BEYOND THE EXTENDED SELBERG CLASS: \( d_F \leq 1 \)

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Abstract. We will introduce two new classes of Dirichlet series which are monoids under multiplication. The first class \( \mathfrak{A}^# \) contains both the extended Selberg class \( \mathcal{F}^# \) of Kaczorowski and Perelli as well as many \( L \)-functions attached to automorphic representations of \( \text{GL}_n(\mathbb{A}_K) \), where \( \mathbb{A}_K \) denotes the adèles over the number field \( K \) (these representations need not be unitary or generic). This is in contrast to the class \( \mathcal{F}^# \) which is smaller and is known to contain, very few of these \( L \)-functions. The larger class is obtained by weakening the requirement for absolute convergence, allowing a finite number of poles, allowing more general gamma factors and by allowing the series to have trivial zeros to the right of \( \text{Re}(s) = 1/2 \), while retaining the other axioms of the extended Selberg class. We will classify series in \( \mathfrak{A}^# \) of degree \( d \) when \( d \leq 1 \) (when \( d = 1 \), we will assume absolute convergence in \( \text{Re}(s) > 1 \)). We will further prove a primitivity result for the \( L \)-functions of cuspidal eigenforms on \( \text{GL}_2(\mathbb{A}_Q) \) and a theorem allowing us to compare the zeros of tensor product \( L \)-functions of \( \text{GL}_n(\mathbb{A}_K) \) which cannot be deduced from previous classification results. The second class \( \mathfrak{G}^# \subset \mathfrak{A}^# \), which also contains \( \mathcal{F}^# \), more closely models the behaviour of \( L \)-functions of unitarily generic representations of \( \text{GL}_n(\mathbb{A}_K) \).

1. Introduction

The purpose of this paper is to introduce two classes of Dirichlet series \( \mathfrak{A}^# \) and \( \mathfrak{G}^# \) and their arithmetic counterparts \( \mathfrak{A} \) and \( \mathfrak{G} \) which we believe provide the correct setting for the study of the analytic theory of automorphic \( L \)-functions. These classes are obtained by weakening the hypotheses used to define the extended Selberg class \( \mathcal{F}^# \) studied by Kaczorowski and Perelli in a series of foundational papers (see [KP99, KP02, KP11] among many others). Our aim is to prove many of their most important results for our classes. The class \( \mathfrak{A}^# \) contains all the standard \( L \)-functions of automorphic representations of \( \text{GL}_n \) over number fields as well as the symmetric square, exterior square and tensor product \( L \)-functions, among others. Only some of these are even expected to belong to \( \mathcal{F}^# \), and very few of those have actually been proven to do so. Finally, the class \( \mathfrak{G}^# \) contains a number of series that are not part of the class \( \mathcal{L} \) defined by A. Booker in [Bool5] and also contains Dirichlet series known to belong to \( \mathcal{L} \).

In Theorems 4.3, 4.7 and 5.1 of this paper we classify series in \( \mathfrak{A}^# \) and \( \mathfrak{G}^# \) of small degrees (the notion of degree will be defined below, just after we define the classes) and present some other results generalising the work of Kaczorowski and Perelli and Booker. We will apply our results to establish

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the primitivity of \( L \)-functions attached to Maass cusp forms in Theorem 6.3 and to compare the zero sets of pairs of \( L \)-functions associated to (the tensor products of) representations of \( \text{GL}_n(\mathbb{A}_K) \) in Theorem 7.1. The latter result improves on the results of Booker.

Our first task is to define our new classes of Dirichlet series below, deferring a more detailed examination of our motivations to the next section.

Let \( F(s) \) be a non-zero meromorphic function on \( \mathbb{C} \). We consider the following conditions on \( F(s) \).

(P1) The function \( F(s) \) is given by a Dirichlet series \( \sum_{n=1}^{\infty} \frac{a_{n}}{n^{s}} \) with abscissa of absolute convergence \( \nu \geq 1/2 \).

(P2) There is a polynomial \( P(s) \) such that the function \( P(s)F(s) \) extends to an entire function.

(P3) There exist a real number \( Q > 0 \), a complex number \( \omega \) such that \( |\omega| = 1 \), and a function \( G(s) \) of the form

\[
G(s) = \prod_{j=1}^{r} \frac{\Gamma(\lambda_{j}s + \mu_{j})}{\prod_{j'=1}^{r'} \Gamma(\lambda_{j'}s + \mu_{j'})^{-1}},
\]

where \( \lambda_{j}, \lambda_{j'} > 0 \) and \( \mu_{j}, \mu_{j'} \in \mathbb{C} \), such that

\[
\Phi(s) := Q^{s}G(s)F(s) = \omega\Phi(1-s).
\]

(P4) The function \( F(s) \) can be expressed as a product \( F(s) = \prod_{p} F_{p}(s) \), where

\[
\log F_{p}(s) = \sum_{k=1}^{\infty} \frac{b_{p,k}}{p^{ks}}
\]

with \( |b_{p,k}| \leq Cp^{k\theta} \) for some \( \theta > 0 \) and some constant \( C > 0 \).

We will denote by \( \mathfrak{A}^{\#} \) the class of Dirichlet series satisfying (P1)-(P3). The class of series satisfying (P1)-(P4) will be denoted \( \mathfrak{A} \). We note that the multiplication of Dirichlet series gives \( \mathfrak{A}^{\#} \) the structure of a monoid and \( \mathfrak{A} \) the structure of a submonoid. The class of series \( \mathfrak{A}^{\#}(\nu_0) \) is defined as the those series in \( \mathfrak{A}^{\#} \) with abscissa of absolute convergence \( \nu \) for some \( \nu \leq \nu_0 \). We see that \( \mathfrak{A}^{\#}(\nu_0) \) also forms a submonoid of \( \mathfrak{A}^{\#} \).

Given \( F(s) \) satisfying an equation of the form (1.2), we define the degree of \( F(s) \) to be \( d_F = 2(\sum_{j=1}^{r} \lambda_{j} - \sum_{j'=1}^{r'} \lambda_{j'}) \). We see (in Theorem 4.1) that this notion is well defined, that is, it does not change if we take some other functional equation of the form (1.2) satisfied by \( F(s) \). This justifies the notation \( d_F \).

Remark 1.1. We can actually allow an even more general functional equation than (1.2). With all the other notation being the same as in (P3), we define \( \Psi(s) = G(s)B(s) \), where \( B(s) \) is a Dirichlet series convergent in some right half-plane. If we assume a weaker functional equation \( \Phi(s) = Q^{s}\Psi(1-s) \) (the constant \( \omega \) can be absorbed into \( B(s) \)), it can be checked that the theorems and the proofs in this paper go through in this generality with minor modifications. We avoid doing this for notational simplicity.
We may strengthen the condition (P3) to a stronger condition (P3') by further requiring
\[ \Re \left( -\frac{\mu_j}{\lambda_j} \right), \Re \left( -\frac{\mu_{j'}}{\lambda_{j'}} \right) \leq \frac{1}{2}, \quad 1 \leq j \leq r, \quad 1 \leq j' \leq r', \quad (1.4) \]
We denote the class of series in \( \mathfrak{A}(1) \) satisfying (P3') by \( \mathfrak{G} \). When \( \theta < 1/2 \) in (P4), we obtain the new condition (P4'), and the class of series satisfying this additional hypothesis will be denoted \( \mathfrak{F} \). Occasionally, we may work with the hybrid class \( \mathfrak{A} \) satisfying (P1), (P2) and (P3') or the hybrid class \( \mathfrak{F} \) satisfying (P1), (P2), (P3) and (P4').

We recall that if we assume \( F(s) \) satisfies (P1) with abscissa of convergence \( \nu \leq 1 \), (P2) for \( P(s) = (s - 1)^m \) for some \( m \geq 0 \), and (P3) with \( r' = 0 \), and the even stronger requirement that
\[ \Re(\mu_j) \geq 0, \quad 1 \leq j \leq r, \quad (1.5) \]
we obtain the extended Selberg class \( \mathfrak{F} \). We denote this stronger version of (P3) by (P3''). This class in turn contains the original Selberg class \( \mathfrak{F} \) introduced by Selberg which consists of all series in \( \mathfrak{F} \) satisfying (P4').

