Basis-free Solution to Sylvester Equation in Clifford Algebra of Arbitrary Dimension

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Abstract. The Sylvester equation and its particular case, the Lyapunov equation, are widely used in image processing, control theory, stability analysis, signal processing, model reduction, and many more. We present basis-free solution to the Sylvester equation in Clifford (geometric) algebra of arbitrary dimension. The basis-free solutions involve only the operations of Clifford (geometric) product, summation, and the operations of conjugation. To obtain the results, we use the concepts of characteristic polynomial, determinant, adjugate, and inverse in Clifford algebras. For the first time, we give alternative formulas for the basis-free solution to the Sylvester equation in the case \( n = 4 \), the proofs for the case \( n = 5 \) and the case of arbitrary dimension \( n \). The results can be used in symbolic computation.

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1. Introduction

This paper is an extended version of the short note in Conference Proceedings [24]. We present for the first time the alternative formulas for the basis-free solution to the Sylvester equation in the case \( n = 4 \) (see the remarks after Theorem 3.1), the proofs of Theorems 4.1 and 5.1, and the simplification of the statement of Theorem 5.1 in the case of odd \( n = p + q \) (see the remarks after Theorem 5.1).

The Sylvester equation [26] is a linear equation of the form \( AX - XB = C \) for known \( A, B, C \) (quaternions, matrices, or multivectors depending on the formalism) and unknown \( X \). The Sylvester equation and its particular

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case, the Lyapunov equation (with $B = -A^H$), are widely used in different applications—image processing, control theory, stability analysis, signal processing, model reduction, and many more. In this paper, we study the Sylvester equation in Clifford’s geometric algebra $\mathbb{C}_p,q$ and present basis-free solution to this equation in the case of arbitrary $n = p + q$.

The Sylvester equation over quaternions corresponds to the Sylvester equation in geometric algebra of a vector space of dimension $n = 2$, because we have the isomorphism $\mathbb{C}_0,2 \cong \mathbb{H}$. Thus the basis-free solution to the Sylvester equation in $\mathbb{C}_p,q$, $p + q = 2$, is constructed similarly to the basis-free solution to the Sylvester equation over quaternions. The same ideas as in the case $n = 2$ work in the case $n = 3$. The cases $n \leq 3$ are also discussed by Acus and Dargys [5].

In this paper, we present basis-free solutions in the cases $n = 4$ and $n = 5$, which are the most important cases for the applications. The geometric algebra $\mathbb{C}_1,3$ (the space-time algebra [9]) of a space of dimension 4 is widely used in physics. The conformal geometric algebra $\mathbb{C}_4,1$ of a space of dimension 5 is widely used in geometry, robotics, and computer vision (see [3,7,11,12,17]). Also we present recursive basis-free formulas to the Sylvester equation in $\mathbb{C}_p,q$ in the case of arbitrary $n = p + q$. They can be used in symbolic computation. We use our previous results on explicit and recursive formulas for the characteristic polynomial coefficients and inverse in Clifford algebras [25]. Note also the papers on the characteristic polynomial [8] and inverse [2,4,13,14,23].

An arbitrary linear quaternion equation with two terms

$$KXL + MXN = P$$

for known $K, L, M, N, P \in \mathbb{H}$ and unknown $X \in \mathbb{H}$ can be reduced to the Sylvester equation. Any nonzero quaternion $Q = a + bi + cj + dk \neq 0$, where $a, b, c, d \in \mathbb{R}$ are real numbers, and $i, j, k$ are the quaternion units, is invertible and the inverse is equal to

$$Q^{-1} = \frac{\overline{Q}}{QQ},$$

where $\overline{Q} := a - bi - cj - dk$ is the conjugate of $Q$. Multiplying both sides of (1.1) on the left by $M^{-1}$ and on the right by $L^{-1}$, we obtain $M^{-1}KX + XNL^{-1} = M^{-1}PL^{-1}$. Denoting $A := M^{-1}K$, $B := -NL^{-1}$, $C := M^{-1}PL^{-1}$, we get the Sylvester equation

$$AX - XB = C$$

for known $A, B, C \in \mathbb{H}$ and unknown $X \in \mathbb{H}$ (see also [15,21]).

Multiplying both sides of (1.2) on the right by $\overline{B}$, we get

$$-AX\overline{B} + XB\overline{B} = -C\overline{B}.$$  \hspace{1cm} (1.3)

Multiplying both sides of (1.2) on the left by $A$, we get

$$A^2X - AXB = AC.$$  \hspace{1cm} (1.4)
Summing (1.3) and (1.4) and using $B + \overline{B} \in \mathbb{R}$, $B\overline{B} \in \mathbb{R}$, we obtain
\[ A^2 X - (B + \overline{B})AX + B\overline{B}X = AC - C\overline{B}. \]
If
\[ D := A^2 - BA - \overline{B}A + B\overline{B} \neq 0, \]
then $D$ is invertible and we get the basis-free solution to (1.2):
\[ X = D^{-1}(AC - C\overline{B}) = \frac{\overline{D}(AC - C\overline{B})}{\overline{DD}}. \]

2. The Cases $n \leq 3$

Let us consider the Clifford’s geometric algebra $\mathbb{C}_p,q$, $p + q = n$, [6,10,16,18,19,22] with the identity element $e$ and the generators $e_a$, $a = 1, \ldots, n$, satisfying
\[ e_a e_b + e_b e_a = 2\eta_{ab} e, \quad a, b = 1, \ldots, n, \]
where $\eta = (\eta_{ab}) = \text{diag}(1, \ldots, 1, -1, \ldots, -1)$ is the diagonal matrix with its first $p$ entries equal to 1 and the last $q$ entries equal to $-1$ on the diagonal.

We call the subspace of $\mathbb{C}_p,q$ of geometric algebra elements, which are linear combinations of basis elements with multi-indices of length $k$, the subspace of grade $k$ and denote it by $\mathbb{C}_k^{p,q}$, $k = 0, 1, \ldots, n$. We identify elements of the subspace of grade 0 with scalars: $\mathbb{C}_0^{p,q} \equiv \mathbb{R}$, $e \equiv 1$. Denote the operation of projection onto the subspace $\mathbb{C}_k^{p,q}$ by $\langle \rangle_k$. The center of $\mathbb{C}_p,q$ is $\text{cen}(\mathbb{C}_p,q) = \mathbb{C}_0^{p,q}$ in the case of even $n$ and $\text{cen}(\mathbb{C}_p,q) = \mathbb{C}_0^{p,q} \oplus \mathbb{C}_n^{p,q}$ in the case of odd $n$.

We use the following two standard operations of conjugation in $\mathbb{C}_p,q$: the grade involution $\hat{}$ and the reversion (an anti-involution) $\tilde{}$
\[ \hat{U} = \sum_{k=0}^{n} (-1)^k \langle U \rangle_k, \quad \hat{UV} = \hat{U}\hat{V}, \quad \forall U, V \in \mathbb{C}_p,q, \] (2.1)
\[ \tilde{U} = \sum_{k=0}^{n} (-1)^{\frac{k(k-1)}{2}} \langle U \rangle_k, \quad \tilde{UV} = \tilde{V}\tilde{U}, \quad \forall U, V \in \mathbb{C}_p,q. \] (2.2)

Let us consider the Sylvester equation in geometric algebra
\[ AX - XB = C \] (2.3)
for known $A, B, C \in \mathbb{C}_p,q$ and unknown $X \in \mathbb{C}_p,q$.

