ON THE GENERIC COMPLETE SYNCHRONIZATION OF THE DISCRETE KURAMOTO MODEL

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Abstract. We study the emergent behavior of discrete-time approximation of the finite-dimensional Kuramoto model. Compared to Zhang and Zhu’s recent work in [38], we do not rely on the consistency of one-step forward Euler scheme but analyze the discrete model directly to obtain sharper and more explicit result. More precisely, we present the optimal condition for the convergence and order preserving for identical oscillators with generic initial data. Then, we give the exact convergence rate of the identical oscillators to their limit under the reasonable assumption on time step. Finally, we provide an alternative proof of the asymptotic phase-locking of nonidentical oscillators which can be applied whenever the given Lyapunov functional is continuous and all zeros are isolated.

1. Introduction. Synchronization is one of the novel type of emergent behavior that has been frequently observed in nature. After Dutch physicist Christiaan Huygens’ observation in 17th century, many types of synchronization of weakly coupled oscillators, such as firing neurons and flashing fireflies, were observed by several physicists and biologists [3, 14, 31]. Then, the systematic studies using explicit mathematical models were begun by Arthur Winfree [37] and Yoshiki Kuramoto [26]. The well-known phase synchronization model proposed by Kuramoto consists of the ODEs for finitely many weakly coupled oscillators:

\[ \dot{\theta}_i = \nu_i + \lambda \sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad t > 0, \]

\[ \theta_i(0) = \theta_i^{in}, \quad i = 1, \ldots, N, \]

where the system parameter \( \lambda > 0 \) denotes the universal scale of coupling strengths between each pair of particles, and \( \theta_i, \nu_i \in \mathbb{R} \) represent phase and natural frequency of \( i \)-th Kuramoto particle, respectively. The Kuramoto model and its variations have been extensively studied in the literature as a prototype model for synchronization, especially for the phase-transition behavior. To name a few, [1, 23, 33] provided comprehensive investigations on the Kuramoto model including numerical

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examples from the perspective of physicists. In [7, 9, 10, 11, 18, 22], authors studied the Kuramoto model with extra structure such as inertia or frustration, and [20, 21] studied an asymptotic complete phase/frequency synchronization for various initial data. Moreover, [5, 12, 34, 35, 36] estimated a critical coupling strength to guarantee the existence of phase locked states under several conditions, such as all-to-all and bipartite graph network topology, and [15] provided some sufficient conditions for the Kuramoto model with static network topology to exhibit asymptotic phase locking. For the kinetic description of the Kuramoto model, the well-posedness, spectral analysis, critical coupling strength and synchronization estimate of Kuramoto-Sakaguchi equation are proposed in [2, 4, 13, 27, 28, 29, 32], and [24, 25] studied a nonlinear stability of Kuramoto-Sakaguchi equation with or without diffusion.

However, the visualization of those results can only be done by discretization, which inevitably approximates the true model (1). For this reason, the applicability of the theory of (continuous) Kuramoto model (1) to the Euler type discrete time approximation scheme has also been studied. Here, the discrete-time version of (1) can be written as

$$\theta_i(n+1) = \theta_i(n) + \nu_i h + \frac{\kappa h}{N} \sum_{j=1}^{N} \sin(\theta_j(n) - \theta_i(n)), \quad i = 1, \ldots, N, \quad (2)$$

where $h := \Delta t > 0$ is a given fixed time step.

In [6], Choi and Ha first proved the complete consensus of the discrete-time Kuramoto model with identical oscillators ($\nu_i \equiv \nu$) when initial phase configurations are confined in a half circle and $h$ is sufficiently small. More precisely, (2) exhibits a complete consensus (complete phase synchronization) if

$$0 < \kappa h < 1, \quad \max_{1 \leq i, j \leq N} |\theta_i(0) - \theta_j(0)| < \pi,$$

which is analogous to the result in [16]. Then, Ha et al. [17] verified the uniform-in-time convergence of (2) to (1) during $h \to 0$ and exponential synchronization of (discrete-time) identical oscillators, whose corresponding result was also provided in [8, 16, 19]. After [20] studied a synchronization and phase-locked state of the Kuramoto model for more generic setting, Zhang and Zhu established a corresponding stability theory in [38] for discrete gradient flow to show that (2) exhibits an asymptotic phase-locking for sufficiently small $\frac{\nu}{\kappa}$ and $h$ (see Definition 2.3 for the definition of phase-locked state). A notable remark is that [38] did not specify an explicit size of $h$ to show the desired emergent behavior, while other previous results always provided the upper bound of $h$. This is because the main idea of [38] relies on the consistency of one-step forward Euler discretization to its continuous model, and the necessary $h$ depends on the implicit converging speed of the discrete Kuramoto model to continuous Kuramoto model. Therefore, in this paper, we study a sufficient framework for the synchronization of the model (2) for generic initial data without using the classical consistency and convergence result of one-step forward Euler scheme. Then, the criteria for the synchronization of (2), especially the upper bounds for possible time step $h$ are explicitly provided, and the result becomes more applicable in practice. Since system (2) depends only on $\kappa h, \{\nu_i h\}_{i=1}^{N}$, and initial configurations $\{\theta_i(0)\}_{i=1}^{N}$, it is clear that all criteria must be able to be characterized by these independent variables, and all of our results are given by this way.

The main results of this paper are three-fold. First, we proceed a Lyapunov functional approach for discrete system (2) to characterize the asymptotic behavior
for identical oscillators and find a critical size of $h$. (see Theorem 3.3). Then, we provide a sufficient criterion to preserve the order of identical oscillators and get an exact convergence rate to either complete phase synchronization state or bipolar state for generic initial data (see Lemma 3.4 and Theorem 3.7). Finally, we combine the Lyapunov functional approach with the finiteness of phase-locked states (see Theorem 4.4 and Theorem 4.6) to (2) to prove the emergence of phase-locked state for generic initial data. Although we follow the idea in [4, 20] of the continuous Kuramoto model, the proof is completely parallel to their continuous version and never use the consistency with their discretization as in [38].

The rest of this paper is organized as follows. In Section 2, we briefly review some previous results on the continuous-time Kuramoto model (1), such as order parameters, asymptotic behavior and finiteness of phase-locked states. In Section 3, we provide a synchronization estimate and exponential convergence rate for the identical discrete Kuramoto model by using the monotoneness of order parameter and order preserving property of identical oscillators. In Section 4, we prove an asymptotic phase-locking for the nonidentical discrete Kuramoto model for generic initial data, when the coupling strength $\kappa$ is sufficiently large compared to $\nu$. Finally, Section 5 is devoted to a brief summary of our results.

2. Preliminaries. In this section, we review some basic concepts and properties of (1)–(2) without proof. First of all, concerning the dynamics of deviations $\hat{\theta}_i := \theta_i - \frac{1}{N} \sum_{j=1}^{N} \theta_j$, if necessary, we may assume $\nu_e := \frac{1}{N} \sum_{i=1}^{N} \nu_i = 0$ without loss of generality. In particular, it suffices to consider the case $\nu_1 \equiv 0$ to see the behaviors of identical Kuramoto model. Note that this statement holds for both continuous and discrete Kuramoto model (1)–(2).

2.1. Continuous-time Kuramoto model. In this part, we present some previous results on the continuous-time Kuramoto model (1). First, we introduce the definition of order parameters and their properties, which are frequently used in our whole discussion.

Definition 2.1. Let $\Theta = (\theta_1, \cdots, \theta_N) \in \mathbb{R}^N$ be a given phase configuration. Then, the order parameter $(r, \phi) \in \mathbb{R}^2$ of the configuration $\Theta$ is defined as

$$r(\Theta) := \left| \frac{1}{N} \sum_{k=1}^{N} e^{i\theta_k} \right|, \quad e^{i\phi(\Theta)} := \frac{1}{N r} \sum_{k=1}^{N} e^{i\theta_k} \quad (r \neq 0).$$ (3)

Now, for notational simplicity, we suppress $\Theta$-dependence on $r$ and $\phi$, i.e.,

$$r := r(\Theta), \quad \phi := \phi(\Theta).$$

Then, we introduce the following well-known properties of order parameter $(r, \phi)$ independent to the dynamics of $\Theta$, and thus hold for both continuous and discrete Kuramoto model.

Lemma 2.2. The order parameters $(r, \phi)$ satisfy

$$r \cos(\phi - \theta) = \frac{1}{N} \sum_{k=1}^{N} \cos(\theta_k - \theta), \quad r \sin(\phi - \theta) = \frac{1}{N} \sum_{k=1}^{N} \sin(\theta_k - \theta) \quad \forall \theta \in \mathbb{R}.$$
In particular, we have

\[
\begin{align*}
    r^2 &= \left( \frac{1}{N} \sum_{k=1}^{N} \cos(\theta_k - \theta) \right)^2 + \left( \frac{1}{N} \sum_{k=1}^{N} \sin(\theta_k - \theta) \right)^2 \quad \forall \theta \in \mathbb{R}, \\
    r &= \frac{1}{N} \sum_{k=1}^{N} \cos(\theta_k - \phi), \quad 0 = \frac{1}{N} \sum_{k=1}^{N} \sin(\theta_k - \phi), \quad r^2 = \frac{1}{N^2} \sum_{i,j=1}^{N} \cos(\theta_i - \theta_j).
\end{align*}
\]

Note that the uniqueness of \( \phi \) in (3) is only guaranteed up to modulo \( 2\pi \). However, once the dynamics of \( \Theta \) is given as (1), we can specify the angle \( \phi \in \mathbb{R} \) continuous in time and satisfying following differential equations:

**Proposition 1.** Let \( \Theta(t) = \{ \theta_i(t) \}_{i=1}^{N} \) be a solution to the continuous time Kuramoto model (1). Then, \( (r, \phi) \) satisfy

\[
\begin{align*}
    \dot{r} &= -\frac{1}{N} \sum_{j=1}^{N} \nu_j \sin(\theta_j - \phi) + \frac{\kappa r}{N} \sum_{j=1}^{N} \sin^2(\theta_j - \phi), \\
    \dot{\phi} &= \frac{1}{Nr} \sum_{j=1}^{N} \nu_j \cos(\theta_j - \phi) - \kappa \frac{2}{2N} \sum_{j=1}^{N} \sin(2\theta_j - 2\phi).
\end{align*}
\]

In fact, this is a key estimate for the convergence of identical Kuramoto model \( (\nu_i \equiv 0) \). If \( \nu_i \equiv 0 \), (4) simply gives a non-decreasingness of the order parameter \( r \), and since it has an upper bound 1, each \( \sin(\theta_j - \phi) \) vanishes at infinity provided that \( r(\Theta(0)) > 0 \). Then, we have the following:

**Proposition 2.** [4] Let \( \Theta(t) = \{ \theta_i(t) \}_{i=1}^{N} \) be a solution to (1), where the natural frequencies \( \nu_i \) and initial configuration \( \{ \theta_{in} \}_{i=1}^{N} \) satisfy

\[ \nu_i \equiv 0, \quad r(\Theta^{in}) > 0. \]

Then, for each \( i, j = 1, \ldots, N \), the limit

\[ \theta_i^\infty := \lim_{t \to \infty} \theta_i(t) \]

exists and

\[ \sin(\theta_i^\infty - \theta_j^\infty) = 0. \]

Moreover, if \( \theta_{in}^i \neq \theta_{jn}^j \) mod \( 2\pi \) when \( i \neq j \), there exists \( \phi^\infty \in \mathbb{R} \) satisfying

\[ |\{ i : \theta_i^\infty \neq \phi^\infty \mod 2\pi \}| \geq N - 1, \quad |\{ i : \theta_i^\infty = \phi^\infty \mod 2\pi \}| \leq 1. \]

In [38], Zhang and Zhu provided an analogous result for Proposition 2 for the discrete Kuramoto model (2). The discrete gradient flow structure (see Proposition 6) and the convergence, consistency of one-step forward Euler scheme were used for the existence of \( \Theta^\infty \) and the classification of asymptotic states. The weakness of this argument is that the consistency argument is only valid for uncertainly small \( h \) and therefore has a difficulty to apply the result to real numerical experiments. We here provide an alternative proof of the asymptotic synchronization by using (2) itself, so that the sufficient framework for the synchronization becomes more explicit (see Section 3).