We recall that Booker has introduced a class \( \mathcal{L} \) of Dirichlet series [Boo15] similar to the Selberg class. In the next section we explain how our classes contrast with \( \mathfrak{F} \) and \( \mathcal{L} \).

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2. Motivation for introducing \( \mathfrak{A}(1) \) and \( \mathfrak{G}(1) \)

From the definitions we have made, it is obvious that
\[ \mathfrak{F} \subset \mathfrak{G} \subset \mathfrak{A} \subset \mathfrak{A}(1) \quad \text{and} \quad \mathfrak{F} \subset \mathfrak{G} \subset \mathfrak{A}(1) \subset \mathfrak{A} \]
The passage from the extended Selberg class to the class \( \mathfrak{A}(1) \) involves relaxing four hypotheses. We will analyse these hypotheses in turn. In what follows we will use the notation \( K \) for a number field and \( A_K \) for its ring of ad` eles over \( K \).

We let \( \pi \) and \( \pi_i, i = 1, 2 \) denote unitary cuspidal automorphic representations of \( \text{GL}_n(A_K) \) and \( \text{GL}_{n_i}(A_K) \) respectively, while \( \sigma \) will denote an automorphic (not necessarily cuspidal or unitary) representation of \( \text{GL}_n(A_K) \).

2.1. Allowing an arbitrary abscissa of absolute convergence. The class \( \mathfrak{A}(1) \) admits series with an arbitrary abscissa of convergence \( \nu_0 \). This yields a much larger class of \( L \)-functions than \( \mathfrak{F}(1) \). Let \( \sigma \) be as above and denote by \( L(s, \sigma) \) the standard \( L \)-function associated to \( \sigma \). It is known by [GJ72] that these \( L \)-functions lie in \( \mathfrak{A}(1) \) (and, in fact in \( \mathfrak{A} \)). However, even in some of the simplest cases, and even when the representation \( \sigma \) is unitary, \( L(s, \sigma) \) does not always belong to \( \mathfrak{F}(1) \). For instance, the \( L \)-function \( \zeta(s - 1/2)\zeta(s + 1/2) \), which is attached to the trivial representation of \( \text{GL}_2(A_K) \), does not lie in \( \mathfrak{F}(1) \). This representation is not **globally generic**. Indeed non-generic automorphic representations provide a large class of \( L \)-functions that belong to \( \mathfrak{A}(1) \) but not
to $\mathcal{F}^\#$. The $L$-functions attached to Siegel modular forms provide further examples of such series.

The difference between the submonoid $\mathfrak{A}^\#(1)$ and $\mathcal{F}^\#$ is less obvious, and indeed, it is conceivable that these classes coincide. Nonetheless, there are many examples of series that are known to lie in $\mathfrak{A}^\#(1)$ but have not been proven to belong to $\mathcal{F}^\#$ as we discuss below. These examples include the symmetric and exterior square $L$-functions $L(s, \sqrt{2}(\pi))$ and $L(s, \wedge^2(\pi))$, and the Rankin-Selberg $L$-function $L(s, \pi_1 \times \pi_2)$. One would expect all $L$-functions associated to generic representations of $\text{GL}_n(\mathbb{A}_K)$ to lie in $\mathfrak{A}^\#(1)$. This is the Generalised Ramanujan Conjecture which we discuss in Subsection 2.3.

2.2. Allowing a finite number of poles. Our condition (P2) allows the polynomial $P(s)$ to be arbitrary, in contrast to the requirement that $P(s) = (s - 1)^m$ for some integer $m \geq 0$ for functions in $\mathcal{F}^\#$. This allows our series to have a finite number of poles, a priori of arbitrary orders and locations. Indeed, general automorphic $L$-functions can and do have a finite number of poles - the example of $\zeta(s - 1/2)\zeta(s + 1/2)$ is once again instructive. Under suitable normalisations, one can expect that the $L$-functions associated to globally generic unitary automorphic representations have poles only at $s = 1$, but there remain cases (using the Langlands-Shahidi method, see [Sha88]) where the finiteness of poles has been established without the stronger expectation having been proved. Even when it has been achieved, the passage from the finiteness of poles to holomorphy when $s \neq 1$ has not come easily.

2.3. Bounds at the archimedean factors. The condition (P3") is the assertion that all the trivial zeros of $F(s)$ lie in the left half-plane. Conjecturally, the standard $L$-functions $L(s, \pi)$ satisfy this condition, as do the $L$-functions $L(s, \sqrt{2}(\pi))$, $L(s, \wedge^2(\pi))$ and $L(s, \pi_1 \times \pi_2)$. But proving this is equivalent to proving the Generalised Ramanujan Conjectures at infinity, which even in the simplest case of representations associated to Maass cusp forms is the deep and unproven Selberg Eigenvalue Conjecture. Thus, the class $\mathcal{F}^\#$ currently excludes many of the $L$-functions of the greatest interest, given our current state of knowledge.

It is possible that an element in $\mathfrak{A}^\#$ satisfies more than one functional equation and (P3") would appear to be the weakest assumption we can impose to ensure that the factor $G(s)$ is unique (upto multiplication by a scalar). This motivates, in part, our consideration of the class $\mathfrak{G}^\#$.

The class $\mathfrak{G}^\#$ (and, in fact, the class $\mathfrak{G}$) contains the standard $L$-function $L(s, \sigma)$ when $\sigma$ is a unitary automorphic representation, since these are known to satisfy (P3") by a theorem of Jacquet and Shalika (see [JS81a] and [JS81b]). These include the the $L$-functions of Maass cuspidal eigenforms, for instance. However, even when both representations are globally generic, the functions $L(s, \pi_1 \times \pi_2)$ are only conjecturally in $\mathfrak{G}$. The class $\mathfrak{G}^\#$ is perhaps the largest class of series for which one may expect the Generalised Ramanujan Conjectures to hold.
2.4. **Allowing** $r' > 0$. For $L$-functions in $S^\#$, and for the $L$-functions of automorphic representations, we always have $r' = 0$. The motivation for allowing $r' \geq 1$, arises in the following setting. Let $F_j(s)$, $j = 1, 2$, $F_1(s) \neq F_2(s)$ be elements of $\mathfrak{A}(1)$. Consider the quotient $F(s) = F_1(s)/F_2(s)$, and suppose that $d_{F_1} - d_{F_2} = 1$. If we assume that this quotient has at most finitely many poles, then under certain further assumptions we can use Theorem 5.1 to get a contradiction. Hence, in a number of situations we will be able to prove that the zero sets of distinct $L$-functions are very different - that there are infinitely many zeros (counted with multiplicity) of $F_2(s)$ which are not zeros of $F_1(s)$. This idea was first pursued in special cases (when $r' = 0$) in [Rag99] and has been generalised by Booker to his class $\mathcal{L}$ in [Boo15]. Theorem 7.1 of this paper further generalises Booker’s theorem to our class $\mathfrak{A}(1)$. Thus, the motivation for weakening this hypothesis comes partly from having a specific application in mind. Conjecturally, of course, we believe that any element of $\mathfrak{A}^\#$ will satisfy a functional equation in which $r' = 0$.

2.5. **The Euler product axioms.** The weakening of (P4') to (P4) allows the presence of non-trivial degree 0 elements which is often a minor irritant, especially when one is interested in the factorisation of series. For this reason, we prefer to retain the condition $\theta < 1/2$ and work with the class $\mathfrak{G}$ in this paper when discussing the primitivity of elements in these monoids. We may also work in $\mathfrak{A}^f$, when the condition at infinity is not known to hold or is not required for the proofs.

The condition (P3') is exactly the archimedean analogue of the condition $\theta < 1/2$ in (P4') and is known to hold $L(s, \pi)$ by [JS81a, JS81b]. Thus, in the class $\mathfrak{G}$, the bounds for local parameters at both the archimedean and nonarchimedean places are assumed to be less than 1/2. From a purely aesthetic point of view, the imposition of the same Jacquet-Shalika type bound at both archimedean and non-archimedean places seems to make $\mathfrak{G}$ a natural subclass in the theory. Further, $\mathfrak{G} \subset \mathfrak{A}^f \cap \mathfrak{A}^\infty$ so theorems proved for these hybrid classes remain valid for $\mathfrak{G}$.

Finally, we must add that analogues of (various versions of) the Generalised Ramanujan Conjectures would assert that the classes $\mathcal{F}$, $\mathfrak{G}$ and $\mathfrak{A}^f$, coincide, while the most general Converse Theorem would assert that $\mathfrak{A}$ coincides with the class of standard automorphic $L$-functions of $GL_n(\mathbb{A}_\mathbb{Q})$, $n \geq 1$. In addition, one could also conjecture $\mathcal{F}^\# = \mathfrak{G}^\# = \mathfrak{A}^\#(1)$. Until these are proven however, it is obviously best to work in the largest possible class.

2.6. **A comparison with the class $\mathcal{L}$.** In [Boo15], Booker introduces what he calls an $L$-datum using the framework of explicit formulas with which he associates a Dirichlet series. This class of Dirichlet series forms a monoid under multiplication. We will not describe his hypotheses in detail but make the following observations. A requirement for a series to belong to the class $\mathcal{L}$ is that the mean square of the coefficients $a_n$ satisfies the Ramanujan bound on average. This is stronger than the average bound $(\nu = 1)$ that we have assumed for the class $\mathfrak{G}^\#$. For the standard $L$-function $L(s, \pi)$ this stronger
bound follows from the papers of Jacquet and Shalika referenced above. However, as in Subsection 2.3, the $L$-functions $L(s, \sqrt{2}(\pi))$ and $L(s, \wedge^2(\pi))$ and $L(s, \pi_1 \times \pi_2)$ have not been proven to lie in $\mathcal{L}$.

Although Booker does not assume the existence of an Euler product explicitly, his series are defined as exponentials of other series, and thus satisfy a non-vanishing condition, implicitly invoking a condition very close to that of the existence of an Euler product. Thus our class $\mathfrak{G}^#$ is not contained in $\mathcal{L}$, since we may take linear combinations of elements in $\mathfrak{G}$ with the same gamma factors to produce elements in $\mathfrak{G}^#$ which are not in $\mathcal{L}$. In light of the Generalised Ramanujan Conjecture at infinity, one might expect the class $\mathcal{L}$ to coincide with our class $\mathfrak{G}$. However, the uniqueness of the functional equation for elements in $\mathcal{L}$ does not seem to follow immediately.

A priori, the kernels in an $L$-datum allow for more general gamma factors at infinity than the quotients we allow for the class $\mathfrak{A}^#$. It seems likely that any Dirichlet series in $\mathcal{L}$ will satisfy a functional equation with the simpler quotients of gamma factors that arise in $\mathfrak{A}^#$, probably with $r' = 0$. Perhaps establishing a suitable local functional equation will allow the passage from the more general condition to the more restrictive one.

To summarise, one expects that $\mathcal{L} = \mathfrak{G}$. There are no known elements of $\mathcal{L}$ which do not lie in $\mathfrak{G}$. However, there are many elements of $\mathfrak{A}(1)$ which have not been proven to belong to $\mathcal{L}$. Finally, not imposing the condition $\nu_0 = 1$ allows us to include the naturally occurring non-generic automorphic $L$-functions in the class $\mathfrak{A}$, and these do not belong to $\mathcal{L}$. Linear combinations of series in $\mathfrak{G}$ and $\mathfrak{A}$ will produce series in $\mathfrak{G}^#$ and $\mathfrak{A}^#$ which will also not belong to $\mathcal{L}$.

2.7. Nomenclature. The notation $\mathfrak{A}^#$ is supposed to be suggestive of the the word “automorphic”, while $\mathfrak{G}^#$ suggests “generic” for the generic automorphic representations that are expected to be the source of all functions in this class. The Ramanujan conjecture is roughly the statement that globally generic representations should be tempered. The condition $\nu_0 = 1$ is consistent with this expectation.

3. Preliminaries

In the rest of this paper, for any function $a : \mathbb{C} \to \mathbb{C}$, we will use the notation $\tilde{a}(s) := a(\overline{s})$.

We will require two different variants of Stirling’s formula applied to the quotients of the gamma functions appearing in the functional equation. Let $w = u + iv$ be in $\mathbb{C}$. The simplest avatar of the formula that we will use is

$$
\frac{\tilde{G}(1/2 - it - w)}{G(1/2 + it + w)} \ll (1 + |t + v| + |u|)^{-u}, \quad (3.1)
$$
Lemma 3.1. Let $w = u + iv$ and $0 < \eta < 1$, and assume that $F(s) \in \mathcal{A}_1^\#(\nu_0)$. We have
\begin{equation}
F(1/2 + it) = \sum_{n=1}^{\infty} a_n e^{-n/X_{n/12 + it}} + r_1 + r_2
\end{equation}
where $r_1 := r_1(t, X) = O(X^{-\nu_0 - 1/2} e^{-|t|})$ is identically zero if $F(s)$ is entire and
\begin{equation}
r_2 = \frac{1}{2\pi i} \int_{u = -1 + \eta} F(1/2 + it + w)X^w \Gamma(w) dw \ll O((1 + |t|)^{-\eta} X^{-1 + \eta}),
\end{equation}
and where $u = -1 + \eta$ is a line on which none of the poles of $F(s)$ lie.

Proof. When $c > \nu_0$, we have
\begin{equation}
\frac{1}{2\pi i} \int_{u = c} F(1/2 + it + w)X^w \Gamma(w) dw = \sum_{n=1}^{\infty} a_n e^{-n/X_{n/12 + it}}.
\end{equation}

When we move the line of integration from $u = c$ to $u = -1 + \eta$ we will cross the poles of the integrand of the form $w = \beta - 1/2 - it$, where $\beta$ is a pole of $F(s)$, and also the pole at $w = 0$. The residue at $w = 0$ is $F(1/2 + it)$, and the
residue at $\beta - 1/2 - it$ is majorised up to a constant by the $X^{\beta-1/2}e^{-|t|}$, since $\Gamma(\beta - 1/2 - it) = O(e^{-|t|})$. For any pole $\beta$, we must have $\text{Re}(\beta) \leq \nu_0$, since $\nu_0$ is the abscissa of absolute convergence. Hence, the residue at $\beta - 1/2 - it$ will also be majorised by $X^{\nu_0-1/2}e^{-|t|}$. We denote the sum of the residues by $r_1$. It is obviously identically zero if $F(s)$ is entire.

From the functional equation we have

$$F(1/2 + it + w) = \omega Q^{-2it} \frac{\tilde{G}(1/2 - it)}{G(1/2 + it)} \tilde{F}(1/2 - it - w).$$

Substituting this in $r_2$, the estimate (3.6) follows immediately from (3.1).

4. Classifying series of small degree: $0 \leq d < 1$

We first show that the notion of degree is well-defined for functions in the class $A^\#$.

Indeed, suppose that $F(s)$ in $A^\#$ satisfies two different functional equations

$$\Phi_j(s) := Q_j^* G_j(s) F(s) = \omega_j \Phi_j(1 - s), \quad j = 1, 2.$$ 

Let $d_j$, $j = 1, 2$, be the degrees of $G_j(s)$. Taking the quotient $\Phi_1(s)/\Phi_2(s)$, we see that

$$H(s) = \frac{G_1(s)}{G_2(s)} = \frac{\omega_1 \tilde{G}_1(1 - s)}{\omega_2 \tilde{G}_2(1 - s)} = c \cdot \tilde{H}(1 - s).$$

The left-hand side is holomorphic and non-vanishing for $\text{Re}(s) \gg 0$, while the right-hand side is holomorphic and non-vanishing for $\text{Re}(s) \ll 0$. Further, all the zeros and poles (on both sides) lie on a finite number of rays parallel to the real (horizontal) axis. Thus, the zeros and poles of $H(s)$ must lie on a finite number of line segments contained in a bounded vertical strip, whence it follows that $H(s)$ has at most finitely many zeros and poles. The number of poles of $G_j(s)$ with $\text{Re}(s) > -T$ is asymptotic to $d_jT$ for $j = 1, 2$. Hence, if $d_1 \neq d_2$, $H(s)$ must have infinitely many zeros or poles. It follows that we must have $d_1 = d_2$. Hence, we see that the degree of an element in $A^\#$ is well-defined. We are thus justified in using the notation $d_F$ to denote the degree of $F$.

