Gravitational wave forms for a three-body system in Lagrange’s orbit: parameter determinations and a binary source test

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Abstract

Continuing work initiated in an earlier publication [Torigoe et al. Phys. Rev. Lett. 102, 251101 (2009)], gravitational wave forms for a three-body system in Lagrange’s orbit are considered especially in an analytic method. First, we derive an expression of the three-body wave forms at the mass quadrupole, octupole and current quadrupole orders. By using the expressions, we solve a gravitational-wave inverse problem of determining the source parameters to this particular configuration (three masses, a distance of the source to an observer, and the orbital inclination angle to the line of sight) through observations of the gravitational wave forms alone. For this purpose, the chirp mass to a three-body system in the particular configuration is expressed in terms of only the mass ratios by deleting initial angle positions. We discuss also whether and how a binary source can be distinguished from a three-body system in Lagrange’s orbit or others.

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I. INTRODUCTION

“Can one hear the shape of a drum?” is a famous question posed by Mark Kac in 1966, where to “hear” the shape of a drum is to infer information about the shape of the drumhead from the sound it makes. Such a question can be traced back to Hermann Weyl. Now, it is interesting to pose a gravitational-wave inverse problem for the forthcoming gravitational-wave astronomy by ground-based or space-borne detectors. To “hear” a source through gravitational-wave observations is to extract the information about the source from the gravitational waves it makes.

It is of general interest to ask “can one tell how many apples are falling in the dark of night?” One simpler question is how and whether two-body and three-body gravitating systems can be distinguished through observations of gravitational waves that are made by these sources. Recently, this issue has been addressed. They found that there is a case that quadrupole wave forms for two-body and three-body gravitating systems (in a particular configuration) cannot be distinguished even with observing the chirp, though different numbers of self-gravitating particles (in different types of periodic motion) are most likely to generate very different shapes of gravitational waves. In order to break this degeneracy between two-body and three-body sources in the particular configuration for the quadrupolar wave (even with observing frequency sweep), they suggested that higher multipolar wave forms (octupole for their cases) are needed.

The purpose of the present paper is to investigate gravitational wave forms for a three-body system in the particular configuration (as Lagrange’s orbit) in more detail, especially in analytic manners up to the mass octupole and current quadrupole orders. In particular, we shall discuss whether and how the source parameters (three masses, a distance of the sources to an observer, and their orbital inclination angle to the line of sight) can be determined through observations of the gravitational waveforms alone.

Inspiraling and finally merging binary compact stars are most likely astrophysical sources for the forthcoming direct detections of gravitational ripples (and consequently gravitational waves astronomy) by a lot of efforts by the on-going or designed detectors. Merging neutron stars and black holes have been successfully simulated by numerical relativity. Furthermore, analytic methods also have provided accurate waveform templates for inspiraling compact binaries, by post-Newtonian approaches (See
for reviews) and also by black hole perturbation techniques especially at the linear order in mass ratio (See also [18] for reviews). Bridges between the inspiraling stage and final merging phase are currently under construction (e.g., [19, 20]).

There is a growing interest in potential astrophysical sources of gravitational waves involving 3-body interactions (e.g., [10, 21, 22, 23] and references therein). Even the classical three-body (or N-body) problem in Newtonian gravity admits an infinite number of solutions, some of which express regular orbits and the others are chaotic, and in fact an increasing number of periodic orbits are found [24, 25]. For the sake of simplicity, we focus on Lagrange’s equilateral triangle solution, mostly because it offers a nice model tractable completely by hand (See below for more details).

The linear perturbation analysis in Newton gravity [24] shows that Lagrange triangular points $L_4$ and $L_5$ to the restricted three-body problem, in which one of three bodies is assumed as a test mass (e.g., an asteroid in the solar system), is stable if the mass ratio of the remaining two bodies is less than 0.0385, though relativistic corrections to stability of the system are poorly understood. Indeed, the ratio of the Jovian mass to the solar mass is $O(10^{-3})$.

Here, it is worthwhile to mention previous works. Nakamura and Oohara [26] studied numerically the luminosity of gravitational radiation by N test particles orbiting around a Schwarzschild black hole, as an extension of Detweiler’s analysis of the $N = 1$ case [27] by using Teukolsky equation [28], in order to show the phase cancellation effect, which had been pointed out by Nakamura and Sasaki [29]. It should be noted that their N particles are test masses but not self-gravitating. The aim and setting of the present paper are different from those previous works.

In this paper, we shall present an expression of gravitational waves for a three-body system in Lagrange’s orbit, especially as a function of mass ratios. In addition, we shall discuss how to determine the source parameters (three masses, a distance of the source to an observer, and the orbital inclination angle to the line of sight) through observations of the gravitational waveforms alone. We also discuss a possible test; which source is a binary, a three-body system in Lagrange’s orbit or others?

This paper is organized as follows. In section 2, we shall briefly summarize a notation and formulation for describing a three-body system in Lagrange’s orbit and its gravitational waves. Wave forms at the mass quadrupole, mass octupole and current quadrupole orders
are obtained in an analytic method. In section 3, we discuss a method of determining the source parameters from gravitational-wave observations alone. A binary source test is also discussed. Section 4 is devoted to the conclusion. In Appendix, we shall present calculations for the chirp mass and some useful relations. Throughout this paper, we take the units of $G = c = 1$. Latin indices take 1, 2, 3, except for $p$ which labels each body.

