NOETHER-LEFSCHETZ THEOREM WITH BASE LOCUS

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Abstract. For an arbitrary curve $Z \subset \mathbb{P}^3$ (possibly reducible, non-reduced, unmixed) lying on a normal surface, the general surface $S$ of high degree containing $Z$ is also normal, but often singular. We compute the class groups of the very general such surface, thereby extending the Noether-Lefschetz theorem (the special case when $Z$ is empty). Our method is an adaptation of Griffiths and Harris’ degeneration proof, simplified by a cohomology and base change argument. We give applications to computing Picard groups.

Dedicated to Robin Hartshorne on his 70th birthday

1. Introduction

The algebraic surfaces in $\mathbb{P}^3_\mathbb{C}$ of degree $d$ are parametrized by the projective space $\mathbb{P} H^0(\mathcal{O}_{\mathbb{P}^3}(d))$ via their equations. The Noether-Lefschetz locus $NL(d) \subset \mathbb{P} H^0(\mathcal{O}_{\mathbb{P}^3}(d))$ corresponds to the smooth surfaces $S$ with $\text{Pic} S \neq \langle \mathcal{O}_S(1) \rangle$. The original Noether-Lefschetz theorem, suggested by Noether in the 1880s and proved by Lefschetz in the 1920s, says that $NL(d)$ is a countable union of proper subvarieties if $d > 3$; in other words, the very general such surface $S$ satisfies $\text{Pic} S = \langle \mathcal{O}_S(1) \rangle$. Work of Ciliberto, Green, Harris, Lopez, Miranda and Voisin around 1990 dramatically increased our understanding of the components of $NL(d)$. Mumford’s challenge to write an explicit equation of a such a quartic surface $S$ was finally met by van Luijk [22] in the last few years.

Carlson, Green, Griffiths and Harris proved the infinitesimal Noether theorem using variations of Hodge structures [2]; this method was used by Ein to compute the Picard group of the dependency locus of generic subspaces of sections of sufficiently ample vector bundles on arbitrary projective manifolds [6] and by Green to explicitly bound the codimension of components of $NL(d)$ [9]. An approach from unpublished notes of Kumar and Srinivas led to Joshi’s version for ambient projective threefolds [16] and the very recent extension to normal ambient varieties by Ravindra and Srinivas [21].

While these recent generalizations are powerful and interesting, we have been impressed by Griffiths and Harris’ degeneration method [8], which relies on neither cohomological vanishings nor deformation theories, making it applicable to singular surfaces of low degree. Lopez used it to compute Picard groups of general surfaces $S \subset \mathbb{P}^3$ containing a smooth connected curve $Z$ [18 II.3.8], a result with many applications [3, 5, 7]. We extend this result, replacing $Z$ with an arbitrary subscheme (possibly reducible, non-reduced, or of mixed dimension) which properly lies on a normal surface. Since the surfaces containing $Z$ are often necessarily singular, it is more natural to compute their class groups:
Theorem 1.1. Let $Z \subset \mathbb{P}_k^n$ be a closed subscheme of dimension $\leq 1$ with embedding dimension $\leq 2$ at all but finitely many points and fix $d \geq 4$ with $\mathcal{I}_Z(d-1)$ generated by global sections. Suppose that either

1. $Z$ is reduced of embedding dimension two or
2. $H^0(\mathcal{I}_Z(d-2)) \neq 0$.

Then the very general surface $S \in |H^0(\mathcal{I}_Z(d))|$ is normal with $\text{Cl} S$ freely generated by $\mathcal{O}_S(1)$ and supports of the curve components of $Z$.

Remarks 1.2. (a) The hypotheses imply that $Z$ lies on a normal surface $T$ of degree $d-1$ with finitely generated class group. In case (1) this is because $Z$ actually lies on a smooth such surface and in case (2) this follows from Theorem 1.1. Thus Theorem 1.1 follows from our more general Theorem 4.5 and Proposition 4.3.

(b) With weaker hypothesis on $Z$, the general surface $S \in |H^0(\mathcal{I}_Z(d))|$ is not normal. Here one may consider the group $\text{APic } S$ of almost Cartier divisors from Hartshorne’s theory of generalized divisors [12], but we would expect $\text{APic } S$ to be infinitely generated due to the behavior along the curve part of the singular locus [12, Example 5.4 and Proposition 6.3].

As an application, we compute some Picard groups:

Corollary 1.3. With the hypotheses of Theorem 1.1, let $Z_1, Z_2, \ldots, Z_r$ be the curve components of $Z$. Then

(a) If $\dim Z = 0$ (i.e. $r = 0$), then $\text{Pic } S = \langle \mathcal{O}_S(1) \rangle$.

(b) If $Z$ is a reduced l.c.i. curve and the $Z_i$ intersect at points of embedding dimension 2 of $Z$, then $\text{Pic } S = \langle \mathcal{O}_S(1), Z_1, \ldots, Z_r \rangle$.

(c) If $Z$ is an integral l.c.i. curve, then $\text{Pic } S = \langle \mathcal{O}_S(1), \mathcal{O}_S(Z) \rangle$.

(d) If $Z$ has embedding dimension 2, then $\text{Pic } S = \langle \mathcal{O}_S(1), Z_1, \ldots, Z_r \rangle$.

Moreover, the generating sets given in (b), (c), and (d) freely generate the given Picard groups.

Proof: (a). Here $Z$ has no irreducible curve components, so $\text{Cl } S = \langle \mathcal{O}_S(1) \rangle$, but $\mathcal{O}_S(1)$ is Cartier on $S$. This strengthens Joshi’s result extending the Noether-Lefschetz theorem to very general singular surfaces [13, 4.4].

(b) and (c). Here $\text{Cl } S$ is generated as in the theorem, but the $Z_i$ intersect at smooth points of a general such surface $S$, so each $Z_i$ is Cartier on $S$ and we have $\text{Cl } S = \text{Pic } S$. The special case (c) where $r = 1$ strengthens [18, Cor. II.3.8].

For part (d), $\text{Cl } S = \langle \mathcal{O}_S(1), W_1, \ldots, W_r \rangle$ by Theorem 1.1 where $W_i$ are the supports of the curve components of $Z$. To compute $\text{Pic } S$, we use the exact sequence

$$0 \to \text{Pic } S \to \text{Cl } S \to \bigoplus_{\text{codim } p = 2} \text{APic}(\text{Spec } \mathcal{O}_{S,p})$$

introduced by Jaffe [14] and developed by Hartshorne [12, 2.15]. The general surface $S$ is smooth where the components of $Z$ intersect and are singular at a finite number of points $p$ along the $Z_i$ of multiplicity $m_i > 1$, each singularity having local equation $xy - z^{m_i}$ by Proposition 2.2(b). For this type of singularity Hartshorne has shown that $\text{APic } \text{Spec } \mathcal{O}_{S,p} \cong \mathbb{Z}/m_i \mathbb{Z}$ generated by the class of $W_i$ [14, Proposition 5.2]. Assembling the kernels of the pieces, we see that $\text{Pic } S$ is generated by $\mathcal{O}(1)$ and $m_i W_i = Z_i$. 


Example 1.4. The conclusion of Corollary 1.3 (b) can fail if the $Z_i$ do not meet at points of embedding dimension 2. The cone $Z \subset \mathbb{P}^3$ over 4 planar points in general position consists of four lines $Z_i$ meeting at a point $p$ and is a complete intersection of two reducible quadrics, we may write $I_Z = (l_1l_2, l_1l_4)$. A very general degree $d \geq 4$ surface $S$ containing $Z$ is singular only at $p$ and has equation $Fl_1l_2 - Gl_3l_4 = 0$ with $F(p), G(p) \neq 0$, hence the local ring of $S$ centered at $p$ can be written $\mathbb{C}[x, y, z]/(ul_1l_2 - vl_3l_4)$, where $u, v$ are units and $l_i$ are general linear forms. This is isomorphic to the local ring of the vertex of a quadric cone, so $\text{APic} S \cong \mathbb{Z}/2\mathbb{Z}$ generated by the class of a ruling \cite{10} II, Example 6.5.2 and any of the lines $Z_i$ will do. Theorem \cite{10} says that $\text{Cl} S = (\mathcal{O}(1), Z_1, Z_2, Z_3, Z_4)$ and the map $\text{Cl} S \rightarrow \text{APic} S \cong \mathbb{Z}/2\mathbb{Z}$ in sequence \cite{10} takes $0(1)$ to zero and each $Z_i$ to the generator, hence

$$\text{Pic} S = \{\mathcal{O}(a) + \sum a_iL_i : 2|\sum a_i\} \subset \text{Cl} S.$$ 

Remark 1.5. One of our motivations is an application to Franco and Lascu’s characterization of contractable curves \cite{7}: if $Y \subset \mathbb{P}^3$ is an integral local complete intersection curve, then the following are equivalent:

1. $Y$ is $\mathbb{Q}$-subcanonical.
2. $Y$ is $\mathbb{Q}$-Gorenstein contractable to a point in a normal surface in $\mathbb{P}^3$ containing $Y$ as a Cartier divisor.
3. $Y$ is contractable on general surfaces of high degree.

The implication (3) $\Rightarrow$ (1) in their proof requires knowing that the Picard group of the general high degree surface $S$ containing $Y$ is generated by $Y$ and $\mathcal{O}_S(1)$, which is Corollary 1.3 (b).

We also extend the Grothendieck-Lefschetz theorem for divisors with base locus, using results of Ravinda and Srinivas \cite{20}. First we note the weakest conditions that allow a subscheme to lie on a normal hypersurface.

