Accuracy of the laminar boundary layer on a flat plate in an immersed boundary–lattice Boltzmann simulation

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Abstract
We investigate the accuracy of the laminar boundary layer on a flat plate in the simulation by an immersed boundary–lattice Boltzmann method (IB-LBM). In this study, we use the single relaxation time-lattice Boltzmann method combined with the multi direct forcing method, which can enforce the no-slip boundary condition accurately by determining the body force iteratively. The simulations of the laminar boundary layer on a flat plate at the Reynolds number of 1000 are performed by using the IB-LBM, and it is found that in order to obtain a reasonably accurate result such that the error from the solution of the boundary layer equations is within 5%, the boundary layer has to be resolved by about 50 lattice spacings. In addition, it is found that the IB-LBM has the same accuracy whether the flat plate is coincident with the lattice or not. In order to improve the accuracy of the boundary layer calculated by the IB-LBM, we discuss the effective thickness of the boundary caused by the body force distributed near the boundary. Also, we find that the accuracy is much improved by using a finer lattice only around the flat plate.

Key words: Immersed boundary method, Lattice Boltzmann method, Laminar boundary layer, No-slip boundary condition, Single relaxation time, Direct forcing

1. Introduction

One of important issues in computational fluid dynamics is to satisfy the no-slip boundary condition on boundaries immersed in viscous fluid accurately and efficiently. Body-fitted or unstructured-grid methods might be the best ways to accurately satisfy the no-slip boundary condition. However, the algorithms of the methods are generally complicated, and also the computation costs of the methods are high, especially for boundaries with complex geometry and for moving boundaries. On the other hand, the immersed boundary method (IBM), which was proposed by Peskin (1972, 1977) in 1970s in order to simulate blood flows in the heart, is an efficient method for satisfying the no-slip boundary condition. The IBM can enforce the no-slip boundary condition on arbitrary-shaped boundaries immersed in the fixed Cartesian grid by applying an appropriate body force near the boundaries. While the accuracy of the IBM is somewhat inferior to that of the body-fitted or unstructured-grid methods, the efficiency of the IBM is so attractive that a lot of variations of the IBM have been proposed and successfully applied to various practical problems. Various approaches and applications using the concept of the IBM were reviewed by Mittal and Iaccarino (2005).

As for the computation of viscous fluid flows, the lattice Boltzmann method (LBM) has been developed into an alternative and promising numerical scheme. The main advantage of the LBM is its simple algorithm without solving the Poisson equation for the pressure field (Chen and Doolen, 1998). In addition, the LBM can be easily combined with the IBM, since both methods are based on the fixed Cartesian grid. Therefore, the LBM combined with the IBM (so-called immersed boundary–lattice Boltzmann method, IB-LBM) is expected to be an efficient method for viscous fluid flows with the no-slip boundary condition. In the past decade, a lot of variations of the IB-LBM have been proposed,
and their accuracy and efficiency have been validated successfully through many benchmark problems where analytical solutions, experimental data, and other numerical results are available. Feng and Michaelides (2005) proposed an IB-LBM called Proteus, in which the discrete delta function (see Peskin, 2002) is used for both the interpolation of the velocity and the distribution of the body force, and the method was validated by comparing its results with experimental results of the sedimentation of a sphere by ten Cate et al. (2002). Dupuis et al. (2008) presented the direct forcing approach similar to the Proteus and the interpolating force approach based on the work of Fadlun et al. (2000), and the methods were validated by comparing their results with other numerical results of the flow past an impulsively started cylinder (e.g., Koumoutsakos and Leonard, 1995, Li et al., 2004). Wu and Shu (2009) proposed an implicit velocity correction-based immersed boundary–lattice Boltzmann method which can enforce the no-slip boundary condition exactly, and the numerical results had a good agreement with other numerical results of flows around a circular cylinder and an airfoil (e.g., Dennis and Cheng, 1970, Gresho et al., 1984, Johnson and Tezduyar, 1994). Le and Zhang (2009) proved that the numerical error in the velocity (so called boundary slip) occurs at a large relaxation time by applying the direct forcing approach proposed by Dupuis et al. (2008) to mathematical analyses and numerical simulations of planar and cylindrical Couette flows. Rojas et al. (2011) proposed an immersed boundary–finite difference lattice Boltzmann method and validated through many benchmark problems, i.e., flows past a circular cylinder, a falling particle, interaction between two falling particles, and Couette flows between a stationary cylinder and a rotating one. Suzuki and Inamuro (2011) proposed the LBM combined with the multi direct forcing method proposed by Wang et al. (2008) (hereafter called multi direct forcing–lattice Boltzmann method, MDF-LBM) which can reduce the error from the no-slip boundary condition by calculating the body force iteratively, and the method was validated by comparing their results with numerical results of flows around an oscillating circular cylinder by Dütsch et al. (1998), with numerical results of the sedimentation of an elliptical cylinder by Xia et al. (2009), and with experimental results of the sedimentation of a sphere by ten Cate et al. (2002). In addition, Suzuki et al. (2015) have shown that the MDF-LBM gives good results in comparison with experimental results of an inclined flat plate in a uniform flow by Taira and Collonius (2009) and flapping wings in a stationary fluid by Dickinson et al. (1999).

Although the IB-LBM has been validated widely as mentioned above, there have been few examples where the IB-LBM is applied to an important benchmark problem, i.e., a semi-ininitely long flat plate fixed in a uniform flow (Blasius’s problem). The Blasius’s problem is a well-known classical problem for understanding the property of the laminar boundary layer. It is important to calculate the boundary layer accurately, since the accuracy of the boundary layer should determine the accuracy of the force acting on the boundary and of the flow not only near the boundary but also far from the boundary due to the vortex separation from the boundary layer. Our primary motivation of this study is the following question: How accurately can the boundary layer be calculated by the IB-LBM? In order to answer this question, we calculate the Blasius’s problem by using the MDF-LBM proposed by Suzuki and Inamuro (2011). Since the MDF-LBM has been validated quite extensively in the previous works by Suzuki and Inamuro (2011), Ota et al. (2012), and Suzuki et al. (2015), the MDF-LBM should give a reliable result even for the Blasius’s problem. In addition, the MDF-LBM without iteration is almost the same as other IB-LBMs which have been commonly used by many researchers, e.g., the Proteus proposed by Feng and Michaelides (2005) and the direct forcing approach proposed by Dupuis et al. (2008). Furthermore, while Lin et al. (2011) applied their proposed IB-LBM (direct-forcing pressure-based lattice Boltzmann method) to the Blasius’s problem only in the case where the flat plate coincides with the lattice, the MDF-LBM can easily calculate the problem even in the case where the flat plate does not coincide with the lattice. Therefore, it can be expected that the results of the MDF-LBM should probably be a good reference for users of other IB-LBMs.

The paper is organized as follows. In Section 2, we formulate the system of the Blasius’s problem. We describe the numerical method in Section 3. Results and discussions are given in Sections 4 and 5. We finally conclude in Section 6.

2. Blasius’s problem

In this section, we formulate the Blasius’s problem, which is a well-known classical problem in boundary layer theory (see Schlichting, 1979).

We consider a semi-ininitely long flat plate in two-dimensional space \((x, y)\). The flat plate lies along the \(x\)-axis, and the leading edge of the plate is at \((x, y) = (0, 0)\). We consider steady flow with a free-stream speed \(U_\infty\) which is parallel to the \(x\)-axis. The governing equations of the system in dimensional form are the equation of continuity and the
Navier–Stokes equations for incompressible viscous fluid as follows:

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \]
\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \]
\[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right), \]

where \( u = (u, v) \) is the fluid velocity, \( p \) is the pressure, \( \rho \) is the density of the fluid, and \( \nu \) is the kinematic viscosity of the fluid.

The equation of continuity (4) is satisfied identically by introducing a stream function with the velocity

\[ \psi = \sqrt{\nu U_{\infty} x} f(\eta), \]

where \( f(\eta) \) denotes the dimensionless stream function of a variable \( \eta \) defined by

\[ \eta = \frac{y}{x} \left( \frac{y}{x} \right)^{1/2}, \quad Re_x = \frac{U_{\infty} x}{\nu}. \]

By using \( f(\eta) \), Eq. (5) reduces to the following equation (Blasius's equation):

\[ 2 \frac{d^3 f}{d\eta^3} + f \frac{d^2 f}{d\eta^2} = 0. \]

The boundary conditions become \( f(0) = 0, df/d\eta(0) = 0, \) and \( df/d\eta \to 1 \) as \( \eta \to \infty \).

3. Numerical method

In this study, we calculate not the approximate equations (4) and (5) but the original Navier–Stokes equations (1)–(3) by the MDF-LBM. Although the exact solution of the system of the Navier–Stokes equations for this problem has not been known, the solution of the Blasius’s equation (9) is well known and should be a very good approximation to the exact solution. Therefore, we compare the numerical results with the solution of the Blasius’s equation by Howarth (1938).

In this section, we describe the lattice Boltzmann method (LBM) and the multi direct forcing method (MDFM). Although the present MDF-LBM is the same method as presented in the previous work by Suzuki and Inamuro (2011), we describe it again for helping readers to understand the results well. Hereafter, we use non-dimensional variables as explained in the following Section 3.1, although the same notations as in Section 2 are used.

3.1. Lattice Boltzmann method

In the LBM, a modeled gas, which is composed of identical particles whose velocities are restricted to a finite set of vectors, is considered (see Chen and Doolen, 1998). In this study, we use the single relaxation time-lattice Boltzmann method with the two-dimensional nine velocity vectors (D2Q9 model). The D2Q9 model has the velocity vectors \( c_i (i = 0, 0, (0, \pm 1), (\pm 1, 0), (\pm 1, \pm 1)) \) for \( i = 1, 2, \cdots, 9 \). The evolution of the particle distribution function \( f_i(x, t) \) with the velocity \( c_i \) at a lattice point \( x \) and time \( t \) is computed by the following equation:

\[ f_i(x + c_i \Delta t, t + \Delta t) = \frac{1}{\tau} \left[ f_i(x, t) - f_{eq}^i(p(x, t), u(x, t)) \right], \]

where \( \tau \) is the relaxation time.
where $\Delta x$ is a lattice spacing, $\Delta t$ is the time step during which the particles travel one lattice spacing, $f_i^{eq}$ is an equilibrium distribution function, $\tau$ is a relaxation time of $O(1)$, and $p$ and $u$ are the pressure and the fluid velocity, respectively, given below. In Eq. (10), $x$ is a non-dimensional position normalized by a characteristic length $H_0$, $t$ is a non-dimensional time normalized by a diffusive time scale $t_0 = H_0/U_0$ where $U_0$ is a characteristic flow speed, and $c_i$ is a non-dimensional particle velocity normalized by a characteristic particle speed $c$. In the Blasius’s problem, $U_0$ is the same order of magnitude as the free-stream speed $U_\infty$, and $H_0$ is the same order of magnitude as the $x$-distance over which $u$ changes by amount of $U_0$. In addition, we assume that $U_0/c$ is of $O(\Delta x)$. Note that $\Delta t = Sh \Delta x$ where $Sh = H_0/(t_0c) = U_0/c = O(\Delta x)$. The equilibrium distribution function $f_i^{eq}$ of the incompressible model (He and Luo, 1997) is given by

$$f_i^{eq}(p, u) = E_i \left[ 3p + 3c_i \cdot u + \frac{9}{2}(c_i \cdot u)^2 - \frac{3}{2}u \cdot u \right],$$  

where $E_i = 4/9$, $E_2 = \cdots = E_5 = 1/9$, and $E_6 = \cdots = E_9 = 1/36$. The pressure and the fluid velocity are calculated by

$$p = \frac{1}{3} \sum_{i=1}^{9} f_i,$$  

$$u = \frac{9}{3} \sum_{i=1}^{9} f_i c_i.$$  

It is found that the asymptotic expansions of $u$ and $p$ with respect to $\Delta x$ can be expressed by $u = (\Delta x)u^{(1)} + (\Delta x)^2u^{(2)} + (\Delta x)^3u^{(3)} + \cdots$ and $p = 1/3 + (\Delta x)^2 p^{(2)} + (\Delta x)^3 p^{(3)} + (\Delta x)^4 p^{(4)} + \cdots$, and $u^{(1)}$ and $p^{(2)}$ satisfy the system of the Navier–Stokes equations (1), (2), and (3) for incompressible viscous fluid with the kinematic viscosity $\nu$ given by

$$\nu = \frac{1}{3} \left( \tau - \frac{1}{2} \right) \Delta x,$$  

while $u^{(2)}$ and $p^{(3)}$ are zero with appropriate initial and boundary conditions (see Junk et al., 2005). That is, the solutions of Eqs. (10)–(13) give the pressure and the fluid velocity for incompressible viscous fluid flows with relative errors of $O((\Delta x)^2)$ (see Inamuro et al., 1997).

