Homology of some surfaces with \( p_g = q = 0 \) isogenous to a product

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Abstract. Bauer and Catanese found four families of surfaces of general type with \( p_g = q = 0 \), each of which is the quotient of a product of curves by the action of a finite Abelian group. We compute the integral cohomology groups of these surfaces.

Keywords: surfaces of general type, surfaces isogenous to a product, fake quadrics, branched coverings, fundamental group, homology groups.

§ 1. Introduction

One of the most important problems in algebraic geometry is the classification of algebraic varieties up to birational equivalence. We recall that a rational map \( f: X \dasharrow Y \) between varieties \( X \) and \( Y \) is a morphism \( U \to Y \) defined on an open subset \( U \subseteq X \). Two varieties \( X \) and \( Y \) are said to be birationally equivalent if there are two mutually inverse rational maps between them.

The spaces of sections \( H^0(X, \Omega_X^k) \) of the sheaves of algebraic \( k \)-forms are birational invariants of \( X \). Another invariant is the canonical ring \( \mathbb{R} = \bigoplus H^0(X, nK) \), where \( K \) is the divisor associated with the line bundle of top-degree forms. The Kodaira dimension \( \kappa(X) \) of \( X \) is by definition equal to \( \text{tr. deg } R - 1 \) (where \( \text{tr. deg } R \) is the transcendence degree of \( R \) over \( \mathbb{C} \)) if \( R \) does not coincide with \( \mathbb{C} \), and to \( -\infty \) otherwise. For every variety \( X \) we have \( \kappa(X) \leq \dim X \). For example, let \( C \) be a curve. Then we have \( \kappa(C) = -\infty \) if \( C \cong \mathbb{P}^1 \), \( \kappa(C) = 0 \) if \( C \) is an elliptic curve, and \( \kappa(C) = 1 \) if \( C \) has genus \( g \geq 2 \). A variety \( X \) is said to be of general type if \( \kappa(X) = \dim X \). Otherwise it is of special type.

A surface \( X \) is said to be minimal if every birational morphism \( X \to Y \) is an isomorphism. If \( S \) is birationally equivalent to a minimal surface \( X \), then \( X \) is called a minimal model of \( S \). A minimal model always exists but is not necessarily unique. There is a complete birational classification of surfaces of special type: their minimal models and moduli spaces have been described.

But the question of classification of surfaces of general type is still open. The geometric genus \( p_g \) of a surface \( S \) is \( h^0(S, \Omega_S^2) \), where \( h^0 = \dim H^0 \). The irregularity \( q \) is \( h^0(S, \Omega_S^1) \). Minimal surfaces with \( p_g = 0 \) necessarily have \( q = 0 \) and \( 1 \leq K^2 \leq 9 \) (the upper bound follows from [1]–[4]), where \( K^2 \) is the self-intersection of a canonical divisor. In the case \( K^2_S = 9 \), the universal covering of \( S \) is the unit
ball $\mathbb{B}_2 \subset \mathbb{C}^2$. A complete classification of such surfaces is given in [5]–[7]. In the case when $K_S^2 = 8$, it is conjectured [8] that $S$ is uniformized by the bidisc $\mathbb{B}_1 \times \mathbb{B}_1$. Bauer, Catanese and Grunewald [9] (see also [8]) considered the case when the surface has an unramified covering by a product of curves. Surfaces of this kind form 18 irreducible components in the moduli space. A list of such surfaces that are quotients of a product of two curves by the diagonal action of a finite Abelian group had already been given in [10]. These surfaces form four irreducible families.

One of the sources of interest in surfaces with $p_g = q = 0$ stems from the properties of their derived categories. The category $D^b(\text{coh}(S))$ can admit an exceptional sequence of maximal length. Its orthogonal will be a quasi-phantom category, that is, a triangulated category whose Hochschild homology is equal to zero and $K_0$ is a torsion group. For the four families mentioned above, such exceptional sequences were constructed in [11]–[13].

Our main result is a computation of the homology groups with integral coefficients of the surfaces in these families. In the next section we recall some basic facts about such surfaces. The last section is devoted to the computation of the homology groups.

