Singularities and Avalanches in Interface Growth with Quenched Disorder

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A simple model for an interface moving in a disordered medium is presented. The model exhibits a transition between the two universality classes of interface growth phenomena. Using this model, it is shown that the application of constraints to the local slopes of the interface produces avalanches of growth, that become relevant in the vicinity of the depinning transition. The study of these avalanches reveals a singular behavior that explains a recently observed singularity in the equation of motion of the interface.

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The problem of interface motion in disordered media has attracted considerable attention recently. By means of numerical, analytical, and experimental studies, it has been observed that scaling theory can be used as the framework to understand the dynamical properties of the interface. In a typical realization of the problem a d-dimensional interface moves in a (d + 1)-dimensional disordered medium, driven by an external force $F$. The interface is assumed to be oriented along the longitudinal $x$-direction, and is specified by the height $y(x, t)$. For small forces the interface is pinned by the random quenched impurities of the medium, and the average velocity is zero. Above a critical value $F_c$, the external force overcomes the effect of the impurities, and the interface moves with a finite velocity. Near the threshold, the velocity scales as $v \sim f^\lambda$, where $f = F/F_c - 1$ is the reduced force, and $\lambda$ the velocity exponent.

The roughness exponent $\alpha$ is defined by the relation $W(L) \sim L^\alpha$, where $W$ is the width of the interface and $L$ is the system size. The exponent $\alpha$ characterizes the different universality classes of interface growth phenomena. The universality classes can also be classified according to the behavior of the coefficient $\lambda$ of a non-linear term of the form $\lambda(\nabla y)^2$ in the equation of motion of the interface. One of the universality classes is described by an equation of the Kardar-Parisi-Zhang (KPZ) type with a quenched random field $\eta(x, y)$.

$$\partial_t y(x, t) = \nabla^2 y + \lambda(\nabla y)^2 + \eta(x, y) + F,$$

with the coefficient $\lambda$ diverging at the depinning transition as

$$\lambda(f) \sim f^{-\phi},$$

where $\phi \simeq 0.64$. The roughness exponent at the depinning transition in $(1+1)$-dimensions is $\phi = 0.63$, and can be obtained by a mapping to directed percolation. We refer to this universality class as directed percolation depinning (DPD).

A second universality class is described, at the depinning transition, by an equation of the Edwards-Wilkinson (EWZ) type with quenched disorder

$$\partial_t y(x, t) = \nabla^2 y + \eta(x, y) + F.$$ Models with $\lambda = 0$ (for any force), or $\lambda \to 0$ (when $F \to F_c^+$, corresponding to a negative exponent $\phi$ in Eq. (2)), belong to the universality class of Eq. (3). Analytical and numerical studies in $(1+1)$-dimensions yield a roughness exponent $\alpha \simeq 1.23 - 1.25$. This universality class is referred to as quenched Edwards-Wilkinson (QEW).

In this paper we present a simple model for an elastic interface that exhibits a transition between the two universality classes of interface growth in a disordered medium. The DPD result is obtained when lateral or longitudinal fluctuations of the interface height are allowed and the growth rule is restricted by a generalized solid-on-solid (SOS) condition. On the other hand, when the constraint is absent, the QEW universality class is obtained. The generalized SOS condition is a universal feature appearing in all the models of the DPD universality class. It is analogous to the restricted SOS condition ($|y(x \pm 1, t) - y(x, t)| \leq 1$) presents in simple models of growth with time-dependent noise. In these type of models, the constraint is commonly associated with lateral propagation of growth that generates nonlinearities in the equation of motion. Moreover, it is shown that, in the presence of quenched disorder, the generalized SOS condition also generates avalanches of growth. These avalanches are irreversible growth events that become relevant near the depinning transition. We show that they present a singular behavior that is responsible for the divergence in the coefficient $\lambda$.

