ADDITIVITY FOR THE PARAMETRIZED TOPOLOGICAL EULER CHARACTERISTIC AND REIDMEISTER TORSION

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Abstract. Dwyer, Weiss, and Williams have recently defined the notions of the parametrized topological Euler characteristic and the parametrized topological Reidemeister torsion which are invariants of bundles of compact topological manifolds. We show that these invariants satisfy additivity formulas paralleling the additive properties of the classical Euler characteristic and the Reidemeister torsion of CW–complexes.

1. Introduction

In recent years various attempts have been made to generalize the classical Reidemeister torsion (which is an invariant of non-simply connected finite CW complexes) to a parametrized version, i.e. to an invariant of fiber bundles. One such generalization is the notion of an analytic torsion introduced by Bismut and Lott [1]. It is defined for bundles of smooth manifolds satisfying some additional conditions. Another definition was proposed by Igusa and Klein [12],[11] who construct parametrized Reidemeister torsion - also for bundles of smooth manifolds - using generalized Morse functions.

Our main interest in this paper lies in yet another definition of torsion developed by Dwyer, Weiss and Williams [5]. One of the main features of their construction is that it is described in the language of the homotopy theory which makes it relatively simple. Briefly, it proceeds as follows. Given a manifold $M$ and a locally constant sheaf of $R$-modules $\rho: V \to M$ such that the homology groups $H_*(M, \rho)$ vanish, the classical Reidemeister torsion of $M$ can be defined as an element of a group $\text{Wh}(\rho)$ which is a certain quotient of $K_1(R)$. Let $Q(M_\infty)$ denote the infinite loops space associated with the suspension spectrum of $M$. The starting point for the construction of Dwyer, Weiss and Williams is the observation that the group $\text{Wh}(\rho)$ can be identified with $\pi_0 \Phi_\infty(\rho)$ where $\Phi_\infty(\rho)$ is the homotopy fiber of a certain map $\lambda: Q(M_\infty) \to K(R)$ defined in terms of the sheaf $\rho$. It follows that the classical torsion of $M$ determines a connectedness component of $\Phi_\infty(\rho)$. In fact, more is true: one can construct the torsion invariant as a unique point $\tau_\rho^s(M) \in \Phi_\rho^s(M)$. Assume now that we are given a smooth bundle $p: E \to B$ and a sheaf of $R$-modules $\rho: V \to E$ such that the homology groups of the fibers $M_b$ of $p$ with coefficients in $\rho|_{M_b}$ vanish. For every $b \in B$ we obtain a point $\tau_\rho^s(M_b) \in \Phi_\rho^s(M_b)$. The spaces $\Phi_\rho^s(M_b)$ can be glued together using the topology of the bundle $p$ in such way that we obtain a fibration $\Phi_\rho^s(p) \to B$. The assignment $b \mapsto \tau_\rho^s(M_b)$ defines then a section $\tau_\rho^s(p)$ of this fibration which can be interpreted as a parametrized smooth torsion of the bundle $p$.

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An advantage of the construction of torsion sketched above is its flexibility: while $\tau^s_\rho(p)$ is defined for smooth bundles of manifolds, its variants can be used to extend the notion of parametrized torsion to more general settings. In fact, Dwyer, Weiss and Williams gave two additional versions of their definition. The homotopy Reidemeister torsion $\tau^h_\rho(p)$ is defined for any fibration $p : E \to B$ with homotopy finitely dominated fibers and is obtained just as the smooth torsion, but with $Q(M_+)$ replaced by $A(M)$ – the Waldhausen $A$-theory of $M$. The topological Reidemeister torsion $\tau^t_\rho(p)$ exists whenever $p : E \to B$ happens to be a bundle of compact topological manifolds and is constructed using $A^\infty(M)$ - the exciscive version of the $A$-theory.

Each of these invariants takes values in a different infinite loop space; thus topological torsion of a smooth bundle is not the same as its smooth torsion, but it can be seen as an approximation of the smooth torsion. Similarly, homotopy torsion can be considered as an approximation of the topological torsion when both are defined.

The work of Goette [8], [9] and Igusa extended our understanding of the relationship between the analytical torsion of Bismut–Lott and the torsion of Igusa–Klein. It is far less clear, however, how these notions relate to the smooth torsion of Dwyer–Weiss–Williams. Our goal here is to bring these constructions closer together. The starting point is an axiomatization of parametrized Reidemeister torsion proposed by Igusa [10]. He showed that if a cohomological version of torsion of smooth bundles satisfies two conditions, then it is unique up to a scalar multiple. The first condition is the product formula relating the torsion of a composition of two fibrations $p_2 \circ p_1$ to the torsion of $p_1$ and $p_2$. The second condition is additivity, which describes the torsion of a pushout of bundles over a space $B$. In [2] the second author showed that a homotopy theoretical analog of additivity is satisfied by the homotopy torsion of Dwyer, Weiss and Williams. The main result of our present paper shows that additivity holds also for the topological torsion $\tau^t_\rho$.

1.1. Definition. Let $p : E \to B$ be a bundle of closed topological manifolds. We say that $p$ admits a fiberwise codimension one splitting if there are subbundles of manifolds $p_i : E_i \to B$ ($i = 0, 1, 2$) such that $E = E_1 \cup_{E_0} E_2$, fibers of $p_1, p_2$ are compact submanifolds (with boundary) of the fibers of $p$, and the fibers of $p_0$ are the common boundary of the fibers of $p_1$ and $p_2$.

1.2. Theorem. Suppose that $p : E \to B$ is a bundle of closed manifolds which admits a fiberwise codimension one splitting into subbundles $p_i : E_i \to B$ for $i = 0, 1, 2$. Let $R$ be a ring and $\rho : V \to E$ be a locally constant sheaf of finitely generated projective left $R$-modules. Finally, assume that $H_*(p^{-1}(b); \rho) = 0$ and $H_*(p_i^{-1}(b); \rho|_{E_i}) = 0$ for $i = 0, 1, 2$ and all $b \in B$. Then there exists a preferred homotopy class of paths in the topological Whitehead space $\Phi^t_\rho(p)$ joining $\tau^t_\rho(p)$ with $j_1*\tau^t_\rho(p_1) + j_2*\tau^t_\rho(p_2) - j_0*\tau^t_\rho(p_0)$, where $j_\ast$ is the map induced by the inclusion $j_i : E_i \hookrightarrow E$.

The definitions of $\Phi^t_\rho(p)$ and $\tau^t_\rho(p)$ are recalled in Section 2. We note that this property of $\tau^t_\rho$ parallels the additivity of the classical combinatorial Reidemeister torsion relating torsion of a finite CW–complex $X \cup_Z Y$ to the torsions of $X$, $Y$ and $Z$. 
While the proof of Theorem 1.2 is more subtle than in the case of homotopy torsion, the essential idea is to reduce the problem to additivity of $\tau^h_\rho$ and then use the arguments of [2]. We expect that, similarly, a proof of the additivity for smooth torsion may be obtained by reduction to topological case and application of Theorem 1.2.

The main component of the proof of Theorem 1.2 is the additivity theorem for the topological Euler characteristic:

1.3. Theorem. Let $p : E \to B$ be a fiber bundle of closed topological manifolds admitting a fiberwise codimension one splitting as in Theorem 1.2. There exists a preferred homotopy class of paths in $A^\% (p)$ joining $\chi^t (p)$ with $j_1 \ast \chi^t (p_1) + j_2 \ast \chi^t (p_2) - j_0 \ast \chi^t (p_0)$.

The space $A^\% (p)$ denotes here the parametrized excisive $A$-theory of $p$, and $\chi^t (p) \in A^\% (p)$ is the topological Euler characteristic of $p$. Again, definitions of these notions are sketched in Section 2.

In the present paper Theorem 1.3 serves as a step in establishing the additivity of topological torsion, but it has also other potential applications. One of the main results of [5] says that a bundle of compact manifolds $p : E \to B$ is fiber homotopy equivalent to a smooth bundle if and only if $\chi^t (p)$ can be lifted to the infinite loop space associated with the parametrized suspension spectrum of $p$. From this perspective additivity of the topological Euler characteristic provides a tool for computing an obstruction for smoothing of the bundle $p$.

1.4. Organization of the paper. In Section 2 we recall the constructions of Dwyer, Weiss, and Williams. In their setting the topological (resp. homotopy) Reidemeister torsion can be thought of as a lift of the parametrized topological (resp. homotopy) Euler characteristic. In order to prove additivity for $\tau^h_\rho$ it is then enough to verify that additivity holds for the topological Euler characteristic, and then show that we can lift the resulting path. As the first step we demonstrate (§3) that excisive Euler characteristic is additive in non-parametrized case, that is for bundles over a one-point space. In Section 4 we show how to extend this result to bundles of manifolds with a discrete structure group. Subsequently in §5 we show that additivity of the Euler characteristic for arbitrary bundles follows from additivity for certain universal bundles. Then, in Section 6 we show that additivity for the universal bundles follows from additivity of the topological Euler characteristic for bundles with a discrete structure groups. This completes the proof of Theorem 1.3. Finally, in Section 7 we prove Theorem 1.2.

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2. Dwyer-Weiss-Williams constructions

The purpose of this section is to provide a quick review of constructions leading to the definition of the topological Reidemeister torsion. We refer to [5] for a detailed treatment of this subject. We also set here the notation which we will use throughout the paper. In general we tried to preserve the notation of [5], although some differences occur.
2.1. Non-parametrized Euler characteristics. By a Waldhausen category we will mean a category $\mathcal{C}$ together with a choice of two subcategories: a subcategory of cofibrations and a subcategory of weak equivalences satisfying the axioms of [21] (in the terminology of [21] such a category $\mathcal{C}$ is called a category with cofibrations and weak equivalences). Applying the $S_*$-construction to $\mathcal{C}$ one obtains $\Omega[wS_\bullet\mathcal{C}]$ – the $K$-theory space of $\mathcal{C}$ [22, 1.3]. This is an infinite loop space which we will denote by $K(\mathcal{C})$. Every object $c \in \mathcal{C}$ represents a point $[c] \in K(\mathcal{C})$, and a weak equivalence $\varphi: c \to c'$ determines a path from $[c]$ to $[c']$.

