FEKETE-SZEGÖ PROBLEM FOR CERTAIN CLASSES OF MA-MINDA BI-UNIVALENT FUNCTIONS

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Abstract. In the present work, we propose to investigate the Fekete-Szegö inequalities for certain classes of analytic and bi-univalent functions defined by subordination. The results in the bounds of the third coefficient which improve many known results concerning different classes of bi-univalent functions. Some interesting applications of the results presented here are also discussed.

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1. INTRODUCTION

Let \( \mathcal{A} \) denote the class of functions of the form
\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\] (1.1)
which are analytic in the open unit disc \( \mathbb{D} = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \} \). Further, by \( \mathcal{S} \) we will show the family of all functions in \( \mathcal{A} \) which are univalent in \( \mathbb{D} \).

For two functions \( f \) and \( g \), analytic in \( \mathbb{D} \), we say that the function \( f(z) \) is subordinate to \( g(z) \) in \( \mathbb{D} \), and write
\[
f(z) \prec g(z) \quad (z \in \mathbb{D})
\]
if there exists a Schwarz function \( w(z) \), analytic in \( \mathbb{D} \), with
\[w(0) = 0 \text{ and } |w(z)| < 1 \quad (z \in \mathbb{D})\]
such that
\[f(z) = g(w(z)) \quad (z \in \mathbb{D}) .
\]
In particular, if the function \( g \) is univalent in \( \mathbb{D} \), the above subordination is equivalent to
\[f(0) = g(0) \quad \text{and} \quad f(\mathbb{D}) \subset g(\mathbb{D}).
\]

It is well known that every function \( f \in \mathcal{S} \) has an inverse \( f^{-1} \), defined by
\[
f^{-1}(f(z)) = z \quad (z \in \mathbb{D})
\]
and
\[f(f^{-1}(w)) = w \quad \left( |w| < r_0(f); \quad r_0(f) \geq \frac{1}{4} \right),
\]
where
\[
f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \ldots \quad (1.2)
\]
A function \( f \in \mathcal{A} \) is said to be bi-univalent in \( \mathbb{D} \) if both \( f(z) \) and \( f^{-1}(z) \) are univalent in \( \mathbb{D} \). Let \( \Sigma \) denote the class of bi-univalent functions in \( \mathbb{D} \) given by (1.1). For a brief history and interesting examples of functions which are in (or which are not in) the class \( \Sigma \), together with various other properties of the bi-univalent function class \( \Sigma \) one can
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Some of the important and well-investigated subclasses of the univalent function class \( S \) include (for example) the class \( S^*(\alpha) \) of starlike functions of order \( \alpha \) in \( \mathbb{U} \) and the class \( K(\alpha) \) of convex functions of order \( \alpha \) in \( \mathbb{U} \). By definition, we have

\[
S^*(\alpha) := \left\{ f : f \in \mathcal{A} \text{ and } \Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha; \ z \in \mathbb{U}; \ 0 \leq \alpha < 1 \right\} \tag{1.3}
\]

and

\[
K(\alpha) := \left\{ f : f \in \mathcal{A} \text{ and } \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha; \ z \in \mathbb{U}; \ 0 \leq \alpha < 1 \right\}. \tag{1.4}
\]

For \( 0 \leq \alpha < 1 \), a function \( f \in \Sigma \) is in the class \( S^*_\Sigma(\alpha) \) of bi-starlike function of order \( \alpha \), or \( K_{\Sigma,\alpha} \) of bi-convex function of order \( \alpha \) if both \( f \) and \( f^{-1} \) are respectively starlike or convex functions of order \( \alpha \). For \( 0 < \beta \leq 1 \), a function \( f \in \Sigma \) is strongly bi-starlike of order \( \beta \), if both the functions \( f \) and \( f^{-1} \) are strongly starlike of order \( \beta \). We denote the class of all such functions is denoted by \( S^*_{\Sigma,\beta} \).

Let \( \varphi \) be an analytic and univalent function with positive real part in \( \mathbb{U} \) with \( \varphi(0) = 1, \varphi'(0) > 0 \) and \( \varphi \) maps the unit disk \( \mathbb{U} \) onto a region starlike with respect to 1, and symmetric with respect to the real axis. The Taylor’s series expansion of such function is of the form

\[
\varphi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \ldots, \tag{1.5}
\]

where all coefficients are real and \( B_1 > 0 \). Throughout this paper we assume that the function \( \varphi \) satisfies the above conditions one or otherwise stated.

By \( S^*(\varphi) \) and \( K(\varphi) \) we denote the following classes of functions

\[
S^*(\varphi) := \left\{ f : f \in \mathcal{A} \text{ and } \frac{zf'(z)}{f(z)} < \varphi(z); \ z \in \mathbb{U} \right\} \tag{1.6}
\]

and

\[
K(\varphi) := \left\{ f : f \in \mathcal{A} \text{ and } 1 + \frac{zf''(z)}{f'(z)} < \varphi(z); \ z \in \mathbb{U} \right\}. \tag{1.7}
\]

The classes \( S^*(\varphi) \) and \( K(\varphi) \) are the extensions of a classical sets of a starlike and convex functions and in a such form were defined and studied by Ma and Minda [7]. A function \( f \) is bi-starlike of Ma-Minda type or bi-convex of Ma-Minda type if both \( f \) and \( f^{-1} \) are respectively Ma-Minda starlike or convex. These classes are denoted respectively by \( S^*_\Sigma(\varphi) \) and \( K_{\Sigma,\alpha}(\varphi) \) (see [1]).

In order to derive our main results, we will need the following lemma.

**Lemma 1.1.** (see [1]) If \( p \in \mathcal{P} \), then \( |p_i| \leq 2 \) for each \( i \), where \( \mathcal{P} \) is the family of all functions \( p \), analytic in \( \mathbb{U} \), for which

\[
\Re \{p(z)\} > 0 \quad (z \in \mathbb{U}),
\]
where
\[ p(z) = 1 + p_1 z + p_2 z^2 + \cdots \quad (z \in \mathbb{U}). \]

Motivated by the aforementioned works (especially \cite{20} and \cite{10,14}), we consider the following subclass of the function class \( \Sigma \) (see also, \cite{17}).

A function \( f \in \Sigma \) given by \((1.1)\) is said to be in the class \( \mathcal{N}_{\Sigma}^{\mu,\lambda}(\varphi) \) if the following conditions are satisfied:
\[
(1 - \lambda) \left( \frac{f(z)}{z} \right)^\mu + \lambda f'(z) \left( \frac{f(z)}{z} \right)^{\mu-1} \prec \varphi(z) \quad (\lambda \geq 1, \mu \geq 0, z \in \mathbb{U})
\]
and
\[
(1 - \lambda) \left( \frac{g(w)}{w} \right)^\mu + \lambda g'(w) \left( \frac{g(w)}{w} \right)^{\mu-1} \prec \varphi(w) \quad (\lambda \geq 1, \mu \geq 0, w \in \mathbb{U}),
\]
where \( g(w) = f^{-1}(w) \).

**Remark 1.2.** From among the many choices of \( \mu, \lambda \) and the function \( \varphi \), we would provide the following known subclasses:

1. \( \mathcal{N}_{\Sigma}^{1,1}(\varphi) = \mathcal{H}_{\Sigma}^\varphi \) \( \text{[1] p.345} \).
2. \( \mathcal{N}_{\Sigma}^{1,1}(\frac{1 + i z}{1 - z}) = \mathcal{H}_{\Sigma}^\beta (0 < \beta \leq 1) \quad \text{and} \quad \mathcal{N}_{\Sigma}^{1,1}(\frac{1 + (1 - 2\alpha)z}{1 - z}) = \mathcal{H}_{\Sigma}^\alpha (0 \leq \alpha < 1) \) \( \text{[15] Definitions 1 and 2} \).
3. \( \mathcal{N}_{\Sigma}^{\mu,\lambda}(\varphi) = \mathcal{R}_{\Sigma}(\lambda, \varphi) \quad (\lambda \geq 0) \) \( \text{[12] Definition 1.1} \).
4. \( \mathcal{N}_{\Sigma}^{\mu,\lambda}(\frac{1 + i z}{1 - z}) = \mathcal{B}_{\Sigma}(\beta, \lambda) \quad (\lambda \geq 1; 0 < \beta \leq 1) \quad \text{and} \quad \mathcal{N}_{\Sigma}^{\mu,\lambda}(\frac{1 + (1 - 2\alpha)z}{1 - z}) = \mathcal{B}_{\Sigma}(\alpha, \lambda) \quad (\lambda \geq 1; 0 \leq \alpha < 1) \) \( \text{[5] Definitions 2.1 and 3.1} \).
5. \( \mathcal{N}_{\Sigma}^{\mu,\lambda}(\varphi) = \mathcal{F}_{\Sigma}(\varphi) \quad (\mu \geq 0) \) \( \text{[12] Definition 2.1} \).
6. \( \mathcal{N}_{\Sigma}^{0,1}(\frac{1 + i z}{1 - z}) = \mathcal{S}_{\Sigma,\beta}^\alpha (0 < \beta \leq 1) \quad \text{and} \quad \mathcal{N}_{\Sigma}^{0,1}(\frac{1 + (1 - 2\alpha)z}{1 - z}) = \mathcal{S}_{\Sigma}(\alpha) (0 \leq \alpha < 1) \) \( \text{[3] Definitions 2.1} \).
7. \( \mathcal{N}_{\Sigma}^{\mu,\lambda}(\frac{1 + i z}{1 - z}) = \mathcal{N}_{\Sigma}^{\mu,\lambda}(\beta) \quad (\lambda \geq 1; \mu \geq 0; 0 < \beta \leq 1) \) \( \text{[3] Definitions 2.1} \).
and
\( \mathcal{N}_{\Sigma}^{\mu,\lambda}(\frac{1 + (1 - 2\alpha)z}{1 - z}) = \mathcal{N}_{\Sigma}^{\mu,\lambda}(\alpha) \quad (\lambda \geq 1; \mu \geq 0; 0 \leq \alpha < 1) \) \( \text{[3] Definitions 3.1} \).

