Fractal Properties of the Distribution of Earthquake Hypocenters

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We investigate a recent suggestion that the spatial distribution of earthquake hypocenters makes a fractal set with a structure and fractal dimensionality close to those of the backbone of critical percolation clusters, by analyzing four different sets of data for the hypocenter distributions and calculating the dynamical properties of the geometrical distribution such as the spectral dimension $d_s$. We find that the value of $d_s$ is consistent with that of the backbone, thus supporting further the identification of the hypocenter distribution as having the structure of the percolation backbone.

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Two important and related phenomena in natural rock masses are earthquakes and the nucleation and propagation of fractures. Earthquakes are the result of a series of complex phenomena involving the interaction between stress concentration and fluid flow, and have been the subject of considerable recent interest [1–7]. They have been proposed [3–5] to be related to self-organized critical phenomena (SOCP), in that they are the product of a dynamical many-body system that reaches a critical state without fine-tuning its parameters. Such systems reach a stationary critical state which is characterized by spatial and temporal correlations that follow power-laws without any intrinsic length or time scales. Although the connection between earthquakes and SOCP has led to a deeper understanding of earthquake phenomenon, a clear geometrical interpretation of the spatial distribution of earthquakes, which is essential for constructing realistic spatial and temporal correlation functions for earthquakes, was lacking until recently. On the other hand, most natural rock masses contain large fractures, in the form of a complex and interconnected network [8], the presence of which is crucial to the higher production of oil from underground reservoirs, generation of heat and vapor from geothermal reservoirs, and the development of groundwater resources. It had been argued [1] that the spatial distribution of earthquakes is closely related to the structure of fracture networks in rocks. However, this connection had not been quantitatively established.

In a recent paper [9] a quantitative connection was proposed between the structure of fracture networks and the spatial distribution of earthquakes. Analyzing extensive geological data and using computer simulation models of fracture [10], it was proposed that large scale fracture networks (order of kilometers) of heterogeneous rocks are fractal sets whose structures are similar to critical percolation clusters with a fractal dimensionality \( d_f \approx 2.5 \). Moreover, since earthquake hypocenters are usually on fracture and fault networks of rocks, they have to belong to the active part of the networks where large scale deformations and stress transmission take place, i.e., earthquake hypocenters have to belong to the backbone of fracture networks. Indeed, the analysis [9] of four different sets of data for the spatial distribution of earthquake hypocenters indicated that the centers are on a fractal set with a fractal dimensionality \( d_f \approx 1.8 \), close to that of the backbone [12] of three-dimensional critical percolation cluster.
Although the closeness of the fractal dimensionalities suggests the connection between the earthquake hypocenter distribution and the percolation backbone, it is not entirely conclusive, since two fractal sets may have the same fractal dimensionality but rather different structures. In this article we further explore this connection by calculating some dynamical properties of the fractal structures of the earthquake hypocenter distribution and by comparing them with those of the percolation backbone.

We have taken four seismic data sets from four different regions in Southern California, namely, San Andreas–Elsinore (SA–EL), Parkfield, Whittier, and Upland (see Table [1]). Two of these are the same ones as analyzed in [9] for the fractal dimensionality $d_f$ and the other two are also essentially the same except for the minor additions of data points. In order to test the hypothesis that the distribution of hypocenters forms a structure similar to that of, say, the lattice nodes contained in a percolation backbone created on that lattice, what we need to do first is to place the earthquake centers on a fictitious lattice network. Then we can investigate the dynamical (and static) properties of this connected network, the nodes of which are the earthquake hypocenters. Physically, we would expect this fictitious network to correspond in some way to the actual fracture network on which the earthquake locations must lie; however, for the present analysis, it is immaterial whether there is a direct correspondence between the supposed connected network and the actual fractures in the rocks.

