A CHANCE-CONSTRAINED STOCHASTIC MODEL PREDICTIVE CONTROL PROBLEM WITH DISTURBANCE FEEDBACK

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(Communicated by Bin Li)

Abstract. In this paper, we develop two algorithms for stochastic model predictive control (SMPC) problems with discrete linear systems. Participially, chance constraints on the state and control are considered. Different from the state-of-the-art robust model predictive control (RMPC) algorithm, the proposed is less conservative. Meanwhile, the proposed algorithms do not assume the full knowledge of the disturbance distribution. It only requires the mean and variance of the disturbance. Rigorous computational analysis is carried out for the proposed algorithms. Numerical results are provided to demonstrate the effectiveness and the superior of the proposed SMPC algorithms.

2020 Mathematics Subject Classification. 91B70.
Key words and phrases. Stochastic model predictive control, chance constraint, robust optimization.

This work was partially supported by a grant from National Natural Science Foundation of China under number 61701124, a grant from Science and Technology on Space Intelligent Control Laboratory, No. KGJZDSYS-2018-03, a grant from Sichuan Province Government under the application number 2019YJ0105, and a grant from Fundamental Research Funds for the Central Universities (China).

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1. **Introduction.** Robust model predictive control (RMPC) \([22, 28, 3, 29, 24]\) algorithms have been criticized since it can only handle the bounded noise by adopting the worst-case robust philosophy, which usually leads to conservativeness.  

To tackle these problems, stochastic model predictive control (SMPC) algorithms are developed. When disturbances occur, constraint violations are allowed with SMPC algorithms by using the chance constraint formulation. Chance constraint can also handle the infeasibility caused by the unbounded noise.

Different SMPC methods and their applications was reviewed in \([9]\). For example, stochastic tube approach \([4]\) utilizes probabilistic tubes with fixed or variant cross sections to replace the chance constraints with linear constraints on the nominal state predictions and to construct terminal sets for ensuring the recursive feasibility. However, since this method is based on the off-line design, then the hard constraints cannot be satisfied in presence of unbounded noise during the on-line implementation. For this, the idea of saturation function is adopted into the affine feedback control policy in \([15]\). Another branch of SMPC is the so-called sample-based approach (also known as scenario-based approaches). This type of algorithms represents the stochastic system by using a finite set of realizations of the uncertainties sampled from the continuous distribution, which are used to solve the optimal control problem in one shot \([3]\).

Another type SMPC algorithm is based on the Cheyshev-Cantelli inequality method which has been used to solve chance-constrained state feedback and output feedback SMPC problems \([10, 8, 6]\). Convex approximation (also known as safe approximation) is a favourable approach to solve the chance constraints optimization problems. However, it has not been yet used for solving SMPC problems. The most attractive feature for this method is that it can always provide a tractable and feasible solution.

In this paper, we consider a chance-constrained SMPC problem with disturbance feedback. Participially, the disturbance is unbounded. Two SMPC algorithms are developed which are based on the convex approximation method and the Cheyshev-Cantelli inequality method. The corresponding tractable reformulations are derived. Rigorous computational complexity analysis are also provided. Numerical results has shown that the proposed algorithms are effective and are superior than the state-of-the-art RMPC algorithm.

The rest of this paper is organized as follows. The problem are stated in Section 2. In Section 3, two computational methods are developed by reformulating the SMPC problem into deterministic conic optimization problems. Simulation examples are reported in Section 4 to illustrate the effectiveness of the proposed SMPC algorithms. Finally, we conclude this paper by making some concluding remarks in Section 5.

2. **Problem statement.**

2.1. **System model.** In this paper, we consider a linear discrete system

\[
x_{t+1} = Ax_t + Bu_t + G\omega_t
\]

where \(t \in \mathbb{N}, x_t \in \mathbb{R}^n\) denotes the state with initial value \(x_0\), \(u_t \in \mathbb{R}^m\) is the input, \(\omega_t \in \mathbb{R}^n\) is an unbounded random noise, and \(A, B\) and \(G\) are known matrices.

Then, we assume that the following assumptions are hold.

