CRITICAL AGE DEPENDENT BRANCHING MARKOV PROCESSES AND THEIR SCALING LIMITS

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Abstract. This paper studies: (i) the long time behaviour of the empirical distribution of age and normalised position of an age dependent critical branching Markov process conditioned on non-extinction; and (ii) the super-process limit of a sequence of age dependent critical branching Brownian motions.

1. Introduction

Consider an age dependent branching Markov process where i) each particle lives for a random length of time and during its lifetime moves according to a Markov process and ii) upon its death it gives rise to a random number of offspring. We assume that the system is critical, i.e. the mean of the offspring distribution is one.

We study three aspects of such a system. First, at time $t$, conditioned on non-extinction (as such systems die out w.p. 1) we consider a randomly chosen individual from the population. We show that asymptotically (as $t \to \infty$), the joint distribution of the position (appropriately scaled) and age (unscaled) of the randomly chosen individual decouples (See Theorem 2.1). Second, it is shown that conditioned on non-extinction at time $t$, the empirical distribution of the age and the normalised position of the population converges as $t \to \infty$ in law to a random measure characterised by its moments (See Theorem 2.2). Thirdly, we establish a super-process limit of such branching Markov processes where the motion is Brownian (See Theorem 2.4).

The rest of the paper is organised as follows. In Section 2.1 we define the branching Markov process precisely and in Section 2.2 we state the three main theorems of this paper and make some remarks on various possible generalisations of our results.

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In Section 3 we prove four propositions on age-dependent Branching processes which are used in proving Theorem 2.1 (See Section 4). In Section 3 we also show that the joint distribution of ancestoral times for a sample of \( k \geq 1 \) individuals chosen at random from the population at time \( t \) converges as \( t \to \infty \) (See Theorem 3.5). This result is of independent interest and is a key tool that is needed in proving Theorem 2.2 (See Section 5).

In Section 6, we prove Theorem 2.4, the key idea being to scale the age and motion parameters differently. Given this, the proof uses standard techniques for such limits. Theorem 2.1 is used in establishing the limiting log-Laplace equation. Tightness of the underlying particle system is shown in Proposition 6.4 and the result follows by the method prescribed in [7].

2. Statement of Results

2.1. The Model.

Each particle in our system will have two parameters, age in \( \mathbb{R}_+ \) and location in \( \mathbb{R} \). We begin with the description of the particle system.

(i) **Lifetime Distribution** \( G(\cdot) \): Let \( G(\cdot) \) be a cumulative distribution function on \([0, \infty)\), with \( G(0) = 0 \). Let \( \mu = \int_0^\infty sdG(s) < \infty \).

(ii) **Offspring Distribution** \( p \): Let \( p \equiv \{p_k\}_{k \geq 0} \) be a probability distribution such that \( p_0 < 1 \), \( m = \sum_{k=0}^\infty kp_k = 1 \) and that \( \sigma^2 = \sum_{k=0}^\infty k^2p_k - 1 < \infty \).

(iii) **Motion Process** \( \eta(\cdot) \): Let \( \eta(\cdot) \) be a \( \mathbb{R} \) valued Markov process starting at 0.

**Branching Markov Process** \((G,p,\eta)\): Suppose we are given a realisation of an age-dependent branching process with offspring distribution \( p \) and lifetime distribution \( G \) (See Chapter IV of [5] for a detailed description). We construct a branching Markov process by allowing each individual to execute an independent copy of \( \eta \) during its lifetime \( \tau \) starting from where its parent died.

Let \( N_t \) be the number of particles alive at time \( t \) and

\[
C_t = \{(a^i_t, X^i_t) : i = 1, 2, \ldots, N_t\}
\]

(2.1)

denote the age and position configuration of all the individuals alive at time \( t \). Since \( m = 1 \) and \( G(0) = 0 \), there is no explosion in finite time (i.e. \( P(N_t < \infty) = 1 \)) and consequently \( C_t \) is well defined for each \( 0 \leq t < \infty \) (See [5]).
Let $\mathcal{B}(\mathbb{R}^+)$ (and $\mathcal{B}(\mathbb{R})$) be the Borel $\sigma$-algebra on $\mathbb{R}^+$ (and $\mathbb{R}$). Let $M(\mathbb{R}^+ \times \mathbb{R})$ be the space of finite Borel measures on $\mathbb{R}^+ \times \mathbb{R}$ equipped with the weak topology. Let $M_\alpha(\mathbb{R}^+ \times \mathbb{R}) := \{\nu \in M(\mathbb{R}^+ \times \mathbb{R}) : \nu = \sum_{i=1}^n \delta_{a_i,x_i}(\cdot,\cdot), n \in \mathbb{N}, a_i \in \mathbb{R}^+, x_i \in \mathbb{R}\}$. For any set $A \in \mathcal{B}(\mathbb{R}^+)$ and $B \in \mathcal{B}(\mathbb{R})$, let $Y_t(A \times B)$ be the number of particles at time $t$ whose age is in $A$ and position is in $B$. As pointed out earlier, $m < \infty$, $G(0) = 0$ implies that $Y_t \in M_\alpha(\mathbb{R}^+ \times \mathbb{R})$ for all $t > 0$ if $Y_0$ does so. Fix a function $\phi \in C_b^+(\mathbb{R}^+ \times \mathbb{R})$, (the set of all bounded, continuous and positive functions from $\mathbb{R}^+ \times \mathbb{R}$ to $\mathbb{R}^+$), and define

$$
\langle Y_t, \phi \rangle = \int \phi \, dY_t = \sum_{i=1}^{N_t} \phi(a_i^t, X_i^t).
$$

(2.2)

Since $\eta(\cdot)$ is a Markov process, it can be seen that $\{Y_t : t \geq 0\}$ is a Markov process and we shall call $Y \equiv \{Y_t : t \geq 0\}$ the $(G, p, \eta)$-branching Markov process.

Note that $C_t$ determines $Y_t$ and conversely. The Laplace functional of $Y_t$, is given by

$$
L_t \phi(a, x) := E_{a,x}[e^{-\langle \phi, Y_t \rangle}] \equiv E[e^{-\langle \phi, Y_t \rangle} | Y_0 = \delta_{a,x}].
$$

(2.3)

From the independence intrinsic in $\{Y_t : t \geq 0\}$, we have:

$$
E_{\nu_1 + \nu_2}[e^{-\langle \phi, Y_t \rangle}] = (E_{\nu_1}[e^{-\langle \phi, Y_t \rangle}]) (E_{\nu_2}[e^{-\langle \phi, Y_t \rangle}]),
$$

(2.4)

for any $\nu_i \in M_\alpha(\mathbb{R}^+ \times \mathbb{R})$ where $E_{\nu_i}[e^{-\langle \phi, Y_t \rangle}] := E[e^{-\langle \phi, Y_t \rangle} | Y_0 = \nu_i]$ for $i = 1, 2$. This is usually referred to as the branching property of $Y$ and can be used to define the process $Y$ as the unique measure valued Markov process with state space $M_\alpha(\mathbb{R}^+ \times \mathbb{R})$ satisfying $L_{t+s} \phi(a, x) = L_t(L_s(\phi))(a, x)$ for all $t, s \geq 0$.

2.2. The Results.

In this section we describe the main results of the paper. Let $A_t$ be the event $\{N_t > 0\}$, where $N_t$ is the number of particles alive at time $t$. As $p_0 < 1$, $P(A_t) > 0$ for all $0 \leq t < \infty$ provided $P(N_0 = 0) \neq 1$.

**Theorem 2.1.** (Limiting behaviour of a randomly chosen particle) On the event $A_t = \{N_t > 0\}$, let $(a_t, X_t)$ be the age and position of a randomly chosen particle from those alive at time $t$. Assume that $\eta(\cdot)$ is such that for all $0 \leq t < \infty$

$$
E(\eta(t)) = 0, v(t) \equiv E(\eta^2(t)) < \infty, \sup_{0 \leq s \leq t} v(s) < \infty,
$$

(2.5)

and $\psi \equiv \int_0^\infty v(s)G(ds) < \infty$. 
Then, conditioned on $A_t$, $(a_t, \frac{X_t}{\sqrt{t}})$ converges as $t \to \infty$, to $(U,V)$ in distribution, where $U$ and $V$ are independent with $U$ a strictly positive absolutely continuous random variable with density proportional to $(1 - G(\cdot))$ and $V$ is normally distributed with mean 0 and variance $\frac{\psi}{\mu}$.

Next consider the scaled empirical measure $\tilde{Y}_t \in M_a(\mathbb{R}_+ \times \mathbb{R})$ given by $\tilde{Y}_t(A \times B) = Y_t(A \times \sqrt{t}B)$, $A \in \mathcal{B}(\mathbb{R}_+), B \in \mathcal{B}(\mathbb{R})$.