When an external body force $g(x, t)$ is applied, the evolution equation (10) of the particle distribution function $f_i(x, t)$ can be calculated in a stepwise fashion as follows:

(i) $f_i(x, t)$ is evolved without the body force by

$$f_i^{\prime}(x + c_i \Delta x, t + \Delta t) = f_i(x, t) - \frac{1}{\tau} \left[ f_i(x, t) - f_i^{eq}(p(x, t), u(x, t)) \right].$$  

(ii) $f_i^{\prime}$ is corrected by the body force:

$$f_i(x, t + \Delta t) = f_i^{\prime}(x, t + \Delta t) + 3\Delta x E_i c_i \cdot g(x, t + \Delta t).$$  

3.2. Multi direct forcing method

In the immersed boundary method (IBM), body forces are applied on lattice points near the boundary in order to enforce the no-slip condition on the boundary. The scheme for determining the body forces is different among variations of the IBM. In this study, we use the multi direct forcing method (MDFM) proposed by Wang et al. (2008).

Supposing that $f_i(x, t)$, $u(x, t)$, and $p(x, t)$ are known, the temporal $f_i^{\prime}(x, t + \Delta t)$ and $u_i^{\prime}(x, t + \Delta t)$ can be calculated by Eqs. (15) and (13), respectively. Let $X_k(t + \Delta t)$ and $U_k(t + \Delta t)$ ($k = 1, 2, \cdots, N$) be the points on the flat plate and the
The weighting function

\[ u^*(x, t + \Delta t) = \sum_x u(x, t + \Delta t) W(x - X_k) (\Delta x)^2, \]

(17)

where \( \sum_x \) describes the summation over all the lattice points \( x \) and \( W \) is a weighting function proposed by Peskin (2002).

The body force \( g(x, t + \Delta t) \) is determined by the following iterative procedure.

**Step 0.** Compute the initial value of the body force at the boundary points by

\[ g_0(X_k, t + \Delta t) = Sh \frac{U_k - u^*(X_k, t + \Delta t)}{\Delta t}, \]

(20)

where it is noted that \( Sh/\Delta t = 1/\Delta x \) as defined in Section 3.1.

**Step 1.** Compute the body force at the lattice points of the \( \ell \)-th iteration by

\[ g_\ell(x, t + \Delta t) = \sum_{k=1}^N g_\ell(X_k, t + \Delta t) W(x - X_k) \Delta V, \]

(21)

where \( \Delta V = S/N \times \Delta x \) where \( S \) is the area of the body surface, and \( S/N \) is taken to be approximately equal to \( \Delta x \).

**Step 2.** Correct the velocity at the lattice point by

\[ u_\ell(x, t + \Delta t) = u^*(x, t + \Delta t) + \frac{\Delta t}{Sh} g_\ell(x, t + \Delta t). \]

(22)

**Step 3.** Interpolate the velocity at the boundary point with

\[ u_\ell(X_k, t + \Delta t) = \sum_x u_\ell(x, t + \Delta t) W(x - X_k) (\Delta x)^2. \]

(23)

**Step 4.** Correct the body force with

\[ g_{\ell+1}(X_k, t + \Delta t) = g_\ell(X_k, t + \Delta t) + Sh \frac{U_k - u_\ell(X_k, t + \Delta t)}{\Delta t}, \]

(24)

and go to **Step 1.**

From preliminary computations, we found that \( g_{\ell=5}(x, t + \Delta t) \) is enough to keep the no-slip boundary condition on the boundary points (see Suzuki and Inamuro, 2011). Therefore, we iterate the above procedure until \( \ell = 5 \) in the following computations.

![Illustration of an arrangement of boundary points Xk and lattice points x.](image)
calculation using the bounce-back method can be seen in Appendix A. We define the Reynolds number as
were determined so that the bounce-back method (see Succi, 2001) can give a stable and accurate result. The preliminary
and the velocity determined by the boundary conditions (25). It should be noted that the domain and the conditions
assume all the distribution functions
In the above Neumann conditions, the first-order one-sided differencing approximations are used for the derivatives. We
body forces are added (see Section 3.2) coincide with the lattice points, that is,
\[ \Delta x = \frac{H}{2}, \Delta y = \frac{H}{2} \]
for \( x \) and \( y \) in the boundary conditions (25). It should be noted that the area of \( 8H < x \leq 8H + 3\Delta x \) is a margin for the distributed body force around the trailing edge. The following boundary
conditions are imposed in the boundaries of the domain:
\[ p = p_0, \quad u = U_\infty, \quad v = 0, \quad \text{at } x = -2H, \]
\[ p = p_0, \quad \partial u / \partial x = 0, \quad \partial v / \partial x = 0, \quad \text{at } x = 8H + 3\Delta x, \]
\[ p = p_0, \quad u = U_\infty, \quad \partial v / \partial y = 0, \quad \text{at } y = \pm H, \]
In the above Neumann conditions, the first-order one-sided difference approximations are used for the derivatives. We
we simulate the laminar boundary layer on a flat plate by using the MDF-LBM presented in Section 3 and compare the numerical
results with the solution of the Blasius’s problem.

4. Results

In this study, in order to estimate how accurately the boundary layer can be calculated by the IB-LBM, we simulate
a red line segment.

Table 1 The parameters used in the case where the flat plate coincides with the lattice.

| H   | \( U_\infty \) | \( \epsilon \) |
|-----|---------------|---------------|
| 20\Delta x | 0.1 | 0.524 |
| 40\Delta x | 0.1 | 0.548 |
| 80\Delta x | 0.1 | 0.596 |

4.1. Case where the flat plate coincides with the lattice

First, we consider the case where the flat plate coincides with the lattice as the simplest case. We use a computational
domain with \([-2H, 8H + 3\Delta x] \times [-H, H] \) where \( H \) is the distance between the plate and the top or bottom of the domain
(Fig. 3). The flat plate is set on the line of \( y = 0 \) for \( 0 \leq x \leq 8H \). On the flat plate, the boundary points around which the
body forces are added (see Section 3.2) coincide with the lattice points, that is, \( \Delta V = (\Delta x)^2 \). It should be noted that the
area of \( 8H < x \leq 8H + 3\Delta x \) is a margin for the distributed body force around the trailing edge. The following boundary
conditions are imposed in the boundaries of the domain:

\[ p = p_0, \quad u = U_\infty, \quad v = 0, \quad \text{at } x = -2H, \]
\[ p = p_0, \quad \partial u / \partial x = 0, \quad \partial v / \partial x = 0, \quad \text{at } x = 8H + 3\Delta x, \]
\[ p = p_0, \quad u = U_\infty, \quad \partial v / \partial y = 0, \quad \text{at } y = \pm H, \]

We calculate the velocity profile at the middle point of the flat plate. Fig. 4 shows the flow velocity \((u, v)\) normalized
by the velocity \((U, V)\) at \( y = H \) against the variable \( \eta \) defined by Eq. (8). This figure includes the solution of the Blasius’s
equation by Howarth (1938) and the results of the MDF-LBM. We can see from Fig. 4 that the results of the MDF-LBM
for \( H = 20\Delta x \) have a large error from the solution of the Blasius’s equation. However, the results of the MDF-LBM
gradually tend to the solution of the Blasius’s equation as the resolution increases, and those for \( H = 80\Delta x \) have a
reasonable agreement with the solution. Table 2 shows the thickness of the boundary layer \( \delta = 5.0 \sqrt{\nu x / U_\infty} \) and the errors
of the results of the MDF-LBM from the solution of the Blasius’s equation for each resolution. The error norms in this
table are defined as below:

\[ e_u = \frac{1}{H} \sum_{0 \leq y < H} \frac{|u - u_{ex}|}{u_{ex}} \times 100 \text{ (%)}, \quad e_v = \frac{1}{H} \sum_{0 \leq y < H} \frac{|v - v_{ex}|}{v_{ex}} \times 100 \text{ (%)}, \]

FIG. 3 Computational domain in the case where the flat plate coincides with the lattice. The flat plate is shown as
a red line segment.

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FIG. 3 Computational domain in the case where the flat plate coincides with the lattice. The flat plate is shown as
a red line segment.
Fig. 4 Velocity profiles at the middle point of the flat plate in the case where the flat plate coincides with the lattice for (a) $H = 20\Delta x$, (b) $H = 40\Delta x$, and (c) $H = 80\Delta x$. The solid line indicates the solution of the Blasius’s equation by Howarth (1938) and the circles indicate the results of the MDF-LBM.

Table 2 The errors from the solution of the Blasius’s equation in the case where the flat plate coincides with the lattice.

| $H$   | $\delta$ | $e_u$ | $e_v$ |
|-------|----------|-------|-------|
| $20\Delta x$ | 12.6$\Delta x$ | 13.89 | 25.63 |
| $40\Delta x$ | 25.3$\Delta x$ | 7.391 | 15.32 |
| $80\Delta x$ | 50.6$\Delta x$ | 3.972 | 9.396 |

where $\sum_{0\leq y\leq H}$ means the summation over the lattice points on the line of $x = 4H$ for $0 \leq y \leq H$, $(u_{ex}, v_{ex})$ is the velocity calculated by the MDF-LBM, and $(u_c, v_c)$ is the solution of the Blasius’s equation. We can see from Table 2 that in order to obtain a reasonably accurate result such that the error $e_u$ in the direction of the uniform flow is within 5%, we have to use a quite high resolution, i.e., the boundary layer is resolved by about $50\Delta x$. It should be noted that since the relaxation time $\tau$ is sufficiently small, the boundary slip reported by Le and Zhang (2009) can be neglected. In addition, the compressibility errors due to the weak-compressibility of the LBM (see Ohwada et al., 2011) should be negligibly small, since the result of the Blasius’s problem is steady and consequently the compressibility errors decay and vanish as time advances. The large error in a lower resolution is considered due to a discontinuity of the velocity gradient on the boundary (see Suzuki and Inamuro, 2013).

We calculate the local shear stress along the flat plate. In the MDF-LBM, the local shear stress $\tau_0$ can be calculated
as follows:

\[ \tau_0(x) = -0.5g_x(x, 0) \Delta x, \quad (27) \]

where \( g_x(x, 0) \) is the \( x \)-component of the body force \( g \) of the immersed boundary method (see Section 3.2) at a boundary point \((x, 0)\). It should be noted that \(-g(x, 0) \Delta x\) means the force per unit length around \((x, 0)\) that the flat plate receives from the fluid in both sides of the plate, i.e., \( y > 0 \) and \( y < 0 \). Since the computational domain is symmetric with respect to the \( x \)-axis, the forces of the fluid in \( y > 0 \) and \( y < 0 \) are the same, that is, the half of \(-g(x, 0) \Delta x\) should be the force only from one side of the plate. Therefore, Eq. (27) can be regarded as the local shear stress at \((x, 0)\). The local shear stress coefficient \( C_f \) is defined as below:

\[ C_f(x) = \frac{\tau_0(x)}{\frac{1}{2} \rho U^2 \Delta x}. \quad (28) \]

On the other hand, the local shear stress coefficient based on the solution of the Blasius’s equation is given by

\[ C_{\text{lex}}(x) = \frac{0.664}{\sqrt{Re_x}}. \quad (29) \]

where \( Re_x \) is given by Eq. (8). Fig. 5 shows the local shear stress coefficient \( C_f \) against \( Re_x \). We can see from Fig. 5 (a) that although the results of the MDF-LBM for \( H = 20\Delta x \) have a large error from the solution of the Blasius’s equation, the results of the MDF-LBM gradually tend to the solution of the Blasius’s equation as the resolution increases. However, large oscillations occur around the leading and trailing edges. Although the oscillation tends to be small as the resolution increases, it remains even for a higher resolution \( H = 80\Delta x \) as shown in Fig. 5 (b). One might consider that the oscillations should be removed by increasing the number of the boundary points on the plate. However, even if we increased the number of the boundary points on the plate while fixing the number of the lattice points, the results did not change. The oscillation is attributed to the fact that the leading and trailing edges of a flat plate are the singular points. Actually, in the Blasius problem the stress at the leading edge is infinitely large as shown in Eq. (29). Since the body force for enforcing the no-slip boundary condition in the immersed boundary method is equivalent to the stress as shown in Eq. (27), the body force should be very large at the leading edge. The large body force at the leading edge affects the neighboring boundary point through the streaming of the particle distribution function and through the distribution of the body force. In order to cancel the effect of the large body force at the leading edge, the body force at the neighboring boundary point should be in the opposite direction to that at the leading edge. Therefore, the body force oscillates around the leading edge. In the same way, the body force has an oscillatory behavior around the trailing edge. Table 3 shows the drag coefficient acting on the flat plate defined by

\[ C_D = \frac{1}{8H} \int_0^{8H} C_f(x) dx. \quad (30) \]

It can be seen from Table 3 that the boundary layer has to be resolved by about \( 50\Delta x \) for a reasonably accurate result in terms of the drag coefficient, too.
Table 3  The drag coefficient $C_D$ in the case where the flat plate coincides with the lattice. The values in parentheses in the column of $C_D$ mean the relative error from the solution of the Blasius’s equation by Howarth (1938).

| $H$       | $\delta$     | $C_D$          |
|-----------|--------------|----------------|
| $20\Delta x$ | 12.6$\Delta x$ | $3.268 \times 10^{-2}$ (10%) |
| $40\Delta x$ | 25.3$\Delta x$ | $3.156 \times 10^{-2}$ (6.3%) |
| $80\Delta x$ | 50.6$\Delta x$ | $3.100 \times 10^{-2}$ (4.4%) |
| Howarth (1938) | | $2.969 \times 10^{-2}$ |

4.2. Case where the flat plate does not coincide with the lattice

Next, we consider the case where the flat plate does not coincide with the lattice for examining the effect of the arrangement of the boundary points and the lattice points. We use a computational domain with $[-2H, 8H + 3\Delta x] \times [-1.25H, 1.25H]$. The flat plate is inclined by an angle $\theta$ with its leading edge fixed to (0, 0) (Fig. 6a). The free stream is also inclined so that it can be parallel to the flat plate. The distance between the boundary points is set to $\Delta x$, which is the same as that in the previous case. We define the coordinate system $(x', y')$ so that $x'$ is parallel to the flat plate, i.e., $x' = x \cos \theta + y \sin \theta$ and $y' = -x \sin \theta + y \cos \theta$. Let the $x'$- and $y'$-components of the velocity be $u'$ and $v'$, respectively.

The following boundary conditions are imposed in the boundaries of the domain:

\[
\begin{align*}
  p &= p_0, \quad u' = U_{\infty}, \quad v' = 0, \quad \text{at } x = -2H, \\
  p &= p_0, \quad \partial u'/\partial x = 0, \quad \partial v'/\partial x = 0, \quad \text{at } x = 8H + 3\Delta x, \\
  p &= p_0, \quad u' = U_{\infty}, \quad \partial u'/\partial y' = 0, \quad \text{at } y = 1.25H, \\
  p &= p_0, \quad \partial u'/\partial y = 0, \quad \partial v'/\partial y = 0, \quad \text{at } y = -1.25H,
\end{align*}
\]

In this simulation, we consider the boundary layer on $y' \geq 0$ for $0^\circ \leq \theta \leq 5^\circ$. It should be noted that the boundary conditions in the outlet and the bottom of the domain are not important for the boundary layer on $y' \geq 0$. Fig. 6 (b) shows the positions of the boundary points relative to the lattice points for $H = 20\Delta x$. We can see from this figure that although the range of the angle $\theta$ is very small due to the narrow computational domain, the arrangement of the boundary points relative to the lattice points significantly changes as $\theta$ changes. Therefore, it can be considered that the range of $0^\circ \leq \theta \leq 5^\circ$ is sufficient to investigate the accuracy for any arrangement of the boundary points. All other computational conditions are the same as those in the previous case.

We calculate the velocity profile at the middle point of the flat plate for various angles $\theta$ in the range of $0^\circ$ to $5^\circ$. We arrange the monitoring points by an interval of $\Delta x$ along the line normal to the plate at $x' = 4H$ (see Fig. 6b), and we interpolate the velocity on the monitoring points by Eq. (17). Fig. 7 shows the fluid velocity along the line normal to the plate. Note that variables observed in the inclined coordinate $(x', y')$ have a prime symbol. We can see from Fig. 7 that the results are almost independent of the angle $\theta$. This means that the arrangement of the boundary points relative to the lattice points does not affect the accuracy of the boundary layer calculated by the IB-LBM. It should be noted that
Fig. 7 Velocity profiles at the middle point of the flat plate in the case where the flat plate does not coincide with the lattice for (a) $H = 20\Delta x$, (b) $H = 40\Delta x$, and (c) $H = 80\Delta x$. The solid line indicates the solution of the Blasius’s equation by Howarth (1938) and the symbols indicate the results of the MDF-LBM.

the body force has a small deviation periodically along the $x'$-axis because the area where the body force is distributed changes along the $x'$-axis. The periodical deviation of the body force should affect the velocity profile. However, the effect on the velocity profile should be small and at most comparable with the difference between the velocity profiles for $0^\circ \leq \theta \leq 5^\circ$ shown in Fig. 7, since the body force is distributed on different areas in the present case.

