The Viterbi process, decay-convexity and parallelized maximum a-posteriori estimation

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SUMMARY

The Viterbi process is the limiting maximum a-posteriori estimate of the unobserved path in a hidden Markov model as the length of the time horizon grows. The existence of such a process suggests that approximate estimation using optimization algorithms which process data segments in parallel may be accurate. For models on state-space $\mathbb{R}^d$ satisfying a new “decay-convexity” condition, we develop an approach to existence of the Viterbi process via fixed points of ordinary differential equations in a certain infinite dimensional Hilbert space. Bounds on the distance to the Viterbi process show that approximate estimation via parallelization can indeed be accurate and scaleable to high-dimensional problems because the rate of convergence to the Viterbi process does not necessarily depend on $d$. The results are applied to a factor model with stochastic volatility and a model of neural population activity.

Some key words: Convex Optimization, Gradient Flow, State-space Models.

1. INTRODUCTION

1.1. Background and motivation

Consider a process $(X_n, Y_n)_{n \in \mathbb{N}_0}$ where $(X_n)_{n \in \mathbb{N}_0}$ is a Markov chain with state space $\mathbb{R}^d$ whose initial distribution and transition kernel admit densities $\mu(x)$ and $f(x, x')$ with respect to Lebesgue measure, and $(Y_n)_{n \in \mathbb{N}_0}$, each valued in a measurable space $(\mathcal{Y}, \mathcal{Y})$, are conditionally independent given $(X_n)_{n \in \mathbb{N}_0}$ and such that for any $A \in \mathcal{Y}$, the conditional probability of $\{Y_n \in A\}$ given $(X_n)_{n \in \mathbb{N}_0}$ can be written in the form $\int_A g(X_n, y) \rho(dy)$, where $g : \mathbb{R}^d \times \mathcal{Y} \to [0, +\infty)$ and $\rho$ is a measure on $\mathcal{Y}$. Models of this form, under names of hidden Markov or state-space models, are applied in a wide variety of fields including econometrics, engineering, ecology, machine learning and neuroscience (Durbin & Koopman, 2012; Douc et al., 2009; West & Harrison, 2006).

Throughout this paper and unless specified otherwise, a distinguished data sequence $(y_n^*)_{n \in \mathbb{N}_0}$ is considered fixed and suppressed from much of the notation. Define:

$$U^n(x_0, \ldots, x_n) = -\log \mu(x_0) - \log g(x_0, y_0^*)$$

$$- \sum_{m=1}^n \log f(x_{m-1}, x_m) - \sum_{m=1}^n \log g(x_m, y_m^*).$$

(1)
The posterior density at the path \((x_0, \ldots, x_n)\) given \((y_0^*, \ldots, y_n^*)\), say \(\text{pr}(x_0, \ldots, x_n|y_0^*, \ldots, y_n^*)\), is then proportional to \(e^{-U^*(x_0, \ldots, x_n)}\). The maximum a-posteriori path estimation problem given \((y_0^*, \ldots, y_n^*)\) is to find:

\[
(\xi_0^n, \ldots, \xi_n^n) = \arg \min_{x_0, \ldots, x_n} U^m(x_0, \ldots, x_n).
\]

In addition to serving as a point estimate of the hidden trajectory, the solution of (2) is of interest when calculating the Bayesian information criterion (Schwarz, 1978) with non-uniform priors over the hidden trajectory, can be used in initialization of Markov chain Monte Carlo algorithms to sample from \(\text{pr}(x_0, \ldots, x_n|y_0^*, \ldots, y_n^*)\), and for log-concave posterior densities is automatically accompanied by universal bounds on highest posterior density credible regions thanks to concentration of measure inequalities (Pereyra, 2017; Bobkov & Madiman, 2011).

The Viterbi process is a sequence \(\xi\) for an integer \(\delta > 0\) such that the maximum a-posteriori path estimation problem given \((y_0^*, \ldots, y_n^*)\) to the authors which considers the Viterbi process with state-space \(\mathbb{R}^d\) is a set of a finite number of states and the convergence in (3) is with respect to the discrete metric. The “Viterbi process” name appeared later, in (Lember & Koloydenko, 2010), inspired by the famous Viterbi decoding algorithm (Viterbi, 1967). We focus on the case of state-space \(\mathbb{R}^d\). The only other work known to the authors which considers the Viterbi process with state-space \(\mathbb{R}^d\) is (Chigansky & Ritov, 2011), where convergence in (3) is with respect to Euclidean distance.

In these studies the Viterbi process appears to be primarily of theoretical interest. Here we consider practical motivation in a similar spirit to distributed optimization methods, e.g., (Moallemi & Van Roy, 2010; Rebeschini & Tatikonda, 2016a,b); the existence of the limit in (3) suggests that (2) can be solved approximately using a collection of optimization algorithms which process data segments in parallel. To sketch the parallelization idea, with \(\Delta\) and \(\ell = (n + 1)/\Delta\) assumed to be integers, consider the partition:

\[
(0, \ldots, n) = \bigcup_{k=1}^{\ell} A_k, \quad A_k = \{(k-1)\Delta, \ldots, k\Delta - 1\},
\]

and for an integer \(\delta > 0\) consider the \(\delta\)-enlargement of each \(A_k\),

\[
A_k(\delta) = \{m \in (0, \ldots, n) : \exists a \in A_k : |m - a| \leq \delta\}.
\]

Suppose the \(\ell\) optimization problems:

\[
\arg \max_{x} \text{pr}(x_{A_k(\delta)}|y_{A_k(\delta)}^*), \quad k = 1, \ldots, \ell,
\]

where \(x_{A_k(\delta)} = \{x_m : m \in A_k(\delta)\}\), \(y_{A_k(\delta)}^* = \{y_m^* : m \in A_k(\delta)\}\), are solved in parallel. Then in a post-processing step, for each \(k\), the components indexed by \(A_k(\delta) \setminus A_k\) of the solution to \(\arg \max_{x} \text{pr}(x_{A_k(\delta)}|y_{A_k(\delta)}^*)\) are discarded, and what remains concatenated across \(k\) to give an approximation to the solution of (2).

If it takes \(T(n)\) time to solve (2) the speed-up from parallelization could be as much as a factor of \(T(n)/T(\Delta + 2\delta)\). The main problem addressed in this paper is to study the rate of convergence to the Viterbi process in (3), and as a corollary we shall quantify the approximation error which trades off against the speed-up from parallelization as a function of \(\delta, \Delta\), the ingredients of the statistical model and properties of the observation sequence.
1.2. Relation to existing works

We approach the solutions of (2) indexed by \( n \in \mathbb{N}_0 \) and there tendency to the Viterbi process in an infinite dimensional Hilbert space, \( l_2(\gamma) \), where \( \gamma \in (0, 1] \) is a parameter related to the rate of convergence to the Viterbi process. This approach is new and has two benefits. Firstly, it allows interpretable quantitative bounds to be obtained which measure the distance to the Viterbi process in a norm on \( l_2(\gamma) \) which gives a stronger notion of convergence than the pointwise convergence in (3). Secondly, via a new “decay-convexity” property of \( U^n \) which may be of independent interest, our approach provides a characterization of the Viterbi process as the fixed point of an infinite dimensional ordinary differential equation which arises in the limit \( n \to \infty \).

In totality, the collection of assumptions we make is neither stronger nor weaker than the collection of assumptions of (Chigansky & Ritov, 2011, Thm 3.1). Comparisons between some individual assumptions are discussed in section A-5. One commonality is that both our assumptions (see the decay-convexity condition in Theorem 1 combined with Lemma 1) and the assumptions of (Chigansky & Ritov, 2011, Thm 3.1) imply that \( (x_0, \ldots, x_n) \mapsto \Pr(x_0, \ldots, x_n|y_0^*, \ldots, y_n^*) \) is strongly log-concave, in the sense of (Saumard & Wellner, 2014).

From a statistical modelling perspective, this strong log-concavity might seem restrictive. However, the merit of assuming strong log-concavity must also take into account its attractive mathematical and computational consequences: strong-convexity of objective functions and strong-log concavity of target probability densities endows gradient-descent algorithms and certain families of diffusion Markov chain Monte Carlo algorithms with dimension-free convergence rates (Bubeck, 2015; Dalalyan, 2017) and plays a role in dimension-free contraction rates for the filtering equations of hidden Markov models (Whiteley, 2017). The notion of decay-convexity introduced here extends these dimension-free phenomena, in particular under our assumptions we shall illustrate that the parameter \( \gamma \) controlling the rate of convergence to the Viterbi process does not necessarily depend on \( d \).

The proof techniques of (Chigansky & Ritov, 2011, Thm 3.1) are quite different to ours. There the existence of the limit (3) is established using a converging series argument to bound terms in a dynamic programming recursion. A quantitative bound on the Euclidean distance between \( \xi^n_0 \) and \( \xi^{n+1}_0 \) is given in (Chigansky & Ritov, 2011, eqs. (3.13) and (3.15)); we address a stronger notion of convergence on the Hilbert space \( l_2(\gamma) \). The proof of (Chigansky & Ritov, 2011, Thm 3.1) is given only in the case \( d = 1 \), but the same approach may be applicable more generally.

Earlier works concerning discrete-state hidden Markov models (Caliebe & Rosler, 2002; Lember & Koloydenko, 2008) establish the existence of the limit in (3) by identifying stopping times which divide the optimal path into unrelated segments. (Chigansky & Ritov, 2011, Example 2.1) illustrates that this approach to existence of the limit can also be made to work when the state-space is \( \mathbb{R}^d \), but it seems not to yield quantitative bounds easily.

In the broader literature on convex optimization, a theory of sensitivity of optimal points with respect to constraints in convex network optimization problems has been introduced by (Rebeschini & Tatikonda, 2016a,b). The notions of scale-free optimization developed there are similar in spirit to the objectives of the present paper, but the results are not directly comparable to ours since they concern a constrained optimization problem. In the context of unconstrained convex optimization problems with separable objective functions which allow for the structure of (2), (Moallemi & Van Roy, 2010) addressed the convergence of a min-sum message passing algorithm. Again some aspects of their analysis are similar in spirit to ours, but their aims and results are quite different.

Amongst our main assumptions will be continuous differentiability of the terms on the right of (1). Considering (3) as a regularized maximum likelihood problem, where the regularization
comes from $\mu$ and $f$, it would be particularly interesting to relax the differentiability assumption in order to accommodate sparsity inducing Lasso-type regularizers (Tibshirani, 1996), but this is beyond the scope of the present paper.

Whilst we restrict our attention to the objective functions $U^n$ in (1) associated with hidden Markov models where $n$ represents time, the techniques we develop could easily be generalized to objective functions which are additive functionals across tuples (rather than just pairs) of variables, and to situations where the arguments of the objective function are indexed over some set with a spatio-temporal (rather than just temporal) interpretation. Indeed many of the techniques presented here are not specific to hidden Markov models at all and may be of wider interest.

2. MAIN RESULTS

2.1. Definitions and assumptions

With $d \in \mathbb{N}$ considered fixed, we shall associate with a generic vector $x \in \mathbb{R}^N$ the vectors $x_0, x_1, \ldots$, each in $\mathbb{R}^d$, such that $x = (x_0^T, x_1^T, \ldots)^T$. With $\langle \cdot, \cdot \rangle$ and $|| \cdot ||$ the usual Euclidean inner product and norm on $\mathbb{R}^d$, define the inner product and norm on $\mathbb{R}^N$ associated with a given $\gamma \in (0, 1]$,

$$\langle x, x' \rangle_\gamma = \sum_{n=0}^{\infty} \gamma^n \langle x_n, x_n' \rangle, \quad ||x||_\gamma = \langle x, x \rangle_\gamma^{1/2} = \left( \sum_{n=0}^{\infty} \gamma^n ||x_n||^2 \right)^{1/2}.$$

Let $l_2(\gamma)$ be the Hilbert space consisting of the set $\{x \in \mathbb{R}^N : ||x||_\gamma < \infty\}$ equipped with the inner-product $\langle \cdot, \cdot \rangle_\gamma$ and the usual element-wise addition and scalar multiplication of vectors over field $\mathbb{R}$. For each $n \in \mathbb{N}_0$, $l_2^n(\gamma)$ denotes the subspace consisting of those $x \in l_2(\gamma)$ such that $x_m = 0$ for $m = n + 1, n + 2, \ldots$, with the convention that $l_2^\infty(\gamma) = l_2(\gamma)$. Note that for $n < \infty$ the set of vectors $l_2^n(\gamma)$ does not actually depend on $\gamma$; the notation $l_2^n(\gamma)$ is used to emphasize that $l_2^n(\gamma)$ is a subspace of $l_2(\gamma)$.

