On the error exponent of variable-length block-coding schemes over finite-state Markov channels with feedback

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Abstract

The error exponent of Markov channels with feedback is studied in the variable-length block-coding setting. Burnashev’s [5] classic result is extended and a single letter characterization for the reliability function of finite-state Markov channels is presented, under the assumption that the channel state is causally observed both at the transmitter and at the receiver side. Tools from stochastic control theory are used in order to treat channels with intersymbol interference. In particular the convex analytical approach to Markov decision processes [4] is adopted to handle problems with stopping time horizons arising from variable-length coding schemes.

1 Introduction

The role of feedback in channel coding is a long studied problem in information theory. In 1956 Shannon [24] proved that noiseless causal output feedback does not increase the capacity of a discrete memoryless channel (DMC). Feedback, though, can help in improving the trade-off between reliability and delay of DMCs at rates below capacity. This trade-off is traditionally measured in terms of error exponent; in fact, since Shannon’s work, much research has focused on studying error exponents of channels with feedback. Burnashev [5] found a simple exact formula for the reliability function (i.e. the highest achievable error exponent) of a DMC with perfect causal output feedback in the variable-length block-coding setting. The present paper deals with a generalization of Burnashev’s result to a certain class of channels with memory. Specifically, we shall prove a simple single-letter characterization of the reliability function of finite-state Markov channels (FSMCs), in the general case when intersymbol-interference (ISI) is present. Under mild ergodicity assumptions, we will prove that, when one is allowed variable-length block-coding with perfect causal output feedback and causal state knowledge both at the transmitter and at the receiver end, the reliability function has the form

\[ E_B(R) = D \left( 1 - \frac{R}{C} \right), \quad R \in (0, C). \] (1)

In (1), \( R \) is the transmission rate, measured with respect to the average transmission time. The capacity \( C \) and the coefficient \( D \) are quantities which will be defined as solution of finite

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dimensional optimization problems involving the stochastic kernel describing the FSMC. The former will turn out to equal the maximum, over all choices of the channel input distributions as a function of the channel state, of the conditional mutual information between channel input and the pair of channel output and next channel state given the current state, whose marginal distribution coincides with the induced ergodic state measure (see (6)). The latter will instead equal the average, with respect to the induced ergodic state measure, of the Kullback-Leibler information divergence between the joint channel output and next channel state distributions associated to the pair of most distinguishable choices of a channel input symbol as a function of the current state (see (12)).

The problem of characterizing error exponents of memoryless channels with feedback has been addressed in the information theory literature in a variety of different frameworks. Particularly relevant are the choice of block versus continuous transmission, the possibility of allowing variable-length coding schemes, and the way delay is measured. In fact, much more than in the non-feedback case, these choices lead to very different results for the error exponent, albeit not altering the capacity value. In continuous transmission systems information bits are introduced at the encoder, and later decoded, individually. Continuous transmission with feedback was considered by Horstein [15], who was probably the first showing that variable-length coding schemes can give larger error exponents than fixed-length ones. Recently, continuous transmission with fixed delay has attracted renewed attention in the context of anytime capacity [23]. In this paper, however, we shall restrict ourselves to block transmission, which is the framework considered by the largest part of the previous literature.

In block transmission systems the information sequence is partitioned into blocks of fixed length which are then encoded into channel input sequences. When there is no feedback these sequences need to be of a predetermined, fixed length in order to guarantee that transmitter and receiver remain synchronized. When there is feedback, instead, the availability of common information shared between transmitter and receiver makes it possible to use variable-length schemes. Here the transmission time is allowed to dynamically depend on the channel output sequence. It is known that exploiting the possibility of using variable-length block-coding schemes guarantees high gains in terms of error exponent. In fact, Dobrushin [11] showed that the sphere-packing bound still holds for fixed-length block-coding schemes over symmetric DMCs even when perfect output feedback is causally available the encoder (a generalization to nonsymmetric DMCs was addressed in [14]). Even though fixed-length block-coding schemes with feedback have been studied (see [34, 10]) the above-mentioned results pose severe constraints on the performance such schemes can achieve. Moreover, no closed form for the reliability function at all rates is known for fixed-length block coding with feedback, but for the very special class of symmetric DMCs with positive zero-error capacity (cf. [7, pag.199]). It is worth to mention that the situation can be much different for continuous alphabet channels. For the additive white Gaussian noise channel (AWGNC) with average power constraint, Shalkwijk and Kailath [26] proved that a doubly exponential error rate is achievable by fixed-length block codes. However, when a peak power constraint to the input of an AWGNC is added, then this phenomenon disappear as shown in [32]. At the same time it has been also well-known that, if variable length coding schemes are allowed, then the sphere-packing exponent can be beaten even when no output feedback is available but for a single una tantum bit guaranteeing synchronization between transmitter and receiver. This situation is traditionally referred to as decision feedback and was studied in [12] (see also [7, pag.201]).