In the case $n = 1$, the geometric algebra $\mathbb{C}_p,q$ is commutative and we get $(A - B)X = C$. Denoting $D := A - B$ and using
\[ \text{Adj}(D) = \hat{D}, \quad \text{Det}(D) = D\hat{D} \in \mathbb{C}_0^{p,q} \equiv \mathbb{R}, \quad D^{-1} = \frac{\text{Adj}(D)}{\text{Det}(D)}, \]
we conclude that if
\[ Q := D\hat{D} \neq 0, \] (2.4)
1The definitions of adjugate $\text{Adj}(D)$, determinant $\text{Det}(D)$, and inverse $D^{-1}$ in $\mathbb{C}_p,q$ for an arbitrary $n$ are given in [25].
then
\[ X = \frac{\hat{D}C}{Q}. \]

In the case \( n = 2 \), we can do the same as for the Sylvester equation over quaternions (see Introduction). Multiplying both sides of (2.3) on the right by \(-\hat{B}\) and on the left by \( A \), we get
\[ -AX\hat{B} + XB\hat{B} = -C\hat{B}, \quad A^2X - AXB = AC. \]

Summing and using \( \text{Det}(B) = B\hat{B} \in \mathcal{C}_{p,q}^0 \equiv \mathbb{R}, \) \( B + \hat{B} \in \mathcal{C}_{p,q}^0 \equiv \mathbb{R} \), we get
\[ (A^2 - (B + \hat{B})A + B\hat{B})X = AC - C\hat{B}. \] (2.5)

Using
\[ \text{Adj}(D) = \hat{D}, \quad \text{Det}(D) = D\hat{D} \in \mathcal{C}_{p,q}^0 \equiv \mathbb{R}, \quad D^{-1} = \frac{\text{Adj}(D)}{\text{Det}(D)}, \]
we conclude that if
\[ Q := D\hat{D} \neq 0, \] (2.6)
then for \( D := A^2 - (B + \hat{B})A + B\hat{B} \), we get
\[ X = \frac{\hat{D}(AC - C\hat{B})}{Q}. \] (2.7)

In the case \( n = 3 \), we have \( B\hat{B} \in \mathcal{C}_{p,q}^0 \oplus \mathcal{C}_{p,q}^3 = \text{cen}(\mathcal{C}_{p,q}) \) and \( B + \hat{B} \in \mathcal{C}_{p,q}^0 \oplus \mathcal{C}_{p,q}^3 = \text{cen}(\mathcal{C}_{p,q}) \) and obtain again (2.5). Using
\[ \text{Adj}(D) = \hat{D}\tilde{D}\hat{D}, \quad \text{Det}(D) = D\hat{D}\tilde{D}\hat{D} \in \mathcal{C}_{p,q}^0 \equiv \mathbb{R}, \quad D^{-1} = \frac{\text{Adj}(D)}{\text{Det}(D)} \]
for \( D := A^2 - (B + \hat{B})A + B\hat{B} \), we conclude that if
\[ Q := D\hat{D}\tilde{D}\hat{D} \neq 0, \] (2.8)
then
\[ X = \frac{\hat{D}\tilde{D}\hat{D}(AC - C\hat{B})}{Q}. \]

3. The Case \( n = 4 \)

Let us consider one additional operation of conjugation \( \triangle \) (compare with the grade involution (2.1) and the reversion (2.2), see also [25])
\[ U^\triangle = \sum_{k=0}^{4} (-1)^{k(k-1)(k-2)(k-3)/4!} \langle U \rangle_k \]
\[ = \sum_{k=0,1,2,3 \mod 8} \langle U \rangle_k - \sum_{k=4,5,6,7 \mod 8} \langle U \rangle_k, \quad \forall U \in \mathcal{C}_{p,q}. \] (3.1)
In the general case, we have \((UV)^\Delta \neq U^\Delta V^\Delta\) and \((UV)^\Delta \neq V^\Delta U^\Delta\).

**Theorem 3.1.** Let us consider the Sylvester equation in \(\mathcal{C}_{p,q}\), \(p + q = 4\)

\[AX - XB = C,\]  
(3.2)

for known \(A, B, C \in \mathcal{C}_{p,q}\) and unknown \(X \in \mathcal{C}_{p,q}\).

If

\[Q := \tilde{D}\tilde{D}(\tilde{D}\tilde{D})^\Delta \neq 0,\]  
(3.3)

then

\[X = \frac{\tilde{D}(\tilde{D}\tilde{D})^\Delta F}{Q},\]  
(3.4)

where

\[D := A^4 - A^3(B + \tilde{B} + \tilde{B}^\Delta + \tilde{B}^\Delta) + A^2(B\tilde{B} + B\tilde{B}^\Delta + \tilde{B}B^\Delta + \tilde{B}^\Delta + (\tilde{B}\tilde{B})^\Delta) - A(B\tilde{B}\tilde{B}^\Delta + \tilde{B}B\tilde{B}^\Delta + B\tilde{B}\tilde{B}^\Delta + \tilde{B}\tilde{B}\tilde{B}^\Delta + (\tilde{B}\tilde{B})^\Delta),\]

(3.5)

\[F := A^3C - A^2C(\tilde{B} + \tilde{B}^\Delta + \tilde{B}^\Delta) + AC(\tilde{B}\tilde{B}^\Delta + \tilde{B}B^\Delta + (\tilde{B}\tilde{B})^\Delta) - C\tilde{B}(\tilde{B}\tilde{B})^\Delta.\]

(3.6)

As one of the anonymous reviewers of this paper noted, the formulas (3.3), (3.4), (3.5), and (3.6) can be rewritten using the new operation

\[B^\Delta := (\tilde{B}\tilde{B})^\Delta\]  
(3.7)

in the following form

\[Q = D\tilde{D}D^\Delta, \quad X = \frac{\tilde{D}D^\Delta F}{Q},\]

\[D = A^4 - A^3(B + \tilde{B} + \tilde{B}^\Delta + \tilde{B}^\Delta) + A^2(B\tilde{B} + B\tilde{B}^\Delta + \tilde{B}B^\Delta + \tilde{B}^\Delta + (\tilde{B}\tilde{B})^\Delta)
\quad + \tilde{B}\tilde{B}^\Delta + B^\Delta) - A(B\tilde{B}\tilde{B}^\Delta + \tilde{B}B\tilde{B}^\Delta + BB^\Delta + \tilde{B}B^\Delta + \tilde{B}\tilde{B}^\Delta),\]

\[F = A^3C - A^2C(\tilde{B} + \tilde{B}^\Delta + \tilde{B}^\Delta) + AC(\tilde{B}\tilde{B}^\Delta + \tilde{B}B^\Delta + B^\Delta) - C\tilde{B}(\tilde{B}\tilde{B})^\Delta.\]

**Proof.** Multiplying both sides of (3.2) on the right by \(-\tilde{B}(\tilde{B}\tilde{B})^\Delta\), we get

\[-AX\tilde{B}(\tilde{B}\tilde{B})^\Delta + XB\tilde{B}(\tilde{B}\tilde{B})^\Delta = -C\tilde{B}(\tilde{B}\tilde{B})^\Delta.\]

(3.8)

Multiplying both sides of (3.2) on the right by \(\tilde{B}\tilde{B}^\Delta + \tilde{B}B^\Delta + (\tilde{B}\tilde{B})^\Delta\) and on the left by \(A\), we get

\[A^2X(\tilde{B}\tilde{B}^\Delta + \tilde{B}B^\Delta + (\tilde{B}\tilde{B})^\Delta) - AXB(\tilde{B}\tilde{B}^\Delta + \tilde{B}B^\Delta + (\tilde{B}\tilde{B})^\Delta)
\quad = AC(\tilde{B}\tilde{B}^\Delta + \tilde{B}B^\Delta + (\tilde{B}\tilde{B})^\Delta).\]

