Vibration of functionally graded plate resting on viscoelastic elastic foundation subjected to moving loads

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Abstract. The dynamics of plates subjected to a moving load must be considered by engineering mechanics and design structures. This paper deals with the dynamic responses of functionally graded (FG) rectangular plates resting on a viscoelastic foundation under moving loads. It is assumed that material properties of the plate vary continuously in the thickness direction according to the power-law. The governing equations are derived by using Hamilton’s principle, which considers the effect of the higher-order shear deformation in the plate. Transient responses of simply supported FG rectangular plates are employed by using state-space methods. Several examples are given for displacement and stresses in the plates with various structural parameters, and the effects of these parameters are discussed.

1. Introduction

The dynamics of beams and plates resting on foundations have been investigated during the past several decades. Beams and plate structures supported by soil and rock are widespread in civil engineering. To describe the interactions of beams, plates, and foundations, various models have been proposed. The simplest one is the Winkler model for elastic foundations, in which the foundation comprises separated springs with constant stiffness. Pasternak [1] proposed a new model by taking into account the shear interactions between adjacent Winkler spring elements. In addition, more efficient foundation models have been proposed; for instance, the modified Pasternak model [2] for reinforced soil, and the nonlinear foundation models [3, 4].

Numerous studies have solved the dynamics of plates resting on foundations using analytical, numerical or semi-analytical methods. Tang [5] investigated an infinite plate strip on a foundation of Kelvin material in terms of a moving load using the integral transform technique. Zaman et al. [6] proposed the finite element model for the dynamic response of thick plates resting on a viscoelastic foundation.

Huang et al. [7] used the finite strip method to develop a model for the dynamic response of plates resting on an elastic foundation in terms of moving loads. Trung et al. developed a cell-based smoothed three-node element for determining dynamics of composite [8] and Midlin plates [9] resting on a viscoelastic foundation. Barati [10] studied the analysis of functionally graded nanoplates with nanovoids on a viscoelastic substrate under hygro-thermo-mechanical loading.

In this work, analytical solutions for the dynamic response of FG rectangular plates resting on a viscoelastic foundation are employed. The FG rectangular plate is subjected to a concentrated moving load at the upper surface of the plate. The governing equations are solved by using state-space methods, and the effects of different parameters on the response of the plate are studied.
2. Governing equation of vibration of FG plate

2.1. FG plate model

We consider an FG plate resting on a viscoelastic foundation; the geometry of the plate and coordinate system is shown in Fig.1:

![Figure 1. The Geometry of FG plate.](image)

The viscoelastic foundation is modeled as a Winkler model with stiffness coefficient $K_e$ and damping coefficient $K_v$.

Applying the higher-order shear deformation plate theory with another ‘refined plate theory’ proposed by Shimpi [11], the displacement components at an arbitrary point $(x, y, z)$ in the plate are as follows:

$$
U(x, y, z, t) = u(x, y, z, t) - z \frac{\partial w_t}{\partial x} + \left[ \frac{1}{4} z - \frac{5}{3} \left( \frac{z}{h} \right)^2 \right] \frac{\partial^2 w_t}{\partial x^2}
$$

$$
V(x, y, z, t) = v(x, y, z, t) - z \frac{\partial w_t}{\partial y} + \left[ \frac{1}{4} z - \frac{5}{3} \left( \frac{z}{h} \right)^2 \right] \frac{\partial^2 w_t}{\partial y^2}
$$

$$
W(x, y, z, t) = w_t(x, y, z, t) + w_{i}(x, y, t)
$$

(1)

where $(u, v, w_t, w_i)$ are the displacement components on the mid-plane.

The linear strains can be obtained by differentiating Eq. as:

$$
\begin{bmatrix}
\varepsilon_{xx}^0 \\
\varepsilon_{yy}^0 \\
\gamma_{xy}^0
\end{bmatrix} =
\begin{bmatrix}
\frac{\partial u}{\partial x} \\
\frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}
\end{bmatrix} +
\begin{bmatrix}
K_e^0 \\
K_e^0 \\
K_e^0
\end{bmatrix} +
\begin{bmatrix}
K_v^0 \\
K_v^0 \\
K_v^0
\end{bmatrix} +
\begin{bmatrix}
\gamma_{xx}^0 \\
\gamma_{yy}^0 \\
\gamma_{xy}^0
\end{bmatrix} =
\begin{bmatrix}
\varepsilon_{xx}^1 \\
\varepsilon_{yy}^1 \\
\gamma_{xy}^1
\end{bmatrix}
$$

