EVIDENCE FOR DEVIATIONS FROM FERMI-LIQUID BEHAVIOUR IN (2+1)-DIMENSIONAL QUANTUM ELECTRODYNAMICS AND THE NORMAL PHASE OF HIGH-\(T_c\) SUPERCONDUCTORS

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We provide evidence that the gauge-fermion interaction in multiflavour quantum electrodynamics in (2 + 1)-dimensions is responsible for non-fermi liquid behaviour in the infrared, in the sense of leading to the existence of a non-trivial (quasi) fixed point (cross-over) that lies between the trivial fixed point (at infinite momenta) and the region where dynamical symmetry breaking and mass generation occurs. This quasi-fixed point structure implies slowly varying, rather than fixed, couplings in the intermediate regime of momenta, a situation which resembles that of (four-dimensional) ‘walking technicolour’ models of particle physics. The inclusion of wave-function renormalization yields marginal \(O(1/N)\)-corrections to the ‘bulk’ non-fermi liquid behaviour caused by the gauge interaction in the limit of infinite flavour number. At low temperatures there appear to be logarithmic scaling violations of the linear resistivity of the system of order \(O(1/N)\). Connection with the anomalous normal-state properties of certain condensed matter systems relevant for high-temperature superconductivity is briefly discussed. The relevance of the large (flavour) \(N\) expansion to the fermi-liquid problem is emphasized.

1 Introduction

One of the most striking phenomena associated with the novel high-temperature superconductors is their abnormal normal-state properties. In particular, these substances are known to exhibit deviations from the known Fermi-liquid behaviour, which are remarkably stable with respect to variations in the relevant parameters.

Recently, Shankar and Polchinski have presented an intuitively appealing idea of using the Renormalization-Group (RG) approach, so powerful in particle and statistical physics, to systems of interacting electrons with a Fermi surface in order to understand, at least qualitatively, how deviations from Fermi liquid behaviour can appear naturally (as opposed to being fine-tuned). From this point of view Landau’s fermi liquid is nothing else but a system of free electrons, which has no relevant perturbations, in the RG sense, that can drive it away from its trivial infrared fixed point. In general, however, as we integrate out certain modes of our original theory, some interactions may become relevant in the RG sense, i.e. their effective coupling may grow as one lowers the momentum scale. Then, two interesting possibilities arise: (i) Fermion bound states are formed, symmetries are spontaneously broken, and the low-energy spectrum bears little resemblance to that of the original theory. In such a case one has to re-write the effective theory in terms of the new degrees of freedom: for instance, in the superconducting case this is the Landau-Ginzburg effective action expressed in terms of the fermion condensate. (ii)
Alternatively, the growth of the coupling is cut off by quantum effects at a certain low energy scale, and in this way a non-trivial fixed point structure emerges. The low energy fluctuations still correspond to fields of the original theory despite their non-trivial interactions. This case leads to observable deviations from the Fermi-liquid behaviour.

In the case of the high-$T_c$ materials, the physically interesting question is whether one model theory can be found with a structure rich enough to describe both the non-fermi liquid behaviour of the normal phase and the transition to (and phenomenology of) the superconducting phase. In this article we shall put forward a candidate model which, as we shall argue, seems to us to fulfill this rôle.

It is known that possibility (i) above can be caused by relevant interactions of superconducting (BCS) or charge-density-wave (CDW) type, both of which are accompanied by the formation of fermion condensates. Possibility (ii) has only recently begun to be seriously explored. It has been known for a long time that the electromagnetic interaction of the vector potential can cause deviation from fermi-liquid behaviour, but its effects are suppressed by terms of $O[(v_F/c)^2]$, with $v_F$ the Fermi velocity and $c$ the light velocity. Its effects occur only at much lower energies than those relevant to the high-$T_c$ materials. Nevertheless, the electromagnetic example is suggestive enough, perhaps, to motivate a search for other (non-electromagnetic) gauge interactions in which the effective signal velocity would be of order $v_F$, and which might be responsible for a non-trivial fixed point behaviour. It was precisely this sort of ("statistical") gauge-fermion interaction that was studied (in different forms) in and , and which led to non-trivial fixed point structure in the infrared.

Returning now to possibility (i), we recall that it has been shown that a variant of $QED$ in $(2 + 1)$-dimensions ($QED_3$) leads to superconductivity, characterized as appropriate to two space dimensions - by the absence of a local order parameter (Kosterlitz-Thouless mode). Thus the exciting possibility arises that a single fermion-gauge theory could describe both non-fermi-liquid behaviour in the normal phase and the transition to the superconducting phase.

The main purpose of the talk is to review an (approximate) renormalization group analysis of a simplified version of this model, namely $QED_3$ itself, which indicates that $QED_3$ exhibits two quite different behaviours depending on the momentum scale. At very low momenta $QED_3$ enters a regime of dynamical mass generation (d.m.g.), which in the full theory leads to superconductivity; but at "intermediate" momenta (see below) d.m.g. does not occur and the dynamics is controlled by a non-trivial fixed point, leading to non-fermi liquid behaviour. Thus we have the possibility - for the first time, to our knowledge - of one theory encompassing both the normal and the superconducting phases of the high-$T_c$ cuprates.

At this point the reader might worry that applying renormalization group techniques to a super-renormalizable theory like $QED_3$ is redundant, since the theory
has no ultraviolet divergencies. However, this is a mistaken view. In the modern approach to the RG and effective field theories, one considers quite generally how a theory evolves as one integrates out degrees of freedom above a certain momentum scale, moving progressively down in scale. From this point of view an effective field theory description is equally applicable to non-renormalizable, renormalizable, and super-renormalizable theories. However, there are some crucial new features in the case of a super-renormalizable theory (which, to our knowledge, have not been identified hitherto). First, the $QED_3$ coupling $e$ introduces an intrinsic intermediate scale $e^2$ which has the dimension of mass, this being directly related to the super-renormalizability of the theory. The physical effect of this will be the existence of an intrinsic mass scale and we can expect different physics in different regimes of momenta relative to this mass scale ($p >> e^2$, $p \simeq e^2$, $p << e^2$).

The second distinctive feature of our RG analysis of $QED_3$, concerns the way in which we introduce a running coupling. Conventionally, such running couplings are dimensionless - so, once again the dimensionfulness of $e^2$ presents a new feature. The way in which an effective dimensionless running coupling can be introduced into $QED_3$ has been shown by Kondo and Nakatani (KN) building on work by Higashijima for $QCD_4$. The crucial step is to consider the effect of wavefunction renormalization in the Schwinger-Dyson (SD) equations, as controlled by a large-$N$ approximation. In this case, one considers the theory at large $N$ with $\alpha = e^2 N$ held fixed, and the dimensionless coupling that runs is essentially $1/N$.

