Synthetic Dimension in Photonics

Luqi Yuan\textsuperscript{1,4}, Qian Lin\textsuperscript{2}, Meng Xiao\textsuperscript{1}, and Shanhui Fan\textsuperscript{1,3}

\textsuperscript{1}Department of Electrical Engineering, and Ginzton Laboratory, Stanford University, Stanford, CA 94305, USA

\textsuperscript{2}Department of Applied Physics, Stanford University, Stanford, CA 94305, USA

\textsuperscript{3}E-mail: shanhui@stanford.edu and

\textsuperscript{4}E-mail: yuanluqi@stanford.edu

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Abstract

The physics of a photonic structure is commonly described in terms of its apparent geometric dimensionality. On the other hand, with the concept of synthetic dimension, it is in fact possible to explore physics in a space with a dimensionality that is higher as compared to the apparent geometrical dimensionality of the structures. In this review, we discuss the basic concepts of synthetic dimension in photonics, and highlighting the various approaches towards demonstrating such synthetic dimension for fundamental physics and potential applications.
I. INTRODUCTION

The physics of a photonic structure is commonly described in terms of its apparent geometric dimensionality. A few 0, 1, 2 and 3 dimensional examples are microcavities [1, 2], waveguides [3], two-dimensional photonic crystals [4], and three-dimensional metamaterials [5, 6], respectively.

Using these same photonic structures, however, it is in fact possible to explore physics in a space with a dimensionality that is higher as compared to the apparent geometrical dimensionality of these structures. The basic idea is to configure synthetic dimensions, and to combine such synthetic dimensions with the geometric dimensions to form higher dimensional synthetic space [7–12].

In photonics, there are two approaches for creating such a synthetic space.

1. Forming a lattice. Consider a system as described by coupling a set of physical states together. The dimensionality of the system is then determined by the nature of coupling. As an illustration, suppose we label the states as consecutive integers, and therefore we can visualize the states as being placed on a line [Fig. 1(a)]. The coupling in Fig. 1(b), where consists of a nearest neighbor coupling between the states, results in a one-dimensional lattice. On the other hand, with the same set of states, one can in fact form higher-dimensional lattices, with the longer-range coupling as indicated in Fig. 1(c).

2. Exploiting the parameter dependency the system. As a general illustration, consider a system in an \( n \)-dimensional space, as described by a Hamiltonian of the form 
\[
H(p_1, \ldots, p_m, r_1, \ldots, r_n),
\]
where \( p_1, \ldots, p_m \) are external parameters, and \( r_1, \ldots, r_n \) are the spatial coordinates. One can think of each parameter of the Hamiltonian as an extra synthetic dimension.

In additional to photonics, both of these approaches for creating synthetic dimensions have been also extensively explored in other physical systems, such as cold atoms in optical lattices [13–36] and superconducting qubits [37–39]. The development of the concept of synthetic dimension in photonics share some of the motivations. For examples, theoretically it is known that there are rich physics, and in particular, rich topological physics in systems beyond three dimensions [40–43]. The development of synthetic dimension provides an
experimental approach to explore such physics. Also, while three dimensional physics can in principle be explored in three dimensional structures, constructing such structures may be challenging and it may be more advantageous to explore such a physics in one or two-dimensional structures that are easier to construct.

FIG. 1: (a) The physical states labelled by consecutive integers. (b) Introducing coupling between nearest-neighbor physical states creates a one-dimensional system. (c) Introducing long-range coupling between different states generates a two-dimensional system [9, 49].

On the other hand, there are aspects of synthetic dimension concepts that are unique in photonics. In particular, in forming a lattice, the states that are being used can have different frequencies, or different orbital angular momentums, corresponding to different internal degrees of freedom of photons. The construction of the synthetic space thus enables new possibilities for manipulating these internal degrees of freedom, which are of significant potential importance for applications such as communications and information processing.

In this paper, we provide a short review of the current developments of the concepts of synthetic dimension in photonics. The rest of the paper is organized as follows: In Sec. II, we review the approach to create synthetic dimension by designing the coupling between various photonic states to form a synthetic lattice. In Section III, we discuss some of the physics effects in these photonic synthetic lattices, focusing in particular on photonic gauge potential and topological photonics effects. In Sec. IV, we review the approach of exploring high dimensional photonic phenomena in the parameter space. We conclude in Sec. V.
II. FORMING A SYNTHETIC LATTICE OF PHOTONIC STATES

In order to create a synthetic space using the approach of forming a lattice, one needs a set of physical states, as well as mechanisms to specifically configure the coupling among these states. Photonics offers a rich variety of possibilities for forming lattices. For the states, one can use photonic modes with different frequencies, or with different spatial distributions such as different orbital angular momentums, or alternatively one can use multiple temporal pulses. Photonic structures also offer great flexibilities in configuring the coupling of these different states. In this section, we brief review these various approaches for creating synthetic space by forming a lattice.

