A CHARACTERIZATION OF SCHMIDT GROUPS BY THEIR ENDOMORPHISM SEMIGROUPS

Peeter Puusemp
Department of Mathematics
Tallinn Technical University
Ehitajate tee 5
Tallinn 19086, Estonia
e-mail: puusemp@edu.ttu.ee

November 6, 2018

Abstract

A Schmidt group is a non-nilpotent finite group in which each proper subgroup is nilpotent. Each Schmidt group \( G \) can be described by three parameters \( p, q \) and \( v \), where \( p \) and \( q \) are different primes and \( v \) is a natural number, \( v \geq 1 \). Denote by \( S \) the class of all Schmidt groups which have the same parameters \( p, q \) and \( v \). It is shown in this paper that the class \( S \) can be characterized by the properties of the endomorphism semigroups of the groups of this class. It follows from this characterization that if \( G \in S \) and \( H \) is another group such that the endomorphism semigroups of \( G \) and \( H \) are isomorphic, then \( H \in S \), too.

Keywords: group, Schmidt group, endomorphism, endomorphism semigroup

2000 MSC numbers: 20F99, 20M20
1. Introduction

The question, whether two algebraic structures are isomorphic when their automorphism groups or endomorphism semigroups are isomorphic, is a fundamental but generally difficult problem in algebra. For example, H. Leptin [3] showed in 1960 that if \( p \geq 5 \), then abelian \( p \)-groups are determined by their automorphism groups. Only after 25 years W. Liebert [4] proved that it is so if \( p = 3 \). Finally, in 1998 P. Schultz [13] showed that Leptin’s result holds for \( p = 2 \), too. We have found many examples of groups which are determined by their endomorphism semigroups in the class of all groups. Some of such groups are: finite abelian groups ([5], Theorem 4.2), non-torsion divisible abelian groups ([6], Theorem 1), generalized quaternion groups ([7], Corollary 1). In this paper we give a characterization of Schmidt groups by their endomorphism semigroups.

A Schmidt group is a non-nilpotent finite group in which each proper subgroup is nilpotent. This notion is dedicated to O. J. Schmidt, who first described some properties of the mentioned groups ([12]). The structure of Schmidt groups is well-known (L. Redei [9, 10]). Each Schmidt group \( G \) can be described by three natural numbers \( p, q \) and \( v \), where \( p \) and \( q \) are different primes and \( v \geq 1 \). These numbers are called parameters of \( G \). A Schmidt group is not uniquely determined by its parameters. But there exists only one Schmidt group \( G \) (up to an isomorphism) with parameters \( p, q \) and \( v \) such that \( G \) is a group of Miller and Moreno. Let us call it the group of Miller and Moreno with parameters \( p, q \) and \( v \). Remember that a group of Miller and Moreno is a non-abelian finite group in which each proper subgroup is abelian.

Fix parameters \( p, q \) and \( v \) and denote by \( S \) the class of all Schmidt groups which have these parameters. In this paper we shall show that the class \( S \) can be characterized by the properties of the endomorphism semigroups of the groups of this class (Theorem 3.1). It follows from this characterization that if \( G \) and \( H \) are groups such that \( G \in S \) and the endomorphism semigroups \( \text{End}(G) \) and \( \text{End}(H) \) are isomorphic, then \( H \in S \) (Theorem 3.2). Let \( u \) be the order of \( p \) in the group of units of the residue-class ring \( Z_q \) modulo \( q \). As corollary from the previous two theorems, it follows that the group of Miller and Moreno with parameters \( p, q \) and \( v \) is determined by its endomorphism semigroup in the class of all groups if and only if \( u \) is odd (Theorem 3.3). Remark that the endomorphism semigroups of the groups of the class \( S \) were
found in [8].

We shall use the following notations: $G$ - a group; $\text{End}(G)$ - the endomorphism semigroup of the group $G$; $Z(G)$ - center of $G$; $[G : H]$ - index of the subgroup $H$ in the group $G$; $C_G(H)$ - the centralizer of $H$ in $G$; $N_G(H)$ - the normalizer of $H$ in $G$; $I_0(G)$ - the set of all idempotents of $\text{End}(G)$ different from 0 and 1; $[x] = \{ y \in \text{End}(G) \mid y^2 = y, \ xy = y, \ yx = x \}$ ($x \in \text{End}(G)$); $\hat{g}$ - the inner automorphism of $G$, generated by an element $g \in G$; $p, q, v$ - parameters of a Schmidt group; $S$ - the class of all Schmidt groups with parameters $p, q$ and $v$; $Z_r$ - the residue-class ring modulo $r$; $u$ - the order of $p$ in the group of units of the ring $Z_q$; $Z_r[x]$ - the polynomial ring over $Z_r$; $[a, b] = a^{-1}b^{-1}ab$; $A \ltimes B$ - the (internal) semidirect product of $A$ and $B$; $K_G(x) = \{ y \in \text{End}(G) \mid xy = xy = y \}; \ V_G(x) = \{ y \in \text{Aut}(G) \mid xy = x \}; \ D_G(x) = \{ y \in \text{Aut}(G) \mid xy = xy = x \}; \ H_G(x) = \{ y \in \text{End}(G) \mid xy = y, \ yx = 0 \}$;

Remark that $V_G(x), D_G(x), K_G(x)$ and $H_G(x)$ are subsemigroups of $\text{End}(G)$, however, $V_G(x)$ and $D_G(x)$ are subgroups of $\text{Aut}(G)$. Note that a map is written in this paper right from the element on which it acts.

2. Preliminaries

Let us cite some easy facts useful in the proofs of the main theorems.

If $x$ is an idempotent of $\text{End}(G)$, then $G$ decomposes into the semidirect product $G = \text{Ker}x \ltimes \text{Im}x$. Clearly, if $x$ and $y$ are two idempotents of $\text{End}(G)$, then $x = y$ if and only if $\text{Im}x = \text{Im}y$ and $\text{Ker}x = \text{Ker}y$.

