A generalization of Kummer’s identity

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Abstract

The well-known Kummer’s formula evaluates the hypergeometric series $2\! F_1\left(\frac{A, B}{C}; -1\right)$ when relation $C - A + B = 1$ holds. This paper deals with evaluation of $2\! F_1(-1)$ series in the case when $C - A + B$ is an integer. Such a series is expressed as a sum of two $\Gamma$-terms multiplied by terminating $3\! F_2(1)$ series. A few such formulas were essentially known to Whipple in 1920’s. Here we give a simpler and more complete overview of this type of evaluations. Additionally, algorithmic aspects of evaluating hypergeometric series are considered. We illustrate Zeilberger’s method and discuss its applicability to non-terminating series, and present a couple of similar generalizations of other known formulas.

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1 The generalization

The subject of this paper is a generalization of Kummer’s identity (see [Kum36, Bal35 2.3], or Cor. 3.1.2 in [AAR99]):

$$2\! F_1\left(\frac{a, b}{1 + a - b}; -1\right) = \frac{\Gamma(1 + a - b) \Gamma(1 + \frac{a}{2})}{\Gamma(1 + a) \Gamma(1 + \frac{a}{2} - b)}. \quad (1)$$

The hypergeometric series on the left is defined if $a - b$ is not a negative integer, and it is absolutely convergent for $\text{Re}(b) < 1/2$. After analytic continuation of $2\! F_1\left(\frac{a, b}{1 + a - b}; z\right)$ on $\mathbb{C} \setminus [1, \infty)$ and after division of both sides by $\Gamma(1 + a - b)$ the formula has meaning and is correct for all complex $a, b$. In this paper, whenever $2\! F_1\left(\frac{A, B}{C}; z\right)$ denotes a well-defined hypergeometric series, it also denotes its analytic continuation on $\mathbb{C} \setminus [1, \infty)$.

The generalization to be considered evaluates the hypergeometric series $2\! F_1\left(\frac{A, B}{C}; -1\right)$ whenever $C - A + B$ is any integer. In the terminology of [AAR99], our generalization applies to $2\! F_1(-1)$ series which are contiguous to a series for

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Kummer’s formula \([\text{[1]}]\). As it is known (see \([\text{AAR99, 2.5]}\)), the 15 classical Gauss contiguity relations can be iterated to produce a linear relation between any three contiguous \(2F1\) series, with coefficients being rational functions in the parameters of those series. This also applies to their analytic extensions. The generalized formula is such a relation in explicit form between contiguous 
\[
2F1\left(\frac{a+n, b}{a-b} \middle| -1\right), 2F1\left(\frac{a, b}{1+a-b} \middle| -1\right) \text{ and } 2F1\left(\frac{a-1, b}{a-b} \middle| -1\right),
\]
where \(n\) is an integer, and the last two series are evaluated using Kummer’s identity \([\text{[1]}]\). The coefficient to the first series cannot be the zero function because the quotient of the other two series is not in \(C(a, b, n)\). In the generalized formula these coefficients are written as terminating \(3F2(1)\) series.

We write the generalization in the form
\[
2F1\left(\frac{a+n, b}{a-b} \middle| -1\right) = P(n) \frac{\Gamma(a-b) \Gamma\left(\frac{a+b}{2}\right)}{\Gamma(a) \Gamma\left(\frac{a+1}{2} - b\right)} + Q(n) \frac{\Gamma(a-b) \Gamma\left(\frac{a+1}{2}\right)}{\Gamma(a) \Gamma\left(\frac{a+1}{2} - b\right)}. \quad (2)
\]
Here the two \(\Gamma\)-terms are equal to 
\[
2F1\left(\frac{a-1, b}{a-b} \middle| -1\right) \text{ and } \frac{a-b}{a-2b} 2F1\left(\frac{a+b}{1+a-b} \middle| -1\right)
\]
respectively, and \(P(n), Q(n)\) are rational functions in \(a, b\) for every integer \(n\). The most convenient expressions for \(P(n)\) and \(Q(n)\) are summarized in the three theorems below. In fact, expressions of \(2F1(-1)\) series in \((2)\) in terms of terminating series and \(\Gamma\)-function were known to Whipple, see \([\text{Whi30]}\). His formulas \((8.3)\) and \((8.41)\) would express the \(2F1(-1)\) series in \((2)\) in terminating series for negative or positive \(n\), respectively. Whipple’s formulas \((11.5,51)\) form the statement of Theorem \([\text{[1]}]\) below. Whipple derived them as a consequence of transformations of \(3F2(1)\) series allied to general \(2F1(-1)\) series, and from \([\text{Fox27, (2.6,7)}]\) where some \(2F1(1/2)\) series are expressed in terms of terminating series. However, Whipple’s main concern was the relations of general \(2F1(-1)\) and \(3F2(1)\) series. As we will see, his approach is not convenient when some of those series terminate.

