FIRST INTEGRALS OF HAMILTONIAN SYSTEMS: THE INVERSE PROBLEM

Rehana Naz
Centre for Mathematics and Statistical Sciences
Lahore School of Economics
Lahore 53200, Pakistan

Fazal M Mahomed*
DST-NRF Centre of Excellence in Mathematical and Statistical Sciences
School of Computer Science and Applied Mathematics
University of the Witwatersrand
Johannesburg, Wits 2050, South Africa

Azam Chaudhry
Department of Economics
Lahore School of Economics
Lahore 53200, Pakistan

Abstract. There has, to date, been much focus on when a Hamiltonian operator or symmetry results in a first integral for Hamiltonian systems. Very little emphasis has been given to the inverse problem, viz. which operator arises from a first integral of a Hamiltonian system. In this note, we consider this problem with examples mainly taken from economic growth theory. We also provide an example from classical mechanics.

1. Introduction. Hamilton’s canonical equations have been a focal point for substantial research since its initial introduction (see e.g., the classical work of Arnold [1]).

The classical link between Hamiltonian symmetries and their first integrals for Hamilton’s canonical equations was uncovered in the landmark work of Levi-Civita in 1899 (Levi-Civita [7]; see also the translation by Saccomandi and Vitolo [21]). Many books and papers are also devoted to this important area (see e.g., Whittaker [23], Smale [22], Marsden and Weinstein [10], Kozlov [6], Olver [20], Dorodnitsyn and Kozlov [3], Mahomed and Roberts [8]).

In [7, 20], a Hamiltonian symmetry in evolutionary form is the standard way to find the first integral of the Hamilton’s equations up to a time-dependent function. There are disadvantages which are discussed in Dorodnitsyn and Kozlov [3] and in Mahomed and Roberts [8]. Dorodnitsyn and Kozlov [3] present a direct method to obtain the first integrals. This method requires the gauge terms that need integration. In the recent contribution [8], the direct connection between the symmetries and first integrals of the canonical Hamiltonian equations are unified in order to...
find which symmetries are Hamiltonian and the conditions which yield the related first integrals in a precise manner.

The nonstandard Hamiltonian formalism has been productively used in economic models, epidemics, mechanics and other areas of applied mathematics. The Hamiltonian framework for arbitrary control, costate and state variables has been investigated in [11] - this is analogous to the method given in [3]. The methodology has been successfully applied to economic growth models [13, 14, 15], epidemics [16, 4], mechanics and other areas [17, 9, 5]. Moreover, the characterization of which operators are Hamiltonian operators that result in first integrals for such nonstandard Hamiltonian equations were recently determined in [18]. Naz [19] studied the characterization of approximate Partial Hamiltonian operators as well.

The inverse problem of when a first integral arises from a Hamiltonian operator has not been addressed thus far. This is the focus of the present contribution. This approach works if there is an underlying Hamiltonian which results in either a symmetry or an integral. If the first integral is deduced by another method then our present approach gives a symmetry for this first integral. Therefore our method can determine the first integral if we work out its symmetry by our approach. We consider examples mainly from economics. However, we do provide one from mechanics. This is done in the conclusion to show the relevance of the proposed method to a variety of fields.

In the appendix A we present the definitions and results that are used in the sequel.

2. Applications. In this section, we solve the inverse problem for an economic growth model with a logarithmic utility function, for a model of optimal growth with an environmental asset and Lucas-Uzawa model with externalities.

Example 1. It is important to mention that there are examples of first integrals in Economics that were found using other approaches, see e.g. Naz et al [12]. By connecting a symmetry we are saying that our approach given here can work for these examples as well provided they have a Hamiltonian. Naz et al [12] studied the economic growth model

\[
\text{Max}_k \int_0^\infty \ln \left( k^\beta - \delta k + Ak - \dot{k} \right) e^{-\rho t} dt, \; k(0) = k_0. \tag{1}
\]

In [12], the following, partial Lagrangian was considered:

\[
L = - \ln \left( k^\beta - \delta k + Ak - \dot{k} \right) = \ln \left( \frac{1}{k^\beta - \delta k + Ak - \dot{k}} \right) \tag{2}
\]

which yields the following first integral

\[
I = \frac{\beta \rho t}{\beta - 1} + \ln(k^\beta - \delta k + Ak - \dot{k}) + k^\beta(k^\beta - \delta k + Ak - \dot{k})^{-1} - 1, \tag{3}
\]

provided

\[
\beta = \frac{\rho + \delta - A}{\delta - A}. \tag{4}
\]

