Abstract

We address in this paper the following two closely related problems: 1. How to represent functions with singularities (up to a prescribed accuracy) in a compact way? 2. How to reconstruct such functions from a small number of measurements? The stress is on a comparison of linear and non-linear approaches. As a model case we use piecewise-constant functions on \([0, 1]\), in particular, the Heaviside jump function \(H_t = \chi_{[0,t]}\). Considered as a curve in the Hilbert space \(L^2([0,1])\) it is completely characterized by the fact that any two of its disjoint chords are orthogonal. We reinterpret this fact in a context of step-functions in one or two variables.

Next we study the limitations on representability and reconstruction of piecewise-constant functions by linear and semi-linear methods. Our main tools in this problem are Kolmogorov’s \(n\)-width and \(\epsilon\)-entropy, as well as Temlyakov’s \((N, m)\)-width.

On the positive side, we show that a very accurate non-linear reconstruction is possible. It goes through a solution of certain specific non-linear systems of algebraic equations. We discuss the form of these systems and methods of their solution, stressing their relation to Moment Theory and Complex Analysis.

Finally, we informally discuss two problems in Computer Imaging which are parallel to the problems 1 and 2 above: compression of still images and video-sequences on one side, and image reconstruction from indirect measurement (for example, in Computer Tomography), on the other.

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Linear problems are all linear alike; every non-linear problem is non-linear in its own way.

M. Livshitz

1. Introduction

In this paper we discuss the following two basic problems:

1. How to represent functions with singularities (up to a prescribed accuracy) in a compact way?

2. How to reconstruct such functions from a small number of measurements?

We consider both the problems mainly from the point of view of a comparison between linear and non-linear approaches.

We study in detail a model case of piecewise-constant functions on \([0, 1]\), which, as we believe, reflects many important issues of a general situation. Considered as curves (or surfaces of higher dimension) in the Hilbert space \(L^2([0, 1])\) the families of piecewise-constant functions with variable jump-points form a nice geometric object: so called “crinkled arks”. They are characterized by the fact that any two their disjoint chords are orthogonal. A remarkable classical fact is that any two such curves are isometric, up to a scale factor. We reinterpret this fact in a context of step-functions in one or two variables.

Next we study the problem of representability of piecewise-constant functions by linear and semi-linear methods. Our main tools in this problem are Kolmogorov’s \(n\)-width and \(\epsilon\)-entropy (\([33, 58]\)), as well as Temlyakov’s \((N,m)\)-width (\([56]\)). See also (\([49, 41, 54]\)) and references there for similar estimates.

Then we turn to the reconstruction problem. We start with a negative result: based on our computation of Kolmogorov’s \(n\)-width of piecewise-constant functions, we provide limitations on the accuracy of linear methods of reconstruction of such functions from measurements.

On the contrary, we show, following \([12, 13, 43, 44, 45, 26, 18, 19, 31, 32, 51, 52]\), that a very accurate non-linear reconstruction is possible. It goes through a solution of certain specific non-linear systems of algebraic equations. We discuss a typical form of these systems and certain approaches to their solution, stressing the relations with Moment Theory and Complex Analysis. See also \([59, 40, 42]\) where a similar approach is presented from a quite different point of view.

We believe that the key to a successful application of the “algebraic reconstruction methods” presented in this paper to real problems in Signal Processing lies in a “model-based” representation of signals and especially of images. This is a very important and difficult problem
by itself (see \[38, 35, 21, 4, 20\] and references there). In the last section we informally discuss this problem together with two other closely related problems in Computer Imaging (which are parallel to the problems 1 and 2 above): compression of still images and video-sequences on one side, and image reconstruction from indirect measurement (for example, in Computer Tomography), on the other.

Our main conclusions are as follows:

1. If we insist on approximating all the family of the piecewise-constant functions, with variable positions of jumps, \textit{by the same linear subspace (Kolmogorov n-width)} then the Fourier expansion is essentially optimal. Any other linear method will provide roughly the same performance: with \( n \) terms linear combinations we get an approximation of order \( \frac{1}{\sqrt{n}} \). This concerns both the “compression” and the “reconstruction from measurements” problems.

2. If for each individual piecewise-constant function we are allowed to take \textit{its own “small” linear combination} of elements of a certain fixed “large” basis (“sparse approximations") then with \( n \) terms linear combination we get an approximation of order \( q^n, q < 1 \).

3. The “non-linear width” approach (Temlyakov’s \((N,m)\)-width) provides a natural interpolation between the Fourier expansion, the sparse approximations and the direct non-linear representation.

4. The “naive” direct non-linear representation of piecewise-constant functions, where we explicitly memorize the positions of the jumps \( 0 < x_i < 1, \ i = 1, \ldots, N \), and the values \( A_i \) of the function between the jumps, provides the best possible compression (not a big surprise!). However, \textit{these parameters can be reconstructed from a small number of measurements (Fourier coefficients) in a robust way, via solving non-linear systems of algebraic equations}.

5. Extended to piecewise-polynomials, and combined with a polynomial approximation, the last result provides an approach to an important and intensively studied problem of a noise-resistant reconstruction of piecewise-smooth functions from their Fourier data.

Let us stress that the problem of an efficient reconstruction of “simple” (“compressible") functions from a small number of measurements has been recently addressed in a very convincing way in the “compressed sensing”, “compressive sampling”, and “greedy approximation” approaches (see \[8, 9, 10, 16, 17, 14, 55, 57\] and references there). Our approach is different, but some important similarities can be found
via the notion of “semi-algebraic complexity” ([60] [61]). We plan to present some results in this direction separately.

The third author would like to thank G. Henkin for very inspiring discussions of some topics related to this paper. Both the complexity of approximations and the moment inversion problem intersect with Henkin’s fields of interest, and we hope that some of his results (see especially [15] [29] [30]) may turn out to be directly relevant to the non-linear representation and reconstruction problems discussed here.

2. Families of piecewise-constant functions in $L^2([0,1])$

In this paper we mostly concentrate on one specific case of a piecewise-constant functions, namely, on the family of step (or Heaviside) functions $H_t(x)$ defined on $[0,1]$ by $H_t(x) = 1$, $x \leq t$ and $H_t(x) = 0$, $x > t$. All the results in Section 2 below remain valid (with minor modifications) for any family of piecewise-constant functions on $[0,1]$ with a fixed number $N$ of variable jumps.

A remarkable geometric fact about the curve $\mathcal{H} = \{H_t(x), t \in [0,1]\} \subset L^2([0,1])$ is that any two its disjoint chords are orthogonal. So the curve $\mathcal{H}$ changes instantly its direction at each of its points: it is as “non-straight” as possible. Such curves are called “crinkled arcs” and we study them in more detail in Section 2.1.

Notice that a general family of piecewise-constant functions on $[0,1]$ with a fixed number $N$ of variable jumps forms what can be called a “crinkled higher-dimensional surface” in $L^2([0,1])$, at least with respect to the jump coordinates: any two chords from the same point, corresponding to the jumps shifts in opposite directions, are orthogonal.

2.1. “Crinkled arcs”. As above, we define the curve $\mathcal{H} : [0,1] \rightarrow L^2([0,1])$ by $\mathcal{H}_t = \chi_{[0,t]}$. This curve is continuous, and it has the following geometric property: any two disjoint chords of it are orthogonal in $L^2([0,1])$. Indeed, such chords are given by the characteristic functions of two non-intersecting intervals. Intuitively, the curve $\mathcal{H}$ exhibits a “very non-linear” behavior: its direction in $L^2([0,1])$ rapidly changes.

Now let $\mathcal{X}$ be a general Hilbert space.

**Definition 2.1.** A curve $\psi : [0,1] \rightarrow \mathcal{X}$ in a Hilbert space $\mathcal{X}$ is called a crinkled arc if:

- it is continuous
- any two disjoint chords of it are orthogonal, namely that for $0 \leq s < t \leq s' < t' \leq 1$ we have:

\[(\psi_t - \psi_s, \psi_{t'} - \psi_{s'}) = 0\]
More details are given in the classical book of Halmos [28]. See, in particular, [28, problems 5-6]. The curve $H$ provides the main example of a crinkled arc.

Crinkled curves are preserved by certain natural transformations. Namely, one can perform

- translation
- scaling
- reparametrization
- application of a unitary operator.

Then the result would still be a crinkled arc. A simple and surprising theorem is that these are the only possibilities to obtain a crinkled arc, and any two arcs are connected by this transformations:

**Theorem 2.2.** Let $\psi : [0, 1] \rightarrow X_1$ and $\phi : [0, 1] \rightarrow X_2$ be two crinkled arcs in two different Hilbert spaces. Then there are two vectors $v_i \in X_i$, $i = 1, 2$, a reparametrization $f : [0, 1] \rightarrow [0, 1]$, a positive number $\alpha$ and a (partial) isometry $U : X_1 \rightarrow X_2$ of the Hilbert spaces s.t.

$$U(\psi_f(t) - v_2) = \alpha \phi_t - v_1$$

**Proof:** See [28, p.169]

**Corollary 2.3.** Let $\psi : [0, 1] \rightarrow \mathcal{X}$ be a crinkled arc. Then it can be obtained from $\mathcal{H}$ by a translation, scaling, reparametrization, and an application of a unitary operator between the appropriate Hilbert (sub-)spaces.

Therefore, if we consider only geometric properties of curves inside the Hilbert space then the curve $\mathcal{H}$ can be taken as a model for any crinkled ark.

While any two Hilbert spaces are isomorphic, their "functional" realizatioms may be quite different. Consider, for example, the space $L^2(Q^2)$ of the square integrable functions on the unit two-dimensional cell $Q^2 = [0, 1] \times [0, 1]$ (we shall later refer to such functions as “images”).

The two families of functions, which are shown in Figure 1, clearly represent crinkled arks in $L^2(Q^2)$. Indeed, their disjoint chords are given by the characteristic functions of certain concentric non-intersecting domains in $Q^2$, and hence they are orthogonal. By Corollary 2.3, each of these curves is isomorphic to the curve $\mathcal{H}$ in $L^2([0,1])$. Let

$^1$A partial isometry $U : X_1 \rightarrow X_2$ between Hilbert spaces is an isometry between $KerU^\perp$ and $ImU$.
us state a general proposition in this direction. Consider a family $D_t \subset Q^n$, $t \in [0,1]$, of “expanding domains” in the n-dimensional cell $Q^n = [0,1]^n$, $D_{t_1} \subset D_{t_2}$ for any $t_1 < t_2$. Consider the curve $S(t)$ in $L^2(Q^n)$ defined by $S(t) = \chi_{D_t} \in L^2(Q^n)$.