Definition 1.6. A closed subscheme $Z$ of a smooth ambient variety $M$ is superficial if (i) $\text{codim}(Z, M) \geq 2$ and (ii) the closed subset $F \subset Z$ of points where the embedding dimension of $Z$ is equal to $\dim M$ satisfies $\text{codim}(F, M) \geq 3$.

Theorem 1.7. Let $Z \subset \mathbb{P}^n_\mathbb{C}$ be superficial closed subscheme with $n \geq 3$. If $I_Z(d)$ is generated by global sections and $H^0(I_Z(d-1)) \neq 0$, then

1. The Zariski general $S \in |H^0(I_Z(d))|$ is normal with finitely generated class group.
2. If $n > 3$, then

$$\text{Cl} S = \{\mathcal{O}_S(1), W_1, \ldots, W_r\}$$

where $W_i$ are the supports of the codimension-2 components of $Z$.

Example 1.8. The conclusions can fail if $H^0(I_Z(d-1)) = 0$: let $Z \subset \mathbb{P}^3$ be the complete intersection of cones over two smooth plane curves of degree $d > 2$ with common vertex. The general surface $V$ of degree $d$ containing $Z$ is a cone over such a curve $C$, so $\text{Cl} V \cong \text{Cl} C = \text{Pic} C$ \cite{10} II, Exer. 6.3 (a) is infinitely generated. The cone over this example in $\mathbb{P}^4$ shows that part (b) also fails for $n > 3$ without the $h^0(I_Z(d-1)) \neq 0$ condition.

Remark 1.9. A few words are in order comparing our main theorem to related results.
When $Z$ is a smooth connected curve, theorem [17] delivers exactly [18, II, Corollary 3.8] of Lopez. Moreover, since the blow-up $M = \mathbb{P}^3 \to \mathbb{P}^1$ at $Z$ in diagram (3) below embeds into $\mathbb{P} H^0(I_Z(d))^*$ by strict transforms of the linear system of degree-$d$ surfaces containing $Z$, and the general such surface maps isomorphically onto its image in $\mathbb{P}^3$, this result also follows from results of Ein [6], Joshi [14], or the very recent theorem of Ravindra and Srinivas [21] applied to $M$.

The hypotheses for the results of Ein, Joshi, and Ravindra-Srinivas do not hold on $M$ for more general $Z$ (for example if $Z$ has some isolated points), so our result is of independent interest. On the other hand, these results apply to surfaces on more general threefolds than $\mathbb{P}^3$, provided that some conditions (vanishing of cohomology, respectively global generation of a certain sheaf) can be verified. The main theorem of Lopez [18, II. Theorem 3.1] is also independent, as it relies on less restrictive hypotheses (the corresponding line bundle on $M$ need not be ample).

In section 2 we blow up the base locus of a linear system to interpret our results in terms of divisors on blow-ups and prove Theorem 1.7. Section 3 is an adaption of the degeneration method used by Griffiths, Harris and Lopez [8, 18] to the case of families of singular surfaces. For this we smooth the surfaces and form an étale cover of the family where we can sort out the exceptional divisors. In the last section we prove the main theorem. Throughout we work over the field $\mathbb{C}$ of complex numbers, since characteristic zero Bertini theorems, generic smoothness, and monodromy arguments are used in the last two sections.\hspace{1em}**Acknowledgments:** As well as wishing him a happy birthday, we thank Robin Hartshorne for his teachings, helpful comments and Example 4.2. The second named author thanks Rosa Maria Miró-Roig for asking a question which inspired this work and Andrew Sommese for useful conversations.

2. Finite generation of the class group

Let $L$ be a line bundle on a smooth variety $M$ and $V \subset H^0(M, L)$ a linear system defining a rational map $\phi : M \to \mathbb{P}V^*$. The image of the natural map $V \otimes L^{-1} \to O_M$ defines the ideal of the base locus $Z \subset M$ for $V$. If $f : \widetilde{M} \to M$ is the blow-up at $Z$, there is a closed immersion $i : \widetilde{M} \hookrightarrow M \times \mathbb{P}V^*$ whose image is the graph of $\phi$ and we have a diagram

$$
\begin{array}{ccc}
E & \subset & \widetilde{M} \\
\downarrow & & \downarrow f \\
Z & \subset & M \\
\end{array}
$$

with exceptional divisor $E$ and the map $\sigma = i \circ \pi_2$ given by the invertible sheaf $\sigma^*(O(1)) = f^*(L) \otimes O_{\widetilde{M}}(-E)$ ([10, II, Example 7.17.3] and [1, Theorem 1.3]).

**Proposition 2.1.** In the setting of diagram (3), assume that $Z$ is superficial (Definition [4,0]) with codimension-2 irreducible components $Z_i$. Then

(a) The general member $X \in |V|$ is normal.

(b) Let $F \subset Z$ be the closed set where $I_Z$ is not 2-generated or $Z$ has embedding dimension equal to $\dim M$. Then $\widetilde{M} - f^{-1}(F)$ is normal with group class

$$\text{Cl}(\widetilde{M} - f^{-1}(F)) = \langle f^*(\text{Pic} M), W_i \rangle$$

where $W_i = \text{Supp} f^{-1}(Z_i - F)$.
Proof: Since $Z$ is a codimension two local complete intersection off of $F$, the projection $\tilde{M} - f^{-1}(F) \to M - F$ is a $\mathbb{P}^1$-bundle over $Z - F$ and an isomorphism elsewhere. Letting $\Sigma \subset \tilde{M}$ denote the singular locus, $\Sigma - f^{-1}(F)$ is a set-theoretic section over the non-smooth locus of $Z - F$ because the embedding dimension of $Z$ is less than dim $M$ away from $F$ [19, Theorem 2.1], hence $\tilde{M} - f^{-1}(F)$ is regular in codimension one away from $\Sigma$. Therefore the singularities of $X$ are isolated points of $Z$, $M - f^{-1}(F)$ (because codim$(F, M) > 2$), hence by repeated application of [10, II, 6.5].

For $X \in |V|$, view $\tilde{X} = f^{-1}(X)$ as a hyperplane section $\sigma^{-1}(H)$ with $H \in (\mathbb{P}V^*)^* = |V|$. Bertini theorems tell us that $\tilde{X} - f^{-1}(F)$ is regular in codimension one. For $z \in Z_i - F$, the fibres $f^{-1}(z) \cong \mathbb{P}^1$ map isomorphically to straight lines in $\mathbb{P}V^*$ because $-E$ is the relative $\mathcal{O}(1)$ in the construction of the blow-up: since $f^{-1}(z)$ contains at most one singular point of $\tilde{M}$ [19, 2.1], the general hyperplane $H$ meets this line transversely in a reduced point so the map $f : \tilde{X} \to X$ is a generic isomorphism along $Z$ (and an isomorphism away from $f^{-1}(Z)$). Thus $X$ is regular in codimension one away from $F$, and hence regular in codimension one because codim$(F, M) > 2$. Divisors on smooth $M$ satisfy $S_2$, so general $X \in |V|$ are normal [10, II, Proposition 8.23 (b)].

Proposition 2.2. In the setting of Proposition 2.1 with dim $M = 3$, the general surface $X \in |V|$ has singularities of two types:

(a) Fixed: Points $F = \{z\}$ where $Z$ has embedding dimension 3.

(b) Moving: Away from $F$, there are a constant number of singularities along each $Z_i$ with multiplicity $m_i > 1$; these move with $X$. For $X$ general, they have local equation $xy = zm_i = 0$.

Proof: Resuming the previous proof, we’ve seen that for $z \in Z_i - F$, the general hyperplane $H \subset (\mathbb{P}V^*)^*$ meets the line $\sigma(f^{-1}(z))$ once, but we can say more. Consider the incidence

$$I = \{(z, H) : z \in \bigcup Z_i - F, \sigma(f^{-1}(z)) \subset H\}.$$ 

The fibres over the first projection to $\bigcup Z_i - F$ have dimension dim $\mathbb{P}V - 2$, so dim $I = $ dim $\mathbb{P}V - 1$ and dim $\pi_2(I) \leq $ dim $\mathbb{P}V - 1$, therefore the general hyperplane $H \in (\mathbb{P}V^*)^*$ meets every such line $\sigma(f^{-1}(z))$ once. It follows that $f : \tilde{X} - f^{-1}(F) \to X - F$ is an isomorphism in a neighborhood of $\bigcup Z_i - F$, at least away from the isolated points of $Z$ which have embedding dimension two, where $X$ is already smooth. Therefore the singularities of $X$ away from $F$ are identified with those of $\tilde{X}$.

If $G$ is the finite singular set of the support of $\bigcup Z_i - F$, $Z_i$ is locally a multiple of a smooth curve on a smooth surface away from $G$, so the ideal of $Z_i$ has the form $(x, y^{m_i})$, where $(x, y)$ is the ideal of the support, $m_i$ is the multiplicity, and $(x, y, z)$ is a regular sequence of parameters for the local ring $R = \mathcal{O}_{M, z}$. The blow-up of this ideal is covered by two affines, one being Spec $R[u]/(ux - y^{m_i})$.
which is singular exactly at the origin (the other is smooth). We conclude that $\Sigma_i = \Sigma \cap f^{-1}(Z_i - F - G)$ is a section over $Z_i - F - G$. If the image $\sigma(\Sigma_i)$ is a point, then $H$ misses $\sigma(\Sigma_i)$ and $X = \sigma^{-1}(H)$ is smooth along $f^{-1}(Z_i - F - G)$. If $\sigma(\Sigma_i)$ is an integral curve of degree $d$, and the map $\Sigma_i \to \sigma(\Sigma_i)$ has degree $e$, then $X'$ has exactly $(e \cdot d_i)$ singularities along $\Sigma_i$ for general $H$. Since general $H$ meets $\sigma(\Sigma_i)$ transversely at each point, $X = \sigma^{-1}(H)$ has singularities with the same equation as above.

