An SOS counterexample to an inequality of symmetric functions

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Abstract

It is known that differences of symmetric functions corresponding to various bases are nonnegative on the nonnegative orthant exactly when the partitions defining them are comparable in dominance order. The only exception is the case of homogeneous symmetric functions where it is only known that dominance of the partitions implies nonnegativity of the corresponding difference of symmetric functions. It was conjectured by Cuttler, Greene, and Skandera in 2011 that the converse also holds, as in the cases of the monomial, elementary, power-sum, and Schur bases. In this paper we provide a counterexample, showing that homogeneous symmetric functions break the pattern. We use a semidefinite program to find a positive semidefinite matrix whose factorization provides an explicit sums of squares decomposition of the polynomial \(H_{44} - H_{521}\) as a sum of 41 squares. This rational certificate of nonnegativity disproves the conjecture, since a polynomial which is a sum of squares of other polynomials cannot be negative, and since the partitions 44 and 521 are incomparable in dominance order.

1 Introduction

In the article *Inequalities for Symmetric Means* [11], by Cuttler, Greene, and Skandera, Muirhead-type inequalities are classified for the different common bases of symmetric functions. We briefly provide some definitions in order to state our main Theorem 2. First, let \(m_\lambda, e_\lambda, p_\lambda, h_\lambda,\) and \(s_\lambda\) denote the monomial, elementary, power-sum, homogeneous, and Schur polynomials, respectively, associated to a partition \(\lambda\). Given a symmetric polynomial \(g(x)\), the *term-normalized symmetric polynomial* is

\[
G(x) := \frac{g(x)}{g(1)}
\]

where \(g(1)\) is the symmetric polynomial evaluated on the all ones vector.

By \(G_\lambda \geq G_\mu\), we mean \(G_\lambda(x_1, \ldots, x_n) \geq G_\mu(x_1, \ldots, x_n)\), on the nonnegative orthant. That is, the inequality holds for all \(n\) (any number of variables), but only for \(x_i \geq 0, \ i = 1, \ldots, n\). We denote the term-normalized symmetric polynomials for monomial, elementary, power-sum, homogeneous, and Schur polynomials, respectively, associated to a partition \(\lambda\).

The following theorem is a summary of known results (special cases of which go back to Maclaurin, Muirhead, Newton, and Schur, for example), which are proven in [11], [23], and [27]:

**Theorem 1.** Let \(\lambda\) and \(\mu\) be partitions such that \(|\lambda| = |\mu|\). Then

\[
\begin{align*}
M_\lambda & \leq M_\mu & \iff & \mu \geq \lambda \\
E_\lambda & \leq E_\mu & \iff & \lambda \geq \mu \\
P_\lambda & \leq P_\mu & \iff & \mu \geq \lambda \\
S_\lambda & \leq S_\mu & \iff & \mu \geq \lambda
\end{align*}
\]
whereas \( \mu \geq \lambda \) implies that \( H_\lambda \leq H_\mu \), i.e.,

\[
H_\lambda \leq H_\mu \iff \mu \geq \lambda
\]

The converse for the homogeneous symmetric functions statement was conjectured in [11] in 2011. The authors of [11] also reported that for \( d = |\lambda| = |\mu| = 1, 2, \ldots, 7 \) their conjecture had been proven. For \( d = 8 \) and higher, the question was unresolved.

**Theorem 2.** A degree-minimal counterexample exhibiting a polynomial \( H_\mu - H_\lambda \geq 0 \) with \( \lambda, \mu \) incomparable in dominance order is provided by \( H_{44} - H_{521} \). We certify the nonnegativity of this polynomial on \( \mathbb{R}_{\geq 0} \) by writing a related polynomial explicitly as a sum of 41 squares with rational coefficients.

