the full generalization, Belinschi and Bercovici used the subordination function to construct the measure $\mu^{\boxplus p}$, $p > 1$, and obtained certain regularity properties. In [15], an explicit formula for the density of $(\mu^{\boxplus p})^{\boxplus q}$ was provided and the relation between the supports of $\mu$ and $\mu^{\boxplus p}$ was analyzed. As a consequence, the number $n(p)$ of components in the support of $\mu^{\boxplus p}$ was shown to be a decreasing function of $p$.

An important class of measures in $\mathcal{M}$ is the set of $\boxplus$-infinitely divisible measures $\mu$. Recall that $\mu$ is $\boxplus$-infinitely divisible if for any $n \in \mathbb{N}$ there exists a measure $\mu_n \in \mathcal{M}$ such that $\mu^{\boxplus n} = \mu$. Another operation of convolution is the Boolean convolution $\boxplus$ introduced by Speicher and Woroudi [22]. The connection among free, Boolean, and classical infinite divisibilities was thoroughly studied by Bercovici.

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and Pata. An aspect of this connection between infinite divisibility with respect to $\boxplus$ and $\uplus$ is the Boolean Ber covici-Pata bijection $\boxtimes$.

Another map $\mathbb{B}_t : \mathcal{M} \to \mathcal{M}$ connecting free and Boolean convolutions is defined by

$$\mathbb{B}_t(\mu) = \left(\mu^{\boxplus(t+1)}\right)^{\uplus 1}, \quad t \geq 0, \ \mu \in \mathcal{M}.$$  

This map introduced by Belinschi and Nica satisfies $\mathbb{B}_t \circ \mathbb{B}_s = \mathbb{B}_{t+s}$, $s, t \geq 0$. More importantly, the map $\mathbb{B}_1$ coincides with the Boolean Ber covici-Pata bijection $\boxtimes$. As a result, $\mathbb{B}_t(\mu)$ is $\boxplus$-infinitely divisible for any $\mu \in \mathcal{M}$ and $t \geq 1$. This led the authors to associate to each measure $\mu \in \mathcal{M}$ a nonnegative number $\text{Ind}(\mu)$, which is called $\boxplus$-divisibility indicator. For instance, the semicircular and Cauchy distributions have $\boxplus$-divisibility indicators $1$ and $\infty$, respectively. It was also shown that $\mu$ is $\boxplus$-infinitely divisible if and only if $\text{Ind}(\mu) > 1$. For any measure $\mu \in \mathcal{M}$ with mean zero and unit variance, denote by $\Phi(\mu)$ the unique measure in $\mathcal{M}$ such that $E_{\mu} = G_{\Phi(\mu)}$. Recall that the free Brownian motion started at $\nu \in \mathcal{M}$ is the process $\{\nu \boxplus \gamma_t : t > 0\}$, where $\gamma_t$ is the centered semicircular distribution of variance $t$. The connection between this process and the map $\mathbb{B}_1$ is via the identity $E_{\mathbb{B}_1(\mu)} = G_{\Phi(\mu)^{\boxplus \gamma_1}}$. These authors also studied the regularity of measures in $\mathbb{B}_1(\mathcal{M})$. In $\mathcal{B}_l$, the same authors studied the map $\mathbb{B}_t$ on the space $D_t(k)$ of distributions of $k$-tuples of self-adjoint elements in a $C^*$-probability space based on moments and combinatorics. As in $\mathcal{B}_1$, they showed that $\mathbb{B}_1$ is the multi-variable Boolean Ber covici-Pata bijection and investigated the relation between $\mathbb{B}_t$ and free Brownian motion. Later, for measures $\mu, \nu \in D_t(k)$, Nica studied the so-called subordination distribution of $\mu \boxplus \nu$ with respect to $\nu$, in which a property related to the present paper is that $(\mu^{\boxplus p})^{\boxplus (p-1)/p}$ is $\boxplus$-infinitely divisible for any $p > 1$. For other further developments on $\mathbb{B}_1$ and the $\boxplus$-divisibility indicator of the measure $(\mu^{\boxplus p})^{\boxplus q}$, we refer the reader to $\cite{2}$.

In the present paper, we mainly use the subordination functions for the $\boxplus$-convolution powers to study $\boxplus$-infinitely divisible measures. We show that measures of the form $(\mu^{\boxplus p})^{\boxplus q}$ are $\boxplus$-infinitely divisible for $\mu \in \mathcal{M}$, $p > 1$, and $0 < q \leq (p - 1)/p$. We also provide explicit formulas for the Voiculescu transforms of $\boxplus$-infinitely divisible measures, particularly, the compound free Poisson distribution with the rate $\lambda$ and jump distribution $\nu \in \mathcal{M}$ is shown to be of the form $(\nu^{\boxplus (\lambda+1)})^{\boxplus \lambda/(\lambda+1)}$, $\nu_1 \in \mathcal{M}$, and its $\boxplus$-divisibility indicator is calculated as well. In the study of the measures with mean zero and finite variance $\sigma^2$, we reformulate their $\boxplus$-divisibility indicators in terms of free Brownian motion:

$$\text{Ind}(\mu) = \sup \{t \geq 0 : E_{\mu} = \sigma^2 G_{\nu_1 \boxplus \gamma_t} \text{ for some } \nu_1 \in \mathcal{M}\}.$$  

As a consequence of this reformulation, a measure $\nu \in \mathcal{M}$ can be written as $\nu_1 \boxplus \gamma_t$ for some $\nu_1 \in \mathcal{M}$ and $t > 0$ if and only if $\text{Ind}(\Phi^{-1}(\nu)) > 0$. Moreover, we have $\text{Ind}(\mu) > 1$ if and only if $\phi_\mu = \sigma^2 G_{\nu \boxplus \gamma_t}$ for some $\nu \in \mathcal{M}$ and $t > 0$. The work $\cite{4}$ provides solid foundations for the current research and leads us to investigate the supports and regularity for the measures $(\mu^{\boxplus p})^{\boxplus q}$, $p \geq 1$, $q > 0$. We prove that the nonatomic parts of this type of measure are absolutely continuous and the densities are analytic wherever they are positive. More importantly, the number of components in the support of $(\mu^{\boxplus p})^{\boxplus q}$ is independent of $q$ and a decreasing function of $p$. Particularly, $(\mu^{\boxplus p})^{\boxplus q}$ contains the same number of components in the support for any $q > 0$ provided that $\text{Ind}(\mu) > 0$. 

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The paper is organized as follows. Section 2 contains definitions and basic facts in free probability theory. Section 3 provides complete descriptions about the connections among free, Boolean convolutions, and \( \text{III}- \) infinitely divisible measures. Section 4 investigates the set of \( \text{III} - \) infinitely divisible measures with mean zero and finite variance. Section 5 contains results about the supports and regularity for the measures \( (\mu^{\text{III}p})^{\text{III}q} \), where \( p \geq 1 \) and \( q > 0 \).

2. Preliminary

For any complex number \( z \) in \( \mathbb{C} \), let \( \Re z \) and \( \Im z \) be the real and imaginary parts of \( z \), respectively. Denote by \( \mathbb{C}^+ = \{ z \in \mathbb{C} : \Re z > 0 \} \) the complex upper half-plane. Consider the set \( \mathcal{G} \) defined as

\[
\mathcal{G} = \left\{ G \in \mathbb{C}^+ \rightarrow \mathbb{C}^- \text{ is analytic and } \lim_{y \to \infty} iyG(iy) = 1 \right\}.
\]

It is known that a function \( G \) is in \( \mathcal{G} \) if and only if there exists some measure \( \mu \in \mathcal{M} \) such that \( G \) can be written as

\[
G(z) = G_\mu(z) := \int_\mathbb{R} \frac{1}{z-s} \, d\mu(s), \quad z \in \mathbb{C}^+.
\]

The function \( G_\mu \) is called the Cauchy transform of \( \mu \). The measure \( \mu \) can be recovered from \( G_\mu \) as the weak limit of the measures

\[
d\mu_\epsilon(s) = -\frac{1}{\pi} \Im G_\mu(s + i\epsilon) \, ds
\]

as \( \epsilon \to 0^+ \). This is the Stieltjes inversion formula. Particularly, if \( \Im G \) extends continuously to an open interval containing some point \( x \in \mathbb{R} \) then the density of the absolutely continuous part of \( \mu \) at \( x \) is given by \( -\Im G(x)/\pi \).

Another class of functions which is closely related to \( \mathcal{G} \) and plays a significant role in free probability theory is the following set

\[
\mathcal{F} = \left\{ F \in \mathbb{C}^+ \rightarrow \mathbb{C}^+ \text{ is analytic and } \lim_{y \to \infty} \frac{F(iy)}{iy} = 1 \right\}.
\]

A function \( F \) belongs to \( \mathcal{F} \) if and only if \( F = F_\mu := 1/G_\mu \) for some \( \mu \in \mathcal{M} \). The function \( F_\mu \) is called the reciprocal Cauchy transform of \( \mu \). Any function \( F \in \mathcal{F} \) has the property \( \Im F(z) \geq \Im z \) for \( z \in \mathbb{C}^+ \) and has a Nevanlinna representation of the form

\[
F(z) = \Re F(i) + z + \int_\mathbb{R} \frac{1+sz}{s-z} \, d\rho(s),
\]

where \( \rho \) is some finite positive Borel measure on \( \mathbb{R} \). Moreover, the function \( F \) has a right inverse \( F_\mu^{-1} \) with respect to composition, which is defined on the truncated cone

\[
\Gamma_{\alpha,\beta} = \{ x + iy \in \mathbb{C}^+ : |x| \leq \alpha y, \ |y| \geq \beta \}
\]

of the upper half-plane for some \( \alpha, \beta > 0 \). The function \( \phi_\mu : \Gamma_{\alpha,\beta} \rightarrow \mathbb{C}^- \cup \mathbb{R} \) defined by

\[
\phi_\mu(z) = F_\mu^{-1}(z) - z, \quad z \in \Gamma_{\alpha,\beta},
\]

is called the Voiculescu transform of \( \mu \). As indicated in the introduction, for \( \mu, \nu \in \mathcal{M} \) and \( z \) in some truncated cone in \( \mathbb{C}^+ \) the following identity holds:

\[
\phi_{\mu \boxplus \nu}(z) = \phi_{\mu}(z) + \phi_{\nu}(z).
\]

Particularly, the identity \( F_{\mu \boxplus \delta_a}(z) = F_\mu(z - a) \) holds for \( z \in \mathbb{C}^+ \) and \( a \in \mathbb{R} \).
The reciprocal Cauchy transform $F_\mu$ can be used to locate the atoms of $\mu$. A point $\alpha$ is an atom of $\mu$ if and only if $F_\mu(\alpha) = 0$ (that is, $F_\mu$ is defined and takes the value 0 at the point $\alpha$) and the Julia-Carathéodory derivative $F'_\mu(\alpha)$ (which is the limit of $\frac{F_\mu(z) - F_\mu(\alpha)}{z - \alpha}$ as $z \to \alpha$ nontangentially, i.e., $(\Re z - \alpha)/\Im z$ stays bounded and $z \in \mathbb{C}^+$) is finite, in which case $\mu(\{\alpha\}) = 1/F'_\mu(\alpha)$.

Given any measure $\mu \in \mathcal{M}$, the function $E_\mu(z) = z - F_\mu(z)$ is called the energy function associated with $\mu$ and belongs to the following set

$$E = \left\{ E : \mathbb{C}^+ \to \mathbb{C}^- \cup \mathbb{R} \text{ is analytic and } \lim_{y \to \infty} \frac{E(iy)}{iy} = 0 \right\}.$$

Conversely, any function $E$ in $E$ is the energy function of some $\mu \in \mathcal{M}$ whose Nevanlinna representation is given by

$$E(z) = \Re E(i) + \int_{\mathbb{R}} \frac{1 + s z}{z - s} \, d\mu(s),$$

where $\rho$ is some finite positive Borel measure on $\mathbb{R}$. Observe that we have the inclusion $G \subseteq E$. Indeed, for any measure $\mu \in \mathcal{M}$ it was proved in [19] that $\mu$ has mean zero and finite variance $\sigma^2$, i.e.,

$$\int_{\mathbb{R}} s \, d\mu(s) = 0 \quad \text{and} \quad \int_{\mathbb{R}} s^2 \, d\mu(s) = \sigma^2$$

if and only if there exists some unique $\nu \in \mathcal{M}$ such that

$$E_\mu = \sigma^2 G_\nu.$$ 

If $\sigma^2 = 1$, let $\Phi(\mu)$ be the unique measure satisfying $E_\mu = G_{\Phi(\mu)}$. The Eq. (2.3) particularly shows that $\mu^{1/\sigma^2}$ has mean zero and unit variance, i.e., $E_\mu = \sigma^2 G_{\Phi(\mu^{1/\sigma^2})}$.

