CONFORMAL DEFORMATION OF SPACELIKE SURFACES IN MINKOWSKI SPACE

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Abstract. We address the problem of second order conformal deformation of spacelike surfaces in compactified Minkowski 4-space. We explain the construction of the exterior differential system of conformal deformations and discuss its general and singular solutions. In particular, we show that isothermic surfaces are singular solutions of the system, which implies that a generic second order deformable surface is not isothermic. This differs from the situation in 3-dimensional conformal geometry, where isothermic surfaces coincide with deformable surfaces.

1. Introduction

The surfaces in the conformal 3-sphere which admit second order deformations with respect to the group of conformal transformations coincide with isothermic surfaces [7], [16]. This is no longer true in higher dimensional conformal spaces. Actually, isothermic surfaces are deformable to second order [14], but generically a deformable surface is not isothermic. This result was originally stated without proof by E. Cartan in his address at the 1920 International Congress of Mathematicians [6]. More precisely, as an illustration of the general deformation theory of submanifolds in homogeneous spaces, Cartan indicated that isothermic surfaces in conformal 4-space are singular solutions of the exterior differential system (EDS) which defines deformable surfaces. The work of Cartan on the deformation of submanifolds in homogeneous spaces and the related questions of contact and rigidity were taken up and further developed by P. Griffiths and G. Jensen [10], [11]. In particular, that the problems of $k$th order deformation are equivalent to solving certain EDSs on appropriate spaces of frames was established in [11]. Still, for each concrete geometric situation there is a specific problem to solve.

In this paper, we investigate the problem of conformal deformation for the case of spacelike surfaces in compactified Minkowski 4-space. We give...
a detailed description of the Pfaffian differential system (PDS) of conformal
deformations and then provide the tools to discuss its general and singular
solutions within the theory of EDSs.\footnote{We recall that an integral manifold $f : N \to M$ of an EDS $\mathcal{I}$ on $M$ is a general solution if the integral element $df(T_p N)$ is ordinary, for every $p \in N$; it is a singular solution if the integral element $df(T_p N)$ fails to be ordinary, for every $p \in N$. So, singular solutions are not given by the Cartan–Kähler theorem. They involve additional equations (see \cite{2} for more detail).} We specify the appropriate space which supports the differential system of a conformal deformation and find
the equations of the integral elements of the system. We show that the
differential system of deformations is in involution and that its general solu-
tions depend on one arbitrary function in two variables. We then determine
the equations of the variety defining the singular solutions of the system and
show that isothermic surfaces are indeed singular solutions. In particular,
isothermic surfaces depend on six arbitrary functions in one variable, which
implies that a generic second order deformable surface is not isothermic.

The paper is organized as follows. In Section 2, we present the background
material and set up the basic constructions. In Section 3, we discuss the
questions of conformal deformation and rigidity for spacelike surfaces in
compactified Minkowski space and describe isothermic surfaces as examples
of deformable surfaces. In Section 4, we study the involutiveness of the PDS
of a conformal deformation and discuss its general and singular solutions.

We use \cite{2} as basic reference for the theory of EDSs. For a general account
on submanifold theory in conformal differential geometry, we refer to \cite{14}.
The summation convention on repeated indices is used throughout the paper.

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paper.

2. Preliminaries and basic constructions

2.1. The conformal completion of Minkowski 4-space. Let $\mathbb{R}^{4,2}$ denote $\mathbb{R}^6$ with the symmetric bilinear form

\begin{equation}
\langle x, y \rangle = -(x^0 y^5 + x^5 y^0) + x^1 y^1 + x^2 y^2 + x^3 y^3 - x^4 y^4 = g_{IJ} x^I y^J
\end{equation}


of signature $(4, 2)$, where $x^0, \ldots, x^5$ are the coordinates with respect to the
standard basis $e_0, \ldots, e_5$ of $\mathbb{R}^6$. The Lie quadric is the hypersurface

$$Q = \{ [k] \in \mathbb{RP}^5 \mid \langle k, k \rangle = 0 \}.$$ 

A pair of points $[k_1], [k_2]$ in $Q$ satisfying $\langle k_1, k_2 \rangle = 0$ defines a line $[k_1, k_2]$ in $Q$. The set of all lines in $Q$ forms a smooth manifold of dimension 5, which
we denote by $\Lambda$. Let $O(4, 2)$ denote the pseudo-orthogonal group of $(2.1)$. The standard action of $O(4, 2)$ on $\mathbb{RP}^5$ maps the quadric $Q$ into itself and
induces an action on $\Lambda$ which is transitive.
The quadric $Q$ is diffeomorphic to $(S^1 \times S^3)/\mathbb{Z}_2$ and inherits a locally conformally flat metric of signature $(3,1)$ from the flat metric on $\mathbb{R}^{4,2}$ corresponding to (2.1). This implies that $O(4,2)/\mathbb{Z}_2$ acts on $Q$ as a group of conformal transformations. In fact, $O(4,2)/\mathbb{Z}_2$ coincides with the conformal group of $Q$. The Lie quadric $Q$ can be regarded as the conformal compactification of flat Minkowski spacetime $\mathbb{R}^{3,1}$ (see [15], [13]). The conformal embedding of $\mathbb{R}^{3,1}$ with flat Lorentz metric $(\cdot,\cdot)$ is given by

$$\mathbb{R}^{3,1} \ni v \mapsto \left[ \begin{array}{c} 1, v, \frac{1}{2}(v,v) \end{array} \right]^T \in Q.$$  

**Remark 2.1 (Lie sphere geometry).** The points in the Lie quadric $Q$ are in bijective correspondence with the set of all oriented spheres and point spheres in the unit sphere $S^3 \subset \mathbb{R}^4$. A line $[k_1,k_2]$ in $Q$ corresponds to a family of spheres in oriented contact. This family of oriented spheres contains a unique point sphere, which is the common point of contact, and determines the common unit normal vector at this point. The set $\Lambda$ can then be identified with $T_1S^3 = \{(u,v) \in S^3 \times S^3 \subset \mathbb{R}^4 \times \mathbb{R}^4 \mid u \cdot v = 0\}$. A Lie sphere transformation is a projective transformation induced by a transformation in the group $O(4,2)/\mathbb{Z}_2$. In terms of $S^3$, a Lie sphere transformation is a map on the space of oriented spheres which preserves oriented contact. The group $O(4,2)/\mathbb{Z}_2$ acts on $\Lambda$, and hence on $T_1S^3$, as a group of contact transformations. See [8] for more detail.

We now introduce moving frames to study surface theory in $Q$. Let $G$ be the identity component of $O(4,2)$ and $\mathfrak{g} = \{B \in \mathfrak{gl}(6,\mathbb{R}) \mid B^Tg + gB = 0\}$ its Lie algebra, where $g = (g_{IJ})$. By a frame is meant a basis $A_0,\ldots,A_5$ of $\mathbb{R}^{4,2}$ such that $(A_0,\ldots,A_5) \in G$. Up to the choice of a reference frame, the manifold of frames identifies with $G$. For $A \in G$, let $A_J = Ae_J$ be the column vectors of $A$ and regard the $A_J$ as $\mathbb{R}^{4,2}$ valued functions on $G$. Since the $A_J$ form a basis of $\mathbb{R}^{4,2}$, there exist unique left invariant 1-forms $\omega^I_J$ ($I,J = 0,1,\ldots,5$) such that

$$dA_J = \omega^I_JA_I \quad (J = 0,\ldots,5).$$  

Differentiating (2.2) yields the Cartan structure equations

$$d\omega^I_J = -\omega^K_J \wedge \omega^K_I.$$  

Differentiating $\langle A_I, A_J \rangle = g_{IJ}$ gives the symmetry equations

$$\omega^K_I g_{KJ} + \omega^K_J g_{KI} = 0.$$
The 1-forms \( \omega^I \) are the components of the Maurer–Cartan form \( \omega = A^{-1} dA \) of \( G \), which accordingly takes the form

\[
\omega = \begin{pmatrix}
\omega_0^0 & \omega_0^1 & \omega_0^2 & \omega_0^3 & \omega_0^4 \\
\omega_1^0 & 0 & -\omega_1^2 & -\omega_1^3 & \omega_1^4 \\
\omega_2^0 & \omega_2^1 & 0 & -\omega_2^2 & \omega_2^3 \\
\omega_3^0 & \omega_3^1 & \omega_3^2 & 0 & -\omega_3^3 \\
0 & \omega_4^1 & \omega_4^2 & \omega_4^3 & -\omega_4^4 & -\omega_4^5
\end{pmatrix}.
\]

(2.3)

The standard action of \( G \) on \( \mathbb{RP}^5 \) restricts to a transitive action on the quadric \( Q \). This action defines a principal \( K_Q \)-bundle

\[
\pi_Q : G \to Q \cong G/K_Q, \quad A \mapsto A[e_0] = [A_0],
\]

where \( K_Q \) is the isotropy subgroup at \([e_0] \). It is easy to compute that

\[
K_Q = \left\{ \begin{pmatrix} r & y^T B & \frac{1}{2r} (y, y) \\ 0 & B & y/r \\ 0 & 0 & 1/r \end{pmatrix} : B \in \text{SO}(3, 1), y \in \mathbb{R}^4, r > 0 \right\}.
\]

From this it follows that the forms \( \{\omega_0^1, \omega_0^2, \omega_0^3, \omega_0^4\} \) span the semibasic forms of the projection \( \pi_Q \). The conformal structure on \( Q \) is determined by the quadratic form

\[
(\omega_0^1)^2 + (\omega_0^2)^2 + (\omega_0^3)^2 - (\omega_0^4)^2.
\]

Next, we will introduce two additional homogeneous spaces which will play a role in the discussion of the conformal deformation problem.

2.2. The Grassmannian of parabolic 3-planes. Let \( \mathcal{P} \subset G_3(\mathbb{R}^{4,2}) \) be the set of degenerate 3-planes \( V \subset \mathbb{R}^{4,2} \) of signature \((0,+,+)\), i.e.,

- \( \dim V \cap V^\perp = 1 \);
- \( \langle \cdot, \cdot \rangle \) restricted to \( V/V \cap V^\perp \) is positive definite.

The natural action of \( G \) on \( \mathcal{P} \) is transitive and the map

\[
\pi_P : G \to \mathcal{P}, \quad A \mapsto A \cdot [e_0 \wedge e_1 \wedge e_2] = [A_0 \wedge A_1 \wedge A_2]
\]

makes \( G \) into a principal fiber bundle over \( \mathcal{P} \) with fiber

\[
H_P = \{ A \in G : A \cdot [e_0 \wedge e_1 \wedge e_2] = [e_0 \wedge e_1 \wedge e_2] \}.
\]

\footnote{We recall that a differential form \( \varphi \) on the total space of a fiber bundle \( \pi : P \to B \) is said to be \emph{semibasic} if its contraction with any vector field tangent to the fibers of \( \pi \) vanishes, or equivalently, if its value at each point \( p \in P \) is the pullback via \( \pi_p^* \) of some form at \( \pi(p) \in B \). Some authors call such a form \emph{horizontal}. A stronger condition is that \( \varphi \) be \emph{basic}, meaning that it is locally the pullback via \( \pi^* \) of a form on the base \( B \).}
where \( H_P \subset G \) is the isotropy subgroup at \([e_0 \wedge e_1 \wedge e_2]\). A direct computation shows that \( H_P \) consists of matrices of the form

\[
A(r, x, y, a, b) = \begin{pmatrix}
  r & xT a & yT \mathbb{I}_{1,1} b & \frac{1}{2r} (x^T x + y^T \mathbb{I}_{1,1} y) \\
  0 & a & 0 & x/r \\
  0 & 0 & b & y/r \\
  0 & 0 & 0 & 1/r
\end{pmatrix},
\]

where \( a \in \text{SO}(2) \), \( b \in \text{SO}(1, 1) \), \( x, y \in \mathbb{R}^2 \), \( r > 0 \), and

\[
\mathbb{I}_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

This implies that \( \mathcal{P} \) is an 8-dimensional homogeneous space of \( G \) and that

\[
\{\omega_0^1, \omega_4^0, \omega_3^0, \omega_3^1, \omega_2^0, \omega_1^0, \omega_2^1, \omega_2^2\}
\]

is a basis for the space of semibasic 1-forms of the projection \( \pi_P \).

2.3. The configuration space and some relevant PDSs. Let \( \mathcal{D} \) be the submanifold of \( \mathcal{P} \times \mathcal{P} \times G \) defined by

\[
\mathcal{D} := \{(V_1, V_2, A) \in \mathcal{P} \times \mathcal{P} \times G : A \cdot V_1 = V_2\}.
\]

We call \( \mathcal{D} \) the configuration space of deformations. The Lie group \( G \times G \) acts on the left on \( \mathcal{D} \) by

\[
(A, B) \cdot (V_1, V_2, F) := (A \cdot V_1, B \cdot V_2, BFA^{-1}),
\]

for each \((A, B) \in G \times G\), and \((V_1, V_2, F) \in \mathcal{D}\). This action is transitive and the isotropy subgroup of \( G \times G \) at

\[
([e_0 \wedge e_1 \wedge e_2], [e_0 \wedge e_1 \wedge e_2], e_G) \in \mathcal{D}
\]

(where \( e_G \) is the identity element of \( G \)) is the closed subgroup

\[
(G \times G)_\mathcal{D} = \{(A, B) \in G \times G : A = B \in H_P\}.
\]

Thus \( \mathcal{D} \) is a 23-dimensional homogeneous space of \( G \times G \) and the natural projection \( \pi_\mathcal{D} : G \times G \to \mathcal{D} \) is given by

\[
(A, B) \mapsto ([A_0 \wedge A_1 \wedge A_2], [B_0 \wedge B_1 \wedge B_2], BA^{-1}).
\]

Next, let \((\omega, \Omega)\) denote the Maurer–Cartan form of \( G \times G \). The set of left-invariant forms

\[
(\omega^0_0 - \Omega^0_0, \omega^0_1 - \Omega^0_1, \omega^2_1 - \Omega^2_1, \omega^4_1 - \Omega^4_1, \omega^0_2, \Omega^0_2, \omega^0_3, \Omega^0_3, \omega^0_4, \Omega^0_4),
\]

where \( I = 1, 2, 3, 4; i = 1, 2; a = 3, 4 \), is a basis for the space of semibasic forms of the fibration \( \pi_\mathcal{D} \). Let

\[
\alpha^0_0 - \beta^0_0, \alpha^0_1 - \beta^0_1, \alpha^2_1 - \beta^2_1, \alpha^4_1 - \beta^4_1, \alpha^0_2, \beta^0_2, \alpha^0_3, \beta^0_3, \alpha^0_4, \beta^0_4,
\]

be the 1-forms on \( \mathcal{D} \) obtained by pulling back the forms (2.5) via a local section of \( \pi_\mathcal{D} \).
Out of these forms, we can construct three invariant Pfaffian systems \( I_1, I_2, I_3 \subset \Gamma(T^*D) \) given by

\[
I_1 = \text{span}\{\alpha_0^i - \beta_0^i, \alpha_0^a, \beta_0^a\},
\]

\[
I_2 = \text{span}\{\alpha_0^i - \beta_0^i, \alpha_0^a, \beta_0^a, \alpha_0^0 - \beta_0^0, \alpha_1^2 - \beta_1^2, \alpha_1^a - \beta_1^a\},
\]

\[
I_3 = \text{span}\{\alpha_0^i - \beta_0^i, \alpha_0^a, \beta_0^a, \alpha_0^0 - \beta_0^0, \alpha_1^2 - \beta_1^2, \alpha_1^a - \beta_1^a, \alpha_3^4 - \beta_3^4, \alpha_0^0 - \beta_0^0\},
\]

\( i = 1, 2; a = 3, 4 \). In Section\(^3\) we will explain the geometric meaning of the Pfaffian differential systems defined by the differential ideals \( \mathcal{I}_1, \mathcal{I}_2, \) and \( \mathcal{I}_3 \) generated by \( I_1, I_2, \) and \( I_3 \), respectively.

