The incommensurate ground state of double-layer quantum Hall systems

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Double-layer quantum Hall systems possess interlayer phase coherence at sufficiently small layer separations, even without interlayer tunneling. When interlayer tunneling is present, application of a sufficiently strong in-plane magnetic field $B_\parallel > B_\perp$ drives a commensurate-incommensurate (CI) transition to an incommensurate soliton-lattice (SL) state. We calculate the Hartree-Fock ground-state energy of the SL state for all values of $B_\parallel$ within a gradient approximation, and use it to obtain the anisotropic SL stiffness, the Kosterlitz-Thouless melting temperature for the SL, and the SL magnetization. The in-plane differential magnetic susceptibility diverges as $|B_\parallel - B_\perp|^{-1}$ when the CI transition is approached from the SL state.

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I. INTRODUCTION

At sufficiently low particle densities and small layer separations, double-layer quantum Hall (2LQH) systems at total filling factor unity ($\nu_T = 1$) can be described as broken-symmetry states that possess interlayer phase coherence, even in the absence of interlayer tunneling. The 2LQH system can be mapped to an equivalent spin-1/2 system by equating “up” (“down”) pseudospins with electrons in the upper (lower) layer. (The electrons are assumed to have their real spins polarized.) The exchange energy between the electrons produces a pseudospin stiffness $\rho_0$ that seeks to keep the pseudospins aligned locally. At finite layer separation $d$, the direct (Hartree) energy produces a local capacitive charging energy that is minimized when the two layers have equal electron density. Thus the expectation value of the $z$ component of the pseudospin vanishes and the pseudospin system has an “easy-plane” anisotropy that gives the itinerant ferromagnet an $XY$ symmetry. (in the absence of interlayer tunneling). The expectation value of a pseudospin at location $\mathbf{r}$ can therefore be specified by its angle $\theta(\mathbf{r})$ in the $xy$ plane.

In the absence of interlayer tunneling, the 2LQH system picks out a particular global value of $\theta$ in the ground state, spontaneously breaking the $U(1)$ symmetry of the $XY$ ferromagnet. This gives rise to a linearly dispersing Goldstone mode at long wavelengths. Recent measurements of the zero-bias tunneling conductance in 2LQH systems show a huge resonant enhancement when interlayer coherence is present. This enhancement is related to the Goldstone mode of the broken $U(1)$ symmetry, and it has been proposed that the dispersion of the Goldstone mode can be observed in tunneling conductance measurements by applying a weak parallel magnetic field. The $XY$ pseudomagnet also possesses vortex excitations called “merons”; unlike those in an ordinary ferromagnet, these vortices are electrically charged, and the lowest-energy charged excitations of the system consist of vorticity-neutral meron pairs. There is also a finite-temperature Kosterlitz-Thouless (KT) phase transition due to the $XY$ symmetry and the finite pseudospin stiffness of the ferromagnet.

In the presence of interlayer tunneling, the $U(1)$ invariance associated with conservation of the charge difference between the two layers is lost. The finite interlayer tunneling $t$ of the electrons acts as an effective Zeeman pseudofield of magnitude $2t$ along the pseudospin $x$ axis and aligns the pseudospins, so that $\theta = 0$ in the ground state. The Goldstone mode disappears, and the collective mode becomes gapped. In the presence of interlayer tunneling, merons of opposite vorticity are bound together by a “string” that produces a linear attraction between the merons, eliminating the finite-temperature KT transition.

Murphy et al. have investigated the effect of an in-plane magnetic field $B_\parallel$ on 2LQH systems. By varying $B_\parallel$ and studying the energy gap obtained from activation energy measurements of the longitudinal resistivity, they find evidence for a phase transition between two competing QH ground states at a critical value $B_\parallel = B_\perp$. These two ground states are understood in the pseudospin language as being due to a competition between the tunneling energy $t$ and the exchange energy $\rho_0$. Read has presented an appealing analysis of charged (meron pair) excitations in this system, focusing on the value of the charge gap near the commensurate-incommensurate transition.

The presence of $B_\parallel$ periodically shifts the phase of the tunneling matrix elements, resulting in an effective Zeeman field for the pseudospins that rotates along the planar direction perpendicular to $B_\parallel$, with a wavelength $\phi_0/B_\parallel d$, which is the distance required to contain one flux quantum $\phi_0 = h/e$ between the layers. The net result is that for gradual variations of the pseudospin texture (gradual on the scale of the magnetic length $\ell \equiv$
balance, and \( \sqrt{\hbar/eB_{\perp}} \), where \( B_{\perp} \) is the strength of the magnetic field normal to the plane), the energy of the \( XY \) pseudospin system has the Pokrovsky-Talapov (PT) form:

\[
\mathcal{E} = \int d^2 r \left[ \frac{1}{2} \rho_s (\nabla \theta)^2 - \frac{t}{2\pi \ell^2} \cos(\theta + Q \cdot r) \right],
\]

up to a constant, where \( Q \equiv (2\pi d/\phi_0) \hat{b} \times \mathbf{z} \) defines the parallel magnetic-field wave vector,

\[
t = t_0 e^{-Q^2 \ell^2 / 4} \sqrt{1 - m_z^2}
\]

is the tunneling energy (where \( t_0 \) is the tunneling energy when \( Q = 0 \)) and

\[
\rho_s = (1 - m_z^2) \rho_E
\]

is the pseudospin stiffness within the Hartree-Fock Approximation (HFA). Here \( m_z = \nu_1 - \nu_2 \) is the layer imbalance, and

\[
\rho_E = \frac{e^2}{4\pi \ell e} \frac{1}{16\pi} \int_0^{\infty} dx x^2 e^{-x^2/2} e^{-x d / \ell}
\]

is the interlayer exchange stiffness when the layers are balanced: \( \nu_1 = \nu_2 = 1/2, \) or \( m_z = 0 \). The layer separation is \( d \), and \( \nu_j \) is the filling factor of layer \( j \).

For small \( Q \) (small \( B_{\perp} \)), the ground-state energy is minimized by having the pseudospins point in the direction of the local (rotating) pseudospin Zeeman field, so that \( \theta(r) = -\mathbf{Q} \cdot \mathbf{r} \). This is the commensurate ground state, and it minimizes the pseudospin Zeeman (tunneling) energy. However, it does so at the expense of the exchange energy, since neighboring pseudospins are no longer parallel. In the limit of large \( Q \), the cost in exchange energy for the pseudospins to align with the rapidly rotating Zeeman field is prohibitive, and the pseudospins become (nearly) uniformly polarized (constant \( \theta \)), just as if \( t \to 0 \). The state with uniformly polarized pseudospins is the large-\( Q \) limit of the incommensurate state. It turns out that for all finite \( B_{\parallel} > B_c \), the translational symmetry of the pseudospin polarization is broken, and a soliton-lattice (SL) state is obtained in the incommensurate phase.

The SL phase of the PT model can also undergo a separate finite-temperature dislocation-mediated KT transition that restores the translational symmetry. This work focuses on calculating the ground-state properties of the SL state, for all \( B_{\parallel} > B_c \). Interestingly, it is not necessary to determine \( \theta(r) \) in order to calculate the total energy of the system. From the total energy, we can calculate thermodynamic quantities such as the SL stiffnesses, extending the results of Ref. 1 for the stiffnesses and the KT temperature to all \( B_{\parallel} \). We also calculate the SL contribution to the magnetization and susceptibility, and discuss some possibilities of measuring these quantities experimentally.