II. NOTATION AND FORMULATION

A. Linearized gravitational waves

For a wave propagation direction denoted by a unit vector $n^a$, we define the transverse-traceless projection operator as

$$P^b_a = \delta^b_a - n_a n^b,$$  \hspace{1cm} (1)

where $\delta^b_a$ denotes the Kronecker’s delta symbol. The linearized waves in the wave zone are expressed in terms of mass and current multipole moments denoted by $I^A_\ell$ and $S^A_\ell$, respectively [30]. These radiative multipole moments at the Newtonian order are related with the source position $x^a$, mass density $\rho$ and velocity $v^a$ as

$$I^A_\ell = \left[ \int \rho X^A_\ell d^3 x \right]^{\text{STF}},$$  \hspace{1cm} (2)

$$S^A_\ell = \left[ \int \epsilon^{a_\ell b c} x^b \rho v^c X^{A_{\ell-1}} d^3 x \right]^{\text{STF}},$$  \hspace{1cm} (3)

where $\text{STF}$ denotes the symmetric tracefree part, $\epsilon_{abc}$ denotes the Levi-Civita symbol in a three-dimensional Euclidean space, and we define the product of $\ell$ spatial coordinates as $X^A_\ell \equiv x^{a_1} x^{a_2} \ldots x^{a_\ell}$. We consider a system of spherical bodies approximated by massive particles. Then, Eqs. (2) and (3) become

$$I^A_\ell = \left[ \sum_{p=1}^{N} m_p X^A_p \right]^{\text{STF}},$$  \hspace{1cm} (4)

$$S^A_\ell = \left[ \sum_{p=1}^{N} m_p \epsilon^{a_\ell b c} x^b v^c X^{A_{\ell-1}}_p \right]^{\text{STF}},$$  \hspace{1cm} (5)

where $N$ denotes the number of the particles and the subscript $p$ denotes the $p$-th body.

In terms of the multipole moments, the linearized waves at the wave zone are expressed
as

\[ h_{jk}^{TT}(t, x) = \frac{1}{r} \left[ \sum_{\ell=2}^{\infty} \left( \frac{4}{\ell!} I_{jk}^{(\ell)}(x) \right) (t-r) N^{:\ell-2} \right. \]

\[ + \left. \sum_{\ell=2}^{\infty} \left( \frac{8\ell}{(\ell+1)!} c^{(\ell)} \right) (t-r) n^{(\ell)} N^{:\ell-2} \right] + O \left( \frac{1}{r^2} \right), \]

where \( t \) and \( r \) mean time and source distance, respectively, in the Minkowskian spherical coordinates \((t, r, \Theta, \Phi)\), \( TT \) denotes the transverse-traceless part, \((\ell)\) denotes the \( \ell \)-th time derivative, and we define the tensor product of \( \ell - 2 \) unit radial vectors as \( N^{:\ell-2} \equiv n^{a_1} n^{a_2} \cdots n^{a_{\ell-2}} \).

For a binary case, mass quadrupolar, octupolar and current quadrupolar waves were considered fully by Blanchet and Schafer [31], where they showed that the octupolar part is linearly proportional to a mass difference.

**B. Lagrange’s solution**

Let us consider the Lagrange’s solution for a three-body system (on \( x-y \) plane), where each mass is denoted by \( m_p \) \((p = 1, 2, 3)\). The initial positions of each mass are expressed as \( x_1 = (0, 0), \ x_2 = a(\sqrt{3}/2, 1/2), \) and \( x_3 = a(0, 1), \) where the side of a regular triangle is denoted as \( a \) [10, 24].

We choose the spatial coordinates such that the center of mass (COM) is at rest as \((x_{\text{COM}}, y_{\text{COM}}) = a(\sqrt{3}\nu_2/2, (\nu_2 + \nu_3)/2))\), where the total mass and mass ratio are denoted as \( m_{\text{tot}} \equiv \sum_p m_p \) and \( \nu_p \equiv m_p/m_{\text{tot}} \), respectively. We have an identity as

\[ \nu_1 + \nu_2 + \nu_3 = 1. \]

The orbital frequency \( \omega \) for the triangle satisfies

\[ \omega^2 = \frac{m_{\text{tot}}}{a^3}, \]

which takes the same form as Kepler’s third law for a binary system but with the total mass of three masses.

Henceforth, it is convenient to employ the COM coordinates \((X, Y)\) that can be obtained by a translation from \((x, y)\). In the COM coordinates, the location of each mass is expressed as

\[ X_p = a_p(\cos(\omega t + \theta_p), \sin(\omega t + \theta_p)), \]
where $a_p$ is defined as $a_1 = \sqrt{x_{COM}^2 + y_{COM}^2}$, $a_2 = \sqrt{(3^{1/2}a/2 - x_{COM})^2 + (a/2 - y_{COM})^2}$, and $a_3 = \sqrt{x_{COM}^2 + (a - y_{COM})^2}$, respectively, and $\theta_p$ denotes the angle between the new $X$-axis and the direction of each mass at $t = 0$ (See Fig. 1). In practice, computations can be simplified by using complex variables. In particular, variables for the triangle configuration $\theta_p$ are written in terms of the mass ratios (See Appendix for more detail).

III. WAVE FORMS FOR THREE BODIES IN LAGRANGE’S ORBIT

Let $i$ denote the orbital inclination angle with respect to the line of sight.

