COUNTING CHAMBERS IN RESTRICTED COXETER ARRANGEMENTS

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Abstract. Solomon showed that the Poincaré polynomial of a Coxeter group $W$ satisfies a product decomposition depending on the exponents of $W$. This polynomial coincides with the rank-generating function of the poset of regions of the underlying Coxeter arrangement. In this note we determine all instances when the analogous factorization property of the rank-generating function of the poset of regions holds for a restriction of a Coxeter arrangement. It turns out that this is always the case with the exception of some instances in type $E_8$.

1. Introduction

Much of the motivation for the study of arrangements of hyperplanes comes from Coxeter arrangements. They consist of the reflecting hyperplanes associated with the reflections of the underlying Coxeter group. Solomon showed that the Poincaré polynomial $W(t)$ of a Coxeter group $W$ satisfies a product decomposition depending on the exponents of $W$, see (1.2). This polynomial coincides with the rank-generating function of the poset of regions of the underlying Coxeter arrangement, see §1.2. The aim of this note is to classify all cases when the analogous factorization property of the rank-generating function of the poset of regions holds for an arbitrary restriction of a Coxeter arrangement. It turns out that this is always the case with the exception of some instances in type $E_8$, see Theorem 1.3.

The analogous factorization property for a localization of a Coxeter arrangement is an immediate consequence of Solomon’s theorem and a theorem of Steinberg [Ste60, Thm. 1.5], see Remark 1.5(iv).

1.1. The Poincaré polynomial of a Coxeter group. Let $(W, S)$ be a Coxeter group with a distinguished set of generators, $S$, see [Bou68]. Let $\ell$ be the length function of $W$ with respect to $S$. The Poincaré polynomial $W(t)$ of the Coxeter group $W$ is the polynomial in $\mathbb{Z}[t]$ defined by

$$W(t) := \sum_{w \in W} t^{\ell(w)}.$$  

(1.1)

The following factorization of $W(t)$ is due to Solomon [Sol66]:

$$W(t) = \prod_{i=1}^{n} (1 + t + \ldots + t^{e_i}),$$

(1.2)

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1.2. The rank-generating function of the posets of regions. Let $\mathcal{A} = (\mathcal{A}, V)$ be a hyperplane arrangement in the real vector space $V = \mathbb{R}^n$. A region of $\mathcal{A}$ is a connected component of the complement $V \setminus \bigcup_{H \in \mathcal{A}} H$ of $\mathcal{A}$. Let $R := R(\mathcal{A})$ be the set of regions of $\mathcal{A}$. For $R, R' \in R$, we let $S(R, R')$ denote the set of hyperplanes in $\mathcal{A}$ separating $R$ and $R'$. Then with respect to a choice of a fixed base region $B$ in $R$, we can partially order $R$ as follows:

$$R \leq R' \quad \text{if} \quad S(B, R) \subseteq S(B, R').$$

Endowed with this partial order, we call $R$ the poset of regions of $\mathcal{A}$ (with respect to $B$) and denote it by $P(\mathcal{A}, B)$. This is a ranked poset of finite rank, where $\text{rk}(R) := |S(B, R)|$, for $R$ a region of $\mathcal{A}$, [Ed84, Prop. 1.1]. The rank-generating function of $P(\mathcal{A}, B)$ is defined to be the following polynomial in $\mathbb{Z}[t]$

$$\zeta(P(\mathcal{A}, B), t) := \sum_{R \in R} t^{\text{rk}(R)}.$$

Let $W = (W, S)$ be a Coxeter group with associated reflection arrangement $\mathcal{A} = \mathcal{A}(W)$ which consists of the reflecting hyperplanes of the reflections in $W$ in the real space $V = \mathbb{R}^n$, where $|S| = n$. Note that the Poincaré polynomial $W(t)$ associated with $W$ given in (1.1) coincides with the rank-generating function of the poset of regions of the underlying reflection arrangement $\mathcal{A}(W)$ with respect to $B$ being the dominant Weyl chamber of $W$ in $V$; see [BEZ90] or [JP95].

Thanks to work of Björner, Edelman, and Ziegler [BEZ90, Thm. 4.4] (see also Paris [Pa95]), respectively Jambu and Paris [JP95, Prop. 3.4, Thm. 6.1], in case of a real arrangement $\mathcal{A}$ which is supersolvable (see see §2.3), respectively inductively factored (see §2.4), there always exists a suitable base region $B$ so that $\zeta(P(\mathcal{A}, B), t)$ admits a multiplicative decomposition which is equivalent to (1.2) determined by the exponents of $\mathcal{A}$, see Theorem 2.2.

