Determinization of ω-automata unified

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Abstract

We present a uniform construction for converting ω-automata with arbitrary acceptance conditions to equivalent deterministic parity automata (DPW). Given a non-deterministic automaton with n states, our construction gives a DPW with at most $2^{O(n^2 \log n)}$ states and $O(n^2)$ parity indices. The corresponding bounds when the original automaton is deterministic are $O(n!)$ and $O(n)$, respectively. Our algorithm gives better asymptotic bounds on the number of states and parity indices vis-a-vis the best known technique when determinizing Rabin or Streett automata with $Ω(2^n)$ acceptance pairs, where $n > 1$. We demonstrate this by describing a family of Streett (and Rabin) automata with $2^n$ non-redundant acceptance pairs, for which the best known determinization technique gives a DPW with at least $Ω(2^{(n^3)})$ states, while our construction constructs a DRW/DPW with $2^{O(n^2 \log n)}$ states. An easy corollary of our construction is that an ω-language with Rabin index k cannot be recognized by any ω-automaton (deterministic or non-deterministic) with fewer than $O(\sqrt{k})$ states.

Keywords: ω-automata, determinization, infinity sets

1. Introduction

The literature contains several interesting constructions for obtaining deterministic Rabin/parity automata from nondeterministic ω-automata with different accepting conditions [1, 2, 3, 4, 5, 6, 13, 14, 17, 11, 7, 9]. However, all known constructions are tailor-made to work for nondeterministic automata with a specific kind of accepting condition. For example, Safra’s celebrated Büchi determinization construction [13, 14] can be used to convert non-deterministic Büchi automata over words (NBW) to deterministic Rabin automata over words (DRW). Piterman showed that Safra’s construction can be augmented with additional machinery to obtain deterministic parity automata (DPW) over words from NBW [11, 9]. It requires the use of a completely different technique (once again, originally due to Safra [15, 16] and subsequently improved by Piterman [11]) to convert non-deterministic Streett automata over words (NSW) to equivalent DRW or DPW. We are unaware of any construction for directly converting non-deterministic Müller automata over words (NMW) to DRW or DPW. A two-step approach would involve first converting an NMW to NBW, and then using Safra’s/Piterman’s determinization construction for NBW to obtain a DRW/DPW. In this backdrop, we propose a uniform determinization construction for all ω-automata for which the acceptance condition is based on infinity sets, i.e., the set of states visited infinitely often in a run of the automaton. It is worth noting that the acceptance conditions for all important classes of ω-automata studied in the literature are based on infinity sets.

We begin by quickly reviewing different acceptance conditions of ω-automata used in the literature. Let $A = (Σ, Q, Q_0, δ, φ)$ be a (possibly non-deterministic) ω-automaton, where Σ is the alphabet, Q is the set of states, $Q_0 \subseteq Q$ is the set of initial states, $δ : Q × Σ \rightarrow 2^Q$ is the transition relation, and φ is the acceptance condition. An acceptance condition φ based on infinity sets specifies properties of the set of states visited infinitely often in an accepting run of the automaton. Hence, φ can be thought of as defining a predicate $P_φ$ over $2^Q$. Formally, for every $X \subseteq Q$, we say $P_φ(X) = \text{True}$ iff $X$, viewed as the infinity set of a run of $A$, satisfies the properties specified by φ. This is a re-statement of the fact that any ω-automaton with acceptance condition based on infinity sets can be converted to a Muller automaton by preserving...
the transition structure of the automaton and by listing all subsets of states that satisfy \( \phi \) in the Muller acceptance set. We list below acceptance conditions of some important classes of \( \omega \)-automata and indicate the interpretation of \( P_\phi \) in each case. In all cases, we assume that \( X \) is a subset of \( Q \).

- **Büchi condition:** \( \phi \) is given by \( F \subseteq Q \), and \( P_\phi(X) = \text{True} \) iff \( X \cap F \neq \emptyset \).
- **Muller condition:** \( \phi \) is given by a collection \( F = \{F_1, F_2, \ldots, F_k\} \), where \( F_i \subseteq Q \) for all \( i \in \{1, \ldots, k\} \), and \( P_\phi(X) = \text{True} \) iff \( X \in F \).
- **Rabin condition:** \( \phi \) is given by a table of pairs \( \mathcal{T} = \{(E_1, F_1), (E_2, F_2), \ldots, (E_h, F_h)\} \), where \( E_i, F_i \subseteq Q \) for all \( i \in \{1, \ldots, h\} \), and \( P_\phi(X) = \text{True} \) iff there exists an \( i \in \{1, 2, \ldots, h\} \) such that \( X \cap E_i = \emptyset \) and \( X \cap F_i \neq \emptyset \).
- **Streett condition:** \( \phi \) is given by a table of pairs, similar to that used for Rabin condition. However, in this case \( P_\phi(X) = \text{True} \) iff for all \( i \in \{1 \ldots h\} \), \( X \cap E_i \neq \emptyset \) whenever \( X \cap F_i \neq \emptyset \).
- **Parity condition:** \( \phi \) is given by a sequence of sets \( F = (F_0, F_1, \ldots, F_h) \), where \( F_i \subseteq Q \) for all \( i \in \{0, \ldots, h\} \). Here, \( P_\phi(X) = \text{True} \) iff for some even number \( j \in \{0, \ldots, h\} \), \( X \cap F_j \neq \emptyset \) and for all \( m \in \{0, \ldots, j-1\} \), \( X \cap F_m = \emptyset \).
- **Emerson-Lei condition [5]:** \( \phi \) is given by a fairness condition, expressed as a Boolean combination \( f \) of special linear-time temporal logic formulae over atomic propositions labeling states of the \( \omega \)-automaton. The sub-formulae of \( f \) are such that their truth can be determined simply by knowing the set of sets visited infinitely often along a path (or run) of the automaton, and from the labels of these states. Therefore, \( P_\phi(X) = \text{True} \) iff every run of the automaton with infinity set \( X \) satisfies the temporal logic formula \( f \).

It follows from the above discussion that to determine if an \( \omega \)-word \( \alpha \) is accepted by \( A \), it suffices to determine the set of infinity sets for all runs of \( A \) on \( \alpha \), and to check if \( P_\phi \) evaluates to \text{True} for any of these infinity sets. This observation forms the basis of our construction for determinizing \( \omega \)-automata with arbitrary acceptance conditions based on infinity sets.

The primary contribution of this paper is a uniform construction for converting \( \omega \)-automata with arbitrary acceptance conditions based on infinity sets to deterministic parity automata. Given a non-deterministic automaton with \( n \) states, our construction gives a DPW with at most \( 2^{O(n^2 \log n)} \) states and \( O(n^2) \) parity indices. The corresponding bounds when the original automaton is deterministic are \( O(n!) \) and \( O(n) \), respectively. Our algorithm gives better asymptotic bounds on the number of states and parity indices vis-a-vis the best known technique when determining Rabin or Streett automata with \( O(n^k) \) acceptance pairs, where \( k > 1 \). We demonstrate this by describing a family of Streett (and Rabin) automata with \( 2^{O(n^2)} \) non-redundant acceptance pairs, for which the best known determination technique gives a DPW with at least \( 2^{O(n^3)} \) states and \( 2^{O(n)} \) parity indices. An easy corollary of our construction is that an \( \omega \)-language with Rabin index \( k \) cannot be recognized by any \( \omega \)-automaton (deterministic or non-deterministic) with fewer than \( O(\sqrt{n}) \) states.

The remainder of this paper is organized as follows. We begin by revisiting Schwoon’s version of Safra’s NSW determination construction and Piterman’s optimization of it. We then describe our uniform construction for determinization of \( \omega \)-automata along with intuition behind the construction and an example that demonstrates steps of the construction. We then prove the correctness of our construction and compute its complexity. Finally, we demonstrate the existence of a family of NSW for which our construction provides better upper bounds for determinization than any of the existing methods.

2. Determinizing NSW: A Recap of Safra’s and Piterman’s Constructions

Since our construction is obtained by adapting Safra’s determination construction for NSW [13, 14] and borrows some key optimization ideas from Piterman’s construction [11], we provide an overview of Safra’s
and Piterman’s constructions below. Additional details of Safra’s construction can be found in [15, 16, 19], and those of Piterman’s construction can be found in [11].

Safra’s determinization construction for NSW is based on the idea of witness sets and hierarchically related decompositions. Since we will use a different notion of witness sets later in the paper, we will henceforth call witness sets as defined by Safra as Streett Safra witness sets. For a Streett automaton $A_S = (\Sigma, Q, Q_0, \delta, \phi)$, the acceptance condition $\phi$ is given by a Streett pairs table $T = \{(E_1, F_1), \ldots, (E_h, F_h)\}$. Let $H = \{1, 2, \ldots, h\}$ be the set of indices of Streett pairs in $T$. A subset $J$ of $H$ is called a Streett Safra witness set for a run $\rho$ of $A_S$ if for every $j \in J$, some state in $E_j$ is visited infinitely often in $\rho$, and for every $j \notin J$, no state in $F_j$ is visited infinitely often in $\rho$. It is easy to see that every accepting run of $A_S$ has at least one Streett Safra witness set, and any run of $A_S$ with a Streett Safra witness set is an accepting run. Note, however, that an accepting run of $A_S$ can have multiple Streett Safra witness sets. The decompositions used in Safra’s construction can be viewed as hierarchically related processes, each of which tracks a subset of runs of $A_S$ on a given word, and checks if a certain subset of $H$ is a Streett Safra witness set for all the tracked runs. While Safra’s original exposition [15, 16] represents the hierarchy between decompositions using the notion of sub-decompositions, Schwoon’s exposition of Safra’s construction [19] explicitly represents the hierarchical relation between decompositions as a tree. Each node in this tree represents a decomposition as defined by Safra, and children of a node represent sub-decompositions in Safra’s terminology. We will use the tree representation of decompositions, called $(Q, H)$-trees by Schwoon [19], in the following discussion for clarity of exposition.

Following the definition given by Schwoon [19], a $(Q, H)$-tree over $A_S$ is a finitely branching rooted tree with the following properties.

- Every leaf node is labeled with a non-empty subset of $Q$ (states of the Streett automaton $A_S$).
- State labels of leaf nodes are pairwise disjoint.
- Every node is assigned a name from the set $V = \{1, 2, \ldots, 2 \cdot |Q| \cdot (|H| + 1)\}$.
- No two nodes have the same name.
- Every edge is annotated with an element of $H \cup \{0\}$.
- No edge annotation other than 0 occurs more than once on any path from the root to a leaf.
- Every non-leaf node has at least one child connected by an edge with a non-zero annotation.
- The children of every node are ordered from left to right.

Every node $v$ in a $(Q, H)$-tree can be thought of as being associated with a Streett Safra witness set, $W(v)$, defined as follows. If $v$ is the root node, then $W(v) = \{1, 2, \ldots, h\} = H$. Otherwise, if $v'$ is the parent of $v$ and if the edge from $v'$ to $v$ is annotated with $j$, then $W(v) = W(v') \setminus \{j\}$. Let $\lambda(v)$ denote the set of Streett states labeling the leaves of the sub-tree rooted at $v$. Thus, if $v$ is a leaf node, $\lambda(v)$ is the state label of $v$. However, if $v$ has children $v_1, v_2, \ldots, v_l$, then $v$ itself does not have a state label but $\lambda(v)$ is the disjoint union of $\lambda(v_1), \lambda(v_2), \ldots, \lambda(v_l)$. A node $v$ in a $(Q, H)$-tree represents a process that tracks the runs represented by states in $\lambda(v)$, and checks if $W(v)$ is a Streett Safra witness set for all these runs. This is done by waiting until all $E_j$ for $j \in W(v)$ are visited in order along the runs, without visiting any $F_l$ for $l \notin W(v)$. If this happens, the process represented by $v$ is said to have “succeeded”; it is then “reset” and the check starts all over again. Clearly, if the process represented by $v$ is reset infinitely often, then $W(v)$ is a Streett Safra witness set for the runs tracked by this process have, and hence these are accepting runs of $A_S$. On the other hand, if some state in $F_l$ for $l \notin W(v)$ is seen in a run being tracked by the process represented by $v$, then that run is removed from this process, and a new process is started for that run. The hierarchical relation between processes is explicitly represented by the parent-child relation between nodes in a $(Q, H)$-tree. Intuitively, if $v'$ is the parent of $v$ and if the edge from $v'$ to $v$ is annotated with $j$, the process represented by $v$ tracks a subset of the runs tracked by $v'$ after giving up hope that it will see a state from $E_j$ ever in the future. While the parent $v'$ keeps alive the hope that $W(v')$ is the Streett Safra witness state that
set for all runs tracked by $v'$, the child $v$ refines and corrects that hope by expecting $W(v) = W(v') \setminus \{j\}$ to be the Safra Streett witness set for the subset of runs tracked by $v$.

The DRW obtained by applying Safra's construction to a Streett automaton $A_S = (\Sigma, Q, Q_0, \delta, \phi)$ is given by $A_R = (\Sigma, Q', Q_0', \delta', \phi')$, where $Q'$ is the set of all $(Q, H)$-trees over $A_S$, and $Q_0'$ is a singleton set containing the $(Q, H)$-tree consisting of only a root node with name 1 and labeled with $Q_0$ (set of initial states of $A_S$). Since $A_R$ is a deterministic automaton, $\delta'$ can be thought of as a function that takes a state (i.e., $(Q, H)$-tree) $t$ and a letter $\sigma \in \Sigma$ and returns the next state (i.e., $(Q, H)$-tree) $t'$. The computation of $t'$ from $t$ and $\sigma$ is detailed in algorithm SafraNext given below (adapted from Schwoon's exposition [19] and Piterman’s correction [14] of an erroneous step in [16, 19]). Note that algorithm SafraNext calls a recursive procedure SafraNextRecursive that is parameterized by the root node of a $(Q, H)$-sub-tree and the corresponding Streett Safra witness set. If $|Q| = n$ and $|H| = h$, the Rabin acceptance condition $\phi'$ is given by a table $T' = \{(E_i', F_i') \mid 1 \leq i \leq 2 \cdot n \cdot (h + 1)\}$, where $E_i'$ is the set of all $(Q, H)$-trees with no node named $i$, and $F_i'$ is the set of all $(Q, H)$-trees in which a leaf node named $i$ occurs.

**Algorithm : SafraNext**

**Input:** $t : (Q, H)$-tree over $A_S$, $\sigma :$ letter in $\Sigma$

**Output:** $t' : (Q, H)$-tree over $A_S$

1. **[Initialization]** For every leaf node $u$ of $t$, set the state label of $u$ to $\delta(\lambda(u), \sigma)$.
2. **[Recursive transformation]** Let root be the root node of $t$.
   - Invoke SafraNextRecursive(root, H).
3. Return $t'$ as the $(Q, H)$-tree rooted at root.

End Algorithm : SafraNext

**Algorithm : SafraNextRecursive**

**Input:** $v$ : root of a $(Q, H)$-sub-tree, $J$ : subset of $H$

**Output:** $t'$ : Transformed $(Q, H)$-sub-tree rooted at $v$

1. If $v$ is a leaf and $J = \emptyset$, return $t'$ as the $(Q, H)$-sub-tree rooted at $v$.
2. If $v$ is a leaf and $J \neq \emptyset$, create a new child $v'$ of $v$ with state label $\lambda(v)$, remove $\lambda(v)$ from the state label of $v$ (since $v$ is no longer a leaf) and annotate the edge from $v$ to $v'$ with max $W(v)$. Assign an unused name from $V = \{1, 2, \ldots, 2 \cdot |Q| \cdot (|H| + 1)\}$ to $v'$.
3. If, after the execution of Steps 1 and 2, $v$ is not a leaf, then let $v_1, \ldots, v_l$ be the children of $v$ ordered from left to right. Let the edge from $v$ to $v_i$ be annotated with $j_i$ for all $i \in \{1, 2, \ldots, l\}$.
   a. For all $i$ from 1 to $l$, invoke SafraNextRecursive($v_i, J \setminus \{j_i\}$)
   b. For every child $v_i$ of $v$ and every $q \in \lambda(v_i)$, do the following
      i. If $q \in F_{j_i}$, remove $q$ from the state labels of all leaves of the sub-tree rooted at $v_i$, create a new rightmost child $v''$ of $v$ with state label $\{q\}$, and annotate the edge from $v$ to $v''$ with $j_i$. Assign an unused name from $V = \{1, 2, \ldots, 2 \cdot |Q| \cdot (|H| + 1)\}$ to $v''$.
      ii. If $q \notin F_{j_i}$, create a new rightmost child $v''$ of $v$ with state label $\{q\}$ and annotate the edge from $v$ to $v''$ with $max((J \cup \{0\}) \cap \{0, 1, \ldots, j_i - 1\})$. In other words, the edge is annotated with the largest integer less than $j_i$ but in $J$, if it exists. Otherwise, it is annotated with 0. Assign an unused name from $V = \{1, 2, \ldots, 2 \cdot |Q| \cdot (|H| + 1)\}$ to $v''$. 

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4. Let \( v_1, v_2, \ldots, v_t \) be the children of \( v \) after the above steps. Let \( j_1, j_2, \ldots, j_t \) be the annotations of the corresponding edges from \( v \) to its children. For every \( q \in \lambda(v_j) \cap \lambda(v_k) \), where \( j \neq k \) and \( j, k \in \{1, 2, \ldots, t\} \), do the following.
   (a) If \( j_1 < j_k \), remove \( q \) from the state labels of all leaves of the sub-tree rooted at \( v_k \).
   (b) If \( j_1 = j_k \) and \( v_i \) is to the left of \( v_k \), remove \( q \) from the state labels of all leaves of the sub-tree rooted at \( v_k \).
5. For every descendant \( u \) of \( v \) such that \( \lambda(u) = \emptyset \), delete \( u \) and all its descendants.
6. If, after the previous steps, all edges from \( v \) to its children are annotated with 0, then the process represented by \( v \) has “succeeded” and needs to be “reset”. Let \( S = \lambda(v) \). Make \( v \) a leaf node by deleting all its children and its descendants, and set the state label of \( v \) to \( S \).
7. Return \( t' \) as the \((Q, H)\)-sub-tree rooted at \( v \).

