GENERALIZED PERMUTAHEDRA AND SCHUBERT CALCULUS

AVERY ST. DIZIER AND ALEXANDER YONG

ABSTRACT. We connect generalized permutahedra with Schubert calculus. Thereby, we
give sufficient vanishing criteria for Schubert intersection numbers of the flag variety. Our
argument utilizes recent developments in the study of Schubitopes, which are Newton
polytopes of Schubert polynomials. The resulting tableau test executes in polynomial time.

1. INTRODUCTION

1.1. Background. Let $X = \text{Flags}(\mathbb{C}^n)$ be the variety of complete flags of vector spaces
$F_i : \langle 0 \rangle \subset F_1 \subset F_2 \cdots \subset F_i \subset \cdots \subset F_{n-1} \subset \mathbb{C}^n, \dim(F_i) = i.$

$X$ has a left-action of $GL_n$, and hence also by lower triangular invertible matrices $B_-$. The
$B_-$-orbits $X_w$ are indexed by permutations $w$ in the symmetric group $S_n$. Let $\leq$ denote
Bruhat order. The Schubert varieties are the closures

$$X_w = \bigcap_{v \geq w} X_v,$$

this is codimension $\ell(w) = \#\{(i, j) : 1 \leq i < j \leq n, w(i) > w(j)\}$. Thus, $X = X_{id}$ and $X_{w_0}$
is the Schubert point, where $w_0 = n \ n-1 \ n-2 \ \cdots \ 2 \ 1$.

The Poincaré duals $\sigma_w := [X_w]$ form the Schubert basis of $H^*(X)$, the cohomology ring
of $X$. A Schubert problem is $(w^{(1)}, w^{(2)}, \ldots, w^{(k)}) \in S_n^k$ with $\sum_{i=1}^k \ell(w^{(i)}) = \binom{n}{2} = \dim_{\mathbb{C}}(X)$. The Schubert intersection number is

$$C_{w^{(1)}, w^{(2)}, \ldots, w^{(k)}} := \text{multiplicity of } \sigma_{w_0} \text{ in } \prod_{i=1}^k \sigma_{w^{(i)}} \in H^*(X)$$

$$= \text{number of points in } \bigcap_{i=1}^k g_i X_{\sigma^{(i)}},$$

where $(g_1, \ldots, g_k)$ are elements of a dense open subset $\mathcal{O}$ of $GL_n^k$ (whose existence is guar-
anteed by Kleiman transversality). A textbook is [8]; expository papers include [11, 10].

Algorithms exist for computing these numbers; see, e.g., [5, 15, 13] and the references
therein. It is the famous open problem of Schubert calculus to find a combinatorial
counting rule that computes $C_{w^{(1)}, w^{(2)}, \ldots, w^{(k)}}$. Such a rule would generalize the classical
Littlewood-Richardson rule governing Schubert calculus of Grassmannians.

This paper explores a related, but not necessarily easier, open problem:

Find an efficient algorithm to decide if $C_{w^{(1)}, w^{(2)}, \ldots, w^{(k)}} = 0$. 

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Known algorithms to compute $C_{w^{(1)}, w^{(2)}, \ldots, w^{(k)}}$ do not provide a solution (being inefficient). In the Grassmannian setting, neither does the Littlewood-Richardson rule, \textit{per se}. However, the \textit{saturation theorem} \cite{vanishing} permits a polynomial-time algorithm in that case \cite{flag} \cite{schubert}, by way of linear programming results. For (generalized) flag varieties, criteria were found by A. Knutson \cite{knutson} and K. Purbhoo \cite{purbhoo}; no efficiency guarantees were stated.

1.2. \textbf{Vanishing criterion.} Our main goal is to connect the theory of generalized permutahedra to Schubert calculus. We give a sufficient test for $C_{w^{(1)}, w^{(2)}, \ldots, w^{(k)}} = 0$ and prove it executes in polynomial-time. The starting point is a simple consideration about Schubert polytopes \cite{permutahedron} \cite{flag} \cite{schubert}, as instances of generalized permutahedra.

The \textit{diagram} of $w \in S_n$, denoted $D(w)$, is the subset of boxes of $[n] \times [n]$ given by

$$D(w) := \{(i, j) : 1 \leq i, j \leq n, j < w(i), i < w^{-1}(j)\}.$$  

Let $\text{code}(w) = (c_1(w), c_2(w), \ldots, c_n(w))$, where $c_i$ counts boxes of $D(w)$ in row $i$. Define

$$D := D(w^{(1)}, \ldots, w^{(k)})$$

by concatenating $D(w^{(1)}), \ldots, D(w^{(k)})$, left to right. Set $\text{Tab} := \text{Tab}_{w^{(1)}, \ldots, w^{(k)}}$ to be the set of fillings of $D$ with nonnegative integers such that:

(a) Each column is strictly increasing from top to bottom.
(b) Any label $\ell$ in row $r$ satisfies $\ell \leq r$.
(c) The number of $\ell$'s is $n - \ell$, for $1 \leq \ell \leq n$.

The first version of our test is:

\textbf{Theorem 1.1.} Let $(w^{(1)}, \ldots, w^{(k)})$ be a Schubert problem. If $\text{Tab} = \emptyset$ then $C_{w^{(1)}, w^{(2)}, \ldots, w^{(k)}} = 0$. There is an algorithm to determine emptiness in $O(\text{poly}(n, k))$.