Next, consider the set

$$\mathcal{H} = \left\{ H : H : \mathbb{C}^+ \to \mathbb{C} \text{ is analytic, } \Im H(z) \leq \Im z, \ z \in \mathbb{C}^+, \ \text{and } \lim_{y \to \infty} \frac{H(iy)}{iy} = 1 \right\},$$

which plays an important role in the investigation of the free convolution powers of measures in $\mathcal{M}$. Indeed, for any $H \in \mathcal{H}$ the function $2z - H(z) \in \mathcal{F}$ is the reciprocal Cauchy transform of some measure in $\mathcal{M}$. More importantly, the right inverses of the functions in $\mathcal{H}$ can be used to construct the $p$-th $\boxplus$-convolution power $\mu^\boxplus_p$, $p \geq 1$, of any measure $\mu \in \mathcal{M}$. We list below the properties needed in this paper. For more details, we refer the reader to [34] and [18].

**Proposition 2.1.** For any $\mu \in \mathcal{M}$ and $p > 1$, define the function

$$H_p(z) = pz + (1 - p)F_\mu(z), \quad z \in \mathbb{C}^+,$$

the set $\Omega_p = \{ z \in \mathbb{C}^+ : \Im H_p(z) > 0 \}$, and the function $f_\mu : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ as

$$f_\mu(x) = \int_{\mathbb{R}} \frac{s^2 + 1}{(s - x)^2} \, d\rho(s), \quad x \in \mathbb{R},$$

where $\rho$ is the measure in the Nevanlinna representation (2.2) of $F_\mu$. 

The function $H_p$ is in $\mathcal{H}$ and the set $\Omega_p$ is a simply connected domain whose boundary is the graph of the continuous function $f_p : \mathbb{R} \to [0, \infty)$, where

$$f_p(x) = \inf \left\{ y > 0 : \frac{\Im E_\mu(x + iy)}{y} > \frac{-1}{p-1} \right\}, \quad x \in \mathbb{R}. \quad (1)$$

For $x \in \mathbb{R}$, $f_p(x) = 0$ if and only if $f_\mu(x) \leq 1/(p-1)$, while $z \in \Omega_p$ if and only if

$$\int_{\mathbb{R}} \frac{s^2 + 1}{|s - z|^2} \, d\rho(s) < \frac{1}{p-1}. \quad (2)$$

Consequently, the functions $E_\mu$ and $H_p$ have continuous extensions to $\overline{\Omega_p}$ which are Lipschitz continuous with Lipschitz constants $1/(p-1)$ and 2, respectively. Moreover, Eq. (2.2) holds for $z \in \Omega_p$.

There exists an analytic function $\omega_p : \mathbb{C}^+ \to \mathbb{C}^+$ extending continuously to $\mathbb{C}^+ \cup \mathbb{R}$ such that $H_p(\omega_p(z)) = z$ holds for $z \in \mathbb{C}^+ \cup \mathbb{R}$. Consequently, $\Omega_p = \omega_p(\mathbb{C}^+)$, $\omega_p(H_p(z)) = z$ holds for $z \in \Omega_p$, and

$$\frac{|z_1 - z_2|}{2} \leq |\omega_p(z_1) - \omega_p(z_2)|, \quad z_1, z_2 \in \mathbb{C}^+ \cup \mathbb{R}. \quad (3)$$

The function $\omega_p$ is analytic in a neighborhood of $x$ wherever $\omega_p(x) \not\in \mathbb{R}$.

Let $\mu^\mathbb{H}_p$ be the unique measure in $\mathcal{M}$ whose reciprocal Cauchy transform satisfies

$$F_{\mu^\mathbb{H}_p}(z) = \frac{p\omega_p(z) - z}{p-1}, \quad z \in \mathbb{C}^+. \quad (2.6)$$

Then there exist some $\alpha, \beta > 0$ such that

$$\phi_{\mu^\mathbb{H}_p}(z) = p\phi_p(z), \quad z \in \Gamma_{\alpha, \beta}. \quad (2.7)$$

Moreover, the function $\omega_p$ is the subordination function of $\mu^\mathbb{H}_p$ with respect to $\mu$, i.e.,

$$F_{\mu^\mathbb{H}_p}(z) = F_\mu(\omega_p(z)), \quad z \in \mathbb{C}^+ \cup \mathbb{R},$$

and consequently

$$F_{\mu^\mathbb{H}_p}(H_p(z)) = F_\mu(z), \quad z \in \overline{\Omega_p}. \quad (2.8)$$

Complete characterizations of the supports of $\mu^\mathbb{H}_p$ were given in [18]. Following the notations in Proposition 2.1 we give below the results needed in the current research.

**Theorem 2.2.** For $\mu \in \mathcal{M}$, define the function $\psi_p : \mathbb{R} \to \mathbb{R}$ by $\psi_p(x) = H_p(x + i\phi_p(x))$, $x \in \mathbb{R}$, and the set $V_p^+ = \{x \in \mathbb{R} : f_p(x) > 0\}$. Then the following statements are true.

1. The function $\psi_p$ is a homeomorphism on $\mathbb{R}$.
2. The measure $(\mu^\mathbb{H}_p)^{ac}$ is concentrated on the set $\psi_p(V_p^+)$ with density

$$\frac{d(\mu^\mathbb{H}_p)^{ac}}{dx}(\psi_p(x)) = \frac{(p-1)p\phi_p(x)}{\pi|x - \psi_p(x) + ip\phi_p(x)|}, \quad x \in V_p^+. \quad (2.7)$$

3. The number of the components in the support of $\mu^\mathbb{H}_p$ is a decreasing function of $p$. 


The set of $\boxplus$-infinitely divisible measures in $\mathcal{M}$ is closed under weak convergence of probability measures. As shown in [9], a necessary and sufficient condition for $\mu$ to be $\boxplus$-infinitely divisible is that $\phi_\mu$ belong to $\mathcal{E}$.

The Boolean convolution introduced in [22] was defined via the functions in $\mathcal{E}$. Given $\mu_1$ and $\mu_2$ in $\mathcal{M}$, the measure $\nu$ satisfying the relation
$$E_\nu = E_{\mu_1} + E_{\mu_2}$$
is called the Boolean convolution of $\mu_1$ and $\mu_2$, and it is denoted $\mu_1 \boxplus \mu_2$. For $\mu \in \mathcal{M}$ and a positive integer $n$, the $n$-fold Boolean convolution $\mu \boxplus \cdots \boxplus \mu$ denoted by $\mu^{\boxplus n}$ satisfies $E_{\mu^{\boxplus n}} = nE_\mu$. This can be extended naturally to the case when the exponent $n$ is not an integer. That is, for every $q \geq 0$ the $q$-th $\boxplus$-convolution power $\mu^{\boxplus q}$ is defined as the unique measure in $\mathcal{M}$ satisfying
$$E_{\mu^{\boxplus q}} = qE_\mu.$$

The following theorem builds the connection between $\boxplus$-infinitely divisible measures and the Boolean convolution, which was thoroughly investigated in [7].

**Theorem 2.3.** Let $\{\mu_n\}$ be a sequence in $\mathcal{M}$ and let $k_1 < k_2 < \cdots$ be a sequence of positive integers. Then the following statements (1)-(3) are equivalent:

1. $\mu_n^{\boxplus k_n} \to \mu_{c,\rho}$ weakly as $n \to \infty$;
2. $\mu_n^{k_n} \to \mu_{c,\rho}$ weakly as $n \to \infty$;
3. the measures
   $$k_n \frac{s^2}{s^2 + 1} \, d\mu_n(s) \to d\rho(s)$$
   weakly as $n \to \infty$ and
   $$\lim_{n \to \infty} \int_R \frac{k_n s}{s^2 + 1} \, d\mu_n(s) = c.$$

If (1)-(3) hold then $\mu_{c,\rho}$ is $\boxplus$-infinitely divisible and
$$\phi_{\mu_{c,\rho}}(z) = E_{\mu_{c,\rho}}(z) = c + \int_R \frac{1 + sz}{z - s} \, d\rho(s), \quad z \in \mathbb{C}^+.$$

For $\mu \in \mathcal{M}$, Theorem 2.3 shows that $(\mu_{1/n})^{\boxplus n}$ converges weakly to some $\boxplus$-infinitely divisible measure $\mathbb{B}(\mu)$ satisfying $\phi_{\mathbb{B}(\mu)} = E_\mu$. Conversely, for any $\boxplus$-infinitely divisible measure $\nu$ the sequence $(\nu^{\boxplus1/n})^{\boxplus n}$ converges weakly to some $\mu$ satisfying $\phi_\mu = E_\mu$. Since $E$ determines the measure uniquely, the map $\mathbb{B}$ induces a bijective map from $\mathcal{M}$ onto the set of $\boxplus$-infinitely divisible measures. This map $\mathbb{B}$ is called the Boolean Bercovici-Pata bijection, which coincides with $\mathbb{B}_1$ as indicated in the introduction.

In the study of $\boxplus$-infinitely divisible measures, there is one useful tool introduced in [5] called $\boxplus$-divisibility indicator:
$$\text{Ind}(\mu) = \sup\{t \geq 0 : \mu \in \mathbb{B}_t(\mathcal{M})\}, \quad \mu \in \mathcal{M}.$$ 
Any measure $\mu \in \mathcal{M}$ with finite support has $\text{Ind}(\mu) = 0$, while $\mu$ is $\boxplus$-infinitely divisible if and only if $\text{Ind}(\mu) \geq 1$. In general, for any $t \geq 0$ and $\mu \in \mathcal{M}$ we have
$$\text{Ind}(\mathbb{B}_t(\mu)) = t + \text{Ind}(\mu) \quad (2.8)$$
For $\boxplus$-divisibility indicators of some specific measures, we refer the reader to [5].
Proposition 2.1 to investigate the measure $B$ and $C$. This yields that the Voiculescu transform $\phi_{\mu,p,q}(z)$, which is an analytic continuation to $\mathbb{C}^+$, is given by

$$\phi_{\mu,p,q}(z) = \frac{p \omega_p(z)}{p'(p-1)} = \omega_p(z), \quad z \in \mathbb{C}^+ \cup \mathbb{R}.$$

As a special case of (3.1), if $1 + pq - p = 0$, i.e., $q = 1/p^*$ then

$$F_{B_{p,1/p^*}(\mu)}(z) = \frac{p \omega_p(z)}{p'(p-1)} = \omega_p(z), \quad z \in \mathbb{C}^+ \cup \mathbb{R}.$$

This yields that the Voiculescu transform $\phi_{B_{p,1/p^*}(\mu)}$ of the measure $B_{p,1/p^*}(\mu)$ has an analytic continuation to $\mathbb{C}^+$, which is given by

$$\phi_{B_{p,1/p^*}(\mu)}(z) = F_{B_{p,1/p^*}(\mu)}^{-1}(z) - z = H_p(z) - z = E_{\mu^w(p-1)}(z), \quad z \in \mathbb{C}^+.$$

These observations are recorded in the following result.

**Proposition 3.1.** For any measure $\mu \in \mathcal{M}$ and number $p > 1$, the measure $B_{p,1/p^*}(\mu)$ is $\mathbb{N}$-infinitely divisible, the function $F_{B_{p,1/p^*}(\mu)}$ extends continuously to $\mathbb{C}^+ \cup \mathbb{R}$,

$$F_{\mu^w}(z) = F_{\mu} \left( F_{B_{p,1/p^*}(\mu)}(z) \right), \quad z \in \mathbb{C}^+ \cup \mathbb{R},$$

and the Voiculescu transform of $B_{p,1/p^*}(\mu)$ can be expressed as

$$\phi_{B_{p,1/p^*}(\mu)} = E_{\mu^w(p-1)}.$$

In particular, the above statements hold for $B_1$.