(3.9)
Multiplying both sides of (3.2) on the right by $-(\tilde{B} + \tilde{B}^\wedge + \tilde{B}^\wedge)$ and on the left by $A^2$, we get

$$-A^3X(\tilde{B} + \tilde{B}^\wedge + \tilde{B}^\wedge) + A^2XB(\tilde{B} + \tilde{B}^\wedge + \tilde{B}^\wedge)$$

$$=-A^2C(\tilde{B} + \tilde{B}^\wedge + \tilde{B}^\wedge).$$

(3.10)

Multiplying both sides of (3.2) on the left by $A^3$, we get

$$A^4X - A^3XB = A^3C.$$  (3.11)

Summing (3.8), (3.9), (3.10), and (3.11), and using the following explicit formulas for the characteristic polynomial coefficients from [25]

$$b_{(1)} := B + \tilde{B} + \tilde{B}^\wedge + \tilde{B}^\wedge \in \mathcal{O}^0_{p,q},$$

$$b_{(2)} := -(B\tilde{B} + B\tilde{B}^\wedge + B\tilde{B}^\wedge + \tilde{B}B^\wedge + \tilde{B}B^\wedge + (\tilde{B}B)^\wedge) \in \mathcal{O}^0_{p,q},$$

$$b_{(3)} := B\tilde{B}B^\wedge + B\tilde{B}B^\wedge + B(\tilde{B}B^\wedge) + \tilde{B}(B\tilde{B}^\wedge) \in \mathcal{O}^0_{p,q},$$

$$b_{(4)} := -\text{Det}(B) = -B\tilde{B}(\tilde{B}B^\wedge) \in \mathcal{O}^0_{p,q},$$

we get

$$(A^4 - A^3(B + \tilde{B} + \tilde{B}^\wedge + \tilde{B}^\wedge)$$

$$+A^2(B\tilde{B} + B\tilde{B}^\wedge + B\tilde{B}^\wedge + \tilde{B}B^\wedge + \tilde{B}B^\wedge + (\tilde{B}B)^\wedge)$$

$$-A(B\tilde{B}B^\wedge + B\tilde{B}B^\wedge + B(\tilde{B}B^\wedge) + \tilde{B}(B\tilde{B}^\wedge) + B\tilde{B}(B\tilde{B}^\wedge))X$$

$$= A^3C - A^2C(\tilde{B} + \tilde{B}^\wedge + \tilde{B}^\wedge)$$

$$+AC(\tilde{B}B^\wedge + \tilde{B}B^\wedge + (\tilde{B}B)^\wedge) - C\tilde{B}(\tilde{B}B^\wedge).$$

Denoting (3.5) and (3.6), and using the formula for the inverse in $\mathcal{O}^0_{p,q}$ with $n = p + q = 4$

$\text{Adj}(D) = \hat{D}(\hat{D}\hat{D})^\wedge$, \quad $\text{Det}(D) = D\hat{D}(\hat{D}\hat{D})^\wedge$, \quad $D^{-1} = \frac{\text{Adj}(D)}{\text{Det}(D)}$, we obtain (3.4). \hfill \Box

Let us present other formulas for the characteristic polynomial coefficients $b_{(1)}$, $b_{(2)}$, $b_{(3)}$, $b_{(4)}$ in the case $n = 4$. We use the same expressions in the case $n = 5$ (see Theorem 4.1). We have

$$B_{(1)} := B,$$

$$b_{(1)} = 4(B_{(1)})_0 = B + \tilde{B} + \tilde{B}^\wedge + \tilde{B},$$

$$B_{(2)} := B(B - b_{(1)}) = -B(\tilde{B} + \tilde{B}^\wedge + \tilde{B}),$$

$$b_{(2)} = 2(B_{(2)})_0 = -2(B(\tilde{B} + \tilde{B}^\wedge + \tilde{B}))_0$$

$$= -\frac{1}{2}(B\tilde{B} + B\tilde{B}^\wedge + B\tilde{B}^\wedge + B\tilde{B} + \tilde{B}^\wedge + \tilde{B}^\wedge + \tilde{B}^\wedge + (\tilde{B}B)^\wedge$$

$$+(\tilde{B}B^\wedge)^\wedge + (\tilde{B}B^\wedge)^\wedge + (\tilde{B}B)^\wedge + (\tilde{B}^\wedge B)^\wedge + (B\tilde{B})^\wedge + (B\tilde{B})^\wedge).$$
to the Sylvester equation in the case \( n \) using the new operation \( \triangle \).

We use the same expressions in the case \( n \) where

\[
\bar{D} : = \bar{D} \bar{B} \bar{D} \triangle \neq 0, \tag{3.12}
\]

then

\[
X = \frac{\bar{D} (\bar{D} \bar{D}) \triangle F}{Q}, \tag{3.13}
\]

where

\[
D : = A^4 - A^3 (B + \bar{B} + \hat{B} \triangle + \bar{B} \triangle) + A^2 (B \bar{B} + B \bar{B} \triangle + B \bar{B} \triangle + B \bar{B} \triangle + \bar{B} \bar{B} \triangle) - A (B \bar{B} \bar{B} \triangle + B \bar{B} \bar{B} \triangle + B \bar{B} \bar{B} \triangle + B \bar{B} \bar{B} \triangle) + B \bar{B} \bar{B} \bar{B} \bar{B} \triangle, \tag{3.14}
\]

\[
F : = A^3 C - A^2 C (\bar{B} + \hat{B} \triangle + \bar{B} \triangle) + A C (B \bar{B} \bar{B} \triangle + B \bar{B} \bar{B} \triangle + B \bar{B} \bar{B} \triangle) - C \bar{B} \bar{B} \bar{B} \bar{B} \triangle. \tag{3.15}
\]

We use the same expressions in the case \( n = 5 \) (see the next section).

Note that the formulas (3.12), (3.13), (3.14), and (3.15) can be rewritten using the new operation

\[
B^\sharp : = (\hat{B} \bar{B}) \triangle \tag{3.16}
\]

in the form

\[
Q = D \bar{D} D^\sharp, \quad X = \frac{\bar{D} D^\sharp F}{Q},
\]

\[
D = A^4 - A^3 (B + \bar{B} + \hat{B} \triangle + \bar{B} \triangle) + A^2 (B \bar{B} + B \bar{B} \triangle + B \bar{B} \triangle + \bar{B} \bar{B} \triangle)
+B \bar{B} \hat{B} \triangle + B^\sharp - A (B \bar{B} \bar{B} \triangle + B \bar{B} \bar{B} \triangle + B \bar{B} \bar{B} \triangle + B \bar{B} \bar{B} \triangle) + B \bar{B} \bar{B} \bar{B} \bar{B} \triangle,
\]

\[2\text{Analytic proof is also possible using the methods from [1].}\]
\[ F = A^3C - A^2C(\bar{B} + \bar{B}^\Delta + \bar{B}^\Delta) + AC(\bar{B}\bar{B}^\Delta + \bar{B}\bar{B}^\Delta) + B^2 - C\bar{B}B^2. \]

**Example.** Let us consider the Sylvester equation \((3.2)\) in \(\mathcal{C}_1,3\) with
\[
A = 3e - 5e_1 + 2e_2 - 2e_3 - 4e_4 + e_12 + 3e_13 + 5e_14 + 2e_23 + 2e_24 - 5e_34 + 2e_12 - 4e_{12} + e_13 + 4e_{23} + 2e_{124},
\]
\[
B = 2e + 5e_1 - e_2 - 2e_3 - e_4 + e_12 + 2e_{13} + 5e_{14} - 5e_{23} + 2e_{24} - 3e_{34} + 4e_{12} - 3e_{124} + 4e_{134} + 3e_{234} + e_{1234},
\]
\[
C = 4e + e_1 - 3e_2 - 2e_3 + 4e_4 - e_{12} - 5e_{14} + 3e_{23} + e_{24} - 4e_{34} + 2e_{12} - 3e_{124} - 2e_{134} - 5e_{234} + 5e_{1234}.
\]

