A quadratic potential in a light cone QCD inspired model

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Abstract. The general equation from previous work is specialized to a quadratic potential \( V(r) = -a + \frac{f}{2} r^2 \) acting in the space of spherically symmetric S wave functions. The fine and hyperfine interaction creates then a position dependent mass \( \tilde{m}(r) \) in the effective kinetic energy of the associated Schrödinger equation. The results are compared with the available experimental and theoretical spectral data on the \( \pi \) and \( \rho \). Solving the eigenvalue problem within the usual oscillator approach induces a certain amount of arbitrariness. Despite of this, the agreement with experimental data is within the experimental error and better than other calculations, including Godfrey and Isgur [9] and Baldicchi and Prosperi [10]. The short coming can be removed easily in more elaborate work.

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1 The S-state Hamiltonian

For spherically symmetric S states the previously derived Hamiltonian reduces in Fourier approximation to [1,2,3]

\[
H = \frac{p^2}{2m} + V + V_{hf} + V_K + V_D ,
\]

\[
V_{hf} = \sigma_1 \sigma_2 \frac{1}{m_1 m_2} \nabla^2 V ,
\]

\[
V_K = \frac{1}{m_1 m_2} p^2 ,
\]

\[
V_D = - \left( \frac{V}{16 m_1 m_2} p^2 + \frac{\nabla^2 V}{4 m_1 m_2} \right) (m_1 + m_2) .
\]

(1)

There are no more interactions than the central potential, the hyperfine, the kinetic, and the Darwin interaction, but also no less. For s-states the total spin squared is a good quantum number \( S^2 = (\sigma_1 + \sigma_2)/2 = S(S+1) \), thus

\[
\sigma_1 \sigma_2 = 2S(S+1) - 3 = \begin{cases} +1, & \text{for } S = 1, \text{ triplet}, \\ -3, & \text{for } S = 0, \text{ singlet}. \end{cases}
\]

(2)

Because it is shorter, \( \sigma_1 \sigma_2 \) is kept explicit in the equations as an abbreviation for Eq. (2). With a quadratic potential,

\[
V(r) = -a + \frac{f}{2} r^2 ,
\]

the spring constant is \( f \), the Hamiltonian \( 1 \) becomes the non-local Schrödinger equation

\[
H = \left[ \frac{1}{2m} + \frac{V(r)}{m_1 m_2} - \frac{V(r)}{16 m_1 m_2 (m_1 + m_2)} \right] \frac{p^2}{2} + \frac{f}{2} r^2 ,
\]

since

\[
\nabla^2 V(r) = \frac{1}{r} \frac{d^2}{dr^2} r V(r) = 3f .
\]

(4)

Shaping notation, the Hamiltonian is written as

\[
H = \frac{p^2}{2\tilde{m}(r)} + \frac{f}{2} r^2 - \tilde{\sigma} + \tilde{c} \sigma_1 \sigma_2 .
\]

(5)

The non locality of the Hamiltonian resides in the position dependent mass

\[
\frac{m_r}{\tilde{m}(r)} = 1 + \frac{V(r)}{8 (m_1 + m_2)} \left[ 16 - \frac{m_1}{m_2} - \frac{m_2}{m_1} \right] .
\]

(6)

To solve this Hamiltonian, one must go on a computer.

The Hamiltonian in Eq. (5) looks like a conventional instant form Hamiltonian as obtained by quantizing the system at equal usual time. But it must be emphasized that it continues to be a genuine front form or light cone Hamiltonian [5], derived from the latter by a series of exact unitary transformations [1,13].

