The Enriched Grothendieck Construction

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Abstract

We investigate the properties of opfibrations of $V$-enriched categories when $V$ is any semi-cartesian monoidal category pullbacks preserve coproducts. This includes sets, simplicial sets, topological spaces, categories, marked simplicial sets, and locally Cartesian closed categories. In particular we show that if $B$ is an ordinary category then there is an equivalence of 2-categories between $V$-enriched opfibrations over the free $V$-category on $B$ and pseudofunctors from $B$ to the 2-category of $V$-categories. This generalizes the classical ($Set$-enriched) Grothendieck correspondence.

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The Grothendieck construction and its inverse relate stacks on a Grothendieck site to fibrations, or fibered categories, over that site. More generally, for any category $B$, there is an equivalence between pseudofunctors $B^{op} \to \text{Cat}$ and fibrations over $B$ [7, Theorem 1.3.6]; by duality, there is also an equivalence between pseudofunctors $B \to \text{Cat}$ and opfibrations over $B$. This paper generalizes the latter equivalence to categories enriched over a suitable monoidal category $V$: we show an equivalence between pseudofunctors $B \to \text{Cat}_V$ and $V$-enriched opfibrations over the free $V$-category on $B$. Despite the title, the focus of this paper is this equivalence or correspondence, rather than the Grothendieck construction itself which is fairly easy to describe.

The motivation for this work goes back to the PhD thesis of the first author (some of which is described in [2]). One of the goals of that thesis was to describe certain coalgebraic structures, e.g. bialgebras and comodules, in quasicategories (which are one of several models for $(\infty,1)$-categories). It would be an unenlightening detour to describe the nature of these problems here, but the first author found that the $(\infty,1)$-categorical Grothendieck construction described in [10] and expanded upon in [11] were not rigid enough for these purposes. Recalling from [3] that we may think of simplicially enriched categories as a model for $(\infty,1)$-categories, we may then think of the enriched Grothendieck construction given in this paper, when $V = sSet$, to be a rigidified and truncated version of Lurie’s quasicategorical Grothendieck construction.
Unfortunately, for the time being, the above must be taken with a large grain of salt. There are at least two major shortcomings in this work. First, our Grothendieck construction only allows for pseudofunctors from a discrete base, i.e. the base category of our opfibrations (equivalently the source of our pseudofunctors) must be an ordinary category, and we only consider pseudofunctors or 2-functors (as opposed to $\infty$-functors, say) from this base into the category $\mathcal{C}at_{\mathcal{V}}$ of $\mathcal{V}$-enriched categories. In a certain sense this is inevitable when working in this level of generality: while we can describe enriched opfibrations over an enriched base, it does not even make sense to talk about $\mathcal{V}$-functors $B \to \mathcal{C}at_{\mathcal{V}}$ from an arbitrary $\mathcal{V}$-category $B$ to the category $\mathcal{C}at_{\mathcal{V}}$ of $\mathcal{V}$-enriched categories, because $\mathcal{C}at_{\mathcal{V}}$ is not necessarily $\mathcal{V}$-enriched.

However, when the $\mathcal{V}$-enrichment has the effect of giving higher categorical structure to $\mathcal{V}$-enriched categories – such as when $\mathcal{V}$ is $\mathcal{C}at$ or $s\mathcal{S}et$ – there is a natural sense in which $\mathcal{C}at_{\mathcal{V}}$ is $\mathcal{V}$-enriched, so that one can talk about $\mathcal{V}$-enriched functors $B \to \mathcal{C}at_{\mathcal{V}}$. Work by Hermida [6], Bakovic [1] and Buckley [4] gives a fully enriched Grothendieck construction in the case when $\mathcal{V} = \mathcal{C}at$. There is also work by Harpaz and Prasma [5] that gives a Grothendieck construction in the case that all of the categories involved are (unenriched) Quillen model categories, which can also serve as presentations for $(\infty, 1)$-categories. In future work we hope to address this shortcoming in the case that $\mathcal{V} = \mathcal{C}at$ or $s\mathcal{S}et$.

The second shortcoming to point out is that our work does not address the existence of Quillen model structures on $\mathcal{V}$ or $\mathcal{C}at_{\mathcal{V}}$ which would be necessary for giving a complete reformulation of the quasicategorical Grothendieck construction of [10]. Thus we also hope to show, in future work, that the adjoint functors described in this paper can be enhanced to Quillen functors on model categories of opfibrations and pseudofunctors. The success of that work, when complete, will give a more rigid, and elementary, alternative to existing $(\infty, 1)$-categorical Grothendieck constructions.

We make use of standard notions and techniques from 2-category theory and enriched category theory, and as far as possible, try to relate our constructions back to the classical Grothendieck construction (i.e. the $s\mathcal{S}et$-enriched case). In fact, one of the strengths of the construction given here is that it is effectively a lengthy exercise in 2-category theory. A young mathematician well versed in the definitions of enriched and 2-category theory, who also perhaps has a tolerance for tedium, could understand all of our proofs.

With an eye toward generalizing beyond Cartesian monoidal categories like $\mathcal{C}at$ and $s\mathcal{S}et$, we try in this paper to pinpoint the properties of $\mathcal{V} = \ldots$
Set that make the classical Grothendieck construction, and the equivalence between opfibrations and pseudofunctors, possible. We hope to do away with some of these properties in future work. For instance, the assumptions we impose on \( \mathbf{V} \) allow our result to be applied to \( \mathbf{V} = \mathbf{Cat}, \mathbf{Top} \) and \( \mathbf{sSet} \), but interestingly, not \( \mathbf{Vect}_k \). The second author has work in progress that gives a generalization of the notion of fibration, which seems likely to allow for a generalized \( \mathbf{Vect}_k \)-enriched Grothendieck construction.

We note that there is already a preprint of Tamaki [15] which discusses an enriched Grothendieck construction with the aim of applying it to situations such as \( \mathbf{V} = \mathbf{Vect}_k \). However, the definition of an (op)fibration given there is not equivalent to ours, and is also not equivalent to the classical definition when \( \mathbf{V} = \mathbf{Set} \) (i.e. it cannot be considered a generalization of the classical Grothendieck construction). While we have taken some inspiration from [15], our work is significantly different. We will elaborate more on these differences as they occur.

This paper is structured as follows: In §2, we recall some notions from enriched category theory and 2-category theory that will be used in this paper. In §3, we define enriched opfibrations and show that such opfibrations \( p: \mathcal{E} \to \mathcal{B} \) give rise to pseudofunctors \( \mathcal{B}_0 \to \mathbf{Cat}_\mathbf{V} \), where \( \mathcal{B}_0 \) is the underlying category of the \( \mathbf{V} \)-category \( \mathcal{B} \). Along the way, we give an enriched version of the classical result that every opfibration is equivalent to a split one. In §4, we show that pseudofunctors \( \mathcal{B} \to \mathbf{Cat}_\mathbf{V} \) give rise to opfibrations over \( \mathcal{B}_\mathbf{V} \), the free \( \mathbf{V} \)-category on the ordinary category \( \mathcal{B} \). Finally, in §5, we show that the constructions in the previous two sections are mutual inverses, yielding the desired equivalence.

## 2 Preliminaries

We begin by recalling a few notions from enriched category theory and 2-category theory that will be used in this paper.

Throughout, we work over a symmetric monoidal category \( (\mathbf{V}, \otimes, \mathbf{1}) \). The 2-category of \( \mathbf{V} \)-categories, \( \mathbf{V} \)-functors and \( \mathbf{V} \)-natural transformations (defined in A.1) will be denoted \( \mathbf{Cat}_\mathbf{V} \). This is a symmetric monoidal category with unit \( \mathbf{1} \), the \( \mathbf{V} \)-category with

\[
\text{Ob}(\mathbf{1}) := \{ *, \}
\]

\[
\mathbf{1}(\ast, \ast) := \mathbf{1}.
\]
2.1 Properties of $V$

We describe a few additional properties of $V$ that we will later require.

1. If $V$ has coproducts, we say that $\otimes$ preserves coproducts if

   $$A \otimes \prod_{i \in I} B_i \cong \prod_{i \in I} A \otimes B_i.$$ 

2. If $V$ has pullbacks and coproducts, we say that pullbacks preserve coproducts if given a set $I$, an $I$-indexed family $(X_i)_{i \in I}$ where $I_i \in V$, maps $f_i : X_i \to Z$ and a map $g : Y \to Z$, we have

   $$Y \times_Z \left( \prod_i X_i \right) \cong \prod_i (Y \times_Z X_i),$$

   where these fibered products are given by the following pullback diagrams:

   $Y \times_Z \left( \prod_i X_i \right) \xrightarrow{\gamma} \prod_i X_i \xrightarrow{\Pi_i f_i} Z$  
   $Y \xrightarrow{g} Z$

3. If the monoidal product $\otimes$ is the cartesian product $\times$, we say that $V$ is cartesian. This implies that the monoidal unit $1$ is terminal. If only this last condition holds, we say that $V$ is semicartesian.

4. Finally, if $X$ a set and $V$ has coproducts, we may ask for an isomorphism of sets

   $$V \left( 1, \bigsqcup_{x \in X} 1 \right) \cong X.$$ 

Remark 2.1. The above assumptions hold if $V$ is any of the categories of sets, simplicial sets, strict $n$-categories or CW complexes, but not if $V$ is the category of vector spaces. In cases where all these conditions hold, we get that $V$ has finite limits and these preserve coproducts.

In §3.5, we will only require that $V$ has pullbacks, while in §4, we will require all the above but with semicartesian instead of cartesian $V$. 

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2.2 Underlying categories and free $V$-categories

Throughout this paper, $B$ will denote a $V$-category with hom-objects $B(b, c) \in V$, while $B$ will denote an unenriched category with hom-sets $B(b, c) \in \text{Set}$. The underlying category of $B$ is the functor category $B_0 := \text{Cat}_V(1, B)$ whose objects are $V$-functors $b : 1 \to B$ and morphisms are $V$-natural transformations $f : b \Rightarrow c$. Equivalently, $B_0$ is the category with the same objects as $B$ and morphisms $f : 1 \to B(b, c)$. The assignment $B \mapsto B_0$ extends to a functor $(-)_0 : \text{Cat}_V \to \text{Cat}$.

When $V$ has coproducts and $\otimes$ preserves them, the free $V$-category on an unenriched category $B$ is the $V$-category $B_V$ with the same objects as $B$ and hom-objects $B_V(b, c) := \coprod_{f \in B(b, c)} 1 \in V$. This extends to a functor $(-)_V : \text{Cat} \to \text{Cat}_V$ which is left adjoint to $(-)_0$.

\[
\begin{array}{c}
\text{Cat} \\
\downarrow \sigma_B \\
\text{Cat}_V \\
\end{array}
\]  
\( (1) \)

The components of the unit of this adjunction will be denoted $\iota_B : B \to (B_V)_0$, while the components of the counit will be denoted $\sigma_B : (B_0)_V \to B$.

When $V$ satisfies Property 4 in the previous subsection, the unit $\iota_B$ is an isomorphism of categories $B \cong (B_V)_0$.

We elaborate on this further in Remark 4.2.

Some examples in the one-object case might be instructive. A one-object category may be identified with a monoid $M$, while a one-object $V$-category may be identified with a monoid $M$ in $V$.

**Example 2.2.** When $V = \text{Top}$, the free topological monoid $M_V$ on a monoid $M$ is the same monoid given the discrete topology, while the underlying category $M_0$ of a topological monoid $M$ is the same monoid forgetting its topology.

In this case, $M = (M_V)_0$, so $\iota_M$ is the identity. The map $\sigma_M$ is also the identity on the underlying sets, but its domain has the discrete topology.
Example 2.3. When $V = \text{Vect}_k$, the $k$-algebra $M_V$ is the monoid-algebra $k[M]$, while the monoid $M_0$ is the $k$-algebra $M$ treated simply as a monoid (forgetting its $k$-linear structure).

Unlike for $\text{Top}$, in this case $M \neq k[M]$, but $\iota_M: M \hookrightarrow k[M]$ is the inclusion of $M$ as a basis. The map $\sigma_M: k[M] \rightarrow M$ sends formal linear combinations of objects in $M$ to their actual sum in $M$.

2.3 Composition and isomorphisms

Let $\mathcal{B}$ be a $V$-category. Each $f \in \mathcal{B}_0(b,c)$ is a $V$-morphism $f: 1 \rightarrow \mathcal{B}(b,c)$, and thus induces pre- and post-composition $V$-morphisms:

\[ - \circ f: \mathcal{B}(c,d) \cong \mathcal{B}(c,d) \otimes 1 \xrightarrow{1 \otimes f} \mathcal{B}(c,d) \otimes \mathcal{B}(b,c) \xrightarrow{\circ} \mathcal{B}(b,d) \]

\[ f \circ -: \mathcal{B}(a,b) \cong 1 \otimes \mathcal{B}(a,b) \xrightarrow{f \otimes 1} \mathcal{B}(b,c) \otimes \mathcal{B}(a,b) \xrightarrow{\circ} \mathcal{B}(a,c) \]

We say that $f$ is an isomorphism if the above composites are $V$-isomorphisms for all $a, d \in V$, and that $b$ and $c$ are isomorphic. This is equivalent to $\mathcal{B}(-, b)$ and $\mathcal{B}(-, c)$ being isomorphic functors $\mathcal{B}_0 \rightarrow V$, with $\mathcal{B}(-, f)$ the natural isomorphism between them.

