A ternary diophantine inequality by primes with one of the form \( p = x^2 + y^2 + 1 \)

S. I. Dimitrov

Abstract

In this paper we solve the ternary Piatetski-Shapiro inequality with prime numbers of a special form. More precisely we show that, for any fixed \( 1 < c < \frac{427}{400} \), every sufficiently large positive number \( N \) and a small constant \( \varepsilon > 0 \), the diophantine inequality

\[
|p_1^c + p_2^c + p_3^c - N| < \varepsilon
\]

has a solution in prime numbers \( p_1, p_2, p_3 \), such that \( p_1 = x^2 + y^2 + 1 \). For this purpose we establish a new Bombieri–Vinogradov type result for exponential sums over primes.

Keywords: Diophantine inequality · Exponential sum · Bombieri–Vinogradov type result · Primes.

2020 Math. Subject Classification: 11D75 · 11L07 · 11L20 · 11P32

1 Introduction and statement of the result

In 1960 Linnik \cite{15} showed that there exist infinitely many prime numbers of the form \( p = x^2 + y^2 + 1 \), where \( x \) and \( y \) are integers. More precisely he proved the asymptotic formula

\[
\sum_{p \leq X} r(p-1) = \pi \prod_{p > 2} \left(1 + \frac{\chi_4(p)}{p(p-1)}\right) \frac{X}{\log X} + O\left(\frac{X (\log \log X)^7}{(\log X)^{1+\theta_0}}\right),
\]

where \( r(k) \) is the number of solutions of the equation \( k = x^2 + y^2 \) in integers, \( \chi_4(k) \) is the non-principal character modulo 4 and

\[
\theta_0 = \frac{1}{2} - \frac{1}{4} \varepsilon \log 2 = 0.0289... \tag{1}
\]

In 1992 Tolev \cite{20} proved that for any fixed \( 1 < c < \frac{\text{15}}{\text{14}} \), for every sufficiently large positive number \( N \) and a small constant \( \varepsilon > 0 \), the diophantine inequality

\[
|p_1^c + p_2^c + p_3^c - N| < \varepsilon
\]
has a solution in prime numbers $p_1, p_2, p_3$.

Subsequently the result of Tolev was improved several authors [1], [2], [3], [4], [5], [6], [13], [14]. The best result up to now belongs to Baker [1] with $1 < c < \frac{6}{5}$.

Motivated by these results in this paper we solve inequality (2) with prime numbers of a special type. More precisely we shall prove solvability of (2) with Linnik primes. In order to achieve our goal we establish a new Bombieri – Vinogradov type result for exponential sums over primes.

Recall that Siegel-Walfisz and Bombieri–Vinogradov theorems are extremely important results in analytic number theory and have various applications.

Siegel-Walfisz theorem is a refinement both of the prime number theorem and of Dirichlet’s theorem on primes in arithmetic progressions. It states that for any fixed $A > 0$ there exists a positive constant $c$ depending only on $A$ such that

$$\sum_{\substack{p \leq x \atop p \equiv a \pmod{d}}} \log p = \frac{x}{\varphi(d)} + O\left(\frac{x}{e^{\sqrt{\log x}}}\right),$$

whenever $x \geq 2$, $(a, d) = 1$, $d \leq (\log x)^A$ and $\varphi(n)$ is Euler’s function.

The celebrated Bombieri – Vinogradov theorem concerns the distribution of primes in arithmetic progressions, averaged over a range of moduli and states the following. Let $A > 0$ be fixed. Then

$$\sum_{d \leq \sqrt{X}/(\log X)^{A+5}} \max_{\chi \leq X} \max_{(a, d) = 1} \left| \sum_{\substack{p \leq y \atop p \equiv a \pmod{d}}} \log p - \frac{y}{\varphi(d)} \right| \ll \frac{X}{\log^A X}.$$

In 2017 Tolev [21] proved a Siegel-Walfisz type result for exponential sums over primes. It states the following. Let $\delta, \xi$ and $\mu$ be positive real numbers depending on $c > 1$, such that

$$\xi + 3\delta < \frac{12}{25}, \quad \mu < 1.$$

Let $D = X^\delta$ and $\lambda(d)$ are real numbers satisfying

$$|\lambda(d)| \leq 1, \quad \lambda(d) = 0 \quad \text{if} \quad 2|d \quad \text{or} \quad \mu(d) = 0,$$

where $\mu(d)$ is Möbius’ function. If

$$L(t, X) = \sum_{d \leq D} \lambda(d) \sum_{\substack{\mu X < p \leq X \atop p + 2 \equiv 0 \pmod{d}}} e(tp^\nu) \log p,$$
then for $|t| < X^{\frac{1}{4} - c}$ the asymptotic formula

$$L(t, X) = \left( \int_{X}^{X} e(ty^c) \, dy \right) \sum_{d \leq D} \frac{\lambda(d)}{\varphi(d)} + O\left( \frac{X}{\log^A X} \right),$$

holds. Here $A > 0$ is an arbitrary large constant.

Motivated by these investigations in this paper we establish a new Bombieri–Vinogradov type result for exponential sums over primes. More precisely we establish the following upper bound. Let $1 < c < 3$, $c \neq 2$, $0 < \mu < 1$ and $A > 0$ be fixed. Then for $|t| < X^{\frac{1}{4} - c}$ the inequality

$$\sum_{d \leq \sqrt{X/(\log X)}} \max_{y \leq X} \max_{(a, d) = 1} \left| \sum_{\mu y \leq p \leq y \atop p \equiv a \pmod{d}} e(tp^c) \log p - \frac{1}{\varphi(d)} \int_{\mu y}^{y} e(tx^c) \, dx \right| \ll \frac{X}{\log^A X} \quad (3)$$

holds.

Using (3) as a main weapon we are able to attack the following theorem.

**Theorem 1.** Let $1 < c < \frac{427}{300}$. For every sufficiently large positive number $N$, the diophantine inequality

$$|p_1^c + p_2^c + p_3^c - N| < \frac{(\log \log N)^6}{(\log N)^{\theta_0}}$$

has a solution in prime numbers $p_1, p_2, p_3$, such that $p_1 = x^2 + y^2 + 1$. Here $\theta_0$ is defined by (1).

In addition we have the following tasks for the future.

**Conjecture 1.** Let $\varepsilon > 0$ be a small constant. There exists $c_0 > 1$ such that for any fixed $1 < c < c_0$, and every sufficiently large positive number $N$, the diophantine inequality

$$|p_1^c + p_2^c + p_3^c - N| < \varepsilon$$

has a solution in prime numbers $p_1, p_2, p_3$, such that $p_1 = x_1^2 + y_1^2 + 1$, $p_2 = x_2^2 + y_2^2 + 1$, $p_3 = x_3^2 + y_3^2 + 1$.

**Conjecture 2.** Let $\varepsilon > 0$ be a small constant. There exists $c_0 > 1$ such that for any fixed $1 < c < c_0$, and every sufficiently large positive number $N$, the diophantine inequality

$$|p_1^c + p_2^c - N| < \varepsilon$$

has a solution in prime numbers $p_1, p_2$, such that $p_1 = x_1^2 + y_1^2 + 1$, $p_2 = x_2^2 + y_2^2 + 1$.

Conjecture 2 is analogous to the binary Goldbach problem and probably quite difficult.
2  Notations

Assume that \( N \) is a sufficiently large positive number. The letter \( p \) with or without subscript will always denote prime numbers. The notation \( m \sim M \) means that \( m \) runs through the interval \((M/2, M] \). Moreover \( e(t) = \exp(2\pi it) \). We denote by \((m, n)\) the greatest common divisor of \( m \) and \( n \). The letter \( \eta \) denotes an arbitrary small positive number, not the same in all appearances. As usual \( \varphi(n) \) is Euler's function, \( \mu(n) \) is Möbius' function, \( \tau(n) \) denotes the number of positive divisors of \( n \) and \( \Lambda(n) \) is von Mangoldt's function. We shall use the convention that a congruence, \( m \equiv n \pmod{d} \) will be written as \( m \equiv n \pmod{d} \). The letter \( \chi \) denotes a Dirichlet character to a given modulus. The sums \( \sum_{\chi(d)} \) and \( \sum_{\chi(d)}^* \) denotes respectively summation over all characters and all primitive characters modulo \( d \). Throughout this paper unless something else is said, we suppose that \( \mu, c \) be fixed with \( 0 < \mu < 1 \) and \( 1 < c < \frac{427}{400} \). Denote

\[
X = \left( \frac{N}{2} \right)^{\frac{1}{c}} ;
\]

\[
D = \frac{X^{\frac{1}{2}}}{(\log X)^{\frac{3\cdot c}{4}}}, \quad A > 3 ;
\]

\[
\Delta = X^{\frac{1}{4} - c} ;
\]

\[
\varepsilon = \frac{(\log \log X)^6}{(\log X)^{6_0}} ;
\]

\[
H = \frac{\log^2 X}{\varepsilon} ;
\]

\[
S_{l,d,j}(t) = \sum_{\substack{\mu \in J \\ p \equiv l \pmod{d}}} e(tp^c) \log p ;
\]

\[
S(t) = S_{1,1,(X/2,X]}(t) ;
\]

\[
I_{J}(t) = \int_{J} e(ty^c) \, dy ;
\]

\[
I(t) = I_{(X/2,X]}(t) ;
\]

\[
\Psi(y, \chi, t) = \sum_{\mu y < n \leq y} \Lambda(n)\chi(n)e(tn^c) ;
\]

\[
E(y, t, d, a) = \sum_{\mu y < n \leq y \atop n \equiv a \pmod{d}} \Lambda(n)e(tn^c) - \frac{1}{\varphi(d)} \int_{\mu y}^{y} e(tx^c) \, dx .
\]
3 Preliminary lemmas

Lemma 1. Let \( a, \delta \in \mathbb{R} \), \( 0 < \delta < a/4 \) and \( k \in \mathbb{N} \). There exists a function \( \theta(y) \) which is \( k \) times continuously differentiable and such that

\[
\begin{align*}
\theta(y) &= 1 & \text{for } |y| &\leq a - \delta ; \\
0 < \theta(y) &< 1 & \text{for } a - \delta &< |y| < a + \delta ; \\
\theta(y) &= 0 & \text{for } |y| &\geq a + \delta .
\end{align*}
\]

and its Fourier transform

\[
\Theta(x) = \int_{-\infty}^{\infty} \theta(y)e(-xy)dy
\]
satisfies the inequality

\[
|\Theta(x)| \leq \min \left( 2a, \frac{1}{\pi |x|}, \frac{1}{\pi |x|} \left( \frac{k}{2 \pi |x| \delta} \right)^k \right) .
\]

Proof. See ([16]).

