Third order quasi-compact schemes for space tempered fractional diffusion equations

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Abstract

Power-law probability density function (PDF) plays a key role in both subdiffusion and Lévy flights. However, sometimes because of the finite of the lifespan of the particles or the boundedness of the physical space, tempered power-law PDF seems to be a more physical choice and then the tempered fractional operators appear; in fact, the tempered fractional operators can also characterize the transitions among subdiffusion, normal diffusion, and Lévy flights. This paper focuses on the quasi-compact schemes for space tempered fractional diffusion equations, being much different from the ones for pure fractional derivatives. By using the generation function of the matrix and Weyl’s theorem, the stability and convergence of the derived schemes are strictly proved. Some numerical simulations are performed to testify the effectiveness and numerical accuracy of the obtained schemes.

Keywords: Tempered fractional calculus, Compact schemes, Stability, Convergence.

1. Introduction

The CTRW model, composed of waiting times and jump lengths, is a pillar of statistical physics to characterize the anomalous dynamics. The power-law waiting time distribution is generally used to describe the subdiffusion; and the power-law jump length distribution is applied to Lévy flights. Based on the corresponding CTRW models, the time, space, or time-space fractional diffusion equations are derived to govern the probability density function (PDF) of the particles [1, 2]. Even though this kind of models find wide applications, sometimes the tempered power-law distribution seems to be a more appropriate choice for some biological processes because of the finite lifespan of the particles or the bounded physical space.

The simplest way to do the tempering is to directly cut the very large jump sizes or very long waiting times. Mantegna and Stanley show that the truncated Lévy flight ultraslowly converges to a Gaussian [3]. Replacing the way of truncation in [3] with exponentially truncating the Lévy flight, some analytic results to the problem of convergence of truncated Lévy flights towards the Gaussian stochastic process are presented [4]. Exponentially tempering the power-law PDF of waiting times or jump lengths seems to become popular nowadays, since it can bring many technical conveniences [5], e.g., making the tempered stochastic process still be Lévy process. For capturing the slow convergence of sub-diffusion to a diffusion limit for passive tracers in heterogeneous media, the model with exponentially tempered power-law waiting time distribution is introduced in [4]. By exponentially truncating the Lévy jump distribution, Cartea and del-Castillo-Negrete propose the partial differential equation (PDE) to describe the solute transport in natural systems [7], and the truncation effects in superdiffusive front propagation is discussed in [8].

Compared with the tempered fractional PDEs, the finite difference methods for fractional PDEs have been much more well developed, e.g., [9, 10, 11, 12]. A nature idea is to use the Grünwald-Letnikov formula [13] to approximate the Riemann-Liouville fractional derivative. Unfortunately, it is unconditionally unstable for the space fractional derivative. Meerschaert and his partners firstly propose the modified version of the Grünwald-Letnikov formula, i.e., the shift Grünwald formula, to effectively solve the space fractional diffusion equation [14, 15]. More recently, a

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series of high order schemes, including the compact ones, for space fractional diffusion equation are designed. In [16], the authors use the superconvergent point to get the second order scheme for the Riemann-Liouville fractional derivative. The second and third order WSGD operators are provided in [17], and a third order CWSGD operator is given in [18]. The related more high order schemes can be seen, e.g., [19, 20].

For the numerical solution of the tempered fractional PDEs, based on simulating the trajectories of the particles and then performing their average, the algorithms are provided in [21] and [22]. By directly discretizing the tempered fractional operators, a series of second order schemes are given in [23]. Here we further design the third order quasi-compact schemes for the tempered fractional diffusion equation; and note that not only the designing of the scheme but the proof of its numerical stability and convergence is much different from the the ones of the fractional diffusion equation. Using Weyl’s theorem by decomposing the matrix and the generation function of the matrix, we strictly prove the numerical stability and convergence of the derived schemes. And the extensive numerical experiments are performed to confirm the convergence order of the schemes.

The outline of this paper is as follows. In the next section, together with the left and right shifted Grünwald-Letnikov tempered operator, we develop the third order quasi-compact approximations for the left and right Riemann-Liouville tempered fractional derivatives, respectively. And then we focus on discussing the stability and convergence of implicit difference schemes with third order accuracy in space in Section 3. In Section 4, some numerical experiments are carried out to confirm the reliability of the obtained results. We conclude the paper with some remarks in the last section.

2. Derivation of the quasi-compact approximations for the tempered fractional derivatives

We begin with the definitions of \( \alpha \)-th order left and right Riemann-Liouville (RL) normalized tempered fractional derivatives [8, 2, 21].

**Definition 2.1.** If the function \( u(x) \) defined in the finite interval \([a, b]\) is regular enough, then for any \( \lambda \geq 0 \) the \( \alpha \)-th order left and right Riemann-Liouville normalized tempered fractional derivatives are, respectively, defined as

\[
aD_{a}^{\alpha, \lambda} u(x) = \frac{e^{-\alpha x}}{\Gamma(2 - \alpha)} \int_{a}^{x} (x - s)^{1-\alpha} e^{\lambda s} u(s) ds - \lambda^\alpha u(x) - \alpha \lambda^{\alpha-1} \frac{du(x)}{dx} \tag{1}
\]

and

\[
xD_{b}^{\alpha, \lambda} u(x) = \frac{e^{\lambda x}}{\Gamma(2 - \alpha)} \int_{x}^{b} (s - x)^{1-\alpha} e^{-\lambda s} u(s) ds - \lambda^\alpha u(x) + \alpha \lambda^{\alpha-1} \frac{du(x)}{dx} \tag{2}
\]

where \( 1 < \alpha < 2 \). Moreover, if \( \lambda = 0 \), then the Riemann-Liouville normalized tempered fractional derivatives \( aD_{a}^{\alpha, 0} u(x) \) and \( xD_{b}^{\alpha, 0} u(x) \) reduce to the Riemann-Liouville fractional derivatives \( aD_{a}^{\alpha} u(x) \) and \( xD_{b}^{\alpha} u(x) \), respectively.

For the convenience of presentation, we denote

\[
aD_{a}^{(\alpha, \lambda)} u(x) = e^{-\alpha x} aD_{a}^{\alpha} (e^{\lambda x} u(x))
\]

and

\[
xD_{b}^{(\alpha, \lambda)} u(x) = e^{\lambda x} xD_{b}^{\alpha} (e^{-\lambda x} u(x)).
\]

Then the left and right RL tempered fractional derivatives can be rewritten as

\[
aD_{a}^{\alpha, \lambda} u(x) = aD_{a}^{(\alpha, \lambda)} u(x) - \lambda^\alpha u(x) - \alpha \lambda^{\alpha-1} \frac{du(x)}{dx} \tag{3}
\]

and

\[
xD_{b}^{\alpha, \lambda} u(x) = xD_{b}^{(\alpha, \lambda)} u(x) - \lambda^\alpha u(x) + \alpha \lambda^{\alpha-1} \frac{du(x)}{dx}. \tag{4}
\]
Now we derive the quasi-compact approximations for the derivatives $-\infty D_{x}^{\alpha,1} u(x)$ and $\lambda D_{x}^{\alpha,0} u(x)$, respectively. Firstly, if the function $u(x)$ is regular enough, then $-\infty D_{x}^{(1,\lambda)} u(x)$ and $\lambda D_{x}^{(0,\lambda)} u(x)$ are equivalent to $\lambda u(x) + \frac{du(x)}{dx}$ and $\lambda u(x) - \frac{du(x)}{dx}$, respectively, i.e.,

$$
-\infty D_{x}^{(1,\lambda)} u(x) = e^{-\lambda x} - \infty D_{x}^{(1)} (e^{\lambda x} u(x)) = e^{-\lambda x} \left( \lambda e^{\lambda x} u(x) + e^{\lambda x} \frac{du(x)}{dx} \right) = \lambda u(x) + \frac{du(x)}{dx},
$$

and

$$
\lambda D_{x}^{(0,\lambda)} u(x) = -\lambda e^{-\lambda x} \frac{du(x)}{dx}.
$$

Recalling the definitions of the left and right RL tempered fractional derivatives, we get

$$
-\infty D_{x}^{(1,\lambda)} u(x) = -\infty D_{x}^{(a,\lambda)} u(x) - \lambda^a u(x) - \alpha \lambda^{a-1} \frac{du(x)}{dx}
$$

and

$$
\lambda D_{x}^{(0,\lambda)} u(x) = \lambda D_{x}^{(a,\lambda)} u(x) - \lambda^a u(x) + \alpha \lambda^{a-1} \frac{du(x)}{dx}.
$$

Thus the left and right RL tempered fractional derivatives at $x \in \mathbb{R}$ can be rearranged as

$$
-\infty D_{x}^{(a,\lambda)} u(x) = -\infty D_{x}^{(a,\lambda)} u(x) - \alpha \lambda^{a-1} - \infty D_{x}^{(1,\lambda)} u(x) + \lambda^a u(x) - \lambda \alpha u(x),
$$

and

$$
\lambda D_{x}^{(a,\lambda)} u(x) = \lambda D_{x}^{(a,\lambda)} u(x) - \alpha \lambda^{a-1} + \lambda D_{x}^{(1,\lambda)} u(x) + \lambda^a u(x) - \lambda \alpha u(x),
$$

which play an important role in the following discussions. Secondly, in [21], Baumeier et al. give an asymptotic expansion of the derivative $-\infty D_{x}^{(a,\lambda)} u(x)$ in the shift Grünwald-Letnikov difference formula, which is useful in constructing high order approximations.

**Lemma 2.1.** Let $1 < \alpha < 2$, $u \in W^{a+1}(\mathbb{R})$. For any integer $p$ and $\lambda \geq 0$, if we define the left and right shifted Grünwald-Letnikov (GL) tempered operators by

$$
\Delta_{p}^{(a,\lambda)} \mu(x) := \frac{1}{h} \sum_{k=0}^{n} \delta_{k}^{(a,\lambda)} e^{-\lambda (k-p)h} \mu(x - (k-p)h)
$$

and

$$
\Lambda_{p}^{(a,\lambda)} \mu(x) := \frac{1}{h} \sum_{k=0}^{n} \delta_{k}^{(a,\lambda)} e^{-\lambda (k-p)h} \mu(x + (k-p)h),
$$

where $h$ is stepsize, then there are

$$
\Delta_{p}^{(a,\lambda)} u(x) = -\infty D_{x}^{(a,\lambda)} u(x) + \sum_{k=1}^{n-1} a_{p,k} - \infty D_{x}^{(a+k,\lambda)} u(x) h^k + O(h^n)
$$

and

$$
\Lambda_{p}^{(a,\lambda)} u(x) = \lambda D_{x}^{(a,\lambda)} u(x) + \sum_{k=1}^{n-1} a_{p,k} \lambda D_{x}^{(a+k,\lambda)} u(x) h^k + O(h^n)
$$

uniformly in $x \in \mathbb{R}$, where the weights $\delta_{k}^{(a,\lambda)} = \frac{\Gamma(k-a)}{\Gamma(-a) \Gamma(k+1)}$ and $a_{p,k}$ are the coefficients of the power series expansion of the functions $(1 - z)^a$ and $\varpi_p(z) = \left( \frac{1-z}{1-z} \right)^z \varpi_p(z)$, respectively, and the first three terms of the coefficients of $\varpi_p(z)$ are

$$
a_{p,0} = 1, \quad a_{p,1} = p - \frac{\alpha}{2}, \quad a_{p,2} = \frac{\alpha + 3a^2 - 12\alpha p + 12p^2}{24}.\]
According to the definitions of the shifted GL tempered fractional derivatives and Lemma Appendix A.2, we construct the following third order quasi-compact approximations for the derivatives \(-\infty D_x^{(1,\delta)} u(x)\) and \(sD_{\infty}^{(1,\delta)} u(x)\).

