su_q(2)-Invariant Schrödinger Equation of the Three-Dimensional Harmonic Oscillator

M. IRAC-ASTAUD
Laboratoire de Physique Théorique de la Matière Condensée,
Université Paris VII, 2, Place Jussieu,
F-75251 Paris Cedex 05, France
e-mail: mici@ccr.jussieu.fr

C. QUESNE *
Physique Nucléaire Théorique et Physique Mathématique,
Université Libre de Bruxelles, Campus de la Plaine CP229,
Boulevard du Triomphe, B-1050 Brussels, Belgium
e-mail: cquesne@ulb.ac.be

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Abstract

We propose a \(q\)-deformation of the \(su(2)\)-invariant Schrödinger equation of a spinless particle in a central potential, which allows us not only to determine a deformed spectrum and the corresponding eigenstates, as in other approaches, but also to calculate the expectation values of some physically-relevant operators. Here we consider the case of the isotropic harmonic oscillator and of the quadrupole operator governing its interaction with an external field. We obtain the spectrum and wave functions both for \(q \in \mathbb{R}^+\) and generic \(q \in S^1\), and study the effects of the \(q\)-value range and of the arbitrariness in the \(su_q(2)\) Casimir operator choice. We then show that the quadrupole operator in \(l = 0\) states provides a good measure of the deformation influence on the wave functions and on the Hilbert space spanned by them.

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1 Introduction

Since the advent of quantum groups and quantum algebras \[1, 13, 21\], there has been a lot of interest in deformations of the harmonic oscillator, since the latter plays a central role in the investigation of many physical systems. Most
studies were concerned with the one-dimensional oscillator [2, 3, 17]. Various q-deformed versions of standard quantum mechanics in the Schrödinger representation were proposed for the latter by using either the ordinary differentiation operator (see e.g. [17]), or a q-differentiation one (see e.g. [16, 18, 22]).

The many-body problem for one-dimensional q-oscillators or, equivalently, the deformation of the N-dimensional harmonic oscillator in cartesian coordinates was also considered [11], but relatively few works dealt with the deformation of the same in radial and angular coordinates. In one approach [14], only the radial problem was deformed via a purely algebraic procedure based upon a \( \text{so}_q(2,1) \oplus \text{so}(N) \) dynamical symmetry algebra. Only the \( N = 2 \) case was dealt with in detail, although extensions to higher dimension would in principle be feasible.

In another type of analysis, use was made of a differential calculus on the \( N \)-dimensional non-commutative Euclidean space [4] to construct and solve a \( q \)-deformed Schrödinger equation for the isotropic harmonic oscillator [4, 10, 5, 19]. In such a setting, the underlying \( \text{SO}(N) \) symmetry gets replaced by an \( \text{SO}_q(N) \) symmetry. In still another study [6], the wave functions and energy spectrum of the \( q \)-isotropic oscillator were obtained in a rather indirect way from some previous results for the \( q \)-linear oscillator [3]. The latter were derived by replacing classical Poisson brackets by \( q \)-commutators (instead of standard commutators). In the last two approaches, the wave functions involve some non-commutative objects: the variables of the quantum Euclidean space in the former, and the elements of the \( \text{SU}_q(2) \) quantum group in the latter. This complicates the calculation and interpretation of operator matrix elements.

The purpose of this Letter is to present an entirely different approach to the \( q \)-deformation of the \( \text{su}(2) \)-invariant Schrödinger equation of a spinless particle in a central potential, which allows us not only to determine a deformed spectrum as in other works, but also to easily calculate the expectation values of some physically-relevant operators, thereby evaluating the deformation influence on the corresponding wave functions. Here we consider in detail the case of a particle in an isotropic harmonic oscillator and of the quadrupole operator governing its interaction with an external field.

In our approach, only the angular sector is deformed by using a representation of the \( \text{su}_q(2) \) quantum algebra on the two-dimensional sphere [20]. This gives rise to an appropriate change in the angular part of the scalar product [13], and to the substitution of the \( \text{su}_q(2) \) Casimir operator eigenvalue for the \( \text{su}(2) \) one in the radial Schrödinger equation [12]. The latter step may be performed in various ways since there is no unique rule for constructing the \( \text{su}_q(2) \) Casimir operator. Similarly, the deforming parameter \( q \) may be assumed either real and positive, or on the unit circle in the complex plane (but different from a root of unity), provided different scalar products are used [13]. We will study the effects of these two choices on the solutions of the radial Schrödinger equation.

