THE REGULARITY OF QUOTIENT PARATOPOLOGICAL GROUPS

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Abstract. Let $H$ be a closed subgroup of a regular abelian paratopological group $G$. The group reflexion $G^\#$ of $G$ is the group $G$ endowed with the strongest group topology, weaker that the original topology of $G$. We show that the quotient $G/H$ is Hausdorff (and regular) if $H$ is closed (and locally compact) in $G^\#$. On the other hand, we construct an example of a regular abelian paratopological group $G$ containing a closed discrete subgroup $H$ such that the quotient $G/H$ is Hausdorff but not regular.

In this paper we study the properties of the quotients of paratopological groups by their normal subgroups.

By a paratopological group $G$ we understand a group $G$ endowed with a topology $\tau$ making the group operation continuous, see [ST]. If, in addition, the operation of taking inverse is continuous, then the paratopological group $(G, \tau)$ is a topological group. A standard example of a paratopological group failing to be a topological group is the Sorgefrey line $L$, that is the real line $\mathbb{R}$ endowed with the Sorgefrey topology (generated by the base consisting of half-intervals $[a, b)$, $a < b$).

Let $(G, \tau)$ be a paratopological group and $H \subset G$ be a closed normal subgroup of $G$. Then the quotient group $G/H$ endowed with the quotient topology is a paratopological group, see [Ra]. Like in the case of topological groups, the quotient homomorphism $\pi : G \to G/H$ is open. If the subgroup $H \subset G$ is compact, then the quotient $G/H$ is Hausdorff (and regular) provided so is the group $G$, see [Ra]. The compactness of $H$ in this result cannot be replaced by the local compactness as the following simple example shows.

Example 1. The subgroup $H = \{(-x, x) : x \in \mathbb{Q}\}$ is closed and discrete in the square $G = \mathbb{L}^2$ of the Sorgenfrey line $\mathbb{L}$. Nonetheless, the quotient group $G/H$ fails to be Hausdorff: for any irrational $x$ the coset $(-x, x) + H$ cannot be separated from zero $(0, 0) + H$.

A necessary and sufficient condition for the quotient $G/H$ to be Hausdorff is the closedness of $H$ in the topology of group reflexion $G^\#$ of $G$.

By the group reflexion $G^\# = (G, \tau^\#$) of a paratopological group $(G, \tau)$ we understand the group $G$ endowed with the strongest topology $\tau^\# \subset \tau$ turning $G$ into a topological group. This topology admits a categorial description: $\tau^\#$ is a unique topology on $G$ such that

- $(G, \tau^\#)$ is a topological group;
- the identity homomorphism $id : (G, \tau) \to (G, \tau^\#)$ is continuous;

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for each continuous group homomorphism \( h : G \to H \) into a topological group \( H \) the homomorphism \( h \circ \text{id}^{-1} : G^\flat \to H \) is continuous.

Observe that the group reflexion of the Sorgenfrey line \( L \) is the usual real line \( \mathbb{R} \).

For so-called 2-oscillating paratopological groups \((G, \tau)\) the topology \( \tau^\flat \) admits a very simple description: its base at the origin \( e \) of \( G \) consists of the sets \( UU^{-1} \), where \( U \) runs over open neighborhoods of \( e \) in \( G \). Following [BR] we define a paratopological group \( G \) to be 2-oscillating if for each neighborhood \( U \subset G \) of the origin \( e \) there is another neighborhood \( V \subset G \) of \( e \) such that \( V^{-1}V \subset UU^{-1} \). The class of 2-oscillating paratopological groups is quite wide: it contains all abelian (more generally all nilpotent) as well as saturated paratopological groups. Following I. Guran we call a paratopological group saturated if for each neighborhood \( U \) of the origin in \( G \) its inverse \( U^{-1} \) has non-empty interior in \( G \).

Given a subset \( A \) of a paratopological group \((G, \tau)\) we can talk of its properties in the topology \( \tau^\flat \). In particular, we shall say that a subset \( A \subset G \) is \( \flat \)-closed in \( G \) if it is closed in the topology \( \tau^\flat \). Also with help of the group reflexion many helpful properties of paratopological groups can be defined.

A paratopological group \( G \) is called

- \( \flat \)-separated if the topology \( \tau^\flat \) is Hausdorff;
- \( \flat \)-regular if it has a neighborhood base at the origin, consisting of \( \flat \)-closed sets;
- \( \flat \)-compact if \( G^\flat \) is compact.

It is clear that each \( \flat \)-separated (and \( \flat \)-regular) paratopological group is functionally Hausdorff (and regular). Conversely, each Hausdorff (resp. regular) 2-oscillating group is \( \flat \)-separated (resp. \( \flat \)-regular), see [BR]. On the other hand, there are examples of (nonabelian) Hausdorff paratopological groups \( G \) which are not \( \flat \)-separated, see [Ra], [BR]. The simplest example of a \( \flat \)-compact non-compact paratopological group is the Sorgenfrey circle \( \{ z \in \mathbb{C} : |z| = 1 \} \) endowed with the topology generated by the base consisting of “half-intervals” \( \{ e^{i\varphi} : \varphi \in [a, b) \} \), \( a < b \).

Now we are able to state our principal positive result.

**Theorem 1.** Let \( H \) be a normal subgroup of a \( \flat \)-separated paratopological group \( G \). Then the quotient paratopological group \( G/H \) is

1. \( \flat \)-separated if and only if \( H \) is closed in \( G^\flat \);
2. \( \flat \)-regular if \( G \) is \( \flat \)-regular and the set \( H \) is locally compact in \( G^\flat \).

**Proof.** Let \( \pi : G \to G/H \) denote the quotient homomorphism.

1. If \( H \) is closed in \( G^\flat \) then \( G^\flat / H \) is Hausdorff as a quotient of a Hausdorff topological group \( G^\flat \). Since the identity homomorphism \( G/H \to G^\flat / H \) is continuous, the paratopological group \( G/H \) is \( \flat \)-separated.

Now assume conversely that the paratopological group \( G/H \) is \( \flat \)-separated. Since the quotient map \( \pi^\flat : G^\flat \to (G/H)^\flat \) is continuous its kernel \( H \) is closed in \( G^\flat \).

2. Assume that \( G \) is \( \flat \)-regular and \( H \) is locally compact in \( G^\flat \). It follows that \( H \) is closed in \( G^\flat \) (this so because the subgroup \( H \subset G^\flat \), being locally compact, is complete). Then there is a closed neighborhood \( W_1 \subset G^\flat \) of the neutral element \( e \) such that the intersection \( W_1 \cap H \) is compact in \( G^\flat \). Take any closed neighborhood \( W_2 \subset G^\flat \) of \( e \) such that \( W_2^{-1}W_2 \subset W_1 \). We claim that \( W_2 \cap gH \) is compact for each \( g \in G \). This is trivial if \( W_2 \cap gH \) is empty. If not, then \( gh = w \) for some \( h \in H \).
and \( w \in W_2 \). Hence \( W_2 \cap gH \subset W_2 \cap w^{-1}H = W_2 \cap wH = w(w^{-1}W_2 \cap H) \subset w(W_2^{-1}W_2 \cap H) \subset w(W_1 \cap H) \) and the closed subset \( W_2 \cap gH \) of \( G \) lies in the compact subset \( w(W_1 \cap H) \) of \( G \). Consequently, \( W_2 \cap gH \) is compact for any \( y \in G \).

Let \( W_3 \subset G^p \) be a neighborhood of \( e \) such that \( W_3^{-1}W_3 \subset W_2 \).

To prove the \( b \)-regularity of the quotient group \( G/H \), given any neighborhood \( U \subset G \) of \( e \) it suffices to find a neighborhood \( V \subset U \) of \( e \) such that \( \pi(V) \) is \( b \)-closed in \( G/H \). By the \( b \)-regularity of \( G \), we can find a \( b \)-closed neighborhood \( V \subset U \cap W_3 \).

We claim that \( \pi(V) \) is \( b \)-closed in \( G/H \). Since the identity map \((G/H)^p \to G^p/H\) is continuous, it suffices to verify that \( \pi(V) \) is closed in the topological group \( G^p/H \).

Take any point \( gH \notin \pi(V) \) of \( G^p/H \). It follows from \( gH \cap V = \emptyset \) and the compactness of the set \( W_2 \cap gH \) that there is an open neighborhood \( W_4 \subset W_2 \) of \( e \) in \( G^p \) such that \( W_4(W_2 \cap gH) \cap V = \emptyset \). We claim that \( W_4z \cap V = \emptyset \) for any \( z \in gH \). Assuming the converse, find a point \( v \in W_4 \cap V \). It follows that \( z \notin W_2 \).

On the other hand, \( z \in W_4^{-1}v \subset W_4^{-1}V \subset W_2 \). This contradiction shows that \( W_4gH \cap V = \emptyset \) and thus \( \pi(W_4g) \) is a neighborhood of \( gH \) in \( G^p/H \), disjoint with \( \pi(V) \).

\[ \square \]

**Corollary 1.** If \( H \) is a \( b \)-compact normal subgroup of a \( b \)-regular paratopological group \( G \), then the quotient paratopological group \( G/H \) is \( b \)-regular.

**Proof.** It follows that the identity inclusion \( H^b \to G^p \) is continuous and thus \( H \) is compact in \( G^p \). Applying the preceding theorem, we conclude that the quotient group \( G/H \) is \( b \)-regular.

\[ \square \]

**Remark 1.** It is interesting to compare the latter corollary with a result of [Ra] asserting that the quotient \( G/H \) of a Hausdorff (regular) paratopological group \( G \) by a compact normal subgroup \( H \subset G \) is Hausdorff (regular).

Since for a 2-oscillating paratopological group \( G \) the Hausdorff property (the regularity) of \( G \) is equivalent to the \( b \)-separatedness (the \( b \)-regularity), Theorem [1] implies

**Corollary 2.** Let \( H \) be a normal subgroup of a Hausdorff 2-oscillating paratopological group \( G \). Then the quotient paratopological group \( G/H \) is

1. Hausdorff if \( H \) is closed in \( G^p \);
2. regular if \( G \) is regular and the set \( H \) is locally compact in \( G^p \).

Example [1] supplies us with a locally compact closed subgroup \( H \) of a \( b \)-regular paratopological group \( G = L^2 \) such that the quotient \( G/H \) is not Hausdorff. Next, we construct a \( b \)-regular abelian paratopological group \( G \) containing a locally compact \( b \)-closed subgroup \( H \) such that the quotient is Hausdorff but not regular. This will show that in Theorem [1] and Corollary [2] the local compactness of \( H \) in \( G^p \) cannot be replaced by the local compactness plus \( b \)-closedness of \( H \) in \( G \).

Our construction is based on the notion of a **cone topology**. Let \( G \) be a topological group and \( S \subset G \) be a closed subsemigroup of \( G \), containing the neutral element \( e \in G \). The **cone topology** \( \tau_S \) on \( G \) consists of sets \( U \subset G \) such that for each \( x \in U \) there is an open neighborhood \( W \subset G \) of \( e \) such that \( x(W \cap S) \subset U \). It is clear that the group \( G \) endowed with the cone topology \( \tau_S \) is a regular paratopological groups and its neighborhood base at \( e \) consists of the sets \( W \cap S \), where \( W \) is a neighborhood of \( e \) in \( G \). Moreover, the paratopological group \((G, \tau_S)\) is saturated.
if \( e \) is a cluster point of the interior of \( S \) in \( G \). In the latter case the paratopological group \((G, T_S)\) is 2-oscillating and thus \( b \)-regular, see [BR, Theorem 3].

In the following example using the cone topology we construct a saturated regular paratopological group \( G \) containing a \( b \)-closed discrete subgroup \( H \) with non-regular quotient \( G/H \).

**Example 2.** Consider the group \( \mathbb{Q}^3 \) endowed with the usual (Euclidean) topology. A subsemigroup \( S \) of \( \mathbb{Q}^3 \) is called a cone in \( \mathbb{Q}^3 \) if \( q \cdot \vec{x} \in S \) for any non-negative \( q \in \mathbb{Q} \) and any vector \( \vec{x} \in S \).

Fix a sequence \((z_n)\) of rational numbers such that \( 0 < \sqrt{2} - z_n < 2^{-n} \) for all \( n \) and let \( S \subset \mathbb{Q}^3 \) be the smallest closed cone containing the vectors \((1, 0, 0)\) and \((\frac{1}{n}, 1, z_n)\) for all \( n \). Let \( \tau_S \) be the cone topology on the group \( \mathbb{Q}^3 \) determined by \( S \). Since the origin of \( \mathbb{Q}^3 \) is a cluster point of the interior of \( S \), the paratopological group \( G = (\mathbb{Q}^3, \tau_S) \) is saturated and \( b \)-regular. Moreover, its group reflexion coincides with \( \mathbb{Q}^3 \).

Now consider the \( b \)-closed subgroup \( H = \{(0, 0, q) : q \in \mathbb{Q}\} \) of the group \( G \). Since \( H \cap S = \{(0, 0, 0)\} \), the subgroup \( H \) is discrete (and thus locally compact) in \( G \). On the other hand \( H \) fails to be locally compact is \( \mathbb{Q}^3 \), the group reflexion of \( G \).

We claim that the quotient group \( G/H \) is not regular. Let \( \pi : G \to G/H \) denote the quotient homomorphism. We can identify \( G/H \) with \( \mathbb{Q}^2 \) endowed with a suitable topology.

Let us show that \((0, 1) \notin \pi(S)\). Assuming the converse we would find \( x \in \mathbb{Q} \) such that \((0, 1, x) \in S\). It follows from the definition of \( S \) that \( x \geq 0 \) and there is a sequence \((\vec{x}_i)\) converging to \((0, 1, x)\) such that

\[
\vec{x}_i = \sum_{n} \lambda_{i,n}(n^{-1}, 1, z_n) + \lambda_i(1, 0, 0)
\]

where all \( \lambda_i, \lambda_{i,n} \geq 0 \) and almost all of them vanish. Taking into account that \( \{\vec{x}_i\} \) converges to \((0, 1, x)\) we conclude that

- \( \lambda_i \to 0 \) as \( i \to \infty \);
- \( \lambda_{i,n} \to 0 \) for every \( n \);
- \( \sum_n \lambda_{i,n} \) tends to 1 as \( i \to \infty \).

Let \( \varepsilon > 0 \). Then

\[
\exists N_1(\forall n > N_1)\{|z_n - \sqrt{2}| < \varepsilon\},
\exists N_2(\forall i > N_2)\{\forall n \leq N_1\}{\lambda_{i,n} < \varepsilon/N_1}\text{ and}
\exists N_3(\forall i > N_3)(|\sum \lambda_{i,n} - 1| < \varepsilon).
\]

Put \( N = \max\{N_2, N_3\} \). Let \( i > N \). Then

\[
|\sqrt{2} - \sum_n \lambda_{i,n} z_n| \leq |\sqrt{2} - \sum_n \lambda_{i,n} \sqrt{2}| + |\sum_{n \leq N_1} \lambda_{i,n} (\sqrt{2} - z_n)| + |\sum_{n > N_1} \lambda_{i,n} (\sqrt{2} - z_n)| \leq 
\varepsilon \sqrt{2} + \varepsilon + \sum_{n > N_1} \lambda_{i,n} \varepsilon \leq \varepsilon(\sqrt{2} + 1 + \varepsilon).
\]

So \( x = \sqrt{2} \) which is impossible. This contradiction shows that \((0, 1) \notin \pi(S)\) and thus \((0, \frac{1}{n}) \notin \pi(S)\) for all \( n \in \mathbb{N} \) (since \( S \) is a cone).

It remains to prove that for each neighborhood \( V \subset \mathbb{Q}^3 \) of the origin we get \( \overline{\pi(V \cap S)} \subset \pi(S) \), where the closure is taken in \( G/H \). This will follow as soon as we show that \((0, \frac{1}{m}) \in \overline{\pi(V \cap S)} \) for some \( m \). Since \( V \) is a (usual) neighborhood
of \((0,0,0)\) in \(\mathbb{Q}^3\), there is \(m \in \mathbb{N}\) such that \(\frac{1}{m}(\frac{1}{n}, 1, z_n) \in V\) for all \(n \in \mathbb{N}\). Then \(\frac{1}{m}(\frac{1}{n}, 1) \in \pi(V \cap S)\) for all \(n \in \mathbb{N}\). Observe that the sequence \(\{(\frac{1}{nm}, \frac{1}{m})\}_n\) converges to \((0, \frac{1}{m})\) in \(G/H\) since for each neighborhood \(W \subset \mathbb{Q}^3\) of \((0,0,0)\) the difference \((\frac{1}{nm}, \frac{1}{m}) - (0, \frac{1}{m}) = (\frac{1}{nm}, 0)\) belongs to \(\pi(W \cap S)\) for all sufficiently large \(n\). Therefore \((0, \frac{1}{m}) \in \pi(V \cap S) \subsetneq \pi(S) \neq (0, \frac{1}{m})\), which means that \(G/H\) is not regular.

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