Binary Bell polynomials approach to the integrability of nonisospectral and variable-coefficient nonlinear equations

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Abstract. Recently, Lembert, Gilson et al proposed a lucid and systematic approach to obtain bilinear Bäcklund transformations and Lax pairs for constant-coefficient soliton equations based on the use of binary Bell polynomials. In this paper, we would like to further develop this method with new applications. We extend this method to systematically investigate complete integrability of nonisospectral and variable-coefficient equations. In addition, a method is described for deriving infinite conservation laws of nonlinear evolution equations based on the use of binary Bell polynomials. All conserved density and flux are given by explicit recursion formulas. By taking variable-coefficient KdV and KP equations as illustrative examples, their bilinear formulism, bilinear Bäcklund transformations, Lax pairs, Darboux covariant Lax pairs and conservation laws are obtained in a quick and natural manner. In conclusion, though the coefficient functions have influences on a variable-coefficient nonlinear equation, under certain constraints the equation turn out to be also completely integrable, which leads us to a canonical interpretation of their $N$-soliton solutions in theory.

Keywords: binary Bell polynomial; variable-coefficient equation; bilinear Bäcklund transformation; Lax pair; Darboux covariance; conservation law.

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1. Introduction

In many physical situations, it is often preferable to have an equation with variable-coefficients, which may allow us to describe real phenomena in physical and engineering fields. For example, variable-coefficient nonlinear Schrodinger-typed ones, which describe such situations more realistically than their constant-coefficient counterparts, in plasma physics, arterial mechanics and long-distance optical communications [1]-[5]. Many physical and mechanical situations governed by variable-coefficient KdV (vc-KdV) equation, e.g., the pulse wave propagation in blood vessels and dynamics in the circulatory system, matter waves and nonlinear atom optics enhanced by the observations of Bose-Einstein condensation in the weakly interacting atomic gases, the nonlinear excitations of a Bose gas of impenetrable bosons with longitudinal confinement, the nonlinear waves in types of rods [6]-[11]. In recent years, there has been considerable interest in the study of variable-coefficient nonlinear equations, such as vc-KdV, vc-KP, vc-Schrödinger, vc-Boussinesq and cylindrical KdV equations. Recent progress in the investigation of the complete integrability and exact solutions for such equations via Painleve analysis, inverse scattering transformation, Hirota bilinear method and Darboux transformation has been reported, the details can be seen in reference and references therein [14]-[21]. It is obvious that variable-coefficient equations are often more complicated and difficult to be solved than constant-coefficient ones. As well-known, investigation of integrability for a nonlinear equation can be regarded as a pre-test and the first step of its exact solvability. There are many significant properties, such as
Lax pairs, infinite conservation laws, infinite symmetries, Hamiltonian structure, Painlevé test that can characterize integrability of nonlinear equations. This may pave the way for constructing their exact solutions explicitly in a future. But in contrast to constant-coefficient cases, very little of detail is known about complete integrability of variable-coefficient nonlinear equations.

Among the direct algebraic methods applicable to nonlinear partial differential equations in soliton theory, there is one which has proved particularly powerful: the bilinear method developed by Hirota [27, 28]. Once a nonlinear equation is written in bilinear forms by a dependent variable transformation, then multi-soliton solutions are usually obtained [29]-[34]. The search for a Hirota representation of a given nonlinear equation is generally recognized as an important first step in the construction of multi-soliton solutions. Yet, the construction of such bilinear Bäcklund transformation is not as one would wish. It relies on a particular skill in using appropriate exchange formulas which are connected with the linear representation of the system. Recently, Lembert, Gilson et al proposed an alternative procedure based on the use of Bell polynomials which enable one to obtain parameter families of bilinear Bäcklund transformation for soliton equations in a lucid and systematic way [36]-[38]. The Bell polynomials are found to play an important role in the characterization of bilinearizable equations. As a consequence bilinear Bäcklund transformation with single field can be linearize into corresponding Lax pairs. Their method provides a shortest way to bilinear Bäcklund transformation and Lax pairs of nonlinear equations, which
establishes a deep relation between integrability of a nonlinear equation and the Bell polynomials.

The problem that we consider in this paper is to further develop this method with new applications. We extend the binary Bell polynomials approach to a large class of nonisospectral and variable-coefficient equations, such as nonisospectral and variable-coefficient KdV and KP equations etc. One of the many remarkable properties that deemed to characterize soliton equations is existence of an infinite sequence of conservation laws. Here we propose a approach to construct infinite conservation laws of nonlinear evolution equations through decoupling binary Bell polynomials into a Riccati type equation and a divergence type equation. As illustrative examples, the bilinear representations, bilinear Bäcklund transformations, Lax pairs and infinite conservation laws of the vc-KdV and vc-KP equations are obtained in a quick and natural manner. The integrable constraint conditions on the variable-coefficient functions can be naturally found in the procedure of applying binary Bell polynomials. We can also find that though the coefficient functions have influences on a variable-coefficient equation, under certain constrains the equation still can admit many integrability properties which are similar to those of its standard constant-coefficient equation. The organization of this paper is as follows. In section 2, we briefly present necessary notations on multi-dimensional binary Bell polynomial that will be used in this paper. In the sections 3 and 4, we deal with integrability of non-isospectral vc-KdV equation and vc-KdV equation, respectively. We aim at integrability of nonisospectral vc-KP equation and vc-KP equation in the sections 5 and 6, respectively.
2. Multi-dimensional binary Bell polynomials

The main tool used in this paper is a class of the Bell polynomials, named after E. T. Bell [35]. To make our presentation easy understanding and self-contained, we simply recall some necessary notations on the Bell polynomials, the details refer, for instance, to Lembert and Gilson’s work [36]-[38].

Let \( f = f(x_1, \cdots, x_n) \) be a \( C^\infty \) function with multi-variables, the following polynomials

\[
Y_{n_1x_1, \cdots, n_\ell x_\ell}(f) \equiv Y_{n_1, \cdots, n_\ell}(f_{r_1x_1, \cdots, r_\ell x_\ell}) = e^{-f} \partial_{x_1}^{n_1} \cdots \partial_{x_\ell}^{n_\ell} e^f
\]

is called multi-dimensional Bell polynomials, in which we denote that

\[ f_{r_1x_1, \cdots, r_\ell x_\ell} = \partial_{x_1}^{r_1} \cdots \partial_{x_\ell}^{r_\ell} f, \quad r_1 = 0, \cdots, n_1; \cdots; r_\ell = 0, \cdots, n_\ell. \]

For example, for the simplest case \( f = f(x) \), the associated one-dimensional Bell polynomials read

\[
Y_1(f) = f_x, \quad Y_2(f) = f_{2x} + f_x^2, \quad Y_3(f) = f_{3x} + 3f_x f_{2x} + f_x^3, \cdots.
\]

For \( f = f(x, t) \), the of associated two-dimensional Bell polynomials are

\[
Y_{x,t}(f) = f_{x,t} + f_x f_t, \quad Y_{2x,t}(f) = f_{2x,t} + f_{2x} f_t + 2f_{x,t} f_x + f_x^2 f_t, \cdots.
\]

Based on the use of above Bell polynomials, the multi-dimensional binary Bell polynomials can be defined as follows

\[
Y_{n_1x_1, \cdots, n_\ell x_\ell}(v, w) = Y_{n_1, \cdots, n_\ell}(f) \mid \begin{cases} v_{r_1x_1, \cdots, r_\ell x_\ell}, & r_1 + \cdots + r_\ell \text{ is odd}, \\ w_{r_1x_1, \cdots, r_\ell x_\ell}, & r_1 + \cdots + r_\ell \text{ is even}, \end{cases}
\]

which inherit the easily recognizable partial structure of the Bell polynomials.
The first few lowest order binary Bell Polynomials are

\[ Y_x(v) = v_x, \quad Y_{2x}(v, w) = w_{2x} + v_x^2, \quad Y_{x,t}(v, w) = w_{xt} + v_xv_t. \]  
\[ Y_{3x} = v_{3x} + 3v_xw_{2x} + v_x^3, \ldots. \]  
\[ (2.1) \]

The link between \( Y \)-polynomials and the standard Hirota bilinear equation \( D_{x_1}^{n_1} \cdots D_{x_\ell}^{n_\ell}F \cdot G \) can be given by an identity

\[ Y_{n_1x_1, \ldots, n_\ell x_\ell}(v = \ln F/G, w = \ln FG) = (FG)^{-1}D_{x_1}^{n_1} \cdots D_{x_\ell}^{n_\ell}F \cdot G, \]  
\[ (2.2) \]
in which \( n_1 + n_2 + \cdots + n_\ell \geq 1 \). In the particular case when \( G = F \), the formula (2.2) becomes

\[ F^{-2}D_{x_1}^{n_1} \cdots D_{x_\ell}^{n_\ell}F \cdot F = Y_{n_1x_1, \ldots, n_\ell x_\ell}(0, q = 2 \ln F) = \begin{cases} 
0, & n_1 + \cdots + n_\ell \text{ is odd,} \\
& \\
P_{n_1x_1, \ldots, n_\ell x_\ell}(q), & n_1 + \cdots + n_\ell \text{ is even,} 
\end{cases} \]  
\[ (2.3) \]
in which the \( P \)-polynomials can be characterized by an equally recognizable even part partitional structure

\[ P_{2x}(q) = q_{2x}, \quad P_{x,t}(q) = q_{xt}, \quad P_{4x}(q) = q_{4x} + 3q_{2x}^2, \quad P_{6x}(q) = q_{6x} + 15q_{2x}q_{4x} + 15q_{2x}^3, \ldots. \]  
\[ (2.4) \]
The formulae (2.2) and (2.3) will prove particularly useful in connecting nonlinear equations with their corresponding bilinear equations. This means that once a nonlinear equation is expressible as a linear combination of \( P \)-polynomials, then it can be transformed into a linear equation.
It follows that the binary Bell polynomials $\mathcal{Y}_{n_1 x_1, \ldots, n_\ell x_\ell}(v, w)$ can be separated into $P$-polynomials and $Y$-polynomials

$$(FG)^{-1} D_{x_1}^{n_1} \cdots D_{x_\ell}^{n_\ell} F \cdot G = \mathcal{Y}_{n_1 x_1, \ldots, n_\ell x_\ell}(v, w)\big|_{v=\ln F/G, w=\ln FG} = \mathcal{Y}_{n_1 x_1, \ldots, n_\ell x_\ell}(v, v + q)\big|_{v=\ln F/G, q=2 \ln G}$$

$$= \sum_{n_1 + \cdots + n_\ell = \text{even}} \sum_{r_1 = 0}^{n_1} \cdots \sum_{r_\ell = 0}^{n_\ell} \prod_{i=1}^{\ell} \binom{n_i}{r_i} P_{r_1 x_1, \ldots, r_\ell x_\ell}(q) Y_{(n_1-r_1)x_1, \ldots, (n_\ell-r_\ell)x_\ell}(v). \quad (2.5)$$