The plan of this paper is as follows. In Sec. II, we discuss the single-soliton solutions that follow from the equation of motion obtained from the PT energy. For \( Q > Q_c \), the solitons proliferate; in Sec. III, the interaction between soliton lines is discussed, and the separation between solitons as a function of \( Q/Q_c \) is derived. In Sec. IV, the compressional and shear elastic constants are analyzed, and an estimate is made of the Kosterlitz-Thouless temperature for melting the soliton lattice, as a function of \( Q \). The interlayer phase coherent 2LQH state has a diamagnetic response to an applied in-plane magnetic field. Section V gives a calculation of the in-plane magnetization due to the 2LQH state, as a function of \( Q \). We conclude with a summary of our results.

\section{II. Single Solitons}

When \( Q > Q_c \), it is convenient to define \( \bar{\theta}(r) \equiv \theta(r) + Q \cdot r \), so that Eq. (1.1) becomes

\[
\mathcal{E} = \int d^2 r \left[ \frac{1}{2} \rho_s (\nabla \bar{\theta})^2 - \frac{t}{2\pi \ell^2} \cos(\bar{\theta}) \right].
\]

Minimizing \( \mathcal{E} \) with respect to variations in \( \bar{\theta} \) gives the sine-Gordon equation:

\[
\nabla^2 \bar{\theta} = \frac{1}{\xi^2} \sin \theta,
\]

where \( \xi = \sqrt{2\pi \rho_0 / t} \). We shall give numerical values for our results for a hypothetical “typical” GaAs (effective mass \( m^* \approx 0.07m_e \) and relative dielectric constant \( \epsilon_r \approx 13 \) 2LQH sample, which for the sake of definiteness we assign the following sample parameters: total particle areal density \( n_T = 1.0 \times 10^{11} \) cm\(^{-2} \), layer (midwell to midwell) separation \( d = 20 \) nm, and tunneling energy \( t_0 = 0.1 \) meV. Such a sample would have \( \ell \approx 12.6 \) nm, \( d/\ell \approx 1.6, h\omega_c \approx 6.9 \) meV for \( n_T = 1, \) and \( e^2 / 4\pi \epsilon \approx 8.8 \) meV. In the HFA, \( \rho_0 \approx 0.03 \) meV and \( \xi \approx 17 \) nm.

The commensurate state minimizes the tunneling energy by having \( \theta(\mathbf{r}) = 0 \), so that the phase angle \( \theta(\mathbf{r}) = Q \cdot \mathbf{r} \) follows the tumbling Zeeman pseudofield. The energy per area of this state is \( \rho_0 Q^2 / 2 - t / 2\pi \ell^2 \). In the limit of large \( Q \), the incommensurate state with constant \( \theta \) has a lower energy per area, equal to zero. These energies are plotted as the solid and dashed lines in Fig. 1. Therefore estimate that there is a phase transition near the point where the commensurate-state and the constant-\( \theta \) incommensurate-state energies are equal, at \( Q \xi \approx \sqrt{2} \). It turns out, however, that the incommensurate state lowers its energy by breaking translation invariance, so that at finite \( t \) and \( Q > Q_c \), the value of \( \theta \) depends on position.

Equation (2.2) possesses soliton solutions. To see this, let us seek solutions of the form

\[
\bar{\theta}(\mathbf{r}) = \bar{\theta} [\hat{e}_1 \cdot (\mathbf{r} - \mathbf{r}_0)],
\]

where \( \hat{e}_1 \) could be any unit vector in the \( xy \) plane. Then Eq. (2.2) becomes
\[ \partial_t^2 \hat{\theta} = \frac{1}{\xi^2} \sin \hat{\theta}. \]  

(2.4)

Note that this equation is equivalent to the equation of motion for a pendulum of length \( l \), \( \partial_t^2 \phi = -(g/l) \sin \phi \), if we replace \( \hat{\theta} \rightarrow \phi - \pi \), \( \hat{\mathbf{e}}_1 \cdot \mathbf{r} \rightarrow t \) (time), \( \rho_s \rightarrow \ell \), and (tunneling amplitude) \( t/2\pi \ell^2 \rightarrow g \). This analogy is very useful in finding the soliton solutions for the PT model. In particular, the pendulum can make full circles in a given direction. This corresponds to the SL state in the 2LQH system.

In analogy with the pendulum problem, we may define a conserved quantity analogous to the total (kinetic plus potential) energy of a pendulum:

\[ 2c^2 \equiv \frac{1}{2} \xi^2 \left( \partial_t \hat{\theta} \right)^2 - (1 - \cos \hat{\theta}). \]  

(2.5)

Differentiating Eq. (2.5) with respect to \( \hat{\mathbf{e}}_1 \cdot \mathbf{r} \) and using Eq. (2.4) shows that \( \partial_t f = 0 \), so that \( c \) is a constant of the motion. Defining \( \beta = \hat{\theta}/2 \) then leads to the equation

\[ \partial_t \beta = \pm \frac{1}{\xi} \sqrt{c^2 + \sin^2 \beta}. \]  

(2.6)

It is straightforward to solve Eq. (2.7) when \( c = 0 \) by writing \( f = \tan(\beta/2) \), so that \( \partial_t f = \pm f/\xi \), giving \( \hat{\theta} = \hat{\theta}_{ss}(\mathbf{r}) \), where

\[ \hat{\theta}_{ss}(\mathbf{r}) = 4 \arctan \left[ \frac{e^{\pm \hat{\mathbf{e}}_1 \cdot (\mathbf{r} - \mathbf{r}_0)}/\xi}{1} \right] \]  

(2.7)

represents a single soliton in the \( \hat{\mathbf{e}}_1 \)-direction, centered at \( \hat{\mathbf{e}}_1 \cdot \mathbf{r}_0 \), with width \( \xi \). This is shown in Fig. 3. Note that \( \hat{\theta}_{ss}(\mathbf{r}) \) changes by \( 2\pi \) as \( \hat{\mathbf{e}}_1 \cdot \mathbf{r} \) goes from \(-\infty\) to \(\infty\). This corresponds to the motion of a pendulum that just barely completes a full revolution, and whose period goes to infinity.

The energy per length of a single soliton may be computed by substituting Eq. (2.7) into Eq. (2.1) and subtracting the commensurate-state (\( \hat{\theta} = 0 \)) energy, to obtain

\[ \frac{\xi}{L_2} = \rho_s \left( \frac{8}{\xi} \mp 2\pi \hat{\mathbf{e}}_1 \cdot \mathbf{Q} \right), \]  

(2.8)

where \( L_2 \) is the sample length in the planar direction perpendicular to \( \hat{\mathbf{e}}_1 \). The lowest (soliton) and highest (anti-soliton) energy solutions occur for \( \hat{\mathbf{e}}_1 = \pm \mathbf{Q} \). Since solitons in the \(-\mathbf{Q}\) directions are equivalent to anti-solitons in the \(\mathbf{Q}\), we shall speak only about solitons with orientations defined by \( \hat{\mathbf{e}}_1 \). The lowest-energy soliton state has \( \hat{\mathbf{e}}_1 = \mathbf{Q} \), and its energy per length is

\[ \frac{\bar{\xi}}{L_2} = 2\pi \rho_s (Q_c - Q), \]  

(2.9)

which goes to zero when \( Q = Q_c \), where

\[ Q_c = \frac{4}{\pi \xi} = \frac{4}{\pi} \sqrt{\frac{t/\rho_s}{2\pi \ell^2}}. \]  

(2.10)

The value of the critical wave vector \( Q_c \) will depend on the layer imbalance \( m_z \equiv \nu_1 - \nu_2 \). Equations (1.2) and (1.3) give

\[ Q_c(m_z) = (1 - m_z^2)^{-1/4} Q_c(0) \]  

(2.11)

in the HFA, where \( Q_c(0) \) is the value of \( Q_c \) when the layers are balanced \((m_z = 0)\). Equation (2.11) implies that the value of \( Q_c \) where the CI transition occurs could be fine-tuned by adjusting the layer imbalance \( m_z \) — i.e., by adjusting the gate voltages on the 2LQH sample. Such a procedure might be very useful in investigating the CI transition.

When \( Q < Q_c \) and \( t > 0 \), the lowest-energy charged excitations are finite-length soliton lines with charged meron ends — i.e., charged vortices bound by a soliton “string” whose tension is given by Eq. (2.4). As \( Q \) increases, the soliton-line “string tension” gets weaker, so that the Coulomb repulsion of the merons stretches out the string and lowers the energy of the charged excitations. At \( Q = Q_c \) the soliton-line “string tension” goes to zero, and it costs zero energy to make infinitely long soliton lines. Since the creation energy per length for a soliton decreases linearly with \( Q \) (with \( B \)) for \( Q \geq Q_c \), it becomes energetically favorable to form solitons. The number of solitons created is determined by the competition between the (negative) creation energy per soliton versus the repulsive (positive energy) interactions between solitons. Note that \( Q_c < \sqrt{2}/\xi \) (the value of \( Q \) at which the commensurate-state and constant-\( \theta \) incommensurate state energies are equal), so that for \( Q > Q_c \) it is energetically favorable to create solitons. Because the solitons are weakly repulsive, the result is a soliton-lattice state that we describe below and illustrate in Fig. 3.

An analogous effect occurs in long Josephson junctions, where application of a magnetic field parallel to two superconducting planes in close proximity produces 2\( \pi \) twists in the Josephson phase and generates a SL state.

To summarize, for \( Q < Q_c \), we obtain the commensurate phase in which \( \theta(\mathbf{r}) = 0 \). For \( Q = Q_c \), we introduce a single soliton, corresponding to the marginal case of a pendulum that makes exactly one full revolution and has an infinite period of oscillation. For \( Q > Q_c \), we obtain a soliton lattice, corresponding to a pendulum making complete revolutions in one direction. It has been argued that the commensurate to incommensurate soliton-lattice (CI) transition at \( Q = Q_c \) can be identified with the phase transition between 2LQH ground states seen by Murphy and co-workers. We therefore make the identification \( Q_c = 2\pi B_c d/\phi_0 \).

III. SOLITON LATTICE

We shall now use Eq. (2.6) to determine the SL spacing \( L_s \). We do this by noting that over one period of the
soliton lattice, $\tilde{\theta}$ changes by $2\pi$, so that $\beta$ changes by $\pi$. We therefore express $L_s$ as

$$
\frac{L_s}{\xi} = \frac{1}{\xi} \int_{-\pi/2}^{\pi/2} \frac{d\beta}{\partial \beta} = 2 \int_{0}^{\pi/2} \frac{d\beta}{\sqrt{c^2 + \sin^2 \beta}} = 2\eta K(\eta),
$$

where we have used Eq. (2.4) and have defined $\eta \equiv 1/\sqrt{c^2 + 1}$, and where

$$
K(\eta) \equiv \int_{0}^{\pi/2} \frac{d\beta}{\sqrt{1 - \eta^2 \sin^2 \beta}}
$$

is the complete elliptic integral of the first kind. We define the SL wave vector $Q_s \equiv (2\pi/L_s)\hat{e}_1$, so that Eq. (3.1) may be written in terms of $Q_s \equiv |Q_s|$ as

$$
\frac{Q_s}{Q_c} = \frac{(\pi/2)^2}{\eta K(\eta)}.
$$

Note that $\eta \to 1$ corresponds to the CI transition, where $Q_s \to 0$ and $Q \to Q_c$, whereas $\eta \to 0$ corresponds to the constant-$\theta$ incommensurate state, where $Q_s \to Q \to \infty$. From Eqs. (3.3) and (3.2), it follows that

$$
\eta \to \begin{cases}
1 - 8e^{-2\pi/\xi Q_s}, & Q_s/Q_c \to 0 \\
\left(\frac{\eta}{\eta Q_s} \right)^2 \left[1 - \frac{1}{4} \left(\frac{\eta}{\eta Q_s} \right)^2 \right]^2 + \frac{3}{4} \left(\frac{\eta}{\eta Q_s} \right)^4, & Q_s/Q_c \to \infty
\end{cases}
$$

(3.4)

In order to determine the value $Q_s$ of the SL wave vector that minimizes the total energy, we express the energy per area from Eq. (2.3) as an integral over $\beta$ [cf. Eq. (3.1)] and obtain

$$
\frac{\mathcal{E}}{L_1 L_2} = \frac{\rho_s}{\xi} \left[\frac{1}{2} \eta Q_s^2 - Q \cdot Q_s \right] \xi^2
$$

(3.5)

$$
= \rho_s \left[\frac{1}{2} Q_s^2 - Q \cdot Q_s \right] \xi^2 + Q_s \cdot Q_s E(\eta) \frac{2}{\eta^2 - 1}
$$

$$
\to \begin{cases}
\rho_s \left[Q_s^2 - t/2Q_s^2 \right], & Q < Q_c \\
\rho_s \left[Q_s^2 + 1 + 4e^{-2\pi/\xi Q_s} - 1/\xi^2 \right], & \eta \to 1 \\
\rho_s \left[\frac{1}{2} \left(Q - Q_s \right)^2 - 1/2Q_s^2 \right], & \eta \to 0
\end{cases}
$$

where

$$
E(\eta) \equiv \int_{0}^{\pi/2} d\beta \sqrt{1 - \eta^2 \sin^2 \beta}
$$

(3.6)

$$
\to \begin{cases}
1 + \frac{1}{2} (1 - \eta^2) \ln(\eta) - \frac{1}{2}, & \eta \to 1 \\
\frac{1}{2} (1 - 4\eta^2 - 3/8\eta^4), & \eta \to 0
\end{cases}
$$

is the complete elliptic integral of the second kind, and $L_1$ is the sample length in the planar direction parallel to $\hat{e}_1$. Agreement between Eqs. (2.3) and (3.3) in the thermodynamic limit is obtained by equating $Q_s = (2\pi/L_s)\hat{e}_1$, so that $L_s = L_1$ at $Q = Q_c$. All the terms in the $\eta \to 1$ limit (near the CI transition, where the solitons are well separated) of Eq. (3.3) have simple physical interpretations. The first and last terms, $\rho_s(Q_s^2/2 - 1/\xi^2)$, constitute the (commensurate-phase) energy per area of the PT model in the absence of solitons. The second and third terms, $\rho_s(Q_s Q_s - Q \cdot Q_s)$ are just the creation energy per area for the solitons in terms of interacting soliton lines [see Eq. (2.4)]. The fourth term, $4\rho_s Q_s Q_s \exp(-2\pi/\xi Q_s)$, is the exponentially weak repulsive interaction energy per area between the solitons. Because $Q_s/2\pi = 1/L_s$ is the density of soliton lines, the fourth term says that near the CI transition, the interaction energy per length between two parallel, straight, and infinitely long soliton lines separated by a distance $L_s$ is

$$
\lim_{\eta \to 1} \frac{U}{\int_{Q}^{\eta}} \to 8\rho_s \xi Q_s e^{-L_s/\xi}.
$$

