TENSOR PRODUCTS AND THE SEMI-BROWDER JOINT SPECTRA

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ABSTRACT. Given two complex Banach spaces $X_1$ and $X_2$, a tensor product of $X_1$ and $X_2$, $X_1 \hat{\otimes} X_2$, in the sense of J. Eschmeier ([5]), and two finite tuples of commuting operators, $S = (S_1, \ldots, S_n)$ and $T = (T_1, \ldots, T_m)$, defined on $X_1$ and $X_2$ respectively, we consider the $(n+m)$-tuple of operators defined on $X_1 \hat{\otimes} X_2$, $(S \otimes I, I \otimes T) = (S_1 \otimes I, \ldots, S_n \otimes I, I \otimes T_1, \ldots, I \otimes T_m)$, and we give a description of the semi-Browder joint spectra introduced by V. Kordula, V. Müller and V. Rakočević in [7] and of the split semi-Browder joint spectra (see section 3), of the $(n+m)$-tuple $(S \otimes I, I \otimes T)$, in terms of the corresponding joint spectra of $S$ and $T$. This result is in some sense a generalization of a formula obtained for other various Browder spectra in Hilbert spaces and for tensor products of operators and for tuples of the form $(S \otimes I, I \otimes T)$. In addition, we also describe all the mentioned joint spectra for a tuple of left and right multiplications defined on an operator ideal between Banach spaces in the sense of [5].

1. Introduction

Given a complex Banach space $X$, V. Kordula, V. Müller and V. Rakočević extended in [7] the notion of upper and lower semi-Browder spectrum of an operator to $n$-tuples of commuting operators, and they proved the main spectral properties for this joint spectra, i.e., the compactness, nonemptiness, the projection property and the spectral mapping property.

On the other hand, there are other many joint Browder spectra, for example, we may consider the one introduced by R. E. Curto and A. T. Dash in [2], $\sigma_b$, and the joint Browder spectra defined by A. T. Dash in [3], $\sigma_b^1$, $\sigma_b^2$ and $\sigma_b^T$. By the observation which follows Definition 4 in [3] and the Example in [7], we have that the Browder spectra of V. Kordula, V. Müller and V. Rakočević, $\sigma_{B_u}$ and $\sigma_{B_v}$, differ, in general, from the other mentioned joint Browder spectra. However, if we consider two complex Hilbert spaces $H_1$ and $H_2$, and $S$ and $T$ two operators defined on $H_1$ and $H_2$ respectively, by [2] and [3, Theorem 7] we have that the joint Browder spectra $\sigma_b$, $\sigma_b^1$, $\sigma_b^2$ and $\sigma_b^T$ of the tuple of operators $(S \otimes I, I \otimes T)$ defined on $H_1 \otimes H_2$, coincide with the set

$$\sigma_b(S) \times \sigma(T) \cup \sigma(S) \times \sigma_b(T),$$

where $\sigma$ and $\sigma_b$ denote, respectively, the usual and the Browder spectrum of an operator.

Moreover, if $S = (S_1, \ldots, S_n)$, respectively $T = (T_1, \ldots, T_m)$, is an $n$-tuple, respectively an $m$-tuple, of commuting operators defined on the Hilbert space $H_1$, respectively $H_2$, R. E. Curto and A. T. Dash computed in [2] the Browder
spectrum of the \((n+m)\)-tuple \((S \otimes I, I \otimes T) = (S_1 \otimes I, \ldots, S_n \otimes I, I \otimes T_1, \ldots, I \otimes T_m)\), and they obtained the formula
\[
\sigma_b(S \otimes I, I \otimes T) = \sigma_b(S) \times \sigma_T(T) \cup \sigma_T(T) \times \sigma_b(T),
\]
where \(\sigma_T\) denotes the Taylor joint spectrum (see [9]).

In this article we give in some sense a generalization of the above formulas for commutative tuples of Banach spaces operators and for the semi-Browder joint spectra. Indeed, we consider two complex Banach spaces, \(X_1\) and \(X_2\), a tensor product between \(X_1\) and \(X_2\) in the sense of J. Eschmeier ([5]) \(X_1 \tilde{\otimes} X_2\), \(S\) and \(T\), two commuting tuples of Banach space operators defined on \(X_1\) and \(X_2\) respectively, and we describe the semi-Browder joint spectra introduced in [7], \(\sigma_{B+}\) and \(\sigma_{B-}\), and the split semi-Browder joint spectra \(sp_{B+}\) and \(sp_{B-}\) (see section 3) of the tuple \((S \otimes I, I \otimes T)\), in terms of the corresponding semi-Browder joint spectra and of the defect and the approximate point spectra of \(S\) and \(T\). The results that we have obtained extend in same way the above formulas, see section 5. Furthermore, since for our objective we need to know the Fredholm joint spectra of J.J. Buoni, R. Harte and T. Wickstead of \((S \otimes I, I \otimes T)\) ([1]) and its split versions ([4]) we also describe in section 4 these joint spectra.

In addition, by similar arguments we describe in section 6 all the mentioned joint spectra for a tuple of left and right multiplications defined on an operator ideal between Banach spaces in the sense of [5].

However, in order to give our descriptions, we need to introduce the split semi-Browder joint spectra of a tuple of commuting Banach space operators, and to prove their main spectral properties (see section 3).

The article is organized as follows. In section 2 we recall several definitions and results which we need for our work. In section 3 we introduce the split semi-Browder joint spectra and prove their main spectral properties. In section 4 we compute the semi-Fredholm joint spectra of \((S \otimes I, I \otimes T)\). In section 5 we compute the semi-Browder joint spectra of \((S \otimes I, I \otimes T)\), and in section 6, the semi-Fredholm and the semi-Browder joint spectra of a tuple of left and right multiplications defined on an operator ideal between Banach spaces in the sense of [5].

2. Preliminaries

Let us begin our work by recalling the definitions of the lower semi-Fredholm and of the lower semi-Browder joint spectra of a finite tuple of operators defined on a complex Banach space, for a complete exposition see [1] and [7].

Let \(T = (T_1, \ldots, T_n)\) be an \(n\)-tuple of commuting operators defined on a Banach space \(X\), and for \(k \in \mathbb{N}\) define \(M_k(T) = R(T_1^k) + \ldots + R(T_n^k)\). Clearly \(X \supseteq M_1(T) \supseteq M_2(T) \supseteq \ldots \supseteq M_k(T) \supseteq \ldots\) Let us set \(R^\infty(T) = \cap_{k=1}^{\infty} M_k(T)\). We now may recall the definition of the lower semi-Browder joint spectrum (see [7]).

We say that \(T = (T_1, \ldots, T_n)\) is lower semi-Browder if \(\text{codim } R^\infty(T) < \infty\). The set of all lower semi-Browder \(n\)-tuples is denoted by \(B_{\leq}^{(n)}(X)\), and the lower semi-Browder spectrum is the set
\[
\sigma_{B_{\leq}}(T) = \{\lambda \in \mathbb{C}^n : T - \lambda \notin B_{\leq}^{(n)}(X)\},
\]
where \( \lambda = (\lambda_1, \ldots, \lambda_n) \) and \( T - \lambda = (T_1 - \lambda_1 I, \ldots, T_n - \lambda_n I) \).

As usual (see [1]), we say that \( T = (T_1, \ldots, T_n) \) is lower semi-Fredholm, i.e.,
\( T \in \Phi_-(n)(X) \), if
\[
\text{codim } M_1(T) = \text{codim } (R(T_1) + \ldots + R(T_n)) < \infty,
\]
equivalently, if the operator \( \hat{T} : X^n \to X \) defined by \( \hat{T}(x_1, \ldots, x_n) = T_1(x_1) + \ldots + T_n(x_n) \) is lower semi-Fredholm, i.e., \( R(\hat{T}) \) is closed and has finite codimension.

The lower semi-Fredholm spectrum is the set
\[
\sigma_{\Phi_-}(T) = \{ \lambda \in \mathbb{C}^n : T - \lambda \notin \Phi_-(n)(X) \}.
\]

An easy calculation shows that
\[
\sigma_{\Phi_-}(T) \subseteq \sigma_{\mathcal{B}_-}(T) \subseteq \sigma_{\delta}(T),
\]
where \( \sigma_{\delta}(T) \) is the defect spectrum of \( T \), i.e.,
\[
\sigma_{\delta}(T) = \{ \lambda \in \mathbb{C}^n : \text{codim } M_1(T - \lambda) \neq 0 \}.
\]

Moreover, it is easy to see that the lower semi-Browder spectrum may be decomposed as the disjoint union of two sets,
\[
\sigma_{\mathcal{B}_-}(T) = \sigma_{\Phi_-}(T) \cup \mathcal{A}(T),
\]
where
\[
\mathcal{A}(T) = \{ \lambda \in \mathbb{C}^n : \forall k \in \mathbb{N}, 1 \leq \text{codim } M_k(T - \lambda) < \infty, \text{codim } M_k(T - \lambda) \underset{k \to \infty}{\longrightarrow} \infty \}.
\]

Now, we recall the definition of the upper semi-Fredholm and the upper semi-Browder joint spectra, as above, for a complete exposition see [1] and [7].

If \( T \) is an \( n \)-tuple of commuting operators defined on a Banach space \( X \), then \( T \) is said upper semi-Fredholm, i.e., \( T \in \Phi_+(n)(X) \), if the map \( \hat{T} : X \to X^n \) defined by \( \hat{T}(x) = (T_1(x), \ldots, T_n(x)) \) is upper semi-Fredholm, equivalently, if \( \hat{T} \) has finite dimensional null space and closed range. Moreover, \( T \) is said upper semi-Browder, i.e., \( T \in \mathcal{B}_+(n)(X) \), if \( T \in \Phi_+(n)(X) \) and \( \text{dim } N^\infty(T) < \infty \), where
\[
N^\infty(T) = \bigcup_{k \in \mathbb{N}} [N(T_1^k) \cap \ldots \cap N(T_n^k)].
\]

As above, the upper semi-Fredholm spectrum is the set
\[
\sigma_{\Phi_+}(T) = \{ \lambda \in \mathbb{C}^n : T - \lambda \notin \Phi_+(n)(X) \},
\]
and the upper semi-Browder spectrum is the set
\[
\sigma_{\mathcal{B}_+}(T) = \{ \lambda \in \mathbb{C}^n : T - \lambda \notin \mathcal{B}_+(n)(X) \}.
\]

In addition, it is easy to see that
\[
\sigma_{\Phi_+}(T) \subseteq \sigma_{\mathcal{B}_+}(T) \subseteq \sigma_{\pi}(T),
\]
where \( \sigma_{\pi}(T) \) denotes the approximate point spectrum of \( T \),
\[
\sigma_{\pi}(T) = \{ \lambda \in \mathbb{C}^n : N(T - \lambda) \neq 0 \text{ or } R(T - \lambda) \text{ is not closed } \}.
\]

Moreover, it is easy to see that the upper semi-Browder spectrum may be decomposed as the disjoint union of two sets,
\[
\sigma_{\mathcal{B}_+}(T) = \sigma_{\Phi_+}(T) \cup \mathcal{D}(T),
\]
where \( \mathcal{D}(T) = \{ \lambda \in \mathbb{C}^n : \forall k \in \mathbb{N}, 1 \leq \dim N_k(T - \lambda) < \infty, R(T - \lambda) \) is closed, and \( \dim N_k(T - \lambda) \xrightarrow[k \to \infty]{} \infty \}, \) where \( N_k(T - \lambda) = N((T - \lambda)^k) \) and \( (T - \lambda)^k = ((T_1 - \lambda_1)^k, \ldots, (T_n - \lambda_n)^k). \)

Let us recall that the semi-Fredholm and the semi-Browder joint spectra are compact nonempty subsets of \( \mathbb{C}^n \), which also satisfy the projection property and the analytic spectral mapping theorem for tuples of holomorphic functions defined on a neighborhood of the Taylor joint spectrum [9] (see [4] and [7]).

On the other hand, in order to prove our main results, we have to recall the axiomatic tensor product between Banach spaces introduced by J. Eschmeier in [5]. This notion will be central in this work. For a complete exposition see [5]. We proceed as follows.

A pair \( < X, \tilde{X} > \) of Banach spaces will be called a dual pairing, if

\[
(A) \tilde{X} = X' \text{ or (B) } X = \tilde{X}'.
\]

In both cases, the canonical bilinear mapping is denoted by

\[
X \times \tilde{X} \rightarrow \mathbb{C}, (x, u) \rightarrow < x, u >.
\]

If \( < X, \tilde{X} > \) is a dual pairing, we consider the subalgebra \( \mathcal{L}(X) \) of \( L(X) \) consisting of all operators \( T \in L(X) \) for which there is an operator \( T' \in L(\tilde{X}) \) with

\[
< Tx, u > = < x, T'u >,
\]

for all \( x \in X \) and \( u \in \tilde{X} \). It is clear that if the dual pairing is \( < X, X' > \), then \( \mathcal{L}(X) = L(X) \), and that if the dual pairing is \( < X', X > \), then \( \mathcal{L}(X) = \{ T^* : T \in L(\tilde{X}) \} \). In particular, each operator of the form

\[
f_{y,v} : X \rightarrow X, x \rightarrow < x, v > y,
\]

is contained in \( \mathcal{L}(X) \), where \( y \in X \) and \( v \in \tilde{X} \).

We now recall the definition of the tensor product given by J. Eschmeier in [5].

Given two dual pairings \( < X, \tilde{X} > \) and \( < Y, \tilde{Y} > \), a tensor product of the Banach spaces \( X \) and \( Y \) relative to the dual pairings \( < X, \tilde{X} > \) and \( < Y, \tilde{Y} > \), is a Banach space \( Z \) together with continuous bilinear mappings

\[
X \times Y \rightarrow Z, (x, y) \rightarrow x \otimes y;
\]

\[
\mathcal{L}(X) \times \mathcal{L}(Y) \rightarrow L(Z), (T, S) \rightarrow T \otimes S,
\]

which satisfy the following conditions,

\[
(T1) \| x \otimes y \| = \| x \| \| y \|,
\]

\[
(T2) T \otimes S(x \otimes y) = (Tx) \otimes (Sy),
\]

\[
(T3) (T_1 \otimes S_1) \circ (T_2 \otimes S_2) = (T_1 T_2) \otimes (S_1 S_2), I \otimes I = I,
\]

\[
(T4) \text{Im}(f_{x,u} \otimes I) \subseteq \{ x \otimes y : y \in Y \}, \text{Im}(f_{y,v} \otimes I) \subseteq \{ x \otimes y : x \in X \}.
\]

In this work, as in [5], instead of \( Z \) we shall often write \( X \tilde{\otimes} Y \). In addition, as in [5], we shall have two applications of this definition of tensor product. First of all, the completion \( X \tilde{\otimes}_\alpha Y \) of the algebraic tensor product of Banach spaces \( X \) and \( Y \) with respect to a quasi-uniform crossnorm \( \alpha \) (see [6]) and an operator ideal between Banach spaces (see [5] and section 6).
In section 4 and 5, given two complex Banach spaces $X_1$ and $X_2$, and two tuples of Banach spaces operators, $S$ and $T$, defined on $X_1$ and $X_2$ respectively, we shall describe the semi-Fredholm and the semi-Browder joint spectra of the tuple $(S \otimes I, I \otimes T)$, whose operators, $S_i \otimes I$ and $I \otimes T_j$, $i = 1, \ldots, n$ and $j = 1, \ldots, m$, are defined on $X_1 \otimes X_2$, a tensor product of $X_1$ and $X_2$ relative to $< X_1, X_1' >$ and $< X_2, X_2' >$. However, in the following section, we first introduce the split semi-Browder joint spectra, which will be necessary for our description.

3. The Split Semi-Browder Joint Spectra

In this section we introduce the upper and lower split semi-Browder joint spectra. We also prove their main spectral properties.

Let us consider, as in section 2, a complex Banach space $X$ and $T = (T_1, \ldots, T_n)$ a commuting tuple of operators defined on $X$. We say that $T$ is lower split semi-Browder if $R^\infty(T)$ has finite codimension and $N(\hat{T})$ has a direct complement in $X^n$, where $\hat{T}: X^n \to X$ is the map considered in section 2. We denote by $SB^{(n)}(X)$ the set of all lower split semi-Browder $n$-tuples, and the lower split semi-Browder spectrum is the set

$$sp_{B_-}(T) = \{ \lambda \in \mathbb{C}^n : T - \lambda \notin SB^{(n)}(X) \}.$$

It is clear that

$$sp_{B_-}(T) = \sigma_{B_-}(T) \cup \mathcal{C}_-(T),$$

where $\mathcal{C}_-(T) = \{ \lambda \in \mathbb{C}^n : N(\hat{T} - \lambda) \text{ has not a direct complement in } X^n \}$. In particular, $sp_{B_-}(T)$ is a nonempty set.

On the other hand, if we consider the split defect spectrum and the essential split defect spectrum of $T$ introduced in [4], $sp_\delta(T)$ and $sp_{\delta e}(T)$ respectively, sets that by [4, Theorem 2.7] may be presented as

$$sp_\delta(T) = \sigma_\delta(T) \cup \mathcal{C}_-(T), \quad sp_{\delta e}(T) = \sigma_{\delta e}(T) \cup \mathcal{C}_-(T),$$

then we have that

$$sp_{\delta e}(T) \subseteq sp_{B_-}(T) \subseteq sp_\delta(T).$$

In addition, if we consider the set $\tilde{A}(T) = \{ \lambda \in \mathbb{C}^n : \lambda \notin sp_{\delta e}(T), \forall k \in \mathbb{N}, 1 \leq \text{codim}M_k(T - \lambda) < \infty, \text{codim}M_k(T - \lambda) \xrightarrow{k \to \infty} \infty \}$, then it is clear that

$$\tilde{A}(T) \subseteq A(T) \subseteq \sigma_{\delta e}(T) \subseteq sp_{B_-}(T).$$

In particular

$$sp_{\delta e}(T) \cup \tilde{A}(T) \subseteq sp_{B_-}(T).$$

On the other hand, let us consider $\lambda \in sp_{B_-}(T)$, and let us decompose the lower split semi-Browder spectrum of $T$ as

$$sp_{B_-}(T) = \sigma_{B_-}(T) \cup \mathcal{C}_-(T) = \sigma_{\delta e}(T) \cup \tilde{A}(T) \cup \mathcal{C}_-(T).$$

Now, if $\lambda \in \sigma_{\delta e}(T) \cup \mathcal{C}_-(T)$, then $\lambda \in sp_{\delta e}(T)$. Moreover, if $\lambda \in A(T) \setminus (\sigma_{\delta e}(T) \cup \mathcal{C}_-(T))$, then $\lambda \in A(T) \setminus sp_{\delta e}(T) = \tilde{A}(T)$. Thus, we have that

$$sp_{B_-}(T) = sp_{\delta e}(T) \cup \tilde{A}(T).$$

We now introduce the upper split semi-Browder spectrum.
If $X$ and $T = (T_1, \ldots, T_n)$ are as above, then we say that $T$ is upper split semi-Browder if it is upper semi-Browder and $R(\tilde{T})$ has a direct complement in $X^n$, where $\tilde{T}: X \to X^n$ is the map considered in section 2. We denote by $SB^{(n)}_+(X)$ the set of all upper split semi-Browder $n$-tuples, and the upper split semi-Browder spectrum is the set

$$sp_{B_+}(T) = \{\lambda \in \mathbb{C}^n: T - \lambda \notin SB^{(n)}_+(X)\}.$$ 

It is clear that

$$sp_{B_+}(T) = \sigma_{B_+}(T) \cup \mathcal{C}_+(T),$$

where $\mathcal{C}_+(T) = \{\lambda \in \mathbb{C}^n: R(\tilde{T} - \lambda) \text{ has not a direct complement in } X^n\}$. In particular, $sp_{B_+}(T)$ is a nonempty set.

On the other hand, if we consider the split approximate point spectrum and the essential split approximate point spectrum of $T$ (see [4]), $sp_{\pi}(T)$ and $sp_{\pi e}(T)$ respectively, i.e., the sets

$$sp_{\pi}(T) = \sigma_{\pi}(T) \cup \mathcal{C}_+(T), \quad sp_{\pi e}(T) = \sigma_{\Phi_+}(T) \cup \mathcal{C}_+(T),$$

then we have that

$$sp_{\pi e}(T) \subseteq sp_{B_+}(T) \subseteq sp_{\pi}(T).$$

In addition, if we consider the set $\mathcal{D}(T) = \{\lambda \in \mathbb{C}^n: \lambda \notin sp_{\pi e}(T), \forall k \in \mathbb{N}, 1 \leq \dim N_k(\tilde{T} - \lambda) < \infty, \dim N_k(\tilde{T} - \lambda) \xrightarrow{k\to\infty} \infty\}$, then it is clear that

$$\mathcal{D}(T) \subseteq \mathcal{D}(T) \subseteq \sigma_{B_+}(T) \subseteq sp_{B_+}(T).$$

In particular

$$sp_{\pi e}(T) \cup \mathcal{D}(T) \subseteq sp_{B_+}(T).$$

On the other hand, let us consider $\lambda \in sp_{B_+}(T)$, and let us decompose the upper split semi-Browder spectrum of $T$ as

$$sp_{B_+}(T) = \sigma_{B_+}(T) \cup \mathcal{C}_+(T) = \sigma_{\Phi_+}(T) \cup \mathcal{D}_+(T) \cup \mathcal{C}_+(T).$$

Now, if $\lambda \in \sigma_{\Phi_+}(T) \cup \mathcal{C}_+(T)$, then $\lambda \in sp_{\pi e}(T)$. Moreover, if $\lambda \in \mathcal{D}(T) \setminus (\sigma_{\Phi_+}(T) \cup \mathcal{C}_+(T))$, then $\lambda \in \mathcal{D}(T) \setminus sp_{\pi e}(T) = \mathcal{D}(T)$. Thus, we have that

$$sp_{B_+}(T) = sp_{\pi e}(T) \cup \mathcal{D}(T).$$

We now see that the sets that we have introduced satisfy the main spectral properties.

**Proposition 3.1.** Let $X$ be a complex Banach space and $T = (T_1, \ldots, T_n)$ a commuting tuple of bounded linear operators defined on $X$. Then the sets $sp_{B_+}(T)$ and $sp_{B_+}(T)$ are compact subsets of $\mathbb{C}^n$.

**Proof.** Since $sp_{B_+}(T) = sp_{\delta e}(T) \cup \tilde{A}(T) \subseteq sp_{\delta e}(T) \cup \sigma_{B_+}(T)$, we have that $sp_{B_+}(T)$ is a bounded subset of $\mathbb{C}^n$.

On the other hand, let us consider a sequence $(\lambda_n)_{n \in \mathbb{N}} \subseteq sp_{B_+}(T)$, and $\lambda \in \mathbb{C}^n$ such that $\lambda_n \xrightarrow{n\to\infty} \lambda$. If there exists a subsequence $(\lambda_{n_k})_{k \in \mathbb{N}} \subseteq sp_{\delta e}(T)$, then $\lambda \in sp_{\delta e}(T) \subseteq sp_{B_+}(T)$. Thus, we may suppose that there is $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}, n \geq n_0$, $\lambda_n \in \tilde{A}(T)$. Moreover, we may also suppose that $\lambda \notin sp_{\delta e}(T)$.
In particular, there is an open neighborhood of \( \lambda, U \), such that \( U \cap \text{sp}_B(T) = \emptyset \), and there is \( n_1 \in \mathbb{N} \) such that \( \lambda_n \in U \), for all \( n \geq n_1 \).

However, since for all \( n \geq n_0, \lambda_n \in A(T) \subseteq A(T) \subseteq \sigma_B(T), \) then \( \lambda \in \sigma_B(T) \).

But \( \lambda \notin \sigma_{\Phi}(T) \), for \( \sigma_{\Phi}(T) \subseteq \text{sp}_B(T) \). Then, \( \lambda \in A(T) \setminus \text{sp}_B(T) = A(T) \subseteq \text{sp}_B(T) \).

By means of a similar argument, it is possible to see that \( \text{sp}_B(T) \) is a compact subset of \( \mathbb{C}^n \). \( \square \)

**Proposition 3.2.** Let \( X \) be a complex Banach space and \( T = (T_1, \ldots, T_n, T_{n+1}) \) a commuting tuple of bounded linear operators defined on \( X \). If \( \pi: \mathbb{C}^{n+1} \rightarrow \mathbb{C}^n \) denotes the projection onto the first \( n \)-coordinate, then we have that

(i) \( \pi(\text{sp}_B(T_1, \ldots, T_n, T_{n+1})) = \text{sp}_B(T_1, \ldots, T_n) \),

(ii) \( \pi(\text{sp}_{B_+}(T_1, \ldots, T_n, T_{n+1})) = \text{sp}_{B_+}(T_1, \ldots, T_n) \).

**Proof.** By [7, Corollary 7] we now that \( \pi(\sigma_B(T_1, \ldots, T_n, T_{n+1})) = \sigma_B(T_1, \ldots, T_n) \subseteq \text{sp}_B(T_1, \ldots, T_n) \).

Moreover, since \( C_-(T_1, \ldots, T_n, T_{n+1}) \subseteq \text{sp}_{\delta B}(T_1, \ldots, T_n, T_{n+1}) \), by [4, Corollary 2.6] we have that

\[
\pi(C_-(T_1, \ldots, T_n, T_{n+1})) \subseteq \pi(\text{sp}_{\delta B}(T_1, \ldots, T_n, T_{n+1})) = \text{sp}_{\delta B}(T_1, \ldots, T_n)
\]

Thus, we have that

\[
\pi(\text{sp}_{B_+}(T_1, \ldots, T_n, T_{n+1})) \subseteq \text{sp}_{B_+}(T_1, \ldots, T_n).
\]

On the other hand, by [7, Corollary 7] we also have that

\[
\sigma_B(T_1, \ldots, T_n) = \pi(\sigma_B(T_1, \ldots, T_n, T_{n+1})) \subseteq \pi(\text{sp}_{B_+}(T_1, \ldots, T_n, T_{n+1})).
\]

Furthermore, since \( C_-(T_1, \ldots, T_n) \subseteq \text{sp}_{\delta B}(T_1, \ldots, T_n) \), by [4, Corollary 2.6] we also have that \( C_-(T_1, \ldots, T_n) \subseteq \text{sp}_{\delta B}(T_1, \ldots, T_n) = \pi(\text{sp}_{\delta B}(T_1, \ldots, T_n, T_{n+1}))) \subseteq \pi(\text{sp}_{B_+}(T_1, \ldots, T_n, T_{n+1})). \) Thus,

\[
\text{sp}_{B_+}(T_1, \ldots, T_n) \subseteq \pi(\text{sp}_{B_+}(T_1, \ldots, T_n, T_{n+1})),
\]

i.e., we have proved the first statement of the proposition.

By means of a similar argument it is possible to see the second statement. \( \square \)

In the following proposition we shall see that the split semi-Browder joint spectra satisfy the analytic spectral mapping theorem.

**Proposition 3.3.** Let \( X \) be a complex Banach space and \( T = (T_1, \ldots, T_n) \) a commuting tuple of bounded linear operators defined on \( X \). Then, if \( f \in \mathcal{O}(\text{sp}(T))^m \), we have that

(i) \( f(\text{sp}_{B_+}(T_1, \ldots, T_n)) = \text{sp}_{B_+}(f(T_1, \ldots, T_n)) \),

(ii) \( f(\text{sp}_{B_+}(T_1, \ldots, T_n)) = \text{sp}_{B_+}(f(T_1, \ldots, T_n)) \),

where \( \text{sp}(T) \) denotes the split spectrum of \( T \).

**Proof.** By [4, Corollary 2.6], the split spectrum of \( T, \text{sp}(T) \), satisfies the analytic spectral mapping theorem, i.e., there is an algebra morphism

\[
\Phi: \mathcal{O}(\text{sp}(T)) \rightarrow L(X), \quad f \rightarrow f(T),
\]
such that $1(T) = I$, $z_i(T) = T_i$, $1 \leq i \leq n$, where $z_i$ denotes the projection of $\mathbb{C}^n$ onto the $i$-th coordinate, and such that the equality $sp(f(T)) = f(sp(T))$, holds for all $f \in \mathcal{O}(sp(T))^m$.

Now, as in [4], let us consider the algebra

$$A = \Phi(\mathcal{O}(sp(T))) \subseteq L(X).$$

Then, we have that the split spectrum is a spectral system on $A$, in the sense of [4, section 1].

In order to show this claim, since the split spectrum is a compact set which also satisfies the projection property ([4, Corollary 2.6]), we have only to see that if $a = (a_1, \ldots, a_n)$ is a tuple of commuting operators such that $a_i \in A$, then $sp(a) \subseteq \sigma_{\text{joint}}^a$ (the usual joint spectrum of $a$ with respect to the algebra $A$, see [4, section 1]).

In fact, if $\lambda = (\lambda_1, \ldots, \lambda_n) \in sp(a) \setminus \sigma_{\text{joint}}^a$, then there are $B_1, \ldots, B_n \in A$ such that $\sum_{i=1}^n B_i(a_i - \lambda_i I) = I$, where $I$ denotes the identity map of $X$. In particular,

$$\sum_{i=1}^n L_{B_i}(L_{a_i} - \lambda_i I_{L(X)}) = I_{L(X)}.$$

Then, $\lambda \notin \sigma(L_a)$, the Taylor joint spectrum of the tuple of left multiplication, $L_a = (L_{a_1}, \ldots, L_{a_n})$, defined on $L(X)$. However, by [4, Corollary 2.5], $\lambda \notin sp(a)$, which is impossible by our assumption.

Now, since $sp_{B_-(T_1, \ldots, T_n)}$ and $sp_{B_+(T_1, \ldots, T_n)}$ are contained in $sp(T)$, by Propositions 3.1 and 3.2, $sp_{B_-(T_1, \ldots, T_n)}$ and $sp_{B_+(T_1, \ldots, T_n)}$ are spectral systems on $A$ contained in $sp(T)$. Then, by [4, Theorem 1.2] and [4, Corollary 1.3], since the split spectrum is a spectral system on $A$ which satisfy the analytic spectral mapping theorem, $sp_{B_-(T_1, \ldots, T_n)}$ and $sp_{B_+(T_1, \ldots, T_n)}$ also satisfy the analytic spectral mapping theorem defined on $\mathcal{O}(sp(T))$. \hfill $\square$

In the following section we give a description of the semi-Fredholm joint spectra of the system $(S \otimes I, I \otimes T)$, which will be a central step for one of the main theorems of the present article.

4. THE SEMI-FREDHOLM JOINT SPECTRA

In this section we consider two complex Banach spaces $X_1$ and $X_2$, two tuples of bounded linear operators defined on $X_1$ and $X_2$, $S = (S_1, \ldots, S_n)$ and $T = (T_1, \ldots, T_n)$ respectively, and we describe the semi-Fredholm joint spectra of the $(n + m)$-tuple $(S \otimes I, I \otimes T)$ defined on $X_1 \bar{\otimes} X_2$, a tensor product between $X_1$ and $X_2$ relative to $< X_1, X_1' >$ and $< X_2, X_2' >$, where $(S \otimes I, I \otimes T) = (S_1 \otimes I, \ldots, S_n \otimes I, I \otimes T_1, \ldots, I \otimes T_m)$.

We recall that if $K_1$, $K_2$ and $K$ are the Koszul complexes associated to the tuples $S$, $T$ and $(S \otimes I, I \otimes T)$ respectively (see [9]), i.e., $K_1 = (X_1 \otimes \mathbb{C}^n, d_1)$, $K_2 = (X_2 \otimes \mathbb{C}^m, d_2)$ and $K = (X_1 \otimes X_2 \otimes \mathbb{C}^{n+m}, d_{12})$, then, by [5, section 3] we have that $K$ is isomorphic to the total complex of the double complex obtained from the tensor product of the complexes $K_1$ and $K_2$: we denote this total complex by $K_1 \bar{\otimes} K_2$. Moreover, if we consider the differential spaces associated to $K_1$, $K_2$,
respectively. Then, for the tuple $(\mu / K\ K)$ of the differential spaces $K\ X$, we consider two tuples of commuting operators defined on $K\ X$.

Let us consider $\partial = \partial_1 \otimes I + \eta \otimes \partial_2$, where $\eta$ is the map, $\eta: K_2 \to K_2$, $\eta \mid X_2 \otimes \wedge^m \mathbb{C} = (-1)^m I$ (for a complete exposition see [5, section 3]).

In the following proposition we describe the defect, the approximate point spectrum, and the split version of these spectra for the tuple $(S \otimes I, I \otimes T)$. This result is necessary for our description of the semi-Fredholm joint spectra.

**Proposition 4.1.** Let $X_1$ and $X_2$ be two complex Banach spaces, and $X_1 \tilde{\otimes} X_2$ a tensor product of $X_1$ and $X_2$ relative to $< X_1, X_1' >$ and $< X_2, X_2' >$. Let us consider two tuples of commuting operators defined on $X_1$ and $X_2$, $S$ and $T$ respectively. Then, for the tuple $(S \otimes I, I \otimes T)$, defined on $X_1 \tilde{\otimes} X_2$, we have that

(i) $\sigma_\delta(S) \times \sigma_\delta(T) \subseteq \sigma_\delta(S \otimes I, I \otimes T) \subseteq \sigma_\delta(S \otimes I, I \otimes T)$, and

(ii) $\sigma_\pi(S) \times \sigma_\pi(T) \subseteq \sigma_\pi(S \otimes I, I \otimes T) \subseteq \sigma_\pi(S \otimes I, I \otimes T)$.

In addition, if $X_1$ and $X_2$ are Hilbert spaces, the above inclusions are equalities.

**Proof.** Let us consider $\lambda \in \mathbb{C}^n$, $\mu \in \mathbb{C}^m$ and the Koszul complexes associated to $S - \lambda$, $T - \mu$, and $(S \otimes I, I \otimes T) - (\lambda, \mu) = ((S - \lambda) \otimes I, I \otimes (T - \mu))$, which we denote by $K_1$, $K_2$ and $K$. By the previous observation we have that $K \cong K_1 \tilde{\otimes} K_2$.

Moreover, if we consider the differential spaces associated to these complexes, $K_1$, $K_2$, and $K$, then we have that $K \cong K_1 \tilde{\otimes} K_2$.

Now, we may apply [5, Theorem 2.2] to the differential spaces $K_1$, $K_2$, and $K_1 \tilde{\otimes} K_2$. However, by the definition of the map $\varphi$ in [5, Theorem 2.2], the grading of the differential spaces $K_1$, $K_2$, and $K_1 \tilde{\otimes} K_2$, and of the isomorphism $K \cong K_1 \tilde{\otimes} K_2$, we have the left hand side inclusion of the first statement.

The middle inclusion is clear.

Let us now suppose that $(\lambda, \mu) \notin \sigma_\delta(S) \times \sigma_\delta(T)$. Then, either $\lambda \notin \sigma_\delta(S)$ or $\mu \notin \sigma_\delta(T)$. We shall see that if $\lambda \notin \sigma_\delta(S)$, then $(\lambda, \mu) \notin \sigma_\delta(S \otimes I, I \otimes T)$. By means of a similar argument it is possible to see that if $\mu \notin \sigma_\delta(T)$ then $(\lambda, \mu) \notin \sigma_\delta(S \otimes I, I \otimes T)$.

Now, if $\lambda \notin \sigma_\delta(S)$, there is a bounded linear operator $h: X_1 \to X_1 \otimes \wedge^n \mathbb{C}$ such that $d_{11} \circ h = I$, where $d_{11}: X_1 \otimes \wedge^n \mathbb{C} \to X_1$ is the chain map of the Koszul complex $K_1$ at level $p = 1$.

Let us consider the map $H: X_1 \tilde{\otimes} X_2 \to X_1 \otimes \wedge^n \mathbb{C} \otimes X_2$, $H = h \otimes I$.

Then, by the properties of the tensor product introduced in [5], $H$ is a well defined map which satisfies

$$d_1 \circ H = d_{11} \circ h \otimes I = I \otimes I = I,$$

where $d_1$ is the chain map of the complex $K_1 \tilde{\otimes} K_2$ at level $p = 1$. Since $K \cong K_1 \tilde{\otimes} K_2$, we have that $(\lambda, \mu) \notin \sigma_\delta(S \otimes I, I \otimes T)$.

The second statement may be proved by means of a similar argument. \( \square \)
In the following proposition we state our description of the semi-Fredholm joint spectra.

**Proposition 4.2.** Let $X_1$ and $X_2$ be two complex Banach spaces, and $X_1 \widehat{\otimes} X_2$ a tensor product of $X_1$ and $X_2$ relative to $< X_1, X_1' >$ and $< X_2, X_2' >$. Let us consider two tuples of commuting operators defined on $X_1$ and $X_2$, $S$ and $T$ respectively. Then, for the tuple $(S \otimes I, I \otimes T)$, defined on $X_1 \widehat{\otimes} X_2$, we have that

(i) $\sigma_{\Phi_-}(S) \times \sigma_{\delta}(T) \cup \sigma_{\Phi_-}(S) \times \sigma_{\Phi_-}(T) \subseteq \sigma_{\Phi_-}(S \otimes I, I \otimes T) \subseteq sp_{\delta e}(S \otimes I, I \otimes T) \subseteq sp_{\delta e}(S) \times sp_{\delta}(S) \times sp_{\delta e}(T),$

(ii) $\sigma_{\Phi_+}(S) \times \sigma_{\pi}(T) \cup \sigma_{\pi}(S) \times \sigma_{\Phi_+}(T) \subseteq \sigma_{\Phi_+}(S \otimes I, I \otimes T) \subseteq sp_{\pi e}(S \otimes I, I \otimes T) \subseteq sp_{\pi e}(S) \times sp_{\pi}(S) \times sp_{\pi e}(T).$

In addition, if $X_1$ and $X_2$ are Hilbert spaces, the above inclusions are equalities.

**Proof.** First of all, let us observe that we use the same notations of Proposition 4.1.

With regard to the first statement, in order to prove the left hand side inclusion, it is enough to adapt for this case the argument that we have developed in Proposition 4.1. For this purpose, we may consider the well defined map $\delta: X_1 \to X_1 \otimes \wedge^1 \mathbb{C}^n$, $g: X_2 \to X_2 \otimes \wedge^1 \mathbb{C}^n$, and two compact operators $k_1: X_1 \to X_1$ and $k_2: X_2 \to X_2$ such that

$$d_{11} \circ h = I - k_1, \quad d_{21} \circ g = I - k_2,$$

where $d_{21}$ is the boundary map of the complex $K_2$ at level $p = 1$. Moreover, by an argument similar to [4, Theorem 2.7] or [5, Proposition 2.1], the maps $k_i$, $i = 1, 2$, may be chosen as finite rank projectors.

In addition, by the properties of the tensor product introduced in [5], we may consider the well defined map

$$H: X_1 \widehat{\otimes} X_2 \to (K_1 \widehat{\otimes} K_2)_1, \quad H = (h \otimes I, I \otimes g).$$

Now, an easy calculation shows that $d_1 \circ H = I - k_1 \otimes k_2$, where $d_1$ denotes the chain map of the complex $K_1 \widehat{\otimes} K_2$ at level $p = 1$. However, it is not difficult to see, using in particular [5, Lemma 1.1], that $k_1 \otimes k_2$ is a finite rank projector whose range coincide with $R(k_1) \otimes R(k_2)$. In particular, $k_1 \otimes k_2$ is a compact operator. Thus, since $K \cong K_1 \widehat{\otimes} K_2$, $(\lambda, \mu) \notin sp_{\delta e}(S \otimes I, I \otimes T)$, which is impossible by our assumptions.

By means of a similar argument it is possible to see the second statement. □
5. The semi-Browder joint spectra

In this section we give our description of the semi-Browder joint spectra of the tuple \((S \otimes I, I \otimes T)\). The following theorem is one of the main results of the present article.

**Theorem 5.1.** Let \(X_1\) and \(X_2\) be two complex Banach spaces, and \(X_1 \ard X_2\) a tensor product of \(X_1\) and \(X_2\) relative to \(< X_1, X_1' >\) and \(< X_2, X_2' >\). Let us consider two tuples of commuting operators defined on \(X_1\) and \(X_2\), \(S\) and \(T\) respectively. Then, for the tuple \((S \otimes I, I \otimes T)\), defined on \(X_1 \otimes X_2\), we have that

\[
\begin{align*}
\sigma_{B-}(S) \times \sigma_{\delta}(T) &\subseteq \sigma_{B-}(S \otimes I, I \otimes T), \\
sp_{B-}(S \otimes I, I \otimes T) &\subseteq \sigma_{B-}(S) \times sp_{\delta}(T) \cup sp_{\delta}(S) \times \sigma_{B-}(T),
\end{align*}
\]

\[
\begin{align*}
(i) \quad &\sigma_{B+}(S) \times \sigma_{\pi}(T) \cup \sigma_{\pi}(S) \times \sigma_{B+}(T) \subseteq \sigma_{B+}(S \otimes I, I \otimes T) \\
&sp_{B+}(S \otimes I, I \otimes T) \subseteq \sigma_{B+}(S) \times \sigma_{\pi}(T) \cup \sigma_{\pi}(S) \times \sigma_{B+}(T),
\end{align*}
\]

In addition, if \(X_1\) and \(X_2\) are Hilbert spaces, the above inclusions are equalities.

**Proof.** First of all, as in Proposition 4.2, we use the notations of Proposition 4.1.

Let us consider \((\lambda, \mu) \in \sigma_{B-}(S) \times \sigma_{\delta}(T)\). If \(\lambda \in \sigma_{\Phi-}(S)\), then, by Proposition 4.2, \((\lambda, \mu) \in \sigma_{\Phi-}(S) \times \sigma_{\delta}(T) \subseteq \sigma_{B-}(S \otimes I, I \otimes T) \subseteq \sigma_{B-}(S \otimes I, I \otimes T)\).

Now, if \(\lambda \in \mathcal{A}(S)\), since \(\mu \in \sigma_{\delta}(T)\), by the definition of the map \(\phi\) in [5, Theorem 2.2], the grading of the complex \(K_1, K_2\), and \(K_1 \ard K_2\), and by the isomorphism \(K \cong K_1 \ard K_2\), we have that \(\dim H_0(K) = \dim H_0(K_1 \ard K_2) \geq \dim H_0(K_1) \times \dim H_0(K_2) \geq 1\). In particular, \((\lambda, \mu) \in \sigma_{\delta}(S \otimes I, I \otimes T)\).

Moreover, if \(\dim H_0(K) = \infty\), then \((\lambda, \mu) \in \sigma_{\Phi-}(S \otimes I, I \otimes T) \subseteq \sigma_{B-}(S \otimes I, I \otimes T)\).

On the other hand, if we suppose that \((\lambda, \mu) \notin \sigma_{\Phi-}(S \otimes I, I \otimes T)\). Then, we consider the tuples of operators \((S - \lambda)^l = ((S_1 - \lambda_1), \ldots, (S_n - \lambda_n))\) and \((T - \mu)^l = ((T_1 - \mu_1), \ldots, (T_m - \mu_m))\), and we denote by \(K_1^l\) and \(K_2^l\) the Koszul complexes associated to the tuples \((S - \lambda)^l\) and \((T - \mu)^l\), respectively. Moreover, if we denote by \(K^l\) the Koszul complex associated to the tuple \((S - \lambda)^l \otimes I, I \otimes (T - \mu)^l\), as above, \(K^l\) is isomorphic to the total complex of the double complex of the tensor product of \(K_1^l\) and \(K_2^l\), i.e., \(K^l \cong K_1^l \otimes K_2^l\).

In addition, as we have seen for the complexes \(K_1, K_2, K,\) and \(K_1 \ard K_2\), we have that \(\dim H_0(K_1^l) = \dim H_0(K_1^l \otimes K_2^l) \geq \dim H_0(K_1^l) \times \dim H_0(K_2^l)\). Now, since \(\mu \in \sigma_{\delta}(T)\), by the analytic spectral mapping theorem for the defect spectrum (see \([4, Corollary 2.1]\), we have that \(\dim H_0(K_1^l) = \text{codim} M_{\ell}(S - \lambda)\), and since \(\lambda \in \mathcal{A}(S)\), then \(\dim H_0(K_1^l) \xrightarrow{l \to \infty} \infty\). However, \(\dim H_0(K^l) = \text{codim} M_{\ell}((S - \lambda) \otimes I, I \otimes (T - \mu))\). In particular, \((\lambda, \mu) \in \mathcal{A}(S \otimes I, I \otimes T) \subseteq \sigma_{B-}(S \otimes I, I \otimes T)\).

By means of a similar argument it is possible to see that \(\sigma_{\delta}(S) \times \sigma_{B-}(T) \subseteq \sigma_{B-}(S \otimes I, I \otimes T)\).
The middle inclusion is clear.

In order to see the right hand inclusion, let us consider \((\lambda, \mu) \in sp_{\mathcal{B}}(S \times I, I \otimes T)\). If \((\lambda, \mu) \in sp_{\mathcal{B}}(S \times I, I \otimes T)\), then by Proposition 4.2, \((\lambda, \mu) \in sp_{\mathcal{B}}(S) \times sp_{\mathcal{B}}(T) \subseteq sp_{\mathcal{B}}(S) \times sp_{\mathcal{B}}(T) \cup sp_{\mathcal{B}}(S) \cup sp_{\mathcal{B}}(T)\).

On the other hand, if \((\lambda, \mu) \in \mathcal{A}(S \otimes I, I \otimes T)\), since by Proposition 4.1 \(sp_{\mathcal{B}}(S \times I, I \otimes T) \subseteq sp_{\mathcal{B}}(S) \times sp_{\mathcal{B}}(T) \subseteq sp_{\mathcal{B}}(S) \times sp_{\mathcal{B}}(T)\), then \(\lambda \notin sp_{\mathcal{B}}(S)\) and \(\mu \notin sp_{\mathcal{B}}(T)\). In particular, \(\lambda \notin sp_{\mathcal{B}}(S)\) and \(\mu \notin sp_{\mathcal{B}}(T)\), and there is \(l \in \mathbb{N}\) such that for all \(r \geq l\), \(\dim H_0(K^r_1) = \dim H_0(K^r_2) = \dim H_0(K^l_1)\).

In addition, by the analytic spectral mapping theorem of the essential split defect spectrum ([4, Corollary 2.6]), the complex \(K^r_1\) and \(K^r_2\) are Fredholm split for all \(r \in \mathbb{N}\) at level \(p = 0\). In particular, for all \(r \in \mathbb{N}\) there are bounded linear maps \(h_r: X_1 \to X_1 \otimes \wedge^1 \mathbb{C}^n\) and \(g_r: X_2 \to X_2 \otimes \wedge^1 \mathbb{C}^m\), and finite rank projectors (see Proposition 4.2), \(k_{1r}: X_1 \to X_1\) and \(k_{2r}: X_2 \to X_2\), such that

\[
ed^r_{11} \circ h_r = I - k_{1r}, \quad \ned^r_{21} \circ g_r = I - k_{2r},
\]

where \(\ned^r_{11}\) and \(\ned^r_{21}\) are the maps of the complex \(K^r_1\) and \(K^r_2\) at level \(p = 1\), respectively.

Moreover, since the complexes \(K^r_1\) and \(K^r_2\) are Fredholm split at level \(p = 0\), by [4, Theorem 2.7] the complexes \(K^r_1\) and \(K^r_2\) are Fredholm at level \(p = 0\) and \(N(d^r_{11})\) and \(N(d^r_{21})\) have direct complements in \(\tilde{X}_1 \otimes \wedge^1 \mathbb{C}^n\) and \(X_2 \otimes \wedge^1 \mathbb{C}^m\) respectively.

Now, by an argument similar to [4, Theorem 2.7] or [5, Proposition 2.1], we have that the maps \(h_r, g_r, k_{1r}\), and \(k_{2r}\), may be chosen in the following way. If \(N^r_1\) and \(N^r_2\) are finite dimensional subspaces of \(X_1\) and \(X_2\) respectively, such that \(R(d^r_{11})\oplus N^r_1 = X_1\) and \(R(d^r_{21})\oplus N^r_2 = X_2\) and \(L^r_1\) and \(L^r_2\) are closed linear subspaces of \(X_1 \otimes \wedge^1 \mathbb{C}^n\) and \(X_2 \otimes \wedge^1 \mathbb{C}^m\) respectively, such that \(N(d^r_{11})\oplus L^r_1 = X_1 \otimes \wedge^1 \mathbb{C}^n\), and \(N(d^r_{21})\oplus L^r_2 = X_2 \otimes \wedge^1 \mathbb{C}^m\), then, \(k_{1r}\), respectively \(k_{2r}\), may be chosen as the projector onto \(N^r_1\), respectively \(N^r_2\), whose null space coincide with \(R(d^r_{11})\), respectively \(R(d^r_{21})\), and the map \(h_r\), respectively \(g_r\), may be chosen such that \(h_r \circ d^r_{11} = I - L^r_1\), respectively \(g_r \circ d^r_{21} = I - L^r_2\), \(h_r \mid N^r_1 = 0\), respectively \(g_r \mid N^r_2 = 0\).

In particular, \(R(k_{1r}) = H_0(K^r_1)\) and \(R(k_{2r}) = H_0(K^r_2)\).

Now, as in Proposition 4.2, for all \(r \in \mathbb{N}\) we have a well defined map \(H_r: X_1 \otimes X_2 \to (K^r_1 \otimes K^r_2)_1\) such that

\[
d^r_1 \circ H_r = I - k_{1r} \otimes k_{2r},
\]

where \(d^r_1\) is the boundary map of the complex \(K^r_1 \otimes K^r_2\) at level \(p = 1\).

Then, since for all \(r \in \mathbb{N}\), \(R(k_{1r} \otimes k_{2r}) = R(k_{1r}) \otimes R(k_{2r})\) (see Proposition 4.2), for all \(r \geq l\) we have that

\[
\dim H_0(K^r) = \dim H_0(K^r_1 \otimes K^r_2) \leq \dim R(k_{1r} \otimes k_{2r})
= \dim R(k_{1r}) \times \dim R(k_{2r}) = \dim H_0(K^r_1) \times \dim H_0(K^r_2)
= \dim H_0(K^l_1) \times \dim H_0(K^l_2),
\]

which is impossible for \((\lambda, \mu) \in \mathcal{A}(S \otimes I, I \otimes (T - \mu))\).
By means of a similar argument it is possible to see the second statement. □

6. OPERATOR IDEALS BETWEEN BANACH SPACES

In this section we extend our descriptions of the semi-Fredholm joint spectra and the semi-Browder joint spectra for tuples of left and right multiplications defined on an operator ideal between Banach spaces in the sense of [5]. We first recall the definition of such an ideal and then we introduce the tuples with which we shall work. For a complete exposition see [5].

An operator ideal $J$ between Banach spaces $X_2$ and $X_1$ will be a linear subspace of $L(X_2, X_1)$, equipped with a space norm $\alpha$ such that

(i) $x_1 \otimes x_2' \in J$ and $\alpha(x_1 \otimes x_2') = \| x_1 \| \| x_2' \|$, 

(ii) $SAT \in J$ and $\alpha(SAT) \leq \| S \| \| \alpha(A) \| \| T \|$, 

where $x_1 \in X_1$, $x_2' \in X_2'$, $A \in J$, $S \in L(X_1)$, $T \in L(X_2)$, and $x_1 \otimes x_2'$ is the usual rank one operator $X_2 \to X_1$, $x_2 \mapsto x_2' x_1$.

Examples of this kind of ideals are given in [5, section 1].

Let us recall that such operator ideal $J$ is naturally a tensor product relative to $< X_1, X_1'>$ and $< X_2', X_2>$, with the bilinear mappings

$$X_1 \times X_2' \to J, \ (x_1, x_2') \mapsto x_1 \otimes x_2',$$

$$\mathcal{L}(X_1) \times \mathcal{L}(X_2') \to L(J), \ (S, T') \mapsto S \otimes T',$$

where $S \otimes T'(A) = SAT$.

On the other hand, if $X$ is a Banach space and $U \in L(X)$, we denote by $L_U$ and $R_U$ the operators of left and right multiplication in $L(X)$, respectively, i.e., if $V \in L(X)$, then $L_U(V) = UV$ and $R_U(V) = VU$.

Now, if $S = (S_1, \ldots, S_n)$ and $T = (T_1, \ldots, T_m)$ are tuples of commuting operators defined on $X_1$ and $X_2$ respectively, if $J$ is seen as a tensor product of $X_1$ and $X_2$ relative to $< X_1, X_1'>$ and $< X_2', X_2>$, then the tuple of left and right multilications $(L_S, R_T)$ defined on $L(J)$, $(L_S, R_T) = (L_{S_1}, \ldots, L_{S_n}, R_{T_1}, \ldots, R_{T_m})$, may be identified with the $(n+m)$-tuple $(S \otimes I, I \otimes T')$ defined on $X_1 \otimes X_2'$, where $T' = (T_1', \ldots, T_m')$ and for all $i = 1, \ldots, m$, $T_i'$ is the adjoint map associated to $T_i$ (see [5, Theorem 3.1]).

In addition, if $\lambda \in \mathbb{C}^n$ and $\mu \in \mathbb{C}^m$, and if we denote by $K_1$ and $K_2'$ the Koszul complexes associated to $S$ and $\lambda$ and $T'$ and $\mu$ respectively, then the total complex of the double complex obtained from the tensor product of $K_1$ and $K_2'$, $K_1 \otimes K_2'$ is isomorphic to $K$, the Koszul complex associated to $(S \otimes I, I \otimes T')$ and $(\lambda, \mu)$ on $X_1 \otimes X_2$, which is naturally isomorphic to the Koszul complex of $(L_S, R_T)$ and $(\lambda, \mu)$ on $L(J)$, see [5, section 3].

In order to state our description of the semi-Fredholm and the semi-Browder joint spectra of the tuple $(L_S, R_T)$, as we have done in section 4, we first describe the defect and the approximate point spectra of the mentioned tuple.

**Proposition 6.1.** Let $X_1$ and $X_2$ be two complex Banach spaces, and $J$ and operator ideal between $X_2$ and $X_1$ in the sense of [5]. Let us consider two tuples of commuting operators defined on $X_1$ and $X_2$, $S$ and $T$ respectively. Then, if $(L_S, R_T)$ is the tuple of left and right multiplications defined on $L(J)$, we have that
(i) \( \sigma_\delta(S) \times \sigma_\pi(T) \subseteq \sigma_\delta(L_S, R_T) \subseteq \text{sp}_\delta(L_S, R_T) \subseteq \text{sp}_\delta(S) \times \text{sp}_\pi(T) \),

(ii) \( \sigma_\pi(S) \times \sigma_\delta(T) \subseteq \sigma_\pi(L_S, R_T) \subseteq \text{sp}_\pi(L_S, R_T) \subseteq \text{sp}_\pi(S) \times \text{sp}_\delta(T) \).

In addition, if \( X_1 \) and \( X_2 \) are Hilbert spaces, the above inclusions are equalities.

Proof. As we have said, \( J \) may be seen as the tensor product of \( X_1 \) and \( X_2' \), \( X_1 \otimes X_2' \), relative to \( < X_1, X_1' > \) and \( < X_2, X_2' > \), and \( (L_S, R_T) \) may be identified with the tuple \((S \otimes I, I \otimes T')\). Moreover, if \( K_1 \) and \( K_2' \) denote the differential space associated to \( K_1 \) and \( K_2' \) respectively, then \( \tilde{K} \), the differentiable space associated to \( \tilde{K} \), is isomorphic to \( K_1 \otimes K_2' \) (see [5, section 3]).

In addition, since for all \( i = 1, \ldots, n \) \( S_i \in L(X_1) \) and for all \( j = 1, \ldots, m \) \( T_j \in L(X_2) \), the differential spaces \( K_1 \) and \( K_2' \) satisfy the conditions of [5, Theorem 2.2], and by means of an argument similar to the one of Proposition 4.1 we have that

\[
\sigma_\delta(S) \times \sigma_\delta(T') \subseteq \sigma_\delta(S \otimes I, I \otimes T') = \sigma_\delta(L_S, R_T).
\]

However, by [8, Theorem 2.0], \( \sigma_\pi(T) = \sigma_\pi(T') \). Thus, we have proved the left hand side inclusion of the first statement.

The middle inclusion is clear.

In order to see the right hand inclusion, let us first observe that if \( \mu \notin \text{sp}_\pi(T) \), then \( \mu \notin \text{sp}_\delta(T') \).

In fact, if \( K_2 \) is split at level \( p = m \), then by [8, Lemma 2.2] \( K_2' \) is split at level \( p = 0 \).

Now, by the isomorphism of [8, Lemma 2.2], if we think the homotopy operator which gives the splitting for the complex \( K_2' \) at level \( p = 0 \) as a matrix, then each component of the matrix is an adjoint operator. In particular, by means of the properties of the tensor product of [5], it is possible to adapt the proof of the corresponding inclusion of Proposition 4.1 in order to see that if \( (\lambda, \mu) \notin \text{sp}_\delta(S) \times \text{sp}_\pi(T) \), then \( (\lambda, \mu) \notin \text{sp}_\delta(S \otimes I, I \otimes T') = \text{sp}_\delta(L_S, R_T) \).

The second statement may be proved by means of a similar argument. \( \square \)

In the following proposition we give our description of the semi-Fredholm joint spectra of the tuple \((L_S, R_T)\).

**Proposition 6.2.** Let \( X_1 \) and \( X_2 \) be two complex Banach spaces, and \( J \) and operator ideal between \( X_2 \) and \( X_1 \) in the sense of [5]. Let us consider two tuples of commuting operators defined on \( X_1 \) and \( X_2 \), \( S \) and \( T \) respectively. Then, if \((L_S, R_T)\) is the tuple of left and right multiplications defined on \( L(J) \), we have that

(i) \[ \sigma_{\Phi_-}(S) \times \sigma_\pi(T) \cup \sigma_\delta(S) \times \sigma_{\Phi_+}(T) \subseteq \sigma_{\Phi_-}(L_S, R_T) \subseteq \text{sp}_{\Phi_-}(L_S, R_T) \subseteq \text{sp}_\delta(L_S, R_T) \subseteq \text{sp}_\delta(S) \times \text{sp}_\pi(T) \cup \text{sp}_\delta(S) \times \text{sp}_{\Phi_+}(T), \]

(ii) \[ \sigma_{\Phi_+}(S) \times \sigma_\delta(T) \cup \sigma_\pi(S) \times \sigma_{\Phi_-}(T) \subseteq \sigma_{\Phi_+}(L_S, R_T) \subseteq \text{sp}_{\Phi_+}(L_S, R_T) \subseteq \text{sp}_{\Phi_+}(L_S, R_T) \subseteq \text{sp}_\pi(S) \times \text{sp}_\delta(T) \cup \text{sp}_\pi(S) \times \text{sp}_{\Phi_-}(T). \]
In addition, if $X_1$ and $X_2$ are Hilbert spaces, the above inclusions are equalities.

**Proof.** By means of an argument similar to the one of Proposition 4.2, adapted as we have done in Proposition 6.1, it is possible to see that

$$\sigma_{\Phi_-}(S) \times \sigma_{\delta}(T') \cup \sigma_{\delta}(S) \times \sigma_{\Phi_-}(T) \subseteq \sigma_{\Phi_-}(L_S, R_T).$$

However, by [8, Theorem 2.0] $\sigma_{\delta}(T') = \sigma_{\pi}(T)$, and by elementary properties of the adjoint of an operator it is easy to see that $\sigma_{\Phi_+}(T) \subseteq \sigma_{\Phi_-}(T')$. Thus, we have seen the left hand side inclusion of the first statement.

The middle inclusion is clear.

Let us consider $(\lambda, \mu) \in sp_{\delta e}(L_S, R_T) \setminus (sp_{\delta e}(S) \times sp_{\pi}(T) \cup sp_{\delta}(S) \times sp_{\pi e}(T))$. By Proposition 6.1 we have that $\lambda \in sp_{\delta}(S) \setminus sp_{\delta e}(S)$ and $\mu \in sp_{\pi}(T) \setminus sp_{\pi e}(T)$. However, by [8, Lemma 2.2] and elementary properties of the adjoint of an operator we have that $\mu \in sp_{\delta}(T') \setminus sp_{\delta e}(T')$. Then, as in Proposition 4.2, there are two linear bounded maps $h: X_1 \rightarrow X_1 \otimes \wedge^1 \mathbb{C}^n$, $g': X_2' \rightarrow X_2' \otimes \wedge^1 \mathbb{C}^m$, and two finite rank projectors $k_1: X_1 \rightarrow X_1$ and $k_2': X_2 \rightarrow X_2$ such that

$$d_{11} \circ h = I - k_1, \quad d_{21}' \circ g' = I - k_2',$$

where $d_{21}'$ is the boundary map of the complex $K_2'$ at level $p = 1$.

Now, by the isomorphism of [8, Lemma 2.2], if we think the map $g'$ as a matrix, then each component of the matrix is an adjoint operator. Then, by the properties of the tensor product introduced in [5], we may consider the well defined map

$$H: X_1 \otimes X_2' \rightarrow (K_1 \otimes K_2')_1, \quad H = (h \otimes I, I \otimes g').$$

Now, by an argument similar to the one of Proposition 4.2, it is easy to see that $(\lambda, \mu) \notin sp_{\delta e}(S \otimes I, I \otimes T') = sp_{\delta e}(L_S, R_T)$, which is impossible by our assumptions.

By means of a similar argument it is possible to see the second statement. $\square$

We now give our description of the semi-Browder joint spectra of the tuple of left and right multiplications $(L_S, R_T)$ defined on $L(J)$.

**Theorem 6.3.** Let $X_1$ and $X_2$ be two complex Banach spaces, and $J$ and operator ideal between $X_2$ and $X_1$ in the sense of [5]. Let us consider two tuples of commuting operators defined on $X_1$ and $X_2$, $S$ and $T$ respectively. Then, if $(L_S, R_T)$ is the tuple of left and right multiplications defined on $L(J)$, we have that

\begin{align*}
(i) \quad & \sigma_{B_+}(S) \times \sigma_{\pi}(T) \cup \sigma_{\delta}(S) \times \sigma_{B_-}(T) \subseteq \sigma_{B_-}(L_S, R_T) \\
& \quad \subseteq sp_{B_+}(L_S, R_T) \subseteq sp_{\delta}(S) \times sp_{\pi}(T) \cup sp_{\delta}(S) \times sp_{B_+}(T),
\end{align*}

\begin{align*}
(ii) \quad & \sigma_{B_+}(S) \times \sigma_{\delta}(T) \cup \sigma_{\pi}(S) \times \sigma_{B_-}(T) \subseteq \sigma_{B_+}(L_S, R_T) \subseteq \\
& \quad \subseteq sp_{B_+}(L_S, R_T) \subseteq sp_{B_+}(S) \times sp_{\delta}(T) \cup sp_{\pi}(S) \times sp_{B_-}(T),
\end{align*}

In addition, if $X_1$ and $X_2$ are Hilbert spaces, the above inclusions are equalities.
**Proof.** In order to see the first statement, let us observe that if $K^r_1$ is the Koszul complex associated to the tuple $(S - \lambda)^r = ((S_1 - \lambda_1)^r, \ldots, (S_n - \lambda_n)^r)$, and if $K^r_2$ is the Koszul complex associated to the tuple $(T - \mu)^r = ((T'_1 - \mu_1)^r, \ldots, (T'_m - \mu_m)^r)$, then $\tilde{K}^r$, the Koszul complex associated to the tuple $((S - \lambda)^r \otimes T, I \otimes (T' - \mu)^r)$, is isomorphic to the total complex obtained from the double complex of the tensor product of $K^r_1$ and $K^r_2$, i.e., $\tilde{K}^r \cong K^r_1 \bar{\otimes} K^r_2$ (see [5, section 3]).

Now, we may adapt the proof of the left hand inclusion of Theorem 5.1, as we have done in Proposition 6.1, using in particular Proposition 6.2 instead of Proposition 4.2, in order to see that we have done in Proposition 6.1, using in particular Proposition 6.2 instead of Proposition 4.2, in order to see that the essential split defect spectrum, the complex $K^r_1 \bar{\otimes} K^r_2$, is isomorphic to the total complex obtained from the double complex of the tensor product of $K^r_1$ and $K^r_2$, i.e., $\tilde{K}^r \cong K^r_1 \bar{\otimes} K^r_2$ (see [5, section 3]).

In order to see the first statement, let us observe that if $\mu \in \sigma_{\delta}(S)$, then $\mu \notin sp_{\pi}(T)$. Moreover, if there exists $l \in \mathbb{N}$ such that for all $r \geq l \dim H_m(K^r_2) = \dim H_m(K^r_2)$, then by [7, Theorem 11] it is easy to see that $\dim H_0(K^r_2) = \dim H_0(K^r_2)$, for all $r \geq l$. In addition, if $\mu \notin sp_{\delta}(T')$, by the analytic spectral mapping theorem for the essential split defect spectrum, the complex $K^r_2$ is Fredholm split for all $r \in \mathbb{N}$, i.e., there are operators $g'_r: X'_2 \rightarrow X'_2 \otimes \wedge^1 C^m$ and finite rank projectors $k_{2r'}: X'_{2} \rightarrow X'_{2}$ such that $d'_{21} \circ g'_r = I - k_{2r'}$, where $d'_{21}$ denotes the chain map of the complex $K^r_2$ at level $p = 1$. Furthermore, by [8, Lemma 2.2], if for $r \in \mathbb{N}$ we think the map $g'_r$ as matrix, then each component of the matrix is an adjoint operator, and by elementary properties of the adjoint of an operator, the maps $g'_r$ and $k_{2r}$ may be chosen with the same properties of the maps $g_r$ and $k_{2r}$ of Theorem 5.1. With all this observations it is possible to conclude the proof of the right hand side inclusion of the first statement.

The second statement may be proved by means of a similar argument. □

**REFERENCES**

1. J. J. Buoni, R. Harte and T. Wickstead, Upper and lower Fredholm spectra, Proc. Amer. Math. Soc. 66 (1977), 309-314.
2. R. E. Curto and A. T. Dash, Browder spectral systems, Proc. Amer. Math. Soc. 103 (1988), 407-413.
3. A. T. Dash, Joint Browder spectra and tensor products, Bull. Austral. Math. Soc. 32 (1985), 119-128.
4. J. Eschmeier, Analytic spectral mapping theorems for joint spectra, Oper. Theory. Adv. Appl. 24 (1987), 167-181.
5. J. Eschmeier, Tensor products and elementary operators, J. Reine Angew. Math. 390 (1988), 47-66.
6. T. Ichinose, Spectral properties of tensor product of linear operators I, Trans. Amer. Math. Soc. 235 (1978), 75-113.
7. V. Kordula, V. Müller and V. Rakočević, On the semi-Browder spectrum, Studia Math. 123 (1997), 1-13.
8. Z. Slodkowski, An infinite family of joint spectra, Studia Math. 61 (1977), 239-255.
9. J. L. Taylor, A joint spectrum for several commuting operators, J. Funct. Anal. 6 (1970), 172-191.

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