The key property of the multi-dimensional Bell polynomials

$$Y_{n_1 x_1, \ldots, n_\ell x_\ell}(v)\big|_{v=\ln \psi} = \frac{Y_{n_1 x_1, \ldots, n_\ell x_\ell}}{\psi}, \quad (2.6)$$

implies that the binary Bell polynomials $\mathcal{Y}_{n_1 x_1, \ldots, n_\ell x_\ell}(v, w)$ can still be linearized by means of the Hopf-Cole transformation $v = \ln \psi$, that is, $\psi = F/G$. The formulae (2.5) and (2.6) will then provide the shortest way to the associated Lax system of nonlinear equations.

We start with construction of infinite conservation laws by virtue of binary Bell polynomials. We define a new auxiliary field variable

$$\eta = (q_1' x_k - q x_k)/2,$$

where $q'$ and $q$ are given by $q' = w + v$, $q = w - v$ and $x_k$ is a appropriate variable chosen from $x_1, \cdots, x_\ell$. The two-filed condition

$$C(q', q) = E(q') - E(q) = 0 \quad (2.7)$$

can be regarded as the natural ansatz for a bilinear Bäcklund transformation. By expressing the two-filed condition (2.7) in terms of binary Bell $\mathcal{Y}$-polynomials and their derivatives, we expect that the resulting condition is then decoupled into a pair of constrains, i.e. often a Riccati
type equation with respect to $x_k$,

$$\eta_{x_k} + f(\eta) = 0,$$  \hspace{1cm} (2.8)

and a divergence-type equation

$$\partial_{x_1} F_1(\eta) + \cdots + \partial_{x_l} F_l(\eta) = 0.$$  \hspace{1cm} (2.9)

The recursion formulas of conversed density come from the equation (2.8), the formulas of associated flux are obtained by using the equation (2.9). It is often the case that the first few conservation laws of a non-linear equation have a physical interpretation.

3. Nonisospectral variable coefficient KdV equation

Consider nonisospectral vc-KdV equation [14]

$$u_t + h_1(u_{3x} + 6uu_x) + 4h_2 u_x - h_3(2u + xu_x) = 0,$$  \hspace{1cm} (3.1)

where $h_1 = h_1(t)$, $h_2 = h_2(t)$ and $h_3 = h_3(t)$ are all arbitrary functions with respect to time variable $t$. The equation (3.1) includes some governing physical equations as special reduction, such as celebrated constant-coefficient KdV equation

$$u_t + 6uu_x + u_{3x} = 0,$$

cylindrical KdV equation ($h_1 = 1, \ h_2 = 1/8t, \ h_3 = 0$) [13]

$$u_t + 6uu_x + u_{3x} + \frac{1}{2t} u_x = 0,$$  \hspace{1cm} (3.2)

and the vc-KdV equation ($h_1 = 1, \ h_2 = c_0/4, \ h_3 = -\gamma$)

$$u_t + 6uu_x + u_{3x} + \gamma u + [(c_0 + \gamma x)u]_x = 0,$$
which describes the effect of relaxation inhomogeneous medium \[15\]. It can be observed that the equation (3.1) is invariant under Galiean transformation

\[ u \rightarrow u + \lambda, \quad x \rightarrow x + 6\lambda t, \quad t \rightarrow t. \]

The inverse scattering transformation of the equation (3.1) was considered by Chan and Li \[14\]. Lou and Ruan obtained infinite conservation laws \[17\]. Here we shall investigate the integrability of the equation (3.1) from bilinear representation, Bäcklund transformation, Lax pair, Darboux covariant Lax pair and infinite conservation laws.

**3.1. Bilinear representation**

In order to detect its existence of linearizable representation, we introduce a potential field \( q \) by setting

\[ u = c(t)q_{2x}, \tag{3.3} \]

with \( c = c(t) \) being free function to be the appropriate choice such that the equation (3.1) connect with \( P \)-polynomials. Substituting (3.3) into (3.1) and integrating with respect to \( x \) yields

\[ E(q) \equiv q_{xt} + h_1(q_{4x} + 3cq_{2x}^2) + 4h_2q_{2x} - h_3(q_x + xq_{2x}) + q_x \partial_t \ln c = 0. \tag{3.4} \]

Comparing the second term of this equation together with the formula (2.4), we require \( c(t) = 1 \). The result equation is then cast into a combination form of \( P \)-polynomials

\[ E(q) = P_{xt}(q) + h_1P_{4x}(q) + 4h_2P_{2x}(q) - h_3(xP_{2x}(q) + q_x) = 0. \tag{3.5} \]

Making a change of dependent variable

\[ q = 2 \ln F \quad \leftrightarrow \quad u = cq_{2x} = 2(\ln F)_{2x} \]
and noting the property (2.3), the equation (3.5) gives the bilinear representation as follows

\[(D_x D_t + h_1 D_x^4 + 4h_2 D_x^2 - xh_3 D_x^2 - h_3 \partial_x)F \cdot F = 0,\]

in which we have used the notation \(\partial_x F \cdot F = \partial_x F^2 = 2FF_x\). This equation is easy to be solved for multi-soliton solutions by using Hirota’s bilinear method. For example, the regular one-soliton like solution reads

\[u = \frac{k^2}{2} \text{sech}^2 \left( \frac{kx}{2} + \omega \right),\]

where \(k = k(t)\) and \(\omega = \omega(t)\) are two functions about \(t\), given by

\[k(t) = \alpha e^{\int h_3 dt}, \quad \omega(t) = -\int (h_1 k^3 + 4h_2 k) dt.\]

The multi-soliton solution are omitted here since exactly solving the equation (3.1) is not our main purpose in this paper.

### 3.2. Bäcklund transformation and Lax pair

Next, we search for the bilinear Bäcklund transformation and Lax pair of the vc-KdV equation (3.1). Let \(q\) and \(q'\) be two different solutions of the equation (3.4), respectively, we associate the two-field condition

\[E(q') - E(q) = (q' - q)_{xt} + h_1[(q' - q)_{4x} + 3(q' + q)_{2x}(q' - q)_{2x}] + 4h_2(q' - q)_{2x} - h_3[(q' - q)_x + x(q' - q)_{2x}] = 0.\]  

(3.5)

This two-field condition can be regarded as the natural ansatz for a bilinear Bäcklund transformation and may produce the required transformation under appropriate additional constraints.

To find such constraints, we introduce two new variables

\[v = (q' - q)/2, \quad w = (q' + q)/2,\]

(3.6)
and rewrite the condition (3.5) into the form

\[
E(q') - E(q) = v_{xt} + h_1(v_{4x} + 6v_{2x}w_{2x}) + 4h_2v_{2x} - h_3(v_x + xv_{2x})
= \partial_x[\mathcal{Y}_t(v) + h_1\mathcal{Y}_3(v, w)] + R(v, w) = 0,
\]

with

\[
R(v, w) = 3h_1\text{Wronskian}[\mathcal{Y}_{2x}(v, w), \mathcal{Y}_x(v)] + \partial_x[4h_2\mathcal{Y}_x(v) - xh_3\mathcal{Y}_x(v)].
\]

In order to decouple the two-field condition (3.7) into a pair of constraints, we impose such a constraint which enable us to express \( R(v, w) \) as the \( x \)-derivative of a combination of \( \mathcal{Y} \)-polynomials. The simplest possible choice of such constraint may be

\[
\mathcal{Y}_{2x}(v, w) + \alpha \mathcal{Y}_x(v) = \lambda,
\]

where \( \alpha \) and \( \lambda \) are arbitrary parameters. On account of the equation (3.8), then \( R(v, w) \) can be rewritten in the form

\[
R(v, w) = \partial_x[3h_1\lambda \mathcal{Y}_x(v) + 4h_2\mathcal{Y}_x(v) - xh_3\mathcal{Y}_x(v)].
\]

