CONVEX MULTIVARIABLE TRACE FUNCTIONS

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Abstract. For any densely defined, lower semi-continuous trace $\tau$ on a $C^*$-algebra $A$ with mutually commuting $C^*$-subalgebras $A_1, A_2, \ldots, A_n$, and a convex function $f$ of $n$ variables, we give a short proof of the fact that the function $(x_1, x_2, \ldots, x_n) \mapsto \tau(f(x_1, x_2, \ldots, x_n))$ is convex on the space $\bigoplus_{i=1}^n (A_i)_{sa}$. If furthermore the function $f$ is log-convex or root-convex, so is the corresponding trace function. We also introduce a generalization of log-convexity and root-convexity called $\ell$-convexity, show how it applies to traces, and give some examples. In particular we show that the Kadison-Fuglede determinant is concave and that the trace of an operator mean is always dominated by the corresponding mean of the trace values.

1. Introduction. The fact that several important concepts in operator theory, in quantum statistical mechanics (the entropy, the relative entropy, Gibbs free energy), in engineering and in economics involve the trace of a function of a self-adjoint operator has motivated a considerable amount of abstract research about such functions in the last half century. An important subset of questions involve the convexity of trace functions with respect to their argument.

The convexity of the function $x \mapsto \text{Tr}(f(x))$, when $f$ is a convex function of one variable and $x$ is a self-adjoint operator, was known to von Neumann, cf. [12, V.3. p. 390]. An early proof for $f(x) = \exp(x)$ can be found, e.g. in [19, 2.5.2]. A proof given by the first author some time ago describes the trace $\text{Tr}(f(x))$, where $f$ is convex, as a supremum over all possible choices of orthonormal bases of the Hilbert space of the sum of the values of $f$ at the diagonal elements of the matrix for $x$. This proof was communicated to B. Simon, who used the method to give an alternative proof of the second Berezin-Lieb inequality in [21, Theorem 2.4], see also [22, Lemma II.10.4]. Simon only considers the exponential function, but the argument is valid for any convex function, cf. [10, Proposition 3.1]. The general case for an arbitrary trace on a von Neumann algebra was established by D. Petz in [17, Theorem 4] using the theory of spectral dominance (spectral scale).

The basic fact, for one variable $x$ and a positive convex function $f$, is that

$$\sum_j f((\phi_j, x \phi_j)) \leq \text{Tr}(f(x)), \tag{1}$$

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where the sum – finite or not – is over any orthonormal basis. Equality is obviously achieved if the basis is the set of eigenvectors of $x$. Thus,

$$\text{Tr}(f(x)) = \sup_{\{\phi\}} \sum_j f(\langle \phi_j, x \phi_j \rangle).$$

(2)

Essentially, the proof is the following: If $\{\phi_j\}$ is the eigenvector basis (with eigenvalues $\lambda_j$) and $\psi_j$ is some other basis, then $\psi_j = \sum_k C_{jk} \phi_k$, and the coefficients of the unitary matrix $C$ satisfy $\sum_j |C_{jk}|^2 = 1 = \sum_k |C_{jk}|^2$. Then $(\psi_j, x \psi_j) = \sum_k |C_{jk}|^2 \lambda_k$ and, by Jensen’s inequality, $f(\sum_k |C_{jk}|^2 \lambda_k) \leq \sum_k |C_{jk}|^2 f(\lambda_k)$. Now, summing on $j$ we obtain (1).

Equation (2) implies the convexity of $x \rightarrow \text{Tr}(f(x))$, because any supremum of convex functions is convex. Moreover, if $f(x) = \exp(x)$ then one sees that $\text{Tr}(\exp(x))$ is log-convex (i.e. log( $\text{Tr}(\exp(x))$) is convex), because an ordinary sum of the type $\sum \exp(a_j)$, with $a_j$ in $\mathbb{R}$, is log-convex. Similarly, if $f(x) = |x|^p$ (with $p \geq 1$) we see that $x \rightarrow (\text{Tr}(|x|^p))^{1/p}$ is convex. In particular, the Schatten $p$–norms are subadditive.

Even more is true. If $f(x) = \exp(g(x))$ and $g$ is convex, then $x \rightarrow \text{Tr}(f(x))$ is log-convex. Similarly, if $f(x) = |g(x)|^p$ then $x \rightarrow (\text{Tr}(f(x)))^{1/p}$ is convex.

A natural question that arises at this point is this: Are there other pairs of functions $e, \ell$ of one real variable, beside the pairs $\exp, \log$ and $|t|^p, |t|^{1/p}$, for which $x \rightarrow \ell(\text{Tr}(f(x)))$ is convex whenever $f$ is $\ell$–convex, i.e. $f(x) = e(g(x))$ and $g$ is convex? In the second part of our paper we answer this question completely and give a few examples, which we believe to have some potential value. But first we turn to the question of generalizing (2) to functions of several variables.

We start with a function $f(\underline{\lambda})$ of $n$ real variables (with $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$).

Next we replace the real variables $\lambda_j$, by operators $x_j$, similar to the one-variable case. An immediate problem that arises is how to define $f(x)$ in this case. The spectral theorem, which was used in the one-variable case, fails here unless the $x_j$’s commute with each other. Therefore, we restrict the $x_1, x_2, \ldots, x_n$ to lie in commuting subalgebras $A_1, A_2, \ldots, A_n$, and then $f(x)$ and $\text{Tr}(f(x))$ are well defined and it makes sense to discuss the joint convexity of this trace function under the condition that $f$ is a jointly convex function of its arguments. (We do not investigate the question whether $f(x)$ is operator convex – only the convexity under the trace.)

More generally, we replace the trace $\text{Tr}$ in a Hilbert space setting by $\tau$, a densely defined, lower semi-continuous trace on a $C^*$–algebra $A$; i.e. a functional defined on the set $A_+$ of positive elements with values in $[0, \infty]$, such that $\tau(x^* x) = \tau(xx^*)$ for all $x$ in $A$. We further assume that $A$ comes equipped with mutually commuting $C^*$–subalgebras $A_1, A_2, \ldots, A_n$.

The convexity of the function $\underline{x} \rightarrow \tau(f(\underline{x}))$ on the space of $n$–tuples in $\bigoplus_{i=1}^n (A_i)_{sa}$ was proved by F. Hansen for matrix algebras in [5]. (Here $A_{sa}$ denotes the self-adjoint elements in $A$.) His result was extended to general operator algebras by the second author in [16]. Both these proofs rely on Fréchet differentiability and some rather intricate manipulations with first and second order differentials.

We realized that the argument in (2) will work in this multi-variable case as well, thereby providing a quick proof of the convexity. The key observation is that the mutual commutativity of $x_1, x_2, \ldots, x_n$ implies that there is one orthonormal
basis (in the case that $A$ is a matrix algebra) that simultaneously makes all the $x_j$ diagonal.

For a general $C^*$-algebra (e.g. the algebra of continuous functions on an interval) we have to find something to take the place of an orthonormal basis. This something is just the commutative $C^*$-subalgebra generated by the $n$ commuting elements $x_1, x_2, \ldots, x_n$. It depends on $x$, of course, but that fact is immaterial for computing the trace for a given $x$.