**Assumption 1.** The dynamics (1) satisfy

(i) The pairs \((A, B)\) are stabilizable.
Thus, we can obtain the following compact form

\[
\text{state and the input variables \([7]\) are imposed.}
\]

\[
\text{2.2. Chance constraints. In this paper, the chance constraints on both of the state and the input variables \([7]\) are imposed.}
\]

\[
P [a^T x_t \leq b] \geq 1 - \epsilon_x, \quad \forall t \in \mathbb{N} 
\]

\[
P [c^T u_t \leq d] \geq 1 - \epsilon_u, \quad \forall t \in \mathbb{N} 
\]

where \(P[.]\) denotes the probability, \(a \in \mathbb{R}^{n_x}, \) \(c \in \mathbb{R}^{n_u}, \) \(b \) and \(d \in \mathbb{R}, \) \((\cdot)^T\) denotes the transpose, and \(\epsilon_x\) and \(\epsilon_u \in (0, 1)\) are given constant numbers.

To proceed further, we define

\[
x_t = \begin{bmatrix} x_t \\ x_{t+1} \\ \vdots \\ x_{t+N} \end{bmatrix}, \ u_t = \begin{bmatrix} u_t \\ u_{t+1} \\ \vdots \\ u_{t+N-1} \end{bmatrix}, \ \omega_t = \begin{bmatrix} \omega_t \\ \omega_{t+1} \\ \vdots \\ \omega_{t+N-1} \end{bmatrix}
\]

Thus, we can obtain the following compact form

\[
x_t = T x_t + H u_t + D G \omega_t
\]

where \(T, H, D\) and \(G\) can be obtained according to \([14]\).

Further, (2) and (3) can also be written more compactly as

\[
P [a^T x_t \leq b] \geq 1 - \epsilon_x, \quad k = 1, 2, \ldots, N
\]

\[
P [c^T u_t \leq d] \geq 1 - \epsilon_u, \quad k = 0, 1, \ldots, N - 1
\]
where \(a_k\) is an \((N + 1) \times n_x\) length vector with the \((k + 1)\)th block equal to \(a\) and 0 elsewhere, and \(c_k\) is defined in a similar way. Mean and covariance of \(\omega_t\) can also be adjusted straightforward as

\[
E[\omega_t] = \mu_\omega, \quad E[(\omega_t - \mu_\omega)(\omega_t - \mu_\omega)^T] = \Sigma_\omega
\]

where \(\mu_\omega = 1_N \otimes \mu_\omega\) and \(\Sigma_\omega = I_N \otimes \Sigma_\omega\). Here, \(1_N\) denotes an \(N\) dimensional vector with all the elements being 1, \(I_N\) denotes an \(N \times N\) identical matrix, and \(\otimes\) denotes the Kronecker product.

2.3. Cost function and terminal constraints. Here, the cost function is defined as

\[
J_t(x_t, u_t) = E\left\{\sum_{k=t}^{t+N-1} [x_k^T Q x_k + u_k^T R u_k] + x_{t+N}^T Q_N x_{t+N}\right\}
\]

where \(Q \in \mathbb{R}^{n_x \times n_x}\) and \(R \in \mathbb{R}^{n_u \times n_u}\) are positive definite, specified weight matrices. \(Q_N\) is the solution of the algebraic Lyapunov equation

\[
(A + B\bar{K})^T Q_N (A + B\bar{K}) - Q_N + Q + \bar{K}^T R \bar{K} = 0
\]

where \(\bar{K}\) is the gain computed as the solution of an LQ control problem.

It is well-known that terminal constraints are closely related to the stability of the system [23]. Thus, the following terminal constraints are imposed

\[
x_{t+N} \in \mathbb{X}_T
\]

where set \(\mathbb{X}_T\) is a positively invariant set satisfying

\[
(A + B\bar{K}) x_{t+N} \in \mathbb{X}_T \quad \forall x_{t+N} \in \mathbb{X}_T
\]