A. Using photonic modes with different frequencies

In photonics, a natural way to create a synthetic dimension is to use the frequency of light. Photonic structures naturally support modes at different frequencies. Moreover, a lattice can be formed by coupling these modes together, through either dynamic modulation of the structure, or by nonlinear optics techniques.

As a simple illustration of the use of dynamic modulation to couple modes with different frequencies together, consider a structure with a time-dependent permittivity described by:

\[ \varepsilon(r, t) = \varepsilon_s(r) + \delta \varepsilon(r, t), \]

(1)

where \( \varepsilon_s(r) \) describes a static dielectric structure, and \( \delta \varepsilon(r, t) \) describes a time-dependent modulation. For such a time-dependent system, its electromagnetic properties can be described from the Maxwell’s equation as:

\[ \nabla \times \nabla \times E + \mu_0 \frac{\partial^2}{\partial t^2} \varepsilon_s E = -\mu_0 \frac{\partial^2 P}{\partial t^2}, \]

(2)

where

\[ P = \delta \varepsilon E \]

(3)

is the polarization current density induced by the dynamic modulation.

Since for typical modulation, we have \( |\delta \varepsilon| \ll \varepsilon_s \), one can treat the dynamically modulated structure perturbatively. We first determine the mode of the static structure by solving an eigenvalue problem:

\[ \nabla \times \nabla \times E_m = \mu_0 \varepsilon_s \omega_m^2 E_m, \]

(4)
FIG. 2: (a) A dynamically modulated ring resonator can be described by a tight-binding model of a photon along a one-dimensional lattice in the synthetic frequency dimension [11, 46]. (b) A phase-matched modulation along a dielectric waveguide can achieve a one-dimensional lattice formed from waveguide modes at different frequencies [52, 58].

where $\omega_m$ is the frequency of the mode, and $E_m$ is the eigenmode field distribution. For the dynamic structure, the modulation induces coupling between these modes. Therefore, we expand the field $E$ in terms of the eigenmodes of the static structure:

$$E = \sum_m a_m E_m e^{i\omega_m t},$$

and describe the properties of the dynamic structure in terms of the dynamics of the modal amplitudes $a_m$.

The formalism above can be used to describe a modulated ring resonator [11, 12, 46, 51]. Consider a static ring resonator composed of a single-mode waveguide, which we assume to have zero group velocity dispersion for simplicity. Suppose the ring supports a resonant mode at a frequency $\omega_0$. In the vicinity of the frequency $\omega_0$, resonant modes then form an
equally spaced frequency comb, with the \( m \)-th resonant mode having the frequency

\[
\omega_m = \omega_0 + m\Omega_R. \tag{6}
\]

In Eq. \( \text{(6)} \), \( \Omega_R = 2\pi c/n_0L \) is the free-spectral range of the ring which also defines the modal spacing in frequency, \( n_0 = n(\omega_0) \) is the group index of the waveguide at \( \omega_0 \), and \( L \) is the circumference for the ring.

Suppose we modulate the ring resonator structure above as:

\[
\delta\epsilon(r, t) = \delta(r) \cos(\Omega t + \phi), \tag{7}
\]

where \( \delta(r) \) is the modulation profile, \( \Omega \) is the modulation frequency and \( \phi \) is the modulation phase. For simplicity we set \( \Omega = \Omega_R \). The induced polarization density from the \( m \)-th mode is then

\[
P = \frac{\delta(r)E_m}{2}a_m \left( e^{i\phi}e^{i\omega_{m+1}t} + e^{-i\phi}e^{i\omega_{m-1}t} \right). \tag{8}
\]

Thus, the induced polarization will resonantly excite modes \( m + 1 \) and \( m - 1 \). Hence we expect that the modal amplitudes satisfies:

\[
i\frac{da_m}{dt} = ge^{i\phi}a_{m-1} + ge^{-i\phi}a_{m+1}. \tag{9}
\]

Here \( g \) is the coupling strength. (A detailed derivation can be found in Refs. \[11, 12\].) The Hamiltonian of such a system is then

\[
H = g \sum_m \left( e^{i\phi}c_{m+1}^\dagger c_m + e^{-i\phi}c_m^\dagger c_{m+1} \right). \tag{10}
\]

where \( c_m(c_m^\dagger) \) is the annihilation (creation) operator for the \( m \)-th resonant mode. The Hamiltonian in Eq. \( \text{(10)} \) describes a tight-binding model of a photon in a one-dimensional lattice in the synthetic frequency dimension \[11, 46\].