KN actually considered only the regime in which dynamical mass generation (chiral symmetry breaking) occurs - and of course here the gauge coupling is becoming strong and the use of a large-$N$ expansion is unavoidable. What we did in ref. is to identify the “normal” (no dynamical mass generation) regime of the theory, and extend the RG-type analysis of KN to this normal regime. We argued that there exists a non-trivial (quasi-)fixed point of the effective dimensionless coupling, which governs the dynamics for a range of intermediate momenta $p \simeq \alpha$, lying between the trivial fixed point at $p >> \alpha$, and the region $p << \alpha$ of dynamical mass generation. Important to this analysis will be the introduction (following KN) of an infrared cutoff $\epsilon$, which serves to delineate the different momentum regimes. The analysis of ref. is performed at zero temperature. Some attempts have also been made to connect this to finite-temperature calculations, by interpreting the temperature as an effective infrared cutoff. We presented an approximate computation, at finite temperature, of the electrical resistivity $\rho$ of the fermionic system. We argued that it is the existence of the non-trivial RG fixed point which is responsible for the fact that the non-fermi liquid behaviour ($\rho$ approximately proportional to the temperature $T$) is observed over so large a temperature range. Wavefunction renormalization effects, important at $O(1/N)$, lead to calculable logarithmic deviations from the linear in $T$ behaviour.

At this stage it is useful to compare and contrast our approach with two other recent explorations of gauge theories in $(2 + 1)$ dimensions in a similar context, by Polchinski and by Nayak and Wilczek. Both works deal with fermions interacting
with a statistical gauge field, the latter representing magnetic spin-spin interactions.
In both, the fermions represent spin quasi-particle excitations (spinons), and they
should therefore not be identified with the carriers of ordinary electric charge (holes
or electrons). This is to be sharply contrasted with our own model of refs. 6, in
which the spin-charge separation is done differently, leading to the fermions in our
model carrying both statistical and ordinary charge.

The alert reader might worry at our cavalier use of a relativistic fermion field the-
ory ($QED_3$) to infer conclusions pertaining to complicated condensed matter sys-
tems with non-trivial fermi surfaces, like the ones of relevance to high-temperature
superconductivity. To such objections, we first stress that the results of ref. 7 should
only be viewed as a qualitative attempt at identifying one particular (but important)
source of (cross-over) deviations from fermi liquid behaviour. In support of this we
refer the reader to an important observation by Polchinski 3 according to which,
in such condensed matter systems, kinematics implies that the most important in-
teractions among fermions are those which pertain to fermionic excitations whose
momentum components tangent to the fermi surface are parallel. This is the only
way that the gauge field momentum transfer can still be relatively large as com-
pared to the distance of the fermion momenta from the fermi surface, as required by
special kinematic conditions 3. There are two cases where such conditions are met
in condensed matter physics. The first pertains to nested fermi surfaces, at which
the points with momenta $k_0$ and $-k_0$ have parallel tangents. This is the situation
relevant to BCS or CDW. The other situation, which is the bulk of Polchinski’s
work and will be of interest to us as well, is the case where the fermions are close
to a single point on the fermi surface. This means that the most important fermion
interactions are those which are local on the fermi surface, and hence qualitatively
this situation can be extended to relativistic (Dirac) fermions as well, since the
dispersion relations become effectively linear 6.

It should be stressed that the curvature of the fermi surface plays also a non-trivial
rôle in deviations from fermi liquid behaviour, since any shape distortion appears
as a relevant RG deformation of the model. However, as already pointed out in
ref. 6 the remarkable stability of the observed non-fermi liquid behaviour in the
normal phase of the high-$T_c$ materials, which persists up to temperatures of 600 $K$,
cannot be explained by deformations of the fermi surface, as this would require an
un-natural fine tuning. It is our belief that a dominant rôle in the phenomenon
is played by the statistical gauge interaction among charged holes, which arguably
characterizes magnetic superconductors 6. Support for this conjecture, within the
context of $QED_3$ prototypes, was one of the main results of ref. 7.

Finally, in an attempt to convince the more skeptical formal readers about the
qualitative validity of the relativistic models as prototypes for the description of
such phenomena in condensed matter, we draw his/her attention to the fact that
the quasi-fixed point behaviour that seems to characterize $QED_3$ at $T = 0$, seems
to persist for a wide range of finite temperaturas $T > 0$, where Lorentz invariance is
definitely lost.
Another important point, which was recently pointed out by Shankar in connection with the RG approach to interacting fermions, is the use of an effective large-$N$ expansion in cases where the effective momentum cut-off $\Lambda$ is much smaller than the size of the fermi surface $k_F$, $\Lambda/k_F \to 0$. Such a situation is encountered in a RG study of (deviations from) fermi liquid theories, the Landau fermi-liquid theory being defined as a trivial infrared fixed point in a RG sense. To understand the connection of a large-$N$ expansion with infrared behaviour of excitations one should recall the work of ref. where the RG approach to the theory of the Fermi surface has been studied in a mathematically rigorous way. The basic observation of ref. is that, unlike the case of relativistic field theories, in systems with an extended fermi surface, the fermionic excitation fields exhibiting the correct scaling are not the original excitations, $\psi_x$ ($x$ a configuration space variable), but rather quasiparticle excitations defined as follows:

$$\tilde{\psi}_x = \int \frac{d\Omega}{|\Omega| = 1} e^{ik_F\Omega \cdot \hat{x}} \psi_{\hat{x},\Omega}$$

where for the shake of simplicity we assumed that the fermi surface is spherical with radius $k_F$, $\Omega$ is a set of angular variables defining the orientation of the momentum vector of the excitation at a point on the fermi surface, and the tilde denotes ordinary Fourier transform in a momentum space $K$. These quasiparticle fields have propagators with the correct scaling, which allows ordinary RG techniques, familiar from relativistic field theories, to be applied, such as the appearance of renormalized coupling constants, scaling fields etc. Indeed it is not hard to understand why this is so. For this purpose it is sufficient to observe that for large $k_F$ the exponent of the exponential in (1) is nothing other than the linearization,