\textbf{Example 1.2.} Let $w^{(1)} = 3256147$, $w^{(2)} = 2143657$, $w^{(3)} = 4632175$. Below we depict $D$. The numerically labelled boxes are forced by conditions (a) and (b) for any (putative) $T \in \text{Tab}$.

\begin{center}
\begin{tabular}{c|c|c}
\hline
Height & $1$ & $2$ & $3$ & $4$ & $5$ & $6$ & $7$ \\
\hline
$1$ & \\
$2$ & $1$ & \\
$3$ & \\
$4$ & \\
$5$ & \\
$6$ & \\
$7$ & \\
\hline
\end{tabular}
\end{center}

Condition (b) forces $e \leq 2$, $a, c \leq 3$, $b \leq 4$, $d \leq 5$, $f \leq 6$. Thus, to satisfy (c), $e = 2$ is also forced, which implies $a, c = 3$. So $T$ has at least five 3's, violating (c) for $\ell = 3$.

Our idea (see Section 4) uses that $C_{w^{(1)}, w^{(2)}, w^{(3)}} = 0$ if $\mathcal{S}_{w_0} = x_1^6 x_2^5 x_3^4 x_4^3 x_5^2 x_6$ does not appear in the product of Schubert polynomials $\mathcal{S}_{w^{(1)}} \mathcal{S}_{w^{(2)}} \mathcal{S}_{w^{(3)}}$, combined with an argument that the rule of Theorem 1.1 permits an efficient check of this vanishing condition.

1.3. \textbf{Organization.} Section 2 discusses generalized permutahedra; we derive facts we will use. Section 3 reviews the subfamily of Schubitopes. In Section 4 we state Theorem 4.7, an “asymmetric” version of Theorem 1.1, it is a stronger test, see Proposition 4.8. Theorem 4.10 gives linear inequalities necessary for $C_{w^{(1)}, \ldots, w^{(k)}} > 0$. Theorems 1.1, 4.7, 4.10, and Proposition 4.8 are proved together, as they follow from the same
reasoning. In Section 5, we compare with the vanishing criteria of [12] and [18]. We show examples that our test captures but are not captured by those criteria, and conversely.

2. Newton Polytopes of Products

If $f$ is an element of a polynomial ring whose variables are indexed by some set $I$, the support of $f$ is the lattice point set in $\mathbb{R}^I$ consisting of the exponent vectors of the monomials that have nonzero coefficient in $f$. The Newton polytope $\text{Newton}(f) \subseteq \mathbb{R}^I$ is the convex hull of the support of $f$. A polynomial $f$ has saturated Newton polytope (SNP) if every lattice point in $\text{Newton}(f)$ is a vector in the support of $f$ [16].

The standard permutahedron is the polytope in $\mathbb{R}^n$ whose vertices consist of all permutations of the entries of the vector $(0, 1, \ldots, n-1)$. A generalized permutahedron is a deformation of the standard permutahedron obtained by translating the vertices in such a way that all edge directions and orientations are preserved (edges are allowed to degenerate to points). Generalized permutahedra are uniquely parametrized by submodular functions (see [2] Theorem 12.3]). These are maps

$$z : 2^{[n]} \rightarrow \mathbb{R},$$

such that $z_\emptyset = 0$ and

$$z_I + z_J \geq z_{I\cup J} + z_{I\cap J} \quad \text{for all } I, J \subseteq [n].$$

Given $z$, the associated generalized permutahedron is given by

$$P(z) = \left\{ t \in \mathbb{R}^n : \sum_{i \in I} t_i \leq z_I \text{ for } I \neq [n], \text{ and } \sum_{i=1}^n t_i = z_{[n]} \right\}.$$

The vertices of generalized permutahedra have been determined.

**Proposition 2.1 ([21 Corollary 44.3a]).** Let $P(z)$ be a generalized permutahedron in $\mathbb{R}^n$. The vertices of $P(z)$ are $\{v(w) : w \in S_n\}$ where $v(w) = (v_1, \ldots, v_n) \in \mathbb{R}^n$ is defined by

$$v_{w_k} = z_{\{w_1, \ldots, w_k\}} - z_{\{w_1, \ldots, w_{k-1}\}}.$$ (2)

It is well-known that the class of generalized permutahedra is closed under Minkowski sums (see for instance [3 Lemma 2.2]). We provide a proof for completeness.

**Lemma 2.2.** If $P(z)$ and $P(z')$ are generalized permutahedra, then

$$P(z) + P(z') = P(z + z').$$

**Proof.** Clearly $P(z) + P(z') \subseteq P(z + z')$. For the opposite containment, let $q$ be a vertex of $P(z + z')$. By Proposition 2.1 write $q$ in the form $q = v(w)$ for some $w \in S_n$. Let $p$ and $p'$ be the vertices of $P(z)$ and $P(z')$ respectively corresponding to $w$. By (2), $q = p + p' \in P(z) + P(z')$. Convexity implies $P(z + z') \subseteq P(z) + P(z')$. \[\Box\]

It follows easily from [21 Theorem 46.2] that whenever $z$ and $z'$ are integer-valued, $P(z) \cap P(z')$ is either empty or an integral polytope (all vertices are lattice points). This is used to prove that integer polymatroids [21 Chapter 44] satisfy a generalization of the integer decomposition property. We state and prove (for convenience) the special case that applies to generalized permutahedra:
Theorem 2.3 ([21] Corollary 46.2c). If \(P(z)\) and \(P(z')\) are integral generalized permutahedra in \(\mathbb{R}^n\), then
\[
(P(z) \cap \mathbb{Z}^n) + (P(z') \cap \mathbb{Z}^n) = (P(z) + P(z')) \cap \mathbb{Z}^n.
\]

