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ON THE UNIVERSAL REGULAR HOMOMORPHISM IN CODIMENSION 2

BRUNO KAHN

Abstract. We point out a gap in Murre’s proof of the existence of a universal regular homomorphism for codimension 2 cycles on a smooth projective variety, and offer two arguments to fill this gap.

In [11], Jacob Murre shows the existence of a universal regular homomorphism for algebraically trivial cycles of codimension 2 on a smooth projective variety over an algebraically closed field. This theorem has been largely used in the literature, most lately in [1], [7] and [2]; for example, it is essential in [2] for descending the method of Clemens and Griffiths [6] to non-algebraically closed fields, thus allowing Benoist and Wittenberg to obtain new examples of geometrically rational non-rational 3-folds.

Unfortunately its proof contains a gap, but fortunately this gap can be filled, actually by two different methods. This is the purpose of this note, which is a slight modification of a letter to Murre on December 5, 2018.

Recall the set-up, with the notation of [11]: \( V \) is a smooth projective variety over an algebraically closed field \( k \) and \( A^n(V) \) denotes the group of codimension \( n \) cycles algebraically equivalent to 0 on \( V \), modulo rational equivalence. Following Samuel, given an abelian \( k \)-variety \( A \), a homomorphism

\[
\phi : A^n(V) \to A(k)
\]

is said to be regular if, for any pointed smooth projective \( k \)-variety \((T, t_0)\) and any correspondence \( Z \in CH^n(T \times V) \), the composition

\[
(1) \quad T(k) \xrightarrow{w_Z} A^n(V) \xrightarrow{\phi} A(k)
\]

is induced by a morphism \( f : T \to A \); here \( w_Z \) is the composition

\[
(2) \quad T(k) \to A_0(T) \xrightarrow{Z} A^n(V)
\]
where the first map sends $t$ to $[t] - [t_0]$. (Note that $f$ is then unique, by Zariski density of the rational points in $T$.)

Using fancy language, regular homomorphisms from $A^n(V)$ form a category and a universal regular homomorphism is an initial object of this category, if it exists. This initial object is well-known to exist when $n = 0$, $n = 1$ (the Picard variety) and $n = \dim X$ (the Albanese variety). Murre’s theorem is:

**Theorem 1** ([11, Th. 1.9]). A universal regular homomorphism $\phi_0$ exists when $n = 2$ for any $V$ (of dimension $\geq 2$).

Recall the main steps of his proof. First, given a regular homomorphism $\phi$, its image in $A(k)$ is given by the points of some sub-abelian variety $A' \subseteq A$ [11, Lemma 1.6.2 i)]. From this, one deduces [11, Prop. 2.1] that $\phi_0$ exists if and only if $\dim A$ is bounded when $\phi$ runs through the surjective regular homomorphisms. Now, Murre’s key idea is to bound $\dim A$ by the torsion of $A^2(V)$, which is controlled by the Merkurjev-Suslin theorem (Bloch’s observation).

Let us elaborate a little on this point, to avoid the $l$-adic argument of loc. cit.: it suffices to prove that $\phi$ induces a surjection

$$A^2(V)\{l\} \twoheadrightarrow A(k)\{l\}$$

for some prime $l \neq \text{char} \ k$, where $M\{l\}$ denotes the $l$-primary torsion of an abelian group $M$: indeed, $\text{corank} A(k)\{l\} = 2 \dim A$. Mainly by Merkurjev-Suslin (Diagram in [11, Prop. 6.1])$^1$,

$$\text{corank} CH^2(V)\{l\} \leq \text{corank} H^3_{\acute{e}t}(V, \mathbb{Q}_l/\mathbb{Z}_l(2))(= b_3(V))$$

so the same holds a fortiori for $\text{corank} A^2(V)\{l\}$.

Now, in [11, Lemma 1.6.2 ii)], Murre constructs an abelian variety $B$ (pointed at 0) and a correspondence $Z \in CH^2(B \times V)$ such that (1) is surjective for $T = B$. Since this map is induced by a morphism of abelian varieties sending 0 to 0 (hence a homomorphism), it restricts to a surjection

$$B\{l\} \twoheadrightarrow A\{l\}.$$ (4)

This allows me to explain the gap:

A priori (4) does not imply (3), because $w_Z$ is in general only a set-theoretic map, not a group homomorphism (see e.g. [4, Th. (3.1) a)]).

We now fix a surjective regular homomorphism $\phi$ as above. We shall give two ways to fill this gap:

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$^1$One could replace this diagram by the injection $CH^2(V) \hookrightarrow H^4_{\acute{e}t}(V, \Gamma(2))$ of [9, Th. 2.13 (c)], together with the surjection $H^3_{\acute{e}t}(V, \mathbb{Q}_l/\mathbb{Z}_l(2)) \twoheadrightarrow H^3_{\acute{e}t}(V, \mathbb{Q}_l(2))\{l\}$, cf. loc. cit., proof of Th. 2.15; here, $\Gamma(2)$ is Lichtenbaum’s complex.
(A) construct \((B, Z)\) such that \(w_Z\) is a homomorphism;
(B) prove that \(w_Z\) always sends torsion to torsion.

(A) was my initial idea, and (B) was inspired by a discussion with Murre.

**Explanation of (A).** We have

**Lemma 1.** Take \((T, t_0, z)\) with \(T\) of dimension 1 and \(z \in CH^2(T \times V)\). Let \(J = J(T)\) be the jacobian of \(T\). Then the homomorphism \(\alpha_\ast : A_0(T) = J(k) \to A^2(V)\) is of the form \(w_\alpha\) for some correspondence \(\alpha \in CH^2(J \times V)\) (using \(0 \in J(k)\) as base point).

**Proof.** Let \(g\) be the genus of \(T\). Recall from [10, Ex. 3.12] the universal relative Cartier divisor \(D_{\text{can}}\) on \(T \times T^{(g)}/T^{(g)}\), parametrising the effective divisors of degree \(g\) on \(T\). It defines a correspondence \(D_{\text{can}} : T^{(g)} \to T\). Composing with the graph of the birational map \(J \dashrightarrow T^{(g)}\) inverse to \((t_1, \ldots, t_g) \mapsto \sum t_i - gt_0\), we find a (Chow) correspondence \(D : J \to T\). I claim that \(\alpha = z \circ D\) answers the question. Indeed, one checks immediately that the homomorphism \(D_\ast : A_0(J) \to A_0(T)\) is the Albanese morphism for \(J\); hence the composition

\[
J(k) \to A_0(J) \xrightarrow{D_\ast} A_0(T)
\]

is the identity. \(\square\)

**Remark 1.** On the other hand, the morphism \(T \to A\) given by the regularity of \(\phi\) factors through a homomorphism

\[
(5) J(T) \to A.
\]

This homomorphism coincides with the one underlying \(\phi \circ z_\ast\) in view of Lemma 1. Indeed, by uniqueness, it suffices to see that (5) induces \(\phi \circ z_\ast\) on \(k\)-points; this is clear since \(T(k)\) generates \(J(T)(k)\) as an abelian group.

Consider all triples \((T, t_0, z)\) with \(\dim T = 1\). The homomorphism \(\bigoplus A_0(T) \xrightarrow{(z_\ast)} A^2(V)\) is surjective, hence so is \(\bigoplus A_0(T) \xrightarrow{(z_\ast)} A^2(V) \to A(k)\). As in Remark 1, each summand of this homomorphism is induced by a homomorphism \(\rho_{T,t_0,z} : J(T) \to A\), so

\[
B := \prod_{(T, t_0, z) \in S} J(T) \xrightarrow{(\rho_{T,t_0,z})} A
\]

is surjective (faithfully flat) for a suitable finite set \(S\). For each \((T, t_0, z)\), let \(\alpha = \alpha_z\) be a correspondence given by Lemma 1. Write \(\pi_{T,t_0,z} : B \to A\).
$J(T)$ for the canonical projection, viewed as an algebraic correspondence. The pair given by $B$ and $Z = \sum_{(T,t_0,z)} \alpha_z \circ \pi_{T,t_0,z}$ yields (A).

**Explanation of (B).** It suffices to show that the map

$$f : B(k) \to A_0(B)$$

sends $l$-primary torsion to $l$-primary torsion. Let $d = \dim B$. By Bloch’s theorem [4, Th. (0.1)], we have $A_0(B)^{(d+1)} = 0$, where $*$ denotes Pontrjagin product. In other words, $f$ has “degree $\leq d$” in the sense that its $(d+1)$-st deviation [8, §8] is identically $0$. It remains to show:

**Lemma 2.** Let $f : M \to N$ be a map of degree $\leq d$ between two abelian groups, such that $f(0) = 0$. Let $m_0 \in M$ be an element such that $am_0 = 0$ for some integer $a > 0$. Then

$$a^{(d+1)} f(m_0) = 0.$$

**Proof.** Induction on $d$. The case $d = 1$ is trivial. Assume $d > 1$. By hypothesis, the $d$-th deviation of $f$ is multilinear, which implies that the map

$$g_a(m) = f(am) - a^d f(m)$$

is of degree $\leq d - 1$. By induction, $a^{(d)} g_a(m_0) = 0$, hence the conclusion. □

**Remark 2.** Of course, either argument proves more generally the following: the map $\phi : A^n(V)\{l\} \to A(k)\{l\}$ is surjective for any integer $n$, any surjective regular homomorphism $\phi : A^n(V) \to A(k)$ and any prime $l \neq \text{char } k$.

**Remark 3.** In [3, §6, Lemma and Prop. 11], Beauville gives a different proof that $f$ sends torsion to torsion. Moreover, he observes that Roittman’s theorem [13] then implies that the restriction of $f$ to torsion is actually an isomorphism, hence a homomorphism.

If we apply Roittman’s theorem together with Lemma 2, we obtain the following stronger result: if $m, m_0 \in B(k)$ and $m_0$ is torsion, then $f(m + m_0) = f(m) + f(m_0)$. (Fixing $m$, the map $f_m : m' \mapsto f(m + m') - f(m) - f(m')$ is of degree $< d$, hence $a^{(d)} f_m(m_0) = 0$ if $am_0 = 0$ by Lemma 2, and therefore $f_m(m_0) = 0$ by Roittman’s theorem.)

**Some expectation.** The landmark work of Bloch and Esnault [5] yields the existence of 4-folds $V$ over fields $k$ of characteristic 0 such that the $l$-torsion of $A^3(V)$ is infinite (hence its $l$-primary torsion has infinite corank). One example, used by Rosenschon-Srinivas [14] and Totaro [16] and relying on Nori’s theorem [12] and Schoen’s results [15],
is the following: start from the generic abelian 3-fold $A$, whose field of constants $k_0$ is finitely generated over $\mathbb{Q}$; choose an elliptic curve $E/k_0(t)$, not isotrivial with respect to $k_0$, and take $V = A_{k_0(t)} \times E$, $k =$ algebraic closure of $k_0(t)$.

**Conjecture 1.** For this $V$, a universal regular homomorphism on $A^3(V)$ does not exist.

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