Improving active eigenvector assignment through passive modifications

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Abstract. Specifications on the dynamic behavior of feedback-controlled vibrating systems are often expressed in terms of its eigenstructure, i.e. eigenvalues and eigenvectors. The notion of controllability establishes the possibility to assign eigenvalues through state feedback, but it is not adequate to assure the assignment of arbitrary eigenvectors. Indeed, assignable eigenvectors are just those belonging to the allowable vector subspace, which depends on the physical properties of the vibrating system (mass, damping and stiffness matrices) and of the actuators. To overcome this limitation, this paper proposes a hybrid approach that exploits passive modification of the system physical parameters to modify the allowable subspace in such a way that it spans (or closely approximates) the desired eigenvectors. Then, once that the system modifications have been computed, standard techniques for control synthesis can be employed to compute the gains assigning the desired poles and the eigenvectors. The modification of the allowable subspace is cast in this work as a rank minimization problem, which can be efficiently tackled through semi-definite programming. The proposed method is numerically validated on a lumped parameter system, by proving that the assignment of eigenvectors by hybrid control is significantly enhanced compared with sole active control.

1. Introduction

The dynamic response of a vibrating system is determined by its eigenstructure, namely the eigenvalues and eigenvectors. In particular, the eigenvalues are related to properties such as stability, damping and settling time. Eigenvectors are of great importance, as well, since they affect the spatial distribution of the vibrations, and also define eigenvalue sensitivity to model parameters. In the process of controlling, designing or optimizing vibrating systems, the specifications on the dynamics are therefore often expressed as requirements on the system eigenvalues and eigenvectors. Over the last years several techniques for solving this problem, usually named Eigenstructure Assignment (EA), have appeared in the literature. Such techniques can be, basically, classified into passive and active approaches.

Passive methods, also called Dynamic Structural Modification techniques (DSM), consist in a wise adjustment of the physical parameters of the system (such as the mass and stiffness distribution) in order to attain the desired eigenstructure. Passive methods have been successfully applied to several systems, such as for example in the optimization of a linear vibratory feeder [1, 2] or to a helicopter tailcone [3]. The popularity of such methods is due, other than to their effectiveness, to the inherent stability of the obtained system (since the modifications of the model matrices are necessarily symmetric) and the moderate cost of implementation, as long as the alteration of the physical parameters are small. Indeed, no additional sensors, actuators or control electronic devices are
required. However, for economical and technical reasons constraining remarkably the feasible modifications, sometimes Dynamic Structural Modification is not a feasible solution, and this motivates the need of different approaches.

Active methods employ instead feedback control. In control theory it is usually emphasized the assignment of eigenvalues only, and in fact a well-established theoretical background and numerous computational tools are available in such field. The potential of feedback control for EA has been first noted in [4] and afterwards many techniques for synthesis of control gains have been proposed in the literature. Given the limitations of the passive approaches previously pointed out, it is natural to wonder if active control can actually overcome such limitations and if it constitutes a feasible alternative. The state-of-the-art of Eigenstructure Assignment by active control is indeed quite satisfactory, as proved by numerous papers [5, 6, 7]. However, not all the eigenpairs can be attained through such approach: also state feedback control has some inherent limitations, even in the case of fully controllable system. Thus, while eigenvalues can be arbitrarily assigned in fully controllable systems, such a condition does not ensure the eigenvectors to be exactly assigned. Indeed, each assignable eigenvector must belong to a certain subspace, that depends on the system model, on the desired corresponding eigenvalue and also on the number and distribution of the actuation forces [5]. In particular, the number of independent entries of the eigenvectors that can be assigned is equal to the number of independent actuation forces. This issue is particularly serious when the system is highly underactuated, such as for example in the case of single-input control.

To overcome the weaknesses of both passive and active methods, in this paper a hybrid method that combines Dynamic Structural Modification and feedback control is proposed. Hybrid approaches have been rarely employed so far, but the results obtained are very encouraging. For example in [8] vibration confinement is attained exploiting mechanical dampers to reduce active power requirements for active control. Similarly, in [9] the same objective is pursued using piezoelectric materials both as active actuators and passive dampers. In [10], instead, the modification of the link in a flexible manipulator enable to improve the performances of input shaping. In a more recent work [11], structural modification and feedback control are consecutively employed for eigenvalue assignment to asymmetric systems. The general problem of EA has been instead tackled for the first time in [12], where a method for simultaneous structural modifications computation and control gains synthesis is proposed and applied to vibration confinement, by applying a concurrent and integrated approach.