**Lemma 2.1** If $x$ is an idempotent of $\text{End}(G)$, then the semigroups $K_G(x)$ and $\text{End}(\text{Im}x)$ are isomorphic. This isomorphism is given by the correspondence $z \mapsto z|\text{Im}x$ where $z \in K_G(x)$.

**Lemma 2.2** If $x \in \text{End}(G)$ and $\text{Im}x$ is abelian, then $\hat{g} \in V_G(x)$ for each $g \in G$.

**Lemma 2.3** If $x \in \text{End}(G)$, $g \in C_G(\text{Im}x)$ and the subgroup $\text{Im}x$ of $G$ is abelian, then $\hat{g} \in D_G(x)$.

**Lemma 2.4** If $x, y \in \text{End}(G)$ and $xy = yx$, then $(\text{Im}x)y \subset \text{Im}x$ and $(\text{Ker}x)y \subset \text{Ker}x$. 

3
The proofs of lemmas 2.1-2.4 are easy exercises.

In the next part of this section $G$ will everywhere denote an arbitrary Schmidt group with parameters $p$, $q$ and $v$. By L. Reidei [9, 10], the group $G$ satisfies following properties: 1) $G$ decomposes into a semidirect product $G = G' \rtimes C(q^v)$ of the derived subgroup $G'$ of $G$ and a cyclic subgroup $C(q^v) = \langle b \rangle$ of the order $q^v$; 2) the derived subgroup $G'$ is a $p$-group; 3) the second derived subgroup $G''$ of $G$ is a subgroup of the center $Z(G)$ of $G$; 4) the factor-group $G'/G''$ is an elementary abelian $p$-group of the order $p^u$ (the integer $u$ is defined later in this section) 5) $Z(G/G'') = \langle b^pG'' \rangle$.

Denote $G^* = G/G''$. Next we remember some properties of $\text{End}(G)$ and $\text{End}(G^*)$ which are proved in [8].

The map $* : \text{End}(G) \rightarrow \text{End}(G^*)$, $\tau \mapsto \tau^*$, where $(gG'')\tau^* = (g\tau)G''$ ($g \in G$, $\tau \in \text{End}(G)$), is a monomorphism. This monomorphism induces an isomorphism between the semigroups $\text{End}(G) \setminus \text{Aut}(G)$ and $\text{End}(G^*) \setminus \text{Aut}(G^*)$ of all proper endomorphisms of $G$ and $G^*$.

Assume that $\psi(x)$ is an arbitrary irreducible normalized divisor of the polynomial
\[
\frac{x^q - 1}{x - 1} = x^{q-1} + x^{q-2} + \ldots + x + 1 \in \mathbb{Z}_p[x].
\]
Denote by $u$ the degree of $\psi(x)$. Otherwise, $u$ is equal to the order of the element $p$ in the group of units of the ring $\mathbb{Z}_q$. Remark that the lowest natural number $n$ such that $\psi(x)$ is a divisor of $x^n - 1$ in $\mathbb{Z}_p[x]$ is equal to $q$ ([9], Proposition 6). Denote by $\overline{\mathbb{Z}}_p[x]$ the factor-ring $\mathbb{Z}_p[x]/(\psi(x))$ of $\mathbb{Z}_p[x]$ over the principal ideal $(\psi(x))$, generated by $\psi(x)$. Proper endomorphisms of $G^*$ are given as pairs $[n; f(x)]$, where $n \in Z(q^v)$ and $f(x) \in \overline{\mathbb{Z}}_p[x]$. Two pairs $[n_1; f_1(x)]$ and $[n_2; f_2(x)]$ are equal if and only if 1) $n_1 = n_2 (= n)$; 2) $f_1(x) = f_2(x)$ or $q$ is a divisor of $n$. Automorphisms of $G^*$ are given as triplets $[n; a(x); b(x)]$, where $n \in Z(q^v)$; $a(x), b(x) \in \overline{\mathbb{Z}}_p[x]$; $b(x) \neq 0$ and $q$ is not a divisor of $n$. If two triplets are distinct, then the corresponding automorphisms of $G^*$ are distinct, too. The composition rule of elements of $\text{End}(G^*)$ is given by the following equalities:
\[
[n; f(x)] \cdot [m; g(x)] = [nm; g(x)],
\]
\[
[n; a(x); b(x)] \cdot [m; f(x)] = [nm; f(x)],
\]
\[
[m; f(x)] \cdot [n; a(x); b(x)] = [nm; a(x) \cdot (x - 1)^{-1} + b(x) \cdot f(x^n)].
\] (2.1)
3. Main theorems

Suppose that \( u \) is the degree of the polynomial \( \psi(x) \), mentioned in the previous section. Then \( u \) is equal to the order of the element \( p \) in the group of units of the ring \( \mathbb{Z}_q \). The following theorem gives a characterization of Schmidt groups by their endomorphism semigroups.

**Theorem 3.1** Let \( G \) be a finite group. Then \( G \) is isomorphic to a Schmidt group with parameters \( p, q \) and \( v \) if and only if there exists \( x \in I_0(G) \) such that

1. \( K_G(x) \cong \text{End}(C(q^v)) \);
2. \( H_G(x) = \{0\} \);
3. \( I_0(G) = [x] \), where \([x] = \{ y \in \text{End}(G) \mid y^2 = y, \ xy = y, \ yx = x \} \);
4. \( |I_0(G)| = p^v \);
5. \( \text{End}(G) \setminus \text{Aut}(G) = \cup_{y \in I_0(G)} K_G(y) \);
6. \( z \in \cap_{y \in I_0(G)} K_G(y) \) if and only if \( z^v = 0 \);
7. \( D_G(x) \) is a \( p' \)-subgroup of \( V_G(x) \);
8. each Sylow \( p \)-subgroup of \( V_G(x) \) is an elementary abelian group of the order \( p^v \).

**Proof of the necessity of Theorem 3.1.**

Assume that \( G \) is a Schmidt group with parameters \( p, q \) and \( v \). Then

\[
G = G' \ast < b >,
\]

where \( < b > \cong C(q^v) \). Denote by \( x \) the projection of \( G \) onto its subgroup \( < b > \). By lemma 2.1 and \( \text{Im} \ x = < b > \), property 1 holds.

Suppose that \( z \in H_G(x) \). Then \( xz = 0 \) and \( xz = z \). Hence, \( \text{Im} \ z \subset \text{Ker} \ x = G' \subset \text{Ker} \ z \) and \( bz \in G' \). Since \( b \) is a \( q \)-element and \( G' \) is a \( p \)-subgroup of \( G \), then \( bz = 1 \) and \( \text{Im} \ z = (G' \ast < b >)z = < 1 > \). Therefore, \( z = 0 \), \( H_G(x) = \{0\} \) and property 2 is true.

As the monomorphism \( * : \text{End}(G) \longrightarrow \text{End}(G^*) \), defined in the previous section, induces an isomorphism between the semigroups of all proper endomorphisms of \( G \) and \( G^* = G/G'' \), then it is sufficient to prove properties 3 to 6 for \( G = G^* \) and \( x = x^* \).

By (2.1), \([0; f(x)]\) is the zero of the semigroup \( \text{End}(G^*) \) and

\[
I_0(G^*) = \{ [1; f(x)] \mid f(x) \in \mathbb{Z}_p[x] \}. \tag{3.2}
\]
Since \([1; f(x)] \cdot [1; g(x)] = [1; g(x)]\), then \(I_0(G^*) = [x^*]\) and property \(3^0\) is true.

The idempotents \([1; f(x)]\) and \([1; g(x)]\) of \(\text{End}(G^*)\) are equal if and only if \(f(x) = g(x)\). Hence, \(|I_0(G^*)| = |\mathbb{Z}_p[x]| = p^n\) and property \(4^0\) holds.

It follows from (2.1) that
\[
[n; f(x)] \cdot [1; f(x)] = [1; f(x)] \cdot [n; f(x)] = [n; f(x)],
\]
i.e. \([n; f(x)] \in K_{G^*}([1; f(x)])\) and, therefore, property \(5^0\) is true.

Assume that
\[
[n; g(x)] \in \bigcap_{y \in I_0(G^*)} K_{G^*}(y).
\]
By (3.2), \([n; g(x)] \in K_{G^*}([1; f(x)])\) for each \(f(x) \in \mathbb{Z}_p[x]\), i.e.
\[
[n; g(x)] = [n; g(x)] = [1; f(x)] \cdot [n; g(x)] = [n; g(x)].
\]
Since the first product in the last equalities is equal to \([n; f(x)]\), then \([n; g(x)] = [n; f(x)]\) for each \(f(x) \in \mathbb{Z}_p[x]\). Hence, \([n; g(x)] = [n; 0]\) and \(q\) is a divisor of \(n\). Conversely, if \(q\) is a divisor of \(n\), then \([n; 0] \in \bigcap_{y \in I_0(G^*)} K_{G^*}(y)\).

Therefore,
\[
\bigcap_{y \in I_0(G^*)} K_{G^*}(y) = \{ [n; 0] \mid n \in q \cdot \mathbb{Z}_{q^*} \}.
\]
(3.3)

It follows from (3.3) that
\[
z = [n; f(x)] \in \bigcap_{y \in I_0(G^*)} K_{G^*}(y)
\]
if and only if \(z^n = [n^n; f(x)] = [0; f(x)]\). Property \(6^0\) is proved.

For the proof of properties \(7^0\) and \(8^0\) we find first \(|V_{G^*}(x^*)|\) and \(|D_{G^*}(x^*)|\).

Assume that \(z \in \text{Aut}(G^*)\). Then \(z = [n; a(x); b(x)]\) for some \(n \in Z(q^*); a(x), b(x) \in \mathbb{Z}_p[x]\), where \(b(x) \neq 0\) and \(q\) is not a divisor of \(n\). It was proved in [8] that \(x^* = [1; 0]\). The automorphism \(z\) of \(G^*\) belongs to \(V_{G^*}(x^*)\) if and only if \(z \cdot x^* = x^*\), i.e.,
\[
[n; a(x); b(x)] \cdot [1; 0] = [n; 0] = [1; 0].
\]
Hence, \(n = 1\) and
\[
V_{G^*}(x^*) = \{ [1; a(x); b(x)] \mid a(x), b(x) \in \mathbb{Z}_p[x]; b(x) \neq 0 \}.
\]
Since \(|\mathbb{Z}_p[x]| = p^n\), then
\[
|V_{G^*}(x^*)| = p^n(p^n - 1).
\]
(3.4)
Let us calculate $|D_{G^*}(x^*)|$. The group $D_{G^*}(x^*)$ consists of $z = [1; a(x); b(x)] \in V_{G^*}(x^*)$ such that $x^* \cdot z = x^*$, i.e.,

$$[1; 0][1; a(x); b(x)] = [1; \frac{a(x)}{x - 1}] = [1; 0].$$

Hence, $a(x) = 0$ and

$$|D_{G^*}(x^*)| = p^u - 1. \tag{3.5}$$

Clearly, $(D_{G^*}(x^*))^* \subset D_{G^*}(x^*)$, and, therefore, property $7^0$ is true.