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\section{Surfaces with $p_g = q = 0$ isogenous to a higher product with an Abelian group}

We consider smooth projective surfaces over $\mathbb{C}$. One way of constructing surfaces with $p_g = q = 0$ is by taking quotients by the action of a finite group. A surface $S$ is said to be isogenous to a higher product [14] if it admits a finite unramified covering of the form $C_1 \times C_2$, where $C_1$, $C_2$ are curves of genera $\geq 2$. As shown in [14], this means that $S$ is the quotient $(C_1 \times C_2)/G$, where $g(C_i) \geq 2$, $G$ is a finite group acting freely on $C_1 \times C_2$. The action is said to be mixed (resp. unmixed) if $G$ interchanges the two curves (resp. $G$ acts separately on each curve).

In what follows, $S$ will be the quotient $(C_1 \times C_2)/G$, where the finite group $G$ acts on each curve $C_i$ of genus $\geq 2$ and the action on the product is free. We shall also assume that $S$ has $p_g = q = 0$. The surface $S$ is a ramified covering of $C_1/G \times C_2/G$. If there were non-zero differential forms on $C_1/G \times C_2/G$, we could pull them back to $S$. Thus the assumption $p_g = q = 0$ implies that each $C_i/G$ is isomorphic to $\mathbb{P}^1$.

Consider a ramified covering $p : C \rightarrow C/G \cong \mathbb{P}^1$. Let $B$ be the branching locus of $p$: $B = \{p(x) \mid x \in C$, the stabilizer of $x$ is a non-trivial subgroup of $G\}$. We have an exact sequence

$$1 \rightarrow \pi_1(C \setminus p^{-1}(B)) \rightarrow \pi_1(\mathbb{P}^1 \setminus B) \rightarrow G \rightarrow 1.$$ 

Let $x_1, \ldots, x_r$ be the distinct points of $B$, and let $\gamma_i$ be a simple geometric loop around $x_i$. Each $\gamma_i$ is mapped to an element $g_i \in G$. We denote the order of $g_i$ by $m_i$. The orbifold fundamental group of $p$ is defined to be the quotient of $\pi_1(\mathbb{P}^1 \setminus B)$ by the normal subgroup generated by the elements $\gamma_i^{m_i}$. 
In our situation, let $\Pi(i)$ for $i = 1, 2$ be the orbifold fundamental group of the covering $C_i \to C_i/G$. There are exact sequences
\[
1 \to \pi_1(C_i) \to \Pi(i) \to G \to 1.
\]
Multiplying the two sequences, we get
\[
1 \to \pi_1(C_1) \times \pi_2(C_2) \to \Pi(1) \times \Pi(2) \to G \times G \to 1.
\]
Here $\pi_1(S)$ is the pre-image in $\Pi(1) \times \Pi(2)$ of the diagonal subgroup $G$ in $G \times G$ \cite{14}. Thus, for Abelian $G$, we get an exact sequence
\[
1 \to \pi_1(S) \to \Pi(1) \times \Pi(2) \to G \to 1,
\]
where the second arrow is the composite of $\Pi(1) \times \Pi(2) \to G \times G$ and the map $G \times G \to G$ given by $(a, b) \mapsto a - b$.

The covering $C \to \mathbb{P}^1$ is determined by the choice of branching points $x_1, \ldots, x_r$ on $\mathbb{P}^1$ and the images $g_i \in G$ of loops around $x_i$ such that $g_1 \cdots g_r = 1$ and $g_1, \ldots, g_r$ generate $G$. If $\{g_i\}$ is the system of generators corresponding to the covering $C_1 \to \mathbb{P}^1$, and $\{h_j\}$ similarly corresponds to $C_2 \to \mathbb{P}^1$, then the action of $G$ on $C_1 \times C_2$ is free if and only if
\[
(\bigcup \langle g_i \rangle) \cap (\bigcup \langle h_j \rangle) = \{1\},
\]
where $\langle g_i \rangle$ is the subgroup generated by $g_i$. Bauer and Catanese classified surfaces with $p_g = q = 0$ isogenous to a higher product with an Abelian group $G$ \cite{10}. They found all Abelian groups $G$ admitting two systems of generators $\{g_i\}, \{h_i\}$ as above. The only possible groups $G$ turn out to be $(\mathbb{Z}/2\mathbb{Z})^3$, $(\mathbb{Z}/2\mathbb{Z})^4$, $(\mathbb{Z}/3\mathbb{Z})^2$ and $(\mathbb{Z}/5\mathbb{Z})^2$. Varying the positions of the branching points on the quotients $C_1/G$, $C_2/G$ corresponds to varying a surface $S$ in a family. These families have dimensions 5, 4, 2 for the groups $(\mathbb{Z}/2\mathbb{Z})^3$, $(\mathbb{Z}/2\mathbb{Z})^4$, $(\mathbb{Z}/3\mathbb{Z})^2$ respectively. For $(\mathbb{Z}/5\mathbb{Z})^2$ there is only one surface (see \cite{15}; it is mistakenly stated in \cite{10} that there are two). In the next section we shall find the homology groups of these surfaces.

