The eta invariant on two-step nilmanifolds

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The eta invariant appears regularly in index theorems but is known to be directly computable from the spectrum only in certain examples of locally symmetric spaces of compact type. In this work, we derive some general formulas useful for calculating the eta invariant on closed manifolds. Specifically, we study the eta invariant on nilmanifolds by decomposing the spin Dirac operator using Kirillov theory. In particular, for general Heisenberg three-manifolds, the spectrum of the Dirac operator and the eta invariant are computed in terms of the metric, lattice, and spin structure data. There are continuous families of geometrically, spectrally different Heisenberg three-manifolds whose Dirac operators have constant eta invariant. In the appendix, some needed results of L. Richardson and C. C. Moore are extended from spaces of functions to spaces of spinors.

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1. Introduction

The eta invariant was introduced in the famous paper of M. F. Atiyah, V. K. Patodi, and I. M. Singer (see [5]), in order to produce an index theorem for manifolds with boundary. The eta invariant of a linear self-adjoint operator is roughly the difference between the number of positive eigenvalues and the number of negative eigenvalues, which of course is undefined when these numbers are both infinite. However, this quantity may be regularized to make it well-defined for classical pseudodifferential operators, using methods similar to the zeta-function regularization of the determinant of the Laplacian and methods used by physicists to regularize divergent integrals. The eta function is analogous to Dirichlet $L$-functions in the same way that the zeta function of elliptic operators is analogous to the Riemann zeta function.

Let $D : C^\infty (E) \to C^\infty (E)$ be an essentially self-adjoint elliptic classical pseudodifferential operator of order $d$ on sections of a vector bundle $E \to M$, where $M$ is a closed (compact, without boundary) Riemannian manifold of dimension $n$. Let $\{\lambda\}$ be the collection of eigenvalues with multiplicity. The eta function is defined as

$$\eta(s) = \sum_{\lambda \neq 0} \text{sgn} (\lambda) |\lambda|^{-s}.$$ 

This reduces to the zeta function if $D$ has only nonnegative eigenvalues. The eta function is holomorphic in $s$ for large $\text{Re} (s)$ and can be analytically continued to a meromorphic function using heat kernel techniques. It is true but not obvious that $\eta(s)$ is regular at $s = 0$, and $\eta(0)$ is always real; the eta invariant is defined as $\eta(0)$. See [5], [6], [22] for general information about the eta invariant.

The eta function and its generalizations have been studied and utilized in index theorems for noncompact manifolds and for families of operators and in gluing formulas. The sign of the eta invariant of the boundary signature operator on a 4-manifold with boundary has important geometric content; in the case of a ball, it determines whether the conformal class of the boundary metric contains a metric induced from a self-dual Einstein metric on the
interior (see [27]). In physics, the eta invariant of the spin Dirac operators has practical importance, for example in the regularization of Feynman path integrals (see [10]). Recently, in the work of J. Brüning, F. W. Kamber, and K. Richardson, the eta invariant is utilized in a new equivariant index formula for $G$-manifolds and an index formula for Riemannian foliations (see [13], [14], [15]).

It is very difficult to calculate the eta invariant for a given operator such as a Dirac operator on a Riemannian manifold; much work has been done to calculate this invariant for space forms, lens spaces and flat tori (see, for example, [19], [21], [10]). More recently, S. Goette has calculated formulas for the eta invariant and equivariant eta invariants on homogeneous spaces of the form $G/H$ with $G$ compact (see [23]). In [1], M. Atiyah, H. Donnelly and I. Singer computed the eta invariant of the boundary signature operator of a framed solvmanifold in terms of the signature defect of a manifold whose boundary is that solvmanifold. In [18], C. Deninger and W. Singhof computed eta invariants of modified versions of Dirac operators on Heisenberg manifolds and were able to compute the eta invariants up to local correction terms. In [31], P. Loya, S. Moroianu and J. Park studied the spectrum of the Dirac operator on a certain three-dimensional circle bundle over a noncompact Riemann surface with cusps, that is, a noncompact manifold that is a cofinite quotient of $PSL(2, \mathbb{R})$. They also study the adiabatic limit of the eta invariant as the fibers are collapsed. The first explicit computations of eigenvalues of Dirac operators on homogeneous spaces corresponding to noncompact Lie groups has been done by B. Ammann and C. Bär (see [1], [8]), where the eigenvalues of the spin Dirac operator on certain (rectangular) Heisenberg manifolds were computed explicitly. In [33], R. Miatello and R. Podestá compute the eta invariant on compact flat spin manifolds with cyclic holonomy of odd prime order (see also [34] for related work). While different techniques are employed, the Miatello-Podestá result has a similar flavor to our main result, in that the final state relies on metric data, spin structure data, lattice data and prominently exploits group actions.

A Riemannian nilmanifold is a closed manifold of the form $(\Gamma \backslash G, g)$ where $G$ is a simply connected nilpotent Lie group, $\Gamma$ is a cocompact (i.e., $\Gamma \backslash G$ is compact) discrete subgroup of $G$, and $g$ is a left-invariant metric on $G$, which descends to a Riemannian metric on $\Gamma \backslash G$ that is also denoted by $g$. A Heisenberg manifold is a two-step Riemannian nilmanifold whose covering Lie group $G$ is one of the $(2n + 1)$-dimensional Heisenberg Lie groups (see, for example, [26]). The study of nilmanifolds and nilpotent Lie groups has long been relevant to inverse spectral problems (see [25] for a survey). Nilmanifolds play an important role in the study of Dirac eigenvalues, as
was shown in a paper of Ammann and C. Sprouse (see [3]). They show that if a Riemannian spin manifold with bounded sectional curvature and finite diameter has scalar curvature bounded from below by a sufficiently small negative number and if the smallest Dirac eigenvalue \( \lambda \) is sufficiently close to zero, then the manifold is diffeomorphic to a nilmanifold.

In this paper, we prove several results concerning the computation of the eta invariant on closed manifolds. In Section 2.1, we discuss the interesting relationships between the zeta and eta functions of operators, which can be derived from [7, Proposition 2.10]. The main point is Proposition 1, the formula

\[
\frac{\partial}{\partial c} \eta_c(s) = -s \zeta((D + c)^2) \left( \frac{s + 1}{2} \right),
\]

where \( \eta_c \) is the eta function corresponding to the operator \( D + c = D + c\mathbf{1} \), where \( c \) is a real number, and where \( \zeta((D + c)^2) \) is the zeta function corresponding to the operator \((D + c)^2\). From this we see that changes in the eta invariant of an elliptic first order operator on a closed, odd-dimensional manifold is related to a particular residue of a pole of the zeta function corresponding to the second order operator \((D + c)^2\).

This residue is, up to a constant, a coefficient in the asymptotic expansion of the trace of the heat operator \( \exp\left(-t(D + c)^2\right) \). In Section 2.2, this coefficient is computed explicitly as a function of \( c \).

Using these general results about \( \frac{\partial}{\partial c} \eta_c(0) \), if \( \eta_c(0) \) is known at a single value of \( c \), the heat kernel asymptotic formula and knowledge of small eigenvalues determine \( \eta_0(0) \), the eta invariant of \( D \). In Theorem 5, we prove a general formula for the eta invariant of a Dirac-type operator on a closed manifold in the case that the spectrum of the operator is symmetric about a certain real number \( \overline{\lambda} \). We deduce from this formula a more specific formula for Dirac-type operators on three-manifolds with spectral symmetry about \( \overline{\lambda} \) in Section 2.4, which calculates the eta invariant in terms of the volume, the total scalar curvature, the total trace of the twisting curvature, and small eigenvalues of the Dirac-type operator (notation defined in that section):

\[
\eta(0) = -\frac{n\overline{\lambda}^3}{6\pi^2} \text{vol}(M) + \frac{\overline{\lambda}}{4\pi^2} \left( \frac{n}{12} \int_M \text{Scal} + \int_M Tr\ (F^W) \right) + \text{sgn}(\overline{\lambda}) \left( 2\#(\sigma(D) \cap (0, \overline{\lambda})) + \#(\sigma(D) \cap \{0, \overline{\lambda}\}) \right).
\]

Using Kirillov theory, the spin Dirac operator on two-step nilmanifolds is decomposed explicitly in terms of irreducible subspaces of the right quasi-regular representation in Section 3.2. To that end, occurrence and multiplicity conditions for Dirac eigenspinors are developed in Section 3.3 in analogy to Pesce’s known work [36] concerning the Laplacian. It is here that we utilize analogues of the work of C.C. Moore [35] and L. Richardson [38].
developed in the appendix, Section 7. Explicit formulas for the Dirac operator are computed in terms of a special basis of spinors for each invariant subspace.

For general Heisenberg three-manifolds, the spectrum of the spin Dirac operator and the eta invariant are computed in terms of the metric, the lattice and spin structure in Section 5.2. The formula for the eta invariant has the form

$$\eta(0) = \frac{r^2 m_v}{96\pi^2 A^2} - N(A, r, w_2, m_v, m_w, \varepsilon),$$

where $N(A, r, w_2, m_v, m_w, \varepsilon)$ is a nonnegative integer specified in terms of $A, r, w_2, m_v, m_w, \varepsilon$, the metric, lattice, and spin structure data. In this section, we exhibit continuous families of geometrically, spectrally different Heisenberg three-manifolds whose spin Dirac operators have constant eta invariant. Computations for a general Heisenberg nilmanifold are done in Section 5.3 in particular, we show how to calculate the Dirac spectrum for any example. We explore symmetries of the Dirac spectrum in higher-dimensional Heisenberg manifolds in Section 5.4. In Section 6, we compute the Dirac operator of a particular five-dimensional non-Heisenberg nilmanifold, and we show that the techniques used in previous sections do not yield explicit formulas for the eigenvalues in this case.

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2. The eta invariant

2.1. Eta and zeta functions of perturbed operators

In this section, we exhibit some general results relating families of eta and zeta functions that may be well-known to experts. In particular, Proposition 2.10 in [7] relates the derivative of the eta invariant of a family of operators to a trace that can be identified with the zeta function in our particular application. Also, in [11, Lemma 2.1] and in [12, Lemma 9], the researchers use the same idea to relate the residues at the poles of the eta function to the asymptotics of a heat kernel. For the sake of exposition and completeness, we include the proofs of very specific results that have not been previously stated in this form, which will be needed in later sections.
regard $c$ in this expression as $c$ times the identity. We also use the notation $\sigma(D)$ to denote the spectrum of $D$, with multiplicities.

**Proposition 1.** Let $D$ be any self-adjoint operator for which $\eta(s)$ is defined and analytic at $s = 0$. Suppose in addition that there exists an interval $I \subset \mathbb{R}$ and a constant $B > 0$ such that for all $c \in I$,

1) $\sum_{\lambda} \text{sgn}(\lambda + c)|\lambda + c|^{-s}$ and $\sum_{\lambda} \left( (\lambda + c)^2 \right)^{-\frac{s+1}{2}}$ converge absolutely for $\text{Re}(s) > B$, and

2) $-c$ is not an eigenvalue of $D$.

Then the eta function $\eta_c(s)$ corresponding to the operator $D + c$ satisfies, on its domain,

$$\frac{d}{dc} \eta_c(s) = -s \zeta((D+c)^2) \left( \frac{s + 1}{2} \right),$$

where $\zeta((D+c)^2)$ is the zeta function corresponding to the nonnegative operator $(D + c)^2$, that is

$$\zeta((D+c)^2)(s) = \sum_{\mu > 0} \mu^{-s},$$

where the sum is over all positive eigenvalues with multiplicity $\{\mu\}$ of the operator $(D + c)^2$. In particular, if $D$ is a first-order, elliptic, essentially self-adjoint differential operator, then $\frac{d}{dc} \eta_c(0)$ is the residue of the simple pole of the meromorphic function $\zeta((D+c)^2) \left( \frac{s + 1}{2} \right)$ at $s = 0$. (If $\zeta((D+c)^2) \left( \frac{s + 1}{2} \right)$ is regular at $s = 0$, then $\frac{d}{dc} \eta_c(0) = 0$.)

**Remark:** It is known that second-order essentially self-adjoint elliptic differential operators such as $(D + c)^2$ on a manifold of dimension $n$ yield zeta functions with at most simple poles, and they are located at $s = \frac{n}{2}$, $s = \frac{n}{2} - 1$, $s = \frac{n}{2} - 2$, ..., for $n$ odd and at $s = \frac{n}{2}$, $s = \frac{n}{2} - 1$, ..., $s = 1$ for $n$ even. See [22] for specifics. Further, the residues at these poles are given by explicitly computable integrals of locally-defined functions.

**Proof.** We know that for each eigenvalue $\lambda$ of $D$, $\text{sgn}(\lambda + c)$ does not vary with $c \in I$. Then for large $\text{Re}(s)$,

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\[ \eta_c(s) = \sum_\lambda \text{sgn} (\lambda + c) \left( (\lambda + c)^2 \right)^{-s/2} \]

\[ \frac{d}{dc} \eta_c(s) = \sum_\lambda \text{sgn} (\lambda + c) \left( -\frac{s}{2} \left( (\lambda + c)^2 \right)^{-s/2-1} \right) 2(\lambda + c) \]

\[ = -s \sum_\lambda \text{sgn} (\lambda + c) |\lambda + c|^{-s-2} (\lambda + c) \]

\[ = -s \sum_\lambda \left( (\lambda + c)^2 \right)^{-\frac{s+1}{2}} = -s \zeta_{(D+c)^2} \left( \frac{s}{2} + 1 \right). \]

Since both sides are analytic in \( s \) for large \( \text{Re} \ (s) \), the statement must remain true after analytic continuation. \( \square \)

We are interested in the eta invariant, which is \( \eta_c(0) \). By the formula in the proposition above, the relevant information is the residue of the pole of the zeta function \( \zeta_{(D+c)^2}(z) \) at \( z = \frac{1}{2} \). For odd-dimensional manifolds, this is a constant times one of the heat invariants. If the manifold is even-dimensional, there is no pole at \( z = \frac{1}{2} \), so that \( \frac{d}{dc} \eta_c(0) = 0 \).

**Corollary 2.** If the manifold is even-dimensional, then \( \frac{d}{dc} \eta_c(0) = 0 \), so that the eta invariant is constant with respect to \( c \) on intervals where \( D + c \) has trivial kernel, and then it changes by integral jumps in general.

We also have the following result about perturbations of zeta functions.

**Proposition 3.** With the assumptions of Proposition 4,

\[ \frac{d}{dc} \zeta_{(D+c)^2}(s) = -2s \eta_c(2s + 1). \]

**Proof.** For large \( \text{Re} \ (s) \),

\[ \frac{d}{dc} \zeta_{(D+c)^2}(s) = \frac{d}{dc} \sum_\lambda \left( (\lambda + c)^2 \right)^{-s} \]

\[ = \sum_\lambda -s \left( (\lambda + c)^2 \right)^{-s-1} 2(\lambda + c) \]

\[ = -2s \sum_\lambda |\lambda + c|^{-2s-2} (\lambda + c) \]

\[ = -2s \sum_\lambda \text{sgn} (\lambda + c) |\lambda + c|^{-2s-1} = -2s \eta_c(2s + 1). \]
Since both sides are analytic in $s$ for large $\text{Re}(s)$, the statement must remain true after analytic continuation. □

2.2. Heat Kernel Asymptotics

Because of Proposition 1, we will be interested in the residues of $\zeta_{(D+c)^2}(s)$ at its poles, which are determined by the heat kernel asymptotics (see Section 2.3). Specifically, we need the asymptotics as $t \to 0^+$ of

$$
\text{Tr} \left( \exp \left( -t(D+c)^2 \right) \right) = \int_M \text{Tr} K_c(t,x,x) \, d\text{vol},
$$

where we assume $D = \sum (e_j \circ) \nabla e_j : C^\infty(E) \to C^\infty(E)$ is a Dirac-type operator and $c \in \mathbb{R}$. That is, the Leibniz rule $\nabla_X (v \circ s) = (\nabla^M_X v) \circ s + v \circ \nabla_X s$ is satisfied for all vector fields $X$ and $v$ and sections $s \in C^\infty(E)$, where $\nabla^M$ is the Levi-Civita connection. We will let $n$ be the dimension of the manifold $M$, and we will let $\hat{n}$ be the rank of the vector bundle $E$. Here and in what follows, we use the $\circ$ symbol to denote Clifford multiplication. The element $K_c(t,x,x) \in \text{End}(E_x)$ is

$$
K_c(t,x,x) = e^{-t(D+c)^2}(x,x),
$$

which satisfies

$$
\left( \frac{\partial}{\partial t} + (D+c)^2 \right) K_c(t,x,y) = 0
$$

$$
\lim_{t \to 0^+} K_c(t,x,y) = \delta_{xy},
$$

where $\delta_{xy}$ is the Dirac delta distribution. To find the asymptotics as $t \to 0^+$, we need to solve for $u_k(x,y) \in \text{Hom}(E_y,E_x)$, where

$$
K_c(t,x,y) \sim \frac{1}{(4\pi t)^{n/2}} e^{-r^2/4t} \left( u_0(x,y) + tu_1(x,y) + t^2 u_2(x,y) + \cdots \right)
$$

where $r = \text{dist}(x,y)$. Such an asymptotic expansion exists, since $(D+c)^2$ is a generalized Laplacian (see [9, 22, 39]).

We will assume that we have chosen geodesic normal coordinates $x = (x_1, \ldots, x_n)$ centered at $y = 0$ and that the frame field $(e_1, \ldots, e_n)$ is parallel.
translated radially from the origin (i.e. \( y \)) such that

\[ e_j(0) = \partial_j. \]

Then in these coordinates, we may map \( E_x \) to \( E_y \) via radial parallel translation, so that each \( u_k(x, y) \) may be regarded as a matrix-valued function of \( x \), with \( \mathbb{R}^\tilde{n} \) identified with \( E_y \). Observe that the Dirac operator may be expressed as

\[
D = \sum_j e_j \circ \nabla e_j = \sum_{p,q} g^{pq} \partial_p \circ \nabla \partial_q,
\]

where in the first case we are summing over an orthonormal frame, and in the second case we are using the coordinate vector fields, with \((g^{pq})\) the inverse of the metric matrix \((g_{ij})\).

We have, using the Einstein summation convention,

\[
-(D + c)^2 = -(e_i \circ \nabla e_i + c)^2
= -(e_i \circ \nabla e_i) (e_j \circ \nabla e_j) - 2c (e_i \circ \nabla e_i) - c^2
= -(e_i \circ) (e_j \circ) \nabla e_i \nabla e_j
+ \left[ -(e_i \circ) ((\nabla e_i e_j) \circ) - 2c (e_j \circ) \right] \nabla e_j - c^2
= \nabla e_i \nabla e_i - \sum_{i<j} (e_i \circ) (e_j \circ) \left[ \nabla e_i, \nabla e_j \right]
+ \left[ -(e_i \circ) ((\nabla e_i e_j) \circ) - 2c (e_j \circ) \right] \nabla e_j - c^2
= \nabla e_i \nabla e_i - \sum_{i<j} (e_i \circ) (e_j \circ) (\nabla [e_i, e_j])
+ \left[ -(e_i \circ) ((\nabla e_i e_j) \circ) - 2c (e_j \circ) \right] \nabla e_j
- \sum_{i<j} (e_i \circ) (e_j \circ) (\nabla e_i, \nabla e_j) - \nabla [e_i, e_j] - \nabla [e_i, e_j] - c^2.
\]

Further, let \( K = \sum_{i<j} K_{ij} \in \text{End} (E_x) \).

Next, let \( s \) be a bundle endomorphism, and let \( f \) be any function. Let \( h = \frac{1}{(4\pi t)^{n/2}} e^{-r^2/4t} \), and let \( g = \det (g_{ij}) \), where \( r \) is the geodesic distance to \( y = 0 \). Then from the formulas in [39, pp. 99-100] (extended, as is common,
to endomorphisms),
\[ \nabla h = - \frac{h}{2t} r \partial_r, \]
\[ \frac{\partial h}{\partial t} + \Delta h = \frac{rh \partial_r g}{4gt}, \]
\[ D (fs) - fDs = (\nabla f) \circ s \]
\[ D^2 (fs) - fD^2 s = (\Delta f) s - 2 \nabla \nabla f s, \]
so
\[ \left( -(D + c)^2 \right) (fs) = -(D^2 + 2cD + c^2) (fs) \]
\[ = -(fD^2 s + (\Delta f) s - 2 \nabla \nabla f s) - 2c (fDs + (\nabla f) \circ s) - c^2 f s \]
\[ = -f (D + c)^2 s - (\Delta f) s + 2 \nabla \nabla f s - 2c (\nabla f) \circ s. \]

Then
\[
\frac{1}{h} \left( \partial_t + (D + c)^2 \right) (hs) \\
= \left( \frac{1}{h} \Delta h + \frac{r \partial_r g}{4gt} \right) s + \partial_t s + (D + c)^2 s \\
+ \left( \frac{\Delta h}{h} \right) s - \frac{r}{h} \nabla \nabla h s + \frac{2c}{h} (\nabla h) \circ s \\
= \partial_t s + (D + c)^2 s + \frac{r}{4gt} \partial_r gs + \frac{1}{t} \nabla r \partial_r s - \frac{c}{t} (r \partial_r) \circ s.
\]
Writing
\[ s = u_0 + tu_1 + t^2 u_2 + \cdots, \]
we solve \( \left( \partial_t + (D + c)^2 \right) (hs) = 0 \) and get the equations
\[
\nabla_{r \partial_r} u_j + \left( j + \frac{r \partial_r g}{4g} - c (r \partial_r \phi) \right) u_j = -(D + c)^2 u_{j-1} ,
\]
or
\[
\nabla_{\partial_r} u_j + \left( j + \frac{\partial_r g}{4g} - c (\partial_r \phi) \right) u_j = -\frac{1}{r} (D + c)^2 u_{j-1}.
\]
This is an ordinary differential equation along a geodesic emanating from \( y \),
the center of the geodesic coordinates.
Note that for any smooth function $f$,

$$\exp (f(r)(\partial_r \phi)) = \sum_{k \geq 0} \frac{1}{(2k)!} f(r)^{2k} (\partial_r \phi)^{2k}$$

$$+ \sum_{k \geq 0} \frac{1}{(2k + 1)!} f(r)^{2k+1} (\partial_r \phi)^{2k+1}$$

$$= \sum_{k \geq 0} (-1)^k \frac{(\partial_r \phi)^k}{(2k)!} f(r)^{2k} + \left( \sum_{k \geq 0} (-1)^k \frac{(\partial_r \phi)^k}{(2k + 1)!} f(r)^{2k+1} \right)$$

$$= \cos (f(r)) \mathbf{1} + \sin (f(r)) (\partial_r \phi) .$$

We also have the operator equation

$$\nabla_{\partial_r} [\cos (f(r)) \mathbf{1} + \sin (f(r)) (\partial_r \phi)]$$

$$= -f'(r) \sin (f(r)) \mathbf{1} + \cos (f(r)) \nabla_{\partial_r} + f'(r) \cos (f(r)) (\partial_r \phi)$$

$$+ \sin (f(r)) (\partial_r \phi) \nabla_{\partial_r}$$

$$= [\cos (f(r)) \mathbf{1} + \sin (f(r)) (\partial_r \phi)] \nabla_{\partial_r} + -f'(r) \sin (f(r)) \mathbf{1}$$

$$+ f'(r) \cos (f(r)) (\partial_r \phi)$$

$$= [\cos (f(r)) \mathbf{1} + \sin (f(r)) (\partial_r \phi)] (\nabla_{\partial_r} + f'(r) (\partial_r \phi)) .$$

Thus we multiply $[4]$ by $r^j g^{1/4} [\cos (-c r) \mathbf{1} + \sin (-c r) (\partial_r \phi)]$. Then observe that

$$\nabla_{\partial_r} \left( r^j g^{1/4} [\cos (-c r) \mathbf{1} + \sin (-c r) (\partial_r \phi)] u_j \right)$$

$$= r^{j} g^{1/4} [\cos (-c r) \mathbf{1} + \sin (-c r) (\partial_r \phi)] \left( \nabla_{\partial_r} + \left( \frac{j}{r} + \frac{\partial_r g}{4g} - c (\partial_r \phi) \right) \right) u_j$$

$$= -\frac{1}{r} r^{j} g^{1/4} [\cos (-c r) \mathbf{1} + \sin (-c r) (\partial_r \phi)] (D + c)^2 u_{j-1} ,$$

so the new recursion formula is

$$\nabla_{\partial_r} \left( r^j g^{1/4} [\cos (-c r) \mathbf{1} + \sin (-c r) (\partial_r \phi)] u_j \right)$$

$$= -r^{j-1} g^{1/4} [\cos (-c r) \mathbf{1} + \sin (-c r) (\partial_r \phi)] (D + c)^2 u_{j-1} .$$

Substituting $j = 0$, we see that $g^{1/4} [\cos (-c r) \mathbf{1} + \sin (-c r) (\partial_r \phi)] u_0$ is parallel along radial geodesics, which means that

$$u_0 (r) = g^{-1/4} [\cos (-c r) \mathbf{1} - \sin (-c r) (\partial_r \phi)]$$

$$= g^{-1/4} [\cos (c r) \mathbf{1} + \sin (c r) (\partial_r \phi)] .$$
In other words, \( u_0 (r) \) is the linear map from \( E_y \) to \( E_x \) (with \( y \) being the origin of the geodesic coordinate system and \( r \) being the distance from \( y \) to \( x \)) defined by

\[
s (y) \mapsto g^{-1/4} [\cos (cr) \mathbf{1} + \sin (cr) (\partial_r \circ)] s (x),
\]

where \( s (x) \) is the radial parallel translate of \( s (y) \) along the geodesic connecting \( y \) to \( x \).

