ON THE DIMENSION OF THE SUBFIELD SUBCODES OF 1-POINT HERMITIAN CODES
SABIRA EL KHALFAOUI AND GÁBOR P. NAGY

Abstract. Subfield subcodes of algebraic-geometric codes are good candidates for the use in post-quantum cryptosystems, provided their true parameters such as dimension and minimum distance can be determined. In this paper we present new values of the true dimension of subfield subcodes of 1–point Hermitian codes, including the case when the subfield is not binary.

1. Introduction

The oldest and best known proposal for post-quantum cryptography schemes are the cryptosystems due to McEliece and Niederreiter. Their security is based on the NP-completeness of the decoding of binary linear codes. Hence, an essential ingredient of their schemes is a binary linear code $C$ which has an efficient decoding algorithm and which cannot be distinguished from the random linear code. McEliece originally proposed the class of extended binary Goppa codes, which are subfield subcodes of the generalized Reed-Solomon codes. For background see [11, 18, 19] and on McEliece schemes [25].

Recently, some other classes of codes have been proposed as well, such as LDPC codes and algebraic-geometric codes over larger fields. However, these classes turned out to have serious security flaws, see [10, 24] and on McEliece schemes [25].

The Berlekamp-Massey algorithm and its variants provide an efficient decoding for Reed-Solomon codes, which can be used to decode subfield subcodes of generalized Reed-Solomon codes, as well. For the binary linear code $C$ in use, the error correcting bound $t$ is determined by these algorithms. Beyond this bound, list-decoding methods are known, cf. [7, 3, 1, 12]. Therefore, it is an important problem to find the true minimum distance of subfield subcodes of generalized Reed-Solomon codes. It is equally important (and difficult) to find the true dimension of such codes, cf. [6] and the series of papers [21, 22, 23].

The class of algebraic-geometry (AG) codes was introduced by V.D. Goppa. This class is a natural generalization of Reed-Solomon codes. The famous Riemann-Roch Theorem provides theoretical bounds for the dimension and minimum distance of AG codes. The ideas of the Berlekamp-Massey algorithm can be used to design efficient decoding algorithms up to the half of the designed minimum distance of AG codes, and beyond [14, 8, 17]. Hence, the subfield subcodes of AG codes are also good candidates for the McEliece and Niederreiter cryptosystems. The determination of the true dimension and the true minimum distance of the subfield subcodes of
AG codes seems to be a hard problem, the attempts so far focused mainly at 1-point Hermitian codes and their subcodes, with some further restrictions on the parameters [20, 15].

In this paper, we prove new results on the true dimension of the subfield subcodes of 1-point Hermitian codes. Our approach deals also with non-binary subfields. The paper is structured as follows. In section 2, we describe the backgrounds with some important properties of algebraic plane curves, function fields and Riemann-Roch spaces, in order to define AG codes. In section 3, we present AG codes, and we give a brief introduction to AG codes over a genus zero curve, which can be considered as generalization of Reed-Solomon codes. Section 4 summarizes the definition of the subfield subcodes of AG codes and techniques used to improve their dimensions, where the famous Delšarte’s result is one of them. In section 5, we present the class of Hermitian curves and their basic properties. Section 6 is dedicated to prove our result concerning the true dimension of the subfield subcodes of 1-point Hermitian codes for specific parameters.

2. Algebraic curves, divisors and Riemann-Roch spaces

2.1. Algebraic curves, places, divisors. Let \( f(X, Y) \in K[X, Y] \) be a polynomial of degree \( n \), the associated algebraic plane curve \( \mathcal{X} \) over the field \( K \) is denoted by \( \mathcal{X} : f(X, Y) = 0 \). The affine points of \( \mathcal{X} \) are defined by the set:

\[
\{(x, y) \in L^2 \mid f(x, y) = 0\}
\]

where \( L \) is the extension of the field \( K \). A point \((x, y)\) is said to be smooth if \((\frac{\partial f}{\partial X}(x, y), \frac{\partial f}{\partial Y}(x, y)) \neq (0, 0)\). \( \mathcal{X} \) is a smooth or nonsingular curve if all its points are smooth, which implies that \( f \) is absolutely irreducible.