In this paper we strive for a clear overview of possible expressions for \(P(n)\) and \(Q(n)\) in terms of terminating \(3F2(1)\) series, with simpler proofs. Another aim is to consider algorithmic aspects of evaluating hypergeometric series. In particular, we specialize formula \((2)\) to two-term identities, which however seem to be beyond Zeilberger’s approach. Also a few evaluations similar to \((2)\) are presented. Specifically, we evaluate hypergeometric series which are contiguous to the \(2F1(1/4)\) and \(3F2(1)\) series in Gosper’s and Dixon’s identities, see \([\text{BC76, 30}].\)

In the following theorems we summarize the most convenient expressions for \(P(n)\) and \(Q(n)\). A few more such expressions are presented in \([\text{BC76, 31}].\)

**Theorem 1** Suppose that \(n\) is a non-negative integer (or \(-1\)), and \(a, b\) are complex numbers such that \((a)_n \neq 0\) and \(a-b\) is not zero or a negative integer. Then the coefficients \(P(n)\) and \(Q(n)\) in formula \((2)\) can be written as:
\[
P(n) = \frac{1}{2^{n+1}} \text{S}_2\left(\begin{array}{c} n, \frac{n+1}{2}, a, b \end{array} \right),
\]
\[
Q(n) = \frac{n+1}{2^{n+1}} \text{S}_2\left(\begin{array}{c} n-1, \frac{n}{2}, a+1, b \end{array} \right). \quad (3)
\]
Theorem 2 Suppose that \( n \) is a non-negative integer, and \( a, b \) are complex such that \( (a)_n \neq 0 \), and \( a - b \) is not zero or a negative integer. Then the coefficients \( P(n) \) and \( Q(n) \) in formula (2) can be written as:

\[
P(n) = \frac{1}{2} _3F_2 \left( -\frac{n}{2}, -\frac{n+1}{2}, b \right), \quad Q(n) = \frac{1}{2} _3F_2 \left( -\frac{n-1}{2}, -\frac{n}{2}, a \right).
\] (5)

The hypergeometric sums should be interpreted as terminating series with (up to \( \pm 1 \)) \( \lfloor n/2 \rfloor \) terms.

Theorem 3 Let \( P(n, a, b) \) and \( Q(n, a, b) \) denote the coefficients \( P(n) \) and \( Q(n) \) in (2) as functions of \( a, b \) as well. If \( n \) is a non-negative integer, and \( a, b \not\in \{0, 1, \ldots, n\} \) then

\[
P(-n-1, a, b) = 2^{2n} \frac{(1-\frac{n}{2})_n}{(1-b)_n} P(n-1, a-2n, b-n),
\] (6)

\[
Q(-n-1, a, b) = -2^{2n} \frac{(\frac{1-n}{2})_n}{(1-b)_n} Q(n-1, a-2n, b-n).
\] (7)

Because of the last theorem we do not give expressions for \( P(n) \) and \( Q(n) \) for a negative \( n \), except (13-14) in the proof of Theorem 3.

These theorems are proved in section 2. There we also overview transformations between other expressions for \( P(n) \) and \( Q(n) \), and give a survey of Whipple’s approach in [Whi30]. In section 3 Theorem 2 is proved using the more universal Zeilberger’s method. The key observation is that the sequences \( P(n) \) and \( Q(n) \) satisfy the same recurrence relation as the left-hand side of (2). Theorem 3 can also be proved in this way. Notice that any different expressions for \( P(n) \) and \( Q(n) \) must represent the same rational functions in \( a, b \) for every \( n \), because the quotient of the \( \Gamma \)-terms in (2) is not in \( \mathbb{C}(a, b) \). Section 4 is devoted to algorithmic aspects of evaluation of hypergeometric series, with similar generalizations of Dixon’s and Gosper’s identities.

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2 Classical proof

We assume here \( \text{Re}(a/2) > \text{Re}(b) > 0 \). One can simply check that Theorems 2 and 3 must hold for the analytic continuation of the \( _2F_1(-1) \) series as well.

To prove Theorem 2 we recall Whipple’s identity [Whi30] (8.41)

\[
_2F_1 \left( A, B \left| \frac{C}{A} \right. \right) = \frac{\Gamma(C)}{2 \cdot \Gamma(A)} \sum_{k=0}^{\infty} (-1)^k \frac{(C-A+B-1)_k}{k!} \frac{\Gamma\left( \frac{A}{2} + \frac{k}{2} \right)}{\Gamma\left( \frac{C}{2} + \frac{k}{2} \right)}.
\] (8)
As it was communicated by Askey, this identity can be easily proved using Euler’s integral representation ([Erd53, 2.12(1)]) for the \( _2F_1(z) \) series. One has to rearrange the integrand as

\[
(1-t)^{C-A-1}(1+t)^{-B} = t^{A-1}(1+t)^{-C+A-B+1}(1-t^2)^{C-A-1},
\]

(9)

expand \((1+t)^{-C+A-B+1}\) as series, interchange integration and summation, change the variable \(t\) into \(\sqrt{s}\), and recognize the beta-integral \([\text{Erd53 1.5(1)}]\). We apply\(^1\) formula \( [5] \) to the right-hand side of the identity \([\text{Erd53 2.9(2)}] \):

\[
_2F_1\left(\begin{array}{l}a + n, b \\ a - b \end{array} \right) - 1 = 2^{-2b-n} \left( \begin{array}{l}a - 2b - n \\ a - b \end{array} \right) - 1.
\]

(10)

After this we sum up the terms with even and odd indexes separately, transform the \(\Gamma\)-factors slightly and get formula \( [6] \) with \(P(n), Q(n)\) defined by \( [6][5] \).

