HÖRMANDER’S THEOREM FOR PARABOLIC EQUATIONS WITH COEFFICIENTS MEASURABLE IN THE TIME VARIABLE

N. V. KRYLOV

Abstract. We are dealing with possibly degenerate second-order parabolic operators whose coefficients are infinitely differentiable with respect to the space variables and only measurable with respect to the time variable. We impose the Hörmander condition on the diffusion coefficients and prove that the solutions of the corresponding equations with right-hand sides which are infinitely differentiable in the space variables in a space-time domain have also this property.

1. Introduction

In this article we are dealing with possibly degenerate second-order parabolic operators whose coefficients are infinitely differentiable with respect to the space variables and only measurable with respect to the time variable. Such operators arise, in particular, in the theory of stochastic diffusion processes and in filtering theory of partially observable diffusion processes. We impose the Hörmander condition on the diffusion coefficients and prove that the solutions of the corresponding equations with right-hand sides which are infinitely differentiable in the space variables in a space-time domain have also this property. One can say that we are proving a restricted hypoellipticity for our operators. The author intends to use this result to prove some kind of restricted hypoellipticity for stochastic partial differential equations.

The problem of hypoellipticity was solved by Hörmander (see the references in [5]) and attracted attention of very many researchers. In particular, there is a probabilistic approach to proving Hörmander’s hypoellipticity theorem initiated by Malliavin and extremely well presented in [4].

The exposition below basically follows the lines designed by Hörmander with substantial impact of Kohn and Oleinik and Radkevic (see [5, 7, 10, 11]). It would be very interesting to find a probabilistic proof of our results. So far, a few attempts by the author failed although the author of [12] succeeded in doing that in case the coefficients and their spatial derivatives are of class $C^1$ in $(t, x)$. Later the probabilistic approach allowed the authors of [2] to weaken the continuity hypotheses with respect to $t$ to just Hölder continuity.

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On the other hand, it is worth mentioning an interesting article [1] where the authors do basically some of the same steps as we do here but under global (restricted) Hörmander’s condition and with some of the arguments which the author of the present article could not quite follow, because some steps seem to be missing (see, for instance, our comment in parentheses below (5.2)). Another difference between our results and those in [1] is that we prove infinite differentiability of any generalized solution and not only of measure-valued ones.

Our main result is stated in Section 2 and proved in Section 6 preceded by Section 5 where we prove the main a priori estimate. Section 3 consists of one-page collection of well-known facts from the theory of pseudo-differential operators. In Section 4 we give estimates for the commutators of some operators and also prove a simple and rather weak a priori estimate for parabolic degenerate equations.

In conclusion we introduce some basic notation. By $\mathbb{R}^d$ we denote a Euclidean space of dimension $d$, $x = (x_1, \ldots, x_d)$ is a generic point of $\mathbb{R}^d$. All functions are assumed to be real valued. We denote by $Du$ the gradient of $u$, $D^2 u$ its Hessian, $D_i u = \partial u / \partial x^i$. If $a \in \mathbb{R}^d$, we denote $L_a = a^i D_i$ (the summation convention is always enforced). If $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a multi-index (meaning $\alpha_i = 0, 1, \ldots$), then

$$D^\alpha := D_1^{\alpha_1} \cdots D_d^{\alpha_d}, \quad |\alpha| := \alpha_1 + \ldots + \alpha_d.$$ 

Finally $\partial_t = \partial / \partial t$.

2. First steps, main ideas, and the main result

Introduce $BC_b^\infty$ as the set of real-valued or $\mathbb{R}^d$-valued measurable vector-fields $\sigma$ on

$$Q = \{(t, x) : t \in (0, 1), x \in \mathbb{R}^d\}$$

such that for each $t \in (0, 1)$, $\sigma(t, x)$ is infinitely differentiable with respect to $x$, and for any multi-index $\alpha$ we have

$$\sup_{t, x \in Q} |D^\alpha \sigma(t, x)| < \infty.$$ 

Let $d_1 \geq 1$ be an integer and let

$$L_k = \sigma^{ik}(t, x) D_i = \sum_{i=1}^d \sigma^{ik}(t, x) D_i, \quad k = 0, 1, \ldots, d_1,$$

be some given operators with coefficients $\sigma^{ik} \in BC_b^\infty$. Define

$$L = \partial_t - \sum_{k=1}^{d_1} L_k^2 + L_0.$$ 

We use $(u, v)_0$ and $\|u\|_0$ for the scalar product and the norm in $L_2 = L_2(Q)$. Set

$$H^{1,2} = \{ u \in L_2 : \partial_t u, Du, D^2 u \in L_2, u(0+, \cdot) = 0 \}. \quad (2.1)$$
To explain what we mean by $u(0^+, \cdot)$ recall that if $u, \partial_1 u \in L^2$, then there exists a $v$ such that $u = v$ (a.e.) in $Q$ and $v(t, \cdot)$ is a continuous $L^2(\mathbb{R}^d)$-valued function defined on $[0, 1]$. Therefore $u(0^+, \cdot)$ is well defined as $v(0, \cdot)$. In the same way, $u(1^-, \cdot)$ is well defined. The following simple fact is true.

**Lemma 2.1.** There is a constant $N$ such that for any $u \in H^{1,2}$

$$\sum_{k=1}^{d_1} \|L_k u\|_0^2 \leq (Lu)_0 + N \|u\|_0^2, \quad (2.2)$$

or, equivalently, for any $u \in H^{1,2}$ such that $Lu = f$

$$\sum_{k=1}^{d_1} \|L_k u\|_0^2 \leq (f, u)_0 + N \|u\|_0^2. \quad (2.3)$$

**Proof.** We multiply $Lu = f$ through by $u$ and integrate. We get

$$- \sum_{k=1}^{d_1} (L_k^2 u, u)_0 + \frac{1}{2} \sum_{i=1}^d \int_Q \sigma^{i0} D_i (u^2) + \partial_t (u^2) \, dx \, dt = (f, u)_0.$$ 

Next, we integrate by parts and use that the derivatives of $\sigma$ are assumed to be bounded. Then we obtain

$$\int_Q \sigma^{i0} D_i (u^2) \, dx \, dt \leq N \|u\|_0^2, \quad \int_Q \partial_t (u^2) \, dx \, dt = \int_{\mathbb{R}^d} u^2 (1 -, x) \, dx \geq 0,$$

$$- \sum_{k=1}^{d_1} (L_k^2 u, u)_0 = \sum_{k=1}^{d_1} (L_k u, L_k u)_0 + \int_{\mathbb{R}^d} u L'u \, dx \, dt,$$

where $L'u = (D_i \sigma^{ik}) L_k u$. As above

$$| \int_Q u L'u \, dx \, dt | \leq N \|u\|_0^2,$$

and to get (2.3), it only remains to combine the above results. The lemma is proved.

