Some aspects on cubic fuzzy $\beta$-subalgebra of $\beta$-algebra

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Abstract. The theory of fuzzy set was introduced by Zadeh and he has shown meaningful applications of this notion in many fields of Engineering and Mathematical studies. Later Atanosov extended the fuzzy set into Intuitionistic fuzzy set by introducing a non-member. Recently the idea of cubic sets and related properties are investigated by Jun et al. All these fuzzy concepts were applied in various algebraic structures. Neggers and Kim presented the thought of $\beta$-algebras in which two operations are coupled in such a way that as to reflect the natural coupling which exists between the usual group operation. This study is to propose a new approach on a $\beta$-algebra using Cubic fuzzy set.

Keywords: $\beta$-algebra, $\beta$-subalgebra, fuzzy set, Cubic fuzzy set, Cubic fuzzy $\beta$-subalgebra

1. Introduction

In the presentation of Zadeh’s [12] fuzzy sets, there have been various speculations of this crucial idea. Zadeh [13] likewise the author presented the idea of interval valued fuzzy subsets where the estimations of the membership functions are intervals of numbers rather than the numbers. The notion of Intuitionistic fuzzy sets were explored by Atanassov [3] in 1986. The study of fuzzy subgroups with interval valued membership functions has been initiated by Biswas et al. [4] in 1994. The thought of $\beta$-algebra was started by Neggers et al. [8]. Renugha et al. [9] proposed the concept of Cubic BF-algebra in 2014. In 2013, Aub Ayub Ansari et al. [1] proposed the concept of fuzzy $\beta$-subalgebras of $\beta$-algebra. Hemavathi et al.[5,6] discussed the concept of interval valued fuzzy $\beta$-subalgebras and also they have extended the concept of interval valued intuitionistic fuzzy $\beta$-subalgebras with interesting results. Sujatha et al.[10] depicted the thought of intuitionistic fuzzy $\beta$-subalgebras of $\beta$-algebras. In 2012, Jun et al. [7] presented cubic sets, and then this notion is applied to many algebraic structures. Akram et al.[2] proposed the concept of Cubic KU-subalgebras and provided useful results. Young Bae Jun et al.[11] applied Cubic interval valued intuitionistic fuzzy sets into BCK/BCI algebras. With all these inspiration, this paper intends to study about the cubic fuzzy $\beta$-subalgebra. The paper is classified into the following sections:

Section 1 shows the introduction, section 2 gives some basic definitions and properties of $\beta$-algebra, cubic fuzzy set and so on, Section 3 deals the concept and operations of cubic fuzzy
Remark 2.10 Let Definition 2.9 Dζ be a fuzzy set in a universal set X, then the union of ζ1 and ζ2, denoted by ζ1 ∪ ζ2 is defined by,
(ζ1 ∪ ζ2)(x) = max{ζ1(x), ζ2(x)} ∀ x ∈ X.

Definition 2.3 If ζ1 and ζ2 are fuzzy sets in X, then the intersection of ζ1 and ζ2, denoted by ζ1 ∩ ζ2, is defined by,
(ζ1 ∩ ζ2)(x) = min{ζ1(x), ζ2(x)} ∀ x ∈ X.

Definition 2.4 For any fuzzy set ζ of X, its complement, ζc is defined by
ζc(x) = 1 − ζ(x) ∀ x ∈ X.

Definition 2.5 Let ζ1 and ζ2 be two fuzzy sets of X and Y respectively. Then the direct product ζ1 × ζ2 of ζ1 and ζ2 is defined by
(ζ1 × ζ2)(x, y) = min{ζ1(x), ζ2(y)} ∀ (x, y) ∈ X × Y and ζ1 × ζ2 is a fuzzy set of X × Y.

Definition 2.6 Let ζ be a fuzzy set of X. For any δ ∈ [0, 1], the set ζδ = {x ∈ X : ζ(x) ≥ δ} is called a level subset of ζ.
The level subset ζδ of a fuzzy set ζ is a crisp subset of the set X.

Definition 2.7 Let ζ be a fuzzy set of X. For δ ∈ [0, 1], the set ζδ = {x ∈ X : ζ(x) ≤ δ} is called a lower level subsets of ζ.

Definition 2.8 The supremum property of the fuzzy set ζ for the subset A in X is defined as
ζ(a0) = Sup ζ(a) if there exist a, a0 ∈ A a∈A

Definition 2.9 Let D[0,1] denote the family of all closed sub intervals of [0,1]. Consider two elements D1,D2 ∈ D[0,1]. If D1 = [a, b1] and D2 = [a2, b2], then rmax(D1, D2) = [max(a1, a2), max(b1, b2)] which is denoted by D1 ∪ D2 and rmin(D1, D2) = [min(a1, a2), min(b1, b2)] which is denoted by D1 ∩ D2.