\section{Computation of the homology groups}

We want to find the first homology groups with integer coefficients of the surfaces $S$ in the four families described in \cite{10}. These surfaces are quotients $(C_1 \times C_2)/G$ of a product of two curves $C_i$ by the product action of a finite Abelian group $G$. Both curves are ramified coverings of $C_i/G \cong \mathbb{P}^1$. Since $H_1(S, \mathbb{Z}) \cong \pi_1(S)^{ab} := \pi_1(S)/[\pi_1(S), \pi_1(S)]$, the problem reduces to a group-theoretic computation: given two homomorphisms $\Pi(i) \to G$, find the abelinization of the kernel in the exact sequence (1).

The group $\Pi(1)$ has a presentation $\langle a_1, \ldots, a_n \mid a_1^{k_1}, \ldots, a_n^{k_n}, a_1 \cdots a_n \rangle$, where $n$ is the number of branching points of the covering $C_1 \to \mathbb{P}^1$, $a_j$ is the class of a loop around $x_j \in B$, and $k_j$ is the order of the stabilizer of each pre-image of $x_j$. The number $k_j$ is equal to the order of the image of $a_j$ in $G$. In all cases,
$G$ is a product of cyclic groups of the same order. Hence all the numbers $k_j$ are equal to some $k$. The group $\Pi(2)$ is of the same form, with generators denoted by $b_1, \ldots, b_m$. We denote the generators of the cyclic factors of $G$ by $e_i$. We now describe the homomorphisms $\varphi: \Pi(1) \to G$ and $\psi: \Pi(2) \to G$. They are the same for all the surfaces in each family.

**Case 1:** $G = (\mathbb{Z}/2\mathbb{Z})^3$. We have $k = 2$, $n = 5$, $m = 6$, and the homomorphism $\varphi$ is given by

$$a_1 \mapsto e_1, \quad a_2 \mapsto e_2, \quad a_3 \mapsto e_3, \quad a_4 \mapsto e_1, \quad a_5 \mapsto e_2 + e_3.$$

The homomorphism $\psi$ is

$$b_1 \mapsto e_1 + e_2, \quad b_2 \mapsto e_1 + e_3, \quad b_3 \mapsto e_1 + e_2 + e_3, \quad b_4 \mapsto e_1 + e_2, \quad b_5 \mapsto e_1 + e_3, \quad b_6 \mapsto e_1 + e_2 + e_3.$$

**Case 2:** $G = (\mathbb{Z}/2\mathbb{Z})^4$. We have $k = 2$, $n = m = 5$. The homomorphism $\varphi$ is given by

$$a_1 \mapsto e_1, \quad a_2 \mapsto e_2, \quad a_3 \mapsto e_3, \quad a_4 \mapsto e_4, \quad a_5 \mapsto e_1 + e_2 + e_3 + e_4,$$

and $\psi$ is

$$b_1 \mapsto e_2 + e_3 + e_4, \quad b_2 \mapsto e_1 + e_3 + e_4, \quad b_3 \mapsto e_1 + e_3, \quad b_4 \mapsto e_2 + e_4, \quad b_5 \mapsto e_3 + e_4.$$

**Case 3:** $G = (\mathbb{Z}/3\mathbb{Z})^2$. We have $k = 3$, $n = m = 4$. The homomorphism $\varphi$ is given by

$$a_1 \mapsto e_1, \quad a_2 \mapsto e_2, \quad a_3 \mapsto -e_1, \quad a_4 \mapsto -e_2,$$

and $\psi$ is

$$b_1 \mapsto e_1 + e_2, \quad b_2 \mapsto e_1 - e_2, \quad b_3 \mapsto -e_1 - e_2, \quad b_4 \mapsto -e_1 + e_2.$$