(2)

where:

$$
\begin{bmatrix}
\varepsilon_{xx}^0 \\
\varepsilon_{yy}^0 \\
\gamma_{xy}^0
\end{bmatrix} =
\begin{bmatrix}
\frac{\partial^2 w_t}{\partial x^2} \\
\frac{\partial^2 w_t}{\partial y^2} \\
\frac{\partial^2 w_t}{\partial x \partial y}
\end{bmatrix},
\begin{bmatrix}
\varepsilon_{xx}^1 \\
\varepsilon_{yy}^1 \\
\gamma_{xy}^1
\end{bmatrix} =
\begin{bmatrix}
\frac{\partial^3 w_t}{\partial x^3} \\
\frac{\partial^3 w_t}{\partial y^3} \\
\frac{\partial^3 w_t}{\partial x^2 \partial y} - 2 \frac{\partial^2 w_t}{\partial x \partial y^2}
\end{bmatrix},
\begin{bmatrix}
\varepsilon_{xx}^0 \\
\varepsilon_{yy}^0 \\
\gamma_{xy}^0
\end{bmatrix} =
\begin{bmatrix}
\frac{\partial^2 w_i}{\partial x^2} \\
\frac{\partial^2 w_i}{\partial y^2} \\
\frac{\partial^2 w_i}{\partial x \partial y}
\end{bmatrix},
\begin{bmatrix}
\varepsilon_{xx}^1 \\
\varepsilon_{yy}^1 \\
\gamma_{xy}^1
\end{bmatrix} =
\begin{bmatrix}
\frac{\partial^3 w_i}{\partial x^3} \\
\frac{\partial^3 w_i}{\partial y^3} \\
\frac{\partial^3 w_i}{\partial x^2 \partial y} - 2 \frac{\partial^2 w_i}{\partial x \partial y^2}
\end{bmatrix}.
The mechanical properties of FGM, such as Young's modulus $E$ and mass density $\rho$, are assumed as

$$E(z) = (E_m - E_c) \left( \frac{z}{h} + \frac{1}{2} \right)^n + E_c$$

$$\rho(z) = (\rho_m - \rho_c) \left( \frac{z}{h} + \frac{1}{2} \right)^n + \rho_c$$

where the subscripts $m$ and $c$ represent the metallic and ceramic constituents, respectively, and $n$ is the power index of the volume fraction.

Substituting Eq. (1) into Eq. (2) to obtain a strain expression. They can be used in constructing the strain energy and kinetic energy, and are dissipation expressions. Using Hamilton’s principle to derive the equations of motion; we obtain the equations of motion of a plate as follows:

$$\delta u : \frac{\partial N_{zz}}{\partial x} + \frac{\partial N_{zz}}{\partial y} = I_u \ddot{u} - I_u \frac{\partial \dot{u}}{\partial x} - J_u \frac{\partial \dot{w}}{\partial x}$$

$$\delta v : \frac{\partial N_{zz}}{\partial x} + \frac{\partial N_{zz}}{\partial y} = I_v \ddot{v} - I_v \frac{\partial \dot{v}}{\partial x} - J_v \frac{\partial \dot{w}}{\partial x}$$

$$\delta w : \frac{\partial^3 M_{xx}}{\partial x^2} + 2 \frac{\partial^3 M_{xx}}{\partial x \partial y} + \frac{\partial^3 M_{xx}}{\partial y^2} + K_z (w_x + w_y) - K_z (\dot{w}_x + \dot{w}_y) + f$$

$$= I_w (\ddot{w}_x + \ddot{w}_y) + J_w \frac{\partial \dot{w}}{\partial x} + J_w \frac{\partial \dot{w}}{\partial y} - I_z \nabla^2 \ddot{w}_x - J_z \nabla^2 \ddot{w}_y$$

$$\delta w : \frac{\partial^3 M_{yy}}{\partial x^2} + 2 \frac{\partial^3 M_{yy}}{\partial x \partial y} + \frac{\partial^3 M_{yy}}{\partial y^2} + \frac{\partial Q_{xx}}{\partial x} + \frac{\partial Q_{xx}}{\partial y} - K_z (w_x + w_y) - K_z (\dot{w}_x + \dot{w}_y) + f$$

$$= I_w (\ddot{w}_x + \ddot{w}_y) + J_w \frac{\partial \dot{w}}{\partial x} + J_w \frac{\partial \dot{w}}{\partial y} - I_z \nabla^2 \ddot{w}_x - J_z \nabla^2 \ddot{w}_y$$

where

$$I_w = \int_{-h/2}^{h/2} \rho(z) \varepsilon \, dz, \quad I_z = -\frac{1}{4} I + \frac{5}{3h^2} I_{zz}$$

$$K_z = \frac{1}{16} I - \frac{5}{6h^2} I_{zz} + \frac{25}{9h^4} I_1$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

The moving load is a concentrated moving load with constant speed and can be represented by the Delta Dirac function, as follows:

$$P = P(t) \delta \left( x - x_{mov}(t) \right) \delta \left( y - y_{mov}(t) \right)$$

where $P(t)$ is the magnitude of the moving load, and $x_{mov}(t)$ and $y_{mov}(t)$ are the coordinates of the location of the load. In this work, the magnitude of the moving load is assumed to be constant $P = P_0$, $x_{mov}(t) = V_0 t$.

Also, the moving load is assumed to move along a $y$-$y$ direction at the central plate: $y_{mov}(t) = b / 2$.

### 3. Analytical solution for vibration of FGM plate

In this work, the state-space representations of the dynamic systems will be used to analyze the transient response of a simply supported rectangular FG plate. The Navier approach is used to derive the closed-form solutions of equations of motion. The sinusoidal function is chosen to satisfy all boundary conditions, as follows:
\[
\begin{align*}
\alpha &= \frac{m\pi}{a}, \quad \beta = \frac{n\pi}{b} \\
\text{Substitution Eq. (6) into Eq. (4), the forced vibration of the functionally graded plate can be written as follows:} \\
\begin{bmatrix}
s_{11} & s_{12} & s_{13} & s_{14} \\
s_{21} & s_{22} & s_{23} & s_{24} \\
s_{31} & s_{32} & s_{33} & s_{34} \\
s_{41} & s_{42} & s_{43} & s_{44}
\end{bmatrix}
\begin{bmatrix}
U_{mn} \\
V_{mn} \\
W_{mn}
\end{bmatrix}
=
\begin{bmatrix}
0 & 0 & 0 & 0 & \hat{U}_{mn} \\
0 & 0 & 0 & 0 & \hat{V}_{mn} \\
0 & 0 & \hat{c}_{44} & 0 & \hat{W}_{mn}
\end{bmatrix}
\end{align*}
\]

where:
\[
F_{mn} = \frac{4P_0}{ab} \sin(\alpha V_{t}) \sin\left(\frac{\beta b}{2}\right)
\]

Eq. (8) must be rewritten in order to find a solution, as follows:
\[
Z = AZ + b
\]

where:
\[
Z = \begin{bmatrix} U_{mn} & V_{mn} & W_{mn} & U_{mn} & V_{mn} & W_{mn} \end{bmatrix}^T
\]
\[
b = \begin{bmatrix} 0 & 0 & 0 & b_1 & b_2 & b_3 & b_4 \end{bmatrix}^T
\]

and \(b_i\) are the terms of the column matrix
\[
\begin{bmatrix} b_1 & b_2 & b_3 & b_4 \end{bmatrix}^T = M^{-1} F^*
\]

and block matrix A is
\[
A = \begin{bmatrix} 0 & I \\
-M^{-1} S & 0 \end{bmatrix}
\]

The solution of is obtained as
\[
\dot{Z}(t) = e^{A(t-t_0)} Z(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} b(\tau) d\tau
\]

where \(t_0\) is the initial time, \(Z(t_0)\) is the initial response, and the exponential matrix is \(e^{A(t-t_0)}\).
This exponential matrix can be formulated in terms of the matrix of eigenvectors and eigenvalues associated with matrix $A$.

4. Examples

We consider a simply supported rectangular FG plate with side-to-thickness ratio $a/h=40$, rectangular dimensions of $a=0.4\text{m}$, $b=0.3\text{m}$, and power index $p=2$. The elastic moduli and mass density are chosen to be the same as [12]: $E_a = 70\text{GPa}$, $\rho_a = 2707\text{kg/m}^3$, $E_c = 380\text{GPa}$, $\rho_c = 3800\text{kg/m}^3$, with Poisson’s ratio being 0.3. The moving load is written as $P_0 = 5000(\text{N})$, $v = 50\left(\frac{\text{m}}{\text{s}}\right)$.

For convenience, the normalized stiffness coefficient and the normalized damping coefficient of the foundation are defined as follows:

$$k_n = \frac{K_n}{D_n}, \quad k_d = \frac{K_d}{a^2D_n}$$

where $D_n = \frac{E_n h^3}{12(1-\nu^2)}$ is the flexural rigidity of a full-metal plate.

In order to investigate the effect of the damping coefficient of the viscoelastic foundation, we investigate two cases of viscoelastic foundation:

$$k_n = 2, \quad k_d = 0.2, \quad k_n = 2, \quad k_d = 0.8$$

Fig.2 illustrates the transverse deflection as functions of time with a parameter $k_n = 0.2$, $k_d = 0.8$, respectively. The response of stress at the center of the FG plate is shown in Figs. 3–4. Two cases with different damping coefficients of viscoelastic foundation are employed (i.e., $k_n = 0.2$ and $k_n = 0.8$). In the forced vibration regime, it can be seen in Figs. 2–4 that the deflection and stresses predicted for FG plates with $k_n = 0.2$ are moderately larger than for the plate with $k_n = 0.8$. It is observed that when the damping coefficient increases, the deflection and stresses become smaller, as expected.

![Figure 2. The deflection response at the central plate.](image-url)
5. Conclusions
This work applied a state-space method to determine forced vibration of FG plates resting on a viscoelastic foundation. The analytical solution for a dynamic FG plate was found to be a double sine series based on the Navier approach. The dynamic responses are considered for both forced and free vibrations. The results indicate that the damping coefficient of the foundation has significant effects on the dynamic response of the FG plate. The damping coefficient of the foundation causes the dissipation of energy, reducing ambient vibration at forced and free vibrations. This work contributes significantly to the field of engineering structures, and the results can be used in practical design structures.

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