**Theorem 2.2. (Empirical Measure)**

Assume (2.5). Then, conditioned on $A_t = \{N_t > 0\}$, the random measures $\{\tilde{Y}_t\}$ converges as $t \to \infty$ in distribution to a random measure $\nu$, characterised by its moment sequence $m_k(\phi) \equiv E[\nu(\phi)^k]$ for $\phi \in C_b(\mathbb{R}_+ \times \mathbb{R})$, $k \geq 1$.

An explicit formula for $m_k(\phi)$ is given in (5.2) below.

Our third result is on the super-process limit. We consider a sequence of branching Markov processes $(G_n, p_n, \eta_n)_{\{n \geq 1\}}$ denoted by $\{Y^n_t : t \geq 0\}_{\{n \geq 1\}}$ satisfying the following:

(a) **Initial measure:** For $n \geq 1$, $Y^n_0 = \pi n\nu$, where $\pi n\nu$ is a Poisson random measure with intensity $n\nu$, for some $\nu = \alpha \times \mu \in M(\mathbb{R}_+ \times \mathbb{R})$.

(b) **Lifetime** $G^n(\cdot)$: For all $n \geq 1$, $G^n$ is an exponential distribution with mean $\frac{1}{\lambda}$.

(c) **Branching** $p_n$: For $n \geq 1$, let $F_n(u) = \sum_{k=0}^{\infty} p_{n,k} u^k$ be the generating function of the offspring distribution $p_n \equiv \{p_{n,k}\}_{k \geq 0}$. We shall assume that $F_n$ satisfies,

\[
\lim_{n \to \infty} \sup_{0 \leq u \leq N} \| n^2(F_n(1 - u/n) - (1 - u/n)) - u^2 \| \to 0,
\]

for all $N > 0$.

(d) **Motion Process** $\eta_n(\cdot)$: For all $n \geq 1$,

\[
\eta_n(t) = \frac{1}{\sqrt{n}} \int_0^t \sigma(u) dB(u), \quad t \geq 0,
\]

where $\{B(t) : t \geq 0\}$ is a standard Brownian motion starting at 0 and $\sigma : \mathbb{R}_+ \to \mathbb{R}$ is a continuous function such that $\int_0^\infty \sigma^2(s) dG(s) < \infty$. It follows that for each $n \geq 1$, $\eta_n$ satisfies (2.5).

**Definition 2.3.** Let $\mathcal{E}$ be an independent exponential random variable with mean $\frac{1}{\lambda}$, $0 < \lambda < \infty$. For $f \in C^+_l(\mathbb{R}_+ \times \mathbb{R})$ let $U_t f(x) = E(f(\mathcal{E}, x + \sqrt{\lambda} \psi B_t))$ where $\psi$ is defined in (2.5). For $t \geq 0$, let $u_t(f)$ be the unique solution of
the non linear integral equation

\begin{equation}
\tag{2.8}
\frac{du_t}{dt} f(x) = U_t f(x) - \lambda \int_0^t U_{t-s} (u_s(f))^2(x) ds.
\end{equation}

Let \( \{Y_t : t \geq 0\} \) be a \( M(\mathbb{R}_+ \times \mathbb{R}) \) valued Markov process whose Laplace functional is given by

\begin{equation}
\tag{2.9}
E_{\mathcal{E} \times \mu} \left[ e^{-\langle f, Y_t \rangle} \right] = e^{-\langle V_t f, \mu \rangle},
\end{equation}

where \( f \in C^+_f(\mathbb{R}_+ \times \mathbb{R}^d) \) (the set of all continuous functions from \( \mathbb{R}_+ \times \mathbb{R} \) to \( \mathbb{R} \) with finite limits as \( (a, x) \to \infty \)) and \( V_t(f)(x) \equiv u_t(f(x)) \) for \( x \in \mathbb{R} \) (See [7] for existence of \( Y \) satisfying (2.9)).

Note that in the process \( \{Y_t : t \geq 0\} \) defined above, the distribution of the age (i.e. the first coordinate) is deterministic. The spatial evolution behaviors like that of a super-process where the motion of particles is like that of a Brownian motion with variance equal to the average variance of the age-dependent particle displacement over its lifetime. Also, \( u_s(f) \) in second term of (2.8) is interpreted in the natural way as a function on \( \mathbb{R}_+ \times \mathbb{R} \) with \( u_s(f)(a,x) = u_s(f)(x) \) for all \( a > 0 \), \( x \in \mathbb{R} \).

**Theorem 2.4. (Age Structured Super-process)**

Let \( \epsilon > 0 \). Let \( \{Y^n_t : t \geq 0\} \) be the sequence of branching Markov processes defined above (i.e. in (a), (b), (c), (d)). Then as \( n \to \infty \), \( \{Y^n_t \equiv \frac{1}{n} Y^n_{nt}, t \geq \epsilon\} \) converges weakly on the Skorokhod space \( D([\epsilon, \infty), M(\mathbb{R}_+ \times \mathbb{R})) \) to \( \{Y_t : t \geq \epsilon\} \).

**2.3. Remarks.**

(a) If \( \eta(\cdot) \) is not Markov then \( \tilde{C}_t = \{a^i_t, X^i_t, \tilde{\eta}^i_t(u) : 0 \leq u \leq a^i_t \} : i = 1, 2, \ldots, N_t \} \) is a Markov process where \( \{\tilde{\eta}^i_t(u) : 0 \leq u \leq a^i_t \} \) is the history of \( \eta(\cdot) \) of the individual \( i \) during its lifetime. Theorem 2.1 and Theorem 2.2 extends to this case.

(b) Most of the above results also carry over to the case when the motion process is \( \mathbb{R}^d \) valued (\( d \geq 1 \)) or is Polish space valued and where the offspring distribution is age-dependent.

(c) Theorem 2.1 and Theorem 2.2 can also be extended to the case when \( \eta(L_1) \), with \( L_1 \overset{d}{=} G \), is in the domain of attraction of a stable law of index \( 0 < \alpha \leq 2 \).
In Theorem 2.4 the convergence should hold on $D([0, \infty), M(\mathbb{R}_+ \times \mathbb{R}))$ if we take $\alpha$ in the sequence of branching Markov processes to be $E$ (i.e. Exponential with mean $\frac{1}{\lambda}$).

(e) The super-process limit obtained in Theorem 2.4 has been considered in two special cases in the literature. One is in [6] where an age-dependent Branching process is rescaled (i.e. the particles do not perform any motion). The other is in [8] where a general non-local super-process limit is obtained when the offspring distribution is given by $p_1 = 1$. In our results, to obtain a super-process limit the age-parameter is scaled differently when compared to the motion parameter giving us an age-structured super-process.

(f) Limit theorems for critical branching Markov processes where the motion depends on the age does not seem to have been considered in the literature before.

3. Results on Branching Processes

Let $\{N_t : t \geq 0\}$ be an age-dependent branching process with offspring distribution $\{p_k\}_{k \geq 0}$ and lifetime distribution $G$ (see [5] for detailed discussion). Let $\{\zeta_k\}_{k \geq 0}$ be the embedded discrete time Galton-Watson branching process with $\zeta_k$ being the size of the $k$th generation, $k \geq 0$. Let $A_t$ be the event $\{N_t > 0\}$. On this event, choose an individual uniformly from those alive at time $t$. Let $M_t$ be the generation number and $a_t$ be the age of this individual.

**Proposition 3.1.** Let $A_t, a_t, M_t$ and $N_t$ be as above. Let $\mu$ and $\sigma$ be as in Section 2.4. Then

(a) \( \lim_{t \to \infty} tP(A_t) = \frac{2\mu}{\sigma^2} \)

(b) \( \lim_{t \to \infty} P(\frac{N_t}{t} > x|A_t) = e^{-\frac{2\mu x}{\sigma^2}} \),

(c) \( \lim_{t \to \infty} P(|\frac{M_t}{t} - \frac{1}{\mu}| > \epsilon|A_t) = 0 \)

(d) \( \lim_{t \to \infty} P(a_t \leq x|A_t) = \frac{1}{\mu} \int_0^x (1 - G(s))ds \).

**Proof:** For (a) and (b) see chapter 4 in [5]. For (c) see [9] and for (d) see [9]. \( \square \)

**Proposition 3.2.** (Law of large numbers) Let $\epsilon > 0$ be given. For the randomly chosen individual at time $t$, let $\{L_{ti} : 1 \leq i \leq M_t\}$, be the lifetimes
of its ancestors. Let $h : [0, \infty) \rightarrow \mathbb{R}$ be Borel measurable and $E(|h(L_1)|) < \infty$ with $L_1 \overset{d}{=} G$. Then, as $t \rightarrow \infty$

$$P\left(\left|\frac{1}{M_t} \sum_{i=1}^{M_t} h(L_{ti}) - E(h(L_1))\right| > \epsilon|A_t\right) \rightarrow 0.$$ 

Proof: Let $\epsilon$ and $\epsilon_1 > 0$ be given and let $k_1(t) = t(\frac{1}{\mu} - \epsilon)$ and $k_2(t) = t(\frac{1}{\mu} + \epsilon)$. By Proposition 3.1 there exists $\delta > 0$, $\eta > 0$ and $t_0 > 0$ such that for all $t \geq t_0$,

(3.1) $tP(N_t > 0) > \delta$ and $P(N_t \leq t\eta|A_t) < \epsilon_1$;

(3.2) $P(M_t \in [k_1(t), k_2(t)]^c|A_t) < \epsilon_1$.