**Corollary 2.3.** Let $Z \subset \mathbb{P}^3$ be superficial and assume that $I_Z(d)$ is generated by global sections. Then the general surface $S$ of degree $d$ containing $Z$ is normal with constant number of singularities, as described in Proposition 2.2.

**Example 2.4.** For a concrete example, let $Z$ be the double structure on the line $L : x = y = 0$ contained in the smooth cubic surface $T \subset \mathbb{P}^3$ with equation $x^3 + y^3 + xw^2 + yz^2 = 0$ so that $I_Z = (x^2, xy, y^2, xw^2 + yz^2)$. A degree $d \geq 3$ surface $S$ containing $Z$ has equation

$$Ax^2 + Bxy + Cy^2 + H(xw^2 + yz^2) = 0$$

with $\deg A = \deg B = \deg C = d - 2$ and $\deg H = d - 3$: computing partial derivatives shows that this surface is singular at a point $q = (0, 0, z_0, w_0)$ on $L$ precisely when $H(q) = 0$, so for general $H$ there are exactly $d - 3$ singular points.

(a) If $d = 3$, then the general surface $S$ is smooth since the special surface $T$ is. Here the constant number of moving singularities is $0$, even though $m_1 > 1$. In this case $\sigma(\Sigma_i)$ collapses to a point in the proof above.

(b) For $d > 3$ and $H$ general, $S$ has exactly $d - 3$ type-$A_1$ singularities along the line where $H = x = y = 0$ as in Proposition 2.2. For special $H$ meeting $L$ with higher multiplicity, these singularities can collide.

We close this section with a variant of the Grothendieck-Lefschetz theorem for linear systems with base locus, which implies Theorem 1.7.

**Theorem 2.5.** Let $L$ be a line bundle on a smooth projective variety $M$ and $V \subset H^0(M, L)$ a linear system defining a rational map $\phi : M \to \mathbb{P}^n$ birational onto its image with superficial base locus $Z$. Then the general $X \in |V|$ is normal and

(a) If $\dim M > 3$, then $\ker(\text{Pic}M \to \text{Cl}X)$ is generated by the supports of the codimension-2 components of $Z$.

(b) If $\dim M = 3$, then $\ker(\text{Pic}M \to \text{Cl}X)$ is finitely generated.

**Proof:** Normality of $X$ is Proposition 2.1 (a). For the additional statements, let $\overline{M} \to \overline{M}_{\text{norm}} \to \tilde{M}$ be the normalization followed by a desingularization, with corresponding maps $\overline{f}, \tilde{f}$ to $M$ and $\overline{\sigma}, \tilde{\sigma}$ to $\mathbb{P}V^*$. Let $\overline{X}$ (resp. $\overline{X}_{\text{norm}}$) be a general hyperplane section of $\overline{\sigma}$ (resp. $\tilde{\sigma}$) and let $E_{\overline{M}}$ (resp. $E_{\overline{X}}$) be the union of exceptional divisors for the desingularization $\overline{M} \to \overline{M}_{\text{norm}}$ (resp. $\overline{X} \to \overline{X}_{\text{norm}}$). We have a commutative diagram

$$
\begin{array}{ccc}
\text{Pic}(\overline{M}) & \xrightarrow{\sigma} & \text{Pic}(\overline{X}) \\
\downarrow & & \downarrow \\
\text{Pic}(\overline{M} - E_{\overline{M}}) & \xrightarrow{\tilde{\sigma}} & \text{Pic}(\overline{X} - E_{\overline{X}}) \\
\end{array}
$$

$$
\begin{array}{ccc}
\text{Pic}(\overline{M} - E_{\overline{M}} - \overline{f}^{-1}(F)) & \xrightarrow{\tilde{\sigma}} & \text{Pic}(\overline{X} - E_{\overline{X}} - \overline{f}^{-1}(F)) \\
\end{array}
$$
Since $\hat{\sigma}$ is birational, $\hat{\sigma}^*O(1)$ is a big invertible sheaf, hence $\rho$ has finitely generated cokernel if $\dim M \geq 3$ [20, Theorem 2 (a)]; If $\dim M > 3$, the cokernel of $\rho$ is generated by divisors supported in $E_0$ [20, Theorem 2 (c)] so the middle horizontal map is surjective. Noting that the lower right vertical map is surjective, we conclude that $\tilde{\rho}$ is surjective for $\dim M > 3$ and has finitely generated cokernel if $\dim M = 3$.

Now because $\tilde{M} - f^{-1}(F)$ is normal (Proposition [21(b)], the desingularization $\tilde{M} - \tilde{f}^{-1}(F) \rightarrow \tilde{M} - f^{-1}(F)$ is obtained by blowing up smooth centers in the singular loci (no normalization is required away from $f^{-1}(F)$) and we have the identifications $\tilde{M} - E_{\tilde{M}} - \tilde{f}^{-1}(F) \cong \tilde{M} - \Sigma - f^{-1}(F)$ and similarly $X - E_X - f^{-1}(F) \cong X - \Sigma - f^{-1}(F)$. Thus $\tilde{\rho}$ may be identified with the restriction map $r$

$$\text{Cl}(\tilde{M} - f^{-1}(F)) \xrightarrow{\sim} \text{Cl}(X - f^{-1}(F)).$$

If $G \subset Z$ is the set over which $f : \tilde{X} - f^{-1}(F) \rightarrow X - F$ fails to be an isomorphism (see proof of Proposition [21], composing $r$ with the surjection

$$\text{Cl}(\tilde{X} - f^{-1}(F)) \rightarrow \text{Cl}(\tilde{X} - f^{-1}(F) - f^{-1}(G)) \cong \text{Cl}(X - F - G) \cong \text{Cl} X$$

shows that the last group is generated by $\text{Pic} M$ and the supports of the $Z_t$ (use equation [14] and note that the classes $W_i$ map to $\text{Supp} Z_t$, so we draw conclusions (a) and (b).

**Proof of Theorem 14.7.** If $Z \subset \mathbb{P}^n$ is superficial, $\mathcal{I}_Z(d)$ is generated by global sections and $H^0(\mathcal{I}_Z(d - 1)) \neq 0$, then $0 \neq f \in H^0(\mathcal{I}_Z(d - 1))$ implies that the rational map $\mathbb{P}^3 \rightarrow \mathbb{P}H^0(\mathcal{I}_Z(d))^*$ is an isomorphism away from the hypersurface $f = 0$, hence birational onto its image and Theorem [23] applies.

3. Two Families of Surfaces

We now turn to the harder problem of computing class groups of degree $d$ surfaces in $\mathbb{P}^3$ with fixed base locus $Z$. In Proposition 3.2 we produce an open set $U \subset \mathbb{P}H^0(\mathcal{I}_Z(d))$ and a family of desingularizations $\tilde{X}_t \rightarrow X_t$ for $t \in U$ such that the kernels of the natural maps $\text{Pic} \tilde{X}_t \rightarrow \text{Cl} X_t$ are represented by irreducible divisors over an étale cover $U' \rightarrow U$. We restrict this family to a general pencil containing a reducible surface $T \cup P$, where (after modification) we compute the Picard group of the central fibre (Proposition [5.6]).

Fix $Z \subset \mathbb{P}^3$ superficial with $\mathcal{I}_Z(d - 1)$ globally generated. Interpreting construction [3] with $M = \mathbb{P}^3$, $L = \mathcal{O}(d)$ and $V = H^0(\mathcal{I}_Z(d)) \subset H^0(L)$ yields a closed immersion $\sigma_d : \mathbb{F}^3 \rightarrow \mathbb{P}H^0(\mathcal{I}_Z(d))^*$, where $\mathbb{F}^3 \rightarrow \mathbb{P}^3$ is the blow-up at $Z$ [14, Theorem 2.1]. Let $h : \mathbb{F}^3 \rightarrow \mathbb{F}^3$ be a desingularization having smooth exceptional divisors with normal crossings [26] with composite maps $\sigma = \sigma_d \circ h : \mathbb{F}^3 \rightarrow \mathbb{P}H^0(\mathcal{I}_Z(d))^*$ and $\tilde{f} = f \circ h : \mathbb{F}^3 \rightarrow \mathbb{P}^3$. There is a similar map $\sigma_{d-1}$ for degree $(d - 1)$ surfaces which need not be a closed immersion.

**Remark 3.1.** If $P \in (\mathbb{P}^3)^*$ is a general plane, we can describe its strict transform $\tilde{P} \subset \mathbb{P}^3$ and the map $\tilde{P} \rightarrow P$. Following the proof of Proposition [26] let $F \subset Z$ be the finite set where $Z$ has embedding dimension three or $\mathcal{I}_Z$ is not 2-generated and let $G$ be the singularities of the support of $Z$. If $Z_t$ are the curve components of $Z$, then $Z_t$ has local ideal $(x, y^m)$ away from $F \cup G$, where $m_t$ is the multiplicity of
$Z_i$. Therefore $\tilde{P} \to P$ is the blow-up at $Z \cap P$ with exceptional divisors $W_{i,j}$ over the points in $Z_i$; these are the components of $W_i \cap \tilde{P}$ with $W_i$ as in Proposition 2.4. Moreover, $\tilde{P}$ has exactly one singularity in each $W_{i,j}$ with $m_i > 1$, which has equation $xy - z^{m_i}$. These singularities have canonical resolution compatible with the corresponding singular locus of $\mathbb{P}^3$ [13, 5.1 and 5.3] obtained by repeatedly blowing up points, so the fibres of $\tilde{P} \to P$ over points in $Z_i \cap P$ consist of a connected chain of $m_i \mathbb{P}^1$s, which include $W_{i,j}$. In particular, Pic $\tilde{P}$ is freely generated by $O(1), W_{i,j}$ and the exceptional divisors of the map $\tilde{P} \to \tilde{P}$ [10, V, Cor. 5.4].

