SEMISTABLE REDUCTION FOR MULTI-FILTERED VECTOR SPACES

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1. Introduction

In this paper, \( \mathcal{O} \) will be a valuation ring. We denote \( \text{Frac}(\mathcal{O}) \) by \( K \) and \( \mathcal{O}/m \) by \( k \). For an \( \mathcal{O} \) module \( M \), we denote \( M_K = M \otimes \mathcal{O} K \) and \( \overline{M} = M \otimes \mathcal{O} k \). Also for an \( \mathcal{O}/\pi \) module \( \tilde{M} \) we still denote \( \overline{M} = \tilde{M} \otimes \mathcal{O}/\pi \ k \) for some \( \pi \in m \). Unless stated otherwise, all the modules considered in this paper will be finitely presented over corresponding ring.

Definition 1.1. A multi-filtered vector space \((V, \Fil^i)\) over a field \( F \) is a finite dimensional \( F \)-vector space \( V \) together with finitely many filtrations \( \Fil^i, 1 \leq i \leq n \), such that for all \( i \) the filtration \( \Fil^i \) is

1. decreasing,
2. indexed by natural numbers,
3. exhaustive and
4. separated:

\[ V = \Fil^0 \supseteq \Fil^1 \supseteq \cdots \supseteq \Fil^n = 0. \]

For any subspace (resp. quotient) \( W \) of \( V \), we give the obvious induced multi-filtration on them.

Definition 1.2.

- The weight of a multi-filtered vector space is
  \[ w(V, \Fil^i) = \sum_i \sum_j j \cdot \dim(\Fil^i_j / \Fil^i_{j+1}). \]

- The slope of a multi-filtered vector space is defined as
  \[ \mu(V, \Fil^i) = \frac{w(V, \Fil^i)}{\dim(V)}. \]

- A multi-filtered vector space \((V, \Fil^i)\) is called semistable if for any nonzero subspace \( W \subseteq V \), we have \( \mu(W) \leq \mu(V) \).

If \( M \) is an \( \mathcal{O} \)-lattice of a multi-filtered \( K \)-vector space \((M \otimes \mathcal{O} K, \Fil^i)\), we will still denote the induced filtration \( \Fil^i \cap M \) on \( M \) by \( \Fil^i \). And we use the notation \( \Fil^i \) (resp. \( \tilde{\Fil}^i \)) to denote the induced filtration on \( \overline{M} \) (resp. \( \tilde{\Mathcal{O}} = \mathcal{O}/\pi \) for some \( \pi \in m \)). We call such a \( M \) an integral model of \((M \otimes \mathcal{O} K, \Fil^i)\).

The main theorem of this paper is the following analogue of Langton’s theorem in the setting of multi-filtered vector spaces.

Theorem 1.3 (Main Theorem). Let \( K \) be a valued field. For any semistable multi-filtered vector space \((V, \Fil^i)\), there exits an integral model \((M, \Fil^i)\) such that its reduction \( \overline{M} \) with its induced filtration is again semistable.
2. Proof of the Main Theorem

For technical reason, let us make the following definition.

**Definition 2.1.** A submodule of $\mathcal{O}^n$ (resp. $(\mathcal{O}/\pi)^n$) is said to be saturated if the quotient is flat over $\mathcal{O}$ (resp. $\mathcal{O}/\pi$).

**Definition 2.2.** Let $M$ be an $\mathcal{O}$-lattice of a multi-filtered $K$-vector space $(M \otimes_{\mathcal{O}} K, \text{Fil}^i_k)$ with its induced filtrations. Given a short exact sequence of $k$-vector spaces

$$0 \rightarrow \overline{B} \rightarrow \overline{M} \rightarrow \overline{G} \rightarrow 0$$

we say it is liftable modulo $\pi \in \mathfrak{m}$ if there exists a short exact sequence of $\mathcal{O}/\pi$-modules.

$$0 \rightarrow \tilde{B} \rightarrow \widetilde{M} = M/(\pi M) \rightarrow \tilde{G} \rightarrow 0$$

such that

1. $\tilde{G}$ is flat over $\mathcal{O}/\pi$
2. this sequence reduces to the above sequence and
3. $\text{Fil}^i_k \cap \overline{B}$ surjects onto $\text{Fil}^i_k \cap \overline{B}$.

