S-wave contributions to \( B_s^0 \to (D^0, \bar{D}^0)\pi^+\pi^- \) in the perturbative QCD framework

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Abstract: \( B_s^0 \to (D^0, \bar{D}^0)\pi^+\pi^- \) is induced by the \( b \to c\bar{u}s/b \to u\bar{c}s \) transitions, which can interfere if a CP-eigenstate \( D_{CP} \) is formed. The interference contribution is sensitive to the CKM angle \( \gamma \). In this work, we study the S-wave \( \pi^+\pi^- \) contributions to the process in the perturbative QCD factorization. In the factorization framework, we adopt two-meson light-cone distribution amplitudes, whose normalization is parametrized by the S-wave time-like two-pion form factor with resonance contributions from \( f_0(500), f_0(980), f_0(1500), f_0(1790) \). We find that the branching ratio of \( B_s^0 \to (D^0, \bar{D}^0)(\pi^+\pi^-)_S \) is of the order of \( 10^{-6} \), and that significant interference exists in \( B_s^0 \to D_{CP}(\pi^+\pi^-)_S \). Future measurement could not only provide useful constraints on the CKM angle \( \gamma \), but would also be helpful for exploring the multi-body decay mechanism of heavy mesons.

Keywords: PQCD, three body decays, \( \gamma \) angle

PACS: 12.38-t DOI: 10.1088/1674-1137/43/7/073103

1 Introduction

In recent years, three-body hadronic \( B/B_s \) meson decays have attracted considerable attention of the experiments [1-3]. These processes provide new ways of studying the phenomenology of the Standard Model and of probing new physics effects. For instance, the LHCb collaboration has measured sizable direct CP asymmetries in the phase space of the three-body \( B \) decays [4, 5]. In addition, these processes are also valuable for understanding the mechanism of multi-body heavy meson decays.

On the theoretical side, the perturbative QCD (PQCD) framework, based on the \( k_T \) factorization, has been applied to analyze the \( B/B_s \) semi-leptonic and two-body decays processes [6-30]. The PQCD framework has also been used to study three-body decays [31-41]. Generally, the multi-scale decay amplitude may be written as a convolution, including the nonperturbative wave functions, hard kernel at the intermediate scale and short-distance Wilson coefficients. The factorization is greatly simplified if two of the final hadrons move collinearly. In this case, the three-body decays are reduced to quasi-two-body processes. Therefore, nonperturbative wave functions include two-meson light-cone distributions, which contain both resonant and nonresonant contributions. For instance, the measurement of \( B_s \to J/\psi(\pi^+\pi^-)_S \) by LHCb [5] indicates that the resonances \( f_0(500), f_0(980), f_0(1500), f_0(1790) \) of the S-wave \( \pi\pi \)-pair are dominant, which is confirmed by the theoretical calculation in the framework of PQCD [42-47]. In this work, we focus on \( B_s^0 \to D^0(\bar{D}^0)\pi^+\pi^- \) and include the \( B_s \to D(f_0(500)+f_0(980)+f_0(1500)+f_0(1790)) \to D(\pi^+\pi^-)_S \) contributions. More explicitly, a Breit-Wigner (BW) model is used for the resonances \( f_0(500), f_0(1500), f_0(1790) \) [48], and the Flatté model is adopted for the resonance \( f_0(980) \) [49]. \( B_s^0 \to D^0(\bar{D}^0)\pi^+\pi^- \), with the CP eigenstate containing the interference amplitude from \( b \to c\bar{u}s \) (\( b \to u\bar{c}s \)), is sensitive to the angle \( \gamma \) of the CKM Unitarity Triangle, whose precise measurement is one of the primary objectives in flavour physics.

The paper is organized as follow: in Sec. 2, we introduce the wave functions of \( B_s, D \) and of the two pions. Sec. 3 contains our perturbative calculation within the...
PQCD framework. In Sec. 4, we give the numerical results, and a conclusion is presented in the last section.

2 Wave functions

In general, the wave function \( \Phi_{aB} \) with Dirac indices \( \alpha, \beta \) can be decomposed into 16 independent components, \( \Phi_{aB} = \gamma_{\mu} \Phi_{aB}^\mu \). For the pseudoscalar \( B_s \) meson, the light-cone matrix element is defined as

\[
\int d^4k_1 (2\pi)^4 \phi_{Bs}(k_1) = \frac{f_{Bs}}{2p^2 Nc} \int d^4k_1 (2\pi)^4 \bar{\phi}_{Bs}(k_1) = 0.
\]

This condition follows the above equation with \( \bar{\phi}_{Bs}(k_1) \) as the conjugate space coordinate of \( \phi_{Bs}(k_1) \), i.e.,

\[
\Phi_{Bs} = \frac{i}{\sqrt{2Nc}} (P_{Bs} + m_{Bs}) \gamma_5 \phi_{Bs}(k_1).
\]

(3)

Usually, the hard part is independent of \( k^+ \) or/and \( k^- \), thus one can integrate out one of them as the light quark in \( B_s \) meson. With \( b \) as the conjugate space coordinate of \( k_\perp \), we can express \( \phi_{Bs}(x, k_\perp) \) as follows:

\[
\phi_{Bs}(x, b)_{aB} = \frac{i}{\sqrt{2Nc}} [(P_{Bs} + m_{Bs}) \gamma_5]_{aB} \phi_{Bs}(x, b).
\]

(4)

where \( x \) is the momentum fraction of the light quark in the \( B_s \) meson. In this paper, we adopt the following expression for \( \phi_{Bs}(x, b) \)

\[
\phi_{Bs}(x, b) = N_{Bs} x^2 (1-x)^2 \exp \left[ \frac{m_{Bs}^2 x^2 - (\omega_b b)^2}{2 \omega_b} - \frac{\omega_b b^2}{2} \right].
\]

(5)

where \( N_{Bs} \) is the normalization factor, which is determined by the above equation with \( b = 0 \). In our calculation, we adopt \( \omega_b = 0.228 \pm 0.004 \text{GeV} \) [10], from which we determine \( N_{Bs} = 63.02 \).

