Full justification of a new Green-Naghdi system for internal waves propagation over large topography variation

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February 25, 2021

Abstract

We consider here asymptotic models describing the evolution of internal waves propagating between a flat rigid-lid and a variable topography. In this paper, we derive and fully justify (in the sense of consistency, well-posedness and convergence) a new Green-Naghdi model in the Camassa-Holm regime taking into account large amplitude topography variations, thus relaxing the smallness assumption on the topographic variation parameter used in [Communications on Pure & Applied Analysis, 2015, 14 (6): 2203-2230] and hence improving the result of the aforementioned paper.

1 Introduction

1.1 Motivation

Ocean water is not uniform when it comes to mass density. In fact, the temperature and salinity of water in the ocean vary according to depth which create a stratification effect dividing the water into layers with different densities. The disturbance of these layers by tidal flows over variable topography generates internal waves. The absorbed solar radiation make the upper surface water warmer with a lower density lying above a colder denser water. Internal waves play an important role in underwater biological life and navigation, thus understanding their behavior is very essential. In this paper, we are interested in the one-dimensional motion of internal waves located between two layers of fluids with different densities. Simplifying assumptions on the nature of the fluids are commonly used in oceanography. Namely, the fluids are supposed to be homogeneous, immiscible, inviscid and affected only by gravitational force. Moreover, the fluids are assumed to be irrotational and incompressible.

The mathematical aspects of the internal waves have been the subject of many studies in the literature. The evolution equations describing the two-layers flow are easy to derive. These equations are commonly called the “full Euler system”. In this paper, we omit the detailed derivation of this system for the sake of readability, instead, we briefly recall the system in section 2. For more details, the interested reader can see for instance [3, 4, 9]. Solving the “full Euler system” mathematically is very difficult due to the free boundary problem i.e the domain is moving with time. Indeed, the interface deformation boundary function is one of the unknowns. To overcome this problem, many researchers searched for approximate solutions of the exact system. To this end, the derivation of simpler asymptotic models had attract a lot of attention. These approximate models are derived in much simpler settings where dimensionless variables and unknowns are introduced, allowing a fair description of the exact behavior of the full system in particular physical regimes. In this paper, we derive an asymptotic model for the two-layers flow over strong variations in bottom topography using an additional smallness assumption on the interface deformation. More precisely, we restrict ourselves to the Camassa-Holm regime where we consider medium amplitude deformations at the interface level.

The two-layers flow has been widely studied in the literature setting an important theoretical framework. Many asymptotic models describing the evolution of the interface between two layers of fluid bounded from above by a flat rigid lid and from below by a flat topography have been previously derived

2010 Mathematics subject classification. 76B55, 35C20, 35Q35, 35L60.
Key words and phrases. Internal waves, asymptotic model, full justification, variable topography, medium amplitude..
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and studied, see for instance [16, 17, 8, 6, 5, 12, 11] and references therein. Two-layers flow over variable topography has been also investigated. Many two-layers systems describing the propagation of internal waves over variable topography have been derived and studied in some significant works [9, 2, 8, 3, 10]. These models were proved to be consistent with the exact system, however they are not supported with a rigorous justification (i.e., well posedness, stability, convergence). More recently, a full justification result for a new Green-Naghdi type model in the Camassa-Holm regime (medium amplitude deformation at the interface) dealing with medium amplitude topography variations has been obtained in [16].

In this paper, we derive and fully justify a new Green-Naghdi model in the same scaling of the Camassa-Holm regime but taking into account large amplitude topography variations, thus improving the result obtained in [16]. In fact, large amplitude topography variations is a quite more reasonable assumption in oceanography, yet arise serious difficulties. Indeed, relaxing any smallness assumption on the parameter $\beta$ prompt some terms that cannot be controlled by the energy norm associated to our model and thus do not always allow its full justification. Consequently, we establish a specially designed model allowing to deal with these terms and possessing a satisfying hyperbolic symmetrizable quasilinear structure. This structure is suitable to the application of the hyperbolic systems classical theory and in particular studying energy estimates and hence allowing its full justification. The obtained model is valid under some key restrictions on the bottom deformation, see [2, 8] and remark 3.14. Moreover, one can easily deduce a direct result of full justification of the model in the one-layer case with uneven bottoms and a small amount of surface tension.

The paper is organized as follows: the full Euler system is briefly introduced in section 2. In section 3, we precisely derive the new asymptotic model starting from the original Green-Naghdi model. In section 4, we state some preliminary results on the properties of the specially designed symmetric differential operator. In section 5, we present the linear analysis of the asymptotic model. Section 6 contains the main components of the full justification of the asymptotic model.

**Notations.** In the following, $C_0$ denotes any nonnegative constant whose exact expression is of no importance.

The notation $a \lesssim b$ means that $a \leq C_0 b$ and we write $A = O(B)$ if $A \leq C_0 B$. We denote by $C(\lambda_1, \lambda_2, \ldots)$ a nonnegative constant depending on the parameters $\lambda_1, \lambda_2, \ldots$ and whose dependence on the $\lambda_j$ is always assumed to be nondecreasing.

We use the condensed notation

$$A_s = B_s + \langle C_s \rangle_{s>2},$$

and denote $A_s = B_s$ if $s \leq 8$ and $A_s = B_s + C_s$ if $s > 8$.

Let $p$ be any constant with $1 \leq p < \infty$ and denote $L^p = L^p(\mathbb{R})$ the space of all Lebesgue-measurable functions $f$ with the standard norm

$$|f|_{L^p} = \left( \int_\mathbb{R} |f(x)|^p dx \right)^{1/p} < \infty.$$ 

The real inner product of any functions $f_1$ and $f_2$ in the Hilbert space $L^2(\mathbb{R})$ is denoted by

$$(f_1, f_2) = \int_\mathbb{R} f_1(x)f_2(x)dx.$$ 

The space $L^\infty = L^\infty(\mathbb{R})$ consists of all essentially bounded, Lebesgue-measurable functions $f$ with the norm

$$|f|_{L^\infty} = \text{ess sup}|f(x)| < \infty.$$ 

Let $k \in \mathbb{N}$, we denote by $W^{k,\infty} = W^{k,\infty}(\mathbb{R}) = \{ f \in L^\infty, |f|_{W^{k,\infty}} < \infty \}$, where $|f|_{W^{k,\infty}} = \sum_{\alpha \in \mathbb{N}_0^n, \alpha \leq k} |\partial_\alpha^\Lambda f|_{L^\infty}$.

For any real constant $s \geq 0$, $H^s = H^s(\mathbb{R})$ denotes the Sobolev space of all tempered distributions $f$ with the norm $|f|_{H^s} = |\Lambda^s f|_{L^2} < \infty$, where $\Lambda$ is the pseudo-differential operator $\Lambda = (1 - \partial^2_t)^{1/2}$.

For a given $\mu > 0$, we denote by $H^{\mu+1}_H(\mathbb{R})$ the space $H^{\mu+1}(\mathbb{R})$ endowed with the norm

$$|f|_{H^{\mu+1}_H} \equiv \left| |f|_{H^\mu} + \mu \right| |f|_{H^{\mu+1}}.$$ 

For any function $u = u(t, x)$ and $v(t, x)$ defined on $[0, T) \times \mathbb{R}$ with $T > 0$, we denote the inner product, the $L^p$-norm and especially the $L^2$-norm, as well as the Sobolev norm, with respect to the spatial variable
\( (u, v) = (u(t, \cdot), v(t, \cdot)) \), \( |u|_{L^p} = |u(t, \cdot)|_{L^p} \), \( |u|_{L^2} = |u(t, \cdot)|_{L^2} \), and \( |u|_{H^s} = |u(t, \cdot)|_{H^s} \), respectively. We denote \( L^\infty([0, T); H^s(\mathbb{R})) \) the space of functions such that \( u(t, \cdot) \) is controlled in \( H^s \), uniformly for \( t \in [0, T) \):

\[
\|u\|_{L^\infty([0, T); H^s(\mathbb{R}))} = \text{ess sup}_{t \in [0, T)} |u(t, \cdot)|_{H^s} < \infty.
\]

Finally, \( C^k(\mathbb{R}) \) denote the space of \( k \)-times continuously differentiable functions.

For any closed operator \( T \) defined on a Banach space \( X \) of functions, the commutator \([T, f]\) is defined by \([T, f]g = T(fg) - fT(g)\) with \( f, g \) and \( fg \) belonging to the domain of \( T \). The same notation is used for \( f \) an operator mapping the domain of \( T \) into itself.

\section{Full Euler system}

In what follows, we derive briefly the governing equations of the two-layers flow commonly known as the \textit{full Euler system}. One can see \cite{2, 4, 9, 10} for more details. The main interest of this system is in its reduced form that is of two evolution equations coupling two unknowns so-called Zakharov’s canonical variables.

We restrict our study to horizontally one-dimensional case. Let \( \zeta(t, x) \) and \( b(x) \) be two functions that represent respectively the interface and bottom variation from their rest place (resp. \((x, 0), (x, -d_2)\)). Moreover, we assume a flat and rigid top surface. The domain of study of the upper fluid is denoted by \( \Omega_1 \) while the one of the lower fluid is denoted by \( \Omega_2 \), see figure 1.

![Figure 1: Domain of study](image)

We assume that there exists \( h_0 > 0 \) such that \( h_1, h_2 \geq h_0 > 0 \), where \( h_1 \) and \( h_2 \) stand for the heights of the top and low fluids respectively. That is the two domains are supposed to stay connected.

Now let us specify the assumptions on the nature and domain of the fluids. This type of reasonable hypothesis is commonly used in oceanography when determining the governing equations of two-layers flow.

First, we consider both fluids are homogeneous, therefore the mass density of the top and low fluids denoted by \( \rho_1 \) and \( \rho_2 \) respectively are constant. Each layer of fluid is incompressible so the corresponding velocity field has a zero divergence. Assuming irrotational flows, there exists velocity potentials denoted by \( \phi_i \) (\( i = 1, 2 \)) that satisfy the Laplace equation. Assuming ideal fluids with no viscosity, one obtains two Bernoulli equations. The surface, interface and bottom are all assumed to be bounding surfaces, that is to say no particle of fluid can cross the surface, interface or bottom. One may close the set of equations using one last condition on the pressure, more precisely we assume the continuity of the stress
tensor at the interface. At this stage, one obtains:

\[
\begin{align*}
\Delta_{x, z} \phi_i &= 0 & \text{in } \Omega_i^1, & i = 1, 2, \\
\rho_i \partial_t \phi_i + \frac{\rho_i}{2} |\nabla_{x, z} \phi_i|^2 &= -P_i - \rho_i g z & \text{in } \Omega_i^2, & i = 1, 2, \\
\partial_n \phi_i &= n_i \nabla_{x, z} \phi_i = 0 & \text{on } \Gamma_i, \\
\partial_t \zeta &= \sqrt{1 + |\partial_z \zeta|^2} \partial_t \phi_1 = \sqrt{1 + |\partial_z \phi_2|^2} \partial_t \phi_2 & \text{on } \Gamma, \\
\partial_n \phi_2 &= n_b \nabla_{x, z} \phi_2 = 0 & \text{on } \Gamma_b, \\
|P(t, x)| &= \sigma \partial_x \left( \frac{1}{\sqrt{1 + |\partial_z \zeta|^2}} \partial_x \zeta \right) & \text{on } \Gamma,
\end{align*}
\]

where \( \Gamma_i \equiv \{(x, z), z = d_i\}, \Gamma \equiv \{(x, z), z = \zeta(t, x)\}, \) and \( \Gamma_b \equiv \{(x, z), z = -d_2 + b(x)\} \).

\[ [P(t, x)] \equiv \lim_{\chi \rightarrow 0} \left( P(t, x, \zeta(t, x) + \chi) - P(t, x, \zeta(t, x) - \chi) \right) \] and \( \partial_n = n_i \nabla_{x, z} \) is the upward normal derivative in the direction of the vector \( n \) under consideration. We denote by \( n_t = (0, 1)^T \), \( n_\zeta = \frac{1}{\sqrt{1 + |\partial_z \zeta|^2}} \) \( (-\partial_z \zeta, 1)^T \) and \( n_b = \frac{1}{\sqrt{1 + |\partial_z b|^2}} \) \( (-\partial_z b, 1)^T \) the unit outward normal vectors at the top rigid surface, interface and bottom respectively.

Here \( \sigma \) denotes the surface (or interfacial) tension coefficient. Studying both theoretical and numerical aspects of system \( (2.1) \) remains difficult as the domain is one of the unknowns. At this point, extra assumptions are made on some parameters with no dimensions in order to obtain reduced asymptotic models suitable for numerical implementation. Thus, we introduce the following dimensionless parameters:

\[ \gamma = \frac{\rho_1}{\rho_2}, \quad \epsilon \equiv \frac{a}{d_1}, \quad \beta \equiv \frac{a_b}{d_1}, \quad \mu \equiv \frac{d_1^2}{\lambda^2}, \quad \delta \equiv \frac{d_1}{d_2}, \quad \text{bo} = \frac{g(\rho_2 - \rho_1)d_1^2}{\sigma}, \]

where we denote by \( a \) (resp. \( a_b \)) the maximal elevation of the internal wave (resp. bottom topography) and \( \lambda \) the horizontal length scale of the wave at the interface.

Written in its dimensionless form, system \((2.1)\) can be reduced to two equations with two unknowns \((\zeta, \psi) \equiv \phi_1(t, x, \zeta(t, x))\), see [20], [7],

\[
\begin{cases}
\partial_t \zeta - \frac{1}{\mu} G^{\mu} \psi = 0, \\
\partial_t \left( H^{\mu, \delta} \psi - \gamma \partial_x \psi \right) + (\gamma + \delta) \partial_x \zeta + \frac{\epsilon}{2} \partial_x \left[ \left( H^{\mu, \delta} \psi \right)^2 - \gamma \left( \partial_x \zeta \right)^2 \right] = \mu \epsilon \partial_x N^{\mu, \delta} - \gamma \frac{\delta}{\text{bo}} \left( k(\epsilon \sqrt{\mu} \zeta) \right),
\end{cases}
\]

where we denote:

\[ N^{\mu, \delta} \equiv \frac{\frac{1}{\mu} G^{\mu} \psi + \epsilon (\partial_x \zeta) H^{\mu, \delta} \psi \left( \partial_x \phi_1 \right)}{2(1 + \mu |\epsilon \partial_x \zeta|^2)}. \]

The system \((2.2)\) is the so-called full Euler system. We define below the two Dirichlet-Neumann operators \( G^{\mu} \psi \) and \( H^{\mu, \delta} \psi \).

**Definition 2.3** Let \( \zeta, b \in H^{1+1}(\mathbb{R}) \), \( t_0 > 1/2 \), such that there exists \( h > 0 \) with \( h_1 \equiv 1 - \epsilon \zeta \geq h > 0 \) and \( h_2 \equiv 1 - \epsilon \zeta - \beta b > h > 0 \), and let \( \psi \in L^2_{\text{loc}}(\mathbb{R}) \), \( \partial_x \psi \in H^{1/2}(\mathbb{R}) \). Then we define:

\[ G^{\mu} \psi \equiv G^{\mu}[\epsilon \zeta] \psi \equiv \sqrt{1 + \mu |\epsilon \partial_x \zeta|^2} \left( \partial_x \phi_1 \right)_{|z=\epsilon \zeta} = -\mu \epsilon (\partial_x \zeta)(\partial_x \phi_1)_{|z=\epsilon \zeta} + (\partial_x \phi_1)_{|z=\epsilon \zeta}, \]

\[ H^{\mu, \delta} \psi \equiv H^{\mu, \delta}[\epsilon \zeta] \psi \equiv \partial_x \left( \phi_2 \right)_{|z=\epsilon \zeta} = (\partial_x \phi_2)_{|z=\epsilon \zeta} + \epsilon (\partial_x \zeta)(\partial_x \phi_2)_{|z=\epsilon \zeta}, \]

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where, \(\phi_1\) and \(\phi_2\) are uniquely deduced from \((\zeta, \psi)\) as solutions of the following Laplace’s problems:

\[
\begin{align*}
\begin{cases}
(\mu \partial_x^2 + \partial_z^2) \phi_1 = 0 & \text{in } \Omega_1 \equiv \{(x, z) \in \mathbb{R}^2, \epsilon \zeta(x) < z < 1\}, \\
\partial_z \phi_1 = 0 & \text{on } \Gamma_1 \equiv \{(x, z) \in \mathbb{R}^2, z = 1\}, \\
\phi_1 = \psi & \text{on } \Gamma \equiv \{(x, z) \in \mathbb{R}^2, z = \epsilon \zeta\}, \\
(\mu \partial_x^2 + \partial_z^2) \phi_2 = 0 & \text{in } \Omega_2 \equiv \{(x, z) \in \mathbb{R}^2, -\frac{1}{\delta} + \beta b(x) < z < \epsilon \zeta\}, \\
\partial_z \phi_2 = \partial_z \phi_1 & \text{on } \Gamma, \\
\partial_z \phi_2 = 0 & \text{on } \Gamma_b \equiv \{(x, z) \in \mathbb{R}^2, z = -\frac{1}{\delta} + \beta b(x)\}.
\end{cases}
\end{align*}
\] (2.4)

\[
\begin{align*}
\begin{cases}
\partial_x \phi_2 = 0 & \text{on } \Gamma_1 \equiv \{(x, z) \in \mathbb{R}^2, z = \epsilon \zeta\}, \\
\phi_2 = \psi & \text{on } \Gamma \equiv \{(x, z) \in \mathbb{R}^2, z = \epsilon \zeta\}.
\end{cases}
\end{align*}
\] (2.5)

**Definition 2.6 (Regimes)** Firstly, the commonly known shallow water regime is defined. We consider the two-layers to be of similar depths:

\[
P_{SW} \equiv \left\{ (\mu, \epsilon, \gamma, \beta, b_0) : 0 < \mu \leq \mu_{max}, 0 \leq \epsilon \leq 1, \delta \in (\delta_{min}, \delta_{max}), 0 \leq \gamma < 1, 0 \leq \beta \leq \beta_{max}, b_0_{min} \leq b_0 \leq \infty \right\},
\] (2.7)

with given \(0 \leq \mu_{max}, \delta_{min}^{-1}, \delta_{max}, b_0_{min}^{-1}, \beta_{max} < \infty\).

Moreover, the model \((3.19)\) is valid under the following extra restrictions:

\[
P_{CH} \equiv \left\{ (\mu, \epsilon, \delta, \gamma, \beta, b_0) \in P_{SW} : \epsilon \leq M\sqrt{\mu}, \beta \leq \beta_{max} \text{ and } \nu(b) \equiv \frac{1 + \gamma \delta}{3\delta(\gamma + \delta - \gamma \delta \beta b)} - \frac{1}{\beta b_0} \geq \nu_0 \right\},
\] (2.8)

with given \(0 \leq M, \nu_0^{-1} < \infty\).

