HEAT-TRACE ASYMPTOTICS FOR EDGE LAPLACIANS WITH ALGEBRAIC BOUNDARY CONDITIONS

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Abstract. We define a class of algebraic self-adjoint extensions for the Hodge Laplacian on manifolds with incomplete edge singularities. We construct the associated heat kernels on edge manifolds, reviewing the method of signaling solutions by Mooers [Moo99] for isolated conical singularities. The arguments in the conical case do not carry over ad verbatim to the setup of edges. We then establish the heat kernel asymptotics for the algebraic extensions of the Hodge operator on edges, and elaborate on the exotic phenomena in the heat trace asymptotics which appear in case of a non-Friedrichs extension.

1. Introduction

Unusual new phenomena in the heat trace asymptotics in the setup of singular spaces have attracted a considerable interest since the explicit observations by Falomir, Muschietti, Pisani and Seeley in [FMPS03] as well as by Kirsten, Loya and Park in [KLP08], [KLP06] for certain explicit regular-singular operators on a line segment. A general discussion of trace expansions for elliptic cone operators has been provided by Gil, Krainer and Mendoza in [GKM10].

For the Hodge-Laplace operator on a manifold with an isolated conical singularity, new usual phenomena have already been hinted at by Mooers in [Moo99], [Moo96]. However, Mooers did not elaborate in detail on the actual heat trace asymptotics, but rather observed certain unexpected non-polyhomogeneity properties of the heat kernel. The present work is aimed at closing this gap and a derivation of a full heat trace asymptotics for certain self-adjoint extensions of the Hodge-Laplacian in a more general setup of incomplete edge singularities.

A complete characterization of self-adjoint extensions for the Laplacian requires a full scale elliptic theory of edge degenerate operators, see [Maz91] and [Sch91]. However, in this paper we present a class of algebraic boundary conditions, which define self-adjoint realizations of the Hodge-Laplacians on edge manifolds, as already employed by the author jointly with Bahuaud and Dryden [BDV11] in context of semi-classical non-linear parabolic equations on edge manifolds.

We then review and extend the heat kernel construction method by Mooers [Moo99] to the setup of incomplete edge singularities and algebraic boundary conditions, based on the construction of the Friedrichs heat kernel on edges by the author jointly with
Mazzeo in [MaVe12]. Hereby, we present (simpler) alternative arguments at various steps in the construction.

In this general geometric setup we recover the unusual new phenomena in the heat trace asymptotics, observed in [FMPS03], [KLP08], [KLP06] and [GKM10]. It should be noted, however, that there the analysis has been performed irrelated to the earlier work by Mooers [Moo99], and in the first three instances relies on a very specific exact operator structure and Bessel analysis.

This paper is organized as follows. We first review the basic geometry of incomplete edge spaces in §2. We then classify certain algebraic self-adjoint realizations for the Hodge Laplacian in §3 and recall from [MaVe12] the asymptotic properties of the heat kernel for the Friedrichs self-adjoint extension in §4. We study the signaling problem in §5 and §6. The solution to the signaling problem is the central ingredient in the construction of the heat kernel for algebraic self-adjoint boundary conditions, which is explained in §7 and is basically a revision of [Moo99]. Finally, in §8 we derive the heat trace expansion directly from the heat kernel structure.

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Contents

1. Introduction 1
   Acknowledgements 2
2. Hodge Laplacian on incomplete edge spaces 2
3. Algebraic boundary conditions on edges 5
4. Asymptotics of the heat kernel on edge manifolds 8
5. Solution to the model signaling problem 13
6. Solution to the signaling problem 16
7. Heat kernel for algebraic boundary conditions 21
8. Heat-trace asymptotic expansions 24
References 29
List of Figures 30

2. HODGE LAPLACIAN ON INCOMPLETE EDGE SPACES

Consider a compact stratified space $\overline{M}$ with a single top-dimensional stratum $\overline{M}^m$ and only one other lower dimensional stratum $B^b$, which is therefore a smooth closed manifold. By stratification hypothesis there is an open neighbourhood $U \subset \overline{M}$ of $B$ and a radial function $x : U \cap M \to \mathbb{R}$, such that $U \cap M$ is the total space of a smooth bundle over $B$ with fibre $\mathcal{C}(F) = (0, 1) \times F$, an open truncated cone over a compact
smooth manifold $F^I$. The restriction of $x$ to each fibre is the radial function of that cone.

Resolution of the startum $B$ in $\overline{M}$ defines a compact manifold $M$ with boundary $\partial M$, where $\partial M$ is the total space of a fibration $\phi : \partial M \to B$ with the fibre $F$. The resolution process is described in detail for instance in [MAZ91]. The neighborhood $U$ lifts to a collar neighborhood $\mathcal{U}$ of the boundary, which is a smooth fibration of cylinders $[0,1) \times F$ over $B$ with the radial function $x$.

**Definition 2.2.** A Riemannian manifold $(M \setminus \partial M, g) := (M, g)$ has an incomplete edge singularity at $B$ if over $\mathcal{U}$ the metric $g$ equals $g = g_0 + h$, where $g_0$ attains the form

$$g_0 \upharpoonright \mathcal{U} \setminus \partial M = dx^2 + x^2 \mathcal{g}^F + \phi^* g^B,$$

where $g^B$ is a Riemannian metric on the closed manifold $B$, $\mathcal{g}^F$ is a symmetric 2-tensor on the fibration $\partial M$ restricting to a Riemannian metric on each fibre $F$, and $|h|_{g_0} = O(x)$ as $x \to 0$.

Similarly to other discussions in the singular edge setup, see [ALB07], [BDV11], [BaVe11] and [MAVE12], we consider a slightly restricted class of edge metrics and require $\phi : (\partial M, g^F + \phi^* g^B) \to (B, g^B)$ to be a Riemannian submersion. If $p \in \partial M$, then the tangent bundle $T_p \partial M$ splits into vertical and horizontal subspaces as $T_p \partial M = T^V_p \partial M \oplus T^H_p \partial M$, where $T^V_p \partial M$ is the tangent space to the fibre of $\phi$ through $p$ and $T^H_p \partial M$ is the orthogonal complement of this subspace. The requirement for $\phi$ to be a Riemannian submersion is the condition that the restriction of the tensor $g^F$ to $T^H_p \partial M$ vanishes. Moreover, we require that the Laplacians associated to $g^F$ at each $b \in B$ are isospectral, and summarize these additional conditions in the definition below.

**Definition 2.2.** Let $(M, g)$ be a Riemannian manifold with an edge metric. This metric $g = g_0 + h$ is said to be admissible if

(i) $\phi : (\partial M, g^F + \phi^* g^B) \to (B, g^B)$ is a Riemannian submersion;

(ii) the Laplacians associated to $g^F$ at each $b \in B$ are isospectral;

(iii) $|h|_{g_0} = O(x^2)$ as $x \to 0$.

In order to explain the reason behind the admissibility assumptions, let $y = (y_1, \ldots, y_b), f = \dim B$ be the local coordinates on $B$ lifted to $\partial M$ and then extended inwards. Let $z = (z_1, \ldots, z_f), f = \dim F$ restrict to local coordinates on $F$ along each fibre. Then $(x, y, z)$ are the local coordinates on $M$ near the boundary. Consider the normal operator $N(x^2 \Delta_p)_{g_0}$, defined as the limit of $x^2 \Delta_p$ on $p$-forms with respect to the local family of dilatations $(x, y, z) \to (\lambda x, \lambda (y - y_0), z)$ as $\lambda \to \infty$. Under the first admissibility assumption, $N(x^2 \Delta_p)_{g_0}$ is naturally identified with $s^2$ times the Hodge Laplacian on $p$-forms on the model edge $\mathbb{R}^+ \times F \times \mathbb{R}^b$ with incomplete edge metric $g_{\text{ne}} = ds^2 + s^2 \mathcal{g}^F + |du|^2$.

The second condition on isospectrality is posed to ensure polyhomogeneity of the associated heat kernels when lifted to the corresponding parabolic blowup space. More precisely we only need that the eigenvalues of the Laplacians on fibres are constant in a fixed range $[0, 1]$, though we still make the stronger assumption for a clear and
convenient representation. The final condition is of technical origin, placed in several other instances, compare [BaVe11] and [MAVe12].

At any hypersurface $S_a = \{s = a\}$ of the model edge, $TS_a \equiv T(\mathbb{R}^b \times F)$ splits into the sum of a ‘vertical’ and ‘horizontal’ subspace, where the first is the tangent space to the $F$ factor and the second is the tangent space to the Euclidean factor $\mathbb{R}^b$. This splitting is orthogonal, and induces a bigrading

\begin{equation}
\Lambda^p(S_a) = \bigoplus_{j+l=p} \Lambda^j(\mathbb{R}^b) \otimes \Lambda^l(F) := \bigoplus_{j+l=p} \Lambda^{j,l}(S_a).
\end{equation}

Let $\Omega^{j,l}(S)$ denote the space of sections of the corresponding summand in this bundle decomposition. We write the normal operator $N(x^2\Delta_p)x_0$ with respect to a rescaling of the form bundles, employed also in [BrSe87]. More precisely, for each $j,l$ with $j + l = p$, we define

$$
\Phi_{j,l} : C_0^\infty(\mathbb{R}^+, \Omega^{j,l-1}(S) \oplus \Omega^{j,l}(S)) \rightarrow \Omega^p_0(\mathbb{R}^b \times \mathcal{C}(F)),
$$

where the lower index refers to the compact support of functions and differential forms, away from $\{x = 0\}$. We denote by $\Phi_p$ the sum of these maps over all $j + l = p$. Like in the special case of conical singularities, the resulting transformation

$\Phi_p : L^2([0, 1], L^2(\bigoplus_{j+l=p} \Omega^{j,l-1}(S) \oplus \Omega^{j,l}(S), \kappa(0) + |du|^2, ds) \rightarrow L^2(\Omega^p(\mathbb{R}^b \times \mathcal{C}(F)), g_{\text{ie}}),$

is an isometry, and a calculation yields

\begin{equation}
\Phi_p^{-1} [s^{-2}N(x^2\Delta_p)] \Phi_p = \left( -\frac{\partial^2}{\partial s^2} + \frac{1}{s^2}(A - 1/4) \right) + \Delta_{\mathbb{R}^b},
\end{equation}

where $A$ is the nonnegative self-adjoint operator on $\Omega^{l-1}(F) \oplus \Omega^l(F)$ given by

\begin{equation}
A = \begin{pmatrix}
\Delta_{l-1,F} + (l - (f + 3)/2)^2 & 2(-1)^l \delta_{l,F} \\
2(-1)^l \delta_{l,F} & \Delta_{l,F} + (l - (f - 1)/2)^2
\end{pmatrix}.
\end{equation}

Under this transformation the indicial roots of $\Delta_p^{\phi}$ have a particularly simple form. Writing the eigenvalues of $A$ as $\nu_j^2, \nu_j \geq 0$, with corresponding eigenform $\phi_j$, the corresponding indicial roots of $\Delta_p$ are given by

\begin{equation}
\gamma_j^+ = \nu_j + \frac{1}{2}, \quad \gamma_j^- = -\nu_j + \frac{1}{2}.
\end{equation}

A similar rescaling $\Phi_p$ using powers of the defining function $x$ makes sense in each local coordinate chart near the boundary $\partial M$. Rescalings on different local coordinate charts are equivalent up to a diffeomorphism.

Conjugating by $\Phi_p$ as before, the rescaled Hodge-Laplacian $\Delta_p^{\Phi}$ is a perturbation of (5.4) with higher order correction terms determined by the curvature of the Riemannian submersion $\phi : \partial M \rightarrow B$ and the second fundamental forms of the fibres $F$. By an abuse of notation we denote the rescaled operator by $\Delta_p$ again, if there is no danger of confusion.
3. Algebraic boundary conditions on edges

The exposition of this section has been presented in the recent work [BDV11] of the author jointly with Bahnaud and Dryden, and inserted here for convenience of the reader and notation fixing. The Hodge Laplacian of an admissible incomplete edge space \((M,g)\) need not be essentially self-adjoint on its core domain \(C^0_0\Omega^p(M)\), and boundary conditions at the edge are posed to obtain self-adjoint extensions.

For spaces with isolated conic singularities, this was first studied by Cheeger [CHE83] using the language of ideal boundary conditions. Further and more systematic studies were presented by Brünig-Lesch [BRLE87], see also [MOO99] and [KLP08] with an appendix by the author, for a full classification of self-adjoint boundary conditions in the conical setting and for results about the associated heat equation.

In case of isolated conical singularities the extension problem is finite dimensional. When the edge \(B\) has positive dimension, the requisite analysis is more intricate and here we can specify only a class of algebraic boundary conditions for the Hodge Laplacian. Let \(\Delta^\Phi_p = \Delta_p\) denote the rescaled Hodge Laplace operator acting on differential forms of degree \(p\) on the incomplete edge space \((M,g)\) with an admissible incomplete edge metric \(g\). Consider the space of square-integrable forms \(L^2\Omega^p(M)\), with respect to \(g\). The maximal and minimal closed extensions of \(\Delta_p\) are defined by the domains

\[
\mathcal{D}_{\text{max}}(\Delta_p) := \{ u \in L^2\Omega^p(M) \mid \Delta_p u \in L^2\Omega^p(M) \},
\]

\[
\mathcal{D}_{\text{min}}(\Delta_p) := \{ u \in \mathcal{D}_{\text{max}}(\Delta_p) \mid \exists u_j \in C^\infty C_0^\infty \Omega^p \text{ such that } u_j \to u \text{ and } \Delta_p u_j \to \Delta_p u \text{ both in } L^2\Omega^p \},
\]

where \(\Delta_p u \in L^2\) is initially understood in the distributional sense.