The wave function of the charmed \( D \) meson, treated as the heavy-light system, is defined by the light-cone matrix element as follows [11]:

\[
\int d^4z (2\pi)^4 \phi_{D}(0) \phi_{D}(z) = \frac{f_D}{2\sqrt{2N_c}}.
\]

which satisfies the normalization

\[
\int d^4k_2 (2\pi)^4 \phi_D(k_2) = \frac{f_D}{2\sqrt{2N_c}}.
\]

(7)

Here, \( f_D \) is the decay constant and the chiral \( D \) meson mass is taken as \( m_D = m_{D_s}^0 = m_{D_s} + m_{D} + \mathcal{O}(A) \). For the numerical calculation, we adopt the parametrization [50],

\[
\phi_D(x_2, b_2) = \frac{f_D}{2\sqrt{2N_c}} 6x_2(1-x_2)[1 + C_D(1 - 2x_2)] \times \exp \left[ -\frac{\omega_D^2 b_2^2}{2} \right].
\]

(8)

where the free shape parameter \( C_D = 0.5 \pm 0.1 \) [14], and \( f_D \), \( \omega_D \) read as \( f_D = 0.209 \pm 0.002 \) [10] and \( \omega_D = 0.1 \) [14].

The S-wave two-pion distribution amplitude is then given as [46]

\[
\Phi^{S-\text{wave}}_{\pi\pi} = \frac{1}{2\sqrt{2N_c}} 1\!b \Phi^{s}_{\pi\pi}(z, \xi, m_{\pi}^2) + m_{\pi} \Phi^{f}_{\pi\pi}(z, \xi, m_{\pi}^2) + m_{\pi}(\xi - 1) \Phi^{T}_{\pi\pi}(z, \xi, m_{\pi}^2).
\]

(9)

where \( z \) is the momentum fraction carried by the spectator quark, \( \Phi^{s}_{\pi\pi} \), \( \Phi^{f}_{\pi\pi} \) and \( \Phi^{T}_{\pi\pi} \) are twist-2 and twist-3 distribution amplitudes. \( m_{\pi} \) is the invariant mass of the pion pair. We consider that the two-pion system moves in the \( n \) direction, \( \xi \) as the momentum fraction of \( \pi^+ \) in the pion pair. The asymptotic forms are parametrized as [51-53]

\[
\Phi^{s}_{\pi\pi} = F_s(m_{\pi}^2) \frac{m_{\pi}^2}{2\sqrt{2N_c}} 6z(1-z)3(2z-1),
\]

\[
\Phi^{f}_{\pi\pi} = F_f(m_{\pi}^2), \quad \Phi^{T}_{\pi\pi} = F_t(m_{\pi}^2) \frac{m_{\pi}^2}{2\sqrt{2N_c}} (1-2z).
\]

(10)

Here, \( F_s(m_{\pi}^2) \) and \( a_2 \) are the timelike scalar form factor and the Gegenbauer coefficient, respectively. As a first approximation, the S-wave resonances used to parametrize \( F_s(m_{\pi}^2) \) include both the resonant and nonresonant parts of the S-wave two-pion distribution amplitude. Therefore, we take into account \( f_0(980), f_0(1500) \) and \( f_0(1790) \) in the \( s\bar{s} \) density operator, and \( f_5(500) \) in the \( u\bar{u} \) density operator:

\[
F^{s}_{\pi\pi}(m_{\pi}^2) = \frac{c_1 m_{1500}^2 10^{10}}{m_{f_0(980)}^2 - m_{1500}^2 - im_{1500}} (g_{f_0(980)} \Gamma_{f_0(980)} + g_{o_0} K_{f_0(980)} K_{f_0(980)})
\]

\[
+ \frac{c_2 m_{1500}^2 10^{10}}{m_{f_0(1500)}^2 - m_{1500}^2 - im_{1500}} (g_{f_0(1500)} \Gamma_{f_0(1500)} (m_{1500}^2))
\]

\[
+ \frac{c_2 m_{1790}^2 10^{10}}{m_{f_0(1790)}^2 - m_{1790}^2 - im_{1790}} (g_{f_0(1790)} \Gamma_{f_0(1790)} (m_{1790}^2))
\]

(11)
$c_0$, $c_1$ and $\theta_i$, $i = 1, 2, 3$, are tunable parameters. $m_S$ is the pole mass of the resonance, and $\Gamma_S(m_{\pi\pi})$ is the energy dependent width of the $S$-wave resonance which decays into two pions. For the contribution of $f_0(980)$, an anomalous structure was found around 980 MeV in the $\pi^+\pi^-$ scattering [54, 55]. This was accompanied by the observation of a narrow anomaly (less than 100 MeV wide) in the $S$-wave phase shift associated with an enhancement in the $(l = 0)$ $K\bar{K}$ system at threshold. It was shown that the anomaly could be understood as a narrow two-channel resonance, which combines the $\pi\pi$ and $K\bar{K}$ channels [56]. Generally, the Breit-Wigner (BW) model can be applied to describe an unstable particle as an isolated resonance. Since the resonance $f_0(980)$ is near the threshold of $K\bar{K}$ of about 992 MeV, the model should be modiﬁed to include the coupled channels $f_0(980) \rightarrow \pi\pi$ and $f_0(980) \rightarrow K\bar{K}$ [56]. Therefore, the Breit-Wigner form proposed by Flatté and adopted widely in many studies of the $\pi-\pi$ and $K\bar{K}$ system is also used in this work. In the Flatté model, the phase space factors $\rho_{\pi\pi}$ and $\rho_{K\bar{K}}$ are given as [48]

$$\rho_{\pi\pi} = \frac{2}{3} \sqrt{1 - \frac{4m_\pi^2}{m_{\pi\pi}^2}} + \frac{2}{3} \sqrt{1 - \frac{4m_\pi^2}{m_{\pi\pi}^2}},$$

$$\rho_{K\bar{K}} = \frac{1}{2} \sqrt{1 - \frac{4m_K^2}{m_{K\bar{K}}^2}} + \frac{1}{2} \sqrt{1 - \frac{4m_K^2}{m_{K\bar{K}}^2}}. \quad (12)$$

3 Perturbative calculations

According to the factorization theorem, the amplitude of a process can be calculated as an expansion in $\alpha_s(Q)$ and $\Lambda/Q$, where $Q$ denotes a large momentum transfer, and $\Lambda$ is a small hadronic scale. Usually, the factorization formula for the nonleptonic $b$-meson decay can be expressed as

$$A \sim \int_0^1 dx_1 dx_2 dx_3 \int d^2 b_1 d^2 b_2 d^2 b_3 \frac{C(t)\phi_b(x_1,b_1,t)}{H(x_1,x_2,x_3,b_1,b_2,b_3,t)\phi_\pi(x_2,b_2,b_3,t)}.$$

where the Wilson coefficients and the typical scale $t$. The hard kernel $H(x_1,b_i,t)$, representing $b$-quark decay subamplitude, and the nonperturbative meson wave function $\phi_\pi(x_2,b_i,t)$, describe the evolution from scale $t$ to the lower hadronic scale $\Lambda_{QCD}$. For a review of this approach, see Ref. [7].

The effective Hamiltonian for $B^0 \rightarrow D^0(\bar{D}^0)\pi^+\pi^-$ is given as

$$\mathcal{H}_{eff} = \frac{G_F}{\sqrt{2}} V_{tb} V_{qc} \left( C_1 O_1 + C_2 O_2 \right), \quad \left( q = c, u, q = u, c \right). \quad (14)$$

with $O_1 = \langle \bar{c}_a b_\nu \rangle \bar{\phi}_{(s,\bar{s})}\bar{\phi}_{(s,\bar{s})} \phi_{(s,\bar{s})}$ for the $B^0 \rightarrow D^0(\bar{D}^0)\pi^+\pi^-$ process, and $O_2 = \langle \bar{c}_a b_\nu \rangle \bar{\phi}_{(s,\bar{s})}\bar{\phi}_{(s,\bar{s})} \phi_{(s,\bar{s})}$ for the process $B^0_s \rightarrow \bar{D}^0(\bar{D}^0)\pi^+\pi^-$. In particular, the penguin operators do not contribute to the process. Using the above effective Hamiltonian, we obtain the typical Feynman diagrams for the $B^0 \rightarrow D^0(\bar{D}^0)\pi^+\pi^-$ process shown in Fig. 1, in which the first row represents the color-suppressed emission process, and the second row indicates the $W$-exchange process. In the factorization framework, the factorizable diagrams in Fig. 1 (a,b,e,f) are relevant for $\rho_2$, and the non-factorizable diagrams in Fig. 1 (c,d,g,h) are proportional to $C_2$ [57], where

$$a_1 = C_2 + C_1/N_c, \quad a_2 = C_2 + C_1/N_c.$$ 

We will work in the light-cone coordinates. The moments of the mesons are defined as follows:

$$P_{B} = (p_{1}^2, p_{11}^2), \quad P_{\pi\pi} = (p_{2}^2, 0, 0), \quad P_{D} = (p_{2}^2 - p_{1}^2, m_B^2 / (2p_{1}^2), 0). \quad (15)$$

Accordingly, the momentum transfer and the light-cone components can be obtained as $q^2 = (P_{B} - P_{\pi\pi})^2 = (1 - \rho)m_B^2$, $\rho = 1 - \frac{m_B^2}{m_B^2}$, $p_{11}^2 = m_B^2 / (2p_{1}^2)$, and $p_{2}^2 = (m_B^2 - q^2)p_{1}^2 / m_B^2$. In the heavy quark limit, the mass difference between the $b$-quark (c-quark) and $B_s(D)$ meson is negligible, $m_{B_s,D} = m_{b,c} + \Lambda(\Lambda$ is of the order of the QCD scale). Since $m_{B_s,D} \gg m_B \gg \Lambda$, we expand the amplitudes in terms of $m_B^2 / m_{B_s,D}^2$, and for high order of the leading order of the expansion, $\rho \sim 1, q^2 \sim 0$. The moments of the light quarks in the mesons $(k_1, k_3$ represent the momenta of the light quarks in $B_s$ and $D$ mesons, $k_2$ is the momentum of the positive quark in the pion-pair system) are given as

$$k_1 = (0, x_1 P_{B}, k_{11}), k_2 = (x_2 P_{\pi\pi}, 0, k_2), k_3 = (0, x_3 P_{D}, k_{11}). \quad (16)$$

In the $k_T$-factorization, the color-suppressed emission Feynman diagrams can be calculated out, with the formulas labeled as $e_k (x = 1, 2, 3, 4)$ in the subscript. Thus, the factorization formulas for the color-suppressed $D^0$-emission diagrams are given as

$$M_{e_1} = 8\pi C_F m_B^4 f_{D} \int_0^1 dx_1 dx_2 dx_3 \int_0^1 b_1 db_2 b_3 \phi_b(x_1,b_1) \phi_{\pi}(x_2,b_2) \phi_{\pi}(x_3,b_3) \phi_b(x_1,x_2,x_3),$$

$$M_{e_2} = 16 \pi C_F m_B^4 f_{D} \int_0^1 dx_1 dx_2 dx_3 \int_0^1 b_1 db_2 b_3 \phi_b(x_1,b_1) \phi_{\pi}(x_2,b_2) \phi_{\pi}(x_3,b_3) \phi_b(x_1,x_2,x_3),$$

$$M_{e_3} = 32 \pi C_F m_B^4 f_{D} \int_0^1 dx_1 dx_2 dx_3 \int_0^1 \phi_{\pi}(x_2,b_2) \phi_{\pi}(x_3,b_3) \phi_b(x_1,x_2,x_3),$$

$$M_{e_4} = 32 \pi C_F m_B^4 f_{D} \int_0^1 dx_1 dx_2 dx_3 \int_0^1 \phi_{\pi}(x_2,b_2) \phi_{\pi}(x_3,b_3) \phi_b(x_1,x_2,x_3).$$

(17)
where \( r_0 = \frac{m_{\pi}}{m_{B}} \), \( C_F \) is the color factor. \( \phi_{\pi}(s\bar{s}, x_2) \) represents the two-pion distribution amplitude defined by the \( s\bar{s} \) operator. The hard kernels \( E_e \) and \( h_e \) are given in the following.

The factorization formulas for the \( W \)-exchange \( D^0 \) diagrams \( M_{w12} \) and \( M_{w34} \) are given as

\[
M_{w12} = 8\pi C_F m_B^4 f_B \int_0^1 dx_2 dx_3 \int_0^{1/\Lambda} b_2 db_2 b_3 db_3 \phi_D(x_3,b_3) \times [E_w(t_w) h_{u_1}(x_3,x_2, b_2,b_3)](x_3 \phi_{\pi}(u\bar{u}, x_2) + 2r_0 r_D(x_3 + 1) \phi_{\pi}^T(u\bar{u}, x_2) - r_0 r_D(2x_3 + 1) \phi_{\pi}^T(u\bar{u}, x_2) + r_0 r_D(1 - 2x_3) \phi_{\pi}^T(u\bar{u}, x_2)] E_w(t_w) h_u(x_3,x_2,b_2, b_3) x_{2}(x_3) h_w(x_1, x_2, x_3, b_1, b_2) C_2(t_w),
\]

\[
M_{w34} = \frac{32\pi C_F m_B^4}{\sqrt{2N_c}} \int_0^1 dx_1 dx_2 dx_3 \int_0^{1/\Lambda} b_1 db_1 b_2 db_2 \phi_B(x_1,b_1) \times \phi_D(x_3,b_2) [E_w(t_w) h_u(x_1,x_2,x_3, b_1, b_2) C_2(t_w) - r_0 r_D(x_2 - x_3) \phi_{\pi}^T(u\bar{u}, x_2) + r_0 r_D(x_2 - x_3)] + r_0 r_D(x_2 - x_3) \phi_{\pi}^T(u\bar{u}, x_2) + r_0 r_D(x_2 - x_3) \phi_{\pi}^T(u\bar{u}, x_2) + r_0 r_D(x_2 - x_3),
\]

where \( r_D = \frac{m_D}{m_B} \), \( \phi_{\pi}(u\bar{u}, x_2) \) represents the distribution amplitude of the \( u\bar{u} \) operator. Due to the helicity suppression, the contribution of the factorizable diagrams \( M_{w12} \) is suppressed significantly. Therefore, the dominant contribution comes from the non-factorizable diagrams \( M_{w34} \).

In the \( \bar{D}^0 \)-emission process, the two factorizable diagrams have the same factorization \( M_{w12} = M_{w12} \). Accordingly, we give the factorization formulas for the non-factorizable emission diagrams \( M_{w34} \), the factorizable \( W \)-exchange diagrams \( M_{w12} \) and the non-factorizable \( W \)-ex-
change diagrams $M_{w',34}$ as follows:

\[
M_{w,34} = \frac{32cF_m^4_B}{\sqrt{2N_c}} \int_0^1 dx_1 dx_2 dx_3 \int_0^{1/\Lambda} b_1 db_1 b_2 db_2 \phi_B(x_1, b_1) \\
\times \phi_B(x_2, b_2) [E_{w,}(t_x), h_{w,}(x_1, x_2, x_3, b_1, b_2) C_2(t_x) \\
\times \{r_0 x_2 (\phi_B(x_1, b_1) + \phi_B^T(x_2, b_2)) x_1 \phi_B(x_1, b_1) \\
- E_{w,}(t_x) h_{w,}(x_1, x_2, x_3, b_1, b_2) C_2(t_x) [r_0 x_2 (\phi_B(x_1, b_1) + \phi_B^T(x_2, b_2)) \\
- \phi_B^T(x_2, b_2)] (x_1 + x_2) \phi_B(x_1, b_1)] \\
\times \phi_B^T(x_2, b_2) [E_{w,}(t_x), h_{w,}(x_1, x_2, x_3, b_1, b_2) C_2(t_x),]
\]

\[
M_{w,12} = 8\pi c F_m^2_B f_B \int_0^1 dx_1 dx_2 dx_3 \int_0^{1/\Lambda} b_2 db_2 b_3 db_3 \phi_B(x_3, b_3) \\
\times \phi_B(x_1, b_1) [E_{w,}(t_x), h_{w,}(x_1, x_2, b_2, b_3) x_2(t_x) \\
\times [(1-x_2) \phi_B(x_1, b_1) + r_0 T (2 x_2 - 3) \phi_B^T(x_1, b_1)] (u_1, x_2) \\
+ r_0 T (1-x_3) \phi_B^T(x_1, b_1) + [-x_3 \phi_B^T(x_1, b_1) \\
+ 2 r_0 T (x_3 + 1) \phi_B^T(x_1, b_1) E_{w,}(t_x) \\
\times h_{w,}(x_1, x_2, b_2, b_3) x_2(t_x),]
\]

\[
M_{w,34} = \frac{32cF_m^4_B}{\sqrt{2N_c}} \int_0^1 dx_1 dx_2 dx_3 \int_0^{1/\Lambda} b_1 db_1 b_2 db_2 \phi_B(x_1, b_1) \\
\times \phi_B(x_2, b_2) [E_{w,}(t_x), h_{w,}(x_1, x_2, x_3, b_1, b_2) C_2(t_x) \\
\times [x_1 \phi_B^T(u_1, x_2) - r_0 x_2 (1-x_3) \phi_B^T(u_1, x_2) \\
+ r_0 x_2 (2 x_3 + 1) \phi_B^T(u_1, x_3)] (u_2, x_3) \\
+ r_0 x_2 (2 x_3 + 3) \phi_B^T(u_1, x_3) + r_0 x_2 (x_3 + 1) \\
\times \phi_B^T(u_1, x_3) E_{w,}(t_x), h_{w,}(x_1, x_2, x_3, b_1, b_2) C_2(t_x),]
\]

In the following, we give the forms for the offshellness of the intermediate gluon $\beta_{e,}/\beta_{w,}$ and quarks $\beta_{e,}/\beta_{w,}$.

\[
h_{w,}(x_1, x_2, b_1, b_2) = \theta(b_1 - b_2) I_0(\sqrt{\alpha_s} b_2) K_0(\sqrt{\alpha_s} b_1) + (b_1 \leftrightarrow b_2) K_0(\sqrt{\alpha_s} b_2) S_i(\alpha_s(1/m_B^2)), \\
h_{w,}(x_1, x_2, x_3, b_1, b_3) = \theta(b_1 - b_3) I_0(\sqrt{\alpha_s} b_3) K_0(\sqrt{\alpha_s} b_1) + (b_1 \leftrightarrow b_3) K_0(\sqrt{\alpha_s} b_3) H^{(1)}_0(\sqrt{\alpha_s} b_1), \\
h_{w,}(x_1, x_2, x_3, b_2, b_3) = \frac{\pi^2}{2} I_0(\sqrt{\alpha_s} b_2) [h_{w,}(b_2 - b_3) H^{(1)}_0(\sqrt{\alpha_s} b_3) + (b_2 \leftrightarrow b_3)] S_i(\alpha_s(1/m_B^2)), \\
h_{w,}(x_1, x_2, x_3, b_1, b_2) = \frac{\pi^2}{2} \theta(b_1 - b_2) h_{w,}(b_1) H^{(1)}_0(\sqrt{\alpha_s} b_2) + (b_1 \leftrightarrow b_2) K_0(\sqrt{\alpha_s} b_1), H^{(1)}_0(\sqrt{\alpha_s} b_1) + (b_1 \leftrightarrow b_2) K_0(\sqrt{\alpha_s} b_1), H^{(1)}_0(\sqrt{\alpha_s} b_1), \beta_{e,} > 0, \\
\beta_{w,} > 0, \beta_{w,} > 0.
\]

where $i, k = 1, 2$ and $j, l = 3, 4,$ and $I_0, K_0$ and $H_0$ are the Bessel functions. The threshold re-summation factor $S_\gamma(x)$ is parametrized as

\[
S_\gamma(x) = \frac{2^{1+2\epsilon/3}/(1+c)}{\sqrt{1+c}} |x(1-x)|^c, \quad \epsilon = 0.4
\]

with the parameter $c = 0.4$ in this work. The evolution factors $E_\gamma(t)$ in the factorization formulas are given by

\[
E_\gamma(t) = \alpha_s(\mu) \exp(-S_B(t) - S_{\gamma}(t)), \\
E_{\gamma}(t) = \alpha_s(\mu) \exp(-S_B(t) - S_{\gamma}(t) - S_D(t)|b=bs), \\
E_{\gamma}(t) = \alpha_s(\mu) \exp(-S_{\gamma}(t) - S_D(t)|b=bs), \\
E_{\gamma}(t) = \alpha_s(\mu) \exp(-S_B(t) - S_{\gamma}(t) - S_D(t)|b=bs),
\]

where

\[
S_B(t) = s(x, m_B, b_1) + \frac{5}{3} \int_{1/b_1}^{m_B} \frac{d\bar{y}}{\bar{y}} \gamma_2(\alpha_s(\bar{y})).
\]
\[ S_D(t) = s(x_3 m_B, b_3) + 2 \int_{1/b_3}^{t} \frac{d\mu}{\mu^2} \gamma_q(\alpha_j(\mu)), \]
\[ S_x(t) = s(x_2 m_B, b_2) + s(1 - x_2) m_B, b_2 \]
\[ + 2 \int_{1/b_2}^{t} \frac{d\mu}{\mu^2} \gamma_q(\alpha_j(\mu)), \]  
(26)

with the quark anomalous dimension \( \gamma_q = -\alpha_s/\pi \). The explicit expression for \( s(Q, b) \) can be found, for example, in Appendix A of Ref. [9].

The hard scales are chosen as
\[ t_e = \max(\sqrt{m_c^2}, \sqrt{m_b^2}, 1/b_1, 1/b_2), \]
\[ t_w = \max(\sqrt{m_c^2}, \sqrt{m_b^2}, 1/b_1, 1/b_3), \]
\[ t_{w_0} = \max(\sqrt{m_c^2}, \sqrt{m_b^2}, 1/b_1, 1/b_3), \]
\[ t_{w_0} = \max(\sqrt{m_c^2}, \sqrt{m_b^2}, 1/b_1, 1/b_2). \]  
(27)

Therefore, we obtain the total decay amplitudes,
\[ \mathcal{A}(B_s \to D_s^0 \pi^+ \pi^-) = \frac{G_F}{\sqrt{2}} V_{cb} V_{us}^* (M_{12} + M_{34}), \]
\[ + M_{12} + M_{34}). \]

The differential branching ratio for the decays \( B_s^0 \to D_s^0(D_s^0) \pi^+ \pi^- \) follows the formula given in [58, 59]
\[ \frac{d\Gamma}{dm_{\pi\pi}} = \frac{\tau_B}{m_{\pi\pi}^2} \frac{|\mathcal{P}_1||\mathcal{P}_2|^2}{4(2\pi)^4 m_B^5} \]  
(29)

with the \( B_s \) meson mean lifetime \( \tau_B \). The kinematic variables \( |\mathcal{P}_1| \) and \( |\mathcal{P}_2| \) denote the magnitudes of the \( \pi^+ \) and \( \pi^- \) momenta in the center-of-mass frame of the pion pair,
\[ |\mathcal{P}_1|^2 = \frac{1}{2} \sqrt{m_{\pi\pi}^2 - 4m_{\pi}^2}, \]
\[ |\mathcal{P}_2|^2 = \frac{1}{4m_{\pi\pi}^2} \left( m_{B_s}^2 - (m_{\pi\pi} + m_D)^2 \right) \left( m_{B_s}^2 - (m_{\pi\pi} - m_D)^2 \right). \]  
(30)

4 Numerical results

We adopt the following inputs (in units of GeV) [58, 59]
\[ \Lambda_{QCD}^\pi = 0.250, \quad m_B = 5.367, \quad m_D = 1.869, \quad m_{\pi} = 0.140, \]
\[ m_{\pi} = 0.135, \quad m_K = 0.494, \]
\[ m_K = 0.498, \quad m_b = 4.66, \quad m_s = 0.095. \]
\[ \tau_B = 1.512 \times 10^{-12}s, \quad G_F = 1.166 \times 10^{-5}, \]  
and the CKM matrix elements are taken as:
\[ |V_{ub}| = 0.2252, \quad |V_{ub}| = 3.89 \times 10^{-3}, \]
\[ |V_{us}| = 0.97345, \quad |V_{us}| = 40.6 \times 10^{-3}. \]

The parameters of the scalar form factor \( F_s(m_{\pi\pi}^2) \) are extracted from the LHCb data for the process \( B_s \to J/\psi \pi^+ \pi^- \), given in [48, 60] (mass and widths are given in units of GeV):
\[ m(f_0(500)) = 0.5, \quad m(f_0(980)) = 0.97, \]
\[ m(f_0(1500)) = 1.5, \quad m(f_0(1790)) = 1.81, \]
\[ \Gamma(f_0(500)) = 0.4, \quad \Gamma(f_0(1500)) = 0.12, \quad \Gamma(f_0(1790)) = 0.32, \]
\[ g_{\pi\pi} = 0.167, \quad g_{K\pi} = 3.47g_{\pi\pi}, \]
\[ c_0 = 3.500, \quad c_1 = 0.900, \quad c_2 = 0.106, \quad c_3 = 0.066, \]
\[ \theta_1 = \frac{\pi}{2}, \quad \theta_2 = \frac{\pi}{4}, \quad \theta_3 = 0. \]

We calculate the branching ratios for the different resonances in the S-wave pion-pair wave function, which are given in Table 1. In this table, the first uncertainties are from \( \omega_b = 0.50 \pm 0.05 \) in the \( B_s \) wave function, the second arise from \( \alpha_2 = 0.2 \pm 0.2 \) in the pion-pair wave function, and the third are from the QCD scale \( \Lambda = 0.25 \pm 0.05 \). The errors from the parameter \( C_D \) in the D meson wave function, the variations of the CKM matrix elements and the mean lifetime of \( B_s \) are small and have been omitted. However, the above results are sensitive to \( \omega_b \) and \( \alpha_2 \), namely the \( B_s \) and S-wave two-pion wave functions. Future measurements of decay branching ratios will be valuable for understanding \( B_s \) physics and the S-wave two-pion resonances.

Including all S-wave resonances \( f_0(500), f_0(980), f_0(1500) \) and \( f_0(1790) \) in the scalar form factor, we obtain the total branching ratio

| Resonances | Branching ratio (x10^-6) |
|------------|--------------------------|
| \( B_s^0 \to D_s^0 f_0(500)(f_0(500) \to \pi^+ \pi^-) \) | 0.13^{+0.04}_{-0.02}(\omega_b) \cdot 0.01(\alpha_2) \cdot 0.01(\Lambda_{QCD}) |
| \( B_s^0 \to D_s^0 f_0(980)(f_0(980) \to \pi^+ \pi^-) \) | 0.45^{+0.15}_{-0.12}(\omega_b) \cdot 0.13(\alpha_2) \cdot 0.11(\Lambda_{QCD}) |
| \( B_s^0 \to D_s^0 f_0(1500)(f_0(1500) \to \pi^+ \pi^-) \) | 0.11^{+0.04}_{-0.03}(\omega_b) \cdot 0.03(\alpha_2) \cdot 0.01(\Lambda_{QCD}) |
| \( B_s^0 \to D_s^0 f_0(1790)(f_0(1790) \to \pi^+ \pi^-) \) | 0.03^{+0.02}_{-0.01}(\omega_b) \cdot 0.02(\alpha_2) \cdot 0.001(\Lambda_{QCD}) |
| \( B_s^0 \to D_s^0 f_0(980)(f_0(980) \to \pi^+ \pi^-) \) | 0.16^{+0.08}_{-0.05}(\omega_b) \cdot 0.11(\alpha_2) \cdot 0.01(\Lambda_{QCD}) |
| \( B_s^0 \to D_s^0 f_0(1500)(f_0(1500) \to \pi^+ \pi^-) \) | 0.03^{+0.04}_{-0.02}(\omega_b) \cdot 0.02(\alpha_2) \cdot 0.001(\Lambda_{QCD}) |
| \( B_s^0 \to D_s^0 f_0(1790)(f_0(1790) \to \pi^+ \pi^-) \) | 0.011^{+0.004}_{-0.005}(\omega_b) \cdot 0.003(\alpha_2) \cdot 0.001(\Lambda_{QCD}) |
\[ \mathcal{B}(\bar{B}_s^0 \rightarrow D^{0}(\pi^+\pi^-)_{S}) = 0.77_{-0.18}^{+0.19}(\omega_{b})^{+0.00}_{-0.20}(\alpha_{S})^{+0.11}_{-0.12} \]
\[ \times (\Lambda_{\text{QCD}}) \times 10^{-6}, \]
\[ \mathcal{B}(\bar{B}_s^0 \rightarrow \bar{D}^{0}(\pi^+\pi^-)_{S}) = 0.47_{-0.15}^{+0.19}(\omega_{b})^{+0.00}_{-0.33}(\alpha_{S})^{+0.02}_{-0.05} \]
\[ \times (\Lambda_{\text{QCD}}) \times 10^{-6}. \]

We found the contributions of $\bar{B}_s^0 \rightarrow D^0 f_0(500)[f_0(500) \rightarrow \pi^+\pi^-]$, $\bar{B}_s^0 \rightarrow D^0 f_0(980)[f_0(980) \rightarrow \pi^+\pi^-]$, $\bar{B}_s^0 \rightarrow D^0 f_0(1500)$, $[f_0(1500) \rightarrow \pi^+\pi^-]$ and $\bar{B}_s^0 \rightarrow D^0 f_0(1790)[f_0(1790) \rightarrow \pi^+\pi^-]$ to be respectively 16.4\%, 59.3\%, 14.6\% and 4.5\% of the total $\bar{B}_s^0 \rightarrow D^0(\pi^+\pi^-)_{S}$ decay rate. For the $\bar{B}_s^0 \rightarrow D^0(\pi^+\pi^-)_{S}$ process, the corresponding rates are respectively 24.6\%, 35.2\%, 8.3\% and 2.4\%. This indicates that the $f_0(500)$ and $f_0(980)$ contributions are dominant, and that the contribution from $f_0(980)$ is larger than $f_0(500)$ in the $D^0(\bar{D}^0)$ final state. LHCb collaboration measured the upper limit of the branching ratio of $\mathcal{B}(\bar{B}_s^0 \rightarrow \bar{D}^0 f_0(980)) < 3.1 \times 10^{-6}$ [61], which roughly agrees with our value.