Thus, we first transform the hypocenter distribution data by a form of coarse graining; i.e., the data points expressed in $(x, y, z)$ coordinates in integer units of 100$m$ are linearly scaled to bring the outlying points closer and in the process those points that fall within a given distance from each other are replaced by a single point. Arriving at integer coordinates in units of arbitrary lattice constant (in all cases within the volume of $50 \times 50 \times 50$), we then overlay the connectivity of a simple cubic lattice, either only to the first neighbors or up to further neighbors. When this is done, the network of lattice nodes typically breaks up into many disconnected clusters, most of which are tiny but there is always one cluster which is composed of the bulk of the nodes in the network. We focus on this largest connected cluster in all cases and study its dynamical and static properties. Of course, we must make
sure that this transformation of the original data has not distorted their geometrical characteristics; we will discuss some checks on this point later on. The parameters of this transformation are also given in Table 1.

By the dynamical properties, we mean the properties associated with diffusion on the connected cluster if the cluster were used as the channel for diffusion (or random walk). By the mapping between diffusion and vibration [13], we may equivalently characterize this work as studying the vibrational spectrum of an elastic network having the same geometric structure as our cluster. Thus, e.g., the probability $P(t)$ of a random walk on this cluster to return to its starting point after $t$ steps is related to the vibrational density of states of the corresponding elastic network by a Laplace transform [13]. (However, it is important to keep in mind that all of this is simply a tool in this case to characterize the geometrical structure of the distribution of the earthquake hypocenters and has nothing to do with the vibration of the rocks per se.)

In terms of the random walk problem, the spectral dimension $d_s$ and the random walk dimension $d_w$ [11, 12] can be defined by the relations:

$$P(t) \sim t^{-d_s/2}$$

$$R(t) \sim t^{1/d_w},$$

where $R(t)$ is the root-mean-square displacement of the random walk in time $t$. However, rather than simulating random walks to calculate $d_s$ and $d_w$, it is in many ways simpler to perform the spectral analysis as suggested by the vibration–diffusion mapping. We do this following the method of Ref. [14, 15] by first constructing the hopping probability matrix $W$ where $W_{ij}$ is the probability for the random walker to hop from site $j$ to $i$ per time step and then diagonalizing $W$ to obtain eigenvalues $\lambda$ and eigenvectors near the maximum eigenvalue with high accuracy. To be specific, we use the so-called blind ant model of the random walk [16], for which $W$ is symmetric and the diagonal terms are generally non-zero, but the specific choice of the random walk kinetics is irrelevant for our purposes. Once the diagonalization is done, we compute two quantities, the density of eigenvalues $n(\lambda)$ and a certain function $\pi(\lambda)$ (which is the product of $n(\lambda)$ and some coefficient determined when the stationary initial state distribution is expanded in terms of the eigenvectors of $W$ [17]). These
functions are expected to behave, asymptotically near $\lambda = 1$ \cite{15}, as

\begin{align}
  n(\lambda) & \sim |\ln \lambda|^{d_s/2-1} \\
  \pi(\lambda) & \sim |\ln \lambda|^{1-2/d_w} .
\end{align}

The results of fitting the transformed data to Eq.\,(3) for $d_s$ are shown in Fig. 1, where $n(\lambda)$ from two of the four data sets (SA--EL and Whittier) is plotted against $|\ln \lambda|$ in a double logarithmic plot and the respective linear least squares fits are also drawn. Clearly the data scatter fairly widely and the exponent estimates are not expected to be very accurate. (The remaining two data sets have slightly greater data scatter but with comparable slopes.) Nonetheless, the central estimates of the slopes from the four sets (only two are shown for clarity) point to a value in the range of $d_s \simeq 1.18$ to 1.29 with the fit to the largest data set by far (SA--EL) yielding $d_s \simeq 1.19 \pm 0.13$. On the other hand, estimating $d_w$ from these data is much more difficult because of the much greater data scatter for $\pi(\lambda)$. Consequently, we do not make numerical estimates of $d_w$ but rather only state the result that the widely scattered data are nonetheless consistent with the backbone values of $d_w$ in the sense that similar fitting procedure yields exponent ranges well encompassing the backbone value. We summarize the exponent estimates in Table 2, where the error estimates are simply from the least squares fitting and do not take into account any finite size effects or other systematic errors that may be present.