Observe that Proposition 3.1 provides an easy way to prove that $B_1(\mu)$, $\mu \in \mathcal{M}$, is identically equal to the image $\mathbb{B}(\mu)$ of $\mu$ under the Boolean Bercovici-Pata bijection. Indeed, by Theorem 2.3 and Proposition 3.1 we obtain

$$\phi_{B(\mu)} = E_{\mu} = \phi_{B_1(\mu)}.$$

In [20], results similar to Proposition 3.1 for the joint distributions for $k$-tuples of selfadjoint elements in a $C^*$-probability space were obtained by combinatorial tools. We refer the reader to the same paper for the so-called $k$-tuple Boolean Bercovici-Pata bijection and related results.

The following lemma contains some basic properties of the map $B_{p,q}$ which is frequently used in the sequel. The identity in (3.3) can be obtained by [5 Proposition 3.1]. Here we provide an alternative proof using Proposition 3.1.

**Lemma 3.2.** If $\mu \in \mathcal{M}$, $p, p_1 \geq 1$, and $q, q_1 > 0$ then

$$(\mu^{pq})^{\oplus p} = B_{1+pq-q_1, \frac{pq}{1+pq-q_1}}(\mu),$$

where $\oplus$ denotes Boolean convolution.
\[ B_{p_1,q_1} \circ B_{p,q} = B_{p(1+p_1-q), \frac{p_1q_2}{1+p_1-q}}, \]  
and

\[ B_{p,q} = B_t \circ B_{p(1-qt), \frac{1}{p^*q}}, \quad 0 \leq t \leq \frac{1}{p^*q}. \]

**Proof.** It suffices to show the lemma for \( p > 1 \). By Proposition 2.1, we have

\[ F_{(\mu^{\text{new}})}(z) = \frac{p\omega(z) - z}{p - 1}, \quad z \in \mathbb{C}^+, \]

where the function \( \omega \) is the right inverse of the function

\[ H(z) = pz + (1 - p)F_{\mu^{\text{new}}} = (1 + pq - q)z + (q - pq)F_{\mu}(z), \quad z \in \mathbb{C}^+. \]

On the other hand, since the number \( 1 + pq - q > 1 \) whose conjugate exponent is

\[ (1 + pq - q)^* = 1 + \frac{pq - q}{pq}, \]

by Proposition 3.1 we see that \( \omega = F_{\nu} \), where

\[ \nu = B_{1+pq-q, \frac{pq}{1+pq-q}^*}(\mu). \]

Then using the definition of the Boolean convolution power and (3.6) gives

\[ \frac{p\omega(z) - z}{p - 1} = \frac{pF_{\nu}(z) - z}{p - 1} = F_{\nu^{\text{new}}}(z), \quad z \in \mathbb{C}^+, \]

and \( \nu^{\text{new}} = B_{1+pq-q, \frac{pq}{1+pq-q}^*}(\mu) \), whence the formula in (3.3) follows. The equality in (3.4) follows directly from (3.3). Finally, note that if \( t \in [0, 1/(p^*q)] \) then

\[ p(1 - qt) \geq 1 \quad \text{and} \quad \frac{q}{1 - qt} > 0, \]

whence the measure \( B_{p(1-qt),q/(1-qt)}(\mu) \) is defined and (3.5) holds by (3.4). \( \square \)

If \( p > 1 \) and \( 0 < q < 1/p^* \) (or, equivalently, \( 1 + pq - p < 0 \)) then (3.5) yields

\[ B_{p,q} = B_1 \circ B_{p_1,q_1}, \]

where

\[ p_1 = p(1-q) > 1 \quad \text{and} \quad q_1 = \frac{q}{1-q} > 0. \]

Using Proposition 3.1 and the preceding discussions gives the following result.

**Theorem 3.3.** If \( \mu \in \mathcal{M} \), \( p > 1 \), and \( 0 < q \leq 1/p^* \) then the following statements hold.

1. The measure \( B_{p,q}(\mu) \) is \( \boxplus \)-infinitely divisible.
2. For any \( n \in \mathbb{N} \),

\[ B_{p,q}(\mu) = \left( B_{p^{(n+q-nq)}}, \frac{q}{n(1-q) + q} \right)(\mu) \]

3. The Voiculescu transform of \( B_{p,q}(\mu) \) can be expressed as

\[ \phi_{B_{p,q}(\mu)}(z) = E_{B_{p_1,q_1}}(\mu). \]

4. For \( r > 0 \), let \( \nu_r = B_{p_1,r}(\mu) \). Then

\[ F_{B_{p,(1-q)r+q}}(z) = F_{\nu_r}(F_{B_{p,q}(\mu)}(z)), \quad z \in \mathbb{C}^+ \cup \mathbb{R}. \]
Particularly, for any $t \geq 1$ the measure $\mathbb{B}_t(\mu)$ is $\boxplus$-infinitely divisible,

$$\phi_{\mathbb{B}_t(\mu)} = F_{\mathbb{B}_{t-1}(\mu)};$$

and $F_{\mathbb{B}_t(\mu)}$ is the subordination function of $\mu^{\boxplus(t+1)}$ with respect to $\mu^{\boxplus t}$, that is,

$$F_{\mu^{\boxplus(t+1)}}(z) = F_\mu (F_{\mathbb{B}_t(\mu)}(z)), \quad z \in \mathbb{C}^+ \cup \mathbb{R}.$$  

**Proof.** The assertions (1) and (3) were proved (particularly, $p = t + 1$ and $q = (t + 1)^{-1}$ satisfy the condition $1 + pq - p \leq 0$ if and only if $t \geq 1$). Next, observe that

$$\frac{p(n + q - nq)}{n} > p - pq \geq 1 \quad \text{and} \quad \frac{q}{n(1 - q) + q} > 0,$$

whence the assertion (2) follows from (3.4). By (3), $F^{-1}_{\mathbb{B}_{p,q}(\mu)}$ can be expressed as

$$F^{-1}_{\mathbb{B}_{p,q}(\mu)}(z) = \left( 1 + \frac{q_1}{r} \right) z - \frac{q_1}{r} F_{\nu_r}(z), \quad z \in \mathbb{C}^+.$$  

(3.7)

Since $\nu_r^{\boxplus(1+q_1/r)} = \mathbb{B}_{p,(1-q)+q}(\mu)$ by (3.3), Proposition 2.1(4) and (3.7) imply the assertion (4). Letting $r = 1$ in (4) yields the last assertion.

Observe that if $\mu$ is $\boxplus$-infinitely divisible then $\mu \in \mathbb{B}(\mathcal{M})$, i.e., the measure $\mathbb{B}^{-1}(\mu) = (\mu^{\boxplus 2})^{\boxplus 1/2}$ is defined. In order to investigate the measure of the form $\mu^{\boxplus p}$, $0 < p < 1$ (that is, $\mu = \nu^{\boxplus 1/p}$ for some $\nu \in \mathcal{M}$), we need the following lemma. This lemma was also provided in [2]; however, the case $\text{Ind}(\mu) = \infty$ was not considered there.

**Lemma 3.4.** For any measure $\mu \in \mathcal{M}$ and any number $q > 0$, we have

$$\text{Ind}(\mu^{\boxplus q}) = \frac{\text{Ind}(\mu)}{q}.$$

**Proof.** First claim the inequality $\text{Ind}(\mu^{\boxplus q}) \geq \text{Ind}(\mu)/q$ holds. It clearly holds if $\text{Ind}(\mu) = 0$. Next, consider the case $\text{Ind}(\mu) > 0$. Then for any finite $r$ with $0 < r < \text{Ind}(\mu)$ pick a measure $\nu \in \mathcal{M}$ such that $\mu = \mathbb{B}_r(\nu)$, from which we obtain that

$$\text{Ind}(\mu^{\boxplus q}) = \text{Ind}\left( \left( \mu^{\boxplus (r+1)} \right)^{\boxplus \frac{q}{r+1}} \right) = \text{Ind}\left( \mathbb{B}_{r/q}(\nu^q) \right) \geq \frac{r}{q},$$

where $\nu^q \in \mathcal{M}$ and (3.4) is used in the second equality above. If $\text{Ind}(\mu) = \infty$ then letting $r \uparrow \infty$ gives $\text{Ind}(\mu^{\boxplus q}) = \infty$ as well, which implies the desired inequality. Otherwise, letting $r \uparrow \text{Ind}(\mu) < \infty$ yields the claim. By considering the identity $\mu = (\mu^{\boxplus q})^{\boxplus 1/q}$ and using the claim, we obtain the opposite inequality, and then the proof is complete.

Now we are able to determine for what value of $p \in (0, 1)$ the $p$-th $\boxplus$-convolution power of a measure is defined. The following implication that (2) implies (1) was proved in [3].

**Proposition 3.5.** Suppose that $\mu \in \mathcal{M}$ and fix a number $p \in (0, 1)$. Then the following statements (1)-(3) are equivalent:

1. $\mu^{\boxplus p}$ is defined, that is, $\mu = \nu^{\boxplus 1/p}$ for some $\nu \in \mathcal{M}$;
2. $1 - p \leq \text{Ind}(\mu)$;
3. $\mu^{\boxplus (1-p)}$ is $\boxplus$-infinitely divisible.

If (1)-(3) hold and $\Omega_p = F_{\mu^{\boxplus (1-p)}}(\mathbb{C}^+) \supseteq \mathbb{C}^+ \cup \mathbb{R}$ then the following statements are true.

(a) The reciprocal Cauchy transform $F_{\mu}$ extends continuously to $\mathbb{C}^+ \cup \mathbb{R}$.
(b) The identity $F_{\mu^{\frac{1}{p}}} (F_{\mu^{(1-p)}}(z)) = F_{\mu}(z)$ holds for $z \in \mathbb{C}^+ \cup \mathbb{R}$.

(c) The identity $F_{\mu^{\frac{1}{p}}} (z) = F_{\mu} \left( F_{\mu^{(1-p)}}^{-1}(z) \right)$ holds for $z \in \mathbb{P}$.

(d) The Voiculescu transform of $\mu^{\frac{1}{p}}$ can be expressed as

$$\phi_{\mu^{(1-p)}} = E_{\left(\mu^{\frac{1}{p}}\right)^{w(1/p-1)}} = E_{B^{-1}(\mu^{w(1-p)})}.$$  

Proof. If $\nu^{\frac{1}{p}} = \mu$ then by Lemma [3.4] we have

$$\text{Ind}(\mu) = p \text{Ind}(\nu^{(1/p-1)}) = p \left( \frac{1}{p} - 1 \right) = 1 - p,$$

which shows that (1) implies (2). Applying Proposition 3.1 to $\nu$ and $1/p$ yields the assertions (3). If (3) holds, i.e., $\phi_{\mu^{w(1-p)}} \in \mathcal{E}$ then the following function

$$F(z) = \frac{-p\phi_{\mu^{w(1-p)}}(z) + (1-p)z}{1-p}, \quad z \in \mathbb{C}^+,$$

belongs to $\mathcal{F}$, i.e., $F = F_{\nu}$ for some $\nu \in \mathcal{M}$, by which $F_{\mu^{w(1-p)}}^{-1}$ can be written as

$$F_{\mu^{w(1-p)}}^{-1}(z) = \frac{1}{p} z + \left( 1 - \frac{1}{p} \right) F_{\mu}(z), \quad z \in \mathbb{C}^+. \quad (3.8)$$

Then by Proposition 2.1 and the definition of the $\psi$-convolution power we have

$$F_{\mu^{\frac{1}{p}}}(z) = \frac{1}{p} F_{\mu^{w(1-p)}}^{-1}(z) - \frac{1}{p} - 1 = F_{\mu}(z), \quad z \in \mathbb{C}^+,$$

whence (1) holds. The assertion (a) holds since $\nu = \nu^{\frac{1}{p}}$ and $1/p > 1$, while (b)-(d) follow from the preceding discussions, (3.2), and Proposition 2.1(5). □

The proof of the preceding proposition also gives the construction of the measure $\mu^{\frac{1}{p}}$ whenever it is defined for $p \in (0,1)$. Indeed, by (3.8) the right inverse $\omega_p$ of the function $H_p(z) = pz + (1-p)F_{\mu}(z)$ ($H_p = F_{\mu^{w(1-p)}}$) satisfies the relation

$$F_{\mu^{\frac{1}{p}}}(z) = \frac{p\omega_p(z) - z}{p - 1}, \quad z \in \mathbb{C}^+, \quad (3.9)$$

and we have $F_{\mu^{\frac{1}{p}}}(z) = F_{\mu}(\omega_p(z))$ for $z \in H_p(\mathbb{C}^+)$. The following proposition can be proved by [5, Proposition 3.1]. It can be also obtained by using [3, 31], and [33], and we leave the proof for the reader.

