RPAE versus RPA for the Tomonaga model with quadratic energy dispersion

K. Schönhammer

Institut für Theoretische Physik, Universität Göttingen, Friedrich-Hund-Platz 1, D-37077 Göttingen

(Dated: November 24, 2018)

Recently the damping of the collective charge (and spin) modes of interacting fermions in one spatial dimension was studied. It results from the nonlinear correction to the energy dispersion in the vicinity of the Fermi points. To investigate the damping one has to replace the random phase approximation (RPA) bare bubble by a sum of more complicated diagrams. It is shown here that a better starting point than the bare RPA is to use the (conserving) linearized time dependent Hartree-Fock equations, i.e. to perform a random phase approximation (with) exchange (RPAE) calculation. It is shown that the RPAE equation can be solved analytically for the special form of the two-body interaction often used in the Luttinger liquid framework. While (bare) RPA and RPAE agree for the case of a strictly linear disperser there are qualitative differences for the case of the usual nonrelativistic quadratic dispersion.

I. INTRODUCTION

In a seminal paper Tomonaga presented the exact solution for the long wavelength density response of a system of interacting fermions in one spatial dimension. In order to simplify the problem Tomonaga studied the high density limit where the range of the interaction is much larger than the interparticle distance. Then the Fourier transform $\tilde{v}(k)$ of the two-body interaction is nonzero only for values $|k| \leq k_c$, where the cut-off $k_c$ is much smaller than the Fermi momentum $k_F \ll k_c$. This implies that for not too strong interaction the ground state and the low energy excited states have negligible admixtures of holes deep in the Fermi sea and particles with momenta $|k| - k_F \gg k_c$. Therefore Tomonaga linearized the nonrelativistic quadratic dispersion in the vicinity of the two Fermi points $\pm k_F$. He then used Bloch’s method of sound waves, now called “bosonization” to obtain the ground state of the interacting system as well as its low energy excitations. As an application Tomonaga calculated the density response of the (spinless) system in the long wavelength limit. This result was obtained before the pioneering paper by Bohm and Pines in which the authors introduced the “random phase approximation” (RPA) for the (linear) density response of the (three-dimensional) interacting electron gas in order to obtain e.g. the collective plasmon mode. The RPA is a good approximation in the high-density limit. Various alternative derivations of this approximation can be given, e.g. using diagramatic techniques or linearizing the time dependent Hartree-Fock (TDHF) equations and neglecting the exchange contribution and replacing the HF-eigenvalues by the noninteracting ones.

If these additional approximations in the TDHF equation are not made the corresponding response functions are called RPAE, where the “E” stands for exchange. While the RPA result is a simple expression in terms of the response function of the noninteracting system, the calculation of the RPAE response function usually requires the solution of an integral equation.

For spin 1/2 fermions collective modes usually occur in the charge and the spin density response. They were studied by Dzyaloshinski and Larkin using standard many body techniques (avoiding bosonzation). They showed that for linear dispersion the exact results for the collective modes for the Tomonaga model are identical to the RPA result. Diagramatically this can be understood in terms of the “closed loop theorem” which holds if one assumes strictly linear energy dispersion around the two Fermi points.\(^\text{6,7}\)

Recently various activities started to study the charge and spin density response when deviations from a strictly linear dispersion for one-dimensional fermions is taken into account\(^\text{8,9}\). Then the approximation using the “bare bubble” is no longer exact.

It is one purpose of this paper to point out that the bare RPA is not even a good starting point for a perturbative calculation as e.g. it has the one-particle one-hole (1p-1h) continuum in the wrong place. It is much better to use TDHF which is a conserving approximation\(^\text{10,11,12,13}\) as the starting point to calculate the damping of the collective mode(s). We show that it is possible to solve the RPAE equation for the Tomonaga model analytically when the two-body interaction $\tilde{v}(q)$ takes a constant value for $|q| < k_c$ and is zero otherwise. This exact solution reduces to the RPA result with the noninteracting response function (“bubble”) for strictly linear energy dispersion but differs markedly for nonlinear dispersion. The continua are in different places and the dispersion of the collective modes differ. This is most drastically seen for the collective spin mode which exists in RPAE but not in the bare RPA.

