Dirac-Coulomb scattering with plane wave energy eigenspinors on de Sitter expanding universe

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Abstract

The lowest order contribution of the amplitude of Dirac - Coulomb scattering in de Sitter spacetime is calculated assuming that the initial and final states of the Dirac field are described by exact solutions of the free Dirac equation on de Sitter spacetime with a given energy and helicity. We find that the total energy is conserved in the scattering process.

1 Introduction

Recently a new time-evolution picture of the Dirac quantum mechanics was defined in charts with spatially flat Robertson-Walker metrics, under the name of Schrödinger picture [1]. Using the advantage offered by this picture in [2] was found a new set of Dirac energy eigenspinors which behave as polarized plane waves.

Also recently the Coulomb scattering on de Sitter expanding universe was studied [3], using the plane wave solutions derived in [1]. The main results in [3] was that the modulus of total momentum is not conserved but there is a tendency of helicity conservation.

On the other hand, the expansion of the Universe is accelerating, and this could increase the interest of studying the scattering processes on de Sitter backgrounds. In the present paper we would like to analyse the Coulomb scattering using the method of [3] with the energy eigenspinors derived in [2], pointing out a series of aspects in comparison with the study presented in [3]. We shall see that in this case the scattering has new important features since this time the energy is conserved.

The paper is organized as follows. In section 2, we present a short review of the Schrödinger picture introduced in [1] and we write the form of the energy eigenspinors derived in [2]. In Section 3 we define the lowest order contribution for scattering amplitude in the potential $A^\mu$ in the new Schrödinger picture and then we calculate the scattering amplitude, showing that the total energy
conservation holds in this case. Section 4 is dedicated to the problem of cross section. Our conclusions are summarized in section 5 pointing out a series of aspects that remain to be clarified elsewhere.

We use elsewhere natural units i.e. $\hbar = c = 1$.

2 Polarized plane wave in the Schrödinger picture

We start with the results of [1], where was shown that two time evolution pictures can be identified in the case of the Dirac theory on backgrounds with spatially flat Robertson-Walker metrics. The idea there was to define the natural picture as that in which the free Dirac equation is written directly as it results from its Lagrangean, in a diagonal gauge and Cartesian coordinates. In addition the Schrödinger picture was introduced in which the free Dirac equation is transformed such that its kinetic part takes the same form as in special relativity while the gravitational interaction is separated in a specific term.

Let us take the local chart with cartesian coordinates of a flat Robertson-Walker manifold, in which the line element reads:

$$ds^2 = dt^2 - \alpha(t)^2 d\vec{x}^2,$$

(1)

where $\alpha$ is an arbitrary function. One knows that for defining spinor fields on curved backgrounds it requires to introduce the tetrad fields $e_\mu(x)$ and $\tilde{e}_\mu(x)$, fixing the local frames and corresponding coframes which are labelled by the local indices $\hat{\mu}, \hat{\nu} = 0, 1, 2, 3$. The form of the line element allows one to choose the simple diagonal gauge where the tetrad fields have the non-vanishing components [2], [4]:

$$e_0^0 = 1, \quad e_j^i = \frac{1}{\alpha(t)} \delta_i^j, \quad \tilde{e}_0^0 = 1, \quad \tilde{e}_j^i = \alpha(t) \delta_i^j.$$

(2)

The Dirac field $\psi$ of mass $m$ satisfy the free Dirac equation which can be easily written using the tetrad fields (2) (see [2]). If $\psi(x)$ is the Dirac field in natural picture then the Dirac field of the Schrödinger picture $\psi_S(x)$, can be obtained using the transformation $\psi_S(x) = W(x)\psi(x)$ produced by the operator of time dependent dilatations [1],

$$W(x) = \exp \left[ -\ln(\alpha(t))(\vec{x} \cdot \vec{\partial}) \right],$$

(3)

which has the property

$$W(x)^+ = \sqrt{-g(t)}W(x)^{-1}.$$  

(4)

