Tri-Bimaximal Mixing and the Neutrino Oscillation Data.

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Abstract
Following recent results from the SNO solar neutrino experiment and the K2K long-baseline neutrino experiment, the combined existing data on neutrino oscillations now point strongly to a specific form for the lepton mixing matrix, with effective bimaximal mixing of $\nu_\mu$ and $\nu_\tau$ at the atmospheric scale and effective tri-maximal mixing for $\nu_e$ with $\nu_\mu$ and $\nu_\tau$ at the solar scale (hence ‘tri-bimaximal’ mixing). We give simple mass-matrices leading to tri-bimaximal mixing, and discuss its relation to the Fritzsch-Xing democratic ansatz.

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1. Motivation

The first data from the Sudbury solar neutrino experiment (SNO) \[1\] have dramatically confirmed the long-standing HOMESTAKE \[2\] solar neutrino result with regard to the high-energy charged-current \(\nu_e\)-rate (SNO/BP2001 = 0.347 ± 0.029, HOMESTAKE/BP2001 = 0.337 ± 0.030). At the same time, the comparison with the existing rather precise SUPER-K (SK) result for solar-neutrino electron elastic scattering (SK/BP2001 = 0.459 ± 0.007 \[3\]) (which includes a neutral-current contribution) has provided a significant first cross-check of the Bahcall-Pinsonneault (BP2001) \[4\] standard solar model calculation of the \(^8\)B flux in these experiments, so that it now seems reasonable to conclude that the suppression for \(^8\)B neutrinos is probably close to \(\sim 1/3\). In detail, the SNO, HOMESTAKE and SUPER-K experiments have different thresholds and response functions (eg. HOMESTAKE is expected to include a \(\sim 15\%\) contribution from \(^7\)Be neutrinos) but such effects are readily taken into account \[5\].

By comparison, the low energy charged-current \(\nu_e\) rate as sampled in the gallium-based experiments SAGE \[6\], GALLEX \[7\] and GNO \[8\] is known to be less suppressed (SAGE/BP2001 = 0.59 ± 0.07, \(\langle\text{GALLEX GNO}\rangle/BP2001 = 0.58 ± 0.07\)). The gallium experiments are sensitive to neutrinos from the pp chain and are only marginally affected by \(^8\)B neutrinos (with expected signal contributions of \(\sim 60\%\) pp, \(\sim 30\%\) \(^7\)Be and \(\sim 10\%\) \(^8\)B in the standard model \[4\]). We have previously emphasised, within the context of the original trimaximal mixing scenario \[9\], the consistency of the gallium suppression with \(5/9 \sim 0.56\) (this consistency survives today at the 1.2\(\sigma\) level, even allowing for the reduced \(^8\)B contribution in gallium in the LMA solution, see below).

Thus energy-dependence of the solar suppression is implicit, with the latest general fits \[3\] to the solar data favouring the so-called large-angle (LMA) MSW \[10\] solution. The long-standing small-angle (SMA) MSW solution is now essentially ruled out, while the so-called LOW and VO solutions are of marginal significance only \[2\]. The LMA solution is illustrated in Fig. 1 for several possible mixing angles. In the LMA solution the base of the MSW ‘bathtub’ is arranged to account for the strong suppression at high-energy where matter effects dominate. At lower energies (for the same solar core density) the suppression reverts to its vacuum level outside the bathtub, accounting for the gallium data (the far high-energy end of the bathtub plays no role in the LMA solution). No significant day-night asymmetry is observed \[3\] so that the latest LMA fits \[3\] now prefer a mass-squared difference towards the higher end of the range \(\sim 10^{-5} - 10^{-4}\) eV\(^2\) (the curves of Fig. 1 are for a representative \(\Delta m^2 = 5 \times 10^{-5}\) eV\(^2\)).

Interestingly, the trimaximal model \[3\] is known \[11\] to predict a ‘5/9 – 1/3 – 5/9’ LMA bathtub (see Fig. 3 of Ref. \[11\]) which could certainly be exploited to fit the
current solar data in isolation. However, within the trimaximal model, the associated mass-squared difference would then necessarily be the larger mass-squared difference (compare Figs. 2 and 3 of Ref. [11]) and would thus be inconsistent with the data on atmospheric neutrinos, which seem to require a mass-squared difference some two orders of magnitude greater \( \sim 10^{-3} - 10^{-2} \) eV\(^2\) [12].