End Algorithm : SafraNextRecursive

It was shown by Safra that given an NSW with \(|Q| = n\) and \(|H| = h\), the above construction gives a deterministic Rabin automaton with \(2^{O(n \cdot h \cdot \log(n \cdot h))}\) states and \(O(n \cdot h)\) Rabin acceptance pairs. Although a proof of correctness of the construction was provided in [15, 16, 19], Piterman pointed out a minor error in the construction and rectified it in [11]. Fortunately, Piterman’s correction affects only a single step of Safra’s construction and does not change the asymptotic count of states or Rabin acceptance pairs. The fact that this erroneous step evaded the scrutiny of researchers for almost 14 years is testimony to the intricate nature of arguments used in Safra’s construction. Piterman also proposed an adaptation of Safra’s construction that uses only \(n \cdot (h + 1)\) names (instead of \(2 \cdot n \cdot (h + 1)\) names used by Safra) and gives a deterministic parity automaton with \(2^{O(n \cdot h \cdot \log(n \cdot h))}\) states and \(2 \cdot n \cdot h\) parity indices. Currently, Piterman’s construction is the best known determinization construction for NSW.

Piterman’s adaptation of Safra’s construction involves two key ideas: (i) a new strategy for naming nodes, and (ii) addition of two integer-valued components, \( e \) and \( f \), to every state of the constructed automaton that allows a parity acceptance condition to be defined. In the new naming strategy, whenever a new node is created in steps (2), (3(b)) or (3(bii)) of algorithm \textit{SafraNextRecursive}, it is assigned the smallest name larger than all names used so far in the construction of \( t' \) from \( t \). In addition, after algorithm \textit{SafraNext} has finished computing \( t' \), a name-compaction step is performed. In this step, for each node \( v \) with name \( i \) in \( t' \), we determine the count, \( \text{rem}(v) \), of nodes that were removed during the construction of \( t' \) from \( t \) and had names less than \( i \). The name of \( v \) is then reduced from \( i \) to \( i - \text{rem}(v) \). This ensures that there are no gaps in the set of names assigned to nodes in a \((Q, H)\)-tree after the name-compaction step. Piterman’s naming strategy also ensures that the name of a node \( v \) is less than that of node \( u \) iff \( v \) was created before \( u \). Since the name of a node that stays back in a run (sequence of \((Q, H)\)-trees) can only reduce finitely many times, it follows that all nodes that eventually stay back in a run get fixed names that are smaller than the names of all other nodes that keep getting created and removed.

The new state components \( e \) and \( f \) in Piterman’s construction keep track of the smallest names of a node removed and the smallest name of a node that represents a successful process (see step (3) of algorithm \textit{SafraNextRecursive}) respectively in the construction of \( t' \) from \( t \). A state in the resulting automaton is therefore a \((Q, H)\)-tree coupled with a pair of integers \( e, f \in \{1, \ldots, n \cdot (h + 1) + 1\} \), with the restriction that the root node is always named 1 and all nodes are assigned names from \(\{1, \ldots, n \cdot (h + 1)\}\). Piterman calls these states \textit{compact Streett Safra trees} over \(A_S\), and obtains a deterministic parity automaton by defining a parity acceptance condition as follows. Let \( D \) denote the set of all compact Streett Safra trees over \(A_S\). Piterman’s parity acceptance condition is given by \( F = \langle F_0, F_1, \ldots, F_{2m-1} \rangle \), where \( m = 2 \cdot n \cdot (h + 1) \) and \( F_i \)'s are defined as follows.

- \( F_0 = \{d \in D \mid f = 1 \text{ and } e > 1\} \)
- \( F_{2i+1} = \{d \in D \mid e = i + 2 \text{ and } f \geq e\} \), for all \( i \in \{0, \ldots, m - 1\} \)
- \( F_{2i+2} = \{d \in D \mid f = i + 2 \text{ and } e > f\} \), for all \( i \in \{0, \ldots, m - 2\} \)
A proof of correctness of the above construction is given in [11]. It is also shown there that the DPW obtained using this construction has at most $2 \cdot n^n \cdot (k + 1)^{n(k+1)} \cdot (n \cdot (k + 1))!$ states and $2 \cdot n \cdot (k + 1)$ parity indices.

3. A uniform determinization construction for ω-automata

We now describe a construction for converting ω-automata with arbitrary acceptance conditions based on infinity sets to deterministic parity automata. Our construction can be viewed as an adaptation of Safra’s NSW determinization construction that works for arbitrary acceptance conditions. As part of our construction, we use Piterman’s naming strategy and his idea of using $e, f$ components of states to get a parity acceptance condition. Interestingly, although our construction is based on Safra’s and Piterman’s constructions, we are able to sharpen the asymptotic upper bound for Streett and Rabin determinization beyond those obtainable by Safra’s and Piterman’s constructions.

Let $A = (\Sigma, Q, Q_0, \delta, \phi)$ be an ω-automaton, where $\phi$ is an arbitrary acceptance condition based on infinity sets. Let $P_{\phi}$ denote the predicate corresponding to $\phi$. Without loss of generality, we will assume that $Q = \{q_1, q_2, \ldots q_n\}$, where $n = |Q|$. For notational clarity, we will henceforth refer to states of $A$ as $A$-states, and use $[p]$ to denote the set $\{1, 2, \ldots p\}$ for every natural number $p > 0$. For every $W \subseteq [n]$, we also define $Q_W$ to be the set $\{q_i \mid q_i \in Q, i \in W\}$.

Motivated by the role played by Streett Safra witness sets in Safra’s NSW determinization construction, we now define generalized witness sets for ω-automata with arbitrary acceptance conditions based on infinity sets.

**Definition 1 (Generalized Witness Set).** A set $W \subseteq [n]$ is a generalized witness set for a run $\rho$ of $A$ iff $\inf(\rho) = Q_W$ and $P_{\phi}(Q_W) = \text{True}$.

Note that Streett Safra witness sets are distinct from generalized witness sets even when $A$ is a Streett automaton. By definition, a Streett Safra witness set is a subset of indices of Streett acceptance pairs, while a generalized witness set is a subset of indices of $A$-states. Thus, if $A$ has $n$ states and $h$ pairs in its acceptance table, and if $n \ll h$ (examples of NSW with this property are given in Section 6), there can be many more Streett Safra witness sets than generalized witness sets. The situation is reversed if $h \ll n$. It follows from the definition above that a run $\rho$ of $A$ can have at most one generalized witness set, although it may have multiple Streett Safra witness sets. Furthermore, the generalized witness set of $\rho$ uniquely determines $\inf(\rho)$, while a Streett Safra witness set for $\rho$ does not necessarily determine $\inf(\rho)$ uniquely. Finally, if $A$ is a Streett automaton and if a run $\rho$ of $A$ has a generalized witness set, then it has at least one (and perhaps more) Streett Safra witness sets. Conversely, if $\rho$ has at least one Streett Safra witness sets, then it has exactly one generalized witness set.

The use of generalized witness sets allows us to adapt Safra’s construction to obtain a uniform determinization construction for ω-automata with arbitrary acceptance conditions. We detail this construction in the following subsections.

3.1. Intuition

The intuition behind our construction parallels that behind Safra’s NSW determinization construction, with some key differences stemming from the use of generalized witness sets instead of Streett Safra witness sets. The overall idea is to construct a deterministic automaton that simulates all runs of $A$ on an ω-word $\alpha$, and uses a Rabin acceptance condition to simultaneously identify the set of state indices in the $\inf$-set of a run and check if this set is a generalized witness set. The construction of the Rabin automaton can be adapted to give a deterministic parity automaton using techniques employed by Piterman [11]. Although there are an exponential number of potential generalized witness sets, we use Safra’s idea of building a process decomposition (represented as a tree), in which each process tracks a subset of runs and checks if a given subset of $A$-state indices is a generalized witness set for these runs. Using the same reasoning as used by Safra, we can show that only a polynomial number of generalized witness sets need to be examined at any time in order to determine if a run has a generalized witness set.
As in Safra’s and Piterman’s constructions \cite{15, 16, 14, 11}, each state of the DPW obtained by our construction is a tree of hierarchically related processes, with additional book-keeping information. The process represented by a node in the tree tracks a subset of runs of the automaton $\mathcal{A}$. Each process is also associated with a set of indices of $\mathcal{A}$-states, called the hope set for the process. A process hopes that its hope set gives the indices of states in the inf-set of all runs tracked by it. This is checked by waiting for all states with indices in the hope set to be visited in turn by every run tracked by the process, without visiting any state with index outside the hope set. If this happens, the process is said to have “succeeded” locally; it is then “reset” and the check starts all over again. Clearly, if the process represented by a node $v$ is reset infinitely often, its hope set gives the indices of states in the inf-set of all runs tracked by it. If, in addition, the set of states with indices in the hope set causes $P_b$ to evaluate to True, the hope set must be a generalized witness set of all runs tracked by the process. In this case, there exists at least one accepting run of $\mathcal{A}$ on the input word. On the other hand, if some state with an index outside the hope set is seen in a run tracked by a process, the corresponding run is removed from the process, and a new process is initiated for that run. As in Safra’s and Piterman’s constructions, we use an acceptance condition that checks for the existence of a node $u$ that is eventually never deleted in the sequence of trees (states) in an infinite run of the constructed automaton, but is reset infinitely often. Unlike Safra’s and Piterman’s construction, we also require that the hope set of the process corresponding to node $u$ be such that the corresponding set of $\mathcal{A}$-states renders $P_b$ True. In the remainder of the discussion, we will refer to a node and the process represented by it interchangeably when there is no confusion.

3.2. The determinization construction

Piterman used compact Streett Safra trees to represent states of the deterministic parity automaton in his NSW determinization construction \cite{11}. We follow the same approach and use a variant of compact Streett Safra trees, called compact generalized Safra trees, or CGS trees. Formally, a CGS tree $t$ over $\mathcal{A} = (Q, \Sigma, Q_0, \delta, \phi)$ is a 9-tuple $(N, M, r, p, \lambda, h, e, f)$, where

- $N$ is the set of nodes.
- $M : N \rightarrow \|Q\|^2 + |Q| + 1$ is the naming function.
- $r$ is the root node.
- $p : N \rightarrow N$ is the parenthood function defined for $N \setminus \{r\}$. Thus, $p(v)$ is the parent of $v \in N \setminus \{r\}$.
- $\lambda : N \rightarrow 2^Q$ is a state labeling function that associates a subset of $Q$ with each node. The state label of every node is equal to the union of state labels of its children. Furthermore, the state labels of two siblings are disjoint.
- $h : N \rightarrow 2^{\|Q\|}$ is an annotation of nodes with a subset of $\|Q\|$. The root is always annotated with $\|Q\|$. The annotation of every node is contained in that of its parent and differs by at most one element from the annotation of its parent. Every non-leaf node $v$ has at least one child with an annotation that is a strict subset of $h(v)$. For a node $v$ with annotation $J$ and child $v'$ with annotation $J' = J \setminus \{j\}$, we will say that the edge from $v$ to $v'$ is annotated with $j$. If $J' = J$, we will say that the edge from $v$ to $v'$ is annotated with $0$.
- $e, f \in \|Q\|^2 + |Q| + 2$ are two integers used to define the parity acceptance condition.

Note that CGS trees differ from compact Streett Safra trees \cite{11} only in the annotation of nodes. In a compact Streett Safra tree, each node is annotated with a potential Streett Safra witness set, while in a CGS tree, the annotations are potential generalized witness sets. As discussed earlier, generalized witness sets can differ significantly from Streett Safra witness sets even when $\mathcal{A}$ is a Streett automaton. Intuitively, each node $v$ in a compact generalized Safra tree represents a process that tracks the runs of $\mathcal{A}$ currently represented by $\lambda(v)$, and hopes that $Q_{h(v)}$ is the inf-set of these runs. The set $h(v)$ may therefore be viewed as the hope set for the process represented by $v$. 

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Given $A = (\Sigma, Q, Q_0, \delta, \phi)$, we now construct a deterministic parity automaton (DPW) $D = (\Sigma, T, t_0, \delta^p, \mathcal{P})$ such that $L(A) = L(D)$. In the following, we assume that $n = |Q|$ and $m = |Q|^2 + |Q| + 1$. The different components of $D$ are as defined below.

- $T$ is the set of all CGS trees over $A$.
- $t_0$ is the CGS tree with a single (root) node $r_0$, with $\lambda(r_0) = Q_0$, $M(r_0) = 1$ and $h(r_0) = [n]$. For $t_0$, we set $e = f = m + 1$.
- The parity acceptance condition $\mathcal{P} = \langle F_0, F_1, \ldots, F_{2m-1} \rangle$ is defined in the same manner as done by Piterman [1]. Specifically,
  
  - $F_0 = \{ t \in T \mid f = 1, e > 1 \}$
  - $F_{2i+1} = \{ t \in T \mid e = i + 2, f \geq e \}$ for $0 \leq i < m - 1$
  - $F_{2i+2} = \{ t \in T \mid f = i + 2, e > f \}$ for $0 \leq i < m - 1$
  - $F_{2m-1} = \{ t \in T \mid e, f > m \}$

  For reasons to be seen later, no CGS tree that arises in our construction can have $e = 1$; hence CGS trees with $e = 1$ are excluded from the $F_i$ sets defined above.

- $\delta^p$ is a deterministic transition function that returns a unique next state (CGS tree) $t'$ for every current state $t \in T$ and input symbol $\sigma \in \Sigma$. The computation of $t'$ from $t$ and $\sigma$ is accomplished by invoking algorithm $\text{GeneralizedNext}(t, \sigma)$, as detailed below.

Recall that a CGS tree has named, state-labeled and annotated nodes hierarchically arranged as a rooted tree, along with two integer valued components named $e$ and $f$. Computing $t'$ from $t$ and $\sigma$ therefore involves transforming the hierarchical arrangement of nodes and determining new values for $e$ and $f$, in general. Component $e$ of $t'$ is intended to record the smallest name of a node that was deleted during the transformation of the hierarchical arrangement. Similarly, component $f$ is meant to record the smallest name of a node that was “reset” (in the sense described in Section 3.1), had a hope set such that the corresponding set of $A$ states satisfies $P_0$, and was not deleted subsequently during the transformation of the hierarchical arrangement. Since a node can be deleted in a step after being reset, algorithm $\text{GeneralizedNext}$ uses a set $U$ to remember all nodes that were reset and had hope sets such that the corresponding set of $A$ states satisfies $P_0$, in some step during the transformation. Finally, component $f$ is set to the smallest name of a node in $U$ that survives the transformation. The task of transforming the hierarchical arrangement of nodes is accomplished by invoking algorithm $\text{GeneralizedNextRecursive}$, as described below. As the transformation proceeds through recursive calls to $\text{GeneralizedNextRecursive}$ and nodes are reset and/or deleted from the CGS tree, component $e$ and the set $U$ described above are updated. After the transformation of the hierarchical arrangement is completed, a name-compaction step is performed on the nodes of the resulting CGS tree in the same way as is done in [11]. Although intermediate steps of algorithm $\text{GeneralizedNextRecursive}$ may use names of nodes outside the set $[m]$, the name-compaction step ensures that all names used in the final CGS tree $t'$ are within $[m]$. The pseudocode of algorithms $\text{GeneralizedNext}$ and $\text{GeneralizedNextRecursive}$ are presented below.

**Algorithm : GeneralizedNext**

**Input:** $t$ : CGS tree over $A$, $\sigma$ : letter in $\Sigma$

**Output:** $t'$ : CGS tree over $A$

1. [Initialization] Initialize $e$ and $f$ to $m + 1$. Initialize $U$ to $\emptyset$. For every node $u$ in $t$, set $\lambda(u)$ to $\delta(\lambda(u), \sigma)$.
2. [Recursive transformation] Let $\text{root}$ be the root node of $t$.
   - Invoke $\text{GeneralizedNextRecursive}(\text{root})$. 

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3. **[Name-compaction]** Let \( \tilde{t} \) be the CGS tree rooted at \( \text{root} \) after Step 2. Let \( Z \) be the set of CGS tree nodes removed during the execution of Step 2. For every node \( u \) in \( t \), let \( \text{rem}(u) = |\{u' \in Z \mid M(u') < M(u)\}| \). Update \( M(u) \) to \( M(u) - \text{rem}(u) \).

4. **[Update of component]** Let \( \tilde{t} \) be the CGS tree rooted at \( \text{root} \) that results after Step 3. Let \( \tilde{N} \) be the set of nodes in \( \tilde{t} \). Set \( f \) to the minimum of its current value and \( \min(M(v') \mid v' \in U \cap \tilde{N}) \).

5. Return \( \tilde{t}' \) as the CGS tree rooted at \( \text{root} \) with \( e \) and \( f \) components as calculated above.

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**Algorithm**: GeneralizedNextRecursive

**Input**: \( v \): root of a CGS sub-tree

**Output**: \( \tilde{t}' \): Transformed CGS sub-tree rooted at \( v \), updated values of \( e \) and \( U \)

1. If \( v \) is a leaf and \( h(v) = 0 \), return \( \tilde{t}' \) as the CGS sub-tree rooted at \( v \).
2. If \( v \) is a leaf and \( h(v) \neq 0 \), create a new child \( v' \) of \( v \). Set \( \lambda(v') = \lambda(v) \), \( h(v') = h(v) \setminus \{\max(h(v))\} \) and \( M(v') \) to the smallest name greater than all names already used. Note that this may require using names not in \([n]\).
3. If, after the execution of Steps 1 and 2, \( v \) is not a leaf, then let \( v_1, \ldots, v_l \) be the children of \( v \) ordered according to their names. Let \( j_1, \ldots, j_l \) be indices such that \( j_i = \max((h(v) \cup \{0\}) \setminus h(v_i)) \).