Proof. Let \(r \in (P(z) + P(z')) \cap \mathbb{Z}^n\). Set \(Q = r + (-1)P(z')\). Clearly, \(Q\) is a generalized permutahedron (by the deformation description). Also note that \(r = p + p'\) for some \(p \in P(z)\) and \(p' \in P(z')\), so \(p \in P \cap Q\) and \(P \cap Q \neq \emptyset\). Since both \(r\) and \(z'\) are integral, \(Q\) is an integral polytope. Thus \(P \cap Q\) contains an integer point \(q\). By definition of \(Q\), the lattice point \(r - q\) is in \(P(z')\). Finally, we have
\[
r = q + (r - q) \in (P(z) \cap \mathbb{Z}^n) + (P(z') \cap \mathbb{Z}^n).
\]

Therefore, in the realm of generalized permutahedra, SNP carries through products.

Proposition 2.4. If \(f, g \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]\) have SNP and Newton\((f)\), Newton\((g)\) are generalized permutahedra then

(i) Newton\((fg)\) is a generalized permutahedron;
(ii) \(fg\) has SNP.

Proof. For any polynomials \(f\) and \(g\), Newton\((fg)\) = Newton\((f) + Newton(g)\). Statement (i) follows from Lemma 2.2. Statement (ii) follows from Lemma 2.2 and Theorem 2.3.

3. SCHUBITOPES, AND AN INTEGER LINEAR PROGRAM

We are interested in a particular family of generalized permutahedra. For \(D \subseteq [n] \times [m]\), the Schubertope \(S_D\) was defined by C. Monical, N. Tokcan, and the second author [16]. Fix \(S \subseteq [n]\) and a column \(c \in [m]\). Let \(\omega_{c,S}(D)\) be formed by reading \(c\) from top to bottom and recording

- ( if \((r, c) \notin D\) and \(r \in S\),
- ) if \((r, c) \in D\) and \(r \notin S\), and
- \(*\) if \((r, c) \in D\) and \(r \in S\).

Let
\[
\theta_D^c(S) = \#\text{paired } (\text{'s in } \omega_{c,S}(D)) + \#\text{*'s in } \omega_{c,S}(D).
\]
Set \(\theta_D(S) = \sum_{c \in [m]} \theta_D^c(S)\). Define the Schubitope as
\[
S_D = \left\{ (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}_{\geq 0}^n : \sum_{i=1}^n \alpha_i = \#D \text{ and } \sum_{i \in S} \alpha_i \leq \theta_D(S) \text{ for all } S \subseteq [n] \right\}.
\]

Example 3.1 (cf. [16] Section 1). Let \(w = 21543\). The Schubert polynomial of \(w\) is
\[
\mathcal{S}_w = x_1^3x_2 + x_1^3x_3 + x_2^3x_4 + x_1^2x_2^2 + x_1^2x_3^2 + 2x_1^2x_2x_3 + x_1^2x_2x_4 + x_1^2x_3x_4
+ x_1x_2x_3^2 + x_1x_2^2x_3 + x_1x_2^2x_4 + x_1x_3^2x_4 + x_1x_3x_4x_3x_4.
\]

As stated in Theorem 4.3, \(S_{D(w)} = \text{Newton}(\mathcal{S}_w)\). This generalized permutahedron and a minimal set of defining inequalities are shown in Figure 1.

We need some notions from A. Adve, C. Robichaux, and the second author’s paper [11]. Given \(D \subseteq [n] \times [m]\) and \(\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n\), Let
\[
P(D, \alpha) \subset \mathbb{R}^{n \times m}
be the polytope whose points
\[(\alpha_{ij})_{1 \leq i \leq n, 1 \leq j \leq m} = (\alpha_{11}, \ldots, \alpha_{n1}, \ldots, \alpha_{1m}, \ldots, \alpha_{nm})\]
satisfy the inequalities (I),(II),(III) below.

(I) Column-injectivity: For all \(i, j \in [n]\),
0 \leq \alpha_{ij} \leq 1.

(II) Content: For all \(i \in [n]\),
\[\sum_{j=1}^{m} \alpha_{ij} = \alpha_i.\]

(III) Row bounds: For all \(s, j \in [n]\),
\[\sum_{i=1}^{s} \alpha_{ij} \geq \# \{(i, j) \in D : i \leq s\}.\]

Define \(\text{Tab}(D, \alpha)\) to be the set of fillings of \(D\) with nonnegative integers such that
(a) Each column is strictly increasing from top to bottom.
(b) Any label \(\ell\) in row \(r\) satisfies \(\ell \leq r\).
(c) The number of \(\ell's\ is \(\alpha_\ell\).

**Theorem 3.2 ([II]).** Suppose \(D \subseteq [n] \times [m] \) and \(\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n\). Then
\[\alpha \in S_D \iff \text{Tab}(D, \alpha) \neq \emptyset.\]

The map \(f : \text{Tab}(D, \alpha) \to \mathcal{P}(D, \alpha)\), that sets \(\alpha_{ij} = 1\) if the label \(i\) appears in column \(j\) of \(D\), and set \(\alpha_{ij} = 0\) otherwise, is a bijection. Therefore \(\text{Tab}(D, \alpha) \neq \emptyset\) if and only if \(\alpha_1 + \cdots + \alpha_n = \#D\) and \(\mathcal{P}(D, \alpha) \cap \mathbb{Z}^{n^2} \neq \emptyset\).