In this paper a similar problem is solved and it is proposed a new strategy to solve the EA problem, in which DSM is exploited to make possible the assignment of eigenvectors by active control. In fact, the physical parameters of the system (e.g. mass and/or stiffness distribution) are altered to modify the vector space of eigenvectors assignable by active control. The objective is to make such a space contain the desired eigenvectors, or at least to make it span a close approximation. The proposed solution takes advantage of some recent results in algebraic computing and tackles the problem as a matrix rank minimization problem, whose solution relies on semi-definite programming.

In the paper, the ability of the proposed hybrid method to attain a more accurate Eigenstructure Assignment with respect to sole active or passive control is also evaluated. In particular, the numerical validation has been performed on the model of a simple 3-dofs lumped-parameters mechanical system, to which two eigenpairs are assigned.

2. Problem formulation

2.1. EA through state feedback active control
In this Section the Eigenstructure Assignment problem is introduced and the issues of active control are discussed. In the present formulation, it is supposed that the dynamics of the considered $N$-dofs system is described by the second order differential equation:

$$M\ddot{q}(t)+C\dot{q}(t)+Kq(t)=Bu(t)$$  \hspace{1cm} (1)
where $\mathbf{M}, \mathbf{C}, \mathbf{K} \in \mathbb{R}^{N \times N}$ are the mass, damping and stiffness matrices, \( \mathbf{B} \in \mathbb{R}^{N \times N_B} \) represents the distribution of control forces (\( N_B \) is the number of independent actuation forces), \( \mathbf{q}(t) \in \mathbb{R}^N \) is the displacement vector and \( \mathbf{v}_a(t) \in \mathbb{R}^{N_a} \) is the control force vector.

In the hypothesis of position and velocity feedback (state feedback), the control law is defined as
\[
\mathbf{v}_a(t) = -(\mathbf{F}^T \dot{\mathbf{q}}(t) - \mathbf{G}^T \mathbf{q}(t) - \mathbf{q}_{ref}(t))
\]  
(2)

where \( \mathbf{F}, \mathbf{G} \in \mathbb{R}^{N \times N_a} \) are the control gains and \( \mathbf{q}_{ref}(t) \) is the state reference. The terms that contain \( \mathbf{q}_{ref}(t) \) can be collected in \( \mathbf{v}_{ref}(t) = \mathbf{F}^T \dot{\mathbf{q}}_{ref}(t) + \mathbf{G}^T \mathbf{q}(t) = \mathbf{B} \mathbf{v}_{ref}(t) \).

The equations of motion of the controlled system thus become:
\[
\mathbf{M} \ddot{\mathbf{q}}(t) + (\mathbf{C} + \mathbf{B} \mathbf{F}^T) \dot{\mathbf{q}}(t) + (\mathbf{K} + \mathbf{B} \mathbf{G}^T) \mathbf{q}(t) = \mathbf{B} \mathbf{v}_{ref}(t).
\]  
(3)

Since only the homogeneous solution is relevant in EA, the right hand side will be henceforth neglected. Under such circumstances, the pair \( (\lambda, \mathbf{u}) \) is an eigenpair if (and only if) it is a solution of
\[
\lambda \mathbf{u} = \left( \mathbf{M} \lambda^2 + \mathbf{C} \lambda + \mathbf{K} \right) \mathbf{u} = 0.
\]  
(4)

The classical active approach to Eigenstructure Assignment consists in finding the appropriate controller gains \( \mathbf{F}, \mathbf{G} \in \mathbb{R}^{N \times N_a} \) such that the closed loop system features a prescribed set of eigenpairs \( (\lambda_i, \mathbf{u}_i) \) for \( i = 1, \ldots, N_e \leq 2N \). It is well known that if \( (\lambda_i, \mathbf{u}_i) \) is a complex eigenpair of the system, then it is complex-conjugate is still an eigenpair, thus, to avoid redundancy it is assumed that only one element for each complex-conjugate pair belongs to the set of \( N_e \) desired eigenpairs.

