Theoretical Foundations of Defeasible Description Logics

Katarina Britz*  Giovanni Casini†  Thomas Meyer‡  Kody Moodley§
Uli Sattler¶  Ivan Varzinczak∥

Abstract

We extend description logics (DLs) with non-monotonic reasoning features. We start by investigating a notion of defeasible subsumption in the spirit of defeasible conditionals as studied by Kraus, Lehmann and Magidor in the propositional case. In particular, we consider a natural and intuitive semantics for defeasible subsumption, and investigate KLM-style syntactic properties for both preferential and rational subsumption. Our contribution includes two representation results linking our semantic constructions to the set of preferential and rational properties considered. Besides showing that our semantics is appropriate, these results pave the way for more effective decision procedures for defeasible reasoning in DLs. Indeed, we also analyse the problem of non-monotonic reasoning in DLs at the level of entailment and present an algorithm for the computation of rational closure of a defeasible ontology. Importantly, our algorithm relies completely on classical entailment and shows that the computational complexity of reasoning over defeasible ontologies is no worse than that of reasoning in the underlying classical DL $\mathcal{ALC}$.

Keywords: Description logics, non-monotonic reasoning, defeasible subsumption, preferential semantics, rational closure.

1 Introduction

Description logics (DLs) [1] are central to many modern AI and database applications since they provide the logical foundation of formal ontologies. Yet, as classical formalisms, DLs...
do not allow for the proper representation of and reasoning with defeasible information, as shown up in the following example, adapted from Giordano et al.'s [46]: Students do not get tax invoices; employed students do; employed students who are also parents do not. From a naïve (classical) formalisation of this scenario, one concludes that the notion of employed student is an oxymoron, and, consequently, the concept of employed student is unsatisfiable. But while concept unsatisfiability has been investigated extensively in ontology debugging and repair [61] [71], our research problem here goes beyond that, as will become clear in the upcoming sections.

Endowing DLs with defeasible reasoning features is therefore a promising endeavour from the point of view of applications of knowledge representation and reasoning. Indeed, the past 25 years have witnessed many attempts to introduce defeasible-reasoning capabilities in a DL setting, usually drawing on a well-established body of research on non-monotonic reasoning (NMR). These comprise the so-called preferential approaches [21] [22] [24] [37] [40] [46] [47] [51] [52] [66] [67], circumscription-based ones [9] [10] [72], amongst others [2] [3] [8] [43] [53] [54] [55] [62] [63] [74].

Preferential extensions of DLs turn out to be particularly promising, mainly because they are based on an elegant, comprehensive and well-studied framework for non-monotonic reasoning in the propositional case proposed by Kraus, Lehmann and Magidor [56] [59] and often referred to as the KLM approach. Such a framework is valuable for a number of reasons. First, it provides for a thorough analysis of some formal properties that any consequence relation deemed as appropriate in a non-monotonic setting ought to satisfy. Such formal properties, which resemble those of a Gentzen-style proof system (see Section 3.1), play a central role in assessing how intuitive the obtained results are and enable a more comprehensive characterisation of the introduced non-monotonic conditional from a logical point of view. Second, the KLM approach allows for many decision problems to be reduced to classical entailment checking, sometimes without blowing up the computational complexity compared to the underlying classical case. Finally, it has a well-known connection with the AGM-approach to belief revision [45] [69] and with frameworks for reasoning under uncertainty [7] [44]. It is therefore reasonable to expect that most, if not all, of the aforementioned features of the KLM approach should transfer to KLM-based extensions of DLs, too.

Following the motivation laid out above, several extensions to the KLM approach to description logics have been proposed recently [21] [24] [27] [29] [32] [33] [37] [40] [46] [47] [51] [52] [63] [75], each of them investigating particular constructions and variants of the preferential approach. However, here our aim is to provide a comprehensive study of the formal foundations of preferential defeasible reasoning in DLs. By that we mean (i) defining a general and intuitive semantics; (ii) showing that the corresponding representation results (in the KLM sense of the term) hold, linking our semantic constructions with the KLM-style set of properties, and (iii) presenting an appropriate analysis of entailment in the context of ontologies with defeasible information with an associated decision procedure that is implementable.
In the remainder of the paper, we shall take the following route: After providing the required background on the DL we consider in this work as well as fixing the notation (Section 2), we introduce the notion of defeasible subsumption along with a set of KLM-inspired properties it ought to satisfy (Section 3). In particular, using an intuitive semantics for the idea that “usually, an element of the class $C$ is also an element of the class $D$”, we provide a characterisation (via representation results) of two important classes of defeasible statements, namely preferential and rational subsumption. In Section 4, we start by investigating two obvious candidates for the notion of entailment in the context of defeasible DLs, namely preferential and modular entailment. These turn out not to have all properties seen as important in a non-monotonic DL setting, mimicking a similar result in the propositional case [59]. Therefore, we propose a notion of rational entailment and show that it is the definition of consequence we are looking for. We take this definition further by exploring the relationship that rational entailment has with both Lehmann and Magidor’s [59] definition of rational closure and the more recent algorithm by Casini and Straccia [37] for its computation (Section 5). After a discussion of, and comparison with, related work (Section 6), we conclude with a summary of our contributions and some directions for further exploration. Proofs of our results can be found in the appendix.

2 Logical preliminaries

Description Logics (DLs) [1] are decidable fragments of first-order logic with interesting properties and a variety of applications. There is a whole family of description logics, an example of which is $\mathcal{ALC}$ and on which we shall focus in the present paper. The (concept) language of $\mathcal{ALC}$ is built upon a finite set of atomic concept names $C$, a finite set of role names $R$ (a.k.a. attributes) and a finite set of individual names $I$ such that $C$, $R$ and $I$ are pairwise disjoint. In our scenario example, we can have for instance $C = \{\text{Employee, Company, Student, EmpStud, Parent, Tax}\}$, $R = \{\text{pays, empBy, worksFor}\}$, and $I = \{\text{john, ibm, mary}\}$, with the respective obvious intuitions. With $A, B, \ldots$ we denote atomic concepts, with $r, s, \ldots$ role names, and with $a, b, \ldots$ individual names. Complex concepts are denoted with $C, D, \ldots$ and are built using the constructors $\neg$ (complement), $\sqcap$ (concept conjunction), $\sqcup$ (concept disjunction), $\forall$ (value restriction) and $\exists$ (existential restriction) according to the following grammar rules:

$$C ::= T \mid \bot \mid C \mid (\neg C) \mid (C \sqcap C) \mid (C \sqcup C) \mid (\exists r.C) \mid (\forall r.C)$$

With $\mathcal{L}$ we denote the language of all $\mathcal{ALC}$ concepts, which is understood as the smallest set of symbol sequences generated according to the rules above. When writing down

---

1For the reader not conversant with Description Logics but familiar with modal logics, there are results in the literature relating some families of description logics to systems of modal logic. For example, a well-known result is the one linking the DL $\mathcal{ALC}$ with the normal modal logic $\mathcal{K}$ [70].
concepts of $\mathcal{L}$, we follow the usual convention and omit parentheses whenever they are not essential for disambiguation. Examples of ALC concepts in our scenario are $\text{Student} \sqcap \text{Employee}$ and $\neg \exists \text{pays. Tax}$.

The semantics of ALC is the standard set-theoretic Tarskian semantics. An interpretation is a structure $\mathcal{I} = \langle \Delta^\mathcal{I}, \cdot^\mathcal{I} \rangle$, where $\Delta^\mathcal{I}$ is a non-empty set called the domain, and $\cdot^\mathcal{I}$ is an interpretation function mapping concept names $A$ to subsets $A^\mathcal{I}$ of $\Delta^\mathcal{I}$, role names $r$ to binary relations $r^\mathcal{I}$ over $\Delta^\mathcal{I}$, and individual names $a$ to elements of the domain $\Delta^\mathcal{I}$, i.e., $A^\mathcal{I} \subseteq \Delta^\mathcal{I}$, $r^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$, and $a^\mathcal{I} \in \Delta^\mathcal{I}$.

Figure 1 depicts an interpretation for our scenario example with domain $\Delta^\mathcal{I} = \{x_i \mid 0 \leq i \leq 10\}$, and interpreting the elements of the vocabulary as follows: $\text{Employee}^\mathcal{I} = \{x_1, x_2, x_5, x_9\}$; $\text{Company}^\mathcal{I} = \{x_6, x_{10}\}$; $\text{Student}^\mathcal{I} = \{x_1, x_5, x_7, x_8\}$; $\text{EmpStud}^\mathcal{I} = \{x_1, x_5\}$; $\text{Parent}^\mathcal{I} = \{x_1, x_2, x_3\}$; $\text{Tax}^\mathcal{I} = \{x_4\}$; $\text{pays}^\mathcal{I} = \{(x_1, x_0), (x_5, x_4)\}$; $\text{empBy}^\mathcal{I} = \{(x_9, x_{10})\}$; $\text{worksFor}^\mathcal{I} = \{(x_5, x_6), (x_9, x_{10})\}$; $\text{john}^\mathcal{I} = x_5$, $\text{ibm}^\mathcal{I} = x_6$, $\text{mary}^\mathcal{I} = x_2$.

Let $\mathcal{I} = \langle \Delta^\mathcal{I}, \cdot^\mathcal{I} \rangle$ be an interpretation and define $r^\mathcal{I}(x) = \{y \in \Delta^\mathcal{I} \mid (x, y) \in r^\mathcal{I}\}$, for $r \in \mathcal{R}$. We extend the interpretation function $\cdot^\mathcal{I}$ to interpret complex concepts of $\mathcal{L}$ as
follows:
\[ \top^\mathcal{I} = \text{def } \Delta^\mathcal{I}; \]
\[ \bot^\mathcal{I} = \text{def } \emptyset; \]
\[ (\neg C)^\mathcal{I} = \text{def } \Delta^\mathcal{I} \setminus C^\mathcal{I}; \]
\[ (C \sqcap D)^\mathcal{I} = \text{def } C^\mathcal{I} \cap D^\mathcal{I}; \]
\[ (C \sqcup D)^\mathcal{I} = \text{def } C^\mathcal{I} \cup D^\mathcal{I}; \]
\[ (\exists r.C)^\mathcal{I} = \text{def } \{ x \in \Delta^\mathcal{I} \mid r^\mathcal{I}(x) \cap C^\mathcal{I} \neq \emptyset \}; \]
\[ (\forall r.C)^\mathcal{I} = \text{def } \{ x \in \Delta^\mathcal{I} \mid r^\mathcal{I}(x) \subseteq C^\mathcal{I} \}. \]

For the interpretation \( \mathcal{I} \) in Figure 1, we have \((\text{Parent} \sqcap \text{Employee})^\mathcal{I} = \{ x_1, x_2 \}\) and \((\exists \text{pays.Tax})^\mathcal{I} = \{ x_5 \}\).

Given \( C, D \in \mathcal{L} \), a statement of the form \( C \sqsubseteq D \) is called a subsumption statement, or general concept inclusion (GCI), read “\( C \) is subsumed by \( D \)”. A concrete example of GCI is \( \text{EmpStud} \sqsubseteq \text{Student} \sqcap \text{Employee} \). \( C \equiv D \) is an abbreviation for both \( C \sqsubseteq D \) and \( D \sqsubseteq C \).

An \( \text{ALC} \) TBox \( \mathcal{T} \) is a finite set of GCIs. Given \( C \in \mathcal{L} \), \( r \in \mathcal{R} \) and \( a, b \in \mathcal{I} \), an assertional statement (assertion, for short) is an expression of the form \( a : C \) or \( (a, b) : r \), read, respectively, “\( a \) is an instance of \( C \)” and “\( a \) is related to \( b \) via \( r \)”. Examples of assertions are \( \text{john} : \text{EmpStud} \) and \( \langle \text{john}, \text{ibm} \rangle : \text{worksFor} \). An \( \text{ALC} \) ABox \( \mathcal{A} \) is a finite set of assertional statements. We denote statements with \( \alpha, \beta, \ldots \). Given \( \mathcal{T} \) and \( \mathcal{A} \), with \( \mathcal{KB} = \text{def } \mathcal{T} \cup \mathcal{A} \) we denote an \( \text{ALC} \) knowledge base, a.k.a. an ontology, an example of which is given below:

\[
\mathcal{T} = \begin{cases}
\text{EmpStud} \sqsubseteq \text{Student} \sqcap \text{Employee}, \\
\text{Student} \sqsubseteq \neg \exists \text{pays.Tax}, \\
\text{EmpStud} \sqsubseteq \exists \text{pays.Tax}, \\
\text{EmpStud} \sqcap \text{Parent} \sqsubseteq \neg \exists \text{pays.Tax}, \\
\text{Employee} \sqsubseteq \exists \text{worksFor.Tax}.
\end{cases}
\]

\[ \mathcal{A} = \{ \text{john} : \text{EmpStud}, \text{john} : \text{Parent}, \langle \text{john}, \text{ibm} \rangle : \text{worksFor} \} \]

An interpretation \( \mathcal{I} \) satisfies a GCI \( C \sqsubseteq D \) (denoted \( \mathcal{I} \vDash C \sqsubseteq D \)) if \( C^\mathcal{I} \subseteq D^\mathcal{I} \). (And then \( \mathcal{I} \vDash C \equiv D \) if \( C^\mathcal{I} = D^\mathcal{I} \).) \( \mathcal{I} \) satisfies an assertion \( a : C \) (respectively, \( (a, b) : r \)), denoted \( \mathcal{I} \vDash a : C \) (respectively, \( \mathcal{I} \vDash (a, b) : r \)), if \( a^\mathcal{I} \in C^\mathcal{I} \) (respectively, \( (a^\mathcal{I}, b^\mathcal{I}) \in r^\mathcal{I} \)).

In the interpretation \( \mathcal{I} \) in Figure 1, we have \( \mathcal{I} \vDash \text{EmpStud} \sqsubseteq \text{Student} \sqcap \text{Employee}, \mathcal{I} \vDash \text{john} : \exists \text{pays.Tax} \) and \( \mathcal{I} \not\vDash \langle \text{john}, \text{ibm} \rangle : \text{empBy} \).

We say that an interpretation \( \mathcal{I} \) is a model of a TBox \( \mathcal{T} \) (respectively, of an ABox \( \mathcal{A} \)), denoted \( \mathcal{I} \vDash \mathcal{T} \) (respectively, \( \mathcal{I} \vDash \mathcal{A} \)) if \( \mathcal{I} \vDash \alpha \) for every \( \alpha \) in \( \mathcal{T} \) (respectively, in \( \mathcal{A} \)). We say that \( \mathcal{I} \) is a model of a knowledge base \( \mathcal{KB} = \mathcal{T} \cup \mathcal{A} \) if \( \mathcal{I} \vDash \mathcal{T} \) and \( \mathcal{I} \vDash \mathcal{A} \). It can be verified that the interpretation in Figure 1 is not a model of the example knowledge base above. (Actually, it is not hard to see that the knowledge base above admits no model.)
A statement $\alpha$ is (classically) entailed by a knowledge base $KB$, denoted $KB \models \alpha$, if every model of $KB$ satisfies $\alpha$. If $I \models \alpha$ for all interpretations $I$, we say $\alpha$ is a validity and denote this fact with $\models \alpha$.

The focus of the present paper being on defeasibility for description logic TBoxes only, we henceforth assume the ABox is empty. (We are currently in the process of extending our approach to description logic knowledge bases, with ABoxes included into the mix.) It is easy to see that, for $T$ as above, we have $T \models \text{EmpStud} \sqsubseteq \bot$.

For more details on Description Logics in general and on $\mathcal{ALC}$ in particular, the reader is invited to consult the Description Logic Handbook [1] and the introductory textbook on Description Logic [4].

3 Foundations for defeasibility in DLs

In this section, we lay the formal foundations of our approach to defeasible reasoning in DL ontologies. For the most part, we build on the so-called preferential approach to non-monotonic reasoning [56, 59, 73].

3.1 Defeasible subsumption relations and their KLM-style properties

In a sense, class subsumption (alias concept inclusion) of the form $C \sqsubseteq D$ is the main notion in DL ontologies. Given its implication-like intuition, subsumption lends itself naturally to defeasibility: “provisionally, if an object falls under $C$, then it also falls under $D$”, as in “usually, students are tax exempted”. In that respect, a defeasible version of concept inclusion is the starting point for an investigation of defeasible reasoning in DL ontologies. (We shall also address defeasibility of the entailment relation in later sections.)

Definition 1 (Defeasible Concept Inclusion) Let $C, D \in \mathcal{L}$. A defeasible concept inclusion axiom (DCI, for short) is a statement of the form $C \sqsubseteq \neg D$.

A defeasible concept inclusion of the form $C \sqsubseteq \neg D$ is to be read as “usually, an instance of the class $C$ is also an instance of the class $D$”. For instance, the DCI $\text{Student} \sqsubseteq \neg \exists \text{pays.Tax}$ formalises the example above. Paraphrasing Lehmann [57], the intuition of $C \sqsubseteq \neg D$ is that “if [the fact it belongs to] $C$ were all the information about an object available to an agent, then [that it also belongs to] $D$ would be a sensible conclusion to draw about such an object”. It is worth noting that $\sqsubseteq$, just as $\subseteq$, is a ‘connective’ sitting between the concept language (object level) and the meta-language (that of entailment) and it is meant to be the defeasible counterpart of the classical subsumption $\sqsubseteq$.

Being (defeasible) statements, DCIs will also be denoted by $\alpha, \beta, \ldots$. Whenever a distinction between GCIs and DCIs is in order, we shall make it explicitly.
Definition 2 (Defeasible TBox) A defeasible TBox (DTBox, for short) is a finite set of DCIs.

Given a TBox $\mathcal{T}$ and a DTBox $\mathcal{D}$, we let $KB_{\text{def}} = \mathcal{T} \cup \mathcal{D}$ and refer to it as a defeasible knowledge base (alias defeasible ontology).

Example 1 The following defeasible knowledge base gives a formal specification for our student scenario:

$$\mathcal{T} = \{ \text{EmpStud} \sqsubseteq \text{Student} \}$$

$$\mathcal{D} = \left\{ \begin{array}{l}
\text{Student} \sqsubseteq \neg \exists \text{pays.Tax}, \\
\text{EmpStud} \sqsubseteq \exists \text{pays.Tax}, \\
\text{EmpStud} \sqcap \text{Parent} \sqsubseteq \neg \exists \text{pays.Tax} \\
\end{array} \right\}$$

In our semantic construction later on, it will also be useful to be able to refer to infinite sets of concept inclusions. Let $KB_{\text{inf}}$ therefore denote a defeasible theory, defined as a defeasible knowledge base but without the restriction on $\mathcal{T}$ and $\mathcal{D}$ to finite sets.

In order to assess the behaviour of the new connective and check it against both the intuition and the set of properties usually considered in a non-monotonic setting, it is convenient to look at a set of $\sqsubseteq$-statements as a binary relation of the ‘antecedent-consequent’ kind.

Definition 3 (Defeasible Subsumption Relation) A defeasible subsumption relation is a binary relation $\sqsubseteq \subseteq L \times L$.

The idea is to mimic the analysis of defeasible entailment relations carried out by Kraus et al. [56] in the propositional case, where entailment is seen as a binary relation on the set of propositional sentences. Here we shall adopt the view of subsumption as a binary relation on concepts of our description language.

Sometimes (e.g. in the structural properties below) we shall write $(C, D) \in \sqsubseteq$ in the infix notation, i.e., as $C \sqsubseteq D$. The context will make clear when we will be talking about elements of a relation or statements (DCIs) in a defeasible knowledge base. Whenever disambiguation is in order, we shall flag it to the reader.

Definition 4 (Preferential Subsumption Relation) A defeasible subsumption relation is a preferential subsumption relation if it satisfies the following set of properties,
which we refer to as the (DL versions of the) preferential KLM properties:

\[
\begin{align*}
(\text{Cons}) & \quad \top \not\sqsubseteq \bot \\
(\text{Ref}) & \quad C \sqsubseteq C \\
(\text{LLE}) & \quad \frac{C \equiv D, C \sqsubseteq E}{D \sqsubseteq E} \\
(\text{And}) & \quad \frac{C \sqsubseteq D, C \sqsubseteq E}{C \sqsubseteq D \cap E} \\
(\text{Or}) & \quad \frac{C \sqsubseteq E, D \sqsubseteq E}{C \sqcup D \sqsubseteq E} \\
(\text{RW}) & \quad \frac{C \sqsubseteq D, D \sqsubseteq E}{C \sqsubseteq E} \\
(\text{CM}) & \quad \frac{C \sqsubseteq D, C \sqsubseteq E}{C \cap D \sqsubseteq E}
\end{align*}
\]

The (Cons) property is a consequence of the adoption of a DL-based semantics, which enforces the non-emptiness of the domain, as will become clear in the next section. The rest of the properties in Definition 4 result from a translation of the properties for preferential consequence relations proposed by Kraus et al. [56] in the propositional setting. They have been discussed at length in the literature for both the propositional and the DL cases [21, 24, 48, 49, 56, 59] and we shall not repeat so here.

If, in addition to the preferential properties above, the relation $\sqsubseteq$ also satisfies rational monotonicity (RM) below, then it is said to be a rational subsumption relation:

\[
(\text{RM}) \quad \frac{C \sqsubseteq D, C \not\sqsubseteq \neg E}{C \cap E \sqsubseteq D}
\]

Rational monotonicity is often considered a desirable property to have, one of the reasons stemming from the fact it is a necessary condition for the satisfaction of the principle of presumption of typicality [58, Section 3.1]. Such a principle is a simple yet intuitive formalisation of a form of reasoning we carry out when facing lack of information: we reason assuming that we are in the most typical possible situation, compatible with the information at our disposal. (More details will be provided in Section 4).

### 3.2 Preferential semantics and representation results

In this section, we present our semantics for preferential and rational subsumption by enriching standard DL interpretations $\mathcal{I}$ with an ordering on the elements of the domain $\Delta^\mathcal{I}$. The intuition underlying this is simple and natural, and extends similar work done for the propositional case by Shoham [73], Kraus et al. [56], Lehmann and Magidor [59] and Booth et al. [12, 13, 14] to the case for description logics. This is not the first extension of this kind, as evidenced by the work of Boutilier [16], Baltag and Smets [5, 6], Giordano et al. [46, 48, 49, 50, 51, 52], Britz et al. [19, 20, 21, 22, 24] and Britz and Varzinczak [26, 27, 30, 29, 31, 32]. However, this is the first comprehensive semantic account of both preferential and rational subsumption relations, with accompanying representation results, based on the standard semantics for description logics.
Informally, our semantic constructions are based on the idea that objects of the domain can be ordered according to their degree of normality or typicality. Paraphrasing Boutilier, surely there is no inherent property of objects that allows them to be judged to be more or less normal in absolute terms. These orderings are purely subjective (in the sense that they can be thought of as part of an agent’s belief state) and the space of orderings deemed plausible by the agent may (among other things) be determined by e.g. empirical data. By using orderings in this way, we encode our (or the agent’s) expectations about the objects corresponding to their perceived regularity or typicality. Those objects not violating our expectations are considered to be more normal than the objects that violate some.

Hence we do not require that there exists something intrinsic about objects that makes one object more normal than another. Rather, the intention is to provide a framework in which to express all conceivable ways in which objects, with their associated properties and relationships with other objects, can be ordered in terms of typicality, in the same way that the class of all standard DL interpretations constitute a framework representing all conceivable ways of representing the properties of objects and their relationships with other objects. Just as the latter are constrained by stating subsumption statements in a knowledge base, the possible orderings that are considered plausible are encoded by writing down DCIs.

That said, we are ready for the definition of the first semantic construction the present work relies on.

**Definition 5 (Preferential Interpretation)** A preferential interpretation is a tuple \( P = \langle \Delta^P, \cdot^P, \prec^P \rangle \), where \( \langle \Delta^P, \cdot^P \rangle \) is a (standard) DL interpretation (which we denote by \( \mathcal{I}_P \) and refer to as the classical interpretation associated with \( P \) ), and \( \prec^P \) is a strict partial order on \( \Delta^P \) (i.e., \( \prec^P \) is irreflexive and transitive) satisfying the smoothness condition (for every \( C \in \mathcal{L} \), if \( C^P \neq \emptyset \), then \( \min_{\prec^P} C^P \neq \emptyset \)).

Figure 2 depicts a preferential interpretation in our scenario example where \( \Delta^P \) and \( \cdot^P \) are as in the interpretation \( \mathcal{I} \) shown in Figure 1 and \( \prec^P = \{ (x_7, x_5), (x_8, x_5), (x_9, x_5), (x_5, x_1), (x_7, x_1), (x_8, x_1), (x_9, x_1), (x_9, x_2), (x_10, x_6) \} \), represented by the dashed arrows in the picture. (For the sake of presentation, in the picture we omit the transitive \( \prec^P \)-arrows.)

Preferential interpretations provide us with a simple and intuitive way to give a semantics to DCIs.

**Definition 6 (Satisfaction)** Let \( P \) be a preferential interpretation, \( C, D \in \mathcal{L} \), \( r \in \mathcal{R} \) and \( a, b \in 1 \). The satisfaction relation \( \models \) is defined as follows:

\[ \text{Given } X \subseteq \Delta^P, \text{ with } \min_{\prec^P} X \text{ we denote the set } \{ x \in X \mid \text{for every } y \in X, y \not\prec^P x \}. \]
Figure 2: A preferential interpretation.

- $\mathcal{P} \models C \subseteq D$ if $C^\mathcal{P} \subseteq D^\mathcal{P}$;
- $\mathcal{P} \models C \subsetneq D$ if $\min_{\prec^\mathcal{P}} C^\mathcal{P} \subseteq D^\mathcal{P}$.