Theorem \( [5] \) follows from Theorem \( [3] \) by the following transformation of terminating \( _3F_2(1) \) series (see \([AAR99]\), proof of Cor. 3.3.4):

\[
_3F_2\left(\begin{array}{l}-m, A, B \\ E, F \end{array} \right) = \frac{(E-A)_m}{(E)_m} _3F_2\left(\begin{array}{l}-m, A, F-B \\ 1+A-E-m, F \end{array} \right),
\]

(11)

where \(m\) must be a non-negative integer. To make sure that the interpretation of ill-defined hypergeometric series in \( [5] \) is correct for this transformation, one may specialize \(A\) to \(-\nu/2\) or \(-(\nu\pm1)/2\) with complex \(\nu\) (instead of \(-n/2\), etc.) and take the limit \(\nu \to n\).

To prove Theorem \( [5] \) we use Euler’s integral again. After rearranging the integrand in \( [5] \) as \(t^{A-1}(1-t)^{C-A+B-1}(1-t^2)^{-B} \) and expanding \((1-t)^{C-A+B-1}\) we finally get formula:

\[
_2F_1\left(\begin{array}{l}A, B \\ C \end{array} \right) - 1 = \frac{1}{2} \frac{\Gamma(C) \Gamma(1-B)}{\Gamma(A) \Gamma(C-A)} \sum_{k=0}^{\infty} \frac{(A-B-C+1)_k}{k!} \frac{\Gamma\left(\frac{1}{2} \nu + \frac{k}{2}\right)}{\Gamma\left(\frac{1}{2} \nu + \frac{k}{2} + 1 - B\right)}.
\]

(12)

Like in the proof of Theorem \( [3] \) we apply this formula to \( _2F_1\left(\begin{array}{l}a-n-1, b \\ a-b \end{array} \right) - 1 \) transformed by \( [10] \), and add the terms with even and odd indexes separately. The result is:

\[
P(-n-1) = 2^n \frac{(1-a/2)_n}{(1-b)_n} \left( -\frac{a}{2}, -\frac{n+1}{2}, \frac{a}{2} - b \right),
\]

(13)

\(^1\)The same could be done directly to \( _2F_1\left(\begin{array}{l}a+n, b \\ a-b \end{array} \right) - 1 \), of course. We would get less-convenient formula

\[
_2F_1\left(\begin{array}{l}a+n, b \\ a-b \end{array} \right) - 1 = \frac{1}{2} \frac{\Gamma(a-b) \Gamma(a+n)}{\Gamma(a+n) \Gamma(a-b)} _3F_2\left(\begin{array}{l}a-b, -\frac{a}{2}, -\frac{n+1}{2}, \frac{a+n}{2}, \frac{a}{2} - b \\ a-n+1, -\frac{a}{2}, -\frac{n+1}{2}, -\frac{n}{2}, -\frac{a+n+1}{2}, -\frac{a}{2} - b \end{array} \right).
\]

Here for each positive integer \(n\) the two \(\Gamma\)-terms are \(C(a,b)\)-multiples of the \(\Gamma\)-terms in \( [5] \), so the coefficients \(P(n), Q(n)\) are equal to \(C(a,b)\)-multiples of the \( _3F_2(1) \) series in this formula. But the correspondence depends on whether \(n\) is even or odd.
\[ Q(-n - 1) = -n 2^n \frac{(\frac{1-n}{2})_n}{(1-b)_n} {}_3F_2 \left( -\frac{n-1}{2}, -\frac{n-3}{2}, \frac{a+1}{2}, \frac{a+1}{2} + n - b \right). \]  

(14)

Comparing these expressions with (3-4) gives Theorem 3. Q.E.D.

To get more expressions for \( P(n) \) and \( Q(n) \) one can use standard transformations of terminating \( {}_3F_2(1) \) series. For example, one may repeatedly apply (11) or rewrite a terminating series in the reverse order. In general, a terminating \( {}_3F_2(1) \) series can be transformed to 17 other terminating \( {}_3F_2(1) \) series, see [Whi24, sect. 8], [Bai35, 3.9]. To give these transformations a group structure one has to consider transpositions of the two lower and two upper parameters as non-trivial transformations. Then one gets a group of 72 elements which acts on the set of 18 hypergeometric series, see [RvdJR]. The action of this group can be summarized as follows. Let \( y_0, \ldots, y_5 \) be six parameters satisfying \( y_0 + y_1 + y_2 = y_3 + y_4 + y_5 = 1 - m \). Then the expression

\[ (y_0+y_4)_m (y_0+y_5)_m \ {}_3F_2 \left( -m, y_0+y_1-y_3, y_0+y_2-y_3 \right) \]

(15)

is invariant under the permutations within the sets \( \{y_0, y_1, y_2\} \) and \( \{y_3, y_4, y_5\} \), and gets multiplied by \((-1)^m\) when these two sets are interchanged. For instance, formula (11) corresponds to the permutation \( y_0 \leftrightarrow y_5, y_1 \leftrightarrow y_4, y_2 \leftrightarrow y_3 \).

Application of these transformations to the series (14) or (15) gives eight sets of 18 terminating \( {}_3F_2(1) \) series, one set for a choice of \( P(n) \) or \( Q(n) \), positive or negative and even or odd \( n \). The number of different hypergeometric series turns out to be 96. Here we summarize a few interesting expressions for \( n \geq 0 \):

\[ P(n) = \frac{\lfloor \frac{n}{2} \rfloor!}{2 \cdot n!} \left( \frac{1-\frac{n}{2}}{\frac{n}{2}} + b \right)_{\lfloor n/2 \rfloor} \ {}_3F_2 \left( -\left\lfloor \frac{n}{2} \right\rfloor, \frac{a+1}{2} + \left\lfloor \frac{n}{2} \right\rfloor, \frac{a}{2} - b \right) \]  

(16)

\[ Q(n) = \frac{\lfloor \frac{n}{2} \rfloor!}{2 \cdot n!} \left( 1 - \frac{a}{2} + b \right)_{\lfloor n/2 \rfloor} \ {}_3F_2 \left( -\left\lfloor \frac{n}{2} \right\rfloor, \frac{a}{2} + \left\lfloor \frac{n}{2} \right\rfloor, \frac{a+1}{2} - b \right) \]  

(17)

\[ \frac{2^{n+1}}{2^{n+1}} \left( \frac{a+1}{2} + b \right)_{\lfloor n/2 \rfloor} \ {}_3F_2 \left( -\left\lfloor \frac{n}{2} \right\rfloor, \frac{a}{2} + \left\lfloor \frac{n}{2} \right\rfloor, \frac{a+1}{2} - b \right) \]  

(18)

To get expressions for negative \( n \) one may use Theorem 3. Notice that series in (17) and (19) terminate for all \( n \).

In the rest of this section we follow Whipple’s approach in [Whi30], where transformations of not necessarily terminating \( {}_3F_2(1) \) series are used to derive various identities with general \( {}_2F_1(-1) \) series. We concentrate on the \( {}_2F_1(-1) \) series which are contiguous to the series in Kummer’s formula (1). Notice that proofs of Theorems 1 and 3 are valid for any complex values of \( n \), so that formula
with \( P(n) \) and \( Q(n) \) defined by (3-4) or (13-14) is true for any complex \( n \). Formula (2) with \( P(n) \), \( Q(n) \) defined by (3) is also true for all \( n \), see Whipple’s formulas (13-14) below. But one may check that in general these \( P(n) \) and \( Q(n) \) are not the same.

Transformations of general \( 3F2(1) \) series were first derived by Thomae, see [Tho79]. Whipple introduced notation (see [Whi24],[Bai35, 3.5-7]) which gives a group-theoretical insight into those formulas. To begin with, there is an action of the symmetric group \( S_5 \) on \( 3F2(1) \)'s. Hardy described it in the notes to lecture VII in [Har40] by saying that the function

\[
\frac{1}{\Gamma(E)\Gamma(F)\Gamma(E+F-A-B-C)} \binom{A, B, C}{E, F} \binom{1}{0, 1, \ldots, 5} \text{is invariant under the permutations of } E, F, E+F-A-B-C, E+F-A-C \text{ and } E+F-A-B. \quad \text{(20)}
\]

is invariant under the permutations of \( E, F, E+F-A-B-C, E+F-A-C \) and \( E+F-A-B \). For example, we have (see [AAR99], Cor. 3.3.5):

\[
3F2\left(\begin{array}{c} A, B, C \\ E, F \end{array}\right) = \frac{\Gamma(F)\Gamma(E+F-A-B-C)}{\Gamma(F-C)\Gamma(E+F-A-B)} 3F2\left(\begin{array}{c} E-A, E-B, C \\ E, E+F-A-B \end{array}\right). \quad \text{(21)}
\]

An orbit of general \( 3F2(1) \) consists of 10 different series. Note that the series in (20) converge when \( \text{Re}(E+F-A-B-C) > 0 \), and the whole expression is well-defined and analytic for any parameters under this condition. Function (20) can be analytically continued to the region in the parameter space where at least one of the 10 series converges.

Further, a general \( S_5 \) orbit of \( 3F2(1) \)'s is associated to 11 other orbits, so that we get sets of 120 allied \( 3F2(1) \) series, see [Whi24]. For example\(^2\), the series in (20) is allied to

\[
3F2\left(\begin{array}{c} A, 1+A-E, 1+A-F \\ 1+A-B, 1+A-C \end{array}\right) \quad \text{and} \quad 3F2\left(\begin{array}{c} E-A, E-B, E-C \\ E, 1+F-E \end{array}\right). \quad \text{(22)}
\]

In general, two allied series are not related by a two-term identity like (21). But for any three allied series there is a linear relation between them, with coefficients being \( \Gamma \)-terms. This also gives three-term relations for the 12 functions of type (20), and even defines their analytic continuation to the whole space of parameters. Indeed, if the series in (20) diverges then its ally \( 3F2\left(\begin{array}{c} 1-A, 1-B, 1-C \\ 2-D, 2-E \end{array}\right) \) converges; for the third term one can take convergent series from a similar pair of functions from other \( S_5 \)-orbits. Besides, all allied series converge in a neighborhood of \( A=B=C=1/2, E=F=1 \).

In [Whi30] Whipple applies the relations of allied series to a general \( 2F1(-1) \) series by expressing it as \( 3F2(1) \) series and considering it as a member of the

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\(^2\)Whipple introduced for \( 3F2(1) \) series six parameters \( r_0, \ldots, r_5 \) related by condition \( \sum r_i = 0 \) so that: all allied series can be obtained by the permutations of the six parameters and/or changing the sign of them all; an \( S_5 \)-orbit is determined by fixing a parameter and an element of the set \( \{+, -\} \), and \( S_5 \) permutes the remaining five parameters. Specifically, one may choose that the \( S_5 \) action on (20) fixes \( r_0 \), and take \( E=1+r_1-r_0 \).
corresponding allied family. In particular, his formulas (3.1) and (3.51) read as:

\[ 2F1 \left( \begin{array}{c} a + \nu, b \\ a - b \end{array} \right) = 1 = \frac{\Gamma(a - b) \Gamma(\nu - a)}{\Gamma(a) \Gamma(a - b)} \left\{ \begin{array}{c} \frac{\nu - 1}{2}, -\frac{3}{2}, b \\ -\nu, \frac{a + 1}{2} \end{array} \right\} \]