The main result in the first part of this paper – proved in section 6 – is

2. Theorem (Multivariable Convex Trace Functions). Let $f$ be a continuous convex function defined on a cube $I = I_1 \times \cdots \times I_n$ in $\mathbb{R}^n$. If $A_1, \ldots, A_n$ are mutually commuting $C^*$-subalgebras of a $C^*$-algebra $A$ and $\tau$ is a finite trace on $A$, then the function

$$(x_1, \ldots, x_n) \mapsto \tau(f(x_1, \ldots, x_n)),$$

(3)

defined on commuting $n$-tuples such that $x_i \in (A_i)_{sa}^{I_i}$ for each $i$, is convex on $\bigoplus (A_i)_{sa}^{I_i}$. (Here, $(A_i)_{sa}^{I_i}$ denotes the set of self-adjoint elements in $A_i$ whose spectra are contained in $I_i$.)

If $\tau$ is only densely defined, but lower semi-continuous, the result still holds if $f \geq 0$, even though the function may now attain infinite values.

In the second part we explore the natural generalization of the concept of log-convexity mentioned before and explained in detail in section 7. We find a necessary and sufficient condition on a concave function $\ell$ that ensures that $\ell$--convexity of a function $f = e \circ g$, with $e = \ell^{-1}$ and $g$ convex, implies $\ell$--convexity of the function $x \mapsto \tau(f(x))$ for a tracial state $\tau$ on a $C^*$-algebra $A$. (A tracial state is a trace satisfying $\tau(1) = 1$.) The main result there – proved in section 9 – is

3. Theorem ($\ell$--Convex Trace Functions). Let $f$ be a continuous function defined on a cube $I = I_1 \times \cdots \times I_n$ in $\mathbb{R}^n$, and assume furthermore that $f$ is $\ell$--convex relative to a pair of functions $e, \ell$ as described in section 7, where $\ell'$/$\ell''$ is convex. If $A_1, \ldots, A_n$ are mutually commuting $C^*$-subalgebras of a $C^*$-algebra $A$ and $\tau$ is a tracial state on $A$, then the function

$$(x_1, \ldots, x_n) \mapsto \tau(f(x_1, \ldots, x_n)),$$

(4)

defined on commuting $n$-tuples such that $x_i \in (A_i)_{sa}^{I_i}$ for each $i$, is also $\ell$--convex on $\bigoplus (A_i)_{sa}^{I_i}$. If moreover $\ell'/\ell''$ is homogeneous and $f \geq 0$ the result holds for any densely defined, lower semi-continuous trace $\tau$ on $A$.

In sections 4, 5 and 7 we set up some necessary machinery, whereas the key lemmas are in sections 6 and 8.

The third part of the paper, sections 9–22, consists of examples where we apply the preceding results. In particular we investigate the n-fold harmonic mean of positive operators and show how its trace behaves under certain concave transformations. As a corollary we prove in Proposition 23 that the trace of any mean (in the sense of Kubo and Ando [8]) is dominated by the corresponding mean of the trace values.

Throughout the paper we have chosen a $C^*$-algebraic setting with densely defined, lower semi-continuous traces, this being the more general theory. We might
as well have developed the theory for von Neumann algebras with normal, semi-finite traces; in fact we need this more special setting in the proof of Lemma 6. However, the Gelfand-Naimark-Segal construction effortlessly transforms the $C^*$-algebra version into the von Neumann algebra setting, so there is no real difference between the two approaches.

4. Spectral Theory. We consider a $C^*$-algebra $A$ of operators on some Hilbert space $\mathcal{H}$ and mutually commuting $C^*$-subalgebras $A_1, \ldots, A_n$, i.e. $A_i \subset A_j^*$ for all $i \neq j$. For each interval $I_i$ we let $(A_i)_{I_i}$ denote the convex set of self-adjoint elements in $A_i$ with spectra contained in $I_i$. If $\mathcal{I} = I_1 \times \cdots \times I_n \subset \mathbb{R}^n$ and $f$ is a continuous function on $\mathcal{I}$ we can for each $\mathbf{x} = (x_1, \ldots, x_n)$ in $\bigoplus (A_i)_{I_i}$ define an element $f(\mathbf{x})$ in $A$. To see this, let $x_i = \int \lambda dE_i(\lambda)$ be the spectral resolution of $x_i$ for $1 \leq i \leq n$. Since the $x_i$’s commute, so do their spectral measures. We can therefore define the product spectral measure $E$ on $\mathcal{I}$ by $E(S_1 \times \cdots \times S_n) = E_1(S_1) \cdots E_n(S_n)$ and then write

$$f(\mathbf{x}) = \int f(\lambda_1, \ldots, \lambda_n) dE(\lambda_1, \ldots, \lambda_n). \quad (5)$$

Of course, if $f$ is a polynomial in the variables $\lambda_1, \ldots, \lambda_n$ we simply find $f(\mathbf{x})$ by replacing each $\lambda_i$ with $x_i$. The map $f \to f(\mathbf{x})$ so obtained is a $^*$-homomorphism of $C(\mathcal{I})$ into $A$ and generalizes the ordinary spectral mapping theory for a single (self-adjoint) operator. In particular, the support of the map (the smallest closed set $S$ such that $f(\mathbf{0}) = 0$ for any function $f$ that vanishes off $S$) may be regarded as the “joint spectrum” of the elements $x_1, \ldots, x_n$.

This theory applies readily in the situation where $A = A_1 \otimes \cdots \otimes A_n$, but, curiously enough, the tensor product structure (used extensively in [5] and [16]) is not needed in our arguments.

5. Conditional Expectations. Let $\tau$ be a fixed, densely defined, lower semi-continuous trace on $A$, and let $C$ be a fixed commutative $C^*$-subalgebra of $A$. By Gelfand theory we know that each commutative $C^*$-subalgebra of $A$ has the form $C_0(T)$ for some locally compact Hausdorff space $T$. Note now that if $y \in C_+$, the positive part of $C$, and has compact support as a function on $T$, then $y = yz$ for some $z$ in $C_+$. Since the minimal dense ideal $K(A)$ of $A$ is generated (as a hereditary $^*$-subalgebra) by elements $a$ in $A_+$ such that $a = ab$ for some $b$ in $A_+$, cf. [14, 5.6.1], and since $\tau$ is densely defined, hence finite on $K(A)$, it follows that $\tau(y) < \infty$. Restricting $\tau$ to $C$ we therefore obtain a unique Radon measure $\mu_C$ on $T$ such that

$$\int y(t) \, d\mu_C(t) = \tau(y), \quad y \in C, \quad (6)$$

cf. [15, Chapter 6]. Furthermore, if $x \in A_+$ the positive functional $y \to \tau(yx)$ on $C$ determines a unique Radon measure on $T$ (by the Riesz representation theorem) which is absolutely continuous with respect to $\mu_C$, in fact dominated by a multiple of $\mu_C$ (by the Cauchy-Schwarz inequality). By the Radon-Nikodym theorem, there is a positive function $\Phi(x)$ in $L^\infty_{\mu_C}(T)$ such that

$$\int y(t)\Phi(x)(t) \, d\mu_C(t) = \tau(yx), \quad y \in C. \quad (7)$$

Extending by linearity, this defines a map $\Phi$ from $A$ to $L^\infty_{\mu_C}(T)$ which is linear, positive, norm decreasing (and unital if both $A$ and $C$ have the same unit). Moreover,
\[ \Phi(y) = y \text{ almost everywhere if } y \in C \] (with respect to the natural homomorphism of \( C = C_0(T) \) into \( L^\infty_{\mu^C}(T) \)).

When \( \tau \) is faithful and \( C \) and \( A \) are von Neumann algebras, the map \( \Phi \) is a classical example of a conditional expectation, cf. [7, Exercise 8.7.28].

### 6. Lemma

With notations as in sections 4 and 5, take \( \mathbf{x} = (x_1, \ldots, x_n) \) in \( \bigoplus_{i=1}^n (A_i)_{\text{sa}} \) and let \( f \) be a continuous, real and convex function on \( I = I_1 \times \cdots I_n \). If \( \tau \) is unbounded we assume, moreover, that \( f \) is positive. For each commutative \( C^*\)-subalgebra \( C \) of \( A \) set

\[ \varphi_C(f, \mathbf{x}) = \int f(\Phi(x_1)(t), \ldots, \Phi(x_n)(t)) \, d\mu_C(t). \tag{8} \]

Then \( \varphi_C(f, \mathbf{x}) \leq \tau(f(\mathbf{x})) \), with equality whenever \( x_i \in C \) for all \( i \).

**Proof.** If \( x_i \in C \) for all \( i \) then also \( f(\mathbf{x}) \in C \), and since \( \Phi(x_i) = x_i \) almost everywhere we get by (6) that

\[ \varphi_C(f, \mathbf{x}) = \int f(x_1(t), \ldots, x_n(t)) \, d\mu_C(t) = \int f(\mathbf{x}(t)) \, d\mu_C(t) = \tau(f(\mathbf{x})). \tag{9} \]

To prove the inequality for a general \( n \)-tuple we first assume that \( \tau \) is a normal, semi-finite trace on a von Neumann algebra \( M \), [7, 8.5] or [20, 2.5.1], and put

\[ M_\tau = \{ x \in M \mid \tau(|x|) < \infty \}. \tag{10} \]

We then assume that \( A = M_\tau^\square \), the norm closure of \( M_\tau \), so that \( A \) is a two-sided ideal in \( M \). We further assume that each \( A_i \) is relatively weakly closed in \( A_i \), i.e. \( A_i = A \cap M_i \) for some von Neumann subalgebra \( M_i \) of \( M \). This has the effect that every self-adjoint element \( x_i \) in \( A_i \) can be approximated in norm by an element \( y_i \) in \( A_i \) with finite spectrum and \( \text{sp}(y_i) \subset \text{sp}(x_i) \). Under these assumptions we notice that since both \( f \) and \( \Phi \) are norm continuous it suffices to establish the inequality for the norm dense set of \( n \)-tuples \( \mathbf{x} = (x_1, \ldots, x_n) \), such that each \( x_i \) has finite spectrum. Since the \( x_i \)'s commute mutually there is in this case a finite family \( \{ p_k \} \) of pairwise orthogonal projections in \( A \) with sum \( 1 \) such that

\[ x_i = \sum_k \lambda_{ik} p_k, \quad 1 \leq i \leq n, \tag{11} \]

where \( \lambda_{ik} \in I_i \) (and repetitions may occur). If we set \( \lambda_k = (\lambda_{1k}, \ldots, \lambda_{nk}) \) this means that

\[ f(\mathbf{x}) = \sum f(\lambda_k)p_k. \tag{12} \]

As \( \sum p_k = 1 \) in \( A \) also \( \sum \Phi(p_k) = 1 \) in \( L^\infty_{\mu^C}(T) \), so that \( \sum \Phi(p_k)(t) = 1 \) for (almost) every \( t \) in \( T \). Since \( f \) is convex this implies that

\[ f(\Phi(x_1)(t), \ldots, \Phi(x_n)(t)) \leq \sum f(\lambda_{1k}\Phi(p_k)(t), \ldots, \lambda_{nk}\Phi(p_k)(t)) \tag{13} \]

where the sum is taken over all \( k \) such that \( \lambda_{ik} \neq 0 \).
Consequently, by (8), (13) and (6),
\[
\varphi_C(f, \underline{x}) \leq \int \sum_{k} f(\Delta_k) \Phi(p_k)(t) \, d\mu_C(t) = \sum_{k} f(\Delta_k) \int \Phi(p_k)(t) \, d\mu_C(t) \\
= \sum_{k} f(\Delta_k) \tau(p_k) = \tau(\sum_{k} f(\Delta_k)p_k) = \tau(f(\underline{x})).
\]

To prove the inequality for a general $C^*$--algebra $A$ with commuting $C^*$--subalgebras $A_i$ consider the GNS representation $(\pi_{\tau}, \mathcal{H}_\tau)$ associated with $\tau$. By construction, cf. [14, 5.1.5], we obtain a normal, semi-finite trace $\tilde{\tau}$ on the von Neumann algebra $M = \pi_{\tau}(A)'$ such that $\tilde{\tau}(\pi_{\tau}(x)) = \tau(x)$ for every $x$ in $A_+$, and we may define the two-sided ideal $M_\tau$ and its norm closure $M_\tau^\infty$ as in (10). If we now put $\tilde{A} = M_\tau^\infty$ and $\tilde{A}_i = \tilde{A} \cap (\pi_{\tau}(A_i))^{-w}$ (closure in the weak operator topology) for $1 \leq i \leq n$, we have exactly the setup above. Our argument, therefore, shows that the inequality holds in the setting of $\tilde{A}$, $\tilde{A}_i$ and $\tilde{\tau}$, and an arbitrary commutative $C^*$--subalgebra $\tilde{C}$ of $\tilde{A}$.

Note now that since $\tau$ is densely defined on $A$ we have $\pi_{\tau}(A) \subset \tilde{A}$. If $C$ is a commutative $C^*$--subalgebra of $A$ we put $\tilde{C} = \pi_{\tau}(C)$, and observe that $\tilde{C}$ has the form $\tilde{C} = C_0(\tilde{T})$ for some closed subset $\tilde{T}$ of $T$ with $\mu_C(T \setminus \tilde{T}) = 0$. The map $\tilde{\Phi}$ from $\tilde{A}$ to $L_{\infty}(\tilde{T})$ defined in section 3 will therefore satisfy the restriction formula
\[
\tilde{\Phi}(\pi_{\tau}(x)) = \Phi(x)|\tilde{T}
\]
for every $x$ in $A$. But then, by our previous result in (14),
\[
\varphi_C(f, \underline{x}) = \int f(\Phi(x_1), \ldots, \Phi(x_n)) \, d\mu_C = \int f\left(\tilde{\Phi}(\pi_{\tau}(x_1)), \ldots, \tilde{\Phi}(\pi_{\tau}(x_n))\right) \, d\mu_C \\
= \varphi_{\tilde{C}}(f, \pi_{\tau}(\underline{x})) \leq \tilde{\tau}(f(\pi_{\tau}(\underline{x}))) = \tilde{\tau}(\pi_{\tau}(f(\underline{x}))) = \tau(f(\underline{x})).
\]

\[\square\]

Proof of Theorem 2. It is evident that the function $\underline{x} \to \varphi_C(f, \underline{x})$, defined in Lemma 6, is convex on $\bigoplus (A_i)_{sa}^1$ for each $C$, being composed of a linear operator $\Phi(= \Phi_C)$, a convex function $f$, and a positive linear functional - the integral. Moreover, if $C$ denotes the set of commutative $C^*$--subalgebras of $A$, it follows from Lemma 6 that
\[
\tau(f(\underline{x})) = \sup_C \varphi_C(f, \underline{x}),
\]
the supremum being attained at every $C$ that contains the commutative $C^*$--subalgebra $C^*(\underline{x})$ generated by the (mutually commuting) elements $x_1, \ldots, x_n$. Thus $\underline{x} \to \tau(f(\underline{x}))$ is convex as a supremum of convex functions.

This concludes the first part of our paper and we now turn to Theorem 3.

7. $\ell$--Convexity. We consider a strictly increasing, convex and continuous function $e$ on some interval $I$, and denote by $\ell$ its inverse function (so that $\ell(e(s)) = s$ for every $s$ in $I$ and $e(\ell(t)) = t$ for every $t$ in $e(I)$). If $g$ is a convex function defined on a convex subset of a linear space then $f = e \circ g$ is convex as well, but in some sense $f$ is "much more" convex, since even $e \circ f$ is convex, whereas $\ell$ is concave. We
say in this situation that the function \( f \) is \( \ell \)-convex. This terminology is chosen to agree with the seminal example, where \( e = \exp \) and \( \ell = \log \).

Our aim is to show – under mild restrictions on \( \ell \) – that \( \ell \)-convexity has some remarkable structural properties, being preserved under integrals and traces. By contrast, the concept of \( \ell \)-concavity – with the obvious definition – seems to be less interesting.

8. Lemma. If \( I \) is an interval in \( \mathbb{R} \) and \( e \) is a strictly increasing, strictly convex function in \( C^2(I) \) with inverse function \( \ell \), then for each probability measure \( \mu \) on a locally compact Hausdorff space \( T \), and for each \( \ell \)-convex function \( f \) defined on some cube \( I \) in \( \mathbb{R}^n \), the function

\[
(u_1, \ldots, u_n) \mapsto \int f(u_1(t), \ldots, u_n(t)) \, d\mu(t), \quad u_i \in L^\infty_\mu(T),
\]

is also \( \ell \)-convex on the appropriate \( n \)-tuples in \( L^\infty_\mu(T) \) if and only if the function \( \varphi \), defined by \( \varphi(e(s)) = (e''(s))^{-1}(e'(s))^2 \), is concave.

Proof. Since \( f = e \circ g \) for some convex function \( g \) on \( I \), we see that to prove the Lemma it suffices to show that the increasing function

\[
k(u) = \ell \left( \int e(u(t)) \, d\mu(t) \right), \quad u \in L^\infty_\mu(T),
\]

is convex on \((L^\infty_\mu(T))^I\). Clearly, this is also a necessary condition. Considering instead the scalar functions

\[
h(s) = \ell \left( \int e(u(t) + sv(t)) \, d\mu(t) \right)
\]

for arbitrary elements \( u, v \) in \( L^\infty_\mu(T) \), where the range of \( u \) is contained in the interior of \( I \), we notice that convexity of \( k \) is equivalent to convexity at zero for all functions of the form \( h \), and we therefore only have to show that \( h''(0) \geq 0 \). Setting \( r(s) = \int e(u + sv) \, d\mu \) we compute

\[
h'(s) = \ell'(r) \int e'(u + sv)v \, d\mu;
\]

\[
h''(s) = \ell''(r) \left( \int e'(u + sv)v \, d\mu \right)^2 + \ell'(r) \int e''(u + sv)v^2 \, d\mu.
\]

On the other hand, since \( \ell(e(t)) = t \) we also have

\[
\ell'(e(t))e'(t) = 1;
\]

\[
\ell''(e(t))(e'(t))^2 + \ell'(e(t))e''(t) = 0. \tag{22}
\]

In our case we can let \( e(t) = r(s) \), whence \( t = \ell(r(s)) = h(s) \). We can therefore eliminate \( \ell'(r) \) in (21) to get the expression

\[
h''(s) = -\ell'(r)\ell''(h(s))(e'(h(s)))^{-2} \left( \int e'(u + sv)v \, d\mu \right)^2 + \ell'(r) \int e''(u + sv)v^2 \, d\mu
\]

\[
= \ell'(r) \left( \int e''(u + sv)v^2 \, d\mu - e''(h(s))(e'(h(s)))^{-2} \left( \int e'(u + sv)v \, d\mu \right)^2 \right). \tag{23}
\]
Since $e$ is strictly increasing, so is $\ell$, which implies that $\ell'(r) > 0$. It follows from (23) that $h''(0) \geq 0$ if and only if
\[
e''(h(0))e'(h(0))^{-2} \left( \int e'(u)v \, d\mu \right)^2 \leq \int e''(u)v^2 \, d\mu.
\] (24)

Now define the function $\varphi$ on $e(I)$ by $\varphi(e(s)) = (e'(s))^2(e''(s))^{-1}$. Since $e(h(0)) = r(0) = \int e(u) \, d\mu$ and $e''(h(0)) > 0$ because $e$ is strictly convex we see that (24) is equivalent to the inequality
\[
\left( \int e'(u)v \, d\mu \right)^2 \leq \varphi \left( \int e(u) \, d\mu \right) \int e''(u)v^2 \, d\mu.
\] (25)

For ease of notation put $\varphi \left( \int e(u) \, d\mu \right) = \widetilde{\varphi}(e(u))$, and choose a function $w$ in $L^1_\mu(T)$ with $\int w \, d\mu = 1$. Then consider the quadratic form
\[
\lambda^2 \int e''(u)v^2 \, d\mu - 2\lambda \int e'(u)v \, d\mu + \widetilde{\varphi}(e(u)) \int w \, d\mu
\] (26)

By construction this form is positive if and only if (25) is satisfied. But (26) expresses the integral of a function which is itself a quadratic form. The minimum in (26) therefore occurs for $\lambda ve''(u) = e'(u)$ and equals
\[
\int (\widetilde{\varphi}(e(u))w - (e''(u))^{-1}(e'(u))^2) \, d\mu
\]
\[
= \int (\widetilde{\varphi}(e(u))w - \varphi(e(u))) \, d\mu = \varphi \left( \int e(u) \, d\mu \right) - \int \varphi(e(u)) \, d\mu.
\] (27)

Evidently this expression is non-negative if and only if $\varphi$ is concave. \square

9. Remark. Using the equation $e(\ell(t)) = t$ as in (22) we easily find that $\varphi(t) = -(\ell''(t))^{-1}\ell'(t)$, so that the condition in Lemma 8 translates to the demand:

The (negative) function $t \to \ell'(t)/\ell''(t)$ must be convex. (28)

Note also that the condition in (27), viz.
\[
\int \varphi(e(u)) \, d\mu \leq \varphi \left( \int e(u) \, d\mu \right)
\] (29)

makes sense for an arbitrary measure $\mu$ and provides a necessary and sufficient condition for the function defined in (18) to be $\ell$–convex. However, in order to satisfy (29) for an arbitrary (point) measure, the function $\varphi$ must have $s\varphi(t) \leq \varphi(st)$ for all $s > 0$, which forces it to be homogeneous, i.e. $\varphi(st) = s\varphi(t)$ for $s > 0$. It follows that the function defined by (18) in Lemma 8 is $\ell$–convex for an arbitrary measure $\mu$ if and only if
\[
\ell'(t)/\ell''(t) = \gamma t \quad \text{for some non-zero number } \gamma.
\] (30)

Of course, this can only happen when the domain of $\ell$ is stable under multiplication with positive numbers, so it is either a half-axis or the full line. But since the expression in (30) must be negative, only half-axes can occur.

