A SYMBOLIC APPROACH TO THE POLY-BERNOULLI NUMBERS

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Abstract. We present a symbolic representation for the poly-Bernoulli numbers. This allows us to prove several new iterated integral representations for the poly-Bernoulli numbers, including an integral transform of the Bernoulli-Barnes numbers. We also deduce some new recurrences for the poly-Bernoulli numbers. Finally, we use these results to present a new iterated integral representation for the Arakawa-Kaneko zeta function, including a nonlinear integral transform of the Barnes zeta function.

1. Introduction

In their seminal 1999 work, Arakawa and Kaneko [1] introduced their namesake zeta function, defined by

\[
\zeta_k(s) := \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} \operatorname{Li}_k(1 - e^{-t}) \, dt
\]

where \( \operatorname{Li}_k(z) \) is the polylogarithm function defined by

\[
\operatorname{Li}_k(z) = \sum_{n=1}^{+\infty} \frac{z^n}{n^k}.
\]

The importance of this function is that when \( s = m \) is a positive integer, we have the evaluation

\[
\zeta_k(m) = \zeta^*(k+1, \{1\}^{m-1}),
\]

where

\[
\zeta^*(s_1, \ldots, s_k) := \sum_{n_1 \geq n_2 \geq \cdots \geq n_k \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}}, s_1 \geq 2, s_i \geq 2, 2 \leq i \leq k
\]

is a multiple zeta starred value (MZSV). Therefore, we can study \( \zeta_k(s) \) complex analytically, which specializes to nontrivial relations for the discretized MZSVs. MZSVs are natural generalizations of the Riemann zeta function which have been systematically studied since the 1990s [12]. They occur naturally in the calculation of higher order Feynman diagrams and renormalization constants in physics [4], and characterizing all linear dependence relations between MZSVs and the closely related multiple zeta values (MZVs)

\[
\zeta(s_1, \ldots, s_k) := \sum_{n_1 > n_2 \geq \cdots \geq n_k \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}}, s_1 \geq 2, s_i \geq 1, 2 \leq i \leq k,
\]

has become a hot topic of recent research. For an excellent survey of this topic, we recommend the review article of Zagier [12].

We now consider the poly-Bernoulli numbers \( B_n^{(k)} \), defined by the generating function

\[
\sum_{n \geq 0} \frac{B_n^{(k)}}{n!} t^n = \frac{\operatorname{Li}_k(1 - e^{-t})}{1 - e^{-t}},
\]

and their polynomial extension, the poly-Bernoulli polynomials with generating function

\[
\sum_{n \geq 0} \frac{B_n^{(k)}(z)}{n!} t^n = \frac{\operatorname{Li}_k(1 - e^{-t})}{1 - e^{-t}} e^{zt}.
\]
The importance of this definition is that for a positive integer \( n \), we have the explicit value for the analytic continuation \( \zeta_k(-n) = (-1)^n B^{(k)}_n \), generalizing the classical result \( \zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1} \) for the Riemann zeta function \(^2\). We wish to study relations for the poly-Bernoulli numbers; these systematically translate to relations for the analytic continuation of the Arakawa-Kaneko zeta function.

In Section 2 we provide the necessary background on umbral calculus and the Bernoulli symbol. In Section 3 we provide our main results: a symbolic representation for the poly-Bernoulli numbers, and a new integral representation for the poly-Bernoulli numbers in terms of Bernoulli-Barnes numbers. This allows us to derive several old and new recurrences for the poly-Bernoulli numbers in Section 4. Finally, in Section 5 we provide new integral and symbolic expressions for the Arakawa-Kaneko zeta function.

2. Umbral Background

The results in this paper depend on umbral calculus, a symbolic computation method for which we provide a short introduction. The key idea is that we can express Bernoulli numbers in terms of moments: more precisely \(^11\),

\[
B_n = \pi^2 \int_{-\infty}^{\infty} \left( t - \frac{1}{2} \right)^n \sech^2 (\pi t) \, dt, \quad n \geq 0.
\]

Given a random variable \( L \) distributed according to the secant square law \( L \sim \frac{\pi}{2} \sech^2 (\pi t) \), the Bernoulli numbers are therefore the moments \( B_n = E \left( \left( iL - \frac{1}{2} \right)^n \right) \), where \( E \) denotes expectation value. This interpretation extends to polynomials; the Bernoulli polynomials are the expectation

\[
B_n(x) = E \left( iL + x - \frac{1}{2} \right)^n = \pi^2 \int_{-\infty}^{\infty} \left( t - \frac{1}{2} + x \right)^n \sech^2 (\pi t) \, dt.
\]

Alternatively, the Bernoulli polynomials are defined by the generating function

\[
\sum_{n=0}^{\infty} B_n(z) \frac{t^n}{n!} = \frac{e^{zt}}{e^t - 1},
\]

and the Bernoulli numbers as \( B_n = B_n(0) \). Therefore, any occurrence of a Bernoulli number may be replaced with an equivalent Bernoulli symbol (or Bernoulli umbra) \( B \), such every Bernoulli number \( B_n \) is mapped to a power \( B^n \). We then perform whatever operations we want, then apply the “evaluation map” \( \text{eval}(B^n) = B_n \), a linear functional. This corresponds to replacing any occurrence of \( B_n \) with \( (it - \frac{1}{2})^n \), performing some manipulations, then multiplying by \( \sech^2(\pi t) \) and integrating across \( \mathbb{R} \). For more details about this approach, we recommend \( \text{[6]} \).

