Strategic Inventory Management of Deteriorating Products with Demand Disruptions

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Abstract
The retailer’s strategic inventory management can convince the supplier to present lower wholesale prices in later orders. In practice, there are always unavoidable factors that cause disruption in the supply chain with undeniable impacts on strategic inventories, such as COVID-19. This is of greater significance in regard to goods that lose their freshness by time. The impacts of disruption in such products on managed strategic inventories and the retailer’s and supplier’s consequent behavior have not been addressed in the literature. Therefore, this research investigated the impacts of disruption in demand for deteriorating items on strategic inventories managed by the retailer. The results demonstrated that strategic inventory management is not always a threat for the supplier, and it can even be useful in certain conditions. Changes in goods deterioration levels were found to affect profit more seriously if disruption increased demand in the first period. We also specified the consequences of a decision to or not to manage strategic inventories in different cases of disruption and inventory management cost.

Keywords Demand disruption · Deteriorating products · Strategic inventory

1 Introduction
A supply chain disruption is any sudden change or crisis—whether local or global—that negatively impacts that process. Sometimes, these disruptions are small and localized, and may affect only a few businesses or industries. Other times, they are globally felt, especially when the hard-hit region is a source of a diverse range of supplies, as with many Asian markets. It has been seen in recent weeks and months how COVID-19 has impacted global supply chains. Global pandemics and other public health crises can have massive supply chain impacts because of the number of people, regions, and global companies that are affected, which can disturb normal operations at any stage of the production line. The COVID-19 global pandemic has caused a major upheaval in the supply chains of many businesses in the USA and North America. Dr. Bob Novack, Associate Professor of Business Logistics at Penn State, explains, “This pandemic is like a natural disaster, but it’s also not. The results of a natural disaster on the supply chain are usually localized and short-lived. In a pandemic it’s global. It’s the rapid increase in demand that a pandemic creates that puts additional strain on suppliers.”

COVID-19 has disrupted supply chains around the world. There is no doubt that the pandemic has tested the ingenuity, resilience, and flexibility of supply chain leaders globally, as they have sought to manage essential operations. 94% of Fortune 1000 companies are seeing supply chain disruptions from COVID-19.2

COVID-19 may accelerate companies’ search for ways to reduce supply chain risk and the likelihood of future disruption. More than 90% of Fortune 1000, including many technology companies, have already suffered supply chain disruption.3

The influence of COVID-19 on the supply chains of deteriorating items is undeniable, and food supply chains have suffered plenty of disruption in the past few months due to its outbreak. Farmers all over the world are letting their crops rot in the fields as the coronavirus pandemic disrupts supply and
demand for a wide variety of deteriorating goods including flowers, fruit, and milk.\footnote{www.cnbc.com.}

COVID-19 has led to a drastic shift in consumer demand away from restaurants, food services, and other types of “food away from home” towards food consumed at home, requiring important changes in the way food supply chains operate. As the COVID-19 pandemic speeded up, sales of food away from home (consumed in hotels, restaurants, catering, and cafés) collapsed. Restaurant reservations declined sharply in early March, and fell to zero in practice as lockdowns were enforced.\footnote{http://www.oecd.org.}

The lockdown on restaurants and food services due to the outbreak eliminated the major purchasers of meat from the beef industry. According to National Cattleman’s Beef Association, cattle ranchers have faced more than $13 billion of losses through 2021 as a result of the coronavirus. An overall decline in demand for dairy products from schools and the restaurant industry—including cheese, butter, and ice cream—has saddled dairy farmers with more raw milk than they can sell, forcing millions of gallons of milk to be dumped every day.

Potatoes are the most popular restaurant vegetable in the USA, thanks to all the ways they can be cut, sliced, sautéed, and fried. Like with meat and dairy products, demand for potatoes from restaurants sharply declined when in-service dining was reduced.\footnote{www.nationalgeographic.com.}

At the same time, sales of frozen and packaged foods in particular increased dramatically; at their peak in the second half of March, the weekly sales of frozen foods were 63% higher than in the year before in France, while the sales of packaged foods were 56% higher year-on-year in Germany. Similar demand spikes were seen in other countries (see footnote 5).

Based on the above statements, it has become more significant to adopt appropriate inventory control policies, where inventory level needs to be decreased for certain products to prevent more loss, while it should be increased for others.

Inventory is commonly carried for a variety of reasons, such as to address uncertainty in demand or supply, to achieve economies of scale, or to hedge against price fluctuation. Importantly, it may also be carried to play a strategic role in multi-period vertical competition environments [1].

In fact, the supplier will need to reduce price to an amount lower than in the first period once the extra inventory in that period is purchased by the retailer. Therefore, strategic inventories are regarded as a competitive tool for the retailer. The quality of products like fresh food is usually reduced over time, so warehoused products are not as fresh as new ones. That is why customers find older products less valuable. If the demand for a product is disrupted (as caused by COVID-19), therefore, it can be a retailer’s concern whether or not to manage the strategic inventories of deteriorating products. Thus, this research investigates the impacts of disruption in demand on the decision to or not to manage strategic inventories. In fact, it extends Mantin and Jiang [1] by postulating disruption in the first-period demand.

2 Literature Review

There are two mainstreams relevant to the topic under investigation: strategic inventories and supply chain disruption (See Table 1), reviewed below.

2.1 Strategic Inventories in the Supply Chain

Anand et al. \cite{13} were the first to identify the strategic role of inventories in a multiperiod supply chain setting. Their results demonstrated that the buyer’s optimal strategy was to carry inventories, and the supplier was unable to prevent this, although the wholesale price was lower in the second period in equilibrium.

Arya et al. \cite{19} used a two-period strategic inventory model to study decentralized decision-making in procurement and inventory control. Arya and Mittendorf \cite{14} explored the role of rebates offered by manufacturers directly to consumers in mediation of the effect of strategic inventories. Essentially, they found that manufacture-to-consumer rebates made the retailer less aggressive in carrying inventories and the manufacturer less exploitative in setting wholesale prices. The retailer, manufacturer, and consumers were all found to be better-off due to manufacture-to-consumer rebates. Hartwig et al. \cite{15} conducted the first empirical study to examine the effect of strategic inventories on supply chain performance based on the theoretical models of Anand et al. \cite{13}. Mantin and Jiang \cite{1} investigated strategic inventories in a bilevel supply chain including a supplier and a retailer. Their research was focused on the strategic inventories of deteriorating goods. Moon et al. \cite{16} studied the impacts of strategic inventories on a two-period supply chain. Demand was based in their model on price and investment effort. The results demonstrated that the retailer’s strategic inventories were not always a threat, and could even be useful.

Mantin and Veldman \cite{17} investigated the issue of strategic inventories in presence of process improvement efforts. They demonstrated that strategic inventories could be harmful for the supplier and the retailer under such conditions. In their investigation of the interaction between a supplier’s centralized or decentralized encroachment and a retailer’s use of strategic inventory, Li et al. \cite{18} demonstrated that decentralized encroachment outperformed centralized encroachment for both the supplier and the retailer.
Table 1  Contribution of our work

|                      | Strategic inventory | Demand disruption | Product type |
|----------------------|---------------------|-------------------|--------------|
|                      |                     |                   | Deteriorating | Other        |
| Huang et al. [2]     | √                   |                   |              |              |
| Cao [3]              | √                   |                   |              |              |
| Wang et al. [4]      | √                   |                   |              |              |
| Qi et al. [5]        | √                   |                   |              |              |
| Tang et al. [6]      | √                   |                   |              |              |
| Wu et al. [7]        | √                   |                   |              |              |
| Pi et al. [8]        | √                   |                   |              |              |
| Zhao et al. [9]      | √                   |                   |              |              |
| Yan et al. [10]      | √                   |                   |              |              |
| Zhang et al. [11]    | √                   |                   |              |              |
| Chen and Xiao [12]   |                     | √                 |              |              |
| Anand et al. [13]    |                     |                   |              |              |
| Arya et al. [19]     |                     |                   |              |              |
| Arya and Mittendorf [14] |                |                   |              |              |
| Hartwig et al. [15]  |                     |                   |              |              |
| Mantin and Jiang [1] |                     |                   |              |              |
| Moon et al. [16]     |                     |                   |              |              |
| Mantin and Veldman [17] |                 |                   |              |              |
| Li et al. [18]       |                     |                   |              |              |
| Our work             | √                   | √                 |              |              |

2.2 Supply Chain Disruption

Qi et al. [5] investigated optimal price and quantity decision-making in both modes with and without demand disruption. They found that a quantity discount contract could coordinate the supply chain in particular situations. Chen and Xiao [12] extended Qi et al. [5] considering the linear quantity discount demand contract and wholesale price contract for supply chain coordination upon disruption in demand. They demonstrated that the former contract would be better for the manufacturer for very high increased demand and very low manufacturing costs. Wang et al. [4] investigated optimal price and quality improvement considering a quantity discount contract in a fashion supply chain. They found that the contract could coordinate the supply chain with no disruption in demand. Huang et al. [2] extended the model presented by Qi et al. [5] considering a dual-channel supply chain. The manufacturer would sell the product via a retailer and a direct channel. They assessed the optimal manufacturing and pricing decisions, and found that optimal price appeared on the channels as affected by customer preferences. Cao [3] extended Huang et al. considering a revenue-sharing contract, and found that the contract could coordinate the supply chain without disruption in demand.

Zhang et al. [11] investigated how a dual-channel supply chain could be coordinated upon disruption in manufacturing costs or demand. Yan et al. [10] studied the impacts of risk aversion and demand disruption on optimal strategies in a dual-channel supply chain in centralized and decentralized mode. Tang et al. [6] extended Cao [3] considering collaboration and competition in a dual-channel supply chain, and investigated the impacts of disruption in demand and costs on optimal strategies. Wu et al. [7] studied pricing and manufacturing decisions with a revenue-sharing contract considering demand disruption and limited capacity. Pi et al. [8] investigated pricing and service strategies in a dual-channel supply chain consisting of one manufacturer, two competitive retailers, and a direct sale channel. Upon disruption in demand for each channel, the customer would be referred to the other. They found that collaboration among retailers could improve their performance, while reducing the profit gained by the manufacturer and by the entire system. Zhao et al. [9] studied the impacts of disruption in demand on collaboration in a fashion supply chain. They utilized revenue-sharing and linear quantity discount contracts for supply chain coordination, and found that the revenue-sharing contract failed in some cases to coordinate the supply chain, whereas the linear quantity discount contract succeeded.
As clear from the literature and Table 1, presence of strategic inventories has been disregarded before occurrence of demand disruption. Therefore, our research serves to contribute to development of the works available in the area of disruption in demand considering disruption for deteriorating goods and strategic inventories for the retailer.

