TENSOR TRIANGULAR GEOMETRY FOR QUANTUM GROUPS

BRIAN D. BOE, JONATHAN R. KUJAWA, AND DANIEL K. NAKANO

ABSTRACT. Let $\mathfrak{g}$ be a complex simple Lie algebra and let $U_\zeta(\mathfrak{g})$ be the corresponding Lusztig $\mathbb{Z}[q,q^{-1}]$-form of the quantized enveloping algebra specialized to an $\ell$th root of unity. Moreover, let $\text{mod}(U_\zeta(\mathfrak{g}))$ be the braided monoidal category of finite-dimensional modules for $U_\zeta(\mathfrak{g})$. In this paper we classify the thick tensor ideals of $\text{mod}(U_\zeta(\mathfrak{g}))$ and compute the prime spectrum of the stable module category associated to $\text{mod}(U_\zeta(\mathfrak{g}))$ as defined by Balmer.

1. Introduction

1.1. The geometry of the nilpotent cone $\mathcal{N}$ plays a central role in the seminal work of Arkhipov, Bezrukavnikov, and Ginzburg [ABG] on the representation theory for quantum groups at a root of unity. In particular, they showed that there are derived equivalences between the principal block for the large quantum group $U_\zeta(\mathfrak{g})$, equivariant coherent sheaves on the Springer resolution of $\mathcal{N}$, and perverse sheaves on the loop Grassmannian. One consequence of these equivalences of triangulated categories is a proof of Lusztig’s character formula for quantum groups when $\ell > h$ (where $\ell$ is the order of the root of unity and $h$ is the Coxeter number for $\mathfrak{g}$). An important underlying idea in [ABG] is the lifting of the support variety theory for the small quantum group to obtain the equivalence between the principal block and coherent sheaves on the Springer resolution. This approach illustrates that to calculate characters of simple modules at the representation theoretic level one needs to examine the structure of the underlying tensor triangulated category at the derived level where the role of geometry is more transparent.

As further evidence of this connection, Drupieski, Nakano, and Parshall [DNP], using the validity of the Lusztig character formula and the positivity of the coefficients of the Kazhdan-Lusztig polynomials for the affine Weyl group, calculated the support varieties for all irreducible representations for the small quantum group at an $\ell$th root of unity. Lusztig demonstrated that there is a bijection between two-sided cells of the affine Weyl group and nilpotent orbits in $\mathcal{N}$. Humphreys conjectured that the support variety of a tilting module for $U_\zeta(\mathfrak{g})$, when restricted to the small quantum group, would be given by the nilpotent orbit whose corresponding two-sided cell contains the high weight of the tilting module. Bezrukavnikov [Bez] verified a version of this conjecture using the tools from [ABG] described above. Bezrukavnikov’s result involves so-called “relative support varieties.” In the case of tilting modules with regular highest weight, we verify Humphreys’ conjecture for “absolute support varieties” in Proposition 7.2.1.

Date: January 14, 2022.

2000 Mathematics Subject Classification. Primary 17B56, 17B10; Secondary 13A50.

Research of the second author was partially supported by NSF grant DMS-1160763 and NSA grant H98230-16-0055.

Research of the third author was partially supported by NSF grant DMS-1402271 and DMS-1701768.
Let $\mathbf{T}$ be the category of finite-dimensional tilting modules for $U_\zeta(g)$. This category is a braided monoidal category via the coproduct on the Hopf algebra $U_\zeta(g)$. Ostrik showed that the thick tensor ideals in $\mathbf{T}$ are in one-to-one correspondence with two-sided cells for the affine Weyl group, or equivalently (via Lusztig’s bijection), nilpotent orbits in $\mathcal{N}$. In particular, two tilting modules generate the same thick tensor ideal in $\mathbf{T}$ if and only if they have the same support varieties. In contrast, the classification of the thick tensor ideals in the category of all finite-dimensional $U_\zeta(g)$-modules has proven elusive.

1.2. Given a braided monoidal category, a fundamental question is to classify its thick tensor ideals (see Section 2.1 for definitions). In the same setting, Balmer showed that one can extract the ambient geometry of the category by using the tensor structure to treat the category like a commutative ring. He defined a notion of prime tensor ideals in the category and used this to construct a categorical spectrum (referred to in this paper as the Balmer spectrum). Determining the Balmer spectrum is thus important because it reveals a natural geometric object that governs the structure and interaction of objects in the category. Furthermore, Balmer proved that determining the Balmer spectrum is intimately related to the classification of the thick tensor ideals. An example closely related to the results of this paper is that the Balmer spectrum for restricted Lie algebras (for the stable module category) identifies with the restricted nilpotent cone [Bal]. Another example, in the same spirit, is in an earlier paper of the authors which determined the Balmer spectrum for classical Lie superalgebras by utilizing their detecting subalgebras [BKN].

The main goals of this paper are to (i) classify the thick tensor ideals in $\text{mod}(U_\zeta(g))$ and (ii) determine the Balmer spectrum for the stable module category associated to $\text{mod}(U_\zeta(g))$ (see Section 3.2 for definitions). One can view our results as complementary to Ostrik’s classification of tensor ideals in $\mathbf{T}$. Both (i) and (ii) will involve much of the machinery developed by the authors in [BKN]. In particular, this entails constructing a support data from the stable module category to an appropriate Zariski space. The geometric object of central importance turns out to be the collection of $G$-invariant ideals in the coordinate algebra of $\mathcal{N}$. Furthermore, we rely heavily on the use of infinitely generated modules for $U_\zeta(g)$ and localization theorems in a stable module category which contains these infinitely generated modules.

A fundamental obstacle we need to overcome is to establish various versions of the tensor product property for the support theories for modules over the large and small quantum groups. This is addressed as follows. First, we define the notion of quasi support data whose axioms involve a weaker form of the tensor product condition. Second, to obtain an analogue of a result of Hopkins, a key assumption (i.e., Assumption 2.5.1) is identified that involves the projectivity of infinitely generated modules. We then reduce, via line bundle cohomology techniques, this issue of projectivity to modules for the small quantum group attached to the Borel subalgebra $b \subseteq g$. The next step is to pass to a designated associated graded algebra, $\text{gr } u_\zeta(b)$, for $u_\zeta(b)$. It is in this category that we prove a tensor product theorem (cf. Theorem 6.2.1). The main idea of the proof is to apply work of Benson, Erdmann, and Holloway to relate supports of modules in $\text{gr } u_\zeta(b)$ to supports for a quantum complete intersection (with equal parameters) where the theory is better understood via rank varieties. By carefully keeping track of the relationship between the support theories between $u_\zeta(b)$ and $\text{gr } u_\zeta(b)$ we can successfully verify the key assumption. Much of our
analysis uses key facts in the local support theory as developed by Benson, Iyengar, and Krause \cite{BIK1}.

In Proposition 2.7.2 and Theorem 7.5.1, as an application of the classification of thick tensor ideals for \(\text{stmod}(U_{\zeta}(\mathfrak{g}))\), we prove the tensor product property for supports of \(U_{\zeta}(\mathfrak{g})\)-modules. The tensor product property is then applied in the construction of a homeomorphism to compute the Balmer spectrum of \(\text{stmod}(U_{\zeta}(\mathfrak{g}))\) (cf. Theorem 7.6.1). The problem of proving the tensor product property for modules over the small quantum group \(u_{\zeta}(\mathfrak{g})\) remains open.

1.3. Connections to Affine Lie Algebras. Our results have analogues for affine Lie algebras which we now describe. Let \(\mathfrak{g}\) be a finite-dimensional complex simple Lie algebra, \(\hat{\mathfrak{g}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}c \oplus \mathbb{C}d\) be the corresponding untwisted affine Lie algebra, and \(\tilde{\mathfrak{g}}\) be the subalgebra \(\tilde{\mathfrak{g}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}c \subseteq \hat{\mathfrak{g}}\). For \(\kappa \in \mathbb{C}\) we let \(O_{\kappa}\) be the full subcategory of all \(\tilde{\mathfrak{g}}\)-modules, \(M\), for which the central element \(c\) acts by \(\kappa\) and \(M\) satisfies certain category \(O\)-type finiteness conditions.

Now we set \(\kappa = (-\ell)/2D - h^\vee\), where \(h^\vee\) is the dual Coxeter number for \(\mathfrak{g}\) and \(D\) is the maximal number of edges between two nodes in the Dynkin diagram of \(\mathfrak{g}\). The category \(O_{\kappa}\) admits a tensor product and duality which makes it into a braided monoidal category which is rigid (i.e., with duals). Under mild assumptions on \(\ell\), Kazhdan-Lusztig \cite{KL1, KL2, KL3} and Lusztig \cite{Lus3} prove in the simply-laced and non-simply-laced cases, respectively, that there exists a functor

\[F_\ell: O_{\kappa} \to \text{mod}(U_{\zeta}(\mathfrak{g}))\]

which is an equivalence of braided monoidal categories. See \cite{Tan} or \cite{CP, Section 16.3} for an overview of these results. From this it immediately follows that the classification of thick tensor ideals in the two categories coincides and that the Balmer spectra for the corresponding stable module categories are homeomorphic. That is, the appropriate analogues of Theorems 7.5.1 and 7.6.1 also hold for \(O_{\kappa}\).

1.4. Acknowledgements. The second and third authors are pleased to acknowledge the hospitality and support of the Mittag-Leffler Institute during the special semester in Representation Theory in Spring 2015. The formulations of several of the key ideas in the paper were discovered at this time. The authors thank Cris Negron for spotting a gap in an earlier version of Theorem 7.3.3, and for his insights into the questions of naturality of supports for quantum groups.

2. Preliminaries

2.1. Tensor Triangulated Categories. For the purposes of this paper we will summarize the notion of tensor triangulated categories as defined by Balmer (cf. \cite{Bal, Definition 1.1}). We will not state all the definitions but instead refer the reader to \cite{BKN} for a full treatment. The main idea is to use the tensor structure as a “multiplication” and define categorical analogues of prime ideals and the spectrum.

**Definition 2.1.1.** A tensor triangulated category (TTC) is a triple \((\mathcal{K}, \otimes, 1)\) such that

1. \(\mathcal{K}\) is a triangulated category, and
(ii) \( \mathbf{K} \) has a braided\(^1\) monoidal tensor product \( \otimes : \mathbf{K} \times \mathbf{K} \to \mathbf{K} \) which is exact in each variable with unit object \( 1 \).

A (tensor) ideal in \( \mathbf{K} \) is a triangulated subcategory \( \mathbf{I} \) of \( \mathbf{K} \) such that \( M \otimes N \in \mathbf{I} \) for all \( M \in \mathbf{I} \) and \( N \in \mathbf{K} \), and an ideal is called thick if it is closed under the taking of direct summands. A prime ideal, \( \mathbf{P} \), of \( \mathbf{K} \) is a proper thick tensor ideal such that if \( M \otimes N \in \mathbf{P} \), then either \( M \in \mathbf{P} \) or \( N \in \mathbf{P} \) (see [Bal, Definition 2.1]).

The Balmer spectrum of \( \mathbf{K} \) is defined to be

\[
\text{Spc}(\mathbf{K}) = \{ \mathbf{P} \subset \mathbf{K} \mid \mathbf{P} \text{ is a prime ideal} \}.
\]

The topology on \( \text{Spc}(\mathbf{K}) \) is given by closed sets of the form

\[
Z(C) = \{ \mathbf{P} \in \text{Spc}(\mathbf{K}) \mid C \cap \mathbf{P} = \emptyset \}
\]

where \( C \) is a family of objects in \( \mathbf{K} \).

For a given TTC, \( \mathbf{K} \), let \( \mathbf{K}^c \) denote the full triangulated subcategory of compact objects. When we say \( \mathbf{K} \) is a compactly generated TTC we mean that \( \mathbf{K} \) is closed under arbitrary set indexed coproducts, the tensor product preserves set indexed coproducts, \( \mathbf{K} \) is compactly generated, the tensor product of compact objects is compact, \( 1 \) is a compact object, and every compact object is rigid (i.e. strongly dualizable) as in, for example, [HPS]. In particular we have an exact contravariant duality functor \((-)^* : \mathbf{K}^c \to \mathbf{K}^c \) such that

\[
\text{Hom}_\mathbf{K}(M \otimes N, Q) = \text{Hom}_\mathbf{K}(N, M^* \otimes Q)
\]

for \( M \in \mathbf{K}^c \) and \( N, Q \in \mathbf{K} \). We refer the reader to [BKN, Section 2.1] for further discussion about compactness.

2.2. Zariski Spaces. A Noetherian topological space \( X \) is a Zariski space if, in addition, any irreducible closed set \( Y \) of \( X \) has a unique generic point (i.e., there exists a unique \( y \in Y \) such that \( Y = \{ y \} \)). Note that for a Noetherian topological space \( X \) any closed set in \( X \) is the union of finitely many irreducible closed sets. In particular we will be primarily interested in the cases when \( X = \text{Spec}(R) \) where \( R \) is a commutative Noetherian ring, or when \( R \) is graded, \( X = \text{Proj}(R) \). We set the following notation for a Zariski space \( X \):

(i) \( X \): the collection of all subsets of \( X \),
(ii) \( X_{\text{sp}} \): the collection of all specialization closed subsets (i.e., sets which are arbitrary unions of closed sets),
(iii) \( X_{\text{cl}} \): the collection of all closed subsets,
(iv) \( X_{\text{irr}} \): the collection of all irreducible closed subsets.

We will be interested in cases where we have an algebraic group \( G \) acting rationally on a graded commutative ring \( R \) by automorphisms which preserve the grading. This action induces an action of \( G \) on \( X = \text{Proj}(R) \). Let \( X_G = G \cdot \text{Proj}(R) \) be the set of homogeneous \( G \)-prime ideals of \( R \) (e.g., as defined in [Lor1]). The topology in \( X_G \) is induced from the map \( \rho : X \to X_G \) with \( \rho(P) = \cap_{g \in G} gP =: \cap_{g \in G} gP \) by declaring \( W \subseteq X_G \) closed if and only if \( \rho^{-1}(W) \) is closed in \( X \). We have \( \cap_{g \in G} gP_1 = \cap_{g \in G} gP_2 \) for \( P_1, P_2 \in X \) if and only if \( G \cdot P_1 = G \cdot P_2 \) in \( X \). In [BKN, Section 2.3] it was shown that \( X_G \) is a Zariski space.

---

\(^1\)To be precise, Balmer assumes the tensor product is symmetric but his definitions and results apply equally well in the braided setting.
2.3. Localization functors. In this section we briefly recall a key tool: the localization and colocalization functors as given in [BIK2, Proposition 2.16]. For details, the reader is referred to [BIK2, Section 2.2], and the general properties given in [BIK1].

Assume that the triangulated category $K$ admits arbitrary set indexed coproducts. A localizing subcategory of $K$ is a triangulated subcategory which is closed under taking set indexed coproducts. For $C$ a collection of objects of $K$, let $\text{Loc}(C)$ be the smallest localizing subcategory containing $C$.

The result we need is the following restatement of [BIK2, Theorem 2.32].

**Theorem 2.3.1.** Let $K$ be a compactly generated triangulated category which admits arbitrary set indexed coproducts. Given a thick subcategory $C$ of $K^c$ and an object $M$ in $K$, there exists a functorial triangle in $K$,

$$\Gamma_C(M) \to M \to L_C(M) \to$$

which is unique up to isomorphism, such that $\Gamma_C(M)$ is in $\text{Loc}(C)$ and there are no non-zero maps in $K$ from $C$ or, equivalently, from $\text{Loc}(C)$ to $L_C(M)$.

2.4. Quasi Support Data. Balmer [Bal] provided a definition of support data as a method to relate objects in a TTC to subsets in a given Zariski space. For our purposes we will need a weaker notion which relaxes the condition on the tensor products of objects. We will call this notion a quasi support data.

Let $K$ be a TTC (possibly $K = K^c$), and $X$ be a Zariski space. A *quasi support data* is an assignment $V : K \to X$ which satisfies the following six properties (for $M, M_i, N, Q \in K$):

(2.4.1) $V(0) = \emptyset$, $V(1) = X$;
(2.4.2) $V(\oplus_{i \in I} M_i) = \bigcup_{i \in I} V(M_i)$ whenever $\oplus_{i \in I} M_i$ is an object of $K$;
(2.4.3) $V(\Sigma M) = V(M)$, where $\Sigma$ is the shift functor for $K$;
(2.4.4) for any distinguished triangle $M \to N \to Q \to \Sigma M$ we have

$$V(N) \subseteq V(M) \cup V(Q);$$

(2.4.5) $V(M \otimes N) \subseteq V(M) \cap V(N)$;

We note that using the above properties and [HPS, Lemma A.2.6] one can verify, when $K$ is a compactly generated TTC, that

(2.4.6) $V(M) = V(M^*)$ for $M \in K^c$.

We will be interested in quasi support data which satisfy one or both of two additional properties:

(2.4.7) $V(M) = \emptyset$ if and only if $M = 0$;
(2.4.8) for any $W \in \mathcal{X}_d$ there is an $M \in K^c$ such that $V(M) = W$ (Realization Property).

2.5. An Additional Assumption on a TTC. Let $V : K^c \to \mathcal{X}_d$ be a quasi support data, as above. Given an object $M \in K^c$, let $\text{Tensor}(M) \subseteq K^c$ be the thick tensor ideal in $K^c$ generated by $M$. For $W \in \mathcal{X}_d$, set

$$I_W = \{ Q \in K^c \mid V(Q) \subseteq W \}.$$  

By (2.4.2)–(2.4.5), $I_W$ is a thick tensor ideal of $K^c$. We state the following key assumption on $K$ (which will be verified later for quantum groups).
Assumption 2.5.1. Suppose $M \in K^c$ with $M \neq 0$, $N \in I_{V(M)} \subseteq K^c$, and $I' = \text{Tensor}(M)$. If $M \otimes L_V(N) = 0$, then $L_V(N) = 0$.

In many circumstances (e.g., finite group schemes, Lie superalgebras) Assumption 2.5.1 is verified by showing the existence of an extension of support data from $K^c$ to $K$ which satisfies the tensor product property (see [BKNN] Section 3.2) for a discussion of extending support data). For quantum groups, suitably extending the quasi support data to non-compact objects is an open problem. In this situation alternative methods will be necessary to verify Assumption 2.5.1.

2.6. Hopkins’ Theorem. We will now explain how one can use Assumption 2.5.1 to prove a version of Hopkins’ theorem. A proof is included for the reader to see where Assumption 2.5.1 is used as a replacement for the extension of support data.

Proposition 2.6.1. Let $K$ be a compactly generated TTC, $X$ be a Zariski space, and $\mathcal{X}_{cl}$ be the collection of closed subsets of $X$. Let $V : K^c \to \mathcal{X}_{cl}$ be a quasi support data defined on $K^c$ satisfying the additional condition (2.4.7), with Assumption 2.5.1 also holding. Given $M \in K^c$, set $W = V(M)$.

Then $I_W = \text{Tensor}(M)$.

Proof. First note that if $M = 0$ then $V(M) = \emptyset$. Moreover, $\text{Tensor}(M)$ and $I_{\emptyset}$ just consist of $0$, by (2.4.7). Therefore, we can assume that $M \neq 0$. Set $I = I_W$ and $I' = \text{Tensor}(M)$.

(⊇) Since $\text{Tensor}(M)$ is the smallest thick tensor ideal of $K^c$ containing $M$, from the definition of $I_W$ and the properties of a quasi support data it follows that $I \supseteq I'$.

(⊆) Let $N \in K$. Apply the exact triangle of functors $I_V \to \text{Id} \to L_V \to \Gamma_1(N)$:

$$I_V \Gamma_1(N) \to \Gamma_1(N) \to L_V \Gamma_1(N) \to$$ (2.6.1)

Since $I' \subseteq I$, the first term belongs to $\text{Loc}(I') \subseteq \text{Loc}(I)$. The second term also belongs to $\text{Loc}(I)$, a triangulated subcategory, and hence so does the third term: $L_V \Gamma_1(N) \in \text{Loc}(I)$.

By Theorem 2.3.1 (applied to $I'$), there are no non-zero maps from $I'$ to $L_V \Gamma_1(N)$. Thus, for any object $S \in K^c$, the duality property implies that

$$0 = \text{Hom}_K(M \otimes S, L_V \Gamma_1(N)) \cong \text{Hom}_K(S, M^* \otimes L_V \Gamma_1(N)).$$ (2.6.2)

Since $K$ is compactly generated it follows that $M^* \otimes L_V \Gamma_1(N) = 0$ in $K$.

Now specialize to $N \in I$. Then we have $\Gamma_1(N) \cong N$, and $M^* \otimes L_V(N) = 0$ in $K$. Therefore, by Assumption 2.5.1 $L_V \Gamma_1(N) = L_V(N) = 0$ in $K$. By (2.6.1) it follows that $\Gamma_1(N) \cong I_V \Gamma_1(N)$, so $I_V(N) \cong N$, whence $N \in \text{Loc}(I')$. Applying [Nee, Lemma 2.2] we see that in fact $N \in I'$. This shows $I \subseteq I'$ and completes the proof.