2.4. Feedback control law. In this paper, an affine disturbance feedback control policy \(u_t\) is adopted. An alternative parametrization is an affine function of the sequence of past disturbances [26, 27]

\[
u_{t+i} = \sum_{j=0}^{i+1} M_{t,i,j} G \omega_{t+j} + v_{t+i}, \quad \forall i = 0, \ldots, N - 1
\]

where \(M_{t,i,j} \in \mathbb{R}^{n_u \times n_x}\) and \(v_{t+i} \in \mathbb{R}^{n_u}\). Using (13), the control policy \(u_t\) can be written as

\[
u_t = M_t G \omega_t + v_t
\]

where the block lower triangular matrix \(M_t \in \mathbb{R}^{n_u N \times n_x N}\) and stacked vector \(v_t\) are given by

\[
M_t = \begin{bmatrix}
0_{n_u \times n_x} & \cdots & \cdots & 0_{n_u \times n_x} \\
M_{t,1,0} & 0 & \cdots & 0_{n_u \times n_x} \\
\vdots & \ddots & \ddots & \vdots \\
M_{t,N-1,0} & \cdots & M_{t,N-1,N-2} & 0_{n_u \times n_x}
\end{bmatrix}
\]

\[
v_t = [v_t^T, v_{t+1}^T, \ldots, v_{N-1}^T]^T
\]

The pair \((M_t, v_t)\) then comprises the decision variables in Problem 1.
2.5. The optimization problem. Now, the SMPC problem are formally stated as follows:

Problem 1

\[
\begin{align*}
\min_{\mathbf{M}_t, \mathbf{v}_t} & \quad J_t (x_t, \mathbf{M}_t, \mathbf{v}_t) \\
\text{s.t.} & \quad \mathbb{P} [\mathbf{a}_k^T \mathbf{x}_t \leq b] \geq 1 - \epsilon_x, \; k = 1, 2, \ldots, N - 1 \\
& \quad \mathbb{P} [\mathbf{c}_k^T \mathbf{u}_t \leq d] \geq 1 - \epsilon_u, \; k = 0, 1, \ldots, N - 1 \\
& \quad x_{t+N} \in \mathcal{X}_T
\end{align*}
\]

3. Two computational methods. In this section, we shall reformulate Problem 1 into a cone optimization problem, which is computationally tractable. To this end, we shall show that the objective function is, in fact, a convex function of the decision variables. The chance constraints (18) and (19) will be converted into deterministic cone constraints by using a convex approximation method and the Chebyshev-Cantelli inequality method.

3.1. Convex reformulation of (9). By substituting the system equations (5) and the feedback control law (14) into the cost function (9) and noting that \( E[\omega_t] = 0 \), we have

\[
J_t (x_t, \mathbf{M}_t, \mathbf{v}_t) = \text{tr} (\Delta_1 \mathbf{M}_t \mathbf{G} \Sigma_\omega + \mathbf{G}^T \mathbf{M}_t^T \Delta_2 \mathbf{M}_t \mathbf{G} \Sigma_\omega) + \mathbf{v}_t^T \Delta_2 \mathbf{v}_t + \mathbf{g}^T \mathbf{v}_t + c
\]

where \( \text{tr}(\cdot) \) denotes the matrix trace and

\[
\begin{align*}
\mathbf{g}^T &= (\mathbf{Tx}_t)^T \mathbf{Q} \\
c &= (\mathbf{Tx}_t)^T \mathbf{Q} (\mathbf{Tx}_t) + \mathbf{G}^T \mathbf{D}^T \mathbf{Q} \mathbf{D} \Sigma_\omega \\
\Delta_1 &= 2 \mathbf{G}^T \mathbf{D}^T \mathbf{Q} \\
\Delta_2 &= \mathbf{R} + \mathbf{H}^T \mathbf{Q} \mathbf{H}
\end{align*}
\]

Obviously, \( J_t (x_t, \mathbf{M}_t, \mathbf{v}_t) \) is convex with \( \mathbf{M}_t, \mathbf{v}_t \). However, it is not in form of disciplined convex programming [30]. For this, we introduce an auxiliary variable \( \mathbf{P}_t \geq 0 \) to deal with the first term of (21), which yields

\[
\begin{align*}
\min_{\mathbf{v}_t, \mathbf{M}_t, \mathbf{P}_t} & \quad \text{tr} (\Delta_1 \mathbf{M}_t \mathbf{G} \Sigma_\omega + \mathbf{P}_t \Sigma_\omega) + \mathbf{v}_t^T \Delta_2 \mathbf{v}_t + \mathbf{g}^T \mathbf{v}_t + c \\
\text{s.t.} & \quad \begin{bmatrix} \mathbf{P}_t & \mathbf{G}^T \mathbf{M}_t^T \\ \mathbf{M}_t \mathbf{G} & \Delta_2^{-1} \end{bmatrix} \succeq 0, \quad \mathbf{P}_t \succeq 0
\end{align*}
\]

(22)-(23) are in form of disciplined convex programming [30], which can be solved by using the standard software package CVX [13].