These estimates are clearly consistent with the corresponding exponent $d_s^B$ for the backbone of the three dimensional critical percolation cluster: the latter can be obtained from the scaling relation \cite{16}

\begin{equation}
  d_s^B = d_f^B = \frac{2d_f^B}{2 + d_f^B - d + \mu/\nu} \simeq 1.16 \pm 0.02.
\end{equation}

Here the superscript $B$ denotes the backbone, $\mu$ and $\nu$ are the DC conductivity and correlation length exponents, respectively, and the error in the numerical value is from the uncertainty in $\mu/\nu$ ($\simeq 2.27 \pm 0.03$ \cite{18}) and in $d_f^B$ ($\simeq 1.75 \pm 0.04$ \cite{21}). Although there are more recent estimates of $\mu/\nu$, e.g., by combining the results of Duering and Roman \cite{19} and of Grassberger \cite{20}, this would give $\mu/\nu \simeq 2.34 \pm 0.08$, consistent with but less accurate than the result we use. Also a more recent estimate
of \( d_f^B \) leads to a slightly higher \( d_s^B \) but still clearly distinguishable from the full percolation cluster value of \( d_s^P \simeq 1.328 \pm 0.006 \). (Unfortunately, we do not know of any direct calculations of \( d_s^B \), and as is often the case with quantities obtained through complicated scaling relations, the error given above may be significantly underestimated.)

The transformed data form relatively small connected clusters and thus it may be prudent to compare these results with similarly sized single percolation backbones obtained by direct simulation on a simple cubic lattice. In Fig. 2, the density of states \( n(\lambda) \) from three data sets are plotted for: (a) a backbone cluster of 287 sites, (b) a cluster of 270 sites obtained by scaling by a factor of 0.5 from a larger backbone cluster of 619 sites, and (c) a full percolation cluster of 344 sites (all at the percolation threshold \( p_c \)). It can be seen that in both cases (a) and (b), the scatter of data are comparable to the transformed earthquake data in Fig. 1 and the slopes are also very similar. Moreover, the data (c) has significantly less scatter and shows a clearly different slope, corresponding to the full percolation cluster value of \( d_s \). This figure thus shows that our transformed earthquake data have a very similar behavior characteristic of a small backbone cluster and that the scaling transformation used to transform the data apparently does not affect the characteristic power-law of the density of states.

As a further check of the possible effects of the transformation applied to the earthquake data, we have measured the fractal dimension \( d_f \) of the earthquake data before and after the transformation. For this purpose, we use the box counting method \([11, 24]\) where the minimum number \( N(L) \) of cubes of side \( L \) required to cover the data points completely are measured. We then obtain an estimate of \( d_f \) from the relation \( N(L) \sim L^{-d_f} \). The numerical estimates of \( d_f \) are given in Table 2, where the fitting regions roughly correspond to those used in Ref. [9] for the original data and the lower cutoffs in the transformed data are obtained by using the scale factors of Table 1. From these, we can see clearly that the fractal dimension of the earthquake data is not significantly affected by the scaling and identification of a single connected cluster from among the data points based on imposed connectivity to either the first (for the SA–EL data) or the second neighbor distances (for the remaining data).

It would be interesting, as a complementary task, to look for the dependence of
the behavior of $n(\lambda)$ on the cluster size for the small critical backbone clusters. For an asymptotically large backbone, we obviously expect results consistent with Eq. (8). However, systematic tendency in finite size effects is usually observable only when a large number of realizations of the finite size systems are averaged. Thus, for single small clusters, the cluster to cluster fluctuations are very large and moreover $n(\lambda)$ for an individual cluster is not a very smooth function of $|\ln \lambda|$, so that it is difficult to analyze for any systematic finite size effects. Indeed, two particular backbone clusters of 287 and 297 sites (on $24^3$ grids) gave the slope of $-0.39 \pm 0.06$ and $-0.43 \pm 0.03$, respectively, in a plot like Fig. 2, while those of 619 and 588 sites (on $36^3$ grids) gave the slope of $-0.39 \pm 0.03$ and $-0.43 \pm 0.03$, respectively. For larger clusters of 1018 and 1005 sites (on $48^3$ grids), and taking the range to include the maximum $\lambda$ after unity, the corresponding slopes were $-0.45 \pm 0.05$ and $-0.43 \pm 0.04$, respectively. Since the purpose of the present work is to demonstrate the similarity of the earthquake data and single backbone clusters of corresponding size, we defer the systematic study of such finite size effects to a future work.