   As discussed earlier (in the definition of compact generalized Safra trees), we will say that the edge from \( v \) to \( v_i \) is annotated with \( j_i \).

   i. For every \( i \) in 1 through \( l \), invoke GeneralizedNextRecursive(\( v_i \)).
   
   ii. If \( q \neq q_i \) and \( q \notin Q_{h(v_i)} = \{q_j \mid q_j \in Q, j \in h(v_i)\} \), remove \( q \) from \( \lambda(v_i) \) and also from \( \lambda(u) \) for all descendants \( u \) of \( v_i \).

4. Let \( v_1, v_2, \ldots, v_l \) be the children of \( v \) after the above steps. Let \( j_1, \ldots, j_l \) be the annotations of the corresponding edges from \( v \) to its children. In other words, let \( j_i = \max((h(v_i) \cup \{0\}) \setminus h(v_i)) \) for \( i \in \{1, 2, \ldots, l'\} \). Then for every \( q \in \lambda(v_i) \) \( \cap \lambda(v_k) \), where \( i \neq k \) and \( i, k \in \{1, \ldots, l'\} \), do the following.

   i. If \( j_i < j_k \), remove \( q \) from \( \lambda(v_k) \) and from \( \lambda(u) \) for all descendants \( u \) of \( v_k \).
   
   ii. If \( j_i > j_k \) and \( M(v_i) < M(v_k) \), remove \( q \) from \( \lambda(v_k) \) and from \( \lambda(u) \) for all descendants \( u \) of \( v_k \).

5. For every descendant \( u \) of \( v \) such that \( \lambda(u) = \emptyset \), delete \( u \) and all its descendants.

6. If, after the previous steps, all children of \( v \) have annotation \( h(v) \), then the process represented by \( v \) is said to “succeed” locally and needs to be “reset”. Delete all descendants of \( v \), so that \( v \) becomes a leaf node. Additionally, if \( P_\phi(Q_{h(v)}) = \text{True} \), then update \( U \) to \( U \cup \{v\} \).

7. Update \( e \) to the minimum of its previous value and the smallest name among all descendants of \( v \) that were deleted.

8. Return \( \tilde{t}' \) as the CGS sub-tree rooted at \( v \).

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\(^1\) Note that if \( h(v) = h(v_i) \), then \( j_i = 0 \).
The similarity of algorithms GeneralizedNext and GeneralizedNextRecursive to the corresponding algorithms in Safra’s and Piterman’s NSW determinization constructions is striking. Yet, there are important differences that enable our construction to achieve something different, and even better Safra’s and Piterman’s constructions when the number of Streett pairs is large compared to the number of Streett states.

The computation of \( \delta^*(t, \sigma) \) starts by determining the successors of all \( \mathcal{A} \)-states appearing in state labels of nodes in the CGS tree \( t \), under the input symbol \( \sigma \). Algorithm GeneralizedNextRecursive is then invoked on the resulting tree rooted at \( \text{root} \). This recursively “extends” the tree (in Steps 1, 2 and the recursive call in Step 3 of algorithm GeneralizedNextRecursive) by adding new leaf nodes with successively smaller hope sets until each leaf node has an empty hope set. As the recursive calls return, algorithm GeneralizedNextRecursive determines in a bottom-up manner which nodes in the extended CGS tree must have their hope sets invalidated and/or hierarchical relations modified. We explain below the reasoning behind this crucial step in the computation of \( t' \).

Suppose the hope set of a node \( v \) is \( h(v) \) and that of its child \( v'' \) is \( h(v'') \). Suppose further that the edge from \( v \) to \( v'' \) is annotated with \( j_i \), i.e., \( h(v) \setminus h(v'') = \{j_i\} \). This represents a situation wherein the process represented by \( v \) is waiting to see \( q_{j_i} \) in the subset of runs being tracked by its child \( v'' \), but the process represented by \( v'' \) has given up hope of seeing any further \( q_{j_i}'s \) in the runs it is tracking. Now, suppose after reading an input symbol \( \sigma \), the initialization step of algorithm GeneralizedNext places \( q_{j_i} \) in \( \lambda(v'') \) (and hence also in \( \lambda(v) \)). This implies that \( v'' \) has seen a state along a run it was tracking, such that the corresponding state index is outside its own hope set but is in the hope set of its parent. Since every node expects to see all and only states with indices in its hope set in all runs being tracked by it, the above situation warrants two actions: (i) invalidating the hope set of \( v'' \) for the run represented by \( q_{j_i} \), and (ii) registering progress towards the realization of \( v's \) hope set as the set of state indices in the inf-set of the run represented by \( q_{j_i} \). Accordingly, \( q_{j_i} \) is removed from \( \lambda(v'') \) by the sequence of steps 3(b) and 3 of algorithm GeneralizedNextRecursive. In addition, step 3(b) creates a new child \( v' \) of \( v \) with \( \lambda(v') = \{q_{j_i}\} \), and annotates the edge from \( v \) to \( v'' \) with the next index (after \( j_i \) in decreasing order), say \( j_k \), in the hope set of \( v \). This represents the new situation wherein the process represented by \( v \) has seen \( q_{j_i} \) and is waiting to see the next \( \mathcal{A} \)-state in its hope set, i.e. \( q_{j_k} \), in the run (currently) represented by \( q_{j_i} \). The new child \( v' \) however hopes to see no further \( q_{j_k}'s \) in the run represented by \( q_{j_i} \); hence its hope set is set to \( h(v) \setminus \{j_k\} \).

A special situation arises if \( j_i \) is the lowest indexed \( \mathcal{A} \)-state in \( Q_{h(v)} \). In this case, node \( v \) has seen all states with indices in its hope set in the run represented by \( q_{j_i} \), since the last time \( v \) was “reset”. The edge from \( v \) to \( v' \) is annotated with a special index, i.e. 0, to represent this situation. The newly created child \( v' \) retains the same hope set as \( v \), i.e. \( h(v) \), and is now delegated the task of checking if \( Q_{h(v)} \) is the inf-set of the run currently represented by \( q_{j_i} \). Meanwhile, the parent node \( v \) continues to check if all states with indices in its hope set, i.e., \( h(v) \), are seen in the remaining runs (other than the one currently represented by \( q_i \)) that it was tracking.

A different situation arises if the initialization step of algorithm GeneralizedNext places \( q_{j_i} \) in \( \lambda(v'') \) for a child \( v'' \) of \( v \), but \( j_i \) is neither the annotation of the edge from \( v \) to \( v'' \), nor is in the hope set of \( v'' \). This represents a situation wherein the process represented by \( v \) was waiting to see some \( \mathcal{A} \)-state other than \( q_{j_i} \) next in the runs being tracked by \( v'' \), and the process represented by \( v'' \) was expecting to never see \( q_{j_i} \) in any run being tracked by it. Since \( q_{j_i} \) is in \( \lambda(v'') \), the hope set of \( v'' \) must be invalidated for the run currently represented by \( q_{j_i} \). This is done in step 3(b) of algorithm GeneralizedNextRecursive by removing \( q_{j_i} \) from the state label of \( v'' \) and all its descendants. Note, however, that we cannot remove the run represented by \( q_{j_i} \) from the state label of \( v \) yet. Although \( v \) was not expecting \( q_{j_i} \) to be the next \( \mathcal{A} \)-state in the runs being tracked by \( v'' \), the hope set of \( v \) may still contain \( j_i \). Therefore, the hope set of \( v \) need not be invalidated yet for the run corresponding to \( q_{j_i} \). As the recursive calls to algorithm GeneralizedNextRecursive return, the hope set of \( v \) will be examined in turn to determine if a run being tracked by \( v \) has encountered a state with index outside \( v's \) hope set. If so, the run will then be removed from the set of runs being tracked by \( v \).

Since runs tracked by different nodes in a CGS tree may merge, we may encounter a situation wherein the same \( \mathcal{A} \)-state \( q \) appears in the state labels of multiple nodes that are not related as ancestors or descendants.
in the tree. However, by definition, two nodes in a CGS tree can have overlapping state labels only if one is an ancestor (or descendant) of the other. Algorithm \texttt{GeneralizedNextRecursive} rectifies this situation by ensuring that whenever an \(A\)-state \(q\) appears in the state labels of multiple children of a node \(v\), at most one child eventually gets to retain \(q\) in its state label. The chosen child is the one that represents the maximum progress (since \(v\) was last reset) towards realisation of the hope set of \(v\) as the set of state indices in the \textit{inf}-set of the run represented by \(q\). This choice can be made by examining the annotations on the edges from \(v\) to the subset of its children containing \(q\) in their state labels. Specifically, the child that represents the most progress is the one that has the smallest annotation on the edge from \(v\). This is because a child with an edge from \(v\) annotated with \(i\) represents the situation wherein all \(A\)-states with indices greater than \(i\) and in the hope set of \(v\) have been seen since \(v\) was last reset. In the event that an \(A\)-state \(q\) appears in the state labels of two siblings with the same annotation on the edges from their parent, we choose to retain \(q\) in the state label of the node that was created earlier, i.e. has a smaller name. As the recursive calls to algorithm \texttt{GeneralizedNextRecursive} return, step 4 examines the nodes of the CGS tree in a bottom-up manner and applies the above criterion to ensure that two nodes not related as ancestor and descendant do not share any \(A\)-state in their state labels in the final tree.

Step 5 of algorithm \texttt{GeneralizedNextRecursive} deletes all nodes with empty state labels from the CGS tree constructed thus far, since the processes represented by these nodes no longer track any runs. In Step 6 we examine the annotations on the edges to all children of the current node \(v\). If these annotations are all 0, we have a situation wherein all runs being tracked by \(v\) have seen all states with indices in \(v\)'s hope set since the last time \(v\) was reset. This constitutes a step of progress in establishing that the hope set of \(v\) is indeed the set of state indices in the \textit{inf}-set of all runs being tracked by it. Node \(v\) is therefore said to have "succeeded" locally, and is "reset" in step 6 of algorithm \texttt{GeneralizedNextRecursive} by deleting all its descendants. If, in addition, \(Q_{h(v)} \models \phi\) then we have a step of progress in establishing that \(Q_{h(v)}\) is the generalized Safra witness set of all runs being tracked by \(v\). Step 6 of algorithm \texttt{GeneralizedNextRecursive} keeps track of this fact by updating the set \(U\). As explained earlier, \(U\) is eventually used to obtain the value of component \(f\) of the CGS tree \(t'\). Finally, step 7 of algorithm \texttt{GeneralizedNextRecursive} updates component \(e\) of \(t'\) by recording the smallest name of a node deleted in the recursive transformation of the CGS tree.

3.3. An Example

We now illustrate the working of our determinization construction using the non-deterministic Müller automaton (NMW) \(A\) shown in Figure (1). The Müller acceptance condition of this automaton is given by \(F = \{\{q_1\}\}\). Let \(D\) be the corresponding deterministic parity automaton obtained by our construction.
To see how different states and transitions in $D$ are obtained, we will follow the construction of states encountered in $D$ on reading a short prefix of the word $bbbcw^2$ that is accepted by $A$. Since $A$ has 5 states, we have $n = 5$ and $m = 5^2 + 5 + 1 = 31$. Thus, every node in the CGS tree representing a state of $D$ has a name in $[31]$, and a hope set that is a subset of $[5]$. Every edge in the tree is annotated with an element of $\{0, 1, \ldots, 5\}$. Since the hope set of the root node is always $[5]$, and since the hope set of any other node $v$ can be obtained by eliminating from $[5]$ the annotations of edges on the path from the root to $v$, we will simply annotate edges with elements of $[5]$ and not explicitly represent hope sets. Similarly, since the state label of every node is the union of the state labels of its children, we will simply label leaves of the CGS tree with subsets of $A$-states. To help illustrate the intermediate steps of the construction, we will also indicate the updated values of $e$ and $f$ (components of the CGS tree) in the following discussion.

![Figure 2: Steps in determinization construction](Image)

We start in the initial state consisting of a CGS tree having a single node named 1 and labeled $\{q_1\}$, as shown in Figure (2-a1). The values of $e$ and $f$ are both $m + 1 = 32$ in this state. On reading the letter $b$, the state label of the node named 1 (also a leaf in this case) is first changed to $\{q_1\}$, since $q_1$ transitions to $q_5$ on reading $b$ in automaton $A$. The CGS tree consisting of only the root node is then extended in Steps (1), (2) and through the recursion in Step (3a) of algorithm GeneralizedNextRecursive to give the tree shown in Figure (2-b1). As the recursive calls return in sequence, all nodes other than the ones named 1 and 2 are deleted. When the recursion returns to the topmost level with the root named 1 as the current node $v$, the condition in Step (3(b)) of algorithm GeneralizedNextRecursive is satisfied. Consequently, a new node named 7 is created as a child of the root, and assigned the state label $\{q_5\}$. The edge from the root to this child is annotated with 4, as shown in Figure (2-b2). Subsequently, Step (4) of algorithm GeneralizedNextRecursive removes $q_5$ from the state label of the leaf named 2 in Figure (2-b2). This is because the annotation of the edge from the root to this node is larger than that of the edge from the root to its sibling having the same $A$-state, $q_5$, in its state label. Removing $q_5$ from its state label causes the leaf named 2 in Figure (2-b2) to acquire an empty state label; hence this node is deleted in Step (5) of algorithm GeneralizedNextRecursive. This gives a tree with only two nodes – a root named 1 and a leaf named 7 with state label $\{q_5\}$. The condition in Step (6) is not satisfied; hence no nodes are “reset” and $U$ continues to be the empty set. In Step (7), the component $e$ finally acquires the value 2, since that is the smallest name of a node that is deleted. Once we return from algorithm GeneralizedNextRecursive to algorithm GeneralizedNext, the name-
compaction step assigns the name 2 to the leaf node that was named 7 earlier. Since no node is reset and the set \( U \) is empty, the updated value of the component \( f \) remains at 32. The resulting CGS tree obtained after reading the first \( b \) from the input word is shown in Figure (2-b)3).

On reading the next \( b \), a sequence of transformations similar to that described above results in a CGS tree with a root named 1 and a leaf named 2 with state label \( \{q_1\} \) and edge annotation 3. Here too, the component \( e \) acquires the value 2 and \( U \) remains empty, causing \( f \) to have the value 32. Figures (2-c1) to (2-c3) illustrate the steps in the construction of this CGS tree.

When the third \( b \) in the input word is read, the tree in Figure (2-c3) is extended in Steps (1), (2) and through the recursion in Step (5a) of algorithm \texttt{GeneralizedNextRecursive} to give the tree shown in Figure (3-d1) sans the nodes named 7 and 8. As the recursive calls to algorithm \texttt{GeneralizedNextRecursive} return in sequence, Step (3(b)) creates two new leaf nodes (albeit in different recursive calls) named 7 and 8, with state labels \( \{q_1\} \) and \( \{q_5\} \) respectively. The edges from the respective parents to the new leaves named 7 and 8 are annotated 0 and 4, respectively. The resulting tree is as shown in Figure (3-d1), except that the node named 6 no longer has \( q_1 \) or \( q_5 \) in its state label. In fact, Step (1) of algorithm \texttt{GeneralizedNextRecursive} removes both \( q_1 \) and \( q_5 \) (once again, in different recursive calls) from the state label of this node, leaving it with an empty state label. Subsequently, this node is removed in Step (6), giving the intermediate CGS tree shown in Figure (3-d2). Observe that the node named 5 in this tree has the edge to its sole child annotated 0. Therefore, this node is “reset” in Step (6) of algorithm \texttt{GeneralizedNextRecursive} and the child named 7 is deleted. Additionally, since the hope set for the node named 5 in Figure (3-d2) is \( \{1, 2, 3, 4, 5\} \setminus \{3, 5, 4, 2\} = \{1\} \), and since \( \{q_1\} \in \mathcal{F} \), we have \( P_\phi(\{q_1\}) = \text{True} \). Therefore, 5 is added to the set \( U \) in Step (6) of algorithm \texttt{GeneralizedNextRecursive}. Since the smallest name of a node that is deleted is 6, component \( e \) finally acquires the value 6 in Step (7). Once we return to algorithm \texttt{GeneralizedNext}, the name-compaction step renames the leaf node named 8 to 6, as shown in Figure (3-d3). The value of \( f \) is updated to \( \min(32, 5) = 5 \). The final CGS tree obtained after reading \( bbb \) is shown in Figure (3-d3). Figures (3-p) and (3-q) show the final CGS trees (states) obtained after reading \( bbbc \) and \( bbbcc \) respectively. For all subsequent \( c \)'s that are read from the input word, the CGS tree in Figure (3-q) is obtained. Therefore, the automaton \( \mathcal{D} \) loops infinitely in the state represented by Figure (3-q) after reading \( bbbcc \). Note that nodes named 5 and 8 are deleted only finitely often but appear as leaves infinitely often in the sequence of CGS trees (states) visited on reading the word \( bbbcc \). Interestingly, the hope sets of the nodes named 5 and

![Figure 3: Steps in determinization construction](image-url)
Consider the tree $T$ in Figure 14. We prove that the inf-sets of the runs of $A$ on the word $bbbe^\omega$. As we will see subsequently, this is not a coincidence, but a consequence of our construction.

Let $T$ be the set of all CGS tree over $A$. The parity acceptance condition for automaton $D$ is $P = \{F_0, F_1, \ldots, F_{30}\}$, where $F_0 = \{t \in T \mid f = 1, e > 1\}$, $F_{2i+1} = \{t \in T \mid e = i + 2, f \geq e\}$ for $0 \leq i < 30$, $F_{2i+2} = \{t \in T \mid f = i + 2, e > f\}$ for $0 \leq i < 30$, and $F_{61} = \{t \in T \mid e, f > 31\}$. If we let $\rho$ denote the run of $D$ on the word $bbbe^\omega$, then clearly $\text{inf}(\rho) \cap F_0 = \emptyset$, while $\text{inf}(\rho) \cap F_i = \emptyset$ for $0 \leq i < 8$. Therefore, $bbbe^\omega$ is accepted by $D$.