**Lemma 3.1 ([MAVE12]).** Let \((M,g)\) be an incomplete edge space with an admissible edge metric. Consider the increasing sequence of eigenvalues \(\nu_j^2 \in [0,1), j = 1, \cdots, p\), of the tangential operator \(A\) in \((2.3)\), counted with their multiplicities. The associated indicial roots of \(\Delta_p\) are given by \(\gamma_j^\pm = \pm \nu_j + 1/2\). Any \(u \in \mathcal{D}_{\text{max}}(\Delta_p)\) then admits a weak asymptotic expansion as \(x \to 0\)

\[
\Phi_p^{-1} u \sim \sum_{j=1}^q (c_j^+ [u]|\psi_j^+(x,z;y) + c_j^- [u]|\psi_j^-(x,z;y)) + \bar{u}, \ \bar{u} \in \mathcal{D}_{\text{min}}(\Delta_p),
\]

where the leading order term of each \(\psi_j^\pm\) is the corresponding solution of the indicial operator. More precisely, let \(\phi_j\) denote the normalized \(\nu_j^2\)-eigenforms of the tangential operator \(A\) at \(y\). Then

\[
\psi_j^+(x,z;y) = x^{\gamma_j^+} \phi_j(z;y), \quad \psi_j^-(x,z;y) = \begin{cases} \sqrt{x} (\log x) \phi_j(z;y), & \nu_j = 0, \\ x^{\gamma_j^-} (1 + a_j x) \phi_j(z;y), & \nu_j > 0, \end{cases}
\]

with \(a_j \in \mathbb{R}\) uniquely determined by \(\Delta_p\). The coefficients \(c_j^\pm [u]\) are of negative regularity in \(y\) and the asymptotic expansion holds only in a weak sense, i.e. there is an expansion of the pairing \(\int_B \langle u(x,y,z) | \chi(y) \rangle \text{dy for any test function } \chi \in \Omega^*(B)\).
The set of all closed extensions of $\Delta_p$ is in bijective correspondence with the closed subspaces of the quotient $\mathcal{D}_{\max} / \mathcal{D}_{\min}$; furthermore, since $\Delta_p$ is symmetric on the core domain $C^0_0 \Omega^p(M)$, self-adjoint extensions are in bijective correspondence with the Lagrangian subspaces of this quotient with respect to certain natural symplectic form induced from the boundary contributions in an integration by parts formula. However, in contrary to the case of isolated conical singularities, the expansion in Lemma 3.1 holds only in the weak sense, so that self-adjointness of such domains follows only by the a mollification argument, as we provide below.

We proceed with the definition of algebraic boundary conditions which define a subclass of self-adjoint extensions of $\Delta_p$ by specifying algebraic relations between the coefficients $c_j^\pm$. Note that though we performed the discussion above under rescalings $\Phi_p$ in each local coordinate neighborhood, the partial weak asymptotic expansion of the rescaled solutions in $\mathcal{D}_{\max}(\Delta)$ is invariant under coordinate changes $x'(x, y, z), y'(x, y, z)$ and $z'(x, y, z)$, with $x'(0, y, z) = 0$ and $y'$ lifted from the base, so that $y'((0, y, z)$ is independent of $z$. Thus, any specification of algebraic relations between the coefficients $c_j^\pm$ and hence also the characterization of algebraic boundary conditions for $\Delta_p$ will be globally well-defined.

Lemma 3.1 asserts that the quotient $\Lambda_q = \mathcal{D}_{\max} / \mathcal{D}_{\min}$ can be identified with the $2q$-dimensional vector space spanned by solutions $\{\psi_j^\pm\}_{j=1}^q$. The symplectic form $\omega_q$ on $\Lambda_q$ is defined by $\omega_q(u, v) := (u, \Delta_p v)_{L^2} – (\Delta_p u, v)_{L^2}$ on polyhomogeneous $u, v \in \mathcal{D}_{\max}$, and explicitly amounts to

\begin{equation}
\omega_q(\psi_j^+, \psi^-_j) = -\omega_q(\psi^-_j, \psi_j^+) = c_j = \begin{cases} 2\nu_j, & \nu_j > 0, \\
1, & \nu_j = 0,
\end{cases}
\omega_q(\psi_j^+, \psi_j^+) = \omega_q(\psi^-_j, \psi_j^-) = \omega_q(\psi_i^+, \psi_j^+) = 0.
\end{equation}

Following ([Moo96], Section 7), consider now a $q \times q$-matrix $\Gamma = (\Gamma_{ij}) \in \text{Matr}(q, \Lambda_q)$ with diagonal entries given by $\Gamma_{jj} = b_{ij} \psi_j^+ + \theta_{ij} \psi_j^+$, and the off-diagonal entries $\Gamma_{ij} = \theta_{ij} \psi_j^+$. The coefficients $b_{ij}, \theta_{ij} \in \mathbb{R}$ are such that either $b_{ii} = 1$, or $b_{ii} = 0$, where in the latter case we require $\theta_{ii} = 1$ and $\theta_{ij} = 0$ for $i \neq j$. We refer to such $\Gamma = (\Gamma_{ij}) \in \text{Matr}(q, \Lambda_q)$ as the Lagrangian matrix.

**Definition 3.2.** For any Lagrangian matrix $\Gamma = (\Gamma_{ij}) \in \text{Matr}(q, \Lambda_q)$ we define the associated algebraic domain of the Hodge Laplacian $\Delta_p$ by

$$\mathcal{D}_\Gamma(\Delta_p) := \{u \in \mathcal{D}_{\max}(\Delta_p) | \forall j = 1, \ldots, q: \sum_{j=1}^q \omega_q(c_j^+ [u] \psi_j^+ + c_j^- [u] \psi_j^-, \Gamma_{ij}) = 0\}.$$

The algebraic boundary conditions provide a full characterization of self-adjoint extensions of the Hodge Laplacian on cones, cf. ([Moo96], Section 7) and also [KLP08]. In the setup of incomplete edge, algebraic boundary conditions define only a subclass of self-adjoint extensions. A full classification of self-adjoint extensions of the Hodge Laplacian on incomplete edges requires a detailed analysis of the elliptic theory of edge differential operators and is out of the scope of the present discussion.
Below we provide a proof of self-adjointness of $\mathcal{D}_T(\Delta_p)$. Self-adjoint extensions of $\Delta_p$ are in one to one correspondence with the Lagrangian subspaces of $(\Lambda_q, \omega_q)$. However, standard arguments from the conical setup do not apply here directly, since due to weakness of the asymptotic expansion in Lemma 3.1 the symplectic form $\omega_q$ can be evaluated explicitly only on polyhomogeneous $u, v \in \mathcal{D}_{\text{max}}$ and not on the full $\Lambda_q$. Hence, even symmetry of $\mathcal{D}_T(\Delta_p)$ is not obvious here and requires a mollification argument.

Proposition 3.3. $\mathcal{D}_T(\Delta_p)$ defines a self-adjoint extension of $\Delta_p$.

Proof. We first prove that $\Delta_p$ is symmetric on $\mathcal{D}_T(\Delta_p)$. Consider any solution $w \in \mathcal{D}_T(\Delta_p)$. Let $\phi$ be a cut-off function supported in a local coordinate neighborhood $(x, y, z)$. Then, still $u := w \cdot \phi \in \mathcal{D}_T(\Delta_p)$. Our first observation is that these locally supported elements $u \in \mathcal{D}_T(\Delta_p)$ can be approximated by $\mathcal{D}_T(\Delta_p) \cap \mathcal{A}_{\text{phg}}$-elements in the graph norm. Consider a coefficient $u_I$ of the form-valued $u$, a test function $\psi \in C^\infty(B)$ and write

$$(u_I \ast \psi)(x, y, z) = \int_B u_I(x, y - \tilde{y}, z)\psi(\tilde{y})d\tilde{y}.$$ 

We can assemble functions $u_I \ast \psi$ locally back into a differential form, which we denote by $u \ast \psi$. The convolution $u \ast \psi$ inherits the expansion as $x \to 0$ from $u$, and hence $u \ast \psi \in \mathcal{D}_T(\Delta_p)$. Moreover, due to pairing with $\psi \in C^\infty(B)$, the coefficients $c_j^\pm[u \ast \psi]$ are now smooth in $y$ and hence $u \ast \psi \in \mathcal{D}_T(\Delta_p) \cap \mathcal{A}_{\text{phg}}$. Specify now $\psi$ to be a bump function, compactly supported around the coordinate origin of $(y)$ with $\hat{\psi}(0) = 1$, and consider a sequence $\psi_\epsilon(y) := e^{-b}(y/\epsilon)$. Then with respect to $\| \cdot \|_\infty$

$$\hat{\psi}_\epsilon(\cdot) = \hat{\psi}(\cdot / \epsilon) \to \hat{\psi}(0) = 1, \text{ as } \epsilon \to 0.$$ 

Set $u_\epsilon = u \ast \psi_\epsilon$. Observe for each coefficient $u_{\epsilon,I} = u_I \ast \psi_\epsilon$

$$\|u_{\epsilon,I} - u_I\|_{L^2} = \|u_I(\hat{\psi}(\epsilon \cdot - 1))\|_{L^2} \leq \|\hat{\psi}(\epsilon \cdot - 1)\|_\infty \|u_I\|_{L^2} \to 0, \text{ as } \epsilon \to 0.$$ 

This proves $u_\epsilon \to u$ in $L^2$ as $\epsilon \to 0$. Moreover, $\Delta_p u_\epsilon = (\Delta_p u) \ast \psi_\epsilon$, and hence by exactly the same argument as above, $\Delta_p u_\epsilon \to \Delta_p u$ in $L^2$ as $\epsilon \to 0$. Thus, we can indeed approximate any locally supported $u \in \mathcal{D}_T(\Delta_p)$ by a sequence $(u_\epsilon) \subset \mathcal{D}_T(\Delta_p) \cap \mathcal{A}_{\text{phg}}$ in the graph norm.

By construction of the Lagrangian matrix $\Gamma$, the Hodge Laplacian $\Delta_p$ is symmetric on $\mathcal{D}_T(\Delta_p) \cap \mathcal{A}_{\text{phg}}$ by exactly the same arguments as in the case of an isolated conical singularity, see ([Moo96], Theorem 7.6) and also [KLP08]. This is due to the fact that polyhomogeneous elements admit an asymptotic expansion in the strong sense and integration by parts arguments apply. 

Now symmetry of $\Delta_p$ on $\mathcal{D}_T(\Delta_p)$ is obtained as follows. Consider a partition of unity $(\phi_\alpha)_{\alpha \in A}$ subordinate to coordinate charts of $M$. For any $f, g \in \mathcal{D}_T(\Delta_p)$ we can write

$$\langle \Delta_p f, g \rangle_{L^2} - \langle f, \Delta_p g \rangle_{L^2} = \sum_{\alpha \in A} \langle \langle \Delta_p f, g \cdot \phi_\alpha \rangle_{L^2} - \langle f, \Delta_p (g \cdot \phi_\alpha) \rangle_{L^2} \rangle = 0,$$

where the last equality follows by approximating each $f$ and $g \cdot \phi_\alpha$ locally by elements of $\mathcal{D}_T(\Delta_p) \cap \mathcal{A}_{\text{phg}}$ in the graph norm, and $\Delta_p$ is symmetric on $\mathcal{D}_T(\Delta_p) \cap \mathcal{A}_{\text{phg}}$. 
In order to deduce self-adjointness on $\mathcal{D}(\Delta_p)$ it now suffices to show

$$\mathcal{D}(\Delta^*_p, \Gamma) = \{ f \in \mathcal{D}_{\max}(\Delta_p) \mid \forall g \in \mathcal{D}(\Delta_p) : (\Delta_p f, g)_{L^2} = (f, \Delta_p g)_{L^2} \} \subseteq \mathcal{D}(\Delta_p).$$

Let $f \in \mathcal{D}(\Delta^*_p, \Gamma)$. Then in any local coordinate chart we may consider a locally supported $g \in \mathcal{D}(\Delta_p) \cap \mathcal{A}_{phg}$, with its coefficients $c_j^I[g]$ being smooth with compact support in $\mathbb{R}^b$. Then repeating ([Moo96], Proposition 7.4) ad verbatim, we deduce from $(\Delta_p f, g)_{L^2} = (f, \Delta_p g)_{L^2}$ that the coefficients $c_j^I[f]$ in the weak expansion of $f$ in that coordinate neighborhood, satisfy the algebraic conditions of $\mathcal{D}(\Delta_p)$. Consequently, $f \in \mathcal{D}(\Delta_p)$. \hfill \Box

4. ASYMPTOTICS OF THE HEAT KERNEL ON EDGE MANIFOLDS

In ([MAVe12], Proposition 2.5) jointly with Rafe Mazzeo we have identified the Friedrichs extension of $\Delta_p$ as the algebraic self-adjoint extension associated to $\Gamma = \text{diag}((\psi^1_+, ..., \psi^m_+))$. We denote the Friedrichs extension of the Hodge Laplacian by $\Delta_p^\mathcal{F}$ and explain here the polyhomogeneity properties of its heat kernel $H_\mathcal{F}$ near the edge.

We begin by recalling the definition of conormal and polyhomogeneous distributions on a manifold with corners, see [Mel93] and [Mel92].

**Definition 4.1.** Let $\mathcal{W}$ be a manifold with corners, with embedded boundary faces, and $\{(H_i, \rho_i)\}_{i=1}^N$ an enumeration of its boundaries and the corresponding defining functions. For any multi-index $b = (b_1, ..., b_N) \in \mathbb{C}^N$ we write $\rho^b = \rho_1^{b_1} \cdots \rho_N^{b_N}$. Denote by $\mathcal{V}_b(\mathcal{W})$ the space of smooth vector fields on $\mathcal{W}$ which are tangent to all boundary faces. A distribution $w$ on $\mathcal{W}$ is said to be conormal if $w \in \rho^b L^\infty(\mathcal{W})$ for some $b \in \mathbb{C}^N$ and $V_1 \cdots V_l w \in \rho^b L^\infty(\mathcal{W})$ for all $V_j \in \mathcal{V}_b(\mathcal{W})$ and for every $l \geq 0$. An index set $E_i = \{ (\gamma, p) \} \subset \mathbb{C} \times \mathbb{N}$ satisfies the following hypotheses:

(i) $\Re(\gamma)$ accumulates only at $+\infty$;

(ii) for each $\gamma$ there exists $P_\gamma \in \mathbb{N}_0$ such that $p \leq P_\gamma$ for each $(\gamma, p) \in E_i$;

(iii) if $(\gamma, p) \in E_i$ then $(\gamma + j, p') \in E_i$ for all $j \in \mathbb{N}_0$ and $0 \leq p' \leq p$.

An index family $E = (E_1, ..., E_N)$ is an $N$-tuple of index sets. Finally, we say that a conormal distribution $w$ is polyhomogeneous on $\mathcal{W}$ with index family $E$, denoted $w \in \mathcal{A}_E^{ph}(\mathcal{W})$, if $w$ is conormal and if near each $H_i$ we have $w \sim \sum_{(\gamma, p) \in E_i} a_{\gamma, p} \rho^\gamma (\log \rho_i)^p$, as $\rho_i \to 0$, with coefficients $a_{\gamma, p}$ conormal on $H_i$ and polyhomogeneous with index $E_j$ at any $H_i \cap H_j$.