If $F(s)$ is in $A^\infty$, we follow the arguments of [CG93]. We see that $H(s)$ is an entire function without zeros, since the poles and zeros of $G_1(s)/G_2(s)$ lie in the half-plane $\text{Re}(s) < 1/2$, while those of $\tilde{G}_1(1 - s)/\tilde{G}_2(1 - s)$ lie in $\text{Re}(s) > 1/2$. Since $H(s)$ is of order 1 (being the quotient of functions of order 1), it must have the form $e^{as+b}$ for constants $a$ and $b$. Because of the functional equation, we see that the constant $a$ is purely imaginary. Stirling’s formula shows that the quotient $|H(it)/H(-it)| \to 1$ as $t \to \infty$, which shows that $a = 0$. We can summarise our arguments as

**Theorem 4.1.** For $F(s)$ in $A^\#$, the degree $d_F$ is well defined. If $F(s)$ in $A^\infty$ satisfies two different functional equations

$$\Phi_j(s) := Q_j^* G_j(s) F(s) = \omega_j \Phi_j(1 - s), \quad j = 1, 2,$$

there is a constant $c$ such that $Q_1^* G_1(s) = c Q_2^* G_2(s)$.
In view of the theorem above, we can define $\mathcal{A}_d^\#$ to be the subset consisting of series $F(s)$ in $\mathcal{A}^\#$ with $d_F = d$.

We begin by classifying the elements in $\mathcal{A}_d^\#$ when $0 \leq d \leq 1$. To this end we first prove the following proposition following the proof for $\mathcal{F}$ in [CG93].

**Proposition 4.2.** If $0 \leq d < 1$, the series $F(s)$ is absolutely convergent on the whole complex plane and hence gives rise to an entire function.

**Proof.** For $\text{Re}(s) = c > \nu$, we know that

$$h(y) = \sum_{n=1}^\infty a_n e^{-2\pi ny} = \int_{\text{Re}(s)=c} F(s) \Gamma(s)(2\pi y)^{-s} ds.$$  

Shifting the line of integration to the left and letting $c \to -\infty$, we get

$$\sum_{n=1}^\infty a_n e^{-2\pi ny} = \sum_{n=0}^\infty \frac{(-1)^n F(-n)(2\pi y)^n}{n!} + \sum_{k=1}^l P_k(\log y) y^{\beta_k}.$$  

where $-\beta_k$ are the poles of $F(s)$ and the $P_k$ are polynomials (with degree one less than the order of the pole at $\beta_k$) for $1 \leq k \leq l$. Using the functional equation (1.2), we obtain

$$\sum_{n=1}^\infty a_n e^{-2\pi ny} = \sum_{n=0}^\infty \frac{(-1)^n G(n+1)F(n+1)(2\pi y)^n}{G(-n)n!} + \sum_{k=1}^l P_k(\log y) y^{\beta_k}.$$  

Using (3.1) one checks easily that $G(n+1)/G(-n)n! = O(n^{-(1-d)n}K^n)$ for some constant $K$, so the infinite sum on the right above converges. It follows that $h(y)$ extends to a holomorphic function on $\mathbb{C} \setminus (-\infty, 0]$. But we also know that $h(y)$ is periodic with period $i$, so it extends holomorphically to the real axis as well, and thus to an entire function $h(z)$. Thus, we can write

$$a_n e^{-2\pi ny} = \int_0^1 h(ix - y)e^{-2\pi inx} dx.$$  

Differentiating repeatedly with respect to $y$ shows that $a_n \ll n^{-k}$, for any $k \geq 0$. It follows that the Dirichlet series $\sum_{n=1}^\infty a_n n^{-s}$ converges absolutely (and uniformly on compact subsets) everywhere in the complex plane. It is thus an entire function. \hfill $\square$

**Theorem 4.3.** The class $\mathcal{A}_0^\#$ consists of Dirichlet polynomials of the form

$$F(s) = \sum_{n \mid Q_1} \frac{a_n}{n^s},$$  

for some $Q_1 > 0$ an integer. In fact, $\mathcal{A}_0^\# = \mathcal{F}_0^\#$. We also have $\mathcal{G}_0 = \{1\}$.

**Proof.** We first assume that $r' \neq 0$ for every non-constant factor $G(s)$ that arises in a functional equation of the form (1.2) for $F(s)$. We write $G(s) = g_1(s)/g_2(s)$ where

$$g_1(s) = \prod_{j=1}^r \Gamma(\lambda_j s + \mu_j) \quad \text{and} \quad g_2(s) = \prod_{j'=1}^{r'} \Gamma(\lambda_{j'} s + \mu_{j'}).$$  

Our first task is to show that \( G(s) \) has only finitely many zeros and poles. Let \( \alpha > 0 \) be a real number and \( \beta \in \mathbb{C} \). We will refer to a subset of \( \mathbb{C} \) of the form \( W = \{-(n+\beta)/\alpha \mid n \in \mathbb{N} \cup \{0\} \} \) as a \( \gamma \)-set, or more precisely, as a \( \gamma(\alpha, \beta) \)-set. We see that the multiset of zeros of \( G(s) \) is the finite sum of \( \gamma \)-sets. Notice that the elements of a \( \gamma \)-set lie on a horizontal ray parallel to the negative real axis.

Suppose that \( G(s) \) has infinitely many zeros. These will lie on a finite number of rays parallel to the negative real axis. Since \( F(s) \) has at most finitely many poles, all but finitely many of the zeros of \( G(s) \) will be poles of \( \Phi(s) \), and hence, of \( \Phi(1-s) \). The zeros of \( \tilde{G}(1-s) \) lie on finitely many rays parallel to the positive real axis. Hence, only finitely many of these will lie in any left half-plane. It follows that \( \tilde{F}(1-s) \) has infinitely many zeros lying on some ray parallel to the negative real axis. This is impossible, since \( \tilde{F}(1-s) \), being absolutely convergent in some left half-plane, is dominated by its first non-zero term and has no zeros for \( \text{Re}(s) < -c_0 \) for some \( c_0 > 0 \). Thus \( G(s) \) has only finitely many zeros.

We now show that \( G(s) \) has only finitely many poles, that is, that all but finitely many of the poles of \( g_1(s) \) are poles of \( g_2(s) \). Let \( H_T \) denote the half-plane \( \text{Re}(s) > -T \), and let \( N_S(T) = |S \cap H_T| \) for any discrete subset \( S \) of \( \mathbb{C} \). We denote the \( \lim_{T \to \infty} N_S(T)/T \) by \( D(S) \), if it exists, and call it the density of \( S \). Note that \( D(W) = \alpha \) for the \( \gamma(\alpha, \beta) \)-set \( W \). The notion of density generalises naturally to multisets, in particular to sums of \( \gamma \)-sets. If \( V = \sum_{i=1}^r W_i \) is a multiset which is the sum of the sets \( W_i \) each with density \( \alpha_i \), then \( D(V) = \sum_{i=1}^r D(W_i) \). We will need the following lemma.

**Lemma 4.4.** Let \( W_i = \{-(n-\beta_i)/\alpha_i \mid n \in \mathbb{N} \cup \{0\} \}, i = 1, 2 \) be \( \gamma \)-sets. Then either \( |W_1 \setminus W_2| < \infty \) or \( D(W_1 \setminus W_2) > 0 \). Further, if \( D(W_1 \setminus W_2) > 0 \), \( W_1 \setminus W_2 \) is a union of \( \gamma \)-sets and a finite set.

