Primal-dual Estimator Learning: an Offline Constrained Moving Horizon Estimation Method with Feasibility and Near-optimality Guarantees

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Abstract—This paper proposes a primal-dual framework to learn a stable estimator for linear constrained estimation problems leveraging the moving horizon approach. To avoid the online computational burden in most existing methods, we learn a parameterized function offline to approximate the primal estimate. Meanwhile, a dual estimator is trained to check the suboptimality of the primal estimator during execution time. Both the primal and dual estimators are learned from data using supervised learning techniques, and the explicit sample size is provided, which enables us to guarantee the quality of each learned estimator in terms of feasibility and optimality. This in turn allows us to bound the probability of the learned estimator being infeasible or suboptimal. Furthermore, we analyze the stability of the resulting estimator with a bounded error in the minimization of the cost function. Since our algorithm does not require the solution of an optimization problem during runtime, state estimates can be generated online almost instantly. Simulation results are presented to show the accuracy and time efficiency of the proposed framework compared to online optimization of moving horizon estimation and Kalman filter. To the best of our knowledge, this is the first learning-based state estimator with feasibility and near-optimality guarantees for linear constrained systems.

I. INTRODUCTION

Estimating the state of a stochastic system is a long-lasting issue in the areas of engineering and science. It draws much attention in different domains such as signal processing, robotics, and econometrics [1]. For linear systems, the Kalman filter gives the optimal estimate when the process and the measurement noise obey Gaussian distributions [2]. However, it is difficult to be applied in one typical case where states or disturbances are subjected to inequality constraints, especially for nonlinear constraints [3]. Considering these constraints is crucial for bounded disturbances modeling, which will greatly facilitate the improvement of state estimation accuracy.

In contrast to the Kalman filter, moving horizon estimation (MHE) offers the possibility of incorporating constraints on the estimated systems [4]–[6]. At each instant, it is required to find a trajectory of state estimates online by solving a finite-horizon constrained optimization problem relying on recent measurements. It is shown that model predictive control (MPC) and MHE share symmetric structures [7]. This means that, similar to MPC, implementing MHE on fast dynamical systems with limited computation capacity remains generally challenging due to the heavy online computational burden. To accelerate the online solving of MHE, a variety of fast optimization techniques have been proposed, including the interior-point nonlinear programming technique [8]–[10] and real-time iteration-based automatic code generation [11].

Compared with online MHE solvers, learning approximating MHE estimation laws offline can significantly improve the online estimation efficiency [12]–[14]. In particular, one can parameterize the MHE estimator using neural networks or a linear combination of basis functions, and then find a parameterized estimator that minimizes the MHE cost using supervised learning techniques. Some studies also apply reinforcement learning or variational inference to obtain an offline estimator [15]–[18]. However, existing offline estimation methods lack the ability to verify the estimation accuracy in real-time during the online application process. Nevertheless, in practice, it is critical to verify a state estimate before it is utilized by a controller to ensure control performance. Besides, these methods also fail to certify the feasibility of the estimation law when considering constrained disturbances.

Inspired by the recently proposed primal-dual MPC framework [19], [20], this paper presents a primal-dual estimator learning method to learn an offline primal estimator with feasibility and stability guarantees, whose online optimality can be quantitatively evaluated in real-time using an offline dual estimator. Specifically, our contributions can be summarized as follows:

1) Given a general constrained MHE problem, we establish the explicit form of its dual problem by introducing a minimum distance Euclidean projection function. Existing forms derived in [21], [22] can be deemed as a special case of our setting, which considers both the discounted factor and disturbance constraints.

2) In the offline phase, we employ a supervised learning scheme to train the primal and dual estimators and evaluate the feasibility and near-optimality of the trained estimator using a randomized verification methodology. Given an admissible probability of feasibility and suboptimality violation, the minimum sample sizes are provided for the verification step. In the online phase, the primal estimator outputs a state estimate. In the meantime, we use the dual estimator to check the near-
optimality of the current estimate using ideas from weak duality theory. If the check fails, we implement a backup estimator (such as an online MHE method) to guarantee the estimation accuracy. Therefore, in contrast to most existing offline methods [12]–[18], our learning scheme guarantees the feasibility and near-optimality of the primal estimator.

3) Finally, we analyze the stability of the learned estimator, which shows that an upper bound of the state estimation error exists for any possible value of the estimator learning error under moderate assumptions.

The remainder of this paper is organized as follows. Section II presents the problem statement of the constrained MHE problem, and Section III derives the explicit form of the dual problem and formulates the primal and dual learning problems. Section IV proposes the algorithm for primal-dual learning to guarantee performance. Section V provides the stability analysis. Finally, we provide numerical results in Section VI and draw conclusions in Section VII.

**Notation:** The Euclidean norm of the vector $x$ is denoted as $\|x\|_2$, and $x^T Ax$ is denoted as $\|x\|_A^2$. A vector $x \geq 0$ means that all the elements are greater than or equal to 0. We use $I_\Omega$ to represent the integer lies in the set $\Omega$. For example, an integer $i \in I_{[a,b]}$ represents $a \leq i \leq b$. $\lambda_{\text{max}}(P_2, P_1)$ is the largest generalized eigenvalue of $P_2$ and $P_1$. $I_{m \times m}$ represents the identity matrix.

**II. PROBLEM FORMULATION**

This section formulates a constrained estimation problem using the moving horizon scheme. We consider the stochastic system with process noise and measurement noise

$$
x_{t+1} = A_t x_t + \xi_t + \zeta_t,
$$

where $x_t \in \mathbb{R}^n$ is the state, $y_t \in \mathbb{R}^m$ is the measurement, $\xi_t$ is the process noise, and $\zeta_t$ is the measurement noise. $\{\xi_t\}$ and $\{\zeta_t\}$ are both i.i.d sequences and independent of the initial state $x_0$. We suppose both the system noise and the measurement noise obey the truncated Gaussian distribution, i.e.,

$$
p(\xi_t) = \begin{cases} \frac{C_\xi}{\sqrt{(2\pi)^m|Q|}} e^{-\frac{1}{2} \xi_t^T Q^{-1} \xi_t} & \xi_t \in \Xi_\xi \\ 0 & \xi_t \notin \Xi_\xi \end{cases}
$$

$$
p(\zeta_t) = \begin{cases} \frac{C_\zeta}{\sqrt{(2\pi)^n|R|}} e^{-\frac{1}{2} \zeta_t^T R^{-1} \zeta_t} & \zeta_t \in \Xi_\zeta \\ 0 & \zeta_t \notin \Xi_\zeta \end{cases}
$$

Note that $C_\xi$ and $C_\zeta$ are the constant factors to normalize the probability density function and $Q$, $R > 0$. The reason behind (2) is that an optimal estimator dealing with inequality constraints can be formulated under the assumption that the probability distributions are truncated Gaussian distributions [23].