For $x \in \mathbb{R}^N$ define

$$\phi_n(x) = \log f(x_{n-1}, x_n) + \log f(x_n, x_{n+1}) + \log g(x_n, y^*_n), \quad n \geq 1, \quad (6)$$

$$\tilde{\phi}_n(x) = \begin{cases} \log \mu(x_0) + \log f(x_0, x_1) + \log g(x_0, y_0), & n = 0, \\ \log f(x_{n-1}, x_n) + \log g(x_n, y^*_n), & n \geq 1. \end{cases} \quad (7)$$

and let $\nabla_n \phi_n(x)$ and $\nabla_n \tilde{\phi}_n(x)$ be the vectors in $\mathbb{R}^d$ whose $i$th entries are the partial derivatives of $\phi_n(x)$ and $\tilde{\phi}_n(x)$ with respect to the $i$th entry of $x_n$ (the existence of such derivatives is part of Condition 1 below).

For each $n \in \mathbb{N}_0$, define the vector field $\nabla U^n : \mathbb{R}^N \to \mathbb{R}^N$,

$$\nabla U^n(x) = -\{\nabla_0 \tilde{\phi}_0(x)^T \nabla_1 \phi_1(x)^T \ldots \nabla_{n-1} \phi_{n-1}(x)^T \nabla_n \tilde{\phi}_n(x)^T \ 0 \ 0 \ldots \}^T. \quad (8)$$

With these definitions, the first $(n+1)$ elements of the vector $\nabla U^n(x)$ are the partial derivatives of $U^n(x_0, \ldots, x_n)$ with respect to the elements of $(x_0^T, x_1^T, \ldots, x_n^T)^T$, whilst the other elements of the vector $\nabla U^n(x)$ are zero. This zero-padding of $\nabla U^n(x)$ to make an infinitely long vector is a mathematical convenience which will allow us to treat $(\nabla U^n)_{n \in \mathbb{N}_0}$ as a sequence of vector fields on $l_2(\gamma)$.

Define also

$$\alpha_{\gamma,n} = \sum_{m=0}^{n} \gamma^{n-m} \beta_m, \quad \beta_m = \|\nabla_m \phi_m(0)\|^2, \quad \gamma \in (0, 1]. \quad (9)$$
\[ \eta_n(r) = \sup_{\|x\|_{\mathcal{V}_n} \leq r} \|\nabla_n \phi_n(x)\|^2 \vee \|\nabla_n \tilde{\phi}_n(x)\|^2, \quad \|x\|_{\gamma,n} = \left( \sum_{m=0}^{\infty} \gamma^{m-n} \|x_m\|^2 \right)^{1/2}. \] (10)

**Condition 1.**

a) \( \mu, f, \) and \( g(\cdot, y^*_n), n \in \mathbb{N}_0, \) are everywhere strictly positive and continuously differentiable.

b) there exist constants \( \zeta, \tilde{\zeta}, \theta \) such that for all \( x, x' \in \mathbb{R}^N, \)
\[
\langle x_n - x'_n, \nabla \tilde{\phi}_n(x) - \nabla \tilde{\phi}_n(x') \rangle 
\leq -\tilde{\zeta} \|x_n - x'_n\|^2 + \tilde{\zeta} \|x_n - x'_n\| \|x_{n-1} - x'_{n-1}\| + \|x_{n+1} - x'_{n+1}\|, \quad \text{for all } n \geq 1,
\]
and
\[
\langle x_n - x'_n, \nabla \tilde{\phi}_n(x) - \nabla \tilde{\phi}_n(x') \rangle 
\leq \begin{cases} 
-\tilde{\zeta} \|x_0 - x'_0\|^2 + \tilde{\zeta} \|x_0 - x'_0\| \|x_1 - x'_1\|, & n = 0, \\
-\tilde{\zeta} \|x_n - x'_n\|^2 + \tilde{\zeta} \|x_n - x'_n\| \|x_{n-1} - x'_{n-1}\|, & \text{for all } n \geq 1.
\end{cases}
\]

2.2. **Viterbi process as the limit of a Cauchy sequence in \( l_2(\gamma) \)**

**Theorem 1.** Assume that Condition 1 holds, and with \( \zeta, \tilde{\zeta}, \theta \) as therein, let \( \gamma \) be any value in \( (0, 1] \) such that:
\[ \zeta > \theta \frac{(1 + \gamma)^2}{2\gamma}, \quad \tilde{\zeta} > \theta \frac{(1 + \gamma)}{2\gamma}. \] \tag{11}

Then with any \( \lambda \) such that:
\[ 0 < \lambda \leq \left\{ \zeta - \theta \frac{(1 + \gamma)^2}{2\gamma} \right\} \cap \left\{ \tilde{\zeta} - \theta \frac{(1 + \gamma)}{2\gamma} \right\}, \] \tag{12}
and any \( n \in \mathbb{N}_0, \)
\[ \langle x - x', \nabla U^n(x) - \nabla U^n(x') \rangle_{\gamma} \geq \lambda \|x - x'\|^2_{\gamma}, \quad \text{for all } x, x' \in l_2^n(\gamma). \] \tag{13}

Amongst all the vectors in \( l_2^n(\gamma), \) there is a unique vector \( \xi^n \) such that \( \nabla U^n(\xi^n) = 0, \) and
\[ \sup_{m \in \mathbb{N}_0, m \geq n} \|\xi^n - \xi^m\|^2 \leq \frac{\gamma^n}{\lambda^2} \eta_n \left( \lambda^{-2} \alpha_{\gamma,n} \right) + \frac{\gamma^{n+1}}{\lambda^2} \eta_{n+1} \left( \gamma \lambda^{-2} \alpha_{\gamma,n} \right) + \frac{1}{\lambda^2} \sum_{k=n+2}^{\infty} \gamma^k \beta_k. \] \tag{14}

The proof of Theorem 1 is in section A.3.

**Remark 1.** Since \( \nabla U^n(\xi^n) = 0, \) the first \( d(n + 1) \) elements of the vector \( \xi^n \) solve the estimation problem (A.1), and since \( \xi^n \in l_2^n(\gamma), \) the remaining elements of \( \xi^n \) are zero.

**Remark 2.** When Condition 1 holds, there always exists \( \gamma \in (0, 1) \) satisfying (11) and \( \lambda \) satisfying (12) because \( \lim_{\gamma \to 1} (1 + \gamma)^2/2\gamma = \lim_{\gamma \to 1} (1 + \gamma)/\gamma = 2 \) and Condition 1 requires \( 0 < \theta < \zeta/2 \wedge \tilde{\zeta}. \) The case \( \gamma \in (0, 1) \) is of interest because if the right hand side of (14) converges to zero as \( n \to \infty, \) then \( (\xi^n)_{n \in \mathbb{N}_0} \) is a Cauchy sequence in \( l_2(\gamma), \) yielding the existence of the Viterbi process, as per the following corollary.

**Corollary 1.** If in addition to the assumptions of Theorem 1, \( \eta_n \left( \lambda^{-2} \alpha_{\gamma,n} \right) \vee \eta_{n+1} \left( \gamma \lambda^{-2} \alpha_{\gamma,n} \right) = o(\gamma^n) \) and \( \sum_{n=0}^{\infty} \gamma^n \beta_n < \infty, \) then there exists \( \xi^\infty \) in \( l_2(\gamma) \) such that \( \lim_{n \to \infty} \|\xi^n - \xi^\infty\|_{\gamma} = 0. \)
The assumptions of Corollary 1 on $\eta_n, \alpha_{\gamma,n}$ and $\beta_n$ implicitly involve the observation sequence $(y_n^*)_{n \in \mathbb{N}_0}$. An explicit discussion of the impact of $(y_n^*)_{n \in \mathbb{N}_0}$ is given in section 3.

2.3. Interpretation of the decay-convexity condition

From hereon (13) will be referred to as “decay-convexity” of $U^n$. To explain the “convexity” part of this term, note that when $\gamma = 1$, (13) says exactly that $(x_0, \ldots, x_n) \mapsto \Pr(x_0, \ldots, x_n | y_0^*, \ldots, y_n^*)$ is $\lambda$-strongly log-concave, in the sense of (Saumard & Wellner, 2014).

To explain the “decay” part of decay-convexity, let us now address the case $\gamma < 1$. It is well known that strong convexity of a continuously differentiable function is closely connected to exponential contraction properties of the associated gradient-flow differential equation. This connection underlies convergence analysis of gradient-descent algorithms, see for example (Nesterov, 2013, chapter 2). The inequality (13) can be interpreted similarly for any $\gamma \in (0, 1)$: using standard arguments for finite-dimensional ODE’s (a more general Hilbert space setting is fully treated in Proposition A1 in section A2), it can be shown that when Condition 1a) and (13) hold, there is a unique, globally-defined flow which solves:

$$\frac{d}{dt} \Phi^n_t(x) = -\nabla U^n \{ \Phi^n_t(x) \}, \quad \Phi^n_0 = \text{Id.} \quad (15)$$

Here $\Phi^n_t(x)$ is a vector in $\mathbb{R}^N$, and the derivative with respect to time is element-wise. Noting the zero-padding of $\nabla U^n$ in (8), the first $d(n+1)$ elements of $\Phi^n_t(\cdot)$ together constitute the gradient flow associated with $U^n$, whilst each of the remaining elements is the identity mapping on $\mathbb{R}^{d'}$.

Thus for $x, x' \in l^n_2(\gamma)$, $\| \Phi^n_t(x) - \Phi^n_t(x') \|^2_\gamma$ can be written as a sum of finitely many terms and by simple differentiation,

$$\frac{d}{dt} \| \Phi^n_t(x) - \Phi^n_t(x') \|^2_\gamma = -2 \langle \Phi^n_t(x) - \Phi^n_t(x'), \nabla U^n \{ \Phi^n_t(x) \} - \nabla U^n \{ \Phi^n_t(x') \} \rangle_\gamma,$$

for all $x, x' \in l^n_2(\gamma)$. Since $x \in l^n_2(\gamma)$ implies $\Phi^n_t(x) \in l^n_2(\gamma)$, it follows from (13) that

$$\| \Phi^n_t(x) - \Phi^n_t(x') \|_\gamma \leq e^{-\lambda t} \| x - x' \|_\gamma, \quad \text{for all } x, x' \in l^n_2(\gamma). \quad (16)$$

To see the significance of the case $\gamma < 1$, suppose that the initial conditions $x, x'$ are such that $x_m = x'_m$ for all $m = 0, \ldots, n - 1$. Then writing $\Phi^n_{t,0}(x)$ for the first $d$ elements of the vector $\Phi^n_t(x)$, it follows from (16) that

$$\| \Phi^n_{t,0}(x) - \Phi^n_{t,0}(x') \| \leq e^{-\lambda t} \gamma^{n/2} \| x_n - x'_n \|.$$  

Thus when $\gamma < 1$, (13) ensures that as $n \to \infty$, the influence of $x_n$ on $\Phi^n_{t,0}(x)$ decays as $n \to \infty$ with rate given by $\gamma$.

Turning to the inequalities in (11), observe that if $\gamma \in (0, 1)$ is fixed, these inequalities are satisfied if $(\zeta / 2 \wedge \bar{\zeta}) / \theta$ is large enough. Further observe that if Condition 1b) is satisfied in the extreme case $\theta = 0$, then each $\phi_n$ (respectively $\bar{\phi}_n$) is $\zeta$ (respectively $\bar{\zeta}$)-strongly concave in $x_n$ and then it is immediate that (13) holds. Discussion of how condition 1b) relates to the model ingredients $\mu, f, g$ is given in section 3.