**Proposition 2.4.** $S(t)$ is a crinkled curve.

**Proof:** Any two disjoint chords of the curve $S$ are given by the characteristic functions of certain concentric non-intersecting domains in $Q^n$, and hence they are orthogonal in $L^2(Q^n)$.

By Corollary 2.3, each of the curves $S$ obtained as above, is isomorphic to the curve $\mathcal{H}$ in $L^2([0,1])$.

If the domains evolve in time in a more complicated way (in particular, their boundaries are deformed in a non-rigid manner), then the corresponding curve formed in $L^2(Q^n)$ may be not exactly a crinkled arc. However, the following proposition shows that *typically* such trajectories look like crinkled arcs “in a small scale”.

**Proposition 2.5.** Let $C_t$, $t \in [0,1]$, be a generic smooth family of closed non-intersecting curves in $Q^2$. Consider a corresponding curve $C_t \subset \mathcal{L}$. Then the angle between any two disjoint chords of $C_t$ tends to $\frac{\pi}{2}$ as these chords tend to the same point.
Proof: Let us assume that the curves $C_t(\tau)$ are parametrized by $\tau \in [0,1]$, $C_t(0) = C_t(1)$. Because of the genericity assumption we can assume that for each $t \in [0,1]$ the derivative $\frac{\partial C_t(\tau)}{\partial \tau}$ has a finite number of zeroes $\tau_1, \ldots, \tau_m$ and it preserves its sign between these zeroes. Therefore, the chords of $C_t$ are the characteristic functions of the domains as shown on Figure 2. Specifically, the intersections of these domains are concentrated near the zeroes $\tau_1, \ldots, \tau_m$ of $\frac{\partial C_t(\tau)}{\partial \tau}$. Clearly, the area of the possible overlapping parts of these domains is of a smaller order than the area of the domains themselves.

![Figure 2. Two chords of the family $C_t$](image)

2.2. $\epsilon$-Entropy of $\mathcal{H}$. From now on we compute the $\epsilon$-entropy, the linear and non-linear width only for the curve $\mathcal{H}$ in the space $L^2([0,1])$. All these quantities depend only on the curve, and not on its parametrization, and they are preserved by the isometries of the ambient Hilbert spaces. To exclude the influence of the scalar rescaling we can normalize our curves, for example, assuming that the distance between the end-points is one. Then by Corollary 2.3 the $\epsilon$-entropy, the linear and non-linear width are exactly the same for each crinkled curve.

Let us remind now a general definition of $\epsilon$-entropy. Let $A \subset X$ be a relatively compact subset in a metric space $X$.

**Definition 2.6.** For $\epsilon > 0$ the covering number $M(\epsilon, A)$ is the minimal number of closed $\epsilon$-balls in $X$ covering $A$. The binary logarithm of the covering number, $H(\epsilon, A) = \log M(\epsilon, A)$ is called the $\epsilon$-entropy of $A$.
See [33][34] and many other publications for computation of \( \epsilon \)-entropy in many important examples. Intuitively, \( \epsilon \)-entropy of a set \( A \) is the minimal number of bits we need to memorize a specific element of this set with the accuracy \( \epsilon \). Thus it provides a lower bound for the “compression” of \( A \), independently of the specific compression method chosen.

**Proposition 2.7.** For the curve \( \mathcal{H} \) in the space \( L^2([0,1]) \) we have

\[
M(\epsilon, \mathcal{H}) \simeq \left( \frac{1}{\epsilon} \right)^2, \quad H(\epsilon, \mathcal{H}) \sim 2 \log\left( \frac{1}{\epsilon} \right).
\]

Here the sign \( \simeq \) is used as an equivalent to the inequality

\[
C_1 \left( \frac{1}{\epsilon} \right)^2 \leq M(\epsilon, \mathcal{H}) \leq C_2 \left( \frac{1}{\epsilon} \right)^2
\]

for certain \( C_1 \) and \( C_2 \), and for all sufficiently small \( \epsilon \). The sign \( \sim \) shows that \( C_1 \) and \( C_2 \) tend to 1 as \( \epsilon \) tends to zero.

**Proof:** Let us subdivide uniformly the interval \([0,1]\) into \( N \) segments \( \Delta_i \) by the points \( t_i = \frac{i}{N} \). We have

\[
\|\mathcal{H}(t_{i+1}) - \mathcal{H}(t_i)\| = \left( \int_{t_i}^{t_{i+1}} dt \right)^{\frac{1}{2}} = \left( \frac{1}{N} \right)^{\frac{1}{2}}.
\]

Hence for \( \epsilon = \frac{1}{2} \left( \frac{1}{N} \right)^{\frac{1}{2}} \) the \( \epsilon \)-balls covering different points \( \mathcal{H}(t_i), \ i = 1, \ldots, N \) of the curve \( \mathcal{H} \) do not intersect. Thus, we need at least \( N \) such \( \epsilon \)-balls to cover \( \mathcal{H} \), while the \( 2\epsilon \)-balls centered at the points \( \mathcal{H}(t_i), \ i = 1, \ldots, N \) cover the entire curve \( \mathcal{H} \). This completes the proof.

**2.3. Kolmogorov’s n-width of \( \mathcal{H} \).** Let \( A \subset V \) be a centrally-symmetric set in a Banach space \( V \).

**Definition 2.8.** ([39][58]). The Kolmogorov’s \( n \)-width \( W_n(A) \) of the set \( A \subset V \) is defined as

\[
W_n(A) = \inf_{\dim L = n} \sup_{x \in A} \text{dist}(x, L),
\]

where the infimum is taken over all the \( n \)-dimensional linear subspaces \( L \) of \( V \), and \( \text{dist}(x, L) \) denotes the distance of the point \( x \) to \( L \).

Intuitively, \( W_n(A) \) is the best possible approximation of \( A \) by \( n \)-dimensional linear subspaces of \( V \). Let us define also \( N(\epsilon, A) \) as the minimal \( n \) for which \( W_n(A) \leq \epsilon \).

To make the Kolmogorov \( n \)-width comparable with the \( \epsilon \)-entropy, we define the notion of a \textit{linear} \( \epsilon \)-entropy of \( A \), which is the number of bits we need to memorize \( A \) with the accuracy \( \epsilon \), if we insist on a \textit{linear approximation} of \( A \) (and if we “naively” memorize each of the coefficients in this linear approximation):
**Definition 2.9.** A linear $\epsilon$-entropy of $A$, $H_l(\epsilon, A)$, is defined by

\[
H_l(\epsilon, A) = N(\epsilon, A) \log \left( \frac{1}{\epsilon} \right).
\]

Now we state the main result of this section:

**Theorem 2.10.** For the curve $\mathcal{H}$ in $L^2[0,1]$ we have

\[
\frac{1}{4 \sqrt{n}} \leq W_n(\mathcal{H}) \leq \frac{2}{\pi \sqrt{n} - 1}, \quad N(\epsilon, \mathcal{H}) \asymp \left( \frac{1}{\epsilon} \right)^2, \quad H_l(\epsilon, \mathcal{H}) \asymp \left( \frac{1}{\epsilon} \right)^2 \log \left( \frac{1}{\epsilon} \right).
\]

**Proof:** It is enough to prove the bounds for the $n$-width of $\mathcal{H}$. The corresponding bound for $N(\epsilon, \mathcal{H})$ and $H_l(\epsilon, \mathcal{H})$ follow immediately.

Now, the upper bound for the $n$-width we obtain, considering the Fourier series approximation of the Heaviside functions $H_t(x)$.

\[
H_t(x) = \sum_{k \in \mathbb{Z}} a_k e^{2\pi i k x}
\]

Then $a_0 = t$ and $a_n = \frac{1 - e^{-2\pi in}}{2\pi n}$ for $n \neq 0$. We have $|a_n| \leq \frac{1}{\pi n}$. Hence the $L^2$ error $f_n$ of the approximation of any $H_t$ by the first $2n+1$ terms of its Fourier series satisfies

\[
f_n \leq \left[ \sum_{m=n+1}^{\infty} \frac{2}{\pi^2 m^2} \right]^{\frac{1}{2}} < \frac{\sqrt{2}}{\pi \sqrt{n}}.
\]

And therefore

\[
W_{2n+1}(\mathcal{H}) \leq f_n \implies W_n(\mathcal{H}) \leq \frac{2}{\pi \sqrt{n} - 1}.
\]

The proof of the lower bound we split into several steps.

**Lemma 2.11.** For a set $A_k = \{e_i | (e_i, e_j) = \delta_{ij}, 1 \leq i \leq k \} \subseteq L^2[0,1]$ and $n < k$ the following inequality holds

\[
W_n(A_k) \geq \sqrt{\frac{k - n}{k}}.
\]

**Proof:** Denote $W = \text{span}\{e_i\}$ and $P_W : L^2[0,1] \rightarrow L^2[0,1]$ the orthogonal projection on $W$. We take an $n$-dimensional subspace $V$. We can assume that $V \subseteq W$.

This is because for $v \in V$, $a \in A_k$ we have:

\[
||a - v||^2 = ||P_W(a - v)||^2 + ||(I - P_W)(a - v)||^2 = ||a - P_Wv||^2 + ||(I - P_W)v||^2 \geq ||a - P_Wv||^2.
\]
Therefore \( \text{dist}(A_k, V) \geq \text{dist}(A_k, P_W V) \), and in order to minimize the distance we can assume \( V \subseteq W \). Denote \( P_V : W \rightarrow W \) the orthogonal projection on \( V \) in \( W \). We need to compute \( \max_{1 \leq i \leq k} \| (I - P_V) e_i \| \).

