Two-grid method for the stationary Burgers equation

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Abstract. A two-grid method is investigated for solving the boundary value problem for nonlinear ordinary differential equation of the second order with a small parameter at the highest derivative. The solution to the problem has large gradients in the boundary layer region. The application of the central difference scheme on the Shishkin and Bakhvalov grids is investigated. Previously, this scheme was investigated only in the case of a linear equation. It is shown that in the case of the considered nonlinear problem, this scheme on the Shishkin and Bakhvalov grids has a convergence uniform in a small parameter. On the basis of computational experiments, it is shown that the use of the two-grid method leads to a reduction in computational costs when implementing the difference scheme. It is shown that in the two-grid method it is effective to apply the Richardson method to improve the accuracy of the difference scheme.

1. Introduction

Consider the two-grid method for the numerical solution of a boundary value problem

\[ \varepsilon u'' + uu' - b(x)u = f(x), \quad u(0) = 0, \quad u(1) = B, \]  

where \( b(x) \geq 0, \varepsilon \in (0, 1], \) functions \( b(x), f(x) \) are smooth enough.

Problem (1) is singularly perturbed for small values of parameter \( \varepsilon. \) The equation (1), in particular, corresponds to the stationary Burgers equation, and the question of its numerical solution is of interest. Dense in the region of the boundary layer grid can be used to achieve convergence of the difference scheme uniform in a small parameter. Bakhvalov mesh [1] and Shishkin mesh [2] are widely used.

The applied difference scheme is a system of nonlinear algebraic equations due to the nonlinearity of the differential equation. To find a solution to a nonlinear system of equations, an iterative method can be applied. The dimension of this system grows with an increase in the number of mesh nodes.

To reduce computational complexity, we explore the application of the two-grid method, when at first the problem is solved on an auxiliary coarse grid. Next, the found solution is interpolated to the original mesh, the result of the interpolation is taken as the initial approximation for the iterative method. This leads to a decrease in the number of iterations on the initial mesh and to a reduction in computational costs. The interpolation formula is applied on a mesh condensing in the boundary layer, therefore, the interpolation error is uniform in a small parameter. The error of the interpolation formula should correspond to the error of the difference scheme used on the auxiliary mesh.
When applying the two-grid method, the solution of the difference scheme is known on two grids, which allows increasing the accuracy of the difference scheme almost without additional computational costs by eliminating the main term of the error.

The two-grid method for nonlinear equations solving was used, for example, in [4]. To solve a singularly perturbed problem, the two-grid method was first proposed in [5]. Further, this method was investigated in a series of papers, for example, in [6]–[11].

By $C$ and $C_j$ we mean positive constants that do not depend on the parameter $\varepsilon$ and the number of grid nodes.

2. Setting the two-grid method

To solve the boundary value problem $Lu = f$ we write out the difference scheme with $N$ nodes on the mesh $\Omega^h$

$$L^h u^h = f^h.$$  \hspace{1cm} (2)

It is assumed that the solution of the difference scheme converges to the solution of the differential problem for $N \to \infty$,

$$\max_i |u(x_i) - u^h_i| \leq C \Delta_N,$$

where $x_i$ are nodes of the mesh $\Omega^h$.

The solution to the scheme (2) is found by an iterative method

$$u^{k+1} = G(u^k),$$  \hspace{1cm} (3)

where $k$ is the number of iteration and the initial approximation $u^0$ is set.

Let’s set the steps of the two-grid method:

1) On the auxiliary coarse mesh $\Omega^H$ with nodes $X_i, i = 0, 1, \ldots, n, n \ll N$ the difference scheme is written as $L^H u^H = f^H$. The solution of the scheme is found by the iterative method (3). The initial approximation $u^0$ will be given on the basis of linear interpolation of the boundary conditions. Iterations stop if the following condition holds

$$\max_i |L^H u^K - f^H_i| \leq \Delta_n,$$

where $K$ is last iteration number.