A very simple exact formula was found by Burnashev [5] for the reliability function of DMCs with full causal output feedback in the case variable-length block-coding schemes. Burnashev’s analysis combined martingale theory arguments with more standard information
theoretic tools. It is remarkable that in this setting the reliability function is known, in a very simple form, at any rate below capacity, in sharp contrast to what happens in most channel coding problems for which the reliability function can be exactly evaluated only at rates close to capacity. Another important point is that Burnashev exponent of a generic DMC can dramatically exceed the sphere-packing exponent; in particular it approaches capacity with nonzero slope.

Thus, variable-length block coding appears a natural setting for transmission over channels with feedback. In fact, it has already been considered by many authors after Burnashev’s landmark work. A simple two-phase iterative scheme achieving Burnashev exponent was introduced by Yamamoto and Itoh in [33]. More recently, low-complexity variable-length block-coding schemes with feedback have been proposed and analyzed in [21]. The works [28] and [29] dealt with universality issues, addressing the question whether Burnashev exponent can be achieved without exact knowledge of the statistics of the channel but only knowing it belongs to a certain class of DMCs. In [2] a simplification of Burnashev’s original proof [5] is proposed, while [17] is concerned with the characterization of the reliability function of DMCs with feedback and cost constraints. In [22] low-complexity schemes for FSMCs with feedback are proposed. However, to the best of our knowledge, no extension of Burnashev’s theorem to channels with memory has been considered.

The present work deals with a generalization of Burnashev’s result to FSMCs. As an example, channels with memory, and FSMCs in particular, model transmission problems where fading is an important component as for instance in wireless communication. Information theoretical limits of FSMCs both with and without feedback have been widely studied in the literature: we refer to the classic textbooks [16, 31] and references therein for overview of the available literature (see also [13]). It is known that the capacity is strongly affected by the hypothesis about the nature of the channel state information (CSI) both available at the transmitter and at the receiver side. In particular while output feedback does not increase the capacity when the state is causally observable both at the transmitter and at the receiver side (see [27] for a proof, first noted in [24]), it generally does so for different information patterns. In particular, when the channel state is not observable at the transmitter, it is known that feedback may help improving capacity by allowing the encoder in estimating the channel state [27]. However, in this paper only the case when the channel state is causally observed both at the transmitter and at the receiver end will be considered. Our choice is justified by the aim to separate the study of the role of output feedback in channel state estimation from its effect in allowing better reliability versus delay tradeoffs for variable-length block-coding schemes.

In [27] a general stochastic control framework for evaluating the capacity of channels with memory and feedback has been introduced. The capacity has been characterized as the solution of a dynamic programming average cost optimality equation. Existence of a solution to such an equation implies information stability. Also lower bounds à la Gallager to the error exponents achievable with fixed-length coding schemes are obtained in [27]. In the present paper we follow a similar approach in order to characterize the reliability function of variable-length block-coding schemes with feedback. Such an exponent will be characterized in terms of solutions to certain Markov decision problems. The main new feature posed by variable-length schemes is that we have to deal with average cost optimality problems with a stopping time horizon, for which standard results in Markov decision theory cannot be used directly. We adopt the convex analytical approach of [4] and use Hoeffding-Azuma inequality in order to prove a strong uniform convergence result for the empirical measure process. This allows us to find sufficient conditions on the tails of a sequence of stopping times for the solutions of the corresponding average cost optimality problems to be asymptotically approximated by
the solution of the corresponding infinite horizon problem, for which stationary policies are known to be optimal.