Consider a local coordinate chart $(x, y, z)$ in the collar neighborhood $\mathcal{U} \subset M$ of the boundary $\partial M$ and note that as in (2.1), the tangent bundle of the hypersurface $\mathcal{U}_{x_0} = \{ x = x_0 \} \cap \mathcal{W}$ splits into the sum of a ‘vertical’ and ‘horizontal’ subspace, where the first is the tangent space to the fiber $F_y$-factor and the second is the tangent space to the base $B_y$-factor. This splitting is orthogonal, and induces a bigrading

$$\Lambda^p(\mathcal{U}) = \bigoplus_{l+k=p} \Lambda^l(B_y) \otimes \Lambda^k(F_z) := \bigoplus_{l+k=p} \Lambda^{l,k}(\mathcal{W}).$$
Then, locally, under the rescaling transformation $\Phi_\rho$, the heat kernel $H_\mathcal{M}$ is a distribution on $M^2_h := \mathbb{R}^+ \times M^2$, acting over $\mathbb{R}^+ \times \mathcal{M}^2$ on and taking values in the sections

$$\Gamma \left( (0, 1), \bigoplus_{l+k=p} \left( \Lambda^l \mathcal{M}^{-1} \left( \mathcal{M} \right) \oplus \Lambda^k \mathcal{M} \left( \mathcal{M} \right) \right) \right).$$

Consider local coordinates $(t, (x, y, z), (\tilde{x}, \tilde{y}, \tilde{z}))$, where $(x, y, z)$ and $(\tilde{x}, \tilde{y}, \tilde{z})$ are coordinates on the two copies of $M$ near the boundary. The heat kernel $H_\mathcal{M}$ has non-uniform behaviour at the submanifolds

$$P = \{(t, (x, y, z), (\tilde{x}, \tilde{y}, \tilde{z})) \in M^2_h \mid t = 0, x = \tilde{x} = 0, y = \tilde{y}\},$$

$$D = \{(t, (x, y, z), (\tilde{x}, \tilde{y}, \tilde{z})) \in M^2_h \mid t = 0, x = \tilde{x}, y = \tilde{y}, z = \tilde{z}\}.$$

The asymptotic behaviour of $H_\mathcal{M}$ near these submanifolds of $M^2_h$ depends on the angle of approach to these submanifolds. This dependence on the angle in the asymptotics of the heat kernel is conveniently treated by introducing polar coordinates around $P$ and $D$. Geometrically this corresponds to appropriate blowups of the heat space $M^2_h$, such that the corresponding heat kernel lifts to a polyhomogeneous distribution on that blowup space, in the sense of Definition 4.1.

To obtain the correct blowup of $M^2_h$ in the case of an incomplete edge metric, one first blows up of the submanifold $P$ parabolically. The notion parabolic refers here basically to the fact that we treat $\sqrt{t}$ instead of $t$ as a smooth variable.

The resulting heat space $[M^2_h, P]$ is defined as the union of $M^2_h \setminus P$ with the interior spherical normal bundle of $P$ in $M^2_h$. The blowup $[M^2_h, P]$ is endowed with the unique minimal differential structure with respect to which smooth functions in the interior of $M^2_h$ and polar coordinates on $M^2_h$ around $P$ are smooth. This blowup introduces a new boundary hypersurface, which we denote by $\mathcal{M}$ (the front face).

The actual heat-space blowup $\mathcal{M}^2_h$ is obtained by an additional parabolic blowup of $[M^2_h, P]$ along the diagonal $D$ lifted to a submanifold of $[M^2_h, P]$. The resulting blowup $\mathcal{M}^2_h$ is defined as before by cutting out the submanifold and replacing it with its spherical normal bundle, which leads to a new boundary hypersurface, the temporal diagonal (td). The blowup $\mathcal{M}^2_h = [[M^2_h, P], D]$ is a manifold with boundaries and corners as depicted in Figure 1. In addition to the new boundary faces $\mathcal{M}$ and $D$, there are three other boundary faces $\mathcal{F}$, $\mathcal{L}$, $\mathcal{T}$ which arise from the lifts of $\{x = 0\}, \{\tilde{x} = 0\}, \{t = 0\} \subset M^2_h$, respectively.

In Figure 1 we have also written out some of the appropriate projective coordinates on $\mathcal{M}^2_h$. Near the top corner of $\mathcal{M}$ away from $\mathcal{T}$ the projective coordinates are given by

$$\rho = \sqrt{t}, \quad \xi = \frac{x}{\rho}, \quad \tilde{\xi} = \frac{\tilde{x}}{\rho}, \quad u = \frac{y - \tilde{y}}{\rho}, \quad y, z, \tilde{z},$$

where in these coordinates $\rho, \xi, \tilde{\xi}$ are the defining functions of the faces $\mathcal{M}$, $\mathcal{F}$ and $\mathcal{L}$, respectively. For the bottom corner of $\mathcal{F}$ near $\mathcal{L}$, the projective coordinates are given by

$$\tau = \frac{t}{x^2}, \quad s = \frac{\tilde{x}}{x}, \quad u = \frac{y - \tilde{y}}{x}, \quad x, y, z, \tilde{z},$$

respectively.
where in these coordinates $\tau, s, x$ are the defining functions of tf, lf and ff, respectively. For the bottom corner of ff near rf the projective coordinates are obtained by interchanging the roles of $x$ and $\tilde{x}$ and are given by
\begin{equation}
\tau = \frac{t}{x^2}, \quad s = \frac{x}{x}, \quad u = \frac{y - \tilde{y}}{x}, \quad \tilde{x}, \tilde{y}, z, \tilde{z}.
\end{equation}

The projective coordinates on $\mathcal{M}_h^2$ near the top of td away from tf are given by
\begin{equation}
\eta = \sqrt{\tau}, \quad S = \frac{1-s}{\eta}, \quad U = \frac{u}{\eta}, \quad Z = \frac{z - \tilde{z}}{\eta}, \quad x, y, z.
\end{equation}

In these coordinates tf is the face in the limit $|(S, U, Z)| \to \infty$, and ff and td are defined by $\tilde{x}$ and $\eta$, respectively. The blowup heat space $\mathcal{M}_h^2$ is related to the original heat space $\mathcal{M}_h^2$ via the obvious ‘blow-down map’ $\beta : \mathcal{M}_h^2 \to \mathcal{M}_h^2$, which in local coordinates is simply the coordinate change back to $(t, (x, y, z), (\tilde{x}, \tilde{y}, \tilde{z}))$.

We can now state the asymptotic properties of the (rescaled) $H_\mathcal{E}$ as a polyhomogeneous distribution on the blowup $\mathcal{M}_h^2$, studied by the author jointly with Mazzeo in [MaVe12].

**Theorem 4.2.** ([MaVe12], Theorem 1.2) The heat kernel $H_\mathcal{E}$ lifts under the rescaling $\Phi$ via the blowdown map $\beta$ to a polyhomogeneous distribution $\beta^*H_\mathcal{E}$ on $\mathcal{M}_h^2$, with asymptotic expansion of leading order $(-1 - \dim B)$ at the front face ff and $(-\dim M)$ at the diagonal face td, with index sets at the right and left boundary faces given by the indicial roots $\gamma \geq 1/2$ (see (2.4)) of the Hodge Laplacian.

In fact, [MaVe12] went beyond the heat kernel construction, establishing an analogue of the even-odd calculus for edges with consequences for metric invariance of analytic torsion. Below we require a rather detailed understanding of the heat kernel asymptotics and are led to provide a short overview of the heat kernel construction in [MaVe12] for the Friedrichs extension.

**Definition 4.3.** Let $\mathcal{E} = (E_{lf}, E_{rf})$ be an index family for the two side faces of $\mathcal{M}_h^2$. We define $\Psi_{e-h}^{l,p,\mathcal{E}}(M)$ to be the space of all (rescaled, i.e. considered under the rescaling $\Phi$) operators $P$ with Schwartz kernels $K_P$ which lift to polyhomogeneous functions $\tilde{K}_P$ on $\mathcal{M}_h^2$, with index family $\{(-b - 3 + l + j, 0) : j \in \mathbb{N}_0\}$ at ff, $\{(-m + p + j, 0) : j \in \mathbb{N}_0\}$ at td.
at \( \text{td}, \emptyset \) at \( \text{tf} \) and \( E \) for the two side faces \( \text{lf} \) and \( \text{rf} \) of \( M_h^2 \). When \( p = \infty, E_{\text{td}} = \emptyset \). For simplicity we usually denote the lifted Schwartz kernel simply by \( K_p \) again.

As the name suggests, kernels in the calculus \( \Psi^{l,p,E}_{\text{e-h}}(M) \) may be composed and we state the corresponding composition result from [MaVe12].

**Theorem 4.4.** For index sets \( E_{\text{lf}} \) and \( E_{\text{rf}}' \) such that \( E_{\text{lf}} + E_{\text{rf}}' > -1 \), we have

\[
\Psi^{l,p,E_{\text{lf}},E_{\text{rf}}}(M) \circ \Psi^{l',p',E_{\text{lf}}',E_{\text{rf}}'}(M) \subset \Psi^{l+l',p+p',E_{\text{lf}}+E_{\text{rf}}}(M),
\]

where the index sets at the side faces of \( M_h^2 \) amount to

\[
\begin{align*}
P_{\text{lf}} &= E_{\text{lf}}^+ \cup (E_{\text{lf}} + l') \cup \{(z, p + q + 1) : \exists (z, p) \in E_{\text{lf}}^+, \text{ and } (z, q) \in (E_{\text{lf}} + l')\}, \\
P_{\text{rf}} &= E_{\text{rf}} \cup E_{\text{rf}}' \cup \{(z, p + q + 1) : \exists (z, p) \in E_{\text{rf}}, \text{ and } (z, q) \in (E_{\text{rf}}' + l)\}.
\end{align*}
\]

We continue under the rescaling \( \Phi_\ast \) introduced in \( \S 2 \) and do not explify the transformation in the notation below.

The heat kernel is a solution operator to \( \mathcal{L} = \partial_t + \Delta \). We lift \( t\mathcal{L} \) to \( M_h^2 \) in projective coordinates \((4.5)\) and \((4.4)\) and obtain for its restrictions to the temporal diagonal and the front face

\[
N_{\text{td}}(t\mathcal{L}) = \frac{1}{2} \eta \partial_\eta + \Delta_{S,U,Z} - \frac{1}{2}(S \partial_S + U \partial_U + Z \partial_Z),
\]

\[
N_{\text{lf}}(t\mathcal{L}) = \tau \left( \partial_\tau - \partial_\tau^s + s^{-2}(A - 1/4) + \Delta^R_u \right),
\]

where \( A \) is the rescaled operator \((2.3)\) acting in the variable \( z \). Note that \( N_{\text{lf}}(t\mathcal{L}) \) acts tangentially to the fibres \( S^{n+1}_{++} \) of \( \text{ff} \), i.e. it involve no derivatives with respect to \((\tilde{x}, \tilde{y}, \tilde{z})\).

In order to define a parametrix which solves the heat equation to first order at the front face, note that for any \( H \in \Psi^{l,p,E}_{\text{e-h}}(M) \) with \( N_{\text{ff}}(H) \) as its leading coefficient in the front face expansion, we have

\[
(4.6) \quad N_{\text{lf}}(t\mathcal{L} \circ H) = N_{\text{lf}}(t\mathcal{L}) \circ N_{\text{ff}}(H).
\]

To make this vanish, we choose \( N_{\text{ff}}(H) \) to equal the fundamental solution for the heat operator \( N_{\text{lf}}(t\mathcal{L}) \), which is simply the heat operator for the Friedrichs extension of the Hodge Laplacian \((5.4)\) on the product \((\mathbb{R}_s^+ \times F \times \mathbb{R}_u^b, ds^2 + s^2 g_F + du^2)\). In other words, we set

\[
N_{\text{ff}}(H) = H^\varphi(\tau, s, z, \tilde{s}, 1, \tilde{z}) H^{R^b}(\tau, \omega, \tilde{\omega} = 0),
\]

\[
H^\varphi(\tau, s, z, \tilde{s}, \tilde{z}) = \sum_\nu \frac{\sqrt{s s}}{2\tau} I_\nu \left( \frac{s s}{2\tau} \right) \exp \left( -\frac{s^2 + \tilde{s}^2}{4\tau} \right) \phi_\nu(z) \otimes \phi_\nu^*(z),
\]

where we sum over eigenvalues \( \nu^2 \) of the tangential operator \( A \) in \((2.3)\) and \( \phi_\nu \) denote the corresponding normalized eigenforms. \( I_\nu \) denotes the modified Bessel function of first kind. The factor \( H^\varphi \) is the Friedrichs heat kernel of the rescaled operator

\[
(4.7) \quad \Delta^\varphi = \frac{d^2}{ds^2} + \frac{1}{s^2} \left( A - \frac{1}{4} \right),
\]
as in [Che83], [Moo99] and [Les97]. The expansion of $N_{\Phi}(H)$ near $td$ is of order $(-m)$, where $m = \dim M$. The leading order of the heat kernel $H$ at $ff$ does not follow from (4.6) but rather from the initial condition that the heat kernel reduces to the delta function as $t = 0$. Because of the homogeneity of the delta function, the leading exponent of the heat kernel at $td$ is $(-m)$. Similar considerations hold at the front face, where taking into account the incomplete edge volume form the leading exponent at $ff$ is $(-b - 1)$.

Thus we may define an initial heat parametrix $H^{(0)} \in \Psi^{2,0,\mathcal{E}}_{e^{-h}}(M)$ by setting it to equal $\rho_{\Phi}^{-1-b}N_{\Phi}(H)$ near the front face and extending it smoothly to zero the interior of $\mathcal{M}^2_{\Phi}$. The expansion of $H^{(0)}$ near $td$ has a universal (euclidean) form and the error of the initial parametrix at the temporal diagonal $td$ is solved away following [Mel93], uniformly in $\rho_{\Phi}$. This defines $H^{(1)} \in \Psi^{2,0,\mathcal{E}}_{e^{-h}}(M)$ solving the heat equation to first order at the front face and infinite order at the temporal diagonal $td$. In view of the explicit formula for $N_{\Phi}(H)$ and $H^\Theta$ this proves the following

**Proposition 4.5.** There exists an element $H^{(1)} \in \Psi^{2,0,\mathcal{E}}_{e^{-h}}(M)$, where $\mathcal{E} = (E_{\Phi}, E_{rf})$ and the index sets $E_{\Phi} = E_{rf}$ are given by \( \{(\nu + 1/2 + k, 0) \mid \nu^2 \in \text{spec}(A), k \in \mathbb{N}\} \). Moreover $tLH^{(1)} = P^{(1)} \in \Psi^{3,\mathcal{E},\mathcal{E}^{(1)}}_{e^{-h}}(M)$, where $\mathcal{E}^{(1)} = (E_{\Phi}, E_{rf} - 1)$, and

$$\lim_{t \to 0} H^{(1)}(t, x, y, z, \tilde{x}, \tilde{y}, \tilde{z}) = \delta(x - \tilde{x}) \delta(y - \tilde{y}) \delta(z - \tilde{z}).$$

In the next construction step we choose a slightly finer parametrix $H^{(2)}$ with an error which vanishes to infinite order along rf as well.