**Theorem 3.3 ([II]).** Let \(D \subseteq [n] \times [m]\) and \(\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n\) with \(\alpha_1 + \cdots + \alpha_n = \#D\). Then \(\mathcal{P}(D, \alpha) \cap \mathbb{Z}^{n \times m} \neq \emptyset\) if and only if \(\mathcal{P}(D, \alpha) \neq \emptyset\).

The above two theorems, combined with the ellipsoid method and/or interior point methods in linear programming, implies:
Corollary 3.4 ([1]). Deciding if $\alpha \in S_D$, or equivalently, if $\text{Tab}(D, \alpha) = \emptyset$, can be determined in $O(\text{poly}(n, m))$-time.

As explained in [11], by using the codes of $w^{(i)}$ as the encoding of the decision problem, or “compressing” $D$, one can reduce the upper bound on the complexity. We will not describe these technical improvements here, although they may be applied.

4. SCHUBERT POLYNOMIALS AND SCHUBITOPES

4.1. Schubert polynomials. Our reference for Schubert polynomials is [13]. They are recursively defined; the initial condition is that for $w_0 \in S_n$,

$$\mathcal{S}_{w_0} := x_1^{n-1}x_2^{n-2}\cdots x_{n-1}.$$  

The divided difference operator on polynomials in $\text{Pol} := \mathbb{Z}[x_1, x_2, \ldots]$ is

$$\partial_i : \text{Pol} \to \text{Pol}, \ f \mapsto \frac{f(\ldots, x_i, x_{i+1}, \ldots) - f(\ldots, x_{i+1}, x_i, \ldots)}{x_i - x_{i+1}}.$$  

If $w \neq w_0$, let $i$ satisfy $w(i) < w(i + 1)$, then $\mathcal{S}_w := \partial_i \mathcal{S}_{w_s_i}$. Since the divided difference operators satisfy the braid relations

$$\partial_i \partial_j = \partial_j \partial_i \quad \text{for } |i - j| \geq 2; \quad \partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1},$$

it follows that $\mathcal{S}_w$ only depends on $w$, and not the choices of $i$ in the recursion.

Schubert polynomials are stable under the inclusion of $S_n \hookrightarrow S_{n+1}$ that sends $w$ to $w$ with $n + 1$ appended. Thus, one defines $\mathcal{S}_w$ for $w \in S_\infty = \bigcup_{n \geq 1} S_n$. The set of Schubert polynomials $\{\mathcal{S}_w : w \in S_\infty\}$ forms a $\mathbb{Z}$-linear basis of $\text{Pol}$.

Borel’s isomorphism ([8] Chapter 9; Prop. 3) asserts

$$H^*(X) \cong \mathbb{Q}[x_1, \ldots, x_n]/I^{S_n} \quad \text{where } I^{S_n} = \langle e_d(x_1, \ldots, x_n) : 1 \leq d \leq n \rangle,$$

and

$$e_d(x_1, \ldots, x_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_d \leq n} x_{i_1}x_{i_2}\cdots x_{i_d}$$

is the $d$-th elementary symmetric polynomial. Under this isomorphism,

$$\sigma_w \mapsto \mathcal{S}_w + I^{S_n}. \quad (3)$$

One has the polynomial identity

$$\mathcal{S}_u \mathcal{S}_v = \sum_{w \in S_\infty} C_{u,v}^{w} \mathcal{S}_w \in \text{Pol}.$$  

Define $C_{w^{(k)}}^{w^{(1)}, \ldots, w^{(k-1)}}$ to be the multiplicity of $\sigma_{w^{(k)}}$ in $\prod_{i=1}^{k-1} \sigma_{w^{(i)}} \in H^*(X)$, which we also write with the coefficient operator as $[\sigma_{w^{(k)}}] \prod_{i=1}^{k-1} \sigma_{w^{(i)}}$.

Lemma 4.1. $C_{w^{(k)}}^{w^{(1)}, \ldots, w^{(k-1)}} = C_{w^{(1)}, \ldots, w^{(k-1)}, w^{(k)}}^{w^{(k)}}$. Also, $C_{w^{(k)}}^{w^{(1)}, \ldots, w^{(k-1)}} = [\mathcal{S}_{w^{(k)}}] \prod_{i=1}^{k-1} \mathcal{S}_{w^{(i)}}$. In particular $C_{u,v}^{w} = C_{u,u_0,v}^{w}$.  

Proof. Duality in Schubert calculus (see, e.g., [15] Proposition 3.6.11) states that if $\ell(u) + \ell(v) = \binom{n}{2}$ then

$$\sigma_u \succeq \sigma_v = \begin{cases} \sigma_{w_0} & \text{if } v = w_0u \\ 0 & \text{otherwise.} \end{cases}$$
Now,
\[ \prod_{i=1}^{k-1} \sigma_{w(i)} = C_{w_1,\ldots,w(k-1)}^{w(k)} + \sum_{w \in S_n, w \neq w(k)} C_{w_1,\ldots,w(k-1)}^{w} \sigma_{w(k)} \]

Multiply both sides by \( \sigma_{w_0w(k)} \) and apply duality. Then use (1) to obtain the first statement. The second assertion follows from (3). The final claim is merely the \( k = 3 \) case. \( \square \)

**Lemma 4.2.** If \( (w^{(1)}, \ldots, w^{(k)}) \) is a Schubert problem then
\[ C_{w^{(1)},\ldots,w^{(k)}} = [x_1^{n-1}x_2^{n-2}\cdots x_{n-1}] \prod_{i=1}^{k} \mathfrak{S}_{w(i)}. \]

