On the Cauchy problem for integro-differential operators in Sobolev classes and the martingale problem

R. Mikulevicius and H. Pragarauskas
University of Southern California, Los Angeles
Institute of Mathematics and Informatics, Vilnius

January 24, 2012

Abstract

The existence and uniqueness in Sobolev spaces of solutions of the Cauchy problem to parabolic integro-differential equation of the order $\alpha \in (0, 2)$ is investigated. The principal part of the operator has kernel $m(t, x, y)/|y|^{d+\alpha}$ with a bounded nondegenerate $m$, Hölder in $x$ and measurable in $y$. The lower order part has bounded and measurable coefficients. The result is applied to prove the existence and uniqueness of the corresponding martingale problem.

MSC classes: 45K05, 60J75, 35B65

Key words and phrases: non-local parabolic equations, Sobolev spaces, Lévy processes, martingale problem.

1 Introduction

In this paper we consider the Cauchy problem

$$
\begin{aligned}
\partial_t u(t, x) &= Lu(t, x) + f(t, x), (t, x) \in E = [0, T] \times \mathbb{R}^d, \\
u(0, x) &= 0
\end{aligned}
$$

(1)

in fractional Sobolev spaces for a class of integrodifferential operators $L = A + B$ of the order $\alpha \in (0, 2)$ whose principal part $A$ is of the form

$$
\begin{aligned}
Av(t, x) &= A_{t, x} v(x) = A_{t, z} v(x)|_{z=x}, \\
A_{t, z} v(x) &= \int [v(x + y) - v(x) - \chi_\alpha(y)(\nabla v(x), y)] m(t, z, y) \frac{dy}{|y|^{d+\alpha}},
\end{aligned}
$$

(2)
\((t, z) \in E, x \in \mathbb{R}^d\), with \(\chi_{\alpha}(y) = 1_{\alpha > 1} + 1_{\alpha = 1} 1_{\{|y| \leq 1\}}\). We notice that the operator \(A\) is the generator of an \(\alpha\)-stable process. If \(m = 1\), then \(A = c(-\Delta)^{\alpha/2}\) (fractional Laplacian) is the generator of a spherically symmetric \(\alpha\)-stable process. The part \(B\) is a perturbing, subordinated operator.

In [10], the problem was considered assuming that \(m\) is Hölder continuous in \(x\), homogeneous of order zero and smooth in \(y\) and for some \(\eta > 0\)

\[
\int_{S^{d-1}} |(w, \xi)|^\alpha m(t, x, w) \mu_{d-1}(dw) \geq \eta, \quad (t, x) \in E, |\xi| = 1, \tag{3}
\]

where \(\mu_{d-1}\) is the Lebesgue measure on the unit sphere \(S^{d-1}\) in \(\mathbb{R}^d\). In [1], the existence and uniqueness of a solution to (1) in Hölder spaces was proved analytically for \(m\) Hölder continuous in \(x\), smooth in \(y\) and such that

\[
K \geq m \geq \eta > 0 \tag{4}
\]

without assumption of homogeneity in \(y\). The elliptic problem \((L - \lambda)u = f\) with \(B = 0\) and \(m\) independent of \(x\) in \(\mathbb{R}^d\) was considered in [4]. The equation (1) with \(\alpha = 1\) can be regarded as a linearization of the quasigeostrophic equation (see [2]).

In this note, we consider the problem (1), assuming that \(m\) is measurable, Hölder continuous in \(x\) and

\[
K \geq m \geq m_0, \tag{5}
\]

where the function \(m_0 = m_0(t, x, y)\) is smooth and homogeneous in \(y\) and satisfies (3). So, the density \(m\) can degenerate on a substantial set.

A certain aspect of the problem is that the symbol of the operator \(A\),

\[
\psi(t, x, \xi) = \int \left[ e^{i(\xi, y)} - 1 - \chi_{\alpha}(y)i(\xi, y) \right] m(t, x, y) \frac{dy}{|y|^{d+\alpha}}
\]

is not smooth in \(\xi\) and the standard Fourier multiplier results (for example, used in [10]) do not apply in this case. We start with equation (1) assuming that \(B = 0\), the input function \(f\) is smooth and the function \(m = m(t, y)\) does not depend on \(x\). In [12], the existence and uniqueness of a weak solution in Sobolev spaces was derived. In this paper we show that the main part \(A : H^\alpha_p \to L^p\) is bounded. Contrary to [4], where Hölder estimates were used, we give a direct proof based on the classical theory of singular integrals (see Lemmas 9, 10 below). The case of variable coefficients is based on the a priori estimates using Sobolev embedding theorem and the method in [9].
As an application, we consider the martingale problem associated to \( L \). Since the lower part of \( L \) has only measurable coefficients, we generalize the results in [13].

The note is organized as follows. In Section 2, the main theorem is stated. In Section 3, the essential technical results are presented. The main theorem is proved in Section 4. In Section 5 we discuss the embedding of the solution space. In Section 6 the existence and uniqueness of the associated martingale problem is considered.

2 Notation and main results

Denote \( E = [0, T] \times \mathbb{R}^d \), \( N = \{0, 1, 2, \ldots \} \), \( \mathbb{R}_0^d = \mathbb{R}^d \backslash \{0\} \). If \( x, y \in \mathbb{R}^d \), we write
\[
(x, y) = \sum_{i=1}^{d} x_i y_i, |x| = (x, x)^{1/2}.
\]

For a function \( u = u(t, x) \) on \( E \), we denote its partial derivatives by \( \partial_t u = \partial u/\partial t, \partial_i u = \partial u/\partial x_i, \partial^2_{ij} u = \partial^2 u/\partial x_i \partial x_j \) and \( D^\gamma u = \partial^{|\gamma|} u/\partial x_1^{\gamma_1} \cdots \partial x_d^{\gamma_d} \), where multiindex \( \gamma = (\gamma_1, \ldots, \gamma_d) \in \mathbb{N}^d \), \( \nabla u = (\partial_1 u, \ldots, \partial_d u) \) denotes the gradient of \( u \) with respect to \( x \).

Let \( L^p_p(E) \) be the space of \( p \)-integrable functions with norm
\[
|f|_p = \left( \int_0^T \int |f(t, x)|^p dx dt \right)^{1/p}.
\]

Similar space of functions on \( \mathbb{R}^d \) is denoted \( L^p_p(\mathbb{R}^d) \).

Let \( \mathcal{S}(\mathbb{R}^d) \) be the Schwartz space of smooth real-valued rapidly decreasing functions. We introduce the Sobolev space \( H^\beta_p(\mathbb{R}^d) \) of \( f \in \mathcal{S}'(\mathbb{R}^d) \) with finite norm
\[
|f|_{\beta, p} = |F^{-1}((1 + |\xi|^2)^{\beta/2} F f))|_p,
\]
where \( F \) denotes the Fourier transform. We also introduce the corresponding spaces of generalized functions on \( E = [0, T] \times \mathbb{R}^d \): \( H^\beta_p(E) \) consist of all measurable \( \mathcal{S}'(\mathbb{R}^d) \)-valued functions \( f \) on \([0, T]\) with finite norm
\[
|f|_{\beta, p} = \left\{ \int_0^T |f(t)|^p_{\beta, p} dt \right\}^{1/p}.
\]

For \( \alpha \in (0, 2) \) and \( u \in \mathcal{S}(\mathbb{R}^d) \), we define the fractional Laplacian
\[
\partial^{\alpha} u(x) = \int \nabla_y^{\alpha} u(x) \frac{dy}{|y|^{d+\alpha}}, \quad (6)
\]
where
$$\nabla^\alpha_y u(x) = u(x + y) - u(x) - (\nabla u(x), y) \chi_\alpha(y)$$
with \(\chi^{(\alpha)}(y) = \mathbf{1}_{\{|y| \leq 1\}} + \mathbf{1}_{\{\alpha = 1\}}\).

We denote \(C^\infty_b(E)\) the space of bounded infinitely differentiable in \(x\) functions whose derivatives are bounded.

\(C = \mathcal{C}(\cdot, \ldots, \cdot)\) denotes constants depending only on quantities appearing in parentheses. In a given context the same letter is (generally) used to denote different constants depending on the same set of arguments.

Let \(\alpha \in (0, 2)\) be fixed. Let \(m : E \times \mathbb{R}^d_0 \to [0, \infty), b : E \to \mathbb{R}^d\) be measurable functions. We also introduce an auxiliary function \(m_0 : [0, T] \times \mathbb{R}^d_0 \to [0, \infty)\) and fix positive constants \(K\) and \(\eta\). Throughout the paper we assume that the function \(m_0\) satisfies the following conditions.

**Assumption A_0.** (i) The function \(m_0 = m_0(t, y) \geq 0\) is measurable, homogeneous in \(y\) with index zero, differentiable in \(y\) up to the order \(d_0 = \left[\frac{d}{2}\right] + 1\) and

$$|D^\gamma m_0^{(\alpha)}(t, y)| \leq K$$

for all \(t \in [0, T], y \in \mathbb{R}^d_0\) and multiindices \(\gamma \in \mathbb{N}^d_0\) such that \(|\gamma| \leq d_0\);

(ii) If \(\alpha = 1\), then for all \(t \in [0, T]\)

$$\int_{S^{d-1}} w m_0(t, w) \mu_{d-1}(dw) = 0,$$

where \(S^{d-1}\) is the unit sphere in \(\mathbb{R}^d\) and \(\mu_{d-1}\) is the Lebesgue measure on it;

(iii) For all \(t \in [0, T]\)

$$\inf_{|\xi|=1} \int_{S^{d-1}} |(w, \xi)|^{\alpha} m_0(t, w) \mu_{d-1}(dw) \geq \eta > 0.$$

**Remark 1** The nondegenerateness assumption \(A_0\) (iii) holds with certain \(\eta > 0\) if, e.g.

$$\inf_{t \in [0, T], w \in \Gamma} m_0(t, w) > 0$$

for a measurable subset \(\Gamma \subset S^{d-1}\) of positive Lebesgue measure. Therefore \(m_0\) can be zero on a substantial set.

Further we will use the following assumptions.

**Assumption A.** (i) For all \((t, x) \in E, y \in \mathbb{R}^d_0\)

$$K \geq m(t, x, y) \geq m_0(t, y),$$
where the function $m_0$ satisfies Assumption $A_0$;

(ii) There is $\beta \in (0, 1)$ and a continuous increasing function $w(\delta)$ such that

$$|m(t, x, y) - m(t, x', y)| \leq w(|x - x'|), t \in [0, T], x, x', y \in \mathbb{R}^d,$$

and

$$\int_{|y| \leq 1} w(|y|) \frac{dy}{|y|^{d+\beta}} < \infty, \lim_{\delta \to 0} w(\delta)\delta^{-\beta} = 0.$$

(iii) If $\alpha = 1$, then for all $(t, x) \in E$ and $r \in (0, 1),$

$$\int_{r < |y| \leq 1/r} ym(t, x, y) \frac{dy}{|y|^{d+\alpha}} = 0.$$

We define the lower order operator $Bu(t, x) = B_{t, z}u(x)|_{z=x}$, $(t, x) \in E$, with

$$B_{t, z}u(x) = (b(t, z)\nabla u(x))1_{1<\alpha<2} + \int [u(x + y) - u(x) - (\nabla u(x), y))1_{|y| \leq 1}1_{1<\alpha<2}]\pi(t, z, dy),$$

where $(\pi(t, z, dy))$ is a measurable family of nonnegative measures on $\mathbb{R}^d_0$ and $b(t, z) = (b^i(t, z))_{1 \leq i \leq d}$ is a measurable function.

We will assume the following assumptions hold.

**Assumption B.** (i) For all $(t, x) \in E$,

$$|b(t, x)| + \int |v|^\alpha \wedge 1\pi(t, x, dv) \leq K;$$

(ii)

$$\lim_{\varepsilon \to 0} \sup_{t, x} \int_{|v| \leq \varepsilon} |v|^\alpha \pi(t, x, dv) = 0;$$

(iii) For each $\varepsilon > 0$,

$$\int_0^T \int \pi(t, x, \{|v| > \varepsilon\})dxdt < \infty.$$

We write

$$Au(t, x) = A_{t, x}u(x), Bu(t, x) = B_{t, z}u(x), L = A + B.$$
According to Assumptions A, B, the operator A represents the principal part of \( L \) and the operator B is a lower order operator.

We consider the following Cauchy problem

\[
\begin{align*}
\partial_t u(t, x) &= (L - \lambda)u(t, x) + f(t, x), (t, x) \in H, \\
u(0, x) &= 0, x \in \mathbb{R}^d,
\end{align*}
\]

in Sobolev classes \( H^\alpha_p(E) \), where \( \lambda \geq 0 \) and \( f \in L_p(E) \). More precisely, let \( H^\alpha_p(E) \) be the space of all functions \( u \in H^\alpha_p(E) \) such that \( u(t, x) = \int_0^t F(s, x) \, ds, 0 \leq t \leq T \), with \( F \in H^\alpha_p(E) \). It is a Banach space with respect to the norm

\[
\|u\|_{\alpha,p} = \|u\|_{\alpha,p} + |F|_p.
\]

**Definition 2** Let \( f \in L_p(E) \). We say that \( u \in H^\alpha_p(E) \) is a solution to (8) if \( Lu \in L_p(E) \) and

\[
 u(t) = \int_0^t ((L - \lambda)u(s) + f(s)) \, dt, 0 \leq t \leq T,
\]

in \( L_p(\mathbb{R}^d) \).