Also by the law of large numbers for any $\{L_i\}_{i \geq 1}$ i.i.d. $G$ with $E|h(L_1)| < \infty$

(3.3) $\lim_{k \rightarrow \infty} P(\sup_{j \geq k} \frac{1}{j} \sum_{i=1}^{j} h(L_i) - E(h(L_1))| > \epsilon) = 0$.

Let $\{\zeta_k\}_{k \geq 0}$ be the embedded Galton-Watson process. For each $t > 0$ and $k \geq 1$ let $\zeta_{kt}$ denote the number of lines of descent in the $k$-th generation alive at time $t$ (i.e. the successive life times $\{L_i\}_{i \geq 1}$ of the individuals in that line of descent satisfying $\sum_{i=1}^{k} L_i \leq t \leq \sum_{i=1}^{k+1} L_i$). Denote the lines of descent of these individuals by $\{\zeta_{ktj} : 1 \leq j \leq \zeta_{kt}\}$. Call $\zeta_{ktj}$ bad if

(3.4) $\left|\frac{1}{k} \sum_{i=1}^{k} h(L_{ktji}) - E(h(L_1))\right| > \epsilon$,

where $\{L_{ktji}\}_{i \geq 1}$ are the successive lifetimes in the line of descent $\zeta_{ktj}$ starting from the ancestor. Let $\zeta_{kt,b}$ denote the cardinality of the set $\{\zeta_{ktj} : 1 \leq
\[ j \leq \zeta_{kt} \text{ and } \zeta_{ktj} \text{ is bad} \}. \text{ Now,} \]

\[
P(\left\lfloor \frac{1}{M_t} \sum_{i=1}^{M_t} h(L_{ti}) - E(h(L_1)) \right\rfloor > \epsilon | A_t) = P( \text{The chosen line of descent at time } t \text{ is bad} | A_t)
\]

\[
\leq P( \text{The chosen line of descent at time } t \text{ is bad, } M_t \in [k_1(t), k_2(t)] | A_t)
+ P(M_t \in [k_1(t), k_2(t)]^c | A_t)
\]

\[
= \frac{1}{P(N_t > 0)} \sum_{j=k_1(t)}^{k_2(t)} E(\zeta_{jt,b} | N_t > \eta)
+ \frac{1}{P(N_t > 0)} \sum_{j=k_1(t)}^{k_2(t)} E(\zeta_{jt,b} | N_t \leq \eta)
+ P(M_t \in [k_1(t), k_2(t)]^c | A_t)
\]

\[
\leq \frac{1}{P(N_t > 0)} \sum_{j=k_1(t)}^{k_2(t)} E(\zeta_{jt,b} | N_t > \eta)
+ \frac{P(N_t \leq \eta | N_t > 0)}{P(N_t > 0)}
+ P(M_t \in [k_1(t), k_2(t)]^c | A_t)
\]

\[
= \frac{1}{t\eta P(N_t > 0)} \sum_{j=k_1(t)}^{k_2(t)} E(\zeta_{jt,b})
+ P(N_t \leq \eta | N_t > 0) + P(M_t \in [k_1(t), k_2(t)]^c | A_t)
\]

(3.5)

For \( t \geq t_0 \) by (3.2) and (3.3), the last two terms in (3.5) are less than \( \epsilon_1 \).

The first term is equal to

\[
\frac{1}{t\eta P(N_t > 0)} \sum_{j=k_1(t)}^{k_2(t)} E(\zeta_{jt,b}) = \frac{1}{t\eta P(N_t > 0)} \sum_{j=k_1(t)}^{k_2(t)} E(\sum_{i=1}^{k_2(t)} \sum_{i=1}^{k_2(t)} 1_{\{\zeta_{jit} \text{ is bad}\}})
\]
\[ t \eta P(N_t > 0) \sum_{j=k_1(t)}^{k_2(t)} E(\zeta_j) \times \]
\[ \times P \left( \sum_{i=1}^{j} L_i \leq t < \sum_{i=1}^{j+1} L_i, \frac{1}{j} \sum_{i=1}^{j} h(L_i) - E(h(L_1)) > \epsilon \right), \]

where the \( \{L_i\}_{i \geq 1} \) are i.i.d. \( G \).

Using (3.1) and (since \( m = 1 \)) \( E(\zeta_j) = E(\zeta_0) \) we can conclude that

\[ \frac{1}{t \eta P(N_t > 0)} \sum_{j=k_1(t)}^{k_2(t)} E(\zeta_{j.t,b}) \]
\[ \leq E(\zeta_0) \frac{P(\sup_{j \geq k_1(t)} \frac{1}{j} \sum_{i=1}^{j} h(L_i) - E(h(L_1)) > \epsilon)}{t \eta P(N_t > 0)} \]
\[ \leq E(\zeta_0) \frac{P(\sup_{j \geq k_1(t)} \frac{1}{j} \sum_{i=1}^{j} h(L_i) - E(h(L_1)) > \epsilon)}{\eta \delta}, \]

(3.6)

which by (3.3) goes to zero. So we have shown that for \( t \geq t_0 \),

\[ P\left( |1 \frac{1}{M_t} \sum_{i=1}^{M_t} h(L_{ti}) - E(h(L_1))| > \epsilon | A_t \right) < 3 \epsilon_1. \]

Since \( \epsilon_1 > 0 \) is arbitrary, the proof is complete.

\[ \square \]

**Proposition 3.3.** Assume (2.5) holds. Let \( \{L_i\}_{i \geq 1} \) be i.i.d \( G \) and \( \{\eta_i\}_{i \geq 1} \) be i.i.d copies of \( \eta \) and independent of the \( \{L_i\}_{i \geq 1} \). For \( \theta \in \mathbb{R}, t \geq 0 \) define \( \phi(\theta, t) = Ee^{i \theta \eta(t)} \). Then there exists an event \( D \), with \( P(D) = 1 \) and on \( D \) for all \( \theta \in \mathbb{R} \),

\[ \prod_{j=1}^{n} \phi \left( \frac{\theta}{\sqrt{n}}, L_j \right) \rightarrow e^{-\frac{\theta^2 \psi}{2}}, \quad \text{as } n \to \infty, \]

where \( \psi \) is as in (2.5).

**Proof:** Recall from (2.5) that \( v(t) = E(\eta^2(t)) \) for \( t \geq 0 \). Consider

\[ X_{ni} = \frac{\eta_i(L_i)}{\sqrt{\sum_{j=1}^{n} v(L_j)}} \quad \text{for } 1 \leq i \leq n \]

and \( \mathcal{F} = \sigma(L_i : i \geq 1) \). Given \( \mathcal{F} \), \( \{X_{ni} : 1 \leq i \leq n\} \) is a triangular array of independent random variables such that for \( 1 \leq i \leq n \), \( E(X_{ni}|\mathcal{F}) = 0 \), \( \sum_{i=1}^{n} E(X_{ni}^2|\mathcal{F}) = 1 \).
Let $\epsilon > 0$ be given. Let

$$L_n(\epsilon) = \sum_{i=1}^{n} \mathbb{E} \left( X_{ni}^2 : X_{ni}^2 > \epsilon | \mathcal{F} \right).$$

By the strong law of large numbers,

$$\sum_{j=1}^{n} v(L_j) / n \to \psi \quad \text{w.p. 1.}$$

Let $D$ be the event on which (3.7) holds. Then on $D$

$$\limsup_{n \to \infty} L_n(\epsilon) \leq \limsup_{k \to \infty} \frac{\psi}{2} \mathbb{E} \left( |\eta_1(L_1)|^2 : |\eta_1(L_1)|^2 > k \right) = 0.$$

Thus the Linderberg-Feller Central Limit Theorem (see [4]) implies, that on $D$, for all $\theta \in \mathbb{R}$

$$\prod_{i=1}^{n} \phi \left( \frac{\theta}{\sqrt{\sum_{j=1}^{n} v(L_j)}}, L_j \right) = E(e^{i\theta \sum_{j=1}^{n} X_{nj} | \mathcal{F}}) \to e^{-\frac{\epsilon^2}{2}}.$$

