Investigation of the physics phenomena of weakly damped wave equations with forced force: theory and simulation

A Jufriansah\textsuperscript{1,4*}, A Hermanto\textsuperscript{2}, M Toifur\textsuperscript{1} and E Prasetyo\textsuperscript{3,4}

\textsuperscript{1}Magister Physics Education Dept., Ahmad Dahlan University, Yogyakarta Indonesia
\textsuperscript{2}Physics Dept., Gadjah Mada University, Yogyakarta Indonesia
\textsuperscript{3}Science Education Dept., Surabaya State University, Surabaya, Indonesia
\textsuperscript{4}Physics Education Dept. IKIP Muhammmadiyah Maumere, Maumere Indonesia

*Corresponding author: saompu@gmail.com

Abstract. The perturbation method is used to see changes in solutions that occur when attenuation is of little value, as in the phenomenon of waves with weak attenuation and which subjected to coercive force. Therefore this study aims to find solutions to solve the wave equations that experience weak attenuation and which subjected to coercive force. The method used is the study of literature and computation with MatLab. Based on the research results obtained that analytically non-homogeneous waveforms can not provide general solutions for differential resolution. Whereas computationally the results obtained are, the wave model with weak attenuation and the wave model with coercive force have amplitude values that change for time and for time wave models that subjected to coercive force has an amplitude value that increases compared to without coercive force.

1. Introduction
Wave phenomenon is a phenomenon that can be formulated into a mathematical form \cite{1,2}. Mathematical solutions to wave phenomena can use differential solutions \cite{3,4}. In this done determining the dominant factor that will be creat as specific parameters that determined in an event, such as in the field of physics. To overcome this, it can be done by determining unknown boundary values \cite{5}. This physical model must be able to reflect dynamic system characteristics. Where the physical model can be obtained by deriving a mathematical model that connects the system with the input \cite{6}.

Waves and vibrations are two interrelated things, and this can be seen in the phenomenon of sea airwaves, earthquake waves, sound waves that propagate in the air \cite{7}. In connection with the phenomenon of the wave phenomenon, it is known that the phenomenon originates from its vibration. Examples of other phenomena that occur in waves are the phenomenon of weak vibrations, vibrations on a string or a weak swinging rope.

A particular wave system, if continuously vibrated, will form a wave that is perpendicular to the wave direction. In a Closed Wave system, the waves formed will produce stationary waves. According to Ohene, et al. \cite{8} the wave model in the system is

\begin{equation}
m_1 \frac{\partial^2 u(x,t)}{\partial t^2} + T \frac{\partial^2 u(x,t)}{\partial x^2} + b_t \frac{\partial u(x,t)}{\partial t} = 0,
\end{equation}

where the model is a non-linear differential equation \cite{9,10}. 
If the object at the initial condition has experienced vibration, then one day it will stop because it caused by a damping factor or so-called damped vibration [11], this also applies to the case of waves. For the case of waves with weak attenuation can be solved correctly by the Laplace or Fourier transform method [12]. However, the results presented in the form of an integral convolution that cannot provide a suitable solution [13]. Therefore the perturbation method is used to see changes in solutions that occur when attenuation is of little value [14]. It occurs in wave equations with weak attenuation [15,16].

Based on this, the study will discuss the solution of wave equations that experience weak attenuation and which are subjected to force. This research also presents wave visualization using programming languages with MatLab software.

2. Methods

This research is theoretical and computational. Theoretically, this research was conducted using literature studies related to wave equations with weak attenuation that subjected to coercive force. While computationally done by making coding with the Matlab programming language that requires a PC. The steps taken are determining the boundary conditions and initial conditions of the system. The first step is to determine the parameters used to solve analytic wave equations concerning wave equations with weak attenuation and those subject to force. The second stage is determining the boundary conditions \(u(x,0) = 0, u(L,t) = 0\) and at intervals, the third stage iterates with the initial conditions \(u(x,0) = \exp[-0.999(3x-1)^2]\).

At the numerical solution, a parameter value is given to explain the system. The parameters used are mass \(m = 500\) kg, system voltage \(T = m g = 4,900\) N, attenuation coefficient \(b_1 = 0.01\), and gravity \(g = 9.8\) m/s\(^2\). Whereas for weak attenuation given force so that the wave equation form becomes non-homogeneous.