**Remark 2.2.** Later on we will use the fact that the above proof can be organized differently. We have

$$\sum_{k=1}^{d_1} \|L_k u\|_0^2 = \sum_{k=1}^{d_1} (L_k u, L_k u)_0 = (u, v)_0,$$

where $v = \sum_{k=1}^{d_1} L_k^* L_k u$. Obviously,

$$v = - \sum_{k=1}^{d_1} L_k^2 u + \sum_{k=1}^{d_1} [L_k + L_k^*] L_k u = Lu - \partial_t u + L'_0 u,$$

where $L'_0$ is a first-order differential operator with respect to $x$ whose coefficients are in $BC_c^{\infty}$. Furthermore,

$$(L'_0 u, u)_0 = (u, (L'_0)^* u)_0 = \frac{1}{2} ([L'_0 + (L'_0)^*] u, u)_0,$$
where \([L_0' + (L_0')^*]\) is an operator of multiplying by a \(BC_\infty^\infty\)-function. Hence,
\[
\|(L_0'u, u)_0 \| \leq N \|u\|_0^2, \quad (u, v)_0 \leq (Lu, u)_0 + N \|u\|_0^2.
\]

**Corollary 2.3.** We have
\[
\|L_k u\|_0 \leq N \|u\|_0^{1/2} (\|Lu\|_0^{1/2} + \|u\|_0^{1/2}) \leq N (\|Lu\|_0 + \|u\|_0) \quad \forall k \geq 1. \tag{2.4}
\]

Indeed, it suffices to use (2.2) and the inequality
\[
(Lu, u)_0 \leq \|Lu\|_0 \|u\|_0.
\]

If we knew that for any \(\xi \in \mathbb{R}^d\) there exist \(b_1, ..., b_{d^1}\) in \(BC_\infty^\infty\) such that for all \(i = 1, ..., d\)
\[
\xi^i = b_k \sigma^{ik}
\]
in \(Q\), then (2.4) would imply
\[
\|Du\|_0 \leq N (\|f\|_0 + \|u\|_0), \tag{2.6}
\]
where \(f = Lu\). Next, one can hope to estimate second-order derivatives of \(u\) by differentiating the equation \(Lu = f\) (if \(u, f\) are smooth enough), hopefully getting a “good” equation for \(u_x\), so that
\[
\|D^2 u\|_0 \leq N (\|Df\|_0 + \|Du\|_0) \leq N (\|Df\|_0 + \|f\|_0 + \|u\|_0).
\]

Keeping dreaming along the same lines, one arrives at
\[
\|D^\alpha u\|_0 \leq N (\sum_{|\beta| \leq |\alpha|} \|D^\beta f\|_0 + \|u\|_0) \tag{2.7}
\]
for any multi-index \(\alpha\). This shows that one has a control on smoothness of \(u\) in terms of \(L_2\) given that \(f\) is smooth. Sobolev’s embedding theorems show that one has a control on smoothness of \(u\) in the uniform norm as well. This turns out to be quite sufficient for proving that \(u\) is infinitely differentiable in \(x\) with the derivatives square integrable in \(t\). Then the fact that, for each \(t\), \(u(t, x)\) is infinitely differentiable in \(x\) follows after integrating in \(t\) the relation \(Lu = f\).

It turns out that the assumption related to (2.5) can be relaxed and the estimate as strong as (2.4) is not needed. It suffices to have, say \(m \mathbb{R}^d\)-valued functions \(a_1, ..., a_m \in BC_\infty^\infty\) such that, for any \(\xi \in \mathbb{R}^d\), one could find real-valued \(b_1, ..., b_m \in BC_\infty^\infty\) satisfying
\[
\xi = b_1 a_1 + ... + b_m a_m
\]
in \(Q\) and such that for each \(k = 1, ..., m\) and any \(u \in H^{1,2}\) we have
\[
\|L_{a_k} u\|_{-\delta} \leq N (\|Lu\|_0 + \|u\|_0), \tag{2.8}
\]
where \(\delta \in (0, 1)\) and \(N\) are independent of \(k\) and \(u\), and \(\|\cdot\|_{-\delta}\) is the negative norm of order \(-\delta\). In that case instead of (2.6) we would have
\[
\|Du\|_{-\delta} \leq N (\|f\|_0 + \|u\|_0), \quad \|u\|_{1-\delta} \leq N (\|f\|_0 + \|u\|_0).
By interpolation inequalities, if $\delta \leq 1/2$, $\|u\|_0 \leq \varepsilon \|u\|_{1-\delta} + N(\varepsilon)\|u\|_{-\delta}$ for any $\varepsilon > 0$. This yields
\[
\|u\|_{1-\delta} \leq N(\|f\|_0 + \|u\|_{-\delta}), \quad \|Du\|_{-\delta} \leq N(\|f\|_0 + \|u\|_{-\delta}).
\] (2.9)
and this allows for iterations as is outlined above. Then in case $Lu = f \in BC_{b}^{\infty}$ one could again differentiate this equation and obtain estimates of higher-order derivatives.

Hörmander discovered that in his situation of elliptic operators with smooth coefficients one can obtain (2.8) for $L_k, k = 0, \ldots, d_1$, for their commutators, and then for the commutators of higher order. Then under the condition that, a finite number of thus obtained vector fields generates the whole space, one comes to global estimates like (2.7) and some additional but almost standard effort is needed in order to show that if, say, $u$ is a generalized function in a domain such that $Lu = 0$ then $u$ is infinitely differentiable with respect to $x$ in this domain.

As we have mentioned, we are following arguments in [7] and [10]. However unlike [10], in our setting we could not obtain (2.8) for $L_0$ and, therefore, we are basically bound to the restricted version of arguments in [10] mimicking those in [7]. Accordingly, set $\mathbb{L}_0 = \{L_1, \ldots, L_{d_1}\}$,

$$\mathbb{L}_{n+1} = \mathbb{L}_n \cup \{[L_k, M] : k = 1, \ldots, d_1, M \in \mathbb{L}_n\}, \quad n \geq 0,$$

where $[L_k, M] = L_k M - M L_k$. Also we denote by $\text{Lie}_n$ the set of (finite) linear combinations of elements of $\mathbb{L}_n$ with real-valued coefficients of class $BC_{b}^{\infty}$. Observe that the operator $L_0$ is not explicitly included into $\text{Lie}_n$.

Fix a domain $G \subset Q$. Everywhere in the paper we impose the following.

**Assumption 2.4.** There exists an $n \in \{0, 1, \ldots\}$ such that for any $\zeta \in C_0^\infty(G)$ we have $\zeta D_i \in \text{Lie}_n$ for any $i = 1, \ldots, d$.

In order to state our main result we introduce the necessary function spaces. Let $\mathcal{D}(Q)$ be the set of generalized functions on $Q$. We work in the usual scale of Sobolev-Hilbert spaces defined for any $m \in \mathbb{R}$ by

$$H^m = \Lambda^{-m} L_2(\mathbb{R}^d), \quad \|u\|_{H^m} = \|\Lambda^m u\|_{H^0},$$

where

$$\Lambda = (1 - \Delta)^{1/2}.$$ 

For $m \in \mathbb{R}$ define

$$\mathbb{H}^m = \{u \in \mathcal{D}(Q) : \Lambda^m u \in L_2\}, \quad \|u\|_m = \|\Lambda^m u\|_0.$$ 

Also introduce $\mathbb{H}^{1,m}$ as the set of functions $u \in \mathbb{H}^m$ such that $u(t, \cdot) \in H^{m-1}$ for any $t \in (0, 1)$ and there exists an $f \in \mathbb{H}^{m-2}$ such that for any $\phi \in C_0^\infty(\mathbb{R}^d)$ and any $t \in (0, 1)$ we have

$$u(t, \cdot), \phi\rangle = \int_0^t (f(s, \cdot), \phi) \, ds.$$ (2.10)
In case (2.10) holds we, naturally, write $\partial_t u = f$. Briefly, one can write that

$$\mathbb{H}^{1,m} = \{ u \in L_2((0,1), H^m) : \partial_t u \in L_2((0,1), H^{m-2}), u(0+,\cdot) = 0 \}. \quad (2.11)$$

The above more detailed definition just makes it precise what we mean by $\partial_t u$ and also emphasizes the fact that for $u \in \mathbb{H}^{1,m}$ the distributions $u(t,\cdot)$ are uniquely defined for any $t \in (0,1)$. Observe that what was said above about functions in $H^{1,2}$ remains true for functions in $\mathbb{H}^{1,2}$, the latter being just the collection of the modifications of elements of $H^{1,2}$ as described before Lemma 2.1 or in the following remark.