### 2.4. Spacelike surfaces in the Lie quadric

Let \( X \) be a 2-dimensional oriented manifold and let \( f : X \to Q \subset \mathbb{R}\mathbb{P}^5 \) be a smooth spacelike conformal immersion.

**Definition 2.2.** A zeroth order frame field along \( f \) is a smooth map \( A : U \subset X \to G \), defined on an open subset \( U \subset X \), such that \( f = [A_0] = \pi_Q \circ A \).

For any such a frame we put \( \theta = A^* \omega \). The totality of zeroth order frames along \( f \) is the bundle \( \pi_0 : P_0(f) \to X \), where

\[
P_0(f) = \{(q, A) \in X \times G : [A_0] = f(q)\} .
\]

**Definition 2.3.** A first order frame field along \( f \) is a zeroth order frame field \( A : U \subset X \to G \) such that

\[
\theta_0^0 = \theta_0^4 = 0.
\]

The totality of first order frames gives rise to a subbundle

\[
\pi_1 : P_1(f) \to X
\]

of \( P_0(f) \), referred to as the first order frame bundle of \( f \). The structure group of \( P_1(f) \) consists of matrices of the form \((2.4)\).

The map

\[
P_1(f) \ni (q, A) \mapsto [A_0 \wedge A_1 \wedge A_2] \in \mathcal{P}
\]

is constant along the fibers of \( \pi_1 : P_1(f) \to X \), and therefore induces a well-defined map

\[
\tau_f : X \to \mathcal{P}
\]

and a corresponding rank 3 vector bundle

\[
\tau(X) = \{(q, W) \in X \times \mathbb{R}^{4,2} : W \in \tau_f(q)\} \to X.
\]

The tautological line bundle

\[
K_X = \{(q, W) \in X \times \mathbb{R}^{4,2} : W \in f(q)\} \to X
\]

is a line subbundle of \( \tau(X) \) such that \( K_X = \tau(X) \cap \tau(X)^\perp \). Thus the quotient bundle

\[
\mathcal{T}(X) = \tau(X)/K_X \to X
\]

inherits from \( \mathbb{R}^{4,2} \) a Riemannian metric, say \( g^\tau \). Note that the tensor product \( \mathcal{T}(X) \otimes K_X^* \) can be canonically identified with the tangent bundle \( \mathcal{T}(X) \).
Similarly, the map
\[ P_1(f) \ni (q, A) \mapsto [A_0 \wedge A_3 \wedge A_4] \]
is constant along the fibers of \( \pi_1 : P_1(f) \to X \), and induces a well-defined map
\[ \nu_f : X \to G_3(\mathbb{R}^{4,2}). \]
We denote by \( \nu(X) \to X \) the corresponding rank 3 vector bundle
\[ \nu(X) = \{(q, W) \in X \times \mathbb{R}^{4,2} : W \in \nu_f(q)\} \to X. \]
Again, \( K_X = \nu(X) \cap \nu(X)^\perp \) and the quotient bundle
\[ N(X) = \nu(X)/K_X \to X \]
inherits from \( \mathbb{R}^{4,2} \) a pseudo-Riemannian metric \( g' \) of signature \((1, 1)\). We call \( N(X) \) the conformal normal bundle of the spacelike immersion \( f : X \to Q \).

The conformal normal bundle is equipped with a metric covariant derivative \( D' \) defined by
\[ D'(x, A_3 + yA_4) = (dx + y\theta^4_3)A_3 + (dy + x\theta^4_3)A_4, \]
where \( A : U \to G \) is a first order frame and \((\tilde{A}_3, \tilde{A}_4)\) denotes the induced local trivialization of the conformal normal bundle. Note that \( g' \) induces a Riemannian metric on the symmetric tensor product \( S^2T^*(X) \). Let \( A : U \to G \) be a fixed first order frame along \( f \) and denote by \( \tilde{A}_0, (\tilde{A}_1, \tilde{A}_2) \) and \((\tilde{A}_3, \tilde{A}_4)\) the corresponding trivializations of the bundles \( K_X, \mathcal{T}(X) \) and \( N(X) \), respectively. Differentiating \( \theta^a_0 = \theta^4_0 = 0 \) and applying Cartan’s Lemma yield
\[ \theta^a_i = h^a_{ij}\theta^j_0, \quad h^a_{ij} = h^a_{ji} \quad (a = 3, 4; i, j = 1, 2) \]
for smooth functions \( h^a_{ij} : U \to \mathbb{R} \).

The conformal second fundamental form of \( f \) is the trace-free quadratic form \( \mathcal{A} \), taking values in \( N(X) \otimes K_X \), given locally by
\[ \mathcal{A} = \left[ h^a_{ij} - \left( \sum_{l=1,2} \tilde{h}^a_{il} \right) \delta^l_{ij}\right] \theta^i_0 \theta^j_0 \otimes A_a \otimes A_0. \]
The form \( \mathcal{A} \) is independent of the choice of first order frames and is a conformal invariant of the immersion \( f \).

**Definition 2.4.** A second order frame field along \( f \) is a first order frame field \( A : U \subset X \to G \) such that
\[ \sum_{l=1,2} \tilde{h}^a_{il} = 0 \quad (a = 3, 4). \]
Locally there exist second order frames. The totality of second order frame fields gives rise to a subbundle \( \pi_2 : P_2(f) \to X \) of \( P_1(f) \), referred
to as the second order frame bundle of $f$, whose structure group is the subgroup $G_2 \subset G$ given by

$$G_2 = \left\{ \begin{pmatrix} r & x^T a & 0 & \frac{x^T x}{r} \\ 0 & a & 0 & 2r \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & 1/r \end{pmatrix} : a \in \text{SO}(2), \quad b \in \text{SO}(1, 1), \quad x \in \mathbb{R}^2, \quad r > 0 \right\}.$$ 

**Remark 2.5.** The frame bundles associated with an immersed surface in $Q$ considered above are the analogs of the bundles considered by Bryant for an immersed surface in the conformal 3-sphere [3].

**Remark 2.6.** Observe that the mappings $[A_3 - A_4], [A_3 + A_4] : X \to Q$ are independent of the choice of the second order frame $A$. Further,

$$F_1 = [A_0 \wedge (A_3 - A_4)] : X \to \Lambda,$$
$$F_2 = [A_0 \wedge (A_3 + A_4)] : X \to \Lambda$$

are two Legendrian immersions with respect to the canonical contact structure of $\Lambda$ (see Remark 2.1). If $p : \Lambda \to \mathbb{R}^3 \cup \{\infty\}$ denotes the projection of $\Lambda$ onto the 3-sphere, the mappings

$$\phi_1 = p \circ F_1 : X \to \mathbb{R}^3 \cup \{\infty\},$$
$$\phi_2 = p \circ F_2 : X \to \mathbb{R}^3 \cup \{\infty\}$$

are the two envelopes of the congruence of spheres represented by the spacelike immersion $f$ (see Remark 2.1). Note that $(A_3 - A_4)$ and $(A_3 + A_4)$ generate the isotropic line subbundle of the conformal normal bundle $\mathcal{N}(X)$.

### 3. Contact and deformation

We start by recalling the notion of deformation (see [10], [11]).

**Definition 3.1.** Let $G/H$ be a homogeneous space and let $f, \hat{f} : X \to G/H$ be smooth maps. Two such maps $f$ and $\hat{f}$ are $k$th order $G$-deformations of each other if there exists a smooth map $D : X \to G$ such that, for each point $p \in X$, the maps $\hat{f}$ and $D(p)f$ have analytic contact of order $k$ at $p$; that is, if they have the same $k$th order jets at $p$. The map $D$ is called the infinitesimal displacement of the deformation. When $D(p)$ does not depend on $p \in X$, the deformation is called trivial, and then $\hat{f} = Df$ is $G$-congruent to $f$. A given map $f : X \to G/H$ is rigid to $k$th order deformations if there are no nontrivial $k$th order deformations of it; it is deformable of order $k$ if it admits a nontrivial $k$th order deformation.
3.1. Analytic Contact. Let $X$ be a 2-dimensional manifold and $f, \hat{f} : X \to Q$ smooth maps. To express the condition of $k$th order analytic contact for $f$ and $\hat{f}$, we need to introduce some notation. Let $\{U; x^1, x^2\}$ be a local coordinate system, where $U \subset X$ is an open set. Let $S^h(U)$ be the space of symmetric $h$-forms on $U$ and denote the symmetric product of $S \in S^h(U)$ and $T \in S^k(U)$ by $S \cdot T$. An element $L$ of $S^h(U)$ has a local expression

$$L = L_{i_1...i_h} dx^{i_1} ... dx^{i_h},$$

where the coefficients $L_{i_1...i_h}$ are smooth functions, which are totally symmetric in the indices $i_1, ..., i_h$. We then define the $k$th order derivative of $L$ to be the symmetric form of order $h + k$ given by

$$\delta^k(L) := \frac{\partial^k L_{i_1...i_h}}{\partial x^{i_{h+1}} ... \partial x^{i_{h+k}}} dx^{i_1} ... dx^h dx^{i_{h+1}} ... dx^{i_{h+k}}.$$

The definition depends on the choice of the local coordinates.

We have the following.

**Lemma 3.2.** Let $f : X \to Q$ and $\hat{f} : X \to Q$ be smooth maps. Then, $f$ and $\hat{f}$ have analytic contact of order $k$ at $p_0$ if and only if, for every local coordinate system $\{U; x^1, x^2\}$ about $p_0$, there exist symmetric forms $\rho_r \in S^r(U)$ such that

$$(3.1) \quad \delta^r(\hat{F})|_{p_0} = \sum_{h=0}^{r} \binom{r}{h} \rho_{r-h}|_{p_0} \delta^h(F)|_{p_0} \quad (r = 0, ..., k),$$

for arbitrary $F, \hat{F} : U \to \mathbb{R}^6$ such that

$$f(p) = [F(p)], \quad \hat{f}(p) = [\hat{F}(p)], \quad \text{for each } p \in U.$$

**Proof.** Let $\{U; x^1, x^2\}$ be a coordinate system about $p_0$. As $G$ acts transitively on $Q$, we may assume that

$$f(p_0) = \hat{f}(p_0) = [e_0].$$

The map

$$y = (y^1, ..., y^4) \in \mathbb{R}^4 \mapsto [X_0(y)] \in Q$$

defined by

$$(3.2) \quad X_0(y) = \left(1, y, \frac{1}{2}(y, y)\right)^T,$$

is a local coordinate system of $Q$ centered at $[e_0]$. Then, there exist an open neighborhood $U' \subset U$ of $p_0$ and smooth maps $h, \hat{h} : U' \to \mathbb{R}^4$ such that

$$f|_{U'} = [(X_0 \circ h)], \quad \hat{f}|_{U'} = [(X_0 \circ \hat{h})].$$

Thus, $f$ and $\hat{f}$ have analytic contact of order $k$ at $p_0$ if and only if the maps $\xi := X_0 \circ h$ and $\zeta := X_0 \circ \hat{h}$ satisfy

$$(3.3) \quad \delta^r(\xi)|_{p_0} = \delta^r(\zeta)|_{p_0}, \quad (r = 0, ..., k).$$
If $F$ and $\hat{F}$ are lifts of $f$ and $\hat{f}$, respectively, we can write
\[(3.4) \quad F = F^0 \xi, \quad \hat{F} = \hat{F}^0 \zeta\]
for smooth functions $F^0, \hat{F}^0 : U' \to \mathbb{R}$. Now, using (3.3) and (3.4), we compute
\[
\delta^r(\hat{F})|_{p_0} = \sum_{h=0}^r \binom{r}{h} \delta^h(\hat{F}^0)|_{p_0} \delta^{r-h}(\zeta)|_{p_0} = \sum_{h=0}^r \binom{r}{h} \delta^h(F^0)|_{p_0} \delta^{r-h}(\xi)|_{p_0}
\]
\[
= \sum_{h=0}^r \binom{r}{h} \delta^h(F^0)|_{p_0} \delta^{r-h} \left(\frac{F}{F^0}\right)|_{p_0}
\]
\[
= \sum_{h=0}^r \sum_{m=0}^{r-h} \binom{r}{h} \binom{r-h}{m} \delta^h(F^0)|_{p_0} \delta^{r-h-m} \left(\frac{1}{F^0}\right)|_{p_0} \delta^m(F)|_{p_0}
\]
\[
= \sum_{m=0}^r \sum_{h=0}^{r-m} \binom{r}{m} \binom{r-m}{h} \delta^h(F^0)|_{p_0} \delta^{r-m-h} \left(\frac{1}{F^0}\right)|_{p_0} \delta^m(F)|_{p_0}
\]
\[
= \sum_{m=0}^r \binom{r}{m} \delta^{r-m} \left(\frac{\hat{F}^0}{F^0}\right)|_{p_0} \delta^m(F)|_{p_0},
\]
and hence the conditions (3.4).

Conversely, if conditions (3.1) hold true for arbitrary lifts $F$ and $\hat{F}$, by choosing $F = \xi$ and $\hat{F} = \zeta$, we get that $f$ and $\hat{f}$ have analytic contact of order $k$. \hfill \square

**Corollary 3.3.** In particular, we have proved that:

- $f$ and $\hat{f}$ have first order contact if and only if
\[(3.5) \quad \hat{F}(p_0) = \rho_0(p_0)F(p_0), \quad \delta(\hat{F})|_{p_0} = \rho_1(p_0)F(p_0) + \rho_0(p_0)\delta(F)|_{p_0}.
\]
- $f$ and $\hat{f}$ have second order contact if and only if (3.5) holds and
\[(3.6) \quad \delta^2(\hat{F})|_{p_0} = \rho_2(p_0)F(p_0) + 2\rho_1(p_0)\delta(F)|_{p_0} + \rho_0(p_0)\delta^2(F)|_{p_0}.
\]
- $f$ and $\hat{f}$ have third order contact if and only if (3.5) and (3.6) hold and
\[(3.7) \quad \delta^3(\hat{F})|_{p_0} = \rho_3(p_0)F(p_0) + 3\rho_2(p_0)\delta(F)|_{p_0} + 3\rho_1(p_0)\delta^2(F)|_{p_0} + \rho_0(p_0)\delta^3(F)|_{p_0}.
\]

### 3.2. Conformal deformation.

In analogy with the characterization of conformal deformation of surfaces in the conformal 3-sphere (see also [11], [12], [17]), we can state the following result.