We now compare the class groups of surfaces $S \in \mathbb{P}H^0(\mathcal{I}_Z(d))$ and Picard groups of their strict transforms $\tilde{S} \subset \tilde{\mathbb{P}^3}$. Letting $X \subset \mathbb{P}^3 \times \mathbb{P}H^0(\mathcal{I}_Z(d))$ be the universal family and $\tilde{X} \subset \tilde{\mathbb{P}^3} \times \mathbb{P}H^0(\mathcal{I}_Z(d))$ be the family of hyperplane divisors of $\tilde{S}$, we have a diagram

\[
\begin{array}{ccc}
\tilde{X} & \subset & \tilde{\mathbb{P}^3} \times \mathbb{P}H^0(\mathcal{I}_Z(d)) \\
\downarrow & & \downarrow \\
X & \subset & \mathbb{P}^3 \times \mathbb{P}H^0(\mathcal{I}_Z(d)) \\
\downarrow & & \downarrow \\
\mathbb{P}H^0(\mathcal{I}_Z(d)).
\end{array}
\]

**Proposition 3.2.** In the setting of diagram 2.4, there is a non-empty open set $U' \subset \mathbb{P}H^0(\mathcal{I}_Z(d))$, an étale cover $U' \to U$, and effective irreducible divisors $A_i \in \text{Pic} \tilde{X} \times_U U'$ such that

(a) There is an open subset $V \subset U$ for which $\tilde{X}_V \to V$ is smooth and each surface $S_v$ with $v \in U' \times_U V$ satisfies $\text{Ker}(\text{Pic} S_v \to \text{Cl} S_v) = \langle A_i \rangle$.

(b) The set $U$ contains points corresponding to general reducible surfaces $X = T \cup P$ with $T \in \mathbb{P}H^0(\mathcal{I}_Z(d-1))$ and $P \in (\mathbb{P}^3)^*$. For these, $A_i \cap T = \emptyset \iff A_i \cap \mathbb{P}^3 \neq \emptyset$ and

1. $\text{Ker}(\text{Pic} \tilde{T} \to \text{Cl} T) = \langle A_i \rangle$
2. $\text{Pic} \tilde{P}$ is freely generated by $O(1)$, the strict transforms of the components $W_{i,j}$ of $W_i \cap \mathbb{P}^3$ and the $A_i$ for which $A_i \cap \mathbb{P}^3 \neq \emptyset$.

**Proof:** For $S \in \mathbb{P}H^0(\mathcal{I}_Z(d))$, we view its strict transform $\tilde{S} \subset \tilde{\mathbb{P}^3}$ as a hyperplane section of the map $\tilde{\sigma}_d$, so $\tilde{S}$ is smooth and irreducible by Bertini’s theorem [15]. If $\Sigma \subset \tilde{S}$ is the singular locus, the kernel of the map

\[
\text{Pic} \tilde{S} \to \text{Pic}(\tilde{S} - \tilde{f}^{-1}(\Sigma)) \cong \text{Pic}(S - \Sigma) \cong \text{Cl} S
\]

is generated by the irreducible divisors in $\tilde{f}^{-1}(\Sigma)$, which appear as intersections with the following divisors in $\tilde{\mathbb{P}^3}$: let $M_{i,j} \subset \tilde{\mathbb{P}^3}$ be the irreducible divisors with $h(M_{i,j}) = \Sigma_i \subset \tilde{\mathbb{P}^3}$ (recall from Proposition 2.2 that the singularities of $\tilde{\mathbb{P}^3}$ away from $f^{-1}(F)$ are sections $\Sigma_i$ of the curve components $Z_i \subset Z$, these give the moving singularities) and let $F_k \subset \tilde{\mathbb{P}^3}$ be the irreducible divisors with $\tilde{f}(F_k) \subset F$ corresponding to fixed singularities. The kernels of the maps 10 are generated by the components of $M_{i,j} \cap \tilde{S}$ and $F_k \cap \tilde{S}$. Similar statements apply to $T$ via the map $\tilde{\sigma}_{d-1}$.

Let $Q \subset \tilde{\mathbb{P}^3}$ be a divisor $M_{i,j}$ or $F_k$ as described above.

If $\dim h(Q) = 0$, let $U_Q \subset \mathbb{P}H^0(\mathcal{I}_Z(d))$ be the open subset of $H$ which miss $\sigma_d(h(Q))$. Clearly $U_Q$ is non-empty and contains reducible surfaces $T \cup P$, for
general $H \in \mathbb{P}H^0(I_Z(d-1))$ misses $\sigma_{d-1}(h(Q))$ and general $P \in (\mathbb{P}^3)^*$ misses $f(h(Q))$.

If $\dim h(Q) = 2$, let $U_Q \subset \mathbb{P}H^0(I_Z(d))$ be the non-empty open subset of $H$ for which $\sigma_d^{-1}(H) \cap Q$ is integral. Again $U_Q$ contains reducible surfaces $T \cup P$, for the general plane $P$ misses the point $f(h(Q)) \in F$. The closed immersion $\mathbb{P}^3 \to \mathbb{P}^3 \times \mathbb{P}H^0(I_Z(d-1))$ shows that $\sigma_{d-1}$ embeds $h(Q)$ into $\mathbb{P}H^0(I_Z(d-1))^*$, so Bertini’s theorem tells us that $\overline{T} = \sigma_d^{-1}(H)$ meets $Q$ irreducibly. We will select $Q$ for one of the divisors $A_i$.

Finally, if $\dim h(Q) = 1$, then $h(Q) = C$ is an integral curve and there is the Stein factorization of $Q \to C$

$$Q \xrightarrow{\alpha} C', \beta \xrightarrow{\sigma} \mathbb{P}H^0(I_Z(d))^*$$

in which the fibres of $\alpha$ are connected [10, III, Cor. 11.5]. Since $Q$ is an exceptional divisor for $h$, it is smooth by construction and we may apply generic smoothness to the map $Q \xrightarrow{\beta\circ\alpha} C$ to find an open set $C^0 \subset C$ over which each fibre consists of exactly $d = \deg\beta$ smooth connected curves. Let $U_Q \subset \mathbb{P}H^0(I_Z(d))$ be the open set of $H$ which meet $\sigma(C^0)$ in $e = \deg(C \xrightarrow{\sigma} \mathbb{P}H^0(I_Z(d)))$ reduced points, so that $\overline{S} \cap Q = \sigma^{-1}(H) \cap Q$ consists of exactly $de$ smooth connected curves. Then the lower horizontal map in diagram

$$\begin{array}{ccc}
Q = \overline{X} \cap (Q \times U_Q) & \to & U_Q \\
\downarrow & & \downarrow \\
I = \{(x', H) : x' \in C', H \in U_Q, \sigma(\beta(x')) \in H\} & \to & U_Q \\
\downarrow & & \\
C' & &
\end{array} (7)$$

is an étale cover of degree $de$.

We check that the open set $U_Q$ contains reducible surfaces $T \cup P$. If $Q = F_k$, then general $P$ misses $z$ and $\sigma_{d-1}$ embeds $h(Q)$ (as when $\dim h(Q) = 2$ above). The map $\sigma_{d-1} : C \to \mathbb{P}H^0(I_Z(d-1))^*$ is given by the line bundle $f^*O(d-1) \otimes I_E$, where $E \subset \mathbb{P}^3$ is the exceptional divisor for the blow-up $f$, and $f^*O(d) \otimes I_E|C \cong f^*O(d-1) \otimes I_E|C$ because $f^*(O(d-1)|C$ is trivial, so the map has degree $e$. If $Q = M_{i,j}$, then $f(C) = W_i$ is the support of a curve component of $Z$ and the general plane $P$ meets $f(C)$ in $\deg W_i$ reduced points. In this case $C$ is a section of $W_i$, so $\deg f^*O(1)|C = \deg W_i$, and $\deg f^*O(d) \otimes I_E|C = \deg f^*O(d-1) \otimes I_E|C + \deg W_i$, so again $T \cup P$ will be in the degree $de$ étale locus $U_Q$.

As an open subset of a projective bundle over $C'$, $I$ is integral. To separate the $de$ connected components, base extend $I \to U$ by itself to obtain

$$\begin{array}{ccc}
I \times_U I & \to & I \\
\varphi \downarrow & & \downarrow \\
I & \to & U
\end{array}$$

in which $\varphi$ is an étale cover of degree $de$ with the canonical diagonal section, so $I \times_U I$ is not connected. If $I' \subset I \times_U I$ is any connected component for which the map $I' \to I$ has degree $> 1$, we can base extend by $I' \to I$ to split it up. We continue until we arrive at an integral base extension $U' \to U$ for which the induced map $U' \times_U I \to U'$ is a trivial étale cover of $de$ sheets. To finish, we base extend diagram [10] by $U' \to U$. Since $I \times_U U'$ has $de$ components, so does
implies sets \( U \)' which we will take as the \( A_i \) in Proposition 3.2.

We carry out this procedure for each \( Q \). Intersecting the resulting Zariski open sets \( U_Q \) and composing the finite étale covers, we obtain effective divisors \( A_i \) on \( \mathcal{X} \times_U U' \) which sort out the components of the intersections of the surfaces with the \( Q \). Thus the \( A_i \) generate the kernels of the maps \( \text{Pic} \mathcal{S} \to \text{Cl} S \). To finish part (a), use generic smoothness to find an open subset \( V \subset U \) where the map \( \mathcal{X}_V \to V \) is smooth. Statement 2 of part (b) follows from Remark 3.1.

**Remark 3.3.** If \( T_0 \in \mathbb{P}H^0(I_Z(d-1)) \) is normal with finitely generated class group, then a reducible surface \( T \cup P \) as in Proposition 3.2(b) can be chosen with \( \text{Cl} T \) finitely generated as well. If \( T_1 \cup P_1 \) is any surface in \( U \), consider the linear deformation \( T \to \mathbb{A}^3 \) given by equation \( (1-t)f_0 + tf_1 = 0 \) in \( \mathbb{P}^3 \times \mathbb{A}^3 \), where \( f_1 = 0 \) is the equation of \( T_1 \). Letting \( T \to T \) be a desingularization in which the central fibre \( T_0 \) is smooth, the family \( T \to \mathbb{A}^3 \) is flat and \( \text{Pic} T_0 \) is finitely generated because \( \text{Cl} T_0 \) is, hence \( H^1(\mathcal{O}_{T_0}) = 0 \). By semicontinuity, \( H^1(\mathcal{O}_{T_u}) = 0 \) for \( u \) near 0, hence \( \text{Cl} T_u \) is finitely generated for \( u \) near 0.

In computing the Picard group of a very general degree \( d \) surface \( S \subset \mathbb{P}^3 \) containing a smooth connected curve \( Z \), Lopez [18] adapted Griffiths and Harris’ degeneration argument [8]. Our construction follows that of Lopez, except that we work in a blow-up of \( \mathbb{P}^3 \) where the surfaces become smooth and must make a base extension to spread out divisors to compute the class groups. We extend his [18, Lem. II.3.3] for these purposes.

**Lemma 3.4.** Let \( Z \subset \mathbb{P}^3 \) be superficial with curve components \( Z_i \). Assume \( I_Z(d-1) \) is globally generated for some \( d \geq 4 \), and fix a normal surface \( T \in \mathbb{P}H^0(I_Z(d-1)) \) with finitely generated class group. Then the very general pair \( (P, S) \in (\mathbb{P}^3)^* \times \mathbb{P}H^0(I_Z(d)) \) with \( D = T \cap P \) smooth satisfies

(a) The restriction map \( \text{Cl} T \to \text{Pic} D \) is injective.
(b) \( p \neq q \in Z_i \cap P \) is not torsion in \( \text{Pic} D \) for each \( i \).
(c) Let \( p_{i,j} \) be the points in \( Z_i \cap P \) and \( q_k \) the remaining points in \( (T \cap S) \cap P \). If \( L \in \text{Cl} T \) such that \( \text{Cl} L \cong \mathcal{O}_D(\sum a_{i,j}p_{i,j} + \sum b_kq_k) \), then there are \( \alpha_i \) and \( \beta \) such that \( a_{i,j} = \alpha_i \) for each \( j \) and \( b_k = \beta \) for each \( k \).

**Proof:** We follow the outline of [18, Lem. II.3.3]. First note that \( T \subset \mathbb{P}^3 \) is not ruled by straight lines, for if \( T \) is a cone over a plane curve \( C \), then normality of \( T \) implies \( C \) smooth and \( \text{Cl} T \cong \text{Pic} C \) [10, II, Example 6.3 (a)], but the latter group is not finitely generated because \( C \) is not rational. If \( T \) is ruled but not a cone, then only finitely many rulings pass through each singularity of \( T \) and the general line \( L \subset T \) is contained in the smooth locus. Here \( \mathcal{O}_T(L)|_L \cong K_L \otimes K_L^* \cong \mathcal{O}_L(d-2) \) [10, II, 8.20]. The exact sequence \( 0 \to \mathcal{O}_T \to \mathcal{O}_T(L) \to \mathcal{O}_T(L)|_L \to 0 \) shows that \( h^0(L, \mathcal{O}_T(L)|_L) \leq 1 \) (depending on whether \( L \) is fixed on \( T \) or moves) and we conclude that \( d < 3 \), a contradiction.

For part (a), it is enough that \( J(L) = \{ D \in \mathcal{O}_T(1) : L_D = \mathcal{O}_D \} \) is a proper closed subset for \( D \neq L \subset \text{Cl} T \), since then the countable union \( \cup_{L \neq D} J(L) \) cannot be all of \( \mathcal{O}_T(1) \); thus, we show that for fixed \( L \subset \text{Cl} T \), \( L_D \cong \mathcal{O}_D \) for general \( D \in \mathcal{O}_T(1) \) implies \( L \cong \mathcal{O}_T \).

Because \( T \) is not ruled by lines, the reducible plane sections in \( \mathcal{O}_T(1) \) form a family of codimension \( \geq 2 \) [18, Lemma II.2.4], so there is a pencil \( \mathbb{P}^1 \rightharpoonup \mathbb{P}H^0(\mathcal{O}_T(1)) \)
of irreducible curves whose base points lie in the smooth locus $T^0$. The total family

$$T \subset T \times \mathbb{P}^1 \xrightarrow{f} \mathbb{P}^1 \xrightarrow{g} T$$

is isomorphic to the blow-up of $T$ at the base points and the exceptional divisors map isomorphically onto $\mathbb{P}^1$ under $f$ by [11, Theorem 1.3]. Since $\tilde{T} \xrightarrow{\delta} T$ is an isomorphism near the singularities, $\tilde{L} = g^*(L)$ is reflexive on $\tilde{T}$ and it suffices to show that $\tilde{L} \cong O_{\tilde{T}}$. The push-forward $B = f_*(\tilde{L})$ is a line bundle on $\mathbb{P}^1$, for it is reflexive by [11, Cor. 1.7] and has rank 1 because $h^0(\tilde{L}_t) = 1$ for general $t \in \mathbb{P}^1$. Since $f_*(\tilde{L} \otimes f^*(B')) \cong O_{\tilde{T}}$ by the projection formula, there is $0 \neq s \in H^0(\tilde{L} \otimes f^*(B'))$. The effective divisor $(s)_0$ does not map dominantly to $\mathbb{P}^1$ because $\tilde{L} \otimes f^*(B')_t$ is trivial for general $t$. It follows that $(s)_0$ is a union of components of fibres of $f$, but since these are irreducible and $f_*(s)$ is nonvanishing, we conclude that $(s)_0 = \emptyset$, hence $\tilde{L} \otimes f^*(B') \cong O_{\tilde{T}}$ and $\tilde{L} \cong f^*(B)$. If $x \in T$ is any base point for our pencil, $f^*B|_{g^{-1}(x)} \cong \tilde{L}|_{g^{-1}(x)}$ is trivial, but $f : g^{-1}(x) \to \mathbb{P}^1$ is an isomorphism, so $B = O_{\mathbb{P}^1}$ and $L = O_{\tilde{T}}$.

For part (b), note that for $n > 1$ the set of planes $H$ for which there are $p \neq q \in Z_i \cap H$ with $n\mathcal{O}_D(p-q)$ trivial is closed by semi-continuity, since this condition is given by non-vanishing of a line bundle on a flat family; therefore it suffices to show the set of these planes is proper in $(\mathbb{P}^3)^*$. For this we choose points $p, q \in Z_i$ such that the line $L$ through $p, q$ meets $T$ at $d-1$ smooth points of $T$ and the general plane $H$ containing $L$ yields a smooth curve $D = T \cap H$. As in part (a), the pencil of planes $H$ containing $L$ gives rise to a total family $\tilde{T}$ isomorphic to the blow-up of $T$ at the points in $T \cap L$ and we again obtain diagram (8). Let $E_p, E_q \cong \mathbb{P}^1$ be the exceptional divisors on $\tilde{T}$ over $p, q$.

The divisor $A_n = n\mathcal{O}_\tilde{T}(E_p - E_q)$ is non-trivial on $\tilde{T}$ and restricts to $n\mathcal{O}_D(p-q)$ on the general fibre over $\mathbb{P}^1$: if the general such restriction is trivial on $D$, then the argument in part (a) shows that there is a line bundle $B \in \text{Pic} \mathbb{P}^1$ such that $A_n \cong f^*(B)$. Since $d > 3$, there is a point $r \in L \cap T$ with $r \neq p, q$. The restriction of $A_n$ to $E_r$ is trivial, but $E_r \cong \mathbb{P}^1$ via the map $f$, so we see that $B$ itself is trivial and therefore $A_n$ is trivial on $\tilde{T}$, a contradiction. We conclude that for each $n > 1$, $n\mathcal{O}_D(p-q)$ is only trivial for finitely many $D$. Taking the union over $n > 1$ shows that the divisors $\mathcal{O}_D(p-q)$ are not torsion for very general $H$.

For part (c), we sketch Lopez’ argument [18, Lemma II.3.3 (3)] with some improvements. Let $Y$ be the (integral) curve linked to $Z$ by $S \cap T$ and let

$$I = \{(p_{i,j}, q_k, P) : \sum p_{i,j} + \sum q_k = P \cap (Z \cup Y)\}$$

be the incidence set inside

$$W_1^{d_1} \times W_2^{d_2} \times \cdots \times W_r^{d_r} \times Y^{(d-1) - \text{deg } Z} \times (\mathbb{P}^3)^*$$

with projection $\pi$ onto the last component. Letting $U \subset (\mathbb{P}^3)^*$ be the family of planes $P$ meeting $Z \cup Y$ transversely, $J = \pi^{-1}(U)$ is smooth and connected (the plane monodromy acts as the product of symmetric groups [18, Proposition II.2.6]), so $J$ is irreducible. For fixed $L \in \text{Cl} T$ and $a_{i,j}, b_k \in \mathbb{Z}$, the sets

$$J(L, a_{i,j}, b_k) = \{(p_{i,j}, q_k, P) : \mathcal{O}_D(\sum a_{i,j}p_{i,j} + \sum b_kq_k) \cong L|_D\} \subset J$$
are closed by semicontinuity. Suppose that \( J(L, a_{i,j}, b_k) = J \). Using the plane monodromy to permute the points \( p_{i,j} \) for fixed \( i \), we arrive at the equation \( (a_{i,s} - a_{i,t})(p_{i,s} - p_{i,t}) = 0 \) in Pic \( D \). By part (b), \( J(L, a_{i,j}, b_k) \) is a proper closed set if \( a_{i,s} \neq a_{i,t} \) for any \( s \neq t \), and so has proper closed image in \((\mathbb{P}^3)^*\). The countable union of all such images does not fill \((\mathbb{P}^3)^*\), so for very general \( P \) we have \( \mathcal{O}_D(\sum a_{i,j}p_{i,j} + \sum b_kq_k) \cong L|_D \Rightarrow a_{i,j} = a_{i,j'}, j \neq j' \). For very general \( S \) we have \( b_k = b_k' \) for \( k \neq k' \) because for \( D \) fixed, we can vary \( S \) to miss pairs \((p, q)\) with \( \mathcal{O}_D(p - q) \) torsion.

Now we construct the second family. Fix an integer \( d \geq 4 \), a superficial scheme \( Z \subset \mathbb{P}^3 \) with \( \mathcal{I}_Z(d - 1) \) globally generated and assume that \( Z \) lies on a normal surface of degree \( d - 1 \). Letting \( U \subset H^0(\mathcal{I}_Z(d)) \) be the open set constructed in Proposition 3.2. Remark 3.3 tells us that there is a point \( 0 \in U \) corresponding to a reducible surface \( T \cup P \) with \( T \) normal and \( \text{Cl}_T \) finitely generated. Fix such a surface \( T \) with equation \( F = 0 \) and choose a plane \( P \subset \mathbb{P}^3 \) with equation \( L = 0 \) and a degree \( d \) surface \( S \) with equation \( G = 0 \) as in Lemma 3.4 such that \( T \cup P \) and \( S \) are both in the set \( U \subset H^0(\mathcal{I}_Z(d)) \) from Proposition 3.2. These define the pencil of surfaces \( S_t \) by equation \( FL - tG = 0 \) for \( t \in \mathbb{A}^1 \). Set \( K = \{ t \in \mathbb{A}^1 : S_t \in U \} \). The universal property gives an embedding \( K \hookrightarrow U \). Now base extend diagram (\ref{diagram}) to obtain families

\[
\begin{array}{ccc}
\mathcal{X}_K & \subset & \mathbb{P}^3 \times K \\
\downarrow & & \downarrow \\
\mathcal{X}_K & \subset & \mathbb{P}^3 \times K
\end{array}
\]

and an étale cover \( K' \to K \) with divisors \( A_i \) as in Proposition 3.2 (a). If \( \mathcal{G}, \mathcal{T}, \mathcal{L} \) are the local equations of the strict transforms \( \mathcal{S}, \mathcal{T}, \mathcal{P} \subset \mathbb{P}^3 \), then the equation of \( \mathcal{X} \) is given by \( \mathcal{F}L - \mathcal{G}t = 0 \), which is singular exactly at \( t = 0 \) along the intersection \( \mathcal{S} \cap \mathcal{T} \cap \mathcal{P} \). For \( T, S \) general, Bertini assures us that \( \mathcal{Y} = \mathcal{S} \cap \mathcal{T} \subset \mathbb{P}^3 \) is a smooth connected curve, the strict transform of the curve \( Y \subset \mathbb{P}^3 \) linked to \( Z \) by the complete intersection \( S \cap T \). Let \( \mathbb{P}^3 \to \mathbb{F}^3 \) be the blow-up along \( \mathcal{Y} \), giving an associated family

\[
\begin{array}{ccc}
\mathcal{X} & \subset & \mathbb{P}^3 \times K \\
\downarrow & & \downarrow \\
K & & K
\end{array}
\]

which agrees with \( \mathcal{X}_K \) away from \( t = 0 \) because \( \mathcal{Y} \) is Cartier on \( \mathcal{X}_t \) for \( t \neq 0 \). At the central fibre \( \hat{T} \cong T \) and \( \hat{P} \to \mathcal{P} \) is the blow-up along the reduced set of points \( \mathcal{P} \cap \mathcal{Y} \). The resulting surfaces in \( \mathbb{F}^3 \) no longer intersect, so family (\ref{diagram}) is smooth.

**Remark 3.5.** Griffiths, Harris, and Lopez smoothed the family by blowing up the quadratic singularities at the central fibre and blowing down the rulings on the resulting quadrics. Local coordinate calculations show this is equivalent to blowing up \( \mathcal{Y} \) as above.

We would like to compute \( \text{Pic} \hat{X}_0 \), but the monodromy of the moving singularities causes ambiguity so instead we compute in the étale cover. Since \( K \subset U \), Proposition 3.2 gives an étale cover \( \epsilon : K' \to K \) and divisors \( A_i \) on \( \hat{X} \times_K K' \) which generate the kernels of the maps \( \text{Pic} \hat{X}_t \to \text{Cl} X_t \) for \( t \neq 0 \) and \( \text{Pic} \hat{T} \to \text{Cl} T \) at the central fibre. The exceptional divisor \( \hat{Y} \) for the blow-up \( \mathbb{P}^3 \to \mathbb{F}^3 \) has the structure
of a $\mathbb{P}^1$-bundle over $\mathcal{Y}$. Let $\hat{W}_i \subset \hat{\mathbb{P}}^3$ be the strict transforms of $W_i \subset \mathbb{P}^3$ from Observation 24(b) and $\hat{P} \subset \hat{\mathbb{P}}^3$ be the strict transform of the plane $P$ supported in the central fibre. With this notation, we compute:

**Proposition 3.6.** In the setting of family 10, let $e : K' \to K$ be the étale cover given in Proposition 3.2 and $A_i$ the corresponding divisors on $\hat{X} \times_K K'$. For $p \in e^{-1}(0)$, we set $N = \mathcal{O}_{\hat{X} \times_K K'}(\hat{P})|_{(\hat{X} \times_K K')_p}$, where $\hat{P}$ is the irreducible component of $(\hat{X} \times_K K')_p$ corresponding to $P$. Then

$$
\text{Pic}(\hat{X} \times_K K')_p = \langle \mathcal{O}(1), \hat{W}_1, \ldots, \hat{W}_r, A_1, N \rangle.
$$

**Proof:** The fibre $(\hat{X} \times_K K')_p$ is the union $\hat{T} \cup \hat{P}$, where $\hat{T} \cong \mathcal{T}$, $\hat{D} = \hat{T} \cap \hat{P}$ is isomorphic to $\mathcal{T} \subset \mathcal{T}$ and $\hat{P} \to \mathcal{T}$ is the blow-up at the $d(d-1) - \deg Z$ reduced points $Y \cap \mathcal{T}$: let $Y_k \subset \hat{P}$ be the corresponding exceptional divisors. If $W_{i,j} \subset \hat{P}$ are strict transforms of the supports of the components of $W_i|_\hat{P}$ (Proposition 24), then $\text{Pic} \hat{P}$ is freely generated by $\mathcal{O}(1), W_{i,j}, Y_k$ and the $A_i$ which meet $\hat{P}$ by Remark 3.4 and Proposition 3.2(b). Thus an arbitrary divisor

$$
Q \in \text{Pic}(\hat{X} \times_K K')_p \cong \text{Pic} \hat{T} \times_{\text{Pic} \hat{D}} \text{Pic} \hat{P}
$$

may be uniquely written as a pair

$$
Q = (A, \mathcal{O}(a) + \sum a_{i,j}W_{i,j} + \sum b_kY_k + \sum_{A_i \cap \hat{P} \neq \emptyset} c_iA_i)
$$

with $A \in \text{Pic} \hat{T}$ and common restriction to $\hat{D}$. Since $A_i|_{\hat{D}}$ are trivial, tensoring with $\mathcal{O}(-a)$ and applying Lemma 3.4(c) shows that $a_{i,j} = \alpha_i$ and $b_k = \beta$ for some $\alpha_i$ and $\beta$ and $Q$ becomes

$$
(A, \mathcal{O}(a) + \sum_{i,j} \alpha_iW_{i,j} + \beta \sum Y_k + \sum c_iA_i) = (A, \mathcal{O}(a) + \sum \alpha_i\hat{W}_i + \beta \hat{Y} + \sum c_iA_i).
$$

Let us describe the divisor $N$ as a pair. Clearly $N|_{\hat{P}}$ is represented by the curve $\hat{D} \in |\mathcal{O}_{\hat{D}}(1)|$. The restriction of the divisor $\hat{T} \cup \hat{P}$ to itself is trivial, therefore $N|_{\hat{P}} = -\hat{T}|_{\hat{P}} = -\hat{D}$. As a divisor on $\hat{P}$, $\hat{D}$ takes the form $\mathcal{O}(d-1) - \sum m_iW_i - \sum Y_k - \sum_{A_i \cap \hat{P} \neq \emptyset} n_iA_i$ because the total transform of $D \subset P$ includes all the exceptional divisors in the desingularization of $\hat{P}$. Since $Z_1$ meets $D$ at $m_i$-fold points, the supports $W_i$ have multiplicity $m_i$ in the total transform; the $A_i$ have positive multiplicity $n_i > 0$ whose exact values we will not need. Therefore

$$
Q - \beta N = (A(-\beta), \mathcal{O}(a + \beta(1-d)) + \sum (\alpha_i + \beta m_i)\hat{W}_i + \sum_{A_i \cap \hat{P} \neq \emptyset} (c_i - \beta)A_i).
$$

Finally, the kernel of $\text{Pic} \hat{T} \to \text{Cl} \hat{T}$ is generated by the $A_i$ meeting $\hat{T}$ and $A(-\beta)|_{\hat{D}}$ has form $\mathcal{O}(a + \beta(1-d)) + \sum (\alpha_i + \beta m_i)\hat{W}_i$. Since $\text{Cl} \hat{T} \to \text{Pic} \hat{D}$ is injective by Lemma 3.4(a), we conclude that

$$
A(-\beta) = \mathcal{O}(a + \beta(1-d)) + \sum (\alpha_i + \beta m_i)\hat{W}_i + \sum_{A_i \cap \hat{T} \neq \emptyset} d_iA_i
$$

and we have expressed $Q$ in terms of the generators stated.
4. The Main Theorem

Now we are now ready to prove Theorem 1.1. We define the relevant Noether-Lefschetz locus (12) and show that it is closed in the Hilbert-flag scheme (Proposition 4.4). We show it is proper by reducing to families dominating the particular family (10) in Claim 1.6 where the proof is not difficult. We begin with a useful consequence of cohomology and base change.