**Remark 2.5.** One knows (see, for instance, Theorem 3 in §5.9.2 of [3]) that if $u \in L_2((0,1), H^{m})$ and there is an $f \in \mathbb{H}^{m-2}$ such that, for any $\phi \in C_0^\infty(\mathbb{R}^d)$, equation (2.10) holds for almost all $t \in (0,1)$, then there exists $v \in \mathbb{H}^{1,m}$ such that $v(t,\cdot)$ is a uniformly continuous $H^{m-1}$-valued function on $(0,1)$, the distributions $v$ and $u$ coincide on $Q$ and (2.10) holds for all $t$ and $\phi \in C_0^\infty(\mathbb{R}^d)$ if we replace $u$ with $v$. In particular, this explains that the condition $u(0+,\cdot) = 0$ in (2.11) makes sense.

**Remark 2.6.** The space $\mathbb{H}^{1,m}$ is a Hilbert space with squared norm $\|u\|_n^2 + \|\partial_t u\|_{m-2}^2$. One may wonder why the $\mathbb{H}^{m-2}$-norm and not a different norm of $\partial_t u$ is entering the $\mathbb{H}^{1,m}$-norm of $u$. The reason is that we are going to deal with equations $Lu = f$ and with $L_2$-estimates of their spacial derivatives, say of order $m$, in terms of $f$ and lower order norms of $u$. In such situation the $L_2$-norm of spacial derivatives of $\partial_t u$ of order $m - 2$ is obtained from the equation itself.

By the way, also observe that almost obviously $\Lambda^n \mathbb{H}^{1,m} = \mathbb{H}^{1,m-n}$ and $\Lambda^n \partial_t = \partial_t \Lambda^n$ for all $m,n \in \mathbb{R}$.

Here is our main result. If $G$ is an open subset of $\mathbb{R}^d$, then by $C_0^\infty(G)$ we mean the set of infinitely differentiable functions on $G$ each of whose derivatives of any order is bounded in $D$.

**Theorem 2.7.** Let $u$ be a generalized function on $G$ such that for an $m \in \mathbb{R}$ we have $u \zeta \in \mathbb{H}^{1,m}$ for any $\zeta \in C_0^\infty(G)$. Take a $c \in BC_0^\infty$ and assume that

$$\zeta(L + c)u \in \bigcap_n \mathbb{H}^n$$

for any $\zeta \in C_0^\infty(G)$. Then

$$\zeta u \in \bigcap_n \mathbb{H}^{1,n} \quad (2.12)$$

for any $\zeta \in C_0^\infty(G)$. Furthermore, if, for some $a,b,r \in (0,1)$ and $\Gamma = (a,b) \times B_r$, where $B_r = \{ x \in \mathbb{R}^d : |x| < r \}$, we have $\bar{\Gamma} \subset G$ then for any $t \in (a,b)$ we have $u(t,\cdot) \in C_0^\infty(B_r)$ and for any multi-index $\alpha$

$$\sup_{(t,x) \in \Gamma} |D^\alpha u(t,x)| + \sup_{(t,x),(s,x) \in \Gamma} \frac{|D^\alpha u(t,x) - D^\alpha u(s,x)|}{|t-s|^{1/2}} < \infty. \quad (2.13)$$
Remark 2.8. It is easy to understand from our arguments how the left-hand side of (2.13) can be estimated in terms of \((L + c)u\) and \(u\). Let \(\zeta \in C_0^\infty (G)\) be such that \(\zeta = 1\) on an open set containing \(\bar{\Gamma}\). Then it turns out that for any \(\alpha\) and \(k\) such that \(2(k - |\alpha| - 2) > d\) the left-hand side of (2.13) is less than a constant independent of \(u\) times
\[
\|\zeta(L + c)u\|_k + \|\zeta u\|_m.
\]

3. Pseudo–differential operators

Let \(m \in \mathbb{R}\) and let \(A\) be a linear operator defined on \(\bigcup_n H^n\), mapping it into itself, and such that, for any \(n \in \mathbb{R}\), it is a bounded operator mapping \(H^n\) into \(H^{n+m}\). Then we say that \(A\) is an operator of order (at most) \(m\) and write \(\text{ord } A = m\).

There is a theory of so-called pseudo–differential operators (see [5]). We will be most interested in particular cases of such operators given by
(i) \(\Lambda^m\), which is a pseudo–differential operator of order \(m\),
(ii) the pseudo–differential operator of order zero which is multiplication by an infinitely differentiable function on \(\mathbb{R}^d\), whose any derivative of any order is bounded,
(iii) the first order pseudo–differential operators \(D_i, i = 1, ..., d\),
(iv) products of not more than seven of the above operators, and their finite linear combinations.

Denote by \(S^m\) the set of pseudo-differential operators of order \(m\) and recall a few facts from the theory of pseudo–differential operators. We borrow the next lemma from [5].

Lemma 3.1. (i). If \(A \in S^m\), then \(\Lambda^{-m} A, AA^{-m} \in S^0\), that is they are bounded operators on \(H^s\) for any \(s \in \mathbb{R}\).

(ii). If \(A_1 \in S^{m_1}\) and \(A_2 \in S^{m_2}\), then \(A_1 A_2 \in S^{m_1+m_2}\) and \([A_1, A_2] := A_1 A_2 - A_2 A_1 \in S^{m_1+m_2-1}\).

We also use a result on pointwise multipliers (see, for instance, [13]).

Lemma 3.2. Let \(m > 0\) and \(a\) be a real-valued function of class \(C_0^\infty (\mathbb{R}^d)\). Then for any \(n \in (-m, m)\) there exists a constant \(N\) such that for any \(u \in H^n\) we have
\[
\|a u\|_{H^n} \leq N\|a\|_{C_0^\infty (\mathbb{R}^d)} \|u\|_{H^n}.
\]

4. Preliminary estimates

Here is a result of simple manipulations.