If $\mathcal{P} \models \alpha$, then we say $\mathcal{P}$ satisfies $\alpha$. $\mathcal{P}$ satisfies a defeasible knowledge base $\mathcal{KB}$, written $\mathcal{P} \models \mathcal{KB}$, if $\mathcal{P} \models \alpha$ for every $\alpha \in \mathcal{KB}$, in which case we say $\mathcal{P}$ is a preferential model of $\mathcal{KB}$. We say $C \in \mathcal{L}$ is satisfiable w.r.t. $\mathcal{KB}$ if there is a model $\mathcal{P}$ of $\mathcal{KB}$ s.t. $C^\mathcal{P} \neq \emptyset$.

It is easy to see that the addition of the $\prec^\mathcal{P}$-component preserves the truth of all classical subsumption statements holding in the remaining structure:

Lemma 1 Let $\mathcal{P}$ be a preferential interpretation. For every $C, D \in \mathcal{L}$, $\mathcal{P} \models C \subseteq D$ if and only if $\mathcal{I}_\mathcal{P} \models C \subseteq D$.

It is worth noting that, due to smoothness of $\prec^\mathcal{P}$, every (classical) subsumption statement is equivalent, with respect to preferential interpretations, to some DCI.

Lemma 2 For every preferential interpretation $\mathcal{P}$, and every $C, D \in \mathcal{L}$, $\mathcal{P} \models C \subseteq D$ if and only if $\mathcal{P} \models C \cap \neg D \subsetneq \bot$.

The following result, of which the proof can be found in Appendix A, will come in handy later on.

Lemma 3 Preferential interpretations are closed under disjoint union.
An obvious question that can now be raised is: “How do we know our preferential semantics provides an appropriate meaning to the notion of defeasible concept inclusion?” The following definition will help us in answering this question:

**Definition 7 (P-Induced Defeasible Subsumption)** Let $\mathcal{P}$ be a preferential interpretation. Then $
abla\mathcal{P} =_{df} \{ (C, D) \mid \mathcal{P} \models C \sqsubseteq D \}$ is the defeasible subsumption relation induced by $\mathcal{P}$.

The first important result we present here, which also answers the above raised question, shows that there is a full correspondence between the class of preferential subsumption relations and the class of defeasible subsumption relations induced by preferential interpretations. It is the DL analogue of a representation result proved by Kraus et al. for the propositional case [56, Theorem 3] and its proof can be found in Appendix B.

**Theorem 1** [Representation Result for Preferential Subsumption] A defeasible subsumption relation $\sqsubseteq \subseteq \mathcal{L} \times \mathcal{L}$ is preferential if and only if there is a preferential interpretation $\mathcal{P}$ such that $\nabla\mathcal{P} = \sqsubseteq$.

What is perhaps surprising about this result is that no additional properties based on the syntactic structure of the underlying DL are necessary to characterise the defeasible subsumption relations induced by preferential interpretations. We provide below a few properties involving the use of quantifiers that are satisfied by all preferential subsumption relations. (See Section 3 for more on properties explicitly mentioning DL-specific constructs.)

The first two are ‘existential’ and ‘universal’ versions of cautious monotonicity (CM):

\[
(CM_\exists) \quad \exists r.C \sqsubseteq E, \exists r.C \sqsubseteq \forall r.D \quad \frac{\exists r.(C \cap D) \sqsubseteq E}{}\]
\[
(CM_\forall) \quad \forall r.C \sqsubseteq E, \forall r.C \sqsubseteq \forall r.D \quad \frac{\forall r.(C \cap D) \sqsubseteq E}{}\]

The third one is a rephrasing of the Rule of Necessitation in modal logic [42]. It guarantees the absence of so-called spurious objects [25] in the original preferential semantics for DLs by Britz et al. [23, 24]. That is, if $C$ is unsatisfiable, then so is $\exists r.C$ (cf. Lemma 2).

\[
(Norm) \quad C \sqsubseteq \bot \quad \frac{\exists r.C \sqsubseteq \bot}{}\]

In addition to preferential interpretations, we are also interested in the study of modular interpretations, which are preferential interpretations in which the $\lt$-component is a modular ordering:

**Definition 8 (Modular Order)** Given a set $X$, $\lt \subseteq X \times X$ is modular if it is a strict partial order, and its associated incomparability relation $\sim$, defined by $x \sim y$ if neither $x \lt y$ nor $y \lt x$, is transitive.
If $\prec$ is modular, then $\sim$ is an equivalence relation.

**Definition 9 (Modular Interpretation)** A modular interpretation is a preferential interpretation $\mathcal{R} = \langle \Delta^\mathcal{R}, \cdot^\mathcal{R}, \prec^\mathcal{R} \rangle$ such that $\prec^\mathcal{R}$ is modular.

Intuitively, modular interpretations allow us to compare any two objects w.r.t. their plausibility. Those that are incomparable are viewed as being equally plausible. As such, modular interpretations are special cases of preferential interpretations, where plausibility can be represented by any smooth strict partial order.

The main reason to consider modular interpretations is that they provide the semantic foundation of rational subsumption relations. This is made precise by our second important result below, which shows that the defeasible subsumption relations induced by modular interpretations are precisely the rational subsumption relations. Again, this is the DL analogue of a representation result proved by Lehmann and Magidor for the propositional case [59, Theorem 5] and its proof can be found in Appendix C.

**Theorem 2** [Representation Result for Rational Subsumption] A defeasible subsumption relation $\sqsubseteq \subseteq \mathcal{L} \times \mathcal{L}$ is rational if and only if there is a modular interpretation $\mathcal{R}$ such that $\sqsubseteq^\mathcal{R} = \sqsubseteq$.

Analogous to the case for cautious monotonicity above, the following ‘existential’ and ‘universal’ versions of rational monotonicity are satisfied by all rational subsumption relations:

- $(\text{RM}_3)\, \frac{\exists r. C \sqsubseteq E, \exists r. C \not\sqsubseteq \forall r. \neg D}{\exists r. (C \cap D) \sqsubseteq E}$
- $(\text{RM}_4)\, \frac{\forall r. C \sqsubseteq E, \forall r. C \not\sqsubseteq \forall r. \neg D}{\forall r. (C \cap D) \sqsubseteq E}$

It is worth pausing for a moment to emphasise the significance of these two results (Theorems 1 and 2). They provide exact semantic characterisations of two important classes of defeasible subsumption relations, namely preferential and rational subsumption, in terms of the classes of preferential and modular interpretations, respectively. As we shall see in Section 4, these results form the core of the investigation into an appropriate notion of entailment for defeasible DL ontologies.

## 4 Rationality in entailment

From the standpoint of knowledge representation and reasoning, a pivotal question is that of deciding which statements are entailed by a knowledge base. We shall devote the remainder of the paper to this matter, and in this section we lay out the formal foundations for that.
4.1 Preferential entailment

In the exploration of a notion of entailment for defeasible ontologies, an obvious starting point is to consider a Tarskian definition of consequence:

**Definition 10 (Preferential Entailment)** A statement \( \alpha \) is preferentially entailed by a defeasible knowledge base \( KB \), written \( KB \models_{\text{pref}} \alpha \), if every preferential model of \( KB \) satisfies \( \alpha \).

As usual, this form of entailment is accompanied by a corresponding notion of closure.

**Definition 11 (Preferential Closure)** Let \( KB \) be a defeasible knowledge base. With \( KB^*_{\text{pref}} = \{ \alpha \mid KB \models_{\text{pref}} \alpha \} \) we denote the preferential closure of \( KB \).

Intuitively, the preferential closure of a defeasible knowledge base \( KB \) corresponds to the ‘core’ set of statements, classical and defeasible, that should hold given those in \( KB \). Hence, preferential entailment and preferential closure are two sides of the same coin, mimicking an analogous result for preferential reasoning in the propositional [56] case.

Recall (cf. the discussion following Definition 2) that a defeasible theory \( KB_{\text{inf}} \) is a defeasible knowledge base without the restriction to finite sets. When assessing how appropriate a notion of entailment for defeasible ontologies is, the following definitions turn out to be useful, as will become clear in the sequel:

**Definition 12 (\( KB_{\text{inf}} \)-Induced Defeasible Subsumption)** Let \( KB_{\text{inf}} \) be a defeasible theory. Then (1) \( D_{KB_{\text{inf}}} = \{ C \sqsubseteq D \mid C \sqsubseteq D \in KB_{\text{inf}} \} \cup \{ C \sqcap \neg D \sqsubseteq \bot \mid C \subseteq D \in KB_{\text{inf}} \} \) is the DTBox induced by \( KB_{\text{inf}} \) and (2) \( \sqsubseteq_{KB_{\text{inf}}} = \{ (C, D) \mid C \sqsubseteq D \in D_{KB_{\text{inf}}} \} \) is the defeasible subsumption relation induced by \( KB_{\text{inf}} \).

So, the DTBox induced by \( KB_{\text{inf}} \) is the set of defeasible subsumption statements contained in \( KB_{\text{inf}} \), together with the defeasible versions of the classical subsumption statements in \( KB_{\text{inf}} \). The defeasible subsumption relation induced by \( KB_{\text{inf}} \) is simply the defeasible subsumption relation corresponding to \( D_{KB_{\text{inf}}} \).

**Definition 13** A defeasible theory \( KB_{\text{inf}} \) is called preferential if the subsumption relation induced by it satisfies the preferential properties in Definition 4.

It turns out that the defeasible subsumption relation induced by the preferential closure of a defeasible knowledge base \( KB \) is exactly the intersection of the defeasible subsumption relations induced by the preferential defeasible theories containing \( KB \).

**Lemma 4** Let \( KB \) be a defeasible knowledge base. Then

\[
\sqsubseteq_{KB^*_\text{pref}} = \bigcap \{ \sqsubseteq_{KB_{\text{inf}}} \mid KB \subseteq KB_{\text{inf}} \text{ and } KB_{\text{inf}} \text{ is preferential} \}.
\]
It follows immediately that the preferential closure of a defeasible knowledge base $KB$ is preferential, and induces the smallest defeasible subsumption relation induced by a preferential defeasible theory containing $KB$.

Preferential entailment is not always desirable, one of the reasons being that it is monotonic, courtesy of the Tarskian notion of consequence it relies on (see Definition 10). In most cases, as witnessed by the great deal of work in the non-monotonic reasoning community, a move towards rationality is in order. Thanks to the definitions above and the result in Theorem 2, we already know where to start looking for it.

Definition 14 (Modular Entailment) A statement $\alpha$ is modularly entailed by a defeasible knowledge base $KB$, written $KB \models_{\text{mod}} \alpha$, if every modular model of $KB$ satisfies $\alpha$.

As is the case for preferential entailment, modular entailment is accompanied by a corresponding notion of closure.

Definition 15 (Modular Closure) Let $KB$ be a defeasible knowledge base. With $KB^{\ast}_{\text{mod}} = \{ \alpha \mid KB \models_{\text{mod}} \alpha \}$ we denote the modular closure of $KB$.

Definition 16 A defeasible theory $KB_{\text{inf}}$ is called rational if it is preferential and $K_{\text{inf}}$ is also closed under the rational monotonicity rule (RM).

For modular closure we get a result similar to Lemma 4.

Lemma 5 Let $KB$ be a defeasible knowledge base. Then

$$K_{\text{inf}} \subseteq KB^{\ast}_{\text{mod}} = \bigcap \{ K_{\text{inf}} \subseteq KB_{\text{inf}} \mid KB_{\text{inf}} \subseteq KB_{\text{inf}} \text{ and } KB_{\text{inf}} \text{ is rational} \}.$$  

That is, the modular closure of a defeasible knowledge base $KB$ induces the smallest defeasible subsumption relation induced by a rational defeasible theory containing $KB$. However, the modular closure of a defeasible knowledge base $KB$ is not necessarily rational. That is, if one looks at the set of statements (in particular the $K_{\text{inf}}$-ones) modularly entailed by a knowledge base as a defeasible subsumption relation, then it need not satisfy the rational monotonicity property. This is so because modular entailment coincides with preferential entailment, as the following result, adapted from a well-known similar result in the propositional case [59, Theorem 4.2], shows.

Lemma 6 $KB^{\ast}_{\text{mod}} = KB^{\ast}_{\text{pref}}$.

As a result, modular entailment unfortunately falls short of providing us with an appropriate notion of non-monotonic entailment. In what follows, we overcome precisely this issue.
4.2 Semantic rational entailment

In this section, we introduce a definition of semantic entailment which, as we shall see, is appropriate in the light of the discussion above. The constructions we are going to present are inspired by the semantic characterisation of rational closure by Booth and Paris [15] in the propositional case. We shall give a corresponding proof-theoretic characterisation of our version of semantic entailment in Section 5.1.

We focus our attention on a subclass of modular orders, referred to as ranked orders:

Definition 17 (Ranked Order) Given a set \( X \), the binary relation \( \prec \subseteq X \times X \) is a ranked order if there is a mapping \( h_{\mathcal{R}} : X \rightarrow \mathbb{N} \) satisfying the following convexity property:

- for every \( i \in \mathbb{N} \), if for some \( x \in X \) \( h_{\mathcal{R}}(x) = i \), then, for every \( j \) such that \( 0 \leq j < i \), there is a \( y \in X \) for which \( h_{\mathcal{R}}(y) = j \),

and s.t. for every \( x, y \in X \), \( x \prec y \) iff \( h_{\mathcal{R}}(x) < h_{\mathcal{R}}(y) \).

It is easy to see that a ranked order \( \prec \) is also modular: \( \prec \) is a strict partial order, and, since two objects \( x, y \) are incomparable (i.e., \( x \sim y \)) if and only if \( h_{\mathcal{R}}(x) = h_{\mathcal{R}}(y) \), \( \sim \) is a transitive relation. By constraining our preference relations to the ranked orders, we can identify a subset of the modular interpretations we refer to as the ranked interpretations.

Definition 18 (Ranked Interpretation) A ranked interpretation is a modular interpretation \( \mathcal{R} = \langle \Delta^\mathcal{R} , \cdot^\mathcal{R} , \prec^\mathcal{R} \rangle \) s.t. \( \prec^\mathcal{R} \) is a ranked order.

We now provide two basic results about ranked interpretations. First, all finite modular interpretations are ranked interpretations.

Lemma 7 A modular interpretation \( \mathcal{R} = \langle \Delta^\mathcal{R} , \cdot^\mathcal{R} , \prec^\mathcal{R} \rangle \) s.t. \( \Delta^\mathcal{R} \) is finite is a ranked interpretation.

Next, for every ranked interpretation \( \mathcal{R} \), the function \( h_{\mathcal{R}}(\cdot) \) is unique.

Proposition 1 Given a ranked interpretation \( \mathcal{R} = \langle \Delta^\mathcal{R} , \cdot^\mathcal{R} , \prec^\mathcal{R} \rangle \), there is only one function \( h_{\mathcal{R}} : X \rightarrow \mathbb{N} \) satisfying the convexity property and s.t. for every \( x, y \in X \), \( x \prec y \) iff \( h_{\mathcal{R}}(x) < h_{\mathcal{R}}(y) \).

Proposition 1 allows us to use the function \( h_{\mathcal{R}}(\cdot) \) to define the notions of height and layers.

Definition 19 (Height & Layers) Given a ranked interpretation \( \mathcal{R} = \langle \Delta^\mathcal{R} , \cdot^\mathcal{R} , \prec^\mathcal{R} \rangle \), its characteristic ranking function \( h_{\mathcal{R}}(\cdot) \), and an object \( x \in \Delta^\mathcal{R} \), \( h_{\mathcal{R}}(x) \) is called the height of \( x \) in \( \mathcal{R} \).
For every ranked interpretation \( R = \langle \Delta^R, \cdot^R, \prec^R \rangle \), we can partition the domain \( \Delta^R \) into a sequence of layers \( (L_0, \ldots, L_n, \ldots) \), where, for every object \( x \in \Delta^R \), we have \( x \in L_i \) iff \( h_R(x) = i \).

Intuitively, the lower the height of an object in an interpretation \( R \), the more typical (or normal) the object is in \( R \). We can also think of a level of typicality for concepts: the height of a concept \( C \in \mathcal{L} \) in \( R \) is the index of the layer to which the restriction of the concept’s extension to its \( \prec^R \)-minimal elements belong, i.e., \( h_R(C) = i \) if \( \emptyset \subset \min_{\prec^R} C^R \subseteq L_i \). As a convention, if \( \min_{\prec^R} C^R = \emptyset \), that is, if \( C^R = \emptyset \), then \( h_R(C) = \infty \).

The following result (proved in Appendix D) will be useful for some of the proofs in later sections of the paper:

**Theorem 3 (Finite-Model Property)** Defeasible \( \mathcal{ALC} \) has the finite-model property. In particular, every defeasible \( \mathcal{ALC} \) knowledge base that has a modular model, has also a finite ranked model.

Given a set of ranked interpretations, we can introduce a new form of model merging, ranked union.

**Definition 20 (Ranked Union)** Given a countable set of ranked interpretations \( \mathcal{R} = \{ R_1, R_2, \ldots \} \), a ranked interpretation \( R^\mathcal{R} = \text{def} \langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}}, \prec^{\mathcal{R}} \rangle \) is the ranked union of \( \mathcal{R} \) if the following holds:

- \( \Delta^{\mathcal{R}} = \text{def} \bigsqcup_{R \in \mathcal{R}} \Delta^R \), i.e., the disjoint union of the domains from \( \mathcal{R} \), where each \( R \in \mathcal{R} \) has the elements \( x, y, \ldots \) of its domain renamed as \( x^R, y^R, \ldots \) so that they are all distinct in \( \Delta^{\mathcal{R}} \);
- \( x^R \in A^{\mathcal{R}} \) iff \( x \in A^R \);
- \((x^R, y^{R'}) \in r^{\mathcal{R}} \) iff \( R = R' \) and \((x, y) \in r^R \);
- for every \( x^R \in \Delta^{\mathcal{R}} \), \( h_{\mathcal{R}}(x^R) = h_R(x) \).

The latter condition corresponds to imposing that \( x^R \prec^{\mathcal{R}} y^{R'} \) iff \( h_R(x) < h_{R'}(y) \).

Informally, the ranked union of a set of ranked interpretations is the result of merging all their layers of height \( i \) into a single layer of height \( i \), for all \( i \).

**Lemma 8** Given a set of ranked models of a defeasible knowledge base \( KB \), their ranked union is itself a ranked model of \( KB \).

Let \( KB \) be a defeasible knowledge base and let \( \Delta \) be a fixed countably infinite set. Define

\[
\text{Mod}_\Delta(KB) = \text{def} \{ R = \langle \Delta^R, \cdot^R, \prec^R \rangle \mid R \models KB, R \text{ is ranked and } \Delta^R = \Delta \}.
\]
The following result shows that the set $\text{Mod}_\Delta(\mathcal{KB})$ suffices to characterise modular entailment (the proof is in Appendix D):

**Lemma 9** For every $\mathcal{KB}$ and every $C, D \in \mathcal{L}$, $\mathcal{KB} \models_{\text{mod}} C \sqsubseteq D$ iff $\mathcal{R} \models C \sqsubseteq D$, for every $\mathcal{R} \in \text{Mod}_\Delta(\mathcal{KB})$.

Therefore, we can use just the set of interpretations in $\text{Mod}_\Delta(\mathcal{KB})$ to decide the consequences of $\mathcal{KB}$ w.r.t. modular entailment.

We can now use the set $\text{Mod}_\Delta(\mathcal{KB})$ as a springboard to introduce what will turn out to be a canonical modular interpretation for $\mathcal{KB}$. Using $\text{Mod}_\Delta(\mathcal{KB})$ and ranked union we can define the following relevant model.

**Definition 21 (Big Ranked Model)** Let $\mathcal{KB}$ be a defeasible knowledge base. The **big ranked model** of $\mathcal{KB}$ is the ranked model $\mathcal{O} = \text{def} \langle \Delta^\mathcal{O}, \cdot^\mathcal{O}, \prec^\mathcal{O} \rangle$ that is the ranked union of the models in $\text{Mod}_\Delta(\mathcal{KB})$.

Given Lemma 8 we can state the following:

**Corollary 1** $\mathcal{O}$ is a ranked model of $\mathcal{KB}$.

Armed with the definitions and results above, we are now ready to provide an alternative definition of entailment in the context of defeasible ontologies:

**Definition 22 (Rational Entailment)** A statement $\alpha$ is **rationally entailed** by a knowledge base $\mathcal{KB}$, written $\mathcal{KB} \models_{\text{rat}} \alpha$, if $\mathcal{O} \models \alpha$.

That such a notion of entailment indeed deserves its name is witnessed by the following result, a consequence of Corollary 1 and Theorem 2:

**Corollary 2** Let $\mathcal{KB}$ be a defeasible knowledge base. $\{C \sqsubseteq D \mid \mathcal{O} \models C \sqsubseteq D\}$ is rational.

In conclusion, rational entailment is a good candidate for the appropriate notion of defeasible consequence we have been looking for. Of course, a question that arises is whether a notion of closure, in the spirit of preferential and modular closures, that is equivalent to it can be defined. In the next section, we address precisely this matter.

## 5 Rational closure for defeasible knowledge bases

We now turn our attention to the exploration, in a DL setting, of the well-known notion of **rational closure** of a defeasible knowledge base as studied by Lehmann and Magidor [69] for propositional logic. For the most part, we base our constructions on the work by Casini and Straccia [37, 40], amending it wherever necessary. (An alternative semantic characterisation of rational closure in DLs has also been proposed by Giordano et al. [51, 52].) As we shall see, rational closure provides a proof-theoretic characterisation of rational entailment and
the complexity of its computation is no higher than that of computing entailment in the underlying classical DL.

5.1 Rational closure and a correspondence result

Rational closure is a form of inferential closure based on modular entailment $\models_{\text{mod}}$, but it extends its inferential power. Such an extension of modular entailment is obtained by formalising the already mentioned principle of *presumption of typicality* \[58, \text{Section 3.1}]. That is, under possibly incomplete information, we always assume that we are dealing with the most typical possible situation that is compatible with the information at our disposal. We first define what it means for a concept to be *exceptional*, a notion that is central to the definition of rational closure:

**Definition 23 (Exceptionality)** Let $KB$ be a defeasible knowledge base and $C \in L$. We say $C$ is *exceptional* in $KB$ if $KB \models_{\text{mod}} T \subseteq \neg C$. A DCI $C \subseteq D$ is exceptional in $KB$ if $C$ is exceptional in $KB$.

A concept $C$ is considered exceptional in a knowledge base $KB$ if it is not possible to have a modular model of $KB$ in which there is a typical object (i.e., an object at least as typical as all the others) that is in the interpretation of $C$. Intuitively, a DCI is exceptional if it does not concern the most typical objects, i.e., it is about less normal (or exceptional) ones. This is an intuitive translation of the notion of exceptionality used by Lehmann and Magidor \[59\] in the propositional framework, and has already been used by Casini and Straccia \[37\] and Giordano et al. \[52\] in their investigations into defeasible reasoning for description logics.

Applying the notion of exceptionality iteratively, we associate with every concept $C$ a *rank* in $KB$, which we denote by $\text{rank}_{KB}(C)$. We extend this to DCIs and associate with every statement $C \subseteq D$ a rank, denoted $\text{rank}_{KB}(C \subseteq D)$:

1. Let $\text{rank}_{KB}(C) = 0$, if $C$ is not exceptional in $KB$, and let $\text{rank}_{KB}(C \subseteq D) = 0$ for every DCI having $C$ in the LHS, with $\text{rank}_{KB}(C) = 0$. The set of DCIs in $D$ with rank 0 is denoted as $D_{0}^\text{rank}$.

2. Let $\text{rank}_{KB}(C) = 1$, if $C$ does not have a rank of 0 and it is not exceptional in the knowledge base $KB^1$ composed of $T$ and the exceptional part of $D$, that is, $KB^1 = \langle T, D \setminus D_0^\text{rank} \rangle$. If $\text{rank}_{KB}(C) = 1$, then let $\text{rank}_{KB}(C \subseteq D) = 1$ for every DCI $C \subseteq D$. The set of DCIs in $D$ with rank 1 is denoted $D_1^\text{rank}$.

3. In general, for $i > 0$, a concept $C$ is assigned a rank of $i$ if it does not have a rank of $i - 1$ and it is not exceptional in $KB^i = \langle T, D \setminus \bigcup_{j=0}^{i-1} D_j^\text{rank} \rangle$. If $\text{rank}_{KB}(C) = i$, then $\text{rank}_{KB}(C \subseteq D) = i$, for every DCI $C \subseteq D$ having $C$ in the LHS. The set of DCIs in $D$ with rank $i$ is denoted $D_i^\text{rank}$.
4. By iterating the previous steps, we eventually reach a subset $\mathcal{E} \subseteq \mathcal{D}$ such that all the DCIs in $\mathcal{E}$ are exceptional (since $\mathcal{D}$ is finite, we must reach such a point). If $\mathcal{E} \neq \emptyset$, we define the rank of the DCIs in $\mathcal{E}$ as $\infty$, and the set $\mathcal{E}$ is denoted $\mathcal{D}_{\text{rank}}^{\infty}$. Moreover, we set $\text{rank}_{KB}(C) = \infty$ for every $C$ in the LHS of some DCI in $\mathcal{D}_{\text{rank}}^{\infty}$.

The notion of rank can also be extended to GCIs as follows: $\text{rank}_{KB}(C \sqsubseteq D) = \text{rank}_{KB}(C \sqcap \neg D)$.

Following on the procedure above, the DTBox $\mathcal{D}$ is partitioned into a finite sequence $\langle \mathcal{D}_{\text{rank}}^{0}, \ldots, \mathcal{D}_{\text{rank}}^{n}, \mathcal{D}_{\text{rank}}^{\infty} \rangle$ $(n \geq 0)$, where $\mathcal{D}_{\text{rank}}^{\infty}$ may possibly be empty. So, through this procedure we can assign a rank to every DCI.

