Localization and absence of Breit-Wigner form for Cauchy random band matrices

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(submitted May 24, 2000)

We analytically calculate the local density of states for Cauchy random band matrices with strongly fluctuating diagonal elements. The Breit-Wigner form for ordinary band matrices is replaced by a Levy distribution of index $\mu = 1/2$ and the characteristic energy scale $\alpha$ is strongly enhanced as compared to the Breit-Wigner width. The unperturbed eigenstates decay according to the non-exponential law $\propto e^{-\sqrt{\alpha \eta}}$. We analytically determine the localization length by a new method to derive the supersymmetric non-linear $\sigma$ model for this type of band matrices.

PACS numbers: 72.15.Rn, 05.40.Fb, 05.60.Gg

Band matrices with random elements appear in a variety of physical problems in the context of quantum chaos and localization. A detailed analytical investigation of the properties of such matrices was performed by Fyodorov and Mirlin using Efetov’s supersymmetry technique. Later, motivated by the localization problem of two interacting particles, Shpelyansky introduced and studied random band matrices superimposed with strongly fluctuating diagonal matrix elements. Subsequent work on this type of matrices showed the existence of the Breit-Wigner regime where the eigenstates have a very peaked structure inside an eventual localization domain.

In all these cases the matrix elements were drawn from a regular, typically gaussian, distribution with finite variance. In the present work, we extend the band matrix ensembles of refs. to the case of Cauchy distributed matrix elements. In an early work, Lloyd already introduced and studied random band matrices with diagonal Cauchy distributed disorder. A detailed study of the level statistics of full random matrices using Cizeau and Bouchaud. Band matrix ensembles with Cauchy distributed matrix elements were recently argued to be relevant and studied numerically in the context of the localization problem of two interacting particles. In this work, we present analytical results concerning the wave function properties for a model similar as in ref. but which differs from that of ref. by two important features: it concerns a banded and not full matrix, and its diagonal elements are typically much larger than the off-diagonal coupling elements.

Actually, we think that this case is of particular interest for a generic disordered fermionic many particle problem (for instance recently studied in where non-interacting eigenstates are coupled by quasi-random two-particle interaction matrix elements. There is an interesting and subtle regime where the effective two-particle level spacing is smaller than the typical interaction matrix element while typical eigenstates are nevertheless composed of many non-interacting eigenstates. In this regime, we can diagonalize the problem for two particles by perturbation theory and apply the corresponding unitary transformation to the full many-particle Hamiltonian. This provides a new type of random matrix with residual three-body interaction matrix elements typically given by perturbative expressions with denominators, containing independent random non-interacting energy differences. Iterating this procedure one also obtains higher order contributions. Flambaum et al. have argued that such denominators give effective distributions for these matrix elements characterized by long tails with the same power law as the Cauchy distribution. Even though the real situation is more complicated, this argument indeed shows the relevance of Cauchy random matrix ensembles.

The model we consider is a random band matrix of dimension $N \times N$ and of width $b \gg 1$ whose matrix elements are of the form $H_{kl} = \eta_k \delta_{kl} + U_{kl}$ with $U_{kl} = 0$ if $k = l$ or $|k - l| > b$. These matrix elements are statistically independent and distributed according to a Cauchy distribution $p_a(x) \equiv \pi^{-1}a/(a^2 + x^2)$ where the width $a$ is given in terms of two parameters $W$ and $U_0$ via $a \equiv W/\pi$ for $\eta_k$ and $a \equiv U_0$ for $U_{kl}$. In this paper, we consider the orthogonal symmetry class where $H$ is real symmetric. The generalization to the other symmetry classes is straightforward. For the following analytical calculations, we will furthermore concentrate on the case where $U_0$ is sufficiently large to avoid a simple perturbative situation but still so small that the total density of states will essentially be dominated by the unperturbed diagonal elements only.