**Proposition 3.6.** Let $\mu \in \mathcal{M}$ and for $p, q > 0$ let $q' = 1 + pq - p$ and $p' = pq/q'$. Then

1. If $\mu^{\frac{1}{p}}$ is defined and $q' > 0$ then we have the following identity

$$\left( \mu^{\frac{1}{p}} \right)^{wq} = \left( \mu^{\frac{1}{pq}} \right)^{wq'}. \quad (3.10)$$

2. The formula (3.8) holds for either

   (a) $p \geq 1$ or
   (b) $1 - \text{Ind}(\mu^{wq}) \leq p < 1$ and $1 + pq - q > 0$.

It was proved in [5] that $\mu \in \mathcal{B}_t(\mathcal{M})$ for any finite $t$ with $0 \leq t \leq \text{Ind}(\mu)$. In the following proposition we give an explicit expression for the measure $\mu_t$ so that $\mu = \mathcal{B}_t(\mu_t)$. The reader should be aware of that this conclusion holds under the essential condition that $t$ has to be finite and this condition may not be noticed without caution.
Proposition 3.7. If \( \mu \in \mathcal{M} \) then for any finite number \( t \) with \(-1 < t \leq \text{Ind}(\mu)\) there exists a unique measure \( \mu_t \in \mathcal{M} \) such that
\[
\mu = \left( \mu^{(1+t)} \right)^{\frac{1}{1+t}}, \tag{3.11}
\]
in which case \( \mu_t \) can be expressed as
\[
\mu_t = \left( \mu^{(t+1)} \right)^{\frac{1}{t+1}}. \tag{3.12}
\]
Particularly, we have \( \mu = \mathbb{B}_t(\mu) \) if \( t \geq 0 \). In addition, if \( 0 < t \leq \text{Ind}(\mu) \) then the measure \( \mu^{ult} \) is \( \mathbb{B} \)-infinitely divisible and its Voiculescu transform can be expressed as
\[
\phi_{\mu^{ult}} = E_{\mu}^{ult} = E_{(\mu^{ult})^{\frac{1}{2}}}. \tag{3.13}
\]

Proof. It is clear that the measure \( \mu_t \) in (3.12) is defined and (3.11) holds if \(-1 < t \leq 0 \). If \( t = \text{Ind}(\mu) < \infty \) then \( 1 - \text{Ind}(\mu^{ult}) = 1/(t+1) \) by Lemma 3.3 whence the measure \( \mu_t \) in (3.12) is defined by Proposition 3.5. The same result also holds for \( 0 < t < \text{Ind}(\mu) \in (0, \infty) \), and hence the identity \( \mu = \mathbb{B}_t(\mu) \) holds. Next, observe that \( \text{Ind}(\mu^{ult}) \geq 1 \) for \( 0 < t \leq \text{Ind}(\mu) \), and therefore \( \mu^{ult} \) is \( \mathbb{B} \)-infinitely divisible. The first equality in (3.13) follows by replacing \( \mu \) with \( \mu^{ult} \) and letting \( p = 1/(t+1) \) in Proposition 3.5 while the second equality follows from (3.12).

Next, we relate the limit laws to \( \mathbb{B} \)-divisibility indicators.

Proposition 3.8. Let \( q > 0 \) and \( \{\mu_n\} \) be a sequence of measures in \( \mathcal{M} \) such that \( \mu_n \to \mu \) weakly as \( n \to \infty \) for some \( \mu \in \mathcal{M} \). Then the following statements hold.

1. The inequality \( \lim \sup_n \text{Ind}(\mu_n) \leq \text{Ind}(\mu) \) holds.
2. For any \( p > 0 \) with \( 1 - \inf_n \text{Ind}(\mu_n) \leq p \), \( \mu_n^{\mathbb{B}p} \to \mu^{\mathbb{B}p} \) weakly as \( n \to \infty \).
3. The measures \( \mu_n^{ult} \to \mu^{ult} \) weakly as \( n \to \infty \).

Proof. The measure \( \mu_n^{\mathbb{B}p} \) in (2) is defined for all \( n \) by Proposition 3.5 whence (2) holds by \( \mathbb{B} \) Proposition 5.7. The assertion (3) holds by \( \mathbb{B} \) Proposition 6.2. To prove (1), first consider the case that \( 0 < t := \lim \sup_n \text{Ind}(\mu_n) < \infty \). Then for sufficiently small \( \epsilon > 0 \) we have \( 1 < \lim \sup_n \text{Ind}(\mu_n^{ult}) \) by Lemma 3.4 whence there exists a subsequence \( \{\mu_{n_k}\} \) such that \( \mu_{n_k}^{ult} \) is \( \mathbb{B} \)-infinitely divisible for all \( k \). Since the set of \( \mathbb{B} \)-infinitely divisible measures is weakly closed, we see that \( \text{Ind}(\mu_{n_k}^{ult}) \geq 1 \) by (3), which yields \( t - \epsilon \leq \text{Ind}(\mu) \). Letting \( \epsilon \to 0 \) shows \( t \leq \text{Ind}(\mu) \). If \( t = \infty \) then by similar arguments it is easy to see that \( m \leq \text{Ind}(\mu) \) for any \( m > 0 \), and therefore \( \text{Ind}(\mu) = \infty \). The assertion (1) clearly holds if \( t = 0 \), and hence the proof is complete.

It was shown in Proposition 3.5 that the subordination function for the \( \mathbb{B} \)-convolution power appearing in (2.6) is in fact the reciprocal Cauchy transform of some \( \mathbb{B} \)-infinitely divisible measure. The following theorem states that the converse is also true. For other related results about the \( \mathbb{B} \)-infinite divisibility of the subordination functions, we refer the reader to [10] and [26].

Theorem 3.9. If \( \mu \in \mathcal{M} \) then the following statements (1) and (2) are equivalent.

1. The measure \( \mu \) is \( \mathbb{B} \)-infinitely divisible.
2. The function \( F_\mu \) is the right inverse of some function \( H \) in \( \mathcal{H} \).

If (1) and (2) hold then \( F_\mu \) extends continuously to \( \mathbb{C}^+ \cup \mathbb{R} \), \( H \) can be written as
\[
H(z) = F_\mu^{-1}(z) = pz + (1-p)F_{\mu p}(z), \quad z \in \mathbb{C}^+, \tag{3.14}
\]
Theorem 3.10. Suppose that $\mu \in M$ is \(\mathbb{H}\)-infinitely divisible.

1. The function $\phi_\mu$ has a continuous extension to $\partial \Omega$ and for any $z_1, z_2 \in \Omega$,
\[
|\phi_\mu(z_1) - \phi_\mu(z_2)| \leq |z_1 - z_2|.
\]

2. For any $z_1, z_2 \in \mathbb{C}^+ \cup \mathbb{R}$,
\[
\frac{|z_1 - z_2|}{2} \leq |F_\mu(z_1) - F_\mu(z_2)|.
\]

Consequently, $G_\mu$ has a continuous extension to $\mathbb{R}$ except one point and the measure $\mu$ has at most one atom.

3. The measure $\mu$ has an atom if and only if $0 \in \partial \Omega$ and
\[
\lim_{\epsilon \downarrow 0} \frac{F_\mu^{-1}(i\epsilon) - F_\mu^{-1}(0)}{i\epsilon} = m > 0,
\]
in which case the point $F_\mu^{-1}(0)$ is an atom of $\mu$ with mass $m$.

4. The nonatomic part of $\mu$ is absolutely continuous (with respect to Lebesgue measure).
The measure $\mu^{ac}$ is concentrated on the set $\psi(V^+)$, where $V^+ = \{ x : f(x) > 0 \}$.

At the point $\psi(x)$, $x \in V^+$, the density of $\mu^{ac}$ is analytic and given by
\[
\frac{d\mu^{ac}}{dx}(\psi(x)) = \frac{f(x)}{\pi(x^2 + f^2(x))},
\]

The measure $\mu$ is compactly supported if and only if so is $f$.

Proof. By letting $p = 2$ in Proposition 2.1 and Theorem 3.9, we have $H(z) = F^{-1}_\mu(z) = 2z - F^{-1}_B(\mu)(z)$, $z \in \mathbb{C}^+$. Then it follows from 2.1(2) that
\[
\int_{\mathbb{R}} \frac{s^2 + 1}{|s - z|^2} d\sigma(s) \leq 1, \quad z \in \mathbb{C}^+.
\]
where $\sigma$ is the measure in the Nevanlinna representation of $F^{-1}_B(\mu)$. Since $\phi_\mu = E^{-1}_{B^{-1}}(\mu)$, the inequality in (1) holds for $z \in \Omega$ by Hölder inequality and 3.10, whence (1) holds by continuous extension. The assertion (2) follows from Proposition 2.1 (3). Observer that $\mu$ has an atom at $\alpha$ if and only if $F_\mu(\alpha) = 0$ and the Julia-Carathéodory derivative $F'_\mu(\alpha) < \infty$, which happens if and only if $0 \in \partial \Omega$ and
\[
0 < \frac{1}{F'_\mu(\alpha)} = (F^{-1}_\mu)'(0),
\]
where $(F^{-1}_\mu)'(0)$ is the Julia-Carathéodory derivative of $F^{-1}_\mu$ at 0. Hence $\mu(\{\alpha\}) = (F^{-1}_\mu)'(0)$ and (3) holds. Next, note that for any $x \in \mathbb{R}$ we have $F_\mu(\psi(x)) = x + if(x)$. Since $F_\mu$ extends continuously to $\mathbb{C}^+ \cup \mathbb{R}$, applying the inversion formula (2.1) gives
\[
\frac{d\mu^{ac}}{dx}(\psi(x)) = \frac{-1}{\pi} \frac{\Im G_\mu(\psi(x))}{\pi(x^2 + f^2(x))}, \quad x \in V^+,
\]
which, along with 2.1(4) gives (5) and (6). As noted above, $F_\mu(x) = 0$ a.e. relative to the singular part of $\mu$, from which we deduce that the singular part of $\mu$ is atomic, which gives (4). That (7) follows from (5) and the fact that $\mu$ has at most one atom.

The constants appearing in 3.10(1) and (2) are sharp. Indeed, by considering the standard semicircular distribution we have $\Omega = \{ z \in \mathbb{C}^+ : |z| > 1 \}$, and then taking $z_1 = 1$ and $z_2 = -1$ shows that 1 is the best constant in (1), whence the same conclusion for (2) follows immediately.

Recall that the compound free Poisson distribution $p(\lambda, \nu)$ with the rate $\lambda > 0$ and jump distribution $\nu$ is defined as the weak limit as $n \to \infty$ of $\mu_n^{\Xi_n}$, where
\[
\mu_n = \left( 1 - \frac{\lambda}{n} \right) \delta_0 + \frac{\lambda}{n} \nu
\]
and $\nu$ is compactly supported. The next proposition generalizes the jump distribution with compact support to any measure in $\mathcal{M}$.