Throughout this paper we denote by \( \theta(y) \) the function from Lemma 1 with parameters \( a = \frac{9 \varepsilon}{10}, \delta = \frac{\varepsilon}{10}, k = \lfloor \log X \rfloor \) and by \( \Theta(x) \) the Fourier transform of \( \theta(y) \).

Lemma 2. Let \( 1 < c < 3 \), \( c \neq 2 \) and \( |t| \leq \Delta \). Then the asymptotic formula

\[
\sum_{\mu X < p \leq X} e(tp^c) \log p = \int_{\mu X}^{X} e(ty^c) dy + O \left( \frac{X}{e(\log X)^{1/5}} \right)
\]

holds.

Proof. See ([20], Lemma 14).

Lemma 3. Let \( \delta, \xi \) and \( \mu \) be positive real numbers depending on \( c > 1 \), such that

\[
\xi + 3\delta < \frac{12}{25}, \quad \mu < 1 .
\]

Let \( D = X^\delta \) and \( B > 0 \) is an arbitrarily large constant. Then for \( |t| < X^{\xi-c} \) the upper bound

\[
\sum_{1 < d \leq D} \frac{1}{\varphi(d)} \sum_{\chi(d)} \max_{y \leq X} |\Psi(y, \chi, t)| \ll \frac{X}{\log^B X}
\]

holds. Here \( \Psi(y, \chi, t) \) is denoted by ([13]).
Proof. See ([21], Lemma 10).

**Lemma 4.** (Pólya – Vinogradov inequality) Suppose that $M, N$ are positive integers and $\chi$ is a non-principal character modulo $q$. Then

$$\left| \sum_{M < n \leq M+N} \chi(n) \right| \leq 6\sqrt{q}\log q.$$  

Proof. See ([12], Theorem 12.5)

**Lemma 5.** We have

$$\int_{\mu_X} y^{\beta-1+i\gamma} e(ty^c) \, dy \ll \begin{cases} X^\beta \sqrt{|t| X^c} & \text{for } |\gamma| < 4\pi c|t| X^c, \\ X^\beta |\gamma| & \text{for } |\gamma| \geq 4\pi c|t| X^c. \end{cases}$$

Proof. See ([21], Lemma 10).

**Lemma 6.** (Large Sieve) For any complex numbers $a_n$ and positive integers $M, N, Q$ we have

$$\sum_{q \leq Q} \frac{q}{\varphi(q)} \sum_{\chi(q)}^* \left| \sum_{n=M+1}^{M+N} a_n \chi(n) \right|^2 \ll (N + Q^2) \sum_{n=M+1}^{M+N} |a_n|^2.$$  

Proof. See ([12], Theorem 7.13).

**Lemma 7.** For the sum denoted by (10) and the integral denoted by (12) we have

(i) $$\int_{-\Delta}^{\Delta} |S(t)|^2 \, dt \ll X^{2-c} \log^3 X,$$

(ii) $$\int_{-\Delta}^{\Delta} |I(t)|^2 \, dt \ll X^{2-c} \log X,$$

(iii) $$\int_{n+1}^{n} |S(t)|^2 \, dt \ll X \log X.$$  

Proof. It follows from the arguments used in ([20], Lemma 7).

**Lemma 8.** For the sum denoted by (9) uniformly for $l$ and $J$ we have

$$\int_{-\Delta}^{\Delta} |S_{l,d,J}(t)|^2 \, dt \ll \frac{X^{2-c} \log^3 X}{d^2}.$$  

6
Proof. It follows by the arguments used in ([7], Lemma 6 (i)). □

Lemma 9. Assume that $F(x), G(x)$ are real functions defined on $[a, b]$, $|G(x)| \leq H$ for $a \leq x \leq b$ and $G(x)/F'(x)$ is a monotonic function. Set

$$I = \int_a^b G(x)e(F(x))dx.$$ 

If $F'(x) \geq h > 0$ for all $x \in [a, b]$ or if $F'(x) \leq -h < 0$ for all $x \in [a, b]$ then

$$|I| \ll H/h.$$ 

If $F''(x) \geq h > 0$ for all $x \in [a, b]$ or if $F''(x) \leq -h < 0$ for all $x \in [a, b]$ then

$$|I| \ll H/\sqrt{h}.$$ 

Proof. See ([19], p. 71). □

Lemma 10. For any complex numbers $a(n)$ we have

$$\left| \sum_{a<n \leq b} a(n) \right|^2 \leq \left( 1 + \frac{b-a}{Q} \right) \sum_{|q| \leq Q} \left( 1 - \frac{|q|}{Q} \right) \sum_{a<n, n+q \leq b} a(n+q)a(n),$$

where $Q \geq 1$.

Proof. See ([12], Lemma 8.17). □

Lemma 11. Let $3 < U < V < Z < X$ and suppose that $Z - \frac{1}{2} \in \mathbb{N}$, $X \gg Z^2U$, $Z \gg U^2$, $V^3 \gg X$. Assume further that $F(n)$ is a complex valued function such that $|F(n)| \leq 1$. Then the sum

$$\sum_{n \sim X} \Lambda(n)F(n)$$

can be decomposed into $O\left( \log^{10} X \right)$ sums, each of which is either of Type I

$$\sum_{m \sim M} a(m) \sum_{l \sim L} F(ml),$$

where

$$L \gg Z, \quad LM \approx X, \quad |a(m)| \ll m^n,$$

or of Type II

$$\sum_{m \sim M} a(m) \sum_{l \sim L} b(l)F(ml),$$

where

$$U \ll L \ll V, \quad LM \approx X, \quad |a(m)| \ll m^n, \quad |b(l)| \ll v^n.$$

7
Proof. See ([10], Lemma 3).

**Lemma 12.** Let \(|f(m)(u)| \asymp XY^{1-m}\) for \(1 \leq X < u < X_0 \leq 2X\) and \(m \geq 1\). Then

\[
\left| \sum_{X < n \leq X_0} e(f(n)) \right| \ll Y^{\varkappa}X^\lambda + Y^{-1},
\]

where \((\varkappa, \lambda)\) is any exponent pair.

**Proof.** See ([9], Ch. 3).

**Lemma 13.** Let \(\theta, \lambda\) be real numbers such that

\[
\theta(\theta - 1)(\theta - 2)\lambda(\lambda - 1)(\theta + \lambda - 2)(\theta + \lambda - 3)(\theta + 2\lambda - 3)(2\theta + \lambda - 4) \neq 0.
\]

Set

\[
\Sigma_I = \sum_{m \sim M} a(m) \sum_{l \in I_m} e(Bm^\lambda l^\theta),
\]

where

\[
B > 0, \quad M \geq 1, \quad L \geq 1, \quad |a(m)| \leq 1, \quad I_m \subset (L/2, L).
\]

Let

\[
F = BM^\lambda L^\theta.
\]

Then

\[
\Sigma_I \ll \left( F^{\frac{3}{16}}M^{\frac{11}{80}}L^{\frac{29}{80}} + F^{\frac{1}{5}}M^{\frac{3}{10}}L^{\frac{11}{10}} + F^{\frac{1}{7}}M^{\frac{3}{10}}L^{\frac{11}{10}} + M^{\frac{3}{2}}L + ML^{\frac{3}{2}} + F^{-1}ML \right) (ML)^{\eta}.
\]

**Proof.** See ([2], Theorem 2).

**Lemma 14.** Let \(\alpha, \beta\) be real numbers such that

\[
\alpha\beta(\alpha - 1)(\beta - 1)(\alpha - 2)(\beta - 2) \neq 0.
\]

Set

\[
\Sigma_{II} = \sum_{m \sim M} a(m) \sum_{l \sim L} b(l)e \left( F \frac{m^\alpha l^\beta}{M^\alpha L^\beta} \right),
\]

where

\[
F > 0, \quad M \geq 1, \quad L \geq 1, \quad |a(m)| \leq 1, \quad |b(l)| \leq 1.
\]
Then
\[ \Sigma_I(FML)^{-\eta} \ll (F^4M^{31}L^{34})^{\frac{1}{32}} + (F^6M^{53}L^{51})^{\frac{1}{66}} + (F^6M^{46}L^{41})^{\frac{1}{66}} + (F^2M^8L^{29})^{\frac{1}{66}} + (F^3M^{43}L^{32})^{\frac{1}{66}} + (FM^9L^{6})^{\frac{1}{10}} + (F^2M^7L^{6})^{\frac{1}{10}} + (FM^6L^{6})^{\frac{1}{10}} + ML^{\frac{1}{2}} + F^{-\frac{1}{2}}ML. \]

Proof. See ([17], Theorem 9).

The next two lemmas are due to C. Hooley.

**Lemma 15.** For any constant \( \omega > 0 \) we have
\[ \sum_{p \leq X} \left| \sum_{d | p-1 \atop \sqrt{X}(\log X)^{-\omega} < d < \sqrt{X}(\log X)^{\omega}} \chi_4(d) \right|^2 \ll \frac{X(\log \log X)^{7}}{\log X}, \]
where the constant in Vinogradov’s symbol depends on \( \omega > 0 \).

**Lemma 16.** Suppose that \( \omega > 0 \) is a constant and let \( F_\omega(X) \) be the number of primes \( p \leq X \) such that \( p - 1 \) has a divisor in the interval \((\sqrt{X}(\log X)^{-\omega}, \sqrt{X}(\log X)^{\omega})\). Then
\[ F_\omega(X) \ll \frac{X(\log \log X)^{3}}{(\log X)^{1+2\theta_0}}, \]
where \( \theta_0 \) is defined by (11) and the constant in Vinogradov’s symbol depends only on \( \omega > 0 \).

The proofs of very similar results are available in ([13], Ch.5).

**Lemma 17.** We have
\[ \int_{-\infty}^{\infty} I^3(t)\Theta(t)e(-Nt) \, dt \gg \varepsilon X^{3-c}. \]

Proof. See ([20], Lemma 6).

