On \((2n/3 - 1)\)-Resilient \((n, 2)\)-Functions

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Abstract—A \{00, 01, 10, 11\}-valued function on the vertices of the \(n\)-cube is called a \(t\)-resilient \((n, 2)\)-function if it has the same number of 00s, 01s, 10s and 11s among the vertices of every subcube of dimension \(t\). The Friedman and Fon-Der-Flaass bounds on the correlation immunity order say that such a function must satisfy \(t \leq 2n/3 - 1\); moreover, the \((2n/3 - 1)\)-resilient \((n, 2)\)-functions correspond to the equitable partitions of the \(n\)-cube with the quotient matrix \([0, r, r, r], [r, 0, r, r], [r, r, 0, r], [r, r, r, 0]\), \(r = n/3\). We suggest constructions of such functions and corresponding partitions, show connections with Latin hypercubes and binary 1-perfect codes, characterize the non-full-rank and the reducible functions from the considered class, and discuss the possibility to make a complete characterization of the class.

Index Terms—vectorial Boolean functions, resilient Boolean functions, correlation-immune functions, equitable partitions, Latin hypercubes.

I. INTRODUCTION

The set \(Q_n := \{0, 1\}^n\) of all \(n\)-words over \(\{0, 1\}\) is called the \(n\)-cube. This set forms a linear \(n\)-dimensional space over the field \(GF(2)\) with the standard basis \(e_1 = 10\ldots0, e_2 = 010\ldots0, \ldots, e_n = 00\ldots01\). The same term \(Q_n\) is used to denote the graph on \(\{0, 1\}^n\), where two words are adjacent if and only if they differ in exactly one position. The number of ones in a word \(v\) from \(Q_n\) is referred to as the weight of \(v\), \(wt(v)\). A \(k\)-subcube of \(Q_n\) is a subgraph isomorphic to \(Q_k\) (and also the corresponding subset of vertices). A (vectorial Boolean) function from \(Q_n\) to \(Q_m\) is called an \((n, m)\)-function. The \((n, 1)\)-functions correspond to the usual, non-vectorial, Boolean functions. An \((n, m)\)-function is called balanced if it possesses each of the \(2^m\) values exactly \(2^{m-n}\) times. An \((n, m)\)-function from \(Q_n\) to some finite set \(S\) is called correlation immune of order \(t\) if the proportion of occurrences of the values is the same in all \(l\)-subcubes of \(Q_n\) with \(l \geq n-t\). So,

\[|f^{-1}(s) \cap Q_l| = |f^{-1}(s)|/2^{n-l}\]

for every \(s \in S\) and every \(l\)-subcube \(Q_l\), \(l \geq n-t\). An \((n, m)\)-function is called \(t\)-resilient if it is both balanced and correlation immune of order \(t\). Equivalently, if for every \(y \in Q_m\), the characteristic function

\[f_y(x) := \begin{cases} 1 & \text{if } f(x) = y \\ 0 & \text{otherwise} \end{cases} \quad (1)\]

of \(f^{-1}(y)\) is correlation immune of order \(t\) and has exactly \(2^{m-n}\) ones. Resilient functions play important role in cryptography, see e.g. [1].

In [4], Friedman derived the bound

\[n - t - 1 \geq \frac{2^{m-1} - 1}{2^m - 1} \quad (2)\]

for \(t\)-resilient \((n, m)\)-functions or, more generally, for \((n, 1)\)-functions of correlation-immunity order \(t\) with \(2^{n-m}\) ones. For \(m = 2\), this bound turns to

\[t \leq \frac{2n}{3} - 1, \quad (3)\]

and in the current thesis we are interested in the \(t\)-resilient \((n, 2)\) functions that meet (3) with equality. Fon-Der-Flaass [2] proved that the correlation-immunity order \(t\) of any non-constant unbalanced Boolean \((n, 1)\) function meets (3). (Note that there are interesting classes of unbalanced Boolean \((n, 1)\) functions with more than \(2^{n-2}\) and less than \(2^{n-1}\) ones that attend the Fon-Der-Flaass bound, see [12], [3].) Since for \(m \geq 2\) the functions \(f_y\), see (1), are not balanced, we see that the class of functions we study lies on both the Friedman bound and the Fon-Der-Flaass bound. It occurs that attending any of these two bounds implies that the function belongs to the class of very regular objects, known as equitable partitions.

