Thermoelectric properties of an ultra-thin topological insulator

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Abstract
Thermoelectric coefficients of an ultra-thin topological insulator are presented here. The hybridization between top and bottom surface states of a topological insulator plays a significant role. In the absence of a magnetic field, the thermopower increases and thermal conductivity decreases with an increase in the hybridization energy. In the presence of a magnetic field perpendicular to the ultra-thin topological insulator, thermoelectric coefficients exhibit quantum oscillations with inverse magnetic field, whose frequency is strongly modified by the Zeeman energy and whose phase factor is governed by the product of the Landé g-factor and the hybridization energy. In addition to the numerical results, the low-temperature approximate analytical results for the thermoelectric coefficients are also provided. It is also observed that for a given magnetic field these transport coefficients oscillate with hybridization energy, at a frequency that depends on the Landé g-factor.

Keywords: thermoelectric, thin topological insulator, quantum transport

1. Introduction
Recently a new class of material, called a topological insulator, has had much attention from condensed matter physicists [1–6]. A topological insulator (TI) shows the conduction of electrons on the surface of 3D materials but otherwise behaves as an insulator. This is due to the time-reversal symmetry possessed by materials such as Bi2Se3, Sb2Te3 and Bi2Te3 [6]. The conducting surface states of these materials show a single Dirac cone, in which spin is always locked perpendicular to its momentum. Angle resolved photoemission spectroscopy [7–9] or scanning tunneling microscopy [10] has been used to realize the single Dirac cone in TIs. In two-dimensional electron systems, under the presence of a perpendicular magnetic field, electrons conduct along the boundary due to the circular orbits bouncing off the edges, leading to skipping orbits. However, in 3D materials, even in the absence of magnetic field electron conduction takes place on the surface. Here, strong Rashba spin–orbit coupling (RSOC) plays the role of magnetic field. The RSOC originates from the lack of structural inversion symmetry of the sample [11, 12].

Though there have been several experimental works on the surface states of TIs, one of the major obstacles in studying the transport properties of the surface is the unavoidable contribution of the bulk. One of the best methods to minimize this problem is to grow TI samples in the form of ultra-thin films, in which the bulk contribution becomes very small in comparison to the surface contribution [13–15]. The transition from 3D to 2D TIs leads to several effects, which have been studied for different thickness [15, 16]. The ultra-thin TI not only reduces the bulk contribution, but also possesses some new phenomena such as possible excitonic superfluidity [17], unique magneto-optical response [18–20] and better thermoelectric performances [21]. Moreover, the small thickness leads to the overlap of the wave functions between top and bottom surfaces, which introduces a new degree of freedom hybridization [22, 23]. However, it happens to a certain thickness of five to ten quintuple layers [15, 24], i.e. of the order of 10 nm. The oscillating exponential decay of hybridization induced band gap with reducing thickness in Bi2Te3 has been also reported theoretically [25]. The formation of Landau levels has been confirmed by several experiments [26, 27] in thin TIs. Moreover, several theoretical studies on
low-temperature transport properties have been reported in a series of works [24, 28–31].

Thermoelectric properties of materials [32] have always been interesting topic for providing an additional way to explore the details of an electronic system. When a temperature gradient is applied across the two ends of the electronic system, the migration of electrons from the hotter to the cooler side leads to the development of a voltage gradient across these ends. This voltage difference per unit temperature gradient is known as longitudinal thermopower. In addition to this temperature gradient, if a perpendicular magnetic field is applied to the system, due to Lorentz force a transverse electric field is also established and gives transverse thermopower. In a conventional 2D electronic system, Landau level induced quantum oscillation (Shubnikov–de Haas) in thermoelectric coefficients has been reported theoretically as well as experimentally in a series of works [33–37]. In 3D TIs, improvement of thermoelectric performance without magnetic field has been predicted theoretically in a series of paper [38–41]. In the newly emerged relativistic-like 2D electron system graphene, thermoelectric effects have also been studied [42–45].

