Dynamics of Quantum Vorticity in a Random Potential

Bennett Link
Montana State University, Department of Physics, Bozeman MT 59717

(Dated: March 27, 2009)

I study the dynamics of a superfluid vortex in a random potential, as in the inner crust of a neutron star. Below a critical flow velocity of the ambient superfluid, a vortex is effectively immobilized by lattice forces even in the limit of zero dissipation. Low-velocity, translatory motion is not dynamically possible, a result with important implications for understanding neutron star precession and the dynamical properties of superfluid nuclear matter.

PACS numbers: 97.60.Jd, 26.60.-c, 47.37.+q, 97.60.Gb

A neutron star (NS) is expected to comprise over a solar mass of distinct quantum liquids. In the inner crust (between the drip density and approximately half nuclear density), a $^{1}$S neutron superfluid (SF) threaded by an array of quantized vortices coexists with the ionic lattice. Forces exerted by the vortex array on the lattice could have important observable effects on NS spin and thermal evolution (e.g.,[1]2,3,4,5), especially if vortices pin to the lattice as suggested long ago[6]. Evidence for long-period precession in some NSs (e.g.,[7]), an interpretation supported by quantitative modeling[8], is difficult to explain if vortices pin in the inner crust[2] and presents challenges to the standard picture of the outer core[9]. The critical question of whether or not pinning occurs in the inner crust has been studied (e.g.,[10,11]) but not yet satisfactorily answered. Here I show that a vortex is trapped by the inner-crust lattice under very general conditions if the ambient SF velocity is below a critical value. The problem of vortex dynamics and pinning in a lattice potential is also of considerable interest in laboratory Bose-Einstein condensates (see, e.g.,[12]).

The vortex drag description.—Most hydrodynamic studies of the coupling problem begin by including a term in the SF acceleration equation for the “mutual friction” force in a homogeneous medium. For a SF of mass density $\rho$, flowing at velocity $v$, coupled to a medium moving at velocity $v_m$, and neglecting the collective tension of the vortex lattice, the mutual friction force density in the non-rotating frame is[13,14]

$$F_{mf} = \beta \rho S \nu \times (v - v_m) + \beta \rho S \nu \times (\nu \times (v - v_m)),$$  \hspace{1cm} (1)

where $\nu \equiv \nabla \times v$ is the SF vorticity, $\nu$ is a unit vector in the direction of $\nu$, and $\beta$ and $\beta'$ are coefficients to be determined by a microscopic calculation for the application of interest. This drag term, or extensions of it to bulk neutron matter in beta equilibrium, has been widely used in studies of NS hydrodynamics and precession; the “medium” could be taken to be, for example, the lattice of the NS inner crust, magnetic flux tubes of the outer core with which the vortices interact, and the charged fluid of the outer core. The mutual friction force of eq. 1 is directly related to the drag force per unit length on a vortex

$$f_d = -\eta \nu - \eta' \nu \times v,$$ \hspace{1cm} (2)

where $v$ is the vortex velocity with respect to the medium, and the coefficients $\eta$ and $\eta'$ are related to $\beta$ and $\beta'$. The second term in eq. 2 is non-dissipative and is usually assumed to be zero, as I also assume; in this case $\beta = \eta / (1 + \eta^2)$ and $\beta' = \eta \beta$, where $\eta r \equiv \eta / \rho \kappa$ is the reduced drag coefficient and $\kappa$ is the vorticity quantum $h/2m_n$ ($m_n$ is the neutron mass for a neutron SF). Dissipative processes that act on a translating vortex include electron scattering[15,16], and the excitation of vortex waves (Kelvin modes) through the vortex-nucleus interaction[17,18]. All of these calculations give $\eta r << 1$.

Neglecting vortex bending and local non-dissipative forces, the motion of a vortex segment follows from equating the sum of the Magnus and drag forces to zero:

$$\rho S \kappa \times (v - v_s) - \eta v = 0,$$ \hspace{1cm} (3)

where $\kappa$ is aligned with the vortex (the $z$ axis) and $v_s$ is now the velocity of the ambient SF in the rest frame of the medium, taken to be along the $-x$ axis. In this description (henceforth, the “drag description”), the vortex segment moves at velocity $v \approx v_s$ for low drag ($$\eta r << 1$$) and $v \approx 0$ for high drag ($$\eta r >> 1$$). The large drag limit has been taken as corresponding to the pinning of vortices to defects, such as nuclei in the inner crust or magnetic flux tubes in the outer core[19,20,21].

