HIGHER ORDER SEIBERG-WITTEN FUNCTIONALS AND THEIR ASSOCIATED GRADIENT FLOWS

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Abstract. We define functionals generalising the Seiberg-Witten functional on closed spin$^c$ manifolds, involving higher order derivatives of the curvature form and spinor field. We then consider their associated gradient flows and, using a gauge fixing technique, are able to prove short time existence for the flows. We then prove energy estimates along the flow, and establish local $L^2$-derivative estimates. These are then used to show long time existence of the flow in sub-critical dimensions. In the critical dimension, we are able to show that long time existence is obstructed by an $L^{k+2}$ curvature concentration phenomenon.

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1. Introduction

In their investigations into the gauge theory of 4-manifolds, N. Seiberg and E. Witten introduced a set of equations \([16], [17]\), now known as the Seiberg-Witten equations, which they then used to construct new differential invariants of 4-manifolds. The invariants defined by Seiberg and Witten, through these equations, were closely related to the Donaldson invariants \([1]\), but rose to prominence when it was observed that they were much simpler to work with, and at times could lead to stronger results than could be obtained through Donaldson theory.

The Seiberg-Witten equations are a system of first order equations, and have a naturally associated energy functional, the Seiberg-Witten functional. Given a spin\(^c\) manifold \(M\), the Seiberg-Witten functional is the functional

\[
SW(\phi, A) = \int_M \left( |F_A|^2 + |\nabla_A \phi|^2 + \frac{S}{4} |\phi|^2 + \frac{1}{8} |\phi|^4 \right) d\mu + \pi^2 c_1(L^2)
\]

where \(\phi\) is a positive spinor, \(A\) a unitary connection on the determinant line bundle \(L^2\) (associated to the spin\(^c\) structure on \(M\)), and \(\nabla_A\) the spin\(^c\) connection induced by \(A\). The importance of this functional comes from the fact that solutions to the Seiberg-Witten equations are absolute minima of the Seiberg-Witten functional. This leads to a variational approach to study the equations.

The variational aspects of the Seiberg-Witten equations were first studied by Jost, Peng, and Wang in \([7]\). In that paper, they considered the Seiberg-Witten functional, and proved regularity for weak solutions to the Euler-Lagrange equations associated to the functional. Furthermore, they proved that the Seiberg-Witten functional satisfies the Palais-Smale condition.

In \([5]\) Hong and Schabrun introduced the Seiberg-Witten flow, which is the gradient flow associated to the Seiberg-Witten functional. They were able to demonstrate long time existence of the flow and showed that, up to gauge transformations, the solution converged to a unique limit, which was then a solution of the Euler-Lagrange equations associated to the Seiberg-Witten functional. This behaviour is analogous to the behaviour of the Yang-Mills flow in dimensions 2 and 3, and the results can be seen as similar to those obtained by Råde \([14]\) for the Yang-Mills flow. Schabrun then generalised these results to the higher dimensional case in \([15]\).

In this paper, we study higher order variants of the Seiberg-Witten functional. Given a spin\(^c\) Riemannian manifold \(M\) of dimension \(n\), and a positive integer \(k\), we consider the functionals

\[
SW^k(A, \phi) = \int_M \left( \frac{1}{2} |\nabla_M^{(k)} F_A|^2 + |\nabla_A^{(k)} \nabla_A \phi|^2 + \frac{S}{4} |\phi|^2 + \frac{1}{8} |\phi|^4 \right) d\mu + \pi^2 c_1(L^2)
\]
defined on pairs \((\phi, A)\), where \(\phi\) is a positive spinor, and \(A\) is a unitary connection on the determinant line bundle \(L^2\), associated to the \(spin^c\) structure on \(M\). We are using \(\nabla_M\) to denote the Levi-Civita connection on \(M\), \(\nabla^{(k)}_M\) and \(\nabla^{(k)}_A\) mean \(k\) iterations of these covariant derivatives.

Critical points of the above functionals satisfy Euler-Lagrange equations, that are higher order generalisations of those coming from the Seiberg-Witten functional. In view of the work of Hong and Schabrun [5], we consider the negative gradient flow associated to the functionals, which takes the form

\[
\frac{\partial \phi}{\partial t} = -\nabla^*_A \nabla_A^{(k+1)} \phi - \frac{1}{4}(S + |\phi|^2)\phi
\]

\[
\frac{\partial A}{\partial t} = (-1)^{k+1} d^* \Delta_M F_A - \sum_{v=0}^{2k-1} P_1^{(v)} [F_A] - 2i Im\left( \sum_{i=0}^{k} C_i \nabla_M^{(i)} \langle \nabla_A^{(k)} \nabla_A \phi, \nabla_A^{(k-i)} \phi \rangle \right)
\]

where \(\Delta_M\) denotes the Bochner Laplacian associated to \(\nabla_M\), \(P_1^{(v)} [F_A]\) denotes a lower order curvature term (see 2.1 for an explicit definition).

As the above gradient flow has order \(2(k+1)\), the technique of using maximum principles and Harnack inequalities to understand the behaviour of solutions is no longer available. It is in this regard that these higher order flows become significantly more difficult to analyse than their second order counterparts. The usual approach one takes, is to obtain localised \(L^2\)-derivative estimates, and energy estimates for solutions along the flow. Together with the Sobolev embedding theorem, these estimates are often robust enough to conclude long time existence for sub-critical dimensions.

Higher order functionals have been studied by a few authors, in different settings. In [3], [4] E. De Giorgi studies compact \(n\)-dimensional hypersurfaces in \(\mathbb{R}^{n+1}\), evolving via the gradient flow of a functional involving higher order derivatives of the curvature. He conjectures that the flow does not develop singularities in finite time. This was part of his program to study singular flows by approximating them by sequences of smooth ones, which involved higher order derivatives. Mantegazza in [10] studies higher order generalisations of the mean curvature flow, by introducing a family of higher order functionals. He is able to show (theorem 7.8 [10]), that provided the derivatives in his functionals are large enough, singularities in finite time do not occur. Inspired by this, Kelleher in [8] considers a higher order variant of the Yang-Mills flow. She proves long time existence in sub-critical dimensions (theorem A in [8]), and is able to prove a curvature concentration phenomenon in the critical dimension (theorem B [8]), analogous to the result obtained by Struwe (theorem 2.3 [18]) for the Yang-Mills flow in dimension four.

A recurrent feature in the study of higher order functionals is that the critical dimension increases with respect to the order of derivatives. Thus, provided the order of the derivatives are sufficiently high (depending on the dimension of the manifold), the associated gradient flows will not develop singularities in finite time.

Our main results are that in dimension \(n < 2(k+2)\) (sub-critical dimension), finite time singularities do not occur, and solutions to the flow exist for all time, see theorem 8.2. Furthermore, when \(n = 2(k+2)\) we cannot rule out finite time singularities, but we show that if present, they are due to an \(L^{k+2}\) curvature concentration phenomenon, see proposition 8.3 and theorem 8.4. This is analogous to what Kelleher observes for the higher order Yang-Mills flow (theorem B [8]). However, this is in contrast with the work of Hong and Schabrun (theorem 1 in [5]) and Schabrun (theorem 1 in [15]), on the Seiberg-Witten flow, who are able to show that an \(L^2\) curvature concentration phenomenon can obstruct long time existence, but are able to rule out such concentration by a careful rescaling
argument together with an $L^2$ energy estimate. In our case, we observe that curvature is concentrating in $L^{k+2}_k$, and while we are able to obtain $L^2$ energy estimates, these are not sufficient enough to prove long time existence via the methods of Hong and Schabrun.

The paper is organised as follows. In section 2, we outline the notation we will be using and explain some basic theory on the action of the gauge group. In section 3, we derive some variational formulae for time dependent connections and spinor fields, and then compute the Euler-Lagrange equations. The section ends by introducing the higher order Seiberg-Witten gradient flow. In section 4, we prove short time existence using a gauge fixing technique. Sections 5 and 6 consider energy and local $L^2$-derivative estimates for solutions of the flow, which are then used to prove estimates of Bernstein-Bando-Shi type, and show that the only obstruction to long time existence is curvature blow up. In section 7, we construct a blow up solution for finite time solutions admitting a singularity in finite time. In Section 8, we prove long time existence in the sub-critical dimension, and then show that in the critical dimension, long time existence is obstructed due to the $L^{k+2}$-norm of the curvature form concentrating in smaller and smaller balls. Finally, in section 9 we end with some concluding remarks.

2. Preliminaries

2.1. Background and notation. In this short section, we outline the setup and notation we will be using throughout the paper.

We will let $(M, g)$ denote a smooth, closed Riemannian manifold of dimension $n$. Its canonical Levi-Civita connection will be denoted by $\nabla_M$, and the Riemannian volume form will be denote by $d\mu$. The metric $g$ will be extended to define a metric on all tensor powers $\bigotimes_r T^*M \otimes \bigotimes_s TM$. We remind the reader that the Levi-Civita connection can also be extended to all tensor powers $\bigotimes_r T^*M \otimes \bigotimes_s TM$, and we will denote any such extension by $\nabla_M$ as well.

As we will be dealing with complex bundles, we will normally be working with the complexification $TM_C$ and $T^*M_C$. The metric $g$ can be canonically extended to these complexified spaces. We will also extend the connection $\nabla_M$, to be $\mathbb{C}$-linear, on these complexified spaces.

Throughout the paper, we will assume $M$ is a spin$^c$ manifold, with a fixed spin$^c$-structure $s$. We will denote the spinor bundle by $S = W \otimes \mathcal{L}$, and by $S^\pm = W^\pm \otimes \mathcal{L}$ the half spinor bundles, with $L^2$ denoting the corresponding determinant line bundle. The spinor bundles, and the half spinor bundles, will all be assumed to have fixed Hermitian metrics. As we will primarily deal with $S^+$, we will call sections of this bundle spinor fields (we should really be calling them positive spinor fields, but as we will never be considering the negative spinor fields, it seems unnecessary to need to distinguish them by using the adjective “positive”). Denote smooth sections of this bundle by $\Gamma(S^+)$ (see [12] and [7], for more on the background of these constructions).

A unitary connection on $L^2$ will be denoted by $A$, recall that $A \in \Lambda^1(M)$. We denote the curvature 2-form associated to $A$ by $F_A = dA \in \Lambda^2(M)$. The space of smooth unitary connections on $L^2$ will be denoted by $\mathfrak{A}$. The spin$^c$ connection defined on $S$, $S^\pm$, and coming from the spin$^c$ structure $s$ and the unitary connection $A$, will be denoted by $\nabla_A$. Locally, we can express $\nabla_A$ by

$$\nabla_A = d + (\omega + A)$$

where $\omega$ is induced by the Levi-Civita connection and Clifford multiplication (see [12]). The curvature of $\nabla_A$ will be denoted by $\Omega_A$. Furthermore, with respect to the hermitian metrics on $S$ and $S^\pm$, $\nabla_A$ is metric compatible.
Once we have connections on our bundles, we can define their $L^2$-adjoints. We will denote the $L^2$-adjoints by $\nabla^*_M$, for the Levi-Civita connection, and $\nabla^*_A$, for the adjoint of the spin$^c$ connection. Thus for example, we have that locally we can write $\nabla^*_M = -g^{ij}(\nabla_M)_i(\nabla_M)_j$.

Using the connection $\nabla_M$, we can extend $\nabla_A$ to any tensor power $\otimes_r, T^r M \otimes S^\pm$. We will denote this extended connection again simply by $\nabla_A$, as is the usual practice in the literature. Once one has extended the connection to all such tensor powers, it is then possible to define composed operators of the form: $\nabla_A \circ \cdots \circ \nabla_A$, we will often denote such a composition by $\nabla_A^{(j)}$, where $j$ is supposed to indicate that we compose $j$ times.

We also point out that the complexified Riemannian metric, together with the Hermitian metric on $S^\pm$, allow us to naturally define an inner product on any tensor power $\otimes_r, T^r M \otimes S^\pm$, which in the course of proofs we will simply denote by $\langle \ , \ \rangle$.

Given a spinor $\phi$, and $p, q \in \mathbb{N}$, with $p \geq q$, we will often use the notation $\langle \nabla^{(p)} \phi, \nabla^{(q)} \rangle$, which will represent a $p-q$ tensor. To see this, write $p = q + r$, then $\langle \nabla^{(p)} \phi, \nabla^{(q)} \rangle = \langle \nabla^{(r)} \nabla^{(q)} \phi, \nabla^{(q)} \phi \rangle$. We can then define a multilinear map

$$\langle \nabla^{(r)} \nabla^{(q)} \phi, \nabla^{(q)} \phi \rangle : T^* M \otimes \cdots \otimes T^* M \to \mathbb{C}$$

by $\langle \nabla^{(r)} \nabla^{(q)} \phi, \nabla^{(q)} \phi \rangle(X_1, \ldots, X_r) = \langle \nabla X_1 \cdots \nabla X_r, \nabla^{(q)} \phi, \nabla^{(q)} \phi \rangle$.

The connections also give rise to Laplacian operators. We will denote the Bochner (or rough) Laplacians associated to $\nabla_M$ and $\nabla_A$ by, $\Delta_M = \nabla^*_M \nabla_M$, and $\Delta_A = \nabla^*_A \nabla_A$ respectively. Furthermore, we will need the Hodge Laplacian on $M$, which we denote by $\Delta_M = dd^* + d^* d$, where $d$ denotes the exterior derivative and $d^*$ its adjoint.

Given tensors $S$, and $T$ on $M$, we let $S \ast T$ denote any multilinear form obtained from $S \otimes T$ in a universal bilinear way. Therefore, $S \ast T$ is obtained by starting with $S \otimes T$, taking any linear combination of this tensor, raising and lowering indices, taking any number of metric contractions (i.e. traces), and switching any number of factors in the product. We then have that

$$|S \ast T| \leq C|S||T|$$

where $C > 0$ will not depend on $S$ or $T$. Furthermore, we have $\nabla(S \ast T) = \nabla(S) \ast \nabla(T)$, and in general we can write $\nabla^{(k)}(S \ast T) = \sum_{i=0}^{k} C_i (\nabla^{(i)} S \ast \nabla^{(k-i)} T)$, for some constants $C_i > 0$. For example, the tensor $\langle \nabla^{(p)} \phi, \nabla^{(q)} \rangle$, defined above, can be written as $\nabla^{(p)} \phi \ast \nabla^{(q)} \phi$.

We will also make use of the $P$ notation, as introduced in [9] p. 314. Given a tensor $\omega$, we denote by

$$P^{(k)}_n[\omega] = \nabla^{(i_1)}_M \omega \ast \nabla^{(i_2)}_M \omega \ast \cdots \ast \nabla^{(i_k)}_M \omega \ast T,$$

where $i_1 + \cdots + i_n = k$, and $T$ is any background tensor depending on only on the metric $g$. In our case, most of the time $T$ will be the curvature tensor $Rm$ associated to $\nabla_M$ (or some contraction of it).

Finally, during the course of many estimates, constants will change from line to line. We will often use the practise of denoting these new constants by the same letter. We will also have many constants depending on the metric $g$. We will often denote such a constant by $C(g)$, and will also use this notation to denote constants that depend on any derivatives of the metric. For example, if we obtain a constant $C$ that depends on the Riemann curvature tensor, we will simply denote this constant by $C(g)$. As the metrics are not changing with respect to time, this notation should not cause any confusion.

2.2. **Action of the gauge group.** The gauge group on $L^2$ is given by $Aut(L^2)$. As $L^2$ is a line bundle, we can identify the gauge group with $\mathcal{G} = \{ g : M \to U(1) \}$. 


Given the connection $\nabla = d + A$ on $\mathcal{L}^2$ (remember $A \in i\Lambda^1(M)$), we define the action of the gauge group $\mathcal{G}$ on $\nabla$ as follows. Let $\zeta \in \mathcal{G}$, then we define a new connection $\zeta^*\nabla$ by

$$\zeta^*\nabla = \zeta^{-1} \circ \nabla \circ \zeta.$$  

Locally, we can express the connection $\zeta^*\nabla$ as

$$\zeta^*\nabla = d + \zeta^{-1}d\zeta + A.$$

We claim that the curvature $F_{\zeta^*\nabla}$ associated to $\zeta^*\nabla$ is actually equal to $F_\nabla$, that is the curvature is invariant under the gauge group. To see this, recall that given $\nabla = d + A$, we have that locally $F_\nabla = dA$. Therefore, using the formula above, we find that

$$F_{\zeta^*\nabla} = d(\zeta^{-1}d\zeta + A) = d(\zeta^{-1}d\zeta) + dA = 0 + A$$

where we have used the fact that $d(\zeta^{-1}d\zeta) = d(\zeta^{-1}) \wedge d\zeta = 0$, as $d\zeta$ is a 1-form. In particular, for any $k > 0$ we have that $\nabla_M^{(k)}F_{\zeta^*\nabla} = \nabla_M^{(k)}F_\nabla$.

On the spin$^c$ connection $\nabla_A$, the gauge group $\mathcal{G}$ acts by

$$\zeta^*\nabla_A = \zeta^{-1} \circ \nabla_A \circ \zeta.$$  

Writing $\nabla_A = d + (\omega + A)$, we find that

$$\zeta^{-1} \circ \nabla_A \circ \zeta = d + (\omega + A\zeta) + \zeta^{-1}d\zeta = \nabla_A + \zeta^{-1}d\zeta.$$  

We point out that the gauge group acts in a similar way on the adjoint: $\zeta^*\nabla^*_A = \zeta^{-1} \circ \nabla^*_A \circ \zeta$.

The action of the gauge group on a spinor field $\phi$ is defined by $\zeta^*\phi = \zeta^{-1}\phi$.

The higher order Seiberg-Witten functional is invariant under the action of the gauge group $\mathcal{G}$. In fact, it is precisely due to this symmetry that the associated higher order Seiberg-Witten gradient flow is not parabolic. As we will see, in order to prove short time existence of the flow one has to resort to a gauge fixing procedure.

3. The higher order Seiberg-Witten gradient flow

In this section, we start our analysis of a family of higher order functionals generalising the Seiberg-Witten functional. Given a pair $(\phi, A)$, in the configuration space $\Gamma(\mathcal{S}^+) \times \mathfrak{A}$, we define the higher order Seiberg-Witten functionals by

$$\text{SW}^k(A, \phi) = \int_M \left( \frac{1}{2} |\nabla_M^{(k)}F_A|^2 + |\nabla_A \nabla A\phi|^2 + \frac{S}{4} |\phi|^2 + \frac{1}{8} |\phi|^4 \right) d\mu + \pi^2 c_1(\mathcal{L}^2).$$  

When considering the gradient flow associated to these functionals, we note that the term $\pi^2 c_1(\mathcal{L}^2)$ does not change along the flow. Therefore, we will simply leave it out.

The main difference between these functionals, and the usual Seiberg-Witten functional, is the higher order derivatives $\nabla_M^{(k)}F_A$, and $\nabla_A \nabla A\phi$ present in the functional. The presence of such higher order derivatives makes the associated gradient flow a higher order system, and this in turn makes their analysis much more involved. In this section, we will start by deriving variational formulas for the above functional, and then move on to working out their associated Euler-Lagrange equations. This will then allow us to define their associated gradient flow, which will be the main topic of this paper.

3.1. Formulas for variations. We start by deriving formulas for variations in the configuration space $\Gamma(\mathcal{S}^+) \times \mathfrak{A}$. These formulas will prove useful when computing the Euler-Lagrange equations.

**Lemma 3.1.** \[
a_M \nabla_A^{(k)}\phi = \nabla_A^{(k)}\phi + \sum_{i=0}^{k-1} C_i \nabla_M^{(i)}dA \otimes \nabla_A^{(k-1-i)}\phi, \text{ for some constants } C_i > 0.\]
Proof. We prove this by induction. For the case $k = 1$, observe that locally we can write $\nabla_A = d + (\omega + IA)$. Differentiating this equation with respect to time we obtain, $\frac{\partial}{\partial t}(\nabla_A) = \frac{\partial A}{\partial t} \otimes \phi + \nabla_A \frac{\partial \phi}{\partial t}$. It then follows that

$$\frac{\partial}{\partial t}(\nabla_A \phi) = \frac{\partial A}{\partial t} \otimes \nabla_A \phi + \nabla_A \left( \frac{\partial}{\partial t}(\nabla_A \phi) \right).$$

This proves the formula for $k = 1$. For the general case, assume the formula is true for $k-1$. We then have

$$\frac{\partial}{\partial t}(\nabla^{(k)}_A \phi) = \frac{\partial A}{\partial t} \otimes \nabla^{(k-1)}_A \phi + \nabla_A \left( \frac{\partial}{\partial t}(\nabla^{(k-1)}_A \phi) \right).$$

Applying the $k = 1$ case and the induction hypothesis, we have that the right hand side of the above equation can be written as

$$\frac{\partial A}{\partial t} \otimes \nabla^{(k-1)}_A \phi + \nabla_A \left( \frac{\partial A}{\partial t} \otimes \nabla^{(k-2)}_A \phi + \sum_{i=0}^{k-2} C_i \nabla_M \frac{\partial A}{\partial t} \otimes \nabla^{(k-2-i)}_A \phi \right)$$

which then simplifies to

$$\frac{\partial A}{\partial t} \otimes \nabla^{(k-1)}_A \phi + \nabla_A \left( \frac{\partial A}{\partial t} \otimes \nabla^{(k-2)}_A \phi + \sum_{i=0}^{k-2} C_i \nabla_M \frac{\partial A}{\partial t} \otimes \nabla^{(k-2-i)}_A \phi \right).$$

Collecting terms we then arrive at the required formula

$$\nabla^{(k)}_A \frac{\partial \phi}{\partial t} + \sum_{i=0}^{k-1} C_i \nabla_M \frac{\partial A}{\partial t} \otimes \nabla^{(k-1-i)}_A \phi.$$  

\[ \square \]

Note that as $F_A = dA$, we have that $\frac{\partial F_A}{\partial t} = d\frac{\partial A}{\partial t}$.

3.2. The Euler-Lagrange equations and the associated gradient flow. In this subsection we compute the Euler-Lagrange equations associated to the higher order functionals (3.0.1). Towards the end of this subsection, we will define their associated gradient flow.

**Proposition 3.2.** The Euler-Lagrange equations associated to the functional

$$SW^k(A, \phi) = \int_M \left( \frac{1}{2} |\nabla^{(k)}_M F_A|^2 + |\nabla^{(k)}_A \nabla_A \phi|^2 + S \frac{1}{4} |\phi|^2 + \frac{1}{8} |\phi|^4 \right) d\mu + \pi^2 c_1(L^2)$$

are given by

$$\nabla^{(k+1)}_A \frac{\partial \phi}{\partial t} + \frac{1}{4} (S + |\phi|^2) \phi = 0$$

$$(-1)^k d^* \Delta^{(k)}_M F_A + \sum_{i=0}^{2k-1} P_i^{(k)} [F_A] + 2i \text{Im} \left( \sum_{i=0}^{k} C_i \nabla_M^{(k)} (\nabla^{(k)}_A \nabla_A \phi, \nabla^{(k-i)}_A \phi) \right) = 0.$$ 

**Proof.** We start with the term $\int |\nabla^{(k)}_A \nabla_A \phi|^2$. We have to obtain formulas for variations in the unitary connection $A$, and variations in the spinor field $\phi$.

Let $A_t$ be a path of unitary connections, with $A(0) = A$, on $L^2$. We then compute:
\[ \frac{\partial}{\partial t} \bigg|_{t=0} \int \langle \nabla_{A_t}^{(k)} \nabla_{A_t} \phi, \nabla_{A_t}^{(k)} \nabla_{A_t} \phi \rangle = \int \nabla_{A_t}^{(k)} \nabla_{A_t} \phi, \nabla_{A_t}^{(k)} \nabla_{A_t} \phi \rangle \bigg|_{t=0} \]

\[ = \int \frac{\partial}{\partial t} \nabla_{A_t}^{(k)} \nabla_{A_t} \phi, \nabla_{A_t}^{(k)} \nabla_{A_t} \phi \rangle \bigg|_{t=0} + \langle \nabla_{A_t}^{(k)} \nabla_{A_t} \phi, \frac{\partial}{\partial t} \nabla_{A_t}^{(k)} \nabla_{A_t} \phi \rangle \bigg|_{t=0}. \]

Using lemma 3.1 we can then write this latter integral (forgetting about the evaluation at \( t = 0 \) for a moment) as

\[ \int \sum_{i=0}^{k} C_i \nabla_M^{(i)} \frac{\partial}{\partial t} \otimes \nabla_{A_t}^{(k-i)} \phi, \nabla_{A_t}^{(k-i)} \nabla_{A_t} \phi \rangle + \langle \nabla_{A_t}^{(k)} \nabla_{A_t} \phi, \sum_{i=0}^{k} C_i \nabla_M^{(i)} \frac{\partial}{\partial t} \otimes \nabla_{A_t}^{(k-i)} \phi \rangle \]

which we can then express as

\[ \int \sum_{i=0}^{k} \langle \nabla_M^{(i)} \frac{\partial}{\partial t} \cdot C_i \langle \nabla_{A_t}^{(k)} \nabla_{A_t} \phi, \nabla_{A_t}^{(k-i)} \phi \rangle + \sum_{i=0}^{k} \langle C_i \nabla_{A_t}^{(k-i)} \phi, \nabla_{A_t}^{(k)} \nabla_{A_t} \phi \rangle, \nabla_{A_t}^{(i)} \frac{\partial}{\partial t} \rangle. \]

Taking adjoints, and simplifying, we have that the above integral can be written as

\[ \int \sum_{i=0}^{k} \langle \frac{\partial}{\partial t} \cdot C_i \nabla_M^{(i)} \langle \nabla_{A_t}^{(k)} \nabla_{A_t} \phi, \nabla_{A_t}^{(k-i)} \phi \rangle + \langle C_i \nabla_{A_t}^{(k-i)} \phi, \nabla_{A_t}^{(k)} \nabla_{A_t} \phi \rangle, \frac{\partial}{\partial t} \rangle \]

\[ = \int \sum_{i=0}^{k} \langle \frac{\partial}{\partial t} \cdot C_i \nabla_M^{(i)} \langle \nabla_{A_t}^{(k)} \nabla_{A_t} \phi, \nabla_{A_t}^{(k-i)} \phi \rangle + \frac{\partial}{\partial t} \cdot C_i \nabla_M^{(i)} \langle \nabla_{A_t}^{(k)} \nabla_{A_t} \phi, \nabla_{A_t}^{(k-i)} \phi \rangle \rangle \]

\[ = \int \sum_{i=0}^{k} \langle \frac{\partial}{\partial t} \cdot C_i \nabla_M^{(i)} \langle \nabla_{A_t}^{(k)} \nabla_{A_t} \phi, \nabla_{A_t}^{(k-i)} \phi \rangle + \frac{\partial}{\partial t} \cdot C_i \nabla_M^{(i)} \langle \nabla_{A_t}^{(k)} \nabla_{A_t} \phi, \nabla_{A_t}^{(k-i)} \phi \rangle \rangle \]

where, in order to get the last line, we have used the fact that \( A_t \) are unitary connections, hence we can write \( A_t = i \alpha t \), with \( \alpha t \) a real valued one form on \( M \).