In this paper we shall obtain the Fekete-Szegö inequalities for \( \mathcal{N}_{\Sigma}^{\mu,\lambda}(\varphi) \) and its special classes. These inequalities will result in bounds of the third coefficient which are, in some cases, better than those obtained in \( \text{[1] [3] [14] [15] [17]} \).

**2. Main Results**

**Theorem 2.1.** Let \( f \) of the form \((1.1)\) be in \( \mathcal{N}_{\Sigma}^{\mu,\lambda}(\varphi) \) and \( \delta \in \mathbb{R} \). Then
\[
|a_3 - \delta a_2^2| \leq \begin{cases} \frac{B_1}{2\lambda + \mu} & ; |\delta| \leq \frac{\mu + 1}{2} \\ \frac{2B_1^2|\delta - 1|}{(2\lambda + \mu)(1 + \mu)(2B_1 - 2(B_1 - B_2)(\lambda + \mu)^2)} & ; |\delta - 1| \geq \frac{\mu + 1}{2} \end{cases} \bigg| 1 + \frac{2(B_1 - B_2)(\lambda + \mu)^2}{B_1^2(2\lambda + \mu)(1 + \mu)} \bigg|.
\]

**Proof.** Since \( f \in \mathcal{N}_{\Sigma}^{\mu,\lambda}(\varphi) \), there exists two analytic functions \( r, s : \mathbb{U} \to \mathbb{U} \), with \( r(0) = 0 = s(0) \), such that
\[
(1 - \lambda) \left( \frac{f(z)}{z} \right)^\mu + \lambda f'(z) \left( \frac{f(z)}{z} \right)^{\mu-1} = \varphi(r(z)) \quad \text{(2.2)}
\]
and
\[(1 - \lambda) \left( \frac{g(w)}{w} \right)^\mu + \lambda g'(w) \left( \frac{g(w)}{w} \right)^{\mu-1} = \varphi(s(z)). \tag{2.3}\]

Define the functions \(p\) and \(q\) by
\[p(z) = \frac{1 + r(z)}{1 - r(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots \tag{2.4}\]
and
\[q(z) = \frac{1 + s(z)}{1 - s(z)} = 1 + q_1 z + q_2 z^2 + q_3 z^3 + \ldots \tag{2.5}\]
or equivalently,
\[r(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2} \left( p_1 z + \left( p_2 - \frac{p_1^2}{2} \right) z^2 + \left( p_3 + \frac{p_1}{2} \left( \frac{p_1^2}{2} - p_2 \right) - \frac{p_1 p_2}{2} \right) z^3 + \ldots \right) \tag{2.6}\]
and
\[s(z) = \frac{q(z) - 1}{q(z) + 1} = \frac{1}{2} \left( q_1 z + \left( q_2 - \frac{q_1^2}{2} \right) z^2 + \left( q_3 + \frac{q_1}{2} \left( \frac{q_1^2}{2} - q_2 \right) - \frac{q_1 q_2}{2} \right) z^3 + \ldots \right). \tag{2.7}\]