For ordinary band matrices with a finite variance of $U_{jk}$, this case corresponds to the Breit-Wigner regime where the eigenstates are in addition to an eventual space localization also localized in the (unperturbed) energy space, i. e. only the sites $j$ with $|E - \eta_j| \lesssim \Gamma$ essentially contribute to an eigenstate of energy $E$ where $\Gamma \approx 2\pi(2b)/U_{jk}^2/W \ll W$ is the Breit-Wigner width. This can be seen in the local density of states at site $j$ which is a Lorentzian in $(E - \eta_j)$ of width $\Gamma/2$.

To study the Cauchy band matrix ensemble introduced above, we therefore start with this quantity,
the limit

\[ \rho_j(E) = -\frac{1}{\pi} \text{Im} \left\langle G_{jj}(E) \right\rangle_j, \]

where \(G(E) = (E \pm i0 - H)^{-1}\) and the average \(\langle \cdots \rangle_j\) is taken with respect to all random matrix elements except the diagonal element \(\eta_j\) at the site \(j\) under consideration. As in the original Lloydl model \[\text{[4]}\], the average over the other diagonal elements \(\eta_k, k \neq j\) can be exactly performed by replacing in the definition of the Green function these elements according to: \(\eta_k + i0 \rightarrow iW/\pi\). Using a general algebraic identity for the block inverse of a matrix, we obtain

\[ \rho_j(E) = -\frac{1}{\pi} \text{Im} \left\langle \frac{1}{E + i0 - \eta_j + i\Gamma_j(U)/2} \right\rangle_U, \]

\[ \Gamma_j(U) = 2i \sum_{k,l \neq j} U_{jk} \tilde{G}_{kl}^{(+)} U_{lj}, \]

where \(\langle \cdots \rangle_U\) denotes the average over \(U_{jk}\) only, and \(\tilde{G}^{(+)} = (E + iW/\pi - \hat{U})^{-1}\) with \(\hat{U}\) being the \((N-1) \times (N-1)\)-matrix obtained from \(U\) after elimination of the \(j\)-th row and \(j\)-th column. Eq. \(2\) reproduces the Lorentzian Breit-Wigner form of width \(\text{Re}[\Gamma_j(U)]/2\) for a given realization of the coupling matrix \(U\). For ordinary band matrices this width is a self-averaging quantity and the further \(U\)-average does not modify the Breit-Wigner form. However, for the case of Cauchy band matrices \(\Gamma_j(U)\) is strongly fluctuating and the \(U\)-average will considerably modify the Breit-Wigner form. Now, we consider the limit \(|E| \ll W\) and we neglect \(\hat{U}\) in the definition of \(\tilde{G}^{(+)}\) which is possible for the regime we consider. Using, \(\tilde{G}^{(+)} \approx (-i \pi/W) I_N\), we obtain

\[ \rho_j(E) = \frac{1}{\pi} \text{Im} \left\langle i \int_0^\infty dt \, e^{it(E-\eta_j)} \left[ \phi \left( \frac{2\pi U^2 t}{W} \right) \right]^{2b} \right\rangle, \]

with the function \(\phi(z)\) defined by

\[ \phi(z) \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} du \frac{1}{1 + u^2} e^{-zu^2/2}, \quad z \geq 0. \]

In the limit \(b \gg 1\), the integral \([8]\) is dominated by the behavior at small \(t\). Using the approximation \(\ln(\phi(z)) \approx -\sqrt{2\pi} z + O(z)\), we obtain

\[ \rho_j(E) = \frac{1}{\alpha} L_{1/2} \left( \frac{E - \eta_j}{\alpha} \right), \quad \alpha = \frac{16 U^2 b^2}{W}, \]

\[ L_\mu(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \, e^{it^2 - |t|^\mu}. \]