By writing

\[
u_1 = u_1 (0) + \mathcal{O} (r),
\]

from (3) we see

\[
u_1 + r \left( \nabla_{\partial_r} u_1 + \left( \frac{\partial_r g}{4g} - c (\partial_r \circ) \right) u_1 \right) = -(D + c)^2 u_0.
\]

In particular,

\[
u_1 (0) = \left( -(D + c)^2 u_0 \right) (0).
\]

We have \( r^2 = x_j x_j \), \( r \partial_r = x_j \partial_j \), and \( g = 1 + \frac{1}{2} R_{ijpi} x_p x_j + \mathcal{O} (r^3) \) in geodesic normal coordinates in terms of the Riemann curvature tensor \( R_{ijkl} \) at \( x = 0 \) (see [39, p. 104]), using the convention that

\[
R_{ijkl} = \langle (\nabla_{\partial_k} \nabla_{\partial_l}^M - \nabla_{\partial_l} \nabla_{\partial_k}^M) \partial_j, \partial_i \rangle.
\]

Using the binomial expansion,

\[
u_0 = g^{-1/4} [\cos (cr) \mathbf{1} + \sin (cr) (\partial_r \circ)]
\]

\[= 1 + cr (\partial_r \circ) - \frac{c^2 r^2}{2} \mathbf{1} - \frac{1}{12} R_{ijkl} x_j x_k \mathbf{1} + \mathcal{O} (r^3)
\]

\[= 1 + cx_j (\partial_j \circ) - \frac{c^2 x_j x_j}{2} \mathbf{1} - \frac{1}{12} R_{ijkl} x_j x_k \mathbf{1} + \mathcal{O} (r^3).
\]

Then at 0,

\[
(Du_0) (0) = g^{pq} (\partial_p \circ) \nabla_{\partial_q} u_0
\]

\[= (\partial_p \circ) \nabla_{\partial_q} u_0
\]

\[= (\partial_p \circ) c (\partial_p \circ) = -nc \mathbf{1}.
\]
At $0$, $\nabla_{\partial_p} \partial_q = 0$ for all $p, q$; thus, from (2) and the above,

$$(D^2 u_0) (0) = (- \nabla_{\partial_p} \nabla_{\partial_q} + K) u_0 = \left( nc^2 + \frac{1}{6} R_{ijji} + K \right) 1 = \left( nc^2 - \frac{1}{6} \text{Scal} + K \right) 1,$$

where $\text{Scal}$ denotes the scalar curvature. Then

$$(7) \quad u_1 (0) = - \left( - \left( D^2 + 2cD + c^2 \right) u_0 \right) (0) = - \left( nc^2 - \frac{1}{6} \text{Scal} + K - 2nc + c^2 \right) 1 = \left( n - 1 \right) c^2 + \frac{1}{6} \text{Scal} 1 - K.$$

We have shown that the heat kernel for $(D + c)^2$ has the expansion

$$K_c (t, x, x) := \exp \left( -t \left( D + c \right)^2 \right) (x, x) = \frac{1}{\left( 4 \pi t \right)^{n/2}} \left( 1 + t \left( \left( n - 1 \right) c^2 + \frac{1}{6} \text{Scal} \right) 1 - K \right) + O \left( t^2 \right),$$

$$\text{Tr} \exp \left( -t \left( D + c \right)^2 \right) = \frac{1}{\left( 4 \pi t \right)^{n/2}} \left( \hat{n} \text{vol} (M) + t \left( \hat{n} \left( n - 1 \right) c^2 \text{vol} (M) + \frac{\hat{n}}{6} \int_M \text{Scal} - \int_M \text{Tr} (K) \right) + O \left( t^2 \right) \right).$$

Here, $n$ is the dimension of the manifold, and $\hat{n}$ is the rank of the bundle $E$.

The Clifford contracted curvature term $K$ has the form (see [39, pp. 48–49], [38, Thm. 3.52])

$$K = \frac{\text{Scal}}{4} + F^{E/S}.$$

On a spin manifold, if $S$ is the spinor bundle, then $E \cong S \otimes W$ with connection $\nabla^{S\otimes W} = \nabla^W \otimes 1 + 1 \otimes \nabla^S$, and $F^{E/S}$ is the twisting curvature of $E$, meaning

$$F^{E/S} = F^W = \sum_{i<j} F^W (e_i, e_j) \left( e^i \circ \right) \left( e^j \circ \right),$$

with $F^W$ the curvature of $\nabla^W$. In particular, if $D$ is the spin Dirac operator on a spin manifold, then $F^W = 0$ and
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$$\text{Tr} \exp \left( - t (D + c)^2 \right) = \frac{1}{(4\pi t)^{n/2}} \left( \hat{n} \text{vol} (M) + t \left[ \hat{n} (n-1) c^2 \text{vol} (M) - \frac{\hat{n}}{12} \int_M \text{Scal} \right] + O \left( t^2 \right) \right).$$

Observe that our first recursion formula (4) for the heat invariant endomorphism $u_j$ corresponding to $(D + c)^2$ is

$$\nabla \partial_r u_j + \left( \frac{j}{r} + \frac{\partial_r g}{4g} - c (\partial_r \circ) \right) u_j = - \frac{1}{r} (D + c)^2 u_{j-1},$$

where $r$ is the distance from the origin of the coordinate system, and the differential equation holds along a geodesic from 0 to $x$. For $j \geq 0$, we expand

$$u_j = \sum_{k=0}^{K} c^k u_{j,k} + O \left( c^{K+1} \right),$$

where each $u_{j,k}$ is independent of $c \in \mathbb{R}$. For consistency we declare that $u_{j,k} = 0$ if either $j$ or $k$ is negative. Our recursive formula above implies that (collecting powers of $c$

$$\nabla \partial_r u_{j,k} + \left( \frac{j}{r} + \frac{\partial_r g}{4g} \right) u_{j,k} = (\partial_r \circ) u_{j,(k-1)} - \frac{1}{r} D^2 u_{j,(k-1)},$$

$$- \frac{2}{r} Du_{j,(k-1)},(k-1) - \frac{1}{r} u_{j,(k-1),(k-2)}.$$ (8)

**Proposition 4.** We have

$$u_{j,k} = O \left( r^{\max \{k-2j,0\}} \right).$$

In particular,

$$u_{j,k} (0) = 0$$

if $k > 2j$, so that $u_j$ is a polynomial in $c$ of degree at most $2j$.

**Proof.** Clearly, $u_{j,0} = O \left( 1 \right)$ for all $j \geq 0$, as these refer to the standard heat invariants (with $c = 0$). Also, the formula holds for $u_{0,k}$ by Taylor analysis of the explicit formula (6). We prove the general case by induction; assume that the theorem holds for all $(j, k)$ such that $0 \leq j < J$ and $k \geq 0$ or $j = J$
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and \(0 \leq k \leq K\), with \(J \geq 1\) and \(K \geq 0\). Then the formula preceding the statement implies that

\[
\begin{align*}
\frac{r}{\Delta_j} u_{j,K+1} + \left( J + \frac{\partial_j g}{4g} \right) u_{j,K+1} &= r (\partial_j \phi) u_{j,K} - D^2 u_{(J-1),K+1} \\
&= -2D u_{(J-1),K} - u_{(J-1),(K-1)}.
\end{align*}
\]

Note that, given \(A(r) = O(r^p)\) is smooth in \(r\), we have \(r \partial_j A(r) = O(r^p)\) if \(p \neq 0\) and \(r \partial_j A(r) = O(r)\) if \(p = 0\). Similarly, \(r (\partial_j \phi) A(r) = O(r^{p+1})\) since \(\partial_j \phi\) is bounded and has constant norm. Then, by the induction hypothesis,

\[
\begin{align*}
u_{j,K+1} &= O(r^{\max(K-2J,0)+1}) + O(r^{\max(K-2J+3-2,0)}) \\
&+ O(r^{\max(K-2J+2-0,0)}) + O(r^{\max(K-2J+1,0)}) \\
&= O(r^{\max(K-2J+1,0)}),
\end{align*}
\]

since \(D(O(r^p)) = O(r^{\max(p-1,0)})\) as long as the quantities are smooth in \(r\). \(\square\)

Because \((\partial_j \phi)^{2j} = (-1)^j\) and \((\partial_j \phi)^{2j+1} = (-1)^j (\partial_j \phi)\), from [6] we have

\[
\begin{align*}
u_{0,k} &= \frac{1}{k!} g^{-1/4} r^k (\partial_j \phi)^k.
\end{align*}
\]

Also, since all of the \(u_{j,0}\) are known (the standard heat invariants), we may use [8] to calculate \(u_{j,k}\) for all \(j \geq 0, k \geq 0\). That is,

\[
\nabla_j \left( r^j g^{1/4} u_{j,k} \right) = r^j g^{1/4} \left[ (\partial_j \phi) u_{j,(k-1)} - \frac{1}{r} D^2 u_{(j-1),k} \right]
- \frac{2}{r} D u_{(j-1),(k-1)} - \frac{1}{r} u_{(j-1),(k-2)},
\]

and so the expression may be integrated along a radial geodesic to solve for \(u_{j,k}\). We note that the formulas above and below for \(u_{j,k}\) are well-known for the case \(k = 0\) (see, for example, [39, pp. 101ff], [22]); they are not easily found in the literature for general \(k\) but may be known to experts. From the formulas for \(u_{0,k}\) and [7] we have

\[
\begin{align*}
u_{0,0}(0) &= 1, \quad u_{1,0}(0) = \left( \frac{1}{6} \text{Scal} \right) 1 - K, \quad u_{1,1}(0) = 0, \quad u_{1,2}(0) = (n-1) 1.
\end{align*}
\]
Let
\[ a_{j,k} = \int_M \text{tr} (u_{j,k} (x,x)) \, d\text{vol}, \]
where \( u_{j,k} (x,x) \) is the expression at \( r = 0 \) of \( u_{j,k} \) found above. In particular, if \( n \) is the dimension of the manifold \( M \) and \( \hat{n} \) is the rank of the bundle \( E \),
\begin{align*}
  a_{0,0} &= \hat{n} \text{vol} (M), \\
  a_{1,0} &= \frac{\hat{n}}{6} \int_M \text{Scal} - \int_M \text{Tr} (K), \\
  a_{1,1} &= 0, \\
  a_{1,2} &= \hat{n} (n-1) \text{vol} (M).
\end{align*}

Then the heat invariants \( a_j (c) \) corresponding to \( (D + c)^2 \) satisfy
\begin{align*}
  a_j (c) &= \int_M \text{tr} (u_j (x,x)) \, d\text{vol} = \sum_{k=0}^{2j} c^k a_{j,k}. \quad (10)
\end{align*}

### 2.3. The eta invariant for arbitrary manifolds with spectral symmetry

Suppose that \( M \) is a closed Riemannian manifold of dimension \( n \). Recall from Proposition 1 we wish to calculate \( \lim_{s \to 0} -s \zeta_{(D + c)^2} (\frac{s+1}{2}) \), at a particular value of \( c \) where \( \text{dim ker} (D + c)^2 = \{0\} \). From (1), as \( t \to 0^+ \),
\[ \sum_{\mu} e^{-t \mu} = \int_M \text{tr} K_c (t,x,x) \, dV (x) \sim \frac{1}{(4\pi t)^{n/2}} (a_0 + ta_1 + t^2 a_2 + \cdots), \]
where \( \{\mu\} \) are the eigenvalues of \( (D + c)^2 \) with multiplicities. The standard derivation of the analytic continuation of the zeta function is as follows. For large \( \text{Re} (s) \),
\[ \zeta_{(D + c)^2} (s) = \sum_{\mu} \mu^{-s} = \frac{1}{\Gamma (s)} \int_0^\infty t^{s-1} \left( \sum_{\mu} e^{-t \mu} \right) dt \]
\[ = \frac{1}{\Gamma (s)} \int_0^1 t^{s-1} \left( \frac{1}{(4\pi t)^{n/2}} (a_0 + a_1 t + \cdots + a_N t^N) \right) dt \]
\[ + \frac{1}{\Gamma (s)} \int_1^\infty t^{s-1} \left( \sum_{\mu} e^{-t \mu} - \frac{1}{(4\pi t)^{n/2}} (a_0 + a_1 t + \cdots + a_N t^N) \right) dt \]
\[ + \frac{1}{\Gamma (s)} \int_1^\infty t^{s-1} \left( \sum_{\mu} e^{-t \mu} \right) dt \]
The eta invariant on two-step nilmanifolds

\[
\eta(s) = \frac{1}{(4\pi)^{n/2} \Gamma(s)} \sum_{j=0}^{N} a_j \int_{0}^{1} t^{s-\frac{n}{2}+j} dt + \phi_N(s) = \frac{1}{(4\pi)^{n/2} \Gamma(s)} \sum_{j=0}^{N} a_j \frac{s}{2} + j + \phi_N(s),
\]

where \(\phi_N(s)\) is holomorphic for \(\text{Re} s > \frac{n}{2} - N - 1\), \(\Gamma(\cdot)\) is the Gamma function, and \(a_j\) is the heat invariant corresponding to \((D+c)^2\):

\[
a_j = \int_{M} \operatorname{tr}(u_j(x,x)) \, d\text{vol}.
\]

Then, since \(\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\),

\[
\lim_{s \to 0} -s \zeta(D+c)^2 \left(\frac{s+1}{2}\right) = \lim_{s \to 0} -s \frac{a_{\pm1}}{(4\pi)^{n/2} \Gamma(s+1/2) \left(\frac{s+1}{2} - \frac{1}{2}\right)} = -2^{1-n} \pi^{-(n+1)/2} a_{\pm1},
\]

or

\[
\frac{d}{dc} \eta_c(0) = -2^{1-n} \pi^{-(n+1)/2} a_{\pm1}(c).
\]

Note that if \(n\) is even, \(\frac{d}{dc} \eta_c(0) = 0\).

Now, suppose that there is a point of symmetry, \(\lambda < 0\), in the spectrum \(\sigma(D)\) of \(D\), meaning that \(\sigma(D) - \lambda\) is symmetric about 0 in \(\mathbb{R}\). Then \(\eta_{-\lambda}(0) = 0\). We then integrate the formula above from \(c = 0\) to \(c = -\lambda\). We have a discontinuity (a jump of +2) at each \(c \in (0, -\lambda)\) that is an eigenvalue of \(-D\), due to the fact that \(c \mapsto \text{sgn}(\lambda + c)\) has a similar discontinuity near \(c = -\lambda\). Also, if either 0 or \(-\lambda\) are contained in the spectrum of \(-D\), then we will have a jump discontinuity of +1 at those points. Let \(c_1 \leq \cdots \leq c_k\) be the points of \((0, -\lambda)\) that are eigenvalues of \(-D\). Let \(n_0\) be the multiplicity of 0 in \(\sigma(D)\), \(n_{-\lambda}\) be the multiplicity of \(-\lambda\) in \(\sigma(D)\). Then the fundamental theorem of calculus yields

\[
\int_{0}^{c_1} \frac{d}{dc} \eta_c(0) \, dc = \eta_{c_1}(0) - \eta_0(0) - 1 - n_0,
\]

\[
\int_{c_j}^{c_{j+1}} \frac{d}{dc} \eta_c(0) \, dc = \eta_{c_{j+1}}(0) - \eta_{c_j}(0) - 2,
\]

\[
\int_{c_k}^{0} \frac{d}{dc} \eta_c(0) \, dc = \eta_{-\lambda}(0) - \eta_k(0) - 1 - n_{-\lambda},
\]

\[
\int_{c_k}^{-\lambda} \frac{d}{dc} \eta_c(0) \, dc = \eta_{-\lambda}(0) - \eta_k(0) - 1 - n_{-\lambda},
\]
which add to
\[
\int_{0}^{-\lambda} dc \frac{d}{dc} \eta_c(0) = \eta_{-\lambda}(0) - \eta_0(0) - 2k - n_0 - n_{-\lambda}.
\]

Therefore, since \( \eta_{-\lambda}(0) = 0 \) and \( \eta_0(0) = \eta(0) \),
\[
\eta(0) = -\int_{0}^{-\lambda} dc \frac{d}{dc} \eta_c(0) - 2k - n_0 - n_{-\lambda}.
\]

In the case where the point of symmetry is positive (\( \lambda > 0 \)), the calculation above may be adapted in the following ways. We integrate the formula for \( \frac{d}{dc} \eta_c(0) \) from \( c = -\lambda \) to \( c = 0 \), and if \( \epsilon_1 \leq \cdots \leq \epsilon_k \) are the points of \((-\lambda, 0)\) that are eigenvalues of \(-D\), we have
\[
\int_{-\lambda}^{\epsilon_1} \frac{d}{dc} \eta_c(0) dc = \eta_{\epsilon_1}(0) - \eta_{-\lambda}(0) - 1 - n_{-\lambda},
\]
\[
\int_{\epsilon_j}^{\epsilon_{j+1}} \frac{d}{dc} \eta_c(0) dc = \eta_{\epsilon_{j+1}}(0) - \eta_{\epsilon_j}(0) - 2,
\]
\[
\int_{\epsilon_k}^{0} \frac{d}{dc} \eta_c(0) dc = \eta_0(0) - \eta_k(0) - 1 - n_0,
\]
which yields
\[
\eta_0(0) = \int_{-\lambda}^{0} dc \frac{d}{dc} \eta_c(0) + 2k + n_0 + n_{-\lambda},
\]
with \( n_0, n_{-\lambda} \) defined above.

In general, if \( \lambda \) is the point of symmetry of \( \sigma(D) \),
\[
\eta_0(0) = \eta_{-\lambda}(0) + \int_{-\lambda}^{0} dc \frac{d}{dc} \eta_c(0) dc
\]
\[
+ \text{sgn}(\lambda) \left( 2\#(\sigma(D) \cap I_\lambda) + \#(\sigma(-D) \cap \{0, -\lambda\}) \right)
\]
\[
= -2^{1-n} \pi^{-(n+1)/2} \int_{-\lambda}^{0} \frac{d}{dc} \eta_c(0) dc + \text{sgn}(\lambda) \left( 2\#(\sigma(D) \cap I_\lambda) + \#(\sigma(D) \cap \{0, \lambda\}) \right),
\]
where \( I_\lambda \) is \((0, \lambda)\) or \((\lambda, 0)\), depending on the sign of \( \lambda \), and where the last two terms include multiplicities.
Thus, from the formula above and the expression for the heat invariant coefficients \( a_{j,k} \) in (10), we have the following formula for \( \eta(0) = \eta_0(0) \).

**Theorem 5.** Let \( \sigma(D) - \lambda \) be symmetric about 0 in \( \mathbb{R} \). Then the eta invariant satisfies

\[
\eta(0) = -2^{1 - n} \pi^{-(n+1)/2} \left( \sum_{k=0}^{n-1} \frac{(-1)^k}{k+1} \lambda^{k+1} a_{n-1,k} \right) + \sgn(\lambda) 2\#(\sigma(D) \cap I_\lambda) + \sgn(\lambda) \#(\sigma(D) \cap \{0, \lambda\}) ,
\]

where \( I_\lambda \) is the open interval between 0 and \( \lambda \), and where implicitly the last two terms include multiplicities.

### 2.4. The zeta function and the eta invariant for three-manifolds

By Theorem 5 for \( n = 3 \) we have

\[
\eta(0) = -2^{-2} \pi^{-2} \left( \lambda^1 a_{1,0} - \frac{1}{2} \lambda^2 a_{1,1} + \frac{1}{3} \lambda^3 a_{1,2} \right) + \sgn(\lambda) 2\#(\sigma(D) \cap I_\lambda) + \sgn(\lambda) \#(\sigma(D) \cap \{0, \lambda\}) ,
\]

From (9),

\[
\eta(0) = -\frac{\hat{n}}{6 \pi^2} \vol(M) - \frac{\lambda}{4 \pi^2} \left( \frac{\hat{n}}{6} \int_M \Scal - \int_M Tr(K) \right) + \sgn(\lambda) (2\#(\sigma(D) \cap (0, \lambda)) + \#(\sigma(D) \cap \{0, \lambda\})) ,
\]

where implicitly the last two terms include multiplicities. Note that every three-manifold is spin, and thus if we let \( F^W \) be the twisting curvature, then

\[
\int_M Tr(K) = \int_M \frac{\hat{n} \Scal}{4} + \int_M Tr(F^W) .
\]

Then we have

\[
\eta(0) = -\frac{\hat{n}}{6 \pi^2} \vol(M) + \frac{\lambda}{4 \pi^2} \left( \frac{\hat{n}}{12} \int_M \Scal + \int_M Tr(F^W) \right) + \sgn(\lambda) (2\#(\sigma(D) \cap (0, \lambda)) + \#(\sigma(D) \cap \{0, \lambda\})) .
\]
3. Two-step Nilmanifolds and Dirac operators

3.1. Two-step Nilmanifolds and the Laplace-Beltrami operator

We review known results about the Laplacian on two-step nilmanifolds in this section. A Lie algebra \( g \) is two-step nilpotent if its derived algebra \( g' = [g, g] \neq 0 \) is contained in its center; i.e., \([g, [g, g]] = 0 \) but \([g, g] \neq 0 \). A Lie group \( G \) is two-step nilpotent if its Lie algebra is. Let \( G \) be a simply connected two-step nilpotent Lie group of dimension \( n \) with Lie algebra \( g \). Let \( \Gamma \) be a cocompact (i.e., \( \Gamma \setminus G \) compact), discrete subgroup of \( G \), and denote \( M = \Gamma \setminus G \). Fix an inner product \( \langle \cdot, \cdot \rangle \) on \( g \), which corresponds to a left-invariant metric on \( G \), and which descends to a Riemannian metric on \( M \). Note that left translation by noncentral elements is no longer an isometry on \( M \). Let \( \{X_i\} \) be an orthonormal basis of left-invariant vector fields of \( g \).

All nilpotent Lie groups are unimodular [17, Proposition 1.2.10], so that the Laplace-Beltrami operator acting on smooth functions on \( G \) can be expressed as

\[
\Delta = - \sum \lambda_i^2 X_i^2.
\]

Denote by \( \rho \) the (right) quasi-regular representation of \( G \) on \( L^2(\Gamma \setminus G) \); i.e., for \( g \in G, f \in L^2(\Gamma \setminus G) \),

\[
(\rho (g) f) (x) = f (xg).
\]

This is a unitary representation of \( G \), and \( \rho \) is the induced representation of the trivial representation of \( \Gamma \). Denote by \( \rho_* \) the associated unitary action of \( g \) on \( C^\infty(\Gamma \setminus G) \subset L^2(\Gamma \setminus G) \); i.e., for \( X \in g, f \in C^\infty(\Gamma \setminus G) \),

\[
(\rho_* (X) f) (x) = \frac{d}{dt} \bigg|_0 f (x \exp (tX)).
\]

Because on smooth functions \( \rho_* (X) f = Xf \), we may rewrite the Laplacian as

\[
\Delta = - \sum (\rho_* X_i)^2.
\]

By expressing the Laplace-Beltrami operator in terms of the representation \( \rho \), we see that irreducible subspaces of the representation are also invariant subspaces of the Laplacian. By restricting \( \Delta \) to an irreducible subspace of \( L^2(\Gamma \setminus G) \), Gordon, Wilson, and Pesce (26, [36]) have been able in the two-step nilpotent case to explicitly solve for its eigenvalues and eigenfunctions. The Laplace spectrum of \( \Gamma \setminus G \) is then the union over all irreducible
subspaces of the spectrum of the restricted Laplacian. The multiplicity of an eigenvalue is the sum over the irreducible subspaces of $L^2(\Gamma \backslash G)$ of the eigenvalue’s multiplicity in the irreducible subspace times the multiplicity of the irreducible subspace in $L^2(\Gamma \backslash G)$. The key ingredient that distinguishes the nilpotent case in general, and the two-step nilpotent case in particular, is that occurrence conditions, eigenvalues, eigenfunctions, and multiplicities can be explicitly expressed in terms of $\log \Gamma$ and $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ using Kirillov theory. For more details, see [25].

Kirillov ([28], [29]) proved that equivalence classes of irreducible unitary representations of nilpotent Lie groups $G$ are in 1-1 correspondence with the orbits of the coadjoint action of $G$ on $\mathfrak{g}^*$. The coadjoint action is defined by, for $x \in G$, $\alpha \in \mathfrak{g}^*$,

$$x \cdot \alpha = \alpha \circ \text{Ad} \left( x^{-1} \right).$$

Given a fixed representative $\alpha \in \mathfrak{g}^*$ corresponding to a coadjoint orbit, let $\pi_{\alpha}$ denote the associated irreducible unitary representation of $G$ with representation space $W_{\alpha}$. The possible dimensions of $W_{\alpha}$ are either 1 (characters) or infinite. L. F. Richardson ([38]) computed the decomposition of $\rho$ into irreducibles.

**Notation:** Given $\alpha \in \mathfrak{g}^*$, let $B_{\alpha} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be defined by

$$B_{\alpha}(X,Y) = \alpha(\left[ X,Y \right]).$$

Let $\mathfrak{g}_\alpha = \ker(B_{\alpha}) = \{ X \in \mathfrak{g} : B_{\alpha}(X,Y) = 0 \text{ for all } Y \in \mathfrak{g} \}$, let $B_{\alpha}$ be the nondegenerate skew-symmetric bilinear form induced by $B_{\alpha}$ on $\mathfrak{g} / \mathfrak{g}_\alpha$, and denote by $\pm i d_1, \ldots, \pm i d_r$ the eigenvalues of $B_{\alpha}$. Note $\log \Gamma$ generates a lattice $\mathcal{L}$ in $\mathfrak{g}$ [20] proof of Thm 2.4. Let $\mathcal{A}_\alpha = \mathcal{L} / (\mathcal{L} \cap \mathfrak{g}_\alpha)$. Let

$$\Delta_{\alpha} = \Delta|_{W_{\alpha}}.$$ 

In the two-step nilpotent case, H. Pesce explicitly calculated the spectrum of the restricted Laplace-Beltrami operator $\Delta_{\alpha}$ as follows.