Combining this with (3.7) yields the result. \qed

**Proposition 3.4.** For the randomly chosen individual at time $t$, let

$\{L_{ti}, \{\eta_i(u) : 0 \leq u \leq L_{ti}\} : 1 \leq i \leq M_t\}$, be the lifetimes and motion processes of its ancestors. Let $Z_{t1} = 1 / \sqrt{M_t} \sum_{i=1}^{M_t} \eta_i(L_{ti})$, and $\mathcal{L}_t = \sigma\{M_t, L_{ti} : 1 \leq i \leq M_t\}$. Then

$$E \left( |E(e^{i\theta Z_{t1}} | \mathcal{L}_t) - e^{-\frac{\epsilon^2}{2}}| | A_t \right) \to 0.$$

**Proof:** Fix $\theta \in \mathbb{R}, \epsilon_1 > 0$ and $\epsilon > 0$. Replace the definition of “bad” in (3.4) by

$$\left| \prod_{i=1}^{k} \phi \left( \frac{\theta}{\sqrt{k}}, L_{ktj} \right) - e^{-\frac{\epsilon^2}{2}} \right| > \epsilon$$

By Proposition 3.3 we have,

$$\lim_{k \to \infty} \mathbb{P}(\sup_{j \geq k} \left| \prod_{i=1}^{j} \phi \left( \frac{\theta}{\sqrt{j}}, L_i \right) - e^{-\frac{\epsilon^2}{2}} \right| > \epsilon) = 0.$$
Using this in place of (3.3) and imitating the proof of Proposition 3.2 (since the details mirror that proof we avoid repeating them here), we obtain that for \( t \) sufficiently large

\[
\begin{align*}
P(\prod_{i=1}^{M_t} \phi(\frac{\theta}{\sqrt{M_t}}, L_{ti}) - e^{-\frac{\theta^2 \psi^2}{2}} | A_t) < \epsilon.
\end{align*}
\]

Now for all \( \theta \in \mathbb{R} \),

\[
E(e^{i\theta Z_t} | L_t) = \prod_{i=1}^{M_t} \phi(\frac{\theta}{\sqrt{M_t}}, L_{ti}).
\]

So,

\[
\limsup_{t \to \infty} E(\sum_{i=1}^{M_t} \eta_i(L_{ti}) - e^{-\frac{\theta^2 \psi^2}{2}} | A_t)
\]

\[
= \limsup_{t \to \infty} E(\prod_{i=1}^{M_t} \phi(\frac{\theta}{\sqrt{M_t}}, L_{ti}) - e^{-\frac{\theta^2 \psi^2}{2}} | A_t)
\]

\[
< \epsilon_1 + 2 \limsup_{t \to \infty} P(\prod_{i=1}^{M_t} \phi(\frac{\theta}{\sqrt{M_t}}, L_{ti}) - e^{-\frac{\theta^2 \psi^2}{2}} > \epsilon_1 | A_t)
\]

\[
= \epsilon_1 + 2 \epsilon.
\]

Since \( \epsilon > 0, \epsilon_1 > 0 \) are arbitrary we have the result. \( \square \)

The above four Propositions will be used in the proof of Theorem 2.1.

For the proof of Theorem 2.2 we will need a result on coalescing times of the lines of descent.

Fix \( k \geq 2 \). On the event \( A_t = \{ N_t > 0 \} \), pick \( k \) individuals \( C_1, C_2, \ldots, C_k \) from those alive at time \( t \) by simple random sampling without replacement.

For any two particles \( C_i, C_j \), let \( \tau_{C_j, C_i, t} \) be the birth time of their most recent common ancestor. Let \( \tau_{k-1, t} = \sup\{ \tau_{C_j, C_i, t} : i \neq j, 1 \leq i, j \leq k \} \).

Thus \( \tau_{k-1, t} \) is the first time there are \( k-1 \) ancestors of the \( k \) individuals \( C_1, C_2, \ldots, C_k \). More generally, for \( 1 \leq j \leq k-1 \) let \( \tau_{j, t} \) as the first time there are \( j \) ancestors of the \( k \) individuals \( C_1, C_2, \ldots, C_k \).

**Theorem 3.5.**

(i) For any \( i, j \), \( \lim_{t \to \infty} P(\tau_{C_i, C_j, t} \leq x | A_t) \equiv H(x) \) exists for all \( x \geq 0 \) and \( H(\cdot) \) is an absolutely continuous distribution function on \( [0, \infty] \).

(ii) Conditioned on \( A_t \) the vector \( \tilde{\tau}_t = \frac{1}{t}(\tau_{j, t} : 1 \leq j \leq k-1) \) as \( t \to \infty \) converges in distribution to a random vector \( \tilde{T} = (T_1, \ldots, T_{k-1}) \).
with $0 < T_1 < T_2 < \ldots < T_{k-1} < 1$ and having an absolutely
continuous distribution on $[0, 1]^{k-1}$.

Proof: The proof of (i) and (ii) for cases $k = 2, 3$ is in [9]. The following
is an outline of a proof of (ii) for the case $k > 3$ (for a detailed proof see [3]).

Below, for $1 \leq j \leq k-1$, $\tau_{j,t}$ will be denoted by $\tau_j$. It can be shown that
it suffices to show that for any $1 \leq i_1 < i_2 \ldots < i_p < k$ and $0 < r_1 < r_2 < \ldots < r_p < 1$,

$$\lim_{t \to \infty} P\left(\frac{\tau_{i_1}}{t} < r_1 < \frac{\tau_{i_2}}{t} < r_2 < \ldots < \frac{\tau_{i_p}}{t} < r_p < \frac{\tau_{k-1}}{t} < r_{k-1} < 1\right| A_t)$$

exists. We shall now condition on the population size at time $tr_1$. Suppose
that at time $tr_1$ there are $n_{11}$ particles of which $k_{11}$ have descendants that
survive till time $tr_2$. For each $1 \leq j \leq k_{11}$, suppose there are $n_{2j}$ descendants
alive at time $tr_2$ and for each such $j$, let $k_{2j}$ out of the $n_{2j}$ have descendants
that survive till time $tr_3$. Let $k_2 = (k_{21}, \ldots, k_{2|k_{11}|})$ and $|k_2| = \sum_{j=1}^{|k_{11}|} k_{2j}$.
Inductively at time $tr_i$, there are $n_{ij}$ descendants for the $j$-th particle, $1 \leq j \leq |k_{i-1}|$. For each such $j$, let $k_{ij}$ out of $n_{ij}$ have descendants that survive
up till time $tr_{i+1}$ (See Figure 3 for an illustration).

It will be useful to use the following notation: Let

$$n_{11}, k_{11} \in \mathbb{N}, k_{11} \leq n_{11}, |k_1| = k_{11}, n_1 = (n_{11}).$$

For $i = 2, \ldots, i_p$ let $(n_i, k_i) \in \mathbb{N}_i$, where $N_i \equiv \mathbb{N}^{[k_{i-1}]} \times \mathbb{N}^{[k_{i-1}]}$

$$k_{ij} \leq n_{ij}, |k_i| = \sum_{j=1}^{|k_{i-1}|} k_{ij}, \binom{n_i}{k_i} = \prod_{j=1}^{|k_{i-1}|} \binom{n_{ij}}{k_{ij}}.$$ 

Let $f_s = P(N_s > 0)$. Now,

$$P\left(\frac{\tau_{i_1}}{t} < r_1 < \frac{\tau_{i_2}}{t} < r_2 < \ldots < \frac{\tau_{i_p}}{t} < r_p < \frac{\tau_{k-1}}{t} < r_{k-1} < t\right| A_t) =$$

$$= \frac{f_{tr_1}}{f_t} \sum_{(n_i,k_i) \in \mathbb{N}_i} \left(\binom{n_{11}}{k_{11}} (f_{tr_1})^{k_{11}} (1 - f_{tr_1})^{n_{11} - k_{11}}\right) \frac{P(N_{tr_1} = n_1)}{f_{tr_1}} \times$$

$$\times \prod_{i=1}^{p+1} \prod_{j=1}^{|k_{i-1}|} \binom{n_{ij}}{k_{ij}} (f_{tu_i})^{k_{ij}} (1 - f_{tu_i})^{n_{ij} - k_{ij}} P(N_{tu_i} = n_{ij} | N_{tu_i}^j > 0) \times$$

$$\times g(k) E \frac{\prod_{j=1}^k x_j}{S_k}.$$ 

with $u_i = r_{i+1} - r_i, i = 1, 2, \ldots, p - 1, u_p = 1 - r_p, N_{tu_i}^j$ is number of
particles alive at time $tu_i$ of the age-dependent branching process starting.
with one particle namely \( j \), \( g(k) = g(k_1, \ldots, k_p) \) is the proportion of configurations that have the desired number of ancestors corresponding to the given event, \( X^j \overset{d}{=} N^j_{tu} | N^j_{tu} > 0 \) and \( S = \sum_{j=1}^{k_{p+1}} X^j \).

