DIFFERENTIAL FORMS AND QUADRICS OF THE CANONICAL IMAGE

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ABSTRACT. Let $\pi: X \to B$ be a family over a smooth connected analytic variety $B$, not necessarily compact, whose general fiber $X$ is smooth of dimension $n$, with irregularity $\geq n+1$ and such that the image of the canonical map of $X$ is not contained in any quadric of rank $\leq 2n+3$. We prove that if the Albanese map of $X$ is of degree 1 onto its image then the fibers of $\pi: X \to B$ are birational under the assumption that all the 1-forms and all the $n$-forms of a fiber are holomorphically liftable to $X$. Moreover we show that generic Torelli holds for such a family $\pi: X \to B$ if, in addition to the above hypothesis, we assume that the fibers are minimal and their minimal model is unique. There are counterexamples to the above statements if the canonical image is contained inside quadrics of rank $\leq 2n+3$. We also solve the infinitesimal Torelli problem for an $n$-dimensional variety $X$ of general type with irregularity $\geq n+1$ and such that its cotangent sheaf is generated and the canonical map is a rational map whose image is not contained in a quadric of rank less or equal to $2n+3$.

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1. Introduction

Given an $n \times n$ matrix $T \in \text{Mat}(n, \mathbb{K})$, there always exists its adjugate matrix, also called classical adjoint, $T^\vee$ such that by row-column product we obtain $T \cdot T^\vee = \det(T)I_n$, where

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$I_n \in \text{Mat}(n, \mathbb{K})$ is the identity matrix. In this paper we consider the above construction in the context of locally free sheaves.

1.1. General theory. Let $\xi \in \text{Ext}^1(\mathcal{F}, \mathcal{O}_X)$ be an extension class associated to the following exact sequence of locally free sheaves over an $m$-dimensional smooth variety $X$:

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{F} \to 0. \tag{1.1}$$

Assume that $\mathcal{F}$ is of rank $n$ and that the kernel of the connecting homomorphism $\partial_\xi : H^0(X, \mathcal{F}) \to H^1(X, \mathcal{O}_X)$ has dimension $\geq n + 1$. Take an $n+1$-dimensional subspace $W \subset \text{Ker} \partial_\xi$ and an ordered basis $\mathcal{B} = \{\eta_1, \ldots, \eta_{n+1}\}$. By choosing a lifting $s_i \in H^0(X, \mathcal{E})$ of $\eta_i$, where $i = 1, \ldots, n+1$, we have a top form $\Omega \in H^0(X, \det \mathcal{E})$ from the element $s_1 \wedge \ldots \wedge s_{n+1} \in \bigwedge^{n+1} H^0(X, \mathcal{E})$. Since $\det \mathcal{E} = \det \mathcal{F}$ we actually obtain from $\Omega$ a top form $\omega$ of $\mathcal{F}$, which depends on the chosen liftings and on $\mathcal{B}$. We call such an $\omega \in H^0(X, \det \mathcal{F})$ an adjoint form of $\xi, W, \mathcal{B}$. Indeed consider the $n+1$ top forms $\omega_1, \ldots, \omega_{n+1} \in H^0(X, \det \mathcal{F})$ where $\omega_i$ is obtained by the element $\eta_1 \wedge \ldots \wedge \eta_i \wedge \ldots \wedge \eta_{n+1} \in \bigwedge^n H^0(X, \mathcal{F})$. It is easily seen that the subscheme of $X$ where $\omega$ vanishes is locally given by the vanishing of the determinant of a suitable $(n+1) \times (n+1)$ matrix $T$ and the local expressions of $\omega_i$, $i = 1, \ldots, n+1$, give some entries of $T^\lor$. This idea to relate extension classes to adjoint forms was first introduced in $[CP]$ for the case of smooth curves and later the theory was extended to any smooth algebraic variety; see: $[PZ]$. Since then it has been fruitfully applied in $[Ra], [PR], [CNP], [G-A1], [G-A2]$ and $[BGN]$.

In this paper we go deeper along the direction indicated in $[PZ]$.

To the $n+1$-dimensional subspace $W \subset \text{Ker} \partial_\xi$ taken above, we associate $D_W$ and $Z_W$ which are respectively the fixed part and the base loci of the sublinear system given by $\lambda^n W \subset H^0(X, \det \mathcal{F})$, where $\lambda^n : \bigwedge^n H^0(X, \mathcal{F}) \to H^0(X, \det \mathcal{F})$ is the natural homomorphism obtained by the wedge product. We prove:

**Theorem [A].** Let $X$ be an $m$-dimensional compact complex smooth variety. Let $\mathcal{F}$ be a rank $n$ locally free sheaf and let $\xi \in \text{Ext}^1(\mathcal{F}, \mathcal{O}_X)$ be the class of the sequence (1.1). Let $\omega$ be an adjoint form associated to a subspace $W \subset H^0(X, \det \mathcal{F})$. If $\omega \in \lambda^n W$ then $\xi \in \text{Ker} (\text{Ext}^1(\mathcal{F}, \mathcal{O}_X) \to \text{Ext}^1(\mathcal{F}(-D_W), \mathcal{O}_X))$. Viceversa. Assume $\xi \in \text{Ker} (\text{Ext}^1(\mathcal{F}, \mathcal{O}_X) \to \text{Ext}^1(\mathcal{F}(-D_W), \mathcal{O}_X))$. If $h^0(X, \mathcal{O}_X(D_W)) = 1$, then $\omega \in \lambda^n W$.

See theorem 2.0.7 and theorem 3.0.11 The first part of the above theorem, but under the assumption $m = n$, is in $[PZ]$, Theorem 1.5.1.

By abuse of notation we denote by $|\det \mathcal{F}|$ the linear system $\mathbb{P}(H^0(X, \det \mathcal{F}))$ and we denote by $D_{\det \mathcal{F}}$ the fixed part of $|\det \mathcal{F}|$. We set $|\det \mathcal{F}| = D_{\det \mathcal{F}} + |M_{\det \mathcal{F}}|$. We prove a relation between liftability of adjoint forms and quadrics vanishing on the image of the map $\phi_{|M_{\det \mathcal{F}}|} : X \to \mathbb{P}(H^0(X, \det \mathcal{F}))$ associated to $|\det \mathcal{F}|$. We recall that for a top form $\omega \in H^0(X, \det \mathcal{F})$ of $\mathcal{F}$ to be liftable to an $n$-form of $\mathcal{E}$ is equivalent to have that $\omega$ belongs to the kernel of the connecting homomorphism

$$\partial^n_\xi : H^0(X, \det \mathcal{F}) \to H^1(X, \bigwedge^n \mathcal{F})$$

of the short exact sequence obtained by the wedge of the sequence (1.1); see: Section 2.

Denote by $\lambda^n H^0(X, \mathcal{F})$ the image of $\bigwedge^n H^0(X, \mathcal{F})$ inside $H^0(X, \det \mathcal{F})$ and consider the linear system $\mathbb{P}(\lambda^n H^0(X, \mathcal{F}))$; denote by $D_{\mathcal{F}}$ its fixed component. Note that $D_{\det \mathcal{F}} < D_{\mathcal{F}}$, but
in general \( D_{\text{det}F} \neq D_F \). We say that \( \xi \) is a supported deformation on \( D_F \) if it belongs to the kernel of \( \text{Ext}^1(F, O_X) \to \text{Ext}^1(F(-D_F), O_X) \). We prove:

**Theorem [B].** Let \( X \) be an \( m \)-dimensional smooth variety where \( m \geq 2 \). Let \( F \) be a generically globally generated locally free sheaf of rank \( n \) such that \( h^0(X, F) \geq n + 1 \). Assume that \( \phi|_{\text{Mfd}_F} : X \to \mathbb{P}(H^0(X, \text{det} F)^\vee) \) is a non trivial rational map and call \( Y \) its schematic image. Let \( \mathcal{I}_Y \) be the ideal sheaf of \( Y \). Assume that \( H^0(Y, \mathcal{I}_Y(2)) \) contains no quadric of rank less or equal to \( 2n + 3 \). If \( \xi \in \text{Ext}^1(F, O_X) \) is such that the homomorphisms \( \partial_\xi = 0 \) and \( \partial_\xi^n = 0 \), then \( \xi \) is a supported deformation on \( D_F \).

See corollary [4.0.13]. Note that the above theorem applies straightly to the particular case where \( Y = \phi|_{\text{Mfd}_F}(X) \) is a hypersurface of \( \mathbb{P}(H^0(X, \text{det} F)^\vee) \) of degree \( > 2 \); see corollary [4.0.14] for a more general claim.

Theorem [A] and theorem [B] give a general criteria for a family \( \pi : \mathcal{X} \to B \) to have birational fibers, if the condition on the rank of the quadrics containing the canonical image of a fiber given in theorem [B] is satisfied; see: [5.3.2].

We recall that in this paper a family of relative dimension \( n \) is a smooth proper surjective morphism of smooth varieties \( \pi : \mathcal{X} \to B \) where any fiber \( X_y := \pi^{-1}(y) \) is a manifold of complex dimension \( n \) and \( B \) is a connected analytic variety, for example a complex polydisk. It is important to have criteria, depending on the local or even infinitesimal variation of some data on the fiber, to understand if the fibers are reciprocally birational.

We say that a family \( \pi : \mathcal{X} \to B \) of relative dimension \( n \) satisfies extremal liftability conditions over \( B \) if

\[
\begin{align*}
(i) & \quad H^0(\mathcal{X}, \Omega^1_{\mathcal{X}}) \to H^0(X_b, \Omega^1_{X_b}); \\
(ii) & \quad H^0(\mathcal{X}, \Omega^n_{\mathcal{X}}) \to H^0(X_b, \Omega^n_{X_b}).
\end{align*}
\]

Clearly if \( H^0(X_b, \Omega^1_{X_b}) = H^0(X_b, \Omega^n_{X_b}) = 0 \) the above conditions are trivially satisfied. Instead the families \( \pi : \mathcal{X} \to B \) which satisfy extremal liftability conditions and such that \( H^0(X_b, \Omega^n_{X_b}) \neq 0 \), that is with irregular fiber, have the advantage that the Albanese variety of the fiber is a fixed Abelian variety. Now by a result called the Volumetric theorem, see: [PZ] Theorem 1.5.3], cf. theorem [5.3.1] we obtain the main application of our general theory that we present here:

**Theorem [C].** Let \( \pi : \mathcal{X} \to B \) be a family of \( n \)-dimensional irregular varieties which satisfies extremal liftability conditions. Assume that for every fiber \( X \) the Albanese map of \( X \) has degree \( 1 \) and that no quadric of rank \( \leq 2n + 3 \) contains the canonical image of \( X \), then the fibers of \( \pi : \mathcal{X} \to B \) are birational.

