The small parameter method in the problem of the lateral surface shape of the vertical liquid bridge between planes

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Abstract. The article is devoted to the study of the shape of the vertical liquid bridge between two solid planes with take into account the gravity force. The variation problem statement is given. The parametric form of solution presentation is used. The solution is constructed as a series in powers of a small Bond number. An algorithm for finding zeroth and the subsequent approximations is proposed. A non-uniqueness of the solution is detected. There may be several different profiles of its lateral surface for a fixed height of the liquid bridge. The dependence of the solutions number on the liquid bridge height has been analyzed.

1. Introduction
The study of the lateral surface shape of a small liquid bridge in zone of a three-phase contact is an urgent task when solving many scientific and technological problems (in particular, when studying the shape of the liquid meniscuses formed in the process of crystal growth by the Stepanov method [1]). In this case, the liquid melt drop is between two solid surfaces with different properties (for example, between a molybdenum shaper and a crystalline seed). To obtain a solution to the problem, both numerical and asymptotic methods for solving equations are used. In [2] the asymptotic of the solution of the problem of the surface shape of a horizontal liquid bridge between vertical walls with small Bond number was constructed, the solution is presented in parametric form. And in [3] the problem of the lateral surface shape of a vertical axisymmetric catenoidal liquid bridge between two planes with take into account the gravity force is solved by the method of successive approximations under the assumption that the Bond number is small. A detailed bibliography of works devoted to the study of liquid bridges is given in [2] and [4].

In the present work the solution of this problem is constructed in the form of a series in powers of the Bond number in contrast to the work [3]. A parametric form of representation of the solution is used which helps to avoid problems associated with the infinity of the first derivative in the neck of the liquid bridge and allows us to build a main iterative process.
2. Formulation of the problem

Let us consider the case of a vertical liquid bridge in contact with two solid planes (bottom and top) (figure 1). In this case, the liquid bridge is a drop of liquid of small volume between two parallel solid planes with given properties (oblate drop). In view of the supposed axial symmetry we will solve the problem of finding the profile of the liquid bridge in the cylindrical coordinate system \((r, z)\). Let the surface tensions between media are \(\alpha_{13}, \alpha_{14}, \alpha_{34}, \alpha_{23}, \alpha_{24}\), respectively.

The contact region between the drop and plane \(z = 0\) (bottom) is a circle with radius \(r_1\), while the contact region between the drop and plane \(z = \hat{h}\) (top) is a circle of radius \(r_2\). We denote the sought functions that describe the profiles of the lower and upper parts of the drop as by \(u_1(r)\) and \(u_2(r)\), respectively. The region separating these parts (neck) is the circle of radius \(r^*\) \((r^* \geq 0)\). Since there are no physical reasons for the appearance of sharpened bridge profile, the tangent to the drop profile at the point with abscissa \(r = r^*\) must be vertical: \(u_1'(r^*) = -\infty\), \(u_2'(r^*) = +\infty\), \(u_1(r^*) = u_2(r^*)\). The only exception is the contact between the sessile and pendent drop at the same point.

In addition, we assume that wetting angles \(\theta_1\) and \(\theta_2\) do not exceed 90 degrees. Then, our liquid bridge has a catenoidal shape \(r^* < \min(r_1, r_2)\), and \(u_1(r)\) and \(u_2(r)\) are single-valued functions. The volume of the liquid bridge is considered fixed: \(I\{u_1(r), u_2(r)\} = V\). Let us introduce the functional \(J\{u_1(r), u_2(r)\}\), which includes the surface energy and the energy of the gravity force. The surface energy in turn consists of the part that corresponds to the free surface of the bridge and the part that corresponds to its contact with a solid. Therefore, we obtain the isoperimetric problem: find the minimum of functional \(J\{u_1(r), u_2(r)\}\) provided that functional \(I\{u_1(r), u_2(r)\}\) takes the preset value \(V\). In accordance with the Euler theorem on isoperimetric problems, we introduce the extended functional \(J\{u_1(r), u_2(r)\} + \lambda I\{u_1(r), u_2(r)\}\) \((\lambda\) is a Lagrange multiplier).