**Proposition 3.11.** Given $\nu \in \mathcal{M}$, define
\[
d\rho(s) = \frac{s^2}{s^2 + 1} d\nu(s).
\]
Then \( p(\lambda, \nu) = B_{1+\lambda,1/(1+\lambda)}(\mu_0) \), where \( \mu_0 \) is a measure in \( \mathcal{M} \) whose reciprocal Cauchy transform satisfies

\[
F_{\mu_0}(z) = -\int_{\mathbb{R}} \frac{s}{s^2 + 1} d\nu(s) + z + \int_{\mathbb{R}} \frac{1 + sz}{s - z} d\rho(s).
\]

Consequently, \( p(\lambda, \nu) \) is an \( \mathbb{H} \)-infinitely divisible measure with an atom at 0 of mass \( 1 - \lambda \) for \( \lambda < 1 \) and no atom for \( \lambda \geq 1 \),

\[
\phi_{p(\lambda, \nu)}(z) = \lambda E_{\mu_0}(z) = \lambda z \int_{\mathbb{R}} \frac{s}{s - z} d\nu(s),
\]

and

\[
\text{Ind}(p(\lambda, \nu)) = \frac{\text{Ind}(\mu_0) + \lambda}{\lambda}.
\]

**Proof.** Since \( s^2/(s^2 + 1) \in L^1(\nu) \), the measure \( \rho \) is finite and positive, and the limit

\[
\lim_{n \to \infty} \int_{\mathbb{R}} \frac{ns}{s^2 + 1} d\mu_n(s) = \int_{\mathbb{R}} \frac{\lambda s}{s^2 + 1} d\nu(s)
\]

exists. Moreover, it is easy to see that

\[
\frac{ns^2}{s^2 + 1} d\mu_n(s) \to \lambda d\rho(s)
\]

weakly. By Theorem 2.8 the measure \( \mu_n^{\infty} \) converges weakly to \( p(\lambda, \nu) \), which satisfies

\[
\phi_{p(\lambda, \nu)}(z) = \int_{\mathbb{R}} \frac{\lambda s}{s^2 + 1} d\nu(s) + \int_{\mathbb{R}} \frac{\lambda(1 + sz)}{s - 1} d\rho(s) = \lambda z \int_{\mathbb{R}} \frac{s}{s - z} d\nu(s).
\]

On the other hand, the definition of \( \mu_0 \) and Proposition 3.1 show that

\[
\phi_{B_{1+\lambda,1/(1+\lambda)}(\mu_0)} = \lambda E_{\mu_0} = \phi_{p(\lambda, \nu)}.
\]

Then by Lemma 3.4 and 2.8 we have

\[
\text{Ind}(p(\lambda, \nu)) = \text{Ind}(B_{\lambda}(\mu_0)^{\omega \lambda}) = \frac{\text{Ind}(\mu_0) + \lambda}{\lambda}.
\]

Next, we apply Theorem 3.10(3) to locate the atom of \( p(\lambda, \nu) \). Since \( \phi_{p(\lambda, \nu)}(0) = 0 \), \( 0 \in \partial F_{p(\lambda, \nu)}(\mathbb{C}^+) \). Moreover, by the dominated convergence theorem we obtain

\[
\lim_{\epsilon \to 0} \frac{\phi_{p(\lambda, \nu)}(i\epsilon) - \phi_{p(\lambda, \nu)}(0)}{i\epsilon} = \lambda \lim_{\epsilon \to 0} \int_{\mathbb{R}} \frac{s}{i\epsilon - s} d\nu(s) = -\lambda,
\]

which gives the desired result. This completes the proof. \( \square \)

Since the \( \mathbb{H} \)-divisibility indicator is zero for any measure with finite support, we have the following result.

**Corollary 3.12.** We have \( p(\lambda, \delta_\alpha) = B_{1+\lambda,1/(1+\lambda)}(\mu_0) \), where \( \mu_0 = (\delta_0 + \delta_\alpha)/2 \),

\[
\text{Ind}(p(\lambda, \delta_\alpha)) = 1, \text{ and } \phi_{p(\lambda, \delta_\alpha)}(z) = a \lambda z/(z - a).
\]

We finish this section with an interesting observation. If \( \text{Ind}(\mu) > 1 \) then \( \text{Ind}(B^{-1}(\mu)) = \text{Ind}(\mu) - 1 > 0 \) by (2.8) and (3.2). This implies that \( \phi_\mu = E_{B^{-1}(\mu)} \) has a continuous extension to \( \mathbb{C}^+ \cup \mathbb{R} \) by Proposition 3.3, whence we have the following proposition.

**Proposition 3.13.** If \( \mu \in \mathcal{M} \) with \( \text{Ind}(\mu) > 1 \) then \( \phi_\mu \) has a continuous extension to \( \mathbb{C}^+ \cup \mathbb{R} \).
4. Measures with mean zero and finite variance

Recall that for \( \mu \in \mathcal{M} \) with mean zero and unit variance, \( \Phi(\mu) \) is the unique measure in \( \mathcal{M} \) satisfying the Eq. (2.4) with \( \sigma^2 = 1 \), i.e., \( E_\mu = G_{\Phi(\mu)} \). In general, a measure \( \mu \) has mean \( m \) and finite variance \( \sigma^2 \) if and only if \( \mu \boxplus \delta_{-m} \) has mean zero and variance \( \sigma^2 \) because \( d(\mu \boxplus \delta_{-m})(s) = d\mu(s + m) \), and hence \( E_{\mu \boxplus \delta_{-m}} = \sigma^2 G_{\Phi((\mu \boxplus \delta_{-m})\mathbb{1}_{-\sigma^2})} \). Since \( \text{Ind}(\mu) = \text{Ind}(\mu \boxplus \delta_a) \) for any \( a \in \mathbb{R} \) by [2, Proposition 3.7], in what follows we only consider measures with mean zero and finite variance.

Recall that the free Brownian motion started at \( \nu \in \mathcal{M} \) is the process \( \{\nu \boxplus \gamma_t : t \geq 0\} \). The connection among this process, the map \( \Phi \), and the subordination function of the \( \boxplus \)-convolution powers is described in the following theorem, which was proved in [4] and [12]. For the completeness, we provide its statement and proof.

**Theorem 4.1.** If \( \mu \in \mathcal{M} \) has mean zero and variance \( \sigma^2 \), and \( \nu = \Phi(\mu^{\omega_1}/\sigma^2) \) then

\[
G_{\nu \boxplus \gamma_t,2}(z) = G_{\nu}(F_{\beta_{t+1/t}(\mu)}(z)) = \frac{E_{\beta_{t+1/t}(\mu)}(z)}{t\sigma^2}, \quad z \in \mathbb{C}^+ \cup \mathbb{R}, \quad (4.1)
\]

where \( t > 0 \). Consequently, we have \( \phi_{\beta_{t+1/t}(\mu)} = t\sigma^2 G_{\nu} \) and

\[
E_{\beta_t}(\mu) = \sigma^2 G_{\nu \boxplus \gamma_{ta^2}}. \quad (4.2)
\]

**Proof.** Let \( p = t + 1 > 1 \). Since \( E_\mu = \sigma^2 G_\nu \), it follows that

\[
H_p(z) := pz + (1 - p)F_\mu(z) = z + (p - 1)\sigma^2 G_\nu(z), \quad z \in \mathbb{C}^+.
\]

If \( \omega_p \) is the right inverse of \( H_p \) then [12, Proposition 2] shows that

\[
G_{\nu \boxplus \gamma_{(p-1)a^2}}(z) = G_{\nu}(\omega_p(z)) = \frac{z - \omega_p(z)}{(p - 1)\sigma^2}, \quad z \in \mathbb{C}^+ \cup \mathbb{R}.
\]

Since \( \omega_p = F_{\beta_{p+1/p^2}(\mu)} \) by Proposition 3.1, the above identity yields (4.1). Finally, the rest assertions follow from \( \phi_{\beta_p \boxplus \gamma_{(p-1)a^2}} = E_{\mu^{\omega_{(p-1)}}} = (p-1)\sigma^2 G_\nu \) and (4.1). \( \square \)

The identity (4.2) indicates that \( \mu^{\boxplus p} \) has mean zero and finite variance \( p\sigma^2 \) if \( p \geq 1 \). The next result shows that this is also true for the measure \( \mu^{\boxplus p} \) whenever it is defined.

**Lemma 4.2.** Suppose that \( \mu \in \mathcal{M} \) has mean zero and variance \( \sigma^2 \). If the measure \( \mu^{\boxplus p} \) is defined for some \( p > 0 \) then it has mean zero and variance \( p \), in which case

\[
\Phi \left( \left( \mu^{\boxplus p} \right)^{\omega_1/\sigma^2} \right) = \Phi \left( \mu^{\omega_1/\sigma^2} \right) \boxplus \gamma_{(p-1)\sigma^2}, \quad p \geq 1,
\]

\[
\Phi \left( \left( \mu^{\boxplus p} \right)^{\omega_1/\sigma^2} \right) \boxplus \gamma_{(1-p)\sigma^2} = \Phi \left( \mu^{\omega_1/\sigma^2} \right), \quad p < 1.
\]

**Proof.** By (4.2), it suffices to show the lemma for the case \( 1 - \text{Ind}(\mu) \leq p < 1 \). Let \( \nu = \Phi(\mu^{\omega_1/\sigma^2}) \), \( \mu_p = \mu^{\boxplus p} \), and \( H(z) = z/p + (1 - 1/p)F_{\mu_p}(z) \). Then it follows from (3.10) that \( E_{\mu_p}(iy) = F_p(H(iy)) \) or, equivalently, \( E_{\mu_p}(iy) = iy - H(iy) + \sigma^2 G_\nu(H(iy)) \) for sufficiently large \( y > 0 \). Since \( z - H(z) = (1 - 1/p)E_\mu(z) \), we see that \( E_{\mu_p}(iy) = p\sigma^2 G_\nu(H(iy)) \) for sufficiently large \( y > 0 \). Next, we claim that \( E_{\mu_p}((p\sigma^2) \in \mathcal{C} \). Indeed, since \( \lim_{y \to \infty} H(iy)/(iy) = 1 \), for any \( \alpha > 0 \) there exists a number \( \beta > 0 \) such that

\[
\left| \frac{H(iy)}{iy} - 1 \right| < \epsilon_\alpha := \frac{\alpha}{\sqrt{1 + \alpha^2}}, \quad y > \beta.
\]
from which we deduce that $H(iy) \in \Gamma_{\alpha, \beta'}$ for $y > \beta$, where $\beta' = (1 - c_\alpha)\beta$. By [9, Proposition 5.1], we obtain
\[
\lim_{y \to \infty} iyE_{\mu_p}(iy) = p\sigma^2 \left( \lim_{y \to \infty} \frac{iH(iy)}{H(iy)} \right) \left( \lim_{y \to \infty} H(iy)G_\nu(H(iy)) \right) = p\sigma^2,
\]
which yields that $E_{\mu_p}/(p\sigma^2) \in \mathcal{G}$, as desired. Finally, let $\nu_p = \Phi((\mu_p)^{u_1/2})$.
Then $E_{\mu_p} = p\sigma^2G_{\nu_p}$ and \((1.2)\) show that
\[
p\sigma^2G_{\Phi(\mu_p^{u_1/2})} = E_{\mu_p} = E_{\mathcal{B}_{1/p-1}(\mu_p^{u_1/2})} = p\sigma^2G_{\nu_p}\mathcal{B}_{1/p-1}(\mu_p^{u_1/2})^*,
\]
which gives the last assertion. If $p > 1$ then $E_\mu = \sigma^2G_{\Phi(\mu_p^{u_1/2})}$ and \((4.2)\) yield
\[
E_{\mathcal{B}_{p-1}(\mu)} = \sigma^2G_{\Phi(\mu_p^{u_1/2})}\mathcal{B}_{p-1}(\mu_p^{u_1/2})^*,
\]
as desired. This completes the proof. \qed

**Proposition 4.3.** Suppose that $\mu \in \mathcal{M}$ has mean zero and finite variance $\sigma^2$. If $t$ is a finite number with $0 \leq t \leq \varphi(\mu)$ and $\mu_t$ is the measure defined in \((3.12)\) then
\[
E_{\mu_t} = \sigma^2G_{\nu_t}, \quad \text{and} \quad E_\mu = \sigma^2G_{\nu_t}\mathcal{B}_{\gamma_{(1-t)}^*},
\]
where $\nu_t = \Phi\left(\mu_t^{u_1/2}\right)$.

**Proof.** By Lemma \((1.2)\), it is clear that $\mu_t$ has mean zero and variance $\sigma^2$, whence the conclusions follows from \((1.2)\). \qed

The preceding proposition gives a reformulation for the $\mathcal{B}$-divisibility indicator of measures with mean zero and finite variance.