If $f$ is an isomorphism, its inverse may be found by setting $a = c$ or $d = b$ and composing the identity $1_c$ or $1_b$ with $(f \circ -)^{-1}$ or $(- \circ f)^{-1}$, resp.

Conversely, if we have $f^{-1}$, we may define $(f \circ -)^{-1} := f^{-1} \circ -$ and so on, to verify that $f$ is an isomorphism in the above sense.

2.4 Comma categories

Given $V$-functors $A \xrightarrow{F} \mathcal{C} \xleftarrow{G} \mathcal{B}$, the (strict) comma category $F \downarrow G$ is the $V$-category fitting into the (non-commuting) square

\[
\begin{array}{ccc}
F \downarrow G & \xrightarrow{\pi_A} & A \\
\pi_B & \downarrow & \leftarrow \\
\mathcal{B} & \xrightarrow{G} & \mathcal{C}
\end{array}
\]
that is universal with this property (see Definition A.6). We are particularly interested in comma categories of the form \( p \downarrow B := p \downarrow 1_B \).

\[
\begin{array}{c}
p \downarrow B \\
\pi_E \downarrow B \\
\end{array}
\quad \begin{array}{c}
\pi_B \\
\varphi \\
\end{array}
\quad \begin{array}{c}
\varepsilon \\
\downarrow \\
\phi \\
\end{array}
\quad \begin{array}{c}
E \\
\downarrow \\
B \\
\end{array}
\]

The objects of \( p \downarrow B \) are tuples \((e, f, b)\) where \( e \in E, b \in B \) and \( f : 1 \to B(pe, b) \) (so \( f \) is an element of \( B_0(pe, b) \)), and the hom-objects are pullbacks:

\[
\begin{array}{c}
p \downarrow B((e, f, b), (e', f', b')) \\
\downarrow \\
B(b, b') \\
\end{array}
\quad \begin{array}{c}
\varepsilon(e, e') \\
\downarrow \\
B(pe, pe') \\
\downarrow \\
B(PE, B) \\
\downarrow \\
B(PE, B) \\
\end{array}
\quad (2)
\]

Now suppose we have functors \( p : E \to B \) and \( q : F \to B \), and let \( k : E \to F \) be a functor such that \( qk = p \).

\[
\begin{array}{c}
E \\
\downarrow \\
B \\
\end{array}
\quad \begin{array}{c}
k \\
\downarrow \\
q \\
\end{array}
\quad \begin{array}{c}
F \\
\downarrow \\
B \\
\end{array}
\]

Together with the diagram defining \( p \downarrow B \), we obtain a natural transformation

\[
\begin{array}{c}
p \downarrow B \\
\downarrow \\
B \\
\end{array}
\quad \begin{array}{c}
\varepsilon \\
\varphi \\
\downarrow \\
\phi \\
\end{array}
\quad \begin{array}{c}
E \\
\downarrow \\
F \\
\downarrow \\
B \\
\end{array}
\quad \begin{array}{c}
k \\
\downarrow \\
q \\
\end{array}
\]

which, by the universal property of \( q \downarrow B \), induces a unique functor \( K : p \downarrow B \to q \downarrow B \) such that the above natural transformation is equal to the one below,
all of whose faces commute:

\[
\begin{array}{ccc}
p \downarrow B & \rightarrow & E \\
\downarrow^K & & \downarrow^k \\
q \downarrow B & \rightarrow & F \\
\end{array}
\]

On objects, \( K \) sends \((e, f, b)\) in \( p \downarrow B \) to \((ke, f, b)\) in \( q \downarrow B \), and it is easy to see that the following diagram also commutes:

\[
\begin{array}{ccc}
E & \xrightarrow{k} & F \\
\downarrow^i_p & & \downarrow^i_q \\
p \downarrow B & \xrightarrow{K} & q \downarrow B \\
\end{array}
\]

where \( i_p \) and \( i_q \) are the respective canonical inclusions. On morphisms, \( K \) is induced by the maps \( k_{e, e'} : E(e, e') \rightarrow F(ke, ke') \) and the identity on \( B(b, b') \).

**Remark 2.4.** Uniqueness of \( K \) arises because we have *strict* comma objects. As a consequence, the process outlined above that sends \( k \) to \( K \) is functorial: if \( k \) is the identity, then so is \( K \), and if \( k \circ k' = \ell \), then \( K \circ K' = L \) as well.

### 2.5 Pseudofunctors, natural transformations, modifications

Finally, we recall some notions from 2-category theory. A 2-category \( \mathcal{K} \) is a category enriched in \( \text{Cat} \), so we have objects (or 0-cells) \( c, d \in \mathcal{K} \), morphisms (1-cells) \( f : c \rightarrow d \) and morphisms between morphisms (2-cells) \( \alpha : f \Rightarrow g \), and the various composites between them are strictly associative and unital.

These are sometimes called *strict* 2-categories, but since all our 2-categories will be strict, we omit the adjective.

However, the functors between 2-categories that we consider will be weaker than \( \text{Cat} \)-enriched functors. A *pseudofunctor* \( F : \mathcal{K} \rightarrow \mathcal{L} \) between 2-categories is a functor that is only associative and unital up to coherent isomorphism:

\[
\begin{align*}
F 1_c & \cong 1_{Fc}, \\
F(gf) & \cong Fg \circ Ff.
\end{align*}
\]
More precisely, $F$ consists of a map on objects $c \mapsto Fc$, a functor between hom-categories

$$K(c,d) \xrightarrow{F_{c,d}} \mathcal{L}(Fc, Fd)$$

which we abbreviate as $F$, and invertible 2-cells $\xi(c)$ and $\theta(f,g)$ for every $c \xrightarrow{f} d \xrightarrow{g} e$ in $K$

$$\begin{array}{ll}
F c & \xrightarrow{\xi(c)} F c \\
\xRightarrow{F 1_c} & \\
F c & \xleftarrow{\theta(f,g)} F e \\
\xRightarrow{F (gf)} & \\
F d & \xrightarrow{\theta(f,g)} F e
\end{array}$$

which we abbreviate as $\xi$ and $\theta$, satisfying the following relations:

$$\begin{array}{ll}
F c & \xrightarrow{\xi} F d \\
\xRightarrow{\theta} & \\
F f & \xleftarrow{\theta} F d
\end{array} = \begin{array}{ll}
F c & \xrightarrow{\theta} F d \\
\xRightarrow{\theta} & \\
F f & \xleftarrow{\theta} F d
\end{array}$$

$$\begin{array}{ll}
F c & \xrightarrow{F f} F d \\
\xRightarrow{\xi} & \\
F d & \xleftarrow{\theta} F d
\end{array} = \begin{array}{ll}
F c & \xrightarrow{F f} F d \\
\xRightarrow{\xi} & \\
F d & \xleftarrow{\theta} F d
\end{array}$$

$$\begin{array}{ll}
F c & \xrightarrow{F f} F d \\
\xRightarrow{\theta} & \\
F h & \xleftarrow{\theta} F e
\end{array} = \begin{array}{ll}
F c & \xrightarrow{F f} F d \\
\xRightarrow{\theta} & \\
F h & \xleftarrow{\theta} F e
\end{array}$$

A pseudonatural transformation (or simply a transformation) $\alpha: F \Rightarrow G$ between pseudofunctors consists of 1-cells $\alpha_c: Fc \to Gc$ for each $c \in K$ and invertible 2-cells

$$\begin{array}{ll}
F c & \xrightarrow{\alpha_c} F d \\
\xRightarrow{\alpha_f} & \\
G c & \xrightarrow{\alpha_d} G d
\end{array}$$

for each $f: c \to d$ in $K$, satisfying further coherence rules given in [9, §1.2].
Finally, a modification $\Gamma: \alpha \Rightarrow \beta$ between transformations consists of 2-cells $\Gamma_c: \alpha_c \Rightarrow \beta_c$ for each $c \in \mathcal{K}$ such that the following equality holds:

$$F_c F d \alpha_c \Gamma_d \beta_d = F_c F d \alpha_c \Gamma_d \beta_c \beta_f \Gamma_f \beta_d$$

Pseudofunctors, transformations and modifications assemble to form a 2-category $\text{Fun}^{ps}(\mathcal{K}, \mathcal{L})$. We will be particularly interested in the 2-category of pseudofunctors $\text{Fun}^{ps}(B, \mathcal{Cat}_V)$, where $B$ is an ordinary category treated as a 2-category with only identity 2-cells.

3 Opfibrations and the Inverse Grothendieck Construction

In this section, we develop the theory of opfibrations in the enriched setting. We define opfibrations over a base $\mathcal{B}$, and the 2-category $\text{OpFib}(\mathcal{B})$ that they form. The inverse Grothendieck construction is then a 2-functor from $\text{OpFib}(\mathcal{B})$ to the category of pseudofunctors $\text{Fun}^{ps}(\mathcal{B}_0, \mathcal{Cat}_V)$, where $\mathcal{B}_0$ is treated as a 2-category whose hom-categories are all discrete (i.e. sets).

Throughout, we will assume that $V$ has pullbacks. This implies that $\mathcal{Cat}_V$ has pullbacks and comma objects.

3.1 Opfibrations

Given a $V$-functor $p: \mathcal{E} \to \mathcal{B}$, we can form the comma category $p\downarrow \mathcal{B}$ as in the previous section. By the universal property of $p\downarrow \mathcal{B}$, the functors $\mathcal{B} \overset{p}{\leftarrow} \mathcal{E} \overset{i}{\rightarrow} \mathcal{E}$ induce a canonical functor $i: \mathcal{E} \rightarrow p\downarrow \mathcal{B}$. 
On objects, we have $ie = (e, 1_{pe}, pe)$, while on morphisms, $i$ is given by the universal property of the pullback:

$$
\begin{array}{c}
\mathcal{E}(e, e') \\
p_{e,e'} \\
\mathcal{B}(pe, pe') \\
\mathbb{B}
\end{array} \xrightarrow{\ell_{e,e'}} \begin{array}{c}
p \downarrow \pi_B \\
\mathcal{E}(e, e') \\
\mathcal{B}(pe, pe') \\
\mathbb{B}
\end{array} $$

It is easy to see that $i$ is full and faithful, and thus $\mathcal{E}$ may be treated as a full subcategory of $p \downarrow \mathcal{B}$ on objects of the form $(e, 1_{pe}, pe)$.

**Definition 3.1.** An opfibration is a $\mathbf{V}$-functor $p: \mathcal{E} \to \mathcal{B}$ along with a left adjoint $\ell$ in $\mathbf{Cat}_{\mathbf{V}/\mathcal{B}}$ to the canonical inclusion $i: \mathcal{E} \hookrightarrow p \downarrow \mathcal{B}$

$$
\begin{array}{c}
p \mathcal{B} \\
\pi_B \\
\mathbb{B}
\end{array} \xrightarrow{i} \begin{array}{c}
\mathcal{E} \\
p
\mathbb{B}
\end{array} $$

We call $\mathcal{E}$ the **total category**, $\mathcal{B}$ the **base category** and $\ell$ the **cleavage** of the opfibration.

**Remark 3.2.** Being a left adjoint in $\mathbf{Cat}_{\mathbf{V}/\mathcal{B}}$ means that $p\ell = \pi_B, \pi_B \eta = 1_{\pi_B}$ and $pe = 1_p$, where $\eta$ and $\varepsilon$ are the unit and counit of the adjunction.

**Remark 3.3.** Our definition of an opfibration follows Street’s definition/characterization of an opfibration in a 2-category found in [14] and [16, Theorem 2.7], applied to the 2-category $\mathbf{Cat}_{\mathbf{V}}$. In [15, Definition 4.15], Tamaki gives a non-equivalent definition of an enriched opfibration. We revisit this difference in Remark 3.18.

**Remark 3.4.** Most other texts use ‘opfibration’ to mean just a functor $p$ such that there exists the requisite adjoint, and use ‘cloven opfibration’ to refer to $p$ along with a cleavage. Since we will be dealing solely with cloven opfibrations, we will drop the adjective ‘cloven’. All our opfibrations are assumed equipped with a cleavage, and we write ‘an opfibration $p: \mathcal{E} \to \mathcal{B}$’ to mean the pair $p$ and $\ell$. 

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The next Proposition gives a more concrete description of an opfibration.

**Proposition 3.5.** A \( \mathbf{V} \)-functor \( p \colon \mathcal{E} \to B \) is an opfibration if and only if for all \( e \in \mathcal{E} \), \( b \in B \) and \( f \colon 1 \to B(pe, b) \), there exists \( f_ie \in \mathcal{E} \) over \( b \) and \( q(f, e) \colon 1 \to \mathcal{E}(e, f_ie) \) over \( f \) such that the following square is a pullback

\[
\begin{array}{c}
\mathcal{E}(f_ie, d) \xrightarrow{-\circ q(f,e)} \mathcal{E}(e, d) \\
p \downarrow \quad \quad \quad \quad \downarrow p \\
B(b, pd) \xrightarrow{-\circ f} B(pe, pd)
\end{array}
\]

for all \( d \in \mathcal{E} \).

**Proof.** Note that the data of \( e \in \mathcal{E}, b \in B \) and \( f \colon 1 \to B(pe, b) \) is precisely the data of an object \((e, f, b)\) in \( p \downarrow B \).

Suppose \( p \colon \mathcal{E} \to B \) is an opfibration. For each tuple \((e, f, b)\) as above, let \( f_ie := \ell(e, f, b) \).