We can check that for a concept $C$ has a rank of $\infty$ iff it is not satisfiable in any model of $KB$, that is, $KB \models_{\text{mod}} C \sqsubseteq \bot$.

Lemma 10 For every knowledge base $KB$ and every concept $C$, $\text{rank}_{KB}(C) = \infty$ iff $KB \models_{\text{mod}} C \sqsubseteq \bot$.

Example 2 Let $KB = T \cup D$, where $T$ and $D$ are as in Example 1, i.e., $T = \{\text{EmpStud} \sqsubseteq \text{Student}\}$ and

$$D = \left\{ \begin{array}{l}
\text{Student} \nless \neg \exists \text{pays}. \text{Tax}, \\
\text{EmpStud} \nless \exists \text{pays}. \text{Tax}, \\
\text{EmpStud} \sqcap \text{Parent} \nless \neg \exists \text{pays}. \text{Tax} \end{array} \right\}$$

Examining the concepts on the LHS of each DCI in $KB$, one can verify that $\text{Student}$ is not exceptional w.r.t. $KB$. Therefore, $\text{rank}_{KB}(\text{EmpStud}) = 0$. We also find that $\text{rank}_{KB}(\text{EmpStud} \sqcap \text{Parent}) \neq 0$ because both concepts are exceptional w.r.t. $KB$. Hence $D_{\text{rank}}^{0} = \{\text{Student} \nless \neg \exists \text{pays}. \text{Tax}\}$ and $KB^{0} = T \cup D_{\text{rank}}^{0}$.

$KB^{1}$ is composed of $T$ and $D \setminus D_{\text{rank}}^{0}$. We find that $\text{EmpStud}$ is not exceptional w.r.t. $KB^{1}$ and therefore $\text{rank}_{KB}(\text{EmpStud}) = 1$. Since $\text{EmpStud} \sqcap \text{Parent}$ is exceptional w.r.t. $KB^{1}$, $\text{rank}_{KB}(\text{EmpStud} \sqcap \text{Parent}) \neq 1$. Thus $D_{\text{rank}}^{1} = \{\text{EmpStud} \nless \exists \text{pays}. \text{Tax}\}$. Similarly, $KB^{2}$ is composed of $T$ and $\{\text{EmpStud} \sqcap \text{Parent} \nless \neg \exists \text{pays}. \text{Tax}\}$. We have that $\text{EmpStud} \sqcap \text{Parent}$ is not exceptional w.r.t. $KB^{2}$ and therefore $\text{rank}_{KB}(\text{EmpStud} \sqcap \text{Parent}) = 2$. Finally, for this example, $D_{\text{rank}}^{\infty} = \emptyset$. ■

Adapting Lehmann and Magidor’s construction for propositional logic, the rational closure of a defeasible knowledge base $KB$ is defined as follows:

**Definition 24 (Rational Closure)** Let $KB$ be a defeasible knowledge base and $C, D \in \mathcal{L}$.

1. $C \nless D$ is in the rational closure of $KB$ if

$$\text{rank}_{KB}(C \sqcap D) < \text{rank}_{KB}(C \sqcap \neg D) \text{ or } \text{rank}_{KB}(C) = \infty.$$  

2. $C \sqsubseteq D$ is in the rational closure of $KB$ if $\text{rank}_{KB}(C \sqcap \neg D) = \infty$.  

Informally, the definition above says that $C \sqsubseteq D$ is in the rational closure of $KB$ if the modular models of $KB$ tell us that some instances of $C \cap D$ are more plausible than all instances of $C \cap \neg D$, while $C \sqsubseteq D$ is in the rational closure of $KB$ if the instances of $C \cap \neg D$ are impossible.

Example 2 (continued) Applying the definition above to the knowledge base in Example 2, we can verify that $\text{Student} \sqsubseteq \neg \exists \text{pays.Tax}$ is in the rational closure of $KB$ because $\text{rank}_{KB}(\text{Student} \cap \neg \exists \text{pays.Tax}) = 0$ and $\text{rank}_{KB}(\text{Student} \cap \exists \text{pays.Tax}) > 0$. The latter can be derived from the fact that $\text{Student} \cap \exists \text{pays.Tax}$ is exceptional w.r.t. $KB$. Similarly, one can derive that both DCIs $\text{EmpStud} \sqsubseteq \exists \text{pays.Tax}$ and $\text{EmpStud} \cap \text{Parent} \sqsubseteq \neg \exists \text{pays.Tax}$ are in the rational closure of $KB$ as well.

We now state the main result of the present section, which provides an answer to the question raised at the end of Section 4.2. (The proof can be found in Appendix E.)

Theorem 4 Let $KB$ be a defeasible knowledge base having a modular model. A statement $\alpha$ is in the rational closure of $KB$ iff $KB \models_{\text{rat}} \alpha$.

An easy corollary of this result is that rational closure preserves the equivalence between GCIs of the form $C \sqsubseteq D$ and their defeasible counterparts ($C \cap \neg D \sqsubseteq \bot$).

Corollary 3 $C \sqsubseteq D$ is in the rational closure of a defeasible knowledge base $KB$ iff $C \cap \neg D \sqsubseteq \bot$ is in the restriction of the closure of $KB$ under rational entailment to defeasible concept inclusions.

Rational entailment from a knowledge base can therefore be formulated as membership checking of the rational closure of the knowledge base. Of course, from an application-oriented point of view, this raises the question of how to compute membership of the rational closure of a knowledge base, and what is the complexity thereof. This is precisely the topic of the next section.

5.2 Rational entailment checking

We now present an algorithm to effectively check the rational entailment of a DCI from a defeasible knowledge base. Our algorithm is a modification of the one given by Casini and Straccia [37] for defeasible $ALC$. Their algorithm had to be modified slightly since it does not always give back the correct result in case $D^\text{rank}_\infty \neq \emptyset$ — cf. Item 4 in the description in Section 5.1.

Let $KB = T \cup D$ be a defeasible knowledge base. The first step of the algorithm is to assign a rank to each DCI in $D$. Central to this step is the exceptionality function $\text{Exceptional}(\cdot)$, which computes the semantic notion of exceptionality of Definition 23. The function makes use of the notion of materialisation to reduce concept exceptionality checking to entailment checking.
**Definition 25 (Materialisation)** Let \( \mathcal{D} \) be a set of DCIs. With \( \overline{\mathcal{D}} = \{ \neg C \sqcup D \mid C \sqsubseteq D \in \mathcal{D} \} \) we denote the **materialisation** of \( \mathcal{D} \).

We can show that, given \( \mathcal{KB} = \mathcal{T} \cup \mathcal{D} \) and \( \mathcal{D}' \subseteq \mathcal{D} \), if \( \mathcal{T} \models \bigcap \overline{\mathcal{D'}} \subseteq \neg C \), where \( \models \) denotes classical ALC entailment, a DCI \( C \sqsubseteq \neg \mathcal{D} \) is exceptional w.r.t. \( \mathcal{T} \cup \mathcal{D}' \), thereby justifying the use of Line 3 of function \( \text{Exceptional}(\cdot) \). The proof of the following lemma can be found in Appendix E.

**Lemma 11** For \( \mathcal{KB} = \mathcal{T} \cup \mathcal{D} \), if \( \mathcal{T} \models \bigcap \overline{\mathcal{D'}} \subseteq \neg C \), then \( C \sqsubseteq \neg \mathcal{D} \) is exceptional w.r.t. \( \mathcal{T} \cup \mathcal{D}' \).

Given a set of DCIs \( \mathcal{D}' \subseteq \mathcal{D} \), \( \text{Exceptional}(\mathcal{T}, \mathcal{D}') \) returns a subset \( \mathcal{E} \) of \( \mathcal{D}' \) such that \( \mathcal{E} \) is exceptional w.r.t. \( \mathcal{T} \cup \mathcal{D}' \).

| Function \( \text{Exceptional}(\mathcal{T}, \mathcal{D}') \) |
|---|
| **Input:** \( \mathcal{T} \) and \( \mathcal{D}' \subseteq \mathcal{D} \) |
| **Output:** \( \mathcal{E} \subseteq \mathcal{D}' \) such that \( \mathcal{E} \) is exceptional w.r.t. \( \mathcal{T} \cup \mathcal{D}' \) |
| 1 \( \mathcal{E} := \emptyset \); |
| 2 foreach \( C \sqsubseteq \mathcal{D}' \) such that \( \mathcal{E} \) is exceptional w.r.t. \( \mathcal{T} \cup \mathcal{D}' \) do |
| 3 if \( \mathcal{T} \models \bigcap \overline{\mathcal{D'}} \subseteq \neg C \) then |
| 4 \( \mathcal{E} := \mathcal{E} \cup \{ C \sqsubseteq \mathcal{D} \} \) |
| 5 return \( \mathcal{E} \) |

While the converse of Lemma 11 does not hold, it follows from Lemma 13 below that this reduction to classical entailment checking, when applied iteratively (lines 4–14 in Algorithm \( \text{ComputeRanking}(\cdot) \)), fully captures the semantic notion of exceptionality of Definition 23.

**Example 2** (continued) If we feed the knowledge base in Example 2 to the function \( \text{Exceptional}(\cdot) \), we obtain the output

\[ \mathcal{E} = \{ \text{EmpStud} \sqsubseteq \exists \text{pays.Tax}, \text{EmpStud} \sqcap \text{Parent} \sqsubseteq \neg \exists \text{pays.Tax} \}. \]

This is because both concepts on the LHS of the DCIs in \( \mathcal{D}' \) are exceptional w.r.t. \( \mathcal{KB} \) in Example 2.

We now describe the overall ranking algorithm, presented in the function \( \text{ComputeRanking}(\cdot) \) below. The algorithm makes a finite sequence of calls to the function \( \text{Exceptional}(\cdot) \), starting from the knowledge base \( \mathcal{KB} = \mathcal{T} \cup \mathcal{D} \). The algorithm terminates with a partitioning of the axioms in the DTBox, from which a ranking of axioms can easily be obtained.

We initialise \( \mathcal{T}^* \) to \( \mathcal{T} \) and \( \mathcal{D}^* \) to \( \mathcal{D} \) (Lines 1 and 2 of \( \text{ComputeRanking}(\cdot) \)). We then repeatedly invoke the function \( \text{Exceptional}(\cdot) \) to obtain a sequence of sets of DCIs \( \mathcal{E}_0, \mathcal{E}_1, \ldots \), where \( \mathcal{E}_0 = \mathcal{D}^* \) and each \( \mathcal{E}_{i+1} \) is the set of exceptional axioms in \( \mathcal{E}_i \) (Lines 4–14 of \( \text{ComputeRanking}(\cdot) \)).

21
Function ComputeRanking(KB)

Input: KB = T ∪ D
Output: KB* = T* ∪ D*, a partitioning R = {D0, . . . , Dn} for D*, and an exceptionality ranking E

1 T* := T;
2 D* := D;
3 R := ∅;
4 repeat
5 i := 0;
6 E0 := D*;
7 E1 := Exceptional(T*, E0);
8 while Ei+1 ≠ Ei do
9 i := i + 1;
10 Ei+1 := Exceptional(T*, Ei);
11 D∞* := Ei;
12 T* := T* ∪ {C ⊑ D | C ⊑ D ∈ D∞*};
13 D* := D* \ D∞*;
14 until D∞* = ∅;
15 E := (E0, . . . , Ei−1);
16 return (KB* = T* ∪ D*, E);

Now, let C D* =def {C | C ⊑ D ∈ D*}, i.e., C D* is the set of all antecedents of DCIs in D*. The exceptionality ranking of the DCIs in D* computed by Exceptional(·) makes use of T*, D*, and C D*. That is, it checks, for each concept C ∈ C D*, whether T* |= ∩ D* ⊑ ¬C. In case C is exceptional, every DCI C ⊑ D ∈ D* is exceptional w.r.t. KB* = T* ∪ D* and is added to the set E1.

If E1 ≠ E0, then we call Exceptional(·) for T* ∪ E1, defining the set E2, and so on. Hence, given KB* = T* ∪ D*, we construct a sequence E0, E1, . . . in the following way:

• E0 := D*;
• Ei+1 := Exceptional(T*, Ei), for i ≥ 0.

Example 2 (continued) Using the knowledge base of Example 2 we initialise T* = {EmpStud ⊑ Student} and

\[ D* = \left\{\begin{array}{l}
\text{Student } \sqsubseteq \neg \exists \text{pays.Tax,} \\
\text{EmpStud } \sqsubseteq \exists \text{pays.Tax,} \\
\text{EmpStud } \sqcap \text{Parent } \sqsubseteq \neg \exists \text{pays.Tax}
\end{array}\right\}. \]
We then obtain the following exceptionality sequence:

\[
E_0 = \begin{cases} 
\text{Student } \subseteq \neg \exists \text{pays.Tax}, \\
\text{EmpStud } \subseteq \exists \text{pays.Tax}, \\
\text{EmpStud } \cap \text{Parent } \subseteq \neg \exists \text{pays.Tax} 
\end{cases}
\]

\[
E_1 = \begin{cases} 
\text{EmpStud } \subseteq \exists \text{pays.Tax}, \\
\text{EmpStud } \cap \text{Parent } \subseteq \neg \exists \text{pays.Tax} 
\end{cases}
\]

\[
E_2 = \{ \text{EmpStud } \cap \text{Parent } \subseteq \neg \exists \text{pays.Tax} \}
\]

Since \( D^* \) is finite, the construction will eventually terminate with a fixed point \( E_{\text{fix}} = \text{Exceptional}(T^*, E_{\text{fix}}) \). If this fixed point is non-empty, then the axioms in there are said to have infinite rank. We therefore set \( D^*_\infty = \text{def } E_{\text{fix}} \) (Line 11 of ComputeRanking(\cdot)), and the classical translations of these axioms are moved to the TBox. Hence we redefine the knowledge base in the following way (Lines 12 and 13 of ComputeRanking(\cdot)):

- \( T^* := T^* \cup \{ C \subseteq D \mid C \subseteq D \in D^*_\infty \} \);
- \( D^* := D^* \setminus D^*_\infty \).

Function ComputeRanking(\cdot) must terminate since \( D \) is finite, and at every iteration, \( D^* \) becomes smaller (hence, we have at most \(|D|\) iterations). In the end, we obtain a knowledge base \( KB^* = T^* \cup D^* \) which is modularly equivalent to the original knowledge base \( KB = T \cup D \) (see Lemma 12 below), in which \( D^* \) has no DCIs of infinite rank (all the classical knowledge implicit in the DTBox has been moved to the TBox). We say that such a knowledge base is in rank normal form.

We also obtain a final exceptionality sequence \( E = (E_0, E_1, \ldots, E_{i-1}) \) (see Line 15 of ComputeRanking(\cdot)). Given \( E \), it is possible to partition the set \( D^* \) into the sets \( D_0, \ldots, D_n \), for some \( n = i - 1 \geq 0 \):

- For every \( j \), \( 0 \leq j \leq n \), \( D_j := E_j \setminus E_{j+1} \);
- \( R := \{ D_0, \ldots, D_n \} \).

The sequence \( R \) is a partition of the DTBox according to the level of exceptionality of each defeasible inclusion in it.

Example 2 (continued) For \( KB \) as in Example 2, we obtain the partition \( R = \{ D_0, D_1, D_2 \} \), where \( D_0 = \{ \text{Student } \subseteq \neg \exists \text{pays.Tax} \} \), \( D_1 = \{ \text{EmpStud } \subseteq \exists \text{pays.Tax} \} \) and \( D_2 = \{ \text{EmpStud } \cap \text{Parent } \subseteq \neg \exists \text{pays.Tax} \} \).
At this stage, we have moved all the classical information implicit the DTBox to the TBox, and ranked all the remaining DCIs, where the rank of a DCI is the index of the unique partition to which it belongs, defined as follows:

**Definition 26 (Ranking)** For every $C, D \in \mathcal{L}$:

- $\text{rk}(C) = \text{def} i, 0 \leq i \leq n$, if $E_i$ is the first element in the sequence $(E_0, \ldots, E_n)$ s.t. $T^* \not\models \bigcap_i E_i \cap C \sqsubseteq \bot$;
- $\text{rk}(C) = \text{def} \infty$, if there is no such $E_i$;
- $\text{rk}(C \sqsubseteq D) = \text{def} \text{rk}(C)$.

**Remark 1** For every $i \leq j \leq n$, $\models \bigcap_i E_j \subseteq \bigcap_i E_i$.

**Remark 2** For every $i < j \leq n$, $D_i \cap D_j = \emptyset$.

To summarise, we transform our initial knowledge base $KB = T \cup D$, obtaining a modularly equivalent knowledge base $KB^* = T^* \cup D^*$ (see Lemma 12 below) and a ranking of DCIs in the form of a partitioning of $D^*$. The main difference between ComputeRanking(·) and the analogous procedure by Casini and Straccia [37] is the reiteration of the ranking procedure until $D^*_\infty = \emptyset$ (lines 4-14 in ComputeRanking(·)). While the two procedures behave identically in the case where there are no DCIs $C \sqsubseteq D$ s.t. $\text{rank}_{KB}(C \sqsubseteq D) = \infty$ in $D$, the procedure by Casini and Straccia [37] did not handle all the cases correctly in which there is classical information implicit in the DTBox. Lemma 45 in Appendix E and Lemma 13 below prove that the procedure here is correct w.r.t. the semantics.

Given the knowledge base $KB^* = T^* \cup D^*$, we can now define the main algorithm for deciding whether a DCI $C \sqsubseteq D$ is in the rational closure of $KB$. To do that, we use the same approach as in the function Exceptional(·), that is, given $KB^* = T^* \cup D^*$ and our sequence of sets $E_0, \ldots, E_n$, we use the TBox $T^*$ and the sets of conjunctions of materialisations $\bigcap E_0, \ldots, \bigcap E_n$.

**Definition 27 (Rational Deduction)** Let $KB = T \cup D$ and let $C, D \in \mathcal{L}$. We say that $C \sqsubseteq D$ is **rationally deducible** from $KB$, denoted $KB \vdash_{\text{rat}} C \sqsubseteq D$, if $T^* \models \bigcap_i E_i \cap C \sqsubseteq D$, where $\bigcap_i E_i$ is the first element of the sequence $\bigcap E_0, \ldots, \bigcap E_n$ s.t. $T^* \not\models \bigcap_i E_i \sqsubseteq \neg C$. If there is no such element, $KB \vdash_{\text{rat}} C \sqsubseteq D$ if $T^* \models C \sqsubseteq D$.

Observe that $KB \vdash_{\text{rat}} C \sqsubseteq D$ if and only if $KB \vdash_{\text{rat}} C \cap \neg D \sqsubseteq \bot$, i.e., if and only if $KB \vdash_{\text{rat}} C \cap \neg D \sqsubseteq \bot$ (that is to say, $T^* \models C \sqsubseteq D$).

The algorithm corresponding to the steps above is presented in the function RationalClosure(·) below.
Function RationalClosure($KB$, $\alpha$)

Input: $KB = T \cup D$ and a query $\alpha = C \sqsubseteq D$.
Output: true if $KB \vdash_{\text{rat}} C \sqsubseteq D$, false otherwise

1. $(KB^* = T^* \cup D^*, \mathcal{E} = (E_0, \ldots, E_n)) := \text{ComputeRanking}(KB)$;
2. $i := 0$;
3. while $T^* \models \bigsqcap E_i \sqcap C \subseteq \bot$ and $i \leq n$ do
   4. $i := i + 1$;
5. if $i \leq n$ then
   6. return $T^* \models \bigsqcap E_i \sqcap C \subseteq D$;
6. else
   7. return $T^* \models C \subseteq D$;

Example 2 (continued) Let $KB$ be as in Example 2 and assume we want to check whether $\text{EmpStud} \subseteq \exists \text{pays}. \text{Tax}$ is in the rational closure of $KB$. Then, the while-loop on Line 2 of function RationalClosure(·) terminates when $i = 1$. At this stage, $\prod E_i = (\neg \text{EmpStud} \sqcup \exists \text{pays}. \text{Tax}) \sqcap (\neg \text{EmpStud} \sqcup \neg \text{Parent} \sqcup \neg \exists \text{pays}. \text{Tax})$. Given this, one can check that $T^* \not\models \prod E_i \sqcap C \subseteq \bot$, i.e., $\{\text{EmpStud} \subseteq \text{Student}\} \not\models (\neg \text{EmpStud} \sqcup \exists \text{pays}. \text{Tax}) \sqcap (\neg \text{EmpStud} \sqcup \neg \text{Parent} \sqcup \neg \exists \text{pays}. \text{Tax}) \sqcap \text{EmpStud} \subseteq \bot$. Finally, it is easy to confirm that $T^* \not\models \prod E_i \sqcap C \subseteq D$, i.e., $\{\text{EmpStud} \subseteq \text{Student}\} \not\models (\neg \text{EmpStud} \sqcup \exists \text{pays}. \text{Tax}) \sqcap (\neg \text{EmpStud} \sqcup \neg \text{Parent} \sqcup \neg \exists \text{pays}. \text{Tax}) \sqcap \text{EmpStud} \subseteq \exists \text{pays}. \text{Tax}$.

Before we state the main theorem of this section, we need to establish the correspondence between the ranking function $\text{rank}_{KB}(\cdot)$ presented in Section 5.1 in the construction of the rational closure of $KB$ and linked by Theorem 4 to the definition of rational entailment, and the ranking function $\text{rk}(\cdot)$ of Definition 26 used in the above algorithm. We also need to establish that the normalisation of a knowledge base by our algorithm maintains modular equivalence. The proofs of the following lemmas, as well as a number of prerequisite results, are in Appendix E.

Lemma 12 Let $KB = T \cup D$ and let $KB^* = T^* \cup D^*$ be obtained from $KB$ through function ComputeRanking(·). Then $KB$ and $KB^*$ are modularly equivalent.

Lemma 13 For every defeasible knowledge base $KB = T \cup D$ and every $C \in \mathcal{L}$, $\text{rank}_{KB}(C) = \text{rk}(C)$.

Now we can state the main theorem, which links rational entailment to rational deduction via Theorem 4. (The proof can be found in Appendix E.)

Theorem 5 Let $KB = T \cup D$ and let $C, D \in \mathcal{L}$. Then $KB \vdash_{\text{rat}} C \subseteq D$ iff $KB \vdash_{\text{rat}} C \subseteq D$. 25
As an immediate consequence, we have that the function $\text{RationalClosure}(\cdot)$ is correct w.r.t. the definition of rational closure in Definition 24.

**Corollary 4** Checking rational entailment is \textsc{exptime}-complete.

Hence entailment checking for defeasible ontologies is just as hard as classical subsumption checking.

We conclude this section by noting that although rational closure is viewed as an appropriate form of defeasible reasoning, it does have its limitations, the first of which is that it does not satisfy the \textit{presumption of independence} [58, Section 3.1]. To consider a well-worn example, suppose we know that birds usually fly and usually have wings, that both penguins and robins are birds, and that penguins usually do not fly. That is, we have the following knowledge base: $\mathcal{KB} = \{\text{Bird} \sqsubseteq \neg \text{Flies}, \text{Bird} \sqsubseteq \text{Wings}, \text{Penguin} \sqsubseteq \text{Bird}, \text{Robin} \sqsubseteq \text{Bird}, \text{Penguin} \sqsubseteq \neg \text{Flies}\}$. Rational closure allows us to conclude that robins usually have wings, since they are viewed as typical birds, thereby satisfying the presumption of typicality. But with penguins being atypical birds, rational closure does not allow us to conclude that penguins usually have wings, thus violating the presumption of independence which, in this context, would require the atypicality of penguins w.r.t. flying to be independent of the typicality of penguins w.r.t. having wings.

This deficiency is well-known, and there are other forms of defeasible reasoning that can overcome this, most notably lexicographic closure [39], relevance closure [35], and inheritance-based closure [38]. But note that the presumption of independence is \textit{propositional} in nature. In fact, the DL version of lexicographic closure is essentially a lifting to the DL case of a propositional solution to the problem [58].

What is perhaps of more interest is the inability of rational closure to deal with defeasibility relating to the \textit{non-propositional} aspects of description logics. For example, Pensel and Turhan [63, 64] have shown that rational closure across role expressions does not always support defeasible inheritance appropriately. Suppose we know that bosses are workers, do not have workers as their superiors, and are usually responsible. Furthermore, suppose we know that workers usually have bosses as their superiors. We thus have the knowledge base $\mathcal{KB} = \{\text{Boss} \sqsubseteq \text{Worker}, \text{Boss} \sqsubseteq \neg \exists \text{hasSuperior}. \text{Worker}, \text{Boss} \sqsubseteq \text{Responsible}, \text{Worker} \sqsubseteq \exists \text{hasSuperior}. \text{Boss}\}$. Since workers usually have bosses as their superiors, and bosses are usually responsible, one would expect to be able to conclude that workers usually have responsible superiors. But rational closure is unable to do so. From the perspective of the algorithm for rational closure, this can be traced back to the use of materialisation (Definition 25) when computing exceptionality, as Pensel and Turhan show. A more detailed semantic explanation for this inability is still forthcoming, though.
6 Related work

In a sense, the first investigations on non-monotonic reasoning in DL-based systems date back to the work by Brewka [17] and Cadoli et al. [34]. Other early proposals to introduce default-style rules into description logics include the work by Baader and Hollunder [2, 3], Padgham and Zhang [62] and Straccia [74], which are essentially based on Reiter’s default logic [68].

Quantz and Royer [66] were probably the first to consider lifting the preferential approach to a DL setting. They propose a general framework for Preferential Default Description Logics (PDDL) based on an \( \mathcal{ALC} \)-like language by introducing a version of default subsumption and proposing a preferential semantics for it. Their semantics is based on a simplified version of standard DL interpretations. They assume all domains to be finite, which means their framework is much more restrictive than ours in this aspect. They also allow for the use of object names (something we don’t do), and assume that the unique-name assumption holds for object names.