Lemma 4.1. Let \(a\) and \(b\) be \(\mathbb{R}^d\)-valued \(C_0^\infty (\mathbb{R}^d)\) functions, \(n \in \mathbb{R}\), and let \(A \in S^n\). Then there is a constant \(N\) such that for any \(u \in H^2 \cap H^n\)
\[
|(L_a L_b u, Au)_{H^n}| \leq N \|L_b u\|_{H^n} (\|L_a u\|_{H^n} + \|u\|_{H^n}),
\]
\[
|(L_b L_a u, Au)_{H^n}| \leq N \|L_a u\|_{H^n} (\|L_b u\|_{H^n} + \|u\|_{H^n}).
\]
Proof. We have
\[ |(L_0 L_0 u, Au)_{H^0}| = |(L_0 u, L_0^* Au)_{H^0}| \leq \|L_0 u\|_{H^0} \|L_0^* Au\|_{H^0}, \]
where, owing to the fact that \( \text{ord } [L_0^*, A] \leq n \) and \( L_0^* u = -L_0 u + cu \) with \( c \in C_b^\infty(\mathbb{R}^d) \),
\[ \|L_0^* Au\|_{H^0} \leq A L_0^* u\|_{H^0} + \|[[L_0^*, A] u\|_{H^0} \leq N \|L_0 u\|_{H^n} + N \|u\|_{H^n}. \]
This proves (4.1).
Next,
\[ |(L_0 L_0 u, Au)_{H^0}| = |(L_0 u, L_0^* Au)_{H^0}| \leq \|L_0 u\|_{H^n} \|L_0^* Au\|_{H^{-n}}, \]
where
\[ \|L_0^* Au\|_{H^{-n}} \leq \|AL_0^* u\|_{H^{-n}} + \|[L_0^*, A] u\|_{H^{-n}} \leq N(\|L_0 u\|_{H^0} + \|u\|_{H^0}) \]
and the lemma is proved.

Now comes the key estimate for \([L_0, L_0]\)u.

**Lemma 4.2.** Let \( a \) and \( b \) be as in Lemma 4.1 and \( \varepsilon \leq 1 \). Then there is a constant \( N \) such that for any \( u \in H^2 \)
\[ \| [L_0, L_0] u \|_{H^{1/2-1}} \leq N(\|L_0 u\|_{H^{1-1}} + \|L_0^* u\|_{H^0} + \|u\|_{H^0}). \]

Proof. We proceed as in Remark 2.2, introduce
\[ A = \Lambda^{\varepsilon-2}[L_0, L_0], \]
and observe that \( \text{ord } A \leq \varepsilon - 1 \leq 0 \) and
\[ \| [L_0, L_0] u \|_{H^{1/2-1}} = ([L_0, L_0] u, Au)_{H^0} = (L_0 L_0 u, Au)_{H^0} - (L_0 L_0 u, Au)_{H^0}. \]
After that it suffices to use (4.1) and (4.2) and the fact that \( \|\cdot\|_{H^0} \leq \|\cdot\|_{H^n} \)
for \( n \leq 0 \). The lemma is proved.

**Corollary 4.3.** If \( a \in BC_b^\infty \) and for a constant \( N \)
\[ \|L_0 u\|_{\varepsilon-1} \leq N(\|L_0 u\|_0 + \|u\|_0) \quad \forall u \in H^{1,2}, \]
then (see Corollary 2.3) there exists a constant \( N \) such that
\[ \| [L_0, L_k] u \|_{\varepsilon/2-1} \leq N(\|L_0 u\|_0 + \|u\|_0) \quad \forall u \in H^{1,2}, k = 1, ..., d_1. \]

Since the operators \( L_k \) satisfy (4.3) (with \( \varepsilon = 1 \)), applying repeatedly
Corollary 4.3 and then using Lemma 3.2, we get the following.

**Theorem 4.4.** Let \( n \in \{0, 1, \ldots\} \) and \( L_0 \in \text{Lie}_n \). Then there are constants \( \varepsilon \in (0, 1) \) and \( N \) such that for all \( u \in H^{1,2} \) we have
\[ \|L_0 u\|_{\varepsilon-1} \leq N(\|L_0 u\|_0 + \|u\|_0). \]

Estimate (4.5) will play the role of (2.8) and will allow us to proceed as it is explained after Corollary 2.3.

We will also use the following lemma which does not require any Hörmander’s condition. The lemma is quite elementary, although as happens often with simple facts, its proof is rather long. Before stating it we remind the reader a classical fact (see, for instance, Section 5 of [8]).
Theorem 4.5. Let $c$ be a real-valued function belonging to $BC^\infty_0$ and $\delta > 0$. Then for any $m \in \mathbb{R}$ and any $f \in \mathbb{H}^m$ there is a unique $u \in \mathbb{H}^{1,m+2}$ such that $cu + Lu + \delta \Delta u = f$.

Here is the lemma.

Lemma 4.6. Let $c$ be a function belonging to $BC^\infty_0$ and $\delta \geq 0$. Then for any $m = 0, \pm 1, \pm 2, ...$ and $u \in \mathbb{H}^{1,m+2}$

$$\|u\|_m \leq N\|cu + Lu + \delta \Delta u\|_m,$$

where $N$ is independent of $u$ and $\delta$, and the set

$$\{cu + Lu + \delta \Delta u : u \in \mathbb{H}^{1,m+2}\}$$

is everywhere dense in $\mathbb{H}^m$.

Proof. First let $m \geq 0$. The usual change of the unknown function $v(t,x) = e^{\lambda t}u(t,x)$ shows that to prove (4.6) it suffices to show that there are $\lambda > 0$ and $N$ (independent of $u$) such that

$$\|u\|_m \leq N\|(c + \lambda)u + Lu + \delta \Delta u\|_m.$$  \hspace{1cm} (4.8)

Take $u \in \mathbb{H}^{1,m+2}$ and define $f = Lu + (c + \lambda)u$. To estimate derivatives of order $\leq m$ of $u$ we differentiate this equation several times and then integrate by parts. Actually, we can make a shortcut using Lemma 2.1. So, let $\alpha$ be a multi-index with $|\alpha| \leq m$. We have

$$D^\alpha Lu + \lambda D^\alpha u + D^\alpha (cu) = D^\alpha f.$$  \hspace{1cm} (4.9)

Here by usual calculus

$$D^\alpha Lu + D^\alpha (cu) = LD^\alpha u + \sum_{k \geq 1} L_k b^\alpha_k u + \sum_{k \geq 1} L_k \tilde{b}^\alpha_k u = LD^\alpha u + \sum_{k \geq 1} L_k b^\alpha_k u + \tilde{b}^\alpha_k u,$$

with $b^\alpha_m$ and $\tilde{b}^\alpha_t$ being certain usual differential operators of order $\leq m$. Also $D^\alpha u \in \mathbb{H}^{1,2}$. Hence from (4.9) by Lemma 2.1 we have

$$\sum_{k \geq 1} \|L_k D^\alpha u\|_0^2 \leq (D^\alpha f - \lambda D^\alpha u - \sum_{k \geq 1} L_k b^\alpha_k u - \tilde{b}^\alpha_k u, D^\alpha u)_0 + N\|u\|_m^2$$

$$\leq N\|f\|_m^2 - \lambda\|D^\alpha u\|_0^2 + N\sum_{k \geq 1} \|L_k D^\alpha u\|_0\|u\|_m + N\|u\|_m^2.$$  \hspace{1cm} (4.10)

By remembering that $ab \leq \delta a^2 + \delta^{-1} b^2$ and using this to estimate the products of norms in (4.10), we get

$$\lambda\|D^\alpha u\|_0^2 \leq N\|f\|_m^2 + N\|u\|_m^2.$$

Upon summing up with respect to $|\alpha| \leq m$, we conclude

$$\lambda\|u\|_m^2 \leq N_1\|f\|_m^2 + N_1\|u\|_m^2,$$

where $N_1$ is independent of $u$ and $\lambda$. By taking $\lambda_0 = 2N_1$, we finish the proof of (4.8) and (4.6) for $m \geq 0$ if $\delta = 0$. From the above argument it is not hard to see that, actually, (4.8) and (4.6) hold for any $\delta > 0$ with the same constants $\lambda_0$ and $N$. 