2) An interpolation formula with an accuracy of at least $O(\Delta_n)$ is chosen. Based on this formula, the mesh function $u^K$ is interpolated to the nodes of the original mesh $\Omega^h$, and the initial approximation for iterations on the mesh $\Omega^h$ is constructed:

$$u^0_h = \text{Int}(u^K, \Omega^H).$$

3) Iterations (3) are performed on the original mesh $\Omega^h$ under the condition $u^0 = u^0_h$. Iterations end at iteration numbered $M$, if

$$\max_i |L^h u^M - f^h_i| \leq \Delta_N.$$

Then $\max_i |u^M_i - u(x_i)| \leq C \Delta_N$.

The reduction in the number of arithmetic operations is achieved due to the fact that, to achieve accuracy $\Delta_n$, iterations are performed for a system of lower dimension, corresponding to the scheme with $(n+1)$ nodes instead the scheme with $(N+1)$ nodes for a one-grid method.

**Richardson Method.** On the basis of iterations on the auxiliary and original meshes, the grid solutions $u^H$ and $u^h$ are approximately found. We suppose that the mesh $\Omega^H$ is nested to
the mesh $\Omega^h$. Using these solutions at common nodes and excluding the main term of the error, we find the mesh function on the mesh $\Omega^H$

$$u^H_{j} = u^H_{j} + \frac{u^H_{j} - u^h_{j}}{\Delta^H_{j} - \Delta_{j}}. \quad (4)$$

Then for all $j$ $u(x_j) = u^H_{j} + o(\Delta_{j})$.

3. Setting the mesh

In numerical experiments, we consider the case when the boundary layer in the solution of the problem (1) is located near the boundary $x = 0$. Then the function $u(x)$ satisfies the representation:

$$u(x) = p(x) + \Phi(x), \quad x \in [0, 1], \quad (5)$$

where

$$|p^{(j)}(x)| \leq C_1, \quad |\Phi^{(j)}(x)| \leq \frac{C_1}{\varepsilon^j} e^{-x/\varepsilon}, \quad j \geq 0, \quad (6)$$

functions $p(x)$ and $\Phi(x)$ are not explicitly set. According to (6), the regular component $p(x)$ has derivatives bounded up to a certain order, and the derivatives of the singular component $\Phi(x)$ grow indefinitely with $\varepsilon$ decreasing. The decomposition (5) is valid for the solution of a differential problem in the presence of an exponential boundary layer [2].

Let’s set the mesh:

$$\Omega^h = \{x_n = x_{n-1} + h_n, n = 1, 2, \ldots, N, \quad x_0 = 0, x_N = 1\}.$$

Bakhvalov mesh. Based on [1], we set the mesh nodes on the interval $[0, 1]$.

Let

$$\sigma = \min \left\{1/2, -2\varepsilon \ln \varepsilon \right\}, \quad \varepsilon \leq e^{-1}.$$

For $\varepsilon > e^{-1}$ we set $\sigma = 1/2$. For $\sigma = 1/2$ we set the mesh $\Omega^h$ as uniform.

For $\sigma < 1/2$ we set the mesh $\Omega^h$ with nodes

$$x_n = -2\varepsilon \ln \left[1 - 2(1 - \varepsilon)n/N\right], \quad n = 0, 1, \ldots, N/2; \quad x_n = \sigma + (2n/N - 1)(1 - \sigma), \quad N/2 \leq n \leq N.$$

Then, outside the boundary layer region, the mesh is uniform.

Shishkin mesh. Let’s define the Shishkin mesh [2] based on the relations

$$\sigma = \min \left\{1/2, 2\varepsilon \ln N \right\}, \quad h_n = \frac{2\sigma}{N}, \quad n \leq N/2; \quad h_n = \frac{2(1 - \sigma)}{N}, \quad n > N/2.$$

4. Difference scheme

Let’s write the central difference scheme on the mesh $\Omega^h$

$$2\varepsilon \frac{h_n(u^n_{n+1} - u^n_{n}) - h_{n+1}(u^n_{n+1} - u^n_{n+2})}{h_nh_{n+1}(h_n + h_{n+1})} + \frac{u^n_{n+1} - u^n_{n-1}}{h_n + h_{n+1}} - b_nu^n_{n} = f_n, \quad u^h_{0} = A, \quad u^h_{N} = B. \quad (7)$$