The following lemma addresses the relationship between the cases $\gamma \in (0, 1)$ and $\gamma = 1$, hence explaining the conjunction of “decay” and “convexity” in the name we give to (13).

**Lemma 1.** If $U^n$ is twice continuously differentiable and (13) holds for some $\lambda > 0$ and $\gamma \in (0, 1)$, then it also holds with that same $\lambda$ and $\gamma = 1$. 
The proof is given in section A-3. Lemma 1 can perhaps be generalized from twice to once continuous differentiability by function approximation arguments, but this is a rather technical matter which it is not our priority to pursue.

2.4. Viterbi process as the fixed point of a differential equation on \( l_2(\gamma) \)

Define the vector field \( \nabla U^\infty : \mathbb{R}^N \to \mathbb{R}^N \),

\[
\nabla U^\infty(x) = -\{\nabla_0 \tilde{\phi}_0(x)^T \nabla_1 \phi_1(x)^T \nabla_2 \phi_2(x)^T \cdots\}^T.
\]

An important point here about notation and interpretation: element-wise, the vector \( \nabla U^\infty(x) \) is the limit as \( n \to \infty \) of the vector \( \nabla U^n(x) \). Indeed it can be read off from (8) that each element of the vector \( \nabla U^n(x) \) is constant in \( n \) for all \( n \) large enough. However, \( \nabla U^\infty \) may not be interpreted as the gradient of the limit of the sequence of functions \( (U^n)_{n\in\mathbb{N}} \), because the pointwise limit \( \lim_{n \to \infty} U^n(x) \) is in general not well-defined. This reflects the fact that on an infinite time horizon, the prior and posterior probability measures over the entire state sequence \( (X_n)_{n\in\mathbb{N}} \) are typically singular, so that a density \( \text{"pr}(x_0, x_1, \ldots | y_0, y_1, \ldots)\) does not exist. Hence there is no sense in characterizing the Viterbi process as: \( \xi^\infty = \arg\max_{x_0, x_1, \ldots} p(x_0, x_1, \ldots | y_0, y_1, \ldots) \), a correct characterization is: \( \nabla U^\infty(\xi^\infty) = 0 \), as Theorem 2 shows via a counter-part of (15)-(16) in the case \( n = \infty \).

**Theorem 2.** In addition to the assumptions of Theorem 1 and with \( \gamma \) as therein, assume a)-c):

a) there exists a finite constant \( \chi \) such that for all \( n \) and \( x \in l_2(\gamma) \),

\[
\|
abla_n \tilde{\phi}_n(x)\|^2 \leq \beta_n + \chi \left( \|x_n\|^2 + \|x_{n-1}\|^2 + \|x_{n+1}\|^2 \right),
\]

b) \( \sum_{n=0}^{\infty} \gamma_n \beta_n < \infty \),

c) \( \nabla U^\infty \) is continuous in \( l_2(\gamma) \).

Then with \( \lambda \) as in Theorem 1,

\[
\langle x - x', \nabla U^\infty(x) - \nabla U^\infty(x') \rangle_\gamma \geq \lambda \|x - x'\|_\gamma^2, \quad \text{for all } x, x' \in l_2(\gamma),
\]

and there exists a unique and globally defined flow \( \Phi^\infty : (t, x) \in \mathbb{R}_+ \times l_2(\gamma) \mapsto \Phi^\infty_t(x) \in l_2(\gamma) \) which solves the Fréchet ordinary differential equation,

\[
\frac{d}{dt} \Phi^\infty_t(x) = -\nabla U^\infty(\Phi^\infty_t(x)), \quad \Phi^\infty_0 = \text{Id}. \tag{19}
\]

This flow has a unique fixed point, \( \xi^\infty \in l_2(\gamma) \), and \( \|\Phi^\infty_t(x) - \xi^\infty\|_\gamma \leq e^{-\lambda t} \|x - \xi^\infty\|_\gamma \) for all \( x \in l_2(\gamma) \) and \( t \geq 0 \). Furthermore with \( (\xi^n)_{n\in\mathbb{N}_0} \) as in Theorem 1,

\[
\sup_{m\in\mathbb{N}_0, m\geq n} \|\xi^n - \xi^m\|_\gamma^2 \leq \frac{1}{\lambda^2} \left( \gamma^{n-1} \alpha_{\gamma, n} \frac{2\chi}{\lambda^2} + \sum_{k=n}^{\infty} \gamma^k \beta_k \right). \tag{20}
\]

The proof of Theorem 2 is in section A-3.

The assumptions a)-b) in Theorem 2 ensure that \( \nabla U^\infty \) maps \( l_2(\gamma) \) to itself. Combined with the continuity in assumption c) and (18), this allows an existence and uniqueness result of (Deimling, 2006) for dissipative ordinary differential equations on Banach spaces to be applied in the proof of Theorem 2. It is from here that the Fréchet derivative (19) arises. Background information about Fréchet derivatives is given in section A-1.
3. Discussion

3.1. Bound on the segment-wise error in the parallelized scheme

The error associated with the first segment in the parallelization scheme described in section 1.1 can be bounded using (14) or (20). In this section we focus on the latter for simplicity of presentation, an application of (14) is discussed in section 3.3. The following is an immediate corollary of (20).

COROLLARY 2. If the assumptions of Theorem 2 hold,

$$\sup_{n \geq \Delta + \delta} \sum_{m=0}^{\Delta} ||\xi_{m}^{\Delta+\delta} - \xi_{m}^{n}||^2 \leq \gamma^{-\Delta} \sup_{n \geq \Delta + \delta} ||\xi_{m}^{\Delta+\delta} - \xi_{m}^{n}||^2$$

$$\leq \frac{1}{\lambda^2} \left( \gamma^{\delta-1} \frac{2\chi}{\lambda^2} \alpha_{\gamma,\Delta+\delta} + \sum_{k=\delta}^{\infty} \gamma^{k} \beta_{\Delta+k} \right).$$

(21)

Recall from (4) that $\delta$ is the “overlap” between segments in the parallelization scheme. To see the impact of the observations $(y_{n}^*)_{n \in \mathbb{N}_0}$, recall from (6), (7) and (9) that $\beta_n$ depends on $n$ only through $||\nabla_x \log g(x, y_{n}^*)|_{x=0}||^2$ where $\nabla_x$ is gradient with respect to $x$. Therefore whether or not the right hand side of (21) converges to zero as $\delta \rightarrow \infty$ depends on the behaviour of $||\nabla_x \log g(x, y_{n}^*)|_{x=0}||$ as $n \rightarrow \infty$. There are a wide variety of assumptions on $(y_{n}^*)_{n \in \mathbb{N}_0}$ which would suffice for convergence to zero. In the particular case of stationary observations, the rate or convergence is exponential:

COROLLARY 3. Let the assumptions of Theorem 2 hold and let $\gamma$ be as therein. Let $(Y_{n}^*)_{n \in \mathbb{N}}$ be any $\mathbb{Y}$-valued, stationary stochastic process such that $E \{ ||\nabla_x \log g(x, Y_{0}^*)|_{x=0}||^2 \} < \infty$. In particular, $(Y_{n}^*)_{n \in \mathbb{N}}$ need not be distributed according to the hidden Markov model described in section 1.1. Then for any $\hat{\gamma} \in (\gamma, 1)$, there exists a stationary process $(C_n)_{n \in \mathbb{N}_0}$ such that $C_0 < \infty$ almost surely, and such that if in (1)-(2) the sequence $(y_{n}^*)_{n \in \mathbb{N}_0}$ is replaced by the random variables $(Y_{n}^*)_{n \in \mathbb{N}_0}$, then for any $\Delta, \delta \in \mathbb{N}_0$,

$$\sup_{n \geq \Delta + \delta} \sum_{m=0}^{\Delta} ||\xi_{m}^{\Delta+\delta} - \xi_{m}^{n}||^2 \leq \hat{\gamma}^{\delta} C_{\Delta}, \text{ almost surely.}$$

The proof is given in section A.3 and the interested reader can deduce an explicit expression for $C_{\Delta}$ from the details there.

Bounds on the errors associated with the other segments in the parallelization scheme can be obtained by very similar arguments to those in proof of Theorem 2 but presenting all the details would involve substantial repetition.

3.2. Verifying Condition 1 and dimension independence of $\gamma$ and $\lambda$

The following lemma provides an example of a model class satisfying Condition 1, allowing us to illustrate that the constants $\gamma$ and $\lambda$ appearing in Theorems 1 and 2 do not necessarily have any dependence on the dimension of the state-space, $\mathbb{R}^d$. Here the smallest and largest eigenvalues of a real, symmetric matrix, say $B$, are denoted $\rho_{\min}(B)$, $\rho_{\max}(B)$.

LEMMA 2. Assume a) and b):

a) The unobserved process satisfies

$$X_n = AX_{n-1} + b + W_n,$$

(22)
where for \( n \in \mathbb{N} \), \( W_n \sim \mathcal{N}(0, \Sigma) \) is independent of other random variables, \( X_0 \sim \mathcal{N}(b_0, \Sigma_0) \), \( \Sigma \) and \( \Sigma_0 \) are positive definite, \( A \) is a \( d \times d \) matrix and \( b \) and \( b_0 \) are length-\( d \) vectors.

b) For each \( n \), \( x_n \mapsto g(x_n, y_n^*) \) is strictly positive, continuously differentiable and there exists \( \lambda_g \in \mathbb{R} \) such that for all \( n \in \mathbb{N}_0 \) and \( x_n, x_n' \),

\[
\langle x_n - x_n', \nabla_n \log g(x_n, y_n^*) - \nabla_n \log g(x_n', y_n^*) \rangle \leq \lambda_g \| x_n - x_n' \|^2. \tag{23}
\]

If the inequality \( \theta < \zeta/2 \land \tilde{\zeta} \) is satisfied by:

\[
\zeta = \frac{1 + \rho_{\min}(A^T A)}{\rho_{\max}(\Sigma)} - \lambda_g, \tag{24}
\]

\[
\tilde{\zeta} = \frac{1}{\rho_{\max}(\Sigma)} \wedge \left\{ \frac{1}{\rho_{\max}(\Sigma_0)} + \frac{\rho_{\min}(A^T A)}{\rho_{\max}(\Sigma)} \right\} - \lambda_g, \tag{25}
\]

\[
\theta = \frac{\rho_{\max}(A^T A)^{1/2}}{\rho_{\min}(\Sigma)}, \tag{26}
\]

then Condition 1 holds.

The proof is in section A-4.

Remark 3. The condition (23) is called semi-log-concavity of \( x \mapsto g(x, y_n^*) \), generalizing log-concavity by allowing \( \lambda_g \in \mathbb{R} \), rather than only \( \lambda_g \leq 0 \). Considering the case \( d = 1 \) for ease of presentation, an example of a likelihood function \( x \mapsto g(x, y_n^*) \) which satisfies (23) for some \( \lambda_g > 0 \), but not for any \( \lambda_g \leq 0 \) is the \( x \)-centered Student’s t-density, for example in the case of 1 degree of freedom: \( g(x, y_n^*) = \pi^{-1/2} \{1 + (y_n^* - x)^2\}^{-1} \). The case \( \lambda_g > 0 \) also allows for multi-modality of \( x \mapsto g(x, y_n^*) \).

Remark 4. The fact that \( \zeta, \tilde{\zeta} \) and \( \theta \) in (24)-(26) depend only on eigenvalues of \( A \), \( \Sigma \) and \( \Sigma_0 \) and the semi-concavity parameter \( \lambda_g \) means they, and consequently \( \lambda \) and \( \gamma \), do not necessarily depend on dimension. As a simple example consider the case: \( \lambda_g \leq 0 \), \( A = aI_d \) and \( \Sigma = \sigma^2 I_d \), with \( |a| < 1 \) and \( \sigma^2 > 0 \). In this situation \( \theta < \zeta/2 \land \tilde{\zeta} \) holds, and \( \gamma \) and \( \lambda \) can be chosen to depend only on \( |a| \) and \( \sigma^2 \) and be such that (12) holds.