**Remark 1.** There are two main obstacles to adapt the above argument to (2).
1. Clearly, the first one is that we have to find an alternative way to get the non-decreasingness of $r$ instead of Proposition 2.2. The other difficulty is that we cannot deduce the convergence of $\theta_j - \phi$ from $\sin(\theta_j - \phi) \to 0$ immediately because of the absence of the continuity of $\theta_j$ and $\phi$.

2. Another unique feature of continuous-in-time model is collision avoidance of identical oscillators, i.e., order preserving. This can be shown by the uniqueness and analyticity of solution $\{\theta_j\}_{j=1}^N$ for (1), which cannot be guaranteed for the discrete Kuramoto model. Still, we can show the order preserving property for the identical discrete Kuramoto model by using different argument (see Section 3).

Now, we review two previous results on the phase-locked states of nonidentical oscillators, namely, emergence of phase-locked state and finiteness of phase-locked states. Below, we introduce a definition of phase-locked states and other relative notions for continuous and discrete Kuramoto model.

**Definition 2.3.** [20, 21]

1. We say that $\Theta^\infty = (\theta_1^\infty, \cdots, \theta_N^\infty)$ is a phase-locked state of continuous (or discrete) Kuramoto model if and only if it is a stationary solution of (1) (or (2)), i.e.,

   $$\nu_i + \frac{\kappa}{N} \sum_{j=1}^N \sin(\theta_j^\infty - \theta_i^\infty) = 0, \quad i = 1, \cdots, N.$$

2. We say that the phase configurations $\Theta = (\theta_1, \cdots, \theta_N)$ and $\tilde{\Theta} = (\tilde{\theta}_1, \cdots, \tilde{\theta}_N)$ are equivalent if and only if for each $1 \leq i, j \leq N$ there exists an integer $m_{ij}$ satisfying

   $$(\theta_i - \tilde{\theta}_i) - (\theta_j - \tilde{\theta}_j) = 2m_{ij}\pi.$$

3. The phase-locked state $\Theta$ is called standardizable if and only if it has a positive order parameter $r(\Theta) > 0$.

4. Let $\Theta = \Theta(t)$ ($\Theta(n)$, resp) be a solution to (1) (2), resp). Then, we say that $\Theta$ exhibits asymptotic phase-locking if and only if it converges to a phase-locked state as $t \to \infty$ ($n \to \infty$, resp).

Then, the following proposition tells us that the asymptotic phase-locking emerges in a strong coupling regime for generic initial configuration.

**Proposition 3.** [20] Suppose that the initial configuration $\Theta^{in}$ and natural frequencies $\nu_i$ satisfy

$$\max_{1 \leq j \leq N} |\nu_j| < \infty, \quad \sum_{j=1}^N \nu_j = 0, \quad r(\Theta^{in}) > 0, \quad \theta_j^{in} = \theta_k^{in} \mod 2\pi, \quad 1 \leq j \neq k \leq N.$$

Then there exists a large coupling strength $\kappa_{\infty} > 0$ such that if $\kappa \geq \kappa_{\infty}$ the solution of (1) with the initial data $\Theta^{in}$ exhibits asymptotic phase-locking.

Although the Kuramoto model (1) can exhibit asymptotic phase-locking for generic initial configuration, the number of possible asymptotic state is finite for given initial data, which comes from the proposition below.

**Proposition 4.** [21] There are at most $2^N$ non-equivalent phase-locked states for the nonidentical Kuramoto model (1).
Remark 2. From the above definition, the phase-locked state of the discrete Kuramoto model (2) is equivalent to that of continuous-in-time Kuramoto model (1). Therefore, the Proposition 4 also tells us that there are at most \( 2^N \) non-equivalent phase-locked states for the nonidentical discrete Kuramoto model (2).

2.2. Discrete Kuramoto model. In this part, we summarize some previous synchronization results on the discrete Kuramoto model (2) provided in [6, 38]. All results in this subsection are at least slightly improved in Section 3 and Section 4.

Proposition 5. [6] (Complete phase synchronization) Let \( \{\Theta(n)\}_{n \geq 0} \) be a solution to the equation (2) satisfying
\[
\nu_1 = \ldots = \nu_N = 0, \quad 0 < \kappa h < 1, \quad \max_{1 \leq i,j \leq N} |\theta_i(0) - \theta_j(0)| < \pi.
\]
Then, we have an asymptotic complete phase synchronization (see Definition 3.1 for the definition of complete phase synchronization):
\[
\lim_{n \to \infty} |\theta_i(n) - \theta_j(n)| = 0, \quad \forall 1 \leq i,j \leq N.
\]

In Section 3, we extend this result to \( 0 < \kappa h \leq 1 \), though the proof is completely different with [6]. In [6], Choi and Ha proved Proposition 5 by adapting the idea in [16] for the discrete Kuramoto model (2). However, we prove the extended result by showing the convergence of the identical discrete Kuramoto model for \( 0 < \kappa h \leq 2 \) and the order preserving property for \( 0 < \kappa h \leq 1 \) (see Lemma 3.6).

Next proposition below and its corollary are the main result of [38], which follows the idea of gradient flow argument in [20] to the discrete Kuramoto model (2).

Proposition 6. [38] (Discrete gradient flow) Let \( f : D \to \mathbb{R} \) be an analytic function in a convex compact domain \( D \subset \mathbb{R}^N \), and \( x : \mathbb{N} \to D \) be a sequence confined in \( D \) satisfying the following gradient-flow type equation in discrete sense:
\[
x(n+1) - x(n) = -\nabla_x f(x(n))h,
\]
where \( h \) is the mesh size. Then, for sufficiently small \( h \), there exists a point \( x^\infty \in D \) such that
\[
\lim_{n \to \infty} x(n) = x^\infty, \quad \nabla_x f(x^\infty) = 0.
\]

Remark 3. For the identical discrete Kuramoto model, the sufficient \( h \) to achieve the convergence given by Proposition 6 is
\[
0 < \kappa h < \frac{2}{N}. \tag{5}
\]
To see this, we here present the brief introduction to the proof of Proposition 6. For the Kuramoto model, the potential function \( f \) is given as
\[
f(x_1, \ldots, x_N) = -\frac{\kappa}{N} \sum_{i,j=1}^{N} \cos(x_i - x_j),
\]
and there exists \( \xi(n) \) satisfying
\[
f(x(n+1)) - f(x(n)) = \nabla_x f(x(n))(x(n+1) - x(n)) + \frac{1}{2}(x(n+1) - x(n))H(\xi(n))(x(n+1) - x(n)),
\]
where each element $H_{ij}$ of the Hessian $H$ is bounded by
\[ |H_{ij}(\xi(n))| \leq \max_{i,j} \max_{x \in D} |\partial_x \partial_x f(x)| = \kappa. \]

Therefore, we have
\[
\begin{align*}
    f(x(n+1)) - f(x(n)) &= -\|\nabla_x f(x(n))\|^2 h + \frac{\kappa h^2}{2} \nabla_x f(x(n)) H(\xi(n)) \nabla_x f(x(n)) \\
    &\leq -\|\nabla_x f(x(n))\|^2 h + \frac{\kappa h^2}{2} |(1, \cdots, 1) \cdot \nabla_x f(x(n))|^2 \\
    &\leq -\|\nabla_x f(x(n))\|^2 h + \frac{N\kappa h^2}{2} \|\nabla_x f(x(n))\|^2 \\
    &= -\|\nabla_x f(x(n))\|^2 h \left(1 - \frac{N\kappa h^2}{2}\right) < 0.
\end{align*}
\]

However, $N$–independent result of Proposition 5 indicates that the sufficient condition (5) might not be optimal, and in fact, this condition is improved in Theorem 3.3.

Finally, as a Corollary of Proposition 6, one has the emergence of phase-locked state for generic initial data.

**Corollary 1.** [38] Let $N \geq 3$ and suppose that the initial configuration $\{\Theta^{\text{in}}\}$ and natural frequencies $\nu_i \in \mathbb{R}$ satisfy
\[
    \sum_{j=1}^{N} \nu_j = 0, \quad \theta_j^{\text{in}} \in [-\pi, \pi), \quad r(\Theta^{\text{in}}) > 0, \quad \theta_j^{\text{in}} \neq \theta_k^{\text{in}}, \quad 1 \leq j \neq k \leq N.
\]

Then, there exists a large coupling strength $\kappa_\infty > 0$ and a small mesh size $h_0 > 0$ such that, if $\kappa > \kappa_\infty$ and $0 < h < h_0$, the emergence of phase-locked state will asymptotically occur, i.e., we can find a phase locked state $\Theta^\infty$ such that the solution to system (2) with initial data $\{\Theta^{\text{in}}\}$ satisfies
\[
    \lim_{n \to \infty} \|\Theta(n) - \Theta^\infty\|_\infty = 0.
\]

3. **Emergent dynamics of identical oscillators.** In this section, we study the emergent dynamics of discrete Kuramoto model (2) with identical oscillators $\nu_i \equiv \nu$:
\[
    \theta_i(n+1) = \theta_i(n) + \nu h + \frac{\kappa h}{N} \sum_{j=1}^{N} \sin(\theta_j(n) - \theta_i(n)). \quad (6)
\]

For this, we consider the alternative sequence:
\[
    \{\tilde{\theta}_i(n) := \theta_i(n) - \nu n h\}_{n \geq 0},
\]
so that the dynamics of $\{\tilde{\Theta}(n)\}_{n \geq 0}$ is given by:
\[
    \tilde{\theta}_i(n+1) = \tilde{\theta}_i(n) + \frac{\kappa h}{N} \sum_{j=1}^{N} \sin(\tilde{\theta}_j(n) - \tilde{\theta}_i(n)). \quad (7)
\]

Therefore, as we pointed out in Section 2, we study the model (7) in the whole Section 3 instead of (6). Note that the total sum of phases $\sum_{i=1}^{N} \tilde{\theta}_i$ is preserved in (7).
3.1. **Asymptotic behavior.** In this subsection, we study a sufficient framework to guarantee the monotonicity of order parameter $r$ for identical oscillators. Then, we show that for generic initial data, the system exhibits either bipolar configuration or complete phase synchronization asymptotically as $n \to \infty$.