2.7. The Tensor Product Property. One can use Hopkins’ Theorem and the Realization Property to prove that a quasi support data has the tensor product property and, hence, is a support data. The tensor product property is used in proving Theorem 2.8.2 to show that the map $f$ takes points in $X$ to prime ideals.

Lemma 2.7.1. Let $K$ be a TTC and let $M, N \in K^c$ be fixed. For any objects $A \in \text{Tensor}(M)$ and $B \in \text{Tensor}(N)$ one has $A \otimes B \in \text{Tensor}(M \otimes N)$.
Proof. By definition Tensor(Q) is the smallest tensor ideal of $K^c$ containing $Q \in K^c$. That
is, any object of Tensor(Q) can be obtained from Q by a finite sequence of the following operations:

1. If $A \in \text{Tensor}(Q)$, then $\Sigma A \in \text{Tensor}(Q)$;
2. If $A \rightarrow B \rightarrow C \rightarrow$ is a triangle in $K$ and two of $A, B, C$ are objects in $\text{Tensor}(Q)$,
   then the third is an object of $\text{Tensor}(Q)$;
3. If $A \in \text{Tensor}(Q)$ and $B \in K$, then $A \otimes B \in \text{Tensor}(Q)$;
4. If $A \oplus B$ is an object of $\text{Tensor}(Q)$, then $A, B$ are objects of $\text{Tensor}(Q)$.

Given $A \in \text{Tensor}(Q)$, write $\ell_Q(A) \in \mathbb{Z}_{\geq 0}$ for the minimal number of operations required to
obtain $A$ from $Q$.

The result is proved by induction on $\ell_M(A) + \ell_N(B)$. If this sum equals zero, then $A \cong M$ and $B \cong N$ and the result holds trivially. Now assume that $A \in \text{Tensor}(M)$ and $B \in \text{Tensor}(N)$ are given. Without loss of generality assume $\ell_M(A) > 0$. Then there are $A’, A” \in \text{Tensor}(M)$ such that $\ell_M(A’), \ell_M(A”) < \ell_M(A)$ and $A$ is obtained from $A’$ and $A”$ by one of the above operations. If $A = \Sigma A’$, then $A’ \otimes B \in \text{Tensor}(M \otimes N)$ and $A \otimes B = (\Sigma A’) \otimes B \cong (A’ \otimes B) \otimes C$, where the isomorphism holds by the defining properties
of a TTC. Thus $A \otimes B \in \text{Tensor}(M \otimes N)$. If $A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow$ is a distinguished triangle
in $K$ with $\{A, A’, A”\} = \{A_1, A_2, A_3\}$, then one can apply the exact functor $- \otimes B$ to obtain the triangle $A_1 \otimes B \rightarrow A_2 \otimes B \rightarrow A_3 \otimes B \rightarrow$. By induction two of the three

Proposition 2.7.2. Let $K$ be a compactly generated TTC, $X$ be a Zariski space, and let $V : K^c \rightarrow X_{cl}$ be a quasi support data defined on $K^c$. If $V$ satisfies both the statement of
Hopkins’ Theorem and $\{2.4.8\}$, then

$$V(M \otimes N) = V(M) \cap V(N)$$

for any $M$ and $N$ in $K^c$.

Proof. Since $V$ is a quasi-support data only the inclusion $V(M) \cap V(N) \subseteq V(M \otimes N)$
needs to be proved. Since $V(M)$ and $V(N)$ are closed sets, the intersection $V(M) \cap V(N)$
is a closed set. Using the Realization Property, fix a $T \in K^c$ with $V(T) = V(T^*) = V(M) \cap V(N)$. Since $V(T) \subseteq V(M)$, Hopkins’ Theorem implies $T \in \text{Tensor}(M)$. Similarly,
$T^* \in \text{Tensor}(N)$. By Lemma 2.7.1 it follows that $T \otimes T^* \in \text{Tensor}(M \otimes N)$ and so $T \otimes T^* \otimes T$
is an object of $\text{Tensor}(M \otimes N)$. By [HPS, Lemma A.2.6], $T$ is a direct summand of $T \otimes T^* \otimes T$.
That is, $T$ is an object of $\text{Tensor}(M \otimes N)$ and so $V(T) \subseteq V(M \otimes N)$. That is, $V(M) \cap V(N) \subseteq V(M \otimes N)$ as required.

2.8. Thick Tensor Ideals in a TTC and $\text{Spc}$. One can now use the version of Hopkins’
Theorem given in the previous section (Proposition 2.6.1) and apply the same arguments
as in [BKN, Theorem 3.4.1] to obtain the following classification of thick tensor ideals of
compact objects in our setting.
Theorem 2.8.1. Let $K$ be a compactly generated TTC. Let $X$ be a Zariski space, $X_{cl}$ be the set of all closed subsets of $X$, and let $V : K^c \rightarrow X_{cl}$ be a quasi support data defined on $K^c$ satisfying the additional conditions (2.4.7) and (2.4.8). Assume Assumption 2.5.1 holds.

Given the above setup there is a pair of mutually inverse maps

$$\begin{align*}
\{ \text{thick tensor ideals of } K^c \} & \xrightarrow{\Gamma} X_{sp}, \\
\Theta & \xleftarrow{\Theta} \{ \text{closed subsets of } X \}.
\end{align*}$$

By Proposition 2.7.2 a quasi support data which satisfies the assumptions of the previous theorem is a support data. One can argue as in the proof of [BKN, Theorem 3.5.1] to obtain the following description of the Balmer spectrum of $K^c$.

Theorem 2.8.2. Let $K$ be a compactly generated TTC, let $X$ be a Zariski space, and let $X_{cl}$ be the set of all closed subsets of $X$. Assume that $V : K^c \rightarrow X_{cl}$ is a quasi support data defined on $K^c$ satisfying the additional conditions (2.4.7) and (2.4.8). Assume Assumption 2.5.1 holds. Then there is a homeomorphism

$$f : X \rightarrow \text{Spc}(K^c).$$

3. Quantum Groups and their Module Categories

3.1. Notation. We will follow the conventions as described in [BNPP, Section 2].

1. $G$: a simple, simply connected algebraic group over $\mathbb{C}$. The assumption of $G$ being simple is for convenience and the results easily generalize to $G$ reductive.
2. $\mathfrak{g} = \text{Lie } G$.
3. $\mathcal{U}(\mathfrak{g})$: the universal enveloping algebra of $\mathfrak{g}$.
4. $\mathcal{N}$: the nullcone (i.e., the set of nilpotent elements of $\mathfrak{g}$).
5. $\mathbb{C}[\mathcal{N}]$: the coordinate algebra of $\mathcal{N}$.
6. $T$: a maximal split torus in $G$.
7. $\Phi$: the corresponding (irreducible) root system associated to $(G,T)$. When a root system has roots of only one length, all roots shall be considered both short and long.
8. $\Phi^\pm$: the positive (respectively, negative) roots.
9. $B$: a Borel subgroup containing $T$ corresponding to the negative roots.
10. $\Pi = \{ \alpha_1, \alpha_2, \ldots, \alpha_{||\Pi||} \}$: an ordering of the simple roots.
11. $\alpha_0$: the highest short root.
12. $\bar{\alpha}$: the highest root.
13. $E$: the Euclidean space $\mathbb{R}\Phi$, with inner product $\langle -, - \rangle$.
14. $\alpha^\vee = 2\alpha / \langle \alpha, \alpha \rangle$ for $0 \neq \alpha \in \mathbb{E}$.
15. $X(T) = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_{||\Pi||}$: the weight lattice, where the fundamental dominant weights $\omega_i \in \mathbb{E}$ are defined by $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$, $1 \leq i, j \leq ||\Pi||$.
16. $X_+ = \mathbb{N}\omega_1 + \cdots + \mathbb{N}\omega_{||\Pi||}$: the dominant weights.
17. $\ell$: a fixed odd positive integer. If $\Phi$ has type $G_2$, then we assume that $3 \nmid \ell$. 

For quantum groups we will use the following notation.

(18) $X_1 = X(T)/\ell X(T)$.

(19) $W$: the Weyl group of $\Phi$.

(20) $W_\ell = W \rtimes \ell \mathbb{Z}\Phi$: the affine Weyl group of $\Phi$.

(21) $\rho$: the Weyl weight defined by $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$.

(22) $h$: the Coxeter number of $\Phi$, given by $h = \langle \rho, \alpha_i^\vee \rangle + 1$.

For quantum groups we will use the following notation.

(23) $\zeta$: a primitive $\ell$th root of unity in $\mathbb{C}$.

(24) $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$, where $q$ is an indeterminate.

(25) $U_q(\mathfrak{g})$: the quantized enveloping algebra over $\mathbb{Q}(q)$.

(26) $U_q^\mathcal{A}(\mathfrak{g})$: Lusztig’s $\mathcal{A}$-form in $U_q(\mathfrak{g})$.

(27) $U_\zeta(\mathfrak{g})$: the (big) quantum group obtained from the specialization at $\zeta$ of Lusztig’s $\mathcal{A}$-form as described below.

(28) $u_\zeta(\mathfrak{g})$: the small quantum group.

(29) $U^0$: the subalgebra of $U_\zeta(\mathfrak{g})$ generated by the binomials in the $K_\alpha$’s, $\alpha \in \Pi$.

(30) $T(\lambda)$: the (indecomposable) tilting $U_\zeta(\mathfrak{g})$-module of highest weight $\lambda \in X_+$. In more detail, let $U_q(\mathfrak{g})$ be the quantized enveloping algebra defined by generators and relations over $\mathbb{Q}(q)$ (where $q$ is an indeterminate) as in, for example, [Jan1, Section 4.3]. Let $U_q^\mathcal{A}(\mathfrak{g})$ denote the $\mathcal{A}$-form of $U_q(\mathfrak{g})$ defined by Lusztig. One can then construct the restricted specialization $\mathbb{C} \otimes_\mathcal{A} U_q^\mathcal{A}(\mathfrak{g})$ by specializing $q$ to $\zeta \in \mathbb{C}$. Within this algebra $1 \otimes K_\alpha^\ell$ is central for all $\alpha \in \Pi$. By definition a module for the restricted specialization is of Type 1 if it is annihilated by $1 \otimes K_\alpha^\ell - 1 \otimes 1$ for all $\alpha \in \Pi$. The discussion in [APW1, Section 1.6] implies that for the modules and questions under consideration here we may assume all modules for the restricted specialization are of Type 1. That is, from this point on we study modules for $U_\zeta(\mathfrak{g})$ where

$$U_\zeta(\mathfrak{g}) := \mathbb{C} \otimes_\mathcal{A} U_q^\mathcal{A}(\mathfrak{g})/(1 \otimes K_\alpha^\ell - 1 \otimes 1 \mid \alpha \in \Pi).$$

Here $(\cdots)$ denotes the two-sided ideal generated by the elements within the parentheses. Following Lusztig, we view $U_q(\mathfrak{g})$ as a Hopf algebra using the maps given in [APW1, BNPP, Jan1]. These maps induce a Hopf algebra structure on the restricted specialization and on $U_\zeta(\mathfrak{g})$.

The elements $E_\alpha, K_\alpha, F_\alpha$, $\alpha \in \Pi$, in $U_\zeta(\mathfrak{g})$ (i.e., the images of $1 \otimes E_\alpha$, etc. in $U_q(\mathfrak{g})$) generate a finite-dimensional Hopf subalgebra, denoted by $u_\zeta(\mathfrak{g})$, of $U_\zeta(\mathfrak{g})$. The Hopf algebra $u_\zeta(\mathfrak{g})$ will be referred to as the small quantum group. The finite-dimensional algebra $u_\zeta(\mathfrak{g})$ is a normal Hopf subalgebra of $U_\zeta(\mathfrak{g})$ such that $U_\zeta(\mathfrak{g})/u_\zeta(\mathfrak{g}) \cong \mathcal{U}(\mathfrak{g})$. [Lus1, Lus2]. We write $u_\zeta(h)$, $u_\zeta(u)$, and $u_\zeta(b)$ for the subalgebras of $u_\zeta(\mathfrak{g})$ generated by $\{K_\alpha^\pm 1 \mid \alpha \in \Pi\}$, $\{E_\alpha \mid \alpha \in \Pi\}$, and $\{K_\alpha^\pm 1, E_\alpha \mid \alpha \in \Pi\}$, respectively. Sometimes we write $K_i$ for $K_{\alpha_i}$.

3.2. Module Categories. Let $\text{Mod}(U_\zeta(\mathfrak{g}))$ be the full subcategory of all $U_\zeta(\mathfrak{g})$-modules which admit a weight space decomposition for $U^0$ with weights lying in $X(T)$ and which are locally finite as $U_\zeta(\mathfrak{g})$-modules. This is precisely the category of integrable Type 1 modules of [BNPP] and [APW1]. Furthermore, let $\text{mod}(U_\zeta(\mathfrak{g}))$ denote the category of all finite-dimensional $U_\zeta(\mathfrak{g})$-modules. By [APW1, Theorem 9.12(i)] every finite-dimensional $U_\zeta(\mathfrak{g})$-module admits a weight space decomposition for $U^0$ with weights lying in $X(T)$; hence, $\text{mod}(U_\zeta(\mathfrak{g}))$ is a full subcategory of $\text{Mod}(U_\zeta(\mathfrak{g}))$. 

$$\zeta = \sum_{\alpha \in \Phi^+} \alpha_j \zeta_j \in \mathbb{C}.$$
Let $A$ denote $u_\zeta(h)$ or $u_\zeta(b)$. Then $\text{Mod}(A)$ denotes the full subcategory of all $A$-modules which admit a weight space decomposition with respect to $u_\zeta(h)$ with weights lying in $X_J$.

Let $\text{mod}(A)$ denote the full subcategory of $\text{Mod}(A)$ consisting of the finite-dimensional modules. In the case of $u_\zeta(u)$ we let $\text{Mod}(u_\zeta(u))$ denote the category of all $u_\zeta(u)$-modules and $\text{mod}(u_\zeta(u))$ the full subcategory of all finite-dimensional $u_\zeta(u)$-modules. Note that if $A$ and $B$ are any of $U_\zeta(g)$, $u_\zeta(g)$, $u_\zeta(b)$, $u_\zeta(h)$, or $u_\zeta(u)$ and $B$ is a subalgebra of $A$, then restriction defines functors from $\text{Mod}(A)$ to $\text{Mod}(B)$ and from $\text{mod}(A)$ to $\text{mod}(B)$.

Our general convention is that by $A$-module we mean an object of $\text{Mod}(A)$ and by finite-dimensional $A$-module we mean an object of $\text{Mod}(A)$.

The Steinberg module is a self-dual simple module which is projective and injective in both $\text{Mod}(U_\zeta(g))$ and $\text{mod}(U_\zeta(g))$. From this it follows that these categories have enough projectives and injectives and they coincide (see [APW1] Section 9 and [APW2] for details). Since the projectives and injectives coincide one can form the stable module categories for $\text{Mod}(U_\zeta(g))$ and $\text{mod}(U_\zeta(g))$. Recall that these have the same objects, but one quotients out the morphisms which factor through a projective module. One thereby obtains the triangulated categories $K := \text{Stmod}(U_\zeta(g))$ and $K^c := \text{stmod}(U_\zeta(g))$, respectively. Since the objects of $\text{mod}(U_\zeta(g))$ are finite dimensional it immediately follows that the subcategory $K^c$ of $K$ consists of compact objects. Using the fact that the objects of $\text{Mod}(U_\zeta(g))$ are locally finite one can argue just as in [BKN] Proposition 4.2.1 to verify that $K$ is compactly generated and $K^c$ is precisely the triangulated subcategory of compact objects.

4. Projectivity Results for Quantum Groups

4.1. Projectivity Results I. In the following two sections we will generalize projectivity tests proved for finite-dimensional $U_\zeta(g)$-modules to arbitrary modules in $\text{Mod}(U_\zeta(g))$. In particular, all modules under discussion will be assumed to be Type 1 and integrable.

Support varieties were used in the proofs for the analogous results in the finite-dimensional cases (cf. [FP] Theorem (1.2), [Dru] Theorem 5.2.1). Our more general results will enable us to verify Assumption 2.5.1 for $K = \text{Stmod}(U_\zeta(g))$.

Given a set $J$ of simple roots we adopt the notation of [BNPP] Section 2.5 and write $p_J$ for the corresponding parabolic subalgebra of $g$, $u_\zeta(p_J)$ for the corresponding small quantum group viewed as a subalgebra of $u_\zeta(g)$, etc.

**Theorem 4.1.1.** Let $Q$ be in $\text{Mod}(U_\zeta(g))$ and $J \subseteq \Pi$. Then $Q$ is projective as a $u_\zeta(g)$-module if and only if $Q$ is projective as a $u_\zeta(p_J)$-module.

**Proof.** Let $Q$ be projective as a $u_\zeta(g)$-module. Then $Q$ is projective as a $u_\zeta(p_J)$-module since $u_\zeta(g)$ is free as a $u_\zeta(p_J)$-module.

On the other hand, suppose that $Q$ is projective as a $u_\zeta(p_J)$-module, and let $N$ be a simple module in $\text{mod}(u_\zeta(g))$. Since $N$ can be regarded as a module in $\text{mod}(U_\zeta(g))$, there exists a spectral sequence (cf. [BNPP] Theorem 5.1.1):

$$E_2^{ij} = R^i \text{ind}_{p_J}^G \text{Ext}^j_{u_\zeta(p_J)}(Q, N)(-1) \Rightarrow \text{Ext}^{i+j}_{u_\zeta(g)}(Q, N)(-1).$$

(4.1.1)

Here $(-1)$ indicate untwisting by the Frobenius morphism so that the cohomology becomes a rational $G$-module. Since $Q$ is projective as a $u_\zeta(p_J)$-module this spectral sequence collapses and yields

$$R^i \text{ind}_{p_J}^G \text{Hom}_{u_\zeta(p_J)}(Q, N)(-1) \cong \text{Ext}^i_{u_\zeta(g)}(Q, N)(-1).$$

(4.1.2)
for $i \geq 0$. By the Grothendieck vanishing theorem,
\[ R^i \text{ind}^G_{P_J} \text{Hom}_{u_\zeta(p_J)}(Q, N)^{(-1)} = 0 \]
for $i > \dim G/P_J = D := |\Phi^+| - |\Phi^-|$. Therefore, $\text{Ext}^i_{u_\zeta(g)}(Q, N) = 0$ for $i > D$.

Let $\cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to Q \to 0$ be a minimal projective resolution of $Q$ in $\text{Mod}(U_\zeta(g))$. At each stage we have a short exact sequence
\[ 0 \to \Omega^{n+1}(Q) \to P_n \to \Omega^n(Q) \to 0. \]
By dimension shifting we see that for $m \geq 1$,
\[ 0 = \text{Ext}^{m+D}_{u_\zeta(g)}(Q, N) \cong \text{Ext}^m_{u_\zeta(g)}(\Omega^{D+1}(Q), N). \]
By interpreting this statement in the stable module category (with the fact that the compact objects are the finite-dimensional $u_\zeta(g)$-modules), one can apply [BIK1, Theorem 2] to conclude that $\Omega^{D+1}(Q)$ is projective as $u_\zeta(g)$-module. Since projective modules are injective in $\text{Mod}(u_\zeta(g))$ the sequence
\[ 0 \to \Omega^{D+1}(Q) \to P_n \to \Omega^D(Q) \to 0 \]
splits. Since $P_n$ is projective as a $u_\zeta(g)$-module, this shows that $\Omega^D(Q)$ is projective as $u_\zeta(g)$-module. By iterating this process we can conclude that $Q$ is projective as $u_\zeta(g)$-module. \hfill $\Box$

4.2. Henceforth we set $A = u_\zeta(b)$. For this subsection we let $N = |\Phi^+|$. Order the positive roots, $\Phi^+ = \{\gamma_1, \ldots, \gamma_N\}$ as in [BNPP, Section 2.4]. Then the algebra $u_\zeta(u)$ has a monomial basis
\[ \{F_{\gamma_1}^{a_1}F_{\gamma_2}^{a_2} \cdots F_{\gamma_N}^{a_N} \mid 0 \leq a_i \leq \ell - 1, \ i = 1, 2, \ldots, N\}. \]
For a shorthand notation, let
\[ F_{\vec{a}} := F_{\gamma_1}^{a_1}F_{\gamma_2}^{a_2} \cdots F_{\gamma_N}^{a_N}. \]
We place a total (lexicographical) ordering $\prec$ on this monomial basis as follows. Set $\vec{a} < \vec{b}$ if and only if there exists $1 \leq i \leq N$ such that $a_i < b_i$ and $a_j = b_j$ for all $j > i$. With this ordering, one can define a filtration on $A$. Given $\vec{a}$, let $A_{\vec{a}}$ be the free $u_\zeta(b)$-module spanned by the $F_{\vec{b}}$ where $\vec{b} \preceq \vec{a}$. By [BNPP, Lemma 2.4.1] we have $A_{\vec{a}} \cdot A_{\vec{b}} \subseteq A_{\vec{a}+\vec{b}}$ and so the filtration is multiplicative.