To produce a lift \( q(f, e) \) of \( f \), note that the unit \( \eta \colon 1_{p \downarrow B} \Rightarrow i\ell \) has components

\[
\eta_{(e, f, b)} \colon 1 \to p \downarrow B((e, f, b), (f_ie, 1_b, b)),
\]

where the codomain \( p \downarrow B((e, f, b), (f_ie, 1_b, b)) \) is given by the pullback:

\[
\begin{array}{c}
p \downarrow B(f, 1_b) \xrightarrow{j} \mathcal{E}(e, f_ie) \\
\downarrow \quad \quad \quad \quad \downarrow p \\
B(b, b) \xrightarrow{-\circ f} B(pe, b)
\end{array}
\]

So \( \eta_{(e, f, b)} \) is determined by a pair \( \eta^E_f \colon 1 \to \mathcal{E}(e, f_ie) \) and \( \eta^B_f \colon 1 \to B(b, b) \) such that the following diagram commutes:

\[
\begin{array}{c}
1 \xrightarrow{\eta^E_f} \mathcal{E}(e, f_ie) \\
\eta^B_f \downarrow \quad \quad \quad \quad \downarrow p \\
B(b, b) \xrightarrow{-\circ f} B(pe, b)
\end{array}
\]

Since the adjunction lies over \( B \), we in fact have \( \eta^B_f = 1_b \colon 1 \to B(b, b) \), which turns the above square into the following triangle:

\[
\begin{array}{c}
1 \xrightarrow{\eta^E_f} \mathcal{E}(e, f_ie) \\
\downarrow f \quad \quad \quad \quad \downarrow p \\
B(pe, b)
\end{array}
\]
Thus $\eta_f^e$ is a lift of $f$, allowing us to set

$$q(f, e) := \eta_f^e.$$  

Using the above notation for the unit $\eta$ and letting the counit be $\varepsilon$, the defining isomorphism of the adjunction $\ell \dashv i$ may be written:

$$E(f e, e') \xrightarrow{\cong} p \downarrow B((e, f, b), (e', 1_{pe'}, pe'))$$  \hspace{1cm} (9)

This isomorphism makes the square in (8) isomorphic to the pullback square:

$$
\begin{array}{ccc}
p \downarrow B(f, 1_{pe'}) & \to & E(e, e') \\
\downarrow & & \downarrow_{p_{e,e'}} \\
B(b, b') & \to & B(pe, pe')
\end{array}
$$

so $q(f, e)$ satisfies the required universal property.

Conversely, suppose we have $f e$ and $q(f, e)$ with the required properties. Define a left adjoint $\ell : p \downarrow B \to E$ by setting

$$\ell(e, f, b) := f e$$
on objects. Since $q(f', e')$ is a lift of $f'$, we obtain a commuting square:

$$
\begin{array}{ccc}
p \downarrow B((e, f, b), (e', f', b')) & \to & E(e, e') \xrightarrow{-\circ q(f', e')} E(e, f' e') \\
\downarrow & & \downarrow \\
B(pe, pe') & \to & B(pe, pe')
\end{array}
$$

Take $\ell_{f,f'} : p \downarrow B(f, f') \to E(f f, f' e')$ to be the map induced from $E(f f, f' e')$ being a pullback. One can check that $\ell$ is a functor.
Finally, \( \ell \) is left adjoint to \( i \) because both \( E(f_1e, e') \) and \( p \downarrow B((e, f, b), ie') \) are pullbacks of the cospan

\[
\begin{array}{c}
E(e, d) \\
\downarrow p \\
B(b, pd) \xrightarrow{-of} B(pe, pd)
\end{array}
\]

hence are isomorphic.

**Remark 3.6.** When \( V = \text{Set} \), the fact that (8) is a pullback may be rephrased thus: for every \( \varphi: e \to d \) and \( g: b \to pd \) such that \( p\varphi = gf \), there exists a unique \( \tilde{g}: f!e \to d \) such that \( p\tilde{g} = g \):

\[
\begin{array}{cc}
e & f_1e \\
\downarrow e & \downarrow \varphi \\
pe & f \\
\downarrow & \downarrow \\
b & d \\
\downarrow & \downarrow \\
gf & pd \\
\end{array}
\]

(10)

Here, the dotted arrows represent \( p \), and indicate which objects and arrows of \( E \) lie over which objects and arrows of \( B \). One says that \( q(f, e) \) is a \( p \)-opcartesian lift of \( f \).

**Remark 3.7.** The previous remark implies that if \( \gamma, \gamma' \) are maps lying over \( g \) such that \( \gamma q(f, e) = \gamma' q(f, e) \), then \( \gamma = \gamma' \). This follows by setting \( \varphi = \gamma q(f, e) = \gamma' q(f, e) \) and invoking the uniqueness of \( \tilde{\gamma} \) (which is thus equal to \( \gamma \) and \( \gamma' \)).

**Remark 3.8.** When \( d = (gf)_1e \) and \( \varphi = q(gf, e) \) for some \( g: b \to b' \), given some other \( h: b' \to b'' \), we may iterate the procedure in Remark 3.6 to
Since both $\widetilde{hg}$ and $\widetilde{h}g$ lie above $hg$ and are equal under composition with $q(f, e)$, we may conclude by the previous Remark that

$$\widetilde{h}g = \widetilde{hg}.$$  \hspace{1cm} (11)

The analogous result holds in the enriched setting.

Finally, we make an observation about the counit $\varepsilon : \ell i \Rightarrow 1_E$ of the adjunction $\ell \dashv i$, whose components are

$$\varepsilon_e : 1 \rightarrow \mathcal{E}((1_{pe})e, e).$$

**Lemma 3.9.** For each $e \in \mathcal{E}$, $\varepsilon_e$ is an isomorphism, with inverse $q(1_{pe}, e)$. Thus

$$e \cong (1_{pe})e.$$  \hspace{1cm} (12)

**Proof.** Recall that the presence of a left adjoint $\ell$ to a full and faithful $i$ means that $\mathcal{E}$ may be treated as a *reflective* subcategory of $p/B$ (with reflector $\ell$), but this is equivalent to the counit $\varepsilon$ being a natural isomorphism.

Writing out the triangle equality $i\varepsilon \cdot \eta i = 1_i$ in components, we get

$$\varepsilon_e \circ q(1_{pe}, e) = 1_e,$$  \hspace{1cm} (12)

which says that $q(1_{pe}, e)$ is the right (and hence left) inverse of $\varepsilon_e$. \hfill \square
3.2 Opfibered functors

Let \( p : E \to B \) and \( q : F \to B \) be opfibrations over \( B \), and let \( k : E \to F \) be a functor such that \( qk = p \). Recall from §2.4 that \( k \) induces \( K : p \downarrow B \to q \downarrow B \) such that the following square commutes:

\[
\begin{array}{ccc}
E & \xrightarrow{k} & F \\
p \downarrow & & \downarrow q \\
p \downarrow & & \downarrow q \\
E \downarrow k & \xrightarrow{\mu} & F \\
\end{array}
\]

**Definition 3.10.** Let \( p, q \) be opfibrations as above, and let \((\ell_p \dashv i_p, \eta_p, \varepsilon_p)\) and \((\ell_q \dashv i_q, \eta_q, \varepsilon_q)\) be their respective adjunctions. An **opfibered functor** from \( p \) to \( q \) is a functor \( k : E \to F \) such that \( qk = p \) and the following composite 2-cell is a natural isomorphism:

\[
\begin{array}{c}
p \downarrow B \\
\xrightarrow{k} q \downarrow B \\
\end{array}
\]

\[
\begin{array}{c}
p \downarrow B \\
\xrightarrow{\ell_p} \quad \xrightarrow{i_p} \quad \xrightarrow{\eta_p} \quad \xrightarrow{\varepsilon_p} \quad \xrightarrow{k} q \downarrow B \\
\end{array}
\]

**Remark 3.11.** The 2-cell on the left of the above equation is also known as the **mate** of the identity 2-cell (commuting square) on the right.

**Remark 3.12.** Note that all this is happening ‘above \( B \)’, in the sense that all functors lie above \( 1_B \) and all natural transformations lie above the identity natural transformation on \( 1_B \).

**Remark 3.13.** When \( V = \text{Set} \), an opfibered functor is precisely one that sends \( p \)-opcartesian maps to \( q \)-opcartesian maps.

**Proposition 3.14.** For a fixed \( V \)-category \( B \) there is a 2-category \( \text{OpFib}(B) \) whose objects are opfibrations over \( B \), morphisms are opfibered functors, and 2-morphisms are \( V \)-enriched natural transformations.

**Proof.** This is a standard construction and we leave it to the reader to check the details.

3.3 Fibers and transport

In this section, we define and study the fibers of an opfibration, and show that arrows in the base \( B_0 \) induce transport functors between fibers. We begin with a more general definition of fibers of any functor.
Definition 3.15. Let \( p: \mathcal{E} \to \mathcal{B} \) be a \( \mathbf{V} \)-functor. For each \( b \in \mathcal{B} \), treated as a functor \( b: \mathbb{1} \to \mathcal{B} \), the fiber of \( p \) over \( b \) is the category \( \mathcal{E}_b \) given by the pullback:

\[
\begin{array}{ccc}
\mathcal{E}_b & \longrightarrow & \mathcal{E} \\
\downarrow & & \downarrow p \\
\mathbb{1} & \longrightarrow & \mathcal{B}
\end{array}
\]

The objects of \( \mathcal{E}_b \) are \( \{ e \in \mathcal{E} \mid pe = b \} \), while the morphisms are given by the pullback:

\[
\begin{array}{ccc}
\mathcal{E}_b(e, e') & \longrightarrow & \mathcal{E}(e, e') \\
\downarrow & & \downarrow p \\
\mathbb{1} & \longrightarrow & \mathcal{B}(b, b)
\end{array}
\] (13)

Remark 3.16. When \( \mathbb{1} \) is terminal, which is the case if \( \mathbf{V} \) if semi-cartesian, we may think of \( \mathcal{E}_b \) as a subcategory of \( \mathcal{E} \) consisting of objects in the pre-image of \( b \) and morphisms in the pre-image of \( 1_b \).

Let

\[
p\downarrow b := (p\downarrow \mathcal{B})_b
\]

denote the fiber of \( \pi_\mathcal{B}: p\downarrow \mathcal{B} \to \mathcal{B} \) over \( b \), given by the pullback:

\[
\begin{array}{ccc}
p\downarrow b & \longrightarrow & p\downarrow \mathcal{B} \\
\downarrow & & \downarrow \pi_\mathcal{B} \\
\mathbb{1} & \longrightarrow & \mathcal{B}
\end{array}
\]

For any functor \( p: \mathcal{E} \to \mathcal{B} \), the canonical \( i: \mathcal{E} \to p\downarrow \mathcal{B} \) restricts to a functor

\[
i_b: \mathcal{E}_b \to p\downarrow b.
\]

Note that \( i_b \) need not be full, even though \( i \) is.

Lemma 3.17. If \( i \) has a left adjoint \( \ell \) in \( \text{Cat}_{\mathbf{V}}/\mathcal{B} \), then \( i_b \) has a left adjoint \( \ell_b: p\downarrow b \to \mathcal{E}_b \).
Proof. We may piece the pullback diagrams for $E_b$ and $p \downarrow b$ into the following diagram:

\[
\begin{array}{c}
\begin{array}{c}
p \downarrow b \\
\downarrow i_b \\
\downarrow b \\
\downarrow \pi_B \\
1 \\
\end{array}
\end{array}
\quad \quad \begin{array}{c}
\begin{array}{c}
E_b \\
\downarrow \ell_b \\
\downarrow i_b \\
\downarrow \pi_B \\
B \\
\end{array}
\end{array}
\quad \quad \begin{array}{c}
\begin{array}{c}
p \downarrow B \\
p \downarrow \pi_B \\
B \\
\end{array}
\end{array}
\quad \quad \begin{array}{c}
\begin{array}{c}
\ell \\
\downarrow i \\
p \\
\end{array}
\end{array}
\end{array}
\]

Then $i_b$ and $\ell_b$ are just pullbacks of $i$ and $\ell$. In more detail, the functor $i$ induces $i_b$ by the universal property of $p \downarrow b$ and the commutativity of the outer diagram:

\[
\begin{array}{c}
\begin{array}{c}
E_b \\
\downarrow i_b \\
\downarrow p \downarrow B \\
\end{array}
\end{array}
\quad \quad \begin{array}{c}
\begin{array}{c}
\ell \\
\downarrow i \\
p \\
\end{array}
\end{array}
\quad \quad \begin{array}{c}
\begin{array}{c}
\ell \\
\downarrow i \\
p \\
\end{array}
\end{array}
\]

In a similar fashion, the left adjoint $\ell$, if it exists, induces $\ell_b$ by the universal property of $E_b$. Since $\ell_b$ is the restriction of $\ell$ to $p \downarrow b$, and $i_b$ is the restriction of $i$ to $E_b$, we may similarly restrict the unit and counit of the adjunction $\ell \dashv i$ to conclude that $\ell_b \dashv i_b$. ☐

Remark 3.18. In [15, Definition 4.15], Tamaki defines an opfibration to be a functor $p : E \to B$ such that each pair $(\ell_b, i_b)$ is an adjoint equivalence. As we noted above, $i_b$ is generally not full, hence is seldom an equivalence. In fact, one can show that if $p$ is a surjective opfibration in Tamaki’s sense when $V = Set$, then $B$ has to be a groupoid. See Remark 3.32 for a possible motivation of Tamaki’s definition.