A natural choice for the optimal estimate $\hat{x}_t^\star$ is the most probable state $x_t$ given the measurement sequence $y_{1:t-1}$, which is known as the maximum a posteriori Bayesian estimation:

$$
\hat{x}_{1:t}^\star = \arg \max_{x_{1:t}} p(x_{1:t}|y_{1:t-1}).
$$

This problem can be formulated as a quadratic optimization problem when applying the logarithm trick [5]. However, this requires all the historical measurements to obtain the estimate, which is called full information estimation. This formulation is generally computationally intractable. To make the problem tractable, we need to bound the problem size. One strategy is to employ an MHE approximation which uses the most recent measurements to perform the estimation. The constrained MHE problem can be formulated as Problem 1.

At time $t$, MHE considers the past measurements in a window of length $M_t \in \mathbb{N}$ and the past optimal estimate $\hat{x}_{t-M_t}^\star$. Thereby, the MHE optimizes over the initial estimate $\hat{x}_{t-M_t}^\star$ and a sequence of $M_t$ estimates of the process noise $\xi_{j[t]} = \{\xi_{j[t]}\}_j$ and a sequence of $M_t$ estimates of the measurement noise $\hat{\zeta}_{j[t]} = \{\hat{\zeta}_{j[t]}\}_j$. Combined, they define a sequence of $M_t + 1$ state estimates $\hat{x}_{j[t]} = \{\hat{x}_{j[t]}\}_j$ through (4b).

**Problem 1 (Constrained MHE problem).**

$$
\begin{array}{ll}
\min & V_{\text{MHE}}(\hat{x}_{j[t]} , \hat{\zeta}_{j[t]} ) \\
\text{subject to} & \hat{x}_{j+1} = A_t \hat{x}_j + \hat{\xi}_j , \\
& \hat{\xi}_j \in \Xi_\xi , \hat{\zeta}_j \in \Xi_\zeta , j \in I_{[t-M_t,t-1]} \\
\end{array}
$$

where $c = \gamma M_t ||\hat{x}_{t-M_t}^\star - \hat{x}_{t-M_t}||^2_{P_{t-M_t}}$ + $\sum_{i=t-M_t}^{t-1} \gamma^{t-i-1}||\hat{\xi}_i||^2_{Q_{i-1}} + \sum_{i=t-M_t}^{t-1} \gamma^{t-i-1}||\hat{\zeta}_i||^2_{R_{i-1}}$.

**Remark 1.** Generally, it is hard to obtain the analytic form of the arrival cost. Notable exceptions are unconstrained linear problems, where $P_{t-M_t}$ can be obtained by solving the matrix Riccati equation:

$$
P_{t+1} = Q + A_t P_t A_t^T - A_t P_t C_t^T (R + C_t P_t C_t^T)^{-1} C_t^T P_t A_t^T.
$$

This results in a recurrent way to obtain the state estimation, which is equivalent to the well-known Kalman filter [2], [5], [7] when horizon length $M_t = 1$.

**Remark 2.** In this paper, we consider the prediction form of the estimation problem to simplify the notation. However, all the results can be directly extended to the filtering form.

**Assumption 1.** Both $\Xi_\xi$ and $\Xi_\zeta$ are convex sets.

**Proposition 1.** Problem 1 is a convex optimization problem.

**Proof.** The objective function is quadratic and thus convex.

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1) This choice is typically called filtering prior.

2) From the definition, $\hat{x}_{1:t}^\star = \hat{x}_t^\star$. 

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The feasible region is also convex because the affine function can preserve convexity.

Assumption 2. We assume that Problem \(\hat{\text{P}}\) is well-posed, i.e., a solution exists to Problem \(\hat{\text{P}}\) for \(t \in [0, \infty)\). The sufficient conditions for the existence of solutions are well studied in [5].

III. DUAL PROBLEM AND ESTIMATOR APPROXIMATION

In this section, we first review some important conclusions about duality theory [25] and then derive the dual problem of Problem \([\text{P}]\). Finally, the supervised learning scheme is used to approximate the primal and dual estimators.

A. Duality theory

Duality is often used in optimization to certify optimality of a given solution. Consider the primal optimization problem:

\[
\begin{align*}
\mathcal{P} & : \ p^* = \min \ F_0(x) \\
& \text{subject to } F_i(x) \leq 0, \ i \in \mathbb{I}_{[0,p)} \\
& H_i(x) = 0, \ i \in \mathbb{I}_{[0,q]}.
\end{align*}
\]

The Lagrange function is defined as

\[
L(x, v, \mu) = F_0(x) + \sum_{i=0}^p v_i F_i(x) + \sum_{i=q}^q \mu_i H_i(x).
\]

Then, the corresponding Lagrange dual function is given by

\[
g(v, \mu) = \inf_x L(x, v, \mu).
\]

The Lagrange dual function gives us a lower bound on the optimal value \(p^*\) of the primal problem \(\hat{\text{P}}\). The calculation of the best lower bound leads to the Lagrange dual problem:

\[
\mathcal{D} : d^* = \max g(v, \mu) \\
\text{subject to } v \geq 0.
\]

The Lagrange dual problem \(\mathcal{D}\) is a convex optimization problem regardless of whether the primal problem \(\mathcal{P}\) is convex. It is well-known that \(d^* \leq p^*\) always holds thanks to the weak duality theory and we refer to the difference \(p^* - d^*\) as the duality gap.

B. Dual problem of Constrained MHE

From (9), we establish the explicit form of the dual problem of Problem \([\text{P}]\) which is given in Problem \([\text{P}^*]\). We defer detailed derivations to Appendix A.

Problem 2 (Duality of Problem \([\text{P}]\). The dual problem of Problem \([\text{P}]\) is

\[
\begin{align*}
\max_\mu & \quad G(\lambda, \mu) \\
\text{subject to} & \quad \begin{align*}
\lambda_{i-1} - A_i^T \lambda_i - C_i^T \mu_i &= 0 \\
\lambda_{i-1} &= 0, \ i \in \mathbb{I}_{[-M, -1, t-1]},
\end{align*}
\end{align*}
\]

where \(\lambda_i\) and \(\mu_i\) are Lagrange multipliers. Besides, \(G(\lambda, \mu)\) is defined as

\[
G(\lambda, \mu) := -\frac{1}{4} \lambda_i^T \left( A_{i-M}^T \lambda_i - M_t \right) + C_{i-M}^T \mu_i \left( P_{i-M}^T \right)^2 + \sum_{i=t}^{t-1} \gamma^{i-1} \left( (i + 1)^{1/2} \lambda_i \right)^2 + \sum_{i=t}^{t-1} \gamma^{i-1} \left( (i + 1)^{1/2} \mu_i \right)^2
\]

and \(\Pi_{\hat{\Xi}}(\cdot)\) and \(\Pi_{\hat{\Xi}}(\cdot)\) are denoted as the minimum distance Euclidean projection onto the sets, i.e.,

\[
\Pi_{\hat{\Xi}}(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n, \ \Pi_{\hat{\Xi}}(x) := \arg \min_{x \in \hat{\Xi}} \left\{ \| x - \frac{1}{2} \|_2^2 \right\}.
\]

