Spherical Structures in the Inflationary Cosmology

N. Riazi

Physics Department and Biruni Observatory, Shiraz University, Shiraz 71454, Iran, and
Institute for Research in Physics and Mathematics (IPM), Farmanieh, Tehran, Iran.

email: riazi@sun01.susc.ac.ir

Abstract

It has been suggested that wormholes and other non-trivial geometrical structures might have been formed during the quantum cosmological era \( t \sim 10^{-43} \text{s} \). Subsequent inflation of the universe might have enlarged these structures to macroscopic sizes. In this paper, spherical geometrical structures in an inflationary RW background are derived from the Einstein equations, using a constraint on the energy-momentum tensor which is an extension of the one expected for inflation. The possibility of dynamical wormholes and other spherical structures are explored in the framework of the solutions.

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1. Introduction

It is now generally accepted that the universe underwent a brief inflationary period at \( t \sim 10^{-35} \text{s} \), during which the scale factor of the universe increased exponentially (or quasi-exponentially). Such a dynamics emerges as a result of an equation of state which resembles that of the vacuum or cosmological constant\(^1\). It has been suggested that non-trivial topolog-
ical structures probably formed during the quantum cosmological era \( (t \sim 10^{-43}s) \). Among these structures are wormholes which are throat-like objects connecting two otherwise disconnected spacetimes or two remote parts of the same spacetime. Wormholes, originally of the Planck length scale, are expected to take part in the cosmological expansion during the inflationary era, and turn into macroscopic objects. Sato et al. investigated the possibility of wormhole formation during the inflationary era. Roman presented a new classical metric which represents a wormhole, embedded in a flat de Sitter spacetime. The throat radius was shown to inflate with time.

Static wormholes are known to violate the weak and strong energy conditions. Even the averaged weak energy conditions are known to be violated by static wormholes. Morris and Thorne discussed the conditions that a wormhole must satisfy in order to be traversable in both directions. These conditions are derived from the requirement that there are no event horizons or curvature singularities. Since the energy-momentum which is required to support static, traversable wormholes does not satisfy the null (and the weak) energy conditions, they called the corresponding matter "exotic". Teo discussed the general form of a stationary, axially symmetric traversable wormhole. He found that the null energy condition is violated by such wormholes, but there can be classes of traversing geodesics which do not cross energy condition violating regions. Evolving wormholes, on the other hand, may alter this unfavorable situation. Although the violation of the energy conditions is unacceptable from the conventional point of view, it has been shown that some quantum field theory effects (such as the Casimir effect) allow such a violation. Moreover, the negative energy densities required for the support of the wormhole geometry might be produced by the gravitational squeezing of the vacuum. The possibility of closed, timelike curves in wormhole geometries is another concern in the wormhole theory. The role of the cosmological constant (i.e. the vacuum energy-momentum tensor) on the dynamics and stability of the Lorentzian wormholes was studied by Kim.

There are two approaches in formulating wormhole spacetimes: 1) By joining two asymptotically flat (or asymptotically RW) spacetimes via a boundary layer with \( \delta \)-function
energy-momentum\textsuperscript{4}, and 2) By smoothly merging the wormhole metrics to a cosmological background\textsuperscript{3}. In the present paper, we employ the latter method to explore the dynamics of spherical structures (including wormholes) which join smoothly to a cosmological background in its inflationary phase. In section 2, we formulate the problem and seek various exact solutions which correspond to different choices of the integration constants. Section 3 is devoted to the corresponding energy-momentum tensor and the exoticity parameter. The last section contains a summary and a few concluding remarks.

2. Spherical solutions in an inflationary background

It is well-known that the most general isotropic and homogeneous metric consistent with the cosmological principle is the Robertson-Walker (RW) metric

\[ ds^2 = -dt^2 + R^2(t) \left[ \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right], \]  

in which \((r, \theta, \phi)\) are the so-called co-moving coordinates, \(R(t)\) is the scale factor, and \(k = 0, \pm 1\) correspond to the flat, closed, and open universes, respectively. This metric has 6 Killing vectors\textsuperscript{5}, representing its high degree of symmetry (Minkowski spacetime has 10 Killing vectors).