As $G'' \subset Z(G)$, $Z(G/G'')$ is a $q$-group and $G'/G''$ is an elementary abelian $p$-group of the order $p^u$, then $G'' \subset G' \cap Z(G)$ and

$$Z(G/G'') \cap (G'/G'') = \{1\}. \tag{3.6}$$

If $g \in G' \cap Z(G)$, then $gG'' \in Z(G/G'') \cap (G'/G'')$ and, in view of (3.6), $gG'' = G''$ and $g \in G''$. Hence, $G' \cap Z(G) \subset G''$, $G'' = G' \cap Z(G)$ and

$$\hat{G}' = \{ \hat{g} \mid g \in G' \} \cong G'/G'' = G'/G''. \tag{3.7}$$

Therefore, $\hat{G}'$ and $(\hat{G}')^*$ are elementary abelian groups of the order $p^u$. By lemma 2.2, $\hat{G}' \subset V_G(x)$. Hence, $(\hat{G}')^* \subset V_{G^*}(x^*)$ and, by (3.4), $(\hat{G}')^*$ is a Sylow $p$-subgroup of $V_{G^*}(x^*)$. Then $\hat{G}'$ is a Sylow $p$-subgroup of $V_G(x)$ and property $8^0$ holds.

The necessity of theorem 3.1 is proved.

Assume now that $G$ is a finite group, $x \in I_0(G)$ and $x$ satisfies properties $1^0, 8^0$, formulated in theorem 3.1. Under these assumptions we shall prove the following lemmas.

**Lemma 3.1** Let $S$ be an arbitrary Sylow $p$-subgroup of $G$. Then there exists $b \in G$, satisfying the following conditions:

$$G = \ker x \times \text{Im} x, \quad \text{Im} x = <b \supset C(q^v), \tag{3.7}$$

$$I_0(G) = \{ x\hat{g} \mid g \in G \}, \tag{3.8}$$

$$I_0(G) = \{ x\hat{g} \mid g \in \ker x \}, \tag{3.9}$$

$$[\ker x : C_{\ker x}(b)] = p^u = [S : C_S(b)], \tag{3.10}$$

$$I_0(G) = \{ x\hat{s} \mid s \in S \}. \tag{3.11}$$
Proof. Since $x$ is an idempotent of $\text{End}(G)$, then $G = \text{Ker} \, x \triangleleft \text{Im} \, x$ and $K_G(x) \cong \text{End}(\text{Im} \, x)$. By $1^0$, $\text{End}(\text{Im} \, x) \cong \text{End}(C(q^n))$. As each finite abelian group is determined by its endomorphism semigroup in the class of all groups ([5], Theorem 4.2), then $\text{Im} \, x \cong C(q^n)$ and there exists $b \in G$ such that $\text{Im} \, x = \langle b \rangle : \cong C(q^n)$.

By property $2^0$, $\text{Ker} \, x$ is a $q'$-group. Indeed, otherwise there exists an element $g \in \text{Ker} \, x$ of the order $q$ and a non-zero $z \in H_G(x)$:

$$(\text{Ker} \, x)z = \{1\}, \, bz = g.$$ 

Therefore, $\langle b \rangle$ is a Sylow $q$-subgroup of $G$. Since $\text{Ker} \, x \lhd G$ and all Sylow subgroups related to a prime are conjugate, then all $q'$-elements belong to $\text{Ker} \, x$.

By property $3^0$, $I_0(G) = [x]$. Therefore, for the proof of (3.8) and (3.9) it is sufficient to prove the inclusions

$$[x] \subset \{x \hat{g} \mid g \in \text{Ker} \, x\} \text{ and } \{x \hat{g} \mid g \in G\} \subset I_0(G).$$

Suppose $h, g \in G$. As $\text{Im} \, x$ is abelian, then the commutator $[hx, g]$ belongs to $\text{Ker} \, x$ and

$$h(x \hat{g})^2 = (g^{-1} \cdot hx \cdot g)(x \hat{g}) = (hx \cdot [hx, g])(x \hat{g}) =$$

$$= (hx^2)\hat{g} = h(x \hat{g}),$$

i.e. $(x \hat{g})^2 = x \hat{g} \in I_0(G)$. Hence, $\{x \hat{g} \mid g \in G\} \subset I_0(G)$.

If $y \in [x]$, then $\text{Ker} \, x = \text{Ker} \, y$ and $\text{Im} \, x \cong \text{Im} \, y$. As $\text{Im} \, x$ is a Sylow $q$-subgroup of $G$, then so is $\text{Im} \, y$ and there exists $g \in \text{Ker} \, x$ such that $\text{Im} \, y = (\text{Im} \, x)\hat{g} = G(x \hat{g}) = \text{Im} \, (x \hat{g})$. Since $x \hat{g} \in I_0(G)$, $\text{Im} \, y = \text{Im} \, (x \hat{g})$ and $\text{Ker} \, y = \text{Ker} \, x = \text{Ker} \, (x \hat{g})$, then $y = x \hat{g}$. Consequently, $[x] \subset \{x \hat{g} \mid g \in \text{Ker} \, x\}$.

Equalities (3.8) and (3.9) are proved.

In view of (3.9), $\text{Im} \, x = \langle b \rangle$ and property $4^0$, the first equality of (3.10) follows. For the proof of the second equality of (3.10) remark that $S \subset \text{Ker} \, x$.

Clearly,

$$[S : C_S(b)] \leq [\text{Ker} \, x : C_{\text{Ker} \, x}(b)] = p^n.$$ 

Assume that $S_0$ is a Sylow $p$-subgroup of $C_{\text{Ker} \, x}(b)$ such that $C_S(b) \subset S_0$. Then $|C_{\text{Ker} \, x}(b)| = |S_0| \cdot t$, where $p \nmid t$. By the first equality of (3.10),

$$|\text{Ker} \, x| = p^n \cdot |C_{\text{Ker} \, x}(b)| = p^n \cdot |S_0| \cdot t$$

8
and, therefore, \( |S| = p^u \cdot |S_0| \). If \([S : C_S(b)] < p^u\), then \(|C_S(b)| > |S| : p^u = |S_0|\). This contradicts the inclusion \(C_S(b) \subset S_0\). Hence, \([S : C_S(b)] = p^u\) and equalities (3.10) hold.

Now equality (3.11) follows already from (3.9) and (3.10). The lemma is proved.

In the next part of this section it will be assumed that \(S\) is an arbitrary Sylow \(p\)-subgroup of \(G\) and \(\text{Im} \ x = < b >\).

**Lemma 3.2** The element \(b\) satisfies the following two properties

\[ C_S(b) \subset Z(G), \quad (3.12) \]

\[ b^{\theta} \in Z(G). \quad (3.13) \]

**Proof.** By lemma 2.3, \(\widehat{C_S(b)} = \{ \hat{s} \mid s \in C_S(b) \} \subset D_G(x)\). Clearly, \(\widehat{C_S(b)}\) is a \(p\)-group. On the other hand, by property 7°, \(\widehat{C_S(b)}\) is a \(p'\)-group. Hence, \(\widehat{C_S(b)} = \{1\}\) and \(C_S(b) \subset Z(G)\). Inclusion (3.12) is proved.

Choose \(g \in G\) and denote \(y = \hat{xg}\). By (3.8), \(y \in I_0(G)\). Define now a map \(z : G \rightarrow G\) as follows:

\[ (cd)z = c^{\theta}; \quad c \in \text{Im} \ y = < g^{-1}bg >, \quad d \in \text{Ker} \ y. \]

As \(G = \text{Ker} \ y \times \text{Im} \ y\), then \(z\) is defined everywhere on \(G\). It is easy to check that \(z\) is an endomorphism of \(G\) and \(z^v = 0\). By property 6°, \(z \in \cap_{u \in I_0(G)} K_G(u)\). Therefore, \(z \in K_G(x)\) and \(zx = xz\). In view of lemma 2.4, \((\text{Im} x)z \subset \text{Im} x = < b >\) and \(bz = b^r\) for some integer \(r\). Since the group \(\text{Im} z = < g^{-1}bg > \cong G/\text{Ker} z\) is abelian, then \(G' \subset \text{Ker} z\) and \([b, g]z = 1\). Hence,

\[ b^r = bz = (bz)[b, g]z = (b[b, g])z = (g^{-1}bg)z = g^{-1}b^{\theta}g = b^{\theta}[b^{\theta}, g] \]

and

\[ [b^{\theta}, g] = b^{r-q} \in \text{Im} x \cap G'. \]

As \(\text{Im} x \cong G/\text{Ker} x\) is abelian, then \(G' \subset \text{Ker} x\) and \([b^{\theta}, g] \in \text{Im} x \cap \text{Ker} x\).

By (3.7), \(\text{Im} x \cap \text{Ker} x = \{1\}\). Therefore, \([b^{\theta}, g] = 1\). Since \(g\) is an arbitrary element of \(G\), then \(b^{\theta} \in Z(G)\). Equality (3.13) is proved, and so is lemma 3.2.
Lemma 3.3 There exists a Sylow $p$-subgroup $S$ of $G$ such that $b \in N_G(S)$ and $<b, S> = S \ltimes <b>$. 

Proof. Suppose that $S_1$ is an arbitrary Sylow $p$-subgroup of $G$. Denote $N = N_G(S_1)$. By (3.13), $<b^q> \subset N$ and, hence, $q^v \mid |N|$. Assume that $q^v \mid |N|$. Then $<b^q>$ is a Sylow $q$-subgroup of $N$. Since $b^q \in Z(G)$, then $N = <b^q> \times N_1$, where $N_1$ is a Hall $q'$-subgroup of $N$. In view of (3.7), $N_1 \subset \text{Ker } x$ and, therefore, $N \subset <\text{Ker } x, b^q>$. Due to [11], Theorem 1.6.18, $<\text{Ker } x, b^q>$ coincides with its normalizer in $G$. This contradicts (3.7). Consequently, $q^v \mid |N|$. 

Since $q^v \mid |N|$ and $<b>$ is a Sylow $q$-subgroup of the order $q^v$ of $G$, then $N$ consists a Sylow $q$-subgroup of $G$ and there exists $g \in G$ such that $g^{-1} \cdot <b> \cdot g \subset N = N_G(S_1)$. Denote $S = gS_1g^{-1}$. Then $<b> \subset N_G(S)$ and $<b, S> = S \ltimes <b>$. The lemma is proved.

In the next part of this section it will be assumed that $S$ is a Sylow $p$-subgroup of $G$ such that $b \in N_G(S)$ and $<b, S> = S \ltimes <b>$. 

Lemma 3.4 The subgroup $S$ of $G$ splits up as follows

$$S = \{[b, s] \mid s \in S\} \cdot C_S(b).$$

(3.14)

Proof. If $s \in S$ then $b^{-1}s^{-1}b \in S$ and $[b, s] = b^{-1}s^{-1}b \cdot s \in S$. Assume that $[b, s_1]$ and $[b, s_2]$ belong to a common left coset of $C_S(b)$ in $S$ ($s_1, s_2 \in S$). Then $[b, s_1] = [b, s_2] \cdot c$ for some $c \in C_S(b) \subset Z(G)$ and

$$s_1^{-1}bs_1 = b \cdot [b, s_1] = b \cdot [b, s_2] \cdot c = s_2^{-1}bs_2 \cdot c.$$ 

As the order of $s_1^{-1}bs_1$ and $s_2^{-1}bs_2$ is $q^v$ and $c$ is a $p$-element which belongs to $Z(G)$, then $c = 1$, $[b, s_1] = [b, s_2]$, and $s_1^{-1}bs_1 = s_2^{-1}bs_2$, i.e., $s_1$ and $s_2$ belong to a common coset of $C_S(b)$ in $S$. Hence, different elements of the form $[b, s]$ ($s \in S$) belong to different left cosets of $C_S(b)$ in $S$ and the number of elements of this form is equal to the number of cosets of $C_S(b)$ in $S$. Consequently, equality (3.14) holds. The lemma is proved.