**Case 4:** $G = (\mathbb{Z}/5\mathbb{Z})^2$. We have $k = 5$, $n = m = 3$. The homomorphism $\varphi$ is given by

$$a_1 \mapsto e_1, \quad a_2 \mapsto e_2, \quad a_3 \mapsto -e_1 - e_2,$$

and $\psi$ is

$$b_1 \mapsto e_1 + 2e_2, \quad b_2 \mapsto 3e_1 + 4e_2, \quad b_3 \mapsto e_1 + 4e_2.$$

We introduce the following notation: $K = \pi_1(S)$, $F = \Pi(1) \times \Pi(2)$. There is an exact sequence

$$1 \to K \to F \to G \to 1.$$

It induces an exact sequence

$$1 \to [F, F]/[K, K] \to F/[K, K] \to F/[F, F] \to 1. \quad (2)$$

Later we shall prove that the group $[F, F]/[K, K]$ is Abelian. We first compute the groups $F/[F, F]$, $[F, F]/[K, K]$ and the 2-cocycle corresponding to the extension (2). Then the extension

$$1 \to [F, F]/[K, K] \to K/[K, K] \to K/[F, F] \to 1$$

will be given by the restriction of this 2-cocycle to the group $K/[F, F]$. 

We use the following convention for the commutator of two elements: \([a, b] = aba^{-1}b^{-1}\). The commutator satisfies the relations

\[
[ab, c] = a[b, c]a^{-1}[a, c], \quad [a, bc] = [a, b][a, c]b^{-1}.
\]

**Lemma 3.1.** We have \([F, [F, F]] \subseteq [K, K]\).

**Proof.** The group \([F, [F, F]]\) is generated by elements of the form \([p_1q_1, [p_2q_2, p_3q_3]]\), where \(p_i\) are elements of the group \(\Pi(1)\) regarded as a subgroup of \(F\), and \(q_i \in \Pi(2)\).

We have

\[
[p_1q_1, [p_2q_2, p_3q_3]] = [p_1, [p_2, p_3]][q_1, [q_2, q_3]].
\]

Since \(G\) is Abelian, \([p_2, p_3], [q_2, q_3] \in K\). The homomorphisms \(\varphi, \psi\) are surjective, so there are \(p'_1 \in \Pi(1), q'_1 \in \Pi(2)\) such that \(p'_1q_1, p_1q'_1 \in K\). Then

\[
[p_1, [p_2, p_3]][q_1, [q_2, q_3]] = [p_1q'_1, [p_2, p_3]][p'_1q_1, [q_2, q_3]] \in [K, K].
\]

Therefore the group \([F, F]/[K, K]\) is Abelian. It follows from (3), (4) that the commutator map

\[
[\cdot, \cdot] : F \times F \to [F, F]/[K, K]
\]

is bilinear and skew-symmetric. It induces a surjective homomorphism

\[
\tau : \Lambda^2 F^{ab} \to [F, F]/[K, K]
\]

which takes \(x \wedge y\) to the class of \([x, y]\). Here \(\Lambda^2 F^{ab}\) is understood as the quotient of the tensor product \(F^{ab} \otimes F^{ab}\) (taken in the category of Abelian groups) by the subgroup generated by the elements of the form \(x \otimes x\).

Suppose that \(p_1 \in K \cap \Pi(1)\) and \(p_2q_2\) is an arbitrary element of \(F\). Then there is a \(q'_2 \in \Pi(2)\) such that \(p_2q'_2 \in K\). We have

\[
\tau(p_1 \wedge p_2q_2) = \tau(p_1 \wedge p_2) = \tau(p_1 \wedge p_2q'_2) = 0.
\]

