ASYMPTOTIC CRITICAL VALUE SET
AND
NEWTON POLYHEDRON

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Abstract - We present an effective method to investigate the asymptotic critical value set of a polynomial map. For this purpose we propose a method to construct rational curves with reduced number of terms present in its parametric representation. In this way we show that the asymptotic critical value set contains the critical value of a polynomial associated to so-called bad face of the Newton polyhedron. Our main technical tool is the toric geometry that has been introduced into this question by A. Némethi and A. Zaharia.

1. Introduction

The bifurcation locus of a polynomial map $f : \mathbb{C}^n \to \mathbb{C}$ is the smallest subset $\mathcal{B}(f) \subset \mathbb{C}$ such that $f$ is a locally trivial $C^\infty$-fibration over $\mathbb{C} \setminus \mathcal{B}(f)$. It is known that $\mathcal{B}(f)$ is the union of the set of critical values $f(\text{Sing} f)$ and the set of bifurcation values at infinity $\mathcal{B}(f)$ which may be non-empty and disjoint from $f(\text{Sing} f)$ even in very simple examples. Finding the bifurcation locus in the cases $n > 2$ is a difficult task and it still remains to be an unreached ideal. Nevertheless one can obtain approximations by supersets of $\mathcal{B}(f)$ from exploiting asymptotic regularity conditions.

Jelonek and Kurdyka [8], [9] established an algorithm for finding the set of asymptotic critical values $\mathcal{K}_\infty(f)$. It is known that in this case $\mathcal{K}_\infty(f)$ is finite and includes $\mathcal{B}(f)$. Under the condition that the projective closure of the generic fibre of $f$ in $\mathbb{P}^n$ has only isolated singularities, Parusiński [12] proved that $\mathcal{B}(f) = \mathcal{K}_\infty(f) \cup f(\text{Sing} f)$. A precedent work [4] established a method to detect the bifurcation set in an efficient way. It gave an answer to a question raised in [9] and [5] about the detection of the bifurcation locus by rational curve with parametric representation.

AMS Subject Classification: 14Q20, 58K05, 32S05. Key words and phrases: regularity at infinity, critical values, Newton polyhedron

Author has been partially supported by Max Planck Institut für Mathematik, CNRS-TÜBİTAK bilateral project 113F007 "Topologie des singularités de la surface complexe," Université Lille 1, TÜBİTAK 1001 Grant No. 116F130 "Period integrals associated to algebraic varieties."
More concretely, for a real polynomial $f: \mathbb{R}^n \to \mathbb{R}$ of degree $\leq d$, authors of [4] consider a real rational curve $X(t), \lim_{t \to 0} \|X(t)\| \to \infty$, with parametric representation with length $(d + 1)d^{n-1} + 1$ to attain the critical value $\lim_{t \to 0} f(X(t)) \in \mathcal{K}_\infty(f)$.

In our present note we propose a method to construct a rational curve with drastically reduced number of terms present in its parametric representation (Theorem 3.1). Further in this article we shall use the terminology "parametric length of a curve" to denote this number. Thus in our Examples 5.1, 5.2 the parametric length of a real rational curve has been reduced to 4 in comparison with 3601 proposed in [4]. In Example 5.3, case ii) the parametric length has been reduced to a number less than $2 \text{deg } f_W$ in comparison with $(1 + 5\text{deg } f_W)(5\text{deg } f_W)^2 + 1$ proposed in [4]. This means that our method requires number of procedures only linearly dependent on $\text{deg } f_W$, while the precedent method required number of procedures of growth rate proportional to $(\text{deg } f_W)^3$.

Starting from Lemma 3.1 we impose a natural condition $(\mu)$ on the toric data $W$ (2.2) and $f_W$ (2.5). We follow [10], [14] as for the use of toric geometry in the investigation of the asymptotically non-regular values of $f$. Our main Theorem 3.1 states the inclusion of critical values of certain polynomial $f_\gamma^W$, with possibly non-isolated singularities, constructed on a "bad face" $\gamma$ of the Newton polyhedron of $f$ with $(\mu)$ into $\mathcal{K}_\infty(f)$. Thus Corollary 3.1 establishes an inclusion relation

$$\bigcup_{\gamma: \text{bad faces}} f_\gamma (\text{Sing } f_\gamma \cap (\mathbb{C}^*)^{\dim \gamma}) \subset \mathcal{K}_\infty(f)$$

For the case where $f$ is non-degenerate at infinity this gives an approximation of $\mathcal{K}_\infty(f)$ together with a result of [2] that determines a superset of $\mathcal{K}_\infty(f)$.

In [13] it is shown that the left hand side set of the above inclusion relation is contained in the bifurcation set $\mathcal{B}(f)$ for $\gamma$ relatively simple bad face (see Definition 4.1) and $f_\gamma^W$ with isolated singularities on $(\mathbb{C}^*)^{\dim \gamma}$. As it is known $\mathcal{B}(f) \subset \mathcal{K}_\infty(f)$ from [8], our Corollary 3.1 represents a new result only for $\gamma$ non-relatively simple bad face if the condition of the isolated singularities at infinity is assumed.

In Section 4 we examine an example of a polynomial in 5 variables with non-relatively simple bad face. Even in this situation we can construct a curve approaching to an asymptotic critical value of $f$. This gives an example to Corollary 3.1 that is not covered by [13]. As [13] imposes the condition of isolated singularity at infinity, it does not concern our Example 5.1 treating the non-isolated singularities.

Our method heavily relies on various kinds of Newton polyhedra constructed in two different chart systems. The core technique is explained in Proposition 3.1 where the key data like the integer vector $q \in \mathbb{Z}^n$ and the integer $\rho > 0$ are introduced. The vector $q \in \mathbb{Z}^n$ is used to calculate the number $L_0$ (3.10) that determines the parametric length together with $\rho$ (3.2).

Finally we remark that efficiency to detect asymptotically non-regular values can be applied to optimisation problems e.g. see [7]. We recall that [1] had recourse to effective use of Newton polyhedra in the investigation of the order of coercitivity of a polynomial $f$. Thus we hope that our approach represents not only purely theoretical interests, but also certain utility in the optimisation problem.
The author expresses gratefulness to Mihai Tibăr for having drawn his attention to the question of asymptotic critical values of a polynomial map and for useful discussions. He thanks Kiyoshi Takeuchi for comments and remarks. He thanks Abuzer Gündüz for having furnished the article with Figures in Section 5 and Example 5.3.

2. Approach with unimodular subdivision of the dual cone

To fix notations and fundamental notions, we follow [10], [14]. Let us consider a polynomial

\[ f(x) = \sum_{\alpha} a_{\alpha} x^{\alpha} \]

with \( f(0) = 0 \) where the multi-index \( \alpha \) runs within the set of integer points \( supp(f) = \{ \alpha \in (\mathbb{Z}_{\geq 0})^{n}; a_{\alpha} \neq 0 \} \). We introduce a convex polyhedron of finite volume \( \Delta(f) \) defined as the convex hull of \( supp(f) \) in \( \mathbb{R}^n \) that is assumed to be of the maximal dimension i.e. \( dim \Delta(f) = n \). We denote the convex hull of \( supp(f) \cup \{0\} \) in \( \mathbb{R}^n \) by \( \tilde{\Delta}_-(f) \).

**Definition 2.1.** For \( a \in (\mathbb{R}^n)^* \), \( \Delta^a \) the face of \( \tilde{\Delta}_-(f) \) such that \( (a,y) \leq (a,x) \) for every pair \( x \in \tilde{\Delta}_-(f) \) and \( y \in \Delta^a \). For a face \( \gamma \subset \Lambda(f) \) of the Newton polyhedron of \( f \) (2.1) we define \( f_{\gamma}(x) = \sum_{\alpha \in \gamma} a_{\alpha} x^{\alpha} \)

**Definition 2.2.** For a set \( \Lambda \in (\mathbb{R}_{\geq 0})^n \) we denote by \( C(\Lambda) = \{ tv; t \in \mathbb{R}_{\geq 0}, v \in \Lambda \} \) the cone with the base \( \Lambda \).

**Definition 2.3.** Let \( K \) be an unimodular simplicial subdivision of \( (\tilde{\Delta}_-(f))^* \) where \( (\tilde{\Delta}_-(f))^* \) is the dual to \( \tilde{\Delta}_-(f) \).

\[ (\tilde{\Delta}_-(f))^* = \{ a \in (\mathbb{R}^n)^*; (a,x) \geq 0, \forall x \in \tilde{\Delta}_-(f) \} \]

\[ = \{ a \in (\mathbb{R}^n)^*; (a,x) \geq 0, \forall x \in C(\tilde{\Delta}_-(f)) \} \]

**Definition 2.4.** ([10]) We call a face \( \gamma \subset \Delta(f) \) bad if it satisfies the following two properties.

(i) The affine subspace of dimension \( = dim \gamma \) spanned by \( \gamma \) contains the origin.

(ii) (\( \pm \) condition for the bad face) There exists an hyperplane \( H \subset \mathbb{R}^n \gamma = H \cap \Delta(f) \) defined by an equation \( \sum_{j=1}^{n} p_j x_j = 0 \) where there exists \( i \neq j \) satisfying \( p_i p_j < 0 \).

If \( a_1, \ldots, a_k \) is an unimodular basis of a \( k \)-dimensional cone \( \sigma \in K \) i.e. \( \sigma = \Sigma_{i=1}^{k} t_i a_i, t_i \geq 0 \), we can choose \( m_1, \ldots, m_n \in \mathbb{R}^n \) a basis of the dual cone \( \sigma^* = \{ x \in \mathbb{R}^n; (x,a) \geq 0, \forall a \in \sigma \} \) such that \( (a_i, m_j) = \delta_{ij}, i \in [1;k], j \in [1;n] \) where \( \delta_{ij} \) is Kronecker Delta. From here on we shall further use the notation \( i \in [r_1;r_2] \Leftrightarrow i \in \{r_1, \ldots, r_2\} \) for two integers \( r_1 < r_2 \). We can further extend the basis \( a_1, \ldots, a_k \) to an \( n \)-dimensional basis \( a_1, \ldots, a_n \) with the aid of supplementary vectors \( a_{k+1}, \ldots, a_n \) in such a way that \( |det(a_1, \ldots, a_n)| = 1 \)

\[ \sigma^* = \{ \sum_{i=1}^{n} \lambda_i m_i; \lambda_1, \ldots, \lambda_k \geq 0 \} \] and \( V_{\sigma^*} = \{ \lambda_{k+1} m_{k+1} + \ldots + \lambda_n m_n, \lambda_j \in \mathbb{R}, j = k+1, \ldots, n \} \).
Assume that $\gamma$ is a bad face and a $n-$dimensional cone $\sigma \in K \subset (\tilde{\Gamma}(-f))^* \subset (\mathbb{R}^n)^*$ satisfying

(A)

$$\gamma \subset \sigma^* = \{ x \in \mathbb{R}^n; \langle \alpha, x \rangle \geq 0, \forall \alpha \in \sigma \}$$

with a basis $(a_1, \ldots, a_k)$ such that

(B)

$$\gamma = \{ v \in \Delta(f); \langle a_i, v \rangle = 0, i = 1, \ldots, k \}.$$ 

Such a basis exists by virtue of Definition 2.4 (ii).