$$k \equiv K - k_F\Omega,$$

about a point on the fermi surface, which makes these quasiparticle excitations identifiable with ordinary field variables of the low-energy limit of these condensed matter systems. The latter is a well-defined field theory, the crucial point in this interpretation is that now the field variables will depend on 'internal degrees of freedom', $\Omega$, which denote angular orientation of the momentum vectors on the fermi surface. In two spatial dimensions, which is the case of interest, $\Omega$ is just the polar angle $\theta$. Following ref. we discretize this angular space into small cells of extent $f(\Lambda/k_F) \ll 1$, e.g. $f = \Lambda/k_F$:

$$\int \frac{d^2k}{4\pi^2} \equiv \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \int_{f(\Lambda/k_F)}^{f(\Lambda/k_F)} k_F d\theta \frac{1}{2\pi}$$

where $k$ denotes a linearizing momentum about a point on the fermi surface. Doing so, we observe that when looking at interaction terms involving fermionic particle-antiparticle pairs, $\bar{\psi}\psi$, the leading interactions are among those fermion-antifermion pairs for which the creation and anihilation operators lie within the same angular cell. This is for purely kinematic reasons in the infrared regime $\Lambda \ll k_F$, similar to those mentioned previously, which implied that the most important fermion interactions on the fermi surface must be among excitations which have their tangents to the fermi surface parallel. It is, then, straightforward to see that interaction terms involving either gauge excitations or just fermions resemble those in large-$N$
relativistic field theories, given that the only \( \Lambda \) dependence appears through proportionality factors \( f(\Lambda/k_F) << 1 \) in front of the interactions, in the infrared. One, then, identifies \( 1/N \) with \( f(\Lambda/k_F) << 1 \), and the only difference from ordinary particle-physics large-\( N \) expansions is the dependence of this effective \( N \) on the cut-off \( \Lambda \): that is to say, \( 1/N \) runs.

As we showed in ref. 7, however, large \( N \) expansions in three dimensional \( QED \) can exhibit such scale dependence. Wave-function renormalization leads to a renormalized ‘running’ \( 1/N \). Furthermore, the running is of a novel nature. Instead of finding a non-trivial infrared fixed point, we shall demonstrate the existence of an (intermediate) regime of momenta, where the effective running of the gauge coupling, which is essentially \( 1/N \) times a spontaneously appearing scale, is slowed down considerably, so that one encounters a quasi-fixed-point situation. This quasi-fixed point structure is sufficient to cause (marginal) deviations from the fermi liquid picture. At finite temperatures, there are indications 7 that this behaviour will lead to logarithmic temperature-dependent corrections to the linear resistivity of the fermion system, the latter being the result of the presence of (statistical) gauge interactions. This makes such theories plausible candidates for a correct qualitative description of deviations from Landau fermi liquid theory, which might be related to the observed anomalous normal phase properties of high-\( T_c \) cuprates.

2 \( QED_3 \): Super-renormalizability, ‘running’ couplings and non-trivial (quasi-)fixed-point structure

Three-dimensional quantum electrodynamics (\( QED_3 \)) has recently received a great deal of attention (11–19) not only as a result of its potential application to the study of planar high-temperature superconductivity (6), mentioned in the introduction, but also because of its use as a prototype for studies of chiral symmetry breaking in higher-dimensional (non-Abelian) gauge theories (20).

Despite the theory’s apparent simplicity the situation is not at all clear at present. A great deal of controversy has arisen in connection with the rôle of the wave-function renormalization. In the early papers (11) the wave-function renormalization \( A(p) \) was argued to be 1 in Landau gauge to leading order in \( 1/N \), where \( N \) is the number of fermion flavours, and thus was ignored. More detailed studies, however, showed (15) that the precise form, within the resummed \( 1/N \) graphs, of \( A(p) \) is

\[
A(p) = \left( \frac{p}{\alpha} \right)^{\frac{8}{3N\pi^{3/2}}}
\]

(3)

where \( \alpha = e^2N \) is the dimensionful coupling constant of \( QED_3 \), which is kept fixed as \( N \to \infty \). It is clear from (13) that, although at energies \( p \approx \alpha \) the wave-function is of order one, however at low momenta \( p << \alpha \), relevant for dynamical generation of mass, the wave-function renormalization yields logarithmic scaling violations which could affect (13) the existence of a critical number of flavours \( N_c \),
below which, as argued in ref. 11, dynamical mass generation occurs. The situation became clearer after the work of ref. 8, who showed that the introduction of an infrared cut-off affects the results severely, depending on the various ansatizes used for the vertex function. In particular, there are extra logarithmic scaling violations in the expression for $N_c$, depending on the form of the vertex function assumed, which render the limit where the infrared cut-off is removed, not well-defined.

For our present purposes, however, we are not so much interested in whether the inclusion of wavefunction renormalization leads to a critical $N_c$ or not, as in the more general point that - as noted by Kondo and Nakatani (KN) 8, following Higashijima - the vacuum polarization contribution to $A$ produces effectively a running coupling, even in the case of the super-renormalizable theory of QED. KN’s analysis was restricted to the regime of dynamical mass generation, and our main purpose in this section is to extend that to the “normal” regime where mass is not dynamically generated. We emphasize now, however, that if $A$ is set equal to unity at the outset, the power of the running coupling concept to unify both regimes is completely lost.

We now proceed to a brief review of our analysis in ref. 7. Following ref. 8, we make the vertex ansatz

$$\Gamma_\mu(q,p) = \gamma_\mu A(p)^n \equiv \gamma_\mu G(p^2)$$

(4)

where $p$ denotes the momentum of the photon. The Pennington and Webb ansatz corresponds to $n = 1$, where chiral symmetry breaking occurs for arbitrarily large $N$. It is this case that was argued to be consistent with the Ward identities that follow from gauge invariance. In this paper we shall concentrate on the generalized ansatz, with $n \neq 1$, and in particular we shall discuss its finite temperature behaviour. We keep the exponent $n$ arbitrary and discuss qualitatively the implications of the vertex ansatz for various ranges of the parameter $n$. As we shall argue below this is crucial for the low-energy renormalization-group structure of the model.