C. Primal and Dual Learning Problems

Given the formulation of Problems \([\text{P}]\) and \([\text{P}^*]\) we are now ready to train primal and dual estimators using supervised learning tools. We define all the information used to train the estimators as

\[
\hat{\mathcal{I}}_t := \{ y_{t-M_t}, \dot{y}_{t-M_t}, \dot{y}_{t-M_t} \}.
\]

We observe that both the optimal estimator \(\hat{\dot{X}}(\hat{\mathcal{I}})\) (the optimal solution of Problem \([\text{P}]\) i.e., \(\hat{\dot{X}}(\hat{\mathcal{I}}) = \{ \hat{\dot{x}}_{t-M_t}^{\text{E}}(\hat{\mathcal{I}}) \} \)) and the optimal dual estimator \(\Lambda_t(\hat{\mathcal{I}})\) (the optimal solution of Problem \([\text{P}^*]\) i.e., \(\Lambda_t(\hat{\mathcal{I}}) = \{ \mu^* \} \)) are determined by \(\hat{\mathcal{I}}_t\).

Suppose the primal and dual estimators are parameterized by approximate functions \(\hat{X}(\mathcal{I} ; \eta)\) and \(\Lambda(\mathcal{I} ; \omega)\) respectively, where \(\eta\) and \(\omega\) are function parameters. Then the primal learning problem is given by

\[
\eta^* = \arg \min_{\eta} \sum_{i=1}^{N} \mathcal{L} \left( \hat{X}(\mathcal{I}_i ; \eta), \hat{X}(\mathcal{I}_i) \right).
\]

Similarly, the parameters of the dual estimator can be optimized by

\[
\omega^* = \arg \min_{\omega} \sum_{i=1}^{N} \mathcal{L} \left( \Lambda(\mathcal{I}_i ; \omega), \Lambda(\mathcal{I}_i) \right).
\]
Note that \( \{ T^i, \hat{X}^i(T^i), \Lambda^i(T^i) \} \) represents the \( i \)-th sample, \( N \) denotes the sample size and \( L \) is the loss function which can be chosen as different formulations such as the \( L_2 \) loss function.

IV. PRIMAL-DUAL ESTIMATOR LEARNING

In this section, we show how the parameterized estimators solved by (15) and (16) can be used to efficiently ensure the feasibility and near-optimality of the estimator during runtime, inspired by [19], [20].

A. Offline Training Performance Guarantees

Given (15) and (16), one natural question is how to verify the feasibility and near-optimality of the learned parameterized estimators. To answer this question, we first review a useful lemma in the field of statistical learning theory.

**Lemma 1** (Smallest Sample Size for Reliable Performance) Suppose \( q \) is a random vector. Let \( \{ q^1, q^2, \ldots, q^N \} \) represents \( N \) i.i.d. samples. Then \( \hat{u}_N = \max_{i=1,2,\ldots,N} u(q^i) \) represents the estimate of the worst-case performance function \( u_{\max} := \max_{q \in Q} u(q) \), where \( Q \) denotes the sample space. The smallest sample size that guarantees

\[
\text{Prob}\{u(q) > \hat{u}_N\} \leq \epsilon
\]

with confidence at least \( 1 - \beta \) is given by

\[
N \geq \frac{\ln \frac{\beta}{\epsilon}}{\ln \frac{1}{1-\epsilon}}. \tag{18}
\]

This lemma provides a powerful tool to test the performance of the primal and dual estimators using the collected finite samples. Specifically, given a desired maximum sub-optimality level, we can use this lemma to verify that the approximated estimator satisfies this suboptimality level with high probability. This comes with the following Theorem [1]

**Theorem 1** (Offline Training With Performance Guarantee). Suppose we have \( N_p \) samples \( \{ T^i, \hat{X}^i(T^i) \} \) for primal estimator learning and \( N_d \) samples \( \{ T^i, \Lambda^i(T^i) \} \) for dual estimator learning, where \( N_p \geq \frac{\ln \frac{\beta}{\epsilon_p}}{\ln \frac{1}{1-\epsilon_p}} \) and \( N_d \geq \frac{\ln \frac{\beta}{\epsilon_d}}{\ln \frac{1}{1-\epsilon_d}} \).

Let \( \epsilon_p, \epsilon_d \in [0, 1) \) be admissible primal and dual violation probabilities, and let \( 0 < \beta_p, \beta_d \ll 1 \) be desired confidence levels. The desired suboptimality level of the learned primal and dual estimators are denoted as \( \Delta_p \) and \( \Delta_d \), respectively. If

\[
V_{\text{MHE}}(\hat{X}(T^i; \eta^*)) \leq V_{\text{MHE}}(\hat{X}^i(T^i)) + \Delta_p, \tag{19}
\]

\( \hat{X}(T^i; \eta^*) \) satisfies (4c), \( i \in [1, N_p] \)

holds, then with confidence at least \( 1 - \beta_p \) the following inequality holds

\[
\text{Prob}\{ V_{\text{MHE}}(\hat{X}(T^i; \eta^*)) \leq V_{\text{MHE}}(\hat{X}^i(T^i)) + \Delta_p, \quad \hat{X}(T^i; \eta^*) \text{ satisfies (4c)} \} \geq 1 - \epsilon_p. \tag{20}
\]

Similarly, if

\[
G(\Lambda(T^i; \omega^*)) \geq G(\Lambda^i(T^i)) - \Delta_d, \quad i \in [1, N_d] \tag{21}
\]

holds, then with confidence at least \( 1 - \beta_d \) the following inequality holds

\[
\text{Prob}\{ G(\Lambda(T^i; \omega^*)) \geq G(\Lambda^i(T^i)) - \Delta_d \} \geq 1 - \epsilon_d. \tag{22}
\]

See Appendix B for detailed proofs. Although we choose the confidence levels \( \beta_p \) and \( \beta_d \) as a small number (< \( 10^{-6} \)), the minimum sample size would not explode due to the logarithm operator in (18). Generally, given the required \( \epsilon_p, \epsilon_d, \Delta_p/d, \Delta_d/p \), Theorem 1 provides an effective way to determine whether the learned estimator needs to be retrained.

B. Online Application Performance Guarantees

Although we have established probabilistic guarantees for the near-optimality of the learned estimator, there still remain some extreme cases where we may get a poor state estimate. To avoid such cases, we use the weak duality property to examine the learned estimator in real-time.