4. Proof of Correctness

Let $A = (\Sigma, Q, Q_0, \delta, \phi)$ be an $\omega$-automaton with acceptance condition based on infinity sets, and let $D$ be the corresponding DPW obtained by our construction. Let $\alpha \in \Sigma^\omega$ be an $\omega$-word, and let $\rho = t_0t_1t_2\ldots$ be the unique run of $D$ on $\alpha$. Here, $t_i = (N_i, M_i, r_i, p_i, \lambda_i, h_i, e_i, f_i)$ is the state (tree) of $D$ reached after reading the prefix $\alpha(0, i − 1)$ of $\alpha$.

We will first show that if $t_0$ is a CGS tree, as defined in Section 3.2, then every $t_i$, for $i \geq 0$, in $\rho$ is also a CGS tree. From algorithms GeneralizedNext and GeneralizedNextRecursive, it is easy to see that if $t_i$ is a rooted tree with nodes labeled by subsets of $Q$ and annotated with subsets of $||Q||$, then so is $t_{i+1}$, for all $i \geq 0$. Since $e_{i+1}$ and $f_{i+1}$ are initialized to $m + 1 = ||Q||^2 + ||Q|| + 2$ and subsequently updated to the smaller of their respective current value and the name of a node in $t_{i+1}$, it follows that $e_{i+1}$ and $f_{i+1}$ are always in $||Q||^2 + ||Q|| + 2$. Given these observations, it suffices to show the following three additional properties of $t_{i+1}$ in order to establish that $t_{i+1}$ is indeed a CGS tree.

1. There are no more than $||Q||^2 + ||Q|| + 1$ nodes in $t_{i+1}$. Since the name-compaction step of algorithm GeneralizedNext ensures the absence of gaps in the set of names eventually assigned to nodes of $t_{i+1}$, proving the above property guarantees that the range of the naming function $M_{i+1}$ is indeed $||Q||^2 + ||Q|| + 1$. We will defer the proof of this property to Section 5.

2. The (hope-set) annotation of every node in $t_{i+1}$ is contained in the annotation of its parent, and differs by at most one element from that of its parent. In addition, every non-leaf node $v$ in $t_{i+1}$ has at least one child with an annotation that is a strict subset of $h_{i+1}(v)$. The first property is proved in Lemma 6 below. The second property is a consequence of Lemma 2 and Step 6 of algorithm GeneralizedNextRecursive.

3. The state label of every node in $t_{i+1}$ is the union of state labels of its children in $t_{i+1}$. In addition, the state labels of sibling nodes in $t_{i+1}$ are mutually disjoint. We will prove the first property in Lemma 6 below. The second property is a consequence of Step 4 of algorithm GeneralizedNextRecursive and the fact that no step of algorithm GeneralizedNextRecursive adds any element to an already existing state label of a node.

Lemma 2. For every $i \geq 0$ and for every node $u$ and its child $v$ in $t_i$, $h_i(v) \subseteq h_i(u)$ and $|h_i(u) \setminus h_i(v)| \leq 1$.

Proof: We will prove the lemma by induction on the indices of $t_0, t_1, \ldots$.

Base Case: For the tree $t_0$ with only the root node $r_0$, the claim in the lemma holds vacuously since there are no nodes with children in $t_0$.

Hypothesis: We assume that the claim in the lemma holds for $t_i$, where $i \geq 0$.

Induction: Consider the tree $t_{i+1}$ obtained by applying algorithm GeneralizedNext to $t_i$. From the pseudocode of algorithms GeneralizedNext and GeneralizedNextRecursive, we observe that the hope set of a node in $t_{i+1}$ can be updated only in Step 2 or Step 6 of algorithm GeneralizedNextRecursive. In both these steps, the node whose hope set is updated is a newly created node that is added as a child of the current node.

Now let $v$ be an arbitrary node in $t_{i+1}$. We consider two cases below.

- Suppose $v \in N_i \cap N_{i+1}$. Thus, $v$ was present in $t_i$ and was not deleted in the process of transforming $t_i$ to $t_{i+1}$. Since deletion of a node (Step 4 or Step 6 of algorithm GeneralizedNextRecursive) entails deletion of all descendants of the node as well, the fact that $v$ was not deleted implies that no ancestor of $v$ was deleted either in the process of transforming $t_i$ to $t_{i+1}$. Thus, both $v$ and its parent, say $u$, in
Suppose $v$ newly created in the process of transforming $t_i$ to $t_{i+1}$. Since new nodes can be created only in Step 2 or Step 3(b)i of the recursive algorithm GeneralizedNextRecursive, $v$ must be created in one of these steps. From the pseudocode of algorithm GeneralizedNextRecursive, it is easy to see that both these steps set $h_{i+1}(v)$ to $h_{i+1}(u) \setminus \{ k \}$, where $u$ is the parent of $v$ in $t_{i+1}$ and $k \in \{ 0 \} \cup h_{i+1}(u)$. It follows that $h_{i+1}(v) \subseteq h_{i+1}(u)$ and $|h_{i+1}(v) \setminus h_{i+1}(u)| \leq 1$.

This can give rise to a situation wherein

$v\text{ label of any child of state labels of its children. However, such a violation is only temporary and is rectified by the time the}$

We will prove the lemma by induction on the indices of $t_0, t_1, \ldots$.

**Base Case:** For the tree $t_0$ with only the root node $r_0$, the claim in the lemma holds vacuously since there are no non-leaf nodes in $t_0$.

**Hypothesis:** We assume that the claim in the lemma holds for $t_i$, where $i \geq 0$.

**Induction:** Consider the tree $t_{i+1}$ obtained by applying algorithm GeneralizedNext to $t_i$. Since the claim in the lemma holds for $t_i$ (by induction hypothesis), and since the initialization step of algorithm GeneralizedNext replaces the state label of every node $v$ with $\delta(\lambda_i(v), \sigma)$, it follows that the state label of every non-leaf node continues to be the union of state labels of its children even after the initialization step. Since no nodes are added or deleted, and the state labels of no nodes are changed in Steps 3 and 4 of algorithm GeneralizedNext (i.e., during name-compaction and updation of component $f$), the inductive claim can be proved by establishing that Step 3 of algorithm GeneralizedNext does not violate the claim. This amounts to showing that algorithm GeneralizedNextRecursive preserves the property that the state label of every node is the union of state labels of its children. We therefore focus on the steps of algorithm GeneralizedNextRecursive below.

Clearly, Step 3 of algorithm GeneralizedNextRecursive preserves the desired property. Although Step 2 results in the creation of a new child $v'$ of $v$, the desired property is preserved, since the state label of $v'$ is set to that of $v$. In Step 3(b)i, new children may be created for $v$, but the union of state labels of children of $v$ remains unchanged. This is because for every new child $v'$ that is created, Step 3(b)i sets the state label of $v'$ to $\{ q \}$, where $q$ is in the state label of an already existing child of $v$. Step 3(b)i presents a more interesting situation. Let $v_k$ be a child of $v$ such that the annotation on the edge from $v$ to $v_k$ is $q_k$. From Lemma 2 and from the definition of edge annotations, we know that $h_{i+1}(v) = h_{i+1}(v_k) \cup \{ q_k \}$. If a state $q$ in the state label of $v_k$ is such that $q \neq q_j$ and $q \notin Q_{h_{i+1}(v_k)}$, Step 3(b)i of algorithm GeneralizedNextRecursive removes $q$ from the state label of $v_k$ and from the state labels of all its descendants. This can give rise to a situation wherein $q$ is in the state label of $v$ (parent of $v_k$) but not in the state label of any child of $v$, potentially violating the property that the state label of every node is the union of state labels of its children. However, such a violation is only temporary and is rectified by the time the recursion of algorithm GeneralizedNextRecursive terminates. To see why this is so, notice that since $q \neq q_j$ and $q \notin Q_{h_{i+1}(v_k)}$, we must have $q \notin Q_{h_{i+1}(v)} = Q_{h_{i+1}(v_k)} \cup \{ q_j \}$. Hence, when the recursion of algorithm GeneralizedNextRecursive returns to the level where the current node is the parent $u$ of node $v$ in $t_{i+1}$, we have two possibilities.

1. Suppose $q = q_r$, where $r$ is the annotation of the edge from $u$ to $v$ in $t_{i+1}$. In this case, Step 3(b)i of algorithm GeneralizedNextRecursive creates a new child $v'$ of $u$ with state label $\{ q \}$, and with an edge annotation that is smaller than $r$. This eventually causes $q$ to be removed from the state label of $v$ in Step 3(b)i of algorithm GeneralizedNextRecursive.

2. Suppose $q \neq q_r$, where $r$ is the annotation of the edge from $u$ to $v$ in $t_{i+1}$, since $q \notin Q_{h_{i+1}(v)}$ as well, Step 3(b)i of algorithm GeneralizedNextRecursive removes $q$ from the state label of $v$. 

Therefore, by the principle of mathematical induction, the claim in the lemma holds for all $t_i$, where $i \geq 0$. \[ \square \]
Therefore, if \( q \) is removed from the state label of a child \( v_q \) of \( v \) by Step (b)(iii) of algorithm GeneralizedNextRecursive, then it is also eventually removed from the state label of \( v \). This ensures that the desired property of the state label of a node being the union of state labels of its children is eventually preserved. Step (1) of algorithm GeneralizedNextRecursive can remove a state \( q \) from the state label of a node, but only if \( q \) is also present in the state label of a sibling node. Hence, Step (1) cannot change the union of state labels of children of a node. Step (2) deletes nodes with an already empty state label, while Steps (3) and (4) do not modify the state label of any node. Step (6) can cause a non-leaf node to turn into a leaf node, but this does not affect the desired property, which relates only to non-leaf nodes.

Thus, if algorithm GeneralizedNextRecursive is invoked on a tree in which the state label of every node is the union of state labels of its children, the algorithm preserves this property after it has transformed the tree recursively. This, coupled with the inductive hypothesis, implies that \( t_{i+1} \) satisfies the inductive claim.

Therefore, by the principle of mathematical induction, the claim in the lemma holds for all \( t_i \), where \( i \geq 0 \).

CGS trees encountered along a run of \( \mathcal{D} \) have several interesting properties that are useful in proving the correctness of our construction. We will prove these properties below by considering an arbitrary run \( \rho = t_0t_1t_2 \ldots \) of \( \mathcal{D} \) and by inductively showing that the respective properties hold for every CGS tree \( t_i \) along \( \rho \).

**Proposition 4.** For every \( i \geq 0 \), for every \( v \in \mathcal{N}_i \) and for every \( q \in \lambda_i(v) \), there is a run of the automaton \( A \) from some \( q_0 \in \mathcal{Q}_0 \) to \( q \) on the prefix \( \alpha(0, i - 1) \).

**Proof:** We will prove this by induction on the indices of \( t_0, t_1, \ldots \).

**Base Case:** For the tree \( t_0 \) with only the root node \( r_0 \), the claim in the lemma holds trivially, since \( \lambda_0(r_0) = \mathcal{Q}_0 \) by definition.

**Hypothesis:** We assume that the claim in the lemma holds for \( t_i \), where \( i \geq 0 \).

**Induction:** Consider the tree \( t_{i+1} \) obtained by applying algorithm GeneralizedNext to \( t_i \). We know from the initialization step (Step (1) of algorithm GeneralizedNext that the state label of \( r_{i+1} \) is initially set to \( \delta(\lambda_i(r_i), \alpha_i) \). We also know from the pseudocode of algorithm GeneralizedNextRecursive that invoking this algorithm on a CGS tree rooted at a node \( v \) does not change the state label of \( v \). Since Step (2) of algorithm GeneralizedNext invokes algorithm GeneralizedNextRecursive on the CGS tree rooted at \( r_{i+1} \), the state label of \( r_{i+1} \) remains unchanged at \( \delta(\lambda_i(r_i), \alpha_i) \) after the call to GeneralizedNextRecursive returns. Subsequently, neither Step (3) nor Step (4) of algorithm GeneralizedNext changes the state label of any node in \( t_{i+1} \).

Therefore, \( \lambda_{i+1}(r_{i+1}) = \delta(\lambda_i(r_i), \alpha_i) \). Now let \( v \) be an arbitrary node in \( t_{i+1} \) and let \( q \in \lambda_{i+1}(v) \). By Lemma 3, we know that \( q \in \lambda_{i+1}(r_{i+1}) = \delta(\lambda_i(r_i), \alpha_i) \). By the inductive hypothesis, for every \( q' \in \lambda_i(r_i) \), there is a run of \( A \) from some \( q_0 \in \mathcal{Q}_0 \) to \( q' \) on the prefix \( \alpha(0, i - 1) \). Therefore, there is a run of \( A \) from some \( q_0 \in \mathcal{Q}_0 \) to \( q \in \delta(\lambda_i(r_i), \alpha_i) \) on the prefix \( \alpha(0, i) \).

By the principle of mathematical induction, the claim in the lemma holds for all \( i \geq 0 \).

**Lemma 5.** For every \( i \geq 0 \) and for every \( v \in \mathcal{N}_i \) such that \( v \) is a non-leaf node of \( t_i \), we have \( h_i(v) \neq \emptyset \).

**Proof:** From Lemma 2 and Step 3 of algorithm GeneralizedNextRecursive, it follows that if \( v \) is a non-leaf node of \( t_i \), it must have a child \( v' \) such that \( h_i(v') \) is a strict subset of \( h_i(v) \). This immediately implies that \( h_i(v) \neq \emptyset \).

**Lemma 6.** Let \( m = |\mathcal{Q}|^2 + |\mathcal{Q}| + 1 \). For every \( i \geq 0 \), if \( f_i < m + 1 \), there exists a leaf node \( v \) in \( t_i \) with name \( M_i(v) = f_i \), such that \( h_i(v) \neq \emptyset \).

**Proof:** We will prove the lemma by induction on the indices of \( t_0, t_1, \ldots \).

**Base Case:** For the CGS tree \( t_0 \) with only the root node \( r_0 \), the claim in the lemma holds vacuously since \( f_0 = m + 1 \).

**Hypothesis:** We assume that the claim in the lemma holds for \( t_i \), where \( i \geq 0 \).

**Induction:** Consider the CGS tree \( t_{i+1} \) obtained by applying algorithm GeneralizedNext to \( t_i \). The value of \( f_{i+1} \) is set in Step 4 of algorithm GeneralizedNext to the smaller of \( m + 1 \) and the smallest name of a
node added to the set $U$ in Step (6) of algorithm GeneralizedNextRecursive. Therefore, if $f_{i+1} < m + 1$, a node $v$ with $M_{i+1}(v) = f_{i+1}$ must have been added to the set $U$ in Step (6) of a recursive call of algorithm GeneralizedNextRecursive. Furthermore, $v$ must have been the root node of the CGS sub-tree transformed by this specific recursive call. The condition in Step (6) of algorithm GeneralizedNextRecursive requires that all children of $v$ must have their hope set equal to $h_{i+1}(v)$ (or alternatively, the annotations on all edges from $v$ to its children must be $\emptyset$). Therefore, $v$ must have been a non-leaf node prior to being “reset” in Step (6) of algorithm GeneralizedNextRecursive. We now consider two cases below depending on whether the node $v$ was present in $t_i$ or not, and show that $h_{i+1}(v) \neq \emptyset$ in both cases.

- Suppose $v \in N_i \cap N_{i+1}$. By the argument given in the proof of Lemma (2), we know that $h_i(v) = h_{i+1}(v)$. If $v$ was a non-leaf node in $t_i$, by Lemma (5), $h_i(v) \neq \emptyset$. Hence, $h_{i+1}(v) \neq \emptyset$ as well. If $v$ was a leaf node in $t_i$, we could either have $h_i(v) = \emptyset$ or $h_i(v) \neq \emptyset$. In the latter case, we easily get $h_{i+1}(v) = h_i(v) \neq \emptyset$. In the former case, we note that $v$ cannot become a non-leaf node prior to Step (6) of algorithm GeneralizedNextRecursive in the process of transforming $t_i$ to $t_{i+1}$. This is because Step (1) of algorithm GeneralizedNextRecursive prevents any children from being added to $v$ if $h_i(v) = \emptyset$. Therefore, $h_i(v)$ must have been non-empty in $t_i$, and the claim in the lemma follows.

- If $v$ is newly created in the process of transforming $t_i$ to $t_{i+1}$, then by the argument used in the proof of Lemma (2), $v$ must have been created either in Step (2) or in Step (3(b)) of algorithm GeneralizedNextRecursive. If $v$ was created as a leaf node in Step (3(b)), it could not have become a non-leaf node prior to execution of Step (6). This is because algorithm GeneralizedNextRecursive is not called recursively on any leaf node created in Step (3(b)). If $v$ was created as a leaf node in Step (2), the only way it could have become a non-leaf node prior to execution of Step (6) is by a recursive invocation of algorithm GeneralizedNextRecursive on this node in Step (3). However, Step (1) of algorithm GeneralizedNextRecursive ensures that such a recursive invocation adds a child to $v$ only if the hope set of $v$ is non-empty. Therefore, we must have $h_{i+1}(v) \neq \emptyset$.

Since node $v$ is “reset” and all descendants of $v$ are deleted in Step (6) of algorithm GeneralizedNextRecursive, $v$ becomes a leaf node at the end of Step (6). Furthermore, since $t_i$ and $t_{i+1}$ are trees, every node has a unique parent in $t_i$ and $t_{i+1}$, and hence, algorithm GeneralizedNextRecursive is recursively invoked at most once on a node in Step (6). It follows that after node $v$ is “reset” and turned into a leaf by a recursive call of algorithm GeneralizedNextRecursive, there are no subsequent recursive calls to GeneralizedNext with $v$ as the root of a CGS sub-tree to be transformed. From the pseudocode of algorithm GeneralizedNext, we note that this implies that no child gets added to $v$ after it is “reset”. Therefore, $v$ either remains as a leaf node in $t_{i+1}$ or is subsequently deleted in the process of transforming $t_i$ to $t_{i+1}$. However, since $f_{i+1}$ is set to $M_{i+1}(v)$, we know from Step (1) of algorithm GeneralizedNext that $v$ is present in $N_{i+1}$. Therefore, $v$ is a leaf node in $t_{i+1}$ with $M_{i+1}(v) = f_{i+1}$ and $h_{i+1}(v) \neq \emptyset$.