Consider a one-dimensional elastic interface moving in a two-dimensional disordered medium of lateral size $L$. A discrete model for such an interface is defined in the square lattice by the height values $\{y_k\}_{k=1, \ldots, L}$. The interface moves under the influence of an external force $\vec{F} = (0, F)$ and fluctuates in the longitudinal or $x$-direction, and transversal or $y$-direction. The strength of these fluctuations is controlled by the elastic constants $\nu_x$ and $\nu_y$, respectively. For the $k$-th column and at a given time, the velocity $\vec{v} = (v_x, v_y)$ is calculated according to

$$v_y = \nu_y (y_{k+1} - y_{k-1} - 2y_k) + \eta(k, y_k) + F,$$

$$v_x = -2\nu_x + \eta(k, y_k).$$

The term $\eta(k, y_k)$ represents the strength of the random force, and mimics the effect of quenched disorder. It is
a random field with a uniform probability distribution
between $[-\delta, \delta]$, and represents a repulsive force for posi-
tive values and an attractive or pinning force for negative
values. A site at the interface moves forward or laterally
only when the respective velocity is positive. Therefore,
the $k$-th column and its nearest neighbors are updated in the following way
\[
\begin{align*}
y_k &= y_k + 1 & \text{if } v_y > 0 \\
y_{k+1} &= y_k & \text{if } v_x > 0 \\
y_{k-1} &= y_k & \text{if } v_x > 0.
\end{align*}
\]

The first equations in (4) and (5), correspond to a
model introduced to study an elastic interface described
by the QEW equation [3], and they are equivalent to the
discretization of Eq. (3). The last equations in (4) and (5)
are the simplest generalization of the model to include
longitudinal motions [11]. A lateral motion to a nearest
neighbor column is allowed if the neighboring column is
smaller than the column considered. After the lateral mo-
tion, the interface becomes a multi-valued function of the
longitudinal coordinate $x$ (see Fig. 3(a)). The generalized
SOS condition is then applied in order to transform the
interface into a single-valued function, by defining the in-
terface with the highest value of the height. This growth
process can be thought as a transversal avalanche of
growth in the neighboring column, as shown in Fig. 3(b).
This growth event occurs regardless the value of the noise
in the neighboring column.

We perform numerical simulations on a square lattice
with $L = 1024$. Helical boundary conditions $y_k = y_1 + m L$, where $m = \langle \nabla y \rangle$ is the average external tilt,
are imposed on the interface in order to study the in-
terface velocity as a function of the tilt [2]. Figure 2
shows the tilt-dependence of the velocity of the interface
for the value $\nu_x = 0.2$. In the following, we use the values
$\nu_y = 1.0$ and $\delta = 3.0$ since our results do not depend
of these parameters. We see that the parabolas become
steeper when $F \rightarrow F_c^+$ and as in Eq. (3), indicating that
the model belongs to the DPD universality class.

The model presents a transition at a critical value $\nu_{x_c}$.
If the value of $\nu_x$ is increased such that $\nu_x \geq \nu_{x_c} = \delta/2$, then the QEW result is found. Specifically, we find that
$v$ is independent of $m$, indicating that $\lambda = 0$ for any
force. The fact that for $\nu_x > \delta/2$ the $x$-component of the velocity $v_x$ is always negative explains this transition:
longitudinal motions cannot occur and one recovers the
model of Ref. [5] with the corresponding behavior of $\lambda$.

The different parabolas obtained for a given value of
$\nu_x < \nu_{x_c}$, can be rescaled using the scaling ansatz
\[
v(m, f) \sim f^{\theta} g(m^2/f^{\theta+\phi}),
\]
where $g$ is a universal scaling function. Support for (6)
is provided by the data collapse shown in Fig. 3, where
we replotted the data of Fig. 2 and the data obtained for
$\nu_x = 0.6$, $\nu_x = 1.0$, and $\nu_x = 1.4$ [3]. The scaling func-
tion $g$ becomes flatter as $\nu_x \rightarrow \nu_{x_c}$, indicating that the
prefactor of $\lambda$ goes to zero, even though, the singularity
associated to $f$ is still present as long as $\nu_x < \nu_{x_c}$.

In the following, we argue that the divergence in $\lambda$
is explained by the singular behavior of the size of the avalanche
of growth.

