An Alternative String Theory in Twistor Space
for N=4 Super-Yang-Mills

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In this letter, an alternative string theory in twistor space is proposed for describing perturbative N=4 super-Yang-Mills theory. Like the recent proposal of Witten, this string theory uses twistor worldsheet variables and has manifest spacetime superconformal invariance. However, in this proposal, tree-level super-Yang-Mills amplitudes come from open string tree amplitudes as opposed to coming from D-instanton contributions.
In a recent paper [1], Witten has shown that the simple form of maximal helicity violating amplitudes of Yang-Mills theory has a natural generalization to the non-maximal helicity violating amplitudes. He also constructed a topological B-model from twistor worldsheet variables and argued that D-instanton contributions in this model reproduce the perturbative super-Yang-Mills amplitudes. For details on this model and the twistor approach to super-Yang-Mills, see the review and references in [1].

The formula for D-instanton contributions of degree \( d \) to \( n \)-gluon tree-level amplitudes is [1] [2]

\[
B(\lambda_r, \bar{\lambda}_r) = \int d^{2d+2}a \int d^{2d+2}b \int d^{4d+4} \gamma \prod_{\sigma=1}^{n} d\sigma_r \left( \text{vol}(GL(2))^{-1} \right) 
\]

\[
\prod_{r=1}^{n-1} (\sigma_r - \sigma_{r+1})^{-1} (\sigma_n - \sigma_1)^{-1} \prod_{r=1}^{n} \delta(\frac{\lambda^2_r}{\lambda^1_r} - \frac{\lambda^2(\sigma_r)}{\lambda^1(\sigma_r)}) \exp\left(i\bar{\lambda}^\alpha_r \lambda^\alpha(\sigma_r) \right) 
\]

\[
Tr[\phi_1(\frac{\psi^A(\sigma_1)}{\lambda^1(\sigma_1)}) \phi_2(\frac{\psi^A(\sigma_2)}{\lambda^1(\sigma_2)}) ... \phi_n(\frac{\psi^A(\sigma_n)}{\lambda^1(\sigma_n)})] 
\]

where \( P^\alpha_r = \lambda^\alpha_r \bar{\lambda}^\alpha_r \) is the momentum of the \( r \)th state,

\[
\lambda^\alpha(\sigma) = \sum_{k=0}^{d} a^\alpha_k \sigma^k, \quad \mu^\hat{\alpha}(\sigma) = \sum_{k=0}^{d} b^\hat{\alpha}_k \sigma^k, \quad \psi^A(\sigma) = \sum_{k=0}^{d} \gamma^A_k \sigma^k, 
\]

\( \phi_r(\psi^A) \) is the N=4 superfield whose lowest component is the positive helicity gluon and whose top component is the negative helicity gluon, and the \( (\text{vol}(GL(2))^{-1} \) factor can be used to remove one of the a integrals and three of the \( \sigma \) integrals.

For maximal helicity violating amplitudes (i.e. \( n - 2 \) positive helicity gluons and 2 negative helicity gluons), the above formula when \( d = 1 \) has been shown to give the correct \( n \)-point amplitude. For non-maximal helicity violating amplitudes, it has been suggested that this formula may also give the correct \( n \)-point amplitude where one has \( n - d - 1 \) positive helicity gluons and \( d + 1 \) negative helicity gluons. Although there is a possibility that the formula of (1) needs to be modified for non-maximal amplitudes by contributions from instantons of lower degree, it has been recently verified that no such modifications are necessary when \( d = 2 \) and \( n = 5 \) [2]. It will be assumed below that the formula of (1) correctly reproduces the super-Yang-Mills tree amplitudes for any \( d \) and \( n \).

In this letter, a new string theory in twistor space is proposed which reproduces the formula of (1) using ordinary open string tree amplitudes as opposed to D-instanton contributions. This string theory shares many aspects in common with the original idea
of Nair in \cite{3}. The worldsheet matter variables in this string theory consist of a left and right-moving set of super-twistor variables,

\[ Z_L^I = (\lambda^\alpha_L, \mu^\alpha_L, \psi^A_L), \quad Z_R^I = (\lambda^\alpha_R, \mu^\alpha_R, \psi^A_R) \]  

(2)

for \( \alpha, \dot{\alpha} = 1 \) to \( 2 \) and \( A = 1 \) to \( 4 \), a left and right-moving set of conjugate super-twistor variables,

\[ Y_L^I = (\bar{\mu}^{L\alpha_L}, \bar{\lambda}^{L\dot{\alpha}_L}, \bar{\psi}^{A_L}), \quad Y_R^I = (\bar{\mu}^{R\alpha_R}, \bar{\lambda}^{R\dot{\alpha}_R}, \bar{\psi}^{A_R}) \]  

(3)

and a left and right-moving current algebra,

\[ j^C_L, \quad j^C_R \]  

(4)

where \( C \) is Lie-algebra valued and \( j^C_L \) and \( j^C_R \) satisfy the usual OPE's of a current algebra, i.e.

\[ j^C_L(y)j^D_L(z) \rightarrow \frac{f^{CDE}_L j^E_L(z)}{y-z} + \frac{kg^{CD}}{(y-z)^2}, \quad j^C_R(\bar{y})j^D_R(\bar{z}) \rightarrow \frac{f^{CDE}_R j^E_R(\bar{z})}{\bar{y}-\bar{z}} + \frac{kg^{CD}}{\bar{y}-\bar{z})^2}. \]  

(5)

The current algebra can be constructed from free fermions, a Wess-Zumino-Witten model, or any other model.

The worldsheet action is

\[ S = \int d^2z(Y_{LI} \nabla_R Z^I_L + Y_{RI} \nabla_L Z^I_R) + S_G \]  

(6)

where \( S_G \) is the worldsheet action for the current algebra and \( (\nabla_R, \nabla_L) \) contains a worldsheet GL(1) connection for which \( Z^I_L \) and \( Z^I_R \) have charge +1, and \( Y_{LI} \) and \( Y_{RI} \) have charge \(-1\).

Quantizing this worldsheet action gives rise to left and right-moving Virasoro ghosts, \((b_L, c_L)\) and \((b_R, c_R)\), as well as left and right-moving GL(1) ghosts, \((u_L, v_L)\) and \((u_R, v_R)\). The untwisted left-moving stress tensor is

\[ T_0 = Y_{LI} \partial_L Z^I_L + T_G + b_L \partial_L c_L + c_L(b_L c_L) + u_L \partial_L v_L \]  

(7)

where \( T_G \) is the left-moving stress tensor for the current algebra, and the left-moving GL(1) current is

\[ J = Y_{LI} Z^I_L. \]  

(8)
To have vanishing conformal anomaly, the current algebra must be chosen such that the central charge contribution from $T_G$ is 28. Note that there is no GL(1) anomaly because of cancellation between bosons and fermions in $J$.

The open string theory is defined using the conditions

$$Z^I_L = Z^I_R, \quad Y_{LI} = Y_{RI}, \quad j^C_L = j^C_R, \quad c_L = c_R, \quad b_L = b_R, \quad v_L = v_R, \quad u_L = u_R \quad (9)$$
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on the open string boundary. Unlike a usual open string theory where Lie algebra indices come from Chan-Paton factors, the Lie algebra indices in this open string theory come from a current algebra.

The physical integrated and unintegrated open string vertex operator for the super-Yang-Mills states is

$$V = \int dz \, j^C(z)\Phi_C(Z(z)), \quad U = c(z)j^C(z)\Phi_C(Z(z)). \quad (10)$$

The superfields $\Phi_C(Z)$ are similar to those defined in [1], namely for a super-Yang-Mills state with momentum $P_r^{\alpha\dot{\alpha}} = \lambda^\alpha_\dot{\alpha}$,

$$\Phi_C(Z(z_r)) = \delta(\frac{\lambda^2}{\lambda^1} - \frac{\lambda^2(z_r)}{\lambda^1(z_r)}) \exp(i\bar{\lambda}_r^\dot{\alpha}\lambda^1_\dot{\alpha}M^\dot{\alpha}(z_r))\phi_C(\psi^A(z_r)) \quad (11)$$

where $\phi_C(\psi^A)$ is the same N=4 superfield as in [1]. Note that $\Phi_C(Z)$ is GL(1)-neutral and has zero conformal weight.

Tree-level open string scattering amplitudes are computed in the usual manner from the disk correlation function

$$A = \langle U_1(z_1)U_2(z_2)U_3(z_3) \int dz_4V_4(z_4)\ldots \int dz_nV_n(z_n) \rangle \quad (12)$$

where different twistings of the stress tensor are used to compute different helicity violating amplitudes. For amplitudes involving $(n - d - 1)$ positive helicity gluons and $d + 1$ negative helicity gluons, the twisted stress tensor is defined as

$$T_d = T_0 + \frac{d}{2} \partial J \quad (13)$$

where $T_0$ and $J$ are defined in [7] and [8]. Note that $T_d$ has no conformal anomaly since $J$ has no GL(1) anomaly.

So after twisting, $Z^I$ has conformal weight $-\frac{d}{2}$ and $Y_I$ has conformal weight $\frac{4d^2+2}{2}$. This means that the disk correlation function of (12) involves an integration over the $4d + 4$
bosonic and 4d+4 fermionic zero modes of $Z^I$, except for the one bosonic zero mode which can be removed using the worldsheet GL(1) gauge invariance. Performing the correlation function for the current algebra gives the contribution\[2\]

$$Tr[\phi_1...\phi_n] \prod_{r=1}^{n-1} (z_r - z_{r+1})^{-1}(z_n - z_1)^{-1},$$  (15)

and the $(b,c)$ correlation function gives the factor $(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)$.

So one obtains the formula

$$A = \int d^{2d+2}a \ d^{2d+2}b \ d^{4d+4}\gamma \int dz_1... \int dz_n (Vol(GL(2)))^{-1}$$  (16)

$$\prod_{r=1}^{n-1} (z_r - z_{r+1})^{-1}(z_n - z_1)^{-1} \prod_{r=1}^{n} \delta(\frac{\lambda^2}{\lambda^1} - \lambda^2(z_r)) \exp(i\bar{\lambda}^\alpha \lambda^1_\alpha(z_r))$$

$$Tr[\phi_1(\frac{\psi^A(z_1)}{\lambda^1(z_1)})\phi_2(\frac{\psi^A(z_2)}{\lambda^1(z_2)})...\phi_n(\frac{\psi^A(z_n)}{\lambda^1(z_n)})]$$

where

$$\lambda^\alpha(z) = \sum_{k=0}^{d} a^\alpha_k z^k, \quad \mu^{\dot{\alpha}}(z) = \sum_{k=0}^{d} b^{\dot{\alpha}}_k z^k, \quad \psi^A(z) = \sum_{k=0}^{d} \gamma^A_k z^k,$$

$(a^\alpha_k, b^{\dot{\alpha}}_k, \gamma^A_k)$ are the zero modes of $Z^I$ on a disk, and the SL(2) part of GL(2) can be used to fix three of the $z_r$ integrals and reproduce the $(b,c)$ correlation function. This formula clearly agrees with the formula of (1) for the D-instanton amplitude where the $\sigma$ variable from the D1-string worldvolume has been replaced with the $z$ variable from the open string boundary.

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2 As was pointed out to me by Edward Witten, one also gets multitrace contributions such as

$$Tr[\phi_1...\phi_m]Tr[\phi_{m+1}...\phi_n](\prod_{r=1}^{m-1} (z_r - z_{r+1})^{-1}(z_m - z_1)^{-1})(\prod_{s=m+1}^{n-1} (z_s - z_{s+1})^{-1}(z_n - z_{m+1})^{-1})$$  (14)

coming from other contractions of the current algebra. These multitrace contributions are also present in the amplitudes coming from D-instantons in [1] and in the proposal of Nair in [3].
References

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[3] V.P. Nair, *A Current Algebra for some Gauge Theory Amplitudes*, Phys. Lett. B214 (1988) 215.