Indeed, the other important new experimental input motivating the present analysis, is the currently emerging data from the K2K long-baseline experiment [13], tending to confirm [14] the mass-squared difference from the atmospheric neutrino experiments \( \Delta m^2 \approx 3 \times 10^{-3} \) eV\(^2\), clearly well above the solar mass-squared difference defined by the LMA solar fits and, in particular, subject to the CHOOZ [15] and PALO-VERDE [16] reactor limits on \( \bar{\nu}_e \)-mixing, which imply \( |U_{e3}|^2 \lesssim 0.03 \), for \( \Delta m^2 \gtrsim 10^{-3} \) eV\(^2\). (\( U \) denotes the lepton mixing or MNS matrix, [17]). Note that, in this last respect, the new K2K results strongly disfavour the original trimaximal model.

An obvious solution is to sacrifice the economy of the original trimaximal model and acknowledge (in line with most other phenomenological analyses) the existence of two distinct mass-squared difference scales, \( \Delta m^2 >> \Delta m^2' \), controlling respectively the behaviour of atmospheric and of solar neutrinos. Within this context, the totality of the data clearly point to a particular form for the lepton mixing matrix, which turns out to be closely related to the trimaximal hypothesis, and which is given below.

2. The Trend of the Data

The atmospheric neutrino results are known to point strongly to twofold maximal \( \nu_\mu - \nu_\tau \) mixing (or at least to effective twofold maximal \( \nu_\mu - \nu_\tau \) mixing at the atmospheric scale). In particular the SUPER-K zenith angle distribution for multi-GeV ‘\( \mu \)-like’ events (Fig. 2a) clearly indicates a suppression of upward \( \nu_\mu \) with respect to downward \( \nu_\mu \) by a factor of \( \sim 1/2 \) (from Fig. 2 the up-to-down ratio for multiGeV muons is \( (U/D)_\mu \approx 0.53 \pm 0.05 \) for zenith angles \( |\cos \theta| > 0.20 \)). By contrast, the corresponding distribution for ‘\( e \)-like’ events (Fig. 2b) appears to be completely unaffected, \( (U/D)_e \approx 1.09 \pm 0.12 \). In Fig. 2a the solid curve is the full oscillation curve for twofold maximal \( \nu_\mu \rightarrow \nu_\tau \) mixing for \( \Delta m^2 = 3 \times 10^{-3} \) eV\(^2\) for a representative neutrino energy, \( E = 3 \) GeV. The dashed curve incorporates energy averaging and muon angular smearing (with respect to the neutrino direction) and clearly fits the data.

More generally, the (locally averaged) survival probability \( < P_\mu > \) for \( \nu_\mu \) at intermediate \( L/E \) scales, \( (\Delta m^2)^{-1} \lesssim L/E \lesssim (\Delta m^2)^{-1} \) (where \( L \) is the neutrino flight path length) is given by the magnitude-squared \( |U_{\mu3}|^2 \) of the MNS matrix element \( U_{\mu3} \) via \( < P_\mu > = (1 - |U_{\mu3}|^2)^2 + |U_{\mu3}|^4 \), whereby \( |U_{\mu3}|^2 = [1 \pm \sqrt{1 - 2(1 - < P_\mu >)}]/2 \)
(clearly $P_\mu = 1/2$ implies $|U_{\mu 3}|^2 = 1/2$, as expected). The overall fit to the SK atmospheric data (ie. not just $P_\mu$ from the multi-GeV muons above) gives $P_\mu = 0.50 \pm 0.04$ whereby $0.36 \lesssim |U_{\mu 3}|^2 \lesssim 0.64$ (68% CL), certainly consistent with $|U_{\mu 3}|^2 = 1/2$ and twofold maximal mixing. Independent evidence for strong $\nu_\tau$ mixing comes from the observation of a substantive charged-current $\nu_\tau$ appearance signal, statistically separated from the neutral-current sample in SUPER-K [18].

As indicated in Section 1, intermediate baseline reactor experiments, such as CHOOZ [15] and PALO-VERDE [16], in fact provide the best limits on $\nu_e$ (actually $\bar{\nu}_e$) mixing at the atmospheric scale (in terms of the vacuum mixing matrix, the interpretation of the atmospheric experiments themselves can be seriously obscured by terrestrial matter effects, which tend to suppress $\nu_e$-mixing and enhance $\nu_\tau$-mixing [19] in the high-energy limit). Reactor experiments, utilising very low energy (anti) neutrinos and with existing baselines much shorter than the matter-oscillation length in the Earth, turn out to be almost completely immune to matter effects [20]. As discussed in Section 1, the best reactor limits give $|U_{e3}|^2 \lesssim 0.03$ (95% CL) for $\Delta m^2 \simeq 3 \times 10^{-3}$ eV$^2$, consistent with $U_{e3} = 0$ and thus with two-fold maximal $\nu_\mu - \nu_\tau$ mixing.