See: theorem [5.3.2] for a slightly more general statement.

1.2. Torelli-type problems. As a partial motivation to understand the above general theory and in particular theorem [C], we point out the reader that, in certain cases, it gives an answer to the generic Torelli problem and to the infinitesimal Torelli problem and, more important, that there are counterexamples to both generic and infinitesimal Torelli claim if the above condition on the quadrics through the canonical image is removed.

As references for the Torelli problems here we quote the following books: [Vo1], [Vo2], [CMP], see also [Re2].
We recall that the global Torelli problem asks whether two compact Kähler manifolds of dimension $n$, $X_1$, $X_2$, with an isomorphism $\psi : H^n(X_1, \mathbb{Z}) \rightarrow H^n(X_2, \mathbb{Z})$ preserving the cup product pairing and with the same Hodge decomposition are biholomorphic. There are some variations of this problem cf. [Do]. Let $\pi : \mathcal{X} \rightarrow B$ be a family. Fix a point $0 \in B$. We can interpret $\pi : \mathcal{X} \rightarrow B$ as a deformation of complex structures on $X := X_0$. Indeed by Ehresmann’s theorem, after possibly shrinking $B$, we have a canonical isomorphism between $H^k(X_y, \mathbb{C})$ and $V := H^k(Y, \mathbb{C})$, for every $y \in B$. Furthermore, $\dim F^p H^k(X_y, \mathbb{C}) = \dim C F^p H^k(X, \mathbb{C}) = b^{p,k}$, where $F^p H^k(X_y, \mathbb{C}) = \bigoplus_{r \geq p} H^{r,k-r}(X_y)$.

The period map $d\pi : B \rightarrow \mathbb{C} = \text{Grass}(b^{p,k}, V)$ (cf. [GS]) is the map which to $y \in B$ associates the subspace $F^p H^k(X_y, \mathbb{C})$ of $V$. In [Gri1], [Gri2], P. Griffiths proved that $P^{p,k}$ is holomorphic and that the image of the differential $dP^{p,k} : T_B,0 \rightarrow T_{G,F^p H^k(X,\mathbb{C})}$ is actually contained in

$$\text{Hom}(F^p H^k(X, \mathbb{C}), F^{p+1} H^k(X, \mathbb{C})/F^p H^k(X, \mathbb{C})).$$

Setting $q = k - p$ and using the canonical isomorphism

$$F^p H^k(X, \mathbb{C})/F^{p+1} H^k(X, \mathbb{C}) \simeq H^q(X, \Omega^n_X)$$

he showed that $dP^{p,k}$ is the composition of the Kodaira-Spencer map $T_{B,0} \rightarrow H^1(X, \Theta_X)$ with the map given by the cup product:

$$d\pi^q : H^1(X, \Theta_X) \rightarrow \text{Hom}(H^q(X, \Omega^n_X), H^{q+1}(X, \Omega^{n+1}_X))$$

where $\Theta_X$ is the tangent sheaf of $X$. Following [Do] we recall that the local Torelli problem asks under which hypothesis the period map of $\pi : \mathcal{X} \rightarrow B$ is an immersion and the generic Torelli problem asks whether the period map is generically injective where $B$ is a connected analytic variety; the n-infinitesimal Torelli problem asks if the differential of the period map $P^{n,n}$ for the local Kuranishi family $\pi : \mathcal{X} \rightarrow B$ ([KM], cf. [KMc]) of a given variety $X$ of dimension $n$ is injective. We say that the n-infinitesimal Torelli claim holds for $X$ if $d\pi^0_n$ is injective.

In the case of smooth curves the answer to the Torelli problems is well-known; see: [To], [An], [We], [OS].

Concerning Torelli problems in higher dimensions here we can recall only that some positive answers were obtained: [Gr1], [Gr2], [Gr3], [Fl], [Co], [Re1], [Pe] and, with a slightly different point of view, [Do]. See also [Re2], [Vo1], [Vo2].

On the other hand, surfaces $X$ of general type with $H^0(X, \Omega^1_X) = H^0(X, \Omega^2_X) = 0$ are known examples of failure for the injectivity of the period map; see: [Ca1].

More recently the problem has been studied in [BC] where the authors construct families such that $d\pi^0_2$ is not injective and the canonical sheaf is quasi very ample, that is the canonical map is a birational morphism and a local embedding on the complement of a finite set. In [GZ] it has been proved that for any natural number $N \geq 5$ there exists a generically smooth irreducible $(N + 9)$-dimensional component $\mathcal{M}_N$ of the moduli space of algebraic surfaces such that for a general element $[X]$ of $\mathcal{M}_N$, the canonical sheaf is very ample and $d\pi^0_2$ has kernel of dimension at least 1. Hence the problem about good hypotheses to obtain the infinitesimal Torelli claim is still quite open. Note that the counterexamples studied in [BC] and in [GZ] are given by fibered surfaces.
On the other hand by [OS], see also: [Cod], it is easy to see that the \( n \)-symmetric product of a hyperelliptic curve gives a counterexample to the infinitesimal Torelli claim for non fibered varieties with Albanese morphism of degree 1 onto the image. Note that for surfaces of general type which are the symmetric product of a hyperelliptic curve of genus 3, \( D_{\Omega_1^X} \) exists and it is a rational \(-2\) curve; see: [CCM] Proposition 3.17, (ii). Hence an infinitesimal Torelli deformation is supported on \( D_{\Omega_1^X} \).

Nevertheless an easy computation shows that these varieties have canonical image contained in certain quadrics of rank less or equal to \( 2n + 3 \). Indeed theorem [B] confirms that for the Torelli problem \( 2n + 3 \) is a critical threshold for the rank of quadrics containing the canonical image. In particular, as a corollary of theorem [B], we easily have that infinitesimal Torelli claim holds for a variety \( X \) whose cotangent sheaf is generated, if we also assume that \( X \) has irregularity \( \geq n + 1 \) and there are no quadrics of rank less or equal to \( 2n + 3 \) containing the canonical image; see: corollary \([4.0.15]\) We point out the reader that \([Po]\) and \([LPW]\) contain the same claim but under quite different hypothesis.

By the above counterexamples the hypothesis of theorem [C] concerning the rank of the quadrics through the canonical image appears to be crucial to show the generic Torelli claim. In dimension \( \geq 3 \) non-uniqueness of the minimal model is naturally another obstruction.

Hence in Corollary \([5.3.3]\) we obtain that the generic Torelli claim holds for families which satisfies the hypothesis of theorem [C] and such that the general fiber is a \( n \)-dimensional irregular minimal variety with unique minimal model. Even in the case of surfaces there are obvious counterexamples to the above claim if we do not assume minimality. It is sufficient to consider the family \( \pi: \mathcal{X} \to B \) where \( B \) is a smooth curve inside a smooth surface \( S \) and \( X_b \) is the surface obtained by the blow-up of \( S \) at the point \( b \in B \). So the hypothesis of our corollary \([5.3.3]\) appears to be quite optimal to have a prove of the generic Torelli claim.

1.3. Examples. We point out the reader that theorem [C], and consequently corollary \([5.3.3]\) applies straightly to the families whose fiber \( X \) has canonical map which is not an isomorphism and \( X \) is an irregular variety such that its canonical image \( Y \) is a (possibly very singular) hypersurface of degree \( > 2 \) or a (possibly very singular) complete intersection of hypersurfaces of degree \( > 2 \); see also our remark \([4.0.16]\). Many of these examples were not well studied in the literature.

As far as other examples are concerned, we remind the reader that the space \( Q_{k,n} \) of quadrics of the projective space \( \mathbb{P}^n \) of rank \( \leq k \) has dimension \( k(n-k+1)+\binom{k+1}{2} - 1 \). If we consider, for example, surfaces whose canonical map is an embedding and the canonical image is 2-normal, then, letting \( n = p_g(S) - 1 \), by Riemann-Roch theorem the dimension of the vector space of quadrics containing \( S = Y \) is \( h^0(\mathbb{P}^n, I_{S/\mathbb{P}^n}(2)) = \binom{n+2}{2} - K_S^2 - \chi(O_S) \). In this case \( k = 7 \).

Now, imposing the natural condition \( K_S^2 + \chi(O_S) \geq 7n - 15 \) it is at least reasonable, by the obvious dimensional estimate, to expect that, for such a generic irregular surface, \( Q_{7,n} \) is a direct summand of \( H^0(\mathbb{P}^n, I_{S/\mathbb{P}^n}(2)) \) and hence theorem [C] applies. We think that it is an interesting problem to find conditions such that this last remark holds.

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2. THE ADJOINT THEOREM

Let $X$ be a compact complex smooth variety of dimension $m$ and let $\mathcal{F}$ be a locally free sheaf of rank $n$. Fix an extension class $\xi \in \text{Ext}^1(\mathcal{F}, \mathcal{O}_X)$ associated to the exact sequence:

\begin{equation}
(2.1) \quad 0 \to \mathcal{O}_X \xrightarrow{d \epsilon} \mathcal{E} \xrightarrow{\rho_1} \mathcal{F} \to 0.
\end{equation}

The Koszul resolution associated to the section $d \epsilon \in H^0(X, \mathcal{E})$

\[ 0 \to \mathcal{O}_X \xrightarrow{d \epsilon} \mathcal{E} \xrightarrow{\rho_1} \mathcal{F} \to 0 \]

splits into $n + 1$ short exact sequences,

\begin{equation}
(2.2) \quad 0 \to \mathcal{O}_X \xrightarrow{d \epsilon} \mathcal{E} \xrightarrow{\rho_1} \mathcal{F} \to 0,
\end{equation}

\begin{equation}
(2.3) \quad 0 \to \mathcal{F} \xrightarrow{d \epsilon} \bigwedge^2 \mathcal{E} \xrightarrow{\rho_2} \bigwedge^2 \mathcal{F} \to 0,
\end{equation}

\[ \ldots \]

\begin{equation}
(2.4) \quad 0 \to \bigwedge^{n-1} \mathcal{F} \xrightarrow{d \epsilon} \bigwedge^n \mathcal{E} \xrightarrow{\rho_n} \det \mathcal{F} \to 0,
\end{equation}

\begin{equation}
(2.5) \quad 0 \to \det \mathcal{F} \xrightarrow{d \epsilon} \det \mathcal{E} \to 0 \to 0,
\end{equation}

each corresponding to $\xi$ via the natural isomorphism

\[ \text{Ext}^1(\mathcal{F}, \mathcal{O}_X) \cong \text{Ext}^1\left(\bigwedge^i \mathcal{F}, \bigwedge^{i-1} \mathcal{F}\right) \]

where $i = 0, \ldots, n$. If we still denote by $\xi$ the element corresponding to $\xi$ by the isomorphism $\text{Ext}^1(\mathcal{F}, \mathcal{O}_X) \cong \text{Ext}^1(\mathcal{O}_X, \mathcal{F}^\vee) \cong H^1(X, \mathcal{F}^\vee)$, then the coboundary homomorphisms

\[ \partial^i : H^0(X, \bigwedge^i \mathcal{F}) \to H^1(X, \bigwedge^{i-1} \mathcal{F}) \]

are computed by the cup product with $\xi$.