**Theorem 2** (Online Application With Performance Guarantee). Assume \( \hat{X}(T^i; \eta^*) \) satisfies (4c), then

\[
V_{\text{MHE}}(\hat{X}(T^i; \eta^*)) - V_{\text{MHE}}(\hat{X}^i(T^i)) \leq V_{\text{MHE}}(\hat{X}(T^i; \eta^*)) - G(\Lambda(T^i; \omega^*)) \tag{23}
\]

Proof. This can be easily verified by weak duality \( G(\Lambda(T^i; \omega^*)) \leq G(\Lambda^i(T^i)) \leq V_{\text{MHE}}(\hat{X}^i(T^i)) \).

We use Theorem 2 in our framework as follows: Let \( \Delta \) be the desired maximum suboptimality level. During the online application process, for a given parameter \( \eta^* \), if the right hand side of (23) is smaller than \( \Delta \), the performance gap between the learned primal estimator \( \hat{X}(T^i; \eta^*) \) and the optimal estimator \( \hat{X}^i(T^i) \) can be bounded by \( \Delta \). So we call this property as “\( \Delta \)-suboptimality”. If the learned primal estimator \( \hat{X}(T^i; \eta^*) \) is guaranteed to be at most \( \Delta \)-suboptimal, its output would be applied in real-time. However, if the right hand side of (23) is larger than the predetermined suboptimality level \( \Delta \), then a backup estimator (such as an online MHE method) will be used to provide the state estimate at this instant. The following corollary bounds the failure probability of the online application.

**Corollary 1** (Violation Probability). Suppose \( \Delta := \Delta_p + \Delta_d + \Delta_{\text{gap}} \), where \( \Delta_{\text{gap}} \) represents the maximum duality gap, i.e., \( \Delta_{\text{gap}} = \max \{ V_{\text{MHE}}(\hat{X}^i(T^i)) - G(\Lambda^i(T^i)) \} \). Under the assumptions in Theorem 1 if (19) and (21) hold, then

\[
\text{Prob}\{ V_{\text{MHE}}(\hat{X}(T^i; \eta^*)) - G(\Lambda(T^i; \omega^*)) \leq \Delta, \quad \hat{X}(T^i; \eta^*) \text{ satisfies (4c)} \} \geq 1 - (\epsilon_p + \epsilon_d)
\]

holds with confidence at least \( 1 - (\beta_p + \beta_d) \). In most cases, Problems (1) and (2) satisfy the strong duality as long as the Slater condition holds [25], which leads to \( \Delta_{\text{gap}} = 0 \).

Proof. Using the union probability inequality \( \text{Prob}\{ A \cup B \} \leq \text{Prob}\{ A \} + \text{Prob}\{ B \} \) and the results in Theorem 1, we can easily end this proof. \( \square \)
Remark 3. Theorem 2 already provides a “hard” certificate to judge the performance $V_{MHE}(X; \eta^*)$ of the learned primal estimator. Besides, we can bring some ideas from the control theory, such as the safety shield [27], to ensure online feasibility.

C. Primal-dual MHE

Based on the above theoretical analysis, our offline MHE method building on the primal-dual estimator learning framework is shown in Algorithm 1. We refer to this method as primal-dual MHE (PD-MHE).

Algorithm 1 Primal-dual MHE

Input: confidence level $0 < \beta < 1$, violation probability $\epsilon > 0$, and suboptimality level $\Delta > 0$

Select: $\beta_p$, $\beta_d$, $\epsilon_p$, $\epsilon_d$, such that $\beta_p + \beta_d = \beta$, $\epsilon_p + \epsilon_d = \epsilon$

Offline Training

1. Learn primal estimator $\hat{X}(I; \eta^*)$ as in (15)
2. Learn dual estimator $\Lambda(I; \omega^*)$ as in (16)
3. Validate $\hat{X}(I; \eta^*)$ and $\Lambda(I; \omega^*)$ using Theorem 1
4. if (19) and (21) are satisfied and $\Delta_d + \Delta_p \leq \Delta$ then
5. End training
6. else
7. Repeat

Online Application (for $t = M_t, M_t + 1, M_t + 2, ...$)

1. Obtain $I_t$
2. if $V_{MHE}(\hat{X}(I_t; \eta^*)) - G(\Lambda(I_t; \omega^*)) \leq \Delta$; $\hat{X}(I_t; \eta^*)$ satisfies (4) then
3. Apply $X(I_t; \eta^*)$ to obtain the estimate
4. else
5. Apply a backup estimator to obtain the estimate

V. Stability Analysis

In this section, we prove the stability of the offline primal estimator solved by (15). Our analysis follows similar ideas in [28], with suitable extensions to account for the “$\Delta$-suboptimality” of the learned estimator. We begin with some useful definitions and lemmas.

Definition 1 (Exponential $\delta$-IOSS [28]). The system has an exponential incremental input/output-to-state stability (IOSS) property if there exists a quadratic $\delta$-IOSS Lyapunov function $W_\delta$ and $P_1, P_2, P, R > 0$ such that

$$\|x - \hat{x}\|_{P_1}^2 \leq W_\delta(x, \hat{x}) \leq \|x - \hat{x}\|_{P_2}^2,$$  (24a)

$$W_\delta(x^+, \hat{x}^+) \leq \gamma W_\delta(x, \hat{x}) + \|\xi - \hat{\xi}\|_Q^2 + \|y - \hat{y}\|_R^2.$$  (24b)

Here $x^+(\hat{x}^+)$ represents the next state of $x(\hat{x})$, $\xi(\hat{\xi})$ represents the process noise, and $y(\hat{y})$ represents the measurement.

Lemma 2 (Quadratic $\delta$-IOSS Lyapunov function [28]). The system (1) admits a Quadratic $\delta$-IOSS Lyapunov function if there exists $\bar{\gamma} \in [0, 1)$ and symmetric matrices $P, Q, R > 0$ such that

$$\begin{bmatrix}
A^T \bar{P} A - \gamma \bar{P} - C^T \bar{R} C & A^T \bar{P} B - C^T \bar{R} D \\
B^T \bar{P} A - D^T \bar{R} C & B^T \bar{P} B - Q_1 - D^T \bar{R} D
\end{bmatrix} \preceq 0$$  (25)

holds, then $W_\delta(x, \hat{x})$ is a $\delta$-IOSS Lyapunov function that satisfies $\bar{P}_1 = \bar{P}_2 = \bar{P}$ in (24a). Here, $B = [I_{m \times n} \ 0_{n \times m}]$, $D = [0_{m \times n} \ I_{m \times m}]$, and $Q_1 = \begin{bmatrix} \bar{Q} & 0_{n \times m} \\ 0_{m \times n} & 0_{m \times m} \end{bmatrix}$.