In Section 2, the \( \text{su}_q(2) \)-invariant Schrödinger equation of the three-dimensional harmonic oscillator is introduced and solved. In Section 3, its spectrum is studied in detail for various choices of \( \text{su}_q(2) \) Casimir operators and \( q \) ranges. The effect of the deformation on the corresponding wave functions is determined
in Section 4 by calculating the quadrupole moment in $l = 0$ states. Finally, Section 5 contains the conclusion.

2 \textit{su}_q(2)-\text{Invariant Schrödinger Equation}

Let

$$H_q = \frac{\hbar^2}{2\mu} \left( \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{C_q}{r^2} \right) + \frac{1}{2} \mu \omega^2 r^2$$

(1)

be the Hamiltonian of a $q$-deformed three-dimensional harmonic oscillator in spherical coordinates $r, \theta, \phi$. Here $C_q$ is the $\text{su}_q(2)$ Casimir operator, which we may take as

$$C_q = J_+ J_- + \left[ J_3 - \frac{1}{2} \right]^2_q - \frac{1}{4}$$

(2)

where $[x]_q \equiv (q^x - q^{-x}) / (q - q^{-1})$, and $q = e^w \in R^+$ or $q = e^{iw} \in S^1$ (but different from a root of unity). The operators $J_3, J_+, J_-$, satisfying the $\text{su}_q(2)$ commutation relations

$$[J_3, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = [2J_0]_q$$

(3)

are defined in terms of the angular variables by

$$J_3 = -i\partial_\phi, \quad J_+ = -e^{i\phi} \left( \tan(\theta/2) [T_1]_q q^{T_2} + \cot(\theta/2) q^{T_1} [T_2]_q \right), \quad J_- = e^{-i\phi} \left( \cot(\theta/2) [T_1]_q q^{T_2} + \tan(\theta/2) q^{T_1} [T_2]_q \right)$$

(4)

with $T_1 = -\frac{1}{2} (\sin \theta \partial_\theta - i \partial_\phi)$, $T_2 = -\frac{1}{2} (\sin \theta \partial_\theta + i \partial_\phi)$ \cite{21, 12}.

Instead of Equation (2), we may alternatively use the operator

$$C'_q = J_+ J_- + [J_3]_q [J_3 - 1]_q$$

(5)

in Equation (1), in which case the corresponding Hamiltonian will be denoted by $H'_q$.

The Hamiltonians $H_q$ and $H'_q$ remain invariant under $\text{su}_q(2)$ since they commute with $J_3, J_+, J_-$, and they coincide with the Hamiltonian of the standard three-dimensional isotropic oscillator when $q = 1$. For simplicity’s sake, we shall henceforth adopt units wherein $\hbar = \mu = \omega = 1$.

The $\text{su}_q(2)$-invariant Schrödinger equation

$$H_q \psi_{n\ell m q}(r, \theta, \phi) = E_{n\ell q} \psi_{n\ell m q}(r, \theta, \phi)$$

(6)

is separable and the corresponding wave functions can be written as

$$\psi_{n\ell m q}(r, \theta, \phi) = R_{n\ell q}(r) Y_{\ell m q}(\theta, \phi)$$

(7)
The $q$-spherical harmonics $Y_{lmq}(\theta, \phi)$, satisfying the equations

\begin{align}
C_q Y_{lmq}(\theta, \phi) &= C_q(l) Y_{lmq}(\theta, \phi), \quad C_q(l) = \left[l + \frac{1}{2}\right]^2 - \frac{1}{4}, \tag{8}
\end{align}

\begin{align}
J_3 Y_{lmq}(\theta, \phi) &= m Y_{lmq}(\theta, \phi), \tag{9}
\end{align}

are given by \cite{20, 13}

\begin{align}
Y_{lmq}(\theta, \phi) &= \mathcal{N}_{lmq} Q_{lq} \left(\cot^2(\theta/2)\right) R_{m\alpha} (\cot^2(\theta/2)) \cos^m(\theta/2) e^{im\phi},
\end{align}

\begin{align}
\mathcal{N}_{lmq} &= (-1)^l \left([2l + 1]_q[l + m]_q\right)^{1/2} (4\pi[l - m]_q)^{-1/2},
\end{align}

\begin{align}
Q_{lq} \left(\cot^2(\theta/2)\right) = \prod_{k=0}^{l-1} (1 + q^{2k-2} \cot^2(\theta/2))^{-1},
\end{align}

\begin{align}
R_{m\alpha} (\cot^2(\theta/2)) = \left[l\right]_q! [l - m]_q! \sum_k \frac{(-\cot^2(\theta/2))^k}{[k]_q! [l-m-k]_q! [l-k]_q! [m+k]_q!},
\end{align}

and $[x]_q \equiv [x]_q [x-1]_q \ldots [1]_q$ if $x \in \mathbb{N}^+$, $[0]_q! \equiv 1$, and $([x]_q!)^{-1} \equiv 0$ if $x \in \mathbb{N}^-$. The functions $Y_{lmq}(\theta, \phi)$, where $l = 0, 1, 2, \ldots$, and $m = -l, -l+1, \ldots, l$, form an orthonormal set with respect to the scalar product \cite{13}

\begin{align}
\langle l' m' | l m \rangle_q &\equiv (q - q^{-1})(4\ln q)^{-1} f_0^\pi d\theta \int_0^{2\pi} d\phi
\times \left(\frac{Y_{l' m' q^{-1}}(\theta, \phi)}{\sin^2(\theta/2) + q^{-2} \cos^2(\theta/2)} q^{\sin \theta \theta_{l' q^{-1}} - 1} Y_{lmq}(\theta, \phi) \right.
\nonumber\left. + \frac{Y_{l' m' q^{1}}(\theta, \phi)}{\sin^2(\theta/2) + q^{2} \cos^2(\theta/2)} q^{- \sin \theta \theta_{l' 1} + 1} Y_{lmq^{-1}}(\theta, \phi) \right) \nonumber
\nonumber\nonumber
= \delta_{l', l} \delta_{m', m}, \tag{11}
\end{align}

where the upper (resp. lower) signs correspond to $q \in \mathbb{R}^+$ (resp. $q \in S^1$). Note that the definitions \cite{8} and \cite{10} slightly differ from those given in \cite{13}. With the present choice, in the $q \to 1$ limit they go over into the definitions of $su(2)$ generators and spherical harmonics used in most quantum mechanics textbooks (see e.g. \cite{8}).

The radial wave functions $R_{n\alpha l}(r) \equiv r^{-1} S_{n\alpha l}(r)$ in Equation (8) are the solutions of the radial equation

\begin{align}
\left(\frac{d^2}{dr^2} - \frac{C_q(l)}{r^2} - r^2 + 2E_{n\alpha l}\right) S_{n\alpha l}(r) &= 0 \tag{12}
\end{align}

that satisfy the condition:

\begin{align}
S_{n\alpha l}(0) &= 0, \tag{13}
\end{align}

and are square integrable with respect to the usual scalar product, i.e.,

\begin{align}
\langle n' l | n l \rangle_q &\equiv \int_0^\infty dr \frac{S_{n' \alpha l}(r)}{S_{n\alpha l}(r)} S_{n\alpha l}(r) = \delta_{n', n}. \tag{14}
\end{align}

They are given by

\begin{align}
S_{n\alpha l}(r) &= \sqrt{2(nl)} \left(\Gamma \left(\alpha_l + n + \frac{1}{2}\right)\right)^{-1/2} e^{-\frac{r^2}{2}} r^{\alpha_l} L_n^{\alpha_l - \frac{1}{2}}(r^2), \tag{15}
\end{align}

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where $L^{\lambda-\frac{1}{2}}_{n}(r^2)$ is an associated Laguerre polynomial [1], and $\alpha_{l}$ is any one of the two solutions

$$\alpha_{l} = \frac{1}{2} \pm \lambda_{q}(l), \quad \lambda_{q}(l) \equiv \sqrt{\frac{1}{4} + C_{q}(l)},$$

of the equation

$$\alpha_{l}(\alpha_{l} - 1) = C_{q}(l),$$

provided it satisfies the condition $\alpha_{l} \in \mathbb{R}^{+}$. The corresponding energy eigenvalues are

$$E_{nlq} = 2n + \alpha_{l} + \frac{1}{2}, \quad n = 0, 1, 2, \ldots$$

The $\frac{1}{2}(N + 1)(N + 2)$-degeneracy of the isotropic oscillator energy levels, where $N = 2n + l$, is therefore lifted.

Similar results hold for the choice (5) for the Casimir operator, the only change being the substitution of $C_{q}'(l) = [l]_{q}[l + 1]_{q}$ for $C_{q}(l)$. To distinguish the latter choice from the former, we shall denote all quantities referring to it by primed letters ($\lambda_{q}'(l), \alpha_{l}', E_{nlq}', R_{nlq}'(r), S_{nlq}'(r), \ldots$).

### 3 Spectrum of the $su_{q}(2)$-Invariant Harmonic Oscillator

In the present section, we will study the condition $\alpha_{l} \in \mathbb{R}^{+}$ for the existence of the radial wave functions (15) and of the corresponding energy eigenvalues (18), as well as the behaviour of the latter as functions of $l$ and $q$ for the two choices (2), (3) of Casimir operators, and for $q \in \mathbb{R}^{+}$ or $q \in S^{1}$.

