On the Product of Real Spectral Triples

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Abstract

The product of two real spectral triples \( \{A_1, \mathcal{H}_1, D_1, J_1, \gamma_1 \} \) and \( \{A_2, \mathcal{H}_2, D_2, J_2(, \gamma_2) \} \), the first of which is necessarily even, was defined by A.Connes [3] as \( \{A, \mathcal{H}, D, J(, \gamma) \} \) given by \( A = A_1 \otimes A_2, \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2, D = D_1 \otimes \text{Id}_2 + \gamma_1 \otimes D_2, J = J_1 \otimes J_2 \) and, in the even-even case, by \( \gamma = \gamma_1 \otimes \gamma_2 \). Generically it is assumed that the real structure \( J \) obeys the relations \( J^2 = \epsilon \text{Id}, JD = \epsilon ' DJ, J\gamma = \epsilon ^" \gamma J \), where the \( \epsilon \)-sign table depends on the dimension \( n \) modulo 8 of the spectral triple. If both spectral triples obey Connes’ \( \epsilon \)-sign table, it is seen that their product, defined in the straightforward way above, does not necessarily obey this \( \epsilon \)-sign table. In this note, we propose an alternative definition of the product real structure such that the \( \epsilon \)-sign table is also satisfied by the product.

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1 Introduction

A real spectral triple \( \mathcal{T} = \{ \mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{J} (\gamma) \} \), is given by a pre-\( C^* \)-algebra \( \mathcal{A} \) with a faithful \(*\)-representation by bounded operators \( \mathcal{B}(\mathcal{H}) \) on a Hilbert space \( \mathcal{H} : \pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H}) : a \rightarrow \pi(a) \). A self-adjoint Dirac operator \( \mathcal{D} \) with compact resolvent acts on this Hilbert space such that \([\mathcal{D}, \pi(a)] \in \mathcal{B}(\mathcal{H})\). The dimension of the spectral triple is given by the integer \( n \) such that the operator \(|\mathcal{D}|^{-n}\) defined on \( \mathcal{H} \setminus \text{Ker}(\mathcal{D}) \) is an infinitesimal of first order. This means that the eigenvalues \( \mu_\alpha \) of the compact positive operator \(|\mathcal{D}|^{-n}\) arranged in decreasing order \( \mu_0 \geq \mu_1 \geq \cdots \) behave asymptotically as \( \mu_\alpha = O(\alpha^{-1}) \) and \( \sigma_n := \sum_{\alpha < N} \mu_\alpha = O(\log N) \), where the sum includes the multiplicities of the eigenvalues. The coefficient of \( \log N \) is, by definition \([1],[5]\), the noncommutative integral of \(|\mathcal{D}|^{-n}\) written as \( \int |\mathcal{D}|^{-n} \). The real structure \( \mathcal{J} \) is an antilinear isometry in \( \mathcal{H} \). It is further assumed that \( \pi^\circ(a) = \mathcal{J}\pi(a)\mathcal{J}^{-1} \) provides a representation of the opposite algebra \( \mathcal{A}^\circ \) commuting with \( \pi(b) \), \( \forall b \in \mathcal{A} \), so that \( \mathcal{H} \) is endowed with an \( \mathcal{A} \)-bimodule structure. The Dirac operator is then assumed to be a first-order operator on this bimodule which entails that \([\mathcal{D}, \pi(a)], \pi^\circ(b) = 0 \). This real structure should further obey the relations :

\[
\mathcal{J}^2 = \epsilon \text{Id} \ ; \ \mathcal{J}\mathcal{D} = \epsilon'\mathcal{D}\mathcal{J} \ ,
\]

(1.1)

where the \textit{epsilon}s are sign factors \( \pm 1 \). Finally, when the dimension is even, there is a grading operator \( \gamma \), i.e. \( \gamma^\dagger = \gamma \) and \( \gamma^2 = \text{Id} \), such that the representation \( \pi(a) \) of \( a \in \mathcal{A} \) is even, \( \pi(a)\gamma - \gamma\pi(a) = 0 \), and the Dirac operator is odd, \( \mathcal{D}\gamma + \gamma\mathcal{D} = 0 \). The real structure \( \mathcal{J} \) and the grading \( \gamma \) obey also a relation of the type :

\[
\mathcal{J}\gamma = \epsilon''\gamma\mathcal{J} \ ,
\]

(1.2)

where again \( \epsilon'' \) is a sign factor \( \pm 1 \).