We emphasise that the atmospheric and reactor data do not require $U_{e3} \equiv 0$ any more than they require $|U_{\mu 3}|^2 \equiv 1/2$ (small non-zero values of $U_{\mu 3}$, and/or somewhat different values of $|U_{\mu 3}|^2$, eg. $|U_{\mu 3}|^2 = 2/3$ [21], are more-or-less equally acceptable experimentally). It is only that $U_{e3} = 0$ and $|U_{\mu 3}|^2 = 1/2$ provide a simple and adequate description of the current trend of the data, making twofold maximal $\nu_\mu - \nu_\tau$ mixing (for now) the accepted ‘default option’ [22] (at the atmospheric scale).

In a similar spirit, we turn again to the solar data displayed in Fig. 1, drawing particular attention now to the solid curve representing the ‘5/9 − 1/3 − 5/9’ bathtub (in the LMA solution the base of the bathtub essentially measures $|U_{e2}|^2$ directly). In Fig. 1 the data are plotted assuming BP2001 fluxes [4]. The SNO, HOME STEAKE and SUPER-K data (after correction for the neutral-current contribution in SUPER-K) then determine the base of the bathtub, with $|U_{e2}|^2 \sim 1/3$ clearly closely preferred. In the flux-independent approach [14] the $^8$B suppression is found to be $\sim 0.33 \pm 0.10$ (based on the measured SK–SNO difference [4]). In Fig. 1 the two broken curves correspond roughly (in the bathtub region) to the $\pm 1\sigma$ errors on the flux-independent suppression. Finally, note that with $|U_{e3}|^2 = 0$ (or small) the $\nu_e$ survival probability outside the bathtub, $P_e$, is (inversely) correlated with the value at its base in the LMA solution. For vacuum mixing $P_e = (1 - |U_{e2}|^2)^2 + |U_{e2}|^4$, so that for $|U_{e3}|^2 = 1/3$ we have $P_e = 5/9 \simeq 0.56$. (Taking account of the $^8$B contribution the expected gallium suppression is actually closer to $\simeq 0.53$, but this is still consistent with the data at the $\sim 1.2\sigma$ level). Thus the gallium data themselves provide an
independent cross-check on the consistency of the LMA solution and on \( |U_{e2}|^2 \sim 1/3 \).

As above, we emphasise that the data do not require \( |U_{e2}|^2 \equiv 1/3 \). If the (implicit) energy dependence of the solar suppression is real, certainly \( |U_{e2}|^2 \neq 1/2 \) [22], since there can be no MSW bathtub in that case [11]. But a somewhat more pronounced (eg. a ‘5/8 − 1/4 − 5/8’) bathtub (corresponding to \( |U_{e2}|^2 = 1/4 \), is clearly far from excluded. As before, we regard \( |U_{e2}|^2 = 1/3 \) as a simple and adequate description of the data, which could usefully come to be seen as the default option at the solar scale.

3. ‘Tri-Bimaximal’ Mixing

In this section we simply take the above ‘default’ values \( U_{e3} = 0 \), \( |U_{\mu 3}|^2 = 1/2 \) and \( |U_{e2}|^2 = 1/3 \) as given, and use them to evaluate the resulting lepton mixing matrix.

The lepton mixing matrix is defined with the rows labelled by the charged-lepton mass-eigenstates (e, µ, τ) and the columns labelled by the neutrino mass-eigenstates (ν₁, ν₂, ν₃). Focussing on the last column (the ν₃ column), we note that with \( U_{e3} = 0 \) and \( |U_{\mu 3}|^2 = 1/2 \), we have \( |U_{\tau 3}|^2 = 1/2 \) from unitarity, so that the last column is just as in the original bimaximal scheme [23]. Moving to the center column (the ν₂ column), again as a consequence of \( U_{e3} = 0 \), orthogonality requires \( |U_{\tau 2}| = |U_{\mu 2}| \). With \( |U_{e2}|^2 = 1/3 \) (above) we then have \( |U_{\tau 2}|^2 = |U_{\mu 2}|^2 = 1/3 \) from unitarity, so that the center column is just as in the original trimaximal scheme (see eg. Ref. [19]). Finally the first column (the ν₁ column) follows from unitarity applied to the rows.