Now, we solve for the inverse problem. We have knowledge of the first integral \( I \) and we want to determine the operator/s that arise. The current value (partial) Hamiltonian function for this model, with the aid of the Legendre transformation, is

\[
H^c(t, c, k, \theta) = \ln(c) + \theta(k^\beta - \delta k + Ak - c), \tag{5}
\]
and this results in the following dynamical system of ODEs:

\[ \dot{\theta} = \frac{1}{c} \]

\[ \dot{k} = k^\beta - \delta k + Ak - c \]  \tag{6}

\[ \dot{\theta} = -\theta(\beta k^{\beta - 1} - \delta + A - \rho) \]  \tag{7}

With the aid of equations (6) and (7), the first integral (3) can be re-written as

\[ I = \beta \rho t - \ln(\theta) + \theta k^\beta - 1, \]  \tag{8}

Now, we compute the \( I_k, I_\theta \) and \( I_t \) as follows:

\[ I_k = \beta \theta k^{\beta - 1}, \]

\[ I_\theta = -\frac{1}{\theta} + k^\beta, \]  \tag{10}

\[ I_t = \frac{\beta \rho}{\beta - 1}. \]

We utilize Theorem 4 which yields \( I_k = -\bar{\zeta}, I_\theta = \bar{\eta} \). Thus we have

\[ \bar{\zeta} = -\beta \theta k^{\beta - 1}, \]

\[ \bar{\eta} = -\frac{1}{\theta} + k^\beta, \]  \tag{11}

and one can show that \( I_t \) is identically satisfied. The condition \( (A-18) \) is satisfied

\[ \frac{\partial \bar{\eta}}{\partial k} + \frac{\partial \bar{\zeta}}{\partial \theta} = 0. \]  \tag{12}

Also \( (A-19) \) and \( (A-20) \) must hold. Next, \( \zeta = \bar{\zeta} + \xi \dot{\theta} \) and \( \eta = \bar{\eta} + \xi \dot{k} \) result in

\[ \zeta = -\beta \theta k^{\beta - 1} - \xi \theta (\beta k^{\beta - 1} - \delta + A - \rho), \]

\[ \eta = -\frac{1}{\theta} + k^\beta + \xi (k^\beta - \delta k + Ak - \frac{1}{\theta}), \]  \tag{13}

where \( \xi \) is arbitrary. One can obtain a point-type operator by setting \( \xi = -1 \) whereby then \( \eta = (\delta - A)k \) and \( \zeta = (A - \delta - \rho)\theta \), where \( \beta = \frac{\rho + \delta - A}{\delta - A} \).

**Example 2.** The current value (partial) Hamiltonian for a model of optimal economic growth with an environmental asset is (see Naz [17])

\[ H^c(t, c, s, p) = \frac{(cs^\phi)^{1-\sigma}}{1-\sigma} + p(ms - c), \]  \tag{14}

and the associated dynamical system of ODEs is

\[ p = c^{-\sigma} k^{\phi(1-\sigma)}, \]  \tag{15}

\[ s = ms - c, \]  \tag{16}

\[ \dot{p} = (\rho - m - \frac{c}{s})p. \]  \tag{17}

The partial Hamiltonian approach in Naz [17]) established the following three first integrals for this model:

\[ I_1 = \frac{\rho pse^{-\rho t}}{(\phi + 1)(1-\sigma)} - e^{-\rho t}[(cs^\phi)^{1-\sigma} - p(ms - c)], \]

\[ I_2 = e^{(\rho - m\rho - m)^{1-\sigma}}[pc - (cs^\phi)^{1-\sigma}]. \]  \tag{18}
Equation (15) yields

\[ I_3 = ps^{-\phi} e^{(m\phi + m - \rho)t}. \]

Equation (15) yields \( c = s^{\frac{(1-\sigma)}{\sigma}} p^{1} \) and these first integrals can be re-written as

\[ I_1 = e^{-pt}\left[ \frac{\rho ps}{(\phi + 1)(1-\sigma)} - \frac{\sigma}{1-\sigma} s^{\frac{(1-\sigma)}{\sigma}} p^{1} - pm \right], \]
\[ I_2 = -\frac{\sigma}{1-\sigma} e^{(m\phi + m - \rho)(1-\sigma)} s^{\frac{(1-\sigma)}{\sigma}} p^{1} \], \hspace{1cm} (19) \]
\[ I_3 = ps^{-\phi} e^{(m\phi + m - \rho)t}. \]

The first integrals \( I_2 \) and \( I_3 \) are functionally dependent as \( I_2 = -\frac{\sigma}{1-\sigma} I_3^{\frac{(1-\sigma)}{\sigma}} \).

Now we solve for the inverse problem. We have knowledge of the first integral \( I_1 \) and we want to determine the operators that arise. First, we compute \( I_{1s} \), \( I_{1p} \) and \( I_{1t} \) as follows:

\[ I_{1s} = e^{-pt}\left[ \frac{\rho p s}{(\phi + 1)(1-\sigma)} - \frac{\phi s^{\frac{(1-\sigma)}{\sigma}}}{1-\sigma} - \frac{1}{1-\sigma} - pm \right], \]
\[ I_{1p} = e^{-pt}\left[ \frac{\rho s}{(\phi + 1)(1-\sigma)} + s^{\frac{(1-\sigma)}{\sigma}} p^{1} - pm \right], \]
\[ I_{1t} = -pe^{-pt}\left[ \frac{\rho ps}{(\phi + 1)(1-\sigma)} - \frac{s^{\frac{(1-\sigma)}{\sigma}}}{1-\sigma} p^{1} - pm \right]. \]