Here \(L_\mu(s)\) represents a Levy distribution of index \(\mu\) \([10]\) with the behavior \(L_\mu(s) \propto s^{-1-\mu} \) for \(|s| \gg 1\). The expressions \([8]\) and \([9]\) provide the first important results of this work. They show that the local density of states is still a peaked function of \((E-\eta_j)\) but with two important modifications. First, the Lorentzian Breit-Wigner form is replaced by the Levy distribution \(L_{1/2}\), and second, the characteristic energy scale behaves as \(\alpha \approx (4/\pi) b \Gamma \gg \Gamma\) where \(\Gamma\) is the Breit-Wigner width for ordinary band matrices (with a finite variance \((U^2_{jk}) \equiv U_0^2\)). We mention that the approximation to neglect the contribution of \(U\) in \(\tilde{G}^{(+)}\) is valid for \(\alpha \ll W\), i.e., \(U_0 b \ll W\).

The first modification has an important implication for the time-evolution of a state \(|\psi(t)\rangle\) initially localized at one site, \(|\psi(0)\rangle = |j\rangle\). Then, the average of the amplitude \(\langle \psi(t) | j \rangle\) obeys a non-exponential decay law \([14]\) of the form \(\propto e^{-\sqrt{\pi} t}\) instead of \(\propto e^{-\Gamma t/2}\). This type of behavior was for instance recently found in the context of many-body effects in cold Rydberg atoms \([17]\). We conclude that it is a very general feature due to the long tail distribution of residual interaction matrix elements in many-body problems \([13]\).

The second modification concerning the large enhancement of the characteristic energy scale can be qualitatively understood by the fact that for a given realization of \(U\) the sum in eq. \(3\) is essentially determined by the typical maximal value \(2bU_0\) of the \(2b\) different matrix elements \(U_{jk}\).

It is pretty obvious to generalize eqs. \([6]\) and \([7]\) to the case where the matrix elements \(U_{jk}\) are drawn from the more general Levy distribution \(U_0^{-1} L_{\mu}(U_{jk}/U_0)\). In this case, we have to replace in \([6]\) \(L_{1/2}\) by \(L_{\mu/2}\) and the characteristic energy scale is enhanced according to: \(\alpha \sim b^2/\mu^{-1} \Gamma\).

We now turn to a field theoretical formulation of the Cauchy band matrix model in terms of a supersymmetric non-linear \(\sigma\) model \([8]\). This formulation provides the access to the correlators of an advanced Green function \(G^{(+)}\) and a retarded Green function \(G^{(-)}\) which are for example useful to study the transport properties, the level statistics or the appearance of dynamical localization. For this, typically, the ensemble average of a generating functional of the form

\[ F(J) = \text{Sdet}^{-1/2} \left( E + (\frac{2}{3} + i\varepsilon) \Lambda - H \otimes 1_8 + J \right) \]

is considered. For the details concerning the supersymmetry method and the notations, we refer to the standard literature \([8]\). Here, we only remind that \(\Lambda = 1_N \otimes \text{diag}(1, 1, 1, 1, -1, -1, -1, -1)\) is the \(8N \times 8N\) supermatrix whose eigenvalues distinguish between the matrix blocks associated to the advanced and retarded Green functions and that \(J\) is a source matrix which is used to generate the Green function correlators by taking suitable derivatives with respect to its matrix elements \([12]\).

Normally, \(H\) contains a certain number of gaussian random variables and it is possible to perform the ensemble average analytically which provides after a some transformations and approximations the non-linear su-
permatrix model \[2\,3\]. For the case of Cauchy band matrices, this approach has to be modified because of the non-gaussian distribution of its matrix elements. Our strategy will be to perform analytically by a new method the ensemble average with respect to the Cauchy distributed diagonal elements \(\eta_j\) while the coupling matrix \(U\) is kept fixed. This will provide a \(\sigma\) model formulation for an arbitrary coupling matrix \(U\). The ensemble average with respect to \(U\) can then be performed at a later stage when possible and convenient. As a first application of this procedure, we will derive an analytic expression for the localization length of the Cauchy band matrix model.