First, we begin this subsection with the definition of bipolar configuration and complete phase synchronization:

**Definition 3.1.** [20] Let $\Theta = (\theta_1, \cdots, \theta_N)$ be a given phase configuration in $\mathbb{R}^N$.

1. The phase configuration $\Theta = (\theta_1, \cdots, \theta_N)$ is a bipolar configuration if and only if the following two conditions hold:
   - $(\theta_i - \theta_j) \equiv 0 \pmod{\pi}, \quad 1 \leq i, j \leq N.$
   - $\exists k, l \in \{1, \cdots, N\}$ such that $(\theta_k - \theta_l) \equiv \pi \pmod{2\pi}.$

2. The phase configuration $\{\Theta(n)\}_{n \geq 0}$ exhibits a (asymptotic) complete phase synchronization if
   \[\lim_{n \to \infty} (\theta_i(n) - \theta_j(n)) \equiv 0 \pmod{2\pi}, \quad 1 \leq i, j \leq N.\]

According to [38], the solution $\{\Theta(n)\}_{n \geq 0}$ of (7) will exhibit either the asymptotic complete synchronization or the convergence to some bipolar configuration for sufficiently small $h$. On top of this, we here provide the critical time step $h$ to achieve the monotonicity of order parameter $r$, which is given as follows:

**Lemma 3.2.** Let $\{\Theta(n)\}_{n \geq 0}$ be a vector-valued sequence satisfying (7). Then, if the system parameters $\kappa$ and $h$ satisfy
\[0 < \kappa h \leq 2,\]

then the order parameter
\[r_n := r(\Theta(n))\]
increases monotonically as $n$ increases.

**Proof.** It suffices to prove that the functional $A_n$ below is always nonnegative:
\[A_n := r_{n+1} \cos(\phi(n+1) - \phi(n)) - r_n, \quad \text{where } \phi(n) := \phi(\Theta(n)).\]

To see this, we first write the functional $A_n$ explicitly by using Lemma 2.2:
\[
A_n = \frac{1}{N} \sum_{j=1}^{N} \left[ \cos(\theta_j(n+1) - \phi(n)) - \cos(\theta_j(n) - \phi(n)) \right]
\]
\[
= \frac{1}{N} \sum_{j=1}^{N} \left[ \cos(\theta_j(n) - \phi(n) - \kappa hr_n \sin(\theta_j(n) - \phi(n))) - \cos(\theta_j(n) - \phi(n)) \right]
\]
\[
= \frac{2}{N} \sum_{j=1}^{N} \left[ \sin \left( \frac{\kappa hr_n}{2} \sin(\theta_j(n) - \phi(n)) \right) \right.
\]
\[
\times \sin \left( \theta_j(n) - \phi(n) - \frac{\kappa hr_n}{2} \sin(\theta_j(n) - \phi(n)) \right).\]

Now, we introduce the function $f_a$ motivated from the summand in the representation of $A_n$ above. More precisely, we define the function $f_a$ as below:
\[f_a(x) := \sin(a \sin x) \sin(x - a \sin x), \quad x \in \mathbb{R}, \quad 0 \leq a \leq 1.\]
Then, one can easily see that $f_a(x)$ is a $2\pi$-periodic even function with respect to $x$ and

$$a \sin x, \ x - a \sin x \in [0, \pi], \quad 0 \leq x \leq \pi, \ 0 \leq a \leq 1.$$ 

Therefore, we obtain the non-negativity of $f_a$ and conclude

$$r_{n+1} \geq r_n + A_n = r_n + \frac{2}{N} \sum_{j=1}^{N} f\frac{\kappa h}{r_n} (\theta_j(n) - \phi(n)) \geq r_n.$$ 

$\square$

**Remark 4.** Let us evaluate the optimality of the criteria (8) by checking a specific case. To do this, assume that $N, \nu, \Theta(0)$ and $\kappa h$ are given as $N = 2, \nu_1 = \nu_2 = 0, \theta_1(0) = -\theta_2(0) = \varepsilon$, $\kappa h > 2$. Then, we have

$$\theta_1(n) = -\theta_2(n), \ \phi(\Theta(n)) = 0, \ n \geq 0,$$

and

$$r_1 - r_0 = A_0 = \frac{4}{N} \sin \left(\frac{\kappa h}{2} r_0 \sin \theta_1(0)\right) \sin \left(\theta_1(0) - \frac{\kappa h}{2} r_0 \sin \theta_1(0)\right)$$

$$= 2 \sin \left(\frac{\kappa h}{2} \cos \varepsilon \sin \varepsilon\right) \sin \left(\varepsilon - \frac{\kappa h}{2} \cos \varepsilon \sin \varepsilon\right) < 0,$$

when $\varepsilon$ is a small positive number satisfying

$$0 < \varepsilon < \frac{\pi}{2}, \quad \frac{\varepsilon}{\pi} < \frac{2}{\kappa h} < \frac{\sin 2\varepsilon}{2\varepsilon}.$$ 

Therefore, condition (8) is optimal to guarantee the non-decreasing property of order parameter $r$ without any information on initial configurations.

Now, we are ready to show our first main result.

**Theorem 3.3.** Let $\{\Theta(n)\}_{n \geq 0}$ be a vector-valued sequence satisfying the recurrence relation (7), where the system parameter $\kappa h$ satisfies the condition $0 < \kappa h \leq 2$. If the initial order parameter $r_0 := r(\Theta(0))$ is strictly positive, then for each $i, j = 1, \cdots, N$,

$$\theta_i^\infty := \lim_{n \to \infty} \theta_i(n)$$

exists and satisfies

$$\sin(\theta_i^\infty - \theta_j^\infty) = 0.$$

**Proof.** Since $\{r_n\}_{n \geq 0}$ is monotonically increasing with upper bound 1, the sequence $\{r_n\}_{n \geq 0}$ converges to some positive real number $r^\infty(\leq 1)$ and therefore

$$\lim_{n \to \infty} \cos(\phi(n + 1) - \phi(n)) = 1, \quad \lim_{n \to \infty} f\frac{\kappa h}{r_n} (\theta_j(n) - \phi(n)) = 0 \ \forall j.$$ 

We now claim:

$$0 \leq a \leq 1 \implies f_a(x) = \sin(a \sin x) \sin(x - a \sin x) \geq \frac{4a(1-a) \cos 1}{\pi(\pi + 2)} \sin^2 x.$$ 

Indeed, since $f_a$ is $2\pi$-periodic even function, it suffices to show for $x \in [0, \pi]$. Then the above inequality holds for $x \in [0, \frac{\pi}{2}]$ since

$$\sin(a \sin x) \sin(x - a \sin x) \geq \sin(a \sin x) \sin((1-a) \sin x) \geq \frac{4a(1-a) \cos 1}{\pi^2} \sin^2 x,$$
where we used the following basic relations:

\[ 0 \leq a \sin x \leq 1 \leq \frac{\pi}{2} \quad \text{and} \quad 0 \leq (1-a) \sin x \leq x - a \sin x \leq \frac{\pi}{2}, \quad \forall \ 0 \leq x \leq \frac{\pi}{2} \]

Similarly, for \( x \in \left[\frac{\pi}{2}, \pi\right] \), we take \( y = \pi - x \) and complete the proof of aforementioned claim

\[ f_a(x) = \sin(a \sin x) \sin(y + a \sin y) \geq \frac{2}{\pi} (a \sin y) \cdot \frac{\sin \left(\frac{\pi}{2} + 1\right)}{\frac{\pi}{2} + 1} (y + a \sin y) \geq \frac{4a(1 + a) \cos 1}{\pi(\pi + 2)} \sin^2 y, \]

where we used the following relations in the first inequality:

\[ 0 \leq y + a \sin y \leq \frac{\pi}{2} + 1 \quad \text{and} \quad \sin(y + a \sin y) \geq \frac{\sin \left(\frac{\pi}{2} + 1\right)}{\frac{\pi}{2} + 1} (y + a \sin y), \quad \forall \ 0 \leq y \leq \frac{\pi}{2} \]

Thus, for each \( 1 \leq j \leq N \), we have

\[ \lim_{n \to \infty} f_{\frac{\theta_j(n) - \phi(n)}{r_n}}(\theta_j(n) - \phi(n)) = 0 \quad \implies \quad r^\infty = 1 \quad \text{or} \quad \lim_{n \to \infty} \sin(\theta_j(n) - \phi(n)) = 0, \]

and here the former case also implies the latter case. To see this, recall that Lemma 2.2 gives a following representation of order parameter \( r_n \):

\[ r_n = \frac{1}{N} \sum_{j=1}^{N} \cos(\theta_j(n) - \phi(n)). \]

Then, the convergence of order parameter \( r_n \) to 1 is equivalent to \( \cos(\theta_j(n) - \phi(n)) \to 1 \) for each \( j \), which implies the latter case in (9). Therefore, in any case, we have

\[ \lim_{n \to \infty} \sin(\theta_j(n) - \phi(n)) = 0, \quad \lim_{n \to \infty} [\theta_j(n+1) - \theta_j(n)] = \lim_{n \to \infty} [N \theta_j(n) - \theta_j(n)] = 0. \]

Now, although \( \phi(n) \) always defined uniquely up to modulo \( 2\pi \), we may employ the inequality

\[ \cos(\phi(n + 1) - \phi(n)) \geq \frac{r_n}{r_{n+1}} > 0 \]

to find a unique \( \phi(n + 1) \) from \( \phi(n) \) satisfying

\[ |\phi(n + 1) - \phi(n)| < \frac{\pi}{2}. \]

Therefore, for this inductively determined sequence \( \{\phi(n)\}_{n \geq 0} \), we have

\[ \limsup_{n \to \infty} \left| \theta_j(n + 1) - \phi(n + 1) \right| = \limsup_{n \to \infty} \left| \theta_j(n) - \phi(n) \right| \leq \frac{\pi}{2}. \]

From this observation, the convergence of \( \sin(\theta_j(n) - \phi(n)) \) to 0 now implies the convergence of \( \theta_j(n) - \phi(n) \), and there exist \( N \) integers \( k_1, \cdots, k_N \) determined by the relation

\[ \lim_{n \to \infty} (\theta_j(n) - \phi(n)) = k_j \pi. \]
Finally, once we obtain the convergence of \( \{ \phi(n) \}_{n \geq 0} \subset \mathbb{R} \) from the conservation of \( \sum_j \theta_j(n) \), i.e.,

\[
\bar{\theta}(0) = \bar{\theta}(n) = \lim_{n \to \infty} \bar{\theta}(n) = \lim_{n \to \infty} (\phi(n) + \bar{k}\pi), \quad \bar{\theta} := \frac{1}{N} \sum_{i=1}^{N} \theta_i, \quad \bar{k} := \frac{1}{N} \sum_{i=1}^{N} k_i,
\]

we conclude that the limit \( \theta_i^\infty := \lim_{n \to \infty} \theta_i(n) \) exists for all \( i \) and \( \theta_i^\infty = \bar{\theta}(0) - \bar{k}\pi + k_i\pi \quad 1 \leq i \leq N. \)

\[\square\]

**Remark 5.** If the order parameter \( r \) is zero at time \( k \), then Lemma 2.2 implies that for \( 1 \leq i \leq N \),

\[
\theta_i(k+1) = \theta_i(k) + \frac{\kappa h}{N} \sum_{j=1}^{N} \sin(\theta_j(k) - \theta_i(k)) = \theta_i(k) + \kappa hr_k \sin(\phi_k - \theta_i(k)) = \theta_i(k).
\]

Therefore, the phase configuration remains constant from the time \( k \) and never changes. From this observation, one can conclude that the identical discrete Kuramoto model (7) always converges if \( 0 < \kappa h \leq 2 \).