Similarly, if \(q_1 \in K \cap \Pi(2)\), then \(\tau(q_1 \wedge p_2q_2) = 0\). We define a map

\[
\sigma : \Lambda^2 G \to [F, F]/[K, K]
\]

by the rule \(\sigma(g_1 \wedge g_2) = \tau(p_1 \wedge p_2)\), where \(p_i \in \Pi(1)\) are arbitrary elements with \(\varphi(p_i) = g_i\). By the above, this map is well defined. If \(p_1q_1, p_2q_2 \in K\), then \(\varphi(p_i) = -\psi(q_i)\) and \(\tau(p_1q_1 \wedge p_2q_2) = 0\), whence \(\tau(p_1 \wedge p_2) = -\tau(q_1 \wedge q_2)\). This means that for all \(p, p' \in \Pi(1)\) and \(q, q' \in \Pi(2)\) we have \(\tau(p \wedge p') = \sigma(\varphi(p) \wedge \varphi(p'))\) and \(\tau(q \wedge q') = -\sigma(\psi(q) \wedge \psi(q'))\). Thus the image of \(\tau\) is generated by the image of \(\sigma\), that is, the homomorphism \(\sigma\) is surjective.

Using the identities (3), (4), we obtain the following equalities in the group \([F, F]/[F, [F, F]]\):

\[
1 = a_1^k \cdots a_n^k = a_1^k \cdots a_{n-1}^k (a_{n-1}^{-1} \cdots a_1^{-1})^k
\]

\[
= \frac{k(k-1)}{2} \prod_{i<j<k} [a_i^{-1}, a_j^{-1}] = \frac{k(k-1)}{2} \prod_{i<j<n} [a_i, a_j].
\]
Hence the elements \( \frac{k(k-1)}{2} \sum_{i<j<n} \varphi(a_i) \wedge \varphi(a_j) \), \( \frac{k(k-1)}{2} \sum_{i<j<m} \psi(b_i) \wedge \psi(b_j) \) lie in the kernel of \( \sigma \). We write \( H \) for the quotient group of \( \Lambda^2 G \) by the subgroup generated by these two elements. By an abuse of notation, we denote the induced homomorphism \( H \to [F,F]/[K,K] \) again by \( \sigma \).

**Lemma 3.2.** The homomorphism \( \sigma : H \to [F,F]/[K,K] \) is an isomorphism.

**Proof.** Let us describe the inverse homomorphism \( \alpha : [F,F]/[K,K] \to H \). We shall use all the generators (except the last) of the groups \( \Pi(1), \Pi(2) \). Let \( \Sigma_1 \) be the set \( \{a_1^\pm 1, \ldots, a_{n-1}^\pm 1\} \) and let \( \Sigma_2 = \{b_1^\pm 1, \ldots, b_{m-1}^\pm 1\} \). Each element of \( [F,F] = [\Pi(1), \Pi(1)] \times [\Pi(2), \Pi(2)] \) can be represented by a pair of words \( (u,v) \) (in letters from \( \Sigma_1, \Sigma_2 \) respectively) such that the total degree of \( u \) in each \( a_i \) is divisible by \( k \) and the same holds for the degree of \( v \) in each \( b_j \). Write \( (u,v) = (x_1 \ldots x_c, y_1 \ldots y_d) \), where \( x_i = a_i^{\pm 1}_{\pi(i)}, \ y_i = b_j^{\pm 1}_{\pi(j)} \). We put

\[
\alpha(u,v) = \sum_{r<s, i(r)>i(s)} \varphi(x_r) \wedge \varphi(x_s) - \sum_{r<s, j(r)>j(s)} \psi(y_r) \wedge \psi(y_s).
\]

If we add \( xx^{-1} \) or \( x^k \) (where \( x \in \Sigma_1 \)) in an arbitrary place of the word \( u \), then the value of \( \alpha(u,v) \) is not changed. If we add \( (a_1 \ldots a_{n-1})^k \), then this value is changed by \( \frac{k(k-1)}{2} \sum_{i>j} \varphi(a_i) \wedge \varphi(a_j) \), which is equal to zero in \( H \). Thus we have shown that \( \alpha(u,v) \) is independent of the choice of the word \( u \) that represents a given element of \( [\Pi(1), \Pi(1)] \). The same holds for \( v \).

For the concatenation of two pairs of words we have

\[
\alpha(uu', vv') = \alpha(u,v) + \alpha(u', v') + R,
\]

where \( R \) is the sum

\[
\sum \varphi(a_i^{\pm 1}) \wedge \varphi(a_j^{\pm 1}) - \sum \psi(b_r^{\pm 1}) \wedge \psi(b_s^{\pm 1})
\]

over all occurrences of the letters with subscripts \( i \) in \( u \), with \( j \) in \( u' \), respectively, with \( r \) in \( v \), with \( s \) in \( v' \) such that \( i > j \) and \( r > s \). Since the total degree of \( u, u', v, v' \) in each generator is divisible by \( k \), the image of \( R \) in \( H \) is equal to zero. Hence \( \alpha \) is a well-defined homomorphism \( [F,F] \to H \).

