A MULTIPLICITY RESULT FOR A CLASS OF STRONGLY INDEFINITE ASYMPOTICALLY LINEAR SECOND ORDER SYSTEMS

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Abstract. We prove a multiplicity result for a class of strongly indefinite nonlinear second order asymptotically linear systems with Dirichlet boundary conditions. The key idea for the proof is to bring together the classical shooting method and the Maslov index of the linear Hamiltonian systems associated to the asymptotic limits of the given nonlinearity.

1. Introduction

In this paper we deal with a second order nonlinear boundary value problem of the form

\[ \begin{cases} \dot{u}''(t) + S(t,u(t))u(t) = 0 \\ u(0) = 0 = u(1), \end{cases} \]

where

\[ J = \begin{pmatrix} \text{Id}_n - \nu & 0 \\ 0 & -\text{Id}_\nu \end{pmatrix} \]

and \( S : [0,1] \times \mathbb{R}^n \to B_{\text{sym}}(\mathbb{R}^n) \) is continuous. It is useful to observe that any gradient system of the form

\[ J\dot{u}''(t) + \nabla V(t,u(t)) = 0 \]

with \( \nabla V(t,0) = 0 \) is of the form \( Ju''(t) + S(t,u(t))u(t) = 0 \). Indeed, it is sufficient to set

\[ S(t,u) := \int_0^1 D^2V(t,su)ds, \quad \forall (t,u) \in [0,1] \times \mathbb{R}^n. \]

We are concerned with the existence and multiplicity of solutions to (1) with prescribed nodal properties.

Before describing the set of assumptions on the nonlinearity and the method of the proofs and developing some remarks with the literature, we focus on some motivations arising from the study of differential equations on manifolds. More precisely, we wish to (very briefly) describe the question of the study of the conjugate points along a perturbed geodesic in a semi-Riemannian manifold in relation with systems of the form \( Ju''(t) + S(t,u(t))u(t) = 0 \). For a general reference, we quote the book [4]; recent results can be found, among others, in [20, 28, 29, 30, 32]. See also [10].

Let \( M \) be a smooth semi-Riemannian manifold, i.e. a \( C^\infty, n \)-dimensional manifold \( M \) endowed with a (semi)-Riemannian metric (i.e. a non-degenerate symmetric two-form \( g \) of constant index \( \nu \in \{0,\ldots,n\} \)). Denoted by \( D \) and by \( \frac{D}{dt} \) respectively the associated Levi-Civita connection and the covariant derivative of a vector field along a smooth curve \( \gamma \), a perturbed geodesic or briefly a \( p \)-geodesic is a smooth curve \( \gamma : [0,1] \to M \) which satisfies the differential equation

\[ \frac{D}{dt}\gamma'(t) + \nabla_g V(t,\gamma(t)) = 0 \]

where \( \nabla_g V \) denotes gradient of \( V(t,-) \) with respect to the metric tensor \( g \).

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Let $\gamma$ be a $p$-geodesic between two fixed points $p, q \in M$. A vector field $\xi$ along $\gamma$ is called a Jacobi field if it verifies the linear second order differential equation

$$\frac{d^2}{dt^2} \xi(t) + R(\gamma'(t), \xi(t)) \gamma'(t) + D_{\xi(t)} \nabla V(t, \gamma(t)) = 0,$$

where $R$ is the curvature tensor of $D$. Given a $p$-geodesic $\gamma$, an instant $t \in (0, 1]$ is said to be a conjugate instant if there exists at least one non zero Jacobi field with $\xi(0) = \xi(t) = 0$. The corresponding point $q = \gamma(t)$ on $M$ is said to be a conjugate point to the point $p = \gamma(0)$ along $\gamma$.

Equation (5) can be then written in the form $Ju''(t) + S(t)u(t) = 0$. Indeed (we refer for more details to [28]), given a perturbed geodesic $\gamma$, a vector field $\xi$ along $\gamma$ can be written as $\xi(t) = \sum_{i=1}^{n} u_i(t)e^i(t)$, being $\{e^1, \ldots, e^n\}$ a $g$-frame along $\gamma$ (this means that the $e^i$’s are pointwise $g$-orthogonal and $g(e^i(t), e^j(t)) = \delta_{ij}$, with $\delta_{ij} = 1$ for $i = 1, \ldots, n - \nu$ and $\delta_{ij} = -1$ for $i = n - \nu + 1, \ldots, n$, for all $t$). Equation (5) can be thus transformed into the second order system of the form:

$$\epsilon_i u''_i(t) + \sum_{j=1}^{n} S_{ij}(t)u_j(t) = 0, \quad i = 1, \ldots, n,$$

where $S_{ij} = g(R(\gamma', e^i)\gamma', e^j) + D_{\gamma'} \nabla V(\gamma, e^i, e^j)$.

Here we deal with a class of nonlinearities $S$ which have an "asymptotically linear" behaviour at zero and infinity; more precisely, we assume

\[ (V_0) \quad S(t, \xi) = S_0(t, \xi) + o(\|\xi\|) \quad \text{for} \quad \|\xi\| \to 0, \quad \text{uniformly in} \quad t; \]
\[ (V_\infty) \quad S(t, \xi) = S_\infty(t, \xi) + o(\|\xi\|) \quad \text{for} \quad \|\xi\| \to +\infty, \quad \text{uniformly in} \quad t. \]

In the study of nonlinear boundary value problems, it is quite frequent to meet this set of hypotheses: they give rise, in the suitable context, to two indices which contain some information on the behaviour of the problem at zero and infinity. Then, roughly speaking, the bigger the gap between these two indices the greater the number of solutions of the nonlinear BVP. As for BVPs with separated boundary conditions, we refer, among others, to [6], [7], [8], [10], [11], [13], [17], [23], [31], [36], [38]. For results in the same spirit but for the periodic problem we refer to the pioneering work of Amann-Zehnder [8] and to the more recent works [5], [15], [22], [24], [26], [37].

We take as a starting point of the present research the paper [8], where it is treated the particular cases $\nu = 0$. By classical shooting methods, when $n = 1$ (a scalar equation) and $\nu = 0$ (the positive definite case) a gap condition expressed in terms of the rotation number is sufficient for the existence of multiple solutions with a prescribed number of zeros. However, also in the positive definite case, when $n \geq 2$ more assumptions on $S$ (together with the use of the Maslov index) are needed (cf. Remark 4.10 in [8]). More problems arise in case $\nu \neq 0$. In this paper, in order to deal with systems we require (on the lines of [8]) a diagonality condition (cf. $V_3$) on the restriction of $S$ to the $(n - 1)$ coordinate hyperplanes of $\mathbb{R}^n$; the difficulty due to the indefiniteness of $J$ is treated by assuming that $S$ is $J$-commuting (the "split" condition $(V_3)$). In order to distinguish the solutions we use a generalized shooting method initiated in the linear case by L. Greenberg in [19] (similar ideas can be found also in [33]). Indeed, taking advantage of the symplectic structure of the linear Hamiltonian system associated to a linear second order system of the form (1), L. Greenberg [19] has generalized the well-known concepts of elementary phase-plane analysis; he has developed the concepts of lagrangian plane, phase angle and crossing, which correspond, in the case of planar systems, to the notions of line through the origin, polar coordinate and zero of a real function, respectively. Thus, in the present paper we adapt the notion of $h$-type solution given in [8] (inspired by [19]) to our more general (indefinite) framework.

For the statement of our main results (Theorems 1, 2, 3), we focus on the Maslov indices $m_0$ and $m_\infty$ of the linear systems $Ju'' + S_0(t)u = 0$ and $Ju'' + S_\infty(t)u = 0$, respectively. By these indices, we define (cf. [40]) a set $\mathcal{I}$ whose non-emptiness is a sufficient condition (Theorem 1) for the existence of multiple solutions to the given BVP. Theorems 2 and 3 provide sufficient conditions for the non-emptiness of $\mathcal{I}$ (cf. also Proposition 4.9 and Proposition 4.11). For the proofs, we employ the concept of phase angle and the generalized shooting method in order to reformulate the problem in terms of the existence of zeros of an $N$-dimensional vector field. Then, we apply a version of the Miranda fixed point theorem given in [8].
Our results extend the main result in [8] to the case $\nu \neq 0$. In particular, they represent a generalization to indefinite systems of one-dimensional results (we refer, among others, to [11], [12], [13], [18], [30]) where multiplicity is obtained through a comparison between the behaviour of the nonlinearity at zero and infinity. It is worth noticing (as it is explained in detail in Remark 4.7 in [8]) that our approach is similar to the one in [8], but our results are not a consequence of Theorem 4.7 in [8].

Some final comments on the literature are in order. Indeed, we wish to point out that multiplicity for asymptotically linear problems has been achieved, with various different methods, by some authors without a diagonality condition of the form (V3); however, other restrictions are needed. For a comprehensive reference, we refer to the book by J.Mawhin-M.Willem [27]. In particular, in [13], [31] and [38] (the second and third in the framework of elliptic PDEs) the potential $V$ in (3) is even; in our work, we do not deal with such kind of restrictions (cf. Proposition 4.9 for more details). It is also worth noticing that no diagonality condition is imposed also in [10], where, in turn, it is required (for the planar case) a sign condition on $\sigma$. Moreover, we have been able to treat indefinite problems, which may have an infinite Morse index (and whose count of non transverse intersections of the family $\mathcal{L}$ is finite).

