A comprehensive theory of the Weyl-Wigner formalism for the canonical pair angle-angular momentum is presented, with special emphasis in the implications of rotational periodicity and angular-momentum discreteness.

I. INTRODUCTION

A quantum system has a dynamical symmetry group $G$ if its Hamiltonian is a function of the generators of $G$. In this case, the Hilbert space of the system splits into a direct sum invariant subspaces (carriers of the irreducible representations of $G$) and the discussion of any physical property can be restricted to one of these subspaces.

The existence of such a symmetry also allows for the explicit construction of a phase space for the system as the coadjoint orbit associated with an irreducible representation of $G$ (in fact, it turns out to be a symplectic manifold). In consequence, to every operator on Hilbert space we can associate a function on phase space, opening the way to formally representing quantum mechanics as a statistical theory on classical phase space. Various aspects of this formalism for basic quantum systems have been developed by a number of authors.

There are, however, important differences with respect to a classical description. They come from the noncommuting nature of conjugate quantities, which precludes their simultaneous precise measurement and, therefore, imposes a fundamental limit to the accuracy with which we can determine a point in phase space. As a distinctive consequence of this, there is no unique rule by which we can associate a classical phase-space variable to a quantum operator and depending on the operator ordering, various functions can be defined. For example, the quantum state (i.e., the density matrix) of the system can be mapped onto a whole family of functions parametrized by a number $s$; the values $+1$, $0$, and $-1$ corresponding to the Husimi $Q$, the Wigner $W$, and the Glauber-Sudarshan $P$ functions, respectively. These phase-space functions are known as quasiprobability distributions, as in quantum mechanics they play a role similar to that of genuine probability distributions in classical statistical mechanics (for reviews, see Refs. [16, 17, 18, 19]).

Apart from the description of the harmonic oscillator (for which $G$ is the Heisenberg-Weyl group and the corresponding phase space is the plane $\mathbb{R}^2$), this formalism has also been successfully applied to spin-like systems (or qubits in the modern parlance of quantum information), for which $G$ is the group SU(2) and the phase space is the two-dimensional Bloch sphere. However, one can rightly argue that this Wigner function, although describing a discrete system, is not defined in a discrete phase space. In fact, the growing interest in quantum information has fueled the search for discrete phase-space counterparts of the Wigner function (see Ref. [20] for a complete and up-to-date review). The main advantage of such a representation consists in that even states from different irreducible representations can be pictured on the same phase space, which is basically a direct product of two-dimensional discrete tori.

There is still another “mixed” canonical pair: angle and angular momentum. Now, the symmetry group $G$ is noncompact and can be taken as the two-dimensional Euclidean group $E(2)$, whereas the associated phase space is the discrete cylinder $\mathbb{Z} \times S^1$, which are discrete tori. Several interesting properties of a number of systems, such as molecular rotations, electron wave packets, Hall fluids, and light fields, to cite only a few examples, can be described in terms of this formalism [21]. In quantum optics, it is the basic tool to deal with the orbital angular momentum of the so-called twisted photons [22, 23], which have been proposed for applications in quantum experiments [24].

The construction of a proper Wigner function for this case is still under discussion. Although some interesting attempts have been published [25, 26, 27], they seem of difficult application to practical problems. Quite interesting group-theoretical approaches to this problem can be also found in Refs. [28, 29]. In this paper, we approach this interesting problem from the perspective of finite-dimensional systems and construct a bona fide Wigner function that fulfills all the reasonable requirements and is easy to handle and to interpret. We also discuss its applications to some relevant quantum states.

II. WIGNER FUNCTION FOR POSITION-MOMENTUM

In this section we briefly recall the relevant structures needed to set up the Wigner function for Cartesian quantum mechanics. This is to facilitate comparison with the angular case later on. For simplicity, we choose one degree of freedom only, so the associated phase space is the plane $\mathbb{R}^2$.