In this paper, we study the effect of hybridization on the thermopower and the thermal conductivity of ultra-thin TIs in the absence/presence of magnetic field. We find that thermopower increases and thermal conductivity decreases with increase of the hybridization energy when magnetic field is absent. In the presence of perpendicular magnetic field, thermoelectric coefficients oscillate with inverse magnetic field. The frequency of the quantum oscillations is strongly modified by the Zeeman energy, and phase factor is determined by the product of the Landé g-factor and the hybridization energy. The analytical expressions of the thermoelectric coefficients are also obtained. It is also shown that these transport coefficients oscillate with frequency depending on hybridization energy and Landé g-factor.

This paper has the following structure. Section 2 briefly discusses the energy spectrum and the density of states of an ultra-thin TI in the absence and presence of magnetic field. In section 3, we study how the hybridization affects the thermoelectric coefficients in zero magnetic field. In section 4, a complete analysis of thermoelectric coefficients in the presence of magnetic field is provided with numerical and analytical results. We provide a summary and conclusion of our work in section 5.

2. Energy spectrum and density of states

2.1 Zero magnetic field case

Let us consider a surface of an ultra-thin TI in the xy-plane with \( L_x \times L_y \) dimensions, where the carriers are Dirac fermions occupying the top and bottom surfaces of the TI. The quantum tunneling between top and bottom surfaces gives rise to hybridization, and consequently the Hamiltonian can be written as the symmetric and anti-symmetric combination of both surface states as [22]

\[
H = \begin{bmatrix}
    h(k) & 0 \\
    0 & h^*(k)
\end{bmatrix}
\]  

with \( h(k) = \Delta_0 \sigma_z + v_F(p_x \sigma_x - p_y \sigma_y) \). Here \( p \) is the two-dimensional momentum operator, \( v_F \) is the Fermi velocity of the Dirac fermion, \( \sigma = (\sigma_x, \sigma_y, \sigma_z) \) are the Pauli spin matrices and \( \Delta_0 \) is the hybridization matrix element between the states of the top and bottom surfaces of the TI. A typical value of \( \Delta_0 \) varies from 0 to \( 10^2 \text{ meV} \) depending on the thickness of the 3D TI [15]. Because of the block-diagonal nature, the above Hamiltonian can be written as

\[
H = v_F(\sigma_x p_x - \sigma_y p_y) + \Delta_0 \sigma_z,
\]  

where \( \sigma_x, \sigma_y, \sigma_z \) denotes symmetric and anti-symmetric surface states, respectively. The energy spectrum of the Dirac electron is given by

\[
E = \lambda \sqrt{(h v_F k)^2 + \Delta_0^2}.
\]  

Here \( \lambda = \pm \) stands for electron and hole bands. The density of states is given by

\[
D_0(E) = \frac{2E}{\pi h^2 v_F^2}.
\]  

2.2 Non-zero magnetic field case

In the presence of magnetic field perpendicular to the surface, the Hamiltonian for a Dirac electron with hybridization is

\[
H = v_F(\sigma_x p_x - \sigma_y p_y) + (\tau_z \Delta_x + \Delta_0) \sigma_z,
\]  

where \( \Pi = p + eA \) is the two-dimensional canonical momentum operator. Using Landau gauge \( A = (0, Bx, 0) \), exact Landau levels can be obtained very easily [28, 31]. For \( n = 0 \), there is only one energy level, which is given as \( E_0 = -\Delta_0 + \tau_z \Delta_x \). When integer \( n \geq 1 \), there are two energy bands denoted by \( + \) corresponding to the electron and \( - \) corresponding to the hole with energy

\[
E_{n,\pm} = \lambda \sqrt{2n(h v_F k)^2 + (\Delta_x + \tau_z \Delta_0)^2},
\]  

where \( \omega_c = v_F l \) is the cyclotron frequency with \( l = \sqrt{\hbar / (eB)} \) the magnetic length, and \( \Delta_0 = g \mu_B B / 2 \) with \( g \) the Landé g-factor.