**Definition 2.5.** Let $\sigma \in K$ be a unimodular simplicial cone with $\dim(\sigma) = k$. We denote $\tilde{\sigma} = \cup_{\sigma' \subset \sigma} \sigma'$ all cones $\sigma' \subset \sigma$.

For $\sigma$ an algebraic torus of dimension $n-k$ is defined as follows $\Phi[\sigma] = (\mathbb{C}^*)^n / \{ (t^{b_1}, \ldots, t^{b_n}); t \in (\mathbb{C}^*)^k, (b_1, \ldots, b_n) \in \sigma \}$ We also consider a disjoint union of tori defined by $M_\sigma = \cup_{\sigma' \subset \sigma} \Phi[\sigma'] \cong \mathbb{C}^k \times (\mathbb{C}^*)^{n-k} \ni (u_1, \ldots, u_k, u_{k+1}, \ldots, u_n)$

**Definition 2.6.** For an integer vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ we denote by $\alpha' = (\alpha_1, \ldots, \alpha_k) \in \mathbb{Z}^k$ and $\alpha'' = (\alpha_{k+1}, \ldots, \alpha_n) \in \mathbb{Z}^{n-k}$ its respective components. In a parallel way we introduce two sets of variables $u' \in \mathbb{C}^k$ (called affine) and $u'' \in (\mathbb{C}^*)^{n-k}$ (called toric), $u = (u', u'') \in \mathbb{C}^n$ where

$$\mathbb{C}_n^k = \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}.$$ 

We introduce the following unimodular matrices $M$ and $W$ with integer entries (i.e. complementary vectors $a_{k+1}, \ldots, a_n \in \sigma^\perp$) associated to a cone $\sigma \in K$.

$$W = (a_1^T, \ldots, a_n^T) = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}, W^{-1} = M = \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{pmatrix}$$

where $(m_1, \ldots, m_n)$ basis of $\sigma^*$ and $\sigma^* = \Sigma_{i=1}^k \mathbb{R}_{\geq 0} m_i + \Sigma_{j=k+1}^n \mathbb{R} m_j$. Especially we shall take the cone $\sigma$ so that $\{ m_1, \ldots, m_n \} \subset (\mathbb{R}^*)^n$. Further we use the following notation also (see Lemma 3.1);

$$M^T = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{pmatrix}.$$

Under the change of variables

$$x_1, \ldots, x_n = (u^{w_1}, \ldots, u^{w_n})$$

we consider

$$f^W(u) = \sum_{\alpha \in \text{supp}(f)} a_{\alpha} u^{\alpha \cdot W},$$
where \( W = M^{-1} \). For \( \alpha \in \mathbb{Z}^n \) we represent \( \alpha.W \in \mathbb{Z}^n \) the integer vector with the aid of its components
\[
\alpha.W = (\lambda_1(\alpha), \cdots, \lambda_n(\alpha)).
\]

Due to the choice of the basis \( a_1, \cdots, a_k \) and the definition of the cone \( \sigma^* \) we have the following.

**Lemma 2.1.** For \( v \in \Delta(f) \) general (not necessarily located on the bad face \( \gamma \))
\[
\lambda_1(v), \ldots, \lambda_k(v) \geq 0
\]

Thus only \( \lambda_{k+1}(v), \ldots, \lambda_n(v) \) may be negative for \( v \in \Delta(f) \) in general.

For \( v \in \gamma \) bad face, by the choice of \( \sigma \), we have \( \lambda_j(v) = \langle a_j, v \rangle \geq 0 \) for \( \forall j = 1, \ldots, n, \forall v \in \gamma \). Because of (A) \( f_\gamma^W(0, u'') \) is a polynomial in \( u'' \) variables. In other words \( \lambda_1(v) = \langle a_1, v \rangle = 0, \ldots, \lambda_k(v) = \langle a_k, v \rangle = 0, \lambda_{k+1}(v) = \langle a_{k+1}, v \rangle \geq 0, \ldots, \lambda_n(v) = \langle a_n, v \rangle \geq 0, \forall v \in \gamma \).

The expression \( f^W(u) \) is a Laurent polynomial with possibly negative power exponents in toric variables \( u'' \), but restricted to affine variables \( u' \) it is a polynomial in \( u' = (u_1, \ldots, u_k) \). We denote
\[
\theta_u f^W(u) = (\theta_{u_1} f^W(u), \ldots, \theta_{u_k} f^W(u)),
\]
with \( \theta_{u_j} = u_j \frac{\partial}{\partial u_j}, j \in [1; n] \). For a critical point \( u^* = (0, u'') \in \mathbb{C}_k^n \) such that \( \theta_u f^W(u^*) = 0 \), we introduce the notation \( u' = (u_1, \cdots, u_k), U'' = (U_{k+1}, \cdots, U_n) = (u_{k+1} - u_{k+1}', \cdots, u_n - u_n') \) and consider a local expansion of the Laurent polynomial \( f^W(u) \) at \( u = u^* = (0, u'') \in \mathbb{C}_k^n \) as follows
\[
(2.6) \quad f^W(u) = \sum_{\beta \in \text{supp}_{u^*}(f^W)} a_\beta^*(u - u^*)^\beta = \sum_{\beta \in \text{supp}_{u^*}(f^W)} a_\beta^* u^\beta U'^\beta',
\]
for \( \text{supp}_{u^*}(f^W) := \{ \beta \in \mathbb{Z}^n; a_\beta^* \neq 0 \} \). Here the expression corresponding to the term \( \alpha.W \in (\mathbb{Z}_{\geq 0}^k \setminus \{0\}) \times \mathbb{Z}_{\geq 0}^{n-k} \) in (2.5) shall produce series in (2.6) with \( (\beta', \beta'') \in (\mathbb{Z}_{\geq 0}^k \setminus \{0\}) \times (\mathbb{Z}_{\geq 0}^{n-k}) \) according to the rule
\[
(2.7) \quad \frac{1}{u_j} = \frac{1}{u_j^*} \sum_{\ell \geq 0} (\frac{U_j}{u_j^*})^\ell.
\]

**Definition 2.7.** We consider a convex polyhedron \( \Delta_{u^*}(f^W) = \text{convex hull of } \text{supp}_{u^*}(f^W) \) that is a \( n \)-dimensional polyhedron due to the condition \( \dim \Delta(f) = n \). For every facet \( (= n - 1 \text{ dimensional face}) \Gamma \) of \( \Delta_{u^*}(f^W) \) we can find an integer vector \( q \in \mathbb{Z}^n \) and an integer \( r \) such that \( \langle q, \alpha \rangle \geq r, \forall \alpha \in \Delta_{u^*}(f^W), \) and \( \langle q, \beta \rangle = r, \forall \beta \in \Gamma \). Here the components of \( q \in \mathbb{Z}^n \) can be chosen coprime.

For the above mentioned cone \( \sigma \) consider the decomposition.

\[
(2.8) \quad f^W(u) = \tilde{f}^W(u) + R(u),
\]
with
\[ f^W(u) = \sum_{\alpha \in \text{supp}(f) \cap (\mathbb{Z}_{\geq 0})^n} a_\alpha u^{\alpha W} \]

while \( R(u) \) corresponds to terms with exponent vectors \( \alpha W \notin (\mathbb{Z}_{\geq 0})^n \) such that negative powers appear i.e. some of \( \lambda_{k+1}(\alpha), \ldots, \lambda_n(\alpha) \) are strictly negative and \( \lambda_1(\alpha), \ldots, \lambda_k(\alpha) \geq 0 \). We shall note that if some of \( \lambda_{k+j}(\alpha) \) is strictly negative for \( \alpha \in \Delta(f) \) then \( \lambda_i(\alpha) \) for some \( i \in [1; k] \) must be strictly positive. If all \( \lambda_i(\alpha) = 0 \) for all \( i \in [1; k] \) (we remark that for every \( \alpha \in \Delta(f) \), \( \lambda_i(\alpha) \geq 0 \) for all \( i \in [1; n] \)) it means that \( \alpha \) is located on the bad face \( \gamma \), thus \( \lambda_{k+j}(\alpha) \geq 0 \) for all \( j \in [1; n-k] \). In other words

**Lemma 2.2.** The Laurent polynomial \( f^W(0, u'') = f^W(u) = \sum_{\alpha \in \gamma} a_\alpha u^{\alpha W} \) is a polynomial (with positive power terms) in \( u'' \) variables.

For the bad face with \( \text{codim} \gamma \leq k \) in \( \Delta(f) \) one can find at least \( k \) different points in \( (\sigma^* \setminus V_{\sigma^*}) \cap \text{supp}(f) \) and thus one can choose at least \( k \) linearly independent points \( \alpha = \sum_{i=1}^k \lambda_i m_i \in (\sigma^* \setminus V_{\sigma^*}) \cap \text{supp}(f) \) with \( \lambda_i \geq 0 \) for \( i \in [1; k] \) that satisfies \( \alpha W \in (\mathbb{Z}_{\geq 0}^k \setminus \{0\}) \times \{0\} \). There may be contributions from the expansion at \( u = u^* \) of the rational function \( R(u) \) obtained after the principle (2.7), but this situation does not influence on the existence of \( k \) linearly independent points in \( \text{supp}_{u^*}(f^W) \cap \mathbb{R}^k \). The fact that \( f^W(0, u'') = \sum_{\alpha \in \gamma} a_\alpha u^{\alpha W} \) is a polynomial depending on all \( u'' \) variables on \( \mathbb{C}^{n-k} \) and the bad face \( \gamma \) contains at least \( n-k+1 \) points that span a \((n-k)\) dimensional linear subspace of \( \mathbb{R}^n \) yields that

\[ \dim \left( \Delta_{u^*}(f^W) \cap \{0, \alpha''; \alpha'' \in (\mathbb{R}_{\geq 0})^{n-k}\} \right) \geq n-k-1. \]

We remark here that if \( f^W(u) = \tilde{f}^W(u) \) the equality \( \Delta_{u^*}(f^W)|_{\mathbb{R}^k} = \Delta(f^W)|_{\mathbb{R}^k} \) holds.

### 3. Curve Construction by Means of Newton Polyhedron

This section is the core part of this note. First of all we introduce a polyhedron \( \Delta^* \) defined as a convex hull of \( \bigcup_{i=1}^n \Delta_{\text{u}^*}(\langle \mu_i, \partial_u f^W(u) \rangle) \). Here the polyhedron \( \Delta_{u^*}(\langle \mu_i, \partial_u f^W(u) \rangle) \) is defined as a convex hull of \( \text{supp}_{u^*}(\langle \mu_i, \partial_u f^W(u) \rangle) \) obtained after the expansion as in (2.6).