If Assumptions A and B are satisfied, \( p > \frac{d}{\alpha} \vee \frac{d}{\beta} \vee 2 \), then \( Lu \in L_p(E) \) (see Corollary 13 below and Lemma 7 in [10]). So, (9) is well defined.

The main result of the paper is the following theorem.

**Theorem 3** Let \( \beta \in (0,1), p > \frac{d}{\beta}, p \geq 2 \), and Assumption A be satisfied.

Then for any \( f \in L_p(E) \) there exists a unique strong solution \( u \in H^\alpha_p(E) \) to (8) with \( B = 0 \). Moreover, there is a constant \( N = N(T, \alpha, \beta, d, K, w, \eta) \) and a positive number \( \lambda_1 = \lambda_1(T, \alpha, \beta, d, K, w, \eta) \geq 1 \) such that

\[
\begin{align*}
|\partial_t u|_p + |u|_{\alpha,p} &\leq N|f|_p, \\
|u|_p &\leq \frac{N}{\lambda}|f|_p \text{ if } \lambda \geq \lambda_1.
\end{align*}
\]

We prove this theorem in Section 3 below.

In order to handle (8) with the lower order part \( Bu \), the following estimate is needed.

**Lemma 4** (see Lemma 3.5 in [10]) Let \( p > d/\alpha \). There is a constant \( N_1 = N_1(p, \alpha, d) \) such that

\[
\left| \sup_{y \neq 0} \frac{\left| \nabla_y \alpha v(\cdot) \right|}{|y|_\alpha} \right|_p \leq N_1|\partial^\alpha v|_p, v \in C^\infty_0(\mathbb{R}^d).
\]
Consider (8) with $Bv(t, x) = B^{\varepsilon_0}v(t, x) = B^{\varepsilon_0}_{t, x}v(x), (t, x) \in E$, where

$$
B^{\varepsilon_0}_{t, x}v(x) = (b(t, z) + 1_{\alpha \in (1, 2)} \int_{\varepsilon_0 \leq |y| \leq 1} y \pi(t, z, dy), \nabla v(x)) \quad (10)
$$

$$
+ \int_{|y| \leq \varepsilon_0} \nabla_y^0 v(x) \pi(t, z, dy), (t, z) \in E, x \in \mathbb{R}^d,
$$

with some $\varepsilon_0 \in (0, 1]$.

In the consideration of an associated martingale problem (see Section 5 below) the following statement is used.

**Theorem 5** Let $\beta \in (0, 1), p > \frac{d}{\beta} \lor \frac{d}{\alpha} \lor 2$ and Assumption A be satisfied. Let

$$
|b(t, x)| + \int_{\varepsilon_0 \leq |y| \leq 1} |y| \pi(t, x, dy) \leq K,
$$

$$
\int_{|y| \leq \varepsilon_0} |y|^\alpha \pi(t, x, dy) \leq \delta_0, (t, x) \in E,
$$

and $\delta_0 NN_1 \leq 1/2$, where $N$ is a constant of Theorem 3. Then for any $f \in L^p(E)$ there exists a unique solution $u \in H^p_\alpha(E)$ to (8) with $B = B^{\varepsilon_0}$. Moreover,

$$
|\partial_t u|_p + |u|_{\alpha, p} \leq 2N|f|_p,
$$

$$
|u|_p \leq \frac{2N}{\lambda} |f|_p \text{ if } \lambda \geq \lambda_1.
$$

Finally, the results can be extended to

**Theorem 6** Let $\beta \in (0, 1), p > \frac{d}{\beta} \lor \frac{d}{\alpha} \lor 2$ and Assumptions A, B be satisfied. Then for any $f \in L^p(E)$ there exists a unique solution $u \in H^p_\alpha(E)$ to (8). Moreover, there is a constant $N_3$ independent of $u$ such that

$$
|\partial_t u|_p + |u|_{\alpha, p} \leq N_3|f|_p.
$$

### 3 Auxiliary results

In this section we present some auxiliary results.
3.1 Continuity of the principal part

First we prove the continuity of the operator $A$ in $L_p$-norm.

We will use the following equality for Sobolev norm estimates.

**Lemma 7** (Lemma 2.1 in [7]) For $\delta \in (0, 1)$ and $u \in \mathcal{S}(\mathbb{R}^d)$,

$$u(x + y) - u(x) = C \int k^{(\delta)}(y, z) \partial^\delta u(x - z) dz,$$

(11)

where the constant $C = C(\delta, d)$ and

$$k^{(\delta)}(z, y) = |z + y|^{-d+\delta} - |z|^{-d+\delta}.$$

Moreover, there is a constant $C = C(\delta, d)$ such that for each $y \in \mathbb{R}^d$

$$\int |k^{(\delta)}(z, y)| dz \leq C|y|^\delta.$$

For $\alpha \in (0, 1)$ and a bounded measurable function $m(y)$, set for $u \in \mathcal{S}(\mathbb{R}^d), x \in \mathbb{R}^d$,

$$L u(x) = \int [u(x + y) - u(x)] m(y) \frac{dy}{|y|^{d+\alpha}}$$

$$= \lim_{\varepsilon \to 0} \int [u(x + y) - u(x)] m_\varepsilon(y) \frac{dy}{|y|^{d+\alpha}}$$

$$= \lim_{\varepsilon \to 0} \int \int k^{(\alpha)}(z, y) \partial^\alpha u(x - z) dz m_\varepsilon(y) \frac{dy}{|y|^{d+\alpha}}$$

$$= \lim_{\varepsilon \to 0} \int \int k^{(\alpha)}(z, y) m_\varepsilon(y) \frac{dy}{|y|^{d+\alpha}} \partial^\alpha u(x - z) dz,$$

where

$$m_\varepsilon(y) = \chi_{\{\varepsilon \leq |y| \leq \varepsilon^{-1}\}} m(y),$$

$$k^{(\alpha)}(x, y) = \frac{1}{|x + y|^{d-\alpha}} - \frac{1}{|x|^{d-\alpha}}$$

(see Lemma [7]).

For $\varepsilon \in (0, 1), u, v \in \mathcal{S}(\mathbb{R}^d)$, consider

$$L^\varepsilon u(x) = L^\varepsilon_\alpha u(x) = \int [u(x + y) - u(x)] m_\varepsilon(y) \frac{dy}{|y|^{d+\alpha}}$$

(12)

$$= \int k_\varepsilon(z) \partial^\alpha u(x - z) dz = \int k_\varepsilon(x - z) \partial^\alpha u(z) dz,$$
and
\[ K_\varepsilon v(x) = \int k_\varepsilon(z) v(x - z) dz = \int k_\varepsilon(x - z) v(z) dz \]
where
\[ k_\varepsilon(x) = \int k^{(\alpha)}(x, y) m_\varepsilon(y) \frac{dy}{|y|^{d+\alpha}}, \quad x \in \mathbb{R}^d. \]

To prove the continuity \( L : H^\alpha_p(\mathbb{R}^d) \rightarrow L^p_p(\mathbb{R}^d) \), we will show that there is a constant \( C \) independent of \( \varepsilon, v \in \mathcal{S}(\mathbb{R}^d) \) such that
\[ |K_\varepsilon v|_p \leq C |v|_p. \quad (13) \]

By \[15\], \( (13) \) will follow provided
\[ |K_\varepsilon v|_2 \leq C |v|_2, \quad (14) \]
and
\[ \int_{|x| > 4|s|} |k_\varepsilon(x - s) - k_\varepsilon(x)| dx \leq C \text{ for all } s \in \mathbb{R}^d. \quad (15) \]

**Remark 8** For any \( t > 0 \), we have \( k^{(\alpha)}(tx, ty) = t^{\alpha - d} k^{(\alpha)}(x, y) \). Therefore
\[ k_\varepsilon(tx) = t^{-d} \int k^{(\alpha)}(x, y) m_\varepsilon(ty) \frac{dy}{|y|^{d+\alpha}} = t^{-d} k_\varepsilon(t, x) \]
with
\[ k_\varepsilon(t, x) = \int k^{(\alpha)}(x, y) m_\varepsilon(ty) \frac{dy}{|y|^{d+\alpha}}. \]

Note that for \( x \neq 0 \),
\[ k_\varepsilon(x) = k_\varepsilon(|x| \hat{x}) = |x|^{-d} \int k^{(\alpha)}(\hat{x}, y) m_\varepsilon(|x| y) \frac{dy}{|y|^{d+\alpha}} = |x|^{-d} k_\varepsilon(|x|, \hat{x}), \]
where \( \hat{x} = x/|x| \).

**Lemma 9** Let \( \alpha \in (0, 1), |m(y)| \leq 1, y \in \mathbb{R}^d \). Then for each \( p > 1 \) there is a constant \( C \) independent of \( u \) and \( \varepsilon \) such that,
\[ |K_\varepsilon u|_p \leq C |u|_p, \quad u \in L_p(\mathbb{R}^d). \]
Proof. It is enough to show that (14) and (15) hold. By Lemma 1 of Chapter 5.1 in [15], it follows

\[
\hat{k}_\varepsilon(\xi) = \int e^{-i(x,\xi)}k_\varepsilon(x)dx = C|\xi|^{-\alpha} \int [e^{-i(\xi,y)} - 1]m_\varepsilon(y)\frac{dy}{|y|^{d+\alpha}},
\]

where \( \hat{\xi} = \xi/|\xi|. \) Therefore, by Parseval’s equality, (14) holds for \( v \in S(\mathbb{R}^d). \) The key estimate is (15). By Remark 8 denoting \( \hat{s} = s/|s|, \) we have

\[
\int_{|x|>4|s|} |k_\varepsilon(x-s) - k_\varepsilon(x)|dx = \int_{|x|/|s|>4} |k_\varepsilon(|s|\frac{x}{|s|} - \hat{s}) - k_\varepsilon(|s|\frac{x}{|s|})|dx
\]

\[
= |s|^d \int_{|x|>4} |k_\varepsilon(|s|(x - \hat{s})) - k_\varepsilon(|s|x)|dx
\]

\[
= \int_{|x|>4} |k_\varepsilon(|s|, x - \hat{s}) - k_\varepsilon(|s|, x)|dx
\]

and it is enough to prove that

\[
\int_{|x|>4} |k_\varepsilon(|s|, x - \hat{s}) - k_\varepsilon(|s|, x)|dx \leq M \text{ for all } s \in \mathbb{R}^d, \hat{s} = s/|s|. \quad (16)
\]

We will estimate for \( |x| \geq 4, s \in \mathbb{R}^d, \hat{s} = s/|s|, \) the difference

\[
|k(|s|, x - \hat{s}) - k(|s|, x)|
\]

\[
= \int [k^{(\alpha)}(x, \hat{s}, y) - k^{(\alpha)}(x, y)]|s|y )\frac{dy}{|y|^{d+\alpha}}
\]

\[
= \int_{|y| \leq |x|/2} \ldots + \int_{|y| > |x|/2} \ldots = A_1 + A_2.
\]

Let

\[
F(t) = \frac{1}{|x - t\hat{s} + y|^{d-\alpha}} - \frac{1}{|x - t\hat{s}|^{d-\alpha}}, 0 \leq t \leq 1.
\]

If a segment connecting \( x \) and \( x - \hat{s} \) does not contain zero, then

\[
|k^{(\alpha)}(x - \hat{s}, y) - k^{(\alpha)}(x, y)|
\]

\[
= |F(1) - F(0)| \leq \int_0^1 |F'(t)|dt
\]

with

\[
F'(t) = (\alpha - d)[\frac{1}{|x - t\hat{s} + y|^{d-\alpha+1}}|x - t\hat{s} + y|^d + \frac{1}{|x - t\hat{s}|^{d-\alpha+1}}|x - t\hat{s}|^d] - \frac{1}{|x - t\hat{s} + y|^{d-\alpha}}.
\]
\[ |F'(t)| \leq C \left| \frac{1}{|x - t \hat{s} + y|^{d-\alpha+1}} - \frac{1}{|x - t \hat{s}|^{d-\alpha+1}} \right| 
+ \frac{1}{|x - t \hat{s}|^{d-\alpha+1}} \left( \frac{1}{|x - t \hat{s}|} (|y| \wedge 1) \right). \] (18)

**Estimate of \( A_1 \).** Let \(|x| \geq 4, z = x - t \hat{s}, t \in [0, 1], \) and \(|y| \leq |x|/2.\) In this case, \( x + y \geq |x| - |y| \geq |x|/2 \geq 2,\)

\[ C|x| \geq |z + y| \geq |x|/4 \geq 1, \]
\[ C|x| \geq |z| \geq |x| - 1 \geq 3|x|/4 \geq 3 \]

and (17) holds. Since

\[ \int_{|y| \leq |x|/2} \left( \frac{1}{|z + y|^{d-\alpha+1}} - \frac{1}{|z|^{d-\alpha+1}} \right) dy \]
\[ \leq \frac{1}{|z|^{d+1}} \int_{|y| \leq 2/3} \left( \frac{1}{|\hat{z} + y|^{d-\alpha+1}} - 1 \right) dy \]
\[ \leq \frac{C}{|x|^{d+1}}, \]

and

\[ \int_{|y| \leq |x|/2} (|y| \wedge 1) \frac{dy}{|y|^{d+\alpha}} \leq C, \]

It follows by (18), that

\[ |A_1| \leq C \int_{|y| \leq |x|/2} \left( \frac{1}{|z + y|^{d-\alpha+1}} - \frac{1}{|z|^{d-\alpha+1}} \right) + \frac{1}{|z|^{d-\alpha+1}} \left( |y| \wedge 1 \right) \frac{dy}{|y|^{d+\alpha}} \]
\[ \leq C \left[ \frac{1}{|x|^{d+1}} + \frac{1}{|x|^{d-\alpha+1}} \right]. \]

**Estimate of \( A_2 \).** Let \(|x| \geq 4, |y| > |x|/2.\) In this case we split

\[ A_2 = \int_{|y| > |x|/2} \left[ k^{(\alpha)}(x - \hat{s}, y) - k^{(\alpha)}(x, y) \right] \frac{dy}{|y|^{d+\alpha}} \]
\[ = \int_{|x| - 3/2 \leq |y| < |x|/2} \left[ \int_{|x| - 3/2 \leq |y| < |x| + 3/2} \right] \frac{dy}{|y|^{d+\alpha}} \]
\[ = B_1 + B_2. \]
If $|x| - 3/2 \geq |y| > |x|/2$ or $|y| > |x| + 3/2$, then we can apply (17) and (18). For $z = x - t\hat{s}$ we have $|z + y| \geq 1/2$ and

$$|F'(t)| \leq C\left[\frac{1}{|z + y|^{d-\alpha + 1}} + \frac{1}{|z|^{d-\alpha + 1}}\right].$$

Therefore

$$|B_1| \leq C\int_{\{\frac{|x|}{2} \leq |y|\}} \frac{1}{|z|^{d-\alpha + 1}} \frac{dy}{|y|^{d+\alpha}} + \int_{\{\frac{|x|}{2} \leq |y| \leq |x| - \frac{3}{2}\}} \frac{1}{|z + y|^{d-\alpha + 1}} \frac{dy}{|y|^{d+\alpha}}$$

$$+ \int_{\{|x| + \frac{3}{2} \leq |y|\}} \frac{1}{|z + y|^{d-\alpha + 1}} \frac{dy}{|y|^{d+\alpha}}$$

$$= B_{11} + B_{12} + B_{13}.$$ 

Now

$$B_{11} = C\int_{\{\frac{|x|}{2} \leq |y|\}} \frac{1}{|z|^{d-\alpha + 1}} \frac{dy}{|y|^{d+\alpha}} \leq \frac{C}{|x|^{d+1}},$$

and

$$B_{12} \leq C|z|^{-d-1} \int_{\{\frac{|x|}{2|z|} \leq |y| \leq (|x| - \frac{3}{2})/|z|\}} \frac{1}{|\hat{z} + y|^{d-\alpha + 1}} \frac{dy}{|y|^{d+\alpha}},$$

$$B_{13} \leq C|z|^{-d-1} \int_{\{\frac{|x|}{|z|} + \frac{3}{2|z|} \leq |y|\}} \frac{1}{|\hat{z} + y|^{d-\alpha + 1}} \frac{dy}{|y|^{d+\alpha}}$$

with $\hat{z} = z/|z|$. If $|\frac{|x|}{2|z|} \leq |y| \leq (|x| - \frac{3}{2})/|z|$ or $|\frac{|x|}{|z|} + \frac{3}{2|z|} \leq |y|$, then $|\hat{z} + y| \geq \frac{1}{2|z|}$ and $|y| \geq 1/3$. Therefore

$$B_{12} \leq C|z|^{-d-1} \int_{\{|\hat{z} + y| \geq \frac{1}{2|z|}\}} \frac{dy}{|\hat{z} + y|^{d-\alpha + 1}}$$

$$\leq C|z|^{-d-\alpha} \leq C|x|^{-d-\alpha}$$

and

$$B_{13} \leq C|z|^{-d-1} \int_{|\hat{z} + y| \geq \frac{1}{2|z|}} \frac{1}{|\hat{z} + y|^{d-\alpha + 1}} dy$$

$$= C|z|^{-d-\alpha} \leq C|x|^{-d-\alpha}.$$

Now we estimate $B_2$. If $|x| - \frac{3}{2} \leq |y| \leq |x| + \frac{3}{2}$, then we estimate directly.
Lemma 10

Let \( \hat{\chi} \) with \( z \) and \( S(z) \leq |x| - \frac{3}{2} \alpha \), \( \alpha \) with \( x \in R^d \), \( y \in R^d \), \( \alpha \in (0,2) \), set for \( u \in S(R^d) \), \( x \in R^d \),

\[
\mathcal{L} u(x) = \mathcal{L}_\alpha u(x) = \int \nabla^\alpha u(x) m(y) \frac{dy}{|y|^{d+\alpha}},
\]

where

\[
\nabla^\alpha u(x) = u(x + y) - u(x) - \chi_\alpha(y) (\nabla u(x), y)
\]

with \( \chi_\alpha(y) = 1_{\alpha \in (1,2)} + 1_{\alpha = 1} 1_{|y| \leq 1} \).

Lemma 10 Let \(|m(y)| \leq K, y \in R^d, p > 1, and \alpha \in (0,2)\). Assume

\[
\int_{r \leq |y| \leq R} y m(y) \frac{dy}{|y|^{d+\alpha}} = 0
\]

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for any $0 < r < R$ if $\alpha = 1$. Then there is a constant $C$ such that

$$|L_\alpha u|_p \leq CK^p u|_p, u \in L_p(\mathbb{R}^d).$$

**Proof.** If $\alpha \in (0, 1)$, then for $u \in \mathcal{S}(\mathbb{R}^d)$ we have

$$L u(x) = \lim_{\varepsilon \to 0} L^\varepsilon u(x), x \in \mathbb{R}^d,$$

and by Lemma 9 there is a constant $C$ independent on $u$ such that

$$|K^{-1} L u|_p \leq C|u|_p,$$ or

$$|L_\alpha u|_p \leq CK^p u|_p, u \in \mathcal{S}(\mathbb{R}^d).$$

If $\alpha \in (1, 2)$, then it follows by Lemma 7 that for $u \in \mathcal{S}(\mathbb{R}^d), x \in \mathbb{R}^d,$

$$L_\alpha u(x) = \int [u(x + y) - u(x) - (\nabla u(x), y)] m(y) \frac{dy}{y^{d+\alpha}}$$

$$= \int \left[ \int_0^1 (\nabla u(x + sy) - \nabla u(x), y) m(y) \frac{dsdy}{y^{d+\alpha}} \right] \frac{dy}{y^{d+\alpha}}$$

$$= \int \left[ \left( \nabla u(x + y) - \nabla u(x), \frac{y}{|y|} \right) \right] M(y) \frac{dy}{y^{d+\alpha-1}}$$

with

$$M(y) = \int_0^1 m(y/s) s^{-1+\alpha} ds, y \in \mathbb{R}^d.$$ 

Therefore, the estimate reduces to the case of $\alpha \in (0, 1)$: there is a constant $C$ independent of $u \in \mathcal{S}(\mathbb{R}^d)$ such that

$$|L_\alpha u|_p \leq CK^{-1} \nabla u|_p \leq C|\partial^\alpha u|_p.$$

If $\alpha = 1$, then for $u \in \mathcal{S}(\mathbb{R}^d), x \in \mathbb{R}^d,$

$$L_1 u(x) = \int [u(x + y) - u(x) - (\nabla u(x), y)]_{|y| \leq 1} m(y) \frac{dy}{y^{d+1}}$$

$$= \lim_{\varepsilon \to 0} \int [u(x + y) - u(x)] m_\varepsilon(y) \frac{dy}{y^{d+1}}$$

with $m_\varepsilon(y) = m(y)1_{|y| \geq \varepsilon}, y \in \mathbb{R}^d$. By Lemma 7 for $u \in \mathcal{S}(\mathbb{R}^d), x \in \mathbb{R}^d,$

$$\int [u(x + y) - u(x)] m_\varepsilon(y) \frac{dy}{y^{d+1}}$$

$$= \int \int k^{(1/2)}(z, y) \partial^{1/2} u(x - z) dzm_\varepsilon(y) \frac{dy}{y^{d+1}}$$

$$= \int \int k^{(1/2)}(z, y) [\partial^{1/2} u(x - z) - \partial^{1/2} u(x)] dzm_\varepsilon(y) \frac{dy}{y^{d+1}}$$
\[ L_1 u(x) = \lim_{\varepsilon \to 0} \int [u(x+y) - u(x)] m_\varepsilon(y) \frac{dy}{|y|^{d+1}} \]
\[ = \lim_{\varepsilon \to 0} \int \int k^{(1/2)}(z,y) m_\varepsilon(y) \frac{dy}{|y|^{d+1}} [\partial^{1/2} u(x - z) - \partial^{1/2} u(x)] dz. \]

Obviously,
\[ \int k^{(1/2)}(z,y) m_\varepsilon(y) \frac{dy}{|y|^{d+1}} = \frac{1}{|z|^{d+\frac{1}{2}}} \left( \frac{1}{|\hat{z} + y|^{d-\frac{1}{2}}} - 1 \right) m_\varepsilon(|z|y) \frac{dy}{|y|^{d+1}} \]
\[ = \frac{1}{|z|^{d+\frac{1}{2}}} M_\varepsilon(z), \]

where \( \hat{z} = z/|z| \). Since for \( \varepsilon \in (0, 1/2) \), we have \( |M_\varepsilon(z)| \leq CK \) and for \( z \in \mathbb{R}^d \),
\[ \lim_{\varepsilon \to 0} M_\varepsilon(z) = M(z), \]
\[ = \int_{\{|y| \leq \frac{1}{2}\}} \left( \frac{1}{|\hat{z} + y|^{d-\frac{1}{2}}} - 1 \right) m(|z|y) \frac{dy}{|y|^{d+1}} \]
\[ + \int_{\{|y| > \frac{1}{2}\}} \left( \frac{1}{|\hat{z} + y|^{d-\frac{1}{2}}} - 1 \right) m(|z|y) \frac{dy}{|y|^{d+1}}, \]

it follows that
\[ \mathcal{L}^1 u(x) = \int [u(x+y) - u(x) - (\nabla u(x), y) 1_{|y| \leq 1}] m(y) \frac{dy}{|y|^{d+1}} \]
\[ = \int [\partial^{1/2} u(x + z) - \partial^{1/2} u(x)] M(-z) \frac{dz}{|z|^{d+\frac{1}{2}}} \]

with \( |M(z)| \leq CK, z \in \mathbb{R}^d \) and the estimate follows from the case \( \alpha = 1/2 \).

Now we investigate the continuity of the main part \( A \) with \( m \) depending on the spacial variable. For a bounded measurable \( m(x,y), x, y \in \mathbb{R}^d \), consider the operator \( Au(x) = A_{z} u(x) |_{z=x}, x \in \mathbb{R}^d \), with \( u \in \mathcal{S}(\mathbb{R}^d) \) and
\[ A_{z} u(x) = A_{z} u(x) = \mathcal{A}_{z}^m u(x) = \int \nabla_y^\alpha u(x) m(z,y) \frac{dy}{|y|^{d+\alpha}}, z, x \in \mathbb{R}^d. \]
Lemma 11 Assume $\beta \in (0, 1), p > d/\beta$. Let for each $y \in \mathbb{R}^d$, $m(\cdot, y) \in H^\beta_p(\mathbb{R}^d)$ and

$$|m(z, y)| + |\partial^\beta_z m(z, y)| < \infty.$$  

Then

$$|Au|^p_p \leq C|\partial^\alpha u|^p_p \int \sup_y [|m(z, y)|^p + |\partial^\beta m(z, y)|^p]dz.$$  

Proof. By Sobolev embedding theorem, there is a constant $C$ such that

$$|A_z u(x)|^p \leq \sup_z |A_z u(x)|^p \leq C \int [|A_z u(x)|^p + |\partial^\beta_z A_z u(x)|^p]dz \quad (x \in \mathbb{R}^d),$$

and by Lemma 11

$$|Au|^p_p \leq C \int [|A_z u|^p_p + |A^\beta z u|^p_p]dz \leq C|\partial^\alpha u|^p_p \int \sup_y [|m(z, y)|^p + |\partial^\beta m(z, y)|^p]dz.$$  

The following statement holds.