This paper is organized as follows. Section 3 presents the model description and formulation and the decision variable equilibrium. The optimal solutions are analyzed, and the model is evaluated in Sect. 4. Section 5 presents the results obtained from our work. The proofs are provided in Appendices A and B.

3 Problem Description and Formulation

The supply chain investigated in this paper is bilevel, consisting of one supplier and one retailer, where the manufactured goods are deteriorating, and lose their freshness and quality over time. It is examined within the framework of a dynamic bilevel model how the supply chain members function. Product demand in the first period is 1, so the retailer can order a quantity of 1+1 to the supplier to manage strategic inventories. Upon disruption in demand, it may increase or decrease. If demand is lower than 1, the retailer will transfer the extra demand to the second period along with strategic inventories. If demand is greater than 1, the retailer will be able to sell all or part of strategic inventories in the first period in order to meet demand.

In the second period, the retailer has two types of product. The first type involves a new product ordered by the retailer to the supplier in the second period. The second type involves the same inventory transferred from the first period to the second (which may not equal strategic inventories due to demand disruption in the first period). Goods of the second type are not as fresh and high-quality as a product of the first type, so the retailer sells them to the customer at a discount. Since strategic inventories are managed by the retailer, the supplier charges a different wholesale price in each period. The purpose of this paper is to investigate the impacts of disruption on strategic inventories of deteriorating items under different conditions. In fact, we would like to know by presenting the model when and under what conditions it is economically justifiable for the retailer to manage strategic inventories upon disruption in demand, and when it is suggested that strategic inventories not be managed. Figure 1 shows an overview of the model.

The following assumptions are made in the model.

1. Disruption in demand occurs only in the first period.
2. 1 is the demand disruption parameter. 1 > 0 (1 < 0) denotes that demand in the first period is greater (less) than 1.
3. Each unit of inventory transferred from the first period to the second is managed at a cost of .
4. The examined two-period model is one with complete information.
5. In the vertical competition between the retailer and the supplier, the latter is the leader.
6. 1 and 2 represent the amounts of product purchased by the retailer in the first and second periods, respectively.
7. The examined deteriorating goods may include different dairy products or fresh vegetables or fruit.
8. 1 indicates the discount factor, where 1 ∈ (0, 1].
9. ϑ denotes customers’ evaluation of the product, where ϑ ∼ U[0, 1]. Therefore, the value of the new goods is ϑ according to customers, and that of the old goods is ϑ.
10. The demand function for the product in the first period is 1 = 1 + 1 - 1.
11. As in Zhao et al. [9] and Mantin and Jiang [1], customer utility function is as follows:

\[ U_{2r} = \vartheta \delta - p_{2r} \]  
\[ U_{2n} = \vartheta - p_{2n} \]

where 1 is the price of the first-type product (new product), and 1 is that of the second-type one (old product) in the second period.

According to (1) and (2), the demand functions are as follows in the second period.

\[ q_{2n} = 1 - (p_{2n} - p_{2o})/(1 - \delta) \]  
\[ q_{2o} = \left\{ p_{2n} - p_{2o} \over 1 - \delta \right\} - (p_{2o}/\delta) \]

The problem is examined independently in two cases: 1 < 0 and 1 > 0. The mathematical model of the retailer’s and supplier’s profit functions in the first and second periods is presented in each case.

3.1 Model Formulation with 1 < 0

It is assumed in this section that actual demand for the product in the first period is less than 1 due to disruption in demand, on which basis the retailer’s and supplier’s problems in the first and second periods are modeled.
3.1.1 Model Formulation in the First Period

(A) Retailer’s model
In the first period, the purpose of the retailer is to specify \( I_s \) and \( p_1 \) so as to maximize profit. The profit function obtains the difference between the revenue gained from sale and inventory management costs. Due to disruption in demand in the first period, \( I_{\Delta} \) of the demand quantity \( q_1 \) is not sold, and is transferred to the second period (\( I_{\Delta} = -\Delta \)). Therefore, the retailer maximizes total profit in the first and second periods by specifying \( I_s \) and \( p_1 \), as shown below.

\[
\begin{align*}
\max \pi_1 &= p_1(q_1 - I_{\Delta}) - (q_1 + I_s)w_1 - h(I_s + I_{\Delta}) + \pi_{2s} \\
\text{s.t.} \quad 0 < I_{\Delta} \leq aq_1 \quad \text{and} \quad I_s \geq 0
\end{align*}
\]  

(B) Supplier’s model
The supplier specifies the wholesale price \( w_1 \) so as to maximize total profit in the first and second periods. Therefore, the supplier’s problem looks as follows.

\[
\max \pi_{1s} = w_1(q_1 + I_s) + \pi_{2s}
\]

3.1.2 Model Formulation in the Second Period

(A) Retailer’s model
The retailer’s purpose is to specify \( p_{2n} \) and \( p_{2o} \), so as to maximize profit in the second period. This means to select \( q_{2n} \) and \( q_{2o} \) so as to maximize the following profit function.

\[
\begin{align*}
\max \pi_2 &= p_{2n}q_{2n} + p_{2o}q_{2o} - q_{2o}w_2 \\
\text{s.t.} \quad 0 \leq q_{2o} \leq I_s + I_{\Delta} \quad \text{and} \quad q_{2n} \geq 0
\end{align*}
\]

(B) Supplier’s model
In the second period, the supplier specifies \( w_2 \) so as to maximize profit. The following equation therefore holds.

\[
\max \pi_{2s} = w_2q_{2n}
\]

It should be noted that the Stackelberg game between the retailer and the supplier is solved using recursive induction. Therefore, we solve the model first in the second period.

The retailer’s profit function in the second period can be seen in Eq. (8). The constraints of the retailer’s problem appear in Eq. (9).

Proposition 1 The optimal values of \( q_{2o} \) and \( q_{2n} \) are as follows.
\begin{equation}
(q_{2o}^*, q_{2n}^*) = \begin{cases} 
(I_{\Delta} + I_s, 0), & \text{if } I_{\Delta} + I_s < \frac{1}{2} \text{ and } w_2 > 1 - 2\delta(I_{\Delta} + I_s) \\
(\frac{1}{2}, 0), & \text{if } I_{\Delta} + I_s \geq \frac{1}{2} \text{ and } w_2 > 1 - \delta \\
\left(\frac{-w_2}{2\delta - 2}, \frac{w_2+\delta-1}{2\delta - 2}\right), & \text{if } w_2 \leq 1 - \delta \text{ and } w_2 \leq (2 - 2\delta)(I_{\Delta} + I_s) \\
(I_{\Delta} + I_s, \frac{1-w_2-2\delta(I_{\Delta}+I_s)}{2}), & \text{if } (2 - 2\delta)(I_{\Delta} + I_s) < w_2 \leq 1 - 2\delta(I_{\Delta} + I_s) 
\end{cases}
\end{equation}

**Proof** Given the concavity of the profit function (See “Appendix A”) and linearity of the constraints in the retailer’s problem, we can extract the optimal values from the first-order derivative of the profit function (presented in (12) and (13)) with respect to the variables. Therefore, the optimal values of \(q_{2o}^*\) and \(q_{2n}^*\) are obtained through addition of the Lagrangian multipliers to (12) and (13) based on the Kuhn–Tucker conditions.

\begin{align}
\frac{\partial \pi_{2r}}{\partial q_{2o}} &= \delta(1 - 2q_{2o} - 2q_{2n}) \quad (12) \\
\frac{\partial \pi_{2r}}{\partial q_{2n}} &= 1 - 2q_{2n} - 2\delta q_{2o} - w_2 \quad (13)
\end{align}

The retailer’s problem can therefore be solved as follows.

\begin{align}
\frac{\partial \pi_{2r}}{\partial q_{2o}} + \lambda_1 - \lambda_2 &= 0 \quad (14) \\
\frac{\partial \pi_{2r}}{\partial q_{2n}} + \lambda_3 &= 0 \quad (15) \\
\lambda_1 q_{2o} &= 0 \quad (16) \\
\lambda_2(I_{\Delta} + I_s - q_{2o}) &= 0 \quad (17) \\
\lambda_3 q_{2n} &= 0 \quad (18)
\end{align}

\(\lambda_1, \lambda_2, \lambda_3 \geq 0\) and \(0 \leq q_{2o} \leq I_{\Delta} + I_s\) and \(q_{2n} \geq 0\) \quad (19)

We need to analyze the following eight cases by solving the system of Eqs. (14) to (18) and considering Constraint (19).

(i) If \(\lambda_1 = 0\) and \(\lambda_2, \lambda_3 > 0\), then \(q_{2o} = I_{\Delta} + I_s\) and \(q_{2n} = 0\), according to (17) and (18), respectively. Based on (14), \(\lambda_2 = \delta(1 - 2q_{2o} - 2q_{2n})\) must be positive. Therefore, \(I_{\Delta} + I_s < \frac{1}{2}\). According to (15), \(\lambda_3 = -1 + 2q_{2n} + 2\delta q_{2o} + w_2\) must be positive. Therefore, \(w_2 > 1 - 2\delta(I_{\Delta} + I_s)\).

(ii) If \(\lambda_1 = \lambda_2 = 0\) and \(\lambda_3 > 0\), then \(w_2 = 0\) according to (18). Given (14), \(q_{2o} = \frac{1}{2}\). Based on (15), \(\lambda_3 = -1 + 2q_{2n} + 2\delta q_{2o} + w_2\) must be positive. Therefore, \(w_2 > 1 - \delta\). According to (12), \(I_{\Delta} + I_s \geq \frac{1}{2}\).

(iii) If \(\lambda_1 = \lambda_2 = \lambda_3 = 0\), then \(0 \leq q_{2o} \leq I_{\Delta} + I_s\) and \(q_{2n} \geq 0\), according to (16) to (18). Given (14), \(q_{2o} = \frac{1}{2} - q_{2n}\). Moreover, \(q_{2n} = \frac{w_2+\delta-1}{2\delta - 1}\), given (15). Since \(0 \leq q_{2o} \leq I_{\Delta} + I_s\) and \(q_{2n} \geq 0\), \(w_2 \leq 1 - \delta\) and \(w_2 \leq (2 - 2\delta)(I_{\Delta} + I_s)\).