### 4 Outline of the proof

Consider the sum
\[ \Gamma(X) = \sum_{X/2 < p_1, p_2, p_3 \leq X \atop |p_1^2 + p_2^2 + p_3^2 - N| < \varepsilon} r(p_1 - 1) \log p_1 \log p_2 \log p_3. \]
Obviously
\[ \Gamma(X) \geq \Gamma_0(X), \] (16)
where
\[ \Gamma_0(X) = \sum_{X/2 < p_1, p_2, p_3 \leq X} r(p_1 - 1) \theta(p_1^c + p_2^c + p_3^c - N) \log p_1 \log p_2 \log p_3. \] (17)

From (17) and well-known identity
\[ r(n) = 4 \sum_{d|n} \chi_4(d) \]
we obtain
\[ \Gamma_0(X) = 4(\Gamma_1(X) + \Gamma_2(X) + \Gamma_3(X)), \] (18)
where
\[ \Gamma_1(X) = \sum_{X/2 < p_1, p_2, p_3 \leq X} \left( \sum_{d|p_1 - 1 \atop d \leq D} \chi_4(d) \right) \theta(p_1^c + p_2^c + p_3^c - N) \log p_1 \log p_2 \log p_3, \] (19)
\[ \Gamma_2(X) = \sum_{X/2 < p_1, p_2, p_3 \leq X} \left( \sum_{d|p_1 - 1 \atop d \leq X/D} \chi_4(d) \right) \theta(p_1^c + p_2^c + p_3^c - N) \log p_1 \log p_2 \log p_3, \] (20)
\[ \Gamma_3(X) = \sum_{X/2 < p_1, p_2, p_3 \leq X} \left( \sum_{d|p_1 - 1 \atop d \geq X/D} \chi_4(d) \right) \theta(p_1^c + p_2^c + p_3^c - N) \log p_1 \log p_2 \log p_3. \] (21)

In order to estimate \( \Gamma_1(X) \) and \( \Gamma_3(X) \) we need to consider the sum
\[ I_{l,d,J}(X) = \sum_{X/2 < p_1, p_3 \leq X \atop p_1 \equiv (d) \atop p_1 \in J} \theta(p_1^c + p_2^c + p_3^c - N) \log p_1 \log p_2 \log p_3, \] (22)
where \( l \) and \( d \) are coprime natural numbers, and \( J \subset (X/2, X] \) is an interval. If \( J = (X/2, X] \) then we write for simplicity \( I_{l,d}(X) \).

Using the inverse Fourier transform for the function \( \theta(y) \) we deduce
\[ I_{l,d,J}(X) = \sum_{X/2 < p_2, p_3 \leq X \atop p_1 \equiv (d) \atop p_1 \in J} \log p_1 \log p_2 \log p_3 \int_{-\infty}^{\infty} \Theta(t)e((p_1^c + p_2^c + p_3^c - N)t) \, dt \]
\[ = \int_{-\infty}^{\infty} \Theta(t) S^2(t) S_{l,d,J}(t)e(-Nt) \, dt. \]
We decompose $I_{l,d,J}(X)$ over major, minor and trivial arcs as follows

$$I_{l,d,J}(X) = I_{l,d,J}^{(1)}(X) + I_{l,d,J}^{(2)}(X) + I_{l,d,J}^{(3)}(X),$$

(23)

where

$$I_{l,d,J}^{(1)}(X) = \int_{-\Delta}^{\Delta} \Theta(t)S^2(t)S_{l,d,J}(t)e(-Nt) \, dt,$$

(24)

$$I_{l,d,J}^{(2)}(X) = \int_{\Delta \leq |t| \leq H} \Theta(t)S^2(t)S_{l,d,J}(t)e(-Nt) \, dt,$$

(25)

$$I_{l,d,J}^{(3)}(X) = \int_{|t| > H} \Theta(t)S^2(t)S_{l,d,J}(t)e(-Nt) \, dt.$$

(26)

We shall estimate $I_{l,d,J}^{(1)}(X)$, $I_{l,d,J}^{(3)}(X)$, $\Gamma_3(X)$, $\Gamma_2(X)$ and $\Gamma_1(X)$, respectively, in the sections 5, 6, 7, 8 and 9. In section 10 we shall finalize the proof of Theorem 1.

5 Asymptotic formula for $I_{l,d,J}^{(1)}(X)$

A key point in the proof of our theorem is the following Bombieri – Vinogradov type result for exponential sums over primes.

Lemma 18. Let $1 < c < 3$, $c \neq 2$, $|t| \leq \Delta$ and $A > 0$ be fixed. Then the inequality

$$\sum_{d \leq \sqrt{X}/(\log X)^{\frac{1}{6}+\frac{1}{34}}} \max_{y \leq X} \max_{(a,d)=1} \left| \sum_{\mu_y \leq p \leq y, \, p \equiv a \, (d)} e(tp^c) \log p - \frac{1}{\varphi(d)} \int_{\mu_y}^{y} e(tx^c) \, dx \right| \ll \frac{X}{\log^4 X}$$

holds.

Proof. In order to prove our lemma we will use the formula

$$\sum_{\mu_y < p \leq y, \, p \equiv a \, (d)} e(tp^c) \log p = \sum_{\mu_y < n \leq y, \, n \equiv a \, (d)} \Lambda(n)e(tn^c) + O \left( \frac{y^{\frac{1}{2}+\epsilon}}{d} \right).$$

(27)

Define

$$\delta(\chi) = \begin{cases} 1 & \text{if } \chi \text{ is principal}, \\ 0 & \text{otherwise}. \end{cases}$$

(28)
By the orthogonality of characters we have
\[
\sum_{\mu y < n \leq y} \Lambda(n)e(tn^c) - \frac{1}{\varphi(d)} \int_{\mu y}^y e(tx^c) \, dx
\]
\[
= \sum_{\mu y < n \leq y} \Lambda(n)e(tn^c) \frac{1}{\varphi(d)} \sum_{\chi(d)} \chi(n) \overline{\chi}(a) - \frac{1}{\varphi(d)} \int_{\mu y}^y e(tx^c) \, dx
\]
\[
= \frac{1}{\varphi(d)} \sum_{\chi(d)} \left( \overline{\chi}(a) \sum_{\mu y < n \leq y} \Lambda(n) \chi(n)e(tn^c) - \delta(\chi) \int_{\mu y}^y e(tx^c) \, dx \right)
\]
and therefore
\[
\left\| \sum_{\mu y < n \leq y} \Lambda(n) - \frac{1}{\varphi(d)} \int_{\mu y}^y e(tx^c) \, dx \right\|_{L^\infty}
\]
\[
\leq \frac{1}{\varphi(d)} \sum_{\chi(d)} \left\| \Psi(y, \chi, t) - \delta(\chi) \int_{\mu y}^y e(tx^c) \, dx \right\|,
\]
where \( \Psi(y, \chi, t) \) is defined by (13). Denote
\[
\Sigma = \sum_{d \leq \sqrt{X}/(\log X)^{6A+34}} \max_{y \leq X} \max_{(a, d)=1} \left| E(y, t, d, a) \right|.
\]
From (14), (28), (29) and (30) we obtain
\[
\Sigma \leq \Sigma' + \Sigma'' ,
\]
where
\[
\Sigma' = \sum_{d \leq \sqrt{X}/(\log X)^{6A+34}} \frac{1}{\varphi(d)} \max_{y \leq X} \left| \sum_{\mu y < n \leq y} \Lambda(n)e(tn^c) - \int_{\mu y}^y e(tx^c) \, dx + O\left( \log^2 y \right) \right|, \quad (32)
\]
\[
\Sigma'' = \sum_{d \leq \sqrt{X}/(\log X)^{6A+34}} \frac{1}{\varphi(d)} \max_{y \leq X} \left| \Psi(y, \chi, t) \right|.
\]
By (27), (32) and Lemma 2 we find
\[
\Sigma' \ll \frac{X}{\varphi(\log X)^{1/5}} \sum_{d \leq \sqrt{X}/(\log X)^{6A+34}} \frac{1}{\varphi(d)} \ll \frac{X}{\log^A X}.
\]
(34)
Next we consider $\Sigma''$. Moving to primitive characters from (33) we deduce

$$\Sigma'' \ll \sum_{d \leq \sqrt{X}/(\log X)} \frac{1}{\varphi(d)} \sum_{r \mid d, r > 1} \sum_{\chi(r)} \max_{y \leq X} |\Psi(y, \chi, t)| + \frac{\sqrt{X}}{(\log X)^{6A+28}}$$

$$\ll \sum_{1 < r \leq \sqrt{X}/(\log X)} \frac{1}{\varphi(r)} \sum_{\chi(r)} \max_{y \leq X} |\Psi(y, \chi, t)| + \frac{\sqrt{X}}{(\log X)^{6A+28}}$$

$$\ll (\log X) \sum_{1 < r \leq \sqrt{X}/(\log X)} \frac{1}{\varphi(r)} \sum_{\chi(r)} \max_{y \leq X} |\Psi(y, \chi, t)| + \frac{\sqrt{X}}{(\log X)^{6A+28}}$$

$$= (\Omega_1 + \Omega_2) \log X + \frac{\sqrt{X}}{(\log X)^{6A+28}} \tag{35}$$

where

$$\Omega_1 = \sum_{r \leq R_0} \frac{1}{\varphi(r)} \sum_{\chi(r)} \max_{y \leq X} |\Psi(y, \chi, t)|, \tag{36}$$

$$\Omega_2 = \sum_{R_0 < r \leq R} \frac{1}{\varphi(r)} \sum_{\chi(r)} \max_{y \leq X} |\Psi(y, \chi, t)|, \tag{37}$$

$$R_0 = (\log X)^{A+5}, \quad R = \frac{\sqrt{X}}{(\log X)^{6A+28}}. \tag{38}$$

Taking into account (36), (38) and Lemma 3 with $B = A + 1$ we obtain

$$\Omega_1 \ll \frac{X}{(\log X)^{A+1}}. \tag{39}$$

Next we consider $\Omega_2$. From (13) and (37) it follows

$$\Omega_2 \ll \Omega_3 + \Omega_4, \tag{40}$$

where

$$\Omega_3 = \sum_{R_0 < r \leq R} \frac{1}{\varphi(r)} \sum_{\chi(r)} \max_{y \leq X} |\Psi_1(y, \chi, t)|, \tag{41}$$

$$\Omega_4 = \sum_{R_0 < r \leq R} \frac{1}{\varphi(r)} \sum_{\chi(r)} \max_{y \leq X} |\Psi_2(y, \chi, t)|, \tag{42}$$

Taking into account (36), (38) and Lemma 3 with $B = A + 1$ we obtain

$$\Omega_1 \ll \frac{X}{(\log X)^{A+1}}. \tag{39}$$

Next we consider $\Omega_2$. From (13) and (37) it follows

$$\Omega_2 \ll \Omega_3 + \Omega_4, \tag{40}$$

where

$$\Omega_3 = \sum_{R_0 < r \leq R} \frac{1}{\varphi(r)} \sum_{\chi(r)} \max_{y \leq X} |\Psi_1(y, \chi, t)|, \tag{41}$$

$$\Omega_4 = \sum_{R_0 < r \leq R} \frac{1}{\varphi(r)} \sum_{\chi(r)} \max_{y \leq X} |\Psi_2(y, \chi, t)|, \tag{42}$$

Taking into account (36), (38) and Lemma 3 with $B = A + 1$ we obtain

$$\Omega_1 \ll \frac{X}{(\log X)^{A+1}}. \tag{39}$$

Next we consider $\Omega_2$. From (13) and (37) it follows

$$\Omega_2 \ll \Omega_3 + \Omega_4, \tag{40}$$

where

$$\Omega_3 = \sum_{R_0 < r \leq R} \frac{1}{\varphi(r)} \sum_{\chi(r)} \max_{y \leq X} |\Psi_1(y, \chi, t)|, \tag{41}$$

$$\Omega_4 = \sum_{R_0 < r \leq R} \frac{1}{\varphi(r)} \sum_{\chi(r)} \max_{y \leq X} |\Psi_2(y, \chi, t)|, \tag{42}$$
and where

\[ \Psi_1(y, \chi, t) = \sum_{u<n \leq y} \Lambda(n) \chi(n) e(tn^c), \]  
\[ \Psi_2(y, \chi, t) = \sum_{u<n \leq \mu y} \Lambda(n) \chi(n) e(tn^c), \]  
\[ u \leq (\log X)^{2A+12}. \]  
\[ (43) \]
\[ (44) \]
\[ (45) \]

The choice of the parameter \( u \) will be made later.