Thus if Di = [ai, bi] ∈ D[0,1] for i=1,2,3,...
rsupi(Di) = [supi(ai), supi(bi)],
i.e. \( \bigvee_i^r D_i = [\bigvee_i a_i, \bigvee_i b_i] \)

similarly
rinfi(Di) = [infi(ai), infi(bi)]
i.e. \( \bigwedge_i^r D_i = [\bigwedge_i a_i, \bigwedge_i b_i] \).
Now D1 ≥ D2 iff a1 ≥ a2 and b1 ≥ b2.
Similarly the relations D1 ≤ D2 and D1 = D2 are defined.

Remark 2.10 Let D1 := [a1, b1] and D2 := [a2, b2] ∈ D[0,1]. Then
Definition 2.11 An interval valued fuzzy set (i.v. fuzzy set) $A$ defined on $X$ is given by $A = \{ (x, [\mu_A^L(x), \mu_A^U(x)]) \mid x \in X \}$ (briefly denoted by $A = [\mu_A^L, \mu_A^U]$), where $\mu_A^L$ and $\mu_A^U$ are two fuzzy sets in $X$ such that $\mu_A^L(x) \leq \mu_A^U(x)$ for any $x \in X$.

Let $\overline{\mu}_A(x) = [\mu_A^L(x), \mu_A^U(x)] \forall x \in X$ and let $D[0,1]$ denotes the family of all closed sub-intervals of $[0,1]$. If $\mu_A^U(x) = \mu_A^L(x) = c$, say, where $0 \leq c \leq 1$, then $\overline{\mu}_A(x) = [c,c]$ also for the sake of convenience, to belong to $D[0,1]$. Thus $\overline{\mu}_A(x) \in D[0,1] \forall x \in X$, and therefore the i.v. fuzzy set $A$ is given by

$$A = \{ (x, \overline{\mu}_A(x)) \mid x \in X, \text{ where } \overline{\mu}_A : X \to D[0,1] \}.$$ 

Now let us define what is known as refined mimimum (rmin) of two elements in $D[0,1]$. Let us define the symbols " $\geq$ " and " $\leq$ " in case of two elements in $D[0,1]$.

Consider two elements $D_1 := [a_1, b_1]$ and $D_2 := [a_2, b_2] \in D[0,1]$. Then $rmin(D_1, D_2) = [\min\{a_1, a_2\}, \min\{b_1, b_2\}]$.

$D_1 \geq D_2$ if and only if $a_1 \geq a_2, b_1 \geq b_2$.

Similarly, $D_1 \leq D_2$ and $D_1 = D_2$.

Definition 2.12 [3] An Intuitionistic fuzzy set (IFS) in a nonempty set $X$ is defined by $A = \{ (x, \zeta_A(x), \eta_A(x)) \mid x \in X \}$ where $\zeta_A : X \to [0,1]$ is a membership function of $A$ and $\eta_A : X \to [0,1]$ is a non-membership function of $A$ satisfying

$0 \leq \zeta_A(x) + \eta_A(x) \leq 1 \forall x \in X$.

Definition 2.13 An intuitionistic fuzzy set $A$ with the degree membership $\zeta_A(x) : X \to [0,1]$ and the degree of non-membership function $\eta_A(x) : X \to [0,1]$ is said to have sup-inf property if for any subset $T$ of $X$ there exists $x_0 \in T$ such that $\zeta_A(x_0) = \text{Sup}_{x \in T} \zeta_A(x)$ and $\eta_A(x_0) = \text{inf}_{x \in T} \eta_A(x)$.

Definition 2.14 Let $A = \{ (x, \zeta_A(x), \eta_A(x)) \mid x \in X \}$ be an intuitionistic fuzzy subset in $X$ and $f$ be a mapping from $X$ into $Y$. Then the image of $A$ under $f$ is defined as $f(A) = \{ (y, \zeta_f(A)(y), \eta_f(A)(y)) \mid y \in Y \}$.

Definition 2.15 Let $f : X \to Y$ be a function. Let $A$ and $B$ be two intuitionistic fuzzy subsets in $X$ and $Y$ respectively. Then inverse image of $B$ under $f$ is defined by $f^{-1}(B) = \{ (x, f^{-1}(\zeta_B(x)), f^{-1}(\eta_B(x))) \mid x \in X \}$ such that $f^{-1}(\zeta_B(x)) = \zeta_B(f(x))$ and $f^{-1}(\eta_B(x)) = \eta_B(f(x))$.

Definition 2.16 An Interval valued intuitionistic fuzzy set (i.v.i.fuzzy set) $A$ over $X$ is an object having the form $A = \{ (x, \overline{\mu}_A(x), \overline{\eta}_A(x)) \mid x \in X \}$ where $\overline{\mu}_A : X \to D[0,1]$ and $\overline{\eta}_A : X \to D[0,1]$, where $D[0,1]$ is the set of all sub-intervals of $[0,1]$.

The intervals $\overline{\mu}_A(x)$ and $\overline{\eta}_A(x)$ denote the intervals of the grade of membership and grade of non-membership of the element $x$ to the set $A$, where $\overline{\mu}_A(x) = [\mu_A^L(x), \mu_A^U(x)]$ and $\overline{\eta}_A(x) = [\eta_A^L(x), \eta_A^U(x)] \forall x \in X$, with the condition $0 \leq \mu_A^L(x) + \eta_A^L(x) \leq 1$ and $0 \leq \mu_A^U(x) + \eta_A^U(x) \leq 1$.

