Research Article

Rational Cohomology Algebra of Mapping Spaces between Complex Grassmannians

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1. Introduction

The complex Grassmannian \(G(k, n)\) is the set of \(k\)-planes through the origin in \(\mathbb{C}^n\). Moreover,

\[
G(k, n) \equiv \frac{U(n)}{U(k) \times U(n-k)},
\]

(1)

where \(U(n)\) is the unitary group ([1], chap. 18). There is a canonical inclusion \(i_{nr}: G(k, n) \hookrightarrow G(k, n + r)\) which is induced by \(i: \mathbb{C}^n \rightarrow \mathbb{C}^{n+r}\) defined by \(i(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, \ldots, 0)\).

The study of the rational homotopy type of function spaces started with Thom in the case where the codomain is an Eilenberg–Maclane space [2]. The first description of a Sullivan model of function spaces is due to Haefliger [3]. Moreover, a model of function spaces between complex projective spaces was given by Moller and Raussen using Postnikov tower [4]. However, there is no explicit and complete description of the homotopy type of the component of the inclusion \(G(k, n) \hookrightarrow G(k, n + r)\) in the space of mappings from \(G(k, n)\) to \(G(k, n + r)\) for \(r \geq 1\) and in particular to show that the cohomology of \(G(k, n)\) contains either a polynomial algebra or a truncated algebra over a generator of degree 2.

In this paper, we consider the inclusion \(G(k, n) \hookrightarrow G(k, n + r), k \geq 3\) [5].

2. Sullivan Models

Henceforth, we work on the field of rational numbers, \(\mathbb{Q}\).

**Definition 1.** A differential graded algebra (dga) is a graded algebra \(A = \oplus_{n \geq 0} A^n\) endowed with a derivation \(d\), of degree +1, such that \(d^2 = 0\). The pair \((A, d)\) is called a cochain algebra. A graded algebra \(A\) is commutative if \(a \cdot b = (-1)^{\text{deg} a \cdot \text{deg} b} b \cdot a\) for \(a, b \in A\) ([6], chap. 3).

**Definition 2.** A Sullivan algebra is a commutative cochain algebra of the form \((\land V, d)\) where \(V = \oplus_{k \geq 1} V_k\) and \(V = \cup_{k \geq 0} V(k), V(0) \subset V(1) \subset \cdots\) such that \(dV(k) \subset \land V(k-1)\). A Sullivan model for a commutative cochain algebra \((A, d)\) is a quasi-isomorphism \(m: (\land V, d) \rightarrow (A, d)\) from a Sullivan algebra \((\land V, d)\). A Sullivan algebra (or
model) is said to be minimal if the differential is decomposable, that is, \( \text{Im} d \subset \wedge^V \cdot \wedge^V \).

Moreover, if \( H^0(A) = \mathbb{Q} \), then \( (A, d) \) has a minimal model which is unique up to isomorphism. If \( X \) is a nilpotent space and \( A_{PL}(X) \) the commutative differential graded algebra (cdga) of piecewise linear forms on \( X \), then a Sullivan model of \( X \) is a Sullivan model of \( A_{PL}(X) \) ([16], chap.12). A space \( X \) is formal if there is a quasi-isomorphism \( (\wedge^V, d) \rightarrow H^* (X, Q) \). Moreover, complex Grassmann manifolds are formal [7].

The cohomology ring \( H^* (Gr(k, n), \mathbb{Q}) \) of \( Gr(k, n) \) has a presentation

\[
H^* (Gr(k, n), \mathbb{Q}) = \Lambda^* (c_1, c_2, \ldots, c_k) / \langle h_{p-k+1}, \ldots, h_n \rangle, \tag{2}
\]

a quotient of the polynomial ring generated by \( c_1, c_2, \ldots, c_k \), \( |c_i| = 2i \) modulo the ideal generated by the elements \( h_j, n - k + 1 \leq j \leq n \). Here, \( h_j \) is defined as the 2\(^j\)th degree term in Taylor’s expansion of \((1 + c_1 + c_2 + \cdots + c_k)^{-1} \), where \((1 + c_1 + c_2 + \cdots + c_k) \) is the total Chern class and \( c_i \in H^{2i} (Gr(k, n), \mathbb{Q}) \) for \( 1 \leq i \leq k \) [8].