A partition \((C_i)_{i \in I}\) (where \(I\) is a finite index set) of the vertices of a graph \(\Gamma\) is called an equitable partition (perfect coloring, regular partition, partition design) with a quotient matrix \([|S_{i,j}|]_{i,j \in I}\) if for every \(i, j \in I\) and for every \(c \in C_i\) one has \(|\Gamma(c) \cap C_j| = S_{i,j}\), where \(\Gamma(c)\) is the neighborhood of \(c\) in \(\Gamma\).

As was proved by Fon-Der-Flaass [2], any non-constant unbalanced Boolean function attaining the bound (3) on the order \(t\) of correlation immunity corresponds to an equitable partition \((f^{-1}(1), f^{-1}(0))\) with a quotient matrix \([a, b], [c, d]\), where \(a + b = c + d = n\) and \(a - c = d - b = -n/3\). On the other hand, Potapov [8], [10] proved that any Boolean function with \(2^{n-m}\) ones whose correlation-immunity order \(t\) satisfies (2) with equality corresponds to an equitable partition \((f^{-1}(1), f^{-1}(0))\) with the quotient matrix \([0, n], [n/(2m-1), n-n/(2m-1)]\). From any of these results, we get the following.

Proposition 1: For any \((2n/3 - 1)\)-resilient \((n, 2)\)-function \(f\), the partition \((f^{-1}(00), f^{-1}(01), f^{-1}(10), f^{-1}(11))\) of \(Q_n\) is equitable with the quotient matrix

\[
S_n^{(4)} := \begin{bmatrix} 0 & r & r & r \\ r & 0 & r & r \\ r & r & 0 & r \\ r & r & r & 0 \end{bmatrix}, \quad r = \frac{n}{3} \quad (4)
\]
Remark 1: In general, one can see from (8) that any $t$-resilient $(n, m)$-function such that (2) holds with equality induces an equitable partition of $Q_n$ into $2^m$ cells with the following quotient matrix: the diagonal elements are 0, while every non-diagonal element is $n/(2^m - 1)$.

In this thesis, we study the $(2n/3 - 1)$-resilient $(n, 2)$-functions or, equivalently, the equitable $4$-partitions of $Q_n$ with the quotient matrix (4), and the related (as we see from (11)) class of order-$(2n/3 - 1)$ correlation-immune $(n, 1)$-functions with $2^{n-2}$ ones, or, equivalently, the equitable $2$-partitions of $Q_n$ with the quotient matrix

$$S^{(2)}_n := \begin{bmatrix} 0 & 3r \\ r & 2r \end{bmatrix}, \quad r = \frac{n}{3}. \quad (5)$$

We will refer to the equitable partitions of $Q_n$ with the quotient matrix $S^{(4)}_n$ or $S^{(2)}_n$ as the $S^{(4)}_n$-partitions or $S^{(2)}_n$-partitions, respectively.

In the next section, we discuss relations between the classes of $S^{(2)}_n$-partitions and $S^{(4)}_n$-partitions and announce the results of computational classification of $5$-resilient $(9, 2)$-functions (Theorem 1). Section III is devoted to the concept of rank of $(n, 2)$-functions and related partitions. The $S^{(2)}_n$-partitions are characterized in terms of multifold $1$-perfect codes (Theorem 2), giving a powerful construction of such partitions and $(2n/3 - 1)$-resilient $(n, 2)$-functions. In Section V we consider connections of the class of $(2n/3 - 1)$-resilient $(n, 2)$-functions with the class of Latin $n/3$-cubes of order $4$; a related concatenation construction presented in Theorem 3. In Section VI we discuss the existence of $(2n/3 - 1)$-resilient $(n, 2)$-functions that cannot be constructed by any way suggested in the previous sections. Section VII contains the conclusion.

