On symmetrical deformation of toroidal shell with circular cross-section

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By introducing a variable transformation \( \xi = \frac{1}{2} (\sin \theta + 1) \), the complicated deformation equation of toroidal shell is successfully transferred into a well-known equation, namely Heun’s equation of ordinary differential equation, whose exact solution is obtained in terms of Heun’s functions. The computation of the problem can be carried out by symbolic software that is able to with the Heun’s function, such as Maple. The geometric study of the Gauss curvature shows that the internal portion of the toroidal shell has better bending capacity than the outer portion, which might be useful for the design of metamaterials with toroidal shell cells.

Keywords: toroidal shell, deformation, Gauss curvature, Heun’s function, hypergeometric function, Maple

Among of most regular shell, such as circular cylindrical shell, conical shell, spherical shell and toroidal shell, the deformation of toroidal shell is one of upmost difficulty problem due to its complicated topology. Up to date, its exact solution has not been obtained yet.

Toroidal shell, in full or partial geometric form as shown in Figures 1 and 2, is widely used in structural engineering and have been extensively investigated [1-72].

FIG. 1: Toroidal shell and cross-sectional view.

FIG. 2: Toroidal shell and geometry. The principal radius of curvature \( R_1 = a \) and \( R_2 = a + \frac{R}{\theta} \); the principal curvature \( K_1 = \frac{1}{\sin \theta} \), \( K_2 = \frac{\sin \theta}{R \sin \theta} \); the Gauss curvature \( K = K_1 K_2 = \frac{\sin \theta}{\pi (1 + \sin \theta)} \).

The Gauss curvature \( K \) changes its sign as principal radius of curvature \( R_\theta \) when the angle \( \theta \) goes from 0 to \( 2\pi \), it means that Gauss curvature has a turning point, namely \( K = 0 \) at \( \theta = \pi \) as shown in Fig. 3. The exist of the turning point in a complete toroidal shell is one source of the difficulty to find a solution.

FIG. 3: The curves of \( RaK = \frac{\sin \theta}{1 + \sin \theta} \), where \( \varepsilon = Ra \). The geometry of toroidal shell surface is elliptic in \( \theta \in [0, \pi] \), parabolic at \( \theta = 0 \) and hyperbolic \( \theta \in [0, -\pi] \).

Since the bending capacity of the shell is proportional to the Gauss curvature, the Fig.3 reveals that the internal part of toroidal shell \((\theta \in [\pi, 2\pi])\) is much stronger than outer part of the shell \((\theta \in [0, \pi])\). Therefore, for given amount of materials, to construct a high bending performance toroidal shell, the internal toroidal shell topology is the best choice. This secret mechanical performance might be useful to the design of metamaterials with toroidal shell cell.

To attack the problem, the high order and complicated governing equations of toroidal shell under symmetric loads is reduced to a single equation of lower order complex equation. The complex form governing equations of toroidal shell of revolution were formulated firstly by Reissner (1912)[1] and later finalized by Novozhilov

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In this paper, we will shoulder this historical re-

Nevertheless, some kind solutions have been proposed

To provide practical solution, thus a various of asym-

To find an exact solution that can be expressed in a

Wissler’s series solution has little practical value due to

beside the turning point issue, it is clear that the chal-

Heun’s equation is an extension of the

and

Nevertheless, some kind solutions have been proposed

the turning point of the Gauss curvature makes al-

Up to date, no exact solution in terms of special func-

Once we obtain the solution in terms

special function is still a open question eben after more

The first exact series solution of toroidal

The exact solution of Eq.1 is given by Wissler [3].

hence Eq.1 can be transferred into

The Eq.4 is a Fuchian type differential equation that

The Heun functions (named after Karl Heun:1859-

The Heun functions generalize the hypergeometric

The exact solution of Eq.4 is summation of homo-

The homogenous solution can be given as follows

where

\begin{align}
\frac{dV}{dx} &= \cos \theta \frac{d\theta}{d\theta} \frac{d^2V}{d\theta^2} + \epsilon \frac{d^2V}{d\theta^2} + 2id^2xV = P(x), \quad (2)
\end{align}

\begin{align}
(1 + \epsilon \sin \theta) \frac{d^2V}{d\theta^2} - \epsilon \cos \theta \frac{dV}{d\theta} + 2id^2 \sin \theta V &= P(\theta), \quad (1)
\end{align}

\begin{align}
\xi = \frac{1}{2} x + \frac{1}{2} = \frac{1}{2} (\sin \theta + 1),
\end{align}

\begin{align}
\xi(\xi - 1)\left[\xi - \left(1 - \frac{1}{2\epsilon} \right)\frac{d^2V}{d\xi^2} + \frac{1}{2\epsilon} \left(\xi - \frac{1}{2} + \frac{1}{2} \epsilon\right) \frac{dV}{d\xi}ight] + \left(-\frac{4i\epsilon d^2}{\epsilon} + \frac{i\epsilon d^2}{\epsilon}\right)V = P(\xi),
\end{align}

\begin{align}
(1 - \epsilon^2)(1 + \epsilon^2)x^2 \frac{d^2V}{dx^2} - (x + \epsilon) \frac{dV}{dx} + 2id^2xV = P(x),
\end{align}

\begin{align}
y_1(x) &= \text{HeunG} \left( \frac{\epsilon - 1}{2\epsilon}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; (x + 1) \right),
\end{align}

\begin{align}
y_2(x) &= \text{HeunG} \left( \frac{\epsilon - 1}{2\epsilon}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; (x + 1) \right).
\end{align}
where the general Heun’s function HeunG can be easily computed by well-know symbolic software package Maple [76].

The particular solution $V^p$ can be easily obtained by Maple as well, however in here we are not going to list the particular solution $V^p$ due to its long and complicated expression.

It must point out here that although we can find the particular solution by Maple, however, its analytical expression can not be obtained due to the fact that integration of the Heun’s function can not be expressed in any special functions, which unfortunately decreases the value of the complex form of toroidal shell. Nevertheless, approximation and or numerical treatment of the particular solution can still be possible because it have been obtained and represented by the Heun’s function even in its integration form.

This research has confirmed that the deformation of all regular shell structure, such cylindrical shell, conical shell, spherical shell and toroidal shell can be solved by hypergeometric functions. This supports the doctrine of Zurich school of shell theory [1, 2, 11], which predicted that bending deformation of all regular shells can be expressed by the hypergeometric functions.

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