**Proposition 3.4.** Let $f, \hat{f} : X \to Q$ be smooth spacelike immersions of the oriented surface $X$ into the Lie quadric $Q$, viewed as a homogeneous space of the conformal group $G$. Then, the following statements hold true:
Since \( p \) and \( \hat{f} \) along \( f \) and \( \hat{f} \), respectively, such that

\[
\hat{A}^*\omega_0^1 = A^*\omega_0^1, \quad \hat{A}^*\omega_0^2 = A^*\omega_0^2,
\]

where \( \omega \) is the Maurer–Cartan form of \( G \).

(2) The immersions \( f \) and \( \hat{f} \) are second order conformal deformation of each other if and only if there exist second order frame fields \( A \) and \( \hat{A} \) along \( f \) and \( \hat{f} \), respectively, such that

\[
\hat{A}^*\omega_0^1 = A^*\omega_0^1, \quad \hat{A}^*\omega_0^2 = A^*\omega_0^2, \quad \hat{A}^*\omega_1^2 = A^*\omega_1^2,
\]

\[
\hat{A}^*\omega_0^a = A^*\omega_0^a, \quad \hat{A}^*\omega_i^a = A^*\omega_i^a, \quad a = 3, 4; i = 1, 2.
\]

(3) The immersions \( f \) and \( \hat{f} \) are third order conformal deformation of each other if and only if there exist second order frame fields \( A \) and \( \hat{A} \) along \( f \) and \( \hat{f} \), respectively, such that

\[
\hat{A}^*\omega = A^*\omega.
\]

Thus any smooth immersion \( f : X \to Q \) is rigid to third order.

**Sketch of the proof.** (1) Suppose that \( f \) and \( \hat{f} \) are first order conformal deformations of each other. Then \( D : U \to G \) exists so that \( \hat{f} \) and \( D(p)\hat{f} \) have first order analytic contact at \( p \), for each \( p \in U \). Let \( A : U \to G \) be a first order frame field along \( f \) and define \( \hat{A} : U \to G \) by \( \hat{A}(p) = D(p)A(p) \), for each \( p \in U \). Then \( \hat{A} \) is a frame field along \( \hat{f} \) and \( A' = D(p_0)A : U \to G \) is a frame field along \( D(p_0)f \), for each \( p_0 \in U \). According to (3.8) in Corollary 3.3, we have

\[
\hat{A}_0(p_0) = \rho_0(p_0)A'_0(p_0)
\]

\[
d\hat{A}_0 | (p_0) = \rho_0(p_0)dA'_0 | (p_0) + \rho_1(p_0)A'_1(p_0).
\]

Equation (3.10) yields

\[
\rho_0 = 1,
\]

since \( \hat{A} \) and \( A' \) agree at \( p_0 \). Now, the structure equations of \( G \) imply

\[
d\hat{A}_0 = \theta_0^J \hat{A}_J, \quad dA'_0 = \theta_0^J A'_J, \quad (J = 0, \ldots, 5),
\]

where we have set \( \theta = A^*\omega \) and \( \hat{\theta} = \hat{A}^*\omega \). Substituting (3.13) into (3.11) yields

\[
\rho_1 = (\hat{\theta}_0^l - \theta_0^l), \quad \hat{\theta}_0^l = \theta_0^l, \quad 1 \leq l \leq 4.
\]

Since \( p_0 \) has been chosen arbitrarily, equations (3.14) are identically satisfied on \( U \). Thus \( \hat{A} \) is a first order frame along \( \hat{f} \) and conditions (3.8) are satisfied.

Conversely, suppose (3.8) hold for first order frame fields \( A \) and \( \hat{A} \) along \( f \) and \( \hat{f} \), respectively. Then define \( D : U \to G \) by

\[
D(p) = \hat{A}(p)A^{-1}(p), \quad \text{for each} \quad p \in U.
\]
By (3.12), (3.13) and (3.14), we see that (3.10) and (3.11) hold. So, \( D \) induces a first order conformal deformation.

(2) Retain the notations of (1) and suppose that \( f \) and \( \hat{f} \) are second order conformal deformations of each other. Then \( \hat{f} \) and \( D(p)f \) have second order analytic contact at \( p \), for each \( p \in U \). Let \( A : U \to G \) be a second order frame field along \( f \) and \( \hat{A} \) and \( A' \) be as in (1). We have to show that \( \hat{A} \) defines a second order frame field along \( \hat{f} \) such that

\[
\begin{align*}
\hat{\theta}^1_0 &= \theta^1_0, & \hat{\theta}^2_0 &= \theta^2_0, & \hat{\theta}^2_1 &= \theta^2_1, \\
\hat{\theta}^0_0 &= \theta^0_0, & \hat{\theta}^a_0 &= \theta^a_0 \quad (a = 3, 4; i = 1, 2).
\end{align*}
\]

By Corollary 3.3 and the discussion in part (1), we know that the frame fields \( A \) and \( A' \) must satisfy (3.10), (3.11) and

\[
\delta^2(\hat{A}_0)|_{p_0} = \rho_2(p_0)A'_0(p_0) + 2(\hat{\theta}^0_0 - \theta^0_0)|_{p_0}dA'_0|_{p_0} + \delta^2(A'_0)|_{p_0}.
\]

Writing out (3.16), using the structure equations (3.13), the equations \( \hat{\theta}_0^i = \theta_i^a, i = 1, 2, \) and \( \hat{\theta}_0^a = \theta_0^a = 0, a = 3, 4, \) and the fact that \( \hat{A}_0(p_0) = A'_0(p_0) \), we find

\[
\begin{align*}
\rho_2 & = \delta(\hat{\theta}_0^0 - \theta_0^0) + (\hat{\theta}_0^0 - \theta_0^0)^2 + \rho_1(\hat{\theta}_1^0 - \theta_1^0) + \theta_0^2(\hat{\theta}_2^0 - \theta_2^0), \\
0 & = \theta_0^2(\hat{\theta}_2^2 - \theta_2^2) - \theta_0^1(\hat{\theta}_2^1 - \theta_2^1), \\
0 & = \theta_0^1(\hat{\theta}_1^2 - \theta_1^2) - \theta_0^2(\hat{\theta}_1^2 - \theta_1^2), \\
0 & = \theta_0^1(\hat{\theta}_1^a - \theta_1^a) + \theta_0^2(\hat{\theta}_2^a - \theta_2^a) \quad (a = 3, 4).
\end{align*}
\]

From these equations it follows that

\[
\begin{align*}
\hat{\theta}_1^2 &= \theta_1^1, & \hat{\theta}_0^0 - \theta_0^0, & \hat{\theta}_0^a &= \theta_0^a \quad (i = 1, 2; a = 3, 4).
\end{align*}
\]

Thus \( \hat{A} \) is a second order frame along \( \hat{f} \) and the conditions (3.15) are satisfied.

Conversely, suppose (3.15) hold for second order frame fields \( A \) and \( \hat{A} \) along \( f \) and \( \hat{f} \), respectively. As above, define \( D : U \to G \) by

\[
D(p) = \hat{A}(p)A^{-1}(p), \quad \text{for each } p \in U.
\]

By reversing the arguments above, we see that (3.10), (3.11) and (3.16) are satisfied. So, \( D \) induces a second order conformal deformation of \( f \) and \( \hat{f} \).

As for (3), writing out (3.7), one can prove after some lengthy computations that \( \theta = A^{-1}dA = \hat{A}^{-1}d\hat{A} = \hat{\theta} \). By the Cartan–Darboux rigidity theorem, one then have \( dD|_p = 0 \), for every \( p \in U \).

\[ \square \]

**Example 3.5** (Isothermic surfaces). Let \( X \) be an oriented 2-dimensional manifold and let \( f : X \to \mathcal{Q} \subset \mathbb{RP}^5 \) be a smooth spacelike conformal immersion. On \( X \), consider the unique complex structure defined by the given orientation and the conformal structure induced by \( f \). We recall the following.
Definition 3.6. The immersion $f : X \to \mathbb{C}$ is isothermic if there exists a non-zero holomorphic quadratic differential $Q$ on $X$ and a section $S$ of $\mathcal{N}(X) \otimes K_X$ such that $A_{|q} = Q_{|q} \otimes S_{|q}$, for each $q \in X$ such that $Q_{|q} \neq 0$.

Locally, this means that

$$A = r^a Q A_0 \otimes A_0$$

for real-valued smooth functions $r^a$, $a = 3, 4$.

We now discuss the nonlinear partial differential equation governing isothermic surfaces; we mainly follow [4] (see also [14] and [5]). First, note that the covariant derivative $D^\nu$ on the conformal normal bundle of an isothermic immersion is flat. Further, fix coordinates $z = x + iy$ such that $Q = dz \, dz$ and choose a flat trivialization $(A_3, A_4)$ of the conformal normal bundle. Then there exists a unique cross section $A : X \to P_2(f)$ such that

$$A^{-1} dA = \begin{pmatrix} 0 & \chi_1 & \chi_2 & \tau_1 & \tau_2 & 0 \\ dx & 0 & 0 & -k_1 dx & k_2 dx & \chi_1 \\ dy & 0 & 0 & k_1 dx & -k_2 dx & \chi_2 \\ 0 & k_1 dx & -k_1 dx & 0 & 0 & \tau_1 \\ 0 & k_2 dx & -k_2 dx & 0 & 0 & \tau_2 \\ 0 & dx & dy & 0 & 0 & 0 \end{pmatrix},$$

where $k_1, k_2 : X \to \mathbb{R}$ are smooth functions and $\chi_1, \chi_2, \tau_1, \tau_2$ 1-forms to be determined. (In general, $A$ depends on the polarization $Q$, since totally umbilical immersions are allowed. This dependence disappears if $f$ is not totally umbilical). From the Maurer–Cartan equations, we have

$$\begin{align*}
\chi_1 &= \frac{1}{2}(u - \lVert k \rVert) dx + \psi dy, \\
\chi_2 &= \psi dx - \frac{1}{2}(u + \lVert k \rVert) dy, \\
\tau_1 &= (k_1)_x dx - (k_1)_y dy, \\
\tau_2 &= -(k_2)_x dx + (k_2)_y dy,
\end{align*}$$

where

$$\lVert k \rVert := (k_1)^2 - (k_2)^2$$

and $\psi$, $u$ are smooth functions satisfying

$$(k_1)_x = \psi k_1, \quad (k_2)_x = \psi k_2, \quad u_x = -2(\psi_y + \lVert k \rVert_x), \quad u_y = 2(\psi_x + \lVert k \rVert_y).$$

The above equations are compatible if and only if $k_1$, $k_2$ and $\psi$ are solutions of the vector Calapso equation

$$(k_1)_x = \psi k_1, \quad (k_2)_y = \psi k_2, \quad \Delta \psi = -2\lVert k \rVert_{xy}.$$
Conversely, if we start with a solution \((k_1, k_2, \psi)\) of the vector Calapso equation, then the differential 1-form

\[
v = -2(\psi_y + \|k\|_x) \, dx + 2(\psi_x + \|k\|_y) \, dy
\]

is closed. Let \(u : X \to \mathbb{R}\) be a primitive of \(v\), and define the 1-forms \(\chi_1, \chi_2\) and \(\tau_1, \tau_2\) as in \([3.22]\). Next, define a \(g\)-valued form \(\alpha\) as in \([3.21]\); \(\alpha\) satisfies the Maurer–Cartan equation and integrates (locally) to a smooth map \(A : X \to G\) such that \(\alpha = A^{-1}dA\). Then \(f = [A_0] : X \to Q\) is an isothermic immersion. Since \(u\) is defined up to a constant, for each \(\lambda \in \mathbb{R}\), there exists a frame \(A_\lambda\) and a corresponding isothermic immersion \(f_\lambda\). Define

\[
D_\lambda = A_\lambda A^{-1} : X \to G.
\]

It is easy to see that \(f\) and \(f_\lambda\) are second order deformations of each other with respect to the infinitesimal displacement \(D_\lambda\). The isothermic immersion \(f_\lambda\) coincides with the classical T-transform (spectral deformation) of the isothermic immersion \(f\), introduced independently by L. Bianchi and P. Calapso at the turn of the 20th century. The displacement \(D_\lambda\) induces the T-transformation of isothermic surfaces.

It follows that isothermic surfaces are (locally) deformable of second order and allow 1-parameter families of second order deformations. Such families correspond to the solutions of the vector Calapso equation which in turn is equivalent to the Gauss-Codazzi equations and arises as integrability condition of a linear differential system containing a free parameter. For more on the relations with the theory of integrable systems and the classical theory of transformations of isothermic surfaces, including the T-transformation, we refer the reader to \([14]\), Chapter 5, and the references therein.

Consider the Pfaffian differential systems \(I_1, I_2\) and \(I_3\) with independence condition \(\alpha_1^0 \wedge \alpha_2^0 \neq 0\), introduced in Section \([2.3]\). According to Proposition \([3.4]\) Example \([3.5]\) and the preceding discussion, we can state the following.

**Theorem 3.7.** (1) The integral manifolds of the Pfaffian differential system \((I_1, \alpha_1^0 \wedge \alpha_2^0)\) arise as maps

\[
\mathcal{D} : X \to \mathcal{D}, \, q \mapsto (\tau_f(q), \tau_f(q), D(q)),
\]

where \(f, \hat{f} : X \to Q\) are spacelike immersions which are first order deformations of each other and \(D : X \to G\) is the infinitesimal displacement of the deformation.

(2) The integral manifolds of the Pfaffian differential system \((I_2, \alpha_1^0 \wedge \alpha_2^0)\) arise as maps

\[
\mathcal{D} : X \to \mathcal{D}, \, q \mapsto (\tau_f(q), \tau_f(q), D(q)),
\]

where \(f, \hat{f} : X \to Q\) are spacelike immersions which are second order deformations of each other and \(D : X \to G\) is the infinitesimal displacement of the deformation.
(3) The integral manifolds of the Pfaffian differential system \((I_3, \alpha_0^1 \wedge \alpha_0^2)\) arise as maps
\[ \mathfrak{d} : X \rightarrow \mathcal{D}, \quad q \mapsto (\tau_f(q), \tau_{\hat{f}}(q), D(q)), \]
where \(f, \hat{f} : X \rightarrow \mathcal{Q}\) are spacelike isothermic immersions which are \(T\)-transforms of each other and \(D : X \rightarrow G\) defines the \(T\)-transformation.

It follows that the study of second order conformal deformations of spacelike surfaces reduces to the study of the integral manifolds of \((I_2, \alpha_0^1 \wedge \alpha_0^2)\). In particular, the study of the class of deformable surfaces given by isothermic surfaces reduces to the study of the integral manifolds of \((I_3, \alpha_0^1 \wedge \alpha_0^2)\).

4. The exterior differential system of a deformation

In this section, we undertake the study of the PDS of second order conformal deformations.

Let \((\alpha, \beta)\) be the \(g \times g\)-valued 1-form on \(\mathcal{D}\) obtained by pulling-back the Maurer-Cartan form \((\omega, \Omega)\) of \(G \times G\) with respect to a local section of the projection \(\pi_\mathcal{D} : G \times G \rightarrow \mathcal{D}\). Let \(\alpha^1 = \alpha_0^1, \alpha^2 = \alpha_0^2\) and set
\[ \begin{align*}
\eta^1 &= \alpha_0^0 - \beta_0^0, & \eta^2 &= \alpha_0^0 - \beta_0^1, & \eta^3 &= \alpha_0^2 - \beta_0^2, \\
\eta^4 &= \alpha_0^0, & \eta^5 &= \alpha_0^0, & \eta^6 &= \beta_0^3, \\
\eta^7 &= \beta_0^4, & \eta^8 &= \alpha_0^1 - \beta_0^1, & \eta^9 &= \alpha_0^1 - \beta_0^2, \\
\eta^{10} &= \alpha_0^2 - \beta_0^3, & \eta^{11} &= \alpha_0^4 - \beta_0^4, & \eta^{12} &= \alpha_0^4 - \beta_0^4.
\end{align*} \tag{4.1} \]

The Pfaffian differential system \((I_2, \alpha^1 \wedge \alpha^2)\) differentially generated by the 1-forms \(\eta^1, \ldots, \eta^{12}\) with independent condition
\[ \alpha^1 \wedge \alpha^2 \neq 0 \]
is called the differential system of a deformation.