Proposition 4.1. Let \( S \xrightarrow{f} T \) be a projective flat family with \( H^1(\mathcal{O}_{S_t}) = 0 \) for each \( t \in T \) and \( L \in \text{Pic} \ S \). Then

(a) The set \( \{ t \in T : L_t = 0 \in \text{Pic} \ S_t \} \) is open in \( T \).

(b) For \( G \subset \text{Pic} \ S \), the set \( G_L = \{ t \in T : L_t \in G \} \) is open in \( T \).

Proof: Suppose that \( L_0 \cong \mathcal{O}_{S_0} \) for \( 0 \in T \). Then \( H^1 L_0 = 0 \) and this continues to hold in a Zariski open neighborhood \( U \subset T \) about \( 0 \) by semi-continuity. Thus the natural map \( R^1 f_* L \otimes k(t) \to H^1(\mathcal{O}_{S_t}, L_t) = 0 \) is surjective over \( U \), hence an isomorphism by cohomology and base change [10, III, Theorem 12.11(a)]. Therefore \( R^1 f_* L = 0 \) is locally free on \( U \), so again by cohomology and base change [10, III, Theorem 12.11(b)] the natural map

\[
f_* L \to H^0(\mathcal{O}_{S_t}, L_t)
\]

is surjective and an isomorphism for all \( t \in U \). Shrinking \( U \) to an open affine if necessary, this gives surjectivity of \( H^0(\mathcal{O}_{S}, L) \to H^0(\mathcal{O}_{S_0}, L_0) \), allowing us to extend the nonvanishing global section \( 1 \in H^0(\mathcal{O}_{S_0}) \cong H^0 L_0 \) to a global section \( s \) on \( S_U \); since \( s \) vanishes on a closed set, we can further shrink \( U \) to avoid this set and obtain the desired result. Part (b) follows because \( G_L = \bigcup_{A \in G} \{ t \in T : (L - A) = 0 \} \) is the union of open sets.

Example 4.2. Proposition 4.1 (a) fails for \( L \in \text{Cl} \ S \) if we interpret \( L_1 = 0 \) as \( (L_A)^{\vee} \cong \mathcal{O}_{S_t} \). For example, if \( S_t \) is a family of smooth quadric surfaces degenerating to the quadric cone \( S_0 \subset \mathbb{P}^3 \) and \( L \in \text{Cl} \ S \) is the divisor which is the difference of opposite rulings on \( S_t \) for \( t \neq 0 \), then \( L_t \neq 0 \) for \( t \neq 0 \) but the limit is the difference of two rulings on the cone \( S_0 \), which is trivial.

Now we prove a stronger version of Theorem 1.1. We first prove that the proposed generators for the class groups have no relations.

Proposition 4.3. Let \( Z \subset \mathbb{P}^3 \) be superficial with curve components \( Z_i \) having respective supports \( W_i \) and assume \( H^1(\mathcal{O}_{\mathbb{P}^3}(d - 1)) \) is generated by global sections for some \( d \geq 4 \). Fix open sets \( V \subset U \subset \mathbb{P}^3 \) as in Proposition 3.3. Then the divisors \( \mathcal{O}(1), W_i \) have no relations in \( \text{Cl} X_t \) for \( t \in V \).

Proof: Suppose that \( \mathcal{O}(c) + \sum b_i W_i = 0 \) in \( \text{Cl} X_t \) for some \( t \in V \). Using the étale cover \( V' \to V \) from Proposition 3.2, we lift \( t \) to \( t' \) and have the same relation in \( X_{t'} \) and we obtain a corresponding relation \( L = \mathcal{O}(c) + \sum b_i W_i + \sum a_i A_i = 0 \) in \( \text{Pic} X_{t'} \) (the \( \sum a_i A_i \) terms are needed here because the pull-back of generalized divisors is not necessarily additive [12, Example 2.18.1]). Since the family is flat, \( L_{t'} \) is trivial in \( X_{t'} \) on a Zariski open set \( V_0 \subset V' \) by Proposition 1.1.

If \( V_0 \) is non-empty, then we can choose the surface \( S \) from Proposition 3.3 to lie in \( V_0 \) so that \( K'' \cap V'' \) is non-empty. Thus the restriction of \( L \) to \( X_{K'' \cap V_0} \) is the trivial line bundle. By [10, II, 6.5] it follows that \( L \) extends to a line bundle on
all of $\overline{X}_{K}$, which is linearly equivalent to a combination of vertical components, hence we may write $O(c) + \sum b_i W_i + \sum a_i A_i + \sum j V_j = 0$ on $\overline{X} \times_K K'$ and after pulling back we have the same equation on $\overline{X} \times_K K'$. Restricting to $\overline{P}$ in the central fibre, the vertical components away from the central fibre become trivial and the remaining combination of $\overline{P}$ and $\overline{T}$ is a multiple of $N$, so the relation in Pic $\overline{P}$ becomes $O(c) + \sum b_i W_i + \sum a_i A_i + eN = 0$. Now Pic $\overline{P}$ is freely generated by $O(1), W_i, A_i$ and $Y_k$, so we see that $e = 0$ because the coefficient of $Y_k$ in $N$ is nonzero (see calculation of $N|_{\overline{P}}$ in the proof of Proposition 3.6), and therefore $c = b_i = 0$ as well.

To finish the proof of Theorem 1.1 we must show that $O(1)$ and the $W_i$ generate the class groups. First we show that the relevant locus in the Hilbert flag scheme is closed. Let

$$\mathcal{H}_V = \{(C, S) : C \subset S, S \in V\} \subseteq V$$

be the Hilbert-flag scheme of locally Cohen-Macaulay curves $C$ on surfaces $S$ from the family $V$ and let

$$\mathcal{B}_V = \{(C, S) \in \mathcal{H}_V : L(C) \notin \langle O_S(1), W_1, \ldots, W_r \rangle \subset ClS, S \in V\}$$

where $L(C) = I_{C, S}$ is the reflexive sheaf associated to $C$, which generalizes the familiar line bundle $O_S(C)$ on a smooth surface $S$ [12].

**Proposition 4.4.** $\mathcal{B}_V$ is closed in $\mathcal{H}_V$.

**Proof:** For each irreducible component $T \subset \mathcal{H}_V$, we show that $\mathcal{B}_V \cap T$ is closed in $T$. Letting

$$C \subset X \subset \mathbb{P}^3_T$$

be the associated family of curves on surfaces, we base extend by a desingularization to assume $T$ is smooth and it suffices that

$$\{t \in T : L(C)_t \notin \langle O(1), W_1, \ldots, W_r \rangle \subset ClX_t\}$$

is open in $T$.

If $V' \to V$ is the étale cover from Proposition 3.2 then $T \times_V V' \to T$ is an étale cover of smooth varieties. Since the total family $\overline{X}_V \to V$ is smooth, so is $\overline{X} \times_V (T \times_V V')$ by base extension, and there are divisors $A_i$ which generate the kernels of the maps Pic $\overline{X}_t = ClX_t$. Now the pullback of $L(C) \in ClX$ to $\overline{X} \times_V (T \times_V V')$ is a line bundle $L$ and

$$L(C)_t \notin \langle O(1), W_1, \ldots, W_r \rangle \iff L_t \notin \langle O(1), W_1, \ldots, W_r, A_i \rangle.$$ 

The latter condition is open by Proposition 4.1(b).

**Theorem 4.5.** Let $Z \subset \mathbb{P}^3$ be a curve lying on a normal degree $(d - 1)$ surface with finitely generated class group and assume that $I_{Z}(d - 1)$ is generated by global sections. Then the very general surface $S \in |H^0(I_{Z}(d))|$ is normal with $ClS = \langle O(1), W_1, \ldots, W_r \rangle$.

**Proof:** Fix $V \subset \mathbb{P}^3 H^0(I_{Z}(d)), \mathcal{H}_V$ and $\mathcal{B}_V$ as in Proposition 4.4 above. The Hilbert-flag scheme $\mathcal{H}_V$ has countably many irreducible components, finitely many for each Hilbert polynomial for the curves $C$ in the family, hence $\mathcal{B}_V$ also has finitely many irreducible components by Proposition 4.4. To prove Theorem 4.5 it suffices to show that if $D \subset \mathcal{B}_V$ is an irreducible component, then $\pi_2(D) \neq V$. Indeed, any surface
show that for $u \in \mathcal{V}$.