The corresponding eigenstates for a symmetric surface state are

\[
\Psi_n^+(r) = \frac{e^{ik_y y}}{L_y} \begin{pmatrix}
    c_1 \phi_{n-1}(x + x_0) \\
    c_2 \phi_n(x + x_0)
\end{pmatrix},
\]  

\[
\Psi_n^-(r) = \frac{e^{ik_y y}}{L_y} \begin{pmatrix}
    c_1 \phi_{n-1}(x + x_0) \\
    -c_2 \phi_n(x + x_0)
\end{pmatrix},
\]  

where \( \phi_n(x) = (1 / \sqrt{\sqrt{\pi} 2^n n!}) e^{-x^2 / 2} H_n(x / l) \) is the normalized harmonic oscillator wave function, \( x_0 = \pm k_y^2, c_1 = \cos(\theta_n / 2) \) and \( c_2 = \sin(\theta_n / 2) \) with \( \theta_n = \tan^{-1} [\sqrt{\hbar v_F l} / (\Delta_x + \tau_z \Delta_0)] \). The anti-symmetric surface state can be obtained by exchanging \( n \) and \( n - 1 \).

We have derived an approximate analytical form of the density of states for \( n > 1 \) by using the Green’s function technique, which is given as (see the appendix)
where $\Delta_\tau = \Delta_\tau + \tau_0$ and $\Gamma_0$ is the impurity induced Landau level broadening.

3. Thermoelectric coefficients

In this section, we shall calculate thermoelectric coefficients of an ultra-thin TI in zero and non-zero magnetic fields.

3.1 Zero-magnetic field case

In this sub-section, the effect of hybridization on thermopower and thermal conductivity is presented. We follow the most conventional approach in the low temperature regime. The electrical current density $\mathbf{J}$ and the thermal current density $\mathbf{J}_\theta$ for Dirac electrons can be expressed under the linear response regime as

$$\mathbf{J} = Q^1\mathbf{E} + Q^2(-\nabla T)$$  \hspace{1cm} (10)$$

and

$$\mathbf{J}' = Q^3\mathbf{E} + Q^2(-\nabla T),$$  \hspace{1cm} (11)

where $\mathbf{E}$ is the electric field, $\nabla T$ is the temperature gradient and $Q^j$ ($i, j = 1, 2$) are the phenomenological transport coefficients. The above equations describe the response of the electronic system under the combined effects of thermal and potential gradients. Moreover, $Q^j$ can be expressed in terms of an integral $I^j$: $Q^1 = I^0$, $Q^2 = TQ^1 = -I^0/e$, $Q^2 = I^2/\epsilon T$ with

$$I^j = \int dE \left[ -\frac{\partial f(E)}{\partial E} \right](E-\eta)^j \sigma(E),$$  \hspace{1cm} (12)

where $\eta = 0, 1, 2$ and $f(E) = 1/[1 + \exp(E - \eta)\beta]$ is the Fermi–Dirac distribution function with $\eta$ the chemical potential and $\beta = (k_B T)^{-1}$. Here, $\sigma(E)$ is the energy-dependent electrical conductivity. When the circuit is open, i.e. for $J = 0$, the thermopower can be defined as $S = Q^1/Q^2$. By using the Sommerfeld expansion in the low temperature regime, the diffusive thermopower $S$ and thermal conductivity $\kappa$ can be obtained from Mott’s relation and the Wiedemann–Franz law as

$$S = -L_0\sigma T \left[ \frac{d}{dE} \ln(\sigma(E)) \right]_{E=E_F},$$  \hspace{1cm} (13)

and

$$\kappa = L_0 T \sigma(E_F).$$  \hspace{1cm} (14)

Here, $L_0 = (\pi^2 k_B^2)/(3e^2) = 2.44\times10^{-8}$ W/K$^2$ is the Lorentz number and $\sigma(E_F)$ is the electrical conductivity at the Fermi energy.

The classical Boltzmann transport equation can be used to calculate the zero magnetic field electrical conductivity, which is given as [49]

$$\sigma_\eta(E) = e^2\tau(E) \int \frac{d^2k}{(2\pi)^2} \delta(E-E(k)) \nu_i(k)\nu_j(k),$$  \hspace{1cm} (15)

where $i, j = x, y$. For an isotropic system $\nu_i^2 = (1/2)(\nu_x^2 + \nu_y^2) = (1/2)v^2$. In our case,

$$v^2 = \left[ 1 - \left( \frac{\Delta_\tau}{E} \right)^2 \right].$$  \hspace{1cm} (16)

Using these in equation (15), the energy dependent conductivity becomes

$$\sigma(E) = e^2\tau(E) \frac{E}{\pi k_B^2} \left[ 1 - \left( \frac{\Delta_\tau}{E} \right)^2 \right].$$  \hspace{1cm} (17)

Assuming the energy dependent scattering time to be

$$\tau = n_0(E/E_F)^m,$$

where $m$ is a constant depending on the scattering mechanism, $E_F = \sqrt{E_F^0 + \Delta_\tau}$ is the Fermi energy with $E_F^0 = \hbar v_F k_F^0$. Here, the Fermi vector $k_F^0 = \sqrt{2m_e\nu}$. Substituting equation (17) into equation (13), the diffusion thermopower is obtained as

$$S = -L_0\sigma T \frac{e^2E}{E_F^0} \left[ (m + 1) + 2\left( \frac{\Delta_\tau}{E_F^0} \right)^2 \right] \left[ 1 + \left( \frac{\Delta_\tau}{E_F^0} \right)^2 \right],$$  \hspace{1cm} (18)

We plot thermopower versus hybridization for different carrier densities in the upper panel of figure 1. It shows that thermopower increases with increasing hybridization for a particular carrier density. However, for higher carrier density, this rate of enhancement with hybridization becomes very slow.

Thermal conductivity can be directly obtained from the Wiedemann–Franz law given in equation (14), where the electrical conductivity $\sigma(E_F)$ is given as

$$\sigma = \sigma_0 \sqrt{1 + \left( \frac{\Delta_\tau}{E_F^0} \right)^2}.$$  \hspace{1cm} (19)

}\text{Figure 1.} Plots of the thermopower versus hybridization constant for $m = 1$ and for various carrier densities.
3.2 Non-zero magnetic field case

In the presence of a magnetic field, the classical approach cannot explain the phenomenon depending on energy quantization. In this sub-section we follow a quantum mechanical approach, based on linear response theory, to study thermal transport coefficients. Thermoelectric coefficients for a two-dimensional electron system in the presence of a magnetic field were derived by modifying the Kubo formula in \[46, 47\].

These phenomenological transport coefficients are

\[\sigma_{\mu\nu} = \mathcal{L}_{\mu\nu}^{(0)} \tag{20}\]
\[S_{\mu\nu} = \frac{1}{eT} \left( \mathcal{L}_{\mu\nu}^{(0)} \right)^{-1} \mathcal{L}_{1\nu} \tag{21}\]
\[\kappa_{\mu\nu} = \frac{1}{eT} \left[ \mathcal{L}_{\mu\nu}^{(2)} - eT \left( \mathcal{L}_{\mu\nu}^{(1)} S_{\mu\nu} \right) \right], \tag{22}\]

where

\[\mathcal{L}_{\mu\nu}^{(\nu)} = \int dE \left[ - \frac{\partial f(E)}{\partial E} \right] (E-\eta)^{\nu} \sigma_{\mu\nu}(E). \tag{23}\]

Here, \(\mu, \nu = x, y\). Also, \(\sigma_{\mu\nu}(E)\), \(S_{\mu\nu}\) and \(\kappa_{\mu\nu}\) are the zero-temperature energy-dependent conductivity, thermopower and thermal conductivity tensors, respectively.

Generally, diffusive and collisional mechanisms play a major role in electron conduction. The quantized energy spectrum of electrons reveals itself through Shubnikov–de Haas oscillation by the collisional mechanism. In our case, electron transport is mainly due to the collisional instead of the diffusive mechanism. The zero drift velocity of the electrons does not allow us to have a diffusive contribution. In the presence of a temperature gradient, thermal transport coefficients can be expressed as \(\mathcal{L}_{\mu\nu}^{(\nu)} = \mathcal{L}_{\mu\nu}^{(\text{col})} = \mathcal{L}_{\mu\nu}^{(\text{col})} \) and \(\mathcal{L}_{\mu\nu}^{(\nu)} = \mathcal{L}_{\mu\nu}^{(\text{diff})} + \mathcal{L}_{\mu\nu}^{(\text{col})} = \mathcal{L}_{\mu\nu}^{(\text{col})} \). In \[31\], the exact form of the finite-temperature collisional conductivity has been calculated for the screened impurity potential \(U(k) = 2\pi e^2 / (\epsilon \sqrt{k^2 + k_s^2}) \approx 2\pi e^2 / (\epsilon k_s) = U_0 \) under the limit of small \(|k| \ll k_s\), with \(k_s\) and \(\epsilon\) being the inverse screening length and dielectric constant of the material, respectively. In this limit, one can use \(\tau_s^2 \approx \pi \hbar^2 / N_l U_0^2\), where \(\tau_s\) is the relaxation time, \(U_0\) is the strength of the screened impurity potential and \(N_l\) is the two-dimensional impurity density. The exact form of the finite-temperature conductivity \[31\] can be reduced to the zero-temperature energy-dependent electrical conductivity as

\[\sigma_{\mu\nu}(E) = \frac{e^2 N_l U_0^2}{\hbar T^2} \sum \frac{l_{\nu}}{l_{\nu} - \epsilon}, \tag{24}\]

where \(l_{\nu} = \left[ n \left[1 + \cos^2(\theta_i)\right] - \cos(\theta_i) \right] \). Here we have used \(-\partial f / \partial E = \delta(E - E_{\nu})\). Using equation \(23\), the finite-temperature diagonal (\(\mathcal{L}_{\mu\nu}^{(\nu)}\)) and off-diagonal coefficients (\(\mathcal{L}_{\mu\nu}^{(\nu)}\)) can be written as

\[\mathcal{L}_{\mu\nu}^{(\nu)} = \frac{e^2 N_l U_0^2}{\hbar T^2} \sum \frac{l_{\nu}}{l_{\nu} - \epsilon}, \tag{25}\]

and

\[\mathcal{L}_{\mu\nu}^{(\nu)} = \frac{e^2}{\hbar} \sum \frac{\sin^2 \theta_i}{\Delta_n} \int_{E_{\nu}}^{E_{\nu+1}} (E-\eta)^{\nu} \left( - \frac{\partial f(E)}{\partial E} \right) dE. \tag{26}\]

Here,

\[\Delta_n = \sqrt{2n + \left( \frac{\Delta_\epsilon}{\hbar \omega_c} \right)^2} - \sqrt{2(n+1) + \left( \frac{\Delta_\epsilon}{\hbar \omega_c} \right)^2}. \]

4. Numerical results and discussion

In our numerical calculations, the following parameters are used: carrier concentration \(n_c = 2 \times 10^{15} \text{ m}^{-2}\), \(g = 60\), \(v_p = 4 \times 10^5 \text{ m s}^{-1}\) and \(T = 0.7 \text{ K}\). These numerical parameters are consistent with \[15, 30, 31\].
In figure 2, the off-diagonal thermopower, $S_{xy}$, is shown as a function of the inverse magnetic field for different values of the hybridization constant. Similarly, the thermal conductivity, $\kappa_{xx}$, is shown versus inverse magnetic field for different values of the hybridization constant in figure 3. Careful observation of these two figures clearly shows that $S_{xy}$ and $\kappa_{xx}$ oscillate with the same frequency, which does not depend on the hybridization energy. The hybridization energy only influences the phase of oscillations.

To determine the frequency and the phase of the quantum oscillations in the thermoelectric coefficients, we shall derive analytical expressions of these coefficients. The components of the thermopower are given by

$$S_{xx} = S_{yy} = \frac{1}{eT} \left[ \frac{\sigma_{xx} \mathcal{L}^{(1)}_{xx} + \mathcal{L}^{(1)}_{xy}}{\sigma_{xy}} \right]$$

and

$$S_{xy} = -S_{yx} = \frac{1}{eT} \left[ \frac{\sigma_{xx} \mathcal{L}^{(1)}_{xy} + \mathcal{L}^{(1)}_{xx}}{\sigma_{xy}} \right].$$

Here, $S_0 = \sigma_{xx} \sigma_{yy} - \sigma_{xy} \sigma_{yx}$. The dominating term in the above two equations is the last term. The analytical form of $\kappa_{xx}$ and $S_{xy}$ can be obtained directly by deriving the analytical form of the phenomenological transport coefficients. The analytical form of the density of states given in equation (9) allows us to obtain asymptotic expressions of $S_{xy}$ and $\kappa_{xx}$. This is done by replacing the summation over discrete quantum numbers $n$ by the integration, i.e. $\sum_n \to 2\pi^2 \int D(E) dE$; then we get