By the principle of mathematical induction, the claim in the lemma holds for all $t_i$, where $i \geq 0$.

**Lemma 7.** Let $\alpha$ be an $\omega$-word and let $p = t_0t_1 \ldots$ be the unique run of $D$ on $\alpha$. Let $i, k$ be indices and let $v$ be a node such that: (i) $i < k$, (ii) for all $z \in \{i, i+1, \ldots k\}$, node $v$ is present in $t_z$ and $h_z(v) \neq \emptyset$, and (iii) node $v$ is a leaf in both $t_i$ and $t_k$, and is a non-leaf node in all $t_z$, where $i < z < k$. Then the following claims hold.

1. Node $v$ is “reset” in the process of transforming $t_{k-1}$ to $t_k$.
2. For every $q' \in \lambda_k(v)$, there is a $q \in \lambda_i(v)$ such that there is a run $\psi$ of $A$ on $\alpha(i, k-1)$ with $\psi(0) = q$, $\psi(k-i) = q'$, and $\psi(z-i) \in \lambda_z(v)$ for all $z \in \{i, i+1, \ldots k\}$.
3. For every run $\psi$ of $A$ on the word segment $\alpha(i, k-1)$ such that $\psi(z-i) \in \lambda_z(v)$ for all $z \in \{i, i+1, \ldots k\}$, all states in $Q_{h_i(v)}$ are visited in $\psi$.

**Proof:**

1. We will prove this claim by contradiction. If possible, suppose $v$ becomes a leaf node in $t_k$ without being “reset” in the process of transforming $t_{k-1}$ to $t_k$. Consider the case when $k = i + 1$. Since $v$ is a
leaf in \( t_k \) and \( h_i(v) \neq \emptyset \), Step (2) of algorithm \texttt{GeneralizedNextRecursive} creates at least one child of \( v \) with the same non-empty state label as that of \( v \) when \texttt{GeneralizedNextRecursive} is invoked with \( v \) as the root of the CGS subtree to be transformed. If \( k > i + 1 \), then since \( v \) is a non-leaf node in \( t_{k-1} \), there is at least one child of \( v \) with a non-empty state label in \( t_{k-1} \). By Lemma (3), the state label of \( v \) in this case is also the union of state labels of its children in \( t_{k-1} \). Thus, in either case, there is an intermediate step during the transformation of \( t_{k-1} \) to \( t_k \) when \( v \) has one or more children with non-empty state labels, and the union of state labels of its children equals the state label of \( v \). All these children must eventually be deleted before \( v \) becomes a leaf node in \( t_k \).

From the pseudocode of algorithm \texttt{GeneralizedNextRecursive}, we note that the only steps that delete nodes from a CGS tree are Step (5) and Step (6). Since \( v \) exists in \( t_k \) and is assumed not to have been “reset” in the process of transforming \( t_{k-1} \) to \( t_k \), its children could not have been deleted in Step (5).

Therefore, all its children must have been deleted in Step (4) of algorithm \texttt{GeneralizedNextRecursive}. This requires all children of \( v \) to acquire the empty state label. We know from above that there exist one or more children of \( v \) with non-empty state labels in an intermediate step during the transformation of \( t_{k-1} \) to \( t_k \). The state labels of all such children must therefore be emptied before they can be deleted in Step (5).

From the pseudocode of algorithm \texttt{GeneralizedNextRecursive}, the only steps that remove states from the state labels of nodes are Step (1(b)(ii)) and Step (4). Unfortunately, Step 4 simply removes duplicates from the state labels of siblings, and cannot render the state labels of all children of \( v \) empty. Therefore, Step (1(b)(ii)) must eventually be responsible for emptying the state labels of all children of \( v \). However, we know from the proof of Lemma (3) that if a state is removed from the state label of a child of \( v \) in Step (1(b)(ii)) of algorithm \texttt{GeneralizedNextRecursive}, then that state is eventually removed from the state label of \( v \) as well. Since the state label of \( v \) equals the union of state labels of all its children at an intermediate step in the transformation of \( t_{k-1} \) to \( t_k \), the above implies that all states in the state label of \( v \) must eventually be removed in the process of transforming \( t_{k-1} \) to \( t_k \). This, in turn, implies that \( v \) is removed from \( t_k \) in Step (4) of algorithm \texttt{GeneralizedNextRecursive} – a contradiction!

2. Since node \( v \) is present in all \( t_z \), for \( z \in \{i, i+1, \ldots, k\} \), it follows from Step (1) that \( \lambda_i(v) \) is always initialized to \( \delta(\lambda_{r-1}(v), \alpha_{r-1}) \), for \( r \in \{i+1, \ldots, k\} \). Since no other step of algorithm \texttt{GeneralizedNext} or algorithm \texttt{GeneralizedNextRecursive} adds states to the state label of an already existing node, the claim now follows from an easy induction on \( z \in \{i, i+1, \ldots, k\} \).

3. From the pseudocodes of algorithms \texttt{GeneralizedNext} and \texttt{GeneralizedNextRecursive}, we note that since node \( v \) exists in \( t_z \) for all \( z \in \{i, i+1, \ldots, k\} \), the hope set of \( v \) must stay unchanged, i.e., \( h_i(v) = h_z(v) \) for all \( z \in \{i, i+1, \ldots, k\} \). Now let \( r \) be an arbitrary index such that \( i \leq r < k \). Suppose node \( v \) has a child \( v' \) in a (possibly intermediate) step of algorithm \texttt{GeneralizedNextRecursive} during the transformation of \( t_r \) to \( t_{r+1} \). Suppose further that the edge from \( v \) to \( v' \) is annotated with \( j \) and the state label of \( v' \) is \( S \) in this step. We will first prove the following claim.

\textbf{Claim 1:} For every run \( \psi \) of \( \mathcal{A} \) on \( \alpha(i, r) \) such that \( \psi(z-i) \in \lambda_z(v) \) for all \( z \in \{i, \ldots, r-1\} \) and \( \psi(r-i) \in S \), all states in \( \{q_n, q_{n-1}, \ldots, q_{j+1}\} \cap Q_{h_i(v)} \) are visited in \( \psi \). The proof is by induction on \( r \).

\textit{Base Case:} We know that \( v \) is a leaf node in \( t_i \), with \( h_i(v) \neq \emptyset \). Therefore, during the transformation of \( t_i \) to \( t_{i+1} \), Step (2) of algorithm \texttt{GeneralizedNextRecursive} creates a child \( v' \) of \( v \) and adds all states in \( \delta(\lambda_i(v), \alpha_i) \) to the state label of \( v' \). In addition, the edge from \( v \) to \( v' \) is annotated with \( j = \max(h_i(v)) > 0 \). This implies that \( \{q_n, q_{n-1}, \ldots, q_{j+1}\} \cap Q_{h_i(v)} = \emptyset \). Hence, the claim follows vacuously. Suppose additional children of \( v \) are subsequently created in Step (1(b)(ii)) of algorithm \texttt{GeneralizedNextRecursive}. Since \( v \) is a leaf in \( t_i \), it can be seen from the pseudocode of algorithm \texttt{GeneralizedNextRecursive} that prior to execution of Step (1(b)(ii)), \( v \) could have had only a single child – the one created in Step (2), with the edge from \( v \) to this child annotated with \( j = \max(h_i(v)) \). In order for a new child of \( v \), say \( v'' \), to be created in Step (1(b)(ii)), we note from the pseudocode of algorithm \texttt{GeneralizedNextRecursive} that the state label of \( v'' \) must be \( \{q_j\} \) and the annotation of the edge from \( v \) to \( v'' \) must be \( l = \max(\{0\} \cup (h_i(v) \cap \{1, 2, \ldots, j-1\})) \). Since \( j = \max(h_i(v)) \), it follows that \( \{q_n, q_{n-1}, \ldots, q_{j+1}\} \cap Q_{h_i(v)} = \{q_j\} \). Since the state label of \( v'' \) is also \( \{q_j\} \), the claim is easily seen to hold for \( v'' \). Since no other step of algorithm \texttt{GeneralizedNextRecursive} or algorithm \texttt{GeneralizedNext}
To complete the proof of Lemma (7-3), we note from Lemma (7-refclaim1a) that $v$ is “reset” during the transformation of $t_{k-1}$ to $t_k$. Therefore, from Step (3) of algorithm GeneralizedNextRecursive, $v$ must have had at least one child with non-empty state label prior to being “reset”. In addition, the annotations of all edges from $v$ to its children with non-empty state labels must have been $0$ prior to the resetting of $v$. It then follows from Claim 1 that for every run $\psi$ of $A$ such that $\psi(z - i) \in \lambda_z(v)$

 adds any state to the state label of $v''$, this proves the base case of the induction.

**Hypothesis:** We assume that the claim is true for $r$, where $i \leq r < k - 1$.

**Induction:** Consider the transformation of $t_{r+1}$ to $t_{r+2}$. Let $v'$ be a child of $v$ in some step of algorithm GeneralizedNextRecursive during this transformation. Suppose further that the edge from $v$ to $v'$ is annotated with $j$ and the state label of $v'$ is $S$ in this step. We consider two cases below.

- **If** $v'$ is present in $t_{r+1}$, then by the argument used in the proof of Lemma (2), $v$ must also have been present in $t_{r+1}$, with $h_{r+1}(v) = h_{r+2}(v)$ and $h_{r+1}(v') = h_{r+2}(v')$. Therefore, by the definition of edge annotations, the edge from $v$ to $v'$ must have been annotated with $j$ in $t_{r+1}$ as well. Step (1) of algorithm GeneralizedNext ensures that the state label of $v'$ is initialized to $\delta(\lambda_r + 1(v), \alpha_{r+1})$ during the transformation of $t_{r+1}$ to $t_{r+2}$. This, along with the inductive hypothesis, and the facts that $h_{r+1}(v) = h_{r+2}(v)$ and the edge annotations from $v$ to $v'$ are the same in $t_{r+1}$ and in $t_{r+2}$, imply that the claim holds for $v'$ after the initialization step during the transformation of $t_{r+1}$ to $t_{r+2}$. Since no other step of algorithm GeneralizedNextRecursive or algorithm GeneralizedNext adds any state to the state label of $v'$, this proves the inductive claim for $v'$.

- **If** $v'$ is not present in $t_{r+1}$, it must have been created as a child of $v$ in Step (2) or in Step (3(b)) of algorithm GeneralizedNextRecursive during the transformation of $t_{r+1}$ to $t_{r+2}$. Since $i < r + 1 < k$ (by the condition in our inductive hypothesis), we know that $v$ is a non-leaf node in $t_{r+1}$. Therefore, $v'$ could not have been created in Step (2) of algorithm GeneralizedNextRecursive (this step requires $v$ to be a leaf node in $t_{r+1}(v)$). Hence, $v'$ must have been created in Step (3(b)).

From the pseudocode of algorithm GeneralizedNextRecursive, we note that when $v'$ is created as a child of $v$ in Step (3(b)), the state label of $v'$ is set to $\{q_j\}$, where $j$ is the annotation of the edge from $v$ to an already existing child $v_x$, and $q_j$ is in the state label of $v_x$ at the time of creation of $v'$. In addition, the annotation of the new edge from $v$ to $v'$ is set to $l = \max\{0 \cup (h_{r+2}(v) \cap \{1, 2, \ldots, j_x - 1\})\}$. Since $v$ is a non-leaf node in $t_{r+1}$, the child $v_x$ itself could not have been created in Step (2) of algorithm GeneralizedNextRecursive during the transformation of $t_{r+1}$ to $t_{r+2}$. It could not have been created in Step (3(b)) of algorithm GeneralizedNextRecursive either, since Step (3(b)) of algorithm GeneralizedNextRecursive iterates over the children of $v$ existing prior to execution of Step (3). Therefore, the child $v_x$ of $v$ must be present in $t_{r+1}$.

Since $v$ and $v_x$ are present in both $t_{r+1}$ and in the intermediate CGS tree at the time of creation of $v'$, the hope sets of $v$ and $v_x$, and the annotation of the edge from $v$ to $v_x$ must be the same in $t_{r+1}$ and in the intermediate CGS tree. This implies that the edge from $v$ to $v_x$ is annotated with $j_x$ in $t_{r+1}$. By virtue of Step (1) of algorithm GeneralizedNext, we also know that there is a state $q' \in \lambda_{r+1}(v_x)$ such that $q_j \in \delta(q', a_{r+1})$. This, along with the inductive hypothesis, and the facts that $h_{r+1}(v) = h_{r+2}(v)$ and the annotation of the new edge from $v$ to $v'$ is $l = \max\{0 \cup (h_{r+2}(v) \cap \{1, 2, \ldots, j_x - 1\})\}$, imply that for every run $\psi$ of $A$ on $\alpha(i, r+1)$ such that $\psi(z - i) \in \lambda_z(v)$ for $z \in \{i, \ldots, r\}$ and $\psi(r + 1 - i) = q_{j_x}$, all states in $\{q_n, q_{n-1}, \ldots, q_{l+1}\} \cap Q_{h_{r+2}(v)}$ are visited.

From the pseudocode of algorithm GeneralizedNextRecursive, no step other than Step (3(b)) adds any state to the state label of $v'$ after it is created in Step (3(b)). Therefore, $v'$ has at most one state, $q_{j_x}$, in its state label in any intermediate step of algorithm GeneralizedNextRecursive during the transformation of $t_{r+1}$ to $t_{r+2}$. We have already considered the case of $q_{j_x}$ in the state label of $v'$ above. Hence, this proves the inductive claim for $v'$ and also completes the proof of Claim 1.
for all $z \in \{i, k - 1\}$ and $\psi(k - i)$ is in the state label of some child of $v$ prior to it being reset, all states in \{q$_{n-1}$, ..., q$_1$\} $\cap Q_{h_i(v)} = Q_{h_i(v)}$ are visited in $\psi$.

This does not prove Lemma (7.3) yet, since we must show the above result for $\psi(k - i) \in \lambda_k(v)$.

We have seen earlier, in the proof of Lemma (3), that the state label of a node $v$ may temporarily contain states that are not in the state labels of any of its children after intermediate steps of algorithm GeneralizedNextRecursive. However, we also saw in the same proof that all such states are eventually removed from the state label of $v$ after all recursive invocations of GeneralizedNextRecursive have returned. Therefore, proving the claim of Lemma (7.3) for $\psi(k - i)$ in the state labels of children of $v$ prior to $v$ being “reset” proves Lemma (7.3) itself.

**Lemma 8.** Let $\alpha$ be an $\omega$-word and let $\rho = t_0 t_1 \ldots$ be the unique run of $D$ on $\alpha$. For every $i \geq 0$ and for every node $v$ in $t_i$, $\lambda_i(v) \subseteq Q_{h_i(v)}$.

**Proof:** We will prove this claim by contradiction. Suppose there exists an $i \geq 0$ and a node $v$ in $t_i$ such that $q_l \in \lambda_i(v)$ although $l \notin h_i(v)$. Clearly, $v$ cannot be the root, $r_i$, of $t_i$, since $h_i(r_i) = h_0[r_0] = [n]$ contains the indices of all states of $A$. Therefore, $v$ must have a parent, say $u$, in $t_i$. Recalling that $t_0$ has only a single node (i.e., $r_0$) without any parent, we can immediately infer that $i > 0$. In other words, there exists a CGS tree $t_{i-1}$ such that $t_i$ is obtained by applying algorithm GeneralizedNext to $t_{i-1}$.

From the pseudocode of algorithm GeneralizedNextRecursive, we observe that during the transformation of $t_{i-1}$ to $t_i$, the only nodes in $t_i$ on which the recursive algorithm GeneralizedNextRecursive is not recursively invoked are those that are generated in Step (3(b)). Furthermore, every node generated in Step (3(b)) is either deleted or survives as a leaf in the transformation of $t_i$ to $t_{i+1}$. Since node $u$ is a non-leaf node in $t_i$, algorithm GeneralizedNextRecursive must have been invoked with $u$ as the root of the CGS subtree to be transformed, during the transformation of $t_{i-1}$ to $t_i$.

Let $j$ be the annotation of the edge from $u$ to $v$ in $t_i$. There are two possibilities that we consider separately below.

- Suppose $v$ is created during the transformation of $t_{i-1}$ to $t_i$. This can happen either in Step (2) or in Step (3(b)) of the recursive invocation of algorithm GeneralizedNextRecursive with $u$ as the root of the CGS subtree to be transformed.
  - If $v$ is created in Step (3(b)), it follows from the pseudocode of algorithm GeneralizedNextRecursive that $\lambda_j(v) = \{q_l\}$, where $l > 0$ is the annotation of an edge from $u$ to an already existing child, say $v''$, of $u$. In addition, $h_i(v)$ is set to $h_i(u) \setminus \{\text{max}(\{h_i(u) \cup \{0\} \cap \{0, 1, 2, \ldots, l - 1\})\}$. By the definition of edge annotations, $l \in h_i(u) \setminus h_i(v'')$ and hence $l \in h_i(u)$. It then follows that $l \in h_i(u) \setminus \{\text{max}(\{h_i(u) \cup \{0\} \cap \{0, 1, 2, \ldots, l - 1\})\} = h_i(v)$ as well. Therefore, $\lambda_i(v) \subseteq Q_{h_i(v)}$. Since no other step of algorithm GeneralizedNextRecursive adds any state to $\lambda_i(v)$ subsequently, we have $\lambda_i(v) \subseteq Q_{h_i(v)}$. This gives us a contradiction!
  - If $v$ is created in Step (2), then Step (3(b)) must subsequently be executed in the same recursive invocation of GeneralizedNextRecursive with $u$ as the root of the CGS subtree to be transformed. This is similar to the case considered below wherein $v$ exists in $t_{i-1}$, and Step (3(b)) is executed in the recursive invocation of GeneralizedNextRecursive with $u$ as the root of the CGS subtree to be transformed.

- Suppose $v$ exists in $t_{i-1}$. It follows that the parent, $u$, of $v$ must also exist in $t_{i-1}$. Consider Step (3(b)) in the recursive invocation of algorithm GeneralizedNextRecursive with $u$ as the root of the CGS subtree, during the transformation of $t_{i-1}$ to $t_i$. We have two sub-cases to consider.
  - If $j = l$, a new child, say $v'$, of $u$ is been created in Step (3(b)), the state label of $v'$ is set to $\{q_l\}$ and the edge from $u$ to $v'$ is annotated with an index $< l$. This implies that in Step (1) of algorithm GeneralizedNextRecursive, $q_l$ is removed from the state label of $v$. Since no other step of
Algorithm \texttt{GeneralizedNextRecursive} adds states to \( \lambda_i(v) \) subsequently, it follows that \( q_l \notin \lambda_i(v) \). This gives a contradiction!

Suppose \( j \neq l \). Since \( l \) is also not in \( h_l(v) \), it follows that in Step [3(b)] of the recursive invocation of \texttt{GeneralizedNextRecursive} with \( u \) as the root of the CGS subtree to be transformed, \( q_l \) is removed from \( \lambda_i(v) \). By the same argument used above, \( q_l \) cannot be subsequently added to \( \lambda_l(v) \). Hence, \( q_l \notin \lambda_l(v) - a \) contradiction again!

We have therefore shown that there is no \( i \geq 0 \) and no node \( v \) in \( t_i \) such that \( q_l \in \lambda_i(v) \) and \( l \notin h_i(v) \). \( \square \)

Armed with the above properties of CGS trees encountered along a run of \( D \), we will now show that the languages accepted by \( D \) and \( A \) are the same. As before, let \( \alpha \) be an \( \omega \)-word in \( L(D) \) and let \( \rho = t_0t_1 \ldots \) be the unique run of \( D \) on \( \alpha \). By definition of the acceptance condition for \( D \), there exists an even index \( 2a + 2 \), where \( 0 \leq 2a + 2 < 2m - 1 \), such that CGS trees from the parity acceptance set \( F_{2a+2} \) are seen infinitely often along \( \rho \), while CGS trees from all parity acceptance sets \( F_y \), where \( 0 \leq y < 2a + 2 \), are seen only finitely often along \( \rho \). Let \( i^* \) be the smallest index \((\geq 0)\) such that all CGS trees \( t_i \) for \( i > i^* \) are outside \( \bigcup_{0 \leq y < 2a+2} F_y \).

The following lemma describes important properties of the suffix \( t_i, t_{i+1}, \ldots \) of \( \rho \), where \( i > i^* \).

**Lemma 9.** Let \( i \) and \( i' \) be indices such that (i) \( 0 \leq i^* < i < i' \), (ii) both \( t_i \) and \( t_{i'} \) are in \( F_{2a+2} \), and (iii) \( t_z \notin F_{2a+2} \) for all \( z \in \{i + 1, \ldots, i' - 1\} \). Then there exists a node \( v \) such that the following hold.

1. \( v \) is present in \( t_z \) for all \( z \in \{i, i + 1, \ldots, i'\} \). In addition, \( M_z(v) = a + 2 \) and \( h_z(v) = h_i(v) \neq 0 \) for all \( z \in \{i, i + 1, \ldots, i'\} \).
2. \( v \) is a non-leaf node in \( t_{i+1} \), for all \( z \in \{i + 1, \ldots, i' - 1\} \).
3. For every state \( q' \in \lambda_{i'}(v) \), there is some state \( q \in \lambda_i(v) \) such that there is a run of \( A \) from \( q \) to \( q' \) on \( \alpha(i, i' - 1) \) that visits all and only states in \( Q_{h_i(v)} \).

**Proof:**

1. Since both \( t_i \) and \( t_{i'} \) are in \( F_{2a+2} \), it follows from the definition of even-indexed parity acceptance sets that \( f_z = f_{i'} = a + 2 \). Also, since \( 0 \leq 2a + 2 < 2m - 1 \), we have \( 1 \leq a + 2 \leq m \). Therefore, by Lemma [2], both \( t_i \) and \( t_{i'} \) contain a leaf node with name \( a + 2 \) and with a non-empty hope set. Since \( i^* < i < i' \), it follows from the definition of \( i^* \) that for all \( z \in \{i, i + 1, \ldots, i'\} \), the CGS tree \( t_z \) is not in \( \bigcup_{0 \leq z < 2a+2} F_z \). Recalling the definitions of \( F_x \) for odd and even indices \( x \), we see that this implies \( e_z > a + 2 \) for all \( z \in \{i, i + 1, \ldots, i'\} \). Hence no node with name \( \leq a + 2 \) is removed in the process of transforming \( t_i \) to \( t_{i+1} \), \( t_{i+1} \) to \( t_{i+2} \), and so on until \( t_{i'} \) is obtained. Therefore, the node \( v \) with name \( a + 2 \) in \( t_i \) continues to be a part of all \( t_z \), where \( i \leq z \leq i' \). Since \( e_z > a + 2 \), the name-compaction step of algorithm \texttt{GeneralizedNext} keeps the name of node \( v \), i.e., \( a + 2 \), unchanged in all of \( t_z \). Hence, node \( v \) is present in \( t_z \) and \( M_z(v) = a + 2 \), for all \( z \in \{i, i + 1, \ldots, i'\} \). Furthermore, since \( h_i(v) \neq 0 \) and since \( v \) is not deleted in the sequence of transformations from \( t_i \) to \( t_{i'} \), it follows that \( h_z(v) = h_i(v) \neq 0 \), for \( i \leq z \leq i' \).

2. Consider an index \( z \) such that \( i < z < i' \). If \( v \) was a non-leaf node in \( t_{z-1} \), then it starts off as a non-leaf node with at least one child having a non-empty state label when algorithm \texttt{GeneralizedNextRecursive} is invoked on \( t_{z-1} \) to transform it to \( t_z \). If \( v \) was a leaf node in \( t_{z-1} \) (as is the case when \( z = i+1 \), for example), then since \( h_{z-1}(v) \neq 0 \) (by Lemma [2] above), Step 2 of algorithm \texttt{GeneralizedNextRecursive} ensures that \( v \) becomes a non-leaf node with at least one child having a non-empty state label in an intermediate step during the transformation of \( t_{z-1} \) to \( t_z \). Thus, in either case, \( v \) becomes a non-leaf node with at least one child having a non-empty state label in some intermediate step of algorithm \texttt{GeneralizedNextRecursive}.

In order for \( v \) to subsequently become a leaf node in \( t_z \), all its children must be deleted. Deletion of nodes can only happen in Step 5 or Step 6 of algorithm \texttt{GeneralizedNextRecursive}. We show that none of these steps can delete all children of \( v \) in \( t_z \).

- Since \( v \) stays back in \( t_z \) (by Lemma [2] above), if the leaves of \( v \) are deleted in Step 6 of algorithm \texttt{GeneralizedNextRecursive}, \( v \) must be “reset” and \( M_z(v) = a + 2 \) must be added to \( U \).
(since \( P_\phi(Q_{h_i(v)}) = P_\phi(Q_{h_i(v)}) = \text{True} \)) in Step 6. Therefore, \( f_z \) must be set to a value no larger than \( a + 2 \) in Step 1 of algorithm GeneralizedNext. Since \( e_z > a + 2 \) (as shown in the proof of Lemma (9-1)), this would imply that \( t_z \in F_x \), where \( 0 \leq x \leq 2a + 2 \). Recalling the definition of \( i^* \), this contradicts the fact that \( z > i > i^* \).

- If all leaves of \( v \) are deleted in Step 5 of algorithm GeneralizedNextRecursive, then the union of state labels of the children of \( v \) must be empty at some intermediate step of algorithm GeneralizedNextRecursive. We have seen above in the proof of Lemma 9 that the state label of a node is eventually no larger than the union of state labels of its children at any intermediate step of algorithm GeneralizedNextRecursive. Therefore, if all leaves of \( v \) are deleted in Step 5 of algorithm GeneralizedNextRecursive, the state label of \( v \) must eventually become empty in \( t_z \). However, \( v \) must then be deleted from \( t_z \) by Step 5 of algorithm GeneralizedNextRecursive. This contradicts Lemma (9-1) proved above.

Therefore, \( v \) must be a non-leaf node in \( t_z \).

3. Lemma (9-3) is an immediate consequence of Lemmas (9-1), (9-2), (7-2), (7-3) and (8).

\[ \square \]

Lemma 10. \( L(D) \subseteq L(A) \).

Proof: We will prove this lemma by constructing a finitely branching infinite tree \( K \) along the lines of Safra’s proof of correctness of his NSW determinization construction, and by showing the existence of an infinite accepting path of \( A \) in this tree.

The vertices of \( K \) are elements of \( \{ r \} \cup (Q \times N) \), where \( r \) is a special vertex representing the root of \( K \). For every \( q_0 \in Q_0 \), we draw an edge from \( r \) to \((q_0,0)\). As defined earlier, let \( i^* \) be the minimum index after which no CGS tree from \( F_x \), for \( x < 2a + 2 \), is visited in the sequence \( t_0, t_1, \ldots \). Let \( i_1 \) be the smallest index greater than \( i^* \) such that \( f_{i_1} = a + 2 \), and let \( v \) be the node in \( t_{i_1} \) identified in Lemma (9-1). From Lemma (9-1), we know that \( M_i(v) = a + 2 \) and \( h_i(v) = h_{i_1}(v) \neq \emptyset \) for all \( i \geq i_1 \). For every state \( q \) in \( \lambda_{i_1}(v) \) we add a vertex \((q,i_1)\) to the tree \( K \). For every such state \( q \), Proposition 1 tells us that there is a state \( q_0 \in Q_0 \) such that there is a run of \( A \) from \( q_0 \) to \( q \) on \( \alpha(0,i_1-1) \). We add an edge from \((q_0,0)\) to \((q,i_1)\) in tree \( K \) for every such \( q_0 \in Q_0 \) and \( q \in \lambda_{i_1}(v) \). Subsequently, we extend the tree \( K \) inductively as follows. Given a tree with a leaf \((q_k,i_z)\), where \( q_k \in \lambda_{i_z}(v) \) and \( i_z \geq i_1 \) is such that \( f_{i_z} = a + 2 \), we find the smallest \( i_{z+1} > i_z \) such that \( f_{i_{z+1}} = a + 2 \). Since CGS trees in \( F_{2a+2} \) are encountered infinitely often in \( t_0, t_1, \ldots \) (by the acceptance condition of \( D \)), such an \( i_{z+1} \) always exists. For every state \( q' \in \lambda_{i_{z+1}}(v) \), we now add a vertex \((q',i_{z+1})\) to the tree \( K \). By Lemma (9-3), there is a state \( q \) in \( \lambda_{i_z}(v) \) such that there is a run of \( A \) from \( q \) to \( q' \) on \( \alpha(i_z,i_{z+1} - 1) \) that visits all and only states in \( Q_{h_{i_{z+1}}(v)} \). For every such \( q' \in \lambda_{i_{z+1}}(v) \) and \( q \in \lambda_{i_z}(v) \), we add an edge from \((q,i_z)\) to \((q',i_{z+1})\) to extend the tree \( K \). It is easy to see that \( K \) is an infinite tree with the branching of each node \((q,i_z)\) restricted by the cardinality of \( \lambda_{i_{z+1}}(v) \), i.e. \( |Q| \). Therefore, it follows from König’s lemma that there is an infinite path in \( K \).

From Proposition 1, every edge \( ((q_0,0),(q',i_1)) \) corresponds to a run of \( A \) on \( \alpha(0,i_1-1) \) that starts at \( q_0 \) and ends at \( q' \). From Lemma (9-3), every edge \( ((q,i_z),(q',i_{z+1})) \) for \( z \geq 1 \) corresponds to a run of \( A \) on \( \alpha(i_z,i_{z+1} - 1) \) that starts at \( q \) and ends at \( q' \) and visits all and only states in \( Q_{h_{i_{z+1}}(v)} \). Therefore, the infinite path in \( K \) identified above corresponds to a run \( \rho \) of \( A \) that starts from some \( q_0 \in Q_0 \) and eventually visits all and only states in \( Q_{h_{i_1}(v)} \). In other words, \( inf(\rho) = Q_{h_{i_1}(v)} \). Furthermore, since \( f_{i_1} = a + 2 \) and \( M_i(v) = a + 2 \), we must have \( P_\phi(Q_{h_{i_1}(v)}) = True \). In other words, \( inf(\rho) = \phi \), and hence \( \rho \) is an accepting run of \( A \). This implies \( \alpha \in L(A) \).

\[ \square \]

Lemma 11. \( L(A) \subseteq L(D) \).

Proof: Consider an \( \omega \)-word \( \alpha \in L(A) \). Let \( \psi = q_{k_0}, q_{k_1}, q_{k_2} \ldots \) be an accepting run of \( A \) on \( \alpha \), and let \( \rho = t_{i_0}, t_{i_1}, t_{i_2}, \ldots \) be the unique run of \( D \) on \( \alpha \), where \( t_i \) is the CGS tree \( \langle N_i, M_i, r_i, p_i, t_i, h_i, f_i, e_i \rangle \). Consider the transformation of \( t_i \) to \( t_{i+1} \) by algorithm GeneralizedNext. Step 1 of algorithm GeneralizedNext
updates the state label of \( r_i \) to \( \delta(\lambda_i(r_i), \alpha_i) \). Subsequently, no step of algorithm \texttt{GeneralizedNext} or algorithm \texttt{GeneralizedNextRecursive} deletes any node from the state label of \( r_i \), deletes \( r_i \), or adds \( r_i \) as the child of any other node. It therefore follows from an easy inductive argument that the root \( r_i \) of \( t_i \) eventually survives as the root \( r_{i+1} \) of \( t_{i+1} \), for all \( i \geq 0 \). Since \( M_0(r_0) = 1 \) and \( h_0(r_0) = [n] \), and since every node in \( t_i \) that is not deleted in transforming \( t_i \) to \( t_{i+1} \) retains its name and hope set in \( t_{i+1} \), we have \( M_{i+1}(r_{i+1}) = 1, h_{i+1}(r_{i+1}) = [n] \) and \( e_{i+1} > 1 \) for all \( i \geq 0 \). Also, by definition, \( e_0 = m + 1 > 1 \). Therefore, \( e_i > 1 \) for all \( i \geq 0 \).

Let \( J \) be the set of indices of all states in \( \inf(\psi) \), i.e., \( J = \{ j \mid q_j \in \inf(\psi) \} \). Let \( i_1 \) be the smallest index such that for all \( i > i_1 \), we have \( q_{i_1} \in \inf(\psi) \). We wish to identify those nodes \( v \) in \( t_i \) that have \( \psi(v) \in \lambda_2 (v) \), for all \( z_i \geq i + 1 \). In other words, we wish to identify nodes in the sequence of CGS trees \( t_{i+1}, t_{i+2}, \ldots \) that track the run \( \psi \) of \( A \) from position \( i_1 + 1 \) onwards.

We have already seen above that \( r_0 \) survives as the root node in all CGS trees in \( \rho \). We also know that \( \lambda_0(r_0) = \lambda_0 \), by definition. Since Step (1) of algorithm \texttt{GeneralizedNext} updates \( \lambda_{i+1}(r_{i+1}) \) to \( \delta(\lambda_i(r_i), \alpha_i) \) for all \( i \geq 0 \), and since no subsequent step during the transformation of \( t_i \) to \( t_{i+1} \) deletes any state from the state label of the root \( r_{i+1} \), it follows from an easy inductive argument that \( \psi(z) \in \lambda_2 (r_2) \), for all \( z \geq 0 \). Now suppose the root node becomes a leaf infinitely often in \( \rho(i_1 + 1, \infty) \). Let \( j \) and \( j' \) be arbitrary indices such that \( i_1 + 1 \leq j < j' \), and the root node is a leaf in \( t_j \) and \( t_{j'} \), but not in any \( t_z \), for \( j < z < j' \). Since we also know that \( h_i(r_i) = [n] \neq \emptyset \) for all \( i \geq 0 \), it follows from Lemma (7-3) and Lemma (8) that the set of states visited in \( \psi(j, j') \) is exactly \( Q_{h_i(r_i)} = Q[\psi] \). By repeating the same argument for all successive pairs of indices \( j, j' \) such that \( i_1 + 1 \leq j < j' \), and the root node is a leaf in \( t_j \) and \( t_{j'} \), but not in any \( t_z \) in between, we get \( \inf(\psi) = Q_{h_i(r_i)} \), for every \( i > i_1 \). Since \( \psi \) is an accepting run of \( A \), we also know that \( P_\psi(\inf(\psi)) = \text{True} \). This implies that \( P_\psi(\lambda_2(r_2)) = \text{True} \) for all those indices \( i > i_1 \) where \( r_i \) becomes a leaf node in \( \rho(i_1 + 1, \infty) \). By Lemma (7-1), we know that \( r_{i_1} \) is “reset” in these steps as well. Hence \( r_1 \) is added to the set \( U \) in Step (3) of algorithm \texttt{GeneralizedNextRecursive} during the transformation of \( t_{i-1} \) to \( t_i \) for each such \( i \). Since the root has the smallest name (\( M(r_1) = 1 \)), the component \( f_i \) of the CGS tree \( t_i \) is set to 1 infinitely often, while \( e_i > 1 \). Hence the set \( F_i \) is visited infinitely often and \( \psi \in L(D) \).