Remark 5. The condition \( \theta < \zeta/2 \land \tilde{\zeta} \) can be interpreted as balancing the magnitude of temporal correlation in (22) against the fluctuations of \( W_n \) and the degree to which the likelihood \( x \mapsto g(x, y_n^*) \) is informative about \( x \). As \( \lambda_g \to -\infty \) the mapping \( x \mapsto g(x, y_n^*) \) becomes more strongly log-concave, and by inspection of (24)-(26) the condition \( \theta < \zeta/2 \land \tilde{\zeta} \) can always be achieved if \( \lambda_g \) takes a negative value large enough in magnitude, with other quantities on the right of the equations (24)-(26) held constant. On the other hand, if \( \rho_{\max}(\Sigma)^{-1} \land \rho_{\max}(\Sigma_0)^{-1} > \lambda_g \), which implies \( \zeta \land \tilde{\zeta} > 0 \) for any value of \( \rho_{\min}(A^T A) \), the condition \( \theta < \zeta/2 \land \tilde{\zeta} \) can be achieved if \( \rho_{\max}(A^T A) \) is small enough.

Remark 6. If the likelihood functions \( x \mapsto g(x, y_n^*) \) are sufficiently strongly log-concave, that is if (23) holds with a negative value of \( \lambda_g \) which is sufficiently large in magnitude, neither log-concavity of the distribution of \( W_n \) nor linearity of the evolution equation (22) is necessary for Condition 1 to hold. An example illustrating this fact is given in section A-4.

3-3. Application to a factor model with stochastic volatility

Factor models are used in economics and finance to represent asset returns in terms of covariates such as market benchmarks or macroeconomic indicators, and idiosyncratic asset-specific fluctuations. Modelling of time-dependent conditional covariances within such models is an area
of active research (De Nard et al., 2019; Engle et al., 2019). One way to instantiate dynamic conditional covariance is via a stochastic volatility model where, for example, the log-variances of the idiosyncratic terms are modelled in terms of a hidden process. See (Asai et al., 2006; Shephard & Andersen, 2009) for overview of stochastic volatility models. The task of maximum a-posteriori estimation of the latent log-variance process in such models was discussed by (Shephard, 1996).

We consider a model in which at time \( n \), the returns on \( d \) assets \( Y_n = (Y_{1n} \cdots Y_{dn})^T \) are modelled in terms of \( q \) factors \( Z_n = (Z_{1n} \cdots Z_{qn})^T \),

\[
Y_n = BZ_n + \sqrt{V_n} \epsilon_n,
\]

where \( B \) is a \( d \) by \( q \) matrix, \( \epsilon_n = (\epsilon_{1n} \cdots \epsilon_{dn})^T \) is a vector of independent and identically distributed, zero-mean, unit-variance normal random variables, and \( V_n \) is a \( d \) by \( d \) diagonal matrix whose \( i \)th diagonal element is \( \exp(X_{in}) \), and \( X_n = (X_{1n} \cdots X_{dn})^T \) follows the vector autoregressive model specified in section 3.2.

In the case that the factors are observed with realized values \( z_n \), the likelihood function for \( x_n \) is:

\[
g(x_n, y_n) = \frac{1}{(2\pi)^{d/2}} \exp \left[ -\frac{1}{2} \sum_{i=1}^{d} x_i^2 - \frac{1}{2} \sum_{i=1}^{d} \left\{ y_i - (B z_n)^i \right\}^2 \exp(-x_i^2) \right]. \tag{27}
\]

For any \( y_n \in \mathbb{R}^d \), \( x \mapsto g(x_n, y_n) \) is log-concave, so (23) holds with \( \lambda_n = 0 \). In this situation, as per remark 5, the condition \( \theta < \zeta/2 \land \zeta \) in Lemma 2 and hence Condition 1 is satisfied for \( \rho_{\max}(AT, A) \) small enough. Hence Theorem 1 applies.

Due to the doubly-exponential dependence on \( x_n \) in (27), assumption a) of Theorem 2 does not hold, so for the existence of the Viterbi process \( \xi^n \) we appeal to the approach described in Corollary 1. For this purpose, let us examine the quantities on the right of inequality (14). Assuming for simplicity of presentation that \( b = b_0 = 0 \),

\[
\beta_n = \frac{1}{4} \sum_{i=1}^{d} \left( \left\{ (y_n^*)^i - (B z_n)^i \right\}^2 - 1 \right)^2.
\]

Also assuming \( \rho_{\max}(AT, A) < 1 \) and \( \Sigma_0 = \sum_{n=0}^{\infty} A^k \Sigma (A^T)^k \) so that a-priori the hidden process is stationary, it can be shown by elementary manipulations that the following rather crude upper bound holds:

\[
\eta_n(r) \leq \frac{3r}{\rho_{\min}(\Sigma)} \left( 1 + \frac{1}{\gamma} \rho_{\max}(A^T A) \right) + \frac{1}{4} \sum_{i=1}^{d} \left\{ \left( (y_n^*)^i - (B z_n)^i \right)^2 \exp(\sqrt{\gamma}) - 1 \right\}.
\]

In a similar spirit to Corollary 3, let \( (Y_{n}^*)_{n \in \mathbb{Z}} \) and \( (Z_{n})_{n \in \mathbb{Z}} \) be stationary stochastic processes such that \( E\{\|(Y_n^*)^4 \vee (Z_0)^4\|\} < \infty \). Then it can be shown that if \( y_n^* \) and \( z_n \) in (27) are replaced by \( Y_n^* \) and \( Z_n \), for any \( \gamma \in (\gamma, 1) \) there exists an almost surely finite random variable, say \( C \), such that:

\[
\gamma^n \left\{ \eta_n (\lambda^{-2} \alpha_{\gamma,n}) \vee \eta_{n+1} (\gamma \lambda^{-2} \alpha_{\gamma,n}) \right\} \leq C \left( \frac{\lambda}{\gamma} \right)^n \quad \text{and} \quad \sum_{n=0}^{\infty} \gamma^n \beta_n < \infty, \quad \text{almost surely}.
\tag{28}
\]

The proof of this claim is very similar to the proof of of Corollary 3 so is omitted. As per Corollary 1, it follows from (28) that on almost any sample path of \( (Y_{n}^*)_{n \in \mathbb{Z}} \) and \( (Z_{n})_{n \in \mathbb{Z}} \), there
exists $\xi^\infty \in \ell_2(\gamma)$ such that $\lim_{n \to \infty} \|\xi^n - \xi^\infty\| = 0$. An error bound of the same form as in Corollary 3 also can be shown to hold, again the details are similar so are omitted.

3.4. Application to a model of neural population activity

State-space models are used in neuroscience to examine time-varying dependence in the firing activity of neural populations (Paninski et al., 2010; Shimazaki et al., 2012; Cunningham & Byron, 2014; Chen & Brown, 2015; Donner et al., 2017; Durstewitz, 2017). In particular, state-space models have been advocated for use in detecting cell assemblies in the brain – ensembles of neurons exhibiting coordinated firing – thought to play a key role in memory formation and learning (Shimazaki et al., 2012; Donner et al., 2017).

Neural spiking data in the form multivariate binary time series are commonly modelled using random fields with time-dependent parameters. Here $y_n = (y_{n,k}^i, 1 \leq i \leq N, 1 \leq k \leq R) \in \{0, 1\}^N$, where $y_{n,k}^i \in \{0, 1\}$ indicates absence or presence of spiking activity of the $i$'th of $N$ neurons during the $n$'th time bin of the $k$'th of $R$ replicated experimental trials.

Similarly to (Shimazaki et al., 2012) we consider a random field model for $y_n$ given $x_n$, where the latent state has components $x_n = (x_n^{ij}, 1 \leq i < j \leq N) \in \mathbb{R}^{N(N-1)/2}$, that is $d = N(N - 1)/2$, and:

$$g(x_n, y_n) = \exp \left\{ \sum_{j>i}^{R} \frac{x_n^{ij} \sum_{k=1}^{R} (y_{n,k}^i - c^i)(y_{n,k}^j - c^j) - A(x_n)} {R} \right\}. \quad (29)$$

Where $c^i$ is the average firing rate of the $i$'th neuron over the $R$ trials. The interest in this model is that the variables $x_n^{ij}$ can be interpreted as a time-dependent statistical coupling between the firing activity of neurons $i$ and $j$.

The normalizing factor $A(x_n)$ is too expensive to compute for anything more than a handful of neurons and following (Donner et al., 2017) we consider the pseudo-likelihood approximation:

$$\tilde{g}(x_n, y_n) = \prod_{i=1}^{N} \prod_{k=1}^{R} \frac{\exp(y_{n,k}^i z_{n,k}^i)} {1 + \exp(y_{n,k}^i z_{n,k}^i)}, \quad (30)$$

$$z_{n,k}^i = \frac{1} {R} \left\{ \sum_{j<i} (y_{n,k}^j (y_{n,k}^i - c^i) + \sum_{i<j} x_n^{ij} (y_{n,k}^j - c^j) \right\}. \quad$$

Both (29) and (30) are log-concave functions of $x_n$. Combining either with a prior model for the signal process as in (22) with $b = 0, A = aI_{N(N-1)/2}, |a| < 1, \Sigma = \sigma^2 I_{N(N-1)/2}$ and the prior distribution for $X_0, \mu$, set to the stationary distribution of (22) gives an instance of the example discussed in Remark 4, i.e. Condition 1 holds and one may take $\gamma, \lambda$ in Theorem 1 independently of $d$ and hence $N$.

For both (29) and (30) with this prior model it holds that for any $(y_n^i)_{i \in \mathbb{N}_0}, \sup_n \beta_n < \infty$, $\sup_n \alpha_n, \gamma < \infty$, and $\sup_n \eta_n(r) < \infty$ for any $r$. The assumptions of Corollary 1 therefore hold. For (30) combined with this prior model, assumptions a)-c) of Theorem 2 hold.

3.5. Numerical results

We consider neural recordings of action potential spike trains from 30 medial prefrontal cortical neurons. The data were recorded using a 384-electrode Neuropixels probe (Jun et al., 2017) while an adult male rat navigated a 3-arm maze. Each recording was over a duration 35 seconds, from -15 to +20 seconds around the rat’s arrival at particular location on the maze called the “reward point”. The presence or absence of spiking per neuron was recorded in bins of width
10 millisecond, so that \( n = 3500 \). The data was divided in two subsets. The first consisted of \( R = 76 \) replicates of “correct” trials in which the rat navigated a particular route on the maze and consequently received a sweet reward at the reward point. The second data subset consisted of \( R = 22 \) “error” trials, in which the rat did not take that route and consequently received no reward. The scientific objective in analyzing these data is to study time-varying statistical interactions across the population of neurons as the rat approaches and passes through the reward point, and to identify differences in these interactions between the “correct” and “error” trials.

We consider the pseudo-likelihood \((30)\) combined with signal model described below that equation, and since there are \( N = 30 \) neurons we have \( d = 435 \).

Our first objective is to numerically evaluate the error and speed-up associated with the parallelization scheme. For this purpose, gradient-descent was used to approximately solve each of the \( \ell \) optimization problems \((5)\) for the neural model described above applied to the data subset of “correct” trials. The \( \ell \) instances of gradient-descent were implemented in parallel using MATLAB’s “parfor” command across \( \ell \) Intel Sandy Bridge cores, each running at 2.6 Ghz, on a single blade of the University of Bristol’s BlueCrystal Phase 3 cluster. \( 1.5 \times 10^4 \) iterations of gradient descent were performed with a constant step size \( 10^{-8} \) in each instance. Let \( \{ \xi_0^n(\ell, \delta), \ldots, \xi_n^n(\ell, \delta) \} \) be the resulting approximation to \((2)\) obtained by combining the approximate solutions to \((5)\) as described in section 1.1. Figure 1 shows the relative error:

\[
\frac{\sqrt{\sum_{m=0}^{n} \| \hat{\xi}_m^n(\ell, \delta) - \hat{\xi}_m^n(1, 0) \|^2}}{\sqrt{\sum_{m=0}^{n} \| \hat{\xi}_m^n(1, 0) \|^2}}
\]

against the overlap parameter \( \delta \). Results are shown for the full data set from \( N = 30 \) neurons and also for a subset consisting of the first \( N = 5 \) neurons. Here \( \Delta \) is determined by \( \Delta = (n + 1)/\ell \). The parameters of the state-space model were set to \( a = 0.95 \) and \( \sigma = 10^{-4} \).