The homogenous equation of the affine curve \( \mathcal{X} : f(X, Y) = 0 \) is \( F(X, Y, Z) = 0 \) with \( F(X, Y, Z) = Z^n f\left(\frac{X}{Z}, \frac{Y}{Z}\right) \). The projective points are defined by vanishing the homogenous polynomial. In particular, the affine point \((x, y)\) of \( \mathcal{X} \) is represented by the point \((x: y: 1)\) in projective coordinates. The points of \( \mathcal{X} \) at infinity are given by the homogeneous equation \( F(X, Y, 0) = 0 \).

We use the notion of place instead of a point for a smooth algebraic curve defined over an algebraic closure of \( K \). A divisor is a formal sum \( D = n_1P_1 + \ldots + n_kP_k \) with integers \( n_1, \ldots, n_k \) and places \( P_1, \ldots, P_k \). The degree of \( D \) is \( n_1 + \cdots + n_k \). The integer \( n_i \) is the valuation \( v_{P_i}(D) \) of \( D \) at \( P_i \); for \( P \neq P_i \) one has \( v_P(D) = 0 \). The support of \( D \) is the set of places \( P \) such that \( v_P(D) \neq 0 \).

2.2. Function fields and Riemann-Roch spaces. Let \( \mathcal{X} : f(X, Y) = 0 \) be a smooth plane algebraic curve. The function field \( K(\mathcal{X}) \) of \( \mathcal{X} \) is generated by the elements \( x, y \) that satisfy the algebraic relation \( f(x, y) = 0 \).

For every non-zero function \( h \in K(\mathcal{X}) \), \( \text{Div}(h) \) is known as the principal divisor associated with \( h \) while \( \text{Div}(h)_0 \) is said to be the divisor of zeros of \( h \), and \( \text{Div}(h)_\infty \) is the divisor of its poles.

Furthermore, for every separable function \( h \in K(\mathcal{X}) \), \( dh \) is the exact differential arising from \( h \), and \( \Omega \) denotes the set of all these differentials. Also, \( \text{res}_P(dh) \) is the residue of \( dh \) at a place of \( P \) of \( K(\mathcal{X}) \).

For any divisor \( A \) of \( K(\mathcal{X}) \), the Riemann-Roch space of \( A \) is

\[
\mathcal{L}(A) = \{h \in K(\mathcal{X}) \setminus \{0\} \mid \text{Div}(h) \geq -A \} \cup \{0\}.
\]
We denote $\ell(A) = \dim(\mathcal{L}(A))$. Furthermore, the differential space of $A$ is
$$\Omega(A) = \{ dh \in \Omega \mid \text{Div}(dh) \supseteq A \} \cup \{0\}.$$  

For a divisor $A$, the index of specialty of $A$ is the integer
$$i(A) = \ell(A) - \deg A + g - 1.$$  

Both the Riemann-Roch and the differential spaces are linear spaces over $K$. Their dimensions are given by the theorem of Riemann-Roch:
$$\ell(A) = \deg(A) + 1 - g + \ell(W - A).$$  

Here, $W$ is a canonical divisor of $X$, and $g$ is the genus of $X$. The latter is the most important birational invariant of an algebraic curve. For smooth curves of degree $n$, the genus formula is
$$g = \frac{(n-1)(n-2)}{2}.$$  

The theorem of Riemann-Roch implies
$$\ell(A) \geq \deg(A) + 1 - g,$$
with equality if $\deg(A) > 2g - 2$.

Notice that if $X$ is a line or an irreducible conic then it has genus 0. Conversely, $g = 0$ implies the function field of $X$ to be the field of rational functions, if $X$ has at least one degree one place. The latter condition is always true for curves over finite fields ([19, Remark 1.6.4]).

2.3. Algebraic plane curves over finite fields. Let $q$ be a prime power of $p$, and $\overline{\mathbb{F}}_q$ the algebraic closure of $\mathbb{F}_q$. We denote by $\text{Frob}_q$ the Frobenius automorphism $x \mapsto x^q$ of $\overline{\mathbb{F}}_q$. The action of $\text{Frob}_q$ can be extended to an action on the coefficients of $\mathbb{F}_q$-polynomials and on the coordinates of an affine and projective points over $\overline{\mathbb{F}}_q$.

$X$ is $\mathbb{F}_q$-rational curve, if it is invariant under the action of $\text{Frob}_q$. Similarly, $\mathbb{F}_q$-rational places and divisors are invariant under the action of the Frobenius automorphism.