We can then further simplify the above integral as follows.

\[ \int \sum_{i=0}^{k} \langle \frac{\partial}{\partial t} \cdot C_i \nabla_M^{(i)} \langle \nabla_{A_t}^{(k)} \nabla_{A_t} \phi, \nabla_{A_t}^{(k-i)} \phi \rangle + \frac{\partial}{\partial t} \cdot C_i \nabla_M^{(i)} \langle \nabla_{A_t}^{(k)} \nabla_{A_t} \phi, \nabla_{A_t}^{(k-i)} \phi \rangle \rangle \]

\[ = \int \sum_{i=0}^{k} \langle \frac{\partial}{\partial t} \cdot C_i \nabla_M^{(i)} \langle \nabla_{A_t}^{(k)} \nabla_{A_t} \phi, \nabla_{A_t}^{(k-i)} \phi \rangle + \frac{\partial}{\partial t} \cdot C_i \nabla_M^{(i)} \langle \nabla_{A_t}^{(k)} \nabla_{A_t} \phi, \nabla_{A_t}^{(k-i)} \phi \rangle \rangle \]

\[ = \int \langle \frac{\partial}{\partial t} \cdot 2i \text{Im} \sum_{i=0}^{k} C_i \nabla_M^{(i)} \langle \nabla_{A_t}^{(k)} \nabla_{A_t} \phi, \nabla_{A_t}^{(k-i)} \phi \rangle \rangle. \]

Putting this together we finally obtain the following formula, for variations with respect to \( A_t \).

\[ \frac{\partial}{\partial t} \bigg|_{t=0} \int \langle \nabla_{A_t}^{(k)} \nabla_{A_t} \phi, \nabla_{A_t}^{(k)} \nabla_{A_t} \phi \rangle = \int \langle \frac{\partial}{\partial t} \cdot 2i \text{Im} \sum_{i=0}^{k} C_i \nabla_M^{(i)} \langle \nabla_{A_t}^{(k)} \nabla_{A_t} \phi, \nabla_{A_t}^{(k-i)} \phi \rangle \rangle \bigg|_{t=0}. \]
The next step is to compute variations with respect to the spinor field. Let $\phi_t$ be a path of spinors, we then need to compute
\[
\frac{\partial}{\partial t} \int_{t=0}^{t} \left( \langle \nabla_A^{(k)} \nabla_A \phi_t, \nabla_A^{(k)} \nabla_A \phi_t \rangle \right)
\]
\[
= \int \frac{\partial}{\partial t} \left( \langle \nabla_A^{(k)} \nabla_A \phi_t, \nabla_A^{(k)} \nabla_A \phi_t \rangle \right)_{t=0}
\]
\[
= \int \langle \nabla_A^{(k)} \nabla_A \phi_t, \nabla_A^{(k)} \nabla_A \phi_t \rangle |_{t=0} + \langle \nabla_A^{(k)} \nabla_A \phi_t, \frac{\partial}{\partial t} \nabla_A^{(k)} \nabla_A \phi_t \rangle |_{t=0}
\]
\[
= \int \langle \nabla_A^{(k)} \nabla_A \phi_t, \nabla_A^{(k)} \nabla_A \phi_t \rangle |_{t=0} + \langle \nabla_A^{(k)} \nabla_A \phi_t, \nabla_A^{(k)} \nabla_A \phi_t \rangle |_{t=0}
\]
\[
= \int \langle \nabla_A^{(k)} \nabla_A \phi_t, \nabla_A^{(k)} \nabla_A \phi_t \rangle |_{t=0} + \langle \nabla_A^{(k)} \nabla_A \phi_t, \frac{\partial}{\partial t} \nabla_A^{(k)} \nabla_A \phi_t \rangle |_{t=0}.
\]

We now move on to deal with the curvature term in the higher order Seiberg-Witten functional. Recall, this term is given by the integral $\int \frac{1}{2} |\nabla_M^{(k)} F_A|^2$, we therefore need to compute a formula for the variation with respect to a path of unitary connections $A_t$. We remind the reader that $F_{A_t} = dA_t$, and the unitary condition on $A_t$ means that we can write $A_t = i\alpha_t$, where $\alpha_t$ is a real valued one form.

\[
\frac{\partial}{\partial t} \int \frac{1}{2} \langle \nabla_M^{(k)} F_{A_t}, \nabla_M^{(k)} F_{A_t} \rangle = \int \frac{1}{2} \langle \nabla_M^{(k)} dA_t \partial A_t, \nabla_M^{(k)} F_{A_t} \rangle + \frac{1}{2} \langle \nabla_M^{(k)} F_{A_t}, \nabla_M^{(k)} \partial A_t \rangle
\]
\[
= \int \langle \nabla_M^{(k)} dA_t \partial A_t, \nabla_M^{(k)} F_{A_t} \rangle
\]
\[
= \int \langle \frac{\partial A_t}{\partial t}, d^* \nabla_M^{(k)} \nabla_M^{(k)} F_{A_t} \rangle.
\]

We can further simplify the integral in the last line above by appealing to corollary 10.6. Using this we obtain
\[
\int \langle \frac{\partial A_t}{\partial t}, d^* \nabla_M^{(k)} \nabla_M^{(k)} F_{A_t} \rangle = (-1)^k \int \langle \frac{\partial A_t}{\partial t}, d^* \Delta_M^{(k)} F_{A_t} \rangle + \int \langle \frac{\partial A_t}{\partial t}, \sum_{v=0}^{2k-1} P_1^{(v)} [F_{A_t}] \rangle
\]
\[
= \int \langle \frac{\partial A_t}{\partial t}, (-1)^k d^* \Delta_M^{(k)} F_{A_t} + \sum_{v=0}^{2k-1} P_1^{(v)} [F_{A_t}] \rangle.
\]

Finally, for the term $\int \frac{S}{4} |\phi|^2 + \frac{1}{2} |\phi|^4$, variations with respect to $\phi$ give
\[
\frac{\partial}{\partial t} \int \frac{S}{4} |\phi|^2 + \frac{1}{2} |\phi|^4 = \int \langle \frac{\partial \phi}{\partial t}, \frac{1}{4} (S + |\phi|^2) \phi \rangle.
\]

It follows that the Euler-Lagrange equations are given by
\[
\nabla_A^{(k+1)} \nabla_A^{(k+1)} \phi + \frac{1}{4} (S + |\phi|^2) \phi = 0
\]
\[
(-1)^k d^* \Delta_M^{(k)} F_A + \sum_{v=0}^{2k-1} P_1^{(v)} [F_A] + 2iIm \left( \sum_{i=0}^{k} C_i \nabla_M^{(i)} \nabla_A \phi, \nabla_A^{(k-i)} \phi \right) = 0
\]
which proves the proposition.

\[ \square \]
the following system
\[
\begin{align*}
\frac{\partial \phi}{\partial t} &= -\nabla_A^{(k+1)} \nabla_A^{(k+1)} \phi - \frac{1}{4} (S + |\phi|^2) \phi \\
\frac{\partial A}{\partial t} &= (-1)^{k+1} d^* \Delta_M^{(k)} F_A - \sum_{v=0}^{2k-1} P_1^{(v)}[F_A] - 2i \text{Im} \left( \sum_{i=0}^k C_i \nabla_M^{(i)} (\nabla_A^{(k)} \nabla_A^{(k)} \phi, \nabla_A^{(k-i)} \phi) \right).
\end{align*}
\] (3.2.1)

On setting \( k = 0 \), we see that the above system becomes the Seiberg-Witten flow (see [5]).

We also note that during the integration by parts, carried out in the proof of proposition 3.2, we used the fact that
\[
d^* \Delta_M^{(k)} F_A = (-1)^k d^* \Delta_M^{(k)} F_A + \sum_{v=0}^{2k-1} P_1^{(v)}[F_A].
\] (3.2.2)

During the course of the paper, there will be times when it is more convenient to use the term \( d^* \nabla_M^{(k)} \nabla_M^{(k)} F_A \), and we shall do so without hesitation.

4. Short time Existence

In this section we begin the study of short time existence of the higher order Seiberg-Witten flow. We start by explaining why the system is not parabolic, and then move on to showing that, via a gauge fixing technique, solutions exist and are unique on some time interval.

The gradient flow system corresponding to the higher order Seiberg-Witten functional is not parabolic due to the term \( d^* \Delta_M^{(k)} dA \).

**Proposition 4.1.** The operator \( d^* \Delta_M^{(k)} dA \) is not elliptic.

**Proof.** We recall that the Weitzenböck identity, proposition 10.4, tells us that \( \Delta_M = \Delta_H + E \), where \( E \) is a lower order derivative term, depending on the curvature. As we will be interested in computing the principal symbol of the operator, we don’t actually need to know \( E \) explicitly.

From this identity, we obtain the following
\[
d^* \Delta_M = d^* (d^* d + dd^*) + d^* E = (d + d^* dd^*) + d^* E = (dd^* + d^* d) d^* + d^* E = \Delta_H d^* + d^* E = \Delta_M d^* + F
\]

where \( F \) has order 2.

Iterating this construction, we find that
\[
d^* \Delta_M^{(k)} d = \Delta_M^{(k)} d^* d + G
\]
where \( G \) is a term of order \( 2k + 1 \).

It follows that the principal symbol of \( d^* \Delta_M^{(k)} d \) is equal to the principal symbol of the operator \( \Delta_M^{(k)} d^* d \). However, it is clear that \( d^* d \) is not an elliptic operator, from which it immediately follows that \( \Delta_M^{(k)} d^* d \) is not elliptic. \( \square \)
Since the gradient system is not parabolic in order to prove the existence of a solution, with a given initial condition, we need to follow the method of gauge fixing.

We start by adding the term \((-1)^k(\Delta_M A)\phi\) to the first equation (3.2.1), and the term \((-1)^{k+1}d\Delta_M A\phi\) to (3.2.2). We then get the new system

\[
\begin{align*}
\frac{\partial \phi}{\partial t} &= -\nabla_A^{(k+1)}\phi - \frac{1}{4}(S + |\phi|^2)\phi + (-1)^k(\Delta_M d^* A)\phi \\
\frac{\partial A}{\partial t} &= (-1)^{k+1}d\Delta_M F_A - \sum_{i=0}^{2k-1} F_i^{(v)}[F_A] - 2iIm(\sum_{i=0}^{k} C_i \nabla_M^{(i)}(\nabla_A^{(k)}\nabla_A\phi, \nabla_A^{(k-1)}\phi)) \\
&\quad + (-1)^{k+1}d\Delta_M A.
\end{align*}
\]

**Proposition 4.2.** The above system (4.0.1)-(4.0.2) is parabolic

**Proof.** Existence: Observe that we can write the term \((-1)^{k+1}d\Delta_M F_A + (-1)^{k+1}d\Delta_M d^* A\) as \((-1)^{k+1}\Delta_M d^* dA + (-1)^{k+1}\Delta_M dd^* A + G\), where \(G\) has order \(2k+1\). Therefore, when computing the principal symbol we can forget about this lower order term. We then note that \((-1)^{k+1}\Delta_M d^* dA + (-1)^{k+1}\Delta_M dd^* A = (-1)^{k+1}\Delta_M \Delta_H A\). The operator \((-1)^{k+1}\Delta_M \Delta_H\) is the highest order part in the second equation of the above system. Using the Weitzenböck identity, proposition 10.4, we see that we can write this as \((-1)^{k+1}\Delta^{(k+1)} + J\), where \(J\) is a lower order term. It is clear that \((-1)^{k+1}\Delta^{(k+1)}\) is elliptic, and hence ellipticity of the highest order term in the above second equation follows.

For the first equation, we observe that the highest order term is given by \(\nabla_A^{(k+1)}\phi\), which we can express as \(\Delta_A^{(k+1)} + T\), where \(T\) is a term of order \(2k+1\). In computing the principal symbol we can forget about \(T\). Furthermore, given \(A_0\) we can write \(\Delta_A^{(k+1)} = \Delta_A^{(k+1)} + T'\), where \(T'\) is again a lower order term. The ellipticity of the operator \(\nabla_A^{(k+1)}\phi\) is then an immediate consequence of these observations.

We thus see that the above system is a quasilinear parabolic system of order \(2k+2\).

\[\square\]

Existence and uniqueness of higher order quasilinear parabolic systems (see [11]) then implies that, given an initial condition \((\phi_0, A_0)\) there exists a unique solution \((\phi(t), A(t))\) to the system, on some time interval \([0, T]\), where \(0 < T \leq \infty\).

We are going to use this solution, to the above parabolic system, to build a solution to the higher order Seiberg-Witten flow, via a gauge fixing procedure. We fix an initial condition \((\phi_0, A_0)\), and from here on in \((\phi(t), A(t))\) will denote the unique solution to the above parabolic system with initial condition \((\phi_0, A_0)\).

**Theorem 4.3.** Given an initial condition \((\phi_0, A_0) \in \Gamma(S^+ \times \mathfrak{A}\), there exists a unique solution to the higher order Seiberg-Witten flow (3.2.1)-(3.2.2), on some time interval \(0 < T \leq \infty\).

**Proof.** We start by defining the gauge we are going to be working in. Define a gauge \(g(t)\) as the solution to the following ODE

\[
\begin{align*}
\frac{\partial g}{\partial t} &= g(t)(-1)^k \Delta_M d^* A(t) \\
g(0) &= I.
\end{align*}
\]
The term $(-1)^k \Delta^{(k)}_M d^* A(t)$ is a function on $M \times [0, T)$. Therefore, solving the above ODE gives
\[ g(t) = e^{\int_0^t (-1)^k \Delta^{(k)}_M d^* A(s) ds}. \]
We know that $A(t) \in \Lambda^1(M)$, because $A(t)$ is unitary, which implies $g(t) = e^{if(t)}$, with $f(t) = \int_0^t (-1)^k \Delta^{(k)}_M d^* a(s) ds$, where we are writing $A(s) = ia(s)$, with $a(s)$ a real valued 1-form. This implies that the solution $g(t)$ is indeed a $U(1)$-gauge.

We then consider $(g^* \phi, g^* A)$. We are going to prove that this is a solution of the higher order Seiberg-Witten flow. In order to do this we are going to make use of the following formula
\[ \frac{\partial g^{-1}}{\partial t} = -g^{-2} \frac{\partial g}{\partial t} = -g^{-2} (-1)^k \Delta^{(k)}_M d^* A = (-1)^{k+1} g^{-1} \Delta^{(k)}_M d^* A. \]

We will start by computing $\frac{\partial g^* \phi}{\partial t}$:
\[
\frac{\partial g^* \phi}{\partial t} = \frac{\partial g^{-1}}{\partial t} \phi + g^{-1} \left( \frac{\partial \phi}{\partial t} \right) = (-1)^{k+1} g^{-1} (\Delta^{(k)}_M d^* A) \phi + g^{-1} \left( -\nabla^{(k+1)}_A \nabla^{(k+1)}_A \phi - \frac{1}{4} (S + |\phi|^2) \phi + (-1)^k (\Delta^{(k)}_M d^* A) \phi \right) = -g^{-1} \nabla^{(k+1)}_A \nabla^{(k+1)}_A \phi - g^{-1} \frac{1}{4} (S + |g^* \phi|^2) g^* \phi.
\]

We move on to computing $\frac{\partial g^* A}{\partial t}$:
\[
\frac{\partial g^* A}{\partial t} = \frac{\partial}{\partial t} (A + g^{-1} dg) = \frac{\partial A}{\partial t} + \frac{\partial g^{-1}}{\partial t} dg + g^{-1} d \left( \frac{\partial g}{\partial t} \right) = \frac{\partial}{\partial t} (A + g^{-1} dg) = \frac{\partial A}{\partial t} + \frac{\partial g^{-1}}{\partial t} dg + g^{-1} d \left( \frac{\partial g}{\partial t} \right) = -d^* \Delta^{(k)}_M k \nabla^{(k)}_M F_A - 2i \text{Im} \left( \sum_{i=0}^k C_i \nabla^{(i)}_M \nabla^{(k-i)}_A \phi, \nabla^{(k-i)}_A \phi \right) + (-1)^{k+1} d \Delta^{(k)}_M d^* A \]
\[ + (-1)^{k+1} g^{-1} (\Delta^{(k)}_M d^* A) dg + g^{-1} \left( (-1)^k (dg) (\Delta^{(k)}_M d^* A) + (-1)^k gd \Delta^{(k)}_M d^* A \right) = -d^* \Delta^{(k)}_M k \nabla^{(k)}_M F_A - 2i \text{Im} \left( \sum_{i=0}^k C_i \nabla^{(i)}_M \nabla^{(k-i)}_A \phi, \nabla^{(k-i)}_A \phi \right). \]

It follows that $(g^* \phi, g^* A)$ is a solution to the higher order Seiberg-Witten flow with initial condition $(g^*(0) \phi(0), g^*(0) A(0)) = (\phi_0, A_0)$, using the fact that $g(0) = I$. This proves existence.

**Uniqueness:** To see that solutions are unique, observe that given a solution $(\phi, A)$ of the higher order Seiberg-Witten flow, with initial condition $(\phi_0, A_0)$, we can then construct a gauge $g(t)$ as we did above. If we then consider $((g^{-1})^* \phi, (g^{-1})^* A)$, then a simple computation shows that this solves the parabolic system (4.0.1)-(4.0.2), with initial condition $(\phi_0, A_0)$.

This means that if we had two solutions to the higher order Seiberg-Witten flow, $(\phi_1, A_1)$ and $(\phi_2, A_2)$, such that $(\phi_1(0), A_1(0)) = (\phi_2(0), A_2(0)) = (\phi_0, A_0)$. Then we find that $((g^{-1})^* \phi_1, (g^{-1})^* A_1)$ and $((g^{-1})^* \phi_2, (g^{-1})^* A_2)$ both solve the parabolic system (4.0.1)-(4.0.2), with the same initial condition $(\phi_0, A_0)$. Uniqueness of this system then gives $((g^{-1})^* \phi_1, (g^{-1})^* A_1) = ((g^{-1})^* \phi_2, (g^{-1})^* A_2)$. Applying $g^*$ to this equation, and using the fact that $(g^*)^* (g^{-1})^* = I$, it follows that $(\phi_1, A_1) = (\phi_2, A_2)$, and uniqueness is established.
5. Energy estimates

In this section we derive energy estimates for solutions of the higher order Seiberg-Witten flow. These estimates will then be used in the study of long time existence in section 8.

We start by showing that the spinor field does not blow up along the flow as you approach the maximal time.

**Proposition 5.1.** Given a solution \((\phi_t, A_t)\) to the higher order Seiberg-Witten flow on some time interval \([0, T]\), where \(T \leq \infty\). We have that \(\sup_{t \in [0,T]} |\phi_t| < \infty\).

**Proof.** We compute

\[
\frac{\partial}{\partial t} \langle \phi, \phi \rangle = \langle \frac{\partial \phi}{\partial t}, \phi \rangle + \langle \phi, \frac{\partial \phi}{\partial t} \rangle
\]

\[
= \langle -\nabla_A^{(k+1)} \nabla_A^{(k+1)} \phi - \frac{1}{4} (S + |\phi|^2) \phi, \phi \rangle + \langle \phi, -\nabla_A^{(k+1)} \nabla_A^{(k+1)} \phi - \frac{1}{4} (S + |\phi|^2) \phi \rangle
\]

\[
= -2 \langle \nabla_A^{(k+1)} \phi, \nabla_A^{(k+1)} \phi \rangle - \frac{1}{2} (S + |\phi|^2) |\phi|^2.
\]

Let \(S_0 = \min \{ S(x) : x \in M \}\), and choose \(0 < \epsilon < \epsilon\). Suppose there exists \((x, t)\) such that \(|\phi(x, t)| \geq \sqrt{|S_0|} + \epsilon\). Let \(t_0\) be the first time when this happens, so that there exists \((x_0, t_0)\) such that \(|\phi(x_0, t_0)| \geq \sqrt{|S_0|} + \epsilon\). Without loss of generality we assume \(t_0 > 0\), for if \(t_0 = 0\), then replace \(\epsilon\) with \(2\epsilon\) and consider \(\sqrt{|S_0|} + 2\epsilon\) instead.

Therefore assuming \(t_0 > 0\), we get, by the continuity of \(\phi\), that \(|\phi(x_0, t_0)| = \sqrt{|S_0|} + \epsilon\). By continuity, we also know that there exists an interval \((t_1, t_2)\) such that \(t_0 \in (t_1, t_2)\) and \(|\phi(x_0, t)| > \sqrt{|S_0|}\), for all \(t \in (t_1, t_2)\).

Then for any such \(t \in (t_1, t_2)\), we have \(|\phi(x_0, t)|^2 + S \geq |\phi(x_0, t)|^2 + S_0 \geq 0\). This in turn implies that

\(|\phi(x_0, t)|^2 + S|\phi(x_0, t)|^2 \geq 0, \forall t \in (t_1, t_2)\).

Substituting this into the formula obtained for \(\frac{\partial}{\partial t} \langle \phi, \phi \rangle\) at the start of this proof, we find that

\[
\frac{\partial}{\partial t} |\phi(x_0, t)|^2 \leq 0, \forall t \in (t_1, t_2).
\]

This implies that \(|\phi(x_0, t)|^2\) is a non-increasing function for \(t \in (t_1, t_2)\). In particular, this implies that

\[|\phi(x_0, t)| \geq |\phi(x_0, t_0)| = \sqrt{|S_0|} + \epsilon, \forall t \in (t_1, t_0)\).

However, this contradicts the fact that \(t_0\) was the first time such that \(|\phi(x, t)| \geq \sqrt{|S_0|} + \epsilon\). It follows that no such time \(t_0\) exists, and that in fact we have that

\[|\phi(x, t)| \leq \sqrt{|S_0|} + \epsilon, \forall t \in (t_1, t_2)\]

which in turn implies that \(\sup_{t \in [0,T]} |\phi_t| < \infty\).

\[\square\]

**Lemma 5.2.**

\[
\frac{\partial}{\partial t} \mathcal{SW}^k(\phi(t), A(t)) = -2 (\|\frac{\partial \phi}{\partial t}(t)\|^2 + \|\frac{\partial A}{\partial t}(t)\|^2 L_2^2) \leq 0.
\]

In particular, the higher order Seiberg-Witten energy remains bounded along the flow.
Lemma 5.3. \[
\frac{\partial}{\partial \tau} \text{SW}^k(\phi(t), A(t)) = \left. \frac{\partial}{\partial \tau} \right|_{\tau=0} \text{SW}^k(\phi(t) + \tau \frac{\partial \phi}{\partial \tau}, A(t)) + \left. \frac{\partial}{\partial \tau} \right|_{\tau=0} \text{SW}^k(\phi(t), A(t) + \tau \frac{\partial A}{\partial \tau}).
\]

We start by computing \[
\frac{\partial}{\partial \tau} \bigg|_{\tau=0} \text{SW}^k(\phi(t) + \tau \frac{\partial \phi}{\partial \tau}, A(t)).
\]
We can write this derivative as
\[
\frac{\partial}{\partial \tau} \bigg|_{\tau=0} \int (\nabla_A^{(k+1)}(\phi(t) + \tau \frac{\partial \phi}{\partial \tau}), \nabla_A^{(k+1)}(\phi(t) + \tau \frac{\partial \phi}{\partial \tau})) + |\nabla_M F_A|^2 + S_4 \langle \phi(t) + \tau \frac{\partial \phi}{\partial \tau}, \phi(t) + \tau \frac{\partial \phi}{\partial \tau} \rangle + \frac{1}{8} \langle \phi(t) + \tau \frac{\partial \phi}{\partial \tau}, \phi(t) + \tau \frac{\partial \phi}{\partial \tau} \rangle^2.
\]

Getting rid of the terms that don’t involve \(\tau\), we can express the above as
\[
\frac{\partial}{\partial \tau} \bigg|_{\tau=0} \int \tau (\nabla_A^{(k+1)} \frac{\partial \phi}{\partial t}, \nabla_A^{(k+1)} \phi(t)) + \tau (\nabla_A^{(k+1)} \frac{\partial \phi}{\partial t}, \nabla_A^{(k+1)} \frac{\partial \phi}{\partial t}) + S_4 \tau (\frac{\partial \phi}{\partial t}, \phi(t)) + \frac{S_4}{8} \tau (\phi(t), \frac{\partial \phi}{\partial t})
\]
\[
+ \frac{1}{8} (\langle \phi(t) \rangle^2 + \tau \langle \phi, \frac{\partial \phi}{\partial t} \rangle + \tau \langle \frac{\partial \phi}{\partial t}, \phi(t) \rangle + \tau^2 \frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial t} \rangle^2).
\]

Computing the above derivative we obtain
\[
\int (\nabla_A \frac{\partial \phi}{\partial t}, \nabla_A^{(k+1)} \phi(t)) + (\nabla_A \frac{\partial \phi}{\partial t}, \nabla_A^{(k+1)} \frac{\partial \phi}{\partial t}) + (\nabla_A^{(k+1)} \frac{\partial \phi}{\partial t}, \nabla_A^{(k+1)} \phi(t)) + (\frac{S_4}{4} + \frac{\phi^2}{4}) \phi, \frac{\partial \phi}{\partial t}(t))
\]
\[
= \int (\nabla_A \frac{\partial \phi}{\partial t}, -\nabla_A \frac{\partial \phi}{\partial t}) + (\nabla_A \frac{\partial \phi}{\partial t}, \nabla_A^{(k+1)} \phi(t))
\]
\[
= -2 ||\nabla_A \frac{\partial \phi}{\partial t}||_{L^2}^2.
\]

A similar computation proves that \[
\frac{\partial}{\partial \tau} \bigg|_{\tau=0} \text{SW}^k(\phi(t), A(t) + \tau \frac{\partial A}{\partial \tau}) = -2 ||\frac{\partial A}{\partial \tau}(t)||_{L^2}^2,
\]
which gives the statement of the lemma.

Recall that the Seiberg-Witten functional is defined as
\[
\text{SW}(\phi, A) = \int (|\nabla_A \phi|^2 + |F_A|^2 + S_4 |\phi|^2 + \frac{1}{8} |\phi|^4) d\mu + \pi^2 c_1(\mathcal{L}^2).
\]

In the previous lemma, we saw how the higher order Seiberg-Witten energy decreased along the flow, and therefore we could conclude that it remains bounded in time. The following lemma proves that given a solution to the higher order Seiberg-Witten flow for finite time \(T < \infty\), its Seiberg-Witten energy is also bounded along the flow.

Lemma 5.3. Let \((\phi(t), A(t))\) be a solution to the higher order Seiberg-Witten flow, on \([0, T]\) for \(T < \infty\). Then the Seiberg-Witten energy \[
\text{SW}(\phi, A) = \int (|\nabla_A \phi|^2 + |F_A|^2 + S_4 |\phi|^2 + \frac{1}{8} |\phi|^4) d\mu + \pi^2 c_1(\mathcal{L}^2)
\]
is bounded along the flow. That is \(\sup_{t \in [0, T]} \text{SW}(\phi(t), A_t) < \infty\).

Proof. We start by computing \[
\frac{\partial}{\partial t} \text{SW}(\phi_t, A_t) = \int (\frac{\partial \phi}{\partial t}, \nabla_A \phi) + (\nabla_A \phi, \frac{\partial \phi}{\partial t}) + (\frac{S_4}{4} + \frac{\phi^2}{4}) (\frac{\partial \phi}{\partial t}, \phi) + (\phi, \frac{\partial \phi}{\partial t})
\]
\[
+ 2 (\frac{\partial A}{\partial t}, F_A) + (\frac{\partial A}{\partial t} \phi, \nabla_A \phi) + (\nabla_A \phi, \frac{\partial A}{\partial t} \phi).
\]
We now explain how we can bound the quantity on the right. In doing so we will need to define the following constant $C := \max\{1, \sup_{M \times [0,T]} \{S/4 + |\phi|^2/4\}\}$. We then have

$$
\int \frac{\partial \phi}{\partial t} (\nabla_A^* \nabla A \phi) + \langle \nabla_A^* \nabla A \phi, \frac{\partial \phi}{\partial t} \rangle + \langle \phi, \frac{\partial (\nabla_A^* \nabla A \phi)}{\partial t} \rangle + 2\langle \nabla A \phi, d^* F_A \rangle + \langle \nabla A \phi, \frac{\partial A}{\partial t} \otimes \phi, \nabla A \phi \rangle
$$

$$
\leq C \left( \int \frac{\partial \phi}{\partial t} (\nabla_A^* \nabla A \phi) + C \langle \nabla_A^* \nabla A \phi, \frac{\partial \phi}{\partial t} \rangle + 2\langle \nabla A \phi, d^* F_A \rangle + \langle \nabla A \phi, \frac{\partial A}{\partial t} \otimes \phi, \nabla A \phi \rangle \right)
$$

On appealing to Young’s inequality, we can further bound the right hand side of this last inequality as follows.

$$
2C \int \left( |\frac{\partial \phi}{\partial t}|^2 + |\nabla_A^* \nabla A \phi| \right) + \langle |\frac{\partial \phi}{\partial t}|, \phi \rangle + \langle |\frac{\partial A}{\partial t}|, d^* F_A \rangle + \langle |\frac{\partial A}{\partial t} \otimes \phi, \nabla A \phi \rangle
$$

$$
\leq 2C \left( |\frac{\partial \phi}{\partial t}|^2 + C_1(g) ||\nabla_A^* \nabla A \phi||^2_{L^2} + ||\phi||^2_{L^2} + ||\frac{\partial A}{\partial t}||^2_{L^2} + C_2(g)||\nabla_M F_A||^2_{L^2}
\right)
$$

where the constant $C(g, \phi)$ depends on $\phi$ through $||\phi||_{\infty}$, which we know is bounded along the flow by proposition 5.1.