2 The model Hamiltonian and its parameters

In this first round, I try to avoid to go on the computer as far as possible, by the following reason. According to renormalization theory, the renormalization group invariants (parameters) must be determined from experiment. This is a strongly non linear problem. In order to get a first and rough estimate, the Hamiltonian is simplified here until it has a form which is amenable to analytical solution. Therefore, all intractable terms in the above will be replaced here by mean values and related to the experimentally accessible mean square radius \( \langle r^2 \rangle \) [8]. In effect, the substitution

\[
\tilde{m}(r) \rightarrow \tilde{m}(r) ,
\]

(7)
is the only true assumption in the present model. I consider thus the model Hamiltonian,

\[ H = \frac{\mathbf{p}^2}{2m_r} + \frac{f}{2} r^2 - \bar{a} + \bar{c}\sigma_1\sigma_2, \quad (8) \]

with the abbreviations

\[ \bar{c} = \frac{f}{2m_1m_2}, \quad (9) \]

\[ \bar{a} = a + \frac{3\bar{c}}{2} \left( \frac{m_1}{m_2} + \frac{m_2}{m_1} \right), \quad (10) \]

\[ \frac{m_r}{m_r} = 1 + \frac{(f\langle r^2 \rangle/2 - a)}{8(m_1 + m_2)} \left[ 16 - \frac{m_1}{m_2} - \frac{m_2}{m_1} \right]. \quad (11) \]

Its eigenvalues are

\[ E_n = -\bar{a} + \omega \xi_0 + \omega \eta_n + \bar{c}\sigma_1\sigma_2, \quad \omega = \left[ \frac{f}{m_r} \right]^\frac{1}{2}, \quad (12) \]

with \( \xi_0 = \frac{3}{7} \) and \( \eta_n = 2n \). The invariant mass squares

\[ M^2_n = (m_1 + m_2)^2 \]

\[ + 2(m_1 + m_2) \left( -\bar{a} + \xi_0\omega + \eta_n\omega + \bar{c}\sigma_1\sigma_2 \right), \quad (13) \]

are then related to experiment.

For equal masses \( m_1 = m_2 = m \), the model has the 3 parameters \( m, f \) and \( a \). One thus needs 3 empirical data to determine them. I choose:

\[ M^2_{d_1,t_1} = 4m^2 + 4m \left( -\bar{a} + \xi_0\omega + \bar{c} + \eta_1\omega \right), \quad (14) \]

\[ M^2_{d_1,s_0} = 4m^2 + 4m \left( -\bar{a} + \xi_0\omega + \bar{c} \right), \]

\[ M^2_{d_1,s_0} = 4m^2 + 4m \left( -\bar{a} + \xi_0\omega - 3\bar{c} \right). \]

The spectrum is labeled self explanatory by the flavor composition \( M_n = M_{d_1,t_n} \) or \( M_n = M_{d_1,s_n} \), for singlets or triplets, respectively. The triple chosen in Eq.(14) exposes a certain asymmetry. The excited \( \rho \) is chosen since its experimental limit of error is very much smaller than the one for the corresponding \( \pi \) state. Only its ground state mass is known very accurately, \( m_{\pi^+} = 139.57018 \pm 0.00035 \text{ MeV} \). In the present work only the first 4 digits are used. For equal masses, the above abbreviations become

\[ \bar{c} = \frac{f}{2m_r}, \quad \bar{a} = a + 3\bar{c}, \quad \frac{m_r}{m_r} = 1 + \frac{7(f\langle r^2 \rangle/2 - a)}{8m}. \quad (15) \]

The experiment defines 2 certainly positive differences:

\[ X^2 = M^2_{d_1,t_1} - M^2_{d_1,s_0} = 4m \eta_1\omega, \quad (16) \]

\[ Y^2 = M^2_{d_1,t_1} - M^2_{d_1,s_0} = 4m 4\bar{c}. \]
In principle, one could determine the heavier quark masses analytically from the hyperfine splittings:

\[
\frac{1}{m_x} = \frac{M_{u,s} v}{4J} \quad \frac{1}{m_u} = \frac{M_{u,s} v}{4J} \quad \frac{1}{m_c} = \frac{M_{u,s} v}{4J} \quad \frac{1}{m_b} = \frac{M_{u,s} v}{4J}.
\]

(20)

The so obtained results are however, not very reasonable, see Table 1. The experimental numbers are insufficiently accurate. Therefore, I determine them numerically from the singlets \(M_{u,s_{1,0}}\) and \(M_{u,c_{1,0}}\) and \(M_{u,c_{3,0}}\) and compile them in Table 2.

### 3 Results and Discussion

**Unflavored light mesons.** The results for the \(\pi-\rho\) system are compiled in Table 3. The experimental points are taken from from Hagiwara et al.\cite{Hagiwara}. It is no surprise that theory and experiment coincide for the \(\pi^+\) and the \(\rho^+\), because these data have been used to determine the parameters. The remarkable thing is that one can perform such a fit at all. The model reproduces the huge mass of the excited pion within the error limit. This solves the long standing puzzle, why a physical system can have a first excited state with a ten times larger mass.

The remaining three calculated masses of the \(\pi-\rho\) sector agree with experiment almost within the error bars. The model underestimates the second \(\pi\)-excitation by only 26 MeV. The second excitation of the \(\rho\) is overestimated by a comparatively large 224 MeV, but the experiment for the \(\rho^+\) (3\(^2S_1\)) needs confirmation. The third excitation of the \(\rho^+\) (4\(^3S_1\)) is overestimated by 224 MeV.

The table includes also a comparison with other theoretical calculations. It includes the results from a recent oscillator model \cite{Zhou}. Their model is even simpler than the present one: it works with a hyperfine splitting, only, and suppresses the mechanism of a position dependent mass. Despite this, the results of \cite{Zhou} coincide practically with the present ones. I have included also the results from the pioneering work of Godfrey and Isgur \cite{Godfrey} as a prototype of a phenomenological model, and from a recent advanced calculation by Baldicchi and Prosperi \cite{Baldicchi}. Neither of these models have much in common with the present one. They fail to reproduce the pion, this mystery particle of QCD.

**Strange mesons.** The \(S\) wave \(K^+\) and \(K^{*+}\) spectra are given in Table 4. The mass of the singlet ground state is used to determine the mass parameter \(m_s\). Except the ground states, the experiments carry many ambiguities about the quantum number assignment for \(K\) and \(K^*\) mesons. The model prediction for the triplet ground state underestimates the experimental value by 20 MeV. Both the first and the second excited state of \(K^*\) (2\(^1S_0\) and 3\(^3S_1\)) are not confirmed. Another unconfirmed resonance with mass 1.629 ± 0.027 GeV lying between 2\(^1S_0\) and 3\(^3S_1\) was assigned to be a singlet \(K\). Apparently there is no position for it in the \(K\) spectrum if it is an \(S\) wave state. However, according to its mass, it might well be the first excited state of \(K^*\) (2\(^1S_0\)). Taken the numbers in the table, the discrepancies are 88 and 69 MeV for the singlet and triplet \(n = 2\) states, respectively. The second excited state of the \(K^*\) (2\(^3S_0\)) differs by only 48 MeV, but the datum needs confirmation.

**Heavy mesons.** The \(S\) wave \(u\bar{c}, u\bar{b}, s\bar{c}, s\bar{b}\) and \(c\bar{b}\) meson spectra are given in Table 5. No excitations were observed for these mesons.

The \(u\bar{c}\) singlet is used to determine the mass parameter \(m_c\). Its ground state mass (\(D^0\)) therefore coincides with experiment. The model overestimates the mass of the triplet \(D^{*0}\) by about 50 MeV. — The \(u\bar{b}\) singlet is used to determine the mass parameter \(m_b\). Its ground state mass (\(B^+\)) therefore coincides with experiment. The model overestimates the mass of the triplet \(B^{*+}\) by 16 MeV only.

No data in the \(s\bar{c}\) mesons are used to determine model parameters. Model and experiment differ by 27 and 40 MeV for singlet and triplet, respectively.