Then from (3.7)-(3.9), we deduce a coupled system of \( \mathcal{Y} \)-polynomials

\[
\begin{align*}
\mathcal{Y}_{2x}(v, w) + \alpha \mathcal{Y}_x(v) - \lambda &= 0, \\
\partial_x\mathcal{Y}_t(v) + \partial_x[h_1\mathcal{Y}_3(v, w) + (3h_1\lambda + 4h_2 - xh_3)\mathcal{Y}_x(v)] &= 0.
\end{align*}
\]

where prefer the second equation in the conserved form without integration with respect to \( x \), which is useful to construct conservation laws later. By application of the identity (2.2), the system (3.10) immediately leads to the bilinear Bäcklund transformation

\[
(D_x^2 + \alpha D_x - \lambda)F \cdot G = 0,
\]

\[
[D_t + h_1D_x^3 + (3h_1\lambda + 4h_2 - xh_3)D_x + \beta]F \cdot G = 0,
\]

where \( \beta \) is a arbitrary parameter.
By transformation \( \nu = \ln \psi \), it follows from the formulae (2.5) and (2.6) that
\[
\mathcal{Y}_x(v) = \psi_x/\psi, \quad \mathcal{Y}_{2x}(v, w) = q_{2x} + \psi_{2x}/\psi,
\]
\[
\mathcal{Y}_{3x}(v, w) = 3q_{2x}\psi_x/\psi + \psi_{3x}/\psi, \quad \mathcal{Y}_t(v) = \psi_t/\psi,
\]
on account of which, the system (3.10) is then linearized into a Lax pair with double parameters about \( \lambda \) and \( \beta \)
\[
L_1 \psi \equiv (\partial_x^2 + \alpha \partial_x + q_{2x})\psi = \lambda \psi, \quad \lambda_t = 2h_3 \lambda, \tag{3.11}
\]
\[
(\partial_t + L_2) \psi \equiv [\partial_t + h_1 \partial_x^3 + 3h_1 (q_{2x} + \lambda) \partial_x + (4h_2 - xh_3) \partial_x] \psi \tag{3.12}
\]
or equivalently replacing \( q_{2x} \) by \( u \),
\[
\psi_{2x} + \alpha \psi_x + (u - \lambda) \psi = 0, \quad \lambda_t = 2h_3 \lambda,
\]
\[
\psi_t + [h_1 (2u + 4\lambda + \alpha^2) + 4h_2 - xh_3] \psi_x + [\beta - h_1 u_x - \alpha h_1 (u - \lambda)] \psi = 0.
\]
Starting from this Lax pair, the Darboux transformation and soliton-like solutions of the vc-KdV equation (3.1) can be established [14]. It is easy to check that the integrability condition
\[
[L_1 - \lambda, \partial_t + L_2] \psi = 0
\]
is satisfied if \( u \) is a solution of the vc-KdV equation (3.1) and nonisospectral condition \( \lambda_t = 2h_3 \lambda \) holds.

### 3.4. Infinite conservation laws

Finally, we show how to derive the infinite conservation laws for vc-KdV equation (3.1) based on the use of the binary Bell polynomials. The conservation laws actually have been hinted in the two-filed constraint system (3.10), which can be rewritten in the conserved form
\[
\mathcal{Y}_{2x}(v, w) + \alpha \mathcal{Y}_x(v) - \lambda = 0,
\]
\[
\partial_v \mathcal{Y}_x(v) + \partial_x [h_1 \mathcal{Y}_{3x}(v, w) + (3h_1 \lambda + 4h_2 - xh_3) \mathcal{Y}_x(v)] = 0. \tag{3.13}
\]
by applying the relation \( \partial_x \mathcal{Y}_t(v) = \partial_t \mathcal{Y}_{x}(v) = \nu_{xt} \).
By introducing a new potential function

$$\eta = (q'_x - q_x)/2,$$

it follows from the relation (3.6) that

$$v_x = \eta, \quad w_x = q_x + \eta. \quad (3.14)$$

Substituting (3.14) into (3.13), we get a Riccati-type equation

$$\eta_x + \eta^2 + q_{2x} = \lambda = \varepsilon^2, \quad (3.15)$$

and a divergence-type equation

$$\eta_t + \partial_x [h_1 \eta_{2x} + 6h_1(\eta + \varepsilon)\varepsilon^2 - 2h_1(\eta + \varepsilon)^3 + (4h_2 - xh_3)(\eta + \varepsilon)] = 0, \quad (3.16)$$

where we have used the equation (3.15) to get the equation (3.16) and set $\lambda = \varepsilon^2$.

To proceed, inserting the expansion

$$\eta = \varepsilon + \sum_{n=1}^{\infty} I_n(q, q_x, \cdots) \varepsilon^{-n}, \quad (3.17)$$

into the equation (3.15) and equating the coefficients for power of $\varepsilon$, we then obtain the recursion relations for $I_n$

$$I_1 = -p_x = -\frac{1}{2} u, \quad I_2 = \frac{1}{4} p_{2x} = \frac{1}{4} u_x,$$

$$I_{n+1} = -\frac{1}{2}(I_{n,x} + \sum_{k=1}^{n} I_k I_{n-k}), \quad n = 2, 3, \cdots, \quad (3.18)$$

By applying the nonisospectral condition

$$\lambda_t = 2h_3 \lambda \implies \varepsilon_t = h_3 \varepsilon,$$
then substituting (3.17) into (3.16) yields

\[
\sum_{n=1}^{\infty} I_{n,t} \varepsilon^{-n} + \partial_x \left[ h_1 \sum_{n=1}^{\infty} I_{n,2x} \varepsilon^{-n} - 6h_1 \varepsilon (\sum_{n=1}^{\infty} I_n \varepsilon^{-n})^2 - 2h_1 (\sum_{n=1}^{\infty} I_n \varepsilon^{-n})^3 \right. \\
+ (4h_2 - xh_3) \sum_{n=1}^{\infty} I_n \varepsilon^{-n} - h_3 \sum_{n=1}^{\infty} n \partial_x^{-1} I_n \varepsilon^{-n} \right] = 0,
\]

which provides us infinite consequence of conservation laws

\[
I_{n,t} + F_{n,x} = 0, \; n = 1, 2, \ldots. \tag{3.19}
\]

In the equation (3.19), the conversed densities \( I'_n \)’s are given by formula (3.15) and the fluxes \( F'_n \)’s are given by recursion formulas explicitly

\[
F_1 = -\frac{1}{2} \left[ h_1 (u_{2x} + 3u^2) + 4h_2 u - h_3 (xu + \partial_x^{-1} u) \right], \\
F_2 = \frac{1}{4} \left[ h_1 (u_{3x} + 6uu_x) + 4h_2 u_x - h_3 (xu_x + 2u) \right], \\
F_n = h_1 I_{n,2x} - 6h_1 \sum_{k=1}^{n} I_k I_{n+1-k} - 2h_1 \sum_{i+j+k=n} I_i I_j I_k + (4h_2 - xh_3) I_n \\
+ nh_3 \partial_x^{-1} I_n, \; n = 3, 4, \ldots. \tag{3.20}
\]

We present recursion formulas for generating an infinite sequence of conservation laws for each equation, the first few conserved density and associated flux are explicit. The first equation of conservation law equation (3.19) is exactly the vc-KdV equation (3.1). The expressions (3.20) indicate that the fluxes \( F'_n \)’s of the vc-KdV equation are not local, which are different from those of standard constant-coefficient KdV equation. In conclusion, the vc-KdV equation (3.1) is complete integrable in the sense that it admits bilinear Bäcklund transformation, Lax pair and infinite conservation laws.
A more general example, we consider vc-KdV equation \[20\]

\[u_t + h_1 u_{3x} + h_2 u u_x + h_3 u_x + h_4 u = 0, \quad (4.1)\]

where \(h_j = h_j(t), \ j = 1, 2, 3, 4\) are all arbitrary functions with respect to time variable \(t\). Special cases of the equation (4.1) include cylindrical equation \[22\]-\[24\]

\[u_t + f(t) u u_x + g(t) u_{3x} = 0\]

and other special variable-coefficient equation \[25, 26\]

\[u_t + a t^n u u_x + b t^m u_{3x} = 0.\]

Recently, Zhang et al obtained the bilinear form, Bäcklund transformation and exact solutions for the equation (4.1) under the constrain \[20\]

\[h_1 = c_0 h_2 e^{- \int h_4 dt}. \quad (4.2)\]

Here we construct bilinear representation, Bäcklund transformation, Lax pair and conservation laws of the equation (4.1) based on the use of binary Bell polynomials technique, which will be seen to be a natural way to find such a constraint (4.2). We find that the bilinear representation of the equation (4.1) existence without need of any constraint. The constraint only need it for construction of the Bäcklund transformation, Lax pair and conservation laws.

4.1. Bilinear representation

As before, we introduce a field \(q\) by setting

\[u = c(t) q_{2x}, \quad (4.3)\]
in which \( c = c(t) \) is free function to be determined. Substituting (4.3) into (4.1) and integrating with respect to \( x \) yields

\[
E(q) \equiv q_{xt} + h_1 q_{4x} + \frac{1}{2} h_2 c q_{2x}^2 + h_3 q_{2x} + (h_4 + \partial_t \ln c) q_x = 0. \tag{4.4}
\]

which can be cast into a combination form of \( P \)-polynomials by using the formula (2.4)

\[
E(q) = P_{xt}(q) + h_1 P_{4x}(q) + h_3 P_{2x}(q) + (h_4 + \partial_t \ln h_1 h_2^{-1}) q_x = 0, \tag{4.5}
\]

if one chooses the function \( c(t) = 6h_1 h_2^{-1} \).

By transformation

\[
q = 2 \ln F \quad \iff \quad u = c(t) q_{2x} = 12h_1 h_2^{-1} (\ln F)_{2x}
\]

and using the property (2.3), then the equation (4.5) implies the bilinear form for the vc-KdV equation (4.1) as follows

\[
[D_x D_t + h_1 D_x^4 + h_3 D_x^2 + (h_4 + \partial_t \ln h_1 h_2^{-1}) \partial_x] F \cdot F = 0, \tag{4.6}
\]

which is obviously more general than that obtained in [20], since we have no any constraint on the \( h_1, h_2, h_3 \) and \( h_4 \). Starting the bilinear equation (4.6), we can get multi-soliton solutions to the vc-KdV equation (4.1).

