On Elevating Free-Fermion $Z_2 \times Z_2$ Orbifolds Models to Compactifications of $F$ Theory

P. Berglund$^1$, J. Ellis$^2$, A.E. Faraggi$^3$, D.V. Nanopoulos$^4$, and Z. Qiu$^5$

$^1$Inst. for Theoretical Physics, University of California, Santa Barbara, CA 93106
$^2$Theory Division, CERN, CH-1211 Geneva, Switzerland
$^3$Department of Physics, University of Minnesota, Minneapolis, MN 55455, USA
$^4$Dept. of Physics, Texas A & M University, College Station, TX 77843-4242, USA, HARC, The Mitchell Campus, Woodlands, TX 77381, USA, and Academy of Athens, 28 Panepistimiou Avenue, Athens 10679, Greece.
$^5$Department of Physics, University of Florida, Gainesville, FL, 32611, USA

Abstract

We study the elliptic fibrations of some Calabi-Yau three-folds, including the $Z_2 \times Z_2$ orbifold with $(h_{1,1}, h_{2,1}) = (27, 3)$, which is equivalent to the common framework of realistic free-fermion models, as well as related orbifold models with $(h_{1,1}, h_{2,1}) = (51, 3)$ and $(31, 7)$. However, two related puzzles arise when one considers the $(h_{1,1}, h_{2,1}) = (27, 3)$ model as an $F$-theory compactification to six dimensions. The condition for the vanishing of the gravitational anomaly is not satisfied, suggesting that the $F$-theory compactification does not make sense, and the elliptic fibration is well defined everywhere except at four singular points in the base. We speculate on the possible existence of $N = 1$ tensor and hypermultiplets at these points which would cancel the gravitational anomaly in this case.
1 Introduction

Important progress has been achieved during the past few years in understanding non-perturbative aspects of superstring theories [1]. However, the ultimate goal of understanding how string theory is relevant to physics in the real world remains elusive. Encouraged by the hope that string theory provides a framework for consistently unifying all of the observed elementary matter particles and interactions [2], many phenomenological string models have been developed [3]. Among the most advanced models are those constructed in the free-fermion formulation [4, 5, 6, 7, 8]. These models have been the subject of detailed studies showing that they can, at least in principle, account for desirable physical features including the observed fermion mass spectrum, the longevity of the proton, small neutrino masses, the consistency of gauge-coupling unification with the experimental data from LEP and elsewhere, and the universality of the soft supersymmetry-breaking parameters [3]. It is plausible that some improved understanding how recent advances in non-perturbative aspects of string theory are relevant in the real world may be gained by studying their application to these realistic free-fermion models.

An important feature of the realistic free-fermion models is their underlying $Z_2 \times Z_2$ orbifold structure. Many of the encouraging phenomenological characteristics of the realistic free-fermion models are rooted in this structure, including the three generations arising from the three twisted sectors, and the canonical $SO(10)$ embedding for the weak hypercharge. To see more precisely this orbifold correspondence, recall that the free-fermion models are generated by a set of basis vectors which define the transformation properties of the world-sheet fermions as they are transported around loops on the string world sheet. A large set of realistic free-fermion models contains a subset of boundary conditions which can be seen to correspond to $Z_2 \times Z_2$ orbifold compactification with the standard embedding of the gauge connection [10]. This underlying free-fermion model contains 24 generations from the twisted sectors, as well as three additional generation/anti-generation pairs from the untwisted sector. At the free-fermion point in the Narain moduli space [11], both the metric and the antisymmetric background fields are non-trivial, leading to an $SO(12)$ enhanced symmetry group. The action of the $Z_2 \times Z_2$ twisting on the $SO(12)$ Narain lattice then gives rise to a model with $(h_{11}, h_{21}) = (27, 3)$, matching the data of the free-fermion model [1].

Recently, we have shown how to construct this $Z_2 \times Z_2$ orbifold model in the Landau–Ginzburg formalism [13]. This was done using a freely-acting $Z_2$ twist on a $Z_2 \times Z_2$ Landau-Ginzburg orbifold with $(h_{11}, h_{21}) = (51, 3)$. In this paper, we

---

*Other approaches are also possible: see for example the $M$-theory three-generation models proposed in [9]. It will be interesting to see whether such models share the attractive features of the perturbative three-generation models.

†We emphasize that the data of this model differs from the $Z_2 \times Z_2$ orbifold on a $SO(4)^3$ lattice with $(h_{11}, h_{21}) = (51, 3)$, which has been more extensively discussed in the literature [12].
extend this analysis to include a formulation of this and related \((51, 3)\) and \((31, 7)\) models in terms of elliptically-fibered Calabi-Yau manifolds, opening the way to non-trivial \(F\)- or \(M\)-theory compactifications. This geometrical approach should allow us to study properties of the moduli space away from the special free-fermion/orbifold point. However, in this paper we are more concerned with the consistency of these manifolds as valid \(F\)-theory compactifications, and thus as six-dimensional vacua, rather than exploring them from the traditional heterotic four-dimensional point of view.

We recall that \(F\) theory is a way of compactifying type-IIB string theory which allows the string coupling to vary over the compact manifold. The key point in compactifications of \(F\) theory to six dimensions is that the models should admit an elliptic fibration and (at least) a global section, in which a Calabi–Yau three-fold is identified as a two complex–dimensional base manifold \(B\) with an elliptic fiber. Among these models are some which have an orbifold interpretation, such as the above-mentioned \(Z_2 \times Z_2\) orbifold with \((h_{11}, h_{21}) = (51, 3)\) [12], denoted by \(X_1\), as we demonstrate with a standard Weierstrass representation. Using the Landau-Ginzburg analysis and \textit{quartic} Weierstrass representations, we construct related freely-acting \(Z_2\) orbifolds with \((h_{11}, h_{21}) = (27, 3)\) and \((h_{11}, h_{21}) = (31, 7)\), denoted by \(X_2\) and \(X_3\). The former admits an elliptic fibration, apart from four singular points in the base \(B\). However, these points prevent us from having a global section and so the six-dimensional theory does not exist. Another sign of this is that the gravitational anomaly equation in six dimensions is not satisfied. This implies that the formulation of this \(F\)-theory compactification is not well defined. This is consistent with the absence of a globally-defined section, but it is possible that there may be a non-trivial contribution of \(N = 1\) tensor and hypermultiplets associated with the singular points in the base \(B\) of \(X_2\), due to the \(Z_2\) quotient, which cancels the gravitational anomaly. On the other hand, we are able to show that the \((31, 7)\) model \(X_3\) does admit a consistent elliptic fibration.