The rest of this paper is organized as follows. In Section 2 causal feedback variable-length block-coding schemes for FSMCs are introduced, and capacity and reliability function are defined as solution of optimization problems involving the stochastic kernel describing the FSMC. The main result of the paper is then stated in Theorem 1. In Section 3 we prove an upper bound to the best error exponent achievable by variable-length block-coding schemes with perfect feedback over FSMCs. The main result of that section is contained in Theorem 7 which generalizes Burnashev result. Section 4 is of a technical nature and deals with Markov decision processes with stopping time horizons. Some stochastic control techniques are reviewed and the main result is contained in Theorem 11 which is then used to prove that the bound of Theorem 7 asymptotically coincides with the reliability function (1). In Section 5 a family of simple iterative schemes based on a generalization of Yamamoto-Itoh’s [33] is proposed and its performance is analyzed showing that this family is asymptotically optimal in terms of error exponent. Finally, in Section 6 an explicit example is studied. Section 7 presents some conclusions and points out to possible topics for future research.

2 Statement of the problem and main result

2.1 Stationary ergodic Markov channels

Throughout the paper \(\mathcal{X}, \mathcal{Y}, \mathcal{S}\) will respectively denote channel input, output and state spaces. All are assumed to be finite.

**Definition 1** A stationary Markov channel is described by:

- a stochastic kernel consisting in a family \(\{P(\cdot, \cdot | s, x) \in \mathcal{P}(\mathcal{S} \times \mathcal{Y}) | s \in \mathcal{S}, x \in \mathcal{X}\}\) of probability measures over \(\mathcal{S} \times \mathcal{Y}\), indexed by elements of \(\mathcal{S}\) and \(\mathcal{X}\);

- an initial state distribution \(\mu_1\) in \(\mathcal{P}(\mathcal{S})\).

For a channel as in Def[1] let

\[
P_S(s_+ | s, x) := \sum_{y \in \mathcal{Y}} P(s_+, y | s, x)
\]

be the \(S\)-marginals. We shall say that a Markov channel as above has no ISI when the \(S\)-marginals do not depend on the chosen channel input, i.e.

\[
P_S(s_+ | s, x_1) = P_S(s_+ | s, x_2), \quad \forall s, s_+ \in \mathcal{S}, \; x_1, x_2 \in \mathcal{X}.
\] (2)

We will consider the associated stochastic kernels

\[
\{Q(\cdot, \cdot | s, u) \in \mathcal{P}(\mathcal{S} \times \mathcal{Y}) | s \in \mathcal{S}, u \in \mathcal{P}(\mathcal{X})\}, \quad \{Q_S(\cdot | s, u) \in \mathcal{P}(\mathcal{S}) | s \in \mathcal{S}, u \in \mathcal{P}(\mathcal{X})\},
\]

where for every channel input distribution \(u\) in \(\mathcal{P}(\mathcal{X})\)

\[
Q(s_+, y | s, u) := \sum_{x \in \mathcal{X}} P(s_+, y | s, x)u(x), \quad Q_S(s_+ | s, u) := \sum_{x \in \mathcal{X}} P_S(s_+ | s, x)u(x).
\] (3)

Given \(\pi : \mathcal{S} \rightarrow \mathcal{P}(\mathcal{X})\) (we shall refer to such a map as a deterministic stationary policy), denote by

\[
Q_\pi := (Q(s_+ | s, \pi(s)))_{s, s_+ \in \mathcal{S}}
\] (4)
the state transition stochastic matrix induced by \( \pi \). With an abuse of notation, for any map \( f: S \to X \) we shall write \( Q_f \) in place of \( Q_{f(\cdot)} \). Throughout the paper we will restrict ourselves to FSMCs satisfying the following ergodicity assumption.

**Assumption 2** For every \( f: S \to X \) the stochastic matrix \( Q_f \) is irreducible.

Assumption 2 can be relaxed or replaced by other equivalent assumptions. Here we limit ourselves to observe that it involves the \( S \)-marginals \( \{P_S\} \) of the Markov channel only. Moreover it is purely discrete condition, since it requires a finite number of finite directed graphs to be strongly connected. Since taking a convex combination does not reduce the support, Assumption 2 guarantees that for every deterministic stationary policy \( \pi: S \to \mathcal{P}(X) \) the stochastic matrix \( Q_\pi \) is irreducible. Then, Perron-Frobenius theorem guarantees that \( Q_\pi \) has a unique invariant measure in \( \mathcal{P}(S) \) which will be denoted by \( \mu_\pi \). Notice that in the non-ISI case Assumption 2 is tantamount to requiring the strict positivity of the \( S \)-marginals of the stochastic channel.