1.3. Restricted Coxeter arrangements. Let $W$ be a Coxeter group with reflection arrangement $\mathcal{A} = \mathcal{A}(W)$ in $V = \mathbb{R}^n$. We consider the following generalization of the Poincaré polynomial $W(t)$ of $W$. Let $X$ be in the intersection lattice $L(\mathcal{A})$ of $\mathcal{A}$, i.e. $X$ is the subspace in $V$ given by the intersection of some hyperplanes in $\mathcal{A}$. Then we can consider the restricted arrangement $\mathcal{A}^X$ which is the induced arrangement in $X$ from $\mathcal{A}$, see §2.1. In a case-by-case study, Orlik and Terao showed in [OT93] that the restricted arrangement $\mathcal{A}^X$ is always free, so we can speak of the exponents of $\mathcal{A}^X$, see [OT92, §4]. In case $W$ is a Weyl group, Douglass [Dou99, Cor. 6.1] gave a uniform proof of this fact by means of an elegant, conceptual Lie theoretic argument.

It follows from the discussion above that in the special instances when either $\mathcal{A}^X$ is supersolvable (which is for instance always the case for $X$ of dimension at most 2) or inductively factored, or else if $X$ is just the ambient space $V$ (so that $\mathcal{A}^V = \mathcal{A}$), then $\zeta(P(\mathcal{A}^X, B), t)$ is known to factor analogous to (1.2) involving the exponents of $\mathcal{A}^X$. 




Fadell and Neuwirth [FN62] showed that the braid arrangement is fiber type and Brieskorn [Br73] proved this for the reflection arrangement of the hyperoctahedral group. This property is equivalent to being supersolvable, see [Ter86]. Therefore, since any restriction of a supersolvable arrangement is again supersolvable, [Sta72], in case of the symmetric or hyperoctahedral group $W$, we see that $\mathcal{A}(W)^X$ is supersolvable for any $X$. Thus in each of these cases the rank generating function of the poset of regions of $\mathcal{A}(W)^X$ factors as in (1.2), thanks to Theorem 2.2.

Therefore, it is natural to study the rank-generating function of the poset of regions of an arbitrary restriction of a Coxeter arrangement. The following gives a complete classification of all instances when $\zeta(P(\mathcal{A}^X, B), t)$ factors analogous to (1.2).

\textbf{Theorem 1.3.} Let $W$ be a finite, irreducible Coxeter group with reflection arrangement $\mathcal{A} = \mathcal{A}(W)$. Let $\mathcal{A}^X$ be the restricted arrangement associated with $X \in L(\mathcal{A}) \setminus \{V\}$. Then there is a suitable choice of a base region $B$ so that the rank-generating function of the poset of regions of $\mathcal{A}^X$ satisfies the multiplicative formula

\begin{equation}
\zeta(P(\mathcal{A}^X, B), t) = \prod_{i=1}^{n} (1 + t + \ldots + t^{e_i}),
\end{equation}

where $\{e_1, \ldots, e_n\}$ is the set of exponents of $\mathcal{A}^X$ if and only if one of the following holds:

(i) $W$ is not of type $E_8$;

(ii) $W$ is of type $E_8$ and either the rank of $X$ is at most 3, but $\mathcal{A}^X \not\cong (E_8, A_2A_3)$ and $\mathcal{A}^X \not\cong (E_8, A_1A_4)$, or else $\mathcal{A}^X \cong (E_8, D_4)$.

We prove Theorem 1.3 in Section 3. For classical $W$, either $\mathcal{A}(W)^X$ is supersolvable and the result follows from Theorem 2.2, or else $W$ is of type $D$ and $\mathcal{A}(W)^X$ belongs to a particular family of arrangements $\mathcal{C}_p^k$ for $0 \leq k \leq p$ studied by Jambu and Terao, [JT84, Ex. 2.6]. We prove Theorem 1.3 for the family $\mathcal{C}_p^k$ in Lemma 3.11.

For $W$ of exceptional type, there are 31 restrictions $\mathcal{A}(W)^X$ of rank at least 3 (up to isomorphism) that need to be considered. These are handled by computational means, see Remark 3.13.

\textbf{Remarks 1.5.} (i). In the statement of the theorem and later on we use the convention to label the $W$-orbit of $X \in L(\mathcal{A})$ by the Dynkin type $T$ of the stabilizer $W_X$ of $X$ in $W$ which is itself a Coxeter group, by Steinberg’s theorem [Ste60, Thm. 1.5]. So we denote the restriction $\mathcal{A}^X$ just by the pair $(W, T)$; see also [OT92, App. C, D].