The associated graded algebra $\text{gr} A$ is generated by $\{K_i \mid i = 1, 2, \ldots, |\Pi|\}$ and $\{X_\alpha \mid \alpha \in \Phi^+\}$ subject to the relations:
\[ K_i K_j = K_j K_i, \quad K_i X_\alpha = \zeta^{(\alpha, \alpha)} X_\alpha K_i, \]
\[ X_\alpha X_\beta = \zeta^{(\alpha, \beta)} X_\beta X_\alpha \text{ if } \alpha < \beta. \]
Here we write $\alpha < \beta$ to denote that $\alpha = \gamma_i$ and $\beta = \gamma_j$ with $i < j$. The generators are also subject to the additional conditions:
\[ X_{\alpha}^\ell = 0 \text{ for } \alpha \in \Phi^+, \quad K_i^\ell = 1 \text{ for } i = 1, 2, \ldots, |\Pi|. \]

The algebra $\text{gr} A = \text{gr} u_\zeta(b)$ is a quantized version of a truncated symmetric algebra which we will denote by $\tilde{A}$. The same total ordering defines a filtration on the subalgebra
u_ζ(u) ⊆ u_ζ(b) which, in turn, defines a subalgebra gr u_ζ(u) ⊆ A. In terms of the above presentation for A this is precisely the subalgebra generated by \{X_α | α ∈ Φ^+\}. See [GK], for example, for details. In particular, this subalgebra is a quantum complete intersection in the sense of [BEH]. We write A' = u_ζ(u) and A' = gr u_ζ(u) for these algebras.

4.3. Graded Module Categories. We observe that the definition of the graded module gr M (given in Section 4.1) depends on the choice of generators for M. For our purposes we need to choose the generators carefully to insure that projective A-modules go to projective A-modules. This can be done by choosing a basis of elements in the module M modulo its radical, and checking that projective indecomposable A-modules go to projective indecomposable gr A-modules under this choice of generators. In particular, by always choosing generators so that projectives go to projectives passing to the associated graded provides a well defined operator on objects between stable module categories.

Let Mod(Ā) be the full subcategory of all Ā-modules which have a weight space decomposition with respect to the commutative subalgebra gr u_ζ(ℏ) generated by the elements K_1, …, K_{|Π|} where all weights lie in X_1. Moreover, let mod(Ā) denote the full subcategory of Mod(Ā) consisting of finite-dimensional modules. Let Mod(gr u_ζ(u)) denote the category of all gr u_ζ(u)-modules and mod(gr u_ζ(u)) denote the full subcategory of all finite-dimensional gr u_ζ(u)-modules. Our convention is that all modules are assumed to lie in these categories. Note that if D is u_ζ(b) or u_ζ(u), then for any module M in Mod(D) (resp. mod(D)) the associated graded module gr M constructed in Section 4 lies in Mod(gr D) (resp. mod(gr D)). Finally, note that in both cases gr D is self-injective. Consequently the projective and injective modules coincide and, hence, their stable module categories are triangulated categories.

4.4. Projectivity Results II. In this section we first analyze the relationships between projectivity for modules of u_ζ(b) and u_ζ(u) and their associated graded algebras. Given λ ∈ X_1, let λ also denote the one-dimensional simple u_ζ(ℏ)-module (resp. u_ζ(b)-module) of that weight. The collection of all such is a complete set of simple u_ζ(ℏ)-modules (resp. u_ζ(b)-modules).

**Proposition 4.4.1.** Let Q be a u_ζ(b)-module.

(a) Q is projective as a u_ζ(b)-module if and only if Q is projective as a u_ζ(u)-module.
(b) gr Q is projective as a gr u_ζ(b)-module if and only if gr Q is projective as a gr u_ζ(u)-module.
(c) Q is projective as a u_ζ(u)-module if and only if gr Q is projective as a gr u_ζ(u)-module.
(d) Q is projective as a u_ζ(b)-module if and only if gr Q is projective as a gr u_ζ(b)-module.

**Proof.** (a) The “only if” direction of the statement follows because u_ζ(b) is a free u_ζ(u)-module. Assume that Q is projective over u_ζ(u). The Lyndon-Hochschild-Serre spectral sequence for u_ζ(u) ⊹ u_ζ(b) collapses because the quotient is isomorphic to u_ζ(ℏ) and by assumption the modules under consideration are semisimple as u_ζ(ℏ)-modules. This yields the isomorphism

\[ \text{Ext}^i_{u_ζ(b)}(Q, N) \cong \text{Hom}_{u_ζ(b)}(\mathbb{C}, \text{Ext}^i_{u_ζ(u)}(Q, N)) \]
for $N$ a $u_\zeta(b)$-module and $i \geq 0$. Since $Q$ is projective over $u_\zeta(u)$ we can conclude from this isomorphism that $\text{Ext}^i_{u_\zeta(b)}(Q, N) = 0$ for $i > 0$. Therefore, $Q$ is projective as a $u_\zeta(b)$-module.

(b) This follows from the same line of reasoning as in (a).

(c) If $Q$ is projective as a $u_\zeta(u)$-module then $Q$ is a direct sum of copies of $u_\zeta(u)$, and thus $\text{gr} \ Q$ is projective as a $u_\zeta(u)$-module.

For the converse, we have an increasing multiplicative filtration on $u_\zeta(u)$ which leads to a spectral sequence from Proposition 9.3.1:

$$E_1^{i,j} = H^{i+j}(\text{gr} u_\zeta(u), \text{gr} Q) \Rightarrow H^{i+j}(u_\zeta(u), Q).$$

Since $\text{gr} \ Q$ is projective as a $u_\zeta(u)$-module we have $E_1^{i,j} = 0$ for $i + j \geq 1$. This shows that $H^{i+j}(u_\zeta(u), Q) = 0$ for $i + j \geq 1$. Consequently, $Q$ is projective as a $u_\zeta(u)$-module.

(d) This is a consequence of parts (a), (b), and (c).

\[\square\]

5. Localization and Supports

For the remainder of the paper we will assume that $\zeta$ is a primitive $\ell$th root of unity where $\ell > h$. Throughout this section set $A = u_\zeta(b)$, $\bar{A} = \text{gr} u_\zeta(b)$, $A' = u_\zeta(u)$, and $\bar{A}' = \text{gr} u_\zeta(u)$.

5.1. Let $R = H^\bullet(A, \mathbb{C}) = H^\bullet(u_\zeta(b), \mathbb{C})$. Since $\ell > h$ we have $R \cong S^\bullet(u^*)$ as an algebra by [GK] Theorem 2.5. Now consider the spectral sequence in Proposition 9.3.1 with $M = \mathbb{C}$. As in the proof of [GK] Theorem 2.5, the spectral sequence collapses and so we have the isomorphisms:

$$S^\bullet(u^*) \cong R = H^2(u_\zeta(b), \mathbb{C}) \cong H^2(\text{gr} u_\zeta(b), \mathbb{C}).$$

These are isomorphisms of graded rings where the generators of $S^\bullet(u^*)$ are in degree two. See Section 5.5 for more details.

Let $R' = H^\bullet(u_\zeta(u), \mathbb{C})$. Then for any $u_\zeta(u)$-module $M$, $R'$ acts on $\text{Ext}^\bullet_{u_\zeta(u)}(\mathbb{C}, M)$ via the Yoneda product. Since $R$ is a subring of $R'$, it follows that $R$ acts on $\text{Ext}^\bullet_{u_\zeta(u)}(\mathbb{C}, M)$.

For any $u_\zeta(b)$-module $M$, one can apply the Lyndon-Hochschild-Serre spectral sequence to prove that

$$\text{Ext}^\bullet_{u_\zeta(b)}(\bigoplus_{\lambda \in X_1, \lambda, M) \cong \text{Ext}^\bullet_{u_\zeta(u)}(\mathbb{C}, M).}$$

In this way, $R$ naturally acts on $\text{Ext}^\bullet_{u_\zeta(b)}(\bigoplus_{\lambda \in X_1, \lambda, M)$. A similar statement on actions can be made when $A$ is replaced by $A$ and $A'$ is replaced by $A'$.

Let $X = \text{Proj}(R) = \text{Proj}(\text{Spec}(R))$. Given an ideal $I$ of $R$ we write $V(I) = \{P' \in X \mid P' \supseteq I\}$ for the closed subset of $X$ determined by $I$. The specialization closure of a subset $U \subseteq X$ is the subset $\text{cl}(U) = \cup_{P \in U} V(P)$. A subset $U \subseteq X$ is called specialization closed if $\text{cl}(U) = U$; that is, if $U$ is the union of closed sets.

For $M \in \text{Stmod}(A)$, let

$$Z_A(M) = \{P \in X \mid \text{Ext}^\bullet_{u_\zeta(b)}(\bigoplus_{\lambda \in X_1, \lambda, M)P \neq 0}\} = \{P \in X \mid \text{Ext}^\bullet_{u_\zeta(u)}(\mathbb{C}, M)P \neq 0\}.$$

For $M \in \text{Stmod}(\bar{A})$ one can similarly define $Z_{\bar{A}}(M)$. Now applying Proposition 9.3.1 and the fact that $R$ acts on the spectral sequence, one sees that for $M \in \text{Stmod}(u_\zeta(b))$,

$$Z_A(M) \subseteq Z_{\bar{A}}(\text{gr } M).$$

(5.1.1)
One can verify directly that \( Z_A (\cdot) \) and \( Z_{\bar{A}} (\cdot) \) satisfy (2.4.1)–(2.4.4); we use these properties without comment in what follows.

5.2. A Tensor Product for \( \bar{A} \)-Modules. Let \( \Delta : u_\mathcal{C} (b) \to u_\mathcal{C} (b) \otimes u_\mathcal{C} (b) \) be the coproduct for \( u_\mathcal{C} (b) \) and let \( \varepsilon : u_\mathcal{C} (b) \to \mathbb{C} \) be the counit. From the defining properties of a Hopf algebra the composition

\[
\Delta u \xrightarrow{1 \otimes \varepsilon} u \otimes \mathbb{C} \xrightarrow{m} u
\]

is the identity, where here \( m \) denotes the multiplication map. Consequently, \( \Delta \) is injective and we can identify \( u_\mathcal{C} (b) \) as a subalgebra of \( u_\mathcal{C} (b) \otimes u_\mathcal{C} (b) \) via \( \Delta \). From Section 4.2 there is a filtration of \( u_\mathcal{C} (b) \) which induces a filtration on \( u_\mathcal{C} (b) \otimes u_\mathcal{C} (b) \) and on the image of \( \Delta \) (see Section 9.2). This in turn induces (via \( \Delta \)) a different filtration on \( u_\mathcal{C} (b) \), which (by construction) makes \( \Delta \) a map of filtered algebras. We write \( \Delta \) for the subalgebra \( \text{gr} \Delta (u_\mathcal{C} (b)) \subseteq \text{gr} (u_\mathcal{C} (b) \otimes u_\mathcal{C} (b)) \).

Given \( \mathcal{A} \)-modules \( Q_1 \) and \( Q_2 \) we write \( Q_1 \otimes Q_2 \) for the \( \bar{A} \otimes \bar{A} \)-module given by taking their outer tensor product. Then \( Q_1 \otimes Q_2 \) can be viewed as a \( \Delta \)-module by restriction to this subalgebra. For example, if \( \lambda \) and \( \mu \) are simple \( \mathcal{A} \)-modules, then \( \lambda \otimes \mu \) is a one-dimensional simple \( \Delta \)-module. In particular, in this way \( \mathbb{C} = \mathbb{C} \otimes \mathbb{C} \) is a module for \( \Delta \). Furthermore, several times in the sequel the following observation is needed. If \( M, N \) are \( \mathcal{A} \)-modules and \( \text{gr} (M \otimes N) \) is projective as an \( \mathcal{A} \)-module, then \( M \otimes N \) is a projective \( \mathcal{A} \)-module by Proposition 4.4.1. In turn, \( \text{gr} M \otimes \text{gr} N \) is projective as a \( \Delta \)-module by the discussion at the beginning of Section 4.3.

Finally, note that the antipode on \( \mathcal{A} \) preserves the filtration given in Section 4.2 and induces an anti-automorphism on \( \bar{A} \). In particular, given a finite-dimensional \( \bar{A} \)-module \( M \) we can consider the corresponding dual module \( M^* \).

5.3. Given a finite-dimensional \( \bar{A} \)-module \( Q_1 \) and an arbitrary \( \bar{A} \)-module \( Q_2 \), \( \text{Ext}^*_\bar{A} (Q_1, Q_2) \) can be made into an \( \mathcal{R} \)-module in the following way. From Theorem 9.4.1 we have the isomorphism,

\[
\text{Ext}^*_\bar{A} (Q_1, Q_2) \cong \text{Ext}^*_\mathcal{A} (\mathbb{C}, Q_1^\ast \otimes Q_2).
\]

One can then use the natural action of \( \mathcal{R} \cong H^2 (\Delta, \mathcal{C}) \) on \( \text{Ext}^*_\mathcal{A} (\mathbb{C}, Q_1^\ast \otimes Q_2) \) to put an action of \( \mathcal{R} \) on \( \text{Ext}^*_\mathcal{A} (Q_1, Q_2) \). For \( Q_1 = \bigoplus_{\lambda \in X_1} \lambda \), this coincides with the aforementioned construction of the action of \( \mathcal{R} \) on \( \text{Ext}^*_\mathcal{A} (\bigoplus_{\lambda \in X_1} \lambda, Q_2) \). In the case when \( Q_1 = Q_2 \), we have the map of rings:

\[
\mathcal{R} \to \text{Ext}^*_\mathcal{A} (\mathbb{C}, Q_1^\ast \otimes Q_1) \cong \text{Ext}^*_\mathcal{A} (Q_1, Q_1).
\]

The results in [BIK1, Section 5] use the assumption that one has a ring map (5.3.1) for an arbitrary module \( Q_1 \). However, for our setting we only have an action of \( \mathcal{R} \) on \( \text{Ext}^*_\mathcal{A} (Q_1, Q_2) \) when \( Q_1 \) is finite-dimensional (i.e., compact). Using this action, the main results (i.e., [BIK1, Theorems 5.2, 5.13]) will hold for \( \mathcal{A} \)-modules. The verification of [BIK1, Theorem 5.13] for \( \bar{A} \) heavily uses the aforementioned action, and the fact that \( Q_1 \) is compact. Furthermore, the map (5.3.1) is used to construct Koszul objects (cf. [BIK1, Definition 5.10]) for finite-dimensional \( \mathcal{A} \)-modules.
For each $M$ in $\text{Stmod}(A)$ (resp. $\text{Stmod}(\bar{A})$) and $P \in X = \text{Proj}(R)$, let $\nabla_P(M)$ be the object constructed via (co)localization functors in Section 5. The following theorem gives an alternate description of $\nabla_P(M)$.

**Theorem 5.3.1.** Let $M \in \text{Stmod}(A)$ and $P \in X$. Then,

$$\nabla_P(M) \cong M \otimes \nabla_P(\mathbb{C})$$

as $A$-modules.

**Proof.** The arguments which prove the analogous Corollary 8.3 apply here as well with a few minor modifications to the proofs of Proposition 8.1 and Theorem 8.2. In particular, as the tensor product may not be symmetric, one must do both left and right versions of the arguments therein to verify that the subcategories in question are two-sided ideals. □

Given an $M \in \text{Stmod}(A)$, set $W_A(M) = \{P \in X \mid \nabla_P(M) \neq 0\}$. (5.3.2)

When $M \in \text{stmod}(A)$ one has

$$W_A(M) = Z_A(M)$$

(cf. Theorem 5.5). With our aforementioned setting for $\bar{A}$, one can analogously define the support $W_A(M)$ for $M \in \text{Stmod}(\bar{A})$. For $M \in \text{stmod}(\bar{A})$,

$$W_A(M) = Z_A(M)$$

(cf. Theorem 5.5). Furthermore, for $M \in \text{mod}(\bar{A})$ there is an isomorphism, $\text{Ext}^*_A(M, M) \cong \text{Ext}^*_A(M^*, M^*)$. It follows by using (5.3.1) that

$$Z_A(M) \cong Z_A(M^*)$$

(cf. Proposition 5.7.3]).

5.4. As mentioned in Section 4.2, the algebra $\bar{A}' = \text{gr}_\zeta(u)$ is a quantum complete intersection and so admits a rank variety. We adapt rank varieties for quantum complete intersections to the algebra $\bar{A}$ following Benson, Erdmann and Holloway Sections 4–5. For details on the various constructions the interested reader should refer to loc. cit. (taking note that our algebra $C'$ is denoted “$A$” therein).

One first fixes a block, $e\mathbb{C}E_\zeta$, of a finite group $E_\zeta$ defined in terms of the various powers of $\zeta$ occurring in the commutation relations for the $X_\alpha$ in Section 4.2, where $e$ is a central idempotent. As the ground field is the complex numbers the algebra $e\mathbb{C}E_\zeta$ is a matrix algebra with a unique simple module $S$. Let

$$B = e\mathbb{C}E_\zeta \otimes \bar{A}.$$ 

Since $e\mathbb{C}E_\zeta$ is a matrix algebra there is a Morita equivalence between $\text{Mod}(\bar{A})$ and $\text{Mod}(B)$ given by $M \mapsto S \boxtimes M$, where we write $\boxtimes$ for the outer tensor product of modules. Set $C$ to be the subalgebra of $B$ generated by

$$\{ee_i \otimes X_{\gamma_j}, e \otimes K_j \mid i = 1, 2, \ldots, N, j = 1, 2, \ldots, n\}.$$ 

(5.4.1)

2Denoted by $\Gamma_P M$ in [BIK1].

3Denoted by $\text{supp}_R M$ in [BIK1] Section 5]
and let $C'$ denote the subalgebra of $C$ generated by
\[
\{ ee_i \otimes X_{i u} \mid i = 1, 2, \ldots, N \},
\] (5.4.2)
where the $e_i$ are certain distinguished generators of the finite group $E_C$ and $N = |\Phi^+|$. Now we have a functor
\[
F : \text{Mod}(\bar{A}) \to \text{Mod}(C)
\] (5.4.3)
which is the composition of the Morita equivalence with the restriction functor.

One has that $C'$ is a normal subalgebra of $C$ with $C'/C' \cong u_\zeta(\mathfrak{h})$. Moreover, $C'$ is naturally a subalgebra of $B$ and is isomorphic to a quantum complete intersection. The crucial point is that while $\bar{A}'$ is a quantum complete intersection with various powers of $\zeta$ occurring in the commutation relations given in Section 4.2, by construction the analogous relations for $C'$ have only $\zeta^1$ and $\zeta^{-1}$ appearing. As shown in [BEII], a quantum complete intersection of this type admits a rank variety.

5.5. Before proceeding we relate cohomology and supports for these algebras. For any quantum complete intersection the cohomology ring is explicitly computed in [BO] and is given by generators and relations with the generators in degrees one and two and where the degree one generators are nilpotent. If $D$ denotes $\bar{A}'$ or $C'$ and we write $\text{Ext}^*_{D}(\mathbb{C}, \mathbb{C})$ for the subalgebra of the cohomology ring generated by the generators which lie in degree two, then the inclusion $\text{Ext}^*_D(\mathbb{C}, \mathbb{C}) \hookrightarrow \text{Ext}^*_D(\mathbb{C}, \mathbb{C})$ induces a homeomorphism on the spectrum. We choose to work with $\text{Ext}^*_D(\mathbb{C}, \mathbb{C})$ and note that it is canonically isomorphic to $S^*(u^*)$.

In the case when $D$ equals $\bar{A}$ or $C$ the cohomology ring is also generated in degree two and is canonically isomorphic to $S^*(u^*)$. The proof of this fact uses the same line of reasoning as in the case of $\text{Ext}^*_u(\mathbb{C}, \mathbb{C})$ and, in particular, identifies the cohomology ring of $\bar{A}$ (resp. $C$) with the aforementioned subalgebra of the cohomology ring of $\bar{A}'$ (resp. $C'$).

Consequently, if $D$ denotes any one of these four algebras $\bar{A}, C, \bar{A}'$ or $C'$, we have a canonical identification between $\text{Ext}^*_D(\mathbb{C}, \mathbb{C})$ and $S^*(u^*)$ and so obtain canonical identifications of the following varieties:

\[
\text{Spec} \left( \text{Ext}^*_D(\mathbb{C}, \mathbb{C}) \right) \cong \text{Spec} \left( S^*(u^*) \right).
\] (5.5.1)

Moreover, for $M$ in Stmod($D$) one can define $Z_D(M)$ as in Section 5.1. Using these canonical identifications with subsets of the spectrum of $S^*(u^*)$ allows one to sensibly compare these sets for various algebras and their modules.