For example, one-soliton solution takes the form

\[
u = 6h_1 h_2^{-1} k^2 \text{sech}^2 \frac{k x + \omega(t)}{2},
\]

where \( k \) is a constant and \( \omega(t) \) given by

\[
\omega(t) = - \int (k^3 h_1 + kh_3 + h_4 + \partial_t \ln h_1 h_2^{-1}) dt.
\]

4.2. Bäcklund transformation and Lax pair

In order to obtain the bilinear Bäcklund transformation and Lax pairs of the equation (4.1), let \( q, q' \) be two solutions of the equation (4.4) and
consider the associated two-field condition
\[ E(q') - E(q) = (q' - q)_t + h_1[(q' - q)_x + 3(q' + q)_2 (q' - q)_2] + h_3(q' - q)_2 + (h_4 + \partial_t \ln h_1 h_2^{-1})(q' - q)_x = 0, \] (4.7)
which may produce the required bilinear Bäcklund transformation under an appropriate additional constraint. By introducing variables
\[ v = (q' - q)/2, \quad w = (q' + q)/2 \] (4.8)
we can rewrite the condition (4.7) as the form
\[ E(q') - E(q) = v_t + h_1(v_{4x} + 6v_{2x}w_{2x}) + h_3 v_{2x} + (h_4 + \partial_t \ln h_1 h_2^{-1})v_x = \partial_x[Y_t(v) + h_1 Y_{3x}(v, w)] + R(v, w) = 0, \] (4.9)
with
\[ R(v, w) = 3h_1 \text{Wronskian}[Y_{2x}(v, w), Y_x(v)] + h_3 v_{2x} + (h_4 + \partial_t \ln h_1 h_2^{-1})v_x. \]
In order to express \( R(v, w) \) as the \( x \)-derivative of a linear combination of \( Y \)-polynomials, we choose a constraint
\[ Y_{2x}(v, w) + \alpha Y_x(v) = \lambda, \] (4.10)
where \( \alpha \) and \( \lambda \) are arbitrary parameters. Direct calculation gives
\[ R(v, w) = 3h_1 \lambda v_{2x} + h_3 v_{2x} + (h_4 + \partial_t \ln h_1 h_2^{-1})v_x, \]
which can be written as \( x \)-derivative of \( Y \)-polynomials
\[ R(v, w) = \partial_x[3h_1 \lambda Y_x(v) + h_3 Y_x(v) + (h_4 + \partial_t \ln h_1 h_2^{-1})v]. \] (4.11)
From (4.10)-(4.11), we infer that
\[ Y_{2x}(v, w) + \alpha Y_x(v) - \lambda = 0, \]
\[ \partial_x Y_t(v) + \partial_x[h_1 Y_{3x}(v, w) + (3h_1 \lambda + h_3) Y_x(v) + (h_4 + \partial_t \ln h_1 h_2^{-1})v] = 0, \] (4.12)
which can be cast into a bilinear Bäcklund transformation by using the property (2.2)

\[(D_x^2 + \alpha D_x - \lambda)F \cdot G = 0,\] (4.13)

\[[D_t + h_1 D_x^3 + (3h_1 \lambda + h_3)D_x + \beta]F \cdot G = 0,\]

if we set the constraint

\[h_4 + \partial_t \ln h_1 h_2^{-1} = 0 \implies h_1 = c_0 e^{-\int h_4 dt} h_2,\]

with \(c_0\) being an arbitrary parameter. Without loss of generality, taking \(c_0 = 1/6\), then

\[c(t) = 6h_1 / h_2 = e^{-\int h_4 dt}.\]

As \(\alpha = \beta = 0\), the Bäcklund transformation (4.13) reduces to the one obtained in [20].

Making use of the Hopf-Cole transformation \(v = \ln \psi\) and the formula (2.5)-(2.6), then the system (4.10) can be linearized into a Lax pair

\[L_1 \psi = (\partial_x^2 + \alpha \partial_x + q_2 x) \psi = \lambda \psi,\] (4.14)

\[(\partial_t + L_2)\psi = [\partial_t + h_1 \partial_x^3 + (3h_1 q_2 x + 3\lambda h_1 + h_3) \partial_x] \psi = 0,\] (4.15)

or equivalently,

\[\psi_{2x} + \alpha \psi_x + (ue^{\int h_4 dt} - \lambda) \psi = 0,\]

\[\psi_t + [h_1(2ue^{\int h_4 dt} + 4\lambda + \alpha^2) + h_3] \psi_x + [\beta + \alpha \lambda - h_1(ue^{\int h_4 dt} - oue^{\int h_4 dt})] \psi = 0.\]

This Lax pair can be used to construct Darboux transformation, inverse scattering transformation for soliton solutions. It is easy to check that the integrability condition

\[[L_1 - \lambda, \partial_t + L_2] \psi = 0\]

is satisfied if \(u\) is a solution of the vc-KdV equation (4.1).

4.3. Darboux covariant Lax pair
Let us go back to the vc-KdV equation (4.1) and the associated Lax pair (4.14)-(4.15). Assume that $\phi$ is a solution eigenvalue equation (4.14) (taking $\alpha = 0$ for simplicity). It is well-known that the gauge transformation

$$T = \phi \partial_x \phi^{-1} = \partial_x - \sigma, \quad \sigma = \partial_x \ln \phi$$

(4.16)

map the operator $L_1 = \partial_x^2 + q_2 x$ onto a similar operator:

$$T(L_1(q) - \lambda)T^{-1} = \tilde{L}_1(\tilde{q}) - \lambda,$$

which satisfies the covariance condition

$$\tilde{L}_1(\tilde{q}) = L_1(q = q + \Delta q), \quad \text{with} \quad \Delta q = 2 \ln \phi.$$  

But it can be verified that similar property does not hold for the evolution equation (4.15). Next step is to find another third order operator $L_{2,\text{cov}}(q)$ with appropriate coefficients, such that $\partial_t + L_{2,\text{cov}}(q)$ be mapped, by gauge transformation (4.16), onto a similar operator $\partial_t + \tilde{L}_{2,\text{cov}}(\tilde{q})$ which satisfies the covariance condition

$$\tilde{L}_{2,\text{cov}}(\tilde{q}) = L_{2,\text{cov}}(\tilde{q} = q + \Delta q).$$

Suppose that $\phi$ is a solution of the following Lax pair

$$L_1 \phi = \lambda \phi,$$

$$(\partial_t + L_{2,\text{cov}}) \phi = 0, \quad L_{2,\text{cov}} = 4h_1 \partial_x^3 + b_1 \partial_x + b_2,$$

(4.17)

where $b_1$ and $b_2$ are functions to be determined. It suffice that we require that the transformation $T$ map the operator $\partial_t + L_{2,\text{cov}}$ onto the similar one

$$T(\partial_t + L_{2,\text{cov}})T^{-1} = \partial_t + \tilde{L}_{2,\text{cov}}, \quad \tilde{L}_{2,\text{cov}} = 4h_1 \partial_x^3 + \tilde{b}_1 \partial_x + \tilde{b}_2,$$

(4.18)

where $\tilde{b}_1$ and $\tilde{b}_2$ satisfy the covariant condition

$$\tilde{b}_j = b_j(q) + \Delta b_j = b_j(q + \Delta q), \quad j = 1, 2.$$  

(4.19)
It follows from (4.16) and (4.18) that
\[ \Delta b_1 = \tilde{b}_1 - b_1 = 12h_1 \sigma_x, \quad \Delta b_2 = \tilde{b}_2 - b_2 = b_{1,x} + \sigma \Delta b_1 + 12h_1 \sigma_{2x}, \] (4.20)
and \( \sigma \) satisfies
\[ \sigma_t + 4h_1 \sigma_{1x} + b_{2,x} + \Delta b_2 \sigma + \tilde{b}_1 \sigma_x = 0. \] (4.21)
According to (4.19), it remains to determine \( b_1 \) and \( b_2 \) in the form of polynomial expressions in terms of derivatives of \( q \)
\[ b_j = F_j(q, q_x, q_{2x}, q_{3x}, \cdots), \quad j = 1, 2 \]
such that
\[ \Delta F_j = F_j(q + \Delta q, q_x + \Delta q_x, q_{2x} + \Delta q_{2x}, \cdots) - F_j(q, q_x, q_{2x}, \cdots) = \Delta b_j, \] (4.22)
with \( \Delta q_{rx} = 2(\ln q)_r, \quad r = 1, 2, \cdots \), the \( \Delta b_j \) being determined by the relations (4.19).

Expanding the left hand of the equation (4.22), we obtain
\[ \Delta b_1 = \Delta F_1 = F_{1,q} \Delta q + F_{1,q_x} \Delta q_x + F_{1,q_{2x}} \Delta q_{2x} + \cdots = 12h_1 \sigma_x = 6h_1 \Delta q_{2x}, \]
which implies that we can choose
\[ b_1 = F_1(q_{2x}) = 6h_1 q_{2x} + c_1(t), \] (4.23)
with \( c_1(t) \) being arbitrary function about \( t \).

From the eigenvalue equation in (4.17), we can find the following relation
\[ q_{3x} = -\sigma_{2x} - 2\sigma \sigma_x. \] (4.24)
Substituting (4.23) and (4.24) into (4.19) leads to
\[ \Delta b_2 = 12h_1 q_{3x} + 12h_1 \sigma \sigma_x + 12h_1 \sigma_{2x} = 6h_1 \sigma_{2x} = 3h_1 \Delta q_{3x}. \]
The second condition
\[ \Delta F_2 = F_{2,q} \Delta q + F_{2,q_x} \Delta q_x + F_{2,q_{2x}} \Delta q_{2x} + F_{2,q_{3x}} \Delta q_{3x} + \cdots = \Delta b_2, \]
can be satisfied, if one chooses
\[ b_2 = F_2(q, q_{2x}, q_{3x}) = 3h_1q_{3x} + c_2(t), \]
in which \( c_2(t) \) is arbitrary constant.

Setting \( c_1(t) = h_3 \), \( c_2(t) = 0 \), we find the following Darboux covariant evolution equation
\[ (\partial_t + L_{2,cov})\phi = 0, \quad L_{2,cov} = 4h_1 \partial_x^3 + (6h_1 q_{2x} + h_3) \partial_x + 3h_1q_{3x}, \]
which is in agreement with the equation (4.21). Moreover, the relation between the operator \( L_{2,cov} \) and the operator \( L_2 \) is given by
\[ L_{2,cov} = L_2 + 3h_1 \partial_x(L_1 - \lambda). \]

The integrability condition of the Darboux covariant Lax pair (4.15) precisely give rise to the equation (4.1) in Lax representation
\[ [\partial_t + L_{2,cov}, L_1] = -[q_{xt} + h_1q_{4x} + 3h_1q_{2x}^2 + h_3q_{2x} + h_4q_{x}]_x. \]
The higher operators can be obtained in a similar way step by step
\[ L_{k,cov}(q) = 4h_1 \partial_x^k + b_1 \partial_x^{k-2} + \cdots + b_p, \quad k = 3, 4, \cdots \]
which are Darboux covariant with respect to \( L_1 \), so as to produce higher order members of the vc-KdV hierarchy.

4.4. Infinite conservation laws

Finally, we construct the conservation laws of vc-KdV equation. The second equation of (4.12) has been conserved form due to the relation
\[ \partial_t \mathcal{Y}(v) = \partial_t \mathcal{Y}_x(v) = v_{xt}. \]
By introducing a new potential function
\[ \eta = (q_x' - q_x)/2, \]
it follows from the relation (4.8) that
\[ v_x = \eta, \quad w_x = q_x + \eta. \tag{4.25} \]
Substituting (4.25) into (4.12), we get a Riccati type equation
\[ \eta_x + \eta^2 + q_{2x} = \lambda = \varepsilon^2, \tag{4.26} \]
and a divergence type equation
\[ \eta_t + \partial_x [h_1 \eta_{2x} + 6h_1(\eta + \varepsilon)^2 - 2h_1(\eta + \varepsilon)^3 + h_3(\eta + \varepsilon)] = 0, \tag{4.27} \]
where we have used the equation (4.26) to get the equation (4.27) and set \( \lambda = \varepsilon^2 \).

Substituting the expansion
\[ \eta = \varepsilon + \sum_{n=1}^{\infty} I_n(q, q_x, \cdots) \varepsilon^{-n}. \tag{4.28} \]
into the equation (4.26) and equating the coefficients of \( \varepsilon^{-1} \), the conserved densities are explicitly obtained by recursion relations
\[ I_1 = -p_x = -\frac{1}{2} e^\int h_4 dt u_x, \quad I_2 = \frac{1}{4} e^\int h_4 dt u_x, \tag{4.29} \]
\[ I_{n+1} = -\frac{1}{2} (I_{n,x} + \sum_{k=1}^{n} I_k I_{n-k}), \quad n = 2, 3, \cdots \]

In addition, substituting (4.28) into (4.27) leads to
\[ \sum_{n=1}^{\infty} I_{n,t} \varepsilon^{-n} + \partial_x \left[ h_1 \sum_{n=1}^{\infty} I_{n,2x} \varepsilon^{-n} - 6h_1 \varepsilon (\sum_{n=1}^{\infty} I_n \varepsilon^{-n})^2 - 2h_1 (\sum_{n=1}^{\infty} I_n \varepsilon^{-n})^3 + h_3 \sum_{n=1}^{\infty} I_n \varepsilon^{-n} \right] = 0 \]
which provides us infinite conservation laws
\[ I_{n,t} + F_{n,x} = 0, \quad n = 1, 2, \cdots. \tag{4.30} \]
In the equation (4.30), the conversed densities $I'_n$s are given by recursion formulas (4.29), and the fluxes $F'_n$s are given by

$$F_1 = -\frac{1}{2} e^{\int h_4 dt} (h_1 u_{2x} + 3h_2 u^2 + h_3 u),$$

$$F_2 = \frac{1}{4} e^{\int h_4 dt} (h_1 u_{3x} + 6h_2 e^{\int h_4 dt} uu_x + h_3 u_x),$$

$$F_n = h_1 I_{n,2x} - 6h_1 \sum_{k=1}^{n} I_k I_{n+1-k} - 2h_1 \sum_{i+j+k=n} I_i I_j I_k + h_3 I_n, \quad n = 3, 4, \cdots.$$ 

The first equation of (4.30) is exactly the vc-KdV equation (4.1). We see that above fluxes $F'_n$s can be obtained from solution $u$ by algebraic and differential manipulation, thus they are local. Taking the boundary condition of $u$ into account, the conservation equation (4.30) implies that $\{I_n, \ n = 1, 2, \cdots, \}$ constitute infinite conserved densities of the vc-KdV equation (4.1). In conclusion, the vc-KdV equation (4.1) is complete integrable under the constraint $h_2 = 6h_1 e^{\int h_7 dt}$ in the sense that it admits bilinear Bäcklund transformation, Lax pair and infinite conservation laws.

5. Nonisospectral variable-coefficient KP equation

Consider nonisospectral and vc-KP equation \[16\]

$$u_t + h_1(u_{3x} + 6uu_x + 3\alpha^2 \partial_x^{-1} u_{yy}) + h_2(u_x - \alpha xu_y - 2\alpha \partial_x^{-1} u_y)$$

$$- h_3(xu_x + 2u + 2yu_y) = 0,$$  \hspace{1cm} (5.1)

where $h_1 = h_1(t), \ h_2 = h_2(t)$ and $h_3 = h_3(t)$ are all arbitrary functions with respect to time variable $t$. The equation (5.1) reduces to the vc-KdV equation (3.1) when $u = u(x,t)$ is independent of the variable $y$. In the case $h_1 = 1, \ h_2 = h_3 = 0$, it reduce to standard KP equation.
5.1. Bilinear representation

By introducing a potential field \( q \)

\[ u = c(t)q_{2x}, \]  

(5.2)

with \( c = c(t) \) is a free function about \( t \) to be determined, the resulting equation (5.1) for \( q \) (integrating with respect to \( x \)) reads

\[
E(q) \equiv q_{xt} + h_1(q_{4x} + 3cq_{2x}^2 + 3\alpha^2 q_{2y}) + h_2(q_{2x} - x\alpha q_{xy} - \alpha q_y) \\
- h_3(q_x + xq_{2x} + 2yq_{xy}) + q_x \partial_t \ln c = 0,
\]  

(5.3)

which can be expressed in the form of \( P \)-polynomials

\[
E(q) = P_{xt}(q) + h_1[P_{4x}(q) + 3\alpha^2 P_{2y}(q)] + h_2[P_{2x}(q) - \alpha x P_{xy}(q) - \alpha q_y] \\
- h_3[x P_{2x}(q) + 2y P_{xy}(q) + q_x] = 0,
\]  

(5.4)

if one chooses \( c(t) = 1 \) and use the formula (2.4).

By application of the variable transformation

\[ q = 2 \ln F \iff u = 2(\ln F)_{2x} \]

and using the property (2.3), then the equation (5.4) give the bilinear form for the vc-KP equation (5.1) as follows

\[
[D_x D_t + h_1(D_x^4 + 3\alpha^2 D_y^2) + h_2(D_x^2 - \alpha x D_x D_y - \alpha \partial_y) - h_3(x D_x^2 + 2y D_x D_y + \partial_x)]F \cdot F = 0,
\]

starting from which, we can get multi-soliton solutions to the equation (5.1). For example, one-soliton solution takes the form

\[ u = \frac{(k + s)^2}{2} \text{sech}^2 \frac{\xi + \zeta + \ln \omega}{2}, \]

in which \( \xi = kx - k^2y/\alpha \), \( \zeta = sx + s^2y/\alpha \), while \( k = k(t), s = s(t) \) and \( \omega = \omega(t) \) are all functions about \( t \), satisfying

\[ k_t = h_3k - bk^2, \quad s_t = h_3s + bs^2, \]

\[ \omega(t) = \exp\left(\int [h_3(k^3 + s^3) - h_2(k + s)]dt\right). \]
5.2. Bäcklund transformation and Lax pair

In following we consider bilinear Bäcklund transformation and Lax pairs of the equation (5.1). Assume \( q \) and \( q' \) are two solutions of the equation (5.3), we consider the corresponding two-field condition

\[
E(q') - E(q) = (q' - q)_{xt} + h_1[(q' - q)_{4x} + 3(q' + q)_{2x}(q' - q)_{2x} + 3\alpha^2(q' - q)_{2y}]
+ h_2[(q' - q)_{2x} - \alpha x(q' - q)_{xy} - \alpha(q' - q)_{y}] - h_3[x(q' - q)_{2x}
+ 2y(q' - q)_{xy} + (q' - q)_{x}] = 0,
\]

which may produce the required bilinear Bäcklund transformation under an appropriate additional constraint.