**Proposition 6.** ([36, Section II and Appendix A]) We continue the notation above.

1) $\pi_{\alpha}$ occurs in the representation $L^2(\Gamma \backslash G)$ if and only if

$$\alpha(\log \Gamma \cap \mathfrak{g}_\alpha) \subset \mathbb{Z}.$$

2) If $\pi_{\alpha}$ occurs and $\alpha([\mathfrak{g}, \mathfrak{g}]) = \{0\}$, then $\pi_{\alpha}$ is one-dimensional and occurs with multiplicity $m_{\alpha} = 1$. The Laplace spectrum associated to this
irreducible subspace is

\[
\text{spec} (\Delta_\alpha) = \left\{ 4\pi^2 \|\alpha\|^2 \right\}.
\]

3) If \( \pi_\alpha \) occurs and \( \alpha ([g, g]) \neq \{0\} \), then \( \pi_\alpha \) is infinite-dimensional and occurs with multiplicity

\[
m_\alpha = \sqrt{\det (B_\alpha)},
\]

where the determinant is computed with respect to (any) lattice basis of \( A_\alpha \subset g / g_\alpha \). The Laplace spectrum associated to this irreducible subspace is

\[
\text{spec} (\Delta_\alpha) = \left\{ \mu (\alpha, p) : p \in (\mathbb{Z}_{\geq 0})^m \right\},
\]

where

\[
\mu (\alpha, p) = 4\pi^2 \sum \alpha (Z_i)^2 + 2\pi \sum (2p_j + 1) d_j,
\]

with \( \{Z_1, \ldots, Z_k\} \) an orthonormal basis of \( g_\alpha \). The multiplicity of \( \mu \) in \( \text{spec} (\Delta_\alpha) \) is the number of \( p \in (\mathbb{Z}_{\geq 0})^m \) satisfying \( \mu (\alpha, p) = \mu \).

**Remark 7.** In other words, the multiplicity of an eigenvalue \( \lambda \) is the sum of the multiplicity of \( \lambda \) as an eigenvalue in each \( \Delta_\alpha \) times the multiplicity of \( \pi_\alpha \) in the representation \( L^2 (\Gamma \backslash G) \).

### 3.2. The Dirac operator on two-step nilmanifolds

As we intend to calculate the eta invariant of the spin Dirac operator, we now extend Pesce’s results to the Dirac setting. Recall that \( G \) is a simply connected \( n \)-dimensional two-step nilpotent Lie group with Lie algebra \( g \) and \( \Gamma \) is a cocompact, discrete subgroup of \( G \). We fix an inner product on \( g \), which corresponds to a left-invariant metric on \( G \), which descends to a Riemannian metric on \( \Gamma \backslash G \).

Let \( \Sigma_n \) be a standard irreducible spinor representation (see [9 Section 3.2]), also considered as a trivial bundle over \( G \). A spin structure and the corresponding spinor bundle \( \Sigma_\epsilon \) over \( \Gamma \backslash G \) are determined by \( \Sigma_n \) and a
The eta invariant on two-step nilmanifolds

The homomorphism \( \varepsilon : \Gamma \to \{ \pm 1 \} \) (see [9, Prop 3.34, p. 114]). We have

\[
L^2(\Gamma \backslash G, \Sigma_\varepsilon) \cong L^2_\varepsilon(\Gamma \backslash G) \otimes \mathbb{C} \Sigma_n,
\]

where \( L^2_\varepsilon(\Gamma \backslash G) \) is defined by

\[
L^2_\varepsilon(\Gamma \backslash G) = \{ f \in L^2_{\text{loc}}(G) : f(\gamma x) = \varepsilon(\gamma) f(x) \text{ for all } \gamma \in \Gamma, x \in G \}.
\]

The isomorphism from \( L^2_\varepsilon(\Gamma \backslash G) \otimes \mathbb{C} \Sigma_n \) to \( L^2(\Gamma \backslash G, \Sigma_\varepsilon) \) is \( f \otimes s \mapsto fs \), where \( \Sigma_n \) is identified with the constant sections \( G \to G \times \Sigma_n \). Clifford multiplication by elements of \( T(\Gamma \backslash G) \cong \Gamma \backslash G \times g \) is given by the standard Clifford action \( \circ \) of \( \mathbb{C}l(g) \) on \( \Sigma_n \). That is, \( \xi \in g \) acts on \( L^2_\varepsilon(\Gamma \backslash G) \otimes \mathbb{C} \Sigma_n \) by

\[
\xi \circ (fs) = f(\xi \circ s).
\]

By construction, \( (\xi \circ) \) is a constant matrix on \( \Gamma \backslash G \) for every left-invariant vector field \( \xi \).

Note that the (Clifford) connection on any spinor bundle is given by

\[
\nabla^\Sigma_{E_i} = \partial_{E_i} + \frac{1}{4} \sum_{j,k} \Gamma_{ij}^k (E_j \circ) (E_k \circ)
\]

according to the Ammann-Bär formula [1, formula 1.1], where \( \{ E_j \} \) is a left-invariant orthonormal basis of the tangent space, \( \Gamma_{ij}^k \) are the Christoffel symbols associated to the metric and frame, and \( \partial_{E_i} \) is a directional derivative. In our case, we use the left-invariant metric on \( g \), yielding a metric on \( \Gamma \backslash G \). Then the Dirac operator \( D \) on \( \Gamma \backslash G \) acts on \( L^2_\varepsilon(\Gamma \backslash G) \otimes \Sigma_n \) by

\[
D = \sum_i (E_i \circ) \nabla^\Sigma_{E_i}
= \sum_i (E_i \circ) \partial_{E_i} + \frac{1}{4} \sum_{i,j,k} \Gamma_{ij}^k (E_i \circ E_j \circ E_k \circ)
\]

If \( \rho_\varepsilon \) denotes right multiplication acting on \( L^2_\varepsilon(\Gamma \backslash G) \), we have

\[
\partial_{E_i} = \left. \frac{d}{dt} \right|_0 \rho_\varepsilon (\exp (tE_i)) = \rho_\varepsilon^* (E_i).
\]

Note that \( \rho_\varepsilon \) is the induced representation of \( \varepsilon : \Gamma \to \{ \pm 1 \} \) to \( G \). The Christoffel symbols are defined by

\[
\nabla_{E_i} E_j = \sum \Gamma_{ij}^k E_k,
\]
and the Koszul formula gives

\[2\Gamma^k_{ij} = - \langle E_i, [E_j, E_k] \rangle + \langle E_j, [E_k, E_i] \rangle + \langle E_k, [E_i, E_j] \rangle.\]

At this point, the formulas given above are completely general for any Lie group \(G\) with a left-invariant metric.

We now assume \(G\) is 2-step nilpotent, so that \(\langle E_i, [E_j, E_k] \rangle = 0\) unless \(E_i\) is in the center of \(g\). If \(g = z \oplus v\) with \(z\) the center and \(v = z^\perp\), its orthogonal complement, then the inner product on \(g\) is determined by and determines the map \(j : z \rightarrow so(v)\) defined as

\[(15) \quad \langle j(Z), X, A \rangle = \langle Z, [X, A] \rangle\]

for all \(Z \in z\) and all \(X, A \in v\). See, for example, [20, p.618ff]. Note that if \(\langle Z, [g, g] \rangle = 0\), then \(j(Z)\) is the zero map.

Let \(k_0\) be the dimension of the center and \(k_0 + m_0\) the dimension of \(g\), and we choose the orthonormal basis \(\{Z_1, \ldots, Z_{k_0}, X_1, \ldots, X_{m_0}\}\) so that \(\{Z_i\}\) is an orthonormal basis of \(z\) and \(\{X_i\}\) is an orthonormal basis of \(v\). Then one easily verifies that

\[\nabla_{Z_i} X_k = \nabla_{X_k} Z_i = -\frac{1}{2} j(Z_i) X_k, \quad \nabla_X X_k = \frac{1}{2} [X_i, X_k], \quad \nabla_Z Z_k = 0.\]

We label \(E_1 = Z_1, \ldots, E_{k_0} = Z_{k_0}, E_{k_0+1} = X_1, \ldots, E_{k_0+m_0} = X_{m_0}\). The Christoffel symbols satisfy \(\Gamma^{pq}_{ab} = 0\) if at least two of \(p, q, r\) are \(\leq k_0\) or if \(p, q, r > k_0\). If \(a \leq k_0, b, q > k_0\),

\[
2\Gamma^{a}_{bq} = -2\Gamma^{a}_{qb} = 2\Gamma^{b}_{aq} = -2\Gamma^{b}_{qa} = -2\Gamma^{q}_{ba} = \langle Z_a, [X_{b-k_0}, X_{q-k_0}] \rangle = \langle j(Z_a) X_{b-k_0}, X_{q-k_0} \rangle.
\]

Letting \(\partial E_i = \partial_i, C_i = (E_i \circ), C_{abq} = (E_a \circ E_b \circ E_q \circ)\), etc., the Dirac operator is

\[
D = \sum_i \partial_i C_i + \frac{1}{4} \sum_{i,j,k} \Gamma^k_{ij} C_{ijk} = \sum_i \partial_i C_i + \frac{1}{4} \sum_{a \leq k_0; b, q > k_0} \left( \Gamma^a_{bq} C_{bqa} + \Gamma^b_{aq} C_{aqb} + \Gamma^q_{ba} C_{baq} \right)
\]
The eta invariant on two-step nilmanifolds

\[
\begin{align*}
&= \sum_i \partial_i C_i + \frac{1}{4} \sum_{a \leq k_0; q > b > k_0} \left( \Gamma_{aq}^b C_{bqa} + \Gamma_{aq}^b C_{qba} + \Gamma_{ba}^b C_{aqb} + \Gamma_{ab}^b C_{qab} + \Gamma_{qba}^b C_{aqb} + \Gamma_{qab}^b C_{baq} \right) \\
&= \sum_i \partial_i C_i + \frac{1}{2} \sum_{a \leq k_0; q > b > k_0} \Gamma_{ba}^b C_{aqb} \\
&= \sum_i \partial_i C_i + \frac{1}{4} \sum_{a \leq k_0; q > b > k_0} \langle Z_a, [X_{b-k_0}, X_{q-k_0}] \rangle (Z_a \circ X_{b-k_0} \circ X_{q-k_0}^\phi),
\end{align*}
\]

so

\[
D = \sum_i (E_i \phi) \partial E_i + \frac{1}{4} \sum_{a \leq k_0; b < i \leq m_0} \langle Z_a, [X_b, X_i] \rangle (Z_a \circ X_b \circ X_i^\phi) \\
= \sum_i (E_i \phi) \rho_{s^*} (E_i) + \frac{1}{2} \sum_{a \leq k_0} Z_a \circ j (Z_a).
\]

In the expression above, we have used the fact that \( j (Z_a) \in \mathfrak{so} (m_0) = \mathfrak{spin} (m_0) \) and have therefore identified \( j (Z_a) \) with the operator

\[
\frac{1}{2} \sum_{b < i \leq m_0} \langle j (Z_a) X_b, X_i \rangle X_b \circ X_i^\phi.
\]

The formula above works for any two-step nilmanifold.

**Example 8.** *In the three-dimensional Heisenberg case, for some constant \( A > 0 \), we let \( \{X_1 = \frac{1}{\sqrt{A}} X, X_2 = \frac{1}{\sqrt{A}} Y, Z\} \) be an orthonormal frame with \([X, Y] = Z\). We choose a basis of \( \Sigma_3 \cong \mathbb{C}^2 \) so that

\[
(Z^\phi) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad (X_1^\phi) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad (X_2^\phi) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

Then

\[
\langle Z, [X_1, X_2] \rangle = \frac{1}{A},
\]

\[
(Z \circ X_1 \circ X_2^\phi) = -1,
\]

so the equation above becomes

\[
D = \sum_{i=1}^{3} (E_i \phi) \partial E_i - \frac{1}{4A}.
\]
as seen in [1, Equation 3.2], with $d^2T = \frac{1}{\lambda}$ in their notation.

### 3.3. Analogue of Pesce’s theorem for spinors

In this section, we decompose $L^2(\Gamma \setminus G)$ as a direct sum of irreducible representations. Let $\alpha \in \mathfrak{g}^*$. Recall $B_\alpha(X,Y) := \alpha([X,Y])$, $\mathfrak{g}_\alpha = \ker B_\alpha$, so that $\alpha([\mathfrak{g}_\alpha,\mathfrak{g}]) = 0$. Let $\mathfrak{g}_\alpha$ be a maximal polarizer of $\alpha$, meaning that it is a subalgebra of $\mathfrak{g}$ such that $\alpha([\mathfrak{g}_\alpha,\mathfrak{g}_\alpha]) = 0$ and there does not exist a subalgebra $\mathfrak{h}$ with the same property such that $\mathfrak{g}_\alpha \subset \mathfrak{h} \subseteq \mathfrak{g}$. Note that for every $\alpha \in \mathfrak{g}^*$ and every choice of $\mathfrak{g}_\alpha$, $\mathfrak{g}_\alpha \subset \mathfrak{g}^\alpha$.

Given $\mathfrak{g}_\alpha$, let $G_\alpha = \exp(\mathfrak{g}_\alpha)$.

**Lemma 9.** (Lemma 4 from [36, Appendix A]) Let $\alpha \in \mathfrak{g}^*$, $\alpha([\mathfrak{g},\mathfrak{g}]) \neq 0$ and $B_\alpha(X,Y) = \alpha([X,Y]) \in \mathbb{Z}$ for all $X,Y \in \log \Gamma$. Then there exists a basis $\{U_1, \ldots, U_m, V_1, \ldots, V_m, W_1, \ldots, W_k\}$ of $\mathfrak{g}$ formed of elements of $\log \Gamma$, and there exist integers $r_1, \ldots, r_k$ such that

1) We have

\[
B_\alpha(U_i, V_i) = \alpha([U_i,V_i]) = r_i,
\]

\[
B_\alpha(U_i, V_j) = 0 \text{ if } i \neq j, \text{ and}
\]

\[
B_\alpha(U_i, U_j) = B_\alpha(V_i, V_j) = 0 \text{ for all } i,j.
\]

2) $\{W_1, \ldots, W_k\}$ is a basis of $\mathfrak{g}_\alpha$, $\{W_1, \ldots, W_{k_1}\}$ is a basis of $[\mathfrak{g},\mathfrak{g}]$, $k_1 \leq k$.

3) $[\mathfrak{g},\mathfrak{g}] \cap \log \Gamma = \operatorname{span}_\mathbb{Z}\{W_1, \ldots, W_{k_1}\}$.

**Remark 10.** It follows from Pesce’s proof of this Lemma that we may also choose $\{W_1, \ldots, W_{k_0}\}$ to be a basis of $\mathfrak{z}$, with $k_1 \leq k_0 \leq k$.

As before, $\log \Gamma$ generates a lattice $\mathcal{L}$ in $\mathfrak{g}$. Let $\mathcal{A}_\alpha = \mathcal{L} / (\mathcal{L} \cap \mathfrak{g}_\alpha)$. When $\pi_\alpha$ occurs, this will be a lattice in $\mathfrak{g} / \mathfrak{g}_\alpha$.

**Proposition 11.** (Version of Pesce Occurrence Condition ([36, Proposition 9 of Appendix A]) for Dirac spinors) The representation $\pi_\alpha$ appears in
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$L^2_\epsilon (\Gamma \backslash G)$ if and only if

(17) \hspace{1cm} \alpha (\log \gamma) \in \mathbb{Z} + \frac{1 - \varepsilon (\gamma)}{4}

for all \( \gamma \in \Gamma \cap G_\alpha \). In this case, the multiplicity of \( \pi_\alpha \) is \( m_\alpha = 1 \) if \( \alpha ([g, g]) = \{0\} \), and otherwise

\[ m_\alpha = \sqrt{\det (B_\alpha)}, \]

where the determinant is computed with respect to (any) lattice basis of \( A_\alpha \subset g / g_\alpha \).

Proof. Items 4 through 8 in [36, Appendix A] apply in this situation.

If \( \alpha ([g, g]) = 0 \), then \( g = g_\alpha = g^{\alpha} \). Then condition (17) is equivalent to Theorem 22. In addition, using Theorem 24,

\[ m (\pi_\alpha, L^2_\epsilon (\Gamma \backslash G)) = \# \left( (G^\alpha \backslash \Gamma_\epsilon) / \Gamma \right) \]

\[ = \# \left( (G \backslash \Gamma) / \Gamma \right) = 1. \]

For the remainder of the proof, we assume \( \alpha ([g, g]) \neq 0 \). First, we assume \( \pi_\alpha \) appears in \( L^2_\epsilon (\Gamma \backslash G) \). Then, by Theorem 22, there exists \( \alpha' \) in the coadjoint orbit of \( \alpha \) such that \( (\alpha', G^{\alpha'}) \) is an \( \varepsilon \)-integral point, where \( \alpha' = \alpha \circ I_x \) (\( I_x \) = conjugation by \( x \)), \( \alpha' = \alpha \circ Ad(x) \) and \( G^{\alpha'} = I_x^{-1} (G^{\alpha}) \) such that

\[ \alpha' (\log \gamma) \in \begin{cases} \mathbb{Z} & \text{if } \varepsilon (\gamma) = 1 \\ \frac{1}{2} + \mathbb{Z} & \text{if } \varepsilon (\gamma) = -1 \end{cases} \]

for all \( \gamma \in \Gamma \cap G^{\alpha'} \). In the two-step case, \( G^{\alpha'} = G^{\alpha} \) since \( I_x (y) y^{-1} \in Z (G) \) for all \( x, y \in G \), and \( Z (G) \subseteq G_{\alpha} \subseteq G^{\alpha} \). Also in the two-step case, if \( \alpha, \alpha' \) lie in the same coadjoint orbit, then there exists \( X \in g \) such that \( \alpha' = \alpha \circ (I + ad (X)) \). Thus

\[ \alpha (\log \gamma + ad (X) \log \gamma) \in \begin{cases} \mathbb{Z} & \text{if } \varepsilon (\gamma) = 1 \\ \frac{1}{2} + \mathbb{Z} & \text{if } \varepsilon (\gamma) = -1 \end{cases} \]

for all \( \gamma \in \Gamma \cap G^{\alpha} \). This implies the same condition is met for all \( \gamma \in \Gamma \cap G_{\alpha} \), in which case \( \alpha (\log \gamma) = \alpha ([X, \log \gamma]) = 0 \), by definition of \( G_{\alpha} \).
On the other hand, suppose
\[ \alpha (\log \gamma) \in \mathbb{Z} + \frac{1 - \varepsilon(\gamma)}{4} \]
for all \( \gamma \in \Gamma \cap G_\alpha \). Note that if \( X, Y \in \log \Gamma \), then \([X, Y] \in \log \Gamma\) since \( [\exp X, \exp Y] = \exp ([X, Y]) \) (since \( G \) is two-step). Therefore, \( \alpha ([X, Y]) \in \mathbb{Z} \), since \( \varepsilon ([\exp X, \exp Y]) = 1 \). We can then use Lemma 9 to construct a basis \( \{U_1, \ldots, U_m, V_1, \ldots, V_m, W_1, \ldots, W_k\} \subset \log \Gamma \) of \( g \) and integers \( r_1, \ldots, r_m \) such that \( \alpha ([U_j, V_j]) = r_j \). Set \( h = \text{span} \{V_1, \ldots, V_m, W_1, \ldots, W_k\} \). Then \( h \) is a rational ideal of \( g \), since \( [h, g] \subseteq z \subseteq h \) (two-step condition), and \( h \) is a polarizer of \( \alpha \). Set \( H = \exp (h) \), which is a normal subgroup of \( G \). Note that
\[
H = \left\{ \prod_{i=1}^{m} \exp (y_i V_i) \prod_{j=1}^{k} \exp (z_j W_j) : y_i, z_j \in \mathbb{R} \right\}.
\]
Define \( \overline{\alpha}(\exp (X)) = \exp (2\pi i \alpha (X)) \) for all \( X \in h \). By Theorem 22 to prove that \( \pi_\alpha \) occurs, we need only construct an \( \varepsilon \)-integral point in the \( G \)-orbit of \( (\overline{\alpha}, H) \). For \( x \in G \), define \( x_i, y_i, z_j \) by the formula
\[
x = \prod_{i=1}^{m} \exp (x_i U_i) \prod_{i=1}^{m} \exp (y_i V_i) \prod_{j=1}^{k} \exp (z_j W_j),
\]
and define \( p_i, q_i, \eta_j \) by
\[
\alpha \left( \sum u_i U_i + v_i V_i + \sum w_j W_j \right) = \sum (p_i u_i + q_i v_i) + \sum \eta_j w_j,
\]
for all \( u_i, v_i, w_j \in \mathbb{R}, 1 \leq i \leq m, 1 \leq j \leq k \). By [26, Lemma 7, Appendix A], [26, Theorem 8, Appendix A],
\[
H \cap \Gamma = \left\{ \prod_{i=1}^{m} \exp (t_i V_i) \prod_{j=1}^{k} \exp (s_j W_j) : t_i, s_j \in \mathbb{Z} \right\}.
\]
We need to show that there exists \( x \in G \) such that \( (\overline{\alpha} \circ I_x)(\gamma) = \varepsilon(\gamma) \) for all \( \gamma \in H \cap \Gamma \). First note that
\[
(\overline{\alpha} \circ I_x)(\exp (W_j)) = \overline{\alpha}(\exp (W_j + [\log (x), W_j]))
= \exp (2\pi i \alpha (W_j + [\log (x), W_j]))
= \exp (2\pi i \alpha (W_j)) \text{ since } W_j \in g_\alpha
= \overline{\alpha}(\exp (W_j)) = \varepsilon(\exp (W_j))
\]
since $W_j \in (\log \Gamma) \cap \mathfrak{g}_\alpha$. Next,
\begin{align*}
(\alpha \circ I_x) (\exp (V_j)) &= \alpha (\exp (V_j + [\log (x), V_j])) \\
&= \exp (2\pi i \alpha (V_j + [\log (x), V_j])) \\
&= \exp (2\pi i (q_j + x_j r_j)).
\end{align*}

By setting $x_j = -\frac{q_j}{r_j}$ or $-\frac{q_j + \frac{1}{2} r_j}{r_j}$ depending on whether $\varepsilon (\exp (V_j)) = \pm 1$, we conclude,
\begin{align*}
(\alpha \circ I_x) (\exp (V_j)) &= \pm (\exp (V_j)).
\end{align*}

We have shown that for all $\gamma \in H \cap \Gamma$ there exists $x \in G$ such that $(\alpha \circ I_x) (\gamma) = \varepsilon (\gamma)$.

We now calculate the multiplicity with which $\pi_\alpha$ appears. In fact, for $x \in G$, the calculations above show that for all $X \in \mathfrak{h}$, $(\alpha \circ I_x) (\exp (X))$ depends only on the $x_i$ and not on $y_i$ or $z_j$, $1 \leq i \leq m, 1 \leq j \leq k$. Thus from (18) the orbit of $(\alpha, H)$ is the set of characters of $H$
\begin{align*}
\{(\chi_{q'}, H) : q' \in \mathbb{R}^m \},
\end{align*}
where where after a bit of calculation identical to [36] p.453, lines -8 through -5,
\begin{align*}
\chi_{q'} \left( \prod_{i=1}^m \exp (t_i V_i) \prod_{j=1}^k \exp (s_j W_j) \right) = \exp \left( 2\pi i \left( \sum_{i=1}^m q'_i t_i + \sum_{j=1}^k \eta_j s_j \right) \right).
\end{align*}

Then $(\chi_{q'}, H)$ is an $\varepsilon$-integer point if and only if
\begin{align*}
q'_i &\in \mathbb{Z} \text{ whenever } \varepsilon (\exp (V_i)) = 1, \\
q'_i &\in \frac{1}{2} + \mathbb{Z} \text{ whenever } \varepsilon (\exp (V_i)) = -1.
\end{align*}

Note also from (18) that two $\varepsilon$-integer points $(\chi_{q'}, H)$ and $(\chi_{q''}, H)$ are in the same $\Gamma$-orbit if and only if $q'_i - q''_i \in r_i \mathbb{Z}$, $i = 1, \ldots, m$. So the number $m_\alpha$ of $\Gamma$-orbits in the $\varepsilon$-integer points is $r_1 r_2 \cdots r_m$. Next, it is clear that the images of $U_1, \ldots, U_m, V_1, \ldots, V_m$ form a basis of $A_\alpha$. So
\begin{align*}
det \left( B_\alpha \right) = det \left( B_\alpha (U_i, V_j) \right)^2 = (r_1 r_2 \cdots r_m)^2 = m_\alpha^2.
\end{align*}
4. Decomposition of the Dirac operator on two-step nilmanifolds

We continue with the notation of the previous section; recall that \( k_0 \) is the dimension of the center \( z \) and \( n = k_0 + m_0 \) is the dimension of \( g = z \oplus v \), and we will choose the orthonormal basis \( \{ E_1, \ldots, E_n \} = \{ Z_1, \ldots, Z_{k_1}, \ldots, Z_{k_0}, X_1, \ldots, X_{m_0} \} \) so that \( \{ Z_j \}_{j=1}^{k_1} \) is an orthonormal basis of \( [g, g] \), \( \{ Z_j \}_{j=1}^{k_0} \) is an orthonormal basis of \( z \) and \( \{ X_j \} \) is an orthonormal basis of \( v \). From formula (16) and this choice of basis, the Dirac operator is now

\[
D = \sum_{i=1}^{n} (E_i \circ \rho_e) (E_i) + \frac{1}{2} \sum_{a \leq k_1} Z_a \circ j(Z_a),
\]

acting on

\[
\mathcal{H} = L^2(\Gamma \backslash G, G \times \varepsilon \Sigma_n) \cong L^2(\Gamma \backslash G) \otimes \Sigma_n,
\]

which we decompose using Kirillov theory.