The drag description embodied in eq. 3, however, incorrectly describes the motion of a vortex through these fundamentally inhomogeneous environments. A correct description of vortex motion in the inner crust requires the inclusion of two additional forces: 1) the local, non-dissipative component of the force exerted on the vortex by the lattice, and, 2) the elastic force of the vortex. These forces qualitatively change the vortex motion. Appropriate to the inner crust (but not the core), I focus on a single-component SF.

Vortex motion through a random lattice.—Consider the motion of a vortex that is dissipatively coupled to a background lattice. The displacement vector of the vortex
with respect to the z-axis is \( \mathbf{u}(z, t) = u_x(z, t) \hat{x} + u_y(z, t) \hat{y} \).

The equations of motion for a vortex moving in the absence of external forces can be found in Sonin [22]; a hydrodynamic description suffices for excitation wavelengths significantly larger than the SF coherence length \( \xi \), about 10 fm in the inner crust. The force per unit length exerted on the vortex by the lattice has a non-dissipative contribution \( f_0 \) and a dissipative contribution taken here to be the drag force of eq. (2) with \( \eta' = 0 \), assumed to hold locally, and approximated as linear in the local vortex velocity (but easily generalized). The equations of motion become

\[
T_v \frac{\partial^2 \mathbf{u}}{\partial z^2} + \rho_s \kappa \times \left( \frac{\partial \mathbf{u}}{\partial t} - v_s \right) + f_0 - \eta \frac{\partial \mathbf{u}}{\partial t} = 0, \tag{4}
\]

where \( T_v = (\rho_s \kappa^2 / 4\pi) \ln(\xi k)^{-1} \) is the vortex self-energy (tension) for an excitation of wave number \( k \). The first term represents the restoring force of tension as the vortex is bent. The second term is the Magnus force per unit length exerted on a vortex that is moving with respect to the ambient SF. In the absence of external forces, the vortex would remain straight and motionless with respect to the solid. An essential feature of vortex dynamics is that the local vortex velocity is determined entirely by external forces and by the shape of the vortex, and not by inertial forces (for temperatures small compared to the pairing gap). In the absence of external forces, the solutions to eqs. (4) are circularly-polarized, diffusive waves (Kelvin waves) with frequencies \( \omega_k = \pm T_0 k^2 / \rho_s \kappa \) [23]. Characteristic bending scales of the vortex in the problem are \( \sim 500 \text{ fm} \) (see below), and fixing \( \ln(\xi k)^{-1} = 3 \) is an adequate approximation. I ignore quantum effects on the vortex motion and thermal excitations.

Nuclei in the solid with which the superfluid coexists exert local forces on the vortex which vary over a length scale of order the average nuclear spacing \( a \), typically 30-50 fm in the inner crust. Calculations of the vortex-nucleus interaction energy \( E_{vn} \) give values of up to \( \sim 1 \text{ MeV} \), attractive in some regions and repulsive in others [24, 25]; the corresponding characteristic force on a vortex segment of length \( a \) is \( F_m \equiv a^{-1} E_{vn} \). The solid is most likely amorphous [26], in which case the vortex interacts with an effectively random potential Even if the lattice does possess long-range crystalline order, vortices will be aligned with the SF rotation axis on average which will not, in general, coincide with any symmetry axis of the solid; a random lattice approximation should be adequate also in this situation. In the denser regions of the NS inner crust, \( T_v \) is of order an \( \text{MeV fm}^{-1} \). Defining a dimensionless strength parameter \( s \equiv F_m / T_v \) (\( > 0 \) for an attractive potential), estimates of \( F_m \) from Refs. [24, 25] give \( 10^{-2} \leq |s| \leq 0.1 \). A vortex is thus a stiff object; the vortex-nucleus interaction can bend a vortex by an amount \( a \) only over a length scale of order \( \gtrsim 10a \) [24]; a vortex segment of length \( a \) can be regarded as straight to a good approximation and the lattice force on the vortex can be taken as acting in the \( x-y \) plane at any given \( z \) along the vortex. For the force exerted by a single nucleus at the origin on a vortex segment of length \( a \), I take a parameterized central force in the \( x-y \) plane:

\[
f_{vn}(\mathbf{u}) = -1.7 F_m \frac{\mathbf{u}}{r_p} e^{-u^2 / 2r_p^2} \tag{5}
\]

