ON COIDEAL SUBALGEBRAS OF COCENTRAL KAC ALGEBRAS AND A GENERALIZATION OF WALL’S CONJECTURE

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Abstract. It shown that any coideal subalgebra of a finite dimensional Hopf algebra is a cyclic module over the dual Hopf algebra. Using this we describe all coideal subalgebras of a cocentral abelian extension of Hopf algebras extending some results from [4].

1. Introduction

Kac algebras were among the first given examples of noncommutative noncocommutative Hopf algebras [5]. They are also called abelian extensions since they satisfy the following long exact sequence of Hopf algebras:

\[(1.1) \quad \mathbb{k} \to \mathbb{k}G \to A \to kF \to \mathbb{k}\]

where \(G\) and \(F\) are finite groups and \(kG\) is the dual Hopf algebra of the group algebra \(kG\). Irreducible representations of abelian extensions are completely characterized in [7].

Our main result describes all left (right) coideal subalgebras of cocentral Kac algebras:

**Theorem 1.2.** Let \(A = k^G \tau \# kF\) be an abelian cocentral extension of \(kF\) by \(k^G\). Then left (right) coideal subalgebras of \(A\) are parameterized by the following data:

1. Two subgroups \(M \leq G\) and \(H \leq F\) with \(H \triangleright M = M\)
2. A twisted bicharacter \(\lambda : M \times H \to U(1)\) satisfying the following properties

\[(1.3) \quad \lambda(ab, h) = \lambda(a, h)\lambda(b, h)\tau_h(a, b)\]

\[(1.4) \quad \lambda(a, hl) = \lambda(a, h)\lambda(h_1 \triangleright a, l)\sigma(a, h, l)\]

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The coideal subalgebra corresponding to the triple \((M, H, \lambda)\) is given by

\[ C(M, H, \lambda) := \bigoplus_{h \in H} C_\lambda(h) \#_\sigma h, \]

where the space \(C_\lambda(h)\) is defined by

\[ C_\lambda(h) := \{ f \in k^G \mid R_{m, \tau_h}(f) = \lambda(m, h)f \text{ for all } m \in M \}. \]

Here \(\triangleright\) is the induced action of \(F\) on \(G\). For the definition of the operator \(R_{m, \tau_h}\) see Equation 3.9.

In [4] the authors described all left (right) coideal subalgebras of Kac algebras of Izumi-Kosaki type. Using this they proved one of the Wall’s conjecture for these algebras in the case that both finite groups \(F\) and \(G\) are solvable. These Kac algebras are introduced in [8] and are Hopf algebras with an additional \(\mathbb{C}^*\)-structure. They are also studied in more details in [9] by considering compositions of group type subfactors.

In this paper we show that one has the same description of the coideal subalgebras as in [4] even without the assumption on the presence of the additional \(\mathbb{C}^*\)-structure. This is compensated by the new characterization of coideal subalgebras given in Theorem 2.1. Also a Hopf algebraic version of the Conjecture 1.1 formulated in [4] follows for cocentral Kac algebras of solvable groups, see Subsection 3.6.

Note that recently, in [2] the authors proved that a cosemisimple Hopf algebra \(A\) is a quantum permutation algebra if and only if it is generated as an algebra by the matrix coefficients of all its left (right) coideal subalgebras of \(A\).

It is well known that any finite dimensional Hopf algebra \(A\) is a cyclic right \(A^*\)-module generated by the left integral of \(\Lambda \in A\). In Theorem 2.1 we prove an analogue of this result for left (right) coideal subalgebras of \(A\). It is shown that any left coideal subalgebra \(S\) of \(A\) is a cyclic \(A^*\)-module generated by an invariant element \(y_S \in S\) introduced by the author in [3].

Shortly, the organization of the paper is as follows. In the second Section we prove the result concerning the structure of coideal subalgebras as cyclic modules over the dual Hopf algebra. Section 3 contains the proof of Theorem 1.2. In the last section using the results from the previous section we also describe all Hopf subalgebras of cocentral abelian extensions of Hopf algebras.

We work over an algebraically closed field of arbitrary characteristic and all the other Hopf algebra notations are those used in [10].
2. Preliminaries

Recall that a left coideal subalgebra $S$ of $H$ is a subalgebra $S$ of $H$ with $\Delta(S) \subset H \otimes S$. A coideal subalgebra $S$ of $H$ is called normal if $S$ is stable under the adjoint action of $H$ on itself, i.e. $h_1sS(h_2) \in S$ for all $s \in S$ and $h \in H$.

2.1. Invariant elements of coideal subalgebras. Let $S$ be any right coideal subalgebra of a finite dimensional Hopf algebra $A$. Then $A$ is free over $S$ both as left and right $S$-module. Let $A = S \oplus (\oplus_{i=2}^{s} Sx_i)$ be a decomposition of $A$ as a free left $S$-modules of rank one. Then the idempotent integral $\Lambda$ of $A$ admits a decomposition:

$$\Lambda = y_S + \sum_{i=2}^{s} y_i$$

with $y_S \in S$ and $y_i \in Sx_i$ for $i \geq 2$. Then equation $s\Lambda = \epsilon(s)\Lambda$ implies that $s y_S = \epsilon(s)y_S$ for all $s \in S$.