Let us first consider the case where $q = e^{w} \in \mathbb{R}^{+}$. Since the spectrum is clearly invariant under the substitution $q \rightarrow q^{-1}$, we may assume $q > 1$, i.e., $w > 0$.

In the $C_{q}$ case, $\lambda_{q}^2(l) = \sinh^2((l + 1/2)w)/\sinh^2 w > 0$ for $l = 0, 1, 2, \ldots$, hence both roots (14) of Equation (17) are real and distinct. However, for $l \neq 0$, $\lambda_{q}(l) = \sinh((l + 1/2)w)/\sinh w > 1$, showing that only $\alpha_{l+}$ is positive and therefore admissible, whereas for $l = 0$, $\lambda_{q}(0) = [2 \cosh(w/2)]^{-1} < 1/2$ if $q \neq 1$, so that $\alpha_{0+}$ and $\alpha_{0-}$ are both admissible. Note that in the undeformed case ($q = 1$), one gets $\lambda_{1}(0) = 1/2$, so that the root $\alpha_{0-}$ has then to be discarded in accordance with known results. For $q \neq 1$, the spectrum therefore comprises the energy eigenvalues

$$E_{nlq+} = 2n + 1 \pm \frac{1}{2 \cosh(w/2)},$$

$$E_{nlq} = 2n + 1 + \frac{\sinh((l + 1/2)w)}{\sinh w}, \quad l = 1, 2, \ldots$$

The appearance of an additional $l = 0$ level was already observed in the deformed Coulomb potential case [12].
In the $C_q'$ case, $\lambda_q^2(l) = [l]_q[l + 1]_q + \frac{1}{4} \geq \frac{1}{4}$ for $l = 0, 1, 2, \ldots$, so that both roots $\alpha_{l+}$ and $\alpha_{l-}$ are real and distinct, but only the former is positive, hence admissible. The spectrum therefore comprises the same levels as in the undeformed case, their energies being now

$$E'_{nlq} = 2n + 1 + \frac{[4\sinh(lw)\sinh((l+1)w) + \sinh^2 w]^{1/2}}{2\sinh w}, \quad l = 0, 1, 2, \ldots$$  (21)

Note that the energy of the $l = 0$ states is left undeformed:

$$E'_{n0q} = E_{n0} = 2n + \frac{3}{2}.\quad (22)$$

Expanding the right-hand sides of Eqs. (19), (20), and (21) into powers of $w$ shows that in the neighbourhood of $q = 1$, i.e., $w = 0$,

$$E_{n0q\pm} \simeq 2n + 1 \pm \frac{1}{2} + \frac{1}{16}w^2 (1 - \frac{5}{38}w^2 + \cdots),$$

$$E_{nlq} \simeq 2n + l + \frac{3}{2} + \frac{1}{48}(2l - 1)(2l + 1)(2l + 3)w^2 \times \left\{1 + \frac{1}{120} [12(l+1) - 25]w^2 + \cdots\right\}, \quad l = 1, 2, \ldots,$$  (23)

$$E'_{nlq} \simeq 2n + l + \frac{3}{2} + \frac{l(l+1)}{60(2l+1)}w^2 \left\{2l(l+1) - 1 + \frac{24(l+1)^3 - 562(l+1)^2 - 100(l+1) + 7}{60(4l+1)l}w^2 + \cdots\right\}, \quad l = 0, 1, 2, \ldots.$$  

Hence, for $l \neq 0$, $E_{nlq}$ and $E'_{nlq}$ are increasing functions of $w$ in the neighbourhood of $w = 0$, whereas for $l = 0$, $E_{n0q+}$ and $E_{n0q-}$ have opposite behaviours, while $E'_{n0q}$ is independent of $w$. Moreover, for a given $w$ value, the influence of the deformation increases with $l$. Such trends are confirmed by Figure 1, where the first few lowest eigenvalues $E_{nlq}$ are plotted in terms of $w$. For the $l$ and $w$ values considered, $E'_{0lq}$ cannot be distinguished from $E_{0lq}$ for $l \neq 0$, or $E_{00q+}$ for $l = 0$.

Let us next consider the case where $q = e^{i\gamma} \in S^1$. Owing to the invariance of the spectrum under the substitution $q \rightarrow q^{-1}$, we may now assume $0 < w < \pi$.