**Theorem 2.1.** Suppose \(u \in W^{4,1}(\mathbb{R})\). Define the difference operators by

\[ L D_h^{(1,\delta)} u(x) = \frac{1}{2} \Delta_0^{(1,\delta)} u(x) + \frac{1}{2} \Delta_1^{(1,\delta)} u(x) = \frac{1}{2h} (e^{\delta h} u(x + h) - e^{-\delta h} u(x - h)) \quad (15) \]

and

\[ r D_h^{(1,\delta)} u(x) = \frac{1}{2} \Delta_0^{(1,\delta)} u(x) + \frac{1}{2} \Delta_1^{(1,\delta)} u(x) = \frac{1}{2h} (e^{\delta h} u(x - h) - e^{-\delta h} u(x + h)). \quad (16) \]

Then we have

\[ L D_h^{(1,\delta)} u(x) = \left(1 + \frac{1}{6} h^2 -\infty D_x^{(2,\delta)} \right) -\infty D_x^{(1,\delta)} u(x) + O(h^3), \quad (17) \]

\[ r D_h^{(1,\delta)} u(x) = \left(1 + \frac{1}{6} h^2 sD_{\infty}^{(2,\delta)} \right) sD_{\infty}^{(1,\delta)} u(x) + O(h^3) \]

uniformly for \(x \in \mathbb{R}\). Furthermore, the compact approximations to the derivatives \(-\infty D_x^{(1,\delta)} u(x)\) and \(sD_{\infty}^{(1,\delta)} u(x)\) are

\[ L B_{-\infty} D_x^{(1,\delta)} u(x) = L D_h^{(1,\delta)} u(x) + O(h^3) \]

(18) and

\[ r B_{s} D_{\infty}^{(1,\delta)} u(x) = r D_h^{(1,\delta)} u(x) + O(h^3), \]

(19)

respectively, where the compact difference operators are

\[ L Bu(x) = u(x) + \frac{h^2}{6} e^{-\delta h} \delta_x^2 (e^{\delta h} u(x)) = \frac{1}{6} e^{\delta h} u(x - h) + \frac{2}{3} u(x) + \frac{1}{6} e^{\delta h} u(x + h), \]

\[ r Bu(x) = u(x) + \frac{h^2}{6} e^{\delta h} \delta_x^2 (e^{-\delta h} u(x)) = \frac{1}{6} e^{\delta h} u(x + h) + \frac{2}{3} u(x) + \frac{1}{6} e^{-\delta h} u(x + h) \]

and \(\delta_x^2 u(x) = (u(x - h) - 2u(x) + u(x + h))/h^2\).

**Proof.** By Lemma 2.1, if we take \(\alpha = 1\) and \(p = 0\) and 1 in (11) and (12), respectively, then there exist

\[ \Delta_0^{(1,\delta)} u(x) = \frac{1}{h} (u(x) - e^{-\delta h} u(x - h)), \quad \Delta_1^{(1,\delta)} u(x) = \frac{1}{h} (e^{\delta h} u(x + h) - u(x)) \quad (20) \]

and

\[ \Delta_0^{(1,\delta)} u(x) = \frac{1}{h} (u(x) - e^{-\delta h} u(x + h)), \quad \Delta_1^{(1,\delta)} u(x) = \frac{1}{h} (e^{\delta h} u(x - h) - u(x)). \quad (21) \]

From (13) and (14), it’s easy to check that

\[ \Delta_p^{(1,\delta)} u(x) = -\infty D_x^{(1,\delta)} u(x) + \sum_{k=1}^{n-1} a_p^{1,k} -\infty D_x^{(1+k,\delta)} u(x) h^k + O(h^n), \quad p = 0, 1 \quad (22) \]

and

\[ \Delta_p^{(1,\delta)} u(x) = sD_{\infty}^{(1,\delta)} u(x) + \sum_{k=1}^{n-1} a_p^{1,k} sD_{\infty}^{(1+k,\delta)} u(x) h^k + O(h^n), \quad p = 0, 1 \quad (23) \]

hold. Combining (15) and (16) with (22) and (23), respectively, we get

\[ L D_h^{(1,\delta)} u(x) = \frac{1}{2} \Delta_0^{(1,\delta)} u(x) + \frac{1}{2} \Delta_1^{(1,\delta)} u(x) \]

\[ = -\infty D_x^{(1,\delta)} u(x) + \frac{1}{6} -\infty D_x^{(2,\delta)} u(x) h^2 + O(h^3) \]

\[ = \left(1 + \frac{1}{6} h^2 -\infty D_x^{(2,\delta)} \right) -\infty D_x^{(1,\delta)} u(x) + O(h^3) \]

(24)
Thus Equations (17) hold.

Next we establish the discretizations of the operators $1 + \frac{1}{h^2} D^{(2,4)}_{x,-\infty}$ and $1 + \frac{1}{h^2} D^{(2,4)}_{x,+\infty}$. The difference operator $\delta^2 u(x) = (u(x-h) - 2u(x) + u(x+h))/h^2 = \frac{\partial^2 u(x)}{\partial x^2} + O(h^2)$ yields that

$$
\left( 1 + \frac{1}{6} h^2 D^{(2,4)}_{x,-\infty} \right) u(x) = u(x) + \frac{1}{6} h^2 e^{-ih} \frac{d^2}{dx^2} (e^{ih} u(x)) = u(x) + \frac{1}{6} h^2 e^{-ih} \frac{d^2}{dx^2} (e^{ih} u(x)) + O(h^4)
$$

$$
= u(x) + \frac{1}{6} h e^{-ih} u(x-h) - 2e^{ih} u(x) + e^{ih} u(x+h) + O(h^4)
$$

$$
= \frac{1}{6} e^{-ih} u(x-h) + \frac{2}{3} u(x) + \frac{1}{6} e^{ih} u(x + h) + O(h^4)
$$

$$
= L B u(x) + O(h^4)
$$

and

$$
\left( 1 + \frac{1}{6} h^2 D^{(2,4)}_{x,+\infty} \right) u(x) = \frac{1}{6} e^{ih} u(x-h) + \frac{2}{3} u(x) + \frac{1}{6} e^{-ih} u(x + h) + O(h^4)
$$

$$
= L B u(x) + O(h^4)
$$

hold. Then we have

$$
\left( 1 + \frac{1}{6} h^2 -\omega D^{(2,4)}_{x} \right) \omega D^{(1,4)}_{x} u(x) = \frac{1}{6} e^{-ih} -\omega D^{(1,4)}_{x} u(x-h) + \frac{2}{3} -\omega D^{(1,4)}_{x} u(x) + \frac{1}{6} e^{ih} -\omega D^{(1,4)}_{x} u(x + h) + O(h^4)
$$

$$
= L B -\omega D^{(1,4)}_{x} u(x) + O(h^4)
$$

and

$$
\left( 1 + \frac{1}{6} h^2 D^{(2,4)}_{x,+\infty} \right) \omega D^{(1,4)}_{x} u(x) = k B +\omega D^{(1,4)}_{x} u(x) + O(h^4).
$$

Together with (24) and (25), we obtain

$$
L B \omega D^{(1,4)}_{x} u(x) = L D^{(1,4)}_{h} u(x) + O(h^3)
$$

and

$$
\omega B +\omega D^{(1,4)}_{x} u(x) = k D^{(1,4)}_{h} u(x) + O(h^3) .
$$

Now using the compact difference operators $L B$ and $k B$, we derive the corresponding third order quasi-compact approximations to the derivatives $-\omega D^{(1,4)}_{x} u(x)$ and $+\omega D^{(1,4)}_{x} u(x)$.

**Theorem 2.2.** Let $1 < \alpha < 2$, $u \in W^{3+\alpha,1}(\mathbb{R})$. Define the difference operators by

$$
L D^{(\alpha,4)}_{h} u(x) = \mu_{-1} \Delta^{(\alpha,4)}_{-1} u(x) + \mu_{0} \Delta^{(\alpha,4)}_{0} u(x) + \mu_{1} \Delta^{(\alpha,4)}_{1} u(x),
$$

$$
k D^{(\alpha,4)}_{h} u(x) = \mu_{-1} \Delta^{(\alpha,4)}_{-1} u(x) + \mu_{0} \Delta^{(\alpha,4)}_{0} u(x) + \mu_{1} \Delta^{(\alpha,4)}_{1} u(x),
$$

respectively, where the coefficients satisfy

$$
\mu_{-1} = \frac{1}{24} (4 - 7\alpha + 3\alpha^2), \quad \mu_{0} = \frac{1}{12} (8 + \alpha - 3\alpha^2), \quad \mu_{1} = \frac{1}{24} (4 + 5\alpha + 3\alpha^2).
$$
Then there exist
\begin{align*}
L D_h^{(p,1)} u(x) &= \left(1 + \frac{1}{6} h^2 D_x^{(2,1)} \right) -\infty D_x^{(p,1)} u(x) + O(h^3), \\
R D_h^{(p,1)} u(x) &= \left(1 + \frac{1}{6} h^2 D_x^{(2,1)} \right) +\infty D_x^{(p,1)} u(x) + O(h^3)
\end{align*}
(28)
uniformly for $x \in \mathbb{R}$. Furthermore, the following two quasi-compact approximations have third order accuracy, i.e.,
\begin{align*}
L \mathcal{B} -\infty D_x^{(p,1)} u(x) &= L D_h^{(p,1)} u(x) + O(h^3) \\
(29)
\end{align*}
and
\begin{align*}
R \mathcal{B} +\infty D_x^{(p,1)} u(x) &= R D_h^{(p,1)} u(x) + O(h^3).
(30)
\end{align*}

**Proof.** Form Lemma 2.1 we know
\begin{align*}
\Delta_p^{(a,\lambda)} u(x) &= -\infty D_x^{(a,\lambda)} u(x) + a_p^1 -\infty D_x^{(a+1,\lambda)} u(x) h^1 + a_p^2 -\infty D_x^{(a+2,\lambda)} u(x) h^2 + O(h^3).
(31)
\end{align*}
By taking $p = 1, 0$ and $-1$, respectively, we have that
\begin{align*}
L D_h^{(a,\lambda)} u(x) &= \mu_1 -\infty D_x^{(a,\lambda)} u(x) + \mu_0 a_0^{\alpha,\lambda} u(x) + \mu_1 a_1^{\alpha,\lambda} u(x) \\
&= (\mu_1 + \mu_0 + \mu_1) -\infty D_x^{(a,\lambda)} u(x) + (\mu_1 a_1^{\alpha,\lambda} + \mu_0 a_0^{\alpha,\lambda}) -\infty D_x^{(a+1,\lambda)} u(x) h^1 \\
&+ (\mu_1 a_1^{\alpha,\lambda} + \mu_0 a_0^{\alpha,\lambda} + \mu_1 a_1^{\alpha,\lambda}) -\infty D_x^{(a+2,\lambda)} u(x) h^2 + O(h^3).
\end{align*}
In order to obtain the results (28), the coefficients $\mu_1, \mu_0$ and $\mu_1$ need to solve
\begin{align*}
\begin{cases}
\mu_1 + \mu_0 + \mu_1 = 1, \\
\mu_1 a_1^{\alpha,\lambda} + \mu_0 a_0^{\alpha,\lambda} + \mu_1 a_1^{\alpha,\lambda} = 0, \\
\mu_1 a_1^{\alpha,\lambda} + \mu_0 a_0^{\alpha,\lambda} + \mu_1 a_1^{\alpha,\lambda} = \frac{1}{6},
\end{cases}
\end{align*}
which implies (27). Thus (28) hold. Similar to the proof of Theorem 2.1, (29) and (31) can be easily got; here we skip them.