Lemma 3.5 The group $G$ is a semidirect product of its subgroups $S$ and $<b>$:

$$G = S \ltimes <b>.$$ 

(3.15)
Proof. First we shall prove that

\[ S \triangleleft < b > \triangleleft G. \]  

(3.16)

In view of (3.12) and (3.14), it is necessary to show that \( g^{-1}bg, g^{-1}[b, s]g \in < b, S > \) for each \( g \in G, s \in S \). By (3.8) and (3.11), \( x\hat{g} = x\hat{s}_1 \) and \( x\hat{s}_g = x\hat{s}_2 \) for some \( s_1, s_2 \in S \). On the other hand, \( bx = b \). Hence,

\[
g^{-1}bg = b\hat{g} = b(x\hat{g}) = b(x\hat{s}_1) = s_1^{-1}bs_1 \in < b, S >, \]

\[
bs\hat{g} = b(x\hat{s}_g) = b(x\hat{s}_2) = b\hat{s}_2, \]

\[
g^{-1}[b, s]g = g^{-1}b^{-1}s^{-1}bsg = g^{-1}b^{-1}g(b\hat{s}_g) = \]

\[
= (g^{-1}bg)^{-1} \cdot (b\hat{s}_2) = (g^{-1}bg)^{-1}(s_2^{-1}bs_2) \in < b, S >. \]

Equality (3.16) is proved.

Let us prove now (3.15). By (3.16) and the theorem of Schur and Zassenhaus ([11], Theorem 9.1.2) there exists a \( \{ p, q \}' \)-subgroup \( C \) of \( G \) such that \( G = (S \triangleleft < b >) \triangleleft C \). Let \( y \) be the projection of \( G \) onto \( C \). Clearly, \( y \neq 1 \). If \( y \neq 0 \), then \( y \in I_0(G) \) and, by (3.8), \( y = x\hat{g} \) for some \( g \in G \), i.e. \( \text{Im} y = C = \text{Im} (x\hat{g}) \cong \text{Im} x \cong C(q^v) \). This contradicts to the fact that \( C \) is a \( \{ p, q \}' \)-group. Therefore, \( y = 0 \) and \( G = \text{Ker} y = S \triangleleft < b > \). The lemma is proved.

Lemma 3.6 Let \( p \) and \( q \) be different primes and \( P \) be finite group such that

a) \( P \) is non-abelian;

b) \( P = H \triangleleft < d > \), where \( H \) is a \( p \)-group and \( < d > \cong C(q^v) \) \( (v \geq 1) \);

c) \( d^a \in Z(P) \);

d) \( H \) is abelian;

e) if \( h \in H \setminus < 1 > \), then \( P = < h, d > \).

Then \( P \) is a group of Miller and Moreno with parameters \( p, q \) and \( v \). If \( P \) satisfies instead of properties a), d) and e) the properties

a') \( P \) is non-nilpotent,

d') \( C_H(d) \subset Z(P) \),

e') if \( h \in H \setminus C_H(d) \), then \( P = < h, d > \),

then \( P \) is a Schmidt group with parameters \( p, q \) and \( v \).
Proof. Assume that $P$ satisfies properties a)-e) (the first case) or a'), b), c), d'), e') (the second case). Since different Sylow $q$-subgroups of $P$ are conjugate and each $q$-element of $P$ belongs to some Sylow $q$-subgroup of $P$, then all $q$-elements of $P$ have a form $g^{-1}d^ig \in \{g \in P, \ i \in Z(q^v)\}.$

Suppose that $N$ is a proper subgroup of $P$. Assume first that $N$ does not contain $q$-elements of the order $q^v$. By c), all $q$-elements of $N$ have a form $g^{-1}d^iq = d^iq$ and, therefore, $N < H, d^i = H \times d^i$. Hence, in the first case $N$ is abelian and in the second case $N$ is nilpotent.

Assume now that $N$ contains a $q$-element of the order $q^v$, i.e., $g^{-1}d^ig \in N$, where $g \in P, \ i \in Z(q^v)$ and $q \nmid i$. Then $g^{-1}dg \in N$ and $d \in gNg^{-1}$.

Let us consider the first case. If $H \cap gNg^{-1} \neq 1$, then, by e), $gNg^{-1} = P, \ N = g^{-1}Pg = P$. It is impossible, because $N$ is a proper subgroup of $P$. Therefore, $H \cap gNg^{-1} = 1$. In view of $d \in gNg^{-1}$ and b), $gNg^{-1} = < d >$, i.e. $N$ is abelian.

Let us now check the second case. It was showed that $d \in gNg^{-1}$. Since all $p$-elements of $P$ belong to $H$, then $H \cap gNg^{-1}$ is a Sylow $p$-subgroup of $gNg^{-1}$. Assume that $H \cap gNg^{-1} \subset C_H(d)$ and choose $s \in (H \cap gNg^{-1}) \setminus C_H(d)$. By e'), $P = < s, d >$. On the other hand, $< s, d > \subset gNg^{-1}$. Hence, $P = gNg^{-1}$ and $N = g^{-1}Pg = P$. It is impossible, because $N$ is a proper subgroup of $P$. Therefore, $H \cap gNg^{-1} \subset C_H(d)$. By d'), $gNg^{-1} = (H \cap gNg^{-1}) \times < d >$ and $gNg^{-1}$ is nilpotent. Hence, $N$ is nilpotent, too.

We have proved that in the first case all proper subgroups of $P$ are abelian and in the second case all proper subgroups of $P$ are nilpotent. Consequently, in the first case $P$ is a group of Miller and Moreno and in the second case $P$ is a Schmidt group. The lemma is proved.