If \( (u,v) \) is a pair of words representing an element of \( [F,F] \), then

\[
\alpha(a_iua_i^{-1}, v) = \alpha(u,v) + R,
\]

where \( R \) is the sum

\[
\sum_{x_j \text{ from } u} \varphi(a_i) \wedge \varphi(x_j) + \sum_{x_j \text{ from } u} \varphi(x_j) \wedge \varphi(a_i^{-1}) = \varphi(a_i) \wedge \varphi(u) = 0.
\]

The same holds for conjugation by the generators of \( \Pi(2) \). We have shown that \( \alpha(xyx^{-1}) = \alpha(y) \) for all \( x \in F, y \in [F,F] \). Thus \( \alpha \) is a homomorphism

\[
[F,F]/[F,[F,F]] \to H.
\]
It follows from the definition of $\alpha$ that $\alpha([a_i, a_j]) = \varphi(a_i) \wedge \varphi(a_j)$ and $\alpha([b_i, b_j]) = -\psi(b_i) \wedge \psi(b_j)$. By the identities (3), (4), this means that for arbitrary $p, p' \in \Pi(1)$ and $q, q' \in \Pi(2)$ we have $\alpha([p, p']) = \varphi(p) \wedge \varphi(p')$ and $\alpha([q, q']) = -\psi(q) \wedge \psi(q')$. It is now clear that $\alpha$ is a homomorphism $[F, F]/[K, K] \rightarrow H$ inverse to $\sigma$. □

We recall [16] that the extensions

$$0 \rightarrow A \rightarrow E \rightarrow G \rightarrow 1,$$  (5)

of an arbitrary group $G$ by an Abelian group $A$ (endowed with the structure of a $G$-module) are parametrized by elements of the cohomology group $H^2(G, A)$. Specifically, with an extension of the form (5) we associate a normalized 2-cocycle $\langle \cdot, \cdot \rangle : G \times G \rightarrow A$ by the formula $\langle g, h \rangle = (g(1) \cdot h) \gamma(g, h)^{-1}$, where $\gamma$ is an arbitrary section of the map $\pi : E \rightarrow G$ such that $\gamma(1) = 1$. By definition, a normalized 2-cocycle is a map $\langle \cdot, \cdot \rangle : G \times G \rightarrow A$ satisfying two conditions:

$$\langle g, 1 \rangle = \langle 1, g \rangle = 0,$$

$$f \cdot \langle g, h \rangle - \langle fg, h \rangle + \langle f, gh \rangle - \langle f, g \rangle = 0.$$

Given a 2-cocycle $\langle \cdot, \cdot \rangle$, one can recover the group $E$ (up to isomorphism) by putting $E = A \times G$ as a set and defining the multiplication by the formula

$$(a, g) \cdot (b, h) = (a + g \cdot b + \langle g, h \rangle, gh).$$

**Lemma 3.3** [16]. Suppose that the extension

$$0 \rightarrow A \rightarrow E \rightarrow G \rightarrow 1$$

is given by a 2-cocycle $\eta \in Z^2(G, A)$. Let $f : H \rightarrow G$ be a homomorphism of groups. Then the extension

$$0 \rightarrow A \rightarrow E \times_G H \rightarrow H \rightarrow 1$$

is given by the pullback $f^* \eta \in Z^2(H, A)$, where the action of $H$ on $A$ is induced by $f$.