3. Algebraic geometry codes (AG codes)

Algebraic geometry codes are a type of linear error correcting block codes, introduced by D.V. Goppa [19]. The construction of the codes relies on choosing an irreducible smooth curve over a finite field $\mathbb{F}_q$ (e.g. Hermitian curves, elliptic curves, hyperelliptic curves, etc.). Evaluating a suitable linear space as the Riemann-Roch space $\mathcal{L}(G)$ of functions on the points of the given curve, disjoint from $\text{supp} \ G$.

Now, we consider a smooth plane curve over $\overline{\mathbb{F}}_q$. We set a divisor $D = P_1 + \ldots + P_n$ where all $P_i$ are pairwise distinct places of $\mathbb{F}_q(X)$ of degree 1. $G$ is another divisor in $\mathbb{F}_q(X)$ such that $\text{supp} \ G \cap \text{supp} \ D = \emptyset$.

Definition 3.1. The algebraic geometry code $C_{\mathcal{L}}(D, G)$ associated with the divisors $D$ and $G$ is defined as
$$C_{\mathcal{L}}(D, G) = \{(f(P_1), \ldots, f(P_n)) \mid f \in \mathcal{L}(G)\} \subseteq \overline{\mathbb{F}}_q^n.$$  

In other words, $C_{\mathcal{L}}(D, G)$ is the image of $\mathcal{L}(G)$ under the evaluation map
$$\mathcal{L}(G) \ni f \mapsto (f(P_1), \ldots, f(P_n)) \in \overline{\mathbb{F}}_q^n.$$
Indeed, determining the functions field and the divisors in a pertinent way can make AG codes viewed as RS codes (see Section 4). AG codes are fascinating, since we can compute their parameters \( n, k \) and positive true minimum distance. However, this case cannot be described by the Riemann-Roch theorem.

**Theorem 3.2.** \( C_L(D, G) \) is a \([n, k, d]\) codes with parameters:

- \( k = \ell(G) - \ell(G - D) \) where \( \ell(G) = \dim \mathcal{L}(G) \),
- \( d \geq n - \deg G \).

Notice that the condition \( n > \deg G \) implies the evaluation map \( \mathcal{L}(G) \to \mathbb{F}_q^n \) to be injective. If \( n \leq \deg G \), then it is possible that \( C_L(D, G) \) has dimension less than \( n \) and positive true minimum distance. However, this case cannot be described by the Riemann-Roch theorem.

Instead of using functions in \( \mathcal{L}(G) \), it is possible to employ the exact differential \( dh \) to define another algebraic geometry code with the divisors \( D \) and \( G \), which is the dual of \( C_L(D, G) \).

**Definition 3.3.** Let \( G \) and \( D \) be divisors as before. Then we define the differential code \( C_\Omega(D, G) \subseteq \mathbb{F}_q \) by

\[
C_\Omega(D, G) = \left\{ (\text{res}_{P_1}(dh), \text{res}_{P_2}(dh), ..., \text{res}_{P_n}(dh)) \mid dh \in \Omega_{\mathbb{F}_q(x)}(G - D) \right\}.
\]

**Theorem 3.4** (Stichtenoth [19]). \( C_\Omega(D, G) \) is an \([n', k', d']\) code with parameters:

- \( k' = i(G - D) - i(G) \) and \( d' \geq \deg G - (2G - 2) \).

If \( \deg G > 2g - 2 \), we have \( k' = i(G - D) \geq n + g - 1 - 1 \deg G \). If moreover \( 2g - 2 < \deg G < n \) then \( k' = n + g - 1 - \deg G \).

**Theorem 3.5** (Stichtenoth [19]). The dual of the code \( C_L(D, G) \) is the code \( C_\Omega(D, G) \)

\[
C_\Omega(D, G) = C_L(D, G)^\perp.
\]

### 3.1. Genus zero AG codes.

The class of genus zero codes is well-known by the name of generalized Reed-Solomon codes. BCH codes and Goppa codes are part of the most important codes used in practice, which can be described as subfield subcodes of generalized Reed-Solomon codes [19 Chapter 2]. Notice that some authors call genus zero AG codes rational, since the corresponding function field is the field of rational functions. In this paper, we reserve the adjective rational for objects which are invariant under the action of the Frobenius automorphism.

**Definition 3.6** (Genus zero AG codes). An AG code \( C_L(D, G) \) associated with divisors \( G \) and \( D \) over an algebraic curve of genus 0 \( \mathbb{F}_q(x) \) is said to be genus zero code.