The main idea is to replace the diagonal matrix elements \(\eta_j\) by random phases \(e^{i\varphi_j}\) via \(\eta_j = (W/\pi) \tan \varphi_j\). These phases are uniformly distributed on the unit circle and can be formally identified with the scattering matrix of a chaotic cavity ideally coupled to one scattering channel. Describing the Hamiltonian of the chaotic cavity by an \(N \times M\) random matrix \(h_{jk}\) drawn from the gaussian orthogonal ensemble (GOE) one can apply the following substitution \[1\]:

\[
\eta_j = (W/\pi) \tan \varphi_j = (W/\pi) A_j \, h_j^{-1} A_j
\]

where \(A_j\) is an \(N\)-dimensional vector with components \((A_j)_k = \delta_{jk}\) and \(h_j\) is a real symmetric gaussian random matrix with variance \((h_{jk})_k^2 = (1 + \delta_{jk})/M\). The substitution \[1\] is valid in the limit \(M \to \infty\) which will be taken at the end of the calculations. We mention that this approach is related to recent work concerning the derivation of the \(\sigma\) model by performing phase averages in models involving unital quantum maps for classically chaotic systems such as the kicked rotator \[4\] or rough billiards \[2\]. The replacement \[1\] and the following steps follow very closely the method described in ref. \[2\].

Inserting \[1\] in the generating functional \[\xi\] and applying suitable transformations inside the superdeterminant, one obtains an effective model defined on a Hilbert space of dimension \(N \times M\) and characterized by an effective Hamiltonian containing the GOE-matrices \(h_j\) in its diagonal blocks and \((-W/\pi) \, A_j (D^{-1})_{jk} A_j\) in its off-diagonal \(jk\)-block. Here \(D\) is the supermatrix appearing inside the superdeterminant of \(\xi\) with \(\eta_j\) put to zero. This model is quite similar to the IWZ-model used in ref. \[2\] and following the standard procedure to derive the \(\sigma\) model, we obtain \(\langle F(J)\rangle_{\eta} = \int DQ \, e^{-\mathcal{L}(Q)}\) with the action

\[
\mathcal{L}(Q) = \frac{1}{2} \text{Str} \, \ln \left( E + \left( \frac{1}{2} + i\varepsilon \right) \Lambda - U + i \frac{W}{\pi} \, Q + J \right),
\]

where \(Q\) contains in its diagonal blocks \(8 \times 8\) supermatrices \(Q_{\delta}\) belonging to Efetov’s coset space for the orthogonal symmetry class \[3\]. We mention some important points concerning this result: (i) it is exact since the saddle-point approximation to derive the \(\sigma\) model becomes exact in the limit \(M \to \infty\); (ii) its range of applicability goes far beyond the particular model considered here since it is valid for an arbitrary coupling matrix \(U\) and its derivation only needs the Cauchy distribution for the diagonal matrix elements; (iii) it generalizes the above mentioned replacement rule for one-point functions to the case of two-point functions: \(\eta_k + i0A \to i(W/\pi)Q_k\) where \(Q_k\) is now a dynamical variable over which we have to integrate.

To analyze the localization properties, we may put \(\omega = 0\) and expand the action \[1\] for the long wavelength limit. In the regime \(U_0 b \ll W\), this gives apart from the source term the action

\[
\mathcal{L}(Q) \approx -\frac{1}{16} \int dx \, \xi_x(U) \, \text{Str}[(\partial_x Q(x))^2]
\]