Similar approach for creating a synthetic dimension along the frequency axis can also be achieved in a waveguide. Consider a static single-mode waveguide with the propagation direction along the \( z \)-direction, its eigenmode has the form

\[
E_k = E_k(x, y)e^{-ikz}. \tag{11}
\]

Here \( E_k(x, y) \) is the modal profile, and \( k \) is the wavevector. The eigenfrequency of the mode is \( \omega(k) \), which also defines the dispersion relation of the waveguide. Suppose we operate in the
vicinity of an operating frequency $\omega_0$ with the corresponding wavevector $k_0$, i.e. $\omega_0 = \omega(k_0)$. Near $\omega_0$ one can expand the dispersion relation as:

$$\omega - \omega_0 = v_g (k - k_0),$$  

(12)

where $v_g$ is the group velocity of the waveguide.

For such a waveguide, we then modulate its permittivity as:

$$\delta \epsilon = \delta(x, y) \cos(\Omega t - Kz + \phi),$$  

(13)

where we choose the modulation to be phase-matched with the waveguide mode, such that

$$\Omega = Kv_g,$$  

(14)

In the presence of such modulation, we expand the field in the waveguide as:

$$E = \sum_m a_m(z) E_m(x, y) e^{i(\omega_m t - k_m z)},$$  

(15)

where $\omega_m = \omega_0 + m\Omega$, and $k_m$ is the corresponding wavevector, i.e. $\omega_m = \omega(k_m)$. The induced polarization from the $m$-th mode has the form

$$P = \frac{\delta(x, y) E_m(x, y)}{2} a_m \left[ e^{i\phi} e^{i(\omega_{m+1} t - k_{m+1} z)} + e^{-i\phi} e^{i(\omega_{m-1} t - k_{m-1} z)} \right],$$  

(16)

We see that the induced polarization would couple with the $m + 1$ and $m - 1$ mode in a phase-matched fashion. We expect that the coupled mode theory to have the form:

$$i \frac{d a_m}{dz} = g e^{i\phi} a_{m-1} + g e^{-i\phi} a_{m+1}.$$  

(17)

We again see a one-dimensional tight binding model for photons along a synthetic frequency dimension.

The modulations in Eqs. (7) and (13) can be achieved using electro-optical modulation [52]. Recent developments of on-chip silicon [53] and LiNbO$_3$ modulators [54, 55] in either ring resonator or waveguide geometries may prove to be quite useful for creating synthetic lattice. In addition, one can also consider the use of acoustic-optical modulators in fiber ring resonators. Similar effects can also be accomplished with the use of nonlinear optical effects [56, 57]. Ref. [58] used two strong pumps differ in frequency by $\Omega$, to create a synthetic lattice along the frequency dimension, for a weak probe wave. Similar four-wave mixing process has also been considered in a Raman medium [59].
B. Using photonic modes with different orbital angular momentum

Instead of creating a synthetic dimension in the spectral domain, one can also create a synthetic dimension exploiting the spatial degree of freedom in optical modes. Consider a light beam, with its transverse profile carrying non-zero orbital angular momentum, circulating around in a ring cavity to form resonant modes of the cavity. In general, the resonant frequency of such a resonant mode should depend on the orbital angular momentum of the corresponding circulating beam. However, with appropriate design, one can in fact create a degenerate optical cavity, in which resonant modes formed by beams with different orbital angular momenta have the same frequency. Inside such degenerate cavity, one can then introduce an auxiliary cavity, which incorporates spatial light modulators, in order to couple a small portion of the amplitude of a beam with an orbital angular moment $l$ to a beam with an orbital angular moment $l - 1$ and $l + 1$. The system then is described by a tight-binding model

$$H = g \sum_l \left( e^{i\phi} c^+_l c_{l-1} + e^{-i\phi} c^+_l c_{l+1} \right),$$

where $l$ corresponds to the orbital angular moment of light. Thus, the cavity structure shown in Fig. 3 can be used to achieve a synthetic dimension based on the orbital angular momentum. While here for simplicity we consider only nearest neighbor coupling along the $l$-axis, long-range coupling can also be achieved with different design of the spatial light modulators (Ref. 60).

C. Using multiple pulses

Another platform to create a synthetic photonic lattice is to exploit the temporal degree of freedom, where the evolution of a sequence of pulses is mapped onto the dynamics of a particle moving on a set of discrete lattice sites. As an illustration, consider two fibre loops with different lengths, connected by a 50/50 coupler [see Fig. (a)]. We assume that the round-trip times for light travelling through the short and long loops are $T_s$ and $T_l$, respectively, with the time difference $T_l - T_s \equiv 2\Delta T$ and the average time $(T_l + T_s)/2 = T$. Within the long loop there is in addition a phase modulator. For a pulse at a particular position in the short (long) loop at the time $t_0$, after a round trip around the short (long) trip, it returns to the same position at the time $t_0 + T - \Delta T$ ($t_0 + T + \Delta T$). Suppose at
FIG. 3: A cavity that is degenerate for optical beams with different angular momentum (solid line). Within this cavity, an auxiliary cavity (dashed line) is incorporated, where two spatial light modulators couple a beam at angular momentum $l$ to beams at angular momentums $l \pm 1$, respectively. Such a cavity can be described by a synthetic lattice along the angular momenta direction [10].