### 3.2. Order preserving and exponential convergence.

In this subsection, we show the order preserving property of the identical discrete Kuramoto model (7) and characterize the bipolar states that can be reached from given initial data. Then, we provide the exact convergence rate of the oscillators to the bipolar state.

**Lemma 3.4.** (Order preserving) Let \( \{ \Theta(n) \}_{n \geq 0} \) be a solution to the equation (7) with

\[
0 < \kappa h \leq 1, \quad r_0 = r(\Theta^{\text{in}}) < 1.
\]

Then, the order of the oscillators \( \{ \theta_i \}_{i=1}^{N} \) is invariant, i.e.,

\[
\theta_i^\infty < \theta_j^\infty \Rightarrow \theta_i(n) < \theta_j(n) \quad \forall n \geq 0, \quad 1 \leq i, j \leq N,
\]

and the complete phase synchronization cannot be achieved in any finite time step.

**Proof.** We split the proof into two steps.

- **Step 1:** We first show that \( r_n \) is strictly less than 1 for every \( n < \infty \).

Suppose in the contrary that there exists an integer \( n_0 \geq 0 \) satisfying

\[
r_{n_0+1} = 1 \quad \text{and} \quad r_k < 1 \quad \text{for every} \quad k \leq n_0.
\]

From the above conditions, there exist \( N \) integers \( (k_1, \cdots, k_N) \) satisfying

\[
\theta_j(n_0 + 1) - 2k_j\pi = \theta_1(n_0 + 1) - 2k_1\pi \quad \forall 1 \leq j \leq N.
\]

Now, consider the following alternative initial configuration \( \tilde{\Theta}(0) = \{ \tilde{\theta}_j(0) \}_{j=1}^{N} \) :

\[
\tilde{\theta}_j(0) := \theta_j(0) - 2k_j\pi \quad \forall 1 \leq j \leq N,
\]

and denote \( \{ \tilde{\Theta}(n) \}_{n \geq 0} \) as a solution to (7) with initial data \( \tilde{\Theta}(0) \). Then, one can easily see from the induction argument that \( \tilde{\theta}_j(n) = \theta_j(n) - 2k_j\pi \) holds for any \( n \geq 0 \) and \( 1 \leq j \leq N \).
Remark 6. Below, we comment on two facts of the order preserving property.

Therefore, we obtain
\[ \hat{\theta}_j(n_0 + 1) - \hat{\theta}_1(n_0 + 1) = \hat{\theta}_j(n_0) - \hat{\theta}_1(n_0) + \kappa hr_n[\sin(\phi(n_0) - \hat{\theta}_j(n_0)) - \sin(\phi(n_0) - \hat{\theta}_1(n_0))] \]
\[ = (1 - \kappa hr_n \cos \xi_{ij}(n_0))(\hat{\theta}_j(n_0) - \hat{\theta}_1(n_0)) \]
\[ = \prod_{k=0}^{n_0}(1 - \kappa hr_k \cos \xi_{ij}(k))(\hat{\theta}_j(0) - \hat{\theta}_1(0)), \]
where the existence of \( \xi_{ij} \) between \( \phi - \hat{\theta}_1 \) and \( \phi - \hat{\theta}_j \) is guaranteed from the mean value theorem. Since \( r_k \) is assumed to be smaller than 1 for \( k \leq n_0 \) and \( 0 < \kappa h \leq 1 \), each multiplicand \( 1 - \kappa hr_k \cos \xi_{ij}(k) \) in the above equation is strictly positive, which implies \( \theta_j(0) = \theta_1(0) \) for all \( 1 \leq j \leq N \) and contradicts to the initial condition \( r_0 < 1 \).

• Step 2: We now prove the order preservin/g property of \( \{\Theta(n)\}_{n \geq 0} \).

Similar to the previous step, the difference of phases \( \theta_i \) and \( \theta_j \) at time \( (n + 1) \) has a multiplicative representation
\[ \theta_j(n + 1) - \theta_i(n + 1) = \theta_j(n) - \theta_i(n) + \kappa hr_n[\sin(\phi(n) - \theta_j(n)) - \sin(\phi(n) - \theta_i(n))] \]
\[ = \theta_j(n) - \theta_i(n) + \kappa hr_n \cos \xi_{ij}(n)(\theta_j(n) - \theta_i(n)) \]
\[ = (1 - \kappa hr_n \cos \xi_{ij}(n))(\theta_j(n) - \theta_i(n)) \]
\[ = \prod_{k=0}^{n}(1 - \kappa hr_k \cos \xi_{ij}(k))(\theta_j(0) - \theta_i(0)), \]
where the existence of \( \xi_{ij} \) between \( \phi - \theta_i \) and \( \phi - \theta_j \) is again guaranteed from the mean value theorem. Then, since \( 0 < \kappa h \leq 1 \), \( r_k < 1 \) and \( |\cos \xi_{ij}| \leq 1 \), the sign of \( \theta_j - \theta_i \) is invariant in time \( n \).

Remark 6. Below, we comment on two facts of the order preserving property.

1. For any \( k \in \mathbb{Z}^N \), if \( \{\Theta(n)\}_{n \geq 0} \) is a solution to (7), \( 2\pi k + \Theta(n) \) is also a solution of (7). Therefore, the order preserving property also implies that
\[ -2\pi < \theta_i(0) - \theta_j(0) \leq 2\pi \Rightarrow -2\pi < \theta_i(n) - \theta_j(n) \leq 2\pi, \quad \forall n \geq 0. \]

2. The order preserving property has been partially shown in [38], when \( \theta_j \)'s are initially highly aggregated.

Now, we classify the asymptotic states of the identical discrete Kuramoto model. We here used a similar argument with [20] for continuous Kuramoto model, and the admissible set of \( h \) is again extended from [38].

Lemma 3.5. Let \( \{\Theta(n)\}_{n \geq 0} \) be a vector-valued sequence satisfying (7) with
\[ 0 < \kappa h \leq 1, \quad r_0 > 0, \quad \theta_i(0) \neq \theta_k(0) \mod 2\pi, \quad \forall i \neq k. \] (10)
Then, up to permutation, there are only two possible \( k := (k_1, \cdots, k_N) \) modulo 2, namely,
\[ (k_1, \cdots, k_N) \equiv (0, \cdots, 0) \quad \text{or} \quad (k_1, \cdots, k_N) \equiv (1, 0, \cdots, 0) \mod 2, \]
so that \((k_1, \cdots, k_N)\) satisfies
\[
\lim_{n \to \infty} (\theta_j(n) - \phi(n)) = k_j \pi \quad \forall \ 1 \leq j \leq N.
\] (11)

Proof. First, recall that the existence of \(k \in \mathbb{Z}^N\) satisfying (11) comes from Theorem 3.3.

Now, suppose that there are at least two odd elements of \(k\). For simplicity, we may assume \(k_1 = k_2 = 1\) by considering \(2\pi\)-translations several times and changing indices if necessary. Since \(\lim_{n \to \infty} (\theta_j(n) - \phi(n)) = \pi, \ j = 1, 2,\) for any given \(\varepsilon < 1,\) there exists \(M = M(\varepsilon) \in \mathbb{Z}^+\) such that
\[
\pi - \varepsilon < \theta_1(n) - \phi(n), \ \theta_2(n) - \phi(n) < \pi + \varepsilon \quad \forall n \geq M.
\] (12)

On the other hand, it follows from (7) and the mean value-theorem that
\[
\theta_2(n+1) - \theta_1(n+1) = \theta_2(n) - \theta_1(n) + khr_n \left[ \sin (\phi(n) - \theta_2(n)) - \sin (\phi(n) - \theta_1(n)) \right]
= \theta_2(n) - \theta_1(n) + khr_n \left[ \cos \xi_{12}(n) \right] (\theta_1(n) - \theta_2(n)).
\]
where \(\xi_{12}(n)\) is between \(\phi(n) - \theta_1(n)\) and \(\phi(n) - \theta_2(n)\) and satisfies
\[
\theta_2(n) - \theta_1(n) = \prod_{k=M}^{n-1} (1 - khr_k \cos \xi_{12}(k)) \left( \theta_2(M) - \theta_1(M) \right)
= \prod_{k=0}^{n-1} (1 - khr_k \cos \xi_{12}(k)) \left( \theta_2(0) - \theta_1(0) \right) \neq 0, \ n \geq M.
\]
Therefore, we have
\[
|\theta_2(n) - \theta_1(n)| = \prod_{k=0}^{n-1} (1 - khr_k \cos \xi_{12}(k)) |\theta_2(0) - \theta_1(0)|
\geq (1 + khr_0 \cos \varepsilon)^{n-M} |\theta_2(M) - \theta_1(M)|, \ n \geq M,
\]
which contradicts to (12). \(\square\)

Next, we will recover another well-known result on the continuous time Kuramoto model under discrete time setting: identical Kuramoto oscillators confined in half circle exhibits complete synchronization. As we noted in Section 2.2, this result was already shown for \(0 < \kappa h < 1\) by Choi and Ha in [6].

Lemma 3.6. Let \(\{\Theta(n)\}_{n \geq 0}\) be a solution to the equations (7) with
\[
0 < \kappa h \leq 1, \quad \theta_1(0) \leq \theta_2(0) \leq \cdots \leq \theta_N(0) < \theta_1(0) + \pi.
\]
Then, \(\{\Theta(n)\}_{n \geq 0}\) exhibits an asymptotic complete phase synchronization.

Proof. From Theorem 3.3 and order preserving property, it suffices to show that \(\{\theta_N - \theta_1(n)\}_{n \geq 0}\) is a nonincreasing sequence of nonnegative real numbers. If \(\{\theta_N - \theta_1(n)\}\) is nonincreasing for all \(n \leq k,\) then we have
\[
\theta_N(k) - \theta_1(k) \leq \theta_N(0) - \theta_1(0) < \pi,
\]
and thus
\[
-\pi < \theta_j(k) - \theta_N(k) \leq 0 \leq \theta_j(k) - \theta_1(k) < \pi, \quad \forall 1 \leq j \leq N.
\] (13)
Now, recall that the diameter $\theta_N - \theta_1$ at time $k + 1$ has a following formula:

\[
\begin{align*}
(\theta_N - \theta_1)(k + 1) &= (\theta_N - \theta_1)(k) \\
&\quad + \frac{\kappa h}{N} \sum_{j=1}^{N} \sin(\theta_j(k) - \theta_N(k)) - \frac{\kappa h}{N} \sum_{j=1}^{N} \sin(\theta_j(k) - \theta_1(k)).
\end{align*}
\]  

(14)

Therefore, we conclude our desired result by applying (13) to (14).