According to (3), the interface can advance in two in-
dependent ways. One way is via a transversal motion (if $v_y > 0$), and the other is via a longitudinal motion (if $v_x > 0$) plus the transversal avalanche in the neighboring
column. In order to determine the relevant growth mech-
anism near the depinning transition, we study the mean
value of the number of longitudinal and transversal mo-
tions per unit time $n_x$ and $n_y$, respectively, as a function
of the force. For $\nu_x = 0.2$, we find
\[
\begin{align*}
n_x(f) &\sim f^{\gamma_x} \\
n_y(f) &\sim f^{\gamma_y},
\end{align*}
\]
with $\gamma_x \simeq 0.60$ and $\gamma_y \simeq 0.78$. Both quantities go to zero
at $F_c$, since the velocity vanishes at the depinning transition.
However, the ratio $n_x/n_y \sim f^{-0.18}$ diverges at $F_c$, indicating the relevance of longitudinal fluctuations for the
motion of the interface at the depinning transition.
This fact can be explained as follows. For forces close to $F_c$ the velocity is almost zero and the interface moves in a very irregular way, jumping from one metastable pin-
ing configuration to another. In this “jerky” motion the
interface takes advantage of longitudinal motions, rather
than transversal ones, to surround and overcome the im-
impediments. Thus, avalanche events become relevant for the
motion of the interface only in the vicinity of the depin-
ning transition.

We also study the mean value of the size of the avalanche produced by the generalized SOS condition per unit time, $\langle s \rangle$, as a function of the tilt and for dif-
frent forces. Figure 4 shows the results. As it turns out,
$\langle s \rangle$, as well as the velocity, has a parabolic dependence
on $m$. These parabolas become steeper as $F \rightarrow F_c^+$, and we can fit $\langle s \rangle$ to
\[
\langle s \rangle = s_0 + \lambda_s m^2
\]
with $\lambda_s \sim f^{-\phi}$, and $\phi \simeq 0.64$ the same exponent as in Eq. (3). The parabolas can also be rescaled using the scaling ansatz of Eq. (3) with the same value of $\theta$ used for the velocity curves.

As shown in Fig. 3(c), the size of an avalanche is larger
for the tilted interface than for the untilted one. This
explains the increase of $\langle s \rangle$ with the tilt, exemplified in Fig. 3 for a given fixed force. Moreover, since the relative occurrence of lateral motions and avalanches is larger near the depinning transition than away from it, the same ex-
ternal tilt will cause a larger increase in the average $\langle s \rangle$
near $F_c$ than for $F \gg F_c$. Thus, the coefficient $\lambda_s$, that
measures the variation experienced by $\langle s \rangle$ due to a change
in the average tilt of the interface, increases its value as $F \rightarrow F_c$ and the parabolas become steeper.

Notice that the relevance of longitudinal motions and
avalanches of growth near $F_c$ implies that the velocity
is determined by the size of the avalanches, \( v \propto (s) \), so that \( \lambda \propto \lambda_s \). Thus, the singularity of the coefficient \( \lambda \) can be explained by the same divergence observed in \( \lambda_s \). The motion of the interface near the depinning transition is entirely dominated by the avalanches of growth produced by the generalized SOS condition.

The same behavior can be predicted for the other models of the DPD universality class. All these models share a constraint in the growth rule of the interface height that generates avalanches analogous to the ones in our model. In the model of Ref. [14] a slope constraint is applied to the interface that implies a readjustment of the height regardless of the value of the noise. The so-called erosion of overhangs in the model of Ref. [15] presents a restricted SOS condition. And the model of Ref. [16] presents a restricted SOS condition that produces avalanches of growth (see Ref. [15]).

Other variants of the model (4) are also studied. First, we study a model that includes lateral motions but removes the generalized SOS condition. The model can be better understood as a set of L-beads fluctuating in both directions and interacting through elastic springs [17].

We calculate the velocity for this model as a function of the average tilt and for different forces. We find a small value of \( \lambda \) for \( F \gg F_c \) and \( \lambda \to 0 \) when \( F \to F_c^+ \). Therefore, the model falls in the QEW universality class. The nonlinearity observed in this model is of kinematic origin (\( \lambda \propto v \)) [16], in contrast to the behavior of \( \lambda \) in the DPD universality class, for which \( \lambda \) is coupled to the external force and diverges at \( F_c \). These results suggest that lateral motions alone generate nonlinearities only when the motion of the interface is so fast that the noise can be regarded as a time-dependent noise. At the depinning transition, where the quenched disorder is relevant, the QEW result is recovered. These results confirm that the generalized SOS condition is the origin of the singularity in \( \lambda \).