with the quantity \(\xi_x(U) = (4\pi^2/W^2) \sum_b j^2 \, U_{x+b}^2\). If \(\xi_x(U)\) were independent of \(x\) the action \[1\] would correspond to the standard \(\sigma\) model for quasi one-dimensional wires with the localization length \(\xi \equiv \xi_x(U)\). This is actually the case for gaussian distributed \(U_{xy}\) for which \(\xi_x(U) \approx \langle \xi_x(U) \rangle_{U} = 4\pi^2 b^2 U_0^2 / (3W^2)\) is a selfaveraging quantity. However, for the Cauchy band matrix model \(\xi_x(U)\) is strongly fluctuating and its average does not even exist. It is therefore necessary to take the \(x\)-dependence properly into account. A similar situation has recently been encountered by Rupp et al. who derived the 1d \(\sigma\) model with an \(x\)-dependent “diffusion constant” from a hierarchical random-matrix model for many-body states \[2\]. According to this work all quantities that can be extracted from the 1d \(\sigma\) model only depend on the rescaled length variable \(s(x) = \int_0^x dy \xi^{-1}_x(U)\). This justifies the intuitive average procedure \(\xi^{-1} \equiv \lim_{L \to \infty} \frac{1}{L} \int_0^L dx \, \xi^{-1}_x(U) \approx \langle \xi^{-1}_x(U) \rangle\). Fortunately, this average is finite and well defined for the Cauchy band matrix case (if \(b \geq 3\)). It can be easily evaluated by the same techniques used above for the local density of states [see eqs. \[2\]–\[4\]]. For \(b \gg 1\), we obtain:

\[
\xi = \frac{2\pi b^4 U_0^2}{W^2} \approx b \gamma, \quad \gamma \equiv \frac{\pi}{16} \left( \frac{\alpha}{\Delta} \right)
\]

where the dimensionless parameter \(\gamma\) counts the number of well coupled levels inside a strip of typical length \(b\), \(\Delta = W/2b\) is the effective level spacing of such a strip, and \(\alpha\) is the characteristic energy scale for the local density of states \[3\]. (For the more general case of Levy distributed \(U_{jk}\), we find: \(\xi \sim b^{2+2/\nu} U_0^2 / W^2\).) The relation \[1\] compares to \(\xi \sim b(\Gamma/\Delta)\) for the case of ordinary band matrices \[1\]. However, it contradicts the expression \(\xi \sim b^{1/2}\) which was with proper translation of notations numerically obtained from a similar Cauchy band matrix model in ref. \[1\]. We attribute this to the extreme numerical difficulty to access the regime \(1 \ll b \ll \gamma\) where we expect \[1\] to be valid. For \(\gamma \lesssim 1\) we enter the
the localization length is likely to saturate at \( \xi / b \) in the perturbative regime with \( \xi \sim b / \ln(\gamma^{-1}) \) while for \( \gamma \gg b \) the localization length is likely to saturate at \( \xi \sim b^2 \).

\[
\frac{\xi}{b} \approx \frac{1.5}{\ln(1 + 1.5/\gamma_{\text{eff}})}, \quad \gamma_{\text{eff}} \approx \frac{\gamma}{1 + 3.2(\gamma/b)^2}
\]

which reproduces the perturbative limit for \( \gamma \ll 1 \), the analytical result (12) for \( 1 \ll \gamma \ll b \) and the behavior \( \xi \sim b^2 \) for \( \gamma \gg b \). Note that eq. (13) provides for the case \( \gamma/b \ll 1 \) relative large corrections of order \( \sqrt{\gamma/b} \) explaining the numerical difficulties to clearly identify this regime.

In summary, we have obtained analytical results for the local density of states and the localization length for Cauchy random band matrices. We find that the Breit-Wigner form is replaced by a Levy distribution with index \( \mu = 1/2 \) and that the Breit-Wigner width \( \Gamma \) for ordinary band matrices is replaced by a new enhanced energy scale \( \alpha \sim b \Gamma \). This energy scale also determines the localization length \( \xi \sim b (\alpha/\Delta) \). From the technical point of view, we have derived a \( \sigma \) model formulation for arbitrary matrix ensembles with Cauchy distributed diagonal disorder.

The author thanks G. Caldara, B. Georgeot, J. Lages and D. Shepelyansky for useful discussions.