**Proof.** This follows from (1), (3), and \( \mathfrak{S}_{w_0} = x_1^{n-1}x_2^{n-2}\cdots x_{n-1}. \) \( \square \)

4.2. **Schubitopes are Newton polytopes.** This result from work of A. Fink, K. Mészáros, and the first author [7] proves conjectures of [16]:

**Theorem 4.3 ([7] Theorems 7,10).** \( S_{D(w)} = \text{Newton}(\mathfrak{S}_w), \text{and } \mathfrak{S}_w \text{ has SNP.} \)

**Theorem 4.4 ([7] Corollary 8]).** \( S_{D(w)} \) is a generalized permutahedron.

**Proposition 4.5.** \( f = \prod_{i=1}^{k-1} \mathfrak{S}_{w(i)} \) has SNP. In addition,
\[ (4) \quad \text{Newton}(f) = \sum_{i=1}^{k-1} S_{D(w(i))} \text{ (Minkowski sum).} \]

**Proof.** This follows from combining Theorems 4.3 and 4.4 with Proposition 2.4. \( \square \)

By the same argument as Proposition 4, any product of key polynomials (see, e.g., [20]) with Schubert polynomials is SNP, and has a similarly described Newton polytope.

**Corollary 4.6.** If \( \alpha \in \mathbb{Z}^n_{\geq 0} \) then
\[ [x^\alpha] \prod_{i=1}^{k-1} \mathfrak{S}_{w(i)} \neq 0 \iff \alpha \in \sum_{i=1}^{k-1} S_{D(w(i))}. \]

**Proof.** Let \( f = \prod_{i=1}^{k-1} \mathfrak{S}_{w(i)}. \) If \([x^\alpha]f \neq 0 \) then \( \alpha \in \text{Newton}(f) \). Now apply (4). Conversely, by (4), \( \alpha \in \text{Newton}(f) \). By Proposition 4.5, \( f \) has SNP. Hence \([x^\alpha]f \neq 0 \). \( \square \)

4.3. **The asymmetric version of Theorem 1.1** Let \( D' := D(w^{(1)}, \ldots, w^{(k-1)}) \) and let \( \text{Tab}' := \text{Tab}_{w^{(1)},\ldots,w^{(k)}} \) be the set of fillings of \( D' \) with nonnegative integers such that:

(a) Each column is strictly increasing from top to bottom.

(b) Any label \( \ell \) in row \( r \) satisfies \( \ell \leq r. \)

(c) The number of \( \ell \)'s is \( c_\ell(w^{(k)}). \)

**Theorem 4.7.** Let \( (w^{(1)}, \ldots, w_0w^{(k)}) \) be a Schubert problem. If \( \text{Tab}' = \emptyset \) then \( C_{w^{(1)},w^{(2)},\ldots,w^{(k-1)}}^{w^{(k)}} = 0. \) There is an algorithm to determine emptiness in \( O(\text{poly}(n,k)). \)

**Proposition 4.8.** If Theorem 1.1's test shows \( C_{w^{(1)},\ldots,w^{(k-1)},w_0w^{(k)}} = 0 \) then Theorem 4.7's test also shows \( C_{w^{(1)},\ldots,w^{(k-1)}}^{w^{(k)}} = 0. \)
Example 4.9. The converse of Proposition 4.8 is false. That is, Theorem 4.7 provides a strictly stronger test than Theorem 1.1. For example,

\[
\mathcal{G}_{123142} \mathcal{G}_{1342} = x_1^4 x_3 + x_1^4 x_2 + x_1^3 x_2 x_3
\]

avoids code(4312) = 3200 as an exponent vector, proving \( C_{w,v} = C_{4312,1342} = 0 \). However,

\[
\mathcal{G}_u \mathcal{G}_v \mathcal{G}_{w_0 w} = x_1^4 x_2 + x_1^4 x_3 + 3 x_1^4 x_2 x_3 + x_1^3 x_2 x_3^2 + x_1^3 x_2^2 x_3 + x_1^5 x_3 + x_1^5 x_2
\]

implies \( T_{ab} \neq \emptyset \), and hence Theorem 1.1 does not show \( C_{u,v,w_0 w} = C_{123142,1342} = 0 \). □

4.4. The Schubitope inequalities and Schubert calculus. The Schubitope inequalities provide necessary conditions for nonvanishing of a Schubert intersection number.

Theorem 4.10. If \( C_{w(1),w(2),\ldots,w(k)} > 0 \) then \((n-1, n-2, \ldots, 2, 1)\) must satisfy the Schubite inequalities defining \( S_D \) where \( D = D(w(1), \ldots, w(k)) \). Similarly, if \( C_{w(1),\ldots,w(k-1)} > 0 \) then code\((w(k))\) must satisfy the Schubite inequalities defining \( S_{D'} \) where \( D' = D(w(1), \ldots, w(k-1)) \).