The length of a genus zero code over \( \mathbb{F}_q \) is bounded by \( q + 1 \), since the genus zero curve is represented by a projective line. The associated function field can be identified with the field of rational functions in one indeterminate \( \mathbb{F}_q(z) \). This function field has \( q + 1 \) places of degree one. The pole \( P_\infty \) of \( z \) and for each \( \alpha \in \mathbb{F}_q \), the zero \( P_\alpha \) of \( z - \alpha \).

**Definition 3.7.** Let \( q \) be a prime power, \( n \) an integers with \( 2 \leq n \leq q \). Let \( \alpha = \{\alpha_1, ..., \alpha_n\} \subseteq \mathbb{F}_q \), \( v = (v_1, ..., v_n) \) a nonzero vector of \( \mathbb{F}_q \). The generalized
Reed-Solomon code is an \([n, k, d]\) linear code which is denoted by \(\text{GRS}_k(\alpha, v)\). For \(F(z) \in \mathcal{L}_k = \{ f(z) \in \mathbb{F}_q[z] \mid \deg(f) < k \}\), \(\text{GRS}_k(\alpha, v)\) consists of all vectors 
\( (v_1 F(\alpha_1), v_2 F(\alpha_2), \ldots, v_n F(\alpha_n)) \). 

Stichtenoth showed in [19] Proposition 2.3.5 how generalized Reed-Solomon code can be viewed as genus zero AG code. It has been studied in several papers, in order to construct their subfield subcodes with good parameters. This construction has used some strategies to improve their dimension and minimum distance, which we describe in the following section.

4. Subfield subcodes of linear codes

Let \(\mathbb{F}_q\) be the extension field of \(\mathbb{F}_r\) where \(q\) is a prime power such that \(q = r^m\). The subfield subcode \(C|\mathbb{F}_r\) of any linear \([n, k, d]\) code \(C\) over \(\mathbb{F}_q\) is defined as follows:
\[ C|\mathbb{F}_r = C \cap \mathbb{F}_r^m. \]

This is a linear \([n, k_0, d_0]\) code with \(d \leq d_0 \leq n\) and \(n - k \leq n - k_0 \leq m(n - k)\). In other words,
\[ k_0 \geq n - m(n - k). \]

Any parity check matrix of \(C\) over \(\mathbb{F}_q\) yields at most \(m(n - k)\) linearly independent parity equations over \(\mathbb{F}_r\) for the subfield subcodes \(C|\mathbb{F}_r\).

In general the true minimum distance of a subfield subcodes is bigger than the minimum distance of the original code, which makes the subfield subcodes the most important in the binary case \(r = 2\) [8, Theorem 4].

4.1. Trace codes. As above let \(q = r^m\) a prime power. One defines the trace polynomial \(\text{Tr}(X) \in \mathbb{F}_r[x]\) with respect to \(\mathbb{F}_q\) as follows:
\[ \text{Tr}_{\mathbb{F}_q/\mathbb{F}_r}(x) = x + x^r + \ldots + x^{r^{m-1}}. \]

The trace polynomial determines the \(\mathbb{F}_r\)-linear trace map \(\mathbb{F}_q \to \mathbb{F}_r\). For any linear \((n, k, d)\) code \(C\) over \(\mathbb{F}_q\), Delsarte defined the trace code \(\text{Tr}(C) = \text{Tr}_{\mathbb{F}_q/\mathbb{F}_r}(C)\) by
\[ \text{Tr}(C) = \{ \text{Tr}_{\mathbb{F}_q/\mathbb{F}_r}(c_1), \ldots, \text{Tr}_{\mathbb{F}_q/\mathbb{F}_r}(c_n) \mid c_i \in C \}. \]

\(\text{Tr}(C)\) is a linear \((n, k_1, d_1)\) code over \(\mathbb{F}_r\), with \(1 \leq d_1 \leq d\) and \(k \leq k_1 \leq mk\). As for subfield subcodes, the most important case is when \(r = 2\).

The following important result by Delsarte related the class of subfield subcodes to trace codes:

**Theorem 4.1** (Delsarte [6]). Let \(C\) be any linear code over an extension field \(\mathbb{F}_q\) of \(\mathbb{F}_r\). Then \((C|\mathbb{F}_r)^\perp = \text{Tr}(C^\perp)\) holds.