A. Mass Quadrupole

By direct calculations, we obtain the plus mode of quadrupole waves as

$$r \times \mathbf{h}_Q^+ = -2 \sum_p m_p a_p^2 \omega^2 (1 + \cos^2 i) \cos 2(\omega t + \theta_p),$$  \hspace{5cm} (10)
where the subscript $Q$ denotes a mass quadrupolar part. By using Eqs. (A4)-(A6) and (A10)-(A12), Eq. (10) is rewritten as

$$ r \times h_{Q}^+ = -m_{\text{tot}} a^2 \omega^2 (1 + \cos^2 i) $$

$$ \times \left[ \nu_1 (\nu_2 + \nu_3) - 2 \nu_2 \nu_3 \right] \cos 2 \omega t + \sqrt{3} \nu_1 (\nu_2 - \nu_3) \sin 2 \omega t \right]. \quad (11) $$

In a similar manner, we obtain the cross-mode as

$$ r \times h_{Q}^\times = -4 \sum_{p} m_p a_p^2 \omega^2 \cos i \sin (\omega t + \theta_p) $$

$$ = -2 m_{\text{tot}} a^2 \omega^2 \cos i $$

$$ \times \left[ \nu_1 (\nu_2 + \nu_3) - 2 \nu_2 \nu_3 \right] \sin 2 \omega t - \sqrt{3} \nu_1 (\nu_2 - \nu_3) \cos 2 \omega t \right]. \quad (12) $$

For a three-body system in Lagrange’s orbit, the frequency sweep due to gravitational radiation reaction has been recently obtained as [10]

$$ \frac{1}{f_{\text{GW}}} \frac{df_{\text{GW}}}{dt} = \frac{96}{5} \pi^{8/3} M_{\text{chirp}}^{5/3} f_{\text{GW}}^{8/3}, \quad (13) $$

where we define a chirp mass as

$$ M_{\text{chirp}} = m_{\text{tot}} \left\{ \sum_{p} \nu_p \left( \frac{M_p^\text{eff}}{m_{\text{tot}}} \right)^{2/3} \right\}^2 - 2 \sum_{p \neq q} \nu_p \nu_q \left( \frac{M_p^\text{eff}}{m_{\text{tot}}} \right)^{2/3} \left( \frac{M_q^\text{eff}}{m_{\text{tot}}} \right)^{2/3} \sin^2 (\theta_p - \theta_q) \right\}^{3/5} \quad (14) $$

and $M_p^\text{eff}$ denotes the effective one-body mass for which the equation of motion becomes [24]

$$ \frac{d^2 X_p}{dt^2} = -M_p^\text{eff} \frac{X_p}{|X_p|^3} \quad (15) $$

For instance, $M_1^\text{eff}$ is defined as

$$ M_1^\text{eff} = \frac{(m_2^2 + m_2 m_3 + m_3^2)^{3/2}}{m_{\text{tot}}^2}. \quad (16) $$

By cyclic permutations, $M_2^\text{eff}$ and $M_3^\text{eff}$ are defined. It is worthwhile to mention that this frequency evolution equation is the same as that for a binary system [10] including the numerical coefficient $96/5$. 

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The chirp mass is expressed in terms of both each mass \( m_p \) and its initial position angle \( \theta_p \). It is rewritten as a function only of the mass ratios by deleting the initial position angles as (See Appendix for detailed calculations)

\[
M_{\text{chirp}} = m_{\text{tot}} \times F(\nu_1, \nu_2, \nu_3),
\]

where we define \( F \) as

\[
F = \left( \frac{\nu_1^2(\nu_2 - \nu_3)^2 + \nu_2^2(\nu_3 - \nu_1)^2 + \nu_3^2(\nu_1 - \nu_2)^2}{\nu_1 \nu_2 + \nu_2 \nu_3 + \nu_3 \nu_1} \right)^{3/5}.
\]

By using this expression of the three-body chirp mass for Eqs. (11) and (12), we obtain

\[
r \times h_Q^+ = -2m_{\text{tot}}^{5/6} \nu_1^{5/6} \omega^{2/3} (1 + \cos^2 i) \nu_1 \nu_2 \nu_3 + \nu_2 \nu_3 \nu_1 \cos(2\omega t - \Psi_Q),
\]

where we define \( \Psi_Q \) as

\[
\tan \Psi_Q = \frac{B_Q}{A_Q},
\]

for

\[
A_Q = \nu_1(\nu_2 + \nu_3) - 2\nu_2 \nu_3,
\]

\[
B_Q = \sqrt{3}\nu_1(\nu_2 - \nu_3).
\]

We rewrite the cross mode as

\[
r \times h_Q^x = -2m_{\text{tot}}^{5/6} \nu_1^{5/6} \omega^{2/3} \cos i \nu_1 \nu_2 \nu_3 + \nu_2 \nu_3 \nu_1 \sin(2\omega t - \Psi_Q).
\]