**Proof.** Either \( |W_1 \cap W_2| \leq 1 \) or \( |W_1 \cap W_2| \geq 2 \). In the first case there is nothing to prove. In the second case, it is easy to see that \( \alpha_1 = \alpha_2 m_1/m_2 \) and \( \beta_1 + k = \beta_2 m_1/m_2 \) for coprime integers \( m_1 \) and \( m_2 \) and some integer \( k \). The elements of \( W_1 \cap W_2 \) have (except possibly for a finite set) the form \( -(m_1 n + a + \beta_1)/\alpha_1 \) for some integer \( a \) with \( 0 \leq a < m_1 - 1 \). If \( m_1 = 1 \), we see that \( |W_1 \setminus W_2| < \infty \). If \( m_1 - 1 > 1 \), \( W_1 \setminus W_2 \) is (upto a finite set) a union of sets of the form \( -(m_1 n + b + \beta_1)/\alpha_1 \), \( n \in \{0\} \cup \mathbb{N} \) and thus a union of \( \gamma \)-sets and a finite set. Further \( D(W_1 \setminus W_2) = (m_1 - 1)D(W_1)/m_1 = (m_1 - 1)\alpha/m_1 > 0 \). This proves the lemma. \( \square \)

**Corollary 4.5.** If \( V_1 = \sum_{i=1}^r W_i^1 \) and \( V_2 = \sum_{j=1}^s W_j^2 \) are the sums of \( \gamma \)-sets viewed as multisets (that is, if an element occurs in \( k \) different sets \( W_i^1 \) or \( W_j^2 \), it is thought of as occurring \( k \) times in the sum \( V_1 \) or \( V_2 \) ), then either \( |V_1 \setminus V_2| < \infty \) or \( D(V_1 \setminus V_2) > 0 \). In the latter case \( V_1 \setminus V_2 \) is a union of \( \gamma \)-sets and a finite set.

**Proof.** Indeed, we simply apply the lemma succesively to the differences \( W_i^1 \setminus W_j^2 \) as \( 1 \leq i \leq r \) and \( 1 \leq j \leq s \). The subtractions are performed in lexicographic order on the pairs \((i, j)\). After each subtraction, the lemma above
shows that what remains of each of $W_1^j$ and $W_2^j$ is a finite set or a finite union of $\gamma$-sets and (possibly) a finite set, and we can thus continue the process with the next subtraction.

We apply our corollary to the following situation. The multiset $V_1$ of poles of $g_1(s)$ is the sum of the $\gamma$-sets $S_j = \{(-n - \mu_j)/\lambda_j\}$, $1 \leq j \leq r$, while the multiset of poles $V_2$ of $g_2(s)$ is the sum of the $\gamma$-sets $S_j' = \{(-n' - \mu_{j'})/\lambda_{j'}\}$, $1 \leq j' \leq r'$, where $n$ and $n'$ run through the non-positive integers. We have already shown that $G(s)$ has only finitely many zeros. Hence, $|V_2 \setminus V_1| < \infty$ so $D(V_1) \geq D(V_2)$. If $V_1 \setminus V_2$ is not finite, the corollary above tells us that $D(V_1 \setminus V_2) > 0$. Thus we find that $D(V_1) - D(V_2) = 0$. This gives a contradiction. Thus $G(s)$ has at most finitely many poles, which is what we have been trying to prove.

Since $G(s)$ has at most finitely many zeros and poles, and is a quotient of functions of order 1, it can be replaced in the functional equation by a factor of the form $AB^*R(s)$ for some rational function $R(s) = p(s)/q(s)$, where $p(s)$ and $q(s)$ are monic polynomials, $A \in \mathbb{C}$ and $B > 0$. Cross multiplying, we get an equation of the form

$$\tilde{p}(1 - s)q(s)F(s) = \omega Q_1^* p(s) \tilde{q}(1 - s) \tilde{F}(1 - s),$$

for $Q_1 > 0$. Using Perron’s formula (as in [KP99]) it follows immediately that $Q_1$ is an integer and that $F(s)$ is a Dirichlet polynomial with nonzero coefficients only when $n \mid Q_1$. It also easily follows as a consequence that $\tilde{p}(1 - s)q(s) = p(s)\tilde{q}(1 - s)$. Thus $F(s)$ actually satisfies a functional equation of the form

$$F(s) = \omega Q_1^* \tilde{F}(1 - s),$$

so $F(s)$ lies in $\mathcal{F}^\#$.

If $F(s) \in \mathcal{S}_0$, we know further that $F_p(s)$ is non-vanishing for $\text{Re}(s) \geq 1/2$, and hence, that $F(s)$ which is a product of at most finitely many $F_p(s)$, is also non-vanishing in this half-plane. By the functional equation it is non-vanishing in $\text{Re}(s) \leq 1/2$ as well. It follows that $F(s)$ is entire and non-vanishing, and since $\alpha_1 = 1$, we must have $F(s) \equiv 1$.

**Remark 4.6.** We have proved the analogue of first part of Theorem 1 of [KP99]. The second part of Theorem 1 of that paper gives a somewhat more precise description of the elements of $\mathcal{F}_0^\#$ in terms of invariants $q$ and $\omega^*$ that Kaczorowski and Perelli associate to elements of $\mathcal{F}_0^\#$. We can thus recover a similar sharper statement for elements of $\mathcal{A}_0^\#$.

**Theorem 4.7.** If $0 < d < 1$, then $\mathcal{A}_d^\# = \emptyset$.

For the Selberg class this is a theorem of Richert [Ric57] and Conrey and Ghosh ([CG93]) and we follow the proof in the latter paper.

**Proof.** By Proposition 4.2 we know that $F(s)$ is uniformly bounded in $\text{Re}(s) > -\nu$ since it is absolutely convergent in every half-plane. The functional equation

$$F(s) = \frac{\tilde{G}(1 - s)}{G(s)} Q^{1-2s} \tilde{F}(1 - s)$$

shows that what remains of each of $W_1^j$ and $W_2^j$ is a finite set or a finite union of $\gamma$-sets and (possibly) a finite set, and we can thus continue the process with the next subtraction. □
and Stirling’s formula show that $F(s)$ cannot be bounded on the vertical line $\text{Re}(s) = -\varepsilon$ for any $\varepsilon > 0$. 

5. The case $d_F = 1$

The main result of this section is

**Theorem 5.1.** Suppose that $F(s)$ is in $\mathcal{A}_1^\#(1)$.

1. There exists $A \in \mathbb{R}$ and an integer $q > 0$ such that $a_n n^{-iA}$ is periodic with period $q$. Further,

$$F(s) = \sum_{\chi \pmod{q}} P_{\chi}(s) L(s + iA, \chi^*),$$  \hspace{1cm} (5.1)

where the sum runs over all Dirichlet characters $\chi \pmod{q}$, $P_{\chi} \in \mathcal{A}_0^\#$ and $\chi^*$ is the primitive Dirichlet character inducing $\chi$.

2. If $F(s)$ is in $\mathcal{A}_1(1)$, there is a Dirichlet character $\chi \pmod{q}$ such that $F_p(s) = (1 - \chi_p p^{-s+iA})^{-1}$ for all $p \nmid q$. If further $F(s)$ is in $\mathcal{G}$, $F(s) = L(s + iA, \chi)$ for a primitive Dirichlet character $\chi \pmod{q}$.

Kaczorowski and Perelli [KP99] proved the theorem above for series in $\mathcal{S}^\#$. Soundararajan gave another proof of their theorem in [Sou05], but assuming $a_n \ll n^\varepsilon$. We present a modified version of Soundararajan’s proof below for class $\mathcal{A}_1^\#(1)$. Note that the theorem above is valid for $\mathcal{G}_1^\# \subset \mathcal{A}_1^\#(1)$. We hope to remove the restriction $\nu_0 = 1$ in future work.

The hard part of proving the theorem above lies in proving the first assertion of the first part of the theorem. The other assertions will follow relatively easily after that.

**Proof.** Since $F(s)$ converges absolutely for $\text{Re}(s) > 1$, we have the estimate

$$\sum_{n < X} \frac{|a_n|}{\sqrt{n}} = O(X^{3/2 + \varepsilon}).$$  \hspace{1cm} (5.2)

Define

$$\mathcal{F}(\alpha, T) = \frac{1}{\sqrt{\alpha}} \int_{aT}^{2\alpha T} F(1/2 + it) e^{it \log \frac{t}{Q \sqrt{\pi}}} \frac{\eta_0 - t}{iA} dt.$$  \hspace{1cm} (5.3)

The proof involves showing that

$$\mathcal{F}(\alpha) := \lim_{T \to \infty} \frac{\mathcal{F}(\alpha, T)}{T^{1+iA}}$$

exists, and that $\mathcal{F}(\alpha + 1) = \mathcal{F}(\alpha)$, that is, $\mathcal{F}(\alpha)$ is periodic of period 1.