Using the ansatz (4), one can analyze the Schwinger-Dyson (SD) equations, in the various regimes of momenta, in terms of a running coupling. For pedagogical purposes, we first concentrate on the (infrared) regime of dynamical mass generation, following 8. The (approximate) SD equation for $A(p)$ is (in Landau gauge)

$$A(p) = 1 - \frac{g_0}{3} \int_\epsilon^\alpha dk \frac{kA(k)G(k^2)}{k^2A^2(k) + B(k^2)} \left( \frac{k}{p} \right)^3 \theta(p - k) + \theta(k - p)$$

(5)

where $g_0 = 8/\pi^2 N$, $N$ is the number of fermion flavours, and $\epsilon$ is an infrared cutoff. In the low-momentum region relevant for dynamical mass generation $p << \alpha$ and

\[ However, this result was not free of ambiguities either, given that the inclusion of wave-function renormalization necessitates the introduction of a non-trivial vertex function. The exact expression for the latter is not tractable, even to order $O(1/N)$, and one has to assume various ansatizes that can be questioned.\]
the first term in the right-hand-side of (5), cubic in \( k^3 \), may be ignored. Then, taking into account that \( G(k^2) = A(k)^n \), and using the bifurcation method in which one ignores the gap function \( B(k) \) in the denominators of the SD equations, one obtains easily

\[
A(t) = 1 - \frac{g_0}{3} \int_0^t ds A^{n-1}(s)
\]

which has the solution

\[
A(t) = (1 + \frac{2 - n}{3} g_0 t)^{\frac{1}{n}} \quad ; \quad t \equiv ln(p/\alpha)
\]

Substituting to the SD equation for the gap, one then obtains a ‘running’ coupling in the low momentum region

\[
g_L = \frac{g_0}{1 + \frac{2 - n}{3} g_0 t}
\]

which, we note, is actually independent of \( \epsilon \). The existence of the dimensionless parameter \( g^2 \) in QED\(_3\) may be associated with the ratio of the gauge coupling \( e^2/\alpha \), given that in the large \( N \) analysis the natural dimensionful scale \( \alpha \) has been introduced. Thus, a renormalized running \( N^{-1} \) might be thought of as expressing ‘charge’ scaling in this super-renormalizable theory. In particular (8) implies that the \( \beta \) function corresponding to \( g_L \) is of ‘marginal’ form

\[
\beta_L \equiv -\frac{dg_L}{dt} = \frac{2 - n}{3} (g_L)^2
\]

Thus, depending of the sign of \( 2 - n \) one might have marginally relevant or irrelevant couplings \( g_L \propto e^2/\alpha \). The first derivative of the \( \beta \) function with respect to the coupling \( g_L \) is

\[
\frac{d}{dg_L}(\beta_L) = \frac{2 - n}{3} g_L
\]

and since \( g_L > 0 \) by construction, its sign depends on the sign of \( n - 2 \). For \( n < 2 \) (the marginally relevant case) the gauge interaction decreases rapidly as one moves away from low momenta, and the theory is “asymptotically free”\(^8\). If \( n > 2 \) (marginally irrelevant), on the other hand, then \( g_L(t) \) tends to zero in the low momentum region, whilst for \( n = 2 \) the coupling is exactly marginal and one recovers the results of ref.\(^{11,16}\) about the existence of a critical flavour number. Gauge invariance, in the sense of the Ward-Takahashi identity, seems to imply \( n \leq 2 \) and this is the range we shall explore in this article.

Our task in ref.\(^7\) was to extend (8) beyond the region \( p << \alpha \). Consider first the true ultraviolet region \( p \to \infty \). Assuming for the moment that (8) were correct for \( p >> \alpha \), one finds a zero of the \( \beta \) function at the point \( t \to \infty \), the trivial fixed point \( g^* = 0 \), which is an ultraviolet fixed point. However, (8) or (9) are not reliable for the range of momenta \( p >> \alpha \). Both formulas have been derived in the regime of momenta relevant to the dynamical mass generation, \( p << \alpha \).
This being so, do we have an alternative argument for a trivial ultraviolet fixed point? The answer is affirmative. To this end we use the results of ref. 22 employing a quenched fermion approximation in large $N$ QED. The result of such an investigation is that once fermion loops are ignored, and hence only tree-level graphs (ladders) are taken into account, the wave-function renormalization is rigorously proved to be trivial in the Landau gauge:

$$A(p)^{\text{quenched}} = 1$$  \hspace{1cm} (11)

This result is a consequence of special mathematical relations of resummed ladder graphs in Schwinger-Dyson equations. Now in our case, one observes that in the high-energy regime, $p \to \infty$, the $\frac{1}{N}$-resummed gauge-boson polarization tensor vanishes as $\Pi(p \to \infty) \simeq \alpha/8p \to 0$. Thus, the situation is similar to the quenched approximation, which implies the absence of any wave-function renormalization (11), and therefore the vanishing (triviality) of the effective (‘running’) coupling constant $g$ in the ultra-violet regime of momenta. This is in qualitative agreement with the naive estimate made above, based on the formulas (8), (9).

The situation is, therefore, as follows. The coupling grows from the trivial fixed-point (ultraviolet regime) where there is no mass-generation, to stronger values as the momenta become lower. According to the naive formula (8), this coupling grows indefinitely for low momenta and the perturbation expansion breaks down. But - to repeat - (8) was derived for the regime $p << \alpha$, and the question now arises whether nothing new happens from this regime all the way up to $p \to \infty$, or whether there is interesting structure at intermediate scales. In particular, we might envisage a “quasi-fixed-point” situation, in which $g$ remains more or less stationary around the value $g(0)$ for a wide range of $t$ below $t = 0$, before commencing to grow rapidly at very low momenta.

The answer to the above question turns out to reside, essentially, in the infrared cutoff $\epsilon$ (which, as we noted above, actually disappeared from (8)). The coupling of (8) is “asymptotically free” (i.e. grows rapidly in the far infrared) for $n < 2$, provided that the ratio $\alpha/\epsilon$ is large enough - and in this case dynamical mass generation (d.m.g.) occurs. To get to the region where d.m.g. does not occur, we must consider smaller values of $\alpha/\epsilon$, tending ultimately to unity. This is the region that will yield the effective non-trivial fixed point structure. In this case, $p \simeq \alpha$ and hence the only allowed region for the momentum $k$ in (8) is $k \leq p$, which now eliminates the second term in (8). Solving then (8) in this approximation (and taking $B = 0$ since d.m.g. does not occur), with the vertex (4), one obtains

$$A(p) = 1 - \frac{g_0}{3} \int_{\epsilon}^{p} \frac{dk}{k} \left(\frac{k}{p}\right)^3 A^{n-1}(k) =$$

$$1 - \frac{g_0}{3} \int_{t_0 - t}^{t_0} dse^{3s} A^{n-1}(s)$$  \hspace{1cm} (12)

which can be easily solved with the result

$$A(t) = (\text{const} + \frac{2 - n}{9} g_0 e^{3t_0 - 3t}) \frac{1}{\pi^n}$$  \hspace{1cm} (13)
where the \( \text{const} \) is a positive one and can be found from the value of the wave function renormalization at \( t = \ln(\epsilon/\alpha) \equiv t_0 \), namely \( A(t_0) = 1 \). From (12) this yields the value \( \text{const} = 1 - \frac{2-n}{9}g_0 \). Substituting (13) back to the gap equation one obtains a ‘running’ coupling constant in this new intermediate regime

\[
g' = \frac{g_0e^{3t}}{(1 - \frac{2-n}{9}g_0)e^{3t} + \frac{2-n}{9}g_0e^{3t_0}} = \frac{g_0}{1 - \frac{2-n}{9}g_0 + \frac{2-n}{9}g_0(\frac{\epsilon}{\alpha})^3} \quad . \tag{14}
\]

We note that just as the “lower scale” \( \epsilon \) disappeared from (8), so the “intermediate scale” \( \alpha \) is absent from (14).