Since pullbacks of comma categories are comma categories, $p \downarrow b$ thus enjoys the universal property of comma categories given in §A.6. This will be used to lift morphisms in $B_0$ to functors between fibers.

Recall from §2.2 that objects of $B_0$ are $V$-functors $b : 1 \to B$ and morphisms are natural transformations between them. Each $f \in B_0(b, b')$ yields

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a composite 2-cell \( f \cdot \varphi \):

\[
\begin{array}{ccc}
p \downarrow b & \xrightarrow{\varphi} & \mathcal{E} \\
p & \downarrow & \\
1 & \xleftarrow{f \downarrow b'} & \mathcal{B}
\end{array}
\]

By the universal property of the comma category \( p \downarrow b' \), this induces a functor \( f_*: p \downarrow b \to p \downarrow b' \) such that \( \varphi' \circ f_* = f \cdot \varphi \):

\[
\begin{array}{ccc}
p \downarrow b & \xrightarrow{f_*} & p \downarrow b' \\
p \downarrow b & \xleftarrow{\varphi'} & \mathcal{B}
\end{array}
\]

On objects, \( f_* \) acts like ‘post-composition with \( f \)’, and sends \((e, g, b)\) to \((e, fg, b')\). Given another \( g \in B_0(b', b'') \), we have

\[
\varphi'' \circ (gf)_* = (gf) \cdot \varphi = g \cdot (f \cdot \varphi) = g \cdot (\varphi' \circ f_*) = (g \cdot \varphi') \circ f_* = \varphi'' \circ g_* \circ f_*,
\]

and by uniqueness, \((gf)_* = g_* \circ f_*\). Similarly, we have \((1_b)^* = 1_{p \downarrow b}\). We thus obtain a functor

\[
\tilde{F}: B_0 \to \mathcal{C}at_{\mathbf{V}}
\]

\[
b \mapsto p \downarrow b
\]

\[
(b \xrightarrow{f} b') \mapsto (p \downarrow b \xrightarrow{f_*} p \downarrow b').
\]

On objects, \( f_* \) sends \((e, g, b)\) to \((e, fg, b')\).

**Definition 3.19.** Let \( p: \mathcal{E} \to \mathcal{B} \) be a fibration. For each \( f \in B_0(b, b') \), the **transport along** \( f \) is the composite functor

\[
f_t: \mathcal{E}_b \xrightarrow{i_b} p \downarrow b \xrightarrow{f_*} p \downarrow b' \xrightarrow{\ell_{b'}} \mathcal{E}_{b'}.
\]

(14)
Remark 3.20. Observe that we have $f\text{e} = \ell(e, f, b')$ for $e \in \mathcal{E}_b$, which agrees with the notation $f\text{e}$ in Proposition 3.5.

We may similarly attempt to define a functor

$$F: \mathcal{B}_0 \xrightarrow{\ell} \text{Cat}_{\mathcal{V}}$$

$$b \mapsto \mathcal{E}_b$$

$$\left( b \xrightarrow{f} b' \right) \mapsto \left( \mathcal{E}_b \xrightarrow{f\text{e}} \mathcal{E}_{b'} \right).$$

Unfortunately, this need not yield a functor, as $F$ may not preserve composites or identities. However, $F$ will be pseudofunctor. We show this after taking a brief detour into split opfibrations.

3.4 Split and normal opfibrations

Definition 3.21. An opfibration is split if $F$ in (15) is a functor.

Lemma 3.22. An opfibration is split iff

$$q(1_{pe}, e) = 1_e, \quad \text{and}$$

$$q(g \circ f, e) = q(g, f\text{e}) \circ q(f, e)$$

for all $e \in \mathcal{E}$, $f \in \mathcal{B}_0(pe, b')$ and $g \in \mathcal{B}_0(b', b'')$.

Proof. This is merely a rephrasing of the functoriality of (15) in terms of opcartesian lifts.

By keeping only the first condition of Lemma 3.22, we obtain the notion of a normal opfibration:

Definition 3.23. An opfibration $p: \mathcal{E} \rightarrow \mathcal{B}$ is normal if the counit $\varepsilon$ of the adjunction $\ell \dashv i$ is the identity.

Lemma 3.24. An opfibration is normal iff $q(1_{pe}, e) = 1_e$ for all $e \in \mathcal{E}$.

Proof. This follows immediately from (12).

Lemma 3.25. Let $p: \mathcal{E} \rightarrow \mathcal{B}$ be an opfibration, with cleavage $\ell: p\downarrow \mathcal{B} \rightarrow \mathcal{E}$. Then there is a cleavage $\ell': p\downarrow \mathcal{B} \rightarrow \mathcal{E}$ such that $(p, \ell')$ is a normal opfibration.

Proof. Take $\ell'$ to be the same as $\ell$ for all objects of $p\downarrow \mathcal{B}$ except those of the form $ie = (e, 1_{pe}, pe)$ for some $e \in \mathcal{E}$, where we instead define $\ell'(e, 1_{pe}, pe)$ to be $e$ and $q'(1_{pe}, e) = 1_e$.

To see that this is an adjunction, recall that $q(1_{pe}, e): e \rightarrow \ell(e, 1_{pe}, pe)$ is always an isomorphism, so $\ell'(e, 1_{pe}, pe)$ and $\ell(e, 1_{pe}, pe)$ are always isomorphic. Thus $\ell$ and $\ell'$ are isomorphic functors, so $\ell' \dashv i$ iff $\ell \dashv i$. 21
Thus, every opfibration is equivalent to a normal one with the same total category \( E \). It will turn out to be true that every opfibration is equivalent to a split one, although not necessarily with the same total category \( E \). In the rest of this section, we define a split opfibration \( \hat{p}: \hat{E} \to B \) associated to an opfibration \( p: E \to B \), and show that \( p \) is equivalent to \( \hat{p} \).

We will see that \( f_! \) from Definition 3.19 factors through the fibers of \( \hat{E} \). This factorization will then be used to show that \( F \) is a pseudofunctor.

**Definition 3.26.** Let \( p: E \to B \) be an opfibration. Define a category \( \hat{E} \) as follows:

\[
\text{Ob}(\hat{E}) = \text{Ob}(p/B),
\]
\[
\hat{E}(f, f') = \ell \left( p_b(f, f') \right).
\]

The category \( \hat{E} \) sits between \( E \) and \( p\downarrow B \): it has objects from \( p/B \) and morphisms from \( E \). The left adjoint \( \ell: p/B \to E \) factors through \( \hat{E} \) as

\[
\ell: p\downarrow B \xrightarrow{\ell^0} \hat{E} \xrightarrow{\ell^1} E
\]

where \( \ell^1 \) is the identity on objects and \( \ell \) on morphisms, and \( \ell^0 \) is \( \ell \) on objects and the identity on morphisms. There is a projection functor \( \hat{p}: \hat{E} \to B \) which is given by \( \pi_B \) on objects and \( p \) on morphisms, and which fits into the diagram:

\[
p\downarrow B \xrightarrow{\ell^1} \hat{E} \xrightarrow{\ell^0} E \xrightarrow{p} B
\]

\[
\pi_B \quad \hat{p}
\]

**Lemma 3.27.** The functor \( \ell^0: \hat{E} \to E \) is an equivalence of categories, with quasi-inverse given by \( \ell^1 i \).

**Proof.** Since \( \ell^0 \) is the identity on morphisms, it is fully faithful. To show that it is essentially surjective (as a functor between the underlying unenriched categories), we need to show that every \( e \in E \) is isomorphic to an object of the form \( \ell(e', f, b) = f_! e' \) for some \( (e', f, b) \). But by Lemma 3.9, \( e \) is isomorphic to \( \ell(e, 1_{pe}, pe) \). Thus \( \ell^0 \) is an equivalence.

In fact, the proof of Lemma 3.9 shows that the desired isomorphism is given by \( q(1_{pe}, e) \), with inverse \( \varepsilon_e \). These form the components of natural isomorphisms \( 1_\hat{E} \Rightarrow (\ell^1 i) \ell^0 \) and \( \ell^0 (\ell^1 i) = \ell i \Rightarrow 1_\hat{E} \), resp., showing that \( \ell^1 i \) is a quasi-inverse. \( \square \)
Proposition 3.28. There is a left adjoint $\hat{\ell}: \hat{p} \downarrow \mathcal{B} \to \hat{\mathcal{E}}$ to the canonical inclusion $\hat{i}: \hat{\mathcal{E}} \to \hat{p} \downarrow \mathcal{B}$ such that $(\hat{p}, \hat{\ell})$ is an opfibration, which is split if $(p, \ell)$ is normal.

Proof. The objects of $\hat{p} \downarrow \mathcal{B}$ are of the form $\left((e, f, b), g, c \right)$, where $e \in \hat{\mathcal{E}}$, $b, c \in \mathcal{B}$, $f \in \mathcal{B}_0(pe, b)$ and $g \in \mathcal{B}_0(b, c)$. Define $\hat{\ell}$ on objects via $\left((e, f, b), g, c \right) \mapsto \left((e, gf, c \right)$.

Obtaining $\hat{\ell}$ on morphisms is more involved, but is ultimately obtained by iterated use of the isomorphism in (9), Remarks 3.6 and 3.8, and the defining pullback squares for morphisms in $p \downarrow \mathcal{B}$ and $\hat{p} \downarrow \mathcal{B}$. Given $\left((e, f, b), g, c \right)$ and $\left((e', f', b'), g', c' \right)$, we seek a map

$$\hat{p} \downarrow \mathcal{B}\left(\left((e, f, b), g, c \right), \left((e', f', b'), g', c' \right)\right) \to \hat{\mathcal{E}}\left((e, gf, c), (e', g'f', c')\right).$$

(16)

Recall the shorthand of writing $g$ for $\left((e, f, b), g, c \right)$ in $\hat{p} \downarrow \mathcal{B}$ and $gf$ for $(e, gf, c)$ in $p \downarrow \mathcal{B}$ and so on, so that the desired map may be abbreviated $\hat{p} \downarrow \mathcal{B}(g, g') \to \hat{\mathcal{E}}(gf, g'f')$. The reader might find it helpful to refer to the following diagram in the case $\mathbf{V} = \text{Set}$ (and the analogous diagram for $\left((e', f', b'), g', c' \right)$) to keep track of how the various objects of $\mathcal{E}$ and $\mathcal{B}$ are related to each other, and by which maps:

(17)

We start by noting that the codomain of (16) is $\mathcal{E}\left((gf)_e, (g'f')_e \right)$ by definition. We have the lower map in (9)

$$p \downarrow \mathcal{B}(gf, 1_e) \xrightarrow{\circ \hat{\ell}} \mathcal{E}\left((gf)_e, (g'f')_e \right),$$

so that it suffices to find a map to $\hat{p} \downarrow \mathcal{B}(g, g') \to p \downarrow \mathcal{B}(gf, 1_e)$. By the universal property of $p \downarrow \mathcal{B}(gf, 1_e)$, this is the same as producing the following
commuting square:

\[
\begin{array}{c}
\hat{p} \downarrow B(g, g') \longrightarrow \mathcal{E}(e, (g' f') e') \\
\downarrow^p \\
B(c, c') \longrightarrow_{o g f} B(pe, c')
\end{array}
\]  

(18)

Observe that \(\hat{p} \downarrow B(g, g')\) fits into the following diagram, where we have abbreviated the upper map in (9) by \((\eta, p)\):

\[
\begin{array}{c}
\hat{p} \downarrow B(g, g') \longrightarrow \mathcal{E}(f, f') = \mathcal{E}(f_1 e, f'_1 e') \longrightarrow p \downarrow B(f, 1_{b'}) \longrightarrow \mathcal{E}(e, f'_1 e') \\
\downarrow^p \\
B(b, b') \longrightarrow_{o f} B(pe, b') \\
\downarrow^g \circ - \\
B(c, c') \longrightarrow_{o g} B(b, c') \longrightarrow_{o f} B(pe, c')
\end{array}
\]

On the right of the above diagram, we may then paste the following square, where \(\tilde{g}'\) is obtained in the manner of (10):

\[
\begin{array}{c}
\mathcal{E}(e, f'_1 e') \longrightarrow \mathcal{E}(e, (g' f') e') \\
\downarrow^p \\
B(pe, b') \longrightarrow_{o g'} B(pe, c')
\end{array}
\]

We thus have the desired square (18) and hence the map in (16). The functoriality of \(\ell\) and the adjunction \(\ell \dashv i\) may then be used to show that \(\hat{\ell}\) is indeed a functor that is left adjoint to \(\hat{i}\), and thus \(\hat{p}\) is an opfibration.