We shall estimate only the sum \( \Omega_3 \). The sum \( \Omega_4 \) can be estimated likewise. Using (43) and Vaughan’s identity (see [22]) we get

\[ \Psi_1(y, \chi, t) = U_1(y, \chi, t) - U_2(y, \chi, t) - U_3(y, \chi, t) - U_4(y, \chi, t), \]  
\[ (47) \]

where

\[ U_1(y, \chi, t) = \sum_{d \leq u} \mu(d) \sum_{u<d \leq y} \chi(dl) e(td^c l^c) \log l, \]  
\[ U_2(y, \chi, t) = \sum_{d \leq u} c(d) \sum_{u<d \leq y} \chi(dl) e(td^c l^c), \]  
\[ U_3(y, \chi, t) = \sum_{u<d \leq u^2} c(d) \sum_{u<d \leq y} \chi(dl) e(td^c l^c), \]  
\[ U_4(y, \chi, t) = \sum_{d>u, l>u \atop dl \leq y} a(d) \Lambda(l) \chi(dl) e(td^c l^c), \]  
\[ (48) \]
\[ (49) \]
\[ (50) \]
\[ (51) \]

and where

\[ |c(d)| \leq \log d, \quad |a(d)| \leq \tau(d). \]  
\[ (52) \]

Now (43), (47) – (51) give us

\[ \Omega_3 \ll \Omega_3^{(1)} + \Omega_3^{(2)} + \Omega_3^{(3)} + \Omega_3^{(4)}, \]  
\[ (53) \]

where

\[ \Omega_3^{(j)} = \sum_{R_0<r \leq R} \frac{1}{\varphi(r)} \sum_{\chi(r)}^{*} \max_{y \leq X} \left| U_j(y, \chi, t) \right|, \quad j = 1, 2, 3, 4. \]  
\[ (54) \]

**Estimation of \( \Omega_3^{(1)} \) and \( \Omega_3^{(2)} \)**
From (38), (48), (54), Abel’s summation formula and Lemma 4 it follows that

\[
\Omega_{3(1)} \ll \sum_{R_0 < r \leq R} \frac{1}{\varphi(r)} \sum_{y \leq X} \max_{d \leq u} \left| \sum_{u < dl \leq y} \chi(dl) e(td^\kappa) \log l \right|
\]

\[
\ll X^{\frac{1}{2}} (\log X) \sum_{R_0 < r \leq R} \frac{1}{\varphi(r)} \sum_{y \leq X} \max_{u/d \leq x} \max_{u/d \leq y/d} \left| \sum_{u < dl \leq y} \chi(l) \right|
\]

\[
\ll X^{\frac{1}{2}} (\log X) u R^{\frac{3}{2}} \log^2 X.
\]  

(55)

Working similarly to \( \Omega_{3(1)} \) we deduce

\[
\Omega_{3(2)} \ll X^{\frac{1}{2}} u R^{\frac{3}{2}} \log^2 X.
\]  

(56)

**Estimation of \( \Omega_{3(3)} \) and \( \Omega_{3(4)} \)**

We split the range of \( l \) of the exponential sum (51) into dyadic subintervals of the form \( L < l \leq 2L \), where \( u < L \leq y/2d \). Further we use (52), Abel’s summation formula, Perron’s formula (see [18], Chapter II.2, Theorem 1) with parameters

\[
\kappa = 1 + \frac{1}{\log X}, \quad T = X^2,
\]  

(57)

Lemma 5 and partial integration to find

\[
U_4(y, \chi, t) \ll (\log X) \left| \sum_{l \sim L} \sum_{u < d \leq y/l} \Lambda(l) a(d) \chi(dl) e(td^\kappa) \right|
\]

\[
= (\log X) e(ty^e) \sum_{l \sim L} \sum_{u < d \leq y/l} \Lambda(l) a(d) \chi(dl)
\]

\[
- \int_{u^2/l}^{y/l} \left( \sum_{l \sim L} \sum_{u < d \leq x} \Lambda(l) a(d) \chi(dl) \right) de(x^e t^\kappa)
\]

\[
= (\log X) e(ty^e) \mathcal{X}_1 - \mathcal{X}_2,
\]  

(58)
where

\[
x_1 = \frac{1}{2\pi i} \int \frac{\sum_{l \sim L} \sum_{u < d \leq X/L} \Lambda(l) \chi(l)(dl)^s}{(dl)^s} y^s \, ds + O \left( \sum_{l \sim L} \sum_{u < d \leq X/L} \frac{y^{\kappa \Lambda(l) \tau(d)}}{(dl)^{\kappa}(1 + T \log \frac{y}{\pi})} \right),
\]

\[
x_2 = \int \frac{1}{2\pi i} \int \frac{\sum_{l \sim L} \sum_{u < d \leq X/L} \Lambda(l) \chi(l)(dl)^s}{(dl)^s} y^s \, ds + O \left( \sum_{l \sim L} \sum_{u < d \leq X/L} \frac{x^{\kappa \Lambda(l) \tau(d)}}{(dl)^{\kappa}(1 + T \log \frac{y}{\pi})} \right) \, \text{de}(tx^e). \quad (60)
\]

Using (58), (59) and (60) we write

\[
U_4(y, \chi, t) \ll \sum_{l \sim L} \frac{\Lambda(l) \chi(l)}{l^{\kappa - 1 + i\gamma_0}} \sum_{u < d \leq X/L} \frac{a(d) \chi(d)}{d^{\kappa - 1 + i\gamma_0}} \log^2 X + X^{\frac{1}{4}} u^{-2} \log^2 X + \log^2 X + \int_0^T \sum_{l \sim L} \sum_{u < d \leq X/L} \Lambda(l) \chi(l)(dl)^s \left( \int \frac{y^{l \frac{1}{2}} x^{\kappa - 2 + i\gamma} e(tx^e) \, dx}{u^{2/l}} \right) \, \text{de}(tx^e) \log X, \quad (61)
\]

for some $|\gamma_0| \leq T$. Now (54) and (61) imply

\[
\Omega_4^{(4)} \ll (\Xi_1 + \Xi_2 + RX^{\frac{1}{4}} u^{-2} + R) \log^2 X, \quad (62)
\]

where

\[
\Xi_1 = \sum_{R_0 < r \leq R} \frac{1}{\varphi(r)} \sum_{\chi(r)^*} \sum_{l \sim L} \frac{\Lambda(l) \chi(l)}{l^{\kappa - 1 + i\gamma_0}} \left| \sum_{u < d \leq X/L} \frac{a(d) \chi(d)}{d^{\kappa - 1 + i\gamma_0}} \right|, \quad (63)
\]

\[
\Xi_2 = \sum_{R_0 < r \leq R} \frac{1}{\varphi(r)} \sum_{\chi(r)^*} \int_0^T \sum_{l \sim L} \sum_{u < d \leq X/L} \frac{\Lambda(l) \chi(dl) a(d)}{(dl)^{\kappa - 1 + i\gamma}} \left( \int_0^y x^{\kappa - 2 + i\gamma} e(tx^e) \, dx \right) \, d\gamma. \quad (64)
\]
First we estimate $\Xi_1$. By (38), (52), (57), Cauchy’s inequality and Lemma 6 we obtain

$$
\sum_{R_0 < r \leq R} \frac{r}{\varphi(r)} \left| \sum_{l \sim L} \frac{\Lambda(l) \chi(l)}{l^{\sigma-1+\gamma_0}} \right| \sum_{u < d \leq X/L} \left| \frac{a(d) \chi(d)}{d^{\sigma-1+\gamma_0}} \right|
$$

$$
\ll \left( \sum_{R_0 < r \leq R} \frac{r}{\varphi(r)} \left| \sum_{l \sim L} \frac{\Lambda(l) \chi(l)}{l^{\sigma-1+\gamma_0}} \right|^2 \right)^{\frac{1}{2}}
$$

$$
\times \left( \sum_{R_0 < r \leq R} \frac{r}{\varphi(r)} \left| \sum_{u < d \leq X/L} \frac{a(d) \chi(d)}{d^{\sigma-1+\gamma_0}} \right|^2 \right)^{\frac{1}{2}}
$$

$$
\ll (L + R^2)^{\frac{1}{2}} \left( \frac{X}{L} + R^2 \right)^{\frac{1}{2}} \left( \sum_{l \sim L} \Lambda^2(l) \right)^{\frac{1}{2}} \left( \sum_{u < d \leq X/L} \tau^2(d) \right)^{\frac{1}{2}}
$$

$$
\ll (X + X Ru^{-\frac{1}{2}} + X^{\frac{1}{2}} R^2) \log^2 X. \tag{65}
$$

From (38), (63), (65) and Abel’s summation formula it follows that

$$
\Xi_1 \ll (XR_0^{-1} + Xu^{-\frac{1}{2}} \log X + X^{\frac{1}{2}} R) \log^2 X. \tag{66}
$$

Next we consider $\Xi_2$. Put

$$
\Gamma : z = f(\gamma) = \int \frac{x^{\gamma-2+i\gamma} e(tx^\gamma)}{\log x} \, dx, \quad f'(\gamma) \neq 0, \quad 0 \leq \gamma \leq T. \tag{67}
$$