In the typical commutative example, the algebra \( \mathcal{A} \) consists of the smooth functions on a compact Riemannian spin-manifold \( M \) and the Hilbert space \( \mathcal{H} \) is made of the square integrable spinors \( \Psi(x) \) on \( M \). The representation of \( \mathcal{A} \) on \( \mathcal{H} \) is just the multiplication of \( \Psi(x) \) by functions \( f(x) \). \( \mathcal{D} \) is then the usual (massless!) Dirac operator \( \mathcal{D} = -i\gamma^\mu\nabla_\mu \), where the hermitian \( \gamma \)-matrices obey \( \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = +2\delta^\mu\nu \) and where \( \nabla_\mu \) is the covariant derivative acting on spinor fields. The dimension \( n \) is then the usual dimension of the manifold \( M \) and \( \int |\mathcal{D}|^{-n} \) is proportionnal to the volume of \( M \). The
real structure $J$ generalizes the (Euclidean!) charge conjugation operation $C\Psi(x) = C\Psi^*(x)$, where $C\gamma^\mu C^{-1} = -\gamma^\mu$. When the dimension of $M$ is even, $n = 2k$, the grading is given by the chirality matrix $\gamma_{2k+1} = (i)^k \gamma^1 \gamma^2 \cdots \gamma^{2k}$, anti-commuting with all $\gamma^\mu$ matrices and with the Dirac operator $D$.

Connes [3] showed that the *epsilons*, defined in [1.1], may be determined by the dimension $n$, modulo 8, due to Bott periodicity of the real Clifford algebras. They are given in table 1.

In applications to particle models, one usually takes the product of a commutative spectral triple as above, where $M$ is a configuration space or a Riemannian (Wick rotated) space-time, with a 0-dimensional genuinely non-commutative spectral triple describing the internal structure of the particles. Since nowadays almost any dimension is on the model-building market, it seems useful to examine in general the definition of the product of two real spectral triples $T_1 = \{A_1, H_1, J_1, \gamma_1\}$ and $T_2 = \{A_2, H_2, D_2, J_2(\gamma)\}$. With the first necessarily even, it is defined [3] as $T = \{A, H, D, J(\gamma)\}$ where $A = A_1 \otimes A_2, H = H_1 \otimes H_2, J = J_1 \otimes J_2$ and $D = D_1 \otimes \text{Id}_2 + \gamma_1 \otimes D_2$.

This implies that $D^2 = (D_1)^2 \otimes \text{Id}_2 + \text{Id}_1 \otimes (D_2)^2$ and the dimensions add: $n = n_1 + n_2$. When $T_2$ is also even, the total grading is given by $\gamma = \gamma_1 \otimes \gamma_2$.

In order that this recipe should work, with $T$ obeying the $\epsilon$-sign table, the following conditions should be met:

$$
J^2 = \epsilon \text{Id} , \quad \text{with } \epsilon = \epsilon_1 \epsilon_2 , \quad (1.3)
$$

$$
JD = \epsilon' DJ , \quad \text{with } \epsilon' = \epsilon'_1 \epsilon'_2 , \quad (1.4)
$$

