INFINITESIMAL DEFORMATION OF DELIGNE CYCLE CLASS MAP

SEN YANG

ABSTRACT. In this note, we study the infinitesimal forms of Deligne cycle class maps. As an application, we prove that the infinitesimal form of a conjecture by Beilinson [1] is true.

CONTENTS

1. Introduction 1
2. Main results 2
References 10

1. INTRODUCTION

In [1], Beilinson made the following conjecture:

**Conjecture 1.1** (Conjecture 2.4.2.1 [1]). Let $X$ be a smooth projective variety defined over a number field $k$, then for each positive integer $p$, the rational Chow group $CH^p(X)_\mathbb{Q}$ injects into Deligne cohomology of $X_\mathbb{C}$, where $X_\mathbb{C} := X \times_k \mathbb{C}$. Concretely, if a class in $CH^p(X)_\mathbb{Q}$ vanishes in Deligne cohomology $H^{2p}_D(X_\mathbb{C}, \mathbb{Z}(p))_\mathbb{Q}$ under the composition

$$CH^p(X)_\mathbb{Q} \to CH^p(X_\mathbb{C})_\mathbb{Q} \xrightarrow{r} H^{2p}_D(X_\mathbb{C}, \mathbb{Z}(p))_\mathbb{Q},$$

where the right arrow $r$ is the cycle class map for Deligne cohomology, then it is 0.

This conjecture is very difficult to approach, and up to now there is not a single example with dimension $X \geq 2$ with large Chow ring $CH(X \times_k \mathbb{C})$ for which this conjecture has been verified. Esnault and Harris [4] suggest a modest conjecture (see Theorem 0.1 [4]) which follows from Conjecture 1.1 and has been proved in a particular case (see Theorem 0.2 [4]) by using $l$-adic cohomology.

2010 Mathematics Subject Classification. 14C25.

1See the first paragraph of page 1 [4].
The main result of this note is to study the infinitesimal form of the Deligne cycles class map, see Lemma 2.9 and Theorem 2.11. As an application, we prove that the infinitesimal form of the Conjecture 1.1 is true, see Theorem 2.13.

In a companion paper [12], as a further application, we show that the infinitesimal form of the following conjecture (due to Griffiths-Harris) is true,

Conjecture 1.2 (19). Let \( X \subset \mathbb{P}_C^4 \) be a general hypersurface of degree \( d \geq 6 \), we use

\[ \psi : CH^2_{hom}(X) \to J^2(X) \]

to denote the Abel-Jacobi map from algebraic 1-cycles on \( X \) homologically equivalent to zero to the intermediate Jacobian \( J^2(X) \), \( \psi \) is zero.

2. Main results

Let \( X \) be a smooth projective variety defined over \( \mathbb{C} \) and \( \mathbb{C}[\varepsilon]/(\varepsilon^2) \) be the ring of dual numbers, we use \( X[\varepsilon] := X \times \text{Spec}(\mathbb{C}[\varepsilon]/(\varepsilon^2)) \) to denote the first order infinitesimal deformation of \( X \). The classical definition Chow groups can not recognize nilpotents, so to overcome this deficiency, for each positive integer \( p \), one uses the following Soulé’s variant of Bloch-Quillen identification to study the infinitesimal deformations of Chow groups,

\[
CH^p(X) = H^p(X, K^M_p(O_X)) \mod \text{torsion,}
\]

where \( K^M_p(O_X) \) is the Milnor K-theory sheaf associated to the presheaf

\[ U \rightarrow K^M_p(O_X(U)). \]

Using the identification (2.1), one considers \( H^p(X, K^M_p(O_X[\varepsilon])) \) as the first order infinitesimal deformation of \( CH^p(X) \) and defines,

