Quintessence Problem and Brans-Dicke Theory

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It has been shown that Brans-Dicke (BD) theory in anisotropic cosmological model can alone solve the quintessence problem and we have accelerated expanding universe without any quintessence matter. Also the flatness problem has been discussed in this context.

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I. INTRODUCTION

For the last few decades, it is generally believed that after a (small) period of inflationary era (when the universe was accelerated) the universe has been evolving in a Big bang scenario (deceleration of the universe) and most of the present day observations are in accord with this cosmological model (known as Standard Cosmological Model). But difficulty has started when recent data from high redshift supernovae [1,2] (in fact from luminosity redshift relation of type I a supernovas upto about $z = 1$) suggest an accelerated universe at the present epoch (known as quintessence problem). So there must be some matter field (known as Quintessence matter or Q-matter) which is either neglected or unknown, responsible for this accelerated universe. From theoretical point of view, a lot of works [3-8] has been done to solve this quintessence problem and possible candidates for Q-matter are Cosmological Constant (or more generally a variable cosmological term), a scalar field [9,10] with a potential giving rise to a negative pressure at the present epoch, a dissipative fluid with an effective negative stress [11] and more exotic matter like a frustrated network of non-abelian cosmic strings or domain walls [12,13]. In these works, it is generally assumed that Q-matter behaves as a perfect fluid with barotropic equation of state and most of them have considered spatially flat model of the universe (only the work of Chimento et al [11] has been done for open model of the universe). Recently, Banerjee et al [14] have shown that it is possible to have an accelerated universe with BD-theory in Friedmann model without any matter.

In this paper we have generalized their work considering anisotropic models of the universe. The paper is organized as follows: In section II, we have studied field equations and with its solutions. In section III, we have discussed the conformal transformations to the model and also solve the flatness problem. In the last section (i.e., section IV) we give some remarks of this paper.

II. FIELD EQUATIONS AND SOLUTIONS

In this paper, we consider anisotropic space-time model described by the line element

$$ds^2 = -dt^2 + a^2 dx^2 + b^2 d\Omega_k^2$$

(1)

where $a$ and $b$ are functions of time alone: we note that
\[
\Omega_k^2 = \begin{cases} 
    dy^2 + dz^2, & \text{when } k = 0 \text{ (Bianchi I model)} \\
    d\theta^2 + \sin^2 \theta d\phi^2, & \text{when } k = +1 \text{ (Kantowski-Sachs model)} \\
    d\theta^2 + \sinh^2 \theta d\phi^2, & \text{when } k = -1 \text{ (Bianchi III model)}
\end{cases}
\]

Here \( k \) is the curvature index of the corresponding 2-space, so that the above three types are described by Thorne [15] as Euclidean, open and semi closed respectively.

Now, in BD theory, assuming a perfect fluid distribution as only matter field, the field equations for the above space-time symmetry are

\[
\frac{\ddot{a}}{a} + 2\frac{\ddot{b}}{b} = -\frac{1}{3(2+\omega)} [(2+\omega)\rho_f + 3(1+\omega)p_f] - \omega \left(\frac{\dot{\phi}}{\phi}\right)^2 - \frac{\ddot{\phi}}{\phi} \tag{2}
\]

\[
\left(\frac{\dot{b}}{b}\right)^2 + 2\frac{\dot{a}}{a} + \frac{\rho_f}{\phi} - \frac{k}{b} - \left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right) \dot{\phi} + \frac{\omega}{2} \left(\frac{\dot{\phi}}{\phi}\right)^2 \tag{3}
\]

and the wave equation for BD scalar field is

\[
\ddot{\phi} + \left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right) \dot{\phi} = \frac{1}{3(2+\omega)} (\rho_f - 3p_f) \tag{4}
\]

Here \( \rho_f \) and \( p_f \) are the density and hydrostatic pressure respectively of the fluid distribution, obeying the barotropic equation of state

\[
p_f = (\gamma_f - 1)\rho_f \tag{5}
\]

\( \gamma_f \) being the constant adiabatic index of the fluid causality demands \( 0 \leq \gamma_f \leq 2 \) and as usual \( \omega \) is the BD coupling parameter. Now the above field equations, via the Bianchi identities lead to the energy conservation equation

\[
\dot{\rho_f} + \left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right) (\rho_f + p_f) = 0 \tag{6}
\]