Indeed, it was pointed out in Ref. [19] (even before the SNO data first appeared) that a mixing scheme with the ν₃ ‘bimaximally’ mixed and the ν₂ ‘trimaximally’ mixed (hence tri-bimaximal mixing) could naturally account for the data, being also discussed in the conference literature under the name of ‘optimised’ bimaximal mixing [24]:

\[
(U_{\nu})^2 = \begin{pmatrix}
\nu_1 & \nu_2 & \nu_3 \\
\frac{2}{3} & 1/3 & 0 \\
1/6 & 1/3 & 1/2 \\
1/6 & 1/3 & 1/2
\end{pmatrix}
\]

(1)

(where the moduli-squared of the elements are given). The name ‘optimised’ bimaximal mixing reflected the scheme’s pedigree as a special case of the Altarelli-Feruglo generalised bimaximal form [23] and its close relationship to the original bimaximal form [23]. We emphasise that the mixing Eq. 1 is entirely determined by unitarity constraints once the above three ‘corner’ elements are fixed to their default values.

We should also point out that the mixing Eq. 1 has much in common with the Fritzsch-Xing democratic ansatz [21] (which might in fact, see Section 4, reasonably be termed ‘bi-trimaximal’ mixing, as opposed to ‘tri-bimaximal’ mixing). Indeed the
Fritzsch-Xing ansatz may be viewed as a permuted form of Eq. 1 (with, somewhat remarkably, the crucial prediction \( U_{e3} = 0 \) made well before the emergence of the CHOOZ data \([15]\)). It should be clear however that the phenomenologies of these two mixing schemes are quite distinct, eg. the Fritzsch-Xing ansatz predicted \( |U_{e2}|^2 = 1/2 \) and hence no energy dependence \([14]\) of the solar suppression, which is now disfavoured by SNO \([1]\). The original bimaximal scheme \([23]\) is likewise now disfavoured \([26]\).

Asymptotic \( (L/E \to \infty) \) predictions, specific to tri-bimaximal mixing, are the (vacuum) survival probabilities \( < P_\mu > = < P_\tau > = 7/18 \). The corresponding \( \nu_\mu \leftrightarrow \nu_\tau \) appearance probability is also 7/18. Note that \( U_{e3} \equiv 0 \) implies no Pantaleone resonance \([27]\) and no CP violation in neutrino oscillations, which might be considered a disappointment experimentally. Nonetheless, it is fair to say that current data point to Eq. 1, and it is therefore of interest to try to understand what it might imply. In the next section we present simple mass-matrices leading to tri-bimaximal mixing.

4. Simple Mass Matrices

Our mass-matrices will be taken to be hermitian (ie. we will throughout be implicitly referring to hermitian-squares of mass-matrices linking left-handed fields, \( MM^\dagger \equiv M^2 \)). Fermion mass-matrices are most naturally considered in a ‘weak’ basis (i.e. a basis which leaves the charged-current weak-interaction diagonal and universal).

Maximal mixing (whether trimaximal or bimaximal) undeniably suggests permutation symmetries \([28]\). We postulate that, in a particular weak basis, the mass-matrices take the following (permutation symmetric) forms:

\[
M^2_l = \begin{pmatrix} a & b & b^* \\ b^* & a & b \\ b & b^* & a \end{pmatrix} \quad M^2_\nu = \begin{pmatrix} x & 0 & y \\ 0 & z & 0 \\ y & 0 & x \end{pmatrix}
\]

where the real constants \( a, x, y, z \) and the complex constants \( b \) and \( b^* \) encode the charged-lepton and neutrino masses as follows:

\[
a = \frac{m^2_e}{3} + \frac{m^2_\mu}{3} + \frac{m^2_\tau}{3} \quad x = \frac{m^2_1}{2} + \frac{m^2_2}{2}
\]

\[
b = \frac{m^2_e}{3} + \frac{m^2_\mu}{3} \omega + \frac{m^2_\tau}{3} \bar{\omega} \quad y = \frac{m^2_1}{2} - \frac{m^2_3}{2}
\]

\[
b^* = \frac{m^2_e}{3} + \frac{m^2_\mu}{3} \bar{\omega} + \frac{m^2_\tau}{3} \omega \quad z = m^2_2
\]

with \( \omega = \exp(i2\pi/3) \) and \( \bar{\omega} = \exp(-i2\pi/3) \) denoting the complex cube roots of unity.