Now a direct calculation verifies that $F(\mathbb{C}) = S \boxtimes \mathbb{C}$ is isomorphic to the direct sum of $\dim_{\mathbb{C}}(S)$ copies of $\mathbb{C}$ as a $C$-module. Also, note that the compact objects for $\bar{A}$ and $C$ are generated by the collection of one-dimensional simple modules $\{ \lambda \mid \lambda \in X_1 \}$. An important property about the relationship between the algebras $B$ and $C$ is the following fact. If $M$ is a module for $B$ then $M$ is a direct summand of $B \otimes_{\mathbb{C}} M$ [BEII, Section 3 (Res1)]. Furthermore, the induction functor $B \otimes_{\mathbb{C}} -$ is exact. A consequence of these facts is that the restriction map
\[
\text{Ext}^*_B(Z_1, Z_2) \hookrightarrow \text{Ext}^*_C(Z_1, Z_2)
\] (5.5.2)
is a monomorphism for any $B$-modules $Z_1$ and $Z_2$.

Let $Q_1$ be a finite-dimensional $\bar{A}$-module. Combining the observations in the previous paragraph we have the following maps:
\[
S^*(u^*) \cong \text{Ext}^*_A(\mathbb{C}, \mathbb{C}) \to \text{Ext}^*_A(Q_1, Q_1) \cong \text{Ext}^*_B(S \boxtimes Q_1, S \boxtimes Q_1)
\] (5.5.3)
The last isomorphism is given by the Morita equivalence. Since under the Morita equivalence every finite-dimensional $B$-module is isomorphic to $S \boxtimes Q_1$ for some finite-dimensional $Q_1$, the existence of this homomorphism allows one to define Koszul objects for compact $B$-modules as defined in [BIK1, Definition 5.10]. These objects will be used below by restricting to $C$ and using (5.5.2).

Now let $N$ be a finite-dimensional $C$-module and $M$ be an $\hat{A}$-module. Then under the equivalence of categories:

$$\operatorname{Ext}^\bullet_C(N, S \boxtimes M) \cong \operatorname{Ext}_B^\bullet(B \otimes_C N, S \boxtimes M) \cong \operatorname{Ext}^\bullet_A(Q, M) \cong \operatorname{Ext}^\bullet_A(C, Q^* \otimes M)$$

for some finite-dimensional $\hat{A}$-module $Q$. This allows one to place an action of $R$ on $\operatorname{Ext}^\bullet_C(N, S \boxtimes M)$ and define $Z_C(S \boxtimes M)$. One can then verify the analogues of the results in [BIK1, Section 5] for the algebra $C$. In particular, taking $N = E := \oplus_{\lambda \in X}, \lambda$, one has $\operatorname{Ext}^\bullet_C(E, S \boxtimes M) \cong \operatorname{Ext}^\bullet_A(Q, M)$. Therefore,

$$Z_C(S \boxtimes M) \subseteq Z_A(M).$$

Now for any $M \in \operatorname{Stmod}(\hat{A})$ one has the following composition of maps:

$$\operatorname{Ext}^\bullet_A(E, M) \cong \operatorname{Ext}^\bullet_B(S \boxtimes E, S \boxtimes M) \hookrightarrow \operatorname{Ext}^\bullet_C(S \boxtimes E, S \boxtimes M). \quad (5.5.4)$$

Note that $\operatorname{Ext}^\bullet_C(S \boxtimes E, S \boxtimes M) \cong \operatorname{Ext}^\bullet_C(E, S \boxtimes M) \otimes S^\bullet$. By localizing at a prime $P \in X$, it follows by definition of support that $Z_A(M) \subseteq Z_C(S \boxtimes M)$. Therefore,

$$Z_A(M) = Z_C(S \boxtimes M) \quad (5.5.5)$$

for any $M \in \operatorname{Stmod}(\hat{A})$.

Next we observe that a similar relationship holds between the supports $W_\hat{A}(-)$ and $W_C(-)$; namely, for any $M \in \operatorname{Stmod}(\hat{A})$,

$$W_\hat{A}(M) = W_C(S \boxtimes M). \quad (5.5.6)$$

In order to see this one can use similar reasoning (via the equivalence of categories) to conclude that for any $P \in X$,

$$\operatorname{Ext}^\bullet_C(S \boxtimes E \parallel P, S \boxtimes M) \cong \operatorname{Ext}^\bullet_A(Q, M)$$

for some finite-dimensional $\hat{A}$-module $Q$. It follows that

$$\min_R \operatorname{Ext}^\bullet_C(S \boxtimes E \parallel P, S \boxtimes M) = \min_R \operatorname{Ext}^\bullet_A(Q, M).$$

The definitions of $\parallel P$ and $\min_R$ are given in [BIK1, Definition 5.10] and op. cit. p. 589, respectively. Note that for $C$, Koszul objects can be defined for objects of the form $S \boxtimes M$ where $M$ is an $\hat{A}$-module, by using the iterative construction in $\operatorname{mod}(B)$ and then restricting to $C$. Since $S \boxtimes E$ is a compact generator for $C$, it follows that $W_C(S \boximes M) \subseteq W_\hat{A}(M)$ by [BIK1, Theorem 5.13].

On the other hand, let $P \in W_\hat{A}(M)$. Then $\operatorname{Ext}^\bullet_A(E, \nabla_P(M)) \neq 0$. Therefore, by [BIK1, Proposition 5.2], $\operatorname{Ext}^\bullet_A(E \parallel P, M)_P \neq 0$, and $\operatorname{Ext}^\bullet_C(S \boxtimes [E \parallel P], S \boxtimes M)_P \neq 0$ (via an injection similar to (5.5.4)). Since by (5.5.5) and

$$Z_C(S \boxtimes [E \parallel P]) = Z_A(E \parallel P) = W_\hat{A}(E \parallel P) = \{P' \in X \mid P \subseteq P'\},$$

it follows that $P \in \min_R \operatorname{Ext}^\bullet_C(S \boxtimes [E \parallel P], S \boxtimes M)$. Consequently, $P \in W_C(S \boxtimes M)$ by [BIK1, Theorem 5.13].
5.6. Using the notation established in Section 5.4, we introduce rank varieties for $\bar{A} = \text{gr} u_\zeta(b)$ following [BEH]. Since $C'$ is a quantum complete intersection for which, by design, only $\zeta$ occurs in the commutation relations, one can define a rank variety as follows. Fix a field extension, $K$, of $\mathbb{C}$ of sufficiently large transcendence degree (larger than $|\Phi^+|$ suffices). If $D$ is one of the algebras we consider, write $D_K = K \otimes_{\mathbb{C}} D$. For brevity set $Y_{\gamma_i} = 1 \otimes ee_i \otimes X_{\gamma_i} \in C'_K$, where $ee_i \otimes X_{\gamma_i}$ is as in (5.4.2). Let $V_{C'}^{\text{rank}}(\mathbb{C})$ denote the subspace of $C'_K$ spanned by $\{Y_{\gamma_i} \mid \gamma_i \in \Phi^+\}$. Given an element $x = \sum_i a_i Y_{\gamma_i}$ of $V_{C'}^{\text{rank}}(\mathbb{C})$, write $\langle x \rangle$ for the subalgebra of $C'_K$ it generates.

Given a $C'$-module $M$ set
\[ V_{C'}^{\text{rank}}(M) = \left\{ x \in V_{C'}^{\text{rank}}(\mathbb{C}) \mid K \otimes_{\mathbb{C}} M \text{ is not projective as an } \langle x \rangle\text{-module} \right\} \cup \{0\}. \]

The field extension $K$ is needed to handle the case when $M$ is infinite-dimensional. The rank variety does not depend on the choice of $K$ and so we often leave it implicit in what follows. See [BEH, Section 5] for further details or [BKN, Section 4.5] for the analogue in the setting of Lie superalgebras. An important property of the rank variety is that it satisfies Dade’s Lemma; that is, $M$ is a projective $C'$-module if and only if $V_{C'}^{\text{rank}}(M) = \{0\}$ (see [BEH, Theorem 5.4]).

A $C$-module $M$ is a $C'$-module by restriction and so one can define the rank variety
\[ V_{C'}^{\text{rank}}(M) = V_{C'}^{\text{rank}}(M). \]

Using the fact that $C'$ is a normal subalgebra of $C$, it follows from a spectral sequence argument that a $C$-module $M$ is projective if and only if it is projective upon restriction to $C'$. Hence, the rank variety of $C$ also satisfies Dade’s Lemma.

Recall from Section 5.5 that $S^*(u^*)$ canonically identifies with a subalgebra of $\text{Ext}^*_C(\mathbb{C}, \mathbb{C})$ and so acts on $\text{Ext}^*_C(\mathbb{C}, N)$ for any compact $C'$-module $N$. In particular, one can define $Z_{C'}(N)$ for any such $N$. By [BEH, Proposition 5.1] there is a bijection,
\[ \beta^*: Z_{C'}(\mathbb{C}) \to V_{C'}^{\text{rank}}(\mathbb{C}), \]

given by taking an irreducible subvariety to the point of $V_{C'}^{\text{rank}}(\mathbb{C})$ which corresponds to the unique generic point of the subvariety. The results of Section 5.5 imply that there is a canonical identification of $Z_{C'}(\mathbb{C})$ and $Z_C(\mathbb{C})$ and so we obtain a bijection
\[ \beta^*: Z_C(\mathbb{C}) \to V_{C}^{\text{rank}}(\mathbb{C}). \tag{5.6.1} \]

Given an $\bar{A}$-module, $M$, we set
\[ V_{\bar{A}}^{\text{rank}}(M) = V_{C}^{\text{rank}}(F(M)), \tag{5.6.2} \]
where $F$ is the functor given in (5.4.3). We also note that combining (5.5.5) and (5.6.1) provides a bijection
\[ \beta^*: W_{\bar{A}}(\mathbb{C}) = Z_{\bar{A}}(\mathbb{C}) \to V_{\bar{A}}^{\text{rank}}(\mathbb{C}). \tag{5.6.3} \]

6. Comparison of Supports for $A$ and $\bar{A}$

6.1. We will first consider the stable module category for $C$. Let $\mathcal{V}$ be a specialization closed set in $X$, and let $L_V$ and $\Gamma_V$ be the (co)localization functors as defined in [BIKT, Definition 4.6]. If $M \in \text{Stmod}(C)$ then by [BIKT, Theorem 5.6]
\[ W_C(\Gamma_V(M)) = \mathcal{V} \cap W_C(M) \tag{6.1.1} \]
\[ W_C(L_V(M)) = (X - \mathcal{V}) \cap W_C(M). \] (6.1.2)

Let \( P \in X \) and set
\[
\mathcal{V}(P) = \{ P' \in X \mid P \subseteq P' \},
\]
\[
\mathcal{Z}(P) = \{ P' \in X \mid P' \not\subseteq P \}, \quad \text{so that}
\]
\[
X - \mathcal{Z}(P) = \{ P' \in X \mid P' \not\subseteq P \}.
\]

The points \( x \) in \( V_C^{\text{rank}}(\mathbb{C}) \) correspond to primes in \( \text{Proj}(S^*(u^*)) \). Denote this correspondence by \( \beta(x) = P_x \), with \( \beta^* = \beta^{-1} \).

The following proposition shows that \( V_C^{\text{rank}}(-) \) shares inclusion properties analogous to one direction of the equalities (6.1.1) and (6.1.2) for \( W_C(-) \).

**Proposition 6.1.1.** Let \( \mathcal{V} \) be a specialization closed set in \( X \). If \( M \in \text{Stmod}(C) \) then
\[
\begin{align*}
(a) & \quad V_C^{\text{rank}}(L_V(M)) \subseteq \beta^*(X - \mathcal{V}) \cap V_C^{\text{rank}}(M), \\
(b) & \quad V_C^{\text{rank}}(I_V(M)) \subseteq \beta^*(\mathcal{V}) \cap V_C^{\text{rank}}(M), \\
(c) & \quad V_C^{\text{rank}}(\nabla_P(M)) \subseteq \beta^*(\{P\}) \cap V_C^{\text{rank}}(M).
\end{align*}
\]

**Proof.** Recall \( E = \oplus_{\lambda \in X_1} \lambda \). Let \( \zeta \in K^n \) and
\[
\text{res} : \text{Ext}_{C_K}^\bullet(E, M) \to \text{Ext}_{\langle \zeta \rangle}^\bullet(E, M)
\]
be the restriction map with \( \text{Ext}_{\langle \zeta \rangle}^\bullet(E, M) \cong \text{Ext}_{C_K}^\bullet(\text{coind}_{\langle \zeta \rangle}^C E, M) \). This induces an injection of supports
\[
Z_{\langle \zeta \rangle}(E, M) \hookrightarrow Z_{C_K}(E, M).
\]

Next we claim:
\[
\text{if } \zeta \in V_C^{\text{rank}}(M) \subseteq V_C^{\text{rank}}(M) \text{ then } \text{Ext}_{C_K}^\bullet(E, M)_{P_\zeta} \neq 0. \tag{6.1.3}
\]

One has an inclusion of rings \( S^*(u_C) \subseteq S^*(u_K) \) which induces a map \( \text{Spec}(S^*(u_K)) \to \text{Spec}(S^*(u_C)) \) defined by \( P \to P \cap S^*(u_C) \). By the construction of the generic point (see [BCR] Sections 2 and 3) for details and notation; here again \( N = |\Phi^+| \),
\[
(x_1 - \zeta_1, \ldots, x_N - \zeta_N) \cap S^*(u_C) = P_\zeta.
\]

Now suppose that \( \text{Ext}_{C_K}^\bullet(E, M)_{P_\zeta} = 0 \). Since \( \zeta \in V_C^{\text{rank}}(M) \), it follows that
\[
\text{Ext}_{C_K}^\bullet(E, M)_{(x_1 - \zeta_1, \ldots, x_N - \zeta_N)} \neq 0.
\]

But, \( \text{Ext}_{C_K}^\bullet(E, M) \cong K \otimes \text{Ext}_C^\bullet(E, M) \). Let \( \alpha \otimes m \in K \otimes \text{Ext}_C^\bullet(E, M) \). There exists \( t \in S^*(u_C) - P_\zeta \) with \( (1 \otimes t)(\alpha \otimes m) = 0 \). Now if \( 1 \otimes t \in (x_1 - \zeta_1, \ldots, x_N - \zeta_N) \cap S^*(u_C) = P_\zeta \), which cannot occur. This shows that \( \text{Ext}_{C_K}^\bullet(E, M)_{(x_1 - \zeta_1, \ldots, x_N - \zeta_N)} = 0 \), which is a contradiction. The claim at the beginning of the paragraph now follows.

The proofs of parts (a) and (b) follow in four steps.

(i) First, we show that \( V_C^{\text{rank}}(L_V(M)) \subseteq \beta^*(X - \mathcal{V}) \). Suppose that \( \zeta \in V_C^{\text{rank}}(L_V(M)) \); then by \( (6.1.3) \), \( \text{Ext}_{C_K}^\bullet(E, L_V(M))_{P_\zeta} \neq 0 \). From [BKI] Theorem 4.7, it follows that
\[
\text{Ext}_C^\bullet(E, L_{Z(P_\zeta)}L_V(M)) \neq 0.
\]

Thus, \( L_{Z(P_\zeta)}L_V(M) \neq 0 \).
Using (6.1.2) we have

\[ W_C(L_Z(P_\zeta)L_V(M)) \subseteq (X - Z(P_\zeta)) \cap (X - V) \cap W_C(M). \]

Suppose that \( P_\zeta \in V \). If \( P \in (X - Z(P_\zeta)) \cap (X - V) \) then \( P_\zeta \subseteq P \). But \( V \) is specialization closed so \( P \in V \), which is a contradiction. Therefore, \( W_C(L_Z(P_\zeta)L_V(M)) = \emptyset \); that is, \( L_Z(P_\zeta)(L_V(M)) = 0 \), which also cannot occur. We can now conclude that \( \zeta \in \beta^*(X - V) \).

(ii) Second, \( V_\zeta^{\text{rank}}(I_V(M)) \subseteq \beta^*(V) \). Suppose that \( \zeta \in V_\zeta^{\text{rank}}(I_V(M)) \); then by (6.1.3), \( P_\zeta \in Z_C(E, I_V(M)) \). It follows by (6.1.1) and [BIK1] Theorem 5.15 that \( P_\zeta \in V = \text{cl}(V) \). Consequently, \( \zeta \in \beta^*(V) \).

(iii) Third, \( V_\zeta^{\text{rank}}(I_V(M)) \subseteq V_\zeta^{\text{rank}}(M) \). Suppose that \( \zeta \in V_\zeta^{\text{rank}}(I_V(M)) \). From (ii), \( \zeta \in \beta^*(V) \). One has the distinguished triangle

\[ \to L_V(M) \to M \to I_V(M) \to \quad (6.1.4) \]

From (i), \( L_V(M)|_{\zeta} \) is projective. Hence \( M|_{\zeta} \cong I_V(M)|_{\zeta} \). Therefore, by using the definition of the rank variety, one has \( \zeta \in V_\zeta^{\text{rank}}(M) \). Now (b) follows from (ii) and (iii).

(iv) Finally, we show that \( V_\zeta^{\text{rank}}(L_V(M)) \subseteq V_\zeta^{\text{rank}}(M) \). Using the distinguished triangle (6.1.4) and (iii), one has

\[ V_\zeta^{\text{rank}}(L_V(M)) \subseteq V_\zeta^{\text{rank}}(I_V(M)) \cup V_\zeta^{\text{rank}}(M) \subseteq V_\zeta^{\text{rank}}(M). \]

Together with (i) this proves (a).

(c) From parts (a) and (b), one has

\[
V_\zeta^{\text{rank}}(\nabla_P(M)) \subseteq \beta^*(V(P)) \cap V_\zeta^{\text{rank}}(L_Z(P)(M)) \\
\subseteq \beta^*(V(P)) \cap \beta^*(X - Z(P)) \cap V_\zeta^{\text{rank}}(M) \\
\subseteq \beta^*(\{P\}) \cap V_\zeta^{\text{rank}}(M).
\]

\[ \square \]

### 6.2. Tensor Product Theorems.

For our purposes we will need versions of tensor product theorems for both \( A \) and \( \bar{A} \). We begin with \( \bar{A} \).

**Theorem 6.2.1.** Let \( Q_1 \in \text{stmod}(\bar{A}) \) and \( Q_2 \in \text{Stmod}(\bar{A}) \). If \( Q_1 \otimes Q_2 = 0 \) in \( \text{Stmod}(\Delta) \) then \( W_{\bar{A}}(Q_1) \cap W_{\bar{A}}(Q_2) = \emptyset \).

**Proof.** We will argue by contradiction. Suppose that \( P \in W_{\bar{A}}(Q_1) \cap W_{\bar{A}}(Q_2) \). Then by (5.5.6),

\[ P \in W_C(S \otimes Q_1) \cap W_C(S \otimes Q_2). \]

It follows that for \( j = 1, 2 \) one has \( \nabla_P(S \otimes Q_j) \neq 0 \). Since Dade’s Lemma is true for \( C \) (see Section 5.6), \( V_C^{\text{rank}}(\nabla_P(S \otimes Q_j)) \neq \emptyset \).

From Proposition 6.1.2(c), one has \( V_C^{\text{rank}}(\nabla_P(S \otimes Q_j)) = \{x\} \), where \( \beta^*(P) = \{x\} \).

Combining this with the previous statement shows that

\[ V_C^{\text{rank}}(\nabla_P(S \otimes Q_1)) \cap V_C^{\text{rank}}(\nabla_P(S \otimes Q_2)) = \{x\}. \]

Also by Proposition 6.1.2(c) one has \( V_C^{\text{rank}}(\nabla_P(S \otimes Q_j)) \subseteq V_C^{\text{rank}}(S \otimes Q_j) \). Consequently, \( x \in V_C^{\text{rank}}(S \otimes Q_1) \cap V_C^{\text{rank}}(S \otimes Q_2) \).
Suppose that $\text{Ext}^1_{(x)}(S \boxtimes [Q_1^* \otimes E], S \boxtimes Q_2) = 0$, where, as before, $E = \oplus_{\lambda \in X_1} \lambda$. Then the summand

$$\text{Ext}^1_{(x)}(S \boxtimes Q_1, S \boxtimes Q_2) = 0.$$  \hfill (6.2.1)

By (5.3.4), $Z_A(Q_1) = Z_A(Q_1^*)$. According to (5.5.5) and the main result of [BE], for a finite-dimensional $A$-module $M$, $Z_A(M) = Z_C(S \boxtimes M) \cong V^\text{rank}_C(S \boxtimes M)$.