We then have the evaluation rules for the Bernoulli symbols, which essentially coincide with those of probabilistic expectation:

- A product of several Bernoulli numbers is replaced with independent Bernoulli umbræ according to \( B_n B_{n_2} \mapsto B_1^n B_2^{n_2} \).
- For two identical symbols, \( B_1^{n_1} B_1^{n_2} = B_1^{n_1 + n_2} \).
- We have the periodicity relation \( B + 1 = -B \).

For example, the simple recurrence

\[
\sum_{k=0}^{n} \binom{n}{k} B_k = (-1)^n B_n, \quad n \geq 0
\]

can be derived through umbral means:

\[
\sum_{k=0}^{n} \binom{n}{k} B_k \mapsto \text{umbralize} \sum_{k=0}^{n} \binom{n}{k} B^k = (B + 1)^n = (-1)^n B^n \mapsto \text{eval} (-1)^n B_n.
\]
If we explicitly write out the integrals underlying this and note that $L$ is even, we have performed the operations

$$
\sum_{k=0}^{n} \binom{n}{k} B_k \text{ umbralize } \sum_{k=0}^{n} \binom{n}{k} \mathbb{E} \left( iL - \frac{1}{2} \right)^k \\
= \mathbb{E} \left[ \sum_{k=0}^{n} \binom{n}{k} \left( iL - \frac{1}{2} \right)^k \right] \\
= \mathbb{E} \left( iL - \frac{1}{2} + 1 \right)^n \\
= (-1)^n \mathbb{E} \left( iL - \frac{1}{2} \right)^n \\
= \text{eval} \ (-1)^n B_n.
$$

Therefore, we see that the umbral method is equivalent to the linearity of expectation.

Furthermore, there is a conjugate symbol to $B$, which is the uniform symbol $U$. This uniform symbol acts on an arbitrary function by

$$
f(x + U) = \int_{0}^{1} f(x + u) \, du
$$

and corresponds to a continuous random variable distributed uniformly over the interval $[0, 1]$. The Bernoulli and the uniform symbol essentially cancel each other, as is seen from the generating function identity

$$
\exp(zB) \exp(zU) = ze^{z - 1} = 1 = \exp(z(U + B)).
$$

This is equivalent to the identity $\mathbb{E}(U + B)^n = \delta_{n,0}$ for all $n \in \mathbb{N}$, where the $\delta$ is the Kronecker symbol. Therefore, for any sufficiently smooth analytic function,

$$
f(z + U + B) = f(z).
$$

3. Poly-Bernoulli Umbra

We now wish to use the $U$ and $B$ symbols to characterize the poly-Bernoulli numbers as moments. We also prove an integral transform in terms of the Bernoulli-Barnes numbers. Therefore, we introduce the following nonlinear symbol below, which is the key innovation of our paper:

**Definition 1.** Consider $k$ independent Bernoulli umbræ $\{B_i\}_{1 \leq i \leq k}$ and $(k - 1)$ independent uniform umbræ $\{U_i\}_{1 \leq i \leq k-1}$. Then define the poly-Bernoulli umbra $B^{(k)}$ as

$$
B^{(k)} := 1 + \sum_{l=1}^{k} B_l \prod_{j=l}^{k-1} U_j = 1 + B_k + U_{k-1}B_{k-1} + \cdots + U_{k-1} \cdots U_1 B_1.
$$

Note that from this definition, we have the symbolic recursion

$$
B^{(k)} = 1 + B_k + U_{k-1} \left( B^{(k-1)} - 1 \right),
$$

and $B^{(1)} = B + 1$ (meaning $B_1^{(1)} = B_n(1)$) reduces to the usual Bernoulli umbra.

Kaneko and Arakawa also defined a companion sequence $C^{(k)}_n$ to the poly-Bernoulli numbers $B^{(k)}$, with generating function

$$
\sum_{n \geq 0} \frac{C^{(k)}_n}{n!} t^n = \frac{\text{Li}_k(1-e^{-t})}{e^t - 1},
$$

where $\text{Li}_k(x)$ is the polylogarithm.
which suggests the definition of the new symbol $C^{(k)}$ as

$$C^{(k)} := B^{(k)} - 1.$$  

Therefore, with the notation of Definition 1,

$$C^{(k)} := \sum_{l=1}^{k} B_l \prod_{j=l}^{k-1} u_j.$$  

In terms of the sequences $\{B_n^{(k)}\}$ and $\{C_n^{(k)}\}$, this implies

$$C_n^{(k)} = \sum_{l=0}^{n} \binom{n}{l} (-1)^{n-l} B_n^{(k)}, \quad B_n^{(k)} = \sum_{l=0}^{n} \binom{n}{l} C_n^{(k)}.$$  

Notice that the symbol $C^{(k)}$ satisfies the recurrence

$$C^{(k)} = B_k + \mathcal{U}_{k-1} C^{(k-1)}$$  

with initial value $C^{(1)} = B$, the Bernoulli symbol.