Using this operator, the Dirac equation of the Schrödinger picture was obtained in [2], as well as the relativistic scalar product $\langle \psi_S, \psi'_S \rangle = \int d^3x \psi_S(x)\gamma^0 \psi'_S(x)$, which is no more dependent of $\sqrt{-g(t)}$. 
Now taking in Eq. (1) $\alpha(t) = e^{\sigma t}$ one obtains the de Sitter metric which is the case of interest here. The form of the Dirac equation on de Sitter spacetime in Schrödinger picture is given in [2] where a complete set of orthonormalized fundamental solutions was written down. These depend on the normalized Pauli spinors, $\xi_{\lambda} (\vec{n})$, of helicity $\lambda = \pm 1/2$ which satisfy

$$ (\vec{n} \cdot \vec{\sigma}) \xi_{\lambda} (\vec{n}) = 2\lambda \xi_{\lambda} (\vec{n}), $$

where $\vec{\sigma}$ are the Pauli matrices while the momentum direction is given by $\vec{n}$ ($\vec{p} = p\vec{n}$). Then the fundamental spinor solutions of positive frequencies with energy $E$, momentum direction $\vec{n}$ and helicity $\lambda$ obtained in [2] read

$$ U^{S}_{E, \vec{n}, \lambda} (t, \vec{x}) = i \frac{\omega e^{-iEt}}{(2\pi)^{3/2} \sqrt{2}} \int_{0}^{\infty} ds \left( \frac{\frac{\pi k}{2} H^{(1)}_{\nu}(s) \xi_{\lambda} (\vec{n})}{\lambda e^{-\pi k/2} H^{(1)}_{\nu_{e}}(s) \xi_{\lambda} (\vec{n})} \right) e^{i\omega s \vec{n} \cdot \vec{x} - i\epsilon \ln s}. $$

The notations used here are $\nu_{\pm} = \frac{1}{2} \pm ik$ with $k = m/\omega$, $s = p/\omega$ and $\epsilon = E/\omega$. The negative frequency modes can be obtained using the charge conjugation as in [2] $U^{S}_{E, \vec{n}, \lambda} (x) \rightarrow V^{S}_{E, \vec{n}, \lambda} (x) = i\gamma^{0}(U^{S}_{E, \vec{n}, \lambda} (x))^{T}$, because the charge conjugation in a curved space is point independent [5]. However the negative frequency modes will be of no interest here.

These spinors are normalized in the energy scale (in generalized sense) with respect to the new relativistic scalar product defined in Schrödinger picture [2]:

$$ \int d^{3}x \bar{U}^{S}_{E, \vec{n}, \lambda} \gamma^{0} U^{S}_{E', \vec{n}', \lambda'} = \int d^{3}x \bar{V}^{S}_{E, \vec{n}, \lambda} \gamma^{0} V^{S}_{E', \vec{n}', \lambda'} = \delta_{\lambda \lambda'} \delta(E - E') \delta^{2}(\vec{n} - \vec{n}'), $$

where $\delta^{2}(\vec{n} - \vec{n}') = \delta(\cos \theta_{n} - \cos \theta'_{n}) \delta(\phi_{n} - \phi'_{n})$. These spinors form a complete system of solutions:

$$ \int_{0}^{\infty} dE \int_{S^{2}} d\Omega_{n} \sum_{\lambda} \left[ U_{E, \vec{n}, \lambda} (t, \vec{x}) U^{+}_{E, \vec{n}, \lambda} (t, \vec{x}') + V_{E, \vec{n}, \lambda} (t, \vec{x}) V^{+}_{E, \vec{n}, \lambda} (t, \vec{x}') \right] = \delta^{3}(\vec{x} - \vec{x}'). $$

### 3 The scattering amplitude

The solutions written in [2] will be the central piece of our calculations. In [3] it was pointed out that the necessary requirement for developing the scattering on de Sitter background is the global hyperbolicity of the spacetime and having a complete set of solutions of the free equation for incident and scattered field (Born approximation). Now for defining the lowest order contribution of the scattering amplitude in the Schrödinger picture let us recall the definition of this quantity from [3] in the natural picture:

$$ A_{i \rightarrow f} = -ie \int d^{4}x [-g(x)]^{1/2} \bar{\psi}_{f} (x) \gamma_{\mu} A^{\mu} (x) \psi_{i} (x). $$

3
This expression was obtained by analogy with Minkowski space [6, 7], but can also be obtained from one reduction formalism on de Sitter spacetime [8]. Using now (4) it is not hard to obtain the analogue of (8) in the Schrödinger picture:

$$A_{i \rightarrow f}^{S} = -ie \int d^{4}x \bar{\psi}_{f}(x) \gamma_{\mu} A^{\mu}_{f}(x) \psi_{i}(x),$$  \hspace{1cm} (10)

where $e$ is the unit charge of the field, $A^{\mu}_{f}(x)$ is the potential in the Schrödinger picture, and the hated indices label the components in local Minkowski frames.