2.1. Transition to a dimensionless view

Let us introduce dimensionless variables \(\xi = r/V^{1/3}, w_i = u_i/V^{1/3}, i = 1, 2\) and dimensionless parameters \(\mu = \lambda V^{1/3}/\alpha_{34}, h = \hat{h}/V^{1/3}, B = g\rho V^{2/3}/\alpha_{34}\). Here \(g\) is the acceleration of gravity, \(\rho\) is the density of the fluid. Dimensionless constant \(B\) is the Bond number.
2.2. A parametric formulation
To avoid problems associated with the verticality of the tangent to the profile of the liquid bridge, we proceed to the parametric form of the solution \( \xi = \xi(t), w_1 = w_1(t), w_2 = w_2(t) \), where \( t \) is a parameter. As a parameter we use the arc length of a flat curve that defines the lateral surface shape of the liquid bridge and we move from the point at the bottom to the point on the top. Therefore, for \( t = 0 \) we have \( \xi(0) = \xi_1, w_1(0) = 0 \), and for \( t = t_p = t_1 + t_2 - \xi(t_p) = \xi_2, w_2(t_p) = h \), where \( t_1 \) is the length of the arc of the curve from the bottom to the neck, and \( t_2 \) is the length of the arc from the neck to the top. Thus, \( \xi = \xi(t), w(t) \equiv w_1(t), t \in [0, t_1] \) is the equation of the lower branch, and \( \xi = \xi(t), w(t) \equiv w_2(t), t \in [t_1, t_p] \) - the equation of the upper branch of the curve describing the lateral surface shape of the liquid bridge. The sought dimensionless quantities are \( \xi(t), w(t), \mu, t_1, t_p \).

2.3. Equations and boundary conditions
We rewrite the extended functional in dimensionless variables and parametric form. By varying the extended functional, we obtain two Euler equations and transversality conditions. These equations have the following form:

for the lower branch
\[
(w''(t)\xi'(t) - w'(t)\xi''(t))\xi(t) + w'(t) + Bw(t)\xi(t) + \mu\xi(t) = 0, \quad t\epsilon(0, t_1],
\]

for the upper branch
\[
(w''(t)\xi'(t) - w'(t)\xi''(t))\xi(t) + w'(t) - B(h - w(t))\xi(t) + \mu\xi(t) = 0, \quad t\epsilon(t_1, t_p).
\]

The equations (1)-(2) are written taking into account the fact that the parameter is the length of the extremal arc and the additional equation connecting the desired functions has the appearance
\[
(\xi'(t))^2 + (w'(t))^2 = 1, \quad t\epsilon(0, t_p).
\]

The boundary conditions of the problem are as follows:

the transversality conditions
\[
\cos \theta_1 \equiv \frac{d\xi}{dt}(0) = -\alpha_1 \quad (\alpha_1 = (\alpha_{14} - \alpha_{13})/\alpha_{34}),
\]

\[
\cos \theta_2 \equiv \frac{d\xi}{dt}(t_p) = \alpha_2 \quad (\alpha_2 = (\alpha_{24} - \alpha_{23})/\alpha_{34});
\]

the conditions of contact of the liquid bridge with the bottom
\[
w(0) = 0,
\]

with the top
\[
w(t_p) = h;
\]

the continuity conditions of the liquid bridge profile in the neck
\[
\xi(t_1 - 0) = \xi(t_1 + 0) = \xi(t_1), \quad w(t_1 - 0) = w(t_1 + 0) = w(t_1),
\]

\[
\xi'(t_1 - 0) = \xi'(t_1 + 0) = \xi'(t_1), \quad w'(t_1 - 0) = w'(t_1 + 0) = w'(t_1);
\]

the verticality condition of the tangent in the neck
\[
\frac{d\xi}{dt}(t_1) = 0;
\]

and finally, the volume conservation law
\[
- \int_0^{t_1} w(t)\xi(t)\xi'(t)dt + \int_{t_1}^{t_p} (h - w(t))\xi(t)\xi'(t)dt + 0.5\xi_2^2 h = \frac{1}{2\pi}.
\]
3. Algorithm for solving the problem

We construct the solution of the problem of the lateral surface shape of a vertical catenoidal liquid bridge for the case of small Bond numbers. So, let $B$ be a small parameter the task at hand. We seek a solution to the problem as a series in powers of a small parameter, namely:

$$
\xi(t) = \xi_0(t) + \xi_1(t)B + ..., \ w(t) = w_0(t) + w_1(t)B + ..., \\
t_1 = t_{10} + t_{11}B + ..., \ t_p = t_{p0} + t_{p1}B + ..., \\
\mu = \mu_0 + \mu_1B + ...$

at the same time

$$
\xi_+ = \xi(t_1) = \xi_0(t_1) + \xi_1(t_1)B + ... = \xi_0(t_{10} + t_{11}B + ...)+ \\
\xi_1(t_{10} + t_{11}B + ...)B + ... = \xi_0(t_{10}) + [\xi_1(t_{10}) + \xi_0(t_{10})t_{11}]B + ... = \xi_0 + \xi_1B + ... \\
\xi_1 = r_1/V^{1/3} = \xi_0(0) + \xi_1(0)B + ... = \xi_0 + \xi_11B + ... \\
\xi_2 = r_2/V^{1/3} = \xi_0(t_{p0}) + [\xi_1(t_{p0}) + \xi_0(t_{p0})t_{p1}]B + ... = \xi_20 + \xi_21B + ...
$$

Substituting these expansions into equations (1)-(2) and boundary conditions (4)-(10) and equating the coefficients with the same powers of the small parameter $B$ in the left and right sides, we obtain the problems of zero, first and subsequent approximations.