We also study other mechanisms, which are believed to generate nonlinearities in the equation of motion [13]: (i) an external fixed force acting in the normal direction of the interface, analogous to the Lorentz force found in the motion of flux lines in type II-superconductors [18]; and (ii) an anisotropic random medium characterized by an anisotropic random force [15]. In order to check this hypothesis, we simulate the model that allows for longitudinal motions but without the generalized SOS condition [2]. In addition, we generalize the definition of the velocity vector by including: (i) an external force, with fixed magnitude \( F \), acting in the direction of the local normal vector \( \hat{n} \) of the interface: \( \vec{F} \equiv (F_x, F_y) = F \hat{n} \); and (ii) we model the effect of the anisotropy in the medium by introducing a noise vector \( \vec{\eta} \equiv (\eta_x, \eta_y) \), where \( \eta_x \) and \( \eta_y \) are independent random fields with amplitudes \( \delta_x \) and \( \delta_y \) respectively (\( \delta_x \neq \delta_y \) for an anisotropic medium). We find that the model does not belong to the DPD universality class. Under the limitations of our model, we find that the anisotropy of the medium does not generate the divergence in \( \lambda \).

In conclusion we present a simple model for an elastic interface moving in a quenched disordered medium, which captures the relevant features of the two universality classes. The origin of the singular behavior in the equation of motion of the DPD universality class is explained by a simple microscopic constraint imposed to the growth of the interface that generates irreversible avalanches of growth.

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[11] The first term that appears in the \( x \)-component of the velocity in Eq. (4) corresponds to the elastic force that opposes the lateral motion.

[12] According to (4), for a tilted interface with tilt \( m \), the average velocity becomes \( v(m) = v_o + \lambda m^2 \), where \( v_o \) is the velocity of the untilded interface (4). Thus, one can gain information on the presence and magnitude of the nonlinear coefficient \( \lambda \), by monitoring the velocity of the interface as a function of the average tilt, and fitting to a parabola the obtained curve.

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We use $\phi \simeq 0.64$ for all values of $\nu_x$ in the data collapse of Fig. 3. For the velocity exponent we use $\theta \simeq 0.64$ for $\nu_x = 0.2$. However, we find somewhat smaller values of $\theta$ as $\nu_x$ approaches 1.5 ($\theta \simeq 0.40$ for $\nu_x = 1.4$). A possible source for this deviation is finite size effects.

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The position of the $k$-th bead on the interface is defined by the vector $\vec{r}_k = (x_k, y_k)$. Without the generalized SOS condition, an elastic term $\nu_x (x_{k+1} + x_{k-1} - 2x_k)$, like the one defined for $v_y$ in Eq. (4), can be introduced in the $x$-component of the velocity. The dynamics of the interface is now defined by $y_k = y_k + 1$ if $v_y > 0$, $x_k = x_k + 1$ if $v_x > 0$, and $x_k = x_k - 1$ if $v_x < 0$.

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We choose this model because we want to test the hypothesis independently of the generalized SOS condition. Since the model allows for longitudinal motions, it does not rule out the possibility of a diverging $\lambda$.

FIG. 1. Sketch of the effect of lateral motions of a site at the interface: (a)-(b) for an untilted interface, and (c) for a tilted interface with $m = 1$ as defined in the text. A lateral motion (a) produces an effective avalanche (b) in the nearest neighbor column due to the generalized SOS condition. The size of the avalanche, $s$, is larger for the tilted interface than for the untilted one, as can be seen comparing (b) and (c).

FIG. 2. Plot of the velocity as a function of the average tilt of the interface for values of the reduced force from $f = 0.19$ (bottom) to $f = 1.38$ (top). The parameters are $\nu_x = 0.2$, $\nu_y = 1.0$ and $\delta = 3.0$. Results are averaged over 70 independent realizations of the disorder. The “closing” of the parabolas shows that a diverging nonlinear coefficient is present in the equation of motion.

FIG. 3. Data collapse of the data of Fig. 2 ($\nu_x = 0.2$) and the data corresponding to $\nu_x = 0.6$, $\nu_x = 1.0$ and $\nu_x = 1.4$ (shown from bottom to top, respectively), plotted according to the scaling relation of Eq.(6). The scaling function becomes flatter as $\nu_x \to \nu_x^c \approx \delta/2$, indicating the transition to the QEW universality class. Each set of curves is shifted for clarity.

FIG. 4. Plot of the average size of the avalanches produced by the generalized SOS condition as a function of the tilt, for the same forces and parameters as in Fig. 2.