It is clear that the existence of a spherical object in a RW background spoils part of its symmetries. The resulting spacetime is no longer homogeneous. Accordingly, we start by assuming the evolving, spherically symmetric spacetime

\[ ds^2 = -dt^2 + R^2(t) \left[ (1 + a(r)) dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right], \]  

where \(a(r)\) is an unknown function. It is clear that the RW metric (1) is a special case of (2), with \(1 + a(r) = \frac{1}{1-kr^2}\). It can be shown that in general, the metric (2) possesses the following Killing vectors, with respect to the \((\frac{\partial}{\partial t}, \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi})\) basis\textsuperscript{6}:

\[ K_1 = (1, 0, 0, 0), \]

\[ K_2 = (0, 0, \cos \phi, -\cot \theta \sin \phi), \]
\( K_3 = (0, 0, \sin \phi, \cot \theta \cos \phi), \)

\( K_4 = (0, 0, 0, 1), \) \hspace{1cm} (3)

The non-vanishing components of the Einstein tensor for the metric (2) read

\[
G_{tt} = -3 \frac{\dot{R}^2}{R^2} - \frac{ra' + a + a^2}{r^2 R^2 (1 + a)^2}, \]  

\[
G_{rr} = (1 + a)(\ddot{R}^2 + 2 R \ddot{R}) + \frac{a}{r^2}, \]  

and

\[
G_{\theta \theta} = 2 r^2 R \ddot{R} + r^2 \dot{R}^2 + \frac{ra'}{2(1 + a)^2} = \frac{1}{\sin^2 \theta} G_{\phi \phi}. \]  

Such a spacetime is supported by an anisotropic energy-momentum tensor:

\[
\rho(r, t) = -\frac{1}{8 \pi G} T^{t}_t = \frac{1}{8 \pi G} \left[ \frac{3 \dot{R}^2}{R^2} + \frac{ra' + a + a^2}{r^2 R^2 (1 + a)^2} \right], \]  

\[
P_r(r, t) = \frac{1}{8 \pi G} T^r_r = -\frac{1}{8 \pi G} \left[ \frac{\dot{R}^2}{R^2} + 2 \frac{\ddot{R}}{R} + \frac{a}{r^2 R^2 (1 + a)} \right], \]  

and

\[
P_t(r, t) = \frac{1}{8 \pi G} T^\theta_\theta = -\frac{1}{8 \pi G} \left[ \frac{\dot{R}^2}{R^2} + 2 \frac{\ddot{R}}{R} + \frac{a'}{2r R^2 (1 + a)^2} \right]. \]  

In these equations, \( P_r \) and \( P_t \) are the radial and transverse pressures, and "dots" and "primes" denote differentiation with respect to \( r \) and \( t \), respectively. It can be easily shown that the special case \( P_r = P_t \) leads to either \( a = 0 \) or \( a(r) = \frac{r^2}{r_0^2 - r^2} \), which upon substituting in (7), lead to the disappearance of \( r \) dependence in \( \rho(r, t) \) as expected for a homogeneous spacetime.

Since we are looking for spherical structures in an inflationary background, \( P_r \) and \( P_t \) should have the asymptotic behavior

\[
P_r \to P_t \to -\rho \to \text{constant}, \]  

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far from the central structure. Recall that for vacuum (i.e. the cosmological constant),

\[ P_r = P_t = -\rho = \text{constant}. \]

In such a case, the background spacetime is expected to expand exponentially (i.e. \( R(t) \propto \exp(Ht) \), with \( H \) constant). As a generalization of the vacuum equation of state, we consider the relation

\[ \rho = -\frac{1}{1 + \gamma} [P_r + \gamma P_t], \quad (11) \]

in which \( \gamma \) is a constant. Using (7)-(9), one obtains from (11)

\[ 3 \frac{\dot{R}^2}{R^2} + \frac{ra'}{r^2R^2(1+a)^2} = \]

\[ \frac{1}{1 + \gamma} \left\{ \frac{\dot{R}^2}{R^2} + \frac{2\dot{R}}{R} + \frac{a}{r^2R^2(1+a)} + \gamma \frac{\dot{R}}{R^2} + \gamma \frac{\dot{R}^2}{R^2} + \gamma \frac{a'}{2rR^2(1+a)^2} \right\}. \quad (12) \]

This equation can be separated into t-dependent and r-dependent parts:

\[ 2\dot{R}^2 - 2\ddot{R}R = \frac{-(1 + \gamma) [ra' + a(1 + a)] + a(1 + a) + \gamma ra'/2}{(1 + \gamma)r^2(1+a)^2}. \quad (13) \]

The lhs and rhs parts, should therefore be constant, independent of \( r \) and \( t \). Let us call the constant of separation \( 2c_1 \). We first seek solutions for the special case \( c_1 = 0 \).