Before proposing the main theorem, we define the maximum of the largest generalized eigenvalue $\lambda_{\text{max}}$ as

$$\lambda_{\text{max}} := \max_{i \in [1, M_t]} \lambda_{\text{max}} \left\{ \lambda_{\text{max}} (\bar{P}_t^{-1}, \bar{P}_{t-1}^{-1}) \right\},$$

and

$$\lambda_{\text{max}} := \max_{i \in [0, M_t] - \{1\}} \lambda_{\text{max}} \left\{ \lambda_{\text{max}} (\bar{P}_t^{-1}, \bar{P}_{t-1}^{-1}) \right\}.$$  (26)

Theorem 3 (Stability of the Primal Estimator). The proposed estimator with “$\Delta$-suboptimality” is a stable estimator if

$$M_t > \frac{1}{4 \lambda_{\text{max}} \ln \gamma} + 1$$

and

$$\begin{bmatrix}
M_{11} & M_{12} \\
M_{12}^T & M_{22}
\end{bmatrix} \preceq 0$$

$$M_{11} = A^T \bar{P}_{t-1}^{-1} A - \gamma \bar{P}_{t-1}^{-1} - 2C^T R^{-1} C$$

$$M_{12} = A^T \bar{P}_{t-1}^{-1} B - 2C^T R^{-1} D$$

$$M_{22} = B^T \bar{P}_{t-1}^{-1} B - \bar{Q} R^{-1} - 2D^T R^{-1} D, t \in [M_t, \infty).$$

In particular, the state estimation error is bounded above by

$$\|\hat{x}_t^\Delta - x_t\|_{P_{t-1}^{-1}} \leq 2\sqrt{\rho^t} \|\bar{x}_0 - x_0\|_{P_0^{-1}} + \sqrt{\frac{2\Delta}{1 - \rho^t}},$$

$$+ 2\sqrt{\frac{1}{1 - \rho^t}} \max_{i \in [0, t - 1]} \left\{ \|\xi_{t-i-1}\|_Q \right\},$$

(28)

where $\hat{x}_t^\Delta$ denotes the estimate obtained by a $\Delta$-suboptimality estimator and $\rho = \langle 4 \lambda_{\text{max}} \rangle \bar{P}_t \gamma < 1$. Besides, $\bar{Q} = \begin{bmatrix} Q^{-1} & 0_{n \times m} \\ 0_{m \times n} & 0_{m \times m} \end{bmatrix}$.

This theorem shows that for the appropriate horizon length, the error sequence of the estimate generated by Algorithm 1 can be bounded by a function of the initial estimation error, the maximum norm of the process noise with time discounted, and the desired suboptimality level $\Delta$. Although this result does not restrict the range of $\Delta$, for a large $\Delta$, such an upper bound becomes meaningless.

Remark 4. Compared to the original definition of robustly globally exponentially stable (RGES) given in [24], [28], the derived error bound in (28) includes an additional term to reveal the effect of estimator suboptimality.

VI. Numerical Results

In this section, we use a simple example to illustrate the performance of the proposed algorithm. We consider the following stochastic system

$$x_{t+1} = Ax_t + \xi_t$$

$$y_t = Cx_t + \zeta_t,$$

where $x_t = [x_t^{(1)}, x_t^{(2)}]^T \in \mathbb{R}^2$, $y_t \in \mathbb{R}$,

$$A = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}, \ C = [1 \ 0].$$

(29)
\( \xi_t \sim \mathcal{N}(0, Q) \land \xi_t \geq 0 \) and \( \zeta_t \sim \mathcal{N}(0, R) \land \zeta_t \leq 0 \) are the noise satisfying truncated Gaussian distributions. The covariance matrix of the noise is set to
\[
Q = \begin{bmatrix} 0.1^2 & 0 \\ 0 & 0.12 \end{bmatrix}, \quad R = 1. \tag{31}
\]
The projection functions defined in [13] can be analytically expressed as
\[
\Pi_{\xi_t}(z) = \begin{cases} 0, & z^{(1)} \geq 0, \ z^{(2)} \geq 0 \\ -z^{(1)}/2, & z^{(1)} \leq 0, \ z^{(2)} \geq 0 \\ -z^{(2)}/2, & z^{(1)} \geq 0, \ z^{(2)} \leq 0 \\ \|z/2\|_2, & z^{(1)} \leq 0, \ z^{(2)} \leq 0. \end{cases}
\]
and
\[
\Pi_{\zeta_t}(z) = \begin{cases} 0, & z \leq 0 \\ z/2, & z \geq 0. \end{cases} \tag{32}
\]
For the sake of comparison, we consider the performance indices given by the root mean square error (RMSE) and asymptotic root mean square error (ARMSE) as in [13], [14]. To demonstrate the performance of Algorithm 1 we take the Kalman filter and online MHE as baselines. We illustrate our proposed PD-MHE using a Deep Neural Network function approximator (3 hidden layers and the number of neurons are \([512, 512, 512] \)) with Rectified Linear Unit. To solve online MHE, we use CasADI [29], the state-of-the-art optimization problem solver. We set \( M_t = 10 \) and performed 200 Monte-Carlo experiments for each method, and the results are given in Fig. 1 and Table I. The simulations are performed on a computer equipped with Intel i9-7980 XE processor and NVIDIA Titan XP GPU.

![Figure 1: Simulation results of the PD-MHE, KF, and online MHE. The solid and dotted lines correspond to the means and the shaded regions correspond to 95% confidence intervals over 200 runs.](image)

**TABLE I: Comparision of ARMSE and one-step computation time**

| Algorithm | ARMSE  | Computation Time (ms) |
|-----------|--------|-----------------------|
| PD-MHE    | 0.9885 | 2.04                  |
| KF        | 1.9989 | 0.067                 |
| MHE       | 0.9871 | 17.88                 |

From the results, we can see that the constraints defined on the process and measurement noise bring asymmetry to the probability density function, so that KF has already diverged in this constrained estimation problem. Compared to KF, online MHE and PD-MHE capture the information given by the constraints, leading to better estimation accuracy. As for runtime, our algorithm allows us to obtain estimates significantly faster than online MHE, with an average speed-up of over 8x compared to CasaDI. In summary, the proposed algorithm succeeds in learning a stable estimator for linear constrained systems, with negligible performance loss with respect to the online MHE.

**VII. Conclusions**

In this paper, we proposed a new method, called primal-dual estimator learning, for approximating the explicit moving horizon estimation for linear constrained systems. We approximated the moving horizon estimation directly using supervised learning techniques, and invoked two verification schemes to ensure the performance of the approximated estimator. Since the proposed verification scheme only requires the evaluation of primal and dual estimators, our algorithm is computationally efficient, and can be implemented even on resource-constrained systems. The future work will consider the iterative offline learning process and the influence of the capacity of the approximation function.