For $C_q$, $\lambda_q^2(l) = \sin^2 ((l+1/2)w)/\sin^2 w$ vanishes for $w = k\pi/(l+1/2)$, $k = 1, 2, \ldots, l$, but these $w$ values are in any case excluded as roots of unity. Hence, for the $w$ values considered here, Equation (17) has two real, distinct roots. If $\lambda_q(l) = |\sin((l+1/2)w)/\sin w| < 1/2$, both roots $\alpha_{l+}$ and $\alpha_{l-}$ are admissible, whereas if $\lambda_q(l) \geq 1/2$, only $\alpha_{l+}$ is so. The conditions can be reformulated in terms of

$$\gamma_q(l) = 4\sin^2 wC_q(l) = \frac{1}{2}[-4\cos((2l+1)w) + \cos 2w + 3].\quad (24)$$

One finds two admissible roots if $\gamma_q(l) < 0$, but only one root $\alpha_{l+}$ if $\gamma_q(l) \geq 0$. For instance, for $l = 0$, there is a single eigenvalue for any $w$ value,

$$E_{n0q} = 2n + 1 + \frac{1}{2\cos(w/2)},\quad (25)$$

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Figure 1: Spectrum of $n = 0$ states in terms of $w$ for $q = e^w \in \mathbb{R}^+$. The solid lines correspond to $E_{00q}^+$ and $E_{0lq}$, $l = 1, 2, \ldots, 6$, and the crossed line to $E_{00q}^-$. 


while for \( l = 1 \), there are either two or one eigenvalues, which are given by

\[
E_{n1q} = \begin{cases} 
2n + 1 + \frac{4\cos^2(w/2)-1}{2\cos(w/2)}, & \text{if } \cos w \geq \frac{-7+\sqrt{17}}{16}, \\
2n + 1 + \frac{4\cos^2(w/2)-1}{2\cos(w/2)}, & \text{if } \cos w \leq \frac{-7-\sqrt{17}}{16},
\end{cases}
\]

\( \text{or } \cos w \geq \frac{-7+\sqrt{17}}{16}, \) (26)

respectively.

By proceeding in a similar way for \( C'_q \), one finds in terms of

\[
\gamma'_q(l) = 4\sin^2(wC'_q(l)) = 4\sin((l + 1)w) \sin(lw)
\]

(27)

that there are two admissible roots \( \alpha'_{l+} \), \( \alpha'_{l-} \) if \( -\sin^2 w \geq \gamma'_q(l) \geq 0 \), only one \( \alpha'_{l+} \) if either \( \gamma'_q(l) \geq 0 \) or \( \gamma'_q(l) = -\sin^2 w \) (in which case \( \alpha'_{l+} = 1/2 \)), or none if \( \gamma'_q(l) \geq \sin^2 w \). For instance, for \( l = 0 \), there is a single eigenvalue \( (22) \) for any \( w \) value and it coincides with the undeformed one, while for \( l = 1 \), there are two or one eigenvalues if \( \cos w \geq -1/8 \),

\[
E'_{n1q} = \begin{cases} 
2n + 1 + \frac{1}{2}\sqrt{1 + 8\cos w}, & \text{if } -\frac{1}{8} < \cos w < 0, \\
2n + 1 + \frac{1}{2}\sqrt{1 + 8\cos w}, & \text{if } \cos w = -\frac{1}{8} \text{ or } \cos w \geq 0,
\end{cases}
\]

(28)

and none if \( \cos w < -1/8 \).

In both the \( C_q \) and \( C'_q \) cases, similar results can be derived for higher \( l \) values, and close enough to \( w = 0 \), for given \( n \) and \( l \) values one always finds a single eigenvalue going into the undeformed one, \( E_{nl} = 2n + l + 3/2 \), for \( q \to 1 \). For all \( l \) values, the expansion of this eigenvalue into powers of \( w \) can be obtained from Equation (23) by substituting \( iw \) for \( w \). Hence, in a small enough neighbourhood of \( w = 0 \), for \( l \neq 0 \), \( E_{nlq} \) and \( E'_{nlq} \) are decreasing functions of \( w \), whereas \( E_{n0q} \) is increasing and \( E'_{n0q} \) remains constant.

The first few eigenvalues \( E_{0lq} \), going into \( E_{0l} \) when \( q \to 1 \), are displayed on Figure 2 for \( 0 < w < \pi \). One should remember that for some \( l \) and \( w \) values, there may exist other eigenvalues, which are not plotted on the figure, and that the discrete set of points, where \( w \) is equal to a root of unity, is excluded. Such is the case, in particular, of the points where \( E_{0lq} \) \( (l \neq 0) \) takes its minimal value 1, and of the point \( w = \pi/2 \), where \( E_{0lq} = 1 + (1/\sqrt{2}) \) for any \( l \) value.

The influence of the deformation on the spectrum is rather striking. For high \( w \) values, the levels get mixed in a very complicated way. It is remarkable that in the neighbourhood of \( w = \pi/2 \), one obtains a spectrum very close to one with equidistant, infinitely-degenerate levels.