For instance, \( H^* (Gr(2, 4), \mathbb{Q}) = \Lambda^* (x_2, x_3) / (h_3, h_4) \), where \( h_3 = 2x_2x_3 - x_2^2 \) and \( h_4 = x_2^2 - 3x_2^3 + x_3^2 \). As \((2x_2x_3 - x_2^2, x_2^2 - 3x_2^3 + x_3^2) \) forms a regular sequence, the Sullivan model of \( Gr(2, 4) \) is hence given by \( (\wedge^V (x_2, x_3), d) \) with \( dx_2 = dx_3 = 0 \), \( dx_4 = 2x_2x_3 - x_2^2 \) and \( dx_5 = x_2^2 - 3x_2^3 + x_3^2, x_4^2 \).

\section{3. L\text{\text{∞}} Models of Function Spaces}

\textbf{Definition 3.} On a graded vector space \( L \), an \( L_\infty \) structure, usually denoted by \((L, \{ l_k \}_{k \in \mathbb{N}}) \), is a collection of linear maps, \( \{ l_k \}_{k \in \mathbb{N}} \), called brackets, where \( l_k : \phi^k L \rightarrow L \) such that the following conditions are satisfied:

(1) \( l_k \) are graded skew symmetric, that is, for any \( k \) permutation \( \sigma \),

\[
l_k (x_{\sigma(1)}, \ldots, x_{\sigma(k)}) = \text{sgn} (\sigma) \epsilon_k l_k (x_1, \ldots, x_k), \tag{3}
\]

where \( \epsilon_k \) is the sign given by the Koszul convention.

(2) The generalised Jacobi identity holds, that is,

\[
\sum_{i+j+k = n} \sum_{\sigma \in S_n} \text{sgn} (\sigma) \epsilon_\sigma (\sigma) (-1)^{i(j-1)} l_i (x_{\sigma(1)}, \ldots, x_{\sigma(i)}) \cdot l_j (x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}),
\]

\[
x_{\sigma(i+1)} \ldots x_{\sigma(n)} = 0, \tag{4}
\]

where \( S(i, n - i) \) denotes the \( (i, n - i) \) shuffles which are permutations \( \sigma \in S_n \) such that \( \sigma(1) < \cdots < \sigma(i) \) and \( \sigma(i + 1) < \cdots < \sigma(n) \).

An \( L_\infty \) algebra \((L, \{ l_k \}_{k \in \mathbb{N}}) \) is minimal if \( l_1 = 0 \). \( L_\infty \) structures are in one-to-one correspondence with codifferentials on the non unital, free commutative coalgebra \( \Lambda^+ sL \) in which \( s \) denotes suspension, that is \( (sL)_0 = L_{k-1} \) [9].

Note that if \( L \) is a graded vector space of finite type, an \( L_\infty \) structure on \( L \) induces a commutative differential graded algebra structure \( \mathcal{C}^\infty (L) = (\wedge (sL)^{h}, d, \sum d_i + d_j, \text{where} \ d_j : \mathcal{C}^\infty (L) \rightarrow \wedge V \) is defined by \( \langle d_j, x \rangle \cdot \wedge x \rangle = (-1)^{j} \langle x \rangle \cdot \wedge x \rangle = (-1)^{j} x \cdot \wedge x \rangle \) and \( x \cdot \wedge x \rangle = (-1)^{|x|+\sum j - (k-1)x_1 \rangle \} \) [10]. This is a generalisation of the Quillen cochain functor to \( L_\infty \) algebras ([16], Chap.23).

Further, an \( L_\infty \) algebra \( L \) is an \( L_\infty \) model of a simply connected space, \( X \), if \( \mathcal{C}^\infty (L) \) is a Sullivan model of \( X \) [10].