Denote by $H^i_{d \epsilon}$ the isomorphism inverse of (2.5) and by $\Lambda^{n+1}$ the natural map

\begin{equation}
(2.6) \quad \Lambda^{n+1} : \bigwedge^{n+1} H^0(X, \mathcal{E}) \to H^0(X, \det \mathcal{E}).
\end{equation}

The composition of these homomorphisms defines

\begin{equation}
(2.7) \quad \Lambda := H^i_{d \epsilon} \circ \Lambda^{n+1} : \bigwedge^{n+1} H^0(X, \mathcal{E}) \to H^0(X, \det \mathcal{F}).
\end{equation}

Let $W \subset \text{Ker}(\partial^1_{\xi}) \subset H^0(X, \mathcal{F})$ be a vector subspace of dimension $n + 1$ and let $B := \{ \eta_1, \ldots, \eta_{n+1} \}$ be a basis of $W$. By definition we can take liftings $s_1, \ldots, s_{n+1} \in H^0(X, \mathcal{E})$ such that $\rho_i(s_i) = \eta_i$. If we consider the natural map

\[ \lambda^n : \bigwedge^n H^0(X, \mathcal{F}) \to H^0(X, \det \mathcal{F}), \]

we can define the subspace $\lambda^n W \subset H^0(X, \det \mathcal{F})$ generated by

\[ \omega_i := \lambda^n(\eta_1 \wedge \ldots \wedge \widehat{\eta_i} \wedge \ldots \wedge \eta_{n+1}) \]
for $i = 1, \ldots, n+1$.

**Definition 2.0.1.** The section
\[
\omega_{\xi, W, B} := \Lambda(s_1 \wedge \ldots \wedge s_{n+1}) \in H^0(X, \det F).
\]
is called an adjoint form of $\xi, W, B$.

**Definition 2.0.2.** The class
\[
[\omega_{\xi, W, B}] \in H^0(X, \det F) / \lambda^n W
\]
is called an adjoint image of $W$ by $\xi$.

**Remark 2.0.3.** The class $[\omega_{\xi, W, B}]$ depends on $\xi, W$ and $B$ only. The form $\omega_{\xi, W, B}$ depends also on the choice of the liftings $s_1, \ldots, s_{n+1}$.

**Remark 2.0.4.** If we consider another basis $B' = \{\eta'_1, \ldots, \eta'_{n+1}\}$ of $W$, then $[\omega_{\xi, W, B}] = k[\omega_{\xi, W, B'}]$ where $k$ is the determinant of the matrix of the basis change. In particular $[\omega_{\xi, W, B}] = 0$ if and only if $[\omega_{\xi, W, B'}] = 0$.

**Remark 2.0.5.** If $[\omega_{\xi, W, B}] = 0$ then we can find liftings $s_i \in H^0(X, E)$, $i = 1, \ldots, n+1$, such that $\Lambda^{n+1}(s_1 \wedge \ldots \wedge s_{n+1}) = 0$ in $H^0(X, \det E)$. In particular with this choice we have $\omega_{\xi, W, B} = 0$ in $H^0(X, \det F)$.

**Proof.** By hypothesis there exist $a_i \in \mathbb{C}$ such that
\[
(2.8) \quad \omega_{\xi, W, B} = \sum_{i=1}^{n+1} a_i \cdot \Lambda^n(\eta_1 \wedge \ldots \wedge \tilde{\eta}_i \wedge \ldots \wedge \eta_{n+1}).
\]
We can define a new lifting for the element $\eta_i$:
\[
\tilde{s}_i := s_i + (-1)^{n-i} a_i \cdot d\epsilon.
\]
Now it is a trivial computation to show that $\Lambda^{n+1}(\tilde{s}_1 \wedge \ldots \wedge \tilde{s}_{n+1}) = 0$.

**Definition 2.0.6.** If $\lambda^n W$ is nontrivial we denote by $[\lambda^n W] \subset \mathbb{P}(H^0(X, \det F))$ the induced sublinear system. We call $D_W$ the fixed divisor of this linear system and $Z_W$ the base loci of its moving part $[M_W] \subset \mathbb{P}(H^0(X, \det F(-D_W)))$.

From the natural map $F(-D_W) \to F$ we have a homomorphism in cohomology
\[
H^1(X, F^\vee) \to H^1(X, F^\vee(D_W));
\]
we call $\xi_{D_W} = \rho(\xi)$. By obvious identifications the natural map
\[
\Ext^1(\det F, \bigwedge F) \to \Ext^1(\det F(-D_W), \bigwedge^n F)
\]
gives an extension $E^{(n)}$ and a commutative diagram:
Theorem 2.0.7 (Adjoint Theorem). Let $X$ be a compact $m$-dimensional complex smooth variety. Let $F$ be a rank $n$ locally free sheaf on $X$ and $\xi \in H^1(X, F^\vee)$ corresponding to the exact sequence (2.1). Let $W \subset \text{Ker } (\partial^1_1 \xi) \subset H^0(X, F)$ and $[w]$ one of its adjoint images. If $[w] = 0$ then $\xi \in \text{Ker } (H^1(X, F^\vee) \to H^1(X, F^\vee(D_W)))$.

Proof. By remark 2.0.4, the vanishing of $[w]$ does not depend on the choice of a particular basis of $W$. Let $B = \{\eta_1, \ldots, \eta_{n+1}\}$ be a basis of $W$.

By hypothesis, $\omega \in \lambda^n W$, hence by remark 2.0.5 we can choose liftings $s_i \in H^0(X, E)$ of $\eta_i$ with $\Lambda^{n+1}(s_1 \wedge \ldots \wedge s_{n+1}) = 0$. Consider

$$s_1 \wedge \ldots \wedge \tilde{s}_i \wedge \ldots \wedge s_{n+1} \in \bigwedge^n H^0(X, E)$$

and define $\Omega_i$ its image in $H^0(X, \bigwedge^n E)$ through the natural map. By construction $\rho_n(\Omega_i) = \omega_i$, where we remind the reader that $\omega_i := \lambda^n(\eta_1 \wedge \ldots \wedge \tilde{\eta}_i \wedge \ldots \wedge \eta_{n+1})$. Since $D_W$ is the fixed divisor of the linear system $|\lambda^n W|$ and the sections $\omega_i$ generate this linear system, then the $\omega_i$ are in the image of

$$\det F(-D_W) \xrightarrow{D_W} \det F,$$

and so we can find sections $\tilde{\omega}_i \in H^0(X, \det F(-D_W))$ such that

$$d \cdot \tilde{\omega}_i = \omega_i,$$

where $d$ is a global section of $\mathcal{O}_X(D_W)$ with $(d) = D_W$. Hence, using the commutativity of (2.9), we can find liftings $\tilde{\Omega}_i \in H^0(X, E^{(n)})$ of the sections $\Omega_i$, $i = 1, \ldots, n + 1$.

The evaluation map

$$\bigoplus_{i=1}^{n+1} \mathcal{O}_X \xrightarrow{\tilde{\mu}} E^{(n)}$$
given by the global sections $\tilde{\Omega}_i$, composed with the map $\alpha$ of (2.9), induces a map $\mu$ which fits into the following diagram

$$
\begin{align*}
\bigoplus_{i=1}^{n+1} \mathcal{O}_X & \longrightarrow \bigoplus_{i=1}^{n+1} \mathcal{O}_X \\
\mu & \longrightarrow \wedge^{n-1} \mathcal{F} \longrightarrow \mathcal{E}(\alpha) \longrightarrow \det \mathcal{F}(-D_W) \longrightarrow 0.
\end{align*}
$$

The morphism $\mu$ is given by the multiplication by $\tilde{\omega}_i$ on the $i$-th component. Consider the sheaf $\text{Im} \tilde{\mu}$. Locally, on an open subset $U$, we can write

$$
(2.11) \quad s_{i|U} = \sum_{j=1}^{n} b_j^i \cdot \sigma_j + c^i \cdot d\epsilon,
$$

where $(\sigma_1, \ldots, \sigma_n, d\epsilon)$ is a family of local generators of $\mathcal{E}$. By $\Lambda^{n+1}(s_1 \wedge \cdots \wedge s_{n+1}) = 0$ we have

$$
\left| \begin{array}{ccc}
  b_1^2 & \cdots & b_n^2 \\
  \vdots & \ddots & \vdots \\
  b_1^{n+1} & \cdots & b_n^{n+1}
\end{array} \right| c^1 + \cdots + (-1)^n \left| \begin{array}{ccc}
  b_1^1 & \cdots & b_n^1 \\
  \vdots & \ddots & \vdots \\
  b_1^n & \cdots & b_n^n
\end{array} \right| = 0.
$$