They focus on a version of entailment which they refer to as preferential entailment, but which is to be distinguished from the version of preferential entailment that we have presented in this paper. In what follows, we shall refer to their version as \textit{QR-preferential entailment}.

QR-preferential entailment is concerned with what ought to follow from a set of DL statements, together with a set of default subsumption statements, and is parameterised by a fixed partial order on (simplified) DL interpretations. (I.e., the ordering is on the set of DL interpretations, not on the elements of their respective domains.) They prove that any QR-preferential entailment satisfies the properties of a preferential consequence relation and, with some restrictions on the partial order, satisfies rational monotonicity as well. QR-preferential entailment can therefore be viewed as something in between the notions of preferential entailment (or modular entailment) and rational entailment. It is also worth noting that although QR-preferential entailment satisfies the properties of a preferential consequence relation, Quantz and Royer do not prove that QR-preferential entailment provides a characterisation of preferential consequence in the spirit of the representation results we have shown here.

Closely related to our work is that of Giordano et al. [49, 51] who use preferential orderings on \( \Delta^I \) to define a typicality operator \( T(\cdot) \) on \( \mathcal{ALC} \) concepts such that the expression \( T(C) \sqsubseteq D \) corresponds to our \( C \sqsubseteq \sim D \). They provide a version of a representation result for preferential orderings in terms of properties on selection functions (functions on the power set of the domain of interpretations), but not a representation result along the lines of those we have shown here. In the same work, the authors define a tableaux calculus for computing preferential entailment that relies on KLM-style rules.

Recently [52], Giordano and colleagues extended the aforementioned work by considering modular orderings on \( \Delta^I \) (i.e., modular interpretations) and then augment the infer-
ential power of their system with a version of a minimal-model semantics, in which some modular interpretations are preferred over others. This is similar in intuition to rational entailment, but their approach also has a circumscriptive flavour to it (see below) since it relies on the specification of a set of concepts for which atypical instances must be minimised.

Outside the family of preferential systems, there are mature proposals based on circumscription for DLs [9, 10, 72]. The main drawback of these approaches is the burden on the ontology engineer to make appropriate decisions related to the (circumscriptive) fixing and varying of concepts and the priority of defeasible subsumption statements. Such choices can have a major effect on the conclusions drawn from the system, and can easily lead to counter-intuitive inferences. Moreover, the use of circumscription usually implies a considerable increase in computational complexity w.r.t. the underlying monotonic entailment relation. The comparison between the present work and proposals outside the preferential family is more an issue about the pros and cons of the different kinds of non-monotonic reasoning, rather than about their DL re-formulation. As stated in the introduction, the preferential approach has a series of desirable qualities that, to our knowledge, no other approach to non-monotonic reasoning shares.

A more recent proposal is the approach by Bonatti et al. [8, 11], which introduces a normality operator \( N(\cdot) \) on concepts. The resulting system, \( \text{DL}^N \), is not based on the preferential approach, though, and as a consequence their closure operation does not allow defeasible subsumption to satisfy the preferential properties, but it satisfies some interesting properties on the meta-level. It also has the advantage of being computationally tractable for any tractable classical DL.

Lukasiewicz [60] proposes probabilistic versions of the description logics \( \text{SHI}, \text{F}(\text{D}) \) and \( \text{SHOIN}(\text{D}) \). As a special case of these logics, he obtains a version of a logic with defeasible subsumption with a semantics based on that of the propositional version of lexicographic closure [58].

Casini and Straccia [37] define KLM-based decision procedures for \( \text{ALC} \). Their proposal has a syntactic characterisation, but lacks an appropriate semantics, a deficiency that the present paper comes to remedy. The semantic approach presented here can be extended also to other forms of entailment proposed by them [38, 39, 40], and recently Casini, Straccia and Meyer have used it also to characterise a decision procedure for defeasible \( \mathcal{EL}_\perp \) [41].

Britz and Varzinczak [26, 30, 31] explore the notion of defeasible modalities, with which defeasible effects of actions, defeasible knowledge, obligations and others can be formalised and given an intuitive preferential semantics. Their approach differs from ours in that they consider preferential entailment only, but the semantic constructions are similar. This was recently extended [19] to a notion of defeasible role restrictions in a DL setting. The idea comprises extending the language of \( \text{ALC} \) with an additional construct \( \forall \). The semantics of a concept \( \forall r.C = \text{def} \{ x \in \Delta^P \mid \min_{r^P} r^P(x) \subseteq C^P \} \) is then given by all objects of \( \Delta^P \).
such that all of their minimal $r$-related objects are $C$-instances. This is useful in situations where certain classical concept descriptions may be too strong.

Recently, Britz and Varzinczak have lifted the preferential semantics to also allow for orderings on role-interpretations\cite{27,29} that, in turn, induce multi-orderings on objects of the domain\cite{28,32}. The latter give us the handle needed to introduce a notion of context in defeasible subsumption relations making typicality a relativised construct. The former provides a semantics for defeasible role inclusions of the form $r \sqsubseteq s$ and for defeasible role assertions such as “$r$ is usually transitive”, “$r$ and $s$ are usually disjoint”, as well as others.

Finally, there is the recent work of Pensel and Turhan\cite{63,64} mentioned in Section 5.2, the aim of which is to extend both rational closure and relevance closure with defeasible inheritance across role expressions in the description logic $\mathcal{EL}_{\bot}$. With their work being restricted to $\mathcal{EL}_{\bot}$, the semantics they propose is based on a form of canonical model similar to those frequently used for the $\mathcal{EL}$ family of DLs, and is therefore quite different from ours. A detailed comparison of their semantics with the one we provide in this paper is left as future work.

7 Concluding remarks

The main contributions of the work reported in the present paper can be summarised as follows:

1. The analysis of a simple and intuitive semantics for defeasible subsumption in description logics that is general enough to constitute the core framework within which to investigate non-monotonic extensions of DLs;

2. A characterisation of preferential and rational subsumption relations, with the respective representation results, evidencing the fact that our semantic constructions are appropriate;

3. An investigation of what an appropriate notion of entailment in a defeasible DL context means and the analysis of a suitable candidate, namely rational entailment, and

4. The formal connection between rational entailment, the notion of rational closure and an algorithm for its computation.

With regard to point 4 above, the main advantages of our approach are as follows: (i) it relies completely on classical entailment, i.e., entailment checking over defeasible ontologies can be reduced to a number of classical entailment checks over a rewritten ontology, (ii) it has computational complexity that is no worse than that of entailment checking in the classical underlying DL, and (iii) it is easily implementable, e.g. as a Protégé plugin\footnote{https://github.com/kodymoodley/defeasibleinferenceplatform}. 


of which the performance has been shown to scale well in practice [36]. In a companion paper [18] the framework described here is extended to include ABox reasoning, with more extensive experimental results confirming the initial promising results on scalability.

Further topics for future research include the integration of notions such as typicality for both concepts and roles [12, 13, 14, 49, 51, 75] and role-based defeasible constructors [27, 29, 32, 33] into the framework here presented. Another avenue for future exploration is the study of belief revision for DLs via our results for rationality, somewhat mimicking the well-known connection between belief revision and rational consequence in the propositional case [45], thereby pushing the frontiers of theory change in logics that are more expressive than the propositional one.

References

[1] F. Baader, D. Calvanese, D. McGuinness, D. Nardi, and P. Patel-Schneider, editors. The Description Logic Handbook: Theory, Implementation and Applications. Cambridge University Press, 2 edition, 2007.

[2] F. Baader and B. Hollunder. How to prefer more specific defaults in terminological default logic. In R. Bajcsy, editor, Proceedings of the 13th International Joint Conference on Artificial Intelligence (IJCAI), pages 669–675. Morgan Kaufmann Publishers, 1993.

[3] F. Baader and B. Hollunder. Embedding defaults into terminological knowledge representation formalisms. Journal of Automated Reasoning, 14(1):149–180, 1995.

[4] F. Baader, I. Horrocks, C. Lutz, and U. Sattler. An Introduction to Description Logic. Cambridge University Press, 2017.

[5] A. Baltag and S. Smets. Dynamic belief revision over multi-agent plausibility models. In W. van der Hoek and M. Wooldridge, editors, Proceedings of LOFT, pages 11–24. University of Liverpool, 2006.

[6] A. Baltag and S. Smets. A qualitative theory of dynamic interactive belief revision. In G. Bonanno, W. van der Hoek, and M. Wooldridge, editors, Logic and the Foundations of Game and Decision Theory (LOFT7), number 3 in Texts in Logic and Games, pages 13–60. Amsterdam University Press, 2008.

[7] S. Benferhat, D. Dubois, and H. Prade. Possibilistic and standard probabilistic semantics of conditional knowledge bases. Journal of Logic and Computation, 9(6):873–895, 1999.

[8] P. Bonatti, M. Faella, I.M. Petrova, and L. Sauro. A new semantics for overriding in description logics. Artificial Intelligence, 222:1–48, 2015.
[9] P. Bonatti, M. Faella, and L. Sauro. Defeasible inclusions in low-complexity DLs. *Journal of Artificial Intelligence Research, 42:719–764, 2011.*

[10] P. Bonatti, C. Lutz, and F. Wolter. The complexity of circumscription in description logic. *Journal of Artificial Intelligence Research, 35:717–773, 2009.*

[11] P. Bonatti and L. Sauro. On the logical properties of the nonmonotonic description logic DL$^N$. *Artificial Intelligence, 248:85–111, 2017.*

[12] R. Booth, G. Casini, T. Meyer, and I. Varzinczak. On the entailment problem for a logic of typicality. In *Proceedings of the 24th International Joint Conference on Artificial Intelligence (IJCAI), pages 2805–2811, 2015.*

[13] R. Booth, T. Meyer, and I. Varzinczak. PTL: A propositional typicality logic. In L. Fariñas del Cerro, A. Herzig, and J. Mengin, editors, *Proceedings of the 13th European Conference on Logics in Artificial Intelligence (JELIA), number 7519 in LNCS, pages 107–119. Springer, 2012.*

[14] R. Booth, T. Meyer, and I. Varzinczak. A propositional typicality logic for extending rational consequence. In E.L. Fermé, D.M. Gabbay, and G.R. Simari, editors, *Trends in Belief Revision and Argumentation Dynamics, volume 48 of Studies in Logic – Logic and Cognitive Systems, pages 123–154. King’s College Publications, 2013.*

[15] R. Booth and J.B. Paris. A note on the rational closure of knowledge bases with both positive and negative knowledge. *Journal of Logic, Language and Information, 7(2):165–190, 1998.*

[16] C. Boutilier. Conditional logics of normality: A modal approach. *Artificial Intelligence, 68(1):87–154, 1994.*

[17] G. Brewka. The logic of inheritance in frame systems. In *Proceedings of the 10th International Joint Conference on Artificial Intelligence (IJCAI), pages 483–488. Morgan Kaufmann Publishers, 1987.*

[18] K. Britz, G. Casini, T. Meyer, K. Moodley, U. Sattler, and I. Varzinczak. Rational defeasible reasoning for description logics. Technical report, University of Cape Town, South Africa. URL: https://tinyurl.com/yc55y7ts, 2017.

[19] K. Britz, G. Casini, T. Meyer, and I. Varzinczak. Preferential role restrictions. In *Proceedings of the 26th International Workshop on Description Logics, pages 93–106, 2013.*

[20] K. Britz, J. Heidema, and W. Labuschagne. Semantics for dual preferential entailment. *Journal of Philosophical Logic, 38:433–446, 2009.*
[21] K. Britz, J. Heidema, and T. Meyer. Semantic preferential subsumption. In J. Lang and G. Brewka, editors, *Proceedings of the 11th International Conference on Principles of Knowledge Representation and Reasoning (KR)*, pages 476–484. AAAI Press/MIT Press, 2008.

[22] K. Britz, J. Heidema, and T. Meyer. Modelling object typicality in description logics. In A. Nicholson and X. Li, editors, *Proceedings of the 22nd Australasian Joint Conference on Artificial Intelligence*, number 5866 in LNAI, pages 506–516. Springer, 2009.

[23] K. Britz, T. Meyer, and I. Varzinczak. Preferential reasoning for modal logics. *Electronic Notes in Theoretical Computer Science*, 278:55–69, 2011. Proceedings of the 7th Workshop on Methods for Modalities (M4M’2011).

[24] K. Britz, T. Meyer, and I. Varzinczak. Semantic foundation for preferential description logics. In D. Wang and M. Reynolds, editors, *Proceedings of the 24th Australasian Joint Conference on Artificial Intelligence*, number 7106 in LNAI, pages 491–500. Springer, 2011.

[25] K. Britz, T. Meyer, and I. Varzinczak. Normal modal preferential consequence. In M. Thielscher and D. Zhang, editors, *Proceedings of the 25th Australasian Joint Conference on Artificial Intelligence*, number 7691 in LNAI, pages 505–516. Springer, 2012.

[26] K. Britz and I. Varzinczak. Defeasible modalities. In *Proceedings of the 14th Conference on Theoretical Aspects of Rationality and Knowledge (TARK)*, pages 49–60, 2013.

[27] K. Britz and I. Varzinczak. Introducing role defeasibility in description logics. In L. Michael and A.C. Kakas, editors, *Proceedings of the 15th European Conference on Logics in Artificial Intelligence (JELIA)*, number 10021 in LNCS, pages 174–189. Springer, 2016.

[28] K. Britz and I. Varzinczak. Context-based defeasible subsumption for $dSROIQ$. In *Proceedings of the 13th International Symposium on Logical Formalizations of Commonsense Reasoning*, 2017.

[29] K. Britz and I. Varzinczak. Toward defeasible $SROIQ$. In *Proceedings of the 30th International Workshop on Description Logics*, 2017.

[30] K. Britz and I. Varzinczak. From KLM-style conditionals to defeasible modalities, and back. *Journal of Applied Non-Classical Logics (JANCL)*, 28(1):92–121, 2018.

[31] K. Britz and I. Varzinczak. Preferential accessibility and preferred worlds. *Journal of Logic, Language and Information (JoLLI)*, 27(2):133–155, 2018.
[32] K. Britz and I. Varzinczak. Rationality and context in defeasible subsumption. In F. Ferrarotti and S. Woltran, editors, *Proceedings of the 10th International Symposium on Foundations of Information and Knowledge Systems (FoIKS)*, number 10833 in LNCS, pages 114–132. Springer, 2018.

[33] K. Britz and I. Varzinczak. Contextual rational closure for defeasible ALC. *Annals of Mathematics and Artificial Intelligence*, 2019. to appear.

[34] M. Cadoli, F. Donini, and M. Schaerf. Closed world reasoning in hybrid systems. In Z.W. Ras, M. Zemankova, and M.L. Emrich, editors, *Proceedings of the 5th International Symposium on Methodologies for Intelligent Systems (ISMIS’90)*, pages 474–481. Elsevier, 1990.

[35] G. Casini, T. Meyer, K. Moodley, and R. Nortjé. Relevant closure: A new form of defeasible reasoning for description logics. In *Logics in Artificial Intelligence*, pages 92–106. Springer, 2014.

[36] G. Casini, T. Meyer, K. Moodley, U. Sattler, and I. Varzinczak. Introducing defeasibility into OWL ontologies. In M. Arenas, O. Corcho, E. Simperl, M. Strohmaier, M. d’Aquín, K. Srinivas, P.T. Groth, M. Dumontier, J. Heflin, K. Thirunarayanan, and S. Staab, editors, *Proceedings of the 14th International Semantic Web Conference (ISWC)*, number 9367 in LNCS, pages 409–426. Springer, 2015.

[37] G. Casini and U. Straccia. Rational closure for defeasible description logics. In T. Janhunen and I. Niemelä, editors, *Proceedings of the 12th European Conference on Logics in Artificial Intelligence (JELIA)*, number 6341 in LNCS, pages 77–90. Springer-Verlag, 2010.

[38] G. Casini and U. Straccia. Defeasible inheritance-based description logics. In *Proceedings of the Twenty-Second International Joint Conference on Artificial Intelligence - Volume Two*, IJCAI’11, pages 813–818. AAAI Press, 2011.

[39] G. Casini and U. Straccia. Lexicographic closure for defeasible description logics. In *Proceedings of the Eighth Australasian Ontology Workshop - AOW 2012*, number 969 in CEUR Workshop Proceedings, pages 28–39. CEUR, 2012.

[40] G. Casini and U. Straccia. Defeasible inheritance-based description logics. *Journal of Artificial Intelligence Research (JAIR)*, 48:415–473, 2013.

[41] G. Casini, U. Straccia, and T. Meyer. A polynomial time subsumption algorithm for nominal safe $\mathcal{ELO}_{\perp}$ under rational closure. *Information Sciences*, https://doi.org/10.1016/j.ins.2018.09.037, 2018.

[42] B. Chellas. *Modal logic: An introduction*. Cambridge University Press, 1980.
[43] F.M. Donini, D. Nardi, and R. Rosati. Description logics of minimal knowledge and negation as failure. *ACM Transactions on Computational Logic*, 3(2):177–225, 2002.

[44] D. Dubois, J. Lang, and H. Prade. Possibilistic logic. In D.M. Gabbay, C.J. Hogger, and J.A. Robinson, editors, *Handbook of Logic in Artificial Intelligence and Logic Programming*, volume 3, pages 439–513. Oxford University Press, 1994.

[45] P. Gärdenfors and D. Makinson. Nonmonotonic inference based on expectations. *Artificial Intelligence*, 65(2):197–245, 1994.

[46] L. Giordano, V. Gliozzi, N. Olivetti, and G.L. Pozzato. Preferential description logics. In N. Dershowitz and A. Voronkov, editors, *Logic for Programming, Artificial Intelligence, and Reasoning (LPAR)*, number 4790 in LNAI, pages 257–272. Springer, 2007.

[47] L. Giordano, V. Gliozzi, N. Olivetti, and G.L. Pozzato. Reasoning about typicality in preferential description logics. In S. Hölldobler, C. Lutz, and H. Wansing, editors, *Proceedings of the 11th European Conference on Logics in Artificial Intelligence (JELIA)*, number 5293 in LNAI, pages 192–205. Springer, 2008.

[48] L. Giordano, V. Gliozzi, N. Olivetti, and G.L. Pozzato. Analytic tableaux calculi for KLM logics of nonmonotonic reasoning. *ACM Transactions on Computational Logic*, 10(3):18:1–18:47, 2009.

[49] L. Giordano, V. Gliozzi, N. Olivetti, and G.L. Pozzato. \texttt{ALC}++T: a preferential extension of description logics. *Fundamenta Informaticae*, 96(3):341–372, 2009.

[50] L. Giordano, V. Gliozzi, N. Olivetti, and G.L. Pozzato. A minimal model semantics for nonmonotonic reasoning. In L. Fariñas del Cerro, A. Herzig, and J. Mengin, editors, *Proceedings of the 13th European Conference on Logics in Artificial Intelligence (JELIA)*, number 7519 in LNCS, pages 228–241. Springer, 2012.

[51] L. Giordano, V. Gliozzi, N. Olivetti, and G.L. Pozzato. A non-monotonic description logic for reasoning about typicality. *Artificial Intelligence*, 195:165–202, 2013.

[52] L. Giordano, V. Gliozzi, N. Olivetti, and G.L. Pozzato. Semantic characterization of rational closure: From propositional logic to description logics. *Artificial Intelligence*, 226:1–33, 2015.

[53] G. Governatori. Defeasible description logics. In G. Antoniou and H. Boley, editors, *Rules and Rule Markup Languages for the Semantic Web*, number 3323 in LNCS, pages 98–112. Springer, 2004.
[54] B.N. Grosof, I. Horrocks, R. Volz, and S. Decker. Description logic programs: Combining logic programs with description logic. In *Proceedings of the 12th International Conference on World Wide Web (WWW)*, pages 48–57. ACM, 2003.

[55] S. Heymans and D. Vermeir. A defeasible ontology language. In R. Meersman and Z. Tari, editors, *CoopIS/DOA/ODBASE*, number 2519 in LNCS, pages 1033–1046. Springer, 2002.

[56] S. Kraus, D. Lehmann, and M. Magidor. Nonmonotonic reasoning, preferential models and cumulative logics. *Artificial Intelligence*, 44:167–207, 1990.

[57] D. Lehmann. What does a conditional knowledge base entail? In R. Brachman and H. Levesque, editors, *Proceedings of the 1st International Conference on Principles of Knowledge Representation and Reasoning (KR)*, pages 212–222, 1989.

[58] D. Lehmann. Another perspective on default reasoning. *Annals of Mathematics and Artificial Intelligence*, 15(1):61–82, 1995.

[59] D. Lehmann and M. Magidor. What does a conditional knowledge base entail? *Artificial Intelligence*, 55:1–60, 1992.

[60] T. Lukasiewicz. Expressive probabilistic description logics. *Artificial Intelligence*, 172(6-7):852–883, 2008.

[61] K. Moodley, T. Meyer, and I. Varzinczak. Root justifications for ontology repair. In S. Rudolph and C. Gutierrez, editors, *Proceedings of the 5th International Conference on Web Reasoning and Rule Systems (RR)*, number 6902 in LNCS, pages 275–280. Springer, 2011.

[62] L. Padgham and T. Zhang. A terminological logic with defaults: A definition and an application. In R. Bajcsy, editor, *Proceedings of the 13th International Joint Conference on Artificial Intelligence (IJCAI)*, pages 662–668. Morgan Kaufmann Publishers, 1993.

[63] M. Pensel and A.-Y. Turhan. Including quantification in defeasible reasoning for the description logic $\mathcal{EL}_{bot}$. In M. Balduccini and T. Janhunen, editors, *Proceedings of the 14th International Conference on Logic Programming and Nonmonotonic Reasoning (LPNMR)*, LNCS, pages 78–84. Springer, 2017.

[64] M. Pensel and A.-Y. Turhan. Reasoning in the defeasible description logic $\mathcal{EL}_\perp$ – computing standard inferences under rational and relevant semantics. *International Journal of Approximate Reasoning*, 103:28–70, 2018.
[65] G. Qi, J.Z. Pan, and Q. Ji. Extending description logics with uncertainty reasoning in possibilistic logic. In K. Mellouli, editor, Proceedings of the 9th European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty, number 4724 in LNAI, pages 828–839. Springer, 2007.

[66] J. Quantz and V. Royer. A preference semantics for defaults in terminological logics. In Proceedings of the 3rd International Conference on Principles of Knowledge Representation and Reasoning (KR), pages 294–305, 1992.

[67] J. Quantz and M. Ryan. Preferential default description logics. Technical report, TU Berlin, 1993.

[68] R. Reiter. A logic for default reasoning. Artificial Intelligence, 13(1-2):81–132, 1980.

[69] H. Rott. Change, Choice and Inference: a study of belief revision and nonmonotonic reasoning. Oxford University Press, 2001.

[70] K. Schild. A correspondence theory for terminological logics: Preliminary report. In Proceedings of the 12th International Joint Conference on Artificial Intelligence (IJCAI), pages 466–471, 1991.

[71] S. Schlobach and R. Cornet. Non-standard reasoning services for the debugging of description logic terminologies. In Proceedings of the 18th International Joint Conference on Artificial Intelligence (IJCAI), pages 355–362, 2003.

[72] K. Sengupta, A. Alfa Krisnadhi, and P. Hitzler. Local closed world semantics: Grounded circumscription for OWL. In L. Aroyo, C. Welty, H. Alani, J. Taylor, A. Bernstein, L. Kagal, N. Noy, and E. Blomqvist, editors, Proceedings of the 10th International Semantic Web Conference (ISWC), number 7031 in LNCS, pages 617–632. Springer, 2011.

[73] Y. Shoham. Reasoning about Change: Time and Causation from the Standpoint of Artificial Intelligence. MIT Press, 1988.

[74] U. Straccia. Default inheritance reasoning in hybrid KL-ONE-style logics. In R. Bajcsy, editor, Proceedings of the 13th International Joint Conference on Artificial Intelligence (IJCAI), pages 676–681. Morgan Kaufmann Publishers, 1993.

[75] I.J. Varzinczak. A note on a description logic of concept and role typicality for defeasible reasoning over ontologies. Logica Universalis, 12(3-4):297–325, 2018.
A Proofs of lemmas in Section 3.2

NB: The results marked (*) are introduced here in the Appendix, while they are omitted in the main text.

Lemma 2 For every preferential interpretation $P$, and every $C, D \in \mathcal{L}$, $P \models C \sqsubseteq D$ if and only if $P \models C \sqcap \neg D \sqsubseteq \bot$.

Proof: From left to right, $P \models C \sqsubseteq D$ implies, by Lemma 1, that $(C \sqcap \neg D)_{P} = \emptyset$. The latter implies that, for every concept $E$, $P \models C \sqcap \neg D \sqsubseteq E$, and, as a particular case, $P \models C \sqcap \neg D \sqsubseteq \bot$. From right to left, if $P \noplus C \sqsubseteq D$, then $(C \sqcap \neg D)_{P} \neq \emptyset$. Let $x$ be an object in $\min_{\prec P}(C \sqcap \neg D)_{P}$: for $P \models C \sqcap \neg D \sqsubseteq \bot$, we should have also $x \in \bot_{P}$, which is a contradiction.