\textbf{Definition 4.} Let \( \phi : (A, d_a) \rightarrow (B, d_B) \) be a morphism of cochain algebras. A \( \phi \)-derivation \( \theta \) of degree \( n \) is a linear map \( A^* \rightarrow A^* \) such that \( \phi(ab) = \phi(a) \phi(b) \). We denote by \( \text{Der}_n (A, B, \phi) \) the vector space of all \( \phi \)-derivations of degree \( n \). Define a differential of chain complexes \( \delta : \text{Der}_n (A, B, \phi) \rightarrow \text{Der}_{n-1} (A, B, \phi) \), by \( \delta \theta = d_B \theta - (-1)^{\theta} d_A \).

If \( \phi : (\wedge V, d) \rightarrow (B, d) \) is a morphism of commutative differential graded algebras, then there is an isomorphism of vector spaces:

\[
\text{Der} (\wedge V, B, \phi) \xrightarrow{\sim} \text{Hom} (V, B). \tag{5}
\]

Moreover, if \( \{ v_1, \ldots, v_n \} \) is a basis for \( V \), we will denote by \( (v_i, b_i) \) the \( \phi \)-derivation \( \theta \) such that

\[
\begin{align*}
\theta (v_i) &= b_i, & b_i \in B, \\
\theta (v_i) &= 0, & i \neq j.
\end{align*}
\tag{6}
\]

We now follow [10] for the definition below. Consider positive derivations defined by

\[
\text{Der}_r (\wedge V, B, \phi) = \begin{cases} 
\text{Der}_r (\wedge V, B, \phi), & \text{if } i \geq 2, \\
Z (\text{Der}_1 (\wedge V, B, \phi)), & \text{if } i = 1.
\end{cases}
\tag{7}
\]

If \( (A, d) = (\wedge V, d) \) is a Sullivan algebra, then given \( j \geq 2 \) and \( \varphi_1, \ldots, \varphi_j \in \text{Der} (\wedge V, B, \phi) \), of degrees \( p_1, \ldots, p_j \), we define their brackets of length \( j \), \( \{ \varphi_1, \ldots, \varphi_j \} = \lambda_j (\varphi_1, \ldots, \varphi_j) \in \text{Der} (\wedge V, B, \phi) \) by

\[
\{ \varphi_1, \ldots, \varphi_j \} (v) = (-1)^{p_1 + \cdots + p_j - 1} \sum_{i_1 < \cdots < i_j} \epsilon_i \varphi_{i_1} \cdots \varphi_{i_j} (v_{i_1} \cdots v_{i_j}),
\]

where \( dv = \sum v_i \cdot v_j \) and \( \epsilon \) is the sign given by the Koszul convention. These operations may be desuspended to define a set of linear maps \( \{ l^*_j \}_{j \geq 1} \) each of degree \( j - 2 \) on \( s^{-1} \text{Der} (\wedge V, B, \phi) \) as follows:

For \( j = 1 \),

\[
l_1 : s^{-1} \text{Der} (\wedge V, B, \phi) \rightarrow s^{-1} \text{Der} (\wedge V, B, \phi), \quad l_1 (s^{-1} \theta) = -s^{-1} \delta \theta.
\tag{9}
\]

For \( j = 2 \), define

\[
l_2 : s^{-1} \text{Der} (\wedge V, B, \phi) \otimes \cdots \otimes s^{-1} \text{Der} (\wedge V, B, \phi) \rightarrow s^{-1} \text{Der} (\wedge V, B, \phi),
\tag{10}
\]
by $l_j(s^{-1}\varphi_1, \ldots, s^{-1}\varphi_j) = (-1)^j s^{-1}\lambda_j(\varphi_1, \ldots, \varphi_j)$ and $\alpha = \sum_{j=1}^{n} (j - n) + [\varphi_j].$