**Lemma 2.3.** If there is only one filtration $F^\bullet$ on $M$, then any short exact sequence $0 \rightarrow \overline{B} \rightarrow \overline{M} \rightarrow \overline{G} \rightarrow 0$ is liftable, i.e., there exists short exact sequence of $\mathcal{O}$-modules $0 \rightarrow \overline{B} \rightarrow \overline{M} \rightarrow \overline{G} \rightarrow 0$ such that

1. $\overline{G}$ is flat over $\mathcal{O}$
2. this sequence reduces to the above sequence
3. $F^\bullet \cap \overline{B}$ surjects onto $F^\bullet \cap \overline{B}$.

Moreover, fix one lifting and splitting $\widetilde{M} = \widetilde{B} \oplus \widetilde{G}$, then all different liftings are graphs of morphisms in $m\text{Hom}_{F^\bullet}(\overline{B}, \overline{G})$.

**Proof.** Choose a basis of $F^\bullet \cap \overline{B}$ and lift them inside $F^\bullet$. Then we lift the rest of a basis of $\overline{B}$ arbitrarily. Let $B$ be the sum of respective liftings. By construction $B$ satisfies (2) and (3). Note that $B$ is a saturated $\mathcal{O}$-submodule in $M$, hence (1) is also satisfied.

If one has another lifting $0 \rightarrow B' \rightarrow M \rightarrow G' \rightarrow 0$, then the projection from $B'$ to $B$ must be an isomorphism as guaranteed by Nakayama’s lemma. Then this lifting $B'$ is just a graph of some morphism $f \in \text{Hom}(\overline{B}, \overline{G})$. (3) implies that this morphism must preserve induced filtration. (2) implies that this morphism must have target $\mathfrak{m}G$. So we see that $f \in m\text{Hom}_{F^\bullet}(\overline{B}, mG) = m\text{Hom}_{F^\bullet}(\overline{B}, \overline{G})$. By the same reasoning any such $f$ would give rise to a lifting satisfying (1), (2) and (3). □

The second half of the lemma above can be generalized to multi-filtered situation in the following way:

**Lemma 2.4.** In the situation of 2.3, suppose $0 \rightarrow \overline{B} \rightarrow \overline{M} \rightarrow \overline{G} \rightarrow 0$ is liftable modulo $\pi \in \mathcal{O}$, and fix a splitting $\widetilde{M} = \widetilde{B} \oplus \widetilde{G}$. Then all different liftings correspond to graphs of morphisms in $m\text{Hom}_{\text{Fil}^i_k}(\overline{B}, \overline{G}) \cap m\text{Hom}(\overline{B}, \overline{G})$.

It’s nice to prove the following lemma directly:

**Lemma 2.5.** In the situation of the lemma above, $m\text{Hom}_{\text{Fil}^i_k}(\overline{B}, \overline{G})$ is always a finitely generated $(\mathcal{O}/\pi)$-module.
Proof. We will prove by induction on the number of filtrations. The homomorphisms preserve all the \( n \) filtrations are exactly those preserve the first \( n - 1 \) filtrations and preserve the last filtration. Hence as a submodule of \( \text{Hom}(\tilde{B}, \tilde{G}) \) (which is abstractly isomorphic to \((\mathcal{O}/\pi)^n\), we see that

\[
\text{Hom}_{\text{Fil}^1, \ldots, \text{Fil}^n}(\tilde{B}, \tilde{G}) = \text{Hom}_{\text{Fil}^1, \ldots, \text{Fil}^{n-1}}(\tilde{B}, \tilde{G}) \cap \text{Hom}_{\text{Fil}^n}(\tilde{B}, \tilde{G})
\]

By induction hypothesis we see that \( \text{Hom}_{\text{Fil}^1, \ldots, \text{Fil}^{n-1}}(\tilde{B}, \tilde{G}) \) is a finitely generated submodule of \( \text{Hom}(\tilde{B}, \tilde{G}) \) and by Lemma 2.3 \( \text{Hom}_{\text{Fil}^n}(\tilde{B}, \tilde{G}) \) is a saturated submodule of \( \text{Hom}(\tilde{B}, \tilde{G}) \). Therefore it suffices to prove that the intersection of a finitely generated submodule \( H_1 \) and a saturated submodule \( H_2 \) of \( H = (\mathcal{O}/\pi)^n \) is again finitely generated. Fix a splitting \( H = H_2 \oplus H_3 \), we see that \( H_1 \cap H_2 \) is the kernel of projection map \( H_1 \to H_3 \). This is finitely generated because \( \mathcal{O}/\pi \) is a coherent ring.

