Algebraic Bethe Ansatz for $N = 4$

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Abstract. In the present paper we explore some useful devices for discussing the exact solutions of XXX isotropic Heisenberg Hamiltonian for the single node spin equal to $\frac{1}{2}$ as the tensor product states. Our aim is to present the monodromy matrix and the Lax operator in the context of the Bethe Ansatz, signed from now on BA. We construct the former, for $N$ equal to 4 nodes of the magnet, using the so-called auxiliary space which is taken as a copy of $\mathbb{C}^2$. The form of this matrix in the basis of orbits of the translation group $C_4$ reveals its block structure of all possible nonzero elements. Each block has its meaning in the language of creation and annihilation of the magnon. This fact implies, that one can think about appropriate operators and create the theory very similar to that of the quantum oscillator.

1. Introduction

The famous BA was introduced by Bethe [1] in 1931 as an exact solution for the isotropic Heisenberg Hamiltonian with the single node spin $\frac{1}{2}$. The method used by Bethe in [1] is today often called a coordinate BA. In this paper we aim to describe the algebraic BA [2], [3], [4] which is a generalization of the former. Due to this method one can construct, for each exactly solvable model [5], its monodromy matrix, which satisfies certain commutation relations, see for example [6]. In this paper we consider the XXX Heisenberg magnet with the single node spin $\frac{1}{2}$. The monodromy matrix is a generating object for both the Hamiltonian and its eigenvectors for our system. The trace of the monodromy matrix gives the main observables such as the total spin, quasimomentum or the Hamiltonian. The elements of the monodromy matrix play the role of creation and annihilation operators, which provide the eigenvectors of the Hamiltonian when acting on the vacuum. Each such operator is a function of so-called spectral parameter. This fact yields that a single exact eigenfunction is determined by the set of spectral parameters given as solutions of a system of algebraic Bethe equations.

2. Tensor product Hilbert space

Let $\mathcal{N} = \{j = 1, 2, \ldots, N\}$ be the set of nodes of a magnetic ring, and $\mathcal{Z} = \{+, -\}$ - the set of single-node spin projections. Then the Hilbert space for such a system has a natural tensor-product form

$$\mathcal{H} = \prod_{j=1}^{N} \otimes h_j = h_1 \otimes h_2 \otimes \ldots \otimes h_j \otimes \ldots \otimes h_N,$$  \hspace{1cm} (1)
where
\[ h_j = \text{lcC}2 \cong \mathbb{C}^2 \]  
(2)
is the linear closure of the set 2 over \( \mathbb{C} \). The set of all mapping \( f : \tilde{N} \rightarrow \tilde{2} \) constitutes \( \tilde{2}^\tilde{N} \) magnetic configurations, which can be written in the form
\[ |f \rangle \equiv |i_1 \rangle \otimes |i_2 \rangle \otimes \cdots \otimes |i_N \rangle \equiv |i_1, i_2, \ldots, i_N \rangle, \quad i_j \in \tilde{2}, \ j \in \tilde{N}. \]  
(3)
Since \( |i_N \rangle \in h_N \) and \( |f \rangle \in H \), the set (3) provides an orthonormal basis in \( H \), initial for quantum computations, i.e.
\[ H = \text{lcC} \tilde{2}^\tilde{N}. \]  
(4)
The space \( H \) of all quantum states of the Heisenberg magnet, with \( \dim H = 2^N \), decomposes as
\[ H = \bigoplus_{r=0}^{N} \mathcal{H}^r, \]  
(5)
into subspaces \( \mathcal{H}^r \), \( \dim \mathcal{H}^r = \binom{N}{r} \), with the fixed number \( r \) of Bethe pseudoparticles (spin deviations). Stated otherwise, \( \mathcal{H}^r \) is the space of all states of the magnet with the projection
\[ M = \frac{N}{2} - r \]  
(6)
of the total spin \( S = \frac{N}{2} - r' \), where \( 0 \leq r' \leq r \), or of all states with the weight \( [7] \)
\[ \mu = \{N - r, r\}. \]  
(7)
From group-theoretical point of view we can express \( \mathcal{H}^r \) as
\[ \mathcal{H}^r = \text{lcC} \mathcal{O}_\mu, \]  
(8)
where \( \mathcal{O}_\mu \) is an orbit of the symmetric group \( \Sigma_N \), given by the weight \( \mu = \{N - r, r\} \). Let
\[ t = (t_1, t_2, \ldots, t_r), \quad \sum_{a \in \tilde{r}} t_a = N \]  
(9)
be the relative distribution of Bethe pseudoparticles within an orbit of the translational symmetry group \( C_N \), of the ring (\( C_N \)-orbit), being a subgroup of the symmetric group \( \Sigma_N \) (\( C_N \subset \Sigma_N \)). The vector \( t \) provides a label for classification of \( C_N \)-orbits in \( \mathcal{O}_\mu \), so that
\[ \mathcal{O}_\mu = \bigcup_{t \in T} \mathcal{O}_t, \]  
(10)
where \( T = \mathcal{O}_\mu / C_N \) is the set of orbits, which we refer to, after [8], as a skeleton of the configuration space \( Q^{(r)} = \mathcal{O}_\mu \), for more details see [8]. The Hamiltonian of the system of interacting spin deviations in the space (1) has the following form
\[ H = J \sum_{j \in \tilde{N}} (s_j \cdot s_{j+1} - 1/4), \]  
(11)
which can be writed as
\[ H = \frac{J}{4} \sum_{i=1}^{N} \left[ \frac{1}{2}(\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+) + \sigma_j^+ \sigma_{j+1}^+ - I_N \right], \]  
(12)
where
\[ \sigma_j = I_1 \otimes I_2 \otimes \cdots \otimes I_{j-1} \otimes \sigma \otimes I_{j+1} \otimes \cdots \otimes I_N, \]  
(13)
\( \sigma \) is one of the Pauli matrices acting in \( h \) and \( \sigma^+ = \frac{1}{2} \sigma_x + i \sigma_y \) (\( \sigma^- = \frac{1}{2} \sigma_x - i \sigma_y \)).
### 3. Some bases in $\mathcal{H} \otimes V$