**Corollary 4.4.** If $\mu \in \mathcal{M}$ has mean zero and finite variance $\sigma^2$ then
\[
\text{Ind}(\mu) = \sup \left\{ t \geq 0 : E_\mu = \sigma^2G_{\nu_t}\mathcal{B}_{\gamma_{(1-t)}^*} \text{ for some } \nu_t \in \mathcal{M} \right\}.
\]

The preceding corollary enables us to associate to each measure $\nu \in \mathcal{M}$ a non-negative number:
\[
C(\nu) = \sup \left\{ t \geq 0 : \nu = \nu_t \mathcal{B}_{\gamma_t} \text{ for some } \nu_t \in \mathcal{M} \right\}.
\]
We will call $C(\nu)$ the semicircular decomposition indicator of $\nu$. The connection between $\mathcal{B}$-divisibility indicator and semicircular decomposition indicator is described in the next result.

**Theorem 4.5.** For any $\nu \in \mathcal{M}$ we have $B(\nu) = \text{Ind}(\Phi^{-1}(\nu))$.

We now characterize $\mathcal{B}$-infinitely divisible measures with mean zero and finite variance.

**Theorem 4.6.** If $\mu \in \mathcal{M}$ and $\sigma \in (0, \infty)$ then the following statements are equivalent:

1. $\mu$ is a $\mathcal{B}$-infinitely divisible measure with mean zero and variance $\sigma^2$;
2. there exists a measure $\nu \in \mathcal{M}$ such that $\phi_\mu = \sigma^2G_\nu$;
3. $F_\mu$ is the right inverse of some $H \in \mathcal{H}$ satisfying $\lim_{y \to \infty} iy(H(iy) - iy) = \sigma^2$;
4. there exists a measure $\nu \in \mathcal{M}$ such that $E_\mu = \sigma^2G_\nu\mathcal{B}_{\gamma_{\sigma^2}}$. 
If (1)-(4) hold and \( p = 1 + \sigma^2 \) then the measure \( \nu \) in (2) and (4) can be expressed as

\[
\nu = \Phi \left( \left( \mu^{1/p} \right)^{\mathbb{H}1/p} \right).
\]

The function \( H \) in (3) can be expressed as

\[
H(z) = z + \sigma^2 G_\nu(z),
\]

and

\[
G_\nu \mathbb{H}_{\gamma, \sigma^2}(z) = G_\nu(F_\mu(z)), \quad z \in \mathbb{C}^+ \cup \mathbb{R}.
\]

Moreover, for any \( r > 0 \) we have

\[
E_{\mu^{1/r}} = r\sigma^2 G_{\nu \mathbb{H}_{\gamma, \sigma^2}}.
\]

**Proof.** First suppose that (1) holds. Then the measure \( (\mu^{\sigma^2})^{\mathbb{H}1/2} = \mathbb{B}^{-1}(\mu) \) has mean zero and variance \( \sigma^2 \) by Lemma 4.2, whence \( \phi_\mu = E_{\mathbb{B}^{-1}(\mu)} = \sigma^2 G_\nu \) for some \( \nu \in \mathcal{M} \) and (2) follows. The definition of \( \Phi \) shows that \( \nu \) can be expressed as

\[
\nu = \Phi \left( \left( \mathbb{B}^{-1}(\mu) \right)^{1/\sigma^2} \right) = \Phi \left( \left( \mu^{1/p} \right)^{\mathbb{H}1/q} \right),
\]

where the Eq. (3.10) is used in the second equality above. If (2) holds then \( H(z) = F_\mu^{-1}(z) = \phi_\mu(z) + z \), which implies (3). If the statement (3) holds then \( H(z) = z + \sigma^2 G_\nu \) for some \( \nu \in \mathcal{M} \). Then [12, Proposition 2] shows that \( F_\mu \) is the subordination function of \( \nu \oplus \gamma_{\sigma^2} \) with respect to \( \nu \), whence we have

\[
G_{\nu \mathbb{H}_{\gamma, \sigma^2}}(z) = G_\nu(F_\mu(z)) = \frac{z - F_\mu(z)}{\sigma^2}, \quad z \in \mathbb{C}^+,
\]

and the assertion (4) holds. The implication that (4) implies (1) follows from Corollary 4.7. Moreover, the identity (4.2) shows that \( E_\mu = E_{\mathbb{B}(\mathbb{B}^{-1}(\mu))} = \sigma^2 G_{\nu \mathbb{H}_{\gamma, \sigma^2}} \), whence the assertions (4.3) and (4.4) hold by the preceding discussions. For the last assertion it suffices to show that \( \nu_r := \Phi((\mu^{1/r})^{1/\sigma^2}) = \nu \oplus \gamma_{r\sigma^2} \). If \( r < 1 \) then \( \nu_r \oplus \gamma_{(1-r)\sigma^2} = \Phi((\mu^{1/r})^{1/\sigma^2}) \) by Lemma 4.2. Since \( \Phi((\mu^{1/\sigma^2})^{1/\sigma^2}) = \nu \oplus \gamma_{\sigma^2} \), the desired equality follows. Similarly, if \( r > 1 \) then \( \nu_r = \Phi((\mu^{1/\sigma^2})^{1/\sigma^2}) \oplus \gamma_{(r-1)\sigma^2} = \nu \oplus \gamma_{r\sigma^2} \), as desired.

Let \( H \) be the function defined as in (4.3) and

\[
\Omega = \{ z \in \mathbb{C}^+ : \Im H(z) > 0 \}.
\]

It was shown in [12] that the function \( G_\nu \) extends continuously to \( \overline{\Omega} \) and this extension is Lipschitz continuous on \( \overline{\Omega} \) with the Lipschitz constant \( 1/\sigma^2 \). Moreover,

\[
|G_\nu(z)| \leq \frac{1}{\sigma^2}, \quad z \in \overline{\Omega}.
\]

Combining these facts and Theorem 4.6 gives the following result.

**Corollary 4.7.** If \( \mu \in \mathcal{M} \) is a \( \mathbb{H} \)-infinitely divisible measure with mean zero and finite variance \( \sigma^2 \) then

\[
|G_\nu \mathbb{H}_{\gamma, \sigma^2}(z_1) - G_\nu \mathbb{H}_{\gamma, \sigma^2}(z_2)| \leq \frac{1}{\sigma^2} |F_\mu(z_1) - F_\mu(z_2)|, \quad z_1, z_2 \in \mathbb{C}^+ \cup \mathbb{R},
\]

and

\[
|\phi_\mu(z)| \leq \sigma, \quad z \in \overline{\Omega},
\]

where \( \nu = \Phi((\mu^{1/\sigma^2})^{1/\sigma^2}) \) and \( \Omega = F_\mu(\mathbb{C}^+) \).
It was shown before that $E_\mu$ has a continuous extension to $C^+ \cup \mathbb{R}$ if $\text{Ind}(\mu) > 0$. In general, the converse is not true. Indeed, let $\mu \in \mathcal{M}$ be so that $E_\mu = G_N = 1/(z + i)$, where $N$ is the Cauchy distribution. Since $\phi_N = -i$, it is easy to see that $N$ cannot be written as a free Brownian motion stated at some measure, whence $B(N) = 0$, which yields $\text{Ind}(\mu) = 0$ by Theorem 4.5. In the following theorem, we improve this result for measures with mean zero and finite variance.

**Theorem 4.8.** If $\mu \in \mathcal{M}$ has mean zero and finite variance $\sigma^2$ then

1. $\text{Ind}(\mu) > 0$ if and only if $E_\mu = \sigma^2 G_\nu \otimes \gamma_0$ for some $\nu \in \mathcal{M}$ and $t > 0$;
2. $\text{Ind}(\mu) > 1$ if and only if $\phi_\mu = \sigma^2 G_\nu \otimes \gamma_0$ for some $\nu \in \mathcal{M}$ and $t > 0$.

**Proof.** The assertion (1) was proved in Proposition 4.3. Since $\phi_\mu = E_{\mathbb{B}^{-1}(\mu)}$ and $\text{Ind}(\mu) = 1 + \text{Ind}(\mathbb{B}^{-1}(\mu))$ if $\mu$ is $\mathbb{E}$-infinitely divisible, the assertion (2) follows (1).

For $a \in \mathbb{R}$, by the fact $(\mu \boxplus \delta_a) \boxplus p = \mu \boxplus \delta_p$ and the identity $(\mu \boxplus \delta_a) \boxplus q = (\mu \boxplus \gamma_0) \boxplus \delta_{(q-1)a}$ shown in Proposition 3.7 we have
\[
\mathbb{B}_{p,q}(\mu \boxplus \delta_a) = (\mathbb{B}_{p,q}(\mu) \boxplus \delta_{pa}) \boxplus \delta_{(q-1)a}.
\]

Next, we use (4.5) to investigate the free compound Poisson distribution $p(\lambda, \nu)$, where $\nu$ has finite variance.

**Proposition 4.9.** Suppose that $\nu \in \mathcal{M}$ has mean $m$ and finite variance $\sigma^2$. Then
\[
E_{p(\lambda, \nu) \boxplus \delta_{-\lambda m}} = \lambda m_2 G_{\nu_0 \boxplus \gamma_{\lambda m_2}}
\]
and
\[
\phi_{p(\lambda, \nu) \boxplus \delta_{-\lambda m}} = \lambda m_2 G_{\nu_0},
\]
where $m_2 = m^2 + \sigma^2$ is the second moment of $\nu$ and $d\nu_0(s) = s^2/m_2 d\nu(s)$. Consequently, $p(\lambda, \nu)$ has mean $\lambda m$ and variance $\lambda m_2$, and $\text{Ind}(p(\lambda, \nu)) > 1$ if $B(\nu_0) > 0$.

**Proof.** If $\mu_0$ is the measure defined in Proposition 3.11 then
\[
E_{\mu_0}(z) = \int s \, d\nu(s) + \int_{\mathbb{R}} \frac{s^2}{z-s} \, d\nu(s) = m + m_2 G_{\nu_0}(z),
\]
from which we obtain
\[
E_{\mu_0 \boxplus \delta_{-\lambda m}}(z) = E_{\mu_0}(z + m) - m = m_2 G_{\nu_0}(z + m) = m_2 G_{\nu_0 \boxplus \delta_{-\lambda m}}(z).
\]

Then Theorem 4.1 shows that for any $p > 1$ we have
\[
E_{\mathbb{B}_{p,1/p}(\mu) \boxplus \delta_{-\lambda m}}(z) = (p-1)E_{\mathbb{B}_{p,1}(\mu_0 \boxplus \delta_{-\lambda m})} = (p-1)m_2 G_{\nu_0 \boxplus \delta_{-\lambda m} \boxplus \gamma_{(p-1)m_2}}.
\]

On the other hand, by (4.5) we have
\[
E_{\mathbb{B}_{p,1/p}(\mu_0 \boxplus \delta_{-\lambda m})}(z) = E_{\mathbb{B}_{p,1/p}(\mu_0)}(z - pm) + (1-p)m,
\]
from which, along with (4.8), we deduce that
\[
E_{\mathbb{B}_{p,1/p}(\mu_0)}(z) + (1-p)m = (p-1)m_2 G_{\nu_0 \boxplus \delta_{-\lambda m} \boxplus \gamma_{(p-1)m_2}}(z - pm)
\]
or, equivalently,
\[
E_{\mathbb{B}_{p,1/p}(\mu_0 \boxplus \delta_{1-p} m)} = (p-1)m_2 G_{\nu_0 \boxplus \delta_{(p-1)m} \boxplus \gamma_{(p-1)m_2}}.
\]
Letting $p = \lambda + 1$ in the above identity gives that $p(\lambda, \nu)$ have mean $\lambda m$ and variance $\lambda m_2$. Since $\phi_{p(\lambda, \nu)} = \lambda E_{\mu_0}$, it follows from (4.1.5) that

$$
\phi_{p(\lambda, \nu)} \mu_{\delta - \lambda m} = \lambda m_2 G_{\nu_0}.
$$

The last assertion follows from [2, Proposition 3.7] and Corollary 4.8.