Since $f(\gamma)$ is a holomorphic function such that $f'(\gamma) \neq 0$ for $\gamma \in [0, T]$, then there exists $f^{-1}(z)$ for $z \in \Gamma$. Using (61), (67) and path independence we get

$$
\Xi_2 = \sum_{R_0 < r \leq R} \frac{1}{\varphi(r)} \left| \sum_{l \sim L} \frac{\Lambda(l) \chi(l)}{(dL)^{\sigma-1+if^{-1}(z)}} \right| \left| \sum_{l \sim L} \sum_{u < d \leq X/L} \frac{\Lambda(l) \chi(l) a(d)}{(dL)^{\sigma-1+if^{-1}(z)}} \right| \left| \sum_{l \sim L} \sum_{u < d \leq X/L} \frac{\Lambda(l) \chi(l) a(d)}{(dL)^{\sigma-1+if^{-1}(z)}} \right|
$$

$$
\ll \left( |f(0)| + |f(T)| \right) \sum_{R_0 < r \leq R} \frac{1}{\varphi(r)} \left| \sum_{l \sim L} \sum_{u < d \leq X/L} \frac{\Lambda(l) \chi(l) a(d)}{(dL)^{\sigma-1+if^{-1}(z)}} \right| \left| \sum_{l \sim L} \sum_{u < d \leq X/L} \frac{\Lambda(l) \chi(l) a(d)}{(dL)^{\sigma-1+if^{-1}(z)}} \right|, \tag{68}
$$

for some $z_0 \in \Gamma$, where $\Gamma$ is the line segment connecting the points $f(0)$ and $f(T)$. Bearing in mind (57), (67), (68) and proceeding as in $\Xi_1$ we find

$$
\Xi_2 \ll (XR_0^{-1} + Xu^{-\frac{1}{2}} \log X + X^{\frac{1}{2}} R) \log^2 X \int \frac{x^{\gamma-2}}{\log x} \, dx
$$

$$
\ll (XR_0^{-1} + Xu^{-\frac{1}{2}} \log X + X^{\frac{1}{2}} R) \log^2 X. \tag{69}
$$
Now (62), (66) and (69) give us
\[
\Omega_3^{(4)} \ll (XR_0^{-1} + Xu^{-\frac{1}{2}} \log X + X^{\frac{1}{2}}R) \log^4 X. \tag{70}
\]

Working similarly to \(\Omega_3^{(4)}\) we deduce
\[
\Omega_3^{(3)} \ll (XR_0^{-1} + Xu^{-\frac{1}{2}} + X^{\frac{1}{2}}R) \log^4 X. \tag{71}
\]

From (53), (55), (56), (70) and (71) we obtain
\[
\Omega_3 \ll (X^{\frac{1}{2}}uR^{\frac{3}{2}} + XR_0^{-1} + Xu^{-\frac{1}{2}} \log X + X^{\frac{1}{2}}R) \log^4 X. \tag{72}
\]

Using (35), (38), (39), (40) and (72) we get
\[
\Sigma'' \ll (X^{\frac{1}{2}}uR^{\frac{3}{2}} + XR_0^{-1} + Xu^{-\frac{1}{2}} \log X + X^{\frac{1}{2}}R) \log^5 X. \tag{73}
\]

Summarizing (31), (34), (38), (73) and choosing
\[
u = (\log X)^{2A+12}
\]
we find
\[
\Sigma \ll \frac{X}{\log^A X}. \tag{74}
\]

Bearing in mind (14), (27), (30) and (74) we establish Lemma \ref{lem:18}. \hfill \Box

Put
\[
S_1 = S(t), \tag{75}
\]
\[
S_2 = S_{l,d,j}(t), \tag{76}
\]
\[
I_1 = I(t), \tag{77}
\]
\[
I_2 = \frac{I_{l}(t)}{\varphi(d)}. \tag{78}
\]

We use the identity
\[
S_1^2 S_2 = I_1^2 I_2 + (S_2 - I_2)I_1^2 + S_2(S_1 - I_1)I_1 + S_1 S_2(S_1 - I_1). \tag{79}
\]

Define
\[
\Phi_{\Delta,j}(X, d) = \frac{1}{\varphi(d)} \int_{-\Delta}^{\Delta} \Theta(t)I^2(t)I_{j}(t)e(-Nt) \, dt, \tag{80}
\]
\[
\Phi_j(X, d) = \frac{1}{\varphi(d)} \int_{-\infty}^{\infty} \Theta(t)I^2(t)I_{j}(t)e(-Nt) \, dt. \tag{81}
\]
Now $\text{(11)}, \text{(12)}, \text{(24)}, \text{(75)} - \text{(80)}, \text{Lemma \[\text{1}\]}, \text{Lemma \[\text{2}\], Lemma \[\text{7}\], Lemma \[\text{8}\]}$ and Cauchy’s inequality imply

\[
I_{l,d;J}^{(1)}(X) - \Phi_{\Delta,J}(X, d) = \int_{-\Delta}^{\Delta} \Theta(t) \left( S_{l,d;J}(t) - \frac{I_J(t)}{\varphi(d)} \right) I^2(t)e(-Nt) dt
\]

+ \int_{-\Delta}^{\Delta} \Theta(t) S_{l,d;J}(t) \left( S(t) - I(t) \right) I(t)e(-Nt) dt

+ \int_{-\Delta}^{\Delta} \Theta(t) S(t) S_{l,d;J}(t) \left( S(t) - I(t) \right) e(-Nt) dt

\ll \varepsilon \left[ \max_{|t| \leq \Delta} \left| S_{l,d;J}(t) - \frac{I_J(t)}{\varphi(d)} \right| \int_{-\Delta}^{\Delta} |I(t)|^2 dt \right.

+ \frac{X}{e^{(\log X)^{1/5}}} \left( \int_{-\Delta}^{\Delta} |S_{l,d;J}(t)|^2 dt \right)^{\frac{1}{2}} \left( \int_{-\Delta}^{\Delta} |I(t)|^2 dt \right)^{\frac{1}{2}}

+ \frac{X}{e^{(\log X)^{1/5}}} \left( \int_{-\Delta}^{\Delta} |S(t)|^2 dt \right)^{\frac{1}{2}} \left( \int_{-\Delta}^{\Delta} |S_{l,d;J}(t)|^2 dt \right)^{\frac{1}{2}} \left. \right]

\ll \varepsilon \left( X^{2-c} (\log X) \max_{|t| \leq \Delta} \left| S_{l,d;J}(t) - \frac{I_J(t)}{\varphi(d)} \right| + \frac{X^{3-c}}{de^{(\log X)^{1/6}}} \right). \quad (82)

Using $\text{(11)}, \text{(12)}$ and Lemma \[\text{9}\] we deduce

\[
I_J(t) \ll \min \left( X, \frac{X^{1-c}}{|t|} \right), \quad I(t) \ll \min \left( X, \frac{X^{1-c}}{|t|} \right). \quad (83)
\]

From $\text{(11)}, \text{(12)}, \text{(80)}, \text{(81)}, \text{(83)}$ and Lemma \[\text{11}\] it follows

\[
\Phi_{\Delta,J}(X, d) - \Phi_J(X, d) \ll \frac{1}{\varphi(d)} \int_{-\Delta}^{\Delta} |I(t)|^2 |I_J(t)||\Theta(t)| dt \ll \varepsilon \frac{X^{3-3c}}{\varphi(d)} \int_{-\Delta}^{\Delta} \frac{dt}{t^3} \ll \frac{\varepsilon X^{3-3c}}{\varphi(d)\Delta^2}
\]

and therefore

\[
\Phi_{\Delta,J}(X, d) = \Phi_J(X, d) + O \left( \frac{\varepsilon X^{3-3c}}{\varphi(d)\Delta^2} \right). \quad (84)
\]

Finally $\text{(6)}, \text{(82)}, \text{(84)}$ and the identity

\[
I_{l,d;J}^{(1)}(X) = I_{l,d;J}^{(1)}(X) - \Phi_{\Delta,J}(X, d) + \Phi_{\Delta,J}(X, d) - \Phi_J(X, d) + \Phi_J(X, d)
\]
yield
\[ I_{l,d;J}^{(1)}(X) = \Phi_J(X, d) + O\left( \varepsilon X^{2-c}(\log X) \max_{|t| \leq \Delta} \left| S_{l,d;J}(t) - \frac{I_J(t)}{\varphi(d)} \right| \right) + O\left( \frac{\varepsilon X^{3-c}}{d \log^2 X} \right). \] (85)

6 Upper bound of $I_{l,d;J}^{(3)}(X)$

By (8), (9), (10), (26) and Lemma 1 we find
\[ I_{l,d;J}^{(3)}(X) \ll X^3 \log X \int_0^\infty \frac{1}{t} \left( \frac{k}{2\pi \delta t} \right)^k dt = X^3 \log X \left( \frac{k}{2\pi \delta H} \right)^k \ll \frac{1}{d}. \] (86)

7 Upper bound of $\Gamma_3(X)$

Consider the sum $\Gamma_3(X)$. Since
\[ \sum_{d|p_1-1 \atop d \geq X/D} \sum_{m|p_1-1 \atop m \leq (p_1-1)D/X} \chi_4(d) = \sum_{j=\pm 1} \chi_4(j) \sum_{m|p_1-1 \atop m \leq (p_1-1)D/X} 1 \equiv 1, \]
then from (21) and (22) we obtain
\[ \Gamma_3(X) = \sum_{m < D \atop 2|m} \sum_{j=\pm 1} \chi_4(j) I_{1+jm,4m;J_m}(X), \]
where $J_m = \max\{1 + mX/D, X/2\}, X\}$. The last formula and (23) give us
\[ \Gamma_3(X) = \Gamma_3^{(1)}(X) + \Gamma_3^{(2)}(X) + \Gamma_3^{(3)}(X), \] (87)
where
\[ \Gamma_3^{(i)}(X) = \sum_{m < D \atop 2|m} \sum_{j=\pm 1} \chi_4(j) I_{1+jm,4m;J_m}(X), \quad i = 1, 2, 3. \] (88)