### 2.2. Existence of solution for active EA

The possibility to assign the closed loop eigenvalues is a topic exhaustively investigated and the controllability condition enables to determine if such task can be accomplished. Indeed, if a system satisfies the following condition for all the eigenvalues of the open loop system \( \lambda_i \) (\( i = 1, \ldots, 2N \)),
\[
\text{rank} \left( \left[ \lambda \mathbf{M} + \lambda \mathbf{C} + \lambda \mathbf{K} \right] \right) = N
\]  
(5)

then it is said to be controllable and the eigenvalues of the closed-loop system can be arbitrarily placed by means of state feedback control.

Controllability, however, does not assure a proper assignment of the eigenvectors. In fact, let \( (\tilde{\lambda}, \tilde{\mathbf{u}}) \) be a desired pair of eigenvalue and eigenvector, then the assignment of such eigenpair is attained if equation (4) holds, that is
\[
\left[ \mathbf{M} \tilde{\lambda}^2 + \mathbf{C} \tilde{\lambda} + \mathbf{K} \right] \tilde{\mathbf{u}} = -\mathbf{B} \left[ \tilde{\lambda} \mathbf{F}^T + \mathbf{G}^T \right] \tilde{\mathbf{u}}.
\]  
(6)

By left multiplying equation (6) with the open loop system receptance matrix at \( \tilde{\lambda} \), \( \mathbf{H}(\tilde{\lambda}) = \left[ \mathbf{M} \tilde{\lambda}^2 + \mathbf{C} \tilde{\lambda} + \mathbf{K} \right]^{-1} \), equation (6) becomes
\[
\tilde{\mathbf{u}} = \mathbf{H}(\tilde{\lambda}) \mathbf{z},
\]  
(7)

where \( \mathbf{z} = -\left[ \tilde{\lambda} \mathbf{F}^T + \mathbf{G}^T \right] \tilde{\mathbf{u}} \in \mathbb{R}^{N_a} \). Since the gains \( \mathbf{F} \) and \( \mathbf{G} \) are arbitrary, the previous equation states that an assignable eigenvector \( \tilde{\mathbf{u}} \) must belong to the vector space spanned by the columns of matrix \( \left[ \mathbf{H}(\tilde{\lambda}) \mathbf{B} \right] \in \mathbb{C}^{N \times N_a} \). Hence the allowable subspace \( \Psi(\tilde{\lambda}) \), which collects the eigenvectors that can be obtained by active assignment, is defined for each eigenvalue \( \tilde{\lambda} \) :
\[
\Psi(\tilde{\lambda}) = \text{span}\left\{ \mathbf{H}(\tilde{\lambda}) \mathbf{B} \right\}.
\]  
(8)
Since both $H(\hat{\lambda})$ and $B$ have maximum rank, the space $\Psi$ is $N_B$-dimensional. Therefore, it is not always possible to assign any arbitrary eigenvector by employing only state feedback control. A common workaround when $\tilde{u} \not\in \Psi(\hat{\lambda})$ is to assign the projection of $\tilde{u}$ onto $\Psi$, rather than $\tilde{u}$ itself [5]. In such a way, however, the closed loop system performances often differ significantly from the expectations, because the projection of $\tilde{u}$ may fail to approximate $\tilde{u}$. To solve this relevant issue, in the following it is proposed an alternative strategy that exploits DSM.

2.3. Proposed hybrid approach

Let us suppose that it is wanted to assign the complex conjugate set of eigenpairs $(\lambda_i, u_i)$ for $i = 1, \ldots, N_e \leq 2N$. The proposed approach consists in altering the allowable subspaces $\Psi(\lambda_i)$ to obtain new subspaces $\hat{\Psi}(\lambda_i)$ for which $u_i \in \hat{\Psi}(\lambda_i)$.

It is evident from the definition of allowable subspace in equation (8) that such spaces depend on the receptance matrix computed at the desired eigenvalues and also on the distribution of the control forces. It is supposed that the matrix $B$ cannot be modified, as it is often the case in practice, for instance because the location and the numbers of actuators is imposed by technical constraints. Therefore, an effective way to shape the allowable subspaces is to exploit DSM to modify the receptance matrix for each eigenvalue of interest. In practice, it is wanted to find modification matrices $\Delta M$, $\Delta C$ and $\Delta K$ such that

$$u_i \in \text{span}\left\{ \left[ \lambda_i^2 (M + \Delta M) + \lambda_i (C + \Delta C) + (K + \Delta K) \right]^{-1} B \right\} = \hat{\Psi}(\lambda_i)$$

for each assigned eigenpair $(\lambda_i, u_i)$.