II. LINEARIZED TIME DEPENDENT HARTREE-FOCK APPROXIMATION

We want to describe the density response of a system of fermions on a ring of circumference $L$. The Hamiltonian is given by
\[ H = \sum_k \epsilon_k c_k^\dagger c_k + \frac{1}{2} \sum_{k_1,k_2,k_3,k_4} v_{k_1,k_2,k_3,k_4} c_{k_1}^\dagger c_{k_2}^\dagger c_{k_4} c_{k_3} . \]  

The index \( k \) in addition to the momentum (possibly) includes a spin index. The first term presents the kinetic energy with the nonrelativistic dispersion \( \epsilon_k = k^2/2m \). The two-body matrix element \( v_{k_1,k_2,k_3,k_4} \) contains a factor \( \delta_{k_1+k_2+k_3+k_4} \) due to total momentum conservation. The remaining \( k \)-dependences are specified later.

In order to study the linear charge or spin density response the Heisenberg equations of motion for \( \langle c_k^\dagger c_k \rangle t \) have to be solved in the presence of a time dependent external one-body potential with matrix elements \( V_{k,k'} \). Using the Hartree-Fock approximation to factorize the four fermion expectation values and linearizing the expectation values around the HF ground state

\[ \langle c_k^\dagger c_k \rangle_t = \delta_{k k'} f(\epsilon_k^\text{HF}) + \delta(\epsilon_k^\dagger c_k^\dagger c_k) , \]

yields

\[ \left[ \frac{d}{dt} - (\epsilon_k^\text{HF} - \epsilon_k^\text{HF}) \right] \langle c_k^\dagger c_k \rangle_t = \left( f(\epsilon_k^\text{HF}) - f(\epsilon_k^\text{HF}) \right) [ V_{k,k'} + \sum_{k_2,k_4} \tilde{v}_{k',k_2,k_4} \delta(\epsilon_k^\dagger c_k^\dagger c_k) \right] , \]

where the HF-energies are given by

\[ \epsilon_k^\text{HF} = \epsilon_k + \sum_{k'} \tilde{v}_{k',k,k'} f(\epsilon_k^\text{HF}) \]

and \( \tilde{v}_{k_1,k_2,k_3,k_4} = v_{k_1,k_2,k_3,k_4} - \tilde{v}_{k_1,k_2,k_3,k_4} \) are the antisymmetrized two-body matrix elements. At \( T = 0 \) the Fermi function is a step function not depending on the interaction \( f(\epsilon_k^\text{HF}) = \Theta(k_F - |k|) \). Eqs. (3) are the linearized TDHF equations for the expectation values.

As we are interested in the long wavelength response we put \( k \to k_a \) and \( k' \to k_a + q \), where \( |q| \ll k_F \) and \( k_a \) with \( a = R, L \) is in the neighborhood of the right or left Fermi point \( \pm k_F \). Fourier transforming Eq. (3) yields

\[ \delta(\epsilon_{k_a}^\dagger c_{k_a+q}) \omega = \frac{f(\epsilon_{k_a}^\text{HF}) - f(\epsilon_{k_a+q}^\text{HF})}{\omega - (\epsilon_{k_a+q}^\text{HF} - \epsilon_{k_a}^\text{HF})} \]

\[ \times \left[ V_{k_a+q,k_a} + \sum_{k'} \tilde{v}_{k_a+q,k',k_a+q} \delta(\epsilon_{k'}^\dagger c_{k'}^\dagger c_{k_a+q}) \right] , \]

where \( \omega = \omega + i0 \). With the assumption \( |q| \ll k_F \) the \( k' \) sum in Eq. (3) only can take values in the neighbourhood of the two Fermi points. This implies that two different types of two-body matrix elements contribute. For \( k' \) in the neighborhood of the same Fermi point as \( k_a \)

\[ \tilde{v}_{k_a+q,q';k_a,q';k_a+q,q'} = \frac{1}{L} \left[ \hat{g}_q(q) - \delta_{q,q'} \hat{g}_q(k_a - k_a') \right] . \]

Here we have used the usual “g-ology” nomenclature and explicitly introduced the spin dependence. For the two-body matrix element with \( k_a \) and \( k' \approx k_a \) where \( a \) differs from \( a \) no exchange contribution occurs using Tomonaga’s assumptions as this would involve a momentum transfer of order \( 2k_F \), for which the two-body interaction is assumed to vanish. Thus

\[ \bar{v}_{k_a+q,q';k_a,q';k_a+q,q'} = \frac{1}{L} \hat{g}_2(q) . \]

