NEW CHECKABLE CONDITIONS FOR MOMENT DETERMINACY OF PROBABILITY DISTRIBUTIONS

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Abstract. We have analyzed some conditions which are essentially involved in deciding whether or not a probability distribution is unique (moment-determinate) or non-unique (moment-indeterminate) by its moments. We suggest new conditions concerning both absolutely continuous and discrete distributions. By using the new conditions, which are easily checkable, we either establish new results, or extend previous ones in both Hamburger case (distributions on the whole real line) and Stieltjes case (distributions on the positive half-line). Specific examples illustrate both the results and the relationship between the new conditions and previously available conditions.

Key words: Probability distributions, Moments, Stieltjes moment problem, Hamburger moment problem, Carleman’s condition, Krein’s condition, Condition (L)

1. Introduction. There are well-known classical conditions for uniqueness of measures/distributions by their moments. These conditions are expressed either in terms of an infinite sequence of ‘large’ Hankel matrices of orders going to infinity (see [1], [12], [10]), or in terms of the sequence of the minimal eigenvalues of these matrices (see [2]). Because of the complexity of these conditions, for many decades a special attention was and is paid to easily checkable conditions which are only sufficient, or only necessary for either uniqueness or non-uniqueness. The reader can consult the recent survey paper [6], or Section 11 in [15].

In this paper we use generally accepted notations and terminology. We write $X \sim F$ for a random variable $X$ with distribution function $F$ and assume that the support of $F$, denoted $\text{supp}(F)$, is unbounded, and that all moments of $X$, and of $F$, are finite, i.e., $E[|X|^k] < \infty$ for all $k = 1, 2, \ldots$ with $m_k = E[X^k]$ being the moment of order $k$ and $\{m_k\} = \{m_k\}_{k=1}^{\infty}$ the moment sequence of $X$ and of $F$. If $\text{supp}(F) \subset \mathbb{R} = (-\infty, \infty)$, this is the Hamburger case, and if $\text{supp}(F) \subset \mathbb{R}_+ = [0, \infty)$, the Stieltjes case. Either $F$ is uniquely determined by the moments $\{m_k\}$ (M-determinate), or it is not unique (M-indeterminate).

In Section 2 we deal with absolutely continuous distributions. We suggest new conditions guaranteeing M-determinacy in both Hamburger and Stieltjes cases; see Theorems 1 and 2. After some comments in Section 3, we turn to the proofs of Theorems 1 and 2 in Section 4. In Section 5 we deal with discrete distributions. Appropriate conditions are suggested for M-determinacy; see Theorems 3 and 4. As far as we can judge, these are the first results of this kind. In Section 6 we present further insights regarding the conditions involved and provide illustrative examples.

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Recall the fact that the classical Carleman’s condition (for M-determinacy) and Krein’s
condition (for M-indeterminacy) play a fundamental role in the Moment Problem, including
characterization of probability distributions. Diverse aspects and several results involving
these conditions can be found in books and papers, to mention here just a few: [1], [12],
[10], [11], [3], [13], [6], [14], [8], [3], [16], [7]. Our results and their proofs presented in this
paper involve essentially both Carleman’s condition and Krein’s condition, hence they fall
well into this group of studies.

2. M-determinacy of absolutely continuous distributions. Consider two random
variables, \( X \sim F \) with values in \( \mathbb{R} \) and \( Y \sim G \) with values in \( \mathbb{R}_+ \). Assume further that they
are both absolutely continuous with densities \( f = F' \) and \( g = G' \). All moments of \( X \) and
\( Y \) are assumed to be finite.

In this section we formulate two results, Theorems 1 and 2. The symbol \( \nearrow \) used below
has its usual meaning for ‘monotone increasing’.

**Theorem 1 (Hamburger case).** Suppose the density \( f \) of \( X \) is symmetric on \( \mathbb{R} \),
continuous and strictly positive outside an interval \( (-x_0, x_0) \), \( x_0 > 1 \), such that the following
condition holds:

\[
K_*[f] = \int_{|x| \geq x_0} \frac{-\ln f(x)}{x^2 \ln |x|} \, dx = \infty. \tag{1}
\]

Let further \( f \) be such that

\[
\frac{-\ln f(x)}{\ln x} \nearrow \infty \quad \text{as} \quad x_0 \leq x \to \infty. \tag{2}
\]

Under conditions (1) and (2), \( X \sim F \) satisfies Carleman’s condition and hence is M-
determinate. Moreover, \( X^2 \) is M-determinate on \( \mathbb{R}_+ \).

**Theorem 2 (Stieltjes case).** Assume that the density \( g \) of \( Y \) is continuous and strictly
positive on \( [a, \infty) \) for some \( a > 1 \) such that the following condition holds:

\[
K_*[g] = \int_{a}^{\infty} \frac{-\ln g(x^2)}{x^2 \ln x} \, dx = \infty. \tag{3}
\]

In addition, let \( g \) be such that

\[
\frac{-\ln g(x)}{\ln x} \nearrow \infty \quad \text{as} \quad a \leq x \to \infty. \tag{4}
\]

Under conditions (3) and (4), \( Y \sim G \) satisfies Carleman’s condition and hence is M-
determinate.

3. Some Comments. All conditions in Theorems 1 and 2 are expressed in terms
of the densities and they are easy to be checked. We now make some specific comments
comparing the new and old conditions. More comments will be given in Section 6.