and, when both are even,

$$
J\gamma = \epsilon'' \gamma J , \quad \text{with } \epsilon'' = \epsilon''_1 \epsilon''_2 . \quad (1.5)
$$
A prompt examination of the ε-sign table shows that, in the even-even case, condition (1.5) is satisfied\(^2\). However it is readily seen from table 2 and 3 that at least one of the conditions (1.3), (1.4) is violated, in the even-even case, when \(n_1 \in \{6, 2\}\) and, in the odd case, when the total dimension \(n \in \{5, 1\}\). In the even-even case, we could transform the Dirac operator \(D\) with the unitary operator \(U = \frac{1}{2} (\Id_1 \otimes \Id_2 + \gamma_1 \otimes \Id_2 + \Id_1 \otimes \gamma_2 - \gamma_1 \otimes \gamma_2)\), so that \(D' = U D U^\dagger = D_1 \otimes \gamma_2 + \Id_1 \otimes D_2\). The condition (1.4) is then obeyed in cases \(n_1 \in \{6, 2\}\) and \(n_2 \in \{4, 0\}\) with \(J = J_1 \otimes J_2\) but, when \(n_1 \in \{6, 2\}\) and \(n_2 \in \{6, 2\}\), it is not. In the even-odd case there is even no \(\gamma_2\) available. Also a modification of the individual Dirac operators with the unitary operator \(V = (\Id_1 + i \gamma_1) / \sqrt{2}\), changing \(D_1\) into \(i \gamma_1 D_1\) will not help much in matching condition (1.3). It appears thus that it is \(J\) that has to be modified.

2 The modified product real structure

The clue in changing the real structure of the even spectral triple lies in the property of \(\tilde{J}_1 = J_1 \gamma_1\) which is such that

\[(\tilde{J}_1)^2 = \tilde{\epsilon}_1 \Id_1 \quad (2.1)\]

where \(\tilde{\epsilon}_1 = \epsilon_1 \epsilon_1''\) remains unchanged for \(n_1 \in \{4, 0\}\) but changes sign for \(n_1 \in \{6, 2\}\). Furthermore

\[\tilde{J}_1 D_1 = \tilde{\epsilon}_1' D_1 , \quad (2.2)\]

with \(\tilde{\epsilon}_1' = -\epsilon_1'\). Finally, since \(\epsilon_1''\) does not change, condition (1.3) remains satisfied. The even-odd cases with \(n \in \{5, 1\}\) are readily cured defining the product real structure as \(J = \tilde{J}_1 \otimes J_2 = J_1 \gamma_1 \otimes J_2\) as table 4 shows. In the even-even case, when \(n_1 \in \{6, 2\}\), the \(\epsilon_1'\) should not change sign since, for all even \(n\), \(\epsilon' = \epsilon_1' = +1\). To recover the original + sign we multiply by \(\gamma = \gamma_1 \otimes \gamma_2\) so that the real structure of the product reads \(J = (J_1 \gamma_1 \otimes J_2) (\gamma_1 \otimes \gamma_2) = J_1 \otimes J_2 \gamma_2\). The \(n_1 \in \{6, 2\}\) cases will then, with this real structure, obey Connes’ ε-sign table as the table 5 shows.

\(^2\)There is a more sophisticated proof \([4]\) available, invoking the image under \(\pi_D\) of the Hochschild \(n_k\)-cycles, instead of the mere inspection of the ε-sign table.
3 Conclusions

In this short note, we have redefined, by elementary algebraic techniques, the real structure of the product of two real spectral triples such that Connes’ $\epsilon$-sign table remains valid for the product if it holds for each factor. This is achieved taking as real structure $\mathcal{J}$ given by:

- $\mathcal{J} = \mathcal{J}_1 \gamma_1 \otimes \mathcal{J}_2$ when $n_1 + n_2 = n \in \{5, 1\}$,
- $\mathcal{J} = \mathcal{J}_1 \otimes \mathcal{J}_2 \gamma_2$ when $n_1 \in \{6, 2\}$ and $n_2$ even
- $\mathcal{J} = \mathcal{J}_1 \otimes \mathcal{J}_2$ in all other cases.