We separate the two contributions in Eq. (5) and express the matrix elements of the external potential as

\[ \delta(\epsilon_{k_a}^\dagger c_{k_a+q}) \omega = \frac{1}{L} \frac{f(\epsilon_{k_a}^\text{HF}) - f(\epsilon_{k_a+q}^\text{HF})}{\omega - (\epsilon_{k_a+q}^\text{HF} - \epsilon_{k_a}^\text{HF})} \]

\[ \times \left[ V_{\sigma,a}(q) + \sum_{k',\sigma'} \hat{g}_q(q) \delta(\epsilon_{k_a}^\dagger c_{k_a+q}) \right] , \]

where \( V_{\sigma,a}(q) \) are the exchange contributions. They considerably simplify for the special choice for the \( \hat{g}_m(q) \), \( m = 2, 4 \) mentioned in the introduction

\[ \hat{g}_m(q) = \hat{g}_m(\theta(k_c - |q|)) . \]

With this assumption only expectation values of the type

\[ \delta(\rho_{q,a}) \omega = \sum_{k'a} \delta(\epsilon_{k'a}^\dagger c_{k'a+q}) \]

and \( k_a \to k_a' \) occur on the rhs of Eq. (5). In Luttinger liquid terminology this are expectation values of the Fourier components of the densities of the right or left movers. Therefore Eq. (6), after summing over \( k_a \), yields a closed set of equations for \( \delta(\rho_{q,a}) \). Dropping the index \( \omega \) it reads

\[ \delta(\rho_{q,a}) = \frac{R^a_{HF}(q)}{1 + \hat{g}_q R^a_{HF}(q)} \]

with

\[ R^a_{HF}(q) = \frac{1}{L} \sum_{k_a} \frac{f(\epsilon_{k_a}^\text{HF}) - f(\epsilon_{k_a+q}^\text{HF})}{\omega - (\epsilon_{k_a+q}^\text{HF} - \epsilon_{k_a}^\text{HF})} . \]

and the \( \delta(\rho_{q,a}) \) are the charge densities for the right and left movers

\[ \delta(\rho_{q,a}) = \sum_{q} \delta(\rho_{q,a}) . \]

With

\[ \hat{R}^a_{HF}(q,\omega) \equiv \frac{R^a_{HF}(q,\omega)}{1 + \hat{g}_q R^a_{HF}(q,\omega)} \]
Eq. (11) reads
\[ \delta \rho(q,a,a') = \tilde{R}^a(q, \omega) \left[ \tilde{V}_{a,a} + \tilde{g}_4 \delta \rho(q,a) + \tilde{g}_2 \delta \rho(q,a') \right] \] (15)
This equation is the starting point for the calculation of the charge and spin response.

The equations for the charge response are obtained by summing Eq. (15) over the spin index
\[ (1 - 2\tilde{g}_4 \tilde{R}^a) \delta \rho(q,a) - 2\tilde{g}_2 \tilde{R}^a \delta \rho(q,a') = 2\tilde{R}^a \tilde{V}^{(c)} \] (16)
with \( \tilde{V}^{(c)} \equiv \sum_\sigma \tilde{V}_{a,a}/2 \). The solution is given by
\[ \delta \rho(q,a,\omega) = \frac{2\tilde{R}^a (1 - 2\tilde{g}_4 \tilde{R}^a) \tilde{V}_{a}^{(c)} + 4\tilde{g}_2 \tilde{R}^a \tilde{R}^a \tilde{V}_{a}^{(c)}}{(1 - 2\tilde{g}_4 \tilde{R}^a) (1 - 2\tilde{g}_4 \tilde{R}^a) - 4\tilde{g}_2^2 \tilde{R}^a \tilde{R}^a} \equiv R_{aa}(q,\omega) \tilde{V}^{(c)} + R_{aa}(q,\omega) \tilde{V}^{(c)} \] (17)
The spin response \( \delta \sigma(q,a,\omega) = \sum_\sigma \sigma \delta \rho(q,a,\sigma) \) is obtained by taking the difference in Eq. (15). With \( \tilde{V}_a^{(s)} \equiv \sum_\sigma \sigma \tilde{V}_{a,a}/2 \) one obtains
\[ \delta \sigma(q,a,\omega) = 2\tilde{R}^a \tilde{V}_a^{(s)} = \frac{2R_{HF}^a(q,\omega)}{1 + \tilde{g}_4 R_{HF}^a(q,\omega)} \tilde{V}_a^{(s)}. \] (18)
For the spinless model originally studied by Tomonaga the two \( \tilde{g}_a \) terms on the rhs of Eq. (11) cancel and one obtains
\[ \delta \rho(q,a,\omega) = \frac{R_{HF}^a \tilde{V}_a + \tilde{g}_2 R_{HF}^a R_{HF}^a \tilde{V}_a}{1 - \tilde{g}_2^2 R_{HF}^a R_{HF}^a}. \] (19)
Equations (17), (18), and (19) are the central results of this paper. The general behaviour and limiting cases of the corresponding response functions are discussed in the following section.