(iv) If \(\lambda_1 > 0\) and \(\lambda_3 = \lambda_2 = 0\), then \(q_{2o} = 0\), given (16). According to (15), \(q_{2n} = \frac{1-w_2}{2}\). Based on (14), \(\lambda_2 = \delta(1 - 2q_{2o} - 2q_{2n})\) must be positive, which requires \(w_2 < 0\). Therefore, \(q_{2o} = 0\) and \(q_{2n} = \frac{1-w_2}{2}\) cannot provide an optimal solution.

(v) If \(\lambda_1, \lambda_2 > 0\) and \(\lambda_3 = 0\), then \(q_{2o} = 0\) and \(q_{2n} = I_{\Delta} + I_s\), which cannot hold at the same time.

(vi) If \(\lambda_1 = \lambda_3 = 0\) and \(\lambda_2 > 0\), then \(q_{2o} = I_{\Delta} + I_s\), given (17). According to (15), \(q_{2n} = \frac{1-w_2-2\delta(I_{\Delta}+I_s)}{2}\). Based on (14), \(\lambda_2 = \delta(1 - 2q_{2o} - 2q_{2n})\) must be positive. Therefore, \((2 - 2\delta)(I_{\Delta} + I_s) < w_2\). Since \(q_{2n} \geq 0\), on the other hand, \(w_2 \leq 1 - 2\delta(I_{\Delta} + I_s)\).

(vii) If \(\lambda_1, \lambda_2, \lambda_3 > 0\), then \(q_{2o} = 0\) and \(q_{2n} = I_{\Delta} + I_s\), which cannot hold at the same time.

(viii) If \(\lambda_1, \lambda_3 > 0\) and \(\lambda_2 = 0\), then \(q_{2o} = 0\) and \(q_{2n} = 0\), given (16) and (18). According to (14), \(\lambda_1 = -\delta(1 - 2q_{2o} - 2q_{2n})\) must be positive, but it is always negative. Therefore, \(q_{2o} = 0\) and \(q_{2n} = 0\) cannot provide an optimal solution.

Equation (10) shows the supplier’s profit function in the second period. We now obtain \(w_2\) by specifying \(q_{2o}^*\) and \(q_{2n}^*\) in terms of \(w_2\).

**Proposition 2** Given Eqs. (10) and (11), the optimal value of \(w_2\) is as follows.

\(w_2^* = \begin{cases} 
\frac{1-\delta}{2}, & \text{if } I_{\Delta} + I_s \geq \frac{1}{4} \\
\frac{1-2\delta(I_{\Delta}+I_s)}{2}, & \text{if } I_{\Delta} + I_s < \frac{1}{4} 
\end{cases}\)

**Proof** There will be the following two cases if (11) is replaced in Eq. (10).
Therefore, the retailer’s problem can be solved as follows.

\[
\begin{align*}
\frac{\partial \pi_{1r}}{\partial q_1} + a\lambda_2 &= 0, \\
\frac{\partial \pi_{1r}}{\partial I_s} + \lambda_1 &= 0, \\
\lambda_1 I_s &= 0, \\
\lambda_2 (aq_1 - I_\Delta) &= 0
\end{align*}
\]

We need to analyze the following four cases by solving the system of Eqs. (24) to (27) and considering Constraint (28).

(i) If \( \lambda_1 = 0 \) and \( \lambda_2 > 0 \), then \( I_s \geq 0 \) and \( q_1 = \frac{1}{a} I_\Delta \), according to (26) and (27), respectively. Therefore, \( I_s = \frac{4h+4w_1+4(3-8I_\Delta-2\Delta I_\Delta)}{2\Delta(4-\delta)} \) is obtained from (25). Based on (24), \( \lambda_2 = \frac{1}{a}(w_1 - 1 + \frac{2}{a} I_\Delta) \) must be positive. Therefore, \( w_1 > 1 - \frac{2}{a} I_\Delta \). Moreover, \( I_s \geq 0 \), so

\[
\begin{align*}
&w_1 > 1 - \frac{2}{a} I_\Delta, \\
&I_\Delta < \frac{w_1 - 4h+4w_1+4(3-8I_\Delta-2\Delta I_\Delta)}{2\Delta(4-\delta)}
\end{align*}
\]

Based on (26), \( \lambda_2 = \frac{1}{a}(w_1 - 1 + \frac{2}{a} I_\Delta) \) must be positive. Therefore, \( w_1 > 1 - \frac{2}{a} I_\Delta \). Moreover, \( I_s \geq 0 \), so

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\begin{align*}
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&I_\Delta < \frac{w_1 - 4h+4w_1+4(3-8I_\Delta-2\Delta I_\Delta)}{2\Delta(4-\delta)}
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\[
\begin{align*}
&w_1 > 1 - \frac{2}{a} I_\Delta, \\
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\begin{align*}
\frac{\partial \pi_{1r}}{\partial q_1} + a\lambda_2 &= 0, \\
\frac{\partial \pi_{1r}}{\partial I_s} + \lambda_1 &= 0, \\
\lambda_1 I_s &= 0, \\
\lambda_2 (aq_1 - I_\Delta) &= 0
\end{align*}
\]
(It should be noted that the above solutions are obtained for both cases where \( I_\Delta + I_s \geq \frac{1}{4} \) and \( I_\Delta + I_s < \frac{1}{4} \).

The supplier’s problem in the first period is stated as Eq. (7), and the following proposition holds to obtain its optimal solution.

**Proposition 4** Given Eqs. (7) and (21), the optimal value of \( w_1 \) is as follows.

\[
\begin{align*}
\frac{2}{32} & \left( 2I_\Delta \delta (4 + \delta)^2 + a (4h(\delta - 4) - \delta (20 + 2I_\Delta (4 + \delta)^2 + 2(\delta - 7))) \right) \quad \text{if} \quad w_1 > A \quad \text{and} \quad w_1 \leq B \\
\frac{1}{2} & , \\
L_1 & , \\
\frac{4h(\delta - 4) - \delta (32I_\Delta - 36) + \delta ((-15 + 16I_\Delta) + (1 + 2I_\Delta) \delta))}{2(16 + \delta(4 + \delta)^2)} & , \\
\end{align*}
\]

where the following hold.

\[
A = 1 - \frac{2}{a} I_\Delta \\
B = -h + \frac{\delta}{4}(3 - 8I_\Delta + 2\delta I_\Delta)
\]

**Proof**

1. According to (20), there will be four cases as follows if \( \{w_1^*, q_2^*, q_3^*\} = \{(1 - 2hI_\Delta + I_s), I_\Delta + I_s, 1 - w_2 - 2h(I_\Delta + I_s)\} \).

   - \( (q_1^*, I_1^*) = \left( \frac{1}{a} I_\Delta, 0 \right) \).

   In that case, \( \frac{\partial \pi_{11}}{\partial w_1} = \frac{1}{a} I_\Delta \). If \( w_1 > \max\{-h + \frac{\delta}{4}(3 - 8I_\Delta + 2\delta I_\Delta), 1 - \frac{2}{a} I_\Delta\} \), the optimal solution will be specified through an upper bound considered for \( w_1 \).

   - \( (q_1^*, I_1^*) = \left( \frac{1}{a} I_\Delta, \frac{4h + 4w_1 + \delta (3 - 8I_\Delta - 2\delta I_\Delta)}{2(\delta - 4)} \right) \).

   In that case, \( \frac{\partial \pi_{11}}{\partial w_1} = \frac{\delta (2\delta - 3) - 4h}{2(\delta - 4)} - \frac{8}{2(\delta - 4)} w_1 + \frac{1}{(\delta + 4)^2} (-2\delta^2 - 2\delta + 4 + 4w_1 + 4h) \). Therefore, \( w_1^* = \frac{1}{32} \left( 2I_\Delta \delta (4 + \delta)^2 + a (4h(\delta - 4) - \delta (20 + 2I_\Delta (4 + \delta)^2 + 2(\delta - 7))) \right) \). Since \( \frac{\partial^2 \pi_{11}}{\partial w_1^2} < 0 \), the value obtained for \( w_1 \) is optimal.

   - \( (q_1^*, I_1^*) = \left( \frac{1 - w_1}{2}, 0 \right) \).

   In that case, solution of \( \frac{\partial \pi_{11}}{\partial w_1} = 0 \) will obtain \( w_1^* = \frac{1}{2} \).

   Since \( \frac{\partial^2 \pi_{11}}{\partial w_1^2} = -1 \), the value obtained for \( w_1 \) is optimal.

2. According to (20), there will be two cases as follows if \( \{w_2^*, q_2^*, q_3^*\} = \{\frac{1 - \delta}{2}, \frac{1}{4}, \frac{1}{4}\} \).

   - \( (q_1^*, I_1^*) = \left( \frac{1 - w_1}{2}, 0 \right) \).

   In that case, solution of \( \frac{\partial \pi_{11}}{\partial w_1} = 0 \) will obtain \( w_1^* = \frac{1}{2} \).

   Since \( \frac{\partial^2 \pi_{11}}{\partial w_1^2} < 0 \), the value obtained for \( w_1 \) is optimal.

   - \( (q_1^*, I_1^*) = \left( \frac{1}{a} I_\Delta, 0 \right) \).

   In that case, \( \frac{\partial \pi_{11}}{\partial w_1} = \frac{1}{a} I_\Delta \). If \( w_1 > \max\{-h + \frac{\delta}{4}(3 - 8I_\Delta + 2\delta I_\Delta), 1 - \frac{2}{a} I_\Delta\} \), the optimal solution will be specified through an upper bound considered for \( w_1 \).

   For \( \Delta < 0 \), therefore, the following final solution is obtained based on (20), (21), and (29).

   - If \( I_\Delta + I_s < \frac{1}{4} \), then \( I_\Delta < \frac{1}{4} \), based on the second and third solutions given by (21). Moreover, \( w_1 < -h + \frac{\delta}{4}(3 + \frac{1}{2}(\delta - 4)) \). According to the first and fourth solutions. Since \( I_\Delta < \frac{1}{4} \), \( B < E \) always holds, the \( w_1 < E \) condition is automatically left out. For \( I_\Delta < \frac{1}{4} \), therefore, there are the following solutions.