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Table 1: The $\epsilon$-sign table

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----|---|---|---|---|---|---|---|---|
| $\epsilon$ | + | + | - | - | - | + | + | |
| $\epsilon'$ | + | - | + | + | + | - | + | + |
| $\epsilon''$ | + | * | - | * | * | - | * | |

Table 2: The even-even case

| $n_1$ | $n_2$ | $n$ | $\epsilon_1$ | $\epsilon_2$ | $\epsilon_1 \epsilon_2$ | $\epsilon$ | $\epsilon'_1$ | $\epsilon'_2$ | $\epsilon''_1$ | $\epsilon''_2$ | $\epsilon'$ | $?_1$ | $?_2$ |
|-------|-------|-----|-------------|-------------|------------------|--------|------------|------------|--------------|--------------|----------|--------|-------|
| 6     | 6     | 4   | +           | +           | +                | -      | N          | +          | -            | +            | N        | +      | +     |
| 6     | 4     | 2   | +           | -           | -                | -      | Y          | +          | -            | +            | N        | +      | +     |
| 6     | 2     | 0   | +           | -           | -                | +      | N          | +          | -            | +            | N        | +      | +     |
| 6     | 0     | 6   | +           | +           | +                | +      | Y          | +          | -            | +            | N        | +      | +     |
| 4     | 6     | 2   | -           | +           | -                | -      | Y          | +          | +            | +            | Y        | +      | +     |
| 4     | 4     | 0   | -           | -           | +                | +      | Y          | +          | +            | +            | Y        | +      | +     |
| 4     | 2     | 6   | -           | -           | +                | +      | +          | +          | +            | +            | Y        | +      | +     |
| 4     | 0     | 4   | -           | +           | -                | -      | Y          | +          | +            | +            | Y        | +      | +     |
| 2     | 6     | 0   | -           | +           | -                | +      | N          | +          | -            | +            | N        | +      | +     |
| 2     | 4     | 6   | -           | +           | +                | -      | +          | -          | +            | +            | N        | +      | +     |
| 2     | 2     | 4   | -           | -           | +                | -      | N          | +          | -            | +            | N        | +      | +     |
| 2     | 0     | 2   | -           | +           | -                | -      | Y          | +          | -            | +            | N        | +      | +     |
| 0     | 6     | 6   | +           | +           | +                | +      | Y          | +          | +            | +            | Y        | +      | +     |
| 0     | 4     | 4   | +           | -           | -                | -      | Y          | +          | +            | +            | Y        | +      | +     |
| 0     | 2     | 2   | +           | -           | -                | Y      | +          | +          | +            | +            | Y        | +      | +     |
| 0     | 0     | 0   | +           | +           | +                | +      | Y          | +          | +            | +            | Y        | +      | +     |
Table 3: The even-odd case

| $n_1$ | $n_2$ | $n$ | $\epsilon_1$ | $\epsilon_2$ | $\epsilon_1 \epsilon_2$ | $\epsilon$ | $\epsilon_1'$ | $\epsilon_1''$ | $\epsilon_2'$ | $\epsilon_1'' \epsilon_2'$ | $\epsilon'$ | $\epsilon'$ ? |
|-------|-------|-----|------------|------------|----------------|--------|-----------|-----------|-----------|----------------|--------|---------|
| 6     | 7     | 5   | +          | +          | +               | -      | N         | +         | -         | +               | -      | N       |
| 6     | 5     | 3   | +          | -          | -               | -      | Y         | +         | -         | -               | +      | + Y     |
| 6     | 3     | 1   | +          | -          | -               | +      | N         | +         | -         | +               | -      | N       |
| 6     | 1     | 7   | +          | +          | +               | +      | Y         | +         | -         | +               | +      | Y       |
| 4     | 7     | 3   | -          | +          | -               | -      | Y         | +         | +         | +               | -      | Y       |
| 4     | 5     | 1   | -          | -          | +               | +      | Y         | +         | -         | +               | -      | Y       |
| 4     | 3     | 7   | -          | -          | +               | +      | Y         | +         | +         | -               | +      | Y       |
| 4     | 1     | 5   | -          | +          | -               | -      | Y         | +         | -         | +               | -      | N       |
| 2     | 7     | 1   | -          | +          | -               | +      | N         | +         | -         | -               | -      | N       |
| 2     | 5     | 7   | -          | -          | +               | +      | Y         | +         | -         | +               | +      | Y       |
| 2     | 3     | 5   | -          | -          | +               | -      | N         | +         | -         | +               | -      | Y       |
| 2     | 1     | 3   | -          | +          | -               | -      | Y         | -         | -         | +               | +      | Y       |
| 0     | 7     | 7   | +          | +          | +               | +      | Y         | +         | +         | +               | +      | Y       |
| 0     | 5     | 5   | +          | -          | -               | +      | Y         | +         | -         | -               | N      | N       |
| 0     | 3     | 3   | +          | -          | -               | Y      | +         | +         | +         | +               | +      | Y       |
| 0     | 1     | 1   | +          | +          | +               | +      | Y         | +         | -         | -               | -      | N       |