This paper is organized as follows. In section 2, we give a brief review of the free-fermion orbifold and Landau-Ginzburg constructions of the \((h_{11}, h_{21}) = \{(51, 3), (27, 3)\}\) \(Z_2 \times Z_2\) orbifolds. In section 3, we first review the elliptic fibration of the former orbifold using a standard Weierstrass representation, and then generalize it to the closely related \((27, 3)\) and \((31, 7)\) models using a quartic Weierstrass representation. In particular, we focus on the question of the existence of the \((27, 3)\) model in six dimensions and the appearance of singular points in the base space, and suggest how the issue of the gravitational anomaly may be resolved. We study in section 4 type-IIB orientifold constructions relevant for the discussion of the \(F\)-theory compactification of \(X_2\). Finally, we end in section 5 with some concluding remarks.
The $Z_2 \times Z_2$ Orbifold Equivalent of Realistic Free-Fermion Models

The purpose of this section is to motivate the choice of the relevant Calabi-Yau three-folds because of their relation to certain free-fermionic models. We first review the construction of the $Z_2 \times Z_2$ fermionic orbifold of interest followed by their realization as Landau-Ginzburg and toroidal orbifolds.

Let us recall that, in the free-fermion formulation [14], a model is defined by a set of boundary-condition basis vectors, together with the related one–loop GSO-projection coefficients, that are constrained by the string consistency constraints. These boundary-condition basis vectors encode the phases of all the world–sheet fermions, when transported around one of the non-contractible loops of the string world sheet. In the case of the heterotic string in the light–cone gauge, there are 20 left–moving and 44 right–moving real Majorana–Weyl world–sheet fermions, whereas for the type-IIA and type-IIB strings there would be 20 left– and 20 right–moving world–sheet fermions. Given the set of boundary-condition basis vectors and the one–loop GSO-projection coefficients, one can then construct the one–loop partition function and extract the physical spectrum.

The $Z_2 \times Z_2$ fermionic orbifold model of interest is generated by the following set of boundary-condition basis vectors, the so-called NAHE set [4, 15]:

\[
\begin{align*}
1, S, \xi = 1 + b_1 + b_2 + b_3, X, b_1, b_2 \end{align*}
\]

The first four vectors in the basis \{1, \(S, \xi, X\)} generate a model with \(N = 4\) space–time supersymmetry, and an \(E_8 \times SO(12)\) × \(E_8\) gauge group. In this construction, the sector \(S\) generates \(N = 4\) space–time supersymmetry. The \(SO(12)\) factor is obtained from \{\(\bar{y}, \bar{\omega}\)} \(1, \ldots, 6\). The first and second \(E_8\) factors are obtained from the world–sheet fermionic states \{\(\bar{\psi}^1, \ldots, 5, \bar{\eta}^1, 2, 3\)} and \{\(\bar{\phi}^1, \ldots, 8\)} , respectively. The sectors \(X\) and \(\xi\) produce the spinorial representations of \(SO(16)\) in the observable and hidden sectors, respectively, and complete the observable and hidden gauge groups to \(E_8 \times E_8\). The Neveu–Schwarz sector produces the adjoint representations of \(SO(16)\) × \(SO(12)\) × \(SO(16)\). The vectors \(b_1\) and \(b_2\) break the gauge symmetry to \(E_6 \times U(1)^2 \times SO(4)^3\) × \(E_8\) and the \(N = 4\) space–time supersymmetry to \(N = 1\). The sectors \((b_1; b_1 + X)\), \((b_2; b_2 + X)\) and \((b_3; b_3 + X)\) each give eight \(27\)'s of \(E_6\). The \((NS; NS + X)\) sector gives, in addition to the vector bosons and spin-two states, three copies of scalar representations in \(27 + 27\) of \(E_6\). The net number of generations in the \(27\) representation of \(E_6\) is therefore 24.

In the toroidal orbifold construction, the same model is obtained by first specifying the background fields, which produce the \(SO(12)\) lattice \([\mathbb{R}]\), and then applying the appropriate \(Z_2 \times Z_2\) identifications. One takes the metric on the six-dimensional compactified manifold to be the Cartan matrix of \(SO(12)\), and the antisymmetric tensor to be \(b_{ij} = g_{ij}\) for \(i > j\) \([\mathbb{R}]\). When all the radii of the six-dimensional compactified manifold are fixed at \(R_I = \sqrt{2}\), it is easily seen that the left– and right–moving momenta \(P^I_{R,L} = [m_i - \frac{1}{2}(B_{ij} \pm C_{ij})n_j]e^I_i\) reproduce all the massless root vectors in the lattice of \(SO(12)\), where the \(e^I_i = \{e^I_i\}\) are six linearly-independent vectors normal-
ized: \((e_i)^2 = 2\). The \(e_i^*\) are dual to the \(e_i\), and \(e_i^* \cdot e_j = \delta_{ij}\). The momenta \(P^I\) of the compactified scalars in the bosonic formulation can be seen to coincide with the \(U(1)\) charges of the unbroken Cartan generators of the four-dimensional gauge group.

The incorporation in the free-fermion model of the two basis vectors \(b_1\) and \(b_2\) as well as \(\{1, S, \xi, X\}\) corresponds to the \(Z_2 \times Z_2\) orbifold model with standard embedding. The fermionic boundary conditions are translated, in the bosonic language, into twists on the internal dimensions and shifts in the gauge degrees of freedom. Starting from the model with \(SO(12) \times E_8 \times E_8\) symmetry, and applying the \(Z_2 \times Z_2\) twisting on the internal coordinates, we then obtain an orbifold model with \(SO(4)^3 \times E_6 \times U(1)^2 \times E_8\) gauge symmetry. There are sixteen fixed points in each twisted sector, yielding the 24 generations from the three twisted sectors mentioned above. The three additional pairs of 27 and \(\overline{27}\) are obtained from the untwisted sector. This orbifold model, which we call \(X_2\), therefore has the same topological data as the free-fermion model with the six-dimensional basis set \(\{1, S, X, 1 = 1 + b_1 + b_2 + b_3, b_1, b_2\}\), since the Euler characteristic of this model is 48, with \(h_{11} = 27\) and \(h_{21} = 3\).

This \(Z_2 \times Z_2\) orbifold, corresponding to the extended NAHE set at the core of the realistic free-fermion models, differs from the one which has usually been examined in the literature \([12]\). In that orbifold model, the Narain lattice is \(SO(4)^3\), yielding a \(Z_2 \times Z_2\) orbifold model, which we call \(X_1\). It has Euler characteristic equal to 96, corresponding to 48 generations, and \(h_{11} = 51, h_{21} = 3\). This \(Z_2 \times Z_2\) orbifold can be constructed in a similar manner to the model \(X_2\) above. First the Narain \(SO(4)^3\) lattice is produced via the diagonal metric \(g_{ij} = 2\delta_{ij}\) and the trivial anti-symmetric tensor field \(b_{ij} = 0\). For \(R_I = \sqrt{2}\), all the roots in the root lattice of \(SO(4)^3\) are again massless. Then, applying the \(Z_2 \times Z_2\) twisting reduces the \(N = 4\) supersymmetry to \(N = 1\). Each twisted sector now produces 16 generations, yielding a total of 48, and three additional generation and anti-generation pairs are obtained from the untwisted sector.

Before proceeding, we note that, at the level of the toroidal compactification, the \(SO(4)^3\) and \(SO(12)\) lattices are continuously connected by varying the parameters of the background metric and antisymmetric tensor. However, this cannot be done while preserving the \(Z_2 \times Z_2\) invariance, because the continuous interpolation cannot change the Euler characteristic. Therefore, the two toroidal models are in the same moduli space, but not the two orbifold models \(X_1\) and \(X_2\).