Now, let us denote
\begin{align*}
L D_h^{(p,1)} u(x) &= L D_h^{(p,1)} u(x) - \alpha x^{p-1} L D_h^{(1,1)} u(x) + L \mathcal{B} \lambda^p (\alpha - 1) u(x) \\
&= \frac{1}{h^p} \sum_{k=0}^{\infty} w_k^{(p,1)} u(x - (k - 1)h) - \frac{\alpha x^{p-1}}{2h} (e^{ih} u(x + h) - e^{-ih} u(x - h)) + \lambda^p (\alpha - 1) L \mathcal{B} u(x)
(32)
\end{align*}
and
\begin{align*}
R D_h^{(p,1)} u(x) &= R D_h^{(p,1)} u(x) - \alpha x^{p-1} R D_h^{(1,1)} u(x) + R \mathcal{B} \lambda^p (\alpha - 1) R u(x) \\
&= \frac{1}{h^p} \sum_{k=0}^{\infty} w_k^{(p,1)} u(x + (k - 1)h) - \frac{\alpha x^{p-1}}{2h} (e^{ih} u(x - h) - e^{-ih} u(x + h)) + \lambda^p (\alpha - 1) R \mathcal{B} u(x).
(33)
\end{align*}
Then together with (21), (10), Theorem 2.1 and Theorem 2.2 we have the third order quasi-compact approximations for the $a$-th order left and right RL tempered fractional derivatives $-\infty D_x^{(a,\lambda)} u(x)$ and $+\infty D_x^{(a,\lambda)} u(x)$:
\begin{align*}
L \mathcal{B} -\infty D_x^{(a,\lambda)} u(x) &= L D_h^{(a,\lambda)} u(x) + O(h^3) \\
(34)
\end{align*}
and
\[ \mathcal{B}_0 D^{\alpha,1}_0 u(x) = \mathcal{B}_0 D^{\alpha,1}_0 u(x) + O(h^3). \] (35)

If \( u(x) \) is defined on \([a, b]\), and \( u \in W^{3+\alpha,1}(-\infty, b) \) after zero extension, then \( \mathcal{B}_0 D^{\alpha,1}_0 u(x) \) has a third order approximation
\[ L\mathcal{D}^{\alpha,1}_h u(x) = \frac{1}{h^\alpha} \sum_{k=0}^{\infty} w^{(\alpha,1)}_k u(x - (k - 1)h) - \frac{\alpha x^{\alpha - 1}}{2h} (e^{\beta h} u(x - h) - e^{-\beta h} u(x - h)) + \lambda^\alpha (\alpha - 1) L\mathcal{B} u(x). \] (36)

If \( u(x) \) is defined on \([a, b]\), and \( u \in W^{3+\alpha,1}(a, +\infty) \) after zero extension, then \( \mathcal{B}_0 D^{\alpha,1}_b u(x) \) has a third order approximation
\[ B\mathcal{D}^{\alpha,1}_h u(x) = \frac{1}{h^\alpha} \sum_{k=0}^{\infty} w^{(\alpha,1)}_k u(x + (k - 1)h) - \frac{\alpha x^{\alpha - 1}}{2h} (e^{\beta h} u(x + h) - e^{-\beta h} u(x + h)) + \lambda^\alpha (\alpha - 1) B\mathcal{B} u(x). \] (37)

Property 2.1. The formulae (26) at the grid points \( x_i = a + ih \) are denoted as
\[ L\mathcal{D}^{(\alpha,1)}_h u(x_i) = \frac{1}{h^\alpha} \sum_{k=0}^{\infty} w^{(\alpha,1)}_k u(x_{i-k+1}), \]
\[ B\mathcal{D}^{(\alpha,1)}_h u(x_i) = \frac{1}{h^\alpha} \sum_{k=0}^{\infty} w^{(\alpha,1)}_k u(x_{i+k-1}), \]
where the weights are given as
\[ w^{(\alpha,1)}_k = \mu_1 g_0^{(\alpha)} e^{\beta h}, \quad w^{(\alpha,1)}_k = \mu_0 g_1^{(\alpha)} + \mu_0 g_0^{(\alpha)}, \quad w^{(\alpha,1)}_k = \left( \mu_1 g_k^{(\alpha)} + \mu_0 g_{k+1}^{(\alpha)} + \mu_1 g_{k-1}^{(\alpha)} \right) e^{(1-k)\beta h}, \quad k \geq 2, \]
and the coefficients have the following properties:
\[ \left\{ \begin{array}{l}
\begin{aligned}
\frac{w^{(\alpha,1)}_0}{\mu_0} > 0, & \quad w^{(\alpha,1)}_1 \leq 0, & \quad w^{(\alpha,1)}_2 = \frac{e^{-\beta h}}{48} (8 - 50 \alpha + \alpha^2 + 14 \alpha^3 + 3 \alpha^4), \\\n\frac{w^{(\alpha,1)}_3}{144} = \frac{e^{-\beta h}}{144} (80 - 86 \alpha - 11 \alpha^2 + 14 \alpha^3 + 3 \alpha^4), & \quad \frac{w^{(\alpha,1)}_k}{\mu_0} \geq 0, & \quad k \geq 4,
\end{aligned}
\end{array} \right. \] (38)
\[ \sum_{k=0}^{\infty} w^{(\alpha,1)}_k = \left( \mu_1 e^{\beta h} + \mu_0 + \mu_1 e^{-\beta h} \right) \left( 1 - e^{-\beta h} \right)^\alpha, \]
where \( \mu_j, j = -1, 0, 1, \) are given in (22).

3. Quasi-compact schemes for the space tempered fractional diffusion equations

Based on the third order quasi-compact discretizations to the space tempered fractional derivatives, we develop the implicit schemes of the space tempered fractional diffusion equations with the left RL tempered fractional derivative and right RL tempered fractional derivative, respectively. And the detailed numerical stability and convergence analyses are provided.

3.1. Quasi-compact scheme of the fractional diffusion equation with the left RL tempered fractional derivative

Here we consider the following initial boundary value problem
\[ \left\{ \begin{array}{l}
\frac{\partial u(x, t)}{\partial t} = K_\alpha D^{\alpha,1}_t u(x, t) + f(x, t), & \quad (x, t) \in (a, b) \times (0, T), \\
u(x, 0) = u_0(x), & \quad x \in [a, b], \\
u(a, t) = 0, & \quad u(b, t) = u_b(t), \quad t \in [0, T],
\end{array} \right. \] (39)
where $1 < \alpha < 2$. The diffusion coefficient $K$ is a nonnegative constant. Assume that (39) has an unique and sufficiently regular solution.

We take the uniform meshes with the time step size $\tau = T/N$ on the interval $[0,T]$ and the space step size $h = (b-a)/M$ on the interval $(a,b)$, respectively. Then

$$[(x_i, t_n)|x_i = a + ih, \ t_i = 0, \ldots, M; \ t_n = n\tau, \ n = 0, \ldots, N],$$

where $N,M$ are two positive integers. For convenience, denote

$$u_i^n = u(x_i, t_n), \quad f_i^{n+1} = f(x_i, t_{n+1}), \quad \delta_i u_i^n = (u_i^{n+1} - u_i^n)/\tau, \quad 0 \leq n \leq N-1.$$ Discretizing the time derivative in (39) yields

$$\delta_i u_i^n = K_i D_i^{\alpha} u_i^{n+1} + f_i^{n+1} + O(\tau).$$

(40) Recalling the third order quasi-compact discretizations (36), we act the invertible operator $\tau I_B$ on both sides of (40) and obtain

$$l_B u_i^{n+1} - l_B u_i^n = K \tau_l D_i^{\alpha} u_i^{n+1} + \tau_l B f_i^{n+1} + \tau R_i^{n+1},$$

(41) where $|R_i^{n+1}| \leq C_1(\tau + h^\alpha)$.

Separating the time layers and replacing $l_B D_i^{\alpha} u_i^{n+1}$ by (32), we have

$$l_B u_i^{n+1} - K \tau (1/h^\alpha \sum_{k=0}^{n-1} \omega_k^{(\alpha)} u_{i-k}^{n+1} - \frac{\alpha \lambda^{n+1}}{2h}(e^{\lambda h} u_{i+1}^{n+1} - e^{-\lambda h} u_{i-1}^{n+1}) + \lambda^n (a-1) l_B u_i^n) = l_B u_i^n + \tau l_B f_i^{n+1} + \tau R_i^{n+1}.$$ (42)

Denoting $U_i^n$ as the numerical approximation of $u_i^n$, we obtain the quasi-compact scheme for (39) as follows

$$l_B U_i^{n+1} - K \tau (1/h^\alpha \sum_{k=0}^{n-1} \omega_k^{(\alpha)} U_{i-k}^{n+1} - \frac{\alpha \lambda^{n+1}}{2h}(e^{\lambda h} U_{i+1}^{n+1} - e^{-\lambda h} U_{i-1}^{n+1}) + \lambda^n (a-1) l_B U_i^n) = l_B U_i^n + \tau l_B f_i^{n+1}.$$ (43)

Then the corresponding matrix form of (43) can be written as

$$(B_i^n - P_i^n) U^{n+1} = B_i^n U^n + \tau B_i^n F_i^{n+1} + H_i^{n+1},$$

(44)

where $U^n = (U_1^n, U_2^n, \ldots, U_{M-1}^n)^T, \quad F_i^{n+1} = (f_1^{n+1}, f_2^{n+1}, \ldots, f_{M-1}^{n+1})^T, \quad B_i^n = \begin{pmatrix}
\frac{1}{h^\alpha} e^{-\lambda h} & \frac{1}{h^\alpha} \frac{1}{2} e^{\lambda h} & \frac{1}{h^\alpha} \frac{1}{2} e^{\lambda h} & \cdots & \frac{1}{h^\alpha} \frac{1}{2} e^{\lambda h} \\
\frac{\alpha \lambda^{n+1}}{2h} e^{-\lambda h} & \frac{\alpha \lambda^{n+1}}{2h} e^{\lambda h} & \cdots & \frac{\alpha \lambda^{n+1}}{2h} e^{\lambda h} & \cdots & \frac{\alpha \lambda^{n+1}}{2h} e^{\lambda h} \\
\frac{\alpha \lambda^{n+1}}{2h} e^{-\lambda h} & \frac{\alpha \lambda^{n+1}}{2h} e^{\lambda h} & \cdots & \frac{\alpha \lambda^{n+1}}{2h} e^{\lambda h} & \cdots & \frac{\alpha \lambda^{n+1}}{2h} e^{\lambda h} \\
\frac{\alpha \lambda^{n+1}}{2h} e^{-\lambda h} & \frac{\alpha \lambda^{n+1}}{2h} e^{\lambda h} & \cdots & \frac{\alpha \lambda^{n+1}}{2h} e^{\lambda h} & \cdots & \frac{\alpha \lambda^{n+1}}{2h} e^{\lambda h} \\
\frac{\alpha \lambda^{n+1}}{2h} e^{-\lambda h} & \frac{\alpha \lambda^{n+1}}{2h} e^{\lambda h} & \cdots & \frac{\alpha \lambda^{n+1}}{2h} e^{\lambda h} & \cdots & \frac{\alpha \lambda^{n+1}}{2h} e^{\lambda h} \\
\frac{\alpha \lambda^{n+1}}{2h} e^{-\lambda h} & \frac{\alpha \lambda^{n+1}}{2h} e^{\lambda h} & \cdots & \frac{\alpha \lambda^{n+1}}{2h} e^{\lambda h} & \cdots & \frac{\alpha \lambda^{n+1}}{2h} e^{\lambda h}
\end{pmatrix},$$