An exact functor of Waldhausen categories $F: \mathcal{C} \to \mathcal{D}$ is a functor preserving the distinguished subcategories and all other relevant structures. Any such functor induces a map of the associated infinite loop spaces $F_*: K(\mathcal{C}) \to K(\mathcal{D})$. The functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which assigns to a pair of objects $(c, c')$ their coproduct $c \vee c'$ in $\mathcal{C}$ is exact and defines a map $K(\mathcal{C}) \times K(\mathcal{C}) \to K(\mathcal{C})$. This equips $K(\mathcal{C})$ with an $H$-space structure such that $[c] + [c'] = [c \vee c']$. If we have defined a suspension functor $\Sigma: \mathcal{C} \to \mathcal{C}$ [22, p. 349] then the map $K(\mathcal{C}) \to K(\mathcal{C})$ induced by $\Sigma$ represents a homotopy inverse with respect to the $H$-space structure on $K(\mathcal{C})$. Thus, for $c \in \mathcal{C}$ we can write $-[c] := [\Sigma c]$. All Waldhausen categories considered here come equipped with suspension functors.

If $c \to c' \to c''$ is a cofibration sequence in $\mathcal{C}$ then there is a path in $K(\mathcal{C})$ joining $[c']$ with $[c] + [c'']$. One way to get such a path is to use Waldhausen’s additivity theorem [22, Prop. 1.3.2]. Another way is to observe that (in the notation of [22, 1.3]) a cofibration sequence $c \to c' \to c''$ defines a point $[c \to c' \to c''] \in |wS_2\mathcal{C}|$. The restriction of the map $|wS_2\mathcal{C}| \times \Delta^2 \to |wS_\bullet\mathcal{C}|$ to $[c \to c' \to c''] \times \Delta^2$ yields the desired path [22, 1.3.3]. This second construction is more explicit and easily adapts to the parametrized setting (see Lemma [1.3]). This is the construction we are using throughout the paper.

Following [5, p. 40] we will denote by $\mathcal{R}_{}^{fd}(X)$ the category of homotopy finitely dominated retractive spaces over $X$. The objects of $\mathcal{R}_{}^{fd}(X)$ are diagrams

$$Y \xrightarrow{r} X \xleftarrow{s}$$

such that $r \circ s = \text{id}_X$, $s$ is a cofibration, and $Y$ is a homotopy finitely dominated space over $X$. The category $\mathcal{R}_{}^{fd}(X)$ can be equipped with a Waldhausen category structure where a morphism in $\mathcal{R}_{}^{fd}(X)$ is a weak equivalence or a cofibration if its underlying map of spaces is a homotopy equivalence or, respectively, a map with the homotopy extension property. Its $K$-theory space is denoted $A(X)$ and called the $A$-theory of $X$.

2.2. Definition. Let $X$ be a finitely dominated space. The characteristic object $X^h \in \mathcal{R}_{}^{fd}(X)$ is the retractive space

$$X \times \{-1,1\} \xrightarrow{r} X \xleftarrow{s}$$

where $s(X) = X \times \{-1\}$, and $r$ is the projection map. The homotopy Euler characteristic of $X$ is the point $\chi(X) \in A(X)$ represented by $X^h$.

The assignment $X \mapsto A(X)$ defines a functor on the category of finitely dominated spaces. It is not a homology theory since it does not satisfy the excision axiom. By
there exists a functor $X \to A^\%_X$ which in a certain sense is the best possible approximation of $A(-)$ by an excisive functor. In [5, §7] the authors show that if $X$ is an Euclidean neighborhood retract (ENR) then $A^\%_X$ can be explicitly constructed using Waldhausen categories. We outline this construction next.

For an ENR space $X$ we have a category $R^{\text{ld}}(JX)$ the objects of which are diagrams

$$Y \xleftarrow{s} X \times [0, \infty)$$

such that $r \circ s = \text{id}_{X \times [0, \infty)}$ and where $Y$ is homotopy locally finitely dominated as a space over $X \times [0, \infty)$ [5, p.48]. Morphisms in $R^{\text{ld}}(JX)$ are retractive maps. Given objects $r_i: Y_i \xleftarrow{s_i} X \times [0, \infty)$ for $i = 1, 2$ and morphisms $f, g: Y_1 \to Y_2$ we have the notion of controlled homotopy between $f$ and $g$. By this we mean a map $H: Y_1 \times [0, 1] \to Y_2$ which gives a homotopy between $f$ and $g$ in the usual sense and which commutes with the maps $s_1, s_2$, but commutes with the retractions $r_1, r_2$ only in a relaxed, controlled way [5, p. 47].

The category $R^{\text{ld}}(JX)$ can be equipped with a Waldhausen category structure where weak equivalences are controlled homotopy equivalences and cofibrations are the maps with the controlled homotopy extension property. The $K$-theory space of $R^{\text{ld}}(JX)$ will be denoted by $A^J(X)$.

We have a functor

$$I: R^{\text{fd}}(X) \to R^{\text{ld}}(JX)$$

which assigns to a retractive space $r: Y \xleftarrow{s} X$ over $X$ a retractive space over $X \times [0, \infty)$

$$\overline{Y} \xleftarrow{s} X \times [0, \infty)$$

where $\overline{Y}$ is a pushout in the diagram

$$\begin{array}{ccc}
X = X \times \{0\} & \longrightarrow & X \times [0, \infty) \\
| & | \\
| & | \\
Y & \xrightarrow{s} & \overline{Y}
\end{array}$$

The functor $I$ is an embedding of categories, and – considered as a functor of Waldhausen categories – it is exact so it induces a map of the $K$-theory spaces

$$I_*: A(X) \to A^J(X)$$

Next, let $V(X)$ denote the category of proper retractive ENRs over $X \times [0, \infty)$. This is a subcategory of $R^{\text{ld}}(JX)$ whose objects $r: Y \xleftarrow{s} X \times [0, \infty)$ satisfy the conditions that $Y$ is an ENR and that $r$ is a proper map [5, 7.8]. Let $J: V(X) \to R^{\text{ld}}(JX)$ be the inclusion functor. The category $V(X)$ admits a Waldhausen category structure such that $J$ becomes an exact functor [5, p. 51]. As a consequence we obtain an infinite loop space

$$V(X) := K(V(X))$$

and a map $J_*: V(X) \to A^J(X)$.

2.3. Definition. Let $X$ be a compact ENR. The space $A^\%_X$ is the homotopy limit

$$A^\%_X := \text{holim} \ (A(X) \xrightarrow{I_*} A^J(X) \xleftarrow{J_*} V(X))$$
We call $A^\%(X)$ the excisive $A$-theory of $X$. The natural map $\alpha: A^\%(X) \to A(X)$ is called the assembly map.

2.4. Remark. 1) In [5, § 7] the space $A^\%(X)$ is defined in somewhat different manner, as a homotopy fiber of a map $V(X) \to K(\mathcal{RG}^{ld}(X))$ where $\mathcal{RG}^{ld}(X)$ is the Waldhausen category with the same objects as $\mathcal{RL}^{id}(JX)$ but with germs of retractive maps as morphisms. The resulting infinite loop space is however homotopy equivalent to the one described above (see [5, Lemma 8.7]).

2) The space $V(X)$ is in fact contractible [5, p. 52], so $A^\%(X)$ is equivalent to a homotopy fiber of the map $I_*$. The above description of $A^\%(X)$ allows us however to perform certain constructions in $A^\%(X)$ combinatorially, on the level of Waldhausen categories as follows. For a compact ENR $X$ let $\mathcal{R}^\%(X)$ denote the pullback of the diagram of categories

$$
\begin{array}{ccc}
\mathcal{R}^{fd}(X) & \xrightarrow{I} & \mathcal{R}^{id}(JX) \\
\downarrow & & \downarrow \\
\mathcal{V}(X) & \xleftarrow{J} & X
\end{array}
$$

Thus, objects of $\mathcal{R}^\%(X)$ are pairs $(a, b)$ where $a \in \mathcal{R}^{fd}(X)$, $b \in \mathcal{V}(X)$, and $I(a) = J(b)$, and morphisms are defined similarly. The category $\mathcal{R}^\%(X)$ has the obvious structure of a Waldhausen category such that the functors $\mathcal{R}^\%(X) \to \mathcal{R}^{fd}(X)$, $\mathcal{R}^\%(X) \to \mathcal{V}(X)$ are exact. It follows that we have a commutative diagram of infinite loop spaces:

$$
\begin{array}{ccc}
K(\mathcal{R}^\%(X)) & \xrightarrow{} & V(X) \\
\downarrow & & \downarrow \\
A(X) & \xrightarrow{} & A^d(X)
\end{array}
$$

As a consequence we obtain a map $K(\mathcal{R}^\%(X)) \to A^\%(X)$. In particular any object $(a, b) \in \mathcal{R}^\%(X)$ determines a point $[a, b] \in A^\%(X)$, a weak equivalence $(a, b) \to (a', b')$ defines a path from $[a, b]$ to $[a', b']$, and a cofibration sequence

$$(a, b) \to (a', b') \to (a'', b'')$$

in $\mathcal{R}^\%(X)$ determines a path in $A^\%(X)$ joining $[a', b']$ with $[a, b] + [a'', b'']$. It will be sometimes convenient to describe points and paths in $A^\%(X)$ in this way.

We are now ready to define the topological Euler characteristic of a space.

2.5. Definition. Let $X$ be an ENR. The characteristic object of $X$ in $\mathcal{V}(X)$ is the object $X^w$ given by the retractive space

$$X \times \{0\} \sqcup X \times [0, \infty) \xrightarrow{\sim} X \times [0, \infty)$$

The topological Euler characteristic of $X$ is the point $\chi^t(X) \in A^\%(X)$ determined by the object $X^t := (X^h, X^w) \in \mathcal{R}^\%(X)$.

Notice that we have $\alpha(\chi^t(X)) = \chi^h(X)$. 
2.6. Remark. Let $f : X \to Y$ be a map of spaces. By abuse of notation by $f_*$ we will denote each of the functors induced by $f$: $\mathcal{R}^{fd}(X) \to \mathcal{R}^{fd}(Y)$, $\mathcal{V}(X) \to \mathcal{V}(Y)$, $\mathcal{R}^{id}(JX) \to \mathcal{R}^{id}(JY)$, and $\mathcal{R}^%(X) \to \mathcal{R}^%(Y)$, as well as the maps of infinite loop spaces: $A(X) \to A(Y)$, $V(X) \to V(Y)$, $A^i(X) \to A^i(Y)$, and $A^%(X) \to A^%(Y)$. Euler characteristics are not preserved in general by the maps $f_*$. However, we have the following

2.7. Lemma. Let $f : X \to Y$ be a homotopy equivalence of finitely dominated spaces. We have a canonical weak equivalence $f^h : f_*(X^h) \to Y^h$ in $\mathcal{R}^{fd}(Y)$. Moreover, if $g : Y \to Z$ is another homotopy equivalence, then $(gf)^h = g^h \circ g_*(f^h)$.