II. $S^{(4)}_n$-PARTITIONS VS $S^{(2)}_n$-PARTITIONS

Given an $S^{(4)}_n$-partition $(C_{00}, C_{01}, C_{10}, C_{11})$, each of its cells, together with the complement, forms a $S^{(2)}_n$-partition.

Problem 1: Is the first cell of an $S^{(2)}_n$-partition always occurs as a cell of some $S^{(4)}_n$-partition?

In the next sections, we will see that the answer is positive for some special classes of $S^{(2)}_n$-partition (semilinear $S^{(2)}_n$-partitions and reducible $S^{(2)}_n$-partitions).

Lemma 1: The second cell $\overline{C}$ of any $S^{(2)}_n$-partition $(C, \overline{C})$ can be split, $\overline{C} = C_1 \cup C_2$, to form an equitable partition $\overline{C} = (C_1, C_2)$ of $Q_n$ with the quotient matrix

$$S^{(3)}_n := \begin{bmatrix} 0 & r & 2r \\ r & 0 & 2r \\ r & r & \frac{n}{3} \end{bmatrix}, \quad r = \frac{n}{3}. \quad (6)$$

Here and below, we use the notation $\overline{C}$ for the complement of $C$.

Proof: Take $C_1 := C + 0\ldots 01$. The set $C$ is independent, hence $C \cap C_1 = \emptyset$. Therefore, the equitability (and the quotient matrix) of $(C, C_1, \overline{C}, \overline{C}_1)$ is straightforward from that of $(C, \overline{C})$ and $(C_1, C \cup \overline{C}_1, \overline{C}_1)$.

Problem 2: Are the first and second cells of an $S^{(3)}_n$-partition always occur as cells of some $S^{(4)}_n$-partition?

A positive answer to Problem 2 would imply a positive answer to Problem 1.

Another connection of $4$- and $2$-partitions from the considered classes is the following fact.

Lemma 2: The $S^{(4)}_n$-partitions are in one-to-one correspondence with the $S^{(2)}_n$-partitions $(C, \overline{C})$ such that $C = C + 0\ldots 0111111$.

A sketch of proof: Given an $S^{(4)}_n$-partition $(C_0, C_1, C_2, C_3)$, define $c_0 = 000$, $c_1 = 001$, $c_2 = 010$, $c_3 = 010$, $c_4 = 100$, $c_5 = 101$, $c_6 = 110$, $c_7 = 110$,

$$C = \sum_{i=0}^{7} c_i \times \{c_i, c_{i+4}\}. \quad (7)$$

The required properties of the partition $(C, \overline{C})$ are straightforward to check. Conversely, given an $S^{(2)}_n$-partition $(C, \overline{C})$, define $C_0, \ldots, C_7$ such that

$$C = \sum_{i=0}^{7} c_i \times \{c_i\}.$$ 

If $C = C + 0\ldots 0111111$, then $C_i = C_{i+4}$, $i = 0, 1, 2, 3$, and we have (7). The required properties of $(C_0, C_1, C_2, C_3)$ are straightforward.

In [7], among other results, it was established that there are exactly $16$ isomorphism classes of $S^{(2)}_n$-partitions. Eight of them, after some coordinate permutation, are invariant under translation by $0\ldots 01111$. However, different coordinate permutations can result in nonisomorphic $S^{(4)}_n$-colorings, in the mean of Lemma 2. In the ordering of [7], nonisomorphic $S^{(2)}_n$-partitions correspond to $1, 1, 2, 1, 1, 2, 0, 1, 0, 0, 0, 0, 0, 0, 0$ nonisomorphic $S^{(4)}_n$-partitions, respectively. The total number of the isomorphism classes is $10$.

Theorem 1: There are exactly $10$ non-equivalent $5$-resilient $(9, 2)$-functions, where two functions $f$ and $g$ are equivalent if $f(x) = g(\alpha(x))$ for some isomorphism (isometry) $\alpha$ of $Q_n$ and some permutation $\beta$ of $Q_2$. Among these $10$ functions, exactly one is linear and exactly one is full rank; the remaining $8$ are semilinear (see the definitions in the next section).

III. NON-FULL-RANK PARTITIONS, CONNECTION WITH MULTIFOLD PERFECT CODES

In this section we characterize the $S^{(4)}_n$-partitions of deficient rank in terms of equitable partitions of $Q_m, m = 2n/3$ with other parameters, which are closely related with the theory of $1$-perfect codes.

A. The non-full-rank $S^{(2)}_n$-partitions

The (affine) rank of a set $C \subset Q_n$ is the dimension of its affine span. By the rank of an $S^{(2)}_n$-partition $(C, \overline{C})$, we will mean the rank of $C$. As $|C| = 2^{n-2}$, the rank can be only $n - 2$ if $C$ is linear, or $n - 1$, in which case we say that $C$ is strictly semilinear, or $N$, if $C$ is full rank. We say that $C$ is semilinear, or non-full-rank, if it is either linear or strictly semilinear. The dual $C^\perp$ of $C$ is defined as the set of all words $v$ in $Q_n$ such that $(c, v) = 0$ for all $c \in C$ or $(c, v) = 1$ for all $c \in C$, where

$$\langle (c_1, \ldots, c_n), (v_1, \ldots, v_n) \rangle := c_1 v_1 + \ldots + c_n v_n.$$
The following fact is well known.

**Lemma 3:** If \((C, \bar{C})\) is an equitable partition of \(Q_n\) with the quotient matrix \([a, b, |c, d] \), then the nonzero words of \(C'\) have weight \((b + c)\). In particular, for the \(S_n^{(2)}\)-partitions, this weight is \(2n/3\).

It immediately follows that there is only one linear \(S_n^{(2)}\)-partition, up to isomorphism. For a semilinear \(S_n^{(2)}\)-partition, there is only exactly one nonzero dual word. W.l.o.g., we can assume that all its zeros are in the last \(n/3\) positions.

**Remark 2:** We defined the rank and the dual of a code in the affine sense, to be invariant with respect to the translations of the space. In general, this approach is convenient while considering nonlinear vector sets whose main properties are invariant with respect to the space isometries. The readers who prefer using the classical (linear-sence) concepts of the rank and the dual can treat the material below assuming that \(C\) always contains the all-zero word.

**Theorem 2:** The \(S_n^{(2)}\)-partitions \((C, \bar{C})\) such that \(v := 1 \ldots 1 \ldots 0 \in C\) are in one-to-one correspondence with the equitable partitions of \(Q_{2n/3}\) with the quotient matrix

\[
\begin{pmatrix}
0 & 0 & 2r \\
r & r & 0 \\
0 & 0 & 2r
\end{pmatrix}, \quad r = \frac{n}{3}, \tag{8}
\]

**A sketch of proof:** Consider the case \(\langle c, v \rangle = 0\) for all \(c \in C\). Define \(C' = \{c \mid \langle c, v \rangle = 0\}\) and \(C'' = \{c \mid \langle c, v \rangle = 1\}\). Since \((C \cup C', C'')\) is an equitable partition with the quotient matrix

\[
\begin{pmatrix}
0 & 0 & 2r \\
r & r & 0 \\
0 & 0 & 2r
\end{pmatrix},
\]

it is easy to check that \((C, C', C'')\) is an \(S_n^{(3)}\)-partition.