To prove (4.6) for \( m \leq 0 \) we first prove the second assertion of the lemma, which we need to do only for \( \delta = 0 \) in light of Theorem 4.5. Furthermore, since the spaces \( \mathbb{H} \) are nested it suffices to prove the denseness only for \( m \geq 0 \). As above we need only find a \( \lambda > 0 \) such that

\[
\left\{ (c + \lambda)u + Lu : u \in \mathbb{H}^{1,m+2} \right\}
\]

(4.11)
is everywhere dense in \( \mathbb{H}^m \).

The number \( \lambda_0 \), found above, depends on \( m \), and we can write \( \lambda_0 = \lambda_0(m) \). Without loss of generality we assume that \( \lambda_0(m) \) is an increasing function of \( m \geq 0 \) and we prove that the set (4.11) is dense in \( \mathbb{H}^m \) for \( m \geq 0 \) if \( \lambda = \lambda_0(m+2) \).

By Theorem 4.5 for \( \delta > 0 \)

\[
\left\{ (c + \lambda)u + Lu + \delta \Delta u : u \in \mathbb{H}^{1,m+4} \right\} = \mathbb{H}^{m+2}.
\]

Therefore, for any \( f \in \mathbb{H}^{m+2} \) and \( \delta > 0 \), one can find \( u_\delta \in \mathbb{H}^{1,m+4} \) such that

\[
Lu_\delta + \delta \Delta u_\delta + (\lambda + c)u_\delta = f.
\]

In addition, (remember \( \lambda = \lambda_0(m+2) \)),

\[
\|u_\delta\|_{m+2} \leq N \|f\|_{m+2}.
\]

Hence,

\[
\|Lu_\delta + (\lambda + c)u_\delta - f\|_m = \delta \|\Delta u_\delta\|_m \to 0.
\]

This along with the fact that \( \mathbb{H}^{m+2} \) is dense in \( \mathbb{H}^m \) shows that (4.11) is dense in \( \mathbb{H}^m \). Thus, (4.7) is also dense in \( \mathbb{H}^m \).

Now we prove (4.6) in the remaining case by using duality. Take \( m \geq 0 \) and observe that for \( v \in \mathbb{H}^0 \)

\[
\|v\|_{-m} = \sup_{\substack{f \in \mathbb{H}^{m} \\|f\|_m \leq 1}} (v, f)_0.
\]

\[
(4.12)
\]

Let \( \mathbb{H}^{1,m} \) be the collection of \( u(1-t,x) \), where \( u \in \mathbb{H}^{1,m} \). By reversing the time variable and using the above result one easily proves that the set

\[
\left\{ cu + L^*u + \delta \Delta u : u \in \mathbb{H}^{1,m+2} \right\}
\]
is everywhere dense in \( \mathbb{H}^m \). Therefore, for any \( f \) with \( \|f\|_m \leq 1 \) we can find a sequence \( u_n \in \mathbb{H}^{1,m+2} \) such that \( f_n := L^*u_n + cu_n \to f \) in \( \mathbb{H}^m \). By (4.6) applied in reversed time we get \( \|u_n\|_m \leq N \|f_n\|_m \) with \( N \) independent of \( f \) and \( u_n \). This proves that there is a constant \( N \) such that the set \( \{ \|f\|_m \leq 1 \} \) is a subset of the closure in \( \mathbb{H}^m \) of

\[
\left\{ L^*u + \delta \Delta u + cu : u \in \mathbb{H}^{1,m+2}, \|u\|_m \leq N \right\}.
\]

Hence, for \( v \in \mathbb{H}^{1,m+2} \)

\[
\|v\|_{-m} \leq \sup_{\substack{u \in \mathbb{H}^{1,m+2}, \|u\|_m \leq N}} (v, L^*u + \delta \Delta u + cu)_0
\]

\[
(4.13)
\]
\[
\sup_{u \in \mathcal{H}^{1,m+2}, \|u\|_m \leq N} (Lv + \delta \Delta v + cv, u)_0 \leq N\|Lv + \delta \Delta v + cv\|_{-m},
\]
and the lemma is proved.

**Remark 4.7.** Actually, the lemma is true for all \( m \) rather than for integers only. However, the proof of this requires more manipulations based on the theory of pseudo-differential operators and is not so elementary as the above one, the result of which is quite sufficient for our purposes.

5. **The main estimate in a particular case**

Throughout this section we suppose that a stronger condition than Assumption 2.4 is satisfied. Namely, we suppose that there exists an integer \( n \) such that, \( D_i \in \text{Lie}_n \) for any \( i = 1, \ldots, d \).

Observe that
\[
\|u\|_\varepsilon^2 = \|u\|_{\varepsilon-1}^2 + \sum_{i=1}^d \|D_i u\|_{\varepsilon-1}^2.
\]

This, together with Theorem 4.4, leads to the following.

**Corollary 5.1.** There are constants \( \varepsilon \in (0,1] \) and \( N \) such that for all \( u \in \mathcal{H}^{1,2} \) we have
\[
\|u\|_\varepsilon \leq N(\|Lu\|_0 + \|u\|_0).
\]

We thus get (2.8) and we may proceed as is explained in Section 2 moving to (2.9) and then starting differentiating the equation in order to obtain a priori estimates of higher order derivatives.

However, in Theorem 2.7 we are only given that \( u \) is in a negative space and we want to show step by step that its smoothness is by at least \( \varepsilon \) better, then by \( 2\varepsilon \) better and so on. That is why we want to derive from Corollary 5.1 that, with the same \( \varepsilon \in (0,1] \) for any \( m \in \mathbb{R} \), there is a constant \( N \) such that for all \( u \in \mathcal{H}^{1,m+2} \)
\[
\|u\|_{m+\varepsilon} \leq N(\|Lu\|_m + \|u\|_m)
\]

(this step is missing in [1] and in [6]).

In Section 2 we explained the idea of proving (5.2) on the basis of (5.1) by differentiating the equation \( Lu = f \). Since we are interested in estimates in \( H^m \) not only for \( m \geq 0 \), we apply the operator \( \Lambda^m \) to both sides of the equation \( Lu = f \). Actually, this amounts to substituting \( \Lambda^m u \) instead of \( u \) in (5.1).

If \( u \in \mathcal{H}^{1,m+2} \), then \( \Lambda^m u \in \mathcal{H}^{1,2} \) and after substituting we get
\[
\|u\|_{m+\varepsilon} \leq N(\|\Lambda^m u\|_0 + \|u\|_m) \leq N(\|Lu\|_m + \|L, \Lambda^m\|_0 + \|u\|_m).
\]

Here we get into some trouble since Lemma 3.1 only says that \( \text{ord } [L, \Lambda^m] \)
may be \( m+1 > m+\varepsilon \), so that we cannot absorb \( \|L, \Lambda^m\|_0 \) into either \( \|u\|_{m+\varepsilon} \) or \( \|u\|_m \). The help comes from “calculus”, which shows that \( [L, \Lambda^m] \) has a special form.
Lemma 5.2. If a is an $\mathbb{R}^d$-valued $C^\infty_b(\mathbb{R}^d)$ function and $b \in \mathbb{S}^m$, then

\[ [L^2_a, b] = b_1 L_a + b_2, \]

where $b_i \in \mathbb{S}^m$.