Let $f : X \to Y$ be a continuous function between 1-connected spaces and map $(X,Y;f)$ the component of $f$ in the space of continuous maps from $X$ to $Y$. If $\phi : (\wedge V, d) \to B$ is a cdga model of $f$, Lemma 3.3 of [10] proves that $(s^{-1}\text{Der}(\wedge V, B, \phi), \{l_j\}_{j=1}^{\infty})$ is an $I_{\text{co}}$ model of $\text{map}(X,Y;f)$. \hfill (11)

4. Inclusion of $Gr(3,n)$ in $Gr(3,n + r)$

**Theorem 1.** Let $i_{n,r} : Gr(3,n) \to Gr(3,n + r)$ be the canonical inclusion. If $r > 2n - 4$, then the rational cohomology algebra of map $(Gr(3,n), Gr(3,n + r); i_{n,r})$ contains a truncated algebra $Q[x]/(x^{2n-4})$, where $|x| = 2$.

**Proof.** The cohomology algebras of $Gr(3,n)$ and $Gr(3,n + r)$ are given by $H^*(Gr(3,n), Q) = \wedge(y_2, y_3, y_6)/(h_{n-2}, h_{n-1}, h_n)$ and $H^*(Gr(3,n + r), Q) = \wedge(x_2, x_3, x_6)/(h_{n-2}, h_{n-1}, h_n)$, respectively. Recall that $Gr(3,n)$ is a smooth manifold of dimension $6n - 18$; hence, $H^m(Gr(3,n), Q) = 0$ for $m \geq 6n - 18$.

The minimal model of $Gr(3,n + r)$ is

$(\wedge V, d) = (\wedge(x_2, x_3, x_6, x_{2n+2r-5}, x_{2n+2r-3}, x_{2n+2r-1}), d), \quad (12)$

with $d(x_2) = d(x_3) = 0$, $d(x_{2n+2r-5}) = h_{n-2}$, $d(x_{2n+2r-3}) = h_{n-1}$, and $d(x_{2n+2r-1}) = h_n$. A model of the inclusion is given by

$\phi : (\wedge V, d) \to B = (\wedge(y_2, y_3, y_6)/(h_{n-2}, h_{n-1}, h_n)) \quad (13)$

where $\phi(x_2) = y_2$, $\phi(x_3) = y_3$, $\phi(x_6) = y_6$, and $\phi(x_{2n+2r-5}) = \phi(x_{2n+2r-3}) = \phi(x_{2n+2r-1}) = 0$. Let $L = \text{Der}(\wedge V, B, \phi)$. Consider $y_3 = x_6, y_4 \in L$. Then $q_1 = \lambda_2(y_2, y_3, y_4) = \lambda_k(y_2, y_3, y_4)$ are polynomials of degree at least $2n + 2r - 2k - 4$. If $k < r - 2n + 7$, then $q_1, q_2, q_3$ will be of degree $> 6n - 18$. Therefore, $\lambda_k(y_2, y_3, y_4) = 0$ for $k = 1, \ldots, r - 2n + 7$. In $C^\infty(sL)$, let $y_3^2 = \rho_2$.

Theorem 2. Consider the inclusion $i_{n,r} : Gr(3,n)Gr(3,n + r)$. If $r > 5n - 14$, then the cohomology algebra of map $(Gr(3,n), Gr(3,n + r); i_{n,r})$ contains a polynomial algebra over a generator of degree 2.

**Proof.** Let

$\phi : (\wedge(x_2, x_3, x_6, x_{2n+2r-5}, x_{2n+2r-3}, x_{2n+2r-1}), d) \to (\wedge(y_2, y_3, y_6)/(h_{n-2}, h_{n-1}, h_n)). \quad (14)$

be a model of the inclusion. Assume $n \geq 6$, in particular, if $r \geq 5n - 14$, then any odd derivation is of degree at least $2n + 2r - 5 - (6n - 18) = 2r - 4n + 13 \geq 13$. Hence, $L_{10} = L_9 = \cdots = L_1 = 0$. If $L_1 = 0$, then the derivation $y_2 = (x_6, y_3)$ is a cycle.

It is sufficient to show that $\lambda_m(y_2, y_3, \ldots, y_r) = 0$ for $m \geq 2$.

We consider the case when $n + r$ is even, the other case is dealt with in a similar way. For $m > n + r/2$ and let

$r = \lambda_m(y_2, y_3, \ldots, y_r)(x_{2n+2r-5}),
\nu = \lambda_m(y_2, y_3, \ldots, y_r)(x_{2n+2r-3}),
\tau = \lambda_m(y_2, y_3, \ldots, y_r)(x_{2n+2r-1}).\quad (15)$

The polynomials $r, s$, and $t$ are of total degree at least $2n + 2r - 2m - 4$. Assume now that $r \geq 5n - 14$, then $2n + 2r - 2m - 4 \geq 6n - 18$. Therefore, $\lambda_m(y_2, y_3, \ldots, y_r) = 0$ for all $m \geq 1$. Hence, $H^*(C^\infty(sL))$ contains a polynomial algebra over a generator of degree 2. \hfill \Box

5. The General Case

We can generalise the above results.

**Theorem 3.** If $r > nk - k^2 - n + 2k - 1$, then the rational cohomology algebra of map $(Gr(n,k), Gr(n,k + r); i_{n,r})$ contains a truncated algebra $Q[x]/(x^{nk-k^2-nk})$, where $|x| = 2$, for $k \geq 2$ and $n \geq 4$.

**Proof.** Consider $i_{n,r} : Gr(k,n)Gr(k,n + r)$.

$(\wedge V, d) = (\wedge(x_2, x_3, x_6, \ldots, x_{nk}, x_{(n-r)k+1}, \ldots, x_{(n-r)r-1}), d) \quad (16)$

be the minimal Sullivan model of $Gr(k,n + r)$ with $d(x_2) = d(x_3) = \cdots = d(x_{nk}) = 0$, $d (x_{nk}) = 1, \ldots, d (x_{(n-r)k+1}) = h_{n-r+1}, \ldots, d (x_{(n-r)r-1}) = h_{n-r}$. Moreover,

$B = H^*(Gr(k,n), Q) = \wedge(y_2, y_3, \ldots, y_{2k})/(h_{n-k+1}, \ldots, h_k) \quad (17)$

A Sullivan model of $i_{n,r}$ is given by $\phi : (\wedge V, d) \to B$, where $\phi(x_2) = y_2, \phi(x_3) = y_3, \phi(x_{nk}) = y_{2k}, \phi(x_{nk+1}) = \cdots = \phi(x_{(n-r)k+1}) = 0$.

Note that the lowest odd degree derivation is of degree $2k - 1$ as $k \geq 2$; then, $L_1 = 0$. Define $y_2 = (x_{2k}, y_{2k-2})$. Then $y_2$ is a cycle. Moreover,

$\lambda_m(y_2, y_3, \ldots, y_r)(x_{2k}, y_{2k-2}), \quad (18)$

are polynomials of degree $\geq 2n + 2r - 2k + 2 - 2m$. As $Gr(k,n) = \text{dim} 2k(n-k)$, if $m < r + n + k^2 + 1 - nk - k$, then $b_1, b_2, \ldots, b_k$ are of degree $\geq 2k(n-k)$.

Therefore, $\lambda_m(y_2, y_3, \ldots, y_r) = 0$ for $m = 2, \ldots, (r + n + k^2 + 1 - nk - k)$. In $C^\infty(sL)$, let $y_3^2 = \rho_2$. Then $[\phi_2^2] \neq 0$ for $i = 1, \ldots, (r + n + k^2 + 1 - nk - k). \hfill \Box
Theorem 4. If \( r > (n - k)(k - 1) + 2 \), then the cohomology algebra of map (\( \text{Gr}(k, n), \text{Gr}(k, n + r); i_r \)) contains a polynomial algebra over a generator of degree 2.