(23)

\[ = \frac{\Gamma(a - b) \Gamma(\nu + 1)}{\Gamma(a) \Gamma(\nu + 1 - b)} 3F2 \left( \begin{array}{c} -\frac{\nu - 1}{2}, -\frac{a + 1}{2}, b \\ -\nu, \frac{a + 1}{2} \end{array} \right) . \]

(24)

If \( \nu \notin \{0, 1, 2, \ldots\} \) we may relate the \( 2F1(-1) \) series with the \( 3F2(1) \) series in (23-24) and get many two- and three-term relations with \( 2F1(-1) \) and \( 3F2(1) \) series. Some of these identities make sense and are correct even if \( \nu \) is a non-negative integer, because singular \( \Gamma \)-factors cancel. For instance, for identities of allied series.

On the other hand, the \( 3F2(1) \) series in (23-24) cannot be identified with the terminating series in the expressions in (2). One has to compute:

\[ \lim_{\nu \to n} 3F2 \left( \begin{array}{c} -\frac{\nu - 1}{2}, \frac{\nu + 1}{2}, b \\ -\nu, \frac{a + 1}{2} \end{array} \right) = 2P(n) - \frac{1}{4n+1} \left( \frac{b}{2} \right)_{n+1} 3F2 \left( \begin{array}{c} \frac{n+2}{2}, \frac{n+1}{2}, b+n+1 \\ n+2, \frac{a + 1}{2} + n+1 \end{array} \right), \]

\[ \lim_{\nu \to n} 3F2 \left( \begin{array}{c} -\frac{\nu - 1}{2}, \frac{\nu + 1}{2}, b \\ -\nu, \frac{a + 1}{2} \end{array} \right) = 2Q(n) + \frac{1}{4n+1} \left( \frac{b}{2} \right)_{n+1} 3F2 \left( \begin{array}{c} \frac{a + 1}{2}, \frac{b + 2}{2}, b+n+1 \\ n+2, \frac{a + 1}{2} + n+1 \end{array} \right). \]

In the sum of these two equalities the non-terminating \( 3F2(1) \) series on the right-hand side cancel, since they are connected by transformation (21). In this way identities (23-24) prove Theorem 2.

Moreover, the \( 3F2(1) \) series (23-24) can be transformed by \( S_5 \) to four series which are well-defined and terminate when \( \nu \) is an (odd or even) positive integer \( n \). Those terminating series are presented in formulas (3) and (5). However, this does not give expressions for \( 2F1 \left( \begin{array}{c} a+n, b \\ a-b \end{array} \right) - 1 \) in terms of one terminating \( 3F2(1) \) series, because the mentioned four series diverge for \( \nu > 1/2 \) (except when they terminate), and we cannot use the \( S_5 \)-invariance of the corresponding function in (24). Notice, for example, that (24) implies a wrong relation between the \( 3F2(1) \) series in (23) and (24). As we see, Whipple’s approach in \( \text{[Whi30]} \) gets complicated in the case \( \nu \) in (23-24) is an integer, and does not directly explain various expressions for our \( P(n) \) and \( Q(n) \).

### 3 A proof by Zeilberger’s method

Here we prove Theorem 2 only. Theorem 1 can be proved in the same way.

Let us define \( S(n) = 2F1 \left( \begin{array}{c} a+n, b \\ a-b \end{array} \right) - 1 \). The contiguity relation [Erd53, 2.8(28)] between \( 2F1 \left( \begin{array}{c} A+1, B \\ C \end{array} \right) \), \( 2F1 \left( \begin{array}{c} A-1, B \\ C \end{array} \right) \) and \( 2F1 \left( \begin{array}{c} A, B \\ C \end{array} \right) \) gives the fol-
following recurrence relation:

\[
2(n+a)S(n+1) - (3n+2a)S(n) + (n+b)S(n-1) = 0. \quad (25)
\]

We claim that the sequences \(P(n)\) and \(Q(n)\) satisfy the same recurrence relation. Following the “creative telescoping” method of Zeilberger (\[PWZ96, Koe98\]), let

\[
p(n, k) = \frac{(-1)^k (n+1) (n-k)!}{2 \cdot 4^k (n-2k+1)! k!} \left( \frac{b}{2} \right)_k \quad (26)
\]

be the \(k\)th summand of \(P(n)\) in (5). We set \(p(n, k) = 0\) for \(k > \lceil n/2 \rceil\). Also define

\[
r_1(n, k) = -\frac{2k(n-k+1)(a+2k-2)}{(n-2k+2)(n-2k+3)}, \quad R_1(n, k) = r_1(n, k)p(n, k).
\]

One can check that

\[
2(n+a)p(n+1, k) - (3n+2a)p(n, k) + (n+b)p(n-1, k) = R_1(n, k+1) - R_1(n, k),
\]

so

\[
2(n+a)P(n+1) - (3n+2a)P(n) + (n+b)P(n-1) = \\
\sum_{k=0}^{\lfloor n/2 \rfloor} (R_1(n, k+1) - R_1(n, k)) - R_1(n, \lfloor n/2 \rfloor) = 0. \quad (27)
\]

Although this looks like an artificial trick, we follow the standard Wilf-Zeilberger method of proving combinatorial identities, see \[PWZ96, Koe98\]. The expression \(r_1(n, k)\) is the certificate of our standardized proof. Given \(p(n, k)\) the recurrence relation for \(P(n)\) and the certificate \(r_1(n, k)\) can be found by Zeilberger’s algorithm. This algorithm is implemented in computer algebra packages EKHAD (see \[Zei99\], command ct) and hsum.mpl (see \[Koe99\], command sumrecursion with option certificate=true). Also check \[VK00\] for a link to a Maple worksheet for this proof. The finite sums in this proof require some attention, since they are not natural according to \(Koe98\).

In the same way:

\[
2(n+a)Q(n+1) - (3n+2a)Q(n) + (n+b)Q(n-1) = \\
\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (R_2(n, k+1) - R_2(n, k)) - R_2(n, \lfloor n/2 \rfloor) = 0, \quad (28)
\]

where

\[
R_2(n, k) = \frac{2k(n-k+1)(a+2k-1)}{(n-2k+1)(n-2k+2)} \frac{(-1)^k}{2 \cdot 4^k} \frac{(n-k)}{k} \left( \frac{b}{2} \right)_k \quad \frac{b}{2} \quad (29)
\]

is the \(k\)th summand of \(Q(n)\) in (3) multiplied by the corresponding certificate.
Note that the condition \((a)_n \neq 0\) ensures that recurrence relation (23) does not degenerate to a first order relation until we evaluate \(P(n)\) and \(Q(n)\). It remains to check that formula (3) holds for two initial values of \(n\). Kummer’s identity (1) suggests \(P(-1) = 1\) and \(Q(-1) = 0\), which fits into the recurrence relation. We may use Gauss’ contiguity relation \(\text{Erd53, 2.8(38)}\) between \(\binom{A,B}{C+1} z\), \(\binom{A,B}{C} z\) and \(\binom{A-1,B}{C} z\) to obtain

\[
(a - 2b) \frac{\Gamma(1 + a - b) \Gamma(1 + \frac{a}{2})}{\Gamma(1 + a) \Gamma(1 + \frac{a}{2} - b)} - 2 (a - b) S(0) + (a - b) S(-1) = 0. \tag{30}
\]

This implies the correct \(P(0) = 1/2\) and \(Q(0) = 1/2\) and completes the proof.

Note that the Gauss contiguity relations hold for analytic extension of hypergeometric functions on \(\mathbb{C} \setminus [1, \infty)\). Therefore this proof does not require convergence of the \(2F_1(-1)\) series. Q.E.D.

In fact, sequences \(P(n)\) and \(Q(n)\) satisfy recurrence relation (25) for all \(n\). The recurrence can be directly verified for \(n = -2, -1, 0\). The values of \(P(n)\) and \(Q(n)\) for \(n = -3, -2, -1, 0, 1\) are

\[
\begin{align*}
2 & \frac{(a-2)(a-b-2)}{(b-1)(b-2)}, \quad \frac{a-2}{b-1}, \quad 1, \quad \frac{1}{2}, \quad \frac{a-b}{2a} \quad \text{and} \\
\frac{2(a-1)(a-3)}{(b-1)(b-2)}, \quad \frac{a-1}{b-1}, \quad 0, \quad \frac{1}{2}, \quad \frac{1}{2} \quad \text{respectively.}
\end{align*}
\]

To compute the same recurrence relation for negative \(n\) one can use Theorem 1. Alternatively, one may choose an expression for \(P(n)\) and \(Q(n)\) for negative \(n\), say (13-14), and compute the recurrence relation with Zeilberger’s algorithm.

To show equalities like in (16-18) by Zeilberger’s method one would have to compute the recurrences for odd and even integers separately. Recurrence relation (25) for any such expression and for all \(n\) can be computed using contiguity relations for \(3F_2(1)\) series. As it is known (see \(\text{AAR99, 3.7}\)), contiguous \(3F_2(1)\) series satisfy three-term relations (with coefficients being rational functions in the parameters of those series), just like contiguous \(2F_1(z)\) series.

4 Algorithmic aspects

The generalized formula (2) can be specialized so that \(P(n)\) or \(Q(n)\) vanishes, giving an evaluation of \(2F_1(-1)\) series with a single \(\Gamma\)-term. For example,

\[
Q(-4) = -4 \frac{(a - 1) (a - 3) (2a - b - 7)}{(b - 1) (b - 2) (b - 3)}, \tag{31}
\]

so if \(b = 2a - 7\) then \(Q(-4) = 0\), which implies

\[
\binom{3-c, 7-2c}{c} | - 1 = 3 \frac{\Gamma(c) \Gamma(3 \frac{b}{2} - 2)}{4 \Gamma(5-c) \Gamma(4 \frac{b}{2} - 2)}. \tag{32}
\]
Further, \( P(-5) = 0 \) if \( 2a^2 - 4ab + b^2 - 12a + 17b + 12 = 0 \). Parameterizing the curve given by this equation we get
\[
2F_1\left(\begin{array}{c}
\frac{-2t^2-7t+6}{2t^2-12}, \frac{t^2+4t-8}{2t^2-12} \\
-1
\end{array}\middle| \frac{t^2+3t-8}{2t^2-12}\right) = \frac{t^2 + 3t - 6}{t(t-1)} \Gamma\left(\frac{3t-4}{t^2-2}\right) \Gamma\left(\frac{t^2+7t-12}{2(t^2-2)}\right) = \frac{\Gamma\left(\frac{3t-4}{t^2-2}\right)}{\Gamma\left(\frac{7t-10}{t^2-2}\right) \Gamma\left(\frac{(t-1)(t+1)}{2(t^2-2)}\right)},
\tag{33}
\]

It could be expected that formulas like (32) can be proved automatically by current computer algebra algorithms, say by Wilf-Zeilberger method. As it is demonstrated in [Koo98], this method or Zeilberger’s algorithm can be adapted to non-terminating hypergeometric series if one can justify the “creative telescoping” trick by dominated convergence, and the hypergeometric series can be evaluated in the limit \( n \to \infty \), where \( n \) is a discrete parameter. In general non-terminating hypergeometric series is given without a discrete parameter, so it must be introduced by an algorithm. For example, after substitution \( a \mapsto a + 2n \) one can prove Kummer’s formula (1) with Wilf-Zeilberger method, see [Gau99].

In the case of equation (32) we may substitute \( c \mapsto c + n \) and apply Zeilberger’s algorithm to get the right first order difference equation. However, we cannot evaluate the hypergeometric series neither in the limit \( n \to \infty \), nor for a finite value of \( n \). What we can do is to combine explicitly Gauss’ contiguity relations in such a way that we “accidentally” get a two-term relation where one of the terms can be evaluated by Kummer’s formula. For example, the relation between contiguous \( 2F_1\left(\begin{array}{c}
A, B \\
C
\end{array}\middle| z\right), \ 2F_1\left(\begin{array}{c}
A+1, B-2 \\
C
\end{array}\middle| z\right) \) and, say, \( 2F_1\left(\begin{array}{c}
A, B-1 \\
C
\end{array}\middle| z\right) \), after the specialization \((A, B, C, z) \mapsto (3-c, 7-2c, c-1)\) becomes
\[
2F_1\left(\begin{array}{c}
3-c, 7-2c \\
c
\end{array}\middle| -1\right) = \frac{3}{4} 2F_1\left(\begin{array}{c}
4-c, 5-2c \\
c
\end{array}\middle| -1\right),
\tag{34}
\]
In this way even the exotic (33) can be proved.

This shows that relations between contiguous hypergeometric series can be useful for finding new “non-standard” evaluations of \( 2F_1 \) series. One may take such a relation and try to find families of its two term specializations with a discrete parameter \( n \). This would give a first order recurrence relation, and if the series can be evaluated in the limit \( n \to \infty \) one gets a (perhaps) new formula! Relations between contiguous series also give a way to compute recurrence relation, alternative to Zeilberger’s algorithm.

In [VK00] there is a link to Maple routines which for given three integer vectors \((k_i, l_i, m_i)\) for \( i = 1, 2, 3 \) derive a \( C(A, B, C, z) \)-linear relation between three contiguous functions \( 2F_1\left(\begin{array}{c}
A+k_i, B+l_i \\
C+m_i
\end{array}\middle| z\right) \). Computer experiments found many first order recurrence relations for some values \( z = 1/4, 1/3, 1/9, \exp(i\pi/3), 3 - 2\sqrt{2}, \ldots \), some of them can be successfully solved. It is an interesting question which \( 2F_1(z) \) series can be evaluated in terms of \( \Gamma \) function. So far produced evaluations can be obtained using quadratic or cubic transformations.
Here we generalize a few known formulas of the same type as \([2]\). They were obtained by considering relations between three contiguous hypergeometric series where two of them can be evaluated by a known formula, and trying to express the coefficients in these relations as hypergeometric series. This was done by considering partial fraction decomposition of those coefficients empirically. The formulas can be proved by showing that all three terms in a formula satisfy the same recurrence relation by Zeilberger’s algorithm, and checking the identity for a couple of values of the discrete parameter.

We start with a generalization of Gosper’s “non-standard” evaluations of \(2F1(1/4)\) series, see [36881 1/4.1–2]. A generalization is

\[
2F1 \left( \begin{array}{c} -a, \frac{1}{4} \\ 2a + \frac{1}{4} + n \end{array} \right) = \frac{2^{n+3/2}}{3^n + 1} \frac{\Gamma(a + \frac{5}{8} + \frac{3}{2}) \Gamma(a + \frac{3}{8} + \frac{3}{2}) \Gamma(a + \frac{1}{8} + \frac{3}{2})}{\Gamma(a + \frac{5}{6} + \frac{3}{2}) \Gamma(a + \frac{3}{6} + \frac{3}{2}) \Gamma(a + \frac{1}{6} + \frac{3}{2})} K(n)
\]

\[
-(-3)^{-n/2} 2^{3/2} \frac{\Gamma(a + \frac{5}{8} + \frac{3}{2}) \Gamma(a + \frac{3}{8} + \frac{3}{2}) \Gamma(a + 1)}{\Gamma(a + \frac{5}{6} + \frac{3}{2}) \Gamma(a + \frac{3}{6} + \frac{3}{2}) \Gamma(a + \frac{1}{6} + \frac{3}{2})} L(n), \quad (35)
\]

where

\[
K(1) = L(0) = 0, \quad K(0) = L(1) = 1,
\]

for \(n > 1:\)

\[
K(n) = (-1)^n \sum_{k=|n/3|}^{[n/2]} \frac{27^k}{4^k} \frac{n (k-1)!}{(n-2k)! (3k-n)!} \frac{(a + 1/2)_k}{(a+1)_k},
\]

\[
L(n) = 4F3 \left( \begin{array}{c} -n-1, -n-2, -n-3, a+1 \\ -n-2, -n-3, -n-3, a+1 \end{array} \right),
\]

and for \(-n < 0:\)

\[
K(-n) = 4F3 \left( \begin{array}{c} -n, -n-1, -n-2, -n-3, -a \\ -n-2, -n-3, -n-2, -a+1/2 \end{array} \right) = \sum_{k=0}^{[n/3]} \frac{(-4)^k n (n-2k-1)!}{27^k (n-3k)! k!} \frac{(a - 1/2)_k}{(-a+1/2)_k},
\]

\[
L(-n) = (-1)^n \sum_{k=|[n+1)/3]}^{[n+1)/2]} \frac{27^k}{4^k} \frac{(n+1) (k-1)!}{(n-2k+1)! (3k-n-1)!} \frac{(a - 1/2)_k}{(-a)_k}.
\]

Gosper has found the special cases \(n = 0, 1\). The \(\Gamma\)-factors to \(K(n)\) and \(L(n)\) are \(\mathbb{C}(a)\)-multiples of these two Gosper’s evaluations (respectively) for each \(n\). All three terms in (35) satisfy the recurrence relation

\[
2(2n+2a+1) (2n+6a+3) S(n+1) + (2n+4a+3) (4n+6a+1) S(n) - 3(2n+4a+1) (2n+4a+3) S(n-1) = 0.
\]

Next we recall the classical Dixon’s identity which evaluates well-poised \(3F2(1)\) series, see [6533 3.1]. We generalize it as follows:

\[
3F2 \left( \begin{array}{c} a+n, b, c \\ a-b, a-c \end{array} \right) = \frac{\tilde{P}(n)}{2} \frac{\Gamma(a+n+1) \Gamma(a-b) \Gamma(a-c) \Gamma(a+n+1-b-c)}{\Gamma(a) \Gamma(a+n+1-b) \Gamma(a+n+1-c) \Gamma(a-b-c)}
\]

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\begin{align}
+ \frac{\tilde{Q}(n)}{2} \frac{\Gamma\left(\frac{a}{2}\right) \Gamma(a-b) \Gamma(a-c) \Gamma\left(\frac{a}{2} - b\right) \Gamma\left(\frac{a}{2} - c\right) \Gamma\left(a - b - c\right)}{\Gamma(a) \Gamma\left(\frac{a}{2} - b\right) \Gamma\left(\frac{a}{2} - c\right) \Gamma(a - b - c)},
\end{align}

(36)

where \( \tilde{P}(1) = 1, \tilde{Q}(-1) = 0, \) then for \( n \geq 0: \)

\[ \tilde{P}(n) = 4F_3\left(-\frac{n}{2}, -\frac{n+1}{2}, b, c \mid -n, \frac{a}{2}, \frac{1-a^2}{2} + b + c\right), \quad \tilde{Q}(n) = 4F_3\left(-\frac{n-1}{2}, -\frac{n}{2}, b, c \mid -n, \frac{1+a}{2}, 1 - \frac{a^2}{2} + b + c\right), \]

and for \( -n < 0: \)

\[ \tilde{P}(-n-1) = 2^{2n} \frac{(1-\frac{n}{2})n (\frac{1+a}{2} - b - c)_n}{(1-b)n(1-c)_n} 4F_3\left(-\frac{n}{2}, -\frac{n-1}{2}, b - n, c - n \mid 1 - n, \frac{a}{2} - n, \frac{1-a}{2} + b + c - n\right), \]

\[ \tilde{Q}(-n-1) = -2^{2n} \frac{(\frac{1-a}{2})n (\frac{a}{2} - b - c)_n}{(1-b)n(1-c)_n} 4F_3\left(-\frac{n-1}{2}, -\frac{n-2}{2}, b - n, c - n \mid 1 - n, \frac{1+a}{2} - n, 1 - \frac{a}{2} + b + c - n\right). \]

Dixon’s identity is the special case \( n = -1. \) This generalized formula is a relation between contiguous \( 3F_2(1) \) series in explicit form. For positive \( n \) it is strikingly similar to generalization \( (\frac{2}{3}) \) of Kummer’s identity. In fact, the generalization in Theorem 3 is the limiting case \( c \to \infty \) of \( (36), \) just as Kummer’s formula is the limiting case of Dixon’s identity. The recurrence relation for the three terms in \( (36) \) is:

\[ (n+a) (a-n+2b+2c+1) S(n+1) + (n+b) (n+c) S(n-1) - (2n^2 + 3bn + 3cn + a^2 + 2ab + 2ac + a) S(n) = 0. \]

More evaluations of the same type can be obtained using standard transformations of \( 2F_1(z) \) series to \( 2F_1(z/(z-1)) \) series, see \( [Erd53, 2.9(3-4)] \). Applying them to the generalized Kummer’s formula \( (\frac{2}{3}) \) gives evaluations of \( 2F_1(1/2) \) which generalize classical formulas of Gauss and Bailey, see \( [Bai35, 2.4] \). The same transformation of \( (\frac{3}{4}) \) gives evaluation of \( 2F_1(-1/3) \). Similarly, one can apply \( (21) \) to identity \( (\frac{3}{5}) \) and get generalizations of Watson’s and Whipple’s formulas \( [Bai35, 3.3-4] \).

All these formulas evaluate hypergeometric series which are contiguous to a series which evaluation is known. In order to find these formulas automatically one needs an algorithm which would find the solutions of a recurrence relation in form of terminating hypergeometric series.

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