Definition 2.1. Let \( X \) be a smooth projective variety defined over \( \mathbb{C} \), for each positive integer \( p \), the formal tangent space to \( CH^p(X) \), denoted \( T_f CH^p(X) \), is defined to be the kernel of the natural map

\[
H^p(X, K^M_p(O_X[\varepsilon])) \xrightarrow{\varepsilon=0} H^p(X, K^M_p(O_X)).
\]

It is known that the formal tangent space \( T_f CH^p(X) \) can be identified with \( H^p(X, \Omega^p_{X/Q}) \), where \( \Omega^p_{X/Q} \) is the absolute differential. Deligne cohomology \( H^{2p}_D(X, \mathbb{Z}(p)) \) is defined to be the hypercohomology of the Deligne complex \( \mathbb{Z}(p)_D \) (in analytic topology):

\[ \mathbb{Z}(p)_D : 0 \to \mathbb{Z}(p) \to O_X \to \cdots \to \Omega^p_{X/C} \to 0, \]
where $Z(p) = (2\pi i)^p \mathbb{Z}$ is in degree 0. The infinitesimal deformation of this complex, denoted $Z(p)_D[\varepsilon]$, has the form,

$$Z(p)_D[\varepsilon] : 0 \to Z(p) \to O_X[\varepsilon] \to \cdots \to \Omega^{p-1}_{X[\varepsilon]/\mathbb{C}[\varepsilon]} \to 0,$$

where $Z(p)$ is still equal to $(2\pi i)^p \mathbb{Z}$.

**Definition 2.2.** Let $X$ be a smooth projective variety defined over $\mathbb{C}$, for each positive integer $p$, the tangent complex to the Deligne complex $Z(p)_D$, denoted $\mathbb{Z}(p)_D$, is defined to be the kernel of the natural map

$$Z(p)_D[\varepsilon] \xrightarrow{\varepsilon=0} Z(p)_D.$$

Since the map $X \to X[\varepsilon]$ has a retraction $X[\varepsilon] \to X$, $\Omega^i_{X[\varepsilon]/\mathbb{C}[\varepsilon]} = \Omega^i_{X/\mathbb{C}} \oplus \varepsilon \Omega^i_{X/\mathbb{C}}$, where $i = 0, \cdots, p-1$. The tangent complex to the Deligne complex is a direct summand of the thickened Deligne complex

$$Z(p)_D[\varepsilon] = Z(p)_D \oplus \mathbb{Z}(p)_D.$$

One easily sees that

**Lemma 2.3.** Let $X$ be a smooth projective variety defined over $\mathbb{C}$, for each positive integer $p$, the tangent complex (to the Deligne complex $Z(p)_D$) $\mathbb{Z}(p)_D$ is of the form

$$0 \to O_X \to \cdots \to \Omega^{p-1}_{X/\mathbb{C}} \to 0,$$

where $O_X$ is in degree 1 and $\Omega^{p-1}_{X/\mathbb{C}}$ is in degree $p$.

We consider the hypercohomology $\mathbb{H}^p(X, Z(p)_D[\varepsilon])$ of the complex $\mathbb{Z}(p)_D[\varepsilon]$ as the infinitesimal deformation of the Deligne cohomology $H^p_D(X, Z(p))$ and define,

**Definition 2.4.** Let $X$ be a smooth projective variety defined over $\mathbb{C}$, for each positive integer $p$, the formal tangent space to the Deligne cohomology $H^p_D(X, Z(p))$, denoted $T_f H^p_D(X, Z(p))$, is defined to be the kernel of the natural map

$$\mathbb{H}^p(X, Z(p)_D[\varepsilon]) \xrightarrow{\varepsilon=0} \mathbb{H}^p(X, Z(p)_D).$$

By the definition, the formal tangent space to the Deligne cohomology is the hypercohomology of the tangent complex $\mathbb{Z}(p)_D$:

$$T_f H^p_D(X, Z(p)) = \mathbb{H}^p(X, \mathbb{Z}(p)_D) = \mathbb{H}^{p-1}(X, \mathbb{Z}(p)_D[-1]).$$