Proof of Theorem 3. The Theorem follows by using Lemma 6, as in the proof of Theorem 2, combined with Lemma 8. \square
10. Examples. Evidently the condition that \( \ell' / \ell'' \) be convex is not very restrictive, and is satisfied by myriads of functions, of which we shall list a few, below. On the other hand, the demand that \( \ell' / \ell'' \) be homogeneous is quite severe, and only four (classes of) functions will meet this requirement:

(i) Let \( \ell(t) = \log(t) \) for \( t > 0 \). We get \( e(s) = \exp(t) \) for \( s \in \mathbb{R} \) and \( \ell' / \ell'' = -t \). This is the classical example, and by far the most important. Clearly \( \ell(t) = c \log(t) \) for any \( c > 0 \) can also be used, but we omit this trivial parameter here and in the following examples.

(ii) Let \( \ell(t) = t^{1/p} \) for \( t \geq 0 \) and some \( p > 1 \). We get \( e(s) = t^p \) for \( s \geq 0 \) and \( \ell' / \ell'' = -\gamma t \), where \( \gamma = p/(p - 1) > 1 \). The root examples are also fairly well known. Indeed, it is a very general fact that whenever \( f \) is a convex (resp. concave) function that is homogeneous of some degree \( p > 0 \) then (i) we must have \( p \geq 1 \) (resp. \( p \leq 1 \)) and (ii) the function \( f^{1/p} \) is automatically convex (resp. concave). This is discussed in detail in the proof of Corollary 1.2 in [9].

(iii) Let \( \ell(t) = -t^{-\alpha} \) for \( t > 0 \) and some \( \alpha > 0 \). We get \( e(s) = (-s)^{-1/\alpha} \) for \( s < 0 \) and \( \ell' / \ell'' = -\gamma t \), where \( \gamma = (1 + \alpha)^{-1} < 1 \).

(iv) Let \( \ell(t) = -(-t)^p \) for \( t \leq 0 \) and some \( p > 1 \). We get \( e(s) = (-s)^{1/p} \) for \( s \leq 0 \) and \( \ell' / \ell'' = \gamma t \), where \( \gamma = (p - 1)^{-1} > 0 \).

Non-homogeneous examples are not hard to come by. Without any apparent order we mention these:

(v) Let \( \ell(t) = -\exp(-\alpha t) \) for \( t \) in \( \mathbb{R} \) and some \( \alpha > 0 \). We get \( e(s) = -\alpha^{-1} \log(-s) \) for \( s < 0 \) and \( \ell' / \ell'' = -1/\alpha \). In applications of Theorem 3 the parameter \( \alpha \) disappears, since \( \ell(\tau(e(a))) = -\exp(-\alpha \tau(-\alpha^{-1} \log(-a))) = -\exp(\tau(\log(-a))) \) for any operator \( a < 0 \), so we may as well assume that \( \alpha = 1 \).

(vi) Let \( \ell(t) = \log(\log(t)) \) for \( t > 1 \). We get \( e(s) = \exp(\exp(s)) \) for \( s \) in \( \mathbb{R} \) and \( \ell'(t)/\ell''(t) = -t \log(t)(1 + \log(t))^{-1} \), which is only convex for \( t \leq e \). So on the intervals \( 1 < t \leq e \) and \( -\infty < s \leq 0 \) we can use the functions \( \log \) and \( \exp \).

(vii) Let \( \ell(t) = (\log(t))^{1/p} \) for \( t \geq 1 \) and some \( p > 1 \). Here \( e(s) = \exp(s^p) \) for \( s \geq 0 \) and by computation we find that \( \ell' / \ell'' \) is convex for \( t \leq \exp(1 + 1/p) \). The allowed interval for \( e \) is \( 0 \leq s \leq (1 + p)p^{-2} \).

(viii) Let \( \ell(t) = (1 - (1 - t)^p)^{1/p} \) for \( 0 \leq t \leq 1 \) and some \( p > 1 \). We get \( e(s) = 1 - (1 - s)^p \) for \( 0 \leq s \leq 1 \) and \( \ell'(t)/\ell''(t) = (p - 1)^{-1} (1 - t)^{p+1} - (1 - t) \), which is a convex function on the unit interval.

(ix) Let \( \ell(t) = t^{1/p}(1 + t^{1/p})^{-1} \) for \( t \geq 0 \) and some \( p \geq 1 \). Here \( e(s) = s^p(1 - s)^{-p} \) for \( 0 \leq s < 1 \). For \( p > 1 \) we find after some computation that \( \ell'(t)/\ell''(t) = 2pt^{1+p}/((p - 1)(p - 1 + (p + 1)t^{1/p}))^{-1} \) plus a linear term, and this is a convex function. For \( p = 1 \) we simply get \( \ell'(t)/\ell''(t) = -\frac{1}{2}(1 + t) \), and we note that this example is just a translation of (iii) (replacing \( t \) by \( 1 + t \) and adding 1).

11. Corollary. If \( f \) is a positive, continuous, log-convex function on a cube \( I \) in \( \mathbb{R}^n \), and \( A_1, \ldots, A_n \) are mutually commuting \( C^* \)-subalgebras of a \( C^* \)-algebra \( A \), then for each densely defined, lower semi-continuous trace \( \tau \) on \( A \) the function

\[
(x_1, \ldots, x_n) \to \tau(f(x_1, \ldots, x_n)),
\]

defined on commuting \( n \)-tuples such that \( x_i \in (A_i)_{sa}^1 \) for each \( i \), is also log-convex on \( \bigoplus (A_i)_{sa}^1 \). If, instead, \( f^{1/p} \) is convex for some \( p > 1 \), or if \( -f^{-\alpha} \) is convex for some \( \alpha > 0 \), or if \( f \) is convex, then we also have convexity of the above function.
respective functions

\[
\begin{align*}
(x_1, \ldots, x_n) &\to (\tau(f(x_1, \ldots, x_n)))^{1/p}, \\
(x_1, \ldots, x_n) &\to -(\tau(f(x_1, \ldots, x_n)))^{-\alpha}, \\
(x_1, \ldots, x_n) &\to -(-\tau(f(x_1, \ldots, x_n)))^p.
\end{align*}
\] (32)

12. Remarks. The Corollary above applies to some unexpected situations. Thus we see that the function

\[
(x, y) \to \log(\tau(\exp((x + y)^2)))
\] (33)

where \(x\) and \(y\) are self-adjoint elements in a pair of commuting \(C^*\)-algebras \(A\) and \(B\), is (jointly) convex. The same can be said of the function

\[
(x, y) \to \log(\tau(\exp(-x^\alpha y^\beta)))
\] (34)

for \(0 < \alpha, \beta\) and \(\alpha + \beta \leq 1\), defined on \(A_+ \times B_+\). Applied to the root functions the Corollary shows that the function

\[
(x, y) \to (\tau((x + y)^q))^{1/p}
\] (35)

is convex for \(1 \leq p \leq q\), and that also

\[
(x, y) \to (\tau((1 - x^\alpha y^\beta)^p))^{1/p}
\] (36)

is convex for \(p \geq 1\) on the product of the positive unit balls of \(A\) and \(B\).

The last two cases in Corollary 11 are perhaps easier to apply in terms of concave functions. By elementary substitutions we find that if \(f\) is a positive concave function on some cube \(I\) in \(\mathbb{R}^n\), then both functions

\[
\begin{align*}
(x) &\to (\tau(f(x)))^{-1/\alpha}; \\
(x) &\to (\tau((f(x))^{1/p}))^p;
\end{align*}
\] (37)

are concave on \(\oplus(A_i)_{i=1}^I\) for \(\alpha > 0\) (and \(f > 0\)), and for \(p \geq 1\). In particular we see that

\[
\left(\tau((x + y)^{1/p})\right)^p \geq \left(\tau(x^{1/p})\right)^p + \left(\tau(y^{1/p})\right)^p,
\] (38)

for all \(x, y\) in \(A_+\), so that the Schatten \(p\)-norms are super-additive for \(p < 1\).

Recall from [4] the definition of the Kadison - Fuglede determinant \(\Delta\) associated with a tracial state \(\tau\) on a \(C^*\)-algebra \(A\):

\[
\Delta(x) = \exp(\tau(\log |x|)) \quad \text{whenever} \quad x \in A^{-1}.
\]

This is a positive, homogeneous and multiplicative map on the set of invertible elements, cf. [4, Theorem 1], closely related to the ordinary determinant from matrix theory. Note, however, that when \(A = \mathbb{M}_n(\mathbb{C})\) then \(\Delta(x) = (\det(|x|))^{1/n}\), because we have to use the normalized trace \(\tau = \frac{1}{n} \text{Tr}\).

The following result is well known for matrices, at least for functions of one variable.
13. Proposition. For each strictly positive concave function \( f \) on a cube \( I \) in \( \mathbb{R}^n \), and mutually commuting \( C^* \)-subalgebras \( A_1, \ldots, A_n \) of a \( C^* \)-algebra \( A \) the operator function

\[
x \rightarrow \Delta(f(x))
\]

is concave on the appropriate \( n \)-tuples of commuting self-adjoint elements from \( \bigoplus (A_i)^I \). In particular, the Kadison - Fuglede determinant is a concave map on the set of positive invertible elements.

Proof. From example (v) in section 10 we see that the function

\[
x \rightarrow -\exp (\tau(\log(-(-f(x)))) = -\Delta(f(x))
\]

is convex, as desired. \( \square \)

14. Remark. The concavity in (39) is closely related to one of the main results (Theorem 6) in [9] namely: \( x \rightarrow \tau(\exp(z + \log(x))) \) is concave for any self-adjoint operator \( z \). In one sense (39) is stronger because it allows a general concave \( f \), but when \( f \) is linear (39) is a corollary of Theorem 6 in [9], as we show now for one variable:

We have to prove that

\[
\exp(\tau(\log(f(x+y)))) \geq \frac{1}{2}\exp(\tau(\log(x))) + \frac{1}{2}\exp(\tau(\log(y))).
\]

Put \( z = -\log(\frac{1}{2}(x+y)) \). Our condition is then that \( 1 \geq \frac{1}{2}\exp(\tau(\log(x))) + \frac{1}{2}\exp(\tau(\log(y))). \) However, \( \tau \) is a state and therefore Jensen’s inequality applies. Thus, \( \exp(\tau(\log(x))) \leq \tau(\exp(\log(x))) \), and similarly for \( y \). By Theorem 6 of [9] we know that \( x \rightarrow \tau(\exp(x+\log(x))) \) is concave, which implies that

\[
\frac{1}{2}\tau(\exp(x+\log(x))) + \frac{1}{2}\tau(\exp(z + \log(y))) \leq \tau(\exp(x + \log(\frac{1}{2}(x+y)))) = \tau(1) = 1,
\]

as desired.

15. Operator Means. We recall from [8], see also [6, 4.1], that a Kubo-Ando mean on the set \( B_+ = \mathbb{B}(\mathfrak{f})_+ \) of positive operators is a function \( \sigma:B_+ \times B_+ \rightarrow B_+ \) such that

(i) \( (1 \sigma 1) = 1 \),

(ii) \( (x_1 \sigma y_1) \leq (x_2 \sigma y_2) \) if \( x_1 \leq x_2 \) and \( y_1 \leq y_2 \),

(iii) \( \sigma \) is (jointly) concave on \( B_+ \times B_+ \),

(iv) \( z^*(x \sigma y)z = (z^*x^*) \sigma (z^*y^*) \) for every invertible \( z \in \mathbb{B}(\mathfrak{f}) \).

Of particular interest are the harmonic mean ! and the geometric mean \# defined on positive, invertible operators by:

\[
x ! y = 2(x^{-1} + y^{-1})^{-1} = 2x(x + y)^{-1}y;
x \# y = x^{1/2}(x^{-1/2}yx^{-1/2})^{1/2}x^{1/2}.
\]

\((x \# y) \) was introduced in [18] and \( \frac{1}{2}(x ! y) \), the parallel sum, was introduced in [1]; see also [2], [6, 4.1] and [8].) The domain of definition for these means can be extended to all positive operators by a simple limit argument (replacing \( x \) and \( y \) with \( x + \varepsilon1 \) and \( y + \varepsilon1 \), and taking the norm limit as \( \varepsilon \rightarrow 0 \), and we shall tacitly use this procedure in the following. Thus we shall state all results for positive operators, but in the proofs assume that they are invertible as well. Note that these two operator means are symmetric in the two variables, and that they reduce...
to the classical harmonic and geometric means when \( x \) and \( y \) are positive scalars, i.e.

\[
x ! y = 2xy(x + y)^{-1} \quad \text{and} \quad x \# y = \sqrt{xy}.
\]  

(42)

A straightforward application of the Cauchy-Schwarz inequality shows that

\[
\tau(x \# y) \leq \tau(x) \# \tau(y)
\]  

(43)

for any trace \( \tau \). The corresponding result for the harmonic mean was proved for the ordinary trace \( \text{Tr} \) in [2]. The general version below (Proposition 21) is somewhat more involved, but also richer. For greater generality, but at little extra cost, we introduce the harmonic mean of an \( n \)-tuple of positive operators as

\[
(x_1 \! x_2 \! \cdots \! x_n) = n(x_1^{-1} + x_2^{-1} + \cdots + x_n^{-1})^{-1},
\]  

(44)

and we note that this mean is symmetric in all the variables and increasing in each variable. The fact that it is also jointly concave may not be widely known, so we present a short proof.