Using (2.6) and (2.7) in (2.2) and (2.3), we have
\[(1 - \lambda) \left( \frac{f(z)}{z} \right)^\mu + \lambda f'(z) \left( \frac{f(z)}{z} \right)^{\mu-1} = \varphi \left( \frac{p(z) - 1}{p(z) + 1} \right) \tag{2.8}\]
and
\[(1 - \lambda) \left( \frac{g(w)}{w} \right)^\mu + \lambda g'(w) \left( \frac{g(w)}{w} \right)^{\mu-1} = \varphi \left( \frac{q(w) - 1}{q(w) + 1} \right). \tag{2.9}\]

Again using (2.6) and (2.7) along with (1.5), it is evident that
\[\varphi \left( \frac{p(z) - 1}{p(z) + 1} \right) = 1 + \frac{1}{2} B_1 p_1 z + \left( \frac{1}{2} B_1 \left( p_2 - \frac{1}{2} p_1^2 \right) + \frac{1}{4} B_2 p_1^2 \right) z^2 + \ldots \tag{2.10}\]
and
\[\varphi \left( \frac{q(w) - 1}{q(w) + 1} \right) = 1 + \frac{1}{2} B_1 q_1 w + \left( \frac{1}{2} B_1 \left( q_2 - \frac{1}{2} q_1^2 \right) + \frac{1}{4} B_2 q_1^2 \right) w^2 + \ldots. \tag{2.11}\]

It follows from (2.8), (2.9), (2.10) and (2.11) that
\[(\lambda + \mu) a_2 = \frac{1}{2} B_1 p_1 \tag{2.12}\]
\[(2\lambda + \mu) [a_3 + \frac{a_2^2}{2} (\mu - 1)] = \frac{1}{2} B_1 \left( p_2 - \frac{1}{2} p_1^2 \right) + \frac{1}{4} B_2 p_1^2 \tag{2.13}\]
\[-(\lambda + \mu) a_2 = \frac{1}{2} B_1 q_1 \tag{2.14}\]
and
\[(2\lambda + \mu) \left[ \frac{a_2^2}{2} (\mu + 3) - a_3 \right] = \frac{1}{2} B_1 \left( q_2 - \frac{1}{2} q_1^2 \right) + \frac{1}{4} B_2 q_1^2. \tag{2.15}\]

From (2.12) and (2.14), we find that
\[a_2 = \frac{B_1 p_1}{2(\lambda + \mu)} = \frac{-B_1 q_1}{2(\lambda + \mu)} \tag{2.16}\]
it follows that
\[ p_1 = -q_1 \] (2.17)
and
\[ 8(\lambda + \mu)^2 a_2^2 = B_1^2(p_1^2 + q_1^2). \] (2.18)
Adding (2.13) and (2.15), we have
\[ a_2^2(2\lambda + \mu)(\mu + 1) = \frac{B_1}{2}(p_2 + q_2) + \frac{(B_2 - B_1)}{4}(p_1^2 + q_1^2). \] (2.19)
Substituting (2.16) and (2.17) into (2.19), we get,
\[ p_1^2 = \frac{B_1 2(\lambda + \mu)^2(p_2 + q_2)}{B_1^2(2\lambda + \mu)(\mu + 1) - 2(B_2 - B_1)(\lambda + \mu)^2}. \] (2.20)
Now, (2.16) and (2.20) yield
\[ a_2^2 = \frac{B_1^2(p_2 + q_2)}{2(\mu + 1)(2\lambda + \mu)B_1^2 + 4(B_1 - B_2)(\lambda + \mu)^2}. \] (2.21)
By subtracting (2.13) from (2.15) and a computation using (2.17) finally lead to
\[ a_3 = a_2^2 + \frac{B_1(p_2 - q_2)}{8\lambda + 4\mu}. \] (2.22)
From (2.21) and (2.22) it follows that
\[ a_3 - \delta a_2^2 = B_1 \left[ \left( h(\delta) + \frac{1}{8\lambda + 4\mu}\right)p_2 + \left(h(\delta) - \frac{1}{8\lambda + 4\mu}q_2 \right) \right], \]
where
\[ h(\delta) = \frac{B_1^2(1 - \delta)}{2(\mu + 1)(2\lambda + \mu)B_1^2 + 4(B_1 - B_2)(\lambda + \mu)^2}. \]
Since all \( B_j \) are real and \( B_1 > 0 \), we conclude that
\[ |a_3 - \delta a_2^2| \leq \left\{ \begin{array}{ll}
\frac{B_1}{2\lambda + \mu} & ; 0 \leq |h(\delta)| < \frac{1}{8\lambda + \mu} \\
\frac{1}{4B_1|h(\delta)|} & ; |h(\delta)| \geq \frac{1}{8\lambda + \mu} 
\end{array} \right. , \]
which completes the proof. \( \square \)

**Remark 2.2.** For \( \lambda = \mu = 1 \) Theorem 2.1 reduces to the results discussed in [20, Theorem 1, p.172].

### 3. Corollaries and Consequences

Taking \( \delta = 1, \delta = 0 \) in Theorem 2.1 we have the following corollaries.