But

\[
(2.7) \quad \max_{1 \leq i \leq k} \| (I - P_V) e_i \| \geq \sqrt{\frac{1}{k} \sum_{i=1}^{k} \| (I - P_V) e_i \|^2}.
\]

On the other hand,

\[
(2.8) \quad \sum_{i=1}^{k} \| (I - P_V) e_i \|^2 = \sum_{i=1}^{k} ((I - P_V) e_i, (I - P_V) e_i) = \sum_{i=1}^{k} ((I - P_V)^2 e_i, e_i) = \sum_{i=1}^{k} (I - P_V) e_i, e_i = \text{trace}_W (I - P_V) = k - n.
\]

The last equality is because \( I - P_V : W \rightarrow W \) is a projection into a \((k - n)\)-subspace - the orthogonal complement of \( V \) in \( W \). Combining equations (2.7), (2.8) we have

\[
\text{dist}(A_k, V) \geq \sqrt{\frac{k - n}{k}}.
\]

**Corollary 2.12.** For any \( d \in \mathbb{R} \),

\[
W_n(d A_k) \geq |d| \sqrt{\frac{k - n}{k}}.
\]

**Proposition 2.13.**

\[
W_n(\mathcal{H}) \geq \frac{1}{4 \sqrt{n}}.
\]

**Proof:** Denote \( B_k = \{ \chi(\frac{i-1}{k}, \frac{1}{k}) | 1 \leq i \leq k \} \). This set is formed by \( k \) orthogonal vectors of length \( \frac{1}{\sqrt{k}} \). Clearly \( \chi(\frac{i-1}{k}, \frac{1}{k}) = \mathcal{H}_{\frac{k}{k}} - \mathcal{H}_{\frac{i-1}{k}} \), therefore

\[
\text{dist}(B_k, V) \leq 2 \text{dist}(\mathcal{H}, V)
\]

for any vector space \( V \), and thus:

\[
W_n(B_k) \leq 2W_n(\mathcal{H}).
\]

The norm of \( \chi(\frac{i-1}{k}, \frac{1}{k}) \) is \( \frac{1}{\sqrt{k}} \), and according to Corollary 2.12 we have

\[
W_n(\mathcal{H}) \geq \frac{1}{2} W_n(B_k) \geq \frac{1}{2} \frac{1}{\sqrt{k}} \sqrt{\frac{k - n}{k}}.
\]
Taking $k = 2n$ provides the required result. This completes the proof of Proposition 2.13 and of Theorem 2.10.

2.4. Sparse representation of a step-function. Our main example of the family $\mathcal{H}$ of the step-functions $H_t(x)$ allows us to illustrate also some important features of “sparse representations”. Consider the Haar frame:

$$HF = \left\{ \phi_{k,j}(x) = 2^{k/2} \phi(2^k (x-j)) | k \in \{0, 1, 2, \ldots \}, j \in \{0, 1, 2, \ldots, 2^k - 1\} \right\}$$

where $\phi = \chi_{[0,1]}$. To get an approximation of a certain fixed step-function $H_{t_0}(x)$ consider the binary representation of $t_0$:

$$t_0 = \sum_{r=1}^{\infty} \frac{\alpha_r}{2^r}, \quad \alpha_r = 0, 1.$$

Then for each $n$ the sum

$$\sum_{r \leq n} \phi_{r,j_r}(x), \quad j_r = \frac{1}{2^r} \sum_{s=1}^{r-1} \frac{\alpha_s}{2^s}$$

leads to the approximation of $H_{t_0}(x)$ in the Haar frame with the $L^2$-error at most $(\frac{1}{2})^{n+1}$. Indeed, the sum in (6.1) is, in fact, a step-function $H_{t_1}$, with $t_1 \leq t_0$ and $t_0 - t_1 \leq (\frac{1}{2})^{n+1}$ (see Figure 3).

So to $\epsilon$-approximate each individual step-function $H_{t_0}$ via the Haar frame in the $L^2$-norm, we need only $2 \log_2(\frac{1}{\epsilon})$ nonzero terms in the linear combination. This provides a natural example of a “sparse representation”.

Notice, however, that if we fix the required approximation accuracy $\epsilon = (\frac{1}{2})^{n+1}$, and then let the jump point $t$ of $H_t$ change, then the elements of the Haar frame, participating in the representation of different $H_t(x)$, eventually cover all the $2^n$ binary step-functions of the $n$-th scale. So altogether, to approximate the entire curve $\mathcal{H} \subset L^2([0,1])$, we need the space of the dimension $2^n = (\frac{1}{2})^2$. This agrees with the value of $W_n(\mathcal{H})$ computed above.

2.5. $n$-term representation. In order to quantify the “sparseness” of different representations (and, in particular, to include the previous example in a more general framework) we call (following [14, chapter 8]) a countable collection $D$ of vectors in a Banach space a dictionary, and define the error of the $n$-term approximation of a single function
\[ t_0 = 0.10101\ldots \]

**Figure 3.** Approximation by Haar frame of \( \chi_{t_0} \)

We use three different dictionaries for \( L_2[0, 1] \):

- Fourier basis: \( \text{FB} = \{ e^{ikx} \mid k \in \mathbb{Z} \} \),
- Haar frame: \( \text{HF} = \{ \phi_{k,j} = 2^{k/2}\phi(2^k(x-j)) \mid k \in \{0, 1, 2, \ldots \}, j \in \{0, 1, 2, \ldots 2^k-1\} \} \), and Haar basis: \( \text{HB} = \{ \psi_{k,j} = 2^{k/2}\psi(2^k(x-j)) \mid k \in \{0, 1, 2, \ldots \}, j \in \{0, 1, 2, \ldots 2^k-1\} \} \cup \{\phi\} \),

where \( \phi = \chi_{[0,1]} \) and \( \psi = \chi_{[0,1/2]} - \chi_{[1/2,1]} \).

Clearly,

\[ \sigma_n(H_t, \text{FB}) \asymp \frac{1}{n^{1/2}}, \]

which means, that the best \( n \)-term approximation in this case is the same as the usual linear Fourier approximation. Also, we have

\[ \sigma_n(H_t, \text{HF}) \leq C2^{-\frac{n}{2}}, \]

\[ \sigma_n(H_t, \text{HB}) \leq C'2^{-\frac{n}{2}}. \]
Remark: It is customary in the Approximation Theory to demand that $n$-term approximation will be "computable" - so that it has polynomial-depth search. This means that we can enumerate our dictionary with a fixed enumeration $D = \{f_1, f_2, \ldots\}$ in such a way that for a certain polynomial $p : \mathbb{N} \to \mathbb{N}$ the $n$-terms of approximation come from the first $p(n)$ terms of the dictionary, see [9]. Clearly, if we consider HF and HB we will need to take each function from a different level of Haar basis/frame, and therefore our search will have an exponential depth.

2.6. Temlyakov’s non-linear width. The following notion of a “non-linear width” was introduced in [56]:

**Definition 2.14.** Let $A$ be a symmetric subset in a Banach space $\mathcal{X}$. Then the $(N, m)$ width $W_{(N,m)}(A)$ is defined as

$$W_{(N,m)}(A) = \inf_{\mathcal{L}_N \subseteq \mathcal{L}(\mathcal{X})_m, |\mathcal{L}_N| = N} \sup_{f \in A} \inf_{L \in \mathcal{L}_N} \text{dist}(f, L),$$

where $\mathcal{L}(\mathcal{X})_m$ denotes the collection of all the linear $m$-dimensional subspaces of $\mathcal{X}$.

The approximation procedure, suggested by this notion, is as follows: given $N$ and $m$, we fix (in an optimal way) a subset $\mathcal{L}_N$ of $N$ different $m$-dimensional linear subspaces $L_1, \ldots, L_N$ in $\mathcal{X}$. Then for each specific function $f \in \mathcal{X}$ we first pick the most suitable subspace $L_i$ in $\mathcal{L}_N$, and then find the best linear approximation of $f$ by the elements of $L_i$.

The notion of a nonlinear width provides a “bridge” between the linear approximation and the approximation based on “geometric models”. Indeed, ultimately the set $\mathcal{L}_N$ may be just the set formed by all the piecewise-constant functions (in our main example), for all the values of the parameters, discretized with the required accuracy. See Section 3 below where we analyze in somewhat more detail this “bridging” for the curve $\mathcal{H}$.

The set $\mathcal{L}_N$ suggests a covering of the $A$ by $n$ sets

$$V_i = \{g \mid \text{dist}(g, L_i) \leq \text{dist}(g, L_k); L_i, L_k \in \mathcal{L}_N\}.$$ 

Namely, the set $V_i$ contains the elements of $A$, that are best approximated by the subspace $L_i$ from the collection $\mathcal{L}_N$. In the next lemma, we prove that we can replace $V_i$ by open sets.

**Lemma 2.15.** Let $\mathcal{O}_N$ denote the set of all the open covers $\mathcal{U} = \{U_1, \ldots, U_N\}$ of $A$ of cardinality $N$. Then

$$W_{(N,m)}(A) = \inf_{\mathcal{U} \in \mathcal{O}_N} \sup_{U_i \in \mathcal{U}} W_m(U_i).$$
In other words, we subdivide $A$ into $N$ open sets and check $m$-width on each of the sets separately. Then the maximum $m$-width over $N$ sets is the $(N, m)$-width of $A$.

**Proof:** Denote by 
$$d = W_{(N, m)}(A),$$
the left-hand side of the equation and by 
$$e = \inf_{U \in \mathcal{O}, \#U = N} \sup_{U_i \in U} W_m(U_i),$$
it’s right-hand side. For $\epsilon > 0$, we interpret the definition of $d$ as existence of $\mathcal{L}_N$ a collection of $N$ $m$-dimensional subspaces of $X$ such that 
$$\inf_{L_i \in \mathcal{L}_N} \inf_{i = 1..N} g \in L_i \| f - g \| < d + \epsilon \quad \forall f \in A$$
Define $U_i = \{ g | g \in A, \| g - L_i \| < d + \epsilon \}$. Clearly, $U_i$ are open in $A$ since the distance is a continuous function. According to the definition of $U_i$, $L_i$ approximates $U_i$ with accuracy $d + \epsilon$ and therefore $W_m(U_i) \leq d + \epsilon$. We conclude that 

$$e \leq \sup_i W_m(U_i) \leq d + \epsilon. \tag{2.11}$$

In the other direction, let $\bigcup_{i=1}^N U_i = A$ such that $W_m(U_i) < e + \epsilon$. For each $i$ we find $L_i$ an $m$-dimensional subspace s.t. 
$$\operatorname{dist}(g, L_i) < e + 2\epsilon, \quad \forall g \in U_i,$$
Then form $\mathcal{L}_N = \{ L_1, .., L_N \}$. Clearly, 
$$\sup_{f \in A} \inf_{L \in \mathcal{L}_N} \operatorname{dist}(f, L) < e + 2\epsilon$$
and therefore 

$$d < e + 2\epsilon. \tag{2.12}$$

Taking $\epsilon \rightarrow 0$ in the inequalities (2.11), (2.12), we get the required equality.