Let \( q_i = \frac{u_i}{u_{i+1}} \) for \( 1 \leq i \leq p - 1 \). Then following [9] and using Proposition 3.1 (i), (ii) repeatedly we can show that \( P(\tau^1 < r_1 < r_2 < \ldots < r_p < \tau^k_{t-1} < r_{k-1} < t | A_t) \) converges to

\[
\frac{1}{q_1} \sum_{k_i \in \mathbb{N}^{k_{i-1}}} \int dx e^{-x} (q_1 x)^{k_{11}} e^{-xq_1} \times 
\prod_{i=2}^{p+1} \prod_{j=1}^{k_{i-1}} \int dx e^{-x} \frac{(q_i x)^{k_{ij}}}{k_{ij}!} e^{-xq_i} g(k) 
\times \int \prod_{i=1}^{k+1} dx_i \left( \prod_{i=1}^{k} \frac{x_i}{x_{i+1}} \right) e^{-\sum_{i=1}^{k+1} x_i \frac{(x_{k+1})^{k_{p+1}-k}}{(k_{p+1}) - k}} 
\frac{1}{q_1} \sum_{k_i \in \mathbb{N}^{k_{i-1}}} \prod_{i=1}^{p+1} \frac{(q_i)^{k_i}}{(1 + q_i)^{k_i - k_{i-1}}} g(k) \times 
\int \prod_{i=1}^{k+1} dx_i \left( \prod_{i=1}^{k} \frac{x_i}{x_{i+1}} \right) e^{-\sum_{i=1}^{k+1} x_i \frac{(x_{k+1})^{k_{p+1}-k}}{(k_{p+1}) - k}}.
\]

Consequently, we have shown that the random vector \( \tilde{\tau}_t \) converges in distribution to a random vector \( \tilde{T} \). From the above limiting quantity, one can show that the \( \tilde{T} \) has an absolutely continuous distribution on \([0, 1]^{k_{-1}}\). See [3] for a detailed proof.

\[\square\]

4. PROOF OF THEOREM 2.1

For the individual chosen, let \((a_t, X_t)\) be the age and position at time \( t \). As in Proposition 3.4 let \( \{L_{ti}, \{\eta_{ti}(u), 0 \leq u \leq L_{ti}\} : 1 \leq i \leq M_t\} \), be the lifetimes and the motion processes of the ancestors of this individual and \( \{\eta_{i(M_t+1)}(u) : 0 \leq u \leq t - \sum_{i=1}^{M_t} L_{ti}\} \) be the motion this individual. Let \( \mathcal{L}_t = \sigma(M_t, L_{ti}, 1 \leq i \leq M_t) \). It is immediate from the construction of the process that:

\[ a_t = t - \sum_{i=1}^{M_t} L_{ti}, \]
whenever $M_t > 0$ and is equal to $a + t$ otherwise; and that

$$X_t = X_0 + \sum_{i=1}^{M_t} \eta_t(L_{ti}) + \eta_{t(M_t+1)}(a_t).$$

Rearranging the terms, we obtain

$$(a_t, \frac{X_t}{\sqrt{t}}) = (a_t, \sqrt{\frac{t}{\mu}} Z_{t1}) + (0, \left(\sqrt{\frac{M_t}{t}} - \sqrt{\frac{t}{\mu}}\right) Z_{t2}) + (0, \frac{X_0}{\sqrt{t}} + Z_{t2}),$$

where $Z_{t1} = \frac{\sum_{i=1}^{M_t} \eta_t(L_{ti})}{\sqrt{M_t}}$ and $Z_{t2} = \frac{1}{\sqrt{t}} \eta_{t(M_t+1)}(a_t)$. Let $\epsilon > 0$ be given.

**Figure 1.** Tracking particles surviving at various times

$k_{11} = 6$, $k_2 = (2, 0, 1, 1, 0, 1), |k_2| = 5, k_3 = (0, 2, 1, 2, 1), |k_3| = 6$  
$n_{11} = 51, n_2 = (12, 7, 5, 4, 1, 7), n_3 = (5, 14, 9, 10, 8)$
\[ P(|Z_{t2}| > \epsilon |A_t) \leq P(|Z_{t2}| > \epsilon, a_t \leq k |A_t) + P(|Z_{t2}| > \epsilon, a_t > k |A_t) \]
\[ \leq P(|Z_{t2}| > \epsilon, a_t \leq k |A_t) + P(a_t > k |A_t) \]
\[ \leq \frac{E(|Z_{t2}|^2 I_{a_t \leq k} |A_t)}{\epsilon^2} + P(a_t > k |A_t) \]

By Proposition 3.1 and the ensuing tightness, for any \( \eta > 0 \) there is a \( k_\eta \)
\[ P(a_t > k |A_t) < \frac{\eta}{2} \]
for all \( k \geq k_\eta, t \geq 0 \). Next,
\[ E(|Z_{t2}|^2 I_{a_t \leq k_\eta} |A_t) = E(I_{a_t \leq k_\eta} E(|Z_{t2}|^2 |L_t) |A_t) \]
\[ = E(I_{a_t \leq k_\eta} \frac{v(a_t)}{t} |A_t) \]
\[ \leq \sup_{u \leq k_\eta} \frac{v(u)}{t} \]

Hence,
\[ P(|Z_{t2}| > \epsilon |A_t) \leq \frac{\sup_{u \leq k_\eta} v(u)}{\epsilon t} + \frac{\eta}{2} \]

Since \( \epsilon > 0 \) and \( \eta > 0 \) are arbitrary this shows that as \( t \to \infty \)
(4.1)
\[ Z_{t2}|A_t \xrightarrow{d} 0, \]

Now, for \( \lambda > 0, \theta \in \mathbb{R} \), as \( a_t \) is \( L_t \) measurable we have
\[ E(e^{-\lambda a_t} e^{-i \theta \sqrt{\frac{2}{\mu}} Z_{t1}} |A_t) = E(e^{-\lambda a_t} (E(e^{-i \theta Z_{t1}} |L_t) - e^{-\frac{\theta^2}{2\mu}}) |A_t) + \]
\[ -\frac{\theta^2}{2\mu} E(e^{-\lambda a_t} |A_t) \]

Proposition 3.3 shows that the first term above converges to zero and using Proposition 3.1 we can conclude that as \( t \to \infty \)
(4.2)
\[ (a_t, \frac{1}{\sqrt{\mu}} Z_{t1}) |A_t \xrightarrow{d} (U, V) \]

As \( X_0 \) is a constant, by Proposition 3.1 (c), (4.2), (4.1) and Slutsky’s Theorem, the proof is complete. \( \square \)
Let \( \phi \in C_b(\mathbb{R} \times \mathbb{R}_+) \). We shall show, for each \( k \geq 1 \), that the moment-functions of \( E\left(\frac{<Y_i,\phi>^k}{N_t}\right) |A_t \) converges as \( t \to \infty \). Then by Theorem 16.16 in [11] the result follows.

The case \( k = 1 \) follows from Theorem 2.1 and the bounded convergence theorem. We shall next consider the case \( k = 2 \). Pick two individuals \( C_1, C_2 \) at random (i.e. by simple random sampling without replacement) from those alive at time \( t \). Let the age and position of the two individuals be denoted by \( (a^i_t, X^i_t) \), \( i = 1, 2 \). Let \( \tau_t = \tau_{C_1,C_2,t} \) be the birth time of their common ancestor, say \( D \), whose position we denote by \( \tilde{X}_{\tau_t} \). Let the net displacement of \( C_1 \) and \( C_2 \) from \( D \) be denoted by \( X^i_{\tau_t - \tau}, i = 1, 2 \) respectively. Then \( X^i_t = \tilde{X}_{\tau_t} + X^i_{\tau_t - \tau}, i = 1, 2 \).

Next, conditioned on this history up to the birth of \( D(\equiv \mathcal{G}_t) \), the random variables \( (a^i_t, X^i_{\tau_t - \tau}), i = 1, 2 \) are independent. By Proposition 3.5 (i) conditioned on \( A_t \), \( \frac{\tilde{X}_t}{t} \) converges in distribution to an absolutely continuous random variable \( T \) (say) in \([0, 1]\). Also by Theorem 2.1 conditioned on \( \mathcal{G}_t \) and \( A_t \), \( \{(a^i_t, X^i_{\tau_t - \tau}), i = 1, 2\} \) converges in distribution to \( \{(U_i, V_i), i = 1, 2\} \) which are i.i.d. with distribution \( (U, V) \) as in Theorem 2.1. Also \( \frac{\tilde{X}_t}{\sqrt{\tau_t}} \) conditioned on \( A_{\tau_t} \) converges in distribution to a random variable \( S \) distributed as \( V \).

