TORIC BUNDLES, VALUATIONS, AND TROPICAL GEOMETRY OVER SEMIFIELD OF PIECEWISE LINEAR FUNCTIONS

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Abstract. We initiate the algebro-geometric study of tropical geometry over the idempotent semifield of piecewise linear functions. One of our main results shows that points on the tropical variety of a linear ideal over this semifield correspond to toric vector bundles. We introduce the notion of a valuation with values in the semifield of piecewise linear functions and we describe Khovanskii bases in this context. Far extending the Klyachko classification of toric vector bundles, we show that torus equivariant families over toric varieties are classified by such valuations. Finally, we see that the Gross-Hacking-Keel-Kontsevich toric degenerations of cluster varieties fit into our picture as a family over the toric scheme of the Fock-Goncharov fan.

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1. Introduction

The purpose of this paper is manifold. We initiate the study of tropical geometry over the semifield of piecewise linear functions. We also introduce and study the notions of prevaluation and valuation with values in this semifield. Finally we see that these are the natural objects to classify toric vector bundles and more generally torus equivariant families over toric varieties.

Tropical geometry can be described as algebraic geometry over the tropical semifield \((\overline{\mathbb{Q}}, \min, +)\) where \(\overline{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}\). This point of view is enhanced by the theory of tropical schemes \([GG16, MR18]\), which allows for the possibility of changing base to other semifields. An example of this type of construction can be seen in the work of Foster and Ranganathan \([FR16]\) on “higher rank tropical varieties”, which can be viewed as tropical varieties over more general totally ordered groups such as \(\mathbb{Q}^n\) with lexicographic order. In this paper we study valuations and tropical varieties over the idempotent semifield of \(\mathbb{Z}\)-valued piecewise
linear functions on a lattice. Our main results (Theorem 1, Theorem 4, and Theorem 5) show that points on these varieties correspond to torus equivariant bundles on toric varieties.

Let $k$ be an algebraically closed field of characteristic 0. In what follows $N \cong \mathbb{Z}^r$ is a finite rank lattice with dual lattice $M = \text{Hom}(N, \mathbb{Z})$, $N_\mathbb{Q} = N \otimes \mathbb{Q}$, $M_\mathbb{Q} = M \otimes \mathbb{Q}$ corresponding vector spaces and $\Sigma$ a polyhedral fan in $N_\mathbb{Q}$. Also $T_N$ denotes the torus over $k$ with $N$ as the lattice of one-parameter subgroups and $Y(\Sigma)$ is the $T_N$-toric variety over $k$ corresponding to $\Sigma$.

A toric vector bundle $\pi : E \to Y(\Sigma)$ is a vector bundle equipped with an action of the torus $T_N$ which lifts the action of $T_N$ on $Y(\Sigma)$. Famously, Klyachko gave a classification of toric vector bundles $E$ by compatible subspace filtrations in the fiber $E$ of $E$ over the identity element of $T_N \subset Y(\Sigma)$ (see [Kly89]).

By a semifield (respectively semialgebra) we mean a set $O$ equipped with two operations $\oplus$ and $\otimes$ that satisfy the usual field (respectively algebra) axioms except that we do not necessarily have additive inverse. It is called idempotent if moreover $a \oplus a = a$, $\forall a \in O$ (see Section 2). The main example of a semifield for us is the semifield $O_N$ of piecewise linear $\mathbb{Z}$-valued functions on a lattice $N$. We also add $\infty$ to $O_N$. The set $O_N$ is a semifield with respect to min and $+$. Let $I$ be an ideal in a polynomial ring $k[x]$ where $x = (x_1, \ldots, x_n)$. For an idempotent semialgebra $O$, one can define the tropical variety $\text{Trop}_O(I) \subset O^n$ (see Section 2).

**Tropical points.** Our first result is a tropical classification of toric vector bundles. Let $E$ be a finite dimensional $k$-vector space. Let $E \hookrightarrow \mathbb{A}^n$ be the linear embedding corresponding to a choice of a spanning set $B \subset E^*$ and let $I \subset k[x]$ be the associated linear ideal. We prove the following (see Proposition 6.10 and Corollaries 6.8 and 6.9).

**Theorem 1** (Toric vector bundles as tropical points). Given a point $\bar{\phi} \in \text{Trop}_{O_N}(I)$ we can construct a complete polyhedral fan $\Sigma \subset N_\mathbb{Q}$ and a toric vector bundle $E(\bar{\phi})$ on $Y(\Sigma)$ with general fiber $E$. Conversely, for every toric vector bundle $E$ with general fiber $E$ over a complete toric variety $Y(\Sigma)$, we can find $B \subset E^*$ such that $E = E(\bar{\phi})$ for a unique point $\bar{\phi} \in \text{Trop}_{O_N}(I)$.

Theorem 1 is the generalization of the well-known fact that toric line bundles are classified by piecewise linear functions (see [CLS11 Theorem 4.2.12]).

**Remark.** In [SS09 p. 165], Speyer and Sturmfels suggest the research problem of developing tropical geometry over the semialgebras of polyhedra. We consider Theorem 1 as an important step in this direction (see paragraph after Theorem 2).

**Remark.** Theorem 1 suggests an algorithmic way to construct toric vector bundles, by finding solutions of a system of tropical equations corresponding to a linear ideal.

**Prevaluations with values in $O_N$.** In order to explain the construction underlying Theorem 1 we introduce the critical notions of prevaluation and valuation with values in $O_N$. Let $E$ be a $k$-vector space. A prevaluation $\psi : E \to O_N$ is a function which satisfies the following properties for any $e, f \in E$ and $C \in k \setminus \{0\}$.

1. $\psi(e + f) \geq \min(\psi(e), \psi(f))$,
2. $\psi(\lambda e) = \psi(e)$,
3. $\psi(0) = \infty$.

Here $\geq$ means that the inequality holds pointwise, i.e. $\phi(x) \geq \psi(x), \forall x \in N$. We say that $\psi$ is a finite prevaluation if: (i) the image of $\psi$ on any finite dimensional subspace is a finite set, (ii) $\psi$ attains $\infty$ only at 0 (see Section 2).
The next result gives a classification of toric vector bundles in terms of prevaluations with values in $O_N$ (Corollary 5.4). Let us say that toric vector bundles $E$, $E'$ over complete $T_N$-toric varieties $Y(\Sigma)$, $Y(\Sigma')$ are equivalent if there is a complete toric variety $Y(\Sigma'')$ and $T_N$-equivariant morphisms $F : Y(\Sigma'') \to Y(\Sigma)$, $F' : Y(\Sigma'') \to Y(\Sigma')$ such that $F^*(E) = F'^*(E')$.

**Theorem 2** (Toric vector bundles as prevaluations). The equivalence classes of $T_N$-toric vector bundles $E$ are in one-to-one correspondence with the set of finite prevaluations $v$ on finite dimensional vector spaces $E$ with values in $O_N$.

To each prevaluation $v : E \to O_N$ there corresponds a subspace arrangement $A_v = \{E_{\phi} \geq \phi \mid \phi \in O_N\}$ where

$$E_{\phi} \geq \phi = \{e \in E \mid v(e) \geq \phi\}.$$ 

One shows that the arrangement $A_v$ is closed under taking intersection (Proposition 2.2)

We point out that to each subspace arrangement one associates a (representable) matroid (see Section 5.4 as well as [Zie, Section I.4]).

There is a well-known correspondence between piecewise linear functions and polytopes. To a polytope $P \subset M_Q$ one associates its support function $\phi_P : N_Q \to \mathbb{R}$, $\phi_P(x) = \min\{\langle u, x \rangle \mid u \in P\}$, and to a piecewise linear function $\phi$ one associates the polytope

$$P_\phi = \{u \in M_Q \mid \langle u, x \rangle \geq \phi(x), \forall x \in N\}.$$ 

This gives a one-to-one correspondence between polytopes in $M_Q$ and concave piecewise linear functions. The set $P_{M_0}$ of polytopes in $M_Q$ is a semialgebra with convex hull of union as $\oplus$ and Minkowski addition as $\otimes$ (this semialgebra is mentioned in [SS09, p. 165]). The set $P_M$ of lattice polytopes in $M_Q$ forms a subsemialgebra of $P_{M_0}$. It is known that the map which sends a polytope $P$ to its support function $\phi_P$, gives a semialgebra isomorphism between $P_M$ and the subsemialgebra of $O_N$ consisting of concave piecewise linear functions on $N$. We thus regard $P_M$ as a subsemialgebra of $O_N$. We pose the following problem.

**Problem.** What toric vector bundles correspond to prevaluations with values in $P_M$ (where we regard $P_M$ as a subsemialgebra of $O_N$)?

In [DRJS18] the authors introduce a collection of polytopes associated to a toric vector bundle $E$ called the parliament of polytopes of $E$. The polytopes in the parliament are indexed by the ground set of a certain representable matroid $M(E)$ (living in $E$). They give a criterion for global generation of $E$ in terms of its parliament. We can immediately recover the matroid and parliament of polytopes of $E$ from its prevaluation (Proposition 6.13). Recall that to a piecewise linear function $\phi$ we associate a polytope $P_\phi$ (defined in [11]).

**Corollary 3.** (Parliament of polytopes from the prevaluation) Let $E$ be the toric vector bundle corresponding to a prevaluation $v : E \to O_N$.

1. The matroid $M(E)$ is the matroid associated to the subspace arrangement $A_v$ (see the paragraph after Theorem 2).

2. The collection of polytopes $\{-P_{-\phi} \mid \phi \in v(E)\}$ is the parliament of polytopes for $E$, where $-P = \{-u \mid u \in P\}$ (see [1]).

In Proposition 6.14 using the criteria in [DRJS18], we observe that if all the piecewise linear functions in the image of $v$ are convex then $E$ is globally generated.

We now would like to give a more precise version of Theorem 2 for a fixed fan $\Sigma$. For this we need some definitions. For a fan $\Sigma$ (not necessarily complete) let $O_\Sigma$ denote the set of
piecewise linear functions on its support $|\Sigma|$, the union of all the cones in $\Sigma$. In particular, for a cone $\sigma$, $O_\sigma$ denotes the set of piecewise linear functions on $\sigma$.

We say that a vector space basis $B \subset E$ is adapted to a prevaluation $v : E \to O_\Sigma$ if for any $e = \sum_{b \in B} c_b b$ we have $v(e) = \min\{v(b) \mid c_b \neq 0\}$. An adapted basis $B_\sigma$ is a linear adapted basis for a prevaluation $v$ with values in $O_\sigma$ if $v(b)$ is linear on $\sigma$ for all $b \in B_\sigma$ (see Section 3).

**Theorem 4.** Let $\Sigma$ be a fan in $N_\mathbb{Q}$. The following are in one-to-one correspondence:

1. Toric vector bundles on $Y(\Sigma)$ with general fiber $E$.
2. Prevalued vector spaces $(E, v)$ such that the restriction $v|_\sigma$ has a linear adapted basis for each $\sigma \in \Sigma$.

Furthermore, the correspondence between (1) and (2) extends to an equivalence of categories between vector bundles with their $T_N$-equivariant sheaf homomorphisms and the category of prevaled vector spaces $(E, v)$ as described in (2).

**Valuations with values in $O_N$.** Let $A$ be a $k$-domain. A valuation $v : A \to O_N$ is a prevaluation that satisfies the following multiplicativity property: for any $e, f \in A$ we have $v(e f) = v(e) + v(f)$.

When $A = \bigoplus_i A_i$ is a graded algebra, we say that $v$ is homogeneous with respect to the grading of $A$, if $v(f) = \min\{v(f_i)\}$, where min runs over all the homogeneous components $f_i$ of $f$.

We are interested in valuations which have a certain “finiteness” property. To formulate this we need an extension of the notion of Khovanskii basis (introduced in [KMa, Section 2.2]). Fix a fan $\Sigma$ and let $v : A \to O_{\Sigma}$ be a valuation. For each $\rho \in |\Sigma|$, one constructs a quasivaluation $v_\rho : A \to \mathbb{Z} = \mathbb{Z} \cup \{\infty\}$ (in the classical sense) with the corresponding graded algebra

$$\text{gr}_\rho A = \bigoplus_a A_{v_\rho \geq a}/A_{v_\rho > a}.$$ 

We say that $B \subset A$ is a Khovanskii basis for the valuation $v$ if for each $\rho \in |\Sigma|$, the image of $B$ in $\text{gr}_\rho A$ is an algebra generating set (see Section 3).

It is easy to see that a prevaluation on $E$ naturally extends to define a homogeneous valuation on the polynomial algebra $\text{Sym}(E^*)$ and hence Theorem 4 can also be stated in terms of homogeneous valuations on $\text{Sym}(E^*)$. Our next result far generalizes this classification to torus equivariant families over a toric variety base and with an arbitrary affine scheme as general fiber (see Section 5).

**Theorem 5** (Classification of toric families). Let $A = \bigoplus_{n \geq 0} A_n$ be a positively graded $k$-domain. The information of a flat $T_N$-equivariant sheaf $\mathcal{A}$ of positively graded algebras of finite type over $Y(\Sigma)$ such that $\text{Spec}(\mathcal{A})$ has reduced and irreducible fibers with general fiber isomorphic to $\text{Spec}(A)$, is equivalent to the information of a homogeneous valuation $v : A \to O_{\Sigma}$ with a finite Khovanskii basis.

The basic construction of deformation to normal cone in algebraic geometry, which is a family over the affine/projective line, is an instance of the baby case of the above Theorem 5. Note that a toric family $\text{Spec}(A)$ as above is trivial over each $T_N$-orbit. In particular, the fibers over the open orbit are $\text{Spec}(A)$. The fibers over other orbits are “degenerations” of $\text{Spec}(A)$.

**Remark.** In fact, Theorem 5 works for more general cases of graded algebras $A$, e.g. when $A$ is the coordinate ring of a reductive algebraic group.
Remark (Newton-Okounkov bodies and Khovanskii bases). Valuations into totally ordered groups, such as \( \mathbb{Z}^n \) ordered lexicographically, are used in the theory of Newton-Okounkov bodies (Krivine and Manin) to produce convex bodies which capture the asymptotic behavior of positively graded algebras \( A = \bigoplus_{i_1 \geq 0} A_{i_1} \). In \([KMa]\) the fundamental notion of Khovanskii basis of an algebra is introduced which allows computational aspects of the polynomial ring \( k[x_1, \ldots, x_n] \) to be extended to more general domains over \( k \). A generating set \( \mathcal{B} \subset A \) is said to be a Khovanskii basis for a valuation \( \nu \) if the equivalence classes \( \mathcal{B} \) in the associated graded algebra \( \text{gr}_\nu(A) = \bigoplus_{a>0} A_{\nu>a}/A_{\nu>a} \) form a \( k \)-algebra generating set (see Section 2). The present paper extends some of the theory of Khovanskii bases developed in \([KMa]\) to valuations with values in piecewise linear functions.

Remark. (1) (Piecewise linear maps to Berkovich space) Let \( X = \text{Spec}(A) \), and let \( X^an \) denote the Berkovich analytification of \( X \) over \( k \) (see [Pay09]). As a topological space, \( X^an \) is the set of all real valuations \( v \) on \( A \) equipped with the coarsest topology which makes each evaluation function \( ev_f : X^an \to \mathbb{R}, ev_f(v) = v(f) \) for \( f \in A \) continuous. As a corollary of Theorem 4.17 we obtain a piecewise-linear map \( \Phi : \Sigma \cap \mathbb{N} \to X^an \), in the sense that \( v(f) = ev_f \circ \Phi : \Sigma \cap \mathbb{N} \to \mathbb{Z} \) is a piecewise-linear function for all \( f \in A \).

(2) (Piecewise linear maps to buildings and toric principal bundles) Let \( G \) be a linear algebraic group and let \( \mathcal{B}(G) \) denote the (underlying space of) Tits building \( \mathcal{B}(G) \) of \( G \). An element of \( \mathcal{B}(G) \) can be regarded as a one-parameter subgroup of \( G \) and one can extend it to define a valuation on \( G \) (see [Ber90, Section 5.4]). When \( X = \text{GL}(E) \) it turns out that the image of \( \Phi \) lies in \( \mathcal{B}(\text{GL}(E)) \) and we recover one of the main results in \([KMB]\), which gives an equivalence of categories between toric vector bundles and piecewise-linear maps \( \Phi : \Sigma \to \mathcal{B}(\text{GL}(E)) \). For a general linear algebraic group \( G \), the above gives a classification of toric principal \( G \)-bundles in terms of certain piecewise linear maps to \( G^an \). We believe that this is related to the classification in \([BDP6]\).

The category \( \text{Vect}_\Sigma \). Let \( \hat{\mathcal{O}}_\Sigma \) be the semifield of piecewise homogeneous functions on the support \( \Sigma \) into \( \mathbb{Z} = \mathbb{Z} \cup \{ \infty \} \) (i.e., \( \hat{\mathcal{O}}_\sigma \) and \( \hat{\mathcal{O}}_N \) are defined accordingly, see Section 3). Let \( \text{Vect}_\Sigma \) be the category of \( k \)-vector spaces equipped with a prevaluation into the semifield \( \hat{\mathcal{O}}_\Sigma \). The central technical idea of the paper is to relate the category of \( T_N \)-equivariant quasicoherent sheaves over \( Y(\Sigma) \) to the category \( \text{Vect}_\Sigma \) (Theorem 1.17). To any coherent sheaf \( \mathcal{F} \) over \( Y(\Sigma) \) we can associate a pair \((E, \nu)\), where \( E \) is finite dimensional and \( \nu \) takes values in \( \mathcal{O}_\Sigma \). However, passing to quasicoherent sheaves requires us to expand \( \mathcal{O}_\Sigma \) to \( \hat{\mathcal{O}}_\Sigma \). Working locally over the toric open affine subvariety \( Y(\sigma) \subset Y(\Sigma) \) for \( \sigma \in \Sigma \), we show the following. Let \( \text{Mod}^M_{S_\sigma} \) denote the category of \( M \)-homogeneous \( S_\sigma \)-modules, where \( S_\sigma \) is the coordinate ring of \( Y(\sigma) \).