Proof. Let
\[
\phi: \Lambda V = \langle x_2, x_4, x_6, \ldots, x_{2k}, x_{2(n-r-k)+1}, \ldots, x_{2(n+r-1)} \rangle, d\rangle
\]
be a model of the inclusion. By hypothesis, \( r > (n - k)(k - 1) + 2 \) and \( n \geq 4 \). Then any odd derivation is of degree at least \( 2n + 2r - 2k - 2nk + 2k^2 + 1 \). Hence, \( L_1 = L_3 = \cdots = L_f = 0 \) where \( f = 2n + 2r - 2k - 2nk + 2k^2 - 1 \). As defined in the proof of Theorem 3, consider \( \gamma_2 = (x_{2k}, y_{2k-2}) \). It suffices to show that \( \lambda_m(y_2, y_2, \ldots, y_2) = 0 \) for \( m \geq 1 \). Consider \( n + r \) even, the other case is dealt with in a similar way.

For \( m > n + r/2 \), \( \lambda_m(y_2, \ldots, y_2) = 0 \). Assume \( 2 \leq m \leq n + r/2 \),
\[
c_1 = \lambda_m(y_2, y_2, \ldots, y_2)(x_{2n+2r-2k+1}),
\]
\[
\vdots
\]
\[
c_k = \lambda_m(y_2, y_2, \ldots, y_2)(x_{2n+2r-1}),
\]
are polynomials of total degree \( \geq 2n + 2r - 2k + 2 - 2m \). If \( r > (n - k)(k - 1) + 2 \), then \( 2n + 2r - 2k + 2 - 2m > -k + 2 - k^2 - 1 + nk \). Hence, \( c_1 = \cdots = c_k = 0 \).

This implies that \( -k + 2 - k^2 - 1 + nk > 2k(n - k) \). Therefore, \( \lambda_m(y_2, \ldots, y_2) = 0 \) for all \( m \geq 0 \). This shows that \( H^*(C^\infty(L)) \) contains a polynomial algebra over a generator of degree 2. \( \square \)

Data Availability
No data were used to support this study.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

References
[1] P. W. Michor, "Topics in differential geometry," in Graduate Studies in Mathematics, vol. 93, American Mathematical Society, Providence, RI, USA, 2008.
[2] R. Thom, "L’homologie des espaces fonctionnels," Colloque de Topologie Algébrique, pp. 29–39, Masson & Cie, Paris, France, 1957.
[3] A. Haefliger, "Rational homotopy of the space of sections of a nilpotent bundle," Transactions of the American Mathematical Society, vol. 273, no. 2, Article ID 667163, 609 pages, 1982.
[4] J. M. Moller and M. Raussen, "Rational homotopy of spaces of maps into spheres and complex projective spaces," Transactions of the American Mathematical Society, vol. 292, no. 2, Article ID 808750, pp. 721–732, 1985.
[5] J. B. Gatsinzi, P. A. Otieno, and V. Onyango-Otieno, "Rational homotopy of mapping spaces between complex Grassmannians," Quaestiones Mathematicae, pp. 1–12, 2019.
[6] Y. Félix, S. Halperin, and J.-C. Thomas, "Rational homotopy theory," in Graduate Texts in Mathematics, vol. 205, Springer-Verlag, New York, NY, USA, 2001.
[7] P. Deligne, P. Griffiths, J. Morgan, and D. Sullivan, "Real homotopy of Kähler manifolds," Inventiones Mathematicae, vol. 29, no. 3, Article ID 0382702, pp. 245–274, 1975.
[8] M. Hoffman, "Endomorphisms of the cohomology of complex Grassmannians," Transactions of the American Mathematical Society, vol. 281, no. 2, Article ID 722772, 745 pages, 1984.
[9] T. Lada and M. Markl, "Strongly homotopy lie algebras," Communications in Algebra, vol. 23, no. 6, Article ID 1327129, pp. 2147–2161, 1995.
[10] U. Buijs, Y. Félix, and A. Murillo, "\( I_{\infty} \) rational homotopy of mapping spaces," Revista Matemática Complutense, vol. 26, no. 2, Article ID 3068613, pp. 573–588, 2013.