Indeed, $b L^2_a = L a b L_a + c L_a$ and $L a b L_a = L^2_a b + L a c$, where $c = [b, L_a]$. Also $L a c = c L_a + [L_a, c]$, where $\text{ord } [L_a, c] \leq \text{ord } c \leq \text{ord } b \leq m$.

Lemma 5.2 allows us to organize (5.3) differently.

Lemma 5.3. Let $m, n \in \mathbb{R}$ and $A_m \in \mathbb{S}^m$. Then there is a constant $N$ such that for any $u \in \mathbb{H}^{1,m+n+2}$ we have

\[ \|L A_m u\|_n \leq N(\|L u\|_{m+n} + \sum_{k \geq 1} \|L_k u\|_{m+n} + \|u\|_{m+n}). \quad (5.4) \]

Proof. Observe that

\[ \|L A_m u\|_n \leq \|A_m L u\|_n + \|[[A_m, L] u\|_n. \]

It follows that it only remains to estimate $[A_m, L] u$. However, by Lemma 5.2

\[ [A_m, L] = \sum_{k \geq 1} b_k L_k + b_0, \]

where $\text{ord } b_r = m, r = 0, \ldots, d_1$. Hence,

\[ \|[[A_m, L] u\|_n \leq N(\sum_{k \geq 1} \|L_k u\|_{m+n} + \|u\|_{m+n}), \]

and the lemma is proved.

An extra term with $L_k u$ on the right in (5.4) suggests that we look back at (2.4). Indeed, it turns out that, by using (2.2), one can get a somewhat stronger estimate of $L_k u$ than what is needed at this stage. We mean

\[ \sum_{k \geq 1} \|L_k u\|_{m+\varepsilon/2} \leq N(\|L u\|_m + \|u\|_m), \quad (5.5) \]

which, along with (5.4) with $n = 0$ and the first inequality in (5.3) would certainly finish the proof of (5.2).

Theorem 5.4. Take $\varepsilon$ from Corollary 5.1 and let $c \in BC^\infty_b$. Then for any $m, n \in \mathbb{R}$, there is a constant $N$ such that for all $u \in \mathbb{H}^{1,m+n+2}$

\[ \|u\|_{m+\varepsilon} + \sum_{k \geq 1} \|L_k u\|_{m+\varepsilon/2} \leq N(\|(L + c) u\|_m + \|u\|_n). \quad (5.6) \]

Proof. We are going to prove that for any $m, p \in \mathbb{R}$, there is a constant $N$ such that for all $u \in \mathbb{H}^{1,m+n+2}$

\[ \|u\|_{m+\varepsilon} + \sum_{k \geq 1} \|L_k u\|_{m+\varepsilon/2} \leq N(\|(L + c) u\|_m + \|u\|_p + \sum_{k \geq 1} \|L_k u\|_{p-\varepsilon/2}). \quad (5.7) \]

This looks like a weaker estimate than (5.6), but actually by taking $p = n - 1$ in (5.7) and observing that

\[ \|L_k u\|_{n-1-\varepsilon/2} \leq \|L_k u\|_{n-1} \leq N\|u\|_n, \quad \|u\|_{n-2} \leq \|u\|_n \]
we obtain (5.6).

Next, if we have (5.7) for \( p = m \), then for larger \( p \) we get it because then \( \| \cdot \|_m \leq \| \cdot \|_p \). On the other hand, iterating (5.7) with \( p = m \), we get it for all \( p \leq m \). Hence it suffices to concentrate on \( p = m \). In this situation the observation that \( \| Lu \|_m \leq \| (L + c)u \|_m + N\|u\|_m \) allows us to assume that \( c \equiv 0 \).

We have

\[
R_{m+\epsilon/2} := \sum_{k \geq 1} \| L_k u \|_{m+\epsilon/2} \leq \sum_{k \geq 1} \| L_k \Lambda^{m+\epsilon/2} u \|_0 \\
+ \sum_{k \geq 1} \| [L_k, \Lambda^{m+\epsilon/2}] u \|_0 =: I_1 + I_2,
\]

where \( \text{ord} [L_k, \Lambda^{m+\epsilon/2}] \leq m + \epsilon/2 \), so that by interpolation

\[
I_2 \leq N\|u\|_{m+\epsilon/2} \leq N\|u\|_m^{1/2}\|u\|_m^{1/2+\epsilon/2}.
\]

Owing to (2.2) and (5.4), we write for \( I_1 \)

\[
I_1^2 \leq N\|L\Lambda^{m+\epsilon/2}u\|_{m+\epsilon/2} \|\Lambda^{m+\epsilon/2}u\|_{m+\epsilon/2} + \|\Lambda^{m+\epsilon/2}u\|_0^2 \leq 0
\]

\[
= N\|L\Lambda^{m+\epsilon/2}u\|_{m+\epsilon/2} \|\Lambda^{m+\epsilon/2}u\|_{m+\epsilon/2} + \|u\|_{m+\epsilon/2}^2
\]

\[
\leq N\|u\|_{m+\epsilon} (\|Lu\|_m + R_m + \|u\|_m).
\]

Thus,

\[
R_{m+\epsilon/2} \leq N(\|Lu\|_m + R_m + \|u\|_m)^{1/2}\|u\|_{m+\epsilon}^{1/2},
\]

which along with the first inequality in (5.3) and (5.4) shows that

\[
\|u\|_{m+\epsilon} + R_{m+\epsilon/2} \leq N(\|Lu\|_m + R_m + \|u\|_m)
\]

\[
+ N(\|Lu\|_m + R_m + \|u\|_m)^{1/2}\|u\|_{m+\epsilon}^{1/2}.
\]

It follows that

\[
\|u\|_{m+\epsilon} + R_{m+\epsilon/2} \leq N(\|Lu\|_m + R_m + \|u\|_m)
\]

and since again by interpolation inequalities

\[
R_m \leq NR_{m+\epsilon/2}^{1/2} R_{m-\epsilon/2}^{1/2},
\]

we obtain (5.7) with \( p = m \). The theorem is proved.
6. Proof of Theorem 2.7

We derive Theorem 2.7 from an “interior” version of Theorem 5.4. There is a way to localize the result of Theorem 5.4 by using methods from the theory of pseudo-differential operators and the specific features of the problem. We prefer to give a more universal and absolutely standard proof which works in a great variety of situations regardless of what kind of global estimates are obtained in Hölder or Sobolev spaces.