In what follows, we set $\{0 \} \subset \{1, \ldots, n\}$. For each $l \in \{0 \} \cup \{1, \ldots, n\}$, we define the poset $(L_{l, l+1})$, where $\{0 \}$ denotes its dual order, we define the poset $(Z_{1, 2})$ as the direct sum of the above defined posets; we also define the poset $(Z_{1, 2})$ as the direct sum of the posets $(Z_{1, 2})$ and $(Z_{2, 2})$. By $\langle \cdot, \cdot \rangle$ we mean the scalar product. We denote by $B_{\text{sym}}(\mathbb{R}^n)$ and $\text{Sp}(n)$ the set of $(n \times n)$ symmetric and symplectic matrices, respectively. Consider the $(2n \times 2n)$ matrix

$$(7) \quad \sigma_n = \begin{pmatrix} 0 & \text{Id}_n \\ -\text{Id}_n & 0 \end{pmatrix},$$

and the standard symplectic form $\omega(z_1, z_2) := \langle \sigma_n z_1, z_2 \rangle$ for $z_i \in \mathbb{R}^{2n}$. We denote by $\mathcal{L} := \mathcal{L}(\mathbb{R}^{2n}, \omega)$ the set of all Lagrangian subspaces of the symplectic space $(\mathbb{R}^{2n}, \omega)$. For $l_0 \in \mathcal{L}$, the set $\Sigma(l_0) = \{ l \in \mathcal{L} : l \cap l_0 \neq \{0\} \}$ is the train or the Maslov cycle of $l_0$. The Lagrangian subspace $\{0\} \oplus \mathbb{R}^n \subset \mathbb{R}^{2n}$ will be denoted by $L_0$ and we refer to it with the name of vertical Lagrangian.

2. Linear symplectic preliminaries

In this section we first recall (according to [34]) the definition of Maslov index; then, we introduce the notion of phase angle (on the lines of [19]). Finally, we state and prove a Sturm-type theorem.

The Maslov index. The Maslov index is a semi-integer homotopy invariant with fixed endpoints of paths $l$ of Lagrangian subspaces of the symplectic vector space $(\mathbb{R}^{2n}, \omega)$ which gives an algebraic count of non transverse intersections of the family $\{l(t)\}_{t \in [0, 1]}$ with a given Lagrangian subspace $l_0$. For each $C^1$-curve $l : [0, 1] \to \mathcal{L}$ of Lagrangian subspaces, we say that $t_0 \in [0, 1]$ is a crossing for $l$ if $l(t_0) \cap l_0 \neq \{0\}$, i.e. $l(t_0) \in \Sigma(l_0)$. Consider $t_0 \in [0, 1]$ and denote by $W$ a lagrangian complement of $l(t_0)$. For $z \in l(t_0)$ and $t$ in a neighbourhood of $t_0$, let $w(t)$ be the unique vector s.t.

$w(t) \in W, \quad z + w(t) \in l(t).$

Let us then set, for $z \in l(t_0),$

$$Q(l, t_0)(z) = \frac{d}{dt} \omega(z, w(t)|_{t = t_0}.$$

Note that $Q$ is independent of $W$. At each crossing $t_0 \in [0, 1]$ we define the crossing form $\Gamma$ as the quadratic form

$$\Gamma(l, t_0) := Q(l, t_0)|_{l(t_0) \cap l_0}.$$
The crossing $t$ is regular if the crossing form is nonsingular. It is easy to check that regular crossings are isolated and therefore on a compact interval they are in a finite number. Assuming that $l$ has only regular crossings, we can give the following

**Definition 2.1.** The Maslov index of the Lagrangian path $l$ relative to the Lagrangian subspace $l_0$ is the semi-integer defined by

$$\mu_{l_0}(l, [0, 1]) := \frac{1}{2} \text{sgn } \Gamma(l, l_0, 0) + \sum_{t \in [0, 1]} \text{sgn } \Gamma(l, l_0, t) + \frac{1}{2} \text{sgn } \Gamma(l, l_0, 1),$$

where $\text{sgn}$ denotes the signature of a quadratic form and the summation runs over all crossings $t$.

It is a standard fact that the above definition can be extended to the case when there exist non-regular crossings. The above definition comes from [34], to which we refer for all the details; we just note that the construction in [34] is developed on the basis of the form $-\sigma_n$ instead of $\sigma_n$.

For other useful references on the Maslov index, we refer to [1], [2], [25].

Given $\psi: [0, 1] \to \text{Sp}(2n)$ a continuous path of symplectic matrices and $l_0$ a lagrangian subspace, then we define the Maslov index of $\psi$ as follows

$$\mu_{l_0}(\psi, [0, 1]) := \mu_{l_0}(\psi(\cdot)(l_0), [0, 1]).$$

We finally give the following

**Definition 2.2.** For $k \in \mathbb{N}$, we say that a crossing $t_0 \in [0, 1]$ has multiplicity $k$ if $k$ is the dimension of the intersection between $l_0$ and $l(t_0)$.

**A phase angle analysis.** Let $S: [0, 1] \to \text{B}_{\text{sym}}(\mathbb{R}^n)$ be a continuous path of symmetric matrices and let us consider the linear second order system

$$Ju''(t) + S(t)u(t) = 0, \quad t \in [0, 1],$$

where $J$ is defined in (2). By performing the change of coordinates $v = Ju'$, (9) can be written as the following first order system

$$\begin{cases}
  u'(t) = Jv(t) \\
  v'(t) = -S(t)u(t).
\end{cases}$$

By taking $w = (u, v)$, (10) takes the Hamiltonian form

$$w'(t) = \sigma_n H(t)w(t),$$

where

$$H(t) = \begin{pmatrix} S(t) & 0 \\ 0 & J \end{pmatrix}$$

for each $t \in [0, 1]$. Under this change of variables the Dirichlet boundary conditions become

$$(w(0), w(1)) \in L_0 \times L_0.$$

In what follows, we shall need the following

**Definition 2.3.** A split matrix is any matrix commuting with $J$.

It is easy to see that any split matrix has the form:

$$S = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

where $A$ and $B$ are symmetric $(n - \nu \times n - \nu)$ and $(\nu \times \nu)$ matrices, respectively. From now on, we will assume the following condition:

(S) the continuous path of symmetric matrices $S: [0, 1] \to \text{B}_{\text{sym}}(\mathbb{R}^n)$ is split.
Let $K$ be the continuous path defined pointwise by
\[
K(t) := \sigma_n H(t) = \begin{pmatrix}
0 & 0 & \text{Id}_{n-\nu} & 0 \\
0 & 0 & 0 & -\text{Id}_\nu \\
-A_{n-\nu}(t) & 0 & 0 & 0 \\
0 & -B_\nu(t) & 0 & 0
\end{pmatrix}
\]
for $S(t) := \begin{pmatrix} A_{n-\nu}(t) & 0 \\ 0 & B_\nu(t) \end{pmatrix}$
and let $U$ be the block matrix
\[
U := \begin{pmatrix}
\text{Id}_{n-\nu} & 0 & 0 & 0 \\
0 & 0 & \text{Id}_{n-\nu} & 0 \\
0 & \text{Id}_\nu & 0 & 0 \\
0 & 0 & 0 & \text{Id}_\nu
\end{pmatrix}.
\]
Denote by $\tilde{K}$ the continuous path defined pointwise by
\[
\tilde{K}(t) := \begin{pmatrix}
0 & \text{Id}_{n-\nu} & 0 & 0 \\
-A_{n-\nu}(t) & 0 & 0 & 0 \\
0 & 0 & 0 & -\text{Id}_\nu \\
0 & 0 & -B_\nu(t) & 0
\end{pmatrix}.
\]
By the change of coordinates $\tilde{w} = Uw$, since $UK = \tilde{K}U$, system (10) reduces to $\tilde{w}' = \tilde{K}\tilde{w}$. If $\tilde{w} := (\tilde{w}_1, \tilde{w}_2)$, the first order system $\tilde{w}' = \tilde{K}\tilde{w}$ can be written as
\[
\begin{cases}
\tilde{w}_1'(t) = k_A(t) \tilde{w}_1(t) \\
\tilde{w}_2'(t) = k_B(t) \tilde{w}_2(t)
\end{cases}
\]
being
\[
k_A(t) := \begin{pmatrix} 0 & \text{Id} \\ -A(t) & 0 \end{pmatrix}, \quad k_B(t) := \begin{pmatrix} 0 & -\text{Id} \\ -B(t) & 0 \end{pmatrix}.
\]
If $L^B_0, L^A_0$ are (respectively) the vertical Lagrangian subspaces in the symplectic spaces $(\mathbb{R}^{2\nu}, \sigma_\nu)$ and $(\mathbb{R}^{2(n-\nu)}, \sigma_{n-\nu})$, we have $UL_0 = L^A_0 \oplus L^B_0$. Thus by setting $\bar{L}_0 := L^A_0 \oplus L^B_0$, the Dirichlet boundary conditions can be written as
\[
(\tilde{w}(0), \tilde{w}(1)) \in \bar{L}_0 \times \bar{L}_0.
\]
For each $t \in [0,1]$, let us consider the $n-\nu$ independent solutions $\tilde{w}_1^1(t), \ldots, \tilde{w}_1^{n-\nu}(t)$ of
\[
\begin{cases}
\tilde{w}_1^j(t) = k_A(t) \tilde{w}_1(t) \\
\tilde{w}_1(0) \in L^A_0
\end{cases}
\]
and the $\nu$ independent solutions $\tilde{w}_2^1(t), \ldots, \tilde{w}_2^\nu(t)$ of
\[
\begin{cases}
\tilde{w}_2^j(t) = k_B(t) \tilde{w}_2(t) \\
\tilde{w}_2(0) \in L^B_0.
\end{cases}
\]
By setting
\[
\tilde{w}_1^j(t) = \begin{pmatrix} u_1^j(t) \\ v_1^j(t) \end{pmatrix}, \quad \tilde{w}_1^{n-\nu}(t) = \begin{pmatrix} u_1^{n-\nu}(t) \\ v_1^{n-\nu}(t) \end{pmatrix}
\]
and
\[
\tilde{w}_2^j(t) = \begin{pmatrix} u_2^j(t) \\ v_2^j(t) \end{pmatrix}, \quad \tilde{w}_2^{\nu}(t) = \begin{pmatrix} u_2^{\nu}(t) \\ v_2^{\nu}(t) \end{pmatrix},
\]
we can define the two matrices $X_1(t) := [u_1^1(t), \ldots, u_1^{n-\nu}(t)], X_2(t) := [u_2^1(t), \ldots, u_2^{\nu}(t)]$ and $X_1'(t) := [v_1^1(t), \ldots, v_1^{n-\nu}(t)], X_2'(t) := [v_2^1(t), \ldots, v_2^{\nu}(t)]$. Since, for each $t \in [0,1]$, the vectors $\{\tilde{w}_1^1, \ldots, \tilde{w}_1^{n-\nu}\}$ and $\{\tilde{w}_2^1, \ldots, \tilde{w}_2^{\nu}\}$ are linearly independent, the matrices $\bar{X}_1(t) = [X_1(t) X_1'(t)]^T$ and $\bar{X}_2(t) = [X_2(t) X_2'(t)]^T$ have rank $n-\nu$ and $\nu$, respectively; hence the matrix $X_j(t) - iX_j(t)$ is invertible for each $j \in 2$. Now we define, for each $t \in [0,1]$, the unitary symmetric matrices
\[
Y_j(t) := (X_j'(t) + iX_j(t))(X_j'(t) - iX_j(t))^{-1} \quad \text{for } j = 1, 2;
\]
and let us denote by $\lambda_j^l(t)$ their spectrum. Here we refer, for instance, to [19, Section 6]. By Kato’s selection Theorem (cf. [21, Chapter II, Section 6]), for each $l \in \{1, \ldots, n-\nu\}$ there exists
a unique continuous map \( \vartheta^1_l : [0, 1] \to \mathbb{R} \) such that \( \lambda^1_l(t) = e^{2i\vartheta^1_l(t)} \) with \( \vartheta^1_l(0) = 0 \) and for each \( l \in \{n-\nu+1, \ldots, n\} \) there exists a unique continuous map \( \vartheta^2_l : [0, 1] \to \mathbb{R} \) such that \( \lambda^2_l(t) = e^{2i\vartheta^2_l(t)} \) with \( \vartheta^2_l(0) = 0 \).

We are now ready for the following

**Definition 2.4.** For \( j \in 2 \), we term phase angles of the system \( (H) \) the continuous functions \( \theta^j_l : [0, 1] \to \mathbb{R} \) obtained by continuously arranging the \( n-\nu \) functions \( \vartheta^1_l \) corresponding to \( Y^1 \) in increasing order and the \( \nu \) functions \( \vartheta^2_l \) corresponding to \( Y^2 \) also in increasing order.

With this setting, we shall write \( \Theta^1 = (\theta^1_1, \ldots, \theta^1_{n-\nu}) \), \( \Theta^2 = (\theta^2_{n-\nu+1}, \ldots, \theta^2_2) \) and \( \Theta = \Theta^1 \oplus \Theta^2 \).

The following lemma explains the relation between the notions of crossing and of phase angle. To this end, we denote by \( \psi \) the fundamental solution associated to the Hamiltonian system \( (L) \).

**Lemma 2.5.** The following facts are equivalent:

1. \( t_0 \in [0, 1] \) is a crossing for \( \psi(\cdot)(\tilde{L}_0) \) (w.r.t. \( \tilde{L}_0 \)) of multiplicity \( \mu \in \mathbb{n} \); 
2. there exist exactly \( \mu \) different integers \( h_1, \ldots, h_\mu \in \mathbb{n} \) and there exist \( j_1, \ldots, j_\mu \in 2 \), \( h_1, \ldots, h_\mu \in \mathbb{Z} \) such that
   \[
   \theta^j_{i_1}(t_0) = h_1\pi, \ldots, \theta^j_{i_\mu}(t_0) = h_\mu\pi.
   \]

In particular, \( h_k \in \mathbb{N} \) if \( j_k = 1 \) and \( h_k \in -\mathbb{N} \) if \( j_k = 2 \).

**Proof.** This result can be proved by arguing as in \([8, \text{Proposition 3.13}]\). The sign of the phase angles can be easily deduced by \([12, \text{Lemma 8.2}]\). \( \Box \)

In what follows we shall write, for each \( t \in [0, 1] \), the phase angles in the following form

\[\theta^j_l(t) = k^j_l(t)\pi + \alpha^j_l(t) , \quad l \in \{1, \ldots, n-\nu\}; \quad \theta^j_l(t) = -k^j_l(t)\pi - \alpha^j_l(t), \quad l \in \{n-\nu+1, \ldots, n\},\]

where, for each \( t \in [0, 1] \), \( k^j_l(t) \in \mathbb{N} \) and \( \alpha^j_l(t) \in (0, \pi) \).

Let us now turn to a step which will be crucial for the proof of our main results.

**Sturm Comparison Principle for scalar equations.** Let us consider the initial value problem

\[
\begin{cases}
    u''(t) + a(t)u(t) = 0 \\
    u(0) = 0, \quad u'(0) = 1
\end{cases}
\]

for some continuous function \( a \). In this case, for every \( t \in [0, 1] \), the matrix \( Y(t) \) reduces to the complex number

\[Y(t) = \frac{\varphi'(t) + i\varphi(t)}{\varphi'(t) - i\varphi(t)}\]

where \( \varphi \) is the solution of \( (20) \). It is easy to show that the unique phase angle \( \theta(t) \) is the argument, in polar coordinates, of the complex number \( \varphi(t) = \varphi'(t) + i\varphi(t) \). We point out that \( \theta(t) \) does not coincide exactly with the usual polar coordinate in the phase-plane; indeed, the phase angle \( \theta(t) \) is measured in the standard Euclidean plane \((x, y, O)\) starting from the \( y \)-axis and in the clockwise sense.

**Lemma 2.6.** Consider

\[u''(t) + a(t)u(t) = 0, \quad t \in [0, 1].\]

Denoting by \( \vartheta^1_a, \vartheta^2_a \) the phase angles associated respectively to the Hamiltonian systems:

\[
\begin{cases}
    u' = v_1 \\
    v'_1 = -au
\end{cases} \quad \begin{cases}
    u' = v_2 \\
    v'_2 = au
\end{cases}
\]

we have

1. \( \vartheta^2_a(t) = -\vartheta^1_a(t) \);
2. if \( a(t) \leq b(t) \) then \( \vartheta^1_a(1) \leq \vartheta^1_b(1) \) (or, equivalently, \( \vartheta^2_a(1) \geq \vartheta^2_b(1) \)).
Proof. We prove (1). By using Prüfer coordinates in the phase plane \((u, u')\), the following relation holds:
\[
\vartheta_a^2(t) = \arctan \left( \frac{u(t)}{v_a(t)} \right) = \arctan \left( \frac{u(t)}{u'(t)} \right) = -\vartheta_a^1(t).
\]
Now the first conclusion in (2) readily follows by the Sturm comparison principle (cf. [9]). By (1), we also deduce that if \(a(t) \leq b(t)\) then \(\vartheta_a^2(1) \geq \vartheta_b^2(1)\) and this concludes the proof. \(\square\)

**Remark 2.7.** Given \(a_i, b_j \in C^0([0, 1])\) with \(i \in \{1, \ldots, n - \nu\}, j \in \{n - \nu + 1, \ldots, n\}\), consider the following uncoupled second order problem
\[
(22) \quad \begin{cases} J u''(t) + \Delta(t) u(t) = 0, & t \in [0, 1] \\ u(0) = 0 = u(1), \end{cases}
\]
where \(\Delta(t) := \text{diag}(a_1(t), \ldots, a_{n-\nu}(t), b_{n-\nu+1}(t), b_n(t))\). Recalling the definition of (not yet arranged) phase angles \(\vartheta^1_i, \vartheta^2_j\), we observe that
\[
\vartheta^1_i \equiv \vartheta^1_{a_i}, \quad i \in \{1, \ldots, n - \nu\} \quad \text{and} \quad \vartheta^2_j \equiv \vartheta^2_{b_j}, \quad j \in \{n - \nu + 1, \ldots, n\}.
\]
Let us now denote (cf. [9]) by \(\eta_j(a)\) the monotone sequence of the simple eigenvalues of the problem
\[
(23) \quad \begin{cases} u''(t) + (a(t) + \eta) u(t) = 0, & \\
\quad u(0) = 0 = u(1). \end{cases}
\]
Recall that \(\lim_{j \to +\infty} \eta_j(a) = +\infty\). Moreover, the eigenfunction corresponding to \(\eta_j(a)\) has exactly \((j - 1)\) zeros on \((0, 1)\). From Sturm’s theory, a relation can be established between the eigenvalues \(\eta_j(a)\) and the phase angle \(\vartheta^1_{a+\eta_j}(t)\) associated to
\[
(24) \quad \begin{cases} u' = v, \quad v' = -(a(t) + \eta) u, & \\
\quad u(0) = 0, \quad v(0) = 1. \end{cases}
\]
More precisely,
\[
(25) \quad \vartheta^1_{a+\eta_j}(1) = j\pi \quad \iff \quad \eta = \eta_j(a),
\]
\[
(26) \quad \vartheta^1_{a+\eta_j}(1) > j\pi \quad \iff \quad \eta > \eta_j(a),
\]
\[
(27) \quad \vartheta^1_{a+\eta_j}(1) < j\pi \quad \iff \quad \eta < \eta_j(a).
\]
We remark that when \(a\) is constant, it is possible to write the explicit expression of \(\eta_j(a)\) for each \(j \in \mathbb{N}\). More precisely, if \(a(t) = a \in \mathbb{R}\) for every \(t \in [0, 1]\), then
\[
(28) \quad \eta_j(a) := j^2 \pi^2 - a \quad \forall j \in \mathbb{N}.
\]

3. **Asymptotically linear Hamiltonian systems**

We shall be concerned with the differential system
\[
(29) \quad Ju''(t) + \nabla V(t, u(t)) = 0,
\]
where \(\nabla V(t, 0) = 0\) for all \(t\). In what follows, we shall assume that \(V : [0, 1] \times \mathbb{R}^n \to \mathbb{R}\) is such that uniqueness and global continuability of solutions to the initial value problems associated to (29) are guaranteed. Since \(u = 0\) is a trivial solution of (29), it follows that the linearized system at zero takes the form \(Ju''(t) + D^2V(t, 0)u = 0\). By setting
\[
(30) \quad S(t, u) := \int_0^1 D^2V(t, su)ds, \quad \forall (t, u) \in [0, 1] \times \mathbb{R}^n,
\]
the system (29) can be written as follows
\[
(31) \quad Ju''(t) + S(t, u(t))u(t) = 0.
\]
In this Section we describe the set of assumptions used for our main results; we then discuss some useful facts, in terms of Maslov index and phase angles, which hold in this framework.