**Proposition 3.1.** Assume that \( \partial_u f^W(u^*) = \partial_u f^W(0, u'') = 0 \). There is a facet \( \Gamma \) of the polyhedron \( \Delta^* \) satisfying \( \dim(\Gamma \cap \mathbb{R}^{n-k}) = n-k-1 \) defined by a vector \( q \in \mathbb{Z}^n \) such that

\[ \Gamma = \{ \beta \in \Delta^*; \langle \beta, q \rangle \leq \langle \tilde{\beta}, q \rangle \text{ for every } \tilde{\beta} \in \Delta^* \}. \]

In other words \( \forall \beta \in \Delta^* \), the inequality \( \langle \beta, q \rangle \leq \langle \tilde{\beta}, q \rangle \) holds with every \( \tilde{\beta} \in \Delta_{u^*}(\langle \mu_i, \partial_u f^W(u) \rangle) \), \( i \in [1; n] \). We shall further denote by \( \rho \) the following integer

\[ \rho = \min_{\bar{\alpha} \in \Delta^*} \langle \bar{\alpha}, q \rangle \]

that is equal to \( \langle \alpha, q \rangle \) for \( \alpha \in \Gamma \).
Proof. (a) First we show the existence of a facet \( \tilde{\Gamma} \) of \( \Delta_\alpha(\langle \mu_j, \partial_u f^W(u) \rangle) \) for (2.8) satisfying \( \dim (\tilde{\Gamma} \cap \mathbb{R}^{n-k}) = n - k - 1 \) and \( \dim (\tilde{\Gamma} \cap \mathbb{R}^k) = k - 1 \) for a fixed \( j \). This follows from (2.9) and the fact that \( \alpha.W \in (\mathbb{Z}_{\geq 0}^k \setminus \{0\}) \times \{0\} \) for at least \( k \) linearly independent points satisfy \( \alpha \in (\sigma^* \setminus V_\alpha) \cap \text{supp}(f) \) as it has been remarked just after Lemma 2.2.

(b) For a polynomial with support located in \( \Delta_\alpha(\langle f^W \rangle) \) we calculate

\[
\langle \mu_j, \partial_u f^W(u) \rangle = \sum_{\beta} a_{\beta}^* \langle \mu_j, \partial_u \rangle \left( u^{\beta'} U^{n\beta''} \right)
\]

with

\[
\langle \mu_j, \partial_u \rangle \left( u^{\beta'} U^{n\beta''} \right) = \left( \langle \mu_j, (\beta', \beta'') \rangle \right) u^{\beta'} U^{n\beta''} + \sum_{\ell=k+1}^{n} \frac{\mu_{j,\ell} \beta_{j,\ell} u^\ell}{U^\ell} u^{\beta'} U^{n\beta''}
\]

(c) The condition \( |\beta''| \geq 2 \) for \( \beta' = 0 \) in the expression (3.3) follows from the fact that \( \partial_u f^W(u^*) = 0 \) at the point \( u^* = (0, u''^*) \in \mathbb{C}_k^n \).

(d) Next we see that for \( \beta = ( (\mathbb{Z}_{\geq 0})^k \setminus \{0\} ) \times (\mathbb{Z}_{\geq 0}^{n-k} ) \) the convex hull of \( \beta \) and \( \Delta_\alpha(\langle \mu_j, \partial \rangle f^W(u)) \) has a facet \( \tilde{\Gamma} \) such that \( \tilde{\Gamma} \cap \mathbb{R}^{n-k} = \tilde{\Gamma} \cap \mathbb{R}^{n-k} \). This is due to the fact if some of \( \lambda_{k+j}(\alpha) \) is strictly negative for \( \alpha \in \Delta(f) \) then \( \lambda_i(\alpha) \) for some \( i \in [1; k] \) must be strictly positive as it has been remarked just before lemma 2.2.

(e) As it has been shown in (2.6), (2.7) the exponent of terms present in the expansion \( R(u) = \sum_{\beta} c_{\beta} u^{\beta'} U^{n\beta''} \) satisfy \( (\beta', \beta'') \in ( (\mathbb{Z}_{\geq 0})^k \setminus \{0\} ) \times (\mathbb{Z}_{\geq 0})^{n-k} \). There are only finite number of power indices \( (\beta', \beta'') \) in the convex hull of \( \{0\} \) and \( \tilde{\Gamma} \) that may cause correction to the facet \( \tilde{\Gamma} \) as we draw the convex polyhedron \( \Delta_\alpha(\langle \mu_j, \partial_u f^W \rangle) \). After finitely many repetitive application of the argument (d) to \( \tilde{\Gamma}, \tilde{\Gamma}' \) etc. we find a facet (3.1) defined for \( q = (q', q'') \in \mathbb{Z}^n \) that satisfies \( \Gamma \cap \mathbb{R}^{n-k} \supset \tilde{\Gamma} \cap \mathbb{R}^{n-k} \).

See Figure 5.1 where the facet \( \Gamma \) is illustrated for the Example 5.1.

We consider the curve

\[
Q(t) = (u'(t), u''(t)) = (c' t^{q'} + \text{h.o.t.}, u'' + c'' t^{q''} + \text{h.o.t.})
\]

where \( q = (q', q'') \) found in Proposition 3.1 and \( u'' \neq 0 \), as \( u'' \in (\mathbb{C}^*)^{n-k} \).

Here \( c' t^{q'} = (c'_1 t^{q'_1}, \ldots, c'_k t^{q'_k}) \) etc.

**Definition 3.1.** ([8], [9]) Consider a curve \( x = X(t) \) that satisfies the following two conditions

\[
\lim_{t \to 0} ||X(t)|| = \infty.
\]

\[
\lim_{t \to 0} x_i \frac{\partial f(X(t))}{\partial x_j} \to 0 \text{ for every pair } i, j \in [1; n]^2.
\]

We call the value \( \lim_{t \to 0} f(X(t)) \) asymptotic critical value of \( f \). We denote by \( K_\infty(f) \) the set of asymptotic critical values of \( f \).
The existence of the value $\text{Proof.}$ By (3.4) and of the polynomial $\text{Compare with [6, 2.3] Claim 1, Claim 2, Exercise.}$ for every $\text{Lemma 3.1. For } q = (q', q'') \in \mathbb{Z}^n \text{ found in Proposition 3.1 the following equivalence holds.} (i) \exists w_i \text{ such that } \langle (q', 0), w_i \rangle < 0 \iff (ii) (q', 0) \notin \sum_{j=1}^{n} \mathbb{R}_{\geq 0} \mu_j. \text{ We call this condition } (\mu).$

**Proof.** $(i) \Rightarrow (ii).$ We show the contraposition. For the vector $r = \sum_{j=1}^{n} t_j \mu_j, t_j \geq 0$ for every $j \in [1; n], \langle r, w_i \rangle = t_i \geq 0$ for every $i.$

$(ii) \Rightarrow (i).$ Also by contraposition. Take $r = \sum_{j=1}^{n} s_j \mu_j \in \mathbb{R}^n$ such that $\langle r, w_i \rangle = s_i \geq 0$ for every $i.$ As $r = (q', 0) \neq (0, 0)$ not every $s_i$ equals to zero, thus $s_j > 0$ for some $j.$ Compare with [6, 2.3] Claim 1, Claim 2, Exercise.

**Lemma 3.2.** The integer $\rho$ (3.2) is strictly positive for $q$ determined for a facet $\Gamma$ constructed in Proposition 3.1.

**Proof.** By Definition 2.7 $\forall \beta \in \Delta^* (\langle \mu_j, f^W \rangle)$ there exists $\tilde{\alpha} \in \{ \langle q, \cdot \rangle = \rho \}$ such that $\beta = t\tilde{\alpha}$ for $t \geq 1.$ The number $\rho$ was defined as the minimal value of the linear function $\langle q, \cdot \rangle$ on $\Delta^*$ and $\langle q, \beta \rangle = t\rho \geq \rho$ thus $\rho$ must be strictly positive.

Let us denote by $X(t)$ the image of the curve $Q(t)$ defined in (3.4) by the map (2.4).

**Lemma 3.3.** The two condition $(\mu)$ of the lemma 3.1 is sufficient so that there exist a curve $\|X(t)\| \to \infty$ with finite limit $\lim_{t \to 0} f(X(t)) = \lim_{t \to 0} f^W(Q(t)).$ The equality $\lim_{t \to 0} \partial_u f^W(Q(t)) = 0$ holds and the limit $\lim_{t \to 0} f^W(Q(t))$ corresponds to a critical value of the polynomial $f^W(u).

**Proof.** By (3.4) and $x_i = u^{w_i},$ we have

$$x_i(t) = c_i t^{\langle (q', 0), w_i \rangle} (1 + h.o.t.).$$

The existence of the value $\lim_{t \to 0} f(X(t)) = \lim_{t \to 0} f^W(Q(t))$ is clear from the definition of the curve (3.4).

By means of the vectors introduced in Lemma 3.1 $(\mu),$ we deduce the following relation

$$(3.7) \begin{pmatrix} \partial_{x_1} f(x) \\ \partial_{x_2} f(x) \\ \vdots \\ \partial_{x_n} f(x) \end{pmatrix} = M^T \begin{pmatrix} \partial_{u_1} f^W(u) \\ \partial_{u_2} f^W(u) \\ \vdots \\ \partial_{u_n} f^W(u) \end{pmatrix}.$$
Let \( \vec{b} = (b_1, \ldots, b_n) \in (\mathbb{R}^*)^n \) be a vector in general position with non-zero components and denote \( \langle \vec{b}, w \rangle = \sum_{j=1}^n b_j w_j \). Then we have

\[
\langle \vec{b}, x \rangle \begin{pmatrix}
\partial_{x_1} f(x) \\
\partial_{x_2} f(x) \\
\vdots \\
\partial_{x_n} f(x)
\end{pmatrix} = \langle \vec{b}, w \rangle \begin{pmatrix}
\frac{\mu_1}{w_1} \\
\frac{\mu_2}{w_2} \\
\vdots \\
\frac{\mu_n}{w_n}
\end{pmatrix} \begin{pmatrix}
\partial_{u_1} f^W(u) \\
\partial_{u_2} f^W(u) \\
\vdots \\
\partial_{u_n} f^W(u)
\end{pmatrix}.
\]

From this equality we see that it is enough to look for a curve \( Q(t) \) given by (3.4) such that

\[
\min_{i \neq j} \langle (q', 0), w_i - w_j \rangle + \text{ord} \left( \langle \mu_j, \partial_{u} f^W \rangle (Q(t)) \right) > 0
\]

for every \( j \in [1; n] \) so that to ensure the condition (3.6). In fact a linear combination of LHS of (3.8) for various vectors \( \vec{b} \) will produce all \( n \times n \) functions present in (3.6).

We define also

\[
L_0 = \max_{i \neq j} \langle (q', 0), w_i - w_j \rangle.
\]

**Definition 3.2.** We shall use the set of indices \( \mathbb{J} \subset [1; n] \) defined by

\[
\mathbb{J} = \{ j \in [1; n] ; \min_{i \neq j} \langle (q', 0), w_i - w_j \rangle < 0 \}.
\]

The cardinality of \( \mathbb{J} \) is at most \( n - 1 \).