We utilize Theorem 4 which yields \( I_s = -\tilde{\zeta} \), \( I_p = \tilde{\eta} \). Thus we have

\[ \tilde{\zeta} = -e^{-pt}\left[ \frac{\rho p}{(\phi + 1)(1-\sigma)} - \phi s^{\frac{(1-\sigma)}{\sigma}} p^{1} - pm \right], \]
\[ \tilde{\eta} = e^{-pt}\left[ \frac{\rho s}{(\phi + 1)(1-\sigma)} + s^{\frac{(1-\sigma)}{\sigma}} p^{1} - pm \right], \] \hspace{1cm} (21)

and one can show that \( I_t \) is identically satisfied. The condition (A-18) is satisfied

\[ \frac{\partial \tilde{\eta}}{\partial k} + \frac{\partial \tilde{\zeta}}{\partial \lambda} = 0. \] \hspace{1cm} (22)

Also (A-19) and (A-20) must hold. Next, \( \zeta = \tilde{\zeta} + \xi \tilde{p} \) and \( \eta = \tilde{\eta} + \xi \tilde{s} \) result in

\[ \zeta = -e^{-pt}\left[ \frac{\rho p}{(\phi + 1)(1-\sigma)} - \phi s^{\frac{(1-\sigma)}{\sigma}} p^{1} - pm \right] \]
\[ + \xi[(\rho - m)p - \phi s^{\frac{(1-\sigma)}{\sigma}} p^{1} - pm], \]
\[ \eta = e^{-pt}\left[ \frac{\rho s}{(\phi + 1)(1-\sigma)} + s^{\frac{(1-\sigma)}{\sigma}} p^{1} - pm \right] \]
\[ + \xi[(\rho - m)p - \phi s^{\frac{(1-\sigma)}{\sigma}} p^{1} - pm], \] \hspace{1cm} (23)

where \( \xi \) is arbitrary. To arrive at a point-type operator, choose \( \xi = e^{-pt} \) - then \( \eta = e^{-pt}\left[ \frac{\rho s}{(\phi + 1)(1-\sigma)} \right] p e^{-pt} \).

A similar procedure is carried out for \( I_2 \) and we deduce the following expressions for \( \zeta \) and \( \eta \):

\[ \zeta = \phi s^{\frac{(1-\sigma)}{\sigma}} p^{\frac{1}{1-\sigma}} e^{(p-m\phi-m)(1-\sigma)t} \]
\[ + \xi[(\rho - m)p - \phi s^{\frac{(1-\sigma)}{\sigma}} p^{\frac{1}{1-\sigma}}], \]
\[ \eta = s^{\frac{(1-\sigma)}{\sigma}} p^{\frac{1}{1-\sigma}} e^{(p-m\phi-m)(1-\sigma)t} + \xi[(\rho - m)p - \phi s^{\frac{(1-\sigma)}{\sigma}} p^{\frac{1}{1-\sigma}}], \] \hspace{1cm} (24)

where \( \xi \) is arbitrary. To obtain a point-type operator, select \( \xi = e^{\frac{(p-m\phi-m)(1-\sigma)}{\sigma} t} \).

Then \( \eta = mse^{\frac{(p-m\phi-m)(1-\sigma)}{\sigma} t} \) and \( \zeta = (\rho - m)pe^{\frac{(p-m\phi-m)(1-\sigma)}{\sigma} t}. \)
We follow a similar procedure for $I_3$ and we determine the following expressions for $\xi$ and $\eta$:

$$\xi = \phi ps^{-\phi-1}e^{(m\phi+m-\rho)t} + \xi[(\rho - m)p - \phi s^{\frac{\phi(1-x)}{\alpha}}],$$
$$\eta = s^{-\phi}e^{(m\phi+m-\rho)t} + \xi ms s^{\frac{\phi(1-x)}{1-\alpha}},$$  \hspace{1cm} (25)

where $\xi$ is arbitrary. To find a point-type operator, we choose $\xi = 0$, thus $\eta = s^{-\phi}e^{(m\phi+m-\rho)t}$ and $\zeta = \phi ps^{-\phi-1}e^{(m\phi+m-\rho)t}$.

**Example 3.** Naz and Chaudhry [15] established the closed-form solutions of the Lucas-Uzawa model with externalities by first transforming the model to the Lucas-Uzawa model with no externalities [13, 14]. The closed-form solutions to the model were then derived by utilising the transformation to the closed-form solutions constructed in [13, 14]. The first integrals for the original model were not constructed in [15] and are computed here.