It now follows that

$$V^\text{rank}_C(S \boxtimes Q_1) = V^\text{rank}_C(S \boxtimes Q_1^*).$$

This shows that $S \boxtimes Q_1$ is projective over $\langle x \rangle$ if and only if $S \boxtimes Q_1^*$ is projective over $\langle x \rangle$. By assumption, $S \boxtimes Q_1$ is not projective over $\langle x \rangle$ so $S \boxtimes Q_1^*$ is not projective over $\langle x \rangle$. Therefore, there exists a non-projective $\langle x \rangle$-summand $Z_1$ of $S \boxtimes Q_1^*$. Similarly, $S \boxtimes Q_2$ has a non-projective $\langle x \rangle$-summand $Z_2$. From (6.2.1), $\text{Ext}^1_{(x)}(Z_1, Z_2) = 0$. This is a contradiction, and hence

$$\text{Ext}^1_{(x)}(S \boxtimes [Q_1^* \otimes E], S \boxtimes Q_2) \neq 0.$$  \hfill (6.2.2)

By the finite-dimensionality of $Q_1$ and Theorem [9.4.1]

$$\text{Ext}^*_A(Q_1^*, Q_2) \cong \text{Ext}^*_A(C, Q_1 \otimes Q_2).$$

Moreover, from the equivalence of categories,

$$\text{Ext}^*_A(Q_1^*, Q_2) \cong \text{Ext}^*_B(S \boxtimes Q_1^*, S \boxtimes Q_2).$$

Therefore, if $Q_1 \otimes Q_2 = 0$ then $\text{Ext}^*_B(S \boxtimes Q_1^*, S \boxtimes Q_2) = 0$.

Let $Q$ be a finite-dimensional $A$-module. Then $\text{Ext}^*_B(S \boxtimes Q^*, S \boxtimes Q^*)$ acts via Yoneda product on $\text{Ext}^*_B(S \boxtimes Q^*, S \boxtimes Q_2)$. Now by using the equivalence and Theorem [9.4.1](b), one has ring isomorphisms

$$\text{Ext}^*_B(S \boxtimes Q^*, S \boxtimes Q^*) \cong \text{Ext}^*_A(Q^*, Q^*) \cong \text{Ext}^*_A(C, Q \otimes Q^*).$$

Moreover, there is a ring homomorphism,

$$S^*(u^*) \cong \text{Ext}^*_A(C, C) \to \text{Ext}^*_A(C, Q \otimes Q^*).$$

Combining these facts, one has a ring homomorphism,

$$S^*(u^*) \to \text{Ext}^*_B(S \boxtimes Q^*, S \boxtimes Q^*)$$

which induces an action of $S(u^*)$ on $\text{Ext}^*_B(S \boxtimes Q^*, S \boxtimes Q_2)$. One also has the following commutative diagram that involves restriction of cohomology:

$$\begin{array}{c}
\text{Ext}^*_B(S \boxtimes Q^*, S \boxtimes Q^*) \\
\cong \not \uparrow \quad \cong \not \uparrow \\
S^*(u^*) \\
\text{res} \quad \text{res} \\
S^*(\langle x \rangle^*)
\end{array}$$

This setup can be now used when $Q = Q_1^*$ (resp. $Q = \oplus_{\lambda \in X_1} \lambda$) to define the topological spaces in the commutative diagram below.
\[
Z_{(x)}(S \boxtimes Q_1^*, S \boxtimes Q_2) \xrightarrow{\text{res}^*} Z_{B_K}(S \boxtimes Q_1^*, S \boxtimes Q_2) \\
\downarrow \\
Z_{(x)}(\oplus_{\lambda \in X_1} S \boxtimes \lambda, S \boxtimes Q_2) \xrightarrow{\text{res}^*} Z_{B_K}(\oplus_{\lambda \in X_1} S \boxtimes \lambda, S \boxtimes Q_2)
\]

The vertical maps are inclusions since \(S \boxtimes Q_1^*\) has a composition series with sections of the form \(S \boxtimes \sigma\) where \(\sigma \in X_1\). The bottom horizontal map

\[
\text{res}^*: Z_{(x)}(\oplus_{\lambda \in X_1} S \boxtimes \lambda, S \boxtimes Q_2) \hookrightarrow Z_{B_K}(\oplus_{\lambda \in X_1} S \boxtimes \lambda, S \boxtimes Q_2)
\]

is an inclusion because

\[
\text{Ext}^\bullet_{(x)}(\oplus_{\lambda \in X_1} S \boxtimes \lambda, S \boxtimes Q_2) \cong \text{Ext}^\bullet_{B_K}(\text{coind}_{(x)}^{B_K}(\oplus_{\lambda \in X_1} S \boxtimes \lambda), S \boxtimes Q_2),
\]

and \(\text{coind}_{(x)}^{B_K}(\oplus_{\lambda \in X_1} S \boxtimes \lambda)\) has a composition series with sections of the form \(S \boxtimes \sigma\) where \(\sigma \in X_1\). Hence the top map \(\text{res}^*\) is also an inclusion. But since \(\text{Ext}^\bullet_{(x)}(S \boxtimes Q_1^*, S \boxtimes Q_2) = 0\), it follows that \(Z_{B_K}(S \boxtimes Q_1^*, S \boxtimes Q_2) = \emptyset\). This yields a contradiction because from (6.2.2), \(Z_{(x)}(S \boxtimes Q_1^*, S \boxtimes Q_2) \neq \emptyset\).

6.3. We also need a version of the tensor product theorem for \(A\). The following theorem generalizes the result stated in [FW, Theorem 2.5] to infinitely generated modules. Recall that for \(\zeta \in H^n(A, \mathbb{C})\), let \(L_\zeta\) be the kernel of the map \(\zeta: \Omega^n(\mathbb{C}) \to \mathbb{C}\) (i.e., the Carlson module). Note that \(L_\zeta\) is a finite dimensional \(A\)-module.

**Theorem 6.3.1.** Let \(M \in \text{Stmod}(A)\), and \(\zeta, \zeta_i \in R\) be homogeneous elements of positive degree \((1 \leq i \leq t)\).

(a) \(W_A(L_\zeta \otimes M) = W_A(L_\zeta) \cap W_A(M)\).

(b) \(W_A(\bigotimes_{i=1}^t L_{\zeta_i} \otimes M) = [\bigcap_{i=1}^t W_A(L_{\zeta_i})] \cap W_A(M)\).

**Proof.** The main verification involves (a) since (b) follows from part (a) by induction. One has

\[
W_A(L_\zeta \otimes M) \subseteq W_A(L_\zeta) \cap W_A(M).
\]

So it remains to prove the other inclusion. Note that

\[
W_A(L_\zeta) = \{ P \in X \mid \zeta \in P\} = \mathcal{V}(\langle \zeta \rangle)
\]

by [FW, Theorem 2.5]. Let \(P \in W_A(L_\zeta) \cap W_A(M)\), and assume that \(P \notin W_A(L_\zeta \otimes M)\).

We will argue by contradiction.

Let \(E = \bigoplus_{\lambda \in X_1} \lambda\) be the direct sum of simple \(A\)-modules. Since \(P \notin W_A(L_\zeta \otimes M)\), \(\nabla_P(L_\zeta \otimes M) = 0\), thus \(\text{Ext}_A^\bullet(E, \nabla_P(L_\zeta \otimes M)) = 0\). From [BKI, Proposition 5.2], it follows that

\[
\text{Ext}_A^\bullet(E \parallel P, \nabla_P(L_\zeta \otimes M))_P = 0.
\]

Consider the short exact sequence \(0 \to L_\zeta \to \Omega^n(\mathbb{C}) \to \mathbb{C} \to 0\) represented by \(\zeta\). One can tensor this sequence with \(M\) and apply the argument in the proof of [FW, Theorem 2.5] to obtain \(\text{Ext}_A^\bullet(E \parallel P, M)_P = \zeta. \text{Ext}_A^\bullet(E \parallel P, M)_P\) for all \(j\). Now using the facts that \(\text{Ext}_A^\bullet(E \parallel P, M)\) is \(P\)-torsion and \(\zeta \in P\) (cf. [BKI, Lemma 5.11]) shows that

\[
\text{Ext}_A^\bullet(E \parallel P, M)_P = 0.
\]
Therefore, Ext^*(E, \nabla_P(M)) = 0, thus P \notin W_A(M), which is a contradiction. Consequently, P \in W_A(L_\xi \otimes M).

6.4. Realizing Supports via Compact Objects in A and A̅. We will use the following statement several times: if M \in Stmod(A) and P \in X then

\[ W_A(\text{gr} \nabla_P(M)) \subseteq \{ P' \in X \mid P' \subseteq P \} = X - \mathcal{Z}(P). \] (6.4.1)

To verify this claim, recall E = \bigoplus_{\lambda \in X_\lambda} \lambda so that W_A(E \parallel P') = \mathcal{V}(P'). Let P' \in W_A(\text{gr} \nabla_P(M)), and suppose that P' is not a subset of P. Then P \notin \mathcal{V}(P'), and

\[ E \parallel P' \otimes \nabla_P(M) = 0. \]

Therefore, gr E \parallel P' \otimes \nabla_P(M) = 0, and by Theorem 6.2.1,

\[ W_A(\text{gr} E \parallel P') \cap W_A(\text{gr} \nabla_P(M)) = \emptyset. \]

This is a contradiction since P' \in W_A(E \parallel P') \subseteq W_A(\text{gr} E \parallel P'). Consequently, P' \subseteq P.

**Proposition 6.4.1.** Let P \in X. Then there exists Y in stmod(A) such that

\[ W_A(Y) = \mathcal{V}(P) = W_A(\text{gr} Y). \]

**Proof.** First we will prove that P \in W_A(\text{gr} \nabla_P(C)) for all P \in X. By (6.4.1),

\[ W_A(\text{gr} \nabla_P(C)) \subseteq \{ P' \mid P' \subseteq P \}. \]

Using the fact that R is Noetherian, we can perform “induction on primes.” If P is a minimal prime then W_A(\text{gr} \nabla_P(C)) = \{ P \} because gr \nabla_P(C) is not projective by Proposition 4.4.1.

Now assume that P' \in W_A(\text{gr} \nabla_P(C)) for all P' \subseteq P, and fix such a P'. Using (6.3.1) and [FW, Theorem 2.5], let Y = \otimes_\xi L_\xi be such that W_A(Y) = \mathcal{V}(P). Using the hypotheses and Theorem 5.3.1 one has Y \otimes \nabla_P(C) = 0. Therefore, gr Y \otimes \nabla_P(C) = 0. By Theorem 6.2.1 it follows that W_A(gr Y) \cap W_A(gr \nabla_P(C)) = \emptyset. This implies that P' \notin W_A(gr Y).

Since by [BIK1, Theorem 5.1.5], P \in W_A(\nabla_P(Y)) = \{ P \} \cap W_A(Y), one has \nabla_P(Y) \cong Y \otimes \nabla_P(C) \neq 0. Therefore, gr Y \otimes \nabla_P(C) \neq 0, and (using [BIK1, Theorem 5.2])

\[ \emptyset \neq W_A(gr Y \otimes \nabla_P(C)) \subseteq W_A(gr Y) \cap W_A(gr \nabla_P(C)) \subseteq \{ P' \in X \mid P' \subseteq P \}. \]

In the previous paragraph, we showed that \{ P' \in X \mid P' \subseteq P \} \cap W_A(gr Y) = \emptyset. Consequently, P \in W_A(gr \nabla_P(C)).

In order to prove the statement of the proposition, note that for Y = \otimes_\xi L_\xi: (i) W_A(Y) = \mathcal{V}(P), and (by 5.1.1) and 5.3.2 (ii) W_A(Y) \subseteq W_A(gr Y). Suppose that P' \in W_A(gr Y) with P' \notin \mathcal{V}(P). Then

\[ W_A(Y \otimes \nabla_{P'}(C)) = \{ P' \} \cap \mathcal{V}(P) = \emptyset. \]

Using Theorem 6.2.1 this implies that

\[ W_A(gr Y) \cap W_A(gr \nabla_{P'}(C)) = \emptyset. \]

This contradicts the fact that P' \in W_A(gr Y) \cap W_A(gr \nabla_{P'}(C)). Hence, W_A(gr Y) = \mathcal{V}(P). □
6.5. The following result will be needed in the verification of Assumption 2.5.1 in Theorem 7.4.1.

**Theorem 6.5.1.** Let $Q \in \text{Stmod}(A)$. Then $W_A(Q) \subseteq W_A(\text{gr } Q)$.

**Proof.** Let $P \in W_A(Q)$. Using Proposition 6.4.1 choose $Y = \otimes_{j=1}^t L_j$ in stmod$(A)$ such that $W_A(Y) = \mathcal{V}(P) = W_A(\text{gr } Y)$. First observe that by Theorems 5.3.1 and 6.3.1

$$W_A(\nabla_P(Y \otimes Q)) = W_A(Y \otimes \nabla_P(Q)) = W_A(Y) \cap W_A(\nabla_P(Q)).$$

Since $P \in W_A(Q)$ and $W_A(Y) = \mathcal{V}(P)$, one has

$$W_A(\nabla_P(Y \otimes Q)) = \{P\}. \quad (6.5.1)$$

On the other hand, by analyzing the graded module and using (6.4.1),

$$W_A(\text{gr } \nabla_P(Y \otimes Q)) = W_A(\text{gr } Y \otimes \nabla_P(Q)) \subseteq W_A(\text{gr } Y) \cap W_A(\text{gr } \nabla_P(Q)) \subseteq \mathcal{V}(P) \cap \{P' \in X \mid P' \subseteq P\} = \{P\}.$$

Since $W_A(\nabla_P(Y \otimes Q)) \neq \emptyset$, it follows that $W_A(\text{gr } Y) \cap W_A(\text{gr } \nabla_P(Q)) = \{P\}$, thus $P \in W_A(\text{gr } \nabla_P(Q))$. Therefore, $P \in W_A(\text{gr } Q)$ because

$$W_A(\text{gr } \nabla_P(Q)) \subseteq W_A(\text{gr } Q) \cap W_A(\text{gr } \nabla_P(\mathcal{C})) \subseteq W_A(\text{gr } Q). \quad \square$$

7. Classification of Tensor Ideals for Quantum Groups

7.1. **Construction of the Support Data.** Let $K = \text{Stmod}(U_\zeta(\mathfrak{g}))$, $K^c = \text{stmod}(U_\zeta(\mathfrak{g}))$ with $\ell > h$. By work of Ginzburg and Kumar [GK Main Theorem], we have

$$R := H^{2\bullet}(u_\zeta(\mathfrak{g}), \mathbb{C}) \cong \mathbb{C}[\mathcal{N}]$$

and $H^{2\bullet+1}(u_\zeta(\mathfrak{g}), \mathbb{C}) = 0$. Let $X = \text{Proj}(\mathbb{C}[\mathcal{N}])$, $X_G = G$-Proj$(\mathbb{C}[\mathcal{N}])$, and let $\rho : X \rightarrow X_G$ be as in Section 2.2. Let $\mathcal{X}_d$ be the set of all closed sets in $X_G$. We will use the cohomology of the small quantum group $u_\zeta(\mathfrak{g})$ to construct a support data from $K^c$ to $\mathcal{X}_d$.

For any $M \in \text{mod}(u_\zeta(\mathfrak{g}))$, set

$$V_{u_\zeta}(M) = \{P \in X \mid \text{Ext}^\bullet_{u_\zeta}(M, M)_P \neq 0\}. \quad (7.1.1)$$

Let $V : K^c \rightarrow \mathcal{X}_d$ be defined as

$$V(M) := \rho\left(V_{u_\zeta}(M)\right), \quad M \in K^c. \quad (7.1.1)$$

Since $M \in K^c$ it follows that $V_{u_\zeta}(M)$ is a closed $G$-stable subset of $X$ and so $V(M) \in \mathcal{X}_d$.

We now proceed to verify that $V : K^c \rightarrow \mathcal{X}_d$ as defined above is a quasi support data in the sense of Section 2.4. For properties (2.4.1)–(2.4.5) one can apply the arguments given in [PW] Section 5).

By [APW2] the Steinberg module and, hence, any projective/injective $U_\zeta(\mathfrak{g})$-module restricts to a projective/injective module for $u_\zeta(\mathfrak{g})$. Conversely, the authors of loc. cit. also prove that any projective/injective $u_\zeta(\mathfrak{g})$-module is the restriction of a projective/injective $U_\zeta(\mathfrak{g})$-module. Consequently, to prove a $U_\zeta(\mathfrak{g})$-module is projective it suffices to verify it
is projective upon restriction to $u_\zeta(g)$. For the sake of self containment, we have included another proof of this fact below.

**Proposition 7.1.1.** Let $M$ be a $U_\zeta(g)$-module. Then $M$ is projective as a $U_\zeta(g)$-module if and only if it is projective as a $u_\zeta(g)$-module.

**Proof.** If $M$ is projective in $\text{Mod}(U_\zeta(g))$ then, as discussed above, it is projective when restricted to $u_\zeta(g)$.

On the other hand, suppose that $M \in \text{Mod}(U_\zeta(g))$ is projective as a $u_\zeta(g)$-module. Let $N$ be an arbitrary module in $\text{Mod}(U_\zeta(g))$. Consider the Lyndon-Hochschild-Serre spectral sequence

$$E_2^{i,j} = \text{Ext}^i_G(\mathbb{C}, \text{Ext}^j_{u_\zeta(g)}(M, N)) \Rightarrow \text{Ext}^{i+j}_{U_\zeta(g)}(M, N).$$

(7.1.2)

Since $M$ is projective over $u_\zeta(g)$ it follows that $E_2^{i,j} = 0$ for $j > 0$. On the other hand, $E_2^{i,j} = 0$ for all $i > 0$ because all rational $G$-modules are completely reducible. Taken together we have that $E_2^{i,j} = 0$ whenever $i + j > 0$. This immediately implies that $\text{Ext}^{n}_{U_\zeta(g)}(M, N) = 0$ for all $n > 0$. That is, $M$ is projective in $\text{Mod}(U_\zeta(g))$. □

Now let $M \in \mathbb{K}^c$. Note that $V(M) = \emptyset$ if and only if $V_{u_\zeta(g)}(M) = \emptyset$, and that $V_{u_\zeta(g)}$ satisfies (2.4.7). Combined with the previous result this justifies (2.4.7) for $V$. That is, the following result holds true.

**Corollary 7.1.2.** Let $M \in \mathbb{K}^c$. Then $V(M) = \emptyset$ if and only if $M = 0$.

Friedlander [Fri] defined the notion of mock injectivity for algebraic group schemes over fields of prime characteristic. Hardesty, Nakano and Sobaje [HNS] proved that there are pure mock injective modules for reductive groups (i.e., mock injective modules that are not injective). It is interesting to note that Proposition 7.1.1 demonstrates that there are no pure mock injective modules for $U_\zeta(g)$.

### 7.2. Realization.

Let $R = \mathbb{H}^2(\zeta(g), \mathbb{C}) \cong \mathbb{C}[\mathcal{N}]$. For $M_1, M_2$ finite-dimensional $u_\zeta(g)$-modules, let

$$V^{\text{max}}(M_1, M_2) = \text{Maxspec}(R/J(M_1, M_2))$$

where $J(M_1, M_2)$ is the annihilator ideal of the action of $R$ on $\text{Ext}^{\bullet}_{u_\zeta(g)}(M_1, M_2)$. The action is obtained by taking an extension class in $R = \text{Ext}^{\bullet}_{u_\zeta(g)}(\mathbb{C}, \mathbb{C})$ and tensoring by $M_2$, then splicing this to an extension class in $\text{Ext}^{\bullet}_{u_\zeta(g)}(M_1, M_2)$. Set $V^{\text{max}}(M_1) = V^{\text{max}}(M_1, M_1)$.

**Proposition 7.2.1.** Let $\mathcal{O} \subset \mathcal{N}$ be a nilpotent $G$-orbit. Then there is a finite-dimensional $U_\zeta(g)$-module $M$ such that $V^{\text{max}}(M) = \mathcal{O}$.

**Proof.** Bezrukavnikov [Bez] proved that there exists a $U_\zeta(g)$-tilting module $T(w \cdot 0)$ (in the principal block) such that $V^{\text{max}}(\mathbb{C}, T(w \cdot 0)) = \mathcal{O}$. We will show that $V^{\text{max}}(T(w \cdot 0)) = \mathcal{O}$.