3.2. Convex reformulation of chance constraints. Letting \( \mathbf{S}_t = \mathbf{H}^T \mathbf{M}_t \mathbf{G} + \mathbf{D} \mathbf{G} \) and \( \mathbf{y}_t = \mathbf{Tx}_t + \mathbf{H}^T \mathbf{v}_t \), (18) and (19) can be rewritten as

\[
\begin{align*}
\mathbb{P} [\mathbf{a}_k^T (\mathbf{y}_t + \mathbf{S}_t \omega_t) \leq b] & \geq 1 - \epsilon_x, \; k = 1, 2, \ldots, N - 1 \\
\mathbb{P} [\mathbf{c}_k^T (\mathbf{M}_t \mathbf{G} \omega_t + \mathbf{v}_t) \leq d] & \geq 1 - \epsilon_u, \; k = 0, 1, \ldots, N - 1
\end{align*}
\]
3.2.1. Conditional value-at-risk (CVaR) approximation. A popular treatment of chance constraints is using convex approximation. Among them, conditional value-at-risk (CVaR) is widely accepted as the tightest convex approximation according to [25]. CVaR is a special class of risk measure introduced in [1] and further discussed in [5] as a tractable alternative for solving value-at-risk (VaR) problems in financial applications.

Inspired by the connections between chance constraints and the bounds on the CVaR measure in [5], we can approximate the chance constraints (18) and (19) with cone constraints. To simplify the notations, we rewrite constraints (24) as

\[ P \{ f_{1,k} \leq 0 \} \geq 1 - \varepsilon, \quad k = 1, \ldots, N, \]  

where \( f_{1,k} = a_k^T(y_t + S_t \omega_t) - b \), \( \varepsilon = \varepsilon_x \).

For \( k = 1, \ldots, N \),

\[ \text{CVaR}_{1-\varepsilon}(f_{1,k}) \leq 0, \]

or equivalently,

\[ \min \left\{ \beta_{1,k} + \frac{1}{\varepsilon} [E(f_{1,k} - \beta_{1,k})^+] \right\} \leq 0. \]  

Define set,

\[ W_{1,k} = \{ f_{1,k} : f_{1,k} = a_k^T(y_t + S_t \omega_t) - b \}. \]

We can converted the chance constraints (18) and (19) into deterministic cone constraints without bounded of the random process noise from the following lemma (Theorem 2.3 in [5]).

**Lemma 1.** Suppose that \( z \) is a random variable with zero mean and positive variance \( \Sigma_z \), then

\[ E(Y_0 + Y_0^T z)^+ \leq \frac{1}{2} Y_0 + \frac{1}{2} \sqrt{Y_0^2 + Y_0^T \Sigma_z Y} \]  

**Theorem 1.** Problem 1 can be approximated as the following the problem which is denoted as **Problem 2**:

\[
\begin{aligned}
& \min_{\beta_{1},\beta_{2},v,M,P_i} \text{tr} \left( \Delta_1 M G \Sigma_\omega + P_i \Sigma_\omega \right) + v_t^T \Delta_2 v_t + g^T v_t + c \\
\text{s.t.} \quad & \beta_{1,k} + \frac{1}{\varepsilon_x} \left[ \frac{1}{2} (a_k^T y_t - b - \beta_{1,k}) + \frac{1}{2} \sqrt{(a_k^T y_t - b - \beta_{1,k})^2 + a_k^T S_t \Sigma_\omega S_t^T a_k} \right] \leq 0 \\
& \quad k = 1, 2, \ldots, N - 1 \\
& \beta_{2,k} + \frac{1}{\varepsilon_u} \left[ \frac{1}{2} (c_k^T v_t - d - \beta_{2,k}) + \frac{1}{2} \sqrt{(c_k^T v_t - d - \beta_{2,k})^2 + c_k^T M_t G \Sigma_\omega G^T M_t^T c_k} \right] \leq 0 \\
& \quad k = 0, 1, \ldots, N - 1 \\
& \begin{bmatrix} P_i & G^T M_t^T \\ M_t G & \Delta_2^{-1} \end{bmatrix} \succeq 0, \quad P_i \succeq 0, \quad x_{t+N} \in \mathcal{X}_T
\end{aligned}
\]  