\textbf{Lemma 2.6.} Let \( H = (\mathcal{O}/\pi)^N \). Let \( H_1 \) be a finitely generated submodule in \( H \) and let \( H_2 \) be a saturated finitely generated submodule in \( H \). Then

\[
H/(H_1 \cap mH + mH_2) = \bigoplus \mathcal{O}/\pi_j \oplus \bigoplus \mathcal{O}/\pi_i^s m \oplus (\mathcal{O}/m)^{s \times s},
\]

where the direct sum is finite and \( \pi_i, \pi_j \in m - \{0\} \).

\textbf{Proof.} Because \( H_2 \) is saturated, we may assume that \( H = H_2 \oplus H_3 \). Then we see \( H' := H/mH_2 = H_2/mH_2 \oplus H_3 \), and let us fix a basis of \( H_3 \) as \( \{f_i\} \). Notice that \( \overline{H_1} \cap m\overline{H} = \overline{H_1} \cap mH \), where “bar” of a module denote its image in \( H/mH_2 \). Let us consider the image of \( H_1 \) (suppose there are \( L \) generators) inside \( H' \). Project further to \( H'_2 := H_2/mH_2 \), we may assume the image of the first \( l \) generators form a basis of the image in \( H'_2 \). Without loss of generality we can now assume the image of the generators are of the form \( e_i + \sum a_{ij}f_j \) and \( \sum a_{ij}f_j \), where \( e_1, \ldots, e_l \) is the image of the first \( l \) generators.

Now let us perform Gauss elimination carefully as following: choose \( a_{ij} \) with smallest valuation (if it is possible then we would prefer to a choice with \( i > l \)), then do Gauss elimination to cancel all the other \( f_j \) coordinates appearing in other generators. Keep doing this procedure. After rearranging basis of both \( H'_2 \) and \( H_3 \), we can assume the image of \( L \) generators of \( H_1 \) in \( H' \) are of the form \( e_i + \pi_i^s f_i \) and \( \pi_j f_j \). Now we can finally conclude that the quotient is of the form as stated in this lemma.

\textbf{Lemma 2.7.} In the situation of 2.2, there exists a \( \pi \in m \) with biggest valuation such that the sequence can be lifted modulo \( \pi \).

\textbf{Proof.} Let us do induction on the number of filtrations. Suppose there exists a \( \pi' \) with biggest valuation such that the first \( n - 1 \) filtrations are lifted, say \( 0 \to \tilde{B} \to \tilde{M} = M/(\pi M) \to \tilde{G} \to 0 \). Let us fix a splitting \( \tilde{M} = \tilde{B} \oplus \tilde{G} \). By Lemma 2.3 the n-th filtration can be lifted modulo \( \pi \) also. By the same reasoning as in Lemma 2.3 these two liftings differ by an \( A \in m\text{Hom}(\tilde{B}, \tilde{G}) := mH \). Both of these two liftings are not unique. They differ by elements in \( H_1 \cap mH \) and \( H_2 \cap mH = mH_2 \), where \( H_2 = \text{Hom}_{\text{Fil}^n}(\tilde{B}, \tilde{G}) \) which is finitely generated by Lemma 2.5 and \( H_2 = \text{Hom}_{\text{Fil}^n}(\tilde{B}, \tilde{G}) \) which is saturated. We see that by Lemma 2.6 \( H/(H_1 \cap mH + mH_2) = \bigoplus \mathcal{O}/\pi_k \oplus \bigoplus \mathcal{O}/\pi_i^s m \) in which residue of \( A \) lives. The coordinates of residue of \( A \) generate a principal ideal \( (\pi) \) in \((\mathcal{O}/\pi)^n\). Because \( A \in mH \) we see that \( \pi \in m \). It is easy to see that \( \pi \) satisfies the lemma.