**Theorem 2.** The poly-Bernoulli polynomials are equal to

$$B_n^{(k)}(z) = \left(z + B^{(k)}\right)^n,$$  

which is equivalent to the multiple integral representation

$$B_n^{(k)}(z) = \left(\frac{\pi}{2}\right)^k \int_{\mathbb{R}^k} \prod_{i=1}^{k} \text{sech}^2(\pi w_i) \, dw_i \int_{[0,1]} \prod_{i=1}^{k-1} \int_0^{t_{i+1}} \left(1 + \sum_{l=1}^{k} \binom{k}{l} \prod_{j=l}^{k-1} u_j\right) \, dt_i.$$  

As a consequence, there exists a measure $\mu_k : \mathbb{R} \to \mathbb{R}$ such that the poly-Bernoulli polynomial can be expressed as the moment

$$B_n^{(k)}(z) = \int (z + w)^n \, d\mu_k(w).$$  

**Proof.** Our starting point is the iterated integral representation of the generating function of the poly-Bernoulli numbers

$$\sum_{n \geq 0} \frac{B_n^{(k)}}{n!} t^n = e^t \frac{1}{e^t - 1} \int_0^t \frac{dt_k}{e^{tk} - 1} \int_0^{tk} \frac{dt_{k-1}}{e^{tk} - 1} \cdots \int_0^{t_{k-1}} \frac{dt_2}{e^{t_{k-2}} - 1} \int_0^{t_{k-2}} \frac{dt_1}{e^{t_{k-3}} - 1},$$

where the operator $D = \frac{1}{e^t - 1} \int_0^t$ appears $k$-times, so that

$$e^{tB^{(k)}} = e^{tD^{k-1}} \frac{t}{e^t - 1}.$$  

Using (2.1), the action of the operator $D$ on the initial function $\frac{t}{e^t - 1}$ is expressed in terms of Bernoulli and uniform symbols as

$$D^{t} \frac{t}{e^t - 1} = De^{tB_1} = \frac{1}{e^t - 1} \int_0^t e^{tB_1} \, dt = \frac{1}{e^t - 1} \int_0^1 te^{tuB_1} \, du = e^{tB_2} e^{t\mathcal{U}_{1}B_1}.$$  

Hence the operator $D$ replaces $B_1 = C^{(1)}$ with $\mathcal{U}_1 B_1 + B_2 = C^{(2)}$. We similarly deduce

$$D^{t} \frac{t}{e^t - 1} = D e^{t(B_2 + \mathcal{U}_1 B_1)} = e^{tB_3} e^{t\mathcal{U}_2(B_2 + \mathcal{U}_1 B_1)}$$  

and more generally

$$e^{tD^{k-1}} \frac{t}{e^t - 1} = e^{t(1 + B_k + \mathcal{U}_{k-1} B_{k-1} + \cdots + \mathcal{U}_{k-1} \cdots \mathcal{U}_1 B_1)} = e^{tC^{(k)}}.$$  

To pass from poly-Bernoulli numbers to poly-Bernoulli polynomials, we multiply both generating functions by $e^{z}$, and then compare coefficients of $t^n$ to produce the desired result. □
Note that this procedure has converted a nested \(k\)-fold integral, in which the limits of the inner integrals depend on variables of integration, into a \((2k-1)\)-fold integral over uniform bounds. As another consequence of the representation (3.4), we have an expression for the poly-Bernoulli polynomials in terms of the Bernoulli-Barnes polynomials. The Bernoulli-Barnes polynomials \(\mathfrak{B}_n^{(k)}\), a vectorized generalization of the Bernoulli polynomials, are defined by the generating function

\[
\sum_{n \geq 0} \frac{\mathfrak{B}_n^{(k)}(a_1, \ldots, a_k; z)}{n!} t^n = e^{zt} \prod_{i=1}^k \frac{t}{e^{a_i t} - 1}.
\]

**Theorem 3.** We have the \((k-1)\)-dimensional integral transform

\[
B_n^{(k)}(z) = \mathfrak{B}_n^{(k)}(\mathcal{U}_{k-1} \cdots \mathcal{U}_1, \mathcal{U}_{k-1} \cdots \mathcal{U}_2, \ldots, \mathcal{U}_{k-1}, 1; z + 1) \prod_{i=1}^{k-1} \mathcal{U}_i
\]

\[
= \int_{[0,1]^{k-1}} \mathfrak{B}_n^{(k)}(u_{k-1} \cdots u_1, u_{k-1} \cdots u_2, \ldots, u_{k-1}, 1; z + 1) u_1 u_2 \cdots u_{k-1} \, du_1 \cdots du_{k-1}.
\]

**Proof.** The main result of [7] was the umbral characterization

\[
\mathfrak{B}_n^{(k)}(a_1, \ldots, a_k; z) = \frac{1}{\prod_{i=1}^k a_i} \left( z + \sum_{i=1}^k a_i \mathcal{B}_i \right)^n,
\]

where the \(\mathcal{B}_i\) are independent Bernoulli umbræ. Comparing this with the umbral representation (3.4)

\[
B_n^{(k)}(z) = \left( z + B(z) \right)^n = \left( z + 1 + \sum_{i=1}^k B_i \prod_{j=l}^{k-1} \mathcal{U}_j \right)^n
\]

and taking \(a_1 = 1, a_i = \prod_{j=1}^{i-1} \mathcal{U}_j, i \geq 2\), gives the first result. The second follows by explicitly specifying the action of the uniform operators \(\mathcal{U}_i\). \(\square\)

The following result shows how a change of variables in the integral representation in Theorem 3 allows us to replace the cumulative products of variables \(\mathcal{U}_i\) by linear terms, at the price of a more complicated integration domain.