The current value (partial) Hamiltonian function and associated dynamical system as given in Naz and Chaudhry [15] is

$$H^c(t, c, u, k, h, \lambda_1, \lambda_2) = \frac{c^{1-\sigma} - 1}{1-\sigma} + \lambda_1[\gamma k^{\alpha}u^{1-\alpha}h^{1-\alpha+\theta} - \pi k - c] + \lambda_2\delta (1-u)h, \hspace{1cm} (26)$$

and

$$\lambda_1 = c^{-\sigma}, \hspace{1cm} (27)$$
$$u^\alpha = \frac{\gamma(1-\alpha)k^{\alpha}h^{-\alpha+\theta}}{\delta} \frac{\lambda_1}{\lambda_2}, \hspace{1cm} (28)$$
$$\dot{k}(t) = \gamma k^{\alpha}u^{1-\alpha}h^{1-\alpha+\theta} - \pi k - c, \hspace{1cm} (29)$$
$$\dot{h}(t) = \delta (1-u)h, \hspace{1cm} (30)$$
$$\dot{\lambda}_1 = -\lambda_1 (\gamma + (\rho - \delta) - \frac{\lambda_2 \delta}{1-\alpha} u, \hspace{1cm} (31)$$
$$\dot{\lambda}_2 = \lambda_2 (\rho - \delta) - \frac{\lambda_2 \delta}{1-\alpha} u, \hspace{1cm} (32)$$
$$\dot{c} = \frac{\sigma\gamma}{\sigma} u^{1-\alpha}k^{\alpha-1}h^{1-\alpha+\theta} - \rho + \frac{\pi}{\sigma}, \hspace{1cm} (33)$$
$$\dot{u} \frac{c}{u} = \frac{(\delta + \pi)(1-\alpha) + \delta \theta}{\alpha} - \frac{c}{k} + \left(\frac{1-\alpha + \theta}{1-\alpha}\right)\frac{\delta u}{\alpha}. \hspace{1cm} (34)$$

For the current value Hamiltonian (26), the partial Hamiltonian determining equation (see [11]) is given by

$$\lambda_1 (\eta^1_h + \dot{k} \eta^1_k + \dot{h} \eta^1_h) + \lambda_2 (\eta^2_h + k \eta^2_k + \dot{h} \eta^2_h) - \eta^1[\alpha \lambda_1 \gamma u^{1-\alpha} k^{\alpha-1} h^{1-\alpha+\theta} - \pi \lambda_1] - \eta^2 \delta \lambda_2 (1 + \frac{\theta u}{1-\alpha})$$

$$- H(\xi_t + k \xi_k + \dot{h} \xi_h) = B_t + k B_k + \dot{h} B_h \hspace{1cm} (35)$$

in which we assume that $\xi, \eta^1, \eta^2, B$ are functions of $t, k$ and $h$. We expand equation (35) and then separate it with respect to powers of the control variables which gives
rise to the following system of equations:

\[
\begin{align*}
\sigma^2 : & \quad \xi_k = 0, \\
\sigma^{-1} u^2 : & \quad \xi_h = 0, \\
\sigma u^{-1} : & \quad \eta^1_h = 0, \\
\sigma^{-1} : & \quad -\eta^1_k \sigma^{-1} \xi + \rho \xi = 0, \\
\sigma u^{-1} : & \quad \eta^1_k \sigma^{-1} \pi + \pi k \xi_t + \rho \eta^1 + \rho \pi k \xi = 0, \\
\sigma^{-1} u^{-1} : & \quad \eta^2_h = 0, \\
\sigma^{-1} u^{-2} : & \quad \eta^2_h - \delta h \eta^2 + \delta h^2 \xi_t + \rho \eta^2 - \rho \xi \delta h = 0, \\
\sigma^{-1} u^{-1} : & \quad \eta^2_k - (1 - \alpha) \eta^2 + \eta^2 \pi - \alpha \xi_t - \rho \xi \alpha - \eta^2 \pi = 0, \\
\sigma : & \quad B_k = 0, \\
u : & \quad B_h = 0, \\
1 : & \quad B_t = \frac{1}{\sigma^2} \xi_t.
\end{align*}
\]

The solution of the system of equations (36) is

\[
\begin{align*}
\xi &= a_1(t), \\
\eta^1 &= \left(\frac{\sigma}{\sigma - 1} a_1'(t) + \rho a_1(t)\right) k, \\
\eta^2 &= \left[\frac{\sigma - 2\alpha \sigma + \alpha}{\sigma - 1} a_1'(t) + \rho (1 - 2\alpha) a_1(t)\right] \frac{h}{1 - \alpha + \theta} + a_2(t) h^{\frac{\alpha - \theta}{\sigma}}, \\
B(t) &= \frac{1}{1 - \sigma} a_1(t) + d_4,
\end{align*}
\]

\[
\begin{align*}
\sigma^{-1} a_1''(t) + (\rho + \pi + \frac{\sigma \rho}{\sigma - 1}) a_1'(t) + \rho (1 - 2\alpha) a_1(t) = 0, \\
\sigma^{-1} a_2''(t) + (\rho - \delta - \frac{\delta \delta}{\alpha - 1}) a_2(t) = 0.
\end{align*}
\]

The differential equation for \(a_2(t)\) yields

\[
a_2(t) = d_1 e^{-(\rho - \delta) t}, \quad (38)
\]

and solution of \(a_1(t)\) is

\[
\begin{align*}
a_1(t) &= d_2 e^{-\rho t} \quad \text{No restrictions on parameters} \quad (39) \\
a_1(t) &= d_2 e^{-\rho t} + d_3 e^{-(\sigma - 1)(\rho + \pi) t} \quad \text{where} \quad \sigma = \frac{\alpha (\rho + \pi)}{2 \alpha - \pi - \delta (1 - \alpha + \theta)}. \quad (40)
\end{align*}
\]

We can choose \(d_4 = 0\), without loss of generality. The partial Hamiltonian operators, gauge terms and first integrals with no restriction on the parameters of the economy are given by

\[
\begin{align*}
\xi &= 0, \\
\eta^1 &= 0, \\
\eta^2 &= e^{-(\rho - \delta) t} h^{\frac{\alpha - \theta}{\sigma}}, \\
B(t) &= 0, \\
I_1 &= \frac{\gamma (1 - \alpha)}{\delta} c^{-\sigma} k \alpha a h^{-\alpha \pi} e^{-(\rho - \delta) t},
\end{align*}
\]

\[
\begin{align*}
I_2 &= \frac{\gamma (1 - \alpha)}{\delta} c^{-\sigma} k \alpha a h^{-\alpha \pi} e^{-(\rho - \delta) t}, \\
I_3 &= \frac{\gamma (1 - \alpha)}{\delta} c^{-\sigma} k \alpha a h^{-\alpha \pi} e^{-(\rho - \delta) t}.
\end{align*}
\]
FIRST INTEGRALS OF HAMILTONIAN SYSTEMS: THE INVERSE PROBLEM

\[ \xi = e^{-\rho t}, \eta^1 = \frac{1}{1-\sigma} \rho e^{-\rho t} k, \]

\[ \eta^2 = \frac{1}{(1-\sigma)\phi} \rho e^{-\rho t} h, \quad B = \frac{1}{1-\sigma} e^{-\rho t}, \]

\[ I_2 = \frac{e^{-\rho t}}{1-\sigma} \left[ (\rho + \pi - \pi \sigma) k - \sigma c - \gamma \alpha (1-\sigma) k^\alpha u^{1-\alpha} h^{1-\alpha} \right. \]

\[ \left. + \frac{1-\alpha}{\delta \phi} \left( \rho - \delta (1-\sigma) \phi \right) \gamma k^\alpha u^{1-\alpha} h^{1-\alpha} \right], \quad (41) \]

where \( \phi = \frac{1-\alpha + \theta}{1-\alpha} \). Under the parameter restriction \( \sigma = \frac{\alpha (\rho + \pi)}{2\pi \sigma - \pi (1-\alpha + \sigma)} \), another first integral exists. The partial Hamiltonian operators, gauge term and first integral are given by

\[ \xi = e^{-\frac{1}{\sigma} \alpha (\rho + \pi) \phi}, \quad \eta^1 = -\pi \rho \alpha - \pi \sigma + 2 \alpha \sigma \pi - \pi \alpha, \quad \eta^2 = \frac{1}{\sigma \phi (1-\alpha)} \left[ -\rho \alpha - \pi \sigma \right. \]

\[ \left. + 2 \alpha \sigma \pi - \pi \alpha \right] e^{-\frac{1}{\sigma} \alpha (\rho + \pi) \phi}, \quad B = \frac{1}{1-\sigma} e^{-\frac{1}{\sigma} \alpha (\rho + \pi) \phi}, \quad (42) \]

\[ I_3 = e^{-\frac{1}{\sigma} \alpha (\rho + \pi) \phi} e^{-\sigma \left[ \frac{\alpha (\rho + \pi)}{(\rho - \pi) \alpha - \pi - \delta \phi (1-\alpha)} e^{-\alpha \gamma k^{1-\alpha} h^{1-\alpha}} \right]. \]

Remark 1. The method outlined here works if there is an underlying Hamiltonian that has associated with it either a symmetry or an integral. If a first integral is found by another method which admits a Hamiltonian, then our approach can provide a symmetry for the said first integral and therefore our method can result in the first integral if we calculate its symmetry via our method.

3. Conclusions. In this work, we have studied the inverse problem for Hamiltonian systems, viz. given a first integral, how one can construct the operator/s associated with this integral. We have taken examples from economic growth theory. There are also a number of examples in mechanics that can be utilized to show the effectiveness of this approach. We consider the following example to illustrate:

The motion of a particle under a power law radially dependent curl force is governed by the equation of motion (see Berry and Shukla [2], Section 4)

\[ \ddot{r} = J^2 r^{-3}, \]

\[ \dot{J} = r^{-2/3}, \quad (43) \]

where \( J = r^2 \dot{\theta} \) is the angular momentum which is not conserved. In [2], the authors, by using an indirect method resorting to the solution of the Emden-Fowler equation, constructed the first integral

\[ I = J \dot{r}^2 - 3 r^{1/3} \dot{r} + J^3 r^{-2}. \quad (44) \]

By insertion of the value of \( J \) into (43), we can write equation (43) as

\[ \dot{r} - r \dot{\theta}^2 = 0, \]

\[ 2r \dot{\theta} + r \ddot{\theta} = r^{-5/3}. \quad (45) \]