     - If \( A > B \) (i.e. \( h > h_1 \), where \( h_1 = \frac{38}{7} - 1 + I_\Delta (\frac{1}{2}(\delta - 4) + \frac{2}{\delta}) \),
and second periods by specifying then. Therefore, the retailer maximizes total profit in the first period, strategic inventories are sold partly or entirely due to disruption in demand. Due to increase in demand in the first period, the purpose of the retailer is to specify Is so as to maximize profit. The profit function obtains

\[
\pi_s = \frac{1}{2} I_s + \frac{4h + 4w_1 + \frac{4h}{4 + \delta}}{2} - \frac{4h(\delta - \delta)}{(2I_\Delta - 36) + \delta((1 + 2I_\Delta)))}{\frac{1}{2}(16 + 4\delta)^2)}
\]

If \( I_\Delta + I_s \geq \frac{1}{4} \), then \( I_\Delta \geq \frac{1}{4} \), since \( I_s = 0 \). Therefore, \((q_1^*, I_1^*, w_1^*) = (\frac{1-w_2}{2}, 0, \frac{1}{2})\).

3.2 Model Formulation with \( \Delta > 0 \)

In this section, actual demand for the product in the first period is assumed to be greater than \( q_1 \) due to disruption in demand, on which basis the retailer’s and supplier’s problems in the first and second periods are modeled. In that case, the retailer is authorized to sell strategic inventories partly or entirely to meet demand in the first period.

3.2.1 Model Formulation in the First Period

(A) Retailer’s model
In the first period, the purpose of the retailer is to specify \( I_s \) and \( p_1 \) so as to maximize profit. The profit function obtains the difference between the revenue gained from sale and inventory management costs. Due to increase in demand in the first period, strategic inventories are sold partly or entirely then. Therefore, the retailer maximizes total profit in the first and second periods by specifying \( I_s \) and \( p_1 \), as shown below.

\[
\begin{align*}
\max \pi_1 &= p_1 (q_1 + I_\Delta) - (q_1 + I_s) w_1 - h(I_s - I_\Delta) + \pi_2 \\
\text{s.t.} &
\end{align*}
\]

\( I_\Delta \) represents part of strategic inventories that is sold by the retailer in the first period.

(B) Supplier’s model
The supplier specifies the wholesale price \( w_1 \) so as to maximize total profit in the first and second periods. Therefore, the supplier’s problem is as in Eq. (7).

3.2.2 Model Formulation in the Second Period

(A) Retailer’s model
The retailer’s purpose is to specify \( p_{2n} \) and \( q_{2o} \) so as to maximize profit in the second period. This means to select \( q_{2n} \) and \( q_{2o} \) so as to maximize the following profit function.

\[
\begin{align*}
\max \pi_2 &= p_{2n} q_{2n} + p_{2o} q_{2o} - q_{2n} w_2 \\
\text{s.t.} &
\end{align*}
\]

where \( I_\Delta \leq q_{2o} \leq I_s \) and \( q_{2n} \geq 0 \) (34) \[35\]

(B) Supplier’s model
In the second period, the supplier specifies \( w_2 \) so as to maximize profit. The supplier’s objective function is as in Eq. (10).

\[
\begin{align*}
\max \pi_2 &= p_{2n} q_{2n} + p_{2o} q_{2o} - q_{2n} w_2 \\
\text{s.t.} &
\end{align*}
\]

The supplier’s profit function in the second period can be observed in Eq. (34). Equation (35) contains the constraints in the retailer’s problem. The following proposition helps to specify the retailer’s optimal solutions in the second period.

Proposition 5 The optimal values of \( q_{2o} \) and \( q_{2n} \) are as follows.

\[
\begin{align*}
(q_{2o}^*, q_{2n}^*) &= \\
\text{s.t.} &
\end{align*}
\]

\[
\begin{align*}
\text{if } I_\Delta < I_s \left( \frac{1}{2} \right) \text{ and } w_2 \geq 1 - 2\delta I_s &
\end{align*}
\]

\[
\begin{align*}
\text{if } I_\Delta \leq I_s \text{ and } w_2 > 1 - \delta &
\end{align*}
\]

\[
\begin{align*}
\text{if } 2I_\Delta (1 - \delta) \leq w_2 \leq 1 - \delta &
\end{align*}
\]

\[
\begin{align*}
\text{if } w_2 \leq \min(2I_\Delta (1 - \delta), 1 - 2\delta I_\Delta) &
\end{align*}
\]

\[
\begin{align*}
\text{if } 2I_\Delta (1 - \delta) < w_2 \leq 1 - 2\delta I_s &
\end{align*}
\]

\[
\begin{align*}
\text{if } \frac{1}{2} \leq I_\Delta \text{ and } w_2 \leq 1 - 2\delta I_s &
\end{align*}
\]

(36)
Proof Given the concavity of the profit function (See “Appendix A”) and linearity of the constraints in the retailer’s problem, we can extract the optimal values from the first-order derivative of the profit function (presented in (12) and (13)) with respect to the variables. Therefore, the optimal values of \(q_{2o}^*\) and \(q_{2n}^*\) are obtained through addition of the Lagrangian multipliers to (12) and (13) based on the Kuhn–Tucker conditions.

The retailer’s problem can therefore be solved as follows.

\[
\frac{\partial \pi_{2r}}{\partial q_{2o}} + \lambda_1 - \lambda_2 = 0 \tag{37}
\]

\[
\frac{\partial \pi_{2r}}{\partial q_{2n}} + \lambda_3 = 0 \tag{38}
\]

\[
\lambda_1 (q_{2o} - I_\Delta) = 0 \tag{39}
\]

\[
\lambda_2 (I_s - q_{2o}) = 0 \tag{40}
\]

\[
\lambda_3 q_{2n} = 0 \tag{41}
\]

\[
\lambda_1, \lambda_2, \lambda_3 \geq 0 \text{ and } I_\Delta \leq q_{2o} \leq I_s \text{ and } q_{2n} \geq 0 \tag{42}
\]

We need to analyze the following eight cases by solving the system of Eqs. (37) to (41) and considering Constraint (42).

(i) If \(\lambda_1 = 0\) and \(\lambda_2, \lambda_3 > 0\), then \(q_{2o} = I_s\) and \(q_{2n} = 0\), according to (40) and (41), respectively. Based on (37), \(\lambda_2 = \delta (1 - 2I_s)\) must be positive. Therefore,\( I_s < \frac{1}{2}\). According to (38), \(\lambda_3 = -1 + 2\delta I_s + w_2\) must be positive. Therefore, \(w_2 > 1 - 2\delta I_s\).

(ii) If \(\lambda_1 = \lambda_2 = 0\) and \(\lambda_3 > 0\), then \(q_{2o} = 0\) according to (41). Given (37), \(q_{2o} = \frac{1}{2}\). Based on (38), \(\lambda_3 = -1 + \delta + w_2\) must be positive. Therefore, \(w_2 > 1 - \delta\). According to (42), \(I_\Delta \leq \frac{1}{2} \leq I_s\).

(iii) If \(\lambda_1 = \lambda_2 = \lambda_3 = 0\), then \(I_\Delta \leq q_{2o} \leq I_s\) and \(q_{2n} \geq 0\), according to (39) and (41). Given (37), \(q_{2o} = 1 - q_{2n}\). Moreover, \(q_{2n} = \frac{w_2 + \delta - 1}{2\delta - 2}\), given (38). Since \(I_\Delta \leq q_{2o} \leq I_s\) and \(q_{2n} \geq 0\), we have \(w_2 \leq 1 - \delta\) and \(2I_\Delta (1 - \delta) \leq w_2 \leq 2I_s (1 - \delta)\).

(iv) If \(\lambda_1 > 0\) and \(\lambda_3 = \lambda_2 = 0\), then \(q_{2o} = I_\Delta\), given (39). According to (38), \(q_{2o} = 1 - \frac{w_2 + 2\delta I_s}{2\delta - 2}\). Based on (37), \(\lambda_1 = -\delta (1 - 2I_\Delta - 2q_{2n})\) must be positive, which requires \(1 < 2I_\Delta + 2q_{2n}\). Therefore, \(w_2 < 2I_\Delta (1 - \delta)\). Moreover, \(w_2 \leq 1 - 2\delta I_\Delta\) since \(q_{2n} \geq 0\).

(v) If \(\lambda_1, \lambda_2 > 0\) and \(\lambda_3 = 0\), then \(q_{2o} = I_s\) and \(q_{2o} = I_\Delta\), which cannot hold at the same time.

(vi) If \(\lambda_1 = \lambda_3 = 0\) and \(\lambda_2 > 0\), then \(q_{2o} = I_s\), given (40). According to (38), \(q_{2n} = \frac{1 - w_2 + \delta - 1}{2\delta - 2}\). Based on (37), \(\lambda_2 = \delta (1 - 2I_s - 2q_{2n})\) must be positive. Therefore, \(w_2 > 2I_s (1 - \delta)\). Since \(q_{2n} \geq 0\), on the other hand, \(w_2 \leq 1 - 2\delta I_s\).

(vii) If \(\lambda_1, \lambda_2, \lambda_3 > 0\), then \(q_{2o} = I_\Delta\) and \(q_{2o} = I_s\), which cannot hold at the same time.

(viii) If \(\lambda_1 \geq 0\) and \(\lambda_2 = 0\), then \(q_{2o} = I_\Delta\) and \(q_{2n} = 0\), given (39) and (41). According to (37), \(\lambda_1 = -\delta (1 - 2I_\Delta)\) must be positive, i.e. \(\frac{1}{2} < I_\Delta\). According to (38), \(\lambda_3 = w_2 + 2\delta I_\Delta - 1\) must be positive. Therefore, \(w_2 > 1 - 2\delta I_\Delta\).

Proposition 6 Given Eqs. (10) and (11), the optimal value of \(w_2\) is as follows.

\[
w_2^* = \begin{cases} 
\frac{1 - \delta}{2}, & \text{if } I_\Delta \leq \frac{1}{4} \leq I_s \\
\frac{1 - 2\delta I_\Delta}{2}, & \text{if } \frac{1}{4} < I_\Delta < I_s \\
\frac{1 - 2\delta I_\Delta}{2}, & \text{if } I_\Delta < I_s < \frac{1}{4}
\end{cases}
\]

Proof There will be the following three cases if (11) is replaced in Eq. (10).

(i) If \(q_{2n} = \frac{w_2 + 2\delta - 1}{2\delta - 2} - 1\), then \(\frac{\partial \pi_{2n}}{\partial q_{2n}} = \frac{2w_2}{2\delta - 1} + \frac{1}{2}\). Therefore, \(w_2 = \frac{1}{2}(1 - \delta)\). Since \(\frac{\partial \pi_{2n}}{\partial q_{2n}} < 0\), \(w_2 = \frac{1}{2}(1 - \delta)\) is an optimal solution. Given the condition \(2I_\Delta (1 - \delta) \leq w_2 \leq 1 - \delta, I_\Delta \leq \frac{1}{4} \leq I_s\).

(ii) If \(q_{2n} = \frac{1 - w_2 - 2\delta I_\Delta}{2}\), then \(\frac{\partial \pi_{2n}}{\partial q_{2n}} = \frac{1}{2}(1 - 2w_2 - 2\delta I_\Delta)\). Therefore, \(w_2 = \frac{1}{2}(1 - 2w_2 - 2\delta I_\Delta)\). Since \(\frac{\partial^2 \pi_{2n}}{\partial w_2^2} < 0\), \(w_2\) is an optimal solution. Given the condition \(w_2 < \min(2I_\Delta (1 - \delta), 1 - 2\delta I_\Delta), \frac{1}{4} < I_\Delta < I_s\).

(iii) If \(q_{2n} = \frac{1 - w_2 - 2\delta I_\Delta}{2}\), then \(\frac{\partial \pi_{2n}}{\partial q_{2n}} = \frac{1}{2}(1 - 2w_2 - 2\delta I_\Delta)\). Therefore, \(w_2 = \frac{1}{2}(1 - 2w_2 - 2\delta I_\Delta)\). Since \(\frac{\partial^2 \pi_{2n}}{\partial w_2^2} < 0\), \(w_2\) is an optimal solution. Given the condition \(2I_s (1 - \delta) < w_2 \leq 1 - 2\delta I_s, I_\Delta < I_s < \frac{1}{4}\).

The following solution thus results from the values obtained for \(w_2\) and Eq. (36).

\[
(w_2^*, q_{2o}^*, q_{2n}^*) = \begin{cases} 
\left(\frac{1 - \delta}{2}, \frac{w_2 + 2\delta - 1}{2\delta - 2}, 1\right), & \text{if } I_\Delta \leq \frac{1}{4} \leq I_s \\
\left(\frac{1 - 2\delta I_\Delta}{2}, I_\Delta, \frac{1 - w_2 - 2\delta I_\Delta}{2}\right), & \text{if } \frac{1}{4} < I_\Delta < I_s \\
\left(\frac{1 - 2\delta I_\Delta}{2}, I_s, \frac{1 - w_2 - 2\delta I_\Delta}{2}\right), & \text{if } I_\Delta < I_s < \frac{1}{4}
\end{cases}
\]

We now specify the problem solutions in the first period. The retailer’s profit function and its constraints appear in Eqs. (32) and (33). The following proposition holds.

Proposition 7 Given Eqs. (32) and (33), the optimal values of \(q_1\) and \(I_s\) are as follows.
\[ (q_1^*, I_s^*) = \begin{cases} 
\left( \frac{1-w_1}{2}, \frac{4(h+w_1)-3\delta}{2\delta(\delta-4)} \right), & \text{if } w_1 \leq \frac{\delta}{4}(3+2I_\Delta-8I_\Delta)-h \\
\left( \frac{1-w_1}{2}, I_\Delta \right), & \text{if } w_1 > \frac{\delta}{4}(3+2I_\Delta-8I_\Delta)-h 
\end{cases} \] 

(44)

**Proof** We specify the optimal values of the retailer’s decision variables in the first period according to the following three cases.

1. If \( I_\Delta \leq \frac{\delta}{4} \leq I_s \), then \((w_1^*, q_2^*, q_3^*) = \left( \frac{1-w_1}{2}, \frac{1}{4}, \frac{1}{4} \right)\).
   
   Therefore, \( \frac{\partial^2 \pi_{1r}}{\partial q_1^2} = 1 - 2q_1 - w_1 \), so \( q_1^* = \frac{1-w_1}{2} \). Since \( \frac{\partial^2 \pi_{1r}}{\partial q_1^2} < 0 \), \( q_1 \) is an optimal solution.

2. We can extract the optimal values of the first-order derivative of the profit function (presented in (45)) with respect to the decision variable considering its concavity (See “Appendix B”) and the linearity of constraints in the retailer’s problem. Therefore, the optimal value of \( I_s^* \) is obtained through addition of the Lagrangian multipliers to (49) based on the Kuhn–Tucker conditions.

\[ \frac{\partial \pi_{1r}}{\partial I_s} = -h - w_1 + \frac{1}{16} (12\delta + 8I_s \delta(\delta - 4)) \] 

(49)

Therefore, the retailer’s problem can be solved as follows.

\[ \frac{\partial \pi_{1r}}{\partial I_s} + \lambda = 0 \] 

(50)

\[ \lambda(I_s - I_\Delta) = 0 \] 

(51)

\[ \lambda \geq 0 \text{ and } I_s \leq I_s \] 

(52)

We need to analyze the following two cases by solving the system of Eqs. (50) and (51) and considering Constraint (52).

(i) If \( \lambda = 0 \), then \( I_s \leq I_s \) according to (51). Moreover, \( h + w_1 = \frac{1}{16} (12\delta + 8I_s \delta(\delta - 4)) \) based on (50). Therefore, \( I_s = \frac{4(h+w_1)-3\delta}{2\delta(\delta-4)} \). According to (52), \( w_1 \leq \frac{\delta}{4}(3+2I_\Delta-8I_\Delta) - h \).

(ii) If \( \lambda > 0 \), then \( I_s = I_\Delta \) according to (51). Based on (50), \( \lambda = h + w_1 - \frac{1}{16} (12\delta + 8I_s \delta(\delta - 4)) \) must be positive, resulting in \( w_1 > \frac{\delta}{4}(3+2I_\Delta-8I_\Delta) - h \).

The supplier’s problem in the first period is stated as Eq. (7), and the following proposition holds to obtain its optimal solution.

**Proposition 8** Given Eqs. (7) and (44), the optimal value of \( w_1 \) is as follows.

\[ w_1^* = \begin{cases} 
\frac{4(h-3 \delta-4a+36-19a^2)}{2(16+8a^2-8a^4)}, & \text{if } w_1 \leq \frac{\delta}{4}(3+2I_\Delta-8I_\Delta) - h \\
\frac{2I_\Delta+w_1}{2}, & \text{if } w_1 > \frac{\delta}{4}(3+2I_\Delta-8I_\Delta) - h 
\end{cases} \] 

(53)

There is one of the following two cases.

- \((q_1^*, I_s^*) = \left( \frac{1-w_1}{2}, \frac{4(h+w_1)-3\delta}{2\delta(\delta-4)} \right)\)

  In that case, solution of \( \frac{\partial \pi_{1s}}{\partial w_1} = 0 \) will obtain \( w_1^* = \frac{4(h-3 \delta-4a+36-19a^2)}{2(16+8a^2-8a^4)} \). Since \( \frac{\partial^2 \pi_{1s}}{\partial w_1^2} < 0 \), the value obtained for \( w_1 \) is optimal.

- \((q_1^*, I_s^*) = \left( \frac{1-w_1}{2}, I_\Delta \right)\)
In that case, \( \frac{\partial \pi_I}{\partial w_1} = I_\Delta + \frac{1}{2}(1 - 2w_1) \). Solution of \( \frac{\partial \pi_I}{\partial w_1} = 0 \) will result in \( w_1^* = \frac{2}{2I_\Delta + 1} \). Since \( \frac{\partial^2 \pi_I}{\partial w_1^2} < 0 \), the value obtained for \( w_1 \) is optimal.

For \( \Delta > 0 \), therefore, the following final solution is obtained based on (43), (44), and (53).

\[
(q_1^*, I_s^*, w_1^*) = \begin{cases} 
\left( \frac{1 - w_1}{2}, \frac{4h + 4w_1 - 3h}{2(6 + 4w_1)} \right), & \text{if } w_1 \leq B \\
\left( \frac{1 - w_1}{2}, I_\Delta, \frac{2I_\Delta - 1}{2} \right), & \text{if } w_1 > B
\end{cases}
\]

Equation (54) holds where \( I_\Delta < I_s < \frac{2}{4} \), while the following solution is obtained for cases where \( I_\Delta \leq \frac{1}{4} \) or \( \frac{1}{4} < I_\Delta < I_s \).

\[
(q_1^*, I_s^*, w_1^*) = \left( \frac{1 - w_1}{2}, I_\Delta, \frac{2I_\Delta + 1}{2} \right)
\]

Since \( w_1 > 0 \) must hold in the above cases, we assume \( w_1 \) to have an upper bound of \( \frac{2I_\Delta + 1}{2} \).

4 Model Analysis

In this section, we analyze the optimal solutions obtained in the previous section for the decision variables.

4.1 Optimal Solutions Analysis for \( \Delta < 0 \)

The following viewpoints can be extracted from Eqs. (30) and (31).

- If \( \Delta < 0 \), the retailer’s decision on strategic inventories will be either not to order them at all or to manage a quantity of \( \frac{4h + 4w_1 - 3h + (3 + 4I_\Delta - 24I_s)}{2(6 + 4w_1)} \). Therefore, \( I_s \neq 0 \) is a function of \( \delta, I_\Delta, \) and \( w_1 \).
- If \( \Delta < 0 \) and \( I_\Delta \geq \frac{1}{4} \), the retailer will not order strategic inventories in the first period. That is, demand has decreased due to disruption to \( I_\Delta \) less than the original \( q_1 \), the quantity transferred by the retailer to the second period as strategic inventories.
- If \( \Delta < 0 \) and \( I_\Delta < \frac{1}{4} \), the retailer will either not order strategic inventories or purchase a quantity of \( \frac{4h + 4w_1 - 3h + (3 + 4I_\Delta - 24I_s)}{2(6 + 4w_1)} \), in accordance with the goods price assumed by the supplier.
- It is clear from (30) and (31) that the optimal value of \( I_s \) depends on the wholesale price set by the supplier. Therefore, the optimal value of strategic inventories will be zero for \( h > h_1 \), if wholesale price is less than \( A \) and greater than \( B \), or is greater than \( A \), whereas it will be \( \frac{4h + 4w_1 + 4h + (3 + 4I_\Delta - 24I_s)}{2(6 + 4w_1)} \) if wholesale price is less than \( B \).
- If \( h < h_1 \), the value will be \( \frac{4h + 4w_1 + 4h + (3 + 4I_\Delta - 24I_s)}{2(6 + 4w_1)} \) if wholesale price is less than \( B \) and greater than \( A \), or is less than \( A \), whereas it will be zero if wholesale price is greater than \( B \).

\[
p_2n = 1 - q_{2n} - \delta q_{2o} = \frac{3 - \delta}{4}
\]

\[
p_{2o} = (1 - q_{2n} - q_{2o}) = \frac{\delta}{2}
\]

As clear from the above equations, the values of retail price in the second period for new and old products depend only on the discounted invoice considered by the retailer for old products, whereas retail price in the first period depends on wholesale price. The greater the value of \( \delta \), the less discount considered by the retailer for old products. This increases old product price in the second period, consequently decreasing new product price (If \( \delta = 1 \), the retailer will consider no discount, and \( p_{2n} = p_{2o} = \frac{1}{2} \)).

- If \( I_\Delta \geq \frac{1}{4} \), the following equations will hold based on (3) and (4).

\[
p_{2n} = 1 - q_{2n} - \delta q_{2o} = \frac{3 - 2\delta(I_\Delta + I_s)}{4}
\]

\[
p_{2o} = (1 - q_{2n} - q_{2o}) = \frac{\delta}{4}(3 + (I_\Delta + I_s)(2\delta - 4))
\]

- For \( h > h_1 \), \( p_{2n} \) and \( p_{2o} \) will be functions of \( w_1 \) if wholesale price in the first period is less than \( B \). In the other cases, i.e. if wholesale price is less than \( A \) and greater than \( B \), or is greater than \( A \), retail price will be a function only of \( I_\Delta \) and \( \delta \). For \( h < h_1 \), the values of \( p_{2n} \) and \( p_{2o} \) will be functions of wholesale price if it is less than \( A \), or is less than \( B \) and greater than \( A \). Otherwise, they will depend only on the values of \( I_\Delta \) and \( \delta \).
Based on Equation B, it can assume a negative or a positive value. If it assumes a negative value, any value of wholesale price will be greater than it. In that case, the retailer will not manage strategic inventories. Where B assumes a positive value, the retailer is likely to purchase strategic inventories. Finally, B assumes a positive value for $h < \frac{\delta}{4}(3 + 2I_A(\delta - 4))$, and strategic inventory management by the retailer will be an optimal policy if $w_1 \leq B$.

It is clear from (45) that the optimal value of strategic inventories is $I_s$, which does not depend on $w_1$ in that case. That is, the retailer purchases strategic inventories exactly as much as the disruption in demand, and sells them all to meet demand in the first period.

According to the equation $p_1 = 1 + \Delta - q_1$, $p_1 = \frac{1 + 2h_A + w_1}{2}$. Clearly, retailer price in the first period is a function of disruption and wholesale price. In absence of disruption in demand, $p_1$ will be a function only of wholesale price. In case of disruption and increase in demand, however, retail price will increase as disruption rises.

If $I_\Delta \leq \frac{1}{4}$, the following equations will hold based on (3) and (4).

\[ p_{2n} = 1 - q_{2n} - \delta q_{2o} = \frac{3 - \delta}{4} \]
\[ p_{2o} = \delta (1 - q_{2n} - q_{2o}) = \frac{\delta}{2} \]

As clear from the above equations, the values of retail price in the second period for new and old products depend only on the discounted invoice considered by the retailer for old products, whereas retail price in the first period depends on wholesale price. The greater the value of $\delta$, the less discount considered by the retailer for old products. This increases old product price in the second period, consequently decreasing new product price (If $\delta = 1$, the retailer will consider no discount, and $p_{2n} = p_{2o} = \frac{1}{2}$).

If $I_\Delta > \frac{1}{4}$, the following equations will hold based on (3) and (4).

\[ p_{2n} = 1 - q_{2n} - \delta q_{2o} = \frac{3 - 2\delta I_\Delta}{4} \]
\[ p_{2o} = \delta (1 - q_{2n} - q_{2o}) = \frac{\delta}{4} (3 + 2I(\delta - 2)) \]

It is clear from the above equations that the values of retail price in the second period for new and old products depend only on the values of $I_s$ and $\delta$. If the value of disruption is greater than $\frac{1}{4}$, therefore, product retail price in the second period will be set by the retailer given the above value and discounted invoice.

If $I_\Delta < I_s < \frac{1}{2}$, the following equations will hold based on (3) and (4).

\[ p_{2n} = 1 - q_{2n} - \delta q_{2o} = \frac{3 - 2\delta I_s}{4} \]
\[ p_{2o} = \delta (1 - q_{2n} - q_{2o}) = \frac{\delta}{4} (3 + 2I_s(\delta - 2)) \]

It is clear from the above equations that the values of retail price in the second period for new and old products depend only on the values of $I_s$ and $\delta$. If $I_\Delta < I_s < \frac{1}{2}$, therefore,
product retail price in the second period will be set by the retailer given the quantity of strategic inventories and discounted invoice. In that case, the value of strategic inventories will be $I_\Delta$ or $\frac{4(h+w_1)-\delta h}{2(h+4)}$. Where $I_1 = \frac{4(h+w_1)-\delta h}{2(h+4)}$, therefore, product retail price in the second period depends on management costs as well as wholesale price and the discounted invoice, since part of strategic inventories has been sold in the first period, and the rest has been transferred to the second period.

- If $I_\Delta \leq \frac{1}{4} \leq I_1$, or $I_\Delta > \frac{1}{4}$, the retailer’s optimal policy will be to purchase strategic inventories as much as disruption in demand, sold entirely in the first period. That is, the retailer will order a quantity of $I_\Delta + \frac{1-w_1}{2}$ in the first period and sell it all in the same period.
- If $I_\Delta < I_1 < \frac{1}{4}$, and wholesale price is greater than $B$, the retailer’s optimal policy will be to purchase strategic inventories in the first period as much as the increase in demand and to sell them all in the same period to meet demand. If wholesale price is less than $B$ in that case, the retailer’s optimal policy will be to assume strategic inventories greater than disruption in demand, i.e. $I_\Delta$, so that $I_\Delta$ of the above quantity can be sold in the first period to meet demand, with the rest transferred as strategic inventories to the second period. The retailer’s order quantity in the first period is $\frac{1-w_1}{2} + \frac{4(h+w_1)-\delta h}{2(h+4)}$, where $\frac{4(h+w_1)-\delta h}{2(h+4)} > I_\Delta$.

### 4.3 Feasible Space for SI

As stated in Sect. 3, it is suggested under certain conditions that the retailer manage strategic inventories. In other words, it is not always to the retailer’s benefit to adopt the policy when demand is disrupted. In this section, we seek to specify parameter ranges where it would be to the retailer’s benefit to manage strategic inventories.

We first consider a case where demand in the first period has decreased due to disruption ($\Delta < 0$). According to (30), it is optimal for the retailer to manage strategic inventories when $w_1 \leq B$. Therefore, the area where the retailer manages strategic inventories is equivalent to that in which function $f_1(I_\Delta, h, \delta) = B - w_1$ assumes positive values. Moreover, Eq. (30) holds when $A > B$, a condition that must also be considered upon specification of the intended area. The condition $A > B$ is equivalent to a positive value for $g_1(I_\Delta, h, \delta) = A - B$. Functions $f_1$ and $g_1$ are shown at the same time in Fig. 2.

Function $f_1$ assumes positive values for the set of points located to the right of the diagram. So does function $g_1$ for the set of points located under the diagram. Therefore, the set of points located to the right of function $f_1$ and under function $g_1$ constitutes our set of intended points. The area to the right of $f_1$ and under $g_1$ is where the retailer can manage strategic inventories. The intersection of the two diagrams is shown in Fig. 3 to specify the parameter ranges within the area. The hatched area represents where $I_\delta \neq 0$, which is acceptable for parameters $\delta$ and $h$. Therefore, $I_\Delta \geq 0.020$.

According to (31), strategic inventory management is optimal for the retailer when the condition $w_1 \leq A$ or $A < w_1 \leq B$ holds. We first consider the former condition. On that basis, the area where the retailer manages strategic inventories is equivalent to that in which function $f_2(I_\Delta, h, \delta) = A - w_1$ assumes positive values. On the other hand, (31) holds when $A < B$, a condition equivalent to a positive value for function $g_2(I_\Delta, h, \delta) = B - A$. Functions $f_2$ and $g_2$ are shown in Fig. 4.

Function $f_2$ assumes positive values for the set of points located under the diagram. So does function $g_2$ for the set of points located above it. Therefore, the set of points located under function $f_2$ and above function $g_2$ constitutes our intended set of points. The area under $f_2$ and above $g_2$ is where the retailer can manage strategic inventories. The intersection of the two diagrams is shown in Fig. 5 to specify the
We now consider the condition $A < w_1 \leq B$ for Eq. (31).

To represent the corresponding area, the two areas $w_1 \leq B$ and $A < w_1$ are diagrammed at the same time. The condition $w_1 \leq B$ is equivalent to the set of points for which function $f_3(I_{\delta}, h, \delta) = B - w_1$ is positive, and the condition $A < w_1$ is equivalent to the area where function $g_3(I_{\delta}, h, \delta) = w_1 - A$ is positive. We also consider the area where function $g_2(I_{\delta}, h, \delta) = B - A$ is positive to let $A < B$ hold. The areas formed by the three functions are shown in Fig. 6. The value of function $f_3$ is positive for the set of points above its diagram. The same conditions hold for function $g_3$.

All functions $f_3$, $g_2$, and $g_3$ assume positive values for the set of points located above the diagram. The area above the diagram is where the retailer can manage strategic inventories. The intersection of functions $f_3$ and $g_2$ is shown in Fig. 7 to specify the parameter ranges. The hatched area represents where $I_s \neq 0$, which is the acceptable area for parameters $\delta$ and $h$, as clear from the figure. Therefore, $I_{\Delta} \geq 0.021$.

Where demand in the first period increases due to disruption ($\Delta > 0$), strategic inventory management is recommended to the retailer in both cases $w_1 \leq B$ and $w_1 > B$, according to (54). Figure 8 shows the function $f_3(I_{\delta}, h, \delta) = B - w_1$ diagram. The set of points for which
the function assumes positive values is equivalent to the condition $w_1 \leq B$, and that for which it assumes negative values is equivalent to the condition $w_1 > B$. Therefore, the optimal quantity of strategic inventories for the retailer is $\frac{4(h+w_1) - 3\delta}{2(\delta + 4)}$ for the set of points located under the function $f_3$ diagram and $I_\delta$ for that of points located above it.