Obviously also

$$
\left| \begin{array}{ccc}
  b_1^2 & \cdots & b_n^2 \\
  \vdots & \ddots & \vdots \\
  b_1^{n+1} & \cdots & b_n^{n+1}
\end{array} \right| b_j^1 + \cdots + (-1)^n \left| \begin{array}{ccc}
  b_1^1 & \cdots & b_n^1 \\
  \vdots & \ddots & \vdots \\
  b_1^n & \cdots & b_n^n
\end{array} \right| = 0
$$

for $j = 1, \ldots, n$, where it is easy to see that

$$
\left| \begin{array}{ccc}
  b_1^2 & \cdots & b_n^2 \\
  \vdots & \ddots & \vdots \\
  b_1^{n+1} & \cdots & b_n^{n+1}
\end{array} \right|, \ldots, \left| \begin{array}{ccc}
  b_1^2 & \cdots & b_n^2 \\
  \vdots & \ddots & \vdots \\
  b_1^{n+1} & \cdots & b_n^{n+1}
\end{array} \right|
$$

are respectively the local expressions of the sections $\omega_1, \ldots, \omega_{n+1}$. Now let $d_U$ be a local equation of $d \in H^0(X, \mathcal{O}_X(D_W))$. By (2.10) we have

$$
\left| \begin{array}{ccc}
  b_1^2 & \cdots & b_n^2 \\
  \vdots & \ddots & \vdots \\
  b_1^{n+1} & \cdots & b_n^{n+1}
\end{array} \right| = d_U \cdot f_1, \\
\left| \begin{array}{ccc}
  b_1^1 & \cdots & b_n^1 \\
  \vdots & \ddots & \vdots \\
  b_1^n & \cdots & b_n^n
\end{array} \right| = d_U \cdot f_{n+1}
$$

where the functions $f_i$ are local expressions of $\tilde{\omega}_i$. Then we have

$$
d_U \cdot (f_1 \cdot c^1 + \cdots + (-1)^n f_{n+1} \cdot c^{n+1}) = 0
$$

and

$$
d_U \cdot (f_1 \cdot b_j^1 + \cdots + (-1)^n f_{n+1} \cdot b_j^{n+1}) = 0
$$
for $j = 1, \ldots, n$. Since by definition $d_U$ vanishes on $D_W \cap U$, then on $U$

$$f_1 \cdot c^1 + \cdots + (-1)^n f_{n+1} \cdot c^{n+1} = 0$$

and

$$f_1 \cdot b_j^1 + \cdots + (-1)^n f_{n+1} \cdot b_j^{n+1} = 0$$

for $j = 1, \ldots, n$. We immediately obtain

$$(2.12) \quad f_1 \cdot s_1 + \cdots + (-1)^n f_{n+1} \cdot s_{n+1} = 0.$$  

By definition the scheme $ZW \cap U$ is given by $(f_1 = 0, \ldots, f_{n+1} = 0)$. Let $P \in U$ be a point not in $\text{supp}(ZW)$. At least one of the functions $f_i$ can be inverted in a neighbourhood of $P$, for example let the germ of $f_1$ be nonzero in $P$. We have then a relation

$$s_1 = g_2 \cdot s_2 + \cdots + g_{n+1} \cdot s_{n+1}$$

and so we can easily find holomorphic functions $h_i$ such that

$$\Omega_i = h_i \cdot \Omega_1$$

for $i = 2, \ldots, n+1$. Since

$$\mathcal{E}^{(n)} \xrightarrow{\psi} \bigwedge^n \mathcal{E}$$

is injective, then we have

$$\tilde{\Omega}_i = h_i \cdot \tilde{\Omega}_1$$

for $i = 2, \ldots, n+1$. The section $\tilde{\Omega}_1$ is nonzero, otherwise $\Omega_i = 0$ for $i = 1, \ldots, n+1$ and then also $\omega_i = 0$ for $i = 1, \ldots, n+1$, but this fact contradicts our hypothesis that $\lambda^n W$ is not trivial. In particular we have proved that the sheaf $\text{Im} \tilde{\mu}$ has rank one outside $Z_W$. Furthermore, since it is a subsheaf of the locally free sheaf $\mathcal{E}^{(n)}$, then $\text{Im} \tilde{\mu}$ is torsion free. Denote $\text{Im} \tilde{\mu}$ by $\mathcal{L}$.

By definition

$$\text{Im} \mu := \text{det} \mathcal{F}(-D_W) \otimes \mathcal{I}_{ZW}.$$  

The morphism

$$\alpha : \mathcal{E}^{(n)} \rightarrow \text{det} \mathcal{F}(-D_W)$$

restricts to a surjective morphism, that we continue to call $\alpha$,

$$\mathcal{L} \xrightarrow{\alpha} \text{Im} \mu,$$

between two sheaves that are locally free of rank one outside $Z_W$. The kernel of $\alpha$ is then a torsion subsheaf of $\mathcal{L}$, which is torsion free, hence $\alpha$ gives an isomorphism $\mathcal{L} \cong \text{Im} \mu$.

Since $X$ is normal (actually is smooth) and $Z_W$ has codimension at least 2, we have that the inclusion

$$\mathcal{E}^{(n)} \supset \mathcal{L}^{\vee} \cong \text{det} \mathcal{F}(-D_W)$$

gives the splitting

$$0 \longrightarrow \bigwedge^{n-1} \mathcal{F} \longrightarrow \mathcal{E}^{(n)} \longrightarrow \text{det} \mathcal{F}(-D_W) \longrightarrow 0.$$  

Since $\xi_{D_W}$ is the element of $H^1(X, \mathcal{F}^{\vee}(D_W))$ associated to this extension, we conclude that $\xi_{D_W} = 0$.  

$\square$
As an easy consequence of the adjoint theorem we have the infinitesimal Torelli theorem for primitive varieties with \( p_g = q = n + 1 \); for the notion of primitiveness see [G-A1, Definition 1.2.4], see also our introduction to section 5 of this paper.

**Corollary 2.0.8.** Let \( X \) be an \( n \)-dimensional primitive variety of general type such that \( p_g = q = n + 1 \). If \( \Omega^1_X \) is generated then the 1-infinitesimal Torelli claim holds for \( X \).

**Proof.** Take \( W = H^0(X, \Omega^1_X) \). Since \( X \) is primitive, \( \lambda^W = \langle \omega_1, \ldots, \omega_{n+1} \rangle = H^0(X, \omega_X) \). By the hypothesis \( W = \text{Ker} \partial_\xi \), we can construct an adjoint form \( \omega_{\xi,W,B} \) and obviously \( \omega_{\xi,W,B} \in \lambda^W \). Hence the claim holds by theorem 2.0.7. \( \square \)

The cases studied in corollary 2.0.8 do occur.

**Corollary 2.0.9.** Let \( X \) be a surface of general type such that \( p_g = q = 3 \) and \( K_X^2 = 6 \). Then the 1-infinitesimal Torelli claim holds for \( X \).

**Proof.** Indeed a surface of general type such that \( p_g = q = 3 \) and \( K_X^2 = 6 \) is the symmetric product of a non-hyperelliptic curve of genus 3 since [CCM, Proposition 3.17, (i)]. Hence our proof of Corollary 2.0.8 applies. \( \square \)

**Remark 2.0.10.** Note that for surfaces of general type which are the symmetric product of a hyperelliptic curve of genus 3, the infinitesimal Torelli claim does not hold by [OS]. Indeed \( D_{\Omega^1_X} \) exists and it is a rational \(-2\) curve; see: [CCM, Proposition 3.17, (ii)]. Hence an infinitesimal Torelli deformation is supported on \( D_{\Omega^1_X} \).

### 3. An inverse of the adjoint theorem

Let \( X \) be a smooth compact complex variety of dimension \( m \) and \( \mathcal{F} \) a locally free sheaf of rank \( n \). Let \( \xi \in H^1(X, \mathcal{F}^\vee) \) be the class associated to the extension (2.1). Fix \( W = \langle \eta_1, \ldots, \eta_{n+1} \rangle \subset \text{Ker} (\partial_\xi) \subset H^0(X, \mathcal{F}) \) a subspace of dimension \( n + 1 \) and take an adjoint form \( \omega = \omega_{\xi,W,B} \) for chosen \( s_1, \ldots, s_{n+1} \in H^0(X, \mathcal{E}) \) liftings of respectively \( \eta_1, \ldots, \eta_{n+1} \). As above set \( |\lambda^nW| = D_W + |M_W| \). Note that by construction \( \omega \in H^0(X, \det \mathcal{F} \otimes \mathcal{O}_X(-D_W)) \).

**Theorem 3.0.11.** Assume that \( h^0(X, \mathcal{O}_X(D_W)) = 1 \). If \( \xi_{D_W} = 0 \) then \([\omega] = 0\).

**Proof.** If \( n = m = 1 \) then [CP, Theorem 1.1.8.] shows that \( \xi_{D_W} = 0 \) iff \([\omega] = 0\).

Let \( (\omega)_0 = D_W + C \) be the decomposition of the adjoint divisor in its fixed component and in its moving one. The first step is to construct a global section

\[
\Omega' \in H^0(X, \bigwedge^n \mathcal{E}(-C))
\]
which restricts, through the natural map $\bigwedge^n \mathcal{E}(-C) \to \det \mathcal{F}(-C)$, to $d \in H^0(X, \det \mathcal{F}(-C))$, where $(d)_0 = D_W$. Indeed consider the commutative diagram:

\[
\begin{array}{ccccccc}
0 & \rightarrow & \bigwedge^{n-1} \mathcal{F}(-C) & \rightarrow & \bigwedge^n \mathcal{E}(-C) & \rightarrow & \det \mathcal{F}(-C) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow G_2 & & \downarrow & \\
0 & \rightarrow & \bigwedge^{n-1} \mathcal{F} & \rightarrow & \bigwedge^n \mathcal{E} & \rightarrow & \det \mathcal{F} & \rightarrow & 0 \\
\downarrow H_1 & & \downarrow d & & \downarrow G_1 & & \downarrow \rho_n & \\
0 & \rightarrow & \bigwedge^{n-1} \mathcal{F} \otimes \mathcal{O}_X \mathcal{O}_C & \rightarrow & \bigwedge^n \mathcal{E} \otimes \mathcal{O}_X \mathcal{O}_C & \rightarrow & \det \mathcal{F} \otimes \mathcal{O}_X \mathcal{O}_C & \rightarrow & 0 \\
\downarrow & & \downarrow H_2 & & \downarrow H_3 & & \downarrow & \\
0 & & 0 & & 0 & & 0 & & 0
\end{array}
\]

The adjoint form $\omega \in H^0(X, \det \mathcal{F})$ vanishes when restricted to $C$. Since $\xi_{D_W} = 0$ there exists a lifting $\Omega \in H^0(X, \bigwedge^n \mathcal{E})$ of $\omega$. Indeed, by [2.3], take a global lifting $c \in H^0(X, \det \mathcal{F}(-D_W))$ of $\omega$. Since $\xi_{D_W} = 0$, $c$ can be lifted to a section $e \in H^0(X, \mathcal{E}^{(n)})$. Define $\Omega := \psi(e)$. By commutativity, $H_3(\Omega|_C) = 0$. Hence there exists $\bar{\mu} \in H^0(X, \bigwedge^{n-1} \mathcal{F} \otimes \mathcal{O}_X \mathcal{O}_C)$ such that $H_2(\bar{\mu}) = \Omega|_C$. To construct $\Omega'$ we first show that $\delta(\bar{\mu}) = \xi_{D_W}$ where

$$
\delta: H^0(X, \bigwedge^{n-1} \mathcal{F} \otimes \mathcal{O}_X \mathcal{O}_C) \rightarrow H^1(X, \bigwedge^{n-1} \mathcal{F}(-C)).
$$

To do that we consider the cocycle $\{\xi_{\alpha \beta}\}$ which gives $\xi$ with respect to an open cover $\{U_\alpha\}$ of $X$. For any $\alpha$ there exists $\gamma_\alpha \in \bigwedge^{n-1} \mathcal{F}(U_\alpha)$ such that:

$$
(3.1) \quad \Omega|_{U_\alpha} = \omega_\alpha + \gamma_\alpha \wedge d\epsilon
$$

where $\omega_\alpha$ is obtained by the analogue expression to the one in equation [2.11] but on $\bigwedge^n \mathcal{E}$. A local computation on $U_\alpha \cap U_\beta$ shows that $\omega_\beta - \omega_\alpha = \xi_{\alpha \beta}(\omega) \wedge d\epsilon$, where the notation $\xi_{\alpha \beta}(\omega)$ indicates the natural contraction. Since we have a global lifting of $\omega$, this cocycle must be a coboundary. Hence by (3.1)

$$
(\gamma_\beta - \gamma_\alpha) \wedge d\epsilon = \omega_\beta - \omega_\alpha = \xi_{\alpha \beta}(\omega) \wedge d\epsilon,
$$

that is $\gamma_\beta - \gamma_\alpha = \xi_{\alpha \beta}(\omega)$. Note that the sections $\gamma_\alpha$ are local liftings of $\bar{\mu}$. Indeed by the injectivity of $H_2$, it is enough to show that $(\gamma_\alpha \wedge d\epsilon)|_C = \Omega|_C$, and this is obvious by (3.1) and by the fact that $\omega_\alpha|_C = 0$. Since the morphism

$$
\bigwedge^{n-1} \mathcal{F}(-C)(U_\alpha \cap U_\beta) \rightarrow \bigwedge^{n-1} \mathcal{F}(U_\alpha \cap U_\beta)
$$

is given by multiplication by $c_{\alpha \beta}$, where $c_{\alpha \beta}$ is a local equation of $C$ on $U_\alpha \cap U_\beta$, then the desired expression of the coboundary $\delta(\bar{\mu})$ is given by the cocycle

$$
\frac{\gamma_\beta - \gamma_\alpha}{c_{\alpha \beta}} = \frac{\xi_{\alpha \beta}(\omega)}{c_{\alpha \beta}}.
$$
By the above local computation, we see that this cocycle gives $\xi_{D_W}$ via the isomorphism

$$H^1(X, \bigwedge^{n-1} \mathcal{F}(-C)) \cong H^1(X, \mathcal{F}^\vee(D_W)).$$

This is easy to prove since $\xi_{D_W}$ is locally given by $\xi_{\alpha\beta} \cdot d_{\alpha\beta}$, where $d_{\alpha\beta}$ is the equation of $D_W$ on $U_{\alpha} \cap U_{\beta}$, so the image of $\xi_{D_W}$ through the isomorphism (3.2) is $\xi_{\alpha\beta}d_{\alpha\beta}c_{\alpha\beta} = \xi_{\alpha\beta}(\omega)$, which is exactly $\delta(\bar{\mu})$. Now we use again our hypothesis $\xi_{D_W} = 0$ to write $\delta(\bar{\mu}) = 0$. Then there exists a global section $\mu \in H^0(X, \bigwedge^{n-1} \mathcal{F})$ which is a lifting of $\bar{\mu}$. This gives the global section

$$\tilde{\Omega} := \Omega - \mu \wedge d\epsilon \in H^0(X, \bigwedge^n \mathcal{E}).$$

By construction $\tilde{\Omega}$ is a new lifting of $\omega$ which now vanishes once restricted to $C$:

$$\tilde{\Omega}|_C = \Omega|_C - \mu \wedge d\epsilon|_C = \Omega|_C - H_2(\bar{\mu}) = 0.$$

The wanted section $\Omega' \in H^0(X, \bigwedge^n \mathcal{E}(-C))$ is the global section which lifts $\tilde{\Omega}$ and by construction satisfies $\rho_2(G_1(\Omega')) = \omega$ and $G_2(\Omega') = d$.

The second step is to use $\Omega'$ to show that $[\omega] = 0$. The global sections $\omega_i := \lambda^a(\eta_1 \wedge \cdots \wedge \tilde{\eta}_i \wedge \cdots \wedge \eta_{n+1}) \in H^0(X, \det \mathcal{F})$ generate $\lambda^aW$ and by definition they vanish on $D_W$, that is there exist global sections $\tilde{\omega}_i \in H^0(X, \det \mathcal{F}(-D_W))$ such that $\omega_i = \tilde{\omega}_i \cdot d$. We consider now the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_X(-C) & \xrightarrow{\alpha} & W \otimes \mathcal{O}_X & \xrightarrow{\gamma} & \tilde{\mathcal{F}} & \longrightarrow & 0 \\
\downarrow c & & \downarrow \beta & & \downarrow \rho_1 & & \downarrow i & & \\
0 & \longrightarrow & \mathcal{O}_X & \xrightarrow{d\epsilon} & \mathcal{E} & \xrightarrow{\rho_1} & \tilde{\mathcal{F}} & \longrightarrow & 0.
\end{array}$$

The homomorphism $\beta$ is locally defined by

$$(f_1, \ldots, f_{n+1}) \mapsto (-1)^n f_1 \cdot s_1 + \cdots + f_{n+1} \cdot s_{n+1}$$

and similary $\rho_1 \circ \beta$ is given by

$$(f_1, \ldots, f_{n+1}) \mapsto (-1)^n f_1 \cdot \eta_1 + \cdots + f_{n+1} \cdot \eta_{n+1}.$$
then
\[
\beta(\alpha(f)) = \beta \left( f_U \left| \begin{array}{ccc}
\begin{array}{ccc}
 b_1^2 & \cdots & b_n^2 \\
 \vdots & \ddots & \vdots \\
 b_1^{n+1} & \cdots & b_n^{n+1}
\end{array}
\end{array} \right| \frac{1}{d_U}, \ldots, f_U \left| \begin{array}{ccc}
\begin{array}{ccc}
 b_1^1 & \cdots & b_n^1 \\
 \vdots & \ddots & \vdots \\
 b_1^{n+1} & \cdots & b_n^{n+1}
\end{array}
\end{array} \right| \frac{1}{d_U} \right) =
\]
\[
= \left(-1\right)^n f_U \left| \begin{array}{ccc}
\begin{array}{ccc}
 b_1^2 & \cdots & b_n^2 \\
 \vdots & \ddots & \vdots \\
 b_1^{n+1} & \cdots & b_n^{n+1}
\end{array}
\end{array} \right| \frac{1}{d_U} s_1 + \cdots + f_U \left| \begin{array}{ccc}
\begin{array}{ccc}
 b_1^1 & \cdots & b_n^1 \\
 \vdots & \ddots & \vdots \\
 b_1^{n+1} & \cdots & b_n^{n+1}
\end{array}
\end{array} \right| \frac{1}{d_U} s_{n+1} =
\]
\[
= f_U \left| \begin{array}{ccc}
\begin{array}{ccc}
 b_1^1 & \cdots & b_1^{c_1} \\
 b_2^1 & \cdots & b_2^{c_2} \\
 \vdots & \ddots & \vdots \\
 b_1^{n+1} & \cdots & b_n^{n+1}
\end{array}
\end{array} \right| \frac{1}{d_U} d\epsilon =
\]
\[
= (f_U \cdot c_U) d\epsilon,
\]

where \( f_U \) and \( d_U \) are local holomorphic functions which represent the sections \( f \) and \( d \) respectively. The first equality uses the fact that the determinants appearing in the first line are the local equations of the sections \( \omega_i \) (see also the proof of theorem 2.0.7); the last equality comes from the fact that the determinant in the second to last line is the local equation of \( \omega \), and \( \omega = d \cdot c \). To dualize diagram (3.3) we recall the sheaves isomorphisms \( \mathcal{F}^\vee \cong \bigwedge^{n-1} \mathcal{F}(-C - D_W) \) and \( \mathcal{E}^\vee \cong \bigwedge^n \mathcal{E}(-C - D_W) \). Moreover we also recall the isomorphism \( W^\vee \cong \bigwedge^n W \), given by

\[
\eta^i \mapsto \eta_1 \wedge \ldots \wedge \eta_i \wedge \ldots \wedge \eta_{n+1}
\]

where \( \eta^1, \ldots, \eta^{n+1} \) is the basis of \( W^\vee \) dual to the basis \( \eta_1, \ldots, \eta_{n+1} \) of \( W \). Define

\[
e_i := \eta_1 \wedge \ldots \wedge \eta_i \wedge \ldots \wedge \eta_{n+1}.
\]

Now we dualize (3.3):

\[
(3.4) \quad \begin{array}{c}
0 \longrightarrow \bigwedge^n W \otimes \mathcal{O}_X \longrightarrow \bigwedge^n \mathcal{E}(-C - D_W) \\
\downarrow \alpha^\vee \quad \downarrow \beta^\vee \quad \downarrow \gamma^\vee \\
0 \longrightarrow \bigwedge^{n-1} \mathcal{F}(-C - D_W) \longrightarrow \bigwedge^n \mathcal{E}(-C - D_W) \longrightarrow \mathcal{O}_X \longrightarrow 0.
\end{array}
\]

Here \( \alpha^\vee \) is the evaluation map given by the global sections \( \tilde{\omega}_i \), not necessarily surjective. Nevertheless we tensor by \( \mathcal{O}_X(D_W) \) and we obtain

\[
\begin{array}{r}
\bigwedge^n W \otimes H^0(X, \mathcal{O}_X(D_W)) \longrightarrow \bigwedge^n \mathcal{E}(-C) \\
\downarrow \alpha^\vee \quad \downarrow \beta^\vee \\
H^0(X, \bigwedge^n \mathcal{E}(-C)) \longrightarrow H^0(X, \mathcal{O}_X(D_W)).
\end{array}
\]

The section \( \Omega' \in H^0(X, \bigwedge^n \mathcal{E}(-C)) \) constructed in the first part of the proof gives in \( H^0(X, \det \mathcal{F}) \) the adjoint \( \omega \). By commutativity

\[
\omega = \overline{\alpha^\vee}(\overline{\beta^\vee}(\Omega')).
\]
By our hypothesis $h^0(X, \mathcal{O}_X(D_W)) = 1$, the section $d$ is a basis of $H^0(X, \mathcal{O}_X(D_W))$, so $(e_1 \otimes d, \ldots, e_{n+1} \otimes d)$ is a basis of $\bigwedge^n W \otimes H^0(X, \mathcal{O}_X(D_W))$. We have then

$$\beta^\vee(\Omega') = \sum_{i=1}^{n+1} c_i \cdot e_i \otimes d$$

where $c_i \in \mathbb{C}$ and

$$\omega = \alpha^\vee(\beta^\vee(\Omega')) = \alpha^\vee(\sum_{i=1}^{n+1} c_i \cdot e_i \otimes d) = \sum_{i=1}^{n+1} c_i \cdot \hat{\omega}_i \cdot d = \sum_{i=1}^{n+1} c_i \cdot \omega_i,$$

and hence $[\omega] = 0$. \hfill $\Box$

### 4. An infinitesimal Torelli-type theorem

Theorem 2.0.7 can be used to show splitting criteria for extension classes of *generically globally generated* locally free sheaves. To this aim we consider a locally free sheaf $\mathcal{F}$ of rank $n$ over an $m$-dimensional smooth variety $X$. We remind the reader that generically globally generated means that $\mathcal{F}$ is generated by the global sections outside a loci of codimension at least 1.

Naturally associated to $\mathcal{F}$ there is the invertible sheaf $\det \mathcal{F}$ and the natural homomorphism:

$$\lambda^n: \bigwedge^n H^0(X, \mathcal{F}) \to H^0(X, \det \mathcal{F}).$$

We denote by $\lambda^n H^0(X, \mathcal{F})$ its image. Consider the linear system $\mathbb{P}(\lambda^n H^0(X, \mathcal{F}))$ and denote by $D_{\mathcal{F}}$ its fixed component and by $|M_{\mathcal{F}}|$ its associated mobile linear system. By [PZ, Proposition 3.1.6], if $W$ is a generic $n+1$-dimensional subspace of $H^0(X, \mathcal{F})$, then $D_{\mathcal{F}} = D_W$. Moreover we denote by $|\det \mathcal{F}|$ the linear system associated to $\det \mathcal{F}$ and by $D_{\det \mathcal{F}}, M_{\det \mathcal{F}}$ respectively its fixed and its movable part; that is: $|\det \mathcal{F}| = D_{\det \mathcal{F}} + |M_{\det \mathcal{F}}|$. Finally note that $D_{\det \mathcal{F}}$ is a sub-divisor of $D_{\mathcal{F}}$.

**Theorem 4.0.12.** Let $X$ be an $m$-dimensional smooth variety where $m \geq 2$. Let $\mathcal{F}$ be a generically globally generated locally free sheaf of rank $n$ such that $h^0(X, \mathcal{F}) \geq n + 1$. Let $Y$ be the schematic image of $\phi|_{M_{\det \mathcal{F}}}: X \dasharrow \mathbb{P}(H^0(X, \det \mathcal{F})^\vee)$. Assume that $H^0(Y, \mathcal{I}_Y(2))$ contains no quadric of rank $\leq 2n + 3$, where $\mathcal{I}_Y$ is the ideal sheaf of $Y$. If $\xi \in H^1(X, \mathcal{F}^\vee)$ is such that $\partial^n_\xi(\omega) = 0$, where $\omega$ is an adjoint form associated to an $n + 1$-dimensional subspace $W \subset \ker \partial_\xi \subset H^0(X, \mathcal{F})$, then $[\omega] = 0$.

**Proof.** Let $B = \{\eta_1, \ldots, \eta_{n+1}\}$ be a basis of $W$. Set $\omega_i := \lambda^n(\eta_1 \wedge \cdots \wedge \hat{\eta}_i \wedge \cdots \wedge \eta_{n+1}) \in H^0(X, \det \mathcal{F})$ where $i = 1, \ldots, n + 1$, and denote by $\hat{\omega}_i \in H^0(\det \mathcal{F}(-D_W) \otimes \mathcal{I}_W)$ the corresponding sections via $0 \to H^0(X, \det \mathcal{F}(-D_W) \otimes \mathcal{I}_W) \to H^0(X, \det \mathcal{F})$. Recall that $\lambda^n W := \langle \omega_1, \ldots, \omega_{n+1} \rangle \subset H^0(X, \det \mathcal{F})$ is the vector space generated by the sections $\omega_i$. The standard evaluating map $\bigwedge^n W \otimes \mathcal{O}_X \to \det \mathcal{F}(-D_W) \otimes \mathcal{I}_W$ given by $\hat{\omega}_1, \ldots, \hat{\omega}_{n+1}$ results in the following exact sequence

$$\begin{array}{cccccc}
0 & \to & \mathcal{K} & \to & \bigwedge^n W \otimes \mathcal{O}_X & \to \det \mathcal{F}(-D_W) \otimes \mathcal{I}_W & \to 0
\end{array}$$

(4.2)
which is associated to the class $\xi' \in \operatorname{Ext}^1(\det \mathcal{F}(-D_W) \otimes \mathcal{I}_Z, \mathcal{K})$. This sequence fits into the following commutative diagram (cf. (3.4))

\[
\begin{array}{ccccccccc}
0 & \rightarrow & K & \rightarrow & \bigwedge^n W \otimes \mathcal{O}_X & \rightarrow & \det \mathcal{F}(-D_W) \otimes \mathcal{I}_Z & \rightarrow & 0 \\
& & f & & & & g & & \\
0 & \rightarrow & \mathcal{F}^\vee & \rightarrow & \mathcal{E}^\vee & \rightarrow & \mathcal{O}_X & \rightarrow & 0,
\end{array}
\]

where $f$ is the map given by the contraction by the sections $(-1)^{n+1-i}s_i$, for $i = 1, \ldots, n + 1$, and $g$ is given by the global section $\sigma \in H^0(X, \det \mathcal{F}(-D_W) \otimes \mathcal{I}_Z)$ corresponding to the adjoint $\omega$. We have the standard factorization

\[
\begin{array}{ccccccccc}
0 & \rightarrow & K & \rightarrow & \bigwedge^n W \otimes \mathcal{O}_X & \rightarrow & \det \mathcal{F}(-D_W) \otimes \mathcal{I}_Z & \rightarrow & 0 \\
& & & & & & \mathcal{L} & & \mathcal{O}_X & 0 \\
0 & \rightarrow & \mathcal{F}^\vee & \rightarrow & \mathcal{E}^\vee & \rightarrow & \mathcal{O}_X & \rightarrow & 0
\end{array}
\]

where the sequence in the middle is associated to the class $\xi'' \in H^1(X, \mathcal{K})$ which is the image of $\xi \in H^1(X, \mathcal{F}^\vee)$ through the map $H^1(X, \mathcal{F}^\vee) \rightarrow H^1(X, \mathcal{K})$. In particular we obtain the commutative square

\[
\begin{array}{ccccccccc}
H^0(X, \det \mathcal{F}(-D_W) \otimes \mathcal{I}_Z) & \rightarrow & H^1(X, \mathcal{K}) \\
\uparrow & & \uparrow \\
H^0(X, \mathcal{O}_X) & \rightarrow & H^1(X, \mathcal{K}).
\end{array}
\]

By commutativity we immediately have that the image of $\sigma$ through the coboundary map $H^0(X, \det \mathcal{F}(-D_W) \otimes \mathcal{I}_Z) \rightarrow H^1(X, \mathcal{K})$ is $\xi''$. Tensoring by $\mathcal{F}$, the map $\mathcal{F}^\vee \rightarrow \mathcal{K}$ gives

\[
\begin{array}{ccccccccc}
\mathcal{F}^\vee \otimes \det \mathcal{F} & \rightarrow & \mathcal{K} \otimes \det \mathcal{F} \\
& & & & & & r & & \\
\bigwedge^{n-1} \mathcal{F}
\end{array}
\]

and, since $\xi \cdot \omega \in H^1(X, \mathcal{F}^\vee \otimes \det \mathcal{F})$ is sent to $\xi'' \cdot \omega \in H^1(X, \mathcal{K} \otimes \det \mathcal{F})$, we have that

\[
H^1(\Gamma)(\xi \cup \omega) = \xi'' \cdot \omega,
\]

where $\xi \cup \omega$ is the cup product.

By hypothesis $\partial_\mathcal{K}^n(\omega) = \xi \cup \omega = 0 \in H^1(X, \bigwedge^{n-1} \mathcal{F})$, so also $\xi'' \cdot \omega = 0 \in H^1(X, \mathcal{K} \otimes \det \mathcal{F})$, hence the global section $\sigma \cdot \omega \in H^0(X, \det \mathcal{F}(-D_W) \otimes \mathcal{I}_Z \otimes \det \mathcal{F})$ is in the kernel of the coboundary map $H^0(X, \det \mathcal{F}(-D_W) \otimes \mathcal{I}_Z \otimes \det \mathcal{F}) \rightarrow H^1(X, \mathcal{K} \otimes \det \mathcal{F})$ associated to the sequence

\[
\begin{array}{ccccccccc}
0 & \rightarrow & K \otimes \det \mathcal{F} & \rightarrow & \bigwedge^n W \otimes \det \mathcal{F} & \rightarrow & \det \mathcal{F}(-D_W) \otimes \mathcal{I}_Z \otimes \det \mathcal{F} & \rightarrow & 0.
\end{array}
\]
This occurs if there exist $L_i^\sigma \in H^0(X, \det F_i)$, $i = 1, \ldots, n + 1$ such that
\begin{equation}
\sigma \cdot \omega = \sum_{i=1}^{n+1} L_i^\sigma \cdot \omega_i.
\end{equation}
This relation gives the following relation in $H^0(X, \det F^\otimes 2)$:
\begin{equation}
\omega \cdot \omega = \sum_{i=1}^{n+1} L_i^\sigma \cdot \omega_i.
\end{equation}
Assume now that the adjoint form $\omega$ is not in the vector space $\lambda^n W$. Then the equation (4.10) gives a nontrivial quadric in $H^0(Y, I_Y(2))$ of rank at most $2n + 3$. Since we have assumed that there are no such quadrics in $H^0(Y, I_Y(2))$ our claim follows immediately by contradiction. 

The following is theorem [B] of the introduction:

**Corollary 4.0.13.** Let $X$ be an $m$-dimensional smooth variety where $m \geq 2$. Let $F$ be a generically globally generated locally free sheaf of rank $n$ such that $h^0(X, F) \geq n + 1$. Let $Y$ be the schematic image of $\phi|_{M_{\det F}}: X \dashrightarrow \mathbb{P}(H^0(X, \det F)^\vee)$. Assume that $H^0(Y, I_Y(2))$ contains no quadric of rank $\leq 2n + 3$, where $I_Y$ is the ideal sheaf of $Y$. If $\xi \in H^1(X, F^\vee)$ is such that $\partial_\xi = 0$ and $\partial_\xi^p(\omega) = 0$, where $\omega$ is an adjoint form associated to a generic $n + 1$-dimensional subspace $W \subset H^0(X, F)$, then $\xi$ is a supported deformation on $D_F$, that is, $\xi_{D_F}$ is trivial.

*Proof.* The claim follows by theorem 2.0.7 and by the fact that $D_W = D_F$, for the general $W$. 

Note that there are cases where our hypothesis applies easily:

**Corollary 4.0.14.** Let $X$ be an $m$-dimensional smooth variety where $m \geq 2$. Let $F$ be a generically globally generated locally free sheaf of rank $n$. Assume that $\phi|_{M_{\det F}}: X \dashrightarrow \mathbb{P}(H^0(X, \det F)^\vee)$ is a non trivial rational map such that its schematic image is a complete intersection of hypersurfaces of degree $> 2$. Let $\xi \in H^1(X, F^\vee)$. If $\partial_\xi = 0$ and $\partial_\xi^p(\omega) = 0$ where $\omega$ is an adjoint form associated to a generic $n + 1$-dimensional subspace $W \subset H^0(X, F)$, then $\xi$ is a supported deformation on $D_F$. In particular if $D_F = 0$ then $\xi = 0$.

*Proof.* The claim follows straightly by corollary 4.0.13. 

**Corollary 4.0.15.** Let $X$ be an $n$-dimensional variety of general type with irregularity $\geq n + 1$ and such that its cotangent sheaf is generated. Suppose also that there are no quadrics of rank less or equal to $2n + 3$ containing the canonical image. Then the infinitesimal Torelli claims hold for $X$.

*Proof.* By corollary 4.0.13 any $\xi \in H^1(X, \Theta_X)$ such that $\partial_\xi = 0$ and $\partial_\xi^p(\omega) = 0$, where $\omega$ is an adjoint form associated to a generic $n + 1$-dimensional subspace $W \subset H^0(X, \Omega_X^1)$, is supported on the branch loci of the Albanese morphism and, since we have assumed it to be trivial, then the trivial infinitesimal deformation is the only possible case.

**Remark 4.0.16.** As a typical application we obtain infinitesimal Torelli for a smooth divisor $X$ of an $n + 1$-dimensional Abelian variety $A$ such that $\rho_g(X) = n + 2$; for explicit examples consider the case of a smooth surface $X$ in polarisation $(1, 1, 2)$ in an abelian 3-fold $A$. The
invariants of $X$ are $p_g(X) = 4$, $q(X) = 3$ and $K^2 = 12$. The canonical map is in general a birational morphism onto a surface of degree 12. See [CS, Theorem 6.4].

5. Families with birational fibers

Theorem 4.0.12 provides us a criterion to understand which families of certain irregular varieties of general type have birational fibers. First we recall some basic definitions.

5.1. Families of Morphisms. A family of $n$-dimensional varieties is a flat smooth proper morphism $\pi: \mathcal{X} \to B$ such that the fiber $X_b := \pi^{-1}(b)$ over a point $b \in B$ has dimension $n$. The variety $B$ is called base of the family. We only assume that $B$ is smooth connected and analytic. We restrict our study to the case where $X_b$ is an irregular variety of general type. We recall that any irregular variety $X$ comes equipped with its Albanese variety $\text{Alb}(X)$ and its Albanese morphism $\text{alb}(X): X \to \text{Alb}(X)$.

A fibration is a surjective proper flat morphism $f: X \to Z$ with connected fibres between the smooth varieties $X$ and $Z$. A fibration $f: X \to Z$ is irregular if $Z$ is an irregular variety.

We recall that a smooth irregular variety $X$ is said to be of maximal Albanese dimension if $\dim \text{alb}(X) = \dim X$. If $\dim \text{alb}(X) = \dim X$ and $\text{alb}: X \to \text{Alb}(X)$ is not surjective, that is $q(X) > \dim X$, $X$ is said to be of Albanese general type. A fibration $f: X \to Z$ is called a higher irrational pencil if $Z$ is of Albanese general type. An irregular variety $X$ is said to be primitive if it does not admit any higher irrational pencil; see: [G-A1, Definition 1.2.4]. Note that irregular fibrations (resp. higher irrational pencils) are higher-dimensional analogues to fibrations over non-rational curves (resp. curves of genus $g \geq 2$).

To study a family $\pi: \mathcal{X} \to B$ of irregular varieties it is natural to consider the case where it comes equipped with a family $p: \mathcal{A} \to B$ of Abelian varieties; that is, the fiber $A_b := p^{-1}(b)$ is an Abelian variety of dimension $a > 0$.

**Definition 5.1.1.** Let $\pi: \mathcal{X} \to B$ be a family of irregular varieties of general type and $p: \mathcal{A} \to B$ a family of Abelian varieties. A morphism $\Phi: \mathcal{X} \to \mathcal{A}$ will be called a family of Albanese type over $B$ if:

(i) $\Phi$ fits into the commutative diagram

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\Phi} & \mathcal{A} \\
\pi \downarrow & & \downarrow p \\
B.
\end{array}
\]

(ii) The induced map $\phi_b: X_b \to A_b$ of $\Phi$ on $X_b$ is birational onto its image $Y_b$.

(iii) The cycle $Y_b$ generates the fiber $A_b$ as a group.

See: [PZ, Definition 1.1.1]. We remark that $a > n$ (cf. see [H, p.311 and Corollary to Theorem 10.12,(i)])]. We shall say that two families over $B$, $\Phi: \mathcal{X} \to \mathcal{A}$ and $\Psi: \mathcal{Y} \to \mathcal{A}$, of Albanese type have the same image if it is true fiberwise, that is $\phi_b(X_b) = \psi_b(Y_b)$ for every $b \in B$.

**Base change.** Albanese type families have a good behaviour under base change. In fact let $\mu: B' \to B$ be a base change, then $\mu^*(\Phi) = \Phi \times \text{id}: \mathcal{X} \times_B B' \to \mathcal{A} \times_B B'$ is an Albanese type family over $B'$. In particular for a connected subvariety $C \hookrightarrow B$ the pull-back restriction of $\Phi: \mathcal{X} \to \mathcal{A}$ to $C$ is well defined and it will be denoted by $\Phi_C: \mathcal{X}_C \to \mathcal{A}_C$. 
Translated family. If $s: B \to A$ is a section of $p: A \to B$, we define the translated family $\Phi_s: X \to A$ of $\Phi$ by the formula:

$$\Phi_s(x) = \Phi(x) + s(\pi(x)).$$

Notice that $\Phi_s: X \to A$ is a family of Albanese type. Two families $\Phi$ and $\Psi$ over $B$ are said to be translation equivalent if there exists a section $\sigma$ of $p$ such that the images of $\Phi_\sigma$ and $\Psi$ (fiberwise) coincide.

We are interested to find a condition to have that, up to restriction to a nontrivial open subset of $B$, the fibers of the restricted family are reciprocally birational. To this aim and to ease reading we recall the following definition given in [PZ, definition 1.1.2]:

**Definition 5.1.2.** Two families of Albanese type $\Phi: X \to A$, $\Phi': X' \to A'$ over, respectively, $B$ and $B'$ will be said locally equivalent, if there exist an open set $U \subset B$ an open set $U' \subset B'$ and a biregular map $\mu: U' \to U := \mu(U') \subset B$ such that the pull-back families $\mu^*(\Phi_{U'})$ and $\Phi'_{U'}$ are translation equivalent. We will say that $\Phi$ is trivial if $X = X \times B$, $A = A \times B$ and $\pi_A(\Phi(X_b)) = \pi_A(\Phi(X_{b_0}))$ for all $b$ where $\pi_A: A \times B \to A$ is the natural projection.

**Polarization.** We recall that a polarised variety is a couple $(X, H)$ where $X$ is a variety and $H$ is a big and nef divisor on it. The Albanese type family $\Phi$ provides the fibers $A_b$ with a natural polarization. Letting $\omega$ a Chern form of the canonical divisor $K_{X_b}$, the Hermitian form of the polarization $\Xi_b$ is defined on $H^{1,0}(A_b)$ by:

$$\Xi_b(\eta_1, \eta_2) = \int_{X_b} \phi_b^* \eta_1 \wedge \phi_b^* \eta_2 \wedge \omega^{n-1}. \quad (5.1)$$

Notice that, in this way, $p: A \to B$ becomes a family of polarized Abelian varieties. There is then a suitable moduli variety $\mathcal{A}_a$ parameterizing polarized Abelian variety of dimension $a$, and a holomorphic map, called the period map, $P: B \to \mathcal{A}_a$ defined by $P(b) = (A_b, \Xi_b)$. Finally we remark that if $\mathcal{P}(B)$ is a point, then, up to shrinking $B$, $p: A \to B$ is equivalent to the trivial family, $A \to A \times B$ where $A$ is the fixed Abelian variety corresponding to $\mathcal{P}(B)$.

**Example.** The standard example of Albanese type family is given by a family $\pi: \mathcal{X} \to B$ with a section $s: B \to \mathcal{X}$. Indeed by $s: B \to \mathcal{X}$ we have a family $p: \text{Alb}(\mathcal{X}) \to B$ whose fiber is $p^{-1}(b) = \text{Alb}(X_b)$; the section gives a family $\Phi: \mathcal{X} \to \text{Alb}(\mathcal{X})$ with fiber:

$$\text{alb}(X_b) : X_b \to \text{Alb}(X_b).$$

If we also assume that $\phi_b = \text{alb}(X_b)$ has degree 1 onto the image then $\Phi: \mathcal{X} \to \text{Alb}(\mathcal{X})$ is an Albanese type family. We will call $\mathcal{X} \xrightarrow{\Phi} \text{Alb}(\mathcal{X})$ an Albanese family.

We will use the following:

**Proposition 5.1.3.** An Albanese type family $\Phi: \mathcal{X} \to A$ is locally equivalent to a trivial family if and only if the fibers $X_b$ are reciprocally birational.