Using the formulas (3.12), (3.13), (3.14), (3.15), and computer calculations in Wolfram Mathematica, we get
\[
D = -3331e + 16960e_1 - 2736e_2 + 5228e_3 + 11276e_4 - 4372e_{12} - 6740e_{13} - 17764e_{14} - 4208e_{23} - 4520e_{24} + 12072e_{34} - 8664e_{123} + 8664e_{124} - 2128e_{134} - 14968e_{234} - 5868e_{1234},
\]
\[
Q = 818014056354052817e \neq 0,
\]
\[
F = -3654e - 3114e_1 - 4238e_2 - 12909e_3 - 629e_4 - 7164e_{12} - 5583e_{13} - 9442e_{14} - 14155e_{23} - 1197e_{24} + 3316e_{34} - 9352e_{123} - 2768e_{124} - 2570e_{134} + 6614e_{234} - 6485e_{1234},
\]
\[
X = \frac{1}{Q}(-119559672248263574e - 243271127103539030e_1 - 45110505690078854e_2 + 102025493907271711e_3 - 237419769499231033e_4 - 230234896037415164e_{12} - 631822395022405163e_{13} + 354830063944470830e_{14} - 248262081322178503e_{23} + 381628355781437695e_{24} + 24227796156695860e_{34} + 175205777213912492e_{123} + 85615763017907532e_{124} - 78264759152759606e_{1234} + 12173556035563862e_{234} + 268142275333252559e_{1234}).
\]

Substituting (3.17) into (3.2), we get equality.

### 4. The Case \(n = 5\)

**Theorem 4.1.** Let us consider the Sylvester equation in \(\mathcal{C}_{p,q}\), \(p + q = 5\),
\[
AX - XB = C
\]
for known \(A, B, C \in \mathcal{C}_{p,q}\) and unknown \(X \in \mathcal{C}_{p,q}\).

If
\[
Q := D\hat{D}(\hat{D}\hat{D})^\Delta(D\hat{D}(\hat{D}\hat{D})^\Delta)^\Delta \neq 0,
\]
then
\[
X = \frac{D\hat{D}(\hat{D}\hat{D})^\Delta(D\hat{D}(\hat{D}\hat{D})^\Delta)^\Delta F}{Q},
\]
where

\[ D := A^4 - A^3(B + \tilde{B} + \tilde{B}^\Delta + \tilde{B}^\Delta) \]
\[ + A^2(B\tilde{B} + B\bar{B}^\Delta + B\tilde{B}^\Delta + B\tilde{B}^\Delta + (\tilde{B}\bar{B})^\Delta) \]
\[ - A(B\tilde{B}B^\Delta + B\tilde{B}^\Delta + B(\tilde{B}\bar{B})^\Delta + B(\tilde{B}\bar{B})^\Delta) + B\tilde{B}(B\bar{B})^\Delta, \quad (4.4) \]
\[ F := A^3C - A^2C(\tilde{B} + \bar{B}^\Delta + \tilde{B}^\Delta) \]
\[ + AC(\tilde{B}\bar{B}^\Delta + \tilde{B}^\Delta + (\tilde{B}\bar{B})^\Delta) - C\bar{B}(\tilde{B}\bar{B})^\Delta. \quad (4.5) \]

Note that the formulas (4.2), (4.3), (4.4), and (4.5) can be rewritten using the operation (3.16) in the form

\[ Q = D\tilde{D}\tilde{D}^d(D\tilde{D}\tilde{D}^d)^\Delta, \quad X = \frac{D\tilde{D}\tilde{D}^d(D\tilde{D}\tilde{D}^d)^\Delta F}{Q}, \]
\[ D = A^4 - A^3(B + \tilde{B} + \bar{B}^\Delta + \tilde{B}^\Delta) + A^2(B\tilde{B} + B\bar{B}^\Delta + B\tilde{B}^\Delta + \tilde{B}\bar{B}^\Delta + \tilde{B}^\Delta) \]
\[ + \tilde{B}^\Delta + B^\Delta) - A(B\tilde{B}B^\Delta + B\tilde{B}^\Delta + BB^\Delta + \tilde{B}^\Delta + \tilde{B}B^\Delta + B^\Delta, \]
\[ F = A^3C - A^2C(\tilde{B} + \bar{B}^\Delta + \tilde{B}^\Delta) + AC(\tilde{B}\bar{B}^\Delta + \tilde{B}^\Delta + B^\Delta) - C\bar{B}(\tilde{B}\bar{B})^\Delta. \]

**Proof.** In the case \( C_{p,q}, p + q = 5 \), we have 8 characteristic polynomial coefficients\(^3\) \( b_{(1)}, \ldots, b_{(8)} \) for an arbitrary element \( B \in C_{p,q} \). Instead of them, let us consider the following 4 expressions (which are scalars in the case \( n = 4 \))

\[ b'_{(1)} = B + \tilde{B} + \bar{B}^\Delta + \tilde{B}^\Delta, \]
\[ b'_{(2)} = -(B\tilde{B} + B\bar{B}^\Delta + B\tilde{B}^\Delta + B\tilde{B}^\Delta + (\tilde{B}\bar{B})^\Delta), \]
\[ b'_{(3)} = B\tilde{B}B^\Delta + B\tilde{B}^\Delta + B(\tilde{B}\bar{B})^\Delta + B(\tilde{B}\bar{B})^\Delta, \]
\[ b'_{(4)} = -B\tilde{B}(\tilde{B}\bar{B})^\Delta. \quad (4.6) \]

We have\(^4\) \( b'_{(1)}, b'_{(2)}, b'_{(3)}, b'_{(4)} \in \text{cen}(C_{p,q}) = C_{p,q}^0 \oplus C_{p,q}^5 \). We can easily verify that

\[ B + \tilde{B} + \bar{B}^\Delta + \tilde{B}^\Delta = 4(\langle B \rangle_0 + \langle B \rangle_5) \in C_{p,q}^0 \oplus C_{p,q}^5 \]

using definitions of the operations (2.1), (2.2), and (3.1). We have \( B\tilde{B}(\tilde{B}\bar{B})^\Delta \in C_{p,q}^0 \oplus C_{p,q}^5 \) (see [25]). We verified \( b'_{(2)}, b'_{(3)} \in C_{p,q}^0 \oplus C_{p,q}^5 \) using computer calculations\(^5\).

\(^3\)Explicit formulas for the coefficients \( b_{(1)}, \ldots, b_{(8)} \) are presented in [1].

\(^4\)Note that the same is not true for the expressions for characteristic polynomial coefficients from Theorem 3.1. For example, \( B\tilde{B}(\tilde{B}\bar{B})^\Delta \in C_{p,q}^0 \oplus C_{p,q}^1 \oplus C_{p,q}^4 \neq \text{cen}(C_{p,q}) \) in the case \( n = 5 \), see the details in [25].