Using the functional equation (1.2) and equation (3.3), we have

$$\mathcal{F}(\alpha, T) = \frac{\omega e^{iB}}{\sqrt{\alpha}} \int_{aT}^{2\alpha T} \tilde{F}(1/2 - it) (C[\pi Q^2 \alpha])^{-it} t^{-iA} [1 + O(1/t)] dt.$$
Now we deviate a little from Soundararajan’s proof. Using Lemma 3.1 for \( \nu_0 = 1 \), we get

\[
F(\alpha, T) = \frac{\omega e^{iB}}{\sqrt{\alpha}} \int_{\alpha T}^{2\alpha T} \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{m}} e^{-nX(n^{-1}C[\pi Q^2 \alpha])^{-it}} \left[ 1 + O(1/t) \right] dt \\
+ O(X^{1/4}e^{-\alpha T}) + O(T^{2-\eta}X^{-1+\eta}).
\]

We have

\[
= \frac{\omega e^{iB}}{\sqrt{\alpha}} \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{m}} e^{-nX} \int_{\alpha T}^{2\alpha T} (n^{-1}C[\pi Q^2 \alpha])^{-it} \left[ 1 + O(1/t) \right] dt \\
+ O(X^{1/4}e^{-T}) + O(T^{2-\eta}X^{-1+\eta}) \tag{5.4}
\]

If we choose \( X = T^{1+\delta} \) for some \( 0 < \delta < 1 \), then both the error terms above are \( O(T^{1-\varepsilon}) \) for some \( \varepsilon > 0 \), if \( \eta \) is chosen small enough. It is easy to see that

\[
\int_{\alpha T}^{2\alpha T} (n^{-1}C[\pi Q^2 \alpha])^{-it} \eta dt = O(1). 
\]

Using (5.2), we see that the contribution of this integral to the sum in (5.4) is \( O(X^{1/2+\varepsilon}) \). With \( X = T^{1+\delta} \) as above, we have \( O(X^{1/2+\varepsilon}) = O(T^{1-\varepsilon}) \), if \( \delta \) and \( \varepsilon \) are chosen small enough. It remains to estimate

\[
= \frac{\omega e^{iB}}{\sqrt{\alpha}} \sum_{m=1}^{\infty} \frac{a_m}{\sqrt{m}} e^{-mX} \int_{\alpha T}^{2\alpha T} (n^{-1}C[\pi Q^2 \alpha])^{-it} dt.
\]

The integral is estimated using integration by parts (as in (4a) of of [Sou05]). Together with the estimate (5.2), we see that for \( n^{-1}C[\pi Q^2 \alpha] \neq 1 \), the sum above is once again majorised by \( O(X^{1/2+\varepsilon}) = O(T^{1-\varepsilon}) \) for some \( \varepsilon > 0 \) and \( \delta \) small enough. Thus, for \( n^{-1}C[\pi Q^2 \alpha] \neq 1 \) we have

\[
F(\alpha, T) = O(T^{1-\varepsilon}) \tag{5.5}
\]

for some \( \varepsilon > 0 \). If \( m^{-1}C[\pi Q^2 \alpha] = 1 \), for some integer \( m \), we get

\[
\int_{\alpha T}^{2\alpha T} (m^{-1}C[\pi Q^2 \alpha])^{-it} dt = \frac{2^{1+iA} - 1}{1 + iA} \alpha T^{1+iA}.
\]

Combining (5.4) with the estimates above, we see that for suitable choices of \( \delta \) and \( \eta \), there is a \( \varepsilon > 0 \) such that

\[
F(\alpha, T) = \omega e^{iB} \frac{\overline{\alpha m}}{(C\pi Q^2)^{1/2}} \frac{2^{1+iA} - 1}{1 + iA} \alpha T^{1+iA} + O(T^{1-\varepsilon}). \tag{5.6}
\]

Dividing by \( T^{1+iA} \) in (5.5) and taking the limit as \( T \to \infty \), we see that \( F(\alpha) = 0 \). Similarly, if we divide (5.6) by \( T^{1+iA} \) and take the limit as \( T \to \infty \),

\[
F(\alpha) = \omega e^{iB} \frac{\overline{a_m \alpha^{iA}}}{(C\pi Q^2)^{1/2}} \frac{2^{1+iA} - 1}{1 + iA}.
\]

Combining these two cases,

\[
F(\alpha) = \omega e^{iB} \delta(C[\pi Q^2 \alpha]) = m \frac{\overline{a_m C[\pi Q^2 \alpha] \alpha^{iA}}}{(C\pi Q^2)^{1/2}} \frac{2^{1+iA} - 1}{1 + iA}. \tag{5.7}
\]
We have thus shown that the desired limit exists. We remark that by choosing \( \delta = 1/3 \) and \( \eta \) small enough, we could have made the error terms \( O(T^{4+\varepsilon}) \), for any \( \varepsilon > 0 \), rather than \( O(T^{1-\varepsilon}) \) for some \( \varepsilon > 0 \) as above. However, we will get larger error terms later in the proof, so there is no particular reason to optimise the choice of exponents at this stage.

We will now show that \( F(\alpha) \) is a periodic function. Once again, we use Lemma 3.1 with \( \nu_0 = 1 \) and \( X = T^{1+\delta} \). With the same arguments as before, we have

\[
F(\alpha) = \frac{1}{\sqrt{\alpha}} \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}} e^{-n/X} \cdot I_n + O(T^{1-\varepsilon}),
\]

for some \( \varepsilon > 0 \), when \( \delta \) is small enough, and where

\[
I_n = \int_{\alpha T}^{2\alpha T} e^{it \log \frac{1}{2\pi n\alpha} - \frac{i}{\pi} t} dt.
\]

The integral \( I_n \) is estimated by means of Lemmas 4.2, 4.6 and 4.4 of Titchmarsh [Tit86] which yield:

\[
\int_{\alpha T}^{2\alpha T} e^{it \log \frac{1}{2\pi n\alpha} - \frac{i}{\pi} t} dt = \begin{cases} 
O(1) & \text{if } 2\pi n \geq 3T
\end{cases} + 2\pi \sqrt{n\alpha e(-n\alpha) + \varphi(T)} & \text{if } T \leq 2\pi n \leq 2T, \\
\varphi(T) & \text{otherwise},
\]

where

\[
\varphi(T) = O \left[ T^{\frac{2}{3}} + \min \left( \sqrt{T}, \frac{1}{\log(T/2\pi n)} \right) + \min \left( \sqrt{T}, \frac{1}{\log(T/\pi n)} \right) \right].
\]

When \( n \geq 3T \), using the first case of \( (5.9) \) together with \( (5.2) \) yields

\[
\frac{1}{\sqrt{\alpha}} \sum_{n=3T}^{T^{1+\delta}} \frac{a_n}{\sqrt{n}} e^{-n/X} \cdot I_n = O(T^{1+\delta + \varepsilon}).
\]

From this point onwards, we need to further modify the arguments of [Sou05].

If \( 2\pi n \) lies in one of the intervals \( P_{1,T} = [1, T - T^{\frac{4}{3}}] \), \( P_{2,T} = [T + T^{\frac{4}{3}}, 2T - T^{\frac{4}{3}}] \) or \( P_{3,T} = [2T + T^{\frac{4}{3}}, 3T] \), we see that \( \varphi(T) = O(T^{\frac{2}{3}}) \). Hence, using \( (5.2) \) we see that

\[
\frac{1}{\sqrt{\alpha}} \sum_{2\pi n \in P_{j,T}} \frac{a_n}{\sqrt{n}} e^{-n/X} \cdot I_n = O(T^{\frac{2}{3} + \varepsilon}),
\]

for \( j \leq i \leq 3 \). When \( 2\pi n \) lies in either \( Q_{1,T} = [T - T^{\frac{4}{3}}, T + T^{\frac{4}{3}}] \) or in \( Q_{2,T} = [2T - T^{\frac{4}{3}}, 2T + T^{\frac{4}{3}}] \), we have

\[
\min \left( \sqrt{T}, \frac{1}{\log(T/2\pi n)} \right), \min \left( \sqrt{T}, \frac{1}{\log(T/\pi n)} \right) = O(\sqrt{T}).
\]