Choose an element \( \alpha \in g^\ast \). Our strategy is as follows. We first construct a subspace \( \mathcal{H}_\alpha \) of \( L^2(\Gamma \backslash G, G \times \varepsilon \mathbb{C}^k) \) that is invariant with respect to \( \rho_e \) and invariant by \( D \). Once we have done this, by Kirillov theory, let \( \mathcal{H}^{\alpha} \) be the irreducible \( \rho_e \)-subspace of \( L^2(\Gamma \backslash G) \) corresponding to the coadjoint orbit of \( \alpha \), and let

\[
\mathcal{H}_\alpha \cong \mathcal{H}^{\alpha} \otimes \Sigma_n
\]

through the isomorphism above. While \( \mathcal{H}^{\alpha} \) is \( \rho_e \)-irreducible, \( \mathcal{H}_\alpha \) is not for \( n \geq 2 \). We express \( D \) acting on \( \mathcal{H}_\alpha \), and because of the two-step structure, we are able to solve explicitly the partial differential equation for eigenvalues via Hermite functions.

Since \( \sum_{i=1}^{n} \rho_e(\rho_e) (E_i) \circ (E_i) \) is independent of the choice of basis \( \{ E_1, \ldots, E_n \} \), the second term is similarly independent of choices and independent of the representation \( \rho_e \). Define

\[
\begin{align*}
D_{\rho_e} &= \sum_{i=1}^{n} (E_i \circ \rho_e) (E_i) \\
M &= \frac{1}{2} \sum_{a \leq k_0} Z_a \circ j(Z_a) = \frac{1}{2} \sum_{a \leq k_1} Z_a \circ j(Z_a),
\end{align*}
\]

so that \( D = D_{\rho_e} + M \) with \( M \) a hermitian linear transformation independent of invariant subspace. Note that \( \rho_e \) and \( (Y \circ) \) commute if \( (Y \circ) \) is a constant.
transformation — that is, if $Y$ is left-invariant. Thus $M$ commutes with $\rho_\varepsilon$ because each $\langle Z_a, [X_b, X_i] \rangle$ is constant on $\Gamma \setminus G$.

As before, we define the symplectic form on $\mathfrak{g}$ by $B_\alpha (U, V) := \alpha ([U, V])$, and let $\mathfrak{g}_\alpha = \ker B_\alpha = \{ U \in \mathfrak{g} : B_\alpha (U, \cdot) = 0 \}$, $k_\alpha = \dim \mathfrak{g}_\alpha$. We have two cases.

4.1. Finite-dimensional $\mathcal{H}_\alpha$-irreducible subspaces: $k_\alpha = n$, i.e. $\alpha ([\mathfrak{g}, \mathfrak{g}]) = 0$.

In this case, $\mathfrak{g}_\alpha = \mathfrak{g}$, and $\mathfrak{g}^\alpha = \mathfrak{g}$ is a maximal polarizer of $\alpha$. Then $G_\alpha = \exp (\mathfrak{g}^\alpha) = G$. Define

$$\mathcal{H}_\alpha = \{ \sigma \in \mathcal{H} : \sigma (hx) = \overline{\alpha} (h) \sigma (x) \text{ for all } h \in G^\alpha, x \in G \}$$

$$= \{ \sigma \in \mathcal{H} : \sigma (hx) = \overline{\alpha} (h) \sigma (x) \text{ for all } h \in G, x \in G \}$$

$$= \overline{\alpha} (\cdot) \otimes \Sigma_n,$$

where

$$\overline{\alpha} (h) = e^{2\pi i \alpha (\log h)}.$$

For $\sigma \in \mathcal{H}_\alpha$, we have, since $\alpha ([\mathfrak{g}, \mathfrak{g}]) = 0$, for $p \in \Gamma \setminus G$,

$$\rho_\varepsilon (U) \sigma (p) = \frac{d}{dt} \bigg|_0 \sigma (p \exp (tU) 1) = \frac{d}{dt} \bigg|_0 e^{2\pi i \alpha \log (p \exp (tU))} \sigma (1)$$

$$= \frac{d}{dt} \bigg|_0 e^{2\pi i \alpha (\log (p) + tU + \frac{1}{2} \log (p), tU))} \sigma (1)$$

$$= \frac{d}{dt} \bigg|_0 e^{2\pi i \alpha (\log (p) + tU)} \sigma (1).$$

We have $\rho_\varepsilon (Z_a) \sigma = 0$, and $\rho_\varepsilon (X_i) \sigma = \frac{\partial}{\partial x_i} \sigma = 2\pi i \alpha (X_i) \sigma$. Thus,

$$D|_{\mathcal{H}_\alpha} = \sum_{i=1}^{m_0} 2\pi i \alpha (X_i) (X_i \diamond) + \sum_{Z_j \notin [\mathfrak{g}, \mathfrak{g}]} 2\pi i \alpha (Z_j) (Z_j \diamond)$$

$$+ \frac{1}{2} \sum_{a \leq k_1} Z_a \diamond j (Z_a),$$

which is a constant matrix. The eigenvalues of $D|_{\mathcal{H}_\alpha}$ are then the eigenvalues of this Hermitian matrix.
4.2. Infinite-dimensional $\mathcal{H}_\alpha$-irreducible subspaces: $k_\alpha < n$, so that $\alpha ([g, g])$ is not identically zero.

Choose a new orthonormal basis of $g$:

$$\{ W_1, \ldots, W_{k_\alpha}, U_1, \ldots, U_m, V_1, \ldots, V_m \},$$

where $n = k_\alpha + 2m$, $\{ W_j \}$ is a basis of $g_\alpha$ with $W_1 = Z_1, \ldots, W_{k_\alpha} = Z_{k_\alpha} \in \mathfrak{z}$, $W_{k_\alpha+1}, \ldots, W_k \in g_\alpha \cap \mathfrak{z}^\perp$.

$$B_\alpha (U_i, V_i) = \alpha ([U_i, V_i]) = d_i > 0, 0 < d_1 \leq d_2 \leq \cdots \leq d_m,$$

$$B_\alpha (U_i, V_j) = 0 \text{ if } i \neq j,$$

$$B_\alpha (U_i, U_j) = B_\alpha (V_i, V_j) = 0 \text{ for all } i, j.$$

Note the similarity with Lemma 9, but we have replaced some of the $V_j$ with their negatives in order to make $d_j$ positive. We may assume $n - k_\alpha$ is even, since the restriction of $B_\alpha$ to $g_\alpha^\perp$ is a symplectic form. Then the polarizing subalgebra $g_\alpha$ (meaning that $g_\alpha$ is a subalgebra of $g$ such that $\alpha ([g_\alpha, g_\alpha]) = 0$ and is maximal with respect to inclusion) will be chosen to be

$$g_\alpha = \operatorname{span} \{ V_1, \ldots, V_m, W_1, \ldots, W_{k_\alpha} \},$$

and again $G_\alpha := \exp (g_\alpha)$. We have, with $\bar{\sigma} (h) = \exp (2\pi i \alpha (\log h))$,

$$\mathcal{H}_\alpha = \{ \sigma \in \mathcal{H} : \sigma (hx) = \bar{\sigma} (h) \sigma (x) \text{ for all } h \in G_\alpha, x \in G \}.$$

Let $\overline{\mathcal{H}}_\alpha$ be the $\rho_\epsilon$-irreducible subspace of $L^2_\mathbb{C} (\Gamma \backslash G)$ such that

$$\mathcal{H}_\alpha \cong \overline{\mathcal{H}}_\alpha \otimes \Sigma_n$$

through the isomorphism \cite{[12]}. Let $\beta : \overline{\mathcal{H}}_\alpha \to L^2_\mathbb{C} (\mathbb{R}^m)$ be the unitary isomorphism defined by $\beta (F) (t) = F (\exp (t_1 U_1) \cdots \exp (t_m U_m))$. Note that the map

$$t \in \mathbb{R}^k \mapsto G_\alpha \exp (t_1 U_1) \cdots \exp (t_m U_m) \in G_\alpha \backslash G$$

pushes the Euclidean metric onto a right $G$-invariant metric on $G_\alpha \backslash G$. Note that $\overline{\mathcal{H}}_\alpha = \beta^{-1} (L^2_\mathbb{C} (\mathbb{R}^m))$, and for $x = h \exp (t_1 U_1) \cdots \exp (t_m U_m)$ an
arbitrary element of $G$ with $h \in G^\alpha$, and $f \in L^2_\mathbb{C} (\mathbb{R}^m)$,
$$
(\beta^{-1} f) (x) = (\beta^{-1} f) (h \exp (t_1 U_1) \cdots \exp (t_m U_m))
= \pi (h) f (t_1, \ldots, t_m).
$$

Here $\pi_\alpha$ is the representation of $G$ on $\mathcal{H}_\alpha$ induced from the character $\pi$ of $G^\alpha$; we have for $f \in \mathcal{H}_\alpha$,
$$
(\pi_\alpha (x) f) (y) = (\rho_{xy} f) (y) = f (yx).
$$

We define the representation $\pi'_\alpha$ of $G$ on $L^2_\mathbb{C} (\mathbb{R}^m)$ by
$$
\pi'_\alpha (x) = \beta \circ \pi_\alpha (x) \circ \beta^{-1}
$$
for all $x \in G$.

For any $x, y \in G$, let $[x, y] = xyx^{-1}y^{-1}$. To compute the action of $G$ on $L^2_\mathbb{C} (\mathbb{R}^m)$, recall that since $G$ is 2-step (following [36, p. 447, proof of Prop. 9]), for any $h_0 \in G^\alpha$,
$$
\begin{align*}
\prod_{j=1}^m \exp (t_j U_j) h_0 &= \left[ \prod_{j=1}^m \exp (t_j U_j), h_0 \right] h_0 \prod_{j=1}^m \exp (t_j U_j) \\
\prod_{\ell=1}^m \exp (t_\ell U_\ell) \prod_{j=1}^m \exp (s_j U_j)
&= \exp \left( - \sum_{1 \leq j < \ell \leq m} t_\ell s_j [U_j, U_\ell] \right) \prod_{j=1}^m \exp ((t_j + s_j) U_j) \\
\left[ \prod_{j=1}^m \exp (t_j U_j), h_0 \right]
&= \exp \left[ \sum_{j=1}^m t_j U_j, \log h_0 \right].
\end{align*}
$$

For any $x \in G$, by the calculations above, there exists $h_0 \in G^\alpha$ and real numbers $s_\ell \in \mathbb{R}$ such that $x = h_0 \prod_{j=1}^m \exp (s_j U_j)$. For any $f \in L^2_\mathbb{C} (\mathbb{R}^m)$, $t, s \in \mathbb{R}^m$
$$
(\pi'_\alpha (x) f) (t) = (\beta^{-1} f) \left( \prod_{\ell=1}^m \exp (t_\ell U_\ell) h_0 \prod_{j=1}^m \exp (s_j U_j) \right).
$$

Since $(\beta^{-1} f) (hg) = \pi (h) (\beta^{-1} f) (g)$, we see
$$
(\pi'_\alpha (x) f) (t) = \pi \left( \left[ \prod_{\ell=1}^m \exp (t_\ell U_\ell), h_0 \right] h_0 \exp \left( - \sum_{1 \leq j < \ell \leq m} t_\ell s_j [U_j, U_\ell] \right) \right) \\
\times f \left( t + s \right).
$$
We have used the fact that $\exp ( [ g, g ] ) \subset Z ( G ) \subset G^\alpha$ and the calculations above. Since the restriction of $B_\alpha$ to $\mathbb{R} U_1 \oplus \cdots \oplus \mathbb{R} U_m \times \mathbb{R} U_1 \oplus \cdots \oplus \mathbb{R} U_m$ is zero, we have

$$\left( \pi_\alpha' ( x ) f \right) ( t ) = f ( t + s ) e^{2 \pi i \alpha \left( \log h_0 + \left[ \sum_{j=1}^m t_j U_j, \log h_0 \right] \right)}.$$ 

Now, define the vector $w \in \mathbb{R}^m$ by

$$w := \left( \frac{\alpha ( V_1 )}{d_1}, \ldots, \frac{\alpha ( V_m )}{d_m} \right).$$

Define the unitary isomorphism $T_w : L^2_\mathbb{C} ( \mathbb{R}^m ) \to L^2_\mathbb{C} ( \mathbb{R}^m )$ by

$$(T_w f) (t) = f ( t - w ),$$

and define $\pi''_\alpha ( x ) = T_w \circ \pi_\alpha' ( x ) \circ T_w^{-1}$ for all $x \in G$. We claim that the representation $\pi''_{\alpha \ast} = \rho_{\varepsilon \ast}$ is given by

$$\pi''_{\alpha \ast} ( U_j ) f ( t ) = \frac{\partial}{\partial t_j} f ( t ),$$

$$\pi''_{\alpha \ast} ( V_j ) f ( t ) = 2 \pi i t_j d_j f ( t ),$$

$$\pi''_{\alpha \ast} ( W_j ) f ( t ) = 2 \pi i \alpha \left( W_j \right) f ( t ).$$

To see this (see also [36, Section 3]), we have for $r \in \mathbb{R}$,

$$\pi''_\alpha ( \exp ( r U_j ) ) f ( t ) = ( T_w \pi_\alpha' ( \exp ( r U_j ) ) T_{-w} f ) ( t )$$

$$= \left( \pi_\alpha' ( \exp ( r U_j ) ) T_{-w} f \right) ( t - w )$$

$$= ( T_{-w} f ) ( t - w + r e_j )$$

$$= f ( t + r e_j ),$$

with $e_j$ the $j^{\text{th}}$ standard unit vector in $\mathbb{R}^m$. Also,

$$\pi''_\alpha ( \exp ( r V_j ) ) f ( t ) = \left( \pi_\alpha' ( \exp ( r V_j ) ) T_{-w} f \right) ( t - w )$$

$$= ( T_{-w} f ) ( t - w ) e^{2 \pi i \alpha \left( r V_j + \left[ t_j - w_j \right] U_j, V_j \right)}$$

$$= f ( t ) e^{2 \pi i \alpha \left( r V_j + t_j d_j r - w_j d_j r \right)}$$

$$= f ( t ) e^{2 \pi i t_j d_j r}.$$
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We have
\[ \pi''(\exp(rW_j)) f(t) = (\pi'_\alpha(\exp(rW_j)) T_w f)(t - w) = f(t) e^{2\pi i \alpha(rW_j)}. \]

With \( W_1 = Z_1, \ldots, W_{k_0} = Z_{k_0} \), equation (16) becomes (see (20))
\[ D|_{\mathcal{H}_\alpha} = \sum_{j=1}^n E_j \circ \rho_{e^z} (E_j) + \frac{1}{2} \sum_{a \leq k_1} Z_a \circ j (Z_a) \]
\[ = \sum_{j=1}^n E_j \circ \rho_{e^z} (E_j) + M \]
\[ = \sum_{j=1}^{k_0} (W_j \circ) \rho_{e^z} (W_j) + \sum_{j=1}^m (U_j \circ) \rho_{e^z} (U_j) + \sum_{j=1}^m (V_j \circ) \rho_{e^z} (V_j) + M \]
\[ = \sum_{j=1}^{k_0} 2\pi i \alpha (W_j) (W_j \circ) + \sum_{j=1}^m (U_j \circ) \frac{\partial}{\partial t_j} + \sum_{j=1}^m 2\pi it_j d_j (V_j \circ) + M \]
\[ = \sum_{j=1}^m (U_j \circ) \frac{\partial}{\partial t_j} + \sum_{j=1}^m 2\pi id_j (V_j \circ) t_j + M'_\alpha, \]

where \( M'_\alpha \) is defined as the constant Hermitian transformation
\[ (23) \quad M'_\alpha = M + \sum_{j=1}^{k_0} 2\pi i \alpha (W_j) (W_j \circ), \]

with \( M \) as in [20].

We have
\[ (24) \quad D|_{\mathcal{H}_\alpha} = M'_\alpha + \sum_{j=1}^m (U_j \circ) \left( \frac{\partial}{\partial t_j} + 2\pi id_j (V_j \circ) t_j \right) \]
\[ = M'_\alpha + \sum_{j=1}^m (U_j \circ) \left( \frac{\partial}{\partial t_j} - 2\pi id_j (U_j \circ) (V_j \circ) t_j \right), \]

so that
\[ (25) \quad D|_{\mathcal{H}_\alpha} = M'_\alpha + \sum_{j=1}^m (U_j \circ) \left( \frac{\partial}{\partial t_j} + \Theta_j t_j \right), \]
where we define

\[ \Theta_j = -2\pi id_j (U_j \circ) (V_j \circ), \]

a Hermitian symmetric linear transformation.

### 4.3. Matrix choices

We now make specific choices of the matrices \((U_j \circ), (V_j \circ)\), where \(U_j, V_r, W_k\) are from the basis chosen at the beginning of Section 4.2 relative to a particular \(\alpha\). We continue to use the positive real numbers \(d_j\) as defined in that section as well. Note that any other choices would yield the same Dirac spectrum. See [30, Part I, section 5] for details on the representations of Clifford algebras, on which much of this material is based.

Let

\[
\begin{align*}
\sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \sigma_2 &= \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \\
1' &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & 1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{align*}
\]

We view \(\Sigma_n = \mathbb{C}^{2\lfloor n/2 \rfloor} = \bigotimes_{[n/2]} \mathbb{C}^2\). Observe that multiplication satisfies

\[
\begin{array}{c|c|c|c}
1 & 1' & \sigma_1 & \sigma_2 \\
1' & 1 & -i\sigma_2 & i\sigma_1 \\
\sigma_1 & i\sigma_2 & 1 & -i1' \\
\sigma_2 & -i\sigma_1 & i1' & 1
\end{array}
\]

with multiplication on the left given by the column items.

Let

\[ (U_1 \circ) = i\sigma_1 \otimes 1 \otimes \cdots \otimes 1, \quad (V_1 \circ) = i\sigma_2 \otimes 1 \otimes \cdots \otimes 1, \]

and in general, for \(1 \leq j \leq m\),

\[ (U_j \circ) = i1' \otimes \cdots \otimes 1' \otimes \sigma_1 \otimes 1 \otimes \cdots \otimes 1, \]

\[ (V_j \circ) = i1' \otimes \cdots \otimes 1' \otimes \sigma_2 \otimes 1 \otimes \cdots \otimes 1, \]

where each \((U_j \circ)\) and each \((V_j \circ)\) has \(j - 1\) leading factors of \(1'\) and a total of \(n' = \lfloor n/2 \rfloor = m + \lfloor k/2 \rfloor\) matrix factors of size \(2 \times 2\). Continuing, each \((W_k \circ)\),
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1 ≤ k ≤ k_α, is chosen to be

\[(W_k) = i1' \otimes \cdots \otimes 1' \otimes \sigma \otimes 1 \otimes \cdots \otimes 1,\]

with \(\sigma\) being \(\sigma_1\) or \(\sigma_2\) according to whether \(k\) is odd or even, such that there are at least \(m\) leading factors of \(1'\) in the above expression. If the dimension \(k_\alpha\) is odd, then the last matrix is chosen to be

\[(W_{k_\alpha}) = i1' \otimes \cdots \otimes 1'.\]

With these choices, observe that from (26),

\[\Theta_j = 2\pi d_j (1 \otimes 1 \otimes \cdots \otimes 1 \otimes 1' \otimes 1 \otimes \cdots \otimes 1),\]

with \(1'\) in the \(j^{th}\) slot. We let

\[v_\ell = e_{\ell_1} \otimes e_{\ell_2} \otimes \cdots \otimes e_{\ell_n},\]

where each \(e_{\ell_n}\) is either \(e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\), \(e_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\), with \(\ell = (\ell_1, \ldots, \ell_n') \in \{1, -1\}^n\), then \(\{v_\ell\}\) forms a basis of \(\Sigma_n\). Then

\[\Theta_j v_\ell = 2\pi d_j \ell_j v_\ell,\]

\[(U_j) v_\ell = i\ell_1 \ell_2 \cdots \ell_{j-1} v_\ell = \pm iv_\ell,\]

with \(\ell' = (\ell_1, \ldots, -\ell_j, \ldots, \ell_n').\)

We see that \(\Theta_j\) commutes with every \(\Theta_{j'}\), and \(\Theta_j^2 = 4\pi^2 d_j^2 I_\text{Id}\). Note that

\[\{(2\pi d_j \ell_j, v_\ell) : \ell \in \{1, -1\}^n\}\]

is the set of eigenvalues and simultaneous orthonormal eigenvectors of every \(\Theta_j, j = 1, \ldots, m\).

Let \(p = (p_1, \ldots, p_m) \in \mathbb{Z}^m\). We let \(h_p(t) = h_{p_1}(t_1) \cdots h_{p_m}(t_m)\) using the Hermite functions

\[h_p(t) = e^{t^2/2} \left( \frac{d}{dt} \right)^p e^{-t^2} \text{ for } p \geq 0,\]

\[h_p(t) = 0 \text{ for } p < 0,\]

which satisfy

\[h'_p(t) = 2p h_p(t) + h_{p+1}(t)\]

\[h_{p+2}(t) + 2th_{p+1}(t) + 2(p + 1) h_p(t) = 0.\]
The first equality is just the chain rule. To see the second equality, note that by the product rule and the binomial theorem,

\[ \left( \frac{d}{dt} \right)^{p+2} e^{-t^2} = \left( \frac{d}{dt} \right)^{p+1} (-2t e^{-t^2}) \]

\[ = -2t \left( \frac{d}{dt} \right)^{p+1} e^{-t^2} - 2(p+1) \left( \frac{d}{dt} \right)^{p} e^{-t^2}, \]

and the result follows. Combining the two

\[ h_p'(t) = th_p(t) - 2th_p(t) - 2ph_p(t) = -th_p(t) - 2ph_p(t). \]

Note that \( \{h_p(t) : p \in (\mathbb{Z}_{\geq 0})^m \} \) is a basis of \( L^2(\mathbb{R}^m, \mathbb{C}) \).

For \( p \in \mathbb{Z}^m \), let

\[ u_{p,\ell}(t) = h_{p_1} \left( \sqrt{2\pi d_1} t_1 \right) h_{p_2} \left( \sqrt{2\pi d_2} t_2 \right) \cdots h_{p_m} \left( \sqrt{2\pi d_m} t_m \right) v_{\ell}, \]

with \( v_{\ell} \) as in (31). Observe that \( u_{p,\ell} = 0 \) if any coordinate of \( p \) is negative.

Then, using the formulas above for \( h_p'(t) \),

\[ \frac{\partial}{\partial t_j} u_{p,\ell}(t) = -2\pi d_j t_j u_{p,\ell}(t) - 2p_j \sqrt{2\pi d_j} u_{p-\varepsilon_j,\ell}(t) \]

\[ = 2\pi d_j t_j u_{p,\ell}(t) + \sqrt{2\pi d_j} u_{p+\varepsilon_j,\ell}(t), \]

\[ \left( \frac{\partial}{\partial t_j} + t_j \Theta_j \right) u_{p,\ell}(t) = (2\pi d_j (\ell_j - 1)) t_j u_{p,\ell}(t) - 2p_j \sqrt{2\pi d_j} u_{p-\varepsilon_j,\ell}(t) \]

\[ = (2\pi d_j (\ell_j + 1)) t_j u_{p,\ell}(t) + \sqrt{2\pi d_j} u_{p+\varepsilon_j,\ell}(t). \]

Recall that \( p \) has dimension \( m \), and \( \ell \) has dimension \( n' = \left\lfloor \frac{n}{2} \right\rfloor \geq m \). Now, from (25) we have

\[ Du_{p,\ell}(t) = \left( M'_n + \sum_{j=1}^{m} (U_j \diamond) \left( \frac{\partial}{\partial t_j} + t_j \Theta_j \right) \right) u_{p,\ell}(t) \]

\[ = -2 \sum_{j \leq m, \ell_j = 1} p_j \sqrt{2\pi d_j} (U_j \diamond) u_{p-\varepsilon_j,\ell}(t) \]

\[ + \sum_{j \leq m, \ell_j = -1} \sqrt{2\pi d_j} (U_j \diamond) u_{p+\varepsilon_j,\ell}(t) + M'_n u_{p,\ell}(t) \]
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\[ = -2 \sum_{j \leq m, \ell_j = 1} ip_j \sqrt{2 \pi d_j} \ell_1 \ell_2 \cdots \ell_{j-1} u_{p-e_j, \ell} (t) \]
\[ + \sum_{j \leq m, \ell_j = -1} i \sqrt{2 \pi d_j} \ell_1 \ell_2 \cdots \ell_{j-1} u_{p+e_j, \ell} (t) + M'_{\alpha} u_{p,\ell} (t). \]

Often the eigensections can be found as linear combinations of the \( u_{p,\ell} (t) \).