Our target is a fixed charge $Ze$ whose Coulomb potential on de Sitter spacetime in the natural picture [3] reads

$$A_{f}^{0}(x) = \frac{Ze}{|x|} e^{-\omega t},$$  \hspace{1cm} (11)

while in the new Schrödinger picture this becomes

$$A_{f}^{0}(x) = \frac{Ze}{|x|}.$$  \hspace{1cm} (12)

Our aim is to calculate the amplitude of Coulomb scattering using the definition (10) in which we replace our quantities of interest (6) and (12). We start with the waves freely propagating in the in and out sectors, $U_{E_{i}, \bar{n}, \lambda_{i}}^{S}(x)$ and $U_{E_{f}, \bar{n}, \lambda_{f}}^{S}(x)$, assuming that the both of them are of positive frequency. If we replace the explicit form of spinors and the Coulomb potential in (10) we observe two remarkable properties. The first one is that we may split the four dimensional integral into a pure spatial integral and a temporal one. In other respects, these integrals have the same form as in Minkowski spacetime, i. e.

$$\int d^{3}x \frac{e^{i(\vec{p}_{i} - \vec{p}_{f}) \cdot \vec{x}}}{|x|} = \frac{4\pi}{|\vec{p}_{f} - \vec{p}_{i}|^{2}},$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{i(E_{f} - E_{i})t} = \delta(E_{f} - E_{i}).$$  \hspace{1cm} (13)

Note that the limits of integration in (13) for the time variable correspond to $t = \pm \infty$, assuming that the interaction extends into the past and future.

In this case the integration after $s = p/\omega$ variable is not quite simple but we can calculate our amplitude as

$$A_{i \rightarrow f}^{S} = -i\alpha \frac{Z \omega^{2} \delta(E_{f} - E_{i})}{8\pi|\vec{p}_{j} - \vec{p}_{i}|^{2}} \xi_{\lambda_{f}}^{+}(\vec{n}) \xi_{\lambda_{i}}(\vec{n})$$

$$\times \left[ e^{\pi k} \int_{0}^{\infty} ds_{f} s_{f}^{1+iE_{f}/\omega} H_{\nu_{+}}^{(2)}(s_{f}) \int_{0}^{\infty} ds_{i} s_{i}^{1-iE_{i}/\omega} H_{\nu_{-}}^{(1)}(s_{i}) + sgn(\lambda_{f} \lambda_{i}) e^{-\pi k} \int_{0}^{\infty} ds_{f} s_{f}^{1+iE_{f}/\omega} H_{\nu_{-}}^{(2)}(s_{f}) \int_{0}^{\infty} ds_{i} s_{i}^{1-iE_{i}/\omega} H_{\nu_{+}}^{(1)}(s_{i}) \right].$$  \hspace{1cm} (14)
where $\alpha = e^2$. The evaluation of the integrals (14) is given in Appendix A, the final result being expressed in terms of Euler gamma functions,

$$A_{i \to f}^S = \frac{-i\alpha Z \omega^2 \delta(E_f - E_i)}{4\pi|p_f - p_i|^2} \xi^+_{\lambda_f} (i\ell) \xi_{\lambda_i} (i\ell)$$

$$\times \left[ f_k(E_f) f_k^*(E_i) + sgn(\lambda_f \lambda_i) f_{-k}(E_f) f_{-k}^*(E_i) \right]$$