In [23], Vron observed that Delsarte’s theorem can be used to improve the general bound (11) on the dimension of subfield subcodes:
\[ k_0 = n - m(n - k) + \dim_{\mathbb{F}_r}(\text{Tr}). \]  

In [16], the authors stated three conjectures on bounds for the true dimension of Goppa codes. Vron proved all three of them in a series of papers [21, 22, 23]. He used different parameters and techniques to reveal that the general bound \(k \geq n - m \deg g(z)\) can be exceeded. In [24] the author has summarized the strategies used to reach new bounds for the true dimension of Goppa codes. The first strategy is
about the link between the parity check matrices $H$ and $\tilde{H}$, of the generalized Reed-Solomon codes and the Goppa codes. The idea is searching for some polynomials and a special basis, when computing $\tilde{H}$ from $H$, in order to find linearly dependent rows. The second strategy used [2] to obtain the formula

$$k_0 = n - m \deg g(z) + \dim_{\mathbb{F}_2} \ker(\text{Tr}),$$

for the dimension of the binary Goppa codes. The dimension of the kernel of the trace map $\text{Tr}$ is equal to the number of solutions of a modular equation referred to as the redundancy equation [23].

For certain choices of the polynomial $g(z)$, permuted binary Goppa codes are believed to be indistinguishable from a binary random codes [10]. That makes some binary Goppa codes a good candidate for McEliece cryptosystem, which is the first public key cryptosystem based on error-correcting codes introduced in 1978.

Niederrieter cryptosystem is a variant of McEliece cryptosystem proposed in 1986 by Harald Niederreiter. It uses the same idea of McEliece cryptosystem to the parity check matrix of a linear code. In the existence of a quantum computer, the most known public key cryptosystems such as RSA or ECC will be broken. However McEliece is a post-quantum cryptosystem that is thought to be secure against a post-quantum computer attack.

As an application of McEliece cryptosystem, the authors of [5] describe a digital signature scheme based on Niederreiter scheme.

5. Hermitian curves over finite fields

The projective Hermitian curve over $\mathbb{F}_{q^2}$ has the form $X^{q+1} - YZ - YZ^q = 0$ with homogenous coordinates in the projective plane over $\mathbb{F}_q$. It is equivalent to the Hermitian curve $\mathcal{H}_{q^2}$ over $\mathbb{F}_{q^2}$ in affine coordinates with the equation:

$$\mathcal{H}_{q^2} : Y^q + Y = X^{q+1}.$$  

It is easy to verify that $\mathcal{H}_q$ is non-singular, thus by the genus formula, the genus of $\mathcal{H}_q$ is $g = \frac{q(q-1)}{2}$.

$\mathcal{H}_q$ has $q^3 \mathbb{F}_{q^2}$-rational points of the form $P = (x, y)$ plus the $\mathbb{F}_{q^2}$-rational point $(0 : 1 : 0)$ at infinity denoted by $P_\infty$. Hermitian curves are an important class of curves, since they attain the maximal number of rational points with respect to the famous Hasse-Weil bound [11].

Let $x^iy^j$ be a monomial in $\mathbb{F}_{q^2}[x, y]$. The weight of $x^iy^j$ is defined as $qi + (q+1)j$ and it is denoted by $\rho(x^iy^j)$. We define the set

$$\{x^iy^j | 0 \leq i \leq q^2 - 1, 0 \leq j \leq q - 1, \rho(x^iy^j) \leq s\}$$

which forms a basis of the Riemann-Roch space $\mathcal{L}(sP_\infty)$.

6. Hermitian codes

6.1. 1-point Hermitian codes. For $n = q^2$, let $P_1, P_2, ..., P_n$ be all the $q^3 \mathbb{F}_{q^2}$-rational points of $\mathcal{H}_q$ different from the point at infinity. We set $D = P_1 + P_2 + ... + P_n$. Goppa’s construction gives us a class of Hermitian codes denoted by $\mathcal{H}(q^2, s) = C_\mathcal{L}(D, sP_\infty)$ [15]. Here

$$\mathcal{H}(q^2, s) = \{(f(P_1), f(P_2), ..., f(P_n)) | f \in \mathcal{L}(sP_\infty)\}.$$
6.2. 1–point Hermitian codes: parameters and dual codes. The dimension $k$ of 1–point Hermitian codes $\mathcal{H}(q^2, s)$ is the dimension of $L(sP_\infty)$, which can be determined from Riemann–Roch Theorem [19]. $\mathcal{H}(q^2, s)$ has length $n = q^3$, if $2g - 2 < s < n$ then the dimension $k = s - g + 1$ and the minimum distance $d = q^3 - s$.