Let

\[
s_\lambda(x_1, \ldots, x_k) = \sum_T x^T
\]

be the Schur polynomial of \( \lambda \), where the sum is over semistandard Young tableaux of shape \( \lambda \) filled using \( \{1, 2, \ldots, k\} \) and \( x^T = \prod_{i=1}^k x_i^{#i \in T} \). Then

\[
s_\lambda(x_1, \ldots, x_k)s_\mu(x_1, \ldots, x_k) = \sum_\nu c^\nu_{\lambda,\mu} s_\nu(x_1, \ldots, x_k),
\]

where \( c^\nu_{\lambda,\mu} \) is the Littlewood-Richardson coefficient. By the proof of \[16\] Proposition 2.9, (5) \( x^\nu \in s_\lambda s_\mu \) if and only if \( \nu \in \text{Newton}(s_{\lambda+\mu}) = \mathcal{P}_{\lambda+\mu} \) (the permutahedron for \( \lambda + \mu \)). By Rado’s theorem [19, Theorem 1], this means \( \nu \leq_{\text{dom}} \lambda + \mu \) (dominance order). That is

\[
c^\nu_{\lambda,\mu} > 0 \implies \sum_{i=1}^t \nu_i \leq \sum_{j=1}^t \lambda_j + \sum_{k=1}^t \nu_k, \text{ for } t \geq 1.
\]

These are instances of the famous Horn’s inequalities; see the survey [9]. (Those are generalized in the “Levi-movable” case of \( X \) in work of P. Belkale-S. Kumar [4].) Our methods are in the same vein. Hence, we speculate Theorem 4.10 is a first glimpse of putative linear inequalities that control \( C_{w(1),\ldots,w(k)} > 0 \). We hope to study this further in a sequel.

4.5. Proof of Theorems 1.1, 4.7, 4.10 and Proposition 4.8: We combine the proofs of these four results since they all stem from the same reasoning.

We prove Theorem 4.7 first. It is known (e.g., follows from [15, Theorem 2.5.1]) that

\[
[x^{\text{code}(w)}] \mathcal{G}_w \neq 0.
\]

Hence

\[
[x^{\text{code}(w(k))}] \prod_{i=1}^{k-1} \mathcal{G}_{w(i)} = 0 \Rightarrow C_{w(1),w(2),\ldots,w(k-1)} = 0.
\]

By one direction of Corollary 4.6,

\[
[x^{\text{code}(w(k))}] \prod_{i=1}^{k-1} \mathcal{G}_{w(i)} = 0 \iff \text{code}(w(k)) \notin \text{Newton} \left( \prod_{i=1}^{k-1} \mathcal{G}_{w(i)} \right) = \sum_{i=1}^{k-1} S_{D(w(i))}.
\]
By Theorem $4.4$, each $S_{D(w^{(i)})}$ is a generalized permutahedron. Hence, by Lemma 2.2

$$\text{Newton} \left( \prod_{i=1}^{k-1} S_{w^{(i)}} \right) = S_{D'}. $$

Now we may apply Theorem $3.2$ in the special case that $D = D'$ and $\alpha = \text{code}(w^{(k)})$ to obtain the second sentence of the theorem. The final sentence follows from Corollary $3.4$. This completes the proof of Theorem $4.7$.

The proof of Theorem $1.1$ is the same, except that we use Lemma $4.2$.

Theorem $4.10$ follows from the above arguments, combined with Theorems $3.2$ and $4.3$.

Finally, we turn to Proposition $4.8$. We prove the contrapositive. Suppose Theorem $4.7$’s test is inconclusive, that is,

$$[x^{\text{code}(w^{(k)})}] S_{w^{(1)}} \cdots S_{w^{(k-1)}} \neq 0. $$

Claim 4.11. If $w \in S_n$ then $\text{code}(w) + \text{code}(w_0 w) = (n - 1, n - 2, \ldots, 3, 2, 1, 0)$.

Proof of Claim 4.11: By definition of $D(w)$,

$$c_r(w) = (w(r) - 1) - \# \{ i < r : w(i) < w(r) \}. $$

On the other hand,

$$c_r(w_0 w) = (w_0 w(r) - 1) - \# \{ i < r : w_0 w(i) < w_0 w(r) \}
= ((n + 1 - w(r)) - 1) - \# \{ i < r : w(r) < w(i) \}. $$

Hence $c_r(w) + c_r(w_0 w) = n - r$, as desired. □

By (8) and (6) combined,

$$[x_{1}^{n-1} x_{2}^{n-2} \cdots x_{n-1}^{2}](S_{w^{(1)}} \cdots S_{w^{(k-1)}}) S_{w_0 w^{(k)}}
= [x_{1}^{n-1} x_{2}^{n-2} \cdots x_{n-1}^{2}] (x^{\text{code}(w^{(k)})} + \cdots) (x^{\text{code}(w_0 w^{(k)})} + \cdots) \neq 0, $$
where inequality is by Claim 4.11. Thus Theorem 1.1’s test is inconclusive. □

4.6. A flexible version of the asymmetric test. The condition (c) in defining Tab’ can be replaced by the exponent vector of any monomial in $S_{w^{(k)}}$. Unfortunately, the number of such exponent vectors is potentially large. Instead, one can sample points from $S_{D(w)}$ as follows. Construct the Rothe diagram $D(w)$. Fix a column $c$ of $D(w)$. Suppose the boxes of $D(w)$ in that column are in rows $r_1, r_2, \ldots, r_z$. Find integers $1 \leq x_1 < x_2 < \ldots < x_z$ such that $x_j \leq r_j$. Repeat for every column $c$. The result is an element of Tab$(D(w), \alpha)$ for some $\alpha$. (Thus one can create a randomized version of Theorem 4.7.)