Also note that $\overline{\mu}_A(x) = [1 - \eta_A^L(x), 1 - \mu_A^L(x)]$ and $\overline{\eta}_A(x) = [1 - \mu_A^U(x), 1 - \eta_A^U(x)]$, where $[\overline{\mu}_A(x), \overline{\eta}_A(x)]$ represents the complement of $x$ in $A$. 

Definition 2.17 Let \( A = \{ \langle x, \zeta_A(x), \eta_A(x) \rangle : x \in X \} \) be an \( i_v \)\( \_ \) fuzzy set in \( X \) and \( f \) be a mapping from a set \( X \) into a set \( Y \), then the image of \( A \) under \( f \), \( f(A) = \{ \langle x, f_{\sup}(\zeta_A), f_{\inf}(\eta_A) \rangle : x \in Y \} \), where

\[
\begin{align*}
    f_{\sup}(\zeta_A)(y) &= \begin{cases} 
        \sup_{x \in f^{-1}(y)} \zeta_A(x), & \text{if } f^{-1}(y) \neq \emptyset \\
        [0,0], & \text{otherwise}
    \end{cases} \\
    f_{\inf}(\eta_A)(y) &= \begin{cases} 
        \inf_{x \in f^{-1}(y)} \eta_A(x), & \text{if } f^{-1}(y) \neq \emptyset \\
        [1,1], & \text{otherwise}
    \end{cases}
\end{align*}
\]

Definition 2.18 [8] A \( \beta - \) algebra is a non-empty set \( X \) with a constant 0 and two binary operations \( + \) and \( - \) satisfying the following axioms:

(i) \( x - 0 = x \)
(ii) \( (0 - x) + x = 0 \)
(iii) \( (x - y) - z = x - (z + y) \) \( \forall x, y, z \in X \).

Example 2.19 The following Cayley table shows \( (X = \{0, 1, 2, 3\}; +, -, 0) \) is a \( \beta - \) algebra.

| \( + \) | 0 | 1 | 2 | 3 |
|-------|---|---|---|---|
| 0     | 0 | 1 | 2 | 3 |
| 1     | 1 | 2 | 3 | 0 |
| 2     | 2 | 3 | 0 | 1 |
| 3     | 3 | 0 | 1 | 2 |

| \( - \) | 0 | 1 | 2 | 3 |
|-------|---|---|---|---|
| 0     | 0 | 1 | 2 | 3 |
| 1     | 1 | 0 | 3 | 2 |
| 2     | 2 | 3 | 0 | 1 |
| 3     | 3 | 1 | 2 | 0 |

Definition 2.20 A non empty subset \( A \) of a \( \beta - \) algebra \( (X, +, -, 0) \) is called a \( \beta - \) subalgebra of \( X \), if

(i) \( x + y \in A \) and
(ii) \( x - y \in A \) \( \forall x, y \in A \).

Definition 2.21 [7] Let \( X \) be a non empty set. By a cubic set in \( X \) we mean a structure

\( C = \{ \langle x, \zeta_C(x), \eta_C(x) \rangle : x \in X \} \)

in which \( \zeta_C \) is an \( i_v \)\( \_ \) fuzzy set in \( X \) and \( \eta_C \) is a fuzzy set in \( X \).

3. Cubic Fuzzy \( \beta - \) Subalgebra

This section introduces the notion of cubic fuzzy \( \beta - \) subalgebra of a \( \beta - \) algebra and discusses some interesting results.

Definition 3.1 Let \( C = \{ \langle x, \zeta_C(x), \eta_C(x) \rangle : x \in X \} \) be a cubic fuzzy set in \( X \). Then the set \( C \) is a cubic fuzzy \( \beta - \) subalgebra if it satisfies the following conditions.

(i) \( \zeta_C(x + y) \geq \min(\zeta_C(x), \zeta_C(y)) \) \& \( \zeta_C(x - y) \geq \min(\zeta_C(x), \zeta_C(y)) \)
(ii) \( \eta_C(x + y) \leq \max(\eta_C(x), \eta_C(y)) \) \& \( \eta_C(x - y) \leq \max(\eta_C(x), \eta_C(y)) \) \( \forall x, y \in X \).