5. Discussions

As shown in Section 4, in order to obtain a reasonably accurate result, we have to use a quite high resolution, i.e., the boundary layer is resolved by about $50\Delta x$. However, it is difficult to use such a high resolution in practical problems such as the sedimentation of many particles. In this section, we discuss how to improve the accuracy of the boundary layer calculated by the IB-LBM.

5.1. Effective thickness of the boundary

In the immersed boundary method, the boundary has an effective thickness due to the body force distributed near the boundary. Although the boundary has no thickness in itself, the effective thickness makes the boundary slightly thicker, i.e., a few lattice spacings. For example, in simulations using the IB-LBM, the diameter of a sphere and that of a circular cylinder are overestimated due to the effective thickness (see ten Cate et al., 2002, Feng and Michaelides, 2009, Krüger et
al., 2011, Suzuki and Inamuro, 2011). The large error in the present simulation of the boundary layer might be attributed to the effective thickness.

Let the boundary have an effective thickness of \( L_\epsilon \), that is, the effective surface of the flat plate is on \( y = 0.5L_\epsilon \). In order to see the results where the effective thickness is eliminated, we simply shift the velocity profiles shown in Fig. 4 by \(-0.5L_\epsilon \) in the \( y \)-direction. Fig. 8 shows the shifted velocity profiles for \( L_\epsilon = 2.5\Delta x \). We can see that the shifted velocity profiles have a good agreement with the solution of the Blasius’s equation both for \( H = 20\Delta x \) and \( 80\Delta x \). This suggests that the effective thickness of the boundary is about \( L_\epsilon = 2.5\Delta x \) independently of the resolution.

One might be interested in the effective thickness for other IB-LBMs. Appendix B shows the results of other IB-LBMs including the sharp direct-forcing IB-LBM proposed by Kang and Hassan (2011), which can represent the boundary sharply. Even for the sharp direct-forcing IB-LBM, the effective thickness exists, though it is smaller than other IB-LBMs.

In order to ensure the accuracy of the boundary layer calculated by the IB-LBM, the effective thickness should be much smaller than the thickness of the boundary layer. For a lower resolution such as \( H = 20\Delta x \), however, the thickness of the boundary layer is \( \delta = 12.6\Delta x \), which is only ten times as large as the effective thickness \( 0.5L_\epsilon = 1.25\Delta x \) on one side of the flat plate. In this sense, it might be obvious that in Section 4.1 the results for a lower resolution have a large error from the solution of the Blasius’s equation.

5.2. Multi-block lattice only around the flat plate

As shown in Section 5.1, the effective thickness of the boundary is only a few lattice spacings and is independent of the resolution. Therefore, the improvement of the accuracy only near the boundary is expected to improve the overall accuracy significantly. In this section, we use the multi-block lattice only near the flat plate and examine how the overall accuracy is improved. The computational process in the multi-block lattice used here is formulated in Inamuro (2012), which is almost the same as that proposed by Yu et al. (2002) and used by Peng et al. (2006).

The computational domain is the same as that in Section 4.1 except that a fine lattice with a lattice spacing \( 0.5\Delta x \) is used in \([-3\Delta x, 8H + 3\Delta x] \times [-3\Delta x, 3\Delta x] \) and a coarse lattice with a lattice spacing \( \Delta x \) is used in the other domain. All other computational conditions are the same as those in Section 4.1.

Fig. 9 shows the flow profiles for \( H = 20\Delta x, 40\Delta x, \) and \( 80\Delta x \). We can see from Fig. 9 that the multi-block lattice only around the flat plate improves the overall accuracy effectively. Table 4 shows the errors of the velocity given by Eq. (26) and the drag coefficient given by Eq. (30). In comparing them with Tables 2 and 3, we can find that the multi-block lattice only around the flat plate is almost equivalent to use of the fine lattice in the whole domain.

6. Conclusions

We have investigated the accuracy of the laminar boundary layer on a flat plate in the simulation by an immersed boundary–lattice Boltzmann method (IB-LBM). In this study, we used the single relaxation time-lattice Boltzmann method combined with the multi direct forcing method, which can enforce the no-slip boundary condition accurately by determining the body force iteratively. We simulated the laminar boundary layer on a flat plate at the Reynolds number of 1000 by using the IB-LBM and compared the numerical results with the solution of the Blasius’s equation.
Fig. 9  Velocity profiles at the middle point of the flat plate in the case where the flat plate coincides with the lattice in the multi-block lattice for (a) $H = 20\Delta x$, (b) $H = 40\Delta x$, and (c) $H = 80\Delta x$. The solid line indicates the solution of the Blasius’s equation by Howarth (1938) and the red squares and the blue circles indicate the results of the MDF-LBM for the single-block lattice and the multi-block lattice, respectively.

Table 4  The errors from the solution of the Blasius’s equation and the drag coefficient $C_D$ in the case where the flat plate coincides with the lattice in the multi-block lattice. The values in parentheses in the columns of $C_D$ mean the relative error from the solution of the Blasius’s equation by Howarth (1938).

| $H$  | $\delta$  | $c_\infty$ | $c_e$ | $C_D$  | $C_D$ (relative error) |
|------|------------|-------------|--------|--------|------------------------|
| $20\Delta x$ | $12.6\Delta x$ | $7.410$ | $15.64$ | $3.162 \times 10^{-2}$ | $(6.5\%)$ |
| $40\Delta x$ | $25.3\Delta x$ | $3.938$ | $9.440$ | $3.115 \times 10^{-2}$ | $(4.9\%)$ |
| $80\Delta x$ | $50.6\Delta x$ | $2.271$ | $6.149$ | $3.084 \times 10^{-2}$ | $(3.4\%)$ |
| Howarth (1938) | | | | $2.969 \times 10^{-2}$ | |

In the case where the flat plate coincides with the lattice, it was found that in order to obtain a reasonably accurate result such that the error from the solution of the Blasius’s equation is within 5%, the boundary layer has to be resolved by about $50\Delta x$. In the case where the flat plate does not coincide with the lattice, it was found that the IB-LBM has the same accuracy whether the flat plate is coincident with the lattice or not.