To formulate the main theorem of this section, we introduce a coordinate system on the (arc) space of rationally parametrised curves of the form (3.4),

\[
Q(t) = (u'(t), u''(t)) = (c'(0)t^q + c'(1)t^{q+1} + h.o.t., u'' + c''(0)t^{q''} + c''(1)t^{q''+1} + h.o.t.)
\]

where \( q = (q', q'') \in \mathbb{Z}^n \) with coprime elements characterised in Proposition 3.1 and Lemma 3.2.

Here we take into account finite number of coefficients \( c'(j) = (c_1(j), \ldots, c_k(j)) \in \mathbb{C}^k \), \( c''(j) = (c_{k+1}(j), \ldots, c_n(j)) \in \mathbb{C}^{n-k}, j \in \mathbb{Z}_{\geq 0} \). We denote the space of coefficients \( \mathbb{C} \) in such a way that \( c = (c', c'') \in \mathbb{C}, c' = (c'(0), c'(1), c'(2), \ldots), c'' = (c''(0), c''(1), c''(2), \ldots) \).

The following theorem tells us that every critical value of the polynomial

\[
f^W_\gamma(u) = \sum_{\alpha \in \gamma \cap \text{supp}(f)} a_\alpha u^{\alpha W}
\]

with \( \gamma \) bad face is an asymptotic critical value under certain conditions (Lemma 3.1). It is worthy noticing that the singular points of \( f_\gamma(x) \) can be non-isolated and no restriction is assumed on the dimension of the bad face \( \gamma \) in question.

**Theorem 3.1.** Let \( f \in \mathbb{C}[x_1, \ldots, x_n] \) be a polynomial whose Newton polyhedron \( \Delta(f) \) has maximal dimension \( n \). Assume that \( \gamma \) is one of its bad faces like in Definition 2.4.

(i) Under the conditions (\( \mu \)) of Lemma 3.1, we can find a curve \( X(t) \) satisfying (3.5), (3.6) of Definition 3.1 such that \( \lim_{t \to 0} f(X(t)) \) equals to a critical value of the polynomial.
f^W(u). (ii) This curve is obtained as a image by the map (2.4) of a curve Q(t) whose coefficients c ∈ C satisfy (L_0 − ρ + 1) | J | − tuple of algebraic equations for L_0 (3.10). (iii) The curve Q(t) mentioned in (ii) has a parametric representation (3.11) of parametric length L_0 − ρ + 2 for ρ > 0 defined in Lemma 3.2 i.e. we can assume its parametrisation coefficients (c′(j), c″(j)) = 0 for j > L_0 − ρ + 1.

Proof. We have already shown in Lemma 3.3 that the curve under question satisfies (3.5) of Definition 3.1.

Now we need to show that there is a curve (3.11) satisfying (3.6). For this purpose we look for a curve that makes the inequality (3.9) valid. Lemma 3.2 tells us that it is enough to verify (3.9) for j ∈ J as ord (⟨μ_j, ϑ_a f^W⟩ (Q(t))) ≥ ρ > 0.

The expansion of ⟨μ_j, ϑ_a f^W⟩ (Q(t)), j ∈ J in t has the following form
\[ g_j^j(c)t^ρ + g_{ρ+1}^j(c)t^{ρ+1} + h.o.t. \]

For each j ∈ J the vector with polynomial entries g_j^j(c) depends on all n variables (c′(0), c″(0)) ∈ C^n ⊂ C in view of the choice of q ∈ Z^n made in Proposition 3.1.

As | J | < n the system of algebraic equations g_j^j(c) = 0, ∀ j ∈ J has non-trivial solutions in C while g_j^j(c) effectively depends on (c′(0), c″(0)).

The vector with polynomial entries g_{ρ+1}^j(c) effectively depends on (c′(0), c″(0), c′(1), c″(1)) ∈ C^{2n} ⊂ C thus the system of equations g_{ρ+1}^j(c) = 0, ∀ j ∈ J has also non-trivial solutions in C.

In this way we can find non-trivial solutions to (L_0 + 1 − ρ) | J | − tuple of algebraic equations
\[ g_j^j(c) = g_{ρ+1}^j(c) = \cdots = g_{L_0}^j(c) = 0, ∀ j ∈ J \]
for L_0 (3.10).

To prove this it is enough to show that g_{ρ+ℓ}^j(c) effectively depends on (c′(ℓ), c″(ℓ)) that are absent in g_j^j(c) for ℓ ∈ [0; ℓ − 1].

First we remark that g_{ρ+ℓ}^j(c) is a sum of monomials of the form
\[ \text{const.} \prod_{ν=1}^{n} \prod_{i_ν ∈ I_ν} c_ν(i_ν)^{m_ν,μ} \]
satisfying the following homogeneity condition
\[ ρ + ℓ = \sum_{ν=1}^{n} \sum_{i_ν ∈ I_ν} (q_ν + i_ν)m_{i_ν,μ} \]
with I_ν ⊂ [0, ℓ], m_{i_ν,μ} ≥ 0, ∀ i_ν ∈ I_ν, ∀ ν ∈ [1; n].

By a simple calculation we see that non vanishing terms of the following form appear in g_{ρ+ℓ}^j(c), ℓ ≥ 1:
\[ \text{const.} \left( \prod_{ν=1, ν ≠ κ}^{k} \prod_{i_ν ∈ I_ν} c_ν(i_ν)^{m_ν,μ} \right) \left( \prod_{i_κ ∈ I_κ \setminus \{ℓ\}} c_κ(i_κ)^{m_κ,κ} \right) c_κ(ℓ) \]
for \( \kappa \in [1; k] \) and

\[
\tag{3.15} \text{const.} \left( \prod_{\nu=\nu_k+1, \nu \in \mathcal{R}_{\kappa}} c_\nu(i_\nu, \kappa) \right) \left( \prod_{i_\kappa \in \mathcal{R}_{\kappa \setminus \{\ell\}}} c_\kappa(i_\kappa, \kappa) \right) c_\kappa(\ell)
\]

for \( \kappa \in [k + 1; n] \). Non vanishing of (3.14) with \( \kappa \in [1; k] \) is due to presence of a term proportional to \( u^{\alpha'}(t) \) such that \( \langle q, (\alpha', 0) \rangle = \rho \) in \( \langle \mu_j, \vartheta_u f_T(u) \rangle \). That of (3.15) with \( \kappa \in [k + 1; n] \) is due to presence of a term proportional to \( U^{\alpha''}(t) \) such that \( \langle q, (0, \alpha'') \rangle = \rho \) in \( \langle \mu_j, \vartheta_u f_T(u) \rangle \) and \( u'' \neq 0 \). These originating monomials \( u^{\alpha'}(t), U^{\alpha''}(t) \) can be uniquely determined from power exponents \( \{m_{i_{\nu, \kappa}}\}_{i_\nu \in \mathcal{R}_{\nu}} \) that can be seen from (3.12), (3.13):

\[
\alpha_\kappa = 1 + \sum_{i_\kappa \in \mathcal{R}_{\kappa}} m_{i_{\nu, \kappa}} \text{ for } \kappa \in [1; k]; \ (\text{resp. } [k + 1; n]),
\]

\[
\alpha_\nu = \sum_{i_\nu \in \mathcal{R}_{\nu}} m_{i_{\nu, \kappa}} \text{ for } \nu \in [1; k] \setminus \{\kappa\} \ (\text{resp. } [k + 1; n] \setminus \{\kappa\}).
\]

Thus no cancellation of terms (3.14), (3.15) happens. As \( \langle \mu_j, \vartheta_u f_T(u) \rangle \), \( j \in \mathcal{J} \) contains monomials \( u^{\alpha'}(t), U^{\alpha''}(t) \) of the above type, the factor \( c_\kappa(\ell), \kappa \in [1; n] \) appears in \( g_0^j(\kappa) \) but it does not appear in \( g_0^j(\kappa), \ell \in [0; \ell - 1] \) because of (3.13). \( \square \)

**Corollary 3.1.** Under assumptions of Theorem 3.1 the following inclusion holds

\[
\tag{3.16} \bigcup_{\gamma} f_\gamma(\text{Sing } f_\gamma \cap (\mathbb{C}^*)^{\dim \gamma}) \subset K_\infty(f),
\]

where \( \gamma \) runs among bad faces of \( \Delta(f) \) for which a cone \( \sigma \) satisfying condition (\( \mu \)) of Lemma 3.1 can be constructed.

**Proof.** Theorem 3.1 tells us

\[
f_\gamma^W(\text{Sing } f_\gamma^W \cap (\mathbb{C}^*)^{\dim \gamma}) \subset K_\infty(f).
\]

It is enough to show that

\[
f_\gamma^W(\text{Sing } f_\gamma^W \cap (\mathbb{C}^*)^{\dim \gamma}) = f_\gamma(\text{Sing } f_\gamma \cap (\mathbb{C}^*)^n)
\]

for \( f_\gamma(x) = \sum_{\alpha \in \gamma \cap \text{supp}(f)} a_\alpha x^\alpha \).

From Lemma 2.2, \( f_\gamma^W(u) \) is a polynomial depending effectively on toric variables \( u'' \) and independent of affine variables \( u' \) (the condition (i) of the Definition 2.4). This means that \( \vartheta_{u_1} f_\gamma^W(u) = \cdots = \vartheta_{u_k} f_\gamma^W(u) = 0 \). Thus for \( u'' \in \text{Sing } f_\gamma^W \cap (\mathbb{C}^*)^{\dim \gamma} \) the vanishing of the logarithmic gradient vector holds: \( \vartheta_{u''} f_\gamma^W(0, u'') = 0 \). By using the map \( u''(x) = (x^{m_{k+1}}, \ldots, x^{m_n}) \) induced by the inverse to (2.4) we see \( f_\gamma(x) = f_\gamma^W(0, u''(x)) \). Taking the relation (3.7) into account, we see that this entails \( \vartheta_x f_\gamma(x) = 0 \) for \( x \in (\mathbb{C}^*)^n \) that satisfies \( u''(x) = u'' \).

Conversely if \( \vartheta_x f_\gamma(x) = 0 \) for \( x \in (\mathbb{C}^*)^n \), by (3.7) \( \vartheta_u f_\gamma^W(0, u'') = 0 \) for \( u'' = u''(x) \) the image of the map (2.4). \( \square \)
In [2] Theorem 1.1, for \( f \) non-degenerate at infinity it is stated that
\[
\mathcal{K}_\infty(f) \subset \{0\} \cup \bigcup_\Delta f_\Delta(\text{Sing } f_\Delta \cap (\mathbb{C}^*)^\dim \Delta),
\]
where the union runs over all "atypical faces" of \( f \) (faces that satisfy our Definition 2.4 (ii)). In general the set \( \bigcup_\Delta f_\Delta(\text{Sing } f_\Delta \cap (\mathbb{C}^*)^\dim \Delta) \) is larger than \( \bigcup_\gamma f_\gamma(\text{Sing } f_\gamma \cap (\mathbb{C}^*)^\dim \Delta) \) as a bad face \( \gamma \) requires conditions (i) and (ii) of Definition 2.4.

In [13] it is shown that the left hand side set of (3.16) is included in the bifurcation set \( \mathcal{B}(f) \) for \( \gamma \) relatively simple bad face (see Section 4). As it is known \( \mathcal{B}(f) \subset \mathcal{K}_\infty(f) \) from [8], our Corollary 3.1 represents a new result for \( \gamma \) non relatively simple bad face if the condition of the isolated singularities at infinity is assumed.

Parusiński [12] established the equality
\[
\mathcal{K}_\infty(f) \cup f(\text{Sing } f) = \mathcal{B}(f)
\]
for the case where the projective closure in \( \mathbb{P}^n \) of the generic fibre of \( f \) has only isolated singularities on the hyperplane at infinity \( H_\infty \subset \mathbb{P}^n \). His main concern was to look at the case with the Newton polyhedron \( \Delta(f_d) \) with full dimension (= \( n - 1 \)) as the polynomial is decomposed into homogeneous terms \( f(x) = \sum_{j=0}^d f_j(x), \deg f_j = j \). In this setting we see

**Corollary 3.2.** Let \( f \) be a polynomial such that the projective closure in \( \mathbb{P}^n \) of the generic fibre of \( f \) have only isolated singularities on the hyperplane at infinity \( H_\infty \subset \mathbb{P}^n \). Assume the conditions imposed in Corollary 3.1. Then we have
\[
\bigcup_\gamma f_\gamma(\text{Sing } f_\gamma \cap (\mathbb{C}^*)^\dim \gamma) \subset \mathcal{B}(f).
\]

4. Non relatively simple face

In [13] the notion of relatively simple face has been introduced.

**Definition 4.1.** ([13, Definition 1.4]) A face \( \gamma \subset \tilde{\Gamma} - (f) \cap \Delta(f) \) is called relatively simple if \( C(\gamma)^* \) that is in the dual fan of \( \tilde{\Gamma} - (f) \) is simplicial or \( \dim C(\gamma)^* \leq 3 \).

The main result Theorem 1.6. of [13] relies heavily on the notion of relatively simple faces. It shows that the set \( \bigcup_\gamma f_\gamma(\text{Sing } f_\gamma \cap (\mathbb{C}^*)^\dim \gamma) \) where \( \gamma \) runs relatively simple bad faces is contained in the bifurcation value set of a polynomial mapping \( f \) under the condition of non-degeneracy and isolated singularities at infinity. We say that \( f \) has isolated singularities at infinity over \( b \in \bigcup_\gamma f_\gamma(\text{Sing } f_\gamma \cap (\mathbb{C}^*)^\dim \gamma) \) if \( (f_\gamma^W)^{-1}(b) \cap (\mathbb{C}^*)^\dim \gamma \) has only isolated singular points for every Laurent polynomial \( f_\gamma^W \) (2.5) constructed on a corresponding bad face \( \gamma \). It is clear that this definition does not depend on the choice of the unimodular matrix \( W \) (2.2) due to the argument in the proof of Corollary 3.1.

In this section we examine an example of a polynomial in 5 variables with non-relatively simple bad face (see (4.2)). Even in this situation we can construct a curve \( X(t) \) satisfying...
newly constructed cone any more). We have again $v_B$ \((1\) \((4.1)\)

$$F_{i,j,k} := C \left( \sum_{s_i+s_j+s_k=1} s_i \bar{v}_i + s_j \bar{v}_j + s_k \bar{v}_k \right)$$

where \(\{i,j,k\} = \{1,2,3,4\} \setminus \{\ell\}\) for \(\ell \neq i,j,k\), is defined as a subset of a plane \(\{v \in \mathbb{R}^4; \langle A_{i,j,k},v \rangle = 0\}\). The orthogonal vector to each of the faces is given by: \(A_{1,2,3} = (-2,0,-4,11)\), \(A_{1,3,4} = (-2,0,1,1)\), \(A_{1,2,4} = (1,-5,2,2)\), \(A_{2,3,4} = (1,1,0,-2)\). We choose the direction of the orthogonal vector in such a way that \(\langle A_{i,j,k},\bar{v}_\ell \rangle > 0\) for every quadruple indices \(\{i,j,k,\ell\} = \{1,2,3,4\}\).

We shall construct a non-simplicial cone by means of an additional cone \(C(\bar{v}_5)\) that will be built with the aid of the vector \(A_{2,3,4}\). Namely we choose \(\bar{v}_5 = \bar{v}_2 + \bar{v}_3 + \bar{v}_4 - A_{2,3,4} = (4,4,10,7)\).

We shall convince ourselves that the new non-simplicial cone generated by five 1 dimensional cones \(C(\bar{v}_1), C(\bar{v}_2), C(\bar{v}_3), C(\bar{v}_4), C(\bar{v}_5)\) is a convex cone with six faces. Here we recall the Definition 2.2. In fact, we calculate the orthogonal vector to each of faces \(F_{i,j,k}\) defined in a manner similar to (4.1) for \(\{i,j,k\} = \{1,2,3,4,5\} \setminus \{\ell, p\}\) such that \(\{i,j,k,\ell,p\} = \{1,2,3,4,5\}\). \(A_{2,3,5} = (17,29,-24,8)\), \(A_{2,4,5} = (2,-1,1,-2)\), \(A_{3,4,5} = (-1,3,2,-4)\) in addition to \(A_{1,2,3}, A_{1,3,4}, A_{1,2,4}\) already known \((F_{2,3,4} \text{ is not a face of the newly constructed cone any more})\). We have again \(\langle A_{i,j,k},\bar{v}_\ell \rangle > 0\), \(\langle A_{i,j,k},\bar{v}_p \rangle > 0\) for every quintuple indices \(\{i,j,k,\ell,p\}\) as above and see thus the newly constructed cone is convex.

2) We consider a shift of the apex of the cone towards a vector \(\bar{v}_0 = (1,2,3,1)\). We denote the face of the shifted cone \(B_{i,j,k} = \bar{v}_0 + F_{i,j,k}\), \(\{i,j,k\} = \{1,2,3\}, \{1,3,4\}, \{1,2,4\}, \{2,3,5\}, \{2,4,5\}, \{3,4,5\}\). The face \(B_{i,j,k}\) is a subset of a plane \(\{v \in \mathbb{R}^4; \langle A_{i,j,k},v \rangle = \langle A_{i,j,k},\bar{v}_0 \rangle \}\). In "homogenizing" the defining equation of a plane containing \(B_{i,j,k}\) we get a plane in \(\mathbb{R}^5\): \(H_{i,j,k} = \{(x,y,z,w,r) \in \mathbb{R}^5; \langle A_{i,j,k},(x,y,z,w) \rangle = \langle A_{i,j,k},\bar{v}_0 \rangle r \}\). In this way we get six planes in \(\mathbb{R}^5\) passing through the origin and the intersection

$$\bar{C} = \cap_{\{i,j,k\}} \{(x,y,z,w,r) \in \mathbb{R}^5; \langle A_{i,j,k},(x,y,z,w) \rangle - \langle A_{i,j,k},\bar{v}_0 \rangle r \geq 0 \}$$

produces a convex cone. By construction every plane \(H_{i,j,k}\) contains a 1 dimensional cone \(C(\bar{v}_0)\) and \(\bar{C} \supset C(\bar{v}_0)\) where \(\bar{v}_0 = (\bar{v}_0,1)\).

If we use the choice done in 1) and \(\bar{v}_0 = (1,2,3,1,1)\), the vectors \(L_{i,j,k}\) orthogonal to the planes \(H_{i,j,k}\) are given by \(L_{1,2,3} = (-2,0,-4,11,3)\), \(L_{1,3,4} = (-2,0,1,1,-2)\), \(L_{1,2,4} = (1,-5,2,2,1)\), \(L_{2,3,5} = (17,29,-24,8,-11)\), \(L_{2,4,5} = (2,-1,1,-2,1)\), \(L_{3,4,5} = (-1,3,2,-4,-7)\). We define vectors \(v_i = (\bar{v}_i,0), i \in [1,4]\) in \(\mathbb{R}^5\). The vector \(L_{i,j,k} \in (\mathbb{R}^5)^*\) is orthogonal to \(v_i, v_j, v_k\) in addition to \(v_0\) in general. We shall check that \(\langle L_{i,j,k}, v_\ell \rangle \geq 0\) for every \(v_\ell\),
ℓ ∈ [0, 5]. Except 6 triples shown above this positivity property is not satisfied for other triples from \{1, 2, 3, 4, 5\}.

The following polynomial
\[
(4.2) \quad f = -3x^0 + x^3v_0 + x^1x^4 + x^2x^v_0 + x^1x^4 + x^3x^v_0 + x^4x^v_0 + x^5x^v_0
\]
has a 1-dimensional bad face contained in \(C(v_0)\) that is not relatively simple in the sense of Definition 4.1. In fact the cone \(\Gamma_{\min} = (\mathbb{R}^5)^*\) of the dual fan \((\Gamma)\) corresponding to the cone \(C(v_0)\) is a 4 dimensional cone with 6 generators \(L_{i,j,k}\) calculated above. In the sequel we shall show the inclusion
\[
(4.3) \quad \{\pm 2\} \subset \mathcal{K}_\infty(f) \subset \{0, \pm 2\}.
\]

3) Now we shall construct a unimodular cone \(\sigma \in K\) of the unimodular simplicial subdivision \((\Gamma)\). For example, if we choose \(a_1 = \frac{1}{6}(L_{1,3,4} + L_{1,2,4} + 2L_{1,2,3}), a_2 = \frac{1}{15}(L_{1,2,4} + 3L_{1,2,3} + 10L_{2,4,5}), a_3 = L_{1,2,3}, a_4 = L_{2,4,5}, a_5 = (1, 1, -1, 1, 0)\), as the column vectors of

\[
W = (a_1^T, a_2^T, a_3^T, a_4^T, a_5^T) = \begin{pmatrix}
 w_1 & w_2 & w_3 & w_4 & w_5 \\
 -1 & 1 & -2 & 2 & 1 \\
 -1 & -1 & 0 & -1 & 1 \\
 -1 & 0 & -4 & 1 & -1 \\
 5 & 1 & 11 & -2 & 1 \\
 1 & 0 & 3 & -1 & 0
\end{pmatrix}.
\]

they are generators of a unimodular cone \(\sigma\). One shall also verify that \(\langle a_5, \alpha \rangle > 0\) for all \(\alpha \in \text{supp}(f)\). As for the method to obtain unimodular simplicial subdivision of a cone see [11].