The scheme (7) on the Shishkin mesh in the case of the linear problem was investigated in [12], where it was proved that

$$\max_n |u^n_{n} - u(x_n)| \leq C \ln^2(N)/N^2.$$

It is of interest to analyze the accuracy of this scheme in the case of the Bakhvalov mesh [1].
Let’s define an iterative method for linearizing the scheme (7)

\[
\frac{h_n}{h_{n+1} + h_{n+1}} (u^k_{n+1} - u^k_n) - b_n u^k_n = f_n, u^0_k = A, u^k_N = B.
\]

Here \( k \) is the number of iteration. The initial approximation \( u^0_n \) is given.

5. Numerical results
Consider the boundary value problem:

\[\varepsilon u'' + uu' = 0, \ u(0) = 0, \ u(1) = \frac{1}{2\varepsilon}.\]  (8)

The solution of problem (8) is of the form \( u(x) = \frac{x}{2\varepsilon} \).

The initial approximation for iterations is given in the form:

\[u^0_n = \frac{n}{N} \frac{1}{2\varepsilon}, n = 0, 1, ..., N.\]

In tables, the error and order of accuracy are calculated by the formulas

\[
\Delta_{N,\varepsilon} = \max_{0 \leq i \leq N} |u^h_i - u(x_i)|, \ M_{N,\varepsilon} = \log_2 \frac{\Delta_{N,\varepsilon}}{\Delta_{2N,\varepsilon}}.
\]

We denote \( 10^{\pm m} \) by \( e^{\pm m} \).

Following notations are used in the tables:

\( A|B \): \( A \) is the number of iterations on a coarse mesh, \( B \) is the number of iterations on the original mesh. Let \( N \) be the number of nodes of the original mesh, \( n \) is the number of nodes of the nested coarse mesh. For the original and coarse Shishkin meshes to be nested, parameter \( \sigma \) for mesh \( \Omega^H \) is taken the same as for \( \Omega^h \).

Table 1 shows the error \( \Delta_{N,\varepsilon} \) and the order of accuracy \( M_{N,\varepsilon} \) of the scheme (7) on the Shishkin mesh. Table 2 presents the results on the Bakhvalov mesh. Experiments on the Shishkin mesh show that the scheme has the error of order \( O(\ln^2(N)/N^2) \) uniformly in \( \varepsilon \) in the case of a nonlinear problem. On the Bakhvalov mesh, the results are consistent with the error estimate \( \max_{\Omega} |u^h_n - u(x_n)| \leq C/N^2. \) Note that the error estimates for the scheme (7) on the Bakhvalov mesh are unknown, we got them numerically.

Tables 3, 4 show the number of iterations of the two-grid method on the Shishkin mesh for \( \varepsilon = 1 \) and for \( \varepsilon = 10^{-2} \). For comparison, the number of iterations of the one-grid method is shown. It can be seen that the number of iterations on the original mesh significantly decreases when the two-grid method is applied. In the case of the Bakhvalov mesh, the results are similar.

Next, consider how the accuracy improves when Richardson’s method is applied in accordance with the formula (4) on the mesh \( \Omega^H \). Table 5 shows the error of the formula (4) on the Shishkin mesh in comparison with the error of the one-grid method on the \( \Omega^H \) and \( \Omega^h \) for \( \varepsilon = 10^{-2} \). It follows from the table that the application of the Richardson method leads to a significant increase in accuracy.

Table 6 similarly shows the maximum error of the Richardson method on the Bakhvalov mesh \( \Omega^H \).