(15)

where we introduced the following notations:

$$f_k(E) = e^{\pi k/2} \left[ 2^{iE/\omega} \frac{\Gamma \left( \frac{5}{4} + \frac{ik}{2} + \frac{iE}{2\omega} \right)}{\Gamma \left( \frac{1}{4} + \frac{ik}{2} - \frac{iE}{2\omega} \right)} - \frac{i2^{iE/\omega}}{\pi} \cos \left( \frac{\pi}{4} + \frac{ik\pi}{2} - \frac{iE\pi}{2\omega} \right) \right]$$

$$\times \Gamma \left( \frac{5}{4} + \frac{ik}{2} + \frac{iE}{2\omega} \right) \Gamma \left( \frac{3}{4} - \frac{ik}{2} + \frac{iE}{2\omega} \right)$$

(16)

and $f_{-k}(E)$ is obtained when $k \to -k$ in (16).

Now let us take a look to our scattering amplitude (15). We obtain that the energy is conserved in the scattering process as in Minkowski case. This is however expected because the form of the external field (12) allows us to consider that the scattering process take place in a constant field. One knows that the energy of a system scattered on a constant field is conserved (but this does not mean that the momentum is conserved too), as we obtained here. It is also remarkable that we obtain the Rutherford denominator $|p_f - p_i|^2$ as in Minkowski scattering. In our previous work [3] the Coulomb scattering was analyzed with spinors having given momentum and helicity. The surprising result was that there exists a nonvanishing probability for a scattering process where the law of conservation of total momentum is lost. Here we obtain the nice result that the total energy is always conserved and the non-linear terms that may broken the energy conservation have no contributions to the amplitude (15).

Let us make an analysis in the helicity space. We observe that the analysis can be done using only the terms from parenthesis in (15). Now the probability of scattering is proportional with the amplitude at square $P \sim |A_{i \to f}|^2$. After a little calculation we obtain that the probability of transition between identical helicity states is bigger that the probability of transition between opposite helicity states ($P_{\lambda_i = \lambda_f} > P_{\lambda_i \neq \lambda_f}$) with the quantity:

$$2 \left[ f_k(E_f) f_k^*(E_f) f_k^*(E_i) f_{-k}(E_i) + f_k^*(E_f) f_{-k}(E_f) f_{-k}(E_i) f_k^*(E_i) \right]$$

(17)

Hereby we conclude that in the scattering process a tendency is manifested for helicity conservation. This conclusion was also obtained in [3] were the analysis was done using the momentum eigenspinors. The obvious conclusion is that in the de Sitter space there is a tendency for total angular momentum conservation.

4 The cross section problem

The first observation here is that in this case we have just linear contributions to the cross section in contrast with [3] where the cross section was calculated
as a sum of a linear contribution and a non-linear one. Moreover, the term \( \delta(E_f - E_i) \) will give us the opportunity to define the transition probability in unit of time like in Minkowski case. Then the definition of cross section here will have the same form as in Minkowski space,

\[
d\sigma = \frac{1}{2} \sum_{\lambda_i\lambda_f} \frac{dP}{dt} \dfrac{1}{j},
\]

where \( \frac{dP}{dt} \) is the transition probability in unit of time, \( j \) is the incident flux while the factor \( \frac{1}{2} \) transforms the sum in mediation.

The problem of calculating the incident flux is identical to that of [3]. First let us introduce the expression of the Dirac current in local frames,

\[
J_\mu^\hat{\mu} = e^\hat{\mu}_{\nu\bar{\lambda}i} \bar{U}_{\vec{p}i,\lambda} \gamma^\nu U_{\vec{p}i,\lambda},(x).
\]

(19)

Then the spatial components can be defined as follows:

\[
j(t) = e^{\omega t} \bar{U}_{E_i,\vec{n},\lambda_i}(x)(\vec{n} \cdot \vec{\gamma})U_{E_i,\vec{n},\lambda_i}(x)
\]

\[
= \frac{e^{\omega t}}{32\pi^3} \left[ 1 + i \cot \left( \frac{\pi}{4} + \frac{ik\pi}{2} \right) \left( \frac{1}{2} + i \right) \right]^2.
\]

(20)

From this equation it is immediate that our incident flux is a time dependent quantity. We know from the well-established picture from Minkowski spacetime that the incident flux does not depend of time. This property is no longer valid in a spacetime where the translational invariance with respect to time is lost, and our result is in agreement with this observation. We note that in [3] it was obtained a similar dependence of time for the incident flux. Another observation is that the incident flux calculated here does not depend on the incident momentum like in [3].