Definition 28 (Disjoint-Union Preferential Interpretation) Let $S$ be a countable set and let $P = \{ P_{s} = (\Delta P_{s}, \cdot P_{s}, \prec P_{s}) \mid s \in S \}$ be a collection of preferential interpretations. The disjoint union of $P$ is a tuple $U = \{ (\Delta U, \cdot U, \prec U) \}$ where:

- $\Delta U = \{ (x, s) \mid x \in \Delta P_{s} \text{ and } s \in S \}$;
- $A U = \{ (x, s) \mid x \in A P_{s} \text{ and } s \in S \}$, for every $A \in \mathcal{C}$;
- $r U = \{ ((x, s), (y, s)) \mid (x, y) \in r P_{s} \text{ and } s \in S \}$, for every $r \in \mathcal{R}$;
- $\prec U = \{ ((x, s), (y, s)) \mid (x, y) \in \prec P_{s} \text{ and } s \in S \}$.

Lemma 14 (*) Let $S$ and $P$ be as in Definition 28 and let $U$ be the latter’s disjoint union. For every $C \in \mathcal{L}$, every $s \in S$, and every $x \in \Delta P_{s}$, $x \in C P_{s}$ if and only if $(x, s) \in C U$.

Proof: For every $s \in S$, define $E_{s} = \{ (x, (x, s)) \mid x \in \Delta P_{s} \}$. We can easily show that $E_{s}$ is a preferential bisimulation between $P_{s}$ and $U$. The lemma is then proved by induction on the structure of concepts in the usual way.

It is easy to see that the following result also holds:

Lemma 15 (*) Let $S$ and $P$ be as in Definition 28 and let $U$ be the latter’s disjoint union. For every $C \in \mathcal{L}$, every $s \in S$, and every $x \in \Delta P_{s}$, $x \in \min_{\prec P} C P_{s}$ if and only if $(x, s) \in \min_{\prec U} C U$.

Lemma 3 Preferential interpretations are closed under disjoint union.

Proof: Let $KB$ be a defeasible knowledge base, let $S$ and $P$ be as in Definition 28 and such that $P_{s} \models KB$, for every $P_{s} \in P$, and let $U$ be the disjoint union of the models in $P$. We have

37
to show that $\mathcal{U} \not\models KB$. Assume that $\mathcal{U} \not\models KB$. Then there must be a DCI $C \subseteq D \in KB$ (recall Lemma [2]) and an object $(x,s) \in \Delta^\mathcal{U}$ such that $(x,s) \in \min_{\times^\mathcal{U}} C^D$ but $(x,s) \notin D^\mathcal{U}$. From Lemmas [14] and [15] above, it follows that $x \in \min_{\times^\mathcal{P}} C^P$, and $x \notin D^P$, and therefore $P_s \not\models C \subseteq D$. Hence, $P_s \not\models KB$, which contradicts our assumption.

\section*{B Proof of Theorem 1}

\textbf{Theorem 1} [Representation Result for Preferential Subsumption] A defeasible subsumption relation $\subseteq \subseteq L \times L$ is preferential if and only if there is a preferential interpretation $\mathcal{P}$ such that $\mathcal{P} = \subseteq$.

\subsection*{B.1 If part}

We show that $\subseteq \subsetneq \subseteq$ is preferential for every preferential interpretation $\mathcal{P} = (\Delta^\mathcal{P}, \succ^\mathcal{P}, \prec^\mathcal{P})$.

(Ref): Let $x \in \Delta^\mathcal{P}$ be such that $x \in \min_{\prec^\mathcal{P}} C^P$. Then clearly $x \in C^P$ and therefore $\mathcal{P} \models C \subseteq C$. Hence $\subseteq \subsetneq \succ^\mathcal{P}$.

(LLE): Assume that $C \subseteq \succ^\mathcal{P}E$ and $\mathcal{P} \models C \equiv D$. Then $\mathcal{P} \models C \subseteq E$, which means $\min_{\prec^\mathcal{P}} C^P \subseteq E^P$. Since $\mathcal{P} \models C \equiv D$, i.e., $C^P = D^P$, we have $\min_{\prec^\mathcal{P}} C^P = \min_{\prec^\mathcal{P}} D^P$. Hence $\min_{\prec^\mathcal{P}} D^P \subseteq E^P$, and therefore $\mathcal{P} \models D \subseteq E$, from which follows $D \subseteq \succ^\mathcal{P}E$.

(And): Assume we have both $C \subseteq \succ^\mathcal{P}D$ and $C \subseteq \succ^\mathcal{P}E$. Then $\mathcal{P} \models C \subseteq D$, i.e., $\min_{\prec^\mathcal{P}} C^P \subseteq D^P$ and $\min_{\prec^\mathcal{P}} C^P \subseteq E^P$, and then $\min_{\prec^\mathcal{P}} C^P \subseteq (D \cap E)^P$. Hence $\mathcal{P} \models C \subseteq D \cap E$, and therefore $C \subseteq \succ^\mathcal{P}D \cap E$.

(Or): Assume we have both $C \subseteq \succ^\mathcal{P}E$ and $D \subseteq \succ^\mathcal{P}E$. Let $x \in \min_{\prec^\mathcal{P}} (C \cup D)^P$. Then $x$ is minimal in $C^P \cup D^P$, and therefore either $x \in \min_{\prec^\mathcal{P}} C^P$ or $x \in \min_{\prec^\mathcal{P}} D^P$. In either case $x \in E^P$. Hence $\mathcal{P} \models C \cup D \subseteq E$ and therefore $C \cup D \subseteq \succ^\mathcal{P}E$.

(RW): Assume we have both $C \subseteq \succ^\mathcal{P}D$ and $\mathcal{P} \models D \subseteq E$. Then $\mathcal{P} \models C \subseteq D$, i.e., $\min_{\prec^\mathcal{P}} C^P \subseteq D^P$, and $D^P \subseteq E^P$. Hence $\min_{\prec^\mathcal{P}} C^P \subseteq E^P$ and then $\mathcal{P} \models C \subseteq E$. Therefore $C \subseteq \succ^\mathcal{P}E$.

(CM): Assume we have both $C \subseteq \succ^\mathcal{P}D$ and $C \subseteq \succ^\mathcal{P}E$. Then $\mathcal{P} \models C \subseteq D$ and $\mathcal{P} \models C \subseteq E$, and therefore $\min_{\prec^\mathcal{P}} C^P \subseteq D^P$ and $\min_{\prec^\mathcal{P}} C^P \subseteq E^P$. Let $x \in \min_{\prec^\mathcal{P}} (C \cap D)^P$. We show that $x \in \min_{\prec^\mathcal{P}} C^P$. Suppose this is not the case. Since $\prec^\mathcal{P}$ is smooth, there must be $x' \in \min_{\prec^\mathcal{P}} C^P$ such that $x' \prec^\mathcal{P} x$. Because $\mathcal{P} \models C \subseteq D$, $x' \subseteq D^P$, and then $x' \in C^P \cap D^P$, i.e., $x' \in (C \cap D)^P$. From this and $x \prec^\mathcal{P} x'$ it follows that $x$ is not minimal in $(C \cap D)^P$, which is a contradiction. Hence $x \in \min_{\prec^\mathcal{P}} C^P$. From this and $\min_{\prec^\mathcal{P}} C^P \subseteq E^P$, it follows that $x \in E^P$. Hence $\mathcal{P} \models C \cap D \subseteq E$, and therefore $C \cap D \subseteq \succ^\mathcal{P}E$.

\subsection*{B.2 Only-if part}

\textbf{NB}: The results marked (*) are introduced here in the Appendix, while they are omitted in the main text.
Let \( \sqsubseteq \subseteq \mathcal{L} \times \mathcal{L} \) be a preferential subsumption relation. We shall construct a preferential interpretation \( \mathcal{P} \) such that \( \sqsubseteq \mathcal{P} \overset{\text{def}}{=} \{ (C, D) \mid \mathcal{P} \vdash C \sqsubseteq D \} = \sqsubseteq \).

**Definition 29** Let \( \mathcal{U} = \text{def} \{ (\mathcal{I}, x) \mid \mathcal{I} = \langle \Delta^\mathcal{I}, \cdot \rangle \text{ and } x \in \Delta^\mathcal{I} \} \).

Intuitively, \( \mathcal{U} \) denotes the universe of objects in the context of their respective DL interpretations, i.e., \( \mathcal{U} \) is a set of first-order interpretations.

**Definition 30** A pair \( (\mathcal{I}, x) \in \mathcal{U} \) is normal for \( C \in \mathcal{L} \) if for every \( D \in \mathcal{L} \) such that \( C \sqsubseteq D \), we have \( x \in D^\mathcal{I} \).

**Lemma 16** (* Let \( \mathcal{U} \subseteq \mathcal{L} \times \mathcal{L} \) satisfy (Ref), (RW) and (And), and let \( C, D \in \mathcal{L} \). Then all normal \( (\mathcal{I}, x) \) for \( C \) satisfy \( D \) if and only if \( C \sqsubseteq D \).

**Proof:**
The if part follows from the definition of normality above. For the only-if part, assume \( C \nsubseteq D \). We build a pair \( (\mathcal{I}, x) \) that is normal for \( C \) but that does not satisfy \( D \). Let \( \Gamma = \text{def} \{ \neg D \} \cup \{ E \mid C \sqsubseteq E \} \). All we need to do is show that there is \( (\mathcal{I}, x) \) such that \( x \in F^\mathcal{I} \) for every \( F \in \Gamma \). Suppose this is not the case. Then by compactness there exists a finite \( \Gamma' \subseteq \Gamma \) such that \( \vdash \top \subseteq \neg \bigwedge_{F \in \Gamma'} F \sqcup D \), and, in particular, we have \( \vdash C \subseteq \neg \bigwedge_{F \in \Gamma'} F \sqcup D \). Now from (Ref) we have \( C \subseteq C \). From this, \( \vdash C \subseteq \neg \bigwedge_{F \in \Gamma'} F \sqcup D \) and (RW) we get \( C \subseteq (\neg \bigwedge_{F \in \Gamma'} F \sqcup D) \). But we also have \( C \subseteq \bigwedge_{F \in \Gamma'} F \) by the (And) rule, and then by applying (And) once more we derive \( C \subseteq \bigwedge_{F \in \Gamma'} F \sqcap (\neg \bigwedge_{F \in \Gamma'} F \sqcup D) \). From this and (RW) we conclude \( C \nsubseteq D \), from which we derive a contradiction.

**Lemma 17** (* If \( \mathcal{U} \) is preferential, the following rule holds:

\[
\frac{C \sqcup D \subseteq C, \ D \sqcup E \subseteq D}{C \sqcup E \subseteq C}
\]

**Proof:**
The proof is analogous to that by Kraus et al. [56, Lemma 5.5].

**Definition 31** Let \( C, D \in \mathcal{L} \). \( C \leq D \) if \( C \sqcup D \nsubseteq C \).

**Lemma 18** (* If \( \mathcal{U} \) is preferential, then \( \leq \) is reflexive and transitive.

**Proof:**
From (Ref) we have \( C \nsubseteq C \). This and (LLE) gives us \( C \sqcup C \nsubseteq C \); therefore we have \( C \leq C \) and \( \leq \) is reflexive. Transitivity follows from Lemma [17].

**Lemma 19** (* If \( \mathcal{U} \) is preferential, the following rule holds:

\[
\frac{C \sqcup D \subseteq C, \ D \subseteq E}{C \nsubseteq \neg D \sqcup E}
\]

39
Proof:
The proof is analogous to that by Kraus et al. [56, Lemma 5.5].

Lemma 20 (*) If $C \leq D$ and $(I, x)$ is normal for $C$, and $x \in D^\mathcal{I}$, then $(I, x)$ is normal for $D$.

Proof:
From $C \leq D$ we get $C \sqcup D \sqsubseteq C$. Assume that $D \sqsubseteq \neg D \sqcup E$ is the case. Then by Lemma 19 we have $C \sqsubseteq \neg D \sqcup E$. Since $(I, x)$ is normal for $C$, we have $x \in (\neg D \sqcup E)^\mathcal{I}$. Given that $x \in D^\mathcal{I}$, we must have $x \in E^\mathcal{I}$.

Lemma 21 (*) If $\sqsubseteq$ is preferential, the following rule holds:

\[
\frac{C \sqcup D \sqsubseteq C, \ D \sqcup E \sqsubseteq D}{C \sqsubseteq \neg E \sqcup D}
\]

Proof:
The proof is analogous to that by Kraus et al. [56, Lemma 5.5].

Lemma 22 (*) If $C \leq D \leq E$ and $(I, x)$ is normal for $C$, and $x \in E^\mathcal{I}$, then $(I, x)$ is normal for $D$.

Proof:
By Lemma 20, it is enough to show that $x \in D^\mathcal{I}$. By Lemma 21 we have $C \sqsubseteq \neg E \sqcup D$. Since $(I, x)$ is normal for $C$ and $x \in E^\mathcal{I}$, then we must have $x \in D^\mathcal{I}$.

We now construct a preferential interpretation as in Definition 5.

Let $\mathcal{L}^\perp = \{ C | C \sqsubseteq \bot \}$ and let $\mathfrak{K} = \{ I = (\Delta^\mathcal{I}, \mathcal{I}^\mathcal{K}) | C^\mathcal{K} = \emptyset \text{ for all } C \in \mathcal{L} \}$. Intuitively, $\mathfrak{K}$ contains all interpretations that are ‘compatible’ with $\sqsubseteq$ in the sense of not satisfying concepts that are defeasibly subsumed by the contradiction.

For each $I \in \mathfrak{K}$, let $\mathcal{I}^+ = \{ (I, x, C) | (I, x) \text{ normal for } C \in \mathcal{L} \}$, and $\mathcal{I}^\perp = \{ (I, x, \bot) | (I, x) \text{ not normal for any } C \in \mathcal{L} \}$;

- $\mathcal{I}^+$ is such that for every $D \in \mathcal{L}$, $(I, x, C) \in D^\mathcal{I}^+$ if and only if $x \in D^\mathcal{I}$, and for every $r \in \mathcal{R}$, $(I, x, C), (I, y, D)) \in r^\mathcal{I}^+$ if and only if $(x, y) \in r^\mathcal{I}$.

Let $\mathcal{P} = \{ (\Delta^\mathcal{P}, \mathcal{P}, \prec^\mathcal{P}) | \text{ such that:} \}$

- $\Delta^\mathcal{P} = \bigcup_{I \in \mathfrak{K}} \Delta^\mathcal{I}^+$;
- $\mathcal{P} = \bigcup_{I \in \mathfrak{K}} \mathcal{I}^+$;
- $\prec^\mathcal{P}$ is the smallest relation such that:
For every \( \langle I, x, C \rangle \in \Delta^P \) such that \( C \neq \bot \), \( \langle I, x, C \rangle \prec^P \langle J, y, \bot \rangle \) for every \( \langle J, y, \bot \rangle \in \Delta^P \);

For every \( \langle I, x, C \rangle, \langle J, y, D \rangle \in \Delta^P \) such that \( C, D \neq \bot \), \( \langle I, x, C \rangle \prec^P \langle J, y, D \rangle \) if and only if \( C \leq D \) and \( x \notin D^I \).

(In the construction of \( P \), note that all pairs \( \langle I, x \rangle \) that are not normal for any concept \( C \) are moved higher up in the ordering so that they correspond to the least preferred objects of the domain.)

In Lemmas 23 to 28 below we show that \( P \) as constructed above is indeed a preferential interpretation.

**Lemma 23 (\(*\))** \( \Delta^P \neq \emptyset \).

**Proof:**
From \( \top \not\preceq \bot \) and Lemma 16, it follows that there is some normal \( \langle I, x \rangle \) for \( \top \) that does not satisfy \( \bot \). Hence \( \langle I, x, \top \rangle \in \Delta^P \) and therefore \( \Delta^P \neq \emptyset \). □

**Lemma 24 (\(*\))** \( C \leq \bot \) for every \( C \in L \).

**Proof:**
By (Ref) we have \( C \sqsubseteq C \). Since \( \models C \equiv C \sqcup \bot \), by (LLE) we get \( C \sqcup \bot \preceq \bot \), and then from the definition of \( \leq \) follows \( C \leq \bot \). □

**Lemma 25 (\(*\))** \( \prec^P \) is a strict partial order on \( \Delta^P \), i.e., \( \prec^P \) is irreflexive and transitive.

**Proof:**
First we show irreflexivity. From the construction of \( \prec^P \), it clearly follows that for every \( \langle I, x, \bot \rangle \in \Delta^P \), \( \langle I, x, \bot \rangle \not\prec \langle I, x, \bot \rangle \). Assume that \( \langle I, x, C \rangle \prec^P \langle I, x, C \rangle \) for some \( C \neq \bot \). Then \( C \leq C \) and \( x \notin C^I \), i.e., \( C \sqsubseteq C \). Then \( C \subseteq C \) by (LLE). This and \( x \notin C^I \) contradicts the fact that \( \langle I, x \rangle \) is normal for \( C \). Hence \( \langle I, x, C \rangle \not\prec \langle I, x, C \rangle \) for every \( \langle I, x, C \rangle \in \Delta^I \).

We now show transitivity. Suppose \( \langle I, x, C \rangle \prec^P \langle I', x', D \rangle \) and \( \langle I', x', D \rangle \prec^P \langle I'', x'', E \rangle \). From the definition of \( \prec^P \) we know that \( C, D \neq \bot \), since all non-normal objects are at the highest level in the ordering and are all incomparable. We then have \( C \leq D \) and \( E \leq D \). (If \( E = \bot \), we also have \( D \leq E \) by Lemma 24.) From transitivity of \( \leq \) (Lemma 18), we conclude \( C \leq E \). Since \( \langle I, x, C \rangle \in \Delta^P \) and \( \langle I, x, C \rangle \prec^P \langle I', x', D \rangle \), we conclude that \( \langle I, x \rangle \) is normal for \( C \) and \( x \notin D^P \). This and Lemma 22 imply that \( x \notin E^P \). □

**Lemma 26 (\(*\))** Given \( \langle I, x, D \rangle \in \Delta^P \), \( \langle I, x, D \rangle \in \min_{\prec^P} C^P \) iff \( x \in C^I \) and \( D \leq C \).

**Proof:**
For the if part, suppose that \( x \in C^I \) and \( D \leq C \). Then it clearly follows that \( \langle I, x, D \rangle \in C^P \) (Lemma 20). Now suppose that \( \langle I, x, D \rangle \) is not \( \prec^P \)-minimal in \( C^P \), i.e., there is \( \langle I', x', E \rangle \)
for some $\mathcal{I}'$ such that $x' \in \Delta^{\mathcal{I}'}$ and some $E \in \mathcal{L}$ such that $(\mathcal{I}', x', E) \prec_{\mathcal{P}} (\mathcal{I}, x, D)$ and $x' \in C^{\mathcal{I}'}$. From this and the definition of $\prec_{\mathcal{P}}$, it follows that $E \leq D$ and $x' \notin D^{\mathcal{I}'}$. Hence $E \leq D \leq C$ and $(\mathcal{I}', x')$ is normal for $E$, and since $x' \in C^{\mathcal{I}'}$, by Lemma 22 we get that $(\mathcal{I}', x')$ is normal for $D$, from which we conclude $x' \in D^{\mathcal{I}'}$, a contradiction.

For the only-if part, suppose that $(\mathcal{I}, x, D)$ is $\prec_{\mathcal{P}}$-minimal in $C^{\mathcal{P}}$. Then clearly $x \in C^{\mathcal{I}}$. Now assume there is some $(\mathcal{I}', x')$ which is normal for $C \sqcup D$ and $x' \notin D^{\mathcal{I}'}$. Since $C \sqcup D \leq D$, we must have $(\mathcal{I}', x', C \sqcup D) \prec_{\mathcal{P}} (\mathcal{I}, x, D)$. Since $(\mathcal{I}', x')$ is normal for $C \sqcup D$ and $x' \notin D^{\mathcal{I}'}$, it follows that $x' \in C^{\mathcal{I}'}$. This contradicts the minimality of $(\mathcal{I}, x, D)$ in $C^{\mathcal{P}}$. Hence all normal $(\mathcal{I}', x')$ for $C \sqcup D$ satisfy $D$. From this and Lemma 16 follows $C \sqcup D \subseteq D$, i.e., $D \leq C$.

Lemma 27 (*) There is no $C \in \mathcal{L}$ such that $C^{\mathcal{P}} \neq \emptyset$ and $\perp \leq C$.

Proof:
Let $C \in \mathcal{L}$ be such that $C^{\mathcal{P}} \neq \emptyset$. Assume that $\perp \leq C$. Then $\perp \cup C \not\subseteq \perp$, i.e., $C \not\subseteq \perp$. Then $C \in \mathcal{C}_\perp$, and then $C^{\mathcal{P}} = \emptyset$ by the construction of $\mathcal{P}$.

Corollary 5 (*) It follows from the two last lemmas that there is no $C \in \mathcal{L}$ for which any $\langle \mathcal{I}, x, \perp \rangle \in \Delta^{\mathcal{P}}$ is minimal.

Lemma 28 (*) For any $C \in \mathcal{L}$, $C^{\mathcal{P}}$ is smooth.

Proof:
Suppose that $\langle \mathcal{I}, x, D \rangle \in C^{\mathcal{P}}$, i.e., $x \in C^{\mathcal{I}}$. If $D \leq C$, then by Lemma 26 $\langle \mathcal{I}, x, D \rangle$ is $\prec_{\mathcal{P}}$-minimal in $C^{\mathcal{P}}$. On the other hand, i.e., if $D \not\subseteq C$, $C \sqcup D \not\subseteq D$, then by Lemma 16 there is a normal $(\mathcal{I}', x')$ for $C \sqcup D$ such that $x \notin D^{\mathcal{I}'}$. But $C \sqcup D \subseteq C \sqcup D$, and then $(C \sqcup D) \sqcup D \not\subseteq C \sqcup D$, and then $C \sqcup D \leq D$. Hence $\langle \mathcal{I}', x', C \sqcup D \rangle \prec_{\mathcal{P}} \langle \mathcal{I}, x, D \rangle$. But $x' \in (C \sqcup D)^{\mathcal{I}'}$ and $x' \notin D^{\mathcal{I}'}$, therefore $x' \in C^{\mathcal{I}'}$. Since $C \sqcup D \leq C$, from Lemma 26 we conclude that $(\mathcal{I}', x', C \sqcup D)$ is $\prec_{\mathcal{P}}$-minimal in $C^{\mathcal{P}}$.

Next we show in Lemma 29 that the abstract relation $\sqsubseteq$ we started off with coincides with the relation $\sqsubseteq_{\mathcal{P}}$ obtained from our constructed preferential interpretation $\mathcal{P}$.

Lemma 29 (*) $C \sqsubseteq D$ if and only if $C \sqsubseteq_{\mathcal{P}} D$.

Proof:
For the only-if part, we show that $\min_{\prec_{\mathcal{P}}} C^{\mathcal{P}} \subseteq D^{\mathcal{P}}$. Let $\langle \mathcal{I}, x, E \rangle$ be $\prec_{\mathcal{P}}$-minimal in $C^{\mathcal{P}}$. Then $\langle \mathcal{I}, x \rangle$ is normal for $E$ and $x \in C^{\mathcal{P}}$, and from Lemma 26 we also have $E \leq C$. From these results and Lemma 20 it follows that $\langle \mathcal{I}, x \rangle$ is normal for $C$. Since $C \sqsubseteq D$, we have $x \in D^{\mathcal{I}}$, and therefore $\langle \mathcal{I}, x, E \rangle \in \Delta^{\mathcal{P}}$.

For the if part, let $C \sqsubseteq_{\mathcal{P}} D$. From the definition of $\prec_{\mathcal{P}}$, it follows that for every $\langle \mathcal{I}, x \rangle$ normal for $C$, $\langle \mathcal{I}, x, C \rangle \in \min_{\prec_{\mathcal{P}}} C^{\mathcal{P}}$. Since $C \sqsubseteq_{\mathcal{P}} D$, then $y \in D^{\mathcal{I}}$ for every $\langle \mathcal{I}', y \rangle$ that is normal for $C$. This and Lemma 10 give us $C \sqsubseteq D$. ■
**Proof of Theorem 1**

Soundness, the if part, is given in Section B.1. For the only-if part, let $\sqsubseteq$ be a preferential subsumption relation and let $\mathcal{P}$ be defined as above. Lemmas 23, 25 and 28 show that $\mathcal{P}$ is a preferential DL interpretation. Lemma 29 shows that $\mathcal{P}$ defines a subsumption relation that is exactly $\sqsubseteq$.

\[
\begin{array}{c}
\end{array}
\]

**C Proof of Theorem 2**

**Theorem 2** [Representation Result for Rational Subsumption] A defeasible subsumption relation $\sqsubseteq \subseteq \mathcal{L} \times \mathcal{L}$ is rational if and only if there is a modular interpretation $\mathcal{R}$ such that $\sqsubseteq \mathcal{R} = \sqsubseteq$.

**C.1 If part**

Satisfaction of the basic KLM properties for preferential subsumption follows from the proof in Section B.1, given the fact that modular interpretations are a special case of preferential interpretations. Below we show that rational monotonicity is satisfied.

Assume that $C \sqsubseteq_R E$ but $C \not\sqsubseteq_R \neg D$. From the latter it follows that there is $x \in \min_{\sqsubseteq_R} C^R$ such that $x \in D^R$, i.e., $x \in (C \cap D)^R$. Let now $x' \in \min_{\sqsubseteq_R} (C \cap D)^R$. Since $x \in (C \cap D)^R$, $x \not\preceq_R x'$. This means that $x' \in \min_{\sqsubseteq_R} C^R$, for if there is $x'' \in C^R$ such that $x'' \preceq_R x'$, then $x'' \preceq_R x$, which is impossible since $x$ is minimal in $C^R$. From $x' \in \min_{\sqsubseteq_R} C^R$ and $\mathcal{R} \models C \sqsubseteq E$ follows $x' \in E^R$. Hence $\mathcal{R} \models C \cap D \sqsubseteq E$ and therefore $C \cap D \sqsubseteq_R E$.