**Proof.** See [PZ, Proposition 1.1.3].
5.2. Families with liftability conditions. To find conditions which force the fibers of a family $\pi: \mathcal{X} \to B$ to be birationally equivalent, it is natural to understand conditions on Albanese type families whose associated family of Abelian varieties is trivial. The easiest condition to think on is given by conditions on the space of 1-forms.

**Proposition 5.2.1.** Let $\Phi: \mathcal{X} \to \mathcal{A}$ be an Albanese type family such that for every $b \in B$ it holds that $H^0(\mathcal{X}, \Omega^1_{\mathcal{X}}) \to H^0(X_b, \Omega^1_{X_b})$. Then up to shrinking $B$ the fibers of $p: \mathcal{A} \to B$ are isomorphic.

**Proof.** Let $\mu_b \in \text{Ext}^1(\Omega^1_{\mathcal{A}_b}, \mathcal{O}_{\mathcal{A}_b})$ be the class given by the family $p: \mathcal{A} \to B$, that is the class of the following extension:

$$0 \to \mathcal{O}_{\mathcal{A}_b} \to \Omega^1_{\mathcal{A}_b|A_b} \to \Omega^1_{\mathcal{A}_b} \to 0.$$ 

Since $\phi_b$ is flat we have the following sequence

$$0 \to \phi^*_b \mathcal{O}_{\mathcal{A}_b} \to \phi^*_b \Omega^1_{\mathcal{A}_b|A_b} \to \phi^*_b \Omega^1_{\mathcal{A}_b} \to 0$$

which fits into the following diagram

$$
\begin{array}{cccccc}
0 & \phi^*_b \mathcal{O}_{\mathcal{A}_b} & \phi^*_b \Omega^1_{\mathcal{A}_b|A_b} & \phi^*_b \Omega^1_{\mathcal{A}_b} & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
0 & \mathcal{O}_{X_b} & \Omega^1_{\mathcal{X}_b|X_b} & \Omega^1_{X_b} & 0.
\end{array}
$$

In cohomology we have

$$H^0(X_b, \phi^*_b \Omega^1_{\mathcal{A}_b}) \to H^1(X_b, \mathcal{O}_{X_b})$$

so, by commutativity and by the hypothesis $H^0(\mathcal{X}, \Omega^1_{\mathcal{X}}) \to H^0(X_b, \Omega^1_{X_b})$, we immediately obtain $H^0(X_b, \phi^*_b \Omega^1_{\mathcal{A}_b|A_b}) \to H^0(X_b, \phi^*_b \Omega^1_{\mathcal{A}_b})$ and hence the coboundary $\partial_{\mu_b} : H^0(A_b, \Omega^1_{A_b}) \to H^1(A_b, \mathcal{O}_{A_b})$ is trivial.

Then by cf. [CP, Page 78] we conclude. \hfill \Box

For an Albanese type family $\Phi: \mathcal{X} \to \mathcal{A}$ such that $H^0(\mathcal{X}, \Omega^1_{\mathcal{X}}) \to H^0(X_b, \Omega^1_{X_b})$ we can say more. Actually up to shrinking $B$ it is trivial to show that for every $b \in B$ it holds that $\text{Alb}(X_b) = A$ where $A$ is a fixed Abelian variety. For later reference we state:

**Corollary 5.2.2.** Let $\Phi: \mathcal{X} \to \mathcal{A}$ be an Albanese type family such that $H^0(\mathcal{X}, \Omega^1_{\mathcal{X}}) \to H^0(X_b, \Omega^1_{X_b})$, where $b \in B$. Then for every $b \in B$ there exists an open neighborhood $U$ such that the restricted family $\Phi_U: \mathcal{X}_U \to \mathcal{A}_U$ is locally equivalent to $\Psi: \mathcal{X}_U \to \hat{A} \times U$, where $\hat{A}$ is an Abelian variety. Moreover there exists another Abelian variety $A$ such that for every $b \in U$ it holds that $A = \text{Alb}(X_b)$ and the natural morphism $A \to \hat{A}$ gives a factorisation of $\Psi: \mathcal{X}_U \to \hat{A} \times U$ via $\text{alb}(\mathcal{X}_U): \mathcal{X}_U \to A \times U$.

We recall a definition given in the Introduction of this paper and suitable to study Torelli type problems for irregular varieties.

**Definition 5.2.3.** We say that a family $f: \mathcal{X} \to B$ of relative dimension $n$ satisfies extremal liftability conditions over $B$ if
(i) $H^0(X, \Omega_X^1) \rightarrow H^0(X_b, \Omega_{X_b}^1)$;
(ii) $H^0(X, \Omega^n_X) \rightarrow H^0(X_b, \Omega^n_{X_b})$.

**Remark 5.2.4.** Let $X$ be a smooth variety such that $\text{alb}(X): X \rightarrow \text{Alb}(X)$ has degree 1. Let $f: X \rightarrow Z$ be a fibration of relative dimension $n$ such that the general fiber $f^{-1}(y) = X_y$ is smooth of general type and with irregularity $q \geq n + 1$. By Abelian subvarieties rigidity, the image of the map $\text{Alb}(X_y) \rightarrow \text{Alb}(X)$ is a (translate of a) fixed abelian variety $A$. If, moreover, we have surjections $H^0(X, \Omega_X^1) \rightarrow H^0(X_y, \Omega_{X_y}^1)$ and $H^0(X, \Omega^n_X) \rightarrow H^0(X, \Omega^n_{X_y})$, then $A = \text{Alb}(X_y)$ and taking a sufficiently small polydisk around any smooth fibre of $f$ we obtain a family which satisfies extremal liftability conditions.

5.3. **The theorem.** Our main theorem is grounded on the Volumetric theorem; see [PZ Theorem 1.5.3]:

**Theorem 5.3.1.** Let $\Phi: \mathcal{X} \rightarrow \mathcal{A}$ be an Albanese type family such that $p: \mathcal{A} \rightarrow B$ has fibers isomorphic to a fixed Abelian variety $A$. Let $W \subset H^0(\mathcal{A}, \Omega^1_{\mathcal{A}})$ be a generic $n+1$-dimensional subspace and $W_b \subset H^0(X_b, \Omega_{X_b}^1)$ its pull-back over the fiber $X_b$. Assume that for every point $b \in B$ it holds that $\omega_{\xi_b, W_b} \in \lambda^n W_b$ where $\xi_b \in H^1(X_b, \Theta_{X_b})$ is the class given on $X_b$ by $\pi: \mathcal{X} \rightarrow B$, then the fibers of $\pi: \mathcal{X} \rightarrow B$ are birational.

**Proof.** See [PZ Theorem 1.5.3].

In the next theorem we do not assume that $|K_X|$ is without fixed component where $X$ is a general fiber of a Torelli type family.

**Theorem 5.3.2.** Let $\Phi: \mathcal{X} \rightarrow \mathcal{A}$ be a family of Albanese type whose associated family of $n$-dimensional irregular varieties $\pi: \mathcal{X} \rightarrow B$ satisfies extremal liftability conditions. Assume that for any fiber $X$ of $\pi: \mathcal{X} \rightarrow B$ no quadric of rank $\leq 2n+3$ contains the canonical image of $X$ then the fibers of $\pi: \mathcal{X} \rightarrow B$ are birational.

**Proof.** Since our claim is local in the analytic category, up to base change, we can assume that $B$ is a 1-dimensional disk and that $\pi: \mathcal{X} \rightarrow B$ has a section. In particular by proposition 5.2.4 $p: \mathcal{A} \rightarrow B$ is trivial. Moreover the Albanese family $\text{alb}(\mathcal{X}): \mathcal{X} \rightarrow \text{Alb}(\mathcal{X})$ exists and by proposition 5.1.3 our claim is equivalent to show that the Albanese family $\text{alb}(\mathcal{X}): \mathcal{X} \rightarrow \text{Alb}(\mathcal{X})$ is locally equivalent to the trivial family. By corollary 5.2.2 we can assume that $\text{Alb}(\mathcal{X}) = \mathcal{A} \times B$ too.

We denote by $\xi_b \in H^1(X_b, \Theta_{X_b})$ the class associated to the infinitesimal deformation of $X_b$ induced by $\pi: \mathcal{X} \rightarrow B$. First we assume that $q > n+1$.

Let $\mathcal{B} := \{dz_1, \ldots, dz_{n+1}\}$ be a basis of an $n+1$-dimensional generic subspace $W$ of $H^0(\mathcal{A}, \Omega^1_{\mathcal{A}})$. For every $b \in B$ let $\eta_i(b) := \text{alb}(X_b)^*dz_i$, $i = 1, \ldots, n+1$. By standard theory of the Albanese morphism it holds that $\mathcal{B}_b := \{\eta_1(b), \ldots, \eta_{n+1}(b)\}$ is a basis of the pull-back $W_b$ of $W$ inside $H^0(X_b, \Omega^1_{X_b})$. Let $\omega_i(b) := \lambda^n(\eta_1(b) \wedge \ldots \wedge \eta_{i-1}(b) \wedge \eta_i(b) \wedge \ldots \wedge \eta_{n+1}(b))$ for $i = 1, \ldots, n+1$. Since $\Phi: \mathcal{X} \rightarrow \mathcal{A}$ is a family of Albanese type, $\dim \lambda^n W_b \geq 1$, actually if $q > n+1$ by [PZ Theorem 1.3.3] it follows that $\lambda^n W_b$ has dimension $n+1$, and we can write: $\lambda^n W_b = \langle \omega_1(b), \ldots, \omega_{n+1}(b) \rangle$. Let $\omega := \omega_{a, W_b, b_b}$ be an adjoint image of $W_b$.

By theorem 4.10.12 it follows that $\omega_b \in \lambda^n W_b$. By theorem 5.3.1 we conclude.

The above theorem gives, in particular, an answer to the generic Torelli problem if the fibers of $\pi: \mathcal{X} \rightarrow B$ are smooth minimal with unique minimal model. Indeed we have:
Corollary 5.3.3. Let $\pi : X \to B$ be a family of $n$-dimensional irregular varieties which satisfies extremal liftability conditions. Assume that every fiber $X$ is minimal and it has a unique minimal model. If the Albanese map of $X$ has degree 1 and no quadric of rank $\leq 2n+3$ contains the canonical image of $X$ then the generic Torelli claim holds for $\pi : X \to B$ (assuming that the Kodaira-Spencer map is generically injective).

Proof. Since the fiber $X$ has a unique minimal model the claim follows straightly by theorem 5.3.2. □

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