Next, we define \(D = \{x \mid x \in C\} \subseteq C\), and similar \(D' = \{x \mid x \in C\}, D'' = \{x \mid x \in C\}\), where \(Q_{2n/3}\) is the set of even-weight words of \(Q_r\).

- \((i)\) \(C'' = \{xy \mid x \in D', y \in Q_{2r}\}\), which is straightforward from the definitions of \(C''\) and \(D''\).
- \((ii)\) \(C = \{xy \mid x \in D', y \in Q_{2r}\} \cup \{xy \mid x \in D', y \in Q_{2r,od}\}\), which comes from the definition of \(D\), the cardinality of \(C\) is \(2^n/4\), and the fact that \(C\) is an independent set.
- \((iii)\) \(C' = \{xy \mid x \in D', y \in Q_{2r}\} \cup \{xy \mid x \in D', y \in Q_{2r,od}\}\), similarly.

Now, since a vertex \((x, 10\ldots 0)\), \(x \in D''\), has \(r\) neighbors from \(C\), we see that \(x\) has \(r\) neighbors from \(D\). Immediately, we get that \((D, D', D'')\) is an equitable partition with the quotient matrix \([a, b, |c, d] \).

Conversely, having an equitable partition \((D, D', D'')\) with the quotient matrix \([a, b, |c, d] \) and defining \(C = C'\), \(C''\) by \((i), (ii), (iii)\), we get an \(S_n^{(2)}\)-partition. If \(C'' = \bar{Q}_{2r}\), then \(\langle c, v \rangle = 0\) for all \(c \in C\). If \(C'' = Q_{2r}\), then \(\langle c, v \rangle = 1\) for all \(c \in C\).

**Corollary 1:** The answer to Problem \(\Pi\) is positive for the class of semilinear \(S_n^{(2)}\)-partitions.

**B. Multifold perfect codes**

In this subsection, we discuss equitable partitions with the quotient matrix \([a, b, |c, d] \) and their connections with perfect codes. A \(1\)-perfect code in \(Q_m\) is a set \(C\) such that \(|C \cap \bar{B}| = 1\) for every ball \(B\) or radius \(1\). More general, a \(t\)-fold \(1\)-perfect code in \(Q_m\) is a set \(C\) such that \(|C \cap \bar{B}| = 1\) for every ball \(B\) or radius \(1\). Equivalently, \((C, \bar{C})\) is an equitable partition with quotient matrix \([|t - m - t + 1|, |t - m - t|] \). After adding the all-parity-check bit to all words of \(C\), we obtain a \(t\)-fold extended \(1\)-perfect code \(C'\) such that \((C' : Q_{m+1} \backslash Q_{m+1})\) is an equitable partition with the quotient matrix \([a, b, |c, d] \, r = (m+1)/2\). So, Theorem \(\Pi\) establishes a connection between the non-full-rank \(S_n^{(2)}\)-partitions and the \(r\)-fold \(1\)-perfect codes of length \(2r - 1\). \(r = n/3\). \(t\)-fold \(1\)-perfect codes can be easily constructed constructed from an arbitrary usual \(1\)-fold \(1\)-perfect code \(C\), as the union of \(t\) translations of \(C\) by vectors of weight \(1\) (it should be noted that not all \(t\)-fold codes can be treated in such a way \([5]\)).

The binary \(1\)-perfect codes have been extensively studied since the first construction of nonlinear such codes by Vasil’ev \([13]\), but currently, even the asymptotic of the log log of the number of such codes is not known. It is hard to expect a constructive characterisation of the class of binary \(1\)-perfect codes. The problem of the characterization of all \(r\)-fold \(1\)-perfect codes of length \(2r - 1\) is not formally harder or simpler of the same problem for \(1\)-fold \(1\)-perfect codes: we cannot construct all \(r\)-fold \(1\)-perfect codes from all \(1\)-perfect or vice versa. But intuitively, the problems belong to the same class: in both cases, we have classes of equitable 2 partitions with doubly-exponential (in length) grows of the number of objects. The main problem in this area is the asymptotic of the log log of the number of objects (note that this asymptotic is the same for the number of different object and the number of nonisomorphic objects), which is known bounded by \(n^2/2\) from below (from \([13]\) and similar approach for \(t > 1\)) and by \(n^2\) from above (a trivial bound), for \(t\)-perfect binary codes of length \(n\), for any \(t\).

**C. The rank of \(S_n^{(4)}\)-partitions**

It remains to discuss the rank of an \(S_n^{(4)}\)-partition, or, equivalently, a \((2n/3 - 1)\)-resilient \((n, 2)\)-function. Given such function \(f\), its (affine) rank can be defined as the dimension of the affine span if its graph

\[G(f) := \{xy \mid x \in Q_n, y = f(x) \in Q_2\}\]

and is obviously one of \(n, n + 1, n + 2\). A nice property of \(Q_2\) is that any permutation \(\pi\) of its element is an affine transform; so, the \((n, 2)\) function \(\pi(f(\cdot))\) has the same rank as \(f(\cdot)\). This means that we can well define the rank of an \(S_n^{(4)}\)-partition as the rank of the associated \((n, 2)\)-function, without caring about the order of the partition cells. Readily, the rank of an \(S_n^{(4)}\)-partition is connected with the rank of the associated \(S_n^{(2)}\)-partition, in the sense of Lemma \(\Pi\).

**Lemma 4:** Let \(C\) be an \(S_n^{(4)}\)-partition, and let \((C, \bar{C})\) be the associated \(S_n^{(2)}\)-partition, in the sense of Lemma \(\Pi\). Then \(\text{rank}(C) + 1 = \text{rank}(C)\).
Proof: Straightforward from \( C = \{x0 | x \in G(f)\} + 0...0111 \), where \( f \) is the \((n,2)\)-function corresponding to \( C \).

In this way, we see that the non-full-rank \( S_n^{(4)} \)-partitions are also related to multifold 1-perfect codes.

IV. CONNECTION WITH LATIN HYPERCUBES OF ORDER 4, REDUCIBLE \( S_n^{(2)} \)- AND \( S_n^{(4)} \)-PARTITIONS

In this sections, we consider the equitable partitions of another graph, \( H(r,4) \), with the same quotient matrices (4), and show how they are connected with \( S_n^{(4)} \)- and \( S_n^{(2)} \)-partitions. The Hamming graph \( H(r,q) \) is the graph on the words of length \( r \) over an alphabet \( \Sigma \) of size \( q \) (to be explicit, \( \Sigma : = \{0,\ldots , q-1\} \) ), two words being adjacent if and only if they differ in exactly one position. For the purposes of our study, we restrict consideration by the case \( q = 4 \) only, including the definitions, which can be easily expanded to an arbitrary \( q \). The graph \( H(n,4) \) is regular of degree 3. A Latin \( r \)-cube or Latin hypercube, of order 4 is an equitable partition of \( H(r,4) \) with the quotient matrix \( S_n^{(4)} \), see (4). A set \( C \) of vertices of \( H(r,4) \) is a 4-ary distance-2 MDS code (for brevity, we will omit “4-ary distance-2”) if \((C, \overline{C})\) is an equitable partition of \( H(r,4) \) with the quotient matrix \( S_n^{(4)} \), see (5). The following known and straightforward correspondence is similar to Lemma 2 but without any additional requirement on \( C \).

Lemma 5: The Latin \( r \)-cubes \((C_0,C_1,C_2,C_3)\) of order 4 are in one-to-one correspondence with the MDS codes \( C \) in \( H(r+1,4) \): \( C = \{v_i | v \in C_i, \ i = 0,1,2,3\} \).

As we see from Lemma 5 below, the similarity between the quotient matrices corresponding to the Latin hypercubes on one side and the \( S_n^{(4)} \)-partitions on the other side is not an accident, and there is simple injective map from the first class of partitions to the second (similarly, for MDS codes and \( S_n^{(2)} \)-partitions). This connection is important because the class of Latin hypercubes of order 4 was deeply studied in previous works and we can say that it is rather well understood. In particular:

1) There is a constructive characterisation of this class (6): every Latin hypercube of order 4 is semilinear or reducible. The semilinear MDS codes correspond to semilinear \( S_n^{(2)} \)-partitions, but in contrast with the class of all semilinear \( S_n^{(2)} \)-partitions, the analog of Theorem 2 reduces the semilinear MDS codes (orthogonal to a fixed vector) to the class of Boolean functions, which is trivially constructive.

2) The asymptotic of the number \( L(r) \) of objects is known (11). In particular, the log-log of this number is \( \log_2 \log_2 L(r) = r + o(r) \); we note that it corresponds to the simple lower bound obtained by a switching approach similar to (13) rather than the trivial upper bound \( \log_2 \log_2 L(r) < 2r \).

3) There are positive answers to the questions similar to Problem 1 and Problem 2. The first is almost trivial; the second has a complicated proof (9), even using the characterisation (6).

All of those is applicable to the classes of \( S_n^{(4)} \)-partitions obtained from the Latin hypercubes.

Our main objects can be constructed from Latin hypercubes or MDS codes utilizing the following fact (for \( S = S_n^{(2)} \), this is a partial case of (8) Theorem 2).

Lemma 6: If \((C_i)_{i=0}^r \) is an equitable partition of \( H(r,4) \) with a quotient matrix \( S \), then \((C'_i)_{i=0}^r \) is an equitable partition of \( Q_{3r} \) with the quotient matrix \( S' \), where

\[
C'_i := \bigcup_{(v_1,\ldots ,v_r) \in C_i} T_{v_1} \times \cdots \times T_{v_r}, \quad T_0 := \{000,111\}, \quad T_1 := \{010,101\}, \quad T_2 := \{100,011\}, \quad T_3 := \{110,001\}.
\]

Remark 3: In (9), we see a correspondence between the vertices \((v_1,\ldots ,v_r)\) of \( H(r,4) \) and vertex subsets of \( Q_{3r} \). This correspondence (to be exact, its inverse) can be treated as a graph covering of \( H(r,4) \) by \( Q_{3r} \). In general, a covering of a graph \( Tar \) (the target) by the graph \( Cov \) (the cover) is an equitable partition of \( Cov \) with the quotient matrix coinciding with the adjacency matrix of \( Tar \). In the case of such covering, any equitable partition of \( Tar \) naturally induces an equitable partition of \( Cov \), called lifted (this is a folklore fact). In particular, for any integer \( q, m \ge 2 \), there is an additive (over \( \mathbb{Z}_q \)) covering of \( H(q^m,n) \) by \( H(q,n(q^m-1)/(q-1)) \).

The previous lemma can also be treated as a kind of concatenated construction for equitable partitions. In the case when \( S \) is one of the quotient matrices considered in the current paper, the construction can be generalized, allowing to concatenate equitable partitions in different dimensions and giving a reach classes of \( S_n^{(2)} \)- and \( S_n^{(4)} \)-partitions including the ones that cannot be represented as in Lemma 5 or Theorem 2.

Theorem 3: Assume that for each \( j \) from 1 to \( r \), \((T^{(j)})_{i=0,1,2,3} = S_n^{(2)} \)-partition, for some \( n_i \). If \((C_i^{(k)})_{i=0}^r \) is an equitable partition of \( H(r,4) \) with a quotient matrix \( S_n^{(2)} \), \( S_{3r} \), or \( S_{4r} \) (where \( k \) is 2, 3, or 4, respectively), then \((C'_i^{(k)})_{i=0}^r \) is an equitable partition of \( Q_{3r}, N := \{n_1 + \cdots + n_r\} \), with the quotient matrix \( S_n^{(2)} \), \( S_n^{(3)} \), or \( S_n^{(4)} \), respectively.

\[
C'_i := \bigcup_{(v_1,\ldots ,v_r) \in C_i} T_{v_1}^{(1)} \times \cdots \times T_{v_r}^{(r)}.
\]

A sketch of proof: The proof is straightforward, but utilizes the following fact, which is often used for equivalent definitions of MDS codes and Latin hypercubes: every cell \( C' \) of an equitable partition of \( H(r,4) \) with one of the considered quotient matrices intersects every clique of size 4 in exactly \(|C'|/4r-1\) vertices. This explains why the lemma cannot be generalized to an arbitrary quotient matrix.

The \( S_n^{(2)} \)- and \( S_n^{(4)} \)-partitions that can be represented as in Theorem 3 with \( r \ge 0 \) will be called reducible. In the remaining part of the paper, we discuss existence of \( S_n^{(2)} \)- and \( S_n^{(4)} \)-partitions that are neither semilinear nor reducible.

V. IRREDUCIBLE FULL-RANK PARTITIONS

Theorems 2 and 3 give two powerful ways to construct \( S_n^{(2)} \)- and \( S_n^{(4)} \)-partitions (and hence, \((2n/3 - 1)\)-resilient \((n,2)\)-functions); moreover, the concatenation construction of Theorem 3 allows to combine concatenated, or semilinear, or
any other $S_n^{(4)}$-partitions constructed in smaller dimensions $n_i$. Any partition that can be constructed using this two ways is semilinear or reducible (or both), and it would be very important to find $S_n^{(2)}$- and $S_n^{(4)}$-partitions that are neither semilinear nor reducible. Since at this moment no general method to construct such partitions is known, we can only claim the following computational result.

**Proposition 2:** There are an $S_{12}^{(2)}$-partition and an $S_{12}^{(4)}$-partition that are neither semilinear nor reducible.

The only $S_{12}^{(2)}$-partition $(C, C)$ with this property is the last partition in [7, Appendix]. There are at least 3 disjoint translations $C, C_1, C_2$ of $C$, providing a $S_{12}^{(4)}$-partition $(C, C_1, C_2, C \cup C_1 \cup C_2)$ with the required property.

**Problem 3:** Do there exist $S_n^{(2)}$- and $S_n^{(4)}$-partitions that are full rank and irreducible (i.e., neither semilinear nor reducible), for any $n \geq 12$, $n \equiv 0 \mod 3$?

VI. Conclusion

In this thesis, we discussed connections of the $(2n/3 - 1)$-resilient $(n, 2)$-functions (which has the best correlation-immunity order $(2n/3 - 1)$ among the balanced $(n, 2)$-functions) with special classes of equitable partitions, including the widely studied classes of 1-perfect codes and Latin hypercubes.

The $(2n/3 - 1)$-resilient $(n, 2)$-functions are equivalent to the equitable partition of a special type, called the $S_n^{(4)}$-partitions. We described the non-full-rank $S_n^{(4)}$-partitions in terms of multifold binary codes (Theorem 2), which gives a construction of at least $2^{2n/3}$ different $S_n^{(4)}$-partitions. Another presented construction (Theorem 3) uses $S_n^{(4)}$-partitions with $n_i < n$ and Latin hypercubes of order 4; we call resulting partitions reducible. No general way is known to construct full-rank irreducible $S_n^{(4)}$-partitions, but computational results say than in the dimension $n = 12$ such partitions exist. We conclude that the current knowledge is not sufficient to make a conjecture about the structure of an arbitrary $(2n/3 - 1)$-resilient $(n, 2)$-function (in contrast to the known characterization of the Latin hypercubes of order 4), but enough to construct double-exponential number of such functions with different properties.

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