THE ADVANCED MAXIMUM PRINCIPLE FOR PARABOLIC SYSTEMS ON MANIFOLDS WITH BOUNDARY

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Abstract. In this short note we extend Chow and Lu’s advanced maximum principles for parabolic systems on closed manifolds to the case of compact manifolds with boundary, which also generalizes a Hopf type theorem of Pulemotov.

1. Introduction

In his important 1982 paper [H1], Hamilton introduced Ricci flow and used it to prove that any closed three-manifolds with positive Ricci curvature is diffeomorphic to a spherical space form. In 1986 Hamilton [H2] introduced an advanced maximum principle for parabolic systems on closed manifolds with evolving metrics, and using it he was able to prove that any closed 4-manifold with positive curvature operator is diffeomorphic to a spherical space form, in addition to giving a simplified proof of his 1982 3-dimensional result.

Later, Hamilton [H3] and Ivey [I] proved independently an important pinching estimate for 3-dimensional Ricci flow using Hamilton’s advanced maximum principle. For recent applications of Hamilton’s advanced maximum principle see for example Böhm and Wilking [BW], Brendle and Schoen [BS].

In [CL] Chow and Lu presented two useful generalizations (see Theorems 3 and 4 in [CL]) of Hamilton’s advanced maximum principle for parabolic systems on closed manifolds. In this short note we extend Chow and Lu’s results to the case of compact manifolds with boundary, which also generalize a Hopf type theorem of Pulemotov (see Theorem 2.1 in [P], which itself is a generalization of Hamilton’s maximum principle in [H2] and Shen’s Hopf type theorem in [S]).

Below we will follow closely the notations in [CL]. Let $M$ be a compact manifold with boundary with a one-parameter family of Riemannian metrics $g_{ij}(t), 0 \leq t \leq T$ with $T < \infty$.

Let $\pi : V \rightarrow M$ be a vector bundle with a fixed (i.e. time-independent) bundle metric $h_{ab}$. We equip $V$ with a family of time-dependent connections $\nabla_t$ compatible with $h_{ab}$. We define the Laplacian $\Delta_t$ acting on a section $\sigma \in \Gamma(V)$ by $\Delta_t \sigma = g^{ij}(x, t)(\nabla_t)_i(\nabla_t)_j \sigma$ as usual.

Let $U$ be an open subset of $V$, and for each $t \in [0, T]$, let $\mathcal{K}(t) \subset U$ be a closed subset such that in each fiber $V_x$ over $x \in M$ the subset $\mathcal{K}_x(t) = \mathcal{K}(t) \cap V_x$ is nonempty, closed and convex, and $\mathcal{K}(t)$ is invariant under parallel translation.

1991 Mathematics Subject Classification. 53C44, 58J35.
Key words and phrases. advanced maximum principle, parabolic systems, Hopf Lemma, manifolds with boundary.
Partially supported by NSFC no.10671018.
defined by the connection $\nabla_t$. We assume further the space-time track $T = \{(v, t) \in V \times [0, T] | v \in K(t), t \in [0, T]\}$ is closed.

Let $F: U \times [0, T] \to V$ be a fiber preserving map, which may be viewed as a time-dependent vector field on $U$ which is tangent to the fibers. We assume $F(x, \sigma, t)$ is continuous in $x, t$ and Lipschitz continuous in $\sigma$ in the sense that the inequalities

$$|F(x, \sigma_1, t) - F(x, \sigma_2, t)| \leq C_B |\sigma_1 - \sigma_2|$$

hold true for all $x \in M, t \in [0, T]$ and $|\sigma_1| \leq B, |\sigma_2| \leq B$, where $C_B$ is a constant depending only on $B$.

We will consider sections of $V$ which satisfy the following

$$\frac{d}{dt}x(t) = \Delta x(t) + u'(x, t)(\nabla_t)^i x(t) + F(x, \sigma(x, t), t),$$

where $u'$ is a time-dependent vector field on $M$.