At present, the universe is considered as matter dominated (i.e., filled with cold matter (dust) of negligible pressure) so considering \( p_f = 0 \), we have from equation (5) (after integration)

\[
\rho_f = \frac{\rho_0}{V} \tag{6}
\]

where \( V = ab^2 \) is the volume index at the present instant and \( \rho_0 \) is an integration constant. Also using \( p_f = 0 \) and equation (6), the wave equation has a first integral

\[
V\ddot{\phi} = \frac{\rho_0^2 t}{3(2+\omega)} + c_o \tag{7}
\]

We now assume a power-law form of the scale factors keeping in mind that we must have an accelerated universe to match the recent observations. So we take

\[
a(t) = a_0 t^\alpha, \quad b(t) = b_0 t^\beta \tag{8}
\]
and consequently

\[ V = V_0 t^{\alpha + 2\beta}, \quad (V_0 = a_0 b_0^2) \]  

(9)

where \((a_0, b_0)\) are positive constants and \((\alpha, \beta)\) are real constants. Thus the matter density and the BD-scalar field takes the form

\[ \rho_f = \rho_0 t^{-(\alpha + 2\beta)}, \quad (\rho_f = \rho_0/V_0) \]  

(10)

and

\[ \phi = \frac{\rho_1 t^{2-\alpha-2\beta}}{(3+2\omega)(2-\alpha-2\beta)} + \frac{c_0 t^{1-\alpha-2\beta}}{V_0(1-\alpha-2\beta)} \]  

(11)

The field equations will be consistent for the above solutions (eqs.(8)-(11)) provided we have the following restrictions on the parameters

\[ c_0 = 0, \quad \beta = 1 \quad \text{and} \]

(12)

either

\[ \alpha = 1, \quad \omega = -2 \left(1 + \frac{k}{3b_0^2}\right) \]  

(13)

or

\[ \omega = -2, \quad \frac{k}{b_0^2} = \alpha - 1 \]  

(14)

It is to be noted that both the cases coincide for flat model of the universe and we have the solution

\[ a = a_0 t, \quad b = b_0 t, \quad \rho_f = \rho_1 t^{-3}, \quad \phi = \rho_1 t^{-1}, \quad 2\omega + 3 = -1 \]  

(15)

so for \(k \neq 0\), the solution in first case is

\[ a = a_0 t, \quad b = b_0 t, \quad \rho_f = \rho_1 t^{-3}, \quad \phi = -\frac{\rho_1 t^{-1}}{2\omega + 3} \]  

(16)

and

\[ 2\omega + 3 = -\left(1 + \frac{4k}{3b_0^2}\right). \]

(for \(k < 0, b_0^2 > 4/3\)) while for the second case the expression for the geometric and physical parameters are

\[ a = a_0 t, \quad b = b_0 t, \quad \rho_f = \rho_1 t^{-(\alpha+2)}, \quad \phi = \frac{\rho_1}{\alpha} t^{-\alpha}, \quad 2\omega + 3 = -1 \]  

(17)

and we have a restriction

\[ \frac{k}{b_0^2} = \alpha - 1 \]
The deceleration parameter has the expression

$$ q = - \left( \frac{\alpha - 1}{\alpha + 2} \right), $$

(18)

(Note that $\alpha = 1$ corresponds to the first two case)

Thus we always have an accelerated model of the universe (in the third case i.e., eq.(17)) (except for $-2 \leq \alpha \leq 1$) as predicted by recent Supernova observation. Hence for the solutions represented by equations (15) and (16) $q = 0$ i.e., at present the universe is in a state of uniform expansion and the conclusion is identical to that of Banerjee etal [14]. Further, the solution corresponds to equation (17) is new as we have a negative deceleration parameter. This solution is valid for closed (or open) model of the universe for $\alpha > 1$ (or $\alpha < 1$). Therefore, it is possible to have an accelerated universe today by considering anisotropy model of the universe and hence the quintessence problem may be solved for closed, open or flat type of model of the universe.