**Theorem 4.** The Arakawa-Kaneko polynomials are expressed as

\[
B_n^{(k)}(z) = \mathfrak{B}_n^{(k)}(\mathcal{V}_1, \ldots, \mathcal{V}_{k-1}; z + 1) \prod_{i=1}^{k-1} \mathcal{V}_i,
\]

where \(\{\mathcal{V}_i\}_{1 \leq i \leq k}\) are random variables with the mutual probability density

\[
f_{\mathcal{V}_1,\ldots,\mathcal{V}_{k-1}}(v_1, \ldots, v_{k-1}) = \begin{cases} \frac{1}{v_2 \cdots v_{k-1}} & 0 \leq v_1 \leq \cdots \leq v_{k-1} \leq 1, \\ 0 & \text{else.} \end{cases}
\]

This yields the integral representation

\[
B_n^{(k)}(z) = \int_{0 \leq v_1 \leq \cdots \leq v_{k-1} \leq 1} v_1^{\mathfrak{B}_n^{(k)}(v_1, \ldots, v_k; z + 1)} \, dv_1 \cdots dv_{k-1}
\]

and, in terms of Bernoulli umbræ,

\[
B_n^{(k)}(z) = \int_{0 \leq v_1 \leq \cdots \leq v_{k-1} \leq 1} (1 + z + B_k + v_{k-1} B_{k-1} + \cdots + v_1 B_1) \, dv_1 \cdots dv_{k-1}
\]

\[
\frac{v_2 \cdots v_{k-1}}{v_2 \cdots v_{k-1}}
\]
Proof. Perform the change of variables

\[
\begin{align*}
&v_{k-1} = u_{k-1} \\
v_{k-2} = u_{k-1}u_{k-2} \\
&\vdots \\
v_1 = u_{k-1}\ldots u_1
\end{align*}
\]

in the integral representation in Theorem 3, the Jacobian is then

\[
J = \frac{\partial v}{\partial u} = \begin{bmatrix}
u_{k-1} & 0 & \cdots & 0 \\
u_{k-1} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
u_{k-1} & 0 & \cdots & 1
\end{bmatrix},
\]

with determinant

\[
|J| = \prod_{i=2}^{k} u_i^{i-1} = \prod_{i=2}^{k-1} v_i.
\]

The product of uniform densities

\[
\mathbb{I}_{[0,1)} (u_1) \times \cdots \times \mathbb{I}_{[0,1)} (u_{k-1})
\]

is transformed into

\[
\mathbb{I}_{[0,1)} \left( \frac{v_1}{v_2} \right) \mathbb{I}_{[0,1)} \left( \frac{v_2}{v_3} \right) \times \cdots \times \mathbb{I}_{[0,1)} \left( \frac{v_{k-2}}{v_{k-1}} \right) \mathbb{I}_{[0,1)} (v_{k-1}) = \begin{cases} 1, & 0 \leq v_1 \leq \cdots \leq v_{k-1} \leq 1 \\ 0, & \text{else} \end{cases},
\]

which is the desired result. \qed

4. Consequences

This symbolic representation enables us to quickly prove various recursions for the poly-Bernoulli polynomials. All the identities in this section have analogs for the companion \(C(n)\) sequence with identical proofs, which we omit.

Theorem 5. The poly-Bernoulli polynomials satisfy the recursions

\[
P_n^{(k)} (z) = \sum_{m=0}^{n} \binom{n}{m} P_{n-m}^{(k-1)} \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} B_l \frac{B_{z+1}}{n-l+1}.
\]

and

\[
C_n^{(k)} (z) = \sum_{m=0}^{n} \binom{n}{m} B_m (z) C_{n-m}^{(k-1)} \frac{B_{z+1}}{n-m+1}.
\]

Proof. Begin with the umbral representation (3.11). We then recursively write

\[
P_n^{(k)} (z) = \left( z + 1 + U_{k-1} B^{(k-1)} + B_k - U_{k-1} \right)^n
\]

\[
= \sum_{m=0}^{n} \binom{n}{m} \left[ U_{k-1} B^{(k-1)} \right]^{n-m} (B_k + z + 1 - U_{k-1})^m
\]

\[
= \sum_{m=0}^{n} \binom{n}{m} \left( B^{(k-1)} \right)^{n-m} \sum_{l=0}^{m} \binom{m}{l} (B_k + z + 1)^l (-1)^{m-l} U_{k-1}^{m-l} U_{k-1}^{n-m}
\]

\[
= \text{(eval)} \sum_{m=0}^{n} \binom{n}{m} P_{n-m}^{(k-1)} \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} B_l \frac{B_{z+1}}{n-l+1}.
\]

Note that the nonlinear \(B^{(k-1)}\) umbra is only dependent on \(\{B_1, \ldots, B_{k-1}\}\) and \(\{U_1, \ldots, U_{k-2}\}\), which means that we can apply the “eval” functional to it independently. \qed
Remark 6. This result appears as [3, Thm 1.2] under the form
\[ B_n^{(k)}(z) = \sum_{m=0}^{n} \binom{n}{m} (-1)^m B_{n-m}^{(k-1)} \sum_{l=0}^{m} \binom{m}{l} \frac{(-1)^l}{n-l+1} B_l(z), \]
which differs by the term \( B_l(z) \) from our result and has been numerically verified as incorrect.

The next several results feature a nonlinear product of symbols of the form \( UB^{(k-1)} \), which, to our knowledge, is the first time such a nonlinear function of symbols has appeared in the literature. It also features the negatively indexed symbol \( (B^{(k)})^{-1} \), which is evaluated as the negative moment
\[ (z + B^{(k)})^{-1} = \left(\frac{n}{2}\right)^k \int_{[0,1]} \prod_{l=1}^{k} \frac{1}{\cosh(\pi w_l)} \sum_{l=0}^{k-1} \frac{1}{1 + z + \sum_{l=1}^{k} (i w_l - \frac{1}{2}) \prod_{j=l}^{k-1} u_j}. \]
The evaluation of these negative moments will be explicitly described in the next section.