**Comment 1.** Condition (2), and also (4), can be considered in parallel with the following
well-known condition (L), introduced in [3]: The density \( f(x) \) is symmetric and positive for
\( x \geq x_0 \geq 0 \), its derivative \( f' \) exists and

\[
\frac{-xf'(x)}{f(x)} \nearrow \infty \quad \text{as} \quad x_0 \leq x \to \infty. \tag{5}
\]
Notice that in Theorems 1 and 2 we do not require \( f \) and \( g \) to be differentiable. However, if the derivative \( f' \) exists and the quantity \( -xf'(x)/f(x) \) has a limit, say \( \ell \), as \( x \to \infty \), then by l’Hopital’s rule we obtain ‘one common property’ between (5) and (2), namely:

\[
\lim_{x \to \infty} \frac{-\ln f(x)}{\ln x} = \lim_{x \to \infty} \frac{(-\ln f(x))'}{(\ln x)'} = \lim_{x \to \infty} \frac{-xf'(x)}{f(x)} = \ell.
\]

Besides this observation, in general, conditions (2) and (4) are different from condition (5). E.g., the monotone convergence in (2) and (4) is not related to a similar property in (5). And, there are ‘so many’ non-differentiable functions for which (2) and (4) hold, while no reason to talk about (5).

**Comment 2.** Conditions (1) and (3) can be considered in parallel with the following ones:

\[
K[f] = \int_{-\infty}^{\infty} \frac{-\ln f(x)}{1 + x^2} \, dx = \infty \quad (6H); \quad K[g] = \int_{0}^{\infty} \frac{-\ln g(x^2)}{1 + x^2} \, dx = \infty \quad (6S). \tag{6}
\]

And, these are the converse to the well-known Krein’s conditions:

\[
K[f] = \int_{-\infty}^{\infty} \frac{-\ln f(x)}{1 + x^2} \, dx < \infty \quad (7H); \quad K[g] = \int_{0}^{\infty} \frac{-\ln g(x^2)}{1 + x^2} \, dx < \infty \quad (7S). \tag{7}
\]

The integration in the four integrals in (6) and (7) can exclude a neighborhood of zero; see [4], [8]. Recall that \( K[f] < \infty \) implies M-indeterminacy of \( F \), and \( K[g] < \infty \) implies M-indeterminacy of \( G \); see [1], [13], or [6, 8]. But this is not so if dealing with \( K_\ast[f] \) and \( K_\ast[g] \). For some densities, the conditions (1) and (3) are stronger than (6), as shown in Section 6 (see Example 1, Lemma 3 and the follow-up comments).

**Comment 3.** The converse Krein’s condition (6H) together with (5) implies M-determinacy of \( X \) on \( \mathbb{R} \), while conditions (6S) and (5) together imply that of \( Y \) on \( \mathbb{R}_+ \) (see [4]). It should be noticed that the argument of the density \( g \) in (3), (6S) and (7S) is \( x^2 \) rather than \( x \).

**4. Proofs of Theorems 1 and 2.** Proof of Theorem 1. Here we do not require existence of \( f' \), so we do not involve condition (5). All our arguments will be based entirely on conditions (1) and (2). We follow basically the same idea used in [5] to analyze the moment \( m_{2k} = \mathbb{E}[X^{2k}] \) as a function of \( k \), derive an appropriate upper bound, and then use Carleman’s condition for uniqueness (see [11]).

Let us start with a preliminary step based on the analysis of condition (2). Since \( m_{2k} = \int_{-\infty}^{\infty} x^{2k} f(x) \, dx \), we focus on the properties of the integrand

\[ w_k(x) = x^{2k} f(x), \quad k = 1, 2, \ldots, \quad x \in \mathbb{R}. \]

Notice that for any \( k \), \( w_k(x) \) is an even function of \( x \) and we want to know how \( w_k(x) \) depends on \( x \) for fixed \( k \) and on \( k \) for fixed \( x \). It is useful to write \( w_k(x) \) as follows:

\[ w_k(x) = x^{2k} f(x) = x^{2k} x^{-u(x)} = x^{2k-u(x)} \quad \text{with} \quad u(x) = \frac{-\ln f(x)}{\ln x}, \quad x \geq x_0. \]

By assumption (2), \( u(x) \) increases to infinity on \( [x_0, \infty) \). Thus, for any fixed \( k \), \( w_k(x) \) eventually decreases to zero on \( [x_0, \infty) \). On the other hand, \( w_{k+1}(x) = x^2 w_k(x) \), hence for any fixed \( x \geq x_0 \), \( w_k(x) \) strictly increases to infinity in \( k \): \( w_k(x) < w_{k+1}(x) < \cdots \).
From here on we go through a few steps as done in [3]. We provide details for two reasons: first, for reader’s convenience, and second, because we are going to follow similar steps in the proof of Theorem 3, when dealing with discrete distributions.

**Step 1.** For $k \geq 1$, define the supremum (maximum) of $w_k(x)$ on $[x_0, \infty)$: $w_k(x_k) \equiv \max\{w_k(x) : x \geq x_0\}$. Then there exists a natural number, say $k_*$, such that for any $k \geq k_*$, $w_k(x_k) \in (1, \infty)$. Indeed, for each fixed $k \geq 1$, because the continuous function $w_k(x)$ on $[x_0, \infty)$ eventually decreases to zero, there exists an $x_0 > x_0$ such that $w_k(x) \leq 1$ for all $x \geq x_0$. Moreover, the interval $[x_0, x_*]$ is compact, hence the maximum of $w_k(x)$ on $[x_0, x_*]$ is finite. Therefore, for each fixed $k \geq 1$, the maximum point $x_k \in [x_0, \infty)$ exists and $w_k(x_k) < \infty$. On the other hand, recall that for each fixed $x \in [x_0, \infty)$, $w_k(x)$ strictly increases to infinity as $k \to \infty$. These together imply that there exists $k_*$ such that