### III. GENERAL RESULTS AND LIMITING CASES

In this section we discuss the RPAE charge and spin response restricting ourselves to zero temperature. The basic building blocks of the RPAE response functions are the functions \( \tilde{R}^q(q,\omega) \) which via Eq. (14) are expressed in terms of the \( R_{HF}^a(q,\omega) \). In order to calculate these functions we first have to determine the HF-eigenvalues \( \epsilon_{HF}^q \) defined in Eq. (4). For the special choice of the two-body matrix elements in Eqs. (9), (17), and (19) one obtains for \( k = \pm k_F + \tilde{k} \) with \( |\tilde{k}| < k_c \) and \( k_c \leq k_F/2 \)
\[ \epsilon_{HF}^{\pm k_F + \tilde{k}} = \frac{(\pm k_F + \tilde{k})^2}{2m} \pm \frac{\tilde{g}_4 k}{2n} + \text{const.} \] (20)
and
\[ \epsilon^{\pm k_F + \tilde{k}} = \pm (v_F + \frac{\tilde{g}_4}{2n}) \tilde{k} + \frac{\tilde{k}^2}{2m} + \tilde{\epsilon} \equiv \pm \nu_{HF} \tilde{k} + \frac{\tilde{k}^2}{2m} + \tilde{\epsilon}. \]
The constant is given by the Hartree contribution and the constant part of the Fock term. It is not needed here, as the Fermi function at \( T = 0 \) is independent of the interaction and and therefore only differences of HF-energies occur in Eq. (12).

For \( |q| < k_c \) the behaviour of the response functions can be expressed in terms of two types of dimensionless quantities using \( m = k_F/v_F \).
\[ \alpha_1 = \frac{\tilde{g}_4}{2\pi v_F}, \quad \beta = \frac{q}{2k_F}. \] (21)
If we define \( s_a = 1 \) for \( a = R \) and \( s_a = -1 \) for \( a = L \) the HF response functions in Eq. (12) are given by
\[ R_{HF}^a(q,\omega) = \frac{-1}{2\beta} \frac{1}{1 + \alpha_4 + \beta - s_a \tilde{\omega}} \log \left( 1 + \alpha_4 - \beta - s_a \tilde{\omega} \right). \] (22)
with \( \tilde{\omega} \equiv \omega/(v_F q) \).
The linear dispersion limit corresponds to the high density limit \( \beta \to 0 \) \( (k_F \to \infty) \)
\[ \beta \to 0 : \quad R_{HF}^a(q,\omega) \rightarrow \frac{s_a}{2\pi v_F} \frac{1}{\omega - s_a(1 + \alpha_4)} \] (23)
In this limit the functions \( \tilde{R}^a \) reduce to the noninteracting ones for linear dispersion
\[ \tilde{R}^a(q,\omega) \rightarrow \frac{q}{2\pi \omega} \frac{s_a}{s_a v_F q} \equiv \nu^a(q,\omega)_{lin}. \] (24)
The well known results for the charge and spin mode of the Tomonaga model for linear dispersion [18,19] then follow from Eqs. (17) and (18).
\[ \omega_{q,c}^2 = (v_F q)^2 \left( 1 + 2\alpha_4 \right)^2 - (2\alpha_2)^2 \]
\[ \omega_{q,s}^2 = (v_F q)^2. \] (25)
For the spinless model Eq. (19) yields the same form for the charge mode as in Eq. (25) but with \( 2\alpha_i \to \alpha_i \).