Remark 4.1. The definition of \(I_2\) is independent of the local sections of \(\pi_\mathcal{D}\).

The integral manifolds of \((I_2, \alpha^1 \wedge \alpha^2)\) are the two-dimensional immersed submanifolds
\[ \mathfrak{d} = ([A_0 \wedge A_1 \wedge A_2], [B_0 \wedge B_1 \wedge B_2], BA^{-1}) : X \rightarrow \mathcal{D}, \]
where \((A, B) : X \rightarrow G \times G\) is an integral manifold of the Pfaffian differential system \(\pi_\mathcal{D}(I_2)\) defined on \(G \times G\) and \(f = [A_0], \hat{f} = [B_0]\) are second order deformations of each other with respect to \(D = BA^{-1}\), i.e., \(\hat{f}\) and \(D(q) \cdot f\) have second order analytic contact at \(q\), for each \(q \in X\).

Using the Maurer-Cartan equations, we compute, modulo the algebraic ideal generated by \(\eta^1, \ldots, \eta^{12}\), the quadratic equations of the system
\[ \begin{align*}
d\eta^1 &= -(\alpha_0^0 - \beta_0^0) \wedge \alpha^1 - (\alpha_0^0 - \beta_0^0) \wedge \alpha^2, \\
d\eta^2 &= d\eta^3 \equiv 0, \\
d\eta^4 &= d\eta^6 \equiv -\alpha_0^1 \wedge \alpha^1 - \alpha_0^3 \wedge \alpha^2, \\
d\eta^5 &= d\eta^7 \equiv -\alpha_0^4 \wedge \alpha^1 - \alpha_0^2 \wedge \alpha^2, \tag{4.2}
\end{align*} \]
Remark its dual basis.

is a general vector in the space $\eta$ if and only if

\begin{equation}
\begin{aligned}
\eta^8 &\equiv (\alpha_1^0 - \beta_1^0) \wedge \alpha^2 - (\alpha_2^0 - \beta_2^0) \wedge \alpha^1, \\
\eta^9 &\equiv - (\alpha_3^0 - \beta_3^0) \wedge \alpha^1 - (\alpha_4^0 - \beta_4^0) \wedge \alpha^2, \\
\eta^{10} &\equiv - (\alpha_3^0 - \beta_3^0) \wedge \alpha^2 - (\alpha_4^0 - \beta_4^0) \wedge \alpha^1, \\
\eta^{11} &\equiv (\alpha_4^0 - \beta_4^0) \wedge \alpha^1 - (\alpha_3^0 - \beta_3^0) \wedge \alpha^2, \\
\eta^{12} &\equiv (\alpha_4^0 - \beta_4^0) \wedge \alpha^2 - (\alpha_3^0 - \beta_3^0) \wedge \alpha^1.
\end{aligned}
\end{equation}

(4.3)

From this, we see that the differential ideal $\mathcal{I}_2$ is algebraically generated by $\eta^1, \ldots, \eta^{12}$ and the 2-forms

\begin{equation}
\begin{aligned}
\Omega^1 &\equiv -(\alpha_1^0 - \beta_1^0) \wedge \alpha^1 - (\alpha_2^0 - \beta_2^0) \wedge \alpha^2, \\
\Omega^2 &\equiv -\alpha_1^0 \wedge \alpha^1 - \alpha_2^0 \wedge \alpha^2, \\
\Omega^3 &\equiv -\alpha_1^0 \wedge \alpha^1 - \alpha_2^0 \wedge \alpha^2, \\
\Omega^4 &\equiv (\alpha_1^0 - \beta_1^0) \wedge \alpha^1 - (\alpha_2^0 - \beta_2^0) \wedge \alpha^2, \\
\Omega^5 &\equiv - (\alpha_3^0 - \beta_3^0) \wedge \alpha^1 - (\alpha_4^0 - \beta_4^0) \wedge \alpha^2, \\
\Omega^6 &\equiv - (\alpha_3^0 - \beta_3^0) \wedge \alpha^2 - (\alpha_4^0 - \beta_4^0) \wedge \alpha^1, \\
\Omega^7 &\equiv (\alpha_3^0 - \beta_3^0) \wedge \alpha^1 - (\alpha_4^0 - \beta_4^0) \wedge \alpha^1, \\
\Omega^8 &\equiv (\alpha_3^0 - \beta_3^0) \wedge \alpha^2 - (\alpha_4^0 - \beta_4^0) \wedge \alpha^2.
\end{aligned}
\end{equation}

(4.4)

Remark 4.2. The Pfaffian differential system $(\mathcal{I}_2, \alpha^1 \wedge \alpha^2)$ is linear, in the sense that

$$d\{\eta^1, \ldots, \eta^{12}\} \subset \{\eta^1, \ldots, \eta^{12}, \alpha^1, \alpha^2\},$$

where $\{\eta^1, \ldots, \eta^{12}\}$ and $\{\eta^1, \ldots, \eta^{12}, \alpha^1, \alpha^2\}$ denote, respectively, the ideals generated algebraically by $\eta^1, \ldots, \eta^{12}$ and $\eta^1, \ldots, \eta^{12}, \alpha^1, \alpha^2$. If a Pfaffian differential system is linear, then its integral elements at a point are determined by inhomogeneous linear equations. In the literature, linear systems are also called quasi-linear systems, systems in good form, or systems in normal form.

We now discuss the involutiveness of the system. For this, we consider the basis

$$(\alpha^1, \alpha^2, \eta^i, \alpha_i^0, \alpha_i^0 - \beta_i^0, \alpha_3^0 - \beta_3^0)$$

($j = 1, \ldots, 12; i = 1, 2; a = 3, 4; I = 1, 2, 3, 4$) for the 1-forms on $\mathcal{D}$ and denote by

$$\left( \frac{\partial}{\partial \alpha^1}, \frac{\partial}{\partial \alpha^2}, \frac{\partial}{\partial \eta^1}, \frac{\partial}{\partial \alpha_i^0}, \frac{\partial}{\partial (\alpha_i^0 - \beta_i^0)}, \frac{\partial}{\partial (\alpha_3^0 - \beta_3^0)} \right)$$

its dual basis.

A 1-dimensional integral element of the system is of the form $E_1 = [V]$, where

$$V = a_i \frac{\partial}{\partial \alpha^1} + b_i \frac{\partial}{\partial \alpha^2} + b_2 \frac{\partial}{\partial \alpha_2^0} + b_3 \frac{\partial}{\partial \alpha_1^0} + b_4 \frac{\partial}{\partial \alpha_2^0} + b_{4+I} \frac{\partial}{\partial (\alpha_i^0 - \beta_i^0)} + b_9 \frac{\partial}{\partial (\alpha_3^0 - \beta_3^0)}$$

is a general vector in the space $\eta^i = 0, j = 1, \ldots, 12$. Thus, the manifold of 1-dimensional integral elements $\mathcal{V}_1 \cong \mathcal{D} \times \mathbb{R}P^{10}$. Moreover, $E_1$ is admissible if and only if $(a_1)^2 + (a_2)^2 \neq 0$. 

The polar equations of a given \( E_1 \in \mathcal{V}_1 \) are \( \eta^j = 0 \) \((j = 1, \ldots, 12)\) and
\[
i_\mathcal{V} \Omega^\beta = 0 \quad (\beta = 1, \ldots, 8),
\]
which read
\[
\begin{align*}
-b_5 \alpha^1 + a_1 (\alpha_1^0 - \beta_1^0) - b_6 \alpha^2 + a_2 (\alpha_2^0 - \beta_2^0) &= 0, \\
b_1 \alpha^1 + a_1 \alpha_1^3 - b_2 \alpha^2 + a_2 \alpha_2^3 &= 0, \\
b_3 \alpha^1 + a_1 \alpha_1^4 - b_4 \alpha^2 + a_2 \alpha_2^4 &= 0,
\end{align*}
\]
\[(4.5)\]
\[
\begin{align*}
b_5 \alpha^2 - a_2 (\alpha_1^0 - \beta_1^0) - b_6 \alpha^3 + a_1 (\alpha_2^0 - \beta_2^0) &= 0, \\
b_7 \alpha^1 + a_1 (\alpha_3^0 - \beta_3^0) - b_9 \alpha_4^1 + b_3 (\alpha_4^4 - \beta_4^3) &= 0, \\
b_7 \alpha^2 + a_2 (\alpha_3^0 - \beta_3^0) - b_9 \alpha_4^2 + b_4 (\alpha_4^4 - \beta_4^3) &= 0, \\
b_8 \alpha^1 - a_1 (\alpha_3^0 - \beta_3^0) - b_9 \alpha_4^3 + b_1 (\alpha_4^4 - \beta_4^3) &= 0, \\
b_8 \alpha^2 - a_2 (\alpha_3^0 - \beta_3^0) - b_9 \alpha_4^3 + b_2 (\alpha_4^4 - \beta_4^3) &= 0.
\end{align*}
\]

If
\[
b_9 \left[ (a_1)^2 - (a_2)^2 \right] [a_1 (b_2 b_7 + b_4 b_8) - a_2 (b_1 b_7 + b_3 b_8) + b_9 (b_2 b_3 - b_1 b_4)] \neq 0,
\]
the polar equations are linearly independent and the polar space \( H(E_1) \) is 3-dimensional. Thus \( c_0 = \dim \mathcal{J}_2^{(1)} = 12 \) and \( c_1 = \text{codim} \ H(E_1) = 20 \). Further, the variety of 2-dimensional integral elements over a point \( x \in \mathcal{D} \) has dimension 10. Hence \( c_0 + c_1 = \dim G_2(T_x \mathcal{D}) - \dim \mathcal{V}_2(x) = 32 \), and Cartan’s test applies. The Cartan characters \( s_k = c_k - c_{k-1} \), \( k = 0, 1, 2 \), are then computed to be \( s_0 = 12, s_1 = 8, s_2 = 1 \).

Summarizing, we can state the following.

**Proposition 4.3.** The Pfaffian differential system \( (\mathcal{J}_2, \alpha^1 \wedge \alpha^2) \) is in involution and its general solutions depend on one arbitrary function in two variables. The singular solutions of the system correspond to the points of the reducible variety defined by the equation
\[
b_9 \left[ (a_1)^2 - (a_2)^2 \right] [a_1 (b_2 b_7 + b_4 b_8) - a_2 (b_1 b_7 + b_3 b_8) + b_9 (b_2 b_3 - b_1 b_4)] = 0.
\]

In particular, isothermic surfaces correspond to the points of the variety defined by
\[
b_9 \left[ (b_7)^2 + (b_8)^2 + (b_9)^2 \right] = 0.
\]

**Remark 4.4.** Note that isothermic surfaces satisfy the additional equations
\[
\alpha_3^4 - \beta_3^4 = 0, \quad \alpha_3^0 - \beta_3^0 = 0, \quad \alpha_4^0 - \beta_4^0 = 0,
\]
and are then integral manifolds of the Pfaffian differential system \( (\mathcal{J}_3, \alpha^1 \wedge \alpha^2) \), differentially generated by \( \mathcal{J}_3 \subset \Gamma(T^* \mathcal{D}) \) (see Section 2.3). A direct computation shows that the system \( (\mathcal{J}_3, \alpha^1 \wedge \alpha^2) \) is in involution and its general solution depends on six arbitrary functions in one variable.

Considering that a generic surface in a \( (2 + r) \)-dimensional space may be locally given as graph of \( r \) arbitrary functions of two variables, deformable surfaces in compactified Minkowski 4-space are then exceptional.
REFERENCES

[1] H. Bernstein, Non-special, non-canal isothermic tori with spherical lines of curvature, *Trans. Amer. Math. Soc.* **353** (2001), 2245–2274.

[2] R. L. Bryant, S.-S. Chern, R. B. Gardner, H. L. Goldschmidt, P. A. Griffiths, *Exterior differential systems*, MSRI Publications, 18, Springer-Verlag, New York, 1991.

[3] R. L. Bryant, A duality theorem for Willmore surfaces, *J. Differential Geom.* **20** (1984), 23–53.

[4] F. Burstall, U. Hertrich-Jeromin, F. Pedit, U. Pinkall, Curved flats and isothermic surfaces, *Math. Z.* **225** (1997), 199–209.

[5] F. Burstall, Isothermic surfaces: conformal geometry, Clifford algebras and integrable systems, in “Integrable systems, Geometry and Topology” (ed. C.-L. Terng), 1-82, *AMS/IP Stud. Adv. Math.*, 36 (2006).

[6] E. Cartan, *Sur le problème général de la déformation*, C. R. Congrès Strasbourg (1920), 397–406; or *Oeuvres Complètes*, III 1, 539–548.

[7] E. Cartan, *Les espaces à connexion conforme*, Ann. Soc. Pol. Math. (1923), 171–221; or *Oeuvres Complètes*, III 1, 747–797.

[8] T. E. Cecil, *Lie sphere geometry: with applications to submanifolds*, Springer-Verlag, New York, 1992.

[9] A. Fujioka, J. Inoguchi, Deformations of surfaces preserving conformal or similarity invariants, in “From geometry to quantum mechanics”, 53–67, *Progr. Math.*, 252, Birkhäuser Boston, Boston, MA, 2007; arXiv:math.DG/0512255.

[10] P. A. Griffiths, On Cartan’s method of Lie groups and moving frames as applied to uniqueness and existence questions in differential geometry, *Duke Math. J.* **41** (1974), 775–814.

[11] G. R. Jensen, Deformation of submanifolds of homogeneous spaces, *J. Differential Geom.* **16** (1981), 213–246.

[12] G. R. Jensen, E. Musso, Rigidity of hypersurfaces in complex projective space, *Ann. Sci. École Norm. Sup.* (4) **27** (1994), 227–248.

[13] V. Guillemin, S. Sternberg, *Variations on a theme by Kepler*, AMS Colloquium Publications, 42, American Mathematical Society, Providence, 1990.

[14] U. Hertrich-Jeromin, *Introduction to Möbius differential geometry*, London Mathematical Society Lecture Note Series, 300, Cambridge University Press, Cambridge, 2003.

[15] N. H. Kuiper, On conformally-flat spaces in the large, *Ann. of Math. (2)* **50**, (1949), 916–924.

[16] E. Musso, Deformazione di superfici nello spazio di Möbius, *Rend. Istit. Mat. Univ. Trieste* **27** (1995), 25–45.

[17] E. Musso, L. Nicolodi, Deformation and applicability of surfaces in Lie sphere geometry, *Tohoku Math. J.* **58** (2006), 161–187.

[18] B. Palmer, Isothermic surfaces and the Gauss map, *Proc. Amer. Math. Soc.* **104** (1988), 876–884.

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