### 2.2 Capacity of ergodic FSMCs

To any ergodic FSMC we associate the mutual information cost function \( c: S \times \mathcal{P}(X) \to \mathbb{R} \),

\[
c(s, u) = \sum_{x \in X} \sum_{y \in Y} \sum_{v \in S} u(x)P(v, y| s, x) \log \frac{P(v, y| s, x)}{\sum_{z \in X} u(z)P(v, y| s, z)} ,
\]

and define its capacity as

\[
C := \max_{\pi: S \to \mathcal{P}(X)} \sum_{s \in S} \mu_\pi(s)c(s, \pi(s)) = \max_{\pi: S \to \mathcal{P}(X)} I(X; Y, S_+| S) .
\]

In the rightmost side of (5) the term \( I(X; S_+, Y| S) \) denotes the conditional mutual information \( [5] \) between \( X \) and the pair \((S_+, Y)\) given \( S \), where \( S \) is an \( S \)-valued r.v. whose marginal distribution is given by the invariant measure \( \mu_\pi \), \( X \) is an \( X \)-valued r.v. whose conditional distribution given \( S \) is described by the policy \( \pi \), while \( S_+ \) and \( Y \) are respectively an \( S \)-valued r.v. and a \( Y \)-valued r.v. whose joint conditional distribution given \( X \) and \( S \) is described by the stochastic kernel \( P(S_+, Y| S, X) \). Notice that in particular the mutual information cost function \( c \) is continuous over \( S \times \mathcal{P}(X) \) and takes values in the bounded interval \([0, \log |X|]\).

The quantity \( C \) defined above is known to equal the capacity of the ergodic Markov channel we are considering when perfect causal CSI is available at both transmission ends, with or without output feedback \([27]\). It is important to observe that, due to the presence of ISI in the channel model we are considering, the policy \( \pi \) plays a dual role in the optimization problem in (6) since it affects both the mutual information cost \( c(s, \pi(s)) = I(X; S_+, Y| S = s) \) and the ergodic channel state distribution \( \mu_\pi \) with respect to which the former is averaged.

In the case when there is no ISI, i.e. when \([2] \) is satisfied, this phenomenon disappears. In fact, since the invariant measure \( \mu_\pi \) is independent of the policy \( \pi \) we have that (6) reduces to

\[
C = \sum_{s \in S} \mu_\pi(s) \max_{p_X \in \mathcal{P}(X)} c(s, p_X) = \sum_{s \in S} \mu_\pi(s) \max_{p_X \in \mathcal{P}(X)} I(X; Y| S = s) ,
\]

where in the rightmost side of (7) the quantity \( \max_{p_X \in \mathcal{P}(X)} I(X; Y| S = s) \) coincides with the capacity of the DMC associated to the state \( s \). The simplest case of FSMCs with no ISI is obtained when the state sequence forms an i.i.d. process independent from the channel input with distribution \( \mu \), i.e. when

\[
P_S(s_+| s, x) = \mu(s_+), \quad \forall s, s_+ \in S, x \in X .
\]
In this case, it is not difficult to check that (6) reduces to the capacity of a DMC with input space $X' = S^X$ -the set of all maps from $S$ to $X$-, output space $Y' := S \times Y$ -the Cartesian product of $S$ times $Y$-, and transition probabilities given by

$$P'(y' | x') := \sum_{s \in S} \mu(s)P(y' | s, x'(s)),$$

(8)

Observe the difference with respect to the case when the state is causally observed at the receiver end, also at the transmitter only, whose capacity was first found in [25]. While the input space of the equivalent DMC is the same in both cases, its output space is larger in the case we are dealing with in this paper with respect to that addressed by Shannon, since we are assuming that the state is causally observable also at the receiver end.

Finally, notice that, when the state space is trivial (i.e. when $|S| = 1$), (6) reduces to the usual definition of the capacity of a DMC.

### 2.3 Burnashev coefficient of FSMCs

Consider now the cost function $d : S \times \mathcal{P}(X) \rightarrow [0, +\infty]$

$$d(s, u) := \sup_{u' \in \mathcal{P}(X)} \sum_{y \in Y} \sum_{x', u \in S} u(x)Q(s_+, y | s, u) \log \frac{Q(s_+, y | s, u)}{Q(s_+, y | s, u')},$$

(9)

Notice that the term to be optimized in the righthand side of (9) equals the Kullback-Leibler information divergence between the probability measures $Q(\cdot, \cdot | s, u)$ and $Q(\cdot, \cdot | s, u')$ in $\mathcal{P}(S \times Y)$. It follows that, if we introduce the quantities

$$\lambda := \min \{ \lambda_s | s \in S \}, \quad \lambda_s := \min \{ \min_{s \in X} P(s_+, y | s, x) | s_+, y : \exists z : P(s_+, y | s, z) > 0 \},$$