A partial Hamiltonian of (45) is

\[ H = \frac{1}{2} \dot{r}^2 + \frac{1}{2} r^{-2} \dot{\theta}^2. \quad (46) \]
with $\Gamma_1 = 0$ and $\Gamma_2 = r^{-2/3}$ and the corresponding first order system
\[
\begin{align*}
\dot{r} &= p_1, \\
\dot{\theta} &= p_2 r^{-2}, \\
\dot{p}_1 &= r^{-3} p_2^2, \\
\dot{p}_2 &= r^{-2/3}.
\end{align*}
\] (47)

One can easily rewrite $I$ as in (44) in terms of the phase space coordinates as
\[
I = p_1^2 p_2 - 3 r^{1/3} p_1 + r^{-2} p_2^3.
\] (48)

Then using the relations (A-21), we arrive at
\[
\begin{align*}
\bar{\zeta}_1 &= r^{-2/3} p_1 + 2 r^{-3} p_2^3, \\
\bar{\zeta}_2 &= 0, \\
\bar{\eta}^1 &= 2 p_1 p_2 - 3 r^{1/3}, \\
\bar{\eta}^2 &= p_2^2 + 3 r^{-2} p_2^2. 
\end{align*}
\] (49)

Here $I_t = 0$ is identically satisfied. There is a liberty of choice for $\xi$ and by choosing $\xi = 0$ we find at once the point-type operator
\[
X = (2 p_1 p_2 - 3 r^{1/3}) \frac{\partial}{\partial r} + (p_2^2 + 3 r^{-2} p_2^2) \frac{\partial}{\partial \theta} + (r^{-2/3} p_1 + 2 r^{-3} p_2^2) \frac{\partial}{\partial p_1}.
\] (50)

This is indeed the operator that results in the first integral (44).

In this paper, we have mainly taken examples from economic growth theory to study the inverse problem for Hamiltonian systems. The direct technique was applied to some models in mechanics and physics in the recent paper by Naz and Mahomed \[18\]. There are also many other examples one can take from epidemiology and other fields. This could be the focus of future work for the inverse problem as outlined in this paper.

**Acknowledgments.** FM is supported by the N.R.F. of South Africa for research.

**Appendix A.** As usual we set $t$ to be the independent variable, taken as the time. Also $(q, p) = (q^1, ..., q^n, p_1, ..., p_n)$ are the phase space coordinates. For equations of economics, $q^1, ..., q^n$ are the state variables and $p_1, ..., p_n$, the costate variables.

The definitions and results hereunder are adapted from the works \[20, 3, 8, 11, 13, 17, 18\].

**Definition 1.** The Euler operator $\delta/\delta q^i$ and the variational operator $\delta/\delta p_i$ are respectively defined to be
\[
\frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} - D_t \frac{\partial}{\partial q^i}, \quad i = 1, \cdots, n,
\] (A-1)
and
\[
\frac{\delta}{\delta p_i} = \frac{\partial}{\partial p_i} - D_t \frac{\partial}{\partial p_i}, \quad i = 1, \cdots, n,
\] (A-2)
wherin
\[
D_t = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + \dot{p}_i \frac{\partial}{\partial p_i} + \cdots
\] (A-3)
is the total derivative operator with respect to $t$. Summation is intended for repeated indices.
The variables \( t, q, p \) are said to be independent and are connected by the differential relations
\[
\dot{p}_i = D_t(p_i), \quad \dot{q}^i = D_t(q^i), \quad i = 1, 2, \ldots, n. \tag{A-4}
\]
The current value Hamiltonian system satisfies [11]
\[
\dot{q}^i = \frac{\partial H}{\partial p_i}, \tag{A-5}
\]
\[
\dot{p}^i = -\frac{\partial H}{\partial q^i} + \Gamma^i, \quad i = 1, \ldots, n,
\]
where \( \Gamma^i \) is generally a nonzero function of \( t, p_i, q^i \) and \( H \) is a current value Hamiltonian function (see e.g. [11, 13]).

**Definition 2.** The operator or generator \( X \) is
\[
X = \xi D_t + \eta^i \frac{\partial}{\partial q^i} + \zeta^i \frac{\partial}{\partial p_i}. \tag{A-6}
\]
This can also be written in characteristic form as
\[
X = \xi D_t + \tilde{\eta}^i \frac{\partial}{\partial q^i} + \tilde{\zeta}^i \frac{\partial}{\partial p_i}, \tag{A-7}
\]
where \( \tilde{\eta}^i \) and \( \tilde{\zeta}^i \) are the characteristic functions given by
\[
\tilde{\eta}^i = \eta^i - \xi \dot{q}^i, \quad \tilde{\zeta}^i = \zeta^i - \xi \dot{p}_i, \quad i = 1, \ldots, n. \tag{A-8}
\]
The operator (A-7) has evolutionary representative
\[
\bar{X} = \bar{\eta}^i \frac{\partial}{\partial q^i} + \bar{\zeta}^i \frac{\partial}{\partial p_i}, \tag{A-9}
\]
or is sometimes called the canonical form of \( X \). The operators (A-7) and (A-9) are called equivalent as \( X - \bar{X} = \xi D_t \).

