FREE LIE ALGEBROIDS AND THE SPACE OF PATHS

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Introduction.

(0.1) The goal of this paper is to construct algebraic and algebro-geometric models for spaces of paths. Paths in a manifold $X$, considered up to reparametrizations and cancellations, form a groupoid $\Pi_X$. We construct a Lie algebroid $\mathcal{P}_X$ which plays the role of the Lie algebra for $\Pi_X$, i.e., describes “infinitesimal paths”.

When $X$ is an algebraic variety, we construct an algebro-geometric model for the formal neighborhood of $X$ (constant paths) in $\Pi_X$. This is a certain ind-scheme, denoted $\hat{\Pi}_X$, which also has a groupoid structure. It is analogous to (but different from) $\mathcal{L}X$, the ind-scheme of formal parametrized loops in $X$, constructed in [KV1].

Among known algebro-geometric constructions that have the flavor of the space of unparametrized paths, one should mention the Kontsevich moduli stack $\overline{M}_{g,2}(X, \beta)$ of stable maps from 2-pointed curves $(C, x, y)$ of genus $g$ into $X$, see [Ma]. Indeed, this stack consists of maps $(C, x, y) \to X$ for variable $(C, x, y)$ modulo isomorphisms of curves, i.e., changes of parametrization. The points $x$ and $y$ serve as the beginning and end of a path. One of our main results, Theorem 7.3.5, constructs, in the case $g = 0$, a morphism from a certain formal neighborhood in $\overline{M}_{0,2}(X, \beta)$ into $\hat{\Pi}_X$. The case of $n$-pointed curves, $n \geq 3$, can be treated similarly by considering the simplicial classifying space of $\hat{\Pi}_X$.

(0.2) Any vector bundle with connection on $X$ gives a representation of $\Pi_X$ by holonomy. In fact, representations of $\Pi_X$ are more or less the same as connections, as was shown by Kobayashi [Ko]. Therefore it is not so surprising that our Lie algebroid $\mathcal{P}_X$ can be seen as the universal receptacle of the curvature data for all connections; these data include the “higher covariant derivatives of the curvature”. The quotes here emphasize the fact that these covariant derivatives are not really defined as tensors; in fact $\mathcal{P}_X$ is filtered, with graded quotients expressed as certain tensor spaces. It appears that $\mathcal{P}_X$ is a fundamental differential-geometric object associated to any manifold, similar in importance to the sheaf of differential operators.
Our other main result, Theorem 4.4.3, identifies formal germs of connections on $X$ at $x$ with representations of a certain infinite-dimensional Lie algebra $\mathcal{P}(X, x)$, which we call the \textit{fundamental Lie algebra} (by analogy with the fundamental group and its relation with flat connections). Combined with some results of Reutenauer, this provides a “Taylor formula” for connections (Corollary 4.4.5).

(0.3) It is known since the work of K.-T. Chen [C] that the group of paths in $\mathbb{R}^n$ can be seen as a certain continuous analog of the free group on $n$ generators, so its Lie algebra is a certain completion of the free Lie algebra. Our construction of $\mathcal{P}_X$ is also a version of the free Lie algebra construction but in the context of Lie algebroids. This construction generalizes that of Casas, Ladra and Pirashvili [CLP]. In the curvature language the appearance of the (free) Lie algebras here reflects the interpretation of the Bianchi identity as the Jacobi identity for the curvature operators $\nabla_i = \partial_i + A_i$.

In fact, transformation rules for sections of $\mathcal{P}_X$ from one coordinate system $(x_1, ..., x_n)$ to another, $(y_1, ..., y_n)$, reflect the transformation rules for the system of Chen’s iterated integrals of 1-forms constant with respect to these systems:

$$\left\{ \int_\gamma d\gamma^1_1 ... d\gamma^p_p \right\} \sim \left\{ \int_\gamma d\gamma^1_1 ... d\gamma^p_p \right\}.$$

Here the path $\gamma$ is assumed to be the same on both sides.

Sections of $\mathcal{P}_X$ give rise to “noncommutative vector fields”, i.e., to natural systems of operators $\{P_{E, \nabla} : E \to E\}$ defined for all bundles with connections $(E, \nabla)$ on $X$ and satisfying the Leibniz rule with respect to the tensor product. By applying the enveloping algebra construction we get a sheaf of rings $\mathcal{D}_X = U(\mathcal{P}_X)$ whose sections are called noncommutative differential operators. Such an operator $P$ gives a natural system of differential operators for all $(E, \nabla)$. By analogy with classical results of Riemannian geometry [Ep] [St], we identify, in Theorem 5.3.4, sections of $\mathcal{D}_X$ with such natural operators satisfying an extra assumption of regularity. It is extremely interesting to study natural systems of pseudo-differential operators because intuitively they are given by kernels which are measures on $\Pi X$, see (8.1).

(0.4) The paper is written on two levels. The first level is that of elementary differential geometry of bundles, connections and curvatures on $C^\infty$-manifolds. Our main constructions make sense and seem interesting already at this level and, with very few exceptions, we stick to this level until §6.

The other level is that of formal geometry and ind-schemes, necessary to construct $\hat{\Pi}_X$. For convenience of the reader, most of the technical issues related to ind-schemes and formal groupoids, were put into the Appendix. It is here that we address the issue of formal integration of Lie algebroids, i.e., prove the groupoid analog of the fact that any Lie algebra over a field of characteristic 0 gives rise to a formal group. Because integration to a Lie groupoid, even locally, is not always possible, see [Mack], we discuss formal integration systematically.
Unlike the Lie algebra case, working directly with the Campbell-Hausdorff series can be confusing here. Our approach, although essentially equivalent, is based on dualization of $U(\mathcal{G})$, the enveloping algebra of a Lie algebroid $\mathcal{G}$. The corresponding fundamental structure on $U(\mathcal{G})$ is that of a bialgebroid as defined by J.-H. Lu [Lu] and P. Xu [Xu]. In our situation we need a particular case when the cosource and cotarget maps coincide and the algebra of “objects” is commutative but not central. This case (but including the antipode which we do not require) was considered in the recent paper of J. Mrčun [Mr]. We call such special bialgebroids left bialgebras and give a self-contained exposition for reader’s convenience.

(0.5) The paper is organized as follows. In §1 we recall main facts about Lie algebroids and their algebraic counterparts, Lie-Rinehart algebras. In §2 we present our main algebraic construction: that of a free Lie-Rinehart algebra generated by an anchored module. In §3 we concentrate on $\mathcal{P}_X$, the free Lie algebroid generated by the tangent bundle of a $C^\infty$-manifold $X$. The stabilizer Lie algebra of $\mathcal{P}_X$ at $x \in X$, is the subject of §4. In §5 we study noncommutative differential operators and more general natural systems of differential operators in bundles with connections. In §6 we extend the results of §5 to the algebro-geometric situation. In §7 we introduce the formal neighborhood $\hat{\Pi}_X$ and relate it to Kontsevich’s moduli spaces. Finally, §8 is devoted to informal discussion of possible further directions motivated by the present work.

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1. Reminder on Lie algebroids.

(1.1) **Lie-Rinehart algebras.** Let $k$ be a field of characteristic 0 and $A$ be a commutative $k$-algebra with unit. We denote by $\text{Der}(A)$ the $A$-module of derivations of $A$ vanishing on $k$. Recall that $\text{Der}(A)$ is naturally a Lie $k$-algebra with respect to the usual commutator.

A Lie-Rinehart $A$-algebra is an $A$-module $L$ equipped with a structure of a $k$-Lie algebra and with a morphism $a : L \rightarrow \text{Der}(A)$ of $A$-modules called the anchor map. These data are required to satisfy the following properties:

(1.1.1) $a$ is a morphism of Lie $k$-algebras.

(1.1.2) For any $f \in A$ and sections $x, y \in L$ we have

$$[x, fy] - f \cdot [x, y] = a(x)(f) \cdot y.$$
This concept goes back to [Ri]. We denote by $\text{Rin}_A$ the category of Lie-Rinehart $A$-algebras.

**Examples.** (a) When $A = k$, a Lie-Rinehart $k$-algebra is the same as a Lie $k$-algebra. More generally, a Lie-Rinehart $A$-algebra with the anchor map being zero is the same as an Lie $A$-algebra in the usual sense.

(b) $\text{Der}(A)$ with $a = \text{Id}$ is a Lie-Rinehart $A$-algebra.

(c) Let $M$ be an $A$-module. The Atiyah algebra of $M$ is the Lie-Rinehart $A$-algebra $A_M$ whose elements are pairs $(\phi, D)$ with $\phi \in \text{End}_k(M)$ and $D \in \text{Der}(M)$ satisfying the following property:

$$\phi(fm) - f \cdot \phi(m) = D(f) \cdot m, \quad f \in A, m \in M.$$ 

One sees that commuting the $\phi$'s and the $m$'s makes $A_M$ into a $k$-Lie algebra. Further, the anchor map $(\phi, D) \mapsto D$ makes $A_M$ into a Lie-Rinehart algebra. Cf. [Kal].

For a Lie-Rinehart algebra $L$ we denote by $L^\circ$ the kernel of the anchor map. This is an Lie $A$-algebra in virtue of (1.1.2). A Lie-Rinehart algebra is called transitive, if $a$ is surjective.

**Definition.** Let $L$ be a Lie-Rinehart $A$-algebra. An $L$-module is an $A$-module $M$ together with a morphism of Lie-Rinehart $A$-algebras $L \to A_M$.

Thus, in particular, $M$ is a module over $L$ as a $k$-Lie algebra.

**Enveloping algebras.** We follow [HS], §3.4, see also [Ri], §2. Let $L$ be a Lie-Rinehart $A$-algebra. Its (twisted) enveloping algebra $U(L) = U_A(L)$ is the associative algebra satisfying the following properties:

(1.2.1) $L$-modules are the same as left modules over $U_A(L)$ as an associative algebra.

(1.2.2) $A$ is a subalgebra of $U_A(L)$; there is a natural algebra filtration $F_\bullet$ of $U_A(L)$ with $F_0U_A(L) = A$ and 

$$\text{gr}^F U_A(L) = S^*_A(L),$$

the symmetric algebra of $L$ considered as an $A$-module.

The explicit construction of $U_A(L)$ given in [HS], §3.5, is as follows. One starts from $U_k(L)$, the usual enveloping algebra of $L$ considered as a Lie $k$-algebra, see [Di] [Re]. Let $U_k^+(L) \subset U_k(L)$ be the augmentation ideal. We consider it as an associative $k$-algebra without unit. Further, let $U_A(L)^+$ be the quotient algebra of $U_k(L)^+$ by the relations

$$x \cdot fy - fx \cdot y = a(x)(f) \cdot y, \quad x, y \in L, f \in A.$$

Then we define $U_A(L) = U_A(L)^+ \oplus A$ with the algebra structure given by

$$f \cdot x = fx; \quad x \cdot f = fx + a(x)(f), \quad f \in A, x \in L.$$

The above properties follow right away from the definition. For example, $F_iU_A(L) = F_iU_k(L)^+ \oplus A$, where $F_iU_k(L)$ is the canonical filtration of the usual enveloping algebra.
(1.2.5) Example. Let \( k = \mathbb{R} \) and \( X \) be a \( C^\infty \)-manifold. Let \( A = C^\infty(X) \) be the algebra of smooth functions on \( X \). Then \( \text{Der}(A) \) is the Lie algebra of vector fields on \( X \), and \( U_A(\text{Der}(A)) \) is the algebra of differential operators on \( X \).

Note that \( A \) is an \( L \)-module (via the anchor map), and thus a left \( U_A(\text{L}) \)-module. We have the augmentation map

\[
\epsilon : U_A(L) \to A, \quad P \mapsto P \cdot 1,
\]

which is a morphism of left \( U_A(L) \)-modules. Alternatively, \( \epsilon \) can be described as the projection to \( A \) along \( U_A(\text{L})^+ \).

(1.3) Lie algebroids: \( C^\infty \)-setting. Let \( X \) be a \( C^\infty \)-manifold. By a vector bundle on \( X \) we mean a locally trivial \( \mathbb{R} \)-vector bundle, possibly of infinite rank. In other words, a vector bundle \( E \) is the same as a locally free sheaf of modules over \( C^\infty_X \), the sheaf of \( C^\infty \)-functions.

Let \( T_X \) be the tangent bundle of \( X \). As before, sections of \( T_X \) (i.e., vector fields) form a Lie-Rinehart \( C^\infty(X) \)-algebra (with \( k = \mathbb{R} \)). So we will call a Lie algebroid on \( X \) a vector bundle \( G \) on \( X \) with a morphism of vector bundles \( \alpha : G \to T_X \) and a structure of a Lie algebra in sections of \( G \) so that \( \alpha \) preserves the bracket and the analog of (1.1.2) is satisfied.

In particular, for a vector bundle \( E \) on \( X \) we have the Atiyah algebroid \( A_E \). It can be defined more classically as a sheaf of differential operators \( P : E \to E \) of order \( \leq 1 \) whose first order symbol lies in \( T_X \otimes 1 \subset T_X \otimes \text{End}(E) \).

A module over a Lie algebroid \( G \) is a vector bundle \( E \), assumed here to be of finite rank, equipped with a morphism of Lie algebroids \( G \to A_E \).

(1.3.1) Example. A module over a Lie algebroid \( T_X \) is just a vector bundle on \( X \) equipped with a flat connection.

If \( G \) is transitive, i.e., if \( \alpha \) is surjective, then \( G^\circ = \text{Ker}(\alpha) \) is a bundle of Lie algebras, so every fiber is of it is an \( \mathbb{R} \)-Lie algebra.

We also have the concept of the enveloping algebra \( U(G) \) of a Lie algebroid. Thus \( U(T_X) = D_X \) is the sheaf of \( C^\infty \)-differential operators on \( X \).

2. Free Lie-Rinehart algebras and Lie algebroids

(2.1) Free Lie-Rinehart algebras. One version of the concept of free Lie-Rinehart algebras was introduced in [CLP]. Here we define a different concept. The relation of our construction with that of [CLP] will be explained in (2.2.7).

Let \( k \) be a field of characteristic zero and \( A \) be a commutative \( k \)-algebra. By an anchored \( A \)-module we mean an \( A \)-module \( M \) together with a homomorphism of \( A \)-modules \( b : M \to \text{Der}(A) \). Anchored \( A \)-modules form a category in an obvious way. We denote this category
by Anch. Thus we have the forgetful functor
\[(2.1.1) \phi : \text{Rin}_A \to \text{Anch}_A,\]
which takes a Lie-Rinehart $A$-algebra $L$ into $L$ considered as just an anchored $A$-module.

**Theorem.** The functor $\phi$ has a left adjoint functor $\text{FR} : \text{Anch}_A \to \text{Rin}_A$ whose value on an anchored $A$-module $M$ is called the free Lie-Rinehart algebra generated by $M$. Thus we have natural isomorphisms
\[\text{Hom}_{\text{Rin}}(\text{FR}(M), L) = \text{Hom}_{\text{Anch}}(M, L), \quad M \in \text{Anch}_A, L \in \text{Rin}_A.\]

**Example.** Suppose that the map $b$ in an anchored $A$-module $M$ is zero. Then $\text{FR}(M) = \text{FL}(M/A)$ is the free Lie $A$-algebra generated by the $A$-module $M$.

Let us recall, for the purpose of the future generalization, some details of the construction of $\text{FL}(M/A)$. To construct it, we start with $\text{FL}(M/k)$, the free Lie $k$-algebra generated by $M$ as a $k$-vector space. See [Re] for background on free Lie algebras over a field. The Lie $A$-algebra $\text{FL}(M/A)$ is obviously the quotient of $\text{FL}(M/k)$ by the “$A$-linearity relations”. In order to impose these relations in a systematic way, we consider the natural grading
\[(2.1.4) \quad \text{FL}(M/k) = \bigoplus_{d=1}^{\infty} \text{FL}_d(M/k),\]
where $\text{FL}_d(M/k)$ is the space spanned by brackets involving exactly $d$ elements of $M$. For example,
\[(2.1.5) \quad \text{FL}_1(M/k) = M, \quad \text{FL}_2(M/k) = \Lambda^2(M).\]
Then we construct $\text{FL}(M/A)$ as a graded $A$-Lie algebra:
\[(2.1.6) \quad \text{FL}(M/A) = \bigoplus_{d=1}^{\infty} \text{FL}_d(M/A)\]
and define the graded components (which are $A$-modules) inductively, starting from $\text{FL}_1(M/A) = M$.

Suppose that for $i = 1, \ldots, d$ we have defined the $A$-modules $\text{FL}_i(M/A)$ and surjections of $k$-vector spaces $p_i : \text{FL}_i(M/k) \to \text{FL}_i(M/A)$. We define $\text{FL}_{d+1}(M/A)$ as the quotient of $\text{FL}_{d+1}(M/k)$ be the following relations:
\[(2.1.7)(a) \quad [x, r], \quad x \in M, r \in \text{Ker}(p_d);\]
\[(2.1.7)(b) \quad [fx, y] = [x, fy], \quad x \in M, y \in \text{FL}_d(M/A).\]
Here the brackets on the left in (2.1.7)(b) are well defined modulo the relations from (2.1.7)(a). Then we make $\text{FL}_{d+1}(M/A)$ into an $A$-module by defining $f[x, y]$ to be the common value of the two brackets in (2.1.7)(b).

**Proof of Theorem 2.1.2.**
We generalize the approach to the construction of $\text{FL}(M/A)$ outlined in Example 2.1.3. The only difference is that we work with filtered and not graded objects.

Thus, we construct $\text{FR}(M)$ as the union of an increasing sequence of anchored $A$-modules

$$\text{FR}_{\leq 1}(M) \subset \text{FR}_{\leq 2}(M) \subset \ldots$$

which are defined inductively starting from $\text{FR}_{\leq 1}(M) = M$. Suppose we have defined an anchored $A$-module $\text{FR}_{\leq d}(M) \xrightarrow{\alpha_d} \text{Der}(A)$ and a surjective homomorphism of $k$-vector spaces

$$\text{FL}_{\leq d}(M/k) = \bigoplus_{i=1}^{d} \text{FL}_i(M/k) \xrightarrow{\theta_d} \text{FR}_{\leq d}(M).$$

We define $\text{FR}_{\leq d+1}(M)$ as the quotient of $\text{FL}_{\leq d+1}(M/k)$ by the following relations:

$$\begin{align*}
(2.2.2)(a) & \quad [x, r], \quad x \in M, r \in \text{Ker}(\theta_d); \\
(2.2.2)(b) & \quad [fx, y] - [x, fy] = a_d(y)(f) \cdot x - b(x)(f) \cdot y, \quad x \in M, y \in \text{FR}_{\leq d}(M).
\end{align*}$$

Here, as before, the brackets on the left in (2.2.2)(b) are well defined modulo the relations from (2.2.2)(a).

The $A$-module structure in $\text{FR}_{\leq d+1}(M)$ is defined by

$$\begin{align*}
(2.2.3) & \quad f[x, y] = -a_d(y)(f) \cdot x + [fx, y] = [x, fy] + b(x)(f) \cdot y, \quad x \in M, y \in \text{FR}_{\leq d}(M).
\end{align*}$$

The anchor map is defined by

$$\begin{align*}
(2.2.4) & \quad a_{d+1}([x, y]) = [b(x), a_d(y)], \quad x \in M, y \in \text{FR}_{\leq d}(M).
\end{align*}$$

The Lie algebra structure on $\text{FR}(M) = \bigcup_d \text{FR}_{\leq d}(M)$ is induced by that on $\text{FL}(M/k)$. We leave to the reader the verification that $\text{FR}(M)$ is a Lie-Rinehart algebra and satisfies the adjunction as in Theorem 2.1.2.

The construction of $\text{FR}(M)$ implies the following:

(2.2.5) **Proposition.** (a) The filtration $\{\text{FR}_{\leq d}(M)\}$ makes $\text{FR}(M)$ into a filtered Lie-Rinehart algebra, and the associated graded Lie-Rinehart algebra is isomorphic to $\text{FL}(M/A)$ with trivial anchor map.

(b) Let $\text{FR}^\circ(M) \subseteq \text{FR}(M)$ be the kernel of the anchor map (so it is an $A$-Lie algebra). Then the induced filtration on $\text{FR}^\circ(M)$ is compatible with the $A$-Lie algebra structure, and the associated graded $A$-Lie algebra is isomorphic to $\text{FL}_{\geq 2}(M/A)$:

$$\bigoplus_{d=2}^{\infty} \text{FR}_{\leq d}^\circ(M) / \text{FR}_{\leq d-1}^\circ(M) = \text{FL}_{\geq 2}(M/A), \quad \text{FR}_{\leq 1}^\circ(M) = 0.$$

Assume that $A$ is finitely generated over $k$ and $A \to A'$ be an étale extension of $k$-algebras. Then, as well known, $A'$ is finitely generated too, and

$$\text{Der}(A') = \text{Der}(A) \otimes_A A'.$$
Because the construction of $\text{FR}(M)$ involves correction terms given by the action of derivations of $A$, we get the following conclusion.

(2.2.6) **Proposition.** The construction of the free Lie-Rinehart algebra is compatible with étale base change. In other words, if $(M,b)$ is an anchored $A$-module, and $(M',b')$ its extension to $A'$, with $M' = M \otimes_A A'$, then

$$\text{FR}(M') = \text{FR}(M) \otimes_A A'.$$

In particular, the construction of $\text{FR}(M)$ is compatible with localization.

(2.2.7) **Relation to [CLP].** Casas, Ladra and Pirashvili define, in *loc. cit.* a different concept of free Lie-Rinehart algebras. They start not with anchored $A$-modules but with $A$-anchored $k$-vector spaces. By definition, an $A$-anchored $k$-vector space is a vector space $V$ together with a $k$-linear map $c : V \to \text{Der}(A)$. Such objects for a category $\text{Anch}_{k/A}$ and we obviously have the forgetful functor

$$\psi : \text{Rin}_A \to \text{Anch}_{k/A}.$$

Their free Lie-Riehart algebra functor is the left adjoint functor to $\psi$. Let us denote it by $\text{FR}_{CLP}$. It is clear that the functor $\psi$ factor through the forgetful functor $\phi$ to the category of anchored $A$-modules. Further, if $(V,c)$ is an $A$-anchored $k$-vector space, then we have an anchored $A$-module $(V \otimes_k A, c \otimes 1)$ freely generated by $V$, and we have:

(2.2.8) **Proposition.** If $(V,c)$ is an $A$-anchored $k$-vector space, then

$$\text{FR}_{CLP}(V) = \text{FR}(V \otimes_k A).$$

Our construction has an additional flexibility in that it is defined for not necessarily free $A$-modules and localizes on $\text{Spec}(A)$ if $A$ is finitely generated.

(2.3) **Free Lie algebroids: $\mathcal{C}^\infty$-setting.** Let $k = \mathbb{R}$. Let $X$ be a $\mathcal{C}^\infty$-manifold and $(V,\beta)$ be an anchored vector bundle on $X$, i.e., a vector bundle (possibly infinite-dimensional) with a morphism of vector bundles $\beta : E \to T_X$. Then we get a Lie algebroid $\mathcal{F}(V)$, called the free Lie algebroid generated by $E$ with properties similar to the ones listed above.

(2.3.1) **Definition.** Let $X$ be a $\mathcal{C}^\infty$-manifold, and $\beta : V \to T_X$ be an anchored vector bundle. A pre-module over $V$ is a vector bundle $E$ together with a morphism of anchored bundles $V \to \mathcal{A}_E$ from $V$ to the Atiyah algebroid of $E$.

(2.3.2) **Example.** A pre-module over $T_X$ is the same as a vector bundle with connection (not necessarily flat).

The universal property of free Lie algebroids implies the following:

(2.3.3) **Proposition.** Modules over $\mathcal{F}(V)$ are the same as pre-modules over $V$. 
3. The path Lie algebroid.

(3.1) Main properties. Let \( X \) be a \( C^\infty \)-manifold. The free Lie algebroid \( \mathcal{F}(T_X) \) generated by \( T_X \) will be called the path algebroid of \( X \) and denoted by \( \mathcal{P}_X \). The kernel of the anchor map of \( \mathcal{P}_X \) will be denoted by \( \mathcal{P}_X^0 \). This is a bundle of Lie algebras on \( X \). Let us summarize the properties of \( \mathcal{P}_X \) that follow from the general results of §2.

(3.1.1) Proposition. (a) Modules over \( \mathcal{P}_X \) are the same as finite vector bundles on \( X \) with connection.
(b) \( \mathcal{P}_X \) is the union of subbundles \( \mathcal{P}_{X, \leq d}, \ d \geq 1 \) of finite rank. The filtration \( \{ \mathcal{P}_{X, \leq d} \} \) is admissible (A.5.7), and
\[
\text{gr}(\mathcal{P}_X) = \text{FL}(T_X)
\]
is the bundle of fiberwise free Lie algebras of \( T_X \).
(c) The induced filtration \( \mathcal{P}_{X, \leq d}^0 = \mathcal{P}_{X, \leq d} \cap \mathcal{P}_X^0 \) has \( \mathcal{P}_{X, \leq 1}^0 = 0 \). It is compatible with the Lie algebra structure in the fibers, and
\[
\text{gr}(\mathcal{P}_X^0) = \text{FL}_{\geq 2}(T_X).
\]
(d) We have a canonical embedding
\[
c : \Lambda^2(T_X) \hookrightarrow \mathcal{P}_{X, \leq 2}^0 \subset \mathcal{P}_{X, \leq 2},
\]
which takes the wedge product \( v \wedge w \) of two vector fields into
\[
c(v \wedge w) = [v, w]_\mathcal{P} - [v, w]_{\text{Lie}}.
\]
Here \( [v, w]_\mathcal{P} \) is the bracket in \( \mathcal{P}_X \), while \( [v, w]_{\text{Lie}} \) is the standard Lie bracket of \( v, w \) as vector fields. \( \Box \)

(3.2) Noncommutative vector fields. Let \( \text{Bun}_\nabla(X) \) be the category of vector bundles of finite rank on \( X \) equipped with connections. It is a tensor category with respect to the usual tensor product \( \otimes \).

(3.2.1) Definition. A noncommutative vector field on \( X \) is a rule \( D \) which to each \( (E, \nabla) \in \text{Bun}_\nabla(X) \) associates a differential operator \( D_{E, \nabla} : E \to E \) satisfying the following properties:
(a) Naturality: for any morphism \( \phi : (E, \nabla) \to (E', \nabla') \) of bundles with connections we have
\[
D_{E', \nabla'} \circ \phi = \phi \circ D_{E, \nabla}.
\]
(b) Leibniz rule with respect to the tensor product: for any two bundles with connections \( (E, \nabla) \) and \( (E', \nabla') \) we have
\[
D_{E \otimes E', \nabla \otimes \nabla'} = D_{E, \nabla} \otimes 1 + 1 \otimes D_{E', \nabla'}.
\]
Clearly, noncommutative vector fields form a sheaf on \( X \) which we denote by \( \text{Vect}_X \). Further, \( \text{Vect}_X \) is a sheaf of Lie algebras with respect to the usual commutator of differential operators in any \( (E, \nabla) \).
If $D$ is a noncommutative vector field, then taking $(E, \nabla) = (E', \nabla') = C^\infty_X$ to be the trivial rank 1 bundle with standard connection, we find from the $\otimes$-Leibniz rule that

$$D_{C^\infty_X} : C^\infty_X \to C^\infty_X$$

is a ring derivation, i.e., it is a vector field in the usual sense. This defines a morphism of sheaves of Lie algebras $\alpha : \text{Vect}_X \to T_X$.

(3.2.2) Proposition. The map $\alpha$ makes $\text{Vect}_X$ into a Lie algebroid (sheaf of Lie-Rinehart algebras).

Further, applying the $\otimes$-Leibniz rule to $C^\infty_X \otimes E = E$, we find that each $D_{E,\nabla} : E \to E$ is a first order differential operator and its first order symbol is $\alpha(D) \otimes 1_E$. In other words:

(3.2.3) Proposition. For each $(E, \nabla) \in \text{Bun}_\nabla(X)$ the correspondence $D \mapsto D_{E,\nabla}$ defines a morphism of Lie algebroids

$$\text{Vect}_X \to \mathcal{A}_E.$$

(3.2.4) Examples. (a) Let $v$ be a usual vector field on $X$. Then the rule $D_v$ which to each $(E, \nabla)$ associates $\nabla_v : E \to E$ (covariant derivative along $v$), is a noncommutative vector field.

(b) Let $\xi$ be a bivector field on $X$, i.e., a section of $\Lambda^2(T_X)$. Then we have the noncommutative vector field $\Phi_\xi$ which to each $(E, \nabla)$ associates the endomorphism $(F_\nabla, \xi) : E \to E$. Here $F_\nabla \in \Omega^2_X \otimes \text{End}(E)$ is the curvature of $\nabla$.

(c) Note that the correspondence $v \mapsto D_v$ from (a) is not a Lie algebra homomorphism. Indeed, we have

$$[D_v, D_w] = D_{[v,w]} + \Phi_{v \wedge w}.$$  

This is just the definition of the curvature.

(3.2.5) Theorem. The correspondence $v \mapsto D_v$ extends to a monomorphism of Lie algebroids

$$h : \mathcal{P}_X \to \text{Vect}_X.$$

The existence of $h$ follows from the universal property of $\mathcal{P}_X$, as we have a morphism of anchored vector bundles $T_X \to \text{Vect}_X$. The proof that $h$ is a monomorphism will be given in §5 below where we will also characterize the image.

(3.3) Example: the case of $\mathbb{R}^n$. Let $X = \mathbb{R}^n$ with coordinates $x_1, \ldots, x_n$. Let us describe $\mathcal{P}(\mathbb{R}^n)$, the space of global sections of $\mathcal{P}_{\mathbb{R}^n}$ explicitly. Let $\partial_i = \partial/\partial x_i$ and $D_i$ be the section of $\mathcal{P}_{\mathbb{R}^n} = \mathcal{F}(T_{\mathbb{R}^n})$ corresponding to the section $\partial_i$ of $T_{\mathbb{R}^n}$.

Consider the associative algebra $\mathbb{D}(\mathbb{R}^n)$ generated by $C^\infty$-functions $f(x_1, \ldots, x_n)$ and the symbols $D_i$ subject only to the relations

$$[D_i, f] = \frac{\partial f}{\partial x_i}.$$  

(3.3.1)

Note that we do not assume the $D_i$ to commute with each other; in fact, they generate the free associative algebra $\mathbb{R}(D_1, \ldots, D_n)$. 

Let $\mathbb{D}(\mathbb{R}^n)_{\text{Lie}}$ be $\mathbb{D}(\mathbb{R}^n)$ considered as a Lie algebra with respect to the usual commutator $[x, y] = xy - yx$. Then the Lie subalgebra in $\mathbb{D}(\mathbb{R}^n)_{\text{Lie}}$ generated by $D_1, \ldots, D_n$ is the free Lie algebra $FL(\mathbb{R}^n)$.

The space $\mathcal{P}(\mathbb{R}^n)$ is identified with the Lie subalgebra in $\mathbb{D}(\mathbb{R}^n)_{\text{Lie}}$ generated by elements of the form

$$\sum_{i=1}^{n} f_i(x_1, \ldots, x_n) D_i.$$

Indeed, such an element corresponds to $D_v$ from Example 3.2.4(a), where $v$ is the vector field $\sum f_i \partial_i$.

We see easily that a general element of $\mathcal{P}(\mathbb{R}^n)$ can be uniquely written in the form (sum with only finitely many nonzero summands)

$$\eta = \sum_{p=1}^{\infty} \sum_{i_1, \ldots, i_p} f_{i_1, \ldots, i_p}(x) [D_{i_1}, [D_{i_2}, \ldots [D_{i_{p-1}}, D_{i_p}]\ldots],$$

where, for each $p$, the $(i_1, \ldots, i_p)$ run over the set of multiindices such that the brackets in the above sum form a Hall basis in $FL_p(\mathbb{R}^n)$, see [Re]. The structure of a vector bundle, i.e., of a module over the ring $C^\infty(\mathbb{R}^n)$, is given by multiplying the $f_{i_1, \ldots, i_p}$ simultaneously with a given function $f$. The anchor map takes a sum above into $\sum f_i(x) \partial_i$, with $f_i$ coming from the sub-sum with $p = 1$.

Note that the higher commutators of the $D_i$ commute with functions in virtue of (4.3.1).

Let us denote by

$$\mathcal{P}_p(\mathbb{R}^n) = C^\infty(\mathbb{R}^n) \otimes_{\mathbb{R}} FL_p(\mathbb{R}^n)$$

the part of $\mathcal{P}(\mathbb{R}^n)$ corresponding to the summand in (3.3.2) with given $p$. Then we have that the degree $\geq 2$ part forms a graded Lie algebra:

$$[\mathcal{P}_p(\mathbb{R}^n), \mathcal{P}_q(\mathbb{R}^n)] \subset \mathcal{P}_{p+q}(\mathbb{R}^n), \quad p, q \geq 2,$$

while

$$[\mathcal{P}_1(\mathbb{R}^n), \mathcal{P}_q(\mathbb{R}^n)] \subset \mathcal{P}_q(\mathbb{R}^n) \oplus \mathcal{P}_{1+q}(\mathbb{R}^n).$$

If $E$ is a trivial rank $N$ bundle on $\mathbb{R}^n$ with connection $\nabla$, we have the connection operators

$$\nabla_i = \partial_i + A_i(x), \quad A_i(x) \in C^\infty(\mathbb{R}^n) \otimes \text{Mat}_N(\mathbb{R}).$$

The element $D_i \in \mathcal{P}(\mathbb{R}^n)$ is sent by the homomorphism $h$ from the formulation of Theorem 3.2.5, to the noncommutative vector field $D_{\partial_i}$ which acts on $(E, \nabla)$ by

$$(D_{\partial_i})_{E, \nabla} = \nabla_i.$$

**Remarks.** (a) The above description of $\mathcal{P}(\mathbb{R}^n)$ is a particular case of the construction in [CLP], Example 2.2 called the transformational Lie-Rinehart algebra. This construction was used in loc. cit. to define the free Lie-Rinehart algebra generated by an anchored vector space (the functor $\text{FR}_{\text{CLP}}$, see Remark 2.2.7). In fact, in our case $T_{\mathbb{R}^n}$ is trivialized,
so $\mathcal{P}(\mathbb{R}^n) = FRCLP(\mathbb{R}^n)$ where $\mathbb{R}^n$ on the right is considered as a $C^\infty(\mathbb{R}^n)$-anchored vector space by sending the $i$th basis vector of $\mathbb{R}^n$ to $\partial_i$.

(b) The algebra $\mathcal{D}(\mathbb{R}^n)$ is identified with the enveloping algebra of the Lie-Rinehart algebra $\mathcal{P}(\mathbb{R}^n)$. It is a particular case of algebras of noncommutative differential operators to be studied later.

4. The fundamental Lie algebra.

(4.1) Interpretation via curvatures. Let $X$ be a $C^\infty$-manifold and $x \in X$. We call the fundamental Lie algebra of $X$ at $x$ and denote $\mathcal{P}(X, x)$ the fiber of $\mathcal{P}_X$ at $x$. It can be seen as the Lie algebra of the group of $x$-based loops in $X$ modulo reparametrizations and cancellations. Proposition 3.1.1 implies the following.