The following isomorphism is a standard fact in complex geometry,

**Lemma 2.5** (cf. Proposition on page 17 of [6]). With the notations above, one has the isomorphism

$$\mathbb{H}^p(X, \mathbb{Z}(p)_D) \cong H^{p-1}(O_X) \oplus H^{2p-2}(\Omega^1_{X/\mathbb{C}}) \oplus \cdots \oplus H^p(\Omega^{p-1}_{X/\mathbb{C}}).$$
Let $\overline{K}_p^M(O_X)$ denote the kernel of the natural map
\[
K_p^M(O_X[\varepsilon]) \xrightarrow{\varepsilon=0} K_p^M(O_X).
\]
Since the map $X \to X[\varepsilon]$ has a retraction $X[\varepsilon] \to X$, $K_p^M(O_X[\varepsilon]) = K_p^M(O_X) \oplus \overline{K}_p^M(O_X)$. By Definition 2.1, the tangent space $T_f CH^p(X)$ is $H^p(X, \overline{K}_p^M(O_X))$. Next, we would like to construct a map between tangent spaces
\[
H^p(X, \overline{K}_p^M(O_X)) \to \mathbb{H}^{2p}(X, \mathbb{Z}(p)_D),
\]
which is the infinitesimal form of the Deligne cycle class map $CH^p(X) \to H_D^{2p}(X, \mathbb{Z}(p))$.

An element of $\overline{K}_p^M(O_X)$ is of the form $\prod_i \{f_1^i + \varepsilon g_1^i, \ldots, f_p^i + \varepsilon g_p^i\} \in K_p^M(O_X[\varepsilon])$ such that $\prod_i \{f_1^i, \ldots, f_p^i\} = 1 \in K_p^M(O_X)$. We are reduced to looking at $\{f_1 + \varepsilon g_1, \ldots, f_p + \varepsilon g_p\} \in K_p^M(O_X[\varepsilon])$ such that $\{f_1, \ldots, f_p\} = 1 \in K_p^M(O_X)$. For simplicity, we assume that $g_2 = \cdots = g_p = 0$ and have
\[
\{f_1 + \varepsilon g_1, f_2, \ldots, f_p\} = \{f_1, f_2, \ldots, f_p\} \{1 + \varepsilon \frac{g_1}{f_1}, f_2, \ldots, f_p\} = \{1 + \varepsilon \frac{g_1}{f_1}, f_2, \ldots, f_p\}.
\]

Applying $\wedge^d \Delta g$ to $\{1 + \varepsilon \frac{g_1}{f_1}, f_2, \ldots, f_p\}$, where $d = d_{C[\varepsilon]}$, one obtains
\[
\frac{d_{C[\varepsilon]}(1 + \varepsilon \frac{g_1}{f_1})}{1 + \varepsilon \frac{g_1}{f_1}} \wedge \frac{d_{C} f_2}{f_2} \wedge \cdots \wedge \frac{d_{C} f_p}{f_p} = (1 - \varepsilon \frac{g_1}{f_1}) d_{C[\varepsilon]}(1 + \varepsilon \frac{g_1}{f_1}) \wedge \frac{d_{C} f_2}{f_2} \wedge \cdots \wedge \frac{d_{C} f_p}{f_p}
\]
\[
= \varepsilon d_{C}(\frac{g_1}{f_1}) \wedge \frac{d_{C} f_2}{f_2} \wedge \cdots \wedge \frac{d_{C} f_p}{f_p}
\]
\[
= \varepsilon \frac{f_1 d_{C} g_1 - g_1 d_{C} f_1}{f_1^2} \wedge \frac{d_{C} f_2}{f_2} \wedge \cdots \wedge \frac{d_{C} f_p}{f_p}.
\]
This gives a map from $\overline{K}_p^M(O_X)$ to $\Omega^p_{X/C}$.