The first extension or prolongation of \( X \) is given by
\[
X^{[1]} = \xi D_t + \bar{\eta}^i \frac{\partial}{\partial q^i} + \bar{\zeta}^i \frac{\partial}{\partial p_i} + D_t(\bar{\eta}^i) \frac{\partial}{\partial q^i} + D_t(\bar{\zeta}^i) \frac{\partial}{\partial p_i}, \tag{A-10}
\]
\[
= \xi \frac{\partial}{\partial t} + \eta^i \frac{\partial}{\partial q^i} + \zeta^i \frac{\partial}{\partial p_i} + (D_t(\eta^i) + \xi \dot{q}^i) \frac{\partial}{\partial q^i} + (D_t(\zeta^i) + \xi \dot{p}_i) \frac{\partial}{\partial p_i}. \tag{A-10}
\]

**Definition 3** (see [11]). An operator \( X \) in the form (A-6) is called a current value Hamiltonian operator corresponding to a Hamiltonian \( H(t, q, p) \) if there exists a function \( B(t, q, p) \) such that
\[
\zeta^i \frac{\partial H}{\partial p_i} + p_i D_t(\eta^i) - X(H) - HD_t(\xi) = D_t(B) + (\eta^i - \xi \frac{\partial H}{\partial p_i})(-\Gamma^i) \tag{A-11}
\]
is satisfied on solutions of (A-5).

The following theorem is of essence in the construction of first integrals for the system (A-5).

**Theorem 1** (see [11]). The first integral corresponding to the system (A-5) related to a current value Hamiltonian operator \( X \) is determined from
\[
I = p_i \eta^i - \xi H - B, \tag{A-12}
\]
where $B(t,p,q)$ is a gauge-like function.

If $\Gamma_i = 0$ and $B = B(t,p,q)$, then (A-12) is valid for an invariant Hamiltonian action up to divergence (see [3]) as well.

**Theorem 2** (see [18]). If $\bar{X}$ is a current value Hamiltonian operator in evolutionary form, then $\bar{X}$ satisfies

$$D_t(\bar{\zeta}_i) + \bar{X}'\left(\frac{\partial H}{\partial \dot{q}_i}\right) = \bar{\eta}_i\Gamma_{q_i},$$  \hspace{1cm} (A-13)

$$D_t(\bar{\eta}_i) - \bar{X}'\left(\frac{\partial H}{\partial \dot{p}_i}\right) = -\bar{\eta}_i\Gamma_{p_i}, \; i = 1, \ldots, n,$$  \hspace{1cm} (A-14)

on the system (A-5).

Next, we present the relevant definitions for the characterization of the Hamilton operator which directly corresponds to a first integral and as a consequence deduce the extra conditions on the first integral and divergence term that arise.

**Theorem 3.** (see [18]) Necessary and sufficient conditions that the operator $X$ of the form (A-7), gives rise to a first integral of the system (A-5) is that the characteristics $\bar{\eta}_i$ and $\bar{\zeta}_i$ of $X$ are also the characteristics of the first integral of (A-5) and additionally satisfies the relations

$$\frac{\partial \bar{\eta}_i}{\partial \dot{q}_i} + \frac{\partial \bar{\zeta}_i}{\partial \dot{p}_i} = 0,$$  \hspace{1cm} (A-15)

$$\frac{\partial \bar{\zeta}_i}{\partial \dot{q}_i} - \frac{\partial \bar{\zeta}_j}{\partial \dot{q}_j} = 0,$$  \hspace{1cm} (A-16)

$$\frac{\partial \bar{\eta}_i}{\partial \dot{p}_j} - \frac{\partial \bar{\eta}_j}{\partial \dot{p}_j} = 0, \; i, j = 1, \ldots, n,$$  \hspace{1cm} (A-17)

as well as the operator conditions (A-13) and (A-14).

Theorem 3 for $n = 1$ yields the conditions (A-15), (A-16), (A-17), (A-13) and (A-14) reduce to just one relation, viz.

$$\frac{\partial \bar{\eta}_1}{\partial \dot{q}} + \frac{\partial \bar{\zeta}_1}{\partial \dot{p}_1} = 0,$$  \hspace{1cm} (A-18)

and

$$D_t(\bar{\zeta}) + \bar{X}'\left(\frac{\partial H}{\partial \dot{q}}\right) = \bar{\eta}\Gamma_{q},$$  \hspace{1cm} (A-19)

$$D_t(\bar{\eta}) - \bar{X}'\left(\frac{\partial H}{\partial \dot{p}}\right) = -\bar{\eta}\Gamma_{p}, \; i = 1, \ldots, n.$$  \hspace{1cm} (A-20)

**Theorem 4** (see [18]). For each current value Hamiltonian operator $X = \bar{X} + \xi D_t$ which satisfies (A-15), (A-16) and (A-17), there corresponds a first integral $I$ which is determined uniquely (up to an ignorable constant) from

$$I_{q_i} = -\bar{\zeta}_i, \; I_{p_i} = \bar{\eta}_i, \; i = 1, \ldots, n$$  \hspace{1cm} (A-21)

$$I_t = \bar{\eta}_i(H_{q_i} - \Gamma_i) + \bar{\zeta}_i H_{p_i}.$$
Proof. If the Hamilton symmetry generator $X$ satisfies (A-15), (A-16), A-17) and operator conditions (A-13), A-14), then
\[ D_t I = \bar{\eta}^i (\dot{p}_i + H q_i - \Gamma_i) - \bar{\zeta}^i (\dot{q}_i - H p_i), \]  
(A-22)
holds. The expansion of the left hand side of (A-22) and equating it to the right hand side of (A-22) manifestly gives the results (A-21). This concludes the proof. □

REFERENCES

[1] V. I. Arnold, Mathematical Methods of Classical Mechanics, Graduate Texts in Mathematics, 60. Springer-Verlag, New York, 1989.
[2] M. V. Berry and P. Shukla, Classical dynamics with curl forces, and motion driven by time-dependent flux, J. Phys. A: Math. Theor., 45 (2012), 305201, 18 pp.
[3] V. Dorodnitsyn and R. Kozlov, Invariance and first integrals of continuous and discrete Hamiltonian equations, Journal of Engineering Mathematics, 66 (2010), 253–270.
[4] B. U. Haq and I. Naeem, First integrals and exact solutions of some compartmental disease models, Zeitschrift für Naturforschung A, 74 (2019), 293–304.
[5] B. U. Haq and I. Naeem, First integrals and analytical solutions of some dynamical systems, Nonlinear Dynamics, 95 (2019), 1747–1765.
[6] V. V. Kozlov, Integrability and nonintegrability in Hamiltonian mechanics, Russ. Math. Surveys, 38 (1983), 3–67, 240.
[7] T. Levi-Civita, Interpretazione gruppale degli integrali di un sistema canonico, Rend. Acc. Lincei, s. 3, 8° sem., 2° sem., 7 (1899), 235–238.
[8] F. M. Mahomed and J. A. G. Roberts, Characterization of Hamiltonian symmetries and their first integrals, International Journal of Non-Linear Mechanics, 74 (2015), 84–91.
[9] K. S. Mahomed and R. J. Moitsheki, First integrals of generalized Ermakov systems via the Hamiltonian formulation, International Journal of Modern Physics B, 30 (2016), 1640019, 12 pp.
[10] J. Marsden and A. Weinstein, Reduction of symplectic manifolds with symmetry, Rep. Math. Phys., 5 (1974), 121–130.
[11] R. Naz, F. M. Mahomed and A. Chaudhry, A partial Hamiltonian approach for current value Hamiltonian systems, Communications in Nonlinear Science and Numerical Simulation, 19 (2014), 3600–3610.
[12] R. Naz, F. M. Mahomed and A. Chaudhry, A partial Lagrangian method for dynamical systems, Nonlinear Dynamics, 84 (2016), 1783–1794.
[13] R. Naz, A. Chaudhry and F. M. Mahomed, Closed-form solutions for the Lucas-Uzawa model of economic growth via the partial Hamiltonian approach, Communications in Nonlinear Science and Numerical Simulation, 30 (2016), 299–306.
[14] R. Naz and A. Chaudhry, Comparison of closed-form solutions for the Lucas-Uzawa model via the partial Hamiltonian approach and the classical approach, Mathematical Modelling and Analysis, 22 (2017), 464–483.
[15] R. Naz and A. Chaudhry, Closed-form solutions of Lucas-Uzawa model with externalities via partial Hamiltonian approach, Computational and Applied Mathematics, 37 (2018), 5146–5161.
[16] R. Naz and I. Naeem, The artificial Hamiltonian, first integrals, and closed-form solutions of dynamical systems for epidemics, Zeitschrift für Naturforschung A, 73 (2018), 323–330.
[17] R. Naz, The applications of the partial Hamiltonian approach to mechanics and other areas, International Journal of Non-linear Mechanics, 86 (2016), 1–6.
[18] R. Naz and F. M. Mahomed, Characterization of partial Hamiltonian operators and related first integrals, Discrete & Continuous Dynamical Systems-Series S (DCDS-S), 11 (2018), 723–734.
[19] R. Naz, Characterization of approximate Partial Hamiltonian operators and related approximate first integrals, International Journal of Non-Linear Mechanics, 105 (2018), 158–164.
[20] P. J. Olver, Applications of Lie Groups to Differential Equations, Graduate Texts in Mathematics, 107. Springer-Verlag, New York, 1993.
[21] G. Saccomandi and R. Vitolo, A translation of the T. Levi-Civita paper: Interpretazione Gruppale degli integrali di un Sistema Canonico, Regul. Chaotic Dyn., 17 (2012), 105–112, arxiv:1201.2388v1.
[22] S. Smale, Topology and mechanics, Invent. Math., 10 (1970), 305–331.
[23] E. T. Whittaker, A Treatise on the Analytical Dynamics of Particles and Rigid Bodies: With an Introduction to the Problem of Three Bodies, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1988.

Received January 2019; revised May 2019.

E-mail address: drrehana@lahoreschool.edu.pk
E-mail address: Fazal.Mahomed@wits.ac.za
E-mail address: azam@lahoreschool.edu.pk