Lemma 12 Let $u \in C_0^\infty(\mathbb{R}^d)$ have its support in a unit ball. Assume $\beta \in (0, 1), p > d/\beta$, and

$$|m(z, y)| + |\partial^\beta_z m(z, y)| \leq K, z, y \in \mathbb{R}^d.$$  

Then there is a constant $C = C(\alpha, p, \beta, d)$ independent of $u$ such that

$$|Au|^p_p \leq CK^p|u|^p_{\alpha, p}.$$  

Proof. Let the support of $u$ is a subset of the ball centered at $x_0$ with radius 1. Then for $x \in \mathbb{R}^d$,

$$Au(x) = \int_{|y| \leq 1} \nabla^\alpha_y u(x)m(x, y)\frac{dy}{|y|^{d+\alpha}} + \int_{|y| > 1} \nabla^\alpha_y u(x)m(x, y)\frac{dy}{|y|^{d+\alpha}}$$

$$= A_1(x) + A_2(x),$$

and

$$|A_2|^p_p \leq C|u|^p_{\alpha, p} \sup_{z, y} |m(z, y)|^p.$$  

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Let \( \varphi \in C_c^\infty(\mathbb{R}^d) \), \( 0 \leq \varphi \leq 1 \), \( \varphi(x) = 1 \) if \( |x| \leq 1 \), and \( \varphi(x) = 0 \) if \( |x| > 2 \). Then

\[
A_1(x) = \int \nabla_y^\alpha u(x) \varphi \left( \frac{x - x_0}{2} \right) m(x, y) \mathbb{1}_{|y| \leq 1} \frac{dy}{|y|^{d + \alpha}}
\]

and by Lemma 11,

\[
|A_1|^p \leq C |\partial^\alpha u|^p \left( \int \sup_y |\varphi \left( \frac{z - x_0}{2} \right) m(z, y) | \right)^p dz.
\]

For each \( y, z \in \mathbb{R}^d \),

\[
\partial_z^\beta \left( \varphi \left( \frac{z - x_0}{2} \right) m(z, y) \right)
\]

\[
= \int_{|v| > 1} \left[ \varphi \left( \frac{z + v - x_0}{2} \right) m(z + v, y) - \varphi \left( \frac{z - x_0}{2} \right) m(z, y) \right] \frac{dv}{|v|^{d + \beta}}
\]

\[
+ \varphi \left( \frac{z - x_0}{2} \right) \int_{|v| \leq 1} [m(z + v, y) - m(z, y)] \frac{dv}{|v|^{d + \beta}},
\]

\[
+m(z, y) \int_{|v| \leq 1} \left[ \varphi \left( \frac{z + v - x_0}{2} \right) - \varphi \left( \frac{z - x_0}{2} \right) \right] \frac{dv}{|v|^{d + \beta}}
\]

\[
+ \int_{|v| \leq 1} [m(z + v, y) - m(z, y)] \left[ \varphi \left( \frac{z + v - x_0}{2} \right) - \varphi \left( \frac{z - x_0}{2} \right) \right] \frac{dv}{|v|^{d + \beta}}
\]

\[
= G_0(z, y) + G_1(z, y) + G_2(z, y) + G_3(z, y).
\]

Obviously,

\[
\int \sup_y |G_0(z, y)|^p dz \leq CK^p \int |\varphi(z)|^p dz.
\]

Since

\[
\int_{|v| \leq 1} |m(z + v, y) - m(z, y)| \frac{dv}{|v|^{d + \beta}}
\]

\[
= \partial_z^\beta m(z, y) - \int_{|v| > 1} [m(z + v, y) - m(z, y)] \frac{dv}{|v|^{d + \beta}},
\]

we have

\[
\int \sup_y |G_1(z, y)|^p dz \leq CK^p \int |\varphi(z)|^p dz.
\]

Also,

\[
\int_{|v| \leq 1} \left| \varphi \left( \frac{z + v - x_0}{2} \right) - \varphi \left( \frac{z - x_0}{2} \right) \right| \frac{dv}{|v|^{d + \beta}}
\]

\[
\leq C \int_0^1 \int_{|v| \leq 1} \left| \nabla \varphi \left( \frac{z + sv - x_0}{2} \right) \right| \frac{dv}{|v|^{d + \beta - 1}}
\]

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and
\[ \int \sup_y |G_2(z,y)|^p dz \leq CK^p |\nabla \varphi|^p. \]

Finally,
\[ \int \sup_y |G_3(z,y)|^p dz \leq C \sup_{y,z} |m(z,y)|^p |\nabla \varphi|^p \]
and the statement follows. ■

**Corollary 13** Assume \( \beta \in (0, 1), p > d/\beta \), and
\[ |m(z,y)| + |\partial_x^\beta m(z,y)| \leq K, z, y \in \mathbb{R}^d. \]
Then there is a constant \( C = C(\alpha, \beta, p, d) \) such that
\[ |A u|^p \leq CK^p |u|^p_{\alpha,p}. \]

**Proof.** Let \( \zeta \in C_0^\infty(\mathbb{R}^d) \) be such that \( \int |\zeta|^p dx = 1 \) and \( \zeta \) has its support in the unit ball centered at the origin. Then for each \( x \in \mathbb{R}^d \),
\[ |A u(x)|^p = \int |\zeta(x-v)A u(x)|^p dv. \]

Obviously,
\begin{align*}
\zeta(x-v)A u(x) & = A (u(\cdot)\zeta(\cdot - v)) - u(x)A \zeta(x-v) \\
& + \int (u(x+y) - u(x)) (\zeta(x+y-v) - \zeta(x-v)) m(x,y) \frac{dy}{|y|^{d+\alpha}}.
\end{align*}

Now
\begin{align*}
A \zeta(x-v) & = \int [\zeta(x+y-v) - \zeta(x-v)] m(x,y) \frac{dy}{|y|^{d+\alpha}} \\
& = \int_{|y| > 1} [\zeta(x+y-v) - \zeta(x-v)] m(x,y) \frac{dy}{|y|^{d+\alpha}} \\
& + \int_{|y| \leq 1} [\zeta(x+y-v) - \zeta(x-v)] m(x,y) \frac{dy}{|y|^{d+\alpha}} \\
& = A_1(x,v) + A_2(x,v).
\end{align*}

Obviously,
\[ |A_1(x,v)| \leq K \int_{|y| > 1} |\zeta(x+y-v)| \frac{dy}{|y|^{d+\alpha}} + |\zeta(x-v)| \int_{|y| > 1} \frac{dy}{|y|^{d+\alpha}}. \]
and
\[ |A_2(x, v)| \leq K \int_{|y| \leq 1} \int_0^1 |\nabla \zeta(x + sy - v)| \frac{dy}{|y|^{d+\alpha-1}}. \]

Therefore,
\[ |A_2(x, v)| \leq CK^p. \]

Denoting
\[ D(x, v) = \int (u(x + y) - u(x)) (\zeta(x + y) - \zeta(x - v)) m(x, y) \frac{dy}{|y|^{d+\alpha}}, \]
we have
\[ |D(x, v)| \leq K \int_0^1 \int_{|y| \leq 1} |u(x + y) - u(x)| |\nabla \zeta(x + sy - v)| \frac{dy}{|y|^{d+\alpha-1}} + K \int_{|y| > 1} (|u(x + y)| + |u(x)| (|\zeta(x + y)| + |\zeta(x - v)|)) \frac{dy}{|y|^{d+\alpha}} \]

and
\[ \int \int |D(x, v)|^p dx dv \leq CK^p |u|^p_{\alpha', p} \]
for some \( \alpha' < \alpha \). Therefore, by Lemma [12]
\[ |Au|^p \leq CK^p \left[ \int |(\cdot)|^{\alpha'}(\cdot - v) \ |_{\alpha', p}^p dv + |u|^p_{\alpha', p} + |u|^p_{\alpha', p} \right]. \]

Since as in (20)
\[ \partial^p_x (u(x)\zeta(x - v)) = \partial^p u(x)\zeta(x - v) + u(x)\partial^p \zeta(x - v) + \int (u(x + y) - u(x)) (\zeta(x + y) - \zeta(x - v)) \frac{dy}{|y|^{d+\alpha}}, \]
we derive in a similar way,
\[ \int |(\cdot)|^{\alpha'}(\cdot - v) \ |_{\alpha', p}^p dv = \int |(\cdot)|^{\alpha'}(\cdot - v) \ |_{\alpha', p}^p dv + \int |\partial^p (\cdot)|^{\alpha'}(\cdot - v) \ |_{\alpha', p}^p dv \leq C |u|^p_{\alpha, p}. \]

The statement follows. ■
3.2 Solution for \( m \) independent of spacial variable

In this section, we consider the following partial case of equation (8):

\[
\begin{align*}
\partial_t u(t, x) &= A^m u(t, x) - \lambda u(t, x) + f(t, x), \\
u(0, x) &= 0,
\end{align*}
\]

(21)

where \( m(t, x, y) = m(t, y) \) does not depend on the spacial variable.

We denote by \( \mathcal{D}_p(E) \), \( p \geq 1 \), the space of all measurable functions \( f \) on \( E \) such that \( f \in \cap_{\kappa > 0} H^\kappa_p(E) \) and for every multiindex \( \gamma \in \mathbb{N}_0^d \)

\[
\sup_{(t,x) \in H} |D^\gamma f(t, x)| < \infty.
\]

The set \( \mathcal{D}_p(E) \) is a dense subset of \( H^\kappa_p(E) \) (see [12]).

**Lemma 14** (see Theorem 14 in [12]) Let \( p \geq 2, f \in \mathcal{D}_p(E) \) and Assumption A be satisfied.

Then there is a unique strong solution \( u \in \mathcal{D}_p(E) \) of (21). Moreover, \( u(t, x) \) is continuous in \( t \), smooth in \( x \) and the following assertions hold:

(i) for every multiindex \( \gamma \in \mathbb{N}_0^d \)

\[
|D^\gamma u|_p \leq C_{\rho, \lambda} |D^\gamma f|_p.
\]

where \( \rho, \lambda = T \wedge \frac{1}{\lambda} \) and the constant \( C = C(\alpha, p, d, K, \eta) \);

(ii) the following estimate holds:

\[
|u|_{\alpha, p} \leq C |f|_p,
\]

where the constant \( C = C(\alpha, p, d, T, K, \eta) \).

Passing to the limit we arrive at

**Proposition 15** Let \( p \geq 2, f \in L_p(E) \) and Assumption A be satisfied.

Then there is a unique strong solution \( u \in H^\alpha_p(E) \) of (21). Moreover, there are constants \( C_0 = C_0(\alpha, p, d, T, K, \eta) \) and \( C_{00} = C_{00}(\alpha, p, d, K, \eta) \) such that

\[
|u|_{\alpha, p} \leq C_0 |f|_p
\]

and

\[
|u|_p \leq C_{00} \rho, \lambda |f|_p
\]

where \( \rho, \lambda = T \wedge \frac{1}{\lambda} \).
Proof. Existence. There is a sequence of input functions $f_n$, $n = 1, 2, \ldots$, such that $f_n \in \mathcal{D}_p(E)$, and

$$|f - f_n|_p \to 0$$  \hspace{2cm} (22)

as $n \to \infty$. By Lemma 14, for every $n$ there is a strong solution $u_n \in \mathcal{D}_p(E)$ of (21) with the input function $f_n$. Since (21) is a linear equation, using the estimate (ii) of Lemma 14 we derive that $(u_n)$ is a Cauchy sequence in $H^0_p(E)$. Hence, there is a function $u \in H^0_p(E)$ such that $|u_n - u|_{\alpha,p} \to 0$ as $n \to \infty$.

Passing to the limit in the inequalities of Lemma 14 with $u, f$ replaced by $u_n, f_n$ ($\gamma = 0$), we get the corresponding estimates for $u$.

Denoting $\langle f, g \rangle = \int fg dx$ and passing to the limit in the equality (see definition (9))

$$\langle u_n(t, \cdot), \varphi \rangle = \int_0^t \left[ \langle (A - \lambda)u_n(s, \cdot) + f(s, \cdot), \varphi \rangle \right] ds, \varphi \in \mathcal{S}(\mathbb{R}^d),$$

as $n \to \infty$, we get that the function $u$ is a weak solution of (21).

Since for each $v \in H^0_p(E)$

$$|Av|_{\alpha,p} \leq C|v|_p,$$

the solution is strong.

Uniqueness. Let $u \in H^0_p(E)$ be a solution of (21) with zero input function $f$. Hence, for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $t \in [0, T]$

$$\langle u(t, \cdot), \varphi \rangle = \int_0^t \langle u(s, \cdot), A(\alpha)\varphi - \lambda\varphi \rangle ds$$  \hspace{2cm} (23)

Let $\zeta_\varepsilon = \zeta_\varepsilon(x), x \in \mathbb{R}^d, \varepsilon \in (0, 1)$, be a standard mollifier. Inserting $\varphi(\cdot) = \zeta_\varepsilon(x - \cdot)$ into (23), we get that the function

$$v_\varepsilon(t, x) = u(t, \cdot) * \zeta_\varepsilon(x)$$

belongs to $\mathcal{D}_p(E)$ and

$$v_\varepsilon(t, x) = \int_0^t (A - \lambda)v_\varepsilon(s, x)ds.$$

By Lemma 14, $v_\varepsilon = 0$ in $E$ for all $\varepsilon \in (0, 1)$. Hence, $u(t, \cdot) = 0$ and the statement holds. $\blacksquare$
4 Proofs of main Theorems

We follow the proof of Theorem 1.6.4 in [9]. In order to use the method of continuity, we derive the a priori estimates first.

Lemma 16 Assume A holds, $\beta \in (0,1), p > d/\beta, p \geq 2$. There are $\varepsilon = \varepsilon(d, \alpha, \beta, K, w, T, \eta), C = C(d, \alpha, \beta, p, K, w, T, \eta)$ and $\lambda_0 = \lambda_0(d, \alpha, \beta, p, K, w, T, \eta) \geq 1$ such that for any $u \in \mathcal{D}_p(E)$ satisfying (8) with $B = 0$ and with support in a ball of radius $\varepsilon$ ($u(t, x) = 0$ for all $t$ if $x$ does not belong to a ball of radius $\varepsilon$),

\[ |u|_{\alpha,p} \leq C|f|_p, \]
\[ |u|_p \leq \frac{C}{\lambda}|f|_p \text{ if } \lambda \geq \lambda_0. \]

Proof. Let the support of $u$ be a subset of the ball centered at $x_0$ with radius $\varepsilon > 0$. Then

\[ \partial_t u = A_{t,x_0}u(t, x) + A_{t,x}u(t, x) - A_{t,x_0}u(t, x) - \lambda u + f, \]
\[ u(0) = 0. \]

Let $\varphi \in C_0^\infty(\mathbb{R}^d), 0 \leq \varphi \leq 1, \varphi(x) = 1$ if $|x| \leq 1$, and $\varphi(x) = 0$ if $|x| > 2$. Denote

\[ \tilde{A} = \varphi\left(\frac{x - x_0}{2\varepsilon}\right)[A_{t,x}u(t, x) - A_{t,x_0}u(t, x)], \]
\[ m_0(t, x, y) = m(t, x, y) - m(t, x_0, y), (t, x) \in E, y \in \mathbb{R}^d. \]

By Corollary [13]

\[ |\tilde{A}|_p \leq C|u|_{\alpha,p}K_\varepsilon, \tag{24} \]

where $C = C(\alpha, \beta, p, d)$ and $K_\varepsilon$ is the constant bounding

\[ M(t, z, y) = |m_0(t, z, y)\varphi\left(\frac{z - x_0}{2\varepsilon}\right)| + |\partial^\beta\left(m_0(t, z, y)\varphi\left(\frac{z - x_0}{2\varepsilon}\right)\right)|. \]
Obviously, \(|m_0(t, z, y)\varphi(\frac{z-x_0}{2\varepsilon})| \leq w(2\varepsilon), z, y \in \mathbb{R}^d, t \in [0, T] ,\) and
\[
|\partial^\beta \left( m_0(t, z, y)\varphi(\frac{z-x_0}{2\varepsilon}) \right)| \\
\leq \int_{|v|>\varepsilon} \left| \frac{m_0(t, z+v, y)}{\varepsilon} - m_0(t, z, y) \right| \, dv + |\varphi(\frac{z-x_0}{2\varepsilon})| \int_{|v|\leq\varepsilon} \left| \frac{m(t, z+v, y)}{\varepsilon} - m(t, z, y) \right| \, dv \\
+ |m_0(t, z, y)| \int_{|v|\leq\varepsilon} \left| \varphi(\frac{z-x_0}{2\varepsilon}) - \varphi(\frac{z-x_0}{2\varepsilon}) \right| \, dv \\
+ \int_{|v|\leq\varepsilon} \left| \varphi(\frac{z+x_0}{2\varepsilon}) - \varphi(\frac{z-x_0}{2\varepsilon}) \right| \left| \frac{m(t, z+v, y)}{\varepsilon} - m(t, z, y) \right| \, dv \\
\leq C[w(2\varepsilon)]^{\varepsilon-\beta} + \int_{|v|\leq\varepsilon} w(|v|) \frac{dv}{|v|^{d+\beta}}.
\]
Therefore
\[
K_\varepsilon \leq C[w(2\varepsilon)]^{\varepsilon-\beta} + \int_{|v|\leq\varepsilon} w(|v|) \frac{dv}{|v|^{d+\beta}}
\]
and \(K_\varepsilon \to 0\) as \(\varepsilon \to 0\). Obviously,
\[
|Au - \tilde{A}|_p \leq C \int_{|v|>\varepsilon} |\nabla_y u(\cdot)|_p \frac{dy}{y^{d+\alpha}} \\
\leq C\varepsilon^{-\alpha} [\|u\| + 1_{\alpha>1} |\nabla u|_p].
\]
So, by Proposition 15 and (24), there are constants \(C_1 = C_1(\alpha, p, d, T, K, \eta)\) and \(C_{11} = C_{11}(\alpha, p, d, K)\) such that
\[
|u|_{\alpha,p} \leq C_1 \left[ |f|_p + K_\varepsilon|u|_{\alpha,p} + \varepsilon^{-\alpha} (|u|_p + 1_{\alpha>1} |\nabla u|_p) \right]
\]
with \(K_\varepsilon \to 0\) as \(\varepsilon \to 0\) and
\[
|u|_p \leq C_{11}\rho_\lambda \left[ |f|_p + K_\varepsilon|u|_{\alpha,p} + \varepsilon^{-\alpha} (|u|_p + 1_{\alpha>1} |\nabla u|_p) \right],
\]
where \(\rho_\lambda = \frac{1}{\varepsilon} \wedge T\). We choose \(\varepsilon\) so that \(C_1K_\varepsilon \leq 1/2, K_\varepsilon \leq 1\). In this case,
\[
|u|_{\alpha,p} \leq 2C_1 \left[ |f|_p + \varepsilon^{-\alpha} (|u|_p + 1_{\alpha>1} |\nabla u|_p) \right],
\]
\[
|u|_p \leq C_{11}(1 + 2C_1)\rho_\lambda \left[ |f|_p + \varepsilon^{-\alpha} (|u|_p + 1_{\alpha>1} |\nabla u|_p) \right].
\]
By interpolation inequality, for \(\alpha > 1\) and each \(\kappa \in (0, 1)\) there is a constant \(C = C(\alpha, p, d)\) such that
\[
|\nabla u|_p \leq \kappa |u|_{\alpha,p} + C\kappa^{-\alpha-1} |u|_p.
\]
Therefore choosing $\kappa$ so that $2C_1\varepsilon^{-\alpha}\kappa \leq \frac{1}{2}$ (if $\alpha > 1$), one can see that there is $\tilde{C}_1 = \tilde{C}_1(\alpha, \beta, p, d, T, K, w, \eta)$ such that
\[
|u|_{\alpha,p} \leq \tilde{C}_1\left(||f||_p + |u|_p\right),
\]
\[
|u|_p \leq \tilde{C}_1\rho_{\alpha}||f||_p + |u|_p.
\]
The statement follows by choosing $\lambda$ so that $\tilde{C}_1\lambda^{-1} \leq \frac{1}{2}$ ($\lambda_0 = (2\tilde{C}_1)^{-1}$).

Now we extend the estimates.

**Lemma 17** Assume A holds and $p > \frac{d}{2}, p \geq 2$. There is a constant $C = C(d, \alpha, \beta, p, K, \lambda, \eta, T) > 1$ such that for any $u \in \mathcal{D}_p(E)$ satisfying (3) with $B = 0$ and $\lambda \geq \lambda_1$,
\[
|u|_{\alpha,p} \leq C|f|_p,
\]
\[
|u|_p \leq \frac{C}{\lambda}f|_p \text{ if } \lambda \geq \lambda_1.
\]

**Proof.** As in [9], Theorem 1.6.4, take $\zeta \in C_c^\infty(\mathbb{R}^d)$ such that $\int |\zeta|^p dx = 1$ and whose support is in a ball of radius $\varepsilon\lambda$ from Lemma 16 centered at 0. Then
\[
|\partial^\alpha u(t,x)|^p = \int |\partial^\alpha u(t,x)\zeta(x-v)|^p dv
\]
and
\[
\partial^\alpha u(t,x)\zeta(x-v) = \partial^\alpha (u(t,x)\zeta(x-v)) - u(t,x)\partial^\alpha_x \zeta(x-v)
\]
\[
+ \int [u(t,x+y) - u(t,x)] [\zeta(x+y-v) - \zeta(x-v)] \frac{dy}{|y|^{d-\alpha}}.
\]
Since
\[
\partial_t (u(t,x)\zeta(x-v)) = \zeta(x-v)Au(t,x) - \lambda\zeta(x-v)u(t,x) + \zeta(x-v)f(t,x)
\]
\[
= A (\zeta(x-v)u(t,x)) - \lambda\zeta(x-v)u(t,x) + \zeta(x-v)f(t,x)
\]
\[
- \lambda u(t,x)A\zeta(x-v)
\]
\[
- \int [u(t,x+y) - u(t,x)] [\zeta(x+y-v) - \zeta(x-v)] m(t,x,y) \frac{dy}{|y|^{d-\alpha}},
\]
it follows by Lemma 16 that there is $C = C(d, \alpha, \beta, p, K, \lambda, \eta)$ and $\lambda_0 = \lambda_0(d, \alpha, \beta, p, K, \lambda, \eta)$ such that
\[
\int |u\zeta(-v)|_{\alpha,p}^p dv \leq C[|f||_{p}^{p} + |u|_{p}^{p} + |u|_{\alpha,p}^{p}],
\]
\[
\int |u\zeta(-v)|_{p}^{p} dv \leq \frac{C}{\lambda^{p}}[|f||_{p}^{p} + |u|_{p}^{p} + |u|_{\alpha,p}^{p}] \text{ if } \lambda \geq \lambda_0,
\]

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for some $\alpha' < \alpha$. According to (26) and (25),

$$
|\partial^\alpha u|_p \leq C[|f|_p^p + |u|_p^p + |u|_{\alpha',p}^p],
$$

(27)

$$
|u|_p^p \leq \frac{C}{\lambda^p}[|f|_p^p + |u|_p^p + |u|_{\alpha',p}^p] \text{ if } \lambda \geq \lambda_0.
$$

By interpolation inequality, for each $\kappa > 0$ there is a constant $K_1 = K_1(\kappa, \alpha', \alpha, p, d)$ such that

$$
|u|_{\alpha',p} \leq \kappa |u|_{\alpha,p} + K_1 |u|_p.
$$

Therefore, choosing $\kappa$ so that $C\kappa \leq \frac{1}{2}$, we get by (27) that there is a constant $C_1 = C_1(d, \alpha, \beta, p, K, w, T, \eta)$ such that

$$
|u|_p^p \leq \frac{C_1}{\lambda^p}[|f|_p^p + |u|_p^p] \text{ if } \lambda \geq \lambda_0.
$$

We finish the proof by choosing $\lambda$ so that $\frac{C_1}{\lambda^p} \leq \frac{1}{2}$ or $\lambda \geq (2C_1)^{1/p} = \lambda_1$. Thus by (28),

$$
|u|_p^p \leq \frac{2C_1}{\lambda^p} |f|_p^p, |u|_{\alpha,p}^p \leq C_1(1 + \frac{2C_1}{\lambda^p}) |f|_p^p.
$$

The statement follows.

**Corollary 18** Assume $A$ holds, $p > \frac{d}{\beta}, p \geq 2$ and $u \in D_p(E)$ satisfies (8) with $B = 0$. Then there is $C = C(d, \alpha, \beta, p, K, w, T, \eta)$ such that

$$
|u|_{\alpha,p} \leq C |f|_p.
$$

**Proof.** For $\lambda \geq \lambda_1$ ($\lambda_1$ is from Lemma 17), the estimate is proved in Lemma 17. If $u \in H_p^\alpha(E)$ solves (8) with $\lambda \leq \lambda_1$, then $\tilde{u}(t, x) = e^{(\lambda_1 - \lambda)t} u(t, x)$ solves the same equation with $\lambda = \lambda_1$ and $f$ replaced by $e^{(\lambda_1 - \lambda)t} f$. Hence

$$
|u|_{\alpha,p} \leq |\tilde{u}|_{\alpha,p} \leq C e^{(\lambda_1 - \lambda)T} |f|_p
$$

with $C = C(d, \alpha, \beta, p, K, w, T, \eta)$ from Lemma 17. So, the estimate holds for all $\lambda \geq 0$. □
4.1 Proof of Theorem 3

We use the a priori estimate and the continuation by parameter argument. Let

$$M_\tau u = \tau Lu + (1 - \tau) \partial^\alpha u, \tau \in [0, 1].$$

We introduce the space $H^\alpha_p(E)$ of functions $u \in H^\alpha_p(E)$ such that for each,

$$u(t, x) = \int_0^t F(s, x) \, ds,$$

where $F \in H^\alpha_p(E)$. It is a Banach space with respect to the norm

$$||u||_{\alpha, p} = |u|_{\alpha, p} + |F|_p.$$

Consider the mappings $T_\tau : H^\alpha_p(E) \to L_p(E)$ defined by

$$u(t, x) = \int_0^t F(s, x) \, ds \mapsto F - M_\tau u.$$

Obviously, for some constant $C$ not depending on $\tau$,

$$|T_\tau u|_p \leq C|u|_{\alpha, p}.$$

On the other hand, there is a constant $C$ not depending on $\tau$ such that for all $u \in H^\alpha_p(E)$

$$||u||_{\alpha, p} \leq C|T_\tau u|_p. \quad (29)$$

Indeed,

$$u(t, x) = \int_0^t F(s, x) \, ds = \int_0^t (M_\tau u + (F - M_\tau u)) (s, x) \, ds,$$

and, according to Corollary 18, there is a constant $C$ not depending on $\tau$ such that

$$|u|_{\alpha, p} \leq C|T_\tau u|_p = C|F - M_\tau u|_p. \quad (30)$$

Thus,

$$||u||_{\alpha, p} = |u|_{\alpha, p} + |F|_p \leq |u|_{\alpha, p} + |F - M_\tau u|_p + |M_\tau u|_p \leq C \left( |u|_{\alpha, p} + |F - M_\tau u|_p \right) \leq C|F - M_\tau u|_p = C|T_\tau u|_p,$$

and (29) follows. Since $T_0$ is an onto map, by Theorem 5.2 in [3] all the $T_\tau$ are onto maps and Theorem 3 follows.
4.2 Proof of Theorem 5

Assume $A$ holds and $p > \frac{d}{\beta} \lor \frac{d}{\alpha}, p \geq 2$ and $u \in \mathcal{D}_p(E)$ satisfies (8) with $B = B^{\varepsilon_0}$. By Theorem [3],

$$|\partial_t u|_p + |u|_{\alpha, p} \leq N[|f|_p + |B^{\varepsilon_0} u|_p],$$

$$|u|_p \leq \frac{N}{\lambda} |f|_p + |B^{\varepsilon_0} u|_p$$

if $\lambda \geq \lambda_1$,

where $\lambda_1 = \lambda_1(T, \alpha, \beta, d, K, w, \eta) \geq 1$. According to Lemma [4],

$$|\partial_t u|_p + |u|_{\alpha, p} \leq 2N|f|_p, \quad (31)$$

$$|u|_p \leq \frac{2N}{\lambda} |f|_p$$

if $\lambda \geq \lambda_1$.

If $u \in \mathcal{D}_p(E)$ satisfies (8) with $B = B^{\varepsilon_0}, \lambda \leq \lambda_1$, then $\tilde{u}(t, x) = e^{(\lambda_1 - \lambda)t} u(t, x)$ satisfies the same equation with $\lambda_1$ with $f$ replaced by $e^{(\lambda_1 - \lambda)t} f$. By (31),

$$|u|_{\alpha, p} \leq |\tilde{u}|_{\alpha, p} \leq 2Ne^{(\lambda_1 - \lambda)t}|f|_p. \quad (32)$$

The statement follows by the a priori estimates (31)-(32) and the continuation by parameter argument, repeating the proof of Theorem [3] for the operators

$$M_\tau = A + \tau B^{\varepsilon_0}, 0 \leq \tau \leq 1.$$

4.3 Proof of Theorem 6

Again we derive the a priori estimates first and use the continuation by parameter argument. There is $\varepsilon_0 \in (0, 1)$ such that

$$\int_{|y| \leq \varepsilon_0} |y|^\alpha \pi(t, x, dy) \leq \delta_0, (t, x) \in E,$$

where $\delta_0$ is a number in Theorem [5]. Let $u \in \mathcal{D}_p(E)$ satisfy (8). Let

$$\tilde{L} v(t, x) = A v(t, x) + B^{\varepsilon_0} v(t, x),$$

$(t, x) \in E$, where $B^{\varepsilon_0} v$ is defined in (10). Applying Theorem [5] to $\tilde{L}$, we have

$$|\partial_t u|_p + |u|_{\alpha, p} \leq N_2[|f|_p + |(L - \tilde{L}) u|_p],$$

$$|u|_p \leq \frac{N_2}{\lambda} |f|_p + |(L - \tilde{L}) u|_p$$

if $\lambda \geq \lambda_2$. 

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There is $\alpha' < \alpha$ such that $p > d/\alpha'$ and by Sobolev embedding theorem there is a constant $C$ such that
\[
|(L - \bar{L})u|_p \leq C|u|_{\alpha',p} \left( \int_0^T \int \pi(t,x,\{|y| > \epsilon_0\})^p \, dx \, dt \right)^{1/p}
\]
By interpolation inequality, for each $\kappa > 0$ there is a constant $\tilde{N} = \tilde{N}(\kappa, K_2, \alpha, \alpha', p, d)$ such that
\[
|(L - \bar{L})u|_p \leq \kappa |u|_{\alpha,p} + \tilde{N}|u|_p,
\]
where $K_2$ is a constant bounding $\int_0^T \int \pi(t, x, \{|y| > \epsilon_0\})^p \, dx \, dt$. Choose $\kappa$ so that $2N\kappa \leq 1/2$. Then
\[
|\partial_t u|_p + |u|_{\alpha,p} \leq 4N(|f|_p + \tilde{N}|u|_p),
\]
\[
|u|_p \leq \frac{4N}{\lambda}(|f|_p + \tilde{N}|u|_p) \text{ if } \lambda \geq \lambda_1.
\]
Choosing $\lambda \geq \lambda_2 = 8N\tilde{N}$, we derive
\[
|u|_p \leq \frac{8N}{\lambda} |f|_p,
\]
\[
|\partial_t u|_p + |u|_{\alpha,p} \leq 8N|f|_p.
\]
Multiplying $u$ by $e^{(\lambda - \lambda_2)t}$, we obtain the a priori estimate for all $\lambda \geq 0$ as in the proof of Corollary 18 above.

The statement follows by the a priori estimates and the continuation by parameter argument, repeating the proof of Theorem 3 for the operators
\[
M_\tau v = \bar{L}v + \tau(L - \bar{L})v, 0 \leq \tau \leq 1.
\]

5 Embedding of the solution space

Following the main steps of Section 7 in [8], we will show that for a sufficiently large $p$, the Hölder norm of the solution is finite. Since the solution of (8) $u \in H^\alpha_p(E)$, we will derive an embedding theorem for $H^\alpha_p(E)$.

Remark 19 If $u \in H^\alpha_p(E)$, then $u \in H^\alpha_p(E)$ and
\[
u(t) = \int_0^t F(s) ds, 0 \leq t \leq T,
\]
with \( F \in L_p(E) \). It is the \( \mathcal{H}^\alpha_p \)-solution to the equation

\[
\begin{align*}
\partial_t u &= \partial^\alpha u + f, \\
u(0) &= 0,
\end{align*}
\]

(33)

where \( f = F - \partial^\alpha u \in L_p(E) \) with \(|f|_p \leq |F|_p + |\partial^\alpha u|_p \leq ||u||_{\alpha,p} \). In addition (e.g., see [12]),

\[
u(t,x) = \int_0^t G_{t-s}(x-y)f(s,y)dyds, 0 \leq t \leq T, x \in \mathbb{R}^d,
\]

(34)

where

\[
G_t = F^{-1}\left[e^{-t|\xi|^\alpha}\right], t > 0,
\]

(35)

(here \( F^{-1} \) is the inverse Fourier transform). The function \( G_t \) is the probability density function of a spherically symmetric \( \alpha \)-stable process whose generator is the fractional Laplacian \( \partial^\alpha \):

\[
\int G_t dx = 1, t > 0.
\]

(36)

**Remark 20** Note that for any multiindex \( \gamma \in \mathbb{N}_0^d \) there is a constant \( C = C(\alpha, \gamma, d) \) such that

\[
|D_\gamma e^{-|\xi|^\alpha}| \leq Ce^{-|\xi|^\alpha} \sum_{1 \leq k \leq |\gamma|} |\xi|^{k\alpha - |\gamma|}.
\]

(37)

**Lemma 21** Let \( K(x) = G_1(x), x \in \mathbb{R}^d \). Then

(i) \( K \) is smooth and for all multiindices \( \gamma \in \mathbb{N}_0^d, \kappa \in (0, 2) \),

\[
\int |\partial^\kappa D^\gamma K(x)| dx < \infty.
\]

(ii) for \( t > 0, x \in \mathbb{R}^d \),

\[
G_t(x) = t^{-d/\alpha} K(x/t^1/\alpha)
\]

and for any multiindex \( \gamma \in \mathbb{N}_0^d \) and \( \kappa \in (0, 2) \), there is a constant \( C \) such that

\[
|\partial^\kappa D^\gamma G_t * v|_p \leq Ct^{-(|\gamma|+\kappa)/\alpha}|v|_p, t > 0, v \in L_p(\mathbb{R}^d).
\]

(iii) Let \( \kappa \in (0, 1) \). There is a constant \( C \) such that for \( v \in \mathcal{S}(\mathbb{R}^d), t > 0 \),

\[
|G_t * v - v|_p \leq Ct^\kappa |\partial^\alpha v|_p.
\]
Proof. (i) For any multiindex $\gamma \in \mathbb{N}_0^d$,

$$
\sup_x |D^\gamma K(x)| \leq \int |(i\xi)^\gamma e^{-|\xi|^\alpha}|d\xi < \infty.
$$

Let $\varphi \in C^\infty_0(\mathbb{R}^d), 0 \leq \varphi \leq 1, \varphi(x) = 1$ if $|x| \leq 1, \varphi(x) = 0$ if $|x| \geq 2$. Then $K(x) = K_1(x) + K_2(x)$ with

$$
K_1 = \mathcal{F}^{-1}\left(e^{-|\xi|^\alpha}\varphi(\xi)\right), \quad K_2 = \mathcal{F}^{-1}\left([1 - \varphi(\xi)]e^{-|\xi|^\alpha}\right).
$$

Since $\psi = \mathcal{F}^{-1} \varphi \in S(\mathbb{R}^d)$, we have $K_1(x) = K \ast \psi(x)$. Therefore, by (36), for any multiindex $\gamma \in \mathbb{N}_0^d, \kappa \in (0, 2),$

$$
\sup_x |\partial^\kappa D^\gamma K_1(x)| \leq \sup_x |\partial^\kappa D^\gamma \psi(x)| < \infty,
$$

$$
\int |\partial^\kappa D^\gamma K_1(x)|dx \leq \int |\partial^\kappa D^\gamma \psi(x)|dx < \infty.
$$

By Parseval’s equality and (37), for any multiindices $\gamma, \mu, \kappa \in (0, 2),$

$$
\int |\partial^\kappa D^\gamma K_2(x)|^2dx = \int |(i\xi)^\gamma |\xi|^\kappa [1 - \varphi(\xi)] e^{-|\xi|^\alpha}|^2d\xi < \infty,
$$

$$
\int |(ix)^\mu \partial^\kappa D^\gamma K_2(x)|^2dx = \int |D^\mu \left((i\xi)^\gamma |\xi|^\kappa e^{-|\xi|^\alpha}\right)|^2 [1 - \varphi(\xi)] d\xi < \infty.
$$

Therefore, by Cauchy-Schwartz inequality with $d_1 = \left[\frac{d}{4}\right] + 1,$

$$
\int |\partial^\kappa D^\gamma K_2(x)|dx \leq \left\{ \int (1 + |x|^2)^{2d_1} |D^\gamma K_2(x)|^2dx \right\}^{1/2} \times
$$

$$
\times \left\{ \int (1 + |x|^2)^{-2d_1}dx \right\}^{1/2}.
$$

(ii) Changing the variable of integration in (35) we get $G_t(x) = t^{-d/\alpha}K(x/t^{1/\alpha}), x \in \mathbb{R}^d, t > 0$. For any $v \in S(\mathbb{R}^d), \gamma \in \mathbb{N}_0^d, \kappa \in (0, 2),$

$$
\partial^\kappa D^\gamma G_t \ast v(x) = \int \partial^\kappa D^\gamma G_t(x - y)v(y)dy
$$

$$
= t^{-d/\alpha - (|\gamma| + \kappa)/\alpha} \int \partial^\kappa D^\gamma K((x - y)/t^{1/\alpha})v(y)dy
$$

$$
= t^{-d/\alpha - (|\gamma| + \kappa)/\alpha} \int \partial^\kappa D^\gamma K(y/t^{1/\alpha})v(x - y)dy
$$

and the statement follows.
(iii) Since for \( v \in \mathcal{S}(\mathbb{R}^d) \)

\[
G_t * v - v = \int_0^t \partial^\alpha G_s * v ds, 0 \leq t,
\]

it follows by part (ii),

\[
|G_t * v - v|_p \leq \int_0^t |\partial^\alpha G_s * v|_p ds = \int_0^t |\partial^\alpha(1-\kappa) G_s * \partial^\alpha v|_p ds \\
\leq \int_0^t s^{\kappa-1} ds |\partial^\alpha v|_p \leq Ct^{\kappa} |\partial^\alpha v|_p.
\]

We will need the following embedding estimate as well.

**Lemma 22** (see Lemma 7.4 in [8]) Let \( \mu \in (0, 1], \mu p > 1, p \geq 1, \kappa \in (0, 1] \).

Let \( h(t) \) be a continuous \( H^\mu_p(\mathbb{R}^d) \)-valued function. Then there is a constant \( C = C(d, \mu) \) such that for \( s \leq t \),

\[
|\partial^\alpha(1-\kappa)[h(t) - h(s)]|_p \leq C(t-s)^{\mu p-1} \int_0^{t-s} \frac{dr}{r^{1+\mu p}} \int_s^{t-r} |\partial^\alpha[1-\kappa] h(v+r) - h(v)|_p dv.
\]

**Proposition 23** Assume \( p > 2, f \in \mathcal{D}_p(E) \), and

\[
u(t) = \int_0^t G_s * f(s) ds, 0 \leq t \leq T.
\]

Let \( 1 - \frac{1}{p} > \kappa \geq \frac{1}{2} \) (note \( \frac{1}{p} < 1 - \kappa \leq \frac{1}{2} \)). Then there is a constant \( C \) such that for all \( 0 \leq s \leq t \leq T \),

\[
|\partial^\alpha(1-\kappa)[u(t) - u(s)]|_p \leq C(t-s)^{\kappa p-1} ||f||_p^p + ||\partial^\alpha u||_p^p.
\]

**Proof.** We apply Lemma 22 to \( u(t) = \int_0^t G_{t-s} * f(s) ds, 0 \leq t \leq T \). Since \( G_{t+s} = G_t * G_s \), it follows for \( v, r \geq 0 \),

\[
\begin{align*}
u(v+r) - \nu(v) &= \int_0^{v+r} G_{v+r-\tau} * f(\tau) d\tau - \int_0^v G_{v-\tau} * f(\tau) d\tau \\
&= \int_0^{v+r} G_{v+r-\tau} * f(\tau) d\tau \\
&+ \int_0^v G_{v+r-\tau} * f(\tau) d\tau - \int_0^v G_{v-\tau} * f(\tau) d\tau \\
&= \int_0^r G_{r-\tau} * f(v+\tau) d\tau + G_r * \nu(v) - \nu(v).
\end{align*}
\]
By Hölder inequality and Lemma 21 for $r > 0$,

$$\left| \partial^{\alpha(1-\kappa)} \int_{0}^{r} G_{r-\tau} * f(v + \tau) d\tau \right|^{p}$$

$$= \left| \int_{0}^{r} v^{\kappa} v^{-\kappa} \partial^{\alpha(1-\kappa)} G_{r-\tau} * f(v + r - \tau) d\tau \right|^{p}$$

$$\leq \left( \int_{0}^{r} v^{-\kappa q} dv \right)^{p/q} \int_{0}^{r} v^{\kappa p} \left| \partial^{\alpha(1-\kappa)} G_{r-\tau} * f(v + r - \tau) \right|^{p} d\tau$$

$$\leq r^{(1-\kappa)p-1} \int_{0}^{r} v^{(2\kappa-1)p} |f(v + r - \tau)|^{p} d\tau$$

$$\leq r^{(1-\kappa)p-1, (2\kappa-1)p} \int_{0}^{r} |f(v + r - \tau)|^{p} d\tau$$

$$= r^{\kappa p-1} \int_{0}^{r} |f(v + r - \tau)|^{p} d\tau.$$

By Lemma 21 for $r, v > 0$,

$$\left| \partial^{\alpha(1-\kappa)} [G_{r} * u(v) - u(v)] \right|^{p} \leq C_{r}^{p-1} \left| \partial^{\alpha} u(v) \right|^{p}.$$