The cross section however must be evaluated using an incident flux that is independent on time. We will follow the formalism presented in [9], where for the calculation of the incident flux one must know the state of the unperturbed system from the approximative moment of collision. In [9], this was taken to be \( t \sim 0 \), which in the case of our incident flux (20) gives:

\[
j = j(0) = \frac{\omega^2}{32\pi^3} \frac{2e^{\pi k}}{\cosh(\pi k)} \left( k^2 + \frac{1}{4} \right).
\]

(21)

The evaluation of the transition probability in unit of time \( \frac{dP}{dt} = \frac{d|A_{ij}|^2}{dt} \frac{d^3p_f}{(2\pi)^3} \) (where we use the fact that \( |\delta(E_f - E_i)|^2 = \frac{1}{4\pi}\delta(E_f - E_i) \)) yields

\[
\frac{dP}{dt} = \frac{(\alpha Z)^2 \omega^4}{32\pi^3|\vec{p}_f - \vec{p}_i|^2} \delta(E_f - E_i) \left[ |f_k(E_f)|^2 |f_k(E_i)|^2 \right.
\]

\[
+ |f_{-k}(E_f)|^2 |f_{-k}(E_i)|^2 + \text{sgn}(\lambda_f \lambda_i) f_k(E_f) f^*_{-k}(E_f) f_k(E_i) f_{-k}(E_i)
\]

\[
+ \text{sgn}(\lambda_f \lambda_i) f^*_{k}(E_f) f_{-k}(E_f) f^*_{-k}(E_i) f_{k}(E_i) \left[ \xi^*_{\lambda_f}(\vec{n}) \xi_{\lambda_i}(\vec{n}) \right]^2 \frac{d^3p_f}{(2\pi)^3}.
\]

(22)
For obtaining the cross section when we have particles with given helicities we must average upon the helicities of incident particles and sum upon the helicities of emergent ones. In our case we obtain:

\[ \frac{1}{2} \sum_{\lambda_i, \lambda_f} \left[ \xi_{\lambda_f}^\dagger(n) \xi_{\lambda_i}(n) \right]^2 = 2. \quad (23) \]

The final expression of the differential cross section after we replace (21), (22) and (23) in (18) turns out to be:

\[
d\sigma = \frac{\alpha Z^2}{4\pi^3} \frac{\omega^2}{p_f^2 - p_i^2} \delta(E_f - E_i) \left[ |f_k(E_f)|^2 |f_k(E_i)|^2 + |f_{-k}(E_f)|^2 |f_{-k}(E_i)|^2 + sgn(\lambda_f \lambda_i) f_k(E_f) f_{-k}(E_i) f_k^*(E_i) f_{-k}(E_f) + sgn(\lambda_f \lambda_i) f_{-k}(E_f) f_k(E_i) f_{-k}(E_f) f_k(E_i) \right] \, d^3p_f . \quad (24)\]

In Minkowski case the factor with \( \delta(E_f - E_i) \) was eliminated after performing the integral with respect to the final momentum, because there the relation between momentum and energy is known. In de Sitter spacetime we do not know this relation, and the factor that contain delta Dirac distribution can not be eliminated when one performs the integration with respect to final momentum in (24).

We note that in [3] integrals of this type were used for evaluating the cross section. These had the form

\[
\int_0^\infty dp_f f(p_f) p_f^2 \delta(p_f - p_i),
\int_0^\infty dp_f f(p_f) p_f^2 \theta(p_f - p_i), \quad (25)
\]

since there we calculated the scattering process using spinors with a definite momentum but unknown energy. The last integral in (25) was discussed in [3] (\( \theta(p_f - p_i) \) is the unit step function), and is solved when the modulus of the momentum is not conserved in the scattering process.