Example 3.2 For the \( \beta - \) algebra \( X \) in the example 2.15, the cubic fuzzy set \( C = \{ \langle x, \zeta_C(x), \eta_C(x) \rangle : x \in X \} \) on \( X \) as follows.

\[ \zeta_C = \begin{cases} 
    [0.3, 0.6] &: x = 0 \\
    [0.2, 0.5] &: x = 2 \\
    [0.1, 0.4] &: x = 1, 3
\end{cases} \]

\[ \eta_C = \begin{cases} 
    0.7 &: x = 0, 1 \\
    0.6 &: x = 3 \\
    0.4 &: x = 2
\end{cases} \]

is a Cubic fuzzy \( \beta - \) sub algebra of \( X \).
Definition 3.3 Let \( A = \{ \langle x, \zeta_A(x), \eta_A(x) \rangle : x \in X \} \) and \( B = \{ \langle x, \zeta_B(x), \eta_B(x) \rangle : x \in X \} \) be two cubic fuzzy sets on \( X \), then the intersection of \( A \) and \( B \) denoted by \( A \cap B \) is defined by
\[
A \cap B = \{ \langle x, \zeta_{A \cap B}(x), \eta_{A \cap B}(x) \rangle \}
\]
\[
= \{ \langle x, r\min\{\zeta_A(x), \zeta_B(x)\}, \max(\eta_A(x), \eta_B(x)) \rangle \} : x \in X \}.
\]

Lemma 3.4 Let \( C = \{ \langle x, \zeta_C(x), \eta_C(x) \rangle : x \in X \} \) cubic fuzzy \( \beta \)-subalgebra of \( X \). Then

(1) \( \zeta_C(x) \leq \zeta_C(0) \) and \( \eta_C(x) \geq \eta_C(0) \) \( \forall \ x \in X \)

(2) \( \zeta_C(x) \leq \zeta_C(x^*) \leq \zeta_C(0) \) and \( \eta_C(x) \geq \eta_C(x^*) \geq \eta_C(0) \) \( \forall \ x \in X \) where \( x^* = 0 - x \)

Proof:

(1) For \( x \in X \),
\[
\zeta_C(0) = \zeta_C(x - x) \\
\geq r\min\{\zeta_C(x), \zeta_C(x)\} \\
= \zeta_C(x)
\]
\[
\therefore \zeta_C(0) \geq \zeta_C(x)
\]
and
\[
\eta_C(0) = \eta_C(x - x) \\
\leq \max\{\eta_C(x), \eta_C(x)\} \\
= \eta_C(x)
\]

(ie) \( \eta_C(0) \leq \eta_C(x) \)

(2) Also for \( x \in X \),
\[
\zeta_C(x^*) = \zeta_C(0 - x) \\
\geq r\min\{\zeta_C(0), \zeta_C(x)\} \\
= \zeta_C(x)
\]
\[
\therefore \zeta_C(x) \leq \zeta_C(x^*) \leq \zeta_C(0)
\]
\( \forall \ x \in X \),
\[
\eta_C(x^*) = \eta_C(0 - x) \\
\leq \max\{\eta_C(0), \eta_C(x)\} \\
= \eta_C(x)
\]

Hence \( \eta_C(x) \geq \eta_C(x^*) \geq \eta_C(0) \)

Theorem 3.5 If \( C = \{ \langle x, \zeta_C(x), \eta_C(x) \rangle : x \in X \} \) be a cubic fuzzy \( \beta \)-subalgebra of \( X \). Let \( \chi_C = \{ x \in X / \zeta_C(x) = \zeta_C(0) \} \) & \( \eta_C(x) = \eta_C(0) \}. \) Then \( \chi_C \) is a \( \beta \)-subalgebra of \( X \).

Proof:

For any \( x, y \in \chi_C \),
\[
\zeta_C(x) = \zeta_C(0), \ \zeta_C(y) = \zeta_C(0) \) and \( \eta_C(x) = \eta_C(0) = \eta_C(y) \)
Now
\[ \zeta_C(x + y) = [\zeta_C^L(x + y), \zeta_C^U(x + y)] \]
\[ \geq [\min\{\zeta_C^L(x), \zeta_C^L(y)\}, \min\{\zeta_C^U(x), \zeta_C^U(y)\}] \]
\[ = r\min\{[\zeta_C^L(x), \zeta_C^L(y)], [\zeta_C^U(x), \zeta_C^U(y)]\} \]
\[ \geq r\min\{\zeta_C(x), \zeta_C(y)\} \]
\[ = r\min\{\zeta_C(0), \zeta_C(0)\} \]
\[ = \zeta_C(0) \]..............................(1)

Consider,
\[ \zeta_C(0) = \zeta_C(0 - 0) \]
\[ = [\zeta_C^L(0 - 0), \zeta_C^U(0 - 0)] \]
\[ \geq [\min\{\zeta_C^L(0), \zeta_C^L(0)\}, \min\{\zeta_C^U(0), \zeta_C^U(0)\}] \]
\[ = r\min\{[\zeta_C^L(0), \zeta_C^L(0)], [\zeta_C^U(0), \zeta_C^U(0)]\} \]
\[ \geq r\min\{\zeta_C(0), \zeta_C(0)\} \]
\[ = r\min\{\zeta_C(x), \zeta_C(y)\} \]
\[ = \zeta_C(x + y) \]..............................(2)

From (1) and (2) \[ \zeta_C(x + y) = \zeta_C(0) \]
Similarly, \[ \zeta_C(x - y) \geq \zeta_C(0) \]
Hence \[ x + y, x - y \in \chi_C \]