Explicitly, the polynomial we exhibit as a sum of squares of polynomials is

\[
(H_{44} - H_{521})(x_1^2, x_2^2, x_3^2) = 17 \left( \frac{9450}{x_1^2} \right) + \frac{1}{1050} x_1^{14} x_2^2 + \frac{1}{9450} x_1^{12} x_2 x_3 + \frac{1}{525} x_1^{10} x_2^6 + \frac{2}{315} x_1^8 x_2^8 + \frac{1}{525} x_1^6 x_2^{10} + \frac{1}{9450} x_1^4 x_2^{12} + \frac{1}{1050} x_1^2 x_2^{14}
\]

\[
+ 17 \left( \frac{9450}{x_2^2} \right) + \frac{1}{1050} x_2^{14} x_1^2 + \frac{16}{4725} x_2^{12} x_2 x_3 + \frac{8}{1575} x_2^{10} x_2^3 + \frac{1}{9450} x_2^8 x_2 x_3 + \frac{1}{525} x_2^6 x_2^{10} + \frac{1}{9450} x_2^4 x_2^{12} + \frac{1}{1050} x_2^2 x_2^{14}
\]

\[
- \frac{8}{1575} x_1^2 x_2^{12} x_3 - \frac{16}{4725} x_1^2 x_2^{10} x_3 - \frac{1}{1050} x_1^2 x_2^8 x_3 - \frac{1}{9450} x_1^2 x_2^6 x_3 - \frac{1}{525} x_1^2 x_2^4 x_3 - \frac{1}{1050} x_1^2 x_2^2 x_3 - \frac{1}{9450} x_1^2 x_2^0 x_3
\]

\[
- \frac{1}{1890} x_1^4 x_2^8 x_3 + \frac{1}{4725} x_1^4 x_2^6 x_3 + \frac{1}{9450} x_1^4 x_2^4 x_3 + \frac{1}{525} x_1^4 x_2^2 x_3 + \frac{1}{315} x_1^4 x_2^0 x_3 - \frac{1}{1575} x_1^4 x_2^10 x_3 - \frac{1}{1575} x_1^4 x_2^8 x_3 - \frac{1}{1575} x_1^4 x_2^6 x_3 - \frac{1}{1575} x_1^4 x_2^4 x_3 - \frac{1}{1575} x_1^4 x_2^2 x_3 - \frac{1}{1575} x_1^4 x_2^0 x_3
\]

\[
+ \frac{1}{525} x_1^6 x_2^{10} + \frac{1}{9450} x_1^4 x_2^{12} - \frac{16}{4725} x_1^2 x_2^{12} x_3 + \frac{1}{9450} x_1^2 x_2^{10} x_3 + \frac{1}{525} x_1^2 x_2^{12} x_3 + \frac{1}{1050} x_1^2 x_2^{14} + \frac{1}{1050} x_2^2 x_2^{14} + \frac{17}{9450} x_1^{16}.
\]

**Remark 1.** Of course, there are other ways to show that a polynomial is nonnegative. However, Theorem 2 states something stronger. Not only is it nonnegative, but also a sum of squares. An ongoing research topic belonging to the general context of Hilbert’s 17th Problem is to understand the difference between sums of squares and nonnegativity, see [1, 2, 4, 5, 6, 10] to name only a few. As an interesting example, the degrees of irreducible components of the boundary of the SOS cone are Gromov-Witten numbers (see [3] and [22]).

## 2 Methods

The methods we have used are well-known, but had to be tailored specifically for this problem in order to be successful. In particular, attempts to use the Macaulay2 package SGS [9, 10], and also the Maple package SPECTRA [18, 21] were unsuccessful. Instead, we took advantage of the symmetric group action and also the real zeros of our polynomial.

We first present two facts that will be fundamental to the proof of our main Theorem 2.

**Lemma 1.** Consider a polynomial \( H(x_1, \ldots, x_n) \). Define another polynomial

\[
h(x_1, \ldots, x_n) = H(x_1^2, \ldots, x_n^2).
\]

If \( h \) can be written as a sum of squares, then \( H \) is nonnegative on the nonnegative orthant.

