About the Closed Quasi Injective S-Acts Over Monoids

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Abstract: The aim of introducing and studying the notion of closed quasi injective S-act is to create a basis facilitate for the exchange ideas between module theory and act theory. As well as it represents a generalization of the quasi-injective act. The quasi-injective act was first introduced and studied by A. M. Lopez, Jr. and J. K. Luedeman, 1979. Then the author was one of the researchers which introduced several generalizations for this notion from several aspects because of its importance. More accurately, the contribution of this paper to the field of competence can be summarized into three points as follows: First: The possibilities for applying the topic of this article helps researchers about how can connect class of injectivity with its generalizations. Second: Study the topic of this article contributes to the improvement of the vision for finding the corresponding between acts theory and module theory. Third: This article has dealt with the important subject in the field of science and knowledge especially in algebra and can take it as a basis for future work for the researchers who work on algebra. Now, in this paper, the concept of closed quasi injective acts over monoids is introduced which represents a generalization of quasi injective. Several characterizations of this concept are given to show the behavior of the property of closed quasi injective. Relationship of the concept of closed quasi injective acts over monoids with Hopfian, co-Hopfian and directly finite property are considered. This work gives the answer to the question of what are the conditions to be met in the subacts in order to inherit the property of closed quasi injectivity. We obtained the main result in this direction in proposition (2.5) and proposition (2.6). A part of this paper was devoted to studying the relationship among the class of closed quasi injective acts with some generalizations of injectivity.

Keywords: Closed Quasi Injective Acts, Extending Acts, Continuous Acts, Noetherian Acts, Hopfian Acts

1. Introduction

Actions of a semigroup have always been interesting to mathematicians. From an algebraic perspective, a semigroup action is a generalization of the notion of group action in group theory. Besides, it is familiar that in the theoretical computer science and in algebra, an action of a semigroup on a set is a rule which associates to each element of the semigroup a transformation of the set in such a way that the product of two elements of the semigroup is associated with composite of two corresponding transformations. The terminology conveys the idea that the elements of the semigroup are acting as transformations of the set. An important special case is a monoid action or act, in which the semigroup is a monoid and the identity element of the monoid acts as the identity transformation of a set. It is recognized that the theory of monoids and systems is a generalization of the theory of rings and modules, which has a number of direct applications in theoretical Computer science, Theory of differential equations and Functional analysis, etc.

Now, by a monoid S we always mean monoid with zero elements 0 and every right S-act M is unitary with zero element 0 which denoted by M₀. A right S-act M₀ with zero is a non-empty set with a function f: M×S→M, f (m, s)→ ms such that the following properties hold: (1) m·1=m (2) m (st) = (ms) t, for all m∈M and s, t∈S, 1 is the identity element of S. It is possible to find the S-act with several names as follows: S-systems, S-sets, S-operands, S-polygons,
Transition systems, S-automata [1]. Note that we will use terminology and notations from [2-5] freely. For more details about injective acts and their generalizations we refer the reader to the references [6-8]. Familiar concepts are good and natural stations for starting. Recall that a non-zero subact N of M is intersection large if for all non-zero subact A of M, A ∩ N ≠ 0, and will be denoted by N is ∩-large in M. Equivalently, if for each θ ∈ M, there exists eθ such that θ ≠ eθ ∈ N [9]. In this case, we call M is ∩-large extension of N. An S-homomorphism f which maps an S-act M into an S-act N is said to be split if there exists S-homomorphism g which maps N into M such that fog = 1M [10].

Let M, N be two S-acts. A S is called M-injective if given an S-monomorphism α: N → M, where N is a subact of M and every S-homomorphism β: N → A can be extended to an S-homomorphism σ: M → A [11]. An S-act A is injective if and only if it is M-injective for all S-acts M. An S-act A is quasi injective if and only if it is A-injective. Quasi injective S-acts have been studied by Lopez and Luedeman [12]. A subact N of S-act M is called closed if it has no proper ∩-large extension in M that is the only solution of N ↪ L such that N ≠ ms ∈ N [9]. In this case, we call M is C-quasi injective act, namely C-quasi injective S-acts, which represent generalizations of the considered under well-known constructions such as product, coproduct, and direct sum.