Proof. Define $f^h := f \sqcup \text{id}: f_*(X^h) = X \sqcup Y \to Y \sqcup Y$ The properties of $f^h$ are straightforward to check. □

2.8. Corollary. If $f : X \to Y$ is a homotopy equivalence of finitely dominated spaces then there is a canonical path $\omega_f$ from $f_*\chi^h(X)$ to $\chi^h(Y)$.

2.9. Lemma. Let $f : X \to Y$ be a cell-like map [15], [12], [13] of compact ENRs. We have a canonical weak equivalence $f^v : f_*(X^v) \to Y^v$ in $\mathcal{V}(X)$. Moreover the weak equivalences $f^h$ and $f^v$ satisfy the equations $I(f^h) = J(f^v)$, and so they define a weak equivalence $f^t := (f^h, f^v) : f_*(X^t) \to Y^t$ in $\mathcal{R}^%(Y)$. The weak equivalences $f^v$ and $f^t$ satisfy a cochain condition analogous to the one described in Lemma 2.7.

Proof. The map $f^v$ is given by $f^v := f \sqcup \text{id}: f_*(X^v) = X \sqcup Y \times [0, \infty) \to Y \sqcup Y \times [0, \infty)$ It is a weak equivalence in $\mathcal{V}(X)$ by [5] p. 53. Verification of the remaining properties of $f^v$ is straightforward. □

2.10. Corollary. If $f : X \to Y$ is a cell-like map of compact ENRs then there is a canonical path $\sigma_f$ joining $f_*(\chi^h(X))$ with $\chi^h(Y)$ Moreover, if $a : A^%(X) \to A(X)$ is the assembly map, then $a(\sigma_f) = \omega_f$ where $\omega_f$ is the path from Lemma 2.8.

We will refer to the properties of $\chi^h(-)$ and $\chi^t(-)$ described in Corollaries 2.8 and 2.10 as the lax naturality of Euler characteristics.

2.11. Parametrization. The constructions sketched above can be generalized to the setting where the space $X$ is replaced by a fibration $E \xrightarrow{p} B$ with a fiber $F$. Intuitively, one can construct in this case a fibration $A_B(E) \to B$ the fiber of which is $A(F)$. An analog of the homotopy Euler characteristic in this context is a section of this fibration which restricts to $\chi^h(F)$ over every point of $B$. A similar idea underlies the notions of the parametrized excisive $A$-theory and the parametrized topological Euler characteristic. For technical reasons it is more convenient, however, to define the parametrized Euler characteristics in different terms. We give these formal definitions first, and then explain how they relate to the above idea.
Let $\mathcal{C}$ be a small category, and let $F: \mathcal{C} \to \text{Spaces}$ be a functor. Recall that $\text{holim}_c F$ is the space of natural transformations

$$\text{holim}_c F := \text{Map}_\mathcal{C}(|\mathcal{C}/-|, F)$$

where $|\mathcal{C}/-|$ is the functor which assigns to $c \in \mathcal{C}$ the nerve of the overcategory $\mathcal{C}/c$. The following fact is implicitly present in \cite{4}.

2.12. Lemma. Let $\text{WCat}$ denote the category of Waldhausen categories with exact functors as morphisms. Assume that for a functor $F: \mathcal{C} \to \text{WCat}$ we have a rule which assigns to $c \in \mathcal{C}$ an object $c^i \in F(c)$, and to $f \in \text{Mor}_\mathcal{C}(c,d)$ a weak equivalence $f^i: F(f)(c^i) \to d^i$ in $F(d)$ in such way that $(g \circ f)^i = g^i \circ F(f)(f^i)$. Then the assignment $|\mathcal{C}/c| \mapsto [c^i]$ defines a point $[c^i, f^i] \in \text{holim}_c K(F(c))$.

Indeed, if $wF(c)$ denotes the subcategory of weak equivalences in the Waldhausen category $F(c)$, then the assignments $c \mapsto c^i$, $f \mapsto f^i$ as in Lemma 2.12 defines a point in $\text{holim}_c K(F(c))$. Using the natural transformation of functors $|wF(c)| \to K(F(c))$ we obtain a point in $\text{holim}_c K(F(c))$ as claimed.

Now, let $\mathcal{B}: \Delta^{op} \to \text{Sets}$ be a simplicial set. Denote by $S(\mathcal{B})$ the category whose objects are all simplices $x \in \mathcal{B}$, and where morphisms $x \to y$ in $S(\mathcal{B})$ come from morphisms $\varphi$ in $\Delta^{op}$ satisfying $\mathcal{B}(\varphi)(y) = x$ (thus $S(\mathcal{B})$ is the opposite category of the Grothendieck construction on the functor $\mathcal{B}$ \cite[p.22]{4}). Let $B$ be the geometric realization of $\mathcal{B}$, and for $x \in \mathcal{B}$ let $\varphi_x: \Delta^{|x|} \to B$ denote the characteristic map of $x$. Assume that we have a fibration $E \to B$ with a homotopy finitely dominated fiber $F$. We can define a functor

$$S(\mathcal{B}) \to \text{Spaces}, \quad x \mapsto E_x$$

where $E_x := \text{lim}(E \to B \xrightarrow{\varphi_x} \Delta^{|x|})$. As a result we get a functor

$$F: S(\mathcal{B}) \to \text{WCat}, \quad x \mapsto \mathcal{R}^{fd}(E_x)$$

For $x \in S(\mathcal{B})$ consider the assignment $x \mapsto E^h_x$ where $E^h_x \in \mathcal{R}^{fd}(X)$ is the characteristic object of $E_x$ \cite{2.2}. Notice that for any morphism $f: x \to y$ in $S(\mathcal{B})$ the map $F(f): E_x \to E_y$ is a homotopy equivalence, so by Lemma 2.7 it defines a weak equivalence $F(f)^h: F(f)_*(E^h_x) \to E^h_y$ in $\mathcal{R}^{fd}(E_y)$. Lemma 2.7 also shows that the assignments $x \mapsto E^h_x$, $f \mapsto F(f)^h$ satisfy the conditions Lemma 2.12 and so they define a point $[E^h_x, F(f)^h] \in \text{holim}_{S(\mathcal{B})} A(E_x)$.

2.13. Definition. The homotopy Euler characteristic of a fibration $E \to B$ is the point $[E^h_x, F(f)^h] \in \text{holim}_{S(\mathcal{B})} A(E_x)$. We will write $\chi^h(p) := [E^h_x, F(f)^h]$. Next, notice that the constant maps $A(E_x) \to \ast$ define a map of homotopy colimits

$$p_*: \text{hocolim}_{S(\mathcal{B})} A(E_x) \to \text{hocolim}_{S(\mathcal{B})} \ast = B$$

which by \cite[p.180]{4} is a quasi-fibration with the fiber $A(F)$. Let $p_*: A_B(E) \to B$ be the fibration associated to this quasi-fibration. For all morphisms $x \to y$ in $S(\mathcal{B})$ the map
$E_x \to E_y$ is a homotopy equivalence, and thus so is the map $A(E_x) \to A(E_y)$. Therefore as an application of [3, Prop. 3.12] we obtain

### 2.14. Proposition. The map

\[ \text{holim}_{x \in \mathcal{S}(\mathcal{B})} A(E_x) = \text{Map}_{\mathcal{S}(\mathcal{B})}(\mathcal{S}(\mathcal{B})/|x|, A(E(-))) \to \text{Map}_B(\text{holim}_{x \in \mathcal{S}(\mathcal{B})} \mathcal{S}(\mathcal{B})/x, \text{holim}_{x \in \mathcal{S}(\mathcal{B})} A(E_x)) \]

\[ f \mapsto \text{holim}_{x \in \mathcal{S}(\mathcal{B})} f \]

is a weak equivalence.

Here $\text{Map}_B(-,-)$ denotes the mapping space of spaces over $B$. Also, in the category of spaces over $B$ we have weak equivalences

\[ \text{holim}_{x \in \mathcal{S}(\mathcal{B})} |\mathcal{S}(\mathcal{B})/x| \simeq B \quad \text{and} \quad \text{holim}_{x \in \mathcal{S}(\mathcal{B})} A(E_x) \simeq A_B(E) \]

which combined with Proposition 2.14 give

\[ \text{holim}_{x \in \mathcal{S}(\mathcal{B})} A(E_x) \simeq \text{Map}_B(B, A_B(E)) \]

It follows that $\chi^h(p) \in \text{holim}_{x \in \mathcal{S}(\mathcal{B})} A(E_x)$ defines a point (unique up to a contractible space of choices) in $\Gamma(p_*)$ – the space of sections of the fibration $p_*: A_B(E) \to B$. This brings us back to the intuitive construction of $\chi^h(p)$ sketched at the beginning of this section.

As we have mentioned above the idea behind the definition of the parametrized topological characteristic $\chi^t(p)$ is similar. The details, however, are more involved. The problem is that one cannot (mimicking Definition 2.13) define $\chi^t(p)$ using Lemmas 2.9 and 2.12 since the maps $E_x \to E_y$ are usually not cell-like. In [5] the authors overcome this difficulty by replacing $\mathcal{B}$ with a new simplicial set $p\mathcal{B}$, and the functor $F: \mathcal{S}(\mathcal{B}) \to \text{Spaces}$ by a new functor $tF: \mathcal{S}(p\mathcal{B}) \to \text{Spaces}$, which sends every morphism in $\mathcal{S}(p\mathcal{B})$ to a homeomorphism. Then the assumptions of Lemma 2.9 are satisfied and we can apply 2.12 in order to define a point in $\chi^t(p) \in \text{holim}_{x \in \mathcal{S}(p\mathcal{B})} A^t(tF)$. The details follow.