[1] G. Casati, I. Guarneri, F. M. Izrailev and R. Scharf, Phys. Rev. Lett. 64, 5 (1990); R. Scharf J. Phys. A 22, 4223 (1989); B. V. Chirikov, F. M. Izrailev, and D. L. Shepelyansky, Physica (Amsterdam) 33D, 77 (1988).
[2] S. Iida, H. A. Weidenmüller, and J. A. Zuk, Ann. Phys. (NY) 200, 219 (1990).
[3] Y. V. Fyodorov and A. D. Mirlin, Phys. Rev. Lett. 67, 2405 (1991); ibid, 69, 1093 (1992); ibid, 71, 412 (1993); A. D. Mirlin and Y. V. Fyodorov, J. Phys. A: Math. Gen. 26, L551 (1993).
[4] K. B. Efetov, Supersymmetry in Disorder and Chaos, Cambridge University Press (1997).
[5] T. Guhr, A. Müller-Groeling, and H. A. Weidenmüller, Phys. Rep. 299, 189 (1998).
[6] D. L. Shepelyansky, Phys. Rev. Lett. 73, 2607 (1994).
[7] P. Jacquod and D. L. Shepelyansky, Phys. Rev. Lett. 75, 3501 (1995); Y. V. Fyodorov and A. D. Mirlin, Phys. Rev. B 52, R11580 (1995); K. Frahm and A. Müller–Groeling, Europhys. Lett. 32, 385 (1995).
[8] P. Lloyd, J. Phys. C: Solid St. Phys. 2, 1717 (1969).
[9] P. Cizeau, and J. P. Bouchaud, Phys. Rev. E 50, 1810 (1994).
[10] J. P. Bouchaud and A. Georges, Phys. Rep. 195, 127 (1990).
[11] F. von Oppen, T. Wettig, and J. Müller, Phys. Rev. Lett. 76, 491 (1996).
[12] D. L. Shepelyansky, Proceedings of les Rencontres de Moriond 1996 on “Correlated Fermions and Transport in Mesoscopic Systems”, edited by T. Martin, G. Montambaux and J. Trân Thanh Vân, 201 (1996).
[13] B. L. Altshuler, Yu. Gefen, A. Kamenev and L.S. Levitov, Phys. Rev. Lett. 78, 2803 (1997); P. Jacques and D. L. Shepelyansky, Phys. Rev. Lett. 79, 1837 (1997); A. D. Mirlin and Y. V. Fyodorov, Phys. Rev. B 56, 13393 (1997); C. Mejía-Monasterio, J. Richert, T. Rupp and H. A. Weidenmüller, Phys. Rev. Lett. 81, 5189 (1998); X. Leyronas, P. G. Silvestrov and C. W. J. Beenakker, Phys. Rev. Lett. 84, 3414 (2000).
[14] B. Georgeot and D. L. Shepelyansky, Phys. Rev. Lett. 79, 4365 (1997).
[15] V. V. Flambaum, A. A. Gribakina, G. F. Gribakin and M. G. Kozlov, Phys. Rev. A 50, 267 (1994); V. V. Flambaum, A. A. Gribakina, G. F. Gribakin and I. V. Ponomarev, Physica D 131, 205 (1999).
[16] We emphasize that the average amplitude (or average local density of states) does not capture any quantum coherence effects such as dynamical localization. These are seen for instance at the average of the absolute square of the amplitudes and only at very long time scales comparable to the Heisenberg time.
[17] V. M. Akuhin, F. de Tomasi, I. Mourachko and P. Pillet, Physica D 131, 125 (1999).
[18] C. H. Lewenkopf and H. A. Weidenmüller, Ann. Phys. (N.Y.) 212, 53 (1991); P. W. Brouwer, Phys. Rev. B 51, 16878 (1995).
[19] A. Altland and M. R. Zirnbauer, Phys. Rev. Lett. 77, 4536 (1996); M. R. Zirnbauer, J. Phys. A 29, 7113 (1996).
[20] K. M. Frahm, Phys. Rev. B 55, R8626 (1997).
[21] T. Rupp, H. A. Weidenmüller, J. Richert, Phys. Lett. B 483, 331 (2000).