In order to improve the accuracy of the boundary layer calculated by the IB-LBM, we discussed the effective thickness of the boundary caused by the body force distributed near the boundary, and it was found that the effective thickness of the boundary is about $L_\epsilon = 2.5\Delta x$ independently of the resolution. Also, we found that the multi-block lattice only around the flat plate is almost equivalent to use of the fine lattice in the whole domain.
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Appendix A: Preliminary calculation using the bounce-back method

In lattice Boltzmann simulations, the bounce-back method (see Succi, 2001) is one of the most standard methods for enforcing the no-slip boundary condition on the boundary coincident with the lattice. As a preliminary calculation, we first calculated the laminar boundary layer on a flat plate by using the bounce-back method for determining the domain and the conditions.

The computational domain is \([2H, 8H + 3\Delta x] \times [0, \text{H}]\) and the flat plate is set in the area of \(y = 0\) and \(0 \leq x \leq 8\text{H}\) (Fig. 10), which is the half of the computational domain in Section 4.1. The following boundary conditions are imposed in the boundaries of the domain:

\[
\begin{align*}
p &= p_0, \quad u = U_\infty, \quad v = 0, \quad \text{at } x = -2\text{H}, \\
p &= p_0, \quad \frac{\partial u}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0, \quad \text{at } x = 8\text{H} + 3\Delta x, \\
p &= p_0, \quad u = U_\infty, \quad \frac{\partial v}{\partial y} = 0, \quad \text{Mirror at } y = 0, -2\text{H} < x < 0, \\
\text{No-slip at } y = 0, 0 \leq x < 8\text{H} + 3\Delta x,
\end{align*}
\]

(32)

where ‘Mirror’ and ‘No-slip’ mean the mirror and no-slip boundary conditions, respectively, explained as below. It should be noted that all of the conditions except at \(y = 0\) are the same as those in Section 4.1. The mirror boundary condition is specified in terms of the particle distribution function as follows:

\[
\begin{align*}
f_5 &= f_5, \\
f_6 &= f_9, \\
f_7 &= f_8.
\end{align*}
\]

(33)

The no-slip boundary condition is satisfied by the bounce-back method as follows:

\[
\begin{align*}
f_5 &= f_5, \\
f_6 &= f_8, \\
f_7 &= f_9.
\end{align*}
\]

(34)

We set \(Re_m = 1000\) in this simulation.

Fig. 11 shows the velocity profiles at the middle point of the flat plate calculated by the bounce-back method for \(H = 20\Delta x\). We can see from this figure that the result of the bounce-back method has a good agreement with the solution of the Blasius’s equation by Howarth (1938) even for a low resolution. One might consider that it should be more appropriate to expand the height of the domain and to use the Neumann condition for the pressure on the top and the outlet of the domain, since the free stream should be achieved at infinity. When we used such computational domain and conditions in our preliminary calculations, however, the maximum flow speed quite exceeded the free-stream speed and, in a worse case, the flow field became unsteady. This is because the pressure significantly changes in both \(x\)- and \(y\)-directions, that is, Eq. (12) in Section 2 is violated, unless the pressure of the free stream is enforced near the flat plate. Therefore, in the present study, we use a narrow computational domain and the Dirichlet condition for the pressure on the boundary of the domain.

Fig. 10 Computational domain for the preliminary calculation using the bounce-back method in the case where the flat plate coincides with the lattice. The flat plate is shown as a red line segment.
different methods should enhance the generality of this study. Here, we consider two methods proposed by Kang and Hassan (2011), i.e., the implicit diffuse IB-LBM and the sharp direct-forcing IB-LBM. Although the split-forcing algorithm proposed by Guo et al. (2002). In the sharp direct-forcing IB-LBM, the body force is imposed on solid nodes which are inside the boundary and closest to the boundary. For the details of these methods, please see Kang and Hassan (2011). In the sharp direct-forcing IB-LBM, the body force is imposed in the sharp direct-forcing IB-LBM are on the boundary. Since the boundary coincides with the lattice, the solid nodes where the body force is imposed in the sharp direct-forcing IB-LBM are the same as those for the present MDF-LBM. For comparison, we set the iteration count to 5 times for the implicit diffuse IB-LBM. Since the boundary coincides with the lattice, the solid nodes where the body force is imposed in the sharp direct-forcing IB-LBM are on the boundary. In this section, we compare the results of the present MDF-LBM with those of other IB-LBMs. The comparison of different methods should enhance the generality of this study. Here, we consider two methods proposed by Kang and Hassan (2011), i.e., the implicit diffuse direct-forcing IB-LBM and the sharp direct-forcing IB-LBM. Although the explicit diffuse direct-forcing IB-LBM is similar to the present MDF-LBM, it uses the split-forcing algorithm proposed by Guo et al. (2002). In the sharp direct-forcing IB-LBM, the body force is imposed on solid nodes which are inside the boundary and closest to the boundary. For the details of these methods, please see Kang and Hassan (2011).

We compare these methods in the case where the flat plate coincides with the lattice. The computational domain is the same as that described in Section 4.1, and we set \( H = 40\Delta x \). The boundary points and the weighting function used for the implicit diffuse direct-forcing IB-LBM are the same as those for the present MDF-LBM. For comparison, we set the iteration count to 5 times for the implicit diffuse direct-forcing IB-LBM, which is the same as that for the present MDF-LBM. In addition, we compare the results of the MDF-LBM for the iteration count of \( \ell = 0 \) with those of the explicit diffuse direct-forcing IB-LBM. Since the boundary coincides with the lattice, the solid nodes where the body force is imposed in the sharp direct-forcing IB-LBM are on the boundary.