**Theorem 6.** There are functors \( \mathcal{L} : \text{Mod}^M_{S_\sigma} \to \text{Vect}_\sigma \) and \( \mathcal{R} : \text{Vect}_\sigma \to \text{Mod}^M_{S_\sigma} \) which give an adjunction of categories:

\[
\mathsf{Hom}_{\text{Mod}^M_{S_\sigma}}(R, \mathcal{R}(E, \nu)) \cong \mathsf{Hom}_{\text{Vect}_\sigma}(\mathcal{L}(R), (E, \nu)).
\]

Notably, \( \mathcal{R} \) and \( \mathcal{L} \) do not give an equivalence, but they do restrict to an equivalence between the image subcategories \( \mathcal{L}(\text{Mod}^M_{S_\sigma}) \subset \text{Vect}_\sigma \) and \( \mathcal{R}(\text{Vect}_\sigma) \subset \text{Mod}^M_{S_\sigma} \). We call the objects of these subcategories eversive. In Section 5 we use these concepts to characterize the images of flat, projective, and free modules under the functor \( \mathcal{L} \). In particular, Proposition 5.2 shows that projective (and therefore free) modules are always eversive, this is the main ingredient in the proofs of Theorems 4.17 and 4.18.
Cluster algebras. Finally, to illustrate applications of our theory we present an extended example from the theory of cluster algebras. We show that the toric degenerations of Gross, Hacking, Keel and Kontsevich [GHKK18] can naturally be explained in our setting of valuations into the semifield of piecewise linear functions. Following the terminology of [GHKK18], we show the following.

**Theorem 7.** Let $\mathcal{A}$ be a cluster variety with canonical algebra $\text{can}(\mathcal{A})$ and canonical basis $\Theta \subset \text{can}(\mathcal{A})$, and let $\Delta_+ \subset N_\mathbb{Q}$ be the Fock-Goncharov fan with top dimensional faces $\sigma_s \in \Delta_+$, then:

1. There is a valuation $v : \text{can}(\mathcal{A}) \to \mathcal{O}_{\Delta_+}$ which is adapted to $\Theta$.
2. For $\theta \in \Theta$, $v|_{\sigma_s}(\theta)$ is the $g$-vector of $\theta$ corresponding to $s$.
3. There is a piecewise linear map $\Phi : \Delta_+ \cap N \to \mathcal{A}_{\text{an}}^n$, the Berkovich analytification of $\mathcal{A}$.
4. For each $\theta$, there is a toric line bundle $\mathcal{O}(\theta)$ on $Y(\Delta_+)$. The direct sum $\bigoplus_{\theta \in \Theta} \mathcal{O}(\theta)$ gives a degeneration of $\text{can}(\mathcal{A})$ over $Y(\Delta_+)$, with a toric fiber over each fixed point.

We expect that the sheaf of algebras $\bigoplus_{\theta \in \Theta} \mathcal{O}(\theta)$ can be realized as a degeneration of a sheaf of algebras over the $X$-type cluster variety by following the toric degeneration constructed in [BPMMC].

The first classification result for toric vector bundles goes back to Kaneyama [Kan75]. Building on Klyachko’s work, Perling [Per04] gives a classification of $\mathcal{T}_N$-equivariant sheaves over $Y(\Sigma)$ using certain directed systems of vector spaces (see Section 4.6 and also [KS98]). We also point to the more recent interesting works of Biswas, Dey, and Poddar on classifying toric principal bundles [BDPa] (in the spirit of Kaneyama’s classification) and [BDPb] (a Tanakian classification in terms of certain filtered algebras). Recently in [KMi], extending Klyachko’s classification, the authors have given a new classification of toric principal $G$-bundles using piecewise linear maps to the (underlying space of) spherical Tits building of a linear algebraic group $G$. In particular, this gives a classification (equivalent to Klyachko’s classification) of toric vector bundles in terms of piecewise linear maps to the spherical Tits buildings of general linear groups. The present paper can be considered as a sister paper to [KMi]. We would like to point out that our proofs here are new and do not rely on Klyachko’s classification.

An example: tangent bundle of $\mathbb{P}^2$. The tangent bundle $T\mathbb{P}^2$ is naturally a toric vector bundle over $\mathbb{P}^2$. Let $N = \mathbb{Z}^2$ and let $\rho_1 = (-1,0)$ and $\rho_2 = (0,-1)$ and $\rho_0 = (1,1)$. The fan $\Sigma$ of $\mathbb{P}^2$ has rays generated by $\rho_0$, $\rho_1$ and $\rho_2$ and has maximal faces $\sigma_0 = \mathbb{Q}_{\geq 0}\{\rho_0,\rho_2\}$, $\sigma_1 = \mathbb{Q}_{\geq 0}\{\rho_0,\rho_1\}$, and $\sigma_2 = \mathbb{Q}_{\geq 0}\{\rho_0,\rho_1\}$ (note that we use opposite convention in the definition of dual cone and thus our fan is inverted compared to the usual convention in the literature for defining fan of a toric variety, see Remark 4.1). The tangent bundle $T\mathbb{P}^2$ has rank 2, so it corresponds to a valuation $v : E \to \mathcal{O}_\Sigma$, where $\text{dim}(E) = 2$. We identify $E$ with $\mathbb{K}^2$, and let $e_1, e_2$ be the standard basis vectors with $e_0 = -e_1 - e_2$. The spanning set $B = \{e_0, e_1, e_2\} \subset E$ is a representation of the matroid $M(\mathbb{P}^2)$, and the set $B_i = B \setminus \{e_i\}$ is a linear adapted basis of $v|_{\sigma_i}$, $i = 0, 1, 2$. In order to describe $v$ it suffices to give the values over each $\sigma_i$. Letting $m_1 = (1,0)^*$, $m_2 = (0,1)^* \in M = (\mathbb{Z}^2)^*$ be the dual standard basis vectors, we have:

| $v$  | $\sigma_0$ | $\sigma_1$ | $\sigma_2$ |
|------|------------|------------|------------|
| $v(e_1)$ | $m_1$ | $(m_2 - m_1) \oplus -m_1$ | $m_1 - m_2$ |
| $v(e_2)$ | $m_2$ | $m_2 - m_1$ | $(m_1 - m_2) \oplus -m_2$ |
| $v(e_0)$ | $m_1 + m_2$ | $-m_1$ | $-m_2$ |
The linear ideal $I$ of relations in $B$ is generated by the relation $e_0 + e_1 + e_2 = 0$. The corresponding tropical ideal is generated by $x_0 \oplus x_1 \oplus x_2$. We note that the piecewise linear functions $v(e_i)$ satisfy the relation $v(e_0) \oplus v(e_1) \oplus v(e_2) = v(e_0) \oplus v(e_1)$. This exactly means that $(v(e_0), v(e_1), v(e_2))$ lies on the tropical variety $\text{Trop}_{O_N}(I)$ which verifies Theorem 1.

The polytopes $P_{v(e_i)}$ are shown in Figure 1 (cf. [DRJS18, Example 3.8]). This verifies Corollary 3.

**Notation.** Throughout the paper we will use the following notation:

- $k$ is an algebraically closed field of characteristic 0 which we take to be our base field.
- $N$ is a finite rank lattice with dual lattice $M = \text{Hom}(N, \mathbb{Z})$ and $N_\mathbb{Q} = N \otimes \mathbb{Q}$, $M_\mathbb{Q} = M \otimes \mathbb{Q}$ the corresponding vector spaces.
- $T_N$ denotes the torus over $k$ with $N$ as the lattice of one-parameter subgroups.
- $\Sigma$ is a fan in $N_\mathbb{Q}$ with corresponding toric variety $Y(\Sigma)$. For each $i$, we denote the set of $i$-dimensional cones in $\Sigma$ by $\Sigma(i)$. In particular, $\Sigma(1)$ is the set of rays in $\Sigma$. Also $|\Sigma|$ denotes the support of $\Sigma$, i.e. the union of cones in it.
- $\rho$ denotes a primitive vector, i.e. the shortest lattice vector, along a ray $\rho$ in $\Sigma(1)$.
- $O_N$ is the semifield of piecewise linear functions on $N$, $O_\Sigma$ is the semifield of piecewise linear functions on $|\Sigma| \cap N$, and $\hat{O}_\Sigma$ is the semialgebra of linearly homogeneous functions on $|\Sigma| \cap N$ with values in $\mathbb{Z} = \mathbb{Z} \cup \{\infty\}$.
- $P_M$ is the semialgebra of lattice polytopes in $M_\mathbb{Q}$ with convex hull of union as $\oplus$ and Minkowski addition as $\otimes$.
- $\text{Vect}_\Sigma$ is the category of $k$-vector spaces equipped with a prevaluation with values in $O_\Sigma$.
- $E$ is a finite dimensional $k$-vector space, and $A$ is a finitely generated $k$-domain.
- $v : E \to O_N$ and $v : A \to O_N$ are a prevaluation and a valuation with values in $O_N$ respectively. From the domain of $v$, that is, a vector space or an algebra, it should be understood that $v$ is a prevaluation or a valuation.
- $A_v$ is the subspace arrangement associated to a prevaluation $v$ on a vector space $E$. 7
2. Background on valuations

In this section we collect the basic definitions and background for prevaluations on vector spaces, and valuations and quasivaluations on algebras. We start by introducing the general notion of a valuation with values in a semilattice. The term valuation appears in [KK12, Section 2.1] where the notion of a valuation with values in a totally ordered group is defined.

Let \( (\Gamma, \preceq, \wedge) \) be a meet-semilattice. That is, \( (\Gamma, \preceq) \) is a partially ordered set (poset) together with a binary operation \( \wedge \) (meet) of greatest lower bound. That is, for any \( \phi, \psi \in \Gamma \), whenever we have \( \phi \preceq \mu, \psi \preceq \mu \), for some \( \mu \in \Gamma \), then \( \phi \wedge \psi \preceq \mu \). We assume \( \Gamma \) has a maximum element which we denote by \( \infty \).

**Definition 2.1.** Let \( v : E \to \Gamma \) be a map that satisfies the following:

1. For every \( \phi \in \Gamma \), \( F_\phi(v) := \{ e \in E \mid v(e) \preceq \phi \} \) is a vector subspace of \( E \).
2. For any \( e \in E \) and any \( 0 \neq C \in k \) we have \( v(Ce) = v(e) \).
3. \( v(0) = \infty \).

We call such a map \( v \) a prevaluation on \( E \) with values in \( \Gamma \). In addition, we say that \( v \) is a finite valuation if: (i) \( v \) attains a finite set of values on each finite dimensional subspace of \( E \), and (ii) \( v(e) = \infty \) only at \( e = 0 \).

**Proposition 2.2.** Let \( v : E \to \Gamma \) be a finite prevaluation on a finite dimensional vector space \( E \).

1. For every \( \phi \in \Gamma \), \( F_\phi(v) := \{ e \in E \mid v(e) \preceq \phi \} \) is a vector subspace of \( E \).
2. For every \( \phi \in \Gamma \), there is \( \psi \) in the image of \( v \) such that \( F_\phi(v) = F_\psi(v) \).
3. Suppose \( (\Gamma, \preceq, \wedge, \vee) \) is a meet-join lattice then \( F_\phi(v) \cap F_\psi(v) = F_{\phi \vee \psi}(v) \), \( \forall \phi, \psi \in \Gamma \).

**Proof.** (1) This is immediate from the definition of a prevaluation. (2) Let \( v(F_\phi(v)) = \{ \phi_1, \ldots, \phi_c \} \). Then \( F_\phi(v) \) is the union of the subspaces \( F_{\phi_i}(v) \). It follows that we have \( F_\phi(v) = F_{\phi_j}(v) \) for some \( j \) which proves the claim. (3) This follows immediately from the definition of join, namely, \( \mu \preceq \phi \) and \( \mu \preceq \psi \) if and only if \( \mu \preceq \phi \vee \psi \). \( \square \)

Let \( A_v = \{ F_\phi(v) \mid \phi \in \Gamma \} \). From Proposition 2.2 it follows that \( A_v \) is a subspace arrangement in \( E \). Moreover, if \( \Gamma \) is a meet-join lattice then \( A_v \) is closed under intersection.

**Remark 2.3.** Unfortunately the term “lattice” is used in the mathematics literature with two different meanings. In this paper, the only time we use the term “lattice” to mean a certain partially ordered set is in the above paragraphs. In the rest of the paper we use the term “lattice” to mean a finite rank free abelian group.

Next we define notions of quasivaluation and valuation with values in a semialgebra. Let \( (O, \oplus, \otimes) \) be an idempotent semialgebra. This means that the operations \( \oplus \) and \( \otimes \) satisfy the same axioms as addition and multiplication in a ring, with the exception that there are not necessarily additive inverses, and \( a \oplus a = a \forall a \in O \). Let \( Z = Z \cup \{ \infty \} \). All semialgebras we consider are sets of \( Z \)-valued functions on (subsets of) a lattice \( N \). Addition \( \oplus \) and multiplication \( \otimes \) are taken to be the pointwise min and + of functions. The additive identity, also denoted \( \infty \), is the function which assigns \( \infty \) to every point.

Any idempotent semialgebra \( O \) has an intrinsic partial ordering \( \succeq \), where \( a \succeq b \) if \( a \oplus b = b \). We denote the neutral element with respect to \( \oplus \) by \( \infty \).

**Definition 2.4.** For a \( k \)-algebra \( A \), a quasivaluation \( v : A \to O \) is a function which satisfies the following:

1. \( v(fg) \succeq v(f) \otimes v(g) \),
A quasivaluation \( \phi \) is said to be a \textit{valuation} if \( \phi(fg) = \phi(f) \odot \phi(g) \). From Definition 2.1 we see that a \textit{prevaluation} on a vector space \( E \) is a function which satisfies (2) – (4) above.

Let \( A \) be a \( k \)-algebra with finite generating set \( \mathcal{B} \subset A \) with \( n = \lvert \mathcal{B} \rvert \). We let \( \pi : k[x] \to A \) be the associated presentation, with ideal \( I_\mathcal{B} = \ker(\pi) \). For any monomial \( x^\alpha \in k[x] \) with \( \alpha = (\alpha_1, \ldots, \alpha_n) \) there is a function \( ev_{x^\alpha} : \mathcal{O}^\mathcal{B} \to \mathcal{O} \) defined by sending \( (\psi_1, \ldots, \psi_n) \) to \( \bigoplus_{i=1}^n \psi_i^{\alpha_i} \). Following [GG16, 5.1], the tropical variety \( \text{Trop}_{\mathcal{B}}(\mathcal{O}) \subset \mathcal{O}^\mathcal{B} \) is defined to be the set of tuples \( (\psi_1, \ldots, \psi_n) \) such that for any polynomial \( \sum_{j=1}^m C_j x^{\alpha(j)} \in I_\mathcal{B} \) we have \( \bigoplus_{j \in [m]} ev_{x^{\alpha(j)}} = \bigoplus_{j \in [m] \setminus \{i\}} ev_{x^{\alpha(j)}} \) for any \( i \in [m] \). The following is a well-known relationship between valuations and the tropical variety, see [Pay09, GG16].

**Proposition 2.5.** Let \( \mathcal{O} \to \mathcal{B} \subset A \) be a valuation and \( \mathcal{B} \subset A \) a generating set, then the tuple \( (\mathcal{O}(b_1), \ldots, \mathcal{O}(b_n)) \in \mathcal{O}^\mathcal{B} \) is a point in the tropical variety \( \text{Trop}_{\mathcal{B}}(\mathcal{O}) \).

When \( \mathcal{O} = \mathbb{Z} \) we can associate a natural filtration \( \{ G_r(\mathcal{O}) \mid r \in \mathbb{Z} \} \) to a prevaluation \( \mathcal{O} : E \to \mathbb{Z} \), where \( G_r(\mathcal{O}) = \{ f \mid \mathcal{O}(f) \geq r \} \). Similarly, we let \( G_{>r}(\mathcal{O}) = \{ f \mid \mathcal{O}(f) > r \} \). The \textit{associated graded} vector space \( \text{gr}_\mathcal{O}(E) \) is defined to be the direct sum:

\[
\text{gr}_\mathcal{O}(E) = \bigoplus_{r \in \mathbb{Z}} G_r(\mathcal{O}) / G_{\geq r}(\mathcal{O}).
\]

If \( \mathcal{O} : \mathcal{B} \subset E \) is a quasivaluation, it is straightforward to show that \( \text{gr}_\mathcal{O}(A) \) is \( k \)-algebra. The function \( \mathcal{O} \) can be recovered as \( \mathcal{O}(f) = \max\{ r \mid f \in G_r(\mathcal{O}) \} \), where this is taken to be \( \infty \) if the maximum is never attained. For the following see [KMa, Section 2.5].