**C.2 Only-if part**

**NB:** The results marked (*) are introduced here in the Appendix, while they are omitted in the main text.

The proof of the only-if part relies on the results for the preferential case (Section B.1), with the main difference being the definition of the preference relation, which is shown to be a smooth modular order. This ensures that the canonical model constructed in the proof is a modular interpretation.

Let $\sqsubseteq \subseteq \mathcal{L} \times \mathcal{L}$ satisfy all the basic properties of preferential subsumption relations together with rational monotonicity.

The proof of the following lemma is analogous to that of Lemma 3.11 by Lehmann and Magidor [59]:

**Lemma 30** (*) *If $\sqsubseteq$ is rational, then the properties below hold:*

\[
\begin{array}{c}
C \cup D \sqsubseteq \neg D & \quad C \sqsubseteq \neg D & \quad C \sqsubseteq \neg D \quad C \sqsubseteq \neg D \quad C \sqsubseteq \neg D
\end{array}
\]
Definition 32 Let \( C \in \mathcal{L} \). We say that \( C \) is consistent w.r.t. \( \sqsubseteq \) if \( C \not\sqsubseteq \bot \). Given \( \mathcal{R} = \langle \Delta^\mathcal{R}, \cdot^\mathcal{R}, \prec^\mathcal{R} \rangle \), we say that \( C \) is consistent w.r.t. \( \sqsubseteq^\mathcal{R} \) if \( C \not\sqsubseteq^\mathcal{R} \bot \), i.e., if there is \( x \in \Delta^\mathcal{R} \) s.t. \( x \in C^\mathcal{R} \).

Let \( C = \{ C \in \mathcal{L} \mid \text{C is consistent w.r.t. } \sqsubseteq \} \).

Lemma 31 (*) Let \( C \in \mathcal{L} \) and let \( \sqsubseteq \) be a rational relation. Then \( C \in C \) iff there is \( (I, x) \in \mathcal{U} \) s.t. \( (I, x) \) is normal for \( C \). (Cf. Definitions 29 and 30 in Appendix B.2.)

Definition 33 Given \( C, D \in C \), \( C \) is not more exceptional than \( D \), written \( C \sqtriangleleft D \), if \( C \sqcup D \sqsubseteq \neg C \). We say that \( C \) is as exceptional as \( D \), written \( C \sim D \), if \( C \sqtriangleleft D \) and \( D \sqtriangleleft C \).

The proof of the lemma below follows those of Lemmas A.4 and A.5 by Lehmann and Magidor [59]:

Lemma 32 (*) \( \mathcal{R} \) is reflexive and transitive.

That \( \sim \) is an equivalence relation follows from the fact that \( \mathcal{R} \) is reflexive and transitive (Lemma 32). With \([C]\) we denote the equivalence class of \( C \). The set of equivalence classes of concepts of \( C \) under \( \sim \) is denoted by \([C]\). We write \([C] \leq [D] \) if \( C \sqtriangleleft D \), and \([C] < [D] \) if \([C] \leq [D] \) and \( C \not\sim D \).

Thanks to Lemma 32 we can state:

Lemma 33 (*) The relation \( < \) is a strict order on \([C]\).

Lemma 34 (*) Let \( C, D \in \mathcal{L} \) be consistent w.r.t. \( \sqsubseteq \). If \([C] < [D] \), then \( C \sqsubseteq \neg D \).

Proof: The assumption implies that \( C \sqtriangleleft D \) is not the case, i.e., \( C \sqcup D \sqsubseteq \neg C \). This and Lemma 30 imply the conclusion.

Lemma 34 warrants the following result:

Lemma 35 (*) Let \( C, D \in \mathcal{L} \) be consistent w.r.t. \( \sqsubseteq \). If there is \( (I, x) \in \mathcal{U} \) s.t. \( (I, x) \) is normal for \( C \) and \( x \in D^\mathcal{R} \), then \([D] \leq [C] \).

Armed with these results, we can then construct an interpretation \( \mathcal{R} \) analogous to the preferential interpretation \( \mathcal{P} \) in Appendix B.2, with the preference relation defined as follows:

- For every \( (I, x, C) \in \Delta^\mathcal{R} \) such that \( C \neq \bot \), \( (I, x, C) \prec^\mathcal{R} (J, y, \bot) \) for every \( (J, y, \bot) \in \Delta^\mathcal{R} \);

- For every \( (I, x, C), (J, y, D) \in \Delta^\mathcal{R} \) such that \( C, D \neq \bot \), \( (I, x, C) \prec^\mathcal{R} (J, y, D) \) if \([C] < [D] \).
It is not hard to see that this definition implies the following result:

**Lemma 36** (*) \( \preceq^R \) is a modular partial order.

The proof of the following lemma follows that of Lehmann and Magidor’s Lemma A.12 [59]:

**Lemma 37** (*) For every \( C \in \mathcal{L} \), if \( C \) is consistent, then \( C^R \) is smooth.

From this point on, a result analogous to Lemma 29 in [3.2] can be shown to hold for the defeasible subsumption \( \precsim_R \) induced by \( R \). From that the result follows. \( \blacksquare \)

## D Proofs of results in Section 4

**Lemma 4** Let \( KB \) be a defeasible knowledge base. Then
\[
\precsim_{KB^*_{\text{pref}}} = \bigcap \{ \precsim_{KB_{\text{inf}}} \mid KB \subseteq KB_{\text{inf}} \text{ and } KB_{\text{inf}} \text{ is preferential} \}.
\]

**Proof:**
By Definitions 10 and 11, \( \alpha \in KB^*_{\text{pref}} \) iff for every preferential model \( P \) of \( KB \), \( P \models \alpha \). Combined with Lemma 2, this implies that, for any defeasible subsumption \( C \subset D \), \( C \precsim D \in KB^*_{\text{pref}} \) iff \( (C, D) \in \precsim_P \) for every preferential model \( P \) of \( KB \). Due to Theorem 1, the latter condition, that is, \( (C, D) \in \precsim_P \) for every preferential model \( P \) of \( KB \), holds iff \( (C, D) \in \precsim_{KB_{\text{inf}}} \) for every preferential theory \( KB_{\text{inf}} \) containing \( KB \). This concludes the proof. \( \blacksquare \)

**Lemma 5** Let \( KB \) be a defeasible knowledge base. Then
\[
\precsim_{KB^*_{\text{mod}}} = \bigcap \{ \precsim_{KB_{\text{inf}}} \mid KB \subseteq KB_{\text{inf}} \text{ and } KB_{\text{inf}} \text{ is rational} \}.
\]

**Proof:**
The proof follows the one of Lemma 4. It is sufficient to refer to Definitions 14 and 15 instead of Definitions 10 and 11, and to Theorem 2 instead of Theorem 1. \( \blacksquare \)

**Lemma 7** A modular interpretation \( R = \langle \Delta^R, \preceq_R, \prec_R \rangle \) s.t. \( \Delta^R \) is finite is a ranked interpretation.

**Proof:**
The preference relation \( \prec_R \) is a strict partial order, hence, since there cannot be cycles, for every finite set \( \emptyset \neq X \subseteq \Delta^R \), \( \min_R X \neq \emptyset \). We can define the function \( h_R(\cdot) \) in the following way:

1. \( \Delta^R_0 = _{\text{def}} \Delta^R \);
2. \( i = _{\text{def}} 0 \);

45
3. If \( \Delta^{R^i} \neq \emptyset \) proceed, else return the function \( h_R \);

4. \( h_R(x) = i \) iff \( x \in \min_{\succeq_R} \Delta^{R^i} \); let \( \Delta^{R^{i+1}} = \text{def} \, \Delta^{R^i} \setminus \min_{\succeq_R} \Delta^{R^i} \);

5. \( i = \text{def} \, i + 1 \);

6. Go back to step 3.

It is easy to check that \( h_R(\cdot) \) satisfies the convexity property and characterises \( \prec_R \) (i.e., \( x \prec_R y \) iff \( h_R(x) < h_R(y) \)).

**Proposition 1** Given a ranked interpretation \( \mathcal{R} = (\Delta^R, \cdot^R, \prec^R) \), there is only one function \( h_R : X \longrightarrow \mathbb{N} \) satisfying the convexity property and s.t. for every \( x, y \in X \), \( x \prec y \) iff \( h_R(x) < h_R(y) \).

**Proof:**
Assume that for a ranked interpretation \( \mathcal{R} = (\Delta^R, \cdot^R, \prec^R) \) there are two distinct functions \( h_R(\cdot) \) and \( h'_R(\cdot) \) satisfying the convexity constraint and characterising \( \prec^R \). Since the two functions are distinct, at a certain point they must diverge; that is, there must be an \( i \in \mathbb{N} \) s.t. for every \( k < i \) and every \( x \in \Delta^R \), \( h_R(x) = k \) iff \( h'_R(x) = k \), but there is a \( y \in \Delta^R \) s.t. \( h_R(y) = i \) and \( h'_R(y) = j \), with \( j > i \). The convexity constraint imposes that there must be a \( z \in \Delta^R \) s.t. \( h'_R(z) = i \); then \( h'_R(\cdot) \) enforces \( z \prec_R y \), while according to \( h_R(\cdot) \) that cannot be the case (it must be \( h_R(y) \leq h_R(z) \)).

Some extra material needs to be introduced to prove Theorem 3 stating the Finite Model Property for Defeasible \( \mathcal{ALC} \). First of all, we will refer to the following semantic construction.

**Definition 34 (Finite Model Construction) (\( \ast \))** Let \( \mathcal{KB} = T \cup D \) be a finite defeasible knowledge base, and let \( \mathcal{R} = (\Delta^R, \cdot^R, \prec^R) \) be a modular model of \( \mathcal{KB} \) (with \( \Delta^R \) possibly infinite). Let \( C, R \) be the sets of names of our language, as from Section 2, and \( \Gamma \) be the set of concepts \( \{C_1, \ldots, C_n\} \subseteq L \) obtained by closing the set of all concepts appearing in the axioms in \( \mathcal{KB} \) under sub-concepts and negation. We define the equivalence relation \( \approx^\Gamma \) as follows: for every \( x, y \in \Delta^R \), \( x \approx^\Gamma y \) if for every \( C \in \Gamma \), \( x \in C^R \) iff \( y \in C^R \).

We indicate with \( [x]_\Gamma \) the equivalence class of the objects that are related to an object \( x \) through \( \approx^\Gamma \):

\[
[x]_\Gamma = \text{def} \{ y \in \Delta^R \mid x \approx^\Gamma y \}
\]

We introduce a new model \( \mathcal{R}' = (\Delta^{R'}, \cdot^{R'}, \prec^{R'}) \), defined as:

- \( \Delta^{R'} = \{ [x]_\Gamma \mid x \in \Delta^R \} \);
- For every \( A \in C \cap \Gamma \), \( A^{R'} = \{ [x]_\Gamma \mid x \in A^R \} \);
- For every \( A \notin C \cap \Gamma \), \( A^{R'} = \emptyset \);
• For every \( r \in R \), \( r^{R'} = \{ ([x]_R, [y]_R) \mid (x, y) \in r^{R} \} \);

• For every \([x]_R, [y]_R \in \Delta^{R'}, [x]_R \prec^{R'} [y]_R \) if there is an object \( z \in [x]_R \) s.t. for all the objects \( v \in [y]_R \), \( z \prec^{R} v \);

Let \( \sim^{R'} \) be the indifference relation, defined as usual:

• \([x]_R \sim^{R'} [y]_R \) if \([x]_R \not\prec^{R'} [y]_R \) and \([y]_R \not\prec^{R'} [x]_R \).

Given that \( \Gamma \) is finite, \( \Delta^{R'} \) is clearly finite. The following results are easy to prove.

**Lemma 38 (**) For every \( C \in \Gamma \) and every \( x \in \Delta^{R} \), \( x \in C^{R} \iff [x]_R \in C^{R'} \).

**Proof:**
The proof is analogous to that for the classical case and is by induction on the structure of concepts.

**Lemma 39 (**) Let \( \prec^{R'} \) and \( \prec^{R} \) be as in Definition 34. Then \( \prec^{R'} \) is a strict partial order.

**Proof:**
We show that \( \prec^{R'} \) is irreflexive and transitive.

Irreflexivity: Assume \([x]_R \prec^{R'} [x]_R \). By the definition of \( \prec^{R'} \), it implies that there is a \( z \in [x]_R \) s.t. \( z \prec^{R} v \) for every \( v \in [x]_R \). That is, we have that \( z \prec^{R} z \) that, since \( \prec^{R} \) is a strict partial order (Definitions 5 and 9), cannot be the case.

Transitivity: Assume \([x]_R \prec^{R'} [y]_R \) and \([y]_R \prec^{R'} [u]_R \). This means that there is a \( z \in [x]_R \) s.t. \( z \prec^{R} v \) for every \( v \in [y]_R \), and there is a \( v' \in [y]_R \) s.t. \( v' \prec^{R} w \) for every \( w \in [u]_R \). Since \( \prec^{R} \) is transitive, it follows that there is a \( z \in [x]_R \) s.t. \( z \prec^{R} w \) for every \( w \in [u]_R \), that is, \([x]_R \prec^{R'} [u]_R \).

**Lemma 40 (**) Let \( \sim^{R'} \) be as in Definition 34. Then relation \( \sim^{R'} \) is transitive.

**Proof:**
Let \([x]_R \sim^{R'} [y]_R \), \([y]_R \sim^{R'} [u]_R \), but \([x]_R \not\prec^{R'} [u]_R \). The latter implies that either \([x]_R \prec^{R'} [u]_R \) or \([u]_R \prec^{R'} [x]_R \); w.l.o.g. let’s assume \([x]_R \prec^{R'} [u]_R \). That is, there is a \( z \in [x]_R \) s.t. \( z \prec^{R} w \) for every \( w \in [u]_R \).

\([x]_R \prec^{R'} [y]_R \) implies that \( z \prec^{R} v \) for some \( v \in [y]_R \). Assume the latter does not hold, then for every \( v \in [y]_R \) either \( z \prec^{R} v \) or \( v \prec^{R} z \). It cannot be that \( z \prec^{R} v \) for every \( v \in [y]_R \), since that would imply \([x]_R \prec^{R'} [y]_R \), so there must be some \( v \in [y]_R \) s.t. \( v \prec^{R} z \). However the latter would also imply, due to the transitivity of \( \prec^{R} \), that there is a \( v \in [y]_R \) s.t. \( v \prec^{R} w \) for every \( w \in [u]_R \), that is, \([y]_R \prec^{R'} [u]_R \), against the hypothesis that \([y]_R \sim^{R'} [u]_R \). Consequently, \( z \prec^{R} v \) for some \( v \in [y]_R \).

So there is a \( z \in [x]_R \) s.t. \( z \prec^{R} w \) for every \( w \in [u]_R \) and there is a \( v \in [y]_R \) s.t. \( v \sim^{R} z \). That implies that \( v \prec^{R} w \) for every \( w \in [u]_R \). To see it, assume that it not the case, that
is, we have that for some \( w' \in [u]|_\Gamma \) either \( w' \prec^R v \) or \( w' \sim^R v \): in the former case we would obtain \( z \prec^R v \), in the latter \( z \sim^R w' \), both taking us to contradiction. Hence \( v \prec^R w \) for every \( w \in [u]|_\Gamma \), that is, \( [y]|_\Gamma \prec'^R [u]|_\Gamma \), against the hypothesis.

**Lemma 41 (\*)** Let \( \mathcal{KB} = \mathcal{T} \cup \mathcal{D} \) be finite. If \( \mathcal{KB} \) has a modular model, then it has a finite ranked model.

**Proof:**
Let \( \mathcal{KB} = \mathcal{T} \cup \mathcal{D} \) be a finite defeasible knowledge base, \( \mathcal{R} \) a model of \( \mathcal{KB} \) and \( \mathcal{R}' \) a finite interpretation constructed from \( \mathcal{R} \) as in Definition 34. \( \mathcal{R}' \) is a finite interpretation, and it is modular, since Lemmas 39 and 40 prove that \( \prec^R \) satisfies Definition 8. Being \( \mathcal{R}' \) a finite modular interpretation, it is a finite ranked interpretation (Lemma 7).

It remains to prove that \( \mathcal{R}' \) is a model of \( \mathcal{KB} \). The proof that \( \mathcal{R}' \) satisfies \( \mathcal{R} \) is straightforward by Lemma 38. With regard to \( \mathcal{D} \), let \( C \not\subseteq D \in \mathcal{D} \). Since \( \mathcal{R} \) is a model of \( \mathcal{D} \), either \( C^\mathcal{R} = \emptyset \), or the height of \( C \cap D \) in \( \mathcal{R} \) is lower than the height of \( C \cap \neg D \), that is, there is at least an object \( y \) in \( (C \cap D)^\mathcal{R} \) s.t. for every object \( x \) in \( (C \cap \neg D)^\mathcal{R} \), \( y \prec^R x \). Since \( C \), \( D \) and \( \neg D \) are in \( \Gamma \), the object \( [y]|_\Gamma \in \Delta^\mathcal{R}' \) (obtained from \( y \in (C \cap D)^\mathcal{R} \)) must be preferred to all the objects in \( (C \cap \neg D)^\mathcal{R} \). That implies that in \( \mathcal{R}' \), \( [y]|_\Gamma \not\prec^\mathcal{R}' [x]|_\Gamma \), that is, \( [x]|_\Gamma \sim^\mathcal{R}' [y]|_\Gamma \) or \( [x]|_\Gamma \prec^\mathcal{R}' [y]|_\Gamma \) for every \( y \) s.t. \( y \in (C \cap D)^\mathcal{R} \), and consequently \( \mathcal{R}' \not\models C \not\subseteq \mathcal{D} \).

**Lemma 42 (\*)** Let \( \mathcal{KB} = \mathcal{T} \cup \mathcal{D} \) be finite and \( C, D \in \mathcal{L} \). If \( \mathcal{KB} \) has a modular model that is a counter-model to \( C \not\subseteq D \), then it has a finite ranked model that is a counter-model to \( C \not\subseteq D \).

**Proof:**
It is sufficient to apply the same construction defined for the finite-model property above. We just need to add \( C \) and \( D \) to the set \( \Gamma \) (and close \( \Gamma \) also under the subconcepts of \( C \) and \( D \) and their negations). If \( \mathcal{R} \models C \not\subseteq D \), then there is an object \( x \) s.t. \( x \in (C \cap \neg D)^\mathcal{R} \) and \( x \prec^\mathcal{R} y \) or \( x \sim^\mathcal{R} y \) for every object \( y \) s.t. \( y \in (C \cap D)^\mathcal{R} \). That implies that in \( \mathcal{R}' \), \( [y]|_\Gamma \not\prec^\mathcal{R}' [x]|_\Gamma \), that is, \( [x]|_\Gamma \sim^\mathcal{R}' [y]|_\Gamma \) or \( [x]|_\Gamma \prec^\mathcal{R}' [y]|_\Gamma \) for every \( y \) s.t. \( y \in (C \cap D)^\mathcal{R} \), and consequently \( \mathcal{R}' \not\models C \not\subseteq D \).

**Corollary 6 (\*)** Let \( \mathcal{KB} = \mathcal{T} \cup \mathcal{D} \) be a finite defeasible knowledge base. If \( \mathcal{KB} \) has a modular model \( \mathcal{R} \), then for every \( C \in \mathcal{L} \) s.t. \( h_{\mathcal{R}}(C) = 0 \) there is also a finite ranked model \( \mathcal{R}' \) of \( \mathcal{KB} \) s.t. \( h_{\mathcal{R}'}(C) = 0 \).

**Proof:**
Given \( \mathcal{KB} = \mathcal{T} \cup \mathcal{D} \), a model \( \mathcal{R} \) of \( \mathcal{KB} \) and a concept \( C \) s.t. \( h_{\mathcal{R}}(C) = 0 \), a finite model \( \mathcal{R}' \) satisfying the constraint above can be defined in the same way as the model \( \mathcal{R}' \) from Definition 34. We just need to add \( C \) to the set \( \Gamma \) (and close \( \Gamma \) also under the subconcepts of \( C \) and their negations). To see that \( \mathcal{R}' \) is a model of \( \mathcal{KB} \), just go again through the proof of the finite-model property above, and check that the addition of \( C \) to \( \Gamma \) does not affect any of the above results.

48
Now, \( h_R(C) = 0 \) implies that there is an object \( x \in \Delta^R \) s.t. \( x \in C^R \) and \( h_R(x) = 0 \). Consider now \([x]_\Gamma\). By Lemma 38, \([x]_\Gamma \in C^R\). Since \( h_R(x) = 0 \), for every \([y]_\Gamma \in \Delta^R\) it cannot be the case that there is an object \( z \in [y]_\Gamma\) s.t. \( z \prec^R v \) for every \( v \in [x]_\Gamma\); hence, the definition of \( \prec^R \) implies that for every \([y]_\Gamma \in \Delta^R\), \([y]_\Gamma \notin^R [x]_\Gamma\), that is, \( h_{R'}([x]_\Gamma) = 0 \), that implies \( h_{R'}(C) = 0 \).

Now we can prove Theorem 3.

**Theorem 3 (Finite-Model Property)** Defeasible \( \mathcal{ALC} \) has the finite-model property. In particular, every defeasible \( \mathcal{ALC} \) knowledge base that has a modular model, has also a finite ranked model.

**Proof:**
The result follows straightforwardly from Lemmas 41 and 42.

**Lemma 8** Given a set of ranked models of a defeasible knowledge base \( \mathcal{KB} \), their ranked union is itself a ranked model of \( \mathcal{KB} \).

**Proof:**
Let \( \mathfrak{R} \) be a set of ranked models of a defeasible knowledge base \( \mathcal{KB} \), and let \( R^{\mathfrak{R}} =_{\text{def}} (\Delta^{\mathfrak{R}}, \mathfrak{R}, \prec^{\mathfrak{R}}) \) be its ranked union. We want to prove that also \( R^{\mathfrak{R}} \) is a ranked model of \( \mathcal{KB} \), and to do that is sufficient to prove that for every DCI \( C \sqsubseteq D \), if \( R \vDash C \sqsubseteq D \) for every \( R \in \mathfrak{R} \), then \( R^{\mathfrak{R}} \vDash C \sqsubseteq D \).

It is easy to prove by induction on the construction of the concepts that for every object \( x_R \in \Delta^{\mathfrak{R}} \) and every concept \( C \), \( x_R \in C^{\mathfrak{R}} \) iff \( x \in C^R \).

This, together with the condition that, for every \( x_R \in \Delta^{\mathfrak{R}} \), \( h_R(x_R) = h_R(x) \), implies that for every concept \( C \), \( h_{R^{\mathfrak{R}}}(C) = \min \{ h_R(C) \mid R \in \mathfrak{R} \} \), implies that for every concept \( C \), \( h_{R^{\mathfrak{R}}}(C) = \min \{ h_R(C) \mid R \in \mathfrak{R} \} \).

Now, let \( C \sqsubseteq D \) be satisfied by every \( R \in \mathfrak{R} \). Hence, for every \( R \in \mathfrak{R} \), either \( h_R(C \sqcap D) < h_R(C \sqcap \neg D) \) or \( h_R(C) = \infty \). Since \( h_{R^{\mathfrak{R}}}(C) = \min \{ h_R(C) \mid R \in \mathfrak{R} \} \), \( h_{R^{\mathfrak{R}}}(C \sqcap D) = \min \{ h_R(C \sqcap D) \mid R \in \mathfrak{R} \} \), and \( h_{R^{\mathfrak{R}}}(C \sqcap \neg D) = \min \{ h_R(C \sqcap \neg D) \mid R \in \mathfrak{R} \} \), it is easy to check that \( R^{\mathfrak{R}} \) satisfies \( C \sqsubseteq D \) too: assume that is not the case, that is, \( h_{R^{\mathfrak{R}}}(C \sqcap \neg D) \leq h_{R^{\mathfrak{R}}}(C \sqcap D) \) and \( h_{R^{\mathfrak{R}}}(C) < \infty \); then we have that \( \min \{ h_R(C \sqcap D) \mid R \in \mathfrak{R} \} \leq \min \{ h_R(C \sqcap \neg D) \mid R \in \mathfrak{R} \} \) and \( \min \{ h_R(C) \mid R \in \mathfrak{R} \} \) is not the case, that is, since for every \( R \in \mathfrak{R} \), either \( h_R(C \sqcap \neg D) < h_R(C \sqcap D) \) or \( h_R(C) = \infty \), cannot be the case.

**Lemma 9** For every \( \mathcal{KB} \) and every \( C, D \in L \), \( \mathcal{KB} \models_{\text{mod}} C \sqsubseteq D \) iff \( R \vDash C \sqsubseteq D \), for every \( R \in \text{Mod}_\Delta(\mathcal{KB}) \).

**Proof:**
Let \( \Delta \) be a countably infinite domain. For the only-if part, if \( \mathcal{KB} \models_{\text{mod}} C \sqsubseteq D \), then obviously \( R \vDash C \sqsubseteq D \) for every \( R \in \text{Mod}_\Delta(\mathcal{KB}) \). For the if part, assume \( \mathcal{KB} \models_{\text{mod}} C \sqsubseteq D \).