In summary, we have presented the analysis of the dynamical properties of the geometric network represented by the four earthquake hypocenter distributions in Southern California. Mainly based on the good agreement between the measured spectral dimension $d_s$ and fractal dimension $d_f$ and those of the critical percolation backbone in three dimensions, we believe the case supporting the idea that these earthquake distributions lie on the percolation backbone has been strengthened. Clearly, it is desirable to establish that the physical network of active fractures is properly represented by the connectivity we imposed in this calculation. Although we have not done this, we believe that this idea has sufficient supporting evidence now to deserve further attention.

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lation backbone), it appears that the box counting method used here and another
common method based on mass scaling tend to give rather different estimates of
d_f. Since our clusters do satisfy this criterion, we use the box counting method
which is closer to the original definition of d_f [1].
Fig. 1: Density of eigenvalues \( n(\lambda) \) from the data sets derived from SA–EL (○) and Whittier (□) regions are shown in double logarithmic plot against \(|\ln \lambda|\). The least square fitted lines have slopes of \(-0.41\pm0.07\) and \(-0.37\pm0.09\), respectively.

Fig. 2: Density of eigenvalues \( n_B(\lambda) \) from comparably sized backbone and full percolation clusters at \( p_c \). The symbols ○, □, and Δ correspond to a backbone cluster of 287 sites, a cluster of 270 sites obtained by linear scaling from a larger backbone cluster, and a full percolation cluster of 344 sites, respectively. The linear least squares fitting yields lines with slopes of \(-0.39\pm0.06\) (solid line for ○), \(-0.40\pm0.11\) (dash-dotted line for □), and \(-0.36\pm0.03\) (dashed line for Δ).
**TABLES**

Table 1: Brief descriptions of the four earthquake hypocenter distributions and the transformations used to obtain the connected lattice clusters used in the analysis. The last column indicates the connectivity imposed up to the indicated neighboring distance. Each event represents an earthquake with magnitude greater than unity.

| Region   | Range (100m) | Events | Scale Factors       | Sites | Connectivity |
|----------|--------------|--------|---------------------|-------|--------------|
| SA–EL    | 934 × 1823 × 210 | 2004   | 0.025 × 0.025 × 0.025 | 419   | 1            |
| Parkfield| 145 × 168 × 154 | 885    | 0.25 × 0.25 × 0.25  | 326   | 2            |
| Whittier | 129 × 145 × 210  | 224    | 0.125 × 0.125 × 0.075| 140   | 2            |
| Upland   | 139 × 156 × 182  | 291    | 0.125 × 0.125 × 0.1  | 129   | 2            |

Table 2: Numerical estimates of the exponents $d_f$ for the earthquake data before and after the scaling/connection transformation, and that of $d_s$ for the transformed cluster. Error estimates are only the least squares fitting errors.

| Region   | $d_f$ (original) | $d_f$ (transformed) | $d_s$        |
|----------|------------------|---------------------|--------------|
| SA–EL    | 1.72 ± 0.03      | 1.75 ± 0.04         | 1.19 ± 0.13  |
| Parkfield| 1.76 ± 0.03      | 1.69 ± 0.08         | 1.29 ± 0.20  |
| Whittier | 1.73 ± 0.03      | 1.92 ± 0.16         | 1.26 ± 0.17  |
| Upland   | 1.79 ± 0.02      | 1.72 ± 0.11         | 1.18 ± 0.37  |