#### A. The \( g_4 \) model

While the spin response function in Eq. (18) only depends on the \( g_4 \) interaction, the charge response functions \( R_{q,a'} \) in Eq. (14) involve a coupling of both Fermi points via the \( g_2 \) interaction. The chiral model obtained by putting \( g_2 = 0 \) is called the \( g_4 \) model [18,19]. We discuss its response functions for the right movers and \( q > 0 \). The charge response follows from Eq. (17) as
\[ \delta \rho(q,R,\omega) = \frac{2\tilde{R}^R(q,\omega)}{1 - 2\tilde{g}_4 \tilde{R}^R(q,\omega)} \tilde{V}^{(c)}(q,\omega) \]
\[ (\beta \to 0) \to \frac{2R_{HF}^R(q,\omega)_{lin}}{1 - 2\tilde{g}_4 R_{HF}^R(q,\omega)_{lin}} \tilde{V}^{(c)}(q,\omega). \] (26)
In the second line we also presented the result for the linear dispersion case. It looks tempting to generalize it to nonlinear dispersion by simply dropping the index “lin” and use instead the noninteracting response function for quadratic dispersion which is obtained by putting
\( \alpha_4 = 0 \) in Eq. \([22]\). As pointed out in the introduction this would be a bad approximation as it e.g. has the non-vanishing imaginary part of the response function in the wrong frequency range. To show this we use Eq. \([14]\) to express \( \tilde{R}^R \) in terms of \( R^R_{\text{HF}} \)

\[
\delta\langle \rho_{q,R} \rangle_\omega = \frac{2R^R_{\text{HF}}(q,\omega)}{1 - \tilde{g}_4R^R_{\text{HF}}(q,\omega)} \tilde{V}^{(c)}_R(q,\omega) \quad (27)
\]

For \( q > 0 \) the continuous part of the imaginary part of this RPAE response function corresponding to the HF 1p-1h excitations is different from zero for \( v_F q (1 + \alpha_4 - \beta) \leq \omega \leq v_F q (1 + \alpha_4 + \beta) \), while the imaginary part of \( R^R_{\text{HF}}(q,\omega) \) is different from zero for \( v_F q (1 - \beta) \leq \omega \leq v_F q (1 + \beta) \).

The RPAE charge mode follows from Eq. \([27]\) by putting the denominator to zero. The solution of the resulting transcendental equation is given by

\[
\omega^\text{RPA}_{q,c} = v_F q \left( 1 + \alpha_4 + \beta \coth \frac{\beta}{\alpha_4} \right) , \quad (28)
\]

which in the limit \( \beta \to 0 \) agrees with the \( \alpha_2 = 0 \) result in Eq. \([25]\). The mode is always above the the HF 1p-1h continuum as also shown in Fig. 1. This is not the case for the RPA-charge mode which is obtained from \( 1 - 2\tilde{g}_4 R^R_{\text{HF}} = 0 \) as

\[
\omega^\text{RPA}_{q,c} = v_F q \left( 1 + \beta \coth \frac{\beta}{2\alpha_4} \right) . \quad (29)
\]

It follows from Eq. \([18]\) that the RPAE collective spin mode is generally independent of \( \tilde{g}_2 \). Its location is given by

\[
\omega^\text{RPA}_{q,s} = v_F q \left( 1 + \alpha_4 - \beta \coth \frac{\beta}{\alpha_4} \right) , \quad (30)
\]

which in the limit \( \beta \to 0 \) agrees with the result in Eq. \([26]\) which is independent of the interaction strength for the Tomonaga model with linear dispersion. The RPA spin response function \( 2\tilde{R}^R_{\text{HF}}(q,\omega) \) has only a continuous non-vanishing imaginary part. This is different for the RPAE result for nonlinear dispersion, where the delta spike of the spin mode is always below the HF 1p-1h continuum.

In Fig. 1 we show the dispersion of the modes relative to the HF-continuum for \( \alpha_4 = 0.5 \).

The imaginary part of the RPAE spin response function as a function of \( \omega \) for fixed \( \beta = 0.1 \) is shown in Fig. 2, where it is compared to the result of the (bare) RPA result, which shows no collective mode. The weight of the RPAE collective spin mode which is slightly below \( \tilde{\omega} = 1 \) carries 94.8 per cent of the total RPAE spectral weight.

B. Galilei invariant model: \( \tilde{g}_2 = \tilde{g}_4 = \tilde{v}_0 \)

In this subsection we discuss the model that describes interacting fermions with a spin independent two-body interaction \( v(|x_i - x_j|) \) on a ring with

\[
\tilde{v}(q) = \tilde{v}_0 \Theta(k_c - |q|) , \quad (31)
\]

and \( k_c \leq k_F/2 \).

Then the expression for the total charge response in
ear dispersion

\[ \alpha \]

including spin for a repulsive interaction of dimensionless strength \( \alpha = \tilde{v}_0/(2\pi v_F) = 0.25 \). The RPAE result (full line) lies above \( 1 + \alpha + \beta \) indicated by the dashed line, marking the top of the HF 1p-1h continuum. The RPA charge velocity (dashed-dotted line) crosses the dashed line and only stays above \( 1 + \beta \) corresponding to the bare 1p-1h continuum (not shown).