**B. Mass Octupole**

We obtain, by direct calculations, the plus mode of quadrupole waves as

\[
r \times h_{\text{Oct}}^+ = -\frac{1}{12} \sum_p m_p a_p^3 \omega^3 \sin i \nu_1 \nu_2 \nu_3 + \nu_2 \nu_3 \nu_1 \sin(2\omega t - \Psi_Q),
\]

where the subscript \( \text{Oct} \) denotes a mass octupolar part.
By using Eqs. (A4)-(A6), (A7)-(A9) and (A13)-(A15), one can rewrite Eq. (24) as

\[
r \times h_{\text{Oct}}^+ = -\frac{1}{12} m_{\text{tot}}^2 \omega \sin i \times \left[ 27(1 + \cos^2 i) \left( \frac{3^3}{2} \nu_1 \nu_2 \nu_3 \cos 3 \omega t + (\nu_1 - \nu_2)(\nu_2 - \nu_3)(\nu_3 - \nu_1) \sin 3 \omega t \right) \\
+ (1 - 3 \cos^2 i) \left( \frac{\sqrt{3}}{2} \nu_1 \left( \nu_2 - \nu_1 \right) + \nu_3 (\nu_3 - \nu_1) \right) \cos \omega t \\
- \frac{1}{2} (\nu_2 - \nu_3) \{ (\nu_1 - \nu_2)(\nu_1 - \nu_3) - 3 \nu_2 \nu_3 \} \sin \omega t \right],
\]

where we use Eq. (8).

We obtain the cross mode of mass octupole waves as

\[
r \times h_{\text{Oct}}^x = -\frac{1}{12} \sum_p m_p a_p^3 \omega^3 \sin 2i \\
\times \left[ 27 \sin(\omega t + \theta_p) - \sin(\omega t + \theta_p) \right].
\]

By using Eqs. (A4)-(A6), (A7)-(A9) and (A13)-(A15), one can rewrite Eq. (26) as

\[
r \times h_{\text{Oct}}^x = -\frac{1}{12} m_{\text{tot}}^2 \omega \sin 2i \\
\times \left[ 27 \left( 3^3/2 \nu_1 \nu_2 \nu_3 \sin 3 \omega t - (\nu_1 - \nu_2)(\nu_2 - \nu_3)(\nu_3 - \nu_1) \cos 3 \omega t \right) \\
- \left( \frac{\sqrt{3}}{2} \nu_1 \left( \nu_2 - \nu_1 \right) + \nu_3 (\nu_3 - \nu_1) \right) \sin \omega t \\
+ \frac{1}{2} (\nu_2 - \nu_3) \{ (\nu_1 - \nu_2)(\nu_1 - \nu_3) - 3 \nu_2 \nu_3 \} \cos \omega t \right],
\]

where we use Eq. (8).

C. Current Quadrupole

Here, we consider current quadrupolar waves. The plus mode becomes

\[
r \times h_{\text{C}}^+ = \frac{4}{3} m_{\text{tot}}^2 \omega \sin i \\
\times \left[ \frac{\sqrt{3}}{2} \nu_1 \left( \nu_2 - \nu_1 \right) + \nu_3 (\nu_3 - \nu_1) \right] \cos \omega t \\
- \frac{1}{2} (\nu_2 - \nu_3) \{ (\nu_1 - \nu_2)(\nu_1 - \nu_3) - 3 \nu_2 \nu_3 \} \sin \omega t \right],
\]

where the subscript \( C \) denotes a current quadrupolar part.
The cross mode becomes
\[ r \times h^+_C = \frac{2}{3} m^2_{\text{tot}} \omega \sin 2i \times \left[ \sqrt{3} \nu_1 \{ \nu_2 (\nu_2 - \nu_1) + \nu_3 (\nu_3 - \nu_1) \} \sin \omega t \right. \\
\left. + \frac{1}{2} (\nu_2 - \nu_3) \{ (\nu_1 - \nu_2) (\nu_1 - \nu_3) - 3\nu_2 \nu_3 \} \cos \omega t \right] . \] (29)

Both the mass octupolar and current quadrupolar parts are proportional to \( m^2_{\text{tot}} \omega \). Hence they can be combined as
\[ r \times h^+_{\text{Oct+C}} = -\frac{1}{4} m^2_{\text{tot}} \omega \sin i \times \left[ 9 (1 + \cos^2 i) \left( 3^{3/2} \nu_1 \nu_2 \nu_3 \cos 3\omega t + (\nu_1 - \nu_2)(\nu_2 - \nu_3)(\nu_3 - \nu_1) \sin 3\omega t \right) \\
- (5 + \cos^2 i) \left( \frac{\sqrt{3}}{2} \nu_1 \{ \nu_2 (\nu_2 - \nu_1) + \nu_3 (\nu_3 - \nu_1) \} \cos \omega t \\
- \frac{1}{2} (\nu_2 - \nu_3) \{ (\nu_1 - \nu_2)(\nu_1 - \nu_3) - 3\nu_2 \nu_3 \} \sin \omega t \right) \right] , \] (30)

and
\[ r \times h^+_{\text{Oct+C}_C} = -\frac{1}{4} m^2_{\text{tot}} \omega \sin 2i \times \left[ 9 \left( 3^{3/2} \nu_1 \nu_2 \nu_3 \sin 3\omega t - (\nu_1 - \nu_2)(\nu_2 - \nu_3)(\nu_3 - \nu_1) \cos 3\omega t \right) \\
- 3 \left( \frac{\sqrt{3}}{2} \nu_1 \{ \nu_2 (\nu_2 - \nu_1) + \nu_3 (\nu_3 - \nu_1) \} \sin \omega t \\
+ \frac{1}{2} (\nu_2 - \nu_3) \{ (\nu_1 - \nu_2)(\nu_1 - \nu_3) - 3\nu_2 \nu_3 \} \cos \omega t \right) \right] . \] (31)

It is natural that one can recover the wave forms to a binary system from Eqs. (30) and (31) when a third mass vanishes, say \( \nu_3 = 0 \).