Fig. 12 shows the comparison of the velocity profiles at \( x = 4H \). We can see from this figure that the present MDF-LBM for the iteration count of \( \ell = 0 \) and \( \ell = 5 \) and the implicit and explicit diffuse direct-forcing IB-LBM give almost the same results, and we cannot distinguish them. On the other hand, the result of the sharp direct-forcing IB-LBM is slightly better than that of the other methods. This suggests that the value of the effective thickness (see Section 5.1) depends on how sharply the boundary is represented. We can see that the effective thickness for the diffuse direct-forcing methods

Table 5 The errors from the solution of the Blasius’s equation and the drag coefficient \( C_D \) calculated from the results of the bounce-back method. The values in parentheses in the column of \( C_D \) mean the relative error from the solution of the Blasius’s equation by Howarth (1938).

| \( H \)     | \( \delta \) | \( e_\alpha \) | \( e_\nu \) | \( C_D \)  |
|-----------|-------------|---------------|-------------|------------|
| 20\Delta x| 12.6\Delta x| 0.5627        | 1.957       | 3.045 \times 10^{-4} (2.6\%) |
| 40\Delta x| 25.3\Delta x| 0.6670        | 2.595       | 3.058 \times 10^{-2} (3.0\%) |
| 80\Delta x| 50.6\Delta x| 0.7045        | 2.742       | 3.057 \times 10^{-2} (2.9\%) |
| Howarth (1938) | | | | 2.969 \times 10^{-2} |

Table 5 shows the errors of the velocity given by Eq. (26) and the drag coefficient given by Eq. (30) for each resolution. In the calculation using the bounce-back method, the local shear stress is calculated by its definition \( \tau_{ij}(x) = \mu \partial u_i/\partial y(x, 0) \), where \( \mu \) is the viscosity, with the second-order one-sided difference approximation. We can find from this table that even if the resolution increases, the errors do not decrease. This is because the solution of the system of the Navier–Stokes equations, which is calculated by the lattice Boltzmann method in the present study, should not exactly be the same as that of the Blasius’s equation due to the simplifications in boundary layer theory. Therefore, we cannot expect that the result of the present simulation converges with the solution of the Blasius’s equation. However, the solution of the Blasius’s equation should be a good approximation of that of the system of the Navier–Stokes equations. The values shown in Table 5 suggest the lower limit of the errors under the simplification.

Appendix B: Comparison with other IB-LBMs

In this section, we compare the results of the present MDF-LBM with those of other IB-LBMs. The comparison of different methods should enhance the generality of this study. Here, we consider two methods proposed by Kang and Hassan (2011), i.e., the implicit diffuse direct-forcing IB-LBM and the sharp direct-forcing IB-LBM. Although the implicit diffuse direct-forcing IB-LBM is similar to the present MDF-LBM, it uses the split-forcing algorithm proposed by Guo et al. (2002). In the sharp direct-forcing IB-LBM, the body force is imposed on solid nodes which are inside the boundary and closest to the boundary. For the details of these methods, please see Kang and Hassan (2011).

We compare these methods in the case where the flat plate coincides with the lattice. The computational domain is the same as that described in Section 4.1, and we set \( H = 40\Delta x \). The boundary points and the weighting function used for the implicit diffuse direct-forcing IB-LBM are the same as those for the present MDF-LBM. For comparison, we set the iteration count to 5 times for the implicit diffuse direct-forcing IB-LBM, which is the same as that for the present MDF-LBM. In addition, we compare the results of the MDF-LBM for the iteration count of \( \ell = 0 \) with those of the explicit diffuse direct-forcing IB-LBM. Since the boundary coincides with the lattice, the solid nodes where the body force is imposed in the sharp direct-forcing IB-LBM are on the boundary.

Fig. 12 shows the comparison of the velocity profiles at \( x = 4H \). We can see from this figure that the present MDF-LBM for the iteration count of \( \ell = 0 \) and \( \ell = 5 \) and the implicit and explicit diffuse direct-forcing IB-LBM give almost the same results, and we cannot distinguish them. On the other hand, the result of the sharp direct-forcing IB-LBM is slightly better than that of the other methods. This suggests that the value of the effective thickness (see Section 5.1) depends on how sharply the boundary is represented. We can see that the effective thickness for the diffuse direct-forcing methods.
such as the present MDF-LBM and the diffuse direct-forcing IB-LBM is almost the same independently of whether implicit or explicit. In addition, the effective thickness for the sharp direct-forcing IB-LBM is slightly smaller than that for the diffuse direct-forcing methods. Therefore, though the effective thickness exists even for the sharp direct-forcing IB-LBM, it can be smaller as the boundary is represented more sharply.

Fig. 13 shows the comparison of the local shear stress coefficient. We can see from Fig. 13 (a) that the present MDF-LBM for the iteration count of \( \ell = 0 \) and \( \ell = 5 \) and the implicit and explicit diffuse direct-forcing IB-LBMs give almost the same results except the leading and trailing edges. Although the result of the sharp direct-forcing IB-LBM is slightly closer to the solution of the Blasius’s equation, it includes a much heavier oscillation around the leading edge compared with the other results as shown in Fig. 13 (b). This is because the body force at the leading edge obtained by sharp direct-forcing IB-LBM is much larger than that by the other methods due to the sharper treatment of the singular point, and consequently its effect on the neighboring boundary points is much heavier.

From the above comparisons, we can conclude that the present MDF-LBM and the diffuse direct-forcing IB-LBM have almost the same accuracy of the boundary layer independently of whether implicit or explicit. Although the sharp direct-forcing IB-LBM gives a more accurate result and a smaller effective thickness than the present MDF-LBM and the diffuse direct-forcing IB-LBM, it suffers a much heavier oscillation of the shear stress around the leading and trailing edges.

![Velocity profiles calculated by the present MDF-LBM with the iteration count of \( \ell = 0 \) and \( \ell = 5 \), the implicit and explicit diffuse direct-forcing IB-LBMs, and the sharp direct-forcing IB-LBM proposed by Kang and Hassan (2011) for \( H = 40\Delta x \). The solid line indicates the solution of the Blasius’s equation by Howarth (1938).](image1)

![The local shear stress coefficient \( C_f \) against \( Re_x \) calculated by the present MDF-LBM with the iteration count of \( \ell = 0 \) and \( \ell = 5 \), the implicit and explicit diffuse direct-forcing IB-LBMs, and the sharp direct-forcing IB-LBM proposed by Kang and Hassan (2011) for \( H = 40\Delta x \) with the solution of the Blasius’s equation by Howarth (1938): (a) for the whole of the flat plate \( (0 \leq Re_x \leq 2000) \); (b) near the leading edge \( (0 \leq Re_x \leq 200) \).](image2)

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