$$1 < w_k(x_k) < w_{k+1}(x_{k+1}) < \infty \quad \text{for all } k \geq k_*.$$ 

**Step 2.** We will focus on the maximum-point sequence $\{x_k\}_{k=k_*}^\infty$, with $k_*$ defined in Step 1, and claim the monotone property: $x_{k_*} \leq x_{k_*+1} \leq \cdots \leq x_k \leq \cdots$. Suppose on the contrary that there exists a $k \geq k_*$ such that $x_0 \leq x_{k+1} < x_k$. Then, one holds, by definition of $w_k$,

$$w_k(x_{k+1}) = x_{k+1}^2 w_k(x_{k+1}) < x_k^2 w_k(x_k) = w_k(x_k),$$

which contradicts the definition of $x_{k+1}$. Therefore, the sequence $\{x_k\}_{k=k_*}^\infty$ increases in $k$. Moreover, for each fixed $x > x_0$, $\lim_{k \to \infty} w_k(x)/w_k(x_0) = \infty$. So, for large $k$, $x_k > x_0$.

**Step 3.** Since the sequence $\{x_k\}_{k=k_*}^\infty$ is increasing, its limit exists, say $\tilde{x} \in (x_0, \infty)$. We claim that $\lim_{k \to \infty} x_k = \tilde{x} = \infty$. Suppose on the contrary that $\tilde{x} \in (x_0, \infty)$. Then for any fixed pair $(\delta, \Delta)$ with $0 < \delta < \Delta < \tilde{x} - x_0$, there exists a $k^* = k^*(\tilde{x}, \delta, \Delta) \geq k_*$ such that

$$2k(\ln(x + \delta) - \ln x) + u(x) \ln x - u(x + \delta) \ln(x + \delta) > 0 \quad \text{for all } k \geq k^*, \ x \in [\tilde{x} - \Delta, \tilde{x}],$$

due to the smooth and monotone properties of the logarithmic function and the function $u$. More precisely, we can take $k^* > \max\{k_*, k_*\}$, where $k_*$ is defined in Step 1 and

$$k_* \geq \max_{x \in [\tilde{x} - \Delta, \tilde{x}]} \{u(x+\delta)\ln(x+\delta) - u(x)\ln x\} / [2(\ln(\tilde{x}+\delta) - \ln \tilde{x})].$$

Equivalently,

$$w_k(x+\delta) > w_k(x) \quad \text{for all } k \geq k^*, \ x \in [\tilde{x} - \Delta, \tilde{x}].$$

Taking $x = x_k$, we have $x_k \in [\tilde{x} - \Delta, \tilde{x}]$ for sufficiently large $k$ and obtain $w_k(x_k+\delta) > w_k(x_k)$, which contradicts the definition of $x_k$. Therefore, the limit $\tilde{x} = \infty$.

**Step 4.** At the point $x_k$ with $k \geq k_*$, the exponent $2k - u(x_k)$ is positive. This follows from the fact that $w_k(x_k) = x_k^2^{2k-u(x_k)} > 1$ (see Step 1).

**Step 5.** The next is to derive an upper bound for the moment $m_{2k}$. Since $x^{2k} f(x)$ is an even function, we have, for $k \geq k_*$, the following:

$$m_{2k} = 2 \int_0^{x_*} x^{2k} f(x) \, dx = 2 \int_0^{x_*} x^{2k} f(x) \, dx + \int_{x_*}^\infty x^{2k+2} \frac{f(x)}{x^2} \, dx \leq 2 \left( \int_0^{x_*} x_k^2 f(x) \, dx + \int_{x_*}^\infty x_k^2 \frac{f(x)}{x^2} \, dx \right) \leq 2 \left( 1 + \int_{x_*}^\infty 2x_k \frac{f(x_k)}{x^2} \, dx \right) x_k^2 \leq 2 (1 + x_k^2 f(x_k) x_k^{-1}) x_k^2 \equiv \tilde{c} x_k^{2k}.$$
By using condition (3) for \( g \),

\[
x^{2k+2}f(x) = w_{k+1}(x) \leq w_{k+1}(x_{k+1}) = x^{2k+2}f(x_{k+1}), \quad x \geq x_0;
\]

\[
x^{2}_{k+1}f(x_{k+1}) = w_1(x_{k+1}) \leq w_1(x_1) = x^{2}_1f(x_1).
\]

Step 6. Because \( w_k(x) = x^{2k-u(x)} \) has a maximum at \( x_k \) and \( 2k - u(x_k) > 0 \) for \( k \geq k_* \), it follows from the monotone property of the function \( u \) that \( 2k - u(x) > 0 \) for all \( x \in [x_{k-1}, x_k] \), where \( k \geq k_* + 1 \) and \( [x_{k-1}, x_k] = \{x_k\} \) if eventually \( x_{k-1} = x_k \). From here we deduce the following relations:

\[
\int_{x_{k_*}}^{x_n} \frac{-\ln f(x)}{x^2 \ln x} \, dx = \int_{x_{k_*}}^{x_n} \frac{u(x)}{x^2} \, dx = \sum_{j=k_*+1}^{n} \int_{x_{j-1}}^{x_j} \frac{u(x)}{x^2} \, dx \leq \sum_{j=k_*+1}^{n} \int_{x_{j-1}}^{x_j} \frac{2j}{x^2} \, dx
\]

\[
= \sum_{j=k_*+1}^{n} 2j \left( \frac{1}{x_{j-1}} - \frac{1}{x_j} \right) \leq (2k_* + 2) \sum_{j=k_*}^{n} \frac{1}{x_j}.
\]