In this situation the polynomial (4.2) will have the following form

\[
f^W(u) = u_1^7u_2^5u_3^{29}u_5^6 + u_1^2u_2^4u_4^5u_5^3 + u_1^2u_2u_3^5u_5^3 + u_1u_5^2 + u_2^2u_4^3u_5^5 + u_3^5 - 3u_5.
\]

Consider the expansion (2.6) around the singular point \(u = (0, 0, 0, 1, 0)\) where \(\partial_uf^W(u) = 0\). Then we see that \(\text{supp}_{\mu}(\langle \mu_j, f^W \rangle) \subset \{v; \langle q, v \rangle \geq 5\}\) for \(j \in [1,4]\) and vector \(q = (5, -20, 3, 15, 5)\). The facet \(\Gamma \subset \text{supp}_{\mu}(\langle \mu_j, f^W \rangle)\) treated in Proposition 3.1, i.e. \(\Gamma \subset \{v; \langle q, v \rangle = 5\}\) can be found as a convex hull of points \((0, 0, 0, 0, 1), (0, 2, 0, 3, 0), (1, 0, 0, 0, 0), (2, 1, 5, 0, 0), (2, 4, 0, 5, 0)\).

The relation \(W(q', 0) = (-1, 0, -2, 8, -1)\) for \((q', 0) = (5, -20, 3, 15, 0)\) shows that the condition \((\mu)\) of Lemma 3.1 is satisfied. From this relation we see that the index set \(J = \{1, 2, 4, 5\}\) and \(\text{min}_{i\neq j} < (q, 0), w_i - w_j >= (q', 0), w_3 - w_4 >= -10 = -L_0, \rho = \text{min}_{\alpha \in \Delta_u, \langle \mu, f^W \rangle} < q, \alpha >= 5\).

We consider a curve \(Q(t)\) with real coefficients of length \(11 = L_0 + 1\) namely

\[
u_1 = \sum_{j=0}^{10} c_1(j)t^{j+5}, \quad u_2 = \sum_{j=0}^{10} c_2(j)t^{j-20},
\]

\[
u_3 = \sum_{j=0}^{10} c_3(j)t^{j+3}, \quad u_4 = \sum_{j=0}^{10} c_4(j)t^{j+15}, \quad u_5 = 1 + \sum_{j=0}^{10} c_5(j)t^{j+5}.
\]
The system of equations (that corresponds to the coefficients of \( t^5 \) terms)

\[
g_1(c) = g_2(c) = g_4(c) = g_5(c) = 0
\]

where

\[
g_1(c) = 4c_1(0)^2c_2(0)^4c_4(0)^5 + 2c_1(0)^2c_2(0)c_3(0)^5 + 2c_1(0) + 4c_2(0)^2c_4(0)^3 + 6c_5(0),
g_2(c) = 3c_1(0)^2c_2(0)^4c_4(0)^5 + 3c_1(0)^2c_2(0)c_3(0)^5 + 5c_1(0) + 5c_2(0)^2c_4(0)^3 + 12c_5(0),
g_4(c) = 3c_1(0)^2c_2(0)^4c_4(0)^5 + 2c_1(0)^2c_2(0)c_3(0)^5 + 3c_1(0) + 3c_2(0)^2c_4(0)^3 + 6c_5(0)
g_5(c) = c_1(0)^2c_2(0)^4c_4(0)^5 + c_1(0)^2c_2(0)c_3(0)^5 + c_1(0) + c_2(0)^2c_4(0)^3 + 6c_5(0)
\]

admits non-trivial solutions because each equation effectively depends on \( c_j(0)_{j=1,5} \). In a similar manner, the system of equations (that corresponds to the coefficients of \( t^{k+1} \) terms)

\[
g_k(c) = g_k(c) = g_k(c) = g_k(c) = 0
\]

for \( k \in [2,6] \) also admits non-trivial solutions by virtue of Theorem 3.1.

In this way we can find non-trivial solutions for a system of 24 algebraic equations \( g_0(c) = 0, c \in C, j = 1, 2, 4, 5, k \in [1,6] \). This means that we can construct a curve \( Q(t) \) of parametric length 7 satisfying the condition \( (3.9) - 10 + ord(\mu_j, \varphi f^W(Q(t))) > 0 \) for \( j \in J = \{1, 2, 4, 5\} \). The image \( X(t) \) of the curve \( Q(t) \) by the map

\[
x_1 = u(-1, 1, -1), x_2 = u(-1, 1, 0), x_3 = u(-1, 0, -1), x_4 = u(0, 1, 0), x_5 = u(1, 0, 1)
\]

satisfies (I), (II) of Definition 3.1 and \( \lim_{t \to 0} f(X(t)) = -2 \in K_\infty(f) \). A similar arguments shows \( 2 \in K_\infty(f) \). We see that the polynomial \( (4.2) \) is non-degenerate at infinity in the sense of [2] and its Theorem 1.1 can be applied to it. There is no contribution in the right hand side superset in (3.17) from "atypical faces" except that from the "strongly atypical face" [2, Definition 3.2] corresponding to the bad face of \( \Delta(f) \) for (4.2). Thus we conclude the inclusion relation (4.3).

For the polynomial \( (4.2) \) the method of [4, Theorem 3.5.] proposes construction of a curve of parametric length \( d(d+1) + 1 = 3360001 \) with \( d = 20 = |v_0 + v_1| \) satisfying the required properties.

5. Examples

We shall give several examples that illustrate our Theorem 3.1.

Example 5.1. (Non-isolated singularity on a two dimensional bad face)

Consider a polynomial \( f(x) = x^{v_1} + (x^{v_2} - x^{v_3} + 1)^2 + (x^{v_2} - x^{v_3} + 1)^3 + x^{v_4} - 2 \) with \( v_1 = (2, 1, 1), v_2 = (2, 2, 1), v_3 = (1, 2, 1), v_4 = (3, 1, 1) \). This case with non-isolated singularities at infinity has not been treated in [13].

We remark that

\[
M = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = (\mu_1 T, \mu_2 T, \mu_3 T) = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}
\]
is unimodular. Thus we can set

\[ M^{-1} = W = (a_1^T, a_2^T, a_3^T) = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 2 & -3 & 2 \end{pmatrix}. \]

The only bad face \( \gamma \) of \( \Delta(f) \) is located on the plane spanned by \( v_2, v_3 \). For the above \( W \) we have

\[ f^W(u) = -2 + u_1 + (u_2 - u_3 + 1)^2 + (u_2 - u_3 + 1)^3 + \frac{u_1 u_2}{u_3}. \]

The polynomial \( f^W_\gamma(0, u_2, u_3) = (u_2 - u_3 + 1)^2 + (u_2 - u_3 + 1)^3 \) has non-isolated singularities along a line \( u_2 - u_3 + 1 = 0 \). We can choose, for example \( u^* = (0, -1/3, 2/3) \).

In the neighbourhood of this point the rational function \( f^W(u) \) has the expansion

\[ f^W(u) = -2 + u_1 + (U_2 - U_3)^2 + (U_2 - U_3)^3 + \frac{3u_1}{2}(U_2 - 1/3)(1 - \frac{3U_3}{2} + (\frac{3U_3}{2})^2 + \cdots), \]

for \( U_2 = u_2 + 1/3, U_3 = u_3 - 2/3 \).

\( \Delta(\langle \mu_i, \vartheta_u f^W(u) \rangle) \) \( i = 1, 2, 3 \) give rise to the facet \( \Gamma \), (3.1).

\[ \langle \mu_3, \vartheta_u f^W(u) \rangle = \frac{u_1}{16} - 2U_2 + 2U_3 + h.o.t. \]

Thus we find the facet \( \Gamma \) located on the plane containing \((1, 0, 0), (0, 0, 1), (0, 1, 0) \) and \( q = (1, 1, 1), (q', 0) = (1, 0, 0) \).

We calculate \( L_0 = 3 \) and \( \rho = 1 \). A curve \( Q(t) \) (3.11) with real coefficients of parametric length 4, namely

\[ u_1 = \sum_{j=0}^{3} c_1(j)t^{j+1}, u_2 = -1/3 + \sum_{j=0}^{3} c_2(j)t^{j+1}, u_3 = 2/3 + \sum_{j=0}^{3} c_3(j)t^{j+1} \]

that satisfies

\[ -3 + \text{ord} \langle \mu_3, \vartheta_u f^W(Q(t)) \rangle > 0 \]

(as we see \( \mathcal{J} = \{3\} \)) can be constructed.
We remark that after the method of [4, Theorem 3.5], the real curve with required property has parametric length $16 \times 15^2 + 1 = 3601$.

In fact if we plug these expressions into $\langle \mu_3, \partial_u f^W(u) \rangle$ we get an expansion with initial term proportional to $t^1$, $(\langle g, \alpha \rangle = 1$ for $\alpha \in \Gamma$):

$$\langle \mu_3, \partial_u f^W(u) \rangle (Q(t)) = (c_1(0)/2 - 2c_2(0) + 2c_3(0)) t + 1/4(2c_1(1) + 6c_1(0)c_2(0) - 4c_2(0)^2 - 8c_2(1) + 3c_1(0)c_3(0) + 8c_2(0)c_3(0) - 4c_3(0)^2 + 8c_3(1))^2 + 1/8(4c_1(2) + 12c_1(1)c_2(0) + 24c_2(0)^3 + 12c_1(0)c_2(1) - 16c_2(0)c_2(1) - 16c_2(2) + 6c_1(1)c_3(0) - 18c_1(0)c_2(0)c_3(0) - 72c_2(0)^2c_3(0) + 16c_2(1)c_3(0) - 9c_1(0)c_3(0)^2 + 72c_2(0)c_3(0)^2 - 24c_3(0)^3 + 6c_1(0)c_3(1) + 16c_2(0)c_3(1) - 16c_3(0)c_3(1) + 16c_3(2))t^3 + \cdots$$

The coefficient of $t$ depends on $(c_1(0), c_2(0), c_3(0), c_1(1), c_2(1), c_3(1))$, that of $t^2$ depends on $(c_1(0), c_2(0), c_3(0), c_1(1), c_2(1), c_3(1))$, that of $t^3$ depends on $(c_i(j))_{i=1,2,3,j=0,1,2}$. Thus the system of algebraic equations imposed on $(c_i(j))_{i=1,2,3,j=0,1,2} \in \mathbb{C}^9$ to make the coefficients of $t, t^2, t^3$ vanish has non-trivial solutions. In fact this system can be solved in $\mathbb{R}^9$.

We can choose as $(c_1(3), c_2(3), c_3(3)) \in \mathbb{C}^3$ arbitrary non-zero vector.

The image of the curve $Q(t)$ by the map

$$x_1 = u_2 u_3^{-1}, x_2 = u_1^{-1} u_2, x_3 = u_1^2 u_2^{-3} u_3^2$$

satisfies (3.5), (3.6) of Definition 3.1 and $\lim_{t \to 0} f(X(t)) = -2 \in \mathcal{K}_\infty(f)$.

As it can be seen in this example, the curve $X(t)$ approaches to the surface $\{x; f(x) = -2\}$ as $t \to 0$.

We obtain a curve $X(t) = (x_1(t), x_2(t), x_3(t))$ asymptotically approaching to the surface $\{x; f(x) = -2\}$ as follows:

$$x_1(t) = \frac{t^4 + t^3 + t^2 + t - \frac{1}{3}}{t^4 + \frac{131t^3}{256} - \frac{t^2}{4} + \frac{3t}{4} + \frac{2}{3}}$$

$$x_2(t) = \frac{t^4 + t^3 + t^2 + t - \frac{1}{3}}{t^4 + t^3 + t^2 + t}$$

$$x_3(t) = \frac{(t^4 + \frac{131t^3}{256} - \frac{t^2}{4} + \frac{3t}{4} + \frac{2}{3})^2 (t^4 + t^3 + t^2 + t)^2}{(t^4 + t^3 + t^2 + t - \frac{1}{3})^3}.$$
On the figure we see two branches of the curve that correspond to the asymptotes $t \to 0$ from $t > 0$ and $t < 0$ respectively. In all examples 5.1, 5.2, 5.3, figures illustrating algebraic surfaces and rational parametric curves are prepared with the aid of the computer programme MATLAB.

**Example 5.2.** (Isolated singularities at infinity) Consider a polynomial $f(x) = -3x^3 + x^2 + x^3$ with $v_0 = (2, 2, 1), v_1 = (1, 0, 1), v_2 = (0, 1, 1)$.

\[
W = (a_1^T, a_2^T, a_3^T) = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ -2 & -1 & -1 \\ 2 & 2 & 1 \end{pmatrix}.
\]

\[
M = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = (\mu_1^T, \mu_2^T, \mu_3^T) = \begin{pmatrix} -1 & -2 & -1 \\ 0 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix}.
\]

The only bad face $\gamma$ of $\Delta(f)$ is located on the cone $\{ t, v_0; t > 0 \}$.

\[
f^W = u_1^3u_2^2u_3^2 + u_2 + u_3^2 - 3u_3.
\]

and $f^W(u) = u_3^2 - 3u_3$ has singular points $u_3^2 = \pm 1$ and critical values $\mp 2$ respectively.

After [13] we see that in this case the bifurcation set $B(f) \subset \mathcal{K}_{\infty}(f)$ contains $\{ \pm 2 \}$.

We shall construct a curve $X(t)$ that satisfies (3.5), (3.6) of Definition 3.1 and $\lim_{t \to 0} f(X(t)) = -2$. A curve satisfying $\lim_{t \to 0} f(X(t)) = 2$ can be also constructed in a parallel way.

First of all we calculate $\langle \mu_j, \vartheta_u f^W(u) \rangle$, $j = 1, 2, 3$ and find the facet $\Gamma$ as in Proposition 3.1. For example $\langle \mu_3, \vartheta_u f^W(u) \rangle$ has the following form with $U_3 = u_3 - 1$.

\[
3u_1^3u_2^2(U_3 + 1)^2 - u_2 + 3U_3(U_3 + 1)(U_3 + 2).
\]

The facet $\Gamma$ is on the plane containing $(3, 2, 0), (0, 1, 0), (0, 0, 1)$ and $q = (-1, 3, 3)$ i.e. $(q', 0) = (-1, 3, 0)$. As we see

\[
\langle (q', 0), w_1 \rangle = -1, \langle (q', 0), w_2 \rangle = -1, \langle (q', 0), w_3 \rangle = 4,
\]

$\mathbb{J} = \{3\}$ and $L_0 = \max_{i \neq j} \langle (q', 0), w_i - w_j \rangle = 5$. The curve (3.11) has the expansion

\[
u_1 = c_1(0)t^{-1} + c_1(1) + h.o.t., \quad u_2 = c_2(0)t^2 + c_2(1)t^4 + h.o.t., \quad u_3 = 1 + c_3(0)t^3 + c_3(1)t^4 + h.o.t.
\]

If we plug these expressions into $\langle \mu_3, \vartheta_u f^W(u) \rangle$ we get an expansion with initial term $t^3$

\[
\langle (c_2(0) + c_1(0)^3c_2(0)^2 + 6c_3(0))t^3 + (3c_1(0)^2c_1(1)c_2(0)^2 + c_2(1) + 2c_1(0)^3c_2(0)c_2(1) + 6c_3(1))t^4 + \ldots \rangle.
\]
\[(3c_1(0)c_1(1)^2c_2(0)^2 + 3c_1(0)^2c_1(2)c_2(0)^2 + 6c_1(0)^2c_1(1)c_2(0)c_2(1) + \\
+ c_1(0)^3c_2(1)^2 + c_2(2) + 2c_1(0)^3c_2(0)c_2(2) + 6c_3(2))t^5 + h.o.t.\]

The coefficient of \(t^3\) depends on \((c_1(0), c_2(0), c_3(0))\) that of \(t^4\) depends on \((c_1(0), c_2(0), c_3(0), c_1(1), c_2(1), c_3(1))\) that of \(t^5\) depends on \((c_1(0), c_2(0), c_1(1), c_2(1), c_2(2), c_2(3))\).

Thus we can construct a curve such that \(-5 + \text{ord} \langle \mu_3, \vartheta f^W \rangle (Q(t)) > 0\). The minimum parametric length of such a curve \(Q(t) (3.11)\) is 4. There coefficients can be chosen to be real. We remark that after the method of [4, Theorem 3.5.], the rational curve with required properties has length 3601.

We get the desired curve \(X(t)\) as the image of the curve \(Q(t)\) by the map

\[x_1 = u_1u_3, x_2 = (u_1^2u_2u_3)^{-1}, x_3 = u_1^2u_2u_3.\]

We obtain a curve \(X(t) = (x_1(t), x_2(t), x_3(t))\) asymptotically approaching to the surface \(\{x; f(x) = -2\}\) as follows:

\[x_1(t) = \left( t^2 + t + \frac{1}{t} + 1 \right) \left( t^6 - \frac{8t^5}{3} - t^4 - \frac{t^3}{3} + 1 \right)\]

\[x_2(t) = \frac{1}{\left( t^2 + t + \frac{1}{t} + 1 \right)^2 \left( t^6 - \frac{8t^5}{3} - t^4 - \frac{t^3}{3} + 1 \right) \left( t^6 + t^5 + t^4 + t^3 \right)}\]

\[x_3(t) = \left( t^2 + t + \frac{1}{t} + 1 \right)^2 \left( t^6 - \frac{8t^5}{3} - t^4 - \frac{t^3}{3} + 1 \right) \left( t^6 + t^5 + t^4 + t^3 \right)^2.\]

On the figure we see two branches of the curve that correspond to the asymptotes \(t \to 0\) from \(t > 0\) and \(t < 0\) respectively.

**Example 5.3.** (\(A_n\) singularity on a 1—dimensional bad face) More generally consider a polynomial \(f(x) = \prod_{j=1}^n (x^{m_j} - z_j)^{m_j} + b_1x^{v_1} + b_2x^{v_2} + b_3x^{v_3}\) with \(b_1b_2b_3 \neq 0, m_1 \geq 2\), and \(v_0 = (2, 2, 1), v_1 = (0, 1, 1), v_2 = (1, 0, 1), v_3 = (1, 1, 3)\).

we can choose \(a_1 \perp \text{span} \{v_0, v_1\} \cap \Delta(f), a_2 \perp \text{span} \{v_0, v_2\} \cap \Delta(f)\) and \(a_3 \perp \text{span} \{v_1, v_2\} \cap \Delta(f)\). Then we get respectively \(a_1 = (1, -2, 2), a_2 = (-2, 1, 2), a_3 = (1, 1, -1)\).

We can take a simplicial unimodular subdivision of the cone generated by \(a_1, a_2, a_3\) as \(a_1^{(1)} = \frac{a_2 + 2a_3}{3} = (0, -1, 2), a_1^{(2)} = \frac{a_2 + a_3^{(1)}}{2} = (-1, 0, 2), a_3^{(1)} = (1, -1, 1)\) (see [11]).
\[ W = (a_1^{(2)T}, a_1^{(1)T}, a_3^{(1)T}) = \begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & -1 \\ 2 & 2 & 1 \end{pmatrix}. \]

\[ M = \begin{pmatrix} m_1 & m_2 \\ m_2 & m_3 \end{pmatrix} = (\mu_1^T, \mu_2^T, \mu_3^T) = \begin{pmatrix} 1 & 2 & 1 \\ -2 & -3 & -1 \\ 2 & 2 & 1 \end{pmatrix}. \]

The only bad face \( \gamma \) of \( \Delta(f) \) is located on the cone \( C(v_0) = \{ t.v_0; t > 0 \} \).

Case i). We assume \( z_1 = 0 \).

\[ f^W = u_3^{m_1} \prod_{j=2}^{q} (u_3 - z_j)^{m_j} + b_1 u_1^2 u_2 + b_2 u_1 u_2^2 u_3^2 + b_3 u_1^5 u_2^5 u_3^3 \]

and \( f^W_\gamma(u) = u_3^{m_1} \prod_{j=2}^{q} (u_3 - z_j)^{m_j} \) has singular point \( u_3^* = 0 \) and the critical value 0 respectively. We shall construct a curve \( X(t) \) that satisfies (3.5), (3.6) of Definition 3.1 and \( \lim_{t \to 0} f(X(t)) = 0 \). Thus \( \{ 0 \} \subset \mathcal{K}_\infty(f) \). For \( (b_1, b_2, b_3) \in \mathbb{C}^3 \) so that \( f \) be non-degenerate at infinity we conclude \( \mathcal{K}_\infty(f) = \{ 0 \} \) thanks to [2, Theorem 1.1].

First of all we calculate \( \langle \mu_3, \partial_u f^W(u) \rangle \) and find the facet \( \Gamma \) as in Proposition 3.1. The polynomial \( \langle \mu_3, \partial_u f^W(u) \rangle \) has the following form with \( U_3 = u_3 - 0 \),

\[ b_1 u_1^2 u_2 + b_2 u_1 u_2^2 u_3^2 + b_3 u_1^5 u_2^5 u_3^3 + u_3 \sum_{j=1}^{q} \frac{m_j}{u_3 - z_j} \prod_{k=1}^{q} (u_3 - z_k)^{m_k} \]

where we assumed that \( \sum_{j=2}^{q} m_j \prod_{k=2}^{q} (-z_k)^{m_k - q} \neq 0 \).

We see that Newton polyhedra \( \Delta(\langle \mu_1, \partial_u f^W(u) \rangle), \Delta(\langle \mu_2, \partial_u f^W(u) \rangle) \) are located in the Newton polyhedron of \( \Delta(\langle \mu_3, \partial_u f^W(u) \rangle) \).