Lemma 3.7 Denote $\overline{G} = G/C_S(b), \overline{S} = S/C_S(b), \bar{g} = g \cdot C_S(b) \ (g \in G)$. Then
1) $\overline{S}$ is a non-trivial elementary abelian $p$-group;
2) $\overline{G} = \overline{S} \times < \bar{b} >$;
3) $G$ does not contain normal subgroups of the index $p$;
4) $\bar{b}$ induces a non-trivial inner automorphism on $\overline{S}$;
5) if $s \in \overline{S} \setminus < 1 >$, then $\overline{G} = < \overline{s}, \overline{b} >$.

Proof. In view of lemma 3.2, $C_S(b) \subset Z(G)$. Therefore, $S \cap Z(G) = C_S(b)$ and $\hat{S} \cong (S \cap Z(G)) = \overline{S}$. By lemma 2.2, $\hat{S} \subset V_G(x)$. Hence, by $8^0, \overline{S}$ is an elementary abelian $p$-group. Suppose $S = C_S(b)$. Then equality (3.15)
implies $G = S \times < b >$, i.e., $G$ is abelian. Therefore, $x\hat{g} = x$ for each $g \in G$ and, by (3.8), $I_0(G) = \{x\}$. This contradicts property 4). Consequently, $S \neq C_S(b)$ and $\overline{S} \neq < 1 >$. Statement 1) of the lemma is proved.

Statement 2) of the lemma follows immediately from equality (3.15).

Let us prove now statement 3). By contradiction, assume that $M$ is a normal subgroup of the index $p$ of $G$, i.e. $G/M \cong C(p)$. Choose a generator $gM$ of $G/M$ and an element $h \in G$ of the order $p$. Then there exists $\tau = \pi \rho \in \text{End}(G)$, where $\pi : G \rightarrow G/M$ is the natural homomorphism and $\rho : G/M \rightarrow < h >$, $(gM)\rho = h$, is the isomorphism. By construction, $\tau$ is a non-zero proper endomorphism of $G$ and $\text{Im} \tau \cong C(p)$. In view of property 5), $\tau \in K_G(y)$ for some $y \in I_0(G)$. Hence, $\tau y = y\tau = \tau$ and $\text{Im} \tau \subset \text{Im} y$. By (3.8), $y = x\hat{g}_1$ for some $g_1 \in G$. Therefore,

$$C(p) \cong \text{Im} \tau \subset \text{Im} y = \text{Im}(x\hat{g}_1) = (\text{Im} x)\hat{g}_1 \cong \text{Im} x \cong C(q^v).$$

It is impossible. Consequently, $G$ does not contain normal subgroups of the index $p$. Statement 3) is proved.

For the proof of statement 4) remark that $S$ is a normal subgroup of $G$ and $C_S(b)$ is invariant with respect to the automorphism $\hat{b}$. Hence, it is correct to construct the automorphism $\mu$ of $\overline{S}$ as follows

$$\bar{s}\mu = \overline{b^{-1}sb} = \overline{sb}, \quad s \in S.$$  

Since $\bar{s}\mu^q = \overline{b^{-q}sb^q} = \bar{s}$, then $\mu^q = 1$ and the order of $\mu$ is $q$ or $\mu = 1$. If $\mu = 1$, then, by 2), $G = \overline{S} \times < \hat{b} >$ and there exists a normal subgroup $M$ of $G$ such that $G/M \cong C(p)$ (we took into account that $\overline{S}$ is a non-trivial elementary abelian $p$-group). This contradicts to statement 3). Consequently, the order of $\mu$ is $q$, and statement 4) is proved.

Let us prove now statement 5). The group $\overline{S}$ can be expanded into the direct product

$$\overline{S} = \overline{S}_1 \times \cdots \times \overline{S}_k,$$

where $\overline{S}_1, \ldots, \overline{S}_k$ are minimal non-trivial subgroups of $\overline{S}$ which are invariant with respect to the automorphism $\mu$ which was defined above ([2], Theorem 3.3.2). Clearly, $< \overline{S}_i, \hat{b} > = \overline{S}_i \times < \hat{b} >$ for each $i \in \{1, \ldots, k\}$. Since $\mu$ is a non-trivial automorphism of $\overline{S}$ then there exists $i$ such that $\mu$ acts on $\overline{S}_i$ non-identically. We can assume that $i = 1$. Denote $P = < \overline{S}_1, \hat{b} >$. By construction and lemma 3.2, the group $P$ satisfies properties a)-d) of lemma
3.6 (take there \( H = S_1 \) and \( d = \bar{b} \)). Since \( S_1 \) is a minimal non-trivial subgroup of \( S \) which is invariant with respect to \( \mu \), then \( P \) satisfies also property e) of lemma 3.6. In view of lemma 3.6, \( S_1 \triangleleft < \bar{b} > \) is a group of Miller and Moreno with parameters \( p, q \) and \( v \). By the characterization of groups of Miller and Moreno with parameters \( p, q \) and \( v \) ([9]), \( |S_1| = p^u \). As \( p^u = |S_1| = |S_1| : |C_S(b)| = [S_1 : C_S(b)] \), then, by (3.10), \( S = S_1, \bar{S} = S_1, k = 1 \) and \( \bar{G} = \bar{S} \triangleleft < \bar{b} > = S_1 \triangleleft < \bar{b} > \). Hence, \( \bar{G} \) can be generated by \( \bar{b} \) and an arbitrary non-trivial element \( \bar{s} \in \bar{S} \). Statement 5) is proved. The lemma is proved.

**Proof of the sufficiency of theorem 3.1.**

Let \( G \) be a finite group. Assume that \( x \in I_0(G) \) and \( x \) satisfies properties \( 1^0 - 8^0 \) of Theorem 3.1. By these assumptions, lemmas 3.1-3.5 and 3.7 hold. Let us preserve the notations of these lemmas. We will prove that \( G \) is a Schmidt group with parameters \( p, q \) and \( v \). For this aim we shall show that \( G \) satisfies statements of lemma 3.6 (take there \( P = G, H = S, d = b \)).

