Paraconformal structures, ODEs and totally geodesic manifolds

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Abstract

We construct point invariants of ordinary differential equations which generalise Cartan’s invariants in the case of order two and three. If the invariants vanish then the solution space of an equation is equipped with a paraconformal structure, an adapted connection and two-parameter family of totally geodesic hypersurfaces.

1 Introduction

The first aim of this paper is to provide an unified approach to the Cartan invariants of second and third order ordinary differential equations (ODEs) given up to point transformations. The second aim is to generalise the invariants for higher order ODEs. The Cartan invariants for equations of second and third order appeared in different contexts in 1924 and 1941 respectively. The invariant for second order case was already known to Tresse [21, 29]. The nature of both of them is very similar. Indeed, the vanishing of the invariants means that the solution space of an equation is equipped with a \(GL(2, \mathbb{R})\)-structure, an adapted connection \(\nabla\) and a two-parameter family of hyper-surfaces which are totally geodesic for \(\nabla\). We call such structures, in any dimension, totally geodesic paraconformal structures. In the case of dimension 2 the structures are just the projective structures whereas in the case of dimension 3 the structures are the Einstein-Weyl geometries of Lorenzian signature.

An unified approach to the projective structures on a plane and to the three-dimensional Einstein-Weyl structures was given in the complex setting by Hitchin [18] in terms of a twistor construction. The construction involves a two-dimensional manifold \(B\) and a curve with a normal bundle \(O(1)\) or \(O(2)\), respectively. The construction clearly can be generalised to curves with normal bundles \(O(k)\), \(k > 2\), and so-obtained structures correspond to higher-dimensional totally geodesic paraconformal structures considered in this paper. In the Hitchin’s paper there is no construction of invariants on the side of ODEs. On contrary, in the present paper we concentrate on this issue (a second order ODE for a projective structure considered in [18] is dual in the sense of Cartan to the one considered in the present paper).

At the end of the present paper we present a general example based on special families of foliations, called Veronese webs. The webs were introduced by Gelfand and Zakharevich [15] in connection to bi-Hamiltonian structures on odd dimensional manifolds. We show that any Veronese web defines a totally geodesic paraconformal structure. This result generalises [11]. We also construct a hierarchy of integrable systems.

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2 Paraconformal structures and connections

Let $M$ be a manifold of dimension $k + 1$. A paraconformal structure on $M$ is a vector bundle isomorphism

$$TM \cong S \otimes S \otimes \cdots \otimes S$$

where $S$ is a rank-two vector bundle over $M$ and $\otimes$ denotes the symmetric tensor product. It follows that any tangent space $T_xM$ is identified with the space of homogeneous polynomials of degree $k$ in two variables. The natural action of $GL(2, \mathbb{R})$ on $S$ extends to the irreducible action on $TM$ and reduces the full frame bundle to a $GL(2, \mathbb{R})$-bundle. Therefore the paraconformal structures are sometimes called $GL(2, \mathbb{R})$-geometries. We refer to [1, 10, 22] for more detailed descriptions of the paraconformal structures.

A paraconformal structure defines the following cone

$$C(x) = \{ v \circ \cdots \circ v \mid v \in S(x) \} \subset T_xM$$

which is the set of pure powers. It is an easy exercise to show that the field of cones $x \mapsto C(x)$ defines the paraconformal structure uniquely. If a basis $e_0, e_1$ in $S(x)$ is chosen then any $v \in S(x)$ can be written as $v = se_0 + te_1$ and then

$$C(x) = \{ s^kV_0 + s^{k-1}V_1 + \cdots + t^kV_k \mid (s, t) \in \mathbb{R}^2 \}$$

where $V_i = \binom{k}{i} e_0^{\circ k-i} \otimes e_1^{\circ i}$. We shall denote

$$V(s, t) = s^kV_0 + s^{k-1}V_1 + \cdots + t^kV_k$$

and refer to the vectors as null vectors. The cone $C(x)$ defines a rational normal curve $(s : t) \mapsto \mathbb{R}V(s, t)$ of degree $k$ in the projective space $P(T_xM)$. Sometimes, for convenience, we will use an affine parameter $t = (1 : t)$ and denote $V(t) = V_0 + tV_1 + \cdots + t^kV_k$. Derivatives of $V(t)$ with respect to $t$ will be denoted $V'(t)$, $V''(t)$ etc.

We will consider connections $\nabla$ on $M$ which are compatible with the projective structure in a sense that the parallel transport preserves the null vectors i.e. it preserves the field of cones $x \mapsto C(x)$. Precisely we have

**Definition 2.1** A connection $\nabla$ is called paraconformal for a given paraconformal structure $x \mapsto C(x)$ on a manifold $M$ if

$$\nabla_Y V(t) \in \text{span}\{V(t), V'(t)\}$$

for any $t \in \mathbb{R}$ and any vector field $Y$ on $M$.

From the point of view of $GL(2, \mathbb{R})$-structures, the connections satisfying the above condition are in a one to one correspondence with the principal connections.

We are interested in the properties of the geodesics of $\nabla$. Therefore, at least at this point, we will not impose any additional assumptions on the torsion of a connection. Let us only remark here that in low dimensions ($k = 1, 2$) there are plenty of torsion-free connections adapted to a paraconformal structure. On the other hand, already in the case $k = 3$ any connection adapted to a generic paraconformal structure has a torsion but in the most interesting case related to ODEs there is a unique torsion-free connection [1].

Let us fix a point $x \in M$. We define the following 1-parameter family of $i$-dimensional subspaces of $T_xM$ for any number $i \in \{1, \ldots, k\}$

$$V_i(t)(x) = \text{span}\{V(t)(x), V'(t)(x), V''(t)(x), \ldots, V^{(i-1)}(x)\}.$$  \hspace{1cm} (1)
The family of subspaces \( \{ \mathcal{V}_i(t)(x) \mid t \in \mathbb{R}, x \in M \} \), for any \( i \), is canonically defined by the paraconformal structure itself, although the choice of the parameter \( t \) is not canonical. In what follows we will consider paraconformal structures with an adapted connection such that the subspaces \( \mathcal{V}_i \) are tangent to totally geodesic submanifolds of \( M \). Two problems arise. First of all the subspaces \( \mathcal{V}_i \) have to be tangent to submanifolds of \( M \) and this issue does not depend on \( \nabla \). The integrability of \( \mathcal{V}_i \) can be guaranteed by the assumption that a paraconformal structure is defined by an ODE as will be explained in the next section. In fact we will consider only these paraconformal structures which comes from ODEs. The second problem is how to make a submanifold totally geodesic with respect to some connection. We will show that there are obstructions for the existence of such connections. In terms of ODEs the obstructions are expressed by new point invariants.

3 ODEs and paraconformal structures

Paraconformal structures can be constructed out of ODEs. We will consider ODEs in the following form

\[
x^{(k+1)} = F(t, x, x', \ldots, x^{(k)}).
\]

The following theorem is a compilation of results of Chern [6], Bryant [1], Dunajski and Tod [10] (see also [14, 16, 25]).

**Theorem 3.1** If the Wünschmann invariants of \( F \) vanish then the solution space of \( F \) possesses a canonical paraconformal structure.

To explain the meaning of the theorem and give an insight into its proof we recall that the geometry of an ODE of order \( k + 1 \) is described on a manifold of \( k \)-jets, denoted \( J^k(\mathbb{R}, \mathbb{R}) \). There is a canonical projection \( \pi \) from \( J^k(\mathbb{R}, \mathbb{R}) \) to the solution space \( M_F \) with one-dimensional fibre which is tangent to the total derivative vector field

\[
X_F = \partial_t + x_1 \partial_0 + x_2 \partial_1 + \cdots + F \partial_k,
\]

where \( t, x_0, x_1, \ldots, x_k \) are standard coordinates on the space of jets and \( \partial_i = \frac{\partial}{\partial x_i} \). It follows that \( M_F = J^k(\mathbb{R}, \mathbb{R})/X_F \). The term **canonical** in the theorem means that the null vectors of the paraconformal structure are tangent to \( \pi_* \partial_k \). An equation of order \( k + 1 \) has \( k - 1 \) Wünschmann invariants (or strictly speaking relative invariants). In particular there are no invariants for equations of order 2. There is one invariant in order three. This invariant was originally defined by Wünschmann [31] and later used by Chern [6]. The two invariants in order four were introduced by Bryant [1]. The general case was treated by Dunajski and Tod [10]. We use the name Wünschmann invariants in all cases for convenience and because all invariants have similar nature. Actually, in the linear case, all of them were defined already by Wilczynski [30]. Doubrov [8] generalised the Wilczynski invariants to non-linear case by computing Wilczynski invariants for the linearised equation. It appears that this procedure also gives the Wünschmann invariants, cf. [10]. In what follows we will sometimes say that the Wünschmann condition (or Bryant condition in the case of order 4) holds if all Wünschmann invariants are zero.

In our work the following approach to the Wünschmann invariants will be useful. One looks for sections of \( \mathcal{V} = \text{span}\{\partial_k\} \) and \( X_F = \text{span}\{X_F\} \), which are necessarily of the form \( g \partial_k \) and \( f X_F \) for some functions \( f \) and \( g \), and imposes the condition

\[
ad_{fX_F}^{k+1} g \partial_k = 0 \mod g \partial_k, \quad \text{ad}_{fX_F} f \partial_k, \quad \text{ad}_{fX_F}^2 g \partial_k, \ldots, \quad \text{ad}_{fX_F}^{k-2} g \partial_k, \quad X_F,
\]

(2)
where \( \text{ad}_X Y = [X, Y] \) is the Lie bracket of vector fields. One can prove that such \( f \) and \( g \) always exist (see Proposition 4.1 [22]) and then
\[
\text{ad}^{k+1}_{X_F} g \partial_k = L_0 g \partial_k + L_1 \text{ad}^{1}_{X_F} g \partial_k + \ldots + L_{k-2} \text{ad}^{k-2}_{X_F} g \partial_k \mod X_F.
\]
for some \( L_i \). Then, there exist rational numbers \( c_{ij} \in \mathbb{Q} \) such that the Wünschmann invariants are given by the formulae
\[
W_i = L_i + \sum_{j>i} c_{ij} f^{j-i} X_F^{j-i}(L_j).
\]
In particular the vanishing of all \( W_i \) is equivalent to the vanishing of all \( L_i \). The construction described above is a non-linear version of a construction of the Halphen normal form and reproduces the Wilczynski invariants for linear equations [30]. Moreover, if the Wünschmann invariants vanish then
\[
\text{ad}^{k+1}_{X_F} g \partial_k = 0 \mod X_F
\]
and it follows that \( g \partial_k \) depends polynomially on a parameter on integral curves of \( X_F \). It implies that the projection of \( g \partial_k \) to the solution space defines a field of rational normal curves in \( P(TM_F) \). This is a sketch of the proof of Theorem 3.1.

The construction of the Wünschmann invariants presented above can be split into two steps. One can look first for a function \( g \) and then for \( f \). Already the first step gives interesting results. Namely, (2) can be weakened to
\[
\text{ad}^{k+1}_{X_F} g \partial_k = 0 \mod g \partial_k, \text{ad}^k_{X_F} g \partial_k, \ldots, \text{ad}^1_{X_F} g \partial_k
\]
and such \( g \) always exists. This gives \( k \) coefficients \( K_0, K_1, \ldots, K_{k-1} \) defined by the formula
\[
\text{ad}^{k+1}_{X_F} g \partial_k = -K_0 g \partial_k + K_1 \text{ad}^{1}_{X_F} g \partial_k - K_2 \text{ad}^2_{X_F} g \partial_k + \ldots + (-1)^{k-1} K_{k-1} \text{ad}^{k-1}_{X_F} g \partial_k
\]
(we add the minus signs for convenience). The coefficients, called curvatures in [19] [20], have well defined geometric meaning. They are invariant with respect to contact transformations that do not change the independent variable \( t \). The class of transformations was called time-preserving contact transformations (or contact-affine transformations) in [20]. The class gives a natural framework in the context of control mechanical systems [19] [20], Finsler geometry (in this case \( K_0 \) is the flag curvature) and webs [23]. In the present paper we will use the invariants \( K_i \) to write down more complicated objects in a simple form (compare [17] in the case of second order). The curvatures \( K_i \) can be explicitly computed in terms of the original equation \( (F) \) using [20] Proposition 2.9. We will provide the formulae in the case of equations of order 2, 3 and 4 in Appendix A.

A function \( g \) defined by (1) is a non-trivial solution to
\[
X_F(g) = \frac{g}{k+1} \partial_k F.
\]
We will use the notation
\[
V = g \partial_k.
\]

In the subsequent sections we will extensively use the Lie derivative \( L_{X_F} \) acting on different objects. If not mentioned otherwise the terms “derivative” or “differentiation” will refer to \( L_{X_F} \). Moreover, we will denote differentiations by adding primes to the objects. In particular we will have
\[
V' = \text{ad}^{0}_{X_F} V, \quad V'' = \text{ad}^{2}_{X_F} V, \quad \ldots, \quad V^{(j)} = \text{ad}^{j}_{X_F} V
\]
for the vector field \( V \) or
\[
K'_i = X_F(K_i), \quad K''_i = X_F^2(K_i), \quad \ldots, \quad K^{(j)}_i = X_F^j(K_i)
\]
for the curvatures \( K_i \).
4 ODEs and connections

We assume that an ODE \((F)\) defines a paraconformal structure on \(M_F\) via Theorem 3.1 i.e. all Wünschmann invariants vanish. Let us introduce on the space of jets \(J^k(\mathbb{R}, \mathbb{R})\) the following integrable distributions

\[
\mathcal{D}_i = \text{span}\{\partial_k, \partial_{k-1}, \ldots, \partial_{k-i+1}\} = \text{span}\{V, V', \ldots, V^{(i-1)}\}
\]

which are tangent to the fibres of the natural projection \(J^k(\mathbb{R}, \mathbb{R}) \rightarrow J^{k-i}(\mathbb{R}, \mathbb{R})\), for \(i = 1, \ldots, k\). The distributions can be projected to the solution space \(M_F\). The projections give exactly the subspaces \(V_i \subset TM_F\) defined before by formula (1) for a paraconformal structure. Therefore, one can ask if the projection of leaves of \(\mathcal{D}_k\) defines a two-parameter family of totally geodesic hypersurfaces in \(M\). If yes, then we shall consider ODEs up to point transformations, i.e. transformations of variables \(t\) and \(x\) only, because in terms of jets we get precisely contact transformations preserving \(\mathcal{D}_k\). It follows that there is a double fibration picture

\[
M_F \leftarrow J^k(\mathbb{R}, \mathbb{R}) \rightarrow B
\]

where \(B = J^0(\mathbb{R}, \mathbb{R})\) is a space on which an equation is defined with a local coordinate system \((t, x)\) and \(M_F\) is the solution space as before.

**Definition 4.1** A class of point equivalent equations admits a totally geodesic paraconformal connection if the projections of the integral manifolds of \(\mathcal{D}_k\) to the solution space \(M_F\) are totally geodesic submanifolds with respect to a paraconformal connection on \(M_F\).

In order to construct a paraconformal connection on \(M_F\) we will construct a connection on \(J^k(\mathbb{R}, \mathbb{R})\) which is “invariant” along \(X_F\) and then we will project it to \(M_F\). Precisely, if \(\nabla\) is a connection on \(J^k(\mathbb{R}, \mathbb{R})\) then we would like to define a connection \(\tilde{\nabla}\) on \(M_F\) by the formula

\[
\tilde{\nabla}_{Y_1}Y_2 = \pi_*\nabla_{\pi^{-1}_*Y_1}\pi^{-1}_*Y_2.
\]

The definition is correct only for special \(\nabla\). There are two difficulties. Firstly, the lifts \(\pi_*^{-1}Y_i\) are given modulo \(X_F\) only. Secondly, \(\nabla\) may depend on a point in the fibre of \(\pi\). To overcome the difficulties we need several additional conditions.

**Lemma 4.2** A connection \(\nabla\) on \(J^k(\mathbb{R}, \mathbb{R})\) defines a connection \(\tilde{\nabla}\) on \(M_F\) via (8) if and only if

1. \(\nabla_YX = 0 \mod X_F\),
2. \(\nabla_XY = [X, Y] \mod X_F\),
3. \(\mathcal{L}_X\nabla Y = \nabla[X, Y] \mod \Omega^1(J^k(\mathbb{R}, \mathbb{R})) \otimes X_F\),

where \(X\) is an arbitrary section of \(X_F\) and \(Y\) is an arbitrary vector field on \(J^k(\mathbb{R}, \mathbb{R})\).

**Proof.** The first two conditions are equivalent to the fact that \(\nabla_{\pi_*^{-1}Y_1}\pi_*^{-1}Y_2 \mod X_F\) does not depend on the lift of \(Y_1\) or \(Y_2\) to \(J^k(\mathbb{R}, \mathbb{R})\). The third condition (together with the first one) is equivalent to the fact that \(\mathcal{L}_X\nabla Y = 0 \mod X_F\) for \(Y\) being a lift of a vector field on \(M_F\). It means that \(\pi_*\nabla Y\) is well defined independently on the point in the fibre of \(\pi\), hence defines a connection on \(M_F\). \(\square\)

If we assume that equation \((F)\) satisfies the Wünschmann condition and a connection \(\nabla\) on \(J^1(\mathbb{R}, \mathbb{R})\) satisfies the three conditions given in Lemma 4.2 then the connection \(\nabla\)
on $M_F$ will be compatible with the paraconformal structure on $M_F$ defined by $(F)$ if and only if
\[ \nabla V = \alpha V + \beta V', \tag{9} \]
for some two one-forms $\alpha$ and $\beta$ on $J^k(\mathbb{R}, \mathbb{R})$.

**Lemma 4.3** The one-forms $\alpha$ and $\beta$ satisfy the following system of differential equations
\[ \alpha' + k\beta'' = 0, \]
\[ \left( \binom{k+1}{j} - \frac{k}{2} \binom{k+1}{j} \right) \beta(k-j+2) + (-1)^{j+1} K_j^\prime \beta + (-1)^{j+1}(k-j+1)K_j \beta' \]
\[ = (-1)^{j+1}dK_j + \sum_{l=j+1}^{k-1} (-1)^{l+1} \left( \binom{l}{j-1} - \frac{k}{2} \binom{l}{j} \right) K_l \beta^{(l-j+1)} \tag{10} \]
for $j = 0, \ldots, k-1$.

**Proof.** A consecutive application of Lemma 4.2 gives $L^i_{X_F} \nabla V = \nabla V^{(i)} \mod X_F$. This written in terms of $\alpha$ and $\beta$ reads
\[ \nabla V^{(i)} = \sum_{j=0}^{i+1} \left( \binom{i}{j} \alpha^{(i-j)} + \binom{i}{j-1} \beta^{(i-j+1)} \right) V^{(j)}. \tag{11} \]
The formula is valid for all $i$. For $i = 1, \ldots, k$ it defines the connection uniquely (note that for $i = k$ the formula involves $K_j$’s via the last term $V^{(k+1)} = \sum_{j=0}^{k-1} (-1)^{j+1}K_j V^{(j)}$) and for $i = k + 1$ it gives a set of conditions that should be satisfied by $\alpha$ and $\beta$. The conditions are as follows
\[ \left( \binom{k+1}{j} \right) \alpha^{(k-j+1)} + \left( \binom{k+1}{j-1} \right) \beta^{(k-j+2)} + (-1)^{j+1} K_j^\prime \beta + (-1)^{j+1}(k-j+1)K_j \beta' \]
\[ = (-1)^{j+1}dK_j + \sum_{l=j+1}^{k-1} (-1)^{l+1} \left( \binom{l}{j-1} \alpha^{(l-j)} + \binom{l}{j-1} \beta^{(l-j+1)} \right) K_l \]
for $j = 0, \ldots, k$. In particular, for $j = k$ we get
\[ 2\alpha' = -k\beta'' \]
and using it we can eliminate derivatives of $\alpha$ from the remaining equations and obtain (10) as a result. \hfill $\square$

We get the following result

**Theorem 4.4** An ODE of order $k+1$ with the vanishing Wünschmann invariants admits a totally geodesic paraconformal connection if and only if there exists a one-form $\beta$ on $J^k(\mathbb{R}, \mathbb{R})$ satisfying
\[ -\frac{1}{2} \binom{k+2}{3} \beta''' + (-1)^k K_{k-1}^\prime \beta + 2(-1)^k K_{k-1} \beta' = (-1)^k dK_{k-1} \tag{12} \]
and
\[ \beta(V) = \beta(V') = \cdots = \beta(V^{(k-1)}) = 0. \tag{13} \]
Proof. Assume first that an ODE admits a totally geodesic paraconformal connection. Then by Lemma 4.3 it satisfies System (10). In particular, for \( j = k - 1 \) one gets (12). Moreover one should have
\[
\nabla_{\nabla V(i)} V^{(k-1)} \in D_k \mod \mathcal{X}_F
\]
for all \( j \leq k - 1 \). But, it follows from (11) that the coefficient of \( \nabla_{\nabla V(i)} V^{(k-1)} \) next to \( V^{(k)} \) is exactly the one form \( \beta \) evaluated on \( V^{(j)} \). Therefore \( \beta(V) = \beta(V') = \cdots = \beta(V^{(k-j)}) = 0 \).

In order to prove the theorem in the opposite direction it is sufficient to show that if (12) has a solution \( \beta \) and the Wünschmann invariants vanish then \( \beta \) solves also all other equations from the System (10). But in Lemma 4.2 one can use an arbitrary section of \( \mathcal{X}_F \) instead of \( X_F \). It is convenient to make all computations using a multiple of \( X_F \) by a function \( f \) as in (3). Such a function \( f \) exists due to the Wünschmann condition. If (3) is satisfied then all \( K_i \) in (10) are zero and the System (10) takes the form \( \beta^{(k-j+2)} = 0 \), \( j = 0, \ldots, k-1 \). The system clearly has a solution. \( \square \)

Remark. A reasoning similar to the proof of Theorem 4.4 implies that the projections to \( M_F \) of the integral manifolds of \( \mathcal{D}_i \) are totally geodesic for a paraconformal connection if \( \beta(V) = \beta(V') = \cdots = \beta(V^{(i-1)}) = 0 \). In particular, if projections of the integral manifolds of \( \mathcal{D}_i \) are totally geodesic then also projections of the integral manifolds of \( \mathcal{D}_j \) for \( j < i \) are totally geodesic. Let us use (11) again and compute the following torsion coefficient
\[
T(\nabla(V^{(i)}), V^{(i+1)}) = \nabla V^{(i)} V^{(i+1)} - \nabla V^{(i+1)} V^{(i)} - [V^{(i)}, V^{(i+1)}] = \beta(V^{(i)} V^{(i+2)} \mod \mathcal{D}_{i+2}.
\]
The last equality holds because \( [V^{(i)}, V^{(i+1)}] \in \mathcal{D}_{i+2} \). The expression has sense for \( i = 0, \ldots, k-2 \) and it follows that the condition \( \beta(V) = \beta(V') = \cdots = \beta(V^{(i)}) = 0 \) is expressed in terms of the torsion \( T(\nabla(V^{(i)}), V^{(i+1)}) \) for \( i = 0, \ldots, k-2 \). However, the condition (13) for \( i = k - 1 \) has a different nature.

Remark. Instead of using in the proof of Theorem 4.4 the vector field \( f X_F \) satisfying (3) one can differentiate (12) sufficiently many times and subtract it from the remaining equations from (10) in such a way that the highest derivatives of \( \beta \) are eliminated. Then one will recover the Wünschmann condition as vanishing of coefficients next to the derivatives of \( \beta \) of lower order. Conditions are given in terms of \( K_i \)'s. In particular in the case of an equation of order 3 we get
\[
K_0 + \frac{1}{2} K_1' = 0
\]
and it can be checked that \( W_0 = K_0 + \frac{1}{2} K_1' \) is really the Wünschmann invariant (\( K_0 \) and \( K_1 \) are given explicitly below in Appendix A).

In the case of order 4 we get
\[
K_0 + \frac{3}{10} K_1' - \frac{9}{100} K_2' = 0, \quad (15)
\]
\[
K_1 + K_2' = 0. \quad (16)
\]
We have computed that the conditions coincide with the conditions in [10] Theorem 1.3] and consequently with (11) (again, \( K_0, K_1 \) and \( K_2 \) are given explicitly below in Appendix A). Namely (16) is exactly the second condition in [10] and \( W_1 = K_1 + K_2' \) is the Wünschmann invariant. The first condition in (10) has the form
\[
K_0 + K_1' + \frac{7}{10} K_2'' - \frac{9}{100} K_2' - \frac{1}{4} \partial_3 F(K_1 + K_2')
\]
which is (15) modulo (16) and the derivative of (16).

In the general case the simplest Wünschmann condition has the form

\[ K_{k-2} + \frac{k-1}{2} K'_{k-1} = 0 \]  (17)

The other are more complicated, but we will not need them in the explicit form.

5 Twistor correspondence

The condition (13) means that the one-form \( \beta \) is a pullback of a one-form defined on the space \( B = J^0(\mathbb{R}, \mathbb{R}) \). One can call \( B \) the twistor space. Indeed, due to the double fibration (7) a point in \( B \) can be considered as a hypersurface in \( M_F \) and a point in \( M_F \) is represented by a curve in \( B \) which is a solution to (F). There is \( (k+1) \)-parameter family of such curves corresponding to different points in \( M_F \).

In the complex setting one can repeat the reasoning of [18, Section 5]. One considers a complex surface \( B \) and a curve \( \gamma \subset B \) with a normal bundle \( N_\gamma \simeq \mathcal{O}(k) \). Then \( H^0(\gamma, N_\gamma) = \mathbb{C}^{k+1} \) and \( H^1(\gamma, N_\gamma) = 0 \). Therefore by the Kodaira theorem one gets a \( (k+1) \)-dimensional complex manifold \( M \) parametrising a family of curves in \( B \) with self intersection number \( k \). One can see a paraconformal structure in this picture, an adapted connection and a set of totally geodesic surfaces. Indeed, if \( \gamma \) is a curve in \( B \) then due to \( N_\gamma \simeq \mathcal{O}(k) \) we get that for any collection of points \( \{y_1, \ldots, y_k\} \), \( y_i \in \gamma \), possibly with multiplicities, there is a one-parameter family of curves in \( B \) which intersect \( \gamma \) exactly at these points. The family of curves defines a geodesic in \( M \) (the fact that such a definition gives geodesics of a connection can be proved exactly as in [1] and follows from the fact \( H^1(\gamma, N_\gamma) = 0 \)). The null geodesics are defined by \( y_i \)'s such that \( y_1 = y_2 = \cdots = y_k \). A totally geodesic hypersurface in \( M \) corresponding to a point \( y \in B \) is defined by all curves which pass through \( y \). It follows automatically from the definition of the geodesics that such hypersurfaces are totally geodesic indeed.

6 Second order

Let

\[ x'' = F(t, x, x') \]

be a second order ODE. All two-dimensional manifolds possess canonical \( GL(2, \mathbb{R}) \)-structures. Hence, any second order equation defines a paraconformal structure on its solution space. However, the existence of a totally geodesic paraconformal connection is a more restrictive condition which is equivalent to the existence of a projective structure. A result due to Cartan [3] says that a class of point equivalent ODEs defines a projective structure on the solution space if and only if the Cartan invariant \( C \) vanishes. In coordinates (see [7])

\[
C = \frac{\partial^2_t F}{2} - \frac{1}{2} F \partial_t \partial^2_1 F - \frac{1}{2} \partial_t F \partial^2_1 F - \frac{2}{3} \partial_t \partial_0 \partial_1 F + \frac{1}{6} \partial^2_0 \partial^2_1 F + \]
\[
\frac{1}{3} x_1 \partial_0 \partial_0 \partial^2_1 F + \frac{1}{6} x_1 \partial_1 \partial^3_1 F + \frac{1}{3} F \partial_0 \partial_1 F - \frac{2}{3} x_1 \partial_0 \partial_1 F + \frac{1}{6} x_1^2 \partial^2_0 \partial^2_1 F + \]
\[
\frac{1}{6} x_1 \partial_0 F \partial^2_1 F + \frac{1}{3} x_1 F \partial_0 \partial^2_1 F + \frac{2}{3} \partial_1 F \partial_0 \partial_1 F - \frac{1}{6} \partial_1 F \partial_0 \partial^2_1 F - \]
\[
\frac{1}{6} x_1 \partial_1 F \partial_0 \partial^2_1 F + \frac{1}{6} F^2 \partial^1_1 F.
\]
On the other hand Theorem 4.4 specified to \( k = 1 \) implies that the existence of a totally geodesic paraconformal connection is equivalent to the existence of a solution to

\[
-\frac{1}{2} \beta''' - K_0' \beta - 2K_0 \beta' = -dK_0
\]

satisfying

\[
\beta(V) = 0.
\]

Thus, we reproduce Cartan’s result in the following form

**Theorem 6.1** A second order ODE defines a projective structure on its solution space if and only if

\[
4V'(K_0) - V(K_0') = 0.
\]

Additionally \( C = 4V'(K_0) - V(K_0') \).

**Proof.** We are looking for a common solution to (18) and (19). Let us denote

\[ \beta(V') = b. \]

Taking into account that \( V'' = -K_0 V \) and differentiating (19) one finds

\[
\beta'(V) = -b, \quad \beta''(V) = -2b', \quad \beta'''(V) = -3b'' + K_0 b
\]

and

\[
\beta'(V') = b', \quad \beta''(V') = b'' - K_0 b, \quad \beta'''(V') = b''' - K_0 b - 3K_0 b'.
\]

Thus, evaluating (18) on \( V \) and \( V' \) one gets

\[
\begin{aligned}
\frac{3}{2} b'' + \frac{3}{2} K_0 b &= -V(K_0), \\
\frac{1}{2} b''' + \frac{1}{2} K_0' b + \frac{1}{2} K_0 b' &= V'(K_0).
\end{aligned}
\]

Differentiating the first equation and substituting to the second one one gets (20). Besides one can check by direct computations using formulae in Appendix A that (20) coincides with \( C \).

\[ \Box \]

7 Third order

Let

\[ x''' = F(t, x, x', x'') \]

be a third order ODE. Its solution space is a three dimensional manifold. A paraconformal structure on a three dimensional manifold is a conformal metric \([g]\) of Lorentzian signature. Moreover, a torsion-free connection adapted to a paraconformal structure is a Weyl connection \( \nabla \) for \([g]\). We recall that if a representative \( g \in [g] \) is chosen then a Weyl connection \( \nabla \) is uniquely defined by a one-form \( \varphi \) such that

\[ \nabla g = \varphi g. \]

According to Cartan [4], a Weyl connection \( \nabla \) is totally geodesic in our sense if and only if the Einstein equation is satisfied

\[ Ric(\nabla)_{sym} = \frac{1}{3} R_g(\nabla)g \]
where $\text{Ric}(\nabla)_{\text{sym}}$ is the symmetric part of the Ricci curvature of $\nabla$ and $R_g(\nabla)$ is the scalar curvature with respect to $g$. The pair $(|g|, \nabla)$ is called an Einstein-Weyl structure in this case. Cartan also proved that there is a one to one correspondence between Einstein-Weyl structures and third order ODEs for which the Wünschmann $W_0$ and Cartan $C$ invariants vanish (see [25, 28]). In coordinates

$$W_0 = \partial_0 F - \frac{1}{2} X_F(\partial_1 F) + \frac{1}{3} \partial_1 F \partial_2 F + \frac{1}{6} X_F^2(\partial_2 F) - \frac{1}{3} X_F(\partial_2 F) \partial_2 F + \frac{2}{27} (\partial_2 F)^3,$$

$$C = X_F^2(\partial_2 F) - X_F(\partial_1 \partial_2 F) + \partial_0 \partial_2 F.$$

On the other hand Theorem 4.3 implies that if $W_0 = 0$ then the existence of a totally geodesic paraconformal connection is equivalent to the existence of a solution to

$$-2\beta'' + K_1' \beta + 2K_1 \beta' = dK_1$$

satisfying

$$\beta(V) = \beta(V') = 0.$$

We reproduce Cartan’s result in the following way

**Theorem 7.1** A third order ODE defines an Einstein-Weyl structure on its solution space if and only if $W_0 = 0$ and

$$2V'(K_1) + V(K_1') = 0.$$  

Additionally $C = -\frac{3}{2} (V'(K_1) - V(K_0))$ and under the Wünschmann condition $2V'(K_1) + V(K_1') = 2(V'(K_1) - V(K_0)) = -\frac{3}{4} C$.

**Proof.** The formula for $C$ in terms of $K_0$ and $K_1$ can be verified by computations using the appropriate formulae given in Appendix A. The invariant meaning of this expression follows from our proof. We are looking for a common solution to (21) and (22). Let us denote

$$\beta(V'') = b.$$

Taking into account that $V''' = -K_0 V + K_1 V'$ and differentiating (22) one finds

$$\beta'(V) = 0, \quad \beta''(V) = b, \quad \beta'''(V) = 3b',$$

$$\beta'(V') = -b, \quad \beta''(V') = -2b', \quad \beta'''(V') = -3b'' - K_1 b,$$

and

$$\beta'(V'') = b', \quad \beta''(V'') = b'' + K_1 b, \quad \beta'''(V'') = b''' + K_1' b + 3K_1 b' + K_0 b.$$

Thus, evaluating (22) on $V$, $V'$ and $V''$ one gets

$$6b' = -V(K_1),$$

$$6b'' = V'(K_1),$$

$$2b''' + K_1' b + 4K_1 b' + 2K_0 b = -V''(K_1).$$

The last equation reduces to $-2b''' - 4K_1 b' = V''(K_1)$ due to the Wünschmann condition $W_0 = 0$ which is equivalent to $K_1' = -2K_0$. Then, differentiating the first equation one gets that a common solution $b$ exists if and only if $V'(K_1) + \frac{1}{2} V(K_1') = 0$ and $K_1 V(K_1) = 2V'''(K_1) + \frac{1}{2} V'_1(K_1')$. We get (23) and using the Wünschmann condition again

$$K_1 V(K_1) = 2V''(K_1) - V_1(K_0).$$  

Now, the theorem follows from the following...
Lemma 7.2 If \((\ref{equation_23})\) holds and the Wünschmann invariant vanishes for a third order ODE then also \((\ref{equation_21})\) holds.

Proof. We can write \([V, V'] = AV + BV'\) for some functions \(A\) and \(B\). Taking the Lie brackets with \(X_F\) and using the fact that \(V''' = -K_0V + K_1V'\) we get formulae for \([V, V''], [V', V''\] in terms of \(A, B\) and their derivatives. Namely \([V, V''] = A'V + (A + B')V' + BV''\) and \([V', V''] = (A'' + V(K_0) - BK_0 - AK_1)V + (2A' + B'' - V(K_1))V' + (A + 2B')V''\).

One more Lie bracket and the Jacobi identity gives the following three equations
\[
\begin{align*}
3A' + 3B'' - V(K_1) &= 0, \\
3A'' + B''' - X_FV(K_1) + V(K_0) - 2BK_0 + 2B'K_1 - V'(K_1) &= 0, \\
A''' - 3B'K_0 - BK_0' - A'K_1 + XV(K_0) + V'(K_0) &= 0.
\end{align*}
\]

Differentiating the first equation and substituting to the second and third one we can eliminate \(A\) and its derivatives. But due to the Wünschmann condition we can also eliminate \(B\) and its derivatives and get one relation
\[
2V'(K_0) + \frac{2}{3}X_F^2V(K_1) + 2X_FV(K_0) - \frac{2}{3}K_1V(K_1).
\]

The subsequent use of the Wünschmann condition and the relation \(\text{ad}_{X_F}V = V^{(i)}\) reads
\[
K_1V(K_1) = 2V'(K_0) + V''(K_1) + V(K_0).
\]

On the other hand, differentiating \((\ref{equation_23})\) we get
\[
0 = V'''(K_1) - 3V'(K_0) - V(K_0)
\]
and adding the last two equations we finally get \((\ref{equation_24})\).

\(\square\)

8 Fourth order

Let
\[
x^{(4)} = F(t, x, x', x'', x''')
\]
be a fourth order ODE. This is the case considered by Bryant \cite{Bryant}. However, it appears that the torsion free connections of \cite{Bryant} are not, in general, totally geodesic in the sense of the present paper. Bryant proved in \cite{Bryant} Theorem 4.1] that a paraconformal structure possesses a torsion free connection if and only if every null plane, i.e. every subspace \(V_2(s : t)(x) \subset T_xM\) for \(x \in M\) and \((s : t) \in \mathbb{R}P^1\) in the notation of Section 2 is tangent to a totally-geodesic surface in \(M\). We saw in Section 3 (the first remark following Theorem 4.4) that this condition is really expressed in terms of the torsion. Actually, according to Bryant, this condition is also equivalent to the fact that a paraconformal structure is defined by an equation and it can be expressed as vanishing of a polynomial of degree 7 in \((s : t)\) (compare \cite{Bryant}). The corresponding ODE satisfies the Bryant-Wünschmann condition.

Our Theorem 4.4 describes paraconformal structures satisfying more restrictive conditions. Namely, any subspace \(V_3(s : t)(x) \subset T_xM\) for \(x \in M\) and \((s : t) \in \mathbb{R}P^1\) is tangent to a totally-geodesic submanifold of \(M\). We get that it happens if and only if the Bryant-Wünschmann condition holds and additionally there is a solution to
\[
5\beta''' + K_2^3\beta + 2K_2\beta'' = dK_2
\]
(25)
satisfying
\[ \beta(V) = \beta(V') = \beta(V'') = 0. \tag{26} \]
Of course, a torsion-free connection also exists in this case, but this connection is not necessarily totally geodesic in our sense as we will see in Section 10. We get the following new result

Theorem 8.1 A fourth order ODE admits a totally geodesic paraconformal connection if and only if the Bryant-Wünschmann condition holds and
\[ 4V'(K_2) + 3V(K'_2) = 0. \tag{27} \]
Additionally under the Bryant-Wünschmann condition
\[ 4V'(K_2) + 3V(K'_2) = 4V'(K_2) - 3V(K_1). \]

Proof. The proof is similar to proofs of Theorems 6.1 and 7.1. Let us denote
\[ \beta(V''') = b. \]
Taking into account that
\[ V^{(4)} = -K_0V + K_1V' - K_2V'' \]
and differentiating (26) one finds
\[ \beta'(V) = 0, \quad \beta''(V) = 0, \quad \beta'''(V) = -b, \]
\[ \beta'(V') = 0, \quad \beta''(V') = b, \quad \beta'''(V') = 3b', \]
\[ \beta'(V'') = -b, \quad \beta''(V'') = -2b', \quad \beta'''(V'') = -3b' + K_2b. \]
and
\[ \beta'(V''') = b', \quad \beta''(V''') = b' - K_2b, \quad \beta'''(V''') = b'' - K'_2b - 3K_2b' - K_1b. \]
Thus, evaluating (25) on \( V, V', V'' \) and \( V''' \) one gets
\[ 5b = -V(K_2), \]
\[ 15b' = V'(K_2), \]
\[ 15b'' - 3K_2b = -V''(K_2), \]
\[ 5b''' - 4K_2b - 13K_2b' - 5K_1b = V'''(K_2). \]
If we differentiate the first equation and substitute it to the second one we get the condition (27). From the last two equations we get two additional conditions of higher order. However, as in the case of order three, they are consequences of (27) and the Wünschmann condition and do not give new conditions on the equation (F) (we have checked it by direct computations in coordinates). The identity \( 4V'(K_2) + 3V(K'_2) = 4V'(K_2) - 3V(K_1) \) follows from (10).

Corollary 8.2 If a fourth order ODE admits a totally geodesic paraconformal connection then a solution \( \beta \) to (25) and (26) is unique.

Proof. The one-form \( \beta \) is uniquely defined by the condition \( 5b = -V(K_2). \)

Remark. It would be nice to have a characterisation of (F) admitting a totally geodesic paraconformal structure in terms of the curvature of the associated torsion-free
Bryant connection. The curvature was explicitly computed in [26]. However, the Bryant connection is an object invariant with respect to the group of contact transformations which is much bigger than the group of point transformations. In fact a class of point equivalent ODEs splits into several classes of point equivalent ODEs. Therefore the problem would be to determine if a given class of contact equivalent ODEs contains a subclass of point equivalent ODEs which admits a totally geodesic paraconformal connection. A priori, the subclass of point equivalent equations is not unique for a given class of contact equivalent ODEs. The problem is similar, in spirit, to the problem considered in [12] where we characterise hyper-CR Einstein-Weyl structures in terms of point invariants. One gets invariants of very high order and the similar result should hold in the present case.

Remark. Let $\nabla$ be a totally geodesic paraconformal connection associated to a fourth order ODE. Direct computations show

$$R(\nabla)(Y_1, Y_2)V = d\alpha(Y_1, Y_2)V + d\beta(Y_1, Y_2)V' - \beta \wedge \alpha'(Y_1, Y_2)V - \beta \wedge \beta'(Y_1, Y_2)V'$$

and one gets

$$Ric(\nabla)(V, V) = d\beta(V', V) - \beta \wedge \beta'(V', V).$$

But, if $\beta(V) = \beta(V') = \beta(V'') = 0$ then the right hand side vanishes (we use here $[V, V'] \in \text{span}\{V, V'\}$) and therefore

$$Ric(\nabla)(V, V) = 0.$$

It follows that the symmetric part of the Ricci tensor of $\nabla$ is a section of the bundle of symmetric 2-tensors annihilating the field of null cones $x \mapsto C(x)$ of the paraconformal structure. The bundle has rank 3 in the case of fourth order.

The situation should be compared with the case of third order. In this lower dimensional case we also have $Ric(\nabla)(V, V) = 0$ (this is true also in dimensions greater than four), but the bundle of symmetric 2-tensors annihilating null cones is one dimensional. Precisely it is the conformal metric $[g]$. It follows that $Ric(\nabla)_{\text{sym}}$ is proportional to $g$. More detailed computations give the Einstein-Weyl equation.

9 General case

In this section we consider an ODE in the form $(F)$. Our main result is as follows

**Theorem 9.1** An ODE of order $k + 1$, where $k \geq 4$ admits a totally geodesic paraconformal connection if and only if the W"unschmann condition holds,

$$V^{(i)}(K_{k-1}) = 0, \quad i = 0, \ldots, k - 4,$$

and

$$4V^{(k-2)}(K_{k-1}) + 3V^{(k-3)}(K_{k-1}') = 0,$$

and

$$\left( \frac{2}{\gamma_k} - 1 \right) K_{k-1}V^{(k-3)}(K_{k-1}) = (-1)^k \left( V^{(k-2)}(K_{k-1}') - 2V^{(k-1)}(K_{k-1}) \right),$$

$$\left( \frac{2}{3\gamma_k} - 1 \right) K_{k-1}V^{(k-2)}(K_{k-1}) + \left( \frac{1}{\gamma_k} + \frac{k - 3}{2} \right) K_{k-1}'V^{(k-3)}(K_{k-1}) = (-1)^{k+1}\left( V^{(k-2)}(K_{k-1}) - V^{(k-1)}(K_{k-1}') + V^{(k)}(K_{k-1}) \right);$$

13
where $\gamma_k = -\frac{1}{2} \binom{k+2}{3}$. Additionally (29) is equivalent to

$$4V^{(k-2)}(K_{k-1}) - \frac{6}{k-1}V^{(k-3)}(K_{k-2}) = 0.$$ 

Proof. Let $\theta_0, \theta_1, \ldots, \theta_k$ be one-forms dual to vector fields $V, V', \ldots, V^{(k)}$. Assume that $\beta(V_k) = b$. Then $\beta = b\theta_k$. We compute $\theta'_k = -\theta_{k-1}$, $\theta''_k =\theta_{k-2} + (-1)^{k+1}K_{k-1}\theta_k$ and $\theta'''_k = -\theta_{k-3} + (-1)^k K_{k-1} + (-1)^{k+1}(K'_{k-1} + K_{k-2})\theta_k$. Thus

$$\beta' = b'\theta_k - b\theta_{k-1}$$

and

$$\beta''' = \left(b''' + 3(-1)^{k+1}b'K_{k-1} + (-1)^{k+1}b(K'_{k-1} + K_{k-2})\right)\theta_k + (-3b'' + (-1)^k K_{k-1})\theta_{k-1} + 3b'\theta_{k-2} - b\theta_{k-3}.$$

Substituting this to (12), evaluating on $V_0, \ldots, V_k$, and using the Wünschmann condition (17) we get the conditions (28)-(30) in a way analogous to lower dimensional cases. □

Corollary 9.2 If an ODE of order $k + 1 > 4$ admits a totally geodesic paraconformal connection then a solution $\beta$ to (12) and (13) is unique.

Proof. The one-form $\beta$ is uniquely defined by the condition $(-1)\binom{k+2}{3}b = (-1)^{k+2}2V^{(k-3)}(K_{k-1})$ which is obtained by evaluation of (12) on $V^{(k-3)}$. □

Remark. Our conjecture is that the equations (30) and (31) are redundant and follow from (28), (29) and the Wünschmann condition. However, we were unable to prove it in full generality.

10 Veronese webs

A particularly simple example of paraconformal structures admitting totally geodesic connections can be obtained from special families of foliations, called Veronese webs. They are one-parameter families of foliations introduced by Gelfand and Zakharevich [15] in connection to bi-Hamiltonian systems on odd-dimensional manifolds. Precisely, a one-parameter family of foliations $\{F_t\}_{t \in \mathbb{R}}$ of co-dimension 1 on a manifold $M$ of dimension $k + 1$ is called Veronese web if any $x \in M$ has a neighbourhood $U$ such that there exists a co-frame $\omega_0, \ldots, \omega_k$ on $U$ such that $T F_t = \ker \left(\omega_0 + t\omega_1 + \cdots + t^k\omega_k\right)$.

In [23] we proved that there is a one to one correspondence between Veronese webs and ODEs for which all curvatures $K_i$ vanish. The equations are given modulo time-preserving contact transformations, mentioned earlier in Section 3. But if all $K_i = 0$ then automatically all conditions given in Theorems 6.1, 7.1, 8.1 and 9.1 are satisfied. Therefore all Veronese webs admit totally geodesic paraconformal connections. The paraconformal structures obtained in this way will be called of Veronese type. The structures are
very specific. They correspond to projective structures defined by connections with skew-symmetric Ricci tensor in the case of order 2 (see [24]), and to Einstein-Weyl structures of hyper-CR type in the case of order 3 (see [11]). The connections in the case of order 2 are projectively equivalent to the Chern connections of classical 3-webs [24]. The hyper-CR Einstein-Weyl structures are connected to integrable equations of hydrodynamic type and have Lax pairs with no terms in the direction of a spectral parameter [9]. A characterisation of this special Einstein-Weyl structures in terms of point invariants of the related ODEs is complicated and involves 4 additional invariants of high order [12]. In any case, Veronese webs define exactly those totally geodesic paraconformal structures for which the corresponding twistor space fibres over \( \mathbb{R} P^1 \) (cf. [9, 11]).

Let \( \omega(t) = \omega_0 + t\omega_1 + \cdots + t^k\omega_k \). Then the curve \( t \mapsto \mathbb{R}\omega(t) \in P(T^* M) \) is a Veronese curve dual to the curve \( t \mapsto \mathbb{R}V(t) \in P(TM) \) of null directions defining a paraconformal structure in Section 2. In the case of Veronese webs the parameter \( t \) is well defined globally uniquely modulo the Möbius transformations \( t \mapsto \frac{at + b}{ct + d} \), where \( a, b, c, d \in \mathbb{R} \) and \( ad - bc \neq 0 \).

Moreover, the distributions \( V_i(t) \) are integrable for any particular choice of \( t \). It means that

\[
\omega(t) \wedge d\omega(t) = 0, 
\]

for any \( t \). In order to get useful formulæ we note that due to the integrability condition one can choose local coordinates \( x_0, \ldots, x_k \) on \( M \) such that \( TF_{t_i} = \ker dx_i \) for some fixed \( t_0, \ldots, t_k \in \mathbb{R} \). If we also assume that

\[
TF_{t_{k+1}} = \ker dw
\]

for a function \( w = w(x_0, \ldots, x_k) \) then one verifies that

\[
\omega(t) = \sum_{i=0}^{k} (t_{k+1} - t_i) \prod_{j \neq i} (t - t_j) \partial_i w dx_i 
\]

and \( \omega \) is given up to a multiplication by a function on \( M \). The following theorem for \( k = 2 \) was proved in [32] and [11]. It is new for \( k > 2 \).

**Theorem 10.1** Let \( M \) be a manifold of dimension \( k + 1 \) with local coordinates \( x_0, \ldots, x_k \) and let \( t_0, \ldots, t_{k+1} \in \mathbb{R} \) be distinct numbers. Then, any \( w \) satisfying the system

\[
\sum_{\text{cyclic}(i,j,l)} a_{ij,l} \partial_i \partial_j w \partial_l w = 0, \quad 0 \leq i < j < l \leq k, 
\]

where

\[
a_{ij,l} = (t_i - t_j)(t_{k+1} - t_l),
\]

defines a paraconformal structure via \( (33) \), and conversely, any paraconformal structure of Veronese type can be locally put in this form. Moreover, if \( k > 2 \) then any totally geodesic paraconformal connection is given by the formula

\[
\nabla_{\partial_i} = \left( \frac{d(\partial_i w)}{\partial_i w} + \alpha \right) \partial_i 
\]

for some one-form \( \alpha \) on \( M \). If \( k = 2 \) then the conformal metric is defined by

\[
g = \sum_{i,j=0}^{2} (t_3 - t_i)(t_3 - t_j) \left( t_i^2 + t_j^2 - t_it_j - \sum_{l=0}^{2} t_l^2 \right) \partial_i w \partial_j w dx_i dx_j
\]
and 
\[ \varphi = \left( \frac{\partial_0 \partial_1 w}{\partial_1 w} + \frac{\partial_0 \partial_2 w}{\partial_2 w} \right) dx_0 + \left( \frac{\partial_0 \partial_1 w}{\partial_1 w} + \frac{\partial_1 \partial_2 w}{\partial_2 w} \right) dx_1 + \left( \frac{\partial_0 \partial_2 w}{\partial_1 w} + \frac{\partial_1 \partial_2 w}{\partial_2 w} \right) dx_2. \]
is a unique one-form such that the pair \((g, \varphi)\) defines an Einstein-Weyl structure. If \(k = 1\) then
\[ \nabla \partial_0 = \left( \frac{\partial_0 \partial_1 w}{\partial_1 w} - \frac{\partial_0 \partial_1 w}{\partial_1 w} \right) dx_0 \partial_0, \quad \nabla \partial_1 = \left( \frac{\partial_1 \partial_1 w}{\partial_1 w} - \frac{\partial_0 \partial_1 w}{\partial_1 w} \right) dx_1 \partial_1. \]
defines a unique torsion-free, totally geodesic paraconformal connection satisfying the additional condition \(\nabla Y V(t) \in \{V(t)\}\) for any vector field \(Y\) and any \(t \in \mathbb{R}\).

**Proof.** The integrability condition (32) written in coordinates and in terms of the function \(w\) takes the form
\[ \sum_{i<j<l} \left( \sum_{\text{cyc}(i,j,l)} (T_i - T_j) T_i \partial_i \partial_j w \partial_l w \right) dx_i \wedge dx_j \wedge dx_l = 0 \]
where
\[ T_i = (t_{k+1} - t_i) \prod_{j \neq i} (t - t_j). \]
Thus, for any \(i < j < l\) one gets the equation
\[ \sum_{\text{cyc}(i,j,l)} (T_i - T_j) T_i \partial_i \partial_j w \partial_l w = 0 \] (35)
which should be satisfied for any \(t \in \mathbb{R}\). But the coefficient \((T_i - T_j) T_i\) equals to
\[ a_{i,j,l} P_{i,j,l}(t, t_1, \ldots, t_k, t_{k+1}) \]
where
\[ P_{i,j,l} = (t - t_i)(t - t_j)(t - t_l)(t - t_{k+1}) \prod_{s \neq i,j,l} (t - t_s)^2 \]
is a polynomial that does not depend on the permutation of indices \((i,j,l)\) and has zeroes (with multiplicities) exactly at points \(t_0, \ldots, t_{k+1}\). Thus, for \(t \in \{t_0, \ldots, t_{k+1}\}\) we get that the condition (35) is void and for \(t \notin \{t_0, \ldots, t_{k+1}\}\) the condition (34) reduces to (33).

The converse statement follows from the fact that any Veronese web can be written down as \((33)\) in some coordinate system.

The formulae for \(\nabla\) can be computed in the following way. We have \(\nabla V(t) = \alpha V(t) + \beta V'(t)\) where the one-forms \(\alpha\) and \(\beta\), a priori, depend on \(t\). However, in the Veronese case system (10) gives \(\alpha' = -\frac{4}{3} \beta''\) and \(\beta'' = 0\). It follows that \(\beta\) is a polynomial of degree 2 in \(t\). Moreover, in the case of Veronese webs \(V(t)\) considered on \(M\) as in Section 2 satisfies (13) and it means that \(\beta = f \omega\) for some function \(f\) on \(M\). Hence \(f = 0\) for \(k > 2\) and consequently \(\alpha\) does not depend on \(t\). Therefore \(\nabla V(t) = \alpha V(t)\) and in coordinates we get \(\nabla \partial_i = \left( \frac{d(\partial_i w)}{\partial_i w} + \alpha \right) \partial_i\). It finishes the proof in the case \(k > 2\).

In the case \(k = 2\) the theorem follows from (11). Here, we present a sketch of a different, more direct, proof. We have \(\beta = f \omega\) and \(\alpha = \tilde{\alpha} - ftk \omega''\), where \(\tilde{\alpha}\) is a one-form on \(M\). Simple but long computations prove that \(\nabla\) is torsion-free if and only if
\[ f = \frac{1}{4} \frac{\partial_0 \partial_1 w - \partial_1 \partial_2 w}{\partial_0 w \partial_1 w} \frac{1}{(t_3 - t_1)(t_0 - t_2)}. \]
and
\[
\tilde{\alpha} = -\frac{1}{4} \sum_{cijkl(0,1,2)} \left( \frac{t_1 - 3t_2}{t_1 - t_2} \frac{\partial_0 \partial_1 w}{\partial_1 w} + \frac{t_2 - 3t_1}{t_2 - t_1} \frac{\partial_0 \partial_2 w}{\partial_2 w} \right) dx_0.
\]

The formula for \( f \) does not depend on the permutation of indices \((0,1,2)\) due to (34). Having \( f \) and \( \tilde{\alpha} \) one has all ingredients necessary for the computation of \( \nabla g \) and consequently \( \varphi \). The so-obtained connection satisfies the Einstein-Weyl equation due to results of Cartan.

In the case \( k = 1 \) the one-form \( \beta \) is linear in \( t \) and equals \( f \omega \) for some function \( f \). It follows that \( \alpha \) does not depend on \( t \) and the vanishing of the torsion gives
\[
\alpha = -\left( \frac{\partial_0 \partial_1 w}{\partial_1 w} - 2f(t_2 - t_0) \partial_0 w \right) dx_0 - \left( \frac{\partial_0 \partial_1 w}{\partial_0 w} - 2f(t_2 - t_1) \partial_1 w \right) dx_1.
\]

It can be shown that the so obtained connection satisfies \( \nabla Y V(t) \in \text{span}\{V(t)\} \) for any \( Y \) if and only if \( f = 0 \). For \( f \neq 0 \) we only have \( \nabla V(t) V(t) \in \text{span}\{V(t)\} \).

**Remark.** Note that \( \sum_{cijkl(i,j,l)} a_{ijkl} = 0 \) and (34) is a Hirota equation (c.f. [11, 13, 32]). It follows that the first part of Theorem [10.1] is a generalisation of unpublished preprint [32], Corollary 3.7 whereas the second part is a generalisation of [11, Theorem 1]. The connection for \( k = 1 \) is exactly the Chern connection of a 3-web [23]. The connection for \( k = 2 \) is exactly the hyper-CR connection from [11] (in [11] the conformal class is defined by our \( g \) multiplied by \((w_0 w_1 w_2)^{-1}\) and consequently the one-form \( \phi \) from [11] equals \( \varphi = d\ln(w_0 w_1 w_2) \)).

In the case of \( k > 2 \) we get the following result, which in particular shows that the totally geodesic paraconformal connections are not the torsion free connections used in [1].

**Corollary 10.2** If \( k > 2 \) then for a non-flat paraconformal structure of Veronese type all totally geodesic paraconformal connections have non-vanishing torsion.

**Proof.** We assume that a Veronese web \( \{F_t\}_{t \in \mathbb{R}} \) is described by a function \( w \) as in Theorem [10.1]. A connection \( \nabla \) is defined by the formula \( \nabla \partial_i = \left( \frac{d(\partial_i w)}{\partial_i w} + \alpha \right) \partial_i \). Thus, if the torsion of \( \nabla \) vanishes then \( \alpha(\partial_i) = -\frac{\partial_j \partial_i w}{\partial_j w} \) for any \( j \neq i \). Taking \( l \neq j \) and computing \( \alpha(\partial_l) \) in two ways we get that \( \partial_l \left( \frac{\partial_i w}{\partial_l w} \right) = 0 \). It implies that \( \text{grad}(w) \) is proportional to a vector field \( (b_0, \ldots, b_k) \) where \( b_i \) is a function of \( x_i \) only. All \( b_i \) are non-vanishing functions because any two foliations from the family \( \{F_t\}_{t \in \mathbb{R}} \) intersect transversally. If we change local coordinates \( \tilde{x}_i := \int b_i dx_i \) then still \( \ker d\tilde{x}_i = TF_{\tilde{x}_i} \) and we get that in new coordinates the web is described by \( w = \tilde{x}_0 + \ldots + \tilde{x}_k \) which means that the corresponding paraconformal structure is flat.

**Remark.** As mentioned before, there is a unique torsion-free paraconformal connection (Bryant connection) in the case of paraconformal structures defined by equations of order 4. It follows from above that in the case of Veronese webs the one-form \( \beta \), involving the torsion-free connection via \( \nabla V(t) = \alpha V(t) + \beta V'(t) \) does not vanish unless the structure is flat. Precisely, \( \beta = \beta_0 + t \beta_1 + t^2 \beta_2 \) for some one-forms \( \beta_i \) which do not depend on \( t \). The one-forms \( \beta_i \) can be computed explicitly. We shall do this in order to show the difference between totally geodesic connections and the Bryant connection.
Assume that a paraconformal structure is given by \( V(t) = V_0 + tV_1(t) + t^2V_2(t) + t^3V_3(t) \) and let us introduce structural functions \( c_{ij} \) by

\[
[V_i, V_j] = \sum_{l=0}^{3} c_{ij} V_l
\]

and let \( \eta_0, \eta_1, \eta_2, \eta_3 \) be the dual one-forms such that \( \eta_i(V_j) = \delta_{ij} \). Then

\[
\beta_0 = \frac{1}{3} c_{02}^3 \eta_0 + \frac{1}{3} c_{12}^3 \eta_1 + (2c_{03}^2 - c_{12}^3) \eta_2 - c_{03}^3 \eta_3,
\]

\[
\beta_1 = (c_{03}^3 - c_{13}^3) \eta_0 + \left( \frac{1}{3} c_{01}^3 + \frac{1}{3} c_{13}^3 - c_{03}^2 \right) \eta_1 + \left( \frac{1}{3} c_{23}^3 + \frac{1}{3} c_{02}^3 - c_{13}^2 \right) \eta_2 + (c_{03}^3 - c_{13}^2) \eta_3,
\]

\[
\beta_2 = -c_{03}^2 \eta_0 + (2c_{03}^2 - c_{13}^2) \eta_1 + \frac{1}{3} c_{12}^3 \eta_2 + \frac{1}{3} c_{13}^3 \eta_3,
\]

and additionally

\[
\alpha = (3c_{02}^2 - 2c_{03}^3) \eta_0 + (3c_{03}^2 - c_{01}^3) \eta_1 - c_{02}^3 \eta_2 - c_{03}^3 \eta_3.
\]

To get these expressions one considers \( \nabla V(t) = \alpha V(t) + \beta V'(t) \) which gives \( \nabla V_i \) in terms of \( \alpha \) and \( \beta_i \). Then the vanishing of the torsion gives 24 linear equations for 16 unknown functions: \( \beta_i(V_j) \) and \( \alpha(V_j) \), \( i = 0, 1, 2, \) and \( j = 0, \ldots, 3 \). However, the structural functions \( c_{ij} \) satisfy eight additional linear relations given explicitly in [22]. The additional relations are exactly obstructions for the vanishing of the torsion.

In the case of Veronese webs

\[
V_i = (-1)^{i+1} \binom{3}{i} \sum_{j=0}^{3} \frac{t_{3-i}}{\partial_{t_j}} \left( \prod_{l \in \{0,1,2,3,4\} \setminus j} \frac{1}{t_l - t_j} \right) \partial_j
\]

and

\[
c_{ij} = \sum_{a,b=0}^{3} d_{ij,ab} \frac{\partial_a \partial_b w}{\partial_{t_j} \partial_{t_j}}
\]

where \( d_{ij,ab} \) are some constants depending on \( t_i \)'s.

The proof of Theorem 10.1 gives also an elementary proof of the following result which was previously proven by Panasyuk [27] in the analytic category and further generalised in [2] (the problem is called Zakharevich conjecture in [27]).

**Corollary 10.3** The integrability condition (32) is satisfied for any \( t \) if and only if it is satisfied for \( k + 3 \) distinct values of \( t \).

**Proof.** If \( t_{k+2} \notin \{t_0, \ldots, t_{k+1}\} \) then \( P_{ij}(t_{k+2}) \neq 0 \) and we can divide (33) by \( P_{ij}(t_{k+2}) \). Then we get that (34) has to be satisfied if the integrability condition holds for \( t_{k+2} \). But then it follows from Theorem 10.1 that the integrability condition holds for any \( t \). \( \Box \)

Finally, we define the following \( t \)-dependent vector fields

\[
L_i(t) = -\frac{\partial w}{\partial t_i} \partial_0 + a_i(t - t_i) \partial_i
\]
where \( a_i = \frac{t_{k+1} - t_0}{t_{k+1} - t_i} \), for \( i = 1, \ldots, k \). The vector fields are linear in \( t \) and satisfy \( \omega(t)(L_i(t)) = 0 \). Therefore they span the distributions \( V_k(t) \) tangent to foliations \( \mathcal{F}_t \), respectively. We will call \( (L_1(t), \ldots, L_k(t)) \) the Lax tuple of the paraconformal structure. The tuple commutes, i.e.

\[
[L_i, L_j] = 0.
\]

Indeed, the condition is equivalent to the Hirota system \([34]\). In this way we recover the hierarchy of integrable systems \([11\text{ Equation 6}]\). The following result extends \([11, \text{ Theorem 4.1}]\) to the case of arbitrary \( k \).

**Proposition 10.4** Let \( N = M \times \mathbb{R}^k \) and let \( y_1, \ldots, y_k \) be coordinates on \( \mathbb{R}^k \). If \( (L_i(t))_t^k \) is a Lax tuple on \( M \) defined by a Veronese web then

\[
t \mapsto \sum_{i=1}^k L_i(t) \wedge \partial_{y_i}.
\]

is a bi-Hamiltonian structure on \( N \) and the construction is converse to the Gelfand-Zakharevich reduction.

\[\text{A Appendix: formulae}\]

All formulae below are either taken from \([20]\) or computed by hands using \([20, \text{ Proposition 2.9}]\). All vector fields \( V^{(i)} \) are given up to a multiplicative factor \( g \) from equation \([55]\) which can be neglected.

Order 2:

\[
K_0 = -\partial_0 F + \frac{1}{2} X_F(\partial_1 F) - \frac{1}{4} (\partial_1 F)^2
\]

\( V = \partial_1, \quad V' = -\partial_0 - \frac{1}{2} \partial_1 F \partial_1. \)

Order 3:

\[
K_0 = \partial_0 F - X_F(\partial_1 F) + \frac{1}{3} \partial_1 F \partial_2 F + \frac{2}{3} X_F(\partial_2 F) - \frac{2}{3} X_F(\partial_2 F) \partial_2 F + \frac{2}{27} (\partial_2 F)^3,
\]

\[
K_1 = \partial_1 F - X_F(\partial_2 F) + \frac{1}{3} (\partial_2 F)^2.
\]

\[
V = \partial_2, \quad V' = -\partial_1 - \frac{2}{3} \partial_2 F \partial_2, \quad V'' = \partial_0 + \frac{1}{3} \partial_2 F \partial_1 + \left( \partial_1 F + \frac{4}{9} (\partial_2 F)^2 - \frac{2}{3} X_F(\partial_2 F) \right) \partial_2.
\]
Order 4:

\[
K_0 = -\partial_0 F + X_F(\partial_1 F) - X_F^2(\partial_2 F) + \frac{3}{4} X_F^3(\partial_3 F) - \frac{9}{16} X_F(\partial_3 F)^2 + \\
\frac{18}{64} X_F(\partial_3 F)(\partial_3 F)^2 - \frac{3}{256} (\partial_3 F)^4 - \frac{1}{4} \partial_1 F \partial_2 F + \frac{1}{2} X_F(\partial_2 F) \partial_3 F - \\
\frac{3}{4} X_F^2(\partial_3 F) \partial_3 F + \frac{1}{4} X_F(\partial_3 F) \partial_2 F - \frac{1}{16} \partial_2 F(\partial_3 F)^2,
\]

\[
K_1 = -\partial_1 F + 2 X_F(\partial_2 F) - 2 X_F^2(\partial_3 F) - \frac{1}{2} \partial_2 F \partial_3 F + \frac{3}{2} X(\partial_3 F) \partial_3 F - \frac{1}{8} (\partial_3 F)^3,
\]

\[
K_2 = -\partial_2 F + \frac{3}{4} X_F(\partial_3 F) - \frac{3}{8} (\partial_3 F)^2.
\]

General case, equations of order \(k + 1\):

\[
K_{k-1} = (-1)^k \left( \partial_{k-1} F - \frac{k}{2} X_F(\partial_k F) + \frac{k}{2(k+1)} (\partial_k F)^2 \right).
\]

\[
V = \partial_k, \quad V' = -\partial_{k-1} - \frac{k}{k+1} \partial_k F \partial_k,
\]

\[
V^{(i)} = \sum_{j=0}^i \binom{i}{j} X_F^j(g) \text{ad}_{X_F}^{i-j} \partial_k.
\]

where \(X_F^j(g)\) can be computed using (5) several times.

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