**Corollary 3.1.** If \( f \in \mathcal{N}_\Sigma^{\mu,\lambda}(\varphi) \) then
\[ |a_3 - a_2^2| \leq \frac{B_1}{2\lambda + \mu}. \]

**Corollary 3.2.** If \( f \in \mathcal{N}_\Sigma^{\mu,\lambda}(\varphi) \) then
\[ |a_3| \leq \left\{ \begin{array}{ll}
\frac{B_1}{2\lambda + \mu} & ; \frac{(B_1 - B_2)}{B_1^2} \in \left( -\infty, \frac{-(3+\mu)(2\lambda + \mu)}{2(\lambda + \mu)^2} \right) \\
\frac{1}{2B_1^2}\left(2\lambda + \mu\right) & ; \frac{(B_1 - B_2)}{B_1^2} \in \left[ \frac{-(3+\mu)(2\lambda + \mu)}{2(\lambda + \mu)^2}, \frac{-1(\lambda + \mu)(2\lambda + \mu)}{2(\lambda + \mu)^2} \right] \\
\frac{1}{2B_1^2}\left(2\lambda + \mu\right) & ; \frac{(B_1 - B_2)}{B_1^2} \in \left( \frac{-(3+\mu)(2\lambda + \mu)}{2(\lambda + \mu)^2}, \frac{-(1+\mu)(2\lambda + \mu)}{2(\lambda + \mu)^2} \right) \\
\frac{1}{2B_1^2}\left(2\lambda + \mu\right) & ; \frac{(B_1 - B_2)}{B_1^2} \in \left[ \frac{-(1+\mu)(2\lambda + \mu)}{2(\lambda + \mu)^2}, \frac{1(\mu)(2\lambda + \mu)}{2(\lambda + \mu)^2} \right] \\
\end{array} \right. . \]
Remark 3.3. Corollary 3.2 provides an improvement of the estimate $|a_3|$ obtained by Tang et al. [17, Theorem 2.1, p.3].

In view of Remark 1.2, Corollaries 3.1 and 3.2 yield the following corollaries.

**Corollary 3.4.** If $f \in \mathcal{N}_{\Sigma}^{\mu,\lambda}(\beta)$ then

$$|a_3| \leq \frac{2\beta}{2\lambda + \mu} \quad \text{and} \quad |a_3 - a_2^2| \leq \frac{2\beta}{2\lambda + \mu}.$$

**Corollary 3.5.** If $f \in \mathcal{N}_{\Sigma}^{\mu,\lambda}(\alpha)$ then

$$|a_3| \leq \frac{2(1-\alpha)}{2\lambda + \mu} \quad \text{and} \quad |a_3 - a_2^2| \leq \frac{2(1-\alpha)}{2\lambda + \mu}.$$

**Remark 3.6.** The bounds $|a_3|$ obtained in Corollaries 3.4 and 3.5 are improvement of the bounds $|a_3|$ estimated by Çağlar et al. [3, Theorems 2.1 and 3.1].

**Remark 3.7.** In view of Remark 1.2, the aforecited work for the subclasses $\mathcal{H}_{\Sigma}^{e}, \mathcal{H}_{\Sigma}^{b}$ and $\mathcal{H}_{\Sigma}^{a}$ are coincide with the results of Zaprawa [20, Corollaries 1 to 4, p.173].

**Corollary 3.8.** If $f \in \mathcal{S}_{\Sigma}(\varphi)$ then

$$|a_3 - a_2^2| \leq \frac{B_1}{2}.$$

**Corollary 3.9.** If $f \in \mathcal{S}_{\Sigma}(\varphi)$ then

$$|a_3| \leq \begin{cases} \frac{B_1}{2} & \frac{(B_1-B_2)}{B_1^2} \in (-\infty, -3] \cup [0, \infty) \\ \frac{B_1}{2(B_1-B_2)} & \frac{(B_1-B_2)}{B_1^2} \in [-2, 1) \cup (1, 1]. \end{cases}$$

**Corollary 3.10.** If $f \in \mathcal{S}_{\Sigma,\beta}$ then

$$|a_3| \leq \beta \quad \text{and} \quad |a_3 - a_2^2| \leq \beta.$$

**Corollary 3.11.** If $f \in \mathcal{S}_{\Sigma}(\alpha)$ then

$$|a_3| \leq 1 - \alpha \quad \text{and} \quad |a_3 - a_2^2| \leq 1 - \alpha.$$

**Remark 3.12.** The inequalities estimated in Corollaries 3.9 to 3.11 are improvement of the inequalities obtained by Zaprawa [20, Corollaries 11 and 12, p.174].