(3.7)

Hence for $Q > Q_c$, soliton lines proliferate rapidly until the repulsion between the solitons become significant. The notion of an effective repulsive interaction between sine-Gordon solitons was discussed by Perring and Skyrme and who obtained the exponentially weak repulsion found above. The arguments of Ref. [3] imply that when the solitons are close together ($L_s/\xi \to 0$) at large $Q/Q_c$, the repulsive potential energy per length between soliton lines is $(\pi^2/2)\rho_s Q_s^2(\xi/L_s)$. This latter repulsion is due to boundary condition that $\tilde{\theta}$ must change by $2\pi$ over the small distance $L_s$, which implies a large gradient energy.

The value of $Q_s$ which minimizes the energy per area [Eq. (3.3)] is found by setting to zero

$$
\frac{\partial}{\partial Q_s} \left( \frac{\mathcal{E}}{L_1 L_2} \right) = \rho_s \left[ Q_s \frac{E(\eta)}{\eta} - Q_s \right]
$$

(3.8)

where we have used the identity (2.4)

$$
\frac{dE}{d\eta} = \frac{E(\eta) - K(\eta)}{\eta}.
$$

(3.9)

It is not difficult to show that the second derivative of the energy per area with respect to $Q_s$ is always positive; thus the extremum calculated above is a minimum. It follows from Eq. (3.3) that the energy is minimized for $Q_s = \bar{Q}$, and for $\eta = \bar{\eta}$ such that

$$
\frac{Q_s}{Q_c} = \frac{E(\bar{\eta})}{\bar{\eta}}.
$$

(3.10)

where $Q_s \equiv |Q_s|$. We define the CI closeness parameter
\[ \epsilon \equiv Q/Q_c - 1, \quad (3.11) \]

which vanishes at the CI transition; from Eqs. (3.6) and (3.10), it follows that for \( \eta \to 1 \),

\[ \epsilon \approx \left( \frac{1 - \eta}{2} \right) \left[ \ln \left( \frac{8}{1 - \eta} \right) + 1 \right], \quad (3.12) \]

so that asymptotically,

\[
\eta \to \begin{cases} 1 - 2\epsilon/\ln(1/\epsilon), & Q_s/Q_c \to 0 \\ \left( \frac{\pi Q}{2} \right)^2 \left[ 1 - \frac{1}{4} \left( \frac{\pi Q}{2} \right)^2 \right]^2 + \frac{\epsilon}{\eta} \left( \frac{\pi Q}{2} \right)^4, & Q_s/Q_c \to \infty \end{cases} \quad (3.13)
\]

Equations (3.3) and (3.10) together allow us to determine the equilibrium SL wave vector \( Q_s(Q) \) that minimizes the energy, in terms of the parallel-field wave vector \( Q_c \). We have plotted this in Fig. 4. From Eqs. (3.3), (3.5), and (3.10), it follows that

\[
\frac{Q_s}{Q_c} = \left( \frac{\pi}{2} \right)^2 \frac{K(\eta)E(\eta)}{K(\eta)E(\eta)} \quad (3.14)
\]

where the \( Q \to Q_c \) limit is true asymptotically. We note however that, as found by Pokrovsky and Talapov and discussed in Ref. 22, the meandering of soliton lines at finite temperature renormalizes the dependence of the soliton-line density on the parallel magnetic field, so that \( Q_s \propto \sqrt{\epsilon} \) sufficiently close to the CI transition.

The minimized value \( \frac{\dot{E}}{L_1 L_2} \) of the energy per area at \( Q_s = Q_c \) is found by using Eqs. (3.5) and (3.10) in Eq. (3.6) to obtain

\[
\frac{\dot{E}}{L_1 L_2} = \frac{1}{2} \rho_s Q^2 - \frac{t}{2\pi \ell^2} \left( \frac{2}{r^2} - 1 \right) \quad (3.15)
\]

\[
\to \begin{cases} \frac{1}{2} \rho_s Q^2 - t/2\pi \ell^2, & Q < Q_c \\ -\frac{1}{2} \rho_s Q_s^2 (\pi/8 - 1), & Q/Q_c \to 1 \\ \frac{t}{2\pi \ell^2} \ln \left( \frac{\pi Q}{2} \right)^2, & Q/Q_c \to \infty \end{cases}
\]

The SL state energy per area is represented by the dots in Fig. 4.

Although it is not needed for calculating the stiffnesses or susceptibility of the SL, the SL solution for \( \dot{\theta}(r) \) is given by

\[ \sin \left[ \frac{1}{2} (\dot{\theta} - \pi) \right] = \sin(\hat{\epsilon}_1 \cdot (r - r_0)/\eta \xi, \eta), \quad (3.16) \]

where \( \eta \) denotes the sine-amplitude Jacobian elliptic function with parameter \( \eta \). Near the CI transition, when \( Q \to Q_c \), \( \dot{\theta} \) is very close to being a periodic superposition of single-soliton solutions, spaced apart by \( L_s \), so that \( \dot{\theta}(r) \approx \sum_j \theta_{ss}(r - jL_s) \), where \( \theta_{ss}(r) \) denotes the single-soliton solution, Eq. (2.7). In the SL state, especially away from the CI transition, it is sometimes useful to work with

\[
\dot{\theta}_{ij}(r) \equiv \dot{\theta}(r) - Q_s \cdot r = \dot{\theta}(r) + (Q - Q_s) \cdot r, \quad (3.17)
\]

because it is periodic in the SL spacing, so that \( \theta_{ij}(r + L_s) = \theta_{ij}(r) \), where \( L_s = Q_s \pi/\tilde{Q}_s \). In the limit \( Q/Q_c \to \infty \), \( Q_s \to Q \) and \( \theta \to \theta/\ell \), so that we may regard \( \theta_{ij}(r) \) as a small quantity. Expressing the sine-Gordon equation [Eq. (2.4)] in terms of \( \dot{\theta} \) and working to lowest order in \( \dot{\theta} \) gives \( \nabla^2 \dot{\theta} \approx 1/\xi^2 \sin(\dot{\theta} \cdot (r - r_0)) \), so that

\[
\lim_{Q \to \infty} \dot{\theta}_{ij}(r) \approx - \left( \frac{\pi Q}{4 Q_s} \right)^2 \sin(\dot{\theta} \cdot (r - r_0)). \quad (3.18)
\]

IV. STIFFNESSES OF THE SOLITON LATTICE

The elastic constants of the soliton lattice are given by the stiffness tensor \( K_{ij} \). The change in the energy due to varying the spacing between the soliton lines is characterized by the compressional stiffness \( K_{11} \). The shear stiffness \( K_{22} \) is associated with the change in energy due to shearing the upper and lower ends of the soliton lines in opposite directions, and is equivalent to a combined rotation and compression. We use the calculated stiffnesses to describe the \( B_{ij} \)-dependence of the KT temperature \( T_{KT} \) for the dislocation-mediated KT melting transition of the soliton-lattice.