Corollary 2 suggests that \((31)\) should decay to zero exponentially fast as \( \delta \) grows. This is apparent in Figure 1. A reduction in error as \( \ell \) decreases can also be observed. Recalling from Corollary 2 that it is \( \gamma \) which gives a rate of convergence to zero and from Remark 4 that \( \gamma \) can be chosen independently of \( d \), and hence \( N \). The plot on the left of figure 2 shows the same results as figure 1 but with relative error on a logarithmic scale. Since the lines in figure 2 are close to parallel it appears that indeed there is no degradation with \( d \) in the exponential rate at which the relative error decays as \( \delta \) grows. In other results which are not shown, it was found that this rate of decay grew as \( a \to 0 \), which is consistent with the setting \( \gamma = |a| \) discussed in remark 4.

The plot on the right of figure 2 also shows the relative execution time of the parallelized scheme, that is the time taken to compute \( \{ \xi_0^n(\ell, \delta), \ldots, \xi_n^n(\ell, \delta) \} \) using \( \ell = 2, 4, \ldots, 16 \) cores divided by the time to compute \( \{ \xi_0^n(1, 0), \ldots, \xi_n^n(1, 0) \} \). Moving from 1 to 2 and 4 cores results in a roughly linear speed-up. Beyond 4 cores the speed-up is sublinear, which may be due to communication overhead associated with parallelization. From figure 2, \( \delta = 100 \) and \( \ell = 4 \) results in a relative error of less that 0.1% and a speed-up from parallelization of \( 1/0.32 = 3.125 \).

Our next objective is to examine the state-estimates obtained for the two data subsets. Figure 3 shows maximum a-posteriori state estimates of the pairwise coupling parameters \( x_{ij}^n \) for the two data subsets, consisting respectively of “correct” and “error” trials. The first and fourth rows in figure 3 display \( \hat{\xi}_m^n(4, 100) \) as a heat map, for \( m \) corresponding to 3 and 1 seconds before arrival at the reward point, at the reward point \( (t = 0) \), and 1 and 3 seconds after pass the reward point. These times are marked by vertical blue lines in the second and third row plots in figure 3.
Signatures of neural population interaction emerge from the estimates of pairwise coupling parameters: on correct trials, the red coloring on the heat-maps corresponds to strong positive influence from a small minority of neurons onto a larger pool, at \( t = -3, t = 1, t = 0 \) and extending to \( t = 1 \). Such anticipatory activity could reflect reward expectancy.

Conversely, on “error” trials the predominantly blue coloring on the heatmaps at \( t = -1, t = 0 \) and \( t = 1 \) indicates negative pairwise interactions shortly before, during and after the reward point. The estimates on the “error” trials also present greater variation over time than in the “correct” trials, possibly reflecting an error signal or the consolidation of trial outcome-related information.

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**A. APPENDIX**

**A-1. Fréchet derivatives**

The following definitions can be found in (Hopper & Andrews, 2011, App. A). For Banach spaces \( V, W \) over \( \mathbb{R} \), with respective norms \( \| \cdot \|_V, \| \cdot \|_W \), a function \( \varphi : V \to W \) has a directional derivative at \( x \in V \) in direction \( v \in V \) if there exists \( \partial \varphi (v; x) \in W \) such that

\[
\lim_{\epsilon \to 0} \left\| \frac{\varphi(x + \epsilon v) - \varphi(x)}{\epsilon} - \partial \varphi (v; x) \right\|_W = 0.
\]
Fig. 3: First and second rows: “correct” trials, third and fourth rows: “error” trials. Second and third row show \{\hat{\xi}^n_0(4,100), \ldots, \hat{\xi}^n_n(4,100)\} with time on the horizontal axis in seconds relative to the reward delivery time. First and fourth rows display \hat{\xi}^n_m(4,100) as a heat map, for m corresponding left-to-right to: 3 and 1 seconds before reward delivery, delivery time itself, and 1 and 3 seconds after reward delivery. These times are marked by the vertical blue lines in the second and third row plots.

The function \(\phi\) is Gâteaux differentiable at \(x\) if \(\partial \phi(v;x)\) exists for all \(v \in V\) and \(D\phi(\cdot;x) : v \mapsto \partial \phi(v;x)\) is a bounded linear operator from \(V\) to \(W\), in which case \(D\phi(\cdot;x)\) is called the Gâteaux derivative at \(x\).

The function \(\phi\) is additionally Fréchet differentiable at \(x\) if

\[
\lim_{\epsilon \to 0} \sup_{\|v\| \leq 1} \left\| \frac{\varphi(x + \epsilon v) - \varphi(x)}{\epsilon} - \partial \varphi(v;x) \right\|_W = 0, \tag{A1}
\]

in which case the operator \(D\phi(\cdot;x)\) is called the Fréchet derivative at \(x\).

**A.2. Ordinary differential equations on the Hilbert space**

In the following proposition the operator of orthogonal projection from \(l^2(\gamma)\) to \(l^n_2(\gamma)\) is written \(\Pi^n\).

**Proposition A1.** For a given triple \((\gamma, F, n)\) consisting of a constant \(\gamma \in (0,1]\), a mapping \(F : l^2(\gamma) \to l^2(\gamma)\) and \(n \in \mathbb{N}_0 \cup \{\infty\}\), assume that a)-c) hold:

a) \(F\) is continuous with respect to the norm \(\|\cdot\|_\gamma\) on \(l^2(\gamma)\),

b) there exists \(\lambda > 0\) such that for all \(x, x' \in l^n_2(\gamma)\),

\[
\langle x - x', F(x) - F(x') \rangle_\gamma \leq -\lambda \|x - x'\|_\gamma^2,
\]

c) \(F(x) = F \circ \Pi^n(x)\) and \(F(x) \in l^n_2(\gamma)\) for all \(x \in l^2(\gamma)\).
The Viterbi process

Then there exists a unique and globally defined flow \( \Phi : (t, x) \in \mathbb{R}_+ \times l_2(\gamma) \mapsto \Phi_t(x) \in l_2(\gamma) \) solving the Fréchet differential equation,

\[
\frac{d}{dt} \Phi_t(x) = F(\Phi_t(x)), \quad \Phi_0 = \text{Id}.
\]

This flow has a unique in \( l_2^0(\gamma) \) fixed point, \( \xi \), and \( \| \Phi_t(x) - \xi \|_\gamma \leq e^{-\lambda t} \| x - \xi \|_\gamma \) for all \( x \in l_2^0(\gamma) \) and \( t \geq 0 \).

The proof is postponed.

The term \( \frac{d}{dt} \Phi_t(x) \) in Proposition A1 is an application of the Fréchet derivative of \( \Phi_t(x) \) with respect to \( t \), that is in (A1), \( V \) is \( \mathbb{R} \) equipped with the Euclidean norm, \( W \) is the Hilbert space \( l_2(\gamma) \), and \( \varphi \) is the map \( t \mapsto \Phi_t(x) \), where in the latter the \( x \) argument is regarded as fixed. Similarly with \( x \) fixed, and denoting the Fréchet derivative of \( t \mapsto \Phi_t(x) \) at \( t \) by \( D\Phi(t; t, x) \), the quantity \( \frac{d}{dt} \Phi_t(x) \) is precisely \( D\Phi(1; t, x) \). Thus in particular,

\[
\lim_{\delta \searrow 0} \frac{\| \Phi_{t+\delta}(x) - \Phi_t(x) \|_\gamma - \frac{d}{dt} \Phi_t(x) \|_\gamma}{\delta} = 0,
\]

which, in general, is a stronger condition than the element-wise convergence of \( \frac{\Phi_{t+\delta}(x) - \Phi_t(x)}{\delta} \) to \( \frac{d}{dt} \Phi_t(x) \).

The following Lemma will be used in the proof of Proposition A1.

**Lemma A1.** If a triple \(( \gamma, F, n)\) satisfies the assumptions of Proposition A1, then with \( \Phi \) as therein and any \( x, x' \in l_2(\gamma) \),

\[
\frac{d}{dt} \| \Phi_t(x) - \Phi_t(x') \|_\gamma^2 = 2\left\langle \Phi_t(x) - \Phi_t(x'), F\{\Phi^n_t(x)\} - F\{\Phi^n_t(x')\}\right\rangle_\gamma. \tag{A2}
\]

**Proof.** In the case \( n < \infty \), assumption c) of Proposition A1 implies that only the first \( d(n + 1) \) elements of the vector \( \Phi_t(x) \) depend on \( t \), and in that case the lemma can be proved by the chain rule of elementary differential calculus. The following proof is valid for any \( n \in \mathbb{N}_0 \cup \{\infty\} \) and uses the chain rule of Fréchet differentiation.

Pick any \( x, v \in l_2(\gamma) \), write them as \( x = (x^T_0, x^T_1, \cdots)^T \), \( v = (v^T_0, v^T_1, \cdots)^T \) with each \( x_k, v_k \in \mathbb{R}^d \).

The first step is to prove that the mapping \( \varphi(x) = \|x\|_\gamma^2 \) is Fréchet differentiable everywhere in \( l_2(\gamma) \), with Fréchet derivative \( D\varphi(v; x) = 2\langle v, x\rangle_\gamma \).

Consider the existence of directional derivatives. For \( m \in \mathbb{N} \) let \( e_m \) denote the vector in \( l_2(\gamma) \) whose \( m \)th entry is 1 and whose other entries are zero. The directional derivative \( \partial \varphi(e_m; x) \) clearly exists.

We now need to check the existence of directional derivatives of \( \varphi \) in arbitrary directions in \( l_2(\gamma) \). To do so we shall validate the following four equalities:

\[
\lim_{\epsilon \to 0} \frac{\varphi(x + \epsilon v) - \varphi(x)}{\epsilon} = \lim_{\epsilon \to 0} \lim_{m \to \infty} \frac{\varphi(x + \epsilon \Pi^m(v)) - \varphi(x)}{\epsilon} \tag{A3}
\]

\[
= \lim_{m \to \infty} \lim_{\epsilon \to 0} \frac{\varphi(x + \epsilon \Pi^m(v)) - \varphi(x)}{\epsilon} \tag{A4}
\]

\[
= \lim_{m \to \infty} 2 \sum_{k=0}^m \gamma^k \langle v_k, x_k \rangle \tag{A5}
\]

\[
= 2 \langle v, x \rangle_\gamma. \tag{A6}
\]

For (A3), we have for any \( \epsilon > 0 \),

\[
|\varphi(x + \epsilon \Pi^m(v)) - \varphi(x + \epsilon v)| \leq \sum_{k \geq m+1} \gamma^k \|x_k\|^2 - \|x_k + \epsilon v_k\|^2 | \leq 3\epsilon \sum_{k \geq m+1} \gamma^k \|x_k\|^2 + \epsilon^2 \sum_{k \geq m+1} \gamma^k \|v_k\|^2 \to 0, \quad \text{as} \quad m \to \infty,
\]

(A7)
where the convergence holds since $x$ and $v$ are members of $l_2(\gamma)$.

Let $\nabla_k \varphi(x)$ be the vector in $\mathbb{R}^d$ whose $i$th entry is the partial derivative of $\varphi(x)$ with respect to the $i$th element of $x_k$, that is $\nabla_k \varphi(x) = 2x_k$. Since $\varphi(x) = \sum_{k=0}^{\infty} \gamma^k \|x_k\|^2$, the directional derivative in direction $\Pi^m(v)$ at $x$ is given by:

$$
\partial \varphi(\Pi^m(v); x) = \lim_{\epsilon \to 0} \frac{\varphi(x + \epsilon \Pi^m(v)) - \varphi(x)}{\epsilon} = \sum_{k=0}^{m} \langle v_k, \nabla_k \varphi(x) \rangle = 2 \sum_{k=0}^{m} \gamma^k \langle v_k, x_k \rangle .
$$

Let us now check that the convergence in (A8) is uniform in $m$ in order to verify the equality in (A4).

By the mean value theorem of elementary differential calculus, for any $\epsilon > 0$ there exists $y^{m,\epsilon}$ on the line segment between $x$ and $x + \epsilon \Pi^m(v)$ (so $y^{m,\epsilon}_k = x_k$ for $k > m$) such that

$$
\sup_m \left| \frac{\varphi(x + \epsilon \Pi^m(v)) - \varphi(x)}{\epsilon} - 2 \langle \Pi^m(v), x \rangle \gamma \right|
= \sup_m \left| \sum_{k=0}^{m} \epsilon \langle v_k, \nabla_k \varphi(y^{m,\epsilon}) \rangle - 2 \langle \Pi^m(v), x \rangle \gamma \right|
= 2 \sup_m \left| \langle \Pi^m(v), y^{m,\epsilon} - x \rangle \gamma \right|
\leq 2 \sup_m \|\Pi^m(v)\|_\gamma \|y^{m,\epsilon} - x\|_\gamma
\leq 2\|v\|_\gamma^2 \epsilon,
$$

so the convergence in (A8) is indeed uniform in $n$. Therefore (A4) holds.

For the two remaining equalities, (A5) is already proved in (A8), and (A6) holds by Cauchy-Schwartz combined with the facts that $x, v \in l_2(\gamma)$ and that absolute convergence of a series in $\mathbb{R}$ implies its convergence.

We have established that the directional derivative of $\varphi$ at an arbitrary $x$ in an arbitrary direction $v$ exists and is given by $2 \langle v, x \rangle_\gamma$. To prove that $\varphi$ is everywhere Gâteaux differentiable, we also need to show that for each $x$, $D \varphi(\cdot; x) : v \mapsto 2 \langle v, x \rangle_\gamma$ is a bounded operator from $l_2(\gamma)$ to $\mathbb{R}$. This follows from Cauchy-Schwartz:

$$
\sup_{v \neq 0} \frac{2|\langle v, x \rangle_\gamma|}{\|v\|_\gamma} \leq 2\|x\|_\gamma < +\infty, \quad \text{for all } x \in l_2(\gamma).
$$

To prove that $\varphi$ is Fréchet differentiable everywhere in $l_2(\gamma)$, it suffices, by (Hopper & Andrews, 2011, App A, Prop A.3), to check that $D \varphi(\cdot; x)$ is operator-norm continuous in $x$. This follows again by the Cauchy-Schwartz inequality:

$$
\sup_{v \neq 0} \frac{2|\langle v, x \rangle_\gamma - \langle v, x' \rangle_\gamma|}{\|v\|_\gamma} = \sup_{v \neq 0} \frac{2|\langle v, x - x' \rangle_\gamma|}{\|v\|_\gamma} \leq 2\|x - y\|_\gamma, \quad \text{for all } x, x' \in l_2(\gamma).
$$

We have proved that $\varphi(x) = \|x\|_\gamma^2$ is Fréchet differentiable everywhere in $l_2(\gamma)$, with Fréchet derivative in direction $v$ given by $D \varphi(v; x) = 2 \langle v, x \rangle_\gamma$.

The proof is completed by an application of the chain rule of Fréchet differentiation:

$$
\frac{d}{dt} \|\Phi_t(x) - \Phi_t(x')\|_\gamma^2 = D \varphi \left( \frac{d}{dt} \{ \Phi_t(x) - \Phi_t(x') \}; \Phi_t(x) - \Phi_t(x') \right)
= 2 \langle F(\Phi_t(x)) - F(\Phi_t(x')), \Phi_t(x) - \Phi_t(x') \rangle_\gamma ,
$$
**Proof of Proposition A1.** For any \( n \in \mathbb{N}_0 \), applying assumptions b) and c) of the proposition, we have for any \( x, x' \in l_2(\gamma) \),
\[
\langle F(x) - F(x'), x - x' \rangle_\gamma = \langle F \circ \Pi^n(x) - F \circ \Pi^n(x'), \Pi^n(x) - \Pi^n(x') \rangle_\gamma
\]
\[
+ \langle F \circ \Pi^n(x) - F \circ \Pi^n(x'), x - \Pi(x) - x' + \Pi(x') \rangle_\gamma
\]
\[
\leq -\lambda\|\Pi^n(x) - \Pi^n(x')\|^2_\gamma + 0
\]
\[
\leq 0.
\]
This global dissipation condition, combined with assumption a) of the proposition allows the application of (Deimling, 2006, Thm 3.4, p.41) on the Hilbert space \( l_2(\gamma) \) to give the existence and uniqueness of the globally defined flow as required.

It follows from Lemma A1 that \( \|\Phi_t(x) - \Phi_t(x')\|_\gamma \leq e^{-\lambda t}\|x - x'\|_\gamma \) for all \( x, x' \in l_2(\gamma) \), and under assumption c) of the proposition, \( F(x) \in l_2(\gamma) \) for any \( x \in l_2(\gamma) \), implying that if \( x \in l_2(\gamma) \), then \( \Phi_t(x) \in l_2(\gamma) \) for all \( t > 0 \). An application of the Banach fixed point theorem to the restriction of \( \Phi \) to the Hilbert space \( l_2(\gamma) \) then gives the existence of the unique (in \( l_2(\gamma) \)) fixed point \( \xi \).

**A.3. Proofs of the main results**

**Proof of Lemma 1.** Let \( \nabla^2 U^n(x) \) be the matrix in \( \mathbb{R}^n \times \mathbb{R}^n \) such that the top-left \( d(n+1) \times d(n+1) \) sub-matrix is the Hessian of \( U^n(x) \) w.r.t. \( (x_0^T x_1^T \ldots x_n^T)^T \), and all other entries of \( \nabla^2 U^n(x) \) are zero. This definition is just the same kind of zero-padding as in (8).

We first claim that for any \( \lambda > 0 \) and \( \gamma \in (0, 1] \), (13) holds if and only if:
\[
\langle v, \nabla^2 U^n(x) \cdot v \rangle_\gamma \geq \lambda\|v\|^2_\gamma,
\]
for all \( x, v \in l_2^n(\gamma) \), (A10)

where \( \cdot \) is standard matrix-vector multiplication. For the “only-if” part, let \( x, v \) be any vectors in \( l_2^n(\gamma) \) and with \( \tau > 0 \), define \( x_\tau = x + \tau v \in l_2^n(\gamma) \). Then:
\[
\|v\|^2_\lambda = \frac{1}{\tau^2}\|\tau v\|^2_\lambda = \frac{1}{\tau^2}\|x_\tau - x\|^2_\lambda
\]
\[
\leq \frac{1}{\tau^2}\langle x_\tau - x, \nabla U^n(x_\tau) - \nabla U^n(x) \rangle_\gamma
\]
\[
= \frac{1}{\tau}\langle v, \nabla U^n(x_\tau) - \nabla U^n(x) \rangle_\gamma
\]
\[
= \frac{1}{\tau}\left\langle v, \int_0^\tau \nabla^2 U^n(x + sv)ds \cdot v \right\rangle_\gamma
\]
\[
= \frac{1}{\tau}\int_0^\tau \langle v, \nabla^2 U^n(x + sv) \cdot v \rangle_\gamma ds
\]
\[
\rightarrow_{\tau \rightarrow 0} \langle v, \nabla^2 U^n(x) \cdot v \rangle_\gamma,
\]
where the first inequality is an application of (13) and the fourth equality is an application of the mean-value theorem for finite-dimensional vector-valued functions, which is valid here because of the zero-padding in the definitions of \( \nabla^2 U^n \) and \( \nabla U^n \).

For the “if” part of the claim:
\[
\langle x - x', \nabla U^n(x) - \nabla U^n(x') \rangle_\gamma = \left\langle x - x', \int_0^1 \nabla^2 U^n(x' + s(x - x'))ds \cdot (x - x') \right\rangle_\gamma
\]
\[
= \int_0^1 \langle x - x', \nabla^2 U^n(x' + s(x - x')) \cdot (x - x') \rangle_\gamma ds
\]
\[
\geq \lambda\|x - x'\|^2_\gamma.
\]

Now let \( \lambda_* \) be any eigenvalue of the Hessian matrix of \( U^n(x) \) w.r.t. \( (x_0^T x_1^T \ldots x_n^T)^T \). Thus \( \lambda_* \) is real. Let \( v_* \) any real eigenvector associated with \( \lambda_* \), and padded with zeros so as to be a vector in \( l_2^n(\gamma) \) (a real
eigenvector associated with \( \lambda_\ast \) always exists, since if \( a + ib \) is an eigenvector associated with a given real eigenvalue of a real matrix then so are \( a \) and \( b \). The dependence of \( \lambda_\ast \) and \( v_\ast \) on \( x \) is not shown in the notation. Then since \( \nabla^2 U^n(x) \cdot v_\ast = \lambda_\ast v_\ast \), and \( v_\ast \neq 0 \) implies \( \|v_\ast\|^2_\gamma \neq 0 \) for all \( \gamma \in (0,1) \), we have

\[
\lambda_\ast = \frac{\langle v_\ast, \nabla^2 U^n(x) \cdot v_\ast \rangle_\gamma}{\|v_\ast\|^2_\gamma} \text{ for all } \gamma \in (0,1].
\] (A11)

To complete the proof, recall we have already shown that if (13) holds for some \( \lambda > 0 \) and \( \gamma \in (0,1) \), then (A10) holds, and comparing (A10) with (A11) we find \( \lambda_\ast \geq \lambda \). Combining \( \lambda_\ast \geq \lambda \) with the variational formula for the smallest eigenvalue of the Hessian matrix of \( U^n(x) \), denoted \( \lambda_{\min} \) (recall this matrix is finite dimensional, where as \( \nabla^2 U^n(x) \) is its zero-padded version), we have

\[
\lambda \leq \lambda_{\min} = \inf_{v \in l_2^\gamma, v \neq 0} \frac{\langle v, \nabla^2 U^n(x) \cdot v \rangle_1}{\|v\|^2_1}.
\] (A12)

But (A12) implies (A10) holds with \( \gamma = 1 \), which we have already shown implies (13) with \( \gamma = 1 \). \( \square \)

In Lemma A2 and the proof of Theorem 1 below we shall need the following generalization of the inner-product \( \langle \cdot, \cdot \rangle_\gamma \) and norm \( \| \cdot \|_\gamma \), for \( n \in \mathbb{N}_0 \),

\[
\langle x, x' \rangle_{\gamma,n} = \sum_{m=0}^{\infty} \gamma^{m-n} \langle x_m, x'_m \rangle, \quad \|x\|_{\gamma,n} = \langle x, x \rangle_{\gamma,n}^{1/2}, \quad x, x' \in l_2^\gamma.
\] (A13)

**Lemma A2.** Assume that Condition 1 holds, and with \( \zeta, \bar{\zeta}, \theta \) as therein and \( \gamma \in (0,1] \) assume the following inequalities hold:

\[
\zeta > \frac{\theta}{2\gamma}(1 + \gamma)^2, \quad \bar{\zeta} > \frac{\theta}{2\gamma}(1 + \gamma).
\] (A14)