We proceed by denoting without difficulty

\[
M_{SW} \equiv \max \{\mu_{max}, \delta_{min}^{-1}, \delta_{max}, b_0_{min}^{-1}, \beta_{max}\}, \quad M_{CH} \equiv \max \{M_{SW}, M, \nu_0^{-1}\}.
\]

3 Green Naghdi systems

In this section, we are interested in the construction and full justification of a new asymptotic model describing the two-layers flow in the Camassa-Holm regime \((\mu \ll 1, \epsilon \leq M\sqrt{\mu}\) with \(M > 0)\) taking into account large amplitude topography variations. To this end, we start our analysis by recalling the commonly known Green-Naghdi model. The latter model is obtained after plugging the asymptotic expansions of the Dirichlet-Neumann operators given in \([8, 10]\) into the system \([2.2]\) and after straightforward computations while neglecting all terms of order \(\mu^2\). At this point, the commonly known “shear mean velocity” variable \(\bar{v}\) is introduced coupling the upper and lower layer vertically integrated horizontal velocities \(u_1\) and \(u_2\), see \([10]\):

\[
\bar{v} \equiv u_2 - \gamma u_1,
\] (3.1)

\[
u_1(t, x) = \frac{1}{h_1(t, x)} \int_{\epsilon \zeta}^{1} \partial_x \phi_1(t, x, z) \, dz
\]

and

\[
u_2(t, x) = \frac{1}{h_2(t, x)} \int_{-\frac{1}{\delta} + \beta b(x)}^{\epsilon \zeta} \partial_x \phi_2(t, x, z) \, dz,
\]

where \(\phi_1\) and \(\phi_2\) are the solutions to the Laplace’s problems \((2.4)-(2.5)\).

In this paper, for the sake of simplicity, we do not give the asymptotic expansions of the Dirichlet-Neumann operators nor the detailed derivation of the Green-Naghdi model. Instead, we recall below
Using the following asymptotic expansion: 1

\[
\begin{aligned}
\frac{\partial \zeta + \partial_x \left( \frac{h_1 h_2}{h_1 + \gamma h_2} v \right)}{v} &= 0, \\
\frac{\partial_t \left( v + \mu Q[h_1, h_2] v \right) + (\gamma + \delta) \partial_x \zeta + \frac{\epsilon}{2} \partial_x \left( \frac{h_1^2 - \gamma h_2^2}{(h_1 + \gamma h_2)^2} v^2 \right)}{\mu} &= \partial_x \left( \frac{T[h_1, h_2] v}{\mu} \right) + \mu^2 \partial_x^2 \zeta,
\end{aligned}
\]

(3.2)

where we denote \( h_1 = 1 - \epsilon \zeta \) and \( h_2 = \frac{1}{\delta} + \epsilon \zeta - \beta b \), as well as

\[
\begin{aligned}
Q[h_1, h_2] v &= T[h_2, \beta b] \left( \frac{h_1 v}{h_1 + \gamma h_2} \right) - \gamma T[h_1, 0] \left( \frac{-h_2 v}{h_1 + \gamma h_2} \right), \\
&= - \frac{1}{3 h_2} \partial_x \left( \frac{h_1^2 \partial_x (h_1 v)}{h_1 + \gamma h_2} \right) + \frac{1}{2 h_2} \beta \left( \partial_x \left( \frac{h_1^2}{h_1 + \gamma h_2} \right) \right) - h_2 \left( \partial_x (h_1 v) \right) + h_2 \left( \partial_x (h_1 v) \right) \\
&= \left( \frac{1}{3 h_1} \partial_x \left( \frac{h_1^2 \partial_x (h_1 v)}{h_1 + \gamma h_2} \right) \right), \\
\end{aligned}
\]

\[
\begin{aligned}
Q[h_1, h_2] v &= \left( \frac{1}{2} \left( - h_2 \partial_x \left( \frac{h_1 v}{h_1 + \gamma h_2} \right) + \beta \left( \partial_x \left( \frac{h_1 v}{h_1 + \gamma h_2} \right) \right) \right)^2 - \gamma \left( \partial_x \left( \frac{-h_2 v}{h_1 + \gamma h_2} \right) \right)^2 \\
&= \left( \frac{1}{h_1 + \gamma h_2} \right) T[h_2, \beta b] \left( \frac{h_1 v}{h_1 + \gamma h_2} \right) + \gamma \left( \frac{-h_2 v}{h_1 + \gamma h_2} \right) T[h_1, 0] \left( \frac{-h_2 v}{h_1 + \gamma h_2} \right),
\end{aligned}
\]

with,

\[
T[h, b] V = - \frac{1}{h} \partial_x (h^3 \partial_x V) + \frac{1}{2} \left[ \partial_x^2 \left( h^2 (\partial_x b) V \right) \right] + \left( \partial_x b \right)^2 V.
\]

The Green-Naghdi model (3.2) has been proved to be consistent with the full Euler system 2.2 in [10]. However, it was not supported with a rigorous justification result (i.e. well posedness, stability, convergence). In this paper, starting from the Green-Naghdi model (3.2), we construct a new equivalent model (having the same precision) using an extra assumption on the interface deformations (that is we assume \( \epsilon = O(\sqrt{\mu}) \)) taking into account large amplitude bottom variations. The resulting model enjoys a hyperbolic quasilinear structure allowing its full justification. The construction of the aforementioned model is detailed in the section below.

### 3.1 Construction of the new model

In this section, we construct a new Green-Naghdi model in the Camassa-Holm regime (\( \epsilon = O(\sqrt{\mu}) \)) but taking into account large variations in topography, thus relaxing any smallness assumption on the amplitude topographic variations parameter \( \beta \). This is more reasonable in ocean beds. In fact, we suppose that there exists \( \beta_{\text{max}} < \infty \) such that

\[
\beta = O(1) \quad \text{with} \quad \beta \in [0, \beta_{\text{max}}].
\]

Using the following asymptotic expansion:

\[
\begin{aligned}
\frac{h_1}{h_1 + \gamma h_2} &= \frac{\delta}{\gamma + \delta - \gamma \delta \beta b} + O(\epsilon), \\
\frac{h_2}{h_1 + \gamma h_2} &= \frac{1 - \delta \beta b}{\gamma + \delta - \gamma \delta \beta b} + O(\epsilon), \\
\frac{h_1}{h_1 + \gamma h_2} &= \frac{\delta}{\gamma + \delta - \gamma \delta \beta b} \left( 1 - \epsilon \zeta + \frac{\epsilon \zeta (1 - \gamma)}{\gamma + \delta - \gamma \delta \beta b} + O(\epsilon^2) \right),
\end{aligned}
\]

6
\[
\frac{h_2}{h_1 + \gamma h_2} = \frac{\delta}{\gamma + \delta - \gamma \delta b} \left( \delta^{-1} + \epsilon \zeta - \beta b + \frac{(1 - \delta \beta b) \epsilon \zeta (1 - \gamma)}{\gamma + \delta - \gamma \delta b} + \mathcal{O}(\epsilon^2) \right).
\]

After replacing these functions by their corresponding approximations in \(\mathcal{Q}[h_1, h_2]v\) and \(\mathcal{R}[h_1, h_2]v\), one gets the following:

\[
\mathcal{Q}[h_1, h_2]v = -\lambda(b) \partial^2_x v + \epsilon \left( \theta(b) v \partial^2_x \zeta + (2\theta(b) + (\gamma - 1)g(b)) \partial_x \zeta \partial_x v + \left( \theta(b) + \frac{2}{3} (\gamma - 1)g(b) \right) \zeta \partial^2_x v \right)
+ \beta \left( \alpha(b) v \partial^2_x b + 2 \alpha(b) \partial_x b \partial_x v + \left( \frac{2}{3} f(b) + \frac{2}{3} \delta^{-1} b f(b) \right) b \partial^2_v \right)
+ \epsilon \beta \left( (\theta_1(b) - \alpha_1(b)) \zeta v \partial^2_x b + (2\theta_1(b) - \alpha_1(b)) \zeta \partial_x b \partial_x v \right)
+ \epsilon \beta \left( (2\theta_1(b) - 2\alpha_1(b)) + \frac{1}{3} (\delta^{-1} - \beta b) f'(b) - \frac{\gamma}{3} g'(b) \right) \partial_x \zeta \partial_x v
+ \beta^2 \left( \eta(b) \partial_x b \partial_x v + \frac{2\gamma}{3} f'(b) \partial_x b \partial_x v + \frac{\gamma}{3} f'(b) b \partial^2_v - \frac{1}{3} f(b) b^2 \partial^2_v \right)
+ \epsilon \beta^2 \left( \eta_1(b) \partial_x b^2 \partial_x v \right)
+ \beta^3 \left( \frac{\gamma}{3} f''(b) \partial_x b^2 v \right)
+ \mathcal{O}(\epsilon^2),
\]

(3.3)

\[
\mathcal{R}[h_1, h_2]v = (1 - \gamma) g(b)^2 \left( \frac{1}{2} (\partial_x v)^2 + \frac{1}{3} \epsilon \partial^2_x v \right) + s(b)v^2 + t(b) v \partial_x v + \mathcal{O}(\epsilon),
\]

(3.4)

with

\[
\lambda(b) = \frac{1 + \gamma \delta}{3 \delta (\gamma + \delta - \gamma \delta b)}, \quad f(b) = \frac{\delta}{(\gamma + \delta - \gamma \delta b)}, \quad g(b) = \frac{1 - \delta \beta b}{(\gamma + \delta - \gamma \delta b)},
\]

\[
\theta(b) = \frac{1}{3} (\delta^{-1} - \beta b)^2 f(b) - \frac{1}{3} (\delta^{-1} - \beta b)^2 (1 - \gamma)f(b)^2 + \frac{2}{3} f(b) g(b)(1 - \gamma),
\]

\[
\alpha(b) = -\frac{1}{3} (\delta^{-1} - \beta b)^2 f'(b) + \frac{(\delta^{-1} - \beta b)}{2} f(b) - \frac{\gamma}{3} \delta^{-1} f'(b) + \frac{4}{3} f(b),
\]

\[
\theta_1(b) = \frac{1}{3} (\delta^{-1} - \beta b)^2 f''(b) - \frac{1}{3} (\delta^{-1} - \beta b)^2 (1 - \gamma) f(b) f'(b) - \frac{(\delta^{-1} - \beta b)}{2} f(b) + \frac{(\delta^{-1} - \beta b)}{2} f''(b) - \frac{\gamma}{3} \delta^{-1} f'(b) + \frac{2}{3} g'(b),
\]

\[
\alpha_1(b) = \frac{\gamma}{3} (1 - \gamma) \left( g(b) f'(b) + g'(b) f(b) \right),
\]

\[
\eta(b) = -\frac{1}{3} (\delta^{-1} - \beta b)^2 f'''(b) + (\delta^{-1} - \beta b) f'(b) - \frac{4}{3} \delta^{-1} f'''(b) + \frac{2}{3} g'(b),
\]

\[
\eta_1(b) = \frac{1}{3} (\delta^{-1} - \beta b)^2 f''''(b) - (\delta^{-1} - \beta b) f''(b) - 2 (\delta^{-1} - \beta b)(1 - \gamma) f'(b) f(b)
- \frac{1}{3} (\delta^{-1} - \beta b)^2 (1 - \gamma) f(b) f''(b) + \frac{(\delta^{-1} - \beta b)}{3} f''(b)
+ 2 f'(b) - (\delta^{-1} - \beta b) f''(b) - \frac{\gamma}{3} f'''(b) + \frac{\gamma (1 - \gamma)}{3} f''(b) g(b) - \frac{(1 - \gamma)}{3} \frac{2 f'(b) g'(b)}{3} - \gamma (1 - \gamma) f''(b) g(b) - \frac{(1 - \gamma)}{3} \frac{2 f'(b) g'(b)}{3}.
\]
A suitable choice of $s(b)$ so that\( \frac{\partial s}{\partial t}(b) \) are functions of $\zeta$ and $\beta b$ follows an equivalent model that can deal with these terms. To this end, we will construct in what follows an equivalent model that can deal with these terms.

Firstly, we introduce a second order symmetric differential operator,

$$\Sigma[\epsilon, \beta b] = q_1(\epsilon, \beta b) V + \mu \epsilon_0 \zeta_x \partial_x \partial_v V - \mu \partial_x (\nu q_2(\epsilon, \beta b) \partial_v V),$$

(3.5)

where $q_1(X, Y) = 1 + \kappa_1 X + \omega_1 Y$ and $q_2(X, Y) = 1 + \kappa_2 X + \omega_2 Y + \eta_2 X Y$ and $\nu, \kappa_0, \kappa_1, \kappa_2, \omega_1, \omega_2, \eta_2$ are functions of $b$ to be determined in an appropriate way treating all terms that cannot be controlled by the intended energy norm. For the sake of readability, we omit here and for the rest of the paper the dependence of these functions on $b$, thus one can write:

$$\Sigma[\epsilon, \beta b] \partial_v = q_1(\epsilon, \beta b) \partial_v \left( v + \mu \Omega[h_1, h_2] v \right) + q_1(\epsilon, \beta b) \mu \gamma + \delta \partial_v^2 \zeta$$

$$= \mu \epsilon \partial_x \partial_v \zeta \partial_x x \partial_v v - \mu \partial_x v \partial_x \partial_v v - \mu \nu \partial_x \zeta \partial_v v - \mu \partial_x (\nu \partial_x \zeta \partial_x v) - \mu \partial_x (\nu \beta \partial_x \zeta \partial_x v) - \mu \partial_x (\nu \partial_x \beta \partial_x \zeta \partial_x v)$$

$$+ \mu \gamma + \delta \partial_v^2 \zeta + \mu \epsilon \partial_x (f(b))^2 - \gamma (g(b))^2 \partial_v \zeta + O(\mu^2).$$

A proper choice of $\nu$ ought to treat the first order $(\mu \partial_x \partial_v v)$ terms by canceling them out. In fact, the second equation of system (3.3) gives

$$\partial_t v = -(\gamma + \delta) \partial_x \zeta - \epsilon 2 \partial_x \left( (f(b))^2 - \gamma (g(b))^2 \right) + O(\epsilon^2).$$

Equivalently, one has

$$\mu \gamma + \delta \partial_v^2 \zeta = \frac{\mu \gamma + \delta}{\gamma (g(b))^2} \partial_v v - \frac{\mu \epsilon}{2 \gamma (g(b))^2} \partial_x \left( (f(b))^2 - \gamma (g(b))^2 \right) + O(\epsilon, \mu^2).$$

Let's consider now equation (3.6). Substituting the term $\mu \gamma + \delta \partial_v^2 \zeta$ of this same equation by its above expression and applying the time partial derivative to the expansion of $\Omega[h_1, h_2] v$ in (3.3), thus one defines

$$\nu = \lambda(b) \frac{1}{\gamma (g(b))^2}.$$ 

(3.7)

Moreover, all terms including $\zeta$ and its derivatives can be canceled with an appropriate choice of the functions $\kappa_0, \kappa_1, \kappa_2, \omega_1, \omega_2$ and $\eta_2$. Indeed, using (3.2), two approximations hold: $\partial_v v = -(\gamma + \delta) \partial_x \zeta + O(\epsilon, \mu)$ and $\partial_t \zeta = -\partial_x (g(b)v) + O(\epsilon, \mu)$. Thus, one can set

$$\kappa_0 = \left( 1 + \beta \omega_1 b \right)^2 \partial_x \zeta,$$

(3.8)

so that $(\partial_x \zeta)^2 \partial_v b$ terms are withdrawn.

A suitable choice of $\kappa_1$ can cancel the $\zeta \partial_v^2 \zeta$ terms,

$$\kappa_1 = \frac{\left( -2 \theta - \frac{\gamma (g(b))}{\gamma (g(b))} \right) (1 + \beta \omega_1 b)}{\nu - \beta \left( \frac{\gamma (g(b))}{2 \gamma (g(b))} \right) - \frac{2 \gamma (g(b))}{3} \partial_v f(b)}.$$ 

(3.9)
\(\kappa_2\) and \(\eta_2\) are determined as a solution of the system below so that \(\partial_z\zeta\partial^2_z\zeta\) and \(\zeta\partial_x b\partial^2_x\zeta\) terms are withdrawn:

\[
\nu\kappa_2 + \nu\beta b\eta_2 = -(3\theta + (\gamma - 1)g(b))(1 + \beta\omega_1 b),
\]

\[
\nu'\kappa_2 + \nu\kappa_2' + (\nu + \nu'\beta b)\eta_2 + \nu\beta b\eta_2' = (-2\theta_1 + \alpha_1)(1 + \beta\omega_1 b) - \beta\nu b \frac{2\gamma}{3}f'(b) - \kappa_2 2\alpha.
\] (3.10)

Treating the \(b\partial^2_x\zeta\) terms is done through determining the function \(\omega_1\) as follows,

\[
\omega_1 = \frac{\nu\omega_2 + \frac{\gamma}{3}f(b) + \frac{2}{3}\delta^{-1}f(b) - \beta\nu b f(b)}{\nu - \beta b (\frac{\gamma}{3}f(b) + \frac{2}{3}\delta^{-1}f(b)) + \frac{2\gamma}{3}f(b)}.
\] (3.11)

Solving the following first order linear differential equation leads to the determination of the function \(\omega_2\) that can cancel the \(\partial_x b\partial^2_x\zeta\) terms.

\[
(\nu + \beta b\nu')\omega_2 + \beta b\nu\omega_2' = -\nu' - \beta b\omega_2 - \beta^2 b^2\omega_1 \frac{2\gamma}{3}f'(b) - 2\alpha - \frac{2\gamma}{3}f'(b)\beta b.
\] (3.12)

**Remark 3.14** A first order linear differential equation has the following form:

\[
y' + n(x)y = p(x), \quad \text{where} \quad y = y(x) \quad \text{and} \quad y' = \frac{dy}{dx},
\]

where \(n(x)\) and \(p(x)\) must be continuous functions. The general solution is given by:

\[
y = Ce^{-F(x)} + e^{-F(x)} \int e^{F(x)} p(x) dx, \quad \text{where} \quad F(x) = \int n(x) dx, \quad \text{and} \quad C \text{ is an arbitrary constant.}
\]