Therefore, since \( k_* \) is fixed, it follows from (1) and Step 3 that

\[
\sum_{j=k_*}^{n} \frac{1}{x_j} \geq \frac{1}{2k_* + 2} \left( \int_{x_{k_*}}^{x_n} \frac{-\ln f(x)}{x^2 \ln x} \, dx \right) \rightarrow \infty \quad \text{as} \quad n \rightarrow \infty.
\]

Step 7. Now we use the relation between the moment \( m_{2k} \) and the numbers \( x_k \) as found in Step 5. Since \( \sum_{k=k_*}^{\infty} x_k^{-1} = \infty \) it follows that \( \sum_{k=k_*}^{\infty} (m_{2k})^{-1/2k} = \infty \), which is exactly Carleman’s condition in the Hamburger case, hence the distribution \( F \) is \( M \)-determinate. Moreover, \( X^2 \) also satisfies Carleman’s condition (Stieltjes case) and is \( M \)-determinate on \( \mathbb{R}_+ \) (see [6], Lemma 4**). This completes the proof of Theorem 1.

Proof of Theorem 2. We deal here with the random variable \( Y \) and use only conditions (3) and (4) for the density \( g \). All moments \( a_k := \mathbb{E}[Y^k], \quad k = 1, 2, \ldots, \) are positive and finite. Our arguments are partly similar to those in the proof of Theorem 4 in [3].

Let \( \tilde{Y} \) be the symmetrization of \( \sqrt{Y} \), and have the density \( h(x) = |x| g(x^2), \quad x \in \mathbb{R} \). Moreover, \( \tilde{Y} \) has all moments finite with

\[
b_{2k} := \mathbb{E}[\tilde{Y}^{2k}] = a_k = \mathbb{E}[Y^k], \quad b_{2k-1} = \mathbb{E}[\tilde{Y}^{2k-1}] = 0, \quad k = 1, 2, \ldots.
\]

By using condition (3) for \( g \), we derive easily that \( h \) satisfies (1):

\[
K[h] = \int_{|x|\geq a} \frac{-\ln h(x)}{x^2 \ln |x|} \, dx = \int_{|x|\geq a} \frac{-\ln |x| - \ln g(x^2)}{x^2 \ln |x|} \, dx = \frac{2}{a} + 2 \int_{a}^{\infty} \frac{-\ln g(x^2)}{x^2 \ln x} \, dx = \infty.
\]

The next useful fact is that condition (4) for \( g \) implies condition (2) for \( h \):

\[
\frac{-\ln h(x)}{\ln x} = \frac{-\ln |x| - \ln g(x^2)}{\ln x} = -1 + 2 \times \frac{-\ln g(x^2)}{\ln(x^2)} \quad \rightarrow \infty \quad \text{as} \quad \sqrt{a} \leq x \rightarrow \infty.
\]

This means that we are exactly within the conditions in Theorem 1 with \( x_0 = a > 1 \). Therefore, \( \tilde{Y} \) on \( \mathbb{R} \) (Hamburger case) satisfies Carleman’s condition (see Step 7 above):

\[
\sum_{k=k_*}^{\infty} \frac{1}{(b_{2k})^{1/2k}} = \infty \quad \text{for some} \quad k_*.
\]
This, however, is equivalent to \( \sum_{k=k_0}^{\infty} (a_k)^{-1/2k} = \infty \), which is exactly Carleman’s condition for \( Y \sim G \) (Stieltjes case). Hence \( Y \) is M-determinate. The proof of Theorem 2 is completed.

5. M-determinacy of discrete distributions. Let \( X \) be a discrete random variable described by the pair \( \{Z, \mathcal{P}\} \): \( X \) takes values in the set \( Z \) of all integer numbers and \( \mathcal{P} = \{p_j, j \in Z\} \) is its probability distribution, \( p_j = \mathbb{P}[X = j], j = 0, \pm 1, \pm 2, \ldots \), with all \( p_j > 0 \) and \( \sum_{j \in Z} p_j = 1 \). We assume that \( X \) is symmetric.

We write \( m_k = \mathbb{E}[X^k] = \sum_{j \in Z} j^k p_j \) for the \( k \)th moment of \( X \) and assume that all \( m_k, k = 1, 2, \ldots \), are finite, hence the moment sequence \( \{m_k\} \) is well-defined. By the symmetry, all \( m_{2k+1} = \mathbb{E}[X^{2k+1}] = 0 \), so later we will be working with \( m_{2k} \).

Our interest here is in the moment determinacy of discrete distributions. It is well-known that many of the popular discrete distributions are M-determinate, however there are discrete distributions which are M-indeterminate; a few explicit examples can be found in [15], Section 11.

We remind first the following result (see [9]): Suppose \( X \sim \{Z, \mathcal{P}\} \) is a discrete random variable with finite moments and the following condition holds:

\[
\sum_{j \in Z} \frac{-\ln p_j}{1 + j^2} < \infty.
\]

Then \( X \) is M-indeterminate.