It is possible that, even with choice, no exponent vector exhibits nonvanishing:

Example 4.12. $C_{231645, 231645}^{451623} = 0$. Now,

$$S_{451623} = x_1^3 x_2^3 x_4^2 + x_1^3 x_2^3 x_3 x_4 + x_1^3 x_2^3 x_3^2. $$

Here $\text{code}(451623) = 3302$. One can check that

$$[x^{\text{code}(451623)}] S_{231645}^2 > 0, [x_{1}^{3} x_{2}^{3} x_{4}^{2}] S_{231645}^2 > 0, \text{ and } [x_{1}^{3} x_{2}^{3} x_{3} x_{4}] S_{231645}^2 > 0. $$

Thus Theorem 4.7’s test is inconclusive using any choice of monomial from $S_{451623}$. □
Individual monomials have no geometric meaning in Schubert calculus. Thus, our tests seem inherently combinatorial, as opposed to being avatars of the geometry.

4.7. Certificate of vanishing. Textbook linear programming results implying efficiency of Theorems 1.1 and 4.7 offer an additional benefit. There is a short certificate when Tab or Tab' is empty. This follows from standard reasoning using Farkas’ lemma.

Theorem 4.10 provides an alternative certification method. Recording one Schubitope inequality defining $S_D$ for which $(n, n - 1, \ldots, 2, 1)$ fails proves $C_{w(1), \ldots, w(k)} = 0$. (A similar statement holds about $S_D'$.)

5. Comparisons to other vanishing tests

We compare our tests to three non-ad hoc vanishing tests. There are examples where our method is successful where the others are not, and vice versa.

5.1. Bruhat order. Bruhat order on $S_n$ is (combinatorially) defined as the reflexive and transitive closure of the covering relations $u \leq ut_{ij}$ if $\ell(ut_{ij}) = \ell(u) + 1$, where $t_{ij}$ is the transposition interchanging $i$ and $j$. There exist efficient tests to determine $u \leq v$, such as the Ehresmann tableau criterion [15, Proposition 2.2.11]. The following is well-known; we include a proof since we do not know where it exactly appears in the literature:

**Fact 5.1 (Bruhat vanishing test).** $C_{w(1), \ldots, w(k)} = 0$ if $w^{(i)} \not\leq w_0 w^{(j)}$ for some $i \neq j$.

**Proof.** We prove the case $k = 3$; the general case is similar. Say $u \not\leq w_0 w$ but $C_{u,v,w} > 0$. By Lemma 4.1, $C_{u,w} = C_{u,w,v} > 0$. Monk’s formula [15, Theorem 2.7.1] states that if $z \in S_n$,

$$\sum_{\sigma_{j \leq m < k}} \sigma_{t_{m,m+1}} = \sum_{\sigma_{zt_{jk}} \in H^*(X)} \sigma;$$

the sum is over all $j \leq m < k$ such that $\ell(zt_{jk}) = \ell(w) + 1$ and $zt_{jk} \in S_n$. Suppose $s_m := t_{m,m+1}$ and $v = s_{m_1} s_{m_2} \cdots s_{m_\ell(v)}$ is a reduced expression for $v$. By (9), for some $\alpha \in \mathbb{Z}_{>0}$,

$$\prod_{i=1}^{\ell(v)} \sigma_{s_{m_i}} = \alpha \sigma_v + \text{(positive sum of Schubert classes)}.$$  

By induction using (9),

$$[\sigma_y] \sigma_u \prod_{i=1}^{\ell(v)} \sigma_{s_{m_i}} \neq 0 \iff y \geq u.$$  

By the positivity of Schubert calculus, and the assumption $C_{u,v,w} > 0$,

$$[\sigma_{w_0 w}] \sigma_u(\alpha \sigma_v + \text{(positive sum of Schubert classes)}) \neq 0.$$  

In view of (10), this contradicts (11).

We give bad news first:

**Example 5.2.** $(u, v, w) = (1243, 1342, 3142)$ is a vanishing problem detected by Fact 5.1 since $1342 = v \not\leq w_0 w = 2413$. Our methods do not detect $C_{u,v,w} = C_{1243,1342}^{2413}$. Since

$$\mathcal{S}_{1243} \mathcal{S}_{1342} = x_2 x_3^2 + x_1 x_3^2 + 3 x_1 x_2 x_3 + x_2 x_3 + x_1 x_2^2 + x_1^2 x_3 + x_1^2 x_2,$$
contains both monomials of $\mathcal{S}_{u,v,w} = \mathcal{S}_{2413} = x_1x_2^2 + x_1^2x_2$, no monomial of $\mathcal{S}_{u,v,w}$ can be used to detect vanishing. In particular, Theorem 4.7 is inconclusive (and hence by Proposition 4.8, the symmetric test is also inconclusive). Since $C_{u,v,w} = C_{v,w,u} = C_{w,u,v}$, one hopes the asymmetric method shows either $C_{w,u,v} = C_{1342,3142} = 0$ or $C_{w,u,v} = C_{1423,1423,3142} = 0$. Unfortunately, both attempts are similarly inconclusive.

Example 5.3. The vanishing of the Schubert problem $(u, v, w) = (1423, 1423, 1423)$ is undetected by Fact 5.1. Now

$$\mathcal{S}_{1423} = x_1^6 + 3x_1x_2^5 + 6x_1^2x_2^4 + 7x_1^3x_2^3 + 6x_1^4x_2^2 + 3x_1^5x_2 + x_1^6$$

does not contain $\mathcal{S}_{w_0} = \mathcal{S}_{4321} = x_1^3x_2^3x_3$ and hence vanishing is seen by Theorem 1.1. □

5.2. A. Knutson’s descent cycling. In [12], A. Knutson introduced a vanishing criterion. Recall, $u \in S_n$ has a descent at position $i$ if $u(i) > u(i+1)$ and has an ascent at position $i$ otherwise. That is, respectively, $us_i \leq u$ and $us_i \geq u$.