Using (3.12), the differential equation (3.13) can be easily rewritten as:

\[
\omega_2 + \left(\frac{1}{\beta b} + \frac{\nu'}{\nu} + \frac{2\alpha + \beta b^2 f'(b)}{\nu - \beta b (\frac{\gamma}{3}f(b) + \frac{2}{3}\delta^{-1}f(b)) + \frac{2\gamma}{3}f(b)}\right)\omega_2
\]

\[
= -\frac{\nu'}{\beta b} - \frac{2\alpha + \beta b^2 f'(b)}{\nu - \beta b (\frac{\gamma}{3}f(b) + \frac{2}{3}\delta^{-1}f(b)) + \frac{2\gamma}{3}f(b)} - \frac{2\alpha}{\beta b} - \frac{2\gamma}{3\nu}f'(b).
\] (3.15)

The continuity of the functions \(n(x)\) and \(p(x)\) is necessary in order to solve (3.15) but requires certain limitation conditions consisting of additional assumptions on the bottom deformation function \(b(x)\). In what follows, we briefly present these conditions:

- **If** \([\gamma^2\delta b^2 - 3\gamma\delta^2 - 4\delta^2 b((1 + \gamma)\delta b - 3\delta(\gamma + \delta))] > 0\) **then:**

\[
\nu(b) \neq 0, \quad \beta b \neq 0
\]

and \(\beta b \neq 2b^2b + \gamma^2\delta b^2 - 3\gamma\delta^2 \pm \sqrt{(\gamma^2\delta b^2 + 2\delta^2 b - 3\gamma^2\delta^2 - 4\delta^2 b((1 + \gamma)\delta b - 3\delta(\gamma + \delta))}} \frac{2\delta^2b}{2\delta^2b}
\]

- **If** \([\gamma^2\delta b^2 - 3\gamma\delta^2 - 4\delta^2 b((1 + \gamma)\delta b - 3\delta(\gamma + \delta))] < 0\) **then:**

\[
\nu(b) \neq 0, \quad \beta b \neq 0.
\]

- **If** \([\gamma^2\delta b^2 - 3\gamma\delta^2 - 4\delta^2 b((1 + \gamma)\delta b - 3\delta(\gamma + \delta))] = 0\) **then:**

\[
\nu(b) \neq 0, \quad \beta b \neq 0 \quad \text{and} \quad \beta b \neq \frac{2b^2b + \gamma^2\delta b^2 - 3\gamma\delta^2}{2\delta^2b}.
\] (H0)

Moreover, \(\partial_z^2\nu\) terms remain in (3.6), as well as in \(\partial_x (\overline{K}[h_1,h_2]v)\). These terms cannot be controlled by the energy space considered hereinafter. For this reason, and in order to cancel these terms, we introduce
a new function of $b$ to be determined to this end, named $\varsigma$ and embedded in the term $\Sigma[\epsilon \varsigma, \beta b](\epsilon \varsigma \partial_x v)$. In fact, one has

\[
\Sigma[\epsilon \varsigma, \beta b](\epsilon \varsigma \partial_x v) + \mu \epsilon q_1(\epsilon \varsigma, \beta b) \partial_x \left( \frac{h_1 h_2}{h_1 + \gamma h_2} v \right) = q_1(\epsilon \varsigma, \beta b)(\epsilon \varsigma \partial_x v) + \mu \epsilon q_2(\epsilon \varsigma, \beta b) \partial_x (\epsilon \varsigma \partial_x v) - \mu \partial_x \left( \nu q_2(\epsilon \varsigma, \beta b) \partial_x (\epsilon \varsigma \partial_x v) \right) + \mu \epsilon q_1(\epsilon \varsigma, \beta b) \partial_x \left( \frac{h_1 h_2}{h_1 + \gamma h_2} \right) v.
\] (3.16)

Adding (3.6) to (3.16), and fixing $\varsigma$ as follows

\[
\nu(1 + \beta \omega_2) \varsigma = \frac{(1 - \gamma) g(b)^2}{3} (1 + \beta \omega_1) b - \frac{1}{b_0} (f(b) - 2 g(b)) \theta(b) g(b)(1 + \beta \omega_1) b + \nu \beta \omega_2 (f(b) - 2 g(b)) (1 + \beta \omega_1) b \left[ \beta b \varepsilon f(b) + \frac{2}{3} \delta - 1 f(b) \right] (f(b)^2 - \gamma g(b)^2)
\]

\[
- (1 + \beta \omega_1) [\beta b^2 (1 + \frac{1}{3} f(b) f(b)^2 - \gamma g(b)^2)] - \beta \omega_1 b \lambda (b) (f(b)^2 - \gamma (g(b)^2),
\] (3.17)

while dropping all terms of order $O(\mu^2, \mu^2)$ yields the following approximation:

\[
\Sigma(\partial_t v + \epsilon \varsigma \partial_x v) - q_1(\epsilon \varsigma, \beta b) \partial_t \left( v + \mu \left( \frac{h_1 h_2}{h_1 + \gamma h_2} \right) v \right) + q_1(\epsilon \varsigma, \beta b) \mu \frac{\gamma + \delta}{\beta_0} \partial_x^2 \varsigma + \mu \epsilon q_1(\epsilon \varsigma, \beta b) \partial_x \left( \frac{h_1 h_2}{h_1 + \gamma h_2} \right) v = q_1(\epsilon \varsigma, \beta b)(\epsilon \varsigma \partial_x v) + \mu \mathcal{A}[\epsilon \varsigma \partial_x v] + \mu \mathcal{B}[\partial_t (\partial_x v)^2] + \mu \mathcal{C}[\epsilon \varnothing^2 v] + \mu \mathcal{D}[\partial_x (\partial_x v)^2] + \mu \mathcal{E} v^2 + \mu \mathcal{F} \partial_x \varsigma + O(\mu^2, \mu^2).
\] (3.18)

We would like to mention that the terms $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$, and $\mathcal{F}$ are functions depending on $\varsigma(t,x)$ and $b(x)$. We do not try to give in here their exact expressions for the sake of simplicity. However, these functions are detailed at the end of this paper in Appendix A. In fact, their exact expressions are of no interest for our present purpose. Although these functions are extensive but they remain easy to control.

Now, the last step to get the new equivalent asymptotic model is to multiply the second equation of (3.2) by $q_1(\epsilon \varsigma, \beta b)$ and include the estimate (3.18). Consequently, one gets the below system:

\[
\begin{align*}
\partial_t \varsigma + \partial_x \left( \frac{h_1 h_2}{h_1 + \gamma h_2} v \right) = 0, \\
\Sigma[\epsilon \varsigma, \beta b](\partial_t v + \epsilon \varsigma \partial_x v) + (\gamma + \delta) q_1(\epsilon \varsigma, \beta b) \partial_x \varsigma + \frac{\epsilon}{2} q_1(\epsilon \varsigma, \beta b) \partial_x \left( \frac{h_1^2 - \gamma h_2^2}{h_1 + \gamma h_2} \right) v - q_1(\epsilon \varsigma, \beta b)(\epsilon \varsigma \partial_x v) = \mu \mathcal{A}[\epsilon \varsigma \partial_x v] + \mu \mathcal{B}[\partial_t (\partial_x v)^2] + \mu \mathcal{C}[\epsilon \varnothing^2 v] + \mu \mathcal{D}[\partial_x (\partial_x v)^2] + \mu \mathcal{E} v^2 + \mu \mathcal{F} \partial_x \varsigma.
\end{align*}
\] (3.19)

**Remark 3.20** It is worth mentioning that one can easily recover the same model obtained and fully justified in [11] just after setting $\beta = 0$ in (3.19). Moreover, one can also recover the same model obtained and fully justified in [10] just after setting $\beta = O(\sqrt{\mu})$ and dropping all terms of order $O(\mu^2, \mu^2, \mu \beta, \mu \beta^2)$ in (3.19).

## 4 Properties of the operator $\Sigma$

The strong ellipticity of the operator $\Sigma[\epsilon \varsigma, \beta b]$, defined in (3.5) and recalled below is necessary to obtain the well-posedness and continuity of the inverse $\Sigma^{-1}$:

\[
\Sigma[\epsilon \varsigma, \beta b] V = q_1(\epsilon \varsigma, \beta b) V + \mu \beta \kappa_0 \partial_x \varsigma \partial_x b V - \mu \partial_x \left( \nu q_2(\epsilon \varsigma, \beta b) \partial_x V \right),
\] (4.1)

with $\nu, \kappa_0, \kappa_1, \kappa_2, \omega_1, \omega_2$ are functions of $b$. 


In what follows, we search for sufficient conditions to guarantee the ellipticity property of the operator $\mathcal{S}$. First, we assume that $\nu(b) > 0$ has a lower bound:
$$\nu(b) = \lambda(b) - \frac{1}{b_0} \geq \nu_0 > 0.$$  
Second, we assume the non-zero depth condition on both respectively upper and lower layer of fluid:

$$\exists h_{01} > 0, \text{ such that } \inf_{x \in \mathbb{R}} h_1 \geq h_{01} > 0, \quad \inf_{x \in \mathbb{R}} h_2 \geq h_{01} > 0, \quad (H1)$$

Briefly, since $\epsilon \leq \epsilon_{\max}$, thus $\epsilon' \leq \epsilon|z|_{L^\infty} = \epsilon_{\max}|z|_{L^\infty} < \min(1, \frac{1}{\delta_{\max}}) < 1$ so $(1 - \epsilon') > 0$, and

$$\epsilon' - \beta b \geq -\epsilon_{\max}|z|_{L^\infty} - \beta_{\max}|b|_{L^\infty} > - \min(1, \frac{1}{\delta_{\max}}) > - \frac{1}{\delta_{\max}} > - \frac{1}{\delta} \geq (1 + \epsilon' - \beta b) > 0.$$  
Thus, one can easily say that the condition $\epsilon_{\max}|z|_{L^\infty} + \beta_{\max}|b|_{L^\infty} < \min(1, \frac{1}{\delta_{\max}})$ is adequate to define $h_{01} > 0$ such that \(H1\) is valid independently of $(\mu, \epsilon, \delta, \gamma, \beta, b_0) \in \mathcal{P}_{\mathcal{CH}}$.

Equivalently, we introduce the condition

$$\exists h_{02} > 0, \text{ such that } \inf_{x \in \mathbb{R}} \left( q_1(\epsilon \zeta, \beta b) + \mu \epsilon h_0 \partial_x \zeta \partial_y b \right) \geq h_{02} > 0; \quad \inf_{x \in \mathbb{R}} q_2(\epsilon \zeta, \beta b) \geq h_{02} > 0. \quad (H2)$$

**Lemma 4.2** Let $\zeta \in L^\infty$, $b \in W^{2, \infty}$ and $\epsilon_{\max} = \min(M \sqrt{\mu_{\max}})$ be such that there exists $h_0 > 0$ with

$$\max(|\kappa_1|_{L^\infty}, |\kappa_2|_{L^\infty}, 1, \delta_{\max}) \epsilon_{\max}|z|_{L^\infty} + \max(|\omega_1|_{L^\infty}, |\omega_2|_{L^\infty}, \delta_{\max}) \beta_{\max}|b|_{L^\infty} + |\eta_2|_{L^\infty} \epsilon_{\max}|z|_{L^\infty} \beta_{\max}|b|_{L^\infty} + \mu_{\max} \epsilon_{\max} \max_{\beta_{\max}} \left( |\kappa_0|_{L^\infty} |\partial_x \zeta|_{L^\infty} |\partial_y b|_{L^\infty} \right) \leq 1 - h_0.$$  

Then there exists $h_{01}, h_{02} > 0$ such that \(H1, H2\) hold for any $(\mu, \epsilon, \delta, \gamma, \beta, b_0) \in \mathcal{P}_{\mathcal{CH}}$.

**Proof.** One can easily check that,

$$\epsilon' \leq \epsilon |z|_{L^\infty} \leq \epsilon_{\max} |z|_{L^\infty} \leq 1 - h_0 \quad \frac{1 - \epsilon'}{1 - \epsilon} \geq h_0, \quad \inf_{x \in \mathbb{R}} h_1 \geq h_0,$$

$$\delta_{\max} \epsilon_{\max} |z|_{L^\infty} + \delta_{\max} \beta_{\max} |b|_{L^\infty} \leq 1 - h_0 \quad \delta_{\max} \epsilon_{\max} |z|_{L^\infty} + \delta_{\max} \beta_{\max} |b|_{L^\infty} \geq h_0 - 1 \quad \delta \epsilon' - \delta \beta b \geq h_0 - 1 \quad \frac{1}{\delta} + \epsilon' - \beta \geq h_0 \quad \inf_{x \in \mathbb{R}} h_2 \geq h_0,$$

and in the same way,

$$|\kappa_1|_{L^\infty} \epsilon_{\max} |z|_{L^\infty} + |\omega_1|_{L^\infty} \beta_{\max} |b|_{L^\infty} + \mu_{\max} \epsilon_{\max} \beta_{\max} \left( |\kappa_0|_{L^\infty} |\partial_x \zeta|_{L^\infty} |\partial_y b|_{L^\infty} \right) \leq 1 - h_0 \quad \kappa_1 \epsilon_{\max} |z|_{L^\infty} + \omega_1 \beta_{\max} |b|_{L^\infty} + \mu_{\max} \epsilon_{\max} \beta_{\max} \left( |\kappa_0|_{L^\infty} |\partial_x \zeta|_{L^\infty} |\partial_y b|_{L^\infty} \right) \leq 1 - h_0 \quad -\kappa_1 \epsilon_{\max} |z|_{L^\infty} - \omega_1 \beta_{\max} |b|_{L^\infty} - \mu_{\max} \epsilon_{\max} \beta_{\max} \left( |\kappa_0|_{L^\infty} |\partial_x \zeta|_{L^\infty} |\partial_y b|_{L^\infty} \right) \geq h_0 - 1 \quad \kappa_1 \epsilon' + \omega_1 \beta b + \mu \beta \left( \kappa_0 \partial_x \zeta \partial_y b \right) \geq h_0 - 1 \quad 1 + \kappa_1 \epsilon' + \omega_1 \beta b + \mu \beta \left( \kappa_0 \partial_x \zeta \partial_y b \right) \geq h_0 \quad \inf_{x \in \mathbb{R}} \left( q_1(\epsilon \zeta, \beta b) + \mu \epsilon h_0 \partial_x \zeta \partial_y b \right) \geq h_0.$$
\[ |\kappa_2| L^\infty \epsilon + |\omega_2| L^\infty \beta_{\text{max}}|b| L^\infty + |\eta_2| L^\infty \epsilon + |\eta_2| L^\infty \beta_{\text{max}}|b| L^\infty \leq 1 - h_0 \]
\[ \kappa_2 \epsilon \max \left[ |\kappa_1| L^\infty + |\omega_2| L^\infty \beta_{\text{max}}|b| L^\infty + |\eta_2| L^\infty \epsilon \max \left[ |\kappa_1| L^\infty \right] \right] \leq 1 - h_0 \]
\[ - (\kappa_2 \epsilon \max \left[ |\kappa_1| L^\infty + |\omega_2| L^\infty \beta_{\text{max}}|b| L^\infty + |\eta_2| L^\infty \epsilon \max \left[ |\kappa_1| L^\infty \right] \right] ) \geq h_0 - 1 \]
\[ \kappa_2 \epsilon \kappa_1 + \omega_2 \beta_{\text{max}} b + \eta_2 \epsilon \beta b \geq h_0 - 1 \]
\[ 1 + \kappa_2 \epsilon \kappa_1 + \omega_2 \beta_{\text{max}} b + \eta_2 \epsilon \beta b \geq h_0 \]
\[ \inf_{x \in \mathbb{R}} (q_2(\epsilon \kappa_1, \beta b)) \geq h_0. \]

\[ \square \]

**Definition 4.3** We define by \( H^1_0(\mathbb{R}) \) the space \( H^1(\mathbb{R}) \) endowed with the norm \( \cdot \mid _{H^1_0} \) that is equivalent to the \( H^1(\mathbb{R}) \)-norm but not uniformly with respect to \( \mu \), defined as
\[ \forall v \in H^1(\mathbb{R}), \quad |v|_{H^1_0}^2 = |v|_{L^2}^2 + \mu |\partial_x v|_{L^2}^2. \]

We define also \( (H^1_0(\mathbb{R}))^* \) the space \( H^{-1}(\mathbb{R}) \) the dual space of \( H^1_0(\mathbb{R}) \).

Let \( \mu, \epsilon, \delta, \gamma, \beta, \) and \( b \in \mathcal{P}_{CH} \) and \( \zeta \in W^{1, \infty}(\mathbb{R}), b \in W^{1, \infty}(\mathbb{R}) \) such that [H2] is satisfied.

Then the operator
\[ \mathfrak{T}[\epsilon \kappa_1, \beta b] : H^1_0(\mathbb{R}) \rightarrow (H^1_0(\mathbb{R}))^* \]
is uniformly continuous and coercive. More precisely, there exists \( c_0 > 0 \) such that
\[ (\mathfrak{T} u, v) \leq c_0 |u|_{H^1_0} |v|_{H^1_0} \]  \hfill (4.5)
\[ (\mathfrak{T} v, v) \geq \frac{1}{c_0} |v|_{H^1_0}^2 \]  \hfill (4.6)

with \( c_0 = C(\mathcal{M}_{CH}, \mu_{\epsilon, \delta} \cdot |\epsilon|_{W^{1, \infty}}, |\beta|_{W^{1, \infty}}) \).

The constant \( c_0 \) also depends on \( \mu, \epsilon, \delta, \gamma, \beta, \) and \( b \in W^{1, \infty}(\mathbb{R}) \).

Moreover, the following estimates hold:

Let \( s_0 > \frac{1}{2} \) and \( s \geq 0 \),

(i) If \( \zeta \in H^{\alpha}(\mathbb{R}) \cap H^s(\mathbb{R}) \) and \( b \in H^{\alpha+2}(\mathbb{R}) \cap H^{s+2}(\mathbb{R}) \) and \( u \in H^{\alpha+1}(\mathbb{R}) \) and \( v \in H^1(\mathbb{R}) \), then:
\[ |(\mathfrak{A}^s \mathfrak{T}[\epsilon \kappa_1, \beta b] u, v)| \leq C_0 \left( |\epsilon|_{H^{\alpha+1}} + |\beta|_{H^{\alpha+s}} \right) |u|_{H^{\alpha+s+1}} + \left( |\epsilon|_{H^{\alpha+1}} + |\beta|_{H^{\alpha+s}} \right) |u|_{H^{\alpha+s+1}}^2 \]  \hfill (4.7)

(ii) If \( \zeta \in H^{\alpha+1} \cap H^s(\mathbb{R}) \) and \( b \in H^{\alpha+3} \cap H^{s+2}(\mathbb{R}) \), \( u \in H^s(\mathbb{R}) \) and \( v \in H^1(\mathbb{R}) \), then:
\[ |([\mathfrak{A}^s \mathfrak{T}[\epsilon \kappa_1, \beta b] u, v)| \leq \max(\epsilon, \beta) C_0 \left( |\epsilon|_{H^{\alpha+1}} + |b|_{H^{\alpha+s+1}} \right) |u|_{H^{\alpha+s}} \]  \hfill (4.8)

where \( C_0 = C(\mathcal{M}_{CH}, \mu_{\epsilon, \delta}^{-1}) \).

**Proof.** We define first the bilinear form
\[ a(u, v) = (\mathfrak{T} u, v) = \left( \left( 1 + \epsilon \kappa_1 \zeta + \beta \omega_1 b \right) u, v \right) + \mu \beta \left( \kappa_0 \partial_x \zeta \partial_x b u, v \right) + \mu \left( \nu \left( 1 + \epsilon \kappa_2 \zeta + \beta \omega_2 b + \epsilon \beta \eta_2 \zeta \zeta b \right) \partial_x u, \partial_x v \right), \]
where \( \left( \cdot, \cdot \right) \) denotes the \( L^2 \)-based inner product. One can easily check that
\[ |a(u, v)| \leq \sup_{x \in \mathbb{R}} |1 + \epsilon \kappa_1 \zeta + \beta \omega_1 b + \mu \beta \kappa_0 \partial_x \zeta \partial_x b( u, v ) + \mu \sup_{x \in \mathbb{R}} |\nu(1 + \epsilon \kappa_2 \zeta + \beta \omega_2 b + \epsilon \beta \eta_2 \zeta \zeta b)( \partial_x u, \partial_x v ). \]
By Cauchy-Schwarz inequality, one can easily obtain (4.5).

Let us now prove the $H^1_μ(\mathbb{R})$-coercivity of $a(\cdot, \cdot)$, inequality (4.6):

$$ ( \mathcal{T} v , v ) = \left( [q_1(\epsilon_z, \beta b) + \mu \epsilon \kappa_0 \partial_x \zeta \partial_z v , v ] + \mu \left( \nu q_2(\epsilon_z, \beta b) \partial_z v , \partial_z v \right) \right). $$

Since (H2) is satisfied one has,

$$ ( \mathcal{T} v , v ) \geq h_{02} |v|^2 + \mu h_{02} \partial_z v |^2 \geq \min ( h_{02} , \nu h_{02} ) \left( |v|^2 + \mu |\partial_z v|^2 \right). $$

Finally one can easily deduce that the inequality (4.9) is now:

$$ ( \mathcal{T} v , v ) \geq h_{02} \min (1, \nu_0) \mathcal{v}_H^1. $$

We prove the product and commutator higher-order estimates of the Lemma, Making use of, $\kappa_0 \partial_x \zeta \partial_z b = \partial_x (\kappa_0 \partial_x \zeta) b - \kappa_0 u \zeta \partial_z b$, one has

$$ ( \Lambda^s \mathcal{T} u , v ) = ( \Lambda^s \{ q_1(\epsilon_z, \beta b) u + \mu \epsilon \kappa_0 \partial_x \zeta \partial_z b \}, v ) + \mu \left( \Lambda^s \nu q_2(\epsilon_z, \beta b) \partial_z u , \partial_z v \right) $$

$$ = ( \Lambda^s \{ ( 1 + \epsilon_k \epsilon_z + \beta_0 \beta_1 ) u , v \} - \mu \beta ( \Lambda^s \{ \kappa_0 \partial_x \zeta \partial_z b \}, \partial_z v ) - \mu \beta ( \Lambda^s \{ \kappa_0 u \zeta \partial_z b \}, v ) \right) $$

Since (H2) is satisfied one has,

$$ \left( \Lambda^s \mathcal{T} u , v \right) \geq h_{02} |v|^2 + \mu |\partial_z v|^2 \geq \min ( h_{02} , \nu h_{02} ) \left( |v|^2 + \mu |\partial_z v|^2 \right). $$

Finally one can easily deduce that the inequality (4.9) is now:

$$ ( \mathcal{T} v , v ) \geq h_{02} \min (1, \nu_0) \mathcal{v}_H^1. $$

We prove the product and commutator higher-order estimates of the Lemma, Making use of, $\kappa_0 \partial_x \zeta \partial_z b = \partial_x (\kappa_0 \partial_x \zeta) b - \kappa_0 u \zeta \partial_z b$, and the fact that $\partial_x (\Lambda^s, f |g) = [\Lambda^s, \partial_x f |g + [\Lambda^s, f] \partial_x g$, one has

$$ ( [\Lambda^s, \mathcal{T}] u , v ) = \epsilon ( \Lambda^s, \kappa_1 u , v ) + \beta ( \Lambda^s, \omega_1 b , v ) + \mu \epsilon ( \Lambda^s, \kappa_0 \partial_x \zeta \partial_z b , v ) + \mu \beta ( \Lambda^s, \nu q_2 \partial_z b , \partial_z v ) $$

$$ + \mu \beta ( \Lambda^s, \nu q_2 \partial_z b , \partial_z v ) $$

Similarly, for the commutator estimates, using $\kappa_0 \partial_x \zeta \partial_z b = \partial_x (\kappa_0 \partial_x \zeta) b - \kappa_0 u \zeta \partial_z b$, one has

Estimates (4.8) follow, using again Cauchy-Schwarz inequality and the commutator estimates. □

The invertibility of $\mathcal{T}$ is asserted in the Lemma below.

**Lemma 4.10** Let $(\mu, \epsilon, \delta, \gamma, \beta, bo) \in \mathcal{P}_{CH}$ and $\zeta \in W^{1, \infty}(\mathbb{R})$, $b \in W^{1, \infty}(\mathbb{R})$ such that (H2) is satisfied.

Hence the operator

$$ \mathcal{T}[\epsilon \zeta, \beta b] : H^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) $$

is bijective.

(i) $(\mathcal{T}[\epsilon \zeta, \beta b])^{-1} : L^2 \rightarrow H^1_\mu(\mathbb{R})$ is continuous. More precisely, one has

$$ \| \mathcal{T}^{-1} \|_{L^2(\mathbb{R}) \rightarrow H^1_\mu(\mathbb{R})} \leq c_0, $$

with $c_0 = C(M_{CH}, h_{02}^{-1}, \epsilon |\zeta|_{W^{1, \infty}}, \beta |b|_{W^{1, \infty}})$.

(ii) Additionally, if $\zeta \in H^{s_0+1}(\mathbb{R})$ and $b \in H^{s_0+3}(\mathbb{R})$ with $s_0 > \frac{1}{2}$, then one has for any $0 < s \leq s_0 + 1$,

$$ \| \mathcal{T}^{-1} \|_{H^s(\mathbb{R}) \rightarrow H^s_\mu(\mathbb{R})} \leq c_{s_0+1}. $$

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We use the Lax-Milgram theorem to prove that the operator \( T \).

Proof.

Dually of \( H \) is proved to be continuous and uniformly coercive on \( H \), for all \( u \in H \) is satisfied with

\[
\mu > \nu_2 \quad \text{and since} \quad u \in H, \quad \nu_2 b \quad \text{is defined in (3.8), (3.9), (3.10), (3.11), (3.12), (3.13), and (3.17)}.
\]

\[
(3.19) \quad \text{Let us recall the system (3.19).}
\]

\[
\left\{ \begin{array}{l}
\partial_t \zeta + \partial_x \left( \frac{h_1 h_2}{h_1 + h_2} v \right) = 0, \\
\tau [\zeta, \zeta] = \partial_t v + \epsilon \nu \partial_x v + \left( \gamma + \delta \right) q_1 (\epsilon, \zeta) \partial_x \zeta + \left( \frac{\epsilon}{2} q_1 (\epsilon, \zeta) \right) \partial_x \left( \frac{h_2 (h_1 + h_2)^2 v^2}{h_1 (h_1 + h_2)^2 v^2} - q_1 (\epsilon, \zeta) \right) \\
\quad \mu [A] v \partial_x v + \mu [B] (\partial_x v)^2 + \mu [C] v \partial_x v + \mu [D] \partial_x (\epsilon \nu \partial_x v) + \mu [E] v^2 + \mu [F] \partial_x \zeta,
\end{array} \right.
\]

with \( h_1 = 1 - \epsilon \), \( h_2 = 1/\delta + \epsilon - \beta \), \( q_1(X, Y) = 1 + \kappa_1 X + \omega Y \), \( q_2(X, Y) = 1 + \kappa_2 X + \omega_2 Y + \eta_2 XY \), where \( \kappa_0, \kappa_1, \kappa_2, \omega_1, \omega_2, \eta_1, \zeta \) defined in (3.8), (3.9), (3.10), (3.11), (3.12), (3.13), (3.14), (3.17), and

\[
(3.17) \quad \tau [\zeta, \zeta] V = q_1 (\epsilon, \zeta) V + \mu \epsilon \kappa_0 \partial_x \zeta \partial_x v - \mu \partial_x \left( \nu q_2 (\epsilon, \zeta) \partial_x v \right).
\]
In order to ease the reading, we define the function

$$H : X \rightarrow \frac{(1 - X)(\delta^{-1} + X - \beta b)}{1 - X + \gamma(\delta^{-1} + X - \beta b)}$$

and

$$G : X \rightarrow \left(\frac{1 - X}{1 - X + \gamma(\delta^{-1} + X - \beta b)}\right)^2.$$ 

One can easily check that

$$H(\epsilon \zeta) = \frac{h_1 h_2}{h_1 + \gamma h_2}, \quad H'(\epsilon \zeta) = \frac{h_1^2 - \gamma h_2^2}{(h_1 + \gamma h_2)^2} \quad \text{and} \quad G(\epsilon \zeta) = \left(\frac{h_1}{h_1 + \gamma h_2}\right)^2.$$

One can rewrite,

$$\begin{cases}
\partial_t \zeta + 2 H(\epsilon \zeta) \partial_x v + \epsilon \partial_x H'(\epsilon \zeta) v - \beta \partial_x bG(\epsilon \zeta) v = 0, \\
\Xi \left(\partial_t v + \frac{\epsilon}{2} \partial_x (v^2)\right) + (\gamma + \delta) q_1(\epsilon \zeta, \beta b) \partial_x \zeta + \epsilon q_1(\epsilon \zeta, \beta b)(H'(\epsilon \zeta) - \zeta) v \partial_x v + \epsilon q_1(\epsilon \zeta, \beta b) \partial_x \left(\frac{H'(\epsilon \zeta)}{2}\right)v^2 \\
= \mu[A] v \partial_x v + \mu[B] (\partial_x v)^2 + \mu[C] v \partial_x^2 v + \mu[D] \partial_x ((\partial_x v)^2) + \mu[E] v^2 + \mu[F] \partial_x \zeta.
\end{cases}$$

with \(\partial_x (\frac{H'(\epsilon \zeta)}{2}) = -\gamma \epsilon \partial_x \zeta (h_1 + h_2)^2 + \gamma \beta \partial_x b h_1 (h_1 + h_2) \).  

The equations can be written after applying \(\Xi^{-1}\) to the second equation in (5.2) as

$$\partial_t U + A[U] \partial_x U + B[U] = 0,$$

with

$$A[U] = \left(\Xi^{-1}(Q_0(\epsilon \zeta, \beta b) + \epsilon^2 Q_1(\epsilon \zeta, \beta b, v)) - H(\epsilon \zeta)\right),$$

$$B[U] = \left(\Xi^{-1}(\frac{-\beta \partial_x b G(\epsilon \zeta) v}{(h_1 + \gamma h_2)^3} (v^2 - \mu[E] v^2)\right)$$

where \(Q_0(\epsilon \zeta, \beta b), Q_1(\epsilon \zeta, \beta b, v)\) are defined as

$$Q_0(\epsilon \zeta, \beta b) = (\gamma + \delta) q_1(\epsilon \zeta, \beta b) - \mu[F], \quad Q_1(\epsilon \zeta, \beta b, v) = -\gamma q_1(\epsilon \zeta, \beta b) (h_1 + h_2)^2 (h_1 + \gamma h_2)^3 v^2$$

and the operator \(\Xi[\epsilon \zeta, \beta b, v]\) defined by

$$\Xi[\epsilon \zeta, \beta b, v] f \equiv (\epsilon q_1(\epsilon \zeta, \beta b)(H'(\epsilon \zeta) - \zeta) - \mu[A]) v f - \mu[B] \partial_x v f - \mu[C] v \partial_x f - \mu[D] \partial_x ((\partial_x v)^2).$$

System (5.3) enjoys a quasilinear hyperbolic structure. We start by studying the properties and the energy estimates for the linearized system to conclude finally the well-posedness of the initial value problem of the system. The linearization is around some reference state \(U = (\zeta, \nu)\) as follows:

$$\begin{cases}
\partial_t U + A[U] \partial_x U + B[U] = 0; \\
U(0) = U_0.
\end{cases}$$

### 5.1 Energy space

First, we introduce a pseudo-symmetrizer of the system:

$$Z[U] = \begin{pmatrix}
Q_0(\epsilon \zeta, \beta b) + \epsilon^2 Q_1(\epsilon \zeta, \beta b, v) & 0 \\
H(\epsilon \zeta) & \Xi(\epsilon \zeta, \beta b)
\end{pmatrix}. \quad (5.9)$$

The pseudo-symmetrizer is defined and positive under the assumption below:

$$\exists h_0 > 0 \ \text{such that} \ \ Q_0(\epsilon \zeta, \beta b) + \epsilon^2 Q_1(\epsilon \zeta, \beta b, v) \geq h_0 > 0. \quad (H3)$$

Now, we define the energy space:
More precisely, there exists \( (H_2) \) and \( \text{Lemma } 5.12 \) Let \( (H_3) \). We assert this equivalency in the Lemma below. We omit in here the proof of this Lemma since \( X \triangleq \mathbb{R} \), endowed with the norm \( \| U \|_{X^*} \equiv \sup_{t \in [0,T]} \| U(t,\cdot) \|_{X^*} + \epsilon \sup_{t \in [0,T]} \| \partial_t U(t,x) \|_{\mathbb{R}} \).

The energy of the initial value problem (5.8) is now given naturally by:

\[
E^s(U) = (\Lambda^* U, Z[U], \Lambda^* U) = (\Lambda^* \zeta, \frac{Q_0(\zeta, \beta b)}{H(\zeta)} + \epsilon^2 Q_1(\epsilon \zeta, \beta b, U) \Lambda^* \zeta, \Lambda^* \xi, U, U, \beta, |b| |U||b|) + (\Lambda^* v, \mathcal{V}[\xi, \beta b |\Lambda^* v|) = (5.11)
\]

The energy of the pseudo-symmetrizer is equivalent to \( X^* \) under the additional assumption given in \( \mathcal{H}^3 \). We assert this equivalency in the Lemma below. We omit in here the proof of this Lemma since it is proved using the same techniques as in \( \mathcal{H}^3 \). Lemma 5.2.

**Lemma 5.12** Let \( p = (\mu, \epsilon, \delta, \gamma, \beta, b_0) \in \mathcal{P}_{CH} \), \( s \geq 0 \), \( U \in W^{1,\infty}(\mathbb{R}) \) and \( b \in W^{2,\infty}(\mathbb{R}) \), satisfying \( \mathcal{H}^3 \), \( \mathcal{H}^2 \), and \( \mathcal{H}^3 \). Then \( E^s(U) \) is equivalent to the \( \|U\|_{X^s}\)-norm.

More precisely, there exists \( c_0 = c_0(M_{CH}, h_{01}, h_{02}, h_{03}, s \in W^{1,\infty}(\beta b) > 0 \) such that

\[
\frac{1}{c_0} E^s(U) \leq |U|_{X^s} \leq c_0 E^s(U).
\]

To complete this section we assert some useful general estimates concerning our new operators.

**Lemma 5.13** Let \( p = (\mu, \epsilon, \delta, \gamma, \beta, b_0) \in \mathcal{P}_{CH} \), and let \( U = (\zeta, u)^T \in W^{1,\infty}, b \in W^{2,\infty} \) satisfies \( \mathcal{H}^3 \), \( \mathcal{H}^2 \) and \( \mathcal{H}^3 \). Then for any \( V, W \in X^0 \), one has

\[
\left\|\left( Z[U] V, W \right) \right\| \leq C \left\| V \right\|_{X^0} \left\| W \right\|_{X^0},
\]

with \( C = C(M_{CH}, h_{01}, h_{02}, h_{03}, \epsilon \left\| U \right\|_{W^{1,\infty}}, \beta \left\| b \right\|_{W^{2,\infty}} \).

Moreover, if \( U \in X^s, b \in H^{s+2}, V \in X^{s-1} \) with \( s \geq s_0 + 1, s_0 > 1/2 \), then one has

\[
\left\|\left( \Lambda^*, Z[U] V, W \right) \right\| \leq C \left\| V \right\|_{X^{s-1}} \left\| W \right\|_{X^0}
\]

\[
\left\|\left( \Lambda^*, Z^{-1}[U] V, Z[U] W \right) \right\| \leq C \left\| V \right\|_{H^{s+1}} \left\| W \right\|_{X^0}
\]

with \( C = C(M_{CH}, h_{01}, h_{02}, h_{03}, \epsilon \left\| U \right\|_{X^0}, \beta \left\| b \right\|_{H^{s+2}} \).

**Proof.** The Lemma 5.13 is proved using Cauchy-Schwartz inequality, Lemma 4.10 and Corollary 4.13. We do not detail the proof, and refer to [11] Lemma 6.4.

### 5.2 Energy estimates

In this section we aim at establishing an \( X^s \) energy estimate regarding our linear system. The well-posedness and stability of the nonlinear system is made possible using linear analysis by considering a modified system of the form:

\[
\begin{align*}
\partial_t U + A[U] \partial_x U + B[U] &= F; \\
U|_{t=0} &= U_0.
\end{align*}
\]

where we added a right-hand-side \( F \), whose properties will be precised in the following Lemmas.

We begin by asserting a basic \( X^s \) \( (s > 3/2) \) energy estimate.
Lemma 5.18 (\(X^s\) energy estimate) Set \((\mu, \epsilon, \delta, \gamma, \beta, \bo) \in \mathcal{P}_{CH},\) and \(s \geq s_0 + 1, \ s_0 > 1/2.\) Let \(U = (\zeta, v)\) and \(U = (\zeta, v)\) be such that \([U, \partial_t U] \in L^\infty([0, T/\max(\epsilon, \beta)]; X^s), \partial_t [U, \partial_t U] \in L^\infty([0, T/\max(\epsilon, \beta)]; X^s)\), \(b \in H^{s+3}\) and \(U\) satisfies \([H1], [H2],\) and \([H3]\) uniformly on \([0, T/\max(\epsilon, \beta)],\) and such that system (5.17) holds with a right hand side, \(F,\) with

\[
(\Lambda^s F, Z[U] \Lambda^s U) \leq C_F \max(\epsilon, \beta)|U|_{X^s}^2 + f(t) |U|_{X^s},
\]

where \(C_F\) is a constant and \(f\) is an integrable function on \([0, T/\max(\epsilon, \beta)]\).

Then there exists \(\lambda, C_2 = C(\|U\|_{X^s}, \|b\|_{H^{s+3}}, C_F)\) such that the following energy estimate holds:

\[
E' (U)(t) \leq e^{2\max(\epsilon, \beta)\lambda t} E^s(U_0) + \int_0^t e^{2\max(\epsilon, \beta)\lambda(t-t')} (f(t') + \max(\epsilon, \beta)C_2) dt',
\]

(5.19)

The constants \(\lambda\) and \(C_2\) are independent of \(p = (\mu, \epsilon, \delta, \gamma, \beta, \bo) \in \mathcal{P}_{CH},\) but depend on \(M_{CH}, h_{01}^{-1}, h_{02}^{-1},\) and \(h_{03}^{-1}\).

Remark 5.20 In this Lemma, and in the proof below, the norm \(\|U\|_{X^s}\) is to be understood as an essential sup:

\[
\|U\|_{X^s} = \text{ess sup}_{t \in [0, T/\max(\epsilon, \beta)]} |U(t, \cdot)|_{X^s} + \text{ess sup}_{t \in [0, T/\max(\epsilon, \beta)], x \in \mathbb{R}} |\partial_t U(t, x)|.
\]

Proof. Let us multiply the system (5.17) on the right by \(\Lambda^s Z[U] \Lambda^s U,\) and integrate by parts. One obtains

\[
(\Lambda^s \partial_t U, Z[U] \Lambda^s U) + (\Lambda^s A[U] \partial_x U, Z[U] \Lambda^s U) + (\Lambda^s B[U], Z[U] \Lambda^s U) = (\Lambda^s F, Z[U] \Lambda^s U),
\]

(5.21)

from which we deduce, using the symmetry property of \(Z(U),\) as well as the definition of \(E'(U)\):

\[
\frac{1}{2} \frac{d}{dt} E'(U)^2 = \frac{1}{2} (\Lambda^s U, [\partial_t, Z[U]] \Lambda^s U) - (Z[U] A[U] \partial_x \Lambda^s U, \Lambda^s U) - ([\Lambda^s, A[U]] \partial_x U, Z[U] \Lambda^s U) - (\Lambda^s B[U], Z[U] \Lambda^s U) + (\Lambda^s F, Z[U] \Lambda^s U).
\]

(5.22)

We now estimate each of the different components of the r.h.s of the above identity.

- **Estimate of** \((\Lambda^s B[U], Z[U] \Lambda^s U),\)

\[
(\Lambda^s B[U], Z[U] \Lambda^s U) = \left( \Lambda^s \left( - G(\epsilon \zeta, \beta b) \partial_x b, \frac{Q_0(\epsilon \zeta, \beta b) + c^2 Q_1(\epsilon \zeta, \beta b, v)}{H(\epsilon \zeta)} \frac{Q_0(\epsilon \zeta, \beta b) + c^2 Q_1(\epsilon \zeta, \beta b, v)}{H(\epsilon \zeta)} \Lambda^s \zeta \right) + \Lambda^s \sum_{h=1}^\infty (c^2 \rho_1(\epsilon \zeta, \beta b) h_1(h_1 + h_2) c^2 \partial_x b) \frac{Q_0(\epsilon \zeta, \beta b) + c^2 Q_1(\epsilon \zeta, \beta b, v)}{H(\epsilon \zeta)} \Lambda^s \zeta b \right)
\]

Using Cauchy-Schwarz inequality, Lemma (4.4) and Lemma (4.10) one has,

\[
|((\Lambda^s B[U], Z[U] \Lambda^s U)| \leq \beta C(\|U\|_{X^s}, \|b\|_{H^{s+3}})\|U\|_{X^s} \leq \max(\epsilon, \beta)C_2\|U\|_{X^s}.
\]

(5.23)

- **Estimate of** \((Z[U] A[U] \partial_x \Lambda^s U, \Lambda^s U).\)

Now we have,

\[
Z[U] A[U] = \left( \frac{Q_0(\epsilon \zeta, \beta b) + c^2 Q_1(\epsilon \zeta, \beta b, v)}{H(\epsilon \zeta)} \frac{Q_0(\epsilon \zeta, \beta b) + c^2 Q_1(\epsilon \zeta, \beta b, v)}{H(\epsilon \zeta)} \Lambda^s \zeta \left( \frac{Q_0(\epsilon \zeta, \beta b) + c^2 Q_1(\epsilon \zeta, \beta b, v)}{H(\epsilon \zeta)} \frac{Q_0(\epsilon \zeta, \beta b) + c^2 Q_1(\epsilon \zeta, \beta b, v)}{H(\epsilon \zeta)} \Lambda^s \zeta b \right) \right)
\]

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One can use the definition of $A[\cdot]$ and $Z[\cdot]$ in (5.4) and (5.9), one deduces that,

$$\left(ZA[\cdot]\partial_s U, U\right) = \frac{1}{2} \left( \frac{\partial_s (\epsilon (\epsilon_s \beta) H'(\epsilon_s) - \zeta u \partial_s v, v) - \mu \left([A]_{\partial_s u, \partial_s v}, v\right) - \mu \left([B]_{\partial_s u, \partial_s v}, v\right)}{\mu \left([C]_{\partial_s u, \partial_s v}, v\right) + \mu \left([D]_{\partial_s u, \partial_s v}, v\right)} + \mu \left([D]_{\partial_s u, \partial_s v}, v\right) \right)$$

One make use of the identities below,

- $\left(\Omega[\epsilon_s, \beta, \eta] \partial_s u, v\right) = \left(\epsilon q_1 (\epsilon_s, \beta) (H'(\epsilon_s) - \zeta u \partial_s v, v) - \mu \left([A]_{\partial_s u, \partial_s v}, v\right) - \mu \left([B]_{\partial_s u, \partial_s v}, v\right)

Using Cauchy-Schwarz inequality we obtain,

$$\left|\left(ZA[\cdot]\partial_s U, U\right)\right| \leq \max(\epsilon, \beta) C\left(\|U\|_L^2 + \|\partial_s U\|_L^2 + \|b\|_{W^{3,\infty}}\right)|U|^2_X. \quad (5.24)$$

Thanks to Sobolev embedding, one has for $s > s_0 + 1, s_0 > 1/2$

$$C(\|U\|_L^2 + \|\partial_s U\|_L^2) \leq C(\|U\|^2_X).$$

One can use the $L^2$ estimate derived in (5.24), applied to $\Lambda^* U$. One deduces

$$\left|\left(ZA[\cdot]\partial_s \Lambda^* U, \Lambda^* U\right)\right| \leq \max(\epsilon, \beta) C\left(\|U\|^2_X + \|b\|_{W^{3,\infty}}\right)|U|^2_X. \quad (5.25)$$

- Estimate of $([\Lambda^*, A[\cdot]]_U, ZA[\cdot]_{\Lambda^* U}$). Using the definition of $A[\cdot]$ and $Z[\cdot]$ in (5.4) and (5.5), one
Here and in the following, we denote $\Sigma \equiv \Sigma[e, b]$ and $Q(e, b, v) = Q_0(e, b) + e^2 Q_1(e, b, v)$.

We can now use Corollary 4.13, and deduce $\Sigma \Lambda^s \equiv B_1 + B_2 + B_3$.

From Cauchy-Schwarz inequality, one has $\|Q(e, b, v)\|_{L^\infty} \leq \max(\epsilon, \beta) C(\|U\|_{X_1}, \|b\|_{H^s}) |U|_{X_1}$.

It follows, using that $Q(e, b, v) = e^2 Q_1(e, b, v)$.

By symmetry of $\Sigma$, one has $B_2 = \left( \Sigma \Lambda^s \equiv \Sigma e^{-1}(Q(e, b, v)) \partial_x \zeta , \Sigma \Lambda^s v \right)$.

Now, one can check that, by definition of the commutator,

$$\Sigma \Lambda^s \equiv \Sigma e^{-1}(Q(e, b, v)) \partial_x \zeta = \Sigma \Lambda^s - Q(e, b, v) \Lambda^s \partial_x \zeta = -[\Lambda^s, Q(e, b, v)] \partial_x \zeta.$$
so that we finally get
\[ |B_2| \leq \max(\epsilon, \beta) C_2 |U|^2_{X^{r,s}}. \]

To control of \( B_3 = \left( [\Lambda^s, \Xi^{-1}(\Sigma e_{\zeta}, \beta b, v)] + \epsilon_2 v^2 \right) \partial_x v, \Xi \Lambda^s v. \)

Let us first use the definition of \( \Sigma e_{\zeta}, \beta b, v \) [37] to expand:
\[ B_3 = \left( [\Lambda^s, \Xi^{-1}(\epsilon q_1(e_{\zeta}, \beta b) (H'(e_{\zeta}) - \zeta) \psi)] \partial_x v, \Xi \Lambda^s v \right) - \mu \left( [\Lambda^s, \Xi^{-1}(\partial_x v)] \partial_x v, \Xi \Lambda^s v \right) - \mu \left( [\Lambda^s, \Xi^{-1}(\partial_x (\partial_x v \cdot)))] \partial_x v, \Xi \Lambda^s v \right) + \epsilon \left( [\Lambda^s, \epsilon_2 v^2 \partial_x v, \Xi \Lambda^s v \right).
\]

In order to estimate \( B_{31}, B_{32}, B_{33}, B_{34} \) and \( B_{35}, \) one proceeds as for the control of \( B_2. \)

One can check, using Cauchy-Schwarz inequality, Corollary 4.13 and the commutator estimate, one obtains
\[ |B_{31}| \leq \max(\epsilon, \beta) C((|U|_{X^{r,s}}^2, |b|_{H^{r + 2}}^2)|U|^2_{X^{r,s}}. \]

In the same way we control \( B_{32}, \)
\[ \Sigma [\Lambda^s, \Xi^{-1}(\partial_x v)] \partial_x v = -[\Lambda^s, \Xi^{-1}(\partial_x v)] \partial_x v + [\Lambda^s, \partial_x v] \partial_x v. \]

Again, using Cauchy-Schwarz inequality, Corollary 4.13 and the commutator estimate, one has
\[ \mu \left( \left( [\Lambda^s, \Xi^{-1}(\partial_x v)] \partial_x v, \Lambda^s v \right) \right) \leq \mu \max(\epsilon, \beta) C((|U|_{X^{r,s}}^2, |b|_{H^{r + 2}}^2)|U|^2_{X^{r,s}}. \]

and
\[ \mu \left( \left( [\Lambda^s, \partial_x v] \partial_x v, \Lambda^s v \right) \right) \leq \max(\epsilon, \beta) C(|U|_{X^{r,s}}^2, |b|_{H^{r + 2}}^2)|U|^2_{X^{r,s}}. \]

Thus we proved
\[ |B_{32}| \leq \max(\epsilon, \beta) C((|U|_{X^{r,s}}^2, |b|_{H^{r + 2}}^2)|U|^2_{X^{r,s}}. \]

To control \( B_{33} \) one can check,
\[ \Sigma [\Lambda^s, \Xi^{-1}(\partial_x v)] \partial_x v = -[\Lambda^s, \Xi^{-1}(\partial_x v)] \partial_x v + [\Lambda^s, \partial_x v] \partial_x v. \]

Again, using Cauchy-Schwarz inequality, Corollary 4.13 and the commutator estimate, one has
\[ \mu \left( \left( [\Lambda^s, \Xi^{-1}(\partial_x v)] \partial_x v, \Lambda^s v \right) \right) \leq \mu \max(\epsilon, \beta) C((|U|_{X^{r,s}}^2, |b|_{H^{r + 2}}^2)|U|^2_{X^{r,s}}. \]

and
\[ \mu \left( \left( [\Lambda^s, \partial_x v] \partial_x v, \Lambda^s v \right) \right) \leq \max(\epsilon, \beta) C(|U|_{X^{r,s}}^2, |b|_{H^{r + 2}}^2)|U|^2_{X^{r,s}}. \]

Thus we proved
\[ |B_{33}| \leq \max(\epsilon, \beta) C((|U|_{X^{r,s}}^2, |b|_{H^{r + 2}}^2)|U|^2_{X^{r,s}}. \]

In the same way we control \( B_{34}, \)
\[ \Sigma [\Lambda^s, \Xi^{-1}(\partial_x v)] \partial_x v = -[\Lambda^s, \Xi^{-1}(\partial_x v)] \partial_x v + [\Lambda^s, \partial_x v] \partial_x v. \]
Thus we proved,

Again, using Cauchy-Schwarz inequality, Corollary 4.13 and the commutator estimate, one has

\[
\mu \left| \left( \Lambda^s, \sum \frac{1}{\lambda} \right) \left( \mathcal{L}_v \partial_v^2 v, \Lambda^s v \right) \right| \leq \mu \max(\epsilon, \beta) C \left( \left\| \mathcal{L}_v \partial_v^2 v \right\|_{H^s}, \left\| b \right\|_{H^{s+2}} \right) \left\| \mathcal{L}_v \partial_v^2 v \right\|_{H^{s-1}} \left\| v \right\|_{H^s}
\]

\[
\leq \max(\epsilon, \beta) C \left( \left\| \mathcal{L}_v \partial_v^2 v \right\|_{H^s}, \left\| b \right\|_{H^{s+2}} \right) v^2
\]

and

\[
\mu \left| \left( \Lambda^s, [\mathcal{L}] \partial_v^2 v, \Lambda^s v \right) \right| \leq \max(\epsilon, \beta) C \left( \left\| \mathcal{L}_v \partial_v^2 v \right\|_{H^s}, \left\| b \right\|_{H^{s+2}} \right) v^2
\]

Thus we proved

\[
\left| B_{34} \right| \leq \max(\epsilon, \beta) C \left( \left\| \mathcal{L}_v \partial_v^2 v \right\|_{H^s}, \left\| b \right\|_{H^{s+2}} \right) U^2
\]

To control \( B_{35} \) one can check,

\[
\sum \frac{1}{\lambda} \left( \mathcal{L}_v \partial_v^2 v \right) \partial_v^2 v = -\left( \Lambda^s, \sum \frac{1}{\lambda} \left( \mathcal{L}_v \partial_v^2 v \right) \partial_v^2 v \right) \partial_v^2 v + \left( \Lambda^s, \sum \frac{1}{\lambda} \left( \mathcal{L}_v \partial_v^2 v \right) \partial_v^2 v \right) \partial_v X,
\]

Thus we proved

\[
\left| B_{35} \right| \leq \max(\epsilon, \beta) C \left( \left\| \mathcal{L}_v \partial_v^2 v \right\|_{H^s}, \left\| b \right\|_{H^{s+2}} \right) U^2
\]

Finally, we turn to \( B_{36} = \epsilon \left( \left[ \Lambda^s, \mathcal{L}_v \partial_v^2 v, \sum \Lambda^s v \right] \right) \).

From Lemma 4.4, one has

\[
\left| B_{36} \right| \leq \left| \left[ \Lambda^s, \mathcal{L}_v \partial_v^2 v \right] \right|_{H^s} \left| \Lambda^s v \right|_{H^s}
\]

\[
\leq \left| \left[ \Lambda^s, \mathcal{L}_v \partial_v^2 v \right] \right|_{L^2} \left| \Lambda^s v \right|_{H^s} + \sqrt{\epsilon} \left| \partial_v \left( \left[ \Lambda^s, \mathcal{L}_v \partial_v^2 v \right] \right) \right|_{L^2} \left| \Lambda^s v \right|_{H^s}
\]

Note the identity

\[
\partial_v \left( \left[ \Lambda^s, \mathcal{L}_v \partial_v^2 v \right] \right) = \left( \left[ \Lambda^s, \partial_v (\mathcal{L}_v \partial_v^2 v) \right] \right) + \left( \left[ \Lambda^s, \mathcal{L}_v \partial_v^2 v \right] \right)
\]

Thus we proved

\[
\left| B_{36} \right| \leq \max(\epsilon, \beta) C \left( \left\| \mathcal{L}_v \partial_v^2 v \right\|_{H^s}, \left\| b \right\|_{H^{s+2}} \right) U^2
\]

Altogether, we proved

\[
\left| \left( \left[ \Lambda^s, \mathcal{L}_v \partial_v^2 v \right] \right) \right| \leq \max(\epsilon, \beta) C_2 \left| U \right|_{H^s}^2
\]

(5.26)

- Estimate of \( \frac{1}{2} \left( \Lambda^s U, [\partial_v, Z[u]] \Lambda^s U \right) \).
One has
\[
\begin{align*}
(L^\ast U, [\partial_t, Z[U]] L^\ast U) &\equiv (L^\ast U, [\partial_t, Z[U]] L^\ast U) + \left( L^\ast \zeta, \left[ \partial_t, \frac{Q_0(\epsilon \zeta, \beta b) + \epsilon^2 Q_1(\epsilon \zeta, \beta b, v)}{H(\epsilon \zeta)} \right] L^\ast \zeta \right) \\
&= \left( L^\ast U, (\partial_t q_1(\epsilon \zeta, \beta b)) L^\ast U \right) + \left( L^\ast U, \mu \epsilon \kappa_0 \partial_x \partial_t \zeta \partial_x b L^\ast U \right) \\
&\quad - \mu \left( L^\ast U, \partial_x (\nu(\partial_t q_2(\epsilon \zeta, \beta b)) \partial_x L^\ast U) \right) \\
&\quad + \left( L^\ast \zeta, \partial_t \left( \frac{Q_0(\epsilon \zeta, \beta b) + \epsilon^2 Q_1(\epsilon \zeta, \beta b, v)}{H(\epsilon \zeta)} \right) L^\ast \zeta \right) \\
&= \epsilon \left( L^\ast U, \kappa_1(\partial_t \zeta) L^\ast U \right) + \epsilon \mu \beta \left( L^\ast U, \kappa_0 \partial_x \partial_t \zeta \partial_x b L^\ast U \right) \\
&\quad + \epsilon \left( L^\ast U, \mu \epsilon \kappa_2(\partial_t \zeta) L^\ast U \right) \\
&\quad - \mu \beta \left( L^\ast U, \mu \epsilon \kappa_0 \partial_t \zeta \partial_t b L^\ast U \right) \\
&\quad + \mu \beta \left( L^\ast U, \mu \epsilon \kappa_2(\partial_t \zeta) L^\ast \partial_x \zeta \right) \\
&\quad + \left( L^\ast \zeta, \partial_t \left( \frac{Q_0(\epsilon \zeta, \beta b) + \epsilon^2 Q_1(\epsilon \zeta, \beta b, v)}{H(\epsilon \zeta)} \right) L^\ast \zeta \right).
\end{align*}
\]