**APPENDIX**

**A. Derivation of Problem 2**

Considering the primal constrained moving horizon estimation problem [1] its Lagrange function is defined as

\[
L(\hat{x}_{t|t}, \hat{\xi}_{t|t}, \hat{\zeta}_{t|t}, \lambda, \mu) = \gamma^M_t \| \hat{x}_{t-M_t|t} - \hat{x}^*_{t-M_t} \|_{P^{-1}_{t-M_t}}^2 + \sum_{i=t-M_t}^{t-1} \gamma^{t-i-1} \| \hat{\xi}_{i|t} \|_{Q^{-1}}^2 + \sum_{i=t-M_t}^{t-1} \gamma^{t-i-1} \| \hat{\zeta}_{i|t} \|_{R^{-1}}^2 \\
+ \sum_{i=t-M_t}^{t-1} \lambda_t^T (\hat{x}_{i+1|t} - A_i \hat{x}_{i|t} - \hat{\xi}_{i|t}) \\
+ \sum_{i=t-M_t}^{t-1} \mu_t^T (y_i - C_i \hat{x}_{i|t} - \hat{\zeta}_{i|t}), 
\]

where \( \lambda_t \) and \( \mu_t \) are Lagrange multipliers. The Lagrange dual function is given by
\[
g(\lambda, \mu) = \inf_{\hat{x}_{i|t}, \hat{\xi}_{i|t}, \hat{\zeta}_{i|t}} L(\hat{x}_{i|t}, \hat{\xi}_{i|t}, \hat{\zeta}_{i|t}, \lambda, \mu). \tag{34}
\]
Because the Lagrange function is convex with respect to \( \hat{x}_{i|t}, \hat{\xi}_{i|t}, \) and \( \hat{\zeta}_{i|t} \), optimal variables \( \hat{x}^*_{i|t}, \hat{\xi}^*_{i|t}, \hat{\zeta}^*_{i|t} \) can be calculated by the necessary condition:

\[
\frac{\partial L(\cdot)}{\hat{x}^*_{i|t} - M_t} = 2 \gamma^M_t P_{t-M_t}^{-1} \left( \hat{x}^*_{t-M_t|t} - \hat{x}^*_{t-M_t} \right) \\
+ A^T_{t-M_t} \lambda_{t-M_t} - C^T_{t-M_t} \mu_{t-M_t} = 0 \tag{35a}
\]

\[
\frac{\partial L(\cdot)}{\hat{\xi}^*_{i|t}} = \lambda_{i-1} - A^T_{i-1} \lambda_t - C^T_{i-1} \mu_t = 0, \quad i \in \{t-M_t, t-1\} \tag{35b}
\]
\[
\frac{\partial L(\cdot)}{\partial t_i(t)} = \lambda_{t-1} = 0
\]  
(35c)

\[
\hat{c}_{i|t} = \arg \min_{\xi_{i|t}} \left\{ \gamma^{t-i-1} \| \hat{\xi}_{i|t} \|^2_{Q^{-1}} - \lambda_t \gamma^{t-i} \right\}
\]
\[
= \arg \min_{\xi_{i|t}} \left\{ \| \hat{\xi}_{i|t} \|^2_{Q^{-1}} - \gamma^{t-i+1} \lambda_t \right\}, \quad i \in \{ t-M_t, t-1 \}
\]  
(35d)

\[
\hat{c}_{i|t} = \arg \min_{\xi_{i|t}} \left\{ \| \hat{\xi}_{i|t} \|^2_{R^{-1}} - \gamma^{t-i+1} \mu_t \right\}, \quad i \in \{ t-M_t, t-1 \}.
\]  
(35e)

First, we observe that
\[
\hat{\alpha}_{t-M_t|t} = \frac{1}{2\gamma_t M_t} \hat{P}_{t-M_t} (A^T_{t-M_t} \lambda_{t-M_t} + C^T_{t-M_t} \mu_{t-M_t}) \hat{\alpha}_{t-M_t},
\]  
(36)

Similar to the method proposed in [21], [22], [30], we express the (35d) and (35e) in the form of projection function. We denote \( \hat{c}_{i|t} = Q^{-1/2} \hat{c}_{i|t} \) and \( \hat{c}_{i|t} = R^{-1/2} \hat{c}_{i|t} \). Then (35d) and (35e) can be rewritten as
\[
\hat{c}_{i|t} = \arg \min_{\xi_{i|t} \in \Sigma} \left\{ \| \hat{\xi}_{i|t} \|^2_{2} - \gamma^{t-i+1} \lambda_t \right\},
\]
\[
\hat{c}_{i|t} = \arg \min_{\xi_{i|t} \in \Sigma} \left\{ \| \hat{\xi}_{i|t} \|^2_{2} - \gamma^{t-i+1} \mu_t \right\},
\]  
(37)

The solution can be formulated as
\[
\hat{c}_{i|t} = Q^{-1/2} \Pi_{\Sigma} \left( \gamma^{t-i+1} Q^{-1/2} \lambda_t \right), \quad i \in \{ t-M_t, t-1 \}
\]
\[
\hat{c}_{i|t} = R^{-1/2} \Pi_{\Sigma} \left( \gamma^{t-i+1} R^{-1/2} \mu_t \right), \quad i \in \{ t-M_t, t-1 \},
\]  
(38)

where \( \tilde{\Sigma} \) and \( \tilde{\Xi} \) are defined by (12). At the meantime, \( \Pi_{\Sigma} \) and \( \Pi_{\Xi} \) are defined by (13). Plugging (36) and (38) into (34), we have (11).

**B. Proof of Theorem 7**

**Proof.** We observe that the constraints defined by (45c) can always be written as several inequalities according to the convex property by Assumption 1 and the affine function defined by (46b). We denote them as \( C_j (X) \leq 0, \quad i \in I_{[0, e]} \). Then for a given \( X(\cdot) \), consider the following function:
\[
u (I) := \max \left\{ \max_j C_j (X (I)) \right\},
\]
\[
V_{\text{MHE}} (X (I)) - V_{\text{MHE}} (\hat{X} (I)) - \Delta_p.
\]  
(39)

Define \( \hat{u}_N := \max_{i=1, \ldots, N} u (I_i) \), where \( \{ I_i \} \) are independent samples. Following the result in Lemma 1 and set \( u_N = 0 \), we can derive the results of the primal learning part in Theorem 1. The proof of the dual learning part can be derived in a similar way. □