### 2.15. Definition (Compare [5, p.13]). Let $E \xrightarrow{p} B$ be a (locally trivial) fiber bundle, where $B$ is the geometric realization of a simplicial set $\mathcal{B}$. By $p\mathcal{B}$ we will denote the simplicial set whose $k$-simplices are pairs $(x, \theta)$ where $x$ a $k$-simplex in $\mathcal{B}$, and $\theta$ is an equivalence relation on $E_x$ such that the quotient map $E_x \to E^\theta_x$ and the projection $E_x \to \Delta^k$ give a homeomorphism $E_x \to E^\theta_x \times \Delta^k$.

Notice that if $F$ is the fiber of the fiber bundle $E \xrightarrow{p} B$ then for every $(x, \theta) \in p\mathcal{B}$ we have $E^\theta_x \cong F$ and the assignment $(x, \theta) \mapsto E^\theta_x$ defines a functor $tF: \mathcal{S}(p\mathcal{B}) \to \text{Spaces}$ which sends every morphism in $\mathcal{S}(p\mathcal{B})$ to a homeomorphism.

### 2.16. Definition. Let $E \xrightarrow{p} B$ be a fiber bundle whose fiber $F$ is a compact ENR. The assignments $(x, \theta) \mapsto (E^\theta_x)_t$ (see 2.5), and $f \mapsto F(f)_t$ (2.9), where $f$ is a morphism in $p\mathcal{B}$, satisfy the conditions of Lemma 2.12. Therefore they define a point

\[ \chi^t(p) := [(E^\theta_x)_t, F(f)_t] \in \text{holim}_{(x, \theta) \in p\mathcal{B}} A^t(E^\theta_x) \]
We call $\chi^t(p)$ the topological Euler characteristic of the bundle $p$.

The following fact lets us compare the homotopy and topological Euler characteristics of bundles.

**2.17. Proposition.** Let $E \xrightarrow{p} B$ be a locally trivial fiber bundle whose fiber is a compact topological manifold $M$ (perhaps with boundary). We have a commutative diagram

$$
\begin{array}{ccc}
\text{holim}_x A% \mathcal{E}_x & \simeq & \text{holim}_{(x,\theta)} A% \mathcal{E}_x \\
\alpha \downarrow & & \alpha \downarrow \\
\text{holim}_x A\mathcal{E}_x & \simeq & \text{holim}_{(x,\theta)} A\mathcal{E}_x \\
\end{array}
$$

The vertical maps are induced by assembly maps. All horizontal maps are weak equivalences by homotopy invariance of homotopy limits.

The maps on the right are weak equivalences by homotopy invariance of homotopy limits. The maps on the left are weak equivalences by [5, Corollary 2.7].

**2.18.** The implications of Proposition 2.17 are twofold. On one hand it lets us think about $\chi^t(p)$ as an element of $\text{holim}_{x \in S(pB)} A% \mathcal{E}_x$ defined uniquely up to a contractible space of choices. Similarly as in 2.14 we then get $\text{holim}_{x \in S(pB)} A% \mathcal{E}_x \simeq \Gamma(p%_\ast)$ where $\Gamma(p%_\ast)$ is the space of sections of the fibration $p%_\ast : A_B% = A% \mathcal{E} \to B$ associated to the quasi-fibration $\text{hocolim}_{x \in S(pB)} A% \mathcal{E}_x \to B$ with the fiber $A% \mathcal{E}(M)$. In particular we can think of $\chi^t(p)$ as a section of $p%_\ast$. On the other hand consider the image of $\chi^h(p)$ under the composition of the bottom maps in the diagram above. One can see that the point $\text{holim}_{(x,\theta)} A\mathcal{E}_x$ defined in this way can be explicitly described by means of Lemma 2.12 as coming from the assignment $(x, \theta) \mapsto (E_x\theta)^ah := E_x \sqcup E_x\theta$ for any object $(x, \theta) \in S(pB)$, and $f \mapsto f^{ah}$ for any morphism $f : (x, \theta) \to (y, \theta')$ in $S(pB)$ where $f^{ah}$ is the map

$$f^{ah} : tF(f)(E_x \sqcup E_x\theta) = E_y \sqcup E_y \xrightarrow{id \cup iF(f)} E_y \sqcup E_y$$

We can consider the element $[(E_x\theta)^ah, f^{ah}] \in \text{holim}_{(x,\theta)} A\mathcal{E}_x$ as a re-definition of $\chi^h(p)$. Unlike the non-parametrized case this element is not equal of the image of $\chi^t(p)$ under the assembly map. However, the homotopy equivalences $E_x \sqcup E_x\theta \to E_x\theta \sqcup E_x\theta$ define a canonical path $\sigma_p$ in $\text{holim}_{(x,\theta)} A\mathcal{E}_x$, joining the image of $\chi^t(p)$ with the image of $\chi^h(p)$.

**2.19. Notation.** We will denote by $A(p)$ the homotopy limit $\text{holim}_x A\mathcal{E}_x$. Similarly, $A%(p)$ will denote $\text{holim}_{(x,\theta)} A% \mathcal{E}_x$.

**2.20. Reidemeister Torsions.** Let $R$ be a (discrete) ring with identity, and let $Ch^{fd}(R)$ denote the category of chain complexes of left projective $R$-modules which are chain homotopy equivalent to finitely generated complexes. The category $Ch^{fd}(R)$ can be equipped
with a Waldhausen category structure where weak equivalences are chain homotopy equivalences and cofibrations are chain maps which are split injective on every level. The infinite loop space $K(Ch^{fd}(R))$ is homotopy equivalent to the $K$-theory space of the ring $R$ [5, p. 43]. Thus, from now on we will denote $K(Ch^{fd}(R))$ by $K(R)$.

Let $\rho: V \to E$ be a locally constant sheaf of projective modules. The sheaf $\rho$ induces a functor $L_\rho: R^{fd}(E) \to Ch^{fd}(R)$ which assigns to any object $(X \looparrowright E) \in R^{fd}(E)$ the relative singular chain complex $C(X, E, \rho)$ with local coefficients given by $\rho$. The functor is $L_\rho$ is not exact. It is however close enough to being exact that it still defines a map $L_\rho: A(E) \to K(R)$ if we slightly modify the construction of $A(E)$ and $K(R)$ using a variant of the $S_\bullet$-construction proposed by Thomason (see [5, p. 43]). Assume that $H_*(E, \rho) = 0$. In this case the complex $L_\rho(E^h)$ is acyclic, so the map $0 \to L_\rho(E^h)$ (where $0$ is the zero chain complex) is a weak equivalence in $Ch^{fd}(E)$. This gives a canonical path $\sigma_\rho(E)$ in $K(R)$ joining $L_\rho(\chi^h(E))$ with $\ast$ the basepoint of $K(R)$ which is represented by the chain complex $0$.

2.21. Definition. The pair $(\chi^h(E), \sigma_\rho(E))$ defines a point in $\Phi_\rho^h(E)$ - the homotopy fiber of the map $L_\rho$. This point is called the homotopy Reidemeister torsion of the space $E$ and is denoted by $\tau^h_\rho(E)$.

If $E$ is a compact ENR then let $\Phi_\rho^t(E)$ denote the homotopy fiber of the map $L_\rho \alpha$ where $\alpha$ is the assembly map. The point $\tau^t_\rho(E) \in \Phi_\rho^t(E)$ defined by the pair $(\chi^t(E), \sigma_\rho(E))$ is the topological Reidemeister torsion of $E$.

We will call $\Phi_\rho^h(E)$ and $\Phi_\rho^t(E)$ the homotopy (resp. topological) Whitehead spaces.

Next, let $E \looparrowright B$ be a fibration with a homotopy finitely dominated fiber $F$, where as before $B = |\mathcal{B}|$, and let $\rho: V \to E$ be a locally constant sheaf of finitely generated projective left $R$-modules. In such case following [5, p. 66] we can define fibrations $\Phi_\rho^h(B) \to B$ and $\Phi_\rho^t(B) \to B$ with fibers $\Phi_\rho^h(F)$ and $\Phi_\rho^t(F)$ respectively. If $H_*(E_x, \rho|_{E_x}) = 0$ for all $x \in \mathcal{B}$ these fibrations admit sections which assign $\tau^h_\rho(F)$ (resp. $\tau^t_\rho(F)$) to every point $b \in B$. We can think of these sections as parametrized versions of the Reidemeister torsions. However, similarly as it was the case for the parametrized Euler characteristics (2.11), this describes torsions only up to a contractible space of choices. We will then again need other, more precise definitions.

For $x \in S(\mathcal{B})$ consider the maps $E_x \to E$. The induced functors of Waldhausen categories $R^{fd}(E_x) \to R^{fd}(E)$ define a natural transformation $\eta$ from the functor $F: S(\mathcal{B}) \to W\text{Cat}$, $F(x) = R^{fd}(E_x)$ to the constant functor over $S(\mathcal{B})$ with the value $R^{fd}(E)$. Therefore we obtain a map

$$\eta_*: A(p) = \lim_{x \in S(\mathcal{B})} A(E_x) \to \lim_{x \in S(\mathcal{B})} A(E)$$

Recall the the homotopy Euler characteristic $\chi^h(p)$ was defined as the point in $A(p) = \lim_{x \in A(E_x)}$ represented by the assignments $x \mapsto E_x^h$, $f \mapsto F(f)^h$ (2.13). The image of $\chi^h(p)$ under $\eta_*$ is in turn represented by the assignments $x \mapsto E_x \cup E$, $f \mapsto (F(f) \cup \text{id}: E_x \cup E \to E_y \cup E)$. Assume that $H_*(E_x, \rho|_{E_x}) = 0$ for all $x \in S(\mathcal{B})$. In this case the relative
chain complexes $C(E_x \sqcup E, E, \rho)$ are acyclic, and the weak equivalences $0 \to C(E_x \sqcup E, E, \rho)$ define a canonical path $\sigma_\rho(p)$ joining the basepoint and $L_\rho, \eta_\rho(\chi^h(p))$ in $\text{holim}_x K(R)$.

2.22. Definition. Let $\Phi^h_\rho(p)$ be the homotopy fiber of the map $L_\rho, \eta_\rho : A(p) \to \text{holim}_x K(R)$ over the basepoint of $\text{holim}_x K(R)$. The homotopy Reidemeister torsion of $p$ is the point $\tau^h_\rho(p) \in \Phi^h_\rho(p)$ given by the pair $(\chi^h(p), \sigma_\rho(p))$.