In the time $t = mT + n\Delta T$, two pulses, denoted as $u_{n}^{m}$ and $v_{n}^{m}$, arrives at the input ends of the coupler in the short and long loops, respectively. Two pulses $u_{n+1}^{m}$ and $v_{n+1}^{m}$, upon passing through the coupler, generates an output pulse in the short loop. Such an output pulse then goes through a round trip in the short loop to generate $u_{n}^{m+1}$, i.e.

$$u_{n}^{m+1} = \frac{1}{\sqrt{2}} \left( u_{n+1}^{m} + iv_{n+1}^{m} \right).$$  \hspace{1cm} (19)

And similarly

$$v_{n}^{m+1} = \frac{1}{\sqrt{2}} \left( iv_{n-1}^{m} + v_{n-1}^{m} \right) e^{i\phi(n)},$$  \hspace{1cm} (20)

In Eqs. (19) and (20), we have used the scattering matrix of the coupler, and have incorporated the effects of the phase modulators. Here we assume that the modulation period is $T$, and hence the time-dependent transmission phase depends on $n$ only. Eqs. (19) and (20) describe the temporal motion (motion along the $m$-axis) of a particle on a one-dimensional synthetic lattice as labelled by $n$.

To summarize Section II, photonics provides a rich set of opportunities to create synthetic lattices. The general idea here is to specifically design the coupling between various photonic modes. In addition to the few examples above, with different spectral, spatial and temporal modes, there exist many other possibilities of utilizing photonic modes. For example, Ref.
FIG. 4: (a) Two fibre loops connected by a 50/50 coupler. (b) An equivalent lattice network which describes a one-dimensional synthetic lattice \((n)\) evolves along the time \((m)\) \([7, 8]\).

Experimentally demonstrated a synthetic lattice that maps into a topological insulator, based on modes in an array of coupled waveguides. While in the examples above for simplicity we have considered one-dimensional synthetic lattices with nearest neighbor coupling, it is possible to achieve higher dimensional synthetic lattices with more complex couplings. For example, Refs. \([9, 49]\) show that by utilizing modulators with a few modulation frequencies, one can achieve a synthetic lattice with dimensions higher than one using the same set of modes in the ring resonator here as we considered in Section II. A. The effects of long-range coupling in synthetic dimensions have been also explored in Ref. \([58]\).

In photonics, the number of distinct lattice sites along the synthetic dimension can potentially be quite large. For ring resonators systems, for example, it is conceivable to have hundreds of different modes coupling together, since the modulation frequency is typically far smaller than the resonant frequencies of the modes. Have such a large space along the synthetic dimension is useful for the demonstration of analogues of bulk physics effects in the synthetic space. In addition, the boundaries along the synthetic dimension can also be introduced, either naturally through the group velocity dispersion for the modulated ring or waveguides \([11]\), or by specifically designed boundaries using a memory effect \([33]\).
III. THE PHYSICS OF SYNTHETIC LATTICE

The photonic synthetic lattices as discussed in Section II provide versatile platforms to explore fundamental physics effects. In addition, since the synthetic lattice is built upon various degrees of freedom of light, the abilities to control the flow of light in the synthetic lattice provides abilities to control properties of light that are important for practical applications.

The description of a dynamically modulated ring in terms of a one-dimensional tight-binding model certainly has a long history. This description, for example, has been used to described the physics of mode-locked lasers [71–74]. In recent years, the concept of the synthetic lattice has been explored to demonstrate a wide range of physics effects including the physics of parity-time symmetry [8, 51, 65, 66], Anderson localization [68], and time-reversal of light [69]. Here we focus on two important emerging directions in the physics of synthetic lattice: creating an effective gauge potential for light, and topological photonics effects.