Assume now that the original opfibration \((p, \ell)\) is normal. Then it is easy to see that \((\tilde{p}, \hat{\ell})\) is also normal. To show that it is split, we need to show that for \(g \in \mathcal{B}_0(b, b')\) and \(h \in \mathcal{B}_0(b', b'')\), we have \(\tilde{h} \circ \tilde{g} = \tilde{g} \circ \tilde{h}\), where \(\tilde{g}\) is defined as per Definition 3.19:

\[
\begin{array}{c}
\tilde{g}: \tilde{\mathcal{E}}_b \longrightarrow \tilde{\mathcal{E}}_b \\
\tilde{i}_b \circ \tilde{g} \longrightarrow \tilde{p} \downarrow b \longrightarrow \tilde{p} \downarrow b' \longrightarrow \tilde{\mathcal{E}}_{b'}
\end{array}
\]

This is tedious but easy to check (and in fact does not require normality of \((p, \ell)\)). We need to show that the following diagram commutes for all
(e, f, b) and (e’, f’, b) in $\hat{E}_b$, where we have rewritten the hom-objects of $\hat{E}$ as their defining hom-objects of $E$:

$$
\begin{array}{ccc}
\mathcal{E}_b(f_e, f'_e) & \xrightarrow{\hat{g}} & \mathcal{E}_{b'}((gf)_e, (gf')_e) \\
& \searrow_{h_{gf}} & \downarrow_{\hat{h}_e} \\
& \mathcal{E}_{b'}((hgf)_e, (hgf')_e) & 
\end{array}
$$

But this follows from repeated applications of (the enriched versions of) (10) and (11), applied to the appropriate maps. Ultimately, the diagram commutes because the two maps into $p \downarrow B(hgf, 1_{b'})$ are the same. \hfill \Box

**Corollary 3.29.** Every opfibration is equivalent to a split one.

**Proof.** By Lemma 3.25, every opfibration is equivalent to a normal one, which is in turn equivalent to a split one by Lemma 3.27 and Proposition 3.28. \hfill \Box

We write $\hat{p}: \hat{E} \to \mathcal{B}$ for the resulting split opfibration, where $p$ is a possibly non-normal, non-split opfibration. The functor that results from the split opfibration will be denoted $\hat{F}$:

$$
\hat{F}: \mathcal{B}_0 \to \text{Cat}_\mathcal{V} \\
b \mapsto \hat{E}_b \\
(b \xrightarrow{f} b') \mapsto (\hat{E}_b \xrightarrow{\hat{f}} \hat{E}_{b'}).
$$

Finally, we record the following lemma, whose proof is similar to the proof that $\hat{p}$ is split.

**Lemma 3.30.** The following diagram commutes:

$$
\begin{array}{ccc}
p \downarrow b & \xrightarrow{f} & p \downarrow b' \\
\hat{E}_b & \xrightarrow{\hat{f}} & \hat{E}_{b'} \\
\hat{E}_b & \xrightarrow{\hat{f}} & \hat{E}_{b'} \\
\end{array}
$$

### 3.5 The Inverse Grothendieck Construction I

We are now ready to show that $F$ as defined in (15) is a pseudofunctor for any opfibration $p: \mathcal{E} \to \mathcal{B}$. 

25
Proposition 3.31. The map $F : B_0 \to \mathbf{Cat}_\mathbf{V}$ can be given the structure of a pseudofunctor.

Proof. By Lemma 3.30, the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{E}_b & \xrightarrow{i_b} & p \downarrow b \\
\downarrow \ell_0^b & & \downarrow \ell_0^b' \\
\hat{\mathcal{E}}_b & \xrightarrow{f_i} & \hat{\mathcal{E}}_{b'} \\
\end{array}
\begin{array}{ccc}
p \downarrow b' & \xrightarrow{p \downarrow b'} & \mathcal{E}_{b'} \\
\downarrow \ell_0^b & & \downarrow \ell_0^b' \\
\mathcal{E}_{b'} & \xrightarrow{\ell_0^b} & \hat{\mathcal{E}}_{b'} \\
\end{array}
\begin{array}{ccc}
\hat{\mathcal{E}}_b & \xrightarrow{f_i} & \hat{\mathcal{E}}_{b'} \\
\end{array}
\begin{array}{ccc}
\end{array}
$$

The top composite is the definition of $f_i$ from (14), so this says that we may write $f_i$ as the bottom composite.

To make $F : B_0 \to \mathbf{Cat}_\mathbf{V}$ a pseudofunctor, we supply $\xi$ and $\theta$ as follows:

$$
\begin{array}{ccc}
\mathcal{E}_b & \xrightarrow{\ell_1^b} & \hat{\mathcal{E}}_b \\
\downarrow \ell_0^b & & \downarrow \ell_0^b \\
\hat{\mathcal{E}}_b & \xrightarrow{f_i} & \hat{\mathcal{E}}_{b'} \\
\end{array}
\begin{array}{ccc}
\mathcal{E}_{b'} & \xrightarrow{\ell_1^{b'}} & \hat{\mathcal{E}}_{b'} \\
\downarrow \ell_0^{b'} & & \downarrow \ell_0^{b'} \\
\hat{\mathcal{E}}_{b'} & \xrightarrow{f_i} & \hat{\mathcal{E}}_{b'} \\
\end{array}
\begin{array}{ccc}
\end{array}
$$

where the natural isomorphisms come from Lemma 3.27. Then $\hat{p} : \hat{\mathcal{E}} \to B$ being split implies that $\hat{F}$ is a functor, and hence $F$ is a pseudofunctor.

Remark 3.32. The Proposition above relies on $\ell_0^b$ and $\ell_1^b$ being mutual quasi-inverses whenever $p : \mathcal{E} \to B$ is an opfibration. In [15, Lemma 4.17, Definition 4.18], Tamaki does something similar, but with $\ell_b$ and $i_b$ instead. This motivates his definition in Remark 3.18, as it is almost immediate to obtain pseudofunctoriality of $F$ if $\ell_b$ and $i_b$ are quasi-inverses. Our Proposition shows that pseudofunctoriality still holds when $\ell_b$ and $i_b$ are not quasi-inverses.

Proposition 3.33. The inverse Grothendieck construction that sends a fibration $p$ to a pseudofunctor $F$ extends to a 2-functor

$$
I : \text{OpFib}(B) \to \mathbf{Fun}^{ps}(B_0, \mathbf{Cat}_\mathbf{V}).
$$
Proof. We have seen above what $I$ does to opfibrations i.e. the 0-cells of $\text{OpFib}(B)$. We need to define what $I$ does to 1-cells and 2-cells. Let $p: \mathcal{E} \to B$ and $q: \mathcal{F} \to B$ be opfibrations over $B$, and let $E, F: B \to \text{Cat}_V$ be the corresponding pseudofunctors.

Suppose we have opfibered functor $k: \mathcal{E} \to \mathcal{F}$. Recall that this induces a functor $K: p\downarrow B \to q\downarrow B$ such that we have the following commuting square and natural isomorphism, where we have written $\ell$ for both $\ell_p$ and $\ell_q$, and $i$ for $i_p$ and $i_q$:

$$
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{k} & \mathcal{F} \\
\downarrow{i} & & \downarrow{i} \\
p\downarrow B & \xrightarrow{K} & q\downarrow B
\end{array}
\quad
\begin{array}{ccc}
p\downarrow B & \xrightarrow{\ell} & q\downarrow B \\
i & \cong & i \\
p\downarrow B & \xrightarrow{\ell} & q\downarrow B
\end{array}
$$

Pulling these back along $b: 1 \to B$ in $B_0$, we obtain commuting squares of fibers:

$$
\begin{array}{ccc}
\mathcal{E}_b & \xrightarrow{k_b} & \mathcal{F}_b \\
\downarrow{i_b} & & \downarrow{i_b} \\
p\downarrow b & \xrightarrow{K_b} & q\downarrow b
\end{array}
\quad
\begin{array}{ccc}
p\downarrow b & \xrightarrow{\ell_b} & q\downarrow b \\
p\downarrow b & \xrightarrow{\ell_b} & q\downarrow b
\end{array}
$$

Given $f: b \to b'$ in $B_0$, it is easy to see that the following diagram also commutes:

$$
\begin{array}{ccc}
p\downarrow b & \xrightarrow{K_b} & q\downarrow b \\
f_* & & f_* \\
p\downarrow b' & \xrightarrow{K_{b'}} & q\downarrow b'
\end{array}
$$

Piecing these diagrams of fibers together, we obtain a natural isomorphism:

$$
\begin{array}{ccc}
\mathcal{E}_b & \xrightarrow{k_b} & \mathcal{F}_b \\
f_i & \cong & f_i \\
\mathcal{E}_{b'} & \xrightarrow{k_{b'}} & \mathcal{F}_{b'}
\end{array}
\quad
\begin{array}{ccc}
Eb & \xrightarrow{k_b} & Fb \\
f & \cong & f \\
Eb' & \xrightarrow{k_{b'}} & Fb'
\end{array}
$$

The functors $k_b$ thus form the 1-components of a pseudonatural transformation which we denote $\kappa: E \Rightarrow F$, while the 2-components are given by the natural isomorphism above.

Given another opfibered functor $h: \mathcal{E} \to \mathcal{F}$ which induces $\lambda: E \Rightarrow F$, and a $V$-natural transformation $\alpha: h \Rightarrow k$, we may once again pull all these
back along $b$ to obtain a $V$-natural transformation

$$
\mathcal{E}_b = E_b \xrightarrow{h_b} F_b = \mathcal{F}_b
\xleftarrow{k_b} \alpha_b
$$

The 2-cells $\alpha_b$ then form the data of a modification $\Gamma : \lambda \Rightarrow \kappa$.

We leave it to the reader to check that the various coherence conditions are satisfied, and that this indeed yields a 2-functor.

Note that while the inverse Grothendieck construction takes opfibrations over an arbitrary enriched $V$-category $\mathcal{B}$, it only returns pseudofunctors from an unenriched $\mathcal{B}_0$. It is thus generally not possible to recover an opfibration $p : \mathcal{E} \to \mathcal{B}$ over an arbitrary base $\mathcal{B}$ from its corresponding pseudofunctor $E : \mathcal{B}_0 \to \text{Cat}_V$. The next section describes the best we can do.

4 The Grothendieck Construction

We now describe an opfibration $p : \text{Gr}F \to B_V$ associated to a pseudofunctor $F : B \to \text{Cat}_V$.

4.1 Assumptions and notation

For what follows, we require the following assumptions described in §2.1:

1. $V$ has coproducts, and these are preserved by $\otimes$;
2. $V$ has pullbacks, and these preserve coproducts;
3. $V$ is semi-cartesian i.e. $1$ is terminal;
4. For $X$ a set, we have an isomorphism of sets

$$
V \left( 1, \coprod_{x \in X} 1 \right) \cong X,
$$

(this implies that $B \cong (B_V)_0$).

Remark 4.1. We briefly sketch where these properties will be used: Co-products are required for the formation of $B_V$ and $\text{Gr}F$, and we require $\otimes$ to commute with coproducts to define composition in these categories. While $1$ need not be terminal to obtain $\text{Gr}F$, we do need it to obtain a functor
Pullbacks are needed for comma categories and fibers, and pullbacks are required to preserve coproducts in order to get the description of the hom-objects of $p_\downarrow B_\mathbf{V}$ in Lemma 4.8. Finally, the last condition is needed to produce a left adjoint $\ell : p_\downarrow B_\mathbf{V} \to GrF$ (see Remark 4.12 for details).

**Remark 4.2.** Property 4 implies that the unit of the adjunction (1) is an isomorphism of categories:

$$B \cong (B_\mathbf{V})_0.$$  

We simplify matters by assuming that this isomorphism is in fact equality, which we justify in the following manner:

Recall that elements of $(B_\mathbf{V})_0$ are $\mathbf{V}$-maps

$$1 \to B_\mathbf{V}(b, c) = \bigsqcup_{f \in B(b,c)} 1.$$  

The set of such maps contains the inclusions $1_g \hookrightarrow B_\mathbf{V}(b, c)$, where $1_g$ denotes the copy of 1 corresponding to $g \in B(b,c)$. Assumption 4 then says that these inclusions account for all $\mathbf{V}$-maps $1 \to B_\mathbf{V}(b, c)$. By abuse of notation, we may identify elements $g \in B(b,c)$ with maps $1 = 1_g \hookrightarrow B_\mathbf{V}(b, c)$, which we also call $g$. Under this identification, we then have $B = (B_\mathbf{V})_0$.

We also declare certain notational conventions that will prevent what follows from becoming completely unreadable.

**Notation 4.3.** Let $B$ be an ordinary (i.e. $\mathbf{Set}$-enriched) category treated as a 2-category and let $F : B \to \mathbf{Cat}_\mathbf{V}$ be a pseudofunctor.

- The image of $b \in B$ under $F$ will be denoted $F_b \in \mathbf{Cat}_\mathbf{V}$.
- The image of $f : b \to c$ under $F$ will be denoted $F_f : F_b \to F_c$. This sends $x \in F_b$ to $F_f x \in F_c$.
- For each $b \in B$, the invertible 2-cell $\xi(b) : F_{1_b} \cong 1_{F_b}$ will be denoted $\xi$. Its component at each $x \in F_b$ will be denoted

$$\xi_x : 1 \to F_b(F_{1_b}x, x).$$

Since $\xi$ is invertible, each $\xi_x$ is an isomorphism in $F_b$. 

For each $b \xrightarrow{f} c \xrightarrow{g} d$, the invertible 2-cell $\theta(f, g) : F_{gf} \cong F_g F_f$ will be denoted $\theta$. Its component at each $x \in F_b$ will be denoted $\theta_x : 1 \to F_d(F_{gf}x, F_g F_f x)$.

Since $\theta$ is invertible, each $\theta_x$ is an isomorphism in $F_d$.