We modify the basis so that it is more convenient. For fixed \( p = (p_1, \ldots, p_m) \) and \( \ell = (\ell_1, \ldots, \ell_m, \ldots, \ell_n') \), let \( E_\ell \) be the \( m \)-tuple defined by
\[
(E_\ell)_a = \begin{cases} 
0 & \text{if } \ell_a = 1 \\
-1 & \text{if } \ell_a = -1.
\end{cases}
\]

Then
\[ \pi_{p,\ell} (t) = \left( \prod_{j \leq m, \ell_j = -1} \sqrt{2p_j} \right) u_{p+E_\ell,\ell} (t). \]

Using the fact that \( p + E_\ell + e_j = p + E_\ell \) if \( \ell_j = -1 \) and \( p + E_\ell - e_j = p + E_\ell \) if \( \ell_j = 1 \), we compute
\[
D\pi_{p,\ell} (t) = -2 \sum_{j \leq m, \ell_j = 1} 2i \sqrt{\pi d_j} p_j \ell_1 \ell_2 \cdots \ell_{j-1} \pi_{p,\ell} (t)
\]
\[ + \sum_{j \leq m, \ell_j = -1} 2i \sqrt{\pi d_j} p_j \ell_1 \ell_2 \cdots \ell_{j-1} \pi_{p,\ell} (t) + M'_{\alpha} \pi_{p,\ell} (t), \]

so that
\[ D\pi_{p,\ell} (t) = -2i \sum_{j \leq m} \sqrt{\pi d_j} p_j \ell_1 \ell_2 \cdots \ell_{j} \pi_{p,\ell} (t) + M'_{\alpha} \pi_{p,\ell} (t). \]

5. Heisenberg examples

Heisenberg Lie algebras are the only two-step nilpotent Lie algebras with one-dimensional center. Let \( n = 2m + 1 \); define the \( n \)-dimensional Heisenberg Lie algebra by \( g = \text{span} \{X_1, \ldots, X_m, Y_1, \ldots, Y_m, Z\} \) with \([X_j, Y_k] = \delta_{jk} Z\) and other basis brackets not defined by skew-symmetry equal to zero. The \( n \)-dimensional Heisenberg Lie group \( G \) is the simply connected Lie group with Lie algebra \( g \). A Heisenberg manifold is a quotient of \( G \) by a cocompact discrete subgroup \( \Gamma \), where the metric comes from a left-invariant metric on \( G \). From [24, Proposition 2.16], we see that every Heisenberg manifold is
isometric to one with the following metric and lattice. The metric may be chosen for $\Gamma \backslash G$ on $(X_1, \ldots, X_m, Y_1, \ldots, Y_m, Z)$ to be

$$g_A = \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{pmatrix} = (\bar{g}_A \ 0 \ 1)$$

where $A = \text{diag}(a_1, \ldots, a_m)$ is a diagonal $m \times m$ matrix with positive non-decreasing entries.

We identify $X_i$ with the matrix $E_{1,i+1}$, which is the matrix with 1 in the $(1, i+1)$-entry and all other entries zero. Similarly, we identify $Y_j$ with $E_{j+1,m+2}$ and $Z$ with $E_{m+2,m+2}$. In this section, we define $\exp(X_i)$ to be the matrix exponential $\exp(E_{1,i+1}) = I + E_{1,i+1}$, and we define $\exp(Y_j)$ and $\exp(Z)$ in a similar way. For $v \in \mathbb{R}^{2m}$ and $z \in \mathbb{R}$, we denote

$$(v, z) = \begin{pmatrix} 1 & v_1 & \cdots & v_m & z \\ 0 & 1 & \cdots & 0 & v_{m+1} \\ \vdots & \vdots & I & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & v_{2m} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

With this notation,

$$\begin{align*}
\exp(x_1X_1 + \cdots + x_mX_m + y_1Y_1 + \cdots + y_mY_m + zZ) &= (x_1, \ldots, x_m, y_1, \ldots, y_m, z + \frac{1}{2}x \cdot y), \\
\log(x_1X_1 + \cdots + x_mX_m + y_1Y_1 + \cdots + y_mY_m + zZ) &= x_1X_1 + \cdots + x_mX_m + y_1Y_1 + \cdots + y_mY_m + \left(z - \frac{1}{2}x \cdot y\right)Z.
\end{align*}$$

To get from the matrix coordinates to the exponential coordinates, we use the change of coordinate mapping

$$(v, z) \mapsto \exp\left(v_1X_1 + \cdots + v_mX_m + v_{m+1}Y_1 + \cdots + v_{2m}Y_m + \left(z - \frac{1}{2}(v_1v_{m+1} + \cdots + v_mv_{2m})\right)Z\right).$$

Every cocompact discrete subgroup $\Gamma$ can be generated by $\exp(L)$ and $\exp(rZ)$, where $L$ is a $2m$-dimensional lattice in $\mathbb{R}^{2m} = \text{span}(X_1, \ldots, X_m)$,
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Y₁, ..., Yₘ, and \exp(rZ), r > 0, generates a one-dimensional lattice in the center of G. We denote \( \Gamma = \Gamma(\mathcal{L}, r) \); note \((\mathcal{L}, r)\) will yield a compact discrete subgroup if and only if for all \( V, V' \in \mathcal{L} \), \([\exp(V), \exp(V')] = \exp(krZ)\) for some \( k \in \mathbb{Z} \) [26, proof of Theorem 2.4]. Two such Heisenberg manifolds determined by \((\mathcal{L}, r, g_{A})\) and \((\mathcal{L}', r', g_{A}')\) are isometric iff \( g_{A} = g_{A}', r = r' \), and there exists a matrix \( \Phi = \tilde{Sp}(m, \mathbb{R}) \cap O(2m, g_{A}) \subset M_{2m}(\mathbb{R}) \) such that

\[ \Phi(\mathcal{L}) = \mathcal{L}' \]

(See [24, Proposition 2.16]). Here, \( O(2m, g_{A}) \) is the orthogonal group, and \( \tilde{Sp}(m, \mathbb{R}) = \{ \beta \in GL(2m, \mathbb{R}) : \beta^{t}J\beta = \pm J \} \), where \( J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \).

5.1. Three-dimensional case

5.1.1. Eigenvalues. For our Heisenberg manifold, we choose \( \{X, Y, Z\} \) so that \([X, Y] = Z\) and \( \{ \frac{1}{\sqrt{A}}X, \frac{1}{\sqrt{A}}Y, Z \} \) is an orthonormal frame, with \( A > 0 \). With notation as in the general case, we choose an element \( \alpha \in \mathfrak{g}^{\ast} \), which fixes a coadjoint orbit.

Finite-dimensional irreducible subspaces: If the one-form \( \alpha(Z) = 0 \), then \( g_{\alpha} = \mathfrak{g} \), and \( g^{\alpha} = \mathfrak{g} \) is the maximal polarizer of \( \alpha \). Then \( G^{\alpha} = \exp(\mathfrak{g}^{\alpha}) = G \). Let

\[ \mathcal{H}_{\alpha} = \{ f : \mathfrak{g} \to \Sigma_{n} \mid \text{for some } s \in \Sigma_{n}, \text{all } h \in G, f(h) = \overline{\alpha}(h)s \} \],

where

\[ \overline{\alpha}(h) = e^{2\pi i \alpha(\log h)}. \]

From [21],

\[ D|_{\mathcal{H}_{\alpha}} = \frac{2\pi i}{A} \alpha(X)(X^{\circ}) + \frac{2\pi i}{A} \alpha(Y)(Y^{\circ}) + \frac{1}{4A^{2}} \langle Z, [X, Y] \rangle (Z \circ X \circ Y^{\circ}) \]

\[ = \frac{2\pi i}{A} \alpha(X)(X^{\circ}) + \frac{2\pi i}{A} \alpha(Y)(Y^{\circ}) + \frac{1}{4A^{2}} (Z \circ X \circ Y^{\circ}), \]

which is a constant matrix. The eigenvalues of \( D|_{\mathcal{H}_{\alpha}} \) are then the eigenvalues of this Hermitian matrix. We set
\[(X_\diamond) = i\sqrt{A}\sigma_1 = \begin{pmatrix} 0 & i\sqrt{A} \\ i\sqrt{A} & 0 \end{pmatrix}, \quad (Y_\diamond) = i\sqrt{A}\sigma_2 = \begin{pmatrix} 0 & -\sqrt{A} \\ \sqrt{A} & 0 \end{pmatrix},\]
\[(Z_\diamond) = i1' = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\]

The matrix is
\[D|_{\mathcal{H}_\alpha} = \begin{pmatrix} -\frac{1}{4A} & -\frac{2\pi}{\sqrt{A}} (\alpha (X) + i\alpha (Y)) \\ -\frac{2\pi}{\sqrt{A}} (\alpha (X) - i\alpha (Y)) & -\frac{1}{4A} \end{pmatrix},\]
eigenvalues: \[\begin{cases} \frac{1}{16A^2} (-4A - 32\sqrt{\pi^2 A^4 \alpha^2 + \pi^2 A^4 \beta^2}) & \text{if } A \neq 0 \end{cases}\]

The eigenvalues are
\[\sigma_\alpha = \left\{ -\frac{1}{4A} + \frac{2\pi}{\sqrt{A}} \|\alpha\|, -\frac{1}{4A} - \frac{2\pi}{\sqrt{A}} \|\alpha\| \right\} .\]

**Infinite-dimensional irreducible subspaces:** On the other hand, suppose \(\alpha (Z) = \alpha ([X,Y]) = d\) is nonzero.

From (36), (23), and (31), with \(U = \frac{X}{\sqrt{A}}, V = \text{sgn}(d) \frac{Y}{\sqrt{A}}, W = Z\), we have, since \(d_1 = \frac{d}{A}\), \(m = 1, \ell = \pm 1\), \(E_\ell = \ell - \frac{1}{2}\),
\[u_{p,\ell} (t) = h_p \left( \sqrt{2\pi d_1} t \right) v_\ell,\]
\[\overline{u}_{p,\ell} (t) = \begin{cases} u_{p,\ell} (t) & \ell = 1 \\ \sqrt{2p} u_{p,-1,\ell} (t) & \ell = -1, \end{cases}\]
\[D\overline{u}_{p,\ell} (t) = -2i\ell \sqrt{\frac{|d_1|}{A}} \overline{u}_{p,-\ell} (t) + M'_\alpha \overline{u}_{p,\ell} (t) .\]

Then
\[M'_\alpha = \frac{1}{4A^2} \langle Z, [X, \text{sgn}(d) Y]\rangle (Z_\diamond X_\diamond \text{sgn}(d) Y_\diamond) + 2\pi i \alpha (Z) (Z_\diamond)\]
\[= \frac{1}{4A^2} \left( \begin{array}{cc} i & 0 \\ 0 & -i \end{array} \right) \left( \begin{array}{cc} 0 & i\sqrt{A} \\ i\sqrt{A} & 0 \end{array} \right) \left( \begin{array}{cc} 0 & -\sqrt{A} \\ \sqrt{A} & 0 \end{array} \right) + 2\pi i d \left( \begin{array}{cc} i & 0 \\ 0 & -i \end{array} \right)\]
\[= \left( \begin{array}{cc} -2\pi d - \frac{1}{4A} & 0 \\ 0 & 2\pi d - \frac{1}{4A} \end{array} \right) ,\]
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so that

\[ M'_\alpha v_\ell = -\left(2\pi d\ell + \frac{1}{4A}\right)v_\ell, \]

and, for \( \ell \in \{-1, 1\}, p \in \mathbb{Z}_{\geq 0} \),

\[ D\pi_{p,\ell}(t) = -2i\ell \sqrt{\frac{\pi |d|}{A}} p \pi_{p,\ell-\ell}(t) + \left(-2\pi d\ell - \frac{1}{4A}\right) \pi_{p,\ell}(t). \]

The \( p = 0 \) case (\( \bar{u}_{0,-1} = u_{-1,-1} = 0 \)) is

\[ D\bar{u}_{0,1}(t) = \left(-2\pi d - \frac{1}{4A}\right) \bar{u}_{0,1}(t). \]

The matrix for \( D \) restricted to the span of \( \{\pi_{p,1}, \pi_{p,-1}\} \) for \( p \geq 1 \) is

\[
\begin{pmatrix}
-2\pi d - \frac{1}{4A} & 2i{\sqrt{\frac{\pi |d|}{A}}p} \\
-2i\sqrt{\frac{\pi |d|}{A}}p & 2\pi d - \frac{1}{4A}
\end{pmatrix},
\]

which has eigenvalues

\[-\frac{1}{4A} \pm \frac{1}{2}\sqrt{\frac{\pi |d| p}{A} + \pi^2 d^2}.\]

Thus, the list of all eigenvalues for the \( \alpha(Z) = d \neq 0 \) case is

\[ \sigma_{\alpha} = \left\{-\frac{1}{4A} - 2\pi d\right\} \cup \left\{-\frac{1}{4A} \pm \frac{1}{2}\sqrt{\frac{\pi |d| p}{A} + \pi^2 d^2} : p \geq 1 \right\}. \]

5.1.2. Occurrence conditions for lattice. Here, the lattice \( \mathcal{L} \) should be a two-dimensional lattice, say spanned by \( v = (v_1, v_2) \) (corresponding to the matrix element \((v_1, v_2, 0)\) ) and \( w = (w_1, w_2) \). The central lattice is spanned by \( r \) (corresponding to \((0, 0, r)\) ). Let \( \tilde{Sp}(1, \mathbb{R}) = \{\beta \in GL(2, \mathbb{R}) : \beta^t J \beta = \pm J\} \). The condition \( \beta^t J \beta = \pm J \) is equivalent to det \( \beta = \pm 1 \), so in fact \( \tilde{Sp}(1, \mathbb{R}) \cap O(2, \mathbb{R}) = O(2, \mathbb{R}) \). This means we can rotate so that \( v = (v_1, 0) \) with \( v_1 > 0 \), and so that \( w = (w_1, w_2) \) with \( w_2 > 0 \). Because \( v, w, \) and \( r \) generate a cocompact discrete subgroup, we must have, for any \( h_1, h_2, h, \)
$k_1, k_2, k \in \mathbb{Z}$,

$$(h_1 v + h_2 w, hr) (k_1 v + k_2 w, kr)$$

$$= ((h_1 + k_1) v + (h_2 + k_2) w, r (h + k) + h_1 k_2 v_1 w_2 + h_2 k_2 v_1 w_2).$$

is an element of the lattice, by closure for multiplication. Thus, for any choice of integers $h_1, h_2, k_1, k_2$, we must have

$$h_1 k_2 v_1 w_2 + h_2 k_2 v_1 w_2 \in r \mathbb{Z},$$

i.e. $v_1 w_1, w_1 w_2 \in r \mathbb{Z}$. Letting $v_1 w_2 = rm_v, w_1 w_2 = rm_w$, we have $v = \left( \frac{rm_v}{w_2}, 0 \right)$, $w = \left( \frac{rm_w}{w_2}, w_2 \right)$. The parameters are

\begin{equation}
A > 0, r > 0, v_2 > 0, m_v \in \mathbb{Z}_{>0}, m_w \in \mathbb{Z}.
\end{equation}

In our matrix coordinate system, from (37) we have

$$\log \left( \frac{rm_v}{w_2} h_1 + \frac{rm_w}{w_2} h_2, w_2 h_2, rh \right) = \left( \frac{rm_v}{w_2} h_1 + \frac{rm_w}{w_2} h_2 \right) X + (w_2 h_2) Y$$

$$+ \left( hr - \frac{1}{2} rh_1 h_2 m_v - \frac{1}{2} rh_2^2 m_w \right) Z.$$  

The commutator satisfies

$$\left[ \left( \frac{rm_v}{w_2} h_1 + \frac{rm_w}{w_2} h_2, w_2 h_2, rh \right), \left( \frac{rm_v}{w_2} k_1 + \frac{rm_w}{w_2} k_2, w_2 k_2, kr \right) \right]$$

$$= (0, 0, rm_v (h_1 k_2 - h_2 k_1)).$$

We now determine a spin structure by fixing $\varepsilon : \Gamma \to \{1, -1\}$. Let

$$\varepsilon_1 = \varepsilon (v, 0) = \varepsilon \left( \frac{rm_v}{w_2}, 0, 0 \right),$$

$$\varepsilon_2 = \varepsilon (w, 0) = \varepsilon \left( \frac{rm_w}{w_2}, w_2, 0 \right),$$

$$\varepsilon_3 = \varepsilon (0, 0, r).$$

Since $\varepsilon_1 \varepsilon_2 \varepsilon_1^{-1} \varepsilon_2^{-1} = (\varepsilon_3)^{m_v} = 1$ is the only relation, the values of $\varepsilon_1$ and $\varepsilon_2$ are arbitrary ($\pm 1$), but it may be that $\varepsilon_3$ is restricted by $\varepsilon_3^{m_v} = 1$. If $m_v$ is even, there is no restriction, but

(40)

if $m_v$ is odd, then $\varepsilon_3 = 1$.

Now we choose an arbitrary element $\alpha \in \mathfrak{g}^*$, we may either choose $\alpha = \alpha_3 Z^*$ or $\alpha = \alpha_1 X^* + \alpha_2 Y^*$, since all possible coadjoint orbits may be parametrized
by such elements. The occurrence condition is calculated on \( v \) and \( w \). In particular, \( \alpha (v) \) must be an integer or half integer depending on whether \( \varepsilon_1 = \pm 1 \). Likewise for \( \alpha (w) \). From Section 7, the occurrence conditions are:

\[
\begin{align*}
\alpha_1 \frac{rm_v}{w_2} &\in \mathbb{Z} + \frac{1 - \varepsilon_1}{4}, \\
\alpha_1 \frac{rm_w}{w_2} + \alpha_2 w_2 &\in \mathbb{Z} + \frac{1 - \varepsilon_2}{4}, \\
\alpha_3 r &\in \mathbb{Z} + \frac{1 - \varepsilon_3}{4}.
\end{align*}
\]

The multiplicities corresponding to these representations are as follows. If we choose \( \alpha \in \mathfrak{g}^* \) such that \( \alpha = \alpha_1 X^* + \alpha_2 Y^* \), then \( m_{\alpha} = 1 \) (see Section 7). If we choose \( \alpha \in \mathfrak{g}^* \) such that \( \alpha = \alpha_3 Z^* \), then \( g_{\alpha} = 3 \). We have

\[
m_{\alpha} = \sqrt{\det (B_\alpha|_{\text{span}\{X,Y\}})}
\]

with respect to a lattice basis of \( \mathcal{L} \), chosen to be \( v = \frac{rm_v}{w_2} X, w = \frac{rm_w}{w_2} X + w_2 Y \), and thus

\[
\begin{pmatrix}
B_\alpha (v,v) & B_\alpha (v,w) \\
B_\alpha (w,v) & B_\alpha (w,w)
\end{pmatrix} = \begin{pmatrix}
0 & \alpha ([v,w]) \\
-\alpha ([v,w]) & 0
\end{pmatrix}.
\]

So

\[
m_{\alpha} = |\alpha ([v,w])| = |\alpha_3| \frac{rm_v}{w_2} \in \mathbb{Z}_{>0}.
\]

The conditions (40) and (43) confirm that \( m_\alpha \) is an integer.

Now we are ready to calculate the spectrum of the Dirac operator on a general Heisenberg 3-manifold with spin structure. Such a manifold with spin structure is given by \( (\mathcal{L}, r, g_A, \varepsilon) \), and it is determined by the lattice basis \( v = \frac{rm_v}{w_2} X, w = \frac{rm_w}{w_2} X + w_2 Y \) for \( \mathcal{L} \) and \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \) as above with conditions (39), (40), (41), (42), (43).

We now calculate the part of the spectrum corresponding to each coadjoint orbit in \( \mathfrak{g}^* \). There are two cases, \( \alpha_3 = 0 \) and \( \alpha_3 \neq 0 \). If \( \varepsilon_3 = -1 \), the condition (43) does not permit \( \alpha_3 = 0 \). As a consequence, finite-dimensional irreducible subspaces do not occur. If \( \varepsilon_3 = 1 \), condition (43) is satisfied and \( \alpha = \alpha_1 X^* + \alpha_2 Y^* \). The conditions (41) and (42) are satisfied if and only if
there exist \( j_1, j_2 \in \mathbb{Z} \) such that
\[
\begin{align*}
\alpha_1 &= \frac{w_2}{r m_v} \left( j_1 + \frac{1 - \varepsilon_1}{4} \right), \\
\alpha_2 &= \frac{1}{w_2} \left[ (j_2 + \frac{1 - \varepsilon_2}{4}) - \frac{m_w}{m_v} \left( j_1 + \frac{1 - \varepsilon_1}{4} \right) \right],
\end{align*}
\]
with eigenvalues
\[
\sigma_\alpha = \left\{ -\frac{1}{4A} + 2\pi \|\alpha\|, -\frac{1}{4A} - 2\pi \|\alpha\| \right\} = \left\{ -\frac{1}{4A} + 2\pi \sqrt{\alpha_1^2 + \alpha_2^2}, -\frac{1}{4A} - 2\pi \sqrt{\alpha_1^2 + \alpha_2^2} \right\},
\]
and the multiplicity of this representation is \( m_\alpha = 1 \). If \( \alpha = 0 \) is permitted — i.e. \( \varepsilon = 1 \) — then \( H_\alpha \) is no longer irreducible, and the eigenspace corresponding to \( -\frac{1}{4A} \) is two-dimensional.

We now consider the case \( \alpha_3 \neq 0 \). By (43) we may choose \( \alpha \in g^* \) in the coadjoint orbit such that \( \alpha = dZ^* = \frac{1}{r} \left( \kappa + \frac{1 - \varepsilon_3}{4} \right) Z^* \neq 0 \) with \( \kappa \in \mathbb{Z} \), with eigenvalues
\[
\left\{ -\frac{1}{4A} - 2\pi d \right\} \cup \left\{ -\frac{1}{4A} + 2\pi \sqrt{\frac{\pi |d| p}{A} + \pi^2 d^2} : p \geq 1 \right\},
\]
or in other words
\[
\sigma_\alpha = \left\{ -\frac{1}{4A} - \frac{2\pi}{r} \left( \kappa + \frac{1 - \varepsilon_3}{4} \right) \right\} \cup \left\{ -\frac{1}{4A} \pm 2 \sqrt{\frac{\pi p}{r A} \kappa + \frac{1 - \varepsilon_3}{4} + \frac{\pi^2}{r^2} \left( \kappa + \frac{1 - \varepsilon_3}{4} \right)^2} : p \in \mathbb{Z}_{>0} \right\},
\]
and the multiplicity of this representation is
\[
m_\alpha = m_v \left| \kappa + \frac{1 - \varepsilon_3}{4} \right| > 0.
\]
A special case occurs in \( \mathbf{1} \), with \( r = T', A = (d')^2 T', m_v = r', p = p', d = \frac{r'}{T'}, m_w = 0 \), where the primes indicate the notation used in \( \mathbf{1} \).
5.2.  Eta invariant of three-dimensional Heisenberg manifolds

From (11), the eta invariant of the spin Dirac operator corresponding to a spin structure on a three-dimensional manifold is ($n = 3, \tilde{n} = 2, W$ is trivial so that $\mathrm{tr}(F^W) = 0$)
\[
\eta(0) = -\frac{\lambda^3}{3\pi^2} \mathrm{vol}(M)
+ \frac{\lambda}{24\pi^2} \int_M \mathrm{Scal} - 2\#(\sigma(D) \cap (\lambda, 0)) - \#(\sigma(D) \cap \{0, \lambda\}),
\]
where $\lambda < 0$ is the point of symmetry of the spectrum, and where the last two terms count multiplicities. (Recall the rank of the spinor bundle is two.)

We have
\[
\lambda = -\frac{1}{4A}
\]
is the point of symmetry, and from ([20 Section 2]),
\[
\mathrm{Scal} = \frac{1}{4} \mathrm{Tr}(j(Z)^2) = \frac{1}{4} \mathrm{Tr} \left( \begin{pmatrix} 0 & -1 \ A \\ \frac{1}{A} & 0 \end{pmatrix} \right)^2 = -\frac{1}{2A^2}.
\]

Also,
\[
\mathrm{vol}(M) = rA \det \begin{pmatrix} \frac{r m_v}{u_2} & \frac{r m_w}{u_2} \\ 0 & 0 \end{pmatrix} = r^2 A m_v.
\]

From the expressions for the eigenvalues of $\sigma(D)$, we see that $\#(\sigma(D) \cap \{-1/4A\})$ is nonzero only if the part of the spectrum corresponding to $\alpha = 0 \in g^*$ is nontrivial. This happens only if $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$. Thus,
\[
\#(\sigma(D) \cap \{-1/4A\}) = \begin{cases} 2 & \text{if } \varepsilon = 1 \\ 0 & \text{otherwise}. \end{cases}
\]

To count $\#(\sigma(D) \cap (-1/4A, 0))$, the toral eigenvalues, i.e. the ones from the finite-dimensional irreducible subspaces, are (see (38))
\[
\left\{ \begin{array}{l}
-\frac{1}{4A} + \tau : 0 < \tau = 2\pi \sqrt{\frac{\alpha_1^2 + \alpha_2^2}{A}} < \frac{1}{4A} \\
-\frac{1}{4A} + \tau : \tau = 2\pi \frac{\|\alpha\|}{\sqrt{A}}, 0 < \|\alpha\| < \frac{1}{8\pi\sqrt{A}}
\end{array} \right\}
\]
With fixed $r > 0, w_2 > 0, m_v \in \mathbb{Z}_{>0}, m_w \in \mathbb{Z}$, by (41), (42) the coadjoint orbit represented by $\alpha = \alpha_1 X^* + \alpha_2 Y^*$ has an associated irreducible representation that occurs with multiplicity one if and only if

$$\frac{\alpha_1 m_v}{w_2} \in \mathbb{Z} + \frac{1 - \varepsilon_3}{4}, \quad \alpha_2 w_2 \frac{r m_w}{w_2} \alpha_1 \in \mathbb{Z} + \frac{1 - \varepsilon_2}{4}.$$ 

The relevant nontoral eigenvalues, i.e. those from the infinite-dimensional irreducible subspaces, are

$$\sigma_\alpha = \left\{ -\frac{1}{4A} - \frac{2\pi}{r} \left( \kappa + \frac{1 - \varepsilon_3}{4} \right) : \kappa \in \mathbb{Z}, -\frac{r}{8\pi A} < \kappa + \frac{1 - \varepsilon_3}{4} < 0 \right\} \cup \left\{ -\frac{1}{4A} + 2\sqrt{\frac{2\pi}{r}} \frac{\kappa + \frac{1 - \varepsilon_1}{4}}{\pi} : p \in \mathbb{Z}_{>0}, \kappa \in \mathbb{Z}, \kappa \geq 0 < rp \frac{\kappa + \frac{1 - \varepsilon_1}{4}}{\pi} + \pi A \left( \kappa + \frac{1 - \varepsilon_1}{4} \right)^2 < \frac{r^2}{64\pi A} \right\},$$

with multiplicity $m_\alpha = m_v \left| \kappa + \frac{1 - \varepsilon_1}{4} \right| > 0$. Letting $\mu = \kappa + \frac{1 - \varepsilon_1}{4} \in \mathbb{Z} + \frac{1 - \varepsilon_1}{4}$, the inequality $0 < rp \frac{\kappa + \frac{1 - \varepsilon_1}{4}}{\pi} + \pi A \mu^2 < \frac{r^2}{64\pi A}$ is equivalent to

$$0 < |\mu| < \frac{r}{2\pi A} \left( \sqrt{\frac{1}{16} + p^2} - p \right),$$

so the relevant nontoral eigenvalues in the open interval $\left( -\frac{1}{4A}, 0 \right)$ associated to $D_{\mu_\alpha}$ are

$$\sigma_\alpha = \left\{ -\frac{1}{4A} - \frac{2\pi}{r} \mu : \mu \in \mathbb{Z} + \frac{1 - \varepsilon_3}{4}, -\frac{r}{8\pi A} < \mu < 0 \right\} \cup \left\{ -\frac{1}{4A} + 2\sqrt{\frac{2\pi}{r}} |\mu| + \frac{\pi^2}{r} \mu^2 : p \in \mathbb{Z}_{>0}, \mu \in \mathbb{Z} + \frac{1 - \varepsilon_1}{4}, \kappa \geq 0 \right\},$$

with multiplicities $m_\alpha = m_v |\mu|$.

In summary, summing over all coadjoint orbits whose associated irreducible representation occurs in $\rho_\varepsilon$,

$$\# \left( \sigma(D) \cap \left( -\frac{1}{4A}, 0 \right) \right) = \# \left\{ (\alpha_1, \alpha_2) : \frac{\alpha_1 m_v}{w_2} \in \mathbb{Z} + \frac{1 - \varepsilon_3}{4}, \alpha_2 w_2 + \frac{r m_w}{w_2} \alpha_1 \in \mathbb{Z} + \frac{1 - \varepsilon_2}{4}, 0 < ||\alpha|| < \frac{1}{8\pi \sqrt{A}} \right\} + m_v \sum_{\mu \in \mathbb{Z} + \frac{1 - \varepsilon_1}{4} \cap \mu < 0} |\mu| + m_v \sum_{p \in \mathbb{Z}_{>0}} \sum_{\mu \in \mathbb{Z} + \frac{1 - \varepsilon_1}{4} } 0 < |\mu| < \frac{r}{2\pi A} \left( \sqrt{\frac{1}{16} + p^2} - p \right) |\mu|.$$
Likewise,

\[
\# (\sigma(D) \cap \{0\}) = \# \left\{ (\alpha_1, \alpha_2) : \frac{\alpha_1 r m_1}{w_1}, \frac{\alpha_2 r m_2}{w_2} \in \mathbb{Z} + \frac{1-\varepsilon_4}{4}, \|\alpha\| = \frac{1}{8\pi\sqrt{A}} \right\} + \left\{ m_v |\mu| \text{ if } \mu = -\frac{r}{8\pi A} \in \mathbb{Z} + \frac{1-\varepsilon_3}{4} \right\} \]

\[+ m_v \sum_{p \in \mathbb{Z}_{>0}} \sum_{|\mu| = \frac{r}{2\pi \sqrt{A}} \left( \sqrt{\frac{1}{16} + p^2} - p \right)} |\mu|.\]

We now show that the last line produces at most two nonzero terms. If \(\mu_1, \mu_2 \in \mathbb{Z} + \frac{1-\varepsilon_4}{4}\) both satisfy \(|\mu_j| = \frac{r}{2\pi A} \left( \sqrt{\frac{1}{16} + p_j^2} - p_j \right) > 0\) and \(|\mu_1| \neq |\mu_2|\), solving for \(\frac{r}{2\pi A}\) yields

\[k \left( \sqrt{1 + 16p_2^2} + 4p_1 \right) - h \left( \sqrt{1 + 16p_2^2} + 4p_2^2 \right) = 0\]

for some positive \(h, k \in \mathbb{Z} + \frac{1-\varepsilon_4}{4}\), and

\[1 - \frac{h^2}{k^2} + 32 \frac{h}{k} p_1 p_2 - 32 \frac{h^2}{k^2} p_2^2 = 8 \frac{h}{k} \left( \frac{h}{k} p_2 - p_1 \right) \sqrt{16p_2^2 + 1}.\]

If \(\frac{h}{k} p_2 = p_1\), then the equation above implies \(p_1 = p_2\). On the other hand, if \((\frac{h}{k} p_2 - p_1)\) is not zero,

\[1 - \frac{h^2}{k^2} + 32 \frac{h}{k} p_1 p_2 - 32 \frac{h^2}{k^2} p_2^2 = 8 \frac{h}{k} \left( \frac{h}{k} p_2 - p_1 \right) = \sqrt{16p_2^2 + 1}.\]

Since the left side is rational and the right side is irrational, this is impossible.

Thus, there are at most two nonzero summands in the expression below.

\[m_v \sum_{p \in \mathbb{Z}_{>0}} \sum_{\substack{\mu \in \mathbb{Z} + \frac{1-\varepsilon_3}{4} \\
|\mu| = \frac{r}{2\pi \sqrt{A}} \left( \sqrt{\frac{1}{16} + p^2} - p \right) }} |\mu| \]

\[= \begin{cases} 
\frac{m_v r}{2\pi A} \left( \sqrt{\frac{1}{16} + p^2} - p \right) & \text{if } \frac{r}{2\pi A} \left( \sqrt{\frac{1}{16} + p^2} - p \right) \in \mathbb{Z} + \frac{1-\varepsilon_3}{4} \\
0 & \text{otherwise}
\end{cases}\]
Then

\[ \left\{ m_v | \mu | \text{ if } \mu = -\frac{r}{8\pi A} \in \mathbb{Z} + \frac{1-\varepsilon_3}{4} \right\} + m_v \sum_{p \in \mathbb{Z}_{>0}} \sum_{\mu \in \mathbb{Z} + \frac{1-\varepsilon_3}{4}} |\mu| \]

\[ = \begin{cases} 
\frac{m_v r}{8\pi A} \left( \sqrt{\frac{1}{16} + p^2} - p \right) & \text{if } \frac{r}{2\pi A} \left( \sqrt{\frac{1}{16} + p^2} - p \right) \in \mathbb{Z} + \frac{1-\varepsilon_3}{4} \\
\frac{m_v r}{8\pi A} & \text{for some } p \in \mathbb{Z}_{>0} \\
0 & \text{otherwise} 
\end{cases} \]

In summary,

\[ \# (\sigma (D) \cap \{0\}) \]

\[ = \# \left\{ (\alpha_1, \alpha_2) : \alpha_1 w_2 + \frac{\alpha_2 m_v}{w_2} \alpha_1 \in \mathbb{Z} + \frac{1-\varepsilon_3}{4}, \ \|\alpha\| = \frac{1}{8\pi A} \right\} \]

\[ + \begin{cases} 
\frac{m_v r}{8\pi A} \left( \sqrt{\frac{1}{16} + p^2} - p \right) & \text{if } \frac{r}{2\pi A} \left( \sqrt{\frac{1}{16} + p^2} - p \right) \in \mathbb{Z} + \frac{1-\varepsilon_3}{4} \\
\frac{m_v r}{8\pi A} & \text{for some } p \in \mathbb{Z}_{>0} \\
0 & \text{otherwise} 
\end{cases} \]

Putting these calculations together, we have

\[ \eta (0) = -\frac{\lambda^3}{3\pi^2} \text{ vol } (M) \]

\[ + \frac{\lambda}{24\pi^2} \int_M \text{Scal} - 2 \# (\sigma (-D) \cap (\lambda, 0)) - \# (\sigma (-D) \cap \{0, \lambda\}) \]

\[ = -\frac{(-1)^3}{3\pi^2} r^2 A m_v + \frac{(-1)^2}{24\pi^2} \left( -\frac{1}{2A^2} \right) (r^2 A m_v) \]

\[ - N (A, r, w_2, m_v, m_w, \varepsilon) \]

\[ = \frac{r^2 m_v}{192\pi^2 A^2} + \frac{r^2 m_v}{192\pi^2 A^2} - N (A, r, w_2, m_v, m_w, \varepsilon) \]

\[ = \frac{r^2 m_v}{96\pi^2 A^2} - N (A, r, w_2, m_v, m_w, \varepsilon) , \]
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where \( N(A, r, w_2, m_v, m_w, \varepsilon) \) is the nonnegative integer defined by

\[
N(\cdot) = 2\# \left( \sigma(D) \cap (\bar{x}, 0) \right) + \# \left( \sigma(D) \cap \{0, \bar{x}\} \right)
\]

\[
= 2\# \left\{ (\alpha_1, \alpha_2) : \frac{\alpha_1}{m_w} \in \mathbb{Z} + \frac{1-\varepsilon_1}{4}, \alpha_2 w_2 + \frac{r m_v}{w_2} \alpha_1 \in \mathbb{Z} + \frac{1-\varepsilon_2}{4}, 0 < \|\alpha\| < \frac{1}{8\pi\sqrt{A}} \right\}
\]

\[
+ \left( \frac{m_v}{\pi A} \right) \left( \sqrt{\frac{1}{16} + p^2} - p \right) \quad \text{if} \quad \frac{r}{\pi A} \left( \sqrt{\frac{1}{16} + p^2} - p \right) \in \mathbb{Z} + \frac{1-\varepsilon_3}{4}
\]

\[
+ \left\{ \begin{array}{ll}
\frac{m_v}{\pi A} & \text{if } \frac{r}{\pi A} \in \mathbb{Z} + \frac{1-\varepsilon_3}{4} \\
0 & \text{otherwise}
\end{array} \right.
\]

All the sums above are finite.

We summarize this result in the following theorem.

**Theorem 12.** The eta invariant of the spin Dirac operator on a three-dimensional Heisenberg manifold with parameters \( A, r, w_2 > 0, m_v \in \mathbb{Z}_{>0}, m_w \in \mathbb{Z} \) with spin structure determined by \( \varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \{\pm 1\}^3 \) satisfies

\[
\eta(0) = \frac{r^2 m_v}{96\pi^2 A^2} - N(A, r, w_2, m_v, m_w, \varepsilon),
\]

where \( N(A, r, w_2, m_v, m_w, \varepsilon) \) is the nonnegative integer given by the expression (44).

**Remark 13.** The expression above is consistent with the calculation of W. Zhang in [41], who calculates the adiabatic limit of the mod1 reduction of the eta invariant on circle bundles \( M \to B \), where the metric on the base \( B \) is blown up \( (A \to \infty) \). The Zhang formula for the case of the spin Dirac operator twisted by a line bundle \( L \) over the base (corresponding to
a different spin structure) is

\[ \lim_{A \to \infty} \eta(D_{M,L}) = \frac{1}{2} \dim \ker (D_{B,L}) + \left( \tilde{A}(TB) \text{ch}(L) \frac{\tanh \left( \frac{e}{2} \right) - \frac{e}{2}}{e \tanh \left( \frac{e}{2} \right)} \right) \mod 1, \]

where \( e \) is the Euler class of the circle bundle. In our case, the base is a flat torus, and the fibers of the circle bundle are the \( \mathbb{Z} \)-parameter curves. First, we consider the right side of the equation. The integer \( \dim \ker (D_{B,L}) \) is either 2 or zero (depending on whether the line bundle \( L \) is trivial or not), so that term is zero. The Euler class of the circle bundle is trivial, since it has a section given by \( \{(x, y, 0) : x, y \in \mathbb{R}/\mathbb{Z}\} \), and even the Euler form is zero since the global angular form (constant) \( dz \) is closed. The relevant characteristic forms are then \( \tilde{A}(TB) = 1 \), \( \text{ch}(L) = 1 + (2\text{-form}) \), \( \frac{\tanh \left( \frac{e}{2} \right) - \frac{e}{2}}{e \tanh \left( \frac{e}{2} \right)} = -\frac{e}{2} = 0 \), so the second term is also 0. The left side of Zhang’s equation is

\[ \lim_{A \to \infty} \left( \frac{r^2 m_w}{96 \pi^2 A^2} \mod 1 \right) = 0 \]

in our case, so we see that our formula is consistent with this result.

**Corollary 14.** From the expressions for \( \eta(0) \), we may consider families of Heisenberg manifolds with constant \( \eta(0) \). For example, if we let

\[ A = b_1 r \]
\[ w_2 = b_2 \sqrt{r} \]

for some constants \( b_1, b_2 > 0 \). Holding \( m_v, m_w, \varepsilon, b_1, b_2 \) constant and letting \( r \) vary, we obtain a family of Heisenberg manifolds with constant \( \eta(0) \) yet with different eigenvalues for \( D \); even the point of symmetry \( -\frac{1}{4A} \) varies with \( r \).

**Corollary 15.** Consider the “rectangular” Heisenberg 3-manifold (i.e. \( m_w = 0 \)). Suppose that the following conditions are met:

1) \( A > \frac{r}{4\pi} \)
2) \( \frac{r m_v}{4 \pi \sqrt{A}} < w_2 < 4 \pi \sqrt{A} \)
Then if the spin structure is nontrivial ($\varepsilon \neq \text{id}$),

$$\eta(0) = \frac{r^2m_v}{96\pi^2A^2}.$$  

Otherwise,

$$\eta(0) = \frac{r^2m_v}{96\pi^2A^2} - 2.$$  

### 5.3. Dirac Operator eigenvalues for general Heisenberg nilmanifolds

We use the notation of Section 4. Suppose that $k_0 = 1$ is the dimension of the center $z$ and $n = 1 + m_0$ is the dimension of $\mathfrak{g} = z \oplus \mathbb{V}$, and we will choose the orthonormal basis $\{Z, X_1, \ldots, X_{m_0}\}$, with $m_0 = 2m$, so that $Z$ is a unit vector and $\{X_j\}$ is an orthonormal basis of $\mathbb{V}$. From formula (16), the Dirac operator is

$$D = \sum_{i=1}^{n} (E_i \circ \rho \varepsilon^*) (E_i) + \frac{1}{4} \sum_{b < i \leq m_0} \langle Z, [X_b, X_i] \rangle (Z \circ X_b \circ X_i \circ),$$

acting on

$$\mathcal{H} = L^2\left(\Gamma \backslash G, G \times \varepsilon, C^k \right) \cong L^2\left(\Gamma \backslash G\right) \otimes \Sigma_n,$$

which we decompose using Kirillov theory. Using notation from Section 4, the cases are:

**Case 1:** $k_\alpha = n$, i.e. $\alpha(Z) = 0$.

As in (21),

$$D|_{\mathcal{H}_\alpha} = \sum_{i=1}^{m_0} 2\pi i \alpha (X_i) (X_i \circ) + \frac{1}{4} \sum_{b < i \leq m_0} \langle Z, [X_b, X_i] \rangle (Z \circ X_b \circ X_i \circ),$$

which is a constant matrix. The eigenvalues of $D|_{\mathcal{H}_\alpha}$ are then the eigenvalues of this Hermitian matrix.

**Case 2:** $k_\alpha < n$, so that $\alpha(Z) \neq 0$.

For every noncentral vector $v$, there exists a vector $w$ such that $B_\alpha(v, w) = \alpha([v, w]) = \alpha(Z) \neq 0$; we must have $\mathfrak{g}_\alpha = \mathfrak{z}$ and $k_\alpha = 1$. From (20), (23), (36), the Dirac operator may be expressed in terms of the basis
\[ \{\pi_{p,\ell}\} \text{ as} \]
\[ D\pi_{p,\ell} = -\sum_j 2i\sqrt{\pi d_j p_j} \ell_1 \ell_2 \cdots \ell_j \pi_{p,\ell} + M'_a \pi_{p,\ell}, \]
where in this case
\[ M'_a \pi_{p,\ell} = 2\pi i \alpha (Z) (Z \circ) \pi_{p,\ell} + \frac{1}{4} \sum_{j=1}^m \langle Z, [U_j, V_j] \rangle (Z \circ U_j \circ V_j \circ) \pi_{p,\ell}. \]

We use the matrix choices of Section 4.3 and for convenience, we let
\[ Z = i 1' \otimes \cdots \otimes 1' \]
and thus, since \( \langle Z, [U_j, V_j] \rangle = d_j \), the formulas (28) and (27) yield
\[ M'_a \pi_{p,\ell} = \left( -2\pi \alpha (Z) \ell_1 \cdots \ell_m - \frac{1}{4} \sum_{j=1}^m d_j \ell_1 \cdots \hat{\ell}_j \cdots \ell_m \right) \pi_{p,\ell}. \]

In summary,
\[ D\pi_{p,\ell} = -\sum_j 2i\sqrt{\pi d_j p_j} \ell_1 \ell_2 \cdots \ell_j \pi_{p,\ell} \]
\[ + \left( -2\pi \alpha (Z) \ell_1 \cdots \ell_m - \frac{1}{4} \sum_{j=1}^m d_j \ell_1 \cdots \hat{\ell}_j \cdots \ell_m \right) \pi_{p,\ell}, \]
and we have the following.

**Proposition 16.** The infinite-dimensional subspace \( H_\alpha \) decomposes on any Heisenberg manifold as a direct sum of finite-dimensional subspaces that are invariant by the Dirac operator. In particular, the Dirac operator acts by the formula (46) on the finite-dimensional invariant subspace
\[ U_p = \text{span} \{\pi_{p,\ell} : \ell \in \{-1, 1\}^m\}. \]

**Remark 17.** For any specific example of a Heisenberg manifold, the formula (45) allows us to calculate the eigenvalues of \( D \) restricted to the finite-dimensional representations spaces \( H_\alpha \) with \( \alpha (Z) = 0 \), and the previous proposition allows us to calculate all other eigenvalues of \( D \) explicitly.
5.4. Symmetries of invariants subspaces of higher dimensional Heisenberg manifolds

It is well-known (see [5, p.61, Remark 3a], [2, p.174, Cor. 2.19]) that the spectrum of the Dirac operator on spin manifolds of dimension congruent to 1 mod 4 is symmetric about 0. In following sections, we explore the symmetry of the spectrum restricted to invariant subspaces.

5.4.1. Symmetry in the toroidal part of the spectrum for Heisenberg manifolds. Suppose that we are given a \((2m+1)\)-dimensional Heisenberg manifold, and \(\alpha \in \mathfrak{g}^*\) is chosen so that \(\alpha(Z) = 0\). Then we may choose an orthonormal basis \(\{A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m\}\) of \(\mathfrak{g}^\perp \subseteq \mathfrak{g}\) with the following properties:

1) \(\alpha(A_j) = 0\) if \(j \geq 2\), \(\alpha(B_j) = 0\) if \(j \geq 1\);
2) \([A_i, B_j] = a_j \delta_{ij}Z\) for some real numbers \(a_j\).

(Simply choose \(A_1\) orthogonal to \(\text{ker} \alpha\) and continue to form a symplectic basis of \(\mathfrak{g}^\perp\).) Then the restriction of \(D\) to the subspace \(\mathcal{H}_\alpha\) is

\[
D|_{\mathcal{H}_\alpha} = \sum_{i=1}^{m_0} 2\pi i \alpha(X_i)(X_i\varphi) + \frac{1}{4} \sum_{b<i \leq m_0} \langle Z, [X_b, X_i] \rangle (Z \circ X_b \circ X_i\varphi) \\
= 2\pi i \alpha(A_1)(A_1\varphi) + \frac{1}{4} \sum_{j=1}^{m} a_j (Z \circ A_j \circ B_j\varphi).
\]

If \(m\) is even, then observe that \(A_1 \circ A_2 \circ \cdots \circ A_m\) anticommutes with \(D|_{\mathcal{H}_\alpha}\) and is also invertible. Thus, it maps the \(\lambda\) eigenspace of \(D|_{\mathcal{H}_\alpha}\) isomorphically onto the \(-\lambda\) eigenspace of \(D|_{\mathcal{H}_\alpha}\), and therefore the spectrum of \(D|_{\mathcal{H}_\alpha}\) is symmetric about zero and does not contribute to the \(\eta\) invariant.

A more complicated argument can be used to show that for all \(m \geq 2\), the spectrum of \(D|_{\mathcal{H}_\alpha}\) is symmetric about zero. Let

\[L_j = Z \circ A_j \circ B_j\]

for \(1 \leq j \leq m\). Observe that \(L_j\) is symmetric, \(L_j^2 = 1\), and \(L_jL_k = L_kL_j\) for all \(j, k\). Also \(A_2\) is invertible and anticommutes with \(L_1\), and \(A_1\) anticommutes with \(L_j\) for \(j > 1\). Thus, the dimension of the +1 eigenspace of \(L_j\) is the same as the dimension of the −1 eigenspace for \(L_j\), and there exists a basis of simultaneous eigenvectors of \(\Sigma_m = \mathbb{C}^{2^m}\). Let \(\{v_1, \ldots, v_{2^m-1}\}\) be the sub-
set of this basis consisting of +1 eigenvectors of \(L_2\). Since \(A_2\) commutes with
$L_2$ and anticommutes with $L_1$, the +1-eigenspace of $L_2$ is a direct sum of +1 and −1 eigenspaces of $L_1$ in equal dimensions. Thus, we may further assume that $\{v_1, \ldots, v_{2m-2}\}$ are +1-eigenvectors of $L_1$ and that $\{v_{2m-2+1}, \ldots, v_{2m-1}\}$ are −1-eigenvectors of $L_1$. Then $\{v_1, \ldots, v_{2m-1}, iA_1v_1, \ldots, iA_1v_{2m-1}\}$ provides a basis of $\mathbb{C}^{2m}$ for which $D_{|\mathcal{H}_\alpha}$ corresponds to a block matrix with $2^{m-2}$-dimensional blocks of the form

$$
x \begin{pmatrix}
Q + R & 0 & I & 0 \\
0 & -Q + R & 0 & I \\
I & 0 & Q - R & 0 \\
0 & I & 0 & -Q - R
\end{pmatrix},
$$

where $x$ is a scalar and $Q$ and $R$ are (commuting) diagonal matrices. A simple argument shows that the characteristic polynomial of such a matrix is an even function, and thus the spectrum of $D_{|\mathcal{H}_\alpha}$ is symmetric about zero and does not contribute to the eta invariant, if $m \geq 2$. We summarize the results in the following theorem.

**Theorem 18.** On any Heisenberg manifold of dimension greater than 3, the restriction of the Dirac operator to any invariant subspace $\mathcal{H}_\alpha$ with $\alpha(Z) = 0$ has spectrum that is symmetric about 0.

**Remark 19.** No Heisenberg three-manifolds have this property; see (38).

5.4.2. Symmetry in the infinite-dimensional irreducible subspaces.

Next, suppose that $\alpha \in \mathfrak{g}^*$ is chosen so that $\alpha(Z) \neq 0$. Let

$$
U = \text{span} \{\pi_{p,\ell} : p = (p_1, \ldots, p_m) \in (\mathbb{Z}_{\geq 0})^m, \ell \in \{-1, 1\}^m\}.
$$

Let $L : U \to U$ be the linear map defined by

$$
L(\pi_{p,\ell}) = \delta_\ell \pi_{p,-\ell},
$$

where $\delta_\ell = \pm 1$ according to an unspecified formula. Note that $L^{-1}(\pi_{p,\ell}) = \delta_\ell \delta_{-\ell} L\pi_{p,\ell} = \delta_{-\ell} L\pi_{p,-\ell}$. Now, we have
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\[ L^{-1}DL\pi_{p,\ell} = \delta_\ell L^{-1}D\pi_{p,-\ell} \]

\[ = \delta_\ell^{-1} \left( -\sum_j 2i/\pi d_j p_j \ell_1 \cdots \ell_j (\ell_{j+1} \cdots \ell_m - 1) p_{p,-\ell} \right) \]

\[ = \delta_\ell \left( -\sum_j 2i/\pi d_j p_j \ell_1 \cdots \ell_j (\ell_{j+1} \cdots \ell_m - 1) \pi_{p,-\ell} \right) \]

\[ = \delta_\ell \left( -\sum_j 2i (\ell_{j+1} \cdots \ell_m - 1) \pi_{p,-\ell} \right) \]

\[ + (-1)^{m-1} \left( 2\pi (Z) \ell_1 \cdots \ell_m - \frac{1}{4} \sum_{j \leq m} d_j \ell_1 \cdots \ell_j \right) \pi_{p,\ell}. \]

**Remark 20.** For the case where \( m \) is even, we define \( \delta_\ell = (\ell_1)^2 (\ell_2)^3 \cdots (\ell_m)^{m+1} \), so that \( \delta_\ell \delta_\ell (-1)^j = (-1)^{j+1} (1)^j = -1 \), and thus, the matrix for \( L^{-1}DL \) is the negative of the matrix for \( D \) with \( \alpha (Z) \) replaced by its negative. Thus the spectrum \( \sigma_\alpha \) satisfies \( \sigma_{-\alpha} = -\sigma_\alpha \) if \( m \) is even. The symmetry of the eigenvalues about 0 follows from the fact that, \( \alpha \) occurs if and only if \( -\alpha \) occurs; see (17). This confirms in this case the known fact mentioned at the beginning of this section that the spectrum of the spin Dirac operator on manifolds of dimension congruent to 1 mod 4 is symmetric.

**Remark 21.** For the case where \( m \) is odd and the dimension is \( 2m + 1 \), we define \( \delta_\ell = (\ell_1)^1 (\ell_2)^2 \cdots (\ell_m)^m \), so that \( \delta_\ell \delta_\ell (-1)^j = 1 \); we see in that case that the matrix for \( L^{-1}DL \) is the same as the matrix for \( D \) with \( \alpha (Z) \) replaced by its negative. Thus the spectrum \( \sigma_\alpha \) satisfies \( \sigma_{-\alpha} = \sigma_\alpha \) if \( m \) is odd. Moreover, the eigenvalues of the Dirac operator need not be symmetric about 0, and the eta invariant need not be zero, as can be seen from the \( m = 1 \) case in Section 2.2. Therefore, for Heisenberg manifolds of dimension \( 2m + 1 \) with \( m \) odd and greater than 1, because of this remark and Theorem 18, the methods of Section 2.1 do not apply and cannot be used to obtain a formula for the eta invariant.