A. Effective gauge potential

Photons are neutral particles. Thus, there is no naturally occurring gauge potential that couples to photons. On the other hand, in the construction of photonic synthetic lattices, the ability for achieving an effective gauge potential for photons naturally emerge. To illustrate the concept of such an effective gauge potential for photons [75–77], consider first the Hamiltonian

$$H = g \sum_{\langle ij \rangle} \left( e^{-i\phi_{ij}(t)} c_i^\dagger c_j + e^{i\phi_{ij}(t)} c_j^\dagger c_i \right),$$

(21)

where $\phi_{ij}(t)$ is the hopping phase between lattice sites $i$ and $j$. The hopping phase in general can be time-dependent. For simplicity we consider only nearest neighbor coupling. We can therefore make the association [78]

$$\int_i^j A \cdot d\mathbf{r} = \phi_{ij}(t).$$

(22)

where $A$ is the effective gauge potential for photons. In a higher dimensional lattice [76],

$$B = \frac{1}{S} \oint_{\text{plaquette}} A \cdot d\mathbf{r}$$

(23)
is the effective magnetic field through a plaquette, here \( S \) is the area of the plaquette. Also, if \( \phi_{ij}(t) \) is time dependent, we then have a time dependent gauge potential \( A(t) \).

\[
E = -\frac{\partial A}{\partial t}
\]  

is therefore an effective electric field for photons. As we see in Section II, the various techniques for creating a photonic synthetic lattice naturally incorporates the capabilities for controlling such hopping phases in the lattice. Therefore, these technique naturally leads to gauge potentials, as well as effective electric or magnetic fields for photons.

As perhaps the simplest illustration of the effect of a gauge potential, we consider the effect of Bloch oscillation in the synthetic space along the frequency axis. Bloch oscillation occurs when a charge particle in a one-dimensional lattice is subject to a constant electric field. This effect, initially proposed in solid state physics [79], has been previously considered for photons in waveguide arrays [80–84] and photonic crystals [85–90]. Here we show that such an effect can occur in the synthetic space as well [46, 56, 57, 91, 92]. Consider the ring resonator incorporating a phase modulator as discussed in Section II. A. By choosing the modulation frequency \( \Omega \) to be slightly different from the mode spacing \( \Omega_R \), the resulting Hamiltonian has the same form as Eq. (10), but with a time-dependent phase \( \phi(t) = (\Omega - \Omega_R)t \). From Eq. (24), then, such a modulation results in an effective time-independent electric field in the synthetic space.

The effect of such a constant effective electric field can be seen in Fig. 5. Suppose at \( t = 0 \) a few spectral components are excited. As time evolves the excited spectral components oscillates, which is precisely the effect of Bloch oscillation in the spectral domain. Moreover, it was noted in Ref. [46] that a periodic switching of the modulation frequency around the mode spacing can give rise to a uni-directional shift of photons along the frequency axis, which is a useful capability for controlling the frequency of light. Related capabilities for controlling the spectrum of light, including negative refraction and focusing of light along the frequency axis, has also been demonstrated by creating a photonic gauge potential in a waveguide [52, 56–58], based on the waveguide system as discussed in Section II. A.

One can also explore the consequences of effective magnetic fields for photons in the synthetic space [11, 12, 59]. For this purpose, consider an array of identical ring resonators, each of which incorporates a phase modulator with the modulation frequency equal to the mode spacing in the ring [Fig. 6(a)]. Based on the discussion in Section II. A, each ring
FIG. 5: The evolution of the light in time for each resonant mode $n$ exhibits the spectral Bloch oscillation [46].

FIG. 6: (a) A one-dimensional array of ring resonators undergoing dynamic modulations [11, 12]. The modulators in the rings have different modulation phases $\phi$. (b) Projected band structure for the system in (a) with 21 rings, assuming that the lattice is infinite along the synthetic frequency dimension [11].
supports a one-dimensional lattice along the frequency axis. The modes in nearest-neighbor rings at the same frequency are also coupled through evanescent tunnelling. The system thus is described by a Hamiltonian:

$$H = \sum_{m,n} g \left( e^{-i\phi} c_{m,n}^\dagger c_{m+1,n} + e^{i\phi} c_{m+1,n}^\dagger c_{m,n} \right) + \kappa \left( c_{m,n}^\dagger c_{m,n+1} + c_{m,n+1}^\dagger c_{m,n} \right),$$

where $m$ denotes the different modes in the same ring, $n$ labels a ring in the array, and $\kappa$ is the coupling constant due to evanescent tunnelling between two nearest-neighbor rings. We assume that the spacing between the rings is $d$. By choosing the modulation phase in Eq. (7) to be $n\phi$ for the modulator in the $n$-th ring, as shown in Eq. (25), the resulting Hamiltonian gives rise to a uniform effective magnetic field $\phi/\Omega R d$ in the synthetic space as described in the Landau gauge.

The Hamiltonian of Eq. (25) is periodic along the frequency axis (i.e. $m$-axis). Thus the wavevector reciprocal to the frequency axis, $k_f \in \left[-\pi/\Omega R, \pi/\Omega R\right)$, is conserved. Such $k_f$ conservation remains true even with a finite number of rings. In Fig. 6(b), we plot the projected bandstructure, which represents the eigenvalues of the Hamiltonian in Eq. (25) as a function of $k_f$, for a system consisting of 21 rings with the choice of $\phi = \pi/2$. The bandstructure exhibits one-way edge states, as expected for such a lattice system in the presence of an effective magnetic field. This system therefore enables one-way frequency translation that is topologically protected. Hamiltonian similar to Eq. (25) can also be created in the synthetic space based on orbital angular momentum of light as discussed in Section II. B. In such a case the system provides a novel platform for controlling and converting the orbital angular moment of light, which is of potential importance for communication applications.

B. Topological Photonics

In the previous section, the Hamiltonian of Eq. (25) in fact is topologically non-trivial. The resulting band structure has non-trivial Chern number that arises from the effective magnetic field. In addition to using such magnetic field, there are other mechanisms in the synthetic systems to achieve non-trivial topological effects. For example, in a system similar to what has been discussed in Section II. B, where modes with different orbital angular momenta are coupled together to form a one-dimensional synthetic lattice, one can realize
the Su-Schriffer-Heeger (SSH) model with a sharp boundary, and hence demonstrating bulk-edge correspondence in the SSH model [62].

FIG. 7: (a) A two-dimensional honeycomb array of ring resonators undergoing dynamic modulations. (b) Bandstructures show Weyl points in three-dimensional synthetic space. Left (right): the band structure in \( k_x(k_y) - k_f \) plane at \( k_y = 0 \) (right): \( k_x = 4\pi/3\sqrt{3}d \) [47].

The effects of topological physics depends strongly on the dimensionality of the physical system. There are rich set of effects that are unique to higher-dimensional system with no lower-dimensional counterparts. The concept of synthetic dimension provides a natural pathway towards exploring these higher-dimensional physics.

Here we illustrate this pathway by considering the exploration of Weyl point physics in synthetic dimension [47, 48, 61]. A Weyl point is a two-fold degeneracy in a three-dimensional band structure with linear dispersion in its vicinity [93]. The Weyl point is of substantial interest in topological physics because it represents a magnetic monopole in the momentum space, and hence its presence is robust to any small perturbation.

Photonic structures that support a Weyl point typically have complex three-dimensional geometries [94–97]. On the other hand, with the concept of synthetic dimension, it is in fact possible to explore Weyl-point physics with two-dimensional geometries that are easier to construct. Consider a two-dimensional array of ring resonators forming a honeycomb
lattice. Each ring resonator incorporates a phase modulator. Therefore, as discussed in Section II. A, each ring supports a one-dimensional lattice along the frequency axis. The system thus is described by a three-dimensional lattice model. It was shown in Ref. [47] that Weyl-point physics emerges by appropriately choosing the modulation phases $\phi_A$ and $\phi_B$, on the $A$ and $B$ sites of the honeycomb lattices (Fig. 7). Along similar directions, Ref. [50] showed that a three-dimensional topological insulator can be constructed using a two-dimensional ring resonator lattice. Four-dimensional quantum Hall effect has been studied in a three-dimensional resonator lattice [12]. Also, it was shown that the two-dimensional Haldane model can be implemented using only three ring resonators [49].

In Section II we have discussed various techniques for achieving long-range coupling in the synthetic space. Such long range coupling can be used to create topological flat bands [98, 99] which is important for simulating of many-body physics including the fractional quantum Hall effect [100]. The presence of long-range coupling can also be used for achieving novel band structure effects such as a single Dirac cone in a two-dimensional system without breaking time-reversal symmetry [101].

Exploration of nonlinear effects in synthetic space is certainly of fundamental interests in the context, for example, of quantum simulations. In many important interacting lattice Hamiltonians, the interaction is local with respect to the lattice sites. On the other hand, for the schemes involving the frequency axis as the synthetic dimension, typical nonlinear optics effects result in a form of interaction that is nonlocal across different lattice sites [102], and thus is not directly suitable for the simulation of local-interacting Hamiltonians. Ref. [103] proposed to achieve local interaction in a system where the synthetic dimension is the geometrical angular coordinate. It is an interesting open question to achieve local interaction for other approaches aiming to create synthetic space.

IV. SYNTHETIC DIMENSION FROM PARAMETER SPACE

A. Physics concept

Instead of forming a synthetic lattice, another common method to create a synthetic space is to utilize the parameter degrees of freedom. Consider any physical system as described by a Hamiltonian $H(p)$ that is parametrically dependent on a continuous variable $p$. The
parameter dependency of the system can be alternatively described in a synthetic space with the $p$-axis as an extra synthetic dimension in addition to the usual physical space. In this way, higher dimensional physics can then manifest in terms of parameter dependency of a lower-dimensional physical system.