3.1. Construction of a zero approximation

Let $B = 0$ (the case of a liquid bridge in zero gravity). The equations of zero approximation are

$$(w_0''(t)\xi_0'(t) - w_0'(t)\xi_0''(t))\xi_0(t) + w_0'(t) + \mu_0\xi_0(t) = 0, \ (\xi_0'(t))^2 + (w_0'(t))^2 = 1, \ \text{te}(0, t_p). \ (11)$$

The solution to this system of equations is reduced to solving a nonlinear ordinary second-order differential equation that does not contain an independent variable with respect to the function $\xi_0(t)$

$$
-\frac{\xi_0(t)\xi_0''(t)}{\sqrt{1 - (\xi_0'(t))^2}} + \sqrt{1 - (\xi_0'(t))^2} + \mu_0\xi_0(t) = 0, \ \text{te}(0, t_p), \ (12)
$$

and function $w_0(t)$ is found from the second equation (11).

However, if we set the height of the liquid bridge, then there are a need inversion of the integral and solution of a system of nonlinear equations to find the functions that determine the lateral surface shape of the bridge. The key point for building an efficient algorithm, devoided of this drawback, was the transition from setting the height $h$ to setting the auxiliary parameter $M = \mu_0\xi_0$. For the selected admissible value of this parameter all required values, including functions describing the liquid bridge surface profile, are easily found. The value that determines the height of the bridge $h_0$ is also located. Next, we cover the entire range of admissible values of parameter $M$ and construct the dependence of the dimensionless height of the liquid bridge on this parameter. To determine the lateral surface shape of the liquid bridge for preset height $h$ we find the corresponding values of parameter $M$, using the plotted dependence $h(M)$, and perform the calculations for these values (see appendix). As can be seen from the appendix, function $\xi_0(t)$ is expressed through elementary functions, and function $w_0(t)$ can be expressed through the elliptic integrals of the first and second kind.

It was found that in range of admissible negative values of the parameter $M$ each parameter value $M$ correspond to four values height of the bridge. Consequently, the maximum possible number of different profiles of the liquid bridge lateral surface, corresponding to a given height value, is four.
3.2. Construction of the first and subsequent approximations
In contrast to the zero approximation, the differential equations for finding the first and subsequent approximations will be linear. The equations of first approximation are

\[
(\xi_0(t)w_0'(t) - \xi_0(t)w_0'(t) - w_0(t)\xi_0'(t) + w_0'(t)\xi_0'(t))\xi_0(t) + \\
(\xi_0'(t)w_0'(t) - \xi_0'(t)w_0'(t))\xi_1(t) + w_1'(t) + \mu_0 \xi_1(t) = -f(t), \quad i = 1, 2,
\]

\[
f(t) = \begin{cases} 
(w_0(t) + \mu_1)\xi_0(t), & 0 < t \leq t_{10} + t_{11}B, \\
(w_0(t) + \mu_1 - h)\xi_0(t), & t_{10} + t_{11}B < t < t_{p0} + t_{p1}B, \\
\xi_0'(t)\xi_1'(t) + w_0'(t)w_1'(t) = 0, & t \in (0, t_{p0} + t_{p1}B).
\end{cases}
\]  

The solution to this system reduces to solving a linear differential equation with a discontinuous right-hand side with respect to the function \( \xi_1(t) \)

\[
\left( \frac{\xi_0(t)\xi_1'(t)}{\sqrt{1 - (\xi_0(t))^2}} \right)' - \left( \frac{\mu_0 - \xi_0'(t)}{\sqrt{1 - (\xi_0(t))^2}} \right)\xi_1(t) = f(t), \quad t \in (0, t_{p0} + t_{p1}B),
\]

and the function \( w_1(t) \) is found from equation (14). The coefficients of these equations depend on the functions \( \xi_0(t), w_0(t) \) and the parameter \( \mu_0 \) found in the zero approximation. Equations of the second and subsequent approximations will differ from the equations of the first approximation only by the right-hand sides.

4. Calculation results
The results of the zero approximation calculating for the variant in which four different profiles of the lateral surface correspond to the same height of the bridge are shown in figure 2. Here \( w_{10} = w_0(\xi) \) for \( \xi = \xi_0(t), \; t \in [0, t_{10}] \), \( w_{20} = w_0(\xi) \) for \( \xi = \xi_0(t), \; t \in [t_{10}, t_{p0}] \). Note that it is a symmetric case \( (\alpha_1 = \alpha_2) \).