where \( \beta_i = [\beta_{i,k}, \ldots, \beta_{i,N}]^T \in \mathbb{R}^N \) for \( i = [1, 2] \).
Proof. According to Lemma 1, we set \( z = \omega_t \), \( y = a_k^T \Sigma \) and \( y_0 = a_k^T y_t - b - \beta_{1,k} \), and obtain

\[
\mathbb{E}(a_k^T y_t - b - \beta_{1,k} + a_k^T \Sigma \omega_t)^+ \leq \frac{1}{2} (a_k^T y_t - b - \beta_{1,k}) + \frac{1}{2} \sqrt{(a_k^T y_t - b - \beta_{1,k})^2 + a_k^T \Sigma \Sigma^T a_k},
\]

where \( \Sigma \omega \) is the variance of \( \omega_t \). Hence, constraint (24) is valid if

\[
\beta_{1,k} + \frac{1}{\varepsilon_x} \left[ \frac{1}{2} (a_k^T y_t - b - \beta_{1,k}) + \frac{1}{2} \sqrt{(a_k^T y_t - b - \beta_{1,k})^2 + a_k^T \Sigma \Sigma^T a_k} \right] \leq 0 \tag{30}
\]

is valid.

Similarly, we can obtain the following set of constraints according to constraint (25)

\[
\beta_{2,k} + \frac{1}{\varepsilon_u} \left[ \frac{1}{2} (c_k^T v_t - d - \beta_{2,k}) + \frac{1}{2} \sqrt{(c_k^T v_t - d - \beta_{2,k})^2 + c_k^T M_t \Sigma \Sigma^T M_t^T c_k} \right] \leq 0 \tag{31}
\]

3.2.2. Chebyshev–Cantelli Inequality. To proceed further, we consider a random variable \( z \) with mean value \( \bar{z} = \mathbb{E}[z] \), variable \( Z = \mathbb{E}(z - \bar{z})(z - \bar{z})^T \), and the following chance constraint

\[
P[h^T z > z_{max}] \leq p \tag{33}
\]

To proceed, we need the following lemma from [8].

**Lemma 2.** Letting \( f(p) = \sqrt{(1 - p)/p} \), constraint (33) is verified if

\[
h^T \bar{z} \leq z_{max} - h^T Z f(p) \tag{34}
\]

**Theorem 2. Problem** 1 can be approximated as the following problem which is denoted as **Problem** 3:

\[
\min_{v_t, M_t, P_t} \text{tr}(\Delta_1 M G \Sigma + P_t \Sigma) + v_t^T \Delta_2 v_t + g^T v_t + c \tag{35a}
\]

s.t. \( f(\varepsilon_x) \sqrt{a_k^T S_t \Sigma S_t^T a_k} \leq b - a_k^T y_t, \ k = 1, 2, \ldots, N - 1 \)

\( f(\varepsilon_u) \sqrt{a_k^T M_t \Sigma G^T M_t^T c_k} \leq d - c_k^T v_t, \ k = 0, 1, \ldots, N - 1 \)

\[
\begin{bmatrix}
P_t & G^T M_t^T \\
M_t G & \Delta_2^{-1}
\end{bmatrix} \succeq 0, \ P_t \succeq 0, \ x_t + N \in X_T
\]

**Proof.** In view of Lemma 2, the probabilistic constraints (18) and (19) at time \( t \) are verified provided that the following (deterministic) inequalities are satisfied.

\[
f(\varepsilon_x) \sqrt{a_k^T S_t \Sigma S_t^T a_k} \leq b - a_k^T y_t, \ k = 1, 2, \ldots, N - 1 \]

\[
f(\varepsilon_u) \sqrt{a_k^T M_t \Sigma G^T M_t^T c_k} \leq d - c_k^T v_t, \ k = 0, 1, \ldots, N - 1 \]
This completes the proof. □

3.3. Optimization algorithm. In this section, we shall present the optimization algorithm for solving SMPC problem at each time $t$. Problem 2 is already a conic optimization problem which is computationally tractable and can be solved by the standard software package such as CVX [13]. We will apply the problem 2 into stochastic MPC. More specifically, our optimization algorithm at each time $t$ is described as Algorithm 1 in Table 1. Algorithm 2 is described as in Table 2.

### Algorithm 1

Find $v^*_t, M^*_t$ for the optimal control policy for any initial condition $\bar{x}_t$.

**Input:** $x_t$, $\epsilon_x$ and $\epsilon_u$.

**Output:** $x_{t+1}$.

1. Solve Problem 2.
2. if Problem 2 is feasible then
3. Return $v^*_t$ and $M^*_t$.
4. end if
5. Compute $u^*_t$ and $x_{t+1}$ with $v^*_t$ and $M^*_t$ according to (14) and (1), respectively.

### Algorithm 2

Find $v^*_t, M^*_t$ for the optimal control policy for any initial condition $\bar{x}_t$.

**Input:** $x_t$, $\epsilon_x$ and $\epsilon_u$.

**Output:** $x_{t+1}$.

1. Solve Problem 3.
2. if Problem 3 is feasible then
3. Return $v^*_t$ and $M^*_t$.
4. end if
5. Compute $u^*_t$ and $x_{t+1}$ with $v^*_t$ and $M^*_t$ according to (14) and (1), respectively.

3.4. Computational complexity analysis. In this section, we shall provide computational complexity analysis of Algorithm 1 and Algorithm 2.

According to [21], the main computation load for Algorithm 1 is solving Problem 2, and the corresponding iteration computational complexity is

$$O \left\{ 2N(n_w N + 1)(n_u n_x N^2 + n_w N + 1)^2 + (n_w + n_u)^2 N^2 (n_x n_u N^2 + n_x^2 N^2)^2 + n_w^6 N^6 \right\}$$

Similarly, for Algorithm 2, the main computation is solving Problem 3, and the corresponding iteration computational complexity is

$$O \left\{ 2N(n_w N + 1)(n_u n_x N^2 + n_u N)^2 + (n_w + n_u)^2 N^2 (n_x n_u N^2 + n_u^2 N^2)^2 + n_u^6 N^6 \right\}$$

We can see that the iteration complexity for Algorithm 2 is lower than that of Algorithm 1.

For the bound of number of iterations, Algorithm 1 and Algorithm 2 both take at most $O \left( \sqrt{(n_w + n_u)N + n_w N} \right)$ iterations to converge according to [21].

4. Numerical examples. To test the effectiveness of the proposed methods, in this section, we test the proposed two algorithms with the following two numerical examples. According to the performance index of the cost function, the state trajectories of system should be driven to the origin, this indicates that the system is stable. Problem 2 and Problem 3 are solved by using CVX [13].
4.1. **Example 1.** In the first example, the proposed algorithms are used for solving the problem in [31]. The matrices in (1) are set as

\[
A = \begin{bmatrix} 1.02 & -0.1 \\ 0.1 & 0.98 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 \\ 0.05 \end{bmatrix}, \quad G = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}
\]

\[
Q = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad R = \begin{bmatrix} 50 \\ 0 \\ 0 \\ 50 \end{bmatrix}, \quad \Sigma_w = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\]

where noise is zero mean, and covariance matrix \( \Sigma_w \). The chance constraints are

\[
P \left\{ \left[ -\frac{1}{\sqrt{5}} - \frac{2}{\sqrt{5}} \right] x_t \leq 2.5 \right\} \geq 1 - \epsilon_x, \quad (36)
\]

\[
P \left\{ \left[ -\frac{0.4}{\sqrt{1.16}} - \frac{1}{\sqrt{1.16}} \right] u_t \leq 0.4 \right\} \geq 1 - \epsilon_u. \quad (37)
\]

We set \( \epsilon_x = \epsilon_u = 0.05 \) and \( N = 10 \), and plot the trajectories of the state and control that are generated by Algorithm 1 and Algorithm 2 for 36 steps in Fig. 1 with an 100,000-run Monte-Carlo simulation. The red lines represent the boundary of the feasible domain. The blue lines represent the trajectory of the mean and light blue circles represent 95% confidence intervals. We can see that the state trajectory cannot be derived into the origin this is because only 36 steps are not
Figure 2. Violated results of 100,000 sample trajectories

Figure 3. 100 sample trajectories of SMPC implementation

enough. The number of constraint violations are plotted in Fig. 2. As it is shown in Fig. 2, the constraint violations for both of the algorithms are less than 5%, which demonstrates the effectiveness of the proposed algorithms.