7.1 Estimation of $\Gamma_3^{(1)}(X)$

From (85) and (88) we get
\[ \Gamma_3^{(1)}(X) = \Gamma^* + O\left( \varepsilon X^{2-c}(\log X) \Sigma_1 \right) + O\left( \frac{\varepsilon X^{3-c}}{e^{(\log X)^{1/6}}} \Sigma_2 \right), \] (89)
where

\[
\Gamma^* = \sum_{m < D \atop 2 | m} \Phi_J(X, 4m) \sum_{j = \pm 1} \chi_4(j),
\]

\[
\Sigma_1 = \sum_{m < D \atop 2 | m} \max_{|t| \leq \Delta} \left| S_{1+ jm, 4m; J}(t) - \frac{I_J(t)}{\varphi(4m)} \right|,
\]

\[
\Sigma_2 = \sum_{m < D} \frac{1}{4m}.
\]

From the properties of \( \chi(k) \) we have that

\[
\Gamma^* = 0.
\]

By (5), (9), (11), (91) and Lemma 18 we deduce

\[
\Sigma_1 \ll \frac{X}{\log^4 X}.
\]

It is well known that

\[
\Sigma_2 \ll \log X.
\]

Bearing in mind (89), (93), (94) and (95) we find

\[
\Gamma_3^{(1)}(X) \ll \frac{\varepsilon X^{3-c}}{\log X}.
\]

### 7.2 Estimation of \( \Gamma_3^{(2)}(X) \)

Now we consider \( \Gamma_3^{(2)}(X) \). From (25) and (88) we have

\[
\Gamma_3^{(2)}(X) = \int_{\Delta \leq |t| \leq H} \Theta(t)S^2(t)K(t)e(-Nt) dt,
\]

where

\[
K(t) = \sum_{m < D \atop 2 | m} \sum_{j = \pm 1} \chi_4(j)S_{1 + jm, 4m; J_m}(t).
\]

**Lemma 19.** Assume that

\[
\Delta \leq |t| \leq H, \quad |a(m)| \ll m^n, \quad LM \asymp X, \quad L \gg X^{\frac{2}{3}}.
\]

21
Set
\[ S_I = \sum_{m \sim M} a(m) \sum_{l \sim L} e(tm^c l^c). \] (100)

Then
\[ S_I \ll X^{\frac{373}{400} + \eta}. \]

**Proof.** We first consider the case when
\[ M \ll X^{\frac{4}{11}}. \] (101)

By (6), (8), (99), (100), (101) and Lemma 12 with the exponent pair \((\frac{1}{2}, \frac{1}{2})\) we obtain
\[
S_I \ll X^\eta \sum_{m \sim M} \left| \sum_{l \sim L} e(tm^c l^c) \right|
\leq X^\eta \sum_{m \sim M} \left( |t|X^c L^{-1} \right)^{\frac{1}{2}} L^{\frac{1}{2}} + \frac{1}{|t|X^c L^{-1}}
\leq X^\eta \left( MH^{\frac{1}{2}} X^c + \Delta^{-1} X^{1-c} \right)
\leq X^\eta \left( MX^{\frac{1}{2}} + X^{\frac{1}{2}} \right)
\ll X^{\frac{373}{400} + \eta}. \] (102)

Next we consider the case when
\[ X^{\frac{4}{11}} \ll M \ll X^{\frac{4}{3}}. \] (103)

Using (100), (103) and Lemma 13 we deduce
\[ S_I \ll X^{\frac{373}{400} + \eta}. \] (104)

Bearing in mind (102) and (104) we establish the statement in the lemma. \(\square\)

**Lemma 20.** Assume that
\[ \Delta \leq |t| \leq H, \quad |a(m)| \ll m^\gamma, \quad |b(l)| \ll l^\gamma, \quad LM \asymp X, \quad X^{\frac{1}{2}} \ll L \ll X^{\frac{1}{2}}. \] (105)

Set
\[ S_{II} = \sum_{m \sim M} a(m) \sum_{l \sim L} b(l)e(tm^c l^c). \] (106)

Then
\[ S_{II} \ll X^{\frac{373}{400} + \eta}. \]
Proof. We first consider the case when
\[ X^{\frac{1}{5}} \ll L \ll X^{\frac{63}{200}}. \] (107)
From (105), (106), Cauchy’s inequality and Lemma 10 with \( Q = X^{\frac{1}{5}} \) it follows that
\[ |S_{II}|^2 \ll X^\eta \left( \frac{X^2}{Q} + \frac{X}{Q} \sum_{1 \leq q \leq Q} \sum_{l \sim L} \left| \sum_{m \sim M} e(f(l, m, q)) \right| \right), \] (108)
where \( f(l, m, q) = tm^c((l + q)^c - l^c) \). Now (6), (8), (105), (107), (108) and Lemma 12 with the exponent pair \( \left( \frac{2}{7}, \frac{4}{7} \right) \) give us
\[
S_{II} \ll X^\eta \left( \frac{X^2}{Q} + \frac{X}{Q} \left( \frac{H^2}{X^{(c-1)}} M^2 Q^2 L + \Delta^{-1} X^{1-c} L \log Q \right) \right)^{\frac{1}{2}}
\ll X^\eta \left( \frac{X^2}{Q} + \frac{X}{Q} \left( \Delta^{-1} X^{1-c} L \log Q \right) \right)^{\frac{1}{2}}
\ll X^{\frac{373}{400} + \eta}. \] (109)
Next we consider the case when
\[ X^{\frac{63}{200}} \ll L \ll X^{\frac{1}{5}}. \] (110)
Using (106), (110) and Lemma 14 we find
\[ S_{II} \ll X^{\frac{373}{400} + \eta}. \] (111)
Taking into account (109) and (111) we establish the statement in the lemma. \( \square \)

Lemma 21. Let \( \Delta \leq |t| \leq H \). Then for the exponential sum denoted by (10) we have
\[ S(t) \ll X^{\frac{373}{400} + \eta}. \]

Proof. In order to prove the lemma we will use the formula
\[ S(t) = S^*(t) + O\left( X^{\frac{2}{5} + \varepsilon} \right), \] (112)
where
\[ S^*(t) = \sum_{X/2 < n \leq X} \Lambda(n)e(tn^c). \]
Let 
\[ U = X^{\frac{1}{5}}, \quad V = X^{\frac{1}{3}}, \quad Z = \left[ X^{\frac{1}{2}} \right] + \frac{1}{2}. \]

According to Lemma 11, the sum \( S^*(t) \) can be decomposed into \( O \left( \log^{10} X \right) \) sums, each of which is either of Type I
\[ \sum_{m \sim M} a(m) \sum_{l \sim L} e(t ml^c), \]
where
\[ L \gg Z, \quad LM \asymp X, \quad |a(m)| \ll m^\eta, \]
or of Type II
\[ \sum_{m \sim M} a(m) \sum_{l \sim L} b(l) e(t ml^c), \]
where
\[ U \ll L \ll V, \quad LM \asymp X, \quad |a(m)| \ll m^\eta, \quad |b(l)| \ll l^\eta. \]

Using Lemma 19 and Lemma 20 we deduce
\[ S^*(t) \ll X^\frac{373}{400} + \eta. \] (113)

Bearing in mind (112) and (113) we establish the statement in the lemma. \( \square \)

**Lemma 22.** For the sum denoted by (98) we have

\[ \int_{\Delta}^{H} |K(t)|^2 |\Theta(t)| \, dt \ll X \log^7 X. \]

**Proof.** By Lemma 1 we get
\[ \int_{\Delta}^{H} |K(t)|^2 |\Theta(t)| \, dt \ll \varepsilon \int_{\Delta}^{1/\varepsilon} |K(t)|^2 \, dt + \int_{1/\varepsilon}^{H} \frac{|K(t)|^2}{t} \, dt \]
\[ \ll \varepsilon \sum_{0 \leq n \leq 1/\varepsilon} \int_{1/n}^{n+1} |K(t)|^2 \, dt + \sum_{1/\varepsilon \leq n \leq H} \frac{1}{n} \int_{1/n}^{n+1} |K(t)|^2 \, dt. \] (114)

On the other hand (9) and (98) yield
\[
\int_{n}^{n+1} |K(t)|^2 dt = \sum_{m_1, m_2 < D, j_1 = \pm 1, j_2 = \pm 1} \chi_4(j_1) \chi_4(j_2) \\
\times \int_{n}^{n+1} S_{1+j_1 m_1, 4m_1; j_1} (t) S_{1+j_2 m_2, 4m_2; j_2} (-t) dt \\
= \sum_{m_1, m_2 < D, j_1 = \pm 1, j_2 = \pm 1} \chi_4(j_1) \chi_4(j_2) \\
\times \sum_{p_i \in J_{m_i}, i = 1, 2} \log p_1 \log p_2 \int_{n}^{n+1} e((p_1^c - p_2^c)t) dt \\
\ll (\log X)^2 \sum_{m_1 \leq D} \sum_{X/2 < n_1, n_2 \leq X} \min \left(1, \frac{1}{|n_1^c - n_2^c|}\right) \sum_{m_1 \leq D} \sum_{m_2 \leq D} 1 \\
\ll (\log X)^2 \sum_{X/2 < n_1, n_2 \leq X} \tau(n_1 - 1) \tau(n_2 - 1) \\
\ll (\log X)^2 \sum_{X/2 < n_1, n_2 \leq X} \tau^2(n_1 - 1) \min \left(1, \frac{1}{|n_1^c - n_2^c|}\right) \\
\ll (\mathcal{G}_1 + \mathcal{G}_2) \log^2 X ,
\]

where
\[
\mathcal{G}_1 = \sum_{X/2 < n_1, n_2 \leq X, |n_1^c - n_2^c| \leq 1} \tau^2(n_1 - 1), \quad \mathcal{G}_2 = \sum_{X/2 < n_1, n_2 \leq X, |n_1^c - n_2^c| > 1} \frac{\tau^2(n_1 - 1)}{|n_1^c - n_2^c|}.
\]

First we shall estimate \(\mathcal{G}_1\). By the mean-value theorem and the well-known inequality
\[
\sum_{n \leq X} \tau^2(n) \ll X \log^3 X
\]
we find

\[ G_1 = \sum_{X/2<n_1 \leq X} \tau^2(n_1 - 1) \sum_{X/2<n_2 \leq X} 1 \]

\[ \ll \sum_{X/2<n \leq X} \tau^2(n - 1) \left( (n^c + 1)^{1/c} - (n^c - 1)^{1/c} + 1 \right) \]

\[ \ll \sum_{X/2<n \leq X} \tau^2(n - 1) (X^{1-c} + 1) \]

\[ \ll \sum_{X/2<n \leq X} \tau^2(n - 1) \]

\[ \ll X \log^3 X. \quad (117) \]