Eq. (17) simplifies to

\[
R(q,\omega) \equiv \sum_{a,a'} R_{aa'}(q,\omega) = \frac{\tilde{R}(q,\omega)}{1 - \tilde{v}_0 R(q,\omega)} \tag{32}
\]

\[
(\beta \to 0) \to \frac{R_0(q,\omega)}{1 - \tilde{v}_0 R_0(q,\omega)}_{\text{lin}}
\]

where in \( \tilde{R} \equiv 2(\tilde{R}^R + \tilde{R}^L) \) and \( R_0 \equiv 2(R_0^R + R_0^L) \) the factor two is due to the spin degeneracy. For the spinless model one obtains the same result but this factor of two is missing.

As in the previous subsection it would be a bad approximation to generalize the bare RPA in the second line in Eq. (32) (which provides the exact result for linear dispersion) by simply dropping the index “lin”.

The transcendental equation which results from putting the denominator in Eq. (32) equal to zero can no longer be solved analytically. In Fig. 3 we show numerical results for \( v_c(q) \equiv \omega_{q,c}/|q| \) for the spinfull model for \( \alpha = \tilde{v}_0/(2\pi v_F) = 0.25 \).

As a comparison we also show the corresponding bare RPA result. For \( 0 < \alpha < 2 \) the charge velocity for linear dispersion \( v_c = v_F \sqrt{1 + 4\alpha} \) is larger than \( v_F^{\text{HF}} = v_F(1 + \alpha) \). For these \( \alpha \) values the RPAE charge mode is above the HF 1p-1h continuum. For larger repulsive interactions \( \alpha > 2 \) the charge velocity for linear dispersion is smaller than \( v_F^{\text{HF}} \) which puts the RPAE charge mode below the HF 1p-1h continuum.

In the spinless model the RPAE charge mode for all \( \alpha > 0 \) lies below the HF 1p-1h continuum. This can easily be seen by examining the denominator in Eq. (19). This is qualitatively different from the RPA result which has the charge mode above the noninteracting continuum.

In an approximation beyond RPAE the collective modes discussed above are expected to acquire a finite lifetime leading to a broadening of the delta peaks in the imaginary part of the response function. Because the higher HF p-h continua have an additional extent for energies above the 1p-1h continuum but not below it, the damping of the collective modes will depend crucially on how they are located relative to the HF 1p-1h continuum.

\[
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\[
R(q,\omega) \equiv \sum_{a,a'} R_{aa'}(q,\omega) = \frac{\tilde{R}(q,\omega)}{1 - \tilde{v}_0 R(q,\omega)} \quad (\beta \to 0) \to \frac{R_0(q,\omega)}{1 - \tilde{v}_0 R_0(q,\omega)}_{\text{lin}}
\]

\[
where \tilde{R} \equiv 2(\tilde{R}^R + \tilde{R}^L) \) and \( R_0 \equiv 2(R_0^R + R_0^L) \) the factor two is due to the spin degeneracy. For the spinless model one obtains the same result but this factor of two is missing.

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\text{IV. SUMMARY}
\]

In this paper we have revisited the old problem of the long wavelength linear charge and spin response of interacting fermions in one spatial dimension. As there is renewed interest in the questions of the lifetime of the ideal bosonic modes of the Tomonaga model due to deviations from the linear electronic dispersion\textsuperscript{10,11,12} we hope to have clarified the often quoted statement that “RPA becomes exact in the long wavelength limit”. We have shown that a naive generalization of this result fails badly.

We have not presented a calculation for the lifetimes but only set the proper stage from where to start a perturbative approach. In order to avoid tadpole diagrams HF-propagators have to be used. The use of bare propagators is useful only for the case of linear dispersion because of the “closed loop theorem”\textsuperscript{5,8}.

For the spinfull model and not too strong repulsive interactions the collective charge and spin modes are above respectively below the HF 1p-1h continuum. As the higher particle-hole continua for fixed \( q \) with particle hole pairs around both Fermi points are higher in energy the damping of the spin mode is expected to be quite different from the charge mode. For the spinless model and repulsive interaction the RPAE charge mode is always below the 1p–1h continuum. Therefore we also expect the spinfull and the spinless model to behave differently concerning the damping of the charge mode for weak repulsive interactions.

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\text{V. ACKNOWLEDGEMENTS}
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The author is grateful to P. Kopietz and V. Meden for valuable discussions.

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