Let us now show how the two \(Z_2 \times Z_2\) orbifold models, constructed above using toroidal compactification, may be represented in the Landau-Ginzburg orbifold construction \([13]\). We start from a non-degenerate quasi-homogeneous superpotential \(W\) of degree \(d\), \(W(\lambda^n X_i) = \lambda^d W(X_i)\), where the \(X_i\) are chiral superfields and the \(q_i = n_i/d\) are their left and right charges under the \(U(1)_{b_0}\) current of the \(N = 2\) algebra. In the Landau–Ginzburg construction one twists by some symmetry group \(G\) of the original superpotential \([10]\). The Landau–Ginzburg potential that mimics
the \( T^3 \) torus, corresponding to the \( SO(4)^3 \) lattice, is given by

\[
W = X_1^4 + X_2^4 + X_3^2 + X_4^4 + X_5^4 + X_6^2 + X_7^4 + X_8^4 + X_9^2 \tag{2.1}
\]

where \( X_{3,6,9} \) are trivial superfields, and the superpotential (2.1) corresponds to a superconformal field theory with \( c = 9 \). The mirror of the \( X_1 \) model is obtained by taking the orbifold \( \mathcal{M}/(Z_2^A \times Z_2^B) \) where

\[
\begin{align*}
Z_2^A : (X_1, \ldots, X_9) & \rightarrow (X_1, X_2, X_3, -X_4, -X_5, X_6, -X_7, -X_8, X_9); \\
Z_2^B : (X_1, \ldots, X_9) & \rightarrow (-X_1, -X_2, X_3, -X_4, -X_5, X_6, X_7, X_8, X_9). \tag{2.2}
\end{align*}
\]

and \( \mathcal{M} = W/j \), where \( j \) is the \( Z_4 \) scaling symmetry of (2.1). It is easy to show that there are 51 \((1,1)\) and 3 \((-1,1)\) states. Using the convention that the deformations of \( W \), which give part of the spectrum of \((1,1)\) states, correspond to the \((2,1)\) forms of the orbifold, and the fact that the \((1,1)\) forms are in one-to-one correspondence with the \((-1,1)\) states, we reproduce the data of the \( Z_2 \times Z_2 \) orbifold on the \( SO(4)^3 \) lattice. We showed in [13] that the mirror of the \( X_2 \) model is obtained from the mirror of the \( X_1 \) model by applying the twist

\[
Z_2^w : (X_1, \ldots, X_9) \rightarrow (-X_1, X_2, -X_3, -X_4, X_5, -X_6, -X_7, X_8, -X_9), \tag{2.3}
\]

i.e., we have the Landau–Ginzburg orbifold \( \mathcal{M}/(Z_2^A \times Z_2^B \times Z_2^w) \). Note that the three trivial superfields are twisted by the \( Z_2^w \) twist. This is to ensure that \( Z_2^w \) acts freely on each of the \( T^2 \) factors in (2.1), thus reproducing the data of the \( Z_2 \times Z_2 \) orbifold on the \( SO(12) \) lattice.

Another way to realize the connection between the \( X_1 \) and \( X_2 \) models is by using a freely-acting shift, rather than a freely-acting twist [1]. For this purpose, let us first start with the compactified \( T_1^3 \times T_2^3 \times T_3^3 \) torus parameterized by three complex coordinates \( z_1, z_2 \) and \( z_3 \), with the identification

\[
z_i = z_i + 1 \quad ; \quad z_i = z_i + \tau_i \tag{2.4}
\]

where \( \tau \) is the complex parameter of each \( T^2 \) torus. We consider \( Z_2 \) twists and possible shifts of order two:

\[
z_i \rightarrow (-1)^{\epsilon_i} z_i + 1/2 \delta_i \tag{2.5}
\]

subject to the condition that \( \Pi_i (-1)^{\epsilon_i} = 1 \). This condition insures that the holomorphic three–form \( \omega = dz_1 \wedge z_2 \wedge z_3 \) is invariant under the \( Z_2 \) twist. Under the identification \( z_i \rightarrow -z_i \), a single torus has four fixed points at

\[
z_i = \{0, 1/2, \tau/2, (1 + \tau)/2\}. \tag{2.6}
\]

*This second way will be instrumental later in trying to find a type-IIB orientifold description of the six-dimensional vacuum corresponding to the \( F \)-theory compactification on \( X_2 \).*
The first model that we consider is produced by the two $Z_2$ twists

$$
\alpha : (z_1, z_2, z_3) \rightarrow (-z_1, -z_2, z_3)
$$
$$
\beta : (z_1, z_2, z_3) \rightarrow (z_1, -z_2, -z_3)
$$

There are three twisted sectors in this model, $\alpha$, $\beta$ and $\alpha\beta = \alpha \cdot \beta$, each producing 16 fixed tori, for a total of 48.

To facilitate the discussion of the subsequent examples, we briefly describe the calculation of the cohomology for this orbifold: a more complete discussion can be found in [17]. Consider first the untwisted sector. The Hodge diamond for a single untwisted torus is given by

$$
\begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}
$$

which displays the dimensions of the $H^{p,q}(T_i)$, with $H^{0,0}$, $H^{0,1}$, $H^{1,0}$ and $H^{1,1}$ being generated by the differential forms $1$, $dz_i$, $d\bar{z}_i$ and $dz_i \wedge d\bar{z}_i$, respectively. Under the $Z_2$ transformation $z \rightarrow -z$, $H^{0,0}$ and $H^{1,1}$ are invariant, whereas $H^{1,0}$ and $H^{0,1}$ change sign.

The untwisted sector of the manifold produced by the product of the three tori $T_1 \times T_2 \times T_3$ is then given by the product of differential forms which are invariant under the $Z_2 \times Z_2$ twists $\alpha \times \beta$. The invariant terms are summarized by the Hodge diamond

$$
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 16 & 16 & 0 \\
0 & 16 & 16 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

For example, $H^{1,1}$ is generated by $dz_i \wedge d\bar{z}_i$ for $i = 1, 2, 3$, and $H^{2,1}$ is produced by $dz_1 \wedge z_2 \wedge z_3, dz_2 \wedge z_3 \wedge \bar{z}_1, dz_3 \wedge z_1 \wedge \bar{z}_2$, etc. We next turn to the twisted sectors, of which there are three, produced by $\alpha$, $\beta$ and $\alpha\beta$, respectively. In each sector, two of the $z_i$ are identified under $z_i \rightarrow -z_i$, and one torus is left invariant. We need then consider only one of the twisted sectors, say $\alpha$, and the others will contribute similarly. The sector $\alpha$ has 16 fixed points from the action of the twist on the first and second tori. Since the action is trivial on the third torus, we get 16 fixed tori. The cohomology is given by sixteen copies of the cohomology of $T_3$, where each $H^{p,q}$ of $T_3$ contributes $H^{p+1,q+1}$ to that of the orbifold theory [17]. The Hodge diamond from each twisted sector then has the form

$$
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 16 & 16 & 0 \\
0 & 16 & 16 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