(45)

$$P_i^n = K \tau (A^n - \alpha I^{n+1} - \lambda^n (a-1) B_i^n)$$

$$= K \tau \frac{1}{h^\alpha} \begin{pmatrix}
\omega_{1,1}^{(\alpha)} & \omega_{1,1}^{(\alpha)} & \cdots & \omega_{1,M-1}^{(\alpha)} \\
\omega_{2,1}^{(\alpha)} & \omega_{2,1}^{(\alpha)} & \cdots & \omega_{2,M-1}^{(\alpha)} \\
\vdots & \vdots & \ddots & \vdots \\
\omega_{M-1,1}^{(\alpha)} & \omega_{M-1,1}^{(\alpha)} & \cdots & \omega_{M-1,M-1}^{(\alpha)} \\
\omega_{M,1}^{(\alpha)} & \omega_{M,1}^{(\alpha)} & \cdots & \omega_{M,M-1}^{(\alpha)} \\
\omega_{M,2}^{(\alpha)} & \omega_{M,2}^{(\alpha)} & \cdots & \omega_{M,M}^{(\alpha)} \\
\omega_{M-1,2}^{(\alpha)} & \omega_{M-1,2}^{(\alpha)} & \cdots & \omega_{M-1,M}^{(\alpha)} \\
\omega_{M-2,2}^{(\alpha)} & \omega_{M-2,2}^{(\alpha)} & \cdots & \omega_{M-2,M}^{(\alpha)} \\
\vdots & \vdots & \ddots & \vdots \\
\omega_{2,M-2}^{(\alpha)} & \omega_{2,M-2}^{(\alpha)} & \cdots & \omega_{2,M-1}^{(\alpha)} \\
\omega_{1,M-2}^{(\alpha)} & \omega_{1,M-2}^{(\alpha)} & \cdots & \omega_{1,M-1}^{(\alpha)} \\
\omega_{1,M-1}^{(\alpha)} & \omega_{1,M-1}^{(\alpha)} & \cdots & \omega_{1,M}^{(\alpha)}
\end{pmatrix} - \frac{\alpha \lambda^{n+1}}{2h} \begin{pmatrix}
0 & e^{\lambda h} & e^{\lambda h} & \cdots & e^{\lambda h} \\
e^{-\lambda h} & 0 & e^{\lambda h} & \cdots & e^{\lambda h} \\
e^{-\lambda h} & -e^{-\lambda h} & 0 & \cdots & e^{\lambda h} \\
e^{-\lambda h} & -e^{-\lambda h} & -e^{-\lambda h} & \cdots & 0 \\
e^{-\lambda h} & -e^{-\lambda h} & -e^{-\lambda h} & \cdots & 0
\end{pmatrix}$$

(46)

$$+ K \tau \lambda^n (a-1) B_i^n.$$
and
\[
H_i^{t+1} = \begin{pmatrix} \frac{1}{6} e^{-(\tau t)} & 0 \\ \vdots & \ddots & 0 \\ 0 & \frac{1}{6} e^{-(\tau t)} \end{pmatrix} (U^m_i - U_{m+1}^i + \tau f_{m+1}^i) + K \tau \begin{pmatrix} \frac{1}{m} w_2^{(\alpha)} + \frac{\alpha h^{\alpha-1}}{2h} e^{-(\tau t)} + \frac{\lambda}{2} \lambda (\alpha-1) e^{-\lambda t} \\ \vdots \\ \frac{1}{M} w_M^{(\alpha)} \end{pmatrix} U_{m+1}^i + \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} U_{m+1}^i.
\]
(47)

**Property 3.1.** Let \( \frac{1}{h} P_i^\alpha = (P_{j, k}^\alpha)_{(M-1) \times (M-1)} \). For \( \alpha \in (1, 2) \) and \( 0 \leq \lambda h \leq 1 \), the elements of \( \{P_{j, k}^\alpha\}_{(M-1) \times (M-1)} \) have the following properties:

1. \( P_{j, j} = \frac{h^{\alpha-1}}{h} w_0^{(\alpha)} + \frac{\lambda^\alpha}{\alpha} h (\alpha - 1) < 0, \quad j = 1, \ldots, M - 1; \)
2. \( P_{j, j+1} = \frac{h^{\alpha-1}}{h} w_0^{(\alpha)} - \frac{h^{\alpha-1}}{2h} e^{-(\tau t)} + \frac{\lambda^\alpha}{\alpha} h (\alpha - 1), \quad j = 1, \ldots, M - 2; \)
\( P_{j+1, j} = \frac{h^{\alpha-1}}{h} w_2^{(\alpha)} - \frac{h^{\alpha-1}}{2h} e^{-(\tau t)} + \frac{\lambda^\alpha}{\alpha} h (\alpha - 1), \quad j = 2, \ldots, M - 1; \)
\( P_{j+1, j+1} + P_{k+1, jk} > 0, \quad j = 1, \ldots, M - 2, k = 1, \ldots, M - 2; \)
3. \( P_{j+n, j} = -\frac{h^{\alpha-1}}{h} w_{n+1}^{(\alpha)} \), \( n = 2, \ldots, M - 2, j = n + 1, \ldots, M - 1; \)
4. \( \sum_{k=0}^{\infty} w_k^{(\alpha)} = -\frac{h^{\alpha-1}}{2} (-e^{-\lambda h} + e^{\lambda h}) + (\lambda h)^\alpha (\alpha - 1)(\frac{1}{2} + (e^{-\lambda h} + e^{\lambda h})/6) \leq 0. \)

**3.2 Quasi-compact scheme of the fractional diffusion equation with the right RL tempered fractional derivative**

We further consider the initial boundary value problem with the right RL tempered fractional derivative
\[
\frac{\partial u(x, t)}{\partial t} = K_i D_{h}^{3\lambda-1} u(x, t) + f(x, t), \quad (x, t) \in (a, b) \times (0, T),
\]
\[
u(x, 0) = u_0(x), \quad x \in [a, b],
\]
\[
u(a, t) = u_a(t), \quad u(b, t) = 0, \quad t \in [0, T],
\]
where \( 1 < \alpha < 2 \). Assume that the solution of (48) is unique and sufficiently regular to guarantee the feasibility of achieving the finite difference scheme and establishing its accuracy.

Recalling the third order quasi-compact discretization (37), we obtain
\[
r B U_i^{n+1} - K_i R_i^{n+1} U_i^{n+1} = r B u_i^{n+1} + r B f_i^{n+1} + \tau R_i^{n+1},
\]
(49)
where
\[
\|R_i^{n+1}\| \leq C_2(\tau + h^3).
\]

Denoting \( U_i^n \) as the numerical approximation of \( u_i^n \) and replacing \( R_i^{n+1} U_i^{n+1} \) by (33), we obtain the quasi-compact scheme of (48) as
\[
r B U_i^{n+1} - K_i \left( \sum_{k=0}^{M-1} w_k^{(\alpha)} U_i^{n+1} - \frac{\alpha h^{\alpha-1}}{2h} (e^{\lambda h} U_{i-1}^{n+1} - e^{-\lambda h} U_i^{n+1}) + \lambda^\alpha (\alpha - 1) r B U_i^{n+1} \right) = r B U_i^n + r B f_i^{n+1}.
\]
(50)
The corresponding matrix form of (50) can be written as

\[(B^\alpha_i - P^\alpha_i^T)U^{\alpha+1} = B^\alpha_iU^\alpha + \tau B^\alpha_iF^{\alpha+1} + H^{\alpha+1},\]

where \(B^\alpha_i = (B^\alpha_i)^T, P^\alpha_i = (P^\alpha_i)^T, H^{\alpha+1} = \text{flipud}(H^{\alpha+1}), B^\alpha_i\) is defined in (45), \(P^\alpha_i\) in (46), and \(H^{\alpha+1}\) in (47). We further discuss the properties of \(P^\alpha_i\).

**Lemma 3.1** (24). Let \(A \in \mathbb{R}^{n \times n}\). And it satisfies \(v^T Av \geq 0\) for all real nonzero vectors \(v\), if and only if its symmetric part \(H = \frac{A + A^T}{2}\) is positive definite.

**Lemma 3.2** (25). Let \(H\) be a Toeplitz matrix with a generating function \(f \in \mathbb{C}[z]\). Let \(\min(H)\) and \(\max(H)\) denote the smallest and largest eigenvalues of \(H\), respectively. Then we have

\[f_{\min} \leq \min(H) \leq \max(H) \leq f_{\max},\]

where \(f_{\min}\) and \(f_{\max}\) denote the minimum and maximum values of \(f(x)\), respectively. In particular, if \(f_{\max} \leq 0\) and \(f_{\min} \neq f_{\max}\), then \(H\) is negative definite.

**Lemma 3.3** (Weyl’s Theorem 25). Let \(A, E \in \mathbb{C}^{n \times n}\) be Hermitian and the eigenvalues \(\varrho_1(A), \varrho_1(E), \varrho_1(A + E)\) be arranged in an increasing order. Then for each \(k = 1, 2, \cdots, n\), we have

\[\varrho_k(A) + \varrho_k(E) \leq \varrho_k(A + E) \leq \varrho_k(A) + \varrho_n(E).\]

**Theorem 3.1.** When \(1 < \alpha < 2\) and \(\lambda \alpha \leq 1\), the matrices \(P^\alpha_i\) and \(P^\alpha_i^T\) satisfy \(v^T P^\alpha_i^T v < 0\) and \(v^T P^\alpha_i v < 0\), respectively, for all real nonzero vectors \(v\).

**Proof.** By Lemma 3.1, we just need to prove that their symmetric part \(\frac{P^\alpha_i + (P^\alpha_i)^T}{2}\) is strictly negative definite. Define a symmetry matrix \(H = \frac{P^\alpha_i + (P^\alpha_i)^T}{2} = [h_{jk}]\).

Next, we discuss the sign of the elements of matrix \(H\). From Property 3.1 we know that the elements in the main diagonal of matrix \(H\) are negative, i.e.,

\[h_{jj} < 0;\]

except \(h_{j,j+2}, h_{j+2,j}\) and \(h_{j,j}\), all the other elements of matrix \(H\) are nonnegative, i.e.,

\[h_{jk} \geq 0, k \neq j - 2, j, j + 2;\]

together with Property 3.1 \(h_{j,j+2} = h_{j+2,j} = \frac{1}{2P^2w^{(2)}(j)} = -\frac{2\alpha 15}{288\alpha}(80 - 86\alpha - 11\alpha^2 + 14\alpha^3 + 3\alpha^4), j = 1, \cdots, M - 2\).

Denote \(g(\alpha) = \frac{1}{2P^2}(80 - 86\alpha - 11\alpha^2 + 14\alpha^3 + 3\alpha^4)\). We can check that \(g(\alpha) = 0\) have two simple roots: \(\alpha_1 = 1\) and \(\alpha_2 = \frac{1}{4}(-17 + (6184 - \sqrt{317190})^\frac{1}{2} + (6184 + \sqrt{317190})^\frac{1}{2})\). Because \(g(2) = -\frac{1}{2} < 0\), \(g(\alpha) \geq 0\) for \(\alpha \in (1, \alpha_2)\) and \(g(\alpha) < 0\) for \(\alpha \in (\alpha_2, 2)\). Then \(h_{j,j+2} \geq 0\) for \(\alpha \in (1, \alpha_2)\) and \(h_{j,j+2} < 0\) for \(\alpha \in (\alpha_2, 2)\). Now we prove that \(H\) is strictly negative definite in both of the two cases.

When \(\alpha \in (1, \alpha_2)\),

\[h_{j,j+2} = h_{j+2,j} \geq 0, j = 1, \cdots, M - 2;\]

Then matrix \(H\) is a strictly diagonally dominant matrix. Combining with the Gerschgorin disk theorem and Property 3.1 we know that the eigenvalues of matrix \(H\) are all negative. So \(H\) is strictly negative definite.