Let us consider the tensor product of $h_j$ or $\mathcal{H}$ and some auxiliary space $V$, namely $h_j \otimes V$ and $\mathcal{H} \otimes V$. The space $V$ is identical with $h_j$, but is not related to any node of the magnetic ring $N$. Thus we have

$$h_j \otimes V \cong \mathbb{C}^2 \otimes V \cong (\mathbb{C}^2)^\otimes 2,$$

and

$$\mathcal{H} \otimes V \cong (\mathbb{C}^2)^\otimes N \otimes V \cong (\mathbb{C}^2)^\otimes N+1.$$  \hfill (14) \hfill (15)

From now on, we will proceed with the case of the magnetic ring with 4 nodes.

As the initial basis for quantum calculations in $\mathcal{H} \otimes V$, we chose the lexical basis (numbered by $J_M$) of the simple tensor product $h^{(4+1)}$ of 4 + 1 spaces $h$ given by (2). The lexical basis for the tensor product of four $h$ spaces, that is for the physical space $\mathcal{H}$ of our magnet, is labelled by the number $J$ what we can see in the Table 1. Each physical state in sixteenth dimensional space $\mathcal{H}$ is doubly degenerated, since the auxiliary space $V$ is two dimensional.

The second basis used in our calculations is given by the structure of orbits of the translational symmetry group of the magnet $C_4$ ($C_4$-orbit) acting on the magnetic configurations given by the formula (3), and is called the basis of orbits. Each vector of this basis is labelled by three quantum numbers (Table 1): $r$, $t$ and $j$, where $j$ counts the magnetic configurations within the $C_4$-orbit. The only difference between the lexical basis and the basis of orbits is ordering of its contents.

### 4. The Lax and the monodromy operators

The Lax operator given, by the formula

$$L_{j,a}(\lambda) = \lambda I \otimes I_a + i \sum_{k=1}^{3} s_j^k \otimes \sigma_a^k,$$  \hfill (16)

acts in $\mathcal{H} \otimes V$, thus $L_{j,a}(\lambda) \in End(\mathcal{H} \otimes V)$, where $a$, $\lambda$ and $I(I_a)$ denotes respectively the auxiliary space $V = h_a$, spectral parameter, and the identity operator in $\mathcal{H}(h_a)$. Thanks to

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**Table 1.** The basis of orbits $|r t j>$ of the group $C_4$ in $\mathcal{H}$.

| $r$ | $t$ | $j$ | $J_M$ for + in $V$ | $J_M$ for - in $V$ |
|-----|-----|-----|-------------------|-------------------|
| 0   | 0   | 1   | 1                 | 2                 |
| 1   | 4   | 1   | 9                 | 17                | 18 |
| 2   | 5   | 9   | 10                |
| 3   | 3   | 5   | 6                 |
| 4   | 2   | 3   | 4                 |
| 2   | 13  | 1   | 13                | 25                | 26 |
| 2   | 7   | 13  | 14                |
| 3   | 4   | 7   | 8                 |
| 4   | 10  | 19  | 20                |
| 22  | 1   | 11  | 21                | 22                |
| 2   | 6   | 11  | 12                |
| 3   | 112 | 1   | 15                | 29                | 30 |
| 2   | 14  | 27  | 28                |
| 3   | 12  | 23  | 24                |
| 4   | 8   | 15  | 16                |
| 4   | 1111| 1   | 16                | 31                | 32 |
the Dirac equivalency $P_{j,a} = \frac{1}{2} I + 2 s_j s_a$ between the product of two single-node spin operators $(s_j, s_a)$ and two-node permutation operator $P_{j,a}$, one can write $L_{j,a}(\lambda)$ in the form

$$L_{j,a}(\lambda) = (\lambda - \frac{i}{2}) I \otimes I_a + i P_{j,a},$$

(17)

where $P_{j,a}$ is the permutation operator between the spaces $h_j$ and $h_a$.

For two different spectral parameters $(\lambda, \mu)$ the following relation is true

$$R_{a_1,a_2}(\lambda - \mu)L_{j,a_1}(\lambda)L_{j,a_2}(\mu) = L_{j,a_2}(\lambda)L_{j,a_1}(\lambda)R_{a_1,a_2}(\lambda - \mu),$$

(18)

where

$$R_{a_1,a_2}(\lambda) = \lambda I_{a_1} \otimes I_{a_2} + i P_{a_1,a_2},$$

(19)

and $R_{a_1,a_2}(\lambda) \in \text{End}(V_1 \otimes V_2)$ with $V_1$ and $V_2$ being some auxiliary spaces isomorphic with $V$ [2], [3], [4]. The equation (18) turns out to be the Yang-Baxter relation [2], [3], [4], which is equivalent with original BA equations given in [1], that is why this approach is called algebraic Bethe Ansatz (ABA). From (17) one can see that the action of $L_{j,a}(\lambda)$ in $H \otimes V$ implies exchange of contents between $j$-th and $a$-th nodes. The product of two Lax operators $L_{j',a}(\lambda)$, $L_{j,a}(\lambda)$ acting in $H \otimes V$ constitutes some translation operator on the nodes $(j, j')$ of the magnetic ring $\tilde{N}$. The product of three Lax operators $L_{j'',a}(\lambda)$, $L_{j',a}(\lambda)$, $L_{j,a}(\lambda)$ acting in $H \otimes V$ constitutes some translation operator on the nodes $(j, j', j'')$ of the magnetic ring $\tilde{N}$. Thus the $N$-th product of the consecutive Lax operators, namely

$$L_{N,a}(\lambda)L_{N-1,a}(\lambda) \ldots L_{2,a}(\lambda)L_{1,a}(\lambda),$$

(20)

defines some translation operator on the nodes $1, 2, \ldots, N$, a of the magnetic ring $\tilde{N}$ plus the content of the auxiliary space $V$. The operator given by (20) is called monodromy operator $T_a(\lambda)$ acting in $H \otimes V$. The general structure of this matrix is presented in the Figure 1, but the Figure 2 presents the structure of this matrix in the basis of orbits.

$B(\lambda)$ and $C(\lambda)$ submatrices stand for creation and annihilation operator, respectively, for magnon labelled by the spectral parameter $\lambda$. The $Tr(T_a(\lambda)) = A(\lambda) + D(\lambda)$ constitutes the generating object for $N$ constants of motion with the operator of XXX Heisenberg Hamiltonian among themselves. For $r$ spectral parameters $\lambda_{j'}$ $(j' = 1, 2, \ldots, r)$, which satisfy Bethe equation in the form

$$(\lambda_{j'} + \frac{i}{2})^N = \prod_{k \neq j'}^{r} \lambda_{j'} - \lambda_k + i \lambda_{j'} - \lambda_k - i,$$

(21)

and for the vacuum state $\Omega$ (consisting only of pluses) one can build the exact BA solution $\phi(\lambda_1, \ldots, \lambda_r)$ for the system of $r$ magnons characterised by the set $\{\lambda\} = (\lambda_1, \ldots, \lambda_r)$ of spectral parameters as follows

$$\phi(\lambda_1, \ldots, \lambda_r) = B(\lambda_1) \ldots B(\lambda_r) \Omega$$

(22)
Figure 2. The structure of the monodromy matrix in the basis of orbits of the translation group $C_N$. The non-zero elements can occur in black boxes only.

[2], [3], [4]. It is very crucial to observe, that the set of operators $\{B(\lambda)\} = \{B(\lambda_{j'}), j' \in \tilde{r}\}$ alone, is not able to generate the complete basis in $\mathcal{H}'$, since the vector (22) is the highest weight vector i.e. with the total spin $S = M = \frac{N}{2} - r$ (6). In order to obtain the remaining basis state in $\mathcal{H}'$ one should use the well known operator $S^- = s_1^- + s_2^- + ... + s_N^-$ where $s_j^- = \frac{1}{2}\sigma_j^-$ ($j \in \tilde{N}$), which generates the magnon with pseudomomentum $p_{j'} = 0$, i.e. so called Goldstone boson.

From the relation between $p_{j'}$ and $\lambda_{j'}$ i.e. $\lambda_{j'} = \frac{1}{2} \cot \frac{p_{j'}}{2}$ it is easy to notice that the operator $B(\lambda_{j'})$ is singular for $p_{j'}$.

5. Final remarks and conclusions

We showed some of details for construction the monodromy matrix for the magnetic ring with four nodes, using the auxiliary space. We emphasis on the fact that it is very crucial to distinguish between the physical and the nonphysical states in case when one has the auxiliary space besides the real one. We localized all possible nonzero elements of this matrix on the Picture 2. Taking the trace of the monodromy matrix over the auxiliary space one can obtain the sum of two operators acting in $\mathcal{H}'$. Two remaining ones, which are placed on the antidiagonal positions of this matrix are responsible for creation and anihilation of one magnon. We presented the role of the $B(\lambda)$ and $S^-$ operators in creation of the basis of the spaces $\mathcal{H}'$ for $r = 0, 1, 2, 3, 4$, where the first operator is responsible for creation of the highest weight vectors ($M = S$), and the second one creates the so called Goldstone bosons.

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