The facet \( \Gamma \) is on the plane containing \( (2, 1, 0), (1, 2, 2), (0, 0, m_1) \), \( \rho = 3 m_1 \) and \( q = (m_1 + 2, m_1 - 4, 3) \) i.e. \( \langle q', 0 \rangle = (m_1 + 2, m_1 - 4, 0) \). As we see

\[ \langle (q', 0), w_2 - w_1 \rangle = 6, \langle (q', 0), w_3 - w_1 \rangle = 5 m_1 - 2, \langle (q', 0), w_3 - w_2 \rangle = 5 m_1 - 8, \]

\( J = \{ 2, 3 \} \) and \( L_0 = \max_{i \neq j} \langle (q', 0), w_i - w_j \rangle = 5 m_1 - 2 \). The curve (3.11) has the expansion

\[ u_1 = c_1(0) t^{m_1+2} + c_1(1) t^{m_1+3} + c_1(2) t^{m_1+4} + \text{h.o.t.}, \]

\[ u_2 = c_2(0) t^{m_1-4} + c_2(1) t^{m_1-3} + c_2(2) t^{m_1-2} + \text{h.o.t.}, \]

\[ u_3 = c_3(0) t^3 + c_3(1) t^4 + c_3(2) t^5 + \text{h.o.t.} \]

If we plug these expressions into \( \langle \mu_3, \partial_u f^W(u) \rangle \) we get an expansion with initial term \( t^{m_1+4} \)

\[ \langle (q, 0), a \rangle = 3 m_1 \text{ for } a \in \Gamma; \]

\[ \{ b_1(1)c_1(0)^2 c_2(0) + b_2 c_1(0) c_2(0)^2 c_3(0)^2 + m_1(c_3(0))^{m_1} \prod_{j=2}^{q} (-z_j)^{m_j} \} t^{m_1} + \]

\[ \{ b_1[(2c_1(1)c_1(0)c_2(0) + (c_1(0))^{2} c_2(1))^2 + b_2[c_1(1) c_2(0) c_3(0)^2] + 2 c_1(0) c_2(0) c_3(1) c_3(0) + (m_1)^2 c_2(1) c_3(0)] t^{m_1+4} + \]

\[ \{ b_1[(c_1(0))^2 c_2(2) + 2 c_1(1) c_1(0) c_2(1) + 2 c_1(2) c_1(0) c_2(0) + (c_1(0))^2 c_2(0)] + b_2 [c_1(1) c_2(0) c_3(0)^2] + 2 c_1(0) c_2(0) c_3(2) c_3(0) + c_1(0) (c_2(1) c_3(0))^2 + \]
\[ c_1(0)c_2(0)c_3(1)^2 + 2c_1(1)c_2(1)c_2(0)(c_3(0)) + 2c_1(1)(c_2(0))^2c_3(1)c_3(0) + 4c_1(0)c_2(1)c_2(0)c_3(1)c_3(0) + \frac{m_1(m_1 - 1)}{2}c_2(0)^2m_1^{-2} + m_1c_3(2)(c_3(0)m_1^{-1})m_1 \prod_{j=2}^{q}(-z_j)^{m_j} t^{3m_1} + h.o.t. \]

For \( \langle \mu_2, \partial_u f^W(Q(t)) \rangle \) we get a similar expansion. The coefficient of \( t^{3m_1} \) depends on \((c_1(0), c_2(0), c_3(0))\) that of \( t^{3m_1 + 1} \) depends on \((c_1(j))_{i=1,2,3,j=0,1,2}\). Let \( L_0 - \rho + 2 = 5m_1 - 2 = 3m_1 + 2 = 2m_1 \), thus we constructed a curve of parametric length \( 2m_1 \) satisfying

\[
2 - 5m_1 + \text{ord} \langle \mu_j, \partial_u f^W(Q(t)) \rangle > 0 \quad j = 2, 3.
\]

Case ii). We assume \( z_1 \cdot z_2 \cdots z_q \neq 0, \; z_i \neq z_j, \; \forall i, j \in [1 : q] \).

\[
f^W = \prod_{j=1}^{q}(u_3 - z_j)^{m_j} + b_1u_1^2u_2 + b_2u_1u_2^2u_3 + b_3u_3^3 u_2^5 u_3^3
\]

and \( f^W_q(u) = \prod_{j=1}^{q}(u_3 - z_j)^{m_j} \) has singular points \( u_3^* = z_1 \) where \( z_1 \neq 0 \) and critical values 0 respectively. We shall construct a curve \( X(t) \) that satisfies (3.5), (3.6) of Definition 3.1 and \( \lim_{t \to 0} f(X(t)) = 0 \). Thus \( \{0\} \subset K_{\infty}(f) \). For \((b_1, b_2, b_3) \in \mathbb{C}^3 \) so that \( f \) be non-degenerate at infinity we conclude \( K_{\infty}(f) = \{0\} \) thanks to [2, Theorem 1.1].

First of all we calculate \( \langle \mu_3, \partial_u f^W(u) \rangle \) and find the facet \( \Gamma \) as in Proposition 3.1. For example \( \langle \mu_3, \partial_u f^W(u) \rangle \) has the following form with \( U_3 = u_3 - z_1 \).

\[
\begin{align*}
&b_1u_1^2u_2 + b_2u_1u_2^2(U_3 + z_1)^2 + (U_3 + z_1)[m_1U_3^{m_1 - 1} - \prod_{k=2}^{q}(U_3 + z_1 - z_k)^{m_k} + \\
&m_1U_3^{m_1}(U_3 + z_1 - z_2)^{m_2} - \prod_{k=3}^{q}(U_3 + z_1 - z_k)^{m_k} + \cdots + 3b_3u_1^5u_2^5u_3^3
\end{align*}
\]

where \( \sum_{j=1}^{q}m_j \sum_{k=1}^{q}(z_1 - z_k)^{m_k} - \delta_{j,k} \neq 0 \).

We see that Newton Polyhedra \( \Delta(\langle \mu_3, \partial_u f^W(u) \rangle), \Delta(\langle \mu_2, \partial_u f^W(u) \rangle) \) are located in the Newton Polyhedron of \( \Delta(\langle \mu_3, \partial_u f^W(u) \rangle) \).

The facet \( \Gamma \) is on the plane containing \((1, 2, 0), (2, 1, 0), (0, 0, m_1 - 1) \) and \( q = (m_1 - 1, m_1 - 1, 3) \) i.e. \( \langle q', 0 \rangle = (m_1 - 1, m_1 - 1, 0) \). As we see \( J = \{3\} \) and \( L_0 = \max_{i \neq j} \langle (q', 0), (w_1 - w_j) \rangle = 5m_1 - 5 \) and \( \rho = 3m_1 - 3 \) The curve (3.11) has the expansion

\[
\begin{align*}
&u_1 = c_1(0)t^{m_1 - 1} + c_1(1)t^{m_1} + c_2(1)t^{m_1 + 1} + h.o.t., \\
&u_2 = c_2(0)t^{m_1 - 1} + c_2(1)t^{m_1} + c_2(2)t^{m_1 + 1} + h.o.t., \\
&u_3 = z_1 + c_3(0)t^3 + c_3(1)t^4 + c_3(2)t^5 + h.o.t.
\end{align*}
\]

If we plug these expressions into \( \langle \mu_3, \partial_u f^W(u) \rangle \) we get an expansion with initial term \( t^3m_1 - 3 \) \( \langle q, \alpha \rangle = 3m_1 - 3 \) for \( \alpha \in \Gamma \);

\[
\begin{align*}
&\left( b_1(c_1(0))^2c_2(0) + b_2z_1^2c_1(0)(c_2(0))^2 + z_1m_1(c_3(0))^{m_1 - 1} \prod_{k=2}^{q}(z_1 - z_k)^{m_k} \right) t^{3m_1 - 3} + \\
&(b_1[(c_1(0))^2c_2(1) + 2c_1(1)c_1(0)c_2(0)] + b_2z_1^2[c_1(0)c_2(1)c_2(0) + c_1(1)(c_2(0))^2] + z_1m_1(c_1(0)c_2(0)^2)m_1^{-2} \prod_{k=2}^{q}(z_1 - z_k)^{m_k} t^{3m_1 - 2} + \\
&(b_1[(c_1(0))^2c_2(2) + 2c_1(1)c_1(0)c_2(1) + 2c_2(0)c_2(1)c_3(0)] + c_1(1)^2c_1(0)c_2(0)) + b_2z_1^2[2c_1(0)c_2(2)c_2(0) + 2c_1(1)c_2(1)c_2(0) + c_1(2)(c_2(0))^2 + c_1(0)(c_2(1))^2c_2(0)]
\end{align*}
\]
+z_1 m_1 \left[ \frac{(m_1-1)(m_1-2)}{2} (c_3(1))^2 (c_3(0))^{m_1-3} + (m_1-1) c_3(2)(c_3(0))^{m_1-2} \right] \prod_{k=2}^q (z_1-z_k)^{m_k} t^{3m_1-1} + h.o.t.

The coefficient of $t^{3m_1-3}$ depends on $(c_1(0), c_2(0), c_3(0))$ that of $t^{3m_1-2}$ depends on $(c_1(0), c_2(0), c_3(0), c_1(1), c_2(1), c_3(1))$ that of $t^{3m_1-1}$ depends on $(c_1(0), c_2(0), c_3(0), c_1(1), c_2(1), c_3(1), c_1(2), c_2(2), c_3(2))$.

Thus we can construct a curve such that $5 - 5m_1 + ord \langle \mu_3, \partial_u f^W \rangle (Q(t)) > 0$. The minimum parametric length of such a curve $Q(t)$ is $2m_1$. We remark that after the method of [4, Theorem 3.5], the rational curve with required properties has parametric length $(1 + 5 \cdot \sum_{j=1}^q m_j)(5 \cdot \sum_{j=1}^q m_j)^2 + 1$.

We get the desired curve $X(t)$ as the image of the curve $Q(t)$ by the map

$$x_1 = u_1^{-1} u_3, x_2 = (u_2 u_3)^{-1}, x_3 = u_1^2 u_2^2 u_3.$$

Similarly as it can be seen in this example for case 2, the $X(t)$ curve asymptotically approaches to the surface $\{x; f(x) = 0\}$ as $t \to 0$.

We illustrate below this case with an example $q = 2$, $(z_1, z_2) = (1, 2)$, $f(x) = (x^{v_0} - 1)^3(x^{v_0} - 2) + x^{v_1} + x^{v_2} + x^{v_3}$.

For this example we obtain a curve $X(t) = (x_1(t), x_2(t), x_3(t))$ asymptotically approaching to the surface $\{x; f(x) = 0\}$ as follows:

$$x_1(t) = \frac{t^8 + \frac{57427 t^7}{20736} + \frac{1645 t^6}{864} + \frac{203 t^5}{144} + \frac{13 t^4}{12} + \frac{t^3}{2} + 1}{t^7 + t^6 + t^5 + t^4 + t^3 + t^2 + \frac{t^2}{2}}$$

$$x_2(t) = \frac{1}{(t^7 + t^6 + t^5 + t^4 + t^3 + t^2) \left( t^8 + \frac{57427 t^7}{20736} + \frac{1645 t^6}{864} + \frac{203 t^5}{144} + \frac{13 t^4}{12} + \frac{t^3}{2} + 1 \right)}$$

$$x_3(t) = \left( t^7 + t^6 + t^5 + t^4 + t^3 + \frac{t^2}{2} \right)^2 \left( t^7 + t^6 + t^5 + t^4 + t^3 + \frac{t^2}{2} \right)^2 \left( t^8 + \frac{57427 t^7}{20736} + \frac{1645 t^6}{864} + \frac{203 t^5}{144} + \frac{13 t^4}{12} + \frac{t^3}{2} + 1 \right).$$

On the figure we see two branches of the curve that correspond to the asymptotes $t \to 0$ from $t > 0$ and $t < 0$ respectively.
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