**Definition 2.6.** A vector space basis \( \mathcal{B} \subset E \) is said to be an adapted basis for a prevaluation \( \mathcal{O} : E \to \mathbb{Q} \) if \( \mathcal{B} \cap G_r(\mathcal{O}) \) is a basis of \( G_r(\mathcal{O}) \) for each \( r \in \mathbb{Z} \).

If \( b_i, i = 1, \ldots, k \) are from an adapted basis then \( \mathcal{O}(\sum_{i=1}^k C_i b_i) = \oplus_{i=1}^k \mathcal{O}(b_i) \). This identity simplifies computations for prevaluations with adapted bases.

We finish this section by recalling the notion of Khovanskii basis for a quasivaluation on an algebra. Khovanskii bases for general quasivaluations are the subject of [KMa].

**Definition 2.7.** Let \( A \) be a \( k \)-algebra, and \( \mathcal{O} : \mathcal{B} \subset \mathbb{Z} \) be a quasivaluation. A subset \( \mathcal{B} \subset A \) is said to be a \textit{Khovanskii basis} of \( \mathcal{O} \) if the equivalence classes \( \mathcal{B} \subset \text{gr}_\mathcal{O}(A) \) generate \( \text{gr}_\mathcal{O}(A) \) as a \( k \)-algebra.

### 3. The category \( \text{Vect}_\Sigma \)

Recall that \( \mathcal{O}_\Sigma \) is the of piecewise linear functions defined on the set \( \Sigma \cap N \) and \( \hat{\mathcal{O}}_\Sigma \) is the of functions \( \psi : \Sigma \cap N \to \mathbb{Z} \) such that \( \psi(\ell \rho) = \ell \psi(\rho) \) for all \( \ell \in \mathbb{Z}_{\geq 0} \) and \( \rho \in \Sigma \cap N \).

**Definition 3.1.** Let \( \text{Vect}_\Sigma \) be the category of pairs \( (E, \mathcal{O}) \), where \( \mathcal{O} : E \to \hat{\mathcal{O}}_\Sigma \) is a prevaluation over \( k \). A morphism \( \phi : (E, \mathcal{O}) \to (D, \mathcal{W}) \) of prevalued vector spaces is a \( k \)-linear map \( \phi : E \to D \) such that \( \mathcal{W}(f) \leq \mathcal{W}(\phi(f)) \forall f \in E \).

Strictly speaking, \( \text{Vect}_\Sigma \) only depends on the support of \( \Sigma \). The fan structure of \( \Sigma \) will be used in Section 4 when it becomes necessary to single out special elements of \( \mathcal{O}_\Sigma \subset \hat{\mathcal{O}}_\Sigma \).
3.1. **Vect$_\Sigma$ is pre-abelian.** The notions of epimorphism, monomorphism, kernel, and cokernel all make sense in Vect$_\Sigma$, but it is not always the case that a monomorphism is the kernel of a map. As a consequence, Vect$_\Sigma$ is not an abelian category. The rest of this section is devoted to showing that Vect$_\Sigma$ is pre-abelian.

**Proposition 3.2.** Let $\Sigma \subset N_Q$ as above, then Vect$_\Sigma$ is a pre-abelian category, in particular:

1. Vect$_\Sigma$ is enriched over the category of $k$-vector spaces.
2. Vect$_\Sigma$ has a biproduct $\oplus$.
3. Every morphism in Vect$_\Sigma$ has a kernel and cokernel.

It is straightforward to show that Hom$_{\text{Vect}_\Sigma}$($(E, v), (D, w)$) is a subspace of Hom$_k(E, D)$, and that composition of morphisms in Vect$_\Sigma$ is just composition of $k$-linear maps; this shows (1). For (2) we make the following definition.

**Definition 3.3.** For $(E, v), (D, w) \in \text{Vect}_\Sigma$ we let $(E, v) \oplus (D, w) = (E \oplus D, v \oplus w)$, where $(v \oplus w)(f + g) = v(f) \oplus w(g) \in \hat{O}_\Sigma$.

The space $(0, -\infty)$ is the additive identity for $\oplus$. The proof that $\oplus$ is both a product and coproduct in Vect$_\Sigma$ (see JLa19a) is straightforward, and we leave it to the reader. The direct sum operation allows us to extend the notion of adapted basis (Definition 2.6) to the context of prevaluations into $\hat{O}_\Sigma$. For a vector space $E$ and $\psi \in \hat{O}_\Sigma$, we let $(E, \psi) \in \text{Vect}_\Sigma$ denote the vector space with prevaluation which assigns every non-zero element of $E$ the function $\psi$.

**Definition 3.4.** We say $(E, v)$ has an adapted basis $\mathbb{B} \subset E$ if the natural maps $(k, v(b)) \to (E, v)$, $1 \to b$ define an isomorphism $(E, v) \cong \bigoplus_{b \in \mathbb{B}} (k, v(b))$. This is said to be a linear adapted basis if $v(b) \in M \subset \mathcal{O}_\Sigma \subset \hat{O}_\Sigma$ for each $b \in \mathbb{B}$.

We show (3) by describing the kernels and cokernels in Vect$_\Sigma$. Let $(E, v)$ be a prevalued vector space, and $\pi : E \to D$ be a surjection. The pushforward prevaluation $\pi_\ast(v) : D \to \hat{O}_\Sigma$ is defined by the formula $\pi_\ast(v)(g)(\rho) = \max\{v(f)(\rho) \mid \pi(f) = g\}$. As above, it is understood that $\pi_\ast(v)(g) = \infty$ if the maximum is never attained. Similarly, given $\phi : F \to E$, we have the pullback $\phi^\ast(v)$, where $\phi^\ast(v)(f) = v(\phi(f))$.

Categorically, a morphism $i : (K, u) \to (E, v)$ is the kernel of $\phi : (E, v) \to (D, w)$ if $\phi \circ i = 0$, and for any other $i' : (K', u') \to (E, v)$ with $\phi \circ i' = 0$ we have a unique $j : (K', u') \to (K, u)$ such that $i \circ j = i'$. Similarly (with arrows reversed), a cokernel $\pi : (D, w) \to (C, u)$ satisfies $\pi \circ \phi = 0$, and if $\pi' : (D, w) \to (C', u')$ also satisfies $\pi' \circ \phi = 0$ then there is a unique $p : (C, u) \to (C', u')$ such that $\pi' = p \circ \pi$.

**Lemma 3.5.** A morphism $\phi : (E, v) \to (D, w)$ is the kernel of a morphism in Vect$_\Sigma$ if and only if it is a kernel in Vect$_k$ and $v = \phi^\ast(w)$. It is a cokernel if and only if it is a cokernel in Vect$_k$ and $w = \phi_\ast(v)$.

**Proof.** Suppose that $\phi$ is the kernel of $\pi : D \to C$ in Vect$_k$ and $v = \phi^\ast(w)$. Without loss of generality, we assume that $\pi$ is surjective. We upgrade $\pi$ to a map in Vect$_\Sigma$ by letting $u = \pi_\ast(v)$. Suppose that $\phi' : (E', v') \to (D, w)$ satisfies $\pi \circ \phi' = 0$. The map $\phi$ is a kernel in Vect$_k$ so there is a linear a map $\psi : E' \to E$ such that $\phi \circ \psi = \phi'$. For any $e' \in E'$ we must have $v'(e') \leq w(\phi'(e')) = w(\phi \circ \psi(e')) = \phi^\ast(w)(\psi(e'))$, so $\phi$ is a kernel. The cokernel case is similar.

The colimit $\lim_{\to}(E_i, v_i)$ of a diagram $I$ is constructed by taking $\lim_{\to} v_i : \lim_{\to} E_i \to \hat{O}_\Sigma$ to be the pushforward of the prevaluation on $\bigoplus_{i \in I}(E_i, v_i)$ under the quotient map $\pi_I : \bigoplus_{i \in I} E_i \to \lim_{\to} E_i$. In particular, $\{\lim_{\to} v_i(f) = \max\{v_i(f_i) \mid \sum_{i=1}^l v_i(f_i) = f\}$.
A prevaluation \( \nu : E \to \hat{\Omega}_\Sigma \) is captured by a family of subspaces \( G^\rho_r(\nu) \subset E \). For \( \rho \in \Sigma \cap N \) and \( r \in \mathbb{Z} \) let:

\[
G^\rho_r(\nu) = \{ f \mid \nu(f)(\rho) \geq r \}.
\]

We have \( \nu(f)(\rho) = \max\{r \mid f \in G^\rho_r(\nu)\} \). A map \( \phi : E \to D \) lifts to a \( \text{Vect}_\Sigma \) morphism \( \phi : (E, \nu) \to (D, \mu) \) if and only if \( \phi(G^\rho_r(\nu)) \subset G^\rho_r(\mu) \forall \rho \in \Sigma \cap N, r \in \mathbb{Z} \). Similarly, an epimorphism \( \phi : (E, \nu) \to (F, \mu) \) is a cokernel if and only if \( \phi(G^\rho_r(\nu)) = G^\rho_r(\mu) \), and a monomorphism \( \phi \) is a kernel if and only if \( \phi(G^\rho_r(\nu)) = G^\rho_r(\mu) \cap \phi(E) \).

### 3.2. Tensor product

We define the tensor product \( \otimes \) in \( \text{Vect}_\Sigma \), making \( \text{Vect}_\Sigma \) into a symmetric monoidal category. This construction allows us to define algebras and Schur functors in \( \text{Vect}_\Sigma \).

**Definition 3.6.** The tensor product \((E_1, \nu_1) \otimes (E_2, \nu_2)\) is the pair \((E_1 \otimes_k E_2, \nu_1 \ast \nu_2)\), where \( \nu_1 \ast \nu_2 : E_1 \otimes_k E_2 \to \hat{\Omega}_\Sigma \) is defined by the following spaces:

\[
G^\rho_r(\nu_1 \ast \nu_2) = \sum_{s + t = r} G^\rho_s(\nu_1) \otimes_k G^\rho_t(\nu_2) \subset E_1 \otimes_k E_2,
\]

in particular \( \nu_1 \ast \nu_2(f)(\rho) = \max\{r \mid f \in G^\rho_r(\nu_1 \ast \nu_2)\} \).

A tensor \( f \in E_1 \otimes_k E_2 \) is in \( G^\rho_r(\nu_1 \ast \nu_2) \) if it can be written \( f = \sum_{i=1}^\ell x_i \otimes y_i \) with \( \nu_1(x_i)(\rho) + \nu_2(y_i)(\rho) \geq r \) for each \( 1 \leq i \leq \ell \). This means that \( \nu_1 \ast \nu_2(f)(\rho) \) is the maximum of the quantities \( \min\{\nu_1(x_i)(\rho) + \nu_2(y_i)(\rho), 1 \leq i \leq \ell\} \), taken over all such expressions.

**Lemma 3.7.** For any simple tensor we have \( (\nu_1 \ast \nu_2)(x \otimes y) = (\nu_1(x) + \nu_2(y)) \).

**Proof.** For any \( \rho \in \Sigma \cap N \) we have \( \nu_1 \ast \nu_2(x \otimes y)(\rho) \geq \nu_1(x)(\rho) + \nu_2(y)(\rho) \). Suppose \( x \otimes y = \sum_{i=1}^\ell x_i \otimes y_i \) with \( \nu_1(x_i)(\rho) + \nu_2(y_i)(\rho) \geq r \) for each \( 1 \leq i \leq \ell \). Choose bases of \( E_1 \) and \( E_2 \) which include \( x \) and \( y \), and are compatible with the subspaces \( G^\rho_{\nu_1(x_i)}(\nu_1) \subset E_1 \) and \( G^\rho_{\nu_2(y_i)}(\nu_2) \subset E_2 \). It follows that \( x_i \otimes y_i = (x + \cdots) \otimes (y + \cdots) \) for some \( i \). As a consequence \( \nu_1(x_i)(\rho) \leq \nu_1(x)(\rho) \) and \( \nu_2(y_i)(\rho) \leq \nu_2(y)(\rho) \), so \( \min\{\nu_1(x_i)(\rho) + \nu_2(y_i)(\rho), 1 \leq i \leq \ell\} \leq \nu_1(x)(\rho) + \nu_2(y)(\rho) \).

The product \( \otimes \) distributes over \( \oplus \), and has \((k, 0)\) as its identity object.

**Proposition 3.8** (Algebra objects). A commutative algebra object \((A, \nu) \in \text{Vect}_\Sigma\) is the same information as an algebra \( A \) equipped with a quasivaluation \( \nu : A \to \hat{\Omega}_\Sigma \).

**Proof.** If \( m : (A, \nu) \otimes (A, \nu) \to (A, \nu) \) is a morphism in \( \text{Vect}_\Sigma \), we must have \( \nu(f) + \nu(g) \leq \nu(f \otimes g) \). Conversely, for any algebra \( A \) with \( \nu : A \to \hat{\Omega}_\Sigma \) a quasivaluation and multiplication \( m \), we must have \( m(G^\rho_r(\nu \ast \nu)) \subset G^\rho_r(\nu) \subset A \) so that \( m \) satisfies the requirements for a morphism in \( \text{Vect}_\Sigma \).

We let \( \text{Alg}_\Sigma \) denote the category of commutative algebra objects in \( \text{Vect}_\Sigma \).

### 3.3. Schur functors

For any vector space \( E \) and partition \( \lambda \vdash n \) we can form the space \( S_\lambda(E) \subset E^\otimes n \). We make the following definition using the tensor product in \( \text{Vect}_\Sigma \).

**Definition 3.9.** The Schur functor \( S_\lambda : \text{Vect}_\Sigma \to \text{Vect}_\Sigma \) takes a prevalued vector space \((E, \nu)\) to \((S_\lambda(E), s_\lambda(\nu)) \subset (E, \nu)^\otimes n\), where \( s_\lambda(\nu) \) is the pullback of \( \nu^\otimes n \) under the inclusion map \( S_\lambda(E) \subset E^\otimes n \).
Showing that $S_\lambda : \text{Vect}_\mathbb{C} \to \text{Vect}_\mathbb{C}$ is a functor is straightforward. Moreover, the categories we deal with in this paper are symmetric monoidal and Cauchy-complete (see [La19]). It follows that the functor $S_\lambda$ makes sense in any of these categories, and all strictly monoidal functors we consider commute with any $S_\lambda$.

4. THE FUNCTORS $\mathcal{L}$ AND $\mathcal{R}$

A strongly convex rational polyhedral cone $\sigma \subset \mathbb{R}_+$ determines an affine semigroup $\sigma^\vee \cap M$, where $\sigma^\vee = \{ u | \langle \rho, u \rangle \leq 0 \ \forall \rho \in \sigma \} \subset M$. We let $S_0 = k[M]$ be the Laurent polynomial ring determined by $M$, and $S_\sigma \subset S_0$ denote the affine semigroup algebra associated to $\sigma^\vee$. 

**Remark 4.1.** Our definition for the dual cone $\sigma^\vee$ is the negative of the convention found in the literature on toric varieties (e.g [CLS11] and [Stu96a]). This is to conform with the min convention for valuations and tropical geometry.

Let $\text{Mod}^M_{S_\sigma}$ denote the category of $M$-graded $S_\sigma$-modules. In this section we set up the functors $\mathcal{R} : \text{Vect}_\sigma \to \text{Mod}^M_{S_\sigma}$ and $\mathcal{L} : \text{Mod}^M_{S_\sigma} \to \text{Vect}_\sigma$, and show that they are adjoint (Theorem 6). The functor $\mathcal{R}$ is recognizable as taking the Rees space (algebra) of a filtration, so $\mathcal{L}$ can be characterized as the left adjoint of the Rees construction.

4.1. The functor $\mathcal{L}$. Fix a module $R = \bigoplus_{m \in M} F_m(R)$. For $u \in \sigma^\vee \cap M$ we can view the element $\chi_u \in S_\sigma$ as a family of linear maps $\chi_m : F_m(R) \to F_{m+u}(R)$. In this way $R$ determines a directed system in the category of $k$-vector spaces (this is a central observation of [Per04, KS98]). We let $E_R = \lim_{\rightarrow} F_m(R)$ be the colimit of this system. There is a natural surjection $\phi_R : R \to E_R$, and for each $m \in M$ there is a subspace $\phi_R(F_m(R)) \subset E_R$. Let $m_0 \subset S_\sigma$ be the maximal ideal generated by the forms $\chi_u - 1 \ \forall u \in \sigma^\vee \cap M$. Similarly, let $m_0 = S_0$ be the ideal generated by $\chi_u - 1 \ \forall u \in M$.

**Proposition 4.2.** The map $\phi_R$ induces an isomorphism $E_R \cong R/mR$. Furthermore, the natural map $i : R \to R \otimes_{S_\sigma} S_0$ induces an isomorphism of vector spaces $R/mR \cong R \otimes_{S_\sigma} S_0/m_0R \otimes_{S_\sigma} S_0$, and $R \otimes_{S_\sigma} S_0$ is isomorphic to the induced module $E_R \otimes_k S_\sigma$.

**Proof.** The second part follows easily from the flatness of $S_0$ as an $S_\sigma$ module. To see the first part, note that the vector space $E_R$ is defined to be the quotient of $R$ by the span of the forms $\{ f - g \mid f, g \in F_m, \exists u, v \in \sigma^\vee \cap M \ | \ \chi_u f = \chi_v g \}$. We have $f - g = (f - \chi_u f) - (g - \chi_v g) = (1 - \chi_u) f - (1 - \chi_v) g \in mR$. Moreover, $mR$ is the span of the forms $f - \chi_v f$, so for any such element we have $(\chi_v f) = (\chi_v) f \in R$. \[ \square \]

Now we define a function $\nu_R : E_R \to \hat{\mathcal{O}}_\sigma$. For $\rho \in \sigma \cap N$ and $r \in \mathbb{Z}$ let $G^\rho_r(R) \subset E_R$ be the sum:

$$ G^\rho_r(R) = \sum_{(\rho, m) \geq r} \phi_R(F_m(R)). $$

Observe that $G^\rho_r(R) \supseteq G^\rho_s(R)$ whenever $r \leq s$, so these spaces define an decreasing filtration on $E_R$ for each $\rho \in \sigma \cap N$.