**Definition 2.6.** One defines a map $\Delta^p \Delta g : \overline{K}_p^M(O_X) \to \Omega^p_{X/C}$ by
\[
(2.2) \quad \{1 + \varepsilon \frac{g_1}{f_1}, f_2, \ldots, f_p\} \to \frac{f_1 d_{C} g_1 - g_1 d_{C} f_1}{f_1^2} \wedge \frac{d_{C} f_2}{f_2} \wedge \cdots \wedge \frac{d_{C} f_p}{f_p}.
\]

On the other hand, let
\[
(2.3) \quad \omega = \frac{g_1}{f_1} \frac{d_{C} f_2}{f_2} \wedge \cdots \wedge \frac{d_{C} f_p}{f_p} \in \Omega^{p-1}_{X/C},
\]
one checks that
\[
d_C(\omega) = d_C\left(\frac{g_1}{f_1 f_2 \cdots f_p}\right) \wedge d_C f_2 \wedge \cdots \wedge d_C f_p
\]
\[
= \left(\frac{1}{f_1 f_2 \cdots f_p}d_C g_1 - g_1 d_C(f_1 f_2 \cdots f_p)\right) \wedge d_C f_2 \wedge \cdots \wedge d_C f_p
\]
\[
= \left(\frac{1}{f_1 f_2 \cdots f_p} f_1 f_2 \cdots f_p d_C f_1 \right) \wedge d_C f_2 \wedge \cdots \wedge d_C f_p.
\]

Comparing \(d_C(\omega)\) with (2.2), one sees that

**Lemma 2.7.** With the notations above, one has
\[
\tilde{\lambda}' d\log(\{1 + \varepsilon g_1/f_1, f_2, \cdots, f_p}\}) = d_C(\omega),
\]
where \(\omega = \frac{g_1 d_C f_2}{f_1 f_2} \wedge \cdots \wedge \frac{d_C f_p}{f_p} \).

The following commutative diagram, which gives a quasi-isomorphism from the upper complex to the bottom one, is the tangent to the commutative diagram in Section 2.7 [5] (page 56),

\[
\begin{array}{ccccccccc}
O_X & \longrightarrow & \cdots & \longrightarrow & \Omega_{X/\mathbb{C}}^{p-2} & \longrightarrow & \Omega_{X/\mathbb{C}}^{p-1} & \longrightarrow & 0 \\
\alpha_1 & \downarrow & & \downarrow & \alpha_{p-1} & \downarrow & \alpha_p & \downarrow & \\
O_X & \longrightarrow & \cdots & \longrightarrow & \Omega_{X/\mathbb{C}}^{p-2} & \longrightarrow & \Omega_{X/\mathbb{C}}^{p-1} \oplus \Omega_{X/\mathbb{C}}^{p-2} & \longrightarrow & \Omega_{X/\mathbb{C}}^{p+1} \oplus \Omega_{X/\mathbb{C}}^{p} \cdots,
\end{array}
\]
where \(\alpha_i(x) = (-1)^{i-1}(x)\) for \(1 \leq i \leq p-1\) and \(\alpha_p(x) = (-1)^{p-1}(d_C x, x)\); \(\delta_i(x) = dx\) for \(1 \leq i \leq p-2\), \(\delta_{p-1}(x) = (0, dx)\) and \(\delta_p(x, y) = (-dx, -x + dy)\). For \((d_C \omega, \omega) \in \Omega_{X/\mathbb{C}}^{p-1} \oplus \Omega_{X/\mathbb{C}}^{p-1}\), there exists the unique \((-1)^{p-1} \omega \in \Omega_{X/\mathbb{C}}^{p-1}\) such that \(\alpha_p((-1)^{p-1} \omega) = (d_C \omega, \omega)\).

**Definition 2.8.** One defines a map \(\beta : K_p^M(O_X) \longrightarrow \Omega_{X/\mathbb{C}}^{p-1}\) by

\[
(2.4) \quad \{1 + \varepsilon g_1/f_1, f_2, \cdots, f_p\} \rightarrow (-1)^{p-1} \omega,
\]
where \(\omega = \frac{g_1 d_C f_2}{f_1 f_2} \wedge \cdots \wedge \frac{d_C f_p}{f_p}\).
We can see the map $\beta$ (2.4) in an alternative way. Firstly, applying $\wedge^pd\log$ to $\{1 + \varepsilon g_1/f_1, f_2, \cdots, f_p\}$, where $d = d_C$, one obtains

$$d_C(1 + \varepsilon g_1/f_1) \wedge \frac{d_C f_2}{f_2} \wedge \cdots \wedge \frac{d_C f_p}{f_p} = (1 - \varepsilon g_1/f_1) d_C(1 + \varepsilon g_1/f_1) \wedge \frac{d_C f_2}{f_2} \wedge \cdots \wedge \frac{d_C f_p}{f_p}$$

$$= (\varepsilon d_C(g_1/f_1) + g_1/f_1 d_C(\varepsilon)) \wedge \frac{d_C f_2}{f_2} \wedge \cdots \wedge \frac{d_C f_p}{f_p}.$$ 

Secondly, applying the truncation map $\frac{\partial}{\partial \varepsilon} \big|_{\varepsilon = 0} : \Omega^p_{X[\varepsilon]/C} \to \Omega^{p-1}_{X/C}$ to $(\varepsilon d_C(g_1/f_1) + g_1/f_1 d_C(\varepsilon)) \wedge \frac{d_C f_2}{f_2} \wedge \cdots \wedge \frac{d_C f_p}{f_p}$, one obtains $(-1)^{p-1} g_1/f_1 d_C f_2 \wedge \cdots \wedge \frac{d_C f_p}{f_p}$. This shows that the composition

$$\overline{K}_p^M(O_X) \xrightarrow{\wedge^p d\log} \Omega^p_{X[\varepsilon]/C} \xrightarrow{\frac{\partial}{\partial \varepsilon} \big|_{\varepsilon = 0}} \Omega^{p-1}_{X/C},$$

agrees with the map $\beta$ (2.4).

The map $\beta$ (2.4) induces a map from the complex

$$0 \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow \overline{K}_p^M(O_X),$$

where $\overline{K}_p^M(O_X)$ is in degree $p$, to the complex $\Omega^p_{X[\varepsilon]/C}$.

This induces a map between (hyper)cohomology groups

$$\lambda : H^p(X, \overline{K}_p^M(O_X)) \rightarrow \mathbb{H}^{2p}(X, \mathbb{Z}(p)_{\mathcal{D}}),$$

which is the infinitesimal form of the Deligne cycle class map

$$r : CH^p(X) \rightarrow H^{2p}_{B}(X, \mathbb{Z}(p)).$$

By Lemma 2.4, $\mathbb{H}^{2p}(X, \mathbb{Z}(p)_{\mathcal{D}}) \cong H^{2p-1}(O_X) \oplus \cdots \oplus H^{p}(\Omega^{p-1}_{X/C})$ and by the diagram (2.5), we note the image of the map (2.6) lies in $H^p(\Omega^{p-1}_{X/C})$. So the map (2.6) is indeed the composition:

$$H^p(X, \overline{K}_p^M(O_X)) \rightarrow H^p(\Omega^{p-1}_{X/C}) \hookrightarrow \mathbb{H}^{2p}(X, \mathbb{Z}(p)_{\mathcal{D}}).$$