The strategic inventory area in Mantin and Jiang [1] looks as in Fig. 9. A comparison of this figure to Figs. 2, 3, 4, 5, 6, 7 and 8 indicates that the parameter $\delta$ range in the plausible space of strategic inventories varies by disruption in demand, demonstrating that the retailer’s policies deciding whether or not to manage strategic inventories are greatly influenced by disruption.

**Table 2** Comparison of the optimal solutions for $\Delta = 0$ to those of Mantin and Jiang [1]

| This study | Mantin and Jiang [1] |
|------------|----------------------|
| Solution 1 | Solution 2           | Solution 3 |
| $\Delta > 0$ | $\Delta < 0$         | $\Delta > 0$ |
| $I_\Delta < I_s$ | $I_\Delta + I_s < \frac{1}{2}$ | $I_\Delta < I_s < \frac{1}{2}$ |
| $w_1 > B$       | $B < w_1$            | $w_1 \leq B$ |
| $w_1^+$         | $\frac{2I_\Delta + 1}{2}$ | $\frac{4h(3h - 4) + 36 + 19\delta + 4\delta^2}{2(16h^2 - 9h^2 + 8\delta^2)}$ |
| $w_2^+$         | $\frac{1 - 2hI_\Delta}{2}$ | $\frac{1 - 2hI_s}{2}$ |
| $q_1^+$         | $\frac{1 - w_1^+}{2}$ | $\frac{1 - w_1^+}{2}$ |
| $q_2^+$         | $\frac{1 - 2hI_\Delta}{2}$ | $\frac{1 - 2hI_s}{2}$ |
| $q_3^+$         | $I_\Delta$             | $I_s$ |
| $I^*$          | $I_\Delta$             | $I^*$ |
| $I^*$          | $0$                    | $\frac{4(h + w_1^+)^2 + 36}{2(\delta + 4)} - \frac{36h^2}{2(\delta + 4)^2}$ |

**4.4 Sensitivity Analysis**

In this research, we extended Mantin and Jiang [1] assuming disruption in demand. For validation of the model, we first compare the optimal solutions obtained from it for $I_\Delta = 0$ to those of the above study. The solutions appear in Table 2. Different optimal solutions are obtained given the different values that $I_\Delta$ can assume. $I_\Delta = 0$ holds for only three of all the solutions. The conditions of solution 1 and solution 2 for $I_\Delta = 0$ require that $h > \frac{3h}{4} - \frac{1}{2}$. Moreover, $\frac{36}{4} - \frac{1}{2} = \delta$ for different values of $\delta$, and those of solution 3 require that $h < \frac{4(h + w_1^+)^2 + 36}{2(\delta + 4)^2}$, where $\frac{4(h + w_1^+)^2 + 36}{2(\delta + 4)^2} < \frac{\delta}{4}$.

**Proposition 9** When the retailer’s optimal policy is to manage strategic inventories, the following relations hold for $\Delta < 0$ and $B < A$.

(i) $\frac{dw_1}{dh} > 0$  \hspace{1cm} (ii) $\frac{dw_2}{dh} < 0$  \hspace{1cm} (iii) $\frac{dp_1}{dh} > 0$

(iv) $\frac{dp_2}{dh} < 0$  \hspace{1cm} (v) $\frac{dp_3}{dh} > 0$  \hspace{1cm} (vi) $\frac{dq_1}{dh} < 0$

(vii) $\frac{dq_2}{dh} < 0$  \hspace{1cm} (viii) $\frac{dq_3}{dh} > 0$  \hspace{1cm} (ix) $\frac{dq_1 + I_s}{dh} > 0$

It should be noted that all the above relations hold also for $\Delta < 0$ and $A < B$ and for $\Delta > 0$.

**Proof** See “Appendix C”.

**Proposition 10** When the retailer’s optimal policy is not to manage strategic inventories, the following relations hold for $\Delta < 0$ and $B < A$.

(i) $\frac{dw_2}{dh} < 0$  \hspace{1cm} (ii) $\frac{dp_2}{dh} < 0$  \hspace{1cm} (iii) $\frac{dp_3}{dh} > 0$

(iv) $\frac{dq_2}{dh} < 0$  \hspace{1cm} (v) $\frac{dq_3}{dh} > 0$  \hspace{1cm} (vi) $\frac{dq_1 + I_s}{dh} > 0$

It should be noted that all the above relations hold also for $\Delta < 0$ and $A < B$ and for $\Delta > 0$.

**Proof** See “Appendix C”.

---

**Fig. 9** Strategic inventory area in Mantin and Jiang [1]
The size changes in the variables with respect to \( \delta \) are zero. It should be noted that all the above relations hold also for \( \Delta < 0 \) and \( A < B \) and for \( \Delta > 0 \).

**Proof** See “Appendix D”.

Figures 10, 11, 12, 13, 14, 15 and 16 express the above propositions, showing the changes in the decision variables and chain members’ profits in both periods.

According to Proposition 9 (Figs. 10, 11, 12, 13), the supplier benefits in the first period from the increase in wholesale price and order quantity as \( \delta \) increases, and is convinced by the profit gained in the first period despite the decrease in profit in the second period due to the decrease in wholesale price and demand for the new product. There are different conditions for the retailer, who benefits from selling old products in the second period as \( \delta \) increases, despite the decrease in profit in the first period due to the decrease in product demand and in profit margin.

According to Proposition 10, only wholesale price, retail price, and new product demand in the second period change as \( \delta \) increases. This decreases the supplier’s profit in the second period by decreasing wholesale price and new product demand, while the profit remains constant in the first period. It also increases the retailer’s profit in the second period due to the increase in old product price and decrease in new product demand, while the profit remains constant in the first period.

The following points can be made through a comparison of Figs. 10, 11, 12, 13, 14, 15 and 16.

- When the retailer’s optimal policy is to manage strategic inventories, the supplier’s profit increases, and the retailer’s decreases as \( \delta \) increases for both \( \Delta < 0 \) and \( \Delta > 0 \). It does not affect the supplier’s or retailer’s profit changes whether or not there are inventory management costs.

- When the retailer’s optimal policy is not to manage strategic inventories, the supplier’s profit decreases, and the retailer’s increases as \( \delta \) increases for both \( \Delta < 0 \) and \( \Delta > 0 \). Moreover, inventory management costs are ineffective on the changes.

- When the retailer manages strategic inventories, demand increase or decrease in the first period due to disruption does not affect the incremental or decremental trends of the decision variables, but it affects their values. This is true also when the quantity of strategic inventories is zero.

- As \( \delta \) increases, the greatest increase or decrease in the supply chain members’ profit is observed when disruption increases demand in the first period, and the retailer sells all the strategic inventories in the same period to meet demand.

- The retailer gains greater profit than the supplier only when disruption increases demand in the first period, and the
Fig. 13 Changes in equilibrium decisions with respect to $\delta$ when $\Delta > 0$, $h \neq 0$, and $I_s > 0$

Fig. 14 Changes in equilibrium decisions with respect to $\delta$ when $\Delta < 0$, $h \neq 0$, and $I_s = 0$

Fig. 15 Profit changes with respect to $\delta$ when $\Delta < 0$, $h = 0$, and $I_s = 0$

- When the retailer sells all the strategic inventories in the same period, where the value of $\delta$ is greater than 0.9.
- When the retailer’s optimal policy is to manage strategic inventories, a small value of $\delta$ is preferable for the retailer, whereas increase in $\delta$ is more profitable for the supplier. This indicates the conflict of interest between the retailer and the supplier when strategic inventories are managed by the retailer. While the supplier prefers products that live longer, the retailer seeks fewer durable products.
- When the retailer’s optimal policy is not to manage strategic inventories, large values of $\delta$ are more profitable for the retailer, whereas the supplier gains greater profit for small values of $\delta$, and therefore prefers them. Thus, the retailer seeks products of higher durability in these conditions, while the supplier prefers products with short lives.
- A comparison of Figs. 11 and 12 and those in Mantin and Jiang [1] indicates that the same changes are made in the supply chain members’ profit and decision variables in the two modes with and without disruption in demand, when strategic inventories are managed, and management cost is zero. The results are not altered by nonzero management costs.
- When the retailer does not manage strategic inventories, the price of the old product is less than that paid for it by the retailer, while retail prices in the second period are all greater than the wholesale price in the first when strategic inventories are managed.

Fig. 16 Changes in equilibrium decisions with respect to $\delta$ when $\Delta > 0$ and $I_s = 0$
– For \( \Delta > 0 \), the rate of changes made in the supply chain members’ profit as \( \delta \) changes are greater than that in the other cases. This demonstrates that change of goods deterioration levels has greater impacts on profit as demand in the first period increases.

– The retailer’s strategic inventories are not always a threat, and can even be useful. When the retailer manages strategic inventories in the case where \( \Delta > 0 \), for instance, the supplier gains greater profit for \( \Delta > 0 \) than for \( \Delta = 0 \) as \( \delta \) increases.

– It can be concluded from the above points that the retailer’s or supplier’s incremental or decremental trend can be altered only through quantitative change of strategic inventories.

5 Conclusion

The notion of disruption has gained far greater significance today with the COVID-19 outbreak. Given the freshness of the notion of strategic inventories, on the other hand, their investigation in conditions of disruption provides a very important topic, which has been disregarded so far in the literature. In our two-period model, demand is disrupted in the first period, affecting strategic inventories. Therefore, this paper investigated the impacts of disruption in demand on the retailer’s quantity of strategic inventories. We found that there is a limited parameter range for the case where the retailer manages strategic inventories. Comparing our findings to those of Mantin and Jiang [1], we observed that disruption is a factor that changes the parameter ranges. The obtained results suggest that the retailer’s strategic inventories are not always a threat, and can even be useful, even where disruption reduces demand.

When the retailer manages strategic inventories, wholesale price in the first/second period is the same for a case where actual demand in that period is equal to or greater than that predicted by the retailer. Where actual demand is less than the predicted demand, wholesale price is greater/less than that in the previous two cases.

Wholesale price in the first period is the same for \( \Delta > 0 \) and \( \Delta = 0 \), and that for \( \Delta < 0 \) is greater than both. Moreover, wholesale price in the second period is the same for \( \Delta > 0 \) and \( \Delta = 0 \), and that for \( \Delta < 0 \) is less than both.

In the case where the retailer manages no strategic inventories for the second period, wholesale price in the first period is the same for a case where actual demand is equal to or less than that predicted by the retailer, and this price is lower than wholesale price for a case where actual demand is greater than the predicted demand.