If the root node becomes a leaf infinitely often, there is an index \( i_2 > i_1 \) such that the root node is a non-leaf node in all \( t_z \) for \( z > i_2 \). By Lemma (5), we know that for all \( z > i_2 \), every state in \( \lambda_i(r_z) \) is also in \( \lambda_2 (v) \) for some child \( v \) of \( r_z \). Since \( \psi(z) \in \lambda_2 (v) \) for all \( z \geq 0 \), it follows that for all \( z > i_2 \), there is a child \( v \) of \( r_z \) such that \( \psi(z) \in \lambda_2 (v) \). Now consider the transformation of \( t_z \) to \( t_{z+1} \) for \( z > i_2 \), and let \( v_z \) be the node in \( t_z \) such that \( \psi(z) \in \lambda_2 (v_z) \). Step (1) of algorithm \texttt{GeneralizeNext} initializes the state label of \( v_z \) with \( \delta(\lambda_2 (v_z), \alpha_z) \), thereby placing \( \psi(z+1) \) in the state label of \( v_z \). Subsequently, if \( \psi(z+1) \) is moved out of the state label of \( v_z \), either Step (3(b)ii) or Step (4) of algorithm \texttt{GeneralizedNextRecursive} must be responsible for this. However, if \( \psi(z+1) \) is removed from the state label of \( v_z \) in Step (3(b)ii), from the argument used in the proof of Lemma (6), we know that \( \psi(z+1) \) must eventually be removed from the state label of the parent of \( v_z \) in \( t_{z+1} \), i.e., from the state label of \( r_{z+1} \). This is a contradiction, since \( \psi(z) \in \lambda_2 (r_z) \) for all \( z \geq 0 \). Therefore, if \( \psi(z+1) \) is removed from the state label of \( v_z \), Step (4) of algorithm \texttt{GeneralizedNextRecursive} must be responsible for the removal. From the pseudocode of \texttt{GeneralizedNextRecursive}, we now observe that if \( v_{z+1} \) is the new node containing \( \psi(z+1) \) in its state label in \( t_{z+1} \), then either \( M_{z+1}(v_{z+1}) < M_{z+1}(v_z) = M_z \) or the annotation of the edge from \( r_{z+1} \) to \( v_{z+1} \) in \( t_{z+1} \) is lesser than the annotation of the edge from \( r_{z+1} \) to \( v_z \) in \( t_{z+1} \). Since both \( r_z = r_{z+1} \) and \( v_z \) existed in \( t_z \), the annotation of the edge from \( r_{z+1} \) to \( v_z \) in \( t_{z+1} \) must be the same as the annotation of the edge from \( r_z \) to \( v_z \) in \( t_z \). Therefore, if the child of the root that tracks \( \psi \) changes from \( t_z \) to \( t_{z+1} \), then either the name of the node reduces or the annotation of the edge from the root to this node reduces during the transformation from \( t_z \) to \( t_{z+1} \). Since neither the name nor the annotation can decrease infinitely, there must be an index \( i_3 > i_z \) such that for all \( z > i_3 \), the child of the root that contains \( \psi(z) \) in its state label has the same name and the same annotation of the edge from the root to this child. In other words, if \( v_z \) and \( v_{z+1} \) are children of the root in \( t_z \) and \( t_{z+1} \) respectively such that \( \psi(z) \in \lambda_2 (v_z) \) and \( \psi(z+1) \in \lambda_2 (v_{z+1}) \), then \( M_{z+1}(v_{z+1}) = M_{z+1}(v_z) \) and \( h_{z+1}(v_{z+1}) = h_{z+1}(v_z) \).

If possible, let \( v_z \) and \( v_{z+1} \) be distinct nodes. As seen above, Step (4) of algorithm \texttt{GeneralizedNextRecursive} is responsible for moving \( \psi(z+1) \) from the state label of \( v_z \) to that of \( v_{z+1} \) during the transformation of \( t_z \) to \( t_{z+1} \). From the pseudocode of algorithm \texttt{GeneralizedNextRecursive}, we note that either the annotation of the edge from the root to \( v_{z+1} \) must be less than the annotation of the edge from the root to \( v_z \), or the
name of \( v_{z+1} \) must be less than the name of \( v_z \) at the time of execution of Step (4). Since the name of \( v_{z+1} \) can only reduce further during the name-compaction step and since the annotation of the edge from the root to \( v_{z+1} \) cannot change subsequently in any step of algorithm \textit{GeneralizedNextRecursive} or algorithm \textit{GeneralizedNext}, we cannot have both the names and the annotations of the edges from the root identical for \( v_z \) in \( t_z \) and for \( v_{z+1} \) in \( t_{z+1} \). Since \( z > i_3 \), this gives us a contradiction! Therefore, \( v_z \) is the same node as \( v_{z+1} \) for all \( z > i_3 \). Since \( M_z(v_z) \) also stays unchanged for all \( z > i_3 \), no node with name \( < M_z(v_z) \) is deleted during the transformation of \( t_z \) to \( t_{z+1} \), for \( z > i_3 \). This implies that \( v_z > M_z(v_z) \) for all \( z > i_3 \).

We now claim that \( h_z(v_z) \neq \emptyset \) for all \( z > i_3 \). To see why this is so, suppose \( h_z(v_z) = \emptyset \) for some \( z > i_3 \) and let \( j \) be the annotation of the edge from \( r_z \) to \( v_z \) in \( t_z \). Consider Step (5) of the recursive invocation of algorithm \textit{GeneralizedNextRecursive} with the parent of \( v_z \), i.e., \( r_z \), as the root of the CGS subtree to be transformed. Let \( q_l \) be a state in the state label of \( v_z \) when Step (5) is executed. If \( l = j \), then Step (3(b)) creates a new sibling \( v' \) of \( v_z \), sets the state label of \( v' \) to \( \{q_l\} \) and sets the annotation of the edge from \( r_z \) to \( v' \) to an index \( \leq l \). Since no further step removes the state label of the newly created leaf \( v' \), state \( q_l \) gets removed from the state label of \( v_z \) in Step (3) of algorithm \textit{GeneralizedNextRecursive}. If, on the other hand, \( l \neq j \), then since \( h_z(v_z) \) is assumed to be \( \emptyset \), Step (3(b)) removes \( q_l \) from the state label of \( v_z \). Thus, in either case, no state eventually remains in the state label of \( v_z \) in \( t_z \) if \( h_z(v_z) = \emptyset \). This implies that \( v_z \) is deleted from \( t_z \) in Step (5) — a contradiction! Therefore, we must have \( h_z(v_z) \neq \emptyset \) for all \( z > i_3 \).

We now consider the case where the node \( v_z \) becomes a leaf infinitely often in \( \rho(i_3 + 1, \infty) \). By using the same argument as above when the root becomes a leaf infinitely often, we find that for every \( z > i_3 \) such that \( v_z \) is a leaf in \( t_z \), the node \( v_z \) is added to the set \( U \) in Step (5) of algorithm \textit{GeneralizedNextRecursive} during the transformation of \( t_{z-1} \) to \( t_z \). Therefore, \( f_z < M_z(v_z) \) for all \( z > i_3 \). We have also seen above that \( e_z > M_z(v_z) \) for all \( z > i_3 \). This implies that a parity acceptance set \( F_z \) with an even index \( x \) is visited infinitely often by the run \( \rho \) of \( D \). Hence \( w \in L(D) \).

If \( v_z \) becomes a leaf only finitely often in \( \rho(i_3 + 1, \infty) \), we can repeat the same argument as used above and show that there is an index \( i_4 > i_3 \) and a child \( v' \) of \( v_z \) such that (i) \( v' \) is present in \( t_{i_4} \), (ii) \( \psi(i) \in \lambda_i(v') \), (iii) \( h_i(v') = h_{i+1}(v') \neq \emptyset \), and (iv) \( M_i(v') = M_{i+1}(v') \), for all \( i > i_4 \). Since all CGS trees \( t_z \) have height \( \leq n \) (as argued in Section (5)), by continuing the above argument, we find that there must exist an even index \( x \) such that \( F_z \) is visited infinitely often by \( \rho \). In other words, \( w \in L(D) \).

\textbf{Theorem 12.} \( L(D) = L(A) \)

\textbf{Proof:} Follows from Lemmas (10) and (11).

\section{5. Complexity}

\textbf{Theorem 13.} \textit{Given an automaton} \( A \) \textit{with} \( n \) \textit{states, the deterministic parity automaton} \( D \) \textit{constructed above has at most} \( n^{O(n^2)} \) \textit{states and} \( O(n^2) \) \textit{parity acceptance sets.}

\textbf{Proof:} The computation for the number of states of the automaton \( D \) is similar to that done by Piterman for his NSW to DPW construction [11]. Since every state of \( D \) is a CGS tree over \( A \), we will count the total number of CGS trees over \( A \) below, assuming \( n = |Q| \) and \( m = n^2 + n + 1 \).

The salient steps in counting the number of CGS trees over \( A \) are as follows.

- Since the state labels of leaves in a CGS tree are pair-wise disjoint, and since every leaf has a nonempty state label, there can be at most \( n \) leaves.

- If we collapse the vertices at the head and tail of every \( 0 \)-annotated edge in a CGS tree, we will get a tree with no \( 0 \)-annotated edges. Since the hope set of the root is always \( [n] \) and since the hope set of a child in the collapsed tree misses exactly one index from the hope set of its parent, the height of the collapsed tree can be at most \( n \). This, along with the fact that there are at most \( n \) leaves, implies that there are at most \( n^2 + 1 \) nodes in the collapsed tree.

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• To count the nodes that were removed due to the collapsing operation described above, we note that each node in the original CGS tree must have a path (possibly of zero length) to a leaf such that each edge along this path has a non-0 annotation. Hence, if u and v are nodes such that the edge from the parent of u to u and that from the parent of v to v are both annotated with 0, the path of non-0 annotated edges from u to a leaf cannot overlap with the corresponding path from v to a leaf. Therefore, there can be at most $n$ nodes in a CGS tree such that the edges from the respective parents to these nodes are annotated with 0. This implies that the total number of nodes in a CGS tree can be at most $m = n^2 + n + 1$.

• By construction, the parent of a node always has a smaller name than the node. Thus the parenthood relation can be represented by a sequence of at most $m - 1$ names where the $i$th name is a value in $\{1, \ldots, i - 1\}$. For a tree with $k$ nodes, the there are at most $(k - 2)!$ such sequences of length $k - 1$. Considering all trees with number of nodes in $\{1, \ldots, m\}$, there are at most $\sum_{k=1}^{m} (k - 2)!$, i.e. $\leq (m - 1)!$ such sequences. Hence, there are at most as many named trees where children have larger names than their respective parents.

• The state label of a node is given by the union of state labels of leaves in the sub-tree rooted at that node. In addition, the labels of leaves are pairwise disjoint. Therefore, the state labels of all nodes in a tree can be obtained by associating each $A$-state with the leaf that contains it in its state label. Since leaves in a tree may not be named with the first few contiguous names, we sort the leaves by names and then use a mapping from $A$-states to positions of leaves in this name-sorted order. If an $A$-state doesn’t appear in any leaf, we associate the position 0 with it. Thus, the number of state labelings of a named tree is at most the number of mappings $Q \to \{0, 1, \ldots, n\}$, i.e. $\leq (n + 1)^n$.

• The (hope set) annotation of a node is represented using edge annotations as follows. Suppose the hope set of a node $v$ is $h(v)$ and that of its child $v'$ is $h(v')$. Then the edge from $v$ to $v'$ is annotated with $h(v) \setminus h(v')$, if $h(v') \subseteq h(v)$, and with 0 if $h(v') = h(v)$. By properties of CGS trees, $h(v') \subseteq h(v)$ and $|h(v') \setminus h(v)| \leq 1$. Therefore, the edge annotation is a unique element in $[n] \cup \{0\}$. Similarly, the hope set for every node is uniquely determined if the annotations of all edges are given. Specifically, the hope set of a node is simply $[n]$ sans the annotations on edges along the path from the root to this node. Therefore, it is sufficient to count the number of edge annotation functions to obtain the count of hope set annotations of nodes. Each edge can be identified by the name of the node it points to. The total number of edge annotation functions is then easily seen to be the number of functions $[m] \to [n] \cup \{0\}$. This is bounded above by $(n + 1)^m$.

• For the acceptance condition, we need to know the value of $e$ when $e \leq f$, and the value of $f$ when $f < e$. Thus we need to keep track of at most $2m$ values.

Combining the above counts, the total number of CGS trees over $A$ is at most

$$(m - 1)! \cdot (n + 1)^{n+m} \cdot (2m) = n^{O(n^2)}$$

. The number of parity acceptance sets is $2m = 2 \cdot (n^2 + n + 1) = O(n^2)$. \hfill \qedsymbol

6. An improved upper bound for $\omega$-automata

The determinization construction proposed above gives a DPW starting from a variety of different non-deterministic automata, all of which have an acceptance condition based on infinity sets. By Theorem 19, the number of states of the DPW is at most $n^{O(n^2)}$ or $2^{O(n^2 \log n)}$, while the number of sets in the parity acceptance condition is at most $O(n^2)$, where $n$ is the number of states of the original automaton $A$. This bound also holds when the input automaton is a pairs automaton viz. a Streett or a Rabin automaton. This is significant since the size of the output DPW, both in terms of number of states and acceptance pairs, is independent of the number of pairs of the input pairs automaton. This is different from the case of Safra’s
determinization construction for NSW, where the output DR W/DPW has at most $2^{O(nh \log(nh))}$ states and $O(nh)$ pairs, where $n$ and $h$ are the count of states and pairs, respectively, of the input NSW.

This naturally leads us to ask if $2^{O(n^2 \log n)}$ is a better bound than $2^{O(nh \log(nh))}$ for determinization of NSW/NR W. The answer to this question is not immediately obvious and requires us to show that there are indeed examples of NSW/NR W with $O(n)$ states and $h$ pairs for which Safra’s and Piterman’s NSW determinization construction will end up constructing automata with state count worse than $2^{O(n^2 \log n)}$. In the following, we present a class of such automata. In the case when $h \geq n^k$, where $k > 1$, this immediately implies an improved worst case complexity bound on NSW/NR W determinization.

**Theorem 14.** There exists a family $A_S$ of NSW where each NSW $A_S \in A_S$ has $3n+1$ states and $2n+1$ accepting pairs for which the Safra-Schwoon (Piterman) construction constructs a DR W (DPW) with $2^{\Omega(n^3)}$ states, while our construction (algorithm GeneralizedNext) constructs a DR W/DPW with $2^{O(n^2 \log n)}$ states.

The proof of Theorem (14) is given in Subsection (6.1) by demonstrating the construction of an automaton from the family $A_S$.

To begin with, a strategy to generate more than $2^{O(n^2 \log n)}$ states for the DR W/DPW using Safra’s/Piterman’s construction is established. The input NSW for such a strategy has $O(n)$ states and $h = 2^n$ pairs. One way to generate a sufficiently large number of $(Q, H)$-trees (as used in Schwoon’s exposition of Safra’s construction) is to obtain different permutations of the edge labels on a path from a leaf to the root, and then repeat this for all paths in the tree. We shall follow the construction of Schwoon[19] described in algorithms SafraNext and SafraNextRecursive (see Subsection (2)) for NSW determinization.

Figure (4) shows three possible $(Q, H)$-trees, $t_x, t_y$, and $t_z$ that can be generated using the Safra-Schwoon construction in algorithm SafraNextRecursive starting from the initial tree $t_0$, where $t_0$ is the CGS tree with a single (root) node $r_0$, with $\lambda(r_0) = Q_0$, $M(r_0) = 1$, $h(r_0) = [n]$ and for $t_0$ we have $e = f = m + 1$.

The first tree $t_x$ is not hard to generate, since Steps (1) and (2) recursively extend a $(Q, H)$-tree at its leaves. If the Streett state label of the leaf node in the first tree $t_x$ is $\{q_1, q_2, \ldots, q_n\}$ and $q_i \in F_h$ for all $i \in [n]$, then in Step (3(b)) a new node is created for each such $q_i \in F_h$ with the edge from the root node...
to the newly created node annotated $h$, giving the second tree $t_y$. An application of Steps 1 and 2 will result in the extension of the second tree $t_y$ at its leaves giving the third tree $t_z$. For each Streett state $q_i, i \in [n]$ that appears in the label of a leaf node in the third tree $t_z$, the path from the leaf to the node is disjoint from every other path in the tree. Each such disjoint path has exactly the same edge annotations. Note that since the number of leaves in a $(Q, H)$-tree can never be more than the total number of Streett states, we cannot expect to get more than $n$ disjoint paths from a leaf to the root. The challenge now is to permute the edge annotations giving a large number of $(Q, H)$-trees.

Since, the maximum length of a disjoint path in a $(Q, H)$-tree depends on the number of pairs of the NSW, one would like to start with an NSW with as many pairs as possible. Suppose, we start out with $h = 2^n$ pairs in the NSW. A permutation of $2^n$ edge annotations would give us $(2^n)!$ possible trees with just one branch and $(2^n)!^n$ trees with all $n$ disjoint branches. With only $n$ states in the NSW and $2^n$ pairs, it is clear that one or more Streett states will be replicated across pairs. This replication of Streett states is a potential problem as the example in Figure (5) shows.

Figure (5) shows different edge annotations for a path of length $h$ (with no 0 edges) in a $(Q, H)$-tree. The different edge annotations are obtained as the Streett state label at the leaf changes. We assume that $h = 2^n$. It is not hard to obtain the edge annotations along $(p_2)$ from the edge annotations along $(p_1)$. In this transformation only the edge annotation of the first edge in $(p_1)$ changes from $2^n$ to $2^n - 1$. This is possible if there is a state $q_k$ in the leaf label that is also in the pair $E_{2^n}$ of the pair $(E_{2^n}, F_{2^n})$. This causes the entire path to be replaced by a pair of nodes - the root node with exactly one child. The edge between the root and its child node is annotated $2^n - 1$. This path is again extended by Steps 1 and 2 of algorithm SafraNextRecursive. We see that repeated application of this change allows us to change the edge annotations of $(p_2)$ to those shown along $(p_3)$, where the edge annotation on the edge from the root to the first child node is $2^n - k$. Note that this requires that the NSW has a path from $q_k$ back to itself on some letter or word segment. Once the first edge annotation is fixed we can apply a similar set of transformations using some other state $q_l$ to fix the second edge annotation to $2^n - l$. But, this immediately implies that the state $q_l$ cannot be in $E_{2^n-k}$ or $F_{2^n-k}$ since that would either change the annotation of the first edge to $2^n - k + 1$ or reset the path back to the third path $(p_3)$ shown in the figure. Hence, every time we fix the
edge annotation for an edge it constrains the possible pairs that a Streett state can belong to. With only \( n \) states and \( 2^n \) pairs, we are soon forced to repeat Streett states across pairs in our example NSW. This in turn forces already fixed edge annotations to change, defeating our purpose. Thus, generating arbitrary permutations of \( 2^n \) pair indices along paths in a \((Q,H)\)-tree is extremely hard with an NSW with just \( n \) states.