Now,
\[ \eta_C(x + y) \leq \max\{\eta_C(x), \eta_C(y)\} \]
\[ = \max\{\eta_C(0), \eta_C(0)\} \]
\[ = \eta_A(0) \]..............................(3)

Consider,
\[ \eta_C(0) = \eta_C(0 - 0) \]
\[ \geq \max\{\eta_C(0), \eta_C(0)\} \]
\[ = \max\{\eta_C(x), \eta_C(y)\} \]
\[ = \eta_C(x + y) \]..............................(4)

From (3) and (4) \[ \eta_C(x + y) = \eta_C(0) \]
Similarly, \[ \eta_C(x - y) = \eta_C(0) \]
Hence \[ x + y, x - y \in \chi_C \]
\[ \therefore \chi_C \text{ is a } \beta-\text{subalgebra of } X. \]

**Theorem 3.6** If \( C = \{\langle x, \zeta_C(x), \eta_C(x) \rangle : x \in X \} \) is a cubic fuzzy \( \beta-\text{subalgebra of } X \), then \[ \zeta_C(x) \leq \zeta_C(x - 0) \text{ and } \eta_C(x) \geq \eta_C(x - 0) \]

**Proof:**
Let \( C \) be a cubic fuzzy \( \beta-\text{subalgebra of } X \).
\[ \zeta_C(x - 0) = [\zeta_C^L(x - 0), \zeta_C^U(x - 0)] \]
\[ \geq [\min\{\zeta_C^L(x), \zeta_C^L(0)\}, \min\{\zeta_C^U(x), \zeta_C^U(0)\}] \]
Definition 3.7 Let \( f : X \to Y \) be a function. Let \( A \) and \( B \) be two cubic fuzzy \( \beta \)-subalgebras in \( X \) and \( Y \) respectively. Then inverse image of \( B \) under \( f \) is defined by 
\[
f^{-1}(B) = \{ f^{-1}(\zeta_B(x)), f^{-1}(\eta_B(x)) : x \in X \} \text{ such that } f^{-1}(\zeta_B(x)) = (\zeta_B(f(x))) \text{ and } f^{-1}(\eta_B(x)) = (\eta_B(f(x))
\]

Theorem 3.8 Let \( X \) and \( Y \) be two cubic fuzzy \( \beta \)-subalgebra. Let \( f : X \to Y \) be a homomorphism. If \( C = \{ (x, \zeta_C(x), \eta_C(x)) : x \in X \} \) cubic fuzzy \( \beta \)-subalgebra of \( Y \), then \( f^{-1}(C) \) is a cubic fuzzy \( \beta \)-subalgebra of \( X \).

Proof:
Let \( C \) be a cubic fuzzy \( \beta \)-subalgebra of \( Y \)
Let \( x, y \in Y \).
Then
\[
f^{-1}(\zeta_C(x + y)) = f^{-1}(\zeta_C(f(x) + f(y)))
\]
\[
= \zeta_C(f(x) + f(y))
\]
\[
= [\zeta_C^L(f(x) + f(y)), \zeta_C^U(f(x) + f(y))]
\]
\[
\geq [\min\{\zeta_C^L(f(x)), \zeta_C^L(f(y))\}, \min\{\zeta_C^U(f(x)), \zeta_C^U(f(y))\}]
\]
\[
= rmin\{\zeta_C^L(f(x)), \zeta_C^U(f(x))\}
\]
\[
= rmin\{\zeta_C(f(x)), \zeta_C(f(y))\}
\]
\[
\geq rmin\{f^{-1}(\zeta_C(x)), f^{-1}(\zeta_C(y))\}
\]

Similarly, \( f^{-1}(\zeta_C(x + y)) \geq rmin\{f^{-1}(\zeta_C(x)), f^{-1}(\zeta_C(y))\} \)
and,
\[
f^{-1}(\eta_C(x + y)) = \eta_C(f(x + y))
\]
\[
= \eta_C(f(x) + f(y))
\]
\[
\leq \max\{\eta_C(f(x)), \eta_C(f(y))\}
\]
\[
= \max\{f^{-1}(\eta_C(x)), f^{-1}(\eta_C(y))\}
\]
Similarly,
\[ f^{-1}(\eta_C)(x - y) \leq \max\{f^{-1}(\eta_C)(x), f^{-1}(\eta_C)(y)\} \]
and
\[ : f^{-1}(C) \text{ is a cubic fuzzy } \beta-\text{subalgebra of } X. \]

**Definition 3.9** Let \( f \) be a mapping from a set \( X \) into a set \( Y \). Let \( C \) be a cubic fuzzy set in \( X \). Then the image of \( C \), denoted by \( f[A] \), is the cubic fuzzy set in \( Y \) with the membership function defined by

\[
\tilde{\zeta}_{fC}(y) = \begin{cases} 
  r sup\tilde{\zeta}_C(z) & : if \ f^{-1}(y) \neq \emptyset \ \forall y \in Y \\
  [0,0] & : Otherwise 
\end{cases} \\
\eta_{fC}(y) = \begin{cases} 
  inf \eta_C(z) & : if \ f^{-1}(y) \neq \emptyset \ \forall y \in Y \\
  1 & : Otherwise 
\end{cases}
\]

where \( f^{-1}(y) = \{x/f(x) = y\} \)

**Theorem 3.10** Let \( X \) and \( Y \) be two cubic fuzzy \( \beta-\)subalgebra. Let \( f : X \rightarrow X \) be an endomorphism. If \( C = \{(x, \zeta_C(x), \eta_A(x)) : x \in X\} \) is a cubic fuzzy \( \beta-\)subalgebra of \( X \), then \( f(C) \) is cubic fuzzy \( \beta-\)subalgebra of \( X \). That is, \( \tilde{\zeta}_f(x) = \tilde{\zeta}(f(x)) \) and \( \eta_f(x) = \eta(f(x)) \).

**Proof:**

Let \( A \) be a cubic fuzzy \( \beta-\)subalgebra of \( Y \)

Let \( a, b \in A \) with \( a \in f^{-1}(a) \) and \( b \in f^{-1}(b) \) such that

\[
\tilde{\zeta}_{C}(x) = sup_{t \in f^{-1}(a)}(\tilde{\zeta}_C(t)), \quad \tilde{\zeta}_C(y) = sup_{t \in f^{-1}(b)}(\tilde{\zeta}_C(t))
\]

and \( \eta_A(x) = inf_{t \in f^{-1}(a)}(\eta_C(t)), \quad \eta_C(y) = inf_{t \in f^{-1}(b)}(\eta_C(t)) \)

Then

\[
\tilde{\zeta}_{f(C)}(a + b) = sup_{t \in f^{-1}(a+b)}(\tilde{\zeta}_C(t)) \\
= \tilde{\zeta}(f(x + y)) \\
= \tilde{\zeta}(f(x) + f(y)) \\
\geq rmin\{\tilde{\zeta}(f(x)), \tilde{\zeta}(f(y))\} \\
= rmin\{sup_{t \in f^{-1}(a)}(\tilde{\zeta}_C(t)), sup_{t \in f^{-1}(b)}(\tilde{\zeta}_C(t))\} \\
= rmin\{\tilde{\zeta}_{f(C)}(a), \tilde{\zeta}_{f(C)}(b)\}
\]

Similarly,
\[ \tilde{\zeta}_{f(C)}(a - b) \geq rmin\{\tilde{\zeta}_{f(C)}(a), \tilde{\zeta}_{f(C)}(b)\} \]

Now,
\[
\eta_{f(C)}(a + b) = inf_{t \in f^{-1}(a+b)}(\eta_C(t)) \\
= \eta(f(x + y)) \\
= \eta(f(x) + f(y)) \\
\leq \max\{\eta(f(x)), \eta(f(y))\} \\
= \max\{inf_{t \in f^{-1}(a)}(\eta_C(t)), inf_{t \in f^{-1}(b)}(\eta_C(t))\} \\
= \max\{\eta_{f(C)}(a), \eta_{f(C)}(b)\}
\]

Similarly,
\[ \eta_{f(C)}(a - b) \leq \max\{\eta_{f(C)}(a), \eta_{f(C)}(b)\} \]

Hence \( f(C) \) is a cubic fuzzy \( \beta-\)subalgebra of \( X \).
4. Product on Cubic Fuzzy $\beta-$subalgebra

This section introduces the notion of product on Cubic fuzzy $\beta-$ subalgebras and provides some elegant results.

**Definition 4.1** Let $(X, +, -, 0)$ and $(Y, +, -, 0)$ be two sets. Let $A = \{ (x, \zeta_A(x), \eta_A(x)) : x \in X \}$ and $B = \{ (y, \zeta_B(y), \eta_B(y)) : y \in Y \}$ be cubic fuzzy sets in $X$ and $Y$ respectively. The Cartesian product of $A$ and $B$ denoted by $A \times B$ is defined to be the set $A \times B = \{ (x, y, \zeta_{A \times B}(x, y), \eta_{A \times B}(x, y)) : (x, y) \in X \times Y \}

where $\zeta_{A \times B} : X \times Y \rightarrow [0, 1]$ is given by $\zeta_{A \times B}(x, y) = \min \{ \zeta_A(x), \zeta_B(y) \}$ and $\eta_{A \times B} : X \times Y \rightarrow [0, 1]$ is given by $\eta_{A \times B}(x, y) = \max \{ \eta_A(x), \eta_B(y) \}.$

**Theorem 4.2** Let $A = \{ (x, \zeta_A(x), \eta_A(x)) \}$ and $B = \{ (y, \zeta_B(y), \eta_B(y)) \}$ be any two Cubic fuzzy $\beta-$subalgebras of $X$ and $Y$ respectively. Then $A \times B$ is also an Cubic fuzzy $\beta-$subalgebra of $X \times Y$.

**Proof:**
Let $A = \{ (x, \zeta_A(x)) : x \in X \}$ and $B = \{ (y, \zeta_B(y)) : y \in Y \}$ be an Cubic fuzzy $\beta-$subalgebras in $X$ and $Y$. Take $(a, b) \in X \times Y$, where $a = (x_1, x_2)$ and $b = (y_1, y_2)$.

Now,

$$
\zeta_{A \times B}(a + b) = \zeta_{A \times B}((x_1, x_2) + (y_1, y_2))
= (\zeta_A \times \zeta_B)((x_1 + y_1), (x_2 + y_2))
= r\min \{ \zeta_A(x_1 + y_1), \zeta_B(x_2 + y_2) \}
\geq r\min \{ r\min \{ \zeta_A(x_1), \zeta_A(y_1) \}, r\min \{ \zeta_B(x_2), \zeta_B(y_2) \} \}
= r\min \{ r\min \{ \zeta_A(x_1), \zeta_B(x_2) \}, r\min \{ \zeta_A(x_1), \zeta_B(x_2) \} \}
= r\min \{ r\min \{ \zeta_A(x_1), \zeta_B(x_2) \}, r\min \{ \zeta_A(x_1), \zeta_B(x_2) \} \}
= r\min \{ \zeta_{A \times B}(a), \zeta_{A \times B}(b) \}
$$

Similarly, $\zeta_{A \times B}(a - b) \geq r\min \{ \zeta_{A \times B}(a), \zeta_{A \times B}(b) \}$

Further,

$$
\eta_{A \times B}(a + b) = \eta_{A \times B}((x_1, y_1) + (x_2, y_2))
= (\eta_A \times \eta_B)((x_1 + y_1), (x_2 + y_2))
= \max \{ \eta_A(x_1 + y_1), \eta_B(x_2 + y_2) \}
\leq \max \{ \max \{ \eta_A(x_1), \eta_A(y_1) \}, \max \{ \eta_B(x_2), \eta_B(y_2) \} \}
\leq \max \{ \max \{ \eta_A(x_1), \eta_B(x_2) \}, \max \{ \eta_A(y_1), \eta_A(y_2) \} \}
= \max \{ \max \{ \eta_A \times \eta_B(x_1, x_2), (\eta_A \times \eta_B)(y_1, y_2) \} \}
= \max \{ \eta_{A \times B}(a), \eta_{A \times B}(b) \}
$$

Similarly, $\eta_{A \times B}(a - b) \leq \max \{ \eta_{A \times B}(a), \eta_{A \times B}(b) \}$

**Theorem 4.3** If $A \times B$ is an i-v.i.-fuzzy $\beta-$subalgebra of $X \times Y$, then either $A$ is a Cubic fuzzy $\beta-$subalgebra of $X$ or $B$ is a Cubic fuzzy $\beta-$subalgebra of $Y$.

**Proof:**
Let $A \times B$ is a Cubic fuzzy $\beta-$subalgebra of $X \times Y$.

Take $(x_1, y_1)$ and $(x_2, y_2) \in X \times Y$. Then,
Similarly, a \( B \) that is a Cubic fuzzy subalgebra of \( A \) subalgebra of \( B \), \( \eta_B(y_1 - y_2) \leq \max\{\eta_B(y_1), \eta_B(y_2)\} \) and \( \eta_{A \times B}\{x_1 + y_1, x_2, y_2\} \geq \min\{\eta_{A \times B}\{x_1, y_1\}, \eta_{A \times B}\{x_2, y_2\}\} \)

\[ x_1 = x_2 = 0 \Rightarrow \eta_{A \times B}\{(0, y_1), (0, y_2)\} \leq \max\{\eta_{A \times B}\{(0, y_1), \varphi_{A \times B}(0, y_2)\}\} \]

Then \( \eta_{A \times B}\{(0 + 0), (y_1 + y_2)\} \leq \max\{\eta_{A \times B}(0, y_1), \eta_{A \times B}(0, y_2)\} \).

\[ \therefore \eta_{A \times B}(y_1 + y_2) \leq \max\{\eta_B(y_1), \eta_B(y_2)\} \]

Similarly, \( \eta_{A \times B}\{x_1, y_1\} \geq \min\{\eta_{A \times B}(x_1, y_1), \eta_{A \times B}(x_2, y_2)\} \)

\[ \therefore \eta_{A \times B}(x_1, y_1) \geq \min\{\eta_{A \times B}(x_1, y_1), \eta_{A \times B}(x_2, y_2)\} \]

\[ \therefore \eta_{A \times B}(y_1 + y_2) \leq \max\{\eta_B(y_1), \eta_B(y_2)\} \]

\[ \therefore \eta_{A \times B}(x_1, y_1) \geq \min\{\eta_{A \times B}(x_1, y_1), \eta_{A \times B}(x_2, y_2)\} \]

\[ \therefore \eta_{A \times B}(y_1 + y_2) \leq \max\{\eta_B(y_1), \eta_B(y_2)\} \]

\[ \therefore \eta_{A \times B}(x_1, y_1) \geq \min\{\eta_{A \times B}(x_1, y_1), \eta_{A \times B}(x_2, y_2)\} \]

\[ \therefore \eta_{A \times B}(y_1 + y_2) \leq \max\{\eta_B(y_1), \eta_B(y_2)\} \]

\[ \therefore \eta_{A \times B}(x_1, y_1) \geq \min\{\eta_{A \times B}(x_1, y_1), \eta_{A \times B}(x_2, y_2)\} \]

\[ \therefore \eta_{A \times B}(y_1 + y_2) \leq \max\{\eta_B(y_1), \eta_B(y_2)\} \]

Hence \( B \) is a Cubic fuzzy \( \beta \)-subalgebra of \( Y \).