The following lemma will give another characterization of liftable modulo \( \pi \).
Lemma 2.8. Given \( 0 \to B \to M \to G \to 0 \) in the situation of 2.2. Then the following are equivalent:

1. the sequence above is liftable modulo \( \pi \)
2. there exists a splitting \( M = B \oplus G \) that reduces to the above sequence and for all \( \nu \in B \cap \text{Fil}_i \) there exists a lifting \( v \in \text{Fil}_i \) such that \( p_2(v) \in \pi G \) where \( p_2 \) is the obvious projection onto \( G \).
3. the same as above except \( \nu \) only have to run over a set of chosen basis of the induced filtration.

Proof. (2) and (3) are obviously equivalent by finiteness of dimension and linearity.

(1) implies (2): liftable means one can find splitting \( \tilde{M} = \tilde{B} \oplus \tilde{G} \), lift this splitting over \( O \) we see that (2) is fulfilled.

(2) implies (1): one just modulo this lifting by \( \pi \).

□

Now we are ready to prove the analogue of Langton’s theorem for multi-filtered vector spaces.

Proof of the Main Theorem. Choose an arbitrary integral model of \((V, \text{Fil}^*_i)\), let us denote it as \((M, \text{Fil}^*_i)\). Then consider its reduction \( \overline{M} \) with induced multi-filtrations. If it is semistable, then we are done. Otherwise, there is a maximal destabilizing subspace \( B \) inside \( M \) (cf. [Eve, Lemma 2.5]): \( 0 \to B \to M \to G \to 0 \). By Lemma 2.7 there exists a \( \pi \) with biggest valuation, such that the sequence above is liftable modulo \( \pi \). Then \( \pi \) is not 0 because \((V, \text{Fil}^*_i)\) is semistable. Let us denote a lifting as \( 0 \to \tilde{B} \to \tilde{M} = M/\pi M \to \tilde{G} \to 0 \). This lifting gives us a splitting \( \tilde{M} = \tilde{B} \oplus \tilde{G} \), we can and do lift this splitting further to get \( M = B \oplus G \). Let \( M' = \ker(M \to \tilde{G}) \) be another integral model.

Now we have the following sequence

\[
0 \to \pi M' \to M' \to \pi M \to \pi M' \to 0
\]
tensoring with \( \kappa \) gives

\[
0 \to \overline{G} \to \overline{M'} \to \overline{B} \to 0
\]

After computing in the single filtered case we see that it is a sequence of multi-filtered vector spaces.

We claim that there is no splitting \( \overline{B} \to \overline{M'} \) as multi-filtered vector spaces. Otherwise there exists such a splitting \( h: \overline{B} \to \overline{M'} \). Then lift this splitting to a morphism \( h: B \to \pi G \).

Now let us consider the splitting \( M = \Gamma_h \oplus G \). Let \( \nu \in B \cap \text{Fil}^*_i \), then by assumption we can lift \( h(\nu) \) to \( v \in F_1 \) and \( h(p_1(v)) - p_2(v) \in \pi mG \). Hence under the new splitting \( M = \Gamma_h \oplus G \), \( p_2(v) \in \pi' G \) for some \( \pi' \in \pi m \) where \( p'_2 = h \circ p_1 - p_2 \) is the new obvious projection onto \( G \). Now by Lemma 2.8 we can lift the original sequence modulo \( \pi' \) for some \( \pi' \in \pi m \) contradicting our assumption that \( \pi \) has the biggest valuation modulo which one can lift the sequence.