For convenience we recall some notions about convex sets. Let $J$ be a closed convex subset of $R^n$ and $v \in \partial J$. We denote by $C_v J$ the tangent cone to $J$ at $v$, which is the smallest closed convex cone containing $J$ with vertex at $v$, and by $S_v J$ the set of support functions $l$ for $J$ at $v$, which are linear functions on $R^n$ satisfying $|l| = 1$ and $l(v) \geq l(w)$ for all $w \in J$.

Now we can state our

**Theorem** Let $M, V, K(t)$ and $F$ be as above. Assume that for any $x \in M$ and any $t_0 \in [0, T]$, any solution $\rho_x(t)$ of the

$$(\text{ODE}) \quad \frac{d}{dt} x(t) = F(x, \rho_x, t),$$

which starts in $K_\varepsilon(t_0)$ at $t_0$ will remain in $K_\varepsilon(t)$ for all later times $t \in [t_0, T]$. Moreover, assume that for any $x \in \partial M, t \in (t_0, T)$ (given $t_0 \in [0, T]$) and any $v \in \partial K_\varepsilon(t)$, the solution $\sigma$ of the (PDE) satisfies $v + (\nabla_t)^i \sigma(x, t) \in C_v K_\varepsilon(t)$, where $\nu$ is the unit outward normal vector to $\partial M$ at $(x, t)$. Then the solution $\sigma$ of the (PDE) will remain in $K(t)$ for all later times $t \in [t_0, T]$ provided it starts in $K(t_0)$ at $t_0$.

In the following section we will give a proof of our theorem. In forthcoming papers we will generalize the Hamiton-Ivey pinching estimate and Hamilton’s 4-dimensional theorem in [H2] to the case of manifolds with boundary as applications of the various generalized maximum principles.

## 2. Proof of Theorem

**Proof** We will adapt Hamilton’s and Chow and Lu’s idea (in [H2] and [CL]) to our case.

We prove by contradiction. Suppose we have a solution $\sigma$ of the (PDE) which starts in $K(t_0)$ at $t_0$ but runs out of $K(t_2)$ at some time $t_2 \in (t_0, T]$. Then there exists $t_0 \leq t_1 < t_2$ such that $\sigma(x, t_1) \in K_\varepsilon(t_1)$ for all $x \in M$ but for any $t \in (t_1, t_2]$ there is some $x \in M$ such that $\sigma(x, t) \notin K_\varepsilon(t)$. Let

$$g(x, t) = d(\sigma(x, t), K_\varepsilon(t)) = \inf \{|\sigma(x, t) - w| | w \in K_\varepsilon(t)|$$

for each $x \in M, t \in [t_1, t_2]$, and

$$f(t) = \sup_{x \in M} g(x, t)$$

for each $t \in [t_1, t_2]$, where we define the distance $d(w_1, w_2)$ between $w_1 \in V_x$ and $w_2 \in V_x$ using the metric $h_{ab}$ and denote it by $|w_1 - w_2|$.

By choosing $r$ large enough we may assume that $\sigma(x, t) \in V(r)$ and $d(\sigma(x, t), \partial (V(r) \cap V_x)) > d(\sigma(x, t), K_\varepsilon(t))$ for all $x \in M, t \in [t_1, t_2]$, where $V(r)$ is the (closed) tubular
neighborhood of the zero section in $V$ whose intersection with each fiber $V_x$ is a ball of radius $r$ (measured by the bundle metric $h_{ab}$) around the origin. Note that

$$f(t) = \sup\{l(\sigma(x, t) - v)|x \in M, v \in (\partial K_x(t)) \cap V(r) \text{ and } l \in S_v K_x(t)\}$$

for each $t \in (t_1, t_2)$.