Assume now that $E \xrightarrow{p} B$ is a bundle of compact manifolds. Let $\mathcal{A}^\%_\rho(p)$ be the pullback of the diagram

$$\xymatrix{ \text{holim}_x A(E_x) & \text{holim}_x A(E^\theta_x) \ar[l] \ar[r]^{\alpha} & \text{holim}_x A^\%_\rho(E^\theta_x) \ar[r] \ar[d]_{\delta_1} & \text{holim}_x K(R) }$$

where the maps are as in Proposition 2.17. Recall (2.18) that we have a canonical path $\sigma_\rho$ in $\text{holim}_x A(E^\theta_x)$ joining the images of $\chi^i(p)$ and $\chi^j(p)$. Thus, the triple $(\chi^i(p), \sigma_\rho, \chi^h(p))$ defines a point $\mathcal{A}^\%(p) \in \mathcal{A}^\%_\rho(p)$.

2.23. Definition. Let $\Phi^\%_\rho(p)$ denote the homotopy fiber of the map

$$\mathcal{A}^\%(p) \to \text{holim}_x A(E_x) \xrightarrow{L_\rho, \eta_\rho} \text{holim}_x K(R)$$

If $H_*(E_x, \rho|E_x) = 0$ for all $x \in \mathcal{S}(\mathfrak{B})$ then the pair $(\mathcal{A}^\%(p), \sigma_\rho(p))$ (where $\sigma_\rho(p)$ is the path as in Definition 2.22) defines a point $\tau^\%_\rho(p) \in \Phi^\%_\rho(p)$. We call it the topological parametrized Reidemeister torsion of the bundle $p$.

Notice that we have a pullback diagram

$$\xymatrix{ \Phi^\%_\rho(p) \ar[r]^{\gamma_2} \ar[d]_{\gamma_1} & \mathcal{A}^\%(p) \ar[d] \ar[r]^{\delta_1} & \text{holim}_x K(R) \ar[r]^{\delta_2} & A(p) }$$

and that $\gamma_1(\tau^\%_\rho(p)) = \tau^h_\rho(p)$, $\gamma_2(\tau^\%_\rho(p)) = \mathcal{A}^\%(p)$.

3. Non-parametrized additivity theorem

The goal of this section is to prove the following

3.1. Theorem (Additivity for the topological Euler characteristic). Let $M$ be a closed topological manifold which admits a splitting along a compact codimension one submanifold $M_0$:

$$M = M_1 \cup_{M_0} M_2$$

There exists a preferred path $\omega$ in $A^\%(M)$ from $\chi^i(M)$ to $k_1, \chi^iM_1 + k_2, \chi^iM_2 - k_0, \chi^iM_0$, where $k_i : M_i \hookrightarrow M$ is the inclusion map ($i = 0, 1, 2$).

Applying the assembly map $\alpha : A^\%(M) \to A(M)$ to the path $\omega$ we obtain

3.2. Corollary. If $M$ is a manifold as in Theorem 3.1, then there exists a path in $A(M)$ joining $\chi^h(M)$ with $k_1, \chi^hM_1 + k_2, \chi^hM_2 - k_0, \chi^hM_0$. 


Thus we recover the additivity theorem for the homotopy Euler characteristic which was proved in [2] by the second author.

**Proof of Theorem 3.1.** Consider a manifold
\[ M = M_1 \cup M_0 \times \{-1, 1\} \cup M_0 \times [0, \infty) \]

For \( i = 1, 2 \) let \( \bar{k}_i : M_i \hookrightarrow M \) denote the inclusion map. We have a map \( f : M \to M \) which restricts to the inclusions \( k_1, k_2 \) on \( M_1, M_2 \) respectively, and which sends \( M_0 \times [-1, 1] \subseteq M \to M \) via the projection map onto the first factor.

We will construct the path \( \omega \) as a concatenation of three paths in \( A^\%(M) \):

1) a path \( \omega_1 \) from \( \chi^t(M) \) to \( f^* \chi^t(M) \),
2) a path \( \omega_2 \) from \( f^* \chi^t(M) \) to \( f^*(\bar{k}_1^* \chi^t(M_1) + \bar{k}_2^* \chi^t(M_1) + C) \) for some \( C \in A^\%(\overline{M}) \),
3) a path \( \omega_3 \) from \( f^* \bar{k}_1^* \chi^t(M_1) + f^* \bar{k}_2^* \chi^t(M_1) + f^* C \) to \( k_1^* \chi^h M_1 + k_2^* \chi^h M_2 - k_0^* \chi^h M_0 \).

**Construction of \( \omega_1 \).** Since \( f \) is a cell-like map, the path \( \omega_1 \) exists by lax naturality of \( \chi^t(2.10) \).

**Construction of \( \omega_2 \).** We will construct a point \( C \in A^\%(\overline{M}) \) and a path \( \sigma \in A^\%(\overline{M}) \) joining \( \chi^t(\overline{M}) \) with \( (\bar{k}_1^* \chi^t(M_1) + \bar{k}_2^* \chi^t(M_1) - C) \). Then we will have \( \omega_2 = f^*(\sigma) \).

First we define the point \( C \) as follows. Let \( C' \in R^d(\overline{M}) \) be the retractive space over \( \overline{M} \) given by
\[ C' := \overline{M} \cup M_0 \times \{-1, 1\} \]
and let \( C'' \in V(\overline{M}) \) be given by the retractive space over \( \overline{M} \times [0, \infty) \)
\[ C'' := \overline{M} \times [0, \infty) \cup M_0 \times \{-1, 1\} \]
where the embedding \( M_0 \times \{-1, 1\} \hookrightarrow \overline{M} \times [0, \infty) \) is given by \( (x, \pm 1) \mapsto ((x, \pm 1), 0) \). One can check that (in the notation of Remark 2.4) we have \( I(C') = J(C'') \), so \( (C', C'') \) is an object of \( R^\%(\overline{M}) \). Take \( C \) to be the point in \( A^\%(\overline{M}) \) represented by this object.

Next, recall (2.5) that for any compact ENR \( X \) the Euler characteristic \( \chi^t(X) \) is represented by the object \( X^t = (X^h, X^v) \in R^\%(X) \). Also, notice that in our case the object
\( \tilde{k}_{i*}M^h_i \in \mathcal{R}^{fd}(\overline{M}) \) is the retractive space \( M_i \sqcup \overline{M} \) over \( \overline{M} \). We have a cofibration sequence in \( \mathcal{R}^{fd}(\overline{M}) \)

\[
\tilde{k}_{1*}M^h_1 \to (M_1 \sqcup M_2 \sqcup \overline{M}) \to \tilde{k}_{2*}M^h_2
\]

Similarly, the object \( \tilde{k}_{i*}M^v_i \in \mathcal{V}(\overline{M}) \) is given by the retractive space \( M_i \sqcup \overline{M} \times [0, \infty) \) over \( \overline{M} \times [0, \infty) \), which gives a cofibration sequence in \( \mathcal{V}(\overline{M}) \)

\[
\tilde{k}_{1*}M^v_1 \to (M_1 \sqcup M_2 \sqcup \overline{M} \times [0, \infty)) \to \tilde{k}_{2*}M^v_2
\]

These two cofibration sequences lift to a cofibration sequence in \( \mathcal{R}^{\%}(M) \):

\[
\tilde{k}_{1*}M^v_1 \to (M_1 \sqcup M_2 \sqcup \overline{M}, M_1 \sqcup M_2 \sqcup \overline{M} \times [0, \infty)) \to \tilde{k}_{2*}M^v_2
\]

which gives us a path in \( \mathcal{A}^\%(\overline{M}) \) joining \( \tilde{k}_{1*}\chi^t(M_1) + \tilde{k}_{2*}\chi^t(M_2) \) with the point \( [M_1 \sqcup M_2 \sqcup \overline{M}, M_1 \sqcup M_2 \sqcup \overline{M} \times [0, \infty]) + C \). As a consequence we only need to construct a path from \( \chi^t(\overline{M}) \) to \( [M_1 \sqcup M_2 \sqcup \overline{M}, M_1 \sqcup M_2 \sqcup \overline{M} \times [0, \infty]) + C \). In order to accomplish this notice that \( C' \) fits into a cofibration sequence in \( \mathcal{R}^{fd}(\overline{M}) \)

\[
(M_1 \sqcup M_2) \sqcup \overline{M} \xrightarrow{(\tilde{k}_1 \sqcup \tilde{k}_2) \sqcup 1} \overline{M}^h = \overline{M} \sqcup \overline{M} \to C'
\]

while \( C'' \) is the cofiber in the following cofibration sequence in \( \mathcal{V}(\overline{M}) \):

\[
(M_1 \sqcup M_2) \sqcup \overline{M} \times [0, \infty) \xrightarrow{(\tilde{k}_1 \sqcup \tilde{k}_2) \sqcup 1} \overline{M}^v = \overline{M} \sqcup \overline{M} \times [0, \infty) \to C''
\]

As before these sequences yield a cofibration sequence in \( \mathcal{R}^{\%}(\overline{M}) \)

\[
(M_1 \sqcup M_2 \sqcup \overline{M}, M_1 \sqcup M_2 \sqcup \overline{M} \times [0, \infty)) \to \overline{M}' = (\overline{M}^h, \overline{M}^v) \to (C', C'')
\]

Passing from \( \mathcal{R}^{\%}(\overline{M}) \) to \( \mathcal{A}^\%(\overline{M}) \) we obtain the desired path.

**Construction of \( \omega_3 \).** We will show that in \( \mathcal{A}^\%(M) \) we have paths \( \delta_i \) \( (i = 1, 2) \) joining \( f_\ast \tilde{k}_{i*}\chi^t(M_i) \) with \( k_{i*}\chi^t(M_i) \), and a path \( \delta_0 \) from \( f_\ast C \) (where \( C \in \mathcal{A}^\%(\overline{M}) \) is defined as above) to \( -k_{0*}\chi^tM_0 \). Then we can take \( \omega_3 = \delta_1 + \delta_2 + \delta_0 \).

For \( i = 1, 2 \) we have \( f \circ \tilde{k}_i = k_i \), so \( f_\ast \tilde{k}_{i*}\chi^t(M_i) = k_{i*}\chi^t(M_i) \), and we can choose \( \delta_1, \delta_2 \) to be the constant paths.