When \(\alpha \in (\alpha_2, 2)\),

\[h_{j,j+2} = h_{j+2,j} < 0, j = 1, \cdots, M - 2;\]

Let us construct a new symmetric Toeplitz matrix \(H^+ \in \mathbb{R}^{(M-1)\times(M-1)}\),

\[
H^+ = \begin{pmatrix}
    h_a^+ & h_a^+ & h_a^+ & \cdots & h_a^+ \\
    h_b^+ & h_a^+ & h_b^+ & \cdots & h_b^+ \\
    h_c^+ & h_b^+ & h_a^+ & \cdots & h_c^+ \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    h_c^+ & h_b^+ & h_a^+ & \cdots & h_a^+
\end{pmatrix}
\]
with

\[ h_n^+ = -h_{n+2} > 0, \quad h_n^+ + 2h_n^+ + 2h_n^+ = 0, \]

which should be positive definite and can make \( H^+ + H \) strictly negative definite. In order to make \( H^+ \) be positive definite, we need the generation function of \( H^+ \) to be positive for any \( x \in [-\pi, \pi] \), i.e.,

\[ f_{H^+}(x) = h_n^+ + 2h_n^+ \cos(x) + 4h_n^+ \cos(x)^2 \geq 0. \]

Let \( y = \cos(x) \). Then the above equation can be rewritten as a quadratic function

\[ f^+(y) = h_n^+ - 2h_n^+y + 2h_n^+y^2. \quad (52) \]

where \( y \in [-1, 1], \ h_n^+ > 0 \). It’s easy to check that the discriminant \( \Delta = (2h_n^+)^2 - 4(h_n^+ - 2h_n^+)(4h_n^+) = (h_n^+ + 2h_n^+)^2 - 4(h_n^+ - 2h_n^+)(4h_n^+) = (h_n^+ - 6h_n^+)^2 \) and \( f^+(1) = 0 \). Then when \( \Delta = 0 \), i.e., \( h_n^+ = 6h_n^+ \), the function \( (52) \) is nonnegative.

From \( h_n^+ + 2h_n^+ + 2h_n^+ = 0 \), we get \( h_n^+ = -4h_n^+ \). So, after taking \( h_n^+ = 6h_n^+ \) and \( h_n^+ = -4h_n^+ \), \( H^+ \) is positive definite.

By some simple calculations, it can be shown that all the elements on the main diagonal of \( H^+ + H \) are negative, and the others are nonnegative; and \( H^+ + H \) is a strictly diagonally dominant matrix. Combining with the Gerschgorin disk theorem, the eigenvalues of matrix \( H^+ + H \) are all negative. Since \( H^+ \) is positive definite, \( -H^+ \) is negative definite.

As \( H = (H^+ + H) + (-H^+) \), together with the Weyl Theorem, the eigenvalues of \( H \) satisfy

\[ \rho(H) \leq \max(\rho(H^+ + H)) + \max(\rho(-H^+)) < 0. \]

Then the Toeplitz matrix \( H \) is strictly negative definite.

From the above, when \( \alpha \in (1, 2) \), the matrix \( H \) is strictly negative definite, which means \( \frac{P^t(H^+)P}{2} \) is also strictly negative definite. By Lemma 3.4, the matrix \( P^t \) satisfies \( v^T P^t v < 0 \) for all real nonzero vectors \( v \). Since \( P^t = (P^t)^T \), the matrix \( P^t \) satisfies \( v^T P^t v < 0 \) for all real nonzero vectors \( v \).

### 3.3. Stability and convergence analysis

In this subsection, we focus on the stability and convergence of the numerical schemes and get that the schemes have third order accuracy in space. Define

\[ V_h = \{ u : u = \{u_i\} \text{ is a grid function in } \{x = ih\}_{i=0}^{M-1} \text{ and } u_0 = u_M = 0 \}. \]

For any \( u = \{u_i\} \in V_h \), we use the discrete \( L^2 \) norm as

\[ ||u||^2 = h \sum_{i=1}^{M-1} u_i^2. \]

Next we probe into some properties of matrix \( B_{ij}^\alpha \).

**Lemma 3.4.** Let \( B_{ij}^\alpha \) be defined in \( (45) \). Then, for any \( h \leq 1/\lambda \), \( B_{ij}^\alpha \) satisfies that

\[ v^T B_{ij}^\alpha v > \frac{1}{12} v^T v, \quad (53) \]

and

\[ v^T B_{ij}^\alpha v < 2v^T v, \quad (54) \]

for all real nonzero vectors \( v \).

**Proof.** It’s easy to check that \( \frac{(B_{ij}^\alpha - \frac{1}{12} I) + (B_{ij}^\alpha - \frac{1}{12} I)^T}{2} \) is positive definite, i.e.,

\[ \rho \left( \frac{(B_{ij}^\alpha - \frac{1}{12} I) + (B_{ij}^\alpha - \frac{1}{12} I)^T}{2} \right) = \rho \left( \frac{B_{ij}^\alpha + (B_{ij}^\alpha)^T - \frac{2}{12} I}{2} \right) = \frac{7}{12} + \frac{1}{6} e^{-\lambda h} + e^{\lambda h} \cos \left( \frac{j\pi}{M} \right) > 0, \quad j = 1, \ldots, M - 1. \]
According to Lemma 3.1, we get \( \nu^T (B_i^\alpha - \frac{1}{12}I)\nu > 0 \) for all real nonzero vectors \( \nu \), which means

\[

v^T B_i^\alpha v > v^T \frac{1}{12} I v. \tag{55}
\]

On the other hand, we know that \( \frac{(B_i^\alpha - 2I) + (B_j^\alpha - 2I)^T}{2} \) is negative definite, i.e.,

\[

\varepsilon \left( \frac{(B_i^\alpha - 2I) + (B_j^\alpha - 2I)^T}{2} \right)_j = \varepsilon B_j^\alpha + \frac{1}{3} \varepsilon (e^{-\beta h} + e^{\beta h}) \cos \left( \frac{j\pi}{M} \right) < 0, \quad j = 1, \cdots, M - 1.
\]

Then \( \nu^T (B_j^\alpha - 2I)\nu < 0 \) for all real nonzero vectors \( \nu \), which means that

\[

\nu^T B_j^\alpha \nu < 2\nu^T I \nu. \tag{56}
\]

**Lemma 3.5** ([18]). Assume that \( \{k_n\} \) and \( \{p_n\} \) are nonnegative sequences, and the sequence \( \phi_n \) satisfies

\[

\phi_0 \leq g_0, \quad \phi_n \leq g_0 + \sum_{l=0}^{n-1} p_l + \sum_{l=0}^{n-1} k_l \phi_l, \quad n \geq 1,
\]

where \( g_0 \geq 0 \). Then the sequence \( \{\phi_n\} \) satisfies

\[

\phi_n \leq \left( g_0 + \sum_{l=0}^{n-1} p_l \right) \exp \left( \sum_{l=0}^{n-1} k_l \right), \quad n \geq 1.
\]

**Theorem 3.2.** Let \( \bar{U}_h^\alpha \) be the exact solution of (43), and \( U_h^\alpha \) the numerical solution of (43) obtained in finite precision arithmetic. Then, when \( 0 < h \leq 1/\alpha \), the difference scheme (43) is stable for all \( 1 < \alpha < 2 \).

**Proof.** Denoting \( \varepsilon_k^\alpha = \bar{U}_j^\alpha - U_j^\alpha \) and \( \varepsilon_k^\alpha = (\varepsilon_1^\alpha, \varepsilon_2^\alpha, \cdots, \varepsilon_{M-1}^\alpha)^T \), from (43), we obtain

\[

(B_i^\alpha - P_i^\alpha)\varepsilon_k^{i+1} = B_j^\alpha \varepsilon_k^j.
\]

where \( k = 0, 2, \cdots, n - 1 \). Multiplying (57) by \( h(\varepsilon_k^{i+1})^T \), we obtain that

\[

h(\varepsilon_k^{i+1})^T B_i^\alpha \varepsilon_k^{i+1} = h(\varepsilon_k^{i+1})^T P_i^\alpha \varepsilon_k^{i+1} + h(\varepsilon_k^{i+1})^T B_j^\alpha \varepsilon_k^j.
\]

By Theorem 3.1, we know that the matrix \( P_i^\alpha \) satisfies \( \nu^T P_i^\alpha \nu < 0 \) for all real nonzero vectors \( \nu \). Thus

\[

(\varepsilon_k^{i+1})^T P_i^\alpha \varepsilon_k^{i+1} \leq 0.
\]

Then (58) leads to

\[

h(\varepsilon_k^{i+1})^T B_i^\alpha \varepsilon_k^{i+1} \leq h(\varepsilon_k^{i+1})^T B_j^\alpha \varepsilon_k^j \leq \frac{1}{2} (h(\varepsilon_k^{i+1})^T B_i^\alpha \varepsilon_k^{i+1} + h(\varepsilon_k^{i+1})^T B_j^\alpha \varepsilon_k^j),
\]

which implies

\[

h(\varepsilon_k^{i+1})^T B_i^\alpha \varepsilon_k^{i+1} \leq h(\varepsilon_k^{i+1})^T B_i^\alpha \varepsilon_k^j.
\]

Denoting \( E^k = h(\varepsilon_k^{i+1})^T B_i^\alpha \varepsilon_k^j \), we have

\[

E^{k+1} \leq E^k.
\]

Taking \( k \) from 0 to \( n \) yields

\[

E^{n+1} \leq E^n \leq E^{n-1} \cdots \leq E^0.
\]

Together with Lemma 3.4, we get

\[

\frac{1}{12} \|e^{n+1}\|^2 \leq E^{n+1} \leq E^n \leq 2\|e^0\|^2.
\]

Then

\[

\|e^{n+1}\|^2 \leq 24\|e^0\|^2.
\]

Therefore, the difference scheme (43) is stable. \( \square \)
Theorem 3.3. Let \( u^n \) be the exact solution of (59), and \( U^n \) the solution of the given finite difference scheme (43). Then we have
\[
\|u^n - U^n\| \leq C(\tau + h^3),
\]
for all \( 1 \leq n \leq N \), where \( C \) is a constant independent of \( n, \tau, \) and \( h \).

Proof. Denoting \( \epsilon_j^k = u(x_j, t_k) - U_j^k \), from (41), we obtain
\[
(B^n_j - P^n_j)\epsilon^{k+1} = B_j^k \epsilon^k + \tau R^{k+1},
\]
where \( \epsilon^k = (\epsilon_1^k, \epsilon_2^k, \ldots, \epsilon_{M-1}^k)^T \), \( R^{k+1} = (R_{1}^{k+1}, R_{2}^{k+1}, \ldots, R_{M-1}^{k+1})^T \) and \( 0 \leq k \leq n - 1 \). Multiplying (59) by \( h(\epsilon^{k+1})^T \), we get
\[
h(\epsilon^{k+1})^T(B^n_j - P^n_j)\epsilon^{k+1} = h(\epsilon^{k+1})^T B_j^k \epsilon^k + \tau h(\epsilon^{k+1})^T R^{k+1},
\]
By Theorem 3.1, we have
\[
\frac{1}{2} h(\epsilon^{k+1})^T B_j^k \epsilon^k \leq \frac{1}{2} h(\epsilon^{k+1})^T B_j^k \epsilon^k + \tau h(\epsilon^{k+1})^T R^{k+1}.
\]
Let \( \epsilon^k = h(\epsilon^{k+1})^T B_j^k \epsilon^k \), and we have
\[
(60) \quad \epsilon^{k+1} \leq \epsilon^k + 2\tau (\epsilon^{k+1})^T R^{k+1}.
\]
Together with Lemma 3.4 summing up (60) for all \( 0 \leq k \leq n - 1 \) shows that
\[
\frac{1}{12} \|\epsilon^0\|^2 \leq \epsilon^0 \leq \epsilon^0 + 2\tau \sum_{k=0}^{n-1} h(\epsilon^{k+1})^T R^{k+1} \leq \frac{1}{48\tau} 2\tau \|\epsilon^0\|^2 + \frac{24\tau}{2} 2\tau \|R^n\|^2 + 2\tau \sum_{k=0}^{n-1} h(\epsilon^{k+1})^T R^{k+1}.
\]
Then
\[
\frac{1}{24} \|\epsilon^0\|^2 \leq 24\tau^2 \|R^n\|^2 + 2\tau \sum_{k=0}^{n-1} h(\epsilon^{k+1})^T R^{k+1} \leq \frac{1}{24} \tau \sum_{k=0}^{n-1} \|\epsilon^k\|^2 + 24\tau \sum_{k=1}^{n-1} \|R^n\|^2 + 24\tau^2 \|R^n\|^2.
\]
Noticing that \( |R_j^{k+1}| \leq c(\tau + h^3) \) for \( 1 \leq k \leq n \) and utilizing the discrete Gronwall’s inequality, we obtain
\[
\|\epsilon^0\|^2 \leq \tau \sum_{k=1}^{n-1} \|\epsilon^k\|^2 + 24\tau \sum_{k=1}^{n-1} \|R^n\|^2 \leq C(\tau + h^3)^2.
\]

By the similar idea, we can prove the following results; and the details are omitted here.