Therefore for a fixed $\mu \in (0, 1/2)$,

$$\int_{0}^{t-s} \frac{dr}{r^{1+\mu p}} \int_{s}^{t-r} \left| \partial^{\alpha(1-\kappa)} [u(v + r) - u(v)] \right|^{p} dv.$$

$$\leq C \left[ \int_{0}^{t-s} \frac{dr}{r^{1+\mu p}} \int_{s}^{t-r} r^{\kappa p-1} \int_{0}^{r} |f(v + r - \tau)|^{p} d\tau dv + \right.$$

$$\left. \int_{0}^{t-s} \frac{dr}{r^{1+\mu p}} \int_{s}^{t-r} r^{\kappa p} \left| \partial^{\alpha} u(v) \right|^{p} dv \right]$$

$$\leq C \left[ \int_{0}^{t-s} \frac{dr}{r^{2+(\kappa-\mu)p}} \int_{s}^{t-r+\tau} \int_{s+\tau}^{t-r+\tau} |f(v)|^{p} dv d\tau + \right.$$

$$\left. \int_{0}^{t-s} \frac{dr}{r^{1+(\kappa-\mu)p}} \int_{s}^{t-r} \left| \partial^{\alpha} u(v) \right|^{p} dv \right]$$

$$\leq C \left[ |f|^{p}_{p} + |\partial^{\alpha} u|^{p}_{p} (t-s)^{(\kappa-\mu)p} \right],$$

and by Lemma 22 applied for $\mu \in (0, 1/2)$,

$$\left| \partial^{\alpha(1-\kappa)} [u(t) - u(s)] \right|^{p} \leq C (t-s)^{\kappa p-1} \left[ |f|^{p}_{p} + |\partial^{\alpha} u|^{p}_{p} \right].$$

$$\blacksquare$$
Corollary 24 Let \( u \in H^{\alpha,p} \), \( p > 2, p > 2d/\alpha, \beta = \frac{\alpha}{2} - \frac{d}{p} \). Then there is a Hölder continuous modification of \( u \) on \( E \) and a constant \( C \) independent of \( u \) such that

\[
\sup_{s,x} |u(s,x)| + \sup_{s,x \neq x'} \frac{|u(s,x) - u(s,x')|}{|x - x'|^{\beta}} \leq C||u||_{\alpha,p}.
\]

**Proof.** By Proposition 23 with \( \kappa = 1/2 \) and Remark 19, \( u \) is Hölder continuous and

\[
\sup_{0 \leq s \leq T} |u(s,\cdot)|_{\alpha/2,p} \leq C||u||_{\alpha,p}.
\]

By Sobolev embedding theorem, there is a constant \( C \) such that

\[
\sup_{0 \leq s \leq T} |u(s,x)| + \sup_{s,x \neq x'} \frac{|u(s,x) - u(s,x')|}{|x - x'|^{\beta}} \leq \sup_{0 \leq s \leq T} |u(s,\cdot)|_{\alpha/2,p} \leq C||u||_{\alpha,p}.
\]

\[\blacksquare\]

6 Martingale problem

In this section, we consider the martingale problem associated with the operator

\[ L = A + B. \]

Let \( D = D([0,T],\mathbb{R}^d) \) be the Skorokhod space of cadlag \( \mathbb{R}^d \)-valued trajectories and let \( X_t = X_t(w) = w_t, w \in D \), be the canonical process on it.

Let

\[ D_t = \sigma(x_t, s \leq t), D = \bigvee_t D_t, \mathbb{D} = (D_{t+}), \quad t \in [0,T]. \]

We say that a probability measure \( \mathbb{P} \) on \( (D, \mathcal{D}) \) is a solution to the \((s,x,L)\)-martingale problem (see [16], [11]) if \( \mathbb{P}(X_r = x, 0 \leq r \leq s) = 1 \) and for all \( v \in C_0^\infty(\mathbb{R}^d) \) the process

\[ M_t(v) = v(X_t) - \int_0^t Lv(r, X_r)dr \quad (38) \]

is a \((\mathbb{D}, \mathbb{P})\)-martingale. We denote \( S(s,x) \) the set of all solutions to the problem \((s,x,L)\)-martingale problem.

A modification of Theorem 5 in [11] is the following statement.
Proposition 25 Let Assumptions A and B(i)-(ii) hold. Then for each \((s,x) \in E\) there is a unique solution \(P_{s,x}\) to the martingale problem \((s,x,L)\), and the process \((X_t, \mathbb{D}, (P_{s,x}))\) is strong Markov.

If, in addition,

\[
\lim_{l \to \infty} \int_0^T \sup_x \pi(t,x,\{|v| > l\})dt = 0, \tag{39}
\]

then the function \(P_{s,x}\) is weakly continuous in \((s,x)\).

6.1 Auxiliary results

We will need the following \(L_p\)-estimate.

Lemma 26 (cf. Lemma 3.6 in [11]) Let Assumptions A and B(i)-(ii) hold. Let \(p > \frac{d}{\beta} \lor \frac{d}{2\alpha} \lor 2\), \((s_0, x_0) \in E, P \in S(s_0, x_0, L)\).

Then there is a constant \(C = C(R, T, K, \eta, \beta, w, p)\) such that for any \(f \in C_0^\infty(E)\),

\[
P \int_{s_0}^T f(r, X_r)dr \leq C|f|_p.
\]

Proof. Let \(\zeta \in C_0^\infty(\mathbb{R}^d), \zeta \geq 0, \zeta(x) = \zeta(|x|), \zeta(x) = 0\) if \(|x| \geq 1\), and \(\int \zeta^p dx = 1\). For \(\delta > 0\) denote \(\zeta_{\delta}(x) = \varepsilon^{-d/p} \zeta(x/\delta), x \in \mathbb{R}^d\). Let

\[
u_{\delta}(t,x) = \int u(t, x-y)\zeta_{\delta}^p(y)dy
= \int u(t,y)\zeta_{\delta}^p(x-y)dy, (t,x) \in E.
\]

Let

\[
\tilde{L}v = Av + B^{\varepsilon_0}v,
\]

where \(B^{\varepsilon_0}\) is defined by (10) with \(\varepsilon_0\) so that the assumptions of Theorem 5 hold. Then

\[
L v = \tilde{L}v + Rv
\]

with

\[
Rv(t,x) = \int_{|y| > \varepsilon_0} [v(x+y) - v(x)]\pi(t,x,dy).
\]

Define

\[
\tilde{L}_{\delta}v = Av + B^{\varepsilon_0,\delta}v,
\]

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where
\[ B^\varepsilon_v(t, x) = \frac{P[\zeta_p^\varepsilon(X_t - x)B_t^\varepsilon_v(x)]}{P\zeta_p^\varepsilon(X_t - x)} \]
(here we assume \( \frac{\mu}{\nu} = 0 \)). Since for \( \tilde{L}^\delta \) the assumptions of Theorem 5 hold uniformly in \( \delta \), there is \( u = u^\delta \in H^\alpha_p(E) \) solving
\[
\partial_t u(t, x) + \tilde{L}^\delta u(t, x) = f(t, x), \quad (t, x) \in E,
\]
\[ u(T, x) = 0, \quad x \in \mathbb{R}^d. \]
Moreover, there is a constant \( C \) independent of \( \delta \) such that
\[ ||u^\delta||_{\alpha, p} \leq C |f|_p. \] (40)
In addition, by Corollary 24 and (40), there is a constant independent of \( \delta \) such that
\[ \sup_{s, x} |u^\delta(s, x)| \leq C |f|_p. \] (41)
Applying Ito formula to \( u^\delta(t, x) = \int \zeta_p^\varepsilon(x - z)u^\delta(t, z)dz = \int \zeta_p^\varepsilon(z)u^\delta(t, x - z)dz \), we have
\[
- u^\delta(s_0, x_0) = \int_{s_0}^{T} [\partial_t u^\delta(r, z) + (A + B^\varepsilon_v)u^\delta(r, z)]\kappa^\delta(t, z)dz \\
+ P \int_{s_0}^{T} Ru^\delta(r, X_r)dr + \int_{s_0}^{T} R_2(r)dr \] (42)
where \( \kappa^\delta(t, z) = P\zeta_p^\varepsilon(X_t - z) \) and
\[
R_2(r) = \int P[\zeta_p^\varepsilon(X_r - z)A_r, X_r, u^\delta(r, z) - \zeta_p^\varepsilon(X_r - z)A_r, z]dz \\
= P \int \int \nabla_y^{\alpha} u^\delta(r, z)[m(r, X_r, y) - m(r, z, y)] \frac{dy}{|y|^{d+\alpha}} \zeta_p^\varepsilon(X_r - z)dz,
\]
satisfies for any \( r \in [s_0, T] \),
\[
|R_2(r)|^p \leq P \int \int \nabla_y^{\alpha} u^\delta(r, z)[m(r, X_r, y) - m(r, z, y)] \zeta_p^\varepsilon(X_r - z) \frac{dy}{|y|^{d+\alpha}} |dz|.
\]
We will show that
\[
\int_{s_0}^{T} |R_2(r)|^p dr \to 0 \quad \text{as} \quad \delta \to 0. \] (43)
According to Lemma [11]
\[ |R_2(r)|^p \leq C|\partial^\alpha u_\delta(r)|^p \int \sup_y ||M(r, z, y)||^p + |\partial^\beta M(r, z, y)||^p|dz, \]
where
\[ M(r, z, y) = [m(r, X_r, y) - m(r, z, y)]\zeta_\delta(X_r - z). \]
Obviously,
\[ |m(r, X_r, y) - m(r, z, y)| \zeta_\delta(X_r - z) \leq w(\delta)\zeta_\delta(X_r - z) \]
and
\[ \int \sup_y ||M(r, z, y)||^p dz \leq \int w(\delta)^p \zeta_\delta(X_r - z) dz = w(\delta)^p. \]
Denoting \( m_0(r, z, y) = m(r, X_r, y) - m(r, z, y) \), we have
\[ |\partial^\beta (m_0(r, z, y)\zeta_\delta(z - X_r))| \]
\[ \leq \int_{|v| > \delta} |m_0(t, z + v, y)\zeta_\delta(z + v - X_r) - m_0(t, z, y)\zeta_\delta(z - X_r)| \frac{dv}{|v|^{d+\beta}} \]
\[ + \zeta_\delta(z - X_r) \int_{|v| \leq \delta} |m(r, z + v, y) - m(r, z, y)| \frac{dv}{|v|^{d+\beta}} \]
\[ + |m_0(t, z, y)| \int_{|v| \leq \delta} |\zeta_\delta(z + v - X_r) - \zeta_\delta(z - X_r)| \frac{dv}{|v|^{d+\beta}} \]
\[ + \int_{|v| \leq \delta} |\zeta_\delta(z + v - X_r) - \zeta_\delta(z - X_r)| |m(r, z + v, y) - m(r, z, y)| \frac{dv}{|v|^{d+\beta}} \]
\[ = (H_1 + H_2 + H_3 + H_4)(r, z, y), (r, z, y) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d. \]
Obviously,
\[ \int \sup_y H_2(r, z, y)^p dz \leq \left( \int_{|v| \leq \delta} w(v) \frac{dv}{|v|^{d+\beta}} \right)^p. \]
It follows, by Hölder inequality,
\[ \int \sup_y H_1(r, z, y)^p dz \leq Cw(\delta)^p \delta^{-\beta}, \]
\[ \int \sup_y H_4(r, z, y)^p dz \leq C \left( \int_{|v| \leq \delta} w(v) \frac{dv}{|v|^{d+\beta}} \right)^p. \]
Changing the variable of integration,
\[ H_3(r, z, y) \leq w(2\delta) \int_{|v| \leq \delta} |\zeta_\delta(z + v - X_r) - \zeta_\delta(z - X_r)| \frac{dv}{|v|^{d+\beta}} \]
\[ = w(2\delta)\delta^{-\beta} \delta^{-d/p} \int_{|v| \leq 1} \left| \zeta \left( \frac{z - X_r}{\delta} + v \right) - \zeta \left( \frac{z - X_r}{\delta} \right) \right| \frac{dv}{|v|^{d+\beta}} \]
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and

\[
\int \sup_y H_3(r, z, y)^p dz \\
\leq \ w(2\delta)^p \delta^{-\beta_1} \int (\int_{|v| \leq 1} |\zeta(z + v) - \zeta(z)| \frac{dv}{|v|^{d+\beta}})^p dz \\
\leq Cw(2\delta)^p \delta^{-\beta_1}.
\]

Therefore by (40), (43) follows. Since

\[
\int T_{s_0} P_t f(r, X_r) dr = - \left( u_\delta(s_0, x_0) + P \int_{s_0}^T R u_\delta(r, X_r) dr + \int_{s_0}^T R_2(r) dr \right)
\]

(see (42)), the statement follows by (41) and (43) passing to the limit as \( \delta \to 0 \).

**Corollary 27** Let Assumptions A and B hold, \((s_0, x_0) \in E\). Then the set \(S(s_0, x_0, L)\) consists of at most one probability measure.