In what follows, we take $X = L_2([0, 1])$.

**Proposition 2.16.**

$$W_{(N, m)}(\mathcal{H}) \simeq \frac{1}{\sqrt{Nm}}.$$
Proof:
To establish an upper bound, define

\[ L_k = \text{span}\{\chi_{\frac{k-1}{N} + \frac{n}{N m}, \frac{k-1}{N} + \frac{n+1}{N m}} : 0 \leq n \leq m - 1\}, \quad k = 1..N \]

Each \( L_k \) approximates \( \mathcal{H}|_{\left[\frac{k-1}{N}, \frac{k}{N}\right]} \) within an error of \( \frac{1}{\sqrt{2N m}} \). Therefore \( W_{(N,m)} \) is bounded above by an error of \( \{L_k\}_{k=1}^N \).

In order to establish the lower bound, we prove a variant of a Lemma 2.11.

Lemma 2.17. For a set \( A_k = \{e_i | (e_i, e_j) = \lambda_i^2 \delta_{ij}, 1 \leq i \leq k, \lambda_i > 0\} \) and \( n < k \) the following inequality holds

\[ W_n(A_k) \geq \sqrt{\frac{k-n}{k}} \min_i \lambda_i. \]

The difference with Lemma 2.11 is that we allow orthogonal vectors with varying lengths.

Proof: Let \( V \) be a \( n \)-dimensional space and \( W = \text{span}\{e_i\} \). Just like in Lemma 2.11 we can assume \( V \subseteq W \). Denoting the orthogonal projection of \( V \) on \( W \) by \( P \), we are required to compute \( \max_i \| (I - P)e_i \| \).

(2.13)

\[ \max_i \| (I - P)e_i \| \geq \sqrt{\frac{1}{k} \sum_{i=1}^{k} \| (I - P)e_i \|^2} \geq \min_i \sqrt{\frac{1}{k} \sum_{i=1}^{k} \| (I - P)\frac{e_i}{\lambda_i} \|^2}. \]

Since \( \frac{e_i}{\lambda_i} \) are orthonormal then

(2.14)

\[ \sum_{i=1}^{k} \| (I - P)\frac{e_i}{\lambda_i} \|^2 = \sum_{i=1}^{k} ((I - P)\frac{e_i}{\lambda_i}, \frac{e_i}{\lambda_i}) = \text{trace}(I - P) = k - n. \]

Combining equations 2.13 and 2.14 we get the required result.

We return to the proof of the Proposition 2.16. We employ Lemma 2.15. Let \( \{U_i\}_{i=1..n} \) be an open cover of \( \mathcal{H} \). Define

\[ V_i = \mathcal{H}^{-1}(U_i). \]

Namely, \( V_i \subseteq [0,1] \) contains all the \( t \)'s such that \( \mathcal{H}(t) \in U_i \). \( V_i \) are open in \([0,1]\), since \( \mathcal{H} \) is continuous. The collection \( \{V_i\}_{i=1..n} \) is a covering of \([0,1]\) since \( \{U_i\} \) is a cover of \( \mathcal{H} \). Because \( V_i \) is open, we can find \( 2m + 1 \) points \( \lambda_k \) in \( V_i \) such that \( \lambda_k - \lambda_{k+1} \geq \frac{\text{meas}(V_i)}{2m} - \epsilon \), for any \( \epsilon > 0 \), where \( \text{meas}(V_i) \) denotes here the Lebesgue measure of \( V_i \). We apply Lemma 2.17 to \( B_{2m} = \{\chi_{[\lambda_k, \lambda_{k+1}]} : 1 \leq k \leq 2m\} \). Since
\[ \|\chi_{[\lambda_k, \lambda_{k+1}]}\| = \sqrt{\lambda_{k+1} - \lambda_k} \geq \sqrt{\frac{\text{meas}(V_i)}{4m}} - \epsilon, \] the application of Lemma 2.17 gives
\[ W_m(B_{2m}) \geq \sqrt{\frac{\text{meas}(V_i)}{4m}} - \epsilon. \]

But \[\chi_{[\lambda_k, \lambda_{k+1}]} = \mathcal{H}_{\lambda_{k+1}} - \mathcal{H}_{\lambda_k}.\] Denote \[S = \{\mathcal{H}_{\lambda_k} : 1 \leq k \leq 2m+1\} \subseteq U_i.\]

For any vector space \(V\), we have
\[\text{dist}(B_{2m}, V) \leq 2\text{dist}(S, V),\]
and therefore
\[(2.15) \quad W_m(U_i) \geq W_m(S) \geq \frac{1}{2} W_m(B_{2m}) \geq \frac{1}{4} \sqrt{\frac{\text{meas}(V_i)}{m}} - \epsilon.\]

But since \(V_i\) cover \([0, 1]\), we have \[\sum_{i=1}^{N} \text{meas}(V_i) \geq 1\] and so
\[\max_i \text{meas}(V_i) \geq \frac{1}{N}.\]

Therefore
\[(2.16) \quad \max_i W_m(U_i) \geq \frac{1}{4\sqrt{Nm}} - \epsilon,\]
for any open cover of \(\mathcal{H}\). And so, according to Lemma 2.15
\[(2.17) \quad W_{(N,m)}(\mathcal{H}) \geq \frac{1}{4\sqrt{Nm}} - \epsilon.\]

Thus we obtain the required lower bound after we take \(\epsilon \rightarrow 0\).

3. Linear versus Non-Linear Compression: some conclusions

In this section we summarize the above results, interpreting them as the estimates of the “compression” of the family \(\mathcal{H}\) (and of other families of piecewise-constant functions): how many bits do we need to memorize an arbitrary jump-function \(H_i\) in \(\mathcal{H}\) with the \(L^2\)-error at most \(\epsilon\), via different representation methods?

3.1. \(\epsilon\)-entropy. Let us start with the \(\epsilon\)-entropy: by Proposition 2.7, \(H(\epsilon, \mathcal{H}) \sim 2\log(\frac{1}{\epsilon})\). This is the lower bound on the number of bits in any compression method.
3.2. “Model-based compression”. Let us consider a “non-linear model-based compression” which in the case of the jump-functions takes an extremely simple form: we use the “library model” $H_t(x)$ to represent itself, and we memorize just the specific value of the parameter $t$. Quite expectedly, “compression” with this model requires exactly the number of bits prescribed by the $\epsilon$-entropy. Indeed, since the $L^2$-norm of $H_{t_2} - H_{t_1}$ is $\sqrt{t_2 - t_1}$, we have to memorize $t$ with the accuracy $\epsilon^2$. This requires exactly $2 \log(\frac{1}{\epsilon})$ bits.

3.3. “Linear” compression. Let us assume now that, given the required accuracy $\epsilon$, we insist on a representation of the functions $H_t(x)$ in a fixed basis, the same for each $t$. On the other hand, we allow the approximating linear space to depend on $\epsilon$. This leads to the Kolmogorov $n$-width, as defined in Section 2.3. We store each coefficient with the maximal error $\epsilon$, so we allow for it $\log(\frac{1}{\epsilon})$ bits (and thus we ignore a very special “sparse” nature of the representation of $H_t(x)$ in some special bases, for instance, in the Haar frame, discussed in Section 2.4). Then the number of bits required is given by the “linear $\epsilon$-entropy” $H_l(\epsilon, \mathcal{H})$, introduced in Section 2.3. By Theorem 2.10, we have

$$H_l(\epsilon, \mathcal{H}) \approx (\frac{1}{\epsilon})^2 \log(\frac{1}{\epsilon}).$$

In fact, to get a representation with this amount of information stored, we do not need all the freedom provided by the definition of $n$-width. It is enough to fix the approximating space to be the space of trigonometric polynomial for any required accuracy $\epsilon$. Then to approximate $H_t$ with the $L^2$-accuracy $\epsilon$ we take the Fourier polynomial $F^n_t$ of $H_t$ of degree $n = \frac{1}{\epsilon^2}$ and memorize its coefficients with the accuracy $\frac{\epsilon}{n}$.

3.4. “Non-linear width” compression. In [56] a notion of a “non-linear $(N,m)$-width” has been introduced (see Section 2.6 above). It suggests the following procedure for approximating functions $H_t(x)$: given the required accuracy $\epsilon$, we fix a subset of $N$ $m$-dimensional linear subspaces $L_1, \ldots, L_N$ in $L^2[0,1]$. Then for each specific function $H_t(x)$ we first pick one of the subspaces $L_i$ (the most suitable), then find the best linear approximation of $H_t(x)$ by the elements of $L_i$, and finally memorize the coefficients of the best linear approximation found.

Let us estimate the number of bits required in this approach. By Proposition 2.16, for the non-linear $N,m$-width of $\mathcal{H}$ we have

$$W_{(N,m)}(\mathcal{H}) \approx \frac{1}{\sqrt{Nm}}.$$
Given the required accuracy \( \epsilon \), we have to fix the parameters \( N \) and \( m \) in such a way that \( \frac{1}{\sqrt{Nm}} \leq \epsilon \). Therefore, for each choice of \( m \) between 1 and \( (\frac{1}{\epsilon})^2 \) we have to take \( N = \frac{1}{m} (\frac{1}{\epsilon})^2 \). To memorize the choice of the space \( L_i \) we need then \( \log N = 2 \log(\frac{1}{\epsilon}) - \log m \) bits. To memorize the coefficients we need \( m \log(\frac{1}{\epsilon}) \) bits. Hence, the total amount of bits is
\[
(m + 2) \log(\frac{1}{\epsilon}) - \log m.
\]
Certainly, the best choice is \( m = 1 \): we just take \( N = (\frac{1}{\epsilon})^2 \) elements \( H_t, \ t = \frac{i}{N} \), and approximate \( H_t \) with the nearest among \( H_t \). This is, essentially, the same as the “model-based” representation in Section 3.2 above.

3.5. “Sparse” representation. Till now the comparison was in favor of a model-based approach. Let us consider now the Haar frame representation of \( H_t(x) \) considered in Section 2.4 above. This is the most natural competitor, both because of its theoretical efficiency, and since many modern practical approximation schemes are based on sparseness considerations (see [8, 9, 55, 56, 57]).