5. Conclusions
A variational formulation of the problem of the lateral surface shape of a vertical catenoidal liquid bridge in parametric form is given. An algorithm is proposed for finding the zero and first approximations in the expansion of the solution of the problem in powers of a small parameter (Bond number). The absence of the solution uniqueness of the problem are found, namely, for a fixed height of the liquid bridge there may be several different profiles of its lateral surface. The dependence of the number of solutions on the height is investigated. It is established that the maximum number of the solutions of problem is four. For the values of the liquid bridge height that exceeds the critical value there are no solutions and for values of the height less than the some value we have only one solution.

Appendix
We present formulas describing the shape of the lateral surface of a vertical liquid bridge in a zero approximation for given value parameter \( M \). The expression for the function \( \xi_0(t) \) is

\[
\xi_0(t) = \begin{cases} 
\left( \frac{\xi_0^2}{4\mu_0} + \frac{1 + M}{0.25\mu_0^2 + \gamma_{10}(t)} \right)^{1/2}, & 0 < t \leq t_{10}, \\
\left( \xi_0^2 + \frac{4(1 + M)}{\mu_0} \sin^2(0.5\mu_0(t - t_{10})) \right)^{1/2}, & t_{10} < t < t_{p0},
\end{cases}
\]  

where

\[
\gamma_{10}(t) = \left\{ \frac{\beta_{10} + 0.5\mu_0 \tan(0.5\mu_0 t)}{1 - 2\beta_{10} \tan(0.5\mu_0 t)/\mu_0} \right\}^2, \quad \beta_{10} = \left\{ \frac{1 + M - 0.25\mu_0^2(\xi_{10}^2 - \xi_{20}^2)}{\xi_{10}^2 - \xi_{20}^2} \right\}^{1/2},
\]  

and

\[
\xi_{10} = \left( \frac{0.25\mu_0^2 + \gamma_{10}(t)}{\xi_0^2 + \frac{4(1 + M)}{\mu_0} \sin^2(0.5\mu_0(t - t_{10}))} \right)^{1/2}.
\]
Here $z$ found by the formula:

$$z^2 = \frac{\alpha_1}{2 \beta_{10}}.$$  \hfill (A.2)

Four profiles of the liquid bridge (zero approximation). Variant: $h = 1.0$, $\alpha_1 = \alpha_2 = 0.5$. (a) - $M = -0.36$; (b), (c) - $M = -0.216$; (d) - $M = -0.019$.

$$\xi_{s0} = M/\mu_0, \quad \xi_{10} = z_{10}/\mu_0, \quad \xi_{20} = z_{20}/\mu_0, \quad t_{10} = \frac{2}{\mu_0} \arctan \left( \frac{\mu_0 (\xi_{20} - \xi_{s0})}{2(1 + M - 0.25\mu_0^2(\xi_{20}^2 - \xi_{s0}^2))^{1/2}} \right).$$

Here $z_{10}$ and $z_{20}$ roots of square equations $z_{10}^2 + 2\sqrt{1 - \alpha_1^2} z_{10} - M(2 + M) = 0$, $z_{20}^2 + 2\sqrt{1 - \alpha_2^2} z_{20} - M(2 + M) = 0$, respectively. And the dimensionless Lagrange multiplier is found by the formula

$$\mu_0 = \left\{ \pi \left[ \frac{1}{M} \int_{z_{10}}^{z_{20}} \frac{\psi_0(z) z^2 dz}{\sqrt{1 - (\psi_0(z))^2}} + \frac{1}{M} \int_{z_{10}}^{z_{20}} \frac{\psi_0(z) z^2 dz}{\sqrt{1 - (\psi_0(z))^2}} \right] \right\}^{1/3},$$  \hfill (A.3)

where $\psi_0(z) = \frac{1}{\pi} \left\{ z^2 - [(1 + M)^2 - 1] \right\}$. The expression for the function $w_0(t)$ is

$$w_0(t) = \begin{cases} \frac{\xi_{0}(t)}{\sqrt{1-(\psi_0(\xi))^2}}, & 0 < t \leq t_{10}, \\ \frac{\xi_{0}(t)}{\sqrt{1-(\psi_0(\xi))^2}} - \frac{\xi_{0}(t)}{\sqrt{1-(\psi_0(\xi))^2}}, & t_{10} \leq t < t_{p0}, \end{cases} \hfill (A.3)$$
where $\varphi_0(\xi) = \frac{1}{\xi}\left[0.5\mu_0(\xi^2 - \xi^2_0) - \xi_0\right]$.

And finally, the height of the liquid bridge is equal to $w_0(t_{\mu_0})$.

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