□

From the preceding proposition, it is easy to see that $p(\lambda, \delta_a)$ has mean $\lambda a$ and variance $\lambda a^2$, and $\text{Ind}(p(\lambda, \delta_a)) = 1$ since $\nu_0 = \delta_a$.

5. Support and regularity for measures in $\mathbb{B}_{p,q}(\mathcal{M})$

If $p, q > 0$ then the measure $\mathbb{B}_{p,q}(\mu)$ (if $\mu_{\mathbb{B}^p}$ is defined) is a Dirac measure $\delta_a$ if and only if $\mu = \delta_{a/(pq)}$. For the rest of the paper we confine our attention to the case of $\mu \in \mathcal{M}$ which is not a point mass and follow the notations used in Proposition 2.1 and 2.2. We denote by $\rho$ the unique nonzero (because $\mu \neq \delta_a$) measure in the Nevanlinna representation (2.2) of $F_{\mu}$. Therefore, the Nevanlinna representation of $F_{\mu, w}$ is

$$
F_{\mu, w}(z) = q \Re F_{\mu}(1) + z + q \int \frac{1 + sz}{s - z} \, d\rho(s), \quad z \in \mathbb{C}^+.
$$

(5.1)

In certain situations, $F_{\mu}$ is defined and takes a real value at some $x \in \mathbb{R}$ (for instance, $x$ is an atom of $\mu$), in which case we write $F_{\mu}(x) \in \mathbb{R}$. The following result shows that for $p > 1, q > 0$, $F_{\mu, w}$ is Lipschitz continuous on $\overline{\Omega_p}$ and takes real values on $\Omega_p \cap \mathbb{R}$.

**Proposition 5.1.** For $p > 1, q > 0$, $F_{\mu, w}$ extends continuously to $\overline{\Omega_p}$ and satisfies

$$
\left| \frac{F_{\mu, w}(z_1) - F_{\mu, w}(z_2)}{z_1 - z_2} \right| \leq 1 + \frac{q}{p - 1}, \quad z_1, z_2 \in \overline{\Omega_p}.
$$

Moreover, (5.1) holds for $z \in \overline{\Omega_p}$ and the Julia-Caratheodory derivative $F_{\mu, w}'$ is

$$
F_{\mu, w}'(z) = 1 + q \int \frac{s^2 + 1}{(s - z)^2} \, d\rho(s), \quad z \in \overline{\Omega_p}.
$$

(5.2)

**Proof.** First, applying Proposition 2.1 (2) and the Hölder inequality to $E_{\mu}$ gives

$$
\left| \frac{E_{\mu}(z_1) - E_{\mu}(z_2)}{z_1 - z_2} \right| \leq \int \frac{(s^2 + 1) \, d\rho(s)}{|s - z_1| |s - z_2|} \leq \frac{1}{p - 1}, \quad z_1, z_2 \in \Omega_p.
$$

Then by the continuous extension, the above inequality holds for $z_1, z_2 \in \overline{\Omega_p}$, and therefore the Nevanlinna representation (2.2) of $E_{\mu}$ holds for $z \in \overline{\Omega_p}$. Using the dominated convergence theorem, the Julia-Caratheodory $E_{\mu}'$ is then given by

$$
E_{\mu}'(z) = \lim_{\epsilon \downarrow 0} \frac{E_{\mu}(z + i\epsilon) - E_{\mu}(z)}{i\epsilon} = -\int \frac{s^2 + 1}{(s - z)^2} \, d\rho(s), \quad z \in \overline{\Omega_p},
$$

whence the desired results follow from the identities $E_{\mu, w} = q E_{\mu}$ and $F_{\mu, w}' = 1 - E_{\mu, w}'$.

The following lemma plays an important role in the investigation of atoms of the measure $\mathbb{B}_{p,q}(\mu)$.

**Lemma 5.2.** Let $x \in \mathbb{R}$ and $f_{\mu}$ be the function defined as in (2.5). Then

1. $F_{\mu}(x) \in \mathbb{R}$ and the Julia-Caratheodory derivative $F_{\mu}'(x) \in (1, \infty)$ if and only if

□
Proposition 5.3. Carathéodory derivative $F$ consequence of Lemma 5.2 and the identity

\[ \mu \text{ atoms of } x \]

in which case (1) implies (2) and the proof is complete.

Proof. First, suppose that (2) holds. Then $x \in \Omega_p$ for some $p > 1$ and (2.2) holds for $z = x$ by Proposition 2.1.2. As shown in Proposition 5.1 we have the Julia-Carathéodory derivative $F'_\mu(x) = f_\mu(x)$, whence Julia-Carathéodory derivative $F'_\mu(x) = 1 + f_\mu(x) \in (1, \infty)$ and (1) follows. On the other hand, if both $F_\mu(x)$ and the Julia-Carathéodory derivative $F'_\mu(x)$ are real numbers then

\[
F'_\mu(x) = \lim_{\epsilon \downarrow 0} \frac{\Re[F_\mu(x + i\epsilon) - F_\mu(x)]}{i\epsilon} + \lim_{\epsilon \downarrow 0} \frac{\Im[F_\mu(x + i\epsilon) - F_\mu(x)]}{i\epsilon} = \lim_{\epsilon \downarrow 0} \frac{\Im[F_\mu(x + i\epsilon) - F_\mu(x)]}{\epsilon} = \lim_{\epsilon \downarrow 0} \Im[F_\mu(x + i\epsilon)]
\]

where the monotone convergence theorem is used in the last equality. This yields the implication that (1) implies (2) and the proof is complete. $\square$

Recall that $\alpha$ is an atom of a measure $\nu$ if and only if $F_\nu(\alpha) = 0$ and the Julia-Carathéodory derivative $F'_\nu(\alpha) \in [1, \infty)$, in which case $\nu(\{\alpha\}) = 1/F'_\nu(\alpha)$. The atoms of $\mu^{\omega q}$, $q > 0$, are characterized in the following proposition, which is a direct consequence of Lemma 5.2 and the identity $F'_\mu^{\omega q} = qF'_\mu + 1 - q$, where $F'_\mu^{\omega q}$ and $F'_\mu$ are the Julia-Carathéodory derivatives.

Proposition 5.3. If $\mu \in \mathcal{M} (\mu \neq \delta_1)$, $q > 0$, and $\alpha \in \mathbb{R}$ then (1)-(3) are equivalent:

1. the point $\alpha$ is an atom of the measure $\mu^{\omega q}$;
2. $F_\mu(\alpha) = \alpha/q^*$ and the Julia-Carathéodory derivative $F'_\mu(\alpha) \in (1, \infty)$;
3. $F_\mu(\alpha) = \alpha/q^*$ and $f_\mu(\alpha) \in (0, \infty)$.

If $r = (1 - \mu^{\omega q}(\{\alpha\}))^{-1} > 1$ then (2.2) holds for $z = \alpha$ and

\[
F'_\mu(\alpha) = 1 + f_\mu(\alpha) = 1 + \frac{1}{q(r - 1)}.
\]

Using the identity $\mu = (\mu^{\omega q})^{\omega 1/q}$, $q > 0$, gives the following corollary.

Corollary 5.4. If $q > 0$, $r > 1$, and $\alpha \in \mathbb{R}$ then the following statements are equivalent:

1. $\alpha$ is an atom of $\mu$;
2. $F_\mu(\alpha) = 0$ and $f_\mu(\alpha) \in (0, \infty)$;
3. $F_{\mu^{\omega q}}(\alpha) = (1 - q)\alpha$ and the Julia-Carathéodory derivative $F'_{\mu^{\omega q}}(\alpha) \in (1, \infty)$;
4. $F_{\mu^{\omega 1/q}}(\alpha) = \alpha/q^*$ and the Julia-Carathéodory derivative $F'_{\mu^{\omega 1/q}}(\alpha) \in (1, \infty)$.

If $\mu(\{\alpha\}) = 1 - r^{-1}$ then $f_\mu(\alpha) = 1/r - 1$, $F'_{\mu^{\omega q}}(\alpha) = 1 + q/(r - 1)$, and $F'_{\mu^{\omega 1/q}}(\alpha) = 1 + [q(r - 1)]^{-1}$.

Next, we characterize the points in $\mathbb{R}$ at which $F_\mu$ is defined, takes real values, and has finite Julia-Carathéodory derivatives.
Proposition 5.5. Let $p > 1$ and let $x, \alpha,$ and $\beta$ be real numbers. If $px+(1-p)\beta = \alpha$ then (1)-(4) are equivalent:

1. $F_p(x) = \beta$ and $0 < f_p(x) < 1/(p-1);$  
2. $F_p(x) = \beta$ and Julia-Carathéodory derivative $F_p'(x) \in (1, p^*);$  
3. $H_p(x) = \alpha$ and the Julia-Carathéodory derivative $H_p'(x) \in (0,1);$  
4. $\omega_p(\alpha) = x$ and the Julia-Carathéodory derivative $\omega_p'(\alpha) \in (1,\infty);$  
5. $F_{\mu\boxplus p}(\alpha) = \beta$ and the Julia-Carathéodory derivative $F_{\mu\boxplus p}'(\alpha) \in (1,\infty).$

If (1)-(5) hold then (2.1) holds for $z = x$ and

$$F_p'(x) = 1 + f_p(x) = \frac{p - H_p'(x)}{p - 1} = \frac{p\omega_p'(\alpha) - 1}{(p-1)\omega_p'(\alpha)} = \frac{pF_{\mu\boxplus p}'(\alpha)}{1 + (p-1)F_{\mu\boxplus p}'(\alpha)}.$$

Proof. The equivalence of (1) and (2) was proved in Lemma 5.2. The equivalence of (2) and (3) and that of (4) and (5) follow from the identities of Julia-Carathéodory derivatives $H_p'(x) = p+(1-p)F_p'(x)$ and $F_{\mu\boxplus p}'(\alpha) = (p\omega_p'(\alpha) - 1)/(p-1)$, respectively.

The implication that (4) implies (3) holds by the fact $\omega_p'(\alpha) = 1/H_p'(x)$, where the Julia-Carathéodory derivative $\omega_p'(\alpha)$ must be understood as $+\infty$ if the Julia-Carathéodory derivative $H_p'(x) = 0$. Conversely, if (3) holds, i.e., (1) holds (because (1) and (3) was proved to be equivalent) then $x \in \partial \Gamma_p$, whence we have $\omega_p(H_p(x)) = x$ by Proposition 2.1 (3), and the statement (4) holds by the equality $\omega_p(\alpha) = 1/H_p'(x)$. The last assertion follows from the preceding discussions.

We are now in a position to characterize the atoms of the measures in $\mathcal{B}_{p,q}(\mathcal{M}).$

Proposition 5.6. Suppose $\alpha \in \mathbb{R}$, $p > 1$, and $q > 0$, and let $p', q'$ be the numbers defined in Proposition 5.5. If $p'q \neq 1$ then the following statements are equivalent:

1. the point $\alpha$ is an atom of $\mathcal{B}_{p,q}(\mu);$  
2. $F_{\mu\boxplus p}(\alpha) = \alpha/q^*$ and the Julia-Carathéodory derivative $F_{\mu\boxplus p}'(\alpha) \in (1,\infty);$  
3. $F_{\mu}(\alpha/p') = \alpha/q^*$ and the Julia-Carathéodory derivative $F_{\mu}'(\alpha/p') \in (1,p^*);$  
4. $F_{\mu}(\alpha/p') = \alpha/q^*$ and $0 < f_{\mu}(\alpha/p') < 1/(p-1).$

Moreover, if $\mathcal{B}_{p,q}(\mu)(\{\alpha\}) = 1 - r^{-1}$ for some $r > 1$ then

$$F_{\mu\boxplus p}'(\alpha) = 1 + \frac{1}{q(r-1)}$$

and

$$F_{\mu}'(\alpha/p') = 1 + f_{\mu}(\alpha/p') = \frac{pq(r-1) + p}{pq(r-1) + p - 1} = 1 + \frac{1}{q'(rp' - 1)}.$$

In addition, if $p'q < 1$ then $\mathcal{B}_{p,q}(\mu)$ has at most one atom. Particularly, the above assertions hold for $\mathcal{B}_t$, $t \in (0,\infty)\setminus\{1\}$, as well.