(10)

we have that the cost function $d$ is bounded and continuous over $S \times \mathcal{P}(X)$ if and only if $\lambda$ is strictly positive, i.e.

$$\lambda > 0 \iff d_{\max} := \sup_{s \in S, u \in \mathcal{P}(X)} d(s, u) < +\infty.$$  

(11)

Define the Burnashev coefficient of a Markov channel as

$$D := \sup_{\pi : S \rightarrow \mathcal{P}(X)} \sum_{s \in S} \mu_\pi(s)d(s, \pi(s)).$$

(12)

Notice that $D$ is finite iff (11) holds. Moreover, a standard convexity argument allows to conclude that both the suprema in (9) and in (12) are achieved in some corner points, so that

$$D = \max_{f_0, f_1 : S \rightarrow X} \sum_{s \in S} \mu_{f_0}(s) \sum_{s \in S} \sum_{y \in Y} P(s_+, y | s, f_0(s)) \log \frac{P(s_+, y | s, f_0(s))}{P(s_+, y | s, f_1(s))}$$

$$= \max_{f_0, f_1 : S \rightarrow X} \sum_{s \in S} \mu_{f_0}(s)D \left( P(\cdot, \cdot | s, f_0(s)) \| P(\cdot, \cdot | s, f_1(s)) \right).$$

(13)

Similarly to what already noted for the role of policy $\pi$ in the optimization problem (6), it can be observed that, due to the presence of ISI, the map $f_0$ has a dual effect in the maximization in (13) since it affects both the Kullback-Leibler information divergence cost $D \left( P(\cdot, \cdot | s, f_0(s)) \| P(\cdot, \cdot | s, f_1(s)) \right)$ and the ergodic state measure $\mu_{f_0}$. Notice the asymmetry with the role of the map $f_1$ whose associated ergodic measure instead does not come
into the picture at all in the definition of the coefficient $D$. Once again, in the absence of ISI, (13) simplifies to

$$D = \sum_{s \in S} \mu(s) \max_{x_0, x_1 \in X} D(P(\cdot, \cdot | s, x_0)) D(P(\cdot, \cdot | s, x_1)).$$

We observe that in the memoryless case (which can be recovered when $|S| = 1$) the coefficient $D$ coincides with the Kullback-Leibler information divergence between the output measures associated to the pair of most distinguishable inputs, the quantity originally denoted with the symbol $C_1$ in [5]. When the state space is nontrivial ($|S| > 1$), and the channel state process forms an i.i.d. sequence independent from the channel input, then the Burnashev coefficient $D$ reduces to that of the equivalent DMC with enlarged input space $X' = X^S$ and output space $Y' = S \times Y$ with transition probabilities defined in [8].

2.4 Causal feedback encoders, sequential decoders, and main result

**Definition 3** A causal feedback encoder is the pair of a finite message set and a sequence of maps

$$\Phi = \left( W, \{ \phi_t : W \times Y^{t-1} \times S^{t} \to X' \}_{t \in \mathbb{N}} \right).$$

With Def[3] we are implicitly assuming that perfect state knowledge as well as perfect output feedback are available at the encoder side.

Given a stationary Markov channel and a causal feedback encoder as in Def[3], we will consider a probability space $(\Omega, \mathcal{A}, P_\Phi)$ ($E_\Phi$ will denote the corresponding expectation operator) over which are defined:

- a $W$-valued random variable $W$ describing the message to be transmitted;
- a sequence $X = (X_t)_{t \in \mathbb{N}}$ of $X$-valued r.v.s (the channel input sequence);
- a sequence $Y = (Y_t)_{t \in \mathbb{N}}$ of $Y$-valued r.v.s (the channel output sequence);
- a sequence $S = (S_t)_{t \in \mathbb{N}}$ of $S$-valued r.v.s (the state sequence).

We shall consider the time ordering

$$W, S_1, X_1, Y_1, S_2, X_2, Y_2, \ldots,$$

and assume that

$$P_\Phi(W = w) = \frac{1}{|\mathcal{W}|}, \quad P_\Phi(S_1 = s|W) = \mu(s),$$

$$P_\Phi(X_t = x|W, S^t_1, X^{t-1}, Y^{t-1}_1) = \delta_{\{\phi_t(W, Y^{t-1}_1, S^t_1)\}}(x), