A double complex construction and discrete Bogomolny equations

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Abstract We study discrete models which are generated by the self-dual Yang-Mills equations. Using a double complex construction we construct a new discrete analog of the Bogomolny equations. Discrete Bogomolny equations, a system of matrix valued difference equations, are obtained from discrete self-dual equations.

1 Introduction

This work is concerned with discrete model of the $SU(2)$ self-dual Yang-Mills equations described in [11]. It is well known that the self-dual Yang-Mills equations admit reduction to the Bogomolny equations [1]. Let $A$ be an $SU(2)$-connection on $\mathbb{R}^3$. This means that $A$ is an $su(2)$-valued 1-form and we can write

$$A = \sum_{i=1}^{3} A_i(x)dx^i,$$

where $A_i : \mathbb{R}^3 \to su(2)$. Here $su(2)$ is the Lie algebra of $SU(2)$. The connection $A$ is also called a gauge potential with the gauge group $SU(2)$ (see [8] for more details). Given the connection $A$ we define the curvature 2-form $F$ by

$$F = dA + A \wedge A,$$

where $\wedge$ denotes the exterior multiplication of differential forms. Let $\Phi : \mathbb{R}^3 \to su(2)$ be a scalar field (a Higgs field). The Bogomolny equations are a set of nonlinear partial differential equations, where unknown is a pair $(A, \Phi)$. These equations can be written as

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\[ F = \ast d_A \Phi, \]  
\[ (3) \]

where \( \ast \) is the Hodge star operator on \( \mathbb{R}^3 \) and \( d_A \) is the covariant exterior differential operator. This operator is defined by the formula

\[ d_A \Omega = d\Omega + A \wedge \Omega + (-1)^{r+1} \Omega \wedge A, \]

where \( \Omega \) is an arbitrary \( su(2) \)-valued \( r \)-form.

Let now consider the connection \( A \) on \( \mathbb{R}^4 \). We define \( A \) to be

\[ A = \sum_{i=1}^{3} A_i(x) dx^i + \Phi(x) dx^4, \]

\[ (4) \]

where \( A_i \) and \( \Phi \) are independent of \( x^4 \). In other word, the scalar field \( \Phi \) is identified with a fourth component \( A_4 \) of the connection \( A \). It is easy to check that if the pair \((A, \Phi)\) satisfies Equation (3), then the connection (4) is a solution of the self-dual equation

\[ F = \ast F, \]

\[ (5) \]

In fact, the Bogomolny equations can be obtained from the self-dual equations by using dimensional reduction from \( \mathbb{R}^4 \) to \( \mathbb{R}^3 \) [1].

The aim of this paper is to construct a discrete model of Equation (3) that preserves the geometric structure of the original continual object. This mean that speaking of a discrete model, we mean not only the direct replacement of differential operators by difference ones but also a discrete analog of the Riemannian structure over a properly introduced combinatorial object. The idea presented here is strongly influenced by book Dezin [3]. Using a double complex construction we construct a new discrete analog of the Bogomolny equations. In much the same way as in the continual case these discrete equations are obtained from discrete self-dual equations. We continue the investigations [10, 11], where discrete analogs of the self-dual and anti-self-dual equations on a double complex are studied. It should be noted that there are many other approaches to discretisation of Yang-Mills theories. As the list of papers on the subject is very large, we content ourselves by referencing the works [2, 4, 5, 6, 7, 9]. In these papers some other discrete versions of the Bogomolny equations are studied.

2 Double complex construction

The double complex construction is described in [10]. For the convenience of the reader we briefly repeat the relevant material from [10] without proofs. Let the tensor product \( C(n) = C \otimes \ldots \otimes C \) of an 1-dimensional complex \( C \) be a combinatorial model of Euclidean space \( \mathbb{R}^n \). The 1-dimensional complex \( C \) is defined in the following way. Let \( C^0 \) denotes the real linear space of 0-dimensional chains generated by basis elements \( x_i \) (points), \( i \in \mathbb{Z} \). It is convenient to introduce the shift operator
\( \tau \) in the set of indices by

\[
\tau i = i + 1.
\]

We denote the open interval \((x_i, x_{i+1})\) by \(e_i\). We can regard the set \(\{e_i\}\) as a set of basis elements of the real linear space \(C^1\) of 1-dimensional chains. Then the 1-dimensional complex (combinatorial real line) is the direct sum of the introduced spaces \(C = C^0 \oplus C^1\). The boundary operator \(\partial\) on the basis elements of \(C\) is given by

\[
\partial x_i = 0, \quad \partial e_i = x_{i+1} - x_i.
\]

(6)

The definition is extended to arbitrary chains by linearity.

Multiplying the basis elements \(x_i\) and \(e_i\) of \(C\) in various way we obtain basis elements of \(C(n)\). Let \(x_k^{(r)} = x_{k_1} \otimes \ldots \otimes x_{k_r}\), where \(k = (k_1, \ldots, k_r)\) and \(k_i \in \mathbb{Z}\), be an arbitrary \(r\)-dimensional basis element of \(C(n)\). The product contains exactly \(r\) of 1-dimensional elements \(e_k\) and \(n-r\) of 0-dimensional elements \(x_k\). The superscript \((r)\) also uniquely determines an \(r\)-dimensional basis element of \(C(n)\). For example, the 1-dimensional \(e_k^1\) and 2-dimensional \(e_k^{(2)}\) basis elements of \(C(3)\) can be written as

\[
e_k^1 = e_{k_1} \otimes x_{k_2} \otimes x_{k_3}, \quad e_k^2 = x_{k_1} \otimes e_{k_2} \otimes x_{k_3}, \quad e_k^3 = x_{k_1} \otimes x_{k_2} \otimes e_{k_3},
\]

\[
e_k^{12} = e_{k_1} \otimes x_{k_2} \otimes x_{k_3}, \quad e_k^{13} = x_{k_1} \otimes e_{k_2} \otimes e_{k_3}, \quad e_k^{23} = x_{k_1} \otimes e_{k_2} \otimes e_{k_3},
\]

where \(k = (k_1, k_2, k_3)\) and \(k_i \in \mathbb{Z}\).

Now we consider a dual object of the complex \(C(n)\). Let \(K(n)\) be a cochain complex with \(gl(2, \mathbb{C})\)-valued coefficients, where \(gl(2, \mathbb{C})\) is the Lie algebra of the group \(GL(2, \mathbb{C})\). We suppose that the complex \(K(n)\), which is a conjugate of \(C(n)\), has a similar structure: \(K(n) = K \otimes \ldots \otimes K\), where \(K\) is a dual of the 1-dimensional complex \(C\). We will write the basis elements of \(K\) as \(x^i\), \(e^i\). Then an arbitrary basis element of \(K(n)\) is given by \(s^k = s_k^1 \otimes \ldots \otimes s_k^n\), where \(s_k^i\) is either \(x_k^i\) or \(e_k^i\). For an \(r\)-dimensional cochain \(\varphi \in K(n)\) we have

\[
\varphi = \sum_k \sum_r \varphi_k^{(r)} s_k^{(r)},
\]

where \(\varphi_k^{(r)} \in gl(2, \mathbb{C})\). We will call cochains forms, emphasizing their relationship with the corresponding continual objects, differential forms.

We define the pairing operation for arbitrary basis elements \(\epsilon_k \in C(n), s^k \in K(n)\) by the rule

\[
< \epsilon_k, a s^k > = \begin{cases} 0, & \epsilon_k \neq s_k \\ a, & \epsilon_k = s_k \\ \end{cases}, \quad a \in gl(2, \mathbb{C}).
\]

(8)

Here for simplicity the superscript \((r)\) is omitted. The operation \(\varphi[8]\) is linearly extended to cochains.

The operation \(\partial\) induces the dual operation \(d^c\) on \(K(n)\) in the following way:
\[ \langle \partial \varepsilon_k, \alpha^k \rangle = \langle \varepsilon_k, a \partial^c \alpha^k \rangle. \]  

(9)

For example, if \( \varphi \) is a 0-form, i.e. \( \varphi = \sum_k \varphi_k x^k \), where \( x^k = x^{k_1} \otimes \ldots \otimes x^{k_n} \), then

\[ d^c \varphi = \sum_k \sum_{i=1}^n (\Delta_i \varphi_k) e_i^k, \]  

(10)

where \( e_i^k \) is the 1-dimensional basis elements of \( K(n) \) and

\[ \Delta_i \varphi_k = \varphi_{\tau_i k} - \varphi_k. \]  

(11)

Here the shift operator \( \tau_i \) acts as

\[ \tau_i k = (k_1, \ldots, \tau_i k_i, \ldots, k_n). \]

The coboundary operator \( d^c \) is an analog of the exterior differentiation operator \( d \).

Introduce a cochain product on \( K(n) \). We denote this product by \( \cup \). In terms of the homology theory this is the so-called Whitney product. For the basis elements of 1-dimensional complex \( K \) the \( \cup \)-product is defined as follows

\[ x^i \cup x^j = x^i, \quad e^i \cup x^j = e^i, \quad x^i \cup e^j = e^j, \quad i \in \mathbb{Z}, \]

supposing the product to be zero in all other case. By induction we extend this definition to basis elements of \( K(n) \) (see [10] for details). For example, for the 1-dimensional basis elements \( e_i^k \in K(3) \) we have

\[ e_1^k \cup e_2^{i_2 k} = e_{12}^k, \quad e_1^k \cup e_3^{i_3 k} = e_{13}^k, \quad e_2^k \cup e_3^{i_3 k} = e_{23}^k, \]

\[ e_2^k \cup e_1^{i_1 k} = -e_{12}^k, \quad e_3^k \cup e_1^{i_1 k} = -e_{13}^k, \quad e_3^k \cup e_2^{i_2 k} = -e_{23}^k. \]  

(12)

To arbitrary forms the \( \cup \)-product be extended linearly. Note that the coefficients of forms multiply as matrices. It is worth pointing out that for any forms \( \varphi, \psi \in K(n) \) the following relation holds

\[ d^c (\varphi \cup \psi) = d^c \varphi \cup \psi + (-1)^r \varphi \cup d^c \psi, \]  

(13)

where \( r \) is the dimension of a form \( \varphi \). For the proof we refer the reader to [3]. Relation (13) is a discrete analog of the Leibniz rule for differential forms.

Let us now together with the complex \( C(n) \) consider its "double", namely the complex \( \tilde{C}(n) \) of exactly the same structure. Define the one-to-one correspondence

\[ * : C(n) \to \tilde{C}(n), \quad * : \tilde{C}(n) \to C(n) \]  

(14)

in the following way:

\[ * : \delta_k^{(r)} \to \pm \delta_k^{(n-r)}, \quad * : \tilde{\delta}_k^{(r)} \to \pm s_k^{(n-r)}. \]  

(15)
where \( \varepsilon^{(n-r)} = \ast s_{k_1} \otimes \cdots \otimes s_{k_m} \) and \( \ast s_{k_i} = \tilde{e}_k \) if \( s_{k_i} = e_k \) and \( \ast s_{k_i} = \tilde{x}_k \) if \( s_{k_i} = x_k \). We let the plus sign in (15) if a permutation of \((1, \ldots, n)\) with \((1, \ldots, n) \rightarrow ((r), \ldots, (n-r))\) is representable as the product of an even number of transpositions and the minus sign otherwise.

The complex of the cochains \( \tilde{K}(n) \) over the double complex \( \tilde{C}(n) \) has the same structure as \( K(n) \). Note that forms \( \phi \in K(n) \) and \( \tilde{\phi} \in \tilde{K}(n) \) have both the same components. The operation (14) induces the respective mapping

\[
\ast : K(n) \rightarrow \tilde{K}(n), \quad \ast : \tilde{K}(n) \rightarrow K(n)
\]

(16)

by the rule:

\[
< \tilde{c}, \ast \phi > = < \ast \tilde{c}, \phi >, \quad < c, \ast \tilde{\psi} > = < \ast c, \tilde{\psi} >,
\]

where \( c \in C(n) \), \( \tilde{c} \in \tilde{C}(n) \), \( \phi \in K(n) \), \( \tilde{\phi} \in \tilde{K}(n) \). For example, for the 2-dimensional basis elements \( e^k_{ij} \in K(3) \) we have

\[
\ast e^k_{12} = \tilde{e}^k_3, \quad \ast e^k_{13} = - \tilde{e}^k_2, \quad \ast e^k_{23} = \tilde{e}^k_1.
\]

(17)

This operation is a discrete analog of the Hodge star operation. Similarly to the continual case we have

\[
\ast \ast \phi = (-1)^{r(n-r)} \phi
\]

for any discrete \( r \)-form \( \phi \in K(n) \).

Finally, for convenience we introduce the following operation

\[
\tilde{i} : K(n) \rightarrow \tilde{K}(n), \quad \tilde{i} : \tilde{K}(n) \rightarrow K(n)
\]

(18)

by setting \( \tilde{i}s^k_{(r)} = \tilde{s}^k_{(r)} \), \( \tilde{i}s^k_{(r)} = s^k_{(r)} \). It is easy to check that the following hold

\[
\tilde{i} \ast = \ast \tilde{i}, \quad \tilde{i}d\phi = d\tilde{\phi}, \quad \tilde{i}\phi = \tilde{\phi}, \quad \tilde{i} \phi = \phi, \quad \tilde{i} (\phi \cup \psi) = \tilde{i} \phi \cup \tilde{i} \psi,
\]

where \( \phi, \psi \in K(n) \).

### 3 Discrete Bogomolny equations

Let us consider a discrete 0-form \( \Phi \in K(3) \) with coefficients belonging to \( su(2) \). We put

\[
\Phi = \sum_k \Phi_k x^k,
\]

(19)

where \( \Phi_k \in su(2) \) and \( x^k = x^{k_1} \otimes x^{k_2} \otimes x^{k_3} \) is the 0-dimensional basis element of \( K(3) \), \( k = (k_1, k_2, k_3) \), \( k_i \in \mathbb{Z} \).

We define a discrete \( SU(2) \)-connection \( A \) to be
\[ A = \sum_{k}^{3} \sum_{i=1}^{3} A_{i}^{k} e_{i}^{k}, \]  
\[ \text{where } A_{k}^{j} \in su(2) \text{ and } e_{i}^{k} \text{ is the 1-dimensional basis element of } K(3). \]

On account of (7) an arbitrary discrete 2-form \( F \in K(3) \) can be written as follows
\[ F = \sum_{k}^{3} \sum_{i<j}^{3} F_{ij}^{k} e_{ij}^{k} = \sum_{k}^{3} \left( F_{12}^{k} e_{12}^{k} + F_{13}^{k} e_{13}^{k} + F_{23}^{k} e_{23}^{k} \right), \]  
\[ \text{where } F_{ij}^{k} \in gl(2, \mathbb{C}) \text{ and } e_{ij}^{k} \text{ is the 2-dimensional basis element of } K(3). \]

Define a discrete analog of the curvature form (2) by
\[ F = d_{c} A + A \cup A. \]  

By the definition of \( d_{c} \) (9) and using (12) we have
\[ d_{c} A = \sum_{k}^{3} \sum_{i<j}^{3} \left( \Delta_{i} A_{j}^{k} - \Delta_{j} A_{i}^{k} \right) e_{ij}^{k}, \]  
\[ \text{and} \]
\[ A \cup A = \sum_{k}^{3} \sum_{i<j}^{3} \left( A_{i}^{k} A_{j}^{k} - A_{j}^{k} A_{i}^{k} \right) e_{ij}^{k}. \]

Recall that \( \Delta_{i} \) is the difference operator (11). Combining (23) and (24) with (21) we obtain
\[ F_{ij}^{k} = \Delta_{i} A_{j}^{k} - \Delta_{j} A_{i}^{k} + A_{i}^{k} A_{j}^{k} - A_{j}^{k} A_{i}^{k}. \]

It should be noted that in the continual case the curvature form \( F \) takes values in the algebra \( su(2) \) for any \( su(2) \)-valued connection form \( A \). Unfortunately, this is not true in the discrete case because, generally speaking, the components \( A_{i}^{k} A_{j}^{k} - A_{j}^{k} A_{i}^{k} \) of the form \( A \cup A \) in (22) do not belong to \( su(2) \). For a definition of the \( su(2) \)-valued discrete curvature form we refer the reader to [11].

Let us define a discrete analog of the exterior covariant differential operator \( d_{c} \) as follows
\[ d_{c}^{+} \varphi = d^{+} \varphi + A \cup \varphi + (-1)^{r+1} \varphi \cup A, \]
\[ \text{where } \varphi \text{ is an arbitrary } r\text{-form (7) and } A \text{ is given by (20). Then for the 0-form (19) we obtain} \]
\[ d_{c}^{+} \Phi = d^{+} \Phi + A \cup \Phi - \Phi \cup A. \]

Using (10) and the definition of \( \cup \) we can rewritten (26) as follows
\[ d_{c}^{+} \Phi = \sum_{k}^{3} \sum_{i=1}^{3} \left( \Delta_{i} \Phi_{k} + A_{i}^{k} \Phi_{\tau_{k}} - \Phi_{\tau_{k}} A_{i}^{k} \right) e_{i}^{k}. \]

Applying the operation \( \ast \) (16) to this expression and by (17) we find
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\[ * d^c A \Phi = \sum_k \left( \Delta_1 \Phi_k + A^1_k \Phi_{\tau^1 k} - \Phi_k A^1_k \right) \hat{e}^k_{12} \]

\[ - \sum_k \left( \Delta_2 \Phi_k + A^2_k \Phi_{\tau^2 k} - \Phi_k A^2_k \right) \hat{e}^k_{13} \]

\[ + \sum_k \left( \Delta_3 \Phi_k + A^3_k \Phi_{\tau^3 k} - \Phi_k A^3_k \right) \hat{e}^k_{23}. \tag{28} \]

Now suppose that \( \Phi \) in the form (19) is a discrete analog of the Higgs field. Then the discrete analog of the Bogomolny equation (3) is given by the formula

\[ F = \tilde{i} * d^c A \Phi, \tag{29} \]

where \( \tilde{i} \) is the operation (17). From (21) and (28) it follows immediately that Equation (29) is equivalent to the following difference equations

\[ F^{12}_k = \Delta_3 \Phi_k + A^3_k \Phi_{\tau^3 k} - \Phi_k A^3_k, \]

\[ F^{13}_k = -\Delta_2 \Phi_k + A^2_k \Phi_{\tau^2 k} + \Phi_k A^2_k, \]

\[ F^{23}_k = \Delta_1 \Phi_k + A^1_k \Phi_{\tau^1 k} - \Phi_k A^1_k. \tag{30} \]

Consider now the discrete curvature form (22) in the 4-dimensional case, i.e. \( F \in K(4) \). The discrete analog of the self-dual equation (5) can be written as follows

\[ F = \tilde{i} * F. \tag{31} \]

By the definition of \( * \) for the 2-dimensional basis elements \( \hat{e}^k_{ij} \in K(4) \) we have

\[ * \hat{e}^k_{12} = \hat{e}^k_{24}, \quad * \hat{e}^k_{13} = -\hat{e}^k_{24}, \quad * \hat{e}^k_{14} = \hat{e}^k_{23}, \]

\[ * \hat{e}^k_{23} = \hat{e}^k_{14}, \quad * \hat{e}^k_{24} = -\hat{e}^k_{13}, \quad * \hat{e}^k_{34} = \hat{e}^k_{12}. \]

Using this we may compute \( *F \):

\[ *F = \sum_k \left( F^{12}_k \hat{e}^k_{24} - F^{13}_k \hat{e}^k_{24} + F^{14}_k \hat{e}^k_{23} + F^{23}_k \hat{e}^k_{14} - F^{24}_k \hat{e}^k_{13} + F^{34}_k \hat{e}^k_{12} \right). \]

Then Equation (31) becomes

\[ F^{12}_k = F^{34}_k, \quad F^{13}_k = -F^{24}_k, \quad F^{14}_k = F^{23}_k. \tag{32} \]

Let the discrete connection 1-form \( A \in K(4) \) be given by

\[ A = \sum_{k} \sum_{i=1}^{3} A^i_k \hat{e}^i_{k} + \sum_k \Phi_k \hat{e}^k_{4}, \tag{33} \]

where \( A^i_k \in su(2), \Phi_k \in su(2) \) and \( k = (k_1, k_2, k_3, k_4), \quad k_i \in \mathbb{Z} \). Note that here we put \( A^4_k, \Phi_k \) are the components of the discrete Higgs field. Suppose that the connection form (33) is independent of \( k_4, \) i.e.
\[ \Delta_4 A^i_k = 0, \quad \Delta_4 \Phi_k = 0 \]  \hspace{1cm} (34)

for any \( i = 1, 2, 3 \) and \( k = (k_1, k_2, k_3, k_4) \). Substituting (34) into (25) yields

\[ F_{ik}^i = \Delta_i \Phi_k + A^i_k \Phi_{\tau k} - \Phi_k A^i_{\tau k}, \quad i = 1, 2, 3. \]

Putting these expressions in Equations (32) we obtain Equations (30).

Thus, if the component \( A^i_k \) of \( A \) is identified with \( \Phi_k \) for any \( k = (k_1, k_2, k_3, k_4) \), \( k_i \in \mathbb{Z} \), then the discrete Bogomolny equations and the discrete self-dual equations are equivalent.

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