Below, we give a procedure to find an equivalent phase configuration (up to index permutation) to make analysis simple. First, we may assume $0 < \kappa h \leq 1$, $-\pi < \theta_1(0) < \cdots < \theta_N(0) \leq \pi$ without loss of generality. Then, Remark 6 implies that

\[
\theta_N(n) - 2\pi < \theta_1(n) < \cdots < \theta_N(n), \quad \forall n \geq 0.
\]  

(15)

For the solution $\{\Theta(n)\}_{n \geq 0}$ of (7), we now define the index set

\[
\mathcal{I}_m := \{i : \lim_{n \to \infty} \theta_i(n) - \theta_1(n) = m\pi\},
\]  

(16)

and denote $\Theta_m(n) := \{\theta_i(n) : i \in \mathcal{I}_m\}$. Note that all $\mathcal{I}_m$ except $m = 0, 1, 2$ are empty sets from the ordering (15) and the order preserving property.

Now, suppose that the solution $\{\Theta(n)\}_{n \geq 0}$ for (7) satisfies

\[
\{1, \cdots, k - 1\} \subset \mathcal{I}_0 \subset \{1, \cdots, N\} \neq \emptyset.
\]

We then consider an alternative initial configuration $\{\Theta_{alt}(0)\}$ instead of $\{\Theta(0)\}$, which gives an equivalent solution to $\{\Theta(0)\}$ up to index permutation. The alternative initial data is defined as following way:

\[
\theta_i^{alt}(0) = \begin{cases} 
-\pi - \theta_k(0) + \theta_{i+k}(0), & 1 \leq i \leq N - k, \\
\pi - \theta_k(0) + \theta_{i+k-N}(0), & N - k + 1 \leq i \leq N.
\end{cases}
\]

Then, the configuration $\{\Theta_{alt}(0)\}$ and its solution $\{\Theta_{alt}(n)\}_{n \geq 0}$ satisfy

\[
\theta_1^{alt}(n) = -\pi < \theta_1^{alt}(n) < \cdots < \theta_N^{alt}(n) = \pi,
\]

\[
\theta_i^{alt}(n) = \begin{cases} 
-\pi - \theta_k(0) + \theta_{i+k}(n), & 1 \leq i \leq N - k, \\
\pi - \theta_k(0) + \theta_{i+k-N}(n), & N - k + 1 \leq i \leq N,
\end{cases}
\]

\[
\lim_{n \to \infty} (\theta_i^{alt}(n) - \theta_1^{alt}(n)) = 0, \quad 1 \leq i \leq N - 1,
\]

\[
\lim_{n \to \infty} (\theta_N^{alt}(n) - \theta_1^{alt}(n)) = 0 \text{ or } \pi.
\]

Therefore, under the setting (10), we can always find an equivalent initial configuration with

\[
\{1, \cdots, N - 1\} \subset \mathcal{I}_0, \quad \mathcal{I}_1 \subset \{N\} \text{ and } \mathcal{I}_2 = \emptyset.
\]

In our second main result, we provide the exact convergence rate to two asymptotic states described in Lemma 3.5.
**Theorem 3.7.** Let \( \{ \Theta(n) \}_{n \geq 0} \) be a solution to the recurrence relation (7) with \( N \geq 3 \) and initial data \( \Theta(0) \) satisfying

\[
0 < \kappa h \leq 1, \quad -\pi < \theta_1(0) < \cdots < \theta_N(0) \leq \pi, \quad r_0 > 0.
\]

Then, for the limit \( \Theta^\infty = (\theta_1^\infty, \cdots, \theta_N^\infty) \) of the solution \( \Theta(n) \), one has

\[
\lim_{n \to \infty} \sqrt[|n|]{\|\Theta(n) - \Theta^\infty\|} = 1 - \kappa h \text{ or } 1 - \frac{(N - 2) \kappa h}{N}.
\]

**Proof.** Without loss of generality, we may assume

\[
\{1, \cdots, N - 1\} \subset \mathcal{I}_0, \quad \mathcal{I}_1 \subset \{N\}, \quad \mathcal{I}_2 = \emptyset,
\]

where \( \mathcal{I}_m : m = 0, 1, 2, \cdots \) is defined as (16). We then consider the two cases, \( \mathcal{I}_1 = \emptyset \) and \( \mathcal{I}_2 = \{N\} \) separately.

- **Case 1 (\( \mathcal{I}_1 = \emptyset \))**: In this case, there is a common limit \( \theta^\infty \) for phase configuration \( \{\theta_j\}_{j=1}^N \):

\[
\theta_1^\infty = \cdots = \theta_N^\infty = \theta^\infty,
\]

and the order preserving property implies

\[
\|\Theta(n) - \Theta^\infty\| = \max \{ |\theta_1(n) - \theta^\infty|, |\theta_N(n) - \theta^\infty| \}.
\]

This common limit \( \theta^\infty \) can be determined by the relation

\[
N \theta^\infty = \sum_{j=1}^N \theta_j(0) = \sum_{j=1}^N \theta_j(n),
\]

since the total sum of phases \( \theta_j \) is preserved. Thus, the \( \ell^\infty \)-distance \( \|\Theta(n) - \Theta^\infty\| \) can be controlled by the diameter \( \theta_N - \theta_1 \):

\[
\|\Theta(n) - \Theta^\infty\| \leq |\theta_1(n) - \theta^\infty| + |\theta_N(n) - \theta^\infty| = \theta_N(n) - \theta_1(n) \leq 2 \|\Theta(n) - \Theta^\infty\|.
\]

On the other hand, following the notation in Lemma 3.4, we have

\[
\frac{|\theta_N(n + 1) - \theta_1(n + 1)|}{|\theta_N(n) - \theta_1(n)|} = 1 - \kappa h r_n \cos \xi_{1N}(n).
\]

Moreover, the condition \( \mathcal{I}_1 = \emptyset \) implies that the order parameter \( r_n \) converges to 1, i.e.,

\[
\lim_{n \to \infty} r_n = 1,
\]

and \( \xi_{1N} \to 0 \) can be obtained from the definition of \( \mathcal{I}_0 \). Therefore, we have

\[
\lim_{n \to \infty} \sqrt[|n|]{\|\Theta(n) - \Theta^\infty\|} = \lim_{n \to \infty} \sqrt[|n|]{\theta_N(n) - \theta_1(n)} = 1 - \kappa h.
\]

- **Case 2 (\( \mathcal{I}_1 = \{N\} \))**: In this case, there exists a common limit \( \theta^\infty \) only for \( \theta_1, \cdots, \theta_{N-1} \):

\[
\theta_1^\infty = \cdots = \theta_{N-1}^\infty = \theta^\infty, \quad \theta_N^\infty = \theta^\infty + \pi,
\]

and this common limit \( \theta^\infty \) can be determined by the relation

\[
N \theta^\infty + \pi = \sum_{j=1}^N \theta_j(0) = \sum_{j=1}^N \theta_j(n).
\]
If $\theta_N(n) - \pi$ is smaller than $\theta_1(n)$ or larger than $\theta_{N-1}(n)$ for some $n$, then the solution $\{\Theta(n)\}_{n \geq 0}$ exhibits an asymptotic complete phase synchronization according to Lemma 3.6. Since this contradicts to the condition $I_1 = \{N\}$, we have

$$\theta_1(n) \leq \theta_N(n) - \pi \leq \theta_{N-1}(n), \quad \forall n \geq 0.$$  

(19)

Now, we combine (17)–(19) and order preserving property to obtain

$$\theta_1(n) - \theta^\infty \leq \theta_j(n) - \theta_j^\infty \leq \theta_{N-1}(n) - \theta^\infty \quad \forall 1 \leq j \leq N, \quad \sum_{j=1}^{N} (\theta_j(n) - \theta_j^\infty) = 0.$$  

Thus, we have

$$\theta_1(n) - \theta^\infty \leq 0 \leq \theta_{N-1}(n) - \theta^\infty,$$

and again the $L^\infty$-distance $\|\Theta(n) - \Theta^\infty\|_\infty = \max \{ |\theta_1(n) - \theta^\infty|, |\theta_{N-1}(n) - \theta^\infty| \}$ can be controlled by $\theta_{N-1} - \theta_1$:

$$\|\Theta(n) - \Theta^\infty\|_\infty \leq |\theta_1(n) - \theta^\infty| + |\theta_{N-1}(n) - \theta^\infty| = \theta_{N-1}(n) - \theta_1(n) \leq 2\|\Theta(n) - \Theta^\infty\|_\infty.$$  

Now, similar to Case 1, we have

$$\frac{|\theta_{N-1}(n+1) - \theta_1(n+1)|}{|\theta_{N-1}(n) - \theta_1(n)|} = 1 - \kappa hr_n \cos \xi_{1,N-1}(n),$$

and the order parameter $r_n$, angle $\xi_{1,N-1}$ converge to

$$\lim_{n \to \infty} r_n = \sqrt{\frac{1}{N^2} \sum_{i,j=1}^{N} \cos(\theta_i^\infty - \theta_j^\infty)} = \frac{N - 2}{N}$$

and

$$\lim_{n \to \infty} \xi_{1,N-1}(n) = 0,$$

where we used Lemma 2.2 and (17) to find the limit of $r_n$. Therefore, we have

$$\lim_{n \to \infty} \sqrt{\|\Theta(n) - \Theta^\infty\|_\infty} = \lim_{n \to \infty} \sqrt{\theta_{N-1}(n) - \theta_1(n)} = 1 - \frac{N - 2}{N} \kappa h.$$  

$\square$

Remark 7. In [38], the above convergence rates are implicitly given as the upper bounds of convergence rates when $h$ is sufficiently small.

4. Phase-locking for nonidentical oscillators. In this section, we provide a Lyapunov functional for the Kuramoto model with nonidentical oscillators and show the asymptotic phase-locking for generic initial data for sufficiently large coupling strength.

Lemma 4.1. Let $\{\Theta(n)\}_{n \geq 0}$ be a vector-valued sequence satisfying the recurrence relation (2) with $0 < \kappa h \leq 2$. Then, the functional

$$F(\Theta(n)) := \sum_{j=1}^{N} \nu_j \theta_j(n) + \frac{\kappa}{2N} \sum_{i,j=1}^{N} \cos(\theta_i(n) - \theta_j(n)) = \sum_{j=1}^{N} \nu_j \theta_j(n) + \frac{\kappa N}{2} r_n$$

increases monotonically as $n$ increases.

Proof. For simplicity, we now set

$$\phi_n := \phi(n), \quad \hat{\theta}_i(n) := \theta_i(n) - \phi_n, \quad 1 \leq i \leq N, \quad n \geq 0.$$  

Below, we compute the increase of the functional $F(\Theta)$ at time step $n$. 
First, similar to Lemma 3.2, we write the difference $F(\Theta(n+1)) - F(\Theta(n))$ explicitly:

$$ F(\Theta(n+1)) - F(\Theta(n)) = \sum_{j=1}^{N} \nu_j (\theta_j(n+1) - \theta_j(n)) + \frac{\kappa N}{2} (r_{n+1}^2 - r_n^2) $$

$$ = \sum_{j=1}^{N} \nu_j \left( \nu_j h - \kappa r_n \sin \hat{\theta}_j(n) \right) + \frac{\kappa N}{2} (r_{n+1}^2 - r_n^2). $$

Then, we split $J_2$ into three terms

$$ J_2 = \frac{\kappa}{2N} \left[ \left( \sum_{j=1}^{N} \cos(\theta_j(n+1) - \phi_n) \right)^2 + \left( \sum_{j=1}^{N} \sin(\theta_j(n+1) - \phi_n) \right)^2 \right] $$

$$ - \frac{\kappa}{2N} \left( \sum_{j=1}^{N} \cos \hat{\theta}_j(n) \right)^2 =: \sum_{i=1}^{3} J_{2i}, $$

and claim:

$$ J_1 + J_{21} + J_{23} \geq 0. \quad (20) $$

To prove this, we again simplify $J_1 + J_{21} + J_{23}$ by using

$$ a_j := \nu_j - \kappa r_n \sin \hat{\theta}_j(n). $$

Then, $J_1$ and $J_{21} + J_{23}$ can be written as:

$$ J_1 = \sum_{j=1}^{N} \nu_j h a_j = \sum_{j=1}^{N} \left( h a_j^2 + \kappa r_n a_j \sin \hat{\theta}_j(n) \right), $$

$$ J_{21} + J_{23} = \frac{\kappa}{2N} \left[ \left( \sum_{j=1}^{N} \cos (\hat{\theta}_j(n) + a_j h) \right)^2 - \left( \sum_{j=1}^{N} \cos \hat{\theta}_j(n) \right)^2 \right] $$

$$ = \kappa r_n \sum_{j=1}^{N} \left\{ \cos (\hat{\theta}_j(n) + a_j h) - \cos \hat{\theta}_j(n) \right\} $$

$$ + \frac{\kappa}{2N} \left( \sum_{j=1}^{N} \left\{ \cos (\hat{\theta}_j(n) + a_j h) - \cos \hat{\theta}_j(n) \right\} \right)^2, $$

where we used $r_n = \frac{1}{N} \sum_{i=1}^{N} \cos \hat{\theta}_i(n)$ in the last equality.

Therefore, it suffices to prove that

$$ h a_j^2 + \kappa r_n a_j \sin \hat{\theta}_j(n) + \kappa r_n \left\{ \cos (\hat{\theta}_j(n) + a_j h) - \cos \hat{\theta}_j(n) \right\} \geq 0, \quad 1 \leq j \leq N. $$

On the other hand, from the Taylor’s theorem, there exists $c_j \in [0, 1]$ satisfying

$$ \cos (\hat{\theta}_j(n) + a_j h) = \cos \hat{\theta}_j(n) - a_j h \sin \hat{\theta}_j(n) - \frac{(a_j h)^2}{2} \cos (\hat{\theta}_j(n) + c_j a_j h). $$
Hence, our claim (20) holds:
\[
J_1 + J_{21} + J_{23} \geq \sum_{j=1}^{N} h a_j^2 \left(1 - \frac{\kappa h}{2} r_n \cos \left(\hat{\theta}_j(n) + c_j a_j h \right) \right) \geq 0.
\] (21)

Now, since \(J_{22} \geq 0\) is clear, we have the monotone increasingness of \(F(\Theta)\).

Once we have a Lyapunov functional \(F(\Theta)\), the uniform boundedness of \(F(\Theta(n))\) in time \(n\) will give a convergence of \(F(\Theta)\). The following lemma says that if coupling strength \(\kappa\) is sufficiently large and the majority of phases \(\theta_i\) are initially confined in some small interval, then all phases \(\theta_i\) are uniformly bounded in time \(n\). In fact, an analogous result of Lemma 4.2 is invented by [20], and this is a first step to achieve the uniform boundedness of \(\Theta\).

**Lemma 4.2.** Let \(\{\Theta(n)\}_{n \geq 0}\) be a vector-valued sequence satisfying the recurrence relation (2) with \(\sum_i \nu_i = 0\) and \(\theta_j(0) \in [-\pi, \pi]\), \(1 \leq j \leq N\). Suppose that \(n_0, \ell\) and \(\kappa\) satisfy \(0 < \kappa h \leq 1\) and
\[
n_0 \in \left(\frac{N}{2}, N \right], \quad \ell \in \left((D(\nu) + 2\kappa) h, \frac{2(1 + \kappa h) \cos^{-1}\left(\frac{\nu - n_0}{\nu} - D(\nu) h / 2\kappa \right) - \nu}{1 + 2\kappa h}\right),
\]
\[
\max_{1 \leq j, k \leq n_0} |\theta_j(0) - \theta_k(0)| < \ell, \quad \kappa > \frac{D(\nu)}{2 \sin \frac{\ell - D(\nu) h}{2(1 + \kappa h)}} \left(\frac{n_0}{\nu} \cos \frac{(1 + 2\kappa h) (\ell + D(\nu) h)}{2(1 + \kappa h)} - \frac{\nu - n_0}{N}\right),
\] (22)

where \(D(\nu) := \max_{1 \leq i, j \leq N} |\nu_i - \nu_j|\). Then, the angle configuration \(\{\Theta(n)\}_{n \geq 0}\) is uniformly bounded in time.

**Proof:** The basic idea of this lemma is similar to [20]. We assume the finite first hitting time of the diameter of majority set to the prescribed diameter \(\ell\). Once we deduce a contradiction, the majority set is then contained in small moving arc whose length is \(\ell\). Then, if the distance from these majorities to one of the other phases is assumed to be unbounded, it will be captured eventually to this length \(\ell\) arc. The continuity of \(\{\theta_i(t)\}_{t \geq 0}\) in the argument in [20] is replaced by the boundness of \(\theta_i(n + 1) - \theta_i(n)\), and the majority diameter bound \(\ell\) has to be larger than this to establish the above capturing argument. Although some of the ideas are similar to [38], we here provide a full proof of our lemma for the completeness of the paper.

(i) (Uniform boundness of \(D(\Theta(n))\)): Let \(\Theta = (\theta_1, \ldots, \theta_N)\) be a solution to the recurrence relation (2) satisfying the conditions (22).

• Case A (Dynamics of \(\{\theta_1, \ldots, \theta_{n_0}\}\) ): For this, we follow the argument in [20]. We set
\[
\Sigma_0(n) := \{\theta_1(n), \ldots, \theta_{n_0}(n)\} \subset \mathbb{R}.
\]

Then, we claim:
\[
\text{diam} \Sigma_0(n) \leq \ell, \quad n \geq 0.
\] (23)

**Proof of claim (23):** For this, we define
\[
m := \inf \{n \in \mathbb{Z}_+ | \max_{1 \leq j, k \leq n_0} |\theta_j(n) - \theta_k(n)| > \ell\}, \quad m^* := m - 1.
\] (24)
First, note that $m$ cannot be zero (i.e., $m^* \geq 0$) due to the condition (22). If $m < \infty$, we can find a index pair $(j_m, k_m)$ satisfying
\[
\theta_{j_m}(m) = \max_{1 \leq i \leq n_0} \theta_i(m), \quad \theta_{k_m}(m) = \min_{1 \leq i \leq n_0} \theta_i(m), \quad \theta_{j_m}(m) - \theta_{k_m}(m) > \ell,
\]
from the definition of $m$. Then, we consider a following inequality
\[
\theta_{j_m}(m^*) - \theta_{k_m}(m^*)
= \theta_{j_m}(m) - \theta_{k_m}(m) - (\nu_{j_m} - \nu_{k_m}) h + \kappa h r_{m^*} (\sin \theta_{j_m}(m^*) - \sin \theta_{k_m}(m^*))
\geq \ell - (\nu_{j_m} - \nu_{k_m}) h + \kappa h r_{m^*} (\sin \hat{\theta}_{j_m}(m^*) - \sin \hat{\theta}_{k_m}(m^*))
\geq \ell - D(\nu) h - \kappa h |\theta_{j_m}(m^*) - \theta_{k_m}(m^*)|,
\]
and condition $0 < \kappa h \leq 1$ to derive a lower bound of $\theta_{j_m} - \theta_{k_m}$ at time $n = m^*$:
\[
\theta_{j_m}(m^*) - \theta_{k_m}(m^*) > \frac{\ell - D(\nu) h}{1 + \kappa h}.
\]
However, we can use (2) and (24) to obtain
\[
0 < (\theta_{j_m}(m) - \theta_{k_m}(m)) - (\theta_{j_m}(m^*) - \theta_{k_m}(m^*))
= (\nu_{j_m} - \nu_{k_m}) h - \frac{\kappa h}{N} \sum_{i=1}^{N} (\sin \theta_{j_m}(m^*) - \theta_i(m^*)) - \sin (\theta_{k_m}(m^*) - \theta_i(m^*))
\leq D(\nu) h - \frac{2\kappa h}{N} \sum_{i=1}^{N} \sin \theta_{j_m}(m^*) - \theta_{k_m}(m^*)
\times \left[ \sum_{i=1}^{n_0} \cos \left( \frac{\theta_i(m^*) - \theta_{j_m}(m^*) + \theta_{k_m}(m^*)}{2} \right) \right]
+ \sum_{i=n_0+1}^{N} \cos \left( \frac{\theta_i(m^*) - \theta_{j_m}(m^*) + \theta_{k_m}(m^*)}{2} \right).
\tag{25}
\]
Here, we have the lower bound of cosines in (25) from the following observations:
\[
\theta_i(m^*) - \frac{\theta_{j_m} + \theta_{k_m}(m^*)}{2} = (\theta_i - \theta_{j_m})(m^*) + \frac{\theta_{j_m} - \theta_{k_m}(m^*)}{2} \geq -\ell + \frac{\ell - D(\nu) h}{2(1 + \kappa h)},
\]
\[
\theta_i(m^*) - \frac{\theta_{j_m} + \theta_{k_m}(m^*)}{2} = (\theta_i - \theta_{k_m})(m^*) - \frac{\theta_{j_m} - \theta_{k_m}(m^*)}{2} \leq \ell - \frac{\ell - D(\nu) h}{2(1 + \kappa h)}.
\tag{26}
\]
Thus, it follows from (25), (26) and the condition on $\kappa$ in (22) that
\[
0 < (\theta_{j_m}(m) - \theta_{k_m}(m)) - (\theta_{j_m}(m^*) - \theta_{k_m}(m^*))
\leq D(\nu) h - \frac{2\kappa h}{N} \sin \theta_{j_m}(m^*) - \theta_{k_m}(m^*)
\times \left[ \sum_{i=1}^{n_0} \cos \left( \frac{\theta_i(m^*) - \theta_{j_m}(m^*) + \theta_{k_m}(m^*)}{2} \right) \right]
+ \sum_{i=n_0+1}^{N} \cos \left( \frac{\theta_i(m^*) - \theta_{j_m}(m^*) + \theta_{k_m}(m^*)}{2} \right)
\leq D(\nu) h - \frac{2\kappa h}{N} \sin \left( \frac{\ell - D(\nu) h}{2(1 + \kappa h)} \right) \frac{1 + 2\kappa h}{2(1 + \kappa h)} - (N - n_0) < 0.
\]
which gives a contradiction. Therefore, we have \( m = \infty \), i.e.,
\[
\operatorname{diam} \Sigma_0(n) \leq \ell, \quad n \geq 0.
\]

- Case B (Dynamics of \( \{ \theta_{n_0+1}, \cdots, \theta_N \} \)): Suppose that the diameter \( D(\Theta(n)) \) is unbounded in \( \mathbb{R} \). Then without loss of generality, we may assume
\[
\limsup_{n \to \infty} (\theta_{n_0+1}(n) - \theta_1(n)) = \infty. \tag{27}
\]
Then, we define
\[
z^* := \sup\{ n \in \mathbb{Z}_+ | \max_{1 \leq i \leq n_0} (\theta_{n_0+1}(k) - \theta_i(k)) \leq 2\pi \quad \forall 0 \leq k \leq n \}, \quad z := z^* + 1.
\]
Note that \( z^* \) satisfies \( 0 \leq z^* < \infty \) from the initial condition in (22) and assumption (27).