The construction of the path \( \delta_0 \) resembles the construction of \( \omega_2 \) above. We have \( f_\ast C = ([f_\ast C'], [f_\ast C'']) \). Notice that \( f_\ast C' \) fits into the following pushout diagram in \( \mathcal{R}^{fd}(M) \):

\[
M_0 \times \{-1, 1\} \longrightarrow M \times [-1, 1] \xrightarrow{f_\ast C'}
\]

Similarly, \( f_\ast C'' \) can be represented as a pushout in \( \mathcal{V}(M) \):

\[
M_0 \times \{-1, 1\} \longrightarrow M \times [-1, 1] \xrightarrow{f_\ast C''}
\]
where the map $M_0 \times \{-1, 1\} \to M \times [0, \infty)$ is given by $(x, \pm 1) \mapsto (x, 0)$. As a consequence we have a cofibration sequence in the category $R^f(M)$:

$$M_0 \sqcup M \to M_0 \times [-1, 1] \cup_{M_0 \times \{-1\}} M \to f_*C'$$

as well as a cofibration sequence in $V(M)$:

$$M_0 \sqcup M \times [0, \infty) \to M_0 \times [-1, 1] \cup_{M_0 \times \{-1\}} M \times [0, \infty) \to f_*C''$$

(we identify here $M_0 \times \{-1\}$ with a subspace of $M \times [0, \infty)$ via the embedding $(x, -1) \mapsto (x, 0)$). Notice that $M_0 \sqcup M = k_0 s M_0$ in $R^f(M)$, and $M_0 \sqcup M \times [0, \infty) = k_0 s M_0$ in $V(M)$.

Again, we can lift these two sequence to a cofibration in $R^\%(M)$

$$k_0 s (M^h, M^v) \to (M_0 \times [-1, 1] \cup_{M_0 \times \{-1\}} M, M_0 \times [-1, 1] \cup_{M_0 \times \{-1\}} M \times [0, \infty)) \to f_* (C', C'')$$

As a consequence we obtain a path $\delta_0'$ in $A^\%(M)$ joining $k_0 s \chi^t(M_0) + f_* C$ with the point

$$[M_0 \times [-1, 1] \cup_{M_0 \times \{-1\}} M, M_0 \times [-1, 1] \cup_{M_0 \times \{-1\}} M \times [0, \infty)] \in A^\%(M)$$

Finally notice that the retractive spaces $M_0 \times [-1, 1] \cup_{M_0 \times \{-1\}} M$ and $M_0 \times [-1, 1] \cup_{M_0 \times \{-1\}} M \times [0, \infty)$ are weakly equivalent to the trivial retractive spaces $M \Rightarrow M$, and (respectively) $M \Rightarrow M \times [0, \infty)$, which implies that $\delta_0'$ can be further extended to the basepoint of $A^\%(M)$. The path $\delta_0'$ can be now obtained shifting $\delta_0'$ by the element $-k_0 \chi^t(M_0) \in A^\%(M)$.

4. Flat bundles

The argument which led us to the proof of Theorem 3.1 can be generalized to give a proof of Theorem 4.1 for a certain class of fiber bundles. Namely, assume that $M$ is a closed topological manifold with a codimension one splitting $M = M_1 \cup_{M_0} M_2$, and let $G$ be a discrete group acting on $M$ on the right by homeomorphisms which preserve the splitting. In such case the bundle $p^G: EG \times_G M \to BG$ splits into subbundles $p^G_i: EG \times_G M_i \to BG$. For $i = 0, 1, 2$ let $j_i: EG \times_G M_i \to EG \times_G M$ denote the inclusion map. We have

4.1. Proposition. For the bundle $p^G$ as above we have a path in $A^\%(p^G)$ joining $\chi^t(p^G)$ with $j_{i*} \chi^t(p^G_i) + j_{i*} \chi^t(p^G_i) - j_{i*} \chi^t(p^G_i)$.

The proof of this fact will rely on two lemmas. The first lemma generalizes lax naturality of the topological Euler characteristic (2.10). Recall (2.9) that if $C$ is a small category and if $F: C \to Spaces$ is a diagram of compact ENRs then we have assignments $c \mapsto F(c)^t$ for $c \in C$ and $f \mapsto F(f)^t$ for a morphism $f$ in $C$, which by Lemma 2.12 define a point $[F(c)^t, F(f)^t] \in \text{holim}_{c \in C} A^\%(F(c))$.

4.2. Lemma. Let $C$ be a small category. Assume that $F, G: C \to Spaces$ are diagrams of compact ENRs and cell-like maps, and let $\eta: F \to G$ be a natural transformation such that for every $c \in C$ the map $\eta_c: F(c) \to G(c)$ is cell-like. Then there is a path in $\text{holim}_{c \in C} A^\%(G(c))$ joining $\eta_c[F(c)^t, F(f)^t]$ with $[G(c)^t, G(f)^t]$. Here

$$\eta_c: \text{holim}_{c \in C} A^\%(F(c)) \to \text{holim}_{c \in C} A^\%(G(c))$$

is the map induced by $\eta$. 
Proof of this fact resembles justification for Lemma 2.12. The natural transformation $\eta$ defines a path in $\operatorname{holim}_{c\in\mathcal{C}}|wR_\%\otimes(G(c))|$. Using the map holim $\in\mathcal{C}|wR_\%\otimes(G(c))| \to \operatorname{holim}_{c\in\mathcal{C}}A^\%(G(c))$ we obtain the required path.

The second lemma describes how one can construct paths in homotopy limits of diagrams of $K$-theory spaces using cofibration sequences.

4.3. Lemma. Let $F : \mathcal{C} \to \mathcal{W}\text{Cat}$ be a functor, and for $i = 1, 2, 3$ let $c \mapsto c_i^1$, $f \mapsto f_i^1$ be assignments as in the Lemma 2.12. Assume also that for every $c \in \mathcal{C}$ we have a cofibration sequence in $F(c)$:

$$c_1^1 \xrightarrow{\varphi c} c_2^1 \xrightarrow{\phi_c} c_3^1$$

such that for any morphism $c \to d$ in $\mathcal{C}$ the following diagram commutes:

$$
\begin{array}{ccc}
F(f)(c_1^1) & \xrightarrow{F(f)\varphi c} & F(f)(c_2^1) & \xrightarrow{F(f)\phi_c} & F(f)(c_3^1) \\
\downarrow f_1^1 & & \downarrow f_2 & & \downarrow f_3^1 \\
d_1^1 & \xrightarrow{\varphi d} & d_2 & \xrightarrow{\phi_d} & d_3
\end{array}
$$

Then there is a path in $\operatorname{holim}_{c\in\mathcal{C}}K(F(c))$ joining the point $[c_2^1, f_3^1]$ with $[c_1^1, f_1^1] + [c_3^1, f_3^1]$.

Indeed, in the notation of [21] the cofibration sequences $c_1^1 \xrightarrow{\varphi c} c_2^1 \xrightarrow{\phi_c} c_3^1$ define a point $[c_1^1, f_1^1, f_3^1] \in \operatorname{holim}_{c\in\mathcal{C}}|wS_2F(c)|$. Also, we have a map

$$\operatorname{holim}_{c\in\mathcal{C}}|wS_2F(c)| \times \Delta^2 \to \operatorname{holim}_{c\in\mathcal{C}}|wS_3F(c)|$$

Restricting this map to $[c_1^1, c_2^1, c_3^1] \times \Delta^2$ we obtain the desired path.

Proof of Proposition 4.1. Consider the group $G$ as a category with one object $\ast$. The action of $G$ on $M$ defines functors $F : G^{\text{op}} \to \text{Spaces}$ and $F_i : G^{\text{op}} \to \text{Spaces}$ (where $G^{\text{op}}$ is the opposite category of $G$) such that $F(\ast) = A^\%(M)$ and $F_i(\ast) = A^\%(M_i)$ for $i = 0, 1, 2$. One can check that we have weak equivalences.

$$A^\%(p^G) \simeq \operatorname{holim}_{G^{\text{op}}} A^\%(M) \text{ and } A^\%(p_i^G) \simeq \operatorname{holim}_{G^{\text{op}}} A^\%(M_i)$$

Moreover, the maps

$$k_i : \operatorname{holim}_{G^{\text{op}}} A^\%(M_i) \to \operatorname{holim}_{G^{\text{op}}} A^\%(M)$$

induced by the inclusions $k_i : M_i \hookrightarrow M$ correspond under these weak equivalences to the maps $j_i : A^\%(p_i^G) \to A^\%(p^G)$. Also, the point of $\operatorname{holim}_{G^{\text{op}}} A^\%(M)$ corresponding to $\chi^i(p^G)$ is (in the notation of [2.5] and [2.9]) the point $[M^i, F(g)]$, and similarly the Euler characteristics $\chi^i(p_i^G)$, $i = 0, 1, 2$ correspond to $[M_i^i, F_i(g)] \in \operatorname{holim}_{G^{\text{op}}} A^\%(M_i)$. As a consequence it is enough to construct a path $\omega$ in $\operatorname{holim}_{G^{\text{op}}} A^\%(M)$ joining the point $[M^i, F(g)]$ with $k_1[M^i_1, F(g)] + k_2[M^i_2, F(g)] - k_0[M^i_0, F(g)]$.

The construction of this path follows the same steps as the proof of Theorem 3.1. As in that proof we construct a manifold $\overline{M} = M_1 \cup_{M_0 \times \{1\}} M_0 \times [-1, 1] \cup_{M_0 \times \{1\}} M_2$. Any
homeomorphism of $M$ extends to a homeomorphism of $\overline{M}$ (which is a product map on $M_0 \times [-1,1]$), thus the group $G$ acts on $\overline{M}$, and we have a functor $\overline{F}: G \to R^\%(\overline{M})$ such that $\overline{F}(s) = R^\%(\overline{M})$. The map $f: \overline{M} \to M$ contracting $M_0 \times [-1,1] \subset \overline{M}$ to $M_0 \subset M$ is $G$-equivariant, so it defines a natural transformation $f_*: \overline{F} \to F$ which in turn induces a map of homotopy limits

$$f_*: \text{holim}_{G^{\text{op}}} A^\%(\overline{M}) \to \text{holim}_{G^{\text{op}}} A^\%(M)$$

Similarly as in the proof of Theorem 3.1 we can now build the path $\omega$ as a concatenation of three paths. The first path in 3.1 (joining $f_*\chi^t(M)$ with $\chi^t(M)$) was obtained using lax naturality of $\chi^t$. In our present context we can construct a path in $\text{holim}_{G^{\text{op}}} A^\%(M)$ joining $f_*[\overline{M}^t, \overline{F}(g)^t]$ with $[M^t, F(g)^t]$ by applying Lemma 4.2 to the natural transformation $f_*: \overline{F} \to F$. The other two paths in the proof of Theorem 3.1 were constructed using certain cofibration sequences in $R^\%(M)$ and $R^\%(M)$. One can start with the same cofibration sequences and then for each retractive space appearing in them choose assignments $\{g \mapsto g\}$ for $g \in G$ in such way that the conditions of Lemma 4.3 are satisfied. As a consequence one obtains the desired paths in $\text{holim}_{G^{\text{op}}} A^\%(\overline{M})$ and $\text{holim}_{G^{\text{op}}} A^\%(M)$. □

5. Universal bundles

In the last section we verified that Theorem 1.3 holds for some bundles whose structure group is discrete. Our strategy of proving Theorem 1.3 in its whole generality is as follows. In this section we show that the bundles which admit the required splitting into subbundles are induced by some universal bundle $p^U$. Moreover, additivity of the topological Euler characteristic for an arbitrary bundle follows from its additivity for this universal bundle. Thus, we only need to show that $\chi_t^t$ is additive for the bundle $p^U$. We prove this special case, using the results of the last section, in §6.