**Definition 4.4** Let \( C_i = \{x \in X_i : \zeta_{A_i}(x), \eta_{C_i}(x)\} \) be a Cubic fuzzy \( \beta \)-subalgebra of \( X_i \), \( i = 1, 2, \ldots, n \). Then \( \prod_{i=1}^{n} C_i \) is called direct product of finite Cubic fuzzy \( \beta \)-subalgebra of \( \prod_{i=1}^{n} X_i \) if

(a) (i) \( \prod_{i=1}^{n} \zeta_{C_i}(x_i + y_i) \geq \min\{\prod_{i=1}^{n} \zeta_{C_i}(x_i), \prod_{i=1}^{n} \zeta_{C_i}(y_i)\} \)

(ii) \( \prod_{i=1}^{n} \zeta_{C_i}(x_i - y_i) \geq \min\{\prod_{i=1}^{n} \zeta_{C_i}(x_i), \prod_{i=1}^{n} \zeta_{C_i}(y_i)\} \)

(b) (i) \( \prod_{i=1}^{n} \eta_{C_i}(x_i + y_i) \leq \max\{\prod_{i=1}^{n} \eta_{C_i}(x_i), \prod_{i=1}^{n} \eta_{C_i}(y_i)\} \)

(ii) \( \prod_{i=1}^{n} \eta_{C_i}(x_i - y_i) \leq \max\{\prod_{i=1}^{n} \eta_{C_i}(x_i), \prod_{i=1}^{n} \eta_{C_i}(y_i)\} \)

**Theorem 4.5** Let \( C_i = \{x \in X_i : \zeta_{C_i}(x), \eta_{C_i}(x)\} \) be a Cubic fuzzy \( \beta \)-subalgebra of \( X_i \), respectively, for \( i = 1, 2, \ldots, n \). Then \( \prod_{i=1}^{n} C_i \) is a Cubic fuzzy \( \beta \)-sub algebra of \( \prod_{i=1}^{n} X_i \).

**Proof**

Let \( C_i = \{x \in X_i : \zeta_{C_i}(x), \eta_{C_i}(x)\} \) be a Cubic fuzzy \( \beta \)-sub algebra of \( X_i \).

Let \( (x_1, \ldots, x_n) \) and \( (y_1, \ldots, y_n) \in \prod_{i=1}^{n} X_i \).

Take \( a = (x_1, \ldots, x_n) \) and \( b = (y_1, \ldots, y_n) \).

Then

\[ (i) \prod_{i=1}^{n} \zeta_{C_i}(a + b) \geq \min\{\prod_{i=1}^{n} \zeta_{C_i}(a + b), \prod_{i=1}^{n} \zeta_{C_i}(a + b)\} \]

\[ = \min\{\prod_{i=1}^{n} \zeta_{C_i}(a), \prod_{i=1}^{n} \zeta_{C_i}(b)\} \]

\[ = \min\{\prod_{i=1}^{n} \zeta_{C_i}(a), \prod_{i=1}^{n} \zeta_{C_i}(b)\} \]

Similarly,
\[
\prod_{i=1}^{n} \zeta_{C_i}(a-b) \geq r_{\text{min}} \{ \prod_{i=1}^{n} \zeta_{C_i}(a), \prod_{i=1}^{n} \zeta_{C_i}(b) \}
\]

\[(ii) \prod_{i=1}^{n} \eta_{C_i}(a+b) \leq \max \{ \eta_{C_1}(a+b), \ldots, \eta_{C_n}(a+b) \}
\]

\[
= \max \{ \max \{ \eta_{C_1}(a), \eta_{C_n}(b) \}, \ldots, \max \{ \eta_{C_n}(a), \eta_{C_1}(b) \} \}
\]

\[
= \max \{ \max \{ \eta_{C_1}(a), \eta_{C_1}(b) \}, \ldots, \max \{ \eta_{C_n}(a), \eta_{C_n}(b) \} \}
\]

Similarly,
\[
\prod_{i=1}^{n} \eta_{C_i}(a-b) \leq \max \{ \prod_{i=1}^{n} \eta_{C_i}(a), \prod_{i=1}^{n} \eta_{C_i}(b) \}
\]

Hence \( \prod_{i=1}^{n} C_i \) is a Cubic fuzzy \( \beta \)-sub algebra of \( \prod_{i=1}^{n} X_i \)

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