The canonical Heisenberg commutation relations between Hermitian coordinate and momentum operators $\hat{q}$ and $\hat{p}$ are (in
units $\hbar = 1$)

\[ \hat{q}, \hat{p} = i , \]  

so that they are the generators of the Heisenberg-Weyl algebra. In the unitary Weyl form this is expressed as

\[ \hat{U}(q)\hat{V}(q) = \hat{V}(q)\hat{U}(q) e^{iq\hat{p}} , \]  

where

\[ \hat{V}(q) = \exp(-i\hat{q}\hat{p}) , \quad \hat{U}(p) = \exp(i\hat{p}\hat{q}) , \]  

are the generators of translations in position and momentum, respectively. In the Cartesian case, these exponentials can be entangled to define a displacement operator

\[ \hat{D}(q, p) = \hat{U}(p)\hat{V}(q)e^{-iqp/2} = \exp[i(p\hat{q} - q\hat{p})] , \]  

with the parameters $(q, p)$ labelling phase-space points. However, this cannot be done for other canonical pairs, as we shall see.

The displacement operators form a complete trace-orthonormal set (in the continuum sense) in the space of operators acting on $\mathcal{H}$ (the Hilbert space of square integrable functions on $\mathbb{R}$):

\[ \text{Tr}[\hat{D}(q, p)\hat{D}^\dagger(q', p')] = 2\pi\delta(q - q')\delta(p - p') . \]  

Note that $\hat{D}^\dagger(q, p) = \hat{D}(-q, -p)$, while $\hat{D}(0, 0) = \mathbb{I}$.

The mapping of the density matrix $\hat{\rho}$ into a Wigner function defined on $\mathbb{R}^2$ is established in a canonical way:

\[ W(q, p) = \text{Tr}[\hat{\rho}\hat{w}(q, p)] , \]  

\[ \hat{w} = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \hat{w}(q, p)W(q, p)\, dqdp , \]  

where the (Hermitian) Wigner kernel $\hat{w}$ (a particular instance of a Stratonovitch-Weyl quantizer) is the double Fourier transform of the displacement operator:

\[ \hat{w}(q, p) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \exp[-i(pq' - qp')\hat{D}(q', p')]\, dq'dp' . \]  

One can immediately check that the Wigner kernels are also a complete trace-orthonormal set. Furthermore, they transform properly under displacements

\[ \hat{w}(q, p) = \hat{D}(q, p)\hat{w}(0, 0)\hat{D}^\dagger(q, p) , \]  

where

\[ \hat{w}(0, 0) = \int_{\mathbb{R}^2} \hat{D}(q, p)\, dqdp = 2\hat{P} , \]  

and $\hat{P}$ is the parity operator.

The Wigner function in (2.6) fulfills all the basic properties required for any good probabilistic description. First, due to the Hermiticity of $\hat{w}(q, p)$, it is real for Hermitian operators. Second, on integrating $W(q, p)$ over one variable, the probability distribution of the conjugate variable is reproduced

\[ \int_{\mathbb{R}} W(q, p)\, dp = \langle q|\hat{\rho}|q \rangle , \quad \int_{\mathbb{R}} W(q, p)\, dq = \langle p|\hat{\rho}|p \rangle . \]  

Third, $W(q, p)$ is covariant, which means that for the displaced state $\hat{\rho}' = \hat{D}(q_0, p_0)\hat{\rho}\hat{D}^\dagger(q_0, p_0)$, one has

\[ W(q', p) = W(q - q_0, p - p_0) , \]  

so that the Wigner function follows displacements rigidly without changing its form, reflecting the fact that physics should not depend on a certain choice of the origin.

Finally, the overlap of two density operators is proportional to the integral of the associated Wigner functions:

\[ \text{Tr}[(\hat{\rho}_1\hat{\rho}_2)\propto \int_{\mathbb{R}^2} W_1(q, p)W_2(q, p)\, dqdp . \]  

This property (often known as traciality) offers practical advantages, since it allows one to predict the statistics of any outcome, once the Wigner function of the measured state is known.

### III. WIGNER FUNCTION FOR DISCRETE SYSTEMS

Many quantum systems can be appropriately described in a finite-dimensional Hilbert space. The previous standard approach can be extended to these discrete systems, since they do have a dynamical symmetry group. However, in a continuous Wigner function for these systems, there is a lot of information redundancy. The goal of this section is to carry out a non-redundant discrete phase-space analysis for this case.

Let us consider a system living in a Hilbert space $\mathcal{H}_d$, of dimension $d$ (a qudit). It is useful to choose a computational basis $|n\rangle$ ($n = 0, \ldots, d - 1$) in $\mathcal{H}_d$ and introduce the basic operators $\hat{X}$ and $\hat{Z}$:

\[ \hat{X}|n\rangle = |n + 1\rangle , \quad \hat{Z}|n\rangle = \omega^n |n\rangle , \]  

where addition and multiplication must be understood modulo $d$ and, for simplicity, we use the notation

\[ \omega^m \equiv \omega^m = \exp(i2\pi m/d) , \]  

$\omega = \exp(i2\pi/d)$ being a $d$th root of the unity. The operators $\hat{X}$ and $\hat{Z}$ generate a group under multiplication known as the generalized Pauli group and obey

\[ \hat{Z}\hat{X} = \omega \hat{X}\hat{Z} , \]  

which is the finite-dimensional version of the Weyl form of the commutation relations.

The monomials $\{\hat{Z}^k\hat{X}^l\}$ ($k, l = 0, 1, \ldots, d - 1$) form a basis in the space of all the operators acting in $\mathcal{H}_d$. It seems then natural to introduce the unitary displacement operators

\[ \hat{D}(k, l) = e^{i\phi(k, l)}\hat{Z}^k\hat{X}^l , \]
where \( \phi(k, l) \) is a phase. The unitarity condition imposes that
\[
\phi(k, l) + \phi(-k, -l) = \frac{2\pi}{d} kl.
\]
Different choices have been analyzed in the literature \[33\]; one of special relevance is
\[
\phi(k, l) = \frac{2\pi}{d} 2^{-1} kl,
\]
where \( 2^{-1} \) is the multiplicative inverse of \( 2 \) in \( \mathbb{Z}_d \) when \( d \) is prime and \( 2^{-1} = 1/2 \) for nonprime dimensions.

In this way, we have got a discrete phase space of the system as a \( d \times d \) grid of points, in such a way that the coordinate of each point \( (k, l) \) define powers of \( Z \) (“position”) and \( X \) (“momentum”) and the whole phase space is isomorphic to a discrete two-dimensional torus.

The following mapping from the Hilbert space into the discrete phase space [equivalent to (2.4)]
\[
W(k, l) = \text{Tr}[\hat{\varrho} \hat{w}(k, l)],
\]
\[
\hat{\varrho} = \frac{1}{d^2} \sum_{k, l} \hat{w}(k, l) W(k, l),
\]
is established in terms of the following (Hermitian) Wigner kernel
\[
\hat{w}(k, l) = \frac{1}{d^2} \sum_{m, n} \omega(kn - lm) \hat{D}(m, n),
\]
which is normalized, satisfies the overlap condition
\[
\text{Tr}[\hat{w}(k, l) \hat{w}(k', l')] = d \delta_{k, k'} \delta_{l, l'},
\]
and it is explicitly covariant:
\[
\hat{w}(k, l) = \hat{D}(k, l) \hat{w}(0, 0) \hat{D}^\dagger(k, l),
\]
where
\[
\hat{w}(0, 0) = \frac{1}{d^2} \sum_{k, l} \hat{D}(k, l).
\]

It is interesting to note that the phase \[34] for prime dimensions leads to \( \hat{w}(0, 0) = \hat{P}, \hat{P} \) being the parity operator. In view of these properties, one can easily conclude that the corresponding Wigner function \( W(k, l) \) fulfills properties fully analogous as those for the continuous harmonic oscillator.

### IV. WIGNER FUNCTION FOR ANGLE-ANGULAR MOMENTUM

In this section, we consider the conjugate pair angle and angular momentum. To avoid the difficulties linked with periodicity, the simplest solution \[34, 35, 36\] is to adopt two angular coordinates, such as, e.g., cosine and sine, we shall denote by \( \hat{C} \) and \( \hat{S} \) to make no further assumptions about the angle itself. One can concisely condense all this information using the complex exponential of the angle \( \hat{E} = \hat{C} + i\hat{S} \), which satisfies the commutation relation
\[
[\hat{E}, \hat{L}] = \hat{E},
\]
or, equivalently,
\[
[\hat{C}, \hat{L}] = i\hat{S}, \quad [\hat{S}, \hat{L}] = -i\hat{C},
\]
\[
[\hat{C}, \hat{S}] = 0.
\]

In mathematical terms, this defines the Lie algebra of the two-dimensional Euclidean group \( E(2) \). Note also, that from the Baker-Campbell-Hausdorff formula, one gets
\[
e^{-i\varphi \hat{L}} \hat{E} = e^{i\varphi} \hat{E} e^{-i\varphi \hat{L}},
\]
which is the unitary Weyl form of (4.1).

The action of \( \hat{E} \) on the angular momentum basis is
\[
\hat{E} |\ell\rangle = |\ell - 1\rangle,
\]
and, since the integer \( \ell \) runs from \(-\infty\) to \(+\infty\), \( \hat{E} \) is a unitary operator whose normalized eigenvectors
\[
|\phi\rangle = \frac{1}{\sqrt{2\pi}} \sum_{\ell \in \mathbb{Z}} e^{i\ell \varphi} |\ell\rangle,
\]
form a complete basis
\[
\langle \phi | \phi' \rangle = \sum_{\ell \in \mathbb{Z}} \delta(\phi - \phi' - 2\ell \pi) = \delta_{2\pi}(\phi - \phi'),
\]
where \( \delta_{2\pi} \) represents the periodic delta function (or Dirac comb) of period \( 2\pi \).

As anticipated in the Introduction, the phase space is now the semi-discrete cylinder \( \mathbb{Z} \times S_1 \). Following the ideas of Sec. \[11\] a displacement operator can be introduced as
\[
\hat{D}(\ell, \varphi) = e^{i\alpha(\ell, \varphi)} \hat{E}^{-\ell} e^{-i\varphi \hat{L}},
\]
where \( \alpha(\ell, \varphi) \) is a phase to be specified. Note that here there is no possibility to rewrite Eq. (4.7) as an entangled exponential, since the action of the operator to be exponentiated would not be well defined. The requirement of unitarity imposes now
\[
\alpha(\ell, \varphi) + \alpha(-\ell, -\varphi) = \ell \varphi.
\]

As desired, the displacement operators form a complete trace-orthonormal set:
\[
\text{Tr}[\hat{D}(\ell, \varphi) \hat{D}^\dagger(\ell', \varphi')] = 2\pi \delta_{\ell, \ell'} \delta_{2\pi}(\phi - \phi'),
\]
whose resemblance with relation (2.5) is evident.

We can introduce then the canonical mapping
\[
W(\ell, \phi) = \text{Tr}[\hat{\varrho} \hat{w}(\ell, \phi)],
\]
\[
\hat{\varrho} = \frac{1}{(2\pi)^2} \sum_{\ell \in \mathbb{Z}} \int_{-\pi}^{\pi} \hat{w}(\ell, \phi) W(\ell, \phi) d\phi,
\]
where the Wigner kernel \( \hat{w} \) is defined, in close analogy to the previous cases, as

\[
\hat{w}(\ell, \phi) = \frac{1}{(2\pi)^2} \sum_{\ell', \phi'} \int_{2\pi} e^{i(\ell' \phi - \ell \phi')} \hat{D}(\ell', \phi') d\phi'.
\]  

(4.11)

The set of Wigner kernels constitutes a complete orthogonal Hermitian operator basis. In addition, they are explicitly covariant:

\[
\hat{w}(\ell, \phi) = \hat{D}(\ell, \phi) \hat{w}(0, 0) \hat{D}^\dagger(\ell, \phi),
\]  

(4.12)

with

\[
\hat{w}(0, 0) = \frac{1}{(2\pi)^2} \sum_{\ell \in \mathbb{Z}, \phi} \int_{2\pi} \hat{D}(\ell, \phi) d\phi,
\]  

(4.13)

although the interpretation of \( \hat{w}(0, 0) \) as the parity on the cylinder is problematic.

All these properties automatically guarantee that we have indeed a well-behaved Wigner function for this canonical pair.

\section{Examples}

To work out the explicit form of the Wigner function for a given state, one first needs to specify the phase \( \alpha(\ell, \phi) \) in Eq. (4.3). For convenience, in this paper the choice

\[
\alpha(\ell, \phi) = -\ell \phi / 2
\]  

(5.1)

shall be used, as it is linear in both arguments, and it appears to be the simplest function fulfilling the unitarity condition and the periodicity in \( \phi \) [21].

In this case, the Wigner kernel (4.11) becomes

\[
\hat{w}(\ell, \phi) = \frac{1}{(2\pi)^2} \sum_{\ell', \phi'} \int_{2\pi} e^{i(\ell' \phi' - \ell \phi')} e^{-i\ell' \phi'}
\times \exp\left[\frac{i(\ell' \phi' - \ell \phi)}{2}\right]
\times e^{i(\ell' \phi' - \ell \phi')} |\ell' + \ell\rangle \langle \ell - \ell'|
\]  

(5.2)

After some manipulations, we obtain

\[
\hat{w}(\ell, \phi) = \frac{1}{2\pi} \sum_{\ell' \in \mathbb{Z}} e^{-i\ell' \phi} |\ell + \ell'\rangle \langle \ell - \ell'|
\]

\[
+ \frac{1}{2\pi^2} \sum_{\ell', \phi' \in \mathbb{Z}} \frac{(-1)^{\ell'}}{\ell' + 1/2} e^{-i(2\ell' + 1) \phi}
\times \exp\left[\frac{i(\ell' \phi' - \ell \phi)}{2}\right]
\times |\ell + \ell' + \ell + 1\rangle \langle \ell + \ell' - \ell' - \ell'|
\]  

(5.3)

which coincides with the kernel derived by Plebanski and coworkers [29] in the context of deformation quantization.

Note that (5.3) splits into “even” and “odd” parts, depending on whether the matrix elements \( \delta_{\ell \ell'} = \langle \ell | \hat{\phi} | \ell' \rangle \) have \( \ell \pm \ell' \) even (first sum) or odd (second sum).

For an angular momentum eigenstate \( |\ell_0\rangle \), one immediately gets

\[
W_{|\ell_0\rangle}(\ell, \phi) = \frac{1}{2\pi} \delta_{\ell, \ell_0},
\]  

(5.4)

which is quite reasonable in this case: it is flat in \( \phi \) and the integral over the whole phase space gives the unity, reflecting the normalization of \( |\ell_0\rangle \).

For an angle eigenstate \( |\phi_0\rangle \), one has

\[
W_{|\phi_0\rangle}(\ell, \phi) = \frac{1}{2\pi} \delta_{2\pi}(\phi - \phi_0).
\]  

(5.5)

Now, the Wigner function is flat in the conjugate variable \( \ell \), and thus, the integral over the whole phase space diverges, which is a consequence of the fact that the state \( |\phi_0\rangle \) is not normalizable.

The coherent states \( |\ell_0, \phi_0\rangle \) (parametrized by points on the cylinder) introduced in Ref. [37] (see also Refs. [38, 39] for a detailed discussion of the properties of these relevant states) are characterized by

\[
\langle \ell | \ell_0, \phi_0 \rangle = \frac{1}{\sqrt{\vartheta_3(0 \frac{1}{\epsilon})}} e^{-i\ell \phi_0} e^{-(\ell - \ell_0)^2/2},
\]

\[
\langle \phi | \ell_0, \phi_0 \rangle = \frac{e^{i\ell_0(\phi - \phi_0)}}{\sqrt{\vartheta_3(0 \frac{1}{\epsilon})}} \vartheta_3 \left( \frac{\phi - \phi_0}{2} \right)^{1/2},
\]

where \( \vartheta_3 \) denotes the third Jacobi theta function [40].

The Wigner function for the state \( |\ell_0, \phi_0\rangle \) splits as

\[
W_{|\ell_0, \phi_0\rangle}(\ell, \phi) = W_{|\ell_0, \phi_0\rangle}^{(+)}(\ell, \phi) + W_{|\ell_0, \phi_0\rangle}^{(-)}(\ell, \phi).
\]  

(5.7)

The “even” part turns out to be

\[
W_{|\ell_0, \phi_0\rangle}^{(+)}(\ell, \phi) = \frac{1}{2\pi \vartheta_3(0 \frac{1}{\epsilon})} e^{-(\ell - \ell_0)^2} \vartheta_3 \left( \frac{\phi - \phi_0}{2} \right)^{1/2}.
\]  

(5.8)

This seems a sensible result, since it is a discrete Gaussian in the variable \( \ell \), and for the continuous angle \( \phi \) it is a Jacobi theta function, which plays the role of the Gaussian for circular statistics [41]. However, the “odd” contribution spoils this simple picture:

\[
W_{|\ell_0, \phi_0\rangle}^{(-)}(\ell, \phi) = \frac{e^{i(\phi - \phi_0) - 1/2}}{2\pi \vartheta_3(0 \frac{1}{\epsilon})} \vartheta_3 \left( \frac{\phi - \phi_0 + i/2}{1} \right)^{1/2}
\times \sum_{\ell' \in \mathbb{Z}} (-1)^{\ell' - \ell + \ell_0} e^{-\ell'^2 - \ell'}
\times \frac{e^{-\ell'^2 - \ell'}}{\ell' + \ell_0 - \ell + 1/2}.
\]  

(5.9)

In Fig. 1 the Wigner function for the coherent state \( |\ell_0 = 0, \phi_0 = 0\rangle \) is plotted on the discrete cylinder. A pronounced peak at \( \phi = 0 \) for \( \ell = 0 \) and slightly smaller ones for \( \ell = \pm 1 \) can be observed. The associated marginal distributions [obtained from Eq. (5.7) by integrating over \( \phi \) or by summing over \( \ell \), respectively] are also plotted. They are strictly positive, as correspond to true probability distributions.

For quantitative comparisons, however, sometimes it may be convenient to “cut” this cylindrical plot along a line \( \phi = \text{constant} \) and unwrap it. This is shown in Fig. 2. Here, the range of \( \ell \) is from -4 to 4, while the angle is plotted between \(-\pi \) to \( \pi \).
A closer look at these figures reveals also a remarkable fact: for values close to \( \phi = \pm \pi \) and \( \ell = \pm 1 \), the Wigner function takes negative values. Actually, a numeric analysis suggests the existence of negativities close to \( \phi = \pm \pi \) for any odd value of \( \ell \).

As our last example, we address the superposition

\[
|\Psi\rangle = \frac{1}{\sqrt{2}} (|\ell_1\rangle + e^{i\phi_0}|\ell_2\rangle) \tag{5.10}
\]

of two angular-momentum eigenstates with a relative phase \( e^{i\phi_0} \). The analysis can be carried out for the superposition of any number of eigenstates, but (5.10) is enough to display the relevant features.

The Wigner function splits again; now the “even” part reads

\[
W^{(+)}_{|\Psi\rangle}(\ell, \phi) = \frac{1}{4\pi} \{ \delta_{\ell,\ell_1} + \delta_{\ell,\ell_2} + 2\delta_{\ell_1+\ell_2,2\ell} \cos[\phi_0 + (\ell_2 - \ell_1)\phi] \} \tag{5.11}
\]
For the “odd” part, the diagonal contributions vanish, and one has
\[
W_{\Psi}^{(-)}(\ell, \phi) = \frac{1}{\pi^2} \cos[\phi + (\ell_2 - \ell_1)\phi] \\
\times \frac{(-1)^{\ell + (\ell_1 + \ell_2 - 1)/2}}{\ell_1 + \ell_2 - 2\ell} \delta_{\ell_1 + \ell_2 = \text{odd}},
\]
where \(\delta_{\ell_1 + \ell_2 = \text{odd}}\) indicates that the sum is nonzero only when \(\ell_1 + \ell_2\) is odd.

In consequence, when \(|\ell_1 - \ell_2|\) is odd, the interference term contains contributions for any \(\ell\), damped as \(1/\ell\). When \(|\ell_1 - \ell_2|\) is an even number, the contribution \((5.12)\) vanishes and we have three contributions: two flat slices coming from the states \(|\ell_1\rangle\) and \(|\ell_2\rangle\) and an interference term located at \(\ell = (\ell_1 + \ell_2)/2\).

These features are illustrated in Figs. 3 and 4. The state \(\Psi\) is plotted for \(\ell_2 = -3\) and \(\ell_1 = 3\) and (Fig. 3) and \(\ell_2 = -3\) and \(\ell_1 = 4\) (Fig. 4). Changing the relative phase \(\phi_0\) results in a global rotation of the cylinder. In can be observed in Fig. 3 that the two rings at \(\ell = -3\) and \(\ell = 4\) (as opposed to the rings at \(\ell = \pm 3\) in Fig. 4), are not flat in \(\phi\), but show a weak dependence on the angle due to the odd contributions added to the flat Kronecker deltas.

VI. CONCLUDING REMARKS

In summary, we have carried out a full program for a complete phase-space description in terms a Wigner function for the canonical pair angle-angular momentum. An experimental demonstration in terms of optical beams is presently under way in our laboratory.

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