In our situation, consider the extension (2):

$$0 \rightarrow H \rightarrow F/[K, K] \rightarrow F^{ab} \rightarrow 1.$$  

Since $[F, [F, F]] \subseteq [K, K]$, the action of $F^{ab}$ by conjugation on $H$ is trivial. The group $F^{ab}$ is a free $\mathbb{Z}/k\mathbb{Z}$-module with basis $\{a_1, \ldots, a_{n-1}, b_1, \ldots, b_{m-1}\}$. We define a section $\gamma$ by the formula

$$\gamma \left( \sum_{i=1}^{n-1} r_i a_i + \sum_{i=1}^{m-1} r'_i b_i \right) = a_1^{r_1} \cdots a_{n-1}^{r_{n-1}} b_1^{r'_1} \cdots b_{m-1}^{r'_{m-1}},$$

where $0 \leq r_i, r'_i \leq k - 1$ for all $i$. Then

$$\langle \sum r_i a_i + \sum s_i b_i, \sum r'_i b_i, \sum s'_i b_i \rangle$$

$$= \sigma(a_1^{r_1} \cdots a_{n-1}^{r_{n-1}} b_1^{r'_1} \cdots b_{m-1}^{r'_{m-1}} a_1^{s_1} \cdots a_{n-1}^{s_{n-1}} b_1^{s'_1} \cdots b_{m-1}^{s'_{m-1}} a_1^{t_1} \cdots a_{n-1}^{t_{n-1}} b_1^{t'_1} \cdots b_{m-1}^{t'_{m-1}} b_1^{t_1} \cdots b_{m-1}^{t_{m-1}}),$$
where \( t_i \) can be equal to \( r_i + s_i \) or \( r_i + s_i - k \), and similarly for \( t'_i \). It follows from the definition of \( \sigma \) that this is equal to

\[
-\sum_{i<j} r_j s_i \varphi(a_i) \wedge \varphi(a_j) + \sum_{i<j} r'_j s'_i \psi(b_i) \wedge \psi(b_j).
\]

In particular, the 2-cocycle is bilinear in our case.

Suppose that we are given an extension in the category of Abelian groups,

\[
0 \to A \to E \to G \to 0,
\]

and a surjective homomorphism \( \mathbb{Z}^n \to G \), \( e_i \mapsto g_i \), whose kernel is generated by some elements \( r_j, j \in J \). Then there is a homomorphism \( A \oplus \mathbb{Z}^n \to E \) which is the identity on \( A \) and maps \( e_i \) to \((0, g_i)\). It is well defined since \( E \) is Abelian. We easily see that this homomorphism is surjective and its kernel is generated by the elements \( (a_j, r_j), j \in J \), where \( a_j \in A \) is the unique element such that \( (a_j, r_j) \) is mapped to the zero element of \( E \).

We have an exact sequence of Abelian groups

\[
0 \to H \to K^{ab} \to K/[F, F] \to 0.
\]

The corresponding 2-cocycle is the restriction of the 2-cocycle \( \langle \cdot, \cdot \rangle \) to the subgroup \( K/[F, F] \leq F^{ab} \). We denote this restriction again by \( \langle \cdot, \cdot \rangle \). The group \( K/[F, F] \) is a free \( \mathbb{Z}/k\mathbb{Z} \)-module (we recall that \( k \) is a prime in all our cases). Hence it is a quotient of some \( \mathbb{Z}' \) by the relations \( k c_i = 0 \). Since the 2-cocycle \( \langle \cdot, \cdot \rangle \) is bilinear, we have the following relations in the group \( K^{ab} \):

\[
(0, c_i)^k = \langle c_i, c_i \rangle + \cdots + \langle (k-1)c_i, c_i \rangle, kc_i) = \left( \frac{k(k-1)}{2} \langle c_i, c_i \rangle, 0 \right).
\]

Therefore, to obtain the group \( K^{ab} \), we must add to the Abelian group \( H \) new generators \( f_i \) with relations \( kf_i = -\frac{k(k-1)}{2} \langle c_i, c_i \rangle \). We shall do this in each case in turn.

**Theorem 3.4.** The first homology groups of our surfaces are given by the following formulae:

1) \( H_1(S, \mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^4 \oplus (\mathbb{Z}/4\mathbb{Z})^2 \) in case \( G = (\mathbb{Z}/2\mathbb{Z})^3 \);

2) \( H_1(S, \mathbb{Z}) \cong (\mathbb{Z}/4\mathbb{Z})^4 \) in case \( G = (\mathbb{Z}/2\mathbb{Z})^4 \);

3) \( H_1(S, \mathbb{Z}) \cong (\mathbb{Z}/3\mathbb{Z})^5 \) in case \( G = (\mathbb{Z}/3\mathbb{Z})^2 \);

4) \( H_1(S, \mathbb{Z}) \cong (\mathbb{Z}/5\mathbb{Z})^3 \) in case \( G = (\mathbb{Z}/5\mathbb{Z})^2 \).