Figure 2 shows wave forms due to mass quadrupole, octupole and current quadrupole parts that are expressed above.

IV. PARAMETER DETERMINATIONS

The frequency \( \omega \) can be directly determined as \( \omega = \pi f_{GW} \) from measurements of the frequency of the mass quadrupolar wave \( f_{GW} \). What we have to do is to determine the remaining six quantities as \( m_{\text{tot}}, \nu_1, \nu_2, \nu_3, r \) and \( i \). We have an identity \( \nu_1 + \nu_2 + \nu_3 = 1 \). Hence, we need to find out five more relations for parameter determinations.
FIG. 2: Gravitational waves for a three-body system in Lagrange’s orbit. Log-dashed (blue), short-dashed (red) and dotted (green) curves denote mass quadrupolar, octupolar and current quadrupolar parts, respectively. The sold (black) one denotes the total wave forms. The vertical axis is in arbitrary units and time in the horizontal axis is normalized by the orbital period. For simplicity, the mass ratio is assumed as $m_1 : m_2 : m_3 = 1 : 2 : 3$, though stability arguments prefer much larger ratios [24]. In order to exaggerate differences between each component of waves, we assume a mildly relativistic case of $a = 100m_{\text{tot}}$, which corresponds to $v/c \sim 0.1$. The mass octupolar part makes a relatively large contribution mostly because of including a large numerical coefficient (as 27 in Eq. (25)), whereas the current quadrupolar part is much smaller and changes slowly with time because it has no $3\omega$ but only the $\omega$ part.

The frequency of mass quadrupolar waves is $2\omega$, whereas that for the combination of mass octupolar and current quadrupolar ones is either $\omega$ or $3\omega$. By using this difference of frequency dependences, therefore, one can pick up the quadrupolar waves from observed signals. By comparing amplitudes of the $+$ and $\times$ modes, we obtain

$$\frac{Amp(h_Q^\times)}{Amp(h_Q^+)} = \frac{2\cos i}{1 + \cos^2 i},$$

(32)

where $Amp$ denotes the amplitude of waves. This relation is the same as the well-known
one for binaries. The L.H.S. of Eq. (32) can be determined by observations and thus the R.H.S. tells us the inclination angle \( i \).

Through observing the frequency sweep of the mass quadrupolar part (See Eq. (13)), one can determine the chirp mass as

\[
M_{\text{chirp}} = \left( \frac{5}{96 \pi^{8/3}} \frac{1}{f_{GW}^{11/3}} \frac{df_{GW}}{dt} \right)^{3/5},
\]

(33)

where \( f_{GW} = 2f \) for \( f = \omega/2\pi \).

Next, let the amplitude of each wave component observed separately. From Eq. (19), first, we obtain that for the mass quadrupolar part as

\[
r \times \text{Amp} \left( h_{Q}^{\pm} \right) = 2m_{\text{tot}}^{5/6} M_{\text{chirp}}^{5/6} \omega^{2/3}(1 + \cos^2 i)
\]

\[
\times \sqrt{\nu_1 \nu_2 + \nu_2 \nu_3 + \nu_3 \nu_1},
\]

(34)

which can be solved for the distance \( r \) as

\[
r = \frac{2m_{\text{tot}}^{5/6} M_{\text{chirp}}^{5/6} \omega^{2/3}(1 + \cos^2 i) \sqrt{\nu_1 \nu_2 + \nu_2 \nu_3 + \nu_3 \nu_1}}{\text{Amp} \left( h_{Q}^{\pm} \right)}.
\]

(35)

For parameter determinations, however, it is useful to rewrite this as

\[
m_{\text{tot}}^{5/6} \sqrt{\nu_1 \nu_2 + \nu_2 \nu_3 + \nu_3 \nu_1} = \frac{\text{Amp} \left( h_{Q}^{\pm} \right)}{2M_{\text{chirp}}^{5/6} \omega^{2/3}(1 + \cos^2 i)},
\]

(36)

where the R.H.S. is known up to this point.

By using Eq. (30), the magnitude of \( h_{\text{Oct+C}}^{\pm} \) at \( 3\omega \) becomes

\[
r \times \text{Amp} \left( h_{\text{Oct+C}}^{\pm} \right) = \frac{9}{4} m_{\text{tot}}^2 \omega \sin i(1 + \cos^2 i)
\]

\[
\times \sqrt{27
\nu_1^2 \nu_2^2 \nu_3^2 + (\nu_1 - \nu_2)^2(\nu_2 - \nu_3)^2(\nu_3 - \nu_1)^2},
\]

(37)

where the subscript of \( 3\omega \) in the L.H.S. means a \( 3\omega \) part. This equation is rewritten as

\[
m_{\text{tot}}^2 \sqrt{27
\nu_1^2 \nu_2^2 \nu_3^2 + (\nu_1 - \nu_2)^2(\nu_2 - \nu_3)^2(\nu_3 - \nu_1)^2}
\]

\[
r = \frac{4}{9 \omega \sin i(1 + \cos^2 i)} \text{Amp} \left( h_{\text{Oct+C}}^{\pm} \right),
\]

(38)

where the R.H.S. is known up to this point.

By using Eq. (30), the magnitude of \( h_{\text{Oct+C}}^{\pm} \) at \( \omega \) becomes

\[
r \times \text{Amp} \left( h_{\text{Oct+C}}^{\pm} \right) = \frac{1}{8} m_{\text{tot}}^2 \omega \sin i(5 + \cos^2 i)
\]

\[
\times \left[ 3\{\nu_1 \nu_2 (\nu_1 - \nu_2) + \nu_2 \nu_3 (\nu_2 - \nu_3) + \nu_3 \nu_1 (\nu_3 - \nu_1)\}^2
\]

\[
+ (\nu_1 - \nu_2)^2(\nu_2 - \nu_3)^2(\nu_3 - \nu_1)^2 \right]^{1/2},
\]

(39)
where the subscript of $\omega$ in the L.H.S. means a $\omega$ part. This equation is rewritten as

$$\frac{m^2_{\text{tot}}}{r} \times \left[ 3(\nu_1\nu_2(\nu_1 - \nu_2) + \nu_2\nu_3(\nu_2 - \nu_3) + \nu_3\nu_1(\nu_3 - \nu_1))^2 + (\nu_1 - \nu_2)^2(\nu_2 - \nu_3)^2(\nu_3 - \nu_1)^2 \right]^{1/2} = \frac{8}{\omega \sin i(5 + \cos^2 i)} \text{Amp}(h^+_{\text{Oct+C}}),$$

(40)

where the R.H.S. is known up to this point.

Equation (30) is rewritten as

$$r \times h^+_{\text{Oct+C}} = -\frac{1}{4}m^2_{\text{tot}} \omega \sin i \times \left[ 9(1 + \cos^2 i) \sqrt{27\nu_1^2\nu_2^2\nu_3^2 + (\nu_1 - \nu_2)^2(\nu_2 - \nu_3)^2(\nu_3 - \nu_1)^2 \cos(3\omega t - \Psi_{3\omega})} 
- \frac{1}{2}(5 + \cos^2 i) \left( 3\{\nu_1\nu_2(\nu_1 - \nu_2) + \nu_2\nu_3(\nu_2 - \nu_3) + \nu_3\nu_1(\nu_3 - \nu_1) \}^2 + (\nu_1 - \nu_2)^2(\nu_2 - \nu_3)^2(\nu_3 - \nu_1)^2 \right)^{1/2} \cos(\omega t - \Psi_\omega) \right],$$

(41)

where phases of $h^+_{\text{Oct+C}}$ at $\omega$ and $3\omega$ are defined as

$$\tan \Psi_\omega = \frac{(\nu_2 - \nu_3)(\nu_1 - \nu_2 - \nu_3)}{\sqrt{3} \nu_1\{\nu_2(\nu_2 - \nu_1) + \nu_3(\nu_3 - \nu_1)\}},$$

(42)

and

$$\tan \Psi_{3\omega} = \frac{(\nu_1 - \nu_2)(\nu_2 - \nu_3)(\nu_3 - \nu_1)}{3^{3/2}\nu_1\nu_2\nu_3},$$

(43)

respectively. These phases are not directly observable. Time lags between the mass quadrupole wave with frequency $2\omega$ and the combination of mass octupole and current quadrupole waves with $\omega$ (or $3\omega$) can be measured. They are defined as

$$\Delta t_\omega \equiv t_\omega - t_Q = \frac{2\Psi_\omega - \Psi_Q}{2\omega},$$

(44)

$$\Delta t_{3\omega} \equiv t_{3\omega} - t_Q = \frac{2\Psi_\omega - 3\Psi_Q}{6\omega},$$

(45)

respectively, where $t_Q$, $t_\omega$ and $t_{3\omega}$ are defined as

$$t_Q \equiv \frac{\Psi_Q}{2\omega},$$

(46)

$$t_\omega \equiv \frac{\Psi_\omega}{\omega},$$

(47)

$$t_{3\omega} \equiv \frac{\Psi_{3\omega}}{3\omega}.$$
It is worthwhile to mention that $\Delta t_\omega$ and $\Delta t_{3\omega}$ are observable and thus gauge-invariant, whereas $t_Q$, $t_\omega$ and $t_{3\omega}$ are gauge-dependent in a sense that they rely on a degree of freedom for choosing an initial time “$t_0$”.

In principle, time lags in the cross mode, phase of which is different from that of the plus mode by 45°, are the same as those in the plus mode, and thus bring no additional information on parameter determinations, though they may play a supplementary role in improving accuracy of a practical data analysis.

As a result, one can determine five quantities $m_{\text{tot}}$, $\nu_1$, $\nu_2$, $\nu_3$ and $r$ from Eqs. (7), (17), (36), (38), (40), (44) and (45). In principle, five out of the seven equations are sufficient for parameter determinations if the source is known as the Lagrange’s solution \textit{a priori}. It should be noted that two remaining equations are \textit{never} redundant but play an important role in checking whether a source is the particular three-body system or not. If the remaining equations are satisfied by the determined parameter values, one can safely say that the source is in Lagrange’s orbit. If not, it could be other systems.

Figure 3 shows a flow chart of the parameter determinations and possible source tests that are discussed above.

V. CONCLUSION

We have considered the three-body wave forms at the mass quadrupole, octupole and current quadrupole orders, especially in an analytical method. By using the derived expressions, we have solved a gravitational-wave inverse problem of determining the source parameters to the particular configuration (three masses, a distance of the source to an observer, and the orbital inclination angle to the line of sight) through observations of the gravitational wave forms alone. We have discussed also whether and how a binary source can be distinguished from a three-body system in Lagrange’s orbit or others, and thus proposed a binary source test.

To be more precise, we should take account of post-Newtonian corrections to both wave generation and propagation beyond the linearized theory. It is interesting to consider different shapes of orbits and different numbers of bodies ($N = 4$ or more). This is a topic of future study.
TABLE I: List of quantities characterizing a system in this paper.

| Symbol | Definition |
|--------|------------|
| $T$    | Orbital period |
| $\omega$ | Angular velocity ($= 2\pi/T$) |
| $a$    | Edge length of a Lagrange’s equilateral triangle |
| $a_p$  | Distance of each body from their center of mass |
| $\theta_p$ | Initial angular position of each body |
| $m_p$  | Mass of each body |
| $\nu_p$ | Mass ratio of each body |
| $m_{tot}$ | Total mass |
| $M_{\text{chirp}}$ | Chirp mass |
| $F$    | Ratio as $M_{\text{chirp}}m_{\text{tot}}^{-1}$ |
| $r$    | Distance of a source from an observer |
| $h_{Q}^{+,\times}$ | $+ \times$ mode of mass quadrupolar waves |
| $h_{\text{Oct}}^{+,\times}$ | $+ \times$ mode of mass octupolar waves |
| $h_{C}^{+,\times}$ | $+ \times$ mode of current quadrupolar waves |
| $h_{\text{Oct}+C}^{+,\times}$ | $h_{\text{Oct}}^{+,\times} + h_{C}^{+,\times}$ |
| $Amp$  | Amplitude of a wave component |
| $\Psi_Q$ | Phase of mass quadrupolar waves with $2\omega$ |
| $\Psi_\omega$ | Phase of waves with $\omega$ |
| $\Psi_{3\omega}$ | Phase of waves with $3\omega$ |
| $t_{Q}$ | Time corresponding to the phase $\Psi_Q$ |
| $t_\omega$ | Time corresponding to the phase $\Psi_\omega$ |
| $t_{3\omega}$ | Time corresponding to the phase $\Psi_{3\omega}$ |
| $\Delta t_\omega$ | Observable time lag defined as $t_\omega - t_{Q}$ |
| $\Delta t_{3\omega}$ | Observable time lag defined as $t_{3\omega} - t_{Q}$ |
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APPENDIX A: LAGRANGE’S SOLUTION IN COMPLEX VARIABLES

1. Correspondence between position angles and mass ratios

Let each mass located at each vertex of a regular triangle with the side length denoted as $a$. For simplicity, the center of the complex plane $z \equiv x + iy$ is chosen tentatively as that of the triangle. Without loss of generality, we assume initial positions of three masses as $a, ae^{i2\pi/3}, ae^{i4\pi/3}$. The center of mass for the three masses is located at $z_{CM} = a (\nu_1 + \nu_2 e^{i2\pi/3} + \nu_3 e^{i4\pi/3})$.

When we consider gravitational waves, it is convenient to choose the coordinates origin as the mass center so that the dipole moment can vanish. In order to do so, we make a translation as $z \rightarrow z - z_{CM}$, for which the initial position of each mass becomes

$$
z_{I1} = \frac{\sqrt{3}}{2} a \left( \sqrt{3} (\nu_2 + \nu_3) - (\nu_2 - \nu_3)i \right), \quad (A1)$$

$$
z_{I2} = -\frac{\sqrt{3}}{2} a \left( \sqrt{3} \nu_1 - (\nu_1 + 2\nu_3)i \right), \quad (A2)$$

$$
z_{I3} = -\frac{\sqrt{3}}{2} a \left( \sqrt{3} \nu_1 + (\nu_1 + 2\nu_2)i \right), \quad (A3)$$

where the subscript $I$ means the initial values. The magnitude of $z_{I1}$, $z_{I2}$ and $z_{I3}$, which are
nothing but three masses’ orbital radii around their center of mass, become

\[ a_1 \equiv |z_{11}| = a \sqrt{3(v_2^2 + v_3^2 + v_2v_3)}, \]
\[ a_2 \equiv |z_{12}| = a \sqrt{3(v_3^2 + v_1^2 + v_3v_1)}, \]
\[ a_3 \equiv |z_{13}| = a \sqrt{3(v_1^2 + v_2^2 + v_1v_2)}, \]

which respect symmetry for cyclic changes as \( 1 \rightarrow 2 \rightarrow 3 \rightarrow 1 \).

We obtain a relation between an position angle \( \theta \) (with respect to the center of mass) and the mass ratios as

\[ e^{i\theta_1} = \frac{z_{11}}{|z_{11}|} = \frac{\sqrt{3}(v_2 + v_3) - (v_2 - v_3)i}{2\sqrt{v_2^2 + v_3^2 + v_2v_3}}, \]  
\[ e^{i\theta_2} = -\frac{\sqrt{3}v_1 - (v_1 + 2v_3)i}{2\sqrt{v_1^2 + v_2^2 + v_1v_2}}, \]  
\[ e^{i\theta_3} = -\frac{\sqrt{3}v_1 + (v_1 + 2v_2)i}{2\sqrt{v_1^2 + v_2^2 + v_1v_2}}. \]