In Eq. \([2]\) the charged-lepton mass-matrix \( M^2_l \) takes the familiar \( 3 \times 3 \) circulant form \([28]\), invariant under cyclic permutations of the three generation indices. The neutrino
mass-matrix $M^2_\nu$ is real (i.e. symmetric, since our mass-matrices are hermitian) and is a $2 \times 2$ circulant in the $1 - 3$ index subset, invariant under the permutation of only two out of the three generation indices (generations $1 \leftrightarrow 3$). The neutrino mass-matrix has four ‘texture zeroes’ \[29\] enforcing the effective block-diagonal form. Note that both mass matrices (Eq. 2) are invariant under the interchange of generation indices $1 \leftrightarrow 3$ performed simultaneously with a complex conjugation. Indeed, in this basis, it is the invariance of all the leptonic terms under this combined involution, which guarantees no CP-violation here (since $\text{Im} \det[M^2_l, M^2_\nu]$ \[30\] changes sign).

The mass-matrices $M^2_l$ and $M^2_\nu$ are diagonalised by a threefold maximal unitary matrix $U_l$ and a twofold maximal unitary matrix $U_\nu$, respectively:

$$
U_l = \begin{pmatrix}
e & \mu & \tau \\
\frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} & \frac{\omega}{\sqrt{3}} \\
\frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} & \frac{\bar{\omega}}{\sqrt{3}} \\
\frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} & \frac{\bar{\omega}}{\sqrt{3}} \\
\end{pmatrix} \quad U_\nu = \begin{pmatrix}
\nu_1 & \nu_2 & \nu_3 \\
\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\
0 & 1 & 0 \\
\frac{\sqrt{1}}{2} & 0 & \frac{\sqrt{1}}{2} \\
\end{pmatrix}
$$

i.e. $U_l^\dagger M^2_l U_l = \text{diag}(m^2_{e}, m^2_{\mu}, m^2_{\tau})$ and $U_\nu^\dagger M^2_\nu U_\nu = \text{diag}(m^2_{1}, m^2_{2}, m^2_{3})$, so that the lepton mixing matrix (or MNS matrix) $U = U_l^\dagger U_\nu$ is given by:

$$
\begin{pmatrix}
e & \mu & \tau \\
\frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} & \frac{\omega}{\sqrt{3}} \\
\frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} & \frac{\bar{\omega}}{\sqrt{3}} \\
\frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} & \frac{\bar{\omega}}{\sqrt{3}} \\
\end{pmatrix} \begin{pmatrix}
\nu_1 & \nu_2 & \nu_3 \\
\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\
0 & 1 & 0 \\
\frac{\sqrt{1}}{2} & 0 & \frac{\sqrt{1}}{2} \\
\end{pmatrix} = \begin{pmatrix}
e & \mu & \tau \\
\frac{1}{\sqrt{6}} & \frac{\sqrt{3}}{\sqrt{6}} & \frac{\sqrt{2}}{\sqrt{6}} \\
\frac{1}{\sqrt{6}} & \frac{\sqrt{3}}{\sqrt{6}} & \frac{-\sqrt{2}}{\sqrt{6}} \\
\frac{1}{\sqrt{6}} & \frac{\sqrt{3}}{\sqrt{6}} & \frac{i}{\sqrt{6}} \\
\end{pmatrix}
$$

where the RHS is the tri-bimaximal form (Eq. 1) in a particular phase convention. For Dirac neutrinos, the factor of $i$ is readily removed by a simple rephasing of the $\nu_3$ mass-eigenstate, yielding tri-bimaximal mixing expressed as an orthogonal matrix:

$$
U = \begin{pmatrix}
e & \mu & \tau \\
\frac{\sqrt{2}}{\sqrt{3}} & \frac{\sqrt{3}}{\sqrt{3}} & 0 \\
\frac{1}{\sqrt{6}} & \frac{\sqrt{3}}{\sqrt{6}} & \frac{i}{\sqrt{6}} \\
\frac{-1}{\sqrt{6}} & \frac{\sqrt{3}}{\sqrt{6}} & \frac{-i}{\sqrt{6}} \\
\end{pmatrix}
$$