It follows that \( \varphi(T) = O(\sqrt{T}) \) in these ranges. By \( (5.2) \), we know that the sets of points \( E_{j,\varepsilon} = \{ U \in \mathbb{R} \mid \sum_{2\pi n \in Q_{i,U}} |a_n| > U^{\frac{4}{3} + \varepsilon} \} \), \( j = 1, 2, \) have density zero as a subset of \( \mathbb{R} \) for any \( \varepsilon > 0 \), that is, \( \lim_{X \to \infty} \mu(E_{\varepsilon_j} \cap [0, X]) / X = 0 \), where \( \mu \)
is the Lebesgue measure on \( \mathbb{R} \). Thus, for all \( T \) outside of a set \( S_\varepsilon = E_1,\varepsilon \cup E_2,\varepsilon \) of density 0 in \( \mathbb{R} \) we have
\[
\frac{1}{\sqrt{\alpha}} \sum_{2\pi n \in Q_{j,T}} a_n e^{-n/X} \cdot I_n = O(T^{2+\varepsilon})
\] (5.12)
for \( j = 1, 2 \). It follows that if \( T \not\in S_\varepsilon \),
\[
\mathcal{F}(\alpha, T) = 2\pi \cdot \sum_{T < 2\pi n < 2T} a_n e^{-2\pi n \alpha} + O(T^{-\varepsilon} + \varepsilon).
\]
Choosing \( \delta < 1/2 \) in (5.10), and combining this with the estimates in (5.11) and (5.12), shows that
\[
\mathcal{F}(\alpha) = \lim_{T \to \infty} \frac{\mathcal{F}(\alpha, T)}{T^{1+iA}} = \lim_{T \to \infty} \frac{1}{T^{1+iA}} \cdot 2\pi \cdot \sum_{T < 2\pi n < 2T} a_n e^{-2\pi n \alpha}
\]
is periodic with period 1, where the limit \( T \to \infty \) is taken in \( \mathbb{R} \setminus S_\varepsilon \). We substitute \( \alpha + 1 \) in (5.7) to conclude that \( C\pi Q^2 = q \) must be a positive integer, and that \( a_n n^{-iA} \) is periodic with period \( q \). This proves the first assertion of the first part of the theorem.

Once the periodicity of \( a_n n^{-iA} \) has been established, the passage to the second assertion of the first part of the theorem made in (5.1) is quite short and easy. Since these arguments are identical to those of the proof of Theorem 8.1 of [KP99], we do not repeat them here. We note that the formulation in [KP99] is actually slightly sharper with a more precise description of the Dirichlet polynomials \( P_X \).

If we further assume that \( F(s) \) satisfies (P4) or the stronger (P4'), the second and third assertions of the theorem follow follow almost immediately (see [Sou05], for instance).

\[\square\]

6. Primivity of cuspidal \( L \)-functions of \( GL_2/\mathbb{Q} \)

Recall that an element \( F(s) \) of \( \mathfrak{A}^\# \) is called primitive if \( F(s) = F_1(s)F_2(s) \) implies that either \( F_1(s) \) or \( F_2(s) \) is a unit. We say that an element of \( \mathfrak{A}^\# \) is almost primitive if \( F(s) = F_1(s)F_2(s) \) implies that either \( d_{F_1} = 0 \) or \( d_{F_2} = 0 \). Using the third part of Theorem 4.3 together with the theorem above, we obtain the following corollary by induction on the degree.

**Corollary 6.1.** Every element of \( \mathfrak{A}^\# \) (resp. \( \mathfrak{G}^\# \)) factors into a product of primitive elements.

**Proof.** By using induction on the degree, we see from Theorem 4.7 that factorization into into a product of primitive elements and elements of degree 0 holds in \( \mathfrak{G}^\# \) and \( \mathfrak{A}^\# \). In [KP03] the elements of \( \mathfrak{f}_0^\# = \mathfrak{g}_0^\# = \mathfrak{a}_0^\# \), are shown to factorise into primitives, whence the proof. \[\square\]

Since \( \mathfrak{g}_0 = \mathfrak{a}_0 = \{1\} \), an even easier version of the proof above gives

**Corollary 6.2.** Every element of \( \mathfrak{G} \) (resp. \( \mathfrak{A}^\dagger \)) factors into a product of primitive elements in \( \mathfrak{G} \) (resp. \( \mathfrak{A}^\dagger \)).
Let $f$ be a cuspidal eigenform on the upper half-plane, and let $L(s, f)$ be its associated $L$-function. Recall that $f$ is either a holomorphic cusp form or Maass cusp form (which is real analytic). In the language of representation theory, $L(s, f)$ is the $L$-function associated to the cuspidal automorphic representation $\pi_f$ of $\text{GL}_2(\mathbb{A}_\mathbb{Q})$ attached to $f$. We will assume that $L(s, f)$ is normalised so that it satisfies a functional equation of the form (1.2). It is well known that $L(s, f)$ lies in $\mathcal{S}$ - this result is classical. When $f$ is a holomorphic form, Deligne's celebrated proof of the Ramanujan conjecture shows that $L(s, f)$ lies in $\mathcal{F}$. Using Theorem 5.1 we can prove the following.

**Theorem 6.3.** The function $L(s, f)$ is primitive in $\mathcal{S}$.

**Proof.** By Theorem 6.2 we know that $L(s, f)$ must factor into a product of primitive elements. Because of Theorem 4.7 we know that only the following two types of factorizations are possible. Either

\[ L(s, f) = F_0(s)F_2(s), \]

with $F_0(s)$ of degree zero and $F_2(s)$ primitive of degree 2, or

\[ L(s, f) = F_0(s)F_1(s)F_2(s), \]

with $F_0(s)$ of degree 0, and $F_1(s)$ and $F_2(s)$ both primitive of degree 1. In either case, we know that $F_0(s) = 1$ by Theorem 4.3.

In the first case there is nothing to prove. In the second case, the second part of Theorem 5.1 shows that $F_1(s) = L(s + it_1, \chi_1)$ and $F_2(s) = L(s + it_2, \chi_2)$ for Dirichlet characters $\chi_1$ and $\chi_2$ and real numbers $t_1$ and $t_2$. Hence, we get

\[ L(s, f) = L(s + it_1, \chi_1)L(s + it_2, \chi_2). \tag{6.1} \]

This can be seen to be impossible as follows. For any Dirichlet series $F(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, we define the twist $F(s, \chi)$ of $F(s)$ by a Dirichlet character $\chi$ by

\[ F(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)a_n}{n^s}. \]

We denote the twist of $L(s, \chi_1)$ by $\chi$ by $L(s, \chi_1\chi)$ and the twist of $L(s, f)$ by $\chi$ by $L(s, f \times \chi)$. One sees easily that if $F(s) = F_1(s)F_2(s)$ then $F(s, \chi) = F_1(s, \chi)F_2(s, \chi)$. Let $S$ denote a finite set of places of $\mathbb{Q}$ containing all the primes dividing the conductors of $f$, $\chi_1$ and $\chi_2$. Twisting both sides of (6.1) by $\bar{\chi}_1$ outside of $S$, we get

\[ L_S(s, f \times \bar{\chi}_1) = L_S(s + it_1, \chi_1\bar{\chi}_1)L_S(s + it_2, \chi_2\bar{\chi}_1), \]

where $F_S(s) = \prod_{p \notin S} F_p(s)$ for elements of $\mathcal{S}$. The left-hand side is holomorphic on the line $\text{Re}(s) = 1$ by the classical work of Hecke. On the right-hand side, $L_S(s + it_1, \chi_1\bar{\chi}_1)$ has a simple pole at $s = 1 - it_1$, while $L_S(s + it_2, \chi_2\bar{\chi}_1)$ is non-vanishing there (it may also have a simple pole there if $\chi_1 = \chi_2$). Thus the right-hand side is not holomorphic on the line $\text{Re}(s) = 1$, giving a contradiction. \square

Since primitivity in $\mathcal{S}$ a fortiori implies primitivity in $\mathcal{F}$, we can recover the following result of Kaczorowski and Perelli as a corollary.
Corollary 6.4. If $L(s, f)$ lies in $\mathcal{F}$, it is primitive in $\mathcal{F}$.

7. Comparing zeros of $L$-functions

We return to a theme taken up first in [Rag99] and more recently in [Boo15]. Since the $L$-functions that are of interest in this section arise as $L$-functions associated to automorphic representations, they come naturally equipped with an Euler product. We will thus work in the class $\mathfrak{A}(1)$.