By the computation of Section 2.4, to approximate each individual step-function \( H_{t_0} \) via the Haar frame in the \( L^2 \)-norm, we need only \( m = 2 \log(\frac{1}{\epsilon}) \) of the nonzero terms in the linear combination. Moreover, each coefficient in this linear combination is 1. So to memorize \( H_{t_0} \) via the Haar frame it is enough to specify the position of \( m = 2 \log(\frac{1}{\epsilon}) \) nonzero elements among the total Haar frame of cardinality \( 2^m = (\frac{1}{\epsilon})^2 \). We need
\[
\log \frac{(2^m)!}{m!(2^m - m)!} \asymp m^2 \asymp (\log(\frac{1}{\epsilon}))^2
\]
bits to do this.

We get a little bit more information to store than in the “model-based” approach. Also, it may look not natural to approximate such a simple pattern as a jump of a step-function with a geometric sum of shrinking signals. However, the main problem is that if we let the jump point \( t \) of \( H_t \) change, then the elements of the Haar frame, participating in the representation of different \( H_t(x) \), jump themselves in a very sporadic way, and eventually cover all the \( 2^m \) binary Haar frame functions of the \( m \)-th scale.

Notice also that from the point of view of the non-linear width (Section 2.6 above) the considered Haar frame representation takes an intermediate position: here \( m = 2 \log(\frac{1}{\epsilon}) \). But any subspace \( L = \text{span}\{\chi_{[t_{ki}, t_{ki} + 2^{-ki}]: i = 1..m}\} \) can cover only \( \mathcal{H}_{t_{ki}} \) and their \( \epsilon \)-neighborhoods and therefore \( L \) covers with the accuracy \( \epsilon \) only a set of measure
$4\epsilon^2(\log(\frac{1}{\epsilon}) + 1)$ out of the entire interval $[0,1]$ of parameters. Thus to cover the entire interval we will need $N \asymp \frac{1}{\epsilon^2 \log(\frac{1}{\epsilon})}$ subspaces. We conclude that $\epsilon \asymp \frac{1}{\sqrt{Nm}}$, in agreement with Proposition 2.16. The required number of bits is

$\left( \log \left( \frac{1}{\epsilon} \right) \right)^2 + 2 \log \left( \frac{1}{\epsilon} \right) - \log \log \left( \frac{1}{\epsilon} \right) + \log 2$.

So it would be much more natural and efficient to represent a “video-sequence” $\mathcal{H} = \{H_t(x), t \in [0,1]\}$ by a moving model than to follow the jumping parameters in a sparse Haar representation for variable $t$. This conclusion certainly is not original. The problem is to get a full quality model-based geometric representation of real life images and video-sequences!

4. Non-linear Fourier inversion

Now we turn to our second main problem: how to reconstruct functions with singularities (piecewise-constant functions) from a small number of measurements? Let us assume that our “measurements” are just the scalar products of the function $f$ to be reconstructed with a certain sequence of basis functions. In particular, below we assume our measurements to be the Fourier coefficients of $f$ or its moments. This is a realistic assumption in many practical problems, like Computer Tomography.

The rate of Fourier approximation of a given function and the accuracy of its reconstruction from partial Fourier data is determined by regularity of this function. For functions with singularities, even very simple, like the Heaviside function, the convergence of the Fourier series is very slow. Hence a straightforward reconstruction of the original function from its partial Fourier data (i.e. forming partial sums of the Fourier series) in this cases is difficult. It also involves some systematic errors (like the so-called Gibbs effect).

Let us show that no linear reconstruction method can do significantly better that the straightforward Fourier expansion.

**Theorem 4.1.** Let the function acquisition process comprise taking $n$ measurements (linear or non-linear) $m_i(f)$, $i = 1, \ldots, n$ of the function $f$, together with a consequents processing $P$ of these measurements. If the processing operator $\hat{f} = P(m_1, \ldots, m_n)$ is a linear operator from $\mathbb{R}^n$ to $L^2([0,1])$ then for some $f \in \mathcal{H}$ the error $\|f - \hat{f}\|$ is at least $C_1\frac{1}{\sqrt{n}}$. 
Proof: This follows directly from Theorem 2.10 above. Indeed, the $n$-dimensional linear subspace $\text{Im}(P)$ cannot approximate all the functions in $\mathcal{H}$ with the error better than the Kolmogorov $n$-width of $\mathcal{H}$.

If we have no a priori information on $f \in L^2([0, 1])$ then probably the straightforward Fourier reconstruction as above remains the best solution. However, in our case we know that $f$ is a piecewise constant function. It is completely defined by the positions of its jumps $0 < x_i < 1$, $i = 1, \ldots, N$ and by its values $A_i$ between the jumps. So let us consider $x_i$ and $A_i$ as unknowns and let us substitute these unknowns into the integral expression for the Fourier coefficients. We get certain analytic expressions in $x_i$ and $A_i$. Equating these expressions to the measured values of the corresponding Fourier coefficients we get a system of nonlinear equations on the unknowns $x_i$ and $A_i$. Let us write down this system explicitly.

4.1. Fourier inversion system. Let $f(x) = \sum_{-\infty}^{\infty} c_k e^{2\pi i k x}$ be the Fourier expansion of $f$. Here $c_k = \frac{1}{2\pi i} \int_0^1 f(t) e^{-2\pi i k t} dt$. Taking into account a special form of $f$ as given above we obtain $c_k = \frac{1}{2\pi i} [-A_0 + \sum_{i=1}^{N} (A_{i-1} - A_i) e^{-2\pi i k x_i} + A_N e^{-2\pi i k}]$. Here $A_0$ is the value of $f$ on the leftmost continuity interval. Denoting $-2\pi i c_k$ by $\hat{c}_k$ and $e^{-2\pi i x_i}$ by $z_i$, we finally get the following infinite system

$$A_0 + \sum_{i=1}^{N} (A_i - A_{i-1}) z_i^k - A_N e^{-2\pi i k} = \hat{c}_k, \ k \in \mathbb{Z}.$$ (4.1)

The unknowns in system (4.1) are $A_j$, $j = 0, \ldots, N$ which enter this system in a linear way, and $z_i$, $i = 1, \ldots, N$, entering it non-linearly.

System (4.1) classically appears in Padé Approximation. Very similar systems appear in a reconstruction of plane polygonal domains from their moments ([43, 44, 45, 26, 18, 19]). A detailed investigation of a larger class of systems similar to (4.1) is given in [31, 32]. In particular, we have the following result:

**Theorem 4.2.** Assume that $c_k$ in the right-hand side of (4.1) are Fourier coefficients of a piecewise-constant function $f$ with $A_i \neq A_{i+1}$, $i = 0, \ldots, N$. Then each subsystem of (2.1) obtained by taking from it certain $2N + 1$ subsequent equations has a unique solution $\{A_j, \ j = 0, \ldots, N\}$, $\{z_i, \ i = 1, \ldots, N\}$, with $x_i = \frac{1}{-2\pi i} \log z_i$ being the jump points of $f$ and $A_j$ being the values of $f$ on its continuity interval.

We give a sketch of the proof, following [31, 32], in Section 4.2 below.

Thus solving an appropriate subsystem of system (4.1) we find the jumps and the intermediate values of $f$, so we reconstruct $f$ exactly. If
$f$ had $N$ jumps we need only $2N + 1$ Fourier coefficients to reconstruct it.

4.2. Other examples of the inversion systems. Let us start with another system which essentially coincides with (4.1). To simplify the presentation we shall consider instead of the Fourier coefficients of the function $g(x)$, $x \in [0, 1]$ the moments $m_k(g) = \int_0^1 x^k g(x) dx$.

4.2.1. Linear combination of δ-functions. Let $g(x) = \sum_{i=1}^n A_i \delta(x - x_i)$.

For this function we have

\[
m_k(g) = \int_0^1 x^k \sum_{i=1}^n A_i \delta(x - x_i) dx = \sum_{i=1}^n A_i x_i^k.
\]

So assuming that we know the moments $m_k(g) = \alpha_k$, $k = 1, \ldots, 2n-1$, we obtain the following non-linear system of equations for the parameters $A_i$ and $x_i$, $i = 1, \ldots, n$, of the function $g$:

\[
\begin{align*}
\sum_{i=1}^n A_i &= \alpha_0, \\
\sum_{i=1}^n A_i x_i &= \alpha_1, \\
\sum_{i=1}^n A_i x_i^2 &= \alpha_2, \\
&\cdots \\
\sum_{i=1}^n A_i x_i^{2n-1} &= \alpha_{2n-1}.
\end{align*}
\]

This system can be solved as follows: consider the moments generating function

\[
I(z) = \sum_{k=0}^{\infty} m_k(g) z^k
\]

. The representation (4.2) of the moments immediately implies that

\[
I(z) = \sum_{i=1}^n \frac{A_i}{1 - z x_i}.
\]

So it remains to find explicitly the rational function $I(z)$ from the first $2n$ its Taylor coefficients $\alpha_0, \ldots, \alpha_{2n-1}$.

To do this we remind that the Taylor coefficients of a rational function satisfy a linear recurrence relation of the form

\[
m_{r+n} = \sum_{j=0}^{n-1} C_j m_{r+j}, \quad r = 0, 1, \ldots
\]
Since we know the first $2n$ Taylor coefficients $\alpha_0, \ldots, \alpha_{2n-1}$, we can write a linear system on the unknown recursion coefficients $C_j$:

\[
\begin{align*}
\sum_{j=0}^{n-1} C_j \alpha_j &= \alpha_n, \\
\sum_{j=0}^{n-1} C_j \alpha_{j+1} &= \alpha_{n+1}, \\
\sum_{j=0}^{n-1} C_j \alpha_{j+2} &= \alpha_{n+2}, \\
&\vdots \\
\sum_{j=0}^{n} C_j \alpha_{j+n} &= \alpha_{2n-1}.
\end{align*}
\]

(4.6)

Solving linear system (4.6) with respect to the recurrence coefficients $C_j$ we find them explicitly. For a solvability of (4.6) see [46, 31, 32, 51, 52]. See also [44, 45, 26, 18].

Now the recurrence relation (4.5) with known coefficients $C_i$ and known initial moments allows us to easily reconstruct the generating function $I(z)$ and hence to solve (4.3).

4.2.2. **Algebraic functions.** Let now $g(x)$ be an algebraic function on $[0, 1]$. By definition, $y = g(x)$ satisfies an equation

\[
\begin{align*}
a_n(x)y^n + a_{n-1}(x)y^{n-1} + \cdots + a_1(x)y + a_0(x) &= 0,
\end{align*}
\]

(4.7)

where $a_n(x), \ldots, a_0(x)$ are polynomials in $x$ of degree $m$. $d = m + n$ is, by definition, the degree of $g$.