While this concludes our derivation of tri-bimaximal mixing starting from Eq. 2, we take this opportunity to remark that one might easily (with perhaps equal \emph{a priori} justification) have interchanged the forms of $M^2_l$ and $M^2_\nu$ in Eq. 2, taking the \textit{neutrino} mass-matrix to be the $3 \times 3$ circulant, and the charged-lepton mass-matrix to be of the $2 \times 2$ block-diagonal circulant form. Note however that this leads to physically distinct mixing which, if the $2 \times 2$ circulant is chosen to be in the $1 - 2$ index subset,
is identically the Fritzsch-Xing democratic ansatz [21] (hence ‘bi-trimaximal’ mixing as a synonym for the Fritzsch-Xing ansatz, see Section 3 above). We emphasise once again that the phenomenologies of the above two mixing schemes are physically distinct, with the Fritzsch-Xing ansatz now essentially ruled out, along with many other schemes involving energy independent solar solutions [31] (including trimaximal mixing [9]) following the SNO results [1].

5. Perspective

Tri-bimaximal mixing is a specific mixing matrix (Eq. 1/Eq. 6) which encapsulates the trends of a broad range of experimental data (the LSND oscillation signal [32] was not considered on the grounds that it still awaits confirmation). Tri-bimaximal mixing is closely related to a number of previously suggested lepton mixing schemes, notably trimaximal mixing [4], bimaximal mixing [23], the Fritzsch-Xing democratic ansatz [21] and the Altarelli-Feruglio scheme [25] (of which tri-bimaximal mixing may be considered a special case). For the future, the couplings of the heavy neutrino, $\nu_3$, are expected to be measured more precisely in long-baseline experiments like MINOS [33] (and other projects [34]). In particular, the limits on $|U_{e3}|^2$ should continue to improve. Regarding the $\nu_2$ couplings, if tri-bimaximal mixing is right, the KAMLAND experiment [35] should confirm the LMA solution, measuring a $\bar{\nu}_e$ survival probability tending to $<P_e> = 5/9 \approx 0.56$ at sufficiently low-energy. Corresponding predictions for $\nu_\mu$ disappearance and $\nu_\mu \rightarrow \nu_\tau$ appearance, see Section 3, look very hard to test (experiments with $\nu_\mu$ necessitate higher energies and hence longer baselines, with matter effects generally dominant over vacuum effects for $\nu_2$). Finally, exact tri-bimaximal mixing would imply no high-energy matter resonance and no (intrinsic) CP-violation in neutrino oscillations, which might be considered a disappointment experimentally.

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Figure Captions

Figure 1. The solar data, including the recent SNO measurements, plotted versus neutrino energy $E$. The gallium points are plotted at $<1/E>_\sim^{-1} \simeq 0.5$ MeV, the SUPER-K point at $<1/E>_\sim^{-1} \simeq 10$ MeV and the HOMESTAKE point (which includes an $\sim 15\%$ contribution from the $^7$Be-line at $\sim 0.9$ MeV) is plotted (somewhat arbitrarily) at $<1/E>_\sim^{-1} = 5$ MeV. The solid curve is the expected energy dependence for tri-bimaximal mixing, Eq. 1, with $\Delta m^2 = 5 \times 10^{-5}$ eV$^2$, which predicts a suppression of $1/3$ in the base of the bathtub and $5/9$ outside.

Figure 2. Atmospheric neutrino data. The multi-GeV zenith angle distributions for a) $\mu$-like and b) $e$-like events in SUPER-K. The solid curve is the full oscillation curve for tri-bimaximal mixing (Eq. 1/Eq. 6) with $\Delta m^2 = 3 \times 10^{-3}$ eV$^2$ for a representative neutrino energy $E = 3$ GeV, and the dashed curve shows the effect of angular smearing with respect to the neutrino direction and averaging over neutrino energies. Tri-bimaximal mixing (or indeed any mixing hypothesis which is effectively twofold maximal $\nu_\mu - \nu_\tau$ mixing at the atmospheric scale) fits the data well.
Figure 1:
Figure 2:

- MultigeV
  - a) μ-like (SK)
    - Data/MC (scaled)
    - $\Delta m^2 = 3.0 \times 10^{-3} \text{eV}^2$
    - 2-fold Max. (μ-τ) (with no smearing)
  - b) e-like (SK)
    - Data/MC (scaled)
    - $\cos \theta$