\footnote{2We thank Spencer Bloch and Jerome Hoffman for comments}
In summary,

**Lemma 2.9.** Let $X$ be a smooth projective variety defined over $\mathbb{C}$, for each positive integer $p$, the infinitesimal form of the Deligne cycle class map

$$r : CH^p(X) \rightarrow H^{2p}_D(X, \mathbb{Z}(p)),$$

is given by (2.6) $\lambda : H^p(X, \overline{K}_p^M(O_X)) \rightarrow H^{2p}(X, \mathbb{Z}(p)_D)$, which is the composition:

$$H^p(X, \overline{K}_p^M(O_X)) \rightarrow H^p(\Omega^{p-1}_{X/\mathbb{C}}) \rightarrow H^{2p}(X, \mathbb{Z}(p)_D).$$

**Remark 2.10.** When we consider the infinitesimal deformation of the Deligne complex $\mathbb{Z}(p)_D$, $\mathbb{Z}(p)$ is fixed so that it does not appear in the tangent complex $\mathbb{Z}(p)_D$. This explains why the construction of $\lambda$ (2.6) is simpler than the known construction $[3, 5, 10]$ of the Deligne cycle class map.

In the following, we consider $H^p(X, \overline{K}_p^M(O_X)) \rightarrow H^p(\Omega^{p-1}_{X/\mathbb{C}})$ as the infinitesimal form of the Deligne cycle class map $r : CH^p(X) \rightarrow H^{2p}_D(X, \mathbb{Z}(p))$, and describe it explicitly.

It is well known that $\overline{K}_p^M(O_X) \cong \Omega^{p-1}_{X/\mathbb{Q}}$ and the isomorphism is given by

$$(2.7) \quad \{1 + \epsilon \frac{g_1}{f_1}, f_2, \cdots, f_p\} \rightarrow (-1)^{p-1} \frac{g_1}{f_1} \frac{d_Q f_2}{f_2} \wedge \cdots \wedge \frac{d_Q f_p}{f_p}.$$ 

Moreover, one has the following commutative diagram,

$$\begin{array}{cccc}
\{1 + \frac{g_1}{f_1}, f_2, \cdots, f_p\} & \xrightarrow{\beta} & (-1)^{p-1} \frac{g_1}{f_1} \frac{d_Q f_2}{f_2} \wedge \cdots \wedge \frac{d_Q f_p}{f_p} \\
\cong \downarrow (2.7) & & \downarrow = & \\
(-1)^{p-1} \frac{g_1}{f_1} \frac{d_Q f_2}{f_2} \wedge \cdots \wedge \frac{d_Q f_p}{f_p} & \xrightarrow{d_Q \rightarrow d_C} & (-1)^{p-1} \frac{g_1}{f_1} \frac{d_C f_2}{f_2} \wedge \cdots \wedge \frac{d_C f_p}{f_p}.
\end{array}$$

This shows the following diagram is commutative,

$$\begin{array}{cccc}
\overline{K}_p^M(O_X) & \xrightarrow{\beta} & \Omega^{p-1}_{X/\mathbb{C}} \\
\cong \downarrow (2.7) & & \downarrow = & \\
\Omega^{p-1}_{X/\mathbb{Q}} & \xrightarrow{d_Q \rightarrow d_C} & \Omega^{p-1}_{X/\mathbb{C}}.
\end{array}$$
Passing to cohomology groups, one has the following commutative diagram,
\[
\begin{array}{ccc}
H^p(X, \mathcal{R}_p^M(O_X)) & \longrightarrow & H^p(\Omega_{X/C}^{p-1}) \\
\cong & & = \\
H^p(\Omega_{X/Q}^{p-1}) & \xrightarrow{d_Q \cdot d_C} & H^p(\Omega_{X/C}^{p-1}).
\end{array}
\]

To summarize,

**Theorem 2.11.** Let \(X\) be a smooth projective variety defined over \(\mathbb{C}\), for each positive integer \(p\), the infinitesimal form of the Deligne cycle class map
\[
r : CH^p(X) \rightarrow H^{2p}_D(X, \mathbb{Z}(p)),
\]
is given by
\[
\delta r : H^p(\Omega_{X/Q}^{p-1}) \rightarrow H^p(\Omega_{X/C}^{p-1}),
\]
where \(\delta r\) is induced by the natural map \(\Omega_{X/Q}^{p-1} \rightarrow \Omega_{X/C}^{p-1}\).