**Proposition 4.6.** There exists an element $\mathcal{J} \in \Psi^{3,\mathcal{E}'}_{e^{-h}}(M)$, where $\mathcal{E}' = (E_{\Phi}, E_{rf} + 1)$ and $H^{(2)} := H^{(1)} + \mathcal{J}$ is such that $tLH^{(2)} = P^{(2)} \in \Psi^{3,\mathcal{E},\mathcal{E}^{(1)}}_{e^{-h}}(M)$ and $\lim_{t \to 0} H^{(2)} = \text{Id}$. The identity operator $\text{Id}$ corresponds to the kernel $\delta(x - \tilde{x}) \delta(y - \tilde{y}) \delta(z - \tilde{z})$.

**Proof.** This step proceeds exactly as in [Moo99]. The error term $P^{(1)}$ from the previous step has an expansion along rf, and in order to eliminate a term $s^\gamma a$ (we are using projective coordinates (4.4)) its asymptotic expansion, it is only necessary to solve the indicial equation

$$(-\partial_s^2 + s^{-2}(A - 1/4))u = s^\gamma (\tau^{-1} a).$$

This is because all other terms in the expansion of $tL$ at rf lower the exponent in $s$ by at most one, while the indicial part lowers exponent by two. Note that $\tau, \omega, \tilde{x}, \tilde{y}$ and $\tilde{z}$ only enter this equation as parameters. We may solve this equation using the Mellin transform. The solution is polyhomogeneous in all variables, including the parameters and has leading order $\gamma + 2$ at rf.

Let $\mathcal{J}$ denote a kernel which has asymptotic sum at the right face with terms equal to the ones obtained iteratively as solutions to to the indicial equation above. Since $\tau$ and $\tilde{x}$ enter the indicial equation as parameters, we can assume that $\mathcal{J}$ vanishes to first order at $ff$ and to infinite order at $tf$, as does the error $P^{(1)} \in \Psi^{3,\mathcal{E},\mathcal{E}^{(1)}}_{e^{-h}}(M)$, but has an expansion at $rf$ of two orders higher. So indeed $\mathcal{J} \in \Psi^{3,0,\mathcal{E}'}_{e^{-h}}(M)$. We see that $H^{(2)} = H^{(1)} + \mathcal{J}$ has all the desired properties, with a new error term $P^{(2)} \in \Psi^{3,\mathcal{E},\mathcal{E}^{(1)}}_{e^{-h}}(M)$. \( \square \)
Consider the kernels as convolution operators in time. Then, our parametrix solves 
\( \mathcal{L}H^{(2)} = \text{Id} + t^{-1}P^{(2)} \). The final stage in the parametrix construction is then to consider the formal Neumann series
\[
(\text{Id} + t^{-1}P^{(2)})^{-1} = \text{Id} + \sum_{j=1}^{\infty} (-t^{-1}P^{(2)})^j := \text{Id} + P^{(3)},
\]
where \( t^{-1}P^{(2)} \in \Psi_{e-h}^{1,\infty} \) and Theorem 4.4 yields \( (t^{-1}P^{(2)})^j \in \Psi_{e-h}^{j,\infty,*\infty} \). In other words, successive composition of \( t^{-1}P^{(2)} \) with itself produces an operator which vanishes to higher and higher order on \( \mathbb{f} \). In fact, a slightly finer analysis, see [Mel93] shows that this formal series is actually convergent. We refer to these sources for the necessary estimates, both in the case of compact manifolds and for manifolds with cylindrical ends, but it is easily seen that everything there transports immediately to this setting (and many other ones as well). Indeed, this is a general feature of such Volterra series.

By the arguments in [MAVE12] the exact heat kernel is then given by
\[
H_{\mathbb{f}} = H^{(2)}(\text{Id} + P^{(3)}) = H^{(2)} - H^{(2)} \ast t^{-1}P^{(2)} + H^{(2)} \ast \sum_{j=2}^{\infty} (-t^{-1}P^{(2)})^j
\]
\[
= H^{(1)} + (\mathcal{J} + H^{(1)} \ast \mathcal{L}H^{(1)}) + O(\rho_f^{1+b}), \ \rho_f \to 0.
\]

Recall the admissibility assumption in Definition 2.2 (iii), which requires \( g = g_0 + h \) with \( |h|_{g_0} = O(x^2) \) as \( x \to 0 \). We can therefore separate \( \mathcal{L} \) into the leading order term \( N_\mathbb{f}(\mathcal{L}) \), second order term \( \mathcal{L}' \), comprised of derivatives \( \{\partial_\nu, \partial_\nu \partial_z\} \) which arise from the curvature of the fibration \( \phi : (\partial M, g^F + \phi^*g^B) \to (B, g^B) \), and the higher order terms \( \mathcal{L}'' \), which arise from \( h \) and do not lower the front face asymptotics. Consequently
\[
(4.8) \quad H_{\mathbb{f}} = H^{(1)} + (\mathcal{J} + H^{(1)} \ast \mathcal{L}'H^{(1)}) + O(\rho_f^{1+b}), \ \rho_f \to 0.
\]

5. Solution to the model signaling problem

We now proceed with the first step in the construction of the heat kernel \( H_\Gamma \) for an algebraic self-adjoint extension \( \Delta_\Gamma \). The fundamental idea is to add further terms to the heat kernel \( H_{\mathbb{f}} \) for the Friedrichs extension of the Hodge Laplacian, which correct the asymptotic behaviour of the kernel at \( \mathbb{rf} \) and \( \mathbb{lf} \) to satisfy the boundary conditions of \( \mathcal{D}(\Delta_\Gamma) \). These additional terms are obtained from the signaling solution, which is explained and constructed out of \( H_{\mathbb{f}} \) below in \S 6. The present section provides a preliminary discussion of the signaling problem for a prototype of a model edge \( l_\nu \oplus \Delta_\mathbb{g}^\nu \), where \( l_\nu := -\partial_x^2 + x^{-2}(\nu^2 - 1/4), \nu \in [0,1) \) is a regular-singular differential operator acting on \( C_0^\infty(0,\infty) \).

The maximal domain \( \mathcal{D}_{\max}(l_\nu) \) and the minimal domain \( \mathcal{D}_{\min}(l_\nu) \) for \( l_\nu \) are defined parallel to (3.1), and provide the maximal and minimal closed extensions of the regular singular operator \( l_\nu \) in \( L^2(0,\infty) \). As a special case of Lemma 3.1, we refer for instance to ([KLP08], Proposition 3.1) for an explicit argument, any \( u \in \mathcal{D}_{\max}(l_\nu) \) admits a
partial asymptotic expansion
\[ u \sim c^+|u| \phi_\nu^+(x) + c^-|u| \phi_\nu^-(x) + \tilde{u}, \text{ as } x \to 0, \]
\[ \phi_\nu^+(x) = x^{\nu+1/2}, \quad \phi_\nu^-(x) = \begin{cases} x^{-\nu-1/2}, & \nu \in (0, 1), \\ \sqrt{x} \log(x), & \nu = 0, \end{cases}, \quad \tilde{u} \in \mathcal{D}(l_\nu). \]

The weak expansion of solutions in \( \mathcal{D}_\max(l_\nu \oplus \Delta_{R^b}) \) is a parallel structure. The signaling solution \( u(\cdot, t) \in \mathcal{D}_\max(l_\nu \oplus \Delta_{R^b}), t \in [0, \infty), \) is defined for any given \( h \in L^1(\mathbb{R}^+ \times \mathbb{R}^b) \) which is \( L^1_{\text{loc}}[0, \infty) \cap C^\infty(0, \infty) \) in the time variable, as a solution to the so-called signaling problem
\[
(\partial_t + l_\nu \oplus \Delta_{R^b})u(x, y, t) = 0, \quad u(x, y, 0) \equiv 0, \\
c^-(u(\cdot, t)) = h(t), \quad t > 0.
\]

Note that \( u(\cdot, t) \) cannot take values in a fixed self-adjoint extension of \( l_\nu \oplus \Delta_{R^b}, \) since by uniqueness of solutions to the heat equation, \( u(x, y, 0) \equiv 0 \) then implies \( u \equiv 0. \) The signaling solution is in fact also unique, since \( \{ u \in \mathcal{D}_\max(l_\nu \oplus \Delta_{R^b}) \mid c^-|u| = 0 \} \) defines the Friedrichs self-adjoint extension of \( l_\nu \oplus \Delta_{R^b}, \) which we denote by \( L_\nu^F \oplus \Delta_{R^b}. \)

The heat kernel of the Friedrichs self-adjoint extension \( L_\nu^F \) of the regular singular operator \( l_\nu \) and its restriction to \( \{ \tilde{x} = 0 \} \) are explicitly given by, see ([Les97], Proposition 2.3.9)
\[
E_\nu(t, x, \tilde{x}) = \frac{\sqrt{xx}}{2t} I_\nu \left( \frac{x \tilde{x}}{2t} \right) \exp \left( -\frac{x^2 + \tilde{x}^2}{4t} \right),
\]
\[ NE_\nu(t, x) := \lim_{x \to 0} (\tilde{x}^{-\nu-1/2} E_\nu(x, \tilde{x}, t)) = \frac{x^{\nu+1/2} \exp(-x^2/4t)}{\Gamma(\nu+1)2^{\nu+1} \nu+1}, \]

where we have used the asymptotic behaviour of the modified Bessel function of first kind, cf. ([AbST92])
\[ I_\nu(r) \sim \frac{(r/2)\nu}{\Gamma(\nu+1)}, \text{ as } r \to 0. \]

We define
\[
F_\nu^N(h)(t, x, y) := c_\nu \cdot \int_0^t \int_{\mathbb{R}^b} NE_\nu(t, x) H_{R^b}(\tilde{t}, y, \tilde{y}) h(t - \tilde{t}, \tilde{y}) \, d\tilde{t} \, d\tilde{y},
\]
\[ c_\nu := \begin{cases} (-1), & \text{for } \nu = 0, \\ 2\nu, & \text{for } \nu \in (0, 1). \end{cases} \]

for any \( h \in L^1(\mathbb{R}^+ \times \mathbb{R}^b) \) which is \( L^1_{\text{loc}}[0, \infty) \cap C^\infty(0, \infty) \) in the time variable. Here \( H_{R^b} \) denotes the Euclidean heat kernel on \( \mathbb{R}^b. \) The asymptotic behavior of \( F_\nu^N(h) \) as \( x \to 0 \) is studied by means of the Laplace transform \( \mathcal{L} \) in the time variable \( t \in \mathbb{R}^+ \) and the Fourier transform \( \mathcal{F} \) in the Euclidean variable \( y \in \mathbb{R}^b. \) For any \( g \in L^1(\mathbb{R}^+ \times \mathbb{R}^b) \) which is \( L^1_{\text{loc}}[0, \infty) \cap C^\infty(0, \infty) \) in the time variable \( t \in \mathbb{R}^+ \), not growing exponentially
as \( t \to \infty \), both transforms are defined as follows

\[
(\mathcal{L} g)(\zeta, y) = \int_{\mathbb{R}^+} g(t, y) \exp(-\zeta t) \, dt, \quad \text{Re}(\zeta) > 0,
\]

\[
(\mathcal{F} g)(t, \omega) = \int_{\mathbb{R}^b} g(t, y) e^{-i\omega y} \, dy.
\]

Hence we assume henceforth that \( h \in L^1(\mathbb{R}^+ \times \mathbb{R}^b) \) and \( L^1_{\log}[0, \infty) \cap C^\infty(0, \infty) \) in the time variable \( t \in \mathbb{R}^+ \), not growing exponentially as \( t \to \infty \). Finally, the inverse Laplace transform is given for any \( \delta > 0 \) and analytic \( L(\zeta) \), integrable over \( \text{Re}(\zeta) = \delta \), by

\[
(\mathcal{L}^{-1} L)(t) = \frac{1}{2\pi i} \int_{\delta+i\mathbb{R}} e^{\zeta t} L(\zeta) \, d\zeta.
\]

**Proposition 5.1.** \( F^N_\nu(h) \) is indeed the signaling solution to (5.1). In particular

\[
c^-(F^N_\nu(h)) = h, \quad c^+(F^N_\nu(h)) = G^N_\nu \ast h(t) := \int_0^t \int_{\mathbb{R}^b} G^N_\nu(t - \tilde{t}, y - \tilde{y}) h(\tilde{t}, \tilde{y}) \, d\tilde{t} \, d\tilde{y},
\]

where \((\mathcal{L} \circ \mathcal{F} G^N_\nu)(\zeta, \omega) \equiv \tilde{G}^N_\nu(\zeta + |\omega|^2)

\[
\begin{cases}
\log \sqrt{\zeta + |\omega|^2} + \gamma - \log 2, & \nu = 0, \\
\frac{\Gamma(-\nu)}{\Gamma(\nu)} 2^{-2\nu} (\zeta + |\omega|^2)^\nu, & \nu \in (0, 1).
\end{cases}
\]

**Proof.** We compute for \( \text{Re}(\zeta) > 0 \)

\[
(\mathcal{L} \circ \mathcal{F} F^N_\nu(h))(x, \omega, \zeta) = c_\nu \cdot \mathcal{L}(NE_\nu \cdot \mathcal{F} H_{\mathbb{R}^b})(x, \omega, \zeta) \cdot (\mathcal{L} \circ \mathcal{F} h)(\zeta, \omega)
\]

\[
= \int_0^\infty NE_\nu(t, x) e^{-t(\zeta + |\omega|^2)} \, dt \cdot (\mathcal{L} \circ \mathcal{F} h)(\zeta, \omega)
\]

\[
= c_\nu \cdot \frac{\sqrt{\zeta + |\omega|^2}^{\nu/2}}{2^\nu \Gamma(\nu + 1)} K_\nu(x \sqrt{\zeta + |\omega|^2}) \cdot (\mathcal{L} \circ \mathcal{F} h)(\zeta, \omega),
\]

where \( K_\nu \) is the modified Bessel function of second kind, and in the definition of \( \sqrt{\zeta} \) we fix the branch of logarithm in \( \mathbb{C} \setminus \mathbb{R}^- \). We have assumed that \( h(t, y) \) is not of exponential growth as \( t \to \infty \), so that \((\mathcal{L} \circ \mathcal{F} h)(\zeta, \omega) \) is well-defined for \( \text{Re}(\zeta) > 0 \). The Bessel function \( K_\nu(z) \) admits an asymptotic expansion, see [AbSt92]

\[
K_\nu(z) = \begin{cases} 
(\log 2 - \gamma) - \log(z), & \nu = 0, \\
2^{-\nu-1} \Gamma(\nu) z^{-\nu} + 2^{-\nu-1} \Gamma(-\nu) z^\nu, & \nu > 0,
\end{cases} + \tilde{K}(z), \quad \tilde{K}(z) = O(z) \text{ as } z \to 0.
\]

where \( \gamma \in \mathbb{R} \) is the Euler constant. Consequently

\[
c^+(\mathcal{L} \circ \mathcal{F} F^N_\nu(h)(\cdot, \zeta, \omega)) = c_\nu \cdot (\mathcal{L} \circ \mathcal{F} h)(\zeta, \omega) \cdot \begin{cases}
(\log 2 - \log \sqrt{\zeta + |\omega|^2} - \gamma), & \nu = 0, \\
\frac{\Gamma(-\nu)(\zeta + |\omega|^2)^\nu}{\Gamma(\nu + 1) 2^{2\nu+1}}, & \nu \in (0, 1),
\end{cases}
\]

\[
c^-(\mathcal{L} \circ \mathcal{F} F^N_\nu(h)(\cdot, \zeta, \omega)) = (\mathcal{L} \circ \mathcal{F} h)(\zeta, \omega).
\]
Taking the inverse Laplace and Fourier transform, we obtain
\[ F^N_{\nu}(h)(x, t) = \phi_{\nu}^+(x) (\mathcal{L} \circ \mathcal{F})^{-1}(c^+(\mathcal{L} \circ \mathcal{F} F^N_{\nu}(h))) + \phi_{\nu}^-(x) (\mathcal{L} \circ \mathcal{F})^{-1}(c^- (\mathcal{L} \circ \mathcal{F} F^N_{\nu}(h))) \]
\[ + \frac{\sqrt{x}}{2^{\nu} \Gamma(\nu + 1)} (\mathcal{L} \circ \mathcal{F})^{-1}((\mathcal{L} \circ \mathcal{F} h) \tilde{K}(x \sqrt{\zeta + |\omega|^2}) (\zeta + |\omega|^2)^{\nu/2}), \]
where each \( \phi_{\nu}^\pm \) coefficient exists and the third summand is \( O(x^{3/2}) \), as \( x \to 0 \). Consequently, indeed
\[ \mathcal{L} \circ \mathcal{F}(c_{\pm}(F^N_{\nu}(h))) = c_{\pm}(\mathcal{L} \circ \mathcal{F} F^N_{\nu}(h)). \]
This yields the stated explicit expression for the Laplace-Fourier transform of \( G^N_{\nu} \)
\[ (\mathcal{L} \circ \mathcal{F} G^N_{\nu})(\zeta, \omega) = \begin{cases} \log \sqrt{\zeta + |\omega|^2} + \gamma - \log 2, \nu = 0, \\ \frac{\Gamma(\nu)}{\Gamma(\nu)} 2^{-2\nu} (\zeta + |\omega|^2)^\nu, \nu \in (0, 1). \end{cases} \]
Finally, we should note, evaluating the Fourier and the Laplace transform integrals explicitly, the following relation
\[ G^N_{\nu}(t, y - \tilde{y}) = H_{\mathbb{R}^b}(t, y - \tilde{y}) \mathcal{L}^{-1} \tilde{G}^N_{\nu}(t). \]

6. Solution to the signaling problem

We expand the lifted heat kernel \( \beta^* H_\mathcal{F} \) (as before rescaled under \( \Phi_\nu \) from §2) asymptotically at the left boundary face, using projective coordinates (4.3), where \( s = \tilde{x}/x \) is the defining function of \( \text{lf} \) and \( x \) the defining function of the front face. We obtain by Theorem 4.2
\[ \beta^* H_\mathcal{F} \sim \sum_{\nu \geq 0} \sum_{k \in \mathbb{N}_0} G^K_{\nu} s^{\nu + 1/2 + k}, \text{ as } s \to 0, \]
where the sum runs over \( \nu \geq 0 \) with \( \nu^2 \) being the eigenvalues of the tangential operator \( A \) in (2.3), and natural numbers \( k \geq 0 \). The coefficient \( G^K_{\nu} \) is a polyhomogeneous function on the left boundary face, of leading order \( (-1 - b) \) at the front face and vanishing to infinite order at \( \text{tf} \). In the projective coordinates (4.3), we find
\[ \beta^* (x \partial_x) = x \partial_x - s \partial_s, \beta^* (x \partial_y) = \partial_u, \beta^*(\partial_z) = \partial_\tau, \beta^*(x^2 \partial_\tau) = \partial_t. \]
Hence the action of \( \beta^* \) on \( x^2 (\partial_t + \Delta_p) \) keeps the order of \( s \) invariant. Consequently, since \( H_\mathcal{F} \) solves the heat equation, so does each coefficient \( G^K_{\nu} s^{\nu + 1/2 + k} \) in the heat kernel expansion at \( \text{lf} \). We define
\[ G^0_{\nu} s^{\nu + 1/2} = G^0_{\nu} \tilde{x}^{\nu + 1/2} \tilde{x}^{-\nu - 1/2} =: H_{\nu} \tilde{x}^{\nu + 1/2}, \]
where \( H_{\nu} \) solves heat equation, since so does \( G^0_{\nu} s^{\nu + 1/2} \). The kernel \( H_{\nu} \) lifts to a polyhomogeneous function on \( \text{lf} \) of leading order \( (-3/2 - b - \nu) \) at the front face and vanishing to infinite order at \( \text{tf} \). The left face \( \text{lf} \) is a parabolic blowup of \( \mathbb{R}^+ \times M_{(x, y, z)} \times \partial M_{(\tilde{y}, \tilde{z})} \) at the highest codimension corner \( \{ t = 0, x = 0, y = \tilde{y} \} \) and we denote the corresponding blowdown map again by \( \beta \).
Globally, \( H_\nu \) is thus defined by the regularized limit in the sense that divergent summands are neglected
\[
\beta^* H_\nu = \text{reg.-lim}_{\rho_\H^* \to 0} (\beta^* \H_\rho \cdot \rho_\H^{-\nu-1/2}) \cdot \rho_\H^{-\nu-1/2}.
\]

The relation between the kernels (6.2) in the model situation and the kernels \( H_\rho, H_\nu \) in the setup of an admissible edge manifold is then as follows. According to the heat kernel construction in §4, the expansion of the lifts \( \beta^* H_\rho \) and \( \beta^* H_\nu \) at \( \rho \) in projective coordinates (4.2) at the top corner of \( \M_h^G \) is given by
\[
\begin{align*}
\beta^* H_\rho(\rho, \xi, \tilde{\xi}, u, y, z, \tilde{z}) &= \rho^{-1-b} \sum_{\nu \geq 0} E_\nu(1, \xi, \tilde{\xi}) P_\nu(z, \tilde{z}) H_{R^b}(1, u) + \beta^* K_1 + \beta^* K_2 \\
\beta^* H_\nu(\rho, \xi, u, y, z, \tilde{z}) &= (\rho^{-3/2-b-\nu} N E_\nu(1, \xi) H_{R^b}(1, u) + \beta^* \kappa_1 + \beta^* \kappa_2) P_\nu(z, \tilde{z}),
\end{align*}
\]
as \( \rho \to 0 \), where \( K_{1,2} \) and \( \kappa_{1,2} \) are the higher order terms, the sum runs over \( \nu \geq 0 \) with \( \nu^2 \) being the eigenvalues of the tangential operator \( A \) in (2.3), and \( P_\nu(z, \tilde{z}) \) is the Schwartz kernel of the fibrewise projection onto the corresponding eigenspaces. Moreover, as \( (\xi, \tilde{\xi}, \rho) \to 0 \), the higher order terms satisfy
\[
\begin{align*}
\beta^* K_1 &= \sum_{\nu \geq 0} \sum_{j=0}^{\infty} O(\rho^{-b}(\xi \tilde{\xi})^{\nu+1/2+j}), & \beta^* \kappa_1 &= O(\rho^{-1/2-b-\nu} \xi^{\nu+1/2}), \\
\beta^* K_2 &= \sum_{\nu \geq 0} \sum_{j=0}^{\infty} O(\rho^{-b+1}(\xi \tilde{\xi})^{\nu+1/2+j}), & \beta^* \kappa_2 &= O(\rho^{1/2-b-\nu} \xi^{\nu+1/2}).
\end{align*}
\]

We define for any \( \nu \in [0, 1) \) and \( h \in L^1_{\text{loc}}([0, \infty) \times \partial M) \cap C^\infty((0, \infty) \times \partial M) \), in standard coordinates in the collar neighborhood \( \U \subset M \)
\[
F_\nu(h)(t, x, y, z) := c_\nu \int_0^t \int_{\partial M} H_\nu(\tilde{t}, x, y, z, \tilde{y}, \tilde{z}) h(t - \tilde{t}, \tilde{y}, \tilde{z}) \ d\tilde{t} \ d\text{vol}_{\partial M}(\tilde{y}, \tilde{z}) =: c_\nu H_\nu \ast h.
\]

Since \( H_\nu \) solves the heat equation, so does \( F_\nu(h) \).

The fundamental component in the heat kernel construction of Mooers in [Moo99] is a solution \( u(t, \cdot) \in \mathcal{D}_{\text{max}}(\Delta_\rho) \) to the signaling problem
\[
\begin{align*}
(\partial_t + \Delta_\rho) u &= 0, & u(0, \cdot) &\equiv 0, \\
c^-(u(t, \cdot)) &= P_\rho h(t),
\end{align*}
\]
where \( P_\rho \) is the fibrewise projection onto the \( \nu^2 \)-eigenspace of the tangential operator \( A \) in (2.3). Note that for \( P_\rho h(t) \neq 0 \), the solution \( u(t, \cdot) \) cannot lie in any self-adjoint domain of \( \Delta_\rho \) since by uniqueness of solutions to the heat equation \( u(0, \cdot) \equiv 0 \) then implies \( u \equiv 0 \). The signaling solution is in fact also unique, since for \( c^-(u(t, \cdot)) = 0 \), \( u(t, \cdot) \in \mathcal{D}(\Delta_\rho^\ast) \) and hence \( u(0, \cdot) \equiv 0 \) then implies \( u \equiv 0 \).
Proposition 5.1. The projective coordinates on $F_{\nu, h}$ were introduced in Proposition 5.1 and $G'_{\nu}, G''_{\nu}$ lift to polyhomogeneous functions on the parabolic blowup of $\mathbb{R}^+ \times B^2$ around $Y := \{(t, y, \tilde{y}) \in \mathbb{R}^+ \times B^2 \mid t > 0, y = \tilde{y}\}$ of leading order $(-2\nu - b)$ at the front face. $\theta_{\nu} \in \mathbb{R}$ is a constant and zero unless $\nu = 1/2$. The projective coordinates on $[\mathbb{R}^+ \times B^2, Y]$ near its front face are

$$\rho = \sqrt{t}, \quad u = \frac{y - \tilde{y}}{\sqrt{t}}, \quad y.$$

Proof. The fundamental argument below is to explain how the higher order terms in the front face expansion of $\beta^* H_{\nu}$ contribute to the $c^\pm(F_{\nu, h}(h))$ coefficients in the asymptotic expansion of $F_{\nu, h}$ as $x \to 0$. Integrating first in $\tilde{z}$ along the fibres, we obtain

$$F_{\nu, h}(t, x, y, z) = c_{\nu} \int_0^t \int_B N E_{\nu}(\tilde{t}, x) H_{\nu}(\tilde{t}, y - \tilde{y}) P_{\nu, h}(t - \tilde{t}, \tilde{y}, z) \, d\tilde{t} \, d\nu B(\tilde{y})$$

$$+ c_{\nu} \int_0^t \int_B \kappa_1(\tilde{t}, x, y, \tilde{y}) P_{\nu, h}(t - \tilde{t}, \tilde{y}, z) \, d\tilde{t} \, d\nu B(\tilde{y})$$

$$+ c_{\nu} \int_0^t \int_B \kappa_2(\tilde{t}, x, y, \tilde{y}) P_{\nu, h}(t - \tilde{t}, \tilde{y}, z) \, d\tilde{t} \, d\nu B(\tilde{y}).$$

Expanding $d\nu B(\tilde{y})$ in the first integral around $y$ we find

$$F_{\nu, h}(t, x, y, z) = c_{\nu} \int_0^t \int_B N E_{\nu}(\tilde{t}, x) H_{\nu}(\tilde{t}, y - \tilde{y}) P_{\nu, h}(t - \tilde{t}, \tilde{y}, z) \, d\tilde{t} \, d\tilde{y}$$

$$+ c_{\nu} \int_0^t \int_B (\kappa_0 + \kappa_1 + \kappa_2)(\tilde{t}, x, y, \tilde{y}) P_{\nu, h}(t - \tilde{t}, \tilde{y}, z) \, d\tilde{t} \, d\nu B(\tilde{y})$$

$$=: F_{\nu, h}^N(h) + F_0 + F_1 + F_2,$$

where $\beta^* \kappa_{01} = O(\rho_{H}^{\nu - b} - \nu + 1/2, \rho_{H}^{\nu + 1/2})$ and $\beta^* \kappa_2 = O(\rho_{H}^{\nu - b + 1/2}, \rho_{H}^{\nu + 1/2})$ when lifted to $Y$ as a parabolic blowup of $\mathbb{R}^+ \times M \times \partial M$ at the highest codimension corner.