In Fig. 3, we plot the state trajectories generated by Algorithm 1 and Algorithm 2 for 100 steps with an 100-run Monte-Carlo simulation. All the state trajectory points are coloured in yellow. We can see that both of the state trajectories are driven to the origin. As expected, all most all of the points satisfy the constraint. This also demonstrates the effectiveness of the proposed algorithms.

4.2. Example 2. In the second example, we compare the performance of the proposed algorithms with the state-of-the-art RMPC method with the problem in [24].
The matrices in (1) are chosen as

\[
A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

The noise is zero mean, and covariance matrix \( \Sigma_w = 0.01I_2 \). The chance constraints are \( \mathbb{P}\{x_2 \geq 2\} \leq \epsilon_x \), and \( \mathbb{P}\{u \geq 1\} \leq 0.1 \). \( Q = I_2, R = 0.01 \), initial condition \( x_0 = (-5,-2)^T \) and \( N = 9 \).

We plot the state trajectory by implementing Algorithm 2 with different \( \epsilon_x \) in Fig. 4. As shown in Fig. 4, the state successfully goes to the origin. As expected, higher probability of violation corresponds to a less conservative performance which corresponds to a trajectory that is closer to the boundary \( x_2 = 2 \). In Fig. 5, the
blue-line is the trajectory generated by the RMPC algorithm in [24], and the red-line and black-line are the trajectories generated by Algorithm 1 and Algorithm 2, respectively. As shown in Fig. 5, we can see that the proposed algorithms outperform the state-of-the-art algorithm [24] since the state trajectories that are generated by using Algorithm 1 and Algorithm 2 converge to the origin faster.

5. Conclusions. Two SMPC algorithms were developed for linear discrete systems with disturbance-feedback. The probability distribution of the disturbance is not fully known. Only the mean and the variance are available. An SDP approximation was obtained by using the Chebyshev–Cantelli inequality and the CVaR approximation. According to the rigorous computational analysis, the CVaR approximation yields a lower iteration computational complexity. Numerical results has shown that the performance of the proposed algorithms were effective and they outperformed the state-of-the-art RMPC method. To further improve the performance, we may consider the distributionally robust approach for future research [11, 12, 16, 17, 20, 19, 32, 18].

REFERENCES

[1] A. Bental and M. Teboulle, Expected Utility, Penalty Functions, and Duality in Stochastic Nonlinear Programming. Mgmt. Science, (2011).

[2] D. Bernardini and A. Bemporad, Scenario-based model predictive control of stochastic constrained linear systems, IEEE Conference on Decision & Control, IEEE, (2009).

[3] G. C. Calafiore and L. Fagiano, Robust model predictive control via scenario optimization, IEEE Transactions on Automatic Control, 58 (2013), 219–224.

[4] M. Cannon, B. Kouvaritakis and D. Ng, Probabilistic tubes in linear stochastic model predictive control, Systems & Control Letters, 58 (2009), 747–753.

[5] W. Chen, M. Sim, J. Sun and C.-P. Teo, From CVaR to uncertainty set: Implications in joint chance-constrained optimization, Ops. Research, 58 (2010), 470–485.

[6] E. Cinquemani, M. Agarwal, D. Chatterjee and J. Lygeros, Convexity and convex approximations of discrete-time stochastic control problems with constraints, Automatica, 47 (2011), 2082–2087.

[7] M. Farina, L. Giulioni and L. Magni, A probabilistic approach to model predictive control, in 52nd IEEE Conference on Decision and Control, IEEE, (2013).

[8] M. Farina, L. Giulioni, L. Magni and R. Scattolini, An approach to output-feedback MPC of stochastic linear discrete-time systems, Automatica, 55 (2015), 140–149.

[9] M. Farina, L. Giulioni and R. Scattolini, Stochastic linear Model Predictive Control with chance constraints - A review, J. of Process Control, 44 (2016), 53–67.

[10] M. Farina and R. Scattolini, Model predictive control of linear systems with multiplicative unbounded uncertainty and chance constraints, Automatica, 70 (2016), 258–265.

[11] Z. H. Gong, C. Y. Liu, K. L. Teo and J. Sun, Distributionally robust parameter identification of a time-delay dynamical system with stochastic measurements, Appl. Math. Modelling, 69 (2019), 685–695.