By change of the variables

\[
v = (q' - q)/2, \quad w = (q' + q)/2
\]

we rewrite the condition (5.5) in the form

\[
E(q') - E(q) = v_{xt} + h_1(v_{4x} + 6v_{2x}w_{2x} + 3\alpha^2v_{2y}) + h_2(v_{2x} - \alpha xv_{xy} - \alpha v_{y})
- h_3(xv_{2x} + 2yv_{xy} + v_{x}) = \partial_x[\mathcal{Y}_t(v) + h_1\mathcal{Y}_{3x}(v, w)] + R(v, w) = 0,
\]

with

\[
R(v, w) = 3h_1\text{Wronskian}[\mathcal{Y}_{2x}(v, w), \mathcal{Y}_x(v)] + 3h_1\alpha^2v_{2y} + h_2(v_{2x} - \alpha xv_{xy} - \alpha v_{y})
- h_3(xv_{2x} + 2yv_{xy} + v_{x}).
\]

In order to express \( R(v, w) \) as the \( x \)-derivative of a linear combination of \( \mathcal{Y} \)-polynomials, we choose the constraint

\[
\mathcal{Y}_{2x}(v, w) + \alpha\mathcal{Y}_y(v) = \lambda,
\]
on account of which
\[ R(v, w) = \partial_x \{ h_1 [3\lambda \mathcal{Y}_x(v) - 3\alpha \mathcal{Y}_{xy}(v, w)] + h_2 [\mathcal{Y}_x(v) - \alpha x \mathcal{Y}_y(v)] \
- h_3 [x \mathcal{Y}_x(v) + 2y \mathcal{Y}_y(v)] \}. \quad (5.9) \]

Combining relations (5.7)-(5.9), we then obtain a pair of constraints in Bell polynomial form
\[ \mathcal{Y}_{2x}(v, w) + \alpha \mathcal{Y}_y(v) - \lambda = 0, \]
\[ \partial_t \mathcal{Y}_x(v) + \partial_x \{ h_1 [\mathcal{Y}_{3x}(v, w) - 3\alpha \mathcal{Y}_{xy}(v, w) + 3\lambda \mathcal{Y}_x(v)] + h_2 [\mathcal{Y}_x(v) - \alpha x \mathcal{Y}_y(v)] \
- h_3 [x \mathcal{Y}_x(v) + 2y \mathcal{Y}_y(v)] \} = 0, \quad (5.10) \]

which result in bilinear Bäcklund transformation
\[ (D_x^2 + \alpha D_y - \lambda) F \cdot G = 0, \]
\[ [D_t + h_1 (D_x^3 - 3\alpha D_x D_y + 3\lambda D_x) + h_2 (D_x - \alpha x D_y)] \
- h_3 (xD_x + 2yD_y) + \beta] F \cdot G = 0 \quad (5.11) \]

by applying the property (2.2).

It only remains to linearize the expression (5.10) with the formulae (2.5) and (2.6). By using the Hopf-Cole transformation \( v = \ln \psi \), the bilinear Bäcklund transformation (5.10) can be linearized into a Lax pair
\[ (\partial_y + L_1) \psi \equiv \psi_y + \alpha^{-1} \psi_{2x} + \alpha^{-1} (q_{2x} - \lambda) \psi = 0, \quad \lambda_t = 2h_3 \lambda, \]
\[ (\partial_t + L_2) \psi \equiv \psi_t + 4h_1 \psi_{3x} + (xh_2 + 2y\alpha^{-1}h_3) \psi_{2x} + (6h_1 q_{2x} + h_2 - xh_3) \psi_x \]
\[ + [3h_1 q_{3x} - 3\alpha h_1 q_{xy} + (q_{2x} - \lambda)(xh_2 + 2y\alpha^{-1}h_3)] \psi = 0, \]

which can be used to construct Darboux transformation, inverse scattering transformation for getting soliton solutions. It is easy to check that the integrability condition
\[ [\partial_y + L_1, \partial_t + L_2] \psi = 0 \]
is satisfied if \( u \) is a solution of the vc-KP equation (5.1) and nonisospectral condition \( \lambda_t = 2h_3\lambda \) holds.

5.3. Infinite conservation laws

Finally, we precede to construct the conservation laws of vc-KP equation. For this purpose, we decompose the two-filed condition (5.7) into \( x \)- and \( y \)-derivative of a linear combination of \( \mathcal{Y} \)-polynomials. Let us go back to the two-field condition (5.7) and consider the following decomposition

\[
R(v, w) = \partial_x[3h_1\lambda v_x - 3\alpha h_1v xv y + h_2(v_x - \alpha xv y) - h_3(xv x + 2yv y)] \\
+ \partial_y(3h_1\alpha^2v y + 3h_1\alpha v^2 x).
\]

(5.11)

It follows from (5.7), (5.8) and (5.11) that

\[
\mathcal{Y}_{2x}(v, w) + \alpha \mathcal{Y}_y(v) - \lambda = 0, \\
(\mathcal{Y}_x(v))_t + \partial_x[h_1\mathcal{Y}_{3x}(v, w) - 3\alpha \mathcal{Y}_y(v, w) + 3\lambda \mathcal{Y}_x(v)] + h_2[\mathcal{Y}_x(v) - \alpha x\mathcal{Y}_y(v)] \\
- h_3[x\mathcal{Y}_x(v) + 2y\mathcal{Y}_y(v)] = 0,
\]

(5.12)

which is slightly different from (5.10) and can produce desired conservation laws. By introducing a new potential function

\[
\eta = (q_x' - q_x)/2,
\]

and it follows from the relation (5.6) that

\[
v_x = \eta, \quad w_x = q_x + \eta.
\]

(5.13)

Substituting (5.13) into (5.12), we decompose the two-field condition (5.7) into a Riccati type equation

\[
\eta_x + \eta^2 + \alpha \partial_x^{-1}\eta_y + q_{2x} - \varepsilon^2 = 0,
\]

(5.14)
and a divergence type equation

\[
\eta_t + \partial_x [h_1(\eta_2x + 6\eta\varepsilon^2 - 2\eta^3 - 6\alpha\eta\partial_x^{-1}\eta_y) + h_2(\eta - x\alpha\partial_x^{-1}\eta_y)] + h_3(\eta + 2y\partial_x^{-1}\eta_y)] + \partial_y(3h_1\alpha^2\partial_x^{-1}\eta_y + 3h_1\alpha\eta^2) = 0,
\]

(5.15)

where we have used the equation (5.14) to get the equation (5.15) and set \( \lambda = \varepsilon^2 \).

Substituting the expansion

\[
\eta = \varepsilon + \sum_{n=1}^{\infty} I_n(q, q_x, \cdots) \varepsilon^{-n}.
\]

(5.16)

Inserting the equation (5.16) into the equation (5.14), equation the coefficients for power of \( \varepsilon \), then we have the recursion relations for \( I_n \)

\[
I_1 = -\frac{1}{2} q_{2x} = -\frac{1}{2} u, \quad I_2 = \frac{1}{4}(u_{2x} + \alpha\partial_x^{-1}u_y), \quad I_3 = -\frac{1}{8}(u_{3x} + u^2 + \alpha u_y),
\]

\[
I_{n+1} = -\frac{1}{2}(I_{n,x} + \sum_{k=1}^{n} I_k I_{n-k} + \alpha\partial_x^{-1}I_{n,y}), \quad n = 3, 4, \cdots,
\]

(5.17)

Again substituting (5.16) into (5.15) and noting the nonisospectral condition

\[
\lambda_t = 2h_3\lambda \implies \varepsilon_t = h_3\varepsilon,
\]

we then obtain the following infinite conservation laws

\[
I_{n,t} + F_{n,x} + G_{n,y} = 0, \quad n = 1, 2, \cdots.
\]

(5.18)
In the equation (5.18), the conversed densities $I'_n$ are given by formula (5.17), the first fluxes $F'_n$ are given by

\begin{align*}
F_1 &= h_1 I_{1,2x} + (h_2 - x h_3) I_1 - (x h_1 + 2 y h_3) \partial_x^{-1} I_{1,y} - 6 \alpha h_1 \partial_x^{-1} I_{2,y} + h_3 \partial_x^{-1} I_1, \\
F_2 &= h_1 I_{2,2x} + (h_2 - x h_3) I_2 - (x h_1 + 2 y h_3) \partial_x^{-1} I_{2,y} - 12 h_1 I_1 I_2 - 12 \alpha h_1 I_1 \partial_x^{-1} I_{2,y} \\
&\quad - 6 \alpha h_1 \partial_x^{-1} I_{3,y} + 2 h_3 \partial_x^{-1} I_2, \\
F_n &= -6 h_1 \sum_{k=1}^{n} I_k (I_{n+1-k} + \alpha \partial_x^{-1} I_{n-k,y}) - 2 h_1 \sum_{i+j+k=n} I_i I_j I_k + (h_2 - x h_3) I_n + h_1 I_{n,2x} \\
&\quad - (x h_2 + 2 y h_3) \partial_x^{-1} I_{n,y} + n h_3 \partial_x^{-1} I_n - 6 \alpha h_1 I_{n+1}, \quad n = 3, 4, \ldots.
\end{align*}

and the second fluxes $G'_n$ are given by

\begin{align*}
G_1 &= 3 h_1 \alpha^2 \partial_x^{-1} I_{1,y}, \quad G_2 = 3 h_1 \alpha I_1^2 + 3 h_1 \alpha^2 \partial_x^{-1} I_{2,y}, \\
G_n &= 3 h_1 \alpha \sum_{k=1}^{n} I_k I_{n-k} + 3 h_1 \alpha^2 \partial_x^{-1} I_{n,y}, \quad n = 2, 3, \ldots.
\end{align*}

The first equation of (5.18) is exactly the vc-KP equation (5.1). The expressions $F'_n$ and $G'_n$ indicate that the fluxes of the vc-KP equation are not local. To summarize, the vc-KP equation (5.1) is complete integrable, since it admits bilinear Bäcklund transformation, Lax pair and infinite conservation laws.

6. General variable-coefficient KP equation

Consider a general vc-KP equation \[39\]

\[(u_t + h_1 u_{3x} + h_2 uu_x) + h_3 u_{2y} + h_4 u_{xy} + (h_5 + h_6 y) u_{2x} + h_7 u_x = 0, \quad (6.1)\]
where \( h_i = h_i(t), \ i = 1, 2, \cdots, 7 \) are arbitrary functions with respect to time variable \( t \). The equation (6.1) include many special variable-coefficient equations in physics, such as cylindrical KdV equation [22]

\[
  u_t + uu_x + u_{3x} + \frac{1}{2t} u_x = 0,
\]

(6.2)
cylindrical KP equation [40, 41]

\[
  (u_t + h_1 u_{3x} + h_2 uu_x)_x + \frac{1}{2t} u_x + \frac{3\sigma^2}{t^2} u_{2y} = 0,
\]

(6.3)
generalized cylindrical KP equation [42, 43]

\[
  (u_t + h_1 u_{3x} + h_2 uu_x)_x + \frac{1}{2t} u_x + \frac{3\sigma^2}{t^2} u_{2y} + r(t) u_{xy} + [f(t) + g(t)y] u_{2x} = 0.
\]

(6.4)

Here we attempt to find the integrability condition that the equation (6.1) possesses bilinear representation, Bäcklund transformation, Lax pair, Darboux covariant Lax pair and infinite conservation laws.

6.1. Bilinear representation

By introducing a potential field \( q \)

\[
  u = c(t) q_{2x},
\]

with \( c = c(t) \) is free function to be determined, the resulting equation (6.1) for \( q \) (integrating with respect to \( x \) twice) reads

\[
  E(q) \equiv q_{xt} + h_1 q_{4x} + \frac{c}{2} h_2 q_{2x}^2 + h_3 q_{2y} + h_4 q_{xy} + (h_5 + h_6 y) q_{2x} + (h_7 + \partial_t \ln c) q_x = 0,
\]

(6.5)

which can be expressible as \( P \)-polynomials

\[
  E(q) = P_{xt}(q) + h_1 P_{4x}(q) + h_3 P_{2y}(q) + h_4 P_{xy}(q) + (h_5 + h_6) P_{2x}(q) + (h_7 + \partial_t \ln h_1 h_2^{-1}) q_x = 0,
\]

(6.6)
if one chooses \( c = 6h_1h_2^{-1} \) and use the formula (2.4). By application of the transformation

\[
q = 2 \ln F \iff u = cq_{2x} = 12h_1h_2^{-1}(\ln F)_{2x}
\]

and using the property (2.3), then the equation (6.6) gives the bilinear representation for the vc-KP equation (6.1) as follows

\[
[D_xD_t + h_1D_x^4 + h_3D_y^2 + h_4D_xD_y + (h_5 + h_6y)D_x^2 + (h_7 + \partial_t \ln h_1h_2^{-1})\partial_x]F \cdot F = 0.
\]

Starting from this bilinear equation, we can get multi-solutions, for example, the regular one-soliton like solution is

\[
u = 6h_1h_2^{-1}k^2 \text{sech}^2 \frac{kx + sy + \omega}{2},
\]

in which \( k \) is an arbitrary constant, while \( s = s(t) \) and \( \omega = \omega(t) \) are two function with respect to \( t \), given by

\[
s(t) = k \int h_6 dt, \quad \omega(t) = - \int (k^3h_1 + k^{-1}s^2h_3 + sh_4 + h_7 + \partial_t \ln h_1h_2^{-1})dt.
\]

### 6.2. Bäcklund transformation and Lax pair

We now consider bilinear Bäcklund transformation and Lax pairs of the equation (6.1). Let \( q' \) and \( q \) be two solutions of the equation (6.5), we consider the following two-field condition

\[
E(q') - E(q) = (q' - q)_{x_t} + h_1(q' - q)_{4x} + 3h_1(q' + q)_{2x}(q' - q)_{2x} + h_3(q' - q)_{2y}
+ h_4(q' - q)_{xy} + (h_5 + h_6y)(q' - q)_{2x} + (h_7 + \partial_t \ln h_1h_2^{-1})(q' - q)_x = 0,
\]

which may produce the required bilinear Bäcklund transformation under an appropriate additional constraint.

On introducing two new variables

\[
v = (q' - q)/2, \quad w = (q' + q)/2
\]
we rewrite the condition (6.7) as the form
\[
E(q') - E(q) = v_{xt} + h_1 v_{4x} + 6h_1 v_{2x} w_{2x} + h_3 v_{2y} + h_4 v_{xy} + (h_5 + h_6 y) v_{2x} \\
+ (h_7 + \partial_t \ln h_1 h_2^{-1}) v_x = \partial_x [\mathcal{Y}_t(v) + h_1 \mathcal{Y}_{3x}(v, w)] + R(v, w) = 0,
\]
with
\[
R(v, w) = 3 h_1 \text{Wronskian}[\mathcal{Y}_{2x}(v, w), \mathcal{Y}_x(v)] + h_3 v_{2y} + h_4 v_{xy} + (h_5 + h_6 y) v_{2x} \\
+ (h_7 + \partial_t \ln h_1 h_2^{-1}) v_x.
\]

In order to express \( R(v, w) \) as the \( x \)- and \( y \)-derivative of \( \mathcal{Y} \)-polynomials, we choose the constraint
\[
\mathcal{Y}_y(v) + \alpha(t) \mathcal{Y}_{2x}(v, w) = \lambda,
\]
where \( \alpha = \alpha(t) \) is to be determined. Direct calculation show that
\[
R(v, w) = 3 h_1 \lambda v_{2x} - \alpha^{-1} [h_3 w_{2x,y} + (2h_3 - 3h_1 \alpha^2) v_x v_{xy} + 3h_1 \alpha^2 v_{2x} v_y] \\
+ h_4 v_{xy} + (h_5 + h_6 y) v_{2x} + (h_7 + \partial_t \ln h_1 h_2^{-1}) v_x,
\]
which can be expressible \( R(v, w) \) as the \( x \)-derivative of a linear combination of \( \mathcal{Y} \)-polynomials
\[
R(v, w) = \partial_x [3 h_1 \lambda \mathcal{Y}_x - 3 \alpha h_1 \mathcal{Y}_{3x}(v, w) + h_4 \mathcal{Y}_y + (h_5 + h_6 y) \mathcal{Y}_x],
\]
if we take a simple constraints
\[
h_7 + \partial_t \ln h_1 h_2^{-1} = 0, \quad h_3 = 2h_3 - 3h_1 \alpha^2 = 3h_1 \alpha^2,
\]

namely,
\[
h_2 = 6h_1 e^{\int h_7 dt}, \quad 3 \alpha^2 = h_3 h_1^{-1}.
\]

From (6.9)-(6.11), one infers that
\[
\mathcal{Y}_{2x}(v, w) + \alpha \mathcal{Y}_y(v) - \lambda = 0, \\
\partial_x \mathcal{Y}_t(v) + \partial_x \{h_1[\mathcal{Y}_{3x}(v, w) - 3 \alpha \mathcal{Y}_{xy}(v, w) + 3 \lambda \mathcal{Y}_x(v)] + h_4 \mathcal{Y}_y \} = 0,
\]
(6.12)
which leads to bilinear Bäcklund transformation of variable coefficient KP equation
\[(D^2_x + \alpha D_y - \lambda)F \cdot G = 0,\]
\[\left[ D_t + h_1(D^3_x - 3\alpha D_x D_y + 3\lambda D_x) + h_4 D_y + (h_5 + h_6 y) D_x + \beta \right] F \cdot G = 0\]
by using the property (2.2).

By using the Hopf-Cole transformation \(v = \ln \psi\) and the formulae (2.5) and (2.6), the system (6.12) can be linearized into a Lax pair
\[(\alpha \partial_y + L_1)\psi \equiv (\alpha \partial_y + \partial^2_x + q_{2x} - \lambda)\psi = 0,\]
\[(\partial_t + L_2)\psi \equiv [\partial_t + 4h_1 \partial^3_x - h_4 \alpha^{-1} \partial^2_x + (6h_1 q_{2x} + h_5 + h_6 y) \partial_x \]
\[+ 3h_1 q_{3x} - 3h_1 \alpha q_{xy} - h_4 \alpha^{-1} q_{2x} + h_4 \alpha^{-1} \lambda] \psi = 0,\]
whose integrability condition
\[[\alpha \partial_y + L_1, \partial_t + L_2] \psi = 0\]
is satisfied if \(u\) is a solution of the vc-KP equation (6.1) and \(\alpha_t = 0\), or equivalently, \(h_3 h_1^{-1} = \text{constant}\). This Lax pair can be used to construct Darboux transformation, inverse scattering transformation for soliton solutions to the vc-KP equation (6.1).