(4.1.1) Proposition. The Lie algebra $\mathcal{P}(X, x)$ comes with a natural filtration $\{\mathcal{P}_{\leq d}(X, x)\}$, $d \geq 2$, compatible with the Lie algebra structure. The lowest term of this filtration, i.e., $\mathcal{P}_{\leq 2}(X, x)$, is canonically identified with $\Lambda^2(T_x X)$. Further, the associated graded Lie algebra is canonically identified with the degree $\geq 2$ part of the free Lie algebra generated by $T_x X$: $\bigoplus_{d \geq 2} \mathcal{P}_{\leq d}(X, x)/\mathcal{P}_{\leq d-1}(X, x) \simeq FL_{\geq 2}(T_x X)$.

Another interpretation of $\mathcal{P}(X, x)$ is that it is the universal receptacle of all the covariant derivatives of the curvatures of all connections in vector bundles on $X$. Indeed, if $(E, \nabla) \in \text{Bun}_\nabla(X)$, then we have the “holonomy morphism” of Lie algebras:

$$H_{E, \nabla} : \mathcal{P}(X, x) \to \text{End}(E_x).$$

Proposition 4.1.1 can be interpreted as follows. The curvature $F = F_\nabla$ of a connection $\nabla$ is a tensor $F \in \Omega^2_X \otimes \text{End}(E)$, and the lowest term of the filtration on $\mathcal{P}(X, x)$ is mapped via $F$:

$$H_{E, \nabla}|_{\Lambda^2 T_x X} = F|_x : \Lambda^2 T_x X \to \text{End}(E).$$

Higher covariant derivatives of $F$ are not invariantly defined, as the bundle $\Omega^2_X \otimes \text{End}(E)$ carries no natural connection induced by $\nabla$. What is naturally defined, is the “de Rham differential”

$$\nabla^{(2)} : \Omega^2_X \otimes \text{End}(E) \to \Omega^3_X \otimes \text{End}(E),$$

which satisfied $\nabla^{(2)}(F_\nabla) = 0$, the Bianchi identity. If we choose local coordinates $x_1, ..., x_n$, we can trivialize both $\Omega^1_X$ and $\Omega^2_X$ and get a map

$$\tilde{\nabla}^{(2)} : \Omega^2_X \otimes \text{End}(E) \to \Omega^1_X \otimes \Omega^2_X \otimes \text{End}(E),$$
of which $\nabla^{(2)}$ is obtained by antisymmetrization. The image $\tilde{\nabla}^{(2)}(F_V)$ lies then in
\[
\text{Ker}\left\{\Omega^1_X \otimes \Omega^2_X \otimes \text{End}(E) \to \Omega^3_X \otimes \text{End}(E)\right\} = \text{Hom}(\text{FL}_3(TX), \text{End}(E)).
\]

The last identification corresponds to the well known interpretation of the Bianchi identity as the Jacobi identity for the connection operators $\nabla_i$. However, the operator $\tilde{\nabla}^{(2)}$ is not invariantly defined, and what we have instead is Proposition 4.1.1.

Similarly to Theorem 3.2.5, elements of $\mathcal{P}(X,x)$ can be viewed as derivations (in the sense of [KM], §1.7) of the fiber functor $(E, \nabla) \mapsto E_x$ from $\text{Bun}_X$ to vector spaces.

(4.2) Homology of $\mathcal{P}(X,x)$: preliminaries. As $\mathcal{P}(X,x)$ is isomorphic (non-canonically) to a subalgebra of a free Lie algebra, namely, to $\text{FL}_{\geq 2}(T_x X) \subset \text{FL}(T_x X)$, it is itself free by the Shirshov-Witt theorem, see [Re], Ch.2. The number of free generators of $\mathcal{P}(X,x)$ is, however, infinite.

Let $k$ be a field of characteristic 0. For an $k$-algebra $g$ we denote by $H^\text{Lie}_{p}(g)$ the Lie algebra homology of $g$ with trivial coefficients $k$. Because $\mathcal{P}(X,x)$ is free (with $k = \mathbb{R}$), we have
\[
H^\text{Lie}_p\mathcal{P}(X,x) = 0, \quad p \geq 2.
\]

Let us determine the space
\[
H^\text{Lie}_1\mathcal{P}(X,x) = \mathcal{P}(X,x)/[\mathcal{P}(X,x), \mathcal{P}(X,x)],
\]

i.e., the maximal abelian quotient. Its importance stems from the following interpretation of $H^\text{Lie}_1$ as “the space of generators”.

(4.2.2) Proposition. Let $\mathfrak{g}$ be an $\mathbb{Z}_+$-graded Lie $k$-algebra which is free (i.e., has a system of homogeneous free generators). Suppose $W \subset \mathfrak{g}$ is a graded $k$-vector subspace such that the composite map
\[
W \to \mathfrak{g} \to H^\text{Lie}_1(\mathfrak{g})
\]
is an isomorphism. Then $W$ is a space of free generators for $\mathfrak{g}$, i.e., the natural morphism of Lie algebras $\text{FL}(W) \to \mathfrak{g}$ is an isomorphism. \hfill \Box.

We start with recalling the description of $H^\text{Lie}_1(\text{FL}_{\geq 2}(V))$ where $V$ is a finite-dimensional $k$-vector space. This homology space is graded:
\[
H^\text{Lie}_1(\text{FL}_{\geq 2}(V)) = \bigoplus_{d=2}^{\infty} H^\text{Lie}_{1,d}(\text{FL}_{\geq 2}(V)),
\]

where $H^\text{Lie}_{1,d}$ is the image of $\text{FL}_d(V) \subset \text{FL}_{\geq 2}(V)$ in the maximal abelian quotient of $\text{FL}_{\geq 2}(V)$.

Let $\text{Vect}_k$ be the category of finite-dimensional $k$-vector spaces. For any sequence of integers $\alpha = (\alpha_1 \geq \ldots \geq \alpha_n \geq 0)$ (with arbitrary $n$) we have the polynomial functor $\Sigma^\alpha : \text{Vect}_k \to \text{Vect}_k$, see [Macd]. For $V = k^n$ the space $\Sigma^\alpha(V)$ is the space of irreducible representation of the algebraic group $GL_n/k$ with highest weight $\alpha$. 
(4.2.4) Theorem. (a) We have an identification of functors of $V$:
\[ H_{1,d}^{Lie}(FL_{\geq 2}(V)) \simeq \Sigma^{d-1,1}(V). \]
(b) Let $z_1, \ldots, z_n$ be a basis of $V$. Then the images of the elements
\[ [z_{i_1}, [z_{i_2}, \ldots [z_{i_{d-1}}, z_{i_d}] \ldots], \quad i_1 \geq i_2 \geq \ldots \geq i_{d-1} < i_d, \]
form a basis of $H_{1,d}^{Lie}(FL_{\geq 2}(V))$.
(c) The elements as above, taken for all $d$, form a system of free generators of $FL_{\geq 2}(V)$.

Parts (a) and (b) are due to Reutenauer [Re], §§8.6.12 and 5.3. Part (c) follows from Proposition 4.2.2.

Let us now give a geometric interpretation of Theorem 4.2.4. Consider $V$ as an affine algebraic variety over $k$ and let $\Omega^p(V)$ be the space of regular $p$-forms on $V$. Thus
\[ \Omega^p(V) = \bigoplus_{d=0}^{\infty} \Lambda^p(V^*) \otimes S^d(V^*). \]
The de Rham differential $d : \Omega^p(V) \to \Omega^{p+1}(V)$ makes $\Omega^\bullet(V)$ into a complex exact everywhere except the 0th term, and the space of closed $p$-forms has the following well known decomposition as a $GL(V)$-module:

(4.2.5) $\Omega^{p,cl}(V) = d(\Omega^{p-1}(V)) = \bigoplus_{d=1}^{\infty} \Sigma^{d,1,\ldots,1}(V^*)$

(with $(p-1)$ occurrences of 1 in the RHS). See, e.g., [GKZ], Proposition 14.2.2. Thus

(4.2.6) $H_1^{Lie}(FL_{\geq 2}(V^*)) = \Omega^{2,cl}(V)$.

The first homology of $FL_{\geq 2}(V)$ is the restricted dual of this space with respect to the decomposition (4.2.5), i.e., the direct sum of the duals of the summands. Let us denote by

(4.2.7) $\Gamma_p(V) = \bigoplus_{d=0}^{\infty} \Lambda^p(V) \otimes S^d(V)$

the restricted dual of $\Omega^p(V)$. The de Rham differential $d$ on $\Omega^\bullet(V)$ induces differentials
\[ \partial : \Gamma_p(V) \to \Gamma_{p-1}(V) \]
by dualization. The dual of
\[ \Omega^{2,cl}(V) = \text{Ker}\{d : \Omega^2(V) \to \Omega^3(V)\} \]
is then
\[ \text{Coker}\{\partial : \Gamma_3(V) \to \Gamma_2(V)\} = \text{Ker}\{\partial : \Gamma_1(V) \to \Gamma_0(V)\}, \]
which we denote by $\Gamma_1^d(V)$.

(4.3) Homology of $\mathcal{P}(X,x)$ and currents. Let now $k = \mathbb{R}$, let $X$ be a $C^\infty$-manifold and $x \in X$. Denote by $\Gamma_p(X,x)$ the space of $p$-currents on $X$ is the sense of de Rham, which are supported at $x$. In other words, an element of $\Gamma_p(X,x)$ is a linear functional $\Omega^p(X) \to \mathbb{R}$ on the space of all $C^\infty$ forms, which depends only on a finite jet of a form at $x$. It is clear...
that for an $\mathbb{R}$-vector space $V$ considered as a $C^\infty$-manifold the space $\Gamma_p(V,0)$ is the same as $\Gamma_p(V)$ defined earlier.

As before, the de Rham differential on forms defines differentials $\partial : \Gamma_p(X, x) \to \Gamma_{p-1}(X, x)$.

**Theorem.** The space $H^1_{\text{Lie}}(\mathcal{P}(X, x))$ is canonically identified with $\Gamma^1_{\text{cl}}(X, x)$, the space of closed 1-currents supported at $x$.

By dualizing, we get an essentially equivalent formulation:

**Corollary.** The space $H^1(\mathcal{P}(X, x))$ of first cohomology with coefficients in $\mathbb{R}$ is canonically identified with $\hat{\Omega}^{2,\text{cl}}_{X, x}$, the space of formal germs of closed 2-forms on $X$ near $x$.

To prove Theorem 4.3.1, we first construct a natural linear map

$$\tau : \mathcal{P}(X, x) \to \Gamma^1_{\text{cl}}(X, x),$$

vanishing on the commutators. Indeed, an element $\zeta \in \Gamma^1_{\text{cl}}(X, x)$ can be seen, as above, as a functional on $\Omega^{2,\text{cl}}(X)$ whose value on a closed form $\omega$ depends only on a finite jet of $\omega$ near $x$. On the other hand, any $\eta \in \mathcal{P}(X, x)$ gives rise to the holonomy element

$$H_{E, \nabla}(\eta) \in \text{End}(E_x)$$

for any bundle with connection $(E, \nabla)$ on $X$. Let us restrict to the case when the rank of $E$ is equal to 1. Then the curvature $F_{\nabla}$ is a scalar closed 2-form: $F_{\nabla} \in \Omega^{2,\text{cl}}(X)$. At the same time $\text{End}(E_x) = \mathbb{R}$.

So by sending $\eta$ to $H_{E, \nabla}(\eta)$ where $(E, \nabla)$ is a line bundle with connection such that $F_{\nabla} = \omega$, we associate to $\eta$ a linear functional of $\omega \in \Omega^{2,\text{cl}}(X)$ which clearly depends only on a finite jet of $\omega$ at $x$. This defines the map $\tau$.

Next, to show that $\tau$ is an isomorphism, it is enough to choose a local coordinate system on $X$ near $x$, in which case we can assume $X = V = \mathbb{R}^n$ and the homomorphism $\tau$ is identified with that of Theorem 4.2.4. □

**Remarks.**

(a) Theorem 4.3.1 has the following group-theoretical analog. Let $F_n$ be the free group on $n$ generators, and $[F_n, F_n]$ be its commutator. Being a subgroup of a free group, it is free itself and has only the first group homology space $H^1_{\text{Gr}}([F_n, F_n], \mathbb{R})$ nontrivial. This space is described as follows. Recall that $BF_n$, the classifying space of $F_n$, is the bouquet of $n$ circles. Let $\mathbb{R}^n_\square$ be the Euclidean space $\mathbb{R}^n$ with CW-decomposition into unit cubes of the standard integer lattice $\mathbb{Z}^n$. Then $B([F_n, F_n]) = \text{Sk}_1(\mathbb{R}^n_\square)$ is the 1-skeleton of this CW-complex. Accordingly,

$$H^1_{\text{Gr}}([F_n, F_n], \mathbb{R}) = H^1_{\text{Top}}(\text{Sk}_1(\mathbb{R}^n_\square), \mathbb{R}) = Z_1(\mathbb{R}^n_\square, \mathbb{R})$$

is the space of cellular 1-cycles of the CW-complex $\mathbb{R}^n_\square$.

(b) The identification in (4.2.6) and thus in Theorem 4.3.1, seems related to the results of Feigin and Shoikhet [FS] on the structure of $[A, A]/[A, [A, A]]$ where $A$ is the free associative algebra generated by $V$. 


(c) An identification equivalent to that of Theorem 4.2.4 (a) was also found by Polyakov ([Po], §3) who was studying “gauge invariant words”, i.e., traces of iterated covariant derivatives of the curvature of an indeterminate connection on $X$. Such iterated derivatives are labelled, as discussed in (4.1), by elements of $\mathcal{P}(X, x)$, while traces vanish on $[\mathcal{P}(X, x), \mathcal{P}(X, x)]$.

(4.4) Formal classification of connections. Let $k$ be a field of characteristic 0, and $\mathfrak{D}_n = \text{Spf } k[[x_1, ..., x_n]]$ be the $n$-dimensional formal disk over $k$, see Example A.2.2(a). We can then speak about (finite rank) vector bundles with connection $(E, \nabla)$ on $\mathfrak{D}_n$. Every vector bundle on $\mathfrak{D}_n$ is trivial but perhaps not canonically. If $E = \mathcal{O}^N$ is the standard trivial rank $N$ bundle, a connection in $E$ is given by operators

\[ \nabla_i = \partial_i + A_i, \quad A_i \in \text{Mat}_N k[[x_1, ..., x_n]], \]

and isomorphisms of such connections are given by gauge transformations

\[ A_i \mapsto g^{-1}A_i g + g^{-1}\partial_i g, \quad g \in GL_N k[[x_1, ..., x_n]]. \]

We denote by $\text{Bun}_\nabla(\mathfrak{D}_n)$ the category of finite rank vector bundles with connections on $\mathfrak{D}_n$.

More generally, let $G$ be an affine algebraic group over $k$ with Lie algebra $\mathfrak{g}$. Then we can speak about principal $G$-bundles with connection on $\mathfrak{D}_n$. For example, gauge transformations of the standard trivial $G$-bundle form the group $G(k[[x_1, ..., x_n]])$.

(4.4.3) Theorem. The category $\text{Bun}_\nabla(\mathfrak{D}_n)$ is equivalent (as a tensor category) to the category of finite-dimensional representations of the Lie algebra $\mathfrak{fl}_{\geq 2}(k^n)$.

This is a corollary of Theorem A.7.3 about transitive Lie algebroids on $\mathfrak{D}_n$. □

(4.4.4) Corollary. Let $G$ be an algebraic group as above. Then gauge equivalence classes of connections in principal $G$-bundles on $\mathfrak{D}_n$ are in bijection with $G$-conjugacy classes of homomorphisms $\mathfrak{fl}_{\geq 2}(k^n) \to \mathfrak{g}$. □

Let us consider the case of the trivial $G$-bundle $G \times \mathfrak{D}_n$, so a connection $\nabla$ is given by $\nabla_i = \partial_i + A_i$ with $A_i \in \mathfrak{g}[[x_1, ..., x_n]]$. By a restricted gauge transformation of $G \times \mathfrak{D}_n$ we mean a transformation

\[ g(x_1, ..., x_n) \in G(k[[x_1, ..., x_n]]) \]

whose value at 0 (i.e., the constant term) is equal to 1. Let

\[ F_{ij} = \partial_i A_j - \partial_j A_i - [A_i, A_j] \]

be the $(i, j)$th component of the curvature of $A$.

(4.4.5) Corollary (Taylor theorem for connections). Given a connection $\nabla$ as above, the set of elements

\[ \nabla_{i_1} \nabla_{i_2} ... \nabla_{i_{d-1}} F_{i_{d-1} i_d} \in \mathfrak{g}, \quad i_1 \geq i_2 \geq ... \geq i_{d-1} < i_d, \quad d \geq 2, \]

determine $\nabla$ uniquely up to restricted gauge equivalence. Conversely, given any system of elements

\[ \gamma_{i_1, ..., i_d} \in \mathfrak{g}, \quad i_1 \geq i_2 \geq ... \geq i_{d-1} < i_d, \quad d \geq 2, \]

there exists a unique (up to restricted gauge equivalence) connection $\nabla$ on $\mathcal{D}_n$ with

$$\nabla_{i_1} \nabla_{i_2} \cdots \nabla_{i_d-2} F_{i_{d-1}, i_d} = \gamma_{i_1, \ldots, i_d}.$$ 

This is a consequence of Corollary 4.4.4 as well as Theorem 4.3.4(c) describing a system of free generators of $FL_{\geq 2}(k^n)$. 

\[ \square \]

5. Noncommutative differential operators.

(5.1) Basic definitions. Let $X$ be a $C^\infty$-manifold. We denote by $\mathbb{D}_X$ the sheaf $U(P_X)$ of enveloping algebras of the Lie algebroid $P_X$. Sections of $\mathbb{D}_X$ will be called noncommutative differential operators.

By Theorem A.3.11, $\mathbb{D}_X$ is a sheaf of left $C^\infty_X$-bialgebras; in particular, we have a morphism of algebras

$$\Delta : \mathbb{D}_X \to \mathbb{D}_X \otimes_{C^\infty_X} \mathbb{D}_X.$$ 

Further, each section $P$ of $\mathbb{D}_X$ gives a differential operator $P_{E,\nabla} : E \to E$ for any $(E, \nabla) \in \text{Bun}_\nabla(X)$. These operators are natural with respect to morphisms of bundles with connections and behave with respect to the tensor product as follows:

$$P_{E \otimes E', \nabla \otimes \nabla'} = \sum (P^{(1)}_{i,\nabla} \otimes (P^{(2)}_{i,\nabla'}), \quad \Delta(P) = \sum P^{(1)}_i \otimes P^{(2)}_i.$$ 

By a natural differential operator on $X$ of order $\leq d$ we will mean a system $\{P_{E,\nabla} : E \to E\}$ of differential operators of order $\leq d$ defined for all bundles with connections $(E, \nabla)$ at once and satisfying the naturality property as above. Compare [St], Def. 3.4 for the case of Riemannian manifolds, not connections in bundles on a given manifold. We denote by $\mathcal{N}D_X^{\leq d}$ the sheaf of natural differential operators on $X$ of order $\leq d$, and by $\mathcal{N}D_X$ the union of these sheaves for all $d$.

By the above, each section of $\mathbb{D}_X$ gives a natural differential operator, so we have a homomorphism of sheaves of algebras on $X$ extending (3.2.5):

$$h : \mathbb{D}_X \to \mathcal{N}D_X.$$ 

(5.1.4) Example. Let $X = \mathbb{R}^n$. Then the algebra $\mathbb{D}(\mathbb{R}^n)$ of global sections of $\mathbb{D}_{\mathbb{R}^n}$ was already described in (3.3). It is generated by functions $f(x_1, \ldots, x_n)$ and symbols $D_1, \ldots, D_n$ subject only to the relations (3.3.1). For example, we have the (noncommutative) Laplace operator $\Delta \in \mathbb{D}(\mathbb{R}^n)$ whose definition and the action on vector bundles with connections are given by:

$$\Delta = \sum_{i=1}^n D_i^2, \quad \Delta_{E,\nabla} = \sum_{i=1}^n \nabla_i^2 : E \to E.$$
Further, we have a similar situation for any Riemannian manifold $X$: the metric defines an element $\Delta_X \in \mathbb{D}(X)$.

(5.2) **Two filtrations on** $\mathbb{D}_X$. As for the enveloping algebra of any Lie algebroid, $\mathbb{D}_X = U(\mathcal{P}_X)$ has the filtration $F$, see (1.2.2) by the number of noncommutative vector fields needed to produce a given section $P$ of $\mathbb{D}_X$. Thus

\[(5.2.1) \quad \text{gr}^F(\mathbb{D}_X) = S^*(\mathcal{P}_X)\]

is a commutative algebra.

Second, recall that $\mathcal{P}_X = \bigcup_{d=1}^{\infty} \mathcal{P}_{\leq d}$ is itself a filtered Lie algebroid, so we can take the filtration $\{U_{\leq d}(\mathcal{P}_X)\}$ on $U(\mathcal{P}_X)$ induced by this filtration on $\mathcal{P}_X$, as in (A.5.8). We call this filtration the *filtration by degree* and write $\mathbb{D}_{\leq d}$ for $U_{\leq d}(\mathcal{P}_X)$. We also write $\text{deg}(P) \leq d$, if $P \in \mathbb{D}_{\leq d}$. We then see easily that

\[(5.2.2) \quad \text{gr}^{\text{deg}}(\mathbb{D}_X) = T^*(T_X) = \bigoplus_{d=0}^{\infty} T^\otimes d_X = U(\text{FL}(T_X)),\]

is the full tensor algebra of $T_X$, with the grading induced by the grading on $\text{FL}(T_X)$ on its enveloping algebra. So it is noncommutative.

For a noncommutative differential operator $P$ on $X$ of degree $\leq d$ we denote by Smbl$_d(P) \in T^\otimes d_X$ the highest symbol of $P$, i.e., the image of $P$ in $\text{gr}^d(\mathbb{D}_X)$. Thus Smbl$_d(P)$ can be seen as a noncommutative homogeneous polynomial on $T^*X$, the cotangent bundle of $X$.

(5.2.3) **Proposition.** Let $(E, \nabla) \in \text{Bun}_\nabla(X)$, and $P$ be a section of $\mathbb{D}_X$ with $\text{deg}(P) \leq d$. Then the linear differential operator $P_{E, \nabla} : E \to E$ has order $\leq d$ in the usual sense and its $d$th symbol (in the usual sense) is equal to the symmetrization of Smbl$_d(P)$ times the identity endomorphism of $E$.

**Proof:** This is true for the case $P = D_v$ for a vector field $v$. The general case follows from multiplicativity. \qed

(5.3) **Classification of natural differential operators.** Similarly to what was done in [Ep] [St] for the Riemannian case, it is possible to use classical invariant theory to show, under certain assumptions, that every natural differential operator $P$ (resp. any noncommutative vector field) comes from a unique section of $\mathbb{D}_X$ (resp. $\mathcal{P}_X$). These assumptions specify the way $P_{E, \nabla}$ is allowed to depend on the connection $\nabla$ in a fixed bundle $E$.

Let $\text{Conn}(E)$ be the sheaf of all $C^\infty$ connections in $E$. As well known, it is a sheaf of torsors over the sheaf of groups $\Omega^1_X \otimes \text{End}(E)$. For $x \in X$ let $J^p_x \text{Conn}(E)$ be the space of all $p$-jets of connections in $E$ near $x$. This space has a natural structure of an affine algebraic variety over $\mathbb{R}$ (isomorphic, non-canonically, to an affine space of suitable dimension). When $x$ varies, we get a nonlinear bundle $J^p \text{Conn}(E) \to X$, with fibers $J^p_x \text{Conn}(E)$. We can speak about $C^\infty$ functions on $J^p \text{Conn}(E)$ that are polynomial on each fiber of this bundle, or about
fiberwise polynomial morphisms of $J^p \text{Conn}(E)$ to another vector bundle. We denote by
\[(5.3.1) \quad j_p : \text{Conn}(E) \to J^p \text{Conn}(E)\]
the “universal differential operator” which to each section $\nabla$ associates the collection of its $p$-jets at each $x \in X$.

Similarly, let $J^p E \to E$ be the bundle of $p$-jets of sections of $E$, and
\[(5.3.2) \quad j_p : E \to J^p E\]
be the universal differential operator of order $p$. Thus any linear differential operator $\Phi : E \to E$ of order $p$ can be represented as the composition, with $j_p$, of a uniquely defined linear map (morphism of vector bundles) $\tilde{\Phi} : J^p E \to E$.

\[\text{(5.3.3) Definition. A natural differential operator } P = (P_{E,\nabla}) \text{ of order } \leq d \text{ is called regular, if there exists } p \text{ such that for each } E \text{ there exists a map of bundles} \]
\[\tilde{P} = \tilde{P}_E : J^p \text{Conn}(E) \times J^d E \to E, \]
\[\text{polynomial in the first variable, linear in second variable and such that for each section } s \text{ of } E \text{ and each connection } \nabla \text{ in } E \text{ we have} \]
\[P_{E,\nabla}(s) = \tilde{P}(j_p(\nabla), j_d(s)). \]

Informally, we require that $P_{E,\nabla}$ depends only on the $p$-jet of $\nabla$, and in a polynomial way.

\[\text{(5.3.4) Theorem. (a) The subalgebra in } \mathcal{N} \mathcal{D}_X \text{ formed by regular operators, is identified with } \mathbb{D}_X, \text{ via the homomorphism } h \text{ from (6.1.1).} \]
\[\text{(b) Under this identification the Lie subalgebra in } \mathcal{V} \text{Vect}_X \text{ formed by regular noncommutative vector fields, is identified with } \mathcal{P}_X. \]
\[\text{(c) The subalgebra in } \mathcal{N} \mathcal{E}_X \text{ formed by regular natural endomorphisms, is identified with } U(\mathcal{P}_X^s). \]

\[\text{Proof: We start with (c), so the } P_{E,\nabla} \text{ are assumed to be endomorphisms of the bundle } E. \]
It is enough to work locally, so we assume $X = \mathbb{R}^n$ with coordinates $x_1, \ldots, x_n$, and we also assume $E$ to be the trivial bundle of rank $N$. For any $x \in \mathbb{R}^n$ notice that
\[(5.3.5) \quad J_x^\infty \text{Conn}(E) := \lim_{\leftarrow p} J_x^p \text{Conn}(E) = \text{Conn}(E|_{\mathcal{D}_{n,x}})\]
is just the set of all connections in (the restriction of) $E$ on $\mathcal{D}_{n,x}$, the formal disk near $x$. Naturality of $P$ means that the map $\tilde{P}$ in (5.3.3) (with $d = 0$) is equivariant with respect to the action of $\text{Aut}(E) = GL_N C^\infty(\mathbb{R}^n)$, the group of $C^\infty$-automorphisms of $E$. We now apply Corollary 4.4.5 about classification of connections on $\mathcal{D}_{n,x} \simeq \mathcal{D}_n$ and conclude that for any $x \in X = \mathbb{R}^n$ the operator
\[(5.3.6) \quad P_{E,\nabla,x} : E_x \to E_x \]
given by the action of $P_{E, \nabla}$ on the fiber at $x$, depends only on finitely many operators
\begin{equation}
R_{i_1, \ldots, i_q}(x) := \nabla_{i_1} \nabla_{i_2} \cdots \nabla_{i_{q-2}} F_{i_{q-1}, i_q} : E_x \to E_x.
\end{equation}

Here "depends" means that each matrix element of $P_{E, \nabla, x}$ can be expressed as a polynomial
in the matrix elements of the operators (5.3.7).

We now use the naturality of $P$ again, which, together with Theorem 4.4.3, implies that
the dependence of (the matrix elements of) $P_{E, \nabla, x}$ on (those of) the $R_{i_1, \ldots, i_q}(x)$ is equivariant
with respect to the simultaneous action of $GL(E_x) = GL_N(\mathbb{R})$ on all of them.

The following statement is one of the many instances of the “main theorem of invariant
theory” [We].

(5.3.8) Proposition. (a) Let $V$ be a finite-dimensional $\mathbb{R}$-vector space of dimension $N$.
Consider $\text{End}(V)$ as an affine space over $\mathbb{R}$ of dimension $N^2$, and let $\mathcal{E}_m(V)$ be the space of
all polynomial maps
\[ F = F(Z_1, \ldots, Z_m) : \text{End}(V)^m \to \text{End}(V), \]
which are equivariant with respect to the simultaneous action of $GL(V)$. Then the natural
map
\[ \mathbb{R}\langle Z_1, \ldots, Z_n \rangle \to \mathcal{E}_m(V), \]
which takes each noncommutative polynomial into the induced map $\text{End}(V)^m \to \text{End}(V)$,
is surjective.

(b) In particular, under the standard embedding $\text{End}(\mathbb{R}^N) \subset \text{End}(\mathbb{R}^{N+1})$ any element of
$\mathcal{E}_m(\mathbb{R}^{N+1})$ takes $\text{End}(\mathbb{R}^N)^m$ to $\text{End}(\mathbb{R}^N)$, so we have a surjection $\mathcal{E}_m(\mathbb{R}^{N+1}) \to \mathcal{E}_m(\mathbb{R}^N)$.

(c) The resulting homomorphism
\[ \mathbb{R}\langle Z_1, \ldots, Z_n \rangle \to \lim_{\leftarrow N} \mathcal{E}_m(\mathbb{R}^N), \]
is an isomorphism. \hfill \square

Applying Proposition 5.3.8 to $m$ being the number of the $R_{i_1, \ldots, i_q}(x)$ on which $P_{E, \nabla, x}$
depends, we conclude that for any $x \in X$ the operator $P_{E, \nabla, x}$ is represented as a noncom-
mutative polynomial in these $R_{i_1, \ldots, i_q}(x)$, and this polynomial is defined uniquely, if we allow
$N \to \infty$. This precisely means that $P$ comes from a uniquely defined element of the envelop-
ing algebra of the bundle of Lie algebras $P_{\mathbb{R}^n}$, since the latter is freely generated by elements
that are in bijection with the $R_{i_1, \ldots, i_q}$. This proves part (c) of the theorem.

(b). If $P$ is a regular noncommutative vector field, look at the ordinary vector field $\alpha(P)$
and its lift $D_{\alpha(P)}$ into $\text{Vect}$. Then $P' = P - D_{\alpha(P)}$ is a regular natural endomorphism, so it
comes from a section of $U(P_{\mathbb{X}}^2)$ by (c). Because $P$ and therefore $P'$ satisfies the $\otimes$-Leibniz
rule, the corresponding section of $U(P_{\mathbb{X}}^2)$ is (fiberwise) primitive, so it is really a section of
$P_{\mathbb{X}}^2$.

(a) This is reduced to (b), similarly to the argument in [St], Th. 3.7. Namely, if $P = (P_{E, \nabla})$
is a natural differential operator of order $d$, then the family of the degree $d$ symbols (in the
usual sense) of the $P_{E, \nabla}$ gives a linear system

$$\sigma_d(P_{E, \nabla}) : E \to E \otimes S^d T_X$$

of natural endomorphisms parametrized by $S^d T_X$. We can work locally in coordinates $x_1, ..., x_n$, and write

$$\sigma_d(P_{E, \nabla}) = \sum_{i_1 + ... + i_n = d} Q_{i_1, ..., i_n} \otimes \partial_1^{i_1} ... \partial_n^{i_n},$$

where each $Q_{i_1, ..., i_n}$ is a natural endomorphism, which is also regular since $P$ is regular.

Lifting each $Q_{i_1, ..., i_n}$ to a section $U_{i_1, ..., i_n}$ of $U(P_X)$, as in (c), we form a section

$$\tau = \sum U_{i_1, ..., i_n} D_{i_1}^{j_1} ... D_{i_n}^{j_n}$$

of $\mathcal{D}_X$. Here $D_i = D_{\partial_i}$ is the noncommutative vector field corresponding to $\partial_i$. We then find, using Proposition 5.2.3, that the natural differential operator $P - h(\tau)$ has order less than $d$, and continue by induction to argue that $P$ comes from a section $\tilde{P}$ of $\mathcal{D}_X$. The uniqueness of $\tilde{P}$ is shown in a similar way: looking at the highest symbols of the $h(\tilde{P})_{E, \nabla}$, expanding them in the components $Q_{i_1, ..., i_n}$, and using the uniqueness already proved for part (c). □

6. Algebro-geometric setting.

(6.1) Lie algebroids. Let $k$ be a field of characteristic 0, and $X$ be a scheme of finite type over $k$ (possibly singular). Then we have a coherent sheaf $T_X$ on $X$, called the tangent sheaf, such that $\Gamma(U, T_X) = \text{Der}(k[U])$, if $U \subset X$ is an affine open subset. Sections of $T_X$ will be called vector fields. A Lie algebroid on $X$ is a sheaf $G$ of Lie-Rinehart algebras on the Zariski topology of $X$, which is quasicoherent over $\mathcal{O}_X$. In other words $G$ is a quasicoherent sheaf of $\mathcal{O}_X$-modules equipped with a morphism of $\mathcal{O}_X$-sheaves $\alpha : G \to T_X$ called the anchor map and with a structure of a sheaf of Lie $k$-algebras such that the analog of (1.1.1-2) are satisfied: first, $\alpha$ is a morphism of sheaves of Lie $k$-algebras, and, second,

$$(6.1.1) \quad [x, f y] - f [x, y] = \text{Lie}_{\alpha(x)}(f) \cdot y.$$ 

where Lie stands for the action of vector fields on functions (Lie derivative). A Lie algebroid is called transitive, if $\alpha$ is surjective.

All the constructions of (1.1-2) extend easily to this setting. In particular, for a quasicoherent sheaf $\mathcal{M}$ on $X$ we have the Atiyah algebroid $\mathcal{A}_\mathcal{M}$, we have the concept of modules and of the enveloping algebra $U(\mathcal{G})$ of a Lie algebroid $\mathcal{G}$ which is a sheaf of associative algebras on $X$.

(6.1.2) Example. Assume $X$ smooth. Then $U(T_X) = \mathcal{D}_X$ is the sheaf of regular differential operators on $X$. 


We denote $\mathcal{G}^\circ = \text{Ker}(\alpha)$. By (6.1.1) we have that $\mathcal{G}^\circ$ is a sheaf of Lie $\mathcal{O}_X$-algebras.

For any quasicoherent sheaf $\mathcal{V}$ on $X$ equipped with a morphism $\beta: \mathcal{V} \to T_X$ (anchor map), we have $\mathcal{F}(\mathcal{V})$, the free Lie algebroid generated by $\mathcal{V}$, by sheafifying the construction of Theorem 2.1.2.

Further, $\mathcal{F}(\mathcal{V}) = \bigcup_{d=1}^\infty \mathcal{F}_{\leq d}(\mathcal{V})$ is a union of quasicoherent subsheaves of $\mathcal{O}_X$-modules (which are coherent if $\mathcal{V}$ is), and this filtration is compatible with the Lie algebra structure of sections. The sheaf of $\mathcal{O}_X$-Lie algebras

$$\mathcal{F}^\circ(\mathcal{V}) = \text{Ker}\{\alpha: \mathcal{F}(\mathcal{V}) \to T_X\}$$

has the induced filtration $\{\mathcal{F}^\circ_{\leq d}(\mathcal{V})\}_{d \geq 2}$ compatible with the Lie algebra structure in the fibers. Its associated graded $\mathcal{O}_X$-Lie algebra is isomorphic to $\mathcal{F}_{\geq 2}(\mathcal{V})$, the degree $\geq 2$ part of the sheaf of fiberwise free Lie algebras.

We denote by $\mathcal{P}_X = \mathcal{F}(T_X)$ the free Lie algebroid generated by $T_X$. For each $k$-point $x \in X$ we denote

$$\mathcal{P}(X, x) = \mathcal{P}^\circ_X \otimes_{\mathcal{O}_X} k_x$$

the fiber of $\mathcal{P}^\circ_X$ at $x$. This is a Lie $k$-algebra. If $x$ is a smooth point, then $\mathcal{P}(X, x)$ is free, and its (co)homology is described as in (4.3.1-2).

**6.2 Noncommutative differential operators.** From now on we assume that $X$ is a smooth algebraic variety over $k$. The constructions of (5.1) extend easily to this setting, giving a sheaf $\mathcal{D}_X = U(\mathcal{P}_X)$ of left $\mathcal{O}_X$-bialgebras, which is locally free as a left or right $\mathcal{O}_X$-module. As before, we have the two filtration on $\mathcal{D}_X$: by order and degree, with associated graded sheaves of algebras described as in (5.2.1-2).

As in the $C^\infty$-case, sections of $\mathcal{P}_X$ and $\mathcal{D}_X$ act in bundles with connections. There may not exist enough such bundles on all of $X$, so we have to work with bundles defined locally. For every Zariski open $U \subset X$ we denote by $\text{Bun}_U(U)$ the tensor category of vector bundles (of finite rank) on $U$ with connections.

Let $S$ be a scheme, and $x: S \to X$ ne a morphism. Then, by (A.3-4), $x^*\mathcal{D}_X$ (inverse image with respect to the left $\mathcal{O}_X$-module structure on $\mathcal{D}_X$) is a quasicoherent sheaf of $\mathcal{O}_S$-modules. In fact, it is a sheaf of unital $\mathcal{O}_S$-coalgebras.

Let $P \in \Gamma(S, x^*\mathcal{D}_X)$. Then, for any open $U \subset X$, and any $(E, \nabla) \in \text{Bun}_U(U)$ we have a morphism of sheaves on the open subscheme $x^{-1}(U) \subset S$:

$$P_{E, \nabla}: x^{-1}E \to x^*E.$$  

Here $x^{-1}E$ is the inverse image as a sheaf, and $x^*E$ is the inverse image as an $\mathcal{O}$-module. The construction of $P_{E, \nabla}$ is similar to the $C^\infty$-case. One starts with vector fields $v$ on $X$ acting by operators $\nabla_v$ of covariant differentiation, and then extends by multiplicativity (to get to $\mathcal{D}_X$) and by $\mathcal{O}_S$-linearity (to get to $x^*\mathcal{D}_S = \mathcal{O}_S \otimes_{x^{-1}\mathcal{O}_X} x^{-1}\mathcal{D}_X$).

Let $E$ be a vector bundle on an open $U \subset X$. Then, as in (5.3.2), we have $J^pE$, the vector bundle of $p$-jets of sections of $E$ and, for each $x: S \to X$ as above, we have a morphism of
sheaves on $x^{-1}(U)$:

(6.2.2) \[ j_{p,x} : x^{-1}E \to x^*J^pE, \]
called the universal differential operator at $x$. By a $p$th order differential operator in $E$ at $x$ we will mean a morphism of sheaves

\[ Q : x^{-1}E \to x^*E \]
on $x^{-1}(U)$ which is a composition of a morphism of $\mathcal{O}_{x^{-1}(U)}$-sheaves $x^*J^pE \to x^*E$ with $j_{p,x}$.

Further, we have the sheaf $\text{Conn}(E)$ of connections in $E$ and an affine bundle $\pi_p \text{Conn}(E) \to U$ of $p$-jets of connections. We consider $J^p \text{Conn}(E)$ as an algebraic variety over $U$, in contrast with $J^pE$ which we consider as a sheaf of $\mathcal{O}_U$-modules. We denote by $\pi_p \text{Conn}(E)$ the sheaf of regular sections of $\pi_p$. We have then the morphism of sheaves

(6.2.3) \[ j_p : \text{Conn}(E) \to J^p \text{Conn}(E). \]
Let also

(6.2.4) \[ x^*J^p \text{Conn}(E) := x^{-1}U \times_U J^p \text{Conn}(E) \xrightarrow{\pi_{p,x}} x^{-1}(U) \]
be the induced family over $U$.

(6.2.4) **Definition.** (a) A natural differential operator at $x$ is a system of differential operators

\[ P = \{ P_{E,\nabla} : x^{-1}E \to x^*E \}, \]
given for each open $U \subset X$ and each $(E, \nabla) \in \text{Bun}_\nabla(U)$, which is compatible with restrictions for open inclusions $U' \subset U$ and satisfies naturality for morphisms of bundles with connections.

(b) A natural differential operator $P$ at $x$ is called regular of order $\leq d$, if there exists $p \geq 0$ such that for any $U, E$ as above there is a morphism

\[ \tilde{P} = \tilde{P}_{U,E} : \pi_{p,x}^*J^dE \to \pi_{p,x}^*E \]
of sheaves of $\mathcal{O}$-modules on $x^*J^p \text{Conn}(E)$ with the following property. For any connection $\nabla$ in $E$ the morphism $P_{E,\nabla}$ is the composition of $j_{d,x}$ and

\[ (\text{Id} \times j_p(\nabla))^*(\tilde{P}) : x^*J^dE \to x^*E. \]

Here $\text{Id} \times j_p(\nabla) : x^{-1}(U) \to x^*J^p \text{Conn}(E)$ is the section induced by $j_p(\nabla) : U \to J^p \text{Conn}(E)$.

(6.2.5) **Theorem.** Sections of $x^*\mathbb{D}_X$ over $S$ are in bijection with regular differential operators at $x$.

**Proof:** Restricting to open $S' \subset S$, we see that regular operators form a sheaf on $S_{\text{Zar}}$, so it is enough to prove the statement locally on $S$. For this, it is enough to work locally on $X$, so we can assume that $X$ is an affine variety admitting an étale coordinate system, i.e., an étale map

\[ (x_1, ..., x_n) : X \to \mathbb{A}^n. \]
By Proposition 2.2.6, this allows us to trivialize $\mathcal{P}_X$ and thus $\mathbb{D}_X$ as left $\mathcal{O}_X$-modules, introducing sections $D_i \in \mathcal{P}(X)$ as in Example 3.3, and writing a general element of $\mathbb{D}(X)$ uniquely in the form (sum with only finitely many nonzero summands):

$$
\sum_{p=0}^{\infty} \sum_{i_1, \ldots, i_p=1}^{n} f_{i_1, \ldots, i_p} D_{i_1} \ldots D_{i_p}, \quad f_{i_1, \ldots, i_p} \in \mathcal{O}(X).
$$

Accordingly, a general section of $x^* \mathbb{D}_X$ is written uniquely in a similar form but with $f_{i_1, \ldots, i_p} \in \mathcal{O}(S)$. After this, the proof is achieved in the same way as for Theorem 5.3.4, with purely notational changes.

7. The path groupoid.

(7.1) $C^\infty$ and analytic cases. Here is a convenient way of defining an equivalence relation on the space of parametrized paths so as to get a groupoid.

(7.1.1) Definition. (cf. [HKK]) Let $X$ be a $C^\infty$-manifold, and $\gamma, \gamma' : [0, 1] \to X$ be two piecewise smooth parametrized paths with the same beginning $x = \gamma(0) = \gamma'(0)$ and end $y = \gamma(1) = \gamma'(1)$. A piecewise smooth homotopy $\sigma : [0, 1]^2 \to X$ between $\gamma$ and $\gamma'$ will be called thin, if, for any $(\lambda, \mu) \in [0, 1]^2$ where $\sigma$ is smooth, the differential $d(\lambda, \mu) \sigma$ has rank $\leq 1$.

Let $\Pi_X$ be the set of piecewise smooth paths in $X$ modulo thin homotopies, and $\Pi_X(x, y)$ be the set of classes of paths with $\gamma(0) = x$ and $\gamma(1) = y$. Then $\Pi_X$ is (the set of morphisms of) a groupoid with $X$ as the set of objects: thin homotopies account for both reparametrizations and cancellations of any segment followed by the same segment run in the opposite direction. We call $\Pi_X$ the path groupoid of $X$.

It is natural to view $\mathcal{P}_X$ as the Lie algebroid of $\Pi_X$. Indeed, for $x \in X$ consider the tensor functor

$$
(7.1.2) \quad \Phi_x : (\text{Bun}_\nabla(X), \otimes) \to (\text{Vect}_\mathbb{R}, \otimes), \quad (E, \nabla) \mapsto E_x,
$$

associating to a vector bundle its fiber at $x$. Then, for any $\gamma \in \Pi_X(x, y)$ and any $(E, \nabla) \in \text{Bun}_\nabla(X)$, we have the holonomy operator $H_{E, \nabla}(\gamma) : E_x \to E_y$, and these operators give a natural transformation of functors $\Phi_x \to \Phi_y$ compatible with the tensor functor structures. On the other hand, by (3.2), sections of $\mathcal{P}_X$ evaluated at any $x$, give similar transformations except they satisfy the $\otimes$-Leibniz rule which is the infinitesimal version of compatibility with the tensor product.

Thus, sections of $\mathcal{P}_X$ can be seen as vertical vector fields on

$$
(7.1.3) \quad \Pi_X \xrightarrow{s} X,
$$

where $s$ is the source map (beginning of the path), see (A.1.3).
Let now $X$ be a complex analytic manifold. A homotopy $\sigma$ as above will be called $\mathbb{C}$-thin, if for any $(\lambda, \mu)$ where $\sigma$ is smooth, the image of $d_{(\lambda, \mu)} \sigma$ is contained in a 1-dimensional $\mathbb{C}$-linear subspace of $T_{\sigma(\lambda, \mu)} X$. We denote by $\Pi_X^C$ the set of piecewise smooth paths in $X$ modulo $\mathbb{C}$-thin homotopies. Thus $\Pi_X^C$ is a groupoid, a quotient of $\Pi_X$.

Given a holomorphic vector bundle $E$ with a holomorphic connection $\nabla$, the holonomy operators along the paths are unchanged under $\mathbb{C}$-thin homotopies, and so give rise to a functor

$$H_{E,\nabla} : \Pi_X^C \to \text{Vect}_{\mathbb{C}}.$$  

(7.2) Kontsevich spaces. Let $k$ be a field of characteristic 0, and $X$ is a smooth projective algebraic variety over $k$. As in [Ma], §V.1.4, let $B(X) \subset \text{Hom}(\text{Pic}(X), \mathbb{Z})$ be the set of homomorphisms which are $\geq 0$ on each ample line bundle $L$. Let $\beta \in B(X)$, and $M_\beta = \mathcal{M}_{0,2}(X, \beta)$ be the moduli stack of stable 2-pointed rational curves in $X$ of degree $\beta$. Recall ([Ma], §V.3.2) that for any $\mathbb{C}$-scheme $S$ the $S$-points of $M_\beta$ form a groupoid $M_\beta(S)$ whose objects are data $(C, x, y, f)$ where:

(7.2.1) $C \to X$ is a flat family of proper curves with every geometric fiber being a union of $P^1$'s whose intersection graph is a tree.
(7.2.2) $x, y : S \to X$ are two sections of $\pi$ which are everywhere disjoint and whose values at any geometric point $s \in S$ are smooth on $\pi^{-1}(s)$.
(7.2.3) $f : C \to X$ is a morphism of schemes such that for any $s \in S$ as above the degree of $f|_{\pi^{-1}(s)}$ is $\beta$.

These data are required to satisfy the stability condition ([Ma] §V.1.2). Morphisms between $(C, x, y, f)$ and $(C', x', y', f')$ are isomorphisms $C \to C'$ preserving all the data.

The disjoint union $M = \bigsqcup M_\beta$ can be seen as an algebro-geometric analog of the path groupoid. Indeed, we have the projections

$$s, t : M_\beta \to X, \quad (C, x, y, f) \mapsto x, y.$$  

Further, gluing the mapped curves together followed by the stabilization morphism discussed in [Ma], §V.1.7, gives morphisms of stacks

$$m_{\beta, \beta'} : M_\beta \times_X M_{\beta'} \to M_{\beta+\beta'}$$  

remindful of the composition map (2.1.2) of a groupoid. However, the $m_{\beta, \beta'}$ do not possess inverses or unit. Still, the following statement expresses the similarity between the two objects.

(7.2.6) Proposition. Let $k = \mathbb{C}$. Then there exist natural morphisms $u_\beta : M_\beta(\mathbb{C}) \to \Pi_X^C$ which take the compositions (7.2.5) into the composition of the groupoid $\Pi_X^C$.

Proof: Let $(C, x, y, f) \in M_\beta(\mathbb{C})$. Thus $C$ is a curve over $\mathbb{C}$ which is a union of a tree of projective lines. In particular, $C$ is simply connected. Next, $x, y$ are two smooth points of $C$ and $f : C \to X$ is a map of degree $\beta$. Since $C$ is simply connected, $x$ and $y$ can be joined
by a path in $C$ which is unique up to homotopy. After applying $f$ any such homotopy will be a $\mathbb{C}$-thin homotopy in $X$. Thus we have a unique $\mathbb{C}$-thin homotopy class of paths in $X$ joining $f(x)$ and $f(y)$. This defines $u_\beta$. The rest is clear. \hfill \Box

\textbf{(7.2.7) Remark.} Similarly, $\mathbb{C}$-points of the stack $\overline{M}_{0,n}(X, \beta)$ of stable $n$-pointed rational curves, $n > 2$, can be mapped into $B_{n-1}\Pi_X$, the $(n-1)$-st component of the simplicial classifying space of the groupoid $\Pi_X$.

\textbf{(7.3) The formal path groupoid.} Let $k$ be a field of characteristic $0$, and $X$ be a smooth algebraic variety over $k$ (not necessarily projective). Denote by $\hat{\Pi}_X = e^{\mathcal{P}_X}$ be the formal groupoid integrating the Lie algebroid $\mathcal{P}_X$, as constructed in (A.6). Thus its scheme of objects is $X$, and the ind-scheme of morphisms (for which we retain the notation $\hat{\Pi}_X$) is

\begin{equation}
\hat{\Pi}_X = \text{Spec}(\mathcal{A}) = \lim_{\rightarrow d} \text{Spec}(\mathcal{A}_d), \quad \mathcal{A}_d = \text{Hom}_{\mathcal{O}_X}(\mathbb{D}_X^{\leq d}, \mathcal{O}_X).
\end{equation}

We call $\hat{\Pi}_X$ the \textit{formal path groupoid} of $X$.

\textbf{(7.3.2) Remark.} For $k = \mathbb{C}$ the ind-scheme $\hat{\Pi}_X$ provides an algebro-geometric model for the formal neighborhood of $X$ in $\Pi_X$. This can be understood as follows. By Theorem A.6.5, a morphism from a scheme $S$ into $\hat{\Pi}_X$ is the same as a morphism $y : S \to X$ together with a groupoid-like section $g$ of $y^*\mathbb{D}_X$. On the other hand, suppose that $y$ is a $\mathbb{C}$-point of $X$, and $\gamma$ be a path in $X$ with endpoint $y$. Assuming that $\gamma$ lies in a coordinate patch on $X$ with coordinates $x_1, \ldots, x_n$ (vanishing at $y$), we can associate to $\gamma$ the formal series

\begin{equation}
E_\gamma = \sum_{p=0}^{\infty} \sum_{i_1, \ldots, i_p=1}^{n} \left( \int_{\gamma} dx_{i_1} \cdots dx_{i_p} \right) D_{i_1} \cdots D_{i_p},
\end{equation}

which can be seen as lying in the formal completion of the fiber of $\mathbb{D}_X$ at $x$. Here the $\int_{\gamma}$ are the iterated integrals of Chen [C], and his fundamental shuffle relations imply that $E_\gamma$ is, formally, a group(oid)-like element. Indeed, the $D_i$, being sections of $\mathcal{P}_X$, are primitive. Although the concept of constant differential forms $dx_i$ and thus of their iterated integrals depends on the choice of coordinates, the transformation rules for sections of sheaves $\mathcal{P}_X$ and $\mathbb{D}_X$ are such that the generating series $E_\gamma$ is invariantly defined.

We now extend this point of view to relate $\hat{\Pi}_X$ to the Kontsevich moduli spaces by providing an algebraic version of Proposition 7.2.6. We need to introduce some terminology.

Let $S$ be a reduced scheme, and $(C, x, y, f) \in M_\beta(S)$. Then $C$ is reduced as well. Denote by $C^0 \subset C$ the minimal closed fiberwise connected relative subcurve containing $x(S)$ and $y(S)$. We say that $(C, x, y, f)$ is contracting, if $f$ is constant on the fibers of the projection $\pi^0 : C^0 \to S$, i.e., $f|_{C^0}$ factors through a morphism $S \to X$.

Let now $S$ be arbitrary, and $(C, x, y, f) \in M_\beta(S)$. We then have $C_{\text{red}} = \pi^{-1}(S_{\text{red}})$, so we have a stable $\pi_{\text{red}} : C_{\text{red}} \to S_{\text{red}}$ with sections $x_{\text{red}}, y_{\text{red}}$ and map $f_{\text{red}} : C_{\text{red}} \to X$ given by restriction. They form an $S_{\text{red}}$-point of $M_\beta$. We say that $(C, x, y, f)$ is \textit{almost contracting}, if
Let \( (C_{\text{red}}, x_{\text{red}}, y_{\text{red}}, f_{\text{red}}) \) be contracting. Let \( \widehat{M}_\beta \subset M_\beta \) be the substack whose \( S \)-points are almost contracting \( (C, x, y, f) \). We can think of \( \widehat{M}_\beta \) as a certain formal neighborhood in \( M_\beta \).