Let \(X\) be a smooth projective variety over \(k\), where \(k\) is a field of characteristic 0. For each positive integer \(p\), one still has the Soulé’s variant of Bloch-Quillen identification
\[
CH^p(X) = H^p(X, K^M_p(O_X)) \text{ modulo torsion,}
\]
where \(K^M_p(O_X)\) is the Milnor K-theory sheaf associated to the presheaf
\[
U \rightarrow K^M_p(O_X(U)).
\]
The formal tangent space to \(CH^p(X)\) is identified with \(H^p(\Omega_{X/Q}^{p-1})\).

**Corollary 2.12.** Let \(X\) be a smooth projective variety defined over \(k\), where \(k\) is a field of characteristic 0, for each positive integer \(p\), the infinitesimal form of the composition
\[
CH^p(X)_Q \rightarrow CH^p(X_C)_Q \xrightarrow{r} H^{2p}_D(X_C, \mathbb{Z}(p))_Q,
\]
has the form
\[
H^p(\Omega_{X/Q}^{p-1}) \rightarrow H^p(\Omega_{X_C/Q}^{p-1}) \xrightarrow{\delta r} H^p(\Omega_{X_C/C}^{p-1}),
\]
which is
\[
H^p(\Omega_{X/Q}^{p-1}) \rightarrow H^p(\Omega_{X_C/C}^{p-1}).
\]

If \(k\) is a number field, \(\Omega_{X/Q} = \Omega_{X/k}\) and \(H^p(\Omega_{X/Q}^{p-1}) = H^p(\Omega_{X/k}^{p-1})\). By base change, \(H^p(\Omega_{X_C/C}^{p-1}) \cong H^p(\Omega_{X/k}^{p-1}) \otimes_k \mathbb{C}\). The map (2.9) can be rewritten as
\[
H^p(\Omega_{X/k}^{p-1}) \rightarrow H^p(\Omega_{X/k}^{p-1}) \otimes_k \mathbb{C},
\]
which is obviously injective. In summary,
Theorem 2.13. The infinitesimal form of Conjecture 1.1 is true. To be precise, let $X$ be a smooth projective variety defined over a number field $k$, for each positive integer $p$, the infinitesimal form of the composition

$$CH^p(X) \rightarrow CH^p(X_\mathbb{C}) \rightarrow H^{2p}(X_\mathbb{C}, \mathbb{Z}(p))_\mathbb{Q}$$

in Conjecture 1.1 is of the form,

$$H^p(\Omega^{p-1}_{X/k}) \rightarrow H^p(\Omega^{p-1}_{X/k}) \otimes_k \mathbb{C},$$

which is injective.

Conjecture 1.1 is part of the Bloch-Beilinson conjecture. Let $X$ be a smooth projective variety over $\mathbb{C}$, for each positive integer $p$, Bloch-Beilinson conjecture predicts that there is a filtration which has the form

$$CH^p(X) = F^0CH^p(X) \supset F^1CH^p(X) \supset \cdots \supset F^pCH^p(X) \supset F^{p+1}CH^p(X) = 0.$$ 

The first two steps are known and $F^2CH^p(X)_\mathbb{Q}$ is the kernel of the Deligne cycle class map

$$r : CH^p(X)_\mathbb{Q} \rightarrow H^{2p}(X, \mathbb{Z}(p))_\mathbb{Q}.$$ 

An alternative way to state Conjecture 1.1 is,

**Conjecture 2.14** (cf. Implication 1.2 [7] page 478). If $X$ is a smooth projective variety over a number field $k$, then

$$F^2CH^p(X)_\mathbb{Q} = 0,$$

where $F^2CH^p(X)_\mathbb{Q}$ is the filtration induced from $F^2CH^p(X_\mathbb{C})_\mathbb{Q}$ under the natural map $CH^p(X) \rightarrow CH^p(X_\mathbb{C})$.