A general method for the non-linear inversion of the moment (Fourier) transforms of algebraic functions is given in [32]. Its “quantitative form is given in [52]. Here we analyze only one special case. Assume that the algebraic curve $y = g(x)$ is a rational one. This means that it allows for a rational parametrization

\[
\begin{align*}
x &= P(t), \\
y &= Q(t).
\end{align*}
\]

(4.8)

The moments $m_k(g)$ given by $m_k(g) = \int_0^1 x^k g(x) dx$, $k = 0, 1, \ldots$, now can be expressed as

\[
\begin{align*}
m_k(g) &= \int_0^1 P^k(t)Q(t)p(t)dt,
\end{align*}
\]

(4.9)

where $p$ denotes the derivative $P'$ of $P$. Moments of this form naturally appear in a relation with some classical problems in Qualitative Theory of ODE’s - see [5, 6], [7, 11, 62].

Our problem can be reformulated now as the problem of explicitly finding $P$ and $Q$ from knowing a certain number of the moments $m_k$ in (4.9). Of course, in general we cannot expect this system of non-linear equations to have a unique solution. Indeed, while the function $y = g(x)$ is determined by its moments in a unique way, the rational
**Parametrization** of this curve in general is not unique. In particular, let $W(t)$ be a rational function satisfying $W(0) = 0$, $W(1) = 1$. Substituting $W(t)$ into $P$ and $Q$ we get another rational parametrization of our curve: $x = \hat{P}(t)$, $y = \hat{Q}(t)$ with $\hat{P}(t) = P(W(t))$, $\hat{Q}(t) = Q(W(t))$. Consequently, the “inversion problem” for system (4.9) is:

*To characterize all the solutions of system (4.9) and to provide an effective way to find these solutions.*

A special case of the inversion problem is the “Moment vanishing problem”:

*To characterize all the pairs $P, Q$ for which the moments $m_{k}$ defined by (4.9) vanish.*

In spite of a very classical setting (we ask for conditions of orthogonality of $Q$ to all the powers of $P$!) this problem has been solved only very recently ([48]). It plays a central role in study of the center conditions for the Abel differential equation (see [5, 6], [7, 11, 62], [47, 48]).

4.2.3. **Functions of two variables.** The approach to reconstruction of piecewise-smooth (piecewise-polynomial) functions of one variable discussed above can be extended to two (and more) variables. The case of characteristic functions of polygonal plane domains is considered in [44, 45, 26, 18]. Some initial instances of the reconstruction problem of piecewise-polynomial functions of two variables are considered in [51]. Even the most initial examples in two dimensions provide an exciting variety of non-linear system bringing us to the very heart of Analysis. Let us mention here only one example and a few of the most directly related references.

We want to reconstruct a function $f(x, y)$ of two variables from its moments

$$m_{kl}(f) = \int \int x^{k}y^{l}f(x, y)dxdy. \quad k, l = 0, 1, \ldots$$

Assume that $f$ is a $\delta$-function along a rational curve $S$, i.e. for any $\psi(x, y)$ we have $\int \int f\psi dxdy = \int_{S} \psi(x, y)dx$.

Let

$$x = P(t), \quad y = Q(t), \quad t \in [0, 1]$$

be a rational parametrization of $S$. The moments now can be expressed as

$$m_{kl}(f) = \int_{0}^{1} P^{k}(t)Q^{l}(t)p(t)dt,$$
where $p$ denotes the derivative $P'$ of $P$. The study of the double moments of this form bring us naturally to the recent work of G. Henkin [15, 29, 30]. Indeed, the vanishing condition for the moments (4.12) is given by Wermer’s theorem ([1]): $m_{kl}(f) \equiv 0$ if and only if $S$ bounds a complex 2-chain in $C^2$. In general, if the moments $m_{kl}(f)$ do not vanish identically, then the local germ of complex analytic curve $\hat{S}$ generated by $S$ in $C^2$ does not “close up” inside $C^2$. G. Henkin’s work ([15, 29, 30]), in particular, analyzes various possibilities of this sort in terms of the “moments generating function”. We expect that a proper interpretation of the results of [15, 29, 30] can help also in understanding of the moment inversion problem.

There are many other results closely related to our problem (see references in [44, 45, 26, 18], [51, 52]. Here we mention in addition only [27, 50] where, in particular, the problem of a reconstruction of plane “quadrature domains” from their double moments is considered, and results on moments on Semi-Algebraic sets and positivity (see [36, 37, 53] and references there).

4.3. **Robustness of solutions of (2.1).** Let us return to functions of one variable. The assumption of $f$ being a piecewise-constant function may look too unrealistic in applications. However, the methods can be extended to piecewise-polynomial and ultimately to piecewise-smooth functions (see [31, 32, 51, 52]). The last class is of major importance in applied Analysis and Signal Processing, and the problem of a reconstruction of such functions from their measurements (Fourier data) is at present actively investigated (see [22]-[25], [49, 41, 54] and references there).

The key issue in the extension of the above methods to piecewise-smooth functions is a robustness of solutions of (4.1), (4.3), (4.9) and similar systems. In particular, what happens if we take more than exactly $2N + 1$ consequent equations in (4.1), and because of the noise in our measurements the right hand side is not exactly a sequence of the Fourier coefficients of a piecewise-constant function? Some important results in this direction can be found in [44, 45, 26, 18].

We further investigate these problems in [52]. Our initial considerations show that one can define a robust procedure for solving systems like (4.1),(4.3) for any right-hand side, taking more equations than $2N + 1$ and replacing the exact solution by the least-square fitting. Notice, however, that we apply this procedure not to the original non-linear system (say, (4.3)) but to a linear system (4.6) for the parameters $C_j$ of the linear recurrence relation, satisfied by the Fourier coefficients (moments) $c_k (m_k)$ of any piecewise-constant function.
We expect also that taking more than the minimal number of the measurements, and hence of the equations in (4.1) (say, twice the minimal number) can strongly improve the robustness of the solution. This conclusion is supported by recent results in [64, 63] where we investigate similar problems for Hermite interpolation and Hermite least-square fitting.

4.4. Piecewise-smooth functions. We expect that applying the above results to the case of piecewise-smooth functions we can get, in particular, the following result:

**Conjecture.** Let \( f \) be a piecewise \( C^k \) function on \([0, 1]\) with \( N \) discontinuity points \( x_i \). Assume that the \( C^k \)-norm of \( f \) on each continuity interval does not exceed \( M \). Assume also that a distance \( |x_i - x_j| \geq D \) for \( i \neq j \), and that a jump of \( f \) at each of its discontinuity points \( x_i \) is at least \( J \). Then for each \( n > 2N + 1 \) the points \( x_i \) and the values of \( f \) between the points \( x_i \) can be reconstructed from the first \( n \) Fourier coefficients of \( f \) with the accuracy \( \frac{C}{N^k} \), where the constant \( C \) depends only on \( M, N, J \) and \( D \).

We expect also that \( x_i \) and the approximating polynomials of \( f \) on its continuity intervals are provided by universal analytic expressions in \( c_k \) (see [31, 32]). In [52] we prove a weaker version of this result (with a weaker estimate of the approximation accuracy). The main steps in the proof are as follows:

1. We fix an approximation accuracy \( \epsilon > 0 \). We approximate \( f \) up to \( \epsilon \) by a piecewise-polynomial \( \Delta P \) of degree \( d \) on each of its continuity intervals. By classical Approximation Theory this can be achieved with \( d = C_1 \left( \frac{1}{\epsilon} \right)^{\frac{1}{k}} \).

2. We consider the jump points \( x_i \) and all the coefficients of the piecewise polynomials constituting \( \Delta P \) as the unknowns, and substitute them into a system (*) similar to (4.1), which is constructed “once forever” for piecewise-polynomials of degree \( d \) (see [31, 32, 51, 52]). As the right-hand side we take the Fourier coefficients \( c_k \) of \( f \). By the choice of \( \Delta P \) we know that its Fourier coefficients \( \hat{c}_k \) satisfy \( |\hat{c}_k - c_k| \leq \epsilon \) for any \( k \).

3. At this step we determine the number of the equations (i.e. of the Fourier coefficients of \( f \)) we need to achieve the prescribed accuracy \( \epsilon \). We pick an appropriate finite subsystem (***) of (*) . Then we solve (***) with respect to the unknown parameters of \( \Delta P \). By the robustness estimates of [52] the solution differs from the true parameters of \( \Delta P \) by at most \( C_2 \epsilon \).
4. We form a piecewise-polynomial $\hat{\Delta}P$ with the parameters found in step 3. By the above estimates, the jump points of $\hat{\Delta}P$ approximate the true jump points of $f$ with the accuracy $C_3\epsilon$ while the partial polynomials of $\hat{\Delta}P$ approximate the values of $f$ on its continuity intervals with the accuracy $C_4\epsilon$. This completes the proof.

In [52] we provide a detailed proof. We also compare the above results with the classical results of Approximation Theory on one side, and with some recent results on linear (or semi-linear) reconstruction methods for piecewise-smooth functions (see [22]-[25], [49, 41, 54], and references there).

5. Digital images

Our considerations in Sections 2-4 above were motivated, in particular, by an attempt to estimate the expected efficiency of linear versus non-linear methods methods of acquisition and compression of still images and video-sequences.

Application of rigorous mathematical tools in Image Analysis is usually difficult, because of an appeal to a “human visual perception” which is central in this field. For example, the main compression requirement is to preserve image’s “visual quality” - the notion which is well known to escape any attempt of a rigorous mathematical definition.

Still, simple characteristics of images approximation, like $L^2$-error, while not completely adequate to the “human visual perception” results, are usually very instructive. In the present section we shall try to translate the rigorous results of Sections 2-4 about piecewise constant functions to the language of images. By the reasons that become clear below we believe that our conclusions (which we call “statements” not to mix with theorems) are as accurate as possible: they can be made rigorous by restricting accurately a set of allowed images we work with.

5.1. Linear space of images. A typical image is represented by a rectangular array of pixels (say, $512 \times 512$). At each pixel the brightness (or the color) discretized value is stored, typically, 8 bits or 256 brightness values, for grey-level images, and 24 bits for three-color RGB images. In this paper we shall ignore the discrete nature of digital images, and consider them as bounded functions on the square $Q^2$. (See [35, 4] for the discussion of some specific problems related to the discrete nature of images).

To make the space of images a linear one, we have to ignore another important feature of true images: the image brightness has always
to be within the prescribed interval (say, \([0,255]\)). So we cannot add images as usual functions. Still, it is convenient to consider images as the elements of the Hilbert space \(\mathcal{L} = L^2(Q^2)\) of functions on the unit two-dimensional cell \(Q^2\).

However, considering images as elements in the linear space \(\mathcal{L}\) stresses their non-linear nature. Let us mention some of the most immediate manifestations of this important fact.

1. First of all, addition (or, more generally, forming linear combinations of images) usually produces a new brightness function, which is difficult to interpret as a meaningful “image”. Indeed, such a sum will show an artificial overlapping of the objects appearing on each one of the original images. Only for images representing exactly the same scene (like, for example, the three color separations R, G, B of the same color picture) their linear combinations have a direct visual meaning.

2. Secondly, only a small fraction of the standard image processing operations (like high-pass and low-pass filtering) are linear transformations of the Hilbert space \(\mathcal{L}\). Most of the usual image processing operations (as represented, for example, in the Adobe’s “Photoshop” package) take into account the visual patterns on the image. Consequently, the processing is subordinated to the geometry of the objects on the image, and in this way it is highly non-linear.

3. Third, individual images depend in a highly non-linear way on the boundaries data of the objects.

4. Finally, the most important time-dependent families of images - video-sequences - turn out to be very complicated curves in \(\mathcal{L}\). In fact, as we shall see below, they behave geometrically as the “crinkled arcs” considered in Section 2.

Let us consider in more details the effect of a motion of objects on the image: this is the main content of typical video-sequences. First, to simplify considerations, let us assume that the objects are perfectly black while the background is perfectly white. Then our images, as the elements of the Hilbert space \(\mathcal{L}\), are just the (negative) characteristic functions of the domains occupied by the objects on the image.

If an object moves in such a way that the occupied domains are expanding (for example, the object approaches the camera) then the corresponding trajectory in the space of images \(\mathcal{L}\) is a crinkled arc by Proposition 2.4.
If the objects move in a more complicated way (in particular, their boundaries are deformed in a non-rigid manner), then the corresponding trajectory in the image space may not be a crinkled arc. However, Proposition 2.5 shows that typically such trajectories look like crinkled arcs “in a small scale”: the angle between any two disjoint chords of the corresponding curve in $L$ tends to $\frac{\pi}{2}$ as these chords tend to the same point.

The conclusion of Proposition 2.5 remains essentially valid also under more realistic assumptions on the color of the moving objects: indeed, near the object boundaries the image brightness in any case behaves as a scalar multiple of the characteristic function of the occupied domain.

Moreover, the occlusions of the moving objects do not change this pattern. Indeed, only at the intersections of the boundaries of the occluded objects we can expect new phenomena, but typically these intersections have nearly zero area. So they do not affect the $L^2$ geometry of the trajectory.

Thus we get a general (and, to our point, rather surprising) conclusion:

**Statement 5.1** A typical video-sequence is metrically similar to a “crinkled arc” in the Hilbert space $L$ of images. In particular, its $\epsilon$-entropy and Kolmogorov $n$-width behave as those of the curve $H$.

As in Sections 2-4 above, this fact provides an immediate limitation on the performance of linear approximation and acquisition methods. Let us assume that we want to represent all the images in a set $\Omega \subset L$ which is “large enough”: namely, it contains together with each image $I$ also images representing a “motion of objects” in $I$ (in particular, their zoom, translations, etc.). Hence the set $\Omega$ in fact contains “video-sequences”, and hence Statement 5.1 implies:

**Statement 5.2** No $n$-dimensional linear subspace $W$ in the Hilbert space $L$ of images can approximate all the images in $\Omega$ at once better than to $C \frac{1}{\sqrt{n}}$.

As for the problem of image acquisition from measurements, we get the following conclusion, analogous to that of Theorem 4.1:

**Statement 5.3** Let the image acquisition process comprise taking $n$ measurements $m_i(I), i = 1, \ldots, n$ (linear or non-linear) of the image $I$, together with a consequents processing $P$ of the measurements. If the processing operator $\hat{I} = P(m_1, \ldots, m_n)$ is a linear one then for some $I \in \Omega$ the error $||I - \hat{I}||$ is at least $C \frac{1}{\sqrt{n}}$. 

The inherent limitation of linear acquisition and representation methods forced development of non-linear approaches. Most of them utilize the fact that wavelet representations of typical images in appropriate wavelet bases are “sparse” - with only few “large” coefficients. Efficient image capturing and representation approaches based on this fact are given, in particular in [9, 49, 41, 54].

Another approach is based on “non-linear model-net approximation” ([4, 20, 63]). In the present paper it was described in a “toy example” of piecewise-constant functions. As the images are concerned, the main problem is whether such a “model-based” representation is possible at all. Let us discuss shortly the “state of the art” here.

5.2. Capturing of images and video-sequences by geometric models. As it was stressed in the introduction, the key to a successful application of the “algebraic reconstruction methods” presented in this paper to real problems in Signal Processing lies in a “model-based” representation of signals and especially of images.

From the point of view pursued in this paper, most of the conventional image representation (“compression”) methods can be considered as “semi-linear”: their starting point is a linear representation of the image in a certain basis (Fourier, local Fourier, Wavelets ...). Then the coefficients of this linear representation are truncated, ordered and finally encoded in a highly non-linear way.

There are “geometric” methods of image representation, based on an approximation by non-linear image models (usually constructed from the edges, ridges and other geometric visual patterns appearing in typical images) - see [38, 35, 21, 4, 20]. Some of these geometric methods have proved themselves to be very efficient in a representation and processing of special types of images (like geographic maps, cartoon animations, etc.).

However, in general the “geometric” methods, as for today, suffer from an inability to achieve a full visual quality for high resolution photo-realistic images of the real worlds. In fact, the mere possibility of a faithful capturing such images with geometric models presents one of important open problems in Image Processing, sometimes called “the vectorization problem”.

Let us stress our strong belief that a full visual quality “geometric” representation for high resolution photo-realistic images of the real worlds is possible. As achieved, it promises to bring a major advance in image compression and capturing, in particular, via the approach of the present paper.
5.3. **Reconstruction of images from measurements.** Let a function \( f(x, y) \) of two variables be the brightness function of an image to be reconstructed from Computer Tomography measurements. The data of the Radon transform can be translated into the Fourier data, so we can assume that our measurements are just Fourier coefficients of \( f \).

Now our general approach to this problem extends the non-linear inversion method presented in Section 4 above. It can be summarized as follows:

1. We obtain the “first approximation” \( \hat{f} \) of the function \( f \) to be reconstructed by one of available conventional methods.

2. We approximate the function \( \hat{f} \) (and hence also the function \( f \) to be reconstructed) by a “model one” \( Mf \). The last comprises “simple geometric models” reflecting the structure of singularities of \( f \) and approximating \( f \) at its regular regions.

   Let us stress once more, as we did in our discussion above, that the mere existence of such a representation for real world images is an important open problem.

3. We memorize the “combinatorial structure” of \( Mf \) (the number of its jumps in one-dimensional case; the topological structure of the edges, ridges, patches etc. for images).

4. We consider specific geometric and brightness parameters of the models as unknowns, which we substitute into a system (***) obtained in the same way as the system (4.1) above. The right-hand side of this system is formed by the measured Fourier coefficients of \( f \).

5. We solve the appropriate subsystem of the system (***)). In the solution process we start with the approximate solution obtained in step 2. The solution provides a set of the improved parameters of the model function \( Mf \). Applying these parameters we finally get an improved approximation \( \hat{Mf} \) of the original function \( f \).

The implementation of this program in real applications of Computer Tomography is now in its initial stages. In “toy problems” where we pretend to know a priori the model structure of the image, the approach works perfectly (not a big surprise! see [51]). We believe however that the time is ripe to study both the Image Processing and the Algebraic-Geometric parts of the problem.

5.4. **Geometric image compression and crinkles arcs.** The following remarks concern a possibility to use directly the universality of the “crinkled arc” in image compression. However, it involves encoding
of certain isometries of Hilbert spaces, and the feasibility and complexity of this task should be considered as an absolutely open problem.

One of the most important tasks in the “geometric image compression” is a compact representation of the systems of curves and points on the plane (see, in particular, [1, 2, 3]). Mostly we can assume these curves and points to be mutually disjoint, and their specific parametrization is not essential.

Let us consider a special case where the family of curves to be memorized is the family of the boundaries of a family of expanding domains in the plane. It turns to be difficult to utilize this special structure in the curve compression methods used in [1, 20]. In fact, according to these methods, each curve will be stored separately. On the other hand, by Propositions 2.4, 2.5 the characteristic functions of the inside domains of our curves form a crinkled arc in $L^2(Q^2)$ isomorphic to $H$. Consequently, we have an alternative approach to memorizing our family of curves: it is enough to memorize the transformations bringing it to $H$. This lead to two mathematical problems which we consider as important by themselves:

**Problem 1.** What is the complexity of the “normalizing transformation” in Theorem 2.2, and specifically, in Propositions 2.4, 2.5? (We can use, for example, the notion of complexity for infinite-dimensional objects, introduced in [60, 61]). How many bits do we need to memorize them?

**Problem 2.** How to use “geometric redundancy” of the expanding family - the fact that the curves do not intersect and “bound one another” - in their “conventional” compression?

**References**

1. H. Alexander, J. Wermer, *Envelopes of holomorphy and polynomial hulls*. Math. Ann. 281 (1988), no. 1, 13–22.
2. E. Arias-Castro, D. Donoho, X. Huo, C. Tovey, *Connect the dots: how many random points can a regular curve pass through?* Adv. in Appl. Probab. 37 (2005), no. 3, 571–603.
3. E. Arias-Castro, D. Donoho, X. Huo, C. Tovey, *Correction: ”Connect the dots: how many random points can a regular curve pass through?”* [Adv. in Appl. Probab. 37 (2005), no. 3, 571–603; MR2156550]. Adv. in Appl. Probab. 38 (2006), no. 2, 579.
4. M. Briskin; Y. Elichai; Y. Yomdin, How can Singularity theory help in Image processing, in *Pattern Formation in Biology, Vision and Dynamix*, M. Gromov, A. Carbone, Editors, World Scientific (2000), 392-423.
5. M. Briskin, J.-P. Francoise, Y. Yomdin, *Center conditions, compositions of polynomials and moments on algebraic curve*, Ergodic Theory Dyn. Syst. 19, no 5, 1201-1220 (1999).
6. M. Briskin, J.-P. Francoise, Y. Yomdin, *Generalized moments, center-focus conditions and compositions of polynomials*, in "Operator theory, system theory and related topics", Oper. Theory Adv. Appl., 123, 161–185 (2001).