[12] Z. H. Gong, C. Y. Liu, J. Sun and K. L. Teo, Distributional robust $L_1$-estimation in multiple linear regression, Optim. Letters, 13 (2019), 935–947.

[13] M. Grant and S. Boyd, CVX: Matlab software for disciplined convex programming, version 2.1, (2014). Retrieved from: http://cvxr.com/cvx.

[14] P. Hokayem, D. Chatterjee and J. Lygeros, On stochastic receding horizon control with bounded control inputs: A vector space approach, IEEE Trans. on Automat. Control, 56 (2011), 2704–2710.

[15] P. Hokayem, E. Cinquemani, D. Chatterjee, F. Ramponi and J. Lygeros, Stochastic receding horizon control with output feedback and bounded controls, Automatica, 48 (2012), 77–88.

[16] B. Li, Y. Rong, J. Sun, and K. L. Teo, A distributionally robust linear receiver design for multi-access space-time block coded MIMO systems, IEEE Trans. on Wireless Comms., 16 (2017), 464–474.

[17] B. Li, Y. Rong, J. Sun and K. L. Teo, A distributionally robust minimum variance beamformer design, IEEE Signal Processing Letters, 25 (2018), 105–109.
[18] B. Li, J. Sun, K. L. Teo, C. J. Yu and M. Zhang, A distributionally robust approach to a class of three-stage stochastic linear programs, *Pacific J. of Optim.*, **15** (2019), 219–230.

[19] B. Li, J. Sun, H. L. Xu, and M. Zhang, A class of two-stage distributionally robust stochastic games, *J. of Indust. and Mgmt. Optim.*, **15** (2019), 387–400.

[20] B. Li, Q. Xun, J. Sun, K. L. Teo, and C. J. Yu, A model of distributionally robust two-stage stochastic convex programming with linear recourse, *Appl. Math. Modelling*, **58** (2018), 86–97.

[21] M. S. Lobo, L. Vandenberghe, S. Boyd and H. Lebret, Applications of second-order cone programming, *Linear Algebra and its Appl.*, **284** (1998), 193–228.

[22] L. Magni, G. D. Nicolao and R. Scattolini, Robust model predictive control for nonlinear discrete-time systems, *Int. J. of Robust & Nonlinear Control*, **13** (2003), 229–246.

[23] D. Q. Mayne, J. B. Rawlings, C. V. Rao and P. O. M. Scokaert, Constrained model predictive control: Stability and optimality, *Automatica*, **36** (2000), 789–814.

[24] D. Q. Mayne, M. M. Seron and S. V. Raković, Robust model predictive control of constrained linear systems with bounded disturbances, *Automatica*, **41** (2005), 219–224.

[25] A. Nemirovski and A. Shapiro, Convex approximations of chance constrained programs, *SIAM J. on Optim.*, **17** (2006), 969–996.

[26] J. A. Paulson, E. A. Buehler, R. D. Braatz and A. Mesbah, Stochastic model predictive control with joint chance constraints, *Int. J. of Control*, (2017), 1–14.

[27] S. Qu, Y. Zhou, Y. Zhang, M. I. M. Wahab, G. Zhang and Y. Ye, Optimal strategy for a green supply chain considering shipping policy and default risk, *Comp. & Indust. Engineering*, **131** (2019), 172–186.

[28] D. M. Raimondo, D. Limon and M. Lazar, Min-max model predictive control of nonlinear systems: A unifying overview on stability, *European J. of Control*, **15** (2009), 5–21.

[29] D. R. Ramirez, T. Alamo and E. F. Camacho, Min-Max MPC based on a computationally efficient upper bound of the worst case cost, *J. of Process Control*, **16** (2006), 511–519.

[30] G. Schildbach, P. Goulart and M. Morari, Linear controller design for chance constrained systems, *Automatica*, **51** (2015), 278–284.

[31] M. Y. Shin, Computation in constrained stochastic model predictive control of linear systems, Ph.D dissertation, Stanford University in California, 2011.

[32] Y. F. Sun, G. Aw, B. Li, K. L. Teo and J. Sun, CVaR-based robust models for portfolio selection, Journal of Industrial and Management Optimization, 2018.

[33] D. P. Tesi, *MS Thesis*, Ph.D thesis, University of Pavia in Italy, 2009.

Received February 2019; revised March 2019.

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