It remains to interpolate to the original mesh \( \Omega^h \) the solution \( u^{HH}_{\varepsilon} \) found by the formula (4) on the auxiliary mesh \( \Omega^H \) by the Lagrange polynomial with sufficient accuracy. In [13] estimates of the interpolation error on the Shishkin mesh by a Lagrange polynomial of arbitrary degree are obtained. These estimates are uniform in the parameter \( \varepsilon.\)
Table 1. Maximum errors and orders of accuracy on the Shishkin mesh

| N  | 10^{-1}    | 10^{-2}    | 10^{-3}    | 10^{-4}    | 10^{-5}    | 10^{-6}    |
|----|------------|------------|------------|------------|------------|------------|
| 16 | 1.45e-02   | 1.92e-02   | 2.39e-02   | 2.52e-02   | 2.54e-02   | 2.54e-02   |
| 32 | 3.61e-03   | 7.04e-03   | 7.77e-03   | 8.86e-03   | 9.04e-03   | 9.06e-03   |
| 64 | 8.92e-04   | 2.54e-04   | 2.55e-04   | 2.88e-04   | 3.03e-04   | 3.05e-04   |
| 128| 2.21e-04   | 8.59e-05   | 8.6e-04    | 8.85e-03   | 9.71e-04   | 9.88e-04   |
| 256| 5.61e-05   | 2.80e-05   | 2.83e-04   | 2.84e-04   | 2.98e-04   | 3.16e-04   |

Table 2. Maximum errors and orders of accuracy on the Bakhvalov mesh

| N  | 10^{-1}    | 10^{-2}    | 10^{-3}    | 10^{-4}    | 10^{-5}    | 10^{-6}    |
|----|------------|------------|------------|------------|------------|------------|
| 16 | 8.28e-03   | 1.16e-02   | 2.23e-02   | 3.29e-02   | 3.65e-02   | 3.84e-02   |
| 32 | 2.07e-03   | 2.75e-03   | 3.36e-03   | 7.03e-03   | 8.38e-03   | 8.89e-03   |
| 64 | 5.53e-04   | 6.61e-04   | 7.89e-04   | 1.31e-03   | 1.96e-03   | 2.15e-03   |
| 128| 1.73e-04   | 1.62e-04   | 1.87e-04   | 2.19e-04   | 4.25e-04   | 5.20e-04   |
| 256| 7.94e-05   | 4.03e-05   | 4.43e-05   | 5.29e-05   | 7.58e-05   | 1.22e-04   |

Table 3. The number of iterations on the Shishkin mesh for ε = 1

| n  | N       |
|----|---------|
| 64 | 128 3 | 31 31 31 32 32 32 32 32 |
| 128| 512 3 | 31 31 31 32 32 32 32 32 |
| 256| 1024 3 | 31 31 31 32 32 32 32 32 |
| 512|       | 4 | 4 4 4 4 4 4 |
| 1024|  | 4 | 4 4 4 4 4 4 |
| one-grid|  | 3 | 3 4 4 5 5 5 |

Table 7 contains the maximum error when interpolating the Richardson combination $u^{Hh}$ from the Shishkin mesh $\Omega^H$ to the Shishkin mesh $\Omega^h$. For interpolation on the intervals $[X_i, X_{i+7}]$, the Lagrange polynomial of the seventh degree was used. The use of a Lagrange polynomial of a lesser degree led to a decrease in the accuracy of the solution $u^{Hh}$ found on a coarse grid by the Richardson extrapolation method. Similar results were obtained using the
Table 4. The number of iterations on the Shishkin mesh for $\varepsilon = 10^{-2}$.

| n   | 128    | 256    | 512    | 1024   | 2048   | 4096   |
|-----|--------|--------|--------|--------|--------|--------|
| 64  | 13|7   | 13|9   | 13|10  | 13|11  | 13|12  | 13|14  |
| 128 | 14|7   | 14|9   | 14|10  | 14|11  | 14|13  |
| 256 | 15|8   | 15|9   | 15|10  | 15|12  |       |
| 512 |       | 17|8   | 17|9   | 17|11  |       |
| 1024|       |       | 18|8   | 18|10  |       |
| one-grid | 14 | 15 | 17 | 18 | 19 | 20 |

