Representation Varieties of Non-orientable Surfaces via Topological Quantum Field Theories

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Abstract

We study the $G$-representation varieties of non-orientable surfaces. By a geometric method using Topological Quantum Field Theories (TQFTs), we compute virtual classes of these $G$-representation varieties in the Grothendieck ring of varieties, for $G$ the groups of complex upper triangular matrices of rank 2 and 3. We discuss various ways to use conjugacy invariance to simplify the computations. Finally, we give a number of remarks on the resulting classes, and observe and explain the zero eigenvalues that appear in the TQFT.

1 Introduction

Let $X$ be a closed path-connected manifold with finitely generated fundamental group $\pi_1(X)$, and $G$ an algebraic group over a field $k$. The set of group representations $X_G(X) = \text{Hom}(\pi_1(X), G)$, carries a natural structure of an algebraic variety over $k$, and is called the $G$-representation variety of $X$. Indeed, given a set of generators $\gamma_1, \ldots, \gamma_n$ of $\pi_1(X)$, the morphism $X_G(X) \to G^n : \rho \mapsto (\rho(\gamma_1), \ldots, \rho(\gamma_n))$ identifies the $G$-representation variety with a subvariety of $G^n$. This structure can be shown to be independent of the chosen generators.

As an example, when $X = \Sigma_g$ is a closed surface of genus $g$, the $G$-representation variety can be written as

$$X_G(\Sigma_g) = \left\{ (A_1, B_1, \ldots, A_g, B_g) \in G^{2g} : \prod_{i=1}^g [A_i, B_i] = \text{id} \right\}.$$

In [4], González-Prieto, Loaeres and Muñoz developed a method to compute the virtual class of the $G$-representation variety $X_G(\Sigma_g)$ in the Grothendieck ring of varieties $K(\text{Var}_k)$, using Topological Quantum Field Theories (TQFTs). By a TQFT we mean a (lax) monoidal functor $Z : \text{Bd}_n \to R\text{-Mod}$ from the category of bordisms to the category of $R$-modules, with $R$ a commutative ring.

One of the useful things about TQFTs is that they associate invariants to closed manifolds. Indeed, given a closed manifold $M$, one can consider it as a bordism $\emptyset \to \emptyset$, which induces a map $Z(M)$:
$R \rightarrow R$, since $Z(\emptyset) = R$ by monoidality. This map is determined by $Z(M)(1) \in R$, which we refer to as the invariant associated to $M$. The TQFT that we will use associates to such $M$ the class $[\mathcal{X}_G(M)]$ in $R = K(\text{Var}_k)$.

This method was used in [3] to compute the virtual class of the (parabolic) $\text{SL}_2(\mathbb{C})$-representation and character varieties of $\Sigma_g$. It was also used in [7, 5] to compute the virtual class of $\mathcal{X}_G(\Sigma_g)$ for $G$ the groups of upper triangular matrices of rank 2, 3 and 4. In this paper we extend this result to non-orientable surfaces, for $G$ the groups of upper triangular matrices of rank 2 and 3.

Using the classification of surfaces, any connected closed surface $\Sigma$ is either a sphere or a connected sum of tori or of projective planes. Let $N_r$ be the surface obtained by a connected sum of $r$ projective planes (i.e. demigenus $r$). Its fundamental group is given by
\[
\pi_1(N_r) = \langle a_1, \ldots, a_r | a_1^2 \cdots a_r^2 = 1 \rangle,
\]
and its $G$-representation variety thus by
\[
\mathcal{X}_G(N_r) = \{ (A_1, \ldots, A_r) \in G^r | A_1^2 \cdots A_r^2 = \text{id} \},
\]
which is a closed subvariety of $G^r$.

We write $U_n$ for the group of upper triangular $n \times n$ matrices. The main results of this paper are given by the following theorem.

**Theorem 1.1.** Let $N_r$ be the connected sum of $r > 0$ projective planes. Let $q = [A_1^\mathbb{C}]$ be the class of the affine line in the Grothendieck ring of varieties. Then the virtual class of the $U_2$-representation variety of $N_r$ is
\[
[X_{U_2}(N_r)] = 4q^{r-1}(q-1)^{2r-2} + 2q^{r-1}(q-1)^r,
\]
and the virtual class of the $U_3$-representation variety is
\[
[X_{U_3}(N_r)] = 4q^{2r-1}(q-1)^{2r-1} + 2q^{3r-3}(q-1)^{r+1} + 8q^{3r-3}(q-1)^{2r-1} + 8q^{3r-3}(q-1)^{3r-3}.
\]

## 2 Construction of the TQFT

Let $G$ be an algebraic group over a field $k$. In this section, we briefly discuss the construction of the Topological Quantum Field Theory (TQFT) that is used to compute the classes of $G$-representation varieties in $K(\text{Var}_k)$. This TQFT will be a lax monoidal functor $Z : \text{Bdp}_n \rightarrow K(\text{Var}_k)-\text{Mod}$ from the category of $n$-bordisms with basepoints to the category of $K(\text{Var}_k)$-modules. For a more elaborate construction of this TQFT, see [4, 2, 7].

In previous papers, bordisms were taken to be orientable, but since we will focus on non-orientable surfaces, we allow a bordism to be unoriented.

**Definition 2.1.** Given two $(n-1)$-dimensional closed manifolds $M$ and $M'$, a bordism from $M$ to $M'$ is an $n$-dimensional compact manifold $W$ with boundary together with inclusions
\[
M' \xrightarrow{i'} W \xleftarrow{i} M
\]
such that $\partial W = i(M) \sqcup i'(M')$. 


Definition 2.2. The category of $n$-bordisms with basepoints, denoted $\text{Bdp}_n$, is the 2-category consisting of:

- **Objects:** pairs $(M, A)$, with $M$ an $(n-1)$-dimensional closed manifold, and $A \subset M$ a finite set of basepoints, intersecting each connected component of $M$.

- **1-morphisms:** a map $(M_1, A_2) \rightarrow (M_2, A_2)$ is given by an equivalence class of pairs $(W, A)$, with $W : M_1 \rightarrow M_2$ a bordism, and $A \subset W$ a finite set of basepoints intersecting each connected component of $W$ such that $A \cap M_1 = A_1$ and $A \cap M_2 = A_2$. Two such pairs $(W, A)$ and $(W', A')$ are said to be equivalent if there is a diffeomorphism $F : W \rightarrow W'$ such that $F(A) = A'$ and such that the following diagram commutes:

$$
\begin{array}{ccc}
M_2 & \xleftarrow{F} & W' \\
\downarrow & & \downarrow \\
M_1 & \xrightarrow{F} & W
\end{array}
$$

(2.1)

The composition of 1-morphisms $(W, A) : (M_1, A_1) \rightarrow (M_2, A_2)$ and $(W', A') : (M_2, A_2) \rightarrow (M_3, A_3)$ is $(W \sqcup W', A \cup A') : (M_1, A_1) \rightarrow (M_3, A_3)$. Although this operation is not well-defined on bordisms (there need not be a unique manifold structure on $W \sqcup W'$), it is well-defined on equivalence classes of bordisms [6]. Also note that unless $M = A = \emptyset$, there is not yet an identity morphism for $(M, A)$. For this reason we allow $(M, A)$ to be seen as a bordism from and to itself.

- **2-morphisms:** a map $(W, A) \rightarrow (W', A')$ is given by a diffeomorphism $F : W \rightarrow W'$ such that $F(A) \subset A'$ and (2.1) commutes.

This category is a monoidal category, whose tensor product is given by taking disjoint unions, and whose unital object is the empty manifold $\emptyset$.

Allowing multiple basepoints directly leads to a generalized notion of a $G$-representation variety: if $X$ is a compact manifold (possibly with boundary), and $A \subset X$ a finite set of basepoints, then the $G$-representation variety of $(X, A)$ is given by

$$
\mathcal{X}_G(X, A) = \text{Hom}_{\text{Grpd}}(\Pi(X, A), G),
$$

where $\Pi(X, A)$ denotes the fundamental groupoid of $(X, A)$.