### Table 4. \(S\) wave spectra in GeV for light unflavored mesons.

| n | \(J^P\) | \(m_{Singlets}\) | \(m_{Triplets}\) |
|---|---|---|---|
| 1 | \(1^1S_0\) | 0.1396 | 0.7685 |
| 2 | \(1^3P_0\) | 1.300 | 1.4650 |
| 3 | \(1^3P_2\) | 1.795 | 1.9240 |
| 4 | \(1^3P_4\) | 2.164 | 2.2929 |

### Table 5. \(S\) wave spectra in GeV for strange mesons.

| n | \(J^P\) | \(m_{Singlets}\) | \(m_{Triplets}\) |
|---|---|---|---|
| 1 | \(1^1S_0\) | 0.4937 | 0.8916 |
| 2 | \(1^3P_0\) | 1.460 | 1.8782 |
| 3 | \(1^3P_2\) | 1.830 | 2.1040 |
| 4 | \(1^3P_4\) | 2.164 | 2.2736 |

1. Hagiwara et al.\cite{Hagiwara}, Zhou and Pauli \cite{Zhou}, Godfrey and Isgur \cite{Godfrey}, Baldicchi and Prosperi \cite{Baldicchi}.  
2. To be confirmed; \(f^P\) not confirmed.
Table 6. Ground state masses in GeV for heavy mesons.

| n  | Experiment | Theory   | n  | Experiment | Theory   |
|----|------------|----------|----|------------|----------|
| 1  | 1.8645(5)  | 1.9961   | 1  | 2.0067(5)  | 2.0178   |
|    | 1.9224(4)  | 1.883    |    | 2.0067(2)  | 2.04(3)  |
| 1  | 5.2790(5)  | 5.2970   | 1  | 5.3250(6)  | 5.3085   |
|    | 5.2965(2)  | 5.313    |    | 5.3250(2)  | 5.37(3)  |
| 1  | 1.9685(6)  | 1.9961   | 1  | 2.1124(7)  | 2.07(3)  |
|    | 2.0201(2)  | 1.983    |    | 2.0655(2)  | 2.13(3)  |
| 1  | 5.3696(24) | 5.3976   | 1  | 5.4166(35) | 5.4214   |
|    | 5.3739(2)  | 5.35(3)  |    | 5.3885(3)  | 5.45(3)  |
| 1  | 6.4(4)     | 6.5071   | 1  | —          | 6.5157   |
|    | —          | 6.273    |    | 6.3458(2)  | 6.34(3)  |

Table 7. The predicted S spectrum in GeV for heavy mesons.

| 1  | n S^0 D^0 (uc) | n S^1 D^1 (uc) |
|----|---------------|---------------|
| 1  | 1.8645        | 1.9961        |
| 2  | 2.5013        | 2.5728        |
| 3  | 3.9661        | 3.0658        |
| 4  | 3.4375        | 3.4899        |
|    | n S^0 B^+(ub) | n S^1 B^+(ub) |
| 1  | 5.2790        | 5.3085        |
| 2  | 5.8473        | 5.8739        |
| 3  | 6.3651        | 6.3896        |
| 4  | 6.8438        | 6.8666        |
|    | n S^0 B^0 (sb) | n S^1 B^0 (sb) |
| 1  | 5.3976        | 5.4214        |
| 2  | 5.9163        | 5.9380        |
| 3  | 6.3930        | 6.4131        |
| 4  | 6.8365        | 6.8553        |
|    | n S^0 B^+ (cb) | n S^1 B^+ (cb) |
| 1  | 6.5077        | 6.5157        |
| 2  | 6.8447        | 6.8523        |
| 3  | 7.1659        | 7.1731        |
| 4  | 7.4733        | 7.4802        |

Hagiwara et al., Zhou and Pauli, Godfrey and Isgur, S. J. Brodsky, H. C. Pauli and S. S. Pinsky, Phys. Lett. C (Physics Reports) 301 (1998) 299-486. B. Povh and J. Hufner, Phys. Lett. 245 (1990) 653. K. Hagiwara et al., Phys. Rev. D 66 (2002) 010001. S. G. Zhou and H. C. Pauli, arXiv: hep-ph/0311305. S. Godfrey and N. Isgur, Phys. Rev. D 32 (1985) 189. D. Baldicchi and G. M. Prosperi, Phys. Rev. D 66 (2002) 074008. A. V. Anisovich, V. V. Anisovich, and A. V. Sarantzev, Phys. Rev. D 62 (2000) 051502(R).