Claim 4.6. Let $\hat{\mathcal{V}}$ in which $N^4.1(a)$. The restrictions of $(13)$ of smooth irreducible curves. If $A$ then the ring homomorphism $\mathcal{Z}$ of these results holds because $A$ as in Theorem 3.2, and using Proposition 3.6 (the hypotheses of these results holds because $Z$ lies on a normal surface, therefore is superficial) we may write

$$\hat{\mathcal{L}}_p = \mathcal{O}(a) + bN + \sum a_iA_i + \sum b_jW_j$$

Note that the hypotheses of Proposition 4.1 apply to the family $\hat{\mathcal{Y}} \to D$ because the long exact cohomology sequence associated to

$$0 \to \mathcal{O}(\hat{\mathcal{Y}}_p) \to \mathcal{O}(\hat{\mathcal{P}}) \to \mathcal{O}(\hat{\mathcal{D}}) \to 0$$

shows that $H^1(\mathcal{O}(\hat{\mathcal{Y}}_p)) = 0$, since $H^0(\mathcal{O}(\hat{\mathcal{P}})) = \mathbb{C}$ and $H^1(\mathcal{O}(\hat{\mathcal{D}})) = H^1(\mathcal{O}(\hat{\mathcal{P}})) = 0$. We apply it to the line bundle

$$\mathcal{M} = \mathcal{O}(a) + bN + \sum a_iA_i + \sum b_jW_j - \hat{\mathcal{L}} \in \text{Pic} \hat{\mathcal{Y}}$$

Since $\mathcal{M}_p$ is trivial, this is also true on an open neighborhood of $p$ by Proposition 4.1(a). The restrictions of $N = \mathcal{O}(\hat{\mathcal{X}}(\hat{\mathcal{P}}))$ and $A_i$ are trivial in $\text{Cl} \hat{\mathcal{Y}}_u$ nearby, so $\hat{\mathcal{E}}_u = \mathcal{O}(a) + \sum b_iW_i$ near $u = p$ and $C_u \in \mathcal{O}(\hat{\mathcal{Y}}_p) \mathcal{O}(\hat{\mathcal{D}})$. bCase 2: Now suppose that $f$ is ramified at $p$. We still have $\hat{\mathcal{Y}}_p = \hat{\mathcal{X}}_0$, but the ramification of $f$ at $p$ causes the total family to be singular along $\hat{\mathcal{D}}$ in the fibre $\hat{\mathcal{Y}}_p$. Specifically, let $y \in \hat{\mathcal{D}} \subset \hat{\mathcal{Y}}_p$ be a point with image $z \in \hat{\mathcal{D}} \subset \hat{\mathcal{X}}_0$. Now $\hat{\mathcal{X}}$ is smooth at $z$ and if $(A_i(t)) = \mathcal{O}(\hat{\mathcal{D}})$ (resp. $(B_s(u)) = \mathcal{O}(\hat{\mathcal{D}},p)$) is the local ring of $\hat{\mathcal{K}}$ at $(0)$ (resp. $D$ at $p$), then the ring homomorphism $A \to B$ sends $t \to u^s$ (up to unit) for some $s > 1$. Since $\hat{\mathcal{X}}$ is locally defined in $\mathbb{P}^3 \times K$ by the equation $LF - t = 0$, the base extension gives that $\hat{\mathcal{Y}}$ is locally defined in $\mathbb{P}^3 \times D$ by $LF - u^s$. Thus locally speaking, $\mathcal{O}(\hat{\mathcal{Z}})$ is a quotient of a regular local ring $R$ in four variables $L, F, H, u$ by the equation $LF - u^s = 0$. 

$S \in \mathcal{V}$ carrying a reflexive sheaf $L \not\in \langle \mathcal{O}(1), W_1, \ldots, W_r \rangle$ also contains such a curve $C$ (the zero section of $L(n)$ for some $n > 0$), so $(C, S) \in \mathcal{B}_V$ and $S \in \pi_2(\mathcal{B}_V)$. To show that $\pi_2(D) \neq \mathcal{V}$, it suffices to show that $K \cap \mathcal{V} \not\subseteq \pi_2(D)$, where $K \subset \mathbb{P} H^0(\mathcal{I}_Z(d))$ is the pencil of surfaces given by family (10). If $K \subset \pi_2(D)$, we can apply Bertini’s theorem to the closure $\pi_2^{-1}(K)$ in the full Hilbert scheme to obtain an integral curve $D$ dominating $K$, so we may assume dim $D = 1$. Base extending by the normalization of $D$, it finally suffices to prove the following claim.

Claim 4.6. Let $X \subset \mathbb{P}^3 \times K$ be the family (13) and $f : D \to K$ a surjective morphism of smooth irreducible curves. If $C \subset Y = X \times_K D$ in family (10). If $C \subset Y = X \times_K D$ is a flat family of smooth curves over $D$, then $C_u \in \langle \mathcal{O}(1), W_1, \ldots, W_r \rangle$ for general $u \in D$.

Fix $p \in f^{-1}(0)$. Let $\hat{\mathcal{Y}} \to D$ be the base extension of the family $\hat{\mathcal{X}} \to K$ from (10). We obtain a diagram

$$\begin{array}{ccc}
\hat{C} & \subset & \hat{\mathcal{Y}} \\
\downarrow & & \downarrow \\
D & \to & K.
\end{array}$$

in which $\hat{C} \subset \hat{\mathcal{Y}}$ is the strict transform of $C \subset Y$. Letting $\hat{\mathcal{L}} = \mathcal{I}_{\hat{\mathcal{C}}} \in \text{Cl} \hat{\mathcal{Y}}$, we will show that for $u$ near $p$, $\hat{\mathcal{L}}_u \in \langle \mathcal{O}(1), W_1, \ldots, W_r \rangle$.

Case 1: If $f$ is unramified at $p$, then we have an isomorphism $\hat{\mathcal{Y}}_p \cong \hat{\mathcal{X}}_0$ and the total family $\hat{\mathcal{Y}}$ is smooth along $\hat{P} \cap \hat{T} = \hat{D}$ in $\hat{Y}_p$. After base extension by $K' \to K$ we obtain divisors $A_i$ as in Theorem 6.2 and using Proposition 5.6 (the hypotheses of these results holds because $Z$ lies on a normal surface, therefore is superficial) we may write

$$\hat{\mathcal{L}}_p = \mathcal{O}(a) + bN + \sum a_iA_i + \sum b_jW_j$$

Now suppose that $f$ is ramified at $p$. We still have $\hat{\mathcal{Y}}_p \cong \hat{\mathcal{X}}_0$, but the ramification of $f$ at $p$ causes the total family to be singular along $\hat{D}$ in the fibre $\hat{\mathcal{Y}}_p$. Specifically, let $y \in \hat{D} \subset \hat{\mathcal{Y}}_p$ be a point with image $z \in \hat{\mathcal{D}} \subset \hat{\mathcal{X}}_0$. Now $\hat{\mathcal{X}}$ is smooth at $z$ and if $(A_i(t)) = \mathcal{O}(\hat{\mathcal{D}})$ (resp. $(B_s(u)) = \mathcal{O}(\hat{\mathcal{D}},p)$) is the local ring of $\hat{\mathcal{K}}$ at $(0)$ (resp. $D$ at $p$), then the ring homomorphism $A \to B$ sends $t \to u^s$ (up to unit) for some $s > 1$. Since $\hat{\mathcal{X}}$ is locally defined in $\mathbb{P}^3 \times K$ by the equation $LF - t = 0$, the base extension gives that $\hat{\mathcal{Y}}$ is locally defined in $\mathbb{P}^3 \times D$ by $LF - u^s$. Thus locally speaking, $\mathcal{O}(\hat{\mathcal{Z}})$ is a quotient of a regular local ring $R$ in four variables $L, F, H, u$ by the equation $LF - u^s = 0$. 

$\pi$
By [13 5.1 and 5.3], successive blowing up of the central curve $\hat{D}$ yields a desingularization $Z \to \hat{Y}$ in which the fibre over $y$ is a chain of $\mathbb{P}^1$s and at the global level we have

$$Z_p = \hat{P} \cup D_0 I_1 \cup D_1 \cup \cdots \cup D_{s-1} \cup D_{s-1} \hat{T},$$

where each $I_i$ is a ruled surface over both $\hat{D}_i \cong \hat{D}_{i-1}$ and these two sections do not meet in $I_i$. Running exact sequences and induction shows that $H^1(\mathcal{O}_{Z_p}) = 0$ as before: for example, the exact sequence

$$0 \to \mathcal{O}_{\hat{P}, I_i} \to \mathcal{O}_{\hat{P}} \oplus I_i \to \mathcal{O}_{D_0} \to 0$$

shows that $H^1(\mathcal{O}_{\hat{P}, I_i}) = 0$ because the induced map $H^1(\mathcal{O}_{I_i}) \to H^1(\mathcal{O}_{\hat{P}})$ is an isomorphism via the section $\sigma : \hat{D} \to I_1$. The total family $Z$ is smooth near the central fibre and (similar to Proposition 3.6) we have

$$\text{Pic}_Z = \langle \mathcal{O}(1), W_1, \ldots, W_r, A_i, N_0 = \mathcal{O}_Z(\hat{P})|_{Z_p}, N_1, N_2, \ldots, N_{s-1} \rangle$$

where $N_i = \mathcal{O}_Z(I_i)|_{Z_p}$ for $1 \leq i < s$, essentially because every divisor on the ruled surface $I_i$ has the same restriction to $D_i$ and $D_{i-1}$ modulo $\mathcal{O}_I(D_i)$.

Now the proof goes through as in the unramified case, the point being that the new divisors $N_i$ have trivial restrictions in the nearby $\text{Cl} Z_p$.

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