We then ask if \( 2^n \) is too many pairs and try to see if \( n \) or \( n^2 \) or some number of pairs polynomial in \( n \) allows us to achieve our objective of obtaining arbitrary permutations of edge annotations. But, with \( n^k \) pairs in the NSW, for some constant \( k \), even if obtaining arbitrary permutations of edge annotations is possible, we can obtain at most \( (n^k)! \) permutations along a path and hence \((n^k)!\) \((Q,H)\)-trees using all the paths. But, \((n^k)!\) is \( 2^{O(n^2 \log n)} \), which matches the bound given by our construction and does not serve our purpose.

We now show a solution to the above dilemma. We start out with \( h = 2^n \) pairs in the NSW, but we partition the \( 2^n \) pairs into \( \lfloor \frac{2^n}{n} \rfloor \) blocks of \( n \) pairs each. Hence \( B_1 = ((L_{2^n}, U_{2^n}), (L_{2^n} - 1, U_{2^n} - 1), \ldots, (L_{2^n} - (n-1), U_{2^n} - (n-1))) \) is the first block, \( B_2 = ((L_{2^n} - (n), U_{2^n} - (n)), (L_{2^n} - (n+1), U_{2^n} - (n+1)), \ldots, (L_{2^n} - (2n-1), U_{2^n} - (2n-1))) \) is the second block and so on. If \( \lceil \frac{2^n}{n} \rceil = k \), then the last or \( k^{th} \) block is \( B_k = (L_{2^n} - ((k-1)n), U_{2^n} - ((k-1)n)), (L_{2^n} - ((k-1)n+1), U_{2^n} - ((k-1)n+1)), \ldots, (L_{2^n} - (kn-1), U_{2^n} - (kn-1))) \). Instead of trying to generate arbitrary permutations of \( 2^n \) pair indices we try to generate permutations of only \( n \) pair indices, but with the following properties for a permutation \((j_1, j_2, \ldots, j_n)\), where \( j_i \in [h] \) for all \( i \in \{1, 2, \ldots, n\}\):

- We pick \( k = \lfloor \frac{2^n}{n} \rfloor \) blocks starting with the last block \( B_k \) and picking successively lower numbered blocks \( B_{k-1}, B_{k-2}, \ldots \).
- From each block we pick exactly one pair index. For example if we pick the \( j^{th} \) pair in block \( B_j \) then pair is \((L_{2(j-1)+n+i-1}, U_{2(j-1)+n+i-1})\). We call this pair index \( \text{id} \).
- If index \( \text{id}_j \) is already picked from block \( j \), then we do not pick \( \text{id}_l \) for \( l \neq j \), for every pair of blocks \( B_j \) and \( B_l \) that are picked.

This system of picking elements of the permutation not only allows us to permute only \( n \) elements along every path from a leaf to the root, but also allows us to choose from \( 2^n \) Streett pairs and at the same time have only \( O(n) \) states for the example NSW. We shall see later that this method ends up generating more than \( 2^{O(n^2 \log n)} \) \((Q,H)\)-trees. We shall call a permutation that satisfies the conditions described above as a block permutation of size \( n \). An example of a NSW with \( O(n) \) states and \( 2^n \) pairs for which the corresponding DPW constructed using the Safra/Piterman construction has more than \( 2^{O(n^2 \log n)} \) states is given below.

### 6.1. An example showing improved worst case bounds

Consider the the NSW \( A_q = (\Sigma, Q^*, q_0, \delta^*, T) \) defined as follows. The NSW \( A_q \) is an automaton in the family \( A_S \) described in Theorem (17).

- \( Q^* \) is the state set containing \( 3n + 1 \) states \( \{q_0\} \cup \{q_{0,1}, q_{1,1}, \ldots, q_{n-1,1}\} \cup \{q_{0,s}, q_{1,s}, \ldots, q_{n-1,s}\} \cup \{q_{0,\top}, q_{1,\top}, \ldots, q_{n-1,\top}\} \). States of the form \( q_{i,\downarrow}, q_{i,s}, q_{i,\top} \) are called \( \downarrow \)-states, \( s \)-states and \( \top \)-states respectively.
- \( q_0 \) is the initial state.
- \( \Sigma \) is the alphabet \( \{a_0\} \cup \{a_{x,a} \mid x \in \{0,1,2,\ldots,n-1\}\} \cup \{a_0,\ldots, a_{n-1}\} \cup \{a_{\downarrow}\} \).
- The transitions for the automaton are defined as follows
  1. \( \delta^*(q_0, a_0) = \{q(0,\top), q(1,\top), \ldots, q(n-1,\top)\} \)
  2. \( \delta^*(q_{i,\top}, a_{\downarrow}) = q_{i,\top} \) for all \( i \in \{0,1,\ldots,n-1\} \)
  3. \( \delta^*(q_{i,\top}, a_j) = q_{i,s} \) for all \( i \in \{0,1,\ldots,n-1\} \)
  4. \( \delta^*(q_{i,s}, a_{j,a}) = q_{j,s} \) for all \( i,j \in \{0,1,\ldots,n-1\} \)
5. $\delta^*(q_{(i,j)}, a_{(i,s)}) = q_{(i,T)}$ for all $i, j \in \{0, 1, \ldots, n - 1\}$
6. $\delta^*(q_{i,s}, a_{\perp}) = q_{(i,\perp)}$ for all $i \in \{0, 1, \ldots, n - 1\}$
7. $\delta^*(q_{(i,\perp)}, a_{(i,s)}) = q_{(i,\perp)}$ for all $i, j \in \{0, 1, \ldots, n - 1\}$
8. $\delta^*(q_{(i,\perp)}, a_{\perp}) = q_{(i,\perp)}$ for all $i \in \{0, 1, \ldots, n - 1\}$
9. $\delta^*(q_{(i,\perp)}, a_{j}) = q_{(i,\perp)}$ for all $i, j \in \{0, 1, \ldots, n - 1\}$

• There are $2^n + 1$ Streett pairs $\mathcal{T} = \{(E_{-1}, F_{-1}), (E_1, F_1), (E_2, F_2), \ldots, (E_{2^n}, F_{2^n})\}$, where $F_i, E_i \subseteq Q^n$, for all $i \in \{-1, 0, 1, \ldots, 2^n\}$ satisfying the following constraints

1. $\{q_{(0,T)}, \ldots, q_{(n-1,T)}\} \subseteq F_{2^n}$ and $q_{(i,T)} \notin E_{2^n}$ for all $i \in \{0, 1, \ldots, n - 1\}$.
2. $q_{(i,T)} \notin E_j$ for all $i \in \{-1, 0, 1, \ldots, n - 1\}$ and for all $j \in \{1, 2, \ldots, 2^n\}$.
3. $q_{(i,\perp)} \notin F_j$ for all $i \in \{0, 1, \ldots, n - 1\}$ and for all $j \in \{1, 2, \ldots, 2^n\}$.
4. $\{q_{(0,\perp)}, \ldots, q_{(n-1,\perp)}\} \subseteq F_{-1}$
5. $\{q_{(i,s)}\} \notin E_{2^n-rn-i}$ for all $r \in \{0, 1, \ldots, k - 1\}$ and $q_{(i,s)} \notin F_{2^n-rn-j}$ for all $j \in \{0, 1, \ldots, n - 1\}$ and $j \neq i$.
6. $\{q_{(i,s)}\} \notin F_{(2^n-rn-i)}$ for all $r \in \{0, 1, \ldots, k - 1\}$ and for all $t \in \{0, 1, \ldots, n - 1\}$.

As discussed earlier our goal is to permute $n$ pair indices chosen carefully from different blocks. For example let $B_1 = \langle 2^n - 2n - 1, 2^n - 5n - 1, 2^n - 3n - 4, \ldots \rangle$ be a block permutation of size $n$. Our goal is to start with an arbitrary assignment of edge annotations along a path in a $(Q,H)$-tree and obtain the permutation $B_1$ along that path. We do not insist that the elements of $B_1$ appear along successive edges along the path, but we insist that they appear along the path in the same order as they appear in $B_1$.

Figures 6, 7 and 8 demonstrate the main steps in the process of generating the required permutations of pair indices for the example automaton. In Figure 6, starting from the initial $(Q,H)$-tree consisting of just the root node, we obtain the tree extended at the root and with Streett state label $\{q_{(0,T)}, q_{(1,T)}, \ldots, q_{(n-1,T)}\}$ using the transition from $q_0$ on letter $a_0$ and Steps 1 and 2 of the Safra-Schwoon construction. This single path changes to the branched tree in which the root has $n$ children with
the edge to each child annotated $2^n$ and the $i^{th}$ child has Streett state label $q_{(i-1, \top)}$. Using a sequence of transitions on the letters $a_{(0,0)}$ and $a_{(1,s)}$ we obtain the final tree that has $n$ leaves and $n$ disjoint paths, one from each leaf to the root node.

Note that the letter $a_{(0,0)}$ causes only state $q_{(0, \top)}$ to change to the next state $q_{(0,s)}$, while Streett state labels for all other leaves remain unchanged. This results in the edge annotation between the root and the leftmost child to change to $2^n - 1$. On reading the letter $a_{(1,s)}$, state $q_{(0,s)}$ changes to $q_{(1,s)}$ giving us the tree $t_1$ in the figure. Note that $t_1$ is only an intermediate tree and will evolve through different steps of the Safra-Schwoon algorithm. We observe that by changing the Streett label of just one path at a time we can systematically generate permutations of edge annotations one path at a time. This will be our general strategy henceforth and we shall see how a path $\pi_1$ in tree $t_1$ evolves with succeeding steps.

The $\top$-states can be thought of as the source states of every path transformation. We change a $\top$-state to an $s$-state only along the path whose edge annotations we need to modify.

Figure (7) shows the transformations of path $\pi_1$ in order to obtain the block permutation $B_1$ in order along the edges in $\pi_1$. It is straightforward to obtain the first element $2^n - 2n - 2$ along the first edge. All it requires is successive applications of letter $a_{(1,s)}$ to $a_{(n-1,s)}$ follows by $a_{(1,s)}$. We now try and change the other edge annotations keeping the first edge annotation fixed. On reading the letter $a_{(0,s)}$, $a_{(1,s)}$ we change the second edge annotation to $2^n - 2$. Here, we need to be careful, since an application of $a_{(2,s)}$ at this point will change $2^n - 2$ to $2^n - 3$ but it will also change $2^n - 2n - 2$ to $2^n - 2n - 3$, because of the way the Streett pairs are organised. Hence, we defer the application of $a_{(2,s)}$ and instead apply letter $a_{(0,s)}$ again, which changes $2^n$ to $2^n - 1$. Now an application of $a_{(1,s)}$ will change $2^n - 1$ to $2^n - 3$, since $2^n - 2$ already appears on the edge above. Using this general strategy of deferring the application of a letter if it changes an edge annotation that is already on an edge above and part of $B_1$, we can obtain the required block permutation $B_1$ along path $\pi_1$. Note that it is possible that all elements $2^n - rn - 1$, for all $r \in \{1, 2, \ldots, k\}$, where $k$ is the number of blocks may appear between the first element $2^n - 2n - 2$ and the second element $2^n - 5n - 1$ of $B_1$ in order.

Once all elements of $B_1$ appears along $\pi_1$, we “seal” path $\pi_1$, by applying the letter $a_{(1,\bot)}$, which affects only $q_{(i,s)}$ at the leaf of $\pi_1$ and does not affect the $\top$-states on the other paths. After this the state $(q_{(i,\bot)})$ and hence the edge annotations for $\pi_1$ do not ever change. We now apply $a_{(0,1)}$ to change $q_{(1,\top)}$ to $q_{(0,s)}$
at the leaf of the second path. We then use our usual strategy discussed above to obtain another block permutation along that path. Continuing this way we can obtain arbitrary block permutations of size \( n \) along every path in \((Q,H)\) trees.

![Diagram of block permutations](image)

Figure 8: Example transformation of path \( \pi_1 \)

Although, we consider only special types of \((Q,H)\)-trees, where the paths of the trees are disjoint from one another, we shall show that this is sufficient to generate enough trees to go beyond the \(2^{O(n^2 \log n)}\) upper bound given by our construction.

There are \( k = \lfloor 2^n \rfloor \) blocks of Streett pairs, with \( n \) elements in each block. Note that if \( 2^n \) mod \( n \neq 0 \) i.e \( n \) is not a power of 2, then some pairs may not appear in any block, but this does not affect our construction. Also, the pair \((E_{-1}, F_{-1})\) is not considered at all and serves only as a placeholder for the \( \perp \) states. Consider a block permutation \( B = (2^n - a_0 n, 2^n - a_1 n - 1, 2^n - a_2 n - 2, \ldots, 2^n - a_{n-1} n - n - 1) \), where \( a_1, \ldots, a_{n-1} \in \{1, \ldots, k\} \). Each element \( 2^n - a_i n - j \), for all \( i, j \in \{0, \ldots, n-1\} \) can be chosen from one of \( k \) blocks. There are \( n! \) ways of ordering the blocks themselves. Hence there are \( n! \times k^n \) ways of choosing a block permutation in each branch. Since, we consider \((Q,H)\)-trees that always have \( n \) disjoint branches/paths there are \((n! \times k^n)^n \) ways of choosing block permutations in all branches. But, \((n! \times k^n)^n = (n!)^n \times k^{n^2}\).

Since \( k = \frac{2^n}{n} \) and Stirling’s approximation gives us \( n! = \Omega\left(\left(\frac{2}{e}\right)^n\right) \), this is equal to \( \Omega\left(\frac{2^{n^2}}{n^{n^2}}\right) \times \frac{2^{n^3}}{n^{n^2}} \) or \( \Omega\left(\frac{2^{n^3}}{n^{n^2}}\right) \), which is \( 2^{\Omega(n^3)} \). Hence, the Safra-Schwoon construction generates \( 2^{\Omega(n^3)} \) \((Q,H)\)-trees, which are states of the DRW, while our construction gives a bound of \( 2^{O(n^2 \log n)} \) on the number of states of the constructed DPW/DRW. Since, the bounds for the Safra-Schwoon construction are obtained by counting \((Q,H)\)-trees without names, the same bounds work when constructing a DPW from an NSW using compact \((Q,H)\)-trees as described by Piterman[11].

Hence, its has been effectively demonstrated that our construction for determinization of \( \omega \)-automata using generalized witness sets, results in an improved worst case complexity bound for NSW determinization when the number of pairs of the NSW is \( h = 2^n \). Since, our construction constructs deterministic parity automata and complementing parity automata is trivial, the same arguments can be used to show an improved upper bound for NBW determinization.

In the following we show another interesting consequence of our construction. We show a new lower bound on the number of states of any \( \omega \)-automaton accepting a given \( \omega \)-regular language. Interestingly, this

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lower bound on the number of states is a function of the Rabin index of the \(\omega\)-regular language.

7. A new lower bound for \(\omega\)-automata

demonstrate a new lower bound on the number of states of any \(\omega\)-automaton that uses an acceptance condition based on infinity sets to accept a given \(\omega\)-regular language \(L\). Interestingly, this lower bound is a function of the Rabin index of the \(\omega\)-regular language. The Rabin index of an \(\omega\)-regular language is defined as follows.

**Definition 15 (Rabin Index).** Let \(\mathcal{L}(k)\) be the set of all \(\omega\)-regular languages that are accepted by DRW with \(k\) or less number of pairs. For any \(\omega\)-regular language \(L\) the smallest \(k\) such that \(L \in \mathcal{L}(k)\) is called the Rabin index of \(L\).

Wagner [22] and Kaminski [8] showed that the Rabin index is a property of an \(\omega\)-regular language and not of the deterministic pairs automaton accepting the given language. They also provided a characterization of the Rabin index in terms of structural properties of deterministic automata accepting a given \(\omega\)-regular language. We provide below a lower bound on the number of states of any \(\omega\)-automaton that uses an acceptance condition based on infinity sets and accepts an \(\omega\)-regular language with a given Rabin index.

**Theorem 16.** Given an \(\omega\)-regular language \(L\) with Rabin index \(k\), any \(\omega\)-automaton (deterministic or non-deterministic) that uses an acceptance condition based on infinity sets and accepts \(L\) must have at least \(\sqrt{k} - 1\) states.

**Proof 17.** Proof : Let \(A\) be an \(\omega\)-automaton with \(n\) states that uses an acceptance condition based on infinity sets and accepts \(L\). Using the construction of Section 3.2, we can obtain an equivalent DPW with at most \(O(n^2)\) states and \(2 \cdot (n^2 + n + 1)\) parity acceptance sets. This DPW can be interpreted as an equivalent DRW with the same number of states and at most \(n^2 + n + 1\) Rabin acceptance pairs. By definition of Rabin index we must have \(n^2 + n + 1 \geq k\). It follows that \(n \geq \sqrt{k} - 1\).

8. Conclusion

In this paper, we presented a new construction for determinization of \(\omega\)-automata whose acceptance condition is based on the notion of infinity sets. We extended the Safra/Piterman construction for NSW determinization using the concept of generalized witness sets to construct an equivalent DPW. We demonstrated, by way of an example, that there are families of NSW with \(O(n)\) states and \(2^n\) pairs for which our construction gives a DPW with better worst case complexity bounds than the Safra/Piterman construction. Effectively, we have improved the worst case complexity for NSW/NRW determinization. Also, there is no known direct determinization procedure for NMW; every known procedure uses an indirect method by first translating the NMW to either an NSW or an NBW and then using determinization on it. Our method provides a direct determinization construction for NMW. As an easy corollary of our construction, we demonstrate a new lower bound on the number of states of an \(\omega\)-automaton accepting a given \(\omega\)-regular language, as a function of the Rabin index of the language.

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