In the \( C'_q \) case, the situation is still more complex as some \( l \) values may disappear on some intervals. In the neighbourhood of \( w = 0 \), however, as in the real \( q \) case, \( E'_{nlq} \) cannot be distinguished from \( E_{nlq} \).
Figure 2: Spectrum of $n = 0$ states in terms of $w$ for $q = e^{iw} \in S^1$. The solid and dashed lines correspond to $E_{0lq}$ (or $E_{0lq+}$ if there exists another eigenvalue with $n = 0$ and the same $l$), where $l = 0, 2$, and $l = 1, 3$, respectively.
4 Quadrupole Moment in $l = 0$ States

The purpose of the present section is to study the effect of the deformation on the wave functions of the $su_q(2)$-invariant harmonic oscillator, given in Equation (7), (10), and (15), by determining the variation with $w$ of the expectation value of some physically-relevant operator. For the latter, we choose the electric quadrupole moment operator, and we consider the quadrupole moment in a state with definite $n, l$ values, which is defined conventionally as

$$Q_{nlq} = \langle nll | (3z^2 - r^2) | nll \rangle_q. \quad (29)$$

Equation (29) corresponds to the choice $C_q$ for the $su_q(2)$ Casimir operator. When using instead $C'_q$, the quadrupole moment will be denoted by $Q'_{nlq}$.

The undeformed counterpart of $Q_{nlq}$ and $Q'_{nlq}$ is given by

$$Q_{nl} = \langle nl | r^2 | nl \rangle \langle 00 | (3 \cos^2 \theta - 1) | 00 \rangle = \left( 2n + l + \frac{3}{2} \right) \left( -\frac{2l}{2l+3} \right). \quad (30)$$

It vanishes for $l = 0$ as a result of the familiar selection rule for the angular momenta $l$ and $2$ coupling. Nonvanishing values of $Q_{n0q}$ or $Q'_{n0q}$ will therefore be a direct measure of the effect of the deformation. Note that since in the $C_q$ case, there are two energy eigenvalues with $l = 0$ and a given $n$ value for $q \in R^+$ (see Equation (15)), we have to distinguish the corresponding quadrupole moments by a $\pm$ subscript.

As in Equation (30), $Q_{n0q}$ can be factorized into radial and angular matrix elements,

$$Q_{n0q} = \langle n0 | r^2 | n0 \rangle_q \langle 00 | (3 \cos^2 \theta - 1) | 00 \rangle_q. \quad (31)$$

For any $l$ value, the former is simply obtained by replacing $l$ by $\alpha l - 1$ in the undeformed radial matrix element. Hence

$$\langle n0 | r^2 | n0 \rangle_q = 2n + \alpha_0 + \frac{1}{2}. \quad (32)$$

The calculation of the latter is more complicated as it implies the use of the deformed angular scalar product (11),

$$\langle 00 | (3 \cos^2 \theta - 1) | 00 \rangle_q = \frac{3(q - q^{-1})}{16 \pi \ln q} \left( I_q + I_{q^{-1}} \right) - 1, \quad (33)$$

where

$$I_q = \int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\phi \frac{1}{\sin^2(\theta/2) + q^{-2} \cos^2(\theta/2)} q^{\sin \theta \bar{\theta} - 1} \cos^2 \theta. \quad (34)$$

This integral can be easily performed by making the changes of variables $z = pe^{i\phi}, \bar{z} = pe^{-i\phi}, \rho = \cot(\theta/2)$, and $\eta = \rho^2$. One gets

$$I_q = \frac{4\pi}{q} \int_0^\infty d\eta \frac{(1 - q^{-2}\eta)^2}{(1 + \eta)(1 + q^{-2}\eta)^3} = \frac{8\pi}{(q+q^{-1})^2} \frac{q+q^{-1}}{(q-q^{-1})^2} \ln q - 1 \quad (35)$$
Introducing Equation (35) into Equation (33), we finally obtain

$$
\langle 00 | (3 \cos^2 \theta - 1) | 00 \rangle_q = \frac{2(q^2 + 4 + q^{-2})}{(q - q^{-1})^2} - \frac{3(q + q^{-1})}{(q - q^{-1}) \ln q}.
$$

(36)

Hence

$$
\langle 00 | (3 \cos^2 \theta - 1) | 00 \rangle_q = \begin{cases} 
2 \cosh^2 2w + 1 - \frac{3 \cosh w}{\sinh w}, & \text{if } q = e^w \in R^+, \\
-2 \cos^2 w + 1 + \frac{3 \cos w}{\sin w}, & \text{if } q = e^{iw} \in S^1.
\end{cases}
$$

(37)

The deformed quadrupole moment $Q_{n0q}$ is therefore an even function of $w$, so that we may again restrict ourselves to $0 < w < \infty$ or $0 < w < \pi$ according to whether $q$ is real or complex.

The deformed radial matrix element (32) (or its counterpart for $Q'_{n0q}$), being just equal to the corresponding energy eigenvalue in the units used, varies in the same way with $w$. From Equations (13), (22) and (25), it follows that for increasing $w$, it decreases (resp. increases) from $2n + \alpha_0^+$ (resp. $2n + \alpha_0^-$), increases from $2n + \alpha_0^+$ to $+\infty$ for $C_q$, $q \in R^+$, and $\alpha_0^+$ (resp. $\alpha_0^-$), increases from $2n + \frac{3}{2}$ to $+\infty$ for $C_q', q \in S^1$, and remains constant for $C_q'^0, q \in R^+$ or $q \in S^1$. The effect of the deformation is therefore not significant, except in the case of $C_q$ when $q \in S^1$.

For real $q$, the deformed angular matrix element, given in Equation (37), increases from 0 to 2 when $w$ goes from 0 to $+\infty$. Hence, it is obvious that $Q_{n0q}$ and $Q'_{n0q}$ are increasing positive functions of $w$. It can be checked that the same is true for $Q_{n0q^+}$, the variation of the angular part of the matrix element dominating that of the radial one. On Figure 3, $Q_{n0q^+}$, $Q_{n0q^-}$, and $Q'_{n0q}$ are displayed in terms of $w$ for $n = 0$ and $n = 1$. One can see some effect of the choice of Casimir operator and $\alpha_0$ root, but it becomes significant only for a very large deformation. It should be stressed that the undeformed quadrupole moments in $l \neq 0$ states and the deformed ones in $l = 0$ states have opposite signs. Both become comparable in absolute value for $w \geq 1$.

For complex $q$, the deformed angular matrix element (37) decreases from 0 to $-\infty$ when $w$ goes from 0 to $\pi$. Hence, both $Q_{n0q}$ and $Q'_{n0q}$ are decreasing negative functions of $w$. On Figure 4, they are displayed in terms of $w$ for $n = 0$ and $n = 1$. Apart from the sign, which is now the same as that of $Q_{nl}$ for $l \neq 0$, the conclusions remain similar to those for the real $q$ case.

5 Conclusion

In the present Letter, we did show that as those of the free particle and of the Coulomb potential [12], the $su_q(2)$-invariant Schrödinger equation of the three-dimensional harmonic oscillator can be easily solved not only for $q \in R^+$, but also for generic $q \in S^1$. It is worth stressing that we have been working in the framework of the usual Schrödinger equation (i.e., with no non-commuting objects contrary to some other approaches [4, 10, 11, 13, 18]), but with wave functions belonging to a Hilbert space different from the usual one, since the
Figure 3: Quadrupole moment of $l = 0$ states in terms of $w$ for $q = e^w \in \mathbb{R}^+$. The solid, crossed, and dashed lines correspond to $Q_{n0q^+}$, $Q_{n0q^-}$, and $Q'_{n0q}$, respectively, for $n = 0$ (three lowest curves) and $n = 1$ (three highest ones).
Figure 4: Quadrupole moment of $l = 0$ states in terms of $w$ for $q = e^{iw} \in S^1$. The solid and crossed lines correspond to $Q_{n0q}$ and $Q'_{n0q}$, respectively, for $n = 0$ (two highest curves) and $n = 1$ (two lowest ones).
angular part of the scalar product has been modified when going from $su(2)$ to $su_q(2)$ \[.\]

In the real $q$ case, we did show that the spectrum is rather similar to the undeformed one, except that the energy levels are no more equidistant and that their degeneracy is lifted. For a given $n$ value, the spacing between adjacent levels corresponding to $l$ and $l + 1$, respectively, increases with $l$ and with the deformation. In addition, there appears a supplementary series of $l = 0$ levels when the Casimir operator $C_q$ is used. Apart from this, for small deformations, the results are rather insensitive to the choice made for the Casimir operator.

In the complex $q$ case, we did show that the spectrum is more complicated as for $l \neq 0$ and any $n$ value, there may exist 0, 1, or 2 levels according to the deformation. The existence or inexistence of levels is also rather sensitive to the choice made for the Casimir operator. Close enough to $q = 1$, there however always exists a single level going into the undeformed one for $q \to 1$. In that region, the spacing between adjacent levels corresponding to $l$ and $l + 1$, respectively, now decreases with $l$ and with the deformation.

The closeness of our approach to the standard one did also allow us to study the effect of the deformation on the wave functions and the Hilbert space spanned by them. We did establish that it is rather strong as the quadrupole moment in the $l = 0$ states, which vanishes in the undeformed case, now assumes a positive (resp. negative) value for $q \in R^+$ (resp. $q \in S^1$) irrespective of the Casimir operator used.
References

[1] Abramowitz, M. and Stegun, I. A.: *Handbook of Mathematical Functions*, Dover, New York, 1965.