Let us first give the following

**Definition 3.1.** Given a path of symmetric matrices $S$, we say that the equation $Ju''(t) + S(t)u(t) = 0$ is non-degenerate if the (linear) Dirichlet boundary value problem

$$
\begin{align*}
Ju''(t) + S(t)u(t) &= 0, \quad t \in [0, 1], \\
u(0) &= 0 = u(1)
\end{align*}
$$

has only the trivial solution.

From now on we always assume that the strongly indefinite system ([29]) is asymptotically linear at zero and infinity meaning that there exist two continuous paths of symmetric and split matrices $S_0$ and $S_\infty$ such that the following conditions hold:

1. $S(t, \xi) \xi = S_0(t) \xi + o(|\xi|)$ for $|\xi| \to 0$, uniformly in $t$;
2. $S(t, \xi) \xi = S_\infty(t) \xi + o(|\xi|)$ for $|\xi| \to +\infty$, uniformly in $t$.

Moreover, in what follows we suppose

1. $S(t, x)$ is split for every $(t, x) \in [0, 1] \times \mathbb{R}^n$;
2. the equations $Ju''(t) + S_\infty(t)u(t) = 0$ and $Ju''(t) + S_0(t)u(t) = 0$ are non-degenerate.

The absence of $(V_2)$ would not affect the possibility of obtaining a multiplicity result like our Theorem [1]; the only difference would then be in the exact count of the solutions with prescribed "nodal properties". The choice we make of using $(V_2)$ depends on the availability of useful formulas for the computation of the Maslov index (cf. [39], [41]).

Finally, denoting by $W_i$ the $i$-th $(n-1)$-dimensional coordinate hyperplane in $\mathbb{R}^n$, we assume (as in [8]) that the following condition is fulfilled

$(V_3)$ for every $i \in \mathbb{n}$, the restriction of the matrix $S$ to $[0, 1] \times W_i$ is diagonal; i.e.

$$S(t, x) := \text{diag}(\lambda_1^i(t, x), \ldots, \lambda_\nu^i(t, x)), \quad \forall (t, x) \in [0, 1] \times W_i.$$  

From $(V_3)$, it follows that $S_0$ and $S_\infty$ are diagonal matrices too.

Observe now that under the assumption $(V_\infty)$ there exists a constant $M > 0$ such that

$$|(u(0), u'(0))| \leq R \Rightarrow \left|(u(t), u'(t))\right| \leq Re^{\max\{1, M\}} \text{ for each } t \in [0, 1].$$

We refer to [8] for more details on this so-called "elastic property".

For the (nonlinear) self-adjoint second order boundary value problem

$$
\begin{align*}
Ju''(t) + S(t, u(t))u(t) &= 0 \\
u(0) &= 0 = u(1),
\end{align*}
$$

let us now consider for every $\alpha \in \mathbb{R}^n$ the Cauchy problem:

$$
\begin{align*}
Ju''(t) + S(t, u(t))u(t) &= 0 \\
u(0) &= 0, \quad Ju'(0) = \alpha.
\end{align*}
$$

Under the regularity assumptions on $S$, we know that, for each $\alpha \in \mathbb{R}^n$, there exists a unique solution $u_\alpha$ to ([55]). Observe that $u_\alpha$ is a solution of ([54]) if and only if $u_\alpha(1) = 0$.

**Definition 3.2.** We call $\mathcal{Z}$-system associated to ([51]) at $u_\alpha$ the linear second order system

$$Ju''(t) + S_\alpha(t)u(t) = 0$$

where $S_\alpha(t) := S(t, u_\alpha(t))$ for each $t \in [0, 1]$.

For every $\alpha \in \mathbb{R}^n$, we can develop the phase angle analysis for the linear system ([60]) and in particular we can define, according to the previous notation, the matrices $X_{\alpha,j}(t), X'_{\alpha,j}(t)$, the unitary symmetric matrices $Y_\alpha^1 := (X'_{\alpha,1}(t) + iX_{\alpha,1}(t))(X'_{\alpha,1}(t) - iX_{\alpha,1}(t))^{-1}, Y_\alpha^2 := (X'_{\alpha,2}(t) + iX_{\alpha,2}(t))(X'_{\alpha,2}(t) - iX_{\alpha,2}(t))^{-1}$, the angles $\vartheta_{l,\alpha}^1, \vartheta_{l,\alpha}^2$ with $l \in \mathbb{n}$ and the poset $(\Theta_\alpha, \preceq)$. Analogous definitions can be given for the "asymptotic linear systems" at zero and infinity $Ju''(t) + S_0(t)u(t) = 0$, and $Ju''(t) + S_\infty(t)u(t) = 0$, respectively.
Definition 3.3. We say that $u_\alpha$ is a nontrivial $\hbar$-type solution if there exists $\hbar \in (\mathbb{N}^{*n}, \leq^{\ominus})$ such that $\theta_{l,\alpha}^{(1)} = h_l \pi$ for all $l \in \{1, \ldots, n - \nu\}$ and $\theta_{l,\alpha}^{(2)} = -h_l \pi$ for all $l \in \{n - \nu + 1, \ldots, n\}$, where $\theta_{l,\alpha}^{(j)}$ are the phase angles associated to \text{[8]} for $j \in \mathbb{R}$. In particular, $(\mathbb{N}^{*n}, \leq^{\ominus})$ means $(\mathbb{N}^{*(n-\nu)}, \leq) \oplus (\mathbb{N}^{*\nu}, \leq^\oplus)$.

We can now state the following

Lemma 3.4. Under the assumptions $(V_0) - (V_\infty)$ we have

1. $S_\alpha$ tends to $S_\infty$ for $|\alpha| \to +\infty$ strongly in the $L^1$-norm topology;
2. $S_\alpha$ tends to $S_0$ for $|\alpha| \to 0$ strongly in the $L^1$-norm topology.

Proof. We refer to \text{[8]} Proposition 4.4, Proposition 4.5.

As a direct consequence of Lemma 3.4 the result below easily follows.

Corollary 3.5. Under the assumptions $(V_1) - (V_0) - (V_\infty)$ we have:

1. $Y_\infty^\nu$ tends to $Y_\infty^\nu$ in the $C^0$-norm topology, for $|\alpha| \to +\infty$;
2. $Y_\alpha^\nu$ tends to $Y_0^\nu$ in the $C^0$-norm topology for $|\alpha| \to 0$.

In what follows we shall associate an index to a linear second order system of the form \text{[9]}; then, we will show how it can be computed in some particular cases.

Consider the fundamental solution $\phi$ of the first order system \text{[10]}. Being $\phi(\cdot)$ a path of symmetric matrices, it has a Maslov index $\mu_{L_0}(\phi, [\varepsilon, 1])$, where $\varepsilon$ is chosen in such a way that there are no crossings in $(0, \varepsilon]$; for brevity in what follows we shall write $m(S) := \mu_{L_0}(\phi, [\varepsilon, 1])$.

Remark 3.6. The existence of the constant $\varepsilon$ is guaranteed by the fact that the crossing instants of $\phi$ cannot accumulate at 0 (cf. \text{[28]}). Thus, the Maslov index $m(S)$ is well defined and it is independent of the choice of $\varepsilon$.

In what follows we shall be concerned with some results on the computation of the Maslov index.

The Maslov index for constant and split matrices. Consider the second order Dirichlet boundary value problem

\begin{equation}
\begin{cases}
J u''(t) + S u(t) = 0 & \forall t \in [0, 1] \\
u(0) = 0 = u(1).
\end{cases}
\end{equation}

For any real number $a$, let us consider the integer

\begin{equation}
N(a) := \# \{ i \in \mathbb{N}^* | i^2 \pi^2 < a \}.
\end{equation}

Assume that \text{[37]} has only the trivial solution. It is shown in \text{[29]} that the following formula holds:

\begin{equation}
m(S) = \sum_{i=1}^{n-\nu} N(\lambda_i) - \sum_{i=1}^{\nu} N(-\mu_i),
\end{equation}

where $\lambda_i$ and $\mu_i$ are the eigenvalues of $A$ and $B$, respectively. In the particular case when $\nu = 0$ the previous formula reduces to:

\begin{equation}
m(S) = \sum_{i=1}^{n} N(\lambda_i).
\end{equation}

On the other hand, if the equation in \text{[37]} is degenerate then, by Definition 2.1 the following estimate holds:

\begin{equation}
\left| m(S) - \sum_{i=1}^{n-\nu} N(\lambda_i) + \sum_{i=1}^{\nu} N(-\mu_i) \right| \leq \frac{n}{2}.
\end{equation}

Remark 3.7. If $\nu = 0$ and in the non-degenerate case, the integer $m(S)$ agrees with the total number of moments of verticality of $S$ used in \text{[8]} Definition 3.6. In particular, formula \text{[10]} agrees with the formula given in \text{[8]} Remark 3.9. Observe that in the degenerate case the integer used in \text{[8]} does not coincide with the Maslov index; indeed if we compute (in the scalar case) the Maslov index from Definition 2.1 an extra term $\pm 1/2$ appears.
The Maslov index for non constant and split matrices. In what follows, for brevity (and according to Remark 3.4) we shall write the Maslov index \( \mu_{L_0}^-(\psi) \) of the fundamental solution \( \psi \) of the Hamiltonian system (15) (with respect to the symplectic form \( \tilde{\sigma} := \sigma_{n,-\nu} \oplus \sigma_{\nu} \)) with \( \mu_{L_0}^-(\psi) \).

**Lemma 3.8.** If \( \psi(1)(\tilde{L}_0) \cap \tilde{L}_0 = \{0\} \), then the Maslov index of \( \psi \) is given by

\[
\mu_{L_0}^-(\psi) = \sum_{i=1}^{n-\nu} k_i^1(1) - \sum_{i=1}^{\nu} k_i^2(1),
\]

where the integers \( k_i^j \) have been defined in (19).

**Proof.** Recall at first that

\[
\mu_{L_0}^-(\psi) = \mu_{L_0^A}^-(\psi_A) + \mu_{L_0^B}^-(\psi_B).
\]

Denote by \( \psi_A \) the fundamental solution of the first order system in \( \mathbb{R}^{2(n-\nu)} \)

\[
\begin{aligned}
\begin{cases}
u' &= v' \\
v_1' &= -Au_1
\end{cases}
\end{aligned}
\]

and by \( \psi_B \) the fundamental solution of the first order system in \( \mathbb{R}^{2\nu} \)

\[
\begin{aligned}
\begin{cases}
u_2' &= -v_2 \\
v_2' &= -Bu_2.
\end{cases}
\end{aligned}
\]

The direct sum property of the Maslov index implies that

\[
\mu_{L_0}^-(\psi) = \mu_{L_0^A}^-(\psi_A) + \mu_{L_0^B}^-(\psi_B).
\]

If we denote by \( \psi_{-B} \) the fundamental solution of the first order system in \( \mathbb{R}^{2\nu} \)

\[
\begin{aligned}
\begin{cases}
u_2' &= z_2 \\
z_2' &= Bu_2.
\end{cases}
\end{aligned}
\]

then it follows that

\[
\mu_{L_0^A}(\psi_A) + \mu_{L_0^B}(\psi_B) = \mu_{L_0^A}(\psi_A) - \mu_{L_0^B}(\psi_B).
\]

 Systems (43) and (45) are equivalent to the second order systems

\[
\begin{aligned}
u''_1 + Au_1 &= 0, \\
u''_2 - Bu_2 &= 0.
\end{aligned}
\]

These systems are of the form studied in [3]. Now the thesis follows by using [3] Proposition 3.12].

**Lemma 3.9.** The following equality holds

\[
\mu_{L_0}^-(\psi) = m(S).
\]

**Proof.** Let us introduce on \( \mathbb{R}^{2n} \) the symplectic form \( \omega_1 \), by setting \( \omega_1(z_1, z_2) := (\tilde{\sigma} z_1, z_2) \) for all \( z_i \in \mathbb{R}^{2n} \) with \( i = 1, 2 \). The proof then immediately follows by the naturality property of the Maslov index (cf. [35]), combined with the fact that the matrix \( U : (\mathbb{R}^{2n}, \omega) \rightarrow (\mathbb{R}^{2n}, \omega_1) \) is a symplectic isomorphism.

We end this section with some preliminary consequences of our assumptions.

**Maslov index and phase angles.** We denote by \( m_0 \) and \( m_\infty \), respectively, the Maslov indices of the fundamental solution of the linear Hamiltonian systems at zero and at \( \infty \). (As a direct consequence of Lemma 3.9 we do not need to specify the symplectic structure we are referring to). Assuming \( m_0 + n < m_\infty \), we define the following set

\[
\mathcal{S} := \left\{ \hat{\lambda} \in (\mathbb{N}^n, \leq^\oplus) : \langle \hat{\lambda}, j \rangle \in (m_0 + n - \nu, m_\infty - \nu) \right\}.
\]

**Remark 3.10.** In the case \( m_\infty + n < m_0 \), it is enough to define the set \( \mathcal{S} \) as follows

\[
\mathcal{S}' := \left\{ \hat{\lambda} \in (\mathbb{N}^n, \leq^\oplus) : \langle \hat{\lambda}, j \rangle \in (n + n - \nu, m_0 - \nu) \right\}.
\]
Lemma 3.11. If $\mathcal{I} \neq \emptyset$ then there exists $\varepsilon > 0$ such that the following inequalities hold:

\begin{align}
(49) \quad & \varepsilon < \langle h, \pi \rangle - (m_0 + n - \nu)\pi, \quad \text{and} \quad (m_\infty - \nu)\pi - \langle h, \pi j \rangle > \varepsilon, \quad \forall \ h \in \mathcal{I}.
\end{align}

Moreover, there exists $\alpha_0 := \alpha_\varepsilon$ small enough such that

\begin{align}
(50) \quad & |\alpha| \leq \alpha_0 \Rightarrow \langle \Theta_\alpha(1), 1 \rangle < \langle \Theta_0(1), 1 \rangle + \varepsilon \quad \text{and} \quad \langle \Theta_\alpha(1), 1 \rangle < (m_0 + n - \nu)\pi + \varepsilon.
\end{align}

Furthermore, there exists $\alpha_\infty > \alpha_0 > 0$ such that

\begin{align}
(51) \quad & |\alpha| \geq \alpha_\infty \Rightarrow \langle \Theta_\alpha(1), 1 \rangle > \langle \Theta_\infty(1), 1 \rangle - \varepsilon \quad \text{and} \quad \langle \Theta_\alpha(1), 1 \rangle > (m_\infty - \nu)\pi - \varepsilon.
\end{align}

Proof. The conclusion follows from Corollary 3.5 and Lemma 3.8. \hfill \Box

Define $R := \alpha_\infty$. By (33), it follows that there exists $M > 0$ such that

\begin{align}
(52) \quad & |\alpha| \leq \alpha_\infty \Rightarrow \langle (u_\alpha(t), u'_\alpha(t)) \rangle \leq \alpha_\infty e^{\max(1, M)} \quad \forall t \in [0, 1].
\end{align}

For $r \in (0, R)$, let $\mathcal{D}_r$ be the conical shell defined by

\begin{align}
\mathcal{D}_r := \{ x \in \mathbb{R}^n : r \leq |x| \leq R, \quad x_i \geq 0, \quad \forall i \in n \}.
\end{align}

Recall that, for each $i \in n$, we denote by $W_i$ the $i$-th $(n-1)$-dimensional coordinate hyperplane in the Euclidean space $\mathbb{R}^n$. Let $\alpha_i := (\alpha_1, \ldots, \alpha_{i-1}, 0, \alpha_{i+1}, \ldots, \alpha_n) \in \mathcal{D}_{\alpha_\infty} \cap W_i$ and let us consider the corresponding $\mathcal{Z}$-system given by:

\begin{align}
Ju''(t) + S_\alpha(t)u(t) = 0,
\end{align}

where $S_\alpha(t) := S(t, u_\alpha(t))$ for the solution $u_\alpha$ of the initial value problem (33) with $\alpha = \alpha_i$.

Consider also the eigenvalues $\lambda_{\alpha_i}^1(t), \ldots, \lambda_{\alpha_i}^m(t)$ of $S_\alpha(t)$.

Let us denote by $D^{n-1}_\alpha$ the $(n-1)$-dimensional closed disk of radius $\alpha_\infty e^{\max(1, M)}$ contained in the hyperplane $W_i$ and by $C_i$ the $n$-dimensional (full) cylinder $[0, 1] \times D^{n-1}_\alpha$. Since $\alpha_i \in \mathcal{D}_{\alpha_\infty} \cap W_i$, by (52) it follows that $u_\alpha(t) \in D^{n-1}_\alpha$ for each $t \in [0, 1]$. Thus, by assumption (V$_3$)

\begin{align}
S_\alpha(t) = \text{diag} (\lambda_1^i(t, u_\alpha(t)), \ldots, \lambda_m^i(t, u_\alpha(t))), \quad \forall t \in [0, 1],
\end{align}

whence we deduce that $\lambda_{\alpha_i}^1(t) = \lambda_1^i(t, u_\alpha(t))$. Define

\begin{align}
\lambda_k^i := \max\{ \lambda_k^i(z) : z \in C_i \} \quad \forall k = 1, \ldots, n.
\end{align}

Notice that

\begin{align}
(55) \quad & \lambda_{\alpha_i}^k(t) \leq \lambda_k^i \quad \forall t \in [0, 1].
\end{align}

For each $i$, we define the two sets of permutations

\begin{align}
(56) \quad & \sigma_1^i : \{1, \ldots, n - \nu \} \rightarrow \{1, \ldots, n - \nu \} \quad \text{and} \quad \sigma_2^i : \{n - \nu + 1, \ldots, n\} \rightarrow \{n - \nu + 1, \ldots, n\}
\end{align}

and the vectors

\begin{align}
(57) \quad & \Lambda^1 := (\Lambda^1_1, \ldots, \Lambda^1_{n-\nu}), \quad \text{and} \quad \Lambda^2 := (\Lambda^2_{n-\nu+1}, \ldots, \Lambda^2_n)
\end{align}

obtained by respectively arranging in increasing order the components of

\begin{align}
(58) \quad & \Lambda^1 := (\lambda_1^i, \ldots, \lambda_{n-\nu}^i), \quad \text{and} \quad \Lambda^2 := (\lambda_{n-\nu+1}^i, \ldots, \lambda_n^i).
\end{align}

Denoting by $\Lambda$ the diagonal matrix given by

\begin{align}
\text{diag} (\Lambda^1_1, \ldots, \Lambda^1_{n-\nu}, \Lambda^2_{n-\nu+1}, \ldots, \Lambda^2_n),
\end{align}

we consider the second order boundary value problem

\begin{align}
(59) \quad & Ju''(t) + \Lambda u(t) = 0, \quad t \in [0, 1],
\end{align}

\begin{align}
(60) \quad & u(0) = 0 = u(1).
\end{align}

For each $h = (h_1, \ldots, h_n) \in (N^n, \leq^\Theta)$ we set (recalling that $\eta_i$ are the eigenvalues of problem (23))

\begin{align}
(61) \quad & \delta_h := (\eta_1 \Lambda^1_1, \ldots, \eta_{n-\nu} \Lambda^1_{n-\nu}, \eta_{n-\nu+1} \Lambda^2_{n-\nu+1}, \ldots, \eta_n \Lambda^2_n).
\end{align}
4. The main results

The main idea in order to prove our results is to use the Miranda’s fixed point theorem. For the sake of completeness, we recall it in a formulation suitable for the situation we are dealing with.

**Theorem 4.1.** ([8 Theorem 2.1]). Let $f: \mathcal{D}_r \to \mathbb{R}^n$ be a continuous vector field and assume that the following conditions hold:

1. $\sum_{i=1}^{n} f_i(\alpha) < 0$ for $|\alpha| = r$; $\sum_{i=1}^{n} f_i(\alpha) > 0$ for $|\alpha| = R$;
2. $f_i(\alpha) < 0$ for $\alpha \in \mathcal{D}_r \cap W_i$ and $i \in n$.

Then there exists at least one point $\tilde{\alpha}$ in the interior of $\mathcal{D}_r$ such that $f(\tilde{\alpha}) = 0$.

**Remark 4.2.** The statement of Theorem 4.1 holds true if we replace condition (2) with

(2’) $f_i(\alpha) > 0$ for $\alpha \in \mathcal{D}_r \cap W_i$ and $i \in n$.

Now, let $\mathcal{D}$ be the conical shell $\mathcal{D}_{\alpha_0}^\infty$; for any $\mathbf{h} \in \mathcal{I}$, define $f: \mathcal{D} \to \mathbb{R}^n$ as the continuous vector field whose components are given by

$$f_i(\alpha) := \begin{cases} \theta_{i,\alpha}(1) - h_i \pi, & i \in \{1, \ldots, n - \nu\}, \\ h_i \pi + \theta_{i,\alpha}^2(1) & i \in \{n - \nu + 1, \ldots, n\} \end{cases}$$

where, for $j \in 2$, $\theta_{i,\alpha}^j$ are the phase angles associated to the $\mathcal{L}$-system (60).

**Lemma 4.3.** Assume $(V_1) - (V_2) - (V_0) - (V_\infty)$. Then the following inequalities hold:

$$\sum_{i=1}^{n} f_i(\alpha) < 0 \text{ for } |\alpha| = \alpha_0; \quad \sum_{i=1}^{n} f_i(\alpha) > 0 \text{ for } |\alpha| = \alpha_\infty.$$

**Proof.** These are consequences of the second and third inequalities in Lemma 3.11 and (19). $\square$

We are now ready to state and prove our main result.

**Theorem 1.** Let $n \geq 2$. Assume that the conditions $(V_0) - (V_\infty) - (V_1) - (V_2) - (V_3)$ hold. Suppose that

$$\mathcal{I} \neq \emptyset.$$

Then the boundary value problem (34) has $2^n$ distinct $\mathbf{h}$-type solutions, for every $\mathbf{h} \in \mathcal{I}$.

**Proof.** We fix $\mathbf{h} \in \mathcal{I}$ and we prove at first the existence of $\alpha = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n) \in \mathcal{D}$ and of a solution $u$ of $\mathbf{h}$-type such that $Ju'(0) = \tilde{\alpha}$. To this end, let $f: \mathcal{D} \to \mathbb{R}^n$ be the continuous vector field whose components are defined in the equation (61). By taking into account Lemma 4.3 it follows that the first condition of Theorem 4.1 holds.

In order to conclude the proof of the Theorem it is enough to show that also the second condition of Theorem 4.1 holds, i.e.

$$\begin{cases} f_i(\omega_\alpha) := \theta_{i,\alpha}^1(1) - h_i \pi < 0, & \text{for each } i \in \{1, \ldots, n - \nu\}, \quad \omega_\alpha \in \mathcal{D} \cap W_i \\ f_i(\omega_\alpha) := h_i \pi + \theta_{i,\alpha}^2(1) < 0, & \text{for each } i \in \{n - \nu + 1, \ldots, n\}, \quad \omega_\alpha \in \mathcal{D} \cap W_i. \end{cases}$$

Let us fix $\omega_\alpha \in \mathcal{D} \cap W_i$. As observed in (63), by assumption $(V_3)$ it follows that for each $i \in n$

$$S_{\omega_\alpha}(t) \text{diag } (\lambda_{\omega_\alpha}^1(t), \ldots, \lambda_{\omega_\alpha}^n(t)).$$
Hence, taking into account Remark 2.7 and the definition of phase angles for (30), we first note that
\[
\theta^1_{k,\omega} \equiv \theta^1_{k,\omega_1} \quad \text{and} \quad \theta^2_{k,\omega} \equiv \theta^2_{k,\omega_1}.
\]

First, we fix \(i \in \{1, \ldots, n - \nu\}\). Observe that (due to the Sturm comparison principle stated in Lemma 2.6) the permutation \(\sigma^1_i\) introduced in (30) to arrange in increasing order the constants \(\lambda_k\) arranges in increasing order the constants \(\lambda_k\) (as well, i.e.
\[
\theta^1_{k,\omega}(1) \leq \theta^1_{k,\omega_1}(1) \quad \text{if} \quad k \leq h.
\]

On the other hand, in general the permutation \(\sigma^1_i\) does not arrange the angles \(\vartheta^1_{k,\omega_1}(1)\). Indeed, in general, \(\theta^1_{k,\omega_1}(1) \neq \theta^1_{\sigma^1_i(k),\omega_1}(1)\). However, recalling that the definition of \(\theta^1_{i,\omega_1}(1)\) comes from the arrangement in increasing order of the angles \(\vartheta^1_{k,\omega_1}(1)\), or, equivalently, of the angles \(\vartheta^1_{\sigma^1_i(k),\omega_1}(1)\), we infer that
\[
\forall i \in \{1, \ldots, n - \nu\} \quad \exists k_i \in \{1, \ldots, i\} : \quad \theta^1_{k,\omega_1}(1) \leq \theta^1_{\sigma^1_i(k),\omega_1}(1).
\]

Now, we fix \(i \in \{n - \nu + 1, \ldots, n\}\). Taking into account that the permutation \(\sigma^2_i\) introduced in (30) arranges in increasing order the constants \(\lambda_k\), by applying Lemma 2.6 it is easy to verify that
\[
\theta^2_{k,\omega_1}(1) \leq \theta^2_{\sigma^2_i(k),\omega_1}(1) \quad \text{if} \quad k \leq h.
\]

Moreover, recalling that also \(\theta^2_{\sigma^2_i(k),\omega_1}(1)\) comes from the arrangement in increasing order of the angles \(\vartheta^2_{\sigma^2_i(k),\omega_1}(1)\), we conclude that
\[
\forall i \in \{n - \nu + 1, \ldots, n\} \quad \exists l_i \in \{n - \nu + 1, \ldots, i\} : \quad \theta^2_{k,\omega_1}(1) \leq \theta^2_{\sigma^2_i(l_i),\omega_1}(1).
\]

As a next step, taking into account the relations (55) and (64), we apply the Sturm comparison principle stated in Lemma 2.6 to prove that
\[
\theta^1_{k,\omega_1}(1) \leq \theta^1_{k,\omega_1}(1), \quad k \in \{1, \ldots, n - \nu\},
\]
\[
\theta^2_{k,\omega_1}(1) \leq \theta^2_{k,\omega_1}(1), \quad k \in \{n - \nu + 1, \ldots, n\}.
\]

Thus, by combining (66), (69) and (65) with the fact that \(k_i \leq i\), we deduce that
\[
\theta^1_{k,\omega_1}(1) \leq \theta^1_{\sigma^1_i(k),\omega_1}(1) \leq \theta^1_{k,\omega_1}(1) \leq \theta^1_{\sigma^1_i(k),\omega_1}(1), \quad i \in \{1, \ldots, n - \nu\}.
\]

To complete the proof, we recall that \(\eta_h \left(\lambda_{\sigma^1_i(h)}\right) > 0\) since \(h \in \mathcal{H}\). Hence, by (24), we obtain
\[
\vartheta^1_{\sigma^1_i(h)}(1) < h_i \pi, \quad i \in \{1, \ldots, n - \nu\},
\]
which implies the validity of the first inequalities in (63), i.e.
\[
f_i(\eta) := \theta^1_{k,\omega_1}(1) - h_i \pi < 0, \quad \forall i \in \{1, \ldots, n - \nu\}.
\]

To prove the validity of the second inequalities of (63), we first combine (68), (70) and (71) with the fact that \(l_i \leq i\) to infer
\[
\theta^2_{k,\omega_1}(1) \leq \theta^2_{\sigma^2_i(l_i),\omega_1}(1) \leq \theta^2_{k,\omega_1}(1) \leq \theta^2_{\sigma^2_i(l_i),\omega_1}(1), \quad i \in \{n - \nu + 1, \ldots, n\}.
\]

Secondly, recall that \(\eta_i \left(-\lambda_{\sigma^2_i(h)}\right) < 0\) since \(h \in \mathcal{H}\). Thus, from (26), we deduce that
\[
\vartheta^2_{-\lambda_{\sigma^2_i(h)}}(1) > h_i \pi, \quad i \in \{n - \nu + 1, \ldots, n\},
\]
which, according to Lemma 2.6, can be equivalently written as
\[
\theta^2_{-\lambda_{\sigma^2_i(h)}}(1) < -h_i \pi, \quad i \in \{n - \nu + 1, \ldots, n\}.
\]
We can finally conclude that
\[ f_i(\alpha) := \theta^2_{\alpha, i}(1) + h_i \pi < 0, \quad \forall i \in \{n - \nu + 1, \ldots, n\}, \]
which completes the proof of (63). Thus, Theorem 4.1 guarantees the existence of \( \tilde{\alpha} \) in the interior of \( \mathcal{D} \) such that \( f_i(\tilde{\alpha}) = 0 \) for every \( i \in n \). Taking into account Lemma 2.5, we deduce that all the solutions of
\[ Ju'' + S_{\tilde{\alpha}}(t)u(t) = 0, \quad u(0) = 0 \]
verify \( u(1) = 0 \). Since \( u_\alpha \) solves (73), we have proved the existence of a solution \( u \) of (54) of \( h \)-type such that \( Ju' = \tilde{\alpha} \) and \( u'(0) \geq 0 \).

In order to prove the existence of the other solutions it is enough to apply the abstract result in the remaining \( 2^{n-1} \) conical shells contained in the remaining hyper-octants determined by the coordinate planes. \( \square \)

**Remark 4.4.** Consider \( h = (h_1, \ldots, h_n) \in \mathcal{T} \). By definition,
\[ \eta_{h_i} \left( -\lambda_{\sigma^2(i)} \right) < 0, \quad \forall i \in \{n - \nu + 1, \ldots, n\}. \]
From (28), we know that \( \eta_{h_1} \left( -\lambda_{\sigma^2(i)} \right) h_i^2 \pi^2 + \lambda_{\sigma^2(i)} \), and, consequently,
\[ \lambda_{\sigma^2(i)} < -h_i^2 \pi^2 \leq -\pi^2 \quad \forall i \in \{n - \nu + 1, \ldots, n\}. \]
In particular, we have shown that \( \lambda_{\sigma^2(i)} \) should be negative to guarantee that \( \mathcal{T} \neq \emptyset \).

**Remark 4.5.** We may redefine \( \tilde{\alpha} \) and \( \mathcal{T} \) by replacing in (59) and (60) the constants \( \lambda^1_k \) introduced in (54) with the following functions
\[ \lambda^1_k(t) := \max\{\lambda^1_k(t, x) : x \in D_i\} \quad \forall k, i \in n. \]
Then, it is easy to prove that (under minor modifications) Theorem 1 holds true in this more general setting.
We remark that the maps \( \lambda_{\sigma^2(i)}(\cdot) \), obtained by arranging in increasing order the maps (75), do not need to be negative in the whole interval \([0, 1]\) to satisfy (74); it is sufficient that they are negative on a subset of \([0, 1]\) of positive measure.

Now we give a sufficient condition in order to guarantee that \( \mathcal{T} \neq \emptyset \).

**Lemma 4.6.** A sufficient condition in order that \( \mathcal{T} \neq \emptyset \) is given by
\[ \lambda_{\sigma^2(i)} < -\pi^2 \quad \forall i \in \{n - \nu + 1, \ldots, n\}, \]
\[ m_0 - \frac{n}{2} + \nu < m(\Delta) < m_{\infty} - \frac{5}{2} n + \nu. \]

**Proof.** Denoting by \( m(\Delta) \) the Maslov index associated to the system (58), by formula (41) it directly follows that
\[ m(\Delta) - \frac{n}{2} \leq \sum_{i=1}^{n-\nu} N(\lambda_{\sigma^2(i)}) - \sum_{i=n-\nu+1}^{n} N(-\lambda_{\sigma^2(i)}) \leq m(\Delta) + \frac{n}{2}. \]
According to (28), we easily observe that
\[ N(\lambda_{\sigma^2(i)}) < h_i - 1 \implies \eta_{h_i}(\lambda_{\sigma^2(i)}) > 0, \quad i \in \{1, \ldots, n - \nu\}, \]
\[ N(-\lambda_{\sigma^2(i)}) \geq h_i \implies \eta_{h_i}(-\lambda_{\sigma^2(i)}) < 0, \quad i \in \{n - \nu + 1, \ldots, n\}, \]
for each $h_l \in \mathbb{N}$, with $l \in \mathbb{n}$. Let us define

$$\tilde{h}_i := N \left( \overrightarrow{\lambda}_{\sigma^1(i)}^+ \right) + 2, \quad i \in \{1, \ldots, n - \nu\}$$

$$\tilde{h}_i := N \left( -\overrightarrow{\lambda}_{\sigma^2(i)}^+ \right), \quad i \in \{n - \nu + 1, \ldots, n\}.$$ 

According to the first assumption in (76), $\tilde{h}_i \geq 1$ for each $i \in \mathbb{n}$.

Our aim consists in proving that the vector $\tilde{\mathbf{h}} := (\tilde{h}_1, \ldots, \tilde{h}_n)$ belongs to $\mathcal{F}$. From (78)-(79), we immediately deduce that

$$\overrightarrow{\delta}_\mathbf{h} \geq 0.$$ 

Moreover, by formula (77), we obtain that

$$m(\overrightarrow{\Sigma}) - \frac{n}{2} + 2(n - \nu) \leq \langle \tilde{\mathbf{h}}, \mathbf{j} \rangle \leq m(\overrightarrow{\Sigma}) + \frac{n}{2} + 2(n - \nu).$$

Hence, taking into account the second assumption in (76), we infer that

$$m_0 + n - \nu = m_0 - \frac{n}{2} + \nu - \frac{n}{2} + 2(n - \nu) < \langle \overrightarrow{\mathbf{u}}, \mathbf{j} \rangle < m_\infty - \frac{5}{2} + \frac{n}{2} + 2(n - \nu) = m_\infty - n.$$

Recalling the definition of $\mathcal{F}$, we conclude that $\tilde{\mathbf{h}} \in \mathcal{F}$. \hfill \Box

Note that the first condition in (76) agrees with Remark 4.4. According to Lemma 4.6, we can thus write

**Theorem 2.** Let $n \geq 2$. We assume that the conditions $(V_0) - (V_\infty) - (V_1) - (V_2) - (V_3)$ hold. Suppose moreover that $m_0 + 2n < m_\infty$. If condition (76) is satisfied, then the boundary value problem (31) has $2^n$ distinct $h$-type solutions, for every $\tilde{\mathbf{h}} \in \mathcal{F}$.

According to formula (39), we note that condition (76) can be refined if the equation in (58) is non-degenerate. More precisely, we can prove the following result.

**Theorem 3.** Let $n \geq 2$. We assume that the conditions $(V_0) - (V_\infty) - (V_1) - (V_2) - (V_3)$ hold. Moreover, suppose that the equation $Ju''(t) + \overrightarrow{\Delta}u(t) = 0$ is non-degenerate, and that the following conditions are satisfied:

$$\overrightarrow{\lambda}_{\sigma^1(i)}^+ < -\pi^2 \quad \forall i \in \{n - \nu + 1, \ldots, n\}, \quad m_0 < m(\overrightarrow{\Sigma}) < m_\infty - n.$$ 

Then the boundary value problem (31) has $2^n$ distinct $h$-type solutions, for every $\tilde{\mathbf{h}} \in \mathcal{F}$.

*Proof.* To prove this result, we can argue exactly as before. We have to use formula (39) instead of (31) and we should take into account that in the non-degenerate case the relation (78) can be relaxed into the following

$$N \left( \overrightarrow{\lambda}_{\sigma^1(i)}^+ \right) < \tilde{h}_i \implies \eta_{\mathbf{i}} \left( \overrightarrow{\lambda}_{\sigma^1(i)}^+ \right) > 0, \quad i \in \{1, \ldots, n - \nu\},$$

since, for each $i \in \{1, \ldots, n - \nu\}$, there is no $k \in \mathbb{N}$ such that $\overrightarrow{\lambda}_{\sigma^1(i)}^+ = k^2 \pi^2$. According to (81), we should now choose $\tilde{h}_i := N \left( \overrightarrow{\lambda}_{\sigma^1(i)}^+ \right) + 1$, for each $i \in \{1, \ldots, n - \nu\}$. Then, the thesis easily follows. \hfill \Box

**Remark 4.7.** A result analogue to Theorem 4 can be written by reversing the first inequality in the definition of the set $\mathcal{F}$; moreover, according to Remark 5.10 an analogous statement can be written by exchanging $m_0$ and $m_\infty$ in the second inequality in the definition of the set $\mathcal{F}$. In both cases, variants of Theorem 2 and Theorem 3 can be obtained accordingly.
Remark 4.8. Note that (82) can be written in the equivalent form
\[
\begin{cases}
u''(t) + JS(t, u(t))u(t) = 0 \\
u(0) = 0 = \nu(1).
\end{cases}
\]

The split assumption \((V_1)\) guarantees that \(JS : [0, 1] \rightarrow B_{\text{sym}}(\mathbb{R}^n)\) remains a continuous path of symmetric matrices, whenever \(S : [0, 1] \rightarrow B_{\text{sym}}(\mathbb{R}^n)\) is continuous. In particular, the presence of this symmetry enables us to handle problem (82) with the methods employed in [8]. Given the equation
\[
u''(t) + JS(t)\nu(t) = 0,
\]
and introduced \(z := (\nu, \nu')\), one can denote by \(\hat{\psi}_{JS}\) the fundamental solution of
\[
z' = \begin{pmatrix} 0 & \text{Id} \\ -JS & 0 \end{pmatrix} z.
\]
Assuming the validity of conditions \((V_0) - (V_3)\) and applying directly Theorem 4.7 in [8], it is possible to prove the existence of \(2^n\) nontrivial \(\mathbb{S}\)-type solutions, whenever \(\mathbb{S}\) belongs to a suitable, non empty subset \(\hat{\mathcal{J}}\) of
\[
\hat{\mathcal{J}} := \{ (\mathbb{S} \in (\mathbb{R}^n, \mathbb{Z}) : (\mathbb{S}, \mathbb{L}) \in (\mu_0 + n, \mu_\infty) \}
\]
where \(\mu_0, \mu_\infty\) are, respectively, the Maslov indices of the fundamental solutions \(\hat{\psi}_{JS_0}\) and \(\hat{\psi}_{JS_\infty}\) relative to \(S_0\).

According to Lemma 3.8 and to the previous notation, we can observe that
\[
m_0 := \mu_{L_0} A_0 (\psi_{-A_0}) - \mu_{B_0} B_0 (\psi_{-B_0}) \quad \text{while} \quad \mu_0 := \mu_{A_0} A_0 (\psi_{-A_0}) + \mu_{B_0} B_0 (\psi_{-B_0})
\]
\[
m_\infty := \mu_{L_\infty} A_\infty (\psi_{-A_\infty}) - \mu_{B_\infty} B_\infty (\psi_{-B_\infty}) \quad \text{while} \quad \mu_\infty := \mu_{A_\infty} A_\infty (\psi_{-A_\infty}) + \mu_{B_\infty} B_\infty (\psi_{-B_\infty})
\]
whenever
\[
S_0 = \begin{pmatrix} A_0 & 0 & 0 \\ 0 & B_0 \end{pmatrix} \quad \text{and} \quad S_\infty = \begin{pmatrix} A_\infty & 0 & 0 \\ 0 & B_\infty \end{pmatrix}.
\]
These relations allow us to conclude that the two approaches are deeply different, since, in general, they provide solutions with different nodal properties. Indeed, if we focus our attention on the sets \(\mathcal{J}\) (whose definition is given in (17)) and \(\hat{\mathcal{J}}\), we notice that the intervals \((m_0 + n - \nu, m_\infty - \nu)\) and \((\mu_0 + n, \mu_\infty)\) coincide in the case \(\nu = 0\), but, in general, they are not comparable.

We end this section with two propositions which deal with the question of the emptiness/nonemptiness of \(\hat{\mathcal{J}}\).

We first consider the case when we are in presence of radial symmetry.

Proposition 4.9.
\[
(83) \quad S(t, x) = S(t, |x|) \quad \text{for every} \quad (t, x) \in [0, 1] \times \mathbb{R}^n \quad \implies \quad \hat{\mathcal{J}} = \emptyset.
\]

Proof. According to assumption \((V_3)\) and definitions (53) - (50), we observe that \(\overline{\lambda}_k\) and, consequently, \(\sigma^1, \sigma^2\) are independent of \(i \in \mathbb{n}\). In particular, let us set
\[
\overline{\lambda}_k := \lambda_k, \quad \sigma^i := \sigma^i, \quad \forall i, k \in \mathbb{n}, \quad \forall j \in \mathbb{2}.
\]
Arising by contradiction, assume the existence of \(\mathbb{S} \in \mathcal{J}\).
Fix \(\mathbb{S} \in \mathcal{J} \cap W_1\). As in (66), we observe that for every \(k \in \{1, \ldots, n - \nu\}\) there exists \(l_k \in \{1, \ldots, k\}\) such that \(\theta^1_{\sigma^1(k)}(1) \leq \theta^1_{\sigma^1(k)}(1) \leq \theta^1_{\sigma^1(k)}(1) \leq \theta^1_{\sigma^1(k)}(1) \leq \theta^1_{\sigma^1(k)}(1), \quad \forall k \in \{1, \ldots, n - \nu\}.
\]
Since \(\mathbb{S} \in \mathcal{J}\), by definition,
\[
\eta_{h_{k_1}} (\overline{\lambda}_{\sigma^1(k)}) > 0, \quad \forall k \in \{1, \ldots, n - \nu\},
\]
which, according to (27), leads to
\[
\overline{\sigma}^1_{\sigma^1(k)}(1) < h_{k_1} \pi, \quad k \in \{1, \ldots, n - \nu\}.
\]
Taking into account the definition of $f$ given in (61), we can conclude that
\[ f_k(\varphi_i) := \theta^1_{i,\varphi}(1) - h_k\pi < 0, \quad \forall k \in \{1,\ldots,n\}, \forall \varphi_i \in \mathcal{D} \cap W_1. \]

Following analogous steps, we can prove that
\[ f_k(\varphi_i) := \theta^2_{i,\varphi}(1) + h_k\pi < 0, \quad \forall k \in \{n - \nu + 1,\ldots,n\}, \forall \varphi_i \in \mathcal{D} \cap W_1. \]

Considering $\alpha^* := (0,\beta)$ with $\beta \in \mathbb{R}^{n-1}$ and $|\beta| = \alpha_\infty$, we observe that $\alpha^* \in \mathcal{D} \cap W_1$. From the previous inequalities, we infer that
\[ \sum_{k=1}^{n} f_k(\alpha^*) < 0, \quad |\alpha^*| = \alpha_\infty, \]

which contradicts Lemma 4.3. This implies that $\mathcal{T} = \emptyset$. \hfill $\square$

**Remark 4.10.** In the radial symmetric case when $S(t,x) = S(t,|x|)$ for every $(t,x) \in [0,1] \times \mathbb{R}^n$, it is possible to prove that
\[ m(\Delta) > m_\infty - \frac{3}{2}n - \frac{\varepsilon}{\pi}. \]

Moreover, if the equation $Ju''(t) + \Delta u(t) = 0$ is non-degenerate, then (85) can be refined into the following
\[ m(\Delta) > m_\infty - n - \frac{\varepsilon}{\pi}. \]

The above inequalities show that in the radial case the sufficient condition (76) for the non-emptiness of the set $\mathcal{T}$ is violated. Notice that in the case $n = \nu$, the contradiction follows from the fact that $m(\Delta)$ is a semi-integer, and, without loss of generality, $\varepsilon$ in Lemma 3.11 can be chosen smaller than $\frac{\varepsilon}{\pi}$.

In order to end our discussion on the emptiness/non-emptiness of the set $\mathcal{T}$, let us first observe that from Corollary 3.3 there exists $\tilde{\alpha}_\infty$ such that
\[ |\alpha| \geq \tilde{\alpha}_\infty \implies \theta^1_{i,\alpha}(1) > \theta^1_{i,\infty}(1) - \frac{\varepsilon}{n}, \forall i \in n, \forall j \in 2. \]

with $\varepsilon$ as in Lemma 3.11.

Remark also that (86) implies
\[ |\alpha| \geq \tilde{\alpha}_\infty \implies \langle \Theta_\alpha(1), \mathbb{1} \rangle \geq \langle \Theta_\infty(1), \mathbb{1} \rangle - \varepsilon \quad \text{and} \quad \langle \Theta_\alpha(1), \mathbb{1} \rangle > \langle m_\infty - \nu \rangle \pi - \varepsilon. \]

On the other hand, if we go back to the constant $\alpha_\infty$ (whose existence is proved in Lemma 3.11) then we have

**Proposition 4.11.**
\[ \alpha_\infty \geq \tilde{\alpha}_\infty \implies \mathcal{T} = \emptyset. \]

**Proof.** Assume, by contradiction, the existence of $h \in \mathcal{T}$. For each $i \in n$ and for every vector $\beta^i \in \mathcal{D} \cap W_i$, let us define
\[ \Theta^*(\beta^1,\ldots,\beta^n) := \left( \theta^1_{1,\beta^1}(1),\ldots,\theta^1_{n-\nu,\beta^{n-\nu}}(1),\theta^2_{n-\nu+1,\beta^{n-\nu+1}}(1),\ldots,\theta^2_{n,\beta^n}(1) \right). \]

Note that we can choose $|\beta^i| = \alpha_\infty$ for each $i \in n$. During the proof of Theorem II we have proved the validity of inequalities (83), which lead to
\[ \langle \Theta^*(\beta^1,\ldots,\beta^n), \mathbb{1} \rangle < \langle \pi^i_{\infty}, \mathbb{1} \rangle, \quad \forall \beta^i \in \mathcal{D} \cap W_i, \quad i \in n. \]

Moreover, by combining (83) with the fact that $\alpha_\infty \geq \tilde{\alpha}_\infty$, we get
\[ \langle \Theta^*(\beta^1,\ldots,\beta^n), \mathbb{1} \rangle > \langle \Theta_\infty(1), \mathbb{1} \rangle - \varepsilon > \langle m_\infty - \nu \rangle \pi - \varepsilon \quad \text{if} \quad |\beta^i| = \alpha_\infty, \forall i \in n, \]

from which we conclude that
\[ \langle \pi^i_{\infty}, \mathbb{1} \rangle > \langle m_\infty - \nu \rangle \pi - \varepsilon, \]

which contradicts (89). The emptiness of $\mathcal{T}$ follows. \hfill $\square$
Thus, we have learnt that the non-emptiness of \( \mathcal{J} \) is possible when (51) (which deals with the whole vector \( \Theta_\alpha \)) is not a consequence of (80). In other words, we are implicitly requiring that (90)

\[
\alpha_\infty < \tilde{\alpha}_\infty.
\]

According to Proposition 19, condition (90) can be interpreted as the requirement that the radial symmetry of \( Ju'' + S_\infty(t)u(t) = 0 \) must not be preserved away from infinity.

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