The force has a maximum magnitude of \( F_m \) at \( u = r_p \), where \( r_p \) is the effective range of the potential, henceforth taken to be \( r_p = a / 2 \). Dividing the vortex into segments of length \( a \), let the total non-dissipative force on the \( i \)-th segment be

\[
f_0(z_i) = \sum_j f_{vn}(\mathbf{u}_i - \mathbf{r}_j), \tag{6}
\]

where the nuclei are randomly placed at locations \( \mathbf{r}_j \) in planes separated in \( a \) and parallel to the \( x-y \) plane. For simplicity, each summation is only over nuclei in the \( i \)-th plane. In the following, lengths will be expressed in units of \( a \), time in units of \( t_D \equiv \rho_s a^2 / T_v \) (the characteristic diffusion time of a Kelvin wave of wavenumber \( k = a^{-1} \) along the segment) and velocities in units of \( a / t_D \). As described below, the sign of \( s \) is unimportant for \( r_p \sim a \); I choose \( s > 0 \), corresponding to attractive nuclear potentials, for illustration. I focus on the regime \( 0.1 \geq \eta_r \geq 0 \) which widely brackets the range found by all drag calculations, and take \( s = 0.1 \).

Comparison of terms in eq. (4) shows that the dynamics are essentially determined by the non-dissipative lattice force and \( s \) in the regime \( v_s < v_c \). To determine the character of the dynamical regimes of vortex motion and the scaling relations that define them, eqs. (4-6) were solved numerically. The vortex was divided into 50 zones of length \( a \), with periodic boundary conditions applied at
the ends, in a random lattice consisting of $10^3$ nuclei per zone. Fig. 1 (left panel) shows that the vortex is trapped by the random potential for finite $v_s$ even for zero drag. The chief result of the numerical analysis is shown in the right panel of Fig. 1. Trapping occurs for $v_s$ below a critical velocity $v_c \simeq s^{3/2}$, independent of drag in the regime $\eta_r \ll 1$. For $v_s > v_c$, the vortex moves at $v \simeq v_s$; in this regime, the drag description of eq. 3 is adequate. Below $v_c$, however, the vortex velocity is effectively zero. The drag description fails to show this transition to the pinned state because it excludes both the non-dissipative lattice force and the finite vortex tension. The value of the critical velocity $v_c \sim s^{3/2}$ follows from consideration of a vortex in static equilibrium, as discussed elsewhere \cite{10,27}. Only if the vortex had infinite tension, one of the assumptions implicit in the drag description and eq. 3, would translatory states exist for any $v_s$.

Dissipative processes will damp the vortex to a stationary, pinned configuration. Fig. 2 shows an example segment of the long vortex moving under drag. For $v_s < v_c$, the trajectory is essentially the same as for $v_s = 0$ since the non-dissipative force dominates the global Magnus force in this velocity regime. Finite $v_s$ (below $v_c$), produces only a small displacement of the vortex segment’s final, damped position. Inspection of eq. 4 suggests that a dragged vortex will damp to a pinned position over a characteristic time $\sim (s\eta_r)^{-1}$, and numerical experiments confirm this. The damping time is $\sim 100$ for the examples shown in Fig. 2. Fig. 3 shows the initial motion and damping of the vortex in the $x-z$ plane. The initially-straight vortex forms kinks immediately in the presence of the random potential, and by $t = 10$ the vortex has been excited to amplitudes of order $a$, the length scale of the random potential. By $t = 10^3$, the vortex has damped to a stationary pinned configuration with bends over a characteristic length scale of $\sim 10a$. Because the vortex has large tension, it cannot bend to intersect every nucleus, but assumes a shape that strikes