Fact 5.4 (dc triviality). If $(u, v, w)$ is a Schubert problem such that $us_i \leq u, vs_i \geq v, ws_i \geq w$ then $C_{u,v,w} = 0$.

Example 5.5. The triple $(1423, 1423, 1423)$ is dc trivial and hence $C_{1423,1423,1423} = 0$. Here, the asymmetric test (Theorem 4.7) is inconclusive (again, thus by Proposition 4.8, the symmetric test is also inconclusive). Indeed, $C_{w,u,v} = C_{1423,1423,1423} = 0$ is not detected since

$$\mathcal{S}_{1423} = x_1^6 + 3x_1x_2^5 + 6x_1^2x_2^4 + 7x_1^3x_2^3 + 6x_1^4x_2^2 + 3x_1^5x_2 + x_1^6$$

but $\mathcal{S}_{w_0} = \mathcal{S}_{4213} = x_1^3x_2^3x_3$ and hence vanishing is seen by Theorem 1.1. □

Define the descent cycling equivalence ∼ on Schubert problems by

(dc.1) $(u, v, w) \sim (us_i, v, ws_i)$, $(u, vs_i, ws_i)$ if $us_i \leq u, vs_i \geq v, ws_i \leq w$;

(dc.2) $(u, v, w) \sim (us_i, v, ws_i)$, $(us_i, vs_i, w)$ if $us_i \leq u, vs_i \geq v, ws_i \geq w$;

(dc.3) $(u, v, w) \sim (us_i, w, ws_i)$, $(us_i, vs_i, w)$ if $vs_i \leq u, vs_i \geq v, ws_i \geq w$.

Fact 5.7 ([12]). $C_{u,v,w} = C_{u',v',w'}$ if $(u, v, w) \sim (u', v', w')$.

In particular, $C_{u,v,w} = 0$ if $(u, v, w)$ is equivalent to a dc trivial problem.

Example 5.8. A reported in [12], for $n = 6$ there is one dc equivalence class of problems $(u, v, w)$ which vanishes but does not contain a dc trivial triple. This is precisely the problem studied in Example 4.12, which our methods also cannot explain.

Example 5.9. Let $(u, v, w) = (3216547, 3216547, 4261573)$ be a problem in $S_7$. Theorem 1.1 shows $C_{u,v,w} = 0$ (any element of Tab must contain at least seven 1’s). The ∼ class contains 9 elements, namely

$$(3216574, 3261574, 4216537), (3216547, 3216574, 4261537), (3261547, 3216574, 4261537),$$
$$(3261547, 3216547, 4261573), (3261574, 3216547, 4216573), (3261547, 3261574, 4216573),$$
$$(3261574, 3261547, 4216537), (3216547, 3261574, 4216573), (3216547, 3261574, 4216573).$$

None are dc trivial and thus Fact 5.7 is inconclusive.
5.3. **K. Purbhoo’s root games.** K. Purbhoo’s root games from [18] give a vanishing criteria. Fix the positive roots $\Phi^+$ associated to $GL_n$ to be $\alpha_{i,j} = \varepsilon_i - \varepsilon_j$ for $1 \leq i < j \leq n$, where $\varepsilon_i$ is the $i$-th standard basis vector. The poset $P$ of positive roots takes the form

\[
\begin{array}{cccccccc}
\alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} & \alpha_{17} \\
\alpha_{23} & \alpha_{24} & \alpha_{25} & \alpha_{26} & \alpha_{27} \\
\alpha_{34} & \alpha_{35} & \alpha_{36} & \alpha_{37} \\
\alpha_{45} & \alpha_{46} & \alpha_{47} \\
\alpha_{56} & \alpha_{57} \\
\alpha_{67}
\end{array}
\]

The maximal element of this poset is the highest root $\alpha_{1n}$. For each $i$ place a token $\bullet$ in square $\alpha_{mn}$ if $w^{(i)}(m) > w^{(i)}(n)$. This is called the initial position. An upper order filter $A$ is an up-closed subset of $P$. This initial position is doomed if there exists an upper order filter $A$ such that there are more tokens in $A$ than $\#A$. This is [18, Theorem 3.6]:

**Fact 5.10 (Doomed root game).** If $(w^{(1)}, \ldots, w^{(k)})$’s initial position is doomed, $C_{w^{(1)}, \ldots, w^{(k)}} = 0$.

This test is quite handy. However, the number of upper order filters for type $A_{n-1}$ is the Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$, which is exponential in $n$.

**Example 5.11.** The vanishing of $(1423, 1423, 1342)$ is seen by Fact 5.10. This is doomed:

As is explained in Example 5.5, our methods are inconclusive here.

**Example 5.12.** Let $u = v = 3216547$ and $w = 1652473$. Below we mark the inversions of $u, v, w$ with $\bullet, \bullet, \bullet$ respectively.

This game is not doomed, so Fact 5.10 is inconclusive here. (Descent cycling doesn’t help either, as the equivalence class of size 9 contains no dc trivial elements.) Also, Theorem 1.1 does not succeed. However, Theorem 4.7’s test shows $C_{w^{(1)}, w^{(2)}} = C_{3216547, 3216547} = 0$.

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DEPT. OF MATHEMATICS, U. ILLINOIS AT URBANA-CHAMPAIGN, URBANA, IL 61801, USA

Email address: stdizie2@illinois.edu, ayong@illinois.edu