From Cauchy-Schwarz inequality and since $\zeta$ and $b$ satisfy (H1), one deduces
\[
\left| \frac{1}{2} (L^\ast U, [\partial_t, Z[U]] L^\ast U) \right| \leq \epsilon C \left( \|\partial_t U\|_{L^\infty}, \|U\|_{L^\infty}, \|\epsilon\|_{W^{2,\infty}} \right) \|U\|_{X^\ast}^2,
\]
and continuous Sobolev embedding yields,
\[
\left| \frac{1}{2} (L^\ast U, [\partial_t, Z[U]] L^\ast U) \right| \leq \max(\epsilon, \beta) C \left( \|\partial_t U\|_{L^\infty}, \|U\|_{L^\infty}, \|\epsilon\|_{W^{2,\infty}} \right) \|U\|_{X^\ast}^2,
\]
and consequently
\[
\left| \frac{1}{2} (L^\ast U, [\partial_t, Z[U]] L^\ast U) \right| \leq \epsilon \beta C_2 \|U\|_{X^\ast}^2.
\]
(5.27)

One can now conclude the proof of the $X^\ast$ energy estimate. Plugging (5.23), (5.25), (5.26) and (5.27) into (5.22), and making use of the assumption of the Lemma on $F$,
\[
\frac{1}{2} \frac{d}{dt} E^\ast(U)^2 \leq \max(\epsilon, \beta) C_2 E^\ast(U)^2 + E^\ast(U) \left( f(t) + \max(\epsilon, \beta) C_2 \right),
\]
with $C_2 = C(\|U\|_{X^\ast}^2, \|\epsilon\|_{H^{-1,3}}, \|U\|_{X^\ast}^2)$, and consequently
\[
\frac{d}{dt} E^\ast(U) \leq \max(\epsilon, \beta) C_2 E^\ast(U) + \left( f(t) + \max(\epsilon, \beta) C_2 \right).
\]
Making use of the usual trick, we compute for any $\lambda \in \mathbb{R}$,
\[
e^{\max(\epsilon, \beta) \lambda t} \partial_t (e^{-\max(\epsilon, \beta) \lambda t} E^\ast(U)) = -\max(\epsilon, \beta) \lambda E^\ast(U) + \frac{d}{dt} E^\ast(U).
\]
Thus with $\lambda = C_2$, one has for all $t \in [0, \frac{T}{\max(\epsilon, \beta)}]$,
\[
\frac{d}{dt} (e^{-\max(\epsilon, \beta) \lambda t} E^\ast(U)) \leq \left( f(t) + \max(\epsilon, \beta) C_2 \right) e^{-\max(\epsilon, \beta) \lambda t}.
\]
Integrating this differential inequality yields,
\[
E^\ast(U)(t) \leq e^{\max(\epsilon, \beta) \lambda t} E^\ast(U_0) + \int_0^t e^{\max(\epsilon, \beta) \lambda (t-t')} \left( f(t') + \max(\epsilon, \beta) C_2 \right) dt'.
\]
\[\square\]
5.3 Well-posedness of the linearized system

Proposition 5.28 Let \( p = (\mu, \epsilon, \delta, \gamma, \beta, \text{bo}) \in \mathcal{P}_{CH} \) and \( s \geq s_0 + 1 \) with \( s_0 > 1/2 \), and let \( U = (\zeta, \nu)^T \in X_T^s \) (see Definition 5.10), \( b \in H^{s+3} \) be such that \([H1],[H2]\), and \([H3]\) are satisfied for \( t \in [0, T/\max(\epsilon, \beta)] \), uniformly with respect to \( p \in \mathcal{P}_{CH} \). For any \( U_0 \in X^s \), there exists a unique solution to \( (5.3) \), \( U^p \in C^\infty([0, T/\max(\epsilon, \beta)]; X^s) \cap C^1([0, T/\max(\epsilon, \beta)]; X^{s-1}) \subset X_T^s \), with \( \lambda_T, C_0 = C([L^\infty_{x_T}, T, M_{CH}, h_0^{-1}, h_0^{-1}, h_0^{-1}, \|b\|_{H^{s+3}}]) \), independent of \( p \in \mathcal{P}_{CH} \), such that one has the energy estimates

\[
0 \leq t \leq \frac{T}{\max(\epsilon, \beta)}, \quad E^*(U^p)(t) \leq e^{\max(\epsilon, \beta) \lambda_T t} E^*(U_0) + \max(\epsilon, \beta) C_0 \int_0^t e^{\max(\epsilon, \beta) \lambda_T (t-t')} dt'
\]

and \( E^{s-1}(\partial_t U^p) \leq C_0 e^{\max(\epsilon, \beta) \lambda_T t} E^*(U_0) + \max(\epsilon, \beta) C_0^2 \int_0^t e^{\max(\epsilon, \beta) \lambda_T (t-t')} dt' + \max(\epsilon, \beta) C_0 \).

Proof. We omit here the proof of Proposition 5.28 since it can be done using the same techniques as in the proof of \( [16] \) Proposition 2. \( \square \)

5.4 Estimate of the difference of two solutions

In what follows, we consider two nonlinear systems with different initial data and right-hand sides and control the difference between their solutions. This estimate is a key ingredient to the proof of the stability result.

Proposition 5.29 Let \( (\mu, \epsilon, \delta, \gamma, \beta, \text{bo}) \in \mathcal{P}_{CH} \) and \( s \geq s_0 + 1 \), \( s_0 > 1/2 \), and assume that there exists \( U_i \) for \( i \in \{1, 2\} \), such that \( U_i = (\zeta_i, \nu_i)^T \in X_T^s \), \( U_2 \in L^\infty([0, T/\max(\epsilon, \beta)]; X^{s+1}) \), \( b \in H^{s+3} \), \( U_i \) satisfy \([H1],[H2]\) and \([H3]\) on \([0, T/\max(\epsilon, \beta)]\), with \( h_0, h_0^1, h_0^2, h_0^3 > 0 \), and \( U_i \) satisfy

\[
\partial_t U_1 + A[U_1] \partial_t U_1 + B[U_1] = F_1,
\]

\[
\partial_t U_2 + A[U_2] \partial_t U_2 + B[U_2] = F_2,
\]

with \( F_i \in L^1([0, T/\max(\epsilon, \beta)]; X^s) \).

Then there exists constants \( C_0 = C(M_{CH}, h_0^{-1}, h_0^{-1}, h_0^{-1}, \max(\epsilon, \beta)|U_1|_{X^s}, \max(\epsilon, \beta)|U_2|_{X^s}, \|b\|_{H^{s+3}}) \) and \( \lambda_T = (C_0 \times C(\|U_2\|_{L^\infty([0, T/\max(\epsilon, \beta)]; X^{s+1})} + C_0) \) such that for all \( t \in [0, T/\max(\epsilon, \beta)] \),

\[
E^*(U_1 - U_2)(t) \leq e^{\max(\epsilon, \beta) \lambda_T t} E^*(U_1 \big|_{t=0} - U_2 \big|_{t=0}) + C_0 \int_0^t e^{\max(\epsilon, \beta) \lambda_T (t-t')} E^*(F_1 - F_2)(t') dt'.
\]

Proof. When multiplying the equations satisfied by \( U_i \) on the left by \( Z[U_i] \), one obtains

\[
Z[U_1] \partial_t U_1 + \Sigma[U_1] \partial_t U_1 + Z[U_1] B[U_1] = Z[U_1] F_1
\]

\[
Z[U_2] \partial_t U_2 + \Sigma[U_2] \partial_t U_2 + Z[U_2] B[U_2] = Z[U_2] F_2;
\]

with \( \Sigma[U] = Z[U] A[U] \). Subtracting the two equations above, and defining \( V = U_1 - U_2 \equiv (\zeta, \nu)^T \) one obtains

\[
Z[U_1] \partial_t V + \Sigma[U_1] \partial_t V + (Z[U_1] B[U_1] - Z[U_2] B[U_2]) = Z[U_1](F_1 - F_2) - (\Sigma[U_1] - \Sigma[U_2]) \partial_t U_2 - (Z[U_1] - Z[U_2])(\partial_t U_2 - F_2).
\]

We then apply \( Z^{-1}[U_1] \) and deduce the following system satisfied by \( V \):

\[
\begin{cases}
\partial_t V + A[U_1] \partial_t V + Z^{-1}[U_1](Z[U_1] B[U_1] - Z[U_2] B[U_2]) = F \\
V(0) = (U_1 - U_2) \big|_{t=0},
\end{cases}
\]

(5.30)

where, \( F \equiv F_1 - F_2 - Z^{-1}[U_1](\Sigma[U_1] - \Sigma[U_2]) \partial_t U_2 - Z^{-1}[U_1](Z[U_1] - Z[U_2])(\partial_t U_2 - F_2) \).

(5.31)
We wish to use the energy estimate of Lemma 5.18 to the linear system \([5.30]\).

The additional term now is \(Z^{-1}[U_1](Z[U_1]B[U_1] - Z[U_2]B[U_2])\).

So we have to control,

\[
(\Lambda^*Z^{-1}[U_1](Z[U_1]B[U_1] - Z[U_2]B[U_2]), Z[U_1]\Lambda^*V) = B.
\]

One has,

\[
B = (\Lambda^*(Z[U_1]B[U_1] - Z[U_2]B[U_2]), \Lambda^*V) + ([\Lambda^*, Z^{-1}[U_1]]Z[U_1]B[U_1] - Z[U_2]B[U_2], Z[U_1]\Lambda^*V)
\]

Now we have to estimate the terms \((B_1)\) and \((B_2)\).

\[
(B_1) = \left(\Lambda^*(-Q_0(\epsilon \zeta_1, \beta b)\beta \partial_x b G(\epsilon \zeta_1)v_1 + \frac{Q_0(\epsilon \zeta_2, \beta b)\beta \partial_x b G(\epsilon \zeta_2)v_2}{H(\epsilon \zeta_2)}), \Lambda^*\zeta v_1\right)
\]

In order to control \((B_1)\) we use the following decompositions,

- \(\beta \left(\frac{-Q_0(\epsilon \zeta_1, \beta b)\beta \partial_x b G(\epsilon \zeta_1)v_1}{H(\epsilon \zeta_1)} + \frac{Q_0(\epsilon \zeta_2, \beta b)\beta \partial_x b G(\epsilon \zeta_2)v_2}{H(\epsilon \zeta_2)}\right)
  = \left(\frac{-Q_0(\epsilon \zeta_1, \beta b)G(\epsilon \zeta_1)}{H(\epsilon \zeta_1)} + \frac{Q_0(\epsilon \zeta_2, \beta b)G(\epsilon \zeta_2)}{H(\epsilon \zeta_2)}\right)(\beta \partial_x b v_1)
  - \beta(v_1 - v_2) \frac{Q_0(\epsilon \zeta_2, \beta b)G(\epsilon \zeta_2)b_x}{H(\epsilon \zeta_2)}.
\]

- \(\beta \left(\frac{-\epsilon^2 Q_1(\epsilon \zeta_1, \beta b, v_1)\beta \partial_x b G(\epsilon \zeta_1)v_1}{H(\epsilon \zeta_1)} + \frac{\epsilon^2 Q_1(\epsilon \zeta_2, \beta b, v_2)\beta \partial_x b G(\epsilon \zeta_2)v_2}{H(\epsilon \zeta_2)}\right)
  = \left(\frac{-\epsilon^2 Q_1(\epsilon \zeta_1, \beta b, v_1)G(\epsilon \zeta_1)}{H(\epsilon \zeta_1)} + \frac{\epsilon^2 Q_1(\epsilon \zeta_2, \beta b, v_2)G(\epsilon \zeta_2)}{H(\epsilon \zeta_2)}\right)(\beta \partial_x b v_1)
  - \beta(v_1 - v_2) \frac{\epsilon^2 Q_1(\epsilon \zeta_2, \beta b, v_2)G(\epsilon \zeta_2)b_x}{H(\epsilon \zeta_2)}.
\]

- \(\beta \left(\frac{\gamma \beta q_1(\epsilon \zeta_1, \beta b)h_1(h_1 + h_2)v_1^2 b_x}{(h_1 + \gamma h_2)^3} - \frac{\gamma \beta q_1(\epsilon \zeta_2, \beta b)h_1(h_1 + h_2)v_2^2 b_x}{(h_1 + \gamma h_2)^3}\right)
  = \left(\frac{\gamma q_1(\epsilon \zeta_1, \beta b)h_1(h_1 + h_2)}{(h_1 + \gamma h_2)^3} - \frac{\gamma q_1(\epsilon \zeta_2, \beta b)h_1(h_1 + h_2)}{(h_1 + \gamma h_2)^3}\right)(\beta \partial_x b v_1^2)
  + \frac{\gamma q_1(\epsilon \zeta_2, \beta b)h_1(h_1 + h_2)b_x}{(h_1 + \gamma h_2)^3} \beta(v_1^2 - v_2^2).
\]

- \(\mu \left(\frac{-|\xi| v_1^2 + |\xi| v_2^2}{2}\right) = \mu |\xi| (v_2 - v_1)(v_2 + v_1).
\]

Using the fact that, \(\epsilon^2 Q_1(\epsilon \zeta_1, \beta b, v_1) = Q_1(\epsilon \zeta_1, \beta b, \epsilon v_1)\), one deduces,
\[ |B_1| \leq C(\epsilon v_1 |H^s|, |b| H^{s+2} \epsilon) |\zeta_1 - \zeta_2| H^s |\zeta_1| H^s + C(\epsilon |\zeta_2| H^s, |b| H^{s+2} \epsilon) v_1 - v_2 |H^s| \zeta_1| H^s + C(\epsilon v_1 |H^s|, |b| H^{s+1} |\zeta_1| H^s + C(\epsilon |\zeta_2| H^s, |v_2| |H^s|, |b| H^{s+1} \epsilon) |v_1 - v_2| |\zeta_1| H^s + C(\epsilon v_1 |H^s|, |v_2| |H^s|, |b| H^{s+1} \epsilon) \zeta_1 - \zeta_2| H^s |v_1 - v_2| |\zeta_1| H^s.
\]

Then one has
\[
\leq \max(\epsilon, \beta) C_0 E^s(U_1 - U_2) E^s(V).
\]

\leq \max(\epsilon, \beta) C_0 E^s(V)^2.

with \( C_0 = C(M_{CH}, h^{-1}, h^{-1}, \max(\epsilon, \beta)|U_1|_{X^s}, \max(\epsilon, \beta)|U_2|_{X^s}, |b| H^{s+3}). \)

With \( Q(\epsilon \zeta_1, \beta b, v_1) = Q_0(\epsilon \zeta_1, \beta b) + \epsilon^2 Q_1(\epsilon \zeta_1, \beta b, v_1) \) for \( i = 1, 2. \)

The contribution of \( (B_2) \) is immediately bounded using Lemma 5.13

\[ |B_2| = \left( [A^*, Z^{-1}[U_1]](Z[U_1]B[U_1] - Z[U_2]B[U_2]), Z[U_1]\right) \leq C|Z[U_1]B[U_1] - Z[U_2]B[U_2]|_{H^{s-1}}|V|_{X^s}.
\]

\[ \leq C\left( \frac{Q(\epsilon \zeta_1, \beta b, v_1) \partial bG(\zeta_1)v_1}{H(\zeta_1)} + \frac{Q(\epsilon \zeta_2, \beta b, v_2) \partial bG(\zeta_2)v_2}{H(\zeta_2)} \right)_{H^{s-1}} + \delta |\epsilon| v_1^2 = \frac{\delta}{(h_1 + \gamma h_2)^3} \zeta_1 - \zeta_2| H^s |v_1 - v_2| |\zeta_1| H^s.
\]

\leq \max(\epsilon, \beta) C_0 E^s(U_1 - U_2) E^s(V).
\]

\leq \max(\epsilon, \beta) C_0 E^s(V)^2.

So we have,

\[ |B| \leq C_0 \max(\epsilon, \beta) E^s(V)^2.
\]

Now, in order to control the right hand side \( F \) we take advantage of the Lemma below.

