Some $p$-Adic Aspects of Superanalysis*

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Abstract

A brief review of a superanalysis over real and $p$-adic superspaces is presented. Adelic superspace is introduced and an adelic superanalysis, which contains real and $p$-adic superanalysis, is initiated.

1 Introduction

Supersymmetry plays very important role in construction of new fundamental models of high energy physics beyond the Standard Model. Especially it is significant in formulation of String/M-theory, which is presently the best candidate for unification of matter and interactions. Supersymmetry transformation can be regarded as transformation in a superspace, which is an ordinary spacetime extended by some anticommuting (odd) coordinates. Spacetime in M-theory is eleven-dimensional with the Planck length as the fundamental one. According to the well-known uncertainty

$$\Delta x \geq \ell_0 = \sqrt{\frac{\hbar G}{c^3}} \approx 10^{-33} \text{cm}, \quad (1)$$

one cannot measure distances smaller than the Planck length $\ell_0$. Since the derivation of (1) is based on the general assumption that real numbers and archimedean geometry are valued at all scales it means that the usual approach is broken and cannot be extended beyond the Planck scale without adequate modification which contains non-archimedean geometry. The very natural modification is to use adelic approach, since it contains real and $p$-adic numbers which make all possible completions of the rational numbers. As a result it follows that one has to consider possible relations between adelic and supersymmetry structures. In

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* Based on the talk presented at the International Workshop Supersymmetries and Quantum Symmetries, 24-29 July 2003, Dubna, Russia
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this report we review some aspects of $p$-adic superanalysis and introduce adelic superanalysis, which is a basis for investigation of the corresponding $p$-adic and adelic supersymmetric models.

2 Some basic properties of $p$-adic numbers and adeles

Let us first recall that numerical experimental results belong to the field of rational numbers $\mathbb{Q}$. On the $\mathbb{Q}$ one can introduce the usual absolute value $| \cdot |_\infty$ and $p$-adic absolute value $| \cdot |_p$ for each prime number $p$. Completion of $\mathbb{Q}$ with respect to $| \cdot |_\infty$ gives the field of real numbers $\mathbb{Q}_\infty \equiv \mathbb{R}$. If we replace $| \cdot |_\infty$ by $| \cdot |_p$ then completion of $\mathbb{Q}$ yields a new number field known as the field of $p$-adic numbers $\mathbb{Q}_p$. Consequently, $\mathbb{Q}$ is dense in $\mathbb{R}$ as well as in $\mathbb{Q}_p$ for every $p$. $\mathbb{R}$ has archimedean metric $d_\infty(x, y) = |x - y|_\infty$ and $\mathbb{Q}_p$ has non-archimedean metric (ultrametric) $d_p(x, y) = |x - y|_p$, i.e. $d_p(x, y) \leq \max\{d_p(x, z), d_p(z, y)\}$. It is worth pointing out that $\mathbb{R}$ and $\mathbb{Q}_p$ exhaust all possibilities to get number fields by completion of $\mathbb{Q}$. Any $p$-adic number $x \in \mathbb{Q}_p$ can be presented in the unique way $x = p^\nu \sum_{k=0}^{\infty} a_k p^k$ where $\nu \in \mathbb{Z}$, $a_k \in \{0, 1, \ldots, p - 1\}$, what resembles representation of a real number $y = \pm 10^\mu \sum_{k=0}^{\infty} b_k 10^k$, $\mu \in \mathbb{Z}$, $b_k \in \{0, 1, \ldots, 9\}$ but with expansion in the opposite way. There are two main types of functions with $p$-adic argument: $p$-adic valued and real-valued (or complex-valued). The reader who is not familiar with $p$-adic numbers and their functions can see, e.g. [1].