Table 4: The cured even-odd case

| $n_1$ | $n_2$ | $n$ | $\tilde{\epsilon}_1$ | $\tilde{\epsilon}_2$ | $\tilde{\epsilon}_1 \tilde{\epsilon}_2$ | $\epsilon$ | $\tilde{\epsilon}_1'$ | $\tilde{\epsilon}_1''$ | $\epsilon_2'$ | $\epsilon_1'' \epsilon_2'$ | $\epsilon'$ | $\epsilon'$ ? |
|-------|-------|-----|---------------------|---------------------|----------------------|--------|----------------|----------------|----------------|----------------------|--------|---------|
| 6     | 7     | 5   | -                   | +                   | -                   | Y      | -              | +              | -              | -                   | -      | Y       |
| 6     | 3     | 1   | -                   | -                   | +                   | +      | -              | +              | -              | -                   | -      | Y       |
| 4     | 5     | 1   | -                   | -                   | +                   | +      | Y              | -              | +              | +                   | -      | Y       |
| 4     | 1     | 5   | -                   | +                   | -                   | Y      | -              | +              | -              | +                   | -      | Y       |
| 2     | 7     | 1   | +                   | +                   | +                   | Y      | -              | +              | -              | -                   | -      | Y       |
| 2     | 3     | 5   | +                   | -                   | -                   | Y      | -              | +              | -              | -                   | -      | Y       |
| 0     | 5     | 5   | +                   | -                   | -                   | Y      | -              | +              | -              | -                   | -      | Y       |
| 0     | 1     | 1   | +                   | +                   | +                   | Y      | +              | -              | -              | -                   | -      | Y       |
Table 5: The cured even-even cases

| \(n_1\) | \(n_2\) | \(n\) | \(\epsilon_1\) | \(\epsilon_2\) | \(\tilde{\epsilon}_1\) | \(\tilde{\epsilon}_2\) | \(\epsilon\) | \(\epsilon'_1\) | \(\epsilon''_1\) | \(\epsilon'_2\) | \(\epsilon''_2\) | \(\epsilon'\) | \(\epsilon''\) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 6 | 6 | 4 | + | - | - | - | Y | + | - | - | + | + | Y |
| 6 | 4 | 2 | + | - | - | - | Y | + | - | - | + | + | Y |
| 6 | 2 | 0 | + | + | + | + | Y | + | - | - | + | + | Y |
| 6 | 0 | 6 | + | + | + | + | Y | + | - | - | + | + | Y |
| 2 | 6 | 0 | - | - | + | + | Y | + | - | - | + | + | Y |
| 2 | 4 | 6 | - | - | + | + | Y | + | - | - | + | + | Y |
| 2 | 2 | 4 | - | + | - | - | Y | + | - | - | + | + | Y |
| 2 | 0 | 2 | - | + | - | - | Y | + | - | - | + | + | Y |