**Theorem 6.1** (Dual codes [11]). For $s \geq 0$ define $s = q^3 + q^2 - q - 2 - s$. The codes $\mathcal{H}(q^2, s)$ and $\mathcal{H}(q^2, s')$ are dual to each other.

In particular, if $q$ is even and $s = (q^3 + q^2 - q - 2)/2$, the code $\mathcal{H}(q^2, s)$ is self-dual.

**Definition 6.2.** Let $\mathcal{H}(q^2, s)$ be a 1–point Hermitian code, the subfield subcode of $\mathcal{H}(q^2, s)$ is:

$$C_{q,r}(s) = \mathcal{H}(q^2, s) |_F$$

In [15], the authors present an algorithm to compute $\dim C_{q,r}(s)$. Using this algorithm, the dimension of $C_{4,2}(s)$ is determined for each $s = 0, \ldots, 71$.

In [20] Proposition 3.2, the author shows

$$\text{Tr}(\mathcal{H}(q^2, q)) = 2m + 1,$$

where $q = 2^m$. In our notation, this means

$$\dim C_{q,r}(q^3 + q^2 - 2q - 2) = q^3 - (2m + 1).$$

In particular, $\dim C_{4,2}(70) = 59$, which is confirmed by [15 Table 2]. In the same table, we find $\dim C_{4,2}(s) = 1$ for $s = 0, \ldots, 31$ and $\dim C_{4,2}(32) = 5$. In the next section, we prove a formula which implies these dimensions.

7. MAIN RESULT

Recall that the notation $\mathcal{H}(q^2, s)$ stands for the AG code $C_E(D, sP_\infty)$, where $D$ is the sum of affine places of the Hermitian curve $X^{q+1} = Y + Y^q$ over the finite field $\mathbb{F}_{q^2}$.

**Theorem 7.1.** Let $C_{q,r}(s)$ be a subfield subcode of the Hermitian code $\mathcal{H}(q^2, s)$, $q = r^m$ is a prime power. Then

$$C_{q,r}(s) = \begin{cases} 1 & \text{for } s < \frac{q^3}{r} \\ 2m + 1 & \text{for } s = \frac{q^3}{r} \end{cases}$$

**Proof.** Since the constant polynomials are in $\mathcal{L}(sP_\infty)$ for all $s \geq 0$, we have $\dim C_{q,r}(s) \geq 1$. We first show that $\dim C_{q,r}(s) = 1$ for $s < \frac{q^3}{r}$. Fix an integer $0 < s < \frac{q^3}{r}$ and take an arbitrary element $(c_1, \ldots, c_{q^3}) \in C_{q,r}(s)$. Then there is an element $f \in \mathcal{L}(sP_\infty)$ such that for all $i = 1, \ldots, q^3$, one has $c_i = f(P_i) \in \mathbb{F}_r$. There is an element $\gamma \in \mathbb{F}_r$ such that $c_i = \gamma$ for at least $q^3/r$ indices $i$. In other words, $f - \gamma \in \mathcal{L}(sP_\infty)$ has at least $q^3/r$ zeros on the Hermitian curve $\mathcal{H}_q$. (In fact, a nonzero element of $\mathcal{L}(G)$ cannot have more than $\deg G$ zeros on the curve.) Therefore, $f - \gamma$ must be the constant zero polynomial, and $c_i = \gamma$ for all $i$. In particular, $C_{q,r}(s)$ consists of the constant vectors.

Now, we suppose that $s = q^3/r$. Recall that

$$\text{Tr}(X) = X + X^r + \cdots + X^{r^{2m-1}}$$

is the trace polynomial of $\mathbb{F}_{q^2}$ over $\mathbb{F}_r$. We define the polynomial

$$f_{d,a}(X) = d + \text{Tr}(aX)$$
where \( d \in \mathbb{F}_r, \alpha \in \mathbb{F}_{q^2} \). As a polynomial in one variable, \( f_{d,\alpha} \) maps \( \mathbb{F}_{q^2} \) to \( \mathbb{F}_r \). If the point \( P_i \) is given with affine coordinates \( P_i = (a_i, b_i) \), then \( f_{d,\alpha}(P_i) = f_{d,\alpha}(a_i) \in \mathbb{F}_r \) for all \( i = 1, \ldots, q^3 \). In other words, the evaluation vector

\[
c_{d,\alpha} = (f_{d,\alpha}(P_1), \ldots, f_{d,\alpha}(P_q^3)) \in \mathbb{F}_r^n.
\]