It remains to find the forms from the $\alpha$ twisted sector which are invariant under the action of the $\beta$ twist. Since $z_3 \rightarrow -z_3$ under $\beta$, it follows that 1 and $dz_3 \wedge d\bar{z}_3$
are invariant, whereas $dz_3$ and $d\bar{z}_3$ are not. Consequently, only the contributions of $H^{1,1}$ and $H^{2,2}$ in (2.10) are invariant under the $\beta$ twist. Therefore, we see that the invariant contribution from each twisted sector is only along the diagonal of (2.10), and that the total Hodge diamond of the $Z_2 \times Z_2$ orbifold is

\[
\begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 51 & 3 & 0 \\
0 & 3 & 51 & 0 \\
1 & 0 & 0 & 1
\end{pmatrix}
\] (2.11)

Next we consider the model generated by the $Z_2 \times Z_2$ twists in (2.7), with the additional shift

\[
\gamma : (z_1, z_2, z_3) \rightarrow (z_1 + \frac{1}{2}, z_2 + \frac{1}{2}, z_3 + \frac{1}{2})
\] (2.12)

This model again has fixed tori from the three twisted sectors $\alpha$, $\beta$ and $\alpha\beta$. The product of the $\gamma$ shift in (2.12) with any of the twisted sectors does not produce any additional fixed tori. Therefore, this shift acts freely. Under the action of the $\gamma$ shift, half the fixed tori from each twisted sector are paired. Therefore, the action of this shift is to reduce the total number of fixed tori from the twisted sectors by a factor of 1/2. Consequently, the Hodge diamond for this model is

\[
\begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 27 & 3 & 0 \\
0 & 3 & 27 & 0 \\
1 & 0 & 0 & 1
\end{pmatrix}
\] (2.13)

with $(h_{11}, h_{21}) = (27, 3)$. This model therefore reproduces the data of the $Z_2 \times Z_2$ orbifold at the free-fermion point in the Narain moduli space. The action of the freely-acting shift (2.12) is seen to be identical to the action of the freely-acting twist (2.3) that connects the (51,3) and (27,3) models in the Landau-Ginzburg representation.

Finally, let us consider the model generated by the twists (2.7) with the additional shift given by

\[
\gamma : (z_1, z_2, z_3) \rightarrow (z_1 + \frac{1}{2}, z_2 + \frac{1}{2}, z_3)
\] (2.14)

This model, denoted by $X_3$, again has three twisted sectors $\alpha$, $\beta$ and $\alpha\beta$. Under the action of the $\gamma$ shift, half of the fixed tori from these twisted sectors are identified. These twisted sectors therefore contribute to the Hodge diamond as in the previous model. However, the $\gamma$ shift in (2.14) does not act freely, as its combination with $\alpha$ produces additional fixed tori, since, under the action of the product $\alpha \cdot \gamma$, we have

\[
\alpha \gamma : (z_1, z_2, z_3) \rightarrow (-z_1 + \frac{1}{2}, -z_2 + \frac{1}{2}, z_3)
\] (2.15)

This sector therefore has 16 additional fixed tori. Repeating the analysis as in the previous cases, we see that, under the identification imposed by the $\alpha$ and $\beta$ twists,
the invariant states from this sector give rise to four additional (1,1) forms and four additional (2,1) forms. The Hodge diamond for this model therefore has the form

\[
\begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 31 & 7 & 0 \\
0 & 7 & 31 & 0 \\
1 & 0 & 0 & 1
\end{pmatrix}
\] (2.16)

with \((h_{11}, h_{21}) = (31, 7)\).

As we discuss in subsequent sections, the various \(Z_2 \times Z_2\) orbifold models discussed in this section display interesting features when one considers the possibility of elliptic fibration in the context of \(F\) theory.

### 3 Elliptic Fibration and \(F\) Theory

Let us now study compactifications of \(F\) theory on the different \(Z_2 \times Z_2\) orbifold models analyzed in the previous section, in particular on \(X_2\). Our strategy is to study first the Weierstrass representation of the elliptic fibration of the \(F\)-theory compactification on the \(Z_2 \times Z_2\) orbifold \(X_1\), and then implement the extra twist, so as to obtain the \(X_2\) orbifold. Finally, we discuss how one may hope to resolve the puzzle of the non-vanishing gravitational anomaly that we advertised earlier.

When compactifying \(F\) theory on a Calabi-Yau threefold, it is essential that it should admit an elliptic fibration, with base \(B\) and a toroidal global section. Elliptically-fibered manifolds are conveniently parameterized by writing the equation for the toroidal fiber in the standard Weierstrass form:

\[y^2 = x^3 + fx + g\] (3.1)

which expresses the torus as a double cover of the complex plane with three finite branch points and one branch point at infinity. The functions \(f\) and \(g\) are polynomials of degree 8 and 12, respectively, in the base coordinates. Compactifying on a Calabi-Yau threefold, Morrison and Vafa [18, 19] have shown that, in terms of the \(h_{1,1}(B)\) of the base manifold and the \(h_{1,1}(X)\) and \(h_{2,1}(X)\) of the Calabi-Yau three-fold \(X\), the number of neutral hypermultiplets is given by

\[H^0 = h_{2,1}(X) + 1\] (3.2)

the number of tensor multiplets is given by

\[T = h_{1,1}(B) - 1\] (3.3)

and the rank of the vector multiplets is given by

\[r(V) = h_{1,1}(X) - h_{1,1}(B) - 1.\] (3.4)
Finally, cancelation of the gravitational anomaly in $N = 1$ supergravity in six dimensions requires the following relation between the numbers of neutral hyper-, vector, and tensor multiplets [20]:

$$H^0 - V = 273 - 29T.$$  (3.5)

Although we will mainly be using the Weierstrass parameterization above, this is not so convenient for some of the models written as orbifolds of toroidal compactifications. In these cases, it is sometimes easier to work with the corresponding orientifold model, where such a model has been identified. Specifically, only some of the orbifolds studied in the previous section have been shown to admit an elliptic fibration, and have already been discussed in the context of $F$-theory compactification to six dimensions [14]. These models are the special classes of Calabi–Yau three-folds that have been analyzed by Voisin [21] and Borcea [22]. They have been further classified by Nikulin [23] in terms of three invariants $(r, a, \delta)$. In the context of our discussion, we note that both the $X_1$ and $X_3$ models are part of this classification, whereas $X_2$ is not.

Returning to the Weierstrass representation (3.1), we consider an elliptically-fibered Calabi-Yau manifold $X$ with base $\mathbb{P}^1 \times \mathbb{P}^1$. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold $X_1$ can then be realized as a singular limit of $X$. We can represent $X$ in Weierstrass form, as a (singular) elliptic fiber which depends on the (inhomogeneous) coordinates $w, \tilde{w}$ of the respective $\mathbb{P}^1$:

$$y^2 = x^3 + f_8(w, \tilde{w})xz^4 + g_{12}(w, \tilde{w})z^6.$$  (3.6)

Here $f_8$ and $g_{12}$ are of bidegree eight and twelve, respectively, in $w, \tilde{w}$. This model has $h_{1,1} = 3$ and $h_{2,1} = 243$, and Sen [24] has shown that, considered as an $F$-theory vacuum in six dimensions, it is equivalent to the Gimon-Polchinski type-IIB orientifold on $T^4/\mathbb{Z}_2$ [25].