To regard simultaneously real and $p$-adic properties of rational numbers and their completions one uses concept of adeles. An adele $x$ (see, e.g. [2]) is an infinite sequence $x = (x_\infty, x_2, \ldots, x_p, \ldots)$, where $x_\infty \in \mathbb{R}$ and $x_p \in \mathbb{Q}_p$ with the restriction that for all but a finite set $S$ of primes $p$ one has $x_p \in \mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\} = \{x \in \mathbb{Q}_p : x = a_0 + a_1 p + a_2 p^2 + \cdots\}$. Componentwise addition and multiplication endow a ring structure to the set of adeles $\mathcal{A}$. $\mathcal{A}$ can be defined as

$$\mathcal{A} = \bigcup_S \mathcal{A}_S, \quad \mathcal{A}_S = \mathbb{R} \times \prod_{p \in S} \mathbb{Q}_p \times \prod_{p \notin S} \mathbb{Z}_p. \quad (2)$$

$\mathbb{Q}$ is naturally embedded in $\mathcal{A}$. Ring $\mathcal{A}$ is also a locally compact topological space. Important functions on $\mathcal{A}$ are related to mappings $f : A \to A$ and $\varphi : A \to \mathbb{R} (\mathbb{C})$.

3 Elements of $p$-adic and adelic string theory

A notion of $p$-adic string and hypothesis on the existence of non-archimedean geometry at the Planck scale were introduced by Volovich [3] and have been investigated by many researchers (reviews of an early period are in [1] and [4]). Very successful $p$-adic analogues of the Veneziano and Virasoro-Shapiro amplitudes were proposed in [5] as the corresponding Gel’fand-Graev [2] beta functions. Using this approach, Freund and Witten obtained [6] an attractive adelic formula.
\( A_\infty(a, b) \prod_p A_p(a, b) = 1 \), which states that the product of the crossing symmetric Veneziano (or Virasoro-Shapiro) amplitude and its all p-adic counterparts equals unity (or a definite constant). This gives possibility to consider an ordinary four-point function, which is rather complicate, as an infinite product of its inverse p-adic analogues, which have simpler forms. The ordinary crossing symmetric Veneziano amplitude can be defined by a few equivalent ways and its integral form is

\[
A_\infty(a, b) = \int_\mathbb{R} |x|_\infty^{a-1} |1 - x|_\infty^{b-1} dx, \tag{3}
\]

where it is taken \( \hbar = 1, T = 1/\pi \), and \( a = -\alpha(s) = -1 - \frac{s}{2}, b = -\alpha(t), c = -\alpha(u) \) with the conditions \( s + t + u = -8 \) and \( a + b + c = 1 \). According to \( \mathfrak{5} \) p-adic Veneziano amplitude is a simple p-adic counterpart of (3), i.e.

\[
A_p(a, b) = \int_{\mathbb{Q}_p} |x|_p^{a-1} |1 - x|_p^{b-1} dx, \tag{4}
\]

where now \( x \in \mathbb{Q}_p \). In both (3) and (4) kinematical variables \( a, b, c \) are real or complex-valued parameters. Thus in (4) only string world-sheet parameter \( x \) is treated as p-adic variable, and all other quantities maintain their usual real values. Unfortunately, there is a problem to extend the above product formula to the higher-point functions. Some possibilities to construct p-adic superstring amplitudes are considered in \( \mathfrak{7} \) (see also \( \mathfrak{8}, \mathfrak{9}, \) and \( \mathfrak{10} \)).

A recent interest in p-adic string theory has been mainly related to an extension of adelic quantum mechanics \( \mathfrak{11} \) and p-adic path integrals to string amplitudes \( \mathfrak{12} \). An effective nonlinear p-adic string theory (see, e.g. \( \mathfrak{4} \)) with an infinite number of space and time derivatives has been recently of a great interest in the context of the tachyon condensation \( \mathfrak{13} \). It is also worth mentioning successful formulation and development of p-adic and adelic quantum cosmology (see \( \mathfrak{14} \) and references therein) which demonstrate discreteness of minisuperspace with the Planck length \( \ell_0 \) as the elementary one.

4 Elements of p-adic and adelic superanalysis

Here it will be first presented some elements of real and p-adic superanalysis along approach introduced by Vladimirov and Volovich \( \mathfrak{15} \) and elaborated by Khrennikov \( \mathfrak{16} \). Then I shall generalize this approach to adelic superanalysis.