(ii). Among the restrictions $\mathcal{A}(W)^X$ all supersolvable and all inductively factored instances are known, see Theorems 3.1 and 3.2 below. Thus, by Theorem 2.2, in each of these cases $\zeta(P(\mathcal{A}^X, B), t)$ factors as in (1.4).

(iii). Hoge checked that the exceptional case $(E_8, A_2A_3)$ from Theorem 1.3 is isomorphic to the real simplicial arrangement “$A_4(17)$” from Grünbaum’s list [Gr71]. It was observed by Terao that the latter does not satisfy the product rule (1.4), [BEZ90, p. 277]. It is rather remarkable that this arrangement makes an appearance as a restricted Coxeter arrangement. In contrast, according to Theorem 1.3, the rank-generating function of the poset of regions
of \((E_8, A_2^2 A_3)\) does factor according to (1.4). In particular, these two arrangements are not isomorphic, as claimed erroneously in [OT92, App. D].

(iv). For \(X\) in \(L(A(W))\) consider the localization \(A(W)_X\) of \(A(W)\) at \(X\), which consists of all members of \(A(W)\) containing \(X\), see §2.1. Then, since the stabilizer \(W_X\) in \(W\) of \(X\) is itself a Coxeter group, by Steinberg’s theorem [Ste60, Thm. 1.5], and since \(A(W)_X = A(W_X)\), by [OT92, Cor. 6.28(2)], it follows from Solomon’s factorization (1.2) that the rank generating function of the poset of regions of \(A(W)_X\) (with respect to the base chamber being the unique chamber of \(A(W)_X\) containing the dominant Weyl chamber of \(W\)) factors analogous to (1.2) involving \(W_X\).

(v). In Lie theoretic terms, for \(W\) a Weyl group, \(W(t^2)\) is the Poincaré polynomial of the flag variety of a semisimple linear algebraic group with Weyl group \(W\). The formula (1.2) then gives a well-known factorization of the Poincaré polynomial of the flag variety.

If \(W\) is of type \(A\) or \(B\), then each restriction \(A(W)X\) is the Coxeter arrangement of the same Dynkin type of smaller rank, cf. [OT92, Props. 6.73, 6.77]. Thus, by the previous paragraph, in these cases, \(\zeta(P(A^X, B), t^2)\) is just the Poincaré polynomial of the flag variety of a semisimple linear algebraic group of the same Dynkin type as \(W\) but of smaller rank.

In view of these examples, it is natural to wonder whether in general there is a suitable projective variety associated with a fixed semisimple group \(G\) with Weyl group \(W\) whose Poincaré polynomial is related to the rank-generating function of the poset of regions for any restriction of \(A(W)\) in the same manner as in these special instances above, relating to and generalizing the flag variety of \(G\).

For general information about arrangements and Coxeter groups, we refer the reader to [Bou68] and [OT92].

2. Recollections and Preliminaries

2.1. Hyperplane arrangements. Let \(V = \mathbb{R}^n\) be an \(n\)-dimensional real vector space. A \((real)\) hyperplane arrangement \(A = (A, V)\) in \(V\) is a finite collection of hyperplanes in \(V\) each containing the origin of \(V\). We denote the empty arrangement in \(V\) by \(\Phi_n\).

The lattice \(L(A)\) of \(A\) is the set of subspaces of \(V\) of the form \(H_1 \cap \ldots \cap H_i\) where \(\{H_1, \ldots, H_i\}\) is a subset of \(A\). For \(X \in L(A)\), we have two associated arrangements, firstly \(A_X := \{H \in A \mid X \subseteq H\} \subseteq A\), the localization of \(A\) at \(X\), and secondly, the restriction of \(A\) to \(X\), \(A^X = (A^X, X)\), where \(A^X := \{X \cap H \mid H \in A \setminus A_X\}\). Note that \(V\) belongs to \(L(A)\) as the intersection of the empty collection of hyperplanes and \(A^V = A\). The lattice \(L(A)\) is a partially ordered set by reverse inclusion: \(X \leq Y\) provided \(Y \subseteq X\) for \(X, Y \in L(A)\).

Throughout, we only consider arrangements \(A\) such that \(0 \in H\) for each \(H\) in \(A\). These are called central. In that case the center \(T(A) := \cap_{H \in A} H\) of \(A\) is the unique maximal element in \(L(A)\) with respect to the partial order. A rank function on \(L(A)\) is given by \(r(X) := \text{codim}_V(X)\). The rank of \(A\) is defined as \(r(A) := r(T(A))\).
2.2. **Free arrangements.** Free arrangements play a fundamental role in the theory of hyperplane arrangements, see [OT92, §4] for the definition and properties of this notion. Crucial for our purpose is the fact that associated with a free arrangement is a set of important invariants, its (multi)set of *exponents*, denoted by \( \exp \mathcal{A} \).