**Corollary 3.13.** If $f \in \mathcal{R}_{\Sigma}(\lambda; \varphi)$ then

$$|a_3 - a_2^2| \leq \frac{B_1}{2\lambda + 1}.$$

**Corollary 3.14.** If $f \in \mathcal{R}_{\Sigma}(\lambda; \varphi)$ then

$$|a_3| \leq \begin{cases} \frac{B_1}{2\lambda+1} & \frac{(B_1-B_2)}{B_1^2} \in (-\infty, \frac{2(2\lambda+1)}{(\lambda+1)^2}] \cup [0, \infty) \\ \frac{B_1}{(2\lambda+1)(B_1-B_2)(\lambda+1)^2} & \frac{(B_1-B_2)}{B_1^2} \in \left[\frac{-2(2\lambda+1)}{(\lambda+1)^2}, -\frac{2(\lambda+1)}{(\lambda+1)^2}\right] \cup \left(-\frac{2(\lambda+1)}{(\lambda+1)^2}, 0\right]. \end{cases}$$

**Remark 3.15.** Corollary 3.14 provides an improvement of $|a_3|$ obtained by Sivaprasad Kumar et al. [12, Theorem 2.1, p.3].

**Corollary 3.16.** If $f \in \mathcal{B}_{\Sigma}(\beta, \lambda)$ then

$$|a_3| \leq \frac{2\beta}{2\lambda + 1} \quad \text{and} \quad |a_3 - a_2^2| \leq \frac{2\beta}{2\lambda + 1}.$$
Corollary 3.17. If $f \in B_{\Sigma}(\alpha, \lambda)$ then
\[ |a_3| \leq \frac{2(1-\alpha)}{2\lambda + 1} \quad \text{and} \quad |a_3 - a_2^2| \leq \frac{2(1-\alpha)}{2\lambda + 1}. \]

Remark 3.18. The bounds $|a_3|$ obtained in Corollaries 3.16 and 3.17 are improvement of the bounds $|a_3|$ estimated by Frasin and Aouf [5, Theorems 2.2 and 3.2, p.1570 and 1572], respectively.

Remark 3.19. If we take $\varphi = \varphi_0 = 1 + \frac{z}{1-z} = 1 + 2z + 2z^2 + \ldots$ (3.1)
in the class $N_{\Sigma}^{\mu,\lambda}(\varphi)$, we are led to the class which we denote, for convenience, by $N_{\Sigma}^{\mu,\lambda}(\varphi_0)$. In particular, $N_{\Sigma}^{1,1}(\varphi_0) =: H_{\Sigma}^{\varphi_0}$, $N_{\Sigma}^{0,\mu}(\varphi_0) =: S_{\Sigma}^{*}(\varphi_0)$ and $N_{\Sigma}^{1,\lambda}(\varphi) =: B_{\Sigma}(\lambda, \varphi_0)$.

In view of Remark 3.19, the Corollaries 3.1 and 3.2 yield the following corollaries.

Corollary 3.20. If $f \in N_{\Sigma}^{\mu,\lambda}(\varphi_0)$ then
\[ |a_3| \leq \frac{2}{2\lambda + \mu} \quad \text{and} \quad |a_3 - a_2^2| \leq \frac{2}{2\lambda + \mu}. \]

Remark 3.21. For $\mu = \lambda = 1$ the estimates in Corollary 3.20 would reduce to a known result in [20, Corollary 5, p.173].

Corollary 3.22. If $f \in S_{\Sigma}^{*}(\varphi_0)$ then
\[ |a_3| \leq 1 \quad \text{and} \quad |a_3 - a_2^2| \leq 1. \]

Corollary 3.23. If $f \in B_{\Sigma}(\lambda, \varphi_0)$ then
\[ |a_3| \leq \frac{2}{2\lambda + 1} \quad \text{and} \quad |a_3 - a_2^2| \leq \frac{2}{2\lambda + 1}. \]

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