We calculate the stiffness tensor by two methods. First, we calculate the stiffness tensor \( K_{ij} \) that is obtained by differentiating \( E/L_1 L_2 \) in Eq. (3.5) with respect to the \( i, j \) components of \( Q_s \) for fixed \( Q \) at the extremal, where Eq. (3.5) is zero. Then we calculate the stiffness tensor \( K_{ij} \) obtained by calculating the effects of fluctuations of the angle variable \( \dot{\theta}(r) \) away from its ground-state value, Eq. (3.16).

We begin by calculating the stiffness tensor \( K_{ij} \) from the dependence of the ground-state energy [Eq. (3.5)] on the soliton-lattice wave vector \( Q_s \). The expressions we obtain for \( K_{ij} \) by this method have been carried out for all values of \( Q \geq Q_c \), and agree with the results obtained in Ref. 11, in the limit \( Q \to Q_c \). Because the stiffnesses involve the second derivative of \( E/L_1 L_2 \) with respect to the components of \( Q_s \) at fixed \( Q \), the terms in Eq. (3.5) that depend explicitly on \( Q \) (including the term \( -\rho_s Q_s Q_s \) that gives the orientational dependence of the energy per area) do not contribute to \( K_{ij} \). Thus

\[
K_{ij} = \lim_{Q_s \to Q_s} \frac{\partial^2}{\partial Q_{si} \partial Q_{sj}} \left( \frac{E}{L_1 L_2} \right)_{Q_s} \quad (4.1)
\]

\[
= \rho_s Q_c \lim_{Q_s \to Q} \left( \frac{\partial}{\partial Q_{si}} \frac{Q_{sj} E(\eta)}{Q_s \eta} \right)_{Q_s} \quad (4.2)
\]
× \left[ Q_s \frac{E(\eta)}{\eta} \right]_Q ,

where we have used Eqs. (3.8) and (3.10). Since \( \bar{Q}_{s2} = 0 \), it follows that \( K_{12} = K_{21} = 0. \)

Using the results of Sec. [11] and the identity [23]

\[
\frac{dK}{d\eta} = \frac{E(\eta)}{\eta(1 - \eta^2)} - \frac{K(\eta)}{\eta},
\]

from which it follows that

\[
\frac{d\eta}{dQ_s} = \left( \frac{\pi}{2\eta} \right)^2 \frac{Q_s}{E(\eta)} \left( \frac{Q_c}{Q_s} \right)^2 , \tag{4.4}
\]

we find that the compressional elastic constant \( K_{11} \) is equal to

\[
\frac{K_{11}}{\rho_s} = \lim_{Q_s \to 0} \frac{\partial}{\partial Q_s} \left[ \frac{Q_s E(\eta)}{\eta} \right]_Q = \frac{\partial Q}{\partial Q_s} \tag{4.5}
\]

\[
= \frac{16(1 - \eta^2)}{(Q_s c^2)(Q\bar{c}^2)\eta^2} \to \left\{ \begin{array}{l}
(2/\pi^2) \ln(1/\epsilon), \quad Q/Q_c \to 1 \\
1 - \frac{3}{32} \left( \frac{\pi Q_c}{4 Q_s} \right)^4, \quad Q/Q_c \to \infty
\end{array} \right.
\]

In the limit \( Q/Q_c \to 1 \) when the soliton lines are far apart, \( K_{11} \) is very small [of order \( \epsilon \sim \exp(-L_\phi/\xi) \), see Eq. (3.14)]. The energy cost of compression very close to the CI transition is due to the exponentially weak inter-soliton interaction energy. The energy per area due to the string tension of the soliton lines (the term \( \rho_s Q_s Q_s \) does not contribute to \( K_{11} \), although it does contribute to \( K_{22} \), as we explain below.

As explained in Ref. [23], soliton lines meander at finite temperature and are no longer straight; collisions between meandering soliton lines produce an effective entropic repulsion between the solitons that dominates the exponential repulsion at any nonzero temperature, for \( L_\phi/\xi \) sufficiently large. This effect renormalizes the compressional stiffness \( K_{11} \) upwards so that it becomes proportional to \( T^2 \). In the limit \( Q/Q_c \to \infty \), the tunneling term in the PT energy (2.1) fluctuates on very short length scale and averages to zero, so that Eq. (2.4) becomes the isotropic XY model; thus one expects \( K_{11} \) to approach the pseudospin stiffness \( \rho_s \) in that limit.

From Eqs. (3.10) and (4.1), it follows that the shear elastic constant \( K_{22} \) is given by

\[
\frac{K_{22}}{\rho_s} = \frac{Q}{Q_s} \to \left\{ \begin{array}{l}
(2/\pi^2) \ln(1/\epsilon), \quad Q/Q_c \to 1 \\
1 + \frac{1}{2} \left( \frac{Q_s}{Q_c} \right)^4, \quad Q/Q_c \to \infty
\end{array} \right.
\]

As expected, the shear stiffness \( K_{22} \) approaches the pseudospin stiffness \( \rho_s \) in the limit \( Q/Q_c \to \infty \). But in the limit \( Q \to Q_c \), \( K_{22} \) diverges as \( Q_c/Q_s \). The origin of this effect is that the shear motion described by \( K_{22} \) increases the total length of the soliton lines. By definition, \( K_{22} \) describes a shear displacement in which \( Q_{s2} \) changes, while \( Q_{s1} \) remains fixed: i.e., the solitons lines tilt away from their equilibrium “vertical” (\( \hat{e}_2 \)) direction by a small angle \( \phi = Q_{s2}/Q_s \), while keeping their “horizontal separation” \( \hat{e}_1 \cdot \hat{L}_s = \bar{L}_s \) constant. This shear motion reduces the mean soliton separation (the separation along the direction perpendicular to the tilted soliton lines) to \( L_s = \bar{L}_s \cos \phi \), so that \( Q_s \to Q_s/\cos \phi \). The \( K_{22} \)-shear corresponds to a global rotation plus a compression of the SL. Packing the solitons lines more closely together in a fixed sample area produces more total soliton line length, which costs more soliton-line creation energy.

Because the term \( -\rho_s Q_c Q_s \) that contains the orientational dependence of the energy per area [Eq. (4.3)] is linearly proportional to \( Q_{s1} \), it cannot contribute to the stiffnesses (which are proportional to second derivatives of the energy per area with respect to the components of \( Q_s \)) at all. It might be supposed that the rotation of the soliton lines that occurs upon shearing should cost energy, but this is not so, because a shear is a combination of rotation and compression, rather than a pure rotation. The creation energy per area of the SL near the CI transition (\( Q \approx Q_s \)) is \( \rho_s (Q_c Q_s - Q \cdot Q_s) \), and consists of two terms. The first term \( \rho_s Q_c Q_s \) is just the total line length of the solitons times the line tension, divided by the total area. The second term \( -\rho_s Q_c \cdot Q_s \) depends explicitly on the angle \( \phi \) between \( Q \) and \( Q_s \), and determines the orientation of the SL because it is minimized by choosing \( Q_c \) along \( Q \) (i.e., \( \phi = 0 \)); thus, a different choice of SL orientation (i.e., a pure rotation of the soliton lines) would cost more energy. Interestingly, the second term in the creation energy is unchanged by a shear, because the energy cost of rotating the soliton lines is exactly offset by the increase in total soliton line length: \( -[Q(Q_c/\cos \phi)] \cos \phi = -Q Q_c \), independent of \( \phi \). The only change in the creation energy comes from the first term, which depends only on the density of soliton lines: \( \rho_s Q_c (Q_c/\cos \phi) \). Sufficiently close to the CI transition (i.e., when \( L_\phi/\xi \gg 1 \)), the exponentially small interactions may be neglected in comparison to the creation energy. For \( Q \to Q_c \) and small shear \( (Q_{s2}/Q_{s1} \ll 1) \),

\[
\frac{\mathcal{E}}{L_1 L_2} \to \rho_s Q_c Q_s - t/2\pi \ell^2 \tag{4.7}
\]

\[
\approx \rho_s Q_c \left[ Q_s + \frac{1}{2} Q_{s2}^2 \right] - t/2\pi \ell^2 ,
\]

so that \( K_{22} \to \rho_s Q_c/Q_s \to \infty \) as \( Q \to Q_c \), in agreement with the results of Ref. [10].