**Proof.** Let \( f \in C_0^\infty(E), p > \frac{d}{\beta} \lor \frac{2d}{\alpha} \lor 2 \). By Theorem 6 there is \( u \in \mathcal{H}_p^\alpha(E) \) solving

\[
\partial_t u(t, x) + Lu(t, x) = f(t, x), \ (t, x) \in E; \\
u(T, x) = 0, x \in \mathbb{R}^d.
\]

Let \( \varphi \in C_0^\infty(\mathbb{R}^d), \varphi \geq 0, \int \varphi dx = 1, \varphi_\varepsilon(x) = \varepsilon^{-d} \varphi(x/\varepsilon), x \in \mathbb{R}^d, \) and

\[
u_\varepsilon(t, x) = \int u(t, x - y) \varphi_\varepsilon(y) dy, \ (t, x) \in \mathbb{R}^d.
\]

Applying Ito formula, we have

\[-u_\varepsilon(s_0, x_0) = P \int_{s_0}^T [\partial_t u_\varepsilon(r, X_r) + Lu_\varepsilon(r, X_r)] dr.
\]

Using Lemma 26 and Corollary 24 to pass to the limit we derive that

\[-u(s_0, x_0) = P \int_{s_0}^T f(r, X_r) dr
\]

and the uniqueness follows by Lemma 2.4 in [11].

Now we can construct a "local" solution of the martingale problem.

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Lemma 28 Let Assumptions A, B(i)-(ii) hold, \( \pi(t, x, dv) = \chi_{\{|x| \leq R\}} \pi(t, x, dv), (t, x) \in E, \) for some \( R > 0. \)

Then for each \((s, x) \in E\) there is a unique solution \( P_{s, x} \in S(s, x, L) \) and \( P_{s, x} \) is weakly continuous in \((s, x)\).

Proof. Let \( \varphi \in C_0^\infty(\mathbb{R}^d), \varphi \geq 0, \int \varphi dx = 1, \varphi_\varepsilon(x) = \varepsilon^{-d} \varphi(x/\varepsilon), x \in \mathbb{R}^d, \) and
\[
\pi_\varepsilon(t, x, dv) = \int \pi(t, x - z, dv) \varphi_\varepsilon(z) dz, (t, x) \in E.
\]

Let \( \varepsilon_n \to 0 \) and let \( L^n \) be an operator defined as \( L \) with \( \pi \) replaced by \( \pi_{\varepsilon_n}. \)

It follows by Theorem IX.2.31 in [6] that the set \( S(s, x, L^n) \neq \emptyset. \) Since by Lemma 28 for \( P^n_{s, x} \in S(s, x, L^n), \)
\[
P^n_{s, x} \int_s^T \pi_{\varepsilon_n}(r, X_r, \{|v| > l\}) dr
\]
\[
\leq C \int_s^T \int \pi(r, x, \{|v| > l\}) dx dr \to 0 \text{ as } l \to \infty,
\]
the sequence \( \{P^n_{s, x}\} \) is tight (see Theorem VI.4.18 in [6]). Obviously, for each \( v \in C_0^\infty(\mathbb{R}^d), L^n v(t, x) \to L v(t, x) \) \( dt dx\)-a.e. Therefore, by Lemma 3.7 in [11] the set \( S(s, x, L) \neq \emptyset. \) By Lemma 28 the solution \( P_{s, x} \in S(s, x, L) \) is unique. Applying Lemma 3.7 in [11] again, we see that \( P_{s, x} \) is continuous in \((s, x)\). \(\blacksquare\)

Corollary 29 Let Assumptions A, B(i)-(ii) hold. Then for each \((s, x) \in E, \) there is at most one solution \( P_{s, x} \in S(s, x, L). \)

Proof. The statement is immediate consequence of Lemma 28 and Theorem 1.6(b) in [11]. \(\blacksquare\)

6.2 Proof of Proposition 25

The uniqueness follows by Corollary 29. In the first part of the proof we assume that \( \text{(39)} \) holds and use weak convergence arguments. In the second part, we cover the general case by putting together measurable families of probability measures.

(i) Assume \( \text{(39)} \) holds. Let \( L^n \) be an operator defined as \( L \) with \( \pi \) replaced by \( \chi_{\{|x| \leq n\}} \pi. \) According to Lemma 28 for each \((s, x) \in E\) there is a unique and \( P^n_{s, x} \in S(s, x, L^n) \) and \( P^n_{s, x} \) is weakly continuous in \((s, x)\).

By Theorem VI.4.18 in [6], \( \{P^n_{s, x}\} \) is tight. Since \( L^n v \to L v \) \( dt dx\)-a.e. and \( L^n v \) is uniformly bounded for any \( v \in C_0^\infty(\mathbb{R}^d), \) by Lemma 3.7 in [11], the
sequence $P^n_{s,x} \to P_{s,x} \in S(s, x, L)$ weakly ($P_{s,x}$ is unique by Corollary 29).

The same Lemma 3.7, [11], implies that $P_{s,x}$ is weakly continuous in $(s, x)$.

(ii) In the general case (without assuming (39)), we split the operator $Lu = \tilde{L}u + \tilde{B}u$, where $\tilde{L}$ is defined as $L$ with $\pi(t, x, dv)$ replaced by $\chi_{\{|v|<1\}} \pi(t, x, dv)$, and

$$\tilde{B}_{t,x} u(x) = \int_{|v|\geq 1} [u(x + v) - u(x)] \pi(t, x, dv), (t, x) \in E, u \in C_0^\infty(\mathbb{R}^d).$$

Let $(\Omega_2, \mathcal{F}_2, P_2)$ be a probability space with a Poisson point measure $\tilde{p}(dt, dz)$ on $[0, \infty) \times (\mathbb{R} \setminus \{0\})$ with

$$\mathbb{E} \tilde{p}(dt, dz) = \frac{dz}{|z|^2}.$$ 

According to Lemma 14.50 in [5], there is a measurable $\mathbb{R}^d \cap \{|v| \geq 1\}$-valued function $c(t, x, z)$ such that for any Borel $\Gamma$

$$\int_{\Gamma} \chi_{\{|v|\geq 1\}} \pi(t, x, dv) = \int_{\Gamma} \chi_{\Gamma}(c(t, x, z)) \frac{dz}{|z|^2}, (t, x) \in E.$$

Consider the probability space

$$(\Omega, \mathcal{F}, P'_{s,x}) = (\Omega_2 \times D, \mathcal{F}_2 \otimes \mathcal{D}, P_2 \otimes P_{s,x}).$$

Let

$$H_t = \int_{s \wedge t} \int c(r, X_{r-}, z) \tilde{p}(dr, dz), s \leq t \leq T,$$

$$\tau = \inf(t > s : \Delta H_t = H_t - H_{t-} \neq 0) \wedge T,$$

$$K_t = \chi_{\{\tau \leq t\}},$$

$$Y_t = X_{t \wedge \tau} + H_{t \wedge \tau}, 0 \leq t \leq T.$$

Note that $\tau = \inf(t > s : \Delta H_t \neq 0) \wedge T = \tau = \inf(t > s : |\Delta H_t| \geq 1) \wedge T$. Let $\hat{D} = D([0, T], \mathbb{R}^d \times [0, \infty))$ be the Skorokhod space of cadlag $\mathbb{R}^d \times [0, \infty]$-valued trajectories and let $Z_t = Z_t(w) = (y_t(w), k_t(w))$ for $w \in R^d \times [0, \infty)$, $w \in \hat{D}$ be the canonical process on it. Let

$$\hat{D}_t = \sigma(Z_s, s \leq t), \hat{D} = \vee_{t \in \hat{D}_t} \hat{D} = \left(\hat{D}_{t+}\right), t \in [0, T].$$

Denote $\hat{P}^1_{s,x}$ the measure on $\hat{D}$ induced by $(Y_t, K_t), 0 \leq t \leq T$. Let

$$\tau_1 = \inf(t > s : \Delta k_t \geq 1) \wedge T, \ldots,$$

$$\tau_{n+1} = \inf(t > \tau_n : \Delta k_t \geq 1) \wedge T,$$

$$\hat{D}_{\tau_n} = \sigma(Z_{t \wedge \tau_n}, 0 \leq t \leq T), n \geq 1.$$
Then \( \hat{P}_{s,x}^1 \) is a measurable family of measures on \((\hat{D}, \mathbb{D})\) and for each \( v \in C_0^\infty(\mathbb{R}^d), \)

\[
\hat{M}_{t \wedge \tau_n}(v) = v(y_{t \wedge \tau_n}) - \int_s^{t \wedge \tau_n} L v(r, y_r) \, dr, s \leq t \leq T,
\]

is \((\hat{P}_{s,x}^1, \mathbb{D})\)-martingale with \( n = 1 \). Let us introduce the mappings

\[ J_{\tau_1}(w, w')_t = \begin{cases} w_t & \text{if } t < \tau_1(w), \\ w'_t & \text{if } t \geq \tau_1(w), \end{cases} \]

and let

\[ Q(dw, dw') = \hat{P}_{\tau_1(w), X_{\tau_1(w)}(w)}^1(dw') \hat{P}_{s,x}^1(dw). \]

Then \( P_{s,x}^2 = J_{\tau_1}(Q) \), the image of \( Q \) under \( J_{\tau_1} \), is a measurable family of measures on \( \hat{D} \), and by Lemma 2.3 in [11], \( \hat{M}_{t \wedge \tau_2} \) is \((\hat{P}_{s,x}^2, \mathbb{D})\)-martingale and \( \hat{P}_{s,x}^2|_{\hat{D}_{\tau_1}} = \hat{P}_{s,x}^1|_{\hat{D}_{\tau_1}} \). Continuing and using Lemma 2.3 in [11], we construct a sequence of measures \( \hat{P}_{s,x}^n \) such that

\[
\hat{P}_{s,x}^{n+1}|_{\hat{D}_{\tau_n}} = \hat{P}_{s,x}^n|_{\hat{D}_{\tau_n}}
\]

and \( \hat{M}_{t \wedge \tau_n} \) is \((\hat{P}_{s,x}^n, \mathbb{D})\)-martingale. Since

\[
\hat{P}_{s,x}^n(\tau_n < T) = \hat{P}_{s,x}^n(k_{T \wedge \tau_n} \geq n) \leq n^{-1} \int_s^{T \wedge \tau_n} \pi(r, y_r, \{|v| \geq 1\}) \, dr
\]

\[
\leq n^{-1} KT \to 0 \text{ as } n \to \infty,
\]

there is a measurable family \( \hat{P}_{s,x} \) on \( \hat{D} \) such that

\[
\hat{P}_{s,x}|_{\hat{D}_{\tau_n}} = \hat{P}_{s,x}^n|_{\hat{D}_{\tau_n}}, n \geq 1,
\]

and \( \hat{M}_{t \wedge \tau_n} \) is \((\hat{P}_{s,x}, \mathbb{D})\)-martingale for every \( n \). Obviously, \( y \), under \( \hat{P}_{s,x} \) gives a measurable family \( P_{s,x} \in S(s, x, L) \). The strong Markov property is a consequence of Lemma 2.2 in [11]. The statement of Proposition 25 follows.
References

[1] Abels, H. and Kassman, M., The Cauchy problem and the martingale problem for integro-differential operators with non-smooth kernels, Osaka J. Math. 46 (2009) 661-683.

[2] Caffarelli, L., Vasseur, A., Drift diffusion equations with fractional diffusion and the quasigeostrophic equation, Annals of Math., Vol. 171, No. 3, 2010, 1903-1930.

[3] Gilbarg, D. and Trudinger, N. S. Elliptic Partial Differential Equations of Second Order. Springer, New York, 1983.

[4] Dong H. and Kim D., On Lp estimates of non-local elliptic equations, arXiv:1102.4073v1 [math.AP], 2011.

[5] Jacod, J., Calcul Stochastique et Problèmes de Martingales, Lecture Notes in Mathematics, 714, Springer Verlag, Berlin New York, 1979.

[6] Jacod, J. and Shiryaev, A.N., Limit Theorems for Stochastic Processes, Springer, 1987.

[7] Komatsu, T., On the Martingale Problem for Generators of Stable Processes with Perturbations, Osaka J. of Math. 22-1 (1984) 113-132.

[8] Krylov, N.V., An analytic approach to SPDEs, In: Stochastic Partial Differential Equations: Six Perspectives, AMS, 1999.

[9] Krylov, N.V., Lectures on Elliptic and Parabolic Equations in Sobolev Spaces, AMS, 2008.

[10] Mikulevičius, R. and Pragarauskas, H., On the Cauchy Problem for Certain Integro-Differential Operators in Sobolev and Hölder Spaces, Lithuanian Mathematical Journal 32-2 (1992) 238-264.

[11] Mikulevičius, R. and Pragarauskas, H., On the Martingale Problem Associated with Nondegenerate Lévy Operators, Lithuanian Mathematical Journal 32-3 (1992) 297-311.

[12] Mikulevičius, R. and Pragarauskas, H., On $L^p$-theory for Zakai equation with discontinuous observation process, arXiv:1012.5816v1 [math.PR], 2010.
[13] Mikulevičius, R. and Pragarauskas, H., On the Cauchy problem for integro-differential operators in Hölder classes and the uniqueness of the martingale problem, arXiv: 1103.3492v2 [math.AP], 2011.

[14] Triebel, H., Theory of Function Spaces II. Birkhäuser Verlag, 1992.

[15] Stein E. M., Singular Integrals and Differentiability Properties of Functions, Princeton University Press, Princeton, 1970.

[16] Stroock, D.W., Diffusion processes associated with Levy generators, Z. Wahrsch. Verw. Gebiete, 32 (1975), 209-244.