Let $M$ be a closed topological manifold which admits a splitting along a codimension one submanifold $M_0$:

$$M \simeq M_1 \cup_{M_0} M_2$$

Let $TOP(M)$ be the simplicial group of homeomorphisms of $M$ and let $T$ denote the subgroup of $TOP(M)$ consisting of homeomorphisms which preserve the splitting. Consider the bundle

$$p^U: ET \times_T M \to BT$$

The bundle $p^U$ admits a fiberwise codimension one splitting into sub-bundles $p^U_i: ET \times_T M_i \to BT$ for $i = 0, 1, 2$, such that the fiber of $p^U_i$ is $M_i$. Moreover, $p^U$ is the universal bundle for bundles $E \xrightarrow{p} B$ with fiber $M$ which admit such a splitting. Thus, if $p$ splits into subbundles $p_i$ with fibers $M_i$ for $i = 0, 1, 2$ then we have a map $c: B \to BT$ which fits into pullback diagrams:
Moreover, these maps commute with all inclusions of sub-bundles. Our goal will be to show that the statement of Theorem 1.3 holds for the bundle $p_U$.

5.1. **Proposition.** Let $\iota_i: ET \times_T M_i \hookrightarrow ET \times_T M$ be the inclusion map ($i = 0, 1, 2$). Then there is a path $\sigma$ in $A\% (p_U)$ joining $\chi (\pi (p_U))$ with $\iota_1^* \chi (\pi (p_U)) + \iota_2^* \chi (p_U) - \iota_0^* \chi (p_U)$.

We postpone the proof of this fact until §6. Meanwhile we will show that, as indicated at the beginning of this section, Theorem 1.3 can be obtained from this special case. We will need the following lemmas which follow directly from the constructions of $A\% (p), \chi (p)$, and from Lemma 4.2.

5.2. **Lemma.** Assume that we have a pullback square

\[
\begin{array}{ccc}
E' & \xrightarrow{f} & E \\
p' & \downarrow & \downarrow p \\
B' & \xrightarrow{f} & B \\
\end{array}
\]

where $p, p'$ are fiber bundles of compact topological manifolds. Let $\bar{f}^*$: $A\% (p) \to A\% (p')$ denote the map induced by the pullback. Then there exists a path in $A\% (p)$ joining $\chi (p')$ with $\bar{f}^* \chi (p)$.

5.3. **Lemma.** Assume that we have a commutative diagram

\[
\begin{array}{ccc}
E_1 & \xrightarrow{f} & E_2 \\
p_1 & \downarrow & \downarrow p_2 \\
B_1 & \xrightarrow{f} & B_2 \\
\end{array}
\]

where for $i = 1, 2$ the maps $p_i$ are bundles of compact topological manifolds, $p'_i$ is a sub-bundle of $p_i$, $j_i: E'_i \to E_i$ is the inclusion map, and both squares in the middle are pullbacks. Then $j_1^* f^* = \bar{f}^* j_2^*$, where $j_1^*: A\% (p'_1) \to A\% (p_1)$, $\bar{f}^*: A\% (p_2) \to A\% (p_1)$, $\bar{f}'^*: A\% (p'_2) \to A\% (p'_1)$ are the maps induced by $j_i, \bar{f}, \bar{f}'$ respectively.

**Proof of Theorem 1.3.** Let $p: E \to B$ be a fiber bundle as in the statement of Theorem 1.3 and let $c, c_i, \bar{c}, \bar{c}_i$ denote the maps as in the pullback squares (1) above. Applying the
map $c^*$ to the path $\sigma_{p^U}$ from Proposition 5.1 we obtain a path $c^*\sigma_{p^U}$ in $A^\pi(p)$ joining $c^*\chi^i(p^U)$ with $c^*\eta_1\chi^i(p^U_1) + c^*\eta_2\chi^i(p^U_2) - c^*\eta_0\chi^i(p^U_0)$. Lemma 5.2 implies existence of a path $\eta$ joining $\chi^i(p)$ with $c^*\chi^i(p^U)$. Similarly for $i = 0, 1, 2$ we have paths $\eta_i$ in $A^\pi(p_i)$ joining $c^*\chi^i(p^U_i)$ with $\chi^i(p_i)$. Take $\eta' := j_1*\eta_1 + j_2*\eta_2 - j_0*\eta_0$. This is a path in $A^\pi(p)$ with endpoints $j_1*c_i^1\chi^i(p^U_1) + j_2*c_2^2\chi^i(p^U_2) - j_0*c_0^0\chi^i(p^U_0)$ and $j_1*\chi^i(p_1) + j_2*\chi^i(p_2) - j_0*\chi^i(p_0)$. By Lemma 5.3 we have $j_is^c_i = c^*t_s$. Therefore we can concatenate $\eta, c^*\sigma_{p^U}$ and $\eta'$ and obtain a path in $A^\pi(p)$ which joins $\chi^i(p)$ with $j_1*\chi^i(p_1) + j_2*\chi^i(p_2) - j_0*\chi^i(p_0)$. \qed

6. From topological to discrete structure group

As the previous section demonstrated, checking additivity for the parametrized topological Euler characteristic reduces to showing that it holds for the universal bundle $p^U: ET \times_T M \rightarrow BT$. The structure group of this bundle is the simplicial group $T$. Our next goal is to show that it is enough to verify additivity for a certain flat bundle, i.e. a bundle whose structure group is discrete.

Recall that a map $f: B' \rightarrow B$ is a homology equivalence if it induces isomorphisms on homology groups with arbitrary local coefficients.

6.1. Lemma. Let $E \xrightarrow{p} B$ be a fiber bundle with a compact topological manifold $M = M_1 \cup M_2$ as a fiber. Assume that $p$ admits a decomposition into sub-bundles as in the statement in Theorem 1.3, and that we have a pullback diagram

$$
\begin{array}{ccc}
E' & \xrightarrow{f} & E \\
\downarrow{p'} & & \downarrow{p} \\
B' & \xrightarrow{f} & B
\end{array}
$$

If the map $f: B' \rightarrow B$ is a homology equivalence, then the additivity path for $\chi^i$ exists for the bundle $p$ if it exists for the bundle $p'$ (where $p'$ comes with decomposition into sub-bundles induced from $p$).

Lemma 6.1 follows from the following

6.2. Proposition. Assume that we have a homotopy pullback square

$$
\begin{array}{ccc}
X' & \xrightarrow{f} & X \\
\downarrow{q'} & & \downarrow{q} \\
B' & \xrightarrow{f} & B
\end{array}
$$

such that $q, q'$ are fibrations whose fibers are nilpotent spaces. Assume also that $f: B \rightarrow B'$ is a homology equivalence. Then the induced map of the spaces of sections

$\Gamma(q) \rightarrow \Gamma(q')$

is a weak equivalence.

Proof. See [5] Proof of Cor. 2.7] \qed
Proof of Lemma 6.1. Let $p_i : E_i \to B$ be the sub-bundles in the decomposition of $p$, and let $j_i : E_i \to E$ be the inclusion maps. Using the assumption that we have an additivity path for $p'$ and Lemma 5.2 we obtain a path in $A(p')$ joining $\bar{f}^* \chi_t(p)$ with $\bar{f}^*(j_1 \chi_t(p_1) + j_2 \chi_t(p_2) - j_0 \chi_t(p_0))$. Thus it suffices to prove that the map $\bar{f}^* : A^\%(p) \to A^\%(p')$ is a weak equivalence. In order to show that consider the homotopy pullback square

\[
\begin{CD}
A^\%(E') @>\bar{f}>>& A^\%(E) \\
p' \downarrow @>>> p \\
B' @>f>>& B
\end{CD}
\]

where the fibrations $p'_*$, $p_*$ are defined as in 2.18. The fiber of $p_*$ and $p'_*$ is $A^\%(M)$ which is an infinite loop space, thus in particular a nilpotent space. By Lemma 6.2 we have a weak equivalence of the spaces of sections $\Gamma(p'_*) \to \Gamma(p_*)$.

On the other hand we have $\Gamma(p_*) \simeq A^\%(p)$, and $\Gamma(p'_*) \simeq A^\%(p')$ (see 2.17, 2.18). It follows that $A^\%(p) \simeq A^\%(p')$. □

Our application of Lemma 6.1 is as follows. Consider the universal bundle $p^U : ET \times_T M \to BT$ as in Section 5. By the collaring theorem [6] the submanifold $M_0$ has a bicollar in $M$ i.e. we have an embedding $c : M_0 \times (-1, 1) \to M$ such that $c(m, 0) = m$ for $m \in M_0$. Let $T_c$ denote the subgroup of $T$ consisting of all these splitting preserving homeomorphisms of $M$ which are product maps on the bicollar $c$. In other words, $f \in T_c$ if there is a homeomorphism $f' : M_0 \to M_0$ such that the following diagram commutes:

\[
\begin{CD}
M @>f>>& M \\
c \downarrow @>>> c \\
M_0 \times (-1, 1) @>f' \times \text{id}>>& M_0 \times (-1, 1)
\end{CD}
\]

Let $T_\varepsilon := \text{colim}_c T_c$ where the colimit is taken over all bicollar neighborhoods of $M_0$, and let $T_\varepsilon$ denote the group $T_\varepsilon$, but equipped with the discrete topology. We have

6.3. Lemma. The homomorphism $T_\varepsilon \to T$ induces a homology equivalence of classifying spaces $BT_\varepsilon \to BT$.