The next step is to choose a particular complex structure for $f_8$ and $g_{12}$. To do so, we let

$$f_8 = \eta - 3h^2, \quad g_{12} = h(\eta - 2h^2).$$  (3.7)

which implies that the discriminant, $\Delta = 4f^3 + 27g^2$, takes the form

$$\Delta = \eta^2(4\eta - 9h^2).$$  (3.8)

We then further restrict $f_8$ and $g_{12}$ by setting

$$h = K \prod_{i,j=1}^{4} (w - w_i)(\tilde{w} - \tilde{w}_j), \quad \eta = C \prod_{i,j=1}^{4} (w - w_i)^2(\tilde{w} - \tilde{w}_j)^2.$$  (3.9)

Thus, as we approach any of $w = w_i$ (or $\tilde{w} = \tilde{w}_j$) we have a $D_4$ singular fiber. This follows from Kodaira’s classification of ADE singularities [18, 19, 26], and has

$$f_8 \sim (w - w_i)^2, \quad g_{12} \sim (w - w_i)^3, \quad \Delta \sim (w - w_i)^6.$$  (3.10)
Thus, we have an enhanced $SO(8)^8$ gauge symmetry, since $i, j = 1, \ldots, 4$.

These $D_4$ singularities intersect in 16 points, $(w_i, \tilde{w}_j)$, $i, j = 1, \ldots, 4$ in the base $\mathbb{P}^1 \times \mathbb{P}^1$. Due to the severity of the individual singularities, at each point of intersection one has to resolve the base, as follows. At each point of intersection, we have the following singular behavior:

\[
  f_8 \sim (w - w_i)^2(\tilde{w} - \tilde{w}_j)^2, \quad g_{12} \sim (w - w_i)^3(\tilde{w} - \tilde{w}_j)^3, \quad \Delta \sim (w - w_i)^6(\tilde{w} - \tilde{w}_j)^6.  
\]

(3.11)

Thus the order of the singularity is $(4, 6, 12)$ respectively for $f_8, g_{12}, \Delta$. However, just resolving the singular fiber is not enough. We also have to blow the base up once at each of $(w_i, \tilde{w}_j)$, $i, j = 1, \ldots, 4$. Thus, in addition to the enhanced gauge symmetry, we also obtain 16 additional tensor multiplets [18, 19].

The resulting Calabi-Yau manifold yields an $F$-theory compactification on the elliptically-fibered Calabi–Yau three-fold corresponding to the $\mathbb{Z}_2 \times \mathbb{Z}_2(51,3)$ orbifold model $X_1$. To see that it has $h_{1,1} = 51$ and $h_{2,1} = 3$, we first note that blowing up the base gives 16 $(1, 1)$ forms, and recall that the $SO(8)^8$ group has rank 32, which contributes another 32 $(1, 1)$ forms to $h_{1,1}$. In addition, out of the original 250 deformations encoded in $f_8, g_{12}$ ($9^2 + 13^2$), we are left with $C, K$ and $w_i, \tilde{w}_i, i = 1, \ldots, 4$. This leaves us with three independent deformations, once we have used the $SL(2, \mathbb{C})$ reparametrization of each of the $\mathbb{P}^1$, as well as an overall rescaling of (3.6). (In this way, we can fix three of the $w_i, \tilde{w}_i$ as well as $K$.) From (3.2), (3.3), and (3.4) we then find that this six-dimensional $F$-theory vacuum has $V = 224$, $T = 17$ and $H^0 = 4$ [19], which is consistent with the formula (3.5) for the vanishing of the gravitational anomaly.

To seek the elliptic fibration of the $(27,3)$ orbifold model, we have to implement the final $\mathbb{Z}_2$ which acts as a freely-acting twist in the Landau–Ginzburg representation of the model, or as a freely-acting shift as in (2.12). It is obvious that the model written in the form (3.6) is not the correct way of representing the covering space of the final $\mathbb{Z}_2$. Rather, we have to rewrite (3.6) in terms of a quartic polynomial

\[
  \hat{y}^2 = \hat{x}^4 + \hat{x}^2 \hat{z}^2 \hat{f}_4 + \hat{x} \hat{z}^3 \hat{g}_6 + \hat{z}^4 \hat{h}_8.  
\]

(3.12)

where $\hat{f}_4, \hat{g}_6, \hat{h}_8$ are of bidegree 4, 6, 8 respectively in $w, \tilde{w}$. The relation between (3.6) and (3.12) is given in terms of

\[
  \hat{f}_4 = -3h, \quad \hat{g}_6 = 0, \quad \hat{h}_8 = -1/4\eta.  
\]

(3.13)

Writing the fibered torus in the quartic form (3.12) amounts to bringing the branch point at infinity to a finite point. The reason for this rewriting of the fibered torus is that in this representation some symmetries of the torus become manifest, thus simplifying the analysis.

\begin{footnote}
This form has appeared before in various contexts in F-theory, see e.g. [28, 29, 30, 31, 32, 33].
\end{footnote}
We now note that (3.12) enjoys a $Z_2$ symmetry: $(\hat{y}, \hat{x}, \hat{z}) \rightarrow (-\hat{y}, -\hat{x}, \hat{z})$. We are, however, interested in an action which extends to the base as well: $(\hat{y}, \hat{x}, \hat{z}, w, \tilde{w}) \rightarrow (-\hat{y}, -\hat{x}, \hat{z}, -w, -\tilde{w})$. In order to carry out this identification, we need to modify (3.9) to

$$h = K \prod_{i,j=1}^{2} (w_i^2 - w_j^2)(\tilde{w}_i^2 - \tilde{w}_j^2), \quad \eta = C \prod_{i,j=1}^{2} (w_i^2 - w_j^2)^2(\tilde{w}_i^2 - \tilde{w}_j^2)^2. \quad (3.14)$$

Note that on each of the $\mathbb{P}^1$ there are just two points where the elliptic fiber acquires a $D_4$ singularity: $w_i \sim -w_i, \ i = 1, 2$ and $\tilde{w}_i \sim -\tilde{w}_j, \ j = 1, 2$. Each of them gives rise to an $SO(8)$ enhanced gauge symmetry, leading to a total $SO(8)^4$ gauge-group enhancement. There are now eight points at which the $D_4$ singularities intersect:

$$(w_i, \tilde{w}_j), \ (w_i, -\tilde{w}_j), \ i, j = 1, 2. \quad (3.15)$$

This gives rise to eight tensor multiplets, and hence we have $h_{1,1} = 27$ and $h_{2,1} = 3$. Note that the $Z_2$ action on the base has restricted the $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ acting on the $\mathbb{P}^1 \times \mathbb{P}^1$ base to a 1+1-parameter family rather than the full 3+3-parameter family. Thus, out of the six parameters in (3.14), we are still left with three parameters after rescaling (3.12) and using the above restricted $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ action. From the discussion above and using (3.2), (3.3) and (3.4), we find that in this model $H_0 = 4$, $T = 9$ and $r(V) = 16$. The gauge group $SO(8)^4$ gives rise to 112 vector multiplets. Inserting these values into (3.5) we see that the gravitational anomaly is apparently not satisfied.