Combining these one can conclude that \( \{(a^i_t, X^i_{\tau_t - \tau}), i = 1, 2\} \) conditioned on \( A_t \) converges in distribution to \( \{(U_i, \sqrt{T}S + \sqrt{(1-T)V_i}), i = 1, 2\} \) where \( U_1, U_2, S, V_1, V_2 \) are all independent. Thus for any \( \phi \in C_b(\mathbb{R}_+ \times \mathbb{R}) \) we have, by the bounded convergence theorem,

\[
\lim_{t \to \infty} E\left(\prod_{i=1}^{2} \phi(a^i_t, X^i_t)\big|A_t\right) = E\left(\prod_{i=1}^{2} \phi(U_i, \sqrt{T}S + \sqrt{(1-T)V_i})\right) = m_2(\phi) \quad \text{(say)}
\]

Now,

\[
E\left(\frac{\tilde{Y}_t(\phi)}{N_t}\right)^2 \big|A_t\right) = E\left(\frac{(\phi(a_t, X^i_t))^2}{N_t}\big|A_t\right)
\]

\[
+ E\left(\prod_{i=1}^{2} \phi(a^i_t, X^i_t)\right) N_t(N_t - 1) \big|A_t\right)
\]

Using Proposition 3.1 (b) and the fact that \( \phi \) is bounded we have

\[
\lim_{t \to \infty} E\left(\frac{\tilde{Y}_t(\phi)^2}{N_t}\big|A_t\right) \text{ exists in } (0, \infty) \text{ and equals } m_2(\phi). \quad \text{The case } k > 2
\]
can be proved in a similar manner but we use Theorem 3.5 (ii) as outlined below. First we observe that as \( \phi \) is bounded,

\[
E \left( \frac{\tilde{Y}_t, \phi > k}{N_t^k} | A_t \right) + = \sum_i Eh(N_t, k) \left( \prod_{j=1}^k \phi(a_{ij}^t, \frac{X_{ij}^t}{\sqrt{t}}) | A_t \right) + g(\phi, C_t, N_t),
\]

where \( h(N_t, k) \to 1 \) and \( g(\phi, C_t, N_t) \to 0 \) as \( t \to \infty \); and \( i = \{i_1, i_2, \ldots, i_k\} \) is the index of \( k \) particles sampled without replacement from \( C_t \) (see (2.1)). Consider one such sample, and re-trace the genealogical tree \( \mathcal{T}_i \in \mathcal{T}(k) \), \( (\mathcal{T}(k) \) is the collection of all possible trees with \( k \) leaves given by \( i \)), until their most common ancestor. For any leaf \( i_j \) in \( \mathcal{T}_i \), let \( 1 = n(i_j, 1) < n(i_j, 2) < \cdots < n(i_j, N_{i_j}) \) be the labels of the internal nodes on the path from leaf \( i_j \) to the root. We list the ancestral times on this by \( \{\tau_1, \tau_{n(i_j, 1)}, \ldots, \tau_{n(i_j, N_{i_j})}\} \).

Finally we denote the net displacement of the ancestors in the time intervals

\[ [0, \tau_1], [\tau_1, \tau_{n(i_j, 2)}], \ldots, [\tau_{n(i_j, N_{i_j} - 1)}, \tau_{n(i_j, N_{i_j})}], [\tau_{n(i_j, N_{i_j})}, t] \]

by

\[ \tilde{\eta}_{ij}^1(\tau_1), \tilde{\eta}_{ij}^2(\tau_{n(i_j, 2)}, \tau_1), \ldots, \tilde{\eta}_{ij}^{N_{i_j}}(\tau_{n(i_j, N_{i_j})}, \tau_{n(i_j, N_{i_j} - 1)}), \tilde{\eta}_{ij}^N(t, \tau_{n(i_j, N_{i_j})}) \].

Given the above notation we have:

\[
E \left( \prod_{j=1}^k \phi(a_{ij}^t, \frac{X_{ij}^t}{\sqrt{t}}) | A_t \right) = E \left( \sum_{T \in \mathcal{T}_i} \prod_{j=1}^k f(\phi, j, t) | A_t \right),
\]

where

\[
f(\phi, j, t) = \phi(a_{ij}^t, \frac{1}{\sqrt{t}}(\tilde{\eta}_{ij}^1(\tau_1) + \sum_{m=2}^{N_{i_j}} \tilde{\eta}_{ij}^m(\tau_{n(i_j, m)}, \tau_{n(i_j, m - 1)}) + \tilde{\eta}_{ij}^N(t, \tau_{n(i_j, N_{i_j})})).
\]

Now by Theorem 3.5

\[
\frac{(\tau_1, \tau_{n(i_j, 2)}, \ldots, \tau_{n(i_j, N_{i_j})})}{\sqrt{t}} | A_t \xrightarrow{d} (T_1, T_{n(i_j, 2)}, \ldots, T_{n(i_j, N_{i_j})}).
\]

So by Theorem 2.1

\[
\lim_{t \to \infty} E \left( \frac{\tilde{Y}_t(\phi)}{N_t} \right)^2 | A_t \right) = E \left( \sum_i \sum_{T \in \mathcal{T}_i} \prod_{j=1}^k g(\phi, j, t) | A_t \right) \equiv m_k(\phi)
\]

(5.2)
where
\[ g(\phi, j, t) = \phi \left( U, S \sqrt{T_1} + \sum_{m=2}^{N_{ij}} Z_{ij}^{m} \sqrt{T_{n(ij,m)} - T_{n(ij,m-1)}} + Z_{ij}' \sqrt{1 - T_{n(ij,N_{ij})}} \right) \]

with \( S, Z_{ij}', Z_{ij}^m, m = 2, \ldots, N_{ij} \), are i.i.d. \( V, U \) is an independent random variable given in Theorem 2.1 and \( T_i \)'s are as in Theorem 3.5 (ii). Since \( \phi \) is bounded, the sequence \( \{ m_k(\phi) = \lim_{t \to \infty} E(\tilde{Y}_t, \phi) \} \) is necessarily a moment sequence of a probability distribution on \( \mathbb{R} \). This being true for each \( \phi \), by Theorem 16.16 in [11] we are done. \( \square \)

6. Proof of Theorem 2.4

Let \( Z \) be the Branching Markov process \( Y \) described earlier, with lifetime \( G \) exponential with mean \( \lambda \), \( p_1 = 1 \) and \( \eta = \eta_1 \) (see (2.7)). Then it is easy to see that for any bounded continuous function, \( S_t(\phi)(a,x) = E_{(a,x)} < Z_t, \phi > = E_{(a,x)} \phi(a_t, X_t) \) satisfies the following equation:

\[
S_t(\phi)(a,x) = e^{-\lambda t} W_t(\phi)(a,x) + \int_0^t ds \lambda e^{-\lambda(s-t)} W_{s-s}(S_{s-s}(\phi)(0,\cdot))(a,x),
\]

where \( W_t \) is the semi-group associated to \( \eta_1 \). Let \( \mathcal{L} \) be the generator of \( \eta_1 \). Making a change of variable \( s \to t - s \) in the second term of the above and then differentiating it with respect to \( t \), we have

\[
\frac{d}{dt} S_t(\phi)(a,x) = -\lambda e^{-\lambda t} W_t(\phi)(a,x) + e^{-\lambda t} \mathcal{L} W_t(\phi)(a,x) + \lambda S_t(\phi)(0,x)
\]

\[
+ \int_0^t ds \lambda e^{-\lambda(t-s)} W_{t-s}(S_{s-}(\phi)(0,\cdot))(a,x)
\]

\[
+ \int_0^t ds \lambda e^{-\lambda(t-s)} \mathcal{L} W_{t-s}(S_{s-}(\phi)(0,\cdot))(a,x)
\]

\[= \lambda S_t(\phi)(0,x) + (\mathcal{L} - \lambda) \left[ e^{-\lambda t} W_t(\phi)(a,x) + \int_0^t ds \lambda e^{-\lambda(t-s)} W_{t-s}(S_{s-}(\phi)(0,\cdot))(a,x) \right] \]

\[= \lambda S_t(\phi)(0,x) + (\mathcal{L} - \lambda) S_t(\phi)(a,x) \]

\[= \frac{\partial S_t(\phi)}{\partial a}(a,x) + \frac{\sigma^2(a)}{2} \Delta S_t(\phi)(a,x) + \lambda (S_t(\phi)(0,x) - S_t(\phi)(a,x)), \]
For each \( n \geq 1 \) define (another semigroup) \( R^n_t \phi(a, x) = E_{a,0}( \phi(a_t, x + \frac{X_t}{\sqrt{n}}) ) \).