Then, thanks to the finite-model property (Theorem 3), there is a modular model \( R_{\text{fin}} \) with a finite domain that is a model of \( \mathcal{KB} \) and a counter-model of \( C \sqsubseteq D \); since the domain is
finite, the modular model $\mathcal{R}_{\text{fin}}$ is a ranked model (Lemma 7). Given $\mathcal{R}_{\text{fin}}$, we can extend it to a model of $\mathcal{KB}$ that is a counter-model of $C \subseteq D$ with a countably infinite domain in the following way: make a countably infinite number of copies of $\mathcal{R}_{\text{fin}}$ and make the ranked union of them. Now, let $\mathcal{R}' = (\Delta^{\mathcal{R}'}, \cdot^{\mathcal{R}'}, \prec^{\mathcal{R}'})$ be the result of such ranked union, that is, a ranked model of $\mathcal{KB}$ and a counter-model of $C \subseteq D$ with $\Delta^{\mathcal{R}'}$ being countably infinite (it is the disjoint union of a countably infinite number of finite domains). It is easy to build an isomorphic interpretation $\mathcal{R} = (\Delta, \cdot, \prec)$, once we have defined a bijection $b: \Delta^{\mathcal{R}'} \rightarrow \Delta$, which must exist, being both $\Delta^{\mathcal{R}'}$ and $\Delta$ countably infinite sets. We can define $\cdot$ and $\prec$ in the following way:

- For every $A \in C$ and every $x \in \Delta^{\mathcal{R}'}$, $b(x) \in A^{\mathcal{R}'}$ iff $x \in A^{\mathcal{R}'}$;
- For every $r \in \mathcal{R}$ and every $x, y \in \Delta^{\mathcal{R}'}$, $(b(x), b(y)) \in r^{\mathcal{R}}$ iff $(x, y) \in r^{\mathcal{R}'}$;
- For every $x \in \Delta^{\mathcal{R}'}$, $h^{\mathcal{R}}(b(x)) = h^{\mathcal{R}'}(x)$.

It is easy to prove by induction on the construction of the concepts that for every $C \in \mathcal{L}$ and every $x \in \Delta^{\mathcal{R}'}$, $x \in C^{\mathcal{R}'}$ iff $b(x) \in C^{\mathcal{R}}$. Moreover, $x \in \min_{<^{\mathcal{R}'}}(C^{\mathcal{R}'})$ iff $b(x) \in \min_{<^{\mathcal{R}}}(C^{\mathcal{R}})$. Hence, there is a ranked $\mathcal{KB}$-model which is a counter-model for $C \subseteq D$ with $\Delta$ as its domain.

### E Proofs of results in Section 5

**NB:** The results marked (*) are introduced here in the Appendix, while they are omitted in the main text.

**Lemma 10** For every knowledge base $\mathcal{KB}$ and every concept $C$, $\text{rank}_{\mathcal{KB}}(C) = \infty$ iff $\mathcal{KB} \models_{\text{mod}} C \subseteq \bot$.

**Proof:**

If $\mathcal{KB}$ does not have a modular model or $C$ is never satisfiable, then the result is straightforward. Let $\mathcal{KB} = T \cup D$ have a modular model, and let $C$ be satisfiable. Also, let $D$ be ranked into $(D_{0}^{\text{rank}}, \ldots, D_{n}^{\text{rank}}, D_{\infty}^{\text{rank}})$.

From left to right, let $\text{rank}_{\mathcal{KB}}(C) = \infty$ but $\mathcal{KB} \not\models_{\text{mod}} C \subseteq \bot$. Together they imply that $T \cup D_{\infty}^{\text{rank}} \models_{\text{mod}} T \subseteq \neg C$ but $T \cup D_{\infty}^{\text{rank}} \not\models_{\text{mod}} C \subseteq \bot$. Hence, due to the FMP (Theorem 3), there is a finite ranked model $\mathcal{R}$ of $T \cup D_{\infty}^{\text{rank}}$ with the domain $\Delta^{\mathcal{R}}$ layered into $(L_{0}^{\mathcal{R}}, \ldots, L_{n}^{\mathcal{R}})$, and s.t. $\mathcal{R} \models T \subseteq \neg C$ but $\mathcal{R} \not\models C \subseteq \bot$, that is, in $\Delta^{\mathcal{R}}$ there is an object $o$ s.t. $o \in L_{i}^{\mathcal{R}}$, with $0 < i \leq n$, and $o \in C^{\mathcal{R}}$.

Now let’s define a new model $\mathcal{R}'$ simply taking the lower layer and putting it at the “top” of our model, that is, we rearrange the interpretation in the following way:

- $\Delta^{\mathcal{R}'} = \Delta^{\mathcal{R}}$;
 Clearly for every concept $D$, $D^{R'} = D^R$ (it is easy to prove by induction on the construction of the concepts), and consequently $R'$ is still a model of $T$. We can prove that is still also a model of $D^\text{rank}_\infty$. Assume that is not the case, that is, there is a some $D \subseteq E \in D^\text{rank}_\infty$ s.t. $R \vDash D \subseteq E$ and $R' \not\vDash D \subseteq E$. $R \vDash D \subseteq E$ if either $h_R(D \cap E) < h_R(D \cap \neg E)$ or $h_R(D) = \infty$. It cannot be the latter, since $h_R(D) = \infty$ corresponds to $D^R = \emptyset$, and we would have also $D^{R'} = \emptyset$ and $h_{R'}(D) = \infty$. Hence it must be $h_R(D \cap E) < h_R(D \cap \neg E)$, while $h_R(D \cap E) \neq h_R(D \cap \neg E)$. Let $h_R(D \cap E) = i$ and $h_R(D \cap \neg E) = j$ with $i < j$. If $i > 0$, then $h_R'(D \cap E) = i - 1$ and $h_R'(D \cap \neg E) = j - 1$, and $h_R(D \cap E) < h_R'(D \cap \neg E)$ again; hence it must be $h_R'(D \cap E) = 0$, that is, $h_R'(D) = 0$, but that is incompatible with $D \subseteq E \in D^\text{rank}_\infty$, since $T \cup D^\text{rank}_\infty \models \mod \top \subseteq \neg D$, that is, $h_R(D) > 0$. Consequently, $R'$ too must be a model of $T \cup D^\text{rank}_\infty$.

We have assumed that in $R$ there is an object $o$ s.t. $o \in C^R$ and $o \in L^R_i$ for some $0 < i \leq n$. Repeating the procedure used to define $R'$ for $i$ times, we obtain a model $R^*$ of $T \cup D^\text{rank}_\infty$ s.t. $o \in C^{R^*}$ and $o \in L^R_{i^*}$. However, since $\text{rank}_{KB}(C) = \infty$ implies $T \cup D^\text{rank}_\infty \models \mod \top \subseteq \neg C$, this cannot be the case. We conclude that if $\text{rank}_{KB}(C) = \infty$ then $KB \models \mod \top \subseteq \bot$.

From right to left, let $KB \models \mod \top \subseteq \bot$ but $\text{rank}_{KB}(C) \neq \infty$. The latter implies that there is a model of $T \cup D^\text{rank}_\infty$ that does not satisfy $\top \subseteq \neg C$, that is, does not satisfy $C \subseteq \bot$. Referring again to the FMP (Theorem 3), we can say that there is a finite ranked model $R$ of $T \cup D^\text{rank}_\infty$ that does not satisfy $C \subseteq \bot$. Let $k$ be the number of layers in $R$.

Now consider $T \cup (D^\text{rank}_n \cup D^\text{rank}_\infty)$. For each $D \subseteq E \in D^\text{rank}_n$ there must be a model in which $D \cap E$ is not exceptional, that is, it is satisfied in the layer 0. As a consequence, still using the FMP (Corollary 6), for each $D \subseteq E \in D^\text{rank}_n$ there must be a finite ranked model $R_{D \subseteq E}^\text{rank}$ of $T \cup (D^\text{rank}_n \cup D^\text{rank}_\infty)$ s.t. $h_{R_{D \subseteq E}^\text{rank}}(D \subseteq E) = 0$.

Build a ranked interpretation $R_n$ as follows:

- for every $D \subseteq E \in D^\text{rank}_n$, let $R_{D \subseteq E}$ be a finite ranked model of $T \cup (D^\text{rank}_1 \cup D^\text{rank}_\infty)$ in which $h_{R_{D \subseteq E}}(D \subseteq E) = 0$.

- Let $R' = \langle \Delta_{R'}, \mathcal{R}', \prec_{R'} \rangle$ be the ranked union of such sets. $R'$ is a model of $D^\text{rank}_n$ (Lemma 8) s.t. for every $D \subseteq E \in D^\text{rank}_n$, $h_{R_{D \subseteq E}}(D \subseteq E) = 0$. Since $D_n^\text{rank}$ is finite, it has been obtained from a finite set of finite models and so it is a finite ranked model. Let $m$ be the number of layers in $R'$.

- From $R'$ and $R$ define a finite ranked interpretation $R_n = \langle \Delta_{R_n}, \mathcal{R}_n, \prec_{R_n} \rangle$ as:

  $\Delta_{R_n} = \Delta_R \cup \Delta_{R'}$;}
Lemma 43 (*) Let $\mathcal{R}_n = \mathcal{A}^\mathcal{R} \cup \mathcal{A}^\mathcal{R}$ for every $A \in \mathcal{C}$;
- $r^\mathcal{R}_n = r^\mathcal{R} \cup r^\mathcal{R}'$ for every $r \in \mathcal{R}$;
- for every $i \leq m$, $L^\mathcal{R}_n = L^\mathcal{R}_i$;
- for every $m < i \leq (m + k)$, $L^\mathcal{R}_n = L^\mathcal{R}_{(i-(m+1))}$.

Informally, we build the model $\mathcal{R}_n$ by adding $\mathcal{R}$ on top of $\mathcal{R}'$. It is easy to prove by induction on the construction of the concepts that every object in $\mathcal{R}_n$ satisfies a concept $\mathcal{D}$ iff it satisfies $\mathcal{D}$ also in the original model, $\mathcal{R}$ or $\mathcal{R}'$. As a consequence, $\mathcal{R}_n \not\models C \subseteq \bot$. Also, it is easy to prove that $\mathcal{R}_n$ is a model of $\mathcal{T} \cup (\mathcal{D}_n^{\text{rank}} \cup \mathcal{D}_\infty^{\text{rank}})$; $\mathcal{R}'$ is a model of $\mathcal{D}_n^{\text{rank}}$ with at layer 0 an object satisfying $\mathcal{D} \cap E$ for each $\mathcal{D} \subseteq E \in \mathcal{D}_n^{\text{rank}}$, and both $\mathcal{R}$ and $\mathcal{R}'$ are models of $\mathcal{T} \cup \mathcal{D}_\infty^{\text{rank}}$.

Now consider $\mathcal{T} \cup (\mathcal{D}_n^{\text{rank}} \cup \mathcal{D}_n^{\text{rank}} \cup \mathcal{D}_\infty^{\text{rank}})$. Using the same procedure defined for $\mathcal{R}_n$, we can build a model $\mathcal{R}_{n-1}$, obtained doing the ranked union of a finite set of finite models of $\mathcal{T} \cup (\mathcal{D}_n^{\text{rank}} \cup \mathcal{D}_n^{\text{rank}} \cup \mathcal{D}_\infty^{\text{rank}})$ and adding on top $\mathcal{R}_n$. $\mathcal{R}_{n-1}$ will be a finite ranked model of $\mathcal{T} \cup (\mathcal{D}_n^{\text{rank}} \cup \mathcal{D}_n^{\text{rank}} \cup \mathcal{D}_\infty^{\text{rank}})$ s.t. $\mathcal{R}_{n-1} \not\models C \subseteq \bot$.

We can go on with this procedure until we define a finite ranked model $\mathcal{R}_0$ of $\mathcal{T} \cup (\mathcal{D}_0^{\text{rank}} \cup \ldots \cup \mathcal{D}_n^{\text{rank}} \cup \mathcal{D}_\infty^{\text{rank}})$. That is, $\mathcal{R}_0$ is a model of $\mathcal{T} \cup \mathcal{D}$ s.t. $\mathcal{R}_0 \not\models C \subseteq \bot$, against the hypothesis that $\mathcal{KB} \models_{\text{mod}} C \subseteq \bot$.

In order to prove Theorem 4, we will use the following lemma.

Lemma 43 (*) Let $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$ be a defeasible knowledge base having a modular model, $\mathcal{O}$ its big ranked model, and $\Delta$ the countably infinite domain used to define $\mathcal{O}$. For every $C \subseteq D \in \mathcal{D}$, $\text{rank}_{\mathcal{KB}}(C \cap D) = i$ iff there is a model $\mathcal{R}_\Delta \in \text{Mod}_\Delta(\mathcal{KB})$ s.t. $h_{\mathcal{R}_\Delta}(C \cap D) = i$.

Proof:
First of all, we observe that the exceptionality function in Definition 23 is correctly captured in the model $\mathcal{O}$, that is, for every $C \in \mathcal{L}$, $\mathcal{KB} \models_{\text{mod}} \top \subseteq \neg C$ iff $\mathcal{O} \models \top \subseteq \neg C$. Indeed, by Lemma 9 a concept $C$ is exceptional w.r.t. $\mathcal{KB}$ iff $\mathcal{R}_\Delta \models \top \subseteq \neg C$, for every $\mathcal{R}_\Delta \in \text{Mod}_\Delta(\mathcal{KB})$, which immediately corresponds to $\mathcal{O} \models \top \subseteq \neg C$.

Since $\mathcal{R}_\Delta \models \mathcal{KB}$ for every $\mathcal{R}_\Delta \in \text{Mod}_\Delta(\mathcal{KB})$, if $h_{\mathcal{R}_\Delta}(C) = i$, it is immediate that $\text{rank}_{\mathcal{KB}}(C) \leq i$, otherwise it would be $h_{\mathcal{R}_\Delta}(C) > i$ for every $\mathcal{R}_\Delta \in \text{Mod}_\Delta(\mathcal{KB})$. We have to prove that for every $C \subseteq D \in \mathcal{D}$, if $\text{rank}_{\mathcal{KB}}(C \cap D) = i$, then there is a $\mathcal{R}_\Delta \in \text{Mod}_\Delta(\mathcal{KB})$ s.t. $h_{\mathcal{R}_\Delta}(C \cap D) = i$. In case $i = \infty$, Lemma 10 guarantees that if $\text{rank}_{\mathcal{KB}}(C \cap D) = \infty$, then for all the $\mathcal{R}_\Delta \in \text{Mod}_\Delta(\mathcal{KB})$, $h_{\mathcal{R}_\Delta}(C \cap D) = \infty$. In case $i < \infty$, we can prove it by induction on the ranking value $i$.

Let $C \subseteq D \in \mathcal{D}$, and let $\text{rank}_{\mathcal{KB}}(C \cap D) = i$. For $i = 0$, we already have all that is needed to prove that there is a $\mathcal{R}_\Delta \in \text{Mod}_\Delta(\mathcal{KB})$ s.t. $\mathcal{R}_\Delta \not\models \top \subseteq \neg (C \cap D)$:

- $\text{rank}_{\mathcal{KB}}(C \cap D) = 0$ iff $\mathcal{KB} \not\models_{\text{mod}} \top \subseteq \neg (C \cap D)$ (Definition 23);
- $\mathcal{KB} \not\models_{\text{mod}} \top \subseteq \neg (C \cap D)$ iff there is a $\mathcal{R}_\Delta \in \text{Mod}_\Delta(\mathcal{KB})$ s.t. $\mathcal{R}_\Delta \not\models \top \subseteq \neg (C \cap D)$ (by Lemma 9);
\[ \mathcal{R}_\Delta \models \top \subseteq \neg (C \cap D) \text{ iff } h_{\mathcal{R}_\Delta}(C \cap D) = 0. \]

For \( i > 0 \), we can define a modular model \( \mathcal{R} \) of \( \mathcal{KB} \) as follows:

Let \( C \subseteq D \in \mathcal{D} \) with \( \text{rank}_{\mathcal{KB}}(C \cap D) = i \), and let \( \mathcal{D}_{\geq i}^{\text{rank}} \) be the subset of \( \mathcal{D} \) containing the DCIs with a ranking value of at least \( i \), and \( \mathcal{D}^{\text{rank}} = \mathcal{D} \setminus \mathcal{D}_{\geq i}^{\text{rank}} \). Let \( \mathcal{R}' \) be a modular model of \( \mathcal{T} \cup \mathcal{D}^{\text{rank}} \) such that \( h_{\mathcal{R}'}(C \cap D) = 0 \). Such a model must exist, since \( \text{rank}_{\mathcal{KB}}(C \cap D) = i \), that is, \( C \cap D \) is not exceptional in \( \mathcal{T} \cup \mathcal{D}^{\text{rank}} \). We can assume that \( \mathcal{R}' \) has a finite domain, given the finite-model property (Corollary 6), and hence it is a ranked model (Lemma 7).

For each DCI \( D \subseteq E \in \mathcal{D}_{<i}^{\text{rank}} \), that is, such that \( \text{rank}_{\mathcal{KB}}(D \cap E) = j \) for some \( j < i \), let \( \mathcal{R}_{D \cap E} \in \text{Mod}_\Delta(\mathcal{KB}) \) be a model of \( \mathcal{KB} \) satisfying \( D \cap E \) s.t. \( h_{\mathcal{R}_{D \cap E}}(D \cap E) = j \). The induction hypothesis guarantees that such a model exists for each such DCI.

Now we define a new interpretation \( \mathcal{R}'' = \langle \Delta^{\mathcal{R}'}, \mathcal{R}'' \rangle \) in the following way:

- \( \Delta^{\mathcal{R}''} = \Delta^{\mathcal{R}'} \cup \bigcup_{(C \subseteq D \in \mathcal{D}_{<i}^{\text{rank}})} \Delta^{\mathcal{R}_{C \cap D}} \).

- For every concept name \( A \in \mathcal{C} \) and every \( x \in \Delta^{\mathcal{R}''} \), \( x \in A^{\mathcal{R}''} \) iff one of the two following cases holds: either \( x \in \Delta^{\mathcal{R}_{D \cap E}} \) for some \( D \subseteq E \in \mathcal{D}^{\text{rank}}_{<i} \) and \( x \in A^{\mathcal{R}_{C \cap D}} \), or \( x \in \Delta^{\mathcal{R}'} \) and \( x \in A^{\mathcal{R}''} \).

- For every role name \( r \in \mathcal{R} \) and every \( x, y \in \Delta^{\mathcal{R}''} \), \( (x, y) \in r^{\mathcal{R}''} \) iff one of the two following cases holds: either \( x, y \in \Delta^{\mathcal{R}_{C \cap D}} \) for some \( C \subseteq D \in \mathcal{D}^{\text{rank}} \) and \( (x, y) \in r^{\mathcal{R}''} \), or \( x, y \in \Delta^{\mathcal{R}'} \) and \( (x, y) \in r^{\mathcal{R}''} \).

- For every \( x \in \Delta^{\mathcal{R}''} \), \( h_{\mathcal{R}''}(x) = j \) iff one of the two following cases holds: either \( x \in \Delta^{\mathcal{R}_{D \cap E}} \) for some \( D \subseteq E \in \mathcal{D}^{\text{rank}}_{<i} \) and \( h_{\mathcal{R}_{D \cap E}}(x) = j \), or \( x \in \Delta^{\mathcal{R}'} \) and \( h_{\mathcal{R}'}(x) = j - i \).

The idea is to create a model of \( \mathcal{KB} \) that guarantees for a specific inclusion \( C \subseteq D \in \mathcal{D} \) that the height of \( C \) in the model corresponds exactly to the rank of \( C \). That is, given an inclusion \( C \subseteq D \) that has rank \( i \), we have built a ranked interpretation \( \mathcal{R}'' \) in which \( C \) has height \( i \). Now we need to:

- Prove that \( \mathcal{R}'' \) is a model of \( \mathcal{KB} \);

- Show that an isomorphic model to \( \mathcal{R}'' \) is in \( \text{Mod}_\Delta(\mathcal{KB}) \).

It can easily be proven that \( \mathcal{R}'' \) is a model of \( \mathcal{KB} \): first we prove by induction on the construction of concepts that, for every \( x \in \Delta^{\mathcal{R}''} \), \( x \in D^{\mathcal{R}''} \) iff the corresponding object falls under \( D \) in the original model; this grants us that \( \mathcal{R}'' \) satisfies \( \mathcal{T} \). About the satisfaction of \( \mathcal{D} \), referring to the height values that have been assigned to each object in \( \mathcal{R}'' \), we can prove that for every \( D \subseteq E \in \mathcal{D}, h_{\mathcal{R}''}(D \cap E) < h_{\mathcal{R}''}(D \cap \neg E) \) (or \( h_{\mathcal{R}''}(D) = \infty \)). Hence, \( \mathcal{R}'' \) is a model of \( \mathcal{KB} \). Also, notice that in \( \mathcal{R}' \) we must have an object \( o \) s.t. \( h_{\mathcal{R}'}(o) = 0 \) and \( o \in (C \cap D)^{\mathcal{R}'} \). The construction of \( \mathcal{R}'' \) implies that \( h_{\mathcal{R}''}(o) = i \) and \( o \in (C \cap D)^{\mathcal{R}''} \). That is, \( \mathcal{R}'' \) is a model of \( \mathcal{KB} \) in which \( h_{\mathcal{R}''}(C \cap D) = i \).
\[ \Delta^{R''} \] has been created unifying a finite number of model with the countably infinite domain \( \Delta \) plus the finite domain \( \Delta^{R'} \), hence \( \Delta^{R''} \) has a countably infinite domain, and there is a model \( \mathcal{R}_{\Delta}'' \) that is isomorphic to \( \mathcal{R}'' \) and has \( \Delta \) as domain.

Using Lemma 43, we can prove Theorem 4.

**Theorem 4** Let \( \mathcal{KB} \) be a defeasible knowledge base having a modular model. A statement \( \alpha \) is in the rational closure of \( \mathcal{KB} \) iff \( \mathcal{KB} \models_{\mathsf{rat}} \alpha \).

**Proof:**
Let \( \mathcal{KB} \) be a defeasible knowledge base with a modular model, \( \mathcal{O} \) the big ranked model of \( \mathcal{KB} \), and \( \Delta \) the countably infinite domain used to define \( \mathcal{O} \). \( \mathcal{KB} \models_{\mathsf{rat}} \alpha \) iff \( \mathcal{O} \models \alpha \) (Definition 22), so we need to prove that \( \mathcal{O} \models \alpha \) is in the rational closure of \( \mathcal{KB} \). We first prove the result where \( \alpha \) is a DCI (of the form \( C \subseteq D \)), that is, we need to prove that for every concept \( \text{Definition 22} \), so we need to prove that \( \mathcal{O} \models C \) if we can prove it by induction on the ranking value \( i \).

Let \( \mathcal{KB} \) be a defeasible knowledge base having a modular model, \( \mathcal{O} \) the big ranked model of \( \mathcal{KB} \), and \( \Delta \) the countably infinite domain used to define \( \mathcal{O} \). \( \mathcal{KB} \models_{\mathsf{rat}} \alpha \) iff \( \mathcal{O} \models \alpha \) (Definition 22), so we need to prove that \( \mathcal{O} \models \alpha \) is in the rational closure of \( \mathcal{KB} \). We first prove the result where \( \alpha \) is a DCI (of the form \( C \subseteq D \)), that is, we need to prove that \( \mathcal{O} \models C \) if we can prove it by induction on the ranking value \( i \).

Let \( \mathcal{KB} \) be a defeasible knowledge base having a modular model, \( \mathcal{O} \) the big ranked model of \( \mathcal{KB} \), and \( \Delta \) the countably infinite domain used to define \( \mathcal{O} \). \( \mathcal{KB} \models_{\mathsf{rat}} \alpha \) iff \( \mathcal{O} \models \alpha \) (Definition 22), so we need to prove that \( \mathcal{O} \models \alpha \) is in the rational closure of \( \mathcal{KB} \). We first prove the result where \( \alpha \) is a DCI (of the form \( C \subseteq D \)), that is, we need to prove that \( \mathcal{O} \models C \) if we can prove it by induction on the ranking value \( i \).

Let \( \mathcal{KB} \) be a defeasible knowledge base having a modular model, \( \mathcal{O} \) the big ranked model of \( \mathcal{KB} \), and \( \Delta \) the countably infinite domain used to define \( \mathcal{O} \). \( \mathcal{KB} \models_{\mathsf{rat}} \alpha \) iff \( \mathcal{O} \models \alpha \) (Definition 22), so we need to prove that \( \mathcal{O} \models \alpha \) is in the rational closure of \( \mathcal{KB} \). We first prove the result where \( \alpha \) is a DCI (of the form \( C \subseteq D \)), that is, we need to prove that \( \mathcal{O} \models C \) if we can prove it by induction on the ranking value \( i \).

An immediate consequence of Lemma 43 is that, for every \( C \subseteq D \in \mathcal{D} \), \( h_{\mathcal{O}}(C \cap D) = \text{rank}_{\mathcal{KB}}(C \cap D) \). Being \( \mathcal{O} \) a model of \( \mathcal{D} \), if \( C \subseteq D \in \mathcal{D} \) then \( h_{\mathcal{O}}(C \cap D) = h_{\mathcal{O}}(C) \) and \( \text{rank}_{\mathcal{KB}}(C \cap D) = \text{rank}_{\mathcal{KB}}(C) \). So, for every \( C \subseteq D \in \mathcal{D} \), \( h_{\mathcal{O}}(C) = h_{\mathcal{O}}(C \cap D) = \text{rank}_{\mathcal{KB}}(C \cap D) = \text{rank}_{\mathcal{KB}}(C) \). Now we extend such a result to any concept \( C \), using a construction that is in line with the one used to prove Lemma 43.

Since \( \mathcal{O} \models \mathcal{KB} \), if \( h_{\mathcal{O}}(C) = i \), it is immediate that \( \text{rank}_{\mathcal{KB}}(C) \leq i \), otherwise it would be \( h_{\mathcal{O}}(C) > i \). We have to prove that for every concept \( C \), if \( \text{rank}_{\mathcal{KB}}(C) = i \), then \( h_{\mathcal{O}}(C) = i \), that is, there is a \( \mathcal{R}_{\Delta} \in \mathcal{Mod}_{\Delta}(\mathcal{KB}) \) s.t. \( h_{\mathcal{R}_{\Delta}}(C) = i \). In case \( i = \infty \), Lemma 10 guarantees that if \( \text{rank}_{\mathcal{KB}}(C) = \infty \), then for all the \( \mathcal{R}_{\Delta} \in \mathcal{Mod}_{\Delta}(\mathcal{KB}) \), \( h_{\mathcal{R}_{\Delta}}(C) = \infty \). In case \( i < \infty \), we can prove it by induction on the ranking value \( i \).