Theorem 2.13 suggests that this Conjecture (and Conjecture 1.1) looks reasonable at the infinitesimal level.

**Remark 2.15.** The assumption “$k$ is a number field” in Theorem 2.13 is crucial, it guarantees $\Omega_{X/\mathbb{Q}} = \Omega_{X/k}$. This suggests that the assumption “$k$ is a number field” in Conjecture 1.1 can not be loosened. In fact, for the ground field of transcendental degree one, Green-Griffiths-Paranjape [8] has found counterexamples, extending earlier examples by Bloch, Nori and Schoen.

To understand algebraic cycles, the transcendental degree of the ground field does matter.

**Acknowledgements.** This note is inspired by Hélène Esnault’s talk on [4] at Tsinghua University (December 2017). The author sincerely
thanks Spencer Bloch, Hélène Esnault, Jerome Hoffman and Jan Stienstra for discussions. Jerome Hoffman has read a preliminary version, his suggestions improve this note a lot.

REFERENCES

[1] A. Beilinson, Higher regulators and values of L-functions, J.Soviet Math. 30 (1985), 2036-2070.
[2] S. Bloch, Lectures on algebraic cycles, Second edition. New Mathematical Monographs, 16. Cambridge University Press, Cambridge, 2010. xxiv+130 pp. ISBN: 978-0-521-11842-2.
[3] F. El Zein and S. Zucker, Extendability of normal functions associated to algebraic cycles, Topics in transcendental algebraic geometry (Princeton, N.J., 1981/1982), 269-288, Ann. of Math. Stud., 106, Princeton Univ. Press, Princeton, NJ, 1984.
[4] H. Esnault and M. Harris, Chern classes of automorphic bundles, Preprint 2017, 18 pages, to appear in the Pure and Applied Mathematics Quarterly issue in honor of Prof. Manin's 80th birthday.
[5] H. Esnault and E. Viehweg, Deligne-Beilinson cohomology, Beilinson's conjectures on special values of L-functions, 43-91, Perspect. Math., 4, Academic Press, Boston, MA, 1988.
[6] M. Green, Infinitesimal methods in Hodge theory, Algebraic cycles and Hodge theory (Torino, 1993), 1-92, Lecture Notes in Math., 1594, Springer, Berlin, 1994.
[7] M. Green and P. Griffiths, Hodge theoretic invariants of algebraic cycles, Internat.Math.Res.Notices 9(2003), 477-510.
[8] M. Green, P. Griffiths and K. Paranjape, Cycles over Fields of Transcendence Degree 1, Michigan Math. J. 52 (2004), 181-187.
[9] P. Griffiths and J. Harris, On the Noether-Lefschetz theorem and some remarks on codimension-two cycles, Math. Ann. 271 (1985), no. 1, 31-51.
[10] U. Jannsen, Deligne homology, Hodge-D-conjecture, and motives, Beilinson's conjectures on special values of L-functions, 305-372, Perspect. Math., 4, Academic Press, Boston, MA, 1988.
[11] C. Soulé, Opérations en K-théorie algébrique, Canad. J. Math. 37 (1985), 488-550.
[12] S. Yang, On the image of the Abel-Jacobi map, preprint.

SHING-TUNG YAU CENTER OF SOUTHEAST UNIVERSITY, SOUTHEAST UNIVERSITY, NANJING, CHINA,

SCHOOL OF MATHEMATICS, SOUTHEAST UNIVERSITY, NANJING, CHINA,

YAU MATHEMATICAL SCIENCES CENTER, TSINGHUA UNIVERSITY, BEIJING, CHINA,

E-mail address: 101012424@seu.edu.cn; syang@math.tsinghua.edu.cn