6.3. Darboux covariant Lax pair

Let us go back to the vc-KP equation (6.1) and the associated Lax pair (6.13). Assume that \(\phi\) is a solution of the following Lax pair
\[(\alpha \partial_y + L_1)\phi = \lambda \phi, \quad L_1 = \partial^2_x + q_{2x},\]
\[(\partial_t + L_2)\phi = 0, \quad L_2 = h_1 \partial^3_x + b_1 \partial^2_x + b_2 \partial_x + b_3,\]
where \(b_1, b_2\) and \(b_3\) are functions to be determined. It is shown that the gauge transformation
\[T = \phi \partial_x \phi^{-1} = \partial_x - \sigma, \quad \sigma = \partial_x \ln \phi\]
\[(6.15)\]
map the operator $\alpha \partial_y + L_1(q)$ onto a similar operator:

$$T(\alpha \partial_y + L_1(q))T^{-1} = \alpha \partial_y + \tilde{L}_1(\tilde{q} = q + \Delta q) \quad \text{with} \quad \Delta q = 2 \ln \phi.$$ 

It is suffices to verify that such transformation (6.15) map the $\partial_t + L_{2,\text{cov}}$ into similar one

$$T(\partial_t + L_{2,\text{cov}})T^{-1} = \partial_t + \tilde{L}_{2,\text{cov}}, \quad \tilde{L}_{2,\text{cov}} = h_1 \partial_x^3 + b_1 \partial_x^2 + b_2 \partial_x + b_3, \quad (6.16)$$

where $\tilde{b}_j, j = 1, 2, 3$ and $\tilde{L}_{2,\text{cov}}$ satisfy the covariant conditions

$$\tilde{b}_j = b_j(q) + \Delta b_j = b_j(q + \Delta q), \quad j = 1, 2, 3.$$ 

$$\tilde{L}_{2,\text{cov}}(\tilde{q}) = L_{2,\text{cov}}(q + \Delta q), \quad \Delta q = 2 \ln \phi.$$ 

It follows from (6.15) and (3.16) that

$$\Delta b_1 = \tilde{b}_1 - b_1 = 0, \quad \Delta b_2 = 12h_1 \sigma_x + b_{1,x} + \Delta b_1 \sigma,$$

$$\Delta b_3 = 12h_1 \sigma_{2x} + 2\sigma_x \tilde{b}_1 + \Delta b_2 \sigma + b_{2,x}, \quad (6.17)$$

We require $b_1, b_2$ and $b_3$ in the differential polynomial form of potential filed $q$

$$b_j = F_j(q, q_x, q_{2x}, q_{3x}, \cdots), \quad j = 1, 2$$

such that

$$\Delta F_j = F_j(q + \Delta q, q_x + \Delta q_x, q_{2x} + \Delta q_{2x}, \cdots) - F_j(q, q_x, q_{2x}, \cdots) = \Delta b_j,$$

with $\Delta q_{rx} = 2(\ln q)_{rx}, \quad r = 1, 2, \cdots.$

From eigenvalue equation in (6.14), we get the relation

$$q_{3x} = -\alpha \sigma_{xy} - (\sigma_x + \sigma^2)_x,$$

on account of which, solving the system (6.17) yields

$$b_1 = c_1(y, t), \quad b_2 = 6h_1q_{2x} + c_2(y, t),$$

$$b_3 = 3h_1(q_{3x} - \alpha q_{xy}) + c_1q_{2x} + c_3(y, t), \quad (6.18)$$
with \(c_1(y, t), c_2(y, t)\) and \(c_3(y, t)\) being arbitrary functions with respect to \(y\) and \(t\). From (6.14) and (6.18), we then find a Darboux covariant evolution equation

\[
(\partial_t + L_{2,\text{cov}})\phi = 0,
\]

\[
L_{2,\text{cov}} = 4h_1 \partial_x^3 + c_1 \partial_x^2 + (6h_1 q_{2x} + c_2) \partial_x + 3h_1 q_{3x} - 3h_1 \alpha q_{2x} + c_3.
\]

In particular, if setting

\[
c_1 = -h_4 \alpha^{-1}, \ c_2 = h_5 + h_6 y, \ c_3 = h_4 \lambda,
\]

the Darboux covariant operator \(L_{2,\text{cov}}\) reduce to operator \(L_2\), namely

\[
L_{2,\text{cov}} = L_2.
\]

Therefore the Lax pair (6.13) is Darboux covariant under constraint \(h_3 h_1^{-1} = \text{constant}\).

In a similar way, we can get higher operators

\[
L_{p,\text{cov}}(q) = h_1 \partial_x^p + b_1 \partial_x^{p-2} + \cdots + b_p, \quad p = 3, 4, \cdots
\]

which are Darboux covariant with respect to \(L_1\) step by step, so as to produce higher order members of the vc-KP hierarchy.

### 6.4. Infinite conservation laws

Finally, we turn to construct the conservation laws of vc-KP equation. For this purpose, we expect re-decompose the two-filed condition (6.7) into \(x\)- and \(y\)-derivative of \(\mathcal{Y}\)-polynomials. We return to revisit \(R(v, w)\) in two-field condition (6.9) and write it as another form

\[
R(v, w) = [3h_1 \lambda v_x - 3\alpha h_1 v_x v_y + (h_5 + h_6 y)v_x]_x + (h_3 v_y + h_4 v_x + 3h_1 \alpha v_x^2)_y.
\] (6.19)
It follows from the relations (6.9), (6.10) and (6.19) that
\[ Y_2(v, w) + \alpha Y_y(v) - \lambda = 0, \]
\[
\partial_t Y_x(v) + \partial_x [h_1 Y_3(v, w) - 3h_1 \alpha Y_y(v, w) + 3h_1 \lambda Y_x(v) + (h_5 + h_6y)Y_x(v)] \\
+ \partial_y [h_3 Y_y(v) + 3h_1 \alpha Y_x(v)^2 + h_4 Y_x(v)] = 0,
\]
(6.20)
which is slightly different from (6.10) and can produce desired conservation laws. Especially we don’t need the constraint \(3\alpha^2 = h_3h_1^{-1}\).

By introducing a new potential function
\[ \eta = (q'_x - q_x)/2, \]
and it follows from the relation (6.8) that
\[
v_x = \eta, \quad w_x = q_x + \eta.
\]
(6.21)
Substituting (6.21) into (6.20) yields a coupled system
\[
\eta_x + \eta^2 + \alpha \partial_x^{-1} \eta_y + q_{2x} - \varepsilon^2 = 0,
\]
(6.22)
\[
\eta_t + \partial_x [h_1(\eta_{2x} + 6\eta \varepsilon^2 - 2\eta^3 - 6\alpha \eta \partial_x^{-1} \eta_y) + (h_5 + h_6y)\eta] \\
+ \partial_y (h_3 \partial_x^{-1} \eta_y + 3h_1 \alpha \eta^2 + h_4 \eta) = 0,
\]
(6.23)
where we have used the equation (6.22) to get the equation (6.23) and set \(\lambda = \varepsilon^2\).

Substituting the expansion
\[ \eta = \varepsilon + \sum_{n=1}^{\infty} I_n(p, p_x, \cdots) \varepsilon^{-n} \]
(6.24)
into the equation (6.22), equating the coefficients for power of \(\varepsilon\), then we obtain the recursion relations for \(I_n\) as follows
\[
I_1 = -\frac{1}{2} q_{2x} = -\frac{1}{2} u e^{\int h_4 dt}, \quad I_2 = \frac{1}{4} e^{\int h_4 dt} (u_{2x} + \alpha \partial_x^{-1} u_y),
\]
\[
I_n = -\frac{1}{2}(I_{n,x} + \sum_{k=1}^{n} I_k I_{n-k} + \alpha \partial_x^{-1} I_{n,y}), \quad n = 2, 3, \cdots,
\]
(6.25)
Again substituting (6.24) into (6.23) and comparing the coefficients for power of $\varepsilon$ provide us infinite conservation laws

$$I_{n,t} + F_{n,x} + G_{n,y} = 0, \quad n = 1, 2, \cdots. \quad (6.26)$$

In the equation (6.26), the conversed densities $I'_n$s obtained by recursion formulas (6.25), and the first fluxes $F'_n$s are expressible in $I'_n$s

$$F_1 = h_1I_{1,2x} - 6\alpha h_1I_1^2 - 6\alpha h_1\partial_x^{-1}I_{2,y} + (h_5 + h_6y)I_1,$$

$$F_2 = h_1I_{2,2x} - 6h_1I_1(2I_2 + \alpha I_1 + \alpha\partial_x^{-1}I_{1,y}) - 6h_1\alpha\partial_x^{-1}I_{3,y} + (h_5 + h_6y)I_2,$$

$$F_n = -6h_1\sum_{k=1}^{n} I_k(I_{n+1-k} + \alpha\partial_x^{-1}I_{n-k,y}) - 2h_1\sum_{i+j+k=n} I_iI_jI_k + h_1I_{n,2x}$$

$$- 6\alpha h_1\partial_x^{-1}I_{n+1,y} + (h_5 + h_6y)I_n, \quad n = 3, 4, \cdots.$$

and the second fluxes $G'_n$s are given by

$$G_1 = h_3\partial_x^{-1}I_{1,y} + 6\alpha h_1I_2 + h_4I_1,$$

$$G_n = 3h_1\alpha\sum_{k=1}^{n} I_kI_{n-k} + 3h_1\alpha^2\partial_x^{-1}I_{n,y}, \quad n = 2, 3, \cdots.$$

The first equation of the conservation law equation (6.26) is exactly the $vc$-KP equation (6.1). Taking the boundary condition of $p$ into account, the equation (6.26) implies that $I'_n$s, $n = 1, 2, \cdots$ constitute infinite conserved densities of the $vc$-KP equation (6.1). To this end, we remark that as application of these results, all equations (6.2)-(6.4) are complete integrable under the constraint $h_2 = 6h_1e^{\int h_7dt}$, since they possess bilinear Bäcklund transformation, Lax pair and infinite conservation laws.

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