Let \( \text{rank}_{\mathcal{KB}}(C) = i \). For \( i = 0 \), we already have all that is needed to prove that there is a \( \mathcal{R}_{\Delta} \in \mathcal{Mod}_{\Delta}(\mathcal{KB}) \) s.t. \( \mathcal{R}_{\Delta} \not\models T \subseteq \neg(C) \):

- \( \text{rank}_{\mathcal{KB}}(C) = 0 \) iff \( \mathcal{KB} \not\models_{\text{mod}} T \subseteq \neg(C) \) (Definition 23);
- \( \mathcal{KB} \not\models_{\text{mod}} T \subseteq \neg(C) \) iff there is a \( \mathcal{R}_{\Delta} \in \mathcal{Mod}_{\Delta}(\mathcal{KB}) \) s.t. \( \mathcal{R}_{\Delta} \not\models T \subseteq \neg(C) \) (by Lemma 9);
- \( \mathcal{R}_{\Delta} \not\models T \subseteq \neg(C) \) iff \( h_{\mathcal{R}_{\Delta}}(C) = 0 \);
- \( h_{\mathcal{O}}(C) = 0 \) iff there is a \( \mathcal{R}_{\Delta} \in \mathcal{Mod}_{\Delta}(\mathcal{KB}) \) s.t. \( h_{\mathcal{R}_{\Delta}}(C) = 0 \).

For \( i > 0 \), we can define a modular model \( \mathcal{R} \) of \( \mathcal{KB} \) as follows:

Let \( \text{rank}_{\mathcal{KB}}(C) = i \), and, as in Lemma 43 let \( \mathcal{D}^{\text{rank}}_{\geq i} \) be the subset of \( \mathcal{D} \) containing the DCIs with a ranking value of at least \( i \), and \( \mathcal{D}^{\text{rank}}_{< i} = \mathcal{D} \setminus \mathcal{D}^{\text{rank}}_{\geq i} \). Let \( \mathcal{R}' \) be a modular model of \( \mathcal{T} \cup \mathcal{D}^{\text{rank}}_{\geq i} \) such that \( h_{\mathcal{R}'}(C) = 0 \). Such a model must exist, since \( \text{rank}_{\mathcal{KB}}(C) = i \), that is,
C is not exceptional in $T \cup D_{\geq i}^{\text{rank}}$. We can assume that $R'$ has a finite domain, given the finite-model property (Corollary 0), and hence it is a ranked model (Lemma 7).

Then we define an interpretation $R''$ exactly as done in the proof of Lemma 43, and, exactly as in Lemma 43 we can prove that $R''$ is a model of $KB$ with a countably infinite domain and s.t. $h_{R''}(C) = i$.

That implies that there is a model $R'_{\Delta} \in \text{Mod}_{\Delta}(KB)$ that is isomorphic to $R''$, with $h_{R'_{\Delta}}(C) = i$.

Since $R'_{\Delta}$ is used in the construction of $O$, $h_{O}(C) \leq i$; since $\text{rank}_{KB}(C) = i$, $h_{O}(C) \geq i$. Hence, $h_{O}(C) = i$.

**Lemma 11** For $KB = T \cup D$, if $T \models \bigcap D \not\subseteq C$, then $C \subseteq D$ is exceptional w.r.t. $T \cup D$.

**Proof:**
It suffices to prove that if $T \cup D \not\models_{\text{mod}} \top \subseteq \neg C$ then $T \not\models \bigcap D \subseteq \neg C$. So, suppose that $T \cup D \not\models_{\text{mod}} \top \subseteq \neg C$. This means there is a modular model $R$ of $T \cup D$ for which we have an $x \in \Delta^{R}$ such that $x \in C^{R}$. Let $I$ be the classical interpretation associated with $R$. It follows immediately that $I$ is a model of $T$ and that $x \in (\bigcap D)^{I}$, but that $x \notin (\neg C)^{I}$. ■

**Lemma 44** (*) Let $KB = T \cup D$. Then (i) $KB \subseteq C_{\text{rat}}(KB)$ and (ii) $C_{\text{rat}}(KB)$ induces a defeasible subsumption relation $\sqsubseteq_{\text{rat}}^{KB} = \text{def } \{(C, D) \mid KB \models_{\text{rat}} C \subseteq D\}$ that is rational.

**Proof:**
Let $KB = T \cup D$.

Proving (i): Assume $C \subseteq D \in T$. $KB \models_{\text{rat}} C \subseteq D$ iff $T^{*} \models C \subseteq D$; since $T \subseteq T^{*}$, $T \subseteq C_{\text{rat}}(KB)$. Assume that $C \subseteq D \in D$. Either $C \subseteq D$ ends up in $D_{\infty}^{\text{rat}}$, or there will be an $i \ (0 \leq i \leq n)$ s.t. $\text{rk}(C) = \text{rk}(C \subseteq D) = i$. In the former case, $C \subseteq D$ is in $T^{*}$, and so $T^{*} \models C \subseteq D$, i.e., $KB \models_{\text{rat}} C \subseteq D$.

In the latter case, $\models \bigcap C \subseteq D$, and so $T^{*} \models \bigcap C \subseteq D$, i.e., $C \subseteq D \in C_{\text{rat}}(KB)$. Hence $T \cup D \subseteq C_{\text{rat}}(KB)$. Hence $T \cup D \subseteq C_{\text{rat}}(KB)$.

Proving (ii): Let $\sqsubseteq_{\text{rat}}^{KB} = \text{def } \{(C, D) \mid KB \models_{\text{rat}} C \subseteq D\}$. We show $\sqsubseteq_{\text{rat}}^{KB}$ satisfies all rationality properties.

- (Ref). Since $\models C \subseteq C$ is valid for any $C \in L$, we have that $T^{*} \models \bigcap C \subseteq C$ for any $T^{*}$ and $E_{i}^{*}$.

- (LLE). $C \subseteq E \in C_{\text{rat}}(KB)$ implies that $T^{*} \models \bigcap C \subseteq E$ for some $i$ (or $T^{*} \models C \subseteq E$, if $\text{rk}(C) = \infty$). Since $\models C \equiv D$, $E_{i}$ is the lowest $i$ s.t. $T^{*} \not\models \bigcap C \subseteq D$, and $T^{*} \models \bigcap C \subseteq D \subseteq E$, too.

- (And). $T^{*} \models \bigcap C \subseteq D$ and $T^{*} \models \bigcap C \subseteq E$ (possibly without $\bigcap E_{i}$, if $C$ has an infinite rank), hence $T^{*} \models \bigcap C \subseteq D \subseteq E$, that is, $C \subseteq D \subseteq E \in C_{\text{rat}}(KB)$. 

55
• (Or). \( T^* \models \bigcap E_i \cap C \subseteq E \) for some \( i \) and \( T^* \models \bigcap E_j \cap D \subseteq E \) for some \( j \). Assume that \( i \leq j \) and \( i < \infty \), that is, \( \models \bigcap E_i \subseteq \bigcap E_j \). Then, since \( T^* \models \bigcap E_i \subseteq \neg C \), we have that \( T^* \models \bigcap E_i \subseteq \neg(C \cup D) \). Moreover \( T^* \models \bigcap E_j \cap D \subseteq E \) and \( \models \bigcap E_i \subseteq \bigcap E_j \) imply that \( T^* \models \bigcap E_i \cap D \subseteq E \). The proof is analogous for \( j \leq i \) with \( j < \infty \), or if \( i \) and \( j \) correspond to \( \infty \).

• (RW). \( C \subseteq D \in Cn_{\text{rat}}(KB) \) if \( T^* \models \bigcap E_i \cap C \subseteq D \) for some \( \bigcap E_i \) (or \( T^* \models C \subseteq D \), if \( \text{rk}(C) = \infty \)). Since \( D \subseteq E, T^* \models \bigcap E_i \cap C \subseteq E \).

• (CM). If \( \text{rk}(C) = i < \infty \), \( T^* \models \bigcap E_i \cap C \subseteq D \) and \( T^* \models \bigcap E_i \cap C \subseteq E \) for some \( \bigcap E_i \). Since \( T^* \models \bigcap E_i \cap C \subseteq D \) and \( T^* \not\models \bigcap E_i \subseteq \neg C \), \( T^* \not\models \bigcap E_i \subseteq \neg(C \cap D) \), otherwise we would have \( T^* \models \bigcap E_i \cap C \subseteq D \cup \neg D \), i.e., \( T^* \not\models \bigcap E_i \subseteq \neg C \). Hence we have \( C \cap D \subseteq E \in Cn_{\text{rat}}(KB) \) since \( T^* \models \bigcap E_i \cap C \cap D \subseteq E \). If \( \text{rk}(C) = \infty \), we have \( T^* \models C \subseteq \perp \), and the proof is trivial.

• (RM). If \( \text{rk}(C) = i < \infty \), \( T^* \models \bigcap E_i \cap C \subseteq E \) and \( T^* \not\models \bigcap E_i \cap C \subseteq \neg D \) for some \( \bigcap E_i \). Since \( T^* \not\models \bigcap E_i \cap C \subseteq \neg D \), \( T^* \not\models \bigcap E_i \subseteq \neg(C \cap D) \), otherwise we would have \( T^* \models \bigcap E_i \cap C \subseteq \neg D \). Hence we have \( C \cap D \subseteq E \in Cn_{\text{rat}}(KB) \) since \( T^* \models \bigcap E_i \cap C \cap D \subseteq E \). If \( \text{rk}(C) = \infty \), then we have \( T^* \models C \subseteq \perp \), and the proof is trivial.

The following lemma states that, as in the propositional case \[59\], our procedure correctly manages the classical information, that is, an axiom \( C \subseteq \perp \) in the rational closure of \( KB \) if and only if it is also a modular consequence of \( KB \).

**Lemma 45** (\( \ast \)) Let \( KB = T \cup D \) and assume \( C \subseteq D \in D \). Then \( KB \models_{\text{mod}} C \subseteq \perp \) iff \( \text{rk}(C) = \infty \) iff \( T^* \models C \subseteq \perp \).

**Proof:**
Let \( KB = T \cup D \).

For the only-if part, \( KB \models_{\text{mod}} C \subseteq \perp \) implies that every rational subsumption relation containing \( KB \) must satisfy also \( C \subseteq \perp \). Hence we have that \( KB \vdash_{\text{rat}} C \subseteq \perp \), since \( Cn_{\text{rat}}(KB) \) induces a rational subsumption relation extending \( KB \) (Lemma \[44\]). From Definition \[27\] we know that \( KB \vdash_{\text{rat}} C \subseteq \perp \) is possible only if \( C \) is always negated in the ranking procedure, i.e., \( T^* \models C \subseteq \perp \).

For the if part, we define from \( KB \) a new knowledge base \( KB^* =_{\text{def}} T^* \cup D^* \), with \( T^* \) obtained from \( T \) by adding all the sets \( \{C \subseteq D \mid C \subseteq D \in D^*_\} \) that we obtain at each iteration of function ComputeRanking(\( \cdot \)). Let us denote with \( D^*_1, \ldots, D^*_n \) such sets. Assume that \( T^* \models C \subseteq \perp \), but \( KB \not\models_{\text{mod}} C \subseteq \perp \), i.e., there is a modular model of \( KB \) in which \( C \) is non-empty. Let \( R \) be such a model, with an object \( x \) falling under \( C^R \). Since \( T^* \models C \subseteq \perp \), there must be a GCI \( E \subseteq F \) in some \( D^*_i \) that is not satisfied, that is, given the nature of the GCI in every \( D^*_i \) \( (T^* \models E \subseteq \perp \) for every \( E \subseteq F \) contained.
in some $\mathcal{D}^n_{\mathcal{E}}$, this means that there is a subsumption $E \subseteq \bot$ that is not satisfied in $\mathcal{R}$. Therefore, there must be an object $y$ falling under $E^\mathcal{R}$. Hence, assuming $E \subseteq F \in \mathcal{D}^1_{\mathcal{E}}$, since $\mathcal{T} \cup \mathcal{D}^1_{\mathcal{E}} \cup \ldots \cup \mathcal{D}^{i-1}_{\mathcal{E}} \models \bigcap \{ \neg G \cup H \mid G \subseteq H \in \mathcal{D}^1_{\mathcal{E}} \} \subseteq \neg E$, either $\mathcal{R} \vDash \mathcal{T} \cup \mathcal{D}^1_{\mathcal{E}} \cup \ldots \cup \mathcal{D}^{i-1}_{\mathcal{E}}$ and $y \in (G \cap \neg H)^\mathcal{R}$ for some $G \subseteq H \in \mathcal{D}^1_{\mathcal{E}}$ (Case 1 below), or $\mathcal{R} \not\vDash \mathcal{T} \cup \mathcal{D}^1_{\mathcal{E}} \cup \ldots \cup \mathcal{D}^{i-1}_{\mathcal{E}}$ (Case 2 below).

**Case 1.** Since $\mathcal{R} \vDash \mathcal{KB}$, $\mathcal{R}$ is also a model of $G \subseteq H$, which is an element of $\mathcal{D}$. Hence there must be an object $y$ such that $y \prec^\mathcal{R} x$ (remember that $x \in C^{\mathcal{R}}$) and $y \in (G \cap \neg H)^\mathcal{R}$. Again, since $G \subseteq H \in \mathcal{D}^1_{\mathcal{E}}$ (which implies $\mathcal{T} \cup \mathcal{D}^1_{\mathcal{E}} \cup \ldots \cup \mathcal{D}^{i-1}_{\mathcal{E}} \models \bigcap \{ \neg G \cup H \mid G \subseteq H \in \mathcal{D}^1_{\mathcal{E}} \} \subseteq \neg G$) and $\mathcal{R} \vDash \mathcal{T} \cup \mathcal{D}^1_{\mathcal{E}} \cup \ldots \cup \mathcal{D}^{i-1}_{\mathcal{E}}$, there must be a GCI $I \subseteq L \in \mathcal{D}^i_{\mathcal{E}}$ such that $y \in (I \cap \neg L)^\mathcal{R}$, and we need an object $z$ such that $z \prec^\mathcal{R} y$ and $z \in (H \cap I)^\mathcal{R}$, and so on... This procedure creates an infinitely descending chain of objects, and, since the number of the antecedents of the axioms in $\mathcal{D}^\infty$ is finite, it cannot be the case since the model would not satisfy the smoothness condition for the concept $\bigcup \{ C \mid C \subseteq D \in \mathcal{D}^i_{\mathcal{E}} \}$ (see Definition 5).

**Case 2.** If $\mathcal{R} \not\vDash \mathcal{T} \cup \mathcal{D}^1_{\mathcal{E}} \cup \ldots \cup \mathcal{D}^{i-1}_{\mathcal{E}}$, then $\mathcal{R}$ does not satisfy some $E \subseteq F \in \mathcal{D}^j_{\mathcal{E}}$ for some $j < i$, and therefore there must be an object falling under $E^\mathcal{R}$. Again, it is either Case 1 or Case 2. Nevertheless, since at every iteration of Case 2 we pick a lower value $j$ for $\mathcal{D}^j_{\mathcal{E}}$ and we have a finite sequence of $\mathcal{D}^i_{\mathcal{E}}$, we know that after some steps (in the worst case when we reach $\mathcal{D}^h_{\mathcal{E}}$) we necessarily fall into Case 1, which cannot be the case.

An immediate consequence of Lemma 12 binds preferential consistency (existence of a preferential model – cf. Definition 6) to classical consistency.

**Corollary 7 (** Let $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$. Then $\mathcal{KB} \vDash_{\text{mod}} \top \subseteq \bot$ iff $\mathcal{T}^* \models \top \subseteq \bot$.

We can now prove that the knowledge bases $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$ and $\mathcal{KB}^* = \mathcal{T}^* \cup \mathcal{D}^*$ (in rank normal form) are modularly equivalent.

**Lemma 12** Let $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$ and let $\mathcal{KB}^* = \mathcal{T}^* \cup \mathcal{D}^*$ be obtained from $\mathcal{KB}$ through function $\text{ComputeRanking}(\cdot)$. Then $\mathcal{KB}$ and $\mathcal{KB}^*$ are modularly equivalent.

**Proof:**

Given $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$, the function $\text{ComputeRanking}(\mathcal{KB})$ outputs a knowledge base $\mathcal{KB}^* = \mathcal{T}^* \cup \mathcal{D}^*$, in which the iteration of lines 5–13 identifies a (possibly empty) set $\{C_1 \subseteq D_1, \ldots, C_n \subseteq D_n\}$ of always exceptional defeasible subsumptions, that is moved from $\mathcal{D}$ to $\mathcal{T}$. That is, we have $\mathcal{T}^* = \mathcal{T} \cup \{C_1 \subseteq D_1, \ldots, C_n \subseteq D_n\}$ and $\mathcal{D}^* = \mathcal{D} \setminus \{C_1 \subseteq D_1, \ldots, C_n \subseteq D_n\}$. It is sufficient to prove that $\mathcal{KB} \models_{\text{mod}} C_i \subseteq \bot$ and $\mathcal{KB}^* \models_{\text{mod}} C_i \subseteq D_i$ for every $C_i \subseteq D_i$ $(1 \leq i \leq n)$.

Let $C_i \subseteq D_i \in \mathcal{D} \setminus \mathcal{D}^*$. It means that, at some iteration through Lines 4–14 of function $\text{ComputeRanking}(\cdot)$, we have $\mathcal{T}^* \models \bigcap \mathcal{D}^\infty_{\mathcal{E}} \models \neg C_i$, which implies that $\mathcal{T}^* \cup \mathcal{D}^\infty_{\mathcal{E}} \models \top \subseteq \bot$.
\neg C_i$, where $D^\infty = \{ C \subseteq D \mid C \subseteq D \in D^\infty \}$). Since every $D^\infty$ created at every iteration is contained in the final $T^\ast$, using such final $T^\ast$ we have that $T^\ast \models C_i \subseteq \bot$. Hence, by Lemma 45 we have that $KB \models \bot \Rightarrow C_i$.

On the other hand, if $C_i \subseteq D_i \in D \setminus D^\ast$, then $C_i \subseteq D_i \in T^\ast$, and hence $KB^\ast \models C_i \subseteq D_i$ by supra-classicality (cf. proof of Lemma 46 below).

Now we are justified in using the rank normal form $KB^\ast = T^\ast \cup D^\ast$ in order to analyse the rational closure of the knowledge base $KB = T \cup D$. Hence, in what follows we shall assume that the knowledge bases we are working with are already in rank normal form (and therefore $D^\infty = \emptyset$).

In the next lemma, we observe that the inference relation $\vdash_{\text{rat}}$ respects the preferential conclusions of $KB$ w.r.t. assertions of the form $\top \subseteq C$, another desideratum proven for the propositional case by Lehmann and Magidor [59].

Lemma 46 (*) For every $C \in \mathcal{L}$, $KB \models \top \subseteq C$ iff $KB \vdash_{\text{rat}} \top \subseteq C$.

Proof: First of all, recall that $KB \vdash_{\text{rat}} \top \subseteq C$ if $T^\ast \models \bigcap D^\ast \subseteq C$ (cf. Definition 27).

For the if part, first we need to prove two properties of $\models_{\text{mod}}$, namely supra-classicality (Sup) and one half of the deduction theorem (S): 

(Sup) $C \subseteq D$

The derivation of Sup is straightforward: remember that $C \subseteq C$ holds (Ref), assume $C \subseteq D$ and then apply (RW).

(S) $C \subseteq D$

To see that (S) holds, assume $C \subseteq D$ and note that $\models D \subseteq \neg C \cup D$; we derive by (RW) $C \subseteq \neg C \cup D$. Since $\models \neg C \subseteq \neg C \cup D$, we obtain $\neg C \subseteq \neg C \cup D$ by (Sup). Then apply (Or) to $C \subseteq \neg C \cup D$ and $\neg C \subseteq \neg C \cup D$, obtaining $\top \subseteq \neg C \cup D$.

Now we have to prove that if $T^\ast \models \bigcap D^\ast \subseteq C$, then $KB \models_{\text{mod}} \top \subseteq C$.

From Lemma 12 we know that $T^\ast \cup D^\ast$ is in the modular consequences of $KB$. Applying (S) to all DCIs $C \subseteq D$ in $D^\ast$, we have $KB \models_{\text{mod}} \top \subseteq \neg C \cup D$ from each of them. Applying (And) to all these DCIs, we have $\top \subseteq \bigcap D'$ and, by (R W'), we obtain $\top \subseteq C$.

The only-if part is an immediate consequence of Lemma 14.

Lemma 47 (*) For every $KB = T \cup D$ and every $C \in \mathcal{L}$, $\text{rank}_{KB}(C) = \infty$ iff $\text{rk}(C) = \infty$.

Proof: Let $KB = T \cup D$ and transform it into a modularly equivalent knowledge base $D'$ composed of only DCIs (see Lemma 2). Since the model $\mathcal{O}$ of the rational closure of $KB$ must also
be a model of $\mathcal{D}'$, we can easily derive from Lemma 9 that $KB \models_{\text{rat}} C \subseteq \bot$ (that is, \(\text{rank}_{KB}(C) = \infty\)) iff $KB \models_{\text{mod}} C \subseteq \bot$. From Lemma 15, we have that $KB \models_{\text{mod}} C \subseteq \bot$ iff $\text{rk}(C) = \infty$, hence the result.

**Lemma 13** For every defeasible knowledge base $KB = T \cup D$ and every $C \in L$, $\text{rank}_{KB}(C) = \text{rk}(C)$.

**Proof:**
From Lemmas 13 and 14 and Lemma 12 we can see that, given a knowledge base $KB = T \cup D$ (possibly with an empty $T$), we can define a modularly equivalent knowledge base $KB^* = T^* \cup D^*$ such that all the classical information implicit in $D$ is moved into $T^*$. $KB^*$ can be defined identifying the elements of $D$ that have $\infty$ as ranking value, and Lemma 17 shows that w.r.t. the value $\infty$, $\text{rank}_{KB^*}(\cdot)$ and $\text{rk}(\cdot)$ are equivalent, while Lemma 12 tells us that $KB$ and $KB^*$ are modularly equivalent. Once we have defined $KB^*$, Lemma 49 implies that a concept $C \in L$ is exceptional w.r.t. $\models_{\text{rat}} (KB \models_{\text{mod}} T \subseteq \neg C)$ iff $KB \models_{\text{rat}} \bot \subseteq \neg C$. Hence the two ranking functions $\text{rank}_{KB^*}(\cdot)$ and $\text{rk}(\cdot)$ give back exactly the same results.

**Theorem 5** Let $KB = T \cup D$ and let $C, D \in L$. Then $KB \models_{\text{rat}} C \subseteq D$ iff $KB \models_{\text{rat}} C \subseteq D$.

**Proof:**
Since we have already proven Lemma 13, here we can use $\text{rk}(\cdot)$ to indicate indifferently the equivalent ranking functions $\text{rank}_{KB^*}(\cdot)$ and $\text{rk}(\cdot)$.

For the only-if part, assume $KB \models_{\text{rat}} C \subseteq D$. That means that either $\text{rk}(C \cap \neg D) > \text{rk}(C)$ or $\text{rk}(C) = \infty$. In the first case, it means that there is some $i$, $0 \leq i \leq n$, such that $T^* \not\models \bigcap \mathcal{E}_i \subseteq \neg C$ and $T^* \models \bigcap \mathcal{E}_i \subseteq \neg(C \cap \neg D)$, hence $T^* \models \bigcap \mathcal{E}_i \cap C \subseteq D$, i.e., $KB \models_{\text{rat}} C \subseteq D$. In the second case, we have $T^* \models C \subseteq \bot$, which implies $KB \models_{\text{rat}} C \subseteq D$.

For the if part, assume $KB \models_{\text{rat}} C \subseteq D$. Then either there is some $i$ which is the lowest number such that $T^* \not\models \bigcap \mathcal{E}_i \subseteq \neg C$ (hence $\text{rk}(C) = i$), or $T^* \models C \subseteq \bot$. In the first case, we have also that $T^* \models \bigcap \mathcal{E}_i \cap C \subseteq D$, which implies $T^* \models \bigcap \mathcal{E}_i \subseteq \neg(C \cap \neg D)$, i.e., $\text{rk}(C \cap \neg D) > i$. In the second case, $\text{rk}(C) = \infty$, which implies $KB \models_{\text{rat}} C \subseteq D$.

**Corollary 4** Checking rational entailment is \textsc{exptime}-complete.

**Proof:**
Observe that function $\text{RationalClosure}(\cdot)$ performs at most $n + 2$ (classical) subsumption checks, where $n$ is the number of ranks assigned to elements of $D$. So the number of subsumption checks performed by function $\text{RationalClosure}(\cdot)$ is $O(|D|)$. Furthermore, we need to call function $\text{ComputeRanking}(\cdot)$ to obtain the knowledge base $KB^* = T^* \cup D^*$ and the sequence $\mathcal{E}_0, \ldots, \mathcal{E}_n$, which are needed as input to function $\text{RationalClosure}(\cdot)$. First bear in mind that function $\text{Exceptional}(\cdot)$, with $\mathcal{E}$ as input, performs at most $|\mathcal{E}|$ classical subsumption checks. From this, and an analysis of function $\text{ComputeRanking}(\cdot)$, it follows that the number of subsumption checks performed by function $\text{ComputeRanking}(\cdot)$
is $O(|\mathcal{D}|^3)$. Since we know that subsumption checking w.r.t. general TBoxes in $\mathcal{ALC}$ is EXPTIME-complete [11, Chapter 3], the result follows. ■