Let us study the fixed-point structure of this renormalization-group flow. To this end, consider the \( \beta \) function obtained from (14):

\[
\beta' = -\frac{dg'}{dt} = -3g' + \frac{3}{g_0}(1 - \frac{2-n}{9}g_0)(g')^2 \tag{15}
\]

Taking into account that \( g_0 = \frac{8}{\pi^2N} \) we observe that the vanishing of \( \beta' \) occurs not only at \( g' = 0 \) but also at the non-trivial point

\[
g' = \frac{8}{\pi^2N}(1 - \frac{2-n}{9}g_0)^{-1} \tag{16}
\]

which indicates the existence of a fixed point lying at a distance of \( O(1/N) \), for \( N \to \infty \), from the trivial one.

For what momenta is this fixed point reached? Accepting (14) at face value, the answer would be that it is reached for \( p \to \infty \). But of course (14) is not valid for \( p >> \alpha \), being appropriate for \( \epsilon < p < \alpha \) where the ratio \( \epsilon/\alpha \) is smaller than unity, though not so very small that \( p \) can enter the region of d.m.g. Referring then to the right hand side of the second equality in (14), we see that when \( p \simeq \alpha \) the quantity \( g' \) will be very close to \( g'_0 \), differing from it by terms of order \( (\epsilon/\alpha)^3 \frac{1}{\pi^2} \), which is negligible. Indeed, as \( p \) moves down to \( p \simeq \epsilon \), \( g' \) arrives at \( g_0 \), which is still within \( (1/N^2) \) of \( g'_0 \). Thus the crucial point is that there is - on the basis of this admittedly approximate analysis - a significant momentum region over which the coupling \( g' \) varies very slowly, and we are in a “quasi-fixed-point” situation. In a sense, this slow variation of \( g' \) in the range \( \epsilon < p < \alpha \) (for not too small \( \epsilon \)) provides a reconciliation between the normalizations adopted in the two different approximations (8) and (14) - namely between \( g_L(p = \alpha) = g_0 \) and \( g'(p = \epsilon) = g_0 \).

The new fixed point occurs at weak coupling for large \( N \). This is consistent with the interpretation that such a fixed point should characterize a regime of the theory, as determined by the ratio \( \alpha/\epsilon \), where dynamical mass generation does not occur.

In summary, then, our analysis in ref. suggests a significant modification of the picture presented by Kondo and Nakatani. Whereas those authors only considered \( \epsilon \ll \alpha \), which is the regime of “asymptotic freedom” and d.m.g., we have explored
also the region of smaller values of $\alpha/\epsilon$, and have concluded that here quantum corrections create a quasi-fixed-point with weak coupling. Both regions of $\alpha/\epsilon$ are important in our application of these results to the cuprates, as we discuss in some detail in ref. 7, where we tried to relate the "$\epsilon$" of this $QED_3$ with the temperature $T$ of $QED_3$ at finite temperature.

At this stage, it is worth pointing out the similarity of the above-demonstrated 'slow running' of the effective gauge coupling $g$ at intermediate scales with (four-dimensional) particle physics models of 'walking technicolour' type. Such models pertain to gauge theories with asymptotic freedom and involve regions of momentum scale at which effective running couplings move very slowly with the scale, exactly as happens in our (asymptotically free) $QED_3$ case. This slow running of the coupling results in such theories in a significant enhancement of the size of the fermion condensate. In our case, such condensates are responsible for an opening of a superconducting gap, and, therefore, one could associate the slow running of the coupling at intermediate scales with the suppression of the coherence length of the superconductor (inverse of the fermion condensate) in the phase where dynamical mass generation occurs. Such a suppression, as compared to the phonon (BCS) type of superconductivity, which is an experimentally observed and quite distinctive feature of the high-$T_c$ cuprates, appears then, in the context of the above gauge theory model, as a natural consequence of the non-trivial quasi-fixed-point renormalization group structure. Note that in ref. 6 the enhancement of the superconducting gap-to-critical-temperature ratio, as compared to the standard BCS case, had been attributed to the super-renormalizability of the theory and the $T$-independence of quantum corrections, features which are both associated with the above quasi-fixed-point (slow running) situation as discussed above. It is understood, of course, that before we arrive at definite conclusions about the actual size of the coherence length in the model, we should be able to perform exact calculations by resumming the higher orders in $1/N$ to see whether these features persist. At present this is impossible analytically, but one could hope for (non-perturbative) lattice simulations of the above systems.

For completeness, we would like to compare our results to other existing results in the literature concerning the infrared structure of $QED_3$, and in particular to the results of refs. 8, 27, 28. In ref. 27, it has been argued, on the basis of a power-counting analysis, which did not make any use of the Ward-Takahashi identities, a similarity of $QED_3$ with walking technicolour had also been pointed out previously but from a different point of view. In ref. 24, a formal analogy of $QED_3$ with walking technicolour models was noted, based on the rôle of fermion loops in softening the logarithmic confining gauge potential to a Coulombic $1/r$ type, in the infrared regime of momenta. This $1/r$ behaviour of the potential, and its relevance to dynamical chiral symmetry breaking, is common in both theories. The formal analogy between $QED_3$ and walking technicolour theories is achieved by replacing the coupling $g^2$ of the four-dimensional theory by $1/N$ of $QED_3$. However $N$ of ref. 24 does not vary with the energy scale, since wave-function renormalization effects have not been discussed in their case. This is the crucial difference in our case, where there is a more precise analogy with walking technicolour theories, due to the slowing-down of the variation of the 'effective' $N$ with the (intermediate) energy scale.
that there is no renormalization of $N$ to any order in $1/N$, in the infrared regime of the model. The arguments were based on the softened Coulombic form of the gauge-boson propagator in the infrared, as a result of fermion vacuum polarization: $D_{\mu\nu} \propto (1/q)\langle g_{\mu\nu} - (1 - \xi)q_{\mu}q_{\nu}/q^2 \rangle$, in an arbitrary $\xi$ gauge, for small momentum transfers $q << \alpha$. It is worth noticing that such arguments appear to apply equally well to Abelian as well as non-Abelian theories, since in the latter case non-Abelian three or four gluon interactions could not contribute to the potential scaling-violating interactions. This analysis has been performed without implementing an infrared cutoff, due to the infrared finiteness of the (zero-temperature) theory. In the work of ref. which is applied to the infrared regime, an infrared cut-off is introduced, which changes the scaling properties of the gauge-boson propagator. In this case, the scale-invariant situation seems to occur only for the value $n = 2$ in the vertex ansatz, which notably does not satisfy the Ward-Takahashi identities. As we have seen, gauge invariance requires $n = 1$, and in that case there exists a running $N$, at infrared momentum scales, as well as a finite critical flavour number, which however is infrared cut-off dependent, and diverges in the limit where the cut-off is removed.

We can also compare this result with that of ref. which claims to have proven the gauge invariance of the critical number of flavours in $QED_3$. There, a non-local gauge fixing was used; this mixes orders in $1/N$ expansion, in the sense that the gap function in SD contains now graphs of $O(1/N^2)$, whilst the wave-function renormalization still remains of $O(1/N)$. In contrast, the analysis of ref. remains consistently at leading order in $1/N$, and in the Landau gauge. The meaning of the non-local gauge fixing is not clear if one stays consistently within an order by order $1/N$ expansion. Nor does gauge invariance make complete sense in the presence of an infrared cutoff.

Thus, the key to a possible explanation of the discrepancy between the works of ref. and ref. seems to be hidden in the higher orders in the large $N$ expansion, as well as the presence of the infrared cut-off. Notice that a naive removal of the infrared cut-off might lead to ambiguities, as becomes clear from the work of ref. for finite temperature field theories, provided that one makes the (physically sensible) identification/analogy of the infrared cut-off with the temperature scale, at least within a condensed matter effective theory framework.

Now we come to our case. As can be seen by the above discussion, our results can offer a way out of the above-mentioned discrepancy. For us, the momentum regime of interest is not the infrared one, where dynamical mass generation occurs, but the intermediate scale. In this regime, the power-counting arguments of ref. do not apply, since the gauge-boson propagator does not have a simple Coulombic behaviour. Thus, the wave-function renormalization effects, that appear to exist in our, admittedly rough, truncation of the SD equations, might not be incompatible with the results of ref. pertaining to the existence of a critical flavour number. From our point of view, this would mean that, although there is a (slow) running of an effective $N$, and thus scale invariance is marginally broken, however, the running
of the coupling is even more suppressed in the infrared, where strong quantum effects cut off the increase of the (asymptotically free) coupling. The infrared cut-off then, appears as the (spontaneous ?) scale, above which a slow running of the (asymptotically free) coupling becomes appreciable. In a condensed-matter-inspired framework, such a spontaneously appearing scale makes perfect sense, if one associates the infrared cut-off with the temperature scale . For momenta slightly above the infrared cut-off, then, the situation of KN seems to be valid. This regime may be viewed as the boundary regime for which dynamical mass generation still can happen. Below the infrared scale, which is a regime that makes perfect sense in an infrared-finite theory such as QED, dynamical mass generation certainly occurs, and the arguments of ref. apply, leading to an effective cut-off of the increase of the coupling constant. In this regime, the gauge-boson propagator assumes a softened Coulombic 1/r form, which has been argued to be important for (superconducting) pairing attraction among fermions (holes) in the model of ref. Such a situation was envisaged in ref. for the case of chiral symmetry breaking in four-dimensional QCD, which in this way was dissociated from the confining properties of the theory.

In the work of KN and ours, all these issues could be confirmed only if a more complete analysis of the SD equations, including higher-order 1/N corrections, is performed. Whether resummation to all orders in 1/N washes out completely the wave-function renormalization effects at intermediate momenta, leading to an exactly marginal (scale invariant) situation, or keeps this effect at a RG marginal level, remains an unresolved issue at present. On the basis of the above discussion, one would expect that marginal deviations from scale invariant behaviour at intermediate momenta, such as the ones studied in the present work, survive higher-order analyses, but they also lead to a critical number of flavours, since the latter is an entity pertaining to the infrared regime of the theory. Moreover, for us, who are interested in performing the analysis in a condensed-matter rather than particle-theory framework, there is the issue of the ambiguous infrared limit of the theory at finite temperatures, which is by no means a trivial matter. It seems to us that all these important questions can only be answered if proper lattice simulations of the pertinent systems are performed. At present, the existing computer facilities might not be sufficient for such an analysis.

However, as pointed out in , the slow running of the coupling constant of the model at intermediate momentum scales, if true, is a desirable effect from a condensed matter point of view, where both infrared and ultraviolet cut-offs should be kept. The wave-function renormalization effects, discussed above, prove sufficient in leading to a (marginal) deviation of the theory from the fermi-liquid fixed point. At finite temperatures, this effect can have observable consequences, and might be responsible for the experimentally observed abnormal normal state properties of the high-Tc cuprates, the physics of which the above gauge theories are believed to simulate. We stress once again that such effects would be absent in an exactly marginal situation, like the one suggested in ref.
3 Linear behaviour of the (normal phase) Resistivity in QED$_3$ with the temperature scale

We would like to conclude this talk by making some remarks on the behaviour of the resistivity of QED$_3$, i.e. its response to an externally applied electromagnetic field. This requires connecting the above picture of the behaviour of QED$_3$ at zero temperature to that of the same theory at finite temperature, $T$. In the absence, again, of anything like an exact solution in the $T \neq 0$ case, approximations (quite possibly severe ones) will have to be made. However, the physical aim is clear: we want to connect the experimental observation that the electrical resistivity in the normal phase of the high-$T_c$ superconductors varies linearly with $T$ over a wide range in $T$ from low temperatures up to a scale of 600 K, to the existence of the non-trivial quasi-fixed-point structure of QED$_3$ found in the previous section. Qualitatively, the way we shall make the connection is to interpret the temperature in finite-T QED$_3$ as (related to) an effective infrared cutoff. This will follow from the form of the gauge boson propagator for $T > 0$, which we analysed in some detail in ref. 7, and we shall not repeat here.