2.3.1. Rank minimization problem. The computation of the appropriate modification matrices can be recast as a rank minimization problem. Such formulation is advantageous because it is not necessary to cope with the receptance matrix and thus no inversions of the matrix collecting the unknowns are needed. Moreover, since rank minimization problems have many applications in different fields, there are several algorithms to perform the numerical computations.

The aforementioned formulation is obtained as a consequence of the well known Rouché-Capelli theorem of Linear Algebra. In fact, relation (9) holds if there are coefficients vectors $x_i \in \mathbb{R}^{N_s}$ such that

$$H(\lambda_i)Bx_i = u_i$$

for $i = 1, \ldots, N_e$. Since the receptance matrix is invertible, supposed that $\lambda_i$ is not an open-loop eigenvalue of the system, such an equation is equivalent to

$$Bx_i = \left[ \lambda_i^2 (M + \Delta M) + \lambda_i (C + \Delta C) + (K + \Delta K) \right] u_i.$$  

By defining

$$A = \begin{bmatrix} B & B & \cdots & B \\ \end{bmatrix} \in \mathbb{R}^{N_e \times N_s \times N_s} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N_e} \end{bmatrix} \in \mathbb{C}^{N_e \times N_s}$$

(12)
the \( N_e \) matrix equations (11) can be conveniently combined into the equation

\[
\mathbf{Ax} = \mathbf{b}.
\]  

(13)

Such system is complex in general, except the notable case in which no damping is assumed and only purely imaginary eigenvalues are desired. However, it is trivial to transform the complex system into an equivalent real system of twice the order. Therefore, from now on it is assumed that the system (13) is real, for clarity of explanation.

According to the Rouché-Capelli theorem, such system can be solved if and only if the rank of the matrix \( \mathbf{A} \) is equal to the rank of the augmented matrix \( [\mathbf{A} | \mathbf{b}] \). Since \( \mathbf{B} \) is assumed with maximum rank, it is easy to prove that the matrix \( \mathbf{A} \) has linearly independent columns, thus the rank is exactly \( N_e \cdot N_B \). The rank of the augmented matrix \( [\mathbf{A} | \mathbf{b}] \), instead, could be either \( N_e \cdot N_B \) or \( N_e \cdot N_B + 1 \), because the vector \( \mathbf{b} \) is not constant as it depends on the modification matrices \( \Delta \mathbf{M}, \Delta \mathbf{C} \) and \( \Delta \mathbf{K} \).

In practice, the computation of the modification of the allowable subspaces consists in finding the matrices \( \Delta \mathbf{M}, \Delta \mathbf{C} \) and \( \Delta \mathbf{K} \) such that \( \text{rank} [\mathbf{A} | \mathbf{b}(\Delta \mathbf{M}, \Delta \mathbf{C}, \Delta \mathbf{K})] = N_e \cdot N_B \). It is reasonable to restrict the research of the structural modifications into a constraint set \( \Gamma \) to take into account the physical feasibility of the solution. In such a way, however, the existence of a solution is not guaranteed. Therefore, it is convenient to use an optimization-based formulation: it is wanted to find such modification matrices that solve the following problem

\[
\text{minimize } \text{rank} [\mathbf{A} | \mathbf{b}(\Delta \mathbf{M}, \Delta \mathbf{C}, \Delta \mathbf{K})] \\
\text{subject to } (\Delta \mathbf{M}, \Delta \mathbf{C}, \Delta \mathbf{K}) \in \Gamma
\]  

(14)

Since the rank of a matrix is numerically difficult to handle in an optimization problem, the algorithm to solve problem (14) must be carefully discussed.

2.3.2. Numerical algorithm. The method presented in this paper had been introduced in [13] and it allows to solve optimization problems that involve the rank of any rectangular matrix by exploiting an intuitive and effective heuristic. First of all the problem is recast as the minimization of the rank of a square (and symmetric) matrix. To do so, the semidefinite embedding lemma is used (see [13] for the proof and a detailed explanation). According to such theorem, \( \text{rank} [\mathbf{A} | \mathbf{b}] \leq N_e \cdot N_B \) if and only if there exist matrices \( \mathbf{Y} = \mathbf{Y}^T \in \mathbb{R}^{N_e \cdot N_B \times N_e} \) and \( \mathbf{Z} = \mathbf{Z}^T \in \mathbb{R}^{N_e \cdot (N_B + 1) \times N_e \cdot (N_B + 1)} \) such that

\[
\text{rank} (\mathbf{Y}) + \text{rank} (\mathbf{Z}) \leq 2N_e \cdot N_B, \\
\begin{bmatrix} \mathbf{Y} & [\mathbf{A} | \mathbf{b}]^T \\ [\mathbf{A} | \mathbf{b}]^T & \mathbf{Z} \end{bmatrix} \succeq 0
\]  

(15)

where the symbol \( \succeq \) means that the matrix must be positive semidefinite.