2.3. **Supersolvable arrangements.** We say that \( X \in L(\mathcal{A}) \) is *modular* provided \( X + Y \in L(\mathcal{A}) \) for every \( Y \in L(\mathcal{A}) \), [OT92, Cor. 2.26].

**Definition 2.1** ([Sta72]). Let \( \mathcal{A} \) be a central arrangement of rank \( r \). We say that \( \mathcal{A} \) is *supersolvable* provided there is a maximal chain

\[
V = X_0 < X_0 < \ldots < X_{r-1} < X_r = T(\mathcal{A})
\]

of modular elements \( X_i \) in \( L(\mathcal{A}) \), cf. [OT92, Def. 2.32].

Note that arrangements of rank at most 2 are always supersolvable, e.g. see [OT92, Prop. 4.29(iv)] and supersolvable arrangements are always free, e.g. see [OT92, Thm. 4.58]. Also, restrictions of a supersolvable arrangement are again supersolvable, [Sta72, Prop. 3.2].

2.4. **Nice and inductively factored arrangements.** The notion of a *nice* or *factored* arrangement is due to Terao [Ter92]. It generalizes the concept of a supersolvable arrangement, e.g. see [OT92, Prop. 2.67, Thm. 3.81]. Terao’s main motivation was to give a general combinatorial framework to deduce tensor factorizations of the underlying Orlik-Solomon algebra, see also [OT92, §3.3]. We refer to [Ter92] for the relevant notions and properties (cf. [OT92, §2.3]).

There is an analogue of Terao’s Addition Deletion Theorem for free arrangements ([OT92, Thm. 4.51]) for the class of nice arrangements, see [HR16, Thm. 3.5]. In analogy to the case of free arrangements, this motivates the notion of an *inductively factored* arrangement, see [JP95], [HR16, Def. 3.8] for further details on this concept.

The connection with the previous notions is as follows. Supersolvable arrangements are always inductively factored ([HR16, Prop. 3.11]) and inductively factored arrangements are always free ([JP95, Prop. 2.2]) so that we can talk about the exponents of such arrangements.

The following theorem due to Jambu and Paris, [JP95, Prop. 3.4, Thm. 6.1], was first shown by Björner, Edelman and Ziegler for \( \mathcal{A} \) supersolvable in [BEZ90, Thm. 4.4] (see also Paris [Pa95]).

**Theorem 2.2.** If \( \mathcal{A} \) is inductively factored, then there is a suitable choice of a base region \( B \) so that \( \zeta(P(\mathcal{A}, B), t) \) satisfies the multiplicative formula

\[
\zeta(P(\mathcal{A}, B), t) = \prod_{i=1}^{n} (1 + t + \ldots + t^{e_i}),
\]

where \( \{e_1, \ldots, e_n\} = \exp \mathcal{A} \) is the set of exponents of \( \mathcal{A} \).
2.5. **Restricted root systems.** Given a root system for $W$, associated with a member $X$ from $L(\mathcal{A}(W))$ we have a *restricted root system* which consists of the restrictions of the roots of $W$ to $X$, see [BG07, §2]. As in the absolute case, bases of the restricted root system correspond bijectively to chambers of the arrangement $\mathcal{A}(W)^X$, [BG07, Cor. 7]. More specifically, let $\Phi$ be a root system for $W$ and let $\Delta \subseteq \Phi$ be a set of simple roots. In view of Remark 1.5(i), choosing $X \in L(\mathcal{A}(W))$ amounts to specifying the Dynkin type $T$ of the parabolic subgroup $W_X$, so that the pair $(W, T)$ characterizes $A(W)^X$. Let $\mathcal{B}_T$ be the set of all subsets of $\Delta$ that generate a root system of Dynkin type $T$. Fixing an element $\Delta_j \in \mathcal{B}_T$, the bases for $\Phi$ containing $\Delta_j$ are in bijective correspondence with the bases for the restricted root system, [BG07, Thm. 10].

Furthermore, the set $\mathcal{B}_T$ characterizes a set of representatives for the action of the restricted Weyl group on the set of chambers of the arrangement $\mathcal{A}(W)^X$, [BG07, Lem. 11]. Thus there is a suitable choice of a base region $B$ such that $\zeta(P(\mathcal{A}(W)^X, B), t)$ factors according to (1.4), if and only if there is such a choice among regions that arise from elements in $\mathcal{B}_T$.

3. **Proof of Theorem 1.3**

It is well known that if $W$ is of type $A$ or $B$, then the Coxeter arrangement $\mathcal{A}(W)$ is supersolvable and so is every restriction thereof. So Theorem 1.3 follows in this case from Theorem 2.2. Therefore, for $W$ of classical type, we only need to consider restrictions for $W$ of type $D$. The restrictions $\mathcal{D}_k^p$ for $0 \leq k \leq p$ of Coxeter arrangements of type $D$ are given by the defining polynomial

$$Q(\mathcal{D}_k^p) := x_{p-k+1} \cdots x_p \prod_{1 \leq i < j \leq p} (x_i^2 - x_j^2),$$

see [JT84, Ex. 2.6] ([OT92, Cor. 6.86]).

In view of Theorem 2.2, we next recall the relevant parts of the classifications of the supersolvable and inductively factored restrictions of reflection arrangements from [AHR14] and [MR17], respectively. Here we focus on such $X$ in $L(\mathcal{A})$ of dimension at least 3, as a restriction to a smaller dimensional member of $L(\mathcal{A})$ is already supersolvable.

**Theorem 3.1** ([AHR14, Thm. 1.3]). Let $W$ be a finite, irreducible Coxeter group with reflection arrangement $\mathcal{A} = \mathcal{A}(W)$ and let $X \in L(\mathcal{A}) \setminus \{V\}$ with $\dim X \geq 3$. Then the restricted arrangement $\mathcal{A}^X$ is supersolvable if and only if one of the following holds:

(i) $\mathcal{A}$ is of type $A$ or of type $B$, or

(ii) $W$ is of type $D_n$ for $n \geq 4$ and $\mathcal{A}^X \cong \mathcal{D}_p^k$, where $p = \dim X$ and $p - 1 \leq k \leq p$;

(iii) $\mathcal{A}^X$ is $\{E_6, A_3\}$, $\{E_7, D_4\}$, $\{E_7, A_2^3\}$, or $\{E_8, A_3\}$.

As noted above, every supersolvable restriction from Theorem 3.1 is inductively factored.

**Theorem 3.2** ([MR17, Thms. 1.5, 1.6]). Let $W$ be a finite, irreducible Coxeter group with reflection arrangement $\mathcal{A} = \mathcal{A}(W)$ and let $X \in L(\mathcal{A}) \setminus \{V\}$ with $\dim X \geq 3$. Then the restricted arrangement $\mathcal{A}^X$ is inductively factored if and only if one of the following holds:

(i) $\mathcal{A}^X$ is supersolvable, or...
(ii) $W$ is of type $D_n$ for $n \geq 4$ and $\mathcal{A}^X \simeq \mathcal{D}^{p-2}$, where $p = \dim X$;

(iii) $\mathcal{A}^X$ is one of $(E_6, A_1 A_2), (E_7, A_4)$, or $(E_7, (A_1 A_3)^\omega)$.

It follows from Theorem 2.2 that in all instances covered in Theorem 3.2, $\zeta(P(\mathcal{A}, B), t)$ satisfies the factorization property of (2.3) with respect to a suitable choice of base region $B$. In particular, Theorem 1.3 holds in all these instances.

It is not apparent that the rank-generating function of the poset of regions of $\mathcal{D}^k_p$ factors according to (1.4) for $1 \leq k \leq p - 3$. For, these arrangements are neither reflection arrangements nor are they inductively factored, by the results above. To show that the factorization property from (1.4) also holds in these instances, we first parameterize the regions $\mathcal{R}(\mathcal{D}^k_p)$ suitably and then prove a recursive formula for $\zeta(P(\mathcal{D}^k_p, B), t)$.

**Remark 3.3.** Since the inequalities given by the hyperplanes do not change within a region, the set of regions is uniquely determined by specifying one interior point for each region. Let

$$M^k_p := \{ (x_1, \ldots, x_p) \in \{ \pm 1, \ldots, \pm p \}^p \mid x_1, \ldots, x_{p-k} \neq -1, \ |x_i| \neq |x_j| \ \forall i \neq j \}.$$  

It is easy to verify that each region in $\mathcal{R} := \mathcal{R}(\mathcal{D}^k_p)$ contains exactly one element of $M^k_p$. So this gives a parametrization for the regions in $\mathcal{R}$. Without further comment, we frequently identify points in $M^k_p$ with their respective regions in $\mathcal{R}$. For $x \in M^k_p$, write $R_x \in \mathcal{R}$ for the unique region containing $x$. Once a base region $B$ in $\mathcal{R}$ is chosen so that $\mathcal{R}$ becomes a ranked poset, we may write

$$\zeta(P(\mathcal{D}^k_p, B), t) = \sum_{x \in M^k_p} t^{\kappa(R_x)}.$$  

Using this notation it is easy to see which regions are adjacent and which hyperplanes are walls of a given region. Let $x = (x_1, \ldots, x_p) \in M^k_p$. If $x_j = x_i \pm 1$, then $\ker(x_i - x_j)$ is a wall of $R_x$ and the corresponding adjacent region is obtained from $x$ by exchanging $x_i$ and $x_j$ in $x$. If $x_j = -(x_i \pm 1)$, then $\ker(x_i + x_j)$ is a wall of $R_x$ and the adjacent region again originates from $x$ by exchanging $x_i$ and $x_j$ but maintaining their respective signs. Finally, if $x_i = \pm 1$ and $p - k < i \leq p$, then $\ker(x_i)$ is a wall of $R_x$ and the adjacent region is obtained by exchanging $x_i$ with $-x_i$.

For our subsequent results, we choose $B_p := R_y \in \mathcal{R}$ for $y = (p, p-1, \ldots, 1)$ as our base chamber independent of $k$.

**Lemma 3.4.** Let $p \geq 3$, $k \in \{0, \ldots, p\}$ and $B_p \in \mathcal{R}$ as above. For an arbitrary $i \in \{1, \ldots, p\}$, we have

$$\sum_{x \in M^k_p \atop x_i = p} t^{\kappa(R_x)} = \begin{cases} t^{i-1} \cdot \zeta(P(\mathcal{D}^k_{p-1}, B_{p-1}), t) & \text{if } i \leq p - k, \\ t^{i-1} \cdot \zeta(P(\mathcal{D}^{k-1}_{p-1}, B_{p-1}), t) & \text{if } i > p - k, \end{cases}$$

and

$$\sum_{x \in M^k_p \atop x_i = -p} t^{\kappa(R_x)} = \begin{cases} t^{2p-i-1} \cdot \zeta(P(\mathcal{D}^k_{p-1}, B_{p-1}), t) & \text{if } i \leq p - k, \\ t^{2p-i} \cdot \zeta(P(\mathcal{D}^{k-1}_{p-1}, B_{p-1}), t) & \text{if } i > p - k. \end{cases}$$

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Proof. Set \( N^- := \{ x \in M^k_p \mid x_i = -p \} \). Thanks to Remark 3.3, no hyperplane involving the coordinate \( x_i \) lies between any two regions of \( N^- \). Setting

\[ z = (z_1, \ldots, z_i, \ldots, z_p) := (p - 1, p - 2, \ldots, p - i + 1, -p, p - i - 1, \ldots, 2, 1) \in N^- , \]

there are only hyperplanes involving \( x_i \) between \( B_p \) and \( R_z \). More precisely, we have

\[ S(B_p, R_z) = \begin{cases} \ker(x_i - x_j) \mid j \leq i \} \cup \ker(x_i + x_j) \mid i < j \leq p \} & \text{for } i \leq p - k , \\ \ker(x_i - x_j) \mid j \leq i \} \cup \ker(x_i + x_j) \mid i < j \leq p \} \cup \ker(x_i) & \text{for } i > p - k . 
\]

So if we choose an arbitrary \( x \in N^- \), we have

\[ S(B_p, R_x) = S(B_p, R_z) \cup S(R_z, R_x) . \]

Consequently, we obtain

\[ \text{rk}(R_x) = |S(B_p, R_z)| + |S(R_z, R_x)| = \begin{cases} t^{2p-i-1} + |S(R_z, R_x)| & \text{for } i \leq p - k , \\ t^{2p-i} + |S(R_z, R_x)| & \text{for } i > p - k . 
\]

Now set

\[ \mathcal{A} := \begin{cases} \mathcal{D}_{p-1}^k & \text{if } i \leq p - k , \\ \mathcal{D}_{p-1}^{k-1} & \text{if } i > p - k , 
\]

and identify the set of regions \( \mathcal{R}(\mathcal{A}) \) of \( \mathcal{A} \) with the corresponding set of \( (p - 1) \)-tuples as in Remark 3.3. Then simply omitting the \( i \)-th coordinate defines a map

\[ h : N^- \rightarrow \mathcal{R}(\mathcal{A}) \]

which is bijective, \( h(R_z) = B_{p-1} \) and if \( \tilde{\text{rk}} \) denotes the rank function on \( P(\mathcal{A}, B_{p-1}) \), then we get \( |S(R_z, R_x)| = \tilde{\text{rk}}(h(R_z)) \). Therefore, by (3.7), (3.8) and the bijectivity of \( h \), we get

\[ \sum_{x \in M^k_p \atop x_i = -p} t^{\tilde{\text{rk}}(R_x)} = t^{\text{rk}(B_p, R_z)} \sum_{x \in N^-} t^{\text{rk}(R_x)} \]

\[ = t^{\text{rk}(B_p, R_z)} \sum_{x \in N^-} t^{\tilde{\text{rk}}(h(R_x))} \]

\[ = t^{\text{rk}(B_p, R_z)} \sum_{x \in \mathcal{R}(\mathcal{A})} t^{\tilde{\text{rk}}(R_x)} \]

\[ = t^{\text{rk}(B_p, R_z)} \zeta(P(\mathcal{A}, B_{p-1}), t) \]

\[ = \begin{cases} t^{2p-i-1} \cdot \zeta(P(\mathcal{D}_{p-1}^k, B_{p-1}), t) & \text{if } i \leq p - k , \\ t^{2p-i} \cdot \zeta(P(\mathcal{D}_{p-1}^{k-1}, B_{p-1}), t) & \text{if } i > p - k . 
\]

So (3.5) follows.

Next let \( N^+ := \{ x \in M^k_p \mid x_i = p \} \) and set

\[ z = (z_1, \ldots, z_i, \ldots, z_p) := (p - 1, p - 2, \ldots, p - i + 1, p, p - i - 1, \ldots, 2, 1) \in N^+ . \]

Then \( S(B_p, R_z) = \{ \ker(x_i - x_j) \mid 1 \leq j < i \} \) has cardinality \( i - 1 \). The proof of this case is similar to the one above, and is left to the reader. So (3.6) follows. \[ \Box \]
The next technical lemma is needed in the proof of Lemma 3.11. For ease of notation, we set

\[ F(e_1, \ldots, e_m) := \prod_{i=1}^{m} (1 + t + \cdots + t^{e_i}) \in \mathbb{Z}[t] \]

for any \( m \geq 1 \) and integers \( e_1, \ldots, e_m \geq 1 \). In particular, \( F(e) = 1 + t + \cdots + t^e \). Also note that for \( j > 0 \), we have

\[ F(j-1)(1 + t^j) = F(2j - 1). \]

**Lemma 3.10.** Let \( p \geq 3 \) and \( 0 \leq k \leq p \). Define

\[ \Delta_p^k := \sum_{i=1}^{p-k} (t^{i-1} + t^{2p-i-1}) F(p + k - 2) + \sum_{i=p-k+1}^{p} (t^{i-1} + t^{2p-i}) F(p + k - 3). \]

Then

\[ \Delta_p^k = F(p + k - 1, 2p - 3). \]

**Proof.** We argue by induction on \( k \). First let \( k = 0 \). Then, using (3.9), we have

\[ \Delta_0^0 = \sum_{i=1}^{p} (t^{i-1} + t^{2p-i-1}) F(p - 2) = (1 + \cdots + t^{p-1}) F(p - 2) + (t^{p-1} + \cdots + t^{2p-2}) F(p - 2) = F(p - 1, p - 2) + t^{p-1} F(p - 1, p - 2) = F(p - 1, p - 2) (1 + t^{p-1}) = F(p - 1, 2p - 3). \]

Now let \( k > 0 \) and assume that the statement is true for \( k' < k \). Then using the inductive hypothesis, we get

\[ \Delta_p^k = \Delta_p^{k-1} + t^{p+k-2} \sum_{i=1}^{p-k} (t^{i-1} + t^{2p-i-1}) + t^{p+k-3} \sum_{i=p-k+1}^{p} (t^{i-1} + t^{2p-i}) - F(p + k - 3) \left( t^{p-k} + t^{p-k-2} \right) + F(p + k - 4) \left( t^{p-k} + t^{p-k+1} \right) = F(2p - 3, p + k - 2) + \left( t^{p+k-2} + \cdots + t^{2p-3} + t^{2p+2k-3} + \cdots + t^{3p+k-4} \right) + \left( t^{2p-3} + \cdots + t^{2p+k-4} + t^{2p+k-3} + \cdots + t^{2p+2k-4} \right) - \left( t^{2p-3} + t^{p+k-2} \right) = F(2p - 3, p + k - 2) + t^{p+k-1} (1 + \cdots + t^{2p-3}) = F(2p - 3) (F(p + k - 2) + t^{p+k-1}) = F(2p - 3, p + k - 2), \]

as claimed. \( \square \)

Finally, armed with Lemmas 3.4 and 3.10, we are able to prove the desired result for the arrangements \( \mathcal{D}_p^k \).

**Lemma 3.11.** The rank-generating function of the poset of regions of \( \mathcal{D}_p^k \) factors according to (1.4) for all \( 1 \leq k \leq p - 3 \) and \( p \geq 4 \).
Proof. We argue by induction on $n = p + k$. For $n = 3$, the result holds vacuously. So let $1 \leq k \leq p - 3$ and $p \geq 4$ and assume that for all $p', k'$, with $1 \leq k' \leq p' - 3$, $p' \geq 4$ and $n > p' + k'$, the arrangement $\mathcal{D}_{p'}^{k'}$ satisfies (1.4). Note that

$$
\exp(\mathcal{D}_p^k) = \exp(\mathcal{D}_{p-1}^{k-1}) \cup \{p + k - 1\},
$$

see [JT84, Ex. 2.6]. Then the inductive hypothesis together with Lemmas 3.4 and 3.10 and (3.12) imply

$$
\zeta(P(\mathcal{D}_p^k, B_p), t) = \sum_{x \in M_p^k} t^{\text{rk}(R_x)} = \sum_{i=1}^p \sum_{x_i = \pm p} t^{\text{rk}(R_x)}
$$

$$
= \sum_{i=1}^{p-k} (t^{i-1} + t^{2p-i-1}) \zeta(P(\mathcal{D}_{p-1}^{k-1}, B_{p-1}), t) + \sum_{i=p-k+1}^{p} (t^{i-1} + t^{2p-i}) \zeta(P(\mathcal{D}_{p-1}^{k-1}, B_{p-1}), t)
$$

$$
= \sum_{i=1}^{p-k} (t^{i-1} + t^{2p-i-1}) F(\exp(\mathcal{D}_{p-2}^{k-2})) + \sum_{i=p-k+1}^{p} (t^{i-1} + t^{2p-i}) F(\exp(\mathcal{D}_{p-1}^{k-1}))
$$

$$
= F(\exp(\mathcal{D}_{p-2}^{k-2})) \left( \sum_{i=1}^{p-k} (t^{i-1} + t^{2p-i-1}) F(p + k - 2) + \sum_{i=p-k+1}^{p} (t^{i-1} + t^{2p-i}) F(p + k - 3) \right)
$$

$$
= F(\exp(\mathcal{D}_{p-2}^{k-2})) \Delta_p^k
$$

$$
= F(\exp(\mathcal{D}_{p-2}^{k-2})) F(2p - 3, p + k - 1)
$$

$$
= F(\exp(\mathcal{D}_p^k)).
$$

This completes the proof of the lemma.

Remark 3.13. In view of Theorems 2.2, 3.1 and 3.2, Lemma 3.11 settles all the remaining classical instances of Theorem 1.3. It follows from Theorems 3.1 and 3.2 that there are 31 instances for $W$ of exceptional type to be checked (here we take the isomorphisms of rank 3 restrictions $\mathcal{A}(W)^X$ into account, cf. [OT92, App. D]). We have verified that $\zeta(P(\mathcal{A}(W)^X, B), t)$ satisfies the factorization property (1.4) precisely in all the instances when $W$ is of exceptional type, as specified in Theorem 1.3. In the listed exceptions, $\zeta(P(\mathcal{A}(W)^X, B), t)$ does not factor according to this rule with respect to any choice of base region. This was checked using the computer algebra package SAGE, [S+09].

We used the SAGE-package HyperplaneArrangements which provides methods to compute $\zeta(P(\mathcal{A}, B), t)$ for given $\mathcal{A}$ and $B$. More specifically, the algorithm is initiated with a list containing the vector space $V$ as a polytope and for each hyperplane in $\mathcal{A}$ splits each polytope in the current list into two polytopes, defined by a positive resp. negative inequality, while discarding all empty solutions. This results in a list of chambers implemented as polytopes. After specifying a base region $B$ the algorithm checks for each region $R$ and each hyperplane $H$ whether $H$ separates $B$ from $R$.

In addition, we used the results from [BG07, §2], as detailed in Section 2.5 to greatly reduce the number of chambers that have to be tested. This method worked for all exceptional restrictions other than $(E_8, A_1)$, as the latter is simply too big for SAGE to compute all
its chambers at once. For this case we instead used the bijective correspondences recalled in 2.5 to compute the chambers directly from the elements of the Weyl group $W(E_8)$. By ordering the group elements by length using a depth-first search algorithm implemented in the SAGE-package ReflectionGroup, we were able to compute the chambers of the restricted arrangement ordered by rank, so we could conclude that the rank-generating polynomial of the poset of regions for the restriction $\mathcal{A}^X = (E_8, A_1)$ does not factor according to (1.4) after computing only a small portion of the entire polynomial $\zeta(P(\mathcal{A}^X, B), t)$.

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