Since

$$V^{\text{max}}(T(w \cdot 0)) = V^{\text{max}}\left(\bigoplus_{\delta \in \mathcal{X}_1} L(\delta), T(w \cdot 0)\right) = \bigcup_{\delta \in \mathcal{X}_1} V^{\text{max}}(L(\delta), T(w \cdot 0)),$$

(7.2.1)

it follows, since $L(0) = \mathbb{C}$, that

$$\mathcal{O} \subseteq V^{\text{max}}(T(w \cdot 0)).$$

(7.2.2)
To show the reverse inclusion, we will use induced modules and translation functors. Let $C_\ell$ be the bottom alcove in $W_\ell$. Since $\ell > h$ one has $0 \in C_\ell$. For $\lambda, \mu \in \mathcal{C}_\ell$ (closure of the bottom alcove), one can define the translation functor $T^\lambda_\mu$ [Jan2 II 7.6]. Since the translation functor is given by the composition of tensoring by a finite-dimensional $u_\zeta(g)$-module followed by projection onto a summand, one has

$$V^{\text{max}}(T^\lambda_\mu(M_1), T^\lambda_\mu(M_2)) \subseteq V^{\text{max}}(M_1, M_2).$$

Also, using the $R$-action described above, one can show that if $N$ is a finite-dimensional $u_\zeta(g)$-module then

$$V^{\text{max}}(M_1 \otimes N, M_2 \otimes N) \subseteq V^{\text{max}}(M_1, M_2).$$

(7.2.3)

We claim that $V^{\text{max}}(L(\delta), T(w \cdot 0)) \subseteq V^{\text{max}}(\mathbb{C}, T(w \cdot 0))$ for all $\delta \in X_+$. First observe that

$$V^{\text{max}}(H^0(\delta), T(w \cdot 0)) \subseteq V^{\text{max}}(\mathbb{C}, T(w \cdot 0))$$

(7.2.4)

for all $\delta \in W_\ell \cdot 0 \cap X_+$. This is proved by induction on $\delta$. For $\delta = 0$, this holds because $H^0(0) \cong \mathbb{C}$. Suppose [7.2.4] holds for all dominant weights $\delta_1 < \delta$. According to [Jan2], there exists a codimension one facet and a dominant weight $\delta_1 < \delta$ which is a reflection of $\delta$ across this facet [Jan2 II 6.8 Proposition]. In fact, there is a sequence of affine reflections from $\delta$ to $0 \in \mathcal{C}_\ell$ such that at each stage the weight decreases and remains in the dominant chamber.

Consider the composition of the translation functors $T^0_\mu T^0_\delta$ corresponding to the wall crossing functor. Here $\mu$ will be on the wall in the bottom alcove obtained after applying the aforementioned sequence of affine reflections. Then $T^0_\mu T^0_\delta(H^0(\delta_1))$ has a good filtration with two composition factors $H^0(\delta_1)$ and $H^0(\delta)$ [Jan2 II 7.13 Proposition]. In fact, there is a short exact sequence,

$$0 \to H^0(\delta_1) \to T^0_\mu T^0_\delta(H^0(\delta_1)) \to H^0(\delta) \to 0.$$
where all composition factors $L(\delta_1)$ of $Q$ have the property that $\delta_1 < \delta$. Hence, by applying the induction hypothesis and [7.2.4],

$$V^{\max}(L(\delta), T(w \cdot 0)) \subseteq V^{\max}(H^{0}(\delta), T(w \cdot 0)) \cup V^{\max}(Q, T(w \cdot 0)) \subseteq V^{\max}(C, T(w \cdot 0)).$$

Finally, note that, using the Steinberg tensor product theorem, one can show that any simple module $L(\delta)$ in the principal block, where $\delta \in X_1$, can be realized as a $u_\zeta(g)$ direct summand of some $L(\delta')$ where $\delta' \in W_{T} \cap X_1$. Moreover, by linkage all summands of $T(w \cdot 0)$ upon restriction of $u_\zeta(g)$ are in the principal block. Consequently, $V^{\max}(L(\delta), T(w \cdot 0)) \subseteq V^{\max}(C, T(w \cdot 0))$ for all $\delta \in X_1$.

Combining this inclusion with (7.2.1) and (7.2.2), we have $V^{\max}(T(w \cdot 0)) = \mathcal{O}$. $\square$

We have $V^{\max}(M) = V(M) \cap \text{Proj}(\text{Maxspec}(\mathbb{C}[N]))$. Now the realization property [2.48] holds by the argument in [BKN] Section 2.4. We also remark that the calculation of $V^{\max}(T(w \cdot 0))$ given in the proof above can be combined with the description of the tensor ideals in the full subcategory of tilting modules (see Section 8 and [Osi]) to compute the support variety of any tilting module.

### 7.3. Naturality.

Let $R := H^{2*}(u_\zeta(g), \mathbb{C})$ and $S := H^{2*}(u_\zeta(b), \mathbb{C})$ with restriction map $\pi : R \to S$. When $\ell > h$, $R \cong \mathbb{C}[N]$, $S \cong S^*(u^*)$, $u \subseteq N$ and $\pi$ is realized by the restriction of functions. Therefore, in this case $\pi$ is surjective, and the induced map $\pi^* : \text{Proj}(S) \to \text{Proj}(R)$ is a closed injection. For $M \in \text{mod}(u_\zeta(b))$, let

$$V^{\zeta}_{u_\zeta(b)}(M) = \{ P \in \text{Proj}(S) \mid \text{Ext}^{\bullet}_{u_\zeta(b)}(M, P) \neq 0 \}.$$ 

Let $J \subseteq \Pi$ and $p_J = l_J \oplus u_J$ be the corresponding parabolic subalgebra with $u_J$ consisting of negative root vectors. Let $a_J = t \oplus u_J$. As in the case with $b$, for $\ell > h$, $H^{2*}(u_\zeta(a_J), \mathbb{C}) \cong S^*(u^*_J)$ and $H^{2*}(u_\zeta(p_J), \mathbb{C}) \cong \mathbb{C}[P_J \times_B \mathbb{C}]$. Furthermore, one can define support varieties for modules over $u_\zeta(a_J)$ and $u_\zeta(p_J)$.

If $M \in \text{stmod}(u_\zeta(g))$ then under $\pi^*$ we will consider $V^{\zeta}_{u_\zeta(b)}(M)$ as a closed set in $\text{Proj}(R)$. Similarly we view $V^{\zeta}_{u_\zeta(p_J)}(M)$ as a closed set in $\text{Proj}(R)$. The restriction maps

$$\text{res} : H^{2*}(u_\zeta(p_J), \mathbb{C}) \to H^{2*}(u_\zeta(a_J), \mathbb{C})$$

and

$$\text{res} : H^{2*}(u_\zeta(p_J), \mathbb{C}) \to H^{2*}(u_\zeta(b), \mathbb{C})$$

are surjective because they are given by restriction of functions from $\mathbb{C}[P_J \times_B \mathbb{C}]$ to $\mathbb{C}[u_J]$ and $\mathbb{C}[P_J \times_B \mathbb{C}] \to \mathbb{C}[u]$, respectively. Consequently for $N \in \text{stmod}(u_\zeta(p_J)))$ restriction again defines inclusions of $V^{\zeta}_{u_\zeta(a_J)}(N)$ and $V^{\zeta}_{u_\zeta(b)}(N)$ in $\text{Proj}(\mathbb{C}[P_J \times_B \mathbb{C}])$.

One can further refine these inclusions with the following result.

**Proposition 7.3.1.** Let $M \in \text{stmod}(u_\zeta(g))$, $N \in \text{stmod}(u_\zeta(p_J))$. Then one has the following inclusions of support varieties.

(a) $V^{\zeta}_{u_\zeta(b)}(M) \subseteq V^{\zeta}_{u_\zeta(g)}(M) \cap \text{Proj}(S)$;
(b) $V^{\zeta}_{u_\zeta(p_J)}(M) \subseteq V^{\zeta}_{u_\zeta(g)}(M) \cap \text{Proj}(\mathbb{C}[P_J \times_B \mathbb{C}])$;
(c) $V^{\zeta}_{u_\zeta(a_J)}(N) \subseteq V^{\zeta}_{u_\zeta(p_J)}(N) \cap \text{Proj}(S^*(u^*_J))$;
(d) $V^{\zeta}_{u_\zeta(b)}(N) \subseteq V^{\zeta}_{u_\zeta(p_J)}(N) \cap \text{Proj}(S)$. 

Proof. (a) First consider the commutative diagram:

\[
\begin{array}{ccc}
R = H^{2\bullet}(u_\zeta(\mathfrak{g}), \mathbb{C}) & \xrightarrow{\pi} & \text{Ext}^{\bullet}_{u_\zeta(\mathfrak{g})}(M, M) \\
\downarrow & & \downarrow \\
S = H^{2\bullet}(u_\zeta(\mathfrak{b}), \mathbb{C}) & \xrightarrow{\sigma} & \text{Ext}^{\bullet}_{u_\zeta(\mathfrak{b})}(M, M),
\end{array}
\]

where each horizontal map is given by the action of the ring on the identity morphism, and each vertical map is restriction. Let \( J \) where each horizontal map is given by the action of the ring on the identity morphism, and \( J_b(M) \) be the annihilator of \( S \) on \( \text{Ext}^{\bullet}_{u_\zeta(\mathfrak{b})}(M, M) \). The commutativity of the diagram implies that \( \pi(J_b(M)) \subseteq J_b(M) \). Therefore, \( J_b(M) \subseteq \pi^{-1}(J_b(M)) \). Now if \( J_b(M) \subseteq P \) where \( P \in \text{Proj}(S) \) then \( J_b(M) \subseteq \pi^{-1}(P) = \pi^*(P) \). This shows that \( V_{u_\zeta(\mathfrak{b})}(M) \subseteq V_{u_\zeta(\mathfrak{b})}(M) \cap \text{Proj}(S) \). The cases (b)–(d) are proved in a similar manner.

The following proposition is our first result which shows support varieties behave well with respect to restriction to certain quantum subalgebras.

**Proposition 7.3.2.** Let \( \mathfrak{p}_J \) be a parabolic subalgebra in \( \mathfrak{g} \) containing \( \mathfrak{b} \), and let \( N \in \text{stmod}(u_\zeta(\mathfrak{p}_J)) \). Then

\[
V_{u_\zeta(\mathfrak{a}_J)}(N) = V_{u_\zeta(\mathfrak{p}_J)}(N) \cap \text{Proj}(S^\bullet(u_\zeta^J)).
\]

**Proof.** From Proposition 7.3.1(c), \( V_{u_\zeta(\mathfrak{a}_J)}(N) \subseteq V_{u_\zeta(\mathfrak{p}_J)}(N) \cap \text{Proj}(S^\bullet(u_\zeta^J)) \).

We next prove the other inclusion. For \( N \in \text{stmod}(u_\zeta(\mathfrak{p}_J)) \) one has a Lyndon-Hochschild-Serre spectral sequence:

\[
E_2^{ij}(N) = H^i(u_\zeta(\mathfrak{l}_J), H^j(u_\zeta(\mathfrak{u}_J), N^\ast \otimes N)) \Rightarrow H^i(u_\zeta(\mathfrak{p}_J), N^\ast \otimes N)
\]

with an action of \( E_2^{\bullet\bullet}(\mathbb{C}) \) on \( E_2^{\bullet\bullet}(N) \). Furthermore, the (vertical) edge homomorphism for \( N = \mathbb{C} \) can be realized as the restriction map:

\[
H^\bullet(u_\zeta(\mathfrak{p}_J), \mathbb{C}) \rightarrow E_2^{\bullet\bullet}(\mathbb{C}) = H^\bullet(u_\zeta(\mathfrak{u}_J), \mathbb{C})^{u_\zeta(\mathfrak{l}_J)}.
\]

From the discussion in [BNPP, Section 5.6] this edge homomorphism factors through the restriction map \( \text{Res} : H^{2\bullet}(u_\zeta(\mathfrak{p}_J), \mathbb{C}) \rightarrow H^{2\bullet}(u_\zeta(\mathfrak{a}_J), \mathbb{C}) \) and so \( H^{2\bullet}(u_\zeta(\mathfrak{a}_J), \mathbb{C}) \) acts on the pages of the spectral sequence \( E_r^{\bullet\bullet}(N) \).

Next observe

\[
\text{Ext}^{\bullet}_{u_\zeta(\mathfrak{a}_J)}(\oplus \lambda, N^\ast \otimes N) \cong \text{Ext}^{\bullet}_{u_\zeta(\mathfrak{a}_J)}(\mathbb{C}, N^\ast \otimes N), \tag{7.3.1}
\]

where \( \oplus \lambda \) is the direct sum of the simple \( u_\zeta(\mathfrak{a}_J) \)-modules. Let \( P \in \text{Proj}(S^\bullet(u_\zeta^J)) \) with \( P \notin V_{u_\zeta(\mathfrak{a}_J)}(N) \). Then from (7.3.1) we have

\[
E_2^{\bullet}(N)_P = H^i(u_\zeta(\mathfrak{l}_J), H^\bullet(u_\zeta(\mathfrak{u}_J), N^\ast \otimes N)_P) = 0
\]

for all \( i \geq 0 \). Consequently, \( H^\bullet(u_\zeta(\mathfrak{p}_J), N^\ast \otimes N)_P = 0 \). This proves \( V_{u_\zeta(\mathfrak{a}_J)}(N) \supseteq V_{u_\zeta(\mathfrak{p}_J)}(N) \cap \text{Proj}(S^\bullet(u_\zeta^J)) \). \( \square \)

Next we claim \( P \in \pi^{-1}(\text{Spec}(S)) \subseteq \text{Spec}(R) \) if and only if \( Z(P) \subseteq u \). Here \( Z(P) \) is the zero locus of \( P \). To see this, suppose that \( P = \pi^{-1}(P') \supseteq \pi^{-1}(\{0\}) \). Then \( Z(P) \subseteq Z(\pi^{-1}(\{0\})) = u \). Conversely, let \( Z(P) \subseteq u \). Since \( P \) is prime \( Z(P) \) is irreducible and there exists a prime ideal \( P' \in \text{Spec}(S) \) with \( Z(P') = Z(\pi^{-1}(P')) = Z(P) \). Now since \( R \) is
reduced, prime ideals are equal to their radicals and so \( P = \pi^{-1}(P') \). The same proof also shows that \( P \in \pi^{-1}(\text{Spec}(S^*(u_t^*))) \subseteq \text{Spec}(S) \) if and only if \( Z(P) \subseteq u_t \).

With our identifications, we have the following result involving the naturality of support varieties.

**Theorem 7.3.3.** Let \( M \in \text{stmod}(U_\zeta(\mathfrak{g})) \). Then

\[ V_{u_\zeta(\mathfrak{b})}(M) = V_{u_\zeta(\mathfrak{g})}(M) \cap \text{Proj}(S). \]

**Proof.** According to Proposition 7.3.1(a), \( V_{u_\zeta(\mathfrak{b})}(M) \subseteq V_{u_\zeta(\mathfrak{g})}(M) \cap \text{Proj}(S) \). In order to prove the other inclusion, it suffices to verify the following statement:

(*) If \( P \in V_{u_\zeta(\mathfrak{b})}(M) \) (regarded in \( V_{u_\zeta(\mathfrak{g})}(M) \)) and \( wP \in \text{Proj}(S) \) for \( w \in W \) then \( wP \in V_{u_\zeta(\mathfrak{b})}(M) \).

Suppose the statement holds. Let \( P \in V_{u_\zeta(\mathfrak{g})}(M) \cap \text{Proj}(S) \). Since \( G \cdot V_{u_\zeta(\mathfrak{b})}(M) = V_{u_\zeta(\mathfrak{g})}(M) \) [Drin], Theorem 6.1] it follows using the Bruhat decomposition that \( P = (b_1wP_2)b' \) where \( P' \in V_{u_\zeta(\mathfrak{b})}(M) \) and \( b_1, b_2 \in B \). Furthermore, \( b_1^{-1}P = wP_2b'P' \in \text{Proj}(S) \). Set \( P'' = b_2P' \). Since \( \text{Proj}(S) \) is \( B \)-stable it follows that \( wP'' = b_1^{-1}P \in \text{Proj}(S) \). Since \( V_{u_\zeta(\mathfrak{b})}(M) \) is \( B \)-stable one has \( P'' \in V_{u_\zeta(\mathfrak{b})}(M) \). Now by (*), \( wP'' \in V_{u_\zeta(\mathfrak{g})}(M) \), thus \( wP_2b'P' \in V_{u_\zeta(\mathfrak{b})}(M) \) and by \( B \)-stability, \( b_1wP_2b'P' \in V_{u_\zeta(\mathfrak{b})}(M) \). Consequently, \( P \in V_{u_\zeta(\mathfrak{b})}(M) \).

We now prove (*) by induction on the length of \( w \in W \). For \( l(w) = 0 \), the statement is trivially true. If \( l(w) = 1 \) then \( w = s_\alpha \) for some \( \alpha \in \Pi \). Suppose that \( P \in V_{u_\zeta(\mathfrak{b})}(M) \) and \( s_\alpha P \in \text{Proj}(S) \). Then \( Z(s_\alpha P) \subseteq u_\alpha \). This implies that \( Z(P) \subseteq u_\alpha \). It follows that \( s_\alpha P \in \text{Proj}(S^*(u_t^*)) \) and \( s_\alpha P \in V_{u_\zeta(\mathfrak{g})}(M) \cap \text{Proj}(S^*(u_t^*)) \). Therefore by Proposition 7.3.1 it follows \( s_\alpha P \in V_{u_\zeta(a_\alpha)}(M) \). On the other hand, arguing as in the proof of Proposition 7.3.1 shows \( V_{u_\zeta(a_\alpha)}(M) \subseteq V_{u_\zeta(\mathfrak{b})}(M) \) and the desired result follows.

Now assume the statement holds for all \( w' \in W \) with \( l(w') < l(w) = t \). Let \( w = s_{\alpha_t}s_{\alpha_{t-1}} \cdots s_{\alpha_1} \) be a reduced expression. Set \( w' = s_{\alpha_t}s_{\alpha_{t-1}} \cdots s_{\alpha_1} \). Assume \( P \in V_{u_\zeta(\mathfrak{b})}(M) \) and \( wP \in \text{Proj}(S) \). Then \( Z(P) \subseteq u_\alpha \) and \( wP \subseteq u_\alpha \). According to [Lum Corollary 10.2C], \( w(\alpha_1) = s_{\alpha_1}s_{\alpha_2} \cdots s_{\alpha_{t-1}}(\alpha_1) < u_\alpha \), thus \( Z(P) \subseteq u_{(\alpha_1)} \). Consequently, \( Z(s_\alpha P) = Z(s_{\alpha_t}P) \subseteq u_{(\alpha_1)} \subseteq u_\alpha \) and so \( s_\alpha P \in \text{Proj}(S) \). From the induction hypothesis, \( s_{\alpha_t}P \in V_{u_\zeta(\mathfrak{b})}(M) \). Now set \( P' = s_{\alpha_t}P \). Then \( P' \in V_{u_\zeta(\mathfrak{b})}(M) \) and \( wP' = wP \in \text{Proj}(S) \). Again by the induction hypothesis, \( wP' \in V_{u_\zeta(\mathfrak{b})}(M) \). That is, \( wP \in V_{u_\zeta(\mathfrak{b})}(M) \) as desired. \( \square \)

7.4. Verifying Assumption 2.5.1. The following theorem demonstrates that Assumption 2.5.1 holds for \( K \). In the proof we freely make use of the fact that if \( M \) is an object of \( \text{stmod}(U_\zeta(\mathfrak{g})) \) (resp. \( \text{stmod}(U_\zeta(\mathfrak{b})) \)), then \( W_{u_\zeta(\mathfrak{g})}(M) = V_{u_\zeta(\mathfrak{g})}(M) = Z_{u_\zeta(\mathfrak{g})}(M) \) (resp. \( W_{u_\zeta(\mathfrak{b})}(M) = V_{u_\zeta(\mathfrak{b})}(M) = Z_{u_\zeta(\mathfrak{b})}(M) \) [BIK1, Theorem 5.5]. In particular, \( W_{u_\zeta(\mathfrak{g})}(\_\alpha) \) and \( W_{u_\zeta(\mathfrak{b})}(\_\alpha) \) satisfy properties (2.4.1)–(2.4.3) whenever all objects involved are compact. When \( M \) is not compact the containment \( W_{u_\zeta(\mathfrak{b})}(M) \subseteq Z_{u_\zeta(\mathfrak{b})}(M) \) still holds by [BIK1 Remark 5.4]. Furthermore, \( Z_{u_\zeta(\mathfrak{b})}(\_\alpha) \) satisfies properties (2.4.1)–(2.4.4).

**Theorem 7.4.1.** Let \( K = \text{Stmod}(U_\zeta(\mathfrak{g})) \), let \( M \in K^c \) with \( M \neq 0 \), \( I' = \text{Tensor}(M) \), and \( N \in I'_V(M) \) (recall 2.5.1). If \( M \otimes L'_V(N) = 0 \) then \( L'_V(N) = 0 \).
Proof. Set \( A = u_{\zeta}(b) \) and let \( \bar{A} = \text{gr} \ A \) be the associated graded algebra as described in Section 4.2. Combining Theorem 4.1.1 and the discussion at the end of Section 3.2, it suffices to show that \( L_{\Gamma}(N) = 0 \) in \( \text{Stmod}(A) \).