Problem (14) can be therefore recast as follows:

\[
\text{minimize } \text{rank diag}(\mathbf{Y}, \mathbf{Z}) \\
\text{subject to } \begin{bmatrix} \mathbf{Y} & [\mathbf{A} | \mathbf{b}]^T \\ [\mathbf{A} | \mathbf{b}]^T & \mathbf{Z} \end{bmatrix} \succeq 0 \\
(\Delta \mathbf{M}, \Delta \mathbf{C}, \Delta \mathbf{K}) \in \Gamma.
\]  

(16)
Such formulation has the significant advantage that allows to use the popular “trace heuristic”, which has been successfully employed in numerous works in the literature, see e.g. [14]. Such approach consists in replacing the rank in the objective function of problem (16) with the trace. The reason is intuitive: the rank is the number of non-zero eigenvalues, while the trace is the sum of the eigenvalues, and since the considered matrix is positive semidefinite all its eigenvalues are greater or equal to 0, thus it is fair to assume that a small trace “forces” the rank to be low. A more rigorous interpretation of the trace heuristic is given in [15].

The obtained optimization problem, provided that the feasibility constraint set \( \Gamma \) is convex, as it is often in practice, is a convex optimization problem with semidefinite constraints, for which many efficient and reliable software packages are available. In general, the developed method does not assure that the allowable subspaces will span the desired eigenvectors after structural modification, especially due to the technical constraints expressed by the set \( \Gamma \). The numerical tests that have been carried out, however, show that the method almost always leads to a significant improvement of the assignment accuracy.

2.3.3. Remarks on active Eigenstructure Assignment. When the modifications matrices \( \Delta M \), \( \Delta C \) and \( \Delta K \) are found, it is necessary to compute the feedback gains \( F \) and \( G \) as in equation (2) that allow the simultaneous assignment of the eigenvalues \( \lambda_i \) and the eigenvectors \( u_i \). If \( u_i \notin \Psi(\lambda_i) \) for some \( i \), that is some allowable subspaces of the modified system \( \Psi(\lambda_i) \) fail to span the desired eigenvectors, then the projections \( \text{proj}_{\Psi(\lambda_i)}(u_i) \) are the best allowable approximation of \( u_i \) and thus are the best target eigenvectors for Eigenstructure Assignment.

The actual computation of the gains can be accomplished with the several effective methods available in the literature of EA through active control. For example, in the tests shown in this paper it has been used the method [7] which is capable of partial Eigenstructure Assignment. Such technique is particularly useful when it is wanted that some open loop eigenpairs are retained in the closed loop system. In such a way it can be avoided the so called spillover, which occurs when modifying some eigenpairs causes the unwanted alteration of the others. Since spillover can even lead to system instability, great care must be taken in the computation of the controller gains.

3. Numerical example

In order to assess the effectiveness of the proposed method, the results obtained in the modal optimization by hybrid Eigenstructure Assignment of a simple mechanical system are shown. In particular, it is considered a lumped parameter 3-degrees of freedom system, whose modal optimization by sole DSM has been object of a previous paper [1]. Such system is pictured in figure 1.

The original values of the masses and springs are shown in the first row of table 1. It is supposed that all the system parameters can be modified, as long as the modifications let the physical quantities be non-negative, thus the constraints set \( \Gamma \) is chosen according to the second row of table 1.