Now note that,

\[
R^n_t \phi(a, x) = E_{a,0} \left( \phi(a_t, x + X_t \sqrt{n}) \right) = \phi(a, \sqrt{n} x),
\]

where \( \phi_n(a, x) = \phi(a, \frac{x}{\sqrt{n}}) \). Differentiating w.r.t. \( t \), we have that the generator of \( R^n_t \) is

\[
R^n \phi(a, x) = \frac{\partial \phi}{\partial a}(a, x) + \frac{\sigma^2(a)}{2n} \Delta \phi(a, x) + \lambda(\phi(0, x) - \phi(a, x)).
\]

**Proposition 6.1.** Let \( \epsilon > 0 \) and \( t \geq \epsilon \). Let \( \phi \in C^+_1(\mathbb{R}_+ \times \mathbb{R}^d) \). Then,

\[
\sup_{(a, x) \in \mathbb{R}_+ \times \mathbb{R}} | R^n_{nt}( \phi )(a, x) - U_t(\phi)(x) | \to 0.
\]

**Proof:** Let \( t \geq \epsilon \). Applying Theorem 2.1 to the process \( Z \), we have \( (a_{nt}, \frac{X_{nt}}{\sqrt{n}}) \overset{d}{\to} (U, V) \). The proposition is then immediate from the bounded convergence theorem and the fact that \( \phi \in C^+_1(\mathbb{R}_+ \times \mathbb{R}) \).

**Proposition 6.2.** Let \( \pi_{\nu} \) be a Poisson random measure with intensity \( \nu \) and \( t \geq 0 \). The log-Laplace functional of \( Y^n_t \),

\[
E_{\pi_{\nu}}[e^{-(\phi, Y^n_t)}] = e^{-(u^n_t \phi, \nu)},
\]

where

\[
u^n_t \phi(a, x) = R^n_{nt} n(1 - e^{-\phi})(a, x) - \lambda \int_0^t ds R^n_{n(t-s)}(n^2 \Psi_n(\frac{u^n_s \phi}{n}))(a, x),
\]

where \( \Psi_n(\phi)(a, x) := [F_n(1 - \phi(0,x)) - (1 - \phi(0,x))] \).

**Proof:** For any \( n \in \mathbb{N} \), let \( Y^n_t \) be the sequence of branching Markov processes defined in Section 2.2. It can be shown that its log-Laplace functional \( L^n_t \) satisfies,

\[
L^n_{nt} \phi(a, x) = e^{-\lambda nt} W^n_{nt}[e^{-\phi}](a, x) + \int_0^{nt} ds \lambda e^{-\lambda s} W^n_s \left[ F_n(L^n_{nt-s}(\phi(0, \cdot))) \right] (a, x) ds,
\]

where \( t \geq 0 \) and \( W^n_t \) is the semigroup associated with \( \eta_n \). Using the fact that \( e^{-\lambda u} = 1 - \int_0^u ds \lambda e^{-\lambda s} \) for all \( u \geq 0 \) and a routine simplification, as
Let Proposition 6.3.

\[ L_n^{t} \phi(a, x) = W_n^{t} [e^{-\phi}] (a, x) + \lambda \int_{0}^{nt} W_n^{t-s} (F_n (L_s^{t} \phi(0, \cdot)) - L_s^{t} \phi)(a, x) ds \]

Therefore \( v_n^{nt}(\phi)(a, x) = 1 - L_t^{n} \phi(a, x), \) satisfies,

\[ v_n^{nt}(\phi)(a, x) = W_n^{nt} (1 - e^{-\phi})(a, x) + \int_{0}^{nt} ds W_n^{t-s} ((1 - v_s^{t} \phi - F_n (1 - v_s^{t} \phi)(0, \cdot)))(a, x) \lambda ds. \]

Let \( \mathcal{L}^{n} \) be the generator of \( \eta_n. \) Then for \( 0 \leq s < t \)

\[ \frac{d}{ds} R_{n(t-s)}^{n} (v_{ns}^{n}(\phi))(a, x) = \]

\[ = -(nR^{n}) R_{n(t-s)}^{n} (v_{ns}^{n}(\phi))(a, x) + R_{n(t-s)}^{n} \left( \frac{\partial}{\partial s} v_{ns}^{n}(\phi) \right)(a, x) \]

\[ = -(nR^{n}) R_{n(t-s)}^{n} (v_{ns}^{n}(\phi))(a, x) + R_{n(t-s)}^{n} \left( n \mathcal{L}^{n} W_{ns}^{n} (1 - e^{-\phi}) + n \lambda (1 - v_{ns}^{n} \phi) - F_n (1 - v_{ns}^{n} \phi)(0, \cdot) \right)(a, x) \]

\[ + R_{n(t-s)}^{n} \left( \int_{0}^{ns} dr n \mathcal{L}^{n} (W_{ns-r}^{n} (1 - v_{r}^{n} \phi) - F_n (1 - v_{r}^{n} \phi)(0, \cdot)) \right)(a, x) \]

\[ = R_{n(t-s)}^{n} (-n \lambda (v_{ns}^{n}(\phi))(0, \cdot) - v_{ns}^{n}(\phi)) + \lambda ((1 - v_{ns}^{n} \phi) - F_n (1 - v_{ns}^{n} \phi)(0, \cdot))(a, x)) \]

\[ = -R_{n(t-s)}^{n} (n \Psi_n (v_{ns}^{n} \phi))(a, x), \]

Integrating both sides with respect to \( s \) from 0 to \( t, \) we obtain that

\[ v_{nt}^{n}(\phi)(a, x) = R_{nt}^{n}(1 - e^{-\phi})(a, x) - \int_{0}^{t} ds R_{n(t-s)}^{n} (n \Psi_n (v_{ns}^{n} \phi))(a, x). \]

If \( \pi_{n\nu} \) is a Poisson random measure with intensity \( n\nu, \) then

\[ E_{\pi_{n\nu}} [e^{-\langle \phi, Y_{n\nu}^{\pi} \rangle}] = E_{\pi_{n\nu}} [e^{-\langle \hat{\phi}, Y_{n\nu}^{\pi} \rangle}] = e^{\langle L_{\pi}^{n}(\hat{\phi}) - 1, n\nu \rangle} = e^{-\langle n\nu, \hat{\phi} \rangle}. \]

Therefore if we set \( u_{n}^{t}(\phi) \equiv n v_{nt}^{n}(\hat{\phi}). \) From (6.9), it is easy to see that \( u_{n}^{t}(\phi) \) satisfies (6.4). \( \square \)

For any \( f : \mathbb{R}_{+} \times \mathbb{R} \to \mathbb{R}, \) we let \( \| f \|_{\infty} = \sup_{(a, x) \in \mathbb{R}_{+} \times \mathbb{R}} | f(a, x) |. \) With a little abuse of notation we shall let \( \| f \|_{\infty} = \sup_{x \in \mathbb{R}} | f(x) | \) when \( f : \mathbb{R} \to \mathbb{R} \) as well.

**Proposition 6.3.** Let \( \epsilon > 0. \) \( \phi \in C_{i}^{+}(\mathbb{R}_{+} \times \mathbb{R}^{d}) \) and \( u_{t}^{n}(\phi) \) be as in Proposition 6.2 and \( u_{t}(\phi) \) be as in Theorem 2.4. Then for \( t \geq \epsilon, \)

\[ \sup_{(a, x) \in \mathbb{R}_{+} \times \mathbb{R}} | u_{t}^{n}(\phi)(a, x) - u_{t}(\phi)(x) | \to 0 \]
Proof: For any real $u \in \mathbb{R}$, define, $\varepsilon_n(u) = \lambda n^2(F_n(1 - \frac{u}{n}) - (1 - \frac{u}{n})) - u^2$. So,

$$u^n_t(\phi)(a, x) = R^n_{nt}n(1 - e^{-\frac{\phi}{n}})(a, x) - \lambda \int_0^t dsR^n_{n(t-s)}(n^2\Psi_n(\frac{u^n_s\phi}{n}))(a, x)$$

$$= R^n_{nt}n(1 - e^{-\frac{\phi}{n}})(a, x) - \int_0^t dsR^n_{n(t-s)}(\varepsilon_n(u^n_s(\phi(0\cdot))))(a, x)$$

$$- \lambda \int_0^t dsR^n_{n(t-s)}(u^n_s(\phi(0\cdot))^2)(a, x)$$

Now

$$u^n_t(\phi)(a, x) - u_t(\phi)(x) =$$

$$= R^n_{nt}(n(1 - e^{-\frac{\phi}{n}}))(a, x) - U_t(\phi)(x)$$

$$- \int_0^t dsR^n_{n(t-s)}(\varepsilon_n(u^n_s(\phi(0\cdot))))(a, x)$$

$$+ \lambda \int_0^t ds \left( U_{t-s}(u^n_s(\phi))^2(a, x) - R^n_{n(t-s)}(u^n_s(\phi(0\cdot))^2)(a, x) \right)$$

$$= R^n_{nt}(n(1 - e^{-\frac{\phi}{n}}))(a, x) - U_t(\phi)(x) - \int_0^t dsR^n_{n(t-s)}(\varepsilon_n(u^n_s(\phi(0\cdot))))(a, x)$$

$$+ \lambda \int_0^t dsR^n_{n(t-s)}(u^n_s(\phi)^2 - u^n_s(\phi(0\cdot))^2)(a, x)$$

$$+ \lambda \int_0^t ds \left( U_{t-s}(u^n_s(\phi))^2(a, x) - R^n_{n(t-s)}(u^n_s(\phi)^2)(a, x) \right)$$