We observe that our cross sections have a complicated dependence of energy which is quite unusual. This dependence of energy was obtained after the integration with respect to \( s = p/\omega \), which means that this dependence translated in physical terms means that our cross section is still dependent of the form of the incident wave. However one can write \( d^3p_f = p_f^2 dp_f d\Omega_{pf} \) and solve the integral with respect to the final momentum in (24) for obtaining \( d\sigma \) but restricting the limits of integration between zero and a maximal value of the final momentum.

Finally it will be interesting to study how the results obtained here in the Schrödinger picture have to be translated in the natural picture. This is because in the natural picture the potential is no longer constant. First of all one knows from [4] that the fundamental spinor solution of positive frequency [5] can be wreathed in the natural picture as \( U_{E_i, n, \lambda_i}(t, \vec{x}) = U_{E_i, n, \lambda_i}(t, \vec{x}e^{\omega t}) \). In addition, the external Coulomb field in the natural picture is given by Eq. (14). Replacing
these quantities in the definition of the scattering amplitude from natural picture \((9)\), one will obtain the same scattering amplitude as \((15)\). This can be checked out passing to a new variable of integration \(y = xe^{\omega t}\) when one solves the spatial integrals. It means that our main conclusions from this paper (energy conservation and the tendency for helicity conservation) will also remain valid in the natural picture.

5 Conclusion

We examined in this paper the Coulomb scattering on de Sitter spacetime using the energy eigenspinors. In our considerations the initial and final states of the field are described by exact solutions (with a given energy and helicity) of the free Dirac equation on de Sitter space, which were written in the Schrödinger picture.

Moreover, we found that the scattering amplitude and the cross sections depend on the expansion factor as \(\omega^2\). In addition we recover the result from our previous work that the amplitude and implicitly the cross section depends on the form of the incident wave. The incident flux was also found as a time dependent quantity. Needless to say that this consequences is the result of the lost translational invariance with respect to time in de Sitter spacetime.

In section 3 we found that the total energy is conserved in the scattering process and, in addition, terms that could broken the energy conservation are absent since the scattering was considered in a constant field of the form \((12)\). In section 3 we recover the tendency for helicity conservation as in \([3]\). For further investigations it will be interesting to obtain the definition of the scattering amplitude \((10)\) in the new Schrödinger picture from one reduction formalism for the Dirac field. This will require to use the form of the Dirac equation in the Schrödinger picture \([2]\) and the fundamental spinor solutions \([10]\), with the distinction between positive/negative frequencies.

6 Appendix A

The integrals that help us to arrive at the scattering amplitude \((15)\) are of the type:

\[
\int_0^\infty dz z^{1-iE/\omega} H^{(1)}_\mu(z) = 2^{1-iE/\omega} \Gamma\left(\frac{\mu}{2} + 1 - \frac{iE}{2\omega}\right) \frac{\Gamma\left(\frac{\mu}{2} + \frac{iE}{2\omega}\right)}{\Gamma\left(\frac{\mu}{2} + iE/2\omega\right)}
\]

and

\[
\int_0^\infty dz z^{1+iE/\omega} H^{(2)}_\mu(z) = 2^{1+iE/\omega} \Gamma\left(\frac{\mu}{2} + 1 + \frac{iE}{2\omega}\right) \frac{\Gamma\left(\frac{\mu}{2} + iE/2\omega\right)}{\Gamma\left(\frac{\mu}{2} - \frac{iE}{2\omega}\right)}
\]
$$-i \frac{2^{1+iE/\omega}}{\pi} \cos \left( \frac{\mu \pi}{2} - \frac{iE \pi}{2\omega} \right) \Gamma \left( -\frac{\mu}{2} + 1 + \frac{iE}{2\omega} \right) \Gamma \left( \frac{\mu}{2} + 1 + \frac{iE}{2\omega} \right)$$

(27)

Now setting $z = p/\omega$ and $\mu = 1/2 \pm ik$ one can see that our result (15) is correct.

For calculating our incident flux we solve integrals of the form:

$$\int_0^\infty dz z H_{\mu}^{(1)}(z) = \mu + i\mu \cot \left( \frac{\mu \pi}{2} \right),$$

$$\int_0^\infty dz z H_{\mu}^{(2)}(z) = \mu - i\mu \cot \left( \frac{\mu \pi}{2} \right).$$

(28)

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