This conflict raises the question whether the elliptic fibration on the $Z_2 \times Z_2$ orbifold $X_2$, with either the additional twist in the Landau-Ginzburg representation or the additional shift of (2.12), gives a consistent $F$-theory compactification. However, it is observed that the action of the additional shift (2.12) on the base coordinates commutes with its action on the fiber. Thus, the additional shift should still preserve the fibration. Hence, if the fibration is consistent for the (51,3) Calabi–Yau three-fold, it should still be consistent for the (27,3) one, which is obtained from the (51,3) model by the additional shift (2.12). Note, however that the additional shift on the base commutes with the shift on the fiber only for the 1/2 shift chosen. That is, any other shift $z_i \rightarrow z_i + a$ with $a \neq 1/2$ will not preserve the fibration. Furthermore, regarded as an orbifold of a flat torus, the $Z_2 \times Z_2$ orbifold $X_2$ with $(h_{1,1}, h_{2,1}) = (27, 3)$ is an orbifold of a $T^6$ lattice, rather than $(T^2)^3$. However, as we saw in the previous section, the $X_2$ model can be obtained from the $X_1$ model by adding the freely-acting shift given in (2.12), which preserves the cyclic permutation of the $Z_2 \times Z_2$ orbifold, and hence the factorization into $(\tilde{T}^2)^3$. This is consistent with the fact that the (27,3) $Z_2 \times Z_2$ orbifold model, for example in its free-fermion realization, still possesses the cyclic permutation symmetry between the three $T^2$ factors, which is the characteristic

---

1We thank Andre Losev for discussions on this point.
property of the $Z_2 \times Z_2$ orbifold. This factorization, and the existence of the cyclic permutation symmetry, would naively suggest that the $(27,3) Z_2 \times Z_2$ orbifold model $X_2$ should still possess a sensible elliptic fibration.

But a general requirement of a consistent $F$-theory compactification is that the elliptic fibration produces a global section. We will now show that this is not fulfilled with the additional shift. Consider the quartic representation \((3.12)\) of the Weierstrass representation. This quartic representation has two global sections, \(\hat{y} = \pm \hat{x}, \hat{z} = 0\), whereas that of \((3.6)\) has only one, \(y^2 = x^3, z = 0\). Under the additional $Z_2$ action, the two sections in the quartic representation are simply identified, except for at four fixed points in the base:

\[
(w, \tilde{w}) = (0,0), \ (0,\infty), \ (\infty,0), \ (\infty,\infty) . \tag{3.16}
\]

Note that because of the action on the fiber these are not fixed points of the Calabi-Yau manifold, which remains smooth. Furthermore, at every point on the base, other than the additional singular points of the fibration, the transformation takes one point on the fiber to another point on the fiber. However, at the singular points of the base the fiber is shrunk to half its original size. The crucial observation is that, whilst the intersection of a generic fiber with the section is 1, it is 1/2 for the special fibers over the fixed points. Thus, the section is not globally defined and $F$ theory on $X_3$ is not well defined.

It is intriguing that this new puzzle in the $F$-theory compactification on the $(27,3)$ model arises precisely because of the action of the additional shift on the $T^2$ fiber. To see how the above applies to the $X_2$ model, we consider a related orbifold in which the $Z_2$ action is restricted to the base, leaving the elliptic fiber invariant. In the examples of the previous sections, this corresponds to the additional shift imposed on the $Z_2 \times Z_2$ model in the form \((2.14)\). Taking the first two coordinates, $z_1$ and $z_2$, to be the coordinates of the base and the third, $z_3$, to be the coordinate of the fiber, we see that only the base coordinates are identified under the additional shift, whereas the fiber is left invariant. In the case of this model, as we saw in \((2.13)\) and \((2.16)\), there is an additional sector producing four additional $(1,1)$ and $(2,1)$ forms. To see how these additional multiplets arise in the Weierstrass representation, note that the additional $Z_2$ action is realized by \((\hat{y}, \hat{x}, \hat{z}, w, \tilde{w}) \rightarrow (\hat{y}, \hat{x}, \hat{z}, -w, -\tilde{w})\). This gives the four fixed points in the base \((3.16)\), and hence four fixed tori in the Calabi-Yau manifold, as there is no action on the elliptic fiber. Each of these tori contributes one K"ahler form, from the two-cycle of the $P^1$ used to blow up the torus, and one complex structure deformation from a three-cycle built of a family of $P^1$s over a one-cycle in the torus. Thus, $h_{1,1}$ and $h_{2,1}$ both increase by four. The rest of the analysis follows that of the previous model. After adding these contributions from the fixed tori, the spectrum is $h_{1,1} = 27 + 4 = 31, h_{2,1} = 3 + 4 = 7$.

We see here the difference between the two models $X_2$ and $X_3$. In $X_3$, the additional shift \((2.14)\) is neither freely-acting on the base, nor on the manifold regarded

\[\footnote{We thank Paul Aspinwall for pointing this out.}\]
as a Calabi–Yau three–fold, thus producing the additional four $h_{1,1}$ and $h_{2,1}$ forms needed to resolve the singularity in the conventional manner. However, in the $X_2$ model the shift (2.12) does act freely on the Calabi–Yau three–fold, although there are four fixed points in the base $B$, and therefore does not produce any additional $h_{1,1}$ and $h_{2,1}$ forms associated with these singularities.

We observe that $F$ theory on $X_3$ has $T = 13$ tensor multiplets, $H_0 = 8$ neutral hyper–multiplets and $V = 112$ vector multiplets, and we see from (3.5) that the gravitational anomaly vanishes in this case. Furthermore, from this example we see that the problem with the gravitational anomaly for the $X_3$ model arises strictly because of the action of the additional shift (2.12) on the fiber. Thus, we would like to argue that a possible resolution of the gravitational anomaly is the existence of one hypermultiplet and one tensor multiplet at each of the fixed points in the base. Still, how and whether the additional singularities of the fibration may be resolved resulting in a non-anomalous theory is an open question. However, it is tempting to speculate that the resolution of these singularities is intimately connected to the cancellation of the gravitational anomaly in this model.

4 Orientifold Representation

In this section, we briefly discuss what would be the orientifold construction corresponding to $F$-theory compactification on the $X_2$ orbifold. Whilst it is not guaranteed that there exists an orientifold model for every $F$-theory compactification, studying the orientifold construction may provide a complementary way to understand physical issues. In particular, in the case of the $(27,3)$ model $X_2$, the key question would be how to couple the additional freely-acting shift to the orientifold projection. In this connection, recall that, in this case, as the complex structure of the fiber is identified with the dilaton of the corresponding type-IIB string theory, the shift in (2.12) acts non–trivially on the dilaton.

We begin our brief discussion by studying the orientifold corresponding to the $F$-theory compactification of the $X_1$ orbifold $[34, 35, 36]$. We follow the analysis in [35], focusing in particular on the closed string sector as it seems to relevant for understanding the missing tensor and hypermultiplets. Starting from $T^6$ given in terms of complex coordinates $z_i$, $i = 1, 2, 3$, and the identifications $z_i \sim z_i + 1 \sim z_i + i$, $i = 1, 2, 3$, the $Z_2 \times Z_2$ action is given by

\begin{align}
Z_2^\alpha : (z_1, z_2, z_3) &\rightarrow (-z_1, -z_2, z_3), \\
Z_2^\beta : (z_1, z_2, z_3) &\rightarrow (z_1, -z_2, -z_3).
\end{align}

(4.1) (4.2)

We then let $z_3$ be the coordinate of the elliptic fiber, and define the type-IIB orientifold on $T^4/Z_2^\alpha$ by the orientifold action $\{1, \Omega(-1)^{FL} R_2\}$. The $R_2$ acts on $T^4$ as $(z_1, z_2 \rightarrow (z_1, -z_2)$, and the remainder of the $Z_2^\beta$ action on the elliptic fiber is represented by $\Omega(-1)^{FL} [37]$.  

13
Clearly, in the absence of the $\Omega(-1)^F \cdot R_2$, we would just have a type-IIB compactification on a $K3$ manifold, $T^4/Z_2^g$, which gives an $N = 2$ theory in six dimensions with $T = 21$ $N = 2$ tensor multiplets, each of which consists of one $N = 1$ tensor multiplet and one $N = 1$ hypermultiplet. These arise from each of the sixteen fixed points of the $T^4/\mathbb{Z}^2$, as well as five from the untwisted sector. However, $\Omega(-1)^F \cdot R_2$ projects out the hypermultiplets in the twisted sectors and leaves one tensor and four hypermultiplets invariant in the untwisted sector \[35\]. Thus, we are left with $T = 16 + 1$ tensor multiplets and $H_0 = 4$ hypermultiplets. This is the spectrum from the closed-string sector. It can be shown \[35\] by a more careful analysis that there is an $SO(8)^8$ gauge enhancement arising from the open-string sector, but no charged matter.

In order to understand how to implement the additional $Z_2$ action required to get the $(h_{1,1} = 27, h_{2,1} = 3) \times Z_2 \times Z_2$ orbifold $X_2$ in the orientifold language, let us first consider a similar situation in eight dimensions. It was shown by Witten \[38\] that there exists an orientifold in which there are three $O_+$ planes, one $O_-$ plane and eight $D7$ branes. This is to be compared with the regular orientifold in eight dimensions, in which there are four $O_+$ planes and 16 $D7$ branes. The latter corresponds to an $F$-theory compactification on an elliptically-fibered $K3$ in which the monodromy of the elliptic fiber is $SL(2, \mathbb{Z})$. By placing groups of four $D7$ branes on each of the $O_+$ planes, cancelling the charge locally, one obtains an $SO(8)^4$ gauge group \[37\]. If we do the same for the former model, we find a reduced gauge group $SO(8)^2$, as there now only are eight $D7$ branes. In this case the $F$-theory compactification is an elliptically-fibered $K3$ with restricted monodromy $\Gamma_0(2)$ \[32, 33\]. Witten argued that the existence of the two sets of orientifold planes is due to a non-zero flux through the NS-NS 2-form $B$, even if the 2-form itself is projected out \[38\].

Let us now study the situation in six dimensions. It was pointed out that just as in eight dimensions, by turning on the NS-NS 2-form $B$, along one of the $T^2$ in the underlying $T^4$, the rank is reduced by a factor of two \[39\]. In addition the contribution from the closed string sector of the twisted sector is changed. Rather than contributing 16 hypermultiplets, one instead obtains 12 hypermultiplets and 4 tensor multiplets \[40\]. Since there is an ambiguity in the action of the world-sheet parity $\Omega \ [41]$, we can consider a case with 4 $O_+$ planes and 12 $O_-$ planes. Compared to the extreme case of 16 $O_-$ planes, the gauge symmetry is reduced from $SO(8)^8$ to $SO(8)^4$, whilst there are 12 tensor and 4 hypermultiplets. We recognize this as the spectrum of the $F$-theory compactification of $X_3$. Thus, we find that it does not seem possible to construct an orientifold with the properties corresponding to the $F$-theory based on the $X_2$ orbifold.

We would like to remark that it is not at all clear that the orientifold construction will capture all of the non-perturbative singularities of the $F$-theory compactification. In particular, in the case of the $(27,3) \times Z_2 \times Z_2$ orbifold, the freely-acting shift (2.12) should act non-trivially on the dilaton multiplet. Understanding the exact nature of this action in the orientifold construction may provide a complementary way to
study the physical process involved.

5 Discussion and Conclusions

We have discussed in this paper \( F \)-theory compactification on the Calabi–Yau three–fold which is associated with the realistic free-fermion models. The motivation to consider \( F \)-theory compactification on this particular manifold is apparent: the realistic free-fermion models have, after over a decade of exploration, yielded the most realistic superstring models to date. On the other hand, over the last few years important progress has been achieved in understanding non-perturbative aspects of string theories. Although there is still a considerable way to go before we have complete understanding what role non-perturbative string effects play in connection with physics as observed in Nature, some of the new tools have been applied to phenomenology. One such example is the proposal \[42\] to use the eleventh dimension in \( M \) theory to resolve the mismatch between the unification scale calculated in the MSSM on the basis of the values of the couplings observed at LEP and estimates of the string unification scale in weak-coupling heterotic string models \[3\].

Among the gaps in our understanding of non-perturbative aspects of string theory is the ultimate mechanism that selects the string vacuum. One important aspect of this paper is that, whilst the new puzzles that we have raised are not well understood, \textit{a priori} they may indicate some non–trivial new physics associated with the dilaton multiplet of the type-IIB string theory. We have exhibited and explored elliptic fibrations corresponding to the orbifold models \( X_1 \) and \( X_3 \). Although a six dimensional \( N = 1 \) theory based on the \( F \)-theory compactification of \( X_2 \) does not exist, we can speculate that the additional multiplets needed to resolve the singularities, as well as the gravitational anomaly, arise in some non–trivial non–perturbative way. The most exciting may be the possibility of a still-unknown physical phenomena that will provide a new view on the dilaton fixing problem.

Acknowledgments

We are pleased to thank Shyamoli Chaudhuri, Lance Dixon, Tristan Hübsch, Sheldon Katz, Peter Mayr, David Morrison, Ergin Sezgin and especially Paul Aspinwall and Andre Losev for very helpful discussions. This work was supported in part by the Department of Energy under Grants No. DE-FG-05-86-ER-40272, DE-FG-03-95-ER-40917 and DE-FG-02-94-ER-40823. The work of P.B. was supported in part by the National Science Foundation grant NSF PHY94-07194. P.B. would also like to thank the Aspen Center for Physics, LBL, Berkeley and TPI, Minneapolis for hospitality during the course of this work.

Note Added

Subsequent to the submission of our paper, a Calabi-Yau compactification with an elliptic fibration has been proposed which has a very similar structure to the \( X_2 \)
model discussed here, namely a freely-acting shift with a bi–section and a non–trivial π1 [43].

References

[1] For reviews and references, see e.g.,
M. Li, [hep-th/9811019];
A. Sen, [hep-th/9802051];
C. Vafa, [hep-th/9702201];
P.K. Townsend, [hep-th/9612121];
M.J. Duff, [hep-th/9611203].

[2] For reviews and references, see e.g.,
M.B. Green, J.H. Schwarz and E. Witten, *Superstring Theory, Vols. 1 & 2* Cambridge University Press, Cambridge, 1987;
J. Polchinski, *String Theory, Vols. 1 & 2* Cambridge University Press, Cambridge, 1998;
A. Sen, [hep-ph/9810356].

[3] For reviews and references, see e.g.,
J. Lykken, [hep-ph/9511456];
J.L. Lopez, [hep-ph/9601208];
F. Quevedo, [hep-th/9603074];
A.E. Faraggi, [hep-ph/9807341].

[4] I. Antoniadis, J. Ellis, J. Hagelin and D.V. Nanopoulos, *Phys. Lett. B231* (1989) 65; see also *Phys. Lett. B205* (1988) 459; *Phys. Lett. B208* (1988) 209;
J.L. Lopez, D.V. Nanopoulos and K. Yuan, *Nucl. Phys. B399* (1993) 654.

[5] A.E. Faraggi, D.V. Nanopoulos and K. Yuan, *Nucl. Phys. B335* (1990) 347;
G.B. Cleaver, A.E. Faraggi and D.V. Nanopoulos, [hep-ph/9811427].

[6] I. Antoniadis, G.K. Leontaris and J. Rizos, *Phys. Lett. B245* (1990) 161;
G.K. Leontaris, *Phys. Lett. B372* (1996) 212.

[7] A.E. Faraggi, *Phys. Lett. B278* (1992) 131; *Phys. Lett. B274* (1992) 47; *Phys. Lett. B339* (1994) 223.

[8] S. Chaudhuri, G. Hockney and J.D. Lykken, *Nucl. Phys. B469* (1996) 357;
G. Cleaver, M. Cvetic, J.R. Espinosa, L. Everett and P. Langacker, [hep-th/9805133];
G. Cleaver, M. Cvetic, J.R. Espinosa, L. Everett, P. Langacker and J. Wang, [hep-ph/9807479].

[9] R. Donagi, A. Lukas, B.A. Ovrut and D. Waldram, [hep-th/9811168].
[10] A.E. Faraggi, *Phys. Lett.* **B326** (1994) 62;  
J. Ellis, A.E. Faraggi and D.V. Nanopoulos, *Phys. Lett.* **B419** (1998) 123.

[11] K. Narain, *Phys. Lett.* **B169** (1986) 41;  
K.S. Narain, M.H. Sarmadi and E. Witten, *Nucl. Phys.* **B279** (1987) 369.

[12] D. Morrison and C. Vafa, *Nucl. Phys.* **B476** (1996) 437;  
R. Gopakumar and S. Mukhi, *Nucl. Phys.* **B479** (1996) 260;  
J.D. Blum and A. Zaffaroni, *Phys. Lett.* **B387** (1996) 71;  
A. Dabholkar and J. Park, *Phys. Lett.* **B394** (1997) 302.

[13] P. Berglund, J. Ellis, A.E. Faraggi, D.V. Nanopoulos and Z. Qiu, *Phys. Lett.* **B433** (1998) 269.

[14] H. Kawai, D.C. Lewellen, and S.H.-H. Tye, *Nucl. Phys.* **B288** (1987) 1;  
I. Antoniadis, C. Bachas, and C. Kounnas, *Nucl. Phys.* **B289** (1987) 87;  
I. Antoniadis and C. Bachas, *Nucl. Phys.* **B298** (1988) 586;  
R. Blum, L. Dolan and P. Goddard, *Nucl. Phys.* **B309** (1988) 330.

[15] A.E. Faraggi and D.V. Nanopoulos, *Phys. Rev.* **D48** (1993) 3288;  
A.E. Faraggi, *Nucl. Phys.* **B387** (1992) 239; [hep-th/9708112]

[16] C. Vafa, *Mod. Phys. Lett.* **A4** (1989) 1169;  
K. Intriligator and C. Vafa, *Nucl. Phys.* **B339** (1990) 95.

[17] C. Vafa and E. Witten, *J. Geom. Phys.* **15**, (1995) 189.

[18] D. Morrison and C. Vafa, *Nucl. Phys.* **B473** (1996) 74.

[19] D. Morrison and C. Vafa, *Nucl. Phys.* **B476** (1996) 437.

[20] M.B. Green, J.H. Schwarz and P.C. West, *Nucl. Phys.* **B254** (1985) 1819.

[21] C. Voisin, in *Journées de Géométrie Algébrique d’Orsay* (Orsay, 1992), *Astérisque* **218** (1993) 273.

[22] C. Borcea, in *Essays on Mirror Manifolds*, Vol 2,(B. Greene and S.-T. Yau, eds.), International Press, Cambridge, 1997, p. 717.

[23] V. Nikulin, in *Proceedings of the International Congress of Mathematicians* (Berkeley, 1986), p. 654.

[24] A. Sen, *Nucl. Phys.* **B498** (1997) 135.

[25] E. Gimon and J. Polchinski, *Phys. Rev.* **D54** (1996) 1667.

[26] M. Bershadsky, K. Intriligator, S. Kachru, D.R. Morrison, V. Sadov and C. Vafa, *Nucl. Phys.* **B481** (1996) 215.
27] P. Aspinwall, *Nucl. Phys.* B496 (1997) 149.

28] P. Aspinwall and M. Gross, *Phys. Lett.* B387 (1996) 735.

29] G. Aldazabal, A. Font, L. E. Ibanez and A. M. Uranga, *Nucl. Phys.* B492 (1997) 119.

30] A. Klemm, P. Mayr and C. Vafa, in the proceedings of the conference "Advanced Quantum Field Theory" (in memory of Claude Itzykson), hep-th/9607139.

31] P. Candelas, E. Perevalov and G. Rajesh, *Nucl. Phys.* B502 (1997) 594.

32] M. Bershadsky, T. Pantev and V. Sadov, hep-th/9805056.

33] P. Berglund, A. Klemm, P. Mayr and S. Theisen, hep-th/9805189.

34] J. Blum and A. Zaffaroni, *Phys. Lett.* B387 (1996) 71.

35] A. Dabholkar and J. Park, *Phys. Lett.* B394 (1997) 302.

36] R. Gopakumar and S. Mukhi, *Nucl. Phys.* B479 (1996) 260.

37] A. Sen, hep-th/9605019.

38] E. Witten, hep-th/9712028.

39] M. Bianchi, G. Pradisi and A. Sagnotti, *Nucl. Phys.* B376 (1992) 365.

40] Z. Kakushadze, G. Shiu, S.-H. H. Tye, hep-th/9804092.

41] J. Polchinski, *Phys. Rev.* D55 (1997) 6423.

42] E. Witten, *Nucl. Phys.* B471 (1996) 135.

43] B. Andreas, G. Curio and A. Klemm, hep-th/9903052.