Recall the left and right action of $A^*$ on $A$ given by $f \mapsto a = f(a_2)a_1$ and $a \leftarrow f = f(a_1)a_2$. Next Theorem shows that any coideal subalgebra $S$ of $A$ is a cyclic right $A^*$-submodule of $A$ generated $y_S \in S$.

**Theorem 2.1.** Let $S$ be a right coideal subalgebra of $A$ and $y_S \in S$ be defined as above. Then $y_S \leftarrow A^* = S$.

**Proof.** From Theorem 6.1 of [12] one has that $S$ is a simple object of the category $sM^A$ of relative modules. Clearly $y_S \leftarrow A^* \subset S$. We will show that $y_S \leftarrow A^* \in sM^A$ and then the proof will be complete.

Since $r y_S = \epsilon(r)y_S$ for all $r \in S$ by applying $\Delta$ it follows that

$$\sum_{r} r_1(y_S)_1 \otimes r_2(y_S)_2 = \epsilon(r)(y_S)_1 \otimes (y_S)_2.$$ 

Thus

$$\sum_{r}(y_S)_1 \otimes r(y_S)_2 = S(r_1)(r_2)(y_S)_1 \otimes (r_2)_2 = S(r)(y_S)_1 \otimes (y_S)_2,$$

for all $r \in S$. Thus $r(y_S \leftarrow f) = f((y_S)_1)r(y_S)_2 = f(S(r)(y_S)_1)(y_S)_2 \in y_S \leftarrow A^*$. This shows that $y_S \leftarrow A^*$ is an $S$-module and therefore an object in the category $sM^A$. \qed

**Remark 2.2.** Suppose that $A$ is a finite dimensional Hopf algebra and let $\{e_i\}_{i=1}^{s}$ be a basis of $A$ and $\{e_i^*\}_{i=1}^{s} \in A^*$ be the dual basis. Then for any $f \in A^*$ one has:

$$\Delta_{A^*}(f) = \sum_{i=1}^{s} e_i^* \otimes f \leftarrow e_i = \sum_{i=1}^{s} e_i \rightarrow f \otimes e_i^*$$
Indeed $f(xy) = \sum_{i=1}^{n} e_i^*(x)f(e_i y) = \sum_{i=1}^{n} e_i^*(x)(f \leftarrow e_i)(y)$ for all $x, y \in B$. This shows the first equality and the second is proven similarly.

2.1.1. On the operators $L$ and $R$ of $k^G$. Let $G$ be a finite group and $k^G$ be the dual group algebra. Consider the following operators on $k^G$ given by $R_m(f) = f \leftarrow m$ and $L_m(f) = m \rightarrow f$ for all $m \in G$.

If $M$ is a subgroup of $G$ let $A = k^{(G/M)}_\times$ be the space of all linear functions on $G$ which are constant on the right cosets of $M$ in $G$. Thus

\[(2.3) \quad k^{(G/M)}_\times = \{f \in k^G \mid f(gm) = f(m) \text{ for all } g \in G \text{ and } m \in M\}\]

Thus $k^{(G/M)}_\times$ is the subspace of all functionals $f \in k^G$ such that $L_m(f) = f$ for all $m \in M$.

**Lemma 2.4.** Suppose that $A \subset k^G$ is a subalgebra of $k^G$ such that $L_g(A) = A$ for all $g \in G$. Then there is a subgroup $M$ of $G$ such that $A = k^{(G/M)}_\times$.

**Proof.** Since $L_g(A) = A$ for all $g \in G$ it follows from Formula 2.2 that $\Delta_{k^G}(A) \subset k^G \otimes A$. Thus $A$ is a normal left coideal subalgebra of $k^G$ since $k^G$ is commutative. Then it follows that the quotient $k^G//A^*$ is a Hopf subalgebra of $kG$. Therefore there is a subgroup $M$ of $G$ such that $(k^G//A)^* \cong kM$. This implies that $A = k^{(G/M)}_\times$. \hfill $\square$

3. Structure of coideal subalgebras of cocentral abelian extensions of Hopf algebras

3.1. The Hopf algebra $A$. Let $A = k^G \rtimes_\sigma kF$ be an arbitrary cocentral abelian extension of $k^G$ via $kF$. Recall that this means that $A$ fits into the following exact sequence of Hopf algebras:

\[k \rightarrow k^G \rightarrow A \rightarrow kF \rightarrow k.\]

Moreover the group $F$ acts by automorphisms via $\triangleright : F \times G \rightarrow G$ on the group $G$. This induces an action of $F$ on the dual Hopf algebra $k^G$ via $f \rightarrow p_a = p_{f \triangleright a}$.

Then the Hopf algebra $A$ has the following multiplication structure

\[(3.1) \quad (p_a \# \sigma h)(p_b \# \sigma l) = \delta_{a,h \triangleright b} \sigma_a(h, l)p_a \# \sigma hl\]

and the comultiplication structure given by

\[(3.2) \quad \Delta(p_a \# \sigma h) = \sum_{b \in G} \tau_a(b, b^{-1}a)(p_b \# \sigma h) \otimes (p_b^{-1}a \# \sigma h).\]

Here $\sigma : F \times F \rightarrow k^G$ is a normalized twisted two cocycle of $G$ with respect to the action of $F$, i.e $\sigma$ satisfies:

\[(3.3) \quad \sigma_a(h, l)\sigma_a(hl, t) = \sigma_a(h, lt)\sigma_{h \triangleright a}(l, t)\]
where by definition \( \sigma(h, l) := \sum_{a \in G} \sigma_a(h, l)p_a \).

The dual cocycle \( \tau : F \to k^G \otimes k^G \) is denoted by
\[
\tau(f) = \sum_{a, b \in G} \tau_f(a, b)p_a \otimes p_b
\]
and satisfies the following two cocycle property:

\[
\tau_f(ab, c)\tau_f(a, b) = \tau_f(a, bc)\tau_f(b, c)
\]
for all \( f \in F \) and \( a, b, c \in G \).

Moreover one of the compatibility conditions between \( \sigma \) and \( \tau \) is called Pentagon Equation \([1]\) and it can be written as
\[
\sigma_{ab}(h, l) = \sigma_a(h, l)\tau_h(a, b)\tau_l(h, h \triangleright a, h \triangleright b)
\]
where by definition \( \sigma_a(h, l) := \sum_{a \in G} \sigma_a(h, l)p_a \).

The dual cocycle \( \tau : F \to k^G \otimes k^G \) it is denoted by
\[
\tau(f) = \sum_{a, b \in G} \tau_f(a, b)p_a \otimes p_b
\]
and satisfies the following two cocycle property:

\[
\tau_f(ab, c)\tau_f(a, b) = \tau_f(a, bc)\tau_f(b, c)
\]
for all \( f \in F \) and \( a, b, c \in G \).

Moreover one of the compatibility conditions between \( \sigma \) and \( \tau \), called Pentagon Equation \([1]\), can be written as
\[
\sigma_{ab}(h, l) = \sigma_a(h, l)\tau_h(a, b)\tau_l(h, h \triangleright a, h \triangleright b)
\]
for all \( a, b \in G \) and \( h \in F \). The antipode of \( A \) is given by
\[
S(p_a \#_\sigma h) = \sigma_{h^{-1} \triangleright a^{-1}}(h^{-1}, h)\tau^{-1}_{a^{-1}, a}(h)p_{h^{-1} \triangleright a^{-1}} \#_\sigma h^{-1}
\]

3.2. **Left coideal subalgebras of** \( A \). In this subsection we give a complete description of all left coideal subalgebras of \( A \).

3.3. **Operators** \( L_{m, \tau_h} \) and \( R_{m, \tau_h} \). Define the linear operators on \( k^G \) by

\[
(L_{a, \tau_h}(f)) = (a \triangleright f)\tau_h(-, a)
\]
and

\[
(R_{a, \tau_h}(f)) = (f \rhd a)\tau_h(a, -)
\]
for all \( a \in G, h \in F \). Note that by their definition \( L_{a, \tau_h} \) and \( R_{a, \tau_h} \) satisfy

\[
L_{a, \tau_h}(f)(b) := f(ba)\tau_h(b, a), \quad \text{and} \quad R_{a, \tau_h}(f)(b) := f(ba)\tau_h(a, b),
\]
for all \( a, b \in G \) and \( h \in F \).

The following lemma summarize the properties of these operators which follow from definitions:

**Lemma 3.10.** With the above notations one has that

\[
L_{a, \tau_h}L_{b, \tau_h} = L_{ab, \tau_h}(a, b) \quad \text{and} \quad R_{a, \tau_h}R_{b, \tau_h} = R_{ab, \tau_h}(b, a)
\]

Also,

\[
L_{a, \tau_h}R_{b, \tau_h} = R_{b, \tau_h}L_{a, \tau_h}
\]
for all \( a, b \in G \) and all \( h \in F \).

**Proof.** The proof is by a straightforward computation. \( \square \)

**Lemma 3.13.** Let \( A = k^G \rtimes_{\sigma} kF \) be a cocentral abelian extension of \( k^G \) via \( kF \). Then the comultiplication in \( A \) is given by

\[
\Delta(u \rtimes_{\sigma} h) = \sum_{a \in G} (L_{a, \tau_h}(u) \rtimes_{\sigma} h) \otimes (p_a \rtimes_{\sigma} h) = \sum_{a \in G} (p_a \rtimes_{\sigma} h) \otimes (R_{a, \tau_h}(u) \rtimes_{\sigma} h)
\]

for any \( u \in k^G \) and any \( h \in F \).

**Proof.** We prove the first formula. Using Formula 3.2 one has

\[
\Delta(u \rtimes_{\sigma} h) = \sum_{a,b \in G} \tau_h(a,b)(u_1 p_a \rtimes_{\sigma} h) \otimes (u_2 p_b \rtimes_{\sigma} h)
\]

\[
= \sum_{b \in G} \left( \sum_{a \in G} \tau_h(a,b)u(ab)p_a \rtimes_{\sigma} h \right) \otimes (p_b \rtimes_{\sigma} h)
\]

\[
= \sum_{b \in G} (L_{b, \tau_h}(u) \rtimes_{\sigma} h) \otimes (p_b \rtimes_{\sigma} h).
\]

The second formula has a similar proof. \( \square \)

### 3.4. **Definition of the left coideal subalgebra** \( C(M, H, \lambda) \).

Let \( M \leq G \) be a subgroup of \( G \) and \( H \leq F \) be a subgroup of \( F \) such that \( M \) is stable under the action of \( H \) on \( G \), i.e \( H \triangleright M = M \). Let also \( \lambda : M \times H \to k^* \) be a twisted bicharacter on \( M \times H \), i.e a function satisfying the following properties:

\[
\lambda(mn, h) = \lambda(m, h)\lambda(n, h)\tau_h^{-1}(m, n)
\]

(3.15)

\[
\lambda(m, hl) = \lambda(m, h)\lambda(h \triangleright m, l)\sigma_m(h, l)
\]

(3.16)

for all \( m, n \in M \) and \( h, l \in H \).

Define the following subspace of \( A \):

\[
C(M, H, \lambda) = \bigoplus_{h \in H} C_{\lambda}(h) \rtimes_{\sigma} h
\]

(3.17)

where

\[
C_{\lambda}(h) = \{ f \in k^G \mid L_{m, \tau_h}(f) = \lambda(m, h)f \text{ for all } m \in M \}.
\]
3.4.1. Description of $C_\lambda(h)$.

**Lemma 3.18.** Let $f \in k^G$. One has that $f \in C_\lambda(h)$ if and only if

$$f(gm) = \frac{\lambda(m, h)}{\tau_h(g, m)} f(g)$$

for all $m \in M$ and $g \in G$. In particular $\dim_k C_\lambda(h) = \frac{|G|}{|M|}$.

**Proof.** Note that

$$L_{m, \tau_h}(f) = (m \mapsto f) \tau_h(-, m)$$

$$= (m \mapsto (\sum_{g \in G} f(g)p_g)) \tau_h(-, m)$$

$$= (\sum_{g \in G} f(g)p_{gm^{-1}}) \tau_h(-, m)$$

$$= \sum_{g \in G} f(g) \tau_h(gm^{-1}, m)p_{gm^{-1}}$$

$$= \sum_{g \in G} f(gm) \tau_h(g, m)p_g.$$ 

Thus $L_{m, \tau_h}(f) = \lambda(m, h)f$ if and only if

$$f(gm) = \frac{\lambda(m, h)}{\tau_h(g, m)} f(g)$$

for all $g \in G$.

Let $b_i$ be a set of right coset representatives of $M$ in $G$. Thus one has $G = \bigsqcup_{i=1}^s b_iM$. For any $g \in G$ define the function

$$f_{[g]} = \sum_{m \in M} \frac{\lambda(m, h)}{\tau_h(g, m)} p_{gm}$$

Using Equation 3.4 and Condition 3.15 it is easy to check that the functional $f_{[g]}$ satisfies $f_{[g]} \in C_\lambda(h)$. Next it will be shown that

$$f_{[gmo]} = \frac{\lambda(m_0^{-1}, h)}{\tau_h(gmo, m_0^{-1})} f_{[g]}$$
for all \(m_0 \in M\). This implies that \(\{f_{[g_i]}\}_{i=1}^s\) is a basis of \(C_\lambda(h)\). One has:

\[
\begin{align*}
f_{[g_{m_0}]} &= \sum_{m \in M} \frac{\lambda(m, h)}{\tau_h(g_{m_0}, m)} p_{g_{m_0}m} = \\
&= \sum_{n \in M} \frac{\lambda(m_0^{-1}n, h)}{\tau_h(g_{m_0}, m_0^{-1}n)} p_{gn} \\
&= \lambda(m^{-1}_0, h) \sum_{n \in M} \frac{\lambda(n, h)}{\tau_h(m_0^{-1}, n)\tau_h(g, n)} p_{gn} \frac{\tau_h(g, n)}{\tau_h(m_0^{-1}, n)\tau_h(g_{m_0}, m_0^{-1}n)}
\end{align*}
\]

Note that by Equation 3.4 one has

\[
(3.23) \quad \frac{\tau_h(g, n)}{\tau_h(m_0^{-1}, n)\tau_h(g_{m_0}, m_0^{-1}n)} = \frac{1}{\tau_h(g_{m_0}, m_0^{-1})},
\]

and therefore

\[
(3.24) \quad f_{[g_{m_0}]} = \frac{\lambda(m_0^{-1}, h)}{\tau_h(g_{m_0}, m_0^{-1})} f_{[g]}
\]

\[\square\]

Lemma 3.25. With the above notations one has that

\[
(3.26) \quad R_{a, \tau_h}(f_{[g]}) = \frac{1}{\tau_h(a, a^{-1}g)} f_{[a^{-1}g]} = \frac{1}{\tau_h(a, a^{-1}g)} f_{[a^{-1}g]}
\]

for all \(a, g \in G\).

\textbf{Proof.} Indeed one has that

\[
R_{a, \tau_h}(f_{[g]}) = (f_{[g]} \mapsto a) \tau_h(a, -)
\]

\[
= \sum_{m \in M} \frac{\lambda(m, h)}{\tau_h(g, m)} p_{a^{-1}gm} \tau_h(a, -)
\]

\[
= \sum_{m \in M} \frac{\lambda(m, h)}{\tau_h(g, m)} \tau_h(a, a^{-1}gm) p_{a^{-1}gm}
\]

\[
= \sum_{m \in M} \left( \frac{\lambda(m, h)}{\tau_h(a^{-1}g, m)} p_{a^{-1}gm} \tau_h(a, a^{-1}gm) \tau_h(a, a^{-1}gm) \frac{\tau_h(a^{-1}g, m)}{\tau_h(g, m)} \right)
\]

Using Equation 3.4 observe that

\[
(3.27) \quad \frac{\tau_h(a^{-1}g, m)\tau_h(a, a^{-1}gm)}{\tau_h(g, m)} = \frac{1}{\tau_h(a, a^{-1}g)}
\]

and then the conclusion of the Lemma follows from here. \[\square\]
Note that for any \( u \in k^G \) and \( f \in F \) one can write
\[
(3.28) \quad (f.u) \leftarrow m = f.(u \leftarrow (f \triangleright m))
\]
for all \( m \in M \).

**Proposition 3.29.** Let \( A \cong k^G \) \( \#_\sigma kF \) be a cocentral abelian extension of \( kF \) by \( k^G \). With the following notations it follows that \( C(M, H, \lambda) \) is a left coideal subalgebra of \( A \).

**Proof.** Suppose that \( f \#_\sigma h \in C_\lambda(h)\#_\sigma h \). Then using Lemma 3.13 it follows that
\[
(3.30) \quad \Delta(f \#_\sigma h) = \sum_{a \in G} (p_a \#_\sigma h) \otimes (R_{a, \tau_a}(f) \#_\sigma h)
\]

But by Equation 3.12 one has \( L_{m, \tau_a}(R_{a, \tau_a}(f)) = R_{a, \tau_a}(L_{m, \tau_a}(f)) = \lambda(m, h)R_{a, \tau_a}(f) \) for all \( m \in M \) it follows that \( R_{a, \tau_a}(f) \in C_\lambda(h) \) for all \( a \in G \). This shows that \( \Delta_A(C_\lambda(h)\#_\sigma h) \subset A \otimes (C_\lambda(h)\#_\sigma h) \) and therefore \( C(M, H, \lambda) \) is a left coideal of \( A \).

In order to show that \( C(M, H, \lambda) \) is an algebra one has to check the following inclusion
\[
(3.31) \quad C_\lambda(h)(h.C_\lambda(l))\sigma_-(h, l) \subset C_\lambda(hl),
\]
for all \( hl \in F \). For all \( f \in C_\lambda(h) \) and all \( g \in C_\lambda(l) \) one has that
\[
R_{m, \tau_a}(f(h.g)\sigma_-(h, l)) = [(f(h.g)\sigma_-(h, l)) \leftarrow m]_{\tau_{hl}}(m, -)
\]
\[
= (f \leftarrow m)((h.g) \leftarrow m)(\sigma_-(h, l) \leftarrow m)_{\tau_{hl}}(m, -)
\]
\[
= R_{m, \tau_a}(f)((h(g \leftarrow (h \triangleright m)))(\sigma_-(h, l) \leftarrow m)_{\tau_{hl}}(m, -)_{\tau_h^{-1}}(m, -)
\]
\[
= [\lambda(m, h)f][h.[(R_{h \triangleright m})(g)]_{\tau_l^{-1}}((h \triangleright m), -)]
\]
\[
[(\sigma_-(h, l) \leftarrow m)_{\tau_{hl}}(m, -)_{\tau_h(m, -)^{-1}}]
\]
\[
= \lambda(m, h)\lambda(h \triangleright m, l)\sigma_m(h, l)f(h.g)\sigma_-(h, l)
\]
\[
= \sigma_m^{-1}(h, l)\sigma_-(h, l)(\sigma_-(h, l) \leftarrow m)
\]
\[
= \tau_l^{-1}((h \triangleright m), h \triangleright -)_{\tau_{hl}}(m, -)_{\tau_h(m, -)^{-1}}
\]
\[
= \lambda(m, hl)f(h.g)\sigma_-(h, l)
\]
We used that
\[
(3.32) \quad \sigma_m^{-1}(h, l)\sigma_-(h, l)(\sigma_-(h, l) \leftarrow m)_{\tau_l^{-1}}((h \triangleright m), h \triangleright -) = 1
\]
by Pentagon Equation 3.5 and
\[
(3.33) \quad \lambda(m, hl) = \lambda(m, h)\lambda(h \triangleright m, l)\sigma_m(h, l)
\]
by Equation 3.16. Then since
\[
R_{m, \tau_a}(f(h.g)\sigma_-(h, l)) = \lambda(m, hl)f(h.g)\sigma_-(h, l)
\]
it follows by definition of \( C_\lambda(hl) \) that \( f(h.g)\sigma_-(h, l) \in C_\lambda(hl) \). \( \square \)
3.5. **Left coideal subalgebras of** $A$. In this subsection we show that any left coideal subalgebra of $A$ is of the type $C(M, H, \lambda)$ as above. This result is inspired by [4] where the left coideal subalgebras were described under the additional assumption of a "*-structure on $A$. We will give in this subsection a proof that does not use this assumption but it uses the structure of coideal subalgebras from Theorem 2.1 instead.

With the notations from the previous Subsection define the following linear functionals \( \delta_{b_i}^{M} : \sum_{m \in M} p_{b_i} m \in k^G \) for all \( i = 1, s \).

**Proof of Theorem 1.2**

Let $B$ be an arbitrary left coideal subalgebra of $A$. Write the elements of $B$ as $b = \sum_{h \in F} f_h \cdot h$ where $f_h \in k^G$. Note that by Lemma 3.13

\[
\Delta_A(b) = \Delta_A(\sum_{h \in F} f_h \cdot h) = \sum_{h \in F, g \in G} (p_{g \cdot h} \cdot h) \otimes (R_{g, \tau_h}(f_h) \cdot h).
\]

Since $B$ is a left coideal, it follows that for each fixed pair of elements $(g, h) \in G \times F$ one has that $R_{g, \tau_h}(f_h) \cdot h \in B$.

Therefore one can write $B = \oplus_{h \in F} (B(h) \cdot h)$ with $B(h)$ a subspace of $k^G$ which is mapped by $R_{g, \tau_h}$ to itself for all $g \in G$. Since $B$ is also an algebra, we have

\[
B(h_1)(B(h_2)) \sigma(h_1, h_2) \subseteq B(h_1 h_2),
\]

for all $h_1, h_2 \in F$.

In particular this implies that $B(1)$ is a subalgebra of $k^G$ which affords a left representation of $G$ via the operators $R_{g, \tau_1}(f) = f \leftarrow g$. It follows from Lemma 2.4 that there is a subgroup $M \leq G$ such that $B(1)$ is the subspace $k^{(G/M)}_1$ of $M$-left invariant functions on $G$. Let $G = \sqcup_i b_i M$, $1 \leq i \leq r$ be the left coset decomposition of $G$ with respect to the subgroup $M$. Here $r = |G|/|M|$ is the index of $M$ in $G$. Then the linear functionals $\delta_{b_i M}$ form a linear basis on the algebra $B(1)$.

Let $H := \{h \in F | B(h) \neq 0\}$. It will be shown next that $H$ is a subgroup of $F$. Let $y_B$ be the left invariant element of $B$ defined in Subsection 2.1. Suppose further that

\[
y_B = \sum_{h \in H'} u_h \cdot h \in B
\]

for some nonzero elements $u_h \in B(h)$ and some non-empty subset $H' \subset H$.

By Theorem 2.1 since $B = y_B \leftarrow A^*$ it follows that $H' = H$. Moreover the same Theorem implies that $B(h) = \{a_{\tau_h}(u_h)\}_{a \in G}$, the
linear span of the functionals \( \{ L_{a, \tau_h}(u_h) \}_{a \in G} \). Indeed, \( A^* \) can be identified as algebras to \( k^F \#_\tau kG \) via
\[
(3.34) \quad < p_x \#_\tau a, p_y \#_\sigma y > = \delta_{a,b} \delta_{x,y}
\]
for all \( x, y \in F \) and \( a, b \in G \). Then using Equation (3.13) one has
\[
(u_h \#_\sigma h) \leftarrow (p_x \#_\tau a) = \sum_{y \in G} < p_x \#_\tau a, p_y \#_\sigma h > R_{y, \tau_h}(u_h) \#_\sigma h
\]
\[
= \delta_{x,h} R_{a, \tau_h}(u_h).
\]
Thus \( y_B \leftarrow p_h \#_\tau a = R_{a, \tau_h}(u_h) \#_\sigma h \).

Next it will be shown that \( H \) is a subgroup of \( F \). Since \( (\delta_M \# 1) y_B = y_B \) it follows that \( \delta_M u_h = u_h \) for all \( h \in H \). Since \( u_h \neq 0 \) it follows that \( \text{supp}(u_h) \subseteq M \). Thus one has \( u_h \in B(h) \delta_M \) for all \( h \in H \). In particular \( u_1 \in B(1) \delta_M = k \delta_M \).

Without loss of generality one may suppose further that \( u_1 = \delta_M \).

Since \( B(h) \neq 0 \) it follows that there is \( u \in B(h) \) and \( x \in G \) such that \( u(x) \neq 0 \). It follows that \( R_{x, \tau_h}(u)(1) = u(x) \neq 0 \). On the other hand since \( R_{x, \tau_h}(f) \in B(h) \) one can conclude that there is \( f := R_{x, \tau_h}(u) \in B(h) \) with \( f(1) \neq 0 \). Therefore for such element \( f \in kG \) one has that \( (f \#_\sigma h) y_B = f(1) y_B \) is a nonzero element of \( B \). On the other hand since
\[
(3.35) \quad (f \#_\sigma h) y_B = \sum_{i \in H} f(h. u_l) \sigma(h, l) \#_\sigma h l
\]
we deduce that \( hH \subseteq H \). Thus \( H \) is a subgroup of \( F \). Moreover from Equation (3.35) we deduce that
\[
(3.36) \quad f(h. u_l) \sigma(h, l) = f(1) u_{hl}
\]
for all \( l \in H \). For \( l = h^{-1} \) this identity becomes
\[
(3.37) \quad f(h. u_{h^{-1}}) \sigma(h, h^{-1}) = f(1) \delta_M
\]

Evaluating both sides of the last identity at \( g = 1 \) it follows that \( u_h(1) = 1 \) for all \( h \in H \). Moreover since \( (\delta_M \#_\sigma 1) y_B = y_B \) it follows that \( \delta_M u_h = u_h \) for any \( h \in H \). This shows that \( \text{supp}(u_h) \subseteq M \). On the other hand Equation (3.34) shows that \( M \subseteq \text{supp}(f) \) for any \( f \in B(h) \) with \( f(1) \neq 0 \). In particular, since \( u_h(1) = 1 \) it follows that \( \text{supp}(u_h) = M \). But evaluating Equation (3.37) at any \( m \in M \) it follows that:
\[
(3.38) \quad f(m) u_{h^{-1}}(h \triangleright m) \sigma_m(h, h^{-1}) = f(1)
\]
Since \( \text{supp}(u_h) = M \) this shows that \( M \) is stable under the action of \( H \) and \( \dim B(h) \delta_M = 1 \). Thus \( B(h) \delta_M = < u_h > \).
Since
\[(u_h \# \sigma h) y_B = y_B\]
this implies that
\[(3.39) \quad u_h(h. u_{h'}) \sigma(h, h') = u_{hh'} \]
for all \(h, h' \in H\). Evaluating both sides of the last equality at \(m \in M\) one has
\[(3.40) \quad u_h(m) u_{h'}(h \triangleright m) \sigma_m(h, h') = u_{hh'}(m)\]
Since for \(m \in M\) the operator \(L_{m, \tau h}\) maps \(B(h) \delta_M\) to itself it follows that there is a function \(\lambda : M \times H \to k^*\) such that
\[(3.41) \quad L_{m, \tau h}(u_h) = \lambda(m, h) u_h.\]
Evaluating both sides of the above identity at 1 one gets that \(u_h(m) = \lambda(m, h)\) for all \(m \in M\). On the other hand evaluating both sides of the same equality at \(m' \in M\) it follows that \(u_h(mm') \tau_h(m, m') = \lambda(m, h) u_h(m')\) and therefore one obtains Equation (3.15) i.e.,
\[(3.42) \quad \lambda(mm', h) \tau_h(m, m') = \lambda(m, h) \lambda(m, h')\]
for all \(m, m' \in M\) and \(h \in H\). For the other Equality (3.16) note that
\[\lambda(m, hh') = u_{hh'}(m) = u_h(m) u_{h'}(h \triangleright m) \sigma_m(h, h') = \lambda(m, h) \lambda(h \triangleright m, h') \sigma_m(h, h')\]
It remains to show that \(B(h)\) coincides to \(C_\lambda(h)\), the subspace of all functions \(f \in k^G\) which verify
\[L_{m, \tau h}(f) = \lambda(m, h) f,\]
for all \(m \in M\). Note that since \(B(h) B(1) \subseteq B(h)\) it follows that \(B(h) = \oplus_{i=1}^s B(h) \delta_{b,i} M\). Since \(L_{b,i, \tau h} (B(h) \delta_M) = B(h) \delta_{b,i} M\) it follows that all the spaces \(B(h) \delta_{b,i} M\) are also one dimensional. Thus \(\text{dim}_k(B(h)) = \frac{|G|}{|M|}\).
On the other hand Lemma 3.18 shows that \(\text{dim}_k(C_\lambda(h)) = \frac{|G|}{|M|}\).

Thus in order to conclude that \(B(h) = C_\lambda(h)\) it is enough to show that \(L_{m, \tau h}(f) = \lambda(m, h) f\) for all \(f \in B(h)\). If \(f \in B(h) \delta_M\) then this relation is satisfied from the definition of \(\lambda\) and Relation (3.41). On the other hand since \(B(h)\) is spanned as vector spaces by \(R_{g, \tau h}(u_h)\)
Equation (3.12) implies that for all \(g \in G\) one has
\[(3.43) \quad L_{m, \tau h}(R_{g, \tau h}(u_h)) = R_{g, \tau h}(L_{m, \tau h}(u_h)) = \lambda(m, h) R_{g, \tau h}(u_h).\]
\(\Box\).
3.6. **On the Wall’s conjecture.** In the spirit of [4] we have the following Theorem which is a Hopf algebra analogue of Wall’s conjecture.

**Theorem 3.44.** Let $A \cong k^G \#_\sigma kF$ be a cocentral extension of Hopf algebras with $G$ and $F$ solvable groups. Then the number of maximal (resp. minimal) left coideal subalgebras of $A$ is less or equal than the dimension of $A$.

With the above characterization of coideal subalgebras the proof of this Theorem is the same as that of Theorem 3.8 from [4].

**Remark 3.45.** In the same paper [4], the authors proposed Conjecture 1.2, as a new version of Wall’s conjecture for semisimple fusion algebras. It was announced that this conjecture is solved for commutative fusion rings. We remark that there are cocentral Kac algebras with noncommutative Grothendieck rings, for example the smash product Hopf algebra $kQ_8 \# kC_2$, dual to the smash coproduct Hopf algebra from Section 8 of [6].

4. **Hopf subalgebras of cocentral abelian extensions of Hopf algebras**

In this section we describe all Hopf subalgebras of an abelian cocentral extension.

**Theorem 4.1.** Let $A = k^G \#_\sigma kF$ be a cocentral abelian extension of $kF$ by $k^G$. Then all Hopf subalgebras of $A$ are of the form $C(M, H, \lambda)$ where $M$ is a normal subgroup of $G$ and $\lambda$ satisfies the additional invariance condition:

\[(4.2) \quad \lambda(a^{-1}ma, h) = \lambda(m, h)\tau_h^{-1}(a, a^{-1}ma)\tau_h(m, a)\]

for all $a \in G$ and $m \in M$.

**Proof.** A Hopf subalgebra $B$ of $A$ is in particular a left coideal subalgebra and by the previous Theorem is of the type $C(M, H, \lambda)$ defined above. Suppose that $B := C(M, H, \lambda)$ is a Hopf subalgebra of $A$ and let $f \#_\sigma h \in B$. Then using Lemma 3.13 one has

\[\Delta_A (f \#_\sigma h) = \sum_{a \in G} (L_{a, \tau_h}(f) \#_\sigma h) \otimes (p_a \#_\sigma h)\]

Then \[\Delta_A (f \#_\sigma h) \in B \otimes A\] if and only if

\[(4.3) \quad L_{a, \tau_h}(f) = \sum_{x \in G} f_x \tau_h(xa^{-1}, a)p_{xa^{-1}} \in C_\lambda(h)\]
for any $a \in G$. This implies that for any $a \in G$ the space $Ma^{-1}$ is also a right coset of $M$ in $G$, i.e $M$ is a normal subgroup of $G$. On the other hand using repeatedly Equation 3.11 it follows that:

$$L_{m, \tau}(L_{a, \tau}(f)) = L_{ma, h}(f)\tau_{h}(m, a)$$

$$= L_{a}(L_{a^{-1}ma, \tau}(f))\tau_{h}^{-1}(a, a^{-1}ma)\tau_{h}(m, a)$$

$$= \lambda(a^{-1}ma, h)\tau_{h}^{-1}(a, a^{-1}ma)\tau_{h}(m, a) L_{a, \tau}(f)$$

This shows that $L_{a, \tau}(f) \in C\lambda(h)$ if and only if

(4.4) $\lambda(a^{-1}ma, h) = \lambda(m, h)\tau_{h}^{-1}(a, a^{-1}ma)\tau_{h}(m, a)$.

□

**Remark 4.5.** Alternatively, for the converse of the above Theorem it can be shown that

$$\Delta_{kG}(f[a]) = \sum_{y \in G} \lambda(y^{-1}, h)\tau_{h}(a, y^{-1}) f[yay^{-1}] \otimes p_y$$

and thus $\Delta_{kG}(C\lambda(h)) \subset C\lambda(h) \otimes k^G$.

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