Wholesale price in the first period is the same for \( \Delta < 0 \) and \( \Delta = 0 \) and less than that for \( \Delta > 0 \).

Moreover, wholesale price in the second period is the same for a case where actual demand is less or greater than that assumed by the retailer, while this price is less than the wholesale price for a case where actual demand is equal to that predicted by the retailer.

Due to different cases of demand disruption, it can be stated that if the retailer maintains a strategic inventory, it will have to pay for a higher wholesale price in only one of the different cases of demand disruption during the two periods (in the first period, where actual demand is less than that predicted by the retailer) than in the other cases. If the retailer does not maintain a strategic inventory, however, there will be two cases (in the first period, where actual demand is greater than that predicted by the retailer, and in the second period, with actually no disorder in demand), where it will incur prices higher than those paid for in the other cases of disruption in the two periods.

The model presented in this research can be extended in some respects. A solution that can be adopted by the retailer as demand increases is to manage all strategic inventories to be sold in the second period and let the system suffer deficit in the first. If deficit is allowed, the profitability of the above solution can be compared to that obtained in this research. Moreover, the impacts of the retailer’s quality of risk aversion in conditions of disruption can be investigated in regard to decision-making on strategic inventory management.

Appendices

Appendix A

In order to prove the concavity of the retailer’s profit function (Eq. (8)), its first- and second-order conditions are studied, and the Hessian matrix is then formed.

\[
\frac{\partial^2 \pi_2(q_{2n}, q_{2o})}{\partial q_{2n}^2} = -2 < 0
\]

\[
\frac{\partial^2 \pi_2(q_{2n}, q_{2o})}{\partial q_{2o}^2} = -2\delta < 0
\]

\[
\frac{\partial^2 \pi_2(q_{2n}, q_{2o})}{\partial q_{2n} \partial q_{2o}} = \frac{\partial^2 \pi_2(q_{2n}, q_{2o})}{\partial q_{2o} \partial q_{2n}} = -2\delta < 0
\]

| \( H(q_{2n}, q_{2o}) \) | = | \begin{pmatrix} -2 & -2\delta \\ -2\delta & -2\delta \end{pmatrix} | = 4\delta - 4\delta^2 = 4\delta(1 - \delta) > 0

Given the negativity of the first element of the Hessian matrix for the retailer’s profit in the second period and the positivity of the matrix determinant, the concavity of the profit function is proven.
Appendix B

In order to prove the concavity of the retailer’s profit function in the first period, its second-order conditions are calculated using the first-order derivatives presented in (22) and (23).

\[
\frac{\partial^2 \pi_{1r}(q_1, I_s)}{\partial q_1^2} = -2 < 0
\]

\[
\frac{\partial^2 \pi_{1r}(q_1, I_s)}{\partial q_1 \partial I_s} = \frac{\partial^2 \pi_{1r}(q_1, I_s)}{\partial I_s^2} = 0
\]

\[
|H(q_1, I_s)| = \left| \begin{array}{cc}
-2 & 0 \\
0 & 2\delta(\frac{\delta}{4} - 1)
\end{array} \right| = 4\delta - 4\delta^2 = 4\delta(1 - \frac{\delta}{4}) > 0
\]

Appendix C

For all \( \delta \in (0, 1) \),

\[
\frac{dw_1}{d\delta} = \frac{64(9 + 5h - 8I_\Delta) + 32(15 + 8h - 16I_\Delta)\delta + 16(h - 6(1 + I_\Delta))\delta^2 - 8(13 + h)\delta^3 - 23\delta^4}{2(16 + \delta(4 + \delta)^2)^2} > 0
\]

\[
\frac{dw_2}{d\delta} = - \frac{-768 + 1152\delta + 3120\delta^2 + 1536\delta^3 + 424\delta^4 + 668\delta^5 + 786\delta^6 + 2I_\Delta(1024 + 1024h + 320\delta^2 + 224\delta^3 + 96\delta^4 + 16\delta^5 + \delta^6)}{(\delta - 4)^2(16 + \delta(4 + \delta)^2)^2} < 0
\]

\[
\frac{dp_1}{d\delta} = \frac{576 + 32I_\Delta(4 + \delta)(4 + 3\delta) + 8h(40 + \delta(-32 + (-2 + \delta)\delta)) + \delta(-480 + \delta(96 + \delta(104 + 23\delta)))}{4(16 + \delta(4 + \delta)^2)^2} > 0
\]

\[
\frac{dp_2}{d\delta} = \frac{256(30 - 16I_\Delta + 21\delta) + \delta^2(1440 + \delta(2656 + \delta(1216 + (212 - 9\delta)\delta)) + 4I_\Delta(832 + \delta(12 + \delta)(24 + \delta(4 + \delta))))}{4h(-512 + \delta(384 + \delta(752 + \delta(272 + \delta(82 + \delta(18 + \delta))))))} > 0
\]

\[
\frac{dq_1}{d\delta} = \frac{-576 + 32I_\Delta(4 + \delta)(4 + 3\delta) + 8h(-40 + \delta(-32 + (-2 + \delta)\delta)) + \delta(-480 + \delta(96 + \delta(104 + 23\delta)))}{4(16 + \delta(4 + \delta)^2)^2} < 0
\]

\[
\frac{dq_2}{d\delta} = \frac{264(-16I_\Delta + 24\delta) + \delta^2(384(696 + \delta(576 + \delta(256 + \delta(128 + \delta(64 + \delta))))))}{4h(-512 + \delta(384 + \delta(752 + \delta(272 + \delta(82 + \delta(18 + \delta))))))} > 0
\]

\[
\frac{dq_2}{d\delta} = \frac{528^5 + 3\delta^6 + 8I_\Delta(-128 + 192\delta + 56\delta^2 + 64\delta^3 + 14\delta^4)}{2(\delta - 4)^2\delta^2(16 + \delta(4 + \delta)^2)^2} > 0
\]

\[
\frac{d(q_1 + I_s)}{d\delta} = \frac{4096h + 6144h\delta + 768(6 + h + 8I_\Delta)\delta^2 + 1024(5 + h + I_\Delta)\delta^3 + 64(114 + h - 18I_\Delta)\delta^4 + 64(17 - 17h + 12I_\Delta)\delta^5 + 4(11 - 76h + 80I_\Delta)\delta^6 - 8(-3 + h - 2I_\Delta)\delta^7 + 29h^8}{4(\delta - 4)^2\delta^2(16 + \delta(4 + \delta)^2)^2} > 0
\]
Appendix D

For all $\delta \in (0, 1]$,
\[
\begin{align*}
\frac{dw_2}{d\delta} &= -I_\Delta, \\
\frac{dp_{2n}}{d\delta} &= -\frac{I_\Delta}{2}, \\
\frac{dp_{2o}}{d\delta} &= \frac{3}{4}I_\Delta(\delta - 1), \\
\frac{dq_{2n}}{d\delta} &= -\frac{I_\Delta}{2}.
\end{align*}
\]

References

1. Mantin, B.; Jiang, L.: Strategic inventories with quality deterioration. Eur. J. Oper. Res. 258(1), 155–164 (2017)
2. Huang, S.; Yang, C.; Zhang, X.: Pricing and production decisions in dual-channel supply chains with demand disruptions. Comput. Ind. Eng. 62(1), 70–83 (2012)
3. Cao, E.B.: Coordination of dual-channel supply chains under demand disruptions management decisions. Int. J. Prod. Res. 52(23), 7114–7131 (2014)
4. Wang, K.; Gou, Q.; Sun, J., et al.: Coordination of a fashion and textile supply chain with demand variations. J. Syst. Sci. Syst. Eng. 21(4), 461–479 (2012)
5. Qi, X.T.; Bard, J.F.; Yu, G.: Supply chain coordination with demand disruptions. Omega-Int. J. Manag. Sci. 32(4), 301–312 (2004)
6. Tang, C.H.; Yang, H.L.; Cao, E.B.; Lai, K.K.: Channel competition and coordination of a dual-channel supply chain with demand and cost disruptions. Appl. Econ. 50(46), 4999–5016 (2018)
7. Wu, J.; Chen, Z.; Ji, X.: Sustainable trade promotion decisions under demand disruption in manufacturer-retailer supply chains. Ann. Oper. Res. 290, 115–143 (2018)
8. Pi, Z.; Fang, W.; Zhang, B.: Service and pricing strategies with competition and cooperation in a dual-channel supply chain with demand disruption. Comput. Ind. Eng. 86, 42–58 (2019)
9. Zhao, T.; Xu, X.; Chen, Y.; Liang, L.; Yu, Y.; Wang, K.: Coordination of a fashion supply chain with demand disruptions. Transp. Res. Part E 134, 1–13 (2020)
10. Yan, B.; Jin, Z.J.; Liu, Y.P.; Yang, J.B.: Decision on risk-averse dual-channel supply chain under demand disruption. Commun. Nonlinear Sci. Numer. Simul. 55, 206–224 (2017)
11. Zhang, P.; Xiong, Y.; Xiong, Z.K.: Coordination of a dual-channel supply chain after demand or production cost disruptions. Int. J. Prod. Res. 53(10), 3141–3160 (2015)
12. Chen, K.; Xiao, T.: Demand disruption and coordination of the supply chain with a dominant retailer. Eur. J. Oper. Res. 197(1), 225–234 (2009)
13. Anand, K.; Anupindi, R.; Bassok, Y.: Strategic inventories in vertical contracts. Manag. Sci. 54(10), 1792–1804 (2008)
14. Arya, A.; Mittendorf, B.: Managing strategic inventories via manufacturer-to-consumer rebates. Manag. Sci. 59(4), 813–818 (2013)
15. Hartwig, R.; Inderfurth, K.; Sadrieh, A.; Voigt, G.: Strategic inventory and supply chain behavior. Prod. Oper. Manag. 24(8), 1329–1345 (2015)
16. Moon, L.; Dey, K.; Saha, S.: Strategic inventory: Manufacturer vs. retailer investment. Transp. Res. Part E Logist. Transp. Rev. 109, 63–82 (2018)
17. Mantin, B.; Veldman, J.: Managing Strategic inventories under investment in process improvement. Eur. J. Oper. Res. 279(3), 782–794 (2019)
18. Li, J.; Yi, L.; Shi, V.; Chen, X.: Supplier encroachment strategy in the presence of retail strategic inventory: centralization or decentralization. Omega (2020)
19. Arya, A.; Frimor, H.; Mittendorf, B.: Decentralized procurement in light of strategic inventories. Manag. Sci. 61(3), 578–585 (2014)