We elaborate a bit on the structure of $\mathcal{X}_G(X, A)$. Writing $\mathcal{G} = \Pi(X, A)$, the groupoid $\mathcal{G}$ has finitely many connected components, where we say objects $a, b \in \mathcal{G}$ are connected if $\text{Hom}_\mathcal{G}(a, b)$ is non-empty. Pick a subset $S = \{a_1, \ldots, a_s\} \subset A$ such that each connected component of $\mathcal{G}$ contains exactly one of the $a_i$, and pick an arrow $f_{a_i} : a_i \rightarrow a$, with $a_i \in S$ in the same connected component as $a$, for each $a \in A - S$. Now a morphism of groupoids $\rho : \mathcal{G} \rightarrow G$ is uniquely determined by group morphisms $\rho_i : \mathcal{G}_i \rightarrow G$, where $\mathcal{G}_i = \text{Hom}_\mathcal{G}(a_i, a_i)$, and a choice of $\rho(f_{a_i}) \in G$ for each $a \in A - S$. Namely, any $\gamma : a \rightarrow b$ in $\mathcal{G}$ can be written as $\gamma = f_b \circ \gamma' \circ (f_a)^{-1}$ for some $\gamma' \in \mathcal{G}_i$, with $a_i \in S$ in the same connected component as $a$ and $b$, taking $f_{a_i} = \text{id}$ if either $a$ or $b$ equals $a_i$. There is no restriction on the choice of $\rho(f_{a_i})$, so we obtain

$$
\mathcal{X}_G(X, A) = \text{Hom}_{\text{Grpd}}(\mathcal{G}, G) \simeq \text{Hom}(\mathcal{G}_1, G) \times \cdots \times \text{Hom}(\mathcal{G}_s, G) \times G^{\#A - s}.
$$

(2.2)
Each of these factors naturally carries the structure of an algebraic variety. Namely, since $X$ is compact, it has the homotopy type of a finite CW-complex, so each $G_i$ is finitely generated, and thus $\text{Hom}(G_i, G)$ can be identified with a subvariety of $G^m$ for some $m > 0$ as explained in the introduction. This gives $\mathcal{X}_G(X, A)$ the structure of an algebraic variety, and this structure can be shown not to depend on the choices made. In particular, when $X$ is connected, we obtain

$$\mathcal{X}_G(X, A) = \mathcal{X}_G(X) \times G^{#A-1}. \quad (2.3)$$

The first step in constructing the TQFT is to define the field theory $\mathcal{F}$, a 2-functor from $\text{Bdp}_n$ to the 2-category of spans of varieties over $k$,

$$\mathcal{F} : \text{Bdp}_n \rightarrow \text{Span}(\text{Var}_k).$$

This functor assigns $\mathcal{X}_G(M, A)$ to an object $(M, A)$, and to a bordism $(W, A) : (M_1, A_1) \rightarrow (M_2, A_2)$ is assigned the span

$$\mathcal{F}(W, A) : \mathcal{X}_G(M_1, A_1) \leftarrow \mathcal{X}_G(W, A) \rightarrow \mathcal{X}_G(M_2, A_2),$$

induced by the inclusions $(M_1, A_1) \rightarrow (W, A)$. Similarly, a 2-morphism $F : (W, A) \rightarrow (W', A')$ between bordisms $(W, A), (W', A') : (M_1, A_1) \rightarrow (M_2, A_2)$ induces the 2-cell $\mathcal{F}(F)$ given by

$$\begin{align*}
\mathcal{X}_G(W, A) &\leftarrow \mathcal{X}_G(W', A') \rightarrow \mathcal{X}_G(M_1, A_1) \\
\mathcal{X}_G(M_2, A_2) &
\end{align*}$$

For a proof that this indeed defines a 2-functor, see [4]. The proof mostly relies on the Seifert–van Kampen theorem for fundamental groupoids [1]. Moreover, it can be shown that this functor is a monoidal functor.

Secondly, we define the quantization functor

$$\mathcal{Q} : \text{Span}(\text{Var}_k) \rightarrow \text{K}(\text{Var}_k)\text{-Mod},$$

which assigns to a variety $X$ the $\text{K}(\text{Var}_k)$-module $\text{K}(\text{Var}/X)$, and to a span $X \leftarrow Z \rightarrow Y$ the map of modules $g \circ f^* : \text{K}(\text{Var}/X) \rightarrow \text{K}(\text{Var}/Y)$. Moreover, $\mathcal{Q}$ can be seen as a 2-functor when we consider $\text{R-Mod}$ with $R = \text{K}(\text{Var}_k)$ as a 2-category in the following way.

**Definition 2.3.** Let $f, g : A \rightarrow B$ be $R$-module morphisms. We say $g$ is an immediate twist of $f$ if there exists a factorization $A \xrightarrow{f_1} C \xrightarrow{f_2} B$ of $f$ and an $R$-module morphism $\psi : C \rightarrow C$ such that $g = f_2 \circ \psi \circ f_1$. We say there is a 2-morphism or a twist from $f$ to $g$ if there exists a sequence of $R$-module morphisms $f = f_0, f_1, \ldots, f_k = g : A \rightarrow B$ such that $f_{i+1}$ is an immediate twist of $f_i$. This gives $\text{R-Mod}$ the structure of a 2-category.

Now given a 2-morphism in $\text{Span}(\text{Var}_k)$,
we have \((g_2); f_2^* = (g_1); h h^* f_1^*\), so indeed there is a 2-morphism
\[
Q(X \leftarrow Z_1 \to Y) \Rightarrow Q(X \leftarrow Z_2 \to Y).
\]
For a proof that \(Q\) is a 2-functor, again see [4]. This functor is not quite monoidal, but it is lax monoidal. The reason for this is that given varieties \(X\) and \(Y\), the \(K(\text{Var}_{k})\)-module \(K(\text{Var}/(X\times Y))\) is not canonically isomorphic to \(K(\text{Var}/X) \otimes_{K(\text{Var}_{k})} K(\text{Var}/Y)\), even though there is a natural map from the latter to the first given by \([V]_X \otimes [W]_Y \mapsto [V \times W]_{X\times Y}\), but this is not an isomorphism.

Finally, the composition of \(F\) and \(Q\) gives us the TQFT
\[
Z = Q \circ F : \text{Bdp}_n \to K(\text{Var}_{k})\text{-Mod},
\]
which is a lax monoidal 2-functor. To see why this TQFT is useful, let \(X\) be a closed manifold of dimension \(n\), choose a basepoint \(*\) on \(X\), and view \((X,*)\) as a bordism \(\emptyset \to \emptyset\). Then \(F(X,*)\) is given by the span
\[
\ast \leftarrow \xymatrix{X_G(X,*) \ar[r]^f & \ast}
\]
and hence \(Z(X,*)(1) = f_1 f^*(1) = f_1([X_G(X)]_{X_G(X)}) = [X_G(X)]\). This means that the invariants produced by this TQFT are precisely the classes of the \(G\)-representation varieties.

## 3 Non-orientable surfaces

We focus on the case of dimension \(n = 2\). By the classification of surfaces, any orientable closed surface can be constructed by a composition of the following bordisms:
\[
\begin{array}{ccc}
D^1 : (S^1,*) & \to \emptyset & L : (S^1,*) \to (S^1,*) & D : \emptyset \to (S^1,*) \\
\end{array}
\]
\[
(3.1)
\]
Indeed, a surface of genus \(g\) can be written as \(D^1 \circ L^g \circ D\) after adding sufficiently many basepoints.

Very similarly, any closed non-orientable surface can be written as the connected sum of projective planes. For this reason, we consider the following bordism. Let \(N\) be the real projective plane, with two holes and two basepoints, \(a\) and \(b\), seen as a bordism \((S^1,a) \to (S^1,b)\).
\[
N : (S^1,*) \to (S^1,*)
\]
\[
(3.2)
\]
The fundamental group of \(S^1\) is \(\pi_1(S^1) = \mathbb{Z}\), which gives \(X_G(S^1,*) = G\). The fundamental group of the real projective plane with two punctures is the free group on two elements \(\pi_1(N,a) = F_2\). We pick generators \(\alpha\) and \(\beta\) depicted in the following image, and a path \(\gamma\) connecting \(a\) and \(b\).
According to (2.2), we obtain $X_G(N, \{a,b\}) = \text{Hom}(F_2, G) \times G = G^3$. A generator for $\pi_1(S^1, b)$ is given by $\gamma \beta \alpha^{-2} \gamma^{-1}$. Therefore, the field theory for the bordism $N$ is given by the span

$$F(N) : \quad G \xleftarrow{\nu} G^3 \xrightarrow{w} G \quad B \xleftarrow{(B, A, C)} \xrightarrow{CBA^2C^{-1}}.$$

Hence, the TQFT for $N$ is given by

$$Z(N) = u_1 \circ v^* : K(\text{Var}/G) \to K(\text{Var}/G).$$

Since the fundamental group of a disk is trivial, it is easy to see that the field theories for $D$ and $D^\dagger$ are given by

$$F(D) : \quad \ast \xleftarrow{\nu} \ast \quad \text{id} \quad \text{and} \quad F(D^\dagger) : \quad \ast \xleftarrow{\nu} \ast \quad \text{id} \xrightarrow{\nu} \ast.$$

Now, using (3.3) we find

$$[X_G(N_r)] = \frac{1}{[G]^r}[X_G(N_r, \{r + 1 \text{ basepoints}\})] = \frac{1}{[G]^r}Z(D^\dagger \circ N^r \circ D)(1). \quad (3.3)$$

**Remark 3.1.** Note that in this expression we divide by $[G]$ even though $[G]$ is often not invertible in $K(\text{Var}_k)$. This can be solved by working in the localization of $K(\text{Var}_k)$ w.r.t. $[G]$. Consequently, the resulting class will only be defined up to an annihilator of $[G]$, but in many cases this causes no problems when extracting algebraic data. For example, the $E$-polynomial of a complex variety can be obtained from its class in the Grothendieck ring via $e : K(\text{Var}_C) \to \mathbb{Z}[u, v]$. If $e([G]) \neq 0$, then we must have $e(x) = 0$ for any annihilator $x$ of $[G]$ since $\mathbb{Z}[u, v]$ is a domain. In particular, to know the $E$-polynomial of a variety it is enough to know its class in the localized Grothendieck ring.

## 4 Reduction of the TQFT

Let $Z : \mathcal{B} \to R\text{-Mod}$ be a TQFT, where $\mathcal{B}$ is some kind of bordism category. Sometimes when there is symmetry in $Z$, like a group acting on the $R$-modules, it can be used to ‘reduce’ the TQFT, simplifying the actual computations. We represent this as follows.

For each object $M$ in $\mathcal{B}$, let $A_M$ be an $R$-module, with $R$-module maps

$$Z(M) \xrightarrow{\alpha_M} \xleftarrow{\beta_M} A_M.$$
Assume \( A_\varnothing = Z(\varnothing) \) with \( \alpha_\varnothing \) and \( \beta_\varnothing \) the identity maps. Let \( V_M \subset A_M \) be \( R \)-submodules such that 
\[
(\alpha_{M'} \circ Z(W) \circ \beta_M)(V_M) \subset V_{M'}
\]
for all bordisms \( W : M \to M' \) in \( \mathcal{B} \). In particular, we have a map 
\[
\eta_M := \alpha_M \circ \beta_M : V_M \to V_M
\]
for each \( M \). If the maps \( \eta_M : V_M \to V_M \) are invertible, then under some conditions it is possible to define a functor
\[
\hat{Z} : \mathcal{B} \to \text{R-Mod}
\]
that assigns \( \hat{Z}(M) = V_M \) and \( \hat{Z}(W) = \alpha_{M'} \circ Z(W) \circ \beta_M \circ \eta_M^{-1} \) for any \( W : M \to M' \). Moreover, this functor will produce the same invariants for closed manifolds as \( Z \), that is, \( \hat{Z}(W)(1) = Z(W)(1) \) for any bordism \( W : \varnothing \to \varnothing \). We make this precise in the following proposition.

**Proposition 4.1.** For each object \( M \) in \( \mathcal{B} \), let \( A_M \) be an \( R \)-module together with \( R \)-module maps 
\[
\alpha_M : Z(M) \to A_M \text{ and } \beta_M : A_M \to Z(M), \text{ and let } V_M \subset A_M \text{ be a submodule. Assume that }
\]

(i) \( V_\varnothing = A_\varnothing = Z(\varnothing) \) and \( \alpha_\varnothing, \beta_\varnothing \) are the identity map,

(ii) \( (\alpha_{M'} \circ Z(W) \circ \beta_M)(V_M) \subset V_{M'} \) for all bordisms \( W : M \to M' \),

(iii) \( \eta_M = \alpha_M \circ \beta_M : V_M \to V_M \) has a right-inverse \( \eta_M^{-1} \),

(iv) \( \ker \alpha_M \subset \ker \alpha_{M'} \circ Z(W) \) for all bordisms \( W : M \to M' \).

Then the map \( \hat{Z} : \mathcal{B} \to \text{R-Mod} \) given by

- \( Z(M) = V_M \) for any object \( M \),
- \( Z(W) = \alpha_{M'} \circ Z(W) \circ \beta_M \circ \eta_M^{-1} \) for any bordism \( W : M \to M' \),
- \( \hat{Z}(F) \) is the twist \( \hat{Z}(W_1) \Rightarrow \hat{Z}(W_2) \) induced by \( Z(F) : Z(W_1) \Rightarrow Z(W_2) \) for any 2-morphism \( F : W_1 \Rightarrow W_2 \),

is a 2-functor, which we refer to as the reduced TQFT. Moreover, this reduced TQFT produces the same invariants for closed manifolds as \( Z \).

**Proof.** Let \( W_1 : M_1 \to M_2 \) and \( W_2 : M_2 \to M_3 \) be bordisms in \( \mathcal{B} \). We find that

\[
(\hat{Z}(W_2) \circ \hat{Z}(W_1))(x) = (\alpha_{M_3} \circ Z(W_2) \circ \beta_{M_2} \circ \eta_{M_2}^{-1} \circ \alpha_{M_2} \circ Z(W_1) \circ \beta_{M_1} \circ \eta_{M_1}^{-1})(x) \\
= (\alpha_{M_3} \circ Z(W_2) \circ Z(W_1) \circ \beta_{M_1} \circ \eta_{M_1}^{-1})(x) \\
= (\alpha_{M_3} \circ Z(W_2 \circ W_1) \circ \beta_{M_1} \circ \eta_{M_1}^{-1})(x) \\
= \hat{Z}(W_2 \circ W_1)(x),
\]

where in the second equality we used (iv) and the fact that

\[
(Z(W_1) \circ \beta_{M_1} \circ \eta_{M_1}^{-1})(x) - (\beta_{M_2} \circ \eta_{M_2}^{-1} \circ \alpha_{M_2} \circ Z(W_1) \circ \beta_{M_1} \circ \eta_{M_1}^{-1})(x) \in \ker \alpha_{M_2}.
\]

Given 2-morphisms \( F_1 : W_1 \Rightarrow W_2 \) and \( F_2 : W_2 \Rightarrow W_3 \) in \( \mathcal{B} \), it follows immediately from \( Z(F_2 \circ F_1) = Z(F_2) \circ Z(F_1) \) that \( \hat{Z}(F_2 \circ F_1) = \hat{Z}(F_2) \circ \hat{Z}(F_1) \) as well.
Indeed $\tilde{Z}$ produces the same invariants as $Z$, as for any bordism $W : \emptyset \to \emptyset$ we find by (i) that

$$\tilde{Z}(W)(1) = \alpha_\emptyset \circ Z(W) \circ \beta_\emptyset \circ \eta_\emptyset^{-1}(1) = Z(W)(1).$$

\qed

**Remark 4.2.** The reason we consider submodules $V_M \subset A_M$ instead of just taking $V_M = A_M$ is because it is quite hard to check (and possibly false) that the maps $\eta_M$ are invertible on the whole of $A_M$. However, often only a submodule of $A_M$ is used in the computations, and it will be easier to check invertibility on this submodule.

**Example 4.3.** In this paper we focus on bordisms in dimension $n = 2$. Let $\mathbf{Tbp}_2$ be the subcategory of $\mathbf{Bdp}_2$ whose objects are disjoint unions of $(S^1, *)$, and whose bordisms are disjoint unions of compositions of the bordisms in (3.1) and (3.2). Now the TQFT of Section 2 can be restricted to a TQFT $Z : \mathbf{Tbp}_2 \to K(\mathbf{Var}_k)-\mathbf{Mod}$, and note that to obtain a reduced TQFT, it suffices to specify an $K(\mathbf{Var}_k)$-module $A = A_{(S^1, *)}$ with maps $\alpha = \alpha_{(S^1, *)} : Z(M) \to A$ and $\beta = \beta_{(S^1, *)} : A \to Z(M)$, and a submodule $V = V_{(S^1, *)} \subset A$, satisfying

(i) $(\alpha \circ Z(W) \circ \beta)(V) \subset V$ for $W = L, N$,

(ii) $\eta = \alpha \circ \beta : V \to V$ is invertible,

(iii) $\ker \alpha \subset \ker \alpha \circ Z(W)$ for $W = L, N$.

In the actual computations, what we will do is the following. We take a stratification $G = \sqcup_i C_i$, where each $C_i$ is conjugacy-closed, and we take morphisms $\pi_i : C_i \to C_i$ that identify conjugate elements. The maps $(\pi_i)_!$ and $\pi_i^*$ now induce maps

$$Z(S^1, *) = K(\mathbf{Var}/G) = \bigoplus_i K(\mathbf{Var}/C_i) \xrightarrow{\pi_i} \bigoplus_i K(\mathbf{Var}/C_i).$$

We take $A = \bigoplus_i K(\mathbf{Var}/C_i)$, $\alpha = \pi_i$ and $\beta = \pi_i^*$. Conditions (i) and (ii) of Example 4.3 will depend on the chosen submodule $V \subset A$, and condition (iii) can be shown using the following lemma.

**Lemma 4.4.** Let $C_i$ be a stratification for $G$, with morphisms $\pi_i : C_i \to C_i$ that identify conjugate elements. Suppose that there exist $\sigma_i : C_i \to C_i$ such that $\pi_i \circ \sigma_i = \text{id}$ and $c_i : C_i \to G$ such that $g = c_i(g) \sigma_i(c_i(g))c_i(g)^{-1}$ for all $g \in C_i$. Then condition (iii) of Example 4.3 holds.

**Proof.** Take $x \in \ker \pi_i$. Since $x = \sum_{i=1}^n x_i$ for $x_i = x|_{C_i}$, it is sufficient to show that $x_i \in \ker \pi_i Z(W)$ for any $i$. Since $(\pi_i)_!x_i = 0$ and $\pi_i \circ \sigma_i = \text{id}_{C_i}$, by Lemma 4.5 (see below) we can write $x_i = [X] - [Y]$ for some varieties $X \xrightarrow{f} C_i$ and $Y \xrightarrow{g} C_i$ that are isomorphic over $C_i$, that is, there exists an isomorphism of varieties $\varphi : X \to Y$ such that $\pi_i \circ g \circ \varphi = \pi_i \circ f$.

We will show the condition holds for $W = N$, but the same argument will work for the case $W = L$. Define the morphism

$$\psi : Z(N)(X) \to Z(N)(Y), \quad (x, B, A, C) \mapsto (\varphi(x), \mu(B, x), \mu(A, x), \mu(C, x)).$$
Now simply take \( x \) such that \( \pi \circ \sigma = \text{id}_S \), and suppose that \( \pi(x) = 0 \) for some \( x \in K(\Var/T) \). Then \( x = [X] - [Y] \) for some varieties \( X \) and \( Y \) over \( T \) that are isomorphic over \( S \).

**Proof.** Write \( x = \sum_i \delta_i[T_i] \) for some varieties \( T_i \) over \( T \), with \( \delta_i = \pm 1 \). Since \( \pi(x) = 0 \), there exist varieties \( S_j \) with a closed subvariety \( Z_j \subset S_j \) and complement \( U_j = S_j - Z_j \), and signs \( \varepsilon_j = \pm 1 \), such that

\[
\sum_i \delta_i[T_i] = \sum_j \varepsilon_j ([S_j] - [Z_j] - [U_j])
\]

holds in \( F(\Var/S) \), the free abelian group of isomorphism classes of varieties over \( S \). Now it follows that

\[
x' = \sum_i \delta_i[T_i] - \sum_j \varepsilon_j ([\sigma_i S_j] - [\sigma_i Z_j] - [\sigma_i U_j])
\]

is a lift of \( x \) to \( F(\Var/T) \) such that \( \pi(x') = 0 \in F(\Var/S) \), where we used that \( \pi \circ \sigma = \text{id}_S \).

Now simply take

\[
X = \left( \bigsqcup_{\delta_i = 1} T_i \right) \sqcup \left( \bigsqcup_{\varepsilon_j = -1} \sigma_i S_j \right) \sqcup \left( \bigsqcup_{\varepsilon_j = 1} \sigma_i Z_j \sqcup \sigma_i U_j \right)
\]

and

\[
Y = \left( \bigsqcup_{\delta_i = -1} T_i \right) \sqcup \left( \bigsqcup_{\varepsilon_j = 1} \sigma_i S_j \right) \sqcup \left( \bigsqcup_{\varepsilon_j = -1} \sigma_i Z_j \sqcup \sigma_i U_j \right)
\]

\( \square \)

### 5 Upper triangular \( 2 \times 2 \) matrices

In this section we will compute the class of the \( G \)-representation variety of any non-orientable surface in \( K(\Var_k) \) for \( G = \tilde{U}_2 \), the group of upper triangular \( 2 \times 2 \) matrices. Actually, using the decomposition \( \tilde{U}_2 = \tilde{U}_2 \times \mathbb{G}_m \), where

\[
\tilde{U}_2 = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \neq 0 \right\},
\]

we find that \( \mathfrak{x}_{\tilde{U}_2}(\Sigma) = \mathfrak{x}_{\tilde{U}_2}(\Sigma) \times \mathfrak{x}_{\mathbb{G}_m}(\Sigma) \). Therefore, we will focus on the case \( G = \tilde{U}_2 \) instead. This group is also known as the rank one general affine group \( \text{AGL}_2(\mathbb{C}) \).

First, we will construct a reduced TQFT following Example 4.3. Consider the following stratification of \( G \),

\[
\mathcal{C}_1 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \quad \mathcal{C}_2 = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \neq 0 \right\}, \quad \mathcal{C}_3 = \left\{ \begin{pmatrix} t & b \\ 0 & 1 \end{pmatrix} : t \neq 0, 1, b \in \mathbb{C} \right\}.
\]
The orbit spaces are

\[ C_1 = \ast, \quad C_2 = \ast, \quad C_3 = \mathbb{A}_C^1 - \{0, 1\}, \]

and the projection morphisms

\[ \pi_1 : C_1 \to C_1, \quad \pi_2 : C_2 \to C_2, \quad \pi_3 : C_3 \to C_3, \]

which identify conjugate elements, where \( \pi_3 \left( \begin{pmatrix} t & b \\ 0 & 1 \end{pmatrix} \right) = t \). Now the maps \( (\pi_i)_i \) and \( \pi_i^* \) induce the module morphisms

\[ K(\text{Var}/G) = \bigoplus_{i=1}^3 K(\text{Var}/C_i) \xrightarrow{\pi_i^*} \bigoplus_{i=1}^3 K(\text{Var}/C_i). \]

For the reduction of the TQFT, we will use \( A = \bigoplus_{i=1}^3 K(\text{Var}/C_i) \) with \( \alpha = \pi_1 \) and \( \beta = \pi^* \). As a shorthand notation, we will write \( Z_\pi(W) = \pi^* \circ Z(W) \circ \pi_1 \) for \( W = L, N \). Before giving the submodule \( V \subset N \), we first compute what would be \( Z_\pi(N)(T_1) = (\pi_1 \circ w_1 \circ v^* \circ \pi^*)(T_1) \), where \( T_1 \) denotes the unit of \( K(\text{Var}/C_1) \). Note that \( (w_1 \circ v^* \circ \pi^*)(T_1) \) is the class of \( G^2 \to G \) given by \( (A, C) \mapsto CA^2C^{-1} \) in \( K(\text{Var}/G) \). Thus,

\[ Z_\pi(N)(T_1)|_{C_i} = [(A, C) \in G^2 : A^2 \in C_i] = [G] \cdot [A \in G : A^2 \in C_i] \in K(\text{Var}_k) \quad \text{for } i = 1, 2, 3. \]

Write \( A = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \), so that \( A^2 = \begin{pmatrix} a^2 & (a+1)b \\ 0 & 1 \end{pmatrix} \). We have \( A^2 \in C_1 \) if \( a^2 = 1 \) and \( (a+1)b = 0 \), i.e. either \( a = -1 \) and \( b \in \mathbb{C} \) or \( a = 1 \) and \( b = 0 \). Hence \( Z_\pi(N)(T_1)|_{C_1} = [G](q + 1) \). Similarly, we have \( A^2 \in C_2 \) if \( a^2 = 1 \) and \( (a+1)b \neq 0 \), i.e. \( a = 1 \) and \( b \neq 0 \). This gives \( Z_\pi(N)(T_1)|_{C_2} = (q-1)[G] \). Finally, we have \( A^2 \in C_3 \) if \( a^2 = t \) for some \( t \in C_3 \) (with no condition on \( b \)). Hence \( Z_\pi(N)(T_1)|_{C_3} = [G]qS \), where we define \( S \) as

\[ S = \{ (a, t) \in \mathbb{C} \times C_3 : a^2 = t \}. \]

This shows that we should include \( S \) in the submodule \( V \) (rather than the unit of \( K(\text{Var}/C_3) \)). Note that \( S \) satisfies the following relation in \( K(\text{Var}/C_3) \),

\[ S^2 = \{ (a, b, t) \in \mathbb{C}^2 \times C_3 : a^2 = b^2 = t \}. \]

\[ = \{ (a, b, t) \in \mathbb{C}^2 \times C_3 : a^2 = t, b = a \}. \]

\[ = 2S. \]

Now we give the submodule \( V \). Let \( T_1 \) and \( T_2 \) be the units in \( K(\text{Var}/C_1) \) and \( K(\text{Var}/C_2) \), respectively, and \( S \in K(\text{Var}/C_3) \) as above. Then we take the submodule

\[ V = \langle T_1, T_2, S \rangle. \]

It is easy to see that the map \( \eta = \pi_1 \circ \pi^* \) is given by

\[ \eta = \begin{bmatrix} T_1 & T_2 & S \\ 1 & 0 & 0 \\ 0 & q-1 & 0 \\ 0 & 0 & q \end{bmatrix}, \]

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which is invertible (when localizing the Grothendieck ring $K(\text{Var}_k)$ w.r.t. $q$ and $q - 1$). To show this indeed gives a reduced TQFT, we will apply Lemma 4.4. Simply take $\sigma_i : C_i \rightarrow C_i$ given by

$$\sigma_1(\ast) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_2(\ast) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \sigma_3(t) = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}.$$ 

It is easy to come up with maps $c_i : C \rightarrow \mathbb{U}_2$ such that $g = c(g)\sigma_i(\pi_i(g))c(g)^{-1}$, e.g. take

$$c_1 \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad c_2 \left( \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}, \quad c_3 \left( \begin{pmatrix} t & b \\ 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} 1 & b(1 - t) \\ 0 & 1 \end{pmatrix}.$$ 

This shows that we obtain a reduced TQFT $\tilde{Z}$.

Now we go back to the computations. The above reasoning shows that

$$Z_\pi(N)(T_1) = |G|(q + 1)T_1 + (q - 1)T_2 + qS.$$ 

Next we compute $Z_\pi(N)(T_2)$. We have $(w_1 \circ v^* \circ \pi^*)(T_2)$ is the class of

$$C_2 \times G^2 \rightarrow G$$

in $K(\text{Var}/G)$ given by $(B, A, C) \mapsto CBA^2C^{-1}$.

Therefore,

$$Z_\pi(N)(T_2)|_{C_i} = |G|[\{(B, A) \in C_2 \times G : BA^2 \in C_i\}] \quad \text{for } i = 1, 2, 3.$$ 

Write $B = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ with $a, x \neq 0$. Then

$$BA^2 = \begin{pmatrix} a^2 & (a + 1)b + x \\ 0 & 1 \end{pmatrix}.$$ 

- We have $BA^2 \in C_1$ if $a^2 = 1$ and $(a + 1)b + x = 0$. Since $x \neq 0$, we must have $a = 1$ and then also $b = -x/2$. Therefore $Z_\pi(N)(T_2)|_{T_1} = |G|(q - 1)$.

- We have $BA^2 \in C_2$ if $a^2 = 1$ and $(a + 1)b + x \neq 0$. Either $a = -1$ and $b \in \mathbb{C}, x \neq 0$, or $a = 1$ and $b \neq -x/2$. Therefore $Z_\pi(N)(T_2)|_{T_2} = |G|(q(q - 1) + (q - 1)^2) = |G|(q - 1)(2q - 1)$.

- We have $BA^2 \in C_3$ if $a^2 = t$ for some $t \neq 0, 1$, and $b \in \mathbb{C}$. Therefore $Z_\pi(N)(T_2)|_S = |G|q$.

Finally we compute $Z_\pi(N)(S)$.

Write $B = \begin{pmatrix} x^2 & y \\ 0 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ with $a \neq 0$ and $x \neq 0, \pm 1$. Then

$$BA^2 = \begin{pmatrix} a^2x^2 & (a + 1)x^2b + y \\ 0 & 1 \end{pmatrix}.$$ 

- We have $BA^2 \in C_1$ if $a^2x^2 = 1$ and $(a + 1)x^2b + y = 0$. That is, $a = \pm x^{-1}$ and $y = -(a + 1)x^2b$. Therefore $Z_\pi(N)(S)|_{T_1} = |G|2q(q - 3)$.

- We have $BA^2 \in C_2$ if $a^2x^2 = 1$ and $(a + 1)x^2b + y \neq 0$. That is, $a = \pm x^{-1}$ and $y \neq -(a + 1)x^2b$. Therefore $Z_\pi(N)(S)|_{T_1} = |G|2q(q - 3)(q - 1)$. 

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We have $BA^2 \in C_3$ if $a^2x^2 = t$ for some $t \neq 0,1$. Using a substitution of variables, $\tilde{a} = ax$, this condition is equivalent to $\tilde{a}^2 = t$. Therefore $Z_\pi(N)(S)|_S = |G|q^2(q - 3)$.

In summary,

$$Z_\pi(N) = |G| \begin{bmatrix} T_1 & T_2 & S \\ T_1 & q+1 & q-1 & 2q(q-3) \\ T_2 & q-1 & (q-1)(2q-1) & 2q(q-3)(q-1) \\ S & q & q(q-1) & q^2(q-3) \end{bmatrix}.$$  (5.1)

so it follows that

$$\tilde{Z}(N) = Z_\pi(N) \circ \eta^{-1} = |G| \begin{bmatrix} T_1 & T_2 & S \\ T_1 & q+1 & 1 & 2(q-3) \\ T_2 & q-1 & 2q-1 & 2q(q-3)(q-1) \\ S & q & q & q(q-3) \end{bmatrix}.$$  (5.2)

Now we have that

$$[X_G(N_r)] = \frac{1}{|G|^r} \tilde{Z}(D^\dagger \circ N^r \circ D)(1) = \frac{1}{|G|^r} \tilde{Z}(N)^r(T_1)|_{T_1}.$$  (5.3)

An expression for $\tilde{Z}(N^r) = \tilde{Z}(N)^r$ is easily obtained by the diagonalization

$$\tilde{Z}(N) = PDP^{-1} \quad \text{with} \quad D = |G| \begin{bmatrix} 0 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q(q-1) \end{bmatrix}, \quad P = \begin{bmatrix} \frac{3-q}{q} & -1 & \frac{2}{q} \\ -q + 4 - \frac{3}{q} & 1 & 2 - \frac{2}{q} \\ 1 & 0 & 1 \end{bmatrix}.$$  (5.4)

Expression (5.2) now yields the following result.

**Theorem 5.1.** Let $r > 0$. The virtual class of the representation variety $X_{\tilde{G}_2}(N_r)$ in the Grothendieck ring of varieties is

$$[X_{\tilde{G}_2}(N_r)] = q^{-1}(q-1)(2q-1)^{r-2} + 1.$$  \hfill \Box

**Remark 5.2.** Multiplying the expression from Theorem 5.1 by $[X_{\tilde{G}_m}(N_r)] = 2(q-1)^r$ gives the expression from Theorem 1.1. For small values of $r$, we find

\begin{align*}
X_{\tilde{G}_2}(N_1) &= q+1, \\
X_{\tilde{G}_2}(N_2) &= 3q(q-1), \\
X_{\tilde{G}_2}(N_3) &= q^2(q-1)(2q-1), \\
X_{\tilde{G}_2}(N_4) &= q^3(q-1)(2q^2-4q+3), \\
X_{\tilde{G}_2}(N_5) &= q^4(q-1)(2q^3-6q^2+6q-1), \\
X_{\tilde{G}_2}(N_6) &= q^5(q-1)(2q^4-8q^3+12q^2-8q+3), \\
X_{\tilde{G}_2}(N_7) &= q^6(q-1)(2q^5-10q^4+20q^3-20q^2+10q-1), \\
X_{\tilde{G}_2}(N_8) &= q^7(q-1)(2q^6-12q^5+30q^4-40q^3+30q^2-12q+3), \\
X_{\tilde{G}_2}(N_9) &= q^8(q-1)(2q^7-14q^6+42q^5-70q^4+70q^3-42q^2+14q-1), \\
X_{\tilde{G}_2}(N_{10}) &= q^9(q-1)(2q^8-16q^7+56q^6-112q^5+140q^4-112q^3+56q^2-16q+3).
\end{align*}
The common factor \((q-1)\) for \(r \geq 2\) can be explained as follows. Let \(D \subset X_{\rho_2}(N_r)\) be the subvariety where all \(A_i\) are diagonal. Then \(D \simeq X_{\rho_2}(N_r)\) and thus \([D] = 2(q-1)^r\). Furthermore, there is a free action of \(G_m\) on \(X_{\rho_2}(N_r) - D\) given by conjugation with \((r_0 1)\) for \(x \in \mathbb{C}^x\). Hence \([X_{\rho_2}]\) will contain a factor \((q-1)\) for \(r > 1\).

**Remark 5.3.** By the results of [5], we have (one can compute that \(\tilde{Z}(L)(S) = [G]q^2(q-1)^2S\))

\[
\tilde{Z}(L) = [G]\begin{bmatrix}
T_1 & T_2 & S \\
T_1 & q^2(q-1) & q^2(q-2) & 0 \\
T_2 & q^2(q-2)(q-1) & q^2(q^2-3q+3) & 0 \\
S & 0 & 0 & q^2(q-1)^2
\end{bmatrix}.
\]

Observe that \(\tilde{Z}(N)\) and \(\tilde{Z}(L)\) indeed commute, which we would expect since \(N \circ L = L \circ N\). Also, note that

\(\tilde{Z}(N) \circ \tilde{Z}(L) = \tilde{Z}(N)^3\),

which reflects the fact that \(\mathbb{R}P^2 \# \mathbb{T}^2 = \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2\). However, since \(\mathbb{R}P^2 \# \mathbb{R}P^2 \neq \mathbb{T}^2\), we expect that \(\tilde{Z}(N)^2 \neq \tilde{Z}(L)\), and therefore it is not surprising that \(\tilde{Z}(N)\) has a zero eigenvalue.

This zero eigenvalue can also be explained in another way. Its corresponding eigenvector, see (5.3), tells us that the image of \(\tilde{Z}(N)\) is generated by the image of the first two generators, that is, \(\tilde{Z}(N)(S)\) lies in the span of \(\tilde{Z}(N)(T_1)\) and \(\tilde{Z}(N)(T_2)\). Now, \(Z_\pi(N)(S)\) is given explicitly by

\([G]_\pi\left([\pi^*S \times G \to G : (B, A) \to BA^2]\right)\).

Write \(B = \begin{pmatrix} x^2 & y \\ 0 & 1 \end{pmatrix}\) and \(A = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\) with \(a \neq 0\) and \(x \neq 0, \pm 1\), as before. Making a substitution of variables, \(\tilde{a} = ax\) and \(\tilde{y} = y + b(x-1)(ax + x + 1)\), we obtain

\[BA^2 = \tilde{B}\tilde{A}^2,\]

where \(\tilde{B} = \begin{pmatrix} 1 & \tilde{y} \\ 0 & 1 \end{pmatrix}\) and \(\tilde{A} = \begin{pmatrix} \tilde{a} & b \\ 0 & 1 \end{pmatrix}\).

Note that using these new coordinates, \(\tilde{B}\tilde{A}^2\) is independent of \(x\). Considering the cases \(\tilde{y} = 0\) and \(\tilde{y} \neq 0\) separately, we find that

\[Z_\pi(N)(S) = \left\{x \in \mathbb{C} : x^2 \neq 0, 1\right\} (Z_\pi(N)(T_1) + Z_\pi(N)(T_2)),\]

agreeing with (5.1). Together with \(\tilde{Z} = Z_\pi(N) \circ \eta^{-1}\), this relation yields the zero eigenvalue of \(\tilde{Z}(N)\).

We will also see this pattern in the next section for the group of upper triangular \(3 \times 3\) matrices.

### 6 Upper triangular \(3 \times 3\) matrices

In this section, we repeat the computations as in the previous section for the group of upper triangular \(3 \times 3\) matrices \(U_3\). To be precise, we let \(G = \hat{U}_3\) be the group

\[\hat{U}_3 = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & 1 \end{pmatrix} : a, d \neq 0 \right\},\]
and then as before, from the decomposition $U_3 = \tilde{U}_3 \times \mathbb{G}_m$, we find that $[\mathcal{X}_{U_3}(\Sigma)] = [\mathcal{X}_{\tilde{U}_3}(\Sigma)][\mathcal{X}_{\mathbb{G}_m}(\Sigma)]$.

Based on the conjugation classes, we use the following strata for $\tilde{U}_3$:

\[ C_1 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}, \quad C_2 = \left\{ \begin{pmatrix} 1 & b & c \\ 0 & 1 & e \\ 0 & 0 & 1 \end{pmatrix} : b, e \neq 0 \right\}, \quad C_3 = \left\{ \begin{pmatrix} 1 & b & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : b \neq 0 \right\}, \]

\[ C_4 = \left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & e \\ 0 & 0 & 1 \end{pmatrix} : e \neq 0 \right\}, \quad C_5 = \left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : c \neq 0 \right\}, \quad C_6 = \left\{ \begin{pmatrix} 1 & b & c \\ 0 & t & e \\ 0 & 0 & 1 \end{pmatrix} : t \neq 0, 1, \ c(t-1) = be \right\}, \]

\[ C_7 = \left\{ \begin{pmatrix} 1 & b & c \\ 0 & t & e \\ 0 & 0 & 1 \end{pmatrix} : t \neq 0, 1, \ c(t-1) \neq be \right\}, \quad C_8 = \left\{ \begin{pmatrix} t & b & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : t \neq 0, 1 \right\}, \quad C_9 = \left\{ \begin{pmatrix} t & b & c \\ 0 & 1 & e \\ 0 & 0 & 1 \end{pmatrix} : t \neq 0, 1, \ e \neq 0 \right\}, \]

\[ C_{10} = \left\{ \begin{pmatrix} t & 0 & c \\ 0 & t & e \\ 0 & 0 & 1 \end{pmatrix} : t \neq 0, 1 \right\}, \quad C_{11} = \left\{ \begin{pmatrix} t & b & c \\ 0 & t & e \\ 0 & 0 & 1 \end{pmatrix} : t \neq 0, 1, \ b \neq 0 \right\}, \quad C_{12} = \left\{ \begin{pmatrix} t & b & c \\ 0 & s & e \\ 0 & 0 & 1 \end{pmatrix} : t, s \neq 0, 1 \right\}. \]

The orbit spaces are given by

\[ C_1 = C_2 = C_3 = C_4 = C_5 = *, \quad C_6 = C_7 = C_8 = C_9 = C_{10} = C_{11} = \mathbb{A}^1_C - \{0, 1\}, \quad C_{12} = \{(t, s) \in \mathbb{A}^2_C : t, s \neq 0, 1 \text{ and } t \neq s\} \]

and the projection morphisms $\pi_i : C_i \to C_i$ are given either by projecting to a point, or to the coordinates $t$ and/or $s$.

As for the submodule $V$, let $T_i$ be the unit of $K(\text{Var}/C_i)$, for $i = 1, 2, 3, 4, 5$. Furthermore, let

\[ S_i = \{(a, t) \in \mathbb{C} \times C_i : a^2 = t\} \subset \mathbb{C}(\text{Var}/C_i), \quad \text{for } i = 6, 7, 8, 9, 10, 11, \]

and $D_{12} = \{(a, b, t, s) \in \mathbb{C}^2 \times C_{12} : t = a^2, s = b^2\} \subset \mathbb{C}(\text{Var}/C_{12})$.

We take

\[ V = \langle T_1, T_2, T_3, T_4, T_5, S_6, S_7, S_8, S_9, S_{10}, S_{11}, D_{12} \rangle. \]

As before, one can find suitable maps $\sigma_i : C_i \to C_i$ and $c_i : C_i \to C$ so that from Lemma 4.4 we obtain a reduced TQFT $\tilde{Z}$.

Although the matrices $Z_{\pi}(N)$ and $\tilde{Z}(N)$ could in principle be computed by hand, we use an algorithm to perform the computations. This algorithm can be found in the Appendix. The source code for the computations can be found at [8].

We make some remarks regarding the computations. First, the map $\eta = \pi_1 \pi^*$ is very easy to compute, as $\pi_i : C_i \to C_i$ is a trivial fibration for all $i$. Denoting the fibers by $F_i$, the restriction $\eta|_{C_i}$ is given by multiplication by $[F_i] \in K(\text{Var}_k)$.

Secondly, to simplify the computation of the entries of $Z_{\pi}(N)$, one can make use of conjugacy invariance even more. As an example, consider the entry $Z_{\pi}(N)(T_j)|_{T_i}$, which is $[G]$ times the class of

\[ X = \{(B, A) \in C_j \times G : BA^2 \in C_i\} \]
in \( K(\text{Var}_k) \). A stratification of \( X \) is given by

\[ X_k = \{(B, A) \in C_j \times C_k : BA^2 \in C_i \}, \quad \text{for } k = 1, \ldots, 12. \]

Since the condition \( BA^2 \in C_i \) is conjugacy invariant, for each \( k \) we have an isomorphism

\[ X_k \simeq \{(B, A, a) \in C_j \times C_k \times C_k : a = \pi_k(A) \text{ and } B\sigma_k(a)^2 \in C_i \} \]

\( (B, A) \mapsto (\mu(B, c_k(A)^{-1}), A, \pi_k(A)) \),

where \( \mu(B, C) = CBC^{-1} \). Indeed, an inverse is given by \( (B, A, a) \mapsto (\mu(B, c_k(A)), A) \). Now \( X_k \) can be written as a fiber product over \( C_k \),

\[ X_k \simeq C_k \times C_k \ Y_k, \quad \text{with } Y_k = \{(B, a) \in C_j \times C_k : B\sigma_k(a)^2 \in C_i \}. \]

As \( \pi_k : C_k \to C_k \) is a trivial fibration for each \( k \), say with fiber \( F_k \), we obtain

\[ [X_k] = [F_k] \cdot [Y_k]. \]

The advantage of this is that the classes \([F_k]\) are computed very easily, and the equations defining the variety \( Y_k \) are much simpler than the equations defining \( X \).

The resulting matrix \([G]^{-1}\hat{Z}(N)\) is given by

\[
\begin{bmatrix}
T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 & T_9 & T_{10} & T_{11} & T_{12} & T_{13} & T_{14} & T_{15}
\end{bmatrix}
\]

\[
\begin{bmatrix}
3q + 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
q^2 q + 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
q^3 q^2 q + 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
q^4 & 2q + 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
q^5 & 2q^2 + 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
q^6 & 2q^3 + 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
q^7 & 2q^4 + 1 & 0 & 0 & 0 & 0 & 0 & 0 & q^2 & 0 & 0 & 0 & 0 & 0 & 0
q^8 & 2q^5 + 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2q & 0 & 0 & 0 & 0 & 0 & 0
q^9 & 2q^6 + 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0
q^{10} & 2q^7 + 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0
q^{11} & 2q^8 + 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0
q^{12} & 2q^9 + 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0
\end{bmatrix}
\]

This matrix can be diagonalized by

\[
P = \begin{bmatrix}
\begin{array}{ccccccccccccc}
-q \cdot (q^2 + q + 1) & q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q \cdot (q^3 + q + 1)^2 & -q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q \cdot (q^4 + q + 1) & -q^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q \cdot (q^5 + q + 1)^2 & -q^4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q \cdot (q^6 + q + 1) & -q^5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q \cdot (q^7 + q + 1)^2 & -q^6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q \cdot (q^8 + q + 1) & -q^7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q \cdot (q^9 + q + 1)^2 & -q^8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q \cdot (q^{10} + q + 1) & -q^9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q \cdot (q^{11} + q + 1)^2 & -q^{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q \cdot (q^{12} + q + 1) & -q^{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q \cdot (q^{13} + q + 1)^2 & -q^{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q \cdot (q^{14} + q + 1) & -q^{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q \cdot (q^{15} + q + 1)^2 & -q^{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q \cdot (q^{16} + q + 1) & -q^{15} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q \cdot (q^{17} + q + 1)^2 & -q^{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q \cdot (q^{18} + q + 1) & -q^{17} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q \cdot (q^{19} + q + 1)^2 & -q^{18} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q \cdot (q^{20} + q + 1) & -q^{19} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q \cdot (q^{21} + q + 1)^2 & -q^{20} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q \cdot (q^{22} + q + 1) & -q^{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q \cdot (q^{23} + q + 1)^2 & -q^{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q \cdot (q^{24} + q + 1) & -q^{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q \cdot (q^{25} + q + 1)^2 & -q^{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\end{bmatrix}
\]
Remark 6.3. Let $r > 0$. The virtual class of the representation variety $X_{G_3}(N_r)$ in the Grothendieck ring of varieties is

$$[X_{G_3}(N_r)] = 2q^{2r-1} (q - 1)^r + q^{3r-3} (q - 1)^2 + 4q^{3r-3} (q - 1)^r + 4q^{3r-3} (q - 1)^{2r-2}.$$ 

The corresponding expression for $G = U_3$ can be found in Theorem 1.1.

**Remark 6.2.** For small values of $r$, we obtain

$$X_{G_3}(N_1) = 3q^2 + 1, \quad X_{G_3}(N_2) = 11q^3 (q - 1)^2, \quad X_{G_3}(N_3) = q^5 (q - 1)^2 \left(4q^3 - 4q^2 + 3q - 2\right),$$

$$X_{G_3}(N_4) = q^7 (q - 1)^2 \left(4q^6 - 16q^5 + 28q^4 - 24q^3 + 11q^2 - 4q + 2\right),$$

$$X_{G_3}(N_5) = q^9 (q - 1)^2 \left(4q^9 - 24q^8 + 60q^7 - 76q^6 + 48q^5 - 12q^4 + 3q^3 - 6q^2 + 6q - 2\right).$$

Similar to Remark 5.2, we see that there is a common factor $(q - 1)^2$ for $r > 1$, which can be explained in a similar way. Let $D_1, D_2, D_3 \subset X_{G_3}(N_r)$ be the subvarieties where all $A_i$ are of the form $\begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$, respectively.