Applying the energy estimate, lemma 5.2, we can write this last quantity as

$$
C(g, \phi) \left( - \frac{\partial}{\partial t} \mathcal{S} \mathcal{W}^k(\phi_t, A_t) + ||\nabla_A^* \nabla A \phi||^2_{L^2} + ||\nabla_M F_A||^2_{L^2} + ||\phi||^2_{L^2} + ||\nabla A \phi||^2_{L^2} \right).
$$

In order to estimate this quantity we are going to apply the interpolation inequality, lemma 10.3. Let $\epsilon_1 > \epsilon_2 > 0$, we then have

$$
C(g, \phi) \left( - \frac{\partial}{\partial t} \mathcal{S} \mathcal{W}^k(\phi_t, A_t) + ||\nabla_A^* \nabla A \phi||^2_{L^2} + ||\nabla_M F_A||^2_{L^2} + ||\phi||^2_{L^2} + ||\nabla A \phi||^2_{L^2} \right)
$$

$$
\leq C(g, \phi) \left( - \frac{\partial}{\partial t} \mathcal{S} \mathcal{W}^k(\phi_t, A_t) + C(\epsilon_1)||\nabla_A^*(k+1) \phi||^2_{L^2} + \epsilon_1||\nabla_A \phi||^2_{L^2} + ||\nabla_M F_A||^2_{L^2} + \epsilon_2||F_A||^2_{L^2}
\right)
$$

Therefore for any $t < T$, we have that

$$
\mathcal{S} \mathcal{W}(\phi_t, A_t) - \mathcal{S} \mathcal{W}(\phi_0, A_0) \leq C_1(g, \phi)(\mathcal{S} \mathcal{W}^k(\phi_0, A_0) - \mathcal{S} \mathcal{W}^k(\phi_t, A_t))
$$

$$
+ C_2(g, \phi, \epsilon_1, \epsilon_2) \int_0^t \left( ||\nabla_A^*(k+1) \phi||^2_{L^2} + ||\nabla_M F_A||^2_{L^2} + \int \frac{S}{4} ||\phi||^2 + \frac{1}{8} ||\phi||^4_{L^2} \right)
$$

$$
+ C(g, \phi) \epsilon_1 \int_0^t \left( ||F_A||^2_{L^2} + ||\nabla_A \phi||^2_{L^2} + \int \frac{S}{4} ||\phi||^2 + \frac{1}{8} ||\phi||^4_{L^2} \right) + C(\phi)
$$

where the constant $C(\phi)$ comes from the fact that we added in the terms $\int \frac{S}{4} ||\phi||^2$ and $\frac{1}{8} ||\phi||^4_{L^2}$, remembering that these quantities are bounded along the flow.
We can rewrite this latter quantity as
\[
C_1(g, \phi)(SW^k(\phi_0, A_0) - SW^k(\phi_t, A_t)) + C_2(g, \phi, \epsilon_1, \epsilon_2) \int_0^t SW^k(\phi_s, A_s)ds \\
+ C(g, \phi)\epsilon_1 \int_0^t SW(\phi_s, A_s)ds + C(\phi).
\]
Using the fact that the higher order Seiberg-Witten energy decreases along the flow, we can estimate this quantity as follows.
\[
C_1(g, \phi)(SW^k(\phi_0, A_0) - SW^k(\phi_t, A_t)) + C_2(g, \phi, \epsilon_1, \epsilon_2) \int_0^t SW^k(\phi_s, A_s)ds \\
+ C(g, \phi)\epsilon_1 \int_0^t SW(\phi_s, A_s)ds + C(\phi)
\leq (C_1(g, \phi) + tC_2(g, \phi, \epsilon_1, \epsilon_2))SW^k(\phi_0, A_0) - C_1(g, \phi)SW^k(\phi_t, A_t) + tC(g, \phi)\epsilon_1 \sup_{s \in [0,t]} SW(\phi_s, A_s) + C(\phi)
\leq C_3(g, \phi, T)SW^k(\phi_0, A_0) + tC(g, \phi)\epsilon_1 \sup_{s \in [0,t]} SW(\phi_s, A_s)
\]
where the constant $C_3(g, \phi, T)$ comes from using $t < T$, and absorbing $C(\phi)$ into $(C_1(g, \phi) + tC_2(g, \phi, \epsilon_1, \epsilon_2))$.

In particular, by taking $\epsilon_1 = \epsilon/tC(g, \phi)$, we get the following inequality
\[
SW(\phi_t, A_t) - SW(\phi_0, A_0) \leq C_3(g, \phi, T)SW^k(\phi_0, A_0) + \epsilon \sup_{s \in [0,t]} SW(\phi_s, A_s).
\]
This implies
\[
SW(\phi_t, A_t) - \epsilon \sup_{s \in [0,t]} SW(\phi_s, A_s) - SW(\phi_0, A_0) \leq C_3(g, \phi, T)SW^k(\phi_0, A_0).
\]  
(5.0.1)
Suppose there exists $t_m \to T$ such that $\lim_{m \to \infty} SW(\phi_{t_m}, A_{t_m}) \to \infty$. By throwing out some of the $t_m$ we can assume that $SW(\phi_{t_m}, A_{t_m}) \geq SW(\phi_{t_i}, A_{t_i})$ for $m \geq n$, and that $t_m \geq t_n$, when $m \geq n$.

Partition $[0, T)$ in the following way, $[0, T) = [t_0, t_1] \cup [t_1, t_2] \cup \ldots [t_k, t_{k+1}] \ldots$, where $t_0 = 0$. Then define $s_i \in [t_i, t_{i+1})$ by $\sup_{t \in [t_i, t_{i+1})} SW(\phi_t, A_t) = SW(\phi_{s_i}, A_{s_i})$. It is easy to see that $s_i \to T$, and $SW(\phi_{s_i}, A_{s_i}) \to \infty$ as $i \to \infty$. Furthermore, we also have that $SW(\phi_{s_i}, A_{s_i}) \leq SW(\phi_{s_j}, A_{s_j})$ when $j \leq i$.

We now substitute $s_i$ for $t$ in the above inequality (5.0.1) to obtain
\[
SW(\phi_{s_i}, A_{s_i}) - \epsilon SW(\phi_{s_i}, A_{s_i}) - SW(\phi_0, A_0) \leq C_3(g, \phi, T)SW^k(\phi_0, A_0)
\]
from which we obtain
\[
SW(\phi_{s_i}, A_{s_i}) \leq \frac{1}{1 - \epsilon} (C_3(g, \phi, T)SW^k(\phi_0, A_0) + SW(\phi_0, A_0)).
\]
The right hand side of the above equation is finite, and independent of $i$. Therefore, taking $i \to \infty$ on the left, we contradict the fact that the left hand side should approach $\infty$. It follows that no such $\{t_m\}$ exists, and that $\sup_{t \in [0,T]} SW(\phi_t, A_t) < \infty$.

\[\square\]

6. Local $L^2$-derivative estimates

In this section we prove local $L^2$-derivative estimates for solutions of the higher order Seiberg-Witten flow. As the system (3.2.1)-(3.2.2) is a higher order system, one cannot appeal to the use of maximum principles and Harnack inequalities to study such systems.
It is in this regard that the obtaining of local derivative estimates become a vital tool for the study of such higher order systems. We shall then put these derivative estimates to use when we consider questions of long time existence.

6.1. Bump functions. In the course of obtaining local $L^2$-derivative estimates, we will need to make use of bump functions. In this brief subsection, we outline the notation we use and prove a simple lemma that will be used in our estimates in the subsections to come.

**Definition 6.1.** Given $\gamma \in C^\infty_c(M)$, we say $\gamma$ is a bump function if $0 \leq \gamma \leq 1$.

In this paper, we will always use the notation $\gamma$ to denote such a bump function.

**Lemma 6.2.** Let $\gamma$ be a bump function. Fix $i \in \mathbb{N}$, and let $s$ be a positive real number such that $s \geq i$. We then have

$$\nabla^{(i)}(s) = \sum_{n_1 + \ldots + n_i = i} C_{(n_1, \ldots, n_i)}(\gamma, s) \gamma^{s-i} \nabla^{n_1} \gamma \ast \cdots \ast \nabla^{n_i} \gamma.$$ 

**Proof.** One simply has to compute derivatives. First observe that $\nabla(\gamma^s) = s \gamma^{s-1} \nabla \gamma$, and

$$\nabla^{(2)}(\gamma^s) = \nabla(s \gamma^{s-1} \nabla \gamma) = \nabla(s \gamma^{s-1}) \otimes \nabla \gamma + s \gamma^{s-1} \nabla^{(2)} \gamma = s(s-1) \gamma^{s-2} \nabla^2 \gamma \otimes \nabla \gamma + s \gamma^{s-1} \nabla^{(2)} \gamma.$$ 

Continuing to take derivatives, we see that we can write

$$\nabla^{(i)}(\gamma^s) = \sum_{n_1 + \ldots + n_i = i} \tilde{C}_{(n_1, \ldots, n_i)}(\gamma, s) \gamma^{s-i} \nabla^{n_1} \gamma \otimes \cdots \otimes \nabla^{n_i} \gamma.$$ 

By swapping some products, and collecting like terms, it is then easy to see that we can write

$$\nabla^{(i)}(\gamma^s) = \sum_{n_1 + \ldots + n_i = i} C_{(n_1, \ldots, n_i)}(\gamma, s) \gamma^{s-i} \nabla^{n_1} \gamma \ast \cdots \ast \nabla^{n_i} \gamma.$$ 

□

In the subsections to come, we will obtain local $L^2$-derivative estimates for the spinor field and curvature form. During these estimates we will obtain constants that will depend on a fixed bump function $\gamma$, and its derivatives. We will denote such a constant by $C(\gamma)$, with the understanding that $C(\gamma)$ may be depending on derivatives of $\gamma$ as well.

6.2. Evolution equations. We start by computing the evolution equations satisfied by the spinor field and the curvature form under the flow.

**Lemma 6.3.** Let $(\phi(t), A(t))$ be a solution to the higher order Seiberg-Witten flow. Then

$$\frac{\partial F_{A(t)}}{\partial t} = (-1)^{k+1} \Delta_M^{(k+1)} F_{A(t)} + \sum_{v=0}^{2k} P_v [F_{A(t)}] - 2i \text{Im} \left( \sum_{i=1}^{k} C_i d\nabla^{s_i} \nabla_A \phi, \nabla^{(k-i)} A \right).$$
Proof. We have that $\frac{\partial F_A(t)}{\partial t} = \frac{\partial A(t)}{\partial t} = d\frac{\partial A(t)}{\partial t}$. As $A$ satisfies the higher order Seiberg-Witten flow, we obtain

$$\frac{\partial F_A(t)}{\partial t} = d\left( (-1)^{k+1} d^* \Delta_{M}^{(k)} F_A - \sum_{v=0}^{2k-1} P_1^{(v)}[F_A] - 2iIm\left( \sum_{i=0}^{k} C_i \nabla_M^{(i)} \langle \nabla_A A \phi, \nabla_A^{(k-i)} \phi \rangle \right) \right)$$

$$= (-1)^{k+1} d^* \Delta_{M}^{(k)} F_A - \sum_{v=0}^{2k} P_1^{(v)}[F_A] - 2iIm\left( \sum_{i=0}^{k} C_i \nabla_M^{(i)} \langle \nabla_A A \phi, \nabla_A^{(k-i)} \phi \rangle \right)$$

$$= (-1)^{k+1} \Delta_{M}^{(k+1)} F_A + \sum_{v=0}^{2k} P_1^{(v)}[F_A] - 2iIm\left( \sum_{i=0}^{k} C_i \nabla_M^{(i)} \langle \nabla_A A \phi, \nabla_A^{(k-i)} \phi \rangle \right)$$

where to obtain the last equality we have used proposition 10.4, and have absorbed the extra lower order derivative terms, arising from this formula, into the quantity $\sum_{v=0}^{2k} P_1^{(v)}[F_A]$. □

Corollary 6.4. Let $(\phi(t), A(t))$ be a solution to the higher order Seiberg-Witten flow. Then

$$\frac{\partial}{\partial t} \nabla_M^{(l)} F_A(t) = (-1)^{k+1} \Delta_{M}^{(k+1)} \nabla_M^{(l)} F_A(t) + \sum_{v=0}^{2k+l} P_1^{(v)}[F_A(t)] - 2iIm\left( \sum_{i=1}^{k} C_i \nabla_M^{(l)} d \nabla_M^{(i)} \langle \nabla_A A \phi, \nabla_A^{(k-i)} \phi \rangle \right).$$

Proof. This follows by using the above lemma

$$\frac{\partial}{\partial t} \nabla_M^{(l)} F_A(t) = \nabla_M^{(l)} \frac{\partial F_A}{\partial t}$$

$$= \nabla_M^{(l)} \left( (-1)^{k+1} \Delta_{M}^{(k+1)} F_A(t) + \sum_{v=0}^{2k} P_1^{(v)}[F_A(t)] - 2iIm\left( \sum_{i=1}^{k} C_i \nabla_M^{(i)} \langle \nabla_A A \phi, \nabla_A^{(k-i)} \phi \rangle \right) \right)$$

$$= (-1)^{k+1} \Delta_{M}^{(k+1)} \nabla_M^{(l)} F_A(t) + \sum_{v=0}^{2k+l} P_1^{(v)}[F_A(t)] - 2iIm\left( \sum_{i=1}^{k} C_i \nabla_M^{(l)} d \nabla_M^{(i)} \langle \nabla_A A \phi, \nabla_A^{(k-i)} \phi \rangle \right)$$

where to obtain the last equality we have used proposition 10.4, and have absorbed the extra lower order derivative terms, arising from this formula, into the quantity $\sum_{v=0}^{2k+l} P_1^{(v)}[F_A]$. □

Lemma 6.5. Let $(\phi(t), A(t))$ be a solution to the higher order Seiberg-Witten flow. Then

$$\frac{\partial}{\partial t} \nabla_A^{(l)} \phi = -\Delta_A^{(k+1)} \nabla_A^{(l)} \phi + \sum_{j=0}^{2k-2+l} \nabla_M^{(j)} Rm \ast \nabla_A^{(2k-2+l-j)} \phi + \sum_{j=0}^{2k+l} \nabla_M^{(j)} Rm \ast \nabla_A^{(2k+l-j)} \phi$$

$$+ \sum_{j=0}^{2k-2+l} \nabla_M^{(j)} F_A \ast \nabla_A^{(2k-2+l-j)} \phi + \sum_{j=0}^{2k+l} \nabla_M^{(j)} F_A \ast \nabla_A^{(2k+l-j)} \phi$$

$$+ \frac{1}{4} \nabla_A^{(l)} ((S + |\phi|^2) \phi) + \sum_{i=0}^{l-1} (-1)^{k+1} C_i \nabla_M^{(i)} d^* \Delta_M^{(k)} F_A \otimes \nabla_A^{(l-1-i)} \phi$$

$$+ \sum_{i=0}^{l-1} \sum_{v=0}^{2k+l} P_1^{(v)}[F_A] \otimes \nabla_A^{(l-1-i)} \phi$$

$$- 2iIm\left( \sum_{j=0}^{l-1} \sum_{i=0}^{k} C_i \nabla_M^{(j)} \nabla_M^{(i)} \langle \nabla_A^{(k)} A \phi, \nabla_A^{(k-i)} \phi \rangle \right) \otimes \nabla_A^{(l-1-j)} \phi.$$
Proof. From lemma 3.1, we have that \( \frac{\partial}{\partial t} \nabla_A^{(l)} \phi = \nabla_A^{(l)} \frac{\partial \phi}{\partial t} + \sum_{i=0}^{l-1} C_i \nabla_A^{(i)} \nabla_A^{(l-i)} \phi \). Since \((\phi(t), A(t))\) is a solution to the higher order Seiberg-Witten flow we find

\[
\frac{\partial}{\partial t} \nabla_A^{(l)} \phi = \nabla_A^{(l)} \frac{\partial \phi}{\partial t} + \sum_{i=0}^{l-1} C_i \nabla_A^{(i)} \nabla_A^{(l-i)} \phi
\]

\[
= \nabla_A^{(l)} \left( - \Delta_A^{(k+1)} \phi + \sum_{j=0}^{2k} \nabla_M^{(j)} Rm * \nabla_A^{(2k-j)} \phi + \sum_{j=0}^{2k} \nabla_M^{(j)} F_A * \nabla_A^{(2k-j)} \phi - \frac{1}{4} (S + |\phi|^2) \phi \right)
\]

\[
+ \sum_{i=0}^{l-1} C_i \nabla_M^{(i)} \left( (-1)^{k+1} d^* \Delta_M^{(k)} F_A - \sum_{v=0}^{2k-1} P_1^{(v)} [F_A] \right)
\]

\[
- 2i \text{Im} \left( \sum_{i=0}^{k} C_i \nabla_M^{(i)} \nabla_A^{(k)} \phi, \nabla_A^{(k-i)} \phi \right) \phi
\]

\[
= - \nabla_A^{(l)} \Delta_A^{(k+1)} \phi + \sum_{j=0}^{2k} \nabla_M^{(j)} Rm * \nabla_A^{(2k-j)} \phi + \sum_{j=0}^{2k} \nabla_M^{(j)} F_A * \nabla_A^{(2k-j)} \phi
\]

\[
- \frac{1}{4} \nabla_A^{(l)} \left( (S + |\phi|^2) \phi \right)
\]

\[
+ \left( \sum_{i=0}^{l-1} (-1)^{k+1} C_i \nabla_M^{(i)} d^* \Delta_M^{(k)} F_A + \sum_{v=0}^{2k-1-l} P_1^{(v)} [F_A] \right)
\]

\[
- 2i \text{Im} \left( \sum_{j=0}^{l-1} \sum_{i=0}^{k} C_i \nabla_M^{(j)} \nabla_M^{(i)} \nabla_A^{(k)} \phi, \nabla_A^{(k-i)} \phi \right) \phi.
\]

Applying the commutation formula, lemma 10.7, then gives the result.

\[\square\]

6.3. Estimates for derivatives of the spinor field. We will prove local \(L^2\)-derivative estimates for the spinor field, and take up the case of the curvature in the next subsection.

We start with the following lemma, which will prove to be very useful in the course of obtaining several local estimates.

**Lemma 6.6.** Let \( \phi \in \Gamma(S^+) \) and \( p, q \in \mathbb{N} \), such that \( p > q \). Given \( k \in \mathbb{N} \), we have

\[
|\nabla_M^{(k)} \langle \nabla_A^{(p)} \phi, \nabla_A^{(q)} \phi \rangle| \leq \sum_{j=0}^{k} C(j) |\langle \nabla_A^{(j)} \nabla_A^{(p)} \phi, \nabla_A^{(k-j)} \nabla_A^{(q)} \phi \rangle|.
\]

**Proof.** For this proof we will denote \( \nabla_A \) by \( \nabla \).

Start with the case \( p = 1 \) and \( q = 0 \), and suppose \( k = 1 \). We want to start by working out a formula for \( \nabla_M \langle \nabla \phi, \phi \rangle \). As everything is tensorial, we can work in coordinates. We fix a point \( x \in M \), and work in normal coordinates centred at \( x \). In these coordinates we write \( \langle \nabla \phi, \phi \rangle \) as \( \langle \nabla \phi, \phi \rangle \, dx^i \). Applying \( \nabla_M \) to this we get (at the point \( x \))

\[
\nabla_M \langle \nabla \phi, \phi \rangle \, dx^i = d \langle \nabla \phi, \phi \rangle \, dx^i + \langle \nabla \phi, \phi \rangle \, dx^i
\]

\[
= (\langle \nabla \nabla \phi, \phi \rangle + \langle \nabla \phi, \nabla \phi \rangle) \, dx^i + \langle \nabla \phi, \phi \rangle \, dx^i
\]

where to get the second equality we have used the fact that \( \nabla \) is metric compatible, and to get the third equality we are using the fact that at \( x \) the Christoffel symbols vanish.
Since we are working with tensors, we thus have the formula

\[ \nabla_M \langle \nabla \phi, \phi \rangle = \langle \nabla \nabla \phi, \phi \rangle + \langle \nabla \phi, \nabla \phi \rangle \]

where we are abusing notation and writing \( \langle \nabla \phi, \nabla \phi \rangle \) to denote the 2-tensor, which in coordinates is given by \( \langle \nabla_i \phi, \nabla_j \phi \rangle \). Note that \( |\langle \nabla \phi, \nabla \phi \rangle| = |\langle \nabla \phi, \nabla \phi \rangle| \). The result for this case then follows.

Still assuming \( k = 1 \), and taking general \( p \) and \( q \), we write \( p = q + r \), and then write \( \nabla^{(p)} \phi = \nabla^{(r)} \nabla^{(q)} \phi \). Then in coordinates we can write \( \langle \nabla^{(r)} \nabla^{(q)} \phi, \nabla^{(q)} \phi \rangle \) as

\[ \langle \nabla_{i_1} \nabla_{i_2} \cdots \nabla_{i_r} \nabla^{(q)} \phi, \nabla^{(q)} \phi \rangle dx^{i_1} \otimes \cdots \otimes dx^{i_r}. \]

Applying what we did above, we can then see that

\[ \nabla_M \langle \nabla^{(r)} \nabla^{(q)} \phi, \nabla^{(q)} \phi \rangle = \langle \nabla \nabla^{(r)} \nabla^{(q)} \phi, \nabla^{(q)} \phi \rangle + \langle \nabla \nabla^{(q)} \phi, \nabla^{(r)} \nabla^{(q)} \phi \rangle \]

and the bound for this case follows as well.

Now, suppose we apply \( \nabla_M \) to the above formula, we get

\[ \nabla_M \nabla_M \langle \nabla^{(r)} \nabla^{(q)} \phi, \nabla^{(q)} \phi \rangle = \nabla_M \langle \nabla \nabla^{(r)} \nabla^{(q)} \phi, \nabla^{(q)} \phi \rangle + \nabla_M \langle \nabla \nabla^{(q)} \phi, \nabla^{(r)} \nabla^{(q)} \phi \rangle. \]

We can then apply what we did above, for the case of just one \( \nabla_M \), to take the \( \nabla_M \) to the inside on the right hand side, and then the bound follows. Iterating this, we get the full bound for all \( k \).

\[ \square \]

Observe that

\[ \frac{\partial}{\partial t} \| \gamma^{s/2} \nabla_A^{(l)} \phi \|^2_{L^2} = \frac{\partial}{\partial t} \int \langle \gamma^{s/2} \nabla_A^{(l)} \phi, \gamma^{s/2} \nabla_A^{(l)} \phi \rangle = \int \frac{\partial}{\partial t} \nabla_A^{(l)} \phi, \gamma^{s} \nabla_A^{(l)} \phi \rangle + \langle \gamma^{s} \nabla_A^{(l)} \phi, \frac{\partial}{\partial t} \nabla_A^{(l)} \phi \rangle. \]

From lemma 6.5, we then get the following proposition.
Proposition 6.7. Let \((\phi(t), A(t))\) be a solution to the generalised Seiberg-Witten flow. Then

\[
\frac{\partial}{\partial t} \| \gamma^{s/2} \nabla_A^{(l)} \phi \|^2_{L^2} = \int -2 \text{Re} \left( \langle \Delta_A^{(k+1)} \nabla_A^{(l)} \phi, \gamma^s \nabla_A^{(l)} \phi \rangle \right) + 2 \text{Re} \left( \sum_{j=0}^{2k+1} \nabla_M^{(j)} Rm \ast \nabla_A^{(2k+1-j)} \phi, \gamma^s \nabla_A^{(l)} \phi \rangle \right) + 2 \text{Re} \left( \sum_{j=0}^{2k+1} \nabla_M^{(j)} F_A \ast \nabla_A^{(2k+1-j)} \phi, \gamma^s \nabla_A^{(l)} \phi \rangle \right) + 2 \text{Re} \left( \sum_{j=0}^{2k+1} \nabla_M^{(j)} \nabla_A^{(2k+1-j)} \phi, \gamma^s \nabla_A^{(l)} \phi \rangle \right) + 2 \text{Re} \left( \sum_{j=0}^{2k+1} \nabla_M^{(j)} \nabla_A^{(2k+1-j)} \phi, \gamma^s \nabla_A^{(l)} \phi \rangle \right) + 2 \text{Re} \left( \sum_{j=0}^{2k+1} \nabla_M^{(j)} \nabla_A^{(2k+1-j)} \phi, \gamma^s \nabla_A^{(l)} \phi \rangle \right) + 2 \text{Re} \left( \sum_{j=0}^{2k+1} \nabla_M^{(j)} \nabla_A^{(2k+1-j)} \phi, \gamma^s \nabla_A^{(l)} \phi \rangle \right)
\]

\[
+ 2 \text{Re} \left( \sum_{j=0}^{2k+1} \nabla_M^{(j)} \nabla_A^{(2k+1-j)} \phi, \gamma^s \nabla_A^{(l)} \phi \rangle \right) + 2 \text{Re} \left( \sum_{j=0}^{2k+1} \nabla_M^{(j)} \nabla_A^{(2k+1-j)} \phi, \gamma^s \nabla_A^{(l)} \phi \rangle \right)
\]

We are now going to estimate each term on the right hand side of the above proposition.

Lemma 6.8. Assume \(\sup_{t \in [0,T]} \| F_A \|_{\infty} < \infty\), and let \(K(\| \phi \|_{\infty}) = \max \{ 1, \sup_{t \in [0,T]} \| \phi \|_{\infty} \} \). Suppose \(\gamma\) is a bump function, and \(s \geq 2(k+1)\). Then for \(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_4 > 0\) sufficiently small, we have the following estimate

\[
\int -2 \text{Re} \left( \langle \Delta_A^{(k+1)} \nabla_A^{(l)} \phi, \gamma^s \nabla_A^{(l)} \phi \rangle \right) \leq \left( -2 + C(g, \gamma)(\epsilon_1 + \epsilon_2) + C(g, \gamma) K(\| \phi \|_{\infty}) \epsilon_3 \right) + \frac{C(\epsilon_2, g, \gamma)}{\epsilon_1^2} + (C(\epsilon_3, g) + C(\epsilon_4, g, \gamma)) K(\| \phi \|_{\infty}) + \frac{C(g, \gamma) \left( \sup_{t \in [0,T]} \| F_A \|_{\infty} \right) (C(\epsilon_4, g, \gamma) + C(\epsilon_4, g, \gamma) K(\| \phi \|_{\infty}))}{\| \phi \|_{L^2}^2, \gamma > 0}
\]

where \(C(g), C(g, \gamma), C(\epsilon_2, g, \gamma), C(\epsilon_3, g, \gamma), C(\epsilon_3, g, \gamma), C(\epsilon_4, g, \gamma), C(\epsilon_4, g, \gamma)\) are constants that do not depend on \(t \in [0, T]\).
Proof. We will start by performing an estimate on the quantity $\int - (\Delta_A^{(k+1)} \nabla_A^{(l)} \phi, \gamma^s \nabla_A^{(l)} \phi)$. Observe that using lemma 10.9, we have

$$
\int - (\Delta_A^{(k+1)} \nabla_A^{(l)} \phi, \gamma^s \nabla_A^{(l)} \phi) = \int - (\nabla_A^{(k+1)} \nabla_A^{(l)} \phi, \nabla_A^{(k+1)} (\gamma^s \nabla_A^{(l)} \phi)) \quad (6.3.1)
$$

$$
+ \int \left( \sum_{w=0}^{2k-2} \nabla_M^{(w)} Rm * \nabla_A^{(2k-2-w)} \nabla_A^{(l)} \phi, \gamma^s \nabla_A^{(l)} \phi \right) \quad (6.3.2)
$$

$$
+ \int \left( \sum_{w=0}^{2k-2} \nabla_M^{(w)} F_A * \nabla_A^{(2k-2-w)} \nabla_A^{(l)} \phi, \gamma^s \nabla_A^{(l)} \phi \right). \quad (6.3.3)
$$

Note that we then have

$$
\int -2Re(\langle \Delta_A^{(k+1)} \nabla_A^{(l)} \phi, \gamma^s \nabla_A^{(l)} \phi \rangle) = \int -2Re(\langle \nabla_A^{(k+1)} \nabla_A^{(l)} \phi, \nabla_A^{(k+1)} (\gamma^s \nabla_A^{(l)} \phi) \rangle)
$$

$$
+ \int 2Re\left( \sum_{w=0}^{2k-2} \nabla_M^{(w)} Rm * \nabla_A^{(2k-2-w)} \nabla_A^{(l)} \phi, \gamma^s \nabla_A^{(l)} \phi \right)
$$

$$
+ \int 2Re\left( \sum_{w=0}^{2k-2} \nabla_M^{(w)} F_A * \nabla_A^{(2k-2-w)} \nabla_A^{(l)} \phi, \gamma^s \nabla_A^{(l)} \phi \right).
$$

We first estimate the quantity on the right hand side of (6.3.1).

$$
\int - \langle \nabla_A^{(k+1)} \nabla_A^{(l)} \phi, \nabla_A^{(k+1)} (\gamma^s \nabla_A^{(l)} \phi) \rangle = \int - \langle \nabla_A^{(k+1)} \nabla_A^{(l)} \phi, \sum_{j=0}^{k+1} C_j \nabla^{(j)} \gamma^s \otimes \nabla_A^{(k+1-j)} \nabla_A^{(l)} \phi \rangle
$$

where $C_j$ is a constant and $C_0 = 1$. We can then split this into two terms, giving

$$
\int - \langle \nabla_A^{(k+1)} \nabla_A^{(l)} \phi, \nabla_A^{(k+1)} (\gamma^s \nabla_A^{(l)} \phi) \rangle = \int - \langle \nabla_A^{(k+1)} \nabla_A^{(l)} \phi, \gamma^s \nabla_A^{(k+1)} \nabla_A^{(l)} \phi \rangle
$$

$$
+ \int - \langle \nabla_A^{(k+1)} \nabla_A^{(l)} \phi, \sum_{j=1}^{k+1} C_j \nabla^{(j)} \gamma^s \otimes \nabla_A^{(k+1-j)} \nabla_A^{(l)} \phi \rangle
$$

$$
= - ||\gamma^s/2 \nabla_A^{(k+1)} \nabla_A^{(l)} \phi||^2_{L^2}
$$

$$
+ \int \sum_{j=1}^{k+1} \nabla^{(j)} (\gamma^s) * \langle \nabla_A^{(k+1)} \nabla_A^{(l)} \phi, \nabla_A^{(k+1-j)} \nabla_A^{(l)} \phi \rangle.
$$
In order to estimate \( \int \sum_{j=1}^{k+1} \nabla_j (\gamma^s) \ast (\nabla_\phi^{(k+1)} \nabla_\phi^{(l)}) \), we proceed as follows.

\[
\left| \int \sum_{j=1}^{k+1} \nabla_j (\gamma^s) \ast (\nabla_\phi^{(k+1)} \nabla_\phi^{(l)}) \right| \\
\leq \int \sum_{j=1}^{k+1} C(g) |\nabla_j (\gamma^s)||\nabla_\phi^{(k+1)} \nabla_\phi^{(l)}||\nabla_A \phi| \\
\leq \int \sum_{j=1}^{k+1} C(g, \gamma) \gamma^{s-j} |\nabla_\phi^{(k+1)} \nabla_\phi^{(l)}||\nabla_A \phi| \\
= \int \sum_{j=1}^{k+1} C(g, \gamma) \gamma^{s/2} \nabla_\phi^{(k+1)} \nabla_\phi^{(l)}||\gamma^{s/2} \nabla_\phi^{(k+1)} \nabla_\phi^{(l)}||\nabla_A \phi|.
\]

We then apply a weighted Young’s inequality to obtain

\[
\int \sum_{j=1}^{k+1} C(g, \gamma) \gamma^{s/2} \nabla_\phi^{(k+1)} \nabla_\phi^{(l)}||\gamma^{s/2} \nabla_\phi^{(k+1)} \nabla_\phi^{(l)}||\nabla_A \phi| \\
\leq (k + 1) C(g, \gamma) \epsilon_1 ||\gamma^{s/2} \nabla_\phi^{(k+1)} \nabla_\phi^{(l)}|| L_2^2 + \sum_{j=1}^{k+1} \frac{C(g, \gamma)}{\epsilon_1} ||\gamma^{s/2} \nabla_\phi^{(k+1)} \nabla_\phi^{(l)}|| L_2^2.
\]

Choose \( \epsilon_1 \) sufficiently small so that \( \frac{C(g, \gamma)}{\epsilon_1} \geq 1 \). By applying lemma 10.3, we can bound the above by

\[
C(g, \gamma)(\epsilon_1 + \epsilon_2) ||\gamma^{s/2} \nabla_\phi^{(k+1)} \nabla_\phi^{(l)}|| L_2^2 + \frac{C(\epsilon_2, g, \gamma)}{\epsilon_1} ||\phi|| L_2^2, \gamma > 0
\]

where we have absorbed the \((k + 1)\) into the constant \(C(g, \gamma)\).