We claim that \( f_{d,\alpha}(x) \in \mathcal{L}\left(\frac{q^3}{r}P_\infty\right) \). In fact,

\[
\rho(x^r) = qr^k,
\]

which is at most \( qr^{2m-1} = q^3/r \) for \( k \leq 2m - 1 \). Hence, all monomials of \( f_{d,\alpha}(x) \) are in \( \mathcal{L}\left(\frac{q^3}{r}P_\infty\right) \), and the claim follows.

From the last two properties of \( f_{d,\alpha} \) follows that the evaluation vector \( c_{d,\alpha} \in C_{q,r}(q^3/r) \). Since the map \((d, \alpha) \mapsto c_{d,\alpha}\) is linear over \( \mathbb{F}_r \), and injective, we have \( \dim C_{q,r}(q^3/r) = 2m + 1 \).

In the last step we show that the elements \( c_{d,\alpha} \) exhaust the subfield subcode \( C_{q,r}(q^3/r) \).

Take an element \( g \in \mathcal{L}\left(\frac{q^3}{r}P_\infty\right) \) whose evaluation vector

\[
(g(P_1), \ldots, g(P_q^3)) \in \mathbb{F}_r^n.
\]

We can reduce the high degree \( y \)-terms by the Hermitian equation \( y^{q+1} = x + x^q \). Thus, we can write \( g \) in this form:

\[
g(x, y) = \sum_{j < q} a_{i,j} x^i y^j.
\]

Moreover, since \( \rho(x^i y^j) \equiv j \pmod{q} \), if \( j \leq q - 1 \) then the value \( \rho(x^i y^j) \) determines \( i \) and \( j \) uniquely. Therefore, each term of \( g = \sum_{j \leq q-1} a_{i,j} x^i y^j \) has a different \( \rho \)-value.

By definition we have

\[
\rho(x^i y^j) = v_{P_\infty}(x^i y^j) = i v_{P_\infty}(x) + j v_{P_\infty}(y) = q i + (q + 1) j.
\]

The valuation of \( g \) at \( P_\infty \) is

\[
v_{P_\infty}(g) = v_{P_\infty}\left(\sum_{j < q} a_{i,j} x^i y^j\right) = \max_{a_{i,j} \neq 0} (v_{P_\infty}(x^i y^j)).
\]

Equality holds since the \( \rho \)-values are different. If \( g \in \mathcal{L}\left(\left(\frac{q^3}{r} - 1\right)P_\infty\right) \) then \( g = f_{d,0} \) for some \( d \in \mathbb{F}_r \) as seen above. Assume now

\[
g \in \mathcal{L}\left(\frac{q^3}{r}P_\infty\right) \setminus \mathcal{L}\left(\left(\frac{q^3}{r} - 1\right)P_\infty\right).
\]

Then, \( v_{P_\infty}(g) = q^3/r \) and \( g \) has a unique term \( \beta x^{q^2} \) with \( \rho \)-value \( q^3/r \), \( \beta \in \mathbb{F}_{q^2}^* \). Define \( \alpha \in \mathbb{F}_{q^2} \) by \( \alpha^{2m-1} = \beta \). Then, \( g - f_{0,\alpha} \in \mathcal{L}\left(\frac{q^3}{r}P_\infty\right) \) and \( g - f_{0,\alpha} \) is again a constant \( d \in \mathbb{F}_r \). This means \( g = f_{d,\alpha} \), and the result follows. \( \square \)

Using similar methods, we can show that for any \( \alpha \in \mathbb{F}_{q^2}^* \),

\[
\text{Tr}(\alpha y) \in \mathcal{L}\left(\left(\frac{q + 1}{r}q^2\right)P_\infty\right).
\]
Hence, \( \dim C_{q,r}((q + 1)q^2/r) \geq 4m + 1 \). By [15, Table 2], we have equality for \( q = 4 \) and \( r = 2 \). Using our GAP package HERmitian [13], we computed the true dimension of \( C_{8,2}(s) \) for all values \( s \) from \( 256 = q^3/r \) to \( 512 = q^3 \), see Table 1 and 2.