A. Solutions for the case \( c_1 = 0 \)

The equation \( \dot{R}^2 = \ddot{R}R \) is immediately solved to obtain

\[ R(t) = R_0 e^{Ht}, \quad (14) \]

in which \( R_0 \) and \( H \) are constants. The equation for \( a(r) \) reads

\[ ra' = -\frac{2\gamma}{2 + \gamma} a(1 + a), \quad (15) \]

which can be integrated to obtain the exact solution

\[ a(r) = \frac{r_o^{\alpha}}{r^{\alpha} - r_o^{\alpha}}, \quad (16) \]
where $\alpha = \frac{2\gamma}{2+\gamma}$, and $r_o$ is the constant of integration, having the same dimensions as $r$. The resulting metric is

$$ds^2 = -dt^2 + R_o^2 e^{2Ht} \left[ \frac{r^\alpha}{r^\alpha - r_o^\alpha} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right].$$  \hspace{1cm} (17)$$

We therefore have a wormhole-centered asymptotically de Sitter spacetime, with the central wormhole having a throat circumference

$$\ell = R(t)r_o \oint d\phi = 2\pi r_o R_o e^{Ht},$$  \hspace{1cm} (18)$$

which expands with time, exponentially.

The energy momentum tensor needed to support this structure has the components

$$\rho(r, t) = \frac{3H^2}{8\pi G} \left[ 1 + \left( \frac{2-\gamma}{2+\gamma} \right) \frac{r_o^{2\gamma}}{3H^2 R_o^2 r^{4(1+\gamma)}} e^{-2Ht} \right], \hspace{1cm} (19)$$

$$P_r(r, t) = -\frac{3H^2}{8\pi G} \left[ 1 + \frac{r_o^{2\gamma}}{3H^2 R_o^2 r^{4(1+\gamma)}} e^{-2Ht} \right], \hspace{1cm} (20)$$

and

$$P_t(r, t) = -\frac{3H^2}{8\pi G} \left[ 1 - \left( \frac{\gamma}{2+\gamma} \right) \frac{r_o^{2\gamma}}{3H^2 R_o^2 r^{4(1+\gamma)}} e^{-2Ht} \right]. \hspace{1cm} (21)$$

It is seen that for reasonable values of $\gamma$, the equation of state rapidly approaches that of the cosmological constant for $r >> r_o$ and/or $t >> 1/H$. Moreover, for the special case $\gamma = 2$, the energy density is homogeneous, independent of both $r$ and $t$. It can be easily shown that this result is consistent with the energy equation

$$D_{\mu}T^\mu_\nu = 0,$$  \hspace{1cm} (22)$$

which leads to

$$\frac{\partial \rho}{\partial t} + [3\rho + P_r + 2P_t] = 0.$$  \hspace{1cm} (23)$$

Note that $3\rho + P_r + 2P_t = 0$ in the $\gamma = 2$ case.
B. The general case $c_1 \neq 0$

Let us proceed with the general case $c_1 \neq 0$;

\[ \ddot{R}^2 - R\ddot{R} = c_1. \]  \hspace{1cm} (24)

This equation can be integrated once to obtain

\[ \frac{R^2}{R_o^2} = \dot{R}^2 - c_1, \]  \hspace{1cm} (25)

where $R_o$ is a constant of integration. Integrating once again, leads to

\[ t = \int \frac{dR}{\left[ c_1 + \frac{R^2}{R_o^2}\right]^{1/2}}, \]  \hspace{1cm} (26)

or

\[ R(t) = R_o \sinh(\gamma t), \quad \text{for} \quad c_1 > 0, \]  \hspace{1cm} (27)

and

\[ R(t) = R_o \cosh(\gamma t), \quad \text{for} \quad c_1 < 0, \]  \hspace{1cm} (28)

where $H = \sqrt{|c_1|}/R_o$.

The radial equation reads

\[ \frac{da(r)}{dr} = Ar(1 + a)^2 - B\frac{a(1 + a)}{r}, \]  \hspace{1cm} (29)

where

\[ A = -\frac{4c_1(1 + \gamma)}{2 + \gamma}, \quad \text{and} \quad B = \frac{2\gamma}{2 + \gamma}. \]  \hspace{1cm} (30)

Fortunately, equation (29) can also be solved exactly to obtain

\[ 1 + a(r) = \left[ 1 + c_o r^{-B} - \frac{A}{2 + B} r^2 \right]^{-1}, \]  \hspace{1cm} (31)

in which $c_o$ is a constant of integration. Note that for the special case $c_1 = 0$, we have $A = 0$, and the solution (31) reduces to (14), with a suitable definition for $c_o$. The function $a(r) + 1$
is plotted against \( r \) in Figure 1, for \( c_1 = \pm 1 \), \( c_o = \pm 1 \), and \( \gamma = 2 \). Note that for \( \gamma = 2 \), \((31)\) can be written as

\[
1 + a(r) = \frac{r}{c_1 r^3 + r + c_o} = \frac{r}{(r - r_1)(c_1 r^2 + c_1 r - \frac{c_o}{r_1})},
\]

where

\[
r_1 = \beta - \frac{1}{3c_1 \beta},
\]

and

\[
\beta = -\frac{c_o}{2c_1} + \sqrt{3} \frac{\sqrt{4 + 27c_o^2c_1}}{18c_1}.
\]

In order for \( r_1 \) to be real, we must have

\[
c_1 > 0,
\]

or

\[
c_1 < 0, \quad \text{and} \quad c_o^2 > \frac{4}{27|c_1|}.
\]

The choice \( c_1 = \pm 1 \) and \( c_0 = 0 \) leads to \( 1 + a(r) = \frac{1}{1 + r^2} \), which correspond to the \( k = \mp 1 \) RW metric. Referring to Figure 1, we see that for \( c_1 = -1 \) and \( c_0 = -1 \), \( 1 + a(r) \) becomes negative and the metric has an improper signature. Such a solution, therefore, is not of cosmological significance.

The \( c_1 = 1 \) and \( c_0 = 1 \) case is a metric which is regular everywhere except at \( r = 0 \). The \( c_1 = -1 \) and \( c_0 = 1 \) case possesses an upper bound on \( r (= r_1) \) and corresponds to a closed cosmological background.

The \( c_1 = 1 \) and \( c_0 = -1 \) and similar cases possess a lower bound at \( r = r_1 \), and correspond to wormhole-centered open universes. Note that in such cases, \( r_1 \) plays the role of the wormhole radius, and we also have

\[
1 + a(r) \to \frac{1}{r^2}, \quad \text{as} \quad r \to \infty,
\]

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in accordance with the $k = -1$ RW model. The function $1 + a(r)$ for the closed and open RW universes are also shown in Figure 1 for comparison.

Note that in the $c_1 = c_0 = 1$ case, there is a singularity at $r = 0$, where the diagonal components of the Einstein tensor diverge.

3. Energy-momentum tensor and exoticity parameter

Using equations (7) to (9), it can be shown that for the general case $c_1 \neq 0$, we have

$$\rho = \frac{1}{8\pi G} \left[ \frac{3}{R_o^2} + \frac{f_1(r)}{R^2} \right],$$

(38)

where

$$f_1(r) = 3c_1 + A + \frac{(1 + B)a}{(1 + a)r^2},$$

(39)

$$P_r = -\frac{1}{8\pi G} \left[ \frac{3}{R_o^2} + \frac{f_2(r)}{R^2} \right],$$

(40)

where

$$f_2(r) = c_1 + \frac{a}{(1 + a)r^2},$$

(41)

and

$$P_t = -\frac{1}{8\pi G} \left[ \frac{3}{R_o^2} + \frac{f_3(r)}{R^2} \right],$$

(42)

where

$$f_3(r) = c_1 + \frac{1}{2}A(1 + a) + \frac{1}{2}Ba \frac{1}{r^2}.$$  

(43)

The explicit form of the function $a(r)$ is given by (31). The exoticity parameter is defined according to

$$\xi = -\frac{P + \rho}{|\rho|},$$

(44)
with $\xi > 0$ corresponding to the exotic matter and $\xi < 0$ to the non-exotic. Since we have an unisotropic medium, we modify (44) according to
\[
\bar{\xi} = -\frac{\bar{P} + \rho}{|\rho|}, \quad (45)
\]
where
\[
\bar{P} = \frac{1}{1 + \gamma}(P_r + \gamma P_t). \quad (46)
\]
Note that for the vacuum equation of state we have $\xi = 0$, which is the boarder line between the exotic and non-exotic matter. From (45), we see that the modified exoticity parameter (45) vanishes everywhere and for all cases.