3. Results and Discussion

Based on equation (1) a general solution can be obtained by assuming \(u(x,t) = X(x)T(t)\). If the boundary conditions apply for as long as L whose edges fixed to the given boundary conditions, then the value \(\lambda = \frac{n\pi \sqrt{mg}}{L}\). So by using the value, \(\lambda\) we can get a general solution for equation (1) is

\[
 u(x,t) = \sum_{n=0}^{\infty} \sin \left( \frac{n\pi x}{L} \right) D_n \sin \left( \frac{-b_1 + \sqrt{b_1^2 - 4mT(n\pi/L)^2}}{2m} t \right) + E_n \cos \left( \frac{-b_1 - \sqrt{b_1^2 - 4mT(n\pi/L)^2}}{2m} t \right)
\]

(2)

3.1. Analytical results of weak damping wave equations

The form of a continuous wave equation is presented in equation (1) if given a definition then,

\[
 \frac{\partial u(x,t)}{\partial x} = u(x_j, t_k) = u^k_j
\]

(3)

The transformation of the centre differential for the second derivative for \(t\) is

\[
 \frac{\partial^2 u(x,t)}{\partial t^2} = u(x_j, I_{k+\frac{1}{2}}) = u^k_{j+\frac{1}{2}} - 2u^k_{j+\frac{1}{2}} + u^k_j
\]

(4)

The transformation of the centre differential for the second derivative for \(x\) is
\[
\frac{\partial^2 u(x,t)}{\partial x^2} = u_x \left(x_j, t_{k+1/2} \right) = \frac{u_{j+1}^{k+1} - 2u_{j+1/2}^k + u_j^k}{\Delta x^2}
\]  

(5)

whereas, the forward differential transformation for the second derivative of \( t \) is

\[
\frac{\partial u(x,t)}{\partial t} = u_t \left(x_j, t_{k+1/2} \right) = \frac{u_{j+1/2}^{k+1} - u_{j+1/2}^k}{2\Delta t}
\]  

(6)

so the discrete form of equation (1) for finite difference

\[
b_j \left( u_{j+1}^{k+1} - u_j^k \right) + \beta u_j^k + 2m \frac{u_{j+1}^{k+1}}{\Delta t} = -2m \left( u_{j+1/2}^{k+1} - 2u_{j+1/2}^k + u_j^k \right) - 2T \Delta t \left( u_{j+1}^{k+1} - 2u_{j+1/2}^k + u_j^k \right) + b_j u_j^k + 2m \frac{u_{j+1}^{k+1}}{\Delta t}
\]  

(7)

The simple form of equation (7) if multiplied by \( \Delta t \) is

\[
u_j^{k+1} \left( b_j \Delta t + 2m \right) = -2m \left( -2u_{j+1/2}^{k+1} + u_j^k \right) - 2T \Delta t^2 \left( u_{j+1}^{k+1} - 2u_{j+1/2}^k + u_j^k \right) + b_j \Delta t \frac{u_j^k}{\Delta t}
\]  

(8)

if \( \lambda = \frac{T \Delta t}{\Delta x^2} \) so equation (8) becomes,

\[
u_j^{k+1} = -\frac{2m}{\left( b_j \Delta t + 2m \right)} \left( -2u_{j+1/2}^{k+1} + u_j^k \right) - \frac{2\Delta t}{\left( b_j \Delta t + 2m \right)} \left( u_{j+1}^{k+1} - 2u_{j+1/2}^k + u_j^k \right) + \frac{b_j \Delta t}{\left( b_j \Delta t + 2m \right)} \frac{u_j^k}{\Delta t}
\]  

(9)

If it starts from \( k-1 \), then the \( k \)-iteration is

\[
u_j = -\frac{2m}{\left( b_j \Delta t + 2m \right)} \left( -2u_{j+1/2}^{k+1} + u_j^{k+1} \right) - \frac{2\Delta t}{\left( b_j \Delta t + 2m \right)} \left( u_{j+1}^{k+1} - 2u_{j+1/2}^{k+1} + u_j^{k+1} \right) + \frac{b_j \Delta t}{\left( b_j \Delta t + 2m \right)} \frac{u_j^{k+1}}{\Delta t}
\]  

(10)

3.2. Numerical settlement of weak damping wave equations

\[
500 \frac{\partial^2 u(x,t)}{\partial t^2} + 4900 \frac{\partial^2 u(x,t)}{\partial x^2} + 0.01 \frac{\partial u(x,t)}{\partial t} = 0
\]  

(11)

if selected value \( \Delta t = 0.01 \), \( \Delta x = 0.25 \) then value \( \lambda = \frac{T \Delta t}{\Delta x^2} = 784 \) so equation (11) can be written

\[
u_j = -\frac{2m}{\left( b_j \Delta t + 2m \right)} \left( -2u_{j+1/2}^{k+1} + u_j^{k+1} \right) - 784 \frac{2\Delta t}{\left( b_j \Delta t + 2m \right)} \left( u_{j+1}^{k+1} - 2u_{j+1/2}^{k+1} + u_j^{k+1} \right) + \frac{b_j \Delta t}{\left( b_j \Delta t + 2m \right)} \frac{u_j^{k+1}}{\Delta t}
\]  