The fact that bilayer phase-coherent 2LQH states can exhibit a finite-temperature Kosterlitz-Thouless (KT) transition in the absence of interlayer tunneling \( (t \to 0) \) has been discussed in earlier work [26]. Although finite \( t \) removes the possibility of a KT transition in the commensurate phase of 2LQH systems by altering the nature of the long-range interaction between vortices ("merons" in this case) [2], the SL phase does support a finite-temperature KT transition due to dislocation-mediated melting of the SL. [3] As discussed in Ref. [10], the KT temperature may be estimated as
where we have used the zero-temperature values for $K_{ij}$ that we calculated previously to make a rough estimate the KT temperature. As mentioned earlier, finite-temperature fluctuation effects can strongly renormalize $K_{11}$ and may also effect $K_{22}$. Our results agree with those of Ref. 10 in the limit $Q \to Q_c$. We have plotted the compressional ($K_{11}$) and shear ($K_{22}$) stiffnesses in Fig. 3, together with the KT temperature.

The KT transition would be most easily measured from the temperature dependence of the linear response to oppositely directed currents in each layer. This would require double-layer electron devices with layers that could be contacted separately. Unfortunately, the leakage currents produced when the interlayer tunneling is not vanishingly small would make it difficult, perhaps impossible, to set up oppositely directed currents in each layer. However, because the SL dislocations are electrically charged, it might be possible that the KT transition could be signalled by an increase in the usual longitudinal resistivity $\rho_x$. We do not include dynamics here, because our focus is on ground-state, rather than excited-state properties. The PT energy for $\theta$ is $E_{PT}[\theta_0 + \delta \theta] = E_{PT}[\theta_0] + \delta H$, where $E_{PT}$, given by Eq. (2.3), is the PT energy from which the ground-state $\theta_0(r)$ is determined via Eq. (2.2), and

$$\delta H \approx \frac{1}{2} \int \frac{d^2r}{2\pi\ell^2} \left[ t \cos \theta_0 \delta \theta^2 + 2\pi \ell^2 \rho_s \nabla \delta \theta \right]^2$$

$$= \frac{1}{2} \int \frac{d^2r}{2\pi\ell^2} \delta \theta \left[ t \cos \theta_0 - 2\pi \ell^2 \rho_s \nabla^2 \right] \delta \theta,$$

where we have kept terms up to quadratic order in $\delta \theta$. There are no terms linear in $\delta \theta$ because $\theta_0(r)$ is determined by minimizing $E_{PT}$ with respect to variations in $\theta$. The total energy is minimized by choosing $\delta \theta$ from among the eigenvalues of the bracketed Schrödinger-like operator in Eq. (2.9), so that

$$\left[ t \cos \theta_0 - 2\pi \ell^2 \rho_s \nabla^2 \right] \delta \theta_q = \epsilon_q \delta \theta_q$$

If we take $B_\parallel = B_0 \hat{y}$ so that $Q = Q \hat{x}$, then $\delta \theta_0(r)$ depends only on $x$, $\delta H$ is translationally invariant in the $y$ direction, and we may write $\delta \theta_q(r) = \exp(iq_y y) \delta \theta_q(x)$. The term $t \cos \theta_0$ in Eq. (4.10) is periodic in the $x$ direction, with a period of $L$. As shown in Ref. 13, when $\theta_0(r)$ has the form (3.10), Eq. (4.10) becomes Lamé's equation, after a simple rescaling of $x$. Lamé's equation has three simple solutions, two of which have low-energy limits. The first type of solution has zero energy and corresponds to a uniform translation of the vortex lines, $\delta \theta \sim \partial \theta_0 / \partial x_0$, where $x_0$ is the $x$ component of $r_0$ in Eq. (3.10). Of greatest interest to us are the type of solutions which have been called “vortex oscillations” in the context of long Josephson junctions in parallel magnetic fields. From Ref. 18, it follows that in the long-wavelength limit,

$$\frac{\delta \mathcal{E}}{L_1 L_2} = \frac{1}{2} \left( K_{11} \epsilon_q^2 + \rho_s \epsilon_q^2 \right),$$

where $K_{11}$ is equal to the compressional stiffness in Eq. (4.9), and $\epsilon_q$ is the crystal momentum along the $x$ direction.

Although $K_{11} = K_{22}$, we find that $K_{22} = \rho_s \neq K_{22}$. The reason for the discrepancy between $K_{22}$ and $K_{22}$ is not obvious. It may be that using $K_{22} = Q/Q_s$ for the transverse stiffness is valid only at very long wavelengths $q_s < 1/Y_{SL}$, where $Y_{SL} = L_s \sqrt{K_{22}/K_{11}}$ is Read's estimate of the minimum distance for SL dislocations to interact logarithmically. It is also possible that $K_{22}$, which is based on a calculation that assumes $\delta \theta$ is everywhere small, is not able to describe uniform shear, which can move solitons lines far from their equilibrium positions on the scale of the soliton line thickness $\xi$. The relationship between $K_{22}$ and $K_{22}$ requires further clarification, including a stronger argument for preferring $K_{22}$ over $K_{22}$ in estimating the KT temperature.

V. IN-PLANE MAGNETIZATION

The interlayer phase coherent 2LQH state exhibits an in-plane magnetization $M_\parallel$ in the presence of an in-plane magnetic field $B_\parallel$. The in-plane magnetization can be calculating by differentiating the minimized ground-state energy per volume with respect to the parallel magnetic field:

$$M_\parallel = -\frac{1}{2L_1 L_2} \frac{\partial \tilde{\mathcal{E}}}{\partial B_\parallel} = -\frac{2\pi}{\phi_0} \hat{z} \times \frac{\partial}{\partial Q} \left( \tilde{\mathcal{E}} / L_1 L_2 \right),$$

where $\tilde{\mathcal{E}} / L_1 L_2$ is given by Eq. (3.12). In order to carry out the differentiation in Eq. (5.1), we first differentiate Eq. (3.10) with respect to $Q$ and make use of Eq. (3.3) to obtain

$$\frac{\partial \tilde{\mathcal{E}}}{\partial Q} = -\left( \frac{2}{\pi} \right)^2 \tilde{\mathcal{E}} Q_s / Q_c (1 + Q^2 \ell^2 / 4).$$
We note here that the $Q^2 \ell^2 / 4$ term in Eq. (5.2) arises from differentiating $Q_c$ in Eq. (8.10) with respect to $Q$; the $Q$-dependence of $Q_c$ is due to the dependence of the tunneling matrix element on the parallel magnetic field, $t = t_0 \exp(-Q^2 \ell^2 / 4)$, which is a single-particle effect.