Now we consider \( G_2 \). We have

\[ G_2 \ll \sum_{l} \sum_{X/2<n_1 \leq X} \frac{\tau^2(n_1 - 1)}{|n_1 - n_2^c|} \]

\[ \ll \sum_{l} \frac{1}{l} \sum_{X/2<n \leq X} \tau^2(n - 1) U(n_1, l), \quad (118) \]

where

\[ l \ll \log X \quad (119) \]

and

\[ U(n_1, l) = \sum_{X/2<n_2 \leq X} 1. \]

By the mean-value theorem and (119) we deduce

\[ U(n_1, l) \ll (n^c_l + 2l)^{1/c} - (n^c_l + l)^{1/c} + 1 \ll lX^{1-c} + 1 \ll 1. \quad (120) \]

Bearing in mind (119) and (118) – (120) we obtain

\[ G_2 \ll (\log X) \sum_{X/2<n \leq X} \tau^2(n - 1) \ll X \log^4 X. \quad (121) \]

The assertion in the lemma follows from (114), (115), (117) and (121). \( \square \)

Using (97) and Cauchy’s inequality we write

\[ \Gamma_3^{(2)}(X) \ll \max_{\Delta \leq t \leq H} |S(t)| \left( \int_{\Delta}^{H} |S(t)|^2 |\Theta(t)| \, dt \right)^{\frac{1}{2}} \left( \int_{\Delta}^{H} |K(t)|^2 |\Theta(t)| \, dt \right)^{\frac{1}{2}}. \quad (122) \]
According to Lemma 11 and Lemma 7 (iii) we find

\[
\int_{\Delta} H \int |S(t)|^2 |\Theta(t)| \, dt \ll \varepsilon \int_{\Delta} |S(t)|^2 \, dt + \int_{1/\varepsilon}^H \frac{|S(t)|^2}{t} \, dt
\ll \varepsilon \sum_{0 \leq n \leq 1/\varepsilon} \int |S(t)|^2 \, dt + \sum_{1/\varepsilon - 1 \leq n \leq H} \frac{1}{n} \int |S(t)|^2 \, dt
\ll X \log^4 X.
\] (123)

Finally (7), (122), (123), Lemma 21 and Lemma 22 imply

\[
\Gamma_3^{(2)}(X) \ll \frac{\varepsilon X^{3-c}}{\log X}.
\] (124)

### 7.3 Estimation of \( \Gamma_3^{(3)}(X) \)

From (86) and (88) we have

\[
\Gamma_3^{(3)}(X) \ll \sum_{m < D} \frac{1}{d} \ll \log X.
\] (125)

### 7.4 Estimation of \( \Gamma_3(X) \)

Summarizing (87), (96), (124) and (125) we get

\[
\Gamma_3(X) \ll \frac{\varepsilon X^{3-c}}{\log X}.
\] (126)

### 8 Upper bound of \( \Gamma_2(X) \)

In this section we need a lemma that gives us information about the upper bound of the number of solutions of the binary Piatetski-Shapiro inequality.

**Lemma 23.** Let \( 1 < c < 3 \), \( c \neq 2 \) and \( N_0 \) is a sufficiently large positive number. Then for the number of solutions \( B_0(N_0) \) of the diophantine inequality

\[
|p_1^c + p_2^c - N_0| < \varepsilon
\] (127)

in prime numbers \( p_1, p_2 \in \left( N_0^{\frac{3}{2}} / 2, N_0^{\frac{3}{2}} \right) \) we have that

\[
B_0(N_0) \ll \frac{\varepsilon N_0^{\frac{c}{2} - 1}}{\log^2 N_0}.
\]
Proof. Define
\begin{equation}
B(X_0) = \sum_{\frac{X_0}{2} < p_1, p_2 \leq X_0} \log p_1 \log p_2,
\end{equation}
where
\begin{equation}
X_0 = \frac{1}{c}.
\end{equation}
Let us take the parameters
\begin{equation}
a_0 = \frac{5\varepsilon}{4}, \quad \delta_0 = \frac{\varepsilon}{4}, \quad k_0 = \lfloor \log X_0 \rfloor.
\end{equation}
According to Lemma 1 there exists a function \( \theta_0(y) \) which is \( k_0 \) times continuously differentiable and such that
\begin{align*}
\theta_0(y) &= 1 \quad \text{for} \quad |y| \leq \varepsilon; \\
0 < \theta_0(y) < 1 \quad &\text{for} \quad \varepsilon < |y| < \frac{3\varepsilon}{2}; \\
\theta_0(y) &= 0 \quad \text{for} \quad |y| \geq \frac{3\varepsilon}{2}.
\end{align*}
and its Fourier transform
\begin{equation}
\Theta_0(x) = \int_{-\infty}^{\infty} \theta_0(y)e(-xy)dy
\end{equation}
satisfies the inequality
\begin{equation}
|\Theta_0(x)| \leq \min \left( \frac{5\varepsilon}{2}, \frac{1}{\pi|x|}, \frac{1}{\pi|x|} \left( \frac{2k_0}{\pi|x|\varepsilon} \right)^k \right).
\end{equation}
By (128), the definition of \( \theta_0(y) \) and the inverse Fourier transformation formula we use decomposition over major, minor and trivial arcs as follows
\begin{align}
B(X_0) &\leq \sum_{\frac{X_0}{2} < p_1, p_2 \leq X_0} \theta_0 \left( p_1^c + p_2^c - N_0 \right) \log p_1 \log p_2 \\
&= \int_{-\infty}^{\infty} \Theta_0(t)S_0^2(t)e(-N_0t)dt \\
&= B_1(X_0) + B_2(X_0) + B_3(X_0),
\end{align}
\( 131 \).
where

\[ S_0(t) = \sum_{X_0/2 < p \leq X_0} e(tp^c) \log p, \]  
\[ \Delta_0 = \frac{(\log X_0)^{A_0}}{X_0^c}, \quad A_0 > 10, \]  
\[ B_1(X_0) = \int_{-\Delta_0}^{\Delta_0} \Theta_0(t) S_0^2(t) e(-N_0 t) \, dt, \]  
\[ B_2(X_0) = \int_{\Delta_0 \leq |t| \leq H} \Theta_0(t) S_0^2(t) e(-N_0 t) \, dt, \]  
\[ B_3(X_0) = \int_{|t| > H} \Theta_0(t) S_0^2(t) e(-N_0 t) \, dt. \]  

First we estimate \( B_1(X_0) \). Put

\[ I_0(t) = \int_{X_0/2}^{X_0} e(ty^c) \, dy, \]  
\[ \Psi_{\Delta_0}(X_0) = \int_{-\Delta_0}^{\Delta_0} \Theta_0(t) I_0^2(t) e(-N_0 t) \, dt, \]  
\[ \Psi(X_0) = \int_{-\infty}^{\infty} \Theta_0(t) I_0^2(t) e(-N_0 t) \, dt. \]  

Using (130), (137), (139) and Lemma 9 we find

\[ \Psi(X_0) = \int_{X_0^{-c}}^{X_0^{-c}} \Theta_0(t) I_0^2(t) e(-N_0 t) \, dt + \int_{|t| > X_0^{-c}} \Theta_0(t) I_0^2(t) e(-N_0 t) \, dt, \]  
\[ \ll \int_{-X_0^{-c}}^{X_0^{-c}} \varepsilon X_0^2 \, dt + \int_{X_0^{-c}}^{\infty} \varepsilon \left( \frac{X_0^{1-c}}{t} \right)^2 \, dt, \]  
\[ \ll \varepsilon X_0^{2-c}. \]  

On the other hand (130), (133), (134), (138), Lemma 2 and the trivial estimations

\[ S_0(t) \ll X_0, \quad I_0(t) \ll X_0 \]  

(141)
\[
B_1(X_0) - \Psi_{\Delta_0}(X_0) \ll \int_{-\Delta_0}^{\Delta_0} |S_0^2(t) - I_0^2(t)||\Theta_0(t)| \, dt \\
\ll \varepsilon \int_{-\Delta_0}^{\Delta_0} |S_0(t) - I_0(t)|\left(|S_0(t)| + |I_0(t)|\right) \, dt \\
\ll \varepsilon \frac{X_0}{e^{(\log X_0)^{1/5}}} \left(\int_{\Delta_0}^{\Delta_0} |S_0(t)| \, dt + \int_{-\Delta_0}^{\Delta_0} |I_0(t)| \, dt\right) \\
\ll \frac{\varepsilon X_0^{2-c}}{e^{(\log X_0)^{1/6}}}.
\]

From (130), (133), (138), (139) and Lemma 9 we deduce

\[
|\Psi(X_0) - \Psi_{\Delta_0}(X_0)| \ll \int_{\Delta_0}^{\infty} |I_0(t)|^2|\Theta_0(t)| \, dt \ll \frac{\varepsilon}{X_0^{2(c-1)}} \int_{\Delta_0}^{\infty} \frac{dt}{t^2} \\
\ll \frac{\varepsilon}{X_0^{2(c-1)}} \Delta_0 \ll \frac{\varepsilon X_0^{2-c}}{\log X_0}.
\]

Now (140), (142) and (143) and the identity

\[
B_1(X_0) = B_1(X_0) - \Psi_{\Delta_0}(X_0) + \Psi_{\Delta_0}(X_0) - \Psi(X_0) + \Psi(X_0)
\]

imply

\[
B_1(X_0) \ll \varepsilon X_0^{2-c}.
\]

Further we estimate \(B_2(X_0)\). Consider the integral

\[
B_2^*(X_0) = \int_{\Delta_0}^{H} \Theta_0(t)S_0^2(t)e(-N_0t) \, dt.
\]

By (129), (130), (141), (145) and partial integration it follows

\[
B_2^*(X_0) = -\frac{1}{2\pi i} \int_{\Delta_0}^{H} \Theta_0(t)S_0^2(t) \frac{d}{N_0} e(-N_0t) \, dt \\
= -\frac{\Theta_0(t)S_0^2(t)e(-N_0t)}{2\pi i N_0} \bigg|_{\Delta_0}^{H} + \frac{1}{2\pi i N_0} \int_{\Delta_0}^{H} e(-N_0t) d\left(\Theta_0(t)S_0^2(t)\right) \\
\ll \varepsilon X_0^{2-c} + X_0^{-c}|\Omega|,
\]

(146)
where
\[ \Omega = \int_{\Delta_0}^{H} e(-N_0 t) \, d\left( \Theta_0(t) S_0^2(t) \right). \] (147)

Next we consider \( \Omega \). Put
\[ \Gamma_0 : z = g(t) = \Theta_0(t) S_0^2(t), \quad g'(t) \neq 0, \quad \Delta_0 \leq t \leq H. \] (148)