Proof. The homomorphism $T_\varepsilon \to T$ is a composition of the inclusion $T_\varepsilon \to T$ and the map $T_\varepsilon \to T$. The map of classifying spaces induced by the first of these homomorphisms $BT_\varepsilon \to BT$ is a homotopy equivalence by Siebenmann’s isotopy extension theorem [19]. The map $BT_\varepsilon \to BT_\varepsilon$ induced by the second homomorphism is a homology equivalence by the results of McDuﬀ [17], Thurston [20], Segal [18], and Mather [16]. □

We are now in position to give a proof of Lemma 5.1 and thus complete the proof of Theorem 1.3.
Proof of Lemma 5.1. We have a pullback diagram

\[
\begin{array}{c}
ET^\delta \times_{\mathcal{T}^\delta} M \arrow{d}[p_{\mathcal{T}^\delta}] \arrow{r} & ET \times_T M \arrow{d}[p^U] \\
BT^\delta \arrow{r} & BT
\end{array}
\]

By Lemma 6.3 the map \(BT^\delta \rightarrow BT\) is a homology equivalence. Moreover, by Proposition 4.1 the additivity of \(\chi^t\) holds for the bundle \(p^T^\delta\). Therefore using Lemma 6.1 we obtain additivity of \(\chi^t\) for the bundle \(p^U\). □

7. ADDITIVITY FOR TOPOLOGICAL Reidemeister torsion

Our final task is to give the proof of additivity of the topological Reidemeister torsion, i.e. Theorem 1.2. Let then \(E \xrightarrow{p} B\) be a bundle of compact topological manifolds and let \(V \xrightarrow{\varphi} E\) be a locally constant sheaf of finitely generated projective left \(R\)-modules such that the assumptions of Theorem 1.2 are satisfied. From the pullback diagram below Definition 2.23 it follows that in order to prove additivity for \(\tau^p_\rho\) we need to construct

1) a path \(\bar{\omega}_h^b\) in \(\widetilde{A}^\%_h(p)\) joining \(\bar{\chi}(p)\) with \(j_1*\bar{\chi}(p_1) + j_2*\bar{\chi}(p_2) - j_0*\bar{\chi}(p_0);\)
2) a path \(\omega^h_\rho\) in \(\Phi^h_\rho(p)\) joining \(\tau^h_\rho(p)\) with \(j_1*\tau^h_\rho(p_1) + j_2*\tau^h_\rho(p_2) - j_0*\tau^h_\rho(p_0)\) and such that \(\delta_1(\omega_\rho^h) = \delta_2(\omega_\rho^h).\)

Construction of the path \(\bar{\omega}_h^b\). Recall that the space \(\widetilde{A}^\%_h(p)\) was obtained as a homotopy pullback of the diagram

\[
\begin{array}{c}
\text{holim}_x A(E_x) \xrightarrow{\beta} \text{holim}_{(x,\theta)} A(E^\theta_x) \xrightarrow{\alpha} \text{holim}_{(x,\theta)} A^\%(E^\theta_x) \\
\end{array}
\]

and that the point \(\bar{\chi}(p)\) \(\in \widetilde{A}(p)\) was represented by the triple \((\chi^h(p), \sigma_p, \chi^t(p))\) where \(\sigma_p\) is the canonical path in \(\text{holim}_{(x,\theta)} A(E^\theta_x)\) joining the images of \(\chi^h(p)\) and \(\chi^t(p)\). As a consequence in order to describe the path \(\omega_\rho^h\) we need to construct

(i) a path \(\omega^h_\rho\) in \(\text{holim}_{(x,\theta)} A^\%(E^\theta_x)\) joining \(\chi^t(p)\) with \(j_1*\chi^t(p_1) + j_2*\chi^t(p_2) - j_0*\chi^t(p_0);\)
(ii) a path \(\omega^h_\rho\) in \(\text{holim}_x A(E_x)\) joining \(\chi^h(p)\) with \(j_1*\chi^h(p_1) + j_2*\chi^h(p_2) - j_0*\chi^h(p_0);\)
(iii) a homotopy \(H: [0,1] \times [0,1] \rightarrow \text{holim}_{(x,\theta)} A(E^\theta_x)\) such that \(H(-,0) = \alpha \circ \omega^h_\rho;\)
\(H(-,1) = \beta \circ \omega^h_\rho, H(0,-) = \sigma_p,\) and \(H(1,-) = j_1*\sigma_{p_1} + j_2*\sigma_{p_2} - j_0*\sigma_{p_0}\).

We take \(\omega_\rho^h\) to be the additivity path for the topological Euler characteristic which we constructed in that proof of Theorem 1.3. In order to construct a suitable path \(\omega^h_\rho\) we will use \(A^{biv}(p)\) - the bivariant \(A\)-theory of the bundle \(p\). The space \(A^{biv}(p)\) is the \(K\)-theory space of the Waldhausen category \(\mathcal{R}^{fd}(p)\) which is a full subcategory of the category of retractive spaces over \(E\). The objects of \(\mathcal{R}^{fd}(p)\) are these retractive spaces \(r: X \rightleftarrows E: s\) which satisfy the condition that \(s\) is a cofibrations and that for every point \(b \in B\) the homotopy fiber \(F^b_{p_{or}}\) of the map \(p \circ r\) over \(b\) is a homotopy finitely dominated space. Notice that \(F^b_{p_{or}}\) is in a natural way a retractive space over \(F^b_p\) - the homotopy fiber of \(p\), thus the
above assumption says that for every \( b \in B \) the fiber \( F^b_{p \circ r} \) is an object of the Waldhausen category \( \mathcal{R}^f(F^b_p) \). The Waldhausen category structure on \( \mathcal{R}^f(p) \) is defined by taking a morphism to be a weak equivalence or a cofibration if its underlying map of spaces is a homotopy equivalence or respectively a cofibration.

The retractive space \( E \sqcup E \cong E \) is an object of \( \mathcal{R}^f(p) \), and so it defines a point \( \chi^{biv}(p) \in A^{biv}(p) \). Let \( j_* : A^{biv}(p_i) \to A^{biv}(p) \), \( i = 0, 1, 2 \), be the maps induced by inclusions of subbundles. Using the same constructions as in the proof of Theorem 3.1 with \( E, E_i \) taken in place of \( M, M_i \) we can construct a path \( \omega^{biv}_p \) in \( A^{biv}(p) \) joining \( \chi^{biv}(p) \) with \( j_1 \chi^{biv}(p_1) + j_2 \chi^{biv}(p_2) - j_0 \chi^{biv}(p_0) \).

For every simplex \( x \in S(\mathcal{B}) \) we have an exact functor \( \mathcal{R}^f(p) \to \mathcal{R}^f(E_x) \) which yields a map of infinite loop spaces \( A^{biv}(p) \to A(E_x) \). These maps can be combined to give a map

\[
\text{holim}_x A^{biv}(p) \to \text{holim}_x A(E_x) = A(p)
\]

where the first homotopy limit is taken over the constant functor with the value \( A^{biv}(p) \). Composing this map with the Bousfield-Kan map \( A^{biv}(p) \simeq \text{lim}_x A^{biv}(p) \to \text{holim}_x A^{biv}(p) \) we obtain

\[
\alpha^* : A^{biv}(p) \to A(p)
\]

(this is the generalized coassembly map of [23]). The image of \( \chi^{biv}(p) \) under the map \( \alpha^* \) does not coincide with \( \chi^h(p) \), but there is a canonical path in \( \text{holim}_x A(E_x) \) which joins these two points. As a consequence the path \( \alpha^* \circ \omega^{biv}_p \) defines an additivity path \( \omega^h_p \) for the homotopy Euler characteristic.

In order to see that we have the required homotopy \( H \) one needs to retrace our construction of the additivity path for the topological Euler characteristic. First, one considers bundles with a discrete structure group where the homotopy \( H \) can described using constructions on the level of Waldhausen categories. For more general bundles one uses the fact that the additivity path \( \omega^h_p \) was obtained from the above special case by means of pullbacks. This is enough to verify that the homotopy \( H \) will still exist.

**Construction of the path \( \omega^h_p \).** Notice that the path \( \omega^h_p \) is an additivity path for the homotopy Reidemeister torsion of the bundle \( p \). The existence of such path was proved in [2]. In our present setting we need the path \( \omega^h_p \) which is compatible with \( \omega^f_p \), thus it is not enough to quote that result. The path \( \omega^h_p \) can be obtained however by mimicking the constructions of [2]. Briefly, one starts by constructing \( A^{ac}(p) \) - the acyclic bivariant \( A \)-theory of the bundle \( p \). The space \( A^{ac}(p) \) is obtained from a Waldhausen category \( \mathcal{R}^{ac}(p) \) which is a full subcategory of \( \mathcal{R}^f(p) \). The objects of \( \mathcal{R}^{ac}(p) \) are these retractive spaces \( X \xrightarrow{r} E \) which satisfy the condition that for every \( b \in B \) the relative chain complex of the pair \( (F^b_{p \circ r}, F^b_r) \) with local coefficients in \( \rho |_{F^b_r} \) is acyclic (as before \( F^b_{p \circ r} \) and \( F^b_r \) denote here the homotopy fibers over \( b \) of the maps \( p \circ r \) and \( p \) respectively). The inclusion of Waldhausen categories \( \mathcal{R}^{ac}(p) \hookrightarrow \mathcal{R}^f(p) \) induces a map \( A^{ac}(p) \to A^{biv}(p) \). Directly from the construction of the path \( \omega^biv_p \) it follows that it admits a lift \( \omega^{ac} \) to \( A^{ac}(p) \). The space \( A^{ac}(p) \) in turn maps into
the homotopy Whitehead space $\Phi^h_\rho(p)$ in such way that we obtain a commutative diagram

$$
\begin{array}{ccc}
A^{ac}(p) & \longrightarrow & \Phi^h_\rho(p) \\
\downarrow & & \downarrow \\
A^{biv}(p) & \longrightarrow & A(p)
\end{array}
$$

We can take $\omega^h_\rho$ to be the image of $\omega^{ac}$ under this map.

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