If $L_1(s) \neq L_2(s)$ are elements in $\mathfrak{A}(1)$ we would like to conclude (in many cases) that $L_1(s)/L_2(s)$ has infinitely many poles, that is, that there are infinitely many zeros (counted with multiplicity) of $L_2(s)$ that are not zeros of $L_1(s)$. Our results in this paper for $\mathfrak{A}(1)$ allow us to consider several examples which were not covered by the results in [Boo15].

As in the previous section $F(s, \chi)$ will denote the twist of the Dirichlet series $F(s)$ by a Dirichlet character $\chi$, and $F_S(s) = \prod_{p \not\in S} F_p(s)$ for elements of $\mathfrak{A}$ and $S$ a finite set of primes.

Theorem 7.1. Suppose $F_j(s)$ $j = 1, 2$ are elements of $\mathfrak{A}(1)$ and assume that $F_1(s) \neq F_2(s)$. Let $S$ be any finite set of primes and suppose that

1. $F_{1,S}(s, \chi)$ is holomorphic on $\text{Re}(s) = 1$, and
2. $F_{2,S}(s, \chi)$ is non-vanishing on $\text{Re}(s) = 1$

for every primitive Dirichlet character $\chi$. If $F(s) = F_1(s)/F_2(s)$ with $d_F \in [0, 1]$, then $F(s)$ must have infinitely many poles.

Proof. The arguments are similar to those in [Rag99] but we now have the more powerful Theorem 5.1. If $F(s)$ has only finitely many poles, it must lie in $\mathfrak{A}(1)$, and, in fact, in $\mathfrak{A}(1)$. It follows from Theorem 5.1 that the coefficients of $F(s)$ are periodic with some period $q \in \mathbb{N}$. We let $S$ be the set of primes dividing $q$. If $d_F = 1$, we know by the second assertion of Theorem 5.1 that

$$F_{1,S}(s) = F_{2,S}(s)L_S(s + iA, \chi_0)$$

for some Dirichlet character $\chi_0$ (mod $q$). We can assume that $S$ includes all the places where $\chi_0$ is ramified, since the equality above holds for any larger set of primes containing $S$. Twisting both sides of the equation above by $\chi_0^{-1}$, we see that

$$F_{1,S}(s, \chi_0^{-1}) = F_{2,S}(s, \chi_0^{-1})\zeta_S(s + iA).$$

But $\zeta_S(s + iA)$ has a simple pole at $s = 1 - iA$, while $F_2(s, \chi_0^{-1})$, and hence, $F_{2,S}(s, \chi_0^{-1})$ is non-vanishing there (by hypothesis). Thus, the right-hand side of the equation above has a simple pole at $s = 1 - iA$. On the other hand, $F_1(s, \chi_0^{-1})$, and hence, $F_{1,S}(s, \chi_0^{-1})$, is holomorphic at $s = 1 - iA$, yielding a contradiction.

If $d_F = 0$, we apply our proof above to $J(s) = F(s)L(s, \chi)$, where $\chi$ is a primitive Dirichlet character such that $L(s, \chi) \neq F_2(s)$. Thus $J(s)$ has degree 1 and satisfies all the hypothesis of theorem, so by our proof above we get the stronger result that $J(s)$ has infinitely many poles.

If $0 < d_F < 1$, it follows immediately from Theorem 4.7 that the result is vacuously true. □
Remark 7.2. Since we have absolute convergence in $\text{Re}(s) > 1$ for both $F_1(s)$ and $F_2(s)$ and non-vanishing on $\text{Re}(s) = 1$ for $F_2(s)$, we see that the infinitely many poles of $F(s)$ lie in the critical strip. Thus, the infinitely many poles of $F(s)$ do not arise because of the trivial zeros of $F_2(s)$.

Remark 7.3. The proof shows that we do not actually require that $F_i(s)$, $i = 1, 2$, individually belong to $\mathfrak{A}$. We require only that the quotient does. Thus, the theorem above applies to Artin $L$-functions which have not yet been proven to lie in $\mathfrak{A}$, but for which the functional equations and relevant holomorphy and non-vanishing results for character twists are known.

Remark 7.4. The proof shows that we require the holomorphy and the non-vanishing only for the incomplete twisted $L$-functions. This is usually easier to obtain in practice. In fact, it is enough to show these properties for a fixed finite set $S$ which contains all the primes dividing $N_i/N_j$, where the $N_j$ are the conductors of the $F_j(s)$, $j = 1, 2$.

Remark 7.5. In [Rag99], a similar theorem was proved for $d_F = 0, 1$, but essentially assuming that the gamma factors at infinity were the same for $F_1(s)$ and $F_2(s)$, since the only classification theorems available at the time assumed $r' = 0$. This was of course a strong restriction. There were also strong restrictions on the conductors.

We apply our theorem to the following pair of functions. Let $\pi_i$, $1 \leq i \leq 4$ be (unitary) cuspidal automorphic representations of $\text{GL}_{n_i}(\mathbb{A}_K)$ respectively. We take the tensor product $L$-functions $F_1(s) = L(s, \pi_1 \times \pi_2)$ and $F_2(s) = L(s, \pi_3 \times \pi_4)$. A series of papers due to Jacquet-Piatetski-Shapiro-Shalika [JS81a, JS81b, JPSS83], as well as Shahidi [Sha81, Sha88, Sha90] and Moeglin-Waldspurger [MW89] show that $F_j(s) \in \mathfrak{A}^\#(1)$, $j = 1, 2$, while the relevant non-vanishing statements for character twists are due to Shahidi [Sha81]. The boundedness of the $L$-functions in vertical strips was proved in [GS01]. It follows that $F_j(s)$, $j = 1, 2$, satisfy all the conditions of the theorem.

One expects that if $\pi_i$ and $\pi_j$ ($1 \leq i, j \leq 2$) are all distinct, then $F_1(s) \neq F_2(s)$ almost always. However, there are exceptions, and in practice, it is extremely difficult to rule out the possibility that $F_1(s) = F_2(s)$. If $\pi_1 \simeq \pi_3$ and $L(s, \pi_2) \neq L(s, \pi_4)$, then we can show that $F_1(s) \neq F_2(s)$. Let us now assume further that $n_2 = n_4$. In this case the quotient $F(s)$ has degree 0 and satisfies all the hypotheses of Theorem 7.1. It follows that $F(s)$ has infinitely many poles, that is, there are infinitely many zeros (counted with multiplicity) of $L(s, \pi_1 \times \pi_4)$ which are not zeros of $L(s, \pi_1 \times \pi_2)$. In view of Remark 7.2 these poles lie in the critical strip. When $\pi_1$ and $\pi_3$ are chosen to the trivial representation of $\text{GL}_4(\mathbb{A}_K)$, we recover the theorem for the standard $L$-functions of $\text{GL}_n(\mathbb{A}_K)$. We can thus record the following corollary to Theorem 7.1.

Corollary 7.6. Let $\pi_1$, $\pi_2$ and $\pi_4$ be (unitary) cuspidal automorphic representations of $\text{GL}_{n_i}(\mathbb{A}_K)$ for $i = 1, 2, 4$. Assume that $n_2 = n_4$ and that $L(s, \pi_2) \neq L(s, \pi_4)$. Then $L(s, \pi_1 \times \pi_2)/L(s, \pi_1 \times \pi_4)$ has infinitely many poles in the critical strip $0 < \text{Re}(s) < 1$. 

The point about the example in this corollary is that the functions $F_j(s)$, $j = 1, 2$ are not known to belong to $L$, and thus Theorem 1.7 of [Boo15] could not have been applied in this case. We give two more examples below outside the purview of Booker’s results.

Let $\pi$ be a (unitary) cuspidal automorphic representation of $\text{GL}_n(\mathbb{A}_K)$. The work of Shahidi and Kim-Shahidi (see [Sha81, Sha88, Sha90, Sha97] and [Kim99]) shows that the symmetric and exterior square $L$-functions, $L(s, ^{\vee} \otimes^2 \pi)$ and $L(s, \wedge^2 \pi)$, lie in $A(1)$. The relevant holomorphy and non-vanishing results for twists are also known by [Sha97], and the boundedness in vertical strips by [GS01], so our theorem applies to quotients of these $L$-functions (and quotients of products of these $L$-functions) as well. Again, these $L$-functions are not known to lie in $L$ and thus give more examples where Theorem 7.1 yields new results.

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