From our assumption, \( M \otimes L_{\Gamma}(N) = 0 \) and so \( \text{gr} (M \otimes L_{\Gamma}(N)) = 0 \) in \( \text{Stmod}(\bar{A}) \). Therefore,

\[
\text{gr} M \otimes \text{gr} L_{\Gamma}(N) = 0
\]
in \( \text{Stmod}(\Delta) \). According to Theorem 6.2.1,

\[
W_{A}(\text{gr} M) \cap W_{A}(\text{gr} L_{\Gamma}(N)) = \emptyset.
\]

Applying Theorem 6.5.1 yields,

\[
W_{A}(M) \cap W_{A}(L_{\Gamma}(N)) = \emptyset. \tag{7.4.1}
\]

Now suppose that \( P \notin W_{A}(M) \). Set \( I = I_{V_{(M)}} \). We first claim that \( P \notin W_{A}(Q) = V_{A}(Q) = Z_{A}(Q) \) for any \( Q \in I \). For suppose that \( P \in W_{A}(Q) = V_{A}(Q) \) for some \( Q \in I \). Then

\[
P \in V_{A}(Q) \subseteq V_{u_{\zeta}(g)}(Q) \subseteq V_{u_{\zeta}(g)}(M) = G \cdot V_{A}(M).
\]

The first containment is Theorem 7.3.3, the second holds because \( Q \in I \) and by definition of \( I_{V_{(M)}} \), and the equality follows from [Dru] Theorem 6.1. Thus there exists \( P' \in V_{A}(M) \) with \( P = g \cdot P' \) for some \( g \in G \). Using the Bruhat decomposition: \( G = \bigcup_{w \in W} BwB \), it follows that \( P = bwbl' \cdot P' \) for some \( b, b' \in B \) and \( w \in W \).

Since \( V_{A}(M) \) and \( V_{A}(Q) \) are \( B \)-invariant, we have \( \bar{P} := b' \cdot P' \in V_{A}(M) \) with \( \bar{w} \bar{P} = b^{-1} \cdot P \in V_{A}(Q) \subseteq \text{Proj}(S) \). On the other hand \( w \bar{P} \in G \cdot V_{A}(M) = V_{u_{\zeta}(g)}(M) \). Taken together with Theorem 7.3.3

\[
w \bar{P} \in V_{u_{\zeta}(g)}(M) \cap \text{Proj}(S) = V_{A}(M).
\]

That is, \( b^{-1} \cdot P \in V_{A}(M) \) and, by the \( B \)-stability of \( V_{A}(M) \), \( P \in V_{A}(M) \). Since by assumption \( P \notin W_{A}(M) \), it must be that \( P \notin W_{A}(Q) \) for any \( Q \in I \).

As in the proof of Proposition 2.6.1 we have \( Y' \subseteq I \) and \( Q \in I \) implies \( L_{\Gamma}(Q) \in \text{Loc}(I) \). Since \( W_{A}(-) \) satisfies (2.4.1)–(2.4.4), the preceding result along with [BKN] Lemma 2.4.1 implies that \( P \notin W_{A}(L_{\Gamma}(Q)) \) for all \( Q \in I \); and in particular, for \( Q = N \). That is, we have shown that \( W_{A}(L_{\Gamma}(N)) \subseteq W_{A}(M) \).

Combining this inclusion with (7.4.1) yields \( W_{A}(L_{\Gamma}(N)) = \emptyset \) and so \( L_{\Gamma}(N) = 0 \) by [BKI] Theorem 5.2. \( \square \)

7.5. Classification of Thick Tensor Ideals. With our verifications in the prior sections we can present a complete classification of the (thick) tensor ideals in the category \( \text{stmod}(U_{\zeta}(\mathfrak{g})) \) using Theorem 2.8.1

**Theorem 7.5.1.** Let \( G \) be a complex simple algebraic group over \( \mathbb{C} \) with \( \mathfrak{g} = \text{Lie} G \). Assume that \( \zeta \) is a primitive \( \ell \)-th root of unity where \( \ell > h \). Let \( K = \text{Stmod}(U_{\zeta}(\mathfrak{g})) \), \( K^{c} = \text{stmod}(U_{\zeta}(\mathfrak{g})) \) and \( X = G \cdot \text{Proj}(\mathbb{C}[N]) \). Adopt the notation of Section 2.2. Let \( V : K^{c} \to X \) be the quasi support data on \( K^{c} \) defined in (7.1.7). There is a pair of mutually inverse maps

\[
\{ \text{thick tensor ideals of } K^{c} \} \overset{\Gamma}{\overset{\sim}{\rightarrow}} X_{sp},
\]
where $I_W = \{ M \in K^c | V(M) \subseteq W \}$.

7.6. Computation of $\text{Spec}(\text{stmod}(U_{\zeta}(g)))$. We can apply Theorem 2.8.2 to identify the Balmer spectrum for the stable module category for the quantum group $U_{\zeta}(g)$.

**Theorem 7.6.1.** Let $G$ be a complex simple algebraic group over $\mathbb{C}$ with $\mathfrak{g} = \text{Lie } G$. Assume that $\zeta$ is a primitive $\ell$th root of unity where $\ell > h$. Then there is a homeomorphism

$$\text{Spec}(\text{stmod}(U_{\zeta}(g))) \cong G-\text{Proj}(\mathbb{C}[\mathcal{N}]).$$

7.7. The category $\text{mod}(U_{\zeta}(g))$. Let $R = \mathbb{C}[\mathcal{N}]$. Following the work in [Lor1] and [Lor2], the rational ideals of $R$ are precisely the maximal ideals of $R$. Furthermore, the points in $G-\text{MaxSpec}(\mathbb{C}[\mathcal{N}])$ are in bijective correspondence with $G$-orbits in $\text{MaxSpec}(\mathbb{C}[\mathcal{N}])$ (i.e., the nilpotent orbits in $\mathcal{N}$). It follows that $G$-\text{MaxSpec}(\mathbb{C}[\mathcal{N}])$ is finite and

$$G-\text{MaxSpec}(\mathbb{C}[\mathcal{N}]) = G-\text{Spec}(\mathbb{C}[\mathcal{N}])$$

by [Lor2] Proposition 1. Furthermore, $\rho$ defines an inclusion preserving bijection between the $G$-stable closed sets of $\mathcal{N}$ and the closed sets of $G-\text{Spec}(\mathbb{C}[\mathcal{N}])$ (see [BKN, Section 2.3]). This allows us to lift results from $K^c$ to $\text{mod}(U_{\zeta}(g))$.

**Proposition 7.7.1.** Let $M$ and $N$ be modules in $\text{mod}(U_{\zeta}(g))$. Then

$$V_{u_{\zeta}(g)}(M \otimes N) = V_{u_{\zeta}(g)}(M) \cap V_{u_{\zeta}(g)}(N).$$

**Proof.** From the fact that $V$ satisfies the tensor product property (Proposition 2.7.2) and $V_{u_{\zeta}(g)}$ satisfies (2.4.4), there is the following chain of containments:

$$\rho \left( V_{u_{\zeta}(g)}(M) \cap V_{u_{\zeta}(g)}(N) \right) \subseteq \rho \left( V_{u_{\zeta}(g)}(M) \right) \cap \rho \left( V_{u_{\zeta}(g)}(N) \right) = V(M) \cap V(N) = V(M \otimes N) = \rho \left( V_{u_{\zeta}(g)}(M \otimes N) \right) \subseteq \rho \left( V_{u_{\zeta}(g)}(M) \cap V_{u_{\zeta}(g)}(N) \right)$$

In particular, $\rho(V_{u_{\zeta}(g)}(M) \cap V_{u_{\zeta}(g)}(N)) = \rho(V_{u_{\zeta}(g)}(M \otimes N))$. However, both $V_{u_{\zeta}(g)}(M) \cap V_{u_{\zeta}(g)}(N)$ and $V_{u_{\zeta}(g)}(M \otimes N)$ are $G$-stable closed sets in $\mathcal{N}$ and $\rho$ is a bijection on such sets and so they must be equal. \(\square\)

We call a full subcategory $I$ of $\text{mod}(U_{\zeta}(g))$ a thick tensor ideal if 1) given any $M \in I$ and $N \in \text{mod}(U_{\zeta}(g))$, $M \otimes N \in I$, and 2) if $M \oplus N \in I$, then $M$ and $N$ are in $I$, and 3) if $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is an exact sequence with two of $A, B, C$ in $I$, then the third is in $I$. We call a proper tensor ideal prime if $M \otimes N \in I$ implies $M \in I$ or $N \in I$ for all $M, N \in \text{mod}(U_{\zeta}(g))$. From Theorem 7.5.1 we can also classify the thick tensor ideals and thick prime ideals in $\text{mod}(U_{\zeta}(g))$. 

\begin{equation}
\Gamma(I) = \bigcup_{M \in I} V(M), \quad \Theta(W) = I_W,
\end{equation}

\begin{equation}
\text{where } I_W = \{ M \in K^c | V(M) \subseteq W \}.
\end{equation}
The thick tensor ideals are ordered by inclusion as are the $G$-stable closed subsets of $\mathcal{N}$. Recall that there is also a partial order on the nilpotent orbits given by declaring $G.x \leq G.y$ (for $x, y \in \mathcal{N}$) if and only if $G.x \subseteq G.y$.

**Corollary 7.7.2.** Let $G$ be a complex simple algebraic group over $\mathbb{C}$ with $\mathfrak{g} = \text{Lie} G$. Assume that $\zeta$ is a primitive $\ell$th root of unity where $\ell > h$. Then there exist order preserving bijective correspondences

$$\{\text{thick tensor ideals of } \text{mod}(U_\zeta(\mathfrak{g}))\} \Leftrightarrow \{G$-stable closed subsets of $\mathcal{N}\},$$

and

$$\{\text{nonzero thick prime tensor ideals of } \text{mod}(U_\zeta(\mathfrak{g}))\} \Leftrightarrow \{\text{nilpotent } G\text{-orbits in } \mathcal{N}\}.$$ 

**Proof.** First, note that the zero ideal is a prime thick tensor ideal of $\text{mod}(U_\zeta(\mathfrak{g}))$ which is contained in every other thick tensor ideal.

Second, the full subcategory $\mathcal{P}$ of all projective modules in $\text{mod}(U_\zeta(\mathfrak{g}))$ is a prime thick tensor ideal. That $\mathcal{P}$ is prime follows from Proposition 7.7.1 the fact that $\mathcal{N}$ has a unique minimal nonzero orbit, and that $V_{U_\zeta(\mathfrak{g})}(M) = \emptyset$ if and only if $M$ is projective.

Furthermore, note that $\mathcal{P}$ is contained in every nonzero thick tensor ideal $I$ of $\text{mod}(U_\zeta(\mathfrak{g}))$. Namely, given any nonzero $M \in I$ and nonzero $P \in \mathcal{P}$, $M \otimes P \in I \cap \mathcal{P}$ and thus $I$ contains nonzero projective modules. Thus it suffices to show that any nonzero $Q$ in $\mathcal{P}$ generates $\mathcal{P}$ as a thick tensor ideal. If we write $\text{ev} : Q^* \otimes Q \rightarrow \mathbb{C}$ for the evaluation homomorphism, then there is a surjective homomorphism $\text{ev} \otimes 1 : Q^* \otimes Q \otimes P \rightarrow P$ for any $P \in \mathcal{P}$. Since $P$ is projective this map splits and so $P$ is isomorphic to a direct summand of $Q^* \otimes Q \otimes P$ and thus every projective module is contained in the thick tensor ideal generated by $Q$ as needed.

We now observe that there is a one-to-one correspondence between the nonzero thick tensor ideals of $\text{mod}(U_\zeta(\mathfrak{g}))$ and the thick tensor ideals of $\mathbf{K}^c = \text{Stab}(\text{mod}(U_\zeta(\mathfrak{g})))$. Namely, given a nonzero thick tensor ideal $I$ of $\text{mod}(U_\zeta(\mathfrak{g}))$, let $I'$ denote the full subcategory of $\mathbf{K}^c$ whose objects are those of $I$. Using that $\mathcal{P} \subseteq I$ and the defining properties of a thick tensor ideal in $\text{mod}(U_\zeta(\mathfrak{g}))$ it follows that $I'$ is a thick tensor ideal of $\mathbf{K}^c$. Conversely, given a thick tensor ideal, $J'$, of $\mathbf{K}^c$, let $J$ be the full subcategory of $\text{mod}(U_\zeta(\mathfrak{g}))$ whose objects are the objects of $J'$. This forms a nonzero thick tensor ideal in $\text{mod}(U_\zeta(\mathfrak{g}))$. These provide mutually inverse, order preserving bijections between the nonzero thick tensor ideals of $\text{mod}(U_\zeta(\mathfrak{g}))$ and the thick tensor ideals of of $\mathbf{K}^c$. Combining this with the discussion at the beginning of this section and the bijection between the closed subsets of $G\text{-Proj}(\mathbb{C}[\mathcal{N}])$ and the nonzero $G$-stable closed subsets of $\mathcal{N}$ given by $\rho$ yields the first correspondence, after modifying the bijection slightly by sending $\mathcal{P}$ to $\{0\} \subset \mathcal{N}$ (instead of to $\emptyset$), and extending it by sending the zero ideal to $\emptyset \subset \mathcal{N}$.

Moreover, the above bijection between the nonzero thick tensor ideals of $\text{mod}(U_\zeta(\mathfrak{g}))$ and the thick tensor ideals of $\mathbf{K}^c$ restricts to a bijection between the set of prime ideals of $\mathbf{K}^c$ and the set of nonzero prime ideals of $\text{mod}(U_\zeta(\mathfrak{g}))$. That is, there is a bijection between the nonzero prime ideals of $\text{mod}(U_\zeta(\mathfrak{g}))$ and the irreducible $G$-stable closed subsets of $\mathcal{N}$ and, hence, with the nilpotent $G$-orbits in $\mathcal{N}$. □
8. Connections with Tilting Modules

8.1. Let $T$ be the category of finite-dimensional tilting modules for the braided monoidal tensor category $\text{mod}(U_\zeta(\mathfrak{g}))$. As it is closed under the tensor product and contains the unit object, $T$ is itself a braided monoidal tensor category. Write $T(\lambda)$ for the indecomposable tilting module of highest weight $\lambda \in X_+$, and $\text{Tensor}_T(T)$ for the thick tensor ideal in $T$. Ostrik provided a classification of the thick tensor ideals in $T$. Combined with Bezrukavnikov’s work on the support varieties of tilting modules we can state the following result.

Theorem 8.1.1. \cite{Ost, Bez} Let $\lambda, \mu \in X_+$.

(a) There exist bijective correspondences

$$\{\text{Tensor}_T(T(\lambda)) \mid \lambda \in X_+\} \leftrightarrow \{\text{nilpotent orbits of } G\} \leftrightarrow \{\text{two-sided cells of } W_{\ell}\}.$$ 

(b) $T(\mu) \in \text{Tensor}_T(T(\lambda))$ if and only if $V(T(\mu)) \subseteq V(T(\lambda))$.

8.2. It is interesting to compare Ostrik’s classification with our results presented in Corollary \ref{corollary-7.7.2}. Indeed, the following results indicates that $R := \text{mod}(U_\zeta(\mathfrak{g}))$ is an “integral extension” of $T$.

Theorem 8.2.1. Let $\lambda, \mu \in X_+$. Then

(a) $\text{Tensor}_T(T(\lambda)) = \text{Tensor}_R(T(\lambda)) \cap T$.

(b) If $I$ is a thick prime tensor ideal in $R$, then $I = \text{Tensor}_R(T(\lambda))$ for some $\lambda \in X_+$.

(c) $\text{Tensor}_R(T(\lambda)) = \text{Tensor}_R(T(\mu))$ if and only if $V(T(\lambda)) = V(T(\mu))$.

Proof. For part (a), by using the definitions, it is clear that

$$\text{Tensor}_T(T(\lambda)) \subseteq \text{Tensor}_R(T(\lambda)) \cap T.$$ 

Now suppose that $T(\mu) \in \text{Tensor}_R(T(\lambda)) \cap T$. Then $V(T(\mu)) \subseteq V(T(\lambda))$. Now one can apply Theorem \ref{theorem-8.1.1}(b) to see that $T(\mu) \in \text{Tensor}_T(T(\lambda))$.

Parts (b) and (c) are consequences of part (a), Theorem \ref{theorem-7.5.1}, Corollary \ref{corollary-7.7.2} and Theorem \ref{theorem-8.1.1}.

9. Appendix: Graded Algebras and Modules

9.1. Let $A$ be a finite-dimensional algebra with an increasing filtration:

$$A_0 \subseteq A_1 \subseteq \cdots \subseteq A_N = A$$

which is multiplicative (i.e., $A_i A_j \subseteq A_{i+j}$). Moreover, let

$$\text{gr } A = A_0 \oplus A_1/A_0 \oplus \cdots \oplus A_N/A_{N-1}$$

be the associated graded algebra.

If $M$ is an $A$-module then one can construct $\text{gr } M$ which is a gr $A$-module as follows. Fix a set of generators $\{m_i \mid i \in I\}$ of $M$ and set $M_i = \sum_{i \in I} A_j m_i$. One obtains an increasing filtration

$$M_0 \subseteq M_1 \subseteq \cdots \subseteq M_N = M,$$

with the property that $A_i \cdot M_j \subseteq M_{i+j}$. Set

$$\text{gr } M = M_0 \oplus M_1/M_0 \oplus \cdots \oplus M_N/M_{N-1}$$
which is a gr $A$-module.

9.2. Now let $A$ and $A'$ be finite-dimensional algebras with multiplicative filtrations given by $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_N = A$ and $A'_0 \subseteq A'_1 \subseteq \cdots \subseteq A'_{N'} = A'$. Let $B = A \otimes A'$ and $B_t = \sum_{i+j=t} A_i \otimes A'_j$. Then we have a multiplicative filtration $B_0 \subseteq B_1 \subseteq \cdots \subseteq B_{N+N'} = B$ for the algebra $B$. One can consider the associated graded algebra $\text{gr } B$.

The multiplication on $\text{gr } B$ (i.e., $B_s/B_{s-1} \times B_t/B_{t-1} \to B_{s+t}/B_{s+t-1}$) is given by the formula

\[
[(x \otimes x') + B_{s-1}][(y \otimes y') + B_{t-1}] = (xy \otimes x'y') + B_{s+t-1}.
\]

Suppose we have $x \in A_i$, $y \in A_j$ with $i + j = s$ and $x' \in A_{i'}$, $y' \in A_{j'}$ with $i' + j' = t$. Then replacing $x$ by another element in $A_i$ modulo $A_{i-1}$ does not change the product. The analogous statement is true for $y$, $x'$ and $y'$. Therefore, one has a canonical isomorphism

\[
\text{gr}(A \otimes A') \cong \text{gr } A \otimes \text{gr } A'.
\]

Similarly, if $M$ (resp. $M'$) is an $A$-module (resp. $A'$-module), using the associated filtrations one obtains

\[
\text{gr}(M \boxtimes M') \cong \text{gr } M \boxtimes \text{gr } M'
\]  \hspace{1cm} (9.2.1)

as $\text{gr } A \otimes \text{gr } A'$-modules, where $\boxtimes$ denotes the outer tensor product.

Let $C$ be a subalgebra of $B$. The multiplicative filtration on $B$ induces a multiplicative filtration on $C$ by letting $C_i = C \cap B_i$ for all $i$. One can consider the associated graded algebras, and there is an injection of subalgebras: $\text{gr } C \subseteq \text{gr } B$.

9.3. Spectral Sequences. Let $A$ be a finite-dimensional augmented algebra with augmentation ideal $A_+$. Following the construction in [BNPP Lemma 5.6.1] and [Ba], let $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_N = A$ be an increasing filtration on $A$, $M$ be an $A$-module with induced filtration $M_0 \subseteq M_1 \subseteq \cdots \subseteq M_N = M$.

Let $C^\bullet(A, M)$ and $C^\bullet(\text{gr } A, \text{gr } M)$ be the complexes obtained by taking duals of the respective reduced bar resolutions. That is $C^n(A, M) = \text{Hom}_C((A_+)^{\otimes n} \otimes M, C)$ and $C^n(\text{gr } A, \text{gr } M) = \text{Hom}_C((\text{gr } A_+)^{\otimes n} \otimes \text{gr } M, C)$. Set $A_{+j} = A_j \cap A_+$.

The filtrations above induce a downward filtration on the complex $C^\bullet(A, M)$ as follows. Define

\[
B^n_{[<t]} = \sum_{\eta + \sum \gamma_i < t} A_{+\gamma_1} \otimes A_{+\gamma_2} \otimes \cdots \otimes A_{+\gamma_n} \otimes M_\eta,
\]

\[
B^n_{[\leq t]} = \sum_{\eta + \sum \gamma_i \leq t} A_{+\gamma_1} \otimes A_{+\gamma_2} \otimes \cdots \otimes A_{+\gamma_n} \otimes M_\eta.
\]

Now observe that $B^n_{[<t]} \subseteq B^n_{[\leq t]}$. Set

\[
C^n(A, M)_{[<t]} = \text{Hom}_C((A_+^{\otimes n} \otimes M)/B^n_{[<t]}, C), \quad C^n(A, M)_{[\leq t]} = \text{Hom}_C((A_+^{\otimes n} \otimes M)/B^n_{[\leq t]}, C);
\]

then $C^n(A, M)_{[<t]} \subseteq C^n(A, M)_{[\leq t]}$. Moreover, if $s, t \in \mathbb{N}$ with $s < t$ then $C^n(A, M)_{[<t]} \subseteq C^n(A, M)_{[<s]}$.