**Lemma 5.32** Let \( \mu, \epsilon, \delta, \gamma, \beta, b0 \in \mathcal{P}_{CH} \) and \( s \geq s_0 > 1/2 \). Let \( V = (\zeta_v, v)^T, W = (\zeta_w, w)^T \in X^s \) and \( U_1 = (\zeta_1, v_1)^T, U_2 = (\zeta_2, v_2)^T \in X^s, b \in H^{s+2} \) such that there exists \( h > 0 \) with

\[ 1 - \epsilon \zeta_1 \geq h > 0, \quad 1 - \epsilon \zeta_2 \geq h > 0, \quad \frac{1}{\delta} + \epsilon \zeta_1 - \beta b \geq h > 0, \quad \frac{1}{\delta} + \epsilon \zeta_2 - \beta b \geq h > 0.
\]

Then one has

\[ A^*(Z[U_1] - Z[U_2])V, W \leq \epsilon C |U_1 - U_2|_{X^s}, |V|_{X^s}, |W|_{X^s} \tag{53.33}
\]

\[ A^*(Z[U_1]A[U_1] - Z[U_2]A[U_2])V, W \leq \epsilon C |U_1 - U_2|_{X^s}, |V|_{X^s}, |W|_{X^s} \tag{53.34}
\]

with \( C = C(M_{CH}, h^{-1}, \epsilon|U_1|_{X^s}, |U_2|_{X^s}, \beta|b| H^{s+2}). \)

**Proof.**

Let \( V = (\zeta_v, v)^T, W = (\zeta_w, w)^T \in X^0 \) and \( U_1 = (\zeta_1, v_1)^T, U_2 = (\zeta_2, v_2)^T \in X^s \). By definition of \( Z[] \) (see (5.9)), one has

\[ A^*(Z[U_1] - Z[U_2])V, W = \left( A^* \left( \frac{Q_0(\epsilon \zeta_1, \beta b) + \epsilon^2 Q_1(\epsilon \zeta_1, \beta b, v_1)}{H(\zeta_1)} - \frac{Q_0(\epsilon \zeta_2, \beta b) + \epsilon^2 Q_1(\epsilon \zeta_2, \beta b, v_2)}{H(\zeta_2)} \right)_{\zeta_v, \zeta_w} \right) + \left( A^* \left( \mathcal{T}[\epsilon \zeta_1, \beta b] - \mathcal{T}[\epsilon \zeta_2, \beta b] \right) \right) v, w.
\]

Now, one can check that

\[ \mathcal{T}[\epsilon \zeta_1, \beta b]v - \mathcal{T}[\epsilon \zeta_2, \beta b]v = (q_1(\epsilon \zeta_1, \beta b) - q_1(\epsilon \zeta_2, \beta b)) v + \mu \beta \partial \alpha \partial b \partial \alpha (\zeta_1 - \zeta_2) v - \mu \partial \alpha \left( \nu \partial \zeta_1(\epsilon \zeta_1, \beta b, v_1) - \nu \partial \zeta_1(\epsilon \zeta_2, \beta b, v_2) \right) v - \mu \partial \zeta_1(\epsilon \zeta_1, \beta b, v_1) - \mu \partial \zeta_1(\epsilon \zeta_2, \beta b, v_2).
\]
Making use of, \( \kappa_0 \partial_x (\zeta_1 - \zeta_2) \partial_x b_v = \partial_x (\kappa_0 v (\zeta_1 - \zeta_2) \partial_x b) - \partial_x (\kappa_0 v) (\zeta_1 - \zeta_2) \partial_x b - \kappa_0 v (\zeta_1 - \zeta_2) \partial_x^2 b, \)
so that, after one integration by part, and using Cauchy-Schwarz inequality and the product estimate in Sobolev spaces (see [11,13,14]), one has

\[
\left| \left( A^* \left( X[\zeta_1, \beta b] - X(\epsilon \zeta_2, \beta b) \right) v, w \right) \right| \leq \epsilon C |\zeta_1 - \zeta_2|_{H^1} |v|_{H^{\epsilon \zeta}_1} |w|_{H^{\epsilon \zeta}_1}. \tag{5.35}
\]

Now, applying Cauchy-Schwarz inequality, one has (again thanks to continuous Sobolev embedding for \( s \geq s_0 > 1/2 \))

\[
\left| \left( A^* \left( Q_0(\zeta_1, \beta b) \right) H'(\epsilon \zeta_1) - \frac{Q_0(\zeta_2, \beta b)}{H(\epsilon \zeta_2)} H'(\epsilon \zeta_2) \right) \zeta_v, \zeta_w \right| \leq \epsilon C |\zeta_1 - \zeta_2|_{H^1} \left| \zeta_v \right|_{H^\epsilon_1} \left| \zeta_w \right|_{L^2}, \tag{5.36}
\]

and since, \( \epsilon^2 Q_1(\zeta_1, \beta b, v_1) = Q_1(\zeta_1, \beta b, v_1) \) for \( (i = 1, 2) \), one has

\[
\left| \left( A^* \left( Q_1(\zeta_1, \beta b, v_1) \right) H'(\epsilon \zeta_1) - \frac{Q_1(\zeta_2, \beta b, v_2)}{H(\epsilon \zeta_2)} H'(\epsilon \zeta_2) \right) \zeta_v, \zeta_w \right| \leq \epsilon C |U_1 - U_2|_{X^\epsilon} \left| \zeta_v \right|_{H^\epsilon_1} \left| \zeta_w \right|_{L^2}. \tag{5.37}
\]

with \( C = C(M_{CH}, h^{-1}, \epsilon |U_1|_{X^\epsilon}, \epsilon |U_2|_{X^\epsilon}, \beta b|_{H^{\epsilon \zeta}_1} \). Estimates \( 5.35 \), \( 5.36 \) and \( 5.37 \) yield \( 5.33 \).

Let us now turn to \( 5.34 \). One has

\[
\left( A^*(Z[U_1]A[U_1] - Z[U_2]A[U_2]) V, W \right) \tag{5.38}
\]

\[
= \epsilon \left( A^* \left( \frac{Q_0(\zeta_1, \beta b)}{H(\epsilon \zeta_1)} H'(\epsilon \zeta_1) v_1 \right) - \frac{Q_0(\zeta_2, \beta b)}{H(\epsilon \zeta_2)} H'(\epsilon \zeta_2) v_2 \right) \zeta_v, \zeta_w \\
+ \epsilon \left( A^* \left( \frac{Q_1(\zeta_1, \beta b, v_1)}{H(\epsilon \zeta_1)} H'(\epsilon \zeta_1) v_1 \right) - \frac{Q_1(\zeta_2, \beta b, v_2)}{H(\epsilon \zeta_2)} H'(\epsilon \zeta_2) v_2 \right) \zeta_v, \zeta_w \\
+ \left( A^* \left( Q_0(\zeta_1, \beta b) - Q_0(\zeta_2, \beta b) \right) v, \zeta_v \right) \\
+ \left( A^* \left( \epsilon^2 Q_1(\zeta_1, \beta b, v_1) - \epsilon^2 Q_1(\zeta_2, \beta b, v_2) \right) v, \zeta_v \right) \\
+ \left( A^* \left( Q_0(\zeta_1, \beta b) - Q_0(\zeta_2, \beta b) \right) \zeta_v, \zeta_w \right) \\
+ \left( A^* \left( \epsilon^2 Q_1(\zeta_1, \beta b, v_1) - \epsilon^2 Q_1(\zeta_2, \beta b, v_2) \right) \zeta_v, \zeta_w \right) \\
+ \left( A^* \left( Q_0(\zeta_1, \beta b) - Q_0(\zeta_2, \beta b) \right) \zeta_v, \zeta_w \right) \\
+ \epsilon \left( A^* \left( X[\zeta_1, \beta b] \zeta_v \right) - \frac{Q_0(\zeta_2, \beta b, v_2)}{H(\epsilon \zeta_2)} \right) \zeta_v, \zeta_w \right) \\
= (I) + (II) + (III) + (IV) + (V) + (VI) + (VII) + (VIII). \tag{5.39}
\]

\( (III) \) and \( (V) \) may be estimated exactly as in \( 5.36 \), the \( (IV) \) and \( (VI) \) terms may be estimated exactly as in \( 5.37 \), and we do not detail the precise calculations. The \( (I) \) and \( (II) \) terms follow in the same way, using the decompositions below,

\[
\epsilon \left( \frac{Q_0(\zeta_1, \beta b)}{H(\epsilon \zeta_1)} H'(\epsilon \zeta_1) v_1 \right) - \frac{Q_0(\zeta_2, \beta b)}{H(\epsilon \zeta_2)} H'(\epsilon \zeta_2) v_2 = \left( \frac{Q_0(\zeta_1, \beta b)}{H(\epsilon \zeta_1)} H'(\epsilon \zeta_1) - \frac{Q_0(\zeta_2, \beta b)}{H(\epsilon \zeta_2)} H'(\epsilon \zeta_2) \right) (v_1 - v_2),
\]

and,

\[
\epsilon \left( \frac{Q_1(\zeta_1, \beta b, v_1)}{H(\epsilon \zeta_1)} H'(\epsilon \zeta_1) v_1 \right) - \frac{Q_1(\zeta_2, \beta b, v_2)}{H(\epsilon \zeta_2)} H'(\epsilon \zeta_2) v_2 = \left( \frac{Q_1(\zeta_1, \beta b, v_1)}{H(\epsilon \zeta_1)} H'(\epsilon \zeta_1) - \frac{Q_1(\zeta_2, \beta b, v_2)}{H(\epsilon \zeta_2)} H'(\epsilon \zeta_2) \right) (v_1 - v_2) + \epsilon (v_1 - v_2) \frac{Q_1(\zeta_2, \beta b, v_2)}{H(\epsilon \zeta_2)} H'(\epsilon \zeta_2),
\]
so that one has
\[ |(I)| \leq C(\varepsilon|v_1|_{H^r}, \beta|h|_{H^{r+2}})\varepsilon|\zeta_1 - \zeta_2|_{H^r}|\zeta_v|_{H^r}|\zeta_w|_{L^2} + C(\varepsilon|\zeta_2|_{H^r}, \beta|h|_{H^{r+2}})\varepsilon|v_1 - v_2|_{H^r}|\zeta_w|_{H^r}|\zeta_v|_{L^2}, \]
and,
\[ |(II)| \leq C(\varepsilon|v_1|_{H^r}, \varepsilon|U_1 - U_2|_{X}, |\zeta_v|_{H^r}, |\zeta_w|_{L^2} + C(\varepsilon|\zeta_2|_{H^r}, \varepsilon|v_2|_{H^{r+2}})\varepsilon|v_1 - v_2|_{H^r}|\zeta_w|_{H^r}|\zeta_v|_{L^2}. \]

Let us detail now (VII). One has
\[
(\Omega[\epsilon_1, \beta, v_1] - \Omega[\epsilon_2, \beta, v_2])v = \varepsilon \left( q_1(\epsilon_1, \beta)b(H'(\epsilon_1) - \zeta)v_1 - q_1(\epsilon_2, \beta)b(H'(\epsilon_2) - \zeta)v_2 \right) v \\
- \mu[A](v_1 - v_2)v - \mu[B]\partial_x(v_1 - v_2)v - \mu[C](v_1 - v_2)\partial_x v \\
- \mu[D]\partial_x (\partial_x(v_1 - v_2)v).
\]

Again, the contribution of the first term in (5.40) is estimated as above (recalling that this term is multiplied by a \( \varepsilon \)-factor), and the contributions of the other terms in (5.40) are easily estimated, so one has,
\[
|\text{VII}| \leq C(\varepsilon|v_1|_{H^r})|\zeta_1 - \zeta_2|_{H^r}|v|_{H^r}|w|_{L^2} + C(\varepsilon|\zeta_2|_{H^r})|v_1 - v_2|_{H^r}|v|_{H^r}|w|_{L^2} \\
+ C(\beta|h|_{H^{r+2}})\varepsilon|v_1 - v_2|_{H^{r+2}}|v|_{H^{r+2}}|w|_{L^2}.
\]

We conclude by estimating (VIII). One has
\[
\Sigma[\epsilon_1, \beta, \beta]v(v_1 v) - \Sigma[\epsilon_2, \beta, \beta](v_2 v) = \left( q_1(\epsilon_1, \beta)b(v_1 - q_1(\epsilon_2, \beta)b)\zeta v + \mu\beta\kappa_0 \partial_x b(\partial_x \zeta v_1 - \partial_x \zeta v_2)\zeta v \right. \\
- \mu\partial_x \left( v_1(\epsilon_2, \beta)b)\partial_x(v_1 v) - q_2(\epsilon_2, \beta)b)\partial_x(v_2 v) \right) \\
= \left( \Sigma[\epsilon_1, \beta, \beta] - \Sigma[\epsilon_2, \beta, \beta] \right)(v_1 v) + (v_1 - v_2)(q_1(\epsilon_2, \beta)b)\zeta v \right. \\
+ \mu\beta\kappa_0 \partial_x b(\partial_x \zeta v_1 - \partial_x \zeta v_2)\zeta v \\
- \mu\partial_x \left( v_1 - v_2)\partial_x^2 b\kappa_0 \partial_x \zeta v \right. \\
- \mu\beta(\zeta v_1 - v_2)\partial_x \zeta v \}
\]

One finally uses Cauchy-Schwarz inequality, the product estimate in Sobolev spaces, as well as (5.35), and obtain
\[
|\text{VIII}| \leq \varepsilon^2 C(\beta|h|_{H^{r+2}})\zeta_1 - \zeta_2|_{H^r}|v_1|_{H^{r+1}}|v|_{H^r} \\
+ \varepsilon C(\varepsilon|\zeta_2|_{H^r}, \beta|h|_{H^{r+1}})|v_1 - v_2|_{H^r}|v|_{H^r} \\
+ \varepsilon C(\varepsilon|\zeta_2|_{H^r}, \beta|h|_{H^{r+1}})|v_1 - v_2|_{H^r}|v|_{H^{r+1}} \\
+ \varepsilon C(\varepsilon|\zeta_2|_{H^r}, \beta|h|_{H^{r+1}})|v_1 - v_2|_{H^r}|v|_{H^{r+1}} \\
+ \varepsilon C(\varepsilon|\zeta_2|_{H^r}, \beta|h|_{H^{r+1}})|v_1 - v_2|_{H^r}|v|_{H^{r+1}} \\
+ \varepsilon C(\varepsilon|\zeta_2|_{H^r}, \beta|h|_{H^{r+1}})|v_1 - v_2|_{H^r}|v|_{H^{r+1}}.
\]

Altogether, we obtain (5.34), and the Lemma is proved. \( \square \)

Let us continue the proof of Proposition (5.29) by estimating \( F \) defined in (5.31).
More precisely we want to estimate
\[
(\Lambda^*F, Z[U_1]\Lambda^*V) = (\Lambda^*F_1 - \Lambda^*F_2, Z[U_1]\Lambda^*V) \\
- (\Lambda^*(\Sigma[U_1] - \Sigma[U_2])\partial_x U_2, \Lambda^*V) \\
- (\Lambda^*[Z[U_1] - Z[U_2]])(\partial_x U_2 - F_2), \Lambda^*V) \\
- (\Lambda^*[Z[U_1] - Z[U_2]])(\partial_x U_2 - F_2), Z[U_1]\Lambda^*V).
\]

Let us estimate each of these terms. The first term is immediately bounded using Lemma 5.13:
\[
| (\Lambda^*F_1 - \Lambda^*F_2, Z[U_1]\Lambda^*V) | \leq C|F_1 - F_2|_{X^r}|V|_{X^s}, \tag{5.41}
\]
with \( C = C(M_{CH}, h^{-1}, \epsilon|U_1|_{W^{1,\infty}}, \beta|b|_{W^{2,\infty}}) \).

The contributions of the second and fourth terms follow from Lemma 5.32. Indeed, recalling that \( V \equiv U_1 - U_2 \), (5.33) yields immediately:
\[
| (\Lambda^*(\Sigma[U_1] - \Sigma[U_2])\partial_x U_2, \Lambda^*V) | \leq C\epsilon|\partial_x U_2 - F_2|_{X^r}|V|_{X^s}, \tag{5.42}
\]
and (5.34) yields:
\[
| (\Lambda^*[Z[U_1] - Z[U_2]])(\partial_x U_2 - F_2), \Lambda^*V) | \leq C\epsilon|\partial_x U_2|_{X^r}|V|_{X^s}, \tag{5.43}
\]
with \( C = C(M_{CH}, h^{-1}, \epsilon|U_1|_{X^r}, \epsilon|U_2|_{X^r}, \beta|b|_{H^{h-2}}) \).

Finally, we control the third and fifth terms using Lemma 5.13, 5.16:
\[
| (\Lambda^*[Z[U_1] - Z[U_2]])U, Z[U_1]\Lambda^*V) | \leq C|U|_{H^{h-1}}|V|_{X^s},
\]
with \( C = C(M_{CH}, h^{-1}, \epsilon|U_1|_{X^r}, \beta|b|_{H^{h-2}}) \).

Thus it remains to estimate \( |U|_{H^{h-1}}|V|_{X^1} \), where \( U \equiv U_{(i)} \equiv (\Sigma[U_1] - \Sigma[U_2])\partial_x U_2 \) or \( U \equiv U_{(ii)} \equiv (Z[U_1] - Z[U_2])(\partial_x U_2 - F_2) \).

We proceed as in Lemma 5.32 helped by the fact that one is allowed lose one derivative in our estimates.

Let \( W \equiv \partial_x U_2 - F_2 \equiv (\zeta_\omega, w)^T \). One has
\[
U_{(ii)} \equiv (Z[U_1] - Z[U_2])W \equiv \begin{pmatrix}
\frac{Q_0(\epsilon_1, \beta b) + \epsilon^2 Q_1(\epsilon_1, \beta b, v_1)}{H(\epsilon_1)} - \frac{Q_0(\epsilon_2, \beta b) + \epsilon^2 Q_1(\epsilon_2, \beta b, v_2)}{H(\epsilon_2)}
\end{pmatrix} \zeta_\omega
\approx \begin{pmatrix}
\zeta_{(ii)}
\end{pmatrix},
\]

Recall that
\[
\Sigma[\epsilon\zeta_1, \beta b]w - \Sigma[\epsilon\zeta_2, \beta b]w = \epsilon \left\{ \nabla_\omega(\zeta_1 - \zeta_2)w + \mu\partial_\omega(\nabla_\omega(\zeta_1 - \zeta_2)w) - \mu\partial_\omega \left\{ \nu \partial_\omega(\zeta_1 - \zeta_2)w \right\} \right\}
\]
so that one has straightforwardly
\[
|u_{(ii)}|_{H^{h-1}} \leq C|\partial_x U_2|_{H^{h-1}}|\zeta_\omega|_{H^{h-1}}.
\]

As for the first component, we apply (5.36), (5.37) and deduce:
\[
|\zeta_{(ii)}|^2_{H^{h-1}} = (\Lambda^{h-1}\zeta_{(ii)}, \Lambda^{h-1}\zeta_{(ii)}) \leq C|U_1 - U_2|_{X^{h-1}}|\zeta_\omega|_{H^{h-1}}|\zeta_{(ii)}|_{H^{h-1}}.
\]

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It follows
\[
\left( (\Lambda^s, Z^{-1}[U_1]) (Z[U_1] - Z[U_2]) (\partial_t U_2 - F_2), Z[U_1] \Lambda^s V \right) \leq C \epsilon |\partial_t U_2 - F_2|_{X^s} |V|_{X^s},
\]
with \( C = C(M_{CH}, h^{-1}, \epsilon |U_1|, \epsilon |U_2|, |\beta b|_{H^{s+2}}) \).

Now, recall \((\Sigma[U_1] - \Sigma[U_2]) \partial_t U_2\). Proceeding as above, one obtains
\[
|U_{(i)}|_{H^{s+1}, x_{H^{s+1}}} \leq \epsilon C |\partial_t U_2|_{X^s} |V|_{X^s},
\]
and thus
\[
\left| (\Lambda^s, Z^{-1}[U_1]) (\Sigma[U_1] - \Sigma[U_2]) \partial_t U_2, Z[U_1] \Lambda^s V \right| \leq C \epsilon |\partial_t U_2|_{X^s} |V|_{X^s}. \tag{5.45}
\]

 Altogether, we proved (using Lemma 5.12 that \( F \), as defined in (5.31), satisfies
\[
\left| (\Lambda^s F, Z[U_1] \Lambda^s V) \right| \leq C (|\partial_t U_2|_{X^s} + |\partial_t U_2 - F_2|_{X^s}) \epsilon e^* (V)^2 + CE^* (V) E^* (F_1 - F_2). \tag{5.46}
\]
with \( C = C(M_{CH}, h^{-1}, h_0^{-1}, \epsilon |U_1|, \epsilon |U_2|, |\beta b|_{H^{s+2}}) \).