The tunneling part of the equilibrium energy per area will also give a contribution to the in-plane magnetization proportional to $\partial \mu / \partial Q$, again due to dependence of $t$ on $Q$. It is convenient to separate $M_||$ into two parts: $M_|| \equiv M_{SL} + M_t$, where $M_{SL}$ is calculated at fixed $t$ (independent of $Q$), and $M_t$ arises from the $Q$-dependence of $t$.

Using Eqs. (5.1) and (5.2) it is straightforward to show that the SL contribution to the parallel magnetization is

$$M_{SL} = -M_0 \hat{z} \times (Q - \tilde{Q}_c)/Q_c, \quad (5.3)$$

where

$$M_0 \equiv 2\pi \rho_s Q_c/\phi_0 \sim 0.5 A/m \quad (5.4)$$

sets the scale of the SL magnetization, and the numerical estimate of $M_0$ is given for the “typical” GaAs sample described in Sec. II. For such a sample,

$$M_0 d = 2\pi \rho_s Q_c d/\phi_0 \sim 10^{-8} A \quad (5.5)$$

sets the scale of the SL magnetic moment per unit area. Thus magnetometers with sensitivities in the range of $10^{-14}$ Am$^2$ to $10^{-12}$ Am$^2$ would require sample areas in the range of $1$ mm$^2$ and $1$ cm$^2$ to measure $M_{SL}$. The magnitude of the SL magnetization behaves like

$$\left| \frac{M_{SL}}{M_0} \right| \rightarrow \left\{ \begin{array}{ll} 1 - (\pi/2) / \ln(1/e), & Q/Q_c \rightarrow 1 \\ \pi / 2 Q_c^3, & Q/Q_c \rightarrow \infty \end{array} \right. \quad (5.6)$$

The SL magnetization may also be calculated directly from the pseudospin supercurrent density.

$$J_{\perp} \equiv J_1 - J_2 = (2\rho_s/h) \nabla \theta, \quad \text{and the definition of the magnetic moment.} \quad \text{The electrical current } I \text{ in layers 1 and 2 is}

$$I = I_1 = -I_2 = L_y (\rho_s/h)(\nabla \theta - Q). \quad (5.7)$$

The magnetization produced by the above current is therefore

$$M_{SL} = \frac{1}{L_x L_y} \hat{z} \times \int\int \frac{I}{c} dx dz = \frac{e \rho_s}{h} \hat{z} \times \int_0^{L_y} (\nabla \theta - Q) dx = -\frac{e \rho_s}{h} \hat{z} \times (Q - \tilde{Q}_c), \quad (5.8)$$

in agreement with Eq. (5.3).

The 2LQH interlayer phase coherent state may be regarded as a pseudospin-channel superconductor. The magnetization $M_||$ is due to pseudospin supercurrents, corresponding to electrical currents traveling in opposite directions in each layer, which partially screen $B_||$.

For $Q < Q_c$, $B_||$ is maximally excluded from the region between the planes; but when $Q \geq Q_c$, additional $B_||$ penetrates the region between the plates in the form of solitons that each carry a single flux quantum $\phi_0 = h/e$ (corresponding to a phase change $\Delta \theta = 2\pi$), leading to a precipitous decline in the magnetization. The direction of $M_||$ is opposite to $B_||$, in accord with Lenz’s law. An exactly analogous effect occurs for magnetic fields applied parallel to narrow insulating regions (Josephson junctions) between superconductors.

The contribution to the magnetization due to the $Q$-dependence of the tunneling matrix element $t$ is

$$M_t = -\frac{1}{2} \left( \frac{e}{\pi} \right)^2 \hat{z} \times \left( \left( \frac{2}{\pi} - 1 \right) \frac{Q}{Q_c} - \frac{1}{2} \left( \frac{4}{\pi} \right)^2 \frac{Q}{Q_c} \right). \quad (5.9)$$

In the commensurate phase, $M_t = -(M_0/2)(\ell/\xi)^2 \hat{z} \times Q/Q_c$, and the magnitude of the tunneling contribution to the parallel magnetization behaves like

$$\left| \frac{M_t}{M_0} \right| \rightarrow \frac{1}{2} \left( \frac{\ell}{\xi} \right)^2 \left[ 1 - 4/\ln(1/e), \quad Q/Q_c \rightarrow 1 \right] \frac{Q/Q_c}{\xi^2 (Q_c/Q)} \rightarrow 0, \quad (5.10)$$

The SL magnetization is plotted in Fig. 3. It is useful to compare the total SL magnetization $M_0$ to the scale of the Landau diamagnetism in a $\nu_T = 1$ 2LQH system:

$$\frac{M_0}{(n/d) \mu_B} = \frac{M_0 \phi_0 d}{\mu_B B_\perp} = \frac{16 d \rho_s}{\xi \hbar \omega_c} \sim 0.1, \quad (5.11)$$

where $\mu_B = \hbar / 2m^*$ is the effective Bohr magneton, and we have made use of the parameters for the “typical” GaAs sample described in Sec. II. This shows that the SL magnetization is expected to be roughly an order of magnitude smaller than the Landau diamagnetism. It is interesting to note that the weak signals associated with orbital diamagnetism in two-dimensional electron systems have been measured at high magnetic fields by high-field magnetometry, SQUID magnetometry, and micromechanical cantilever magnetometry. The torque on the 2LQH sample in the presence of both a perpendicular and parallel magnetic field has both a Landau-diamagnetic component $\tau_\perp = (L_x L_y d) M_\perp B_\perp$ and a SL component, $\tau_\parallel = (L_x L_y d) M_t B_\parallel$. The smallness of the parallel moment $M_t$ is offset by the large perpendicular magnetic field $B_\perp$ in the expression for the torque $\tau_\parallel$. The SL magnetization $M_{SL}$ might also be measured using high-field magnetometry techniques mentioned above.

It follows from Eq. (5.1) that when $M_||$ is plotted against $B_||$, the area under the resulting curve from $B_|| = 0$ to $B_|| = \infty$ is

$$\int_0^\infty M_0 d B_\parallel = \frac{1}{L_x L_y \epsilon} \left[ \tilde{E}(B_\parallel = \infty) - \tilde{E}(B_\parallel = 0) \right] \quad (5.12)$$
where \( \langle m_z \rangle \) is the ground-state expectation value of the \( x \) component of the pseudospin order parameter, which has a value of \( \sqrt{1 - m_z^2} \) in the HFA, so that \( \langle m_z \rangle = 1 \) (in the HFA) when the layers are balanced \( (m_z = 0) \). Thus the area under the \( M_c \) versus \( B_y \) curve may be regarded either as a measurement of \( t \) (if quantum fluctuations in the ground state are neglected), or as a measure of order-parameter suppression \( (m_z) \) due to quantum fluctuations \( \mathcal{Q} \) (if \( t \) can be measured separately).