\(^5\)Analytic proof is also possible using the methods from [1].
Multiplying both sides of (4.1) on the right by \(-\hat{B}(\hat{B}\hat{B})\wedge\), we get
\[
-AX\hat{B}(\hat{B}\hat{B})\wedge + XB\hat{B}(\hat{B}\hat{B})\wedge = -C\hat{B}(\hat{B}\hat{B})\wedge. \tag{4.7}
\]

Multiplying both sides of (4.1) on the right by \(\hat{B}\hat{B}\wedge + \hat{B}\hat{B} + (\hat{B}\hat{B})\wedge\) and on the left by \(A\), we get
\[
A^2X(\hat{B}\hat{B}\wedge + \hat{B}\hat{B} + (\hat{B}\hat{B})\wedge) - AXB(\hat{B}\hat{B}\wedge + \hat{B}\hat{B} + (\hat{B}\hat{B})\wedge) = AC(\hat{B}\hat{B}\wedge + \hat{B}\hat{B} + (\hat{B}\hat{B})\wedge). \tag{4.8}
\]

Multiplying both sides of (4.1) on the right by \(-(\hat{B} + \hat{B}\wedge + \hat{B})\) and on the left by \(A^2\), we get
\[
-A^3X(\hat{B} + \hat{B}\wedge + \hat{B}) + A^2XB(\hat{B} + \hat{B}\wedge + \hat{B}) = -A^2C(\hat{B} + \hat{B}\wedge + \hat{B}). \tag{4.9}
\]

Multiplying both sides of (4.1) on the left by \(A^3\), we get
\[
A^4X - A^3XB = A^3C. \tag{4.10}
\]

Summing (4.7), (4.8), (4.9), and (4.10), we get
\[
(A^4 - A^3(B + \hat{B} + \hat{B}\wedge + \hat{B})) + A^2(B\hat{B} + B\hat{B}\wedge + \hat{B}\hat{B} + \hat{B}\hat{B} + (\hat{B}\hat{B})\wedge) - A(B\hat{B}\hat{B}\wedge + B\hat{B}\hat{B} + B(\hat{B}\hat{B})\wedge + B\hat{B} + (\hat{B}\hat{B})\wedge)X = A^3C - A^2C(\hat{B} + \hat{B}\wedge + \hat{B}) + AC(\hat{B}\hat{B}\wedge + \hat{B}\hat{B} + (\hat{B}\hat{B})\wedge) - C\hat{B}(\hat{B}\hat{B})\wedge.
\]

Denoting (4.4) and (4.5), and using the formula for the inverse in the case \(n = 5\) (see [25]):
\[
D^{-1} = \frac{\Adj(D)}{\Det(D)}, \quad \Det(D) = D\hat{D}(\hat{D}\hat{D})\wedge(D\hat{D}(\hat{D}\hat{D})\wedge)\wedge \in \mathcal{C}^0_{p,q} \equiv \mathbb{R},
\]
\[
\Adj(D) = \hat{D}(\hat{D}\hat{D})\wedge(D\hat{D}(\hat{D}\hat{D})\wedge)\wedge,
\]
we get (4.3). \(\square\)

**Example.** Let us consider the Sylvester equation (4.1) in \(\mathcal{C}_{4,1}\) with
\[
A = -e + e_1 - 3e_2 + 3e_3 + 2e_4 - e_5 + 3e_{12} + 2e_{13} - 2e_{14} - 3e_{15} - e_3
-3e_{24} - e_{25} - e_{34} - 3e_{35} - 3e_{45} - e_{123} - 3e_{124} + e_{125} - e_{134} - 3e_{135}
+e_{145} + 2e_{234} + 2e_{235} - 2e_{245} + 3e_{345} + 3e_{1234} - 2e_{1235} + 2e_{1245}
-e_{1345} - 2e_{2345} - 2e_{12345},
B = -2e - e_1 - 3e_2 - 2e_3 - e_4 + e_5 - 2e_{12} + 2e_{13} - e_{14} - 2e_{15} + 3e_3
+e_{24} - 2e_{25} - 3e_{34} + 2e_{35} - 3e_{45} - e_{123} + e_{124} + 2e_{125} - 2e_{134} + 3e_{135}
\]
Using the formulas (4.2), (4.3), (4.4), (4.5), and computer calculations in Wolfram Mathematica, we get

\[
D = -28e - 2784e_1 + 4088e_2 - 1584e_3 + 1432e_4 - 2528e_5 - 1688e_{12}
-2496e_{13} - 3624e_{14} - 1392e_{15} + 2904e_{23} - 3392e_{24} - 648e_{25}
+1664e_{34} + 3104e_{35} + 3088e_{45} + 1568e_{123} - 3056e_{124} - 4272e_{125}
+2888e_{134} + 4280e_{135} + 296e_{145} + 416e_{234} - 936e_{235} + 560e_{245}
+3064e_{345} - 208e_{1234} + 4648e_{1235} - 1632e_{1245} - 1528e_{1345}
-1712e_{2345} - 1112e_{12345},
\]

\[
F = -4792e - 4250e_1 + 2398e_2 + 1168e_3 - 8208e_4 + 3268e_5 + 784e_{12}
+4594e_{13} + 2108e_{14} - 4948e_{15} - 1454e_{23} + 606e_{24} + 2350e_{25}
+7786e_{34} - 3102e_{35} + 8970e_{45} - 1004e_{123} - 4682e_{124} + 5594e_{125}
+3822e_{134} - 1034e_{135} + 9688e_{145} + 6272e_{234} + 4448e_{235} + 5580e_{245}
-222e_{345} + 3816e_{1234} + 2648e_{1235} + 7042e_{1245} + 6824e_{1345}
+6496e_{2345} - 5290e_{12345},
\]

and

\[
X = \frac{1}{Q} \left( -2542637343026554833978381852032e_1 + 124333161192922434122282795008e_1 
+ 425423260869616089214910464e_2 + 7759061400011677795555576448e_3 
- 27479768936387336589072967168e_4 - 2516615656241405234693942688e_5 
+ 172171077554950010584268800000e_{12} + 45097461083174889585390105600e_{13} 
+ 1253628768599348861083777536e_{14} - 105084562335627847197682171904e_{15} 
- 369396757288051254240066080768e_{23} + 2415280095496629837389037568e_{24} 
+ 404037705534519977662329880576e_{25} + 1540183456802404459413458464e_{34} 
- 757495071696413019806932380e_{35} - 6590783498544571184802606596e_{45} 
- 350045165524614842201912639488e_{123} - 1030454686559123954114190981120e_{124}
+ 590953599976818750339972169728e_{125} + 73707775121065150382774059008e_{134}
- 231126475215498004285258232096e_{135} + 271845290646123969359373860864e_{145}
- 10662454161643779070075025408e_{234} + 14839261172489026004936435252e_{235}
+ 8266582802946342235158582800e_{245} + 37807107639770003501829672960e_{255}
+ 226070526593898040150924263424e_{1234} - 1226107966629572100684545536e_{1235}
+ 2004665186315653844965973504e_{1345} + 2737316751740384861270170368e_{1345}
+ 24967534342810865838275067904e_{2345} - 3455848110307459437641386976e_{12345} \right)
\]

Substituting (4.11) into (4.1), we get equality.
5. The Case of Arbitrary $n$

Let us consider the general case of the real Clifford algebra $\mathcal{C}_p^q$ with arbitrary $n = p + q$. We use the following concepts of characteristic polynomial $\varphi_B(\lambda)$, determinant $\text{Det}(B)$, adjugate $\text{Adj}(B)$, and inverse $B^{-1}$ in $\mathcal{C}_p^q$ (see the details in [25])

$$\varphi_B(\lambda) := \text{Det}(\lambda e - B) = \lambda^N - b(1)\lambda^{N-1} - \cdots - b(N) \in \mathcal{C}_p^q$$

$$B(1) := B, \quad B(k+1) := B(B(k) - b(k)), \quad N := 2^\left\lceil \frac{n+1}{2} \right\rceil,$$

$$b(k) = \frac{N}{k} \langle B(k) \rangle_0 \in \mathcal{C}_p^q \equiv \mathbb{R}, \quad k = 1, \ldots, N,$$

$$\text{Det}(B) = -B(N) = -b(N) = B(b(N-1) - B(N-1)) \in \mathcal{C}_p^q \equiv \mathbb{R},$$

$$\text{Adj}(B) = b(N-1) - B(N-1), \quad B^{-1} = \frac{\text{Adj}(B)}{\text{Det}(B)}.$$

In the following theorem, we present recursive formulas for the basis-free solution to the Sylvester equation in the case of arbitrary $n = p + q$.