The grading on $\text{gr } A$ leads in a natural way to a grading of the complex $C^\bullet(\text{gr } A, \text{gr } M)$. Let $C^\bullet(\text{gr } A, \text{gr } M)_{[t]}$ denote the graded component corresponding to $t$; then we can identify $C^\bullet(A, M)_{[<t]}/C^\bullet(A, M)_{[\leq t]}$ with $C^\bullet(\text{gr } A, \text{gr } M)_{[t]}$. From these filtrations on the cobar complexes we obtain the following result.
**Proposition 9.3.1.** Let $A$ be a finite-dimensional augmented algebra with a multiplicative filtration $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_N = A$, and let $M$ be an $A$-module. There exists a spectral sequence

$$E_1^{ij}(M) = \text{Ext}^{i+j}_{\text{gr}A}(\mathbb{C}, \text{gr}M)_{[i]} \Rightarrow \text{Ext}^{i+j}_{A}(\mathbb{C}, M).$$

Furthermore, for each $r$, $E_r^{\bullet\bullet}(M)$ is a differential graded module over the differential graded algebra $E_r^{\bullet\bullet}(\mathbb{C})$.

9.4. In this section, let $A = u_{\zeta}(b)$ and $\bar{A} = \text{gr} A$. The following theorem can be regarded as an adjointness statement between extension groups for $\bar{A}$ and $\Delta$.

**Theorem 9.4.1.** Let $Q_1$ be a finite-dimensional module for $\bar{A}$ and $Q_2$ be an arbitrary $A$-module. For $n \geq 0$ there exists an isomorphism

$$\phi_n : \text{Ext}^n_A(Q_1^*, Q_2) \rightarrow \text{Ext}^n_{\Delta}(\mathbb{C}, Q_1 \otimes Q_2).$$

**Proof.** First we need to construct the maps $\phi_n$. Let $\zeta$ be an element of $\text{Ext}^n_A(Q_1^*, Q_2)$ which, when $n > 0$, is represented by an $n$-fold $\bar{A}$-extension:

$$0 \rightarrow Q_2 \rightarrow T_1 \rightarrow T_2 \rightarrow \cdots \rightarrow T_n \rightarrow Q_1^* \rightarrow 0.$$

By tensoring with $Q_1$, one gets an $n$-fold $\bar{A} \otimes \bar{A}$-extension that one can regard as a $\Delta$-extension via restriction:

$$0 \rightarrow Q_1 \otimes Q_2 \rightarrow Q_1 \otimes T_1 \rightarrow Q_1 \otimes T_2 \rightarrow \cdots \rightarrow Q_1 \otimes T_n \rightarrow Q_1 \otimes Q_1^* \rightarrow 0.$$

This gives us a map $\tilde{\phi}_n : \text{Ext}^n_A(Q_1^*, Q_2) \rightarrow \text{Ext}^n_{\Delta}(Q_1 \otimes Q_1^*, Q_1 \otimes Q_2)$. We have a $\Delta$-map $\mathbb{C} \hookrightarrow Q_1 \otimes Q_1^*$. This induces a map on extension groups

$$\tilde{\phi}_n : \text{Ext}^n_A(Q_1 \otimes Q_1^*, Q_1 \otimes Q_2) \rightarrow \text{Ext}^n_{\Delta}(\mathbb{C}, Q_1 \otimes Q_2).$$

Now one can set $\phi_n = \tilde{\phi}_n \circ \tilde{\phi}_n$. When $n = 0$, $\zeta \in \text{Hom}_{\bar{A}}(Q_1^*, Q_2)$, and we can define $\phi_0(\zeta)$ to be the composition of $\mathbb{C} \hookrightarrow Q_1 \otimes Q_1^*$ followed by $1 \otimes \zeta : Q_1 \otimes Q_1^* \rightarrow Q_1 \otimes Q_2$.

The proof that $\phi_n$ is injective will follow via the first five steps, below. Surjectivity will be handled with two further steps.

1. We claim that for $Q_1 \cong \lambda$,

$$\phi_0 : \text{Hom}_{\bar{A}}(\lambda^*, Q_2) \rightarrow \text{Hom}_{\Delta}(\mathbb{C}, \lambda \otimes Q_2).$$

is an isomorphism.

Consider $A' = u_{\zeta}(u)$ and $\bar{A}' = \text{gr} A'$. Note that $\bar{A}'$ acts trivially on $\lambda$. By comparing the action of $\bar{A}$ on $Q_2$ and $\Delta$ on $\lambda \otimes Q_2$, one can see that the fixed points under $A'$ are the same. The result now follows by considering the weights on the set of fixed points.

2. Next we prove that for $Q_1$ a finite-dimensional $\bar{A}$-module,

$$\phi_0 : \text{Hom}_{\bar{A}}(Q_1^*, Q_2) \rightarrow \text{Hom}_{\Delta}(\mathbb{C}, Q_1 \otimes Q_2).$$

is injective.

Consider the following diagram with exact rows, induced by the short exact sequences

$$0 \rightarrow \lambda \rightarrow Q_1 \rightarrow N \rightarrow 0$$

and

$$0 \rightarrow N^* \rightarrow Q_1^* \rightarrow \lambda^* \rightarrow 0 :$$
One can verify directly that the diagram commutes, and hence rows are short exact sequences. Since \( Q(3) \) we claim that if \( \dim Q \) will lie in a finite-dimensional submodule of \( f \). Then \( Q \) and the injective hull of \( Q \) a finite-dimensional submodule of \( Q \) that \( \lambda \) is the injective hull of \( \lambda \) is locally finite, we may assume that the image of \( f \) is finite-dimensional. From (2), we know that \( \pi(1) \) is injective. Suppose that \( \phi_1(x) = 0 \). Since \( \pi(1) \) is surjective there exists \( z \) such that \( \pi(1)(z) = x \). Now by commutativity, 

\[
\pi_2(\phi_2'(z)) = \phi_1(\pi_1(z)) = \phi_1(x) = 0.
\]

(3) We claim that if \( Q_1 \) is injective as \( \bar{A} \)-module then

\[
\phi_0 : \text{Hom}_A(Q_1^*, Q_2) \rightarrow \text{Hom}_A(\mathbb{C}, Q_1 \otimes Q_2).
\]

is an isomorphism.

From (2), we know that \( \phi_0 \) is injective. For any \( f \in \text{Hom}_A(Q_1, Q_2) \), the image of \( f \) will lie in a finite-dimensional submodule of \( Q_1 \otimes Q_2 \). Since \( Q_1 \) is finite-dimensional and \( Q_2 \) is locally finite, we may assume that the image of \( f \) is contained in \( Q_1 \otimes \mathbb{C} \) where \( \mathbb{C} \) is a finite-dimensional submodule of \( Q_2 \). So for (3), we may assume without loss of generality that \( Q_2 \) is finite-dimensional.

First one can reduce to the the case when \( Q_1 \) is the injective hull of \( \lambda \in X_1 \) as \( \bar{A} \)-module. Then \( Q_1^* \) is the projective cover of \( -\lambda \). Furthermore, for \( \mu \in X_1 \), \( Q_1 \otimes \mu \) is indecomposable and the injective hull of \( \lambda \otimes \mu \). This shows that \( \phi_0 \) is an isomorphism in the case when \( Q_1 \) is the injective hull of \( \lambda \) and \( Q_2 = \mu \). The functors \( \text{Hom}_A(Q_1^*, -) \) and \( \text{Hom}_A(\mathbb{C}, Q_1 \otimes (-)) \) are exact. One can apply induction on the composition length to prove the statement for arbitrary \( Q_2 \).

(4) Let \( Q_1 \) be a finite-dimensional \( \bar{A} \)-module. Then

\[
\phi_1 : \text{Ext}^1_A(Q_1^*, Q_2) \rightarrow \text{Ext}^1_A(\mathbb{C}, Q_1 \otimes Q_2)
\]

is injective.

Consider the short exact sequences obtained where \( I \) is the (finite-dimensional) injective hull of \( Q_1 \):

\[
0 \rightarrow Q_1 \rightarrow I \rightarrow N \rightarrow 0 \quad \text{and} \quad 0 \rightarrow N^* \rightarrow I^* \rightarrow Q_1^* \rightarrow 0. \tag{9.4.1}
\]

These short exact sequences induce the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \rightarrow & \text{Hom}_A(Q_1^*, Q_2) & \rightarrow & \text{Hom}_A(I^*, Q_2) & \rightarrow & \text{Hom}_A(N^*, Q_2) & \rightarrow & \text{Ext}^1_A(Q_1^*, Q_2) & \rightarrow & 0 \\
\phi_0 & \downarrow & \phi_1 & \downarrow & \phi_0 & \downarrow & \phi_2 & \downarrow & \phi_1 & \downarrow & \\
0 & \rightarrow & \text{Hom}_A(\mathbb{C}, Q_1 \otimes Q_2) & \rightarrow & \text{Hom}_A(\mathbb{C}, I \otimes Q_2) & \rightarrow & \text{Hom}_A(\mathbb{C}, N \otimes Q_2) & \rightarrow & \text{Ext}^1_A(\mathbb{C}, Q_1 \otimes Q_2) & \rightarrow & 0
\end{array}
\]

Now according to (2) and (3), \( \phi_2' \) is injective and \( \phi_0' \) is an isomorphism. Our goal is to show that \( \phi_1 \) is injective. Suppose that \( \phi_1(x) = 0 \). Since \( \pi_1 \) is surjective there exists \( z \) such that \( \pi_1(z) = x \). Now by commutativity,

\[
\pi_2(\phi_2'(z)) = \phi_1(\pi_1(z)) = \phi_1(x) = 0.
\]
Therefore, $\phi'(z) \in \ker \tau_2 = \text{im} \tau_2$, and there exists $y$ such that $\tau_2(y) = \phi'(z)$. Since $\phi'$ is an isomorphism there exists $w$ such that $\phi'(w) = y$. By commutativity,

$$\phi'(\tau_1(w)) = \tau_2(\phi'(w)) = \tau_2(y) = \phi'(z).$$

Consequently, by the injectivity of $\phi'$, it follows that $z = \tau_1(w)$. Thus, $x = \pi_1(z) = \pi_1(\tau_1(w)) = 0$.

(5) Finally, let $Q_1$ be a finite-dimensional $\tilde{A}$-module. For $n \geq 0$,

$$\phi_n : \text{Ext}_A^n(Q_1^*, Q_2) \rightarrow \text{Ext}_\Delta^n(\mathbb{C}, Q_1 \otimes Q_2)$$

is injective.

For $n = 0, 1$, this was proved in (2) and (4). As in (4), the short exact sequences (9.4.1) yield commutative diagrams for $n \geq 1$

$$\begin{array}{ccc}
\text{Ext}_A^n(N^*, Q_2) & \xrightarrow{\epsilon_n} & \text{Ext}_A^{n+1}(Q_1^*, Q_2) \\
\phi_n' \downarrow & & \phi_{n+1} \downarrow \\
\text{Ext}_\Delta^n(\mathbb{C}, N \otimes Q_2) & \xrightarrow{\epsilon_n'} & \text{Ext}_\Delta^{n+1}(\mathbb{C}, Q_1 \otimes Q_2)
\end{array}$$

where $\epsilon_n$ and $\epsilon_n'$ are isomorphisms. Now by induction, $\phi'_n$ is injective which implies that $\phi_{n+1}$ is injective. This completes the proof that $\phi_n$ is injective for all $n$.

(6) We next show that

$$\phi_0 : \text{Hom}_A(Q_1^*, Q_2) \rightarrow \text{Hom}_\Delta(\mathbb{C}, Q_1 \otimes Q_2)$$

is an isomorphism.

We shall prove this by induction on the dimension of $Q_1$. If $Q_1$ is one-dimensional then (6) holds by (1). Consider the short exact sequence as in (2):

$$0 \rightarrow N^* \rightarrow Q_1^* \rightarrow \lambda^* \rightarrow 0.$$
Im $\tau_1$. Since $\ker \sigma_1 = \text{Im} \tau_1$, it follows that $\sigma_1(\phi''_0)^{-1}\tau_2(y) \neq 0$. From the injectivity of $\phi'_1$, one has $\phi'_1 \sigma_1(\phi''_0)^{-1}\tau_2(y) \neq 0$, and by commutativity,

$$0 \neq \sigma_2 \phi''_0(\phi''_0)^{-1}\tau_2(y) = \sigma_2 \tau_2(y).$$

This is a contradiction because $\sigma_2 \tau_2 = 0$.

Combining this with (2) shows that $\phi_0$ must be an isomorphism.

(7) We show that

$$\phi_n : \text{Ext}_A^n(Q_1, Q_2) \to \text{Ext}_\Delta^n(\mathbb{C}, Q_1 \otimes Q_2)$$

is an isomorphism.

Consider the short exact sequence

$$0 \to Q_2 \to I \to N \to 0$$

where $I$ is injective. Then one has a commutative diagram with exact rows

\[\begin{array}{cccccc}
\text{Hom}_A(Q_1, N) & \longrightarrow & \text{Ext}_A^1(Q_1, Q_2) & \longrightarrow & 0 \\
\phi''_0 & \downarrow & \phi_1 & \downarrow & \\
\text{Hom}_A(\mathbb{C}, Q_1 \otimes N) & \longrightarrow & \text{Ext}_\Delta^1(\mathbb{C}, Q_1 \otimes Q_2) & \longrightarrow & 0
\end{array}\]

Here, $\phi''_0$ is an isomorphism by (6) and $\phi_1$ is injective by (5). It follows immediately that $\phi_1$ is surjective, and hence is an isomorphism. One can use a dimension shifting argument, as in (5), to show that $\phi_n$ is an isomorphism for $n \geq 0$. The proof is now complete. \(\square\)

References

[APW1] H. H. Andersen, P. Polo, and K. Wen, *Representations of quantum algebras*, Invent. Math. **104** (1991), no. 1, 1–59.

[APW2] ______, *Injective modules for quantum algebras*, Amer. J. Math. **114** (1992), no. 3, 571–604.

[ABG] S. Arkhipov, R. Bezrukavnikov, and V. Ginzburg, *Quantum groups, the loop Grassmannian, and the Springer resolution*, J. Amer. Math. Soc. **17** (2004), no. 3, 595–678.

[Baj] A. M. Bajer, *The May spectral sequence for a finite p-group stops*, J. Algebra **167** (1994), no. 2, 448–459.

[Bal] P. Balmer, *The spectrum of prime ideals in tensor triangulated categories*, J. Reine Angew. Math. **588** (2005), 149–168.

[BNPP] C. P. Bendel, D. K. Nakano, B. J. Parshall, and C. Pillen, *Cohomology for quantum groups via the geometry of the nullcone*, Mem. Amer. Math. Soc. **229** (2014), no. 1077, x+93.

[Ben] D. J. Benson, *Representations and cohomology. II*, second ed., Cambridge Studies in Advanced Mathematics, vol. 31, Cambridge University Press, Cambridge, 1998.

[BCR] D. J. Benson, J. F. Carlson, and J. Rickard, *Complexity and varieties for infinitely generated modules. II*, Math. Proc. Cambridge Philos. Soc. **120** (1996), no. 4, 597–615.

[BEH] D. J. Benson, K. Erdmann, and M. Holloway, *Rank varieties for a class of finite-dimensional local algebras*, J. Pure Appl. Algebra **211** (2007), no. 2, 497–510.

[BIK1] D. J. Benson, S. Iyengar, and H. Krause, *Local cohomology and support for triangulated categories*, Ann. Sci. Éc. Norm. Supér. (4) **41** (2008), no. 4, 573–619.

[BIK2] ______, *Representations of finite groups: local cohomology and support*, Oberwolfach Seminars, vol. 43, Birkhäuser/Springer Basel AG, Basel, 2012.

[BE] P. A. Bergh and K. Erdmann, *The stable Auslander-Reiten quiver of a quantum complete intersection*, Bull. Lond. Math. Soc. **43** (2011), no. 1, 79–90.

[BO] P. A. Bergh and S. Oppermann, *Cohomology of twisted tensor products*, J. Algebra **320** (2008), no. 8, 3327–3338.
[Bez] R. Bezrukavnikov, *Cohomology of tilting modules over quantum groups and t-structures on derived categories of coherent sheaves*, Invent. Math. **166** (2006), no. 2, 327–357.

[BKN] B. D. Boe, J. R. Kujawa, and D. K. Nakano, *Tensor triangular geometry for classical Lie superalgebras* to appear in Adv. in Math., arXiv:1402.3732.

[CP] V. Chari and A. Pressley, *A guide to quantum groups*, Cambridge University Press, Cambridge, 1995, Corrected reprint of the 1994 original.

[Dru] C. M. Drupieski, *On injective modules and support varieties for the small quantum group* Int. Math. Res. Not. IMRN (2011), no. 10, 2263–2294.

[DNP] C. M. Drupieski, D. K. Nakano, and B. J. Parshall, *Differentiating the Weyl generic dimension formula with applications to support varieties* Adv. Math. **229** (2012), no. 5, 2656–2668.

[FW] J. Feldvoss and S. Witherspoon, *Support varieties and representation type of small quantum groups* Int. Math. Res. Not. IMRN (2010), no. 7, 1346–1362.

[Fri] E. M. Friedlander, *Filtrations, 1-parameter subgroups, and rational injectivity*, 2014, arXiv:1408.2918.

[FP] E. M. Friedlander and B. J. Parshall, *Support varieties for restricted Lie algebras*, Invent. Math. **86** (1986), no. 3, 553–562.

[GK] V. Ginzburg and S. Kumar, *Cohomology of quantum groups at roots of unity*, Duke Math. J. **69** (1993), no. 1, 179–198.

[HNS] W. D. Hardesty, D. K. Nakano, and P. Sobaje, *On the existence of mock injectives for algebraic groups*, 2016, arXiv:1604.03840.

[HPS] M. Hovey, J. H. Palmieri, and N. P. Strickland, *Axiomatic stable homotopy theory* Mem. Amer. Math. Soc. **128** (1997), no. 610, x+114.

[Hum] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, vol. 9, Springer-Verlag, New York-Berlin, 1978, Second printing, revised.

[Jan1] J. C. Jantzen, *Lectures on quantum groups*, Graduate Studies in Mathematics, vol. 6, American Mathematical Society, Providence, RI, 1996.

[Jan2] ______, *Representations of algebraic groups*, second ed., Mathematical Surveys and Monographs, vol. 107, American Mathematical Society, Providence, RI, 2003.

[KL1] D. Kazhdan and G. Lusztig, *Tensor structures arising from affine Lie algebras. I, II* J. Amer. Math. Soc. **6** (1993), no. 4, 905–947, 949–1011.

[KL2] ______, *Tensor structures arising from affine Lie algebras. III* J. Amer. Math. Soc. **7** (1994), no. 2, 335–381.

[KL3] ______, *Tensor structures arising from affine Lie algebras. IV* J. Amer. Math. Soc. **7** (1994), no. 2, 383–453.

[Lor1] M. Lorenz, *Algebraic group actions on noncommutative spectra* Transform. Groups **14** (2009), no. 3, 649–675.

[Lor2] ______, *Rational group actions on affine PI-algebras* Glasg. Math. J. **55** (2013), no. A, 101–111.

[Lus1] G. Lusztig, *Finite-dimensional Hopf algebras arising from quantized universal enveloping algebra* J. Amer. Math. Soc. **3** (1990), no. 1, 257–296.

[Lus2] ______, *Quantum groups at roots of 1* Geom. Dedicata **35** (1990), no. 1-3, 89–113.

[Lus3] ______, *Monodromic systems on affine flag manifolds* Proc. Roy. Soc. London Ser. A **445** (1994), no. 1923, 231–246.

[Nee] A. Neeman, *The connection between the K-theory localization theorem of Thomason, Trobaugh and Yao and the smashing subcategories of Bousfield and Ravenel* Ann. Sci. Ecole Norm. Sup. (4) **25** (1992), no. 5, 547–566.

[Ost] V. Ostrik, *Tensor ideals in the category of tilting modules* Transform. Groups **2** (1997), no. 3, 279–287.

[PW] B. Parshall and J. P. Wang, *Cohomology of quantum groups: the quantum dimension* Canad. J. Math. **45** (1993), no. 6, 1276–1298.

[Tau] T. Tanisaki, *Character formulas of Kazhdan-Lusztig type*. Representations of finite dimensional algebras and related topics in Lie theory and geometry, Fields Inst. Commun., vol. 40, Amer. Math. Soc., Providence, RI, 2004, pp. 261–276.
Department of Mathematics, University of Georgia, Athens, GA 30602
E-mail address: brian@math.uga.edu

Department of Mathematics, University of Oklahoma, Norman, OK 73019
E-mail address: kujawa@math.ou.edu

Department of Mathematics, University of Georgia, Athens, GA 30602
E-mail address: nakano@math.uga.edu