Note that $D_1 \simeq D_2 \simeq D_3 \simeq X_{G_3}(N_r) \times X_{G_m}(N_r)$, whose class contains a factor $(q - 1)^2$ for $r > 1$ by the previous section. Also, the intersection of any two of these is the subvariety $D \subset X_{G_3}(N_r)$ of diagonal $A_i$. As $D \simeq X_{G_m}(N_r) \times X_{G_m}(N_r)$, its class also has a factor $(q - 1)^2$ for $r > 1$. Furthermore, there is a free action of $G_m \times G_m$ on $X_{G_3}(N_r) - \cup D_i$ given by conjugation with $\begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ for $x, y \in \mathbb{C}^\times$. Hence $[X_{G_3}(N_r)] = [X_{G_3}(N_r) - \cup_i D_i] + [D_1] + [D_2] + [D_3] - 2[D]$ will contain a factor $(q - 1)^2$ for $r > 1$.

**Remark 6.3.** As in the case of $G = \tilde{U}_2$, the matrix $\tilde{Z}(N)$ contains zero eigenvalues. Moreover, looking at the diagonalization of $\tilde{Z}(N)$, we can see from the corresponding eigenvectors that the

\[
\begin{pmatrix}
-\frac{2}{q} & -\frac{2}{q} & -\frac{2}{q} & \frac{1}{q} & \frac{1}{q} & \frac{1}{q} & 4(q - 1)^2 \\
0 & -\frac{2}{q} & -\frac{2}{q} & -\frac{2}{q} & \frac{1}{q} & \frac{1}{q} & 4(q - 1)^2 \\
0 & 0 & -\frac{2}{q} & -\frac{2}{q} & -\frac{2}{q} & \frac{1}{q} & 4(q - 1)^2 \\
0 & 0 & 0 & -\frac{2}{q} & -\frac{2}{q} & -\frac{2}{q} & 4(q - 1)^2 \\
0 & 0 & 0 & 0 & -\frac{2}{q} & -\frac{2}{q} & 4(q - 1)^2 \\
0 & 0 & 0 & 0 & 0 & -\frac{2}{q} & 4(q - 1)^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

and the eigenvalues corresponding to the columns are

$$0, 0, 0, 0, 0, q^2(q - 1), q^3(q - 1), q^3, q^3(q - 1)^2,$$

respectively. Expression (5.2) and straightforward matrix multiplication now yield the following theorem.
image of $\tilde{Z}(N)$ is generated by $\{\tilde{Z}(N)(T_i) : i = 1, \ldots, 5\}$. This can be explained in the same way as in Remark 5.3. For any $X = S_6, \ldots, S_{11}, D_{12}$ we have that

$$Z_\pi(N)(X) = [G]\pi_t\left([\pi^*X \times G : (B, A) \mapsto BA^2]_G\right).$$

Making a suitable substitution, one can write $BA^2 = \tilde{B}\tilde{A}^2$ where $\tilde{B}$ only has ones on the diagonal. Then it will indeed follow that $Z_\pi(N)(X)$ is generated by $\{Z_\pi(N)(T_i) : i = 1, \ldots, 5\}$.

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Appendix

Let $S$ be $\mathbb{A}^n_S$ with coordinates $x_1, \ldots, x_n$. In this section we discuss an algorithm for computing the class of certain varieties $X$ over $S$ in $K(\text{Var}/S)$. We use the following notation. Let $F, G$ be finite subsets of $\mathbb{C}[x_1, \ldots, x_n, V]$ with $V = \{y_1, \ldots, y_m\}$ for some $m \geq 0$. Then we write $X(V, F, G)$ for the (reduced) subvariety of $\mathbb{A}^m_S$ given by $f = 0$ for all $f \in F$ and $g \neq 0$ for all $g \in G$. For example,

$$\SL_2(\mathbb{C}) \times_{\mathbb{C}} S = X(\{a, b, c, d\}, \{ad - bc - 1\}, \emptyset) \quad \text{and} \quad \GL_2(\mathbb{C}) \times_{\mathbb{C}} S = X(\{a, b, c, d\}, \emptyset, \{ad - bc\}).$$

Now consider the following recursive algorithm.

**Algorithm A.1.** Let $X = X(V, F, G)$ be a variety over $S$ as above.
1. First simplify $F$ as follows:
   
   (a) Replace $F$ with the reduced Gröbner basis for $F$.
   
   (b) If any $f$ in $F$ is of the form $f = y_i - u$ with $u \in \mathbb{C}[x_1, \ldots, x_n, V - \{y_i\}]$, then substitute $y_i$
   with $u$ in each element of $F$, remove $y_i$ from $V$, and go to (a).
   
   (c) Replace each $f$ in $F$ with its square-free part. If anything changed, go to (a).

2. Reduce each $g \in G$ modulo $F$.

3. If $1 \in F$ or $0 \in G$, then $X = \emptyset$ and return $[X] = 0$.

4. If $F = G = \emptyset$, then $X = \mathbb{A}_S^k$ with $k = \#V$, and return $[X] = [\mathbb{A}_S^k]^k$.

5. If there exist partitions $V = V_1 \sqcup V_2, F = F_1 \sqcup F_2$ and $G = G_1 \sqcup G_2$ such that $V_1, V_2 \neq \emptyset$ and
   $F_i, G_i \subset \mathbb{C}[x_1, \ldots, x_n, V_i]$, then $X = X_1 \times_S X_2$ where $X_i = X(V_i, F_i, G_i)$. Therefore, return
   $[X] = [X_1] \cdot [X_2]$.

6. If any $f \in F$ is univariate in some $x \in \{x_1, \ldots, x_n\} \cup V$, then write $f = (x - \alpha_1) \cdots (x - \alpha_k)$. For
   each $i = 1, \ldots, k$, let $X_i = X(V, F \cup \{x - \alpha_i\}, G)$. Now return $[X] = \sum_{i=1}^k [X_i]$.

7. If any $f \in F$ factors as $f = uv$ with $u, v$ both not constant, then let $X_1 = X(V, F \cup \{u\}, G)$ and
   $X_2 = X(V, F \cup \{v\}, G \cup \{u\})$, and return $[X] = [X_1] + [X_2]$.

8. If any $f \in F$ is of the form $f = y_i u + v$ with $u, v \in \mathbb{C}[x_1, \ldots, x_n, V - \{y_i\}]$, then let $X_1 = X(V, F \cup \{u, v\}, G)$ and
   $X_2 = X(V - \{y_i\}, F_2, G_2)$ where $F_2$ (resp. $G_2$) contains all $f$ in $F$ (resp. in $G$) where $y_i$
   is substituted with $-v/u$ and homogenized by multiplying by a suitable number of factors $u$. Now return $[X] = [X_1] + [X_2]$.

9. If $G$ is non-empty, then take some $g \in G$ and let $X_1 = X(V, F, G - \{g\})$ and $X_2 = X(V, F \cup \{g\}, G - \{g\})$. Return $[X] = [X_1] - [X_2]$.

10. Create a new symbol that represents the class of $X$ in $K(\text{Var}/S)$. Return this symbol.

An implementation of this algorithm in Python can be found at [8].

We give some remarks on this algorithm. First, note that steps 1a and 2 depend on a monomial
order: we choose the degree reverse lexicographic order. Secondly, the factorization done in step
1c to determine the square-free part of polynomials can be stored and reused in step 7 (and even
in later computations). Finally, since the algorithm is heavily recursive, it happens that the same
computations are done multiple times. Therefore, it is more efficient to store intermediate results:
we store any result that comes from steps 6 – 10, and check before step 6 if the computation has
already been done before.