Table 5. The maximum error of the Richardson method on an auxiliary coarse Shishkin mesh, $\varepsilon = 10^{-2}$

| n   | 128    | 256    | 512    | 1024   | 2048   | mesh $\Omega^H$ |
|-----|--------|--------|--------|--------|--------|-----------------|
| 64  | 7.93e-06| 3.74e-06| 1.57e-06| 5.98e-07| 2.18e-07| 2.54e-03        |
| 128 | 7.27e-07| 3.19e-07| 1.36e-07| 5.08e-08| 8.59e-04|                |
| 256 | 2.01e-08| 2.17e-08| 9.04e-09| 2.82e-04|       |                |
| 512 | 8.02e-09| 7.48e-10| 8.86e-05|       |       |                |
| 1024| 4.62e-09| 2.73e-05|       |       |       |                |
| one-grid method on the mesh $\Omega^h$ | 8.59e-04| 2.8e-04| 8.86e-05| 2.73e-05| 3.73e-06|                |

Table 6. The maximum error of the Richardson method on an auxiliary coarse Bakhvalov mesh, $\varepsilon = 10^{-2}$

| n   | 128    | 256    | 512    | 1024   | 2048   | 4096   | mesh $\Omega^H$ |
|-----|--------|--------|--------|--------|--------|--------|-----------------|
| 64  | 3.68e-05| 8.25e-06| 2.01e-06| 5.04e-07| 1.25e-07| 3.12e-08| 6.61e-04        |
| 128 | 2.18e-06| 5.22e-07| 1.29e-07| 3.22e-08| 8.03e-09| 1.62e-04|                |
| 256 | 1.07e-07| 2.65e-08| 6.6e-09 | 1.65e-09| 4.03e-05|       |                |
| 512 | 1.92e-08| 2.82e-09| 7.91e-10| 1.26e-05|       |       |                |
| 1024| 5.4e-09 | 8.27e-10| 3.22e-06|       |       |       |                |
| one-grid, mesh $\Omega^h$ | 1.62e-04| 4.02e-05| 1.26e-05| 3.22e-06| 8.64e-07| 2.25e-07|                |

Bakhvalov mesh.

It follows from experiments that Richardson’s method gives the highest accuracy when $n = N/2$. This value of $n$ is also suitable for computational costs reducing.

6. Conclusion

A two-grid method for solving a boundary value problem for a singularly perturbed second-order ordinary differential equation with nonlinearity corresponding to the Burgers equation is investigated. The application of the central difference scheme on the Shishkin and Bakhvalov meshes is investigated. It is shown numerically that, for the problem under consideration, the central difference scheme has the property of convergence, uniform in a small parameter. It is
Table 7. The maximum error of the Richardson method with interpolation to the Shishkin mesh $\Omega^h$, $\varepsilon = 10^{-2}$

| n  | 128    | 256    | 512    | 1024   | 2048   |
|----|--------|--------|--------|--------|--------|
| 64 | 1.4e-05| 4.5e-06| 8.92e-06| 1.2e-05| 1.99e-05|
| 128| 7.28e-07| 3.38e-07| 2.35e-07| 3.05e-08|
| 256| 2.01e-08| 2.17e-08| 9.09e-09|
| 512|        |        | 8.02e-09| 7.49e-10|
| 1024|       |        |        | 4.62e-09|
| one-grid method| | | | |
| on the mesh $\Omega^h$| 7.59e-04| 2.45e-04| 7.6e-05| 2.29e-05| 3.73e-06|

shown that the use of the two-grid method leads to a decrease in computational costs when finding the solution of the difference scheme. The Richardson extrapolation method is applied to improve the accuracy of the difference scheme with almost no additional computational costs. It is shown numerically that in the presence of areas of high gradients, Lagrange polynomials can be successfully used for interpolation on Shishkin and Bakhvalov meshes. This made it possible to transfer the solution obtained by the Richardson extrapolation method from the auxiliary coarse mesh to the original finer mesh without loss of accuracy.

Acknowledgments
The reported study was funded by RFBR, project number 19-31-60009. The authors are grateful to S.V. Tikhovskaya for some useful comments.

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