Theorem 7. The poly-Bernoulli polynomials satisfy the connection relation
\[ B_n^{(k)}(z) = \frac{1}{n+1} \sum_{l=0}^{n} \binom{n+1}{l+1} B_{n-l}(z+1) B_l^{(k-1)}(-1) \]
and
\[ C_n^{(k)}(z) = \frac{1}{n+1} \sum_{l=0}^{n} \binom{n+1}{l+1} B_{n-l}(z) C_l^{(k-1)}. \]

Proof. Begin with the umbral representation (3.4),
\[ B_n^{(k)}(z) = \left(1 + z + B_k + U_{k-1} \left(B^{(k-1)} - 1\right)\right)^n, \]
and directly apply the \( U_{k-1} \) integration operator to obtain
\[ B_n^{(k)}(z) = \left(\frac{B^{(k-1)} - 1}{n+1}\right)^{-1} \left[\left(1 + z + B_k + B^{(k-1)} - 1\right)^{n+1} - (1 + z + B_k)^{n+1}\right] \]
\[ = \frac{1}{n+1} \sum_{l=1}^{n+1} \binom{n+1}{l+1} (1 + z + B_k)^{n+1-l} \left(B^{(k-1)} - 1\right)^{l-1} \]
\[ = \frac{1}{n+1} \sum_{l=0}^{n} \binom{n+1}{l+1} B_{n-l}(z+1) B_l^{(k-1)}(-1). \]

Remark 8. The Fourier coefficients of the periodic poly-Bernoulli polynomials \( C_n^{(k)}(\{z\}) \), defined by
\[ c_l^{(k,n)} = \int_{0}^{1} C_n^{(k)}(\{z\}) e^{-i2\pi l z} dz, \]
are computed in [3] as
\[ c_l^{(k,n)} = \begin{cases} \frac{n!}{(2\pi i)^{n+1}} \sum_{p=1}^{n} \frac{C_{n-p}^{(k-1)}}{p!} (2\pi l)^p & l \neq 0 \\ \frac{1}{n+1} C_n^{(k-1)} & l = 0. \end{cases} \]
Thus the Fourier expansion of the periodic poly-Bernoulli polynomials \( C_n^{(k)}(\{z\}) \) reads
\[ C_n^{(k)}(z) = \frac{1}{n+1} \sum_{p=1}^{n+1} \binom{n+1}{p} c_{p-1}^{(k-1)} B_{n-p+1}(\{z\}). \]
As a consequence, both identities in Theorem 7 can be interpreted as the Fourier expansion of the poly-Bernoulli polynomials \( B_n^{(k)}(\{z\}) \) and \( C_n^{(k)}(\{z\}) \).
The next result is a higher-order analog of the identity on Bernoulli symbols $f(1+B) = f(B) + f'(0)$, for smooth analytic functions $f$.

**Theorem 9.** The poly-Bernoulli polynomials satisfy the difference identities

$$B_n^{(k)}(z) - B_n^{(k)}(z - 1) = \sum_{l=1}^{n} {n \choose l} z^{n-l} B_{l-1}^{(k-1)}(-1).$$

and

$$C_n^{(k)}(z + 1) - C_n^{(k)}(z) = \sum_{l=1}^{n} {n \choose l} z^{n-l} C_{l-1}^{(k-1)}.$$

**Proof.** We begin by applying the classical relation $f(1+B) = f(B) + f'(0)$ to the outermost $B_k$ symbol, then utilize the trick from the previous theorem and directly apply the $U_{k-1}$ integration operator and invoke symbols at negative indices:

$$B_n^{(k)}(z) = \left(1 + z + B_k + U_{k-1} \left(B^{(k-1)} - 1\right)\right)^n$$

$$= (z + B_k + U_{k-1} \left(B^{(k-1)} - 1\right))^n + n \left(z + U_{k-1} \left(B^{(k-1)} - 1\right)\right)^{n-1}$$

$$= (z - 1 + B^{(k)})^n + \left(B^{(k-1)} - 1\right)^{-1} \left[(z - 1 + B^{(k-1)})^n - z^n\right]$$

$$= (z - 1 + B^{(k)})^n + \sum_{l=1}^{n} \binom{n}{l} z^{n-l} \left(B^{(k-1)} - 1\right)^{l-1}.$$

Applying the eval functional completes the proof. 

This identity can be extended to all smooth analytic functions by linearity, since we’ve verified it for monomials. For $k \geq 2$, this gives the symbolic extension

$$f(1 + z + B^{(k)}) - f(z + B^{(k)}) = \frac{f(z + B^{(k-1)}) - f(z)}{B^{(k-1)}},$$

which converts a forwards difference in $z$ into a discrete derivative with respect to $B^{(k-1)}$. Note that for $k = 1$ we instead have $f(1 + z + B) - f(z + B) = f'(z)$, so that as $k \to 1$ we’re performing a discrete approximation to the continuous derivative $f'(z)$, with respect to $B^{(k-1)}$. This suggests that other functional identities involving the Bernoulli symbol and derivatives ought to have generalizations to the poly-Bernoulli numbers.

Analogously the Bernoulli symbol identity \[6\] extends to the poly-Bernoulli case as follows.

**Theorem 10.** The poly-Bernoulli polynomials satisfy the higher-order difference identity

$$B_n^{(k)}(m + z) - B_n^{(k)}(z) = \sum_{l=1}^{n} \binom{n}{l} B_{l-1}^{(k-1)}(-1) \left(\frac{B_{n-l+1}(m + z + 1) - B_{n-l+1}(z + 1)}{n - l + 1}\right)$$

and

$$C_n^{(k)}(m + z) - C_n^{(k)}(z) = \sum_{l=1}^{n} \binom{n}{l} C_{l-1}^{(k-1)} \left(\frac{B_{n-l+1}(m + z) - B_{n-l+1}(z)}{n - l + 1}\right).$$
**Proof.** We begin by applying the classical relation $f(m + B) = f(B) + \sum_{i=0}^{m-1} f'(i)$:

\[
B_n^{(k)}(m + z) = \left( m + 1 + z + \sum_{i=0}^{m-1} f(i) z^{i+k} \right)^n \\
= \left( 1 + z + \sum_{i=0}^{m-1} f(i) z^{i+k} \right)^n \\
= \left( \sum_{i=0}^{m-1} f(i) z^{i+k} \right)^n.
\]

We now apply the generalized Faulhaber formula

\[
\sum_{i=0}^{m-1} (1 + z + i) = \frac{B_{m+1}(m + z + 1) - B_{m+1}(z + 1)}{n - l + 1}
\]

to eliminate the outer summation, which completes the proof.

\[\square\]

## 5. A NEW APPROACH TO MZVs

A consequence of the symbolic representation (3.1) is that it allows us to represent the analytic continuation of the Arakawa zeta function as a Bernoulli symbol to a negative power. The easiest case of this result is the classical representation

\[
\zeta(n) = (-1)^n \frac{B_{1-n}}{1-n}, \quad n \geq 2,
\]

which should be understood as a consequence of the integral representation

\[
\zeta(n) = \frac{(-1)^n \pi}{2} \int_{-\infty}^{\infty} \left( i t - \frac{1}{2} \right)^{1-n} \text{sech}^2(\pi t) dt.
\]

Remark that Theorem 13 essentially computes the negative moments $(B^{(k)})^{-m}$. For $k = 1$, these negative moments can be separately evaluated; we begin with the nonlinear umbral evaluation [5, Thm 2.5]

\[
\log(B + z) = \psi \left( \frac{1}{2} + \left| z - \frac{1}{2} \right| \right)
\]

where $\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)}$ is the digamma function. Then, by noting $(B + z)^{-1} = \frac{d}{dz} \log(B + z)$, and taking iterated derivatives, we arrive at the evaluation

\[
(B + z)^{-m} = \begin{cases} 
\frac{(-1)^{m-1}}{(m-1)!} \psi^{(m)}(z), & z > \frac{1}{2} \\
-\frac{1}{(m-1)!} \psi^{(m)}(1-z), & z < \frac{1}{2}
\end{cases}
\]

which is nondifferentiable at $z = \frac{1}{2}$. Here, $\psi^{(m)}$ is the $m$-th order polygamma function. At $z = 1$, this specializes to $(B + 1)^{-m} = m \zeta(m + 1)$.

Before proving a new symbolic representation for the Arakawa zeta function, we introduce some lemmas about the *Barnes zeta function*, a generalization of the Riemann zeta function parametrized by a vector of $k$ real variables $(a_1, a_2, \ldots, a_k)$ as

\[
\zeta_k(s, z | a_1, \ldots, a_k) := \sum_{m_1, \ldots, m_k \geq 0} \frac{1}{(z + a_1 m_1 + \cdots + a_k m_k)^s}.
\]

Much of the material from this section is derived from the important manuscript [10] of Ruijsenaars, which we highly recommend. The following integral representations are crucial to our argument.
Lemma 11. [10] Eqn 1.6] With values of the parameters such that both sides converge, we have the iterated integral
\begin{equation}
ζ_k \left( s, w + \frac{1}{2} \sum_{i=1}^{k} a_i \right) = \frac{\pi}{2} \frac{k}{\Gamma(s-k)} \int_{\mathbb{R}^k} \prod_{n=1}^{k} \frac{1}{a_n^2} \sech^2 \left( \frac{\pi u_n}{a_n} \right) \frac{1}{(w - i \sum_{n=1}^{k} a_n) s-k} dx_1 \cdots dx_k.
\end{equation}

Lemma 12. [10] Eqn 3.2] With values of the parameters such that both sides converge, we have the Barnes-Mellin transform
\begin{equation}
ζ_k(s, w) = \frac{1}{\Gamma(s)} \int_{0}^{+\infty} \prod_{j=1}^{k} \frac{1}{1-e^{-a_j t}} t^{s-1} e^{-wt} dt.
\end{equation}

The Barnes zeta function satisfies a typical zeta function–Bernoulli number duality; given the Bernoulli–Barnes numbers defined before Theorem 3, we have the identity on analytic continuations
\[ζ_k(-N, z|a_1, \ldots, a_k) = (-1)^k \frac{N!}{(n+N)!} \mathcal{B}_{n+N}^{(k)} (a_1, \ldots, a_k; z).\]

Therefore, we heuristically expect Theorem 3 to have an analog for the Arakawa zeta, by mapping \( s \mapsto -N \). This is how we proceed; we write the Arakawa zeta function as an integral transform of the Barnes zeta function, and then apply some known results for the Barnes zeta.

Theorem 13. A symbolic representation for the Arakawa zeta function is
\begin{equation}
ζ_k(m) = (-1)^m \mathcal{C}^{(k)}_{-m}.
\end{equation}

Proof. Heuristically, the idea behind this identity is identifying the Arakawa zeta as a Barnes-Mellin transform
\[ζ_k(m, w) = \frac{1}{\Gamma(m)} \int_{0}^{+\infty} t^{m-1} e^{-wt} \text{Li}_k \left( 1 - e^{-t} \right) e^t - 1 dt = \frac{1}{\Gamma(m)} \int_{0}^{+\infty} t^{m-1} e^{-wt} e^{\mathcal{C}(k)} dt = \frac{1}{\Gamma(m)} \int_{0}^{+\infty} t^{m-1} e^{-t(w - \mathcal{C}(k))} dt = \left( w - \mathcal{C}(k) \right)^{-m}.
\]