| \( s \) | dime. of subcode | dim. of Hermitian code | designed min. dist. |
|------|------------------|------------------------|---------------------|
| 256  | 7                | 229                    | 256                 |
| 288  | 13               | 261                    | 224                 |
| 292  | 19               | 265                    | 220                 |
| 320  | 25               | 293                    | 192                 |
| 324  | 28               | 297                    | 188                 |
| 328  | 34               | 301                    | 184                 |
| 336  | 36               | 309                    | 176                 |
| 352  | 42               | 325                    | 160                 |
| 356  | 48               | 329                    | 156                 |
| 360  | 54               | 333                    | 152                 |
| 364  | 60               | 337                    | 148                 |
| 368  | 66               | 341                    | 144                 |
| 376  | 72               | 349                    | 136                 |
| 378  | 74               | 351                    | 134                 |
| 384  | 80               | 357                    | 128                 |
| 392  | 86               | 365                    | 120                 |
| 400  | 92               | 373                    | 112                 |
| 402  | 98               | 375                    | 110                 |
| 408  | 104              | 381                    | 104                 |
| 410  | 110              | 383                    | 102                 |
| 416  | 116              | 389                    | 96                  |
| 418  | 122              | 391                    | 94                  |
| 420  | 128              | 393                    | 92                  |
| 424  | 134              | 397                    | 88                  |
| 428  | 140              | 401                    | 84                  |
| 432  | 146              | 405                    | 80                  |
| 434  | 152              | 407                    | 78                  |
| 436  | 158              | 409                    | 76                  |
| 438  | 164              | 411                    | 74                  |
| 440  | 170              | 413                    | 72                  |
| 442  | 176              | 415                    | 70                  |
| 444  | 182              | 417                    | 68                  |
| 448  | 188              | 421                    | 64                  |

Table 1: Parameters of \( C_{8,2}(s) \) for \( s \in \{256, \ldots, 449\} \)
| $s$ | dim. of subcode | dim. of Hermitian code | designed min. dist. |
|-----|----------------|------------------------|---------------------|
| 450 | 194            | 423                    | 62                  |
| 452 | 200            | 425                    | 60                  |
| 456 | 206            | 429                    | 56                  |
| 457 | 212            | 430                    | 55                  |
| 458 | 218            | 431                    | 54                  |
| 460 | 224            | 433                    | 52                  |
| 462 | 226            | 435                    | 50                  |
| 464 | 232            | 437                    | 48                  |
| 466 | 238            | 439                    | 46                  |
| 468 | 244            | 441                    | 44                  |
| 470 | 250            | 443                    | 42                  |
| 472 | 256            | 445                    | 40                  |
| 473 | 262            | 446                    | 39                  |
| 474 | 268            | 447                    | 38                  |
| 475 | 274            | 448                    | 37                  |
| 480 | 280            | 453                    | 32                  |
| 482 | 286            | 455                    | 30                  |
| 484 | 292            | 457                    | 28                  |
| 486 | 295            | 459                    | 26                  |
| 488 | 301            | 461                    | 24                  |
| 489 | 307            | 462                    | 23                  |
| 490 | 313            | 463                    | 22                  |
| 491 | 319            | 464                    | 21                  |
| 492 | 325            | 465                    | 20                  |
| 493 | 331            | 466                    | 19                  |
| 496 | 337            | 469                    | 16                  |
| 498 | 343            | 471                    | 14                  |
| 500 | 349            | 473                    | 12                  |
| 502 | 355            | 475                    | 10                  |
| 504 | 361            | 477                    | 8                   |
| 505 | 367            | 478                    | 7                   |
| 506 | 373            | 479                    | 6                   |
| 507 | 379            | 480                    | 5                   |
| 508 | 385            | 481                    | 4                   |
| 509 | 391            | 482                    | 3                   |
| 510 | 397            | 483                    | 2                   |
| 511 | 403            | 484                    | 1                   |

Table 2: Parameters of $C_{8,2}(s)$ for $s \in \{450, \ldots, 512\}$

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Bolyai Institute, University of Szeged, Aradi vérteanuk tere 1, H-6720 Szeged, Hungary
E-mail address: sabira@math.u-szeged.hu

Department of Algebra, Budapest University of Technology and Economics, Egry József utca 1, H-1111 Budapest, Hungary