The concept of gauge potential and the associated topological physics effects naturally arise in such synthetic space incorporating the parameter axis. As a simple illustration, consider a Hamiltonian in the parameter space described by a two-dimensional vector $\mathbf{R}$, which satisfies the Schrödinger equation \[ H(\mathbf{R})|\Psi(\mathbf{R})\rangle = E(\mathbf{R})|\Psi(\mathbf{R})\rangle, \] (26)

We assume that as $\mathbf{R}$ varies the Hilbert space does not change. Suppose we consider a closed curve $\mathcal{C}$ in the $\mathbf{R}$-space. Along this closed loop, one can define the Berry’s phase $\gamma$ as

$$\gamma = \oint_{\mathcal{C}} i\langle \Psi(\mathbf{R})|\nabla_\mathbf{R}|\Psi(\mathbf{R})\rangle \cdot d\mathbf{R}.$$ (27)

The integral kernel here gives the Berry connection, or the gauge potential in parameter space:

$$\mathbf{A}(\mathbf{R}) = i\langle \Psi(\mathbf{R})|\nabla_\mathbf{R}|\Psi(\mathbf{R})\rangle.$$ (28)

With the Stokes’s theorem, we obtain the Berry curvature

$$\mathbf{B}(\mathbf{R}) = \nabla_\mathbf{R} \times \mathbf{A}(\mathbf{R}).$$ (29)

The usual topological description of a two-dimensional band structure can be formulated in exactly the same fashion as above, with $\mathbf{R}$ corresponding to the wavevector $k$. Since the wavevector is defined on the first Brillouin zone which is a 2-torus, the integration of the $B$-field is quantized and gives rise to the Chern numbers of the bands. The important observation here, however, is that the topological argument commonly used for band-structure can in fact be used for any parameter dependency. Moreover, if one were to vary the parameter as a function of time adiabatically, the dynamics of such time-dependent system has signature of higher dimensional topological physics. Below, we illustrate these aspects with specific examples.
FIG. 8: (a) A one-dimensional photonic crystal in Ref. [109] with each unit cell including four layers where the thickness of each layer depends on parameters $p$ and $q$. (b) The band structure of the crystal with $p = q = 0$. (c) Bandstructure in the $(p, q)$ space with $k = \pi/2(d_a + d_b)$. (d) The reflection phase for light incident upon such a photonic crystal from air, in the $(p, q)$ space [109].

B. Nontrivial topology in the parameter space

As an illustration, consider a one-dimensional photonic crystal in Ref. [109], the unit cell of which consists of four layers, with thickness $(1 + p)d_a$, $(1 + q)d_b$, $(1 - p)d_a$, and $(1 - q)d_b$. The band structure of such a crystal, in the special case where $p = q = 0$, is shown in Fig. 8(b). In such a case, the four-layer unit cell is in fact not a primitive cell, and thus there is no band gap at the edge of the first Brillouin zone $k = \pi/2(d_a + d_b)$. At this $k$ point the bands are two-fold degenerate with linear dispersion along the $k$-axis. On the other hand, for $p$ and $q$ slightly deviating from 0, the four-layer unit cell becomes the primitive cell, and hence a band gap opens at the Brillouin zone edge, as shown in Fig. 8(c). The size of the gap scales linearly with respect to both $p$ and $q$. Therefore, one can show that in the three dimensional space of $k$, $p$ and $q$, the point $k = \pi/2(d_a + d_b)$, $p = 0$, and $q = 0$ is in
fact a Weyl point. The physical signature of such a Weyl point manifests in the reflection phase for a wave at a frequency of the Weyl point $f = 0.49 \frac{L}{\lambda}$, incident from air onto the photonic crystal along the normal incidence direction. As one varies the parameters $p$ and $q$, the reflection phase winds around $p = q = 0$. Remarkably, in this construction one can explore aspects of three-dimension Weyl-point physics using a simple one-dimensional structure.

C. Adiabatic evolution in the parameter space

FIG. 9: (a) Bandstructure of the Aubry-André model described by Eq. (30) composed of 99 sites with $g = 1$, $V = 0.5$, and $b = (\sqrt{5} + 1)/2$. (b) Intensity distributions in the experiment show the edge state in the gap. Here $\phi = 0.5\pi$. (c) Waveguide array where spacings between waveguides are slowly modified along the $z$-direction, as described by Eq. (31). (d) Intensity distributions versus the propagation distance $z$. Light is injected into the rightmost waveguide into a structure similar to (c). $\phi$ is changed from $0.35\pi$ to $1.75\pi$ adiabatically.