![Figure 1. Lumped parameter 3-dofs system.](image-url)
Table 1. System physical properties and admissible modifications.

|                  | \( m_1 \) (kg) | \( m_2 \) (kg) | \( m_3 \) (kg) | \( k_1 \) (kN/m) | \( k_2 \) (kN/m) |
|------------------|----------------|----------------|----------------|-----------------|-----------------|
| Original values  | 45             | 25             | 25             | 2.50 \( \times \) 10^3 | 2.50 \( \times \) 10^3 |
| Constraints      | \((-45, +\infty)\) | \((-25, +\infty)\) | \((-25, +\infty)\) | \((-2.50 \times 10^3, +\infty)\) | \((-2.50 \times 10^3, +\infty)\) |

The original system features a rigid-body mode that must be retained in the modified system. The other two vibration modes, shown in the first two columns of table 2, are replaced by two modes (represented in the last two columns of table 2) at the frequencies of 32 Hz and 50 Hz, without introducing damping. However, in many cases damping is desirable and it can be conveniently achieved by means of active eigenvalue assignment, thus such topic will be investigated in the future.

Table 2. Original and desired eigenstructure.

| Mode number \( i \) | Original System  | Desired modes  |
|---------------------|------------------|----------------|
|                     | 1                | 2              | 1               | 2               |
| \( u_i \) (1)       | 0.0              | -1.1           | 1.0             | -2.0            |
| \( u_i \) (2)       | 1.0              | 1.0            | 6.5             | 1.5             |
| \( u_i \) (3)       | -1.0             | 1.0            | -8.0            | 1.0             |
| \( f_i \) [Hz]      | 50.33            | 73.13          | 32              | 50              |

Concerning the actuation characteristics of the considered system, it is supposed that only the mass named \( m_2 \) is actuated by a control force. Such hypothesis implies that the dimension of the allowable subspaces is \( N_B = 1 \), thus it is very unlikely that EA could be performed by sole active control with satisfactory results. In fact, the desired eigenvectors do not belong to the allowable subspace of their respective eigenvalue. For the first desired mode, the one whose eigenvalue is \( \lambda_1 = \pm 2\pi \cdot 32i \), the eigenvector differs from its projection onto \( \Psi(\lambda_i) \) by an angle whose cosine is 0.7291, whereas the parallelism would lead to a cosine equal to 1. The second desired mode, for which \( \lambda_2 = \pm 2\pi \cdot 50i \), consists of an eigenvector that differs from its projection onto \( \Psi(\lambda_i) \) by an angle whose cosine is 0.6495.

The method to modify the allowable subspaces described in the previous Section has been implemented in Matlab, exploiting the YALMIP [16] modeling language for optimization problems. The employed numerical solver is SeDuMi [17], which is capable of handling semidefinite constraints. In practice, the modifications of the physical parameters \( \Delta m_1 \), \( \Delta m_2 \), \( \Delta m_3 \), \( \Delta k_1 \) and \( \Delta k_2 \) are computed solving the following problem:

\[
\begin{align*}
\text{minimize} & \quad \text{trace diag} (Y, Z) + \alpha \sum_i \Delta m_i^2 + \beta \sum_i \Delta k_i^2 \\
\text{subject to} & \quad \begin{bmatrix} Y & A | b \\ [A | b]^T & Z \end{bmatrix} \succeq 0 \\
& \quad (\Delta M, \Delta C, \Delta K) \in \Gamma.
\end{align*}
\]

where the cost function also contain two additive terms that serve as Tikhonov regularization terms [18], which are aimed at improving numerical conditioning and favoring small modifications. In particular it has been chosen \( \alpha = 1 \) and \( \beta = 10^{-6} \), in order to compensate the different order of magnitude of masses and stiffnesses.

The computed optimal modifications are shown in table 3. The modified allowable subspaces are much more suitable to the wanted EA task, in fact each desired eigenvector differs by a small angle to
its projection onto the respective allowable subspace. In particular, \( \cos(\mathbf{u}_1, \text{proj}_{\mathbf{v}_{\phi}}(\mathbf{u}_1)) = 0.9983 \) and \( \cos(\mathbf{u}_2, \text{proj}_{\mathbf{v}_{\phi}}(\mathbf{u}_2)) = 0.9987 \).

**Table 3. System modifications.**

| Method          | \( \Delta m_1 \) (kg) | \( \Delta m_2 \) (kg) | \( \Delta m_3 \) (kg) | \( \Delta k_1 \) (kN/m) | \( \Delta k_2 \) (kN/m) |
|-----------------|------------------------|------------------------|------------------------|------------------------|------------------------|
| Hybrid method   | -34.26                 | -14.10                 | -9.58                  | -1.777 \cdot 10^3      | -1.950 \cdot 10^3      |
| DSM [1]         | -4.23                  | +5.74                  | +3.89                  | -1.149 \cdot 10^3      | -1.459 \cdot 10^3      |

The assessment of the achieved performances is done evaluating three quantities: the absolute error of the frequency \( f_i \) for \( i = 1, 2 \), the cosine of the angle between the desired eigenvector \( \mathbf{u}_{\text{des},i} \) and the obtained one \( \mathbf{u}_{\text{obt},i} \) and the control effort, expressed by the norm of the gains (in this particular application only position feedback is necessary, so it will be considered \( \mathbf{F} = 0 \) in the control law (2)).