6. Example of a five-dimensional non-Heisenberg nilmanifold

The purpose of this section is to exhibit an example of a two-step nilmanifold for which the techniques used above fail to produce the Dirac eigenvalues as
eigenvalues of finite-dimensional matrices. Because this manifold is $(4(1) + 1)$-dimensional, we know a priori that the eta invariant vanishes. We use the notation of Section 4 with a specific class of examples. We have that $k_0 = 2$ is the dimension of the center $z$ and $m_0 = 3$, and we have the orthonormal basis $\{Z_1, Z_2, X, Y_1, Y_2\}$ so that each $Z_j$ is a unit vector and $\{X, Y_1, Y_2\}$ is an orthonormal basis of $v$. The only nontrivial bracket relations are $[X, Y_1] = Z_1$, $[X, Y_2] = Z_2$. From formula (16), the Dirac operator is

$$ D = \sum_{i=1}^{5} \rho_{\varepsilon} (E_i) (E_i^\diamond) + \frac{1}{4} \sum_{i=1,2} Z_i \diamond X \diamond Y_i \diamond, $$

acting on

$$ \mathcal{H} = L^2(\Gamma \backslash G, G \times_\varepsilon \mathbb{C}^4) \cong L^2(\Gamma \backslash G) \otimes \Sigma_5, $$

which we decompose as follows. For $\alpha \in \mathfrak{g}^*$, the subspace $\mathcal{H}_\alpha$ of $L^2(\Gamma \backslash G, G \times_\varepsilon \mathbb{C}^k)$ is invariant with respect to $\rho_\varepsilon$ and invariant by $D$. If $\mathcal{H}_\alpha$ is the irreducible $\rho_\varepsilon$-subspace of $L^2(\Gamma \backslash G)$ corresponding to the coadjoint orbit of $\alpha$, we have $\mathcal{H}_\alpha \cong \overline{\mathcal{H}_\alpha} \otimes \Sigma_5$. As before, define the symplectic form on $\mathfrak{g}$ by $B_\alpha (u,v) = \alpha ([u,v])$, and let $\mathfrak{g}_\alpha = \ker B_\alpha = \{u \in \mathfrak{g} : B_\alpha (u, \cdot) = 0\}$, $k_\alpha = \dim \mathfrak{g}_\alpha$.

### 6.1. Finite dimensional $\mathcal{H}_\alpha$-irreducible subspaces: $k_\alpha = 5$, i.e. $\alpha (\mathfrak{z}) = 0$.

As in [21],

$$ D|_{\mathcal{H}_\alpha} = 2\pi i \alpha (X) (X^\diamond) + \sum_{j=1,2} 2\pi i \alpha (Y_j) (Y_j^\diamond) + \frac{1}{4} \sum_{i=1,2} Z_i \diamond X \diamond Y_i \diamond. $$

The eigenvalues of $D|_{\mathcal{H}_\alpha}$ are then the eigenvalues of this constant Hermitian linear transformation.

We make the specific choices of the matrices $(E_j^\diamond)$ as in Section 4.2. Note $\Sigma_5 = \mathbb{C}^{2^2} = \mathbb{C}^2 \otimes \mathbb{C}^2$. We have

- $(X^\diamond) = i1' \otimes 1'$, $(Y_1^\diamond) = i\sigma_1 \otimes 1$, $(Y_2^\diamond) = i\sigma_2 \otimes 1$, $(Z_1^\diamond) = i1' \otimes \sigma_1$, $(Z_2^\diamond) = i1' \otimes \sigma_2$. 


In this case, a typical coadjoint orbit has an element of the form
\[ \{ v, \} \]
where \( v \). Recalling (27), our matrix is (using basis \( \alpha \) )
\[
D|_{\mathcal{H}_\alpha} = 2\pi \alpha (X)(X\circ) + \sum_{j=1,2} 2\pi \alpha (Y_j)(Y_j\circ) + \frac{1}{4} \sum_{i=1,2} Z_i \circ X \circ Y_i \circ \\
= -2\pi \alpha (X) \mathbf{1}' \otimes \mathbf{1}' - \sum_{j=1,2} 2\pi \alpha (Y_j) \sigma_j \otimes \mathbf{1} \\
- \frac{i}{4} \sum_{j=1,2} (\mathbf{1}' \otimes \sigma_j) (\mathbf{1}' \otimes \mathbf{1}') (\sigma_j \otimes \mathbf{1}) \\
= -2\pi \alpha (X) \mathbf{1}' \otimes \mathbf{1}' - \sum_{j=1,2} 2\pi \alpha (Y_j) \sigma_j \otimes \mathbf{1} + \frac{1}{4} (\sigma_1 \otimes \sigma_2 - \sigma_2 \otimes \sigma_1) \\
= \left( \begin{array}{ccc}
-2\pi \alpha (X) & -2\pi \alpha (Y_1) - i2\pi \alpha (Y_2) & 0 \\
-2\pi \alpha (Y_1) + i2\pi \alpha (Y_2) & 2\pi \alpha (X) & 0 \\
0 & 0 & -2\pi \alpha (Y_1) + i2\pi \alpha (Y_2) & 2\pi \alpha (X) & 0 \\
\end{array} \right) .
\]
We may then determine that the four eigenvalues of \( D|_{\mathcal{H}_\alpha} \) are:
\[
\frac{1}{4} \pm \frac{1}{4} \sqrt{64\pi^2 \alpha (X)^2 + 16\pi \alpha (X) + 64\pi^2 \alpha (Y_1)^2 + 64\pi^2 \alpha (Y_2)^2 + 1}, \quad \frac{1}{4} \pm \frac{1}{4} \sqrt{64\pi^2 \alpha (X)^2 - 16\pi \alpha (X) + 64\pi^2 \alpha (Y_1)^2 + 64\pi^2 \alpha (Y_2)^2 + 1}.
\]
Using the \( \alpha \mapsto -\alpha \) symmetry, for a typical nilmanifold, this portion of the spectrum will be symmetric about zero.

### 6.2. Infinite-dimensional \( \mathcal{H}_\alpha \)-irreducible subspaces: \( k_\alpha < n \), so that \( \alpha (\mathfrak{g}) \neq 0 \).

In this case, a typical coadjoint orbit has an element of the form \( \alpha = b_2 Y_2^* + g_1 Z_1^* + g_2 Z_2^* \), with \( g_1, g_2 \) not both zero.

Choose a new orthonormal basis of \( \mathfrak{g} \):
\[
\left\{ W_1 = \frac{g_1 Z_1 + g_2 Z_2}{\sqrt{g_1^2 + g_2^2}}, W_2 = \frac{-g_2 Z_1 + g_1 Z_2}{\sqrt{g_1^2 + g_2^2}}, W_3 = \frac{-g_2 Y_1 + g_1 Y_2}{\sqrt{g_1^2 + g_2^2}}, U = X, V = \frac{g_1 Y_1 + g_2 Y_2}{\sqrt{g_1^2 + g_2^2}} \right\},
\]
where \( \{ W_1, W_2, W_3 \} \) is a basis of \( \mathfrak{g}_\alpha \), \( B_\alpha (U, U) = B_\alpha (V, V) = 0 \), and
\[
d := B_\alpha (U, V) = \alpha ([U, V]) = \alpha \left( \frac{g_1 Y_1 + g_2 Y_2}{\sqrt{g_1^2 + g_2^2}} \right) = \sqrt{g_1^2 + g_2^2}.
\]
Then the polarizing subalgebra $g^\alpha$ will be chosen to be

$$g^\alpha = \text{span}\{V, W_1, W_2, W_3\},$$

and again $G^\alpha := \exp(\mathfrak{g}^\alpha)$.

Equation (24) becomes

$$D = (U \diamond) \frac{\partial}{\partial t} + 2\pi i d (V \diamond) t + M'_\alpha,$$

where $M'_\alpha$ is the constant Hermitian matrix (using $X_1 = U, X_2 = V, X_3 = W_3, W_1, W_2, k_0 = 2, m_0 = 3, k_\alpha = 3, m = 1$)

$$M'_\alpha = \frac{1}{4} \sum_{a \leq 2; \ b < i \leq 3} \langle W_a, [X_b, X_i] \rangle (W_a \odot X_b \odot X_i) + \sum_{j=1}^{3} 2\pi i \alpha(W_j) (W_j),$$

from (20), (23).

We calculate

$$[X_1, X_2] = [U, V] = W_1, \ [X_1, X_3] = W_2, \ [X_2, X_3] = 0,$$

so that

$$M = \frac{1}{4} \sum_{a \leq 2; \ b < i \leq 3} \langle W_a, [X_b, X_i] \rangle (W_a \odot X_b \odot X_i)$$

$$= \frac{1}{4} W_1 \odot X_1 \odot X_2 \odot + \frac{1}{4} W_2 \odot X_1 \odot X_3 \odot.$$

Again we make the specific choices of the matrices $(E_j)$ as in Section 4.3 with

$$(X_1) = i \sigma_1 \otimes 1, \ (X_2) = i \sigma_2 \otimes 1, \ (X_3) = i \mathbf{1}' \otimes \mathbf{1}' ,$$

$$(W_1) = i \mathbf{1}' \otimes \sigma_1, \ (W_2) = i \mathbf{1}' \otimes \sigma_2,$$

Then, since $\alpha(W_2) = 0, \ \alpha(X_3) = \frac{2b_2}{d^2}, \ \alpha(W_1) = d = \sqrt{g_1^2 + g_2^2}$, we use (27) to obtain
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\[ M'_\alpha = \frac{1}{4} W_1 \circ X_1 \circ X_2 \circ + \frac{1}{4} W_2 \circ X_1 \circ X_3 \circ + \frac{1}{4} (W_1) W_1 \circ \\
+ 2\pi i \alpha (X_3) (X_3 \circ) \\
= \frac{1}{4} (i1' \otimes \sigma_1) (i\sigma_1 \otimes 1) (i\sigma_2 \otimes 1) \\
\quad + \frac{1}{4} (i1' \otimes \sigma_2) (i\sigma_1 \otimes 1) (i1' \otimes \sigma_1) \\
\quad + 2\pi i g_1 b_2 d (i1' \otimes 1') \\
= -\frac{1}{4} \otimes \sigma_1 + \frac{1}{4} \sigma_1 \otimes \sigma_1 - 2\pi d1' \otimes \sigma_1 - 2\pi g_1 b_2 d 1' \otimes 1'.
\]

We need to determine what \( M'_\alpha \) does to the basis \( \{ \pi_{p,\ell} \} \). We have

\[ \overline{\pi}_{p,\ell} = \begin{cases} u_{p,\ell} & \text{if } \ell_1 = 0 \\
\sqrt{2p} u_{p-1,\ell} & \text{if } \ell_1 = -1, \end{cases} \]

\[ u_{p,\ell} = h_p \left( \sqrt{2\pi d} \right) v_\ell \]

Then

\[ (1 \otimes \sigma_1) \pi_{p,\ell} = (1 \otimes \sigma_1) \begin{cases} u_{p,\ell} & \text{if } \ell_1 = 0 \\
\sqrt{2p} u_{p-1,\ell} & \text{if } \ell_1 = -1 \end{cases} \]

\[ = \begin{cases} u_{p,\ell} & \text{if } \ell_1 = 0 \\
\sqrt{2p} u_{p-1,\ell} & \text{if } \ell_1 = -1 \end{cases} \]

\[ = \overline{\pi}_{p,\ell}, \]

\[ (\sigma_1 \otimes \sigma_1) \pi_{p,\ell} = (\sigma_1 \otimes \sigma_1) \begin{cases} u_{p,\ell} & \text{if } \ell_1 = 0 \\
\sqrt{2p} u_{p-1,\ell} & \text{if } \ell_1 = -1 \end{cases} \]

\[ = \begin{cases} u_{p,-\ell} & \text{if } \ell_1 = 0 \\
\sqrt{2p} u_{p-1,-\ell} & \text{if } \ell_1 = -1 \end{cases} \]

\[ = \left( \sqrt{2p} \right)^{-\ell_1} \overline{\pi}_{p+\ell_1,-\ell_1}, \]

\[ (i1' \otimes \sigma_1) \pi_{p,\ell} = \ell_1 (1 \otimes \sigma_1) \pi_{p,\ell} = \ell_1 \overline{\pi}_{p,\ell}, \]

\[ (i1' \otimes 1') \pi_{p,\ell} = \ell_1 \ell_2 \overline{\pi}_{p,\ell}. \]

Substituting,
\[ M'_\alpha \pi_{p,\ell} = \left( -\frac{1}{4} \mathbf{1} \otimes \sigma_1 + \frac{1}{4} \sigma_1 \otimes \sigma_1 - 2\pi d \mathbf{1}' \otimes \sigma_1 - 2\pi \frac{g_1 b_2}{d} \mathbf{1}' \otimes \mathbf{1}' \right) \pi_{p,\ell} \]

\[ = -\frac{1}{4} \pi_{p,\ell} + \frac{1}{4} \left( \sqrt{2p} \right)^{-\ell_1} \pi_{p+\ell_1,-\ell} - 2\pi d \ell_1 \pi_{p,\ell} - 2\pi \frac{g_1 b_2}{d} \ell_1 \ell_2 \pi_{p,\ell}. \]

From (36), we have

\[ D \pi_{p,\ell} = -2i \sqrt{\pi dp \ell_1} \pi_{p,\ell} + M'_\alpha \pi_{p,\ell} \]

\[ = -2i \sqrt{\pi dp \ell_1} \pi_{p,\ell} + \left( -\frac{1}{4} - 2\pi d \ell_1 \right) \pi_{p,\ell} \]

\[ + \frac{1}{4} \left( \sqrt{2p} \right)^{-\ell_1} \pi_{p+\ell_1,-\ell} - 2\pi \frac{g_1 b_2}{d} \ell_1 \ell_2 \pi_{p,\ell}. \]

There are no apparent invariant subspaces for \( D \) spanned by a finite number of the \( \pi_{p,\ell} \). The matrix for \( D \) is an infinite band matrix. This shows the difficulty of computing the Dirac eigenvalues for a general nilmanifold.

### 7. Appendix: CCMoore/LenRichardson Papers and Adaptations

#### 7.0.1. Occurrence and Multiplicity Condition.

Let \( \Gamma \) be a cocompact (i.e., \( \Gamma \backslash G \) compact) discrete subgroup of the simply connected nilpotent Lie group \( G \). Let \( \varepsilon : \Gamma \to \{ \pm 1 \} \subset GL(\mathbb{C}^k) \) be a homomorphism. Denote by \( U_\varepsilon \) the representation of \( G \) induced by \( \varepsilon \); in particular,

\[ U_\varepsilon = L^2_\varepsilon(\Gamma \backslash G) = \left\{ f : G \to \mathbb{C}^k : f(\gamma g) = \varepsilon(\gamma) f(g) \text{ for all } g \in G, \gamma \in \Gamma \right\}, \]

where the (left) action of \( G \) on \( U_\varepsilon \) is given by interior right multiplication. Note that if \( \varepsilon = 1 \), then \( U_\varepsilon = L^2(\Gamma \backslash G) \) is the direct sum of \( k \) copies of the quasi-regular representation \( U = L^2(\Gamma \backslash G) \). As in the quasi-regular case, standard results in representation theory imply in general that \( U_\varepsilon \) can be decomposed into the direct sum of irreducible representations of \( G \), each with finite multiplicity. A good reference for the standard representation theory used in this appendix is [17].
Our motivation for this construction is that spin structures over nilmanifolds $\Gamma \backslash G$ correspond exactly to homomorphisms $\varepsilon : \Gamma \to GL(C^k)$, where the image of $\varepsilon$ lies in the set $\{\pm 1\}$, and $k = 2^{\lfloor n/2 \rfloor}$. The resulting spinor bundle is

$$\Sigma_\varepsilon = G \times_\varepsilon C^k = G \times C^k / \{ (g,v) : (g,v) = (\gamma g, \varepsilon(\gamma)v) \text{ for all } \gamma \in \Gamma \};$$

see [9, Prop 3.34, p. 114]. The sections of this bundle are elements of $U_\varepsilon$, on which the Dirac operator acts.

In the quasi-regular case ($\varepsilon = 1$), L. Richardson and R. Howe, building on work of C. C. Moore, independently proved an exact occurrence condition and multiplicity formula; they determined the irreducible representations $\pi$ of $G$ that occur in $U = L^2(\Gamma \backslash G)$ and the corresponding multiplicities $m(\pi, U)$. The purpose of this appendix is to generalize their occurrence and multiplicity formula from the quasi-regular to the case of general $\varepsilon$.

Before stating the main results, we require the following definitions and observations.

Denote by $\hat{G}$ the set of equivalence classes of irreducible unitary representations of $G$. The Kirillov Correspondence is the bijection between the set of orbits of the co-adjoint action of $G$ on $g^*$ and $\hat{G}$. In particular, Kirillov Theory proves that to each $\alpha \in g^*$ corresponds an irreducible unitary representation $\pi_\alpha$ of $G$, every irreducible representation of $G$ is unitarily equivalent to such a $\pi_\alpha$, and two such irreducible unitary representations $\pi_\alpha$ and $\pi_{\alpha'}$ are unitarily equivalent if and only if $\alpha' = \alpha \circ \text{Ad}(x^{-1})$ for some $x \in G$. Kirillov Theory applies mainly to nilpotent Lie groups, with generalizations to some solvable groups.

Choose $\alpha \in g^*$ and let $h$ be any subalgebra of $g$. Let $H = \exp(h)$ be the unique simply connected Lie subgroup of $G$ with Lie algebra $h$. The subalgebra $h$ or the subgroup $H$ is subordinate to $\alpha$ iff $\alpha([h, h]) \equiv 0$. If in addition $h$ is maximal with respect to being subordinate, then $h$ is called a maximal subordinate subalgebra for $\alpha$, or a polarizer for $\alpha$.

The explicit mapping between elements of $g^*$ and $\hat{G}$ is as follows. Since $G$ is nilpotent and simply connected, the exponential map is a diffeomorphism with inverse log. For $\alpha \in g^*$, let $h$ be a maximal subordinate subalgebra of $\alpha$. Define $\overline{\alpha}(\cdot) = e^{2\pi i \alpha (\log(\cdot))}$, which is a character on $H$ — i.e., a (complex) one-dimensional representation. The irreducible unitary representation $\pi_\alpha$ is the representation of $G$ induced by the representation $\overline{\alpha}$ of $H$.

Recall that we have fixed a cocompact, discrete subgroup $\Gamma$ of $G$. We call the pair $(\overline{\alpha}, H)$ rational (with respect to $\Gamma$) if it can be constructed with respect to a rational covector $\alpha$, i.e. $\alpha(\log \Gamma) \subset \mathbb{Q}$. We call the pair $(\overline{\alpha}, H)$ a
special maximal pair if \( \log H = \mathfrak{h} \) is a maximal subordinate subalgebra for \( \alpha \) that is special in the sense that it is algorithmically and inductively constructed from \( \alpha \) and \( \Gamma \) as described in [38, pp. 176-178]. As Kirillov theory dictates that the representation \( \pi_\alpha \) is independent of the maximal subordinate subalgebra (up to unitary equivalence), and as Richardson’s paper shows that any covector \( \alpha \) has a special maximal subordinate subalgebra, this additional property is not a restriction. We call \( (\alpha, H) \) an \( \varepsilon \)-integral point if and only if for all \( \gamma \in \Gamma \cap H \),
\[
\alpha (\log \gamma) = \varepsilon (\gamma).
\]
The equivalent condition on the Lie algebra level is, for all \( \gamma \in \Gamma \cap H \),
\[
\alpha (\log \gamma) \in \begin{cases} 
Z & \text{if } \varepsilon (\gamma) = 1 \\
Z + \frac{1}{2} & \text{if } \varepsilon (\gamma) = -1.
\end{cases}
\]

Let \( \pi \in \hat{G} \) be induced from \( \alpha \in \mathfrak{g}^* \) under the Kirillov correspondence. Let \( F \) be the family of special maximal characters of \( \pi \), that is all possible pairs \( (\overline{\alpha}, H) \) that induce \( \pi \) with \( \mathfrak{h} = \log (H) \) a special maximal subordinate subalgebra. Now L. Richardson proved that \( x \in G \) acts on \( F \) via
\[
(\overline{\alpha}, H) \cdot x = (\overline{\alpha}^x, x^{-1}H),
\]
where \( I_x \) denotes conjugation by \( x \), the function \( \overline{\alpha}^x = \overline{\alpha} \circ I_x \), and \( x^{-1}H = x^{-1}Hx = I_{x^{-1}}(H) \). Note that \((\overline{\alpha}, H) \cdot x\) is an \( \varepsilon \)-integral point if and only if
\[
\overline{\alpha}^x (\gamma) = \varepsilon (\gamma)
\]
for all \( \gamma \in \Gamma \cap (x^{-1}H) \) if and only if \( \overline{\pi} (\gamma) = \varepsilon (\gamma) \) for all \( \gamma \in (x^{-1}\Gamma) \cap H \).

We may now state the following main results of this Appendix.

**Theorem 22.** If \( \pi \) is induced by the special maximal character \((\overline{\alpha}, H)\) under the Kirillov correspondence, then \( m (\pi, L^2_\varepsilon (\Gamma \setminus G)) > 0 \) if and only if there is an \( \varepsilon \)-integral point in the orbit \((\overline{\alpha}, H) \cdot G\).

**Lemma 23.** Assume that \( m (\pi, L^2_\varepsilon (\Gamma \setminus G)) > 0 \), and let the special maximal character \((\overline{\alpha}, H)\) induce \( \pi \) under the Kirillov correspondence. The action satisfies \((\overline{\alpha}, H) \cdot x = (\overline{\alpha}, H), \) iff \( x \in H \), so that we may identify the \( G \)-orbit of \((\overline{\alpha}, H)\) with \( H \setminus G \). If \((\overline{\alpha}, H)\) is an \( \varepsilon \)-integral point and if \( \gamma_0 \in \Gamma \), then \((\overline{\alpha}, H) \cdot \gamma_0\) is also an \( \varepsilon \)-integral point.

Let \( (H \setminus G)_\varepsilon \) be the set of \( \varepsilon \)-integral points in \( H \setminus G \). As a result of the Lemma, \( \Gamma \) acts on \((H \setminus G)_\varepsilon \).
Theorem 24. If the special maximal character \( (\pi, H) \) induces \( \pi \) under the Kirillov correspondence, then the multiplicity of \( \pi \) in the \( \epsilon \)-quasi regular representation \( U_\epsilon = L^2_\epsilon (\Gamma \backslash G) \), denoted \( m(\pi, L^2_\epsilon (\Gamma \backslash G)) \), is the number of \( \Gamma \)-orbits in the set \( (H \backslash G)_\epsilon \) of \( \epsilon \)-integral points in the \( G \)-orbit \( H \backslash G \) of \( (\pi, H) \). That is,

\[
m(\pi, L^2_\epsilon (\Gamma \backslash G)) = \# ((H \backslash G)_\epsilon \backslash \Gamma).
\]

7.0.2. Proof of Occurrence and Multiplicity. The proofs of the Lemma and Theorems follows the outline in L. Richardson’s paper closely. We verify that a few key Lemmas of C. C. Moore extend to the \( \epsilon \)-quasi-regular setting, and from there the proof primarily follows that of L. Richardson verbatim, after substituting our Lemmas for those of Moore, and replacing “integral point” with “\( \epsilon \)-integral point.”

For any \( \pi \in \hat{G} \), suppose there exists \( \gamma_0 \in \Gamma \) such that \( N = \exp (\mathbb{R} \log (\gamma_0)) \) is a one-dimensional rational normal subgroup of \( G \) and \( \pi(N) = 1 \). Let \( \varphi \) be the natural projection of \( G \) onto \( G_0 = G/N \), so \( \Gamma_0 = \Gamma \cdot N/N = \varphi(\Gamma) \) is a cocompact discrete subgroup of \( G_0 \). Then the representation \( \pi_0 \) of \( G_0 \) defined by \( \pi_0(\varphi(g)) = \pi(g) \) is well-defined and irreducible, hence an element of \( \hat{G}_0 \) (see [35, Lemma 2.1]).

Lemma 25. Generalized \( \epsilon \)-Reduction Lemma. (generalization of [35, Lemma 2.2], quoted as [38, Lemma 2.6])

Note that \( \epsilon \) induces a homomorphism of \( \Gamma_0 \) iff \( \epsilon(\gamma_0) = 1 \). With notation as above, denote by \( U_{0\epsilon} \) the representation of \( G_0 \) induced by the \( \epsilon \)-homomorphism of \( \Gamma_0 \), if it exists. If \( m(\pi, U_\epsilon) \neq 0 \) then \( \epsilon(\gamma_0) = 1 \), and the multiplicity \( m(\pi, U_\epsilon) \) of \( \pi \) in \( U_\epsilon \) is equal to the multiplicity \( m(\pi_0, U_{0\epsilon}) \) of \( \pi_0 \) in \( U_{0\epsilon} \).

Proof. By normality, \( N \subset Z(G) \). This follows from the Campbell-Baker-Hausdorff formula, since for vectors \( A \in \log G \), and \( X \in \log N \), we have

\[
\exp(A) \exp(X) \exp(A^{-1}) = \exp(rX)
\]

\[
= \exp(X + [A,X] + c_2 [A,[A,X]] + \cdots).
\]

Let \( \text{ad}(A)^k(X) \) be the first zero element of the sequence

\[
(X, [A,X], [A,[A,X]], \ldots).
\]

Because \( G \) is nilpotent, \( \{X, [A,X], [A,[A,X]], \ldots, \text{ad}(A)^{k-1}(X)\} \) is linearly independent. Since \( rX = X + [A,X] + c_2 [A,[A,X]] + \cdots \), we have
\[ [A, X] = 0. \] Note that since \( \pi(N) = 1 \), if \( m(\pi, U_\varepsilon) \neq 0 \), \( U_\varepsilon(n)f = f \) for all \( n \in N \) and \( f \) in the corresponding invariant subspace \( \mathcal{H}_\pi \). This means that while \( f(\gamma g) = \varepsilon(\gamma)f(g) \) for all \( g \in G, \gamma \in \Gamma \), it must also be true that \( (U_\varepsilon(n)f)(g) = f(gn) = f(ng) = f(g) \) for all \( n \in N \). If \( n \in \Gamma \cap N \), then in addition we have \( f(gn) = f(ng) = f(g) = \varepsilon(n)f(g) \), which implies that \( \varepsilon(\gamma) \) is the identity and \( \varepsilon \) induces a homomorphism of \( \Gamma_0 \). Thus \( \varepsilon \) restricted to \( \Gamma \cap N \) acts trivially on the image of the sections of \( \mathcal{H}_\pi \). Let \( M = \Gamma \setminus G \).

We can project \( M \) onto \( M_0 = \Gamma_0 \setminus G_0 \), and \( M \) becomes a fiber bundle over \( M_0 \) with circle \( T \cong (\Gamma \cap N) \setminus N \) as fiber. Let

\[
\mathcal{H}^N = \left\{ f : G \to \mathbb{C}^k : f(\gamma g) = \varepsilon(\gamma)f(g) \right\},
\]

which is the set of sections on \( M \) that are constant on the fibers of \( M \to M_0 \), i.e. such that \( U_\varepsilon(n)f = f \) for all \( n \in N \). This is an invariant subspace of \( U_\varepsilon \), because for such \( f, U_\varepsilon(n)U_\varepsilon(g)f = U_\varepsilon(g)U_\varepsilon(n)f = U_\varepsilon(g)f \) for all \( g \in G, n \in N \). The projection of the space of all sections onto \( \mathcal{H}_\pi \) lies in the center of the commuting algebra of \( U_\varepsilon \); that is, the projection of \( U_\varepsilon \) onto an invariant subspace must commute everything that commutes with \( U_\varepsilon \), because if \( L \) commutes with \( U_\varepsilon \), then \( \mathcal{H}^N \) is also an invariant subspace of \( L \), and thus the projection onto \( \mathcal{H}_\pi^N \) commutes with \( L \). Let \( U_N \) be the restriction of \( U_\varepsilon \) to \( \mathcal{H}_\pi^N \), and we define \( U_{N_0}(\varphi(n)) = U_N(n) \), the corresponding representation of \( G_0 \). Using the realization of \( U_{N_0} \) on sections of \( M \) that are constant on the fibers, we can also realize \( U_{N_0} \) on the space \( L^2(M_0, \Sigma_\varepsilon) \). It is clear that \( U_{N_0} \) is equivalent to \( U_{0_\varepsilon} \). We also have \( m(\pi, U_\varepsilon) = m(\pi, U_N) \), since \( \pi \) is trivial on \( N \), and \( m(\pi, U_{N_0}) = m(\pi_0, U_{N_0}) = m(\pi_0, U_{0_\varepsilon}) \), as desired.

Lemma 26. (Pukansky, as stated in \cite[Lemma 2.2]{38}) Let \( \mathfrak{g} \) be a nilpotent Lie algebra with one dimensional center \( \mathfrak{z} = \mathbb{R}Z_1 \), with \( G \) and \( \Gamma \) as above. Then \( \mathfrak{g} = \mathbb{R}X_1 \oplus \mathbb{R}Y_1 \oplus \mathbb{R}Z_1 \oplus \mathfrak{g}' \), where \( [X_1, Y_1] = Z_1 \). Let \( \mathfrak{g}_1 = \mathbb{R}Y_1 \oplus \mathbb{R}Z_1 \oplus \mathfrak{g}' = \{ X \in \mathfrak{g} : [X, Y_1] = 0 \} \). The elements \( Y_1, Z_1 \) may be chosen to lie in \( \log \Gamma \); i.e., \( \mathfrak{g}_1 \) may be chosen to be rational with respect to the cocompact discrete subgroup \( \Gamma \) of \( G \).

Theorem 27. Kirillov’s Theorem (as quoted in \cite[Theorem 2.3]{38})

If \( G \) has one dimensional center, then every irreducible representation \( \pi \) of \( G \) such that \( \pi \) is non-constant on the center is induced by a necessarily irreducible representation of \( G_1 = \exp(\mathfrak{g}_1) \), with \( \mathfrak{g}_1 \) as in the previous Lemma.
Definition 28. We call the subgroup $G_1$ from the previous theorem a Kirillov subgroup.

Theorem 29. $\varepsilon$-Generalized Moore’s Algorithm (generalization of Moore’s Algorithm, quoted as [38, Moore’s Algorithm 2.7]).

Let $\pi$ be an irreducible representation of $G$, where $G$ has a one dimensional center $Z(G)$, and $\pi|_{Z(G)} \neq \text{id}$. Let $\pi_1$ be an irreducible representation of $G_1$, a rational Kirillov subgroup of $G$ having codimension one, such that $\pi_1$ induces $\pi$. Define $\pi_1^\dagger(x_1) = \pi_1(xx_1^{-1})$ for $x$ in $G$ and $x_1$ in $G_1$. Let $U_1\varepsilon$ be defined for $G_1$ and $\Gamma_1 = \Gamma \cap G_1$ as $U_\varepsilon$ is defined for $G$ and $\Gamma$. Let $\hat{G}_1$ denote the dual space of equivalence classes of unitary irreducible representations of $G_1$. Let $A' = \{\rho_1 \in \hat{G}_1 : m(\rho_1, U_1\varepsilon) > 0 \text{ and } \rho_1|_{Z(G)} \neq \text{id}\}$. For all $\gamma \in \Gamma$, since $G_1$ is normal and $\gamma\Gamma\gamma^{-1} = \Gamma, U_1\varepsilon \cong U_1\varepsilon$. Thus $m(\rho_1^\dagger, U_1\varepsilon) = m(\rho_1, U_1\varepsilon)$, or $\{\rho_1^\dagger : \rho_1 \in A'\} = A'$. Let $A$ be a subset of $A'$ that meets each $\Gamma$-orbit in $A'$ in exactly one element. Then

$$m(\pi, U_\varepsilon) = \sum_{\rho_1 \in \pi_1^\dagger \cap A} m(\rho_1, U_1\varepsilon).$$

Proof. The proof closely follows that in [38, pp. 151–153].

Let $Z_2(G)$ be the subgroup of $G$ such that $Z_2(G)/Z(G)$ is the center of $G/Z(G)$. The group $Z_2(G)$ is a rational subgroup of $G$ (with respect to any lattice), see for example [17, Chapter 5], and we may choose a rational subgroup $W$ of $Z_2(G)$ (and $G$) of dimension 2 that contains $Z(G)$. The centralizer $G_0$ of $W$ then has codimension 1 in $G$ and is a rational normal subgroup (see [38, Lemma 2.2], quoted from [28]). Finally, since $G_0$ is codimension one and normal, we can find a rational one-parameter group $S$ such that $G = G_0 \times S$.

We now use the following, whose proof can be found in any book on Kirillov theory. Denote by $U_0\varepsilon$ the representation of $G_0$ induced by the homomorphism $\varepsilon$.

Lemma 30. If $\pi \leq \hat{G}$ and if $\pi$ is nontrivial on $Z(G)$, then $\pi$ is induced by some $\pi_0 \leq \hat{G}_0$. The set of all representations of $G_0$ that induce $\pi$ is the orbit of $\pi_0$ under $G$; that is, \{\pi_0^x : x \in G\} and $\pi_0^y = \pi_0^x$ if $x = y \mod G_0$, where $\pi_0^x = \pi_0 \circ i_x$. If $\overline{\pi}$ is the restriction of $\pi$ to $G_0$, then $\overline{\pi} = \int_{G/G_0} \pi_0^x \, dx = \int_S \pi_0^x \, dx'$, where $dx$ and $dx'$ refer to Haar measure in $G/G_0 \cong S$.

Now let $(U_\varepsilon)^\dagger$ be the subspace of $U_\varepsilon$ complementary to the stabilizer of $U_\varepsilon|_{Z(G)}$. The projection onto the subspace corresponding to $(U_\varepsilon)^\dagger$ is in the
center of the commuting algebra of $U_\varepsilon$ (see similar argument in the proof of Lemma 25). Thus if $\pi$ is nontrivial on $Z(G)$ and occurs in $U_\varepsilon$, then it occurs in $(U_\varepsilon)^s$ just as often. Thus,

$$U_\varepsilon = \sum_{\pi \in \hat{G}} m(\pi, U_\varepsilon) \pi$$

$$(U_\varepsilon)^s = \sum_{\pi \in B} m(\pi, U_\varepsilon) \pi,$$

where $B$ is the subset of $\hat{G}$ consisting of those $\pi$ that are nontrivial on $Z(G)$ and such that $m(\pi, U_\varepsilon) > 0$. For each $\pi \in B$, choose a $\pi_0 \in \hat{G}_0$ that induces $\pi$. If $(U_\varepsilon)^s$ is the restriction of $(U_\varepsilon)^s$ to $G_0$,

$$\overline{(U_\varepsilon)^s} = \sum_{\pi \in B} m(\pi, U_\varepsilon) \pi$$

$$= \sum_{\pi \in B} m(\pi, U_\varepsilon) \int_{G/G_0} \pi_0^x dx.$$

On the other hand, we can decompose $U_\varepsilon$, the restriction of $U_\varepsilon$ to $G_0$, by using Mackey’s subgroup theorem. Indeed, let $U_{0e}^\varepsilon$ be the representation of $G_0$ induced by the $\varepsilon$-representation of $x \Gamma x^{-1} \cap G_0 = (\Gamma \cap G_0) x^{-1}$ (since $G_0$ is normal). Note that as $x$ is fixed, we can extend the definition of $\varepsilon$ to $x \Gamma x^{-1}$. It is clear that $U_{0e}^\varepsilon$ is the conjugate by $x$ of $U_{0e}$; i.e., $U_{0e}^\varepsilon(n) = (U_{0e})(nx^{-1})$. Then by Mackey’s Theorem ([32, Theorem 12.1]), $U_{0e}^\varepsilon$ depends only on the double coset $\Gamma \cdot x \cdot G_0$ of $x$. But $G_0$ is normal, and $\Gamma \cdot x \cdot G_0 = \Gamma \cdot G_0 \cdot x$ is a coset of the subgroup $\Gamma G_0$. We know that $\Gamma G_0$ is closed (basic fact about nilpotent groups: $\Gamma$ is cocompact discrete, $G_0$ is normal in $G$), and thus the double cosets fill out the group, allowing us to apply Mackey’s Theorem.

Finally, (also by Mackey)

$$\overline{U_\varepsilon} = \int_{G_0 \setminus G} U_{0e}^y dy.$$}

Now, if $(U_\varepsilon)^s$ is the part of $\overline{U_\varepsilon}$ that is orthogonal to the stabilizer of $Z(G)$, then $(U_\varepsilon)^s = \overline{(U_\varepsilon)^s}$, since the center is in $G_0$. Finally, if $(U_{0e})^s$ is the similar subrepresentation of $U_{0e}$ on which $Z(G)$ acts nontrivially, then one immediately deduces from the above that

$$(U_\varepsilon)^s = \overline{(U_\varepsilon)^s} = \int_{G_0 \setminus G} (U_{0e})^y dy.$$
We write
\[(U_0^\varepsilon)^s = \sum_{\lambda_0 \in A'} m(\lambda_0, U_{0\varepsilon}) \lambda_0,\]
where \(A'\) is the set of elements of \(\hat{G}_0\) that do not vanish on \(Z(G)\) and for which \(m(\lambda_0, U_{0\varepsilon}) > 0\). We are using the fact that \(m(\lambda_0, U_{0\varepsilon}) = m(\lambda_0, (U_{0\varepsilon})^s)\) for \(\lambda_0 \in A'\).

If \(\gamma \in \Gamma\), then \(\gamma \Gamma^{-1} \cap G_0 = \Gamma \cap G_0\), and from this it follows that \(((U_{0\varepsilon})^s)^\gamma = (U_{0\varepsilon})^s\). Therefore, we have \(m(\lambda_0^\gamma, U_{0\varepsilon}) = m(\lambda_0, U_{0\varepsilon})\), and thus \(\gamma : A' = A'\). Now let \(A\) be a subset of \(A'\) such that \(A\) meets each orbit of \(\Gamma\) on \(A'\) in exactly one element. Since \(G_0\) acts trivially on \(\hat{G}_0\) and hence on \(A'\), a \(\Gamma G_0\)-orbit in \(A'\) is just a \(\Gamma\)-orbit in \(A'\). Moreover, \(G_0\) (by Kirillov) is the subgroup of \(\Gamma G_0\) leaving any point in \(A'\) fixed. Therefore, we can write
\[(U_{0\varepsilon})^s = \sum_{\lambda_0 \in A} m(\lambda_0, U_{0\varepsilon}) \sum_{s \in \Gamma \cdot G_0 \setminus G_0} \lambda_0^s,\]
and thus
\[(U_{0\varepsilon})^s = \sum_{\lambda_0 \in A} m(\lambda_0, U_{0\varepsilon}) \int_{\Gamma \cdot G_0 \setminus G_0} \left( \sum_{s \in \Gamma \cdot G_0 \setminus G_0} \lambda_0^s \right)^y \, dy.\]

But since \(G/G_0\) is equivalent as a Borel space and measure space to \(\Gamma \cdot G_0 \setminus G \times (G_0 \setminus \Gamma \cdot G_0)\) by choosing a Borel cross section, the representation in square brackets is just
\[\int_{G/G_0} \lambda_0^y \, dx.\]
Thus,
\[(U_{0\varepsilon})^s = \sum_{\lambda_0 \in A} m(\lambda_0, U_{0\varepsilon}) \int_{G/G_0} \lambda_0^y \, dx.\]

Now, since \(G_0\) is type I and direct integral decompositions are essentially unique, we may equate coefficients in \([18]\) and \([17]\). We find then that the family of orbits \(\{\pi_0^G : \pi_0 \in B\}\) and \(\{\lambda_0^G : \lambda_0 \in A\}\) are the same. Moreover, the orbits of \(\pi_0^G\) are all distinct, whereas some of the orbits of \(\lambda_0^G\) may coincide. Thus, we can equate the multiplicities as follows:
\[m(\pi, U_{\varepsilon}) = \sum_{\lambda_0 \in \pi_0^G \cap A} m(\lambda_0, U_{0\varepsilon}).\]

(End of generalized Moore Algorithm Proof) □
Corollary 31. Under the conditions of Moore’s algorithm, \( m(\pi, U_\varepsilon) > 0 \) if and only if there is an irreducible representation \( \pi_1 \) of the rational Kirillov subgroup \( G_1 \) such that \( m(\pi_1, U_{1\varepsilon}) > 0 \) and \( \pi = \text{Ind}_{G_1}^{G}(\pi_1) \).

Remark 32. Abelian case:
Suppose \( \Gamma \) is a lattice in \( G = \mathbb{R}^n \), given by generators \( \gamma_1, \gamma_2, \ldots, \gamma_n \). The coadjoint orbit of any \( \alpha \in g^* \) is \( \{\alpha\} \), and the maximal abelian subalgebra is \( h = g = \mathbb{R}^n \). By the Kirillov correspondence this implies that irreducible representations of \( G \) are characters \( x \mapsto e^{2\pi i \alpha(x)} \) of \( G \) determined by elements \( \alpha \in g^* = (\mathbb{R}^n)^* \). Such an \( \alpha \) occurs as a representation induced by \( \varepsilon \) if
\[
e^{2\pi i \alpha(\gamma)} = \varepsilon(\gamma)
\]
for all \( \gamma \in \Gamma \).

This condition occurs exactly when \( \alpha(\gamma) \in \mathbb{Z} \) whenever \( \varepsilon(\gamma) = 1 \) and \( \alpha(\gamma) \in \mathbb{Z} + \frac{1}{2} \) when \( \varepsilon(\gamma) = -1 \); i.e., the pair \( (\alpha, H) \) is an \( \varepsilon \)-integral point.

This means that there exists \( k_j, l_j \in \mathbb{Z} \) such that
\[
\alpha = \sum_j k_j \gamma_j^* + \sum_{j, \varepsilon(\gamma_j) = -1} \left( \frac{1}{2} + l_j \right) \gamma_j^*,
\]
where \( \{\gamma_1^*, \gamma_2^*, \ldots, \gamma_n^*\} \) is the basis of \( g^* \) dual to \( \{\gamma_1, \gamma_2, \ldots, \gamma_n\} \). So \( \pi_\alpha \) can be written as
\[
\pi_\alpha(t) = \prod_j e^{2\pi ik_j t_j} \prod_{j, \varepsilon(\gamma_j) = -1} e^{\pi i (2l_j + 1) t_j},
\]
with \( t = \sum t_j \gamma_j \in \mathbb{C}^n \).

We now prove Theorem 22, the \( \varepsilon \)-generalized Richardson occurrence condition.

Proof. Forward Direction:
Suppose \( H \) has codimension zero. This implies that \( \alpha([g, g]) = 0 \), by the definition of maximal subordinate subalgebra. By possibly repeated application of Lemma 25, we can factor out \( [g, g] \), and the occurrence and multiplicity remain unchanged. This reduces the problem to the abelian case, which is proved in Remark 32.

We now proceed inductively on the codimension of \( H \): assume that the theorem is known for codimension \( k - 1 \) or less. Now suppose \( \pi \in \hat{G} \) and that \( m(\pi, U_\varepsilon) > 0 \) if and only if there is an irreducible representation \( \pi_1 \) of the rational Kirillov subgroup \( G_1 \) such that \( m(\pi_1, U_{1\varepsilon}) > 0 \) and \( \pi = \text{Ind}_{G_1}^{G}(\pi_1) \).
Cases:

1) Suppose that \( \pi = 1 \) on \( Z(G) \). Since the center is always a rational subalgebra (for nilpotent groups, for any cocompact lattice), then we pick a one-dimensional rational subgroup \( N \subset Z(G) \) on which \( \pi \) is trivial. Then we can apply Lemma 25 and we have reduced the codimension of \( H \) by one.

2) Suppose that \( \pi \) acts nontrivially on \( Z(G) \neq G \) and that \( \dim (Z(G)) > 1 \). We have that \( U_\varepsilon (z) \) is multiplication by \( \varepsilon (z) \) for all \( z \in \Gamma \cap Z(G) \) by the definition of \( U_\varepsilon \). Write \( \pi = \pi_\lambda \) for some rational \( \lambda \in g^* \). Since the kernel of \( \lambda \) restricted to \( \mathfrak{g} \) is rational and at least dimension one, we can pick a one-dimensional rational subgroup \( N \subset Z(G) \) on which \( \pi \) is trivial. We now apply Lemma 25 and reduce the codimension of \( H \) by one.

3) Suppose that \( \pi \) acts nontrivially on \( Z(G) \neq G \) and that \( \dim (Z(G)) = 1 \). Let \( G_1 \) be the rational Kirillov subgroup of \( G \) corresponding to \( \pi \), and note that the codimension of \( G_1 \) is 1 and \( H \subset G_1 \), by construction. Let \( U_{1\varepsilon} \) be the restriction of \( U_\varepsilon \) to \( G_1 \). By Corollary 31 there is an irreducible representation \( \pi_1 \) of \( G_1 \) such that \( m (\pi_1, U_{1\varepsilon}) > 0 \) and \( \pi_1 \) induces \( \pi \). Let \( \pi_1' = \text{Ind}_{H_1}^G \pi_1 \), which then induces \( \pi \), and \( \pi_1' \) is also an irreducible representation of \( G_1 \) by the Kirillov theory. But \( \pi_1 \) must be equivalent to \( \pi_1''(\cdot) = (\pi_1')^\varepsilon (\cdot) : = \pi_1 (x(\cdot)x^{-1}) \) for some \( x \in G \) by the Kirillov correspondence. Since \( m (\pi_1'', U_{1\varepsilon}) > 0 \), there exists \( g_1 \in G_1 \) such that \( f \circ \text{Ad} (x) \circ \text{Ad} (g_1) : \log (\Gamma \cap G_1) \to \mathbb{Q} \) (again, see [33 Cor. 2, p. 154]). Note that we do not know that \( \langle \pi, H \rangle \cdot x \) is maximal. Write \( \log (xg_1) = aX_1 + P_1 \), where \( P_1 \in \mathfrak{g}_1 \) and \( X_1 \) is the first external vector for \( \mathfrak{h} \), as in the construction of the special maximal subordinate subalgebra in [38 Section 3]. Note that \( Y_1 \in \log (\Gamma) \) from the construction satisfies \( [X_1, Y_1] = Z_1 \in \log (\Gamma) \), which generates \( Z(G) \). Since \( g_1 \) is the centralizer \( C(Y_1, g) \), we have

\[
\alpha \circ \text{Ad} (x) \circ \text{Ad} (g_1) (Y_1)
\]

\[
= \alpha \left( Y_1 + [aX_1 + P_1, Y_1] + \frac{1}{2} [aX_1 + P_1, [aX_1 + P_1, Y_1]] \cdots \right)
\]

\[
= \alpha (Y_1 + a [X_1, Y_1] + 0 + 0 + \cdots)
\]

\[
= \alpha (Y_1 + aZ_1),
\]

by the Campbell-Baker-Hausdorff formula. Since \( Y_1 \in \log (\Gamma) \), \( \alpha (Y_1 + aZ_1) = \alpha (Y_1) + a\alpha (Z_1) \in \mathbb{Q} \), but since \( Y_1, Z_1 \in \log (\Gamma) \) we have \( a \in \mathbb{Q} \).
Let $g_0 = \exp(aX_1)$. Then $(\bar{\alpha}, H) \cdot g_0$ induces $\rho_1$ on $G_1$, which induces $\pi$ on $G$, where $(\bar{\alpha}, H) \cdot g_0$ is a rational maximal character on $G_1$ and $m(\rho_1, U_{1\varepsilon}) > 0$. By construction, $(\bar{\alpha}, H) \cdot g_0$ is maximal.

By the induction hypothesis, there is an $\varepsilon$-integral point in $(\bar{\alpha}, H) \cdot g_0 \cdot G_1$, so that $(\bar{\alpha}, H) \cdot G$ has an $\varepsilon$-integral point.

**Converse:**

Suppose $(\bar{\alpha}, H) \cdot G$ has an $\varepsilon$-integral point $(\bar{\alpha}, H) \cdot g_0$. As above, we reduce to the case where the dimension of the center is 1 and $\pi$ restricted to $Z(G)$ is nontrivial. We know that the Kirillov subgroup $G_1$ is normal in $G$, so our $\varepsilon$-integral point $(\bar{\alpha}, H) \cdot g_0$ induces $\pi_{\varepsilon}^0$, which induces $\pi^0$, which is equivalent to $\pi$. Also, $(\bar{\alpha}, H)$ induces $\pi_1$, which induces $\pi$, and $(\bar{\alpha}, H) \cdot g_0$ is a maximal character in $G_1$. It follows from the induction hypothesis that $m(\pi_{\varepsilon}^0, U_{1\varepsilon}) > 0$ which by Moore’s induction implies that $m(\pi, U_\varepsilon) > 0$. □

Assume that $m(\pi, L^2(\Gamma \backslash G)) > 0$, and $(\bar{\alpha}, H)$ induces $\pi$, where $\log(H)$ is a special maximal subordinate subalgebra to $\alpha$ with respect to $\Gamma$. See [38, Section 3] for the construction for the special subordinate subalgebra.

**Lemma 33.** If $x = \exp(X)$, and if $(\bar{\alpha}, H) \cdot x = (\alpha, H)$, then $x \in H$.

**Proof.** See [38, Section 5], Lemma 5.1. The proof holds verbatim. □

As a result, we may identify the $G$-orbit of $(\bar{\alpha}, H)$ with $H \backslash G$.

**Lemma 34.** If $(\bar{\alpha}, H)$ is an $\varepsilon$-integral point and if $\gamma_0 \in \Gamma$, then $(\bar{\alpha}, H) \cdot \gamma_0$ is an $\varepsilon$-integral point.

**Proof.** Consider $(\bar{\alpha}, H) \cdot \gamma_0$. Note that $\Gamma \cap \gamma_0^{-1} H = \gamma_0^{-1} (\Gamma \cap H)$ since $\gamma_0^{-1} \Gamma = \Gamma$. But if $\gamma_0^{-1} \gamma \gamma_0 \in \Gamma \cap \gamma_0^{-1} H$, then $(\bar{\alpha}, H) (\gamma_0^{-1} \gamma \gamma_0) = (\bar{\alpha}, (\gamma) = \varepsilon(\gamma) = \varepsilon(\gamma_0^{-1} \gamma \gamma_0)$ for every $\gamma \in \Gamma \cap H$. Also, $\gamma_0^{-1} (\Gamma \cap H)$ is uniform in $\gamma_0^{-1} H$. □

Let $(H \backslash G)_\varepsilon$ be the set of $\varepsilon$-integral points in $H \backslash G$. As a result of the second Lemma, $\Gamma$ acts on $(H \backslash G)_\varepsilon$.

We now prove Theorem 24.

**Proof.** The proof of [38, Theorem 5.3] goes through, replacing the reference to Lemma 2.6 with Lemma 25 and the reference to Lemma 2.7 with Theorem 29, and replacing the phrase “integral point” with “$\varepsilon$-integral point”. □
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