FIG. 2: Motion of an example vortex segment through one plane of the lattice for $\eta_r = 0.1$. Left panel: For $v_s > v_c \sim s^{3/2}$, the vortex translates with the superfluid, with significant changes in direction as it interacts with nuclei (denoted by circles). For $v_s < v_c$, the segment damps to a pinned configuration that brings it closer to nuclei. Middle panel: Comparison of the damped motion for $v_s = 0$ and $v_s < v_c$, showing that finite $v_s$ has little effect on the motion in this regime. Right panel: Detail of the box region of the middle figure. Circles on the lines indicate points of equal time. The result of finite $v_s$ is to displace the pinned vortex slightly in the $y$ direction.

the best compromise between the energy gain of being close to a nucleus and the cost of bending the vortex.

Further simulations show that these conclusions are unaffected if the attractive nuclear potentials are replaced by repulsive ones (though $v_c$ is slightly reduced for the force of eq. 5). This result is not surprising, since for $r_p \sim a$ the lattice potential effectively turns inside out. A vortex segment now damps to a position that maximizes its distance from the nearest nuclei to the extent possible against opposing tension forces. For $\eta_r = 0$ the vortex is trapped by the lattice as for the case of attractive nuclei. Since these results are independent of the value of $\eta_r$ provided $\eta_r \ll 1$, they are also independent of the assumption that $\eta_r$ is a constant.

Recent calculations of $E_{vn}$, using the local density approximation \cite{24} and mean field theory \cite{25} are not in complete agreement. Though both calculations predict comparable interaction energies ($\sim 1$ MeV) above an inner-crust density of $\sim 10^{13}$ g cm$^{-3}$, they do not agree
on the densities where the interaction is the strongest. The inevitability of pinning is nevertheless a robust result; it is insensitive to the exact density range in which \( E_{vn} \) is strongest, the length scale of the potential, or even the sign of the potential. Within these uncertainties, the critical velocity above which vortices will translate is

\[
v_c \sim s^{1/2} \frac{F_m}{\rho_s \kappa a} = 10^6 - 10^7 \text{ cm s}^{-1}.
\]

(7)

The lattice will trap the vortex for much smaller values of \( E_{vn} \) as well, though \( v_c \) will be lowered according to eq. (7).

Conclusions and implications.—Pinning of vortices to the lattice of the NS inner crust below a critical value of \( v_s \) appears to be inevitable, whether the interaction is attractive or repulsive (and provided \( E_{vn} \) exceeds the stellar temperature, typically \( \sim 0.01 \text{ MeV} \)). For interactions of order \( \sim 1 \text{ MeV per nucleus} \), the pinned vortex lattice becomes unstable for \( v_s > v_c \sim 10^6 - 10^7 \text{ cm s}^{-1} \), and vortices will translate approximately with the SF if the drag is low. For critical velocities this large, the pinned superfluid can store enough angular momentum to drive the giant glitches seen in pulsars [27]. If pinning in regions of the inner crust does occur, however, interpretation of the putative precession seen in some pulsars becomes problematic; as noted by Shaham [2], if even a small portion of the inner crust vorticity is tightly coupled to the solid, the star will precess much faster than the period of \( \sim 1 \text{ yr} \) indicated by observations [7]. In the outer core, where the protons are predicted to form a type-II superconductor, vortices are expected to pin against a disorganized system of flux tubes and a similar difficulty arises in explaining long-period precession [9]. Anywhere there is pinning, however, vortices can creep \((v \ll v_s)\) through thermal activation or quantum tunneling processes [3][28] not considered here. Vortex creep is incompatible with long-period precession if it is a high-drags process [1][20].