In order to compare the two channels $\bar{B}_s \rightarrow D^0(\pi\pi)_{S}$ and $\bar{B}_s \rightarrow D^0(\pi\pi)_{S}$, we determine the rate of their branching ratios
\[ R_1 = \frac{\mathcal{B}(\bar{B}_s^0 \rightarrow D^{0}(\pi^+\pi^-)_{S})}{\mathcal{B}(\bar{B}_s^0 \rightarrow D^{0}(\pi^+\pi^-)_{S})} \approx 1.64, \tag{32} \]
which significantly deviates from the ratio of the CKM factors:
\[ R_{\text{CKM}} = \frac{V_{cb}V_{us}^{*}}{V_{ub}V_{cb}^{*}} \approx 5.83. \tag{33} \]

In these two decays, there are competition effects from the CKM factors and dynamical decay amplitudes. In these processes, the dominant contributions come from the emission diagrams and non-factorizable $W$-exchange diagrams. Although the emission diagrams result in similar factorization formulas and numerical results for the two channels, the formulas for the non-factorizable $W$-exchange diagrams are different. We found that the non-factorizable $W$-exchange process for $\bar{B}_s^0 \rightarrow D^0\pi^+\pi^-$ is numerically larger than for $\bar{B}_s^0 \rightarrow D^0(\pi^+\pi^-)$, with the CKM factor reversed. As a result, their final branching ratios are similar.

The CKM element for $\bar{B}_s^0 \rightarrow D^0(D^0)(\pi^+\pi^-)_{S}$ is $V_{cb}V_{us}^{*}$ ($V_{ub}V_{cb}^{*}$), where $V_{ub}$ is sensitive to $\gamma$. Therefore, we can get the dependence of our results on $\gamma$ by providing a parameter $D_{\text{CP}}$ defined as [62]
\[ \sqrt{2} A(\bar{B}_s^0 \rightarrow D_{\text{CP}}(\pi^+\pi^-)_{S}) = V_{cb}V_{us}^{*} \]
\[ \pm A(\bar{B}_s^0 \rightarrow D^0(\pi^+\pi^-)_{S}). \tag{34} \]

Fig. 2. (color online) The dependence of the differential branching ratios $\mathcal{B}(\bar{B}_s^0 \rightarrow D_{\text{CP}}(\pi^+\pi^-)_{S})$ on $\gamma$ are shown in panels (a,b). In panels (c,d), the corresponding physical observable that is measured $R_{\text{CP}}$ is shown as function of $\gamma$. The shaded (green) regions denote the current bound $\gamma = 73.5_{-1.9}^{+4.2}$. 

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Accordingly, the dependence of the branching ratio \( \mathcal{B}(B_s^{0} \rightarrow D_{CP}(\pi^+\pi^-)_S) \) on \( \gamma \) is shown in Fig. 2(a,b). The corresponding physical observable measured by the experiments is defined as

\[
R_{CP} = \frac{4\mathcal{B}(B_s^{0} \rightarrow D_{CP}(\pi^+\pi^-)_S)}{\mathcal{B}(B_s^{0} \rightarrow D^{0}(\pi^+\pi^-)_S) + \mathcal{B}(B_s^{0} \rightarrow D^{0}(\pi^-\pi^+)_S)}. ~ (35)
\]

The dependence of \( R_{CP} \) on \( \gamma \) is shown in Fig. 2(c,d). The current bound for \( \gamma \) is \( \gamma = (73.5 \pm 1.2)\% \) \([63]\).

The predicted dependence of the differential branching ratio \( d\mathcal{B}/dm_{\pi\pi} \) on the pion-pair invariant mass \( m_{\pi\pi} \) is presented in Fig. 3(a) and Fig. 3(b) for the resonances \( f_0(500), f_0(980), f_0(1500) \) and \( f_0(1790) \) in the decays \( B_s \rightarrow D^{0}\pi^+\pi^- \) and \( B_s \rightarrow D^{0}\pi^-\pi^+ \). The figures show that the main contribution to the two decays lies in the region around the pole mass \( m_{f_0(980)} = 0.97 \), while \( f_0(500) \) gives a contribution primarily in the region below \( m_{\pi\pi} = 1 \) GeV. The other resonances, \( f_0(1500) \) and \( f_0(1790) \), still give considerable contributions to the processes. Therefore, we hope that more precise data from LHCb and the future KEKB may test our theoretical calculations.

5 Conclusions

In the past decades, two-body \( B \) decays have provided an ideal platform for extracting the Standard Model parameters, and for probing new physics beyond SM \([64, 65]\).

In this work, we studied the three-body decays \( B_s^{0} \rightarrow D^{0}(\pi^+\pi^-) \) within the PQCD framework, and in particular the \( S \)-wave contribution which was explicitly calculated. The \( S \)-wave two-pion light-cone distribution amplitudes can have both resonant \( f_0(500), f_0(980), f_0(1500), f_0(1790) \) and nonresonant contributions. Furthermore, the processes proceed via tree level operators, and the branching ratios were found to be in the range from \( 10^{-7} \) to \( 10^{-6} \). It was found that the branching ratios are sensitive to the parameters \( \omega_0 \) and \( \omega_2 \) in the \( B_s \) and two-pion distribution amplitudes. Therefore, we expect that future measurement could help to better understand the multi-body processes and the \( S \)-wave two-pion resonance and \( B_s \) distribution amplitudes.

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