By using the complex representations, double-angle and triple-angle relations, which are useful for later computations, are obtained as

\[ e^{2i\theta_1} = \frac{(v_2^2 + v_3^2 + 4v_2v_3) - \sqrt{3}(v_2^2 - v_3^2)i}{2(v_2^2 + v_3^2 + v_2v_3)}, \]  
\[ e^{2i\theta_2} = \frac{(v_1^2 - 2v_3^2 - 2v_3v_1) - \sqrt{3}v_1(v_1 + 2v_3)i}{2(v_1^2 + v_2^2 + v_3v_1)}, \]  
\[ e^{2i\theta_3} = \frac{(v_1^2 - 2v_2^2 - 2v_1v_2) + \sqrt{3}v_1(v_1 + 2v_2)i}{2(v_1^2 + v_2^2 + v_1v_2)}, \]

and

\[ e^{3i\theta_1} = \frac{3^{3/2}v_2v_3(v_2 + v_3) - (v_2 - v_3)(v_2 + 2v_3)(2v_2 + v_3)i}{2(v_2^2 + v_3^2 + v_2v_3)^{3/2}}, \]  
\[ e^{3i\theta_2} = \frac{3^{3/2}v_3v_1(v_3 + v_1) + (v_1 - v_3)(v_1 + 2v_3)(2v_1 + v_3)i}{2(v_1^2 + v_3^2 + v_3v_1)^{3/2}}, \]  
\[ e^{3i\theta_3} = \frac{3^{3/2}v_1v_2(v_1 + v_2) - (v_1 - v_2)(v_1 + 2v_2)(2v_1 + v_2)i}{2(v_1^2 + v_2^2 + v_1v_2)^{3/2}}. \]
respectively.

Some useful relations are obtained as

\[
A_C \equiv \sum_p \nu_p |z_{lp}|^3 \cos \theta_p \\
= \frac{9}{2} \nu_1 [\nu_2^2 + \nu_3^2 - \nu_1 (\nu_2 + \nu_3)] a^3, (A16)
\]

\[
A_S \equiv \sum_p \nu_p |z_{lp}|^3 \sin \theta_p \\
= \frac{3^{3/2}}{2} (\nu_2 - \nu_3) [\nu_1 (\nu_1 - \nu_2 - \nu_3) - 2 \nu_2 \nu_3] a^3, (A17)
\]

where we use Eqs. (A7)-(A9). Therefore, we obtain

\[
\sum_p \nu_p |z_{lp}|^3 \cos(\omega t + \theta_p) = \sqrt{A_C^2 + A_S^2} \cos(\omega t + \alpha), (A18)
\]

where we define

\[
\alpha \equiv \arctan \left( \frac{A_S}{A_C} \right). (A19)
\]

Next we consider 3\omega parts. We obtain

\[
B_C \equiv \sum_p \nu_p |z_{lp}|^3 \cos 3\theta_p \\
= 27 \nu_1 \nu_2 \nu_3 a^3, (A20)
\]

\[
B_S \equiv \sum_p \nu_p |z_{lp}|^3 \sin 3\theta_p \\
= -3^{3/2} (\nu_1 - \nu_2) (\nu_2 - \nu_3) (\nu_3 - \nu_1) a^3, (A21)
\]

where we use Eqs. (A13)-(A15). Therefore, we obtain

\[
\sum_p \nu_p |z_{lp}|^3 \cos 3(\omega t + \theta_p) = \sqrt{B_C^2 + B_S^2} \cos(3\omega t + \beta), (A22)
\]

where we define

\[
\beta \equiv \arctan \left( \frac{B_S}{B_C} \right). (A23)
\]

2. Chirp mass

By straightforward but lengthy calculations, one can show several identities as

\[
\sum_p \nu_p \left( \frac{M_{\text{eff}}}{m_{\text{tot}}} \right)^{2/3} = \nu_1 \nu_2 + \nu_2 \nu_3 + \nu_3 \nu_1, (A24)
\]
\[
\sum_{p \neq q} \nu_p\nu_q - \sum_p \nu_p \left( \frac{M_p}{m_{tot}} \right)^{2/3} = \nu_1\nu_2 + \nu_2\nu_3 + \nu_3\nu_1, \quad (A25)
\]

\[
2 \sum_{p \neq q} \nu_p\nu_q \left( \frac{M_p}{m_{tot}} \right)^{2/3} \left( \frac{M_q}{m_{tot}} \right)^{2/3} \sin^2(\theta_p - \theta_q) = 3\nu_1\nu_2\nu_3, \quad (A26)
\]

where Eq. (7) is frequently used.

By substituting these relations into Eq. (14), we obtain

\[
\left( \sum_p \nu_p \left( \frac{M_p}{m_{tot}} \right)^{2/3} \right)^2 - 2 \sum_{p \neq q} \nu_p\nu_q \left( \frac{M_p}{m_{tot}} \right)^{2/3} \left( \frac{M_q}{m_{tot}} \right)^{2/3} \sin^2(\theta_p - \theta_q) = \frac{1}{2} \left[ \nu_1^2(\nu_2 - \nu_3)^2 + \nu_2^2(\nu_3 - \nu_1)^2 + \nu_3^2(\nu_1 - \nu_2)^2 \right]. \quad (A27)
\]

Therefore, we can prove that the chirp mass defined by Eq. (14) is expressed in terms of only the mass ratios as Eq. (17).

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FIG. 3: Flow chart of the parameter determinations and source tests. By using equations derived in this paper, the source parameters can be determined through gravitational-wave observations alone. In addition, a binary source can be distinguished from a three-body system in Lagrange's orbit or others.