16. Proposition. The \( n \)-fold harmonic mean is a jointly concave function.

Proof. As the expression in (44) is homogeneous all we have to show is that

\[
\left( \sum x_i^{-1} \right)^{-1} + \left( \sum y_i^{-1} \right)^{-1} \leq \left( \sum (x_i + y_i)^{-1} \right)^{-1},
\]  

(45)

for any pair of \( n \)-tuples of positive invertible operators. Multiplying left and right by \( \sum (x_i + y_i)^{-1} \) we obtain the equivalent inequality

\[
\left( \sum (x_i + y_i)^{-1} \right) \left( \left( \sum x_i^{-1} \right)^{-1} + \left( \sum y_i^{-1} \right)^{-1} \right) \left( \sum (x_i + y_i)^{-1} \right) \leq \sum (x_i + y_i)^{-1}.
\]  

(46)

We now appeal to the fact that the operator function \((x, y) \rightarrow y^*x^{-1}y\) is jointly convex, [11], hence also jointly subadditive on the space of operators \( x, y \), where \( x \) is positive and invertible. To see this, consider \( n \)-tuples \((x_i)\) and \((y_i)\) and define \( z_j = x_j^{-1/2}(y_j - a) \), with \( a = (\sum x_i)^{-1} \sum y_i \). Then by computation we obtain the desired estimate

\[
0 \leq \sum z_j^*z_j = \sum y_j^*x_j^{-1}y_j - \left( \sum y_j^* \right) \left( \sum x_j \right)^{-1} \left( \sum y_j \right).
\]  

(47)

Breaking the left hand side of (46) into the sum of two terms and using (47) on each we obtain the larger operator

\[
\sum (x_i + y_i)^{-1}x_i(x_i + y_i)^{-1} + \sum (x_i + y_i)^{-1}y_i(x_i + y_i)^{-1}) = \sum (x_i + y_i)^{-1}(x_i + y_i)(x_i + y_i)^{-1} = \sum (x_i + y_i)^{-1},
\]  

(48)

which is precisely the right hand side of (46), as claimed. \( \square \)
17. Remark. Note that (47) also shows that the $n-$fold harmonic mean is dominated by the arithmetic mean (the average). Indeed,

$$x_1 ! x_2 ! \ldots ! x_n = \left( \frac{1}{n} \sum_{k=1}^{n} x_k^{-1} \right)^{-1} = \left( \frac{1}{n} \sum_{k=1}^{n} \frac{1}{x_k} \right)^{-1} \left( \sum_{k=1}^{n} \frac{1}{n} \right)^{-1}$$

$$\leq \sum_{k=1}^{n} \frac{1}{n} (x_k^{-1})^{-1} = \frac{1}{n} \sum_{k=1}^{n} x_k .$$

This result is not surprising, since the harmonic mean (on pairs of operators) is the smallest symmetric mean, whereas the arithmetic mean is the largest, cf. [6, 4.1].

18. Proposition. Given positive, invertible operators $x_1, x_2, \ldots, x_n$ and $y$ in $\mathcal{B}(\mathcal{H})$, let $d = \text{diag}\{x_1, x_2, \ldots, x_n\}$ and $e = \sum_{i,j=1}^{n} y \otimes e_{ij}$ in $\mathcal{M}_n(\mathcal{B}(\mathcal{H}))$. Then the following conditions are equivalent:

(i) $y \leq (\sum_{k=1}^{n} x_k^{-1})^{-1}$,

(ii) $ed^{-1} e \leq e$,

(iii) $e \leq d$.

Proof. Assume first that $y = 1$, and set $a = \sum_{k=1}^{n} x_k^{-1}$, so that condition (i) becomes equivalent with $a \leq 1$.

(i) $\iff$ (ii). By computation

$$ed^{-1} e = \sum_{i,j=1}^{n} a \otimes e_{ij},$$

from which it follows that $a \leq 1$ if and only if $ed^{-1} e \leq e$.

(i) $\implies$ (iii). Let $p = \frac{1}{n} e$, and note that $p$ is a projection in $\mathcal{M}_n(\mathcal{B}(\mathcal{H}))$. For every $\varepsilon > 0$ we have

$$d^{-1} = (p + (1 - p))(1 + \varepsilon)p - (1 + \varepsilon)^{-1}(1 - p)d^{-1}(1 - p) = (1 + \varepsilon)n^{-2}ed^{-1} e + (1 + \varepsilon^{-1})(1 - p)d^{-1}(1 - p).$$

Now $a \leq 1$ by (1) and a fortiori $d^{-1} \leq 1$. Moreover, (i) $\implies$ (ii) and thus we get

$$d^{-1} \leq (1 + \varepsilon)n^{-2}e + (1 + \varepsilon^{-1})(1 - p) = (1 + \varepsilon)n^{-1}p + (1 + \varepsilon^{-1})(1 - p).$$

Taking inverses this means that

$$d \geq (1 + \varepsilon)^{-1}np + \varepsilon(1 + \varepsilon)^{-1}(1 - p),$$

from which the desired inequality follows as $\varepsilon \to 0$.

(iii) $\implies$ (ii). If $e \leq d$, then $e + \varepsilon 1 \leq d + \varepsilon 1$ for every $\varepsilon > 0$, whence

$$(d + \varepsilon 1)^{-1} \leq (e + \varepsilon 1)^{-1} = (n + \varepsilon)^{-1} p + \varepsilon^{-1}(1 - p).$$

But then, since $e(1 - p) = 0$, we have

$$e(d + \varepsilon)^{-1} e \leq (n + \varepsilon)^{-1} np + \varepsilon^{-1}(1 - p).$$
and as $\varepsilon \to 0$ we obtain the desired estimate.

In the general case, where $y$ is arbitrary (positive and invertible), we define $\tilde{x}_k = y^{-1/2} x_k y^{-1/2}$ for $1 \leq k \leq n$. Then with $\tilde{e} = \sum_{i,j=1}^n 1 \otimes e_{ij}$ and $\tilde{d} = \text{diag}\{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n\}$ we observe that conditions (i), (ii) and (iii) become equivalent with the conditions

(i) $\frac{1}{2} \leq \left(\sum_{k=1}^n \tilde{x}_k^{-1}\right)^{-1}$,
(ii) $\tilde{e} \tilde{d}^{-1} \tilde{e} \leq \tilde{e}$,
(iii) $\tilde{e} \leq \tilde{d}$.

But these conditions are exactly the ones we proved to be equivalent above, assuming that $y = 1$.

**19. Corollary.** For any $n$-tuple $(x_1, x_2, \ldots, x_n)$ of positive operators in $\mathcal{B}(\mathcal{H})$ the harmonic mean $(x_1 ! x_2 ! \ldots ! x_n)$ is the largest positive operator of the form $nz$, such that

$$\begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \end{pmatrix} \leq \begin{pmatrix} z & z & \cdots & z \\ z & z & \cdots & z \\ \vdots & \vdots & \ddots & \vdots \\ z & z & \cdots & z \end{pmatrix}$$

(56)

**20. Remark.** Note that Proposition 18 actually says slightly more than Corollary 19, namely that the set of positive operators $nz$, such that $z$ satisfies the matrix inequality in (56), is exactly equal to

$$((x_1 ! x_2 ! \ldots ! x_n) - \mathcal{B}(\mathcal{H})) \cap \mathcal{B}(\mathcal{H})_+.$$

(57)

This should be compared to the analogous result for the geometric mean of pairs of positive operators $x$ and $y$. Here $x \# y$ is the largest positive operator $z$ such that $\left(\begin{smallmatrix} x & z \\ z & y \end{smallmatrix}\right) \geq 0$ in $M_2(\mathcal{B}(\mathcal{H}))$. However, one can find positive operators $z$ such that $z \leq x \# y$, without the matrix being positive. It suffices to take $x = 1$, so that $x \# y = y^{1/2}$. Evidently the matrix $\left(\begin{smallmatrix} 1 & z \\ z & y \end{smallmatrix}\right)$ is positive only if $z^2 \leq y$, and it is easy to find examples where $z^2 \leq y^{1/2}$, but $z^2 \leq y$.