The contribution to $c^\pm(F_{\nu, h}(h))$ coming from the first integral $F_{\nu, h}^N(h)$ is studied in Proposition 5.1. It remains to discuss the latter three integrals $F_{0, 1, 2}$. Since we expand as $x \to 0$, for fixed $t > 0$, we may assume $x^2 < t$ and separate for each $j = 0, 1, 2$

$$F_j = \int_0^t + \int_{x^2}^{t} =: F'_j + F''_j.$$

We describe the integral expressions $F'_j$ in the projective coordinates

$$\tau = \frac{\tilde{t}}{x^2}, \quad u = \frac{y - \tilde{y}}{x}, \quad x, \quad y, \quad z, \quad \tilde{z},$$

Theorem 6.1. $F_{\nu, h}(h, \nu \in [0, 1])$, is a signaling solution to (6.4). More precisely

$$F_{\nu, h}(h)(x) \sim c^+(F_{\nu, h}(h)) \psi_\nu^+(x) + c^-(F_{\nu, h}(h)) \psi_\nu^-(x) + O(x^{3/2}), \quad x \to 0,$$

where $G_{\nu}^N$ was introduced in Proposition 5.1 and $G_{\nu}^0, G_{\nu}^1$ lift to polyhomogeneous functions on the parabolic blowup of $\mathbb{R}^+ \times B^2$ around $Y := \{(t, y, \tilde{y}) \in \mathbb{R}^+ \times B^2 \mid t > 0, y = \tilde{y}\}$ of leading order $(-2\nu - b)$ at the front face. $\theta_{\nu} \in \mathbb{R}$ is a constant and zero unless $\nu = 1/2$. The projective coordinates on $[\mathbb{R}^+ \times B^2, Y]$ near its front face are

$$\rho = \sqrt{t}, \quad u = \frac{y - \tilde{y}}{\sqrt{t}}, \quad y.$$
near the the lower corner of the left face If, where \( x = \rho_{\text{tf}} \) is the front face defining function and \( \tau = \rho_{\text{tf}} \) the defining function of the temporal diagonal. Then, writing \( \text{dvol}_G(\tilde{y}) = v(\tilde{y})d\tilde{y} \) in local coordinates, we find

\[
F''_j = \int_0^1 \int_{\mathbb{R}^b} G'_j(x, \tau, u, y, z) P_\nu h(t - x^2 \tau, y - x u, z) v(y - x u) d\tau du \times \begin{cases} \frac{x^{-\nu+3/2}}{2}, & j = 0, 1, \\ \frac{x^{-\nu+5/2}}{2}, & j = 2, \end{cases}
\]

where \( G'_j \) is bounded, polyhomogeneous and vanishing to infinite order as \( \tau \to 0, |u| \to \infty \).

Expanding \( P_\nu h \) and \( v \) in Taylor series around \((t, y, z)\), as well as expanding \( G'_j \) as \( x \to 0 \), we find no contribution to terms \( x^{\nu+1/2} \) and \( \sqrt{x} \log(x) \) in the asymptotic expansion as \( x \to 0 \), unless \( \nu = 1/2 \).

In case \( \nu = 1/2 \), we have \((-\nu + 3/2) = \nu + 1/2\) and hence \( F'_j \) contributes \( \theta'_j \cdot P_\nu h \) to the coefficient \( c^+(F_\nu(h)) \) if \( j = 0, 1 \), with no contribution to \( c^-(F_\nu(h)) \). In case \( \nu \neq 1/2 \), neither of \( F'_j \) contributes to the coefficients \( c^\pm(F_\nu(h)) \) and we set \( \theta'_j = 0 \).

For the analysis of \( F''_j, j = 0, 1, 2 \), consider the projective coordinates

\[
\rho = \sqrt{t}, \quad \xi = \frac{x}{\rho}, \quad u = \frac{y - \tilde{y}}{\rho}, \quad y, \quad z, \quad \bar{z},
\]

near the upper corner of the left face If, where \( \rho = \rho_{\text{tf}} \) defines the front face in these coordinates and \( \xi = \rho_{\text{tf}} \) the right boundary face.

**Contribution from \( F''_0 \).** By construction we find in the new projective coordinates

\[
F''_0 = \int_{\mathbb{R}^b} \int_{\mathbb{R}^b} NE_\nu(1, \xi) H_{Rb}(1, u) \rho^{-\nu-1/2} (\rho u)^k v^{(k)}(y) P_\nu h(t - \rho^2, y - \rho u, z) d\rho du.
\]

Note that \( u H_{Rb}(1, u) = \frac{1}{2} \partial_u H_{Rb}(1, u) \) and hence integrating by parts in \( u \), we find

\[
F''_0 = \sum_{k=1}^{\infty} \frac{(-1)^k}{2k!} \int_{\mathbb{R}^b} \int_{\mathbb{R}^b} NE_\nu(1, \xi) H_{Rb}(1, u) \rho^{-\nu+1/2+k} v^{(k)}(y) u^{k-1} \times (\partial_y P_\nu h)(t - \rho^2, y - \rho u, z) d\rho du \times (\partial_y P_\nu h)(t - \rho^2, y - \rho u, z) d\rho du.
\]

In view of the explicit structure \( NE_\nu(1, \xi) = C \xi^{\nu+1/2} = C x^{\nu+1/2} \rho^{-\nu-1/2} \) for some explicit constant \( C \in \mathbb{R} \), we obtain

\[
F''_0 = x^{\nu+1/2} \int_{\mathbb{R}^b} \int_{\mathbb{R}^b} \rho^{-\nu+1} C_0''(\rho, u, y, z) (\partial_y P_\nu h)(t - \rho^2, y - \rho u, z) d\rho du = \int_{\mathbb{R}^b} \int_{\mathbb{R}^b} \int_{\mathbb{R}^b},
\]

using the fact that \( \rho^{-2\nu+1} \) is integrable at zero for \( \nu \in [0, 1) \). Here, \( C_0'' \) is bounded in its components and polyhomogeneous in \( \rho \). Obviously, the latter summand does not
contribute to the coefficients of $x^{\pm \nu + 1/2}$ and $\sqrt{x} \log(x)$ and hence we finally obtain (we abuse the notation by incorporating the $\rho$-factors into the kernel $G'_{0}$)

$$c^{-}(F''_{0}) = 0, \quad c^{+}(F''_{0}) = G_{0}' \ast \partial_{y} P_{\rho} h = \int_{\mathbb{R}^{b}} G''_{0}(\tilde{t}, y, \tilde{y}) \partial_{y} P_{\rho} h(t - \tilde{t}, \tilde{y}, z) d\tilde{t} \text{dvol}_{B}(\tilde{y}),$$

where $G''_{0}$ lifts to a polyhomogeneous function on the parabolic blowup of $\mathbb{R}^{+} \times \mathbb{B}^{2}$ around $Y := \{(t, y, \tilde{y}) \in \mathbb{R}^{+} \times \mathbb{B}^{2} \mid t = 0, y = \tilde{y}\}$ of leading order $(-2\nu - b)$ at the front face.

**Contribution from $F''_{1}$**. Here a more detailed information on the structure of $\kappa_{1}$ is necessary. Recall (4.8), which asserts that the second order term in the front face expansion of $H_{\mathscr{J}}$ is given by $\mathcal{J} + H^{(1)} \ast \mathcal{L'} H^{(1)}$, where $\mathcal{J} \in \Psi^{2,0,\mathcal{E}'}(M)$ with $\mathcal{E}' = (E_{\text{lf}}, E_{\text{rt}} + 1)$, and $H^{(1)} \in \Psi^{2,0,\mathcal{E}'}(M)$ with $\mathcal{E} = (E_{\text{lf}}, E_{\text{rt}})$. Consequently we may write

$$\kappa_{1} = \kappa_{\mathcal{J}} + H^{(1)} \ast \partial_{y} \kappa_{L},$$

where $\kappa_{\mathcal{J}}$ lifts to a polyhomogeneous function of lf of leading order $(-1/2 - \nu - b)$ at the front face and of order $(\nu + 3/2)$ at the right boundary face. Similarly, $\kappa_{L}$ lifts to a polyhomogeneous function of lf of leading order $(-3/2 - \nu - b)$ at the front face and of order $(\nu + 1/2)$ at the right boundary face. Consequently we may write

$$F''_{1} = c_{\nu} \int_{\mathbb{R}^{b}} \int_{\mathbb{R}^{b}} \kappa_{\mathcal{J}}(\rho, \xi, u, y) P_{\rho} h(t - \rho^{2}, y - \rho u, z) \rho^{1+b} d\rho d u$$

$$+ c_{\nu} \int_{\mathbb{R}^{b}} \int_{\mathbb{R}^{b}} H^{(1)} \ast (\rho^{-1} \partial_{u} + \partial_{y}) \kappa_{L}(t - \rho^{2}, x, \rho, u, y) P_{\rho} h(t - \rho^{2}, y - \rho u, z) \rho^{1+b} d\rho d u,$$

where we have neglected the factor $\nu(\tilde{y})$ in the volume form $\text{dvol}_{B}(\tilde{y}) = \nu(\tilde{y}) d\tilde{y}$. Integrating by parts in the last integral we arrive by the composition law in Theorem 4.4 at the following expression

$$F''_{1} = x^{\nu+3/2} \int_{\mathbb{R}^{b}} \int_{\mathbb{R}^{b}} \rho^{-2\nu - 1} G_{\mathcal{J}}(\rho, \xi, u, y) P_{\rho} h(t - \rho^{2}, y - \rho u, z) d\rho d u$$

$$+ x^{\nu+1/2} \int_{\mathbb{R}^{b}} \int_{\mathbb{R}^{b}} \rho^{-2\nu + 1} G_{\mathcal{J},a}(\rho, \xi, u, y) P_{\rho} h(t - \rho^{2}, y - \rho u, z) d\rho d u$$

$$+ x^{\nu+1/2} \int_{\mathbb{R}^{b}} \int_{\mathbb{R}^{b}} \rho^{-2\nu + 1} G_{\mathcal{J},b}(\rho, \xi, u, y) (\partial_{y} P_{\rho} h)(t - \rho^{2}, y - \rho u, z) d\rho d u,$$

where the kernels $G_{\mathcal{J}}, G_{\mathcal{J},a}, G_{\mathcal{J},b}$ are bounded and polyhomogeneous in $\rho$.

For the first integral, expanding $P_{\rho} h$ in Taylor series around $(t, y, z)$, as well as expanding $G_{\mathcal{J}}$ as $\rho \to 0$, we find no contribution to terms $x^{\pm \nu + 1/2}$ and $\sqrt{x} \log(x)$ in the asymptotic expansion as $x \to 0$, unless $\nu = 1/2$. In case $\nu = 1/2$, we have $(-\nu + 3/2) = \nu + 1/2$ and hence the first integral contributes $\theta''_{1} \cdot P_{\rho} h$ to the coefficient $c^{+}(F_{\nu}(h))$, with no contribution to $c^{-}(F_{\nu}(h))$. In case $\nu \neq 1/2$, the first integral does not contribute to the coefficients $c^{\pm}(F_{\nu}(h))$ and we set $\theta''_{1} = 0$. 
For the latter two integrals the discussion is parallel to that of \( F''_0 \) and we obtain (we again abuse the notation by incorporating the \( \rho \)-factors into the kernels \( G_* \))

\[
\begin{align*}
  c^-(F''_1) &= 0, \quad c^+(F''_1) = G_{L,a} \ast P_{\nu} h + G_{L,b} \ast \partial_y P_{\nu} h + \text{const} \cdot P_{\nu} h \\
  &= \int_0^t \int_B G_{L,a}(\tilde{t}, y, \tilde{y}) P_{\nu} h(t - \tilde{t}, \tilde{y}, z) \, d\tilde{t} \, d\text{vol}_B(\tilde{y}) \\
  &\quad + \int_0^t \int_B G_{L,b}(\tilde{t}, y, \tilde{y}) \partial_y P_{\nu} h(t - \tilde{t}, \tilde{y}, z) \, d\tilde{t} \, d\text{vol}_B(\tilde{y}) \\
  &\quad + \theta''_1 \cdot P_{\nu} h(t, y, z)
\end{align*}
\]

where \( G_{L,a}, G_{L,b} \) lift to a polyhomogeneous function on the parabolic blowup of \( \mathbb{R}^+ \times B^2 \) around \( \tilde{Y} := \{(t, y, \tilde{y}) \in \mathbb{R}^+ \times B^2 \mid t = 0, y = \tilde{y}\} \) of leading order \((-2\nu - b)\) at the front face.

**Contribution from** \( F''_2 \). As before we obtain by construction

\[
\begin{align*}
  F'_2 &= c_\nu \int_{x^2}^t \int_B \kappa_2(\tilde{t}, x, y, \tilde{y}) P_{\nu} h(t - \tilde{t}, \tilde{y}, z) \, d\tilde{t} \, d\text{vol}_B(\tilde{y}) \\
  &= x^{\nu + 1/2} \int_x^{\sqrt{t}} \int_{\mathbb{R}^b} \rho^{-2\nu + 1} G''_2(\rho, \xi, u, y) P_{\nu} h(t - \rho^2, y - \rho u, z) \, v(y - \rho u) \, d\rho \, du
\end{align*}
\]

where the kernel \( G''_2 \) is bounded and polyhomogeneous in \( \rho \). Its discussion is parallel to that of \( F''_0 \) and we obtain (we again abuse the notation by incorporating the \( \rho \)-factors into the kernel \( G''_2 \))

\[
\begin{align*}
  c^-(F''_2) &= 0, \quad c^+(F''_2) = G''_2 \ast P_{\nu} h = \int_0^t \int_B G''_2(\tilde{t}, y, \tilde{y}) P_{\nu} h(t - \tilde{t}, \tilde{y}, z) \, d\tilde{t} \, d\text{vol}_B(\tilde{y}).
\end{align*}
\]

where \( G''_2 \) lifts to a polyhomogeneous function on the parabolic blowup of \( \mathbb{R}^+ \times B^2 \) around \( \tilde{Y} := \{(t, y, \tilde{y}) \in \mathbb{R}^+ \times B^2 \mid t = 0, y = \tilde{y}\} \) of leading order \((-2\nu - b)\) at the front face. This proves the statement with \( G'_\nu = G_{L,a} + G''_2, \ G''_\nu = G''_0 + G_{L,b} \) and \( \theta'_\nu = \theta''_0 + \theta'_1 + \theta''_1 \).

### 7. Heat kernel for algebraic boundary conditions

In this section we finally employ the signaling solution to construct the heat kernel for a given algebraic boundary conditions \( \Gamma \). Consider the increasing sequence of eigenvalues \( \nu^2 \in [0, 1) \) with \( j = 1, \ldots, p \), of the tangential operator \( A \) in (2.3), counted with their multiplicities. Consider \( \phi \in C^\infty_0(M) \) and put \( u = H_\varphi \phi \). We seek to correct \( u \) to satisfy algebraic boundary conditions \( \Gamma \), i.e.

\[
(7.1) \quad w = u + \sum_{j=1}^p F_{\nu_j}(h_{\nu_j}) \in \mathcal{D}(\Delta_\Gamma).
\]

where each \( h_{\nu_j} \) lies in \( \text{Im} P_{\nu_j} \). Recall \( \Gamma = (\Gamma_{ij}) \in \text{Matr}(q, \Lambda_\varphi) \) with diagonal entries given by \( \Gamma_{jj} = b_{jj} \psi_j + \theta_{jj} \psi_j^\dagger \), and the off-diagonal entries \( \Gamma_{ij} = \theta_{ij} \psi_j^\dagger \). The coefficients \( b_{ij}, \theta_{ij} \in \mathbb{R} \) are such that either \( b_{ii} = 1 \), or \( b_{ii} = 0 \), where in the latter case we require...
\( \theta_{ii} = 1 \) and \( \theta_{ij} = 0 \) for \( i \neq j \). Here we assume that \( b_{jj} = 1 \) for every \( j = 1, \ldots, p \), since in case of \( b_{jj} = 0 \), \( h_{\nu_j} = c^+_{\nu_j}(u) \). Then (7.1) reads as follows

\[
(7.2) \quad c^+_{\nu_i}(u) + c^+(F_{\nu_i}(h_{\nu_i})) = \sum_{j=1}^{p} \theta_{ij} h_{\nu_j}, \quad i = 1, \ldots, p.
\]

We define \( p \times p \) matrix valued operators

\[
G^N := (\theta_{ij})_{ij} - \text{diag}(G^N_{\nu_1}, \ldots, G^N_{\nu_p}) - \text{diag}(\theta_{\nu_1}, \ldots, \theta_{\nu_1}).
\]

\[
G := \text{diag}(G'_{\nu_1} + G''_{\nu_1}, \partial_y, \ldots, G'_{\nu_p} + G''_{\nu_p}, \partial_y).
\]

Then we may rewrite (7.2) as \( H_{\nu}\phi = (G^N - G)_{h_{\nu}} \). We apply the Fourier transform in \( y \in \mathbb{R}^{b} \) and the Laplace transform in \( t \in \mathbb{R}^{+} \), and obtain

\[
(7.3) \quad \mathcal{L} \circ \mathcal{F} H_{\nu}\phi = \mathcal{L} \circ \mathcal{F} G^N \cdot \mathcal{L} \circ \mathcal{F} h_{\nu} - \mathcal{L} \circ \mathcal{F} (Gh_{\nu})
\]

where we have set

\[
\tilde{G}^N := \mathcal{L} \circ \mathcal{F} G^N = (\theta_{ij})_{ij} - \text{diag}(\mathcal{L} \circ \mathcal{F} G^N_{\nu_1}, \ldots, \mathcal{L} \circ \mathcal{F} G^N_{\nu_p}) - \text{diag}(\theta_{\nu_1}, \ldots, \theta_{\nu_1}).
\]

Recall from Proposition 5.1 that \( \tilde{G}^N \) is a function of \((\zeta + |\omega|^2)\). Existence and the structure of the inverse matrix \( \tilde{G}^N(\zeta + |\omega|^2)^{-1} \) is the core of the subsequent section. Put

\[
D := \mathcal{F}^{-1} \circ \mathcal{L}^{-1} \tilde{G}^{-1} \mathcal{L} \circ \mathcal{F}.
\]

Then, applying the inverse \( \tilde{G}^N(\zeta + |\omega|^2)^{-1} \) on both sides of (7.3), we arrive at the following relation

\[
(7.4) \quad DH_{\nu}\phi = h_{\nu} - D \circ Gh_{\nu}.
\]

In the setup of isolated conical singularities, the operator \( G \) is absent and (7.4) provides an explicit result for \( h_{\nu} \), obtained in one step by inverting the corresponding matrix.

Unlike in the case of isolated conical singularities, on edges \( G \) is a non trivial operator on the base manifold \( B \) and the solution is obtained from (7.4) by an iterative procedure.

We define for any \( M \in \mathbb{N} \)

\[
h_{\nu}^M \equiv \left( \begin{array}{c} h_{\nu_1}^M \\ \vdots \\ h_{\nu_p}^M \end{array} \right) = \sum_{k=0}^{M} (D \circ G)^k \circ D H_{\nu}\phi
\]

\[
(7.5) \quad = \sum_{j=1}^{p} (K^M_{ij})_{\nu_j} \ast H_{\nu_j}\phi, \ldots, \sum_{j=1}^{p} (K^M_{ij})_{\nu_p} \ast H_{\nu_j}\phi \right)^t.
\]
This defines an approximate solution to (7.4), solving it up to an error

$$DH_\nu\phi = h^M_\nu - D \circ Gh^M_\nu - (D \circ G)^{M+1} DH_\nu\phi.$$  

Below we show that the error term vanishes and $h^M_\nu$ converges as $M \to \infty$. Anticipating that discussion, the heat kernel for algebraic boundary conditions $\Gamma$ is subsequently given by

$$H_\Gamma = H_\mathcal{F} + \sum_{i,j=1}^p H_{\nu_i} \ast (K^\infty_{\Gamma})_{ij} \ast H_{\nu_j},$$

where the kernels are convolved in $t$ and concatenated over $B$.

The remainder of the section is concerned with the analysis of the operator orders in the definition of $h^M$. The integral kernel of $D$ is explicitly given by $(u = (y - \tilde{y}) / \sqrt{t})$

$$D(t, y, \tilde{y}) = \int_{\mathbb{R}^b} \int_{i\mathbb{R}+\delta} e^{-i\omega(y-\tilde{y})} e^{i\omega(t)} \tilde{G}^{-1}_N(\zeta + |\omega|^2) d\zeta d\omega$$

$$= \int_{\mathbb{R}^b} \int_{i\mathbb{R}+\delta} e^{-i\omega(y-\tilde{y})} e^{-t|\omega|^2} e^{st} \tilde{G}^{-1}_N(s) ds d\omega$$

$$= (\sqrt{t})^{-b} \mathcal{L}^{-1} \tilde{G}^{-1}_N(t) \cdot \int_{\mathbb{R}^b} e^{-i\omega t} e^{-|\omega|^2} dw.$$  

The asymptotics of $\mathcal{L}^{-1} \tilde{G}^{-1}_N(t)$ as $t \to 0$ is established in Theorem 8.2 below. More precisely, noting the $p \times p$ matrix structure of the operators, we obtain for any $(ij) \in \{1, \ldots, p\}^2$

$$\mathcal{L}^{-1} \tilde{G}^{-1}_N(t)_{ij} \sim (\sqrt{t})^{-2+2\nu_{ij}}, \quad t \to 0,$$

where $\nu_{ij} = \nu_i + \nu_j$ if $i \neq j$ and $\nu_{ij} = \nu_i$ if $i = j$. Consequently, cf. the projective coordinates (6.6), the component $D_{ij}$ lifts to a polyhomogeneous function on the parabolic blowup of $\mathbb{R}^+ \times B^2$ around $Y := \{(t, y, \tilde{y}) \in \mathbb{R}^+ \times B^2 \mid t = 0, y = \tilde{y}\}$ of leading order $(-2 - b + 2\nu_{ij})$ at the front face.

**Proposition 7.1.** Let $G_1$ and $G_2$ lift to polyhomogeneous functions on the parabolic blowup of $\mathbb{R}^+ \times B^2$ around $Y := \{(t, y, \tilde{y}) \in \mathbb{R}^+ \times B^2 \mid t = 0, y = \tilde{y}\}$ of leading orders $(-2 - b + \alpha_1)$ and $(-2 - b + \alpha_2)$ at the front face, respectively. Then

$$G_1 \ast_t G_2(t, y, y') = \int_0^t \int_B G_1(t - \tilde{t}, y, \tilde{y}) G_2(\tilde{t}, \tilde{y}, y') \tilde{d}t \, dvol_B(\tilde{y}),$$

lifts to a polyhomogeneous function on $[\mathbb{R}^+_N \times B^2, Y]$ of leading order $(-2 - b + \alpha_1 + \alpha_2)$.

**Proof.** By polyhomogeneity of $G_{1,2}$ we may write for $i = 1, 2$ and any $N \in \mathbb{N}, c > 0$

$$G_i(t, y, \tilde{y}) = \sum_{k=0}^{N-1} G^k_i(t, y, \tilde{y}) + \tilde{G}^N_i(t, y, \tilde{y}),$$

$$\begin{align*}
G^k_i(c^2 t, c y, c \tilde{y}) &= c^{-2+b+\alpha_1+k} G^k_i(t, y, \tilde{y}), \\
\tilde{G}^N_i(c^2 t, c y, c \tilde{y}) &= O(c^{-2-b+\alpha_1+N}), \quad c \to 0.
\end{align*}$$
For a composition of individual homogeneous summands we obtain

\[
G^k_1 \ast t \ G^l_2(c^2 t, c y, c y') = \int_{0}^{c^2 t} \int_{B} G^k_1((c^2 t - \widetilde{t}, c y, \widetilde{y}) G^l_2(\widetilde{t}, \widetilde{y}, c y') \ d\widetilde{t} \ d\text{vol}_B(\widetilde{y}) \\
= c^{2+b} \int_{0}^{t} \int_{B} G^k_1(c^2 t - c^2 \widetilde{t}, c y, c \widetilde{y}) G^l_2(c^2 \widetilde{t}, c \widetilde{y}, c y') \ d\widetilde{t} \ d\text{vol}_B(\widetilde{y}) \\
= c^{-2-b+\alpha_1+\alpha_2+k+l} G^k_1 \ast t \ G^l_2(t, y, y').
\]

Similar estimates for the remainder terms \(\widetilde{G}^N_{1,2}\) prove the statement.

The statement on the leading orders in Proposition 7.1 above holds also without the assumption of polyhomogeneity. Moreover, presence of \(\partial_y\) derivatives is not excluded from that picture, since \(\partial_y\) corresponds to lowering the leading order at the front face of \([\mathbb{R}^+ \times B^2, Y]\) by one. Consequently, \(G\) is zero off diagonal and its diagonal components \(G_{jj} = (G_{ij} + G_{ji} \partial_y)\) may be viewed as of leading order \((-1 - b + 2\nu_j)\) at the front face, when lifted to \([\mathbb{R}^+ \times B^2, Y]\).

Hence, by Proposition 7.1, we find that \((D \circ G)_{ij} = D_{ij} \circ G_{jj}\) lifts to a polyhomogeneous distribution on \([\mathbb{R}^+ \times B^2, Y]\) of leading order \((-2 - b + \alpha_{ij})\), where \(\alpha_{ij} := 1 + 2(\nu_{ij} - \nu_j) \geq 1\). Due to \(\alpha_{ij} \geq 1\), subsequent \(M\)-fold compositions improve the order and hence \(h^M_{\nu}\) converges and the error in (7.6) vanishes as \(M \to \infty\).

8. Heat-trace asymptotic expansions

In view of (7.8), we begin with studying \(\widetilde{G}^{-1}_N\) and its inverse Laplace transform. Fix the branch of logarithm in \(\mathbb{C} \setminus \mathbb{R}^-\) for the definition of powers of complex numbers. We apply the inversion rule using adjuncts

\[
(\widetilde{G}^{-1}_N)_{ij} = (-1)^{i+j} (\text{adj} \widetilde{G}_N)_{ij}(\zeta) / \det \widetilde{G}_N(\zeta),
\]

where \((\text{adj} \widetilde{G}_N)_{ij}(\zeta)\) is the determinant of the reduced matrix, obtained from \(\widetilde{G}_N\) by deleting the \(i\)-th row and \(j\)-th column. Set \(\kappa_{\theta} := 2(\gamma - \log 2 + \theta)\) and introduce the multi-index notation, setting for \(\alpha = (\alpha_1, ..., \alpha_q) \in \mathbb{N}^q\) and \(\beta = (\beta_{q+1}, ..., \beta_p) \in \mathbb{N}^{p-q}\)

\[
(\log \zeta + \kappa_{\theta})^\alpha = \prod_{k=1}^{q} (\log \zeta + \kappa_{\theta_k})^{\alpha_k}, \quad \zeta_{\nu}^\beta = \prod_{k=q+1}^{p} \zeta_{\kappa_{\nu_k}}^{\beta_k}.
\]

Let tuples with each entry given by 1, be denoted by 1. Then in view of the explicit formulas in Proposition 5.1 we obtain

\[
\det \widetilde{G}_N(\zeta) = C \ ((\log \zeta + \kappa_{\theta}) \zeta_{\nu}) \left( 1 + \sum_{|\alpha| = 0}^{q} \sum_{|\beta| = 0}^{p} C_{\alpha\beta}(\log \zeta + \kappa_{\theta})^{-\alpha} \zeta_{\nu}^{-\beta} \right),
\]
for certain coefficients $C, C_{\alpha\beta}$. The summation above excludes $|\alpha| = |\beta| = 0$. For $|\zeta| \gg 0$ we may expand $(\det \tilde{G}_N(\zeta))^{-1}$ in Neumann series and obtain

$$(\det \tilde{G}_N(\zeta))^{-1} = C^{-1} ((\log \zeta + \kappa_\theta) \zeta_\nu)^{-1} \left( 1 + \sum_{|\alpha| + |\beta| = 1}^\infty D_{\alpha\beta} (\log \zeta + \kappa_\theta)^{-\alpha} \zeta_\nu^{-\beta} \right),$$

which is convergent for $|\zeta| \gg 0$ and in particular asymptotic series as $|\zeta| \to \infty$. Similar computations hold for $(\adj F)_{ij}(\zeta)$. Hence overall we obtain, cf. ([Moo96], Lemma 8.6)

**Proposition 8.1.** There exist constants $A_{ij}$ and $B_{\alpha\beta}$ such that for $|\zeta| \gg 0$

$$(\tilde{G}_N^{-1})_{ij}(\zeta) = A_{ij} \left( 1 + \sum_{|\alpha| + |\beta| = 1}^\infty B_{\alpha\beta} (\log \zeta + \kappa_\theta)^{-\alpha} \zeta_\nu^{-\beta} \right) \times \begin{cases} (\log \zeta + \kappa_i)^{-1}(\log \zeta + \kappa_j)^{-1}, & \text{if } \nu_i = \nu_j = 0, i \neq j, \\ (\log \zeta + \kappa_i)^{-1} \zeta^{-\nu_j}, & \text{if } \nu_i = 0, \nu_j \neq 0, \\ (\log \zeta + \kappa_i)^{-1} \zeta^{-\nu_i}, & \text{if } \nu_i \neq 0, i \neq j, \\ \zeta^{-\nu_i}, & \text{if } \nu_i = \nu_j \neq 0, i = j. \end{cases}$$

We point out that no $(\log \zeta)$ factors arise if we have Friedrichs extension on the $\nu = 0$ components of the Laplacian. This case exhibits only a classical expansion without the exotic phenomena and we do not consider it here. We now want to derive an asymptotic expansion for $(\delta > 0$ fixed)

$$\mathcal{L}^{-1}(\mathcal{F}_{\alpha\beta})(t) = \frac{1}{2\pi i} \int_{i\mathbb{R} + \delta} e^{t\zeta} (\log \zeta + \kappa_\theta)^{-\alpha} \zeta_\nu^{-\beta} d\zeta.$$  

**Theorem 8.2.** Write $\nu_\beta := \beta_q + \nu_{q+1} + .. + \beta_p \nu_p$ for the given $\beta \in \mathbb{N}^p - q$. For some $\alpha \in \mathbb{N}^q$ and $\beta \in \mathbb{N}^p - q$ we then have up to smooth additive components

$$\mathcal{L}^{-1}(\mathcal{F}_{\alpha\beta})(t) \sim \sum_{k=0}^\infty E_{\beta k} t^{-1+\nu_\beta} \log^{-|\alpha|-k}(t), \ t \to 0.$$  

**Proof.** As the first step we deform the integration region $(i\mathbb{R} + \delta)$ to $\mu$, concatenated out of three parts, $\mu_1 = i(-\infty, 1]$, the half circle $\mu_2 = \{ z \in \mathbb{C} \mid |z| = 1, \text{Re}(z) \geq 0 \}$ oriented counterclockwise, and $\mu_3 = i[1, \infty)$. The change of the integration contour is possible, since

$$\left| \frac{1}{2\pi i} \int_0^\delta e^{(x+iR)} \mathcal{F}_{\alpha\beta}(iR + x) \, dx \right| \leq (\log R)^{-|\alpha|} R^{-\nu_\beta} \to 0, \text{ as } R \to \infty.$$  

Integration over $\mu_2$ leads to a function $\mathcal{L}^{-1}(\mathcal{F}_{\alpha\beta})_{\mu_2} \in C^\infty[0, \infty)$, so that we may write

$$\mathcal{L}^{-1}(\mathcal{F}_{\alpha\beta})(t) = \mathcal{L}^{-1}(\mathcal{F}_{\alpha\beta})_{\mu_2} + \frac{1}{2\pi i} \int_{\mu_1 \cup \mu_3} e^{t\zeta} \mathcal{F}_{\alpha\beta}(\zeta) \, d\zeta$$

$$= \mathcal{L}^{-1}(\mathcal{F}_{\alpha\beta})_{\mu_2} + \frac{1}{2\pi} \int_1^\infty e^{itx} \mathcal{F}_{\alpha\beta}(ix) \, dx - \frac{1}{2\pi} \int_{-1}^\infty e^{-ix} \mathcal{F}_{\alpha\beta}(-ix) \, dx.$$
For the second summand we deform the integration contour to \( i[1, \infty) \). For the third summand we deform the integration contour to \((-i[1, \infty))\). We explicate the argument for the second summand. Let \( R > 1 \) and the contour \( \eta_R := \{ R \exp(i\phi) \mid \phi \in [0, \pi/2]\} \) be oriented counterclockwise.

\[ \int_{\eta_R} | e^{itx} \mathcal{J}_{\alpha, \beta}(ix) | \ dx = R \int_{0}^{\pi/2} | \exp(-tR \sin \phi + i(\phi + tR \cos \phi)) \mathcal{J}_{\alpha, \beta}(iRe^{i\phi}) | \ d\phi \]

\[ \leq C R (\log R)^{-|\alpha|} R^{-\nu} \int_{0}^{\pi/2} \exp(-tR \sin \phi) \ d\phi \]

\[ = C R (\log R)^{-|\alpha|} R^{-\nu} \int_{\sin \pi/4}^{\pi/2} \exp(-tR y) \ dy + O(R^{-\infty}) \]

\[ \leq C R (\log R)^{-|\alpha|} R^{-\nu} \int_{0}^{\pi/4} \exp(-tR \sin \phi) \cos(\phi) \ d\phi + O(R^{-\infty}) \]

\[ = C R (\log R)^{-|\alpha|} R^{-\nu} \int_{0}^{\pi/4} \exp(-tR y) \ dy + O(R^{-\infty}) \]

\[ = O \left( \frac{\log R}{R^{-\nu}} \right) \rightarrow 0, \quad \text{as} \ R \rightarrow \infty, \]

where in the fourth line we have used the fact that \( \cos \phi \) is bounded from below for \( \phi \in [0, \pi/4] \), and in the fifth line we substituted \( y = \sin \phi \). Subsequently, writing \( \overline{\eta} = \{ \overline{z} \mid z \in \eta \} \) for the clockwise oriented contour, we find

\[ \mathcal{L}^{-1}(\mathcal{J}_{\alpha, \beta})(t) = \mathcal{L}^{-1}(\mathcal{J}_{\alpha, \beta})_{\mu_2} + \frac{1}{2\pi} \int_{\eta} e^{itx} \mathcal{J}_{\alpha, \beta}(ix) \ dx - \frac{1}{2\pi} \int_{\eta} e^{-itx} \mathcal{J}_{\alpha, \beta}(-ix) \ dx \]

\[ + \frac{i}{2\pi} \int_{1}^{\infty} e^{-ty} \mathcal{J}_{\alpha, \beta}(-y)_{+} \ dy - \frac{i}{2\pi} \int_{1}^{\infty} e^{-ty} \mathcal{J}_{\alpha, \beta}(-y)_{-} \ dy, \]

where \((-y)_{\pm}\) denotes negative real numbers with the argument \((\pm \pi)\), and the first three summands define smooth functions \( C^{\infty}[0, \infty) \).
We continue under equivalence up to smooth functions. Then without explicating the smooth summands we obtain

$$\mathcal{L}^{-1}(\mathcal{S}_{\alpha,\beta})(t) = \frac{i}{2\pi} \left( \sum_{\rho \geq \alpha} C_{\beta,\rho}^+ \int_1^\infty e^{-ty} y^{-\nu_\beta} (\log y + i\pi + \kappa_\theta)^{-\rho} dy \right)$$

$$- \frac{i}{2\pi} \left( \sum_{\rho \geq \alpha} C_{\beta,\rho}^- \int_1^\infty e^{-ty} y^{-\nu_\beta} (\log y - i\pi + \kappa_\theta)^{-\rho} dy \right)$$

$$=: \frac{i}{2\pi} (\mathcal{L}^+_\beta(t) - \mathcal{L}^-_\beta(t)).$$

We establish an asymptotic expansion for each $\mathcal{L}^\pm_\beta(t)$ as $t \to 0$. For this we differentiate $[\nu_\beta]$ times in $t$ and obtain by a change of variables $x = ty$

$$\frac{d^{[\nu_\beta]}}{dt^{[\nu_\beta]}} \mathcal{L}^\pm_\beta(t) = \sum_{\rho \geq \alpha} (-1)^{[\nu_\beta]} C_{\beta,\rho}^\pm \int_1^\infty e^{-ty} y^{-\nu_\beta + [\nu_\beta]} (\log y \pm i\pi + \kappa_\theta)^{-\rho} dy$$

$$= \sum_{\rho \geq \alpha} (-1)^{[\nu_\beta]} C_{\beta,\rho}^\pm \int_0^\infty e^{-ty} y^{-\nu_\beta + [\nu_\beta]} (\log y \pm i\pi + \kappa_\theta)^{-\rho} dy + \text{smooth}$$

$$= \sum_{\rho \geq \alpha} (-1)^{[\nu_\beta]} C_{\beta,\rho}^\pm \int_0^\infty e^{-x} x^{-\nu_\beta + [\nu_\beta]} \left( \frac{\log(x) \pm i\pi + \kappa_\theta}{\log(t)} - 1 \right)^{-\rho} dx$$

$$\times t^{-1+\nu_\beta - [\nu_\beta]} \log^{-[\alpha]}(t) + \text{smooth}$$

Since $(1 - r)^{-1} = \sum_{k=0}^M r^k + r^{M+1}(1 - r)^{-1}$ for any $M \in \mathbb{N}$ and $r \neq 1$, we find, up to smooth additive components

$$\frac{d^{[\nu_\beta]}}{dt^{[\nu_\beta]}} \mathcal{L}^\pm_\beta(t) \sim t^{-1+\nu_\beta - [\nu_\beta]} \log^{-[\alpha]}(t) \sum_{k=0}^\infty D_{\beta,k} \log^{-k}(t), \ t \to 0.$$ 

In order to integrate the asymptotic series $[\nu_\beta]$ times, note for any $M > -1$ and $N \in \mathbb{N}$

$$\int_0^t \tau^M \log^{-N}(\tau) d\tau = \int_0^t (M + 1)^{-1} (\tau^{M+1} \log^{-N}(\tau))' d\tau + \int_0^t \frac{N}{M+1} \tau^M \log^{-N-1}(\tau) d\tau$$

$$= (M + 1)^{-1} t^{M+1} \log^{-N}(t) + O(t^{M+1} \log^{-N-1}(t)), \ t \to 0.$$ 

Iterating this argument we finally arrive, up to smooth additive components, at the asymptotic expansion

$$\mathcal{L}^\pm_\beta(t) \sim t^{-1+\nu_\beta} \log^{-[\alpha]}(t) \sum_{k=0}^\infty E_{\beta,k} \log^{-k}(t), \ t \to 0.$$ 

This proves the statement. 

Let $K$ lift to polyhomogeneous function on the parabolic blowup of $\mathbb{R}^+ \times B^2$ around $Y := \{(t, y, \tilde{y}) \in \mathbb{R}^+ \times B^2 \mid t = 0, y = \tilde{y}\}$ of leading order $(-2 - b + \gamma)$. By an ad verbatim extension of Proposition 7.1, compare also ([Moo96], Proposition 8.7), we find that the convolution $H_{\nu_\beta} \ast K \ast H_{\nu_\beta}$ lifts to a polyhomogeneous distribution on the parabolic blowup of $\mathbb{R}^+ \times (\partial M)^2$ around $Y_F := \{(t, (y, z), (\tilde{y}, \tilde{z})) \in \mathbb{R}^+ \times (\partial M)^2 \mid t = 0, y = \tilde{y}\}$,
of leading order \((-1 - b - (\nu_i + \nu_j) + \gamma)\) at the front face, and vanishes to infinite order at \(tf\) and \(td\). By standard pushforward arguments we obtain

\[
(8.2) \quad \text{Tr} H_{\nu_i} \star K \star H_{\nu_j}(t) \sim \sqrt{t} \langle t^{-(\nu_i+\nu_j)} \sum_{k=0}^{\infty} d_k \sqrt{t}^k \rangle, \text{ as } t \to 0.
\]

Combining Theorem 8.2 with (8.2), we obtain the following

**Proposition 8.3.** Let \(K\) lift to polyhomogeneous function on the parabolic blowup of \(\mathbb{R}^+ \times B^2\) around \(Y := \{(t, y, y) \in \mathbb{R}^+ \times B^2 \mid t = 0, y = \tilde{y}\}\) of leading order \((-2 \cdot b + \gamma)\). Write \(\nu_\beta := \beta_q + 1 \nu_{q+1} + \ldots + \beta_p \nu_p\) for the given \(\beta \in \mathbb{N}^{p-q}\). For any \(\alpha \in \mathbb{N}^q\) and \(\beta \in \mathbb{N}^{p-q}\) we put \(\mathcal{T}_{\alpha,\beta} = (\log \zeta + \kappa_\theta)^{-\alpha} \zeta_\nu^{-\beta}\). Then as \(t \to 0\) we obtain

\[
\text{Tr} \left( H_{\nu_i} \star \mathcal{L}^{-1}(\mathcal{T}_{\alpha,\beta}) \star K \star H_{\nu_j} \right)(t) \sim \sqrt{t} \langle t^{-(\nu_i+\nu_j)+2\nu_\beta} \sum_{k,l=0}^{\infty} d_k \sqrt{t}^k \log|\alpha|^{-l}(t) \rangle.
\]

**Proof.** The statement follows by an iterative application of the following identity

\[
\int_0^t \tau^\sigma \log^{-\rho}(\tau)(t-\tau)^\mu d\tau = \sum_{k=0}^{\infty} (-1)^k t^{\mu-k} \binom{\mu}{k} \int_0^t \tau^{\sigma+k} \log^{-\rho}(\tau) d\tau
\]

\[
= \sum_{k=0}^{\infty} \frac{(-1)^k}{\sigma+k+1} \binom{\mu}{k} \left( t^{\sigma+\mu+1} \log^{-\rho}(t) + \rho t^{\mu-k} \int_0^t \tau^{\sigma+k} \log^{-\rho-1}(\tau) d\tau \right)
\]

\[
= \sum_{k=0}^{\infty} \frac{(-1)^k}{\sigma+k+1} \binom{\mu}{k} \left( t^{\sigma+\mu+1} \log^{-\rho}(t) + O(t^{\sigma+\mu+1} \log^{-\rho-1}(t)) \right), \quad t \to 0.
\]

\[\square\]

Combining Proposition 8.1 with Proposition 8.3 we obtain a full asymptotic expansion for each \(\text{Tr} H_{\nu_i} \star t(K^\infty)_{ij} \star t H_{\nu_j}\), complicated only by the intricate combination of the various components after matrix multiplications. The heat trace asymptotic expansion admits logarithmic terms in accordance to [KLP08] and [GKM10]. We explify the leading orders in our final main result.

**Theorem 8.4.**

\[
\text{Tr} H_{\nu_i} \star t (K^\infty)_{ij} \star t H_{\nu_j} \sim_{t \to 0} \begin{cases} 
\log^{-2}(t), & \text{if } \nu_i = \nu_j = 0, i \neq j, \\
\sqrt{t} \nu_j \log^{-1}(t), & \text{if } \nu_i = 0, \nu_j \neq 0, \\
\log^{-1}(t), & \text{if } \nu_i = \nu_j = 0, i = j, \\
\sqrt{t} \nu_i + \nu_j, & \text{if } \nu_i, \nu_j \neq 0, i \neq j, \\
\sqrt{t}^0, & \text{if } \nu_i = \nu_j = 0, i = j.
\end{cases}
\]

This corresponds to ([Moo96], Theorem 8.2), up to the case \(\nu_i = \nu_j \neq 0, i = j\). Note for example that for \(\nu_i, \nu_j \neq 0\) and \(i \neq j\) the correcting kernel lies according to ([Moo96], Theorem 8.2) in the space \(\Phi_{\nu_i+\nu_j+2}\), with the front face leading order \(-3 + \nu_i + \nu_j + 2 = \nu_i + \nu_j - 1\), which leads to \(t^{(\nu_i+\nu_j)/2}\) after taking traces. Here, Theorem 8.4 and ([Moo96], Theorem 8.2) agree.
However, in case $i = j$, asymptotics of $\tilde{G}_N^{-1}(\zeta)$ in Proposition 8.1 is different and hence Theorem 8.4 states $t^0$ expansion, different from the expansion for $i \neq j$, in contrast to ([Moo96], Theorem 8.2) which still asserts a heat trace expansion of order $t^{(v_i + v_j)/2}$.

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**List of Figures**

1 Heat-space blowup $\mathcal{M}_h^2$ for incomplete edge metrics 10
2 The integration contour $\eta_R$. 26

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