$$\mathcal{L}^{(1)}_{xx} \approx \frac{4\pi e^2}{\beta h} \frac{\Gamma_0 \Gamma_D}{(\hbar \omega_c)^2} \Omega_0 G' \left( \mathcal{L}^{(1)}_{xy} \right) \sum \frac{u_i f_i \sin(2\pi f_i)}{\tau_i}$$

and

$$\mathcal{L}^{(2)}_{xx} \approx \frac{L_0 \Phi^2 \gamma_0 \sigma_0}{(\omega_c \tau_0)^2} \sum \frac{u_i f_i [1 - 3\Omega_0 G'(x) \cos(2\pi f_i)]}{\tau_i}.$$

where $\mathcal{L}_{ij} = \frac{(E_i^2 - \Delta_0^2)}{\sqrt{2} \hbar \omega_c}$, $U_0 = [1 + \cos^2(\theta_c)]$, the impurity induced damping factor is

$$\Omega_D = \exp \left\{ -\left( \frac{2\pi \Gamma_0 E_F}{\hbar \omega_c} \right)^2 \right\}$$

and the temperature dependent damping factor is the derivative of the function $G(x)$ with $G(x) = x \sin(\theta_c)$. Here, $x = T \mathcal{T}_\lambda$, where $T_\lambda = (\hbar \omega_c)^2 / (2\pi^2 k_B E_F)$ is the critical temperature, which depends on the strength of hybridization through the Fermi energy. Note that $G(x)$ is the temperature dependent damping factor for the electrical conductivity tensor.

The off-diagonal thermopower $S_{xy}$ and the diagonal thermal conductivity $\kappa_{xx}$ are obtained as given by

$$S_{xy} \approx \frac{k_B}{e} \frac{1}{\hbar \omega_c \tau_0} \frac{16\pi}{1 + \left( \frac{\Delta_0}{E_F} \right)^2} \Omega_D G'(x)$$

and

$$\kappa_{xx} \approx L_0 \frac{\sigma_0}{(\omega_c \tau_0)^2} \sum \frac{u_i f_i}{\tau_i} \left[ 1 - 6\Omega_0 G''(x) \cos(2\pi f_i) \right].$$

where the frequency $f$ is given as

$$f = \frac{1}{2e\hbar \nu} \left( E_F^2 - \Delta_0^2 \right)$$

and the phase factor $\varphi = \pi \nu \mu_B \Delta_0 / (e \hbar \nu f)$. Therefore, the thermopower and the thermal conductivity oscillate with the same frequency $f$, which is independent of $\Delta_0$, which can also be shown from numerical results. The oscillation frequency is strongly reduced by the Zeeman energy $\Delta_0$. On the other hand, the phase factor $\varphi$ is related to the product of the Landé g-factor and $\Delta_0$, and it vanishes if either of them is zero. Although the frequency and the phase of $S_{xy}$ and $\kappa_{xx}$ are the same, the damping factor and amplitude are different.
Now we compare the numerical and analytical results for \( s_{xy} \) and \( \kappa_{xx} \) in figure 4. For better visualization, we have taken weak Landau level broadening \( \Gamma_0 = 0.01 \text{ meV} \) for \( s_{xy} \) and \( \kappa_{xx} \). The analytical results, in particular the frequency \( f \), match very well with the numerical results. We must mention here that for different values of \( \Gamma_0 \) analytical results may differ from numerical ones in the amplitude, but the frequency and phase are always in good agreement.

It is interesting to see from the analytical expressions of the thermopower and the thermal conductivity that these transport coefficients possess weak periodic oscillation with the hybridization energy for a given magnetic field. This oscillation is shown in figure 5. The frequency and phase factor of these oscillations for a fixed \( B \) are \( \nu = \phi f / (2\pi) = g \mu_B f / (2e\hbar \gamma) \) and \( \Phi = 2\pi f/B \), respectively.

In the presence of the magnetic field, these approximate analytical expressions of the thermopower and thermal conductivity can also be used for monolayer graphene by putting \( \Delta_h = 0 \). There are several experimental results [42, 48] for thermoelectric properties of a graphene monolayer, but analytical expressions are not available in the literature.