**Theorem 5.1.** Let us consider the Sylvester equation in $\mathcal{C}_p^q$, $p + q = n$,

$$AX - XB = C \quad (5.1)$$

for known $A, B, C \in \mathcal{C}_p^q$ and unknown $X \in \mathcal{C}_p^q$.

Let us denote $N := 2^\left\lceil \frac{n+1}{2} \right\rceil$. If

$$Q := d(N) \neq 0, \quad (5.2)$$

then

$$X = \frac{(D(N-1) - d(N-1))F}{Q}, \quad (5.3)$$

where

$$D := -\sum_{j=0}^{N} A^{N-j}b(j), \quad (5.4)$$

$$F := \sum_{j=1}^{N} A^{N-j}C(B_{(j-1)} - b_{(j-1)}), \quad (5.5)$$

and the following expressions are defined recursively \(^7\):

$$b(k) = \frac{N}{k} \langle B(k) \rangle_0, \quad B(k+1) := B(B(k) - b(k)), \quad B(1) = B,$$

$$d(k) = \frac{N}{k} \langle D(k) \rangle_0, \quad D(k+1) := D(D(k) - d(k)), \quad D(1) = D,$$

$$B(0) = D(0) := 0, \quad b(0) = d(0) := -1, \quad k = 1, \ldots, N.$$

Note that $D (5.4)$ is the characteristic polynomial of the element $B$ with the substitution of $A$.

---

\(^6\)Here and below we denote the integer part of the number $\frac{n+1}{2}$ by $\lfloor \frac{n+1}{2} \rfloor$.

\(^7\)Note that using the recursive formulas $B(k+1) = B(B(k) - b(k))$, the expression (5.5) can be reduced to the form $\sum_{i,j} b_{ij} A^i C B^j$ with some scalars $b_{ij} \in \mathbb{R}$. 

Proof. Multiplying both sides of (5.1) on the right by \(B_{(N-1)} - b_{(N-1)}\), on the right by \(B_{(N-2)} - b_{(N-2)}\) and on the left by \(A\), on the right by \(B_{(N-3)} - b_{(N-3)}\) and on the left by \(A^2\), \ldots, on the right by \(B_{(2)} - b_{(2)}\) and on the left by \(A^{N-3}\), on the right by \(B - b_{(1)}\) and on the left by \(A^{N-2}\), on the left by \(A^{N-1}\), we get

\[
AX(B_{(N-1)} - b_{(N-1)}) - XB(B_{(N-1)} - b_{(N-1)}) = C(B_{(N-1)} - b_{(N-1)}),
\]

\[
A^2X(B_{(N-2)} - b_{(N-2)}) - AXB(B_{(N-2)} - b_{(N-2)}) = AC(B_{(N-2)} - b_{(N-2)}),
\]

\[
A^3X(B_{(N-3)} - b_{(N-3)}) - A^2XB(B_{(N-3)} - b_{(N-3)}) = A^2C(B_{(N-3)} - b_{(N-3)}),
\]

\[
\ldots
\]

\[
A^{N-2}X(B_{(2)} - b_{(2)}) - A^{N-3}XB(B_{(2)} - b_{(2)}) = A^{N-3}C(B_{(2)} - b_{(2)}),
\]

\[
A^{N-1}X(B - b_{(1)}) - A^{N-2}XB(B - b_{(1)}) = A^{N-2}C(B - b_{(1)}),
\]

\[
A^N X - A^{N-1}XB = A^{N-1}C.
\]

Summing these equations and using \(B_{(k+1)} = B(B_{(k)} - b_{(k)}), \ k = 1, \ldots, N,\)
\(B_{(N)} = B(B_{(N-1)} - b_{(N-1)}) = b_{(N)} = -\text{Det}(B) \in \mathcal{C}_p^0\),
we get

\[
(A^N - b_{(1)})A^{N-1}X - b_{(2)}A^{N-2}X - \cdots - b_{(N-1)}AX - b_{(N)}X = A^{N-1}C + A^{N-2}C(B - b_{(1)}) + \cdots + C(B_{(N-1)} - b_{(N-1)}).\]

Denoting (5.4), (5.5), and using

\[
D^{-1} = \frac{\text{Adj}(D)}{\text{Det}(D)}, \quad \text{Adj}(D) = d_{(N-1)} - D_{(N-1)}, \quad \text{Det}(D) = -d_{(N)},
\]

we get (5.3). \(\square\)

Note that in the case of odd \(n\), the formulas (5.4) and (5.5) can be simplified. We can use instead of \(N\) characteristic polynomial coefficients some other \(\frac{N}{2}\) expressions. We call them generalized characteristic polynomial coefficients. For example, in the case \(n = 5\) (see Theorem 4.1), we use the 4 expressions \(b'_{(k)}, k = 1, 2, 3, 4, (4.6)\), which are in the center of \(\mathcal{C}_p^0\), instead of the 8 ordinary characteristic polynomial coefficients \(b_{(k)}, k = 1, \ldots, 8\), which are in \(\mathcal{C}_p^0\).

The ordinary characteristic polynomial coefficients of Clifford algebra element corresponds to the characteristic polynomial coefficients of the corresponding matrix representation of dimension \(N\) (see the details in [25]). The generalized characteristic polynomial coefficients of Clifford algebra element corresponds to the characteristic polynomial coefficients of the corresponding matrix of dimension \(\frac{N}{2}\) with entries in \(\mathbb{C}\) or \(\mathbb{R} \oplus \mathbb{R}\). In more details, the center of \(\mathcal{C}_p^0\) with odd \(n = p + q\) is \(\text{cen}(\mathcal{C}_p^0) = C_{p,q}^0 \oplus C_{p,q}^n\), which is isomorphic
to $\mathbb{C}$ in the case $e_{1\ldots n}^2 = -e$ (i.e. $p - q = 2, 3 \mod 4$) and to $\mathbb{R} \oplus \mathbb{R}$ in the case $e_{1\ldots n}^2 = e$ (i.e. $p - q = 0, 1 \mod 4$). The Clifford algebra $\mathcal{C}_p,q$ with odd $n = p + q$ can be represented in the form (the same idea is used in [14])

$$\mathcal{C}_p,q = C_{p,q}^{(0)} \oplus C_{p,q}^{(1)} = \mathcal{C}_p,q \oplus e_{1\ldots n}C_{p,q}^{(0)}$$

where

$$C_{p,q}^{(0)} = \bigoplus_{k=0 \mod 2}^k \mathcal{C}_{p,q}^k,$$

$$C_{p,q}^{(1)} = \bigoplus_{k=1 \mod 2}^k \mathcal{C}_{p,q}^k$$

are the even subalgebra and the odd subspace of $\mathcal{C}_{p,q}$. Thus any element $B \in \mathcal{C}_{p,q}$ can be written as an element of the even subalgebra $C_{p,q}^{(0)}$ with complex (in the cases $p - q = 2, 3 \mod 4$) or hyperbolic (in the cases $p - q = 0, 1 \mod 4$) coefficients. Also we use the well-known isomorphisms (see, for example, [18, 22])

$$C_{p,q}^{(0)} \cong C_{p,q-1}, \quad q \geq 1; \quad C_{p,q}^{(0)} \cong C_{q,p-1}, \quad p \geq 1.$$

We obtain the following simplification of the statement of Theorem 5.1 in the case of odd $n = p + q$ (with $N^2$ steps in the corresponding recursive formulas for $D$ and $F$ instead of $N$ steps for these expressions as in the previous theorem).

Let us consider the Sylvester equation in $\mathcal{C}_{p,q}$ with odd $p + q = n$,

$$AX - XB = C$$

(5.6)

for known $A, B, C \in \mathcal{C}_{p,q}$ and unknown $X \in \mathcal{C}_{p,q}$. Let us denote $N := 2^{n+1}$. If

$$Q := d_{(N)} \neq 0,$$

then

$$X = \frac{(D_{(N-1)} - d_{(N-1)})F}{d_{(N)}},$$

where

$$D := \varphi_B(A) = -\sum_{j=0}^{\frac{N}{2}} A^{\frac{N}{2}-j}b_{(j)}',$$

$$F := \sum_{j=1}^{\frac{N}{2}} A^{\frac{N}{2}-j}C(B_{(j-1)}' - b_{(j-1)}'),$$

and the following expressions are defined recursively:

$$b_{(k)}' = \frac{N}{k}\langle B_{(k)}'\rangle_{\text{cen}}, \quad B_{(k+1)}' = B(B_{(k)}' - b_{(k)}'), \quad k = 1, \ldots, \frac{N}{2},$$

$$B_{(1)}' = B, \quad B_{(0)}' := 0, \quad b_{(0)}' := -1,$$

$$d_{(m)} = \frac{N}{m}(D_{(m)})_0, \quad D_{(m+1)} = D(D_{(m)} - d_{(m)}), \quad m = 1, \ldots, N,$$

$$D_{(1)} = D, \quad D_{(0)} := 0, \quad d_{(0)} := -1.$$

---

8In the case of odd $n$, the integer part of the number $\frac{n+1}{2}$ is equal to $\lceil \frac{n+1}{2} \rceil = \frac{n+1}{2} \in \mathbb{Z}$. 

Note that expressions for generalized characteristic polynomial coefficients $b'_(k)$, $k = 1, \ldots, \frac{N}{2}$ coincide with the expressions for the characteristic polynomial coefficients for the previous even $n - 1$ (see the example for the case $n = 5$ in Theorem 4.1).

**Example.** Let us present the explicit expressions for $\langle B \rangle_{\text{cen}}$ in the case of small dimensions $n \leq 15$ using the operations of conjugation $\sim, \hat{\sim}, \bigtriangleup, \boxdot$:

\[
\frac{1}{2}(B + \hat{B}) = \langle B \rangle_{\text{cen}} = \begin{cases} 
\langle B \rangle_0, & \text{if } n = 2, \\
\langle B \rangle_0 + \langle B \rangle_3, & \text{if } n = 3;
\end{cases}
\]

\[
\frac{1}{4}(B + \hat{B} + \hat{B}^\bigtriangleup + \hat{B} \sim) = \langle B \rangle_{\text{cen}} = \begin{cases} 
\langle B \rangle_0, & \text{if } n = 4, \\
\langle B \rangle_0 + \langle B \rangle_5, & \text{if } n = 5;
\end{cases}
\]

\[
\frac{1}{4}(B + \hat{B} + \hat{B}^\bigtriangleup + \hat{B} \sim) = \langle B \rangle_{\text{cen}} = \begin{cases} 
\langle B \rangle_0, & \text{if } n = 6, \\
\langle B \rangle_0 + \langle B \rangle_7, & \text{if } n = 7;
\end{cases}
\]

\[
\frac{1}{8}(B + \hat{B} \boxdot + \hat{B}^\bigtriangleup + B^\bigtriangleup + \hat{B} \boxdot + \hat{B}^\bigtriangleup + \hat{B} \hat{\boxdot} + \hat{B} \boxdot + \hat{B} \hat{\boxdot})
\]

\[
= \langle B \rangle_{\text{cen}} = \begin{cases} 
\langle B \rangle_0, & \text{if } n = 8, \\
\langle B \rangle_0 + \langle B \rangle_9, & \text{if } n = 9;
\end{cases}
\]

\[
\frac{1}{8}(B + \hat{B} \boxdot + \hat{B}^\bigtriangleup + \hat{B} + B^\bigtriangleup + \hat{B} \hat{\boxdot} + \hat{B} \boxdot + \hat{B} \hat{\boxdot} + \hat{B} \hat{\boxdot})
\]

\[
= \langle B \rangle_{\text{cen}} = \begin{cases} 
\langle B \rangle_0, & \text{if } n = 10, \\
\langle B \rangle_0 + \langle B \rangle_{11}, & \text{if } n = 11;
\end{cases}
\]

\[
\frac{1}{8}(B + \hat{B} \boxdot + \hat{B}^\bigtriangleup + \hat{B} + B \hat{\bigtriangleup} + \hat{B} \hat{\boxdot} + \hat{B} \boxdot + \hat{B} \hat{\boxdot} + \hat{B} \hat{\boxdot})
\]

\[
= \langle B \rangle_{\text{cen}} = \begin{cases} 
\langle B \rangle_0, & \text{if } n = 12, \\
\langle B \rangle_0 + \langle B \rangle_{13}, & \text{if } n = 13;
\end{cases}
\]

\[
\frac{1}{8}(B + \hat{B} \boxdot + \hat{B}^\bigtriangleup + \hat{B} + B \hat{\bigtriangleup} + \hat{B} \hat{\boxdot} + \hat{B} \boxdot + \hat{B} \hat{\boxdot})
\]

\[
= \langle B \rangle_{\text{cen}} = \begin{cases} 
\langle B \rangle_0, & \text{if } n = 14, \\
\langle B \rangle_0 + \langle B \rangle_{15}, & \text{if } n = 15,
\end{cases}
\]

where we use the additional operation of conjugation $\boxdot$ (compare with (2.1), (2.2), and (3.1), see also [25])

\[
B^\boxdot = \sum_{k=0}^{n} (-1)^{k(k-1)(k-2)(k-3)(k-4)(k-5)(k-6)(k-7)} \langle U \rangle_k
\]

\[
= \sum_{k=0,1,\ldots,7 \mod 16} \langle B \rangle_k - \sum_{k=8,9,10,11,12,13,14,15 \mod 16} \langle B \rangle_k, \quad \forall B \in \mathfrak{S}_{p,q}.
\]

In the general case, we have $(UV)^\boxdot \neq U^\boxdot V^\boxdot$ and $(UV)^\boxdot \neq V^\boxdot U^\boxdot$.

Let us return to the case of arbitrary $n$. The scalar part operation $\langle B \rangle_0$ and the projection onto the center $\langle B \rangle_{\text{cen}}$ can always be realized as linear
combinations of operations of conjugation

\[ B \mapsto \sum_{k=0}^{n} \lambda_k \langle B \rangle_k, \quad \lambda_k = \pm 1. \]

This fact is discussed in [25] (see Theorem 1 and Footnote 7). We need \( \lfloor \log_2 n \rfloor + 1 \) different operations of conjugation to do this\(^9\).

Let us denote \( m := \lfloor \log_2 n \rfloor + 1 \). The formulas from Theorem 5.1 and the following formulas\(^11\) from [25]

\[ \langle B \rangle_0 = \frac{1}{2^m} (B + B^{\triangle 1} + B^{\triangle 2} + \cdots + B^{\triangle m} + B^{\triangle 1 \triangle 2} + \cdots + B^{\triangle 1 \cdots \triangle m}), \]

\[ B^{\triangle j} = \sum_{k=0}^{n} (-1)^{C_{k}^{j-1}} \langle B \rangle_k, \quad C_{k}^{i} = \frac{k!}{i!(k-i)!}, \quad j = 1, \ldots, m, \]

give us the basis-free solution (which involve only the operations of Clifford product, summation, and the operations of conjugation) to the Sylvester equation in \( \mathcal{C}_p,q \) with arbitrary \( n \). Thus, our approach works in the case of arbitrary \( n \).

6. Conclusions

In this paper, we present the basis-free solution to the Sylvester equation in Clifford (geometric) algebra of arbitrary dimension. Note that we discuss the most important (nondegenerate) case when the element \( Q \) (2.4), (2.6), (2.7), (3.3), (4.2), (5.2), (5.7) is non-zero and the corresponding Sylvester equation (2.3) has a unique solution \( X \). The degenerate case \( Q = 0 \) (with zero divisors) can also be studied.

An interesting task is to generalize results of this paper to the case of general linear equations in geometric algebras

\[ \sum_{j=1}^{k} A_j X B_j = C, \quad A_j, B_j, C, X \in \mathcal{C}_p,q \quad (6.1) \]

in the case of arbitrary \( n = p + q \). The basis-free solution to the equation (6.1) in the case of quaternions \( \mathbb{H} \cong \mathcal{C}_0,2 \) is given in [21] (see also the papers [15, 20]).

Note that all results of this paper remain true for the complexified Clifford algebras \( \mathbb{C} \otimes \mathcal{C}_p,q \) because characteristic polynomial coefficients are the same and are defined for \( \mathcal{C}_p,q \) and \( \mathbb{C} \otimes \mathcal{C}_p,q \) in the same manner using the matrix representation of \( \mathbb{C} \otimes \mathcal{C}_p,q \) of dimension \( N = 2^\lceil \frac{n+1}{2} \rfloor \) (see [25]).

The real Clifford algebras are isomorphic to the matrix algebras over \( \mathbb{R}, \mathbb{R} \oplus \mathbb{R}, \mathbb{C}, \mathbb{H}, \) or \( \mathbb{H} \oplus \mathbb{H} \) depending on \( p - q \mod 8 \) and the complex Clifford algebras (and complexified Clifford algebras) are isomorphic to the matrix algebras over \( \mathbb{C} \) or \( \mathbb{C} \oplus \mathbb{C} \) depending on \( n \mod 2 \). In the opinion of the author, the

\(^9\)Here and below we denote the integer part of \( \log_2 n \) by \( \lfloor \log_2 n \rfloor \).

\(^{10}\)In the above example, we use the three operations \( \hat{B}, \tilde{B} \), and \( B^{\triangle} \) in the cases \( 4 \leq n \leq 7 \), the four operations \( \hat{B}, \tilde{B}, B^{\triangle}, \) and \( B^{\boxtimes} \) in the cases \( 8 \leq n \leq 15 \).

\(^{11}\)Note that \( B^{\triangle 1} = \hat{B}, B^{\triangle 2} = \tilde{B}, B^{\triangle 3} = B^{\triangle}, \) and \( B^{\triangle 4} = B^{\boxtimes} \).
structure of naturally defined fundamental subspaces (the subspaces of fixed grades, the even subalgebra, and the odd subspace) and the corresponding operations of conjugation (the grade involution, the reversion, and the others) favourably distinguishes Clifford algebras from the corresponding matrix algebras, when we use them for different applications – in physics, engineering, robotics, computer vision, control theory, stability analysis, model reduction, image and signal processing.

The explicit formulas for solutions to the Sylvester and Lyapunov equations may be useful in applications, in particular, in solving algebraic non-commutative linear equations in quantum physics.

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References

[1] K. Abdulkahev, D. Shirokov, On explicit formulas for characteristic polynomial coefficients in geometric algebras, In: Advances in Computer Graphics. CGI 2021. Lecture Notes in Computer Science. Springer, 2021, 12 pp. (to appear)
[2] Acus, A., Dargys, A.: The Inverse of a Multivector: Beyond the Threshold $p + q = 5$. Adv. Appl. Clifford Algebras 28, 65 (2018)
[3] E. Bayro-Corrochano, Geometric Algebra Applications Vol. I, Springer, 2019
[4] P. Dadbeh, Inverse and determinant in 0 to 5 dimensional Clifford algebra, arXiv:1104.0067, (2011)
[5] A. Dargys, A. Acus, A note on solution of $ax + xb = c$ by Clifford algebras, arXiv:1902.09194, (2019)
[6] Doran, C., Lasenby, A.: Geometric Algebra for Physicists. Cambridge Univ. Press, Cambridge (2003)
[7] L. Dorst, D. Fontijne, S. Mann, Geom. Alg. for Comp. Sc., Morgan Kaufmann Ser. in Comp. Graph., San Francisco, 2007
[8] Helmstetter, J.: Characteristic polynomials in Clifford algebras and in more general algebras Adv. Appl. Clifford Algebras 29, 30 (2019)
[9] Hestenes, D.: Space-Time Algebra. Gordon and Breach, New York (1966)
[10] Hestenes, D., Sobczyk, G.: Clifford Algebra to Geometric Calculus. Reidel Publishing Company, Dordrecht Holland (1984)
[11] Hildenbrand, D.: Foundations of Geometric Algebra Computing. Springer, New York (2013)
[12] E. Hitzer, Geometric operations implemented by conformal geometric algebra neural nodes, Proc. SICE Symposium on Systems and Information (2008), 357–362
[13] Hitzer, E., Sangwine, S.: Multivector and multivector matrix inverses in real Clifford algebras. Appl. Math. Comput. 311, 375–389 (2017)
[14] Hitzer, E., Sangwine, S.J.: Construction of Multivector Inverse for Clifford Algebras Over 2m+1-Dimensional Vector Spaces from Multivector Inverse for Clifford Algebras Over 2m-Dimensional Vector Spaces. Adv. Appl. Clifford Algebras 29, 29 (2019)
[15] Janovska, D., Opfer, G.: Linear equations in quaternionic variables. Mitt. Math. Ges. Hamburg 27, 223–234 (2008)
[16] Lawson, H.B., Michelsohn, M.-L.: Spin geometry. Princeton Univ. Press, Princeton (1989)
[17] H. Li, Invariant Algebras and Geometric Reasoning, World Scientific Publ., 2008
[18] Lou内科to, P.: Clifford Algebras and Spinors. Cambridge Univ. Press, Cambridge (1997)
[19] Porteous, I.R.: Clifford Algebras and the Classical Groups. Cambridge University Press, Cambridge (1995)
[20] C. Schwartz, Linear equations for noncommutative algebras. Available: http://socrates.berkeley.edu/ schwartz/Linear.pdf, 2013
[21] Shao, C., Li, H., Huang, L.: Basis-free Solution to General Linear Quaternionic Equation. Linear Multilinear Algebra 68(3), 435–457 (2020)
[22] D. S. Shirokov, Clifford algebras and their applications to Lie groups and spinors, Proceedings of the 19 International Conference on Geometry, Integrability and Quantization, Avangard Prima, Sofia. 2018; 11-53. arXiv:1709.06608
[23] D.S. Shirokov, Concepts of trace, determinant and inverse of Clifford algebra elements, Progress in analysis. Proceedings of the 8th congress of ISAAC, Volume 1, Peoples’ Friendship University of Russia (ISBN 978-5-209-04582-3/bbk), 2012, 187-194; arXiv: 1108.5447 (2011)
[24] D. S. Shirokov, On basis-free solution to Sylvester equation in geometric algebra, In: Magenenat-Thalmann N. et al. (eds) Advances in Computer Graphics. CGI 2020. Lecture Notes in Computer Science, vol 12221. Springer, Cham, 541–548
[25] D. S. Shirokov, On computing the determinant, other characteristic polynomial coefficients, and inverse in Clifford algebras of arbitrary dimension, Computational and Applied Mathematics, 40, 173, 29 pp. (2021). arXiv:2005.04015
[26] J. J. Sylvester, Sur l’equations en matrices px = xq, C.R. Acad. Sci. Paris. 99(2): 67-71, 115-116, 1884
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