Note that the compositions (7.2.5) restrict to
\[
\tilde{m}_{\beta, \beta'} : \tilde{M}_\beta \times_X \tilde{M}_{\beta'} \to \tilde{M}_{\beta+\beta'}.
\]

(7.3.5) **Theorem.** There exist morphisms of stacks \( \tilde{u}_\beta : \tilde{M}_\beta \to \tilde{\Pi}_X \) which are compatible with the source, target and the composition maps.

**Proof:** To construct \( \tilde{u}_\beta \) means, by definition, to construct, for any scheme \( S \), a map \( \tilde{u}_\beta,S : \tilde{M}_\beta(S) \to \tilde{\Pi}_X(S) \) constant on isomorphism classes in \( \tilde{M}_\beta(S) \), in a way compatible with the base change for the \( \text{fppf} \) topology (on which the stacks are defined).

Since \( M_\beta \) (and thus \( \tilde{M}_\beta \)) is a stack, and \( \tilde{\Pi}_X \), being an ind-scheme, is a sheaf of sets on the \( \text{fppf} \) topology, we can assume \( S \) is affine. Since \( M_\beta \) is a stack of finite type over \( k \), we can assume \( S \) is of finite type as well.

Let \( p = (C, x, y, f) \) be an \( S \)-point of \( \tilde{M}_\beta \). The fact that it is almost contractible implies that the \( S \)-points \( f_x, f_y : S \to X \) are infinitesimally close, i.e., coincide on \( S_{\text{red}} \). Let \( U \subset X \) be Zariski open, and \((E, \nabla) \in \text{Bun}_\nabla(U)\). Because \( f_x, f_y \) are infinitesimally close, we have the morphism of sheaves (restriction)
\[
\text{Res} : (f_y)^{-1}E \to (f_x)^*E.
\]
On the other hand, by Theorem A.6.5, an element of \( \tilde{\Pi}_X(S) \) is the same as a pair \((z, g)\), where \( z : S \to X \), and \( g \in \Gamma(S, z^*\mathbb{D}_X) \) is a groupoid-like element. By Theorem 6.2.5, a section \( g \) of \( z^*\mathbb{D}_X \) over \( S \) is the same as a system of morphisms of sheaves
\[
g_{E, \nabla} : z^{-1}E \to z^*E,
\]
given for any \( U, E, \nabla \) as above and forming a regular natural differential operator along \( z \). For \( g \) to be groupoid-like is equivalent to the fact that the \( g_{E, \nabla} \) commute with tensor products of bundles with connections.

We are going to construct there two types of data as follows. Given \( p \in \tilde{M}_\beta(S) \) as above, we set \( z = f_y \). We then construct \( g_{E, \nabla} \) as the composition of \( \text{Res} \) with a morphism ("holonomy")
\[
H_{E, \nabla}(p) : (fx)^*E \to (fy)^*E.
\]
In other words, we prove the following fact which will imply the existence of \( \tilde{u}_\beta \).

(7.3.9) **Theorem.** (a) For each \((E, \nabla) \in \text{Bun}_\nabla(U)\) as above there exists an isomorphism of \( \mathcal{O}_S \)-sheaves \( H_{E, \nabla}(p) \) as in (7.3.8), compatible with restrictions to \( \tilde{U} \subset U \) and \( \text{fppf} \) base change for \( \tilde{S} \to S \).

(b) The composition \( H_{E, \nabla}(p) \circ \text{Res} \) is a regular natural differential operator.

(c) For fixed \( p \) the \( H_{E, \nabla}(p) \) are compatible with the tensor product.

Note that it enough to assume (by restricting \( U \) if necessary) that the embedding \( U \hookrightarrow X \) is an affine morphism.
Proof of Theorem 7.3.9. Consider the open subschemes
\[ C_U = f^{-1}(U) \subset C, \quad S_U = (f x)^{-1}(U) = (f y)^{-1}(U) \subset S. \]
By restricting \( S \) if necessary, we can assume that \( S_U = S \). Let \( \pi_U : C_U \to S \) be the restriction of \( \pi \), and \( f_U : C_U \to X \) be the restriction of \( f \). We have then the bundle \( f_U^*E \) on \( C_U \) and a relative connection
\[ \nabla_{C/S} : f_U^*E \to \Omega^1_{C/S} \otimes f_U^*E. \]
Note that \( C \) is of finite type because we assumed \( S \) to be of finite type. Let \( \mathcal{C} \) be the formal neighborhood of \( C_0 \) in \( C \) (or, what is the same, in \( f^{-1}(U) \)). We consider it as a topologically ringed space \((C_0)_{zar} \) with sheaf \( \mathcal{O}_C \). We denote by \( \pi : \mathcal{C} \to S \) the projection and for each quasicoherent sheaf \( F \) on \( C \) write \( \Gamma(\mathcal{C}, F) = \Gamma(F \otimes \mathcal{O}_C) \).

For any geometric point \( s \in S \) the preimage \( \pi^{-1}(s) \) is a formal scheme over the field \( K = k(s) \) consisting of \( \pi^{-1}(s) \cap C_0 \) (a chain of \( P^1 \)'s), to which are attached several “tails” at the points of intersection with other components of \( \pi^{-1}(s) \). Each such tail is isomorphic to \( \text{Spf } K[[t]] \).

Theorem 7.3.9 is a consequence of the following fact.

Lemma. There exists a unique
\[ \Phi_x \in \Gamma(\mathcal{C}, \text{Hom}(\pi_U^*x^*f_U^*E, f_U^*E)) \]
("fundamental solution"), satisfying
\[ \nabla_{C/S} \Phi_x = 0, \quad x^*\Phi_x = \text{Id} \in \text{Hom}_S(x^*f_U^*E, x^*f_U^*E). \]

Indeed, we then define
\[ H_{E,\nabla}(p) = y^*\Phi_x \in \text{Hom}_S(x^*f_U^*E, y^*f_U^*E). \]

Its multiplicativity in tensor products as well as naturality in restrictions and base changes follow from the uniqueness, so parts (a) and (c) of the theorem would be proved. The validity of part (b), i.e., the fact that
\[ H_{E,\nabla}(p) \circ \text{Res} : y^{-1}f^{-1}E \to y^*f^*E \]
depends on sections of \( y^{-1}f^{-1}E \) and on \( \nabla \) only through their finite jets, will follow from the construction of \( \Phi_x \) which we are about to give.

Construction of \( \Phi_x \). We proceed by “induction on the order of nilpotency” of \( S \) (assumed to be of finite type). First, consider the case when \( S \) is reduced, so \( S \) and \( C \) are algebraic varieties over \( k \). Then \( C^0 \subset C \) is a union of irreducible components, and the restriction of \( f \) to \( C^0 \) factors through \( S \). In particular, \( f_U^*E \) is trivial over the fibers of \( C^0 \to S \) and the connection \( \nabla_{C/S} \) is also trivial over such fibers. In this case both the existence and the uniqueness of \( \Phi_x \) is clear. Indeed, over \( C^0 \) it is (and must be) equal to the identity. Further, on the formal neighborhood of \( C^0 \) in \( C \) (which is \( \mathcal{C} \)) it is uniquely extended by solving, over each geometric point \( s \in S \), finitely many initial value problems for
differential equations in the ring of formal power series $k(s)[[t]]$. These equations correspond to the “tails” attached, as above, at the intersection point of $(\pi^0)^{-1}(s)$ with irreducible components of $\pi^{-1}(s)$ not in $C^0$.

For the inductive step, we consider an arbitrary $S$ of finite type and a closed subscheme $S' \subset S$ whose sheaf of ideals $I_{S'} \subset O_S$ satisfies $I_{S'} \cdot \sqrt{O_S} = 0$. This means that $I_{S'}$, as a coherent sheaf on $S$, is supported, scheme-theoretically, on $S_{\text{red}}$. Let $C' = \pi^{-1}(S')$. Then the restriction $\pi' : C' \to S'$ together with $f' = f|_{C'}$, as well as $x' = x|_{S'}, y' = y|_{S'}$ form an $S'$-point $p'$ of $\hat{M}_\beta$. In particular, we have the formal scheme $\hat{C}'$, the formal neighborhood of $C^0_{\text{red}}$ in $C'$ etc.

Let $I_{C'} \subset O_C$ be the sheaf of ideals of $C'$. Because $C/S$ is flat, we have

\begin{equation}
I_{C'} = \pi^* I_{S'}, \quad I_{C'} \cdot \sqrt{O_C} = 0.
\end{equation}

Further, flatness is retained by the completion $\hat{C} \to S$, so the sheaf of ideals $I_{\hat{C}'} \subset O_{\hat{C}}$ satisfies the analog of (7.4.5). In addition, since $C$ is a curve of relative arithmetic genus 0, for any coherent sheaf $\mathcal{F}$ on $S$ we have

\begin{equation}
R^1\pi_*(\hat{\pi}^* \mathcal{F}) = 0.
\end{equation}

We assume by induction Lemma 7.4.2 to be proved for $\hat{C}'$. So let

\begin{equation}
\Phi' = \Phi_{x'} \in \Gamma(\hat{C}', \text{Hom}(\pi_{\hat{C}}^* (x')^* (f_{U})^* E, (f_{U})^* E))
\end{equation}

satisfies the conditions of the lemma. Denote by

\begin{equation}
\mathcal{H} \subset \text{Hom}_{\hat{C}'}(\pi_{\hat{C}}^* x^* f_{U}^* E, f_{U}^* E)
\end{equation}

the sheaf of all $\Phi$ restricting to $\Phi'$ over $\hat{C}'$. This is a torsor over the sheaf of all $\Phi$ restricting to 0, i.e., over

\begin{equation}
\mathcal{A} = \text{Hom}_{\hat{C}'}(\pi_{\hat{C}}^* x^* f_{U}^* E, f_{U}^* E \otimes I_{C'})
\end{equation}

We can assume, by restricting $U$ if necessary, that $E$ is trivial as a vector bundle on $U$. Then $\mathcal{A}$, which is, by the properties of $I_{C'}$, a coherent sheaf on $C_{\text{red}}$, is trivial on fibers of $C_{\text{red}} \to S_{\text{red}}$, so its first direct image is 0 by (7.4.6). This means that locally on $S$ the torsor $\mathcal{H}$ has a global section. Since we work locally on $S$ anyway, we can assume that $\mathcal{H}$ is trivial as a torsor. So there exists $\Phi_0$ extending $\Phi'$. We now modify $\Phi'$ be a section $\Psi$ of $\mathcal{A}$ so as to satisfy the conditions of Lemma 7.4.2. Without such modification we have only that

\begin{equation}
\omega := \nabla_{C/S}(\Phi') \in \Gamma(\hat{C}, \text{Hom}(\pi_{\hat{C}}^* x^* f_{U}^* E, f_{U}^* E) \otimes \Omega^1_{C/S} \otimes I_{C'})
\end{equation}

As $I_{C'}$, and thus $\omega$, is supported on $C_{\text{red}}$, we see that the restriction of $\omega$ to each fiber of $C_{\text{red}}' \to S_{\text{red}}$ must vanish since each such fiber is a union of $P^1$'s. So modification of $\Phi'$ is not necessary over $C_{\text{red}}'$, and we need to look for $\Psi$ as a section of $\mathcal{A}$ which vanishes on $C_{\text{red}}'$. As before, this is done by solving a differential equation in the ring of formal power series over each geometric point $s \in S$ and each intersection point of $(\pi^0)^{-1}(s)$ with other irreducible components.
The uniqueness is proved by the same inductive reasoning, using, at each step, the fact that the solution of an initial value problem for a differential equation in the ring of formal power series is unique. This finishes the proof of Lemma 7.4.2 and thus of Theorems 7.3.9 and 7.3.5.

8. Remarks and further directions.

(8.1) Integral kernels on $\Pi_X$. Of great interest are various versions of the convolution algebra of $\Pi_X$, see [Co], §II.5 for general background. Naively, this algebra should consist of fiberwise measures on (7.1.3), i.e., of “kernels” $K = K(x, \gamma)\mathcal{D}_\gamma$ defined for $x \in X$, $\gamma \in s^{-1}(x)$, which behave as functions in $x$ and measures in $\gamma$. At this naive level, such a $K$ gives, for each $(E, \nabla) \in \text{Bun}_\nabla(X)$, an operator $P^K_{E,\nabla}$ in sections $f$ of $E$ (Feynman-Kac formula):

\[
(P^K_{E,\nabla}f)(x) = \int_{\gamma \in s^{-1}(x)} K(x, \gamma) H_{E,\nabla}(\gamma)(f(t(\gamma))) \mathcal{D}_\gamma.
\]

Here $t(\gamma)$ is the endpoint of $\gamma$ and $H_{E,\nabla}(\gamma) : E_x \to E_{t(\gamma)}$ is the holonomy of $\nabla$ along $\gamma$. For example, “distribution kernels” obtained as iterated derivatives of the delta function on $X \subset \Pi_X$, give rise to $P^K_{E,\nabla}$ being the natural differential operators corresponding to sections of $\mathcal{D}(X)$, as in (6.1).

Formula (8.1.1) involves path integration and so may be difficult to justify rigorously. On the other hand, the result, considered for all $(E, \nabla)$, is manifestly natural with respect to morphisms in $\text{Bun}_\nabla(X)$. Extending the approach of (5.1), one can look at natural (systems of) operators $\{P^K_{E,\nabla} : E \to E\}$ given in all bundles with connections on $X$, but of more general nature: pseudo-differential, Fourier integral operators etc. By the above, various algebras of such natural operators provide regularized versions of the convolution algebra of $\Pi_X$. Intuitively, knowledge of $P^K_{E,\nabla}$ for all $(E, \nabla)$ determines $K$ uniquely.

(8.1.2) Example. Let $X$ be a Riemannian manifold, and $\Delta = \Delta_X \in \mathcal{D}(X)$ be its noncommutative Laplacian (5.1.4). We have then a natural operator $\Theta$ with $\Theta_{E,\nabla} = \exp(-\Delta_{E,\nabla})$ being the heat operator. By the result of Bismut [Bi], $\Theta$ corresponds, as in (7.1.3), to the Wiener (Brownian motion) measure on $\Pi_X$. To be precise, this interpretation requires completing $\Pi_X$ to include continuous paths and understanding $H_{E,\nabla}(\gamma)$ by using stochastic differential equations. Cf. [Kap2] for the discussion of the flat case $X = \mathbb{R}^n$.

(8.2) Algebraic theory of $\mathcal{D}$-modules. Let $X$ be a smooth algebraic variety. By analogy with sheaves of modules over $\mathcal{D}_X$, the sheaf of usual differential operators [Kas], one can study sheaves of modules over $\mathcal{D}_X$, quasicoherent over $\mathcal{O}_X$, in a purely algebraic fashion.
Such modules provide a language for studying connections with singularities. On the other hand, sections of $D_X$ are interpreted as translation invariant differential operators on $\Pi_X$, see (7.1.3), so $D_X$-modules describe some systems of linear PDE on the space of paths.

(8.3) Quotients of $P_X$ and geometric structures. Many differential-geometric structures on $X$ can be formulated in terms of appropriate quotients of $P_X$. For example, a Riemannian metric on $X$ specifies the subcategory of bundles with connections satisfying the Yang-Mills equations and thus gives rise to a quotient groupoid $P^\text{YM}_X$ of $P_X$. If $\dim(X) = 4$, we have a further quotient $P^+_X$ governing self-dual connections. These are curvilinear versions of the (self-dual) Yang-Mills algebras of Nekrasov and Connes-Dubois-Violette [Ne] [CDV]. They will be studied in a subsequent paper.

Appendix. Formal integration of Lie algebroids.

(A.1) Lie groupoids. Recall that a groupoid $\mathcal{G}$ is a category in which all morphisms are isomorphisms. We will write $\mathcal{G}_0 = \text{Ob}(\mathcal{G})$ for the class of objects and $\mathcal{G}_1 = \text{Mor}(\mathcal{G})$ for the class of morphisms of $\mathcal{G}$. In the sequel we assume that $\mathcal{G}$ is small, i.e., that the $\mathcal{G}_i$ are sets. Thus we have maps

\[(A.1.1) \quad s, t : \mathcal{G}_1 \to \mathcal{G}_0, \quad e : \mathcal{G}_0 \to \mathcal{G}_1,\]

where $s, t$ associate to a morphism its source and target objects while $e$ associates to an object its identity (unit) morphism. In addition, the composition of morphisms can be seen as a map

\[(A.1.2) \quad m : \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \to \mathcal{G}_1,\]

where the fiber product is taken with respect to the pair of maps $s, t : \mathcal{G}_1 \to \mathcal{G}_0$.

One can speak about groupoid object in any category $\mathcal{C}$ with fiber products. Taking $\mathcal{C}$ to be the category of $C^\infty$-manifolds, we get a concept of a Lie groupoid. Thus a Lie groupoid is a groupoid $\mathcal{G}$ in which both $\mathcal{G}_0$ and $\mathcal{G}_1$ are equipped with structures of $C^\infty$-manifolds, and the maps (A.1.1-2) are smooth. When $\mathcal{G}_0$ is a point, a Lie groupoid is the same as a Lie group. See [Mack] for more background and examples of Lie groupoids.

We recall the fundamental construction which to any Lie groupoid $\mathcal{G} = \text{Lie}(\mathcal{G})$. Denote the manifold of objects by $X = \mathcal{G}_0$. Then, as a vector bundle,

\[(A.1.3) \quad \mathcal{G} = e^*(T_{\mathcal{G}_1/\mathcal{G}_0}),\]

where the relative tangent bundle of $\mathcal{G}_1/\mathcal{G}_0$ is taken with respect to the projection $s$. In fact, sections of $\mathcal{G}$ can be seen as vertical vector fields on $\mathcal{G}_1 \xrightarrow{s} \mathcal{G}_0$ which are right invariant with respect to the composition map, and the Lie bracket of such vector fields makes $\mathcal{G}$ into
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a sheaf of Lie algebras. Further, the differential of \( t : \mathfrak{g}_1 \to \mathfrak{g}_0 \) defines the anchor map \( \alpha : \mathcal{G} \to T_X \), and these structures make \( \mathcal{G} \) into a Lie algebroid. See [Mack] for more details.

It is known that any finite-dimensional Lie algebra can be integrated to a Lie group. The corresponding problem for Lie algebroids is highly nontrivial, see [Mack], and the integration is not always possible. In the following we present a groupoid version of an easier result: that any Lie algebra can be integrated to a formal (Lie) group.

(A.2) (Formal) ind-schemes. We follow the same conventions on ind-schemes as in [KV1-2], see also [De] for general background on ind-objects.

Thus, by an ind-scheme \( \mathcal{X} \) over \( k \) we mean a formal inductive limit \( \mathcal{X} = \lim_{\rightarrow} X_i \), where \( I \) is a filtered poset, and \( (X_i)_{i \in I} \) is an inductive system of \( k \)-schemes indexed by \( I \). It is further assumed that the structure maps \( X_i \to X_j, \ i \leq j \), are closed embeddings. An ind-scheme \( \mathcal{X} \) can be identified with the corresponding (ind-)representable functor \( h_{\mathcal{X}} \) on the category of \( k \)-schemes:

\[
(A.2.1) \quad h_{\mathcal{X}}(S) = \lim_{\rightarrow} \text{Hom}(S, X_i).
\]

This means that if \( \mathcal{Y} = \lim_{\rightarrow} Y_j \) is another ind-scheme (with a possibly different indexing set \( J \)), then morphisms \( \mathcal{X} \to \mathcal{Y} \) are the same as natural transformations of functors \( h_{\mathcal{X}} \to h_{\mathcal{Y}} \).

(A.2.2) Examples. (a) Let \( A \) be a commutative topological \( k \)-algebra which can be represented as a projective limit \( A = \lim_{\leftarrow} A_i \) of discrete \( k \)-algebras. Then we have the ind-scheme

\[
\text{Spf}(A) = \lim_{\leftarrow} \text{Spec}(A_i).
\]

In particular, we will use the \( n \)-dimensional formal disk over \( k \) which is the ind-scheme

\[
\mathcal{D}_n = \text{Spf} \ k[[x_1, ..., x_n]] = \lim_{\leftarrow} \text{Spec} \ k[x_1, ..., x_n]/(x_1, ..., x_n)^d.
\]

(b) Let \( Y \) be a scheme, and \( X \subset Y \) be a closed subscheme with sheaf of ideals \( I_X \subset \mathcal{O}_Y \). Then we have the ind-scheme

\[
X^{(\infty)}_Y = \lim_{\leftarrow} \text{Spec} \ (\mathcal{O}_Y/I_X^{d+1}),
\]

called the formal neighborhood of \( X \) in \( Y \). The scheme \( X^{(d)}_Y \) is called the \( d \)th infinitesimal neighborhood of \( X \) in \( Y \).

For a scheme \( X \) we denote by \( X_{\text{red}} \subset X \) the maximal reduced (nilpotent-free) subscheme. We then extend this to ind-schemes, putting \( \mathcal{X}_{\text{red}} = \lim_{\leftarrow} X_{i,\text{red}}, \) if \( \mathcal{X} = \lim_{\leftarrow} X_i \).

(A.2.3) Definition. An ind-scheme \( \mathcal{X} \) will be called formal, if \( \mathcal{X}_{\text{red}} \) is an ordinary scheme.

Thus a formal ind-scheme can be seen as a limit of nilpotent extensions of a scheme. This concept is more flexible than that of formal schemes as defined in [Gr].
For a scheme $X$ we denote by $X_{\text{Zar}}$ the underlying topological space of $X$ with the Zariski topology, so $X$ is a ringed space $(X_{\text{Zar}}, \mathcal{O}_X)$. Following Haboush [Hab], we associate to an ind-scheme $\mathcal{X} = \lim_{\rightarrow} X_i$ a topological space $\mathcal{X}_{\text{Zar}}$ with a sheaf of topological rings $\mathcal{O}_{\mathcal{X}}$. Explicitly,

\[(A.2.4) \quad \mathcal{X}_{\text{Zar}} = \lim_{\rightarrow} i \in I X_i, \text{Zar},\]

(inductive limit in the category of topological spaces), while

\[(A.2.5) \quad \mathcal{O}_{\mathcal{X}} = \lim_{\leftarrow} i \in I \epsilon_i, \ast \mathcal{O}_{X_i},\]

where $\epsilon_i : X_i, \text{Zar} \to X_{\text{Zar}}$ is the natural inclusion. It was proved in loc. cit. that the correspondence $X \mapsto (\mathcal{X}_{\text{Zar}}, \mathcal{O}_{\mathcal{X}})$ embeds the category of ind-schemes as a full subcategory of the category of locally topologically ringed spaces.

**(A.2.6) Example.** In the situation of Example A.2.2(b) we have

\[(X^{(\infty)}_Y)_{\text{Zar}} = X, \quad \mathcal{O}_{X_Y^{(\infty)}} = \mathcal{O}_{Y,X} := \lim_{\leftarrow} \mathcal{O}_Y/I^d_X.\]

**(A.2.7) Definition.** By a formal groupoid we will mean a groupoid object $G = (G_0, G_1)$ in the category of formal ind-schemes over $k$ such that each of the morphisms $e, s, t$ induces an isomorphism between $(G_0)_{\text{red}}$ and $(G_1)_{\text{red}}$. A formal group is a formal groupoid with $G_0 = \text{Spec}(k)$.

**(A.2.8) Example.** Let $g$ be a Lie $k$-algebra and $U = U_k(g)$ be its universal enveloping algebra. As well known, $U$ is a cocommutative Hopf algebra with comultiplication $\Delta$ defined uniquely by setting each $x \in g \subset U$ to be primitive:

\[(A.2.9) \quad \Delta(x) = x \otimes 1 + 1 \otimes x.\]

Let $U^\vee = \text{Hom}_k(U, k)$ be the linear dual of $U$ as a vector space. Then $\Delta$ makes $U^\vee$ into a topological commutative algebra, and

\[\epsilon^\mathfrak{g} := \text{Spf}(U^\vee)\]

is a formal group. If $\Lambda$ is any commutative $k$-algebra, then $\Lambda \otimes_k U$ is a Hopf $\Lambda$-algebra, and one has

\[(A.2.10) \quad \text{Hom}(\text{Spec}(\Lambda), \epsilon^\mathfrak{g}) = (\Lambda \otimes_k U)^{gr}.\]

Here the subscript “gr" means the set of invertible group-like elements, i.e., invertible elements $g$ satisfying $\Delta(g) = g \otimes g$. It is this approach that we generalize to Lie algebroids.

**(A.3) Left bialgebras.** Let $A$ be a commutative $k$-algebra, included into a possibly non-commutative algebra $H$. We do not assume $A$ to be central in $H$. Thus there are two commuting $A$-module structures on $H$, which we call left and right. Since $A$ is commutative, each of them can be used to form tensor products over $A$. For examples, there are four possibilities for $H \otimes_A H$. 

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In the sequel we will always understand $H \otimes_A H$ to be taken with respect to the left $A$-module structures on both factors, i.e., to be the quotient of $H \otimes_k H$ by the relations

\[(A.3.1)\quad au \otimes u_2 = u_1 \otimes au_2, \quad a \in A, u_1, u_2 \in H.\]

With this definition, $H \otimes_A H$ has one, “left”, $A$-module structure, with $a(u_1 \otimes u_2)$ given by (A.3.1) and two “right” structures, each commuting with the left one:

\[(A.3.2)\quad (u_1 \otimes u_2) a = u_1 a \otimes u_2, \text{ resp. } u_1 \otimes u_2 a.\]

Let $H \otimes_A H \subset H \otimes_A H$ be the locus of coincidence of these two structures, i.e., the space formed by elements $\sum_i u_{1i} \otimes u_{2i}$ satisfying

\[(A.3.3)\quad \sum_i u_{1i} a \otimes u_{2i} = \sum_i u_{1i} \otimes u_{2i} a \in H \otimes_A H, \quad \forall a \in A.\]

(A.3.4) Example. Let $A = k[X]$ be the coordinate algebra of a smooth affine variety $X$ over $k$, and $H = D(X)$ be the algebra of regular differential operators on $X$. Let $D : A \to A$ be a derivation, so we consider $D$ as an element of $H$. For $a \in A$ write $a' = D(a)$. Then $D \otimes 1 + 1 \otimes D \in H \otimes_A H$. Indeed,

\[Da \otimes 1 + a \otimes D = aD \otimes 1 + a' \otimes 1 + a \otimes D,\]

while

\[D \otimes a + 1 \otimes Da = D \otimes a + 1 \otimes aD + 1 \otimes a',\]

which is the same modulo left $A$-linearity.

(A.3.5) Lemma. (a) $H \otimes_A H$ is an associative algebra with respect to the standard product

\[\left( \sum_i u_{1i} \otimes u_{2i} \right) \cdot \left( \sum_j v_{1j} \otimes v_{2j} \right) = \sum_{i,j} u_{1i} v_{1j} \otimes u_{2i} v_{2j}.\]

(b) $A$ is embedded as a subalgebra in $H \otimes_A H$, and the two $A$-module structures on it are induced by the left and right multiplication with $A$.

(c) If $M, N$ are two left $H$-modules, then $M \otimes_A N$ is a left $H \otimes_A H$-module. \(\square\)

The following concept is a particular case of the concept of a bialgebroid as given in [Lu] and [Xu]. We present here a self-contained exposition of this particular case. A stronger structure, involving the antipode and called a Hopf algebroid, was defined by J. Mrčun ([Mr], Def. 2.1).

(A.3.6) Definition. Let $A$ be a commutative $k$-algebra. A left $A$-bialgebra consists of:

1. A possibly noncommutative algebra $H$ containing $A$.

2. A morphism of left $A$-modules $\epsilon : H \to A$ which is a twisted ring homomorphism:

\[\epsilon(uv) = \epsilon(u \cdot \epsilon(v)).\]
(3) A homomorphism of algebras $\Delta : H \to H \otimes_A H$ which is identical on $A$, coassociative and satisfies the left and right counit properties with respect to $\epsilon$.

Note that the conditions in (2) imply that $A$ is a left $H$-module, via

\[(A.3.7) \quad (u, a) \mapsto u(a) = \epsilon(ua),\]

and with respect to this structure $\epsilon$ is a homomorphism of left $H$-modules. Further, if $\Delta(u) = \sum_i u_i^{(1)} \otimes u_i^{(2)}$, then the counit property implies that the right $A$-module structure on $H$ is expressible through the left structure and $\Delta$:

\[(A.3.8) \quad ua = \sum_i u_i^{(1)}(a) \cdot u_i^{(2)}(b).\]

Compare with Formula (6) of [Lu] and Formula (17) of [Xu]. Similarly, for the $H$-module structure on $A$, we have

\[(A.3.9) \quad u(ab) = \sum_i u_i^{(1)}(a) \cdot u_i^{(2)}(b).\]

The last equality establishes part (b) of the following fact.

**(A.3.10) Proposition.** Let $H$ be a left $A$-bialgebra, and $M, N$ be two left $H$-modules. Then:

(a) $M \otimes_A N$ has a natural structure of a left $H$-module.

(b) If $M = A$, then the $H$-module structure on $A \otimes_A N = N$ is identical to the original one on $N$. Similarly, if $N = A$.

**(A.3.11) Theorem.** Let $A$ be a commutative $k$-algebra, and $L \to \text{Der}(A)$ be a Lie-Rinehart $A$-algebra. Then:

(a) The enveloping algebra $U_A(L)$ has a unique structure of a left $A$-bialgebra, with $\epsilon$ given by (1.2.6) and $\Delta$ making each $x \in L$ primitive.

(b) If $L$ is projective over $A$, then all primitive elements of $U_A(L)$ lie in $L$.

**Proof:** (a) This is Theorem 3.7 of [Xu]. The construction of $\Delta$ (without axiomatization of its properties) was explicitly given in [HS], §3.6. Part (b) is proved using the PBW theorem for $U_A(L)$, see [Ri], §3, similarly to the well known fact for ordinary enveloping algebras ([Bo], §II.5).

**(A.4) Groupoid-like elements.** Let $A$ be a commutative $k$-algebra, and $H$ a left $A$-bialgebra. For any commutative algebra $\Lambda$ and any homomorphism $\xi : A \to \Lambda$ we have the change of scalars

\[(A.4.1) \quad \Lambda^\xi H = \Lambda \otimes_A H\]

(using the left $A$-module structure on $H$ and the $A$-module structure on $\Lambda$ given by $\xi$). This is a $(\Lambda, A)$-bimodule. The counit $\epsilon$ of $H$ induces a morphism of $\Lambda$-modules

\[(A.4.2) \quad \Lambda^\xi \epsilon : \Lambda^\xi H \to \Lambda.\]
Further, $\frac{1}{\xi} \hat{H} \otimes_{\Lambda} \frac{1}{\xi} \hat{H}$ has one left $\Lambda$-module structure and two right $A$-module structures, each commuting with the left one. As in (A.3.3), we denote by $\frac{1}{\xi} \hat{H} \otimes_{\Lambda} \frac{1}{\xi} \hat{H}$ the locus of coincidence of these two $A$-module structures. Then the comultiplication $\Delta$ in $H$ induces a morphism of $(\Lambda, A)$-bimodules

\begin{equation}
\Delta : \frac{1}{\xi} \hat{H} \rightarrow \frac{1}{\xi} \hat{H} \otimes_{\Lambda} \frac{1}{\xi} \hat{H},
\end{equation}

which makes $\frac{1}{\xi} \hat{H}$ into a coassociative $\Lambda$-coalgebra with counit $\frac{1}{\xi} \epsilon$. Further, we get from (A.3.9) the following relation between the left $\Lambda$- and the right $A$-module structures on $\frac{1}{\xi} \hat{H}$:

\begin{equation}
ua = \sum_i \frac{1}{\xi} \epsilon(u_i(1)) \cdot u_i(2), \quad u \in \frac{1}{\xi} \hat{H}, a \in A.
\end{equation}

**A.4.5 Definition.** An element $g \in \frac{1}{\xi} \hat{H}$ will be called groupoid-like, if $g$ does not vanish anywhere on $\text{Spec}(\Lambda)$, and $\Delta(g) = g \otimes g$.

Compare [Mr], p. 270 for a related concept of a weakly group-like element.

Note that the counit property implies $\frac{1}{\xi} \epsilon(g) \cdot g = g$, and so $\frac{1}{\xi} \epsilon(g) = 1$. We will refer to $\xi$ as the target homomorphism of $g$, and write $\xi = t(g)$.

Next, we associate to $g$ another homomorphism $\eta : A \rightarrow \Lambda$, called the source of $g$ and denoted $\eta = s(g)$, by setting

\begin{equation}
\eta(a) = \frac{1}{\xi} \epsilon(ga).
\end{equation}

The groupoid-like property of $g$ implies that $\eta$ is indeed a ring homomorphism:

$$
\eta(a)\eta(b) = (\frac{1}{\xi} \epsilon \otimes \frac{1}{\xi} \epsilon)(ga \otimes gb) = (\frac{1}{\xi} \epsilon \otimes \frac{1}{\xi} \epsilon)((g \otimes g) \cdot ab) = \frac{1}{\xi} \epsilon((\Delta(g) \cdot ab) = \frac{1}{\xi} \epsilon(ab) = \eta(ab).
$$

We now consider the multiplication in $H$ as a morphism $m : H \otimes_k H \rightarrow H$. It is $A$-linear with respect to the $A$-module structure given by left multiplication of $a \in A$ on the first factor in the source and on the target. Therefore it descends to

\begin{equation}
\frac{1}{\xi} m : \frac{1}{\xi} \hat{H} \otimes_k H = \Lambda \otimes_A H \otimes_k H \rightarrow \Lambda \otimes_A H = \frac{1}{\xi} \hat{H}.
\end{equation}

Let $g \in \frac{1}{\xi} \hat{H}$ be a groupoid-like element with the source homomorphism $\eta$.

**A.4.8 Lemma.** The map

$$
\mu_g = \frac{1}{\xi} m(g \otimes -) : H \rightarrow \frac{1}{\xi} \hat{H}, \quad h \mapsto \frac{1}{\xi} m(g \otimes h),
$$

is left $A$-linear with respect to the $A$-module structure on $\frac{1}{\xi} \hat{H}$ given by $\eta : A \rightarrow \Lambda$ and by the $\Lambda$-module structure on $\frac{1}{\xi} \hat{H}$.

**Proof:** For $a \in A$ we have in by associativity of $m$,

$$
\mu_g(av) = \frac{1}{\xi} m(g \otimes av) = \frac{1}{\xi} m(ga \otimes v) \stackrel{(A.4.4)}{=} \frac{1}{\xi} m(\frac{1}{\xi} \epsilon(ga) \cdot g \otimes v) = \frac{1}{\xi} \epsilon(ga) \frac{1}{\xi} m(g \otimes v) = \eta(a) \mu_g(v),
$$

where the equality labelled (A.4.4) uses $\Delta(g) = g \otimes g$ and the formula (A.4.4). \qed
By the lemma, the map $\mu_g$ descends to

\[(A.4.9) \quad \Lambda_{\eta} \mu_g : \Lambda_{\eta} H \to \Lambda_{\xi} H.\]

By arguments similar to the above, we now arrive at the following fact.

\[(A.4.10) \quad \text{Theorem.} \quad (a) \text{ If } h \in \Lambda_{\eta} H \text{ is another groupoid-like element with } t(h) = \eta \text{ and } s(h) = \zeta, \text{ then } \]

\[ g \ast h := \Lambda_{\eta} \mu_g(h) \in \Lambda_{\xi} H \]

is a groupoid-like element with source $\zeta$ and target $\xi$.

(b) The operation $\ast$ defines a category $H(\Lambda)$ with objects being algebra homomorphisms $A \to \Lambda$ and $\text{Hom}(\eta, \xi)$ being the set of groupoid-like elements with source $\eta$ and target $\xi$.

\[(A.5) \quad \text{Dualizing left bialgebras.} \quad \text{Let } A \text{ and } H \text{ be as before.} \]

\[(A.5.1) \quad \text{Definition.} \quad \text{By an admissible filtration on } H \text{ we mean an increasing filtration } H = \bigcup_{d=0}^{\infty} H_{\leq d} \text{ with } H_{\leq 0} = A, \text{ compatible with both algebra and coalgebra structures, i.e.,} \]

\[ H_{\leq d} \cdot H_{\leq d'} \subset H_{\leq d+d'}, \quad \Delta(H_{\leq d}) \subset \sum_{d'+d''=d} H_{d'} \otimes H_{d''}, \]

and such that each $H_{\leq d}/H_{\leq d-1}$ is projective of finite rank as a left $A$-module.

Assume that $H$ is equipped with an admissible filtration and, further, is cocommutative. Then each

\[(A.5.2) \quad H_{\leq d}^\vee = \text{Hom}_{A-\text{Mod}}(H_{\leq d}, A) \]

is a commutative algebra with unit $\epsilon$. We have two homomorphisms of algebras

\[(A.5.3) \quad \sigma_d, \tau_d : A \to H_{\leq d}^\vee, \quad \sigma_d(a)(u) = \epsilon(au) = a\epsilon(u), \quad \tau_d(a)(u) = \epsilon(ua). \]

The projective limit

\[(A.5.4) \quad H^\vee = \varprojlim_d H_{\leq d}^\vee = \text{Hom}_{A-\text{Mod}}(H, A) \]

is then a commutative topological algebra and we have two homomorphisms $\sigma, \tau : A \to H^\vee$. Further, the multiplication in $H$ gives rise to comultiplication

\[(A.5.5) \quad H^\vee \to H^\vee \hat{\otimes}_A H^\vee = \varprojlim_d H_{\leq d}^\vee \otimes_A H_{\leq d}^\vee, \]

with the $A$-module structures on the factors given by $\tau$ and $\sigma$. In other words, $H^\vee$ is a topological commutative Hopf algebroid over $A$ in the sense of [Ra]. Thus the ind-scheme

\[(A.5.5) \quad \mathfrak{G}_1 = \text{Spf}(H^\vee) = \varprojlim_{d} \text{Spec}(H_{\leq d}^\vee) \]

is equipped with two morphisms $s, t : \mathfrak{G}_1 \to \text{Spec}(A)$ and the composition as in (A.1.2). In other terms, $\mathfrak{G} = (\mathfrak{G}_0 = \text{Spec}(A), \mathfrak{G}_1)$ is a category object in ind-schemes. In particular, for each commutative algebra $\Lambda$ we have a category $\mathfrak{G}(\Lambda)$ whose objects are homomorphisms $A \to \Lambda$. 

(A.5.7) Proposition. The category $G(\Lambda)$ is identified with $H(\Lambda)$.

Proof: Consider a morphism $\text{Spec}(\Lambda) \to G_1$, i.e., an algebra homomorphism $f : H^{\leq d}_d \to \Lambda$ for some $d$. Combining it with $\tau$, we get a homomorphism $\xi : A \to \Lambda$. Viewing $\Lambda$ as an $A$-module via $\xi$, we see that $f$ is, in particular, a morphism of $A$-modules, i.e., an element of

$$\text{Hom}_{A-\text{Mod}}(\text{Hom}_{A-\text{Mod}}(H^{\leq d}, A), \Lambda) \simeq \Lambda \otimes_A H^{\leq d} = \Lambda^1 H^{\leq d}.$$

Let $g \in \Lambda \otimes_A H^{\leq d}$ be the element corresponding to $f$. Then the fact that $f$ is a ring (and not just an $A$-module) homomorphism, is translated into the fact that $g$ is groupoid-like. We leave the rest to the reader. □

(A.5.8) Definition. Let $L$ be a Lie-Rinehart $A$-algebra. An admissible filtration in $L$ is an uncreasing filtration $L = \bigcup_{d=1}^{\infty} L^{\leq d}$ by $A$-modules, compatible with the brackets and such that each $L^{\leq d}/L^{\leq d-1}$ is a projective $A$-module of finite rank.

For example, if $L$ is itself projective of finite rank over $A$, we can take $L^{\leq 1} = L$.

Given an admissible filtration on $L$, we define a filtration on $U = U_A(L)$ by

(A.5.9) $U^{\leq d} = \text{Span}_A \left\{ x_1 \ldots x_p, \; x_i \in L^{\leq d_i}, \; \sum_{\nu} d_{\nu} \leq d \right\}$, \quad $d \geq 0$.

This filtration is admissible in the sense of (A.5.1).

(A.5.10) Proposition. Let $L$ be a Lie-Rinehart $A$-algebra with an admissible filtration. Then:

(a) The multiplication in $U_A(L)$ makes $G = (G_0 = \text{Spec}(A), G_1)$ into a formal groupoid. We denote this groupoid by $e^L$.

(b) The identification

$$U_A(L) = \text{Hom}^{\text{cont}}_{A}(U_A(L)^\vee, A)$$

(continuous morphisms of modules) restricts to an identification

$$L = \text{Der}^{\text{cont}}_{A}(U_A(L)^\vee, A)$$

(continuous derivations).

Part (b) is an algebraic analog of the equality (A.1.3) and means that $L$ can be seen as the Lie algebroid of the formal groupoid $e^L$. Part (b) expresses the fact that $L$ coincides with the subalgebra of primitive elements in $U_A(L)$, see (A.3.11)(b).

To prove (a), note that we already have that $G$ is a category object in ind-schemes. Further, $G_0 = \text{Spec}(A)$ is a scheme. Each $U_A^\vee$ is a nilpotent extension of $A$, so $G_1$ is a formal ind-scheme, and $s, t, e$ induce identity on reduced subschemes. This latter fact also implies that for any $\Lambda$ all morphisms in $G(\Lambda)$ are invertible. So $G$ is indeed a formal groupoid.

(A.6) Globalization. The following rather restrictive concept will be sufficient for our purposes.
(A.6.1) Definition. A formal ind-scheme \( \mathcal{X} \) over \( k \) will be called smooth, if it is locally isomorphic to an ind-scheme of the form \( Y \times \mathcal{O}_n \) for some smooth algebraic variety \( Y \) over \( k \) and some \( n \geq 0 \).

Thus, for a smooth formal ind-scheme \( \mathcal{X} \) we have the tangent sheaf \( T_\mathcal{X} \) which is a locally free sheaf of \( \mathcal{O}_\mathcal{X} \)-modules of finite rank. In particular, we can speak about Lie algebroids on \( \mathcal{X} \), as well as about \( U(\mathcal{G}) \), the enveloping algebra of a Lie algebroid \( \mathcal{G} \), see (1.2). We also have the concept of an admissible filtration on \( \mathcal{G} \) by sheafifying (A.5.7).

Let \( \mathcal{G} \) be a Lie algebroid on \( X \) equipped with an admissible filtration \( \{ \mathcal{G}_{\leq d} \}_{d \geq 1} \). We then define an algebra filtration \( \{ U_{\leq d}(\mathcal{G}) \} \) on \( U(\mathcal{G}) \) as in (A.5.8). The considerations of (A.5) extend easily to this situation by working with \( U(\mathcal{G}) \) as a sheaf of topological left bialgebras over \( \mathcal{O}_X \). Thus, we define

\[
(A.6.2) \quad \mathfrak{A}_d(\mathcal{G}) = \text{Hom}_{\mathcal{O}_X}(U_{\leq d}(\mathcal{G}), \mathcal{O}_X).
\]

(Hom with respect to the left module structure.) Then \( \mathfrak{A}_d(\mathcal{G}) \) is a sheaf of commutative \( \mathcal{O}_X \)-algebras with two algebra embeddings \( \sigma, \tau : \mathcal{O}_X \to \mathfrak{A}_d(\mathcal{G}) \). Let us further set

\[
(A.6.3) \quad \mathfrak{A}(\mathcal{G}) = \lim_{\longrightarrow} \mathfrak{A}_d(\mathcal{G}), \quad \mathfrak{G}_1 = \text{Spf} \mathfrak{A}(\mathcal{G}) = "\text{lim}"_d \text{ Spec } \mathfrak{A}_d(\mathcal{G}).
\]

We have then, by globalizing (A.5.10):

(A.6.4) Theorem. The multiplication in \( U(\mathcal{G}) \) makes \( \mathfrak{G} = (\mathfrak{G}_0 = \mathcal{X}, \mathfrak{G}_1) \) into a formal groupoid denoted \( e^\mathfrak{G} \), and we have

\[
\mathcal{G} = \text{Der}_{\text{cont}}^\text{conf}(\mathfrak{A}(\mathcal{G}), \mathcal{O}_X) = e^*T_{\mathfrak{G}_1/\mathfrak{G}_0}.
\]

Further, let \( S \) be a scheme, and \( y : S \to \mathcal{X} \) be a morphism. We have then a quasicoherent sheaf \( y^*U(\mathcal{G}) \) of \( \mathcal{O}_S \)-modules. By sheafifying (A.4.3), we see that \( y^*U(\mathcal{G}) \) is a sheaf of unital \( \mathcal{O}_S \)-coalgebras, so we can speak about its groupoid-like sections. Given any such section \( g \), we have, as in (A.4.6), another morphism \( x : S \to \mathcal{X} \) called the source of \( g \) (while \( y \) is called the target).

(A.6.5) Theorem. Morphisms \( S \to \mathfrak{G}_1 \) are in bijection with pairs \( (y, g) \) where \( y : S \to \mathcal{X} \) is a morphism, and \( g \) is a groupoid-like section of \( y^*U(\mathcal{G}) \). \( \square \)

(A.6.6) Example. Let \( \mathcal{G} = T_\mathcal{X} \) with filtration \( \mathcal{G}_{\leq d} = \mathcal{G} \), \( d \geq 1 \). Then \( U(\mathcal{G}) = \mathcal{D}_\mathcal{X} \), and \( U_{\leq d}(\mathcal{G}) = \mathcal{D}_\mathcal{X}^{\leq d} \) is the sheaf of differential operators of order \( \leq d \). The dual bundle of commutative \( \mathcal{O}_d \)-algebras \( \mathfrak{A}_d \) is then identified with \( J^d(\mathcal{O}_X) \), the bundle of \( d \)-jets of functions. So \( \mathfrak{A} = J^\infty(\mathcal{O}_X) \) is the bundle of infinite jets, which is the same as \( \hat{\mathcal{O}}_{\mathcal{X} \times \mathcal{X}, \mathcal{X}} \), the formal completion of \( \mathcal{O}_{\mathcal{X} \times \mathcal{X}} \) along the diagonal. Thus \( \mathfrak{G}_1 = \mathcal{X}^{(\infty)}_{\mathcal{X} \times \mathcal{X}} \) is the formal neighborhood of the diagonal in \( \mathcal{X} \times \mathcal{X} \), with \( s, t \) being the projections and \( e \) being the diagonal embedding. We will call \( \mathfrak{G} = e^{T_\mathcal{X}} \) the crystalline groupoid of \( \mathcal{X} \).

(A.7) Transitive Lie algebroids on the formal disk. Let

\[
(A.7.1) \quad 0 \to \mathcal{G}^0 \to \mathcal{G} \xrightarrow{\alpha} T_{\mathcal{D}_n} \to 0
\]
be a transitive Lie algebroid on \( \mathbb{D}_n \). We assume that \( \mathcal{G} \) is equipped with an admissible filtration \( \{ \mathcal{G}_\leq d \} \). Then \( \mathcal{G}^o \) is a bundle of Lie algebras (possibly of infinite rank) over \( \mathbb{D}_n \), i.e., a Lie \( k[[x_1, \ldots, x_n]] \)-algebra free as a \( k[[x_1, \ldots, x_n]] \)-module. We denote by
\[
(A.7.2) \quad \mathfrak{g} = \mathcal{G}^o/(x_1, \ldots, x_n)\mathcal{G}^o
\]
the fiber of \( \mathcal{G}^o \) at the origin of \( \mathbb{D}_n \). This is a Lie \( k \)-algebra.

Let \( \text{Mod}(\mathcal{G}) \) be the category of \( \mathcal{G} \)-modules, i.e., of finite rank vector bundles \( \mathcal{E} \) on \( \mathbb{D}_n \) with \( \mathcal{G} \)-action, as in (1.1.4). Let also \( \text{Rep}(\mathfrak{g}) \) be the category of finite-dimensional representations of the Lie algebra \( \mathfrak{g} \) over \( k \).

**(A.7.3) Theorem.** The restriction functor
\[
\text{Res} : \text{Mod}(\mathcal{G}) \to \text{Rep}(\mathfrak{g}), \quad \mathcal{E} \mapsto \mathcal{E}/(x_1, \ldots, x_n)\mathcal{E}
\]
is an equivalence of categories.

**(A.7.4) Example.** Let \( \mathcal{G} = T_{\mathbb{D}_n} \). Then the theorem reduces to the well known fact that the category of finite rank vector bundles on \( \mathbb{D}_n \) with flat connections is equivalent to the category of finite-dimensional \( k \)-vector spaces.

**Proof of Theorem A.7.3:** Let \( G = e^\theta \) be the formal group integrating \( \mathfrak{g} \), and \( \mathfrak{G} = e^\mathcal{G} \) be the formal groupoid integrating \( \mathcal{G} \). Thus \( \mathfrak{G}_0 = \mathbb{D}_n \), and \( \mathfrak{G}_1 = \text{Spf}(\mathfrak{A}) \), where \( \mathfrak{A} = \mathfrak{A}(\mathcal{G}) \) is the topological Hopf algebroid constructed in (A.6.3). Because \( \mathcal{G} \) is transitive, we have surjective homomorphism \( U(\mathcal{G}) \to U(T_{\mathbb{D}_n}) = \mathbb{D}_{\mathbb{D}_n} \). Thus the projection to the crystalline groupoid of \( \mathbb{D}_n \),
\[
(A.7.5) \quad p : \mathfrak{G}_1 \to (\mathbb{D}_n)^{(\infty)}_{\mathbb{D}_n \times \mathbb{D}_n} = \mathbb{D}_n \times \mathbb{D}_n,
\]
makes \( \mathfrak{G}_1 \) into a \( G \)-bitorsor over \( \mathbb{D}_n \times \mathbb{D}_n \), i.e., it has two \( G \)-actions, each of which makes it into a torsor. Let
\[
(A.7.6) \quad \overline{\mathfrak{A}} = \mathfrak{A}/(x_1, \ldots, x_n)\mathfrak{A}
\]
be the fiber of \( \mathfrak{A} \) at \( 0 \in \mathbb{D}_n \) (with respect to the right \( \mathcal{O}_{\mathbb{D}_n} \)-module structure). Then, with respect to the left module structure, \( \overline{\mathfrak{A}} \) is still an \( \mathcal{O}_{\mathbb{D}_n} \)-module. Geometrically,
\[
(A.7.7) \quad \text{Spf}(\overline{\mathfrak{A}}) = p^{-1}(\{0\} \times \mathbb{D}_n).
\]
Restricting the identification of Proposition A.6.4(b), we get a homomorphism
\[
(A.7.8) \quad \mathfrak{g} \to \text{Der}^\text{cont}_{\mathcal{O}_{\mathbb{D}_n}}(\overline{\mathfrak{A}}),
\]
which is an isomorphism, in virtue of \( p \) being a torsor (hence isomorphic to the projection \( G \times \mathbb{D}_n \times \mathbb{D}_n \to \mathbb{D}_n \times \mathbb{D}_n \)). In particular, \( \mathfrak{g} \) acts on \( \overline{\mathfrak{A}} \) by derivations.

Let now \( V \in \text{Rep}(\mathfrak{g}) \). Set
\[
(A.7.9) \quad \text{Ind}(V) = (\overline{\mathfrak{A}} \otimes_k V)^\theta = \text{Hom}_{\mathcal{G}^o}(U(\mathcal{G}), V),
\]
where $V$ is considered as a $\mathcal{G}^e$-module via the projection $\mathcal{G}^e \to \mathfrak{g}$. Then $\mathcal{G}$ acts on $\text{Ind}(V)$ because $\mathcal{G}$ acts on $\overline{\mathfrak{a}}$, the latter action being induced from the left $\mathcal{G}$-action on $\mathfrak{a}$. The natural identification of vector spaces

$$V \to \text{Res}(\text{Ind}(V))$$

is obvious. Let $\mathcal{E} \in \text{Mod}(\mathcal{G})$. Let us construct a natural isomorphism of $\mathcal{G}$-modules

$$(A.7.10) \quad \lambda_{\mathcal{E}} : \mathcal{E} \to \text{Ind}(\text{Res}(\mathcal{E})) = \text{Hom}_{\mathcal{G}^e}(U(\mathcal{G}), \overline{\mathcal{E}}).$$

Indeed, to construct $\lambda_{\mathcal{E}}$, we just use the $U(\mathcal{G})$-action on $\mathcal{E}$:

$$\lambda_{\mathcal{E}}(\epsilon)(u) = u(\epsilon)(0), \quad \epsilon \in \mathcal{E}, u \in U(\mathcal{G}),$$

where the value at 0 means the image in $\overline{\mathcal{E}}$. So we have constructed $\lambda_{\mathcal{E}}$ as a morphism of $\mathcal{G}$-modules. Further, after restriction to the fibers at $0 \in D_n$, the morphism $\lambda_{\mathcal{E}}$ is an isomorphism of $k$-vector spaces. Now, the Nakayama lemma for $D_n = \text{Spf} \ k[[x_1, \ldots, x_n]]$ implies that $\lambda_{\mathcal{E}}$ is an isomorphism of vector bundles over $D_n$. $\square$

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