Let us check the null energy condition
\[
R_{\mu\nu}k^\mu k^\nu \geq 0, \quad (47)
\]
where $R_{\mu\nu}$ is the Ricci tensor corresponding to the metric (2), and $k^\mu$ is a null vector taken to be
\[
k^\mu = (\sqrt{-g^{tt}}, \pm\sqrt{g^{rr}}, 0, 0). \quad (48)
\]
Using the metric (2), we readily obtain
\[
R_{\mu\nu}k^\mu k^\nu = 2\frac{\ddot{R}}{R} - 2\frac{\dot{R}^2}{R^2} - \frac{a'}{R^2 r(1 + a)^2}. \quad (49)
\]
Using the solutions obtained in the previous section for the $c_1 = 0$, $c_1 > 0$, and $c_1 < 0$ cases, we obtain
\[
R_{\mu\nu}k^\mu k^\nu = \text{cosech}^2 Ht \left[ \frac{Bco}{R_0^2 r^{2+B}} - \frac{2A}{(2+B)R_0^2} - 2H^2 \right]; \quad \text{for } c_1 > 0, \quad (50)
\]
\[
R_{\mu\nu}k^\mu k^\nu = \frac{2\gamma}{2 + \gamma} \frac{1}{R_0^2 r^2} e^{-2Ht} \left( \frac{r_0}{r} \right)^{\alpha} \; \text{for } c_1 = 0, \quad (51)
\]
and
\[
R_{\mu\nu}k^\mu k^\nu = \text{sech}^2 Ht \left[ 2H^2 + \frac{Bco}{R_0^2 r^{2+B}} - \frac{2A}{(2+B)R_0^2} \right]; \quad \text{for } c_1 < 0. \quad (52)
\]
It is seen that the matter everywhere satisfies the null energy condition for the $c_1 = 0$ case. For the $c_1 > 0$ and $c_0 < 0$ case which contains wormhole solutions, the null energy condition reduces to

$$\frac{8c_1(1 + \gamma)}{(2 + \gamma)(2 + B)R^2_0} \geq 2H^2 + \frac{B|c_0|}{R^2_0 r_1^{2+B}},$$

where $r_1$ is the wormhole throat radius given by (53).

4. Summary and conclusion

If wormholes are produced spontaneously in the quantum era, as claimed by some quantum cosmology considerations, we would expect them to undergo a rapid expansion during the inflationary era, and turn into macroscopic objects. Motivated by this interesting possibility, we studied the dynamics of a spherical geometrical structure in an inflationary background. Based on a reasonable constraint on the energy-momentum tensor of an anisotropic medium reminiscent of the vacuum, we solved the Einstein equations to obtain a metric which smoothly merges into a rapidly expanding background universe. Solutions were classified into different categories, with distinct geometries for the central object. One class of the solutions was shown to represent Lorentzian wormholes with the throat radius expanding with the universe.

The present work differs from Roman in the following respects. In the ansatz metric used by Roman, the scale factor is pre-assumed to be of the form $e^{2\chi t}$ with $2\chi$ corresponding to the expansion rate ($H$ in the present paper). The scale factor is derived here, with our $c_1 = 0$ case coinciding with the one adopted by Roman with $\Phi(r) = 0$ ($\Phi$ is the redshift function). Moreover, while the shape function is left undetermined in the mentioned reference, it is explicitly (and analytically) calculated here.
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Fig. 1. The radial function $1 + a(r)$ for (a) $c_1 = 1$, $c_0 = 1$, (b) $c_1 = -1$, $c_0 = 1$, (c) $c_1 = 1$, $c_0 = -1$, and (d) $c_1 = -1$, $c_0 = -1$. The radial function $1 + a(r)$ for the Robertson-Walker metric is also shown for comparison: (e) open RW with $k = -1$, and (f) closed RW with $k = +1$. 
Figure 1: