On measuring the absolute scale of baryon acoustic oscillations

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ABSTRACT
The baryon acoustic oscillation (BAO) feature in the distribution of galaxies provides a fundamental standard ruler which is widely used to constrain cosmological parameters. In most analyses, the comoving length of the ruler is inferred from a combination of cosmic microwave background (CMB) observations and theory. However, this inferred length may be biased by various non-standard effects in early universe physics; this can lead to biased inferences of cosmological parameters such as \( H_0 \), \( \Omega_m \) and \( w \), so it would be valuable to measure the absolute BAO length by combining a galaxy redshift survey and a suitable direct low-\( z \) distance measurement. One obstacle is that low-redshift BAO surveys mainly constrain the ratio \( r_s/D_\nu(z) \), where \( D_\nu \) is a dilation scale which is not directly observable by standard candles. Here, we find a new approximation \( D_\nu(z) \simeq \frac{1}{4} D_L \left( \frac{1}{2} z \right) (1 + \frac{4}{3} z)^{-1} (1 - 0.02455 z^3 + 0.0105 z^4) \) which connects \( D_\nu \) to the standard luminosity distance \( D_L \) at a somewhat higher redshift; this is shown to be very accurate (relative error \( <0.2 \) per cent) for all Wilkinson Microwave Anisotropy Probe compatible Friedmann models at \( z < 0.4 \), with very weak dependence on cosmological parameters \( H_0 \), \( \Omega_m \), \( \Omega_k \), \( w \). This provides a route to measure the absolute BAO length using only observations at \( z \lesssim 0.3 \), including Type Ia supernovae, and potentially future \( H_0 \)-free physical distance indicators such as gravitational lenses or gravitational wave standard sirens. This would provide a zero-parameter check of the standard cosmology at \( 10^3 \lesssim z \lesssim 10^5 \), and can constrain the number of relativistic species \( N_{\text{eff}} \) with fewer degeneracies than the CMB.

Key words: cosmic background radiation – distance scale – large-scale structure of Universe.

1 INTRODUCTION
The detection of baryon acoustic oscillations (BAOs) in the large-scale distribution of galaxies in both the Sloan Digital Sky Survey (SDSS; Eisenstein et al. 2005) and 2dF Galaxy Redshift Survey (2dFGRS; Cole et al. 2005) redshift surveys was a triumph for the standard cosmological model; the BAO feature (Peebles & Yu 1970; Bond & Efstathiou 1984; Eisenstein & Hu 1998; Meiksin, White & Peacock 1999) is essentially created by closely related physics to the acoustic peaks in the cosmic microwave background (CMB) temperature power spectrum. Therefore, the observed BAO feature supports the standard cosmology in several independent ways: the existence of the feature supports the basic gravitational instability paradigm for structure formation; the relative weakness of the BAO feature supports the \( \sim 1:5 \) ratio of baryons to dark matter, since a baryon-dominated universe would have a BAO feature much stronger than observed; and the observed length-scale of the feature in redshift space is consistent with the concordance \( \Lambda \) cold dark matter (\( \Lambda \)CDM) model derived from the CMB and other observations, with \( \Omega_m \approx 0.27 \) and \( H_0 \approx 70 \text{ km s}^{-1} \text{ Mpc}^{-1} \) (Komatsu et al. 2011).

Recently, there have been several new independent measurements of the BAO feature in galaxy redshift surveys, e.g. from SDSS Data Release 8 (DR8; Percival et al. 2010), WiggleZ (Blake et al. 2011), 6dFGRS (Beutler et al. 2011) and an angular measurement from SDSS Data Release 9 (DR9; Seo et al. 2012), which are all consistent with the concordance \( \Lambda \)CDM model at the few-per cent level.

The BAO feature is probably the best-understood standard ruler in the moderate-redshift Universe, and therefore in conjunction with CMB observations it offers great power for constraining cosmological parameters including dark energy (Weinberg et al. 2012). A number of theoretical and numerical studies (Seo et al. 2008, 2010) have concluded that the comoving length-scale of the BAO feature evolves by \( \sim 0.5 \) per cent between the CMB era and \( z \sim 0.3 \) due to non-linear growth of structure, but this shift can be corrected to the 0.1 per cent level using reconstruction methods (Padmanabhan et al. 2012). Therefore, there are several very ambitious redshift surveys including the ongoing WiggleZ (Blake et al. 2011), Baryon Oscillation Spectroscopic Survey (BOSS; White et al. 2011) and Hobby–Eberly Telescope Dark Energy Experiment (HETDEX), the
recently approved European Space Agency (ESA) Euclid space mission (Laureijs et al. 2011) and the proposed BigBOSS and Wide-Field Infrared Survey Telescope (WFIRST), which plan to survey ~1–50 million galaxy redshifts over huge volumes, to give sub-per cent measurements of the BAO feature at various redshifts 0.2 \leq z \leq 2.5.

However, one drawback of the BAO feature is that the comoving length \( r_z(z) \) is calibrated at \( z > 1000 \) using a combination of CMB observations and theory; this leaves us vulnerable to systematic errors from possible unknown new physics at early times (see Section 2 for discussion). Low-redshift measurements of the BAO scale essentially measure a ratio of \( r_z \) relative to some distance which is itself dependent on cosmological parameters \( H_0, \Omega_m, w \) etc. Therefore, in a joint fit to CMB+BAO data, a wrong assumption in the CMB measurement of \( r_z \) may be masked by biased values of cosmological parameters, i.e. a ‘precisely wrong’ outcome (see Section 4 for an example).

In this paper we present a new and useful approximation which can accurately anchor the absolute BAO length-scale using only low-redshift measurements at \( z \lesssim 0.3 \), therefore providing a clean zero-parameter test of the standard early-universe cosmology, in particular the density of relativistic particles.

The plan of the paper is as follows. In Section 2 we briefly review the main features of BAO observations, then in Section 3 we present the new approximation for the dilation scale \( D_V(z) \). In Section 4 we review the effects of non-standard radiation density, and in Section 5 we discuss potential observational issues for measuring the absolute BAO length. We summarize our conclusions in Section 6.

Throughout the paper we use the standard notation that \( \Omega_i \) is the present-day density of species \( i \) relative to the critical density; and the physical density \( \omega_i = \Omega_i h^2 \), with \( h = H_0/(100 \text{ km s}^{-1} \text{ Mpc}^{-1}) \). Our default model is flat \( \Lambda \)CDM with \( \Omega_m = 0.27 \); in other cases, \( w \) is the dark energy equation of state, \( \Omega_{\text{tot}} = \Omega_m + \Omega_\Lambda \) is the total density parameter and \( \Omega_i \equiv 1 - \Omega_{\text{tot}} \).

### 2 Observations of the BAO Length

The BAO feature appears as a single hump in the galaxy correlation function \( \xi(r) \), or equivalently a series of decaying wiggles in the power spectrum (see Eisenstein, Seo & White 2007a, for a clear exposition, and Bassett & Hlozek 2010, for a recent review). In the following, we denote \( r_z \) to be the comoving length scale of the BAO feature in a galaxy redshift survey, while \( r_z(z) \) is the comoving sound horizon at the baryon drag epoch \( z_d \approx 1020 \), which is conventionally defined as the fitting formula equation (4) of Eisenstein & Hu (1998). These lengths are not quite identical due to evolution of perturbations after the drag epoch and non-linear growth of structure, but the shifts are predicted to be below the 0.6 per cent level and well correctable from theory (Eisenstein et al. 2007b; Seo et al. 2008, 2010); therefore, measuring the BAO feature at low redshift provides a very robust link to the sound horizon in the CMB era.

In the small angle approximation and assuming we observe a redshift shell which is thin compared with its mean redshift, the angular size of the BAO feature is \( \Delta \theta(z) = r_z/(1+z)D_A(z) \), where \( D_A(z) \) is the usual proper angular diameter distance to redshift \( z \); and the difference in redshift along one BAO length in the line-of-sight direction is \( \Delta z_l(z) = r_z H(z)/c \) (Blake & Glazebrook 2003; Seo & Eisenstein 2003).

In practice, current galaxy redshift surveys are not quite large enough to distinguish the BAO feature separately in angular and radial directions, so the current measurements constrain a spherically averaged length scale; the most model-independent quantity derived from these observations is \( r_z/D_V(z) \), where \( D_V \) is called the dilation scale and is defined by Eisenstein et al. (2005) as

\[
D_V(z) \equiv \left( 1 + z \right)^2 D_A^2(z) \frac{cz}{H(z)} \right)^{1/3}.
\]

This is essentially a geometric mean of two transverse and one radial directions.

Measuring the BAO feature from a galaxy redshift survey requires a mapping from observed galaxy positions and redshifts to galaxy pair separations in comoving coordinates, which is itself dependent on cosmological parameters including \( H_0, \Omega_m, w \) etc. Therefore, extracting the value of \( r_z \) from a galaxy redshift survey is slightly theory dependent; but the above dimensionless ratios \( \Delta \theta(z), \Delta z_l(z) \) and \( r_z/D_V(z) \) are essentially theory independent, since to first order any change in the reference cosmology produces an equal shift in the fitted length \( r_z \).

As above, the comoving length \( r_z(z) \) is defined as the sound horizon at the baryon drag epoch \( z_d \approx 1020 \) (Eisenstein & Hu 1998).

Adopting standard early-universe assumptions, \( r_z(z) \) depends only on the densities of matter, baryons and radiation; the latter is very well constrained by the CMB temperature (assuming standard neutrino content), hence \( r_z \) depends only on \( \omega_m \) and \( \omega_b \), which in turn are well constrained by the heights and morphology of the first three CMB peaks. Fits from the 7-year Wilkinson Microwave Anisotropy Probe (WMAP7) data alone (Komatsu et al. 2011) give \( r_z(z) \approx 153 \text{ Mpc} \) with approximately 1.3 per cent precision, and forthcoming data from the Planck mission (Ade et al. 2011) are expected to improve this prediction of \( r_z \) to \( \approx 0.3 \) per cent precision.

We note that the CMB derivation of \( r_z(z) \) does not rely on assuming a flat universe or details of dark energy, since the observed CMB peak heights constrain \( \omega_m \) and \( \omega_b \) well without assuming flatness. However, the inference of \( r_z(z) \) does rely on many simple but weakly tested assumptions about the \( z > 1000 \) universe, including

(i) standard GR;
(ii) standard radiation content (photons plus an effective number \( N_{\text{eff}} \approx 3.04 \) light neutrinos);
(iii) standard recombination history, including negligible variation in fundamental constants;
(iv) negligible early dark energy;
(v) negligible contribution of isocurvature fluctuations;
(vi) the primordial power spectrum is smooth and almost a power law;
(vii) densities of matter and radiation scale as the standard powers of scale factor; e.g. negligible late-decaying particles at \( z \lesssim 10^6 \) etc.

If one or more of the above assumptions are wrong, this can bias the value of \( r_z \) deduced from the CMB fits, and in turn this will generally result in biased inferences about other cosmological parameters (especially \( H_0 \)) when doing joint fits to CMB and BAO data. Possible violation of (ii) above was analysed by Eisenstein & White (2004), and is discussed later in Section 4; for some other non-standard cases, see for example Linder & Robbers (2008) for early dark energy, Menegoni et al. (2012) for varying \( \omega_b \), and Zunckel et al. (2011) for the effect of isocurvature fluctuations.

For the above reasons, measuring the absolute BAO length scale at low redshift forms a powerful consistency test of the assumptions underlying standard cosmology at \( z > 1000 \).

The well-known route to this is to measure the transverse BAO scale at some effective redshift which gives \( r_z/(1+z)D_A(z) \), and also use a combination of standard candles (e.g. Type Ia supernovae (SNe Ia)) and the local Hubble constant \( H_0 \) to measure the usual
luminosity distance, $D_L(z)$, to the same redshift. Combined with the standard distance–duality relation $D_L(z) = (1 + z)^{2}D_v(z)$, this can give a theory-independent absolute measurement of the BAO length. However, one disadvantage of the above is that it requires a BAO survey of sufficiently large volume to separate the transverse and radial BAO scales, and reaching sufficient cosmic volume requires a survey at significant redshift $z \gtrsim 0.3$; in turn, this means that SNe Ia are likely the only distance indicators bright enough to be useful for measuring $D_L(z)$, and there is a non-negligible time baseline over which SN evolution may bias the measurements of $D_L(z)$.

As a complement to the above, it would be valuable to calibrate $r_s$ using BAO measurements of $r_s/D_v(z)$ at lower redshifts $0.1 \lesssim z \lesssim 0.25$, combined with an accurate calibration of $D_v(z)$ from distance indicators. Although BAO surveys at lower redshift suffer from increased cosmic variance due to the limited available volume, there are several compensating benefits: there is a shorter time baseline for possible evolution of SNe properties; the SNe are brighter and more readily observable in the rest-frame near-infrared (IR); and low redshift offers better prospects for using alternative distance indicators such as gravitational lens time delays, and potentially gravitational-wave standard sirens (Sathyaprakash et al. 2011); and finally we avoid the complication of separating the radial and angular BAO scales in the analysis.

However, this low-$z$ route requires an absolute measurement of $D_L(z)$ rather than $D_v(z)$, which is slightly more challenging; from equation (1), a measurement of $D_L(z)$ tells us $D_v(z)$ apart from an unknown factor of $H(z)^{1/3}$; this is helpful due to the $1/3$ power of $H$, but is not good enough for per cent level precision. At $z \to 0$, $D_L(z) \to cz/H_0$, as with all cosmological distances; however, there is insufficient volume to measure the BAO feature at $z \lesssim 0.05$, and beyond this cosmological distance effects cannot be ignored.

For a concordance $\Lambda$CDM model at an example $z = 0.2$, the crude approximation $D_v(z) \sim cz/H_0$ is 6 per cent too large, while the approximation $D_v(z) \approx D_L(z)/(1 + z)$ is too large by 1.6 per cent; these approximations are clearly not good enough for precision cosmology.

Since $D_v(z)$ is directly related to the comoving volume element per unit redshift, via

$$\frac{dV}{dA} = \frac{c(1 + z)^2 D_v(z)}{H(z)} = \frac{1}{z} D_L(z),$$

(2)

where $dV$ is comoving volume and $dA$ is solid angle, we could measure $D_v$ directly if we could observe a population of ‘standard counters’ of known comoving density. However, our limited understanding of galaxy evolution implies that there is little hope of finding standard counters good enough for a per cent level measurement of $D_v$. Alternatively, a direct measurement of $H(z)$ is possible using differential ages of red galaxies (e.g. Stern et al. 2010), but again it may be very challenging to reach few per cent absolute accuracy with this method.

In the next section, we show a new alternative route for obtaining accurate calibration of $D_v$; we find a much better approximation for $D_v$, which relates $D_v(z)$ to the observable $D_L$ at a slightly higher redshift, specifically $\frac{7}{8}z$.

3 A ROUTE TO MEASURING $D_v$

3.1 Relation between $D_v$ and $D_L$

Here we find an accurate approximation relating the dilation length $D_v(z)$ to the observable luminosity distance $D_L$ at a slightly higher redshift.

We first define as usual the scale factor $a = (1 + z)^{-1}$ with $a_0 = 1$, and the time-dependent Hubble parameter $H(z) = \dot{a}/a$, where dot denotes time derivative. We also have the usual expression for comoving radial distance,

$$D_R(z) = c \int_0^{z} \frac{1}{H(z)} \, dz.$$  

(3)

In the appendix of Sutherland (2012), we found a good approximation at moderate redshift:

$$D_R(z) \approx \frac{cz}{H(z)}.$$  

(4)

This approximation was derived using a Taylor series expansion of $1/H(z)$ around redshift $z/2$; this results in the first derivative $(1/H)'$ cancelling so there is no error of order $z^2$, and uses the convenient fact that the second derivative $(1/H)''$ has a zero-crossing at $z \sim 0.3$ in a concordance $\Lambda$CDM model, so the error of order $z^2$ is small at moderate redshift. In a flat universe this leads to

$$D_L(\frac{1}{2}) \approx (1 + z) \frac{cz}{H(z)}.$$  

(5)

This is still fairly accurate even for weakly curved models, since the multiplicative change in $D_L$ for a non-flat model is of order $1 + \Omega_m z^2/6$ for fixed expansion history; for plausible values of $|\Omega_m| < 0.05$, this is a very small effect at $z \lesssim 0.3$.

In Sutherland (2012) we also found a good approximation for $D_L(z)$ at moderate redshift, which is

$$D_L(z) \approx \frac{cz}{H(z)}.$$  

(6)

Both approximations (5) and (6) are accurate to $\lesssim 0.4$ per cent for $z < 0.5$ for models reasonably close to standard $\Lambda$CDM; the error in approximation (6) is shown in Fig. 1 for some example models. (This will be useful below in Section 4.3).

Both equations (5) and (6) involve $H(z)$ at slightly different redshifts; however, it is clear from the above that if we consider a BAO measurement at effective redshift $z_1$, then $D_v(z_1)$ is closely related to $H(z_2)/3$, while if we consider $D_L(z_3)/3$ this is also related to $H(z_3)/3$; we can therefore combine approximations (5) and (6) to

$$\frac{\Omega_0}{\Omega_V} = 0.27,$$

$$\Omega_m = 0.22,$$

$$\Omega_r = 0.33,$$

$$\omega = -0.85,$$

$$\Omega_{\Lambda} = 0.9,$$

$$\Omega_{\Lambda} = 1,$$

$$\Lambda = 0.$$  

Figure 1. The relative accuracy of approximation (6) for various cosmological models. The solid lines show flat $\Lambda$CDM models with $\Omega_m = 0.22, 0.27, 0.33$ (top to bottom). The dashed lines show non-flat $\Lambda$CDM with $\Omega_m = 0.90$ (lower) and 1.1 (upper). The dash–dot line shows flat $\sigma$CDM with $\omega = -0.85$. The dotted lines show $\Omega_m = 1$ (upper), and open $\Omega_m = 0.27, \Omega_{\Lambda} = 0$ (lower).
cancel the unknown $H(2z/3)$, which gives the approximation

$$D_v(z) \simeq \frac{3}{4} D_L \left( \frac{4}{3} z^3 \right) \left( 1 + \frac{4}{3} z^3 \right)^{-1}. \quad (7)$$

We now explore the error in approximation (7): Fig. 2 shows the ratio of the right-hand side (RHS) of equation (7) to the exact $D_v(z)$ for various example cosmological models. Unless otherwise specified, we take $\Omega_m = 0.27$ for each model. Specifically, Fig. 2 shows three flat $\Lambda$CDM models with $\Omega_m = 0.22, 0.27, 0.33$; one flat $wCDM$ model with $w = -0.85$; and two non-flat $\Lambda$CDM models with $\Omega_m = 0.9$ and 1.1, respectively; finally, an Einstein–de Sitter $\Omega_m = 1$ model, and a zero dark energy model with $\Omega_m = 0.27, \Omega_\Lambda = 0$. (These latter two models are well known to be grossly inconsistent with CMB and other measurements, but are included for comparison purposes.)

It is clear from Fig. 2 that approximation (7) is surprisingly accurate: for the three $\Lambda$CDM models the error is less than 0.2 per cent at $z < 0.4$, and the $w$CDM model is only slightly worse. The two $\Lambda$CDM models are still quite good, with error <0.4 per cent at $z < 0.4$; this is considerably better than the medium-term expected errors $\approx 1$ per cent on the BAO ratio, and also current upper limits on $|\Omega_k|$ are significantly tighter than 0.1; more realistic values $|\Omega_k| \sim 0.02$ give rise to minimal error in approximation (7). Therefore, for any WMAP-allowed model, the error in approximation (7) is several times smaller than the medium-term precision on BAO observables.

The approximation (7) becomes significantly worse for the Einstein–de Sitter and open zero-$\Lambda$ models, with errors, respectively, $+1.25$ and $+2.0$ per cent at $z = 0.4$; however, even these give sub-per-cent error at $z \leq 0.25$.

In fact, equation (7) is exact at all $z$ for a de Sitter model with $\Omega_m = 0, \Omega_\Lambda = 1$, while its accuracy degrades rather slowly with increasing $\Omega_m$ and/or curvature; thus, for near-flat and accelerating models favoured by current data, it is remarkably accurate. An explanation of this property in terms of Taylor series is given in Appendix A: this shows that equation (7) is exact to second order in $z$ independent of all cosmological parameters; while at third order, there is a fortunate coincidence that for deceleration parameter $q_0 \lesssim -0.4$ and small curvature, the difference in the $z^3$ coefficients is also small. This makes equation (7) accurate at $z \lesssim 0.4$ for all near-flat accelerating models, with little dependence on precise values of $\Omega_m, \Omega_\Lambda, w$ etc. (Note that all results in the main body of the paper use the numerical integrals for $D_v$ and $D_L$; the Taylor series in Appendix A are only provided as an aid to intuition.)

We also see from Fig. 2 that approximation (7) is a slight overestimate of the exact $D_v(z)$ for all the flat and open models shown; only the closed model ($\Omega_m = 1.1$) gives an underestimate. Since equation (7) gives a slight overestimate of the exact $D_v$ for all the plausible models fairly close to $\Lambda$CDM, we can get a modest but useful improvement by removing this bias, by multiplying by a polynomial in $z$ chosen to give a good fit for concordance $\Lambda$CDM; we find an excellent fit with small terms in $z^3$ and $z^4$, specifically

$$D_v(z) \simeq \frac{3}{4} D_L \left( \frac{4}{3} z^3 \right) \left( 1 + \frac{4}{3} z^3 \right)^{-1} \left( 1 - 0.0245 z^3 + 0.0105 z^4 \right). \quad (8)$$

By construction, this approximation is excellent for the concordance model, with relative error <0.02 per cent at $z \leq 1$. For other plausible models, the resulting ratio (RHS of 8)/(exact $D_v(z)$) is shown in Fig. 3; in this figure we have used a smaller range of $\Omega_k$ for the non-flat models, and added two models with time-varying $w$ with the common parametrization $w(a) = w_0 + w_a (1 - a)$, to give a set of models roughly spanning the 2$\sigma$ allowed range from current data.

It is clear from Fig. 3 that approximation (8) is very accurate in the WMAP-allowed neighbourhood of $\Lambda$CDM, including generous variations of $\Omega_m$, modest curvature and $w \neq -1$. For all the models shown the relative error is smaller than $(z/200)$ at $z < 1$, thus 0.2 per cent error at $z = 0.4$. This error is substantially smaller than the cosmic variance in BAO measurements, and the expected accuracy in next-decade absolute distance measurements, so is almost negligible for practical measurements of $\kappa$.

This means that a direct measurement of $D_L(4z/3)$ can immediately predict $D_v(z)$ with very little dependence on cosmological parameters $H_0, \Omega_m, \Omega_\Lambda, w$. Multiplying this by a BAO measurement

![Figure 2](https://academic.oup.com/mnras/article-abstract/426/2/1280/974341)
of $r_s/D(z)$ from a galaxy redshift survey thus measures $r_s$ in comoving Mpc, based entirely on low-redshift data.

This can then be compared with a CMB-only prediction of $r_s(z_d)$ for a zero parameter test of our early universe assumptions: if the local $r_s$ measured from BAOs and $D(z)$ as above is not consistent with the $r_s$ inferred from the CMB, something is definitely wrong with one or more measurements, or the early-universe assumptions or the FRW framework; tuning of the late-time cosmological parameters $\Omega_m, \Omega_k, w$ within the 3$\sigma$ WMAP-allowed ranges cannot significantly help. Conversely if a bottom-up measurement of $r_s$ does agree at the $\sim 1$–2 per cent level with the CMB prediction, this would provide simple and compelling support for the standard set of early-universe assumptions.

We next discuss some questions in both the CMB and local methods for measuring $r_s$.

4 NON-STANDARD RADIATION DENSITY

Here we consider the effect of non-standard radiation density, which is an important degeneracy for all CMB-determined length scales. This is moderately well known, first analysed for the BAO case by Eisenstein & White (2004), and also in e.g. de Bernardis et al. (2008), but we give a slightly different and hopefully more intuitive explanation compared to previous work.

4.1 Definitions of radiation density

The value of $r_s(z_d)$ is given from the WMAP data alone as 153 Mpc with 2 per cent precision, just assuming standard radiation content but no assumptions about flatness or dark energy. However, if the assumption of standard radiation content is dropped, the precision degrades radically to $> 10$ per cent (Komatsu et al. 2011). The reason is mainly the strong degeneracy between matter density $\Omega_m$ and radiation density $\Omega_{rad}$ in the CMB fits: the first three CMB peaks determine the redshift of matter/radiation equality $z_{eq}$ well, with $1 + z_{eq} \equiv \Omega_m/\Omega_{rad} \approx 3200 \pm 140$, but converting to the physical matter density $\omega_m$ then relies on an assumption of the total radiation density.

The radiation density is conventionally parametrized by an effective number of neutrino species $N_{eff}$ in the CMB era, defined via

$$\omega_{rad} \equiv \omega_{\nu} \left[ 1 + \frac{4}{3} \left( \frac{4}{11} \right)^{4/3} N_{eff} \right].$$

(9)

(Here for non-negligible neutrino mass, $\omega_{rad}$ is not the present-day radiation density, but the value at $z_{eq}$ rescaled by $(1 + z_{eq})^{-4}$. We set $\omega_m = \omega_k + \omega_{\nu}$ to include dark matter and baryons only, excluding any low-redshift contribution from neutrino mass.)

Most analyses assume a standard value very close to $N_{rad} = 3.046$ effective neutrino species (Mangano et al. 2005) which gives $\omega_{rad} = 1.6918 \omega_{\nu}$; and the photon density $\omega_{\gamma} = (40440)^{-1}$ is set by the very accurate CMB temperature, $T_0 = 2.7255$ K. However, we note that there already some hints of a higher value of $N_{eff}$ from e.g. Keisler et al. (2011); these are not yet decisive, but are very interesting.

For general $N_{eff}$, the above can be rearranged into

$$\omega_m = 0.1339 \left( \frac{1 + z_{eq}}{3201} \right) [1 + 0.134(N_{eff} - 3.046)].$$

(10)

4.2 The $N_{eff}$/scale degeneracy

Here we review in more detail the effect of non-standard $N_{eff}$ on cosmological parameter estimates, and show essentially that this creates a degeneracy in overall scale factor which affects all cosmic distances, times and densities, but has very little effect on dimensionless ratios.

It is helpful to rearrange the expression for the sound horizon (e.g. equation 6 of Eisenstein & Hu 1998) in terms of $z_{eq}$, $\omega_{rad}$ and $\omega_m$.

Figure 3. The relative accuracy of approximation (8) for various cosmological models, roughly consistent with WMAP. The solid lines show flat $\Lambda$CDM models with $\Omega_m = 0.22, 0.27, 0.33$ (bottom to top). The dash-dotted lines show flat $w$CDM with constant $w = -0.85$; the dashed lines show evolving $w$ models with values $(w_0, w_z) = (-1.2, +0.8)$ and $(-0.8, -0.8)$, respectively. The dotted lines show non-flat $\Lambda$CDM with $\Omega_m = 0.98$ (upper) and 1.02 (lower). (Note the axis scales are different from Fig. 2.)

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\( \omega_0 \) as the input parameters, which gives

\[
\rho_s(z_0) = 2998 \text{ Mpc} \frac{2}{\sqrt{\omega_\text{rad}}} \frac{\alpha_{\text{rad}}^{-1/2} (1 + z_{\text{eq}})^{-1} R_{\text{eq}}^{-1/2}}{\ln \left( \frac{\sqrt{1 + R_{\text{b}}} + \sqrt{R_{\text{d}} + R_{\text{eq}}}}{1 + \sqrt{R_{\text{eq}}}} \right)}
\]

where \( R(z) \approx 30330 \omega_0(1 + z) \) is the baryon/photon momentum density ratio, and \( R_{\text{b}}, R_{\text{d}}, R_{\text{eq}} \) are the values at \( z_{\text{eq}}, z_{\text{eq}}, z_{\text{eq}}, \) respectively. This shows that if we vary \( \omega_\text{rad} \) while holding \( z_{\text{eq}}, z_{\text{eq}}, \omega_0 \) all fixed, the sound horizon scales simply \( \propto \omega_\text{rad}^{1/2} \). In more detail, the WMAP best-fitting values show small changes in \( z_{\text{eq}} \) and \( \omega_0 \) with varying \( N_{\text{eff}} \), see section 4.7 of Komatsu et al. (2011) for details; however, the consequential shifts in \( \rho_s(z_0) \) are some 10 times smaller than the dominant \( \omega_\text{rad}^{1/2} \) shift, so we ignore those for simplicity.

Next for illustration we take a specific example of two models, an arbitrary model A with \( N_{\text{eff}} = 3.046 \) and parameters assumed a good fit to WMAP, and a model B with one extra neutrino species (or equivalent in dark radiation), thus \( N_{\text{eff}} = 4.046 \) but the same \( z_{\text{eq}} \). Thus, model B has both \( \omega_\text{rad} \) and \( \omega_0 \), larger by 13.4 per cent, while the sound horizon in B is smaller by a factor close to \((1.134)^{1/2} \approx 0.939 \). This would be severely discrepant with the observed position of the CMB acoustic peaks via the acoustic scale \( \ell_\ast \), if either the distance to last scattering \( D_L(z_0) \) or \( H_0 \) were held fixed.

However, if we also choose model B to have increased \( \omega_\text{DE}, \omega_0 \) by the same factor of 1.134 above, then via

\[
h^2 = \omega_0 + \omega_\text{DE} + \omega_0 \tag{12}
\]

model B has \( H_0 \) increased by 6.5 per cent while all of \( \Omega_m, \Omega_{\Lambda}; \Omega_{\text{rad}}, \Omega_\text{DE} \) are identical in models A and B. Since the expansion function \( E(z) = H(z)/H_0 \) depends only on the upper case \( \Omega \) values above, \( E(z) \) remains unchanged at all redshifts; so all cosmic distance \( z \) and \( \Omega \) functions are reduced by \( \approx 6 \) per cent in model B relative to A, but distance ratios between any two redshifts (related to BAO and SN Ia observables and the CMB acoustic scale \( \ell_\ast \)) remain unchanged. (We note here that \( \omega_\text{DE} \) and \( \omega_0 \) are assumed unchanged between models A and B, so \( \Omega_b \) and \( \Omega_{\Lambda}/\Omega_m \) are reduced by 13 per cent in model B, but these do not appear separately in the Friedmann equation. The implied value of \( \sigma_8 \) will also be slightly different for model B as shown by WMAP, but we do not consider that here.)

What is happening here is simple: apart from \( \omega_0 \), WMAP mainly constrains dimensionless quantities: especially \( z_{\text{eq}} \), the acoustic scale \( \ell_\ast \), at last scattering \( z_{\text{ls}} \), and the shift parameter \( \tilde{R} \). Also, BAO measurements are intrinsically dimensionless ratios such as \( r_{s}/D_L(z), \) while SN measurements anchored to the local Hubble flow also give dimensionless ratios, especially \( H_0, D_L(z)/c \) or \( D_L(z)/D_L(z_{0.03}) \) (while \( H_0 \) is degenerate with the standardized SN luminosity). All these above provide precision measurements of the uppercase \( \Omega \) values and \( \ell_\ast \) with no overall scale needed.

However, there are three dimensionful quantities (lengths, times and densities, or combinations of these) in homogenous cosmology; while there are two interrelations: distances and times are related by the known \( c \), and the Friedmann equation relates densities to expansion rate, via \( G \). This implies that even excellent knowledge of all those dimensionful ratios above is not sufficient to solve for any dimensionful quantity; but adding a measurement of \textit{any one} cosmological length, time, or absolute density of matter, radiation or dark energy (in SI units or equivalent) would be sufficient to constrain all the others. Usually, this dimensionful quantity is (implicitly) set by assuming \( N_{\text{eff}} \approx 3.046 \), which fixes \( \omega_\text{rad} \) and thus all the other scales: but if this assumption is dropped, then WMAP+SNe observations leave us short by one dimensionful quantity, and the \( N_{\text{eff}} \) versus \( H_0 \) degeneracy appears.

Given the above, it is convenient to define the scaled radiation density as

\[
X_{\text{rad}} \equiv \omega_\text{rad}/1.692 \omega_0 = 1 + 0.134(N_{\text{eff}} - 3.046),
\]

so that the standard value is 1; and also to choose a fundamental parameter set including

\[
\Omega_m; \quad z_{\text{eq}}; \quad X_{\text{rad}}; \quad \omega_0 \tag{14}
\]

plus optional parameters \( \Omega_\Lambda, w \) defaulting to 0, −1; as usual \( \Omega_{\Lambda} = 1 - \Omega_m - \Omega_\text{DE} \). This set couples very naturally to the observables, and turns both \( \omega_0 \) and \( H_0 \) into derived parameters, via

\[
\omega_\text{rad} = X_{\text{rad}}/23904,
\]

\[
\omega_m = (1 + z_{\text{eq}})X_{\text{rad}}/23904,
\]

\[
h = \sqrt{(\omega_m/\Omega_m)}.
\]

Currently, the observational uncertainty on \( X_{\text{rad}} \) is substantially larger than on the other major parameters: the central value depends somewhat on choice of data sets, with some data sets favouring \( N_{\text{eff}} \approx 4 \) (e.g. Keisler et al. 2011), while others prefer the standard \( N_{\text{eff}} \approx 3 \) (e.g. Mangano & Speric 2011). There is broad agreement that \( 2 \leq N_{\text{eff}} \leq 5 \), which maps to 0.86 ≤ \( X_{\text{rad}} \) ≤ 1.27. We find from WMAP results that if we allow \( X_{\text{rad}} \neq 1 \), then current cosmological measurements mainly constrain the combinations \( \omega_\text{rad} \approx (0.135 \pm 0.005) X_{\text{rad}}, \omega_m \approx (153 \pm 2)/\sqrt{X_{\text{rad}}} \text{ Mpc}, H_0 \approx (70 \pm 1.5) \sqrt{X_{\text{rad}}} \text{ km s}^{-1} \text{ Mpc}^{-1} \) and \( t_0 \approx (13.75 \pm 0.1)/\sqrt{X_{\text{rad}}} \text{ Gyr} \); all these dimensionful observables have error bars dominated by the uncertainty in \( X_{\text{rad}} \), while most dimensionless ratios are nearly uncorrelated with \( X_{\text{rad}} \) (the main exceptions are \( n \) and \( \sigma_8 \), which both show small positive correlations with \( X_{\text{rad}} \)). This simple scaling accurately reproduces the degeneracy track of \( t_0 \) versus \( N_{\text{eff}} \) shown by de Bernardis et al. (2008).

(We emphasize an important distinction that \( h \) and \( \omega_0 \) do not count as dimensionless in this discussion; they are clearly pure numbers, but they represent dimensionful quantities rescaled by an arbitrary choice of \( H_0 = 100 \text{ km s}^{-1} \text{ Mpc}^{-1} \) and a fiducial density \( \rho_\text{rad} = 1.878 \times 10^{-26} \text{ kg m}^{-3} \). The true dimensionless quantities, such as \( z_{\text{eq}}, \Omega_m, H_0D_L(z)/c, H_0r_c/c, \ell_\ast, H_0d_0 \text{ etc.}, \) have values independent of any system of units.)

To break the above degeneracy, it is sufficient to get an accurate measurement of \( X_{\text{rad}} \) and \( \omega_m \) (say \( X_{\text{rad}} \approx 70 \text{ Mpc s}^{-1} \text{ Mpc}^{-1} \) and \( r_s \approx 153 \text{ Mpc} \) to \( \approx 75 \text{ Mpc s}^{-1} \text{ Mpc}^{-1} \) and 1.44 Mpc, respectively, if we assume \( \Lambda \text{CDM} \).
With $N_{\text{eff}} \approx 4.0$. If the latter is the actual cosmology, an observed lower bound $H_0 \geq 73$ km s$^{-1}$ Mpc$^{-1}$ could be fitted with any of $N_{\text{eff}} \approx 4$, or $N_{\text{eff}} \approx 3$ with $w < -1$ and/or open curvature; but a direct upper bound $r_s \leq 148$ Mpc would exclude all of the currently allowed range for $N_{\text{eff}} \approx 3$, and decisively require extra radiation or some other new physics at $z > 1000$.

Furthermore, it is helpful to compare the $N_{\text{eff}}$/scale degeneracy to the better known geometrical degeneracy affecting parameter fitting from the CMB alone (Efstathiou & Bond 1999). Although both degeneracies affect $H_0$, the geometrical degeneracy involves holding fixed physical densities of both matter and radiation (thus fixed $r_s$), while trading off two of $\Omega_m, \Omega_k, w$ so as to maintain a fixed angular distance to last scattering $D_A(z_e)$; this is well broken by BAO ratios, as we see below. The $N_{\text{eff}}$/scale degeneracy above also holds $z_{eq}$ fixed, but rescales densities and distances by $X_{\text{rad}}$ and $1/\sqrt{X_{\text{rad}}}$, respectively; here both $D_A(z_e)$ and $r_s(z_e)$ shift by a common factor. This $N_{\text{eff}}$/scale degeneracy is not broken by BAO distance ratios, but is broken with an absolute BAO length measurement. Therefore, these two degeneracies are ‘orthogonal’ concerning $r_s$, but get mixed in $H_0$, which explains why $r_s$ is a cleaner test of the early Universe.

### 4.3 An easy route to $\Omega_m$

Here we find a strikingly simple route to $\Omega_m$, accurate to better than 1 per cent: first it is convenient to define

$$1 + \epsilon_V(z; \Omega_m, \Omega_k, w) \equiv \frac{cz}{H(\frac{z}{2}) D_V(z)},$$

so the function $\epsilon_V$ is defined to be the (small) correction to approximation (6), as shown in Fig. 1. Then, taking an observed BAO ratio $r_s/D_V(z)$, substituting equation (11) and using $h/\sqrt{\Omega_m} = \sqrt{(1 + z_{eq})/\Omega_m}$, we obtain

$$\frac{z r_s}{D_V(z)} = \frac{r_s H(\frac{z}{2}) \epsilon_V}{c (1 + \epsilon_V)}$$

$$= (1 + \epsilon_V) \frac{E(\frac{z}{2})}{\sqrt{\Omega_m}} \left( \frac{2}{\sqrt{3}} (1 + z_{eq})^{-1/2} R_{eq}^{-1/2} \right) \times \ln \left( \frac{1 + R_{eq} + \sqrt{R_{eq} + R_{eq}^2}}{1 + R_{eq}} \right).$$

This is exact apart from non-linear shifts of $r_s$. All the terms above are clearly dimensionless; both $H_0$ and $\Omega_m$ have cancelled, and there is only a small implicit dependence on $\Omega_{\text{rad}}$ via very small changes in $z_0$.

The last three factors on the RHS above are well constrained given only $z_{eq}$ and $\Omega_\text{b}$ from WMAP, which are almost independent of dark energy, curvature or radiation density. Adopting $\Omega_{\text{b}} = 0.0225$, the RHS above is very well approximated by

$$\frac{z r_s}{D_V(z)} \approx 0.01868 (1 + \epsilon_V) \frac{E(\frac{z}{2})}{\sqrt{\Omega_m}} \left( \frac{1 + z_{eq}}{3201} \right)^{0.25},$$

with the uncertainty due to $\Omega_{\text{b}}$ below 0.4 per cent.

A precise moderate-redshift BAO measurement from SDSS is given by Padmanabhan et al. (2012) as $D_V(z = 0.35)/r_s = 8.88 \pm 0.17$; thus the left-hand side (LHS) above is 0.0394 \pm 0.0008. This, together with $z_{eq} \approx 3200 \pm 130$ and neglecting the sub-per cent $\epsilon_V$ term gives $E(0.233)/\sqrt{\Omega_m} = 2.11 \pm 0.05$; simply squaring this and rearranging gives a linear relation (for $w = -1$) of $\Omega_m = (0.280 + 0.145 \Omega_\text{r})(1 \pm 0.05)$, in excellent agreement with the full likelihood results.

There is a common rule-of-thumb that ‘CMB measures $\Omega_m$ and the BAOs measure $H_0$’; we see from the above that this is only valid assuming the standard $N_{\text{eff}} \approx 3.0$. For general radiation density, the CMB is really measuring $\Omega_{\text{r}}$, not $\Omega_m$—adding a low-redshift BAO measurement then measures primarily $\Omega_m$, with a small sensitivity to $\Omega_k$ and $w$ creeping in via the $E(z)/\sqrt{\Omega_m}$ term. Combining $z_{eq}$ and $\Omega_m$ gives us a value for $H_0/\sqrt{X_{\text{rad}}}$ from equation (15), again with mild dependence on $\Omega_k$, $w$, but does not give an absolute scale.

(The additional information from the large-scale bend in the galaxy power spectrum is discussed in Appendix B.)

### 5 DISTANCE AND SOUND HORIZON MEASUREMENTS

Here we discuss some considerations on observational issues and the realistic precision available for measurements of the absolute BAO length, both locally and from future CMB measurements.

#### 5.1 Distance ladder measurements

Given a measurement of $r_s/D_V(z)$ from BAOs in a redshift survey, we need an absolute measurement of $D_L(z; 4z/3)$ to apply equation (8) and obtain an absolute measurement of $r_s$ independent of the CMB. The most obvious route to measure $D_L(z; 4z/3)$ is to combine a local distance ladder measurement of $H_0$ with a large sample of SNe Ia centred at redshift near 4z/3 to measure $D_L(z; 4z/3)$; along with approximation (8) this provides a direct calibration of $D_L(z)$ and thus $r_s$.

Doing this at fairly low redshift has several advantages: first, it is observationally much cheaper to accumulate a large sample of SNe at $z < 0.3$ compared with $z > 0.5$, and such a sample should arise naturally from the ongoing Pan-STARRS Medium Deep Survey (Kaiser et al. 2010) and the Dark Energy Survey ( Bernstein et al. 2011). Also, lower redshift provides a smaller lever-arm for possible time evolution of the mean SN brightness, minimizing systematic errors. Given an overabundance of SNe (e.g. several hundred in the relevant redshift bin), one can afford to subdivide the sample by light-curve stretch, host galaxy type etc., to provide consistency checks.

Furthermore, there is growing evidence that SNe Ia are closest to standard candles in the rest-frame near-IR wavelengths, specifically the $J$ and $H$ passbands (Barone-Nugent et al. 2012). At $z \approx 0.3$ these bands redshift into observed $H$ and $K_s$, respectively, so that redshift is a sweet-spot which minimizes $k$-corrections.

We note here that this is significantly different to the more common case of computing dark energy figures of merit; in the dark energy case, breaking degeneracies between $\Omega_m, \Omega_k$ and dark energy parameters $w_0, w_a$ requires relative distance measurements spanning a broad range of redshift $0.2 \ll z \ll 1.5$; for SNe anchored to local samples at $z \sim 0.5$, SNe at higher redshift have greater leverage on $w_0$ and especially $w_a$. Since BAOs are anchored in the CMB, the preference for higher redshift is weaker than for SNe, but the rapid increase in available cosmic volume still favours redshifts $0.5 \ll z \ll 1.5$ for precision measurements of $w_0$ and $w_a$ (Weinberg et al. 2012).

In contrast to the above, for an absolute $r_s$ measurement we only need to measure an absolute distance $D_L$ to one specific redshift matched to a given BAO survey: the overall $r_s$ accuracy is simply the quadrature sum of the BAO and $D_L$ errors, with a small addition from the error in equation (8), but there is no lever-arm...
gain towards higher redshift. Thus, the number of required SNe for given precision on $D_L$ is independent of the target redshift; thus low redshifts are both observationally cheaper, and more robust against systematics such as time evolution and imperfect $k$-corrections.

### 5.2 Physical distance measurements

One major benefit of our approximation (8) is that there is no explicit dependence on $H_0$. Therefore, if we can measure $D_L(z)$ to $z \sim 0.25$ using some physical-based method which does not rely on calibration via the local distance ladder and $H_0$, we automatically bypass the uncertainties in the local distance scale.

There are several current or proposed methods for doing this, including gravitational lens time delays, Sunyaev–Zeldovich measurements of galaxy clusters and the expanding-photons field method applied to Type II SNe; however, all of these methods have some level of model dependence and it is not yet clear whether they can reach the percent level absolute accuracy (e.g. 2 per cent accuracy for a $3\sigma$ detection of one additional neutrino species). The lens time-delay method is especially clear at low lens redshift; while lensing observables involve a combination of lens and source distances $D_L$, $D_s$ and $D_m$, selecting systems with $z_\text{eff} \ll z$, makes the ratio $D_0/D_L$, close to unity and well constrained, which is favourable for absolute measurement of the lens distance.

Potentially the ultimate $D_L$ calibration method is the detection of gravitational waves (GWs) from coalescing compact binaries (Schutz 1986), since the model waveforms can be predicted extremely precisely assuming only Einstein gravity, and the method is completely immune to dust extinction or astrophysical nuisance parameters. Of course, such events have not been directly observed so far, but the observations of binary pulsars (Kramer & Stairs 2008) leave no doubt that the waves exist, and there are ongoing upgrades to Advanced Laser Interferometer Gravitational-Wave Observatory (LIGO) and Virgo which should give a near-certain detection of binary inspirals around 2015, assuming they reach their design sensitivity.

These second-generation GW experiments will probably provide only modest $D_L$ accuracy for most events; however, if we are lucky there may be a few ‘golden events’ with high signal-to-noise ratio (SNR), such as massive black hole events at $z \sim 0.1$. In the longer term, there is an ongoing design study for a third-generation ground-based GW observatory called ‘Einstein Telescope’ (Sathyaprakash et al. 2011) for the post–2025 era; this is projected to detect binary black hole mergers to $z \sim 2$, and neutron star + black hole mergers to $z \sim 4$. For the closer merger events at $z \sim 0.1–0.2$, Einstein Telescope would provide very high SNR and per cent-level absolute accuracy on $D_L$ for each event. If these can be tied to a unique galaxy, or statistically tied to a given cluster or sheet of galaxies, redshift constraints will be quite precise.

The future of GW distance measurements is naturally quite uncertain; however, one feature is generic: since the method is largely limited by instrumental SNR not astrophysical scatter, the closest GW inspirals should always provide the best distance precision per event. Furthermore, the closer inspirals lead to much smaller position error ellipsoids, and make it much easier to identify an optical counterpart, or statistically identify the host galaxy in a group or cluster. (Assuming the relative distance and angular errors for a GW inspiral scale $\propto 1/\text{SNR}$, then the comoving volume of the GW error box scales approximately as $D_L^3$; this results in fewer candidate host galaxies per burst at low redshift, by a very steep factor.)

From the above discussion, it is quite generic that for any cosmological distance estimate, the best prospects for per cent-level absolute accuracy on $D_L(z)$ tend to occur at modest redshift $0.1 \lesssim z \lesssim 0.25$: this is distant enough for galaxy peculiar velocities to be a small effect, but close enough to give high SNR and minimal nuisances from possible time evolution and uncorrectable gravitational lensing effects. Until the distant future when we can get cosmological distance measurements with significantly better than 1 per cent absolute accuracy, then a low redshift will be preferred for anchoring the absolute BAO length.

### 5.3 Planck measurement of $r_s$

In the near future, Planck data are expected to improve the precision on $r_s$ to around 1 per cent; assuming all the ‘standard’ early universe conditions (i.e. GR, standard radiation with $N_{\text{eff}} = 3.046$, negligible early dark energy, etc.), this will determine the sound horizon to $\sim 0.3$ per cent precision, which is significantly better than any foreseen direct distance measurement. So, why bother measuring $r_s$ locally?

If instead $N_{\text{eff}}$ is treated as free, Planck will still measure $r_s \sqrt{\epsilon_{\text{rad}}}$ to 0.3 per cent, but the error on $\epsilon_{\text{rad}}$ will dominate: the Planck measurements of the CMB damping tail (peaks 4, 5, 6) will provide a useful constraint on $N_{\text{eff}}$, but a plausible uncertainty of $\pm 0.3$ in $N_{\text{eff}}$ from Planck is equivalent to 4 per cent in $\epsilon_{\text{rad}}$ and 2 per cent in $r_s$; this is around six times worse than the standard-radiation case, and moderate-redshift BAO and distance measurements can potentially be competitive or better than this accuracy.

Furthermore, the fitting of the radiation density from the CMB relies on fairly subtle and smooth suppression of power in the CMB damping tail (Bashinsky & Seljak 2004; Hou et al. 2011); this effect is significantly degenerate with other possible adjustable parameters, including changes in primordial helium abundance $Y_p$ and non-zero running of the primordial spectral index $d\alpha_s/d\ln \alpha_s$ (Hou et al. 2011) and also with possible experimental systematics such as imperfect modelling of beam sidelobes. In CMB analyses, $N_{\text{eff}}$, $Y_p$, and $d\alpha_s/d\ln \alpha_s$ are generally fitted one-at-a-time with the other two fixed to ‘standard’ values; however, if two or three of these are simultaneously free, the CMB-only constraints on $r_s(z_s)$ may well be significantly worse than 2 per cent; while the local BAO route above can provide a direct measure of $r_s$ which is practically theory independent.

Therefore, although cosmic variance means that local BAO measurements cannot compete with the 0.3 per cent best-case Planck precision on $r_s$, this is not a major drawback: a local measurement of $r_s$ to 1–2 per cent absolute accuracy would still be of major benefit for cosmology, and could detect or exclude various early-universe effects such as non-standard $N_{\text{eff}}$ with high significance; this method is independent of early-universe uncertainties including $Y_p$, and the spectral index running which may potentially hamper the Planck measurement of $N_{\text{eff}}$.

Another motivation is that the value of $N_{\text{eff}}$ from the CMB is somewhat degenerate with the primordial spectral index $n_s$ and $d\alpha_s/d\ln \alpha_s$ (e.g. Hou et al. 2011); this can have major implications for constraining the early universe and inflation theory. If $N_{\text{eff}}$ is fixed to 4.04 rather than the standard value 3.04, the WMAP best-fitting value of $n_s$ moves up from $\approx 0.96$ to $\approx 0.975$ to compensate; this is a small change, but is potentially very important because the scale-invariant value of 1.00 is then no longer ruled out at high significance. A constraint on $N_{\text{eff}}$ directly from the absolute length $r_s$ is almost independent of the primordial power spectrum, and is therefore extremely valuable.

In principle we can achieve better precision by going to a higher redshift BAO survey to reduce cosmic variance, e.g. Euclid should
measure the transverse BAO angle \( r/D_A(z) \) to better than 0.4 per cent in many bins between 0.7 ≤ \( z \) ≤ 1.7 (Laureijs et al. 2011). Adding a 0.4 per cent distance measurement to a matching redshift, this could measure \( N_{\text{eff}} \) to around ±0.1 precision, which is substantially better than \( \text{Planck} \). However, a sub-per cent absolute distance to such a redshift currently appears extremely challenging given the potential systematics: thus the low-redshift route outlined above remains a promising intermediate step.

5.4 An ultimate BAO survey at \( z \sim 0.2 \)

The considerations above on distance measurements and CMB degeneracies provide a very strong motivation for obtaining the best possible BAO measurements at modest \( z \sim 0.2 \), approaching the cosmic variance limit. The ongoing BOSS project is a large step in this direction, but there are several potential improvements:

- first of course BOSS only covers around 1/4 of the entire sky, so adding coverage of the Southern hemisphere is very useful;
- secondly, sampling a somewhat higher space density of galaxies can improve reconstruction of the BAO peak and thirdly we may expand the survey to lower galactic latitudes for maximal sky coverage.

Until recently, galaxy surveys have disfavoured low galactic latitudes due to both extinction problems and increased stellar contamination (e.g. from blended images which are hard to morphologically classify). However, the recently completed Wide-field Infrared Survey Explorer (WISE) mid-IR survey combined with the ongoing Visible and Infrared Survey Telescope for Astronomy (VISTA) Hemisphere Survey should provide a galaxy sample of ample depth, and minimal sensitivity to galactic extinction which could push down to \( |b| \sim 15^\circ \). Availability of optical+near-IR colours can also greatly improve the star–galaxy separation, so stellar contamination should remain manageable. The cosmic-variance limits on BAO measurements have been calculated by Seo & Eisenstein (2007); for 3/4 of the full sky and realistic reconstruction methods, interpolation from their fig. 3 predicts precision ≤1 per cent on \( r/D_A(z = 0.2) \); this accuracy is similar to optimistic projections for local \( H_0 \) measurements. Such a BAO survey is comfortably within reach of proposed high-multiplex multi-object spectrographs such as 4-m Multi-Object Spectroscopic Telescope (4MOST) on the VISTA telescope, or Dark Energy Spectrometer (DESpec) at Cerro Tololo Inter-American Observatory (CTIO). The required area is very large, but the target density ~50 galaxies per deg\(^2\) is rather low, so such an observing program would only take a modest fraction of the total number of fibres, and could be run in a simultaneous mode in parallel with stellar and other surveys.

Furthermore, an accurate low-redshift BAO measurement, when compared to a radial BAO measurement at \( z \sim 0.7 \), can provide a clean smoking-gun test of cosmic acceleration entirely from the two BAO measurements (Sutherland 2012); that test is independent of SNe, CMB data and general relativity. BAO measurements at \( z \gtrsim 0.5 \) are necessary but not sufficient for this test, since very little acceleration happened earlier than \( z = 0.5 \).

6 CONCLUSION

Measuring the absolute rather than relative BAO length scale forms a powerful test of standard early-universe cosmology, especially probing the radiation density along with other possible non-standard effects at \( z > 1000 \).

As a step in this direction, we have found a simple and highly accurate approximation (equation 8) relating the BAO dilation scale \( D_A(z) \) to the luminosity distance \( D_L \) at a slightly higher redshift. This is accurate to ≤0.2 per cent at \( z \leq 0.4 \) for all plausible WMAP-compatible Friedmann models, including modest curvature and time-varying dark energy; the inaccuracy is substantially smaller than the cosmic variance limit for low-redshift BAO measurements. The approximation does not explicitly depend on \( H_0 \), so remains applicable if there is any direct physics-based measurement of \( D_A(z) \) bypassing the local distance ladder. The only ways for equation (8) to have per cent-level errors are very radical, such as violation of the distance–duality relation \( D_L = (1 + z)^2 D_A \), or a sharp phase transition in dark energy at low redshift, e.g. a sharp jump in \( w \) causing a kink feature in \( H(z) \).

We also reviewed the degeneracy between radiation density and cosmic scales, and showed this is close to a rescaling of all dimensionful observables (except baryon and photon densities), while leaving most dimensionless ratios unchanged.

Given realistic future observations, the approximation above can provide a high-precision calibration of the BAO length scale using only low-redshift data, which in turn provides a powerful test of standard \( z > 1000 \) CMB assumptions, and in particular a robust test of the radiation density independent of the CMB damping tail.

A measurement of \( H_0 \) is also useful, but on its own does not fully break degeneracies: e.g. a high-precision measurement of \( H_0 \) significantly greater than 73 km s\(^{-1}\) Mpc\(^{-1}\) would signal a problem for vanilla ΛCDM, but could indicate any one of \( w < -1 \), weak open curvature or increased radiation density, and without an absolute \( r_\text{c} \) measurement it would be hard to discriminate these. In contrast, an absolute BAO length measurement can cleanly detect or constrain non-standard pre-CMB physics, almost independent of late-time effects such as \( w \neq -1 \) or weak curvature, and with minimal degeneracy with \( n_s \) and running spectral index. This may also be important for inflation theory, since the currently strong evidence for \( n_s < 1 \) becomes substantially weaker if \( N_{\text{eff}} \) is larger than the standard value.

We can essentially distinguish two possibilities: if all the standard CMB assumptions are correct, then \( \text{Planck} \) will determine \( r_\text{c}(z_d) \) better than the cosmic variance on the BAO length: then an absolute BAO length measurement essentially provides a strong null test of the standard cosmology at around 1–2 per cent precision, but does not improve our error bars on \( \Omega_m \), \( w \) etc.

However, if one or more of the standard early-universe assumptions is wrong, this can be absorbed into biased values of \( H_0 \), and to a lesser extent \( \Omega_m \) and \( w \), in joint fits to CMB, BAO and SN data alone. Therefore, a direct low-redshift measurement of \( r_\text{c} \) can be very powerful for discriminating early-universe modifications such as extra radiation or early dark energy, from late-time effects such as dark energy \( w \neq -1 \) or small non-zero curvature.

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Appendix A: The Taylor Series for $D_V$

Here we derive the Taylor series for $D_V(z)$ and approximation (7); these are not used in the main part of the paper, but are useful to provide an analytic understanding of the high accuracy of approximations (7) and (8), and the dependence on cosmological parameters.

We start by defining the usual deceleration parameter $q$ and the jerk parameter $j$ (e.g. Alam et al. 2003) as

$$q = -\frac{d^2a}{dt^2}a^{-2}, \quad j = \frac{d^3a}{dt^3}a^{-3}.$$  

(A1)

We can rearrange these in terms of $d/dz$, and using the chain rule we find

$$\frac{d}{dz} \left( \frac{1}{H} \right) = \frac{1 + q}{(1 + z)H}, \quad \frac{d^2}{dz^2} \left( \frac{1}{H} \right) = \frac{2 + 4q + 3q^2 - j}{(1 + z)^2 H}.$$  

(A2)

Using these, with subscript 0 denoting present-day values, we obtain the series

$$\frac{c z}{H(z)} = \frac{c z}{H_0} \left[ 1 - (1 + q_0)z + \frac{2 + 4q_0 + 3q_0^2 - j_0 + \Omega_k}{2} z^2 + \cdots \right],$$

(A3)

and integrating and including the leading-order curvature term gives

$$(1 + z)D_A(z) = \frac{c z}{H_0} \left[ 1 - \frac{1 + q_0}{2} z + \frac{2 + 4q_0 + 3q_0^2 - j_0 + \Omega_k}{6} z^2 + \cdots \right].$$

(A4)

Inserting the above two expressions into the definition of $D_V$, collecting powers of $z$ and using $(1 + y)^{1/3} = 1 + y/3 - y^2/9 + \cdots$, we obtain the Taylor series for $D_V$ as

$$D_V(z) = \frac{c z}{H_0} \left[ 1 - \frac{2(1 + q_0)}{3} z + \frac{19 + 38q_0 + 29q_0^2 - 10j_0 + 4\Omega_k}{36} z^2 + \cdots \right].$$

(A5)

(We note that for concordance ΛCDM, the three-term sum above has error $<0.25$ per cent at $z < 0.3$, but worsens quite rapidly above this.)

Substituting 4z/3 in equation (A4), we have

$$\frac{3}{4} \left( 1 + \frac{4}{3} z \right) D_A \left( \frac{4}{3} z \right) = \frac{c z}{H_0} \left[ 1 - \frac{2(1 + q_0)}{3} z + \frac{16}{54} (2 + 4q_0 + 3q_0^2 - j_0 + \Omega_k) z^2 + \cdots \right].$$

(A6)

Comparing the above two equations, it is clear that approximation (7) is correct to second order in $z$, for any values of cosmological parameters.

Subtracting (A5) from (A6) and dividing by (A5), we then find that the ratio of the RHS to LHS of approximation (7) is

$$\frac{3}{4} \left( 1 + \frac{z}{2} \right) D_A(z) \frac{D_A(z)}{D_V(z)} = 1 + z^2 \left[ \frac{14}{9} - 2j_0 + 9 \left( q_0 + \frac{7}{9} \right)^2 + 20\Omega_k \right] + O(z^3).$$

(A7)
A flat ΛCDM model has \( q_0 = \frac{1}{2} \Omega_m - 1 \) and \( j_0 = +1 \), hence the term in square brackets above simplifies to \( \frac{1}{2} \Omega_m (81 \Omega_m - 24) \); this is less than 1 for plausible values of \( \Omega_m < 0.4 \) (it is \(-0.16\) for the concordance model).

More generally, for conservative ranges of parameters \(-1 < q_0 < -0.4, 0 < j_0 < 2 \) and \( |\Omega_L| < 0.05 \), the square bracket is not significantly bigger than \( \pm 4 \); with the pre-factor of \( 1/108 \), this explains the excellent accuracy of approximation (7) at moderate redshift. This also suggests that approximation (7) should remain fairly accurate for modified-gravity models, as long as they are homogeneous, have weak curvature and \( q_0, j_0 \) not very different from the concordance model.

Finally, we note that the square bracket has value 14.25 for an Einstein–de Sitter model, and approximately 22 for a zero-Λ open model, explaining the low-redshift limit of those models shown in Fig. 2.

### APPENDIX B: LARGE-SCALE STRUCTURE AND \( z_{eq} \)

Here we note that while BAO parameter estimates are strongly dependent on correct deduction of \( z_{eq} \) from the CMB, the overall shape of the large-scale galaxy power spectrum does provide an independent check of this.

The large-scale galaxy clustering pattern actually contains two key length scales, the BAO length discussed above, and also the ‘big bend’ scale which describes the overall broad-band shape of the galaxy power spectrum \( P(k) \) excluding the BAO wiggles; these two are approximately independent observables. Assuming the primordial power spectrum is well described by a power law, \( n_s \sim 0.96 \) and the dark matter is cold or warm (not hot), then fitting the ‘big bend’ in a galaxy power spectrum essentially measures the comoving light horizon size \( r_{\text{DL}} \) at matter-radiation equality, again relative to \( D_L(z) \) at the characteristic redshift of the given survey: this gives

\[
\frac{z}{D_L(z)} = (1 + \epsilon_v) \frac{E(2z/3)}{\sqrt{\Omega_m}} \frac{2(\sqrt{2} - 1)}{(1 + z_{eq})} \tag{B1}
\]

which is not explicitly dependent on \( X_{\text{rad}} \). For \( X_{\text{rad}} = 1 \), at small \( z \) the above has the well-known scaling \( \propto (\Omega_m h)^{-1} \), with a result \( \Omega_m h \approx 0.20 \) which has remained consistent over many large galaxy surveys, since the first reliable estimate from the Automatic Plate Measuring (APM) Galaxy Survey (Efstathiou, Sutherland & Maddox 1990), through 2dFGRS (Percival et al. 2002) and SDSS (Reid et al. 2010).

We see that the big-bend observable in (B1) has the same \( E(2z/3)/\sqrt{\Omega_m} \) factor as the BAO ratio in (18), but has a different dependence on \( z_{eq} \). Taking the ratio of these two characteristic lengths, we have

\[
\frac{r_s(z_d)}{r_{\text{DL}}(z_{eq})} = \frac{r_s(z_d)}{r_s(z_{eq})} \frac{r_s(z_{eq})}{r_{\text{DL}}(z_{eq})} \sim \frac{1}{2} \frac{1 + z_{eq}}{1 + z_d} \left( \frac{1 + z_{eq}}{1 + z_d} \right)^{1/2} \frac{1}{\sqrt{2} - 1} \left( \frac{1}{\sqrt{3}} \frac{1}{0.886} \right) \tag{B2}
\]

here the second factor \( 0.886/\sqrt{3} \) represents the weighted average sound speed \( c_s/c \) prior to the drag redshift; the given value is for the concordance model, but this term is very insensitive to reasonable parameter variations. The first term above depends only on the ratio \( (1 + z_{eq})/(1 + z_d) \), and scales approximately \( \propto (1 + z_{eq})^{0.75} \) around the concordance model; there is no separate dependence on \( \Omega_m, h \) and \( X_{\text{rad}} \), so this ratio is predicted robustly given just \( z_{eq} \) from WMAP alone.

Because of various uncertainties in overall \( P(k) \) shape from possible effects such as scale-dependent bias, non-linearity, redshift-space distortions, neutrino masses, running of \( n_s \), etc., this BAO/bend ratio seems unlikely to independently measure \( z_{eq} \) to a precision comparable to the current 4 per cent precision from WMAP, and still less the 1 per cent expected from Planck. However, the fact that parameters estimated from CMB+BAO also provide a reasonable fit to the overall \( P(k) \) shape provides a valuable consistency check, which the concordance model passes (Reid et al. 2010). This strongly argues that the WMAP-only estimate of \( z_{eq} \) cannot have a gross error from unknown physics, unless there has been a fortuitous cancellation of effects.

The above also shows that measuring \( P(k r_s) \) in dimensionless units rescaled by the BAO length may be helpful for testing for neutrino masses or non-standard physics around \( z \approx z_{eq} \), since this is very robust against shifts in \( \Omega_m, H_0 \) and \( X_{\text{rad}} \).

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