We need a special cut-off function which is used in the statement and the proof of the following lemma bearing on interior estimates. Take an infinitely differentiable function $h(p)$ of one variable $p \in \mathbb{R}$ such that $h(p) = 1$ for $p \leq 1$, $h(p) = 0$ for $p \geq 2$, and $0 \leq h \leq 1$. For any $r > 0$ define

$$
\xi_r(x) = h((2|x| - r)/r), \quad \eta_r(t) = h((2|t| - r)/r), \quad \zeta_r(t, x) = \xi_r(x)\eta_r(t)
$$

and for $(t_0, x_0) \in Q$ set

$$
\zeta^{t_0, x_0}_r(t, x) = \zeta_r(t - t_0, x - x_0).
$$

Observe that $\zeta^{t_0, x_0}_r(t, x) = 1$ for $(t, x) \in Q^{t_0, x_0}_r$ and $\zeta_r(t, x) = 0$ outside $Q^{t_0, x_0}_{3r/2}$, where

$$
Q^{t_0, x_0}_r = \{(t, x) : |x - x_0| < r, |t - t_0| < r\}.
$$

Lemma 6.1. Let $(t_0, x_0) \in G$. Then there exist $\varepsilon, R \in (0, 1]$ such that

$$
Q^{t_0, x_0}_{6R} \subset G
$$

(6.1)

and for any $c \in BC^\infty_b, m, n \in \mathbb{R}$, with $n \leq m$, and $r \leq R$

$$
\|\zeta^{t_0, x_0}_r u\|_{m+\varepsilon} + \sum_{k \geq 1} \|\zeta^{t_0, x_0}_r L_k u\|_{m+\varepsilon/2}
\leq N r^{-\alpha} (\|\zeta^{t_0, x_0}_r (L + c) u\|_m + \|\zeta^{t_0, x_0}_r u\|_n)
$$

(6.2)

whenever $\zeta^{t_0, x_0}_r u \in \mathbb{H}^{1, m+2}$, where $N, \alpha > 0$ are independent of $u$ and $r$ (as a matter of fact, one can take $\alpha = 2\tau + 2\tau(1 + m - n)\varepsilon^{-1}$ with $\tau = \max(|m|, |n|) + 3$).

Proof. Take $R$ so small that (6.1) holds and observe that changing $L$ outside $Q^{t_0, x_0}_{3R}$ does not affect $\zeta_r(L + c) u$ for $r \leq R$ since $\zeta_r = 0$ outside $Q^{t_0, x_0}_{3R}$. Bearing this in mind, take a function $\zeta \in C^\infty_0(\mathbb{R}^{d+1})$ such that $\zeta = 0$ on $Q^{t_0, x_0}_{3R}$, $\zeta = 1$ outside $Q^{t_0, x_0}_{4R}$, and $0 \leq \zeta \leq 1$. Define

$$
L'_i = \zeta D_i, \quad i = 1, \ldots, d, \quad L' = \partial_t - \sum_{k \geq 1} L_k^2 - \sum_{i \geq 1} (L'_i)^2 + L_0.
$$

Observe that owing to Assumption 2.4 for any $i = 1, \ldots, d$ we have $(1 - \zeta) D_i \in \text{Lie}_n$ and the formula $e_i = (1 - \zeta) e_i + \zeta e_i$ shows that $D_i$ are in $\text{Lie}_n$ constructed from $L_k, L'_j$. This modification of $L$ outside $Q^{t_0, x_0}_{3R}$ had only one purpose to be able to formally apply Theorem 5.4.
Now fix \( r \leq R \) and for integers \( j \geq 0 \) define
\[
  r_j = r \sum_{i=0}^{j} 2^{-i}, \quad \xi^j(x) = h(2^{j+1}(|x - x_0| - r_j + r2^{-(j+1)})/r),
\]
\[
  \eta^j(t) = h(2^{j+1}(|t - t_0| - r_j + r2^{-(j+1)})/r), \quad \zeta^j(t, x) = \xi^j(x)\eta^j(t),
\]
so that
\[
  r_0 = r, \quad r_j \uparrow 2r, \quad \zeta^0 = \zeta_{r_0,x_0}^0.
\]
Also, \( \zeta^j \in C_0^\infty(\mathbb{R}^{d+1}) \), \( \zeta^j = 1 \) in \( Q_{r_j}^{l_0,x_0} \), \( \zeta^j = 0 \) outside \( Q_{r_j+1}^{l_0,x_0} \), and for \( \tau := \max(|m|, |n|) + 3 \)
\[
  \sup_{|\alpha| \leq \tau-2, t, x} |D^\alpha \partial_t \zeta^j| + \sup_{|\alpha| \leq \tau, t, x} |D^\alpha \zeta^j| \leq Nr^{-\tau}2^{\tau j}, \quad (6.3)
\]
where \( N \) is independent of \( j \) and \( r \). All such constants below are denoted by \( N \) without specifying each time that they are independent of \( j \) and \( r \).
Actually, (6.3) holds for any \( \tau \) with \( N \) depending on \( \tau \). Our particular choice of it is dictated by Lemma 3.2. To finish with notation, let \( f = (L + c)u \).
Since \( \zeta_{2R}^{l_0,x_0} \in \mathcal{H}^{1,m+2} \) by assumption, we can substitute \( u\zeta^j \) in (5.6).
Then we get
\[
  \|u\zeta^j\|_{m+\varepsilon} + \sum_{k \geq 1} \|uL_k \zeta^j + \zeta^jL_k u\|_{m+\varepsilon/2}
\]
\[
  \leq N(\|\zeta^j f + 2 \sum_{k \geq 1} (L_k u)L_k \zeta^j + uL \zeta^j\|_m + \|u\zeta^j\|_n). \quad (6.4)
\]
Here owing to (6.3) and Lemma 3.2
\[
  \|u\zeta^j\|_n = \|\zeta^j \{\zeta_{2r} u\}\|_n \leq Nr^{-\tau}2^{\tau j}\|\zeta_{2r} u\|_n,
\]
\[
  \|\zeta^j f\|_m = \|\zeta^j \{\zeta_{2r} f\}\|_m \leq Nr^{-\tau}2^{\tau j}\|\zeta_{2r} f\|_m,
\]
\[
  \|uL \zeta^j\|_m = \|u\zeta^{j+1}L \zeta^j\|_m \leq Nr^{-\tau}2^{\tau j}\|u\zeta^{j+1}\|_m,
\]
\[
  \|(L_k u)L_k \zeta^j\|_m = \|\zeta^{j+1}(L_k u)\zeta^j\|_m \leq Nr^{-\tau}2^{\tau j}\|\zeta^{j+1}L_k u\|_m,
\]
\[
  \|uL_k \zeta^j + \zeta^j L_k u\|_{m+\varepsilon/2} \geq \|\zeta^j L_k u\|_{m+\varepsilon/2} - Nr^{-\tau}2^{\tau j}\|u\zeta^{j+1}\|_{m+\varepsilon/2}.
\]
Hence (6.4) implies that
\[
  I_j := \|u\zeta^j\|_{m+\varepsilon} + \sum_{k \geq 1} \|\zeta^j L_k u\|_{m+\varepsilon/2}
\]
\[
  \leq N_1 r^{-\tau}2^{\tau j}(\|\zeta_{2r} f\|_m + \|\zeta_{2r} u\|_n + \|u\zeta^{j+1}\|_{m+\varepsilon/2} + \sum_{k \geq 1} \|\zeta^{j+1} L_k u\|_m). \quad (6.5)
\]
Next, we use the interpolation inequality \( \|v\|_k \leq \gamma^{l-k}\|v\|_l + \gamma^{p-k}\|v\|_p \) for any \( \gamma > 0 \) if \( k \) is between \( l \) and \( p \) (which immediately follows from the inequality \( a^{2k} \leq a^{2l} + a^{2p} \)). Then for any \( \delta > 0 \)

\[
N_1\|\zeta^{j+1}L_k u\|_m \leq \delta^{\varepsilon/2}\|\zeta^{j+1}L_k u\|_{m+\varepsilon/2} + N\delta^{n-m-1}\|\zeta^{j+1}L_k u\|_{n-1},
\]

where again by (6.3)

\[
\|\zeta^{j+1}L_k u\|_{n-1} = \|\zeta^{j+1}L_k (\zeta_2 u)\|_{n-1}
\]

\[
\leq Nr^{-\tau}2^{\tau j}\|L_k (\zeta_2 u)\|_{n-1} \leq Nr^{-\tau}2^{\tau j}\|\zeta_2 u\|_n.
\]