Theorem 3.4. Let \( \bar{U}_j^n \) be the exact solution of (50), and \( U_j^n \) the numerical solution of (50) obtained in finite precision arithmetic. Then when \( 0 < h \leq 1/\lambda \), the difference schemes (50) is stable for all \( 1 < \alpha < 2 \).

Theorem 3.5. Let \( u^n \) be the exact solution of (48), and \( U^n \) the solution of the given finite difference scheme (50). Then we have
\[
\|u^n - U^n\| \leq C(\tau + h^3),
\]
for all \( 1 \leq n \leq N \), where \( C \) is a constant independent of \( n, \tau, \) and \( h \).

4. Numerical examples

In this section, we discuss the effectiveness of the third order quasi-compact difference schemes derived in the above. And the presented numerical results confirm the theoretical ones.

Let
\[
e(\tau, h) = \left( \frac{1}{h} \sum_{i=1}^{M-1} \left( u(x_i, t_N) - U_i^N \right)^2 \right)^{1/2},
\]
where \( u(x, t) \) represents the exact solution and \( U^N_i \) is the numerical solutions at the grid point \((x_i, t_n)\) with the mesh step sizes \( \tau \) and \( h \). Together with the equations

\[
\partial_x D_x^{(\alpha, \beta)}(e^{-xt}x) = \frac{\Gamma(1 + j)e^{-xt}}{\Gamma(1 + j - \alpha)}(x - \alpha)^{-j/\alpha}
\]

and

\[
\partial_x D_b^{(\alpha, \beta)}(e^{xt}b - x) = \frac{\Gamma(1 + j)e^{xt}}{\Gamma(1 + j - \alpha)}(b - x)^{-j/\alpha},
\]

we show the following examples. To test the order of convergence, except Table 5, where \( \tau = h^{3/2} \), for all the other Tables, \( \tau = h^3 \) is taken.

**Example 4.1.** We consider the tempered space fractional diffusion equation

\[
\frac{\partial u}{\partial t} = b D_x^{(\alpha, \beta)}u(x) - e^{-rt\beta} \left( x^\beta + \frac{\Gamma(j + 1)x^{j/\beta}}{\Gamma(1 + j - \beta)} - \alpha\lambda^{j/\beta}(j\lambda^{j/\beta} - \lambda^{j/\beta}) - x^\beta \right), \quad (x, t) \in (0, 1) \times (0, 1), \tag{61}
\]

with the boundary conditions \( u(0, t) = 0, u(1, t) = e^{-t} \) and the initial value \( u(x, 0) = e^{-tx}x^J, \ x \in [0, 1], \) where \( j \in \mathbb{N} \). The exact solution is \( u(x) = e^{-tx}x^J \).

In Table 2, we confirm the convergence orders and show that the regularity of the solution is necessary for obtaining the desired convergence orders, even though it is weaker than \( u \in W^{\alpha+1}((\mathbb{R}) \) required in the proof of Theorem 3.2. Table 3 further confirms the convergence orders and shows that, as in the proof of Theorem 3.2, the condition \( \lambda h < 1 \) is required for ensuring the stability of the schemes.

**Example 4.2.** Consider the following tempered space fractional diffusion equation

\[
\frac{\partial u}{\partial t} = x D_1^{(\alpha, \beta)}u(x) - e^{-\tau x} \left( (1 - x)^\beta + \frac{\Gamma(j + 1)(1 - x)^{j/\beta}}{\Gamma(1 + j - \beta)} - \alpha\lambda^{j/\beta}(j\lambda^{j/\beta} - \lambda^{j/\beta}) - x^\beta \right), \tag{62}
\]

where \((x, t) \in (0, 1) \times (0, 1)\) and \( j \in \mathbb{N} \). The boundary conditions are \( u(0, t) = e^{-t}, u(1, t) = 0 \) and the initial value is \( u(x, 0) = e^{t}(1 - x)^J, \ x \in [0, 1]. \) The exact solution is \( u(x) = e^{-tx}(1 - x)^J \).

Table 3 confirms the convergence orders of the scheme of the right tempered fractional diffusion equation and shows the required regularity for the to be approximated solution. And Table 4 verifies the required stability condition \( \lambda h < 1 \).
Table 2: The errors $e(\tau, h)$ and spatial convergence orders of the quasi-compact scheme \([43]\) by computing Example 4.1 for different $\lambda$ with $j = 5$.

| $\alpha$ | $\lambda = 1$ | $\lambda = 10$ | $\lambda = 50$ |
|----------|----------------|----------------|----------------|
|          | $e(\tau, h)$  | rate           | $e(\tau, h)$  | rate           | $e(\tau, h)$  | rate           |
| 1.1      | 0.1            | 6.0259e-06     | 4.0133e-07     | 1.4755e-09     |
|          | 0.05           | 7.6037e-07     | 6.1913e-08     | 2.6965         | 1.9198e-09     | 3.7982         |
|          | 0.025          | 9.5387e-08     | 8.3494e-09     | 2.8905         | 5.2808e-10     | 1.8622         |
|          | 0.0125         | 1.1945e-08     | 1.0774e-09     | 2.9541         | 8.6959e-11     | 2.6023         |
| 1.5      | 0.1            | 9.1408e-05     | 5.8760e-06     | 1.1852e-02     |
|          | 0.05           | 1.1772e-05     | 9.3548e-07     | 2.6510         | 1.9228e-08     | 1.9234         |
|          | 0.025          | 1.4927e-06     | 1.3004e-07     | 2.8467         | 5.6716e-09     | 1.7614         |
|          | 0.0125         | 1.8791e-07     | 1.7048e-08     | 2.9314         | 9.7853e-10     | 2.5351         |
| 1.9      | 0.1            | 2.7192e-04     | 1.7584e-05     | 1.3153e+04     |
|          | 0.05           | 3.4977e-05     | 2.7367e-06     | 2.6837         | Inf            | NaN            |
|          | 0.025          | 4.4201e-06     | 3.7638e-07     | 2.8622         | NaN            | NaN            |
|          | 0.0125         | 5.5510e-07     | 4.8967e-08     | 2.9423         | NaN            | NaN            |

Table 3: The errors $e(\tau, h)$ and spatial convergence orders of the quasi-compact scheme \([50]\) by computing Example 4.2 with $\lambda = 1$.

| $\alpha$ | $j = 1$  | $j = 3$  | $j = 5$  |
|----------|----------|----------|----------|
|          | $e(\tau, h)$  | rate     | $e(\tau, h)$  | rate     | $e(\tau, h)$  | rate     |
| 1.1      | 0.1      | 1.2169e-04 | 1.9583e-05 | 1.6380e-05 |
|          | 0.05     | 3.9867e-05 | 2.4327e-06 | 3.0090     | 2.0609e-06     | 2.9864     |
|          | 0.025    | 1.0612e-05 | 1.9095     | 3.0398e-07 | 3.0005     | 2.5929e-07     | 2.9948     |
|          | 0.0125   | 2.1140e-06 | 2.3277     | 3.8018e-08 | 2.9992     | 3.2469e-08     | 2.9974     |
| 1.5      | 0.1      | 7.9411e-04 | 6.0140e-05 | 2.4847e-04 |
|          | 0.05     | 1.6852e-04 | 2.2364     | 9.4045e-06 | 2.6769     | 3.2000e-05     | 2.9569     |
|          | 0.025    | 3.2337e-05 | 2.3816     | 1.3392e-06 | 2.8120     | 4.0577e-06     | 2.9794     |
|          | 0.0125   | 5.9527e-06 | 2.4416     | 1.8105e-07 | 2.8869     | 5.1080e-07     | 2.9898     |
| 1.9      | 0.1      | 1.7091e-03 | 3.9956e-04 | 7.3915e-04 |
|          | 0.05     | 4.0278e-04 | 2.0852     | 4.5756e-05 | 2.7582     | 9.5077e-05     | 2.9587     |
|          | 0.025    | 9.2145e-05 | 2.1280     | 6.2106e-06 | 2.8812     | 1.2015e-05     | 2.9842     |
|          | 0.0125   | 2.1019e-05 | 2.1322     | 8.0898e-07 | 2.9405     | 1.5089e-06     | 2.9933     |

Table 4: The errors $e(\tau, h)$ and spatial convergence orders of the quasi-compact scheme \([50]\) by computing Example 4.3 for different $\lambda$ with $j = 5$.

| $\alpha$ | $\lambda = 1$ | $\lambda = 10$ | $\lambda = 50$ |
|----------|----------------|----------------|----------------|
|          | $e(\tau, h)$  | rate           | $e(\tau, h)$  | rate           | $e(\tau, h)$  | rate           |
| 1.1      | 0.1            | 1.6380e-05     | 8.8399e-03     | 7.6498e+12     |
|          | 0.05           | 2.0669e-06     | 2.9864         | 1.3637e-03     | 2.6965         | 9.5388e+12     | 3.7982         |
|          | 0.025          | 2.5929e-07     | 9.8948         | 1.8391e-04     | 2.8905         | 2.7379e+12     | 1.8622         |
|          | 0.0125         | 3.2469e-08     | 2.9974         | 2.3731e-05     | 2.9541         | 4.5086e+11     | 2.6023         |
| 1.5      | 0.1            | 2.4847e-04     | 1.2943e-01     | 6.1451e+19     |
|          | 0.05           | 3.2000e-05     | 2.9569         | 2.0605e-02     | 2.6510         | 9.6902e+13     | 19.234         |
|          | 0.025          | 4.0577e-06     | 2.9794         | 2.8644e-03     | 2.8467         | 2.9406e+12     | 1.7614         |
|          | 0.0125         | 5.1080e-07     | 2.9898         | 3.7550e-04     | 2.9314         | 5.0734e+12     | 2.5351         |
| 1.9      | 0.1            | 7.3915e-04     | 3.8731e-01     | 6.8193e+08     |
|          | 0.05           | 9.5077e-05     | 2.9587         | 6.0281e-02     | 2.6837         | NaN            | NaN            |
|          | 0.025          | 1.2015e-05     | 2.9842         | 8.2903e-03     | 2.8622         | NaN            | NaN            |
|          | 0.0125         | 1.5089e-06     | 2.9933         | 1.0786e-03     | 2.9423         | NaN            | NaN            |
Example 4.3. The following two dimensional fractional diffusion problem

\[
\frac{\partial u(x, y, t)}{\partial t} = \alpha D^\alpha_{x} u(x, y, t) + \beta D^\beta_{y} u(x, y, t) + f(x, y, t)
\]  

(63)

is considered in the domain \( \Omega = (0, 1)^2 \) and \( t \in (0, 1) \). The source term is

\[
f(x, t) = -e^{-\alpha_{1} x - \alpha_{2} y}\left[ x^4 + \frac{\Gamma(5)x^{5-\alpha}}{\Gamma(5-\alpha)} - \alpha_{1}\lambda_{1}^{-1}(4x^3 - \lambda_1 x^4) - \lambda_1^3 x^4 \\
- \left( y^5 + \frac{\Gamma(6)y^{6-\beta}}{\Gamma(6-\beta)} - \alpha_{2}\lambda_{2}^{-1}(5y^4 - \lambda_2 y^5) - \lambda_2^3 y^5 \right) \right] y^4(1-y) + \\
\left( \frac{\Gamma(5)y^{5-\beta}}{\Gamma(5-\beta)} - \beta\lambda_{2}^{-1}(4y^3 - \lambda_2 y^4) - \lambda_2^3 y^4 - \left( \frac{\Gamma(6)y^{6-\beta}}{\Gamma(6-\beta)} - \beta\lambda_{2}^{-1}(5y^4 - \lambda_2 y^5) - \lambda_2^3 y^5 \right) \right) x^4(1-x) \right].
\]

The exact solution is given by \( u(x, y, t) = e^{-\alpha_{1} x - \alpha_{2} y} x^4(1-x)y^4(1-y) \). The boundary conditions are \( u(0, y, t) = u(1, y, t) = u(x, 0, t) = u(x, 1, t) = 0 \) with \((x, y) \in \partial \Omega \) and \( t \in [0, 1] \). The initial value is \( u(x, y, 0) = e^{-\alpha_{1} x - \alpha_{2} y} x^4(1-x)y^4(1-y) \) with \((x, y) \in [0, 1]^2 \).