However, we need to make this approach rigorous. Denote by
\[f_L(x) = \frac{\pi}{2} \sech^2 (\pi x)\]
the square hyperbolic secant distribution. Then, by combining Lemmas 11 and 12 with the change of variables \( u_n \mapsto a_n x_n \) on the left-hand side, we obtain
\[\left( \prod_{i=1}^{k} \frac{1}{(m-i) a_i} \right) \int_{\mathbb{R}^k} \frac{\prod_{i=1}^{k} f_L(x_i)}{(w + \sum_{i=1}^{k} a_i (x_i - \frac{1}{2}))^{m-k}} dx_1 \cdots dx_k = \frac{1}{\Gamma(m)} \int_{0}^{+\infty} t^{m-1} e^{-tu} \prod_{i=1}^{k} \frac{1}{1-e^{-a_i t}} dt.
\]

Replacing \( m - k \) by \( m \) and simplifying produces
\[\int_{\mathbb{R}^k} \frac{\prod_{i=1}^{k} f_L(x_i)}{(w + \sum_{i=1}^{k} a_i (x_i - \frac{1}{2}))^{m}} dx_i = \frac{1}{\Gamma(m)} \int_{0}^{+\infty} t^{m-1} e^{-wt} \left( \prod_{i=1}^{k} \frac{-a_i t}{e^{-a_i t} - 1} \right) dt.
\]
Choosing \( a_k = -1 \) and \( a_{k-i} = -u_{k-1} \ldots u_{k-i}, 1 \leq i \leq k - 1 \), now gives

\[
\int_{\mathbb{R}^k} \left( \frac{\prod_{i=1}^k f_L(x_i)}{(w - (ix_k - \frac{1}{2}) - \sum_{i=1}^{k-1} u_{k-1} \ldots u_i (ix_i - \frac{1}{2}))^m} \prod_{i=1}^k dx_i \right) \left( t^{m-1} e^{-xt} \prod_{i=1}^{k-1} \frac{u_{k-1} \ldots u_i t}{e^{u_{k-1} \ldots u_i t} - 1} \right) dt.
\]

Integrating each \( u_i, 1 \leq i \leq k - 1 \) over the interval \([0, 1]\) gives

\[
\int_{[0,1]^{k-1}} \int_{\mathbb{R}^k} \left( w - (ix_k - \frac{1}{2}) - \sum_{i=1}^{k-1} u_{k-1} \ldots u_i (ix_i - \frac{1}{2}) \right)^m \prod_{i=1}^k dx_i \prod_{i=1}^{k-1} du_i \left( t^{m-1} e^{-wt} \int_{[0,1]^{k-1}} \frac{t}{e^{t - 1} \prod_{i=1}^{k-1} \frac{u_{k-1} \ldots u_i t}{e^{u_{k-1} \ldots u_i t} - 1} du_i} \right) dt.
\]

(5.6)

This step essentially writes the Arakawa zeta function as a \((k - 1)\) dimensional integral transform of the Barnes zeta function, a result parallel to Theorem 3. From the definition of the poly-Bernoulli umbra, the left-hand side of Equation (5.6) is written as

\[
\frac{1}{(w - C(k))^m}.
\]

Meanwhile, we now recognize the inner integral on the right-hand side of Equation (5.6) as the generating function of the \( C^{(k)} \) sequence:

\[
\sum_{n \geq 0} \frac{C^{(k)}_n}{n!} t^n = \frac{\text{Li}_k(1 - e^{-t})}{e^t - 1}.
\]

To see this, begin with Theorem [2]

\[
\frac{\text{Li}_k(1 - e^{-t})}{e^t - 1} = \sum_{n=0}^{\infty} \frac{C^{(k)}_n}{n!} t^n = \exp \left( t (B_k + U_{k-1}B_{k-1} + U_{k-2} \ldots U_1B_1) \right).
\]

Then expand each \( B \) symbol through its generating function as

\[
\frac{t}{e^t - 1} \times \frac{U_{k-1}}{e^{U_{k-1} - 1}} \times \ldots \times \frac{U_1}{e^{U_1 - 1}}.
\]

Now expand each \( U \) operator as an integral, giving the key identity

\[
\frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} = \int_{[0,1]^{k-1}} t^{k-1} \prod_{i=1}^k u_i \prod_{i=1}^{k-1} \frac{1}{e^{t \sum_{j=1}^{k-1} u_i - 1} - 1} du_1 \ldots du_{k-1}.
\]

Therefore, the right-hand side is equal to the Arakawa-Kaneko zeta

\[
\frac{1}{\Gamma(m)} \int_0^{+\infty} t^{m-1} e^{-wt} \frac{\text{Li}_k(1 - e^{-t})}{e^t - 1} dt = \zeta_m(k, w).
\]

Evaluating at \( w = 0 \) yields the result. \( \square \)

As an aside, this proof highlights the fascinating (and highly nontrivial) interplay between Bernoulli numbers and zeta functions. This proof was ultimately based on Theorem 3, a new representation for poly-Bernoulli numbers. This allowed us to prove an integral transform for Arakawa zeta function, which when analytically continued re-specializes to results on the poly-Bernoulli numbers. This result provides new integral representations for the starred MZVs, some of which we record here.
For \( k = 2, m = 1 \), we have

\[
\frac{1}{\mathcal{B}_2 + \mathcal{U}_1 \mathcal{B}_1} = -1.20206 = -\zeta^* (3) = -\zeta (3) = -\zeta_2 (1) \tag{5.7}
\]
so that

\[
\zeta (3) = -\frac{\pi^2}{4} \int_{\mathbb{R}^2} \int_0^1 \frac{\text{sech}^2 (\pi w_1) \text{sech}^2 (\pi w_2)}{w_2 - \frac{1}{2} + u (w_1 - \frac{1}{2})} dw_1 dw_2.
\]