Our last claim is that the maximal destabilizing subspace \( \overline{B'} \) of \( \overline{M'} \) has either slope or dimension less than \( \overline{B} \). Denote the image of \( \overline{B'} \) in \( \overline{B} \) by \( \text{Im}(\overline{B'}) \). We have

\[
\mu(\overline{B'}) \leq \frac{w(\overline{B'} \cap \overline{G}) + w(\text{Im}(\overline{B'}))}{\text{dim}(\overline{B'})} \leq \frac{\text{dim}(\overline{B'} \cap \overline{G}) \times \mu(\overline{B}) + \text{dim}(\text{Im}(\overline{B'})) \times \mu(\overline{B})}{\text{dim}(\overline{B'})} = \mu(\overline{B})
\]
where the first equality can be obtained only if $\mathcal{B}' \cap \mathcal{G} = 0$. In that case, we see immediately that $w(\mathcal{B}) \leq w(\text{Im}(\mathcal{B}'))$ where equality is obtained only if $\mathcal{B}'$ is a section of its image inside $\mathcal{B}$ as a multi-filtered vector space. Therefore if $\mu(\mathcal{B}') = \mu(\mathcal{B})$ then $\mathcal{B}'$ is isomorphic to $\text{Im}(\mathcal{B}')$ (as two multi-filtered vector spaces) which is a proper subspace of $\mathcal{B}$ (by last paragraph). Hence in the situation of $\mu(\mathcal{B}') = \mu(\mathcal{B})$ the dimension of $\mathcal{B}'$ must be strictly less than that of $\mathcal{B}$.

Because the ranges of slope and dimension are finite, after doing the procedure above finitely many times, we see that the reduction would become semistable. \hfill \Box

3. STACKS OF SEMISTABLE MULTI-FILTERED VECTOR BUNDLES

In this section, let us make the following definition.

Definition 3.1. Fix two natural numbers $s$ and $n$ and $s$ non-decreasing functions $l_i : \mathbb{N} \to \{0, \ldots, n\}$ such that $l_i(0) = 0$ and $l_i(\infty) = n$, we call such a datum type and denote it as $(s, n; l_i)$.

Given a type $(s, n; l_i)$, we can consider the stack $\mathcal{M}_{(s,n,l_i)}$ which associates any scheme $X$ the groupoid of multi-filtered rank $n$ vector bundles of type $(s, n; l_i)$ on $X$, i.e., there are $s$ decreasing exhaustive separated filtrations $\text{Fil}^j$ satisfying the following two properties:

1. the graded pieces $\text{Gr}^j_i = \text{Fil}^j_i / \text{Fil}^{j+1}_i$ are flat over $X$;
2. the ranks of $\text{Fil}^j_i$ are $n - l_i(j)$.

It is not hard to see that $\mathcal{M}_{(s,n,l_i)} = \prod_i \text{Fl}(n; l_i) / \text{GL}_n$ where $\text{Fl}(n; l_i)$ is the flag variety with numerical conditions as above. Therefore the stack $\mathcal{M}_{(s,n,l_i)}$ is an algebraic stack and it is quasi-separated and of finite type over $\text{Spec}(\mathbb{Z})$.

There is an open subset $(\prod_i \text{Fl}(n; l_i))^{s.s.}$ in $\prod_i \text{Fl}(n; l_i)$ whose points correspond to semistable multi-filtered vector spaces. As being semistable is independent of choice of basis we see that this open subset defines an open substack $\mathcal{M}_{(s,n,l_i)}^{s.s.} = [(\prod_i \text{Fl}(n; l_i))^{s.s.} / \text{GL}_n]$ in $\mathcal{M}_{(s,n,l_i)}$ which is just the stack of rank $n$ multi-filtered vector bundles of type $(s, n; l_i)$ whose fibers are semistable.

Theorem 3.2. $\mathcal{M}_{(s,n,l_i)}^{s.s.}$ is a quasi-separated algebraic stack which satisfies the existence part of the valuative criterion (c.f. [Sta17, Tag 0CL9]) over $\text{Spec}(\mathbb{Z})$.

Proof. From the discussion before theorem we know that $\mathcal{M}_{(s,n,l_i)}^{s.s.}$ is an algebraic stack quasi-separated and of finite type over $\text{Spec}(\mathbb{Z})$. By our Theorem 1.3 we see the structure morphism $\mathcal{M}_{(s,n,l_i)}^{s.s.} \to \text{Spec}(\mathbb{Z})$ satisfies the valuative criterion in [Sta17, Tag 0CL9]. \hfill \Box

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