Notice also that by the system satisfied by \( U_2 \), one has (see detailed calculations in the proof of [10, Proposition 2])
\[
|\partial_t U_2 - F_2|_{X^s} \equiv -|(A_0[U_2] + A_1[U_2]) \partial_t U_2 + B[U_2]|_{X^s} \leq C (|U_2|_{X^{s+1}} + C).
\]

We can now conclude by Lemma 5.18 and the proof of Proposition 5.29 is complete. \( \square \)

6 Full justification

We start by defining the full justification terminology (initiated in [15]). An asymptotic model is fully justified as an approximation of the full Euler system if the asymptotic model is consistent with the full Euler system, if both exact and approximate models are well-posed for the same class of initial data, and if their solutions remain close uniformly with respect to time. In what follows, we will fully justify the new Green-Naghdi model (3.19) derived in a previous section. To this end, we will state the three main ingredients of full justification. For the sake of simplicity, we do not state the proofs of these results. In fact, following the same techniques as in [10] adjusted to our new model, the following results can be proved without any difficulties.

**Proposition 6.1 (Consistency)** For \( p = (\mu, \epsilon, \delta, \gamma, \beta, b_0) \in P_{SW} \), let \( U^p = (\zeta^p, \psi^p)^T \) be a family of solutions of the full Euler system (2.2) such that there exists \( T > 0 \), \( s \geq s_0 + 1 \), \( s_0 > 1/2 \) for which \( (\zeta^p, \partial_t \psi^p)^T \) is bounded in \( L^\infty ([0, T]; H^{s+N})^2 \) with sufficiently large \( N \), and uniformly with respect to \( p \in P_{SW} \). Moreover assume that \( b \in H^{s+N} \) satisfy (H0) and there exists \( h_0 > 0 \) such that
\[
h_1 = 1 - \epsilon \zeta^p \geq h_0 > 0, \quad h_2 = \frac{1}{\delta} + \epsilon \zeta^p - \beta b \geq h_0 > 0.
\]

Define \( v^p \) as in (3.1). Then \( (\zeta^p, v^p)^T \) satisfies (3.19) up to a remainder term, \( R = (0, r)^T \), bounded by
\[
\|r\|_{L^\infty ([0, T]; H^s)} \leq (\mu^2 + \mu \epsilon^2) C,
\]
with \( C = C(h_0^{-1}, M_{SW}, |b|_{H^{s+N}}, \|\zeta^p, \partial_t \psi^p\|_{L^\infty ([0, T]; H^{s+N})}) \).

**Theorem 6.2 (Existence and uniqueness)** Let \( p = (\mu, \epsilon, \delta, \gamma, \beta, b_0) \in P_{CH} \) and \( s \geq s_0 + 1 \), \( s_0 > 1/2 \), and assume \( U_0 = (\zeta_0, v_0)^T \in X^s \), \( b \in H^{s+3} \) satisfies [10, H1, H2], and [13]. Then there exists a maximal time \( T_{\max} > 0 \), uniformly bounded from below with respect to \( p \in P_{CH} \), such that the system of
equations \(3.19\) admits a unique strong solution \(U = (\zeta, v)\) \(\in C^0([0, T_{\max}); X^s) \cap C^1([0, T_{\max}); X^{s-1})\) with the initial value \(\zeta_{\max} = (\zeta_0, v_0)\), and preserving the conditions \(H1, H2, H3\) (with different lower bounds) for any \(t \in [0, T_{\max})\).

Moreover, there exists \(\lambda, C_0 = C(h_{01}^{-1}, h_{02}^{-1}, h_{03}^{-1}, M_{CH}, T, |U_0|_{X^s}, |b|_{H^{\beta+3}})\), independent of \(p \in P_{CH}\), such that \(T_{\max} \geq T/\max(\epsilon, \beta)\), and one has the energy estimates

\[
\forall 0 \leq t \leq \frac{T}{\max(\epsilon, \beta)},
\]

\[
|U(t, \cdot)|_{X^s} + |\partial_t U(t, \cdot)|_{X^{s-1}} \leq C_0 e^{\max(\epsilon, \beta)\lambda t} + \max(\epsilon, \beta) C_0 \int_0^t e^{\max(\epsilon, \beta)\lambda(t-t')} dt' + \max(\epsilon, \beta) C_0
\]

If \(T_{\max} < \infty\), one has

\[
|U(t, \cdot)|_{X^s} \rightarrow \infty \quad \text{as} \quad t \rightarrow T_{\max},
\]

or one of the conditions \(H1, H2, H3\) ceases to be true as \(t \rightarrow T_{\max}\).

**Theorem 6.3 (Stability)** Let \(p = (\mu, \epsilon, \delta, \gamma, \beta, b_0) \in P_{CH}\) and \(s \geq s_0 + 1\) with \(s_0 > 1/2\), and assume

\[
U_{0,1} = (\zeta_{0,1}, v_{0,1}) \in X^s, \quad U_{0,2} = (\zeta_{0,2}, v_{0,2})' \in X^{s+1}, \quad \text{and} \quad b \in H^{\beta+3} \text{ satisfies } (H0, H1, H2, H3).
\]

Denote \(U_j\) the solution to \((3.19)\) with \(U_j|_{t=0} = U_{0,j}\).

Then there exists \(T, \lambda, C_0 = C(M_{CH}, h_{01}^{-1}, h_{02}^{-1}, h_{03}^{-1}, |U_{0,1}|_{X^s}, |U_{0,2}|_{X^{s+1}}, |b|_{H^{\beta+3}})\) such that \(\forall t \in [0, \frac{T}{\max(\epsilon, \beta)}]\),

\[
|U_1 - U_2| \leq C_0 e^{\max(\epsilon, \beta)\lambda t} |U_{0,1} - U_{0,2}|_{X^s}.
\]

**Theorem 6.4 (Convergence)** Let \(p = (\mu, \epsilon, \delta, \gamma, \beta, b_0) \in P_{CH}(\text{see } (2.8))\) and \(s \geq s_0 + 1\) with \(s_0 > 1/2\), and let \(U^0 \equiv (\zeta^0, v^0)\) \(\in H^{\beta+3}(\mathbb{R}^2)\), \(b \in H^{\beta+3}\) satisfy the hypotheses \(H0, H1, H2, H3\), with \(N\) sufficiently large. We suppose \(U \equiv (\zeta, v)\) a unique solution to the full Euler system \((2.2)\) with initial data \((\zeta^0, v^0)\), defined on \([0, T_1]\) for \(T_1 > \frac{\mu^2}{\epsilon}\), and we suppose that \(U \equiv (\zeta, v)\) satisfies the assumptions of our consistency result, Proposition \(6.1\). Then there exists \(C, T > 0\), independent of \(p\), such that

- There exists a unique solution \(U_a \equiv (\zeta_a, v_a)\) \(\in \) our new model \(3.19\), defined on \([0, T]\) and with initial data \((\zeta^0, v^0)\) \(\in \) \((3.19)\) (provided by Theorem \(6.2)\);

- With \(v\), defined as in \(3.1\), one has for any \(t \in [0, T]\),

\[
|\{\zeta, v\} - (\zeta_a, v_a)|_{L^\infty([0, T]; X^s)} \leq C \mu^2 t.
\]

We conclude with a detailed statement of the functions \(A, B, C, D, E, F\) given in Section \(3.1\).
\[ A = 2 \epsilon \beta (1 + \beta w_2 b) \partial_z b w'(b) \partial_z (f(b)^2 - \gamma g(b)^2) + 2 \epsilon \beta^2 \nu \partial_z b w'_2 \partial_z (f(b)^2 - \gamma g(b)^2) \\
+ 2 \epsilon \beta w_2 \nu \partial_z b \partial_z (f(b)^2 - \gamma g(b)^2) + 3 \epsilon \beta w_2 \nu b \partial_z^2 (f(b)^2 - \gamma g(b)^2) - \frac{3 \epsilon}{bo} \partial_z^2 (f(b)^2 - \gamma g(b)^2) \\
- 3 \epsilon \beta w_1 b \beta \partial_z^2 (f(b)^2 - \gamma g(b)^2) \\
+ \epsilon (1 + \beta w_1 b)[2 \beta \partial_z \partial_z b g'(b)] + \partial_z^2 g(b) + \epsilon (1 + \beta w_1 b)[2 \theta + (\gamma - 1) g(b)] \partial_z (\beta \partial_z b g'(b)) \\
+ \epsilon \beta (1 + \beta w_1 b) \alpha (f(b)^2 - \gamma g(b)^2) \partial_z^2 b + 4 \epsilon \beta (1 + \beta w_1 b) \alpha \partial_z (f(b)^2 - \gamma g(b)^2) \partial_z b \\
+ 3 \epsilon \beta (1 + \beta w_1 b) \left[ \frac{2}{3} (f(b)^2 + \frac{2}{3} \delta - 1) \partial_z^2 (f(b)^2 - \gamma g(b)^2) + \epsilon (1 + \beta w_1 b) \theta (1 - \alpha_1) g(b) \partial_z^2 b \\
+ \epsilon (1 + \beta w_1 b) (2 \theta - \alpha_1) \beta \partial_z b) g'(b) \\
+ \epsilon (1 + \beta w_1 b)[2 \theta_1 - 2 \alpha_1 + \frac{1}{3} (\delta - 1) - \beta b) f'(b) - \frac{\gamma}{3} g'(b)] (\beta \partial_z b g'(b) + \partial_z (g(b))) \partial_z b \\
+ \epsilon \beta^2 (1 + \beta w_1 b) \eta (b) \partial_z b \partial_z^2 (f(b)^2 - \gamma g(b)^2) + \epsilon \beta^2 (1 + \beta w_1 b) \frac{\gamma}{3} f'(b) \partial_z^2 b (f(b)^2 - \gamma g(b)^2) \\
+ 2 \epsilon \beta^2 (1 + \beta w_1 b) \frac{2}{3} f'(b) \partial_z b \partial_z b g(b) + \epsilon \beta (1 + \beta w_1 b) \frac{2}{3} f'(b) \partial_z b \partial_z (f(b)^2 - \gamma g(b)^2) \\
- \beta \partial_z b w'(b) (1 + \beta w_2 b) \partial_z (\xi) - \epsilon \nu (1 + \beta w_1 b) [2 \nu (1 + \beta w_1 b) \partial_z^2 (\nu) b \\
- \nu \beta \partial_z (w_2 b) + w_2 \partial_z b \xi \partial_z (\xi) + \epsilon (1 + \beta w_1 b) [2 \nu (1 + \beta w_1 b) \partial_z (\nu) b] \partial_z (t)]. \quad (A.1) \]

\[ B = \epsilon \beta \partial_z b w'(b) (1 + \beta w_2 b) (f(b)^2 - \gamma g(b)^2) + \epsilon \beta^2 \nu \partial_z b w'_2 (b) f(b)^2 - \gamma g(b)^2) \\
+ \epsilon \beta w_2 \nu \partial_z b (f(b)^2 - \gamma g(b)^2) + 3 \epsilon \beta w_2 \nu b \partial_z (f(b)^2 - \gamma g(b)^2) - \frac{3 \epsilon}{bo} \partial_z (f(b)^2 - \gamma g(b)^2) \\
- 3 \epsilon \beta w_1 b \lambda \partial_z b (f(b)^2 - \gamma g(b)^2) \\
+ \epsilon (1 + \beta w_1 b)[2 \theta + (\gamma - 1) g(b)] (\beta \partial_z \partial_z b g'(b) + \partial_z (g(b))) \\
+ \epsilon \beta (1 + \beta w_1 b) [2 \alpha (f(b)^2 - \gamma g(b)^2) \partial_z b \\
+ 3 \epsilon \beta (1 + \beta w_1 b) \frac{2}{3} f'(b) + \frac{2}{3} \delta - 1) \partial_z (f(b)^2 - \gamma g(b)^2) \partial_z b \\
+ \epsilon \beta (1 + \beta w_1 b) [2 \theta_1 - \alpha_1 g(b) \partial_z b + \epsilon \beta^2 (1 + \beta w_1 b) \frac{2}{3} f'(b) \partial_z b (f(b)^2 - \gamma g(b)^2) \\
- \beta \partial_z b w'(b) (f b)^2 - \gamma g(b)^2) + \epsilon \beta (1 + \beta w_2 b) \partial_z (f(b)^2 - \gamma g(b)^2) - \epsilon \beta (1 + \beta w_2 b) \partial_z b w'(b) + 2 \nu (1 + \beta w_2 b) \partial_z (\nu) \\
- \epsilon \nu \beta \partial_z (w_2 b) + w_2 \partial_z b \xi + \epsilon (1 + \beta w_1 b) \frac{1}{2} \partial_z (f b)^2 - \gamma g(b)^2) + t (b)]. \quad (A.2) \]

\[ C = \epsilon \beta \partial_z b w'(b) (1 + \beta w_2 b) (f(b)^2 - \gamma g(b)^2) + \epsilon \beta^2 \nu \partial_z b w'_2 (b) f(b)^2 - \gamma g(b)^2) \\
+ \epsilon \beta w_2 \nu \partial_z b (f(b)^2 - \gamma g(b)^2) + 3 \epsilon \beta w_2 \nu b \partial_z (f(b)^2 - \gamma g(b)^2) - \frac{3 \epsilon}{bo} \partial_z (f(b)^2 - \gamma g(b)^2) \\
- 3 \epsilon \beta w_1 b \lambda \partial_z b (f(b)^2 - \gamma g(b)^2) \\
+ \epsilon (1 + \beta w_1 b)[\theta] (\beta \partial_z \partial_z b g'(b) + 2 \partial_z (g(b))) + \epsilon (1 + \beta w_1 b) [2 \alpha (f(b)^2 - \gamma g(b)^2) \\
+ \epsilon \beta (1 + \beta w_1 b) [2 \alpha (f(b)^2 - \gamma g(b)^2) \partial_z b \\
+ 3 \epsilon \beta (1 + \beta w_1 b) \frac{2}{3} f'(b) + \frac{2}{3} \delta - 1) \partial_z (f(b)^2 - \gamma g(b)^2) \partial_z b \\
+ \epsilon (1 + \beta w_1 b)[2 \theta_1 - \alpha_1 g(b) \partial_z b + \epsilon \beta^2 (1 + \beta w_1 b) \frac{2}{3} f'(b) \partial_z b (f(b)^2 - \gamma g(b)^2) \\
- \beta \partial_z b w'(b) (1 + \beta w_2 b) \partial_z (\xi) - \epsilon \nu \beta \partial_z (w_2 b) + w_2 \partial_z b \xi + \epsilon \nu (1 + \beta w_1 b) [2 \nu (1 + \beta w_1 b) \partial_z (\nu) b] + t (b)]. \quad (A.3) \]
\[ D = \frac{3\epsilon}{2}\nu w_2 b(f(b))^2 - \gamma g(b)^2) - \frac{3\epsilon}{2b_0}(f(b))^2 - \gamma g(b)^2) - \frac{3\epsilon}{2}\beta w_1 b(\lambda(b))(f(b))^2 - \gamma g(b)^2) \\
+ \frac{\epsilon}{2}(1 + \beta w_1 b)^2 \theta + (\gamma - 1)g(b)g(b) \\
+ \frac{\epsilon}{2}(1 + \beta w_1 b)^2 \theta + \frac{2}{3}(\gamma - 1)g(b)g(b) + 3\frac{\epsilon\beta}{2}(1 + \beta w_1 b)^2 f(b) + \frac{2}{3}\delta^{-1}f(b)(f(b))^2 - \gamma g(b)^2) \\
- \frac{3\epsilon\beta}{2}(1 + \beta w_1 b)^2 f(b)^2 - \gamma g(b)^2) \\
- \frac{3\epsilon}{2}(1 + \beta w_1 b)^2 f(b)^2 - \gamma g(b)^2) - \frac{3\epsilon c^2}{2}(1 + \beta w_2 b) + \frac{2(1 - \gamma)g(b)^2}{3}(1 + \beta w_1 b) \\
= \frac{2\epsilon}{3}(1 + \beta\omega_1 b)(\gamma - 1)g(b)^2. \quad \text{(A.4)}
\]

\[ \mathcal{E} = \frac{\epsilon}{2} \delta \partial_x b \gamma(b) (1 + \beta w_2 b) \frac{\partial_x^2 (f(b)^2 - \gamma g(b)^2)}{2} + \frac{\epsilon}{2} \delta \partial_x b \gamma(b) (1 + \beta w_1 b)^2 \gamma(b) \frac{\partial_x^2 (f(b)^2 - \gamma g(b)^2)}{2b_0} \\
+ \frac{\epsilon}{2} \beta \partial_x b \gamma(b) (1 + \beta w_1 b)\partial_x^2 (f(b)^2 - \gamma g(b)^2) \\
+ \frac{\epsilon}{2} \beta \partial_x b \gamma(b) (1 + \beta w_1 b)\partial_x^2 (f(b)^2 - \gamma g(b)^2) + \frac{\epsilon}{2} \beta \partial_x b \gamma(b) (1 + \beta w_1 b)\partial_x^2 (f(b)^2 - \gamma g(b)^2) \\
+ \frac{\epsilon}{2} \beta \partial_x b \gamma(b) (1 + \beta w_1 b)\partial_x^2 (f(b)^2 - \gamma g(b)^2) + \frac{\epsilon}{2} \beta \partial_x b \gamma(b) (1 + \beta w_1 b)\partial_x^2 (f(b)^2 - \gamma g(b)^2) \\
+ \frac{\epsilon}{2} \beta \partial_x b \gamma(b) (1 + \beta w_1 b)\partial_x^2 (f(b)^2 - \gamma g(b)^2) + \frac{\epsilon}{2} \beta \partial_x b \gamma(b) (1 + \beta w_1 b)^2 \gamma(b) \frac{\partial_x^2 (f(b)^2 - \gamma g(b)^2)}{2} \\
+ \frac{\epsilon}{2} \beta \partial_x b \gamma(b) (1 + \beta w_1 b)^2 \gamma(b) \frac{\partial_x^2 (f(b)^2 - \gamma g(b)^2)}{2} \\
+ \epsilon(1 + \beta w_1 b)\partial_x^2 (b). \quad \text{(A.5)}
\]

\[ \mathcal{F} = \beta q_1(\epsilon, \beta b)(\alpha)[\gamma + \delta] \partial_x^2 b + \epsilon \beta(1 + \beta w_1 b)\partial_x^2 b + \beta^2 q_1(\epsilon, \beta b)[\gamma + \delta] \partial_x^2 b + \beta^2 q_1(\epsilon, \beta b)[\gamma + \delta] \partial_x^2 b + \beta^2(1 + \beta w_1 b)\partial_x^2 b + \epsilon \beta^2(1 + \beta w_1 b)\partial_x^2 b + \epsilon \beta^2(1 + \beta w_1 b)\partial_x^2 b + \epsilon \beta^2(1 + \beta w_1 b)\partial_x^2 b + \epsilon \beta^2(1 + \beta w_1 b)\partial_x^2 b + \epsilon \beta^2(1 + \beta w_1 b)\partial_x^2 b. \quad \text{(A.6)}
\]

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