**21. Proposition.** If $\tau$ is a densely defined, lower semi-continuous trace on a $C^*$-algebra $A$, and $f$ is a strictly positive function on $]0, \infty[$ such that $t \to (f(t^{-1}))^{-1}$ is concave, then for all positive operators $x_1 \ldots, x_n \in A$ and $\alpha > 0$ we have

$$\left(\tau((f(x_1 ! x_2 ! \cdots ! x_n)\alpha)))^{1/\alpha} \leq \tau((f(x_1)\alpha))^{1/\alpha} \tau((f(x_2)\alpha))^{1/\alpha} \cdots \tau((f(x_n)\alpha))^{1/\alpha} \right).$$

(58)

**Proof.** Define $g(t) = (f(t^{-1}))^{-1}$. Then from (37) we see that the function

$$t \to g(e^{-t})$$

(59)
is concave on the set of positive invertible elements in $A$. In particular,

$$
(\tau \left( (g(1/n (x_1^{-1} + \cdots + x_n^{-1})))^{-\alpha} \right))^{-1/\alpha} \\
\geq 1/n \left( \tau \left( (g(x_1^{-1}))^{-\alpha} \right) \right)^{-1/\alpha} + \cdots + 1/n \left( \tau \left( (g(x_n^{-1}))^{-\alpha} \right) \right)^{-1/\alpha} .
$$

Taking the reciprocal values and noting that $1/n (x_1^{-1} + \cdots + x_n^{-1}) = (x_1 ! \cdots ! x_n)^{-1}$ this means that

$$
\left( \tau \left( (g((x_1 ! \cdots ! x_n)^{-1}))^{-\alpha} \right) \right)^{1/\alpha} \\
\leq n \left( \tau \left( (g(x_1^{-1}))^{-\alpha} \right) \right)^{-1/\alpha} + \cdots + \left( \tau \left( (g(x_n^{-1}))^{-\alpha} \right) \right)^{-1/\alpha} \\
= \left( \tau \left( (g(x_1^{-1}))^{-\alpha} \right) \right)^{1/\alpha} ! \cdots ! \left( \tau \left( (g(x_n^{-1}))^{-\alpha} \right) \right)^{1/\alpha} .
$$

Since $g(t^{-1})^{-1} = f(t)$ this is the desired result. \hfill \Box

22. \textbf{Remark.} The result above applies to the functions $t \to t^{1/p}$ for $p \geq 1$, in particular to the identity function (and with $\alpha = 1$); but it also applies to the functions $t \to \log(1 + t^{1/p})$ for $p \geq 1$.

On the abstract level Proposition 21 applies to $C^2$-functions $f$, such that $f$ and $t \to t^2 f(t)^{-2} f'(t)$ are simultaneously increasing or decreasing (strictly) on $]0, \infty[$.

Proposition 21 also applies to other (not necessarily symmetric) means of positive operators. Such means were introduced in [8] (see also [6, 4.1]).

23. \textbf{Proposition.} For every Kubo-Ando mean $\sigma$ and for every densely defined, lower semi-continuous trace $\tau$ on a $C^*$-algebra $A$ we have $\tau(x \sigma y) \leq \tau(x) \sigma \tau(y)$ for all $x, y$ in $A_+$.

\textit{Proof.} For each mean $\sigma$ there is a unique probability measure $\mu$ on $[0, \infty]$ such that

$$
x \sigma y = \frac{1}{2} \int_0^\infty (tx! y) (1 + 1/t) \ d\mu(t) = \frac{1}{2} \int_0^\infty ((1 + t)x! (1 + 1/t)y) \ d\mu(t) ,
$$

cf. [6, 4.1]. Applying Proposition 22 with $f(t) = t$ and $\alpha = 1$ it follows that

$$
\tau(x \sigma y) = \frac{1}{2} \int_0^\infty \tau(tx! y) (1 + 1/t) \ d\mu(t) \\
\leq \frac{1}{2} \int_0^\infty (t \tau(x)! \tau(y)) (1 + 1/t) \ d\mu(t) = \tau(x) \sigma \tau(y).
$$

\hfill \Box

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References

[1] William N. Andersen & Richard J. Duffin, *Series and parallel addition of matrices*, Journal of Mathematical Analysis and Applications **26** (1969), 576–594.

[2] Tsuyoshi Ando, *Concavity of certain maps of positive definite matrices and applications to Hadamard products*, Linear Algebra and its Applications **26** (1979), 203–241.

[3] Huzihiro Araki, *On an inequality of Lieb and Thirring*, Letters in Mathematical Physics **19** (1990), 167–170.

[4] Bent Fuglede & Richard V. Kadison, *Determinant theory in finite factors*, Annals of Mathematics **55** (1952), 520–530.

[5] Frank Hansen, *Convex trace functions of several variables*, Linear Algebra and its Applications, to appear.

[6] Fumio Hiai, *Log-majorizations and norm inequalities for exponential operators*, Banach Center Publications **38** (1997), The Polish Academy of Sciences, Warszawa, 119–181.

[7] Richard V. Kadison & John R. Ringrose, *“Fundamentals of the Theory of Operator Algebras”, vol I-II*, Academic Press, San Diego, 1986 (Reprinted by AMS in 1997).

[8] Fumio Kubo and Tsuyoshi Ando, *Means of positive linear operators*, Mathematische Annalen **246** (1980), 205–224.

[9] Elliott H. Lieb, *Convex trace functions and the Wigner-Yanase-Dyson conjecture*, Advances in Mathematics **11** (1973), 267–288.

[10] Elliott H. Lieb, *The classical limit of quantum systems*, Communications in Mathematical Physics **31** (1973), 327–340.

[11] Elliott H. Lieb & Mary Beth Ruskai, *Some operator inequalities of the Schwarz type*, Advances in Mathematics **26** (1974), 269–273.

[12] John von Neumann, *“Mathematical Foundations of Quantum Mechanics”*, Princeton Press, Princeton NJ, 1955.

[13] Masanori Ohya & Dénes Petz, *“Quantum Entropy and its Use”*, Texts and Monographs in Physics, Springer Verlag, Heidelberg, 1993.

[14] Gert K. Pedersen, *“C*-Algebras and their Automorphism Groups”,* LMS Monographs **14**, Academic Press, San Diego, 1979.

[15] Gert K. Pedersen, *“Analysis Now”*, Graduate Texts in Mathematics **118**, Springer Verlag, Heidelberg, 1989, reprinted 1995.

[16] Gert K. Pedersen, *Convex trace functions of several variables on C*-algebras*, Preprint.

[17] Dénes Petz, *Spectral scale of self-adjoint operators and trace inequalities*, Journal of Mathematical Analysis and Applications **109** (1985), 74–82.

[18] W. Pusz and S. Lech Woronowicz, *Functional calculus for sesquilinear forms and the purification map*, Reports on Mathematical Physics **8** (1975), 159–170.

[19] David Ruelle, *“Statistical Mechanics”*, The Mathematical Physics Monograph Series, Benjamin, New York, 1969.

[20] Shôichirô Sakai, *“C*-Algebras and W*-Algebras*, Springer Verlag, Heidelberg, 1971, reprinted 1997.

[21] Barry Simon, *The classical limit of quantum partition functions*, Communications in Mathematical Physics **71** (1980), 247–276.

[22] Barry Simon, *“The Statistical Mechanics of Lattice Gases”, vol I*, Princeton University Press, Princeton, 1993.

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