Definition (2.1): Let M and N be two S-acts, N is called closed M-injective (for short C-M-injective) if for any homomorphism from a closed subact of M to N can be extended to homomorphism from M to N. Certain classes of subacts which inherit the property of C- quasi injective acts were considered. Also, the characterizations of this new class of S-acts were investigated. An example was given to demonstrate C-quasi injective acts over monoids. Some known results on C-quasi injective acts for general modules were generalized to S-acts. In the second part (section three) has clarified the discussion for our results. The third part (section four) has clarified the conclusions of our work.

2. Main Results

Definition (2.1): Let M and N be two S-acts, N is called closed M-injective (for short C-M-injective) if for any homomorphism from a closed subact of M to N can be extended to homomorphism from M to N. Certain classes of subacts which inherit the property of C- quasi injective acts were considered. Also, the characterizations of this new class of S-acts were investigated. An example was given to demonstrate C-quasi injective acts over monoids. Some known results on C-quasi injective acts for general modules were generalized to S-acts. In the second part (section three) has clarified the discussion for our results. The third part (section four) has clarified the conclusions of our work.

Proposition (2.4): Let N is a closed subact of S-act M. If N is C-M-injective act, then any monomorphism from N into M split (in other words if N is C-M-injective act, then N is a retract subact of M).

Proposition (2.5): Let M is C-quasi injective act. Then every fully invariant closed subact of M is C-quasi injective.

Proposition (2.6): Every retract subact of M is C-M-injective act.

Corollary (2.8): Let M and N be two S-acts. Then, N is C-M-injective act if and only if N is C-X-injective act for every closed subact X of M.

Proof: Suppose that N is C-M-injective act, by proposition (2.7), we have N is C-X-injective for every...
closed subact X of M_S. Conversely, since M is closed subact of M_S and by assumption, we have N_S is C-M-injective act.

Proposition (2.9): Let M_S be an S-act and \{N_i \mid i \in I\} a family of S-acts. Then \prod_{i \in I} N_i is C-M-injective act if and only if N_i is C-M-injective act for every i \in I.

Proof: \(\Rightarrow\) Assume that N_i = \prod_{i \in I} N_i is C-M-injective. Let X is closed subact of M_S and \(f : X \rightarrow M_S\) is S-isomorphism from X to N_i. Since M_S is C-M-injective act then there exists S-homomorphism \(g : M_S \rightarrow N_i\) such that \(g \circ i = \pi \circ f\), where \(i_x\) is the inclusion map of X into M_S and \(j_x\) is the injection map of N_i into N_S. Put \(h = \pi \circ g \circ o \circ f\), where \(\pi_i\) is the projection map of N_S onto N_i. Then \(h = \pi \circ g \circ o \circ f\).

\(\Leftarrow\) Assume that N_i is C-M-injective for each i \in I. Let X be closed subact of M_S and \(f : X \rightarrow M_S\) be S-isomorphism from X into N_i=\prod_{i \in I} N_i. Since N_i is C-M-injective act, then there exists S-homomorphism \(\beta : M_S \rightarrow N_i\) such that \(\beta \circ i = \pi \circ f\). Hence, there exists S-homomorphism \(\beta : M_S \rightarrow N_i\) such that \(\beta \circ i = \pi \circ f\). We claim that \(\beta \circ i = \pi \circ f\). Since \(\beta \circ i = \pi \circ f\), then we obtain \(f = \pi \circ g \circ o \circ f\). Therefore N_S is C-M-injective.

Proposition (2.10): If M_S is C-quasi injective act for any finite index n, then M_S is C-quasi injective act.

Proof: Let M_S be a C-quasi injective act. By corollary (2.8), M_S is C-quasi injective act. Since M_S is retract of M_S, so by proposition (2.6) M_S is C-M-injective. Thus, M_S is C-quasi injective act.

Recall that an S-acts M_S is called relatively C-injective acts if M_S is C-M-injective. By corollary (2.8), we have M_S is C-quasi injective act and \(\beta = 1\) is S-homomorphism acts from X onto N_S. Put \(\pi = \pi \circ g \circ o \circ f\). Therefore N_S is C-M-injective act.

Proposition (2.12): Every C-quasi injective act satisfies C_i-condition.

Proof: Assume that M_S is C-quasi injective act. Let f: B \rightarrow A be an S-isomorphism, where A and B are sub acts of M_S and A is a retract of M_S. Then A is C-M-injective act by proposition (2.6). Thus, by remarks and examples (2.2) (2), B is C-M-injective act. Then, by C-M-injectivity of B the inclusion map i_B: B \rightarrow M_S has left inverse g: M_S \rightarrow B such that \(g \circ i = 1_B\). Hence by proposition (2.4), i_B is splits and then B is a retract subact of M_S. Thus, M_S satisfies C_i-condition.

Now, we will study the relationship among C-M-injective act and injective act, extending act, continuous act.

Proposition (2.13): An S-act M_S is extending act (for short CS act) if and only if every S-act is C-M-injective act.

Proof: \(\Rightarrow\) It is obvious.

\(\Leftarrow\) Let N be a closed subact of S-act M_S. By hypothesis N is C-M-injective, so by proposition (2.4), N is a retract subact of M_S. This follows that M_S is CS act.

Proposition (2.14): If every S-act is C-M-injective, then it is continuous.

Proof: Let M_S be S-act so by hypothesis M_S is C-M-injective act. By proposition (2.13), an S-act M_S satisfies C_i-condition and by proposition (2.12), M_S satisfies C_i-condition. Thus M_S is continuous act.

Recall that an S-act M_S is Noetherian if every subact of M_S is finitely generated. A monoid S is a right Noetherian if S is Noetherian. Equivalently, S is a right Noetherian if and only if S satisfies the ascending chain condition for right ideals (definition 1.1.30) in book of M. Kilp, U. Knauer, and A. V. Mikhailev [4, p. 21].

Before the next theorem which is a generalization of theorem (1.1) in A. K. Tiwary, S. A. Paramhans, and B. M. Pandeya’s study [15], we need the following theorem:

Theorem (2.15): For a monoid S with zero, the following conditions are equivalent:

1. Each direct sum of injective S-acts is injective.
2. Each direct sum of weakly injective S-acts is weakly injective.
3. Each injective S-act is countably Σ-injective.
4. Each finitely injective act is weakly injective.
5. S is Noetherian.

Theorem (2.16): The following conditions are equivalent for an S-act M_S, where S is Noetherian monoid:

1. The direct sum of every two C-quasi injective S-acts are C-quasi injective acts.
2. Every C-quasi injective act is injective.

Proof: (1\Rightarrow 2) Assume that M_S is C-quasi injective act and E (M) is injective envelope of M_S. Then, by assumption N_M = M_S @ E (M) is C-quasi injective. Consider the injection maps i: M_S \rightarrow E (M), j: E (M) \rightarrow M_S @ E (M), \pi: M_S @ E (M) \rightarrow M_S, \pi: M_S @ E (M) \rightarrow E (M), such that \(\pi \circ i = 1_M\). Now, M_S is C-quasi injective act, so this implies there exists S-homomorphism \(g: M_S @ E (M) \rightarrow E (M)\) such that \(g \circ i = 1_M\). Thus \(I_M = \pi \circ \pi @ E (M)\). Put \(f = \pi @ E (M)\). Therefore M_S is a retract subact of E (M) and then it is injective.

(2\Rightarrow 1) Let M_S and N_S be two C-quasi injective S-act. By (2) M_S and N_S are injective which implies that the direct sum of any two injective S-acts is injective whence S is Noetherian monoid by theorem (2.15) and then every injective act is C-quasi injective act. Therefore, the direct sum of two C-quasi injective act is C-quasi injective act.

It is clear that every co-Hopfian is directly finite, but the converse is not true in general (for this, assume that M_S is C-quasi injective act and f: g \in E (M_S) such that \(g = 1\), then g is injective homomorphism. Since M_S is co-Hopfian, then g is homomorphism and this implies that \(g = 1\). Therefore, the following proposition, we give a condition to be the converse is true.

Proposition (2.17): Every C-quasi injective act and directly finite is co-Hopfian.

Proof: Let f be an injective endomorphism of M_S and I_M is an identity homomorphism from M_S to M_S. Since M_S is C-M-injective act, so there exists a homomorphism \(g: M_S \rightarrow M_S\) such that \(g \circ f = 1\), since M_S is directly finite, so \(g = 1\) which implies that f is onto. Hence M_S is co-Hopfian.

The following proposition shows the concepts of Hopfian, co-Hopfian and directly finite are coincide under C-
quasi injectivity condition:

**Proposition (2.18):** Let $M_3$ is C-quasi injective act. Then the following concepts are equivalent:

1. $M_3$ is Hopfian,
2. $M_3$ is co-Hopfian,
3. $M_3$ is directly finite.

Proof: (1 $\Rightarrow$ 2) as every Hopfian is directly finite (For this if for any $\alpha, \beta \in \text{End}(M_3)$ and $\alpha \circ \beta = I$, then this means that $\alpha$ is surjective. Since $M_3$ is Hopfian act and then $\alpha$ is isomorphism and $\beta$ is inverse of $\alpha$. Thus $\alpha \circ \beta = I$ which implies that $M_3$ is directly finite act), so by proposition (2.17), $M_3$ is co-Hopfian.

(2 $\Rightarrow$ 3) By proposition (2.17),

(3 $\Rightarrow$ 1) Let $\beta$ be surjective endomorphism of $M_3$, then the inclusion map $i: f(M) \rightarrow M_3$ is isomorphism (since by proposition (2.17), $M_3$ is co-Hopfian). Thus $i \circ f = I_{M_3}$ again since $M_3$ is directly finite, so $i \circ f = I_{M}$ (since $f(M) \cong M_3$). Thus $f$ is injective and then it is isomorphism. Therefore, $M_3$ is Hopfian.

3. Discussion

In this section, we clarify what’s the meaning of the results which were obtained in this article. One of these results, it is proposition (2.4) where it was demonstrated that every monomorphism from closed subact into S-act is split when the subact is C-M-injective. As for proposition (2.5), it was clarified that closed subacts of C-quasi injective is C-quasi injective if they are fully invariant, while, proposition (2.6) explained important result which is the existence of C-M-injective acts. From proposition (2.9) and proposition (2.10), we deduced that the direct sum of C-M-injective act is also C-M-injective and if $M_3$ is C-quasi injective act for any finite integer $n$, then $M_3$ is C-quasi injective respectively. As for the lemma (2.11), we got that if $M_3 = M_1 \oplus M_2$ is C-quasi injective acts, then $M_i = M_1 \oplus M_2$ are relatively C-injective acts.

In addition, proposition (2.13) and proposition (2.14) gave the relationship among C-M-injective acts with CS-acts and continuous acts respectively where these propositions illustrate that if every S-act is C-M-injective, then it will be a CS and continuous act respectively. In the theorem (2.16), we elucidated important result which is the existence of Noetherian monoid was solved the problem of the identical between the following conditions. I. The direct sum of two C-quasi injective S-acts is C-quasi injective acts and II. Every C-quasi injective act is injective. Finally, the relationship among the notions, C-quasi injective, Hopfian, co-Hopfian, and directly finite was illustrated in proposition (2.17) and proposition (2.18).

4. Conclusions

From previous theorem, examples, remark, and propositions, we can pick out some senior points as follows: We obtained an interesting result in proposition (2.4) which was: a closed subact $N$ of S-act $M_3$ is a retract subact of $M_3$ if $N$ is C-M-injective. Also, proposition (2.5) gave the answer to the question raised early in the abstract which was: what are the conditions to be met in the subacts in order to inherit the property of C-quasi injectivity? In the proposition (2.6), we concluded the characterizations of C-M-injective act. From proposition (2.9) and proposition (2.10), we deduced that the direct sum of C-M-injective act is also C-M-injective and if $M_3$ is C-quasi injective act for any finite integer $n$, then $M_3$ is C-quasi injective respectively. As for the lemma (2.11), we got that if $M_3 = M_1 \oplus M_2$ is C-quasi injective acts, then $M_i = M_1 \oplus M_2$ are relatively C-injective acts.

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