Since \( g(t) \) is a holomorphic function such that \( g'(t) \neq 0 \) for \( t \in [\Delta_0, H] \), then there exists \( g^{-1}(z) \) for \( z \in \Gamma_0 \). Thus (147) and (148) imply
\[ \Omega = \int_{\Gamma_0} e\left(-N_0 g^{-1}(z)\right) \, dz. \] (149)

Using (130), (141), (148) and that the integral (149) is independent of path we obtain
\[ \Omega \ll \int_{\Gamma_0} |dz| \ll |g(\Delta_0)| + |g(H)| \ll \varepsilon X_0^2, \] (150)
where \( \Gamma_0 \) is the line segment connecting the points \( g(\Delta_0) \) and \( g(H) \). Taking into account (135), (145), (146) and (150) we deduce
\[ B_2(X_0) \ll \varepsilon X_0^{2-c}. \] (151)

Finally we estimate \( B_3(X_0) \). By (8), (130), (132), (136), (141) we find
\[ B_3(X_0) \ll X_0^2 \int_{H}^{\infty} \frac{1}{t} \left( \frac{2k_0}{\pi \varepsilon} \right)^{k_0} dt \ll X_0^2 \left( \frac{k_0}{\pi H \varepsilon} \right)^{k_0} \ll 1. \] (152)

Summarizing (131), (144), (151) and (152) we get
\[ B(X_0) \ll \varepsilon X_0^{2-c}. \] (153)

Bearing in mind (128), (129) and (153), for the number of solutions \( B_0(N_0) \) of the diophantine inequality (127) we obtain
\[ B_0(N_0) \ll \frac{\varepsilon N_0^{2-1}}{\log^2 N_0}. \]

The lemma is proved. \( \square \)
Consider the sum $\Gamma_2(X)$. We denote by $\mathcal{F}(X)$ the set of all primes $X/2 < p \leq X$ such that $p - 1$ has a divisor belonging to the interval $(D, X/D)$. The inequality $xy \leq x^2 + y^2$ and $\text{(20)}$ give us

$$
\Gamma_2(X)^2 \ll (\log X)^6 \sum_{\frac{X}{2} < p_1, \ldots, p_6 \leq X \atop |p_1^2 + p_2^2 + p_3^2 - N| < \epsilon} \sum_{d | p_1 - 1 \atop D < d < X/D} \chi_4(d) \sum_{t | p_4 - 1 \atop D < t < X/D} \chi_4(t).
$$

The summands in the last sum for which $p_1 = p_4$ can be estimated with $O(X^{3+\varepsilon})$.

Therefore

$$
\Gamma_2(X)^2 \ll (\log X)^6 \sum_{\frac{X}{2} < p_1 \leq X} \chi_4(d)^2 \sum_{\frac{X}{2} < p_4 \leq X} \sum_{\frac{X}{2} < p_2, p_3, p_6 \leq X} 1.
$$

Now $\text{(155)}$ and Lemma $\text{23}$ imply

$$
\Sigma_0 \ll \frac{X^{4-2\varepsilon}}{\log^4 X} \Sigma_0' \Sigma_0''.
$$

where

$$
\Sigma_0' = \sum_{X/2 < p \leq X} \chi_4(d)^2, \quad \Sigma_0'' = \sum_{\frac{X}{2} < p \leq X} \chi_4(d) \sum_{D < d < X/D} 1.
$$

Applying Lemma $\text{15}$ we get

$$
\Sigma_0' \ll \frac{X \log \log X}{\log X}.
$$

Using Lemma $\text{16}$ we obtain

$$
\Sigma_0'' \ll \frac{X \log \log X}{(\log X)^{1+2\theta_0}},
$$

where $\theta_0$ is denoted by $\text{11}$.

Finally $\text{(154)}, \text{(156)}, \text{(157)}$ and $\text{(158)}$ yield

$$
\Gamma_2(X) \ll \frac{X^{3-\varepsilon}(\log \log X)^5}{(\log X)^{\theta_0}} = \frac{\varepsilon X^{3-\varepsilon}}{\log \log X}.
$$
9 Lower bound for $\Gamma_1(X)$

Consider the sum $\Gamma_1(X)$. From (19), (22) and (23) it follows

$$\Gamma_1(X) = \Gamma_1^{(1)}(X) + \Gamma_1^{(2)}(X) + \Gamma_1^{(3)}(X),$$

where

$$\Gamma_1^{(i)}(X) = \sum_{d \leq D} \chi_4(d) I_1^{(i)}(X), \quad i = 1, 2, 3.$$  \hfill (161)

9.1 Estimation of $\Gamma_1^{(1)}(X)$

First we consider $\Gamma_1^{(1)}(X)$. Using formula (85) for $J = (X/2, X]$, (161) and treating the reminder term by the same way as for $\Gamma_3^{(1)}(X)$ we find

$$\Gamma_1^{(1)}(X) = \Phi(X) \sum_{d \leq D} \chi_4(d) \phi(d) + O\left(\frac{\varepsilon X^{3-c}}{\log X}\right),$$

where

$$\Phi(X) = \int_{-\infty}^{\infty} \Theta(t) I_3(t) e(-Nt) \, dt.$$  \hfill (162)

By Lemma (17) we get

$$\Phi(X) \gg \varepsilon X^{3-c}.$$  \hfill (163)

According to (8, p. 14 – 15) we have

$$\sum_{d \leq D} \chi_4(d) \phi(d) = \frac{\pi}{4} \prod_p \left(1 + \frac{\chi_4(p)}{p(p-1)}\right) + O\left(X^{-1/20}\right).$$

From (162) and (163) we obtain

$$\Gamma_1^{(1)}(X) = \frac{\pi}{4} \prod_p \left(1 + \frac{\chi_4(p)}{p(p-1)}\right) \Phi(X) + O\left(\frac{\varepsilon X^{3-c}}{\log X}\right) + O\left(\Phi(X)X^{-1/20}\right).$$

Now (163) and (165) imply

$$\Gamma_1^{(1)}(X) \gg \varepsilon X^{3-c}.\hfill (166)$$

9.2 Estimation of $\Gamma_1^{(2)}(X)$

Arguing as in the estimation of $\Gamma_3^{(2)}(X)$ we find

$$\Gamma_1^{(2)}(X) \ll \frac{\varepsilon X^{3-c}}{\log X}.$$  \hfill (167)
9.3 Estimation of $\Gamma^3_1(X)$

By (86) and (161) it follows that

$$\Gamma^3_1(X) \ll \sum_{m<D} \frac{1}{d} \ll \log X.$$  (168)

9.4 Estimation of $\Gamma_1(X)$

Summarizing (160), (166), (167) and (168) we obtain

$$\Gamma_1(X) \gg \varepsilon X^{3-c}.$$  (169)

10 Proof of the Theorem

Taking into account (7), (16), (18), (126), (159) and (169) we deduce

$$\Gamma(X) \gg \varepsilon X^{3-c} = \frac{X^{3-c} (\log \log X)^6}{(\log X)^\theta}.$$  

The last lower bound yields

$$\Gamma(X) \to \infty \quad \text{as} \quad X \to \infty.$$  (170)

Bearing in mind (15) and (170) we establish Theorem 1.

References

[1] R. Baker, Some diophantine equations and inequalities with primes, Funct. Approx. Comment. Math., 64 (2), (2021), 203 – 250.

[2] R. Baker, A. Weingartner, A ternary diophantine inequality over primes, Acta Arith., 162, (2014), 159 – 196.

[3] Y. Cai, On a diophantine inequality involving prime numbers (in Chinese), Acta Math Sinica, 39, (1996), 733 – 742.

[4] Y. Cai, On a diophantine inequality involving prime numbers III, Acta Mathematica Sinica, English Series, 15, (1999), 387 – 394.

[5] Y. Cai, A ternary Diophantine inequality involving primes, Int. J. Number Theory, 14, (2018), 2257 – 2268.
[6] X. Cao, W. Zhai, A Diophantine inequality with prime numbers, Acta Math. Sinica, Chinese Series, 45, (2002), 361 – 370.

[7] S. I. Dimitrov, A ternary diophantine inequality over special primes, JP Journal of Algebra, Number Theory and Applications, 39, 3, (2017), 335 – 368.

[8] S. I. Dimitrov, Diophantine approximation with one prime of the form \( p = x^2 + y^2 + 1 \), Lith. Math. J., 61, 4, (2021), 445 – 459.

[9] S. W. Graham, G. Kolesnik, Van der Corput’s Method of Exponential Sums, Cambridge University Press, New York, (1991).

[10] D. R. Heath-Brown, The Piatetski-Shapiro prime number theorem, J. Number Theory, 16, (1983), 242 – 266.

[11] C. Hooley, Applications of sieve methods to the theory of numbers, Cambridge Univ. Press, (1976).

[12] H. Iwaniec, E. Kowalski, Analytic number theory, Colloquium Publications, 53, Amer. Math. Soc., (2004).

[13] A. Kumchev, T. Nedeva, On an equation with prime numbers, Acta Arith., 83, (1998), 117 – 126.

[14] A. Kumchev, A diophantine inequality involving prime powers, Acta Arith., 89, (1999), 311 – 330.

[15] Ju. Linnik, An asymptotic formula in an additive problem of Hardy and Littlewood, Izv. Akad. Nauk SSSR, Ser.Mat., 24, (1960), 629 – 706 (in Russian).

[16] I. Piatetski-Shapiro, On a variant of the Waring-Goldbach problem, Mat. Sb., 30, (1952), 105 – 120, (in Russian).

[17] P. Sargos, J. Wu, Multiple exponential sums with monomials and their applications in number theory, Acta Math. Hungar., 87, (2000), 333 – 354.

[18] G. Tenenbaum, Introduction to Analytic and Probabilistic Number Theory, Cambridge Univ. Press, (1995).

[19] E. Titchmarsh, The Theory of the Riemann Zeta-function (revised by D. R. Heath-Brown), Clarendon Press, Oxford (1986).
[20] D. Tolev, *On a diophantine inequality involving prime numbers*, Acta Arith., 61, (1992), 289 – 306.

[21] D. Tolev, *On a diophantine inequality with prime numbers of a special type*, Proc. Steklov Inst. Math., 299, (2017), 261 – 282.

[22] R. C. Vaughan, *An elementary method in prime number theory*, Acta Arith., 37, (1980), 111 – 115.

S. I. Dimitrov
Faculty of Applied Mathematics and Informatics
Technical University of Sofia
8, St.Kliment Ohridski Blvd.
1756 Sofia, BULGARIA
e-mail: sdimitrov@tu-sofia.bg