Equations (5.3) and (5.13) show that \( M_{SL} = -M_0 \chi_{SL} \mathcal{Q}/Q_c \), where \( \chi_0 \equiv \mu_0 (2\pi/\phi_0)^2 \rho_e d \) sets the scale of the SL contribution to the magnetic susceptibility. The SL contribution to the parallel-field magnetic susceptibility is defined for fixed \( t \) (independent of \( Q \)): \[ \chi_{SL} = \chi_0 \left( 1 - \frac{\partial Q_c}{\partial Q} \right) \] (5.13)

\[ \chi_0 \approx 1 - \frac{1}{137} \]

and the numerical estimate of \( \chi_0 \) is given for the “typical” GaAs sample described in Sec. II. Here \( \alpha \approx 1/137 \) is the fine-structure constant. We have plotted the susceptibility in Fig. 4. Note that near the CI transition, the susceptibility diverges like \( 1/(Q - Q_c) \), with logarithmic corrections.

It might be possible to measure the SL magnetization or even the SL magnetic susceptibility by varying the gate voltages of the sample, in order to adjust \( Q_c \) and tune close to the CI transition. Measuring \( \chi_{SL} \) might be possible using AC modulation of the gate voltages in order to AC modulate the layer imbalance \( m_z \) and therefore the critical wave vector \( Q_c \). By such a method, the ratio \( Q/Q_c \) could be AC modulated just above and below unity, allowing \( \chi_{SL} \) to be determined at or near the CI transition. As an example, we compute here that part of \( \partial M_{SL}/\partial m_z \) which is proportional to \( \chi_{SL} \):

\[ \frac{\partial M_{SL}}{\partial m_z} \sim \frac{\partial Q_c}{\partial m_z \partial Q_c} \frac{\partial M_{SL}}{\partial Q} \sim -\frac{\partial Q_c}{\partial m_z \partial Q_c} \frac{\partial M_{SL}}{\partial Q} \] (5.15)

where we have used Eq (2.11) in the last line. Equation (5.15) shows that \( \partial M_{SL}/\partial m_z \) has a contribution proportional to \( \chi_{SL} \), and diverges like \( \rho_s/K_{11} \) near the CI transition. However, Eq. (5.15) vanishes for balanced layers \( (m_z = 0) \); thus, layer imbalance is required to measure \( \partial M_{SL}/\partial m_z \). It turns out that when the layers are not balanced \( (m_z \neq 0) \), the solitons have a nonzero electric dipole moment per length; this changes the inter-soliton repulsion from being exponentially weak to a power law. The net result is that the stiffness \( K_{11} \) near the CI transition is strengthened from being essentially linear in \( (Q - Q_c) \) to being proportional to \( \sqrt{Q - Q_c} \), so that the divergence in \( \chi_{SL} \) becomes an inverse square-root singularity. In real samples, we expect this singularity to be smoothed out by finite temperature and disorder.

In practice, \( m_z \) is varied by adjusting gate voltages. For a small change \( \delta V_G \) in a gate voltage away from balance \( (m_z = 0) \), the change \( \delta m_z \) in the layer imbalance is linearly proportional to \( \delta V_G \). For the “typical” GaAs sample described in Sec. II, a rough estimate indicates that

\[ \delta m_z \sim \frac{2\delta V_G}{e D_G n_T/(\epsilon_e \epsilon_0)} \sim 3 \delta V_G/\text{volt}, \] (5.16)

if we take \( D_G = 10^{-6} \) m to be the distance between the gate and the double layers. There is an additional complication in changing the layer imbalance by adjusting one of the gate voltages. Unless both the back and front gates are adjusted together in a coordinated way, the change in gate voltage \( \delta V_G \) will also change the total filling factor \( \nu_T \) by an amount \( \delta \nu_T \), where

\[ \delta \nu_T \sim \frac{\delta V_G}{e D_G n_T/(\epsilon_e \epsilon_0)} \sim 1.5 \delta V_G/\text{volt}, \] (5.17)

if \( D_G = 10^{-6} \) m.

VI. SUMMARY

In the presence of a sufficiently strong parallel magnetic field, a \( \nu_T = 1 \) 2LQH device undergoes a transition to a soliton-lattice (SL) state. We have investigated the ground-state properties of the SL state for all values of the parallel magnetic field, with an eye towards possible experimentally measurable effects.

We found that the SL contribution to the orbital magnetization rises in the commensurate phase \( (Q < Q_c) \) with \( Q \), and quickly drops to zero in the incommensurate phase \( (Q > Q_c) \). An estimate of the size of the SL magnetization shows that it could be detected by sensitive magnetometry techniques. The SL magnetic susceptibility shows a singularity at \( Q = Q_c \), and it was proposed that this signature of the CI transition might be detected by varying the gate voltages so as to tune close to the CI transition.
The longitudinal and transverse SL stiffnesses were computed and used to estimate the temperature of the KT transition, which might be indicated experimentally by an increase in $\rho_{xx}$ at the transition. A more sensitive signal of the KT transition would be obtained by measuring the transresistivity (Coulomb drag) as a function of temperature, although this would require separately contactable layers. However, the leakage currents produced by any sizeable interlayer tunneling might make it impractical to set up oppositely directed currents in each layer.

In this paper, we have largely neglected the existence of disorder. As pointed out by Fisher and discussed by Read, randomness in the tunneling $t$ due, for example, to small variations in the barrier thickness, pins the SL domain walls randomly and destroys the long-range order, no matter how weak the randomness. This puts limits on how closely $Q$ can approach $Q_c$—i.e., how small $\epsilon$ can be.

We have not discussed here the dynamics of the solitons—i.e., the motion of individual solitons or the collective motion of the SL— which also produce experimental signatures of the incommensurate SL state. We have focused instead on ground-state properties. One of us has recently found that when the layer densities are made unequal by adjusting the gate voltages of the device (keeping the total filling factor equal to one), the layer densities in the SL state become “rippled”, resulting in a dipole density wave. Preliminary calculations show that the sudden onset of such a rippled SL state may give large contributions to the differential capacitance of 2LQH systems, especially for the interlayer capacitance (“Eisenstein ratio”). Further work on the properties of the rippled state is in progress, but it may be that sensitive measurements of differential capacitance in unbalanced $(\nu_1 \neq \nu_2)$ 2LQH systems could detect the CI transition.

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**FIG. 1.** Energy of the commensurate (C), constant-$\theta$ incommensurate(I), and incommensurate SL (S) phases, versus the parallel magnetic-field wave vector $Q$. For $Q \geq Q_c$, the SL phase (dotted) has the lowest energy.

**FIG. 2.** At $Q = Q_c$, the system admits a single soliton, in which $\tilde{\theta}(r)$ twists by $2\pi$ over a distance $\xi$.

**FIG. 3.** Soliton-lattice for $Q$ slightly larger than $Q_c$.  
FIG. 4. Soliton wave vector $\bar{Q}_s$ versus the in-plane magnetic-field wave vector $Q$. As $Q \to Q_c$, $Q_s \xi \sim 2\pi / \ln[Q_c/(Q - Q_s)]$.

FIG. 5. Soliton-lattice stiffnesses $\rho_x \equiv K_{11}$ and $\rho_y \equiv K_{22}$, and their geometric mean, which is proportional to the Kosterlitz-Thouless temperature, $T_{KT}$.

FIG. 6. SL contribution to the magnetization, which is proportional to $(Q - \bar{Q}_s)$ and therefore drops precipitously for $Q > Q_c$.

FIG. 7. SL contribution to the magnetic susceptibility, which diverges as $1/(Q - Q_c) \ln[Q_c/(Q - Q_c)]$ for $Q \to Q_c$. 

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