Putting this together with the previous estimate, we get the following estimate

\[
\int -2Re \left( (\nabla_\phi^{(k+1)} \nabla_\phi^{(l)}), (\gamma^s \nabla_\phi^{(l)})) \right) \leq -2 ||\gamma^{s/2} \nabla_\phi^{(k+1)} \nabla_\phi^{(l)}|| L_2^2 + C(g, \gamma)(\epsilon_1 + \epsilon_2) ||\gamma^{s/2} \nabla_\phi^{(k+1)} \nabla_\phi^{(l)}|| L_2^2
\]

The next step is to estimate the absolute value of 6.3.2.

\[
\left| \int \sum_{w=0}^{2k-2} \nabla_M^{(w)} Rm \ast \nabla_\phi^{(k-2-w)} \nabla_\phi^{(l)} \right| \\
\leq \int \left| \sum_{w=0}^{2k-2} \nabla_M^{(w)} Rm \ast \nabla_\phi^{(k-2-w)} \nabla_\phi^{(l)} \right| \\
\leq \sum_{w=0}^{2k-2} \gamma^s ||\nabla_M^{(w)} Rm|| ||\nabla_\phi^{(k-2-w)} \nabla_\phi^{(l)}|| \nabla_\phi^{(l)}||\nabla_\phi||.
\]

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In order to estimate the quantity
\[
\int \sum_{w=0}^{2k-2} \gamma^s |\nabla_M Rm| |\nabla_A^{(2k-2-w)} \nabla_A \phi||\nabla_A \phi|
\]
we will split the integrand into two parts, those for which \(w\) is odd. We will then show how to estimate each piece.

1. Fix \(w = 2\alpha\) for \(0 \leq \alpha \leq k - 1\). We then have
\[
\int \gamma^s |\nabla_M Rm| |\nabla_A^{(2k-2-w)} \nabla_A \phi||\nabla_A \phi| = \int \gamma^s |\nabla_M Rm| |\nabla_A^{(2k-2-2\alpha)} \nabla_A \phi||\nabla_A \phi|.
\]
The term \(|\nabla_M Rm|\) does not depend on time, and since \(M\) is compact, is bounded on \(M\). We can therefore view it as a constant \(C(g)\). Applying corollary 10.2, and then lemma 10.3, we have
\[
\int \gamma^s |\nabla_A^{(2k-2-2\alpha)} \nabla_A \phi||\nabla_A \phi| \leq C(g) \gamma^{s/2} |\nabla_A^{(k-1+1-\alpha)} \phi||L^2 + C(g) \phi||L^2, \gamma > 0
\]
\[
\leq C(g) \epsilon_3 \gamma^{s/2} |\nabla_A^{(k+1)} \nabla_A \phi||L^2 + C(\epsilon_3, g) \phi||L^2, \gamma > 0.
\]

2. Fix \(w = 2\alpha + 1\) for \(0 \leq \alpha \leq k - 2\). We then have
\[
\int \gamma^s |\nabla_M Rm| |\nabla_A^{(2k-2-w)} \nabla_A \phi||\nabla_A \phi| = \int \gamma^s |\nabla_M Rm| |\nabla_A^{(2k-2-2\alpha-1)} \nabla_A \phi||\nabla_A \phi|
\]
where we are letting \(T = \nabla_A^{2} Rm\). We remind the reader that \(T\) does not depend on time \(t\), and by compactness of \(M\), is uniformly bounded above by some constant.

Applying Holder’s inequality, we can bound the quantity
\[
\int \gamma^s |\nabla_M T||\nabla_A^{(2k-2-2\alpha-1)} \nabla_A \phi||\nabla_A \phi|
\]
by the quantity
\[
\left( \int \gamma^s |\nabla_M T||\nabla_A^{(2k-1-\alpha+1)} \nabla_A \phi\right)^{1/2k-1-\alpha+1} \left( \int \gamma^s |\nabla_A^{(2k-2-2\alpha-1)} \nabla_A \phi\right)^{2k-2-2\alpha-1-1/2k-1-\alpha+1} \left( \int \gamma^s |\nabla_A \phi\right)^{2k-2-2\alpha-1-1/2k-1-\alpha+1}.
\]
As mentioned before, since \(T\) does not depend on time, and using the compactness of \(M\), we can simply express the term in the first bracket as \(C(g, \gamma)\). We therefore need to estimate the quantity
\[
C(g, \gamma) \left( \int \gamma^s |\nabla_A^{(2k-2-2\alpha-1+1)} \nabla_A \phi\right)^{2k-2-2\alpha-1-1/2k-1-\alpha+1} \left( \int \gamma^s |\nabla_A \phi\right)^{2k-2-2\alpha-1-1/2k-1-\alpha+1}.
\]
Appealing to theorem 10.1, we can bound it above by
\[
C(g, \gamma) \left[ ||\phi||_{k-1-\alpha+1}\left( ||\gamma^{s/2} |\nabla_A^{(k-1+1-\alpha)} \phi||L^2 + ||\phi||L^2, \gamma > 0\right) + \gamma^{s/2} \left( ||\gamma^{s/2} |\nabla_A^{(k-1-\alpha+1)} \phi||L^2 + ||\phi||L^2, \gamma > 0\right)\right].
\]
which simplifies to

\[ C(g, \gamma) \| \phi \|_{L^{2, \gamma > 0}} \leq (\| \gamma s/2 A \|_{L^{2, \gamma > 0}} + \| \phi \|_{L^{2, \gamma > 0}})^{2(2k-1-\alpha + l - 1)/(k-1-\alpha + l)}. \]

Recall we defined \( K(\| \phi \|_{\infty}) = \max\{1, \sup_{t \in [0, T]} \| \phi \|_{\infty}\} \). We can then bound the above by

\[ C(g, \gamma) K(\| \phi \|_{\infty}) \leq (\| \gamma s/2 A \|_{L^{2, \gamma > 0}} + \| \phi \|_{L^{2, \gamma > 0}})^{2(2k-1-\alpha + l - 1)/(k-1-\alpha + l)}. \]

We then have that

\[ C(g, \gamma) K(\| \phi \|_{\infty}) \leq (\| \gamma s/2 A \|_{L^{2, \gamma > 0}} + \| \phi \|_{L^{2, \gamma > 0}} + 1)^{2(2k-1-\alpha + l - 1)/(k-1-\alpha + l)}. \]

\[ \leq C(g, \gamma) K(\| \phi \|_{\infty}) (\| \gamma s/2 A \|_{L^{2, \gamma > 0}} + \| \phi \|_{L^{2, \gamma > 0}} + 1)^{2(2k-1-\alpha + l - 1)/(k-1-\alpha + l)}. \]

where we have used the general fact that, given any three non-negative integers \( a, b, c \) we have \((a + b + c)^2 \leq 2(a^2 + b^2 + c^2)\). In our situation we have absorbed the 2 into the constant \( C(g, \gamma) \).

We then apply lemma 10.3, to the first term in the bracket, obtaining

\[ C(g, \gamma) K(\| \phi \|_{\infty}) (\| \gamma s/2 A \|_{L^{2, \gamma > 0}} + \| \phi \|_{L^{2, \gamma > 0}} + 1) \]

\[ \leq C(g, \gamma) K(\| \phi \|_{\infty}) \tilde{\epsilon}_3 \| \gamma s/2 A \|_{L^{2, \gamma > 0}} + C(\tilde{\epsilon}_3, g, \gamma) K(\| \phi \|_{\infty}) \| \phi \|_{L^{2, \gamma > 0}}^2 \]

where we have absorbed the extra \( C(g, \gamma) K(\| \phi \|_{\infty}) \), coming from taking this into the bracket and multiplying by 1, into the coefficient of \( \| \phi \|_{L^{2, \gamma > 0}}^2 \).

Putting the two estimates together gives the following

\[ \int \left| \sum_{w=0}^{2k-2} \nabla_M^{(w)} Rm * \nabla_A^{(2k-2-w)} \nabla_A^{(l)} \phi, \gamma s \nabla_A^{(l)} \phi \right| \]

\[ \leq C(g) \tilde{\epsilon}_3 \| \gamma s/2 A \|_{L^{2, \gamma > 0}} + C(\tilde{\epsilon}_3, g) \| \phi \|_{L^{2, \gamma > 0}}^2 + C(g, \gamma) K(\| \phi \|_{\infty}) \tilde{\epsilon}_3 \| \gamma s/2 A \|_{L^{2, \gamma > 0}} + C(\tilde{\epsilon}_3, g, \gamma) K(\| \phi \|_{\infty}) \| \phi \|_{L^{2, \gamma > 0}}^2. \]

We have thus obtained the following estimate

\[ \int 2Re \left( \sum_{w=0}^{2k-2} \nabla_M^{(w)} Rm * \nabla_A^{(2k-2-w)} \nabla_A^{(l)} \phi, \gamma s \nabla_A^{(l)} \phi \right) \]

\[ \leq (C(g) \tilde{\epsilon}_3 + C(g, \gamma) K(\| \phi \|_{\infty}) \tilde{\epsilon}_3) \| \gamma s/2 A \|_{L^{2, \gamma > 0}} + C(\tilde{\epsilon}_3, g) \| \phi \|_{L^{2, \gamma > 0}}^2 \]

\[ + (C(\tilde{\epsilon}_3, g) + C(\tilde{\epsilon}_3, g, \gamma) K(\| \phi \|_{\infty})) \| \phi \|_{L^{2, \gamma > 0}}^2. \]

The final step is to estimate the absolute value of 6.3.3. We first observe that we can write the term \( \nabla_M^{(w)} F_A * \nabla_A^{(2k-2-w)} \nabla_A^{(l)} \phi \) as \( \sum_{j=0}^{w} C_j \nabla_A^{(2k-2-j)} \nabla_A^{(l)} \phi \), which follows
from the fact that, for two tensors $S$ and $T$ we have $\nabla^{(k)}(S \ast T) = \sum_{i=0}^{k} C_i \nabla^{(i)} S \ast \nabla^{(k-i)} T$.

We then obtain

$$
\int \left( \sum_{w=0}^{2k-2} \nabla_M^{(w)} F_A \ast \nabla_A^{(2k-2-w)} \nabla_A^{(l)} \phi, \gamma^s \nabla_A^{(l)} \phi \right) = \int \sum_{w=0}^{2k-2} \sum_{j=0}^{w} C_j \langle \nabla^{(j)} (F_A \ast \nabla_A^{(2k-2-j)} \nabla_A^{(l)} \phi), \gamma^s \nabla_A^{(l)} \phi \rangle
$$

$$
= \int \sum_{w=0}^{2k-2} \sum_{j=0}^{w} C_j \langle F_A \ast \nabla_A^{(2k-2-j)} \nabla_A^{(l)} \phi), P^{(j)}(\gamma^s \nabla_A^{(l)} \phi) \rangle
$$

where to get the last equality we have applied integration by parts, and absorbed the constant $(-1)^j$ into the $C_j$. We point out that $C_j$ in general won’t be positive, some of them will be negative.

We then have

$$
\int \left( \sum_{w=0}^{2k-2} \nabla_M^{(w)} F_A \ast \nabla_A^{(2k-2-w)} \nabla_A^{(l)} \phi, \gamma^s \nabla_A^{(l)} \phi \right) \leq \int \sum_{w=0}^{2k-2} \sum_{j=0}^{w} C_j \langle (F_A \ast \nabla_A^{(2k-2-j)} \nabla_A^{(l)} \phi), P^{(j)}(\gamma^s \nabla_A^{(l)} \phi) \rangle.
$$

When we take the absolute value inside to the integrand, in the above inequality, the constants $C_j$ become $|C_j|$, and we have simply called this $C_j$ again. Thus, on the right hand side of the above inequality, the $C_j$ are now all positive. We can then estimate the right hand side of the above inequality by

$$
\int \sum_{w=0}^{2k-2} \sum_{j=0}^{w} C_j \langle (F_A \ast \nabla_A^{(2k-2-j)} \nabla_A^{(l)} \phi), P^{(j)}(\gamma^s \nabla_A^{(l)} \phi) \rangle \leq \int \sum_{w=0}^{2k-2} \sum_{j=0}^{w} C_j \left( \sup_{t \in [0,T)} \| F_A \|_{\infty} \right) \| \nabla_A^{(2k-2-j)} \nabla_A^{(l)} \phi \| \| P^{(j)}(\gamma^s \nabla_A^{(l)} \phi) \|.
$$

The term $P^{(j)}(\gamma^s \nabla_A^{(l)} \phi) = \nabla_A^{(j)}(\gamma^s \nabla_A^{(l)} \phi) \ast S$, where $S$ is some tensor depending on the metric $g$, and in particular does not depend on $t$. Therefore, we have the bound

$$
|P^{(j)}(\gamma^s \nabla_A^{(l)} \phi)| \leq C(g) \| \nabla_A^{(j)}(\gamma^s \nabla_A^{(l)} \phi) \| \leq \sum_{i=0}^{j} C(g) \| \nabla^{(i)}(\gamma^s) \| \| \nabla_A^{(j-i)} \nabla_A^{(l)} \phi \|
$$

where we have used the fact that we can write $\nabla_A^{(j)}(\gamma^s \nabla_A^{(l)} \phi) = \sum_{i=0}^{j} C_i \nabla^{(i)}(\gamma^s) \otimes \nabla_A^{(j-i)} \nabla_A^{(l)} \phi$, for some positive constants $C_i$.

Putting this together, we obtain the bound

$$
\int \sum_{w=0}^{2k-2} \sum_{j=0}^{w} C_j \left( \sup_{t \in [0,T)} \| F_A \|_{\infty} \right) \| \nabla_A^{(2k-2-j)} \nabla_A^{(l)} \phi \| \| P^{(j)}(\gamma^s \nabla_A^{(l)} \phi) \| \leq \sum_{i=0}^{j} C(g) \| \nabla^{(i)}(\gamma^s) \| \| \nabla_A^{(j-i)} \nabla_A^{(l)} \phi \|.
$$

In order to estimate the right hand side of the above inequality, we will split the integrand into two cases based on the parity of $i$.

1. Suppose that $i$ is even. Write $i = 2\alpha$ for $\alpha \geq 0$. Then

$$
\int |\nabla_A^{(2k-2-j+1)} \phi| |\nabla^{(2\alpha)}(\gamma)\otimes\nabla^{(j-2\alpha)} \nabla_A^{(l)} \phi| \leq \int C(\gamma, g) \gamma^s - 2\alpha \| \nabla_A^{(2k-2-j+1)} \phi \| \| \nabla_A^{(j-2\alpha)} \nabla_A^{(l)} \phi \|
$$

where we are using lemma 6.2.
Applying theorem 10.1, we obtain
\[ \int C(\gamma,g) \gamma^{s-2\alpha} \left| \nabla_A^{(2k-2-j+l)} \phi \right| \left| \nabla_A^{(j-2\alpha)} \nabla_A^l \phi \right| \]
\[ \leq C(\gamma,g) \left( \left( \gamma^{s-2\alpha} \nabla_A^{(k-1-\alpha+l)} \phi \right)_{L^2} + \left( \left| \phi \right|_{L^2, \gamma > 0} \right)^2 \right) \]
\[ \leq C(\gamma,g) \left( \left( \gamma^{s-2\alpha} \nabla_A^{(k-1-\alpha+l)} \phi \right)_{L^2} + \left( \left| \phi \right|_{L^2, \gamma > 0} \phi \right)^2 \right) \]
Applying interpolation, lemma 10.3, we then get
\[ C(\gamma,g) \left( \left( \gamma^{s-2\alpha} \nabla_A^{(k-1-\alpha+l)} \phi \right)_{L^2} + \left( \left| \phi \right|_{L^2, \gamma > 0} \phi \right)^2 \right) \]
\[ \leq C(\gamma,g) \left( \left( \gamma^{s/2} \nabla_A^{(k+1)} \phi \right)_{L^2} + C(\epsilon_4,\gamma,g) \left( \left| \phi \right|_{L^2, \gamma > 0} \phi \right)^2 \right) \]

2. We now consider the case that \( i \) is odd. Write \( i = 2\alpha + 1 \) for \( \alpha \geq 0 \). From lemma 6.2, we can write the derivative \( \nabla^{(i)} \gamma^s \) as
\[ \nabla^{(i)} \gamma^s = \sum_{n_1 + \ldots + n_i = i} C(n_1,\ldots,n_i) \gamma^{s-i} \nabla_A^{n_1} \ldots \nabla_A^{n_i} \]
and obtain the bound
\[ |\nabla^{(i)} \gamma^s| \leq \sum_{n_1 + \ldots + n_i = i} C(n_1,\ldots,n_i) \gamma^{s-i} |\nabla_A^{n_1} \ldots \nabla_A^{n_i-1} \gamma| |\nabla_A^{n_i} \gamma| \]
where to get the last inequality we have absorbed the norms \( ||\nabla_A^{n_i} \gamma||_{\infty} \), for \( 1 \leq q \leq i - 1 \), into the constant \( C(\gamma) \).
This gives the integral bound
\[ \int \left| \nabla_A^{(2k-2-j+l)} \phi \right| \left| \nabla_A^{(j-2\alpha-1)} \nabla_A^l \phi \right| \]
\[ \leq \int \sum_{n_1=1}^i C(\gamma) \gamma^{s-i} |\nabla_A^{n_1-1} \gamma| \left| \nabla_A^{(2k-2-j+l)} \phi \right| \left| \nabla_A^{(j-2\alpha-1)} \nabla_A^l \phi \right| \]
We then bound this latter integral by using theorem 10.1. We note that in applying theorem 10.1, we get a term involving \( \gamma \), which we will absorb into the constant \( C(g,\gamma) \).
\[ \int \sum_{n_1=1}^i C(\gamma) \gamma^{s-i} |\nabla_A^{n_1-1} \gamma| \left| \nabla_A^{(2k-2-j+l)} \phi \right| \left| \nabla_A^{(j-2\alpha-1)} \nabla_A^l \phi \right| \]
\[ \leq C(\gamma,g) ||\phi||_{\infty} \left( \left( \gamma^{s-2\alpha-1/2} \left| \nabla_A^{(k-1-\alpha+l)} \phi \right|_{L^2} + \left( \left| \phi \right|_{L^2, \gamma > 0} \phi \right)^2 \right) \right)^{2k+2l-2s-1} \]
\[ \leq C(\gamma,g) K \left( ||\phi||_{\infty} \right)^{2k-1-\alpha+1} \left( \left( \gamma^{s-2\alpha-1/2} \left| \nabla_A^{(k-1-\alpha+l)} \phi \right|_{L^2} + \left( \left| \phi \right|_{L^2, \gamma > 0} \phi \right)^2 \right) \right)^{2k+2l-2s-1} \]
\[ \leq C(\gamma,g) K \left( ||\phi||_{\infty} \right)^{2k-1-\alpha+1} \left( \left( \gamma^{s-2\alpha-1/2} \left| \nabla_A^{(k-1-\alpha+l)} \phi \right|_{L^2} + \left( \left| \phi \right|_{L^2, \gamma > 0} \phi \right)^2 \right) \right)^{2k+2l-2s-1} \]
where to get the last inequality we have applied lemma 10.3.
Substituting the two estimates carried out above back into (6.3.6), we obtain
\[
\int \sum_{w=0}^{2k-2} \sum_{j=0}^{w} \sum_{i=0}^{j} C(g)\left( \sup_{t \in [0,T]} ||F_A||_\infty \right) |\nabla_A^{(2k-2-j)} \nabla_A^{(l)} \phi| ||\nabla^{(i)}(\gamma^s)|| \nabla_A^{(j-i)} \nabla_A^{(l)} \phi| \leq C(g, \gamma) \left( \sup_{t \in [0,T]} ||F_A||_\infty (\epsilon_4 + K(||\phi||_\infty)\bar{\epsilon}_4) ||\gamma^{s/2}\nabla_A^{k+1} \nabla_A^{(l)} \phi||_{L^2}^2 + C(g, \gamma) \left( \sup_{t \in [0,T]} ||F_A||_\infty (C(\epsilon_4, g, \gamma) + C(\bar{\epsilon}_4, g, \gamma) K(||\phi||_\infty)) ||\phi||_{L^2, \gamma > 0}^2.\right.
\]
Finally, we obtain the estimate
\[
\int 2Re\left((\sum_{w=0}^{2k-2} \nabla_M^{(w)} F_A \ast \nabla_A^{(2k-2-w)} \nabla_A^{(l)} \phi, \gamma^s \nabla_A^{(l)} \phi)\right) \leq C(g, \gamma) \left( \sup_{t \in [0,T]} ||F_A||_\infty (\epsilon_4 + K(||\phi||_\infty)\bar{\epsilon}_4) ||\gamma^{s/2}\nabla_A^{k+1} \nabla_A^{(l)} \phi||_{L^2}^2 + C(g, \gamma) \left( \sup_{t \in [0,T]} ||F_A||_\infty (C(\epsilon_4, g, \gamma) + C(\bar{\epsilon}_4, g, \gamma) K(||\phi||_\infty)) ||\phi||_{L^2, \gamma > 0}^2.\right.
\]

Combining the estimates, (6.3.4), (6.3.5), and (6.3.7), we obtain
\[
\int -2Re\left((\nabla_A^{(k+1)} \nabla_A^{(l)} \phi, \gamma^s \nabla_A^{(l)} \phi)\right) \leq \left(-2 + C(g, \gamma)(\epsilon_1 + \epsilon_2) + C(g)\epsilon_3 + C(g, \gamma) K(||\phi||_\infty)\bar{\epsilon}_3\right)
\]
\[
C(g, \gamma) \left( \sup_{t \in [0,T]} ||F_A||_\infty \right) (\epsilon_4 + K(||\phi||_\infty)\bar{\epsilon}_4) ||\gamma^{s/2}\nabla_A^{k+1} \nabla_A^{(l)} \phi||_{L^2}^2
\]
\[
+ C(\epsilon_2, g, \gamma) \left( \left( C(\epsilon_3, g) + C(\bar{\epsilon}_3, g, \gamma) K(||\phi||_\infty)\right) ||\phi||_{L^2, \gamma > 0}^2 \right)
\]

which proves the lemma.

\[\square\]

The next lemma gives estimates for the next four terms in proposition 6.7

**Lemma 6.9.** Assume \(\sup_{t \in [0,T]} ||F_A||_\infty < \infty\), and let \(K(||\phi||_\infty) = \max\{1, \sup_{t \in [0,T]} ||\phi||_\infty\}\). Suppose \(\gamma\) is a bump function, and \(s \geq 2(k+l)\). Then for \(\epsilon_5, \epsilon_6, \bar{\epsilon}_6 > 0\) sufficiently small, we have the following estimate
\[
2Re\left(\sum_{j=0}^{2k-2+l} \nabla_M^{(j)} Rm \ast \nabla_A^{(2k-2-l-j)} \phi, \gamma^s \nabla_A^{(l)} \phi)\right) + 2Re\left(\sum_{j=0}^{2k-2+l} \nabla_M^{(j)} Rm \ast \nabla_A^{(2k-l-j)} \phi, \gamma^s \nabla_A^{(l)} \phi)\right)
\]
\[
+ 2Re\left(\sum_{j=0}^{2k-2+l} \nabla_M^{(j)} F_A \ast \nabla_A^{(2k-2-l-j)} \phi, \gamma^s \nabla_A^{(l)} \phi)\right) + 2Re\left(\sum_{j=0}^{2k-2+l} \nabla_M^{(j)} F_A \ast \nabla_A^{(2k-l-j)} \phi, \gamma^s \nabla_A^{(l)} \phi)\right)
\]
\[
\leq (C(g)\epsilon_5 + C(g, \gamma) K(||\phi||_\infty)\bar{\epsilon}_5) ||\gamma^{s/2}\nabla_A^{(k+1)} \nabla_A^{(l)} \phi||_{L^2}^2 + (C(\epsilon_5, g) + C(\bar{\epsilon}_5, g, \gamma) K(||\phi||_\infty)) ||\phi||_{L^2, \gamma > 0}^2
\]
\[
+ C(g, \gamma) \left( \sup_{t \in [0,T]} ||F_A||_\infty (\epsilon_6 + \epsilon_6 + K(||\phi||_\infty)\bar{\epsilon}_6) ||\gamma^{s/2}\nabla_A^{k+1} \nabla_A^{(l)} \phi||_{L^2}^2 + C(g, \gamma) \left( \sup_{t \in [0,T]} ||F_A||_\infty (C(\epsilon_6, g, \gamma) + C(\bar{\epsilon}_6, g, \gamma) K(||\phi||_\infty)) ||\phi||_{L^2, \gamma > 0}^2.\right.
\]
where \( C(g), C(g, \gamma), C(\epsilon_5, g, \gamma), C(\epsilon_5, g, \gamma), C(\epsilon_6, g, \gamma), C(\epsilon_6, g, \gamma) \) are constants that do not depend on \( t \in [0, T] \).

We won’t give the proof of this lemma, as the four terms on the left of the inequality are exactly analogous to the terms that turned up in the course of the proof of lemma 6.8. Therefore, one needs only to apply exactly the same argument we did to obtain (6.3.5) and (6.3.7).

**Lemma 6.10.** Assume \( \sup_{t \in [0, T]} \|F_A\|_\infty < \infty \), and let \( K(||\phi||_\infty) = \max \{1, \sup_{t \in [0, T]} ||\phi||_\infty\} \).