Now, for the times steps \( z \) and \( z^* \), we introduce two indices \( k_{z^*} \) and \( k_z \) in \( \{ 1, \cdots, n_0 \} \) satisfying
\[
\theta_{k_{z^*}}(z^*) = \min_{1 \leq i \leq n_0} \theta_i(z^*), \quad \theta_{k_z}(z) = \min_{1 \leq i \leq n_0} \theta_i(z),
\]
and we claim:
\[
\theta_{n_0+1}(z) < 2\pi + \theta_{k_z}(z) + \ell.
\]
To see this, we again use (2) and condition \( (D(\nu) + 2\kappa)h < \ell \) in (22). More precisely, we can prove the above claim as below:
\[
\theta_{n_0+1}(z) = \theta_{n_0+1}(z^*) + \nu_{n_0+1}h - \kappa hr_{z^*} \sin \hat{\theta}_{n_0+1}(z^*)
\leq 2\pi + \theta_{k_z}(z^*) + \nu_{n_0+1}h - \kappa hr_{z^*} \sin \hat{\theta}_{n_0+1}(z^*)
= 2\pi + \theta_{k_z}(z) - \nu_{k_z}h + \kappa hr_{z^*} \sin \hat{\theta}_{k_z}(z^*) + \nu_{n_0+1}h - \kappa hr_{z^*} \sin \hat{\theta}_{n_0+1}(z^*)
\leq 2\pi + \theta_{k_z}(z) + D(\nu)h + 2\kappa h
< 2\pi + \theta_{k_z}(z) + \ell.
\]
Therefore, \( \theta_{n_0+1}(z) \) lies on a length-\( \ell \) arc containing \( \Sigma_0(z) \), and by using same argument with Case A, we have
\[
\max_{1 \leq i \leq n_0} |\theta_{n_0+1}(n) - \theta_i(n) - 2\pi| \leq \ell, \quad \forall n \geq z.
\]
This gives a contradiction to (27), and hence the diameter \( D(\Theta(n)) \) is uniformly bounded.

(ii) (Uniform boundness of \( \Theta(n) \)) Note that the condition \( \sum_{i=1}^N \nu_i = 0 \) implies
\[
\sum_{i=1}^N \theta_i(n) = \sum_{i=1}^N \theta_i(0), \quad n \geq 0.
\]
Therefore, we have
\[
\sup_{n \geq 0} \max_{1 \leq k \leq N} |\theta_k(n)| \leq \sup_{n \geq 0} \left\{ \frac{1}{N} \sum_{i=1}^N |\theta_i(n)| + D(\Theta(n)) \right\}
= \left| \frac{1}{N} \sum_{i=1}^N \theta_i(0) \right| + \sup_{n \geq 0} D(\Theta(n)),
\]
which is uniformly bounded in \( n \). \( \square \)
Remark 8. Here, we evaluate the condition (22) more quantitatively.

1. In (22), the condition

\[
(D(\nu) + 2\kappa)h < \frac{2(1 + \kappa h) \cos^{-1} \left( \frac{N-n_0}{n_0} \right) - D(\nu)h}{1 + 2\kappa h}
\]

is satisfied if and only if

\[
\left( D(\nu) + \frac{\kappa(1 + 2\kappa h)}{1 + \kappa h} \right) h < \cos^{-1} \left( \frac{N-n_0}{n_0} \right).
\]

(28)

Therefore, if \( \frac{N-n_0}{n_0} < \cos(\frac{3}{2}) = 0.070737... \), the condition (28) does not imply \( 0 < \kappa h \leq 1 \). In particular, if \( D(\nu) = 0 \) and \( \frac{N-n_0}{n_0} < \cos(\frac{3}{2}) \), the condition \( 0 < \kappa h \leq 1 \) is sharper than (28).

2. If we take a limit \( h \to 0^+ \), the conditions (22) can be simplified as

\[
\ell \in \left( 0, 2\cos^{-1} \left( \frac{N-n_0}{n_0} \right) \right), \quad \max_{1 \leq j,k \leq n_0} |(\theta_j - \theta_k)(0)| < \ell,
\]

and

\[
\kappa > \frac{D(\nu)}{2\sin \frac{\ell}{2} \left( \frac{n_0}{N} \cos \frac{\ell}{2} - \frac{N-n_0}{N} \right)},
\]

which coincides with the corresponding condition for the continuous Kuramoto model ([20], Proposition 4.1).

Note that Lemma 4.2 is a statement to guarantee the convergence of the Lyapunov functional \( F(\Theta) \). We apply the following classical result on the dynamical system to prove the asymptotic phase-locking for generic initial configurations.

Lemma 4.3. Let \( G : \mathbb{R}^m \to \mathbb{R}^k \) be a continuous \( 2\pi \)-periodic function such that the set \( G^{-1}(0) \) has only finitely many elements modulo \( 2\pi \). If a vector-valued sequence \( \{X(n)\}_{n \geq 0} \subset \mathbb{R}^m \) satisfies

\[
\lim_{n \to \infty} G(X(n)) = 0, \quad \lim_{n \to \infty} \|X(n+1) - X(n)\| = 0,
\]

then the sequence \( \{X(n)\}_{n \geq 0} \) converges.

Proof. Define \( P \) as an intersection of \( G^{-1}(0) \) with \((-2\pi, 4\pi)^m\), i.e.,

\[
P := G^{-1}(0) \cap (-2\pi, 4\pi)^m.
\]

Since \( P \) is finite, we may impose the order for the elements of \( P \) and denote as

\[
P = \{x_1, \cdots, x_s\}, \quad R := \min_{1 \leq i,j \leq s} \|x_i - x_j\| > 0.
\]

Then, there exists a positive integer \( M \) such that \( B_{1/M}(x_1), \cdots, B_{1/M}(x_s) \) are disjoint closed sets. Note that the disjoint condition implies that \( R > \frac{2\pi}{M} \).

Now, the set \( G([0, 2\pi]^m - \bigcup_{i=1}^s B_{1/M}(x_i)) \) is a compact set which does not contain \( 0 \), and then the distance \( r_M \) from \( 0 \) is strictly positive. Therefore, we obtain

\[
[0, 2\pi]^m \cap G^{-1}(B_{r_M}(0)) \subset \bigcup_{i=1}^s B_{1/M}(x_i).
\]

Moreover, concerning the periodicity of \( G \), we have

\[
G^{-1}(B_{r_M}(0)) \subset \bigcup_{x \in G^{-1}(0)} B_{1/M}(x),
\]
and \{B_{1/M}(x) : x \in G^{-1}(0)\} is a collection of disjoint open balls.

On the other hand, from the condition (29), there exists an integer \( I = I(M) \) such that
\[
\|G(X(n))\| < r_M, \quad \|X(n + 1) - X(n)\| < R - \frac{2}{M}, \quad \forall n \geq I.
\]
This implies that for \( n \geq I(M) \), \( X(n) \) is contained in \( \bigcup_{x \in G^{-1}(0)} B_{1/M}(x) \), and it stays in exactly one open ball \( B_{1/M}(x_0) \) for some proper \( x_0 \in G^{-1}(0) \). Finally, we can consider any integer \( M' \) larger than \( M \) and do a same process to conclude that \( \{X(n)\}_{n \geq 0} \) converges to \( x_0 \).

Now, we are ready to prove the phase-locking of two oscillator system. In fact, for two oscillator system, there is no need to specify the condition for initial data \( \Theta(0) \).

**Theorem 4.4.** Suppose that the natural frequencies and coupling strength satisfy the following conditions:
\[
\begin{align*}
N &= 2, \quad \nu_1 = -\nu_2 = \nu \neq 0, \quad 0 < \kappa h < 1,
\ell &\in \left(\left[|\nu| + \frac{\kappa}{2}\right] h, \frac{\pi(1 + \kappa h) - 2|\nu|h}{1 + 2\kappa h}\right), \quad \kappa > \frac{|\nu|}{\sin \frac{\pi - 2|\nu|h}{2(1 + \kappa h)} \cos \frac{2|\nu|h + 2|\nu|h}{2(1 + \kappa h)}}. \quad (30)
\end{align*}
\]
Then, each sequence \( \{\theta_1(n)\}_{n \geq 0} \) and \( \{\theta_2(n)\}_{n \geq 0} \) of the system (2) converges:
\[
\theta_1^\infty := \lim_{n \to \infty} \theta_1(n), \quad \theta_2^\infty := \lim_{n \to \infty} \theta_2(n),
\]
i.e., the solution \( \{\Theta(n)\}_{n \geq 0} \) exhibits an asymptotic phase locking.

**Proof.** First of all, consider a following discrete Adler equation for \( \theta_1 - \theta_2 \):
\[
\theta_1(n + 1) - \theta_2(n + 1) = \theta_1(n) - \theta_2(n) + 2\nu h - \kappa h \sin(\theta_1(n) - \theta_2(n)). \quad (31)
\]
Then, if \( \theta_1 - \theta_2 \) is unbounded, there exists an integer \( k \) and time \( m \in \mathbb{Z}_{\geq 0} \) such that
\[
\theta_1(m) - \theta_2(m) \in (2\pi k - \ell, 2\pi k + \ell),
\]
since \(|(\theta_1 - \theta_2)(n + 1) - (\theta_1 - \theta_2)(n)|\) has an upper bound \( 2|\nu|h + \kappa h \). Then, under the condition (30) we can use a similar argument to Lemma 4.2 to obtain
\[
\theta_1(n) - \theta_2(n) \in (2\pi k - \ell, 2\pi k + \ell), \quad \forall n \geq m.
\]
Now, according to Lemma 4.1, the sequence \( \{F(\Theta(n))\}_{n \geq 0} \) increase monotonically with an upper bound:
\[
F(\Theta(n)) = \nu(\theta_1(n) - \theta_2(n)) + \kappa r_n^2 \leq |\nu|(2|\pi|+\ell) + \kappa, \quad n \geq m,
\]
and therefore converges as \( n \to \infty \). Note from (21) that the convergence of \( F(\Theta(n)) \) implies a convergence of \( \nu_j - \kappa r_n \sin \theta_j(n) \) to 0, \( j = 1, 2 \). Moreover, for \( N = 2 \), the order parameter \( (r, \phi) \) can be easily found from \( \theta_1 \) and \( \theta_2 \):
\[
r = \cos \frac{\theta_1 - \theta_2 - 2\pi k}{2}, \quad \phi = \frac{\theta_1 + \theta_2 + 2\pi k}{2} \quad \text{if} \quad |\theta_1 - \theta_2 - 2\pi k| < \pi.
\]
Therefore, we substitute these estimates to \( r_n \) and \( \phi_n \) to conclude
\[
\lim_{n \to \infty} \sin(\theta_1(n) - \theta_2(n)) = \frac{2\nu}{\kappa}, \quad \lim_{n \to \infty} |(\theta_1(n + 1) - \theta_2(n + 1) - (\theta_1(n) - \theta_2(n))| = 0. \quad (32)
\]
where we used the Adler equation (31) again for the second result.
It suffices to show that (32) implies the desired result. In fact, the asymptotic phase-locking can be obtained from the convergence $\theta_1 - \theta_2$, as we already know that the total sum $\theta_1 + \theta_2$ is conserved for all $n \geq 0$. We can obtain the convergence of $\theta_1 - \theta_2$ by using Lemma 4.3 to (32).