To solve (63), we derive the quasi-compact D’Yakonov ADI scheme in matrix form as

\[
\left( B_i^\alpha - \frac{T}{2} P_i^\beta \right) U^* = \left( B_i^\alpha + \frac{T}{2} P_i^\beta \right) U^n + \left( B_i^\alpha + \frac{T}{2} P_i^\beta \right) + \tau B_i^\beta f^{n+1/2} (B_i^\beta)^T.
\]

In this example, we take the uniform meshes with the space step size \( h_x = h_z \) and the time step size \( \tau = h_x^2 \). From Table 5 it can be seen that the numerical results are stable and convergent, and the third order accuracy in space is verified.

| \( D'\text{Yakonov} \) \((\alpha, \beta) = (1, 1.5) \& \lambda = 0.1 \) | \( \alpha, \beta = (1.5, 1.9) \& \lambda = 0.1 \) | \( \alpha, \beta = (1.2, 1.5) \& \lambda = 10 \) |
|---|---|---|
| \( h_x = h_z \) | \( \epsilon(\tau, h) \) | \( \epsilon(\tau, h) \) | \( \epsilon(\tau, h) \) |
| 0.1 | 6.7629e-06 | 9.4666e-06 | 3.6415e-10 |
| 0.05 | 8.4125e-07 | 3.0070 | 1.2061e-06 | 2.9725 | 6.1485e-11 | 2.5662 |
| 0.025 | 1.0543e-07 | 2.9962 | 1.5279e-07 | 2.9807 | 9.6300e-12 | 2.6746 |
| 0.0125 | 1.3164e-08 | 3.0017 | 1.9169e-08 | 2.9947 | 1.3539e-12 | 2.8304 |

Example 4.4. The following two sided fractional diffusion problem

\[
\frac{\partial u(x, t)}{\partial t} = \alpha D^\alpha_{x} u(x, t) + \beta D^\beta_{x} u(x, t) + f(x, t)
\]  

(64)

is considered in the domain \( \Omega = (0, 1) \) and \( t \in (0, 1) \). The exact solution is given by \( u(x, t) = e^{-\lambda x} x^4(1-x)^4 \). The boundary conditions are \( u(0, t) = u(1, t) = 0 \) with \( t \in [0, 1] \). The initial value is \( u(x, 0) = e^{-\lambda x} x^4(1-x)^4 \).

To get the source term, we need to calculate the right fractional derivative \( D^\alpha_{x} u(x, t) \) of the exact solution and
obtain that

\[
\begin{align*}
&\alpha D_1^{(\alpha, \lambda)} e^{-xt}x^4(1 - x)^4 \\
&= e^{l_x} \alpha D_1^{(\alpha, \lambda)} e^{-2xt}x^4(1 - x)^4 \\
&= e^{l(x - 2)} \alpha D_1^{(\alpha, \lambda)} \left( e^{2l(x - 1 - \alpha) \sum_{m=0}^{4} \frac{(-1)^m}{m} (1 - x)^m) \right) \\
&= e^{l(x - 2)} \alpha D_1^{(\alpha, \lambda)} \left( e^{\sum_{j=0}^{\infty} \frac{2l(1 - x)}{j!} \sum_{m=0}^{4} \frac{(-1)^m}{m} (1 - x)^m) \right) \\
&= e^{l(x - 2)} \alpha D_1^{(\alpha, \lambda)} \left( e^{\sum_{j=0}^{50} \frac{2l(1 - x)}{j!} \sum_{m=0}^{4} \frac{(-1)^m}{m} (1 - x)^m) \right) \\
&= e^{l(x - 2)} \alpha D_1^{(\alpha, \lambda)} \left( e^{\sum_{j=0}^{50} \frac{2l(1 - x)}{j!} \sum_{m=0}^{4} \frac{(-1)^m}{m} (1 - x)^m) \right) \\
&= e^{l(x - 2)} \alpha D_1^{(\alpha, \lambda)} \left( e^{\frac{1}{2} \frac{(5 + m + j)}{\Gamma(5 + m + j - \alpha)} (1 - x)^{4 + m - \alpha}} \right) \\
&= e^{l(x - 2)} \alpha D_1^{(\alpha, \lambda)} \left( e^{\frac{1}{2} \frac{(5 + m + j)}{\Gamma(5 + m + j - \alpha)} (1 - x)^{4 + m - \alpha}} \right). 
\end{align*}
\]

Then the source term

\[
f(x, t) = -e^{l_x} \left[ e^{\frac{1}{2} \frac{(5 + m + j)}{\Gamma(5 + m + j - \alpha)} (1 - x)^{4 + m - \alpha}} \right] - 2\lambda \tau x^4(1 - x)^4.
\]

Since the compact difference operator \(\mathcal{D}_x\) commutes with \(\mathcal{B}_t\), to solve (64), we introduce the operator splitting method to derive the numerical schemes and the matrix forms of the schemes are as follows:

\[
B_0 U^0 = (B_0 + \tau P_0)U^0 + \frac{\tau}{2} B_0^2 f^{n+1/2}.
\]

\[
(B_0^t - \tau P_0^t)U^{n+1} = B_0 U^0 + \frac{\tau}{2} B_0^2 f^{n+1/2}.
\]

Table 6 shows the numerical results obtained by using the quasi-compact operator splitting method to solve (64) with \(\tau = h^{\lambda}\). It can be noted that the convergence orders are three in spacial direction which confirms the theoretical estimations.

| Table 6: The errors \(e(\tau, h)\) and spatial convergence orders of the operator splitting scheme by computing Example [13] with \(\lambda = 0.1\). |
|---|---|---|---|---|---|---|
| \(h\) | \(\alpha = 1.2\) | \(\alpha = 1.5\) | \(\alpha = 1.8\) | \(\alpha = 1.2\) | \(\alpha = 1.5\) | \(\alpha = 1.8\) |
| 0.1 | 4.046e-06 | 5.874e-06 | 8.432e-06 | 4.046e-06 | 5.874e-06 | 8.432e-06 |
| 0.05 | 5.6875e-07 | 2.8309 | 7.8215e-07 | 2.9090 | 9.7711e-07 | 3.1093 |
| 0.025 | 7.5560e-08 | 2.9121 | 9.8146e-08 | 2.9944 | 1.0509e-07 | 3.2168 |
| 0.0125 | 9.7675e-09 | 2.9516 | 1.2211e-08 | 3.0067 | 1.1657e-08 | 3.1724 |

5. Conclusions

Sometimes the tempered power-law diffusions, instead of pure power-law diffusions, are more physical/reasonable choice in practical applications. This paper focuses on providing the quasi-compact schemes for the tempered fractional diffusion equations. Not only its derivation but the proof of its numerical stability and convergence are much
Appendix A. Some Lemmas

Define

\[ aI_x^{(p, \lambda)} u(x) = \frac{e^{-\lambda x}}{\Gamma(p)} \int_0^x (x - s)^{p-1} e^{\lambda s} u(s) ds, \]

\[ sI_x^{(p, \lambda)} u(x) = \frac{e^{\lambda x}}{\Gamma(p)} \int_x^b (s - x)^{p-1} e^{-\lambda s} u(s) ds, \]

\[ aD_x^{(p, \lambda)} u(x) = \frac{e^{-\lambda x}}{\Gamma(m-p)} \int_x^b (s - x)^{m-p-1} e^{\lambda s} u(s) ds, \]

\[ sD_x^{(p, \lambda)} u(x) = \frac{(-1)^m e^{\lambda x}}{\Gamma(m-p)} \int_x^b (s - x)^{m-p-1} e^{-\lambda s} u(s) ds, \]

where \( p \) is a positive constant and \( m - 1 < p < m \). Now we show some properties of the tempered fractional calculus.

**Lemma Appendix A.1.** Let \( u(x) \) be continuous on \([a, b]\), \( p, q > 0 \) and \( \lambda > 0 \). Then the integration of arbitrary real order has the properties:

\[ aI_x^{(p, \lambda)} (aI_x^{(q, \lambda)} u(x)) = aI_x^{(p+q, \lambda)} (aI_x^{(q, \lambda)} u(x)) \quad \text{(A.2)} \]

and

\[ sI_x^{(p, \lambda)} (sI_x^{(q, \lambda)} u(x)) = sI_x^{(p+q, \lambda)} (sI_x^{(q, \lambda)} u(x)) \quad \text{(A.3)} \]

**Proof.** Taking into account the definition of the integral \( sI_x^{(p, \lambda)} u(x) \), we have

\[ sI_x^{(p, \lambda)} (sI_x^{(q, \lambda)} u(x)) = \frac{e^{\lambda x}}{\Gamma(p)} \int_x^b (s - x)^{p-1} e^{-\lambda s} sI_x^{(q, \lambda)} u(s) ds = \frac{e^{\lambda x}}{\Gamma(p)} \int_x^b (s - x)^{p-1} e^{-\lambda s} \frac{e^{\lambda \eta}}{\Gamma(q)} \int_x^\eta (\eta - s)^{q-1} e^{-\lambda \eta} u(\eta) d\eta ds \]

\[ = \frac{e^{\lambda x}}{\Gamma(p)\Gamma(q)} \int_x^b e^{-\lambda \eta} u(\eta) d\eta \int_0^\eta (\eta - x)^{q-1} (\eta - s)^{p-1} ds. \]

Here we use the substitution \( s = x + \zeta(\eta - x) \) to evaluate the integration from \( x \) to \( \eta \), and obtain

\[ \int_x^{\eta} (s - x)^{p-1} (\eta - s)^{p-1} ds = \int_0^{\eta} (\zeta(\eta - x))^{p-1} ((1 - \zeta)(\eta - x))^{p-1} d\zeta = (\eta - x)^{p+q-1} B(p, q) \]

\[ = (\eta - x)^{p+q-1} \frac{\Gamma(p)\Gamma(p)}{\Gamma(p + q)}, \]

where \( B(p, q) \) is the Beta function. Therefore,

\[ sI_x^{(p, \lambda)} (sI_x^{(q, \lambda)} u(x)) = \frac{e^{\lambda x}}{\Gamma(p + q)} \int_x^b e^{-\lambda \eta} (\eta - x)^{p+q-1} u(\eta) d\eta = aI_x^{(p+q, \lambda)} u(x). \]

Obviously, \( p \) and \( q \) can be interchanged. Then

\[ sI_x^{(p, \lambda)} (sI_x^{(q, \lambda)} u(x)) = sI_x^{(p+q, \lambda)} u(x) = sI_x^{(q, \lambda)} (sI_x^{(p, \lambda)} u(x)). \]

The proof of (A.2) is omitted here.
Lemma Appendix  A.2. Let $k, m$ be positive integers, $m - 1 < p < m$, $\lambda > 0$ and $u(x) \in C^{m+k-1}[a, b]$. Then we have