Such quantities are computed in four cases: the case of EA by DSM only, which is the approach of reference [1], the case of sole active control, in which the method [7] is applied to the original system, the case of active control applied to the system modified as in the case of DSM and finally the case of hybrid control, in which control synthesis is applied to the system modified according to the proposed method. The results are shown in table 4.

**Table 4. Eigenstructure comparison.**

|                     | DSM [1] | Active control [7] | DSM + Active control | Hybrid method |
|---------------------|---------|--------------------|----------------------|--------------|
| \( | f_{i,\text{des}} - f_{i,\text{obt}} | (Hz) | 0.4741 | 0.0000 | 0.0000 | 0.0000 |
| \( \cos(\mathbf{u}_{\text{des},i}, \mathbf{u}_{\text{obt},i}) | 0.9996 | 0.7291 | 0.9989 | 0.9983 |
| \( | f_{2,\text{des}} - f_{2,\text{obt}} | (Hz) | 0.1942 | 0.0000 | 0.0000 | 0.0000 |
| \( \cos(\mathbf{u}_{\text{des},i}, \mathbf{u}_{\text{obt},i}) | 0.9990 | 0.6495 | 0.3541 | 0.9987 |
| \( \|\mathbf{G}\| | (N/m) | - | 1.058 \cdot 10^7 | 7.842 \cdot 10^4 | 1.686 \cdot 10^6 |

The obtained results provide further evidence that DSM can perform a very accurate assignment of eigenvectors. On the other hand, the performances of DSM in the assignment of eigenvalues can be greatly surpassed employing feedback control, in fact the latter can assign eigenvalues with an absolute error comparable to the machine epsilon.

The consecutive employment of DSM and active control can be considered as a simplified strategy of hybrid control. Although the specifications on the eigenvectors are quite accurately fulfilled by DSM only, adding feedback control even degrades such performances, especially for what concerns the mode at 50 Hz. Therefore, the small improvement in terms of closed loop eigenvalues does not justify the use of this approach. Hence, it is corroborated the need of an integrated approach, as the one proposed in this paper.

Finally, the proposed hybrid method actually succeeds in combining the advantages of passive and active control. In fact, the obtained eigenvalues are extremely close to the desired ones and, at the same time, the closed loop eigenvectors are almost parallel to the desired ones, despite the fact that the system is underactuated. Moreover, the hybrid approach enables to significantly reduce the control effort, as proved by the fact that the norm of the gain is one order of magnitude bigger in the case of sole active control.
4. Conclusions

This paper proposes a novel method for Eigenstructure Assignment that exploits concurrently both Dynamic Structural Modification and feedback control. The desired eigenstructure is obtained in two stages: in the first instance passive modifications alter the set of eigenvectors that can be assigned in correspondence to the desired eigenvalue, thereafter a technique for control synthesis can be employed to compute the feedback gains that actually place the eigenvalues and shape the eigenvectors.

The numerical method that has been developed does not raise any particular concern of computational nature: in fact it relies on a heuristic approach that has been widely tested in the literature and on a semi-definite optimization formulation for which the state-of-the-art algorithms are quite satisfactory. Moreover, the proposed method easily allows to take into account feasibility constraints and also to effortlessly employ a regularization strategy that improves numerical conditioning.

An assessment of the method has also been presented in the paper. In particular, the assignment of two eigenpairs to a 3-dofs lumped parameter system has been performed. The comparison of the proposed hybrid method with other state-of-the-art methods that employ either passive control or active control proves that the proposed method is preferable when an accurate assignment is needed both in terms of eigenvalues and eigenvectors. Moreover, the proposed integrated approach to hybrid control is further justified by the low performances obtained just consecutively employing the existing methods for DSM and feedback control. Future developments will be the assessment of the method with more challenging Eigenstructure Assignment tasks and more complicate vibrating systems.

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