**Proof.** If we can write

\[
h = \sum d_i q_i^2
\]

then

\[
h = \sum d_i H(q_i)
\]

is a square.
for positive $d_i > 0$ and polynomials $q_i(x_1, \ldots, x_n)$, then we know $h$ is nonnegative on all of $\mathbb{R}^n$. By way of contradiction, assume there is some point $(a_1, \ldots, a_n) \in \mathbb{R}_{\geq 0}^n$ where $H(a_1, \ldots, a_n) < 0$. Then there also exist real numbers $\sqrt{a_1}, \ldots, \sqrt{a_n}$. But then $h(\sqrt{a_1}, \ldots, \sqrt{a_n}) = H(a_1, \ldots, a_n) < 0$, contradicting our representation of $h$ as a sum of squares. \hfill \Box

Let $S_+^N$ denote the cone of $N \times N$ symmetric positive semidefinite matrices in the space of $N \times N$ symmetric matrices $S^N$.

**Proposition 1.** Let $h$ be a homogeneous polynomial of degree $2d$ in $n$ variables, $h \in \mathbb{R}[x_1, \ldots, x_n]_d$. Let $m$ be a vector containing all $N = \binom{n+d}{d}$ monomials of degree $d$. Then $h$ is a sum of squares exactly when there exists some $A \in S_+^N$ such that $h(x_1, \ldots, x_n) = m^T Am$.

**Proof.** Every symmetric positive semidefinite matrix admits an $LDL^T$ factorization after the action of a permutation matrix. Therefore if such a matrix $A$ exists we can write

$$PAP^T = LDL^T.$$ 

An $LDL^T$ factorization of a matrix allows it to be written as a sum of rank 1 matrices as

$$PAP^T = \sum d_i l_i l_i^T$$

where the $l_i$ are the columns of $L$. Letting our monomial vector $m$ hit both sides, we have

$$h(x_1, \ldots, x_n) = m^T Am = (Pm)^T PAP^T (Pm) = (Pm)^T LDL^T (Pm) = (Pm)^T \left( \sum d_i l_i l_i^T \right) (Pm) = \sum d_i ((Pm)^T l_i)^2$$

where $(Pm)^T l_i$ is actually a polynomial, since the scalars in the columns $l_i$ become coefficients in front of the monomials from $(Pm)^T$. Thus, if we have $A$, we have a sum of squares.

Next suppose $h$ is a sum of squares,

$$h = \sum_{i=1}^k q_i^2.$$ 

Then for each $q_i$ there exists a vector $\alpha_i \in \mathbb{R}^N$, such that $\alpha_i^T m = q_i$. Define $U$ to be the $k \times N$ matrix with rows $\alpha_i$. Then,

$$m^T U^T Um = (Um)^T Um = \sum_{i=1}^k q_i^2$$

and by construction, $U^T U$ is a positive semidefinite $N \times N$ symmetric matrix. \hfill \Box

**Remark 2.** Quite a bit is known about upper and lower bounds for $k$, the number of squares needed to write a polynomial as a sum of squares. See for example [8] and [26]. If a degree 16 polynomial in 3 variables can be written as a sum of squares, then it is known that at most 10 squares are needed. It would be interesting to see how to reduce our 41 squares to 10.

Proposition 1 tells us that writing a polynomial as a sum of squares is equivalent to solving a semidefinite program (SDP). That is, we must find a matrix $A$ in $S_+^N$ that also satisfies the linear constraints defined by equating the coefficients of $h(x_1, \ldots, x_n)$ and $m^T Am$. However, a significant problem is that a semidefinite program solver returns a matrix with floating point entries. In particular, the matrix will (almost) never exactly reproduce the desired polynomial.
Example 1. An attempt to reproduce the polynomial \((H_{21} - H_{111})(x_1^6, x_2^6, x_3^6)\), whose nonnegativity was proven in [11], produced the following polynomial with floating point coefficients

\[
\frac{1}{54} x_1^6 + \frac{1}{54} x_2^6 + (7.888609052210118 \times 10^{-31}) x_1^3 x_2 x_3 + (7.888609052210118 \times 10^{-31}) x_1^2 x_2^3 x_3 \\
+ (3.944304526105059 \times 10^{-31}) x_1 x_2^2 x_3^2 - 0.05555555555555555 x_1^2 x_2 x_3^2 \\
+ (3.944304526105059 \times 10^{-31}) x_1 x_2^3 x_3^3 + (3.944304526105059 \times 10^{-31}) x_1^2 x_2 x_3^3
\]

even though the desired polynomial is

\[
\frac{1}{54} x_1^6 + \frac{1}{54} x_2^6 - \frac{1}{18} x_1 x_2^2 x_3^2 + \frac{1}{54} x_3^6.
\]

Therefore, in order to find an exact sum of squares certificate of nonnegativity, we must make adjustments. To satisfy Proposition [11] we must replace the entries of the matrix itself, while staying in the PSD cone, and continuing to satisfy the requirements of \(m^T Am = h\) exactly. One approach to this problem is to use continued fractions to find the best rational approximation (with some user-specified bound \(B\) on the size of the denominator) to the entries of the matrix. See, for example, Problem 42 part c of [13], where this is referred to as neighbor fractions. Geometrically, the SDP may return a matrix on or near the boundary of the PSD cone. By rounding the floating point entries to rational numbers, we risk moving outside the cone, resulting in a matrix which is not positive semidefinite. Therefore, many times this rational rounding procedure will fail.

Remark 3. In general, rational certificates for polynomials with rational coefficients do not always exist. This was shown by Scheiderer in [25] where he provided explicit minimal examples of degree 4 polynomials in 3 variables. Since our polynomial is of degree 16, there is no a priori reason to believe it must have a rational sum of squares representation.

Several approaches to this rational rounding problem have been developed. The package SOS has a rational rounding procedure built-in, but for our polynomial their package returned an error stating that the rational rounding had failed. The software RealCertify [20], based on [19], uses a hybrid numeric-symbolic algorithm for finding rational approximations for polynomials lying in the interior of the SOS cone. In correspondence during the writing of this paper, Mohab Safey El Din reported that RealCertify failed to terminate for our problem. However, in that paper they also describe and compare complexity of several different algorithms, including geometric critical point methods. Mohab reported that the geometric critical point methods were successful on our problem, providing a second confirmation of the nonnegativity of our polynomial, although not of its SOS-ness.

Remark 4. Another approach would be to search directly for the matrix using exact arithmetic as in [18] with the package SPECTRA for Maple. However, for a problem of our size, this approach is not promising. Indeed we let SPECTRA run for several days, and it did not terminate. Ours is a feasibility problem, but when optimizing a linear function for a rational SDP, the entries of the optimal solution matrix will be algebraic numbers. In [23] the algebraic degree of an SDP is introduced. For generic inputs, this degree depends only on the rank \(r\) of the solution matrix, the size \(n\) of the symmetric matrices, and the dimension \(m\) of the affine subspace. In [15] the authors give an exact formula for the algebraic degree. If \(n = 45, m = 129, r = 41\), their formula yields the following 74 digit number:

\[
27986928303724394857777762195272647267703276932951767224059513477726952420.
\]

Example 1. (continued) Returning to the example above, the output of the SDP for \((H_{21} - H_{111})(x_1^7, x_2^6, x_3^2)\)
was the following matrix, for which we print only the first four of ten columns:

\[
\begin{pmatrix}
1 & 0 & 0 & -\frac{1}{108} \\
0 & 1 & 0 & -\frac{1}{108} \\
-\frac{1}{108} & 0 & \frac{1}{18} & 0 \\
0 & -\frac{1}{108} & 0 & \frac{1}{18} \\
0 & 0 & 0 & \frac{1}{18} \\
0 & 0 & \frac{1}{18} & 0 \\
-\frac{1}{108} & 0 & 0 & 0 \\
0 & -\frac{1}{108} & \frac{1}{18} & 0 \\
0 & 0 & 0 & \frac{1}{18} \\
0 & 0 & \frac{1}{18} & 0
\end{pmatrix}
\]

Using continued fractions with denominator bound \( B = 150 \) we obtain the following (preferable) matrix:

\[
A = \begin{pmatrix}
\frac{1}{144} & 0 & 0 & -\frac{1}{108} & 0 & -\frac{1}{108} & 0 & 0 & 0 \\
0 & \frac{1}{144} & 0 & 0 & -\frac{1}{108} & 0 & 0 & 0 & 0 \\
-\frac{1}{108} & 0 & \frac{1}{18} & 0 & 0 & -\frac{1}{108} & 0 & 0 & 0 \\
0 & -\frac{1}{108} & 0 & \frac{1}{18} & 0 & 0 & -\frac{1}{108} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{18} & 0 & 0 & -\frac{1}{108} & 0 \\
0 & 0 & \frac{1}{18} & 0 & 0 & \frac{1}{18} & 0 & 0 & -\frac{1}{108} \\
-\frac{1}{108} & 0 & 0 & 0 & -\frac{1}{108} & 0 & \frac{1}{18} & 0 & 0 \\
0 & -\frac{1}{108} & \frac{1}{18} & 0 & 0 & 0 & 0 & \frac{1}{18} & 0 \\
0 & 0 & 0 & 0 & \frac{1}{18} & 0 & \frac{1}{18} & 0 & \frac{1}{18} \\
0 & 0 & \frac{1}{18} & 0 & 0 & \frac{1}{18} & 0 & \frac{1}{18} & 0
\end{pmatrix}
\]

With \( mT = (x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_3, x_1x_2^2, x_1x_3^2, x_2x_3^2) \), we can calculate \( m^T Am \), obtaining

\[
\frac{1}{54} x_1^6 + \frac{1}{54} x_2^6 - \frac{1}{18} x_1^2 x_2^2 x_3^2 + \frac{1}{54} x_3^6
\]

which is exactly \( (H_{21} - H_{111})(x_1^2, x_2^2, x_3^2) \), as desired. Since \( A \) is positive semidefinite, by carrying out the LDL\(^T\) factorization we obtain the following sum of squares representation of \( (H_{21} - H_{111})(x_1^2, x_2^2, x_3^2) \):

\[
\frac{1}{216} \left(2 x_1^2 x_2^2 - x_1 x_2 x_3^2\right)^2 + \frac{1}{216} \left(2 x_1^2 x_2^2 - x_1 x_2 x_3^2\right)^2 + \frac{1}{72} \left(x_1 x_2^2 - x_1 x_3^2\right)^2
\]

\[
+ \frac{1}{72} \left(x_1 x_2^2 - x_1 x_3^2\right)^2 + \frac{1}{216} \left(2 x_1^2 x_2^2 - x_2 x_3^2 - x_3^2\right)^2 + \frac{1}{72} \left(x_2 x_3^2 - x_3^2\right)^2.
\]

As noted in the discussion above, for \( H_{44} - H_{521} \), existing tools do not return a numerical matrix which can be successfully rounded. Our solution to this problem uses Theorem 2 and Lemma 2 described below.

To start, we wish to take advantage of the fact that \( H_{44} - H_{521} \) is a symmetric polynomial. That is to say, it is invariant under the action of the symmetric group. There is a great deal of literature on the subject of symmetric polynomials and sums of squares, including [5, 7, 10, 12, and 14], to name only a few. We specialize Theorem 3.3 of [14] to our setting as follows:

**Theorem 3.** Given an orthogonal linear representation of the symmetric group \( S_n, \sigma : S_n \to Aut(S^N) \), consider a semidefinite program whose objective and feasible matrices are invariant under the group action. Then the optimal value of the SDP is equal to the optimal value of the same SDP restricted to its fixed point subspace, \( \{X \in S^N : X = \sigma(g)X, \forall g \in S_n\} \).

In our case, the \( S_3 \) action on the space of polynomials in three variables induces an action on the decision variable (the symmetric matrix) of our SDP, where group elements act on the symmetric matrix by conjugation. More specifically, for each \( g \in S_3 \), let \( \rho(g) \) be the associated matrix that permutes the monomials of degree 8 in 3 variables. Then the induced action sends a symmetric 45 \( \times \) 45 matrix \( X \to \rho(g)^T X \rho(g) \). Note that \( \rho(g) \) is an orthogonal matrix. Then the fixed-point subspace for our particular SDP is:

\[
\mathcal{F} = \{X : X \rho(g) = \rho(g)X, \forall g \in S_3\}
\]

The above theorem allows us to restrict to this fixed-point subspace. Thus we can force our matrix \( A \) to commute with the elements of our group, obtaining better constraints on our semidefinite program, and increasing our chances of success.
Lemma 2. If $x^*$ is a (nonzero) real root of the polynomial $h = m^T Am$, which we write as a sum of squares using the factorization of $A$, then the monomial vector $m$ evaluated at $x^*$ must be in the nullspace of $A$.

Proof. This follows from $0 = h(x^*) = m(x^*)^T Am(x^*)$ and the fact that $A$ is positive semidefinite. □

3 Proof of Theorem 2

Proof of the main Theorem 2. We ran a semidefinite program to find a symmetric positive semidefinite matrix whose factorization could produce a sums of squares representation for $H_{44} - H_{521}$ evaluated at $x_1^2, x_2^2, x_3^2$. By Lemma 1 this certifies the non-negativity of $H_{44} - H_{521}$ on the non-negative octant. By Proposition 1 the existence of a matrix which exactly reproduces our polynomial is equivalent to the existence of a sums of squares representation. Also by Proposition 1, we impose linear constraints on the entries of our unknown matrix $A$ to require that $m^T Am$ exactly matches the coefficients of our desired polynomial. By Theorem 3, we impose linear constraints to force our unknown matrix $A$ to commute with the action of the symmetric group on the space of symmetric $45 \times 45$ matrices. Finally, by Lemma 2 we incorporate 4 linearly independent vectors $m(x^*)$, coming from real zeros of our polynomial, to obtain more linear conditions on our SDP.

The output is a numerical matrix sufficiently located in the PSD cone such that continued fractions rational approximation yields a matrix with exact, rational entries. This matrix remains positive semidefinite and produces our desired polynomial via $m^T Am$. The entries of this $45 \times 45$ matrix include rational numbers with quite large denominators. Therefore we do not print the matrix here. Rather, we refer the reader to the first author’s website [17] for the explicit matrix.

To give the reader a feel for the matrix, the first row is:

$$
\begin{pmatrix}
17/9450, & 0, & 0, & -4/1433, & 0, & -4/1433, & 0, & 0, & 0, & 5/6699, & 0, & 1/484, & 0, & 5/6699, & 0, & 0, & 0, & 0, & 0, & 0, & 1/5516,
0, & -4/6655, & 0, & -4/6655, & 0, & 1/5516, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 1577475196106984510532373543 & 780042543477364748948750531770400,
1577475196106984510532373543 & 780042543477364748948750531770400 & 0, & -1/2243, & 0, & -1/2243, & 0, & -1/2243, & 0, & 1577475196106984510532373543 & 780042543477364748948750531770400
\end{pmatrix}
$$

The (positive) coefficients and polynomial squares are also available at the aforementioned website, along with code to expand this sum of squares, and verify that it does produce the desired polynomial. By Lemma 1 this sum of squares certifies the polynomial $H_{44} - H_{521}$ as a valid counterexample. □

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