In the previous section, for a system as described by a parametric Hamiltonian, the higher-dimensional physics is revealed by considering the properties of a set of physical structures
with varying parameters. On the other hand, with a single physical structure, one can also explore higher dimensional physics by allowing the parameters to vary as a function of time, and considering the dynamics of such time-dependent parametric system.

As an illustration we consider the Aubry-André model \[110\] which describes a one-dimensional lattice

\[
H = \sum_m g \left( a_m^\dagger a_{m+1} + a_{m-1}^\dagger a_m \right) + \sum_m a_m^\dagger a_m V \cos(2\pi bm + \phi), \tag{30}
\]

where \(a_m(a_m^\dagger)\) is the annihilation (creation) operator on the \(m\)-th site, and \(g\) is the coupling strength. \(V\) is the amplitude of the on-site potential. \(b\) and \(\phi\) are parameters controlling the modulation of the on-site potential with respect to the site locations \[111\]. We note that for an irrational \(b\) there is no periodicity in Eq. (30). In fact, in that case Eq. (30) describes a quasi-crystal.

By considering \(\phi\) as a synthetic dimension, the Aubry-André model describes a synthetic two-dimensional space. In fact, it is known that Aubry-André model is closely related to the Quantum Hall system as described by a two dimensional model of a square lattice under a uniform magnetic field in the Landau gauge, as described by the Hamiltonian of Eq. (25). The \(b\) in Eq. (30) maps to the magnetic field per unit cell in the two-dimensional model, and \(\phi\) maps to the wavevector \(k_y\) along periodic direction with the choice of the Landau gauge. Thus, for a finite structure, the eigenspectrum of the Aubry-André model exhibits gaps [Fig. 9(a)]. For particular values of \(\phi\), an edge state can appear inside the gap [Fig. 9(b)]. The variation of eigenfrequencies as a function of \(\phi\) corresponds directly to the dispersion of the one-way edge state in the Quantum Hall system.

In the finite-size Aubry-André model, suppose at \(t = 0\) the system is at an edge state on one of ends, by varying \(\phi\) adiabatically as a function of time, the mode will evolve according to the edge state dispersion, eventually becomes a bulk state, and then reemerges as an edge state localized on the opposite end. Thus, the adiabatic evolution of the state in the time-dependent one-dimensional system provides a direct probe of the properties of the corresponding two-dimensional system.

Ref. [111] provides a direct experimental demonstration of such adiabatic evolution of the edge state. For this purpose, Ref. [111] made two important modification of the Aubry-
André model. First of all, the Aubry-André model is transformed to:

\[ H = g \sum_m \left[ 1 + V \cos(2\pi bm + \phi) \right] a_m^\dagger a_{m+1} + \left[ 1 - V \cos(2\pi b(m - 1) + \phi) \right] a_m^\dagger a_{m-1}. \]  

(31)

where the site-dependent modulation now appears in the coupling constant between nearest neighbor sites. Secondly, instead of considering time evolution, Ref. \[11\] consider an array of waveguides, where the variation of the field amplitudes along the propagation direction \(z\) then provides a simulation of the temporal dynamics. With these modifications, Ref. \[11\] constructed a structure as shown in Fig. 9(c), where \(\phi\) varies from 0.35\(\pi\) to 1.75\(\pi\) as a function \(z\). The adiabatic dynamics is shown in 9(d). The injected light into the edge state at one end of the waveguide array evolves into a bulk state as light propagates along the \(z\)-direction, and eventually appear as an edge state on the other side, which is precisely the adiabatic evolution of the state as expected for the Aubry-André model.

The use of adiabatic evolution provides a powerful approach to explore higher dimension physics. In the waveguide array platform, this approach has also been used to experimentally explore topological phase transition \[112\], as well as four-dimensional quantum Hall effects \[113\]. The connection of the one-dimensional Aubry-André model, which can be quasiperiodic with an irrational \(b\), to a two dimensional model also in fact points to a general connection between quasi-crystal in lower dimensional space and crystal structure in higher dimensional space. This connection has been previously explored to develop a computational tool for photonic quasicrystal \[114\].

V. SUMMARY AND OUTLOOK

To summarize this article, we provide a brief review of the concept of synthetic dimension in photonics, an area that has been rapidly developing in the past few years, in close connection with the developments of the gauge field and topological concepts in photonics. The initial motivation for exploring the synthetic dimension has been to develop a versatile approach in photonics for demonstrating many important fundamental physics effects, including in particular many important topological physics effects. And indeed, as we have seen in this review, a remarkably rich set of physics effects have been theoretically proposed and/or experimentally demonstrated using the synthetic dimension approach. We also envision that the concept of synthetic dimension will prove to be significant for prac-
tical applications, leading to new possibilities for manipulating and controlling some of the fundamental properties of light.

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