Reinforcement Learning for Integer Programming: Learning to Cut

Abstract

Integer programming (IP) is a general optimization framework widely applicable to a variety of unstructured and structured problems arising in, e.g., scheduling, production planning, and graph optimization. As IP models many provably hard to solve problems, modern IP solvers rely on many heuristics. These heuristics are usually human-designed, and naturally prone to suboptimality. The goal of this work is to show that the performance of those solvers can be greatly enhanced using reinforcement learning (RL). In particular, we investigate a specific methodology for solving IPs, known as the Cutting Plane Method. This method is employed as a subroutine by all modern IP solvers. We present a deep RL formulation, network architecture, and algorithms for intelligent adaptive selection of cutting planes (aka cuts). Across a wide range of IP tasks, we show that the trained RL agent significantly outperforms human-designed heuristics, and effectively generalizes to 10X larger instances and across IP problem classes. The trained agent is also demonstrated to benefit the popular downstream application of cutting plane methods in Branch-and-Cut algorithm, which is the backbone of state-of-the-art commercial IP solvers.

1 Introduction

Integer Programming is a very versatile modeling tool for discrete optimization problems, with applications in logistic, vehicle routing, scheduling, resource allocation, frequency assignment, production inventory, supply chain optimization, and planning problems, among others. In its most general form, an Integer Program (IP) minimizes a linear objective function over a set of integer points that satisfy a finite family of linear constraints. Classical results in polyhedral theory [6] imply that any combinatorial optimization (CO) problem can be formulated as an IP if its feasible region is finite. Hence, IP is a natural model for many graph optimization problems, such as the celebrated Traveling Salesman Problem (TSP), maximum cut, minimum vertex cover, maximum stable set, etc.

Due to the generality of the model, IPs can be very hard to solve in theory (NP-Hard) and in practice. There is no general polynomial time algorithm with guaranteed solutions for all integer programs. It is therefore of crucial importance to develop efficient heuristics for solving specific classes of IPs. Machine learning (ML) arises as a natural tool for tuning those heuristics. Indeed, the application of ML to discrete optimization has been a topic of significant interest in recent years, with a range of different approaches adopted in the literature (see [4] for a survey).

One set of approaches focus on directly learning the mapping from an IP problem instance to an approximate optimal solution [23, 3, 17, 15]. These methods implicitly learn a solution procedure for a problem instance as a single forward pass in a neural network. These approaches are attractive for their blackbox nature and general applicability. At the other end of spectrum are approaches which embed the machine-learned agent as a subroutine in a problem-specific human-designed algorithm for
CO problems \[7, 16\]. ML is used to improve some heuristic parts of that algorithm. These approaches can benefit from algorithm design know-how for many important and difficult classes of problems. For example, Dai et al. \[7\] guide greedy algorithms for TSP, minimum vertex cover, and maximum cut problems using reinforcement learning (RL).

In this paper, we take an approach that has the potential to combine the benefits of both lines of research described above. We design a RL agent to be used as a subroutine in a popular algorithmic framework for IP called the **cutting plane method**, thus building upon and benefiting from decades of research and understanding of this fundamental IP approach. The specific cutting plane algorithm that we choose to focus on is Gomory’s method \[10\], which in theory can solve any IP in finite time. Thus, our approach also enjoys wide applicability. In fact, we demonstrate that our learned agent can even be used, in an almost blackbox manner, as a subroutine in another powerful IP method called Branch-and-Cut (B&C), to obtain significant improvements. A recent line of work closely related to our approach includes \[13, 12, 2\], where supervised learning is used to improve branching heuristics in the Branch-and-Bound (B&B) framework for IP. To the best of our knowledge, no work on focusing on pure selection of (Gomory) cuts has appeared in the literature.

Cutting plane and B&C methods rely on the idea that every IP can be relaxed to a Linear program (LP) by dropping the integrality constraints, and efficient algorithms for solving LPs are available. Cutting plane methods iteratively add cuts to the LP, which are linear constraints that can tighten the LP relaxation by eliminating some part of the feasible region of the LP, while preserving the IP optimal solution. B&C methods are based on combining B&B with cutting plane methods and other heuristics; see Section 2 for further details.

Cutting plane methods have had a tremendous impact on the development of algorithms for IPs, having been employed e.g. to solve the first non-trivial instance of TSP \[8\]. Gomory cuts and other cutting plane methods are today widely employed in modern solvers, most commonly as a subroutine of the B&C methods that are the backbone of state-of-the-art commercial IP solvers like Gurobi and Cplex \[11\]. The systematic introduction of cutting planes has in fact been responsible for the tremendous speedups of IP solvers in the 90s \[10, 5\]. However, despite the amount of research on the subject, deciding which cutting plane to add is a non-trivial task. As reported in \[9\], “several issues need to be considered in the selection process [...] unfortunately the traditional analyses of strength of cuts offer only limited help in understanding and addressing these issues”. We believe ML/RL can therefore not only lead to improvements towards solving real applications, but may also aid researchers in understanding effective selection of cutting planes.

**Our contributions.** We develop a RL based method for intelligent adaptive selection of cutting planes, and use them in conjunction with Branch-and-Cut methods for efficiently solving IPs. Our main contributions are the following:

- **Efficient MDP Formulation.** We introduce an efficient Markov decision process (MDP) formulation for the problem of sequentially selecting cutting planes for an IP. Several trade-offs between the size of state-space/action-space vs. generality of method were navigated to arrive at the proposed formulation. For example, directly formulating B&C as an MDP would lead to a very large state space containing all open branches; another example is the use of Gomory’s cuts (vs. other cutting plane methods) that helped limit the number of actions (available cuts) to the number of IP constraints (note that the number of cuts available to the solver may be very large in general).

- **Deep RL solution architecture design.** We build upon state-of-the-art techniques to design an efficient and scalable deep RL architecture for learning to cut. Our design choices successfully address several unique challenges in this problem, including slow state-transition machine (due to complexity of solving LPs), and the resulting need for an architecture that is easy to parallelize, order and size independent representation, reward shaping in case when the optimal solution may not be reached, and numerical errors arising from inherent nature of cutting plane methods.

- **Empirical results.** We evaluate the quality of cutting plane algorithm learned by the RL agent through a variety of experiments over a range of IP problems (namely, packing, binary packing, planning, and maximum cut). Our empirical results demonstrate significant benefits of our approach, both in terms of efficiency and accuracy of solution compared to human designed heuristics for adding Gomory’s cuts, and in terms of generalizability across different sizes and structures of IP problems. We also demonstrate that using our trained RL policy for adding cuts in conjunction with B&C methods leads to significant improvements in efficiency of those methods; thus illustrating the promise of our approach for improving state-of-the-art IP solvers.
2 Background on Integer Programming methods

Integer Programming. It is well-known that any IP can be written in the following canonical form
\[ \min \{ c^T x : Ax \leq b, x \geq 0, x \in \mathbb{Z}^n \} \] (1)
where \( x \geq 0 \) is the set of \( n \) decision variables, \( Ax \leq b \) with \( A \in \mathbb{Q}^{m \times n} \), \( b \in \mathbb{Q}^m \) represent the set of constraints, and the linear objective function is \( c^T x \) for some \( c \in \mathbb{Q}^n \). \( x \in \mathbb{Z}^n \) implies we are only interested in integer solutions. Let \( x_{IP}^* \) be the optimal solution to (1) and \( z_{IP}^* \) its value.

The cutting plane method for Integer Programming. The cutting plane method starts from solving the LP obtained from (1) by dropping the integrality constraints \( x \in \mathbb{Z}^n \). This LP is called a Linear Relaxation (LR) of (1). We let \( C^{(0)} = \{ x | Ax \leq b, x \geq 0 \} \) be its feasible region, \( x_{IP}^*(0) \) its optimal solution, and \( z_{IP}^*(0) \) its value. Since \( C^{(0)} \) contains the feasible region of (1), \( z_{IP}^*(0) \leq z_{IP}^* \). Let us assume \( x_{IP}^*(0) \notin \mathbb{Z}^n \). The method then finds an inequality \( a^T x \leq \beta \) (a cut) that is satisfied by all integer points from (1), but not by \( x_{IP}^*(0) \) (one can prove that such an inequality always exists). \( a^T x \leq \beta \) is added to \( C^{(0)} \), to obtain the feasible region \( C^{(1)} \subset C^{(0)} \); and the new LP is solved, to obtain \( x_{IP}^*(1) \). This procedure is iterated until \( x_{IP}^*(t) \in \mathbb{Z}^n \). Since \( C^{(t)} \) contains the feasible region from (1), \( x_{IP}^*(t) \) is an optimal solution to (1). In fact, \( x_{IP}^*(t) \) is the only feasible solution to (1) produced throughout the algorithm.

A typical way to compare cutting plane methods is by the number of cuts added throughout the algorithm: a better method is the one that terminates after adding a smaller number of cuts. However, even for methods that are guaranteed to terminate in theory, in practice often numerical errors will prevent convergence to a feasible (optimal) solution. In this case, a typical way to evaluate the performance is the following. For an iteration \( t \) of the method, the value \( g^t := z_{IP}^* - z_{IP}^*(t) \geq 0 \) is called the (additive) integrality gap of \( C^{(t)} \). Since \( C^{(t+1)} \subset C^{(t)} \), we have that \( g^t \geq g^{t+1} \). Hence, the integrality gap decreases during the execution of the cutting plane method. We can then measure the performance of a cutting plane method by computing the factor of integrality gap closed between the first LR, and the iteration \( \tau \) when we decide to halt the method. Specifically, we define the Integrality Gap Closure (IGC) to be the ratio
\[ \frac{g^0 - g^\tau}{g^0} \in [0, 1]. \] (2)
In order to measure the IGC for our learned agent, we will need to know the optimal solution \( x_{IP}^* \), which we will compute using a commercial IP solver. Importantly, note that we will not use this measure for training the learner, but only for evaluating it a posteriori.

Gomory’s Integer Cuts. Cutting plane algorithms differ in how cutting planes are constructed at each iteration. Assume that the LR of (1) with feasible region \( C^{(t)} \) has been solved via the simplex algorithm. At convergence, the simplex algorithm returns a so-called tableau, which consists of a constraint matrix \( \tilde{A} \) and a constraint vector \( \tilde{b} \). Let \( I_t \) be the set of components such that \( [x_{IP}^*(t)]_i \) is fractional. For each \( i \in I_t \), we can generate a Gomory cut [10]
\[ (-A_{(i)} + [\tilde{A}_{(i)}])^T x \leq -\tilde{b}_i + [\tilde{b}_i], \] (3)
where \( A_{(i)} \) is the \( i \)th row of matrix \( A \) and \([\cdot]\) means component-wise rounding down. Gomory cuts can therefore be generated for any LP and, as required, are valid for all integer points from (1) but not for \( x_{IP}^*(t) \). For convenience, denote the set of all candidate cuts as \( D^{(t)} \) such that \( |D^{(t)}| = |I_t| = I_t \).

It is shown in Gomory [10] that a cutting plane method that at each step adds Gomory’s cuts terminate in a finite number of iterations. At each iteration \( t \), we have as many as \( |I_t| \) cuts to choose from. As a result, the efficiency and quality of the solutions depend highly on the sequence of generated cutting planes, which are usually chosen by heuristics [24]. We aim to show that the solutions can be significantly improved with an adaptive RL based cut selection.

Branch and cut. In state-of-the-art solvers, the addition of cutting planes is alternated with a branching phase, which can be described as follows. Let \( x_{IP}^*(t) \) be the solution to the current LR of (1), and assume that some component of \( x_{IP}^*(t) \), say wlog the first, is not integer (else, \( x_{IP}^*(t) \) is the optimal solution to (1)). Then (1) can be split into two subproblems, whose LR is obtained from \( C^{(t)} \) by adding constraint \( x_1 \leq \lfloor x_{IP}^*(t)_1 \rfloor \) and \( x_1 \geq \lceil x_{IP}^*(t)_1 \rceil \). Note that the set of feasible integer points for (1) is the union of the set of feasible integer points for the two new subproblems. Hence,
the solution of minimum cost over those two subproblems gives the optimal solution to \( \Pi \). Several heuristics are used to select which subproblem to solve next, as to minimize the number of subproblems created. An algorithm that alternates between the cutting plane method and branching is called Branch-and-Cut (B&C). When all the other parameters (e.g., the number of cuts added to a subproblem) are kept constant, a typical way to evaluate a B&C method is by the number of subproblems explored before the optimal is found.

3 Deep RL Formulation and Solution Architecture

Here we introduce our formulation of the cutting plane selection problem as an RL problem and present our deep RL network architecture for solving the same.

3.1 Formulating Cutting Plane Selection Problem as RL

The standard RL formulation starts with a MDP: at time step \( t \geq 0 \), an agent is in a state \( s_t \in \mathcal{S} \), takes an action \( a_t \in \mathcal{A} \), receives an instant reward \( r_t \in \mathbb{R} \) and transitions to the next state \( s_{t+1} \sim p(\cdot|s_t, a_t) \). A policy \( \pi : \mathcal{S} \rightarrow \mathcal{P}(\mathcal{A}) \) gives a mapping from any state to a distribution over actions \( \pi(\cdot|s_t) \). The objective of RL is to search for a policy that maximizes the expected cumulative rewards over an horizon \( T \), i.e., \( \max_\pi J(\pi) := \mathbb{E}[\sum_{t=0}^{T-1} r_t \gamma^t; \pi] \), where \( \gamma \in (0, 1] \) is a discount factor and the expectation is w.r.t. randomness in the policy \( \pi \) as well as the environment (e.g., the transition dynamics \( p(\cdot|s_t, a_t) \)). In practice, we consider parameterized policies \( \pi_\theta \) and aim to find \( \theta^* = \arg \max_\theta J(\pi_\theta) \). To formulate the procedure of selecting cutting plane algorithms into an MDP, we specify below the state space \( \mathcal{S} \), action space \( \mathcal{A} \), reward function \( r_t \) and transition dynamics \( s_{t+1} \sim p(\cdot|s_t, a_t) \).

State Space \( \mathcal{S} \). At iteration \( t \), the new LP is defined by the feasible region, or equivalently set of constraints \( C(t) = \{a_i^T x \leq b_i\}_{i=1}^{N_t} \). Solving the resulting LP produces an optimal solution \( x^*_\text{LP}(t) \) along with a set of candidate Gomory’s cuts \( D_t \). We set the numerical representation of the state to be \( s_t = \{C(t), c, x^*_\text{LP}(t), D_t\} \). When all components of \( x^*_\text{LP}(t) \) are integer-valued, \( s_t \) is a terminal state and \( s_{t+1} \sim p(\cdot|s_t, a_t) \) is an empty set.

Action Space \( \mathcal{A} \). At iteration \( t \), the action space consists of all possible \( I_t \) Gomory’s cutting planes that can be added to the LP for the next iteration. The action space is discrete because each action is a discrete choice of cutting plane. However, unlike conventional discrete actions, here each action is parameterized as an inequality \( e_i^T x \leq d_i \) and can be computed from the tableau if there exists fractional components of \( x^*_\text{LP}(t) \). We then set \( s_{t+1} \sim p(\cdot|s_t, a_t) \).

Reward \( r_t \). We encourage the cutting planes to cut aggressively. We therefore set the instant reward in iteration \( t \) to be the objective gap between consecutive LP solutions \( r_t = c^T x^*_\text{LP}(t) - c^T x^*_\text{LP}(t+1) \). With a discount factor \( \gamma < 1 \), this encourages the agent reduce the integrality gap as fast as possible.

Transition \( s_{t+1} \sim p(\cdot|s_t, a_t) \). In our case, it is easier to describe how to algorithmically compute \( s_{t+1} \) from \( s_t, a_t \) than explicitly specifying \( p(\cdot|s_t, a_t) \). Given state \( s_t = \{C(t), c, x^*_\text{LP}(t), D_t\} \), we add the cutting plane to the constraint set \( C(t+1) = C(t) \cup \{e_i^T x \leq d_i\} \). The augmented set of constraints form a new LP, which is efficiently solved using the simplex method to get \( x^*_\text{LP}(t+1) \). A new set of Gomory’s cutting planes \( D_{t+1} \) can be computed from the tableau if there exists fractional components of \( x^*_\text{LP}(t+1) \). We then set \( s_{t+1} = \{C(t+1), c, x^*_\text{LP}(t+1), D_{t+1}\} \).

3.2 Policy Network Architecture

We now describe the policy network architecture for \( \pi_\theta(a_t|s_t) \). Recall from the last section we have in the state \( s_t \) a set of inequalities \( C(t) = \{a_i^T x \leq b_i\}_{i=1}^{N_t} \), and as action space another set \( D(t) = \{e_i^T x \leq d_i\}_{i=1}^{I_t} \). A policy \( \pi_\theta \) specifies a distribution over \( D(t) \), via the following architecture.

Attention Network for Order-Agnostic Cut Selection. Given current LP constraints in \( C(t) \), when computing distributions over the \( I_t \) candidate constraints in \( D(t) \), it is desirable that the architecture is agnostic to the ordering among constraints (both in \( C(t) \) and \( D(t) \)), because the ordering does not reflect the geometry of the feasible set. To achieve this, we adopt ideas from the work on

\footnote{Here we drop the non-zero constraints \( x \geq 0 \) for convenience.}
We can test the performance of a trained policy $\pi_{\theta}$ for a given IP instance, by executing it as described in Algorithm 1. One important design consideration is that a cutting plane method can potentially cut off the optimal integer solution due to the LP solver’s numerical errors. Invalid cutting planes generated by numerical errors is a well-known phenomenon in integer programming. However, in our case, learning can amplify this problem, since when a policy is trained to maximize the IGC, it might add aggressive cuts in order to tighten the LP constraints as much as possible. In particular, if no countermeasure is taken, depending on the family of problems, the RL agent was observed to cut the optimal solution in up to 20% of instances!

To remedy this, we propose to add a simple stopping criterion at test time. The idea is to maintain a running statistics that measures the relative progress made by newly added cuts during execution. When a certain number of consecutive cuts have little effect on the IGC, we simply terminate the episode. This prevents the agent from adding cuts that are likely to induce numerical errors. Indeed, our computational experiments show this modification is enough to completely remove the generation of invalid cutting planes. We postpone the details to the Appendix.
We conduct four sets of experiments to evaluate the learned algorithm on the following aspects:

1. **Efficiency of cuts.** Can the RL agent solve an IP problem using fewer number of Gomory cuts?

2. **Integrality gap closed (for larger problems).** As discussed in Section 2 for larger IP problems cutting plane methods alone are unlikely to solve the problem to optimality. For such problems, can the RL agent be used to close the integrality gap effectively? We use the metric of IGC defined in [2], which measures the factor of integrality gap closed compared to the initial gap.

3. **Generalization properties.**
   - (size) Can an RL agent trained on smaller instances be applied to 10X larger instances to yield performance comparable to an agent trained on the larger instances?
   - (structure) Can an RL agent trained on instances from one class of IPs be applied to a very different class of IPs to yield performance comparable to an agent trained on the latter class?

4. **Impact on the efficiency of B&C methods.** As mentioned in Section 2 in practice, cutting planes are alternated with a branching procedure, leading to Branch-and-Cut (B&C). Will the RL agent trained as a cutting plane method be effective as a subroutine within a B&C method? We evaluate the impact on the efficiency of B&C in terms of reduction in the number of subproblems (aka nodes) created until an optimal solution is reached.

**Benchmark Instances.** We consider four classes of IPs: Packing, Production Planning, Binary Packing and Max-Cut. These represent a wide range of IP problems ranging from resource allocation to graph optimization. We leave the IP formulations of these problems to the appendix. We let \( n, m \) be the number of variables and constraints (other than nonnegativity) in the IP formulation and denote the size of an IP instance as \( n \times m \) in the tables below. The mapping from problem parameters (like number of nodes and edges in maximum-cut) to \( n, m \) depends on the IP formulation used for each problem. For training an RL agent on a given IP problem class, we use 20 randomly generated instances of that type. For every problem class and size tested, a fixed common pool of 30 randomly generated instances is used to evaluate and compare different heuristics with the RL agent.

**Baselines.** We compare the performance of the RL agent with the following commonly used human-designed heuristics for choosing (Gomory) cuts [24]: Random, Max Violation (MV) and Max Normalized Violation (MNV). A precise description of each heuristic is provided in the appendix.

**Implementation Details.** We build the MDP simulation environment for our RL problem using the C-programming interface of Gurobi [11] as the LP solver. This low-level interface allows us to add the cut chosen by RL agent to the current LP model and efficiently solve the modified LP. At training time we set the episodic horizon to be \( T = 50 \). We sample actions from the categorical distribution \( \{p_i\} \) during training; while testing, we instead take actions greedily as \( i^* = \arg \max p_i \).
(see Appendix). Further implementation details, including hyperparameters for the policy network and ES optimization, are provided in the appendix.

**Experiment #1: Efficiency of Cuts (Small-sized Instances).** For small-sized IP instances where cutting planes alone can potentially solve the problems, we evaluate different cutting plane methods on total number of cuts it takes to find an optimal integer solution. Table 1 demonstrates favorable performance of our method when compared to the baselines.

| Tasks         | Packing | Planning | Binary Packing | Max Cut |
|---------------|---------|----------|----------------|---------|
| Size          | 10 × 5  | 13 × 20  | 10 × 20        | 10 × 22 |
| RANDOM        | 48 ± 36 | 44.4 ± 37.3 | 81 ± 32 | 69 ± 34 |
| MV            | 62 ± 40 | 47.8 ± 29.4 | 87 ± 27 | 64 ± 36 |
| MNV           | 53 ± 39 | 60.2 ± 34.3 | 85 ± 29 | 47 ± 34 |
| RL            | 14 ± 11 | 10 ± 12  | 22 ± 27        | 13 ± 4  |

**Experiment #2: Integrality gap closure for larger problems.** Next, we train and test the RL agent on significantly larger problem instances compared to the previous experiment. For these instances, the cutting plane methods is unable to reach optimality in $T = 50$ cuts. We compare different cutting plane methods on integrality gap closed using the IGC metric defined in equation (2), Section 2. Table 2 shows that on average RL agent was able to close a significantly higher fraction of gap. Figure 2 provides a more detailed comparison, by showing a percentile plot - here the instances are sorted in ascending order of IGC and then plotted in order; the $y$-axis shows the IGC and $x$-axis shows the percentile of instances achieving that IGC. The blue curve shows the performance of our RL agent. Closest to the blue curve is the yellow curve for RL/10X which is in fact another RL agent trained on smaller instances, in order to test generalization properties as described next.

| Tasks | Packing | Planning | Binary Packing | Max Cut |
|-------|---------|----------|----------------|---------|
| Size  | 30 × 30 | 61 × 84  | 33 × 66        | 27 × 67 |
| RANDOM| 0.18 ± 0.17 | 0.56 ± 0.16 | 0.39 ± 0.21 | 0.56 ± 0.09 |
| MV    | 0.14 ± 0.08 | 0.18 ± 0.08 | 0.32 ± 0.18 | 0.55 ± 0.10 |
| MNV   | 0.19 ± 0.23 | 0.31 ± 0.09 | 0.32 ± 0.24 | 0.62 ± 0.12 |
| RL    | 0.55 ± 0.32 | 0.88 ± 0.12 | 0.95 ± 0.14 | 0.86 ± 0.14 |
| RL/10X| 0.54 ± 0.31 | 0.86 ± 0.12 | 0.94 ± 0.14 | 0.80 ± 0.18 |

Figure 2: Percentile plots of Integrality Gap Closure. X-axis shows the percentile of instances and y-axis shows the IGC achieved after adding at most $T = 50$ cuts. RL agents achieve significantly better performance in tightening the LP relaxation than baseline heuristics.
Table 3: Number of Nodes expanded in B&C. We show mean ± std nodes across all test instances

| Tasks          | Packing     | Planning   | Binary Packing | Max Cut     |
|----------------|-------------|------------|----------------|-------------|
| Size           | 30 × 30     | 61 × 84    | 33 × 66        | 27 × 67     |
| NO CUT         | 847 ± 261   | 1000 ± 0   | 478 ± 269      | 50 ± 29     |
| RANDOM         | 722 ± 339   | 256 ± 249  | 91 ± 181       | 32 ± 20     |
| MV             | 699 ± 358   | 672 ± 347  | 141 ± 167      | 25 ± 19     |
| MNV            | 710 ± 329   | 407 ± 342  | 70 ± 43        | 26 ± 18     |
| RL             | 452 ± 412   | 12 ± 8     | 10 ± 13        | 14 ± 10     |

Figure 3: Percentile plots of number of B&C nodes expanded. X-axis shows the percentile of instances and y-axis shows the number of expanded nodes before reaching optimal solutions.

**Experiment #3: Generalization.** In Figure 2 and Table 2, we also demonstrate the ability of the RL agent to generalize across size of IP instances. This is illustrated through the extremely competitive performance of the RL/10X agent, which is tested on the same size instances as the RL agent of previous experiment, although it was trained on roughly 10X smaller instances. (Exact sizes used were 10 × 10, 32 × 22, 10 × 20, 20 × 10, respectively). Furthermore, we test generalizability across IP classes by training an RL agent on small sized instances of packing problem, and applying it to add cuts to 10X larger instances of the maximum-cut problem; the latter, being a graph optimization problem, has intuitively a very different structure from the former. Figure 4 shows that the RL/10X agent trained on packing (yellow curve) achieve a performance on larger maximum-cut instances that is comparable to the performance of agent trained on the latter class (blue curve).

**Experiment #4: Impact on the efficiency of B&C methods.** In practice, cutting planes alone are not sufficient to solving large problems. In state-of-the-art solvers, the iterative addition of cutting planes is alternated with a branching procedure, leading to Branch-and-Cut (B&C). To demonstrate the full potential of our learner, we implement a comprehensive B&C procedure but without all the heuristics implemented in the standard solvers. The B&C procedure has two hyper-parameters: number of child nodes (suproblems) to expand $N_{exp}$ and number of cutting planes added to each child node $N_{cuts}$. Here, $N_{exp}$ generally determines the depth of the tree search in B&B and $N_{cuts}$ determines the cutting plane budgets per node. In addition, B&C is determined by the implementation of the Branching Rule, Priority Queue and Termination Condition. We explain these details in the appendix.

Figure 3 gives percentile plots showing the number of child nodes (suproblems) $N_{exp}$ until termination for B&C with $N_{cuts} = 10$ cuts added to each nodes using either RL or one of the heuristics. We also include as a comparator, the B&C method without any cuts, i.e., the branch and bound method. Perhaps surprisingly, the RL agent, though not designed to be used with branching, shows substantial improvements in the efficiency of the latter.

**Conclusions.** The variety of tasks across which the RL agent is demonstrated to generalize without being trained for, provides convincing evidence that it is able to learn an intelligent algorithm for selecting cutting planes. We believe those empirical results to be a convincing step forward towards the integration of IP solvers with ML techniques that guide them in the selection of heuristics. This may lead to a functional answer to the “Hamletic question Branch-and-cut designers often have to face: to cut or not to cut?” [9].
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References

[1] Balas, E., Ceria, S., and Cornuéjols, G. (1993). A lift-and-project cutting plane algorithm for mixed 0–1 programs. Mathematical programming, 58(1-3):295–324.

[2] Balcan, M.-F., Dick, T., Sandholm, T., and Vitercik, E. (2018). Learning to branch. arXiv preprint arXiv:1803.10150.

[3] Bello, I., Pham, H., Le, Q. V., Norouzi, M., and Bengio, S. (2017). Neural combinatorial optimization.

[4] Bengio, Y., Lodi, A., and Prouvost, A. (2018). Machine learning for combinatorial optimization: a methodological tour d’horizon. arXiv preprint arXiv:1811.06128.

[5] Bixby, R. (2017). Optimization: past, present, future. Plenary talk at INFORMS Annual Meeting.

[6] Conforti, M., Cornuéjols, G., and Zambelli, G. (2014). Integer programming, volume 271. Springer.

[7] Dai, H., Khalil, E. B., Zhang, Y., Dilkina, B., and Song, L. (2017). Learning combinatorial optimization algorithms over graphs. arXiv preprint arXiv:1704.01665.

[8] Dantzig, G., Fulkerson, R., and Johnson, S. (1954). Solution of a large-scale traveling-salesman problem. Journal of the operations research society of America, 2(4):393–410.

[9] Dey, S. S. and Molinaro, M. (2018). Theoretical challenges towards cutting-plane selection. Mathematical Programming, 170(1):237–266.

[10] Gomory, R. (1960). An algorithm for the mixed integer problem. Technical report, RAND CORP SANTA MONICA CA.

[11] Gurobi Optimization, I. (2015). Gurobi optimizer reference manual. URL http://www.gurobi.com.

[12] Khalil, E., Dai, H., Zhang, Y., Dilkina, B., and Song, L. (2017). Learning combinatorial optimization algorithms over graphs. In Advances in Neural Information Processing Systems, pages 6348–6358.

[13] Khalil, E. B., Le Bodic, P., Song, L., Nemhauser, G. L., and Dilkina, B. N. (2016). Learning to branch in mixed integer programming. In AAAI, pages 724–731.

[14] Kingma, D. P. and Ba, J. (2014). Adam: A method for stochastic optimization. arXiv preprint arXiv:1412.6980.

[15] Kool, W. and Welling, M. (2018). Attention solves your tsp. arXiv preprint arXiv:1803.08475.

[16] Li, Z., Chen, Q., and Koltun, V. (2018). Combinatorial optimization with graph convolutional networks and guided tree search. In Advances in Neural Information Processing Systems, pages 539–548.

[17] Nowak, A., Villar, S., Bandeira, A. S., and Bruna, J. (2017). A note on learning algorithms for quadratic assignment with graph neural networks. arXiv preprint arXiv:1706.07450.

[18] Pochet, Y. and Wolsey, L. A. (2006). Production planning by mixed integer programming. Springer Science & Business Media.

[19] Salimans, T., Ho, J., Chen, X., Sidor, S., and Sutskever, I. (2017). Evolution strategies as a scalable alternative to reinforcement learning. arXiv preprint arXiv:1703.03864.

[20] Sutskever, I., Vinyals, O., and Le, Q. V. (2014). Sequence to sequence learning with neural networks. In Advances in neural information processing systems, pages 3104–3112.

[21] Tokui, S., Oono, K., Hido, S., and Clayton, J. (2015). Chainer: a next-generation open source framework for deep learning. In Proceedings of workshop on machine learning systems (LearningSys) in the twenty-ninth annual conference on neural information processing systems (NIPS), volume 5, pages 1–6.

[22] Vaswani, A., Shazeer, N., Parmar, N., Uszkoreit, J., Jones, L., Gomez, A. N., Kaiser, Ł., and Polosukhin, I. (2017). Attention is all you need. In Advances in neural information processing systems, pages 5998–6008.
[23] Vinyals, O., Fortunato, M., and Jaitly, N. (2015). Pointer networks. In *Advances in Neural Information Processing Systems*, pages 2692–2700.

[24] Wesselmann, F. and Stuhl, U. (2012). Implementing cutting plane management and selection techniques. Technical report, Tech. rep., University of Paderborn.
A Experiment Details

A.1 Projection into the Original Variable Space

In the following we look at only the first iteration of the cutting plane procedure, and we drop the iteration index \( t \). Recall the LP relaxation of the original IP problem (1), where \( A \in \mathbb{Q}^{m \times n} \), \( b \in \mathbb{Q}^m \)

\[
\begin{align*}
\min & \quad c^T x \\
Ax & \leq b \\
x & \geq 0.
\end{align*}
\]

When a simplex algorithm solves the LP, the original LP is converted to a standard form where all inequalities are transformed into inequalities by introducing slack variables.

\[
\begin{align*}
\min & \quad c^T x \\
Ax + Is & = b \\
x, s & \geq 0,
\end{align*}
\]

where \( I \) is an identity matrix and \( s \) is the set of slack variables. The simple method carries out iteratively operations on the tableau formed by \([A, I], b, c\). At convergence, the simplex method returns a final optimal tableau. We generate a Gomory’s cut using the row of the tableau that corresponds to fractional variable of the optimal solution \( x_{LP}^* \). This will in general create a cutting plane of the following form

\[
e^T x + r^T s \leq d
\]

where \( e, x \in \mathbb{R}^n \), \( r, s \in \mathbb{R}^m \) and \( d \in \mathbb{R} \). Though this cutting plane involves slack variables, we can get rid of the slack variables by multiplying both sides of the linear constraints in (7) by \( r \)

\[
r^T Ax + r^T s = r^T b
\]

and subtract the new cutting plane (8) by the above. This leads to an equivalent cutting plane

\[
(e^T - r^T A)x \leq d - r^T b.
\]

Note that this cutting plane only contains variables in the original variable space. For a downstream neural network that takes in the parameters of the cutting planes as inputs, we find it helpful to remove such slack variables. Slack variables do not contribute to new information regarding the polytope and we can also parameterize a network with smaller number of parameters.

A.2 Integer Programming Formulations of Benchmark Problems

A wide range of benchmark instances can be cast into special cases of IP problems. We provide their specific formulations below. For simplicity, we only provide their general IP formulations (with \( \leq, \geq, = \) constraints). It is always possible to convert original formulations into the standard formulation (1) with properly chosen \( A, b, c, x \). Some problems are formulated within a graph \( G = (V, E) \) with nodes \( v \in V \) and edges \((v, u) \in E\).

Their formulations are as follows:

Max Cut. We have one variable per edge \( y_{u,v} \), \((u, v) \in E\) and one variable per node \( x_u, u \in V \). Let \( w_{u,v} \geq 0 \) be a set of non-negative weights per edge.

\[
\begin{align*}
\max & \quad \sum_{(u,v) \in E} w_{u,v} y_{uv} \\
y_{uv} & \leq x_u + x_v, \forall (u, v) \in E \\
y_{uv} & \leq 2 - x_u - x_v, \forall (u, v) \in E \\
0 & \leq x, y \leq 1 \\
x_u, y_{uv} & \in \mathbb{Z}, \forall u \in V, (u, v) \in E.
\end{align*}
\]

In our experiments the graphs are randomly generated. To be specific, we specify a vertex size \(|V|\) and edge size \(|E|\). We then sample \(|E|\) edges from all the possible \(|V| \cdot (|V| - 1)/2\) edges to form the final graph. The weights \( w_{u,v} \) are uniformly sampled as an integer from 0 to 10.
Packing. The packing problem takes the generic form of \( \text{(1)} \) while requiring that all the coefficients of \( A, b, c \) be non-negative, in order to enforce proper resource constraints.

Here the constraint coefficients \( a_{ij} \) for the \( j \)th variable and \( i \)th constraint is sampled as an integer uniformly from 0 and 5. Then the RHS coefficient \( b_i \) is sampled from 9\( n \) to 10\( n \) uniformly as an integer where \( n \) is the number of variables. Each component of \( c_j \) is uniformly sampled as an integer from 1 to 10.

Binary Packing. Binary packing augments the original packing problem by a set of binary constraints on each variable \( x_i \leq 1 \).

Here the constraint coefficients \( a_{ij} \) for the \( j \)th variable and \( i \)th constraint is sampled as an integer uniformly from 5 and 30. Then the RHS coefficient \( b_i \) is sampled from 10\( n \) to 20\( n \) uniformly as an integer where \( n \) is the number of variables. Each component of \( c_j \) is uniformly sampled as an integer from 1 to 10. We also have \( n \) binary constraints.

Production Planning. Consider a production planning problem \([18]\) with time horizon \( T \). The decision variables are production \( x_i, 1 \leq i \leq T \), along with by produce / not produce variables \( y_i, 1 \leq i \leq T \) and storage variables \( s_i, 0 \leq i \leq T \). Costs \( p_i', h_i', q_i \) and demands \( d_i \) are given as problem parameters. The LP formulation is as follows

\[
\begin{align*}
\min & \quad \sum_{i=1}^{T} p_i' x_i + \sum_{i=1}^{T} h_i' s_i + \sum_{i=0}^{T} q_i y_i \\
\text{s.t.} & \quad s_i + x_i = d_i + s_i, \forall 1 \leq i \leq T \\
& \quad x_i \leq M y_i, \forall 1 \leq i \leq T \\
& \quad s \geq 0, x \geq 0, 0 \leq y \leq 1 \\
& \quad s_0 = s_0^*, s_T = s_T^* \\
& \quad x, s, y \in \mathbb{Z}^T, 
\end{align*}
\]

(12)

where \( M \) is a positive large number and \( s_0^*, s_T^* \) are also given.

The instance parameters are the initial storage \( s_0^* = 0 \), final storage \( s_T^* = 20 \) and big \( M = 100 \). The revenue parameter \( p_i', h_i', q_i \) are generated uniformly random as integers from 1 to 10.

Size of IP formulations. In our results, we describe the sizes of the IP instances as \( n \times m \) where \( n \) is the number of columns and \( m \) is the number of rows of the constraint matrix \( A \) from the LR of \( \text{(1)} \).

For a packing problem with \( n \) items and \( m \) resource constraints, the IP formulation has \( n \) variables and \( m \) constraints; for planning with period \( K, n = 3K + 1, m = 4K + 1 \); for binary packing, there are \( n \) extra binary constraints compared to the packing problem; for max-cut, the problem is defined on a graph with a vertex set \( V \) and an edge set \( E \), and its IP formulation consists of \( n = |V| + |E| \) variables and \( m = 3|E| + |V| \) constraints.

A.3 Heuristics for selecting Gomory cuts

Here \( I_t \) is the number of candidate Gomory cuts available in round \( t \), and \( i_t \) denotes the index of cut chosen by the given method. The three heuristic taken from \([24]\) are the following.

- Random. One cut \( i_t \sim \text{Uniform}\{1, 2...I_t\} \) is chosen uniformly at random from all the candidate cuts.
- Max Violation (MV). Let \( x_B^*(t) \) be the basic feasible solution of the current LP relaxation. MV selects the cut that corresponds to the most fractional component, i.e. \( i_t = \arg\max\{||x_B^*(t)||_i - \text{round}([x_B^*(t)]_i)\} \).
- Max Normalized Violation (MNV). Recall that \( \hat{A} \) denotes the optimal tableau obtained by the simplex algorithm upon convergence. Let \( \hat{A}_i \) be the \( i \)th row of \( \hat{A} \). Then, MNV selects cut \( i_t = \arg\max\{||x_B^*(t)||_i - \text{round}([x_B^*(t)]_i)||/\|\hat{A}_i\|\} \).

A.4 Hyper-parameters

Policy Architecture. The policy network is implemented with Chainer \([21]\). The attention embedding \( F_B \) is a 2-layer neural network with 64 units per layer and tanh activation. The LSTM network encodes variable sized inputs into hidden vector with dimension 10.
During a forward pass, a LSTM + Attention policy will take the instance, carry out embedding into a n-d vector and then apply attention. Such architecture allows for generalization to variable sized instances (different number of variables). We apply such architecture in the generalization part of the experiments.

On the other hand, a policy network can also consist of a single attention network. This policy can only process IP instances of a fixed size (fixed number of variables) and cannot generalize to other sizes. We apply such architecture in the IGC part of the experiments.

**ES Optimization.** Across all experiments, we apply Adam optimizer [14] with learning rate \( \alpha = 0.01 \) to optimize the policy network. The perturbation standard deviation \( \sigma \) is selected from \{0.002, 0.02, 0.2\}. We apply \( N = 10 \) perturbations to construct the policy gradient for each iteration. Empirically, we observe that the training is stable for both policy architectures and the training performance converges in \( \leq 500 \) weight updates.

**B Branch-and-Cut Details**

As mentioned, in the introduction, Branch-and-Cut (B&C) is an algorithmic procedure used for solving IP problems.

We list several critical elements of our implementation of B&C.

**Branching Rule.** We adopt a simple branching rule: at each node, we branch on the most fractional variable of the corresponding LP optimal solution (0.5 being the most fractional).

**Priority Queue.** The efficiency of B&C heavily depends on the order in which the priority queue selects subproblems to explore. We adopt a FIFO queue (Breath first search). FIFO queue allows the B&C procedure to improve the lower bound. Alternatively, B&C can adopt a LIFO queue (Depth first search) to quickly find a feasible solution.

**Termination Condition.** Let \( z_0 = c^T x_{LP}^*(0) \) be the objective of the initial LP relaxation. As B&C proceeds, the procedure finds an increasing set of feasible integer solutions \( X_F \), and an upper bound on the optimal objective \( z^* = c^T x_{IP}^* \) is \( z_{upper} = \min_{x \in X_F} c^T x \). Hence, \( z_{upper} \) monotonically decreases.

Along with B&C, cutting planes can iteratively improve the lower bound \( z_{lower} \) of the optimal objective \( z^* \). Let \( z_0 \) be the objective of the LP solution at node \( i \) and denote \( N \) as the set of unpruned nodes with unexpanded child nodes. The lower bound is computed as \( z_{lower} = \min_{i \in N} z_i \) and monotonically increases as the B&C procedure proceeds.

This produces a ratio statistic \( r = \frac{z_{upper} - z_{lower}}{z_{upper} - z_{LP}} > 0 \).

Note that since \( z_{lower} \geq z_{LP}^* \), \( z_{lower} \) monotonically increases, and \( z_{upper} \) monotonically decreases, \( r \) monotonically decreases. The B&C terminates when \( r \) is below some threshold which we set to be 0.0001.

**C Test Time Considerations**

**Stopping Criterion.** Though at training time we guide the agent to generate aggressive cuts that tighten the LP relaxation as much as possible, the agent can exploit the defects in the simulation environment - numerical errors, and generate invalid cuts which cut off the optimal solution.

This is undesirable in practice. At test time, when we execute the trained policy, we adopt a stopping criterion which automatically determines if the agent should stop adding cuts, in order to prevent from invalid cuts. In particular, at each iteration let \( r_t = |c^T x_{LP}^*(t) - c^T x_{LP}^*(t+1)| \) be the objective gap achieved by adding the most recent cut. We maintain a cumulative ratio statistics such that

\[
s_t = \frac{r_t}{\sum_{t' \leq t} r_{t'}}.
\]
We terminate the cutting plane procedure once the average $s_t$ over a fixed window of size $H$ is lower than certain threshold $\eta$. In practice, we set $H = 5, \eta = 0.001$ and find this work effectively for all problems, eliminating all the numerical errors observed in reported tasks. Intuitively, this approach dictates that we terminate the cutting plane procedure once the newly added cuts do not generate significant improvements for a period of $H$ steps.

To analyze the effect of $\eta$ and $H$, we note that when $H$ is too small or $\eta$ is too large, we have very conservative cutting plane procedure. On the other hand when $H$ is large while $\eta$ is small, the cutting plane procedure becomes more aggressive.

**Greedy Action.** The policy network defines a stochastic policy, i.e. a categorical distribution over candidate cuts. At test time, we find taking the greedy action $i^* = \arg \max_i p_i$ to be more effective in certain cases, where $p_i$ is the categorical distribution over candidate cuts. The justification for this practice is that: the ES optimization procedure can be interpreted as searching for a parameter $\theta$ such that the induced distribution over trajectories has large concentration on those high return trajectories. Given a trained model, to decode the most likely trajectory of horizon $T$ generated by the policy, we need to run a full tree search of depth $T$, which is infeasible in practice. Taking the greedy action is equivalent to applying a greedy strategy in decoding the most likely trajectory.

This approach is highly related to beam search in sequence modeling [20] where the goal is to decode the prediction that the model assigns the most likelihood to. The greedy action selection above corresponds to a beam search with 1-step lookahead.