**Definition 4.3.** For $f \in E_R$, we let $\nu_R(f) : \sigma \cap N \to \mathbb{Z}$ be $\nu_R(f)(\rho) = \max \{ r \mid f \in G^\rho_r(R) \}$.

**Proposition 4.4.** There is a functor $\mathcal{L} : \text{Mod}^M_{S_\sigma} \to \text{Vect}_\sigma$ with $\mathcal{L}(R) = (E_R, \nu_R)$.

**Proof.** We must check that $\nu_R(f) \in \hat{\mathcal{O}}_\sigma$, it is then immediate that $\nu_R$ is a prevaluation. Fix $t \in \mathbb{Z}_{>0}$, and suppose $\nu_R(f)(\rho) = r$, so $f \in G^\rho_r(R) \setminus G^\rho_{r-1}(R)$. By definition, $f \in G^\rho_t(R)$,
and if \( f \in G^\rho_{\tau-1}(R) \), then for some \( n_1, \ldots, n_d \) we have \( f \in F_{n_1}(R) + \cdots + F_{n_d}(R) \) where \( \langle \rho, n_i \rangle > r \). But then \( \langle \rho, n_i \rangle > r \), which contradicts \( v_R(f)(\rho) = r \).

For \( \psi : R_1 \to R_2 \) there is an induced map \( \mathcal{L}(\psi) : E_{R_1} \to E_{R_2} \). For any \( m \in M \) the restriction map \( \psi_m : F_m(R_1) \to F_m(R_2) \) satisfies \( \phi_R \circ \psi_m = \mathcal{L}(\psi) \circ \phi_R \). It follows that \( \mathcal{L}(\psi) \) maps \( G^\rho_{\tau}(R_1) \) into \( G^\rho_{\tau}(R_2) \), so \( \mathcal{L}(\psi) \) is a morphism in \( \Vect_{\tau} \).

\[ \square \]

**Example 4.5.** Consider the localization \( S_{\tau} \subset S \) for \( \tau \subset \sigma \) a face. We have \( S_\tau = \frac{1}{v_S}S_{\tau} \) for some \( v \in \sigma \) such that \( \langle \rho, v \rangle = 0 \) \( \forall \rho \in \tau \) and \( E_{S_\tau} = k \). If \( \rho \in \tau \) then \( \langle \rho, v \rangle \leq 0 \) \( \forall v \in \tau \), so \( v_{S_\tau}(1)(\rho) = 0 \). But if \( \rho \in \sigma \setminus \tau \) then \( \langle \rho, v \rangle > 0 \), so in this case \( v_{S_\tau}(1)(\rho) = \infty \).

**Example 4.6.** Consider an \( S_{\tau} \)-module \( R \) generated by \( f_1, \ldots, f_N \in R \) with \( \deg(f_i) = \lambda_i \). If \( p = \sum_{i=1}^N \chi_{u_i}f_i \), then \( \phi_R(p) = q \in \sum \phi_R(F_{\lambda_i}(R)) \), so \( v_R(q) \geq \sum_{i=1}^N \lambda_i \). Furthermore, \( F_m(R) \neq 0 \) only if \( m \in \bigcup_{i=1}^N \lambda_i + [\sigma \setminus \tau] \). It follows that \( v_R(q) \) is the maximum over a finite number of expressions of the form \( \bigoplus_{i=1}^N \lambda_i \), and only a finite number of such expressions are possible. As a consequence, we see that the image \( v_R(E_R) \subset O_\sigma \) is a finite set.

### 4.2. The functor \( \mathcal{L} \) under localization and pullback

Let \( \tau \subset N_0 \) and \( \sigma \subset N_0 \) be pointed polyhedral cones, and let \( \iota : \tau \to \sigma \) be a linear map induced by a map of lattices \( \iota : N' \to N \). There is an associated map on dual lattices \( \iota^* : M \to M' \) and semigroup algebras \( \iota^* : S_\tau \to S_\sigma \). The map \( \iota^* \) gives an extension functor \( - \otimes_{S_\tau} S_\tau : \Mod_{S_\sigma} \to \Mod_{S_\sigma} \).

We also have a functor \( \iota^! : \Vect_\sigma \to \Vect_\tau \) obtained by composing \( \iota : E \to O_\sigma \) with the map on semialgebras \( \iota^* : O_\sigma \to O_\tau \), given by precomposition with \( \iota \). We show that these two functors coincide under \( \mathcal{L} \). As a consequence, the restriction of \( \mathcal{L}(R) = (E_R, \nu_R) \) to a facet \( \tau \subset \sigma \) is \( \mathcal{L}(R \otimes_{S_\tau} S_\tau) \).

**Proposition 4.7.** The following diagram of functors commutes.

\[
\begin{array}{ccc}
\Mod_{S_\sigma} & \xrightarrow{- \otimes_{S_\tau} S_\tau} & \Mod_{S_\sigma}' \\
\mathcal{L}_{\sigma} \downarrow & & \mathcal{L}_{\tau} \\
\Vect_\sigma & \xrightarrow{\iota^!} & \Vect_{\tau}
\end{array}
\]

**Proof.** Let \( R' = R \otimes_{S_\tau} S_\tau \) and \( R_0 = R \otimes_{S_\tau} k[M] \). We have \( R' \otimes_{S_\tau} k[M] \cong R_0 \otimes_{k[M]} k[M'] \).

By setting \( \chi_{m'} = 1 \) for each \( m' \in M' \) we obtain \( E_{R'} \cong E_{R'} \), so \( \phi_{R'} \circ (\iota^* \otimes 1) = \phi_R \).

Now let \( \rho \in \tau \cap N' \), and consider \( G^\iota_{\tau}(R) = \sum_{(\rho, m) \geq \tau} \phi_R(F_m(R)) \). This space is the image of \( \bigoplus_{(\rho, m) \geq \tau} F_m(R) \subset R \) under \( \phi_R \). Similarly, \( G^\iota_{\tau}(R') \) is the image of \( \bigoplus_{(\rho, m) \geq \tau} F_m(R') \) under \( \phi_{R'} \), where

\[
\langle \rho, m \rangle \geq r \text{ if and only if } \langle \rho, \iota^*(m) \rangle \geq r, \text{ so } \phi_R(F_m(R)) \subset \phi_{R'}(F_{\iota^*(m)}(R')) \text{ and } G^\iota_{\tau}(R) \subset G^\iota_{\tau}(R') \text{ in } E_R \cong E_{R'}.
\]

Similarly, if \( f \in G^\rho_{\iota}(R) \) then we can write \( f = \sum \phi_R(g_i) \), where \( g_i \in \phi_R(F_{n_i}(R)) = \phi_R((\iota^* \otimes 1)(F_{n_i}(R) \otimes_{k} k[\chi_u])) \), where \( \iota^*(n_i) + u = m_i = \langle \rho, m_i \rangle \geq r \). But then \( \langle \rho, \iota^*(n_i) \rangle \geq r \), so that \( \langle \rho, n_i \rangle \geq R \). It follows that \( f \in G^\rho_{\iota^!}(R) \) and \( G^\iota_{\tau}(R) = G^\iota_{\tau}(R') \).

Now fix \( \rho \in \tau \cap N' \) and \( f \in A_R \), then \( \iota^! \phi_R(f)(\rho) = \phi_{R'}(f(\iota(\rho))) \), which is equal to \( \nu_{R'}(f)(\rho) \) by the above calculation. This shows that \( \iota^! \circ \mathcal{L}_{\tau}(R) = (E_{R'}, \nu_{R'}) \) is isomorphic to \( (E_{R'}, \nu_{R'}) = \mathcal{L}_{\tau} \circ (R \otimes_{S_\tau} S_\tau) \) by the map which identifies \( E_R \) with \( E_{R'} \). \( \square \)
4.3. The functor \( \mathcal{R} \). For any \((E, \nu) \in \text{Vect}_\sigma \) and \( m \in M_\sigma \) we can consider the space \( F_m(\nu) = \{ f \mid \nu(f) \geq m \} \). If \( u \in \sigma^r \cap M \) then \( m \geq m + u \) in \( \mathcal{O}_\sigma \), so \( F_m(\nu) \subseteq F_{m+u}(\nu) \).

**Definition 4.8.** For \((E, \nu)\) an object of \( \text{Vect}_\sigma \) we define the Rees module \( \mathcal{R}(E, \nu) \in \text{Mod}^M_{S_\sigma} \) to be the following \( M \)-graded vector space:

\[
\mathcal{R}(E, \nu) = \bigoplus_{m \in M} F_m(\nu).
\]

We let \( \chi_u \in S_\sigma \) act by the inclusion map: \( F_m(\nu) \subseteq F_{m+u}(\nu) \).

**Proposition 4.9.** The Rees module construction extends to a functor \( \mathcal{R} : \text{Vect}_\sigma \to \text{Mod}^M_{S_\sigma} \).

**Proof.** It remains to define the morphism \( \mathcal{R}(\nu) : \mathcal{R}(E_1, \nu_1) \to \mathcal{R}(E_2, \nu_2) \) associated to \( \psi : (E_1, \nu_1) \to (E_2, \nu_2) \). We have \( \nu_1(f) \geq m \) implies \( \nu_2(\psi(f)) \geq \nu_1(f) \geq m \), so \( \psi(F_m(\nu_1)) \subseteq F_m(\nu_2) \). For any \( u \in \sigma^r \cap M \) there is a commuting square:

\[
\begin{array}{ccc}
F_m(\nu_1) & \xrightarrow{\chi_{m,u}} & F_{m+u}(\nu_1) \\
\downarrow & & \downarrow \\
F_m(\nu_2) & \xrightarrow{\chi_{m,u}} & F_{m+u}(\nu_2)
\end{array}
\]

So \( \mathcal{R}(\psi) \) is a morphism in \( \text{Mod}^M_{S_\sigma} \). It is straightforward to show \( \mathcal{R}(\psi \circ \psi') = \mathcal{R}(\psi) \circ \mathcal{R}(\psi') \). \( \square \)

**Example 4.10.** We compute \( \mathcal{R}(k, \nu_{S_\sigma}) \) for \( \nu_{S_\sigma} \) from Example 4.5. We must determine if \( \nu_{S_\sigma}(1) \geq u \) for each \( u \in M \). This inequality always holds for \( \rho \notin \tau \), therefore it holds everywhere if and only if \( \rho(\nu_{S_\sigma}) \leq 0 \forall \rho \in \tau \). It follows that \( \mathcal{R}(k, \nu_{S_\sigma}) = \bigoplus_{u \in \tau \cap M} k \chi_u = S_\tau \).

4.4. Adjunction. Now we show that \( \mathcal{L} \) and \( \mathcal{R} \) form an adjunction.

**Theorem 4.11.** The (bi)functors \( \text{Hom}_{\text{Mod}^M_{S_\sigma}}(-, \mathcal{R}(-)) \) and \( \text{Hom}_{\text{Vect}_\sigma}(\mathcal{L}(-), -) \) are naturally isomorphic.

Recall the universal property of colimit: if we are given a system of maps \( \psi_m : F_m(R) \to E \) which commute with \( \chi_u \) for \( u \in \sigma^r \cap M \), then there is a unique map \( \ell(\psi) = \lim \psi_m : E_R \to E \) such that \( \psi = (\ell(\psi) \circ \phi_R)_m \).

**Proof.** Let \( \phi : \mathcal{L}(R) \to (E, \nu) \) be a morphism in \( \text{Vect}_\sigma \). The image \( \phi_R(F_m(R)) \) is a subspace of \( F_m(\nu_R) \), so we let \( r(\phi) : R \to \mathcal{R}(E, \nu) \) be defined by the maps \( r(\phi)_m = \phi_m \circ \phi_R : F_m(R) \to F_m(\nu) \). The following diagram commutes:

\[
\begin{array}{ccc}
F_m(R) & \xrightarrow{\phi_R} & F_m(\nu_R) & \xrightarrow{\phi_m} & F_m(\nu) \\
\downarrow & & \downarrow & & \downarrow \\
F_{m+u}(R) & \xrightarrow{\phi_R} & F_{m+u}(\nu_R) & \xrightarrow{\phi_{m+u}} & F_{m+u}(\nu),
\end{array}
\]

as the right two vertical arrows are inclusions, with the middle occurring in the colimit \( E_R \).

As a consequence, \( r(\phi) \in \text{Hom}_{\text{Mod}^M_{S_\sigma}}(R, \mathcal{R}(A, \nu)) \).

Given \( \psi : R \to \mathcal{R}(E, \nu) \), we have a family of maps \( \psi_m : F_m(R) \to F_m(\nu) \) which commute with each \( \chi_u \). We get \( \ell(\psi) : E_R \to E \), where \( \psi_m = \ell(\psi) \circ \phi_R \) on \( F_m(R) \). Furthermore, \( G^\rho(\phi) \) is mapped into \( \sum_{(\rho, m) \geq \tau} F_m(\nu) \subseteq G^\rho(\nu) \) so \( \ell(\psi) \in \text{Hom}_{\text{Vect}_\sigma}(\mathcal{L}(R), (E, \nu)) \).
Now we show that ℓ and r are inverses of each other. Given φ : L(R) → (E, ν) and f ∈ E_R, pick ˆf ∈ F_m(R) so that φ_R( ˆf) = f. Then r(φ)( ˆf) = φ( ˆf) ∈ F_m(ν). It follows that ℓ(r(φ))(f) = φ( ˆf). Now if ψ : R → R(E, ν), ψ_m = (ℓ(ψ) ◦ φ_R)_m, so
r(ℓ(ψ))_m = (ℓ(ψ) ◦ φ_R)_m = ψ_m. We omit the proof that r and ℓ are natural, as it is straightforward.

4.5. Eversive objects. We have functors LΣ : Vect σ → Vect σ and RΣ : Mod M Sσ → Mod M Sσ; these are the monad and comonad of the adjunction in Theorem 4.11. For (E, ν) ∈ Vect σ, the space LΣ(E, ν) has underlying space lim F_m(ν) ⊆ E and prevaluation defined by the spaces G^p(Σ(E, ν)) = ∑_{(p, m) ≥ r} F_m(ν) ⊆ G^p(ν). It follows that there is a map η : LΣ(E, ν) → (E, ν). The object RΣ(L) is the M-graded module ⊔_{m∈M} F_m(ν_R), where F_m(ν_R) = ∩_{p∈N} G^p_{r,m}(R). There is a map ϵ : R → RΣ(L) which takes F_m(R) to φ_R(F_m(R)) ⊆ F_m(ν_R).

Example 4.12. Let I ⊆ S, be an M-graded ideal spanned by the characters Ω(I) ⊆ M, then Σ(L(I)) ⊆ S is the ideal with Ω(Σ(L(I))) = conv(Ω(I)) ∩ M, i.e. the lattice points in the convex hull of Ω(I).

Definition 4.13 (Eversive objects). We say that (E, ν) ∈ Vect σ is eversive if G^p(Σ(E, ν)) = G^p{ν}(ν) and if F_m(R) ≅ F_m(ν_R) for any m such that (p, m) ≥ r, but by definition G^p(ν) = ∑_{m>0} G^p_{m} F_m(R). This shows that Σ(L) is eversive. Similarly, they are isomorphisms if and only if the corresponding objects are eversive. The functors Σ and L define an equivalence Vect σ ∼= Mod M Sσ.

Proposition 4.14. For any R ∈ Mod M Sσ and (E, ν) ∈ Vect σ, both Σ(E, ν) and L(R) are eversive. Furthermore, ϵ and η are isomorphisms if and only if the corresponding objects are eversive. The functors Σ and L define an equivalence Vect σ ∼= Mod M Sσ.

Proof. We have φ_R(F_m(R)) ⊆ F_m(ν_R) ⊆ G^p_{r,m}(ν_R) for any m such that (p, m) ≥ r, but by definition G^p_{r,m}(ν_R) = φ_R(F_m(R)). This shows that Σ(L) is eversive. Similarly, F_m(ν) = F_m(Σ(E, ν)) = φ_R(A, ν)(φ_R(L(E, ν))), so Σ(E, ν) is eversive. Finally, by definition eversive conditions hold if and only if η, ϵ are isomorphisms.

We let Alg M Sσ be the category of commutative algebras in Mod M Sσ. If R ∈ Alg M Sσ then R/σ M R ≅ E_R is a k-algebra, and ν_R is a quasivaluation on E_R. Similarly, it is easy to check that Σ(R, ν) is an Sσ-algebra if (A, ν) ∈ Alg σ. We obtain a slightly stronger result below. Recall that a functor F : (Σ, ⊗) → (D, ⊗) between symmetric monoidal categories is said to be strictly monoidal if F(A ⊗ B) is naturally isomorphic to F(A)⊗F(B). It is said to be lax-monoidal if there is a natural transformation (of bifunctors) m : F(A)⊗F(B) → F(A ⊗ B).

Proposition 4.15. The functor Σ is strictly monoidal, and the functor L is lax-monoidal. Furthermore, Σ and L define an equivalence Alg σ ∼= Alg M Sσ.

Proof. Let R_1, R_2 ∈ Mod M Sσ. It is straightforward to show E_{R_1⊗S, R_2} = E_{R_1}⊗k E_{R_2}, and that G^p_{r_1⊗r_2}(R_1 ⊗ R_2) = G^p_{r_1 r_2} φ_{R_1} φ_{R_2} for all ρ ∈ σ ∩ N. This implies that Σ(L(R_1 ⊗ S) ≅ Σ(L(R_1)) ⊗ L(R_2). To show that Σ is lax-monoidal we observe that there is a natural map m : Σ(E_{1, ν_1} ⊗ S) → Σ(E_{2, ν_2}) given by sending the simple tensors in F_{p_1} φ_{R_1} φ_{R_2} into F_{p+q} φ_{R_1} φ_{R_2} using Lemma [3.7]. It follows that the maps η : LΣ(A, ν) → (A, ν) and ϵ : R → Σ(L) are algebra maps if (A, ν) ∈ Alg σ or R ∈ Alg M Sσ. An algebra map is an isomorphism if and only if the underlying module map is an isomorphism, so η and ϵ are isomorphisms of algebras if the corresponding objects are eversive. □
4.6. Global equivalence. Now we consider the category Sh\textsuperscript{M}_Y(Σ) of T\textsubscript{N}-equivariant quasicoherent sheaves on the toric variety Y(Σ) (see [Per04]). If F ∈ Sh\textsuperscript{M}_Y(Σ), then for any t ∈ T\textsubscript{N} there is an isomorphism \(φ_t : t^*F \cong F\), where \(t^*F\) is the pullback along the automorphism \(t : Y(Σ) \to Y(Σ)\) defined by \(t\), such that the following diagram commutes:

\[
\begin{array}{ccc}
t_2^*t_1^*F & \cong & (t_1t_2)^*F \\
\downarrow \phi_{t_1t_2} & & \downarrow \phi_{t_1} \downarrow \phi_t \\
t_2^*F & & \end{array}
\]

A morphism \(φ : E \to F\) in Sh\textsuperscript{M}_Y(Σ) is a map of sheaves which respects these diagrams. For a face \(σ \in Σ\) let \(Y(σ) \subset Y(Σ)\) be the corresponding open subvariety. By applying the section functor \(Γ(Y(σ), -)\) we obtain objects and morphisms in Mod\textsuperscript{M}\textsubscript{Σ,σ}.

**Definition 4.16.** Let Sh\textsuperscript{M, ev}_Y(Σ) ⊂ Sh\textsuperscript{M}_Y(Σ) be the full subcategory on quasicoherent sheaves \(F\) such that \(Γ(Y(σ), F) \in Mod\textsuperscript{M}\textsubscript{Σ,σ}\) for each \(σ \in Σ\).

**Theorem 4.17.** There is a monoidal functor \(ℒ : Sh\textsuperscript{M}_Y(Σ) \to \text{Vect}_Σ\). Under \(ℒ\), Sh\textsuperscript{M, ev}_Y(Σ) is equivalent to a full subcategory of \(\text{Vect}_Σ\).

*Proof.* Proposition 4.2 implies that the fiber \(E_F\) over \(\text{Spec}(S_0/m_0) = id \in T_N \subset Y(Σ)\) carries a prevaluation \(v_{F,σ} : E_F \to \mathcal{O}_σ\) for each face \(σ \in Σ\). By Proposition 4.7 for any \(f \in E_F\), and \(τ = σ_1 \cap σ_2\), the restrictions \(v_{F,σ_1}(f)|_τ\) and \(v_{F,σ_2}(f)|_τ\) coincide. It follows that we obtain an object \((E_F, v_F) ∈ \text{Vect}_Σ\). Proposition 4.2 also implies that a morphism \(φ : F \to G\) gives a map \(ℒ(φ)|_σ : (E_F, v_F|_σ) \to (E_G, v_G|_σ)\) whose underlying vector space map does not depend on \(σ\), so we have a functor \(ℒ : Sh\textsuperscript{M}_Y(Σ) \to \text{Vect}_Σ\). We observe (Sta19 Lemma 17.15.4) that for any \(σ \in Σ\), and \(F, E ∈ Sh\textsuperscript{M, ev}_Y(Σ)\), \(ℒ(F ⊗_Y(Σ) E)\) restricted to \(σ\) is \(ℒ(Γ(Y(σ), E)|_Y(σ) ⊗_Y(Σ) F|_Y(σ)) = ℒ(Γ(Y(σ), E) ⊗_{S_σ} Γ(Y(σ), F))\) since \(E, F\) are quasicoherent and \(Y(σ)\) is affine. The latter is \(ℒ(Γ(Y(σ), E)) ⊗ ℒ(Γ(Y(σ), F))\), so it follows that \(ℒ(E ⊗_Y(Σ) F) = ℒ(E) ⊗ ℒ(F)\).

Now let \(F, G ∈ Sh\textsuperscript{M, ev}_Y(Σ)\), and consider a map \(ψ : (E_F, v_F) \to (E_G, v_G)\). For any \(σ \in Σ\) we obtain a map on modules \(R(ψ)|_σ : R(E_F, v_F|_σ) \to R(E_G, v_G|_σ)\), which in turn induces a map on quasicoherent \(Y(σ)\) sheaves: \(\tilde{R}(ψ)|_σ : \tilde{R}(E_F, v_F|_σ) \to \tilde{R}(E_G, v_G|_σ)\). The sheaves \(F, G\) were chosen in Sh\textsuperscript{M, ev}_Y(Σ), so \(\tilde{R}(E_F, v_F|_σ)\) and \(\tilde{R}(E_G, v_G|_σ)\) coincide with the restrictions \(F|_Y(σ)\) and \(G|_Y(σ)\), and we obtain an induced map on descent data for the open cover of \(Y(Σ)\) by the \(Y(σ)\) for \(σ ∈ Σ\): \(\tilde{R}(ψ) : \{(Y(σ), F|_Y(σ)), σ ∈ Σ\} \to \{(Y(σ), G|_Y(σ)), σ ∈ Σ\}\) (see e.g. Sta19 Lemma 68.3.4). This map induces a unique map between \(F\) and \(G\). It’s straightforward to check that this construction is inverse to \(ℒ\) on Sh\textsuperscript{M, ev}_Y(Σ), in particular \(ℒ(φ) : (E_F, v_F) → (E_G, v_G)\) is an isomorphism if and only if \(φ : F \cong G\). □

In Definition 4.16 we require that \(Γ(Y(σ), F)\) has the additional property that any localization \(S_τ ⊗_{S_σ} Γ(Y(σ), F)\) for \(τ ⊂ σ\) a face is also an eversive module. Proposition 4.2 shows that this holds if \(F\) is locally free or projective. Additionally, the following proposition
shows that this is the case for modules which are eversive and finitely generated or eversive and flat when it is combined with Propositions 3.10 and 5.4.

**Lemma 4.18.** Let $R$ be eversive with the property that $F_m(R) = \bigcap_{i=1}^l G_{\rho_i,m}^\rho(R)$ for some finite set $\rho_1, \ldots, \rho_l \in N \cap \sigma$, then $S_\tau \otimes_{S_r} R$ is eversive as well for any face $\tau \subset \sigma$.

**Proof.** If $R$ is eversive, it is torsion free, and $R' = S_r \otimes_{S_r} R$ is torsion-free as well. In this case, $E_R \cong E_{R'}$ and the graded component $F_0(R')$ is isomorphic to the sum $\sum_{u \in \tau \cap N} F_{n+u}(R)$ in $E_{R'}$. We must show that the inclusion $F_n(R') \subseteq \bigcap_{\rho \in \tau \cap N} G_{\rho,n}^\rho(R)$ is an equality. For each facet containing $\tau$ we select a normal vector $u_j$, and we let $u = \sum u_j$. For all $\rho \in \tau \cap N$ we have $\langle \rho, u \rangle = 0$ and $\langle \rho_i, u \rangle < 0$ for each $\rho_i \notin \tau \cap N$. If $f \in \bigcap_{\rho \in \tau \cap N} G_{\rho,n}^\rho(R)$ then $v_R(f)(\rho_i) \geq \langle \rho_i, n \rangle$ for each $\rho_i \in \tau \cap N$. We select an appropriate multiple $su$ so that $v_R(f)(\rho_i) \geq \langle \rho_i, n + su \rangle$ for each $\rho_i \notin \tau \cap N$. It follows that $f \in F_{n+su}(R) \cap F_{n}(R')$. \hfill $\square$

Now by Proposition 5.10 it follows that if $R$ is flat and eversive, $F_m(R)$ is the intersection of the spaces $G_{\rho_i,m}^\rho(R)$ for $\rho_i$ generators of the rays of $\sigma$. Similarly, if $R$ is finitely generated and eversive, Proposition 5.4 shows that there is a finite decomposition $\sigma = \bigcup_{i=1}^l \sigma_i$ such that the restriction $v_R(f)|_{\sigma_i}$ is linear for each $f \in E_R$, so we get a similar decomposition in this case in terms of the generators of the rays of the $\sigma_i$.

5. **Free, projective, and flat modules**

We show that free and projective modules are eversive, and we prove that their associated prevaluations take values in piecewise linear functions. This proves the equivalence of categories in Theorem 3.4. We also establish a number of properties of $\mathcal{L}(R)$ for $R$ a flat module, and we prove a technical precursor to Theorem 5.4.

**5.1. Free and projective.** In what follows it will be useful to know how the eversive condition behaves with respect to direct sum.

**Proposition 5.1.** The functors $\mathcal{L}$ and $\mathcal{R}$ respect direct sums. Furthermore, in Mod$_{S_r}^M$ and Vect$_\sigma$, a direct sum is eversive if and only if its components are eversive.

**Proof.** It is straightforward to verify that $F_m(v \oplus w) = F_m(v) \oplus F_m(w)$, $E_{P \oplus Q} = E_P \oplus E_Q$ and $v_{P \oplus Q} = v_P \oplus v_Q$, so that $F_m(v_P \oplus v_Q) = F_m(v_P) \oplus F_m(v_Q)$. As a consequence, $F_m(P \oplus Q) = F_m(v_{P \oplus Q})$ if and only if $F_m(P) = F_m(v_P)$ and $F_m(Q) = F_m(v_Q)$. Now consider $(A, v) \oplus (B, w)$. We have $G_\ell(v \oplus w) = \{ f + g \mid v(f) \oplus w(g) \geq \ell \} = G_\ell(v) \oplus G_\ell(w)$, so $G_\ell(v \oplus w) = \sum_{(\rho,n) \geq \ell} F_m(v \oplus w)$ if and only if $G_\ell(v) = \sum_{(\rho,n) \geq \ell} F_m(v)$ and $G_\ell(w) = \sum_{(\rho,n) \geq \ell} F_m(w)$. \hfill $\square$

Let $P$ be a free $M$-graded $S_r$-module, then we can write $P = \bigoplus_{b_i \in B} S_r[b_i]$ for $B$ an $M$-homogeneous basis. We let $\deg(b_i) = \lambda_i \in M$. For any $m \in M$ we have a direct sum decomposition: $F_m(P) = \bigoplus_{u_i \in \lambda_i = m} k \chi_{b_i}[b_i]$. 

**Proposition 5.2.** The functors $\mathcal{L}$ and $\mathcal{R}$ give an equivalence between $M$-homogeneous free modules and prevalued spaces with a linear adapted basis (Definition 4.7). Furthermore, if $P = \bigoplus_{b_i \in B} S_r[b_i]$ is free with $\deg(b_i) = \lambda_i$, then $P$ is eversive, $\mathcal{L}(P) \cong \bigoplus_{b_i \in B}[k,\lambda_i]$, and $F_m(P)$ is isomorphic to the subspace of $E_P$ with basis $B_m = \{ b_i \mid m - \lambda_i \in \sigma \cap N \}$.

**Proof.** By Proposition 5.1 and Theorem 1.17 it suffices to note that if $\deg(b) = \lambda$ then $\mathcal{L}(S_r[b]) = (k,\lambda)$ and $\epsilon : \mathcal{R}\mathcal{L}(S_r[b]) \cong S_r[b]$. \hfill $\square$
Propositions \[prop:5.2\] and \[prop:5.1\] imply that projective modules in \(\text{Mod}_M\) are eversive and correspond under \(\mathcal{L}\) to summands of spaces with a linear adapted basis. Furthermore, if \(P\) is free or projective then so is \(P \otimes_{S_\sigma} S_\tau\) for any semigroup algebra map as in Proposition \[prop:4.7\]. It follows that any localization of one of these modules to the open subset corresponding to a face \(\tau \subset \sigma\) is eversive, so a sheaf of these modules fits into the assumptions of Theorem \[thm:4.17\]. We identify the category of \(T_N\)-vector bundles on \(Y(\Sigma)\) with the full subcategory on locally-free \(T_N\)-sheaves in \(\text{Sh}_Y^M\).

**Corollary 5.3** (Theorem \[thm:4\]). The category of locally projective \(T_N\)-sheaves on \(Y(\Sigma)\) is equivalent to the full subcategory of \(\text{Vect}_\Sigma\) on spaces \((E, v)\) such that \((E, v|_\sigma)\) is a summand of a space with a linear adapted basis for each \(\sigma \in \Sigma\). The category of finite rank \(T_N\)-vector bundles on \(Y(\Sigma)\) is equivalent to the full subcategory of \(\text{Vect}_\Sigma\) on spaces \((E, v)\) such that \(E\) is finite dimensional and \((E, v|_\sigma)\) has a linear adapted basis for each \(\sigma \in \Sigma\).

**Proof.** If \(\mathcal{F}\) is locally projective then Theorem \[thm:4.17\] implies that it is classified by \(\mathcal{L}(\mathcal{F}) = (E, v)\). The latter must have the property that each restriction \((E, v|_\sigma)\) is a summand of a space with a linear adapted basis by Proposition \[prop:5.2\]. Given \((E, v)\) with this property, \(R(E, v|_\sigma)\) is always a projective module, so \(R(E, v|_\sigma) = R(E, v|_\sigma) \otimes_{S_\sigma} S_\tau\) by Proposition \[prop:4.7\]. As in the proof of Theorem \[thm:4.17\], we can glue the corresponding quasi-coherent sheaves \(R(E, v|_\sigma)\) to produce \(\mathcal{F}\), a locally projective on \(Y(\Sigma)\) such that \(\mathcal{L}(\mathcal{F}) = (E, v)\). Now, in the special case that \(\mathcal{E}\) is the sheaf of sections of a finite rank vector bundle, the \(\Gamma(Y(\sigma), \mathcal{E})\) is an \(M\)-homogeneous free module, so each \((E, v|_\sigma)\) has a linear adapted basis.

**Corollary 5.4.** Let \((E, v) \in \text{Vect}_N\) with \(E\) a finite dimensional vector space, then the following are equivalent.

1. The image \(v(E \setminus \{0\})\) is a finite subset of \(O_N \setminus \{\infty\}\).
2. There is a finite complete fan \(\Sigma \subset N_\mathbb{Q}\) on \(Y(\Sigma)\) for \(\mathcal{E}\) a toric vector bundle.
3. There is a finite complete fan \(\Sigma \subset N_\mathbb{Q}\) with \((E, v) = \mathcal{L}(\mathcal{F})\) for \(\mathcal{F} \in \text{Sh}_Y^M(\Sigma)\) coherent.

**Proof.** Given \(v : E \to O_N\) with \(v(E) \subset O_\mathbb{Q}\) finite we find any fan \(\Sigma\) such that \(v(f)|_\sigma \in M_\sigma\) for all \(\sigma \in \Sigma\) and all \(f \in E\). The image \(v|_\sigma\) is a finite set \(\lambda_1, \ldots, \lambda_k \in M_\sigma \subset O_\mathbb{Q}\). By construction, the partial order on \(O_\sigma\) must restrict to a total ordering on the \(\lambda_i\). We let \(\text{gr}_\sigma(E)\) be the associated graded vector space, and we choose a basis \(\mathcal{B}_\sigma \subset \text{gr}_\sigma(E)\). Any lift \(B_\sigma \subset E\) of this basis is adapted to \(v|_\sigma\), this proves \((1) \to (2)\). The case \((2) \to (3)\) is immediate, and \((3) \to (1)\) is a consequence of Example \[ex:4.6\].

**Example 5.5.** If \((E, v)\) and \((F, w)\) are equipped with linear adapted bases \(B_1 \subset E, B_2 \subset F\), one checks that \(B_1 \times B_2 \cong \{b_1 \otimes b_2 \mid b_1 \in B_2\} \subset E \otimes_k F\) gives a linear adapted basis as follows. For any \(\rho \in \Sigma \cap N\), \(G^\rho(v) \cap B_1 \subset G^\rho(v)\) and \(G^\rho(w) \cap B_2 \subset G^\rho(w)\) are bases, so \(G^\rho_{s+t}(v \star w) \cap B_1 \times B_2\) is a basis of \(G^\rho_{s+t}(v \star w)\). This shows that \(B_1 \times B_2\) is adapted to \(v \star w\). To see that this is a linear adapted basis we use Lemma \[lem:3.7\] to conclude that \(v \star w(b_1 \otimes b_2) = v(b_1) + w(b_2)\) for any \(b_1 \times b_2 \in B_1 \times B_2\).