Our aim in this subsection is to exhibit non-fermi liquid behaviour of the resistivity, and associate it with the quasi-fixed-point structure at intermediate scales revealed in the previous section, via the qualitative connection $\alpha/\xi \sim \sqrt{\alpha/T}$. The resistivity of the model is found by first coupling the system to an external electromagnetic field $A$ and then computing the response of the effective action of the system, obtained after integrating out the (statistical) gauge boson and fermion fields, to a change in $A$.

In the case at hand, in the model of ref. 6 ($\tau_3 - QED$) the effective action of the electromagnetic field, after integrating out hole and statistical gauge fields, assumes the form

$$S_{eff} = \int A^\mu(p)\Delta_{\mu\nu}A^\nu(-p) ; \quad \Delta_{\mu\nu} = (\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2})\frac{1}{p^2 + \Pi}$$

(17)

in a resummed $1/N$ framework, with $\Pi$ the one-loop polarization tensor due to fermions. The functional variation of the effective action with respect to $A$ yields the electric current $j$. From (17) this is proportional to the electric field $E(\omega) = \omega A$, in, say, the $A_0 = 0$ gauge, with $\omega$ the energy. In the normal phase of the electron

Due to the $\tau_3$ structure, as a result of the bi-partite lattice structure there are no cross-terms between the statistical and the electromagnetic gauge fields to lowest non-trivial order of a derivative expansion in the effective action. This implies that in this model the resistivity is determined by the polarization tensor of the hole (fermion) loop. On the other hand, in models where only a single sublattice is used, such cross terms arise, which are responsible - after the statistical gauge field integration - for the appearance of a conductivity tensor proportional to $\Pi_{B,F}$ denoting (respectively) polarization tensors for the boson fields of the $CP^1$ model and for the fermions (holes) in a resummed $1/N$ framework. In such a case, the conductivity is determined by the lowest conductivity among the subsystems. In condensed-matter systems of this type, relevant for the physics of the normal state of the high-$T_c$ cuprates, it is the bosonic contribution that determines the total electrical resistivity.
system, the proportionality tensor, evaluated at zero spatial momentum, is $\sigma_f \times \omega$, with $\sigma_f$ the conductivity. From (17) then, we have

$$\sigma_f = \frac{1}{p^2 + \Pi} \bigg|_{p=0}$$

(18)

where $\Pi$ denotes spatial components of the momentum.

If the effective action were real, then the temperature ($T$) dependence of the resistivity of the model would be given by the $T$-dependence of the finite-temperature vacuum polarization of the gauge boson. Thus, following the estimates of ref. $^{19}$ for the polarization tensor in the resummed-$1/N$ framework, we would have immediately obtained a linear $T$-dependence for the resistivity. Such a temperature dependence would actually be valid $^7$ for a wide range of temperatures above the critical temperature of dynamical mass generation $^6$, due to specific features of the ansatzes involved in the analysis of ref. $^{19}$.

However, things are not so simple. As first shown by Landau $^{31}$, the analytic structure of the vacuum polarization graphs entering the effective action (17) is such that there are imaginary parts in a real-time formalism $^{32}$. These imaginary parts are associated with dissipation caused by physical processes involving (on-shell) processes of the type fermion $\rightarrow$ fermion $+$ gauge boson. It turns out that these constitute the major contributions to the (microscopic) resistivity $^{33,34,29}$. In this picture, the latter is determined by virtue of the Green-Kubo formula in the theory of linear response, and it turns out to be inversely proportional to the imaginary part of the two-point function of the “electric” current $j^\nu = \psi \gamma^\nu \psi$, evaluated at zero spatial momentum. In our case, in the leading $1/N$-resummed framework, the two-point function of the electric current is given by the graph of fig. 1. Adopting the ansatz (4) for the vertex function, the result for the current-current correlator is

$$<J_\mu(p)J_\nu(-p) > \propto (A(p))^n \Delta_{\mu\nu}(p)(A(p))^n$$

(19)

To compute the imaginary parts of (19) would require a real-time formalism, taking into account the processes of Landau damping $^{18}$, which are not an easy matter to compute in resummed $1/N$ approximation, especially in the limit of zero-momentum transfer, relevant for the definition of resistivity. Indeed, as shown in ref. $^{18}$, and mentioned briefly above, there is a non-analytic structure of the imaginary parts of the one-loop polarization tensors appearing in the quantum corrections of the gauge boson propagator. Such non-analyticities result in a non-local effective action. This non-locality persists upon coupling the system to an external electromagnetic field $A$. Since the resistivity of the system is defined as the response of the system to a variation of $A$, then the Landau processes, which constitute the major contribution to the (microscopic) resistivity, complicate the situation enormously. At present, only numerical treatment of these non-analyticities is possible $^{18,19}$.

We can circumvent this difficulty, and use only the real parts of the gauge boson polarization tensor to estimate the temperature dependence of the resistivity, by
making use of the fact that in “realistic” many-body systems believed to be relevant for a description of the physics of the cuprates, there is the phenomenon of spin-charge separation of the relevant excitations. According to this picture, the statistical current (responsible for spin transport) is opposite to the hole current (electric charge transport) and this constraint is implemented by the statistical gauge field, $a_\mu$, that plays the rôle of a Lagrange multiplier. The gauge field, on the other hand, is identified, for physical (on-shell) processes, with the bosonic current of the spin excitations. The electric charge is, thus, transported with a velocity which equals the propagation velocity $v_F$ of the statistical gauge fields $a_\mu$. In non-trivial vacua, such as the the one pertaining to our system, the velocity $v_F$ receives quantum corrections from vacuum polarization effects. In a thermal vacuum such corrections are temperature-($T$-) dependent.

If we represent the (observable) average of the electric current as $j_\psi = \text{charge} \times v_F$, and use Ohm’s law to relate it with an ($T$-independent) externally applied electric field $E$, $j_\psi = \sigma E$, then one observes that in this picture the main $T$-dependence of the resistivity $\sigma^{-1}$, comes from $v_F$, as a result of (thermal) vacuum polarization effects. The result is

$$v_F \propto \frac{Q}{T^2}; \quad Q \to \epsilon$$  \hspace{1cm} (20)

Using the association of the momentum infrared cutoff $Q \simeq \epsilon$ with $\sqrt{\alpha/\beta} \propto \sqrt{T}$, one gets from (20) a linear $T$-dependence for $v_F^{-1}$, and thus for the resistivity $\rho$. Such a linear $T$ dependence is a characteristic feature of the gauge interactions, and, as we shall discuss below, is valid for a wide range of $T$.

