Detecting Majorana fermions by nonlocal entanglement between quantum dots

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Nonlocal entanglement between two quantum dots can be generated through Majorana fermions. The two Majorana fermions at the ends of a one-dimensional topological superconductor form a nonlocal fermion level, coupling to the occupation states of two quantum dots put close to the two ends, and the entire system will come into an entangled state. After introducing a charging energy by a capacitor, entanglement of the entire system can manifest itself through the nonlocal entanglement between the two quantum dots. That is, when measuring the electron occupations of the quantum dots, the measurement result of one quantum dot will influence the measurement result of the other quantum dot. This nonlocal entanglement between the two quantum dots is a strong evidence of the nonlocal nature of the fermion level constructed by two Majorana fermions.

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Introduction.– The existence of Majorana fermions (MFs) in solid state systems has been under intense investigation recently. It attracts so much attention mainly because they might be useful for quantum computation. Two well separated MFs define a nonlocal fermion level, which can be either occupied or empty, and thus form a topological qubit storing quantum information nonlocally. This nonlocal topological qubit may well be incorporated with the standard superconducting qubit, and used for decoherence-free quantum computing.

One of the candidates which support MFs is the spinless p-wave superconductor. In these topological superconductors, MFs may arise due to the particle-hole redundancy. MFs have already been predicted in several superconducting proximity systems which resemble the spinless p-wave superconductor, including Sr$_2$RuO$_4$, interface between an s-wave superconductor and a topological insulator, and spin-orbit coupling nanowire in proximity to an s-wave superconductor.

Two obvious issues concerning MFs are how to detect and manipulate them. A number of proposals have been suggested to detect MFs in superconducting systems, mostly by tunneling experiments. One unique property of the MFs is that two of them can form a nonlocal fermion level. Making use of this property, it has been proposed by Fu that there can be a novel electron teleportation across the superconductor when it is connected to the ground through a capacitor, as an evidence of MFs existence. Meanwhile, a quantum dot (QD) close to the end of a one dimensional topological superconductor was introduced by Flensberg to manipulate the MFs. In his setup, an electron can tunnel from the QD into the superconductor and generate a ground state rotation, which is similar to braiding the two MFs. It was also illustrated that quantum information may be transferred between a topological qubit formed by MFs and a spin qubit formed by QDs.

In this work, we adopt the system schematically shown in Fig. 1, where two QDs are coupled by tunneling to two well separated MFs supported by a one-dimensional spin-orbit coupling nanowire. The nanowire is in proximity to a mesoscopic superconductor which is connected to the ground through a capacitor. When the energy levels of the two QDs are tuned close to the fermion level formed by MFs, the entire system will come into an entangled state. We show explicitly that, because of the charging energy, nonlocal entanglement between the two QDs appears, i.e., the measurement result of electron occupation on one QD will influence the possibility of finding one electron on the other QD. Therefore, detecting the nonlocal entanglement of the two QDs will provide an unambiguous evidence of the nonlocal fermion level, and thus prove the existence of MFs.

Model.– The system we consider is a one-dimensional spin-orbit coupling wire in proximity with an s-wave superconductor. Under proper magnetic field, the system resembles the properties of spin-less p-wave topological superconductivity. At the two ends of the wire, there are zero energy states described by the Bogoliubov-de-Gennes equation. These zero energy modes have been identified theoretically as local Majorana bound states. The two Majorana operators $\gamma_L$ and $\gamma_R$ are defined as.

![Fig. 1: Tunneling setup of two quantum dots and a one-dimensional spin-orbit coupling nanowire with two Majorana fermions at its ends.](image-url)
\[ \gamma_{L,R} = \int dx \left( e^{-i\phi/2} [\xi_{L,R}(x) c^\dagger_L(x) + \eta_{L,R}(x) c^\dagger_R(x)] + e^{i\phi/2} [\xi_{L,R}(x) c_L(x) + \eta_{L,R}(x) c_R(x)] \right), \tag{1} \]

with \( \phi \) the superconducting phase, \( \xi \) and \( \eta \) bound state wave functions centered at ends of the wire, \( c^\dagger \) and \( c \) electron creation and annihilation operators. We assume that the distance between these two Majorana bound states is large enough, as such we can ignore their interaction. The existence of these two Majorana bound states indicates that the superconducting ground state of the topological superconductor may have either even or odd number of electrons\(^{20}\).