Proof. The equivalence of (1) and (2) follows from Proposition 5.3. Next, note that the hypothesis $p'q \neq 1$ shows that $p' \neq \infty$. Then letting $x = \alpha/p'$ and $\beta = \alpha/q^*$ gives the equivalence of (2) and (3) by Proposition 5.5. By Lemma 5.2 we see that (3) and (4) are equivalent. By simple computations, the rest desired equalities also follow from Lemma 5.2 Proposition 5.3 and 5.5. That the measure $\mathcal{B}_{p,q}(\mu)$, $p'q < 1$, has at most one atom is a direct consequence of Theorems 5.5 and 3.10.

Proposition 5.6 indicates that the Julia-Carathéodory derivative $F_{\mu}' < p^*$ is one of the necessary conditions to guarantee the existence of an atom of $\mathcal{B}_{p,q}(\mu)$, $p'q \neq 1$. 

Supports, regularity, and $\boxplus$-infinite divisibility for measures of the form $(\mu\boxplus p)^{\omega_q}$
Indeed, consider the symmetric Bernoulli distribution \( \mu = \frac{1}{2}(\delta_{-1} + \delta_1) \) and the arcsine law of distribution \( \mathbb{B}_{1/2}(\mu) \) whose density is given by
\[
d(\mathbb{B}_{1/2}(\mu))(x) = \frac{1}{\pi \sqrt{2 - x^2}} dx, \quad [-\sqrt{2}, \sqrt{2}].
\]
In this case \((p = 3/2, q = 2/3, p^* = 2, q^* = 1/2, p^* = 3, q^* = -2)\), if \( \alpha = \pm \sqrt{2} \) then it is easy to check that \( F^\prime_{\mu}(\alpha/2) = -\alpha/2 \) and the Julia-Carathéodory derivative \( F^\prime_{\mu}(\alpha/2) = 3 \). However, the points \( \pm \sqrt{2} \) fail to be atoms of \( \mathbb{B}_{1/2}(\mu) \). This example also reveals an inaccuracy in the statement of [Proposition 5.1(2), 5], which only requires the Julia-Carathéodory derivative \( F^\prime_{\mu} \leq p^* \).

Now we are ready to state the main theorem in this section whose proof is based on Proposition 2.1 and Theorem 2.2.

**Theorem 5.7.** Suppose that \( \mu \) is a measure \((\mu \neq \delta_\alpha) \) in \( \mathcal{M} \), and that \( p > 1, q > 0 \) such that \( p^*q \neq 1 \) \((q^* = 1 + pq - p \neq 0)\). Using the notations in Proposition 2.1 and Theorem 2.2, the following statements hold.

1. The nonatomic part of the measure \( \mathbb{B}_{p,q}(\mu) \) is absolutely continuous.
2. The measure \( (\mathbb{B}_{p,q}(\mu))^{ac} \) is concentrated on the closure of \( \psi_p(V^+_p) \).
3. The density of \( (\mathbb{B}_{p,q}(\mu))^{ac} \) on the set \( \psi_p(V^+_p) \) is given by
   \[
   \frac{d((\mathbb{B}_{p,q}(\mu))^{ac})}{dx}(\psi_p(x)) = \frac{(p - 1)pqf_p(x)}{\pi|pqx - q'\psi_p(x) + ipqf_p(x)|^2},
   \]
4. The density of \( (\mathbb{B}_{p,q}(\mu))^{ac} \) is analytic on the set \( \psi_p(V^+_p) \).
5. Let \( n(p,q) \) be the number of the components in the support of \((\mathbb{B}_{p,q}(\mu))^{ac} \).
   Then \( n(p_1,q_1) \geq n(p_2,q_2) \) whenever \( p_1 \leq p_2 \) and \( q_1,q_2 > 0 \).

Particularly, the statements (1)-(5) hold for \( \mathbb{B}_t(\mu), t \in (0, \infty) \setminus \{1\} \).

**Proof.** Since the function \( \psi_p \) defined in Theorem 2.2 is a homeomorphism on \( \mathbb{R} \) and \( \omega_p \) extends continuously to \( \mathbb{R} \) by Proposition 2.1(3), it follows from (5.1) that
\[
F_{\mathbb{B}_{p,q}(\mu)}(\psi_p(x)) = \frac{pqx - q'\psi_p(x) + ipqf_p(x)}{p - 1}, \quad x \in \mathbb{R}.
\]
(5.3)
Since \( F_{\mathbb{B}_{p,q}(\mu)}(\mu) \) extends continuously to \( \mathbb{C}^+ \cup \mathbb{R} \), by the inversion formula (2.1) we obtain
\[
\frac{d((\mathbb{B}_{p,q}(\mu))^{ac})}{dx}(\psi_p(x)) = \frac{(p - 1)pqf_p(x)}{\pi|pqx - q'\psi_p(x) + ipqf_p(x)|^2}, \quad x \in V^+_p.
\]
Comparing the above formula with (2.7) shows that the supports of \((\mu^{ac})^{ac} \) and \((\mathbb{B}_{p,q}(\mu))^{ac} \) coincide for any \( q > 0 \). Observe that \( 3\omega_p(\psi_p(x)) = f_p(x) > 0 \) for \( x \in V^+_p \), whence \( \omega_p \) is analytic on \( V^+_p \) by Proposition 2.1(4). From the preceding discussion, we deduce that statements (2)-(5) hold by Theorem 2.2.

Next, let \( p' = pq/q' \). We claim that if a point \( \alpha \in \mathbb{R} \) such that \( F_{\mathbb{B}_{p,q}(\mu)}(\alpha) = 0 \) and the Julia-Carathéodory derivative \( F^\prime_{\mathbb{B}_{p,q}(\mu)}(\alpha) = \infty \) or, equivalently, \( F^\prime_{\mu}(\alpha/p') = \alpha/q^* \) and the Julia-Carathéodory derivative \( F^\prime_{\mu}(\alpha/p') = p^* \), then \( \alpha \) belongs to the set \( \psi_p(V^+_p) \), which is the closure of \( \psi_p(V^+_p) \). Note that we have \( f_p(\alpha/p') = 0 \) by Proposition 2.1(2) and Lemma 5.2 and there does not exist an open interval \( I \) containing \( \alpha/p' \) such that \( f_p(x) = 0 \) for all \( x \in I \). Indeed, if such an interval \( I \) exists then \( p(I) = 0 \) by [Corollary 3.6, 18]. This implies that the second order derivative of \( f_{\mu} \) on \( I \) is positive, whence \( f_{\mu}(x) \) is strictly convex on \( I \). But \( f_{\mu}(x) \leq (p - 1)^{-1} \) for
all \( x \in I \) and \( f_\mu(\alpha/p') = (p - 1)^{-1} \), a contradiction. This particularly implies that the point \( \alpha/p' \in V_p^0 \), whence

\[
\psi_p(\alpha/p') = H_p(\alpha/p') = \frac{p\alpha}{p'} + (1 - p)F_{p'}(\alpha/p') = \alpha \in \psi_p(V_p^0),
\]

and the claim follows. Moreover, we see that the set

\[
\{ x \in \mathbb{R} : f_p(x/p') = 0, \ \psi_p(x/p') = x \ \text{and} \ F_{p'}(x/p') < p^* \}
\]

is the collection of all atoms of \( \mathbb{B}_{p,q}(\mu) \) by Proposition 5.6 and the set

\[
\{ \psi_p(x/p') : x \in \mathbb{R}, \ f_p(x/p') = 0, \ \psi_p(x/p') \neq x \}
\]

has \( \mathbb{B}_{p,q}(\mu) \)-measure zero by the established result (2). Then the preceding discussions and the established results (2) and (3) show that \( \mathbb{R} = \psi_p(\mathbb{R}) \) consists of the atoms of \( \mathbb{B}_{p,q}(\mu) \) and the support of \( (\mathbb{B}_{p,q}(\mu))^{ac} \), and therefore the assertion (1) follows.

For the rest of the paper, we turn the attention to numbers \( p > 1 \) and \( q > 0 \) such that \( p^*q = 1 \). The following proposition follows from Lemma 5.2 Proposition 5.3 and 5.5 and the proof is left to the reader.

**Proposition 5.8.** If \( p > 1 \) and \( \alpha \in \mathbb{R} \) then the following statements are equivalent:

1. The point \( \alpha \) is an atom of the measure \( \mathbb{B}_{p,1/p'}(\mu) \):
2. \( F_{p,0}(\alpha) = \alpha/(1 - p) \) and the Julia-Carathéodory derivative \( F'_{p,0}(\alpha) \in (1, \infty) \).
3. \( F_{p,0}(0) = \alpha/(1 - p) \) and the Julia-Carathéodory derivative \( F'_{p,0}(0) \in (1, p^*) \);
4. \( F_{p,0}(0) = \alpha/(1 - p) \) and \( 0 < f_{p,0}(0) < 1/(p - 1) \).

If \( \mathbb{B}_{p,1/p'}(\mu) \{ \{ \alpha \} \} = 1 - r^{-1} \) for some \( r > 1 \) then

\[
F'_{p,0}(0) = 1 + f_{p,0}(0) = 1 + \frac{1}{r(p - 1)} \quad \text{and} \quad F'_{p,0}(\alpha) = 1 + \frac{p}{(p - 1)(r - 1)}.
\]

Particularly, the above statements also holds for \( \mathbb{E}_1 \).

If \( \mu_0 \) is the measure defined in Proposition 3.11 then it is clear that \( F_{\mu_0}(0) = 0 \) and \( f_{\mu_0}(0) = 1 \). This yields that the compound free Poisson distribution \( p(\lambda, \nu) \) has an atom at 0 of mass \( 1 - \lambda \) for \( 0 < \lambda < 1 \) and no atom for \( \lambda \geq 1 \) by Proposition 5.5.

The following theorem is a reformulation of Theorem 3.10 since \( \Omega = \Omega_p, \ \psi = \psi_p, \) and \( f = f_p \). Therefore, its proof is practically identical with that of Theorem 3.10 or Theorem 5.8 and is omitted.

**Theorem 5.9.** If \( \mu \in \mathcal{M} \) and \( p > 1 \) then the following statements hold.

1. The nonatomic part of \( \mathbb{B}_{p,1/p'}(\mu) \) is absolutely continuous.
2. The measure \( (\mathbb{B}_{p,1/p'}(\mu))^{ac} \) is concentrated on the closure of \( \psi_p(V_p^+). \)
3. The density of \( (\mathbb{B}_{p,1/p'}(\mu))^{ac} \) on the set \( \psi_p(V_p^+) \) is given by

\[
\frac{d(\mathbb{B}_{p,1/p'}(\mu))^{ac}}{dx}(\psi_p(x)) = \frac{f_p(x)}{\pi(x^2 + f_p^2(x))}.
\]

4. The density of \( (\mathbb{B}_{p,1/p'}(\mu))^{ac} \) is analytic on the set \( \psi_p(V_p^+) \).
5. The number of the components in the support of \( (\mathbb{B}_{p,1/p'}(\mu))^{ac} \) is a decreasing function of \( p \).

Particularly, the above statements also holds for \( \mathbb{E}_1 \).
Since $\mu = \nu^\oplus p$ for some $\nu \in \mathcal{M}$ and $p > 1$ if $\text{Ind}(\mu) > 0$ by Proposition 3.5, we have the next result by Theorem 5.7 and 5.9.

**Corollary 5.10.** If $\mu \in \mathcal{M}$ with $\text{Ind}(\mu) > 0$ then $(\mu^{(q)\text{ac}})$ and $\mu^{\text{ac}}$ contain the same number of components in their supports for any $q > 0$.

It was shown in [18] that there exists a measure $\mu \in \mathcal{M}$ such that $\mu^\oplus p$ contains infinitely many components in the support for any $p > 1$. Since $(B_{p,q}(\mu))^{\text{ac}}$ and $(\mu^\oplus p)^{\text{ac}}$ have the number of components in their supports, we have the following result.

**Proposition 5.11.** For any $t > 0$, there exists a measure $\mu_t \in \mathcal{M}$ such that $\text{Ind}(\mu_t) = t$ and the support of $\mu_t$ contains infinitely many components.

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