Incorporating wavefunction renormalization effects in the above analysis one can easily demonstrate the existence of (logarithmic) deviations from this linear $T$ behaviour. This part of the analysis does not require an explicit computation of the imaginary part of the correlator (19). It only requires $A$ evaluated at $p = 0$. The resistivity, which formally is given by the imaginary part of the inverse of (19) as $p \to 0$, turns out to have the following temperature dependence (resummed up to $O(1/N)$):

$$\rho \propto O(T^{1 - \frac{1}{n}})$$  \hspace{1cm} (21)

where we have taken $n = 1$ as in (19). We cannot, in any case, take the precise value of the exponent in (21) seriously in view of the rough approximations made along the way.

However the region $\beta \alpha \gg 1$ is, in fact, that of dynamical mass generation, rather than the “intermediate” region $\beta \alpha \gtrsim 1$ in which we expect the quasi-fixed-point structure to play a rôle. A numerical analysis shows that for a wide range of temperature below $\alpha$, but not so low that the symmetry-breaking phase is entered, the resistivity should have the form (21), where the precise coefficient of the $1/N$ power is not known accurately from the above analysis. The main point, then, is the “stability” of this $T$-dependence which correlates remarkably with the quasi-fixed-point structure discussed above.
4 Conclusions and Outlook

In this talk we have reviewed results of some recent work, which provide evidence for certain interesting effects of the wave-function renormalization in (a variant of) $QED_3$ that is believed to be a qualitatively correct continuum limit of semi-realistic condensed matter (planar) systems simulating high-temperature superconducting cuprates.

Based on an (approximate) Schwinger-Dyson (SD) improved Renormalization Group (RG) analysis, we have argued for the existence of an (intermediate) regime of momenta, where the running of the renormalized dimensionless coupling of multiflavour $QED_3$, which is nothing other than the inverse of the flavour number, is considerably slowed down, exhibiting a behaviour similar to that of ‘walking technicolour’ models of particle physics. This slow running, or (quasi) fixed point structure, has been argued to be responsible for an increase of the chiral-symmetry breaking (superconducting) fermion condensate of the model, as well as for a (marginal) deviation from the Landau fermi-liquid fixed point. In connection with the latter property, we have argued that the large $N$ expansion is fully justified from a rather rigorous renormalization group approach to low-energy interacting fermionic systems with large fermi surfaces. Some experimentally observable consequences of this (marginal) non-fermi liquid behaviour, including logarithmic temperature-dependent corrections to the linear resistivity, have been pointed out, which could be relevant for an explanation of the abnormal normal-state properties of the high-$T_c$ cuprates.

The above RG-SD analysis was, however, only approximately performed at present. To fully justify the above considerations, and to make sure that the above-mentioned effects are not washed out in an exact treatment, one has to perform lattice simulations of the above models. Given that this might not be feasible yet, due to the restricted capacities of the existing computer devices, an intermediate step would be to perform a more complete analytic RG treatment of the relevant large-$N$ SD equations at finite temperatures. Such a treatment is not easy, however, due to the mathematical complexity of the involved equations. In addition, finite-temperature field theory is known to exhibit unresolved ambiguities concerning the low momentum limit, which complicates the situation. Some of these issues constitute the object of intensive research effort of our group at present, and we hope to be able to reach some useful conclusions soon.

Our work made use of relativistic fermion systems. We have provided evidence that this might capture the correct qualitative features responsible for the observed deviation from fermi liquid behaviour in realistic high-temperature superconducting systems, which are known to be characterized by large fermi surfaces. Indeed, the remarkable stability in the observed behaviour up to temperatures of 600 K cannot be ascribed to simple deformations of the fermi surface, which would require an
unnatural fine tuning. The presence of gauge interactions, of the type considered in this work, with subtle wave-function renormalization properties, provides a natural and simple explanation of the phenomena in terms of a (quasi)-fixed point (i.e. cross over) behaviour, rather than a new universality class. This should be contrasted to the works of refs. 4, 3, where the existence of a fixed point was argued. This is a non-trivial point to have in mind for possible experimental searches in the future. Of course, it is understood that in order to explain the complete set of the observed properties of the normal phase of high-temperature superconductors the simple relativistic $QED_3$ picture advocated above is not sufficient, not even qualitatively. One should probably take into account all possible sources of deviation, including the ones arising from the curvature of the fermi surface and its distortions, etc, in order to arrive at a quantitatively satisfactory picture of the situation.

In this context it might be worth pointing out that our results are also of value for cases of condensed matter systems with relativistic spectra around some nodes of their fermi surfaces. At present, we do not have a physical intuition on the microscopic nature of the gauge interactions that might be involved in such situations, neither are we aware of realistic candidate systems that would realize such scenaria. A plausible testing ground for these ideas would be the case of $\nu = 1/2$ fractional quantum Hall systems? Experimentally, there appear to be deviations from fermi liquid behaviour in such systems, and there are recent theoretical attempts to relate this to the existence of new infrared fixed points. From our point of view, if the $\nu = 1/2$ Hall case is to be characterized by a new gauge interaction due to, say, interactions among the magnetic moments of the (planar) electrons, then our work shows that it is more likely to be characterized by a cross-over behaviour rather, than the appearence of a non-trivial infrared fixed point. We hope to study these fundamental problems in the future.

Closing, we would like to stress once again the exciting atmosphere for collaboration between particle-physics and condensed-matter communities triggered by the discovery of fractional quantum Hall systems and high-temperature superconductors. Indeed, as we have heard at this meeting, there is a plethora of striking resemblances between many phenomena that characterize these solid state systems with the corresponding phenomena in particle physics. The very nature of the spin-charge separation, which is essential for magnetic scenario of high-temperature superconductivity, seems to be analogous to the quark fractional charge phenomenon inside the hadron. This ‘constituent-fermion’ picture of the planar holes in magnetic superconductor models, which was also extended recently to Hall systems as an attempt at a microscopic understanding of the fractional quantum Hall effect, is strongly reminiscent of the quark model of hadrons. This is not unrelated to the gauge approach to the high-$T_c$ problem advocated in refs. 6 and 7, and briefly discussed above, where the asymptotic freedom of the abelian three dimensional gauge field plays a crucial rôle in determining the infrared behaviour of the system in connection with either dynamical mass generation, related to the superconduct-

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We thank A. Tsvelik for a discussion on this point.
ing phase, or with the anomalous properties of the normal phase. The situation is analogous to chiral symmetry breaking in four dimensional QCD. In this context, the reader’s attention is drawn to recent numerical evidence for a QCD-like-string (hadronic) Regge-pole structure in the physical spectrum of Hubbard or $t-j$ models, which was associated with the spin-charge separation property. Certainly this line of research appears very interesting and exciting, and should be pursued further.

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