Two problems about calculating the number of palindromes with different numbers of digits

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Abstract. The aim of this work is to obtain two types of regularities for calculating the number of palindromes. The main results of the work are presented by solving two problems. In the first problem describes how to find the total number of different locations of palindromes in binary digits that do not exceed the given numbers of digits. The second problem is devoted to determining the number of palindromes of an arbitrary number system that have the same numbers of digits. The theoretical results of the work are accompanied by examples of the application of the described regularities to obtain the total number of locations of the binary palindromes with the number of digits that do not exceed the specified, as well as to calculate the number of palindromes of different number systems with predetermined numbers of digits. The considered regularities are supposed to be applied in the analysis of numerical and computer data.

1. Introduction
It is known that a natural number that can be read from left to right, as well as from right to left and at the same time it remains unchanged, is called a numerical palindrome. The quantity of numbers in a numerical palindrome can be either even or odd. Note that the location of the same numbers in the palindrome is characterized by a central symmetry [1]. In addition, there are palindrome formulas. These are expressions consisting of the sum or difference of numbers. The result of evaluating these expressions does not change when the expression is read from right to left. For example, the identity $42 + 35 = 53 + 24$ holds true [2]. Moreover, palindromes can be obtained by adding a number and its “inverted” number. For example, the identity $71 + 17 = 88$ holds true. In particular, the author of the article [3] described several ways how to determine the number of mathematical operations for converting a two-digit number into a palindrome. At the same time, it is known that a palindrome is not obtained from arbitrary number after a finite number of mathematical operations. The smallest of these numbers is 196 [4]. The paper [5] describes an algorithm for searching for palindromes in a string.

Numerical palindromes can find application in computer data processing. For example, they are located in the filters of the image processing matrices. These filters are based on convolution matrices containing symmetric rows or columns. For more details on palindromic matrices, see papers [6], [7] и [8]. In [9], the authors consider a particular case of the problems considered here. The article [10] defines the number of palindromes in the binary number system with an even and odd number of digits that have the same predetermined number of digits.
The aim of this work is to obtain two types of formulas for calculating the number of palindromes of the predetermined number of digits. The main results of the work are presented in the form of solving two problems. Both problems describe finding the number of palindromes of the predetermined number of digits. Common to both problems is also a separate consideration of cases with an even and odd order of palindromes.

In the first problem, the regularities are presented that determine the total number of locations of binary palindromes of different discharges in binary digits, limited by the predetermined discharge.

On the contrary, in the second problem, the formulas are given for determining the number of palindromes of an arbitrary number system, but having some number of digits.

The theoretical regularities of this work are accompanied by numerical examples confirming the accuracy of the results obtained when using these regularities in calculating the number of palindromes.

It should be noted that the solution to the first problem was formulated by the first co-author of the article. In this case, the solution to the second problem was obtained by the second co-author of the article.

2. Calculation of the number of binary palindromes that do not exceed a predetermined the numbers of digits

Consider the problem of calculating the number of locations of binary palindromes with different number of digits that do not exceed a predetermined a number of digits equals n. Let us denote the total number of such palindrome locations by \( Q \). Note that two cases should be distinguished: the numbers of digits of the considered palindromes n is odd, or the numbers of digits of the palindromes under consideration n is even.

Let \( n \) be odd. In this case, the total number of binary palindrome locations is

\[
Q = 2^{(n-1)0.5} + \sum_{j=2}^{\frac{n-1}{2}} j \cdot 2^{(n-j)0.5} + \sum_{k=3}^{\frac{n-1}{2}} k \cdot 2^{(n-k)0.5}
\]  

(1)

Here \( j = 2,4,6,...,(n-1) \); \( k = 3,5,7,...,(n-2) \).

Let \( n \) be even. In this case, the total number of binary palindrome locations is

\[
Q = 2^{(n-2)0.5} + \sum_{j=2}^{\frac{n-2}{2}} j \cdot 2^{(n-j)0.5} + \sum_{k=3}^{\frac{n-2}{2}} k \cdot 2^{(n-k)0.5}
\]  

(2)

Here \( j = 2,4,6,...,(n-2) \); \( k = 3,5,7,...,(n-1) \).

Remark 1. It is important to note that the authors were unable to find locations of binary palindromes that would not be included in the number defined by formulas (1) or (2) in the presented problem.

Remark 2. When calculating the number of positions of binary palindromes by means of regularities (1) and (2), palindromes with numbers of digits of at least the second are taken into account. Palindromes consisting only of zeros are also ignored by formulas (1) and (2). For example, formula (1) does not allow to take into account palindromes of the form 010 or 000 with numbers of digits \( n = 3 \).

Remark 3. When calculating the number of positions of binary palindromes using regularities (1) and (2), binary palindromes with fewer characters than the digit under consideration are taken into account only if the positions outside the palindrome contain only zeros. For example, the formula (2) allows to take into account the palindrome 101 in the number 1010. However, the same palindrome 101 is not included in this formula in the number 1011.

Let us give several examples of applying the obtained regularities (1) and (2) to calculate the total number of locations of binary palindromes that do not exceed a given numbers of digits, equal to n.

First of all, we write binary palindromes consisting of several digits, starting with two-digit digits.

A Two-digit palindrome is 11. Three-digit palindromes are 101, 110, 111. Four-digit palindromes are 1001, 1101, 1110, 1111. Five-digit palindromes are 10001, 10101, 11011, 11111, 111111. Six-digit palindromes are 100001, 101101, 110111, 111111.
Example 1. It is required to find the number of binary palindrome locations that do not exceed the palindrome with numbers of digits $n = 3$.

Solution. Since the digit of the number within which the palindromes are recalculated is odd, we use formula (1) to find the number of palindromes locations $Q$. As a result, we get: $Q = 2^1 + 2^1 \cdot 2^{(3-3)/2} = 4$.

Check. Note that the number of required palindrome locations is small. Let us indicate all the locations of palindromes, the number of which is equal to four in this problem. Obviously, these are the following four locations of palindromes: two locations of three-digit palindromes 111, 101, and also twice palindrome 1 is located within three-digit numbers (110 and 011). Finally, we get four locations of binary palindromes, which do not exceed the third-digit palindrome.

Remark 4. Since in this example, the upper limit of the second sum in formula (1) is 3-2=1, and the lower limit in this sum is $k = 3$, then 1 is less than 3. Therefore, when calculating the number of palindrome locations $Q$ in formula (1), no the third term.

Example 2. It is required to find the number of binary palindrome locations that do not exceed the palindrome with with numbers of digits $n = 4$.

Solution. Since the discharge of the number within which the palindromes are recalculated is even, then to find the required number of palindrome locations $Q$, we use formula (2). As a result, we get: $Q = 2^1 + 2^1 \cdot 2^1 + 3^1 \cdot 2^0 = 9$.

Check. Here, too, the number of required palindrome locations is small. Let us indicate all the required locations of palindromes, the number of which is equal to nine in the second example. Obviously, these are the following palindromes: two palindromes: 1001 and 1111 within the fourth discharge. Palindrome 111 is located twice in four-digit numbers: 1110 and 0111. Palindrome 101 is located twice in four-digit numbers: 1010 and 0101. Palindrome 11 is located three times in four-digit numbers: 1100, 0011, 0110. Finally, we get nine palindrome locations that do not exceed the palindrome with numbers of digits $n = 4$.

3. Calculation of the number of palindromes of an arbitrary number system of a predetermined number of digits

Consider the problem of calculating the number of palindromes of an arbitrary number system with a predetermined number of digits equals $n$. Let us denote the number of such palindromes by $S$. It should be noted that the formulas for calculating the number of palindromes for the even and odd number of digits $n$ will differ.

Let the ordinal number of the number of digits $n$ be odd. Then the number of palindromes of an arbitrary number system should be found by the formula:

$$S = (i - 1) \cdot i^{0.5(n-1)},$$

where $i$ is the base of the number system.

Let the ordinal number of the number of digits $n$ be even. In this case, the number of palindromes of an arbitrary number system should be found by the formula:

$$S = (i - 1) \cdot i^{0.5(n-2)}.$$

Remark 5. When calculating the number of palindromes using formulas (4) and (5), the authors assume that only natural numbers are palindromes; that is, for number of digits $n = 1$, the number of palindromes does not include zero.

Remark 6. Regularities (3) and (4) are applicable for $1 \leq i \leq 10$. The case when $i > 10$ is beyond the scope of this article.

Example 3. It is required to find the number of palindromes of the binary number system with the ordinal number of the number of digits $n = 4$.

Solution. To find the number of palindromes, we use formula (4), since the number of digits $n$ is even. The required number system of palindromes is binary, therefore $i = 2$. As a result, we get:

$S = (2-1) 2^{0.5(4-2)} = 2$. 
Let us write down binary four-digit numbers: 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111. It is easy to see that two of them are palindromes: 1001, 1111.

**Example 4.** It is required to find the number of palindromes of the ternary number system with the number of digits \( n = 3 \).

Solution: To find the number of palindromes, we will use formula (3), since the number of digits \( n \) is odd. The required number system for palindromes is ternary, therefore \( i = 3 \). As a result, we get:

\[
S = (3-1) 3^{3/2} (3^0) = 6.
\]

Check. Let us write down ternary three-digit numbers: 100, 101, 102, 110, 111, 112, 120, 121, 122, 200, 201, 202, 210, 211, 212, 220, 221, 222. It is easy to see that six of them are palindromes: 101, 111, 121, 202, 212, 222.

**Example 5.** It is required to find the number of palindromes of the decimal number system with the number of digits \( n = 2 \).

Solution: To find the number of palindromes, we will use formula (4), since the number of digits \( n \) is even. The required number system for palindromes is decimal, so \( i = 10 \). As a result, we get:

\[
S = (10-1) 10^{2/2} (10^0) = 9.
\]

Check. Let us write down decimal two-digit numbers: 10, 11, 20, 22, 30, 33, 40, 44, 50, 55, 60, 66, 70, 77, 80, 88, 90, 99, etc., of which palindromes are nine of them: 11, 22, 33, 44, 55, 66, 77, 88, 99.

4. **Conclusion**

In this work, two types of regularities were formulated for calculating the number of palindromes with different numbers of digits. The authors of the work have solved two different problems. The first task contains finding the total number of locations of binary palindromes of the binary numbers, limited by the bounds of a predetermined number of digits. In the second task, the number of palindromes of an arbitrary number system having the same predetermined numbers of digits is determined. At the same time, the authors found that in both problems for even and odd numbers of digits, the regularities differ from each other. The theoretical results of the work are accompanied by examples of application of the described regularities to obtain the total number of locations of binary palindromes of some different numbers of digits.

In addition, examples of the use of these regularities are given to calculate the number of palindromes of several number systems of the selected discharge. Note that there are no rigorous substantiations of the regularities contained in the work. However, during the numerical simulation, the authors did not find palindromes that are not taken into account in the presented regularities. The authors are convinced that the regularities presented in the work have both scientific and practical significance.

Indeed, the described regularities can be applied both in the analysis of numerical structures in number theory, and in the analysis of computer data written in a binary code. For example, the results of the work can find application in the processing of computer data, in particular, in matrix filters for image processing. It is known that these filters are based on convolution matrices containing symmetric rows or columns.

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