Then for any \( \lambda \) such that:

\[
0 < \lambda \leq \left\{ \frac{\bar{\zeta}}{2\gamma}(1 + \gamma)^2 \right\} \wedge \left\{ \zeta - \frac{\theta}{2\gamma}(1 + \gamma) \right\}.
\] (A15)

any \( n \in \mathbb{N}_0 \) and \( m = 0, \ldots, n \),

\[
\langle x - x', \nabla U^n(x) - \nabla U^n(x') \rangle_{\gamma,m} \geq \lambda \|x - x'\|^2_{\gamma,m}, \text{ for all } x, x' \in l_2^\gamma.
\]
Proof. For any \( x, x' \in l_{2}^{n}(\gamma) \) we have:
\[
\langle x - x', \nabla U^{n}(x) - \nabla U^{n}(x') \rangle_{\gamma, m} = -\gamma^{m} \langle x_{0} - x'_{0}, \nabla_{0} \hat{\phi}_{0}(x) - \nabla_{0} \hat{\phi}_{0}(x') \rangle \\
- \sum_{k=1}^{n-1} \gamma^{k} \langle x_{k} - x'_{k}, \nabla_{k} \phi_{k}(x) - \nabla_{k} \phi_{k}(x') \rangle \\
- \gamma^{n} \langle x_{n} - x'_{n}, \nabla_{n} \hat{\phi}_{n}(x) - \nabla_{n} \hat{\phi}_{n}(x') \rangle \\
\geq \gamma^{m} \left\{ \zeta \| x_{0} - y_{0} \|^{2} - \theta \| x_{0} - x'_{0} \| \| x_{1} - x'_{1} \| \right\} \\
+ \sum_{k=1}^{n-1} \gamma^{k} \left\{ \zeta \| x_{k} - x'_{k} \|^{2} - \theta \| x_{k} - x'_{k} \| \left( \| x_{k-1} - x'_{k-1} \| + \| x_{k+1} - x'_{k+1} \| \right) \right\} \\
+ \gamma^{n} \left\{ \zeta \| x_{n} - x'_{n} \|^{2} - \theta \| x_{n} - x'_{n} \| \| x_{n-1} - x'_{n-1} \| \right\} \\
\geq \gamma^{m} \left\{ \zeta - \frac{\theta}{2} \right\} \| x_{0} - x'_{0} \|^{2} - \frac{\theta}{2} \| x_{1} - x'_{1} \|^{2} \\
+ \sum_{k=1}^{n-1} \gamma^{k} \left\{ \zeta - \frac{\theta}{2} \right\} \| x_{k} - x'_{k} \|^{2} - \frac{\theta}{2} \| x_{k-1} - x'_{k-1} \|^{2} \\
+ \gamma^{n} \left\{ \zeta - \frac{\theta}{2} \right\} \| x_{n} - x'_{n} \|^{2} - \frac{\theta}{2} \| x_{n-1} - x'_{n-1} \|^{2} \\
= \gamma^{m} \left\{ \zeta - \frac{\theta}{2} \right\} \| x_{0} - x'_{0} \|^{2} - \frac{\gamma^{1-m}}{\gamma^{m}} \| x_{0} - x'_{0} \|^{2} \\
\sum_{k=1}^{n-1} \gamma^{k} \left\{ \zeta - \frac{\theta}{2} \right\} \| x_{k} - x'_{k} \|^{2} - \frac{\gamma^{k-1-m}}{\gamma^{k-m}} \| x_{k} - x'_{k} \|^{2} \\
\gamma^{n} \left\{ \zeta - \frac{\theta}{2} \right\} \| x_{n} - x'_{n} \|^{2} - \frac{\gamma^{n-1-m}}{\gamma^{n-m}} \| x_{n} - x'_{n} \|^{2} \\
\geq \gamma^{m} \left\{ \zeta - \frac{\theta}{2} \right\} \| x_{0} - x'_{0} \|^{2} \\
+ \left\{ \zeta - \frac{\theta}{2} \right\} \| x_{n} - x'_{n} \|^{2} \\
\geq \lambda \sum_{k=0}^{n} \gamma^{k} \| x_{k} - x'_{k} \|^{2},
\]
where the first equality and inequality are due to (8) and Condition 1b); the second inequality uses the fact that for any \( a, b \in \mathbb{R}, 2|a||b| \leq |a|^{2} + |b|^{2} \); the third inequality uses \( \frac{\gamma^{k-1-m}}{\gamma^{k-m}} \leq \frac{1}{\gamma} \) for \( 0 \leq m \leq n \) and \( \frac{\gamma^{k-1-m}}{\gamma^{k-m}} + \frac{\gamma^{k+1-m}}{\gamma^{k+m}} \leq \frac{1}{\gamma} + \gamma \); and the final inequality holds under the conditions on \( \lambda, \zeta, \tilde{\zeta}, \theta \) and \( \gamma \) given in (11) and (12). \( \square \)

Proof of Theorem 1. Throughout the proof, \( n \in \mathbb{N}_{0} \) and \( \gamma \in (0, 1] \) are fixed. Considering \((\gamma, -\nabla U^{n}, n)\), let us validate the assumptions of Proposition A1 in the order: c), then a), then b). Assumption c) of Proposition A1 holds due to the definition of \( \nabla U^{n} \) in (8). Validating assumption a) of Proposition A1 requires that if \( x \rightarrow x' \) in \( l_{2}(\gamma) \) then \[ \| \nabla U^{n}(x) - \nabla U^{n}(x') \|_{\gamma} \rightarrow 0. \] But we have already
validated assumption c) of Proposition A1, so \( \nabla U^n \) maps \( l_2(\gamma) \) into \( l_2^2(\gamma) \), and

\[
\|\nabla U^n(x) - \nabla U^n(x')\|^2_2 = \|\nabla_0 \tilde{\phi}_0(x) - \nabla_0 \tilde{\phi}_0(x')\|^2 + \sum_{m=1}^{n-1} \gamma^m \|\nabla_m \phi_m(x) - \nabla_m \phi_m(x')\|^2 + \gamma^n \|\nabla_n \tilde{\phi}_n(x) - \nabla_n \tilde{\phi}_n(x')\|^2.
\]

Also by assumption c) of Proposition A1, \( \nabla U^n(x) \) depends on \( x \) only through \((x_0, \ldots, x_n)\). These observations together with Condition 1a) validate assumption a) of Proposition A1. Assumption b) of Proposition A1 holds by an application of Lemma A2. This completes the verification of the assumptions of Proposition A1 for \((\gamma, -\nabla U^n, n)\) and thus establishes the existence of the fixed point \( \xi^n \).

Our next step is to obtain bounds on \( \|\xi^n\|^2_{\gamma,n} \) and \( \|\xi^n\|^2_{\gamma,n+1} \). An application of Lemma A2 and Cauchy-Schwartz gives:

\[
\|\xi^n\|^2_{\gamma,n} \leq \frac{1}{\lambda} (0 - \xi^n, \nabla U^n(0) - 0)_{\gamma,n} \leq \frac{1}{\lambda} \|\xi^n\|_{\gamma,n} \|\nabla U^n(0)\|_{\gamma,n}.
\]

hence

\[
\|\xi^n\|^2_{\gamma,n} \leq \frac{\alpha_{\gamma,n}}{\lambda^2} \quad \text{and} \quad \|\xi^n\|^2_{\gamma,n+1} = \gamma \|\xi^n\|^2_{\gamma,n} \leq \gamma \frac{\alpha_{\gamma,n}}{\lambda^2},
\]

(A17)

where the equality uses the fact that \( \|\xi_m^n\| = 0 \) for \( m > n \).

Now fix any \( m > n \). An application of Lemma A2 and Cauchy-Schwartz gives:

\[
\|\xi^n - \xi^m\|^2_{\gamma,0} \leq \frac{1}{\lambda} (\xi^n - \xi^m, \nabla U^m(\xi^n) - 0)_{\gamma,0} \leq \frac{1}{\lambda} \|\xi^n - \xi^m\|_{\gamma,0} \|\nabla U^m(\xi^n)\|_{\gamma,0}.
\]

As \( \xi^n \) is the fixed point associated with \( \nabla U^n \) we have \( \nabla_0 \tilde{\phi}_0(\xi^n) = \nabla_k \phi_k(\xi^n) = 0 \) for all \( k < n \). Combining this fact with \( \xi^n \in l_2(\gamma), (8), (10) \) and the bound (A17) gives:

\[
\|\nabla U^m(\xi^n)\|^2_{\gamma,0} = \sum_{k=n}^{m-1} \gamma^k \|\nabla \phi_k(\xi^n)\|^2 + \gamma^m \|\nabla \phi_m(\xi^n)\|^2 \leq \sum_{k=n}^{n} \gamma^k \|\nabla \phi_k(\xi^n)\|^2 + \sum_{k=n+2}^{\infty} \gamma^k \beta_k \leq \gamma^n \eta_n \left( \frac{\alpha_{\gamma,n}}{\lambda^2} \right) + \gamma^{n+1} \eta_{n+1} \left( \frac{\alpha_{\gamma,n}}{\lambda^2} \right) + \sum_{k=n+2}^{\infty} \gamma^k \beta_k.
\]

(A19)

The proof of the theorem is completed by combining this bound with (A18). \( \square \)

Proof of Theorem 2. The first step is to apply Proposition A1 to \((\gamma, -\nabla U^{\infty}, \infty)\). From its definition (17) combined with assumptions b) and c) of the theorem, it is clear that \( \nabla U^{\infty} \) maps \( l_2(\gamma) \) into itself and \( \Pi^{\infty} = \text{Id} \) by definition, so assumption c) of Proposition A1 is satisfied. Assumption a) of the theorem is exactly what is required for assumption a) of Proposition A1 to hold. Let us now verify assumption b) of Proposition A1. For any \( x, x' \in l_2(\gamma) \) and \( n \in \mathbb{N}_0 \),

\[
\langle x - x', \nabla U^{\infty}(x) - \nabla U^{\infty}(x') \rangle_{\gamma} = \langle x - x', \nabla U^n(x) - \nabla U^n(x') \rangle_{\gamma} + \Delta_n(x, x')
\]

(A20)

where

\[
|\Delta_n(x, x')| = |\langle x - x', \nabla U^{\infty}(x) - \nabla U^n(x) + \nabla U^n(x') - \nabla U^{\infty}(x') \rangle_{\gamma}| \leq \|x - y\| \{\|\nabla U^n(x) - \nabla U^{\infty}(x)\|_{\gamma} + \|\nabla U^n(x') - \nabla U^{\infty}(x')\|_{\gamma}\}.
\]

(A21)

Using the facts that \( \nabla U^n = \nabla U^n \circ \Pi^n \) and \( \nabla U^n \) maps \( l_2(\gamma) \) into \( l_2^2(\gamma) \), then applying the instance of assumption b) of Proposition A1 which has already been verified for \((\gamma, -\nabla U^n, n)\) in the proof of
The Viterbi process

Theorem 1. We have for any \(x, x' \in l_2(\gamma)\),
\[
\langle x - x', \nabla U^n (x) - \nabla U^n (x') \rangle_\gamma = \langle \Pi^n (x) - \Pi^n (x'), \nabla U^n \circ \Pi^n (x) - \nabla U^n \circ \Pi^n (x') \rangle_\gamma
\]
\[
\geq \lambda \| \Pi^n (x) - \Pi^n (x') \|_\gamma^2
\]
\[
\to \lambda \| x - x' \|_\gamma^2 \quad \text{as} \quad n \to \infty.
\] (A22)

By (8), (17) and assumptions a) and b) of the Theorem, we have for any \(x \in l_2(\gamma)\),
\[
\| \nabla U^n (x) - \nabla U^\infty (x) \|_\gamma^2 = \gamma \| \nabla \tilde{\phi}_n (x) - \nabla \phi_n (x) \|_\gamma^2 + \sum_{k=n+1}^{\infty} \gamma^k \| \nabla \phi_n (x) \|_\gamma^2
\]
\[
\to 0 \quad \text{as} \quad n \to \infty.
\] (A23)

Combining (A20)–(A23) gives:
\[
\langle x - x', \nabla U^\infty (x) - \nabla U^\infty (x') \rangle_\gamma \geq \lambda \| x - x' \|_\gamma^2,
\] (A24)

which completes the verification of assumption b) of Proposition A1 for \((\gamma, -\nabla U^\infty, \infty)\).

The existence of \(\xi^\infty \in l_2(\gamma)\) such that \(\nabla U^\infty (\xi^\infty) = 0\), together with (A24), implies that by the same arguments as in the proof of Theorem 1, equations (A18) and (A19) hold not only for \(m \in N_0\) but also for \(m = \infty\). Under assumption a) of the theorem, the bound: \(\eta_n (r) \leq \beta_n + \chi r / \gamma \) holds using (10), and plugging this bound in completes the proof.

\[\square\]

Proof of Corollary 3. We shall use the fact that if \((Z_n)_{n \in N_0}\) is a stationary process of nonnegative random variables such that \(E(0 \lor \log Z_0) < \infty\), then for any \(\rho \in (0, 1)\),
\[
\sup_n \rho^n Z_n < \infty, \quad \text{a.s.}
\]

For a proof see (Douc & Moulines, 2012, Lemma 7).

Now observe from (9) and (6)-(7) that for some finite constant \(c\) depending only on \(\mu\) and \(f\),
\[
\alpha_n, \gamma \leq \frac{c}{1 - \gamma} + \sum_{m \in Z} \gamma^{m-n} \| \nabla_x \log g(x, Y^*_m) \|_{x=0}^2, \quad \beta_n \leq c + \| \nabla_x \log g(x, Y^*_n) \|_{x=0}^2.
\]

Take \(Z_n = \sum_{m \in Z} \gamma^{m-n} \| \nabla_x \log g(x, Y^*_m) \|_{x=0}^2\), so that \((Z_n)_{n \in N_0}\) is stationary due to the stationary assumption on \((Y^*_m)_{n \in N_0}\), and \(E(0 \lor \log Z_0) \leq E(Z_0) < \infty\) using the assumption \(E[ \| \nabla_x \log g(x, Y^*_n) \|_{x=0}^2 ] < \infty\). It follows that:
\[
\gamma^d \alpha_{\Delta+d, \gamma} \leq \gamma^d \sup_{n \in N_0} \left( \frac{\gamma}{\lambda} \right)^n \alpha_{\Delta+n, \gamma}
\]
\[
\leq \gamma^d \sup_{n \in N_0} \left( \frac{\gamma}{\lambda} \right)^n \left\{ \frac{c}{1 - \gamma} + \sum_{m \in Z} \gamma^{m-n} \| \nabla_x \log g(x, Y^*_m) \|_{x=0}^2 \right\} < \infty, \quad \text{a.s.}
\]

Similarly, taking \(Z_n = \beta_{\Delta+n}\), it follows that:
\[
\sum_{k=d}^{\infty} \gamma^k \beta_{\Delta+k} \leq \left\{ \sup_{n \in N_0} \left( \frac{\gamma}{\lambda} \right)^n \beta_{\Delta+n} \right\} \sum_{k=d}^{\infty} \gamma^k = \frac{\gamma^d}{1 - \gamma} \left\{ \sup_{n \in N_0} \left( \frac{\gamma}{\lambda} \right)^n \beta_{\Delta+n} \right\} < \infty, \quad \text{a.s.}
\]

Substituting these last two inequalities into (21) gives the result as required.

\[\square\]

A.4. Verifying Condition 1

Proof of Lemma 2. In the setting described in section 3,
\[
\langle x_n - x_n', \nabla_n \log f(x_{n-1}, x_n) - \nabla_n \log f(x_{n-1}', x_n') + \nabla_n \log f(x_n, x_{n+1}) - \nabla_n \log f(x_n', x_{n+1}') \rangle
\]
\[
= -(x_n - x_n')^T (\Sigma^{-1} + A^T \Sigma^{-1} A)(x_n - x_n') + (x_n - x_n')^T \Sigma^{-1} A(x_{n-1} - x_{n-1}')
\]
\[
+ (x_n - x_n')^T A^T \Sigma^{-1} (x_{n+1} - x_{n+1}'),
\]
\[
\langle x_n - x_n', \nabla_n \log f(x_{n-1}, x_n) - \nabla_n \log f(x_{n-1}', x_n') \rangle
\]
\[
= -(x_n - x_n')^T \Sigma^{-1} (x_n - x_n') + (x_n - x_n')^T \Sigma^{-1} A(x_{n-1} - x_{n-1}'),
\]
\[
\langle x_0 - x_0', \nabla_0 \log \mu(x_0) - \nabla_0 \log \mu(x_0') \rangle
\]
\[
= -(x_0 - x_0')^T \Sigma_0^{-1} (x_0 - x_0'),
\]
\[
\langle x_0 - x_0', \nabla_0 \log f(x_0, x_1) - \nabla_0 \log f(x_0', x_1') \rangle
\]
\[
= -(x_0 - x_0')^T A^T \Sigma^{-1} A(x_0 - x_0') + (x_0 - x_0')^T A^T \Sigma^{-1} (x_1 - x_1').
\]

Combining these expressions with (23), (6)-(7) and applying the following bounds:

\[
\inf_{u \neq 0} \frac{u^T (\Sigma^{-1} + A^T \Sigma^{-1} A) u}{\|u\|^2} \geq \rho_{\min}(\Sigma^{-1}) + \rho_{\min}(A^T A) \rho_{\min}(\Sigma^{-1}) = \rho_{\max}(\Sigma)^{-1} \{1 + \rho_{\min}(A^T A)\},
\]

\[
\inf_{u \neq 0} \frac{u^T \Sigma^{-1} u}{\|u\|^2} \wedge \inf_{u \neq 0} \frac{u^T (\Sigma_0^{-1} + A^T \Sigma^{-1} A) u}{\|u\|^2} \geq \rho_{\min}(\Sigma^{-1}) \wedge \{\rho_{\min}(\Sigma_0^{-1}) + \rho_{\min}(A^T A) \rho_{\min}(\Sigma^{-1})\}
\]
\[
= \rho_{\max}(\Sigma)^{-1} \wedge \{\rho_{\max}(\Sigma_0)^{-1} + \rho_{\min}(A^T A) \rho_{\min}(\Sigma)^{-1}\},
\]

\[
\sup_{u,v \neq 0} \frac{|u^T A^T \Sigma^{-1} v|}{\|u\| \|v\|} \leq \rho_{\max}(A^T A)^{1/2} \rho_{\max}(\Sigma)^{-1} = \rho_{\max}(A^T A)^{1/2} \rho_{\min}(\Sigma)^{-1},
\]

gives the expressions for \(\zeta, \tilde{\zeta}, \theta\) in the statement of the lemma.

**Lemma A3.** Assume:
a) the unobserved process satisfies

\[
X_n = A(X_{n-1}) + b + W_n,
\]

where for \(n \in \mathbb{N}\), \(W_n\) is independent of other random variables and has density proportional to \(e^{-\psi(w)}\), and \(\mu(x) \propto e^{-\psi_0(x)}\), where \(\psi, \psi_0 : \mathbb{R}^d \rightarrow \mathbb{R}\) are continuously differentiable and have bounded, Lipschitz gradients in the sense that:

\[
\sup_w \|\nabla \psi_0(w)\| \vee \|\nabla \psi(w)\| \leq L_\psi,
\]
\[
\|\nabla \psi(w) - \nabla \psi(w')\| \vee \|\nabla \psi_0(w) - \nabla \psi_0(w')\| \leq L_{\nabla \psi} \|w - w'\|,
\]

\(A\) is continuously differentiable, and has bounded, Lipschitz gradient in the sense that:

\[
\sup_x \|\nabla A(x)\|_{\text{op}} \leq L_A, \quad \|\nabla A(x) - \nabla A(x')\|_{\text{op}} \leq L_{\nabla A} \|x - x'\|,
\]

where \(\nabla A(x)\) is the Jacobian matrix of \(x \mapsto A(x)\), and \(\|\cdot\|_{\text{op}}\) is the Euclidean operator norm.
b) for each \(n, x \mapsto g(x, y_n)\) is strictly positive, continuously differentiable and there exists \(\lambda_g < 0\) independent of \(n\) such that

\[
\langle x_n - x_n', \nabla_n \log g(x_n, y_n^*) - \nabla_n \log g(x_n', y_n^*) \rangle \leq \lambda_g \|x_n - x_n'\|^2.
\]
If \( \theta < \frac{\zeta}{2} \wedge \tilde{\zeta} \) is satisfied with:

\[
\zeta = \tilde{\zeta} = -(L \nabla \psi + L_A^2 L \nabla \psi + L \psi L \nabla A) - \lambda_g, \\
\theta = L \nabla \psi L_A,
\]

then Condition 1 holds.

**Proof.** From the Lipschitz assumptions we have:

\[
\| \nabla_n \log f(x_{n-1}, x_n) - \nabla_n \log f(x'_{n-1}, x'_n) \| \\
= \| \nabla \psi \{ x_n - A(x_{n-1}) \} - \nabla \psi \{ x'_n - A(x'_{n-1}) \} \| \\
\leq L \nabla \psi \| x_n - x'_n \| + L \nabla \psi L_A \| x_{n-1} - x'_{n-1} \|,
\]

\[
\| \nabla \log f(x_n, x_{n+1}) - \nabla \log f(x'_n, x'_{n+1}) \| \\
\leq \| \nabla A(x_n) \|_{\text{op}} \| \nabla \psi \{ x_n - A(x_n) \} - \nabla \psi \{ x'_{n+1} - A(x'_n) \} \| \\
+ \| \nabla \psi \{ x'_{n+1} - A(x'_n) \} \| \| \nabla A(x_n) - \nabla A(x'_n) \|_{\text{op}} \\
\leq L_A L \nabla \psi \| x_{n+1} - x'_{n+1} \| + L_A^2 L \nabla \psi \| x_n - x'_n \| \\
+ L \nabla \psi L A \| x_n - x'_{n+1} \| \\
= (L_A^2 L \nabla \psi + L \nabla \psi L_A) \| x_n - x'_n \| + L \nabla \psi L A \| x_{n+1} - x'_{n+1} \|,
\]

\[
\| \nabla \log \mu(x_0) - \nabla \log \mu(x'_0) \| = \| \nabla \psi_0(x_0) - \nabla \psi_0(x'_0) \| \leq L \nabla \psi \| x_0 - x'_0 \|.
\]

The proof is completed by combining these estimates with assumption b) of the Lemma and (6)-(7). □

**Remark A1.** In the case \( d = 1 \), an example of a function \( \psi \) (or \( \psi_0 \)) which is continuously differentiable and has a bounded, Lipschitz gradient is the Huber function:

\[
\psi(x) = \begin{cases} 
\frac{1}{2} x^2, & |x| \leq c \\
| |x| - c, & |x| > c.
\end{cases}
\]

**A-5. Comparison to the assumptions of Chigansky & Ritov (2011)**

The assumptions of (Chigansky & Ritov, 2011, Thm 3.1) require that \( x \mapsto \mu(x) \) and \( (x, x') \mapsto f(x, x') \) are log-concave, and that \( x \mapsto g(x, y) \) is strongly log-concave, uniformly in \( y \). As discussed in section 3, our Condition 1b) does not require all these conditions to hold simultaneously.

Assumption (a4) (sic.) of (Chigansky & Ritov, 2011, Thm 3.1) is that with \( f(x, x') \propto e^{-\alpha(x, x')} \), there is a non-decreasing function \( \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \) growing to \( +\infty \) not faster than polynomially, such that for all \( M > 0 \),

\[
\alpha(x, x') \leq M \implies \left| \frac{\partial^2}{\partial x \partial x'} \alpha(x, x') \right| \leq g(M), \quad \text{for all } x, x'.
\] (A27)

Putting aside the issue of once versus twice differentiability, this assumption is related to the terms multiplied by \( \theta \) in our Condition 1b), but allows greater generality because \( g(M) \) can grow with \( M \), where as our Condition 1b) requires a value of \( \theta \) uniform in \( x, x' \). (Chigansky & Ritov, 2011, Thm 3.1) also places an assumption on the asymptotic behaviour of \( n^{-1} U^n(X_0, \ldots, X_n) \) as \( n \to \infty \) with the observations \( (y_n^*)_{n \in \mathbb{N}_0} \) treated as random, which may be regarded as a counterpart to the assumptions of Lemma 3.

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