The core neutron-charge mixture could be unstable to the formation of SF turbulence in a precessing star [19][29], though magnetic stresses might suppress such an instability [30]. [Turbulent instabilities might not occur in the single-component SF of the inner crust]. These analyses used the drag description of vortex motion which does not correctly describe vortex motion through a potential; it is necessary to include non-dissipative pinning forces. The mutual friction force of eq. (4) should be replaced with the total force per unit volume exerted on the SF by the vortex array. For a single-component SF, this force is (see, e.g., [1]):

\[
F = \rho_s \{v(v_s) - v_s\} \times \omega,
\]

(8)

where \( v(v_s) \) is local vortex velocity with respect to the background, averaged over many vortices, in response to a local flow \( v_s \). In the approximations made here, \( v = 0 \) for \( v_s < v_c \), and \( F \) reduces to the Magnus force per unit volume on pinned vortices. At finite \( v_s \), however, vortices can move slowly through vortex creep, a process which dissipates energy at a rate \( \rho_s v \cdot \{v(v_s) \times \omega\} \) per unit volume; the dissipative force in the fluid would be small if \( v(v_s) \) is nearly orthogonal to \( v_s \times \omega \), while the non-dissipative force would be, in any case, nearly equal to the Magnus force on perfectly pinned vortices. It would be interesting to study NS star modes under the assumption that creep proceeds with little dissipation, the opposite limit of that studied so far.

I thank I. Wasserman and C. M. Riedel for useful discussions.

*link@physics.montana.edu*

[1] A. Sedrakian, I. Wasserman, and J. M. Cordes, Astrophys. J. 524, 341 (1999).
[2] J. Shaham, Astrophys. J. 214, 251 (1977).
[3] M. A. Alpar, P. W. Anderson, D. Pines, and J. Shaham, Astrophys. J. 276, 325 (1984).
[4] B. Link and R. I. Epstein, Astrophys. J. 457, 844 (1996).
[5] N. Shibazaki and F. K. Lamb, Astrophys. J. 346, 808 (1989).
[6] P. W. Anderson and N. Itoh, Nature 256, 25 (1975).
[7] I. H. Stairs, A. G. Lyne, and S. L. Shemar, Nature 406, 484 (2000).
[8] T. Akgün, B. Link, and I. Wasserman, Mon. Not. Roy. Astr. Soc. 365, 653 (2006).
[9] B. Link, Phys. Rev. Lett. 91, 101101 (2003).
[10] P. B. Jones, Phys. Rev. Lett. 81, 4560 (1998).
[11] P. B. Jones, Phys. Rev. Lett. 79, 792 (1997).
[12] R. Bhat, M. J. Holland, and L. D. Carr, Phys. Rev. Lett. 96, 060405 (2006).
[13] H. E. Hall and W. F. Vinen, Proc. R. Soc. A 238, 215 (1956).
[14] I. L. Bekarevich and I. M. Khalatnikov, Sov. Phys. JETP 13, 643 (1961).
[15] P. J. Feibelman, Phys. Rev. D 4, 1589 (1971).
[16] L. Bildsten and R. I. Epstein, Astrophys. J. 342, 951 (1989).
[17] R. I. Epstein and G. Baym, Astrophys. J. 387, 276 (1992).
[18] P. B. Jones, Mon. Not. Roy. Astr. Soc. 257, 501 (1992).
[19] K. Glampedakis, N. Andersson, and D. I. Jones, Phys. Rev. Lett. 100, 081101 (2008).
[20] B. Link, Astron. Astrophys. 458, 881 (2006).
[21] A. Sedrakian, Phys. Rev. D 71, 083003 (2005).
[22] E. B. Sonin, Rev. Mod. Phys. 59, 87 (1987).
[23] W. Thompson, Philos. Mag. 10, 155 (1880).
[24] P. Donati and P. M. Pizzochero, Phys. Lett. B 640, 74 (2006).
[25] P. Avogadro, F. Barranco, R. A. Broglia, and E. Vigoggi, Phys. Rev. C 75, 012805 (2007).
[26] P. B. Jones, Mon. Not. Roy. Astr. Soc. 321, 167 (2001).
[27] B. Link and C. Cutler, Mon. Not. Roy. Astr. Soc. 336, 211 (2002).
[28] B. Link, R. I. Epstein, and G. Baym, Astrophys. J. 403, 285 (1993).
[29] C. Peralta, A. Melatos, M. Giacobello, and A. Ooi, Astrophys. J. 635, 1224 (2005).
[30] M. van Hoven and Y. Levin, Mon. Not. R. Astron. Soc.
391, 283 (2008).