We will prove by contradiction that $\frac{d^+ f(t)}{dt} \leq Cf(t)$ for any $t \in (t_1, t_2)$, where

$$\frac{d^+ f(t)}{dt} = \limsup_{h \to 0} \frac{f(t+h) - f(t)}{h},$$

and $C = C_\gamma$ is the Lipschitz constant of $F(x, \sigma, t)$ (w.r.t. $\sigma$) within $V(r)$. Suppose this is not the case. Then we can find $t_\alpha \in (t_1, t_2)$ such that $\frac{d^+ f(t_\alpha)}{dt} > Cf(t_\alpha)$.

From [CL] and the assumptions of our theorem we know that there exist $x_\infty \in M, v_\infty \in (\partial K_{x_\infty}(t_\alpha)) \cap V(r)$, and $l_\infty \in S_{v_\infty} K_{x_\infty}(t_\alpha)$, such that $f(t_\alpha) = l_\infty(\sigma(x_\infty, t_\alpha) - v_\infty) = |\sigma(x_\infty, t_\alpha) - v_\infty|$ and

$$\frac{d^+ f(t_\alpha)}{dt} \leq l_\infty(|\Delta_{t_\alpha} \sigma(x_\infty, t_\alpha)| + |u^i(x_\infty, t_\alpha)(\nabla_{t_\alpha} i)\sigma(x_\infty, t_\alpha)| + l_\infty(F(x_\infty, \sigma(x_\infty, t_\alpha), t_\alpha) - F(x_\infty, v_\infty, t_\alpha))$$

$$\leq l_\infty(|\Delta_{t_\alpha} \sigma(x_\infty, t_\alpha)| + l_\infty(u^i(x_\infty, t_\alpha)(\nabla_{t_\alpha} i)\sigma(x_\infty, t_\alpha)) + C|\sigma(x_\infty, t_\alpha) - v_\infty|.$$ Then we have

$$\frac{d^+ f(t_\alpha)}{dt} \leq l_\infty(|\Delta_{t_\alpha} \sigma(x_\infty, t_\alpha)| + l_\infty(u^i(x_\infty, t_\alpha)(\nabla_{t_\alpha} i)\sigma(x_\infty, t_\alpha)) > 0.$$ Now we extend $v_\infty$ and $l_\infty$ respectively by parallel translation (defined by the connection $\nabla_{t_\alpha}$) along geodesics (w.r.t. $g_{ij}(t_\alpha)$) emanating radially from $x_\infty$, and we still denote what we get by $v_\infty$ and $l_\infty$ respectively. Then by our assumption on $K$ we still have $v_\infty \in (\partial K_{x_\infty}(t_\alpha)) \cap V(r)$, and $l_\infty \in S_{v_\infty} K_{x_\infty}(t_\alpha)$. It is easy to see that the function $l_\infty(\sigma(x_\infty, t_\alpha) - v_\infty)$ has a local maximum at $x_\infty$.

Then from (*) we have

$$\Delta_{t_\alpha} l_\infty(\sigma(x, t_\alpha) - v_\infty) + u^i(x, t_\alpha)(\nabla_{t_\alpha} i)l_\infty(\sigma(x, t_\alpha) - v_\infty) > 0$$

at $x_\infty$ and hence also in a sufficiently small neighborhood of $x_\infty$. It follows that $x_\infty \in \partial M$, and by Hopf’s lemma we have $\frac{\partial}{\partial x^i} l_\infty(\sigma(x, t_\alpha) - v_\infty) > 0$. But on the other hand, by our assumption we have $v_\infty + (\nabla_{t_\alpha} i)\sigma(x_\infty, t_\alpha) \in C_{v_\infty} K_{x_\infty}(t_\alpha)$ which implies $l_\infty((\nabla_{t_\alpha} i)\sigma(x_\infty, t_\alpha)) \leq 0$, so $\frac{\partial}{\partial x^i} l_\infty(\sigma(x, t_\alpha) - v_\infty) \leq 0$, and we arrive at a contradiction. Then by Lemma 7 and Lemma 12 in [CL] $f(t) = 0$ on $(t_1, t_2)$, which contradicts the choice of $\sigma$, and we are done.

**Remark** Similarly one can also generalize Chow and Lu’s Theorem 4 in [CL].

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