Observe that, $R^n$ is a contraction, $\| u^n_t(\phi) \|_\infty \leq \| \phi \|_\infty$ and $\| u.t(\phi) \|_\infty \leq \| \phi \|_\infty$ for $\phi \in C_t(\mathbb{R}_+ \times \mathbb{R})$. Therefore, we have

$$\| u^n_t(\phi) - u_t(\phi) \|_\infty \leq \| R^n_{nt}(n(1 - e^{-\frac{\phi}{n}})) - U_t(\phi) \|_\infty + \int_0^t ds \| u^n_s(\phi) - u_s(\phi) \|_\infty$$

$$+ 2\lambda \| \phi \|_\infty \int_0^t ds \| u^n_s(\phi) - u_s(\phi) \|_\infty$$

$$+ \lambda \int_0^t ds \| (U_{t-s} - R^n_{n(t-s)})(u^n_s(\phi)^2) \|_\infty.$$ 

For $\phi \in C_t(\mathbb{R}_+ \times \mathbb{R}^d)$, note that, $U_t$, is a strongly continuous semi-group implies that $u_s(\phi)$ is a uniformly continuous function. So using Proposition 6.3 the first term and the last term go to zero. By our assumption on $F$, $\| \varepsilon_n(u^n_s(\phi(0\cdot))) \|_\infty$ will go to zero as $n \to \infty$. Now using the standard Gronwall argument we have the result. \qed
Proposition 6.4. Let $\epsilon > 0$. The processes $Y^m$ are tight in $D((\epsilon, \infty), M(\mathbb{R}_+ \times \mathbb{R}))$.

Proof By Theorem 3.7.1 and Theorem 3.6.5 (Aldous Criterion) in [7], it is enough to show

$$(6.11) \quad \langle Y^m_{\tau_n + \delta_n}, \phi \rangle - \langle Y^m_{\tau_n}, \phi \rangle \xrightarrow{d} 0,$$

where $\phi \in C^+_c(\mathbb{R}_+ \times \mathbb{R})$, $\delta_n$ is a sequence of positive numbers that converge to 0 and $\tau_n$ is any stop time of the process $Y^m$ with respect to the canonical filtration, satisfying $0 < \epsilon \leq \tau_n \leq T$ for some $T < \infty$.

First we note that, as $\langle Y^m_t, 1 \rangle$ is a martingale, for $\gamma > 0$ by Chebyshev’s inequality and Doob’s maximal inequality we have

$$(6.12) \quad P(\langle Y^m_{\tau_n}, \phi \rangle > \gamma) \leq \frac{1}{\gamma} c_1 \| \phi \|_\infty E(\sup_{t \leq \tau_n \leq T} \langle Y^m_t, 1 \rangle) \leq \frac{1}{\gamma} c_2 \| \phi \|_\infty$$

By the strong Markov Property applied to the process $Y^m$ we obtain that for $\alpha, \beta \geq 0$, we have

$$L_n(\delta_n; \alpha, \beta) = E(\exp(-\alpha \langle Y^m_{\tau_n + \delta_n}, \phi \rangle - \beta \langle Y^m_{\tau_n}, \phi \rangle))$$

$$= E(\exp(-\langle Y^m_{\tau_n}, u^\alpha_{\delta_n}(\alpha \phi) + \beta \phi \rangle)$$

$$= E(\exp(-\langle Y^m_{\tau_n - \epsilon}, u^\alpha_{\delta_n}(\alpha \phi) + \beta \phi \rangle)))$$

Therefore

$$| L_n(0; \alpha, \beta) - L_n(\delta_n; \alpha, \beta) | \leq$$

$$\leq \| u^\alpha_{\delta_n}(\alpha \phi) + \beta \phi \rangle - u^\alpha_{\delta_n}((\alpha + \beta) \phi) \|_\infty E(\sup_{t \leq T} \langle Y^m_t, 1 \rangle)$$

$$\leq c_1 \| u^\epsilon(\alpha \phi) + \beta \phi \rangle - u^\alpha_{\delta_n}((\alpha + \beta) \phi) \|_\infty$$

where is the last inequality is by Doob’s maximal inequality. Now,

$$\| u^\alpha_{\delta_n}(\alpha \phi) + \beta \phi \rangle - u^\alpha_{\delta_n}((\alpha + \beta) \phi) \|_\infty \leq R_n^\alpha(u^\alpha_{\delta_n}(\alpha \phi) - \alpha \phi) \|_\infty +$$

$$+ c_2 \| \phi \|_\infty \int_0^\epsilon da \| u^\alpha_{\delta_n}(\alpha \phi) + \beta \phi \rangle - u^\alpha_{\delta_n}((\alpha + \beta) \phi) \|_\infty + d_n(\phi),$$
where \( d_n(\phi) = \lambda \int_0^\epsilon da \parallel \epsilon_n(u_n^a(u_{\delta_n}^a(\alpha \phi) + \beta \phi) + \epsilon_n(u_n^a((\alpha + \beta)\phi)) \parallel_\infty \). Observe that

\[
\parallel R_{n\epsilon}^n(u_{\delta_n}^a(\alpha \phi) - \alpha \phi) \parallel_\infty \leq \parallel R_{n\epsilon}^n(u_{\delta_n}^a(\alpha \phi) - R_{n\delta_n}^a(\alpha \phi)) \parallel_\infty \\
+ \parallel R_{n\epsilon}^n(R_{n\delta_n}^a(\alpha \phi) - \alpha \phi) \parallel_\infty
\]

\[
\leq \parallel u_{\delta_n}^a(\alpha \phi) - R_{n\delta_n}^a(\theta_2 \phi) \parallel_\infty + \parallel R_{n(\epsilon + \delta_n)}^a(\alpha \phi) - R_{n\epsilon}^n(\alpha \phi) \parallel_\infty
\]

\[
\leq \parallel R_{n\delta_n}^a(n(1 - e^{\frac{\delta_n \phi}{n}}) - \alpha \phi) \parallel_\infty + \int_0^{\delta_n} da \parallel R_{n(\delta_n - a)}^a(n^2 \Psi(u_n^a \phi)) \parallel_\infty \\
+ \parallel R_{n(\epsilon + \delta_n)}^a(\alpha \phi) - R_{n\epsilon}^n(\alpha \phi) \parallel_\infty
\]

\[
\leq n(1 - e^{\frac{\delta_n \phi}{n}}) - \alpha \phi \parallel_\infty + \delta_n c_2(\parallel \phi \parallel_\infty^2 + 1) + \parallel R_{n(\epsilon + \delta_n)}^a(\alpha \phi) - R_{n\epsilon}^n(\alpha \phi) \parallel_\infty,
\]

\[\equiv \epsilon_n(\phi)\]

Consequently,

\[
\parallel u_{\delta_n}^a(\alpha \phi) + \beta \phi) - u_n^a((r + s)\phi) \parallel_\infty \leq \epsilon_n(\phi) + d_n(\phi)
\]

\[
+ c_2 \parallel \phi \parallel_\infty \epsilon \int_0^\epsilon da \parallel u_n^a(u_{\delta_n}^a(\alpha \phi) + \beta \phi) - u_n^a((r + s)\phi) \parallel_\infty.
\]

By Proposition 6.1, \( \epsilon_n(\phi) \to 0 \) and \( d_n(\phi) \to 0 \) by our assumption \( F_n \). Hence by a standard Gronwall argument we have that,

\[(6.13) \quad |L_n(0; s, r) - L_n(\delta_n; s, r)| \to 0\]

By (6.12), \( \{\langle Y_{n_k} \rangle; n = 1, 2, \ldots\} \) is tight in \( \mathbb{R}_+ \). Take an arbitrary subsequence. Then there is a further subsequence of it indexed by \( \{n_k; k = 1, 2, \ldots\} \) such that \( \langle Y_{n_k} \rangle \) converges in distribution to some random limit \( b \). Thus we get

\[(Y_{n_k}(\phi), Y_{n_k}(\phi)) \to (b, b) \quad \text{as} \quad k \to \infty.
\]

But (6.13) implies that

\[(Y_{n_k + \delta_n}(\phi), Y_{n_k + \delta_n}(\phi)) \to (b, b) \quad \text{as} \quad k \to \infty.
\]

This implies that \( \langle Y_{n_k + \delta_n} \rangle \to \langle Y_{n_k} \rangle \) \( \to 0 \) as \( k \to \infty \). So (6.11) holds and the proof is complete. \( \Box \)

**Proof of Theorem 2.4** Proposition 6.3 shows that the log-Laplace functionals of the process \( Y_t \) converge to \( Y_t \) for every \( t \geq \epsilon \). Proposition 6.4 implies tightness for the processes. As the solution to (2.8) is unique, we are done. \( \Box \)
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