We study the “effective string picture” of confinement, deriving theoretical predictions for the interquark potential at finite temperature. At low temperatures, the leading string correction to the linear confining potential between a heavy quark-antiquark pair is the “Lüscher term”. Assuming a Nambu–Goto effective string action, subleading contributions can be worked out in an analytical way. We also discuss the contribution given by a possible “boundary term” in the effective action, and compare these predictions with high precision results from simulations of lattice $\mathbb{Z}_2$ gauge theory in three dimensions, obtained with an algorithm that exploits the duality of the $\mathbb{Z}_2$ gauge model with the Ising spin model.

1. Effective String Picture of Confinement

The string picture is an effective framework which is expected to provide a good physical description of the infrared behavior of confining gauge theories. The basic idea is simple: two confined color charges behave as if they were joined by a thin flux tube, which can fluctuate like a vibrating string. The dynamics of the world sheet spanned by the string during its time-like evolution is described by an effective action $S_{\text{eff}}$, and for massless string fluctuations, the simplest choice for a candidate effective action is the Nambu–Goto string action: $S_{\text{eff}} = \sigma \cdot A$, which is proportional to the
area $\mathcal{A}$ of the world-sheet surface; $\sigma$ is the \textit{string tension}, appearing as the parameter of the effective theory.

For a three-dimensional system with extension $L_s^2 \times L$ (with $L_s \gg L$) and periodic boundary conditions along the short direction, the temperature $T$ is proportional to $1/L$, and in that case the result for the expectation value of the Polyakov loop correlation function reads:

$$\langle P^\dagger(R)P(0) \rangle = \frac{e^{-\sigma RL + k}}{\eta\left(i\frac{L}{2R}\right)}$$

(1)

where $\eta$ is Dedekind’s function. The term associated with the minimal world sheet surface induces the exponential area-law falloff responsible for the linear rise in the interquark potential $V(R)$, while the first non-trivial contribution in $S_{\text{eff}}$ results in the determinant of the Laplace operator, and the corresponding contribution to the interquark potential $V(R)$ — in a regime of distances shorter than $\frac{L}{2}$ — is the Lüscher term:

$$V(R) = -\frac{1}{L} \ln \langle P^\dagger(R)P(0) \rangle \simeq \sigma R - \frac{\pi}{24R}$$

(2)

Inclusion of further terms in the expansion of the world sheet area results in a contribution involving a combination of Eisenstein functions:

$$-\frac{\pi^2}{1152\sigma R^3} \left[ 2E_4 \left(i\frac{L}{2R}\right) - E_2 \left(i\frac{L}{2R}\right) \right]$$

(3)

However, such a contribution is still under debate.

On the other hand, it is also possible to include a “boundary term” in the effective action: a perturbative expansion in $b$ (a parameter proportional to the coefficient of the “boundary term” in the effective action), induces a leading order correction like:

$$R \rightarrow \frac{R}{\sqrt{1 + \frac{2b}{\pi}}}$$

(4)

with a short distance contribution to $V(R)$ reading: $-\frac{b\pi}{24R^2}$.

---

a. This calculation involves a $\zeta$ function regularization.
b. Such a boundary term is related to derivatives of the $h$ field (which describes transverse displacements with respect to the minimal area surface of the world sheet), evaluated along the Polyakov lines.
2. The Model: $\mathbb{Z}_2$ Lattice Gauge Theory

We run numerical simulations of the $\mathbb{Z}_2$ lattice gauge theory in three space-time dimensions. This choice has various motivations: the effective string picture is believed to be independent of the underlying gauge group; the $\mathbb{Z}_2$ gauge group is interesting from the perspective of the center role in confinement\(^c\); the reduced configuration space of this theory and its duality with respect to the Ising spin model enable one to get high precision results within a reasonable amount of CPU time.

The pure 3D lattice gauge model is described in terms of $\sigma_{x,\mu}$ variables (taking values in $\mathbb{Z}_2$) defined on the lattice bonds; the dynamics is governed by the standard Wilson action, which enjoys $\mathbb{Z}_2$ gauge invariance\(^d\). The partition function reads:

$$Z(\beta) = \sum_c e^{-\beta S} = \sum_c \exp \left[ +\beta \sum_\square \sigma_\square \right]$$

and the system may exist in different phases: a confined, strong coupling phase, with massive string fluctuations for $\beta < 0.47542(1)$\(^4\); a confined, rough phase, with massless string fluctuations (this is the regime we studied in our simulations); a deconfined phase for $\beta > 0.7614134(2)$\(^5\).

This model is dual with respect to the $\mathbb{Z}_2$ spin model in 3D, and we exploited this property to express a ratio between Polyakov loop correlators $G(R) = \langle P^1(R) P(0) \rangle$ of the gauge theory as a product of expectation values of one-link variables in the modified spin ensembles. A similar algorithm is the so-called snake algorithm\(^6\).

We used multi-level updating and a hierarchical organization of sub-lattices, and the CPU time turns out to be roughly proportional to the inverse temperature $L$, and virtually independent of the distance $R$ between the quark sources, thus the algorithm is particularly useful in a regime of very large interquark distances.

3. Numerical Results

Let $F(R, L)$ be the free energy associated with the presence of a heavy quark-antiquark pair at finite temperature: $G(R) = e^{-F(R, L)}$. We studied “quantum contributions” in free energy differences, by measuring the

---

\(^c\) The center $Z$ is the center of continuous gauge groups like $SU(2)$ or $Sp(N)$.

\(^d\) $Z_2$ gauge transformations act as local flips of $\sigma_{x,\mu}$ variables living on the lattice bonds which meet at a given site.
following quantity:

\[ Q(R, L) = F(R + 1, L) - F(R, L) - \sigma L \quad (6) \]

Fig. 1 shows that at “high temperatures” \( (L < 2R) \) our numerical results are in good agreement with the NLO prediction from the Nambu–Goto string, while a pure area law is definitely ruled out, and the LO term alone is not sufficient to describe the data. We also found that the coefficient of a possible “boundary term” for this model seems to be very small, likely zero.

As it concerns the \( (L > 2R) \) regime, Fig. 2 shows the deviation of a quantity proportional to \( [Q(R, L) - Q(R - 1, L)] \) from the free string prediction. The chosen normalization allows a meaningful comparison among different LGT’s in 3d: \( SU(2) \) gauge theory\(^7\) (crosses), \( SU(3) \) gauge theory\(^8\) (white squares), and \( \mathbb{Z}_2 \) gauge theory\(^2\) (black squares). The three models display the same qualitative behavior, and, in particular, the data for \( \mathbb{Z}_2 \) and \( SU(2) \) (which are groups with the same center — namely: \( \mathbb{Z}_2 \) itself) are compatible within errorbars. This may be a signature of the relevance of center degrees of freedom to the confinement mechanism.
Figure 2. Behavior of different gauge models in the low temperature regime.

4. Conclusions

We studied confining gauge theories at finite temperature, and tested the theoretical predictions of the Nambu–Goto effective string for $\mathbb{Z}_2$ lattice gauge theory, both at large and short interquark distances. Our algorithm exploits the duality of the model, and this enabled us to explore a wide range of distances, detecting next-to-leading order effects. Our data seem to rule out a “boundary term” in the effective string action describing the present gauge model. Finally, we also made a comparison with some different gauge models.

References

1. M. Lüscher, K. Symanzik and P. Weisz, *Nucl.Phys.* B173 (1980) 365.
2. M. Caselle, M. Hasenbusch and M. Panero, *JHEP* 0301 (2003) 057. M. Caselle, M. Panero and P. Provero, *JHEP* 0206, 061 (2002) and references therein. M. Caselle, M. Hasenbusch and M. Panero, in preparation.
3. J.P. Serre, “A course in Arithmetic”, Springer–Verlag, New York, 1980.
4. M. Hasenbusch and K. Pinn, *J.Phys.* A30 (1997) 63.
5. H.W.J. Blöte, L.N. Shchur, A.L. Talapov, *Int.J.Mod.Phys.* C10 (1999) 1137.
6. Ph. de Forcrand, M. D’Elia and M. Pepe, *Phys.Rev.Lett.* 86 (2001) 1438. Ph. de Forcrand, M. D’Elia and M. Pepe, *Nucl.Phys.Proc.Suppl.* 94 (2001) 494.
7. M. Caselle, M. Pepe and A. Rago, in preparation.
8. M. Lüscher and P. Weisz, *JHEP* 0207 (2002) 049.