Below, we estimate the difference between identical and nonidentical phase sequences. This is a corresponding result to Lemma 4.2 in [20].

**Lemma 4.5.** Let $\{\Theta^NI(n)\}_{n \geq 0}$ and $\{\Theta^I\}_{n \geq 0}$ be the solutions of recurrence relations (2) and (7), respectively, subject to

$$\sum_{j=1}^{N} \nu_j = 0, \quad \Theta^I(0) = \Theta^NI(0).$$

Then, we have

$$\max_{1 \leq i \leq N} |\theta^NI_i(n) - \theta^I_i(n)| \leq \frac{D(\nu)}{2\kappa} ((1 + 2\kappa h)^n - 1).$$

**Proof.** It follows from (2), (7) that

$$\left(\theta^NI_j(n+1) - \theta^I_j(n+1)\right) - \left(\theta^NI_j(n) - \theta^I_j(n)\right)
= \nu_j h + \frac{\kappa h}{N} \sum_{k=1}^{N} \left(\sin(\theta^NI_k(n) - \theta^NI_j(n)) - \sin(\theta^I_k(n) - \theta^I_j(n))\right)
= \nu_j h + \frac{2\kappa h}{N} \sum_{k=1}^{N} \sin \frac{\theta^NI_k(n) - \theta^NI_j(n) - \theta^I_k(n) + \theta^I_j(n)}{2} \cos \xi_{kj},$$

where $2\xi_{kj} := \theta^NI_k(n) - \theta^NI_j(n) + \theta^I_k(n) - \theta^I_j(n)$.

Therefore, we obtain

$$\max_{1 \leq i \leq N} |\theta^NI_i(n+1) - \theta^I_i(n+1)| \leq (1 + 2\kappa h) \max_{1 \leq i \leq N} |\theta^NI_i(n) - \theta^I_i(n)| + \sum_{j} \nu_j h \leq (1 + 2\kappa h) \max_{1 \leq i \leq N} |\theta^NI_i(n) - \theta^I_i(n)| + D(\nu)h,$$

where we used $\sum \nu_j = 0$ in the last inequality. Since the initial phases $\Theta^I(0)$ and $\Theta^NI(0)$ are same, we have

$$\max_{1 \leq i \leq N} |\theta^NI_i(n) - \theta^I_i(n)| \leq \frac{D(\nu)}{2\kappa} h \cdot \frac{(1 + 2\kappa h)^n - 1}{2\kappa} = \frac{D(\nu)}{2\kappa} \cdot \frac{(1 + 2\kappa h)^n - 1}{2\kappa}.$$

**Remark 9.** If we set $h := \frac{1}{n}$ and $n \to \infty$, we have

$$\max_{1 \leq i \leq N} |\theta^NI_i(t) - \theta^I_i(t)| \leq \frac{D(\nu)}{2\kappa} (e^{2\kappa t} - 1),$$

which coincides with the estimate in [20].

Finally, combining Lemma 7 and Lemma 4.1–4.5, we obtain our last main result on asymptotic phase-locking for $N \geq 3$ and generic initial data.
Theorem 4.6. Let \( \{ \Theta(n) \}_{n \geq 0} \) be a solution of (2) satisfying the following conditions:
\[
N \geq 3, \quad \mathcal{I}_2 = \emptyset, \quad -\pi < \theta_1(0) < \cdots < \theta_N(0) \leq \pi, \quad r_0 > 0,
\]
\[
\sum_{i=1}^{N} \nu_i = 0, \quad 0 < \kappa h < 1, \quad \left( D(\nu) + \frac{\kappa(1 + 2\kappa h)}{1 + \kappa h} \right) h < \cos^{-1} \frac{1}{N-1}.
\]
Then, there exists a function \( L = L(\kappa h, D(\nu) h, \Theta(0)) \) such that under the extra condition
\[
\frac{D(\nu)}{\kappa} < L(\kappa h, D(\nu) h, \Theta(0)),
\]
the solution \( \{ \Theta(n) \}_{n \geq 0} \) exhibits asymptotic phase-locking.

Proof. We split its proof into two steps.

- Step 1 (Boundedness of \( \Theta \)): First, let \( \ell \) be a positive number given by
\[
\ell = \frac{1}{2} \left( (D(\nu) + 2\kappa) h + \frac{2(1 + \kappa h) \cos^{-1} \frac{1}{N-1} - D(\nu) h}{1 + 2\kappa h} \right).
\]
We consider the solution \( \{ \Theta^I(n) \}_{n \geq 0} \) to (7) with initial configuration
\[
\Theta^I(0) = \Theta(0) = (\theta_1(0), \cdots, \theta_N(0)).
\]
Then, according to Lemma 3.7, there exists a positive integer \( N_2 = N_2(\kappa h, \Theta(0)) \) such that
\[
D(\Theta^I_0(n)) < \frac{\ell}{2}, \quad \forall n \geq N_2.
\] (33)
Now, if \( L = L(\kappa h, D(\nu) h, \Theta(0)) \) is smaller than both 1 and
\[
\min \left\{ 2\sin \frac{\ell - D(\nu) h}{N - 1} \cos \left( \frac{1 + 2\kappa h}{2(1 + \kappa h)} \right) \nu \frac{\ell}{2(1 + 2\kappa h)N_2 - 2} \right\},
\]
the condition \( \frac{D(\nu)}{\kappa} < L \) implies
\[
\max_{1 \leq i \leq N} |\theta_i(N_2) - \theta_i^I(N_2)| \leq \frac{D(\nu)}{2\kappa} (1 + 2\kappa h)^{N_2} - 1 < \frac{\ell}{4}.
\] (34)
Therefore, we combine (33) and (34) to obtain
\[
\max_{i,j \leq |\mathcal{I}_0|} |\theta_i(N_2) - \theta_j(N_2)| \leq \max_{i,j \leq |\mathcal{I}_0|} |\theta_i^I(N_2) - \theta_j^I(N_2)| + 2 \max_{1 \leq i \leq N} |\theta_i(N_2) - \theta_i^I(N_2)| < \ell,
\]
and deduce a uniform boundedness of phase sequence \( \{ \Theta(n) \}_{n \geq 0} \) by using Lemma 4.2 to \( \{ \Theta(n) \}_{n \geq N_2} \).

- Step 2 (Phase-locking of \( \Theta \)): Similar to Theorem 4.4, the boundedness of \( \Theta(n) \) implies the convergence of \( \{ F(\Theta(n)) \}_{n \geq 0} \), and therefore
\[
\lim_{n \to \infty} \left( \nu_j - \kappa r_n \sin \hat{\theta}_j(n) \right) = 0, \quad j = 1, \cdots, N.
\]
We substitute these estimate to (2) to deduce the following convergences: for \( j = 1, \cdots, N \),
\[
\lim_{n \to \infty} \frac{1}{N} \sum_{k=1}^{N} \sin(\theta_j(n) - \theta_k(n)) = \frac{\nu_j}{\kappa}, \quad \lim_{n \to \infty} (\theta_j(n + 1) - \theta_j(n)) = 0.
\] (35)
Now, consider the solution set
\[
S' := \left\{ X \in \mathbb{R}^N : \sum_{j=1}^{N} x_j = \sum_{j=1}^{N} \theta_j(0), \quad \frac{1}{N} \sum_{j=1}^{N} \sin(x_j - x_k) = \frac{\nu_j}{\kappa}, \quad j = 1, \ldots, N \right\}.
\]

Then, there are finitely many non-equivalent phase-locked states for given \( \kappa, \nu \) from Proposition 4, and the intersection of each equivalence class and \( S' \) has \( N \) elements modulo \( 2\pi \).

To see this, consider two equivalent configurations \( \Theta_1, \Theta_2 \) in \( S' \). Then, for every \( 1 \leq i, j \leq N \), we have
\[
\theta^1_i - \theta^2_i \equiv \theta^1_j - \theta^2_j \mod 2\pi,
\]
and therefore
\[
N(\theta^1_i - \theta^2_i) = \sum_{k=1}^{N} (\theta^1_k - \theta^2_k) \equiv 0 \mod 2\pi.
\]

Now, in order to apply Lemma 4.3, we consider the following function:
\[
G = (G_1, \cdots, G_{N-1}), \quad 1 \leq j \leq N,
\]
\[
G_j(x_1, \cdots, x_{N-1}) := \sum_{k=1}^{N-1} \sin(x_j - x_k) + \sin \left( x_j + \sum_{k=1}^{N-1} x_k - \sum_{k=1}^{N} \theta_k(0) \right) - \frac{N\nu_j}{\kappa}.
\]

Then, \( G \) is a continuous \( 2\pi \)-periodic function, and for every \( X \in G^{-1}(0) \),
\[
\tilde{X} := \left( X, \sum_{k=1}^{N} \theta_k(0) - \sum_{k=1}^{N-1} x_k \right)
\]
is an element of \( S' \). Since \( \tilde{X}^1 \equiv \tilde{X}^2 \mod 2\pi \) implies \( X^1 \equiv X^2 \mod 2\pi \), \( G^{-1}(0) \) has finitely many elements modulo \( 2\pi \).

Finally, by applying Lemma 4.3 to (35) and function \( G \), we obtain the convergence of the configuration \( (\theta_1(n), \cdots, \theta_{N-1}(n)) \) to an element of \( G^{-1}(0) \), which implies the asymptotic phase-locking.

5. **Conclusion.** In this paper, we have presented a direct approach to see the emergence of synchronization of the discrete-time Kuramoto model. For the identical oscillators, we first show that the condition \( 0 < \kappa h \leq 2 \) is enough to obtain the asymptotic equilibrium for every initial configuration by using the nondecreasing property of order parameter \( r \). This is the sharpest condition to guarantee the nondecreasingness of \( r \). Then, to get the exact convergence rate to the equilibrium, we show the order preserving property and classification of all asymptotic states that can be reached from generic initial data under the condition \( 0 < \kappa h \leq 1 \). In particular, we slightly extended the condition to achieve the complete phase synchronization in [6] to \( 0 < \kappa h \leq 1 \) when the initial configuration is confined in half circle. Combining all results under the condition \( 0 < \kappa h \leq 1 \), we presented the exact convergence rate \( \lim_{n \to \infty} \sqrt[2]{\| \Theta(n) - \Theta^\infty \|} \) in terms of \( N \) and \( \kappa h \). Finally, for non-identical oscillators, we provided a criteria to obtain the asymptotic phase-locking in terms of \( \kappa h, D(\nu)h, \Theta(0) \). We here used a convergence of Lyapunov functional and the finiteness of non-equivalent phase-locked states instead of discrete gradient flow argument in [38].

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