Notice that (8) can be considered as a discrete analogue of Krein’s condition for absolutely continuous distributions, \( K[f] = \int_{-\infty}^{\infty} (-\ln f(x))/(1 + x^2) \, dx < \infty \); see (7H). The latter, as mentioned in Section 3 (see Comment 2), implies M-indeterminacy. Condition (8) is sufficient but not necessary for the M-indeterminacy in the discrete case; see [8]. If, however, we know that \( X \) is M-determinate, then necessarily \( \sum_{j \in Z} \frac{-\ln p_j}{1 + j^2} = \infty \). And, here is a question: What requirement should be added to this condition, \( \sum_{j \in Z} \frac{-\ln p_j}{1 + j^2} = \infty \), or to its appropriate modification in order \( X \) to be M-determinate?

Over the last more than 20 years, despite some attempts, it was not clear how for discrete distributions to write an analogue to the ‘continuous’ condition (5) and how to write a ‘converse’ condition to (8) such that the combination of these two to guarantee M-determinacy.

One pair of such conditions is suggested in Theorem 3 below. It was our new and easy ‘continuous’ condition (2) which gave us the idea of how to write the ‘discrete’ condition (10) below. Moreover, considering conditions (1) and (3) and trying to find an appropriate ‘candidate’ as an opposite to (8), motivated us to introduce condition (9).

Let us formulate and prove the next result.

**Theorem 3 (Hamburger case).** Suppose that the random variable \( X \sim \{Z, \mathcal{P}\} \) is symmetric, all its moments are finite and the following condition holds:

\[
\sum_{|j| \geq j_0} \frac{-\ln p_j}{j^2 \ln |j|} = \infty.
\]

Here \( j_0 \geq 2 \) and we assume further that

\[
\frac{-\ln p_j}{\ln j} \nearrow \infty \text{ as } j_0 \leq j \to \infty.
\]
Step 5. Let us derive now an upper bound for the moment
\[ \{X^{2k} \} \text{ satisfies Carleman's condition and hence is } M-determinate. \text{ Moreover, } X^2 \text{ is } M-determinate on } \mathbb{R}_+.

Proof. The idea is close to the one which we have followed in the proof of Theorem 1. Since \( m_{2k} = E[X^{2k}] = \sum_{j=-\infty}^{\infty} j^{2k} p_j \), we consider and analyze the following double-indexed sequence of numbers:
\[ w_k(j) = j^{2k} p_j, \quad j = 0, \pm 1, \pm 2, \ldots, k = 1, 2, \ldots. \]

Notice that \( j \) corresponds to the value of \( X \) and \( k \) to the order of the moment of \( X \).

Let us show that \( \{w_k(j)\} \) has different behavior for fixed \( k \) as \( j \to \infty \) and for fixed \( j \) as \( k \to \infty \). For \( j \geq j_0 \), we rewrite \( w_k(j) \) as follows:
\[ w_k(j) = j^{2k} p_j = j^{2k-u(j)} \text{ with } u(j) = -\frac{\ln p_j}{\ln j}. \]

(a) Fix the argument \( k \) in \( w_k(j) \). Since by (10) \( u(j) \) increases to infinity on \( \{j_0, j_0 + 1, \ldots\} \), we have that for fixed \( k \), \( w_k(j) \) eventually decreases to zero on \( \{j_0, j_0 + 1, \ldots\} \).

(b) Now we fix \( j \geq j_0 \). Since \( w_{k+1}(j) = j^2 w_k(j) \), \( w_k(j) \) strictly increases to infinity in \( k \).

The properties found in (a) and (b) will be used in the next steps. Mimicking the proof of Theorem 1, we proceed and sketch the rest of the proof as follows.

Step 1. For \( k \geq 1 \), define the supremum (maximum) of \( w_k(j) \) on \( \{j_0, j_0 + 1, \ldots\} \): \( w_k(j_k) \equiv \max \{w_k(j) : j = j_0, j_0 + 1, \ldots\} \). Then there exists a natural number, say \( k_* \), such that for any \( k \geq k_* \), \( w_k(j_k) \in (1, \infty) \). More precisely, \( 1 < w_k(j_k) < w_{k+1}(j_k+1) < \infty \) for \( k \geq k_* \).

Step 2. We claim that \( j_{k_*} \leq j_{k_*+1} \leq \cdots \leq j_k \leq \cdots \), where \( k_* \) is defined in Step 1.

Step 3. In addition to the finding in Step 2, we have that \( \lim_{k \to \infty} j_k = \tilde{j} = \infty \).

Step 4. At the point \( j_k, k \geq k_* \), the exponent \( 2k - u(j_k) \) is positive.

Step 5. Let us derive now an upper bound for the moment \( m_{2k} \) of \( X \). We have
\[ m_{2k} = \sum_{j=-\infty}^{\infty} j^{2k} p_j = \sum_{|j| \leq j_{k_*}} j^{2k} p_j + \sum_{|j| > j_{k_*}} j^{2k+2} p_j/j^2. \]

After some transformations which are similar to those in the proof of Theorem 1, we arrive at the following:
\[ m_{2k} \leq c_* j_{k_*}^{2k}, \quad k \geq k_* \text{, } c_* = \text{const} > 0. \]