Suppose \( \gamma \) is a bump function, and \( s \geq 2(k+l) \). Then for \( \epsilon_7, \bar{\epsilon}_7 > 0 \) sufficiently small, we have the following estimate

\[
\int 2\text{Re} \left( \sum_{i=0}^{l-1} (-1)^{k+1} C_i \nabla_M^{(i)} d^* \Delta_M^{(k)} F_A \otimes \nabla_A^{(l-1-i)} \phi, \gamma^s \nabla_A^{(l)} \phi \right) \leq C(\gamma, g) \left( \sup_{t \in [0, T]} ||F_A||_\infty \right) (\epsilon_7 + K(||\phi||_\infty)) \bar{\epsilon}_7 ||\gamma^{s/2} \nabla_A^{k+1} \nabla_A^{(l)} \phi||_L^2 \\
+ \left( \sup_{t \in [0, T]} ||F_A||_\infty \right) (C(\epsilon_7, \gamma, g) + C(\bar{\epsilon}_7, \gamma, g)K(||\phi||_\infty)) ||\phi||_L^2, \gamma > 0.
\]

**Proof.** We start by observing that, we can write

\[
\int \left( \sum_{i=0}^{l-1} (-1)^{k+1} C_i \nabla_M^{(i)} d^* \Delta_M^{(k)} F_A \otimes \nabla_A^{(l-1-i)} \phi, \gamma^s \nabla_A^{(l)} \phi \right) = \int \left( \sum_{i=0}^{l-1} (-1)^{k+1} C_i \nabla_M^{(i)} d^* \Delta_M^{(k)} F_A, \langle \gamma^s \nabla_A^{(l)} \phi, \nabla_A^{(l-1-i)} \phi \rangle \right) \\
= \int \left( \sum_{i=0}^{l-1} (-1)^{k+1} C_i \nabla_M^{(i)} d^* \Delta_M^{(k)} F_A, \langle \gamma^s \nabla_A^{(l)} \phi, \nabla_A^{(l-1-i)} \phi \rangle \right) \\
= \int \left( \sum_{i=0}^{l-1} (-1)^{k+1} C_i \langle \Delta_M^{(k)} F_A, d\nabla_M^{(i)} \gamma^s \nabla_A^{(l)} \phi, \nabla_A^{(l-1-i)} \phi \rangle \right)
\]

We then integrate, this latter integral, by parts to obtain

\[
\int \left( \sum_{i=0}^{l-1} (-1)^{k+1} C_i \langle \Delta_M^{(k)} F_A, d\nabla_M^{(i)} \gamma^s \nabla_A^{(l)} \phi, \nabla_A^{(l-1-i)} \phi \rangle \right) = \int \left( \sum_{i=0}^{l-1} (-1)^{k+1} C_i \langle F_A, P_1^{(2k)} (d\nabla_M^{(i)} \gamma^s \nabla_A^{(l)} \phi, \nabla_A^{(l-1-i)} \phi) \rangle \right)
\]

We then have the bound

\[
\left| \int \left( \sum_{i=0}^{l-1} (-1)^{k+1} C_i \langle F_A, P_1^{(2k)} (d\nabla_M^{(i)} \gamma^s \nabla_A^{(l)} \phi, \nabla_A^{(l-1-i)} \phi) \rangle \right) \right| \leq \int \left( \sum_{i=0}^{l-1} C_i \left| \langle F_A, P_1^{(2k)} (d\nabla_M^{(i)} \gamma^s \nabla_A^{(l)} \phi, \nabla_A^{(l-1-i)} \phi) \rangle \right| \right) \leq \int \left( \sum_{i=0}^{l-1} C_i \left( \sup_{t \in [0, T]} ||F_A||_\infty \right) |P_1^{(2k)} (d\nabla_M^{(i)} \gamma^s \nabla_A^{(l)} \phi, \nabla_A^{(l-1-i)} \phi) | \right).
\]

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Observe that we can bound
\[ |P_1^{(2k)}(d\nabla_M^{(i)}(\gamma^s \nabla^{(l)}_A \phi, \nabla^{(l-1-i)}_A \phi))| \leq C(g)|\nabla_M^{(2k+i+1)}(\gamma^s \nabla^{(l)}_A \phi, \nabla^{(l-1-i)}_A \phi)|.\]

Applying lemma 6.6, we then obtain
\[
C(g)|\nabla_M^{(2k+i+1)}(\gamma^s \nabla^{(l)}_A \phi, \nabla^{(l-1-i)}_A \phi)| \\
\leq \sum_{j=0}^{2k+i+1} C(g)|\langle \nabla_A^{(j)}(\gamma^s \nabla^{(l)}_A \phi), \nabla_A^{(2k+i+1-j)}\nabla^{(l-1-i)}_A \phi \rangle| \\
\leq \sum_{j=0}^{2k+i+1} \sum_{n=0}^{j} C(g)|\langle \nabla_A^{(n)}(\gamma^s) \otimes \nabla_A^{(j-n)}\nabla^{(l)}_A \phi, \nabla_A^{(2k+i+1-j)}\nabla^{(l-1-i)}_A \phi \rangle| \\
\leq \sum_{j=0}^{2k+i+1} \sum_{n=0}^{j} C(g)|\nabla_A^{(n)}(\gamma^s)||\nabla_A^{(j-n)}\nabla^{(l)}_A \phi||\nabla_A^{(2k+i+1-j)}\nabla^{(l-1-i)}_A \phi|.\]

These computations show that we can estimate
\[
\int 2Re \left( \sum_{i=0}^{l-1} (-1)^{k+1} C_i \nabla_M^{(i)} d^s \Delta_M^{(k)} F_A \otimes \nabla^{(l-1-i)}_A \phi, \gamma^s \nabla^{(l)}_A \phi \right) \\
\leq \sum_{i=0}^{l-1} \sum_{j=0}^{2k+i+1} \sum_{n=0}^{j} C(g)|\langle F_{A\infty} \rangle||\nabla_A^{(n)}(\gamma^s)||\nabla_A^{(j-n)}\nabla^{(l)}_A \phi||\nabla_A^{(2k+i+1-j)}\nabla^{(l-1-i)}_A \phi|. \tag{6.3.8}\]

It therefore suffices to estimate the following integral
\[
\int \sum_{i=0}^{l-1} \sum_{j=0}^{2k+i+1} \sum_{n=0}^{j} C(g)|\langle F_{A\infty} \rangle||\nabla_A^{(n)}(\gamma^s)||\nabla_A^{(j-n)}\nabla^{(l)}_A \phi||\nabla_A^{(2k+i+1-j)}\nabla^{(l-1-i)}_A \phi|. \tag{6.3.9}\]

In order to estimate this integral, we split the integral into two cases, depending on the parity of \( n \).

We start by considering the case that \( n \) is even. Write \( n = 2\alpha \) for \( \alpha \geq 0 \). Then
\[
\int ||\nabla_A^{(n)}(\gamma^s)||\nabla_A^{(j-n)}\nabla^{(l)}_A \phi||\nabla_A^{(2k+i+1-j)}\nabla^{(l-1-i)}_A \phi|| \\
= \int ||\nabla_A^{(2\alpha)}(\gamma^s)||\nabla_A^{(j-2\alpha)}\nabla^{(l)}_A \phi||\nabla_A^{(2k+i+1-j)}\nabla^{(l-1-i)}_A \phi|| \\
\leq C(\gamma)|\gamma^{s-2\alpha}||\nabla_A^{(j-n)}\nabla^{(l)}_A \phi||\nabla_A^{(2k+i+1-j)}\nabla^{(l-1-i)}_A \phi|| \\
\leq C(\gamma)|\gamma^{s-2\alpha}||\nabla_A^{(j-n)}\nabla^{(l)}_A \phi||\nabla_A^{(2k+i+1-j)}\nabla^{(l-1-i)}_A \phi||^2 + ||\phi||_{L^2, \gamma > 0}^2 \\
\leq C(\gamma)|\gamma^{s-2\alpha}||\nabla_A^{(j-n)}\nabla^{(l)}_A \phi||^2 + ||\phi||_{L^2, \gamma > 0}^2 \\
\leq C(\gamma)\epsilon T||\gamma^{s-2\alpha}||\nabla_A^{(k-\alpha+1)}\nabla^{(l)}_A \phi||^2 + C(\epsilon T, \gamma)||\phi||_{L^2, \gamma > 0}^2 \tag{6.3.10}
\]

where to get from the second to the third line we have used theorem 10.1, and to obtain the last inequality we have applied lemma 10.3.

We then consider the case where \( n \) is odd. Write \( n = 2\alpha + 1 \), for \( \alpha \geq 0 \). By lemma 6.2, we can write \( \nabla_A^{(n)}(\gamma^s) \) as
\[
\nabla_A^{(n)}(\gamma^s) = \sum_{0 \leq p_1 < \cdots < p_n \leq n} C(p_1, \ldots, p_n)(\gamma)^{s-i} \nabla^{p_1}_A \ast \cdots \ast \nabla^{p_n}_A \]

where...
and obtain the pointwise bound
\[ |\nabla^{(n)} \gamma^s| \leq \sum_{p_1 + \cdots + p_n = n} C(\gamma) \gamma^{s-n} |\nabla^{p_1} \gamma| \cdots |\nabla^{p_n-1} \gamma| |\nabla^{p_n} \gamma| \]
\[ \leq \sum_{p_1 + \cdots + p_n = n} C(\gamma) \gamma^{s-n} |\nabla^{p_1} \gamma||\nabla^{p_2} \gamma||\cdots||\nabla^{p_n-1} \gamma| |\nabla^{p_n} \gamma| \]
\[ \leq \sum_{p_n = 1} C(\gamma) \gamma^{s-n} |\nabla^{p_n-1} \gamma| \]

where to get the last inequality we have absorbed the norms $||\nabla^{p_q} \gamma||_{\infty}$, for $1 \leq q \leq n-1$, into the constant $C(\gamma)$.

We then estimate
\[
\int |\nabla^{(2\alpha+1)} (\gamma^s)\| \nabla_A^{(j-2\alpha-1)} \nabla_A^{(l)} \phi\| \nabla_A^{(2k+i+1-j)} \nabla_A^{(l-1-i)} \phi| \tag{6.3.11}
\]
\[
\leq C(\gamma) K(||\phi||_{\infty}) ||\gamma^{(s-2\alpha-1)} / 2 \nabla_A^{(k+\alpha+l)} \phi||_{L^2} + ||\phi||_{L^{2,\gamma > 0}} \frac{2(k+\alpha+l)}{k+\alpha} ^{2(k+\alpha+l)}
\]
\[
\leq C(\gamma) K(||\phi||_{\infty}) ||\gamma^{(s-2\alpha-1)} / 2 \nabla_A^{(k+\alpha+l)} \phi||_{L^2} + ||\phi||_{L^{2,\gamma > 0}} \frac{2(k+\alpha+l)}{k+\alpha} ^{2(k+\alpha+l)}
\]
\[
\leq C(\gamma) K(||\phi||_{\infty}) ||\gamma^{(s-2\alpha-1)} / 2 \nabla_A^{(k+\alpha+l)} \phi||_{L^2} + ||\phi||_{L^{2,\gamma > 0}} \frac{2(k+\alpha+l)}{k+\alpha} ^{2(k+\alpha+l)}
\]
\[
\leq C(\gamma) K(||\phi||_{\infty}) \bar{\epsilon}_7 ||\gamma^{s/2} \nabla_A^{(k+1)} \nabla_A^{(l)} \phi||_{L^2} + C(\bar{\epsilon}_7, \gamma) K(||\phi||_{\infty}) ||\phi||_{L^{2,\gamma > 0}}^2
\]

where to get from the second line to the third line we apply theorem 10.1, and to get the last line we apply lemma 10.3.

Substituting the estimate we obtained for $n$ even, (6.3.10), and the one for $n$ odd, (6.3.11), into (6.3.9) gives
\[
\int \sum_{i=0}^{l-1} \sum_{j=0}^{2k+i+1} \sum_{n=0}^{j} C(g) \left( \sup_{t \in [0,T]} ||F_A||_{\infty} \right) ||\nabla_A^{(n)} (\gamma^s)\| \nabla_A^{(j-n)} \nabla_A^{(l)} \phi\| \nabla_A^{(2k+i+1-j)} \nabla_A^{(l-1-i)} \phi| \]
\[
\leq C(g, \gamma) \left( \sup_{t \in [0,T]} ||F_A||_{\infty} \right) (\epsilon_7 + K(||\phi||_{\infty}) \bar{\epsilon}_7) ||\gamma^{s/2} \nabla_A^{(k+1)} \nabla_A^{(l)} \phi||_{L^2}^2
\]
\[
+ \left( \sup_{t \in [0,T]} ||F_A||_{\infty} \right) \left( C(\bar{\epsilon}_7, \gamma, g) + C(\bar{\epsilon}_7, \gamma, g) K(||\phi||_{\infty}) ||\phi||_{L^{2,\gamma > 0}}^2 \right)
\]

Using the estimate (6.3.8), we then obtain the statement of the lemma. 

\[ \square \]

**Lemma 6.11.** Assume $\sup_{t \in [0,T]} ||F_A||_{\infty} < \infty$, and let $K(||\phi||_{\infty}) = \max \{ 1, \sup_{t \in [0,T]} ||\phi||_{\infty} \}$. Suppose $\gamma$ is a bump function, and $s \geq 2(k+1)$. Then for $\epsilon_8, \bar{\epsilon}_8 > 0$ sufficiently small, we
have the following estimate

\[
\int 2\text{Re} \left( \sum_{i=0}^{l-1} \sum_{v=0}^{2k-1+i} P^{(v)}_1[A] \otimes \nabla_A^{(l-1-i)} \phi, \gamma^s \nabla_A^{(l)} \phi \right)
\leq C(\gamma, g) \left( \sup_{t \in [0,T]} ||F_A||_\infty \right) (\epsilon_8 + K(||\phi||_\infty))||\gamma^{s/2} \nabla_A^{k+1} \nabla_A^{(l)} \phi||_{L^2}^2
\]

\[
+ \left( \sup_{t \in [0,T]} ||F_A||_\infty \right) (C(\epsilon_8, \gamma, g) + C(\epsilon_8, \gamma, g) K(||\phi||_\infty))||\phi||_{L^2}^2, \gamma > 0.
\]

**Proof.** The proof of this lemma proceeds in the same way to the proof of lemma 6.10. We start by writing

\[
\int \sum_{i=0}^{l-1} \sum_{v=0}^{2k-1+i} P^{(v)}_1[A] \otimes \nabla_A^{(l-1-i)} \phi, \gamma^s \nabla_A^{(l)} \phi = \int \sum_{i=0}^{l-1} \sum_{v=0}^{2k-1+i} P^{(v)}_1[A], \langle \gamma^s \nabla_A^{(l)} \phi, \nabla_A^{(l-1-i)} \phi \rangle).
\]

Performing an integration by parts, we obtain

\[
\int \sum_{i=0}^{l-1} \sum_{v=0}^{2k-1+i} P^{(v)}_1[A], \langle \gamma^s \nabla_A^{(l)} \phi, \nabla_A^{(l-1-i)} \phi \rangle = \int \sum_{i=0}^{l-1} \sum_{v=0}^{2k-1+i} (-1)^v \langle F_A, P^{(v)}_1(\langle \gamma^s \nabla_A^{(l)} \phi, \nabla_A^{(l-1-i)} \phi \rangle) \rangle.
\]

We then estimate

\[
\left| \int \sum_{i=0}^{l-1} \sum_{v=0}^{2k-1+i} (-1)^v \langle F_A, P^{(v)}_1(\langle \gamma^s \nabla_A^{(l)} \phi, \nabla_A^{(l-1-i)} \phi \rangle) \rangle \right|
\leq \int \sum_{i=0}^{l-1} \sum_{v=0}^{2k-1+i} \langle F_A, P^{(v)}_1(\langle \gamma^s \nabla_A^{(l)} \phi, \nabla_A^{(l-1-i)} \phi \rangle) \rangle
\leq \int \sum_{i=0}^{l-1} \sum_{v=0}^{2k-1+i} \left( \sup_{t \in [0,T]} ||F_A||_\infty \right) ||P^{(v)}_1(\langle \gamma^s \nabla_A^{(l)} \phi, \nabla_A^{(l-1-i)} \phi \rangle) ||.
\]

We have \( |P^{(v)}_1(\langle \gamma^s \nabla_A^{(l)} \phi, \nabla_A^{(l-1-i)} \phi \rangle) \rangle \leq C(g) ||\nabla_A^{(l)} \phi, \nabla_A^{(l-1-i)} \phi ||. \) Applying lemma 6.6, we get

\[
||\nabla^{(v)}(\gamma^s \nabla_A^{(l)} \phi, \nabla_A^{(l-1-i)} \phi) || \leq \sum_{n=0}^{v} C(g) ||\nabla_A^{(m)}(\gamma^s \nabla_A^{(l)} \phi, \nabla_A^{(v-n)} \nabla_A^{(l-1-i)} \phi) ||
\leq \sum_{n=0}^{v} C(g) ||\nabla_A^{(m)}(\gamma^s) ||\nabla_A^{(n-m)} \nabla_A^{(l)} \phi, \nabla_A^{(v-n)} \nabla_A^{(l-1-i)} \phi ||
\leq \sum_{n=0}^{v} C(g) ||\nabla_A^{(m)}(\gamma^s) ||\nabla_A^{(n-m)} \nabla_A^{(l)} \phi ||\nabla_A^{(v-n)} \nabla_A^{(l-1-i)} \phi ||.
\]

Therefore, we get the estimate

\[
\int 2\text{Re} \left( \sum_{i=0}^{l-1} \sum_{v=0}^{2k-1+i} P^{(v)}_1[A] \otimes \nabla_A^{(l-1-i)} \phi, \gamma^s \nabla_A^{(l)} \phi \right) \leq \int \sum_{i=0}^{l-1} \sum_{v=0}^{2k-1+i} \sum_{n=0}^{v} C(g) \left( \sup_{t \in [0,T]} ||F_A||_\infty \right) ||\nabla_A^{(m)}(\gamma^s) ||\nabla_A^{(n-m)} \nabla_A^{(l)} \phi ||\nabla_A^{(v-n)} \nabla_A^{(l-1-i)} \phi ||.
\]

The way to proceed to evaluate an estimate for the above integral is to apply the same technique we used in proving lemma 6.10. That is, we need to set up the integral in a form for which theorem 10.1 is applicable. In order to apply theorem 10.1, we first note that the sum of the exponents of the derivatives of the spinor field \( \phi \) is \((n-m+l)+(v-n+l-1-i) = \)

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\[2l + v - m - i - 1.\] Therefore we split the integral into two parts, \(v - m - i - 1\) is even and \(v - m - i - 1\) is odd. The proof in each case exactly follows what we did in the proof of lemma 6.10, see the proof of (6.3.10), and the proof of (6.3.11). Due to this, we will just state the final result of applying that technique.

1. When \(v - m - i - 1\) is even, we obtain the estimate

\[
\int \nabla^{(m)}(\gamma^s) \left| \nabla_A^{(n-m)} \nabla_A^{(l)} \phi \right| \left| \nabla_A^{(v-n)} \nabla_A^{(l-1-i)} \phi \right| 
\leq C(\gamma) \varepsilon_8 \left| \gamma^{s/2} \nabla_A^{k+1} \nabla_A^{(l)} \phi \right|^2_{L^2} + C(\varepsilon_8, \gamma) \left| \phi \right|^2_{L^2, \gamma > 0}.
\]

2. When \(v - m - i - 1\) is odd, we obtain the estimate

\[
\int \nabla^{(m)}(\gamma^s) \left| \nabla_A^{(n-m)} \nabla_A^{(l)} \phi \right| \left| \nabla_A^{(v-n)} \nabla_A^{(l-1-i)} \phi \right| 
\leq C(\gamma) K \left( \left\| \phi \right\|_\infty \right) \varepsilon_8 \left| \gamma^{s/2} \nabla_A^{k+1} \nabla_A^{(l)} \phi \right|^2_{L^2} + C(\varepsilon_8, \gamma) K \left( \left\| \phi \right\|_\infty \right) \left| \phi \right|^2_{L^2, \gamma > 0}.
\]

Substituting these two estimates into (6.3.12) we obtain the statement of the lemma.

\[\square\]

**Lemma 6.12.** Let \(K(\left\| \phi \right\|_\infty) = \max\{1, \sup_{t \in [0,T]} \left\| \phi \right\|_\infty\}.\) Suppose \(\gamma\) is a bump function, and \(s \geq 2(k + l).\) Then for \(\varepsilon_9 > 0\) sufficiently small, we have the following estimate

\[
\int 2Re \left( \langle -2iM \sum_{j=0}^{l-1} \sum_{i=1}^{k} C_i \nabla_M^{(j)} \nabla_A^{(i)} \nabla_A^{(k)} \nabla_A^{(k-i)} \phi \rangle \rangle \nabla_A^{(l-1-j)} \phi, \gamma^s \nabla_A^{(l)} \phi \right) \right) 
\leq C(g) K \left( \left\| \phi \right\|_\infty \right) \varepsilon_9 \left| \gamma^{s/2} \nabla_A^{k+1} \nabla_A^{(l)} \phi \right|^2_{L^2} + C(\varepsilon_9, g) K \left( \left\| \phi \right\|_\infty \right) \left| \phi \right|^2_{L^2, \gamma > 0}.
\]

**Proof.** We have

\[
\int 2Re \left( \langle -2iM \sum_{j=0}^{l-1} \sum_{i=1}^{k} C_i \nabla_M^{(j)} \nabla_A^{(i)} \nabla_A^{(k)} \nabla_A^{(k-i)} \phi \rangle \rangle \nabla_A^{(l-1-j)} \phi, \gamma^s \nabla_A^{(l)} \phi \right) \right) 
= \int 2Re \left( \langle -2iM \sum_{j=0}^{l-1} \sum_{i=1}^{k} C_i \nabla_M^{(j)} \nabla_A^{(i)} \nabla_A^{(k)} \nabla_A^{(k-i)} \phi \rangle \rangle, \gamma^s \nabla_A^{(l)} \phi, \nabla_A^{(l-1-j)} \phi \right) \right).
\]

We estimate

\[
\int \left| \langle -2iM \sum_{j=0}^{l-1} \sum_{i=1}^{k} C_i \nabla_M^{(j)} \nabla_A^{(i)} \nabla_A^{(k)} \nabla_A^{(k-i)} \phi \rangle \rangle, \gamma^s \nabla_A^{(l)} \phi, \nabla_A^{(l-1-j)} \phi \right) \right| 
\leq \int \sum_{j=0}^{l-1} \sum_{i=1}^{k} 2\gamma^s \left| \langle \nabla_M^{(j)} \nabla_A^{(i)} \nabla_A^{(k)} \nabla_A^{(k-i)} \phi \rangle \rangle, \gamma^s \nabla_A^{(l)} \phi, \nabla_A^{(l-1-j)} \phi \right| \right| \left| \langle \nabla_A^{(l)} \phi, \nabla_A^{(l-1-j)} \phi \right| \right|.
\]

Applying lemma 6.6, we then have

\[
\int \sum_{j=0}^{l-1} \sum_{i=1}^{k} 2\gamma^s \left| \langle \nabla_M^{(j)} \nabla_A^{(i)} \nabla_A^{(k)} \nabla_A^{(k-i)} \phi \rangle \rangle, \gamma^s \nabla_A^{(l)} \phi, \nabla_A^{(l-1-j)} \phi \right| \right| \left| \langle \nabla_A^{(l)} \phi, \nabla_A^{(l-1-j)} \phi \right| \right| 
\leq \int \sum_{j=0}^{l-1} \sum_{i=1}^{k} C(g) \left| \langle \nabla_A^{(n)} \nabla_A^{(k)} \nabla_A^{(k-i-n)} \nabla_A^{(k-i)} \phi \rangle \rangle, \gamma^s \nabla_A^{(l)} \phi, \nabla_A^{(l-1-j)} \phi \right| \right| \left| \langle \nabla_A^{(l)} \phi, \nabla_A^{(l-1-j)} \phi \right| \right|.
\]
Applying theorem 10.1, followed by lemma 10.3, we obtain
\[
\int_{t=0}^{t\leq \epsilon} \sum_{j=0}^{k} \sum_{i=1}^{k+i+j} C(g) \langle \nabla_{n}^{(l)} \nabla_{A}^{(n)} \phi, \nabla_{A}^{(l+j-n)} \nabla_{A}^{(k-j)} \phi \rangle \leq C(g) K \| \phi \|_{\infty}^{2} \| \gamma^{s/2} \nabla_{A}^{(l)} \phi \|_{L^{2}}^{2} + \| \phi \|_{L^{2}, \gamma > 0}^{2}
\]
and the lemma follows.

\[\□\]

**Lemma 6.13.** Assume \( \sup_{t \in [0, T]} \| F_{A} \|_{\infty} < \infty \), and let \( K(\| \phi \|_{\infty}) = \max \{ 1, \sup_{t \in [0, T]} \| \phi \|_{\infty} \} \).

Suppose \( \gamma \) is a bump function, and \( s \geq 2(k + l) \). Then for \( \epsilon_{10}, \bar{\epsilon}_{10} > 0 \) sufficiently small, we have the following estimate
\[
\int \frac{1}{2} \text{Re} \left( \langle \nabla_{A}^{(l)} ((S + |\phi|^{2}) \phi), \gamma^{s} \nabla_{A}^{(l)} \phi \rangle \right) \leq C(g, \gamma) K \| \phi \|_{\infty}^{3} (\epsilon_{10} + K(\| \phi \|_{\infty}) \epsilon_{10}) \| \gamma^{s/2} \nabla_{A}^{(l)} \phi \|_{L^{2}}^{2} + K(\| \phi \|_{\infty}) K(\| \phi \|_{\infty}) C(\epsilon_{10}, g, \gamma) \| \phi \|_{L^{2}, \gamma > 0}^{2}.
\]

**Proof.** We start by observing that
\[
\int \frac{1}{2} \langle \nabla_{A}^{(l)} ((S + |\phi|^{2}) \phi), \gamma^{s} \nabla_{A}^{(l)} \phi \rangle = \int \frac{1}{2} \langle (S + |\phi|^{2}) \phi, \gamma^{s} \nabla_{A}^{(l)} \phi \rangle.
\]
We can then bound
\[
\left| \int \frac{1}{2} \langle (S + |\phi|^{2}) \phi, \gamma^{s} \nabla_{A}^{(l)} \phi \rangle \right| \leq \int C(g) K \| \phi \|_{\infty}^{3} \langle \nabla_{A}^{(l)} (\gamma^{s} \nabla_{A}^{(l)} \phi) \rangle \leq \int \sum_{n=0}^{l} C(g) K \| \phi \|_{\infty}^{3} \langle \nabla^{(n)} \gamma^{s} \nabla_{A}^{(l-n)} \phi \rangle.
\]
We estimate this latter integral by splitting the integrand up into two parts, \( n \) even and \( n \) odd. The proof then proceeds analogously to what was done in the proof of lemma 6.10. For details, see the proofs for (6.3.10) and (6.3.11).

\[\□\]

Using the above lemmas, we can prove the following local \( L^{2} \)-derivative estimate.

**Theorem 6.14.** Let \( (\phi(t), A(t)) \) be a solution to the higher order Seiberg-Witten flow. Assume \( Q(\| F_{A} \|_{\infty}) = \sup_{t \in [0, T]} \| F_{A} \|_{\infty} < \infty \), and let \( K(\| \phi \|_{\infty}) = \max \{ 1, \sup_{t \in [0, T]} \| \phi \|_{\infty} \} \).

Suppose \( \gamma \) is a bump function, and \( s \geq 2(k + l) \). Then
\[
\frac{\partial}{\partial t} \| \gamma^{s/2} \nabla_{A}^{(l)} \phi \|_{L^{2}}^{2} \leq -\lambda \| \gamma^{s/2} \nabla_{A}^{(k+1)} \nabla_{A}^{(l)} \phi \|_{L^{2}}^{2} + C_{s}(Q(\| F_{A} \|_{\infty}), K(\| \phi \|_{\infty}), g, \gamma) \| \phi \|_{L^{2}, \gamma > 0}^{2}
\]
where \( 1 \leq \lambda < 2 \)

**Proof.** Taking \( 0 < \epsilon = \epsilon_{1} = \epsilon_{2} = \epsilon_{3} = \ldots = \epsilon_{8} = \bar{\epsilon}_{8} = \epsilon_{9} \) in lemmas 6.8, 6.9, 6.10, 6.11, 6.12, 6.13, and then using proposition 6.7. We see that we have a bound of the form
\[
\frac{\partial}{\partial t} \| \gamma^{s/2} \nabla_{A}^{(l)} \phi \|_{L^{2}}^{2} \leq ( -2 + C_{1}(Q(\| F_{A} \|_{\infty}), K(\| \phi \|_{\infty}), g, \gamma) \epsilon) \| \gamma^{s/2} \nabla_{A}^{(k+1)} \nabla_{A}^{(l)} \phi \|_{L^{2}}^{2} + C_{2}(\epsilon, Q(\| F_{A} \|_{\infty}), K(\| \phi \|_{\infty}), g, \gamma) \| \phi \|_{L^{2}, \gamma > 0}^{2}
\]

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By choosing ε sufficiently small, we can make it so that
\[ 1 \leq 2 - C_1(Q(||F_A||_{\infty}), K(||\phi||_{\infty}), g, \gamma) \epsilon < 2. \]

Taking λ to be any such \( 2 - C_1(Q(||F_A||_{\infty}), K(||\phi||_{\infty}), g, \gamma) \epsilon \), and defining
\[ C_s(Q(||F_A||_{\infty}), K(||\phi||_{\infty}), g, \gamma) = C_2(\epsilon, Q(||F_A||_{\infty}), K(||\phi||_{\infty}), g, \gamma), \]
we arrive at the statement of the theorem.

The following corollary follows from integrating the above inequality in time.

**Corollary 6.15.** Suppose \((\phi(t), A(t))\) is a solution to the higher order Seiberg-Witten flow, on the time interval \([0, T]\), where \(T < \infty\), with the same assumptions as the above theorem. Then
\[ \|\gamma^{s/2} \nabla^{(l)}_{A(t)} \phi(t)\|_{L^2}^2 \leq TC \sup_{t \in [0, T]} \left( \|\phi\|_{L^2, \gamma > 0}^2 \right) \]
where \(C\) depends on \(C_s(Q(||F_A||_{\infty}), K(||\phi||_{\infty}), g, \gamma)\) and the initial condition \((\phi(0), A(0))\).

### 6.4. Estimates for derivatives of the curvature.

In this subsection, we establish local \(L^2\)-derivative estimates for the curvature form.

**Proposition 6.16.** Let \((\phi(t), A(t))\) be a solution to the higher order Seiberg-Witten flow. Then
\[ \frac{\partial}{\partial t} \|\gamma^{s/2} \nabla^{(l)}_M F_A\|_{L^2}^2 \leq \int 2Re(\langle (-1)^{k+1} \Delta^{k+1} \nabla^{(l)}_M F_A, \gamma^{s} \nabla^{(l)}_M F_A \rangle) + 2Re(\sum_{v=0}^{2k+l} P_v[F_A(t)], \gamma^{s} \nabla^{(l)}_M F_A) \]
\[ + 2Re\left( -2iIm \left( \sum_{i=1}^{k} C_i \nabla^{(l)}_M d \nabla^{(i)}_M (\nabla^{(k)}_A \nabla_A \phi, \nabla^{(k-l)}_A \phi), \gamma^{s} \nabla^{(l)}_M F_A \right) \right). \]

The above proposition immediately follows from corollary 6.4.

In order to obtain local \(L^2\)-derivative estimates for derivatives of the curvature form \(F_A\), associated to the solution \((\phi(t), A(t))\). We will proceed as we did for the case of the spinor field. That is, we will start by stating a string of lemmas that give estimates for the right hand side of the above proposition. These estimates will then suffice to prove a general local estimate. Many of the proofs will follow the exact same techniques that was used in obtaining such estimates for the spinor field. Due to this, we won’t give details but rather refer the reader to those proofs.

**Lemma 6.17.** Assume \(\sup_{t \in [0, T]} ||F_A||_{\infty} < \infty\), and suppose \(\gamma\) is a bump function, and \(s \geq 2(k + l)\). Then for \(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5 > 0\) sufficiently small, we have the following
estimate
\[
\int 2\Re \left( \langle (-1)^{k+1} \Delta_M^{k+1} \nabla_M^{(l)} F_A, \gamma^s \nabla_M^{(l)} F_A \rangle \right)
\]
\[
\leq \left( -2 + C(g, \gamma)(\epsilon_1 + \epsilon_2) + C(g)\epsilon_3 + C(g, \gamma)\epsilon_3 \right)
\]
\[
C(g, \gamma) \left( \sup_{t \in [0, T)} \| F_A \|_\infty^2 \right) (\epsilon_4 + \epsilon_4)
\]
\[
+ \left( \frac{C(\epsilon_2, g, \gamma)}{\epsilon_1^2} + C(\epsilon_3, g) + C(\epsilon_3, g, \gamma) \right)
\]
\[
+ C(g, \gamma) \left( \sup_{t \in [0, T)} \| F_A \|_\infty \right) (C(\epsilon_4, g, \gamma) + C(\epsilon_4, g, \gamma)) \| F_A \|_{L^2, \gamma > 0}^2
\]

where \( C(g, C(g, \gamma), C(\epsilon_2, g, \gamma), C(\epsilon_3, g, \gamma), C(\epsilon_3, g, \gamma), C(\epsilon_4, g, \gamma), C(\epsilon_4, g, \gamma) \) are constants that do not depend on \( t \in [0, T) \).

The proof of the above lemma is exactly analogous to how we proved lemma 6.8. One simply replaces the term \( \phi \), in lemma 6.8, with \( F_A \), and then proceeds in exactly the same way. Therefore, we won’t give details of the proof, and refer the interested reader to lemma 6.8.

**Lemma 6.18.** Suppose \( \gamma \) is a bump function, and \( s \geq 2(k + l) \). For \( \epsilon_5, \tilde{\epsilon}_5 > 0 \) sufficiently small, we have the following estimate
\[
\int 2\Re \left( \sum_{v=0}^{2k+1} P_1^{(v)} [F_A], \gamma^s \nabla_M^{(l)} F_A \right) \leq \left( C(g, \gamma)\epsilon_5 + C(\tilde{\epsilon}_5) \right) \| \gamma^{s/2} \nabla_M^{(l)} F_A \|_{L^2}^2 + \frac{C(\epsilon_5, g, \gamma)}{\epsilon_5^3} \| F_A \|_{L^2, \gamma > 0}^2.
\]

**Proof.** We start by writing
\[
\int \sum_{v=0}^{2k+1} P_1^{(v)} [F_A], \gamma^s \nabla_M^{(l)} F_A = \int \sum_{v=0}^{k+l+1} P_1^{(v)} [F_A], \gamma^s \nabla_M^{(l)} F_A + \sum_{v=k+l+2}^{2k+1} \langle P_1^{(v)} [F_A], \gamma^s \nabla_M^{(l)} F_A \rangle.
\]

Estimating the first time on the right, we have
\[
\int \sum_{v=0}^{2k+1} \langle P_1^{(v)} [F_A], \gamma^s \nabla_M^{(l)} F_A \rangle \leq \int \sum_{v=0}^{2k+1} C(g) \nabla_M^{(v)} F_A \| \gamma^s \nabla_M^{(l)} F_A \|
\]
\[
= \int \sum_{v=0}^{2k+1} C(g) \| \gamma^{s/2} \nabla_M^{(v)} F_A \| \| \gamma^{s/2} \nabla_M^{(l)} F_A \|. \]

For \( v = k + l + 1 \), by applying Young’s inequality, we obtain
\[
\int C(g) \| \gamma^{s/2} \nabla_M^{(k+l+1)} F_A \| \| \gamma^{s/2} \nabla_M^{(l)} F_A \| \leq \int C(g) (\epsilon_5) \| \gamma^{s/2} \nabla_M^{(l)} F_A \|^2 + \frac{1}{\epsilon_5} \| \gamma^{s/2} \nabla_M^{(l)} F_A \|^2
\]
\[
= C(g) \epsilon_5 \| \gamma^{s/2} \nabla_M^{(k+l+1)} F_A \|^2 + \frac{C(g)}{\epsilon_5} \| \gamma^{s/2} \nabla_M^{(l)} F_A \|^2.
\]
For $\epsilon_5$ sufficiently small, we have that $\frac{C(g)}{\epsilon_5} \geq 1$. Therefore, applying lemma 10.3 to the term $\frac{C(g)}{\epsilon_5}||\gamma^{s/2}\nabla_M^{(l)}F_A||^2_{L^2}$, we get the following estimate:

\[
C(g)\epsilon_5 ||\gamma^{s/2}\nabla_M^{k+1}\nabla_M^{(l)}F_A||^2_{L^2} + \frac{C(g)}{\epsilon_5} ||\gamma^{s/2}\nabla_M^{(l)}F_A||^2_{L^2} \leq (C(g)\epsilon_5 + \epsilon_5)||\gamma^{s/2}\nabla_M^{k+1}\nabla_M^{(l)}F_A||^2_{L^2} + \frac{C(\epsilon_5, g)}{\epsilon_5^2} ||F_A||^2_{L^2, \gamma > 0}.
\]

For the case that $0 \leq v \leq k + l$, we have

\[
\int C(g)||\gamma^{s/2}\nabla_M^{(l)}F_A||^2_{L^2} \leq \int C(g)(||\gamma^{s/2}\nabla_M^{(l)}F_A||^2 + ||\gamma^{s/2}\nabla_M^{(l)}F_A||^2)
\]

\[
\leq C(g)\epsilon_5 ||\gamma^{s/2}\nabla_M^{k+1}\nabla_M^{(l)}F_A||^2_{L^2} + C(\epsilon_5, g)||F_A||^2_{L^2, \gamma > 0}
\]

where in order to get the second line, we have applied lemma 10.3 to both terms on the right hand side of the first line.

The next step is to estimate the term $\sum_{v=k+l+2}^{2k+l} \langle P^{(v)}_1[F_A], \gamma^s \nabla_M^{(l)}F_A \rangle$. We write this as

\[
\int \sum_{v=k+l+2}^{2k+l} \langle P^{(v)}_1[F_A], \gamma^s \nabla_M^{(l)}F_A \rangle = \int \sum_{j=1}^{k-1} \langle P^{(k+l+1+j)}_1[F_A], \gamma^s \nabla_M^{(l)}F_A \rangle
\]

\[
= \int \sum_{j=1}^{k-1} (-1)^j \langle P^{(k+l+1)}_1[F_A], P^{(j)}_1(\gamma^s \nabla_M^{(l)}F_A) \rangle
\]

where the second equality follows from integrating by parts.

We estimate

\[
\int \left| \langle P^{(k+l+1)}_1[F_A], P^{(j)}_1(\gamma^s \nabla_M^{(l)}F_A) \rangle \right|
\]

\[
\leq \int C(g)||\nabla_M^{(k+l+1)}F_A|| \left\| \nabla_M^{(j)}(\gamma^s \nabla_M^{(l)}F_A) \right\|
\]

\[
\leq \int \sum_{i=0}^{j} C(g)||\nabla_M^{(k+l+1)}F_A|| \left\| \nabla_M^{(i)}(\gamma^s) \otimes \nabla_M^{(j-i)}(\nabla_M^{(l)}F_A) \right\|
\]

\[
\leq \int \sum_{i=0}^{j} C(g, \gamma) \left| \gamma^{s/2} \nabla_M^{(k+l+1)}F_A \right|^2 + \frac{1}{\epsilon_5} \left| \gamma^{(s-2i)/2} \nabla_M^{(l+j-i)}F_A \right|^2
\]

\[
= C(g, \gamma)\epsilon_5 ||\gamma^{s/2} \nabla_M^{k+1} \nabla_M^{(l)}F_A||^2_{L^2} + \int \sum_{i=0}^{j} \frac{C(g, \gamma)}{\epsilon_5} \left| \gamma^{(s-2i)/2} \nabla_M^{(l+j-i)}F_A \right|^2.
\]

For $\epsilon_5$ sufficiently small, we have that $\frac{C(g, \gamma)}{\epsilon_5} \geq 1$, so we can apply lemma 10.3 to obtain

\[
C(g, \gamma)\epsilon_5 ||\gamma^{s/2} \nabla_M^{k+1} \nabla_M^{(l)}F_A||^2_{L^2} + \int \sum_{i=0}^{j} \frac{C(g, \gamma)}{\epsilon_5} \left| \gamma^{(s-2i)/2} \nabla_M^{(l+j-i)}F_A \right|^2
\]

\[
\leq C(g, \gamma)\epsilon_5 ||\gamma^{s/2} \nabla_M^{k+1} \nabla_M^{(l)}F_A||^2_{L^2} + C\epsilon_5 ||\gamma^{s/2} \nabla_M^{k+1} \nabla_M^{(l)}F_A||^2_{L^2} + \frac{C(\epsilon_5, g, \gamma)}{\epsilon_5^2} ||F_A||^2_{L^2, \gamma > 0}.
\]

Putting the estimates obtained for $0 \leq v \leq k + l + 1$ with those obtained for $k + l + 2 \leq v \leq 2k + l$, we see that we get the statement of the lemma.
Lemma 6.19. Assume $\sup_{t \in [0,T]} \| F_A \|_\infty < \infty$, and let $K(\| \phi \|_\infty) = \max \{ 1, \sup_{t \in [0,T]} \| \phi \|_\infty \}$. Suppose $\gamma$ is a bump function, and $s \geq 2(k+l)$. Then for $\epsilon_6 > 0$ sufficiently small, we have the following estimate
\[
\int 2 \Re \left( \langle -2iM \left( \sum_{i=1}^{k} C_i \nabla_M^{(l)} d\nabla_M^{(i)} (\nabla_A \nabla_A \phi, \nabla_A^{(k-i)} \phi) \right), \gamma^s \nabla_M^{(l)} F_A \rangle \right) \leq C(g) \left( \sup_{t \in [0,T]} \| F_A \|_\infty \right) K(\| \phi \|_\infty) \epsilon_6 \| \gamma^s/2 \nabla_A^{k+1} \nabla_M^{(l)} \phi \|_{L^2}^2 + C(\epsilon_6, g) K(\| \phi \|_\infty) \left( \sup_{t \in [0,T]} \| F_A \|_\infty \right) \| \phi \|_{L^2, \gamma > 0}^2.
\]

Proof. By applying integration by parts, we have
\[
\int \langle -2iM \left( \sum_{i=1}^{k} C_i \nabla_M^{(l)} d\nabla_M^{(i)} (\nabla_A \nabla_A \phi, \nabla_A^{(k-i)} \phi) \right), \gamma^s \nabla_M^{(l)} F_A \rangle \leq 2 \Re \left( \langle -2iM \left( \sum_{i=1}^{k} C_i \nabla_M^{(l)} d\nabla_M^{(i)} (\nabla_A \nabla_A \phi, \nabla_A^{(k-i)} \phi) \right), F_A \rangle \right).
\]

We can then bound
\[
\int \langle -2iM \left( \sum_{i=1}^{k} C_i \nabla_M^{(l)} d\nabla_M^{(i)} (\nabla_A \nabla_A \phi, \nabla_A^{(k-i)} \phi) \right), \gamma^s \nabla_M^{(l)} F_A \rangle \leq 2 \left| \langle -2iM \left( \sum_{i=1}^{k} C_i \nabla_M^{(l)} d\nabla_M^{(i)} (\nabla_A \nabla_A \phi, \nabla_A^{(k-i)} \phi) \right), F_A \rangle \right| \leq 2 \left( \sum_{i=1}^{k} C_i \left| \nabla_M^{(l)} (\gamma^s \nabla_M^{(l)} d\nabla_M^{(i)} (\nabla_A \nabla_A \phi, \nabla_A^{(k-i)} \phi)) \right| \right) \left( \sup_{t \in [0,T]} \| F_A \|_\infty \right).
\]

The next step is to estimate the term $\int \sum_{i=1}^{k} C_i \left| \nabla_M^{(l)} (\gamma^s \nabla_M^{(l)} d\nabla_M^{(i)} (\nabla_A \nabla_A \phi, \nabla_A^{(k-i)} \phi)) \right|$. This estimate follows exactly what was done in the proof of lemma 6.12. We refer the reader to that lemma for the details.

The statement of the lemma then easily follows.

In general, we cannot obtain a local estimate for the curvature term alone, like we did for the spinor field in theorem 6.14, due to the term $\| \gamma^s/2 \nabla_A^{k+1} \nabla_M^{(l)} \phi \|_{L^2}^2$ that appears in the above lemma. In fact, as the higher order Seiberg-Witten flow is a coupled system this shouldn’t be surprising.

We can however prove a local estimate for the sum $\| \gamma^s/2 \nabla_A^{k+1} \nabla_M^{(l)} F_A \|_{L^2}^2$, as we now show.

Theorem 6.20. Let $(\phi(t), A(t))$ be a solution to the higher order Seiberg-Witten flow. Assume $Q(\| F_A \|_\infty) = \sup_{t \in [0,T]} \| F_A \|_\infty < \infty$, and let $K(\| \phi \|_\infty) = \max \{ 1, \sup_{t \in [0,T]} \| \phi \|_\infty \}$. Suppose $\gamma$ is a bump function, and $s \geq 2(k+l)$. Then
\[
\frac{\partial}{\partial t} \left( \| \gamma^s/2 \nabla_A^{k+1} \nabla_M^{(l)} F_A \|_{L^2}^2 + \| \gamma^s/2 \nabla_M^{(l)} F_A \|_{L^2}^2 \right) \leq -\lambda \left( \| \gamma^s/2 \nabla_A^{k+1} \nabla_M^{(l)} \phi \|_{L^2}^2 + \| \gamma^s/2 \nabla_A^{k+1} \nabla_M^{(l)} F_A \|_{L^2}^2 \right) + C \left( Q(\| F_A \|_\infty), K(\| \phi \|_\infty), g, \gamma \right) \left( \| \phi \|_{L^2, \gamma > 0}^2 + \| F_A \|_{L^2, \gamma > 0}^2 \right)
\]

where $1 \leq \lambda < 2$. 

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Proof. We start by taking $0 < \epsilon = \epsilon_1 = \epsilon_2 = \epsilon_3 = \ldots = \epsilon_6$ in lemmas 6.17, 6.18, 6.19. We can then obtain an estimate for the curvature
\[
\frac{\partial}{\partial t} \|\gamma^{s/2} \nabla_M (l) F_A \|_2^2 \leq \left( -2 + \tilde{C}_1 (Q(||F_A||_\infty), K(||\phi||_\infty, g, \gamma), \epsilon) \right) \|\gamma^{s/2} \nabla_M^{k+1} \nabla_M (l) F_A \|_2^2 \\
+ \tilde{C}_2 (\epsilon, Q(||F_A||_\infty), K(||\phi||_\infty, g, \gamma)) ||F_A||_{L^2, \gamma > 0}^2 \\
+ \tilde{C}_3 (Q(||F_A||_\infty), K(||\phi||_\infty, g) \epsilon) \|\gamma^{s/2} \nabla_M^{k+1} \nabla_M (l) \phi \|_2^2 \\
+ \tilde{C}_4 (\epsilon, Q(||F_A||_\infty), K(||\phi||_\infty, g)) ||\phi||_{L^2, \gamma > 0}^2
\]
where the last two terms involving $\phi$ come from lemma 6.19.

We then apply the same argument to the spinor field $\phi$. Namely, take $0 < \epsilon = \epsilon_1 = \epsilon_2 = \epsilon_3 = \ldots = \epsilon_8 = \bar{\epsilon}_8 = \bar{\epsilon}_9$ in lemmas 6.8, 6.9, 6.10, 6.11, 6.12, 6.13. We then have the estimate
\[
\frac{\partial}{\partial t} \|\gamma^{s/2} \nabla_M (l) \phi \|_2^2 \leq \left( -2 + \tilde{C}_5 (Q(||F_A||_\infty), K(||\phi||_\infty, g, \gamma), \epsilon) \right) \|\gamma^{s/2} \nabla_M^{k+1} \nabla_M (l) \phi \|_2^2 \\
+ \tilde{C}_6 (\epsilon, Q(||F_A||_\infty), K(||\phi||_\infty, g, \gamma)) ||\phi||_{L^2, \gamma > 0}^2
\]
Combining the two estimates together, we obtain
\[
\frac{\partial}{\partial t} \|\gamma^{s/2} \nabla_M (l) \phi \|_2^2 + \frac{\partial}{\partial t} \|\gamma^{s/2} \nabla_M (l) F_A \|_2^2 \\
\leq \left( -2 + \tilde{C}_5 (Q(||F_A||_\infty), K(||\phi||_\infty, g, \gamma), \epsilon) + \tilde{C}_3 (Q(||F_A||_\infty), K(||\phi||_\infty, g) \epsilon) \right) \|\gamma^{s/2} \nabla_M^{k+1} \nabla_M (l) \phi \|_2^2 \\
+ \left( -2 + \tilde{C}_1 (Q(||F_A||_\infty), K(||\phi||_\infty, g, \gamma), \epsilon) \right) \|\gamma^{s/2} \nabla_M^{k+1} \nabla_M (l) F_A \|_2^2 \\
+ \left( \tilde{C}_4 (\epsilon, Q(||F_A||_\infty), K(||\phi||_\infty, g) \epsilon) + \tilde{C}_6 (\epsilon, Q(||F_A||_\infty), K(||\phi||_\infty, g, \gamma)) \right) ||\phi||_{L^2, \gamma > 0}^2 \\
+ \tilde{C}_2 (\epsilon, Q(||F_A||_\infty), K(||\phi||_\infty, g, \gamma)) ||F_A||_{L^2, \gamma > 0}^2
\]
By choosing $\epsilon$ sufficiently small, we can make it so that
\[
1 \leq \left( 2 - \tilde{C}_5 (Q(||F_A||_\infty), K(||\phi||_\infty, g, \gamma), \epsilon) - \tilde{C}_3 (Q(||F_A||_\infty), K(||\phi||_\infty, g) \epsilon) \right) < 2
\]
and
\[
1 \leq \left( 2 - \tilde{C}_1 (Q(||F_A||_\infty), K(||\phi||_\infty, g, \gamma), \epsilon) \right) < 2.
\]
For this value of $\epsilon$, we define $\lambda$ to be the minimum of these two constants
\[
\lambda = \min \left\{ 2 - \tilde{C}_5 (Q(||F_A||_\infty), K(||\phi||_\infty, g, \gamma) \epsilon) - \tilde{C}_3 (Q(||F_A||_\infty), K(||\phi||_\infty, g) \epsilon), \\
2 - \tilde{C}_1 (Q(||F_A||_\infty), K(||\phi||_\infty, g, \gamma) \epsilon) \right\}.
\]
and define
\[
C(Q(||F_A||_\infty), K(||\phi||_\infty), g, \gamma) = \max \left\{ \tilde{C}_4 (\epsilon, Q(||F_A||_\infty), K(||\phi||_\infty, g) \epsilon) + \tilde{C}_6 (\epsilon, Q(||F_A||_\infty), K(||\phi||_\infty, g, \gamma) \epsilon), \\
\tilde{C}_2 (\epsilon, Q(||F_A||_\infty), K(||\phi||_\infty, g, \gamma) \epsilon) \right\}.
\]
The theorem then follows.
The following corollary is a simple consequence of integrating the inequality, in the above theorem, in time.

**Corollary 6.21.** Suppose \((\phi(t), A(t))\) is a solution to the higher order Seiberg-Witten flow, on the time interval \([0, T]\), where \(T < \infty\). Assume the conditions of the above theorem, then

\[
||\gamma^{s/2} \nabla_{A(t)}^{(l)} \phi(t)||^2_{L^2} + ||\gamma^{s/2} \nabla_M^{(l)} F_{A(t)}||^2_{L^2} \leq TC_1 \sup_{t \in [0, T]} (||\phi||^2_{L^2, \gamma > 0} + ||F_A||^2_{L^2, \gamma > 0})
\]

where \(C_1\) depends on \(C\left(||F_A||_{\infty}\right), K\left(||\phi||_{\infty}\right), g, \gamma\) and on the initial condition \((\phi(0), A(0))\).

Using this corollary, we can then obtain the following proposition, which will be useful when we want to find obstructions to long time existence.

**Proposition 6.22.** Suppose \((\phi(t), A(t))\) is a solution to the higher order Seiberg-Witten flow on the time interval \([0, T]\), with \(T < \infty\). Assume \(Q(||F_A||_{\infty}) = \sup_{t \in [0, T]} ||F_A||_{\infty} < \infty\), and let \(K(||\phi||_{\infty}) = \max\{1, \sup_{t \in [0, T]} ||\phi||_{\infty}\}\). Suppose \(\gamma\) is a bump function, and \(s \geq 2(2k + l)\). Then

\[
\sup_{M \times [0, T]} (||\gamma^{s/2} \nabla_{A(t)}^{(l)} \phi(t)||^2 + ||\gamma^{s/2} \nabla_M^{(l)} F_{A(t)}||^2) \leq T \tilde{C}_1 \sup_{t \in [0, T]} (||\phi||^2_{L^2, \gamma > 0} + ||F_A||^2_{L^2, \gamma > 0})
\]

where \(\tilde{C}_1\) depends on \(C\left(||F_A||_{\infty}\right), K\left(||\phi||_{\infty}\right), g, \gamma\) and on the initial condition \((\phi(0), A(0))\), where \(C\left(||F_A||_{\infty}\right), K\left(||\phi||_{\infty}\right), g, \gamma\) is the constant coming from theorem 6.20.

**Proof.** We start by noting that, by the Sobolev embedding theorem, we have an embedding \(W^{k,2} \subseteq C^{0}\) provided \(k > n/2\). Therefore, fixing \(t \in [0, T]\), we have

\[
\sup_M ||\gamma^{s/2} \nabla_{A(t)}^{(l)} \phi(t)|| \leq \sum_{j=0}^{k} C_{k,2} ||\nabla_{A(t)}^{(j)} (||\gamma^{s/2} \nabla_{A(t)}^{(l)} \phi(t)||)||_{L^2}
\]

\[
\leq \sum_{j=0}^{k} C_{k,2} C(\gamma) ||\gamma^{(s-2j)/2} \nabla_{A(t)}^{(l)} \phi(t)||_{L^2}
\]

where in order to take the derivative \(\nabla_{A(t)}^{(j)}\) inside the absolute value, we have applied Kato’s inequality, and \(C_{k,2}\) is the Sobolev constant.

A similar computation gives

\[
\sup_M ||\gamma^{s/2} \nabla_M^{(l)} F_{A(t)}|| \leq \sum_{j=0}^{k} C_{k,2} ||\nabla_M^{(j)} (||\gamma^{s/2} \nabla_M^{(l)} F_{A(t)}||)||_{L^2}
\]

\[
\leq \sum_{j=0}^{k} C_{k,2} C(\gamma) ||\gamma^{(s-2j)/2} \nabla_M^{(l)} F_{A(t)}||_{L^2}
\]

Combining these two inequalities, we obtain

\[
\sup_M (||\gamma^{s/2} \nabla_{A(t)}^{(l)} \phi(t)||^2 + ||\gamma^{s/2} \nabla_M^{(l)} F_{A(t)}||^2) \leq \sum_{j=0}^{k} C_{k,2} C(\gamma) (||\gamma^{(s-2j)/2} \nabla_{A(t)}^{(l)} \phi(t)||^2 + ||\gamma^{(s-2j)/2} \nabla_M^{(l)} F_{A(t)}||^2_{L^2}).
\]

We now want to apply corollary 6.21. In order to do this, we observe that we are assuming \(s \geq 2(2k + l)\), which implies \(s - 2j \geq 2(j + l)\), so we are free to apply corollary
6.21. In doing so, we obtain

\[
\sup_M \left( |\gamma^{s/2} \nabla_{A(t)}^{(l)} \phi(t)|^2 + |\gamma^{s/2} \nabla_{M}^{(l)} F_{A(t)}|^2 \right) \\
\leq \sum_{j=0}^{k} C_{k,2} C(\gamma) \left( |\gamma^{(s-2j)/2} \nabla_{A(t)}^{(j+l)} \phi(t)|^2 + |\gamma^{(s-2j)/2} \nabla_{M}^{(j+l)} F_{A(t)}|^2 \right) \\
\leq C_{k,2} C(\gamma) T \left( \sum_{j=0}^{k} C_{j+l} \right) \sup_{t \in [0,T]} \left( |\phi|^2_{L^2,\gamma>0} + |F_A|^2_{L^2,\gamma>0} \right)
\]

where \( C_{j+l} \) are the constants coming from corollary 6.21.

Defining \( \tilde{C}_l = C_{k,2} C(\gamma) \left( \sum_{j=0}^{k} C_{j+l} \right) \), gives the result.

\[ \square \]

6.5. **Estimates of Bernstein-Bando-Shi type.** In this subsection we will obtain estimates of Bernstein-Bando-Shi type, using the results obtained from the previous two subsections.

For the next theorem we will be making use of the constant \( C \left( Q(\|F_A\|_{\infty}), K(\|\phi\|_{\infty}), g, \gamma \right) \) defined in theorem 6.20. To make the notation a little bit easier, we will denote this constant by \( C \).

**Theorem 6.23.** Suppose \((\phi(t), A(t))\) is a solution to the higher order Seiberg-Witten flow on the time interval \([0,T]\), with \( \sup_{t \in [0,T]} |F_A|_{\infty} < \infty \). Let \( K = \max \{1, C\} \), and suppose \( T < \frac{1}{K} \). Suppose \( \gamma \) is a bump function, and \( s \geq 2(2k+l) \). Then for each \( l \in \mathbb{N} \), there exists a positive constant \( C_l = C_l(\dim M, K, g, \gamma, s) \) such that

\[
|\gamma^{s/2} \nabla_{A(t)}^{(l)} \phi(t)|^2_{L^2} + |\gamma^{s/2} \nabla_{M}^{(l)} F_{A(t)}|^2_{L^2} \leq C_l \sup_{t \in [0,T]} \left( |\phi|^2_{L^2,\gamma>0} + |F_A|^2_{L^2,\gamma>0} \right) \frac{1}{t^{1+1}}
\]

**Proof.** Define

\[
G(t) = \sum_{m=0}^{l} a_m t^m \left( |\gamma^{s/2} \nabla_{A(t)}^{(k+1)m} \phi(t)|^2_{L^2} + |\gamma^{s/2} \nabla_{M}^{(k+1)m} F_{A(t)}|^2_{L^2} \right)
\]

where \( a_0 = 1 \), and \( a_m \) for \( 1 \leq m \leq l \) will be determined.
Differentiating \( G \), and applying theorem 6.20, we obtain
\[
\frac{\partial G}{\partial t} = \sum_{m=1}^{l} ma mt^{m-1} (||\gamma^{s/2}\nabla^{(k+1)m}_{A(t)} \phi(t)||_{L^2}^2 + ||\gamma^{s/2}\nabla^{(k+1)m}_{M} F_{A(t)}||_{L^2}^2) \\
+ \sum_{m=0}^{l-1} a mt^{m} \frac{\partial}{\partial t} (||\gamma^{s/2}\nabla^{(k+1)m}_{A(t)} \phi(t)||_{L^2}^2 + ||\gamma^{s/2}\nabla^{(k+1)m}_{M} F_{A(t)}||_{L^2}^2) \\
\leq \sum_{m=0}^{l-1} (m+1) a m+1 t^{m} (||\gamma^{s/2}\nabla^{(k+1)(m+1)}_{A(t)} \phi(t)||_{L^2}^2 + ||\gamma^{s/2}\nabla^{(k+1)(m+1)}_{M} F_{A(t)}||_{L^2}^2) \\
+ \sum_{m=0}^{l-1} (m+1) a m+1 - \lambda a m t^{m} (||\gamma^{s/2}\nabla^{(k+1)(m+1)}_{A(t)} \phi(t)||_{L^2}^2 + ||\gamma^{s/2}\nabla^{(k+1)(m+1)}_{M} F_{A(t)}||_{L^2}^2) \\
+ K \sum_{m=0}^{l} a m t^{m} (||\phi(t)||_{L^2,\gamma>0}^2 + ||F_{A(t)}||_{L^2,\gamma>0}^2).
\]

Define \( a_l = 1 \), and then define \( a_m \), for \( 0 \leq m \leq l-1 \), recursively so that
\[
(m+1)a_{m+1} - \lambda a_m \leq 0.
\]

We then have
\[
\frac{\partial G}{\partial t} \leq K \sum_{m=0}^{l} a m t^{m} (||\phi(t)||_{L^2,\gamma>0}^2 + ||F_{A(t)}||_{L^2,\gamma>0}^2) \\
\leq K \hat{C}_{(k+1)} (||\phi(t)||_{L^2,\gamma>0}^2 + ||F_{A(t)}||_{L^2,\gamma>0}^2),
\]

where to get the second inequality we just note that, by assumption \( t < T < 1/K \leq 1 \), and where we have taken \( \hat{C}_{(k+1)} = \sum_{m=0}^{l} a m \).

Integrating the above gives
\[
G(t) - G(0) \leq K \hat{C}_{(k+1)} \int_0^t (||\phi(s)||_{L^2,\gamma>0}^2 + ||F_{A(s)}||_{L^2,\gamma>0}^2) \, ds \\
\leq K \hat{C}_{(k+1)} \left( \sup_{t \in [0,T]} (||\phi(t)||_{L^2,\gamma>0}^2 + ||F_{A(t)}||_{L^2,\gamma>0}^2) \right) t
\]

which implies
\[
G(t) \leq K \hat{C}_{(k+1)} \left( \sup_{t \in [0,T]} (||\phi(t)||_{L^2,\gamma>0}^2 + ||F_{A(t)}||_{L^2,\gamma>0}^2) \right) t + G(0) \\
\leq \hat{C}_{(k+1)} \left( \sup_{t \in [0,T]} (||\phi(t)||_{L^2,\gamma>0}^2 + ||F_{A(t)}||_{L^2,\gamma>0}^2) \right) + G(0) \\
\leq C_{(k+1)} \left( \sup_{t \in [0,T]} (||\phi(t)||_{L^2,\gamma>0}^2 + ||F_{A(t)}||_{L^2,\gamma>0}^2) \right)
\]

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where to get the second inequality, we have used the fact that \( t < T < 1/K \), and to get the third inequality, we have used \( G(0) = ||\phi(0)||_{L^2,\gamma > 0} + ||F_A(0)||_{L^2,\gamma > 0} \), and defined \( C_{(k+1)q} = C_{(k+1)t} + 1 \).

Using the fact that, \( t^l (||\gamma^{s/2} \nabla_{A(t)}^{(k+1)} \phi(t)||_{L^2}^2 + ||\gamma^{s/2} \nabla_M^{(k+1)} F_A(t)||_{L^2}^2) \leq G(t) \), we obtain
\[
t^l (||\gamma^{s/2} \nabla_{A(t)}^{(k+1)} \phi(t)||_{L^2}^2 + ||\gamma^{s/2} \nabla_M^{(k+1)} F_A(t)||_{L^2}^2) \leq C_{(k+1)t} \left( \sup_{t \in [0, T)} (||\phi||_{L^2,\gamma > 0} + ||F_A||_{L^2,\gamma > 0}) \right)
\]

which implies
\[
||\gamma^{s/2} \nabla_{A(t)}^{(k+1)} \phi(t)||_{L^2}^2 + ||\gamma^{s/2} \nabla_M^{(k+1)} F_A(t)||_{L^2}^2 \leq C_{(k+1)t} \left( \frac{\sup_{t \in [0, T)} (||\phi||_{L^2,\gamma > 0} + ||F_A||_{L^2,\gamma > 0})}{t^l} \right)
\]

This proves the lemma for the case of \((k+1)t\), and more generally the case of \((k+1)r\) for any \( r \geq 0 \).

In the general case, write \( l = (k+1) r + w \), where \( 1 \leq w \leq k \). Then
\[
||\gamma^{s/2} \nabla_{A(t)}^{(k+1)r+w} \phi(t)||_{L^2}^2 + ||\gamma^{s/2} \nabla_M^{(k+1)r+w} F_A(t)||_{L^2}^2
\]

\[
\leq ||\gamma^{s/2} \nabla_{A(t)}^{(k+1)(r+1)} \phi(t)||_{L^2}^2 + ||\gamma^{s/2} \nabla_M^{(k+1)(r+1)} F_A(t)||_{L^2}^2 + C_1 (||\phi||_{L^2,\gamma > 0} + ||F_A||_{L^2,\gamma > 0})
\]

\[
\leq C_{(k+1)(r+1)} \left( \frac{\sup_{t \in [0, T)} (||\phi||_{L^2,\gamma > 0} + ||F_A||_{L^2,\gamma > 0})}{t^{l+1}} \right) + C_1 (||\phi||_{L^2,\gamma > 0} + ||F_A||_{L^2,\gamma > 0})
\]

\[
\leq C_1 \left( \frac{\sup_{t \in [0, T)} (||\phi||_{L^2,\gamma > 0} + ||F_A||_{L^2,\gamma > 0})}{t^{l+1}} \right)
\]

where to get the first inequality, we have used lemma 10.3 with \( \epsilon = 1 \). To get the second inequality, we have applied the theorem to the case \((k+1)r\), and to get the third inequality we have defined \( C_1 = C_{(k+1)(r+1)} + C_1 \).

\[ \square \]

**Proposition 6.24.** Under the same assumptions as theorem 6.23, we have
\[
\sup_M (||\gamma^{s/2} \nabla_{A(t)}^{(j)} \phi(t)||_{L^2}^2 + ||\gamma^{s/2} \nabla_M^{(j)} F_A(t)||_{L^2}^2) \leq B_l \sup_{t \in [0, T)} (||\phi||_{L^2,\gamma > 0} + ||F_A||_{L^2,\gamma > 0})
\]

where \( B_l = B_l(t, \dim M, K, g, \gamma, s) \).

**Proof.** We start by noting that by the Sobolev embedding theorem we have an embedding \( W^{k,2} \subseteq C^0 \) provided \( k > n/2 \). Therefore, fixing \( t \in [0, T) \), we have
\[
\sup_M |\gamma^{s/2} \nabla_{A(t)}^{(j)} \phi(t)| \leq \sum_{j=0}^k C_{k,2} ||\nabla_{A(t)}^{(j)} \left( |\gamma^{s/2} \nabla_{A(t)}^{(j)} \phi(t)\right)||_{L^2}
\]

\[
\leq \sum_{j=0}^k C_{k,2} C(\gamma) ||\gamma^{(s-2)/2} \nabla_{A(t)}^{(j)} \phi(t)||_{L^2}
\]

where in order to take the derivative \( \nabla_{A(t)}^{(j)} \) inside the absolute value, we have applied Kato’s inequality.
A similar computation gives
\[
\sup_M |\gamma^{s/2} \nabla^l_M F_{A(t)}| \leq \sum_{j=0}^k C_{k,2} ||\nabla_{(j)}^l (|\gamma^{s/2} \nabla^l_M F_{A(t)})||_{L^2} \\
\leq \sum_{j=0}^k C_{k,2} C(\gamma) ||\gamma^{(s-2j)/2} \nabla^l_M F_{A(t)}||_{L^2}
\]
Combining these two inequalities we obtain
\[
\sup_M \left( |\gamma^{s/2} \nabla^l_M \phi(t)|^2 + |\gamma^{s/2} \nabla^l_M F_{A(t)}|^2 \right) \\
\leq \sum_{j=0}^k C_{k,2} C(\gamma) \left( ||\gamma^{(s-2j)/2} \nabla^l_{A(t)} \phi(t)||_{L^2}^2 + ||\gamma^{(s-2j)/2} \nabla^l_M F_{A(t)}||_{L^2}^2 \right)
\]
where we have used the general fact that if \(a_1, \ldots, a_n\) are positive numbers, then \((a_1 + \ldots + a_n)^2 \leq C(a_1^2 + \ldots + a_n^2)\). We have absorbed the constant \(C\) into \(C(\gamma)\).

We now want to apply theorem 6.23. In order to do this, we observe that we are assuming \(s \geq 2(2k + l)\), which implies \(s - 2j \geq 2(j + l)\), so we are free to apply theorem 6.23. In doing so, we obtain
\[
\sup_M \left( |\gamma^{s/2} \nabla^l_M \phi(t)|^2 + |\gamma^{s/2} \nabla^l_M F_{A(t)}|^2 \right) \\
\leq \sum_{j=0}^k C_{k,2} C(\gamma) \left( ||\gamma^{(s-2j)/2} \nabla^l_{A(t)} \phi(t)||_{L^2}^2 + ||\gamma^{(s-2j)/2} \nabla^l_M F_{A(t)}||_{L^2}^2 \right)
\]
where \(C_{j+l}\) are the constants coming from 6.23.

Defining \(B_t = C_{k,2} C(\gamma) \left( \sum_{j=0}^k \frac{C_{j+l}}{t^{j+l+1}} \right)\), we obtain the statement of the corollary.

\[\square\]

6.6. Obstructions to long time existence. The estimates from the previous subsections can now be used to study obstructions to long time existence. The purpose of this subsection is to show that the only obstruction to extending a solution past the maximal time is curvature blow up.

**Proposition 6.25.** Let \(A(t)\) denote a sequence of time dependent unitary connections, defined on some time interval \([0, T]\), with \(T < \infty\). Suppose we have uniform bounds
\[
\sup_{M \times [0, T]} \left| \nabla^p_M \frac{\partial A(t)}{\partial t} \right| \leq C_p
\]
for some positive constants \(C_p\).

Then \(\lim_{t \to T} A(t)\) exists, is smooth, and the sequence \(\{A(t)\}\) converges to this limit connection in every \(C^m\)-norm, \(m \geq 0\). We remind the reader that we view \(A(t) \in i\Lambda(M)\), so this convergence is in the sense of 1-forms.

**Proof.** We define \(A_T = A(0) + \int_0^T \frac{\partial A(t)}{\partial t} dt\). The uniform bounds, in the assumption of the theorem, imply that the integral on the right is absolutely convergent. Hence \(A_T\), as defined, is well defined and exists.
We then compute

\[ |A(t) - A_T| = \left| A(t) - A(0) - \int_0^T \frac{\partial A(t)}{\partial t} dt \right| \]
\[ = \left| \int_0^t \frac{\partial A(s)}{\partial s} ds - \int_0^T \frac{\partial A(t)}{\partial t} dt \right| \]
\[ = \left| \int_t^T \frac{\partial A(s)}{\partial s} ds \right| \]
\[ \leq \int_t^T C_0 ds \]
\[ = (T - t)C_0. \]

It follows that \( \lim_{t \to T} |A(t) - A_T| \to 0 \), which implies that \( \{A(t)\} \) converges to \( A_T \) uniformly. This in turn implies that \( A_T \) is continuous.

The next step is to show that the limit connection \( A_T \) is smooth. We have

\[ \nabla_{(p)}^M (A_T) = \nabla_{(p)}^M (A(0)) + \int_0^T \frac{\partial \nabla_{(p)}^M A(t)}{\partial t} dt \]
\[ = \nabla_{(p)}^M A(0) + \nabla_{(p)}^M \int_0^T \frac{\partial A(t)}{\partial t} dt \]
\[ = \nabla_{(p)}^M A(0) + \int_0^T \nabla_{(p)}^M \frac{\partial A(t)}{\partial t} dt \]

where we are able to take \( \nabla_{(p)}^M \) into the integral, because \( \frac{\partial A(t)}{\partial t} \) has uniformly bounded derivatives, by the assumption of the theorem. It follows that \( A_T \) is smooth.

Finally, we show that \( \{A(t)\} \) converges to \( A_T \) in \( C^m \). We compute

\[ |\nabla_{(p)}^M (A_T) - \nabla_{(p)}^M (A(t))| = \left| \nabla_{(p)}^M (A(0)) + \int_0^T \nabla_{(p)}^M \frac{\partial A(t)}{\partial t} dt - \nabla_{(p)}^M (A(t)) \right| \]
\[ = \left| - \int_0^t \nabla_{(p)}^M \frac{\partial A(s)}{\partial s} ds + \int_0^T \nabla_{(p)}^M \frac{\partial A(t)}{\partial t} dt \right| \]
\[ \leq (T - t)C_p. \]

It follows that as \( t \to T \), \( \nabla_{(p)}^M (A(t)) \to \nabla_{(p)}^M (A_T) \) uniformly. This proves the result.

\[ \square \]

We have an analogous proposition for time dependent spinor fields. As the proof is exactly the same as that given above, we won’t give the proof.

**Proposition 6.26.** Let \( \phi(t) \) denote a sequence of time dependent spinor fields, and \( A(t) \) denote a sequence of time dependent unitary connections, defined on some time interval \([0, T]\), with \( T < \infty \). Suppose we have uniform bounds

\[ \sup_{M \times [0, T]} \left| \nabla_{(p)}^M \frac{\partial \phi(t)}{\partial t} \right| \leq C_p \]

for some positive constants \( C_p \).

Then \( \lim_{t \to T} \phi(t) \) exists, is smooth, and the sequence \( \{\phi(t)\} \) converges to this limit spinor in every \( C^m \)-norm, \( m \geq 0 \).

With these two propositions we can now show that the only obstruction to long time existence is curvature blow up.
Theorem 6.27. Suppose \((\phi(t), A(t))\) is a solution to the higher order Seiberg-Witten flow on the maximal time interval \([0, T]\), with \(T < \infty\). Then
\[
\sup_{M \times [0, T]} |F_{A(t)}| = \infty
\]

**Proof.** Suppose on the contrary that \(\sup_{M \times [0, T]} |F_{A(t)}| \leq C < \infty\).

Then by proposition 6.22 we have uniform derivative bounds
\[
\sup_{M \times [0, T]} \left| \nabla^{(i)}_{A(t)} P_{A(t)} \right| \leq C_i
\]
(6.6.1)
\[
\sup_{M \times [0, T]} \left| \nabla^{(i)}_{A(t)} \phi(t) \right| \leq C_i.
\]
(6.6.2)

Looking at the second equation in the higher order Seiberg-Witten flow, we have
\[
\frac{\partial A}{\partial t} = (-1)^{k+1} d^* \Delta^{(k)} F_A - \sum_{v=0}^{2k-1} P_1^{(v)} F_A - 2i Im \left( \sum_{i=0}^{k} C_i \nabla^{(i)}_{A(t)} (\nabla^{(k)}_{A} \nabla_A \phi, \nabla^{(k-1)}_{A} \phi) \right).
\]

The two terms \((-1)^{k+1} d^* \Delta^{(k)} F_A\) and \(\sum_{v=0}^{2k-1} P_1^{(v)} F_A\) both have uniform derivative bounds coming from (6.6.1). Furthermore, appealing to lemma 6.6 and (6.6.2), we see that the term \(\nabla^{(i)}_{A(t)} (\nabla^{(k)}_{A} \nabla_A \phi, \nabla^{(k-1)}_{A} \phi)\) also has uniform derivative bounds. It follows that \(\frac{\partial A}{\partial t}\) has uniform derivative bounds, and hence by proposition 6.25 we can define a smooth limit connection \(\lim_{t \to T} A(t) = A(T)\).

Looking at the first equation in the higher order Seiberg-Witten flow, we have
\[
\frac{\partial \phi}{\partial t} = -\nabla^{(k+1)}_{A} \nabla^{(k+1)}_{A} \phi - \frac{1}{4} (S + |\phi|^2) \phi.
\]

From (6.6.2), we see that the two terms on the right of the above equation have uniform derivative bounds. This implies \(\frac{\partial \phi}{\partial t}\) has uniform derivative bounds. Applying proposition 6.26, we get a smooth limiting spinor \(\lim_{t \to T} \phi(t) = \phi(T)\).

We can then apply short time existence with the initial condition \((\phi(T), A(T))\), and extend the solution \((\phi(t), A(t))\) past the time \(T\). However, this contradicts the maximality of \(T\). Therefore we must in fact have that \(\sup_{M \times [0, T]} |F_{A(t)}| = \infty\), which completes the proof.

\(\square\)

7. Finite time solutions

In the previous section, theorem 6.27 showed us that the obstruction to extending a solution past the maximal time is the curvature \(F_A\) blowing up. In this section, we want to show that under such circumstances one can still obtain information about the singularity present in the flow through a blow up solution.

We start with some basic properties on scaling a connection and a spinor field.

**Definition 7.1.** Given a time dependent connection \(\nabla_t\), with connection coefficient \(\Gamma\). We define the \(\lambda\)-scaled connection, \((\nabla_t)^\lambda\), to be the connection with connection coefficient \(\Gamma^\lambda\), defined by
\[
\Gamma^\lambda(x, t) = \lambda \Gamma(\lambda x, \lambda^{2(k+1)} t).
\]

**Definition 7.2.** Given a time dependent spinor field \(\phi\), we define the \(\lambda\)-scaled spinor field \(\phi^\lambda\) by, \(\phi^\lambda(x, t) = \lambda \phi(\lambda x, \lambda^{2(k+1)} t)\).
These definitions will be employed while working in a local coordinate chart, and in cases were \( \lambda \) is sufficiently small, so that the dilation \( \lambda x \) makes sense within the chart.

We will primarily focus on \( \lambda \)-scaled unitary connections, \( A^\lambda \), on the line bundle \( \mathcal{L}^2 \), where \( A^\lambda(x,t) = \lambda A(\lambda x, \lambda^{2(k+1)} t) \). Recall that associated to a unitary connection \( A \) on \( \mathcal{L}^2 \), we had the connection \( \nabla_A \) defined on the spinor bundle. Locally, \( \nabla_A = d + (\omega + A I) \), where \( \omega \) comes from the Levi-Civita connection on \( M \). Given the scaled connection, \( A^\lambda \), the connection \( \nabla_{A^\lambda} \) will denote the scaled version of \( \nabla_A \). We are abusing notation slightly as locally, \( \nabla_{A^\lambda} = d + (\lambda^\omega + A^\lambda I) \), and we point out to the reader that this is not equal to \( d + (\omega + A^\lambda I) \). Furthermore, we will also be dealing with scaled versions of the Levi-Civita connection. We will denote the \( \lambda \)-scaled Levi-Civita connection by \( \nabla_{A^\lambda} \).

Observe that because \( F_{A^\lambda} = dA^\lambda \), we have that \( F_{A^\lambda}(x,t) = \lambda^2 F_A(\lambda x, \lambda^{2(k+1)} t) \), so the curvature scales quadratically in \( \lambda \).

We now want to understand how the derivative terms in the higher order Seiberg-Witten equations scale. We start by computing time derivatives of the scaled connection and spinor field

\[
\frac{\partial A^\lambda}{\partial t}(x,t) = \lambda^{2k+3} \frac{\partial A}{\partial t}(\lambda x, \lambda^{2(k+1)} t) \\
\frac{\partial \phi^\lambda}{\partial t}(x,t) = \lambda^{2k+3} \frac{\partial \phi}{\partial t}(\lambda x, \lambda^{2(k+1)} t).
\]

We want to show that this scaling by \( \lambda^{2k+3} \) holds for the derivative terms on the right hand side of the higher order Seiberg-Witten flow.

The term \( \nabla^{s(k+1)}_{A^\lambda} \nabla^{(k+1)}_{A^\lambda} \phi \) scales as \( \nabla^{s(k+1)}_{A^\lambda} \nabla^{(k+1)}_{A^\lambda} \phi^\lambda(x,t) = \lambda^{2k+3} \nabla^{s(k+1)}_{A} \nabla^{(k+1)}_{A} \phi(\lambda x, \lambda^{2(k+1)} t) \).

We know that the term

\[
( -1 )^{k+1} d^* \Delta^{(k)}_{A^\lambda} F_A - \sum_{v=0}^{2k-1} P_{1}^{(v)} [ F_A ]
\]

can be written as \( d^* \nabla^{s(k)}_{M^\lambda} \nabla^{(k)}_{M^\lambda} F_A \) (see (3.2.3)). The term \( d^* \nabla^{s(k)}_{M^\lambda} \nabla^{(k)}_{M^\lambda} F_A \) scales as

\[
d^* ( \nabla^{s(k)}_{M^\lambda} ) ( \nabla^{(k)}_{M^\lambda} ) (x,t) = \lambda^{2k+3} d^* ( \nabla^{s(k)}_{M} ) ( \nabla^{(k)}_{M} ) (x,t) = \lambda^{2k+3} \lambda^{2(k+1)} t.
\]

It follows that

\[
( -1 )^{k+1} d^* ( \Delta^{(k)}_{A^\lambda} F_A ) - \sum_{v=0}^{2k-1} P_{1}^{(v)} [ F_A ] = \lambda^{2k+3} ( -1 )^{k+1} d^* ( \Delta^{(k)}_{A} F_A ) - \sum_{v=0}^{2k-1} P_{1}^{(v)} [ F_A ].
\]

Finally, if we look at the term \( \nabla^{s(i)}_{M^\lambda} ( \nabla^{(k)}_{A^\lambda} \phi, \nabla^{(k-i)}_{A^\lambda} \phi ) \) it is easy to see that

\[
\langle ( \nabla^{s} M^\lambda ) ( \nabla^{(k)} A^\lambda \phi, \nabla^{(k-i)} A^\lambda \phi ) \rangle = \lambda^{2k+3} \lambda^{2(k+1)} t.
\]

From this discussion, we immediately get the following proposition

**Proposition 7.3.** Let \( (\phi(t), A(t)) \) be a solution to the higher order Seiberg-Witten flow on \([0, T)\). Then \( (\phi^\lambda, A^\lambda) \) is a solution to the following scaled system

\[
\frac{\partial \phi^\lambda}{\partial t} = - \nabla^{s(k+1)}_{A^\lambda} \nabla^{(k+1)}_{A^\lambda} \phi^\lambda - \frac{\lambda^{2k}}{4} (\lambda^2 S + |\phi^\lambda|^2) \phi^\lambda \\
\frac{\partial A^\lambda}{\partial t} = ( -1 )^{k+1} d^* ( \Delta^{(k)}_{A^\lambda} F_A ) - \sum_{v=0}^{2k-1} P_{1}^{(v)} [ F_A ] - 2 i \lambda^k m \left( \sum_{i=0}^{k} C_i ( \nabla^{s(i)}_{M^\lambda} ) ( \nabla^{(k)}_{A^\lambda} \phi^\lambda, \nabla^{(k-i)}_{A^\lambda} \phi^\lambda ) \right).
\]

on the time interval \([0, \frac{1}{\lambda^{2(k+1)} T})\).
We will call the above scaled system a generalised higher order Seiberg-Witten flow.

We will now show that in the case that the curvature form is blowing up, as one approaches the maximal time, a blow up limit can be extracted. The proof of the theorem will closely follow the proof of proposition 3.24 in [8], and the proof of lemma 4.6 in [5].

**Theorem 7.4.** Let \((\phi(t), A(t))\) be a solution to the higher order Seiberg-Witten flow, on some maximal time interval \([0, T]\), with \(T < \infty\). Then there exists a blow up sequence \((\phi^i(t), A^i(t))\), that converges pointwise, up to gauge transformations, to a smooth solution \((\phi^\infty(t), A^\infty(t))\) of the higher order Seiberg-Witten flow, with domain \(\mathbb{R}^n \times (-\infty, 0)\).

**Proof.** By theorem 6.27, we must have that \(\lim_{t \to T} \sup_M |F_A| = \infty\). Therefore, we can choose a sequence of times \(t_i\), such that \(t_i \to T\), and a sequence of points \(x_i\), such that

\[ |F_{A(t_i)}(x_i)| = \sup_{M \times [0, t_i]} |F_A| \]

By compactness of \(M\), we can assume \(x_i \to x_\infty\).

Fix a chart \(U\) about \(x_\infty\) and, without loss of generality, assume that \(U\) gets mapped to \(B_1(0) \subseteq \mathbb{R}^n\), with \(x_\infty\) mapping to 0. We will be considering the behaviour of the solution for points \((x_i, t_i)\) for \(i\) sufficiently large. Therefore, using this chart, we can assume the points \(x_i\) are in \(\mathbb{R}^n\), and are converging to 0.

We define

\[ A^i(x, t) = \lambda_i^{\frac{1}{2k+1}} A(\lambda_i^{\frac{1}{2k+1}} x + x_i, \lambda_i t + t_i) \]

\[ \phi^i(x, t) = \lambda_i^{\frac{1}{2k+1}} \phi(\lambda_i^{\frac{1}{2k+1}} x + x_i, \lambda_i t + t_i) \]

where \(\lambda_i\) are positive numbers to be determined. The domain of \((\phi^i, A^i)\) is \(B_{\lambda_i^{\frac{1}{2k+1}}}(x_i) \times [\frac{-t_i}{\lambda_i}, \frac{T-t_i}{\lambda_i}]\). Furthermore, it easy to see that the pair \((\phi^i, A^i)\) satisfy a generalised higher order Seiberg-Witten flow, with scale factor \(\lambda_i^{\frac{1}{2k+1}}\). In fact, by defining \((\phi^i, A^i)\) for times \(t \leq \frac{-t_i}{\lambda_i}\) by \(A^i(\frac{-t_i}{\lambda_i}, x_i)\), and similarly for \(\phi^i\), we can extend the domain of \((\phi^i, A^i)\) to \(B_{\lambda_i^{\frac{1}{2k+1}}}(x_i) \times (-\infty, \frac{T-t_i}{\lambda_i}]\).

We then observe that, \(F^i(x, t) = F_{A^i}(x, t) = \lambda_i^{\frac{-1}{2k+1}} F_A(\lambda_i^{\frac{1}{2k+1}} x + x_i, \lambda_i t + t_i)\), which implies

\[
\sup_{t \in [\frac{-t_i}{\lambda_i}, \frac{T-t_i}{\lambda_i}]} |F^i(x, t)| = |\lambda_i^{\frac{-1}{2k+1}}| \sup_{t \in [\frac{-t_i}{\lambda_i}, \frac{T-t_i}{\lambda_i}]} |F_A(\lambda_i^{\frac{1}{2k+1}} x + x_i, \lambda_i t + t_i)| \\
= |\lambda_i^{\frac{-1}{2k+1}}| \sup_{t \in [0, t_i]} |F_A(x, t)| \\
= |\lambda_i^{\frac{-1}{2k+1}}| |F(x_i, t_i)|.
\]

Therefore, defining \(\lambda_i = |F_A(x_i, t_i)|\) we find

\[
\sup_{t \in [\frac{-t_i}{\lambda_i}, 0]} |F_{A^i}(x)| = 1. \quad (7.0.3)
\]

We thus see that the sequence \(A^i\) represents a blow up sequence. We now have to show that we can extract an actual blow up limit. Before we show how to do this, we point out to the reader that, by definition, \(\frac{1}{\lambda_i^{\frac{1}{2k+1}}} \to \infty\), as \(i \to \infty\). This means that the domains, \(B_{\lambda_i^{\frac{-1}{2k+1}}}(x_i) \times (-\infty, \frac{T-t_i}{\lambda_i}]\), will expand to \(\mathbb{R}^n \times (-\infty, 0)\).
We also observe that at each time \( t \leq 0 \) in the domain of definition of \( F_{A'} \), we have uniform derivative bounds. To see this, take \( y \in \mathbb{R}^n \), and take \( i \) large enough so that \( B_{2r}(y) \times [t-1,t] \) is in the domain of definition of \( (\phi^i, A^i) \) for some \( r > 0 \). Then take a bump function \( \gamma \), supported in \( B_{2r}(y) \), so that \( \gamma = 1 \) on \( B_r(y) \). Since \( \sup |F_{A'(t)}| = 1 \), where the sup is taken over the domain of definition of \( A^i \), we have that \( \sup |\gamma^s F_{A'(t)}| \leq 1 \). Applying proposition 6.22, we then see that there exists \( C_l \) so that

\[
\sup_{B_r(y)} |\nabla^{(l)} F_{A'(t)}| \leq \sup_{B_{2r}(y)} |\gamma^{s/2} \nabla^{(l)} F_{A'(t)}| \leq C_l.
\] (7.0.4)

If we had another point \( \tilde{y} \), then we could apply the same argument to \( B_{2r}(\tilde{y}) \), and obtain the exact same uniform derivative bound. This means we have uniform bounds for \( |\nabla^{(l)} F_{A'(t)}| \) for all \( i \), and all \( t \).

Like we did for the curvature above, we want to show that we have derivative bounds for the connections \( A^i \). With these bounds, we can then apply the Arzela-Ascoli theorem to extract a limit connection, which will then serve as the blow up limit. In order to do this, we will need to change gauge, obtain the bounds in that gauge, and then transform back.

Before we explain how to put the above remark into action, let us explain what is going on with the spinor fields \( \phi^i \). We know that \( \phi^i(t) \) is uniformly bounded along the flow by proposition 5.1. Therefore, since \( \lambda_i \to 0 \) as \( i \to \infty \), it follows that \( \phi^i \to 0 \) as \( i \to \infty \). What this means is that, any blow up limit we can obtain from the blow up sequence \( (\phi^i, A^i) \) will necessarily have the limit spinor field being 0. Hence, we need only deal with \( A^i \) when we want to extract a blow up limit.

Fix \( r > 0 \) sufficiently large, fix \( \tau < 0 \), and \( m \in \mathbb{N} \). Then for all \( i \) sufficiently large, we have that the domain of \( A^i \) contains \( B_{2r+m} \times [\tau - m, -\frac{1}{m}] \). The \( F_{A'(t)} \) are all uniformly bounded by 1. Therefore, we can find some \( \delta > 0 \) such that

\[
||F_{A'(t)}||_{L^{n/2}(B_\delta(y))} \leq \kappa_n
\]

where \( i \) is taken so that \( B_\delta(y) \) is in the domain of \( A^i \), and \( y \in \mathbb{R}^n \) is in the domain of \( A^i \). The constant \( \kappa_n \) comes from the statement of the Coloumb gauge theorem, see theorem 10.10. We then map

\[
B_\delta(y) \ni x \to \frac{x - y}{\delta} \in B_1(0)
\]
i.e. we translate \( B_\delta(y) \) to \( B_\delta(0) \) and then scale by \( \frac{1}{\delta} \). What we want to do is use the Coloumb gauge theorem to get good bounds on the \( A^i \). The problem is that the Coloumb gauge theorem, theorem 10.10, requires a curvature bound of the above type on \( B_1(0) \). Therefore, we need to scale everything by \( \frac{1}{\delta} \).

We define \( \delta \)-scaled connections \( \tilde{A}^i(x, t) = \delta A^i(\delta x + y, \delta^{2(k+1)} t) \) for \( x \in B_1(0) \). It is easy to see then that the associated curvature \( F_{\tilde{A}^i} \) satisfy the bound

\[
||F_{\tilde{A}^i(t)}||_{L^{n/2}(B_1(0))} \leq \kappa_n.
\]

Also note, that if we let \( \tilde{\phi}^i \) denote the \( \delta \)-scaled spinor fields, then the pair \( (\tilde{\phi}^i, \tilde{A}^i) \), satisfy a generalised higher order Seiberg-Witten flow, with scaling term \( \delta \). Furthermore, \( (\tilde{\phi}^i, \tilde{A}^i) \) is defined on \( B_1(0) \times [\frac{\tau - m}{\delta^{2(k+1)}}, \frac{1}{\delta^{2(k+1)} m}] \).

We then apply the Coloumb gauge theorem, theorem 10.10, to the connections \( \tilde{A}^i(x, t) \), where \( t \in [\frac{\tau - m}{\delta^{2(k+1)}}, \frac{1}{\delta^{2(k+1)} m}] \). In doing so, we get connections \( \tilde{A}^i(x, t) \) defined on \( B_1(0) \), and by (2) of the Coloumb gauge theorem, we have that there exists \( c_n \) such that

\[
||\tilde{A}^i(x, t)||_{C^{p-1}(B_1(0))} \leq c_n(t)
\]

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where \( p \geq n/2 \). By compactness of the interval \([\frac{r-m}{\delta (k+1)}, \frac{r-1}{\delta (k+1)m}]\), we can get a bound of the form
\[
\sup_{B_1(0) \times \left[ \frac{r-m}{\delta (k+1)}, \frac{r-1}{\delta (k+1)m} \right]} \left| \tilde{A}_i(x, t) \right| \leq c_n(\tau - m).
\]

Note that, because the curvature corresponding to a unitary connection is invariant under gauge transformations, we have that the curvature corresponding to \( \tilde{A}_i \) is equal to \( F_{\tilde{A}_i} \).

Since \( \tilde{A}_i \) is just a scaled version of \( A_i \) we have that \( F_{\tilde{A}_i} \) is just a \( \delta^2 \) scaling of \( F_{A_i} \). This means that the curvatures of \( \tilde{A}_i \) also have uniform derivative bounds, just like \( F_{A_i} \) did. In this gauge, we denote the spinor fields by \( \tilde{\Phi}_i \).

We now want to map \( B_1(0) \) back to \( B_\delta(y) \) by mapping \( B_1(0) \ni x \to \delta x + t \in B_\delta(y) \), and then scale \( \tilde{A}_i \) by defining
\[
A_i(t, x) = \frac{1}{\delta} \tilde{A}_i\left(\frac{x - y}{\delta}, \delta^{-2(k+1)}t\right).
\]

We then have
\[
\sup_{B_\delta(y) \times \left[ \tau - m, \frac{1}{m} \right]} \| A_i \| \leq \delta c_n(\tau - m).
\]

We denote the \( \delta \) dilated \( \Phi_i \), by \( \Phi^i \).

Note that because of its construction, \( A_i \) is gauge equivalent to \( A_i \), and \( \Phi^i \) is gauge equivalent to \( \phi^i \). Therefore, the pair \( (\Phi^i, A^i) \) satisfy a generalised higher order Seiberg-Witten flow.

The connections \( A_i \) are defined on \( B_\delta(\delta) \). However, taking any other point \( \tilde{y} \), we can run the same argument above and obtain a connection satisfying the same bounds on \( B_\delta(\tilde{y}) \).

What this means is that, if we take a collection of points \( y_1, \ldots, y_n \) so that
\[
B_{2r+m}(0) \supset \bigcup_{i=1}^n B_\delta(y_i) \supset B_{r+m}(0)
\]

We then obtain connections \( A_{1,i}, \ldots, A_{n,i} \) on each \( B_\delta(y_i) \). As the Coloumb gauge is defined on \( B_{2r+m}(0) \), we can then apply theorem 10.11, to obtain a single \( A_i \) that is defined on all of \( B_{r+m}(0) \).

This means we have a sequence of connections \( A_i \) admitting uniform \( C^{p,1} \) bounds, for \( p \geq n/2 \), on \( B_{r+m} \times [\tau - m, \frac{1}{m}] \). We now want to show that for each \( m \), we can extract a limit connection, defined on \( B_{r+m} \times [\tau - m, \frac{1}{m}] \).

Fix \( p \geq n/2, m \in \mathbb{N} \), and \( 0 < \alpha < 1 \). From the fact that we have uniform \( C^{p,1} \) bounds for \( A^i \), and the fact that \( \alpha < 1 \). We see that if we apply the Arzela-Ascoli theorem, we can extract a limit \( A_p^{m,\infty} \), which is defined on \( B_{r+m}(0) \times [\tau - m, \frac{1}{m}] \).

If we took another \( q > p \geq n/2 \), and applied the above to obtain limits \( A_q^{m,\infty} \) and \( A_p^{m,\infty} \). Then we would in fact have that \( A_q^{m,\infty} = A_p^{m,\infty} \), as \( C^{q,1} \subseteq C^{p,1} \) as topological spaces. Therefore, applying the above for each \( p \geq n/2 \), we get a limit \( A^{m,\infty} \) in \( C^{\infty} \), defined on \( B_{r+m} \times [\tau - m, \frac{1}{m}] \), for each \( m \in \mathbb{N} \). The final step is to show that we can extract a limit defined on all of \( \mathbb{R}^3 \times (-\infty, 0) \). In order to do this, we apply the same procedure as above, but then extract a diagonal limit.

We start by denoting the sequence \( A^i \) on \( B_{r+m} \times [\tau - m, \frac{1}{m}] \) by \( A^{m,i} \). If we fix \( p \geq n/2 \) and \( 0 < \alpha < 1 \), Arzela-Ascoli tells us that, passing to a subsequence if necessary, \( A^{m,i} \to A^{m,\infty}_p \). Doing this for each \( p \geq n/2 \), we obtain a limit \( A^{m,i} \to A^{m,\infty} \) in \( C^{\infty} \).
We then consider the diagonal sequence: \( A^{1,1}, A^{2,2}, \ldots, A^{m,m}, \ldots \). This sequence converges on any compact subset of \( \mathbb{R}^n \times (-\infty, 0) \) to a connection \( A^\infty \), which is the required blow up limit of the \( A^i \).

We remind the reader that we already handled the structure of the blow up limit of the \( \phi^i \). Namely, we saw that the limit was just 0. Together with the above, we see that our blow up limit is \( (0, A^\infty) \). It is also easy to see that this blow up limit satisfies the higher order Seiberg-Witten flow on \( \mathbb{R}^n \times (-\infty, 0) \).

We also point out that, if we let \( F^\infty \) denote the curvature associated to \( A^\infty \), then by (7.0.3) we have

\[
\lim_{t \to 0} \sup_{\mathbb{R}^n} |F^\infty(x, t)| = 1
\]

and that by (7.0.4), \( F^\infty \) has uniform derivative bounds.

\( \Box \)

8. Long time existence results

We prove long time existence for solutions to the flow in sub-critical dimensions, and then show that in the critical dimension, long time existence is obstructed by an \( L^{k+2} \) curvature concentration phenomenon.

8.1. Long time existence for subcritical dimensions. We start with the following proposition.

**Proposition 8.1.** Let \( \dim M = n < 2p \), and suppose \((\phi(t), A(t))\) is a solution to the higher order Seiberg-Witten flow, on \([0, T]\) where \( T \leq \infty \). Assume \( F_{A(t)} \in L^\infty([0, T); L^p(M)) \), then \( F_{A(t)} \in L^\infty([0, T); L^\infty(M)) \). In particular, \( T = \infty \).

**Proof.** So as to obtain a contradiction, assume \( \sup_{[0,T]} ||F_A||_{L^\infty} = \infty \). As we did in theorem 7.4, we can then construct a blowup sequence \((\phi^i, A^i)\), with blow up limit \((\phi^\infty, A^\infty)\). The curvature of \( A^i \) was given by

\[
F_{A^i} = \lambda_i^{\frac{1}{k+2}} F(\lambda_i^{\frac{1}{2(k+1)}} x_i + x_i, \lambda_i t + t_i)
\]

where \( \lambda_i = |F(x_i, t_i)|^{-(k+1)} \).

We also know, by (7.0.4), that the limit curvature \( F^\infty \) satisfies

\[
||F^\infty||_{L^p} \neq 0.
\]

Applying Fatou’s lemma we have

\[
||F^\infty||_{L^p} \leq \liminf_{i \to \infty} ||F_{A^i}||_{L^p} \leq \lim_{i \to \infty} \lambda_i^{\frac{2p-n}{2(k+2)}} ||F_A||_{L^p}.
\]

We know that, \( \lambda_i \to 0 \) as \( i \to \infty \). Furthermore, because \( 2p > n \), by assumption, we have that the right hand side of the above inequality goes to zero. But this is a contradiction. \( \Box \)

Using this result, we can prove long time existence in the sub-critical dimension i.e. for \( \dim M < 2(k+2) \).
**Theorem 8.2.** Let \((\phi(0), A(0))\) be a given initial condition. Suppose \(\dim M < 2(k + 2)\). Then there exists a unique solution \((\phi(t), A(t))\), with initial condition \((\phi(0), A(0))\), that exists for all time \(t > 0\).

**Proof.** By short time existence, we have that a unique solution \((\phi(t), A(t))\) exists, with initial condition \((\phi(0), A(0))\), on some maximal time interval \([0, T)\). If \(T = \infty\), there is nothing to prove, so assume \(T < \infty\).

By the Sobolev embedding theorem, we have that \(W^{k,2}\) embeds continuously into \(L^p\) if \(\frac{1}{2} - \frac{k}{n} < p\). If also add the condition that \(\frac{n}{2} < p\), then we must have \(n < 2(k + 2)\).

Applying the Sobolev embedding theorem we get

\[
\|F_t\|_{L^p} \leq C_{k,2} \left( \sum_{j=0}^{k} \|\nabla^{(j)} F_A\|_{L^2}^2 \right) 
\leq CC_{k,2} \left( \|F_A\|_{L^2}^2 + \|\nabla^{(k+1)} F_A\|_{L^2}^2 \right),
\]

where to obtain the second inequality, we have applied lemma 10.3.

By lemmas 5.2 and 5.3, we know that the Seiberg-Witten energy and the higher order Seiberg-Witten energy are bounded along the flow. We then have that the left hand side of the above inequality is bounded along the flow.

Proposition 8.1 then implies, \(F_A(t) \in L^\infty([0, T); L^\infty(M))\). This means we can extend this solution past \(T\), but this contradicts maximality of \(T\). Therefore we must in fact have that \(T = \infty\).

\[\square\]

The above theorem can be seen as an analogue of the first part of theorem 7.8 in [10], and theorem \(\Lambda\) in [8], for the case of these higher order Seiberg-Witten functionals.

8.2. **Curvature concentration in the critical dimension.** As was seen in the above subsection, long time existence for the sub-critical dimensions is quite straightforward to prove. Unfortunately, the above technique breaks down in the critical dimension. The main issue, as we will see shortly, is that in the critical dimension curvature can start to concentrate in smaller and smaller balls, and this in turn obstructs one from being able obtain a solution for all time.

**Proposition 8.3.** Suppose \(\dim M = n = 2p\), and \((\phi(t), A(t))\) is a solution to the higher order Seiberg-Witten flow, on \([0, T)\), with \(T < \infty\). If \(x_0 \in M\) is such that, \(\limsup_{t \to T} |F_A(x_0, t)| = \infty\).

Then there exists some \(\epsilon > 0\) such that, for all \(r > 0\) we have

\[
\lim_{t \to T} \|F_A\|_{L^p(B_r(x_0))} \geq \epsilon.
\]

**Proof.** As in the proof of theorem 7.4, we pick a sequence of times \(t_i\) so that \(\sup_{M \times [0, t_i]} |F_A| = |F_A(x_0, t_i)|\), with \(t_i \to T\).

We then let \((\phi^i, A^i)\) be the associated blowup sequence, and \((\phi^\infty, A^\infty)\) the associated blowup limit, defined on \(\mathbb{R}^n \times (-\infty, 0)\). Recall from theorem 7.4, we saw that \(\lim_{t \to 0} |F^\infty(0, t)| = 1\). This means that we can find a \(\delta > 0\) such that, for \((x, t) \in B_\delta(0) \times (-\delta, 0)\) we have

\[
|F^\infty(x, t)| \geq \Lambda
\]
where \( \Lambda \) is any constant slightly less than 1, for example take \( \Lambda = 1 - \lambda \) for \( \lambda > 0 \) sufficiently small.

Using this we find

\[
\lim_{t \to 0} \|F^\infty\|_{L^p(B_\delta(0))}^p = \lim_{t \to 0} \int_{B_\delta(0)} |F^\infty(x,t)|^p dx \\
\geq \Lambda^p \text{Vol}(B_\delta(0)).
\]

Now, fix \( r > 0 \). If \( \lim_{t \to T} \|F_A\|_{L^p(B_r(x_0))} = \infty \), then there is nothing to prove and we are done. Therefore, assume \( \lim_{t \to T} \|F_A\|_{L^p(B_r(x_0))} < \infty \).

We compute

\[
\|F^\infty(x,t)\|_{L^p(B_\delta(0))}^p = \int_{B_\delta(0)} |F^\infty(x,t)|^p dx \\
= \int_{B_\delta(0)} \lim_{i \to \infty} |F_A(x,t)|^p dx \\
= \lim_{i \to \infty} \int_{B_\delta(0)} \left| \frac{1}{\lambda_i^{2(k+1)}} F_A \left( \frac{1}{\lambda_i^{2(k+1)}} x + x_0, \lambda_i t + t_i \right) \right|^p dx \\
= \lim_{i \to \infty} \int_{B_{\delta \lambda_i^{1/(2k+2)}}(x_0)} \left| \frac{2^{-n-k}}{\lambda_i^{2(k+1)}} F(z, \lambda_i t + t_i) \right|^p dz \\
\leq \lim_{i \to \infty} \int_{B_r(x_0)} \left| F(z, \lambda_i t + t_i) \right|^p dz \\
= \lim_{i \to \infty} \|F_A\|_{L^p(B_r(x_0))}^p.
\]

Therefore, we obtain

\[
\lim_{t \to 0} \|F^\infty(x,t)\|_{L^p(B_\delta(0))}^p \leq \lim_{i \to \infty} \|F_A\|_{L^p(B_r(x_0))}^p
\]

which in turn gives

\[
\Lambda(\text{Vol}(B_\delta(0)))^{1/p} \leq \lim_{t \to T} \|F_A\|_{L^p(B_r(x_0))}^p.
\]

Taking \( \epsilon = \Lambda(\text{Vol}(B_\delta(0)))^{1/p} \) finishes the proof.

We can now prove our second main theorem.

**Theorem 8.4.** Let \((\phi(0), A(0))\) be an initial condition, and suppose \( \dim M = 2(k+2) \).

Then

1. there exists a unique solution to the higher order Seiberg-Witten flow, on a maximal time interval \([0, T]\), with \( T \leq \infty \).
2. If \( T < \infty \), then \( \limsup_{t \to T} \|F_A\|_{L^\infty} = \infty \), and there exists \( x_0 \in M \) satisfying the following \( L^{k+2} \)-curvature concentration phenomenon: There exists \( \epsilon > 0 \), such that for all \( r > 0 \) we have

\[
\lim_{t \to T} \|F_A\|_{L^{k+2}(B_r(x_0))} \geq \epsilon.
\]

Moreover, the number of points where such a concentration can occur is finite.
Proof. The proof of 1. follows from short time existence. The first part of 2. follows from theorem 6.27, and the concentration of curvature phenomenon follows from proposition 8.3. Therefore, we need only prove that such a phenomenon can take place at most at a finite number of points.

To see this let \( p = k + 2 \), and apply the Sobolev embedding theorem to get an embedding \( W^{k,2} \subseteq L^p \). Then

\[
\| F_A \|_{L^p} \leq C_{k,2} \left( \sum_{j=0}^{k} \| \nabla_M^{(j)} M \|_{L^2} \right)
\]

\[
\leq S_{k,2} C (\| F_A \|_{L^2} + \| \nabla_M^{(k+1)} F_A \|_{L^2})
\]

where \( C_{k,2} \) denotes the constant in the Sobolev inequality, and where we get the second inequality by applying lemma 10.3.

The right hand side of the above inequality is bounded in time by lemma 5.3, which in turn implies the left hand side is bounded as \( t \to T \). The result follows.

\[\square\]

The above theorem shows that in the critical dimension, long time existence is obstructed by the possibility of the curvature form concentrating in smaller and smaller balls. This is analogous to what Struwe observed for the Yang-Mills flow in dimension four (see theorem 2.3 in [18]), and what Kelleher observed for the higher order Yang-Mills flow in the critical dimension (see theorem B in [8]).

9. Concluding Remarks

Theorem 8.2 tells us that, provided the order of derivatives, appearing in the higher order Seiberg-Witten functional, is sufficiently large, solutions to the associated gradient flow do not hit any finite time singularities. On the other hand, theorem 8.4 tells us that if the dimension of \( M \) is equal to the critical dimension, then there is a possibility of finite time singularities, due to the \( L^{k+2} \) energy of the curvature form concentrating in smaller and smaller balls. The theorem in fact proves that the points where this energy concentration can happen, must be finite in number. The question then remains, is it possible that there are in fact no such points?

In the case of the Seiberg-Witten flow, the critical dimension is dimension four. Hong and Schrabrun show that if long time existence is obstructed then again it is due to an energy concentration phenomenon, but this time the energy is an \( L^2 \) energy. Using a rescaling argument, similar to what we did in 7.4, they are able to show that one can extract a limiting curvature form. They then show, by using an \( L^2 \) energy estimate, that this implies the limiting curvature form must be harmonic. Using the mean value formula for harmonic forms, they are then able to derive a contradiction, and show that the \( L^2 \) energy of the curvature form cannot concentrate in smaller and smaller balls.

The key point to note is that for them, everything is taking place in \( L^2 \). Therefore, the \( L^2 \) energy estimates they derive are robust enough to obtain information about a limiting curvature form. In our case, we have that curvature is potentially concentrating in \( L^{k+2} \). This fact, that in these higher order flows curvature concentration takes place in higher \( L^p \) spaces, makes the approach taken by Hong and Schrabrun inadequate for these higher order flows. It becomes a challenge as to whether one can obtain suitable \( L^{k+2} \) estimates, that could possibly lead to ruling out curvature concentration in the critical dimension.
10. Appendix

In the following appendix, we gather together various theorems from other resources that we will be using in the paper.

10.1. Interpolation inequalities. The following interpolation results will be used in section 6, when proving local derivative estimates.

We will need the following theorem, which is theorem 5.4 in [9].

**Theorem 10.1.** Let \( \phi \) be a section of a vector bundle \( E \) over \( M \), with connection \( \nabla \), and let \( \gamma \) be a bump function on \( M \). For \( k \in \mathbb{N} \), \( 1 \leq i \leq k \) and \( s \geq 2k \) we have the identity

\[
\left( \int_M |\nabla^{(i)} \phi|^{2k} \gamma^s d\mu \right)^{\frac{1}{2k}} \leq C ||\phi||_{L^\infty} \left( \left( \int_M |\nabla^{(k)} \phi|^{2s} d\mu \right)^{\frac{1}{2k}} + ||\phi||_{L^2(\gamma^s)} \right)^{\frac{1}{s}}
\]

where \( C = C(g, \gamma, s, n) \).

An immediate corollary of the above is the following, see corollary 5.5 in [9].

**Corollary 10.2.** Under the same assumptions as the above theorem. Let \( 0 \leq i_1, \ldots, i_r \leq k \), \( i_1 + \ldots + i_r = 2k \), and \( s \geq 2k \). Then we have

\[
\left| \int_M \nabla^{(i_1)} \phi \ast \ldots \ast \nabla^{(i_r)} \phi \gamma^s d\mu \right| \leq C_0 \int_M |\nabla^{(i_1)} \phi| \ldots |\nabla^{(i_r)} \phi| \gamma^s d\mu
\]

\[
\leq C ||\phi||_{L^\infty}^{r-2} \left( \int_M |\nabla^{(k)} \phi|^{2s} d\mu + ||\phi||_{L^2(\gamma^s)}^2 \right)
\]

where \( C_0 = C_0(g) \) depends only on the metric, and \( C = C(n, k, r, s, g, \gamma) \).

Finally, we will need the following interpolation result, see corollary 5.5 in [8], and corollary 5.3 in [9].

**Lemma 10.3.** Let \( E \) be a vector bundle over \( M \), \( \nabla \) a connection on \( E \), and \( \gamma \) a bump function on \( M \). For \( 2 \leq p < \infty \), \( l \in \mathbb{N} \), \( s \geq l p \), there exists \( C(\epsilon) = C(\epsilon, n, \text{rank}(E), p, l, g, \gamma, s) \in \mathbb{R}_{>0} \) such that for \( \phi \) a smooth section we have

\[
||\gamma^{s/p} \nabla^{(l)} \phi||_{L^p(M)} \leq \epsilon ||\gamma^{(s+jp)/p} \nabla^{(l+j)} \phi||_{L^p(M)} + C(\epsilon) ||\phi||_{L^p(M), \gamma^s}.
\]

In particular, for \( p = 2 \) and some constant \( K \geq 1 \), we have

\[
K ||\gamma^{s/2} \nabla^{(l)} \phi||_{L^2(M)}^2 \leq \epsilon ||\gamma^{(s+j)/2} \nabla^{(l+j)} \phi||_{L^2(M)}^2 + C(\epsilon) K^2 ||\phi||_{L^2(M), \gamma^s}^2.
\]

10.2. Commutation formulae for connections. During the study of the higher order Seiberg-Witten flow, there will be times when we need to switch derivatives, leading to the need for various commutation formulas. We collect here various results on formulas for commuting connections.

We start with the Weitzenböck identity, see theorem 9.4.1 in [13].

**Proposition 10.4 (Weitzenböck identity).** Let \((M, g)\) be a Riemannian manifold with Levi-civita connection \(\nabla_M\). We also denote by \(\nabla_M^*\) the differential operator from \(\Omega^p(M) \to T^* M \otimes \Omega^p(M)\) induced by the Levi-Civita connection. Let \(\Delta_H = dd^* + d^* d\) denote the Hodge Laplacian, and let \(\nabla_M^* \nabla_M = \Delta_M\) denote the Bochner Laplacian. Given \(\omega \in \Omega^p(M)\), we have

\[
\Delta_M \omega = \Delta_H \omega + Rm \ast \omega.
\]

The following lemma tells us how to switch derivatives, see lemma 5.12 in [8].
Lemma 10.5. Let $E$ be a Hermitian vector bundle over a Riemannian manifold $(M,g)$, with metric compatible connection $\nabla$. Let $\phi$ denote a section of $E$. We have
\[
\nabla_i \nabla_{i_{k-1}} \cdots \nabla_{i_1} \nabla_{j_1} \cdots \nabla_{j_k} \phi = \nabla_i \nabla_{j_k} \nabla_{i_{k-1}} \cdots \nabla_{i_1} \nabla_{j_1} \phi + \sum_{l=0}^{2k-2} \left( (\nabla_M^l \text{Rm} + \nabla^{(l)} F_V) \ast \nabla^{(2k-2-l)} \phi \right),
\]
where $F_V$ denotes the curvature associated to $\nabla$, and $\text{Rm}$ is the Riemannian curvature.

A simple corollary of this lemma is the following.

Corollary 10.6. Let $E$ be a Hermitian vector bundle over a Riemannian manifolds $(M,g)$, with metric compatible connection $\nabla$. Let $\Delta = \nabla^* \nabla$ denote the Bochner Laplacian. Given a section $\phi$ of $E$, we have
\[
\nabla^* (\nabla^k) \phi = \Delta^k \phi + \sum_{j=0}^{2k-2} \left( \nabla_M^j \text{Rm} + \nabla^{(j)} F_V \ast \nabla^{(2k-2-j)} \phi \right).
\]

We will also need to commute derivatives with Laplacian terms. The following lemma shows us how to do this, see corollary 5.15 in [8].

Lemma 10.7. Let $E$ be a Hermitian vector bundle over a Riemannian manifold $(M,g)$, with metric compatible connection $\nabla$. Let $\Delta = \nabla^* \nabla$ denote the Bochner Laplacian, and let $\phi$ be a section of $E$. We have
\[
\nabla^{(n)} \Delta^k \phi = \Delta^k \nabla^{(n)} \phi + \sum_{j=0}^{2k+n-2} \left( \nabla_M^j \text{Rm} + \nabla^{(j)} F_V \ast \nabla^{(2k+n-2-j)} \phi \right).
\]

Combining corollary 10.6 and lemma 10.7 we obtain

Corollary 10.8. Let $E$ be a Hermitian vector bundle over a Riemannian manifold $(M,g)$, with metric compatible connection $\nabla$. Let $\Delta = \nabla^* \nabla$ denote the Bochner Laplacian, and let $\phi$ be a section of $E$. We have
\[
\nabla^{(n)} \nabla^* (\nabla^k) \phi = \Delta^k \nabla^{(n)} \phi + \sum_{j=0}^{2k+n-2} \left( \nabla_M^j \text{Rm} + \nabla^{(j)} F_V \ast \nabla^{(2k+n-2-j)} \phi \right).
\]

We will also need the following integration by parts formula, see lemma 5.13 in [8].

Lemma 10.9. Let $E$ be a Hermitian vector bundle over a Riemannian manifold $(M,g)$, with metric compatible connection $\nabla$. Let $\Delta = \nabla^* \nabla$ denote the Bochner Laplacian, and let $\phi$ and $\psi$ be sections of $E$. We have
\[
\int_M \langle \nabla^k \phi, \nabla^k \phi \rangle d\mu = \int_M (-1)^k \langle \phi, \Delta^k \psi \rangle d\mu + \sum_{v=0}^{2k-2} \left( \langle \nabla_M^v \text{Rm} + \nabla^{(v)} F_V \ast \nabla^{(2k-2-v)} \phi \rangle \right).
\]

10.3. Theorems from gauge theory. The following two theorems from gauge theory will be used in section 7. We state them here for the convenience of the reader.

The first theorem we will need is the Coloumb gauge theorem, theorem 1.3 in [19].

Theorem 10.10 (Coloumb gauge theorem). Let $M = B_1(0) \subseteq \mathbb{R}^n$, $E = B_1(0) \times \mathbb{R}^m$ be a trivial bundle over $M$, and $n \leq 2p$. Suppose $\nabla = d + A$ is a connection on $E$. Then there exists constants $\kappa(n) > 0$ and $c(n) < \infty$ such that if $||F_V||_{L^{n/2}} \leq \kappa(n)$, then $\nabla$ is gauge equivalent to a connection $d + \tilde{A}$ where $\tilde{A}$ satisfies:
1. \( d^* \tilde{A} = 0 \)
2. \(||\tilde{A}||_{C^p,1} \leq c(n)||F_\nabla||_{C^p,0} \).

The second theorem we will need is a theorem that allows us to glue together a sequence of connections defined on small open sets, see corollary 4.4.8 [2].

**Theorem 10.11.** Suppose \( \{\nabla^i\} \) is a sequence of connections on \( E \) over \( M \) with the following property: For each \( x \in M \) there exists a neighbourhood \( U_x \), and a subsequence \( \{\nabla^{i_j}\} \) with corresponding sequence of gauge transformations \( s^{i_j} \) defined over \( M \) such that \( s^{i_j} \nabla^{i_j} \) converges over \( U_x \). Then there exists a single subsequence \( \{\nabla^{i_{jk}}\} \) defined over \( M \) such that \( s^{i_{jk}} \nabla^{i_{jk}} \) converges over all of \( M \).

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