Let \( \Lambda(Q_v) = \Lambda_0(Q_v) \oplus \Lambda_1(Q_v) \) be \( \mathbb{Z}_2 \)-graded vector space over \( Q_v \), \( (v = \infty, 2, 3, \ldots, p, \ldots) \), where elements \( a \in \Lambda_0(Q_v) \) and \( b \in \Lambda_1(Q_v) \) have even \( (p(a) = 0) \) and odd \( (p(b) = 1) \) parities. Such \( \Lambda(Q_v) \) space is called \( v \)-adic (real and p-adic) superalgebra if it is endowed by an associative algebra with unity and parity multiplication \( p(ab) \equiv p(a) + p(b) (\text{mod} \ 2) \). Supercommutator is defined in the usual way: \( [a, b] = ab - (-1)^{p(a)p(b)} ba \). Superalgebra \( \Lambda(Q_v) \) is called (super)commutative if \( [a, b] = 0 \) for any \( a \in \Lambda_0(Q_v) \) and \( b \in \Lambda_1(Q_v) \). As illustrative examples of commutative superalgebras one can consider finite dimensional
\(v\)-adic Grassmann algebras \(G(\mathbb{Q}_v : \eta_1, \eta_2, \cdots, \eta_m)\) which dimension is \(2^m\) and generators \(\eta_1, \eta_2, \cdots, \eta_m\) satisfy anticommutative relations \(\eta_i \eta_j + \eta_j \eta_i = 0\). The role of norm necessary to build analysis on commutative superalgebra \(\Lambda(\mathbb{Q}_v)\) plays the absolute value \(| \cdot |\) for real case and \(p\)-adic norm \(| \cdot |_p\) for \(p\)-adic cases.

Let \(\Lambda(\mathbb{Q}_v)\) be a fixed commutative \(v\)-adic superalgebra. \(v\)-Adic superspace of dimension \((n, m)\) over \(\Lambda(\mathbb{Q}_v)\) is
\[
\mathbb{Q}_{\Lambda(\mathbb{Q}_v)}^{n,m} = \Lambda_0^n(\mathbb{Q}_v) \times \Lambda_1^m(\mathbb{Q}_v)
\]
and it is an extension of the standard \(v\)-adic space. In the sequel we will mainly have in mind that \(\Lambda_0(\mathbb{Q}_v) = \mathbb{Q}_v\) or that \(\mathbb{Q}_v\) is replaced by \(\mathbb{Q}_v(\sqrt{\tau})\), where \(\sqrt{\tau} \not\in \mathbb{Q}_v\).

Then our \(v\)-adic (i.e. real and \(p\)-adic) superspace can be defined as \(\mathbb{Q}_{\Lambda(\mathbb{Q}_v)}^{n,m} = \mathbb{Q}_v^n \times \Lambda_1^m(\mathbb{Q}_v)\) which points are \(X^{(v)} = (X_1^{(v)}, X_2^{(v)}, \cdots, X_n^{(v)}, X_{n+1}^{(v)}, \cdots, X_{n+m}^{(v)}) = (x_1^{(v)}, x_2^{(v)}, \cdots, x_n^{(v)}, \theta_1^{(v)}, \cdots, \theta_m^{(v)}) = (x^{(v)}, \theta^{(v)})\), where coordinates \(x_1^{(v)}, x_2^{(v)}, \cdots, x_n^{(v)}\) are commutative, \(p(x_i^{(v)}) = 0\), and \(\theta_1^{(v)}, \theta_2^{(v)}, \cdots, \theta_m^{(v)}\) are anticommutative (Grassmann), \(p(\theta_j^{(v)}) = 1\). Since supercommutator \([X_i^{(v)}, X_j^{(v)}] = X_i^{(v)} X_j^{(v)} - (-1)^{p(x_i^{(v)}) p(x_j^{(v)})} X_j^{(v)} X_i^{(v)} = 0\), coordinates \(X_i^{(v)}, (i = 1, 2, \cdots, n + m)\) are called supercommuting. A norm of \(X^{(v)}\) can be defined as \(||X^{(v)}|| = \max \{|x_i^{(v)}|_v, |\theta_j^{(v)}|_v\}\). In the sequel, to decrease number of indices we often omit them when they are understood from the context.

One can define functions \(F_v(X)\) on open subsets of superspace \(\mathbb{Q}_{\Lambda(\mathbb{Q}_v)}^{n,m}\), as well as their continuity and differentiability (for more details, see [13] and [14]). One has to differ the left and the right partial derivatives: \(\frac{\partial_v F_v}{\partial \theta_j} , \frac{\partial_v F_v}{\partial \theta_j} \). It is worth noting that derivatives of \(p\)-adic valued function of \(p\)-adic arguments are formally the same as those for real functions of real arguments. Integral calculus for \(p\)-adic valued functions is more subtle than in the real case, since there is no \(p\)-adic valued Lebesgue measure [17]. One can use antiderivatives, but one has to take care about pseudoconstants, which are some exotic functions with zero derivatives. However, for analytic functions one can well define definite integrals using the corresponding antiderivatives [11]. Integration with anticommutating variables is introduced by axiomatic approach requiring linearity and translation invariance in both real and \(p\)-adic cases. In particular, one obtains the following two indefinite integrals: \(\int d \theta_j^{(v)} = 0\) and \(\int \theta_j^{(v)} d \theta_j^{(v)} = 1\).

When \(\mathbb{Q}_v^n\) corresponds to a \(n\)-dimensional spacetime, functions \(F_v(x, \theta)\) on superspace \(\mathbb{Q}_{\Lambda(\mathbb{Q}_v)}^{n,m}\) are called \(v\)-adic superfields. Due to the fact that there is only finite number of non-zero products with anticommuting variables, expansions of \(F_v(x, \theta)\) over \(\theta_j\), \((j = 1, 2, \cdots, m)\) are finite, i.e. there are \(2^m\) terms in the corresponding Taylor expansion. Description of supersymmetric models by superfields is very compact and elegant [18].

We can now turn to adelic superanalysis. It is natural to define the corresponding \(\mathbb{Z}_2\)-graded vector space over \(\mathcal{A}\) as
\[
\Lambda(\mathcal{A}) = \bigcup_{\mathbf{s}} \Lambda_{\mathbf{s}} , \quad \Lambda_{\mathbf{s}} = \Lambda(\mathbb{R}) \times \prod_{p \notin \mathbf{s}} \Lambda(\mathbb{Q}_p) \times \prod_{p \in \mathbf{s}} \Lambda(\mathbb{Z}_p) ,
\]
where \( \Lambda(Z_p) = \Lambda_0(Z_p) \oplus \Lambda_1(Z_p) \) is a graded vector space over the ring of \( p \)-adic integers \( Z_p \) and \( S \) is a finite set of primes \( p \). Graded vector space \( \bigoplus \bigcap \) becomes adelic superalgebra by requiring that \( \Lambda(\mathbb{R}), \Lambda(\mathbb{Q}_p), \Lambda(\mathbb{Z}_p) \) are superalgebras. Adelic supercommutator may be regarded as a collection of real and all \( p \)-adic supercommutators. Thus adelic superalgebra \( \bigoplus \bigcap \) is commutative. An example of commutative adelic superalgebra is the following adelic Grassmann algebra:

\[
G(\mathcal{A} : \eta_1, \eta_2, \cdots, \eta_m) = \bigcup_S G_S(\eta_1, \eta_2, \cdots, \eta_m)
\]

\[
G_S(\eta_1, \cdots, \eta_m) = G(\mathbb{R} : \eta_1, \cdots, \eta_m) \times \prod_{p \in S} G(\mathbb{Q}_p : \eta_1, \cdots, \eta_m) \times \prod_{p \not\in S} G(\mathbb{Z}_p : \eta_1, \cdots, \eta_m).
\]

Adelic superspace of dimension \((n, m)\) has the form

\[
\mathcal{A}_{\Lambda(\mathcal{A})}^{n,m} = \bigcup_S \mathcal{A}_{\Lambda(\mathcal{A}),S}^{n,m}, \quad \mathcal{A}_{\Lambda(\mathcal{A}),S}^{n,m} = \mathbb{R}^{n,m} \times \prod_{p \in S} \mathcal{A}^{n,m}(\mathbb{Q}_p) \times \prod_{p \not\in S} \mathcal{A}^{n,m}(\mathbb{Z}_p),
\]

where \( \mathcal{A}^{n,m}(\mathbb{Z}_p) \) is \((n, m)\)-dimensional \( p \)-adic superspace over superalgebra \( \Lambda(\mathbb{Z}_p) \). Closer to supersymmetric models is the superspace \( \mathcal{A}_{\Lambda(\mathcal{A}),S}^{n,m} = \bigcup_S \mathcal{A}_{\Lambda(\mathcal{A}),S}^{n,m} \) where

\[
\mathcal{A}_{\Lambda(\mathcal{A}),S}^{n,m} = (\mathbb{R}^n \times \Lambda^m(\mathbb{R})) \times \prod_{p \in S} (\mathcal{A}^{n,m}(\mathbb{Q}_p) \times \Lambda^m(\mathbb{Q}_p)) \times \prod_{p \not\in S} (\mathcal{A}^{n,m}(\mathbb{Z}_p) \times \Lambda^m(\mathbb{Z}_p)).
\]

Points of adelic superspace \( X \) have the coordinate form \( X = (X^{(\infty)}, X^{(2)}, \cdots, X^{(p)}, \cdots) \), where for all but a finite set of primes \( S \) it has to be \( ||X^{(p)}|| = \max |X^{(p)}|_{p} \leq 1 \). The corresponding adelic valued functions (superfields) must satisfy adelic structure, i.e. \( F(X) = (F_\infty, F_2, \cdots, F_p, \cdots) \) with condition \( |F_p|_p \leq 1 \) for all but a finite set of primes \( S \). In the spirit of this approach one can continue to build adelic superanalysis.

5 Concluding remarks

In this report are presented some elements of \( p \)-adic and adelic generalization of superanalysis over real numbers. We have been restricted to superanalysis over the field of \( p \)-adic numbers \( \mathbb{Q}_p \) and the corresponding ring of adeles \( \mathcal{A} \). It is worth noting that algebraic extensions of \( \mathbb{Q}_p \) give much more possibilities than in the real case, where there is only one extension, i.e. the field of complex numbers \( \mathbb{C} = \mathbb{R}(\sqrt{-1}) \). In fact there is at least one \( \tau \in \mathbb{Q}_p \) such that \( \mathbb{Q}_p(\tau^{1/2}) \neq \mathbb{Q}_p \) for a fixed \( p \) and any integer \( n \geq 2 \). Note that there are three and seven \( p \)-adic distinct quadratic extensions \( \mathbb{Q}_p(\sqrt{7}) \) if \( p \neq 2 \) and \( p = 2 \), respectively. Using these quadratic extensions, supersymmetric quantum mechanics with \( p \)-adic valued functions is constructed by Khrennikov [16]. The above approach to adelic superanalysis may be easily generalized to the case.
when $\mathbb{R} \rightarrow \mathbb{C}$, $\mathbb{Q}_p \rightarrow \mathbb{Q}_p(\sqrt{\tau})$, $\mathbb{Z}_p \rightarrow \mathbb{Z}_p(\sqrt{\tau})$. Algebraically closed and ultrametrically complete analogue of $\mathbb{C}$ is $\mathbb{C}_p$, which is an infinite dimensional vector space. Thus $p$-adic algebraic extensions offer enormously rich and very challenging field of research in analysis as well as in superanalysis.

It is also very desirable to find a formulation of superanalysis which would be a basis for supersymmetric generalization of complex-valued $p$-adic and adelic quantum mechanics [11] as well as of related quantum field theory and Superstring/M-theory [12].

Acknowledgements The work on this paper was supported in part by the Serbian Ministry of Science, Technologies and Development under contract No 1426 and by RFFI grant 02-01-01084.

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