Step 6. We involve now condition (9). Because \( w_k(j) = j^{2k-u(j)} \) has a maximum at the point \( j_k \) and \( 2k - u(j_k) > 0 \) for \( k \geq k_* \), it follows that \( u(j) < 2k \) for all \( k \geq k_* + 1 \) and \( j \in \{j_{k_*} - 1, \ldots, j_k\} \) by the monotone property of \( u \). With this in mind, we derive a chain of relations:
\[
\frac{j_{n}}{j_{k_*}+1} - \frac{\ln p_j}{j^2 \ln j} = \sum_{j=j_{k_*}+1}^{j_{n}} \frac{u(j)}{j^2} = \sum_{k=k_*}^{j_{k_*}+1} \sum_{j=j_k+1}^{j_{k+1}} \frac{u(j)}{j^2} \leq \sum_{k=k_*}^{n-1} \sum_{j=j_k+1}^{j_{k+1}} \frac{2k + 2}{j^2} \leq \sum_{k=k_*}^{n-1} (2k + 2) \left( \frac{1}{j_k} - \frac{1}{j_{k+1}} \right) \leq (2k_* + 2) \sum_{k=k_*}^{n-1} \frac{1}{j_k}.
\]
Step 7. Since \( k_s \) is a fixed number and \( j_n \to \infty \) as \( n \to \infty \), from (9) and Step 6 we find that
\[
\sum_{k=k_s}^{n-1} \frac{1}{j_k} \geq \frac{1}{2k_s+2} \sum_{j=j_{k_s}+1}^{\infty} -\ln p_j = \frac{-\ln p_j}{j^2 \ln j} \to \infty \quad \text{as} \quad n \to \infty.
\]
Hence \( \sum_{k=k_s}^\infty j_k^{-1} = \infty \). This together with the result in Step 5 implies that
\[
\sum_{k=1}^\infty \left( \frac{1}{(m_{2k})^{1/(2k)}} \right) \geq \sum_{k=k_s}^\infty \left( \frac{1}{(m_{2k})^{1/(2k)}} \right) \geq \sum_{k=k_s}^\infty \frac{1}{c_k^{1/(2k)}j_{k+1}} = \infty.
\]
Since \( \sum_{k=1}^\infty (m_{2k})^{-1/(2k)} = \infty \) is Carleman’s condition (Hamburger case), we conclude that the random variable \( X \) is \( M \)-determinate on \( \mathbb{R} \), so is \( X^2 \) on \( \mathbb{R}_+ \). The proof is complete.

Finally, we consider the Stieltjes case: \( Y \sim \{N_0, \mathcal{P}\} \), where \( N_0 = \{0,1,2,3,\ldots\} \), \( \mathcal{P} = \{p_n, \ n \in N_0 \} \) with \( p_n = P[Y = n] > 0, \ n \in N_0 \), and \( \sum_{n=0}^\infty p_n = 1 \).

Theorem 4 (Stieltjes case). Suppose that the random variable \( Y \sim \{N_0, \mathcal{P}\} \) has finite moments of all orders, and the following condition holds:
\[
\sum_{n \geq n_0} \frac{-\ln p_n}{n^2 \ln n} = \infty.
\]
(11)

Here \( n_0 \geq 2 \) and we assume further that
\[
\frac{-\ln(\frac{1}{2}p_n)}{\ln n} \nearrow \infty \quad \text{as} \quad n_0 \leq n \to \infty.
\]
(12)

Under these two conditions, \( Y \sim \{N_0, \mathcal{P}\} \) satisfies Carleman’s condition and hence is \( M \)-determinate. Moreover, \( Y^2 \) is also \( M \)-determinate on \( \mathbb{R}_+ \).

Proof. First, note that
\[
\mathbb{E}[Y^k] = \sum_{n=0}^\infty n^k p_n \leq \sum_{n=0}^\infty n^{2k} p_n = \mathbb{E}[Y^{2k}], \quad k \geq 1.
\]
(13)

Second, define the symmetrization of \( Y \) by \( X \sim \{Z, \hat{\mathcal{P}}\} \), where \( \hat{\mathcal{P}} = \{q_j, \ j \in \mathbb{Z}\} \):
\[
q_0 = P[X = 0] = p_0, \quad q_j = P[X = j] = \frac{1}{2} P[Y = j] = \frac{1}{2} p_{|j|}, \quad j \in \mathbb{Z} \setminus \{0\}.
\]

Then for \( k \geq 1 \), \( \mathbb{E}[X^{2k}] = \mathbb{E}[Y^{2k}] \), and, by (11) and (12),
\[
\sum_{|j| \geq n_0} \frac{-\ln q_j}{j^2 \ln |j|} = \infty, \quad \frac{-\ln q_j}{\ln j} \nearrow \infty \quad \text{as} \quad n_0 \leq j \to \infty.
\]

Finally, by Theorem 3, \( X \sim \{Z, \hat{\mathcal{P}}\} \) satisfies Carleman’s condition: \( \sum_{k=1}^\infty (\mathbb{E}[X^{2k}])^{-1/(2k)} \to \infty \) (Hamburger case), equivalently, \( \sum_{k=1}^\infty (\mathbb{E}[Y^{2k}])^{-1/(2k)} = \infty \). This in turn implies that \( \sum_{k=1}^\infty (\mathbb{E}[Y^{2k}])^{-1/(2k)} = \infty \) due to (13). Therefore, both \( Y \) and \( Y^2 \) are \( M \)-determinate.

6. Remarks and illustrations. Some more remarks and examples are given below.

Remark 1. In view of the proof of Theorem 1, the smoothness condition on \( F \) near the origin is not necessary. To see this, we rewrite in Step 5 the \( 2k \)th moment as follows:
\[
m_{2k} = 2 \left( \int_{x_{k+1}}^{x_{k+2}} x^{2k} dF(x) + \int_{x_{k+2}}^\infty x^{2k} dF(x) \right) \leq 2 \left( x_{k+1}^{2k} + \int_{x_{k+1}}^\infty x^{2k+2} f(x) \frac{1}{x^2} dx \right).
\]
Then the rest of the proof remains the same. Therefore, Theorems 1 and 2 can be extended slightly to the following.

**Theorem 1** (Hamburger case). Suppose the random variable \( X \) has a symmetric
distribution \( F \) on \( \mathbb{R} \). Assume further that for some \( x_0 > 1 \), \( F \) is absolutely continuous on
\( [x_0, \infty) \) and that its density \( F' = f \) on \( [x_0, \infty) \) is continuous and strictly positive such that
(1) and (2) hold true. Then \( X \sim F \) satisfies Carleman’s condition and is M-determinate.
Moreover, \( X^2 \) is M-determinate on \( \mathbb{R}_+ \).

**Theorem 2** (Stieltjes case). Let \( Y \sim G \) be a nonnegative random variable. Suppose
that for some \( a > 1 \), the distribution \( G \) is absolutely continuous on \( [a, \infty) \) and that its
density \( G' = f \) on \( [a, \infty) \) is continuous and strictly positive such that (3) and (4) hold true.
Then \( Y \sim G \) satisfies Carleman’s condition and is M-determinate.

Next, we clarify the relationship between different conditions. Lemmas 1 and 2 state the
common points of the two functions in (2) and (5), while Lemma 3 and Example 1 below
show the difference between conditions (1) and (6).

**Lemma 1.** Suppose the random variable \( X \sim F \) with density \( f \) has finite moments of
all orders, and let \( x_0 > 1 \) be a constant.

(i) If \( f \) is differentiable on \( [x_0, \infty) \) and the function \( L(x) = -xf'(x)/f(x) \) in (5) has a
limit, say \( \ell \), as \( x \to \infty \), then \( \ell = \infty \).

(ii) If the function \( u(x) = -[\ln f(x)]/\ln x \) in (2) has a limit, say \( \ell_* \), as \( x \to \infty \), then \( \ell_* = \infty \).

**Proof.** Suppose on the contrary that the limit \( \lim_{x \to \infty} L(x) = \ell < \infty \). Then there exists
an \( x_\ell \) such that \( L(x) < \ell + 1 \) for all \( x \geq x_\ell \). Equivalently, by integration,
\( f(x) > Cx^{-(\ell+1)} \) for all \( x \geq x_\ell \) and for some constant \( C > 0 \), which however is a contradiction to the finiteness
of moments of all orders. This proves part (i). The proof of part (ii) is similar and omitted.

**Lemma 2.** If the density \( f \) satisfies either condition (2) or (5), then for each \( M > 0 \),
\( f(x) = \mathcal{O}(x^{-M}) \) as \( x \to \infty \).

**Proof.** Suppose \( f \) satisfies condition (2), then for each \( M > 0 \), there exists an \( x_M \) such
that \( -\ln f(x) > M \ln x \) for all \( x > x_M \). This in turn implies that \( f(x) < x^{-M} \) for all \( x > x_M \). Therefore,
\( f(x) = \mathcal{O}(x^{-M}) \) as \( x \to \infty \).

Suppose instead the density \( f \) satisfies condition (5). Then for each \( M > 0 \), there exists
an \( x_M \) such that
\[
\frac{-xf'(x)}{f(x)} > M, \text{ or, equivalently, } \frac{f'(x)}{f(x)} < -M/x \text{ for all } x > x_M.
\]
Taking integration from \( x_M \) to \( x \) on both sides leads to
\[
\ln f(x) < \ln x^{-M} + c \text{ for all } x > x_M,
\]
namely,
\[
f(x) < e^c x^{-M} \text{ for all } x > x_M,
\]
where \( c \) is a constant. Therefore, \( f(x) = \mathcal{O}(x^{-M}) \) as \( x \to \infty \). The proof is complete.
Lemma 3. Suppose the symmetric density function \( f(x), x \in \mathbb{R} \), is such that condition (1) holds and \( f(x) < 1 \) for all \( x \geq x_0 \geq 4 \). Then \( f \) satisfies the converse Krein’s condition:

\[
K[f] = \int_{|x| \geq x_0} \left(\frac{-\ln f(x)}{(1 + x^2)}\right) dx = \infty.
\]

Proof. Since \( x \geq x_0 \geq 4 \Rightarrow x^2 + 1 = x^2 \left(1 + 1/x^2\right) \geq x^2 \ln x \), the claim follows from

\[
\int_{|x| \geq x_0} -\frac{\ln f(x)}{x^2 \ln |x|} dx \leq 2 \int_{x_0}^{\infty} -\frac{\ln f(x)}{1 + x^2} dx = \int_{|x| \geq x_0} -\frac{\ln f(x)}{1 + x^2} dx.
\]

This means that condition (1) is strictly stronger than the converse Krein’s condition (Hamburger case) under the reasonable bounded assumption on \( f \) which is satisfied by (2) or (5) (see Lemma 2). Similar statement holds for condition (3) (Stieltjes case).

Example 1. The converse of Lemma 3 is not true in general. As an illustration, consider \( X \sim F \) having the symmetric density \( f(x) = c \exp(-|x|/(\ln |x|)^a), x \in \mathbb{R} \), where \( a \in (0, 1], f(0) = 0 \) and \( c > 0 \) is the normalizing constant. Then

\[
\int_{10}^{\infty} -\frac{\ln f(x)}{x^2 \ln x} dx = c_1 + \int_{10}^{\infty} -\frac{1}{x(\ln x)^{a+1}} dx < \infty, \tag{14}
\]

however,

\[
\int_{x \geq 10} -\frac{\ln f(x)}{1 + x^2} dx = c_2 + \int_{10}^{\infty} -\frac{x}{(1 + x^2)(\ln x)^a} dx = \infty, \tag{15}
\]

where \( c_1 \) and \( c_2 \) are two constants. On the other hand, it can be shown that the above density \( f \) satisfies the condition (5) because

\[
L(x) = \frac{-xf'(x)}{f(x)} = \frac{x}{(\ln x)^a} \left(1 - \frac{a}{\ln x}\right) \nearrow \infty \quad \text{eventually as} \quad x \to \infty.
\]

This together with (15) implies that \( X \sim F \) is M-determinate (see [5], Theorem 2). In other words, unlike Krein’s condition \( K[f] < \infty \) (see (7)), the finiteness of the integral in (1) (i.e. \( K_\ast[f] < \infty \) or (14)) does not imply the moment indeterminacy of \( X \sim F \).

Lemma 4. Let \( 0 \leq Y \sim G \) with density \( g \) satisfy the conditions in Theorem 2. Denote \( g(x) = \exp(-u(x) \ln x), x > 0 \). Assume that for some measurable function \( v(x), x > 0 \), we have \( v(x) \geq u(x), x \geq a > 1 \), and let \( Y_\ast \) be a random variable having the density

\[
g_\ast(x) = c_\ast \exp(-v(x) \ln x), x > 0,
\]

where \( c_\ast > 0 \) is the normalizing constant. Then \( Y_\ast \) satisfies Carleman’s condition and is M-determinate.

Proof. Note that for each integer \( n \geq 1 \), we have

\[
E[Y_\ast^n] = \int_{0}^{\infty} x^n g_\ast(x) dx = \int_{0}^{a} x^n g_\ast(x) dx + \int_{a}^{\infty} x^n g_\ast(x) dx
\]

\[
\leq a^n + c_\ast \int_{a}^{\infty} x^n g(x) dx \leq a^n + c_\ast E[Y^n]. \tag{16}
\]

On the other hand,

\[
E[Y^n] \geq \int_{a}^{\infty} x^n g(x) dx = c_a \int_{a}^{\infty} \frac{x^n g(x)}{c_a} dx \geq c_a a^n,
\]
where \( c_a = \int_a^\infty g(x) \, dx \). This together with (16) leads to

\[
\mathbb{E}[Y^*_n] \leq \hat{c} \mathbb{E}[Y^n],
\]

where \( \hat{c} = 1/c_a + c_* > 0 \). Recall that \( Y \) satisfies Carleman’s condition by Theorem 2, so does \( Y_* \). Therefore, \( Y_* \) is M-determinate.

Remark 2. A large class of densities on \((0, \infty)\) can be written in the form

\[
g(x) = e^{-u(x) \ln x} = x^{-u(x)}, \quad x > 0,
\]

with \( u \) such that \( g \) satisfies the conditions in Theorem 2. We require all moments of the random variable \( Y \sim G \) with density \( g \) to be finite.

Based on \( g \), we define two ‘new’ functions, say \( g_1 \) and \( g_2 \), as follows:

\[
g_1(x) = c_1 \exp[-u_1(x) \ln x], \quad g_2(x) = c_2 \exp[-u_2(x) \ln x], \quad x > 0,
\]

where \( u_1(x) = u([x]), \quad u_2(x) = [u(x)] \), the ‘ceiling’ \([x] = \min\{n : n \geq x, \, n \in \mathbb{N}_0\}\), and \( c_1, c_2 \) are normalizing constants making \( g_1 \) and \( g_2 \) to be proper densities of two random variables, say \( Y_1 \) and \( Y_2 \).

Since \( u_1(x) \) and \( u_2(x) \) are step-wise functions, hence not differentiable, the densities \( g_1 \) and \( g_2 \) are also not differentiable. Despite the fact that condition (3) may imply \( K_*[g_1] = \infty \) and \( K[g_1] = \infty \), we cannot apply Theorem 4 from [3]. However, since \( u_1(x) \geq u(x), \quad x \geq a \), we conclude by Lemma 4 that both \( Y_1 \) and \( Y_2 \) are M-determinate.

Example 2. Start with an exponential random variable \( \xi \sim \text{Exp}(1) \) with density \( e^{-x}, \quad x > 0 \) (Stieltjes case), and consider the random variable \( Y = \xi^{3/2} \). The density of \( Y \) is \( g(x) = \frac{3}{2} x^{-1/3} \exp(-x^{2/3}), \quad x > 0 \), and satisfies the conditions in Theorem 2. Therefore, all the conclusions in Remark 2 follow. Let us see how a ‘small perturbation’ of the density \( g \) reflects on the M-determinacy. Define, e.g., the function \( \tilde{g} \) as follows:

\[
\tilde{g}(x) = \hat{c} g(x) \left[ 1 + \frac{1}{2} \sin x \right], \quad x > 0.
\]

Here \( \hat{c} \) is a normalizing constant to make \( \tilde{g} \) a density of a random variable, denoted by \( \tilde{Y} \). Notice that \( \tilde{g} \) is an oscillating function, so none of conditions (4) and (5) can be considered, although both conditions (3) and (6) are satisfied \((K_*[\tilde{g}] = \infty, \quad K[\tilde{g}] = \infty)\). However, since \( \mathbb{E}[\tilde{Y}^n] \leq (2\hat{c})\mathbb{E}[Y^n] \) for all integer \( n \geq 1 \), we conclude via \( Y \) that \( \tilde{Y} \) satisfies Carleman’s condition and hence is M-determinate.

Similar arguments show that the random variable \( Y_* = [\tilde{Y}] = [\xi^{3/2}] \), where the ‘floor’ \([x] = \max\{k : k \leq x, \, k \in \mathbb{N}_0\}\), is also M-determinate.

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