(12)

Calculation results from the program are obtained as shown in Figure 1.
Figure 1. Discrete graph for the weak attenuation wave equation

The weak damping wave model shows that at the time $0 < t < 0.1$ the amplitude of the displacement changes, with maximum vibration $-2 \times 10^{-9}$ up to $2 \times 10^{-9}$ them later on at intervals $0.1 < t < 2$ confirmed with stable conditions.

3.3. Analytical results of the weak damping wave equation with force force

If equation (1) converted into an inhomogeneous form, the equation becomes

$$m \frac{\partial^2 u(x,t)}{\partial t^2} + T \frac{\partial^2 u(x,t)}{\partial x^2} + b \frac{\partial u(x,t)}{\partial t} = 1$$ (13)

The discrete form of equation (28) for finite difference transformation is

$$b_i (u_{i+1}^{j+1} - u_j^i) + h u_j^i + 2m \frac{u_{i+1}^{j+1}}{\Delta t} = (2\Delta t) - 2m \left( \frac{u_{i+1}^{j+1} - 2u_j^{j+1} + u_j^i}{\Delta t} \right) - 2T \left( \frac{u_{i+1}^{j+1} - 2u_{j+1}^{j+1} + u_j^{j+1}}{\Delta x^2} \right) + b_i u_j^i + 2m \frac{u_{i+1}^{j+1}}{\Delta t}$$ (14)

The simple form of equation (14) if multiplied by $\Delta t$ is

$$u_j^{j+1} (h_i \Delta t + 2m_i) = (2\Delta t) - 2m_i \left( -2a_j^{j+1} + u_j^i \right) - 2T \Delta t \left( \frac{u_{i+1}^{j+1} - 2u_{j+1}^{j+1} + u_j^{j+1}}{\Delta x^2} \right) + h_i \Delta t u_j^i$$ (15)

if $\lambda = \frac{T \Delta t}{\Delta x^2}$ so the equation (15) becomes,

$$u_j^{j+1} = \frac{2\Delta t}{(h_i \Delta t + 2m_i)} - \frac{2m_i}{(h_i \Delta t + 2m_i)} \left( -2a_j^{j+1} + u_j^i \right) - \lambda \left( \frac{u_{i+1}^{j+1} - 2u_{j+1}^{j+1} + u_j^{j+1}}{\Delta x^2} \right) + \frac{h_i \Delta t}{(h_i \Delta t + 2m_i)} u_j^i$$ (16)

3.4. 1. Numerical settlement of weak damping wave equations with coercive force

$$500 \frac{\partial^2 u(x,t)}{\partial t^2} + 4900 \frac{\partial^2 u(x,t)}{\partial x^2} + 0.01 \frac{\partial u(x,t)}{\partial t} = 1$$ (17)
if selected value $\Delta t = 0.01, \Delta x = 0.25$ then value $\lambda = T\Delta t/\Delta x^2 = 784$ so equation (17) can be written

$$u^i_j = \frac{2\Delta t}{(b_i\Delta t + 2m_j)} \left( \frac{2m_j}{(b_i\Delta t + 2m_j)} \left( -2u^{i-1}_{j-1} + u^{i-1}_j \right) - 784 \frac{2\Delta t}{(b_i\Delta t + 2m_j)} \left( u^{i-1}_{j-1} - 2u^{i-1}_j + u^{i-1}_{j+1} \right) \right) + \frac{b_i\Delta t}{(b_i\Delta t + 2m_j)} u^{i-1}_j$$

Calculation results from the program obtained, as shown in Figure 2.

**Figure 2.** Discrete graph for the weak damping wave equation with forced force

The non-homogeneous wave model in Figure 2 shows that at $0 < t < 0.1$ the time the displacement method changes, with a maximum vibration of -0.07 to 0.09 then at a later time in the interval $0.1 < t < 2$ experiencing a stable condition and the system has increased.

4. Conclusion

Based on the results of research on wave models with theory and simulation, the results obtained, namely the difference of wave models with weak attenuation and wave models that are subjected to coercive force have amplitude values that change for time $0 < t < 0.1$, whereas for time $0.1 < t < 2$ waves that are subjected to coercive force have amplitude values experienced increase compared to wave models with weak attenuation.

5. References

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