**Example 5.6.** Example \[ex:5.5\] allows us to find adapted bases of \(S_\lambda(E, v)\) for any Schur functor \(S_\lambda\) and \((E, v)\) with a linear adapted basis \(B\). For any \(\lambda\) we obtain a basis \(\{b_r, \ldots\}\) of \(S_\lambda(E)\) by applying the symmetrizers \(s_\tau\) corresponding to semi-standard fillings \(\tau\) of \(\lambda\) to \(B\). It is straightforward to check that if \(v\) is adapted to \(B \subset E\), then every simple tensor of \(b_r\) has the same value. The simple tensors with entries in \(B\) are an adapted basis of \(E^\otimes |\lambda|\), so we conclude that \(s_\lambda(v)(b_r)\) is linear if \(v(b)\) is linear for all \(b \in B\). For any \(\rho \in \sigma \cap N\), we have \(s_\lambda(v)_\rho(\sum C_\tau b_\tau) \geq \bigoplus s_\lambda(v)_\rho(b_\tau)\). If the sum of the minimum contributions \(\sum_{\min} C_\tau b_\tau\)
is non-zero, then some simple tensor in the elements of $\mathbb{B}$ in this sum achieves the top value, so this is an equality. If $\sum_{\min} C_\tau b_\tau = 0$, then $C_\tau = 0$, as the $b_\tau$ form a basis. It follows that the $b_\tau$ are an adapted basis of $S_{\sigma}(E, v)$.

**Example 5.7.** For a $T_N$-vector bundle $E$, the dual $E^*$ corresponds to the locally free sheaf with $\Gamma(E^*, Y(\sigma)) = \bigoplus_{i=1}^k S_\sigma[b_i^*]$ where $\deg(b_i^*) = -\deg(b_i) = -\lambda_i$. We have $\mathcal{L}(E^*) = (E^*, v^*)$, where locally $(E^*, v^*|_{\sigma}) = \bigoplus_{i=1}^k (k_i, -\lambda_i)$.

### 5.2. Properties of flat modules

We consider the image $\mathcal{L}(R)$ of a flat $S_{\sigma}$-module $R$. We make use of Lazard’s Theorem on flat modules, see [Sta19, Theorem 10.80.4].

**Proposition 5.8.** Let $R \in \text{Mod}_S^M$, then the following are equivalent:

1. The functor $- \otimes_{S_{\sigma}} R$ is exact.
2. $R$ is a direct limit of $M$-homogeneous free modules.
3. For any $\phi : P \to R$, where $P$ is $M$-homogeneous and free, and any $p \in \ker(\phi)$, there are maps $\phi' : P' \to R$ and $\pi : P \to P'$ such that $\phi = \phi' \circ \pi$ with $P'$ $M$-homogeneous and free,

For module $R \in \text{Mod}_S^M$, we can associate two $k$-vector spaces to a point $\rho \in \sigma \cap N$. Since $R$ is $T_N$-homogeneous, we obtain the fiber over a point in the orbit corresponding to the face $\tau$ containing $\rho$ in its relative interior by taking the quotient $R/\mathfrak{m}_{\rho}$, where $\mathfrak{m}_{\rho} = \langle \{\chi_u - 1, \chi_v \mid u \in \tau^1, \langle \rho, v \rangle < 0 \ \forall \rho \in \tau \rangle \rangle$. Alternatively, we have the associated graded space $\text{gr}_p(E_R) = \bigoplus_{r \in \mathbb{Z}} G^p_{\sigma}(R)/G^p_{> r}(R)$. These two spaces are naturally isomorphic if $R$ is flat.

**Proposition 5.9.** The map $\hat{\phi}_p : R \to \text{gr}_p(E_R)$ sending $F_n(R)$ to $G^p_{\langle \rho, m \rangle}(R)/G^p_{< \langle \rho, m \rangle}(R)$ is surjective and factors through the surjection $R \to R/\mathfrak{m}_{\rho}$, giving a map $\phi_p : R/\mathfrak{m}_{\rho} \to \text{gr}_p(E_R)$. If $R$ is flat, then $\phi_p$ is also injective. If $R$ is an algebra, $\phi_p$ is a map of algebras.

**Proof.** Any $f \in G^p_{\sigma}(R)$ is a sum of elements $f_i \in \phi_R(F_{m_i}(R))$ with $\langle \rho, m_i \rangle = r$, and all $\phi_R(F_{m_i}(R))$ with $\langle \rho, m_i \rangle = r$ are in the image of $\hat{\phi}_p$. If $g \in \mathfrak{m}_{\rho}$ then we can write $g = \sum \chi_{u_i} g_{i} + \sum (\chi_{u_i} - 1) h_{i}$ for $\langle \rho, v_i \rangle < 0$ and $\langle \rho, u_i \rangle = 0$, where $g_{i}$ and $h_{i}$ are homogeneous. Let $\deg(g) = m - v$ so that $\chi_{u_i} g_{i} \in F_{m_{i}}(R)$, then $\hat{\phi}_p(\chi_{u_i} g_{i}) \in \phi_R(F_{m_{i}}(R)) \subset G^p_{\langle \rho, m_i \rangle}(R)$. It follows that $\hat{\phi}_p(\chi_{u_i} g_{i}) = 0$. Similarly, since $\langle \rho, u_i \rangle = 0$, $\hat{\phi}_p(\chi_{u_i} h_{i}) = \hat{\phi}_p(h_{i})$. If $R \in \text{Alg}_S^M$ then $\mathcal{L}(R) \in \text{Alg}_{\sigma}$; it is straightforward to check that $\text{gr}_p(E_R)$ is a graded algebra, and $\phi_p$ is a map of algebras.

Now we show that $\phi_p$ is a injective for a free module $P$. Let $p_{i} \in F_{m_{i}}(P)$ with $\langle \rho, m_{i} \rangle = r$, and suppose that $\phi_p(\sum_{i=1}^n p_{i}) = 0$. Let $\mathbb{B} \subset E_P$ be a linear adapted basis with $\deg(b_{j}) = \lambda_{j}$ for $b_{j} \in \mathbb{B}$. Then $p_{i} = \sum_{j=1}^{k} c_{ij} \chi_{v_{ij}} b_{j}$ with $\lambda_{j} = (\rho, v_{ij})$. If $\langle \rho, v_{ij} \rangle < 0$ then $c_{ij} \chi_{v_{ij}} b_{j} \in \mathfrak{m}_{\rho} P$, so without loss of generality we assume that $\langle \rho, v_{ij} \rangle = 0$. This implies that $\langle \rho, \lambda_{j} \rangle = r$ for each $b_{j}$. Since the $b_{j}$ with $\langle \rho, \lambda_{j} \rangle = r$ form a basis of $G^p_{\sigma}(P)/G^p_{< \rho}(P)$, we conclude that $\sum_{i=1}^{n} c_{ij} = 0$ for each $j$. Then $\sum_{i=1}^{n} p_{i} = \sum_{j=1}^{k} (\sum_{i=1}^{n} c_{ij} \chi_{v_{ij}}) b_{j} = \sum_{j=1}^{k} (\sum_{i=1}^{n} c_{ij} (\chi_{v_{ij}} - 1)) b_{j} \in \mathfrak{m}_{\rho} P$. Finally, observe that $\text{gr}_p : \text{Vect}_{\sigma} \to \text{Vect}_k$ is a functor which commutes with colimits (it is a direct sum of cokernels). The functor $\mathcal{L}$ is a left adjoint, so it also commutes with colimits. For any flat module $R$, we can write $\lim P_{i} = R$ for $P_{i}$ free. It follows that $R/\mathfrak{m}_{\rho} R \cong \lim P_{i}/\mathfrak{m}_{\rho} P_{i} \cong \lim \text{gr}_{p}(\mathcal{L}(P_{i})) \cong \text{gr}_{p}(\lim \mathcal{L}(P_{i})) \cong \text{gr}_{p}(\mathcal{L}(R)) = \text{gr}_{p}(E_{R})$. 

A function $\phi \in \hat{\mathcal{O}}_{\sigma}$ is said to be concave if for any $\rho_{1}, \ldots, \rho_{t} \in \sigma$ we have $\phi(\sum_{i=1}^{t} \rho_{i}) \geq \sum_{i=1}^{t} \phi(\rho_{i})$. Next we show that $v_{R} : E_{R} \to \hat{\mathcal{O}}_{\sigma}$ takes concave values when $R$ is flat.
Proposition 5.10. Let $R$ be a flat $S_\sigma$ module, then for any $f \in E_R$, $\nu_R(f)$ is concave. Moreover, for any $m \in M$ we have $F_m(\nu_R) = \bigcap_{\rho_i \in \sigma(1)} G^{p_i}_{(\rho_i, m)}(R)$, where $\varrho_i = \mathbb{Z}_{\geq 0} \rho_i$.

Proof. If $(E, \nu) = \mathcal{L}(R)$, where $R \in \text{Mod}_S^{M_\sigma}$, let $\rho, \rho' \in \sigma \cap N$, and consider $f \in G^{\rho + \rho'}_\sigma(R)$ for some $r \in \mathbb{Z}$. We can write $f = \sum_{i=1}^t f_i$ for $f_i \in \phi_R(F_m(R))$ with $\langle \rho + \rho', m_i \rangle \geq r$. We let $s_i = \langle \rho, m_i \rangle$ and $t_i = \langle \rho', m_i \rangle$ with $s_i + t_i = r$, so that $f_i \in G^{s_i}_{\rho}(R) \cap G^{t_i}_{\rho'}(R)$. It follows that containment holds in the other direction when $f = \sum_{i=1}^t f_i$ for some $r \in \mathbb{Z}$. We have $G^{s_i}_{\rho}(R) \cap G^{t_i}_{\rho'}(R)$ is a full dimensional cone in $M_\rho$. Now we choose a presentation $\pi : R \to P$ with $p_i \in F_m(P)$ and $q_j \in F_{m_j}(P)$ such that $\pi(p_i) = q_j = \langle \rho_i, m_j \rangle \geq r$. This is always possible as $\sigma'$ is a full dimensional cone in $M_\rho$. We can find $\phi : R \to P', \psi : P' \to R$ such that $\pi \circ \phi = \tilde{\pi}$ with $P'$ free and $\pi(\sum_{i=1}^t f_i) = \pi(\sum_{i=1}^t f_i)$ for some $r \in \mathbb{Z}$. We have $\mathcal{L}(\pi)(G^{\rho + \rho'}_\sigma(P')) \subseteq G^{\rho + \rho'}_\sigma(R)$, so $\mathcal{L}(\pi)(p) \in G^{\rho + \rho'}_\sigma(R)$.

In conclusion, if $R$ is flat it is torsion-free; we choose an appropriate $m$ and identify $f \in F_m(R)$ with $f = \sum_{i=1}^t f_i$ for some $r \in \mathbb{Z}$. This is always possible as $\sigma'$ is a full dimensional cone in $M_\sigma$. With the following proposition we have a prominent role in Klyachko's main theorem [Kly89]. With the following proposition we recover the fact that a vector bundle $E$ is captured by its Klyachko spaces. In particular, notice that the spaces $F_m(\nu, \sigma) \subseteq E$ (which determine the modules $\Gamma(Y(\sigma), \mathcal{E})$) are a special case of the hypothesis.

5.3. Klyachko spaces. We say a function $\psi : \Sigma \cap N \to \mathbb{Z}$ is piecewise-linear on the fan $\Sigma$ (abbreviated PL) if the restriction $\psi|_{\sigma}$ is in $M_{\sigma}$ for each face $\sigma \in \Sigma$. It is well-known (see [Ful93, Chapter 3]) that PL functions on $\Sigma$ correspond to $T_N$-Cartier divisors on $Y(\Sigma)$. For such a $\psi$, we let $\mathcal{O}(\psi)$ denote the associated invertible sheaf. In particular, for $m \in M$, $\mathcal{O}(m)$ denotes the linearization of the structure sheaf of $Y(\Sigma)$ corresponding to the $T_N$-character $\chi_m$. Recall that for any $(E, \nu) \in \text{Vect}_\Sigma$ we let $F_{m}(\psi) = \{ f \mid \nu(f) \geq \psi \} = \bigcap_{\sigma \in \Sigma} F_{m}(\nu|_{\sigma}) \subseteq E$.

Proposition 5.11. Let $\psi$ be a PL function on $\Sigma$, and let $\mathcal{F} \in \text{Sh}_{Y(\Sigma)}^{M_{\psi}}$ with $\mathcal{L}(\mathcal{F}) = (E, \nu, \psi)$, then $\mathcal{H}_{\text{Sh}_Y^{M_{\psi}}}(\mathcal{O}(\psi), \mathcal{F})$ is isomorphic to $F_{m}(\nu, \psi)$.

Proof. Since $\mathcal{O}(\psi)$ is locally free and $\mathcal{L}(\mathcal{O}(\psi)) = (k, \psi)$ we may use Theorem 4.17.\hfill $\square$
Proposition 5.12. Let $\mathcal{F} \in \text{Sh}_{\Sigma}^{M,v}$ be flat and let $\psi$ be as in Proposition 5.11 then:

\begin{equation}
F_\psi(v) = \bigcap_{\varphi_i \in \Sigma(1)} G^\varphi_{\psi(\rho_i)}(v).
\end{equation}

**Proof.** By definition, $F_\psi(v) \subset \bigcap_{\varphi_i \in \Sigma(1)} G^\varphi_{\psi(\rho_i)}(v)$, and if $v(\rho_i) \geq \langle \rho_i, m_\sigma \rangle$ for each $\rho_i \in \sigma(1)$ then Proposition 5.10 implies that $v(\rho_i) \geq m_\sigma$, so $F_{m_\sigma}(v) = \bigcap_{\varphi_i \in \Sigma(1)} G^\varphi_{\psi(\rho_i)}(v)$. □

5.4. The matroid associated to a prevaluation. Let $v : E \to \mathcal{O}_N$ be a finite prevaluation. Recall that to $v$ there corresponds a subspace arrangement $A_v$ in $E$ given by $A_v = \{F_\phi(v) \mid \phi \in \mathcal{O}_N\}$. By Proposition 2.2(3) the arrangement $A_v$ is closed under intersection. One can also consider the subspace arrangement in $E$ obtained by taking all the Klyachko spaces.

Proposition 5.13. The subspace arrangement $A_v$ coincides with the Klyachko arrangement, obtained by intersecting all the subspaces in the Klyachko filtrations.

**Proof.** From Proposition 5.12 for any $\phi \in \mathcal{O}$ the corresponding subspace $F_\phi(v)$ is an intersection of the Klyachko spaces which shows that $A_v$ is contained in the Klyachko arrangement. To prove the other inclusion, we note that $\mathcal{O}$ is a join-meet lattice with respect to the min and max operations. Thus by Proposition 2.2(3) the subspace arrangement $A_v$ is closed under finite intersections. It remains to show that for any $r \in \mathbb{Z}$ and any ray $\rho_i \in \Sigma(1)$ the Klyachko space $G^\rho_{\psi}(v)$ belongs to $A_v$. Note that we can assume $r$ is such that $G_r^\rho(\psi) \supseteq G^\rho_{r+1}(\psi)$. To prove the claim, we need to find a piecewise linear function $\phi \in \mathcal{O}$ such that $G^\rho_{\psi}(v) = F_\phi(v)$. That is, we would like to have:

\begin{equation}
G^\rho_{\psi}(v) = \{f \mid v(f)(\rho_j) \geq \phi(\rho_j), \forall \rho_j \in \Sigma(1)\}.
\end{equation}

Let $\Sigma'$ be a simplicial refinement of $\Sigma$ with $\Sigma'(1) = \Sigma(1)$. For (11) to hold, it suffices to take $\phi$ to be a piecewise linear function with respect to $\Sigma'$ such that $\phi(\rho_j) = r$ and $\phi(\rho_j)$ is sufficiently small for all $j \neq i$ so that $v(f)(\rho_j) \geq \phi(\rho_j)$ holds for all $f$. We can always find such $\phi$ since $\Sigma'$ is simplicial. This finishes the proof. □

Following [Zie] Section I.4 we recall the matroid associated to a subspace arrangement. Let $A$ be a subspace arrangement in $E$ such that $A$ is closed under intersection. Let $U_1, \ldots, U_m$ be subspaces in $A$ that are not direct sum of other subspaces in $A$. For each $U_i$ pick a basis $B_i = \{e_{i1}, \ldots, e_{ir_i}\}$, where $r_i = \dim(U_i)$. One says that the spanning set $B = \bigcup_{i=1}^m B_i$ is generic if the following is satisfied: let $B_0 \subset B$ be a subset. Take $e_{ij} \in B_i \setminus B_0$. Then $e_{ij}$ lies in span of $B_0$ only if $U_i$ lies in the span of $B_0$. The following is known (see [Zie] Theorem I.4.9 as well as [DRJS18] Proposition 3.1):

**Theorem 5.14.** Let $B$ be generic with respect to the arrangement $A$. Then the matroid structure of the set of vectors $B$ only depends on $A$ (i.e. is independent of the choice of bases $B_i$ for the $U_i$). We denote this representable matroid by $M(A)$. Moreover, we have the following:

1. The poset defined by $A$ is a meet-subsemilattice of the flats of $M(A)$.
2. Among matroids satisfying (1), the number of elements in the ground set is minimal.
3. Among matroids satisfying (1) and (2), the number of circuits is minimal.
Remark 5.15. We should point out that Theorem 4.14 above is stated in dual language compared to [Zie] Theorem I.4.9. That is, to obtain former from the latter one should replace every subspace with its orthogonal complement.

Let $E$ be a toric vector bundle on a toric variety $Y(\Sigma)$ with associated prevalued vector space $(E, v)$. We denote the matroid $\mathcal{M}(A_v)$ by $\mathcal{M}(E)$. This matroid is introduced and used in [DRJS18] to define the notion of parliament of polytopes of $E$. We point out that in [DRJS18], the subspace arrangement used is the Klyachko arrangement. Proposition 5.13 shows that it coincides with $A_v$.

Proposition 5.16. With notation as above, for any $\sigma \in \Sigma$, $B$ contains a linear adapted basis $B_\sigma$ of $(E, v|_\sigma)$.

Proof. In [DRJS18] Section 3 it is observed that the Klyachko spaces $G^\rho_i(v)$ for $\mathbb{Z}_{\geq} \rho_i = \eta_i\cap N, \eta_i \in \sigma(1)$, generate a distributive lattice under intersection, so there is a “compatible basis” $B_\sigma \subset B$. This means that $B_\sigma \cap G^\rho_i(v)$ is a basis for each Klyachko space. By Proposition 5.12 $B_\sigma \cap F_m(v|_\sigma)$ is also a basis for $m \in M$ and $\sigma \in \Sigma$, and by Proposition 5.2 $B_\sigma \cap G^\rho_i(v)$ is a basis for all $\rho \in \sigma \cap N$. Take $b \in B_\sigma$ and write it as a linear combination $\sum_{i=1}^n C_i b_i$ of members of a linear adapted basis $B'$ of $v|_\sigma$ with $\deg(b'_i) = \lambda_i$. It follows that $b \in \sum_{i=1}^n F_{\lambda_i}(v)$. The set $B_\lambda = B_\sigma \cap F_{\lambda_i}(v)$ is a basis, and $b \in \bigcup_{\lambda \geq \lambda_i} F_{\lambda_j}(v)$, so for some $j$, $b \in F_{\lambda_j}(v)$. A basis is adapted, it follows that each $b'_i \in F_{\lambda_j}(v)$ as well, so $\lambda_i \leq \lambda_j$ for each $i$.

As a consequence, $v(b) = \bigoplus_{i=1}^\ell \lambda_i = \lambda_j$. It follows that $B_\sigma$ is a linear adapted basis.

In Section 6 we show that a spanning set $B$ constructed as above for the dual $\mathcal{E}^*$ of a bundle $\mathcal{E}$ with $\mathcal{L}(\mathcal{E}) = (E, v)$ gives a Khovanskii basis of the valuation $s(v^*) : \text{Sym}(E^*) \to \mathcal{O}_\Sigma$.

5.5. Quasivaluations on graded algebras. We prove a technical theorem classifying $T_N$-equivariant degenerations of $k$-algebras using quasivaluations into $\hat{\mathcal{O}}_\Sigma$. In Section 6 we refine this result by describing the Khovanskii bases of these functions. Fix a $k$-algebra $A$, and a direct sum decomposition $A = \bigoplus_{i \in I} A_i$ (not necessarily a grading) into finite dimension $k$-vector spaces. A homogeneous $Y(\Sigma)$-degeneration of this information is defined to be a flat sheaf of algebras $\mathcal{A} \in \text{Alg}^M_{\mathcal{O}_\Sigma}$ which can be written $\mathcal{A} = \bigoplus_{i \in I} A_i$ where each $A_i$ is coherent, such that $E_A = \bigoplus_{i \in I} E_{A_i} = \bigoplus_{i \in I} A_i = A$. A quasivaluation $v : A \to \hat{\mathcal{O}}_\Sigma$ is said to be homogeneous if $(A, v) \cong \bigoplus_{i \in I} (A_i, v|_{A_i})$ in $\text{Vect}_\sigma$.

Theorem 5.17. Any homogeneous degeneration is eversive, the functor $\mathcal{L}$ takes a homogeneous degeneration $A$ to a homogeneous quasivaluation $v : A \to \hat{\mathcal{O}}_\Sigma$ such that each $v|_{A_i}$ has a linear adapted basis. In particular, $v$ takes values in $\mathcal{O}_\Sigma$, and all such quasivaluations arise this way.

Proof. A degeneration $A = \bigoplus_{i \in I} A_i$ is flat if and only if each $A_i$ is flat, and therefore locally projective, and therefore locally free. It follows that $A$ is eversive and $(A, v|_{A_i})$ has a linear adapted basis. Such a homogeneous quasivaluation $v : A \to \mathcal{O}_\Sigma$ likewise corresponds under Theorem 4.17 to a unique homogeneous degeneration.

6. Khovanskii bases

In this section we define the notion of Khovanskii basis for a valuation $v : A \to \hat{\mathcal{O}}_\Sigma$, and we prove Theorems 4 and 5. For a generating set $B \subset A$ with $B = \{b_1, \ldots, b_n\}$ we let $\pi : k[x] \to A$ be the surjection with $\pi(x_i) = b_i \in B$ and $I = \ker(\pi)$.

We let $N_B = \mathbb{Z}^B$ and $M_B = \text{Hom}(N_B, \mathbb{Z})$. For $w \in N_B$ there is a valuation $\bar{v}_w : k[x] \to \mathbb{Z}$ where $\bar{v}_w(x^a) = \langle w, a \rangle$. We let $v_w = \pi_*(\bar{v}_w)$. As in [KMa] Definition 3.1, we refer to $v_w$ as
the weight quasivaluation on $A$ associated to $w \in N_B$. We let $\text{gr}_w(A)$ denote the associated graded algebra of $v_w$ for $w \in N_B$. For any $w \in N_B$ there is a corresponding initial ideal $\text{in}_w(I) \subset k[x]$. By [KMa] Lemma 3.4 we have $\text{gr}_w(A) \cong k[x]/\text{in}_w(I)$, and $\pi(x) = B \subset A$ is a Khovanskii basis of $v_w$.

Let $\Sigma_B \subset N_B$ denote the Gröbner fan of $I$ (see e.g. [Stu96b] Proposition 2.4). Recall that if $w, w' \in N_B$ lie in the same face of $\Sigma_B$, then $\text{in}_w(I) = \text{in}_{w'}(I)$. For a maximal cone $\tau \in \Sigma$, let $B_\tau \subset A$ be the associated standard monomial basis, see [Stu96b] Proposition 1.1. We make use of the following proposition from [KMa].

**Proposition 6.1.** [KMa] Proposition 4.9 Let $v : A \to \mathbb{Z}$ have Khovanskii basis $B \subset A$, and let $w = (v(b_1), \ldots, v(b_n)) \in \Sigma_B$. Then $v = v_w$, and if $B_\tau \subset A$ is a standard monomial basis for any $\tau$ with $w \in \tau$, then $B_\tau$ is adapted to $v$. Furthermore, $v(b^\rho) = \langle w, \alpha \rangle$ for any $b^\rho \in B_\tau$.

**6.1. Khovanskii bases of algebras in Alg$_\Sigma$.**

**Definition 6.2.** A generating set $B \subset A$ is said to be a Khovanskii basis of $(A,v) \in \text{Alg}_\Sigma$ if it is a Khovanskii basis in the sense of Definition 2.7 of $v_\rho : A \to \mathbb{Z}$ for every $\rho \in \Sigma \cap N$.

For $(A,v) \in \text{Alg}_\Sigma$ and a subset $B \subset A$ we define $\Phi_B : \Sigma \cap N \to N_B$ by $\Phi_B(\rho) = (v_\rho(b) \mid b \in B)$ for $\rho \in \Sigma \cap N$.

**Proposition 6.3.** Suppose for each $\sigma \in \Sigma$ the restriction $(A,v|_\sigma)$ has a finite Khovanskii basis $B_\sigma$, and that the image $\Phi_B(\sigma \cap N) \subset N_B$ lies in a face of $\Sigma_B$, then $(A,v|_\sigma) \in \text{Alg}_\sigma$ has an adapted basis. Moreover, if $v(b_i)|_\sigma \in M_\sigma \subset O_\sigma$ for each $b_i \in B_\sigma$ then $(A,v) = \mathcal{L}(A)$ for a locally free family $A$ on $Y(\Sigma)$.

**Proof.** By Proposition 6.1 $v_\rho = v_{\Phi_B(\rho)}$. It follows that $B_\tau$ is an adapted basis of $v_\rho$ for any $\rho \in \sigma \cap N$, where $\tau \subset \Sigma_B$ is any maximal cone containing the image of $\sigma$. If $v(b_i)|_\sigma \in M_\sigma$ for $b_i \in B_\sigma$, then $v(b^\rho)|_\sigma \in M_\sigma$ for any $b^\rho \in B_\tau$ by Proposition 6.1. It follows that $B_\tau \subset A$ is a linear adapted basis for $(A,v|_\sigma) \in \text{Vect}_\sigma$.

By Theorem 5.17 the family $A$ in Corollary 6.3 is uniquely determined by $(A,v)$. If $(A,v) \in \text{Alg}_\Sigma$, and each function $v(b_i) \in O_\sigma$ is piecewise-linear, then we can always create a refinement $\hat{\Sigma}$ so that $v(b_i)|_\sigma \in M_\sigma$ for each $\hat{\sigma} \in \hat{\Sigma}$. By Corollary 6.3, we obtain a locally free family on the toric resolution $\pi : Y(\Sigma) \to Y(\Sigma)$. Now we show that finite Khovanskii bases appear when a sheaf of algebras $A$ is of finite type and $\Sigma$ is a finite fan.

**Proposition 6.4.** Let $\Sigma \subset N$ be a finite fan, and let $A$ be a quasi-coherent sheaf of algebras of finite type in $\text{Sh}^M_{Y(\Sigma)}$, then $\mathcal{L}(A) \in \text{Alg}_\Sigma$ has a finite Khovanskii basis.

**Proof.** Fix $\sigma \in \Sigma$, and consider $R = \Gamma(Y(\sigma),A) \in \text{Alg}_\Sigma^M$. There is a presentation $\hat{\Sigma} : S_\sigma[x] \to R$, let $\hat{b}_i = \hat{\pi}(x_i) \in F_{\lambda_i}(R)$, and $b_i = \phi_R(\hat{b}_i) \in \phi_R(F_{\lambda_i}(R)) \subset E_R$. By assumption, the images of the $\hat{b}_i$ in $R/m_i R$ generate as a $k$-algebra, and since $\phi_R : R/m_i R \to \text{gr}_R(E_R)$ is a surjection of algebras (Proposition 5.9), this is the case for their images in $\text{gr}_R(E_R)$ as well. By definition, the image $\phi_R(\hat{b}_i)$ is in the image of $\phi_R(F_{\lambda_i}(R))$ in $G^p_{\rho,\lambda_i}(R)/G^p_{\rho,\lambda_i}(R)$, so it coincides with the equivalence class of $b_i$ in $\text{gr}_R(E_R)$. The union of these generating sets over the cones of $\Sigma$ gives a finite Khovanskii basis for $\mathcal{L}(A)$.

For any collection of functions $\tilde{v} = \{\psi_1, \ldots, \psi_n\} \subset \hat{O}_\Sigma$ there is a canonical valuation $\tilde{v}_\Sigma : k[x] \to \hat{O}_\Sigma$ defined by letting the monomials $x^\alpha$ be an adapted basis and setting $\tilde{v}_\Sigma(x^\alpha) = \sum \alpha_i \psi_i$. The pushforward $\pi_* \tilde{v}_\Sigma : A \to \hat{O}_\Sigma$ is then a quasivaluation on $A$. We use this construction to give an alternative characterization of Khovanskii bases.
Proposition 6.5. Let \((A, \mathfrak{v}) \in \text{Alg}_{\Sigma}, \{b_1, \ldots, b_n\} = B \subset A\), and suppose \(\Phi_B(\Sigma \cap N) \subset \Sigma_B\), then \(B\) is a Khovanskii basis for \(\mathfrak{v}\) if and only if \(\mathfrak{v} = \pi_* \tilde{\mathfrak{v}}\), where \(\psi_i = \mathfrak{v}(b_i)\) for \(1 \leq i \leq n\).

Proof. By definition of pushforward, \([\pi_* \tilde{\mathfrak{v}}]_\rho\) for \(\rho \in \Sigma\) is the weight quasivaluation \(\mathfrak{v}_\rho\). If \(B \subset A\) is a Khovanskii basis of \(\mathfrak{v}\), it is a Khovanskii basis of \(\mathfrak{v}_\rho\) in the sense of Definition [2.7].

By Proposition 6.1, \(\mathfrak{v}_\rho = \mathfrak{v}_\omega\), where \(w_i = \mathfrak{v}(b_i)(\rho) = \psi_i(\rho)\). This means \(\mathfrak{v}_\rho = \mathfrak{v}_\psi[\rho]\) for all \(\rho \in \Sigma \cap N\). Conversely, if \(\mathfrak{v} = \pi_* \tilde{\mathfrak{v}}\), then \(\mathfrak{v}_\rho = \mathfrak{v}_\psi[\rho]\), so each \(\mathfrak{v}_\rho\) has Khovanskii basis \(B\).

The condition \(\Phi_B(\Sigma \cap N) \subset \Sigma_B\) is always satisfied if \(I\) is a positively graded homogeneous ideal.

6.2. Proof of Theorem 5. It can happen that distinct \(w, w' \in N_B\) give the same quasivaluation: \(\mathfrak{v}_w = \mathfrak{v}_{w'}\). To keep track of this data we recall the piecewise-linear map \(\iota: \Sigma_B \to \Sigma_B\) introduced in [KMa, Section 3.2], where \(\iota(w) = (\mathfrak{v}_w(b) \mid b \in B)\). By [KMa] Proposition 3.7, \(\iota^2 = \iota\) and \(\iota(w) = \iota(w')\) if and only if \(\mathfrak{v}_w = \mathfrak{v}_{w'}\). It is straightforward to show that the image \(\Sigma_B\) of \(\iota\) is a subfan of \(\Sigma_B\) and that \(\text{Trop}(I) \cap N_B \subset \Sigma_B\). Note that for any \(\rho \in \sigma \cap N\), \(\iota(\Phi_B(\rho)) = \Phi_B(\rho)\). We say a map \(\Phi: \Sigma \to \hat{\Sigma}_B\) is piecewise-linear if \(\Phi|_\sigma\) is a linear map with image in a face of \(\Sigma_B\) for all \(\sigma \in \Sigma\).

Theorem 6.6 (proof of Theorem 5). Let \(\Sigma\) be a finite fan, the following are equivalent pieces of information:

1. A flat, positively graded sheaf \(\mathcal{A}\) of algebras of finite type on \(Y(\Sigma)\).
2. A positively graded \((A, \mathfrak{v}) = \mathcal{L}(\mathcal{A})\) where \((A, \mathfrak{v}|_\sigma)\) has a finite linear Khovanskii basis \(B_\sigma \subset A\) for each \(\sigma \in \Sigma\).
3. A piecewise-linear map \(\Phi: \Sigma \to \hat{\Sigma}_B\), where \(B = \bigcup_{\sigma \in \Sigma} B_\sigma\).

Moreover, the fibers of \(\mathcal{A}\) are reduced and irreducible if and only if \(\mathfrak{v}\) is a valuation.

Proof. By Theorem 5.17 \(A\) and \(\mathcal{L}(\mathcal{A}) = (A, \mathfrak{v})\) determine one another, so it remains to be seen that \((A, \mathfrak{v})\) has a finite Khovanskii basis if and only if \(\mathcal{A}\) is of finite type. This is the case by Propositions 6.4 and 6.3. Furthermore, we have the map \(\Phi_B: \Sigma \to \hat{\Sigma}_B\). We have that \(\mathfrak{v}_\rho = \mathfrak{v}_{\Phi_B(\rho)}\), so \(\mathfrak{v}\) may be recovered from \(\Phi_B\). For a face \(\sigma \in \Sigma\) we consider the restriction \((A, \mathfrak{v}|_\sigma) = \mathcal{L}(\mathcal{R})\), where \(\mathcal{R}\) a finitely generated \(S_\sigma\)-algebra. Following the proofs of Proposition 6.4 and Proposition 5.9 for any \(\rho, \rho' \in \sigma|\) we have \(gr_{\rho}(A) \cong gr_{\rho'}(A) \cong R/m_R R\) as \(k[x]|\) algebras. Here \(k[x] \subset S_\sigma[x]\) presents \(A\) by the Khovanskii basis \(B_\sigma\), and \(S_\sigma[x]\) presents \(R\). It follows that \(\text{in}_{\Phi_B(\rho)}(I) = \text{in}_{\Phi_B(\rho')}(I)\). We conclude that \(\Phi(\rho)\) and \(\Phi(\rho')\) lie in the same face of \(\hat{\Sigma}\). It remains to be seen that any such piecewise-linear map produces a quasivaluation. Given \(\Phi: \Sigma \to \hat{\Sigma}_B\), we can define \(\mathfrak{v}_{\Phi, \rho}: A \to \mathcal{Z}\) to be \(\mathfrak{v}_{\Phi(\rho)}\). We obtain \(\mathfrak{v}_{\Phi} : A \to \hat{O}_\Sigma\) with finite Khovanskii basis \(\pi(x) = B\). For \(b \in B\), let \(e_b\) be the corresponding basis member of \(M_B\), then \(\mathfrak{v}_{\Phi}(b)|_\sigma = e_b \circ \Phi|_\sigma\), which is linear on \(\sigma\) by the piecewise-linearity of \(\Phi\).

Remark 6.7. As a consequence of Theorem 6.6 for any presentation \(\pi: k[x] \to A\) there is an associated canonical \((A, \mathfrak{v}) \in \text{Alg}_{\hat{\Sigma}_B}\).

Corollary 6.8. Quasivaluations \(\mathfrak{v}: A \to \hat{O}_\Sigma\) with finite Khovanskii basis \(B\) are classified by their tuple of values \(\mathfrak{v}(B) = (\mathfrak{v}(b) \mid b \in B) \in \hat{O}_\Sigma^B\). If the underlying family \(\mathcal{A}\) has reduced, irreducible fibers (equivalently if \(\mathfrak{v}\) is a valuation), then \(\mathfrak{v}(B) \in \text{Trop}_{\hat{O}_\Sigma}(I)\).

Proof. Let \(\mathfrak{v}, \mathfrak{v}'\) be quasivaluations with Khovanskii basis \(B\), and let \(\rho \in \Sigma \cap N\). If \(\mathfrak{v}(B) = \mathfrak{v}'(B)\) then \(\mathfrak{v}_\rho(b) = \mathfrak{v}'_\rho(b)\) for all \(b \in B\). By Proposition 6.1 we must have \(\mathfrak{v}_\rho = \mathfrak{v}'_\rho\) for all
\[ \rho \in \Sigma \cap R \], so \( \psi = \psi' \). Moreover, the fibers of \( A \) coincide with the associated graded algebras of the \( \alpha \), so \( \psi' = \psi' \). This shows that fibers are domains if and only if the \( \alpha \) are valuations, so we may use Proposition 2.3.

6.3. Proof of Theorem 4. For any \( T_\Sigma \)-vector bundle \( E \) on \( Y(\Sigma) \) there is a \( \mathbb{Z}_{\geq 0} \)-graded locally-free, \( T_\Sigma \)-sheaf \( \mathcal{O}_E = \text{Sym}(E^*) \). If \( (E, \psi) = (\mathcal{L}(E), \psi) \), then by Section 3.3 and Theorem 5.17 this is the degeneration corresponding to the algebra \( \text{Sym}(E^*, \psi^*) = (\text{Sym}(E^*), s(\psi^*)) \).

In this case, \( s(\psi) : \text{Sym}(E^*) \to \mathcal{O}_\Sigma \subset \mathcal{O}_\Sigma \) is a valuation.

Corollary 6.9 (Theorem 4). Toric vector bundles over \( Y(\Sigma) \) with general fiber \( E \) are classified by homogeneous valuations \( \psi : \text{Sym}(E^*) \to \mathcal{O}_\Sigma \) which have a finite Khovanskii basis \( B \subset E^* \) such that there is a linear adapted basis \( \mathcal{B}_\sigma \subset \mathcal{B} \) for \( \psi : \text{Sym}(E^*) \to \mathcal{O}_\Sigma \).

Proof. We show that (2) and (3) are equivalent in Theorem 4. Given a toric vector bundle \( E \) with \( (E^*, \psi^*) = (\mathcal{L}(E), \psi) \) (Example 5.7), we consider the corresponding generic set \( B \subset E^* \) as in Section 5.4. By Proposition 5.16 \( B \) contains a linearly adapted basis \( \mathcal{B}_\sigma \) for each \( \sigma \in \Sigma \); this induces a linear adapted basis in each \( (\text{Sym}^n(E^*), \psi^n) \) (Example 5.6). It follows that \( \mathcal{B}_\sigma \) is a Khovanskii basis for each \( \psi^*(\mathcal{B})_\sigma \) and \( B \) is a Khovanskii basis for \( \psi^*(B) \).

By construction, \( \mathcal{E} \) can be recovered from \( (E^*, \psi^*) \), so two distinct bundles yield distinct valuations on \( \text{Sym}(E^*) \). Conversely, if \( \psi : \text{Sym}(E^*) \to \mathcal{O}_\Sigma \) has Khovanskii basis \( B \) with linear adapted bases \( \mathcal{B}_\sigma \), then \( (E, \psi) \) classifies a toric vector bundle, where \( \psi \) is the dual of the restriction of \( \psi \) to \( E^* \). Now, \( \psi^*(\mathcal{B})_\sigma \) and \( \psi \) take the same values on a common Khovanskii basis \( B \), so they must coincide by Corollary 6.8.

Recall that a tropical basis \( \{ f_1, \ldots, f_m \} \subset I \) is a set of polynomials such that \( \text{Trop}(I) = \bigcap_{i=1}^m \text{Trop}(\langle f_i \rangle) \). We say that a set of polynomials is a tropical basis over \( \mathcal{O}_\Sigma \) if \( \text{Trop}_{\mathcal{O}_\Sigma}(I) = \bigcap_{i=1}^m \text{Trop}_{\mathcal{O}_\Sigma}(\langle f_i \rangle) \). The next proposition completes the proof of Theorem 4.

Proposition 6.10. Let \( I \subset k[x] \) be a linear ideal, then:

1. if \( \{ f_1, \ldots, f_m \} \subset I \) is a tropical basis, then it is a tropical basis over \( \mathcal{O}_\Sigma \),
2. for any point \( \psi \in \text{Trop}_{\mathcal{O}_\Sigma}(I) \) there is a fan \( \Sigma \subset \mathbb{N} \) and a valuation \( \psi : \text{Sym}(E^*) \to \mathcal{O}_\Sigma \) as in Theorem 4 such that \( \psi = (\psi(b_1), \ldots, \psi(b_n)) \).

Proof. For \( \hat{\psi} \in \text{Trop}_{\mathcal{O}_\Sigma}(I) \) consider \( \hat{\psi} : k[x] \to \mathcal{O}_\Sigma \) and let \( \psi = \psi_\Sigma \hat{\psi} \). The quasivaluation \( \psi_\rho \) for \( \rho \in N \) is then the weight quasivaluation \( \psi_\rho(\psi) \). For any \( \psi \in \text{Trop}(I) \cap N_B \), the initial ideal \( \text{in}_\Sigma(I) \) is prime and monomial-free. It follows that \( \psi \) is a valuation with Khovanskii basis \( \mathcal{B} \) by Proposition 6.5.

By construction, the restriction of \( \psi \) to \( E^* \) is computed on \( f \in E^* \) by taking max of expressions of the form \( \bigoplus_{i=1}^n \phi_i \) where \( \phi_i \in \hat{\phi} \). There are only a finite number of functions of this type, so we conclude that \( \psi \) takes a finite number of values on \( E^* \). By Corollary 5.4 there is a fan \( \Sigma \) such that \( \mathcal{B} \) contains a linear adapted basis for each restriction \( \psi_\sigma \), \( \sigma \in \Sigma \). Now suppose \( \hat{\psi} \in \text{Trop}_{\mathcal{O}_\Sigma}(\langle f_i \rangle) \), then \( \psi(\rho) \in \text{Trop}(I) \) for every \( \rho \in N \). It follows that \( \psi = \psi_\Sigma \hat{\psi} \) is a valuation, so that \( \psi(b_1), \ldots, \psi(b_n) \). □

Question 6.11. The condition that \( \text{in}_\Sigma(I) \) is prime for each \( w \in \text{Trop}(I) \cap N_B \) implies that the associated embedding \( E \subset k^m \) is well-poised (see [LM]). Proposition 6.10 suggests a way to phrase this condition for the \( \mathcal{O}_N \): for any point \( \psi \in \text{Trop}_{\mathcal{O}_\Sigma}(I) \) there is a valuation \( \psi : \text{Sym}(E^*) \to \mathcal{O}_N \) such that \( \psi = \psi_\Sigma \hat{\psi} \). This condition could be formulated for any semialgebra \( \mathcal{O} \) rich enough to have cokernels (pushforwards) in the category \( \text{Vect}_\mathcal{O} \). Are linear embeddings of vector spaces still well-poised for any such \( \mathcal{O} \)?

Remark 6.12. By Theorem 6.6 and Proposition 6.10 a vector bundle \( E \) on \( Y(\Sigma) \) is the same information as a piecewise-linear map from \( \Sigma \) to the tropical linear space \( \text{Trop}(I) \).
6.4. Parliament of polytopes from the prevaluation. We finish this section by showing that the parliament of polytopes defined by Di Rocco, Jabbusch, and Smith in [DRJS18] can be recovered from the point $v(B) \in \text{Trop}_{\mathcal{O}_N}(I)$. We also prove a criterion for global generation of a toric vector bundle in terms of its prevaluation. Recall that for $\phi \in \mathcal{O}_N$, we let $P_\phi$ to be the convex polytope defined by $P_\phi = \{ u \in M_Q \mid \langle u, x \rangle \geq \phi(x), \forall x \in N \}$.

**Proposition 6.13.** With notation as before, the parliament of polytopes of $E$ coincides with $\{-P_{-v(b)} \mid b \in B\}$.

**Proof.** We have shown in Section 5.4 that a generic set $B$ is a realization of the matroid $\mathcal{M}(E)$ defined in [DRJS18], so it remains to show that the $-P_{-v(b)}$ coincides with the parliament of polytopes of $E$ defined in loc. cit. By definition, the polytope in the parliament of $E$ associated to $b$ is:

$$\{u \in M_Q \mid \langle u, \rho_i \rangle \leq \max \{r \mid b \in G^0_r(v)\}, \forall \rho_i \in \Sigma(1)\},$$

where $G^0_r(v)$ is the $r$-th subspace in the Khovanskii filtration of $E$ corresponding to $\rho_i$. We recall that the quantity $\max \{r \mid b \in G^0_r(v)\}$ is in fact equal to $v(b)$. Hence $u$ lies in the set if and only if $\langle -u, \rho_i \rangle \geq -v(b)(\rho_i)$, for all the $\rho_i$. But since $v(b)$ is concave restricted to each cone $\sigma$ and the $\rho_i$ span $N_\mathcal{Q}$, this is equivalent to $\langle -u, x \rangle \geq -v(b)(x)$, for all $x \in N$. That is, $u \in -P_{-v(b)}$. \qed

**Proposition 6.14.** Let $E$ be a toric vector bundle with prevaluation $v : E \to \mathcal{O}_N$. If the image of $v$ consists of convex piecewise linear functions then $E$ is globally generated.

**Remark 6.15.** The prevalued vector space $(E, v)$ determines $E$ up to equivalence, that is up to pull-back under toric maps. Also note that the pull-back of a globally generated vector bundle is globally generated.

**Proof of Proposition 6.14.** Take a full dimensional cone $\sigma \in \Sigma$. We know that for each $b \in B_\sigma$, $v(B) |_{\sigma}$ is linear. Let $m_{\sigma,b} \in M$ be this linear function. Convexity of $v(b)$ implies that for all $x \in N$ we have $(m_{\sigma,b}, x) \leq v(b)(x)$. It follows that the linear function $m_{\sigma,b}$ lies in the polytope $-P_{-v(b)} = \{ u \in M_Q \mid \langle u, x \rangle \leq v(b)(x)\}$. The global generation of $E$ now follows from [DRJS18] Theorem 1.2. \qed

7. An example from cluster algebras

In this section we consider an $A$-type cluster variety $A$ with no frozen variables, we use [GHKK18] as a reference. Fix lattices, $N, M = \text{Hom}(N, \mathbb{Z})$ and equip $N$ with non-degenerate skew-symmetric bilinear form $\langle \cdot, \cdot \rangle$. Choose a sublattice $N^0 \subset N$ with $\{N^0, N\} \subset \mathbb{Z}$ which in turn defines $M \subset M^0$. We have maps: $P^* : N \to M^0$, and dually $P^{**} : N^0 \to M$, where $P^*(n) = \{n, \cdot \}$. For a seed $s = \{e_1, \ldots, e_n\} \subset N$, the chamber $\sigma_s \subset M_Q^* = M_Q$ is the dual cone of $Q_\subseteq 0 s = \sigma_s^0 \subset N \otimes \mathbb{Q}$. The monoid $N_s^- = \sigma_s^0 \cap N$ defines a polynomial ring $k[N^-_s]$ equipped with an action of $T_{N^0}$ by way of the map $P^{**}$. In particular, the $T_{N^0}$ character assigned to a monomial $z^n \in k[N^-_s]$ is $P^*(n) \in M^0$.

The canonical algebra $\text{can}(A)$ of $A$ comes with a distinguished $k$-basis $\Theta \subset \text{can}(A)$. We let $S = \text{can}(A_{prin})$ denote the canonical algebra of the cluster variety with principal coefficients. For each choice of seed $s$ there is a subalgebra $S_{s,-} \subset S$ which can be written as the direct sum $S_{s,-} = \bigoplus_{\theta \in \Theta, n \in N_s^-} k\theta z^n$ as a $k$-vector space. The seed $s$ also determines the assignment of the $g$-vector $g_s(\theta) \in M^0$ for each $\theta \in \Theta$. The algebra $S_{s,-} = M^0$-graded with isotypical components $F_m = \{ \langle \theta z^n \mid g_s(\theta) + P^*(n) = m \rangle \}$. The space $F_m$ can be canonically identified with the subspace $\{ \{ \theta \mid m - g_s(\theta) \in \sigma_s^0 \cap N\} \subset \text{can}(A)$, and multiplication in
$S_{a_\theta}$ is given by the rule induced on these subspaces from $can(A)$. We recognize $S_{a_\theta}$ as the Rees algebra of a $\mathcal{O}_{s_\theta}$-valuation on $can(A)$.

**Proposition 7.1.** The algebra $S_{a_\theta}$ is a free $k[N^-_s]$-module with generators given by the canonical basis $\theta \in \Theta$. Furthermore, $\mathcal{L}(S_{a_\theta}) = (can(A), v_s)$ where $v_s : can(A) \to \mathcal{O}_{s_\theta}$ has a linear adapted basis given by $\Theta$, such that $v_s(\theta) = g_s(\theta)$.

*Proof.* In [GHKK18] it is shown that the fibers of $S_{a_\theta}$ are all domains and that the fiber over the identity in $T_{N^-}$ is $can(A)$, so this is essentially Proposition 5.2.

Fix two seeds $s, s'$ which are related by a mutation $\mu : s \to s'$, so that $s' = \{e_1, \ldots, e_{n-1}, e'_n\}$. The intersection $\tau(\mu) = \sigma_s \cap \sigma'_{s'}$ consists of those $m \in M^\circ$ such that $\langle m, e_i \rangle \leq 0$ for $1 \leq i \leq n-1$ and $\langle m, e_n \rangle = \langle m, e'_n \rangle = 0$. By Proposition 7.1, the canonical basis $\Theta \subset can(A)$ is adapted to both $v_s$ and $v'_{s'}$.

**Proposition 7.2.** Let $s, s'$ be as above, then $v_s |_{\tau(\mu)} = v_{s'} |_{\tau(\mu)}$.

*Proof.* The choice $s$ labels $\theta \in \Theta$ with $g_s(\theta) \in M^\circ$. This defines the linear function $v_s(\theta) = (-, g_s(\theta)) : \sigma_s \to A^\circ$. Similarly, $v'_{s'}(\theta) = \mu |_{\sigma_{s'}} = g_{s'}(\theta)$. The mutation rules imply that $g_s(\theta) - g_{s'}(\theta) \in \mathbb{Q}P^\circ(e_n)$.

We let $\Delta$ denote the union of the $\sigma_s$, in [GHKK18] it is shown that $\Delta$ is the support of a fan in $M_Q$.

**Corollary 7.3.** There is a valuation $v : can(A) \to \mathcal{O}_\Delta$ with a piecewise-linear adapted basis given by the canonical basis $\Theta \subset can(A)$ such that $v(\theta)|_{\sigma_\theta} = g_s(\theta)$.

For each $\theta \in \Theta$ we obtain a piecewise-linear function on $\Delta$, which in turn corresponds to a $T_{N^-}$-equivariant line bundle $\mathcal{O}(\theta)$ on the toric variety $Y(\Delta)$. The direct sum of these line bundles $\bigoplus_{\theta \in \Theta} \mathcal{O}(\theta)$ is an $\mathcal{O}_Y(\Delta)$ algebra, and defines a locally free family of algebras over $Y(\Delta)$. This family integrates generic fiber $can(A)$ together with the various cluster toric degenerations of $can(A)$, which appear over the fixed points of $Y(\Delta)$.

**Theorem 7.4.** There is a piecewise-linear map $\Phi : \Delta \to A^n$.

We briefly indicate the connection with the tropicalization $\theta^t$ of a canonical basis member as in [GHKK18]. For a choice of seed $s$ we get a chart $T_{N,s} \subset A$, and a chart $T_{N,M^+,s} \subset A_{prin}$. Any $\theta$ is a cluster monomial, so it can be written as a Laurent polynomial on $T_{N,s}$ and extended to a Laurent polynomial $\tilde{\theta}$ on $T_{N,M^+,s} \subset A_{prin}$. From $\tilde{\theta}$, we know that $g_s(\tilde{\theta})$ is the exponent of the $T_{N,s}$ monomial in $\tilde{\theta}$ which survives when the $T_{N,s}$ coordinates of $\theta$ are evaluated at the origin of the affine space Spec$(k[N^-_s])$. All monomials in the lift $\tilde{\theta}$ have the same $T_{N,s}$ character, so it follows that the other monomials in $\tilde{\theta}$ (as a $T_{N,s}$ regular function) are of the form $g_s(\tilde{\theta}) - u$ for $u \in \sigma_\theta \cap N$. As a consequence, for any $\rho \in N \cap \Delta$ such that $P^\circ(\rho) \in \sigma_s$, the inner product $\langle \rho, - \rangle$ takes its minimum among these monomial exponents at $g_s(\tilde{\theta})$, so it is equal to $v(\theta)|_{\sigma_s}(P^\circ(\rho))$. This is the value of $\theta^t$ at $\rho$.

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