**Proof.** 1) The group \( H \) is the quotient of \( \Lambda^2 G \) by the subgroup generated by the elements

\[
\sum_{i<j<5} \varphi(a_i) \wedge \varphi(a_j) = e_2 \wedge e_3, \quad \sum_{i<j<6} \psi(b_i) \wedge \psi(b_j) = 0.
\]

Hence \( H \cong (\mathbb{Z}/2\mathbb{Z})^2 \). We choose the following basis in \( K/[F, F] \) over \( \mathbb{Z}/2\mathbb{Z} \):

\[
c_1 = (a_1 + a_4, 0), \quad c_2 = (a_1 + a_2, b_1), \quad c_3 = (a_1 + a_3, b_2), \quad c_4 = (a_1 + a_2 + a_3, b_3),
\]

We shall do this in each case in turn.
\[c_5 = (a_1 + a_2, b_4), \quad c_6 = (a_1 + a_3, b_5)\]. Then the group \(K^{ab}\) is the quotient of \(H \oplus \mathbb{Z}^6\) by the relations

\[
\begin{align*}
2f_1 &= -\langle e_1, c_1 \rangle = \varphi(a_1) \wedge \varphi(a_4) = 0, \\
2f_2 &= e_1 \wedge e_2, \\
2f_3 &= e_1 \wedge e_3, \\
2f_4 &= e_1 \wedge (e_2 + e_3) + e_2 \wedge e_3, \\
2f_5 &= e_1 \wedge e_2, \\
2f_6 &= e_1 \wedge e_3.
\end{align*}
\]

The elements on the right-hand sides generate the whole of \(H\). Therefore \(K^{ab} \cong (\mathbb{Z}/2\mathbb{Z})^4 \oplus (\mathbb{Z}/4\mathbb{Z})^2\).

2) The group \(H\) is the quotient \(\Lambda^2 G\) by the subgroup generated by the elements

\[
\begin{align*}
\sum_{i < j < 4} \varphi(a_i) \wedge \varphi(a_j) &= e_1 \wedge (e_2 + e_3 + e_4) + e_2 \wedge (e_3 + e_4) + e_3 \wedge e_4, \\
\sum_{i < j < 4} \psi(b_i) \wedge \psi(b_j) &= e_1 \wedge e_4 + e_2 \wedge e_3 + e_3 \wedge e_4.
\end{align*}
\]

Hence \(H \cong (\mathbb{Z}/2\mathbb{Z})^4\). We choose the following basis in \(K/[F,F]\) over \(\mathbb{Z}/2\mathbb{Z}\): \(c_1 = (a_2 + a_3 + a_4, b_1), \quad c_2 = (a_1 + a_3 + a_4, b_2), \quad c_3 = (a_1 + a_3, b_3), \quad c_4 = (a_2 + a_4, b_4)\). Then the group \(K^{ab}\) is the quotient of \(H \oplus \mathbb{Z}^4\) by the relations

\[
\begin{align*}
2f_1 &= e_2 \wedge e_3 + e_2 \wedge e_4 + e_3 \wedge e_4, \\
2f_2 &= e_1 \wedge e_3 + e_1 \wedge e_4 + e_3 \wedge e_4, \\
2f_3 &= e_1 \wedge e_3, \\
2f_4 &= e_2 \wedge e_4.
\end{align*}
\]

The elements on the right-hand sides are linearly independent in \(H\). Therefore \(K^{ab} \cong (\mathbb{Z}/4\mathbb{Z})^4\).

3), 4) Since \(k\) is odd, the number \(\frac{k(k-1)}{2}\) is divisible by \(k\). Hence \(H\) is the quotient of \(\Lambda^2 G\) by the trivial elements, \(H = \Lambda^2 G\), and the relations for the generators \(f_i\) are simply \(k\varphi_f = 0\). This means that \(K^{ab} \cong \Lambda^2 G \oplus K/[F,F]\). In case 3) we have \(K^{ab} \cong (\mathbb{Z}/3\mathbb{Z})^5\). In case 4) we have \(K^{ab} \cong (\mathbb{Z}/5\mathbb{Z})^3\). \(\square\)

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