[2] Arik, M. and Coon, D. D.: Hilbert spaces of analytic functions and generalized coherent states, *J. Math. Phys.* 17 (1976), 524–527.

[3] Biedenharn, L. C.: The quantum group SU_q(2) and a q-analogue of the boson operators, *J. Phys. A* 22 (1989), L873–L878.

[4] Carow-Watamura, U., Schlieker, M. and Watamura, S.: SO_q(N) covariant differential calculus on quantum space and quantum deformation of Schrödinger equation, *Z. Phys. C* 49 (1991), 439–446.

[5] Carow-Watamura, U. and Watamura, S.: The q-deformed Schrödinger equation of the harmonic oscillator on the quantum Euclidean space, *Int. J. Mod. Phys. A* 9 (1994), 3689–4008.

[6] Chan, G., Finkelstein, R. and Oganesyan, V.: The q-isotropic oscillator, *J. Math. Phys.* 38 (1997), 2132–2147.

[7] Drinfeld, V. G.: Quantum groups, in: A. M. Gleason, (ed), *Proc. Int. Congr. Mathematicians*, Berkeley, 1986, Amer. Math. Soc., Providence, 1987, pp. 798–820.

[8] Edmonds, A. R.: *Angular Momentum in Quantum Mechanics*, Princeton University, Princeton, 1957.

[9] Finkelstein, R. and Marcus, E.: Transformation theory of the q-oscillator, *J. Math. Phys.* 36 (1995), 2652–2672.

[10] Fiore, G.: SO_q(N,R)-symmetric harmonic oscillator on the N-dim real quantum Euclidean space, *Int. J. Mod. Phys. A* 7 (1992), 7597–7614; The SO_q(N,R)-symmetric harmonic oscillator on the quantum Euclidean space R^N_q and its Hilbert space structure, *Int. J. Mod. Phys. A* 8 (1993), 4679–4729.

[11] Floratos, E. G.: The many-body problem for q-oscillators, *J. Phys. A* 24 (1991), 4739–4750.

[12] Irac-Astaud, M.: Schrödinger equations for su_q(2) invariant quantum systems, *Lett. Math. Phys.* 36 (1996), 169–176; su_q(2) invariant Schrödinger equations, *Czech. J. Phys.* 46 (1996), 179–186.

[13] Irac-Astaud, M. and Quesne, C.: Unitary representations of su_q(2) on the plane for q ∈ R^+ or generic q ∈ S^1, *Czech. J. Phys.* 48 (1998), 1363–1368; Unitary representations of the quantum algebra su_q(2) on a real two-dimensional sphere for q ∈ R^+ or generic q ∈ S^1, *J. Math. Phys.* (in press).
[14] Jarvis, P. D. and Baker, T. H.: $q$-deformation of radial problems: the simple harmonic oscillator in two dimensions, *J. Phys. A* **26** (1993), 883–893.

[15] Jimbo, M.: A $q$-difference analogue of $U(g)$ and the Yang-Baxter equation, *Lett. Math. Phys.* **10** (1985), 63–69; A $q$-analogue of $U(gl(N+1))$, Hecke algebra and the Yang-Baxter equation, *Lett. Math. Phys.* **11** (1986), 247–252.

[16] Kulish, P. P. and Damaskinsky, E. V.: On the $q$ oscillator and the quantum algebra $su_q(1,1)$, *J. Phys. A* **23** (1990), L415–L419.

[17] Macfarlane, A. J.: On $q$-analogues of the quantum harmonic oscillator and the quantum group $SU(2)_q$, *J. Phys. A* **22** (1989), 4581–4588.

[18] Minahan, J. A.: The $q$-Schrödinger equation, *Mod. Phys. Lett. A* **5** (1990), 2625–2632.

[19] Papp, E.: $q$ analogs of the radial Schrödinger equation in $N$ space dimensions, *Phys. Rev. A* **52** (1995), 101–106.

[20] Rideau, G. and Winternitz, P.: Representations of the quantum algebra $su_q(2)$ on a real two-dimensional sphere, *J. Math. Phys.* **34** (1993), 6030–6044.

[21] Reshetikhin, N. Yu., Takhtajan, L. A. and Faddeev, L. D.: Quantization of Lie groups and Lie algebras, *Leningrad Math. J.* **1** (1990), 193–225.

[22] Truong, T. T.: The quantum mechanical Schrödinger picture of a $q$-oscillator, *J. Phys. A* **27** (1994), 3829–3846.