Now we introduce two quantum dots, one close to each end of the wire. The QDs are coupled to the two MFs through tunneling. We set the QD in the Coulomb blockade regime, i.e., its level spacing is large enough so that it may be modeled as one energy level. Since a large magnetic field is applied in order to achieve the MFs, it is enough to consider one spin direction for the electrons on QDs. In this case, the QDs can be described by a spin-less electron energy level Hamiltonian\(^{21}\),

\[ H_{QD} = \sum_{j=L,R} \epsilon_j c^\dagger_j c_j. \tag{2} \]

The QDs are coupled to the wire through electron tunneling between QDs and Majorana bound states, with the tunneling Hamiltonian\(^{20}\),

\[ H_T = \sum_{j=L,R} T_j c^\dagger_j \gamma_j e^{-i\phi/2} + h.c. \tag{3} \]

Here the Majorana operator \( \gamma_{L,R} \) changes the fermionic parity of the superconductor\(^{22}\).

The superconductor is considered of mesoscopic size, and connected to the ground by a capacitor. Then the system will have a charging energy term\(^{20}\),

\[ H_C = (ne - Q_0)^2/2C. \tag{4} \]

In the following, we consider the superconducting energy gap to be the largest energy scale and set it as infinity, thus the superconductor is in its ground state and the superconducting quasiparticles are irrelevant, which is a good approximation as far as low temperature is concerned. Now we have the total Hamiltonian of the system,

\[ H = H_{QD} + H_C + H_T. \tag{5} \]

For simplicity, we set the same energy level for the two QDs \( \epsilon_L = \epsilon_R = \epsilon \), as well as the tunneling strength \( T_L = i T_R = T \).

We initialize the system with \( \epsilon = -\infty \), with both QDs occupied by one electron as denoted by \( |1\rangle_L \) and \( |1\rangle_R \). Without losing generality, we set \( Q_0 = 2 N_0 e \), which enforces the superconductor accommodate \( 2 N_0 \) electrons at the ground state denoted\(^{20}\) by \( |2N_0\rangle_S \). Since the Hilbert space of the whole system is simply the direct product of electron number states of QDs and the superconductor, our initial ground state is \( |1\rangle_L |2N_0\rangle_S |1\rangle_R \).

Now we adiabatically increase the QD energy level \( \epsilon \). When \( \epsilon \) approaches zero, tunneling between QDs and MFs becomes important. Because the QD levels are fully occupied, electron tunneling to the QDs is forbidden. On the other hand, the electron may tunnel from either one of or both QDs to the superconductor via MFs, leading to four possible states:\( |1\rangle_L |2N_0 + 1\rangle_S |1\rangle_R \) for no electron tunneling, \( |1\rangle_L |2N_0 + 1\rangle_S |0\rangle_R \) for an electron on the right QD tunneling to the superconductor, \( |0\rangle_L |2N_0 + 1\rangle_S |1\rangle_R \) for the electron on the left QD tunneling to the superconductor, and finally \( |0\rangle_L |2N_0 + 2\rangle_S |0\rangle_R \) for the two electrons on two QDs tunneling to the superconductor. We notice that since only one electron originally exists on each QD, there is no other possible states due to the charge conservation. When \( \epsilon \) is adiabatically increased, the ground state will gradually evolve away from the initial state \( |1\rangle_L |2N_0\rangle_S |1\rangle_R \) due to the electron tunneling, and achieve a superposition state of the four different electron occupations. Nonlocal entanglement between the two QDs can be achieved as revealed below.

Using the four possible states as the basis,

\[ |1\rangle_L |2N_0\rangle_S |1\rangle_R = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |1\rangle_L |2N_0 + 1\rangle_S |0\rangle_R = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \tag{6} \]

\[ |0\rangle_L |2N_0 + 1\rangle_S |1\rangle_R = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |0\rangle_L |2N_0 + 2\rangle_S |0\rangle_R = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{7} \]

the Hamiltonian (5) is reduced to a 4 by 4 matrix. The matrix elements can be given straightforwardly based on the definitions of states, for example,

\[ H_{24} = \langle 0\rangle_R \langle 2N_0 + 2\rangle_S |0\rangle_L \langle 2N_0 + 1\rangle_L \langle 2N_0 + 1\rangle_S |0\rangle_R = T(1)_R (2N_0)_S^2 \langle 1\rangle_L c^\dagger_L \gamma_L e^{-i\phi/2} e^{-i\phi/2} e^{i\phi/2} e^{i\phi/2} \langle 1\rangle_L |2N_0\rangle_S |1\rangle_R = T. \tag{7} \]

The 4 by 4 matrix is thus given by,

\[ H = \begin{pmatrix} 2\epsilon & T & 0 & 0 \\ T^* & \epsilon + \frac{2\epsilon}{T} & 0 & T \\ T & 0 & \epsilon + \frac{2\epsilon}{T} & 0 \\ 0 & T^* & 0 & \frac{2\epsilon}{T} \end{pmatrix}. \tag{8} \]
Here we note that the same matrix can be obtained for the case \( Q_0 = (2N_0 + 1) \epsilon \) for the corresponding basis.

The lowest eigenvalue of this matrix is the ground state energy, and its corresponding eigenvector is the ground state. It is clear that when \( \epsilon = -\infty \), the ground state is \( |1\rangle_L |2N_0\rangle_S |1\rangle_R \). Now we calculate the ground state with \( \epsilon \) adiabatically increased. The problem is to diagonalize the matrix and find the ground state. In general, this ground state is a superposition state of the four possible states,

\[
|G\rangle = \alpha_1 |1\rangle_L |2N_0\rangle_S |1\rangle_R + \alpha_2 |1\rangle_L |2N_0 + 1\rangle_S |0\rangle_R \\
+ \alpha_3 |0\rangle_L |2N_0 + 1\rangle_S |1\rangle_R + \alpha_4 |0\rangle_L |2N_0 + 2\rangle_S |0\rangle_R ,
\]

where the superposition factor \((\alpha_1, \alpha_2, \alpha_3, \alpha_4)\) is the eigenvector of the matrix associated with the lowest eigenvalue.

**QD entanglement.** The eigenvalues and eigenvectors of a 4 by 4 matrix can be found analytically with lengthy expressions except some special cases. Here we choose to present the result numerically for simplicity. Using \( T \) as the unit and setting \( \frac{\epsilon^2}{2C} = 5T \), we show the four components of the eigenvector of the lowest eigenvalue evolving with \( \epsilon \) in Fig. 2A. It is clear that all factors but \( \alpha_1 \) vanish when \( \epsilon \to -\infty \), whereas all factors but \( \alpha_4 \) vanish when \( \epsilon \to \infty \). In these two limit cases, there is no entanglement in this system. However, when \( \epsilon \) approaches zero energy, all factors are non-zero, thus entanglement appears in the system.

Next, we show explicitly that the entanglement of the entire system can be exposed and detected by nonlocal entanglement between the two QDs. For this purpose, we calculate the probability of finding one electron on the right QD with electron occupation on the left QD \( P_1 = \alpha_1^2 / (\alpha_1^2 + \alpha_2^2) \), and that without electron occupation on the left QD \( P_2 = \alpha_2^2 / (\alpha_1^2 + \alpha_2^2) \), as shown in Fig. 2B. When \( \epsilon \) is positively or negatively large, the two curves overlap, which means the measurement result on the left QD has no influence on the measurement result on the right QD. However, when \( \epsilon \) approaches zero, the two curves deviate from each other, implying that the measurement result of the left QD will influence the measurement result of the right QD. This is the signature for entanglement between these two QDs.

Now, let us discuss the importance of the capacitor. For this purpose, we investigate the case of infinite capacitance, with the results of superposition factors corresponding to the ground state presented in Fig. 3A. At the first glance, the results for \( C = \infty \) seem similar to those in Fig. 2, namely, only \( \alpha_1 \) or \( \alpha_3 \) is non-vanishing when \( \epsilon \) is negatively or positively large; all factors are non-vanishing around zero energy, implying entangled ground state for the entire system.

However, although the entire system will come into an
entangled state around $\epsilon = 0$, there is no direct entanglement between left and right QDs in the present case of $C = \infty$. In order to clarify this point, we calculate the possibility of finding one electron in the right QD, with and without the electron occupation in the left QD. As seen in Fig. 3B, the two curves overlap totally, even when $\epsilon$ approaches zero. It demonstrates that there is no measurable nonlocal entanglement induced between the two QDs, although the entire system is in an entangled state.

The key point here is that, for $C = \infty$ or equivalent a bulk superconductor, one has to detect the fermionic parity of the superconductor to reveal the existence of the Majorana fermions, which is still a difficult task. By introducing the capacitor to a mesoscopic superconductor, the nonlocally entangled MFs can be exposed by simply detecting the nonlocal entanglement between the QDs.

As a measure for the entanglement between the two QDs, we calculate the maximal difference between the measurement results $P_2 - P_1$ as a function of the capacitance. As shown in Fig. 4, the degree of entanglement increases with decreasing capacitance monotonically, with $e^2/2C$ not exceeding the superconducting energy gap.

Lastly we discuss about possible experimental implementations of our idea. Our method does not require any measurement on the state of MFs and the topological superconductor. All we need is the measurement of the electron occupation on QDs. Charge sensing on QD is a well developed area, and the measurement of charge occupation on QD has already been achieved with integrated radio-frequency single-electron transistor.

Summary. – In summary, we have investigated the tunneling system of two quantum dots and a one-dimensional topological superconductor with two Majorana fermions at the two ends. When the quantum dot energy level is close to zero energy the system exhibits an entangled state. When the superconductor of mesoscopic size is connected to the ground through a capacitor, the entanglement of the whole system will show up through the entanglement between the two quantum dots. Namely, the measurement result of occupation on one quantum dot will influence the measurement result on the other quantum dot. This nonlocal entanglement between the two quantum dots is a strong evidence of the nonlocal nature of the fermion level constructed by the two Majorana fermions.

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