### III. CONFORMAL TRANSFORMATIONS : FLATNESS PROBLEM

In cosmology, the technique of conformal transformation is often used (for mathematical simplification) to transform a non-minimally coupled scalar field to a minimally coupled one [16]. Usually, in ‘Jordan Conformal frame’ the scalar field (also BD scalar field) couples non-minimally to the ‘Einstein frame’ in which the transformed scalar field is minimally coupled. In the last section, we have developed the BD theory in Jordan frame and to introduce the Einstein frame we make the following transformations:

$$ d\eta = \sqrt{\phi} \, dt, \quad \bar{a} = \sqrt{\phi} \, a, \quad \bar{b} = \sqrt{\phi} \, b, \quad \psi = \ln \phi, \quad \bar{\rho}_f, \bar{p}_f = \phi^{-2} \rho_f, \rho_f = \phi^{-2} \rho_f, \quad (19) $$

As a result the field equations (2)-(4) transformed to

$$ \frac{\dddot{a}}{a} + 2 \frac{\dddot{b}}{b} = \frac{1}{2} \left( \ddot{\rho}_f + 3 \ddot{p}_f \right) - \frac{(3 + 2\omega)}{2} \psi'^2 \quad (20) $$

$$ \left( \frac{\dddot{b}}{b} \right)^2 + 2 \frac{\dddot{a}}{a} \frac{\dddot{b}}{b} + \frac{k}{b^2} = \ddot{\rho}_f + \frac{(3 + 2\omega)}{4} \psi'^2 \quad (21) $$

and

$$ \psi'' + \left( \frac{\dddot{a}}{a} + 2 \frac{\dddot{b}}{b} \right) \psi' = \frac{1}{3 + 2\omega} \left( \ddot{\rho}_f - 3 \ddot{p}_f \right) \quad (22) $$

where $\prime \equiv \frac{d}{d\eta}$.

These equations are the well known fields equations for the anisotropic cosmological models (describe here) with a minimally coupled scalar field $\psi$ (massless). This scalar field behaves like a ‘stiff’ perfect fluid with equation of state

$$ \bar{\rho}_\psi = \bar{\rho}_\psi = \frac{\psi'^2}{16\pi G} \quad (23) $$

In Einstein frame, the total stress-energy tensor is conserved, but the scalar field and normal matter change energy according to
\[ \ddot{\rho}_f + \left( \frac{\dot{a}'}{a} + 2\frac{\dot{b}'}{b} \right) (\dot{\rho}_f + \dot{\rho}_\psi) = - \left[ \ddot{\rho}_\psi + \left( \frac{\dot{a}'}{a} + 2\frac{\dot{b}'}{b} \right) (\dot{\rho}_\psi + \dot{\rho}_f) \right] = -\frac{\psi'}{2} (\dot{\rho}_f - 3\dot{\rho}_\psi) \] (24)

Thus combining the two energy densities we have from the above equation

\[ \ddot{\rho} + 3\gamma H \dot{\rho} = 0 \] (25)

Here \( H = \frac{1}{3} \left( \frac{\dot{a}'}{a} + 2\frac{\dot{b}'}{b} \right) \) is the Hubble parameter in the Einstein frame and \( \gamma \) is the average barotropic index defined as

\[ \gamma \Omega = \gamma_f \Omega_f + \gamma_\psi \Omega_\psi \] (26)

where

\[ \Omega = \Omega_f + \Omega_\psi = \frac{\dot{\rho}}{3H^2} \] (27)

is the density parameter.

From equations (21) and (25) after some algebra, we have

\[ \Omega' = \Omega(\Omega - 1)[\gamma H_a + 2(\gamma - 1)H_b] \] (28)

where \( H_a = \frac{\dot{a}'}{a} \) and \( H_b = \frac{\dot{b}'}{b} \).

This evolution in \( \Omega \) shows that \( \Omega = 1 \) is a possible solution of it and for stability of this solution, we have

\[ \gamma < \frac{2}{3}, \] (29)

for the solutions given in equations (15) and (16) and the restriction is

\[ \gamma < \frac{2}{\alpha + 3}, \] (30)

for the solution (17).

Since the adiabatic indices do not change due to conformal transformation so we take \( \gamma_f = 1 \) (since \( p_f = 0 \)) and \( \gamma_\psi = 2 \). Hence from (26) and (27), we have

\[ \gamma = \frac{\Omega_f + 2\Omega_\psi}{\Omega_f + \Omega_\psi} \] (31)

Now due to upper limit of \( \gamma \) (given above) we must have the inequalities

\[ \Omega_f < 4|\Omega_\psi| \]

\[ OR \]

\[ \Omega_f < \frac{2(\alpha + 1)}{\alpha} |\Omega_\psi| \] (32)

according as \( \gamma \) is restricted by (29) or (30). Also from the field equation (21), the curvature parameter \( \Omega_k = -k/b^2 \) vanishes for the solution \( \Omega = 1 \), provided we are restricted to \( \alpha = 1 \). Therefore, depending on the relative magnitude of the energies of matter and the BD-scalar field (as in isotropic case) it is possible to have a stable solution corresponding to \( \Omega = 1 \) and hence the flatness problem can be solved to extent.
IV. CONCLUDING REMARKS

In this work, we have considered three anisotropic cosmological models namely, Bianchi III \((k < 0)\), axially symmetric Bianchi I \((k = 0)\) and Kantowski-Sachs \((k > 0)\) space-time. We have shown that the anisotropic character is responsible for getting an accelerated model of the universe. In fact, for the present model the shear scalar is given by \(\sigma^2 = \frac{2}{3}(\alpha - 1)^2\), so the deceleration parameter (see eq. (18)) is proportional to \(\sqrt{\sigma}\) and it shows how anisotropy characterizes the accelerating or decelerating universe. In other words we can say that anisotropic nature of the universe has an effect on the quintessence problem. The problem for negative coupling constant \(\omega\) is same as in isotropic case and there is problem in big-bang nucleosynthesis scenario as claimed by Banerjee et al [14]. Lastly, in this case the modified version of BD-theory (where the coupling parameter \(\omega\) is a function of the scalar field) is very similar to that for isotropic case, so we have not presented here.

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References:

[1] Perlmutter S et al. Nature (London) 391 51 (1998); Astrophys. J. 517 565 (1999).
[2] Riess A G et al. Astrophys. J. 116 1009 (1998); Garnavich P M et al. Astrophys. J. 509 74 (1998).
[3] Ostriker J P and Steinhardt P J Nature (London) 377 600 (1995).
[4] Peebles J P E. Astrophys. J. 284 439 (1984).
[5] Wang L, Caldwell R, Ostriker J P, Ateinhart P J. Astrophys. J. 530 17 (2000).
[6] Caldwell R R, Dave R and Steinhardt P J Phys. Rev. Lett. 80 1582 (1998).
[7] Perlmutter S, Turner M S and White M. Phys. Rev. Lett. 83 670 (1999); Dodelson S, Kaplanhag M and Stewart E “Tracking Oscillating energy” astro-ph/0002360.
[8] Faraoni V. Phys. Rev. D 62 023504 (2000).
[9] The scalar field was introduced by Peebles and Ratra; Peebles J P E and B Ratra Astrophys. J. Lett. 325 L17 (1988); B Ratra and Peebles J P E Phys. Rev. D 37 3406 (1988).
[10] For recent work:
Ott, T. Phys. Rev. D 64 023518 (2001); Hwang, J.-c. and H. Noh, Phys. Rev. D 64 103509 (2001); Ferreira, P. G. and M. Joyce, Phys. Rev. D 58 023503 (1998). For a review see, Peebles J P E and B Ratra, astro-ph/0207347.
[11] Chimento L P, Jakubi A S and Pavon D Phys. Rev. D 62 063508 (2000).
[12] Bucher M and Spergel D Phys. Rev. D 60 043505 (1999).
[13] Battye R A, Bucher M and Spergel D “Domain wall dominated universe”, astro-ph/990847.
[14] Banerjee N and Povon D. Phys. Rev. D 63 043504 (2001).
[15] Thorne K S. Astrophys. J. 148 51 (1967).
[16] Faraoni V, Gunzig E and Nardone P Fundam. Cosm. Phys. 20 121 (1999).