Therefore, by introducing a parameter \( \gamma > 0 \), which will be specified later, defining \( \delta \) from the equation \( r^{-\tau}2^{\tau j}\delta^{\varepsilon/2} = \gamma \), and setting

\[
\alpha = 2\tau + 2\tau(1 + m - n)\varepsilon^{-1},
\]

we find

\[
N_1 r^{-\tau}2^{\tau j}\|\zeta^{j+1}L_k u\|_m \leq \gamma\|\zeta^{j+1}L_k u\|_{m+\varepsilon/2} + N(\gamma)r^{-\alpha}2^{\alpha j}\|\zeta_2 u\|_n,
\]

where \( N(\gamma) \) depends on \( \gamma \) but is independent of \( u, r, j \).

Similarly,

\[
N_1\|u\zeta^{j+1}\|_{m+\varepsilon/2} \leq \delta^{\varepsilon/2}\|u\zeta^{j+1}\|_{m+\varepsilon} + N\delta^{n-m-1}\|u\zeta^{j+1}\|_{n-1+\varepsilon/2},
\]

\[
\|u\zeta^{j+1}\|_{n-1+\varepsilon/2} \leq Nr^{-\tau}2^{\tau j}\|\zeta_2 u\|_{n-1+\varepsilon/2} \leq Nr^{-\tau}2^{\tau j}\|\zeta_2 u\|_n,
\]

\[
N_1 r^{-\tau}2^{\tau j}\|u\zeta^{j+1}\|_{m+\varepsilon/2} \leq \gamma\|u\zeta^{j+1}\|_{m+\varepsilon} + N(\gamma)r^{-\alpha}2^{\alpha j}\|\zeta_2 u\|_n.
\]

Hence coming back to (6.5), we get

\[
I_j \leq I_{j+1} + N(\gamma)r^{-\alpha}2^{\alpha j}M,
\]

(6.6)

where \( M := \|\zeta_2 u\|_m + \|\zeta_2 u\|_n \). Now we chose \( \gamma \) so that

\[
\gamma 2^\alpha = 1/2,
\]

multiply both parts of (6.6) by \( \gamma^j \), sum up for \( j = 0, 1, 2, \ldots \). Then we obtain

\[
I_0 + S \leq S + Nr^{-\alpha}M,
\]

(6.7)

where

\[
S := \sum_{j=1}^{\infty} \gamma^j (\|u\zeta^j\|_{m+\varepsilon} + \sum_{k \geq 1} \|L_k u\|_{m+\varepsilon/2}),
\]

and as above in light of (6.3)

\[
\|u\zeta^j\|_{m+\varepsilon} + \sum_{k \geq 1} \|L_k u\|_{m+\varepsilon/2} \leq Nr^{-\tau}2^{\tau j} (\|u\zeta_2\|_{m+\varepsilon} + \sum_{k \geq 1} \|L_k u\|_{m+\varepsilon/2}),
\]

which along with the inequality \( \gamma 2^\tau < 1 \) and the fact that \( \zeta_0, x_0, u \in \mathbb{P}^{m+2} \) yield that \( S < \infty \). This shows that (6.7) coincides with (6.2) and the lemma is proved.
Proof of Theorem 2.7. To prove the first assertion it suffices to show that for any \((t_0, x_0) \in G\) there is a function \(\zeta \in C^\infty(G)\) which equals one in a neighborhood of \((t_0, x_0)\) and is such that \(\zeta u \in \mathbb{H}^{1,k}\) for all \(k\).

Fix a point \((t_0, x_0) \in G\), take \(R\) and \(\varepsilon\) from Lemma 6.1, and reduce \(R\) if necessary so that

\[
v := \zeta_{t_0,x_0} L u \in \mathbb{H}^{1,m}.
\]

Next, define

\[
f = \nabla v + cv
\]

and observe that, since \(v \in \mathbb{H}^{1,m}\), we have \(f \in \mathbb{H}^{m-2}\). Therefore by Theorem 4.5, for \(\delta > 0\), there exists a unique solution \(v_\delta \in \mathbb{H}^{1,m}\) of the equation

\[
L v_\delta + \delta \nabla v_\delta + cv_\delta = f.
\]

By Lemma 4.6

\[
\sup_\delta \|v_\delta\|_{m-2} < \infty, \quad \text{(6.8)}
\]

\[
\|v - v_\delta\|_{m-4} \leq N \|L (v - v_\delta) + c(v - v_\delta)\|_{m-4} = \|\delta \nabla v_\delta\|_{m-4} \to 0 \quad \text{(6.9)}
\]

as \(\delta \downarrow 0\).

Next, since on \(Q_R^{t_0,x_0}\) we have \(u = v\) and \(Lu = L v\), it holds that \(f = Lu + cu\) on \(Q_R^{t_0,x_0}\) where by assumption the right-hand side on \(Q_R^{t_0,x_0}\) is a restriction of an \(\cap_k \mathbb{H}^k\)-function. By parabolic interior regularity theory for uniformly nondegenerate equations (see, for instance, the proof of Corollary 4.2.1 of [9]), \(\zeta_{t_0,x_0} v_\delta \in \cap_k \mathbb{H}^{1,k}\) for any \(r \in (0, R)\), which along with Lemma 6.1 implies that, for any \(k\),

\[
\|\zeta_{t_0,x_0} v_\delta\|_{k + \varepsilon} \leq N \|\zeta_{t_0,x_0} (Lu + cu)\|_k + N (1 + \delta) \|v_\delta\|_{m-2},
\]

where \(N\) are independent of \(\delta\). Hence, by (6.8) (see also Remark 2.6), we get that \(\zeta_{t_0,x_0} v_\delta\) are uniformly bounded in \(\mathbb{H}^{1,k}\) for \(\delta \in (0, 1)\) and, by (6.9), that \(\zeta_{t_0,x_0} v = \zeta_{t_0,x_0} u \in \mathbb{H}^{1,k}\). This proves the first assertion of the theorem.

The second assertion of the theorem follows from the first one by embedding theorems. Indeed, the fact that \(\zeta u \in \mathbb{H}^{1,n}\) for all \(n \geq 1\) implies that \(\partial_t (\zeta u) \in \mathbb{H}^n\) for all \(n \geq 1\) and then equation (2.10) implies that \(\zeta (\cdot, t) u(t, \cdot) \in H^n\) for any \(t \in (0, 1)\) and is \(1/2\)-H"older continuous with respect to \(t\) in the \(H^n\)-norm. Since this holds for any \(n\), an application of the Sobolev embedding theorem with an appropriate \(\zeta\) yields (2.13).

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E-mail address: krylov@math.umn.edu

127 Vincent Hall, University of Minnesota, Minneapolis, MN, 55455