In view of lemmas 3.1, 3.2 and 3.5, statements b), c) and d') of lemma 3.6 hold. Statement a') of lemma 3.6 is also true. Indeed, otherwise \( G \) is the direct product of its Sylow subgroups and, therefore, \( \bar{G} = \bar{S} \times < \bar{b} > \). This contradicts statements 4) of lemma 3.7.

For the proof of property e') choose \( s \in S \setminus C_S(b) \). Then \( \bar{s} = s \cdot C_S(b) \) is a non-trivial element of \( S \) and, by property 5) of lemma 3.7, \( \bar{G} = \langle \bar{s}, \bar{b} \rangle \) and \( G = \langle b, s, C_S(b) \rangle \). Since \( C_S(b) \subset Z(G) \), then it follows from here that \( H = \langle b, s \rangle \) is a normal subgroup of \( G \). As \( S \) is a \( p \)-group, then \( G/H \) is a \( p \)-group, too. If \( H \neq G \), then \( G/H \) is non-trivial and there exists a normal subgroup \( M \) of \( G \) such that \( H \subset M \) and \( G/M \cong C(p) \). This contradicts property 3) of lemma 3.7. Consequently, \( G = H \) and \( G \) satisfies property e') of lemma 3.6.

We have proved that \( G \) satisfies properties a'), b), c), d') and e') of lemma 3.6. By this lemma, \( G \) is a Schmidt group with parameters \( p, q \) and \( v \). The sufficiency of theorem 3.1 is proved.

Theorem 3.1 is proved.

**Theorem 3.2** Let \( G \) be a Schmidt group with parameters \( p, q, v \) and \( H \) be an arbitrary group such that the semigroups \( \text{End}(G) \) and \( \text{End}(H) \) are isomorphic. Then the group \( H \) is a Schmidt group with parameters \( p, q \) and \( v \).
Proof. Let $G$ and $H$ be groups as assumed in the theorem. Since $G$ is finite, then so is $H$ ([1], Theorem 2). By assumption, there exists an isomorphism $T : \text{End}(G) \rightarrow \text{End}(H)$. In view of theorem 3.1, there exists $x \in I_0(G)$ which satisfies properties $1^0 - 8^0$. By isomorphism $T$, the idempotent $xT \in I_0(H)$ satisfies similar properties of this theorem. Applying theorem 3.1 for $H$, it follows that $H$ is a Schmidt group with parameters $p$, $q$ and $v$. The theorem is proved.

Let $u$ be the order of $p$ in the group of units of residue-class ring $\mathbb{Z}_q$ modulo $q$. In [10], Proposition 3, it was proved that all Schmidt groups with parameters $p$, $q$ and $v$ are isomorphic if and only if $u$ is odd. In [8], Theorem 4.4, it was proved that the endomorphism semigroup of the group of Miller and Moreno with parameters $p$, $q$ and $v$ is isomorphic to the endomorphism semigroup of the Schmidt group of the maximal order with same parameters. Therefore, by theorem 3.2, the following theorem is true.

**Theorem 3.3** The group of Miller and Moreno with parameters $p$, $q$ and $v$ is determined by its endomorphism semigroup in the class of all groups if and only if the order of $p$ in the group of units of the residue-class ring $\mathbb{Z}_q$ is odd.

Acknowledgment. This work was supported in part by the Estonian Science Foundation Research Grant 4291, 2000.

References

[1] Alperin, J.L. *Groups with finitely many automorphisms*. Pacific J. Math., 1962, 12, No 1, 1-5.

[2] Gorenstein, D. *Finite groups*. Harper and Row, New York, 1968.

[3] Leptin, H. *Abelsche $p$-Gruppen und ihre Automorphismengruppen*. Math. Z., 1960, 73, 235-253.

[4] Liebert, W. *Isomorphic automorphism groups of primary abelian groups*. Abelian group theory (Oberwolfach, 1985)(R.Göbel and E.A.Walker, eds.), Gordon and Breach, New York, 1987, 9-31.

[5] Puusemp, P. *Idempotents of the endomorphism semigroups of groups*. Acta et Comment. Univ. Tartuensis, 1975, 366, 76-104 (in Russian).
[6] Puusemp, P. A characterization of divisible and torsion Abelian groups by their endomorphism semigroups. Algebras, Groups and Geometries, 1999, 16, 183-193.

[7] Puusemp, P. Endomorphism semigroups of generalized quaternion groups. Acta et Comment. Univ. Tartuensis, 1976, 390, 84-103 (in Russian).

[8] Puusemp, P. Endomorphism semigroups of Schmidt groups. Algebras, Groups and Geometries, 2001, 18, 139-152.

[9] Redei, L. Das "schiefe Produkt" in der Gruppentheorie mit Anwendungen auf die endlichen nichtkommutativen Gruppen mit lauter kommutativen echten Untergruppen und die Ordnungszahlen, zu denen nur kommutative Gruppen gehören. Comm. Math. Helv., 1947, 20, 225-264.

[10] Redei, L. Die endlichen einstufig nicht-nilpotenten Gruppen. Publ. Math., 1956, 4, 303-324.

[11] Robinson, D.J.S. A course in the Theory of Groups. Graduate Texts in Mathematics 80. Springer-Verlag, 1996.

[12] Schmidt, O.J. Über Gruppen, deren sämtliche Teiler spezielle Gruppen sind. Rec. Math. Moscow, 1924, 31, 366-372.

[13] Schultz, P. Automorphisms which determine an abelian p-group. Abelian groups, module theory, and topology: proceedings in honour of Adalberto Orsatti's 60th birthday/ edited by Dikran Dikranjan, Luigi Salce, (Lecture Notes in Pure and Applied Mathematics, v. 201), Marcel Dekker, inc., New York-Basel-Hong Kong, 1998, 373-379.