7. M. Briskin, Y. Yomdin, *Tangential version of Hilbert 16th problem for the Abel equation*, Mosc. Math. J. 5 (2005), no. 1, 23–53.

8. E. J. Candès. *Compressive sampling*. Proceedings of the International Congress of Mathematicians, Madrid, Spain, 2006. Vol. III, 1433–1452, Eur. Math. Soc., Zrich, 2006.

9. E. J. Candès, D. Donoho, *Curvelets - A Suprisingly Effective Nonadaptive Representation For Objects with Edges*, in Curves and Surfaces, L. L. Schumaker et al. (eds), Vanderbilt University Press (1999).

10. E. Candès, J. Romberg, T. Tao, *Stable signal recovery from incomplete and inaccurate measurements*. Comm. Pure Appl. Math. 59 (2006), no. 8, 1207–1223.

11. C. Christopher, *Abel equations: composition conjectures and the model problem*, Bull. Lond. Math. Soc. 32 (2000), no. 3, 332-338.

12. P.J. Davis, *Triangle formulas in the complex plane*, Math. Comput., vol. 18 (1964), 569-577.

13. P.J. Davis, *Plane regions determined by complex moments*, J. Approximation Theory, vol. 19 (1977), 148-153.

14. R. DeVore, *Nonlinear Approximation*, Acta Numerica (1998), 51-150

15. P. Dolbeault, G. Henkin, *Chaines holomorphes de bord donne dans CP*”. Bull. Soc. Math. France 125 (1997), no. 3, 383–445.

16. D. Donoho, *Compressed sensing*. IEEE Trans. Inform. Theory 52 (2006), no. 4, 1289–1306.

17. D. Donoho, M. Elad, V. Temlyakov, *Stable recovery of sparse overcomplete representations in the presence of noise*. IEEE Trans. Inform. Theory 52 (2006), no. 1, 6–18.

18. M. Elad, P. Milanfar, G. H. Golub, *Shape from moments—an estimation theory perspective*, IEEE Trans. Signal Process. 52 (2004), no. 7, 1814–1829.

19. M. Elad, P. Milanfar, R. Rubinstein, *Analysis versus synthesis in signal priors*, Inverse Problems 23 (2007), no. 3, 947–968.

20. Y. Elichai; Y. Yomdin, Normal Forms representation: a technology for image compression, SPIE vol 1903, *Image and Video Processing*, 1993, 204-214.

21. J. Elder, *Are Edges Incomplete?*, Int. J. of Comp. Vision Vol 34 (2-3), 97-122.

22. A. Gelb, E. Tadmor, *Detection of edges in spectral data*. Appl. Comput. Harmon. Anal. 7 (1999), no. 1, 101–135.

23. A. Gelb, E. Tadmor, *Detection of edges in spectral data. II. Nonlinear enhancement*. SIAM J. Numer. Anal. 38 (2000), no. 4, 1389–1408 (electronic).

24. A. Gelb, E. Tadmor, *Spectral reconstruction of piecewise smooth functions from their discrete data*. M2AN Math. Model. Numer. Anal. 36 (2002), no. 2, 155–175.

25. A. Gelb, E. Tadmor, *Adaptive edge detectors for piecewise smooth data based on the minmod limiter*. J. Sci. Comput. 28 (2006), no. 2-3, 279–306.

26. G. H. Golub, P. Milanfar, J. Varah, *A stable numerical method for inverting shape from moments*, SIAM J. Sci. Comput. 21 (1999/00), no. 4, 1222–1243 (electronic).
27. B. Gustafsson, Ch. He, P. Milanfar, M. Putinar, *Reconstructing planar domains from their moments*. Inverse Problems 16 (2000), no. 4, 1053–1070.
28. P. Halmos, *Hilbert Space Problem Book*, Second edition, Springer, 1982.
29. G. Henkin, *Abel-Radon transform and applications*. The legacy of Niels Henrik Abel, 567–584, Springer, Berlin, 2004.
30. G. Henkin, V. Michel, *On the explicit reconstruction of a Riemann surface from its Dirichlet-Neumann operator*. Geom. Funct. Anal. 17 (2007), no. 1, 116–155.
31. V. Kisun’ko, *Cauchy Type Integrals and a D-moment Problem*, to appear in "Mathematical Reports" of the Academy of Science of the Royal Society of Canada.
32. V. Kisun’ko, *D-moment problem and applications*, in preparation.
33. A. N. Kolmogorov, Asymptotic characteristics of some completely bounded metric spaces, Dokl. Akad. Nauk SSSR 108 (1956), 585-589.
34. A. N. Kolmogorov; V. M. Tihomirov, $\varepsilon$-entropy and $\varepsilon$-capacity of sets in functional space. Amer. Math. Soc. Transl. 17, (1961), 277-364.
35. Weidong Kou, *Digital Image Compression: Algorithms and Standards*, Kluwer Academic Publishers, Boston, 1995.
36. S. Kuhlmann, M. Marshall, *Positivity, sums of squares and the multi-dimensional moment problem*. Trans. Amer. Math. Soc. 354 (2002), no. 11, 4285–4301 (electronic).
37. S. Kuhlmann, M. Marshall, N. Schwartz, *Positivity, sums of squares and the multi-dimensional moment problem. II*. Adv. Geom. 5 (2005), no. 4, 583–606.
38. M. Kunt, A. Ikonomopoulos, M. Kocher, Second-generation image coding techniques, *Proceedings IEEE*, Vol. 73, No. 4 (1985), 549-574.
39. G. G. Lorentz, *Approximation of functions*. Second edition. Chelsea Publishing Co., New York, (1986), 188 p.
40. I. Maravić, M. Vetterli, *Sampling and reconstruction of signals with finite rate of innovation in the presence of noise*, IEEE Trans. Signal Process. 53 (2005), no. 8, part 1, 2788–2805.
41. I. Maravić, M. Vetterli, *Exact sampling results for some classes of parametric nonbandlimited 2-D signals*, IEEE Trans. Signal Process. 52 (2004), no. 1, 175–189.
42. P. Marziliano, M. Vetterli, T. Blu, *Sampling and exact reconstruction of bandlimited signals with additive shot noise*, IEEE Trans. Inform. Theory 52 (2006), no. 5, 2230–2233.
43. P. Milanfar, W.C. Karl, A.S. Willsky, *Reconstructing Binary Polygonal Objects from Projections: A Statistical View*, CVGIP: Graphical Models and Image Processing, vol. 56, no.5 (1994), 371-391.
44. P. Milanfar, G.C. Verghese, W.C. Karl, A.S. Willsky, *Reconstructing Polygons from Moments with Connections to Array Processing*, IEEE Transactions on Signal Processing, vol. 43, no. 2 (1995), 432-443.
45. P. Milanfar, W.C. Karl, A.S. Willsky, *A Moment-based Variational Approach to Tomographic Reconstruction*, IEEE Transactions on Image Processing, vol. 5, no. 3 (1996), 459-470.
46. E. M. Nikishin, V. N. Sorokin, *Rational Approximations and Orthogonality*, Translations of Mathematical Monographs, Vol 92, AMS, 1991.

\[2\] Note that only the second edition contains the result mentioned.
47. F. Pakovich, *On polynomials orthogonal to all powers of a given polynomial on a segment*, Bull. Sci. math. 129 (2005) 749-774.
48. F. Pakovich, *Solution of the polynomial moment problem*, preprint, 2007.
49. P. Prandoni, M. Vetterli, *Approximation and compression of piecewise smooth functions*, R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci. 357 (1999), no. 1760, 2573–2591.
50. M. Putinar, C. Scheiderer, *Multivariate moment problems: geometry and indeterminateness*. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 5 (2006), no. 2, 137–157.
51. N. Sarig, Y. Yomdin *Non-linear inversion of Fourier and Moment transforms*, preprint
52. N. Sarig, Y. Yomdin *Robust non-linear reconstruction of piecewise-smooth functions from their Fourier data*, in preparation.
53. C. Scheiderer, *Sums of squares on real algebraic surfaces*. Manuscripta Math. 119 (2006), no. 4, 395–410.
54. R. Shukla, P.L. Dragotti, M.N. Do, M. Vetterli, *Rate-distortion optimized tree-structured compression algorithms for piecewise polynomial images*, IEEE Trans. Image Process. 14 (2005), no. 3, 343–359.
55. V. N. Temlyakov, *Nonlinear methods of approximation*, Found. Comput. Math. 3 (2003), 33-107.
56. V. N. Temlyakov, *Nonlinear Kolmogorov widths*, Math Notes, 63, No. 5-6, 785-795.
57. V. N. Temlyakov, *Greedy Approximations*, in *Foundations of Computational Mathematics, Santander 2005*, L. Pardo, A. Pinkus, E. Suli, M. Todd, Editors, London Mathematical Society Lecture Notes Series 331, 371-394. Cambridge University Press, 2006.
58. V.M. Tihomirov, *Widths of sets in function spaces and the theory of best approximations*, Uspekhi Mat. Nauk 15 (1960), 81-120; English transl. in Russian Math. Surveys 15 (1960).
59. M. Vetterli, P. Marziliano, T. Blu, *Sampling signals with finite rate of innovation*, IEEE Trans. Signal Process. 50 (2002), no. 6, 1417–1428.
60. Y. Yomdin, *Complexity of functions: some questions, conjectures and results*. J. of Complexity, 7, (1991), 70–96.
61. Y. Yomdin, *Semialgebraic complexity of functions*, Journal of Complexity 21 (2005) 111-148.
62. Y. Yomdin, *Center Problem for Abel Equation, Compositions of Functions and Moment Conditions*, with the Addendum by F. Pakovich, *Polynomial Moment Problem*, Mosc. Math. J. 3 (3) (2003) 1167-1195.
63. Y. Yomdin, G. Zahavi, *High-order processing of singular functions*, preprint, 2007.
64. G. Zahavi, Ph.D thesis, Weizmann Institute, 2007.