For \( k = 2, m = 2 \), we have

\[
\frac{1}{(\mathcal{B}_2 + \mathcal{U}_1 \mathcal{B}_1)^2} = 1.3529 = \zeta^* (3, 1) = \zeta_2 (2) = \zeta (3, 1) + \zeta (4)
\]
so that

\[
\zeta^* (3, 1) = \frac{\pi^2}{4} \int_{\mathbb{R}^2} \text{sech}^2 (\pi w_1) \text{sech}^2 (\pi w_2) \left( w_2 - \frac{1}{2} + u (w_1 - \frac{1}{2}) \right)^2 dw_1 dw_2.
\]

This integral can also be expressed as the double integral

\[
\zeta^* (3, 1) = \frac{\pi^2}{4} \int_{\mathbb{R}^2} \text{sech}^2 (\pi w_1) \text{sech}^2 (\pi w_2) \left( \frac{w_1 - \frac{1}{2}}{i (w_2 + w_1) - 1} \right) dw_1 dw_2.
\]

This motivates the study of whether we can simplify some of the \((k - 1)\) fold integrals corresponding to the \(\mathcal{U}_i\) operators.

For \( k = 2, m = 3 \), we have

\[
\frac{1}{(\mathcal{B}_2 + \mathcal{U}_1 \mathcal{B}_1)^3} = -1.45884 = -\zeta^* (3, 1, 1) = -\zeta_2 (3)
\]
so that

\[
\zeta^* (3, 1, 1) = -\frac{\pi^2}{4} \int_{\mathbb{R}^2} \text{sech}^2 (\pi w_1) \text{sech}^2 (\pi w_2) \left( w_2 - \frac{1}{2} + u (w_1 - \frac{1}{2}) \right)^3 dw_1 dw_2.
\]

6. Bernoulli Symbols and Summation

In this last part, we illustrate the summation mechanism performed by the Bernoulli and uniform umbræ.

6.1. Telescoping. Starting with the identity

\[
\frac{1}{x + \mathcal{B}} = \sum_{k \geq 0} \frac{1}{(x + k)^2},
\]
which holds for \( x > \frac{1}{2} \) as a consequence of (5.2), and when evaluated at \( x = 1 \), gives

\[
\zeta (2) = \frac{1}{B + 1} = -1.
\]

This result can be expressed in integral form as

\[
\frac{\pi}{2} \int_{\mathbb{R}} \text{sech}^2 (\pi w) \frac{dw}{w + \frac{1}{2}} = \zeta (2).
\]

Renaming \( \mathcal{B} \) as \( \mathcal{B}_2 \) and replacing \( x \) by \( x + \mathcal{U}_1 \mathcal{B}_1 \) produces

\[
\frac{1}{x + \mathcal{B}_2 + \mathcal{U}_1 \mathcal{B}_1} = \sum_{k \geq 0} \frac{1}{(x + k + \mathcal{U}_1 \mathcal{B}_1)^2}.
\]

Expressing the \( \mathcal{U}_1 \) symbol as an integral over the interval \([0, 1]\) yields

\[
\frac{1}{x + \mathcal{B}_2 + \mathcal{U}_1 \mathcal{B}_1} = \sum_{k \geq 0} \frac{1}{\mathcal{B}_1} \left[ \frac{1}{(x + k) - (x + k + \mathcal{B}_1)} \right] = \sum_{k \geq 0} \frac{1}{(x + k) (x + k + \mathcal{B}_1)}.
\]
and then applying identity (6.1) again gives
\[
\frac{1}{x + B_2 + U_1 B_1} = \sum_{k \geq 0} \frac{1}{(x + k) \sum_{l \geq 0} \frac{1}{(x + k + l)^2}}.
\]
Evaluating at \( x = 1 \) produces
\[
\frac{1}{1 + B_2 + U_1 B_1} = \sum_{k \geq 1, l \geq 0} \frac{1}{k (k + l)^2} = \zeta^* (2, 1) = \zeta (2, 1) + \zeta (3) = 2 \zeta (3).
\]
This result should be compared with the expression (5.7) above, which we rewrite as
\[
\zeta (3) = \frac{1}{1 + B_2 - U_1 B_1}.
\]

6.2. Computing MZVs. Instead of multiple zeta star values, multiple zeta values can be obtained in the computation above, by noticing that each Bernoulli symbol induces a sum that starts at 0. Replacing each Bernoulli symbol \( B_i \) by \( 1 + B_i \) then induces sums that start at 1, producing
\[
\frac{1}{1 + B_k + U_{k-1} (1 + B_{k-1}) + \cdots + U_{k-1} \cdots U_1 (1 + B_1)} = \zeta \left( 2, \{1\}^{k-1} \right).
\]
Since for each \( i \), \( 1 + B_i = -B_i \), this fraction is identified as
\[
-\frac{1}{C(k)} = \zeta (1) = \zeta^* (k + 1) = \zeta (k + 1)
\]
and we deduce
\[
\zeta (k + 1) = \zeta \left( 2, \{1\}^{k-1} \right),
\]
an identity that is a consequence of the duality property of MZVs. Further studying the effect of the \( B + 1 = -B \) periodicity on relations for MZVs and MZSVs is an intriguing possibility for further study.

7. Acknowledgements

This one goes out to the gorgeous views of Luxembourg City, which inspired much of Section 4.

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