1. $aD_x^{(k,l)}(aD_x^{(p,l)}u(x)) = aD_x^{(k+p,l)}u(x),$

2. $bD_x^{(k,l)}(bD_x^{(p,l)}u(x)) = bD_x^{(k+p,l)}u(x),$

3. $aD_x^{(p,l)}(aD_x^{(k,l)}u(x)) = aD_x^{(k+p,l)}u(x) - \sum_{j=0}^{k-1} \frac{(-1)^j \Gamma(k-j)}{\Gamma(k+1)} (x-a)^{j-p}.$

4. $bD_x^{(p,l)}(bD_x^{(k,l)}u(x)) = bD_x^{(k+p,l)}u(x) - \sum_{j=0}^{k-1} \frac{(-1)^j \Gamma(k-j)}{\Gamma(k+1)} (b-x)^{j-p}.$

Proof. The proofs of 1 and 2:

From (A.4), we get

$$aD_x^{(k,l)}(aD_x^{(p,l)}u(x)) = e^{-\lambda x} \frac{d^k}{dx^k} \left( e^{\lambda x} aD_x^{(p,l)}u(x) \right),$$

$$= e^{-\lambda x} \frac{d^k}{dx^k} \left( e^{\lambda x} \frac{\Gamma(m-p)}{\Gamma(m)} \int_a^x (x-s)^{m-p-1} e^{\lambda s} u(s)ds \right),$$

$$= \frac{e^{-\lambda x}}{\Gamma(m-p)} \frac{d^k}{dx^k} \left( \int_a^x (x-s)^{m-p-1} e^{\lambda s} u(s)ds \right),$$

$$= \frac{e^{-\lambda x}}{\Gamma(m)} \frac{d^{k+m}}{dx^{k+m}} \left( \int_a^x (x-s)^m e^{\lambda s} u(s)ds \right),$$

and

$$xD_x^{(k,l)}(bD_x^{(p,l)}u(x)) = (1)^{k+p} \frac{d^k}{dx^k} \left( e^{\lambda x} (-1)^{m-p} \frac{\Gamma(m+1)}{\Gamma(m)} \int_a^x (s-x)^{m-p-1} e^{-\lambda s} u(s)ds \right),$$

$$= \frac{(-1)^{k+m} e^{\lambda x}}{\Gamma(m+1)} \frac{d^{k+m}}{dx^{k+m}} \left( \int_a^x (s-x)^m e^{-\lambda s} u(s)ds \right),$$

$$= xD_x^{(k+p,l)}u(x).$$

The proof of 3:

To consider the operator $aD_x^{(p,l)}(aD_x^{(k,l)}u(x))$, we must mention that

$$aD_x^{(k,l)}(aD_x^{(p,l)}u(x)) = \frac{e^{-\lambda x}}{(k-1)!} \int_a^x (x-s)^{k-1} e^{\lambda s} aD_x^{(k,l)}u(s)ds,$$

$$= \frac{e^{-\lambda x}}{(k-1)!} \int_a^x (x-s)^{k-1} \frac{d^{k-1}}{ds^{k-1}} e^{\lambda s} u(s)ds,$$

$$= u(x) + \sum_{j=0}^{k-1} \frac{(e^{\lambda x} u(x))^j}{\Gamma(j+1)}.$$
where \((e^{x_0}u(a))^{(j)}\) is the differentiation of order \(j\) of the function \(e^{x_0}u(x)\) at point \(x = a\);

\[
a D^{(p, a)}_x u(x) = \frac{e^{-\lambda x}}{\Gamma(m - p)} \frac{d^m}{dx^m} \int_a^x (x - s)^{m-p-1} e^{\lambda s} u(s) ds
\]

\[
= e^{-\lambda x} \frac{d^m}{dx^m} e^{\lambda s} \frac{e^{-\lambda x}}{\Gamma(m - p)} \int_a^x (x - s)^{m-p-1} e^{\lambda s} u(s) ds
\]

\[
= a D^{(m, a)}_x \left( a J^{(m-p, a)}_x u(x) \right)
\]

(A.5)

and

\[
a D^{(p+k, a)}_x \left( a J^{(k, a)}_x u(x) \right) = a D^{(m+k, a)}_x \left( a I^{(m-p, a)}_x u(x) \right) = a D^{(m+k, a)}_x \left( a J^{(k, a)}_x u(x) \right) = a D^{(p, a)}_x u(x).
\]

(A.6)

Using (A.4), (A.5), (A.6) and

\[
a D^{(p, a)}_x (e^{-\lambda x}(x - a)^j) = \frac{\Gamma(1 + j)}{\Gamma(1 + j - p)} (x - a)^{j-p},
\]

we obtain

\[
a D^{(p, a)}_x \left( a D^{(p, a)}_x u(x) \right) = a D^{(p+k, a)}_x \left( a J^{(k, a)}_x u(x) \right)
\]

\[
= a D^{(p+k, a)}_x \left( u(x) - \sum_{j=0}^{k-1} \frac{(e^{x_0}u(a))^{(j)}}{\Gamma(j+1)} e^{\lambda x} (x - a)^j \right)
\]

\[
= a D^{(p+k, a)}_x u(x) - \sum_{j=0}^{k-1} \frac{(e^{x_0}u(a))^{(j)}}{\Gamma(j+1)} e^{\lambda x} (x - a)^{j+p-k}.
\]

The proof of 4:

Since

\[
x D^{(p, a)}_x \left( a D^{(k, a)}_b u(x) \right) = \frac{(-1)^j e^{\lambda x}}{(k-1)!} \int_a^x (s - x)^{k-1} \frac{d^k}{ds^k} (e^{x_0}u(s)) ds
\]

\[
= u(x) - \sum_{j=0}^{k-1} \frac{(-1)^j \frac{d^j}{d^{k-1}} (e^{x_0}u(b))^{(j)}}{\Gamma(j+1)} (x - b)^j,
\]

\[
x D^{(p, a)}_x u(x) = x D^{(m, a)}_b \left( x J^{(m-p, a)}_b u(x) \right), \quad x D^{(p+k, a)}_x \left( x J^{(k, a)}_x u(x) \right) = x D^{(p, a)}_x u(x)
\]

and

\[
x D^{(p, a)}_x (e^{\lambda x}(b - x))^j = \frac{\Gamma(1 + j)}{\Gamma(1 + j - p)} (b - x)^{j-p},
\]

we obtain

\[
x D^{(p, a)}_x \left( x J^{(k, a)}_x u(x) \right) = x D^{(p+k, a)}_x \left( x J^{(k, a)}_x u(x) \right)
\]

\[
= x D^{(p+k, a)}_x \left( u(x) - \sum_{j=0}^{k-1} \frac{(e^{x_0}u(b))^{(j)}}{\Gamma(j+1)} e^{\lambda x} (b - x)^j \right)
\]

\[
= x D^{(p+k, a)}_x u(x) - \sum_{j=0}^{k-1} \frac{(e^{x_0}u(b))^{(j)}}{\Gamma(j+1)} e^{\lambda x} (b - x)^{j+p-k}.
\]

\[
\square
\]

Lemma Appendix A.3. Let \(k, m\) be positive integers, \(m - 1 < p < m, \lambda > 0\) and \(u(x)\) be \((m+k-1)\)-times continuously differentiable on \((-\infty, +\infty)\). Then we have

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\[ 1. \quad -\infty D_x^{(k,\lambda)} \left( -\infty D_x^{(p,\lambda)} u(x) \right) = -\infty D_x^{(k+p,\lambda)} u(x), \]
\[ 2. \quad D_x^{(k,\lambda)} \left( D_x^{(p,\lambda)} u(x) \right) = D_x^{(k+p,\lambda)} u(x). \]

References

[1] R. Metzler, J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach, *Phys. Rep.*, 339 (2000) 1-77.
[2] A. Compte, Stochastic foundations of fractional dynamics, *Phys. Rev. E*, 53(4) (1996) 4191-4193.
[3] R.N. Mantegna, H.E. Stanley, Stochastic process with ultraslow convergence to a Gaussian: The truncated Lévy flight, *Phys. Rev. Lett.*, 73 (1994) 2946-2949.
[4] I. Koponen, Analytic approach to the problem of convergence of truncated Lévy flights towards the Gaussian stochastic process, *Phys. Rev. E*, 52(1) (1995) 1197-1199.
[5] F. Sabzikar, M.M. Meerschaert, J. Chen, Tempered fractional calculus, *J. Comput. Phys.*, 293 (2015) 14-28.
[6] M.M. Meerschaert, Y. Zhang, B. Baeumer, Tempered anomalous diffusion in heterogeneous systems, Geophys. Res. Lett., 35 (2009) L17403.
[7] A. Cartea, D. del-Castillo-Negrete, Fluid limit of the continuous-time random walk with general Lévy jump distribution functions, *Phys. Rev. E*, 76(4) (2007) 041105.
[8] D. del-Castillo-Negrete, Truncation effects in superdiffusive front propagation with Lévy flights, *Phys. Rev. E*, 79(3) (2009) 031120.
[9] M.M. Meerschaert, C. Tadjeran, Finite difference approximations for two-sided space-fractional partial differential equations, *Appl. Numer. Math.*, 56(1) (2006) 80-90.
[10] G. Gao, Z. Sun, A compact finite difference scheme for the fractional sub-diffusion equations, *J. Comput. Phys.*, 230 (2011) 586-595.
[11] C. Šelikov, M. Duman, Crank-Nicolson method for the fractional diffusion equation with the Riesz fractional derivative, *J. Comput. Phys.*, 231(4) (2012) 1743-1750.
[12] C.M. Chen, F. Liu, V. Anh, I. Turner, Numerical schemes with high spatial accuracy for a variable-order anomalous subdiffusion equation, *Adv. Comput. Math.*, 32 (2010) 1740-1760.
[13] I. Podlubny, Fractional differential equations, Academic Press, New York, 1999.
[14] C. Tadjeran, M.M. Meerschaert, H.P. Scheffler, A second-order accurate numerical approximation for the fractional diffusion equation, *J. Comput. Phys.*, 213(1) (2006) 205-213.
[15] C. Tadjeran, M.M. Meerschaert, Finite difference methods for two-dimensional fractional dispersion equation, *J. Comput. Phys.*, 220(2) (2007) 813-823.
[16] H. Nasir, B.K. Gunawardana, H.M.N.P. Abeyrathna, A second order finite difference approximation for the fractional diffusion equation, *Int. J. Appl. Phys. Math.*, 3 (2013) 237-243.
[17] W.Y. Tian, H. Zhou, W.H. Deng, A class of second order difference approximation for solving space fractional diffusion equations, *Math. Comp.*, 84 (2015), 1703-1727.
[18] H. Zhou, W.Y. Tian, W.H. Deng, Quasi-compact finite difference schemes for space fractional diffusion equations, *J. Sci. Comput.*, 56(1) (2013) 45-66.
[19] M.H. Chen, W.H. Deng, High order algorithms for the fractional substantial diffusion equation with truncated Lévy flights, *SIAM J. Sci. Comput.*, 37(2) (2015) 890-917.
[20] Y.Y. Yu, W.H. Deng, Y.J. Wu, High order quasi-compact difference schemes for space fractional diffusion equations, *Commun. Math. Sci.*, (2015) in press (arXiv:1408.6364).
[21] B. Baeumer, M.M. Meerschaert, Tempered stable Lévy motion and transient super-diffusion, *J. Comput. Appl. Math.*, 233 (2010) 2438-2448.
[22] J. Gajda, M. Magdziarz, Fractional Fokker-Planck equation with tempered α-stable waiting times: Langevin picture and computer simulation, *Phys. Rev. E*, 82(1) (2010) 011117.
[23] C. Li, W.H. Deng, High order schemes for the tempered fractional diffusion equations, *Adv. Comput. Math.*, (2015), doi:10.1007/s10444-015-9434-z.
[24] R. Bhatia, Positive definite matrices, Princeton University Press, 2009.
[25] R.H.F. Chan, X.Q. Jin, An introduction to iterative Toeplitz solvers, SIAM, 2007.