**C. Proof of Theorem 3**

**Proof.** Based on Lemma 2 and Definition 1, we observe that \( ||\dot{x}^\Delta - x||^2_{2P_{t-M_t}^{-1}} \) is a \( \delta - JOSS \) Lyapunov function which satisfies
\[
||\dot{x}^\Delta - x||^2_{2P_{t-M_t}^{-1}} \leq \gamma||\dot{x}^\Delta_{t-1}||^2_{2P_{t-M_t}^{-1}} + ||\dot{x}^\Delta_{t-1} - x_{t-1}||^2_{2P_{t-M_t}^{-1}} + ||\dot{x}^\Delta_{t-1} ||^2_{2Q_{t-M_t}} + ||\dot{x}^\Delta_{t-1} ||^2_{2R_{t-M_t}}.
\]  
(40)

By applying (30) \( M_t \) times, we obtain
\[
||\dot{x}^\Delta_{t} - x||^2_{2P_{t-M_t}^{-1}} 
\]
\[
\leq \gamma^M_t ||\dot{x}^\Delta_{t-M_t} - x_{t-M_t}||^2_{2P_{t-M_t}^{-1}}
\]
\[
+ \sum_{i=1}^{M_t} \gamma^{i-1} ||\dot{x}^\Delta_{t-i} - x_{t-i}||^2_{2Q_{t-M_t}} + ||\dot{x}^\Delta_{t-i} ||^2_{2R_{t-M_t}}.
\]  
(41)

According to the property of “\( \Delta \)-suboptimality”,
\[
V_{\text{MHE}} (\dot{x}^\Delta_{t-M_t}, \dot{x}^\Delta_{t-1}, \dot{x}^\Delta_{t-2}, \ldots) \leq V_{\text{MHE}} (\dot{x}^\Delta_{t-M}, \dot{x}^\Delta_{t-1}, \dot{x}^\Delta_{t-2}, \ldots) + \Delta.
\]

Upon the fact that the true underlying system is a feasible solution, \( V_{\text{MHE}} (x, \xi, \zeta) \) is a trivial upper bound of \( V_{\text{MHE}} (\dot{x}^\Delta_{t-M_t}, \dot{x}^\Delta_{t-1}, \dot{x}^\Delta_{t-2}, \ldots) \):
\[
||\dot{x}^\Delta_{t} - x||^2_{2P_{t-M_t}^{-1}} 
\]
\[
\leq V_{\text{MHE}} (x, \xi, \zeta) + \sum_{i=1}^{M_t} \gamma^{i-1} ||\xi_{t-i}||^2_{Q_{t-M_t}}
\]
\[
+ \gamma^M_t ||\dot{x}^\Delta_{t-M_t} - x_{t-M_t}||^2_{2P_{t-M_t}^{-1}} + \Delta
\]
\[
= 2 \sum_{i=1}^{M_t} \gamma^{i-1} ||\xi_{t-i}||^2_{Q_{t-M_t}} + 2\gamma^M_t ||\dot{x}^\Delta_{t-M_t} - x_{t-M_t}||^2_{2P_{t-M_t}^{-1}} + \Delta
\]
\[
\leq 2\gamma^{M_t} \lambda_{\text{max}} (P_{t-M_t}^{-1}, P_{t-2M_t}^{-1}) ||\dot{x}^\Delta_{t-M_t} - x_{t-M_t}||^2_{2P_{t-2M_t}^{-1}}
\]
\[
+ 2 \sum_{i=1}^{M_t} \gamma^{i-1} ||\xi_{t-i}||^2_{Q_{t-M_t}} + \Delta.
\]  
(42)

By assumption, we define \( \rho_{M_t} := 4\lambda_{\text{max}} \gamma^M_t < 1 \). Consider \( t = kM_t + l, \quad k \in [0, \infty), \quad l \in [0, M_t - 1] \). Similar to (42), we can obtain
\[
||\dot{x}^\Delta_{t} - x||^2_{2P_{t-M_t}^{-1}} 
\]
\[
\leq 2 \sum_{i=1}^{l} \gamma^{i-1} ||\xi_{t-kM_t-i}||^2_{Q_{t-M_t}} + 2\gamma^l ||\dot{x}_{t-kM_t} - x_0||^2_{2P_{t-M_t}^{-1}} + \Delta.
\]  
(43)

\(^4\)In appendix \(^4\) we use the notation \( \Delta \) to represent the \( \Delta \)-suboptimality.
By applying (42) $k$ times, we arrive at
\[
\|\hat{x}_t - x_t\|^2_2 P_{t-M_t}^{-1} \\
\leq \rho^{M_t} \|\hat{x}_t - x_t\|^2_2 P_{t-M_t}^{-1} \\
+ 4 \sum_{i=0}^{k-1} \rho_i M_t \sum_{j=1}^{M_t} \|\xi_{t-M_t-j} - i\|^2_2 Q_{t-j}^{-1} + \Delta \sum_{i=0}^{k-1} \rho_i M_t.
\]
(44)

Plugging (43) into (45), we have
\[
\|\hat{x}_t - x_t\|^2_2 P_{t-M_t}^{-1} \\
\leq \rho^{M_t} \left(4 \sum_{i=1}^{M_t} \|\xi_{t-M_t-i} - i\|^2_2 Q_{t-i}^{-1} + 4 \rho_{M_t} \hat{x}_0 - x_0\|^2_2 P_{t-M_t}^{-1} + \Delta \sum_{i=0}^{k-1} \rho_i M_t \right) \\
+ 4 \sum_{i=0}^{k-1} \rho_i M_t \sum_{j=1}^{M_t} \|\xi_{t-M_t-j} - i\|^2_2 Q_{t-j}^{-1} + \Delta \sum_{i=0}^{k-1} \rho_i M_t.
\]
(45)

By assumption, we have $\rho \geq \eta$, thus
\[
\|\hat{x}_t - x_t\|^2_2 P_{t-M_t}^{-1} \\
\leq 4 \sum_{i=1}^{M_t} \|\xi_{t-M_t-i} - i\|^2_2 Q_{t-i}^{-1} + 4 \rho_{M_t} \hat{x}_0 - x_0\|^2_2 P_{t-M_t}^{-1} \\
+ 2 \sum_{i=0}^{k-1} \rho_{i M_t} \sum_{j=1}^{M_t} \|\xi_{t-M_t-j} - i\|^2_2 Q_{t-j}^{-1} + \Delta \sum_{i=0}^{k-1} \rho_{i M_t} \\
\leq 2 \rho_{M_t} \|\hat{x}_0 - x_0\|^2_2 P_{t-M_t}^{-1} + 2 \sum_{i=0}^{k-1} \rho_{i M_t} \sum_{j=1}^{M_t} \sqrt{i} \max \{i, j-i\} \|\xi_{t-j} - i\|^2_2 Q_{t-j}^{-1} \\
+ \frac{\Delta}{1 - \rho_{M_t}}
\]
(46)

Based on the fact that $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$, and
\[
\sum_{i=0}^{t-1} \sqrt{i} \leq \frac{1}{1 - \rho^2}
\]
(47)

we can easily get (28).