5. Conclusion

We have presented a theoretical study of the thermoelectric coefficients of ultra-thin topological insulators in the presence/absence of magnetic field. In the absence of magnetic field, the thermopower and the thermal conductivity are modified due to the hybridization between top and bottom surface states. The thermopower is enhanced and the thermal conductivity is diminished due to the hybridization. The quantum oscillations in the thermopower and the thermal conductivity for different values of the hybridization constant are also studied numerically. In addition to the numerical results, we obtained the analytical expressions for the thermopower \( (s_{xy}) \) and the thermal conductivity \( (\kappa_{xx}) \). The analytical results match very well with the numerical results. We have also provided analytical expressions for the oscillation frequency and phase. The oscillation frequency is the same for both the thermopower and the thermal conductivity. It is independent of the hybridization constant but strongly suppressed by the Zeeman energy. On the other hand, the hybridization constant plays a very significant role in the phase as well as in the amplitude of the oscillations. From the analytical results, critical temperature \( (T_c) \) is found to be reduced with increasing hybridization constant. Thermoelectric coefficients also show a very low-frequency oscillation with the hybridization constant for a given magnetic field. Moreover, our analytical expressions of the thermopower and thermal conductivity are also applicable for a graphene monolayer by setting \( \Delta_h = 0 \).

Appendix

The derivation of asymptotic analytical expression of density of states of a two-dimensional electronic system in the presence of impurities can be done by calculating the self-energy [50, 51], which is given as

\[
\Sigma^-(E) = \Gamma_0 \sum_n \frac{1}{E - E_n^\epsilon - \Sigma^-(E)}. \quad \text{(A.1)}
\]

The imaginary part of the self-energy is related to the density of states as \( D(E) = \text{Im} \left[ \frac{\Sigma(E)}{\pi E^2 \Gamma_0^2} \right] \).

By using the residue theorem, we calculate the summation in equation (A.1), which gives

\[
\Delta + \frac{i}{2} = \frac{2\pi \Gamma_0^2 E}{(\hbar \omega_c)^2} \cot \left( \frac{(u - i v)}{2} \right). \quad \text{(A.3)}
\]

Here, \( u = \frac{\pi}{(\hbar \omega_c)^2} E^2 - (\Delta_s + \tau, \Delta_h)^2 \) and \( v = \frac{\pi\Gamma E/(\hbar \omega_c)^2}. \)

The imaginary part is \( \frac{\Gamma}{2} \frac{2\pi \Gamma_0^2 E}{(\hbar \omega_c)^2} \sinh v \). Now, this can be re-written by using the following standard relation:

\[
\frac{\sinh v}{\cosh v - \cos u} = 1 + 2 \sum_{s=1}^{\infty} e^{-sv} \cos (su). \quad \text{(A.5)}
\]

Here, the most dominant term is for \( s = 1 \) only. We can write

\[
\frac{\Gamma}{2} \frac{2\pi \Gamma_0^2 E}{(\hbar \omega_c)^2} \left[ 1 + 2 \sum_{s=1}^{\infty} e^{-\pi s \Gamma E/(\hbar \omega_c)^2} \cos (su) \right]. \quad \text{(A.6)}
\]

In the limit of \( \pi\Gamma \gg \hbar \omega_c \), after the first iteration, we have \( \Gamma / 2 = 2\pi \Gamma_0^2 E / (\hbar \omega_c)^2 \). Substituting this in the earlier expression, we get

\[
\frac{\Gamma}{2} = \frac{2\pi \Gamma_0^2 E}{(\hbar \omega_c)^2} \left[ 1 + 2 \sum_{s=1}^{\infty} \exp \left\{ -s \left( \frac{2\pi \Gamma_0^2 E}{\hbar^2 \omega_c^2} \right)^2 \right\} \right]. \quad \text{(A.7)}
\]

Here, \( \Delta_c = \Delta_s + \tau, \Delta_h \).

Finally, the density of states for the two branches can be obtained as

\[
D_c(E) = \frac{D_0(E)}{2} \left[ 1 + 2 \sum_{s=1}^{\infty} \exp \left\{ -s \left( \frac{2\pi \Gamma_0^2 E}{\hbar^2 \omega_c^2} \right)^2 \right\} \right] \cos \left\{ s\pi(E - \Delta_c^2) / (\hbar \omega_c)^2 \right\}. \quad \text{(A.8)}
\]

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