4.2 The category $GrF$

Definition 4.4. Let $B$ be an ordinary (i.e. $Set$-enriched) category treated as a 2-category, and let $F : B \to \mathbf{Cat}_V$ be a pseudofunctor. The Grothendieck construction of $F$ is the $V$-category $GrF$ with objects and morphisms

$$Ob(GrF) := \coprod_{b \in B} Ob(F_b),$$

$$GrF(x_b, y_c) := \coprod_{f : b \to c} F_c(F_f x, y).$$

where $x_b$ indicates that $x \in F_b$. Identity morphisms are given by

$$1_{x_b} := \xi_x : 1 \to F_b(F_{1b} x, x) \subset \coprod_{f : b \to b} F_b(F_f x, x) = GrF(x_b, x_b) \quad (19)$$

while composition is induced by the composite

$$F_c(F_f x, y) \otimes F_d(F_g y, z) \xrightarrow{F_g \otimes 1} F_d(F_g F_f x, F_g y) \otimes F_d(F_g y, z) \xrightarrow{(- \circ \theta_x) \otimes 1} F_d(F_g F_f x, z) \otimes F_d(F_g y, z)$$

where $b \xrightarrow{f} c \xrightarrow{g} d$. This extends to a functor out of $GrF(x_b, y_c) \otimes GrF(y_c, z_d)$ because $\otimes$ commutes with coproducts.

Lemma 4.5. Let $F : B \to \mathbf{Cat}_V$ be a pseudofunctor, $f : b \to c$ an isomorphism in $B$ and $\rho : 1 \to F_c(F_f x, y)$ an isomorphism in $F_c$. Then the corresponding $\rho : 1 \to GrF(x_b, y_c)$ is an isomorphism in $GrF$.

Proof. By §2.3, it suffices to produce an inverse to $\rho$ in $GrF$ i.e. a map

$$1 \to GrF(y_c, x_b) = \coprod_{g : c \to b} F_b(F_g y, x).$$
Let $\rho^{-1}: 1 \to F_c(y, F_f x)$ be the inverse of $\rho$ in $F_c$. Applying the functor $F_{f^{-1}}$, we obtain a map

$$F_{f^{-1}} \rho^{-1}: 1 \to F_b(F_{f^{-1}} y, F_{f^{-1}} F_f x)$$

which we may then postcompose with $\theta^{-1}_x: 1 \to F_b(F_{f^{-1}} F_f x, F_{b^x})$ and $\xi_x: 1 \to F_b(F_{b^x}, x)$ to obtain a map

$$1 \to F_b(F_{f^{-1}} y, x) \subset GrF(y, x)$$

One can then check that this is the desired inverse to $\rho$ in $GrF$.

\[ \blacksquare \]

**Remark 4.6.** We will see in what follows that the $GrF$ admits an opfibration to the free $\mathbb{V}$-category $B_\mathbb{V}$ on $B$. If instead we had $F: B^{op} \to \mathcal{C}at_\mathbb{V}$, we can similarly define the $\mathbb{V}$-category $Gr^\lor F$ where

$$Ob(Gr^\lor F) := \coprod_{b \in B} Ob(F_b),$$

$$Gr^\lor F(x_b, y_c) := \coprod_{f: b \to c} F_b(x, F_f y),$$

and show that $Gr^\lor F$ admits a fibration to $B_\mathbb{V}$. We will not study $Gr^\lor F$ but its properties are formally dual to $Gr$.

We next produce the functor that we want to show is an opfibration.

**Lemma 4.7.** Let $F: B \to \mathcal{C}at_\mathbb{V}$ be a pseudofunctor, and $GrF$ its Grothendieck construction. There is a $\mathbb{V}$-functor $p: GrF \to B_\mathbb{V}$.

**Proof.** We will describe the functor but omit checking all the necessary coherences. On objects, $p$ simply projects down to $B$, sending $x_b$ to $b$. On morphisms, we need to give a $\mathbb{V}$-morphism

$$GrF(x_b, y_c) = \coprod_{f: b \to c} F_c(F_f x, y) \to \coprod_{f: b \to c} 1 = B_\mathbb{V}(b, c).$$

Since $\mathbb{V}$ is semi-cartesian, each $F_c(F_f x, y)$ has a unique map to $1$. Taking the coproduct of these maps over $B(b, c)$, we obtain the desired morphism. \[ \blacksquare \]

## 4.3 The comma category of $p: GrF \to B_\mathbb{V}$

We want to show that $p: GrF \to B_\mathbb{V}$ is an opfibration. To do this, we require a left adjoint $\ell: p^! B_\mathbb{V} \to GrF$. We thus first describe the mapping spaces in the comma category $p^! B_\mathbb{V}$.
Recall that objects of $p \downarrow B_{\mathbf{V}}$ are of the form $(x_b, f, c)$ where $x \in F_b$ and $f : 1 \to B_{\mathbf{V}}(b, c)$. By assumption, $f$ is a canonical inclusion of the form $1_f \sqsubseteq \coprod \mathbf{B}V$,

$$1_f \mapsto \prod_{g : b \to c} 1_g,$$

where we also use $f$ to denote the corresponding element of $B(b, c)$.

**Lemma 4.8.** For $F : B \to \mathbf{Cat}_\mathbb{V}$ and $p : GrF \to B_{\mathbf{V}}$ as above, we have the following isomorphism of $\mathbb{V}$-objects:

$$p \downarrow B_{\mathbf{V}}((x_b, f, c), (y_d, g, e)) \cong \prod_{h : b \to d} \left( \prod_{k : c \to e} \prod_{k f = gh} F_d(F_h x, y) \right).$$

*Proof.* By (2), we have a pullback

$$p \downarrow B_{\mathbf{V}}((x_b, f, c), (y_d, g, e)) \rightarrow GrF(x_b, y_d) = \prod_{h : b \to d} F_d(F_h x, y)$$

$$\downarrow$$

$$B_{\mathbf{V}}(b, d) = \prod_{h : b \to d} 1$$

$$\downarrow$$

$$B_{\mathbf{V}}(c, e) = \prod_{k : c \to e} 1$$

where $- \circ f : 1_k \mapsto 1_{kf}$ and $g \circ - : 1_h \mapsto 1_{gh}$. We may factor this as two pullbacks where the lower pullback is:

$$\prod_{(h, k)} 1 \rightarrow \prod_{h : b \to d} 1$$

$$\downarrow$$

$$\prod_{k : c \to e} 1$$

By rearranging coproducts, we have:

$$\prod_{h : b \to d} \left( \prod_{k : c \to e} 1 \right) \cong \prod_{(h, k)} 1 \cong \prod_{k : c \to e} \left( \prod_{h : b \to d} 1 \right).$$

Applying the next Lemma to the upper pullback completes the proof. □
Lemma 4.9. Let $f : J \to I$ be a function of sets and let $\{V_i\}_{i \in I}$ be a collection of $I$-indexed $V$-objects. Then the following is a pullback diagram:

\[
\begin{array}{ccc}
\prod_{j \in J} V_{f(j)} & \longrightarrow & \prod_{i \in I} V_i \\
\downarrow & & \downarrow \prod! \\
\prod_{j \in J} 1 & \longrightarrow & \prod_{i \in I} 1
\end{array}
\]

where $f : 1_j \mapsto 1_{f(j)}$ and $\prod!$ is the coproduct of the unique maps $V_i \to 1$.

Proof. Since pullbacks preserve coproducts in $V$, we have

\[
\prod_{i \in I} V_i \times \prod_{j \in J} 1 \prod_{i \in I} 1 \cong \prod_{j \in J} \left( \prod_{i \in I} V_i \times \prod_{i \in I} 1 \right).
\]

Considering the term in parentheses in the above formula, we note that for a fixed $j$ it is isomorphic to $V_{f(j)}$. \qed

Remark 4.10. The above description of mapping objects in $p \downarrow B$ makes it clear what the identity morphisms are: we require a choice of $V$-morphisms $1 \to p \downarrow B((x_b, f, c), (x_b, f, c))$, hence a morphism

\[
1 \to \prod_{h : b \to b', k : c \to c} F_b(F_h x, x).
\]

Simply choose $h = 1_b$, $k = 1_c$, and

\[
\xi_x : 1 \to F_b(F_1 x, x)
\]

from the pseudofunctoriality of $F$.

4.4 $GrF$ is an opfibration

Proposition 4.11. If $F : B \to Cat_V$ is a pseudofunctor, then $p : GrF \to B_V$ is an opfibration of $V$-categories.

Proof. We will explicitly construct the left adjoint $\ell_p : p \downarrow B_V \to GrF$ whose existence is required by Definition 3.1. Recall that the objects of $p \downarrow B_V$ are triples $(x_b, f, c)$ where $x \in F_b$ and $f : b \to c$ in $B$. 

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The canonical inclusion \( i: GrF \to p\downarrow B_V \) is given by \( ix_c = (x_c, 1_c, c) \) on objects and the ‘identity’ map on morphisms:

\[
\prod_{h: c \to d} F_d(F_h x, y) \to \prod_{h: c \to d} \left( \prod_{k: c \to d} F_d(F_k x, y) \right) \cong \prod_{h: c \to d} F_d(F_h x, y).
\]

We seek a left adjoint \( \ell: p\downarrow B_V \to GrF \) over \( B_V \). On objects, set

\[
\ell(x_b, f, c) = F_f x_c \in F_c,
\]

where we have written \( F_f x_c \) instead of the more cumbersome \( (F_f x)_c \). For morphisms, we need a map

\[
\ell_{f,g}: p\downarrow B_V ((x_b, f, c), (y_b, g, d)) \to GrF (F_f x_c, F_g y_e).
\]

From Lemma 4.8, the source of this map is isomorphic to

\[
\prod_{k: c \to e} \left( \prod_{h: b \to d} F_d(F_h x, y) \right),
\]

so it suffices to map out of each \( F_d(F_h x, y) \). For a fixed \( k: c \to e \) and \( h: b \to d \) such that \( kf = gh \), take the composite:

\[
\begin{align*}
F_d(F_h x, y) & \xrightarrow{F_g} F_e(F_g F_h x, F_g y) \\
& \cong \circ_{\theta_x} F_e(F_{gh} x, F_g y) \\
& \cong \circ_{\theta_{F_{gh}}^{-1}} F_e(F_{kf} x, F_g y)
\end{align*}
\]

(20)

Now that we have constructed the proposed left adjoint, we need to check that it is indeed adjoint to the canonical functor \( i \). In other words, we need to verify that for \( x_b \in GrF \) and \( (y_c, g, d) \in p\downarrow B_V \) we have a \( V \)-isomorphism

\[
GrF(F_g y_d, x_b) \cong p\downarrow B_V ((y, g, d), (x, 1_b, b)).
\]
On the left, we have \( \prod_{h : d \to b} F_b(F_h F_g y, x) \) while on the right we have, by Lemma 4.8, something isomorphic to

\[
\prod_{h : d \to b} \left( \prod_{k : c \to b} F_b(F_k y, x) \right).
\]

Since \( B \) is an ordinary category there is exactly one \( k \) which is equal to each composite \( h \circ g \), so this last term reduces to

\[
\prod_{h : d \to b} F_b(F_h g y, x) \cong \prod_{h : d \to b} F_b(F_h F_g y, x),
\]

where the isomorphism follows from pseudofunctoriality of \( F \).

**Remark 4.12.** Note that we require \((B\mathbf{V})_0 = B\) in order to define \( \ell \dashv i \): we need a functor \( F_f \) for all \( f \in (B\mathbf{V})_0(b, c) \), but the pseudofunctor \( F : B \to \mathcal{C}at_{\mathbf{V}} \) only gives \( F_f \) for \( f \in B(b, c) \). Imposing the condition that \((B\mathbf{V})_0(b, c) = B(b, c)\) resolves this difficulty.

It will be useful to have explicit formulas for the unit and counit of \( \ell \dashv i \), so we describe them below. We still assume that we are beginning with a pseudofunctor \( F : B \to \mathcal{C}at_{\mathbf{V}} \) and a projection \( p : Gr F \to B \mathbf{V} \) with associated adjunction \( \ell \dashv i : p \downarrow B \mathbf{V} \to Gr F \) as in Proposition 4.11.

**Definition 4.13 (The unit \( \eta \)).** The unit \( \eta \) is induced by the identity natural transformation of each \( F_f \), and has components

\[
\eta(x_b, f, c) : 1 \xrightarrow{1_{F_f x}} F_c(F_f x, F_f x)
\]

landing in the \( h = 1_c \) part of the coproduct

\[
\prod_{h : c \to c} F_c(F_h f x, F_f x) = \prod_{h : c \to c} \left( \prod_{k : b \to c \atop h_f = 1_c k} F_c(F_k x, F_f x) \right) 
\]

\[ \cong p \downarrow B \mathbf{V}( (x_b, f, c), (F_f x_c, 1_c, c) ), \]

where the last isomorphism follow from Lemma 4.8.

**Remark 4.14.** Note that although \( \eta(x_b, f, c) \) was induced by an isomorphism (in fact an *identity*) in \( F_c \), it need not be an isomorphism in \( p \downarrow B \mathbf{V} \), since this requires \( f \) to have an inverse for a reason similar to that in Lemma 4.5.
Definition 4.15 (The counit $\varepsilon$). The counit $\varepsilon$ is induced by $\theta$ and $\xi$, and has components

$$\varepsilon_{xb} : 1 \to F_b(F_1(x), x) \xrightarrow{-\circ \theta_{F_1}^{-1}} F_b(F_1x, x)$$

landing in the $h = 1_b$ part of the coproduct

$$\coprod_{h : b \to h} F_b(F_hF_1x, x) = \text{Gr}F(F_1x, x).$$

One can check that $\varepsilon$ and $\eta$ satisfy the triangle identities for adjunctions, but the check is extremely long and tedious, so we omit it. However, we may compactly describe these identities as saying that the following composites are equal to identities:

$$1 \xrightarrow{\eta \otimes \varepsilon} p_bB_V(\iota x_b, i\ell x_b) \otimes p_bB_V(i\ell x_b, i\iota x_b)$$

$$1 \xrightarrow{\ell \eta \otimes \varepsilon} \text{Gr}F(F_{f\iota}x, F_1F_{fx}x) \otimes \text{Gr}F(F_{\iota x}, F_{fx}x)$$

4.5 Extension to a 2-functor

Theorem 4.16. The construction $F \rightsquigarrow \text{Gr}F$ extends to a 2-functor

$$\text{Gr} : \text{Fun}^{ps}(B, \text{Cat}_V) \to \text{OpFib}(B_V).$$

Proof. From Proposition 4.11 we have that the construction takes pseudofunctors to opfibrations. It remains to show that it takes pseudonatural transformations to opfibered functors, and modifications to natural transformations of opfibered functors.

Let $\alpha : F \Rightarrow G$ be a pseudonatural transformation between pseudofunctors $F, G : B \to \text{Cat}_V$. In particular, for every $b \in B$ we have a $V$-functor $\alpha_b : F_b \to G_b$ such that for any $f : b \to c$ in $B$ we have an invertible 2-cell $\alpha_f : G_f\alpha_b \simeq \alpha_c F_f$ with components

$$\alpha_{f,x} : 1 \to G_c(G_f\alpha_bx, \alpha_c F_fx).$$ (22)
Define an opfibered functor $a: \text{Gr}F \to \text{Gr}G$ as follows: on objects,

$$a x_b := \alpha_b x \in G_b.$$  

On morphisms, take the composite:

$$\text{Gr}F(x_b, y_c) \xrightarrow{a_{x_b, y_c}} \text{Gr}G(a x_b, a y_c)$$

$$\coprod_{f: b \to c} F_c(F_f x, y) \xrightarrow{\alpha_c} \coprod_{f: b \to c} G_c(\alpha_c F_f x, \alpha_c y) \xrightarrow{-\alpha f, x} \coprod_{f: b \to c} G_c(G_f \alpha_b x, \alpha_c y)$$

(23)

Pseudofunctoriality of $F$ and $G$, along with pseudonaturality of $\alpha$, immediately indicate that these data assemble into a $\mathbb{V}$-functor $\text{Gr}F \to \text{Gr}G$ and that this $\mathbb{V}$-functor is compatible with the opfibrations $p: \text{Gr}F \to B_\mathbb{V}$ and $q: \text{Gr}G \to B_\mathbb{V}$ (i.e. $qa = p$). However, we must check that $a$ is actually opfibered in the sense of Definition 3.10.

From §3.2, there is an induced $\mathbb{V}$-functor $A: p\downarrow B_\mathbb{V} \to q\downarrow B_\mathbb{V}$ which is $A(x_b, f, c) = (\alpha_b x, f, c)$ on objects. From Definition 3.10, we need to show that the following three-fold composite of 2-morphisms is an isomorphism, where the center square commutes:

$$\begin{bmatrix}
\text{Gr}F & \xrightarrow{a} & \text{Gr}G \\
\downarrow{\eta_p} & & \downarrow{i_p} \\
\text{Gr}F & \xrightarrow{\varepsilon_a} & \text{Gr}G \\
\end{bmatrix}$$

(24)

We can decompose the above diagram into the vertical composition of three whiskered 2-morphisms:

$$\begin{bmatrix}
\text{Gr}F & \xrightarrow{a} & \text{Gr}G \\
\downarrow{i_q} & & \downarrow{\varepsilon_q} \\
\text{Gr}F & \xrightarrow{\varepsilon_a} & \text{Gr}G \\
\end{bmatrix}$$

$$\begin{bmatrix}
\text{Gr}F & \xrightarrow{a} & \text{Gr}G \\
\downarrow{i_p} & & \downarrow{i_q} \\
\text{Gr}F & \xrightarrow{\varepsilon_a} & \text{Gr}G \\
\end{bmatrix}$$

$$\begin{bmatrix}
\text{Gr}F & \xrightarrow{a} & \text{Gr}G \\
\downarrow{i_p} & & \downarrow{i_q} \\
\text{Gr}F & \xrightarrow{\varepsilon_a} & \text{Gr}G \\
\end{bmatrix}$$
The components of the top diagram are the components of the counit $\varepsilon_q$ at $a\ell_p(x_b,f,c) = \alpha_c F_f x_c$:

$$\varepsilon_{q,\alpha_c F_f x_c} : 1 \to GrG(G_{1_c} \alpha_c F_f x_c, \alpha_c F_f x_c)$$  \hspace{1cm} (25)

The components of the bottom diagram are given by the composite:

$$1 \xrightarrow{\eta_p, (x_b,f,c)} GrG(G_{\alpha_b x_c}, G_{1_b} \alpha_c F_f x_c) \xrightarrow{\ell_q} GrG((\alpha_b x_b, f, c), (\alpha_c F_f x_c, 1_c, c))$$

$$\xrightarrow{\ell_p, (x_b,f,c)} GrF$$

$$\xrightarrow{\eta_p} p\downarrow B V$$

$$\xrightarrow{\ell_p} GrF$$

The components of (24), which we want to show are isomorphisms, are thus the composites of these two morphisms.

By Lemma 3.9 the components of the counit $\varepsilon_q$ are isomorphisms, so the morphism in (25) is an isomorphism. Thus, it suffices to show that the morphism in (26) is an isomorphism as well. By making the relevant substitutions from (21), (23) and (20) in that order, we see that this corresponds to the map

$$1 \xrightarrow{\alpha_f, x} G_c(G_{\alpha_b x_c}, \alpha_c F_f x) \xrightarrow{G_{1_c}} G_c(G_{1_c} G_f \alpha_b x_c, G_{1_b} \alpha_c F_f x)$$

treated as a map in the $1_c$ component of $GrG(G_{f \alpha_b x_c}, G_{1_b} \alpha_c F_f x_c)$.

Since $\alpha_f, x$ from (22) is an isomorphism and $G_{1_c}$ is a functor, this composite is an isomorphism in $G_c$ in the component corresponding to the isomorphism $1_c$ in $B$. By Lemma 4.5, it is thus also an isomorphism in $GrG$. This completes the proof that $Gr$ takes pseudonatural transformations to opfibered functors. It remains to show that $Gr$ takes modifications of pseudonatural transformations to $V$-enriched transformations of opfibered functors.

Let $\Gamma : \alpha \Rightarrow \beta : F \Rightarrow G$ be a modification of pseudonatural transformations. We need to produce a $V$-enriched natural transformation $Gr\Gamma : Gr\alpha \Rightarrow Gr\beta$. In particular, we need to produce a natural collection of maps in $V$ of the form $1 \to GrG(Gr\alpha(x_b), Gr\beta(x_b)) = GrG(\alpha_b x, \beta_b x)$ for all $x_b \in GrF$. Recall that a modification $\Gamma : \alpha \Rightarrow \beta$ includes the data of, for each $b$ in $B$,
a natural transformation $\alpha_b \Rightarrow \beta_b$ of functors $\alpha_b, \beta_b : F_b \rightarrow G_b$. It is almost immediate, in light of the above manipulations, that this data induces the needed natural transformation.

5 The Grothendieck Correspondence

In this section we show that the Grothendieck construction of §4 and the inverse Grothendieck construction of §3.5 do behave as inverses when the base category $B$ is of the form $B_V$. Throughout, we make the same assumptions as §4, including the identification $B = (B_V)_0$ from Remark 4.2.

5.1 $Gr \circ I$

We first prove some properties of opfibrations over an arbitrary $B$, before specializing to opfibrations over $B_V$.

Lemma 5.1. Let $p : E \rightarrow B$ be an opfibration. For each $f \in B_0(pe, pe')$, we have

$$E_f(e, e') \cong E_{pe'}(f!e, e')$$

where the category $E_{pe'}$ is the fiber of $p$ over $pe$ and $E_f(e, e')$ is the pullback:

$$\begin{array}{c}
E_f(e, e') \\
\downarrow \\
1 \\
\downarrow \\
B(pe, pe')
\end{array}$$

$$\begin{array}{c}
E(e, e') \\
p \\
\downarrow \\
p
\end{array}$$

Proof. By Proposition 3.5 and (13), we have a composite of pullbacks:

$$\begin{array}{c}
E_{pe'}(f!e, e') \\
\downarrow \\
1 \\
\downarrow \\
B(pe', pe')
\end{array}$$

But the outer cospan is also the defining cospan for the pullback $E_f(e, e')$, hence these two pullbacks are isomorphic.

Lemma 5.2. Let $p : E \rightarrow B$ be an opfibration, and let $q : F \rightarrow (B_0)_V$ be the opfibration that results from applying $I$ and $Gr$ to $p$. Then $F$ fits into the
where \( \sigma \) is the counit of the adjunction \((\text{1})\). 

**Proof.** Note that \((B_0)_V\) and \(B\) have the same objects and \(\sigma_B\) is the identity on objects. Morphisms of \((B_0)_V\) are given by

\[
(B_0)_V(b, b') = \coprod_{f: 1 \to B(x, y)} 1,
\]

and \(\sigma_B\) is the coproduct of the individual maps \(f: 1 \to B(x, y)\), i.e.

\[
\sigma_B = \coprod_{f: 1 \to B(x, y)} f.
\]

The pullback of \(p\) along \(\sigma_B\) is thus given by a category \(F'B\) with the same objects as \(E\) and morphisms fitting into the pullback:

\[
F'(e, e') \rightarrow \coprod_{f: 1 \to B(pe, pe')} 1 \rightarrow E(pe, pe')(f, e, e') \rightarrow E(e, e').
\]

Since pullbacks preserve coproducts, we obtain an isomorphism

\[
F'(e, e') \cong \coprod_{f: 1 \to B(pe, pe')} E_f(e, e') \cong \coprod_{f: 1 \to B(pe, pe')} E_{pe'}(f, e, e') = F(e, e'),
\]

where the second isomorphism is given by the previous Lemma. So \(F \cong F'\), and is thus also a pullback (and the functor \(F \to E\) is the composite of this isomorphism with \(F' \to E\)).

**Proposition 5.3.** Let \(p: E \to B_V\) be an opfibration, and let \(q: F \to B_V\) be the opfibration that results from applying \(I\) and \(Gr\) to \(p\). Then \(p\) and \(q\) are isomorphic opfibrations.
Proof. This follows almost immediately from the previous Lemma and the identification \((B_V)_0 = B\) from Remark 4.2.

In detail, setting \(B = B_V\) for some category \(B\), we have
\[
((B_V)_0)_V = B_V
\]
so that \(\sigma_{B_V}\) is 1\(_{B_V}\). By the previous Lemma, \(q: F \to B_V\) is a pullback of \(p: \mathcal{E} \to B_V\) along an identity, hence is isomorphic to \(p\).

\[\square\]

Remark 5.4. In fact, as long as pullback preserve coproducts, any functor into a free \(V\)-category \(p: \mathcal{E} \to B_V\) yields a coproduct decomposition
\[
\mathcal{E}(e, e') \cong \bigcoprod_{f \in B(pe, pe')} \mathcal{E}_f(e, e')
\]
simply by pulling \(p\) back along 1\(_{B_V}\). This does not require \(p\) to be an opfibration, nor \((B_V)_0 = B\).

5.2 \(I \circ Gr\)

Proposition 5.5. Let \(F: B \to \text{Cat}_V\) be a pseudofunctor and let \(G: B \to \text{Cat}_V\) be the pseudofunctor obtained by applying \(Gr\) and \(I\) to \(F\). Then \(F\) and \(G\) are naturally isomorphic.

Proof. We first produce functors \(\alpha_b: F_b \to G_b\) for each \(b \in B\), and show that each \(\alpha_b\) is an isomorphism of categories.

Since \(F_b\) and \(G_b\) have the same objects, we may take \(\alpha_b\) to be the identity on objects. The hom-objects \(G_b(x, y)\) are precisely the \(f = 1_b\) part of
\[
GrF(x_b, y_b) = \bigsqcup_{f: b \to b} F_b(F_f x, y).
\]

We then take \(\alpha_b\) on morphisms to be the composite
\[
F_b(x, y) \xrightarrow{-\xi_x \cong} F_b(F_1_b x, y) = G_b(x, y).
\]
This yields an isomorphism of categories \(\alpha_b: F_b \cong G_b\).

Next, for each \(f: b \to c\), we need an invertible \(\alpha_f:\)

\[
\begin{array}{c}
F_b \xrightarrow{F_f} F_c \\
\alpha_b \downarrow \quad \alpha_f \downarrow \cong \quad \alpha_c \\
G_b \xrightarrow{G_f} G_c
\end{array}
\]

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In fact, we may take $\alpha_f$ to be the identity. To see this, first note that $G_f$ is given by

$$G_f x = \ell(x_b, f, c) = F_f x$$
on objects and

$$G_b(x, y) \xrightarrow{G_f} G_c(F_f x, F_f y)$$
on morphisms. Expressing both $\alpha_c F_f$ and $G_f \alpha_b$ in terms of $\xi$ and $\theta$, we see that the relevant diagram is given by

$$\begin{array}{ccc}
1 & \xrightarrow{\xi_{F_f x}} & F_c(F_{1c} F_f x, F_f x) \\
\downarrow & & \uparrow_{\sim} \theta(1_c, f) \\
F_b(F_{1b} x, x) & \xrightarrow{F_f} & F_c(F_f F_{1b} x, F_f x) & \xrightarrow{\sim} F_c(F_f x, F_f x)
\end{array}$$

(27)
tensored with $F_b(x, y) \xrightarrow{F_f} F_c(F_f x, F_f y)$, then applying composition in $F_c$. The upper arrow $\xi_{F_f x}$ gives rise to $\alpha_c F_f$, while the lower composite

$$\theta(1_c, f) \theta(f, 1_c)^{-1} F_f \xi_x$$
gives rise to $G_f \alpha_b$, so that $\alpha_c F_f = G_f \alpha_b$ if (27) commutes.

This is the case, because the following diagram commutes (both composites being equal to $1_{F_f x}$ by (4) and (5))

$$\begin{array}{ccc}
1 & \xrightarrow{\xi_{F_f x}} & F_c(F_{1c} F_f x, F_f x) \\
\downarrow & & \uparrow_{\sim} \theta(1_c, f)^{-1} \\
F_b(F_{1b} x, x) & \xrightarrow{F_f} & F_c(F_f F_{1b} x, F_f x) & \xrightarrow{\sim} F_c(F_f x, F_f x)
\end{array}$$

and $\theta(1_c, f)$ and $\theta(1_c, f)^{-1}$ are mutual inverses. Thus $F \cong G$. \qed

Putting Propositions 5.3 and 5.5 together, we obtain:

**Theorem 5.6.** Let $V$ satisfy the assumptions in §4, and let $B$ be a category. There is a 2-equivalence

$$\text{OpFib}(B_V) \cong \text{Fun}^{ps}(B, \text{Cat}_V).$$
Proof. We require natural isomorphisms from $Gr \circ I$ and $I \circ Gr$ to the respective identity functors on $\text{OpFib}(B_V)$ and $\text{Fun}^pt(B, \text{Cat}_V)$. These are supplied in the previous two propositions.

We have thus proved an enriched version of the Grothendieck construction/correspondence when $V$ satisfies the assumptions in §4, which yields the classical result by Grothendieck when $V = \text{Set}$.

However, this is somewhat unsatisfactory for reasons we have mentioned in the Introduction. First, requiring that 1 is terminal and $B \cong (B_V)_0$ seems rather restrictive, ruling out examples such as $V = \text{Vect}_k$. Next, even when these conditions apply, such as when $V = \text{Top}, s\text{Set}$ or $\text{Cat}$, this result really only considers opfibrations over a ‘discrete’ base $B_V$. Subsequent work will involve removing various assumptions on $V$ and retaining more $V$-structure from an opfibration over a non-discrete base.

A Appendix

We begin by recalling some basic information about (small) $V$-categories (all of which can be found in [13, Ch. 3], or [8]). Throughout, $V$ will denote a locally small monoidal category with tensor product $\otimes : V \times V \to V$ and unit 1.

A.1 The 2-category $\text{Cat}_V$

Definition A.1. A $V$-category $C$ is the data of:

1. A set of objects, which we will denote by $C$ or $\text{Ob}(C)$, where the former is an obvious abuse of notation.
2. For every pair of objects $c, d \in C$, an object $C(c, d)$ of $V$.
3. For every object of $C$ a morphism $1_c : 1 \to C(c, c)$ in $V$.
4. For each triple of objects $c, d, e$ in $C$, a morphism in $V$,

$$\circ_{c,d,e} : C(d, e) \otimes C(c, d) \to C(c, e).$$

We will omit subscripts on $\circ$ when it is clear from context.

Where the following means ‘$c, d \in \text{Ob}(C)$’.
All of which causes the following diagrams to commute in $\mathbf{V}$:

![Diagram](image)

**Definition A.2.** A functor of $\mathbf{V}$-categories, or $\mathbf{V}$-functor, $F: \mathcal{C} \to \mathcal{D}$ consists of a function $F: \text{Ob}(\mathcal{C}) \to \text{Ob}(\mathcal{D})$, and for all $c, d \in \mathcal{C}$ a $\mathbf{V}$-morphism

$$F_{c,d}: \mathcal{C}(c, d) \to \mathcal{D}(Fc, Fd)$$

such that the following diagrams commute in $\mathbf{V}$:

![Diagram](image)

When it is clear from context, we may omit the subscripts in $F_{c,d}$, and use $F$ for the functor $\mathcal{C} \to \mathcal{D}$, the function $\text{Ob}(\mathcal{C}) \to \text{Ob}(\mathcal{D})$ and the $\mathbf{V}$-morphism $\mathcal{C}(c, d) \to \mathcal{D}(Fc, Fd)$.

**Definition A.3.** Let $F, G: \mathcal{C} \to \mathcal{D}$ be $\mathbf{V}$-functors. A natural transformation of $\mathbf{V}$-functors $\alpha: F \Rightarrow G$ is a family of $\mathbf{V}$-morphisms $\alpha_c: 1 \to \mathcal{D}(Fc, Gc)$ for each $c \in \mathcal{C}$ such that the following diagram commutes in $\mathbf{V}$:

![Diagram](image)
Definition A.4. Let $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$ be natural transformations, where $F, G, H : C \to D$. Their vertical composite is denoted $\beta \cdot \alpha : F \Rightarrow H$, and has components $(\beta \cdot \alpha)_c$ given by

$$1 \cong 1 \otimes 1 \xrightarrow{\beta_c \otimes \alpha_c} D(Gc, Hc) \otimes D(Fc, Gc) \xrightarrow{\circ} D(Fc, Hc).$$

Definition A.5. Given $\alpha : F \Rightarrow G$ and $\beta : J \Rightarrow K$ as follows

\[
\begin{array}{ccc}
B & \xrightarrow{F} & C \\
\downarrow{\psi} & & \downarrow{\psi} \\
G & \xrightarrow{\beta} & D
\end{array}
\]

their horizontal composite $\beta \circ \alpha : JF \Rightarrow KG$, or simply $\beta \alpha$, has components $(\beta \alpha)_x$ given by the composite:

\[
\begin{array}{ccc}
B(Fx, Gx) & \xrightarrow{J} & \mathcal{C}(JFx, JGx) \\
\downarrow{\alpha_x} & & \downarrow{\cong} \\
1 & \xrightarrow{(\beta \alpha)_x} & \mathcal{C}(JFx, KGx) \\
\downarrow{1 \otimes \beta Gx} & & \downarrow{\circ} \\
\mathcal{C}(JFx, KGx) & \xrightarrow{\circ} & \mathcal{C}(JFx, KGx) \otimes \mathcal{C}(Jgx, Kgx)
\end{array}
\]

When either $\alpha$ or $\beta$ is an identity natural transformation, the horizontal composite is called whiskering. In detail, given functors $F, G, H, K$, and a natural transformation $\alpha$ fitting into the following diagram,

\[
\begin{array}{ccc}
B & \xrightarrow{H} & C \\
\downarrow{\psi} & & \downarrow{\psi} \\
G & \xrightarrow{\alpha} & D \\
\downarrow{\alpha} & & \downarrow{\circ} \\
C & \xrightarrow{} & \mathcal{E}
\end{array}
\]

we write $K \circ \alpha \circ H$, or simply $K\alpha H$, for the natural transformation $KFH \Rightarrow KH$ whose components for each $b \in B$ are given by the composite

$$(K\alpha H)_b : 1 \xrightarrow{\alpha_H b} \mathcal{D}(FHb, GHb) \xrightarrow{K} \mathcal{E}(KFHb, KGHb).$$

The above data assemble into a strict 2-category of $\mathbf{V}$-categories which we denote by $\mathbf{Cat}_\mathbf{V}$.

A.2 Comma categories

The classical definition of a comma category assumes that we have sets of homomorphisms between objects in a category. Comma categories in
the enriched setting require a bit more work, because we have to distinguish between the $V$-object $C(c, d)$ and the set $C_0(c, d)$. The following is motivated by the definition of comma objects in an arbitrary 2-category $K$ as found in [14], and the characterization of comma categories in [12]. A similar treatment can also be found in [15, §A.4] for slice categories, when one of $A$ or $B$ is terminal.

**Definition A.6.** [14, §1] Given $V$-functors $A \overset{F}{\to} C \overset{G}{\leftarrow} B$, the *comma category* is a $V$-category $F\downarrow G$ equipped with $V$-functors $B \overset{H}{\leftarrow} F\downarrow G \overset{K}{\to} A$ and a $V$-natural transformation $\varphi: FK \Rightarrow GH$

$$
\begin{array}{c}
F\downarrow G \xrightarrow{K} A \\
H \downarrow \notag \varphi \downarrow F \\
B \xrightarrow{G} C
\end{array}
$$

that has the following universal property:

1. Given any other diagram

$$
\begin{array}{c}
\mathcal{D} \xrightarrow{\mathcal{D}} A \\
H' \downarrow \notag \psi \downarrow F \\
B \xrightarrow{G} C
\end{array}
$$

there exists a unique $J: \mathcal{D} \to F\downarrow G$ such that: $KJ = K'$, $HJ = H'$ and

$$
\begin{array}{c}
\mathcal{D} \xrightarrow{\mathcal{D}} A \\
H' \downarrow \notag \psi \downarrow F \\
B \xrightarrow{G} C
\end{array} = \begin{array}{c}
\mathcal{D} \xrightarrow{\mathcal{D}} \mathcal{B} \\
H' \downarrow \notag \psi \downarrow F \\
A \xrightarrow{G} C
\end{array}
$$

2. Given $J, J': \mathcal{D} \to F\downarrow G$, $\xi: HJ \Rightarrow HJ'$ and $\eta: KJ \Rightarrow KJ'$ such that

$$
\begin{array}{c}
\mathcal{D} \xrightarrow{\mathcal{D}} A \\
H' \downarrow \notag \psi \downarrow F \\
B \xrightarrow{G} C
\end{array} = \begin{array}{c}
\mathcal{D} \xrightarrow{\mathcal{D}} B \\
H' \downarrow \notag \psi \downarrow F \\
A \xrightarrow{G} C
\end{array}
$$
there exists a unique $\rho: J \Rightarrow J'$ such that $\xi = H\rho$ and $\eta = K\rho$, so that both diagrams above are equal to

\[
\begin{array}{ccc}
D & \xrightarrow{J} & J' \\
\downarrow{\rho} & & \downarrow{\phi} \\
F \downarrow{G} & \xrightarrow{K} & B \\
\downarrow{H} & & \downarrow{F} \\
A & \xrightarrow{\psi} & C
\end{array}
\]

**Remark A.7.** A more succinct way of expressing the universal property of $F \downarrow G$ is that there is an isomorphism of categories

\[\mathbf{Cat}_V(D, F \downarrow G) \cong \mathbf{Cat}_V(D, F) \downarrow \mathbf{Cat}_V(D, G),\]

where on the right we have the usual comma category in $\mathbf{Cat}$. These are sometimes called strict comma categories. However, as these are the only kinds of comma categories we consider, we will omit ‘strict’.

**Remark A.8.** A concrete description of $F \downarrow G$ may be given as follows: the objects are

\[
\text{Ob}(F \downarrow G) := \coprod_{(a,b) \in A \times B} \{a\} \times C_0(Fa, Gb) \times \{b\} = \left\{ (a, f, b) \mid a \in A, b \in B, f: 1 \to C(Fa, Gb) \right\}.
\]

For brevity, we sometimes write $f$ for $(a, f, b)$. The morphisms between $(a, f, b)$ and $(a', f', b')$ are given by the pullback in $V$:

\[
\begin{array}{ccc}
(F \downarrow G)(f, f') & \xrightarrow{K_{f,f'}} & A(a, a') \\
\downarrow{H_{f,f'}} & & \downarrow{F_{a,a'}} \\
B(b, b') & \xrightarrow{G_{b,b'}} & C(Gb, G'b')
\end{array}
\]

The functor $K$ is given by $(a, f, b) \mapsto a$ on objects and $K_{f,f'}$ on morphisms, while $H$ is $(a, f, b) \mapsto b$ on objects and $H_{f,f'}$ on morphisms. We thus have
\[FK(a, f, b) = Fa \text{ and } GH(a, f, b) = Gb, \text{ and the components of the natural}
\text{transformation } \varphi: FK \Rightarrow GH \text{ at }(a, f, b) \text{ are precisely } f:
\]
\[\varphi(a, f, b) = f: 1 \rightarrow \mathcal{C}(Fa, Gb).\]

Identity morphisms and composition of morphisms are induced by the universal property of the pullback:

\[\begin{array}{ccc}
1 & \xrightarrow{1_b} & \mathcal{A}(a, a) \\
\downarrow & & \downarrow \\
\mathcal{B}(b, b) & \rightarrow & \mathcal{C}(Fa, Gb)
\end{array}\]

\[\begin{array}{ccc}
(F \downarrow G)(f, f) & \rightarrow & \mathcal{A}(a, a) \\
\downarrow & & \downarrow \\
\mathcal{B}(b, b) & \rightarrow & \mathcal{C}(Fa, Gb)
\end{array}\]

\[\begin{array}{ccc}
(F \downarrow G)(f', f'') \otimes (F \downarrow G)(f, f') & \rightarrow & \mathcal{A}(a', a'') \otimes \mathcal{A}(a, a') \\
\downarrow & & \downarrow \\
\mathcal{B}(b', b'') \otimes \mathcal{B}(b', b') & \rightarrow & \mathcal{C}(Fa, Gb'')
\end{array}\]

The commutativity of this last outer square, for example, follows from associativity of composition, functoriality of \(F\) and \(G\), and the commutativity of the squares defining \((F \downarrow G)(f', f'')\) and \((F \downarrow G)(f, f')\).

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