References

[1] H. Musoff and P. Zarchan, Fundamentals of Kalman filtering: a practical approach. American Institute of Aeronautics and Astronautics, 2009.

[2] R. E. Kalman, “A new approach to linear filtering and prediction problems,” Journal of Basic Engineering, vol. 82D, pp. 35–45, 1960.

[3] N. Amor, G. Rasool, and N. C. Bouaynaya, “Constrained state estimation—a review,” arXiv preprint arXiv:1807.03463, 2018.

[4] F. Allgöwer, T. A. Badgwell, J. S. Qin, J. B. Rawlings, and S. J. Wright, “Nonlinear predictive control and moving horizon estimation—an introductory overview,” Advances in control, pp. 391–449, 1999.

[5] C. V. Rao, J. B. Rawlings, and D. Q. Mayne, “Constrained state estimation for nonlinear discrete-time systems: Stability and moving horizon approximations,” IEEE transactions on automatic control, vol. 48, no. 2, pp. 246–258, 2003.

[6] J. B. Rawlings and L. Ji, “Optimization-based state estimation: Current status and some new results,” Journal of Process Control, vol. 22, no. 8, pp. 1439–1444, 2012.

[7] C. V. Rao, Moving horizon strategies for the constrained monitoring and control of nonlinear discrete-time systems. The University of Wisconsin-Madison, 2000.

[8] V. M. Zavala, C. D. Laird, and L. T. Biegler, “A fast computational framework for large-scale moving horizon estimation,” IFAC Proceedings Volumes, vol. 40, no. 5, pp. 19–28, 2007.

[9] V. M. Zavala, C. D. Laird, and L. T. Biegler, “A fast moving horizon estimation algorithm based on nonlinear programming sensitivity,” Journal of Process Control, vol. 18, no. 9, pp. 876–884, 2008.

[10] A. Wächter and L. T. Biegler, “On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming,” Mathematical programming, vol. 106, no. 1, pp. 25–57, 2006.

[11] H. J. Ferreau, T. Kraus, M. Vukov, W. Saeyes, and M. Diehl, “High-speed moving horizon state estimation based on automatic code generation,” in 2012 IEEE 51st IEEE Conference on Decision and Control (CDC), pp. 687–692, IEEE, 2012.

[12] A. Alessandri, M. Baglietto, T. Parisini, and R. Zoppoli, “A neural state estimator with bounded errors for nonlinear systems,” IEEE Transactions on Automatic Control, vol. 44, no. 11, pp. 2028–2042, 1999.

[13] A. Alessandri, M. Baglietto, and G. Battistelli, “Moving-horizon state estimation for nonlinear discrete-time systems: New stability results and approximation schemes,” Automatica, vol. 44, no. 7, pp. 1753–1765, 2008.

[14] A. Alessandri, M. Baglietto, G. Battistelli, and M. Gaggero, “Moving-horizon state estimation for nonlinear systems using neural networks,” IEEE Transactions on Neural Networks, vol. 22, no. 5, pp. 768–780, 2011.

[15] W. Cao, J. Chen, J. Duan, S. E. Li, Y. Lyu, Z. Gu, and Y. Zhang, “Reinforced optimal estimator,” IFAC-PapersOnLine, vol. 54, no. 20, pp. 366–371, 2021.

[16] J. Li, S. E. Li, K. Tang, Y. Lv, and W. Cao, “Reinforcement solver for h-infinity filter with bounded noise,” in 2020 15th IEEE International Conference on Signal Processing (ICSP), vol. 1, pp. 62–67, IEEE, 2020.

[17] R. G. Krishnan, U. Shalit, and D. Sontag, “Deep kalman filters,” arXiv preprint arXiv:1511.09121, 2016.

[18] M. Karl, M. Soelch, J. Bayer, and V. Der Schacht, “Deep variational bayes filters: Unsupervised learning of state space models from raw data,” arXiv preprint arXiv:1605.06432, 2016.

[19] X. Zhang, M. Bujarbaruah, and F. Borrelli, “Safe and near-optimal policy learning for model predictive control using primal-dual neural networks,” in 2019 American Control Conference (ACC), pp. 354–359, IEEE, 2019.

[20] X. Zhang, M. Bujarbaruah, and F. Borrelli, “Near-optimal rapid mpc using neural networks: A primal-dual policy learning framework,” IEEE Transactions on Control Systems Technology, vol. 29, no. 5, pp. 2102–2114, 2020.

[21] G. Goodwin, J. A. De Doná, M. M. Seron, and X. W. Zhuo, “On the duality of constrained estimation and control,” in Proceedings of the 2004 American Control Conference, vol. 3, pp. 2148–2153, IEEE, 2004.

[22] G. C. Goodwin, J. A. De Doná, M. M. Seron, and X. W. Zhuo, “Lagrangian duality between constrained estimation and control,” Automatica, vol. 41, no. 6, pp. 935–944, 2005.

[23] C. Lauvernet, J.-M. Brankart, F. Castruccio, G. Broquet, P. Brasseur, and J. Verron, “A truncated gaussian filter for data assimilation with inequality constraints: Application to the hydrostatic stability condition in ocean models,” Ocean Modelling, vol. 27, no. 1–2, pp. 1–17, 2009.

[24] S. Knüfer and M. A. Müller, “Robust global exponential stability for moving horizon estimation,” in 2018 IEEE Conference on Decision and Control (CDC), pp. 3477–3482, IEEE, 2018.

[25] S. Boyd, S. P. Boyd, and L. Vandenberghe, Convex optimization. Cambridge university press, 2004.

[26] R. Tempo, E.-W. Bai, and F. Dabbene, “Probabilistic robustness analysis: Explicit bounds for the minimum number of samples,” in Proceedings of 35th IEEE Conference on Decision and Control, vol. 3, pp. 3424–3428, IEEE, 1996.

[27] S. E. Li, Reinforcement Learning for Decision-making and Control. Springer, 2022.

[28] J. D. Schiller, S. Muntwiler, J. Köhler, M. N. Zeilinger, and M. A. Müller, “A lyapunov function for robust stability of moving horizon estimation,” arXiv preprint arXiv:2202.12744, 2022.

[29] J. A. E. Andersson, J. Gillis, G. Horn, J. B. Rawlings, and M. Diehl, “CasADi – A software framework for nonlinear optimization and optimal control,” Mathematical Programming Computation, vol. 11, no. 1, pp. 1–36, 2019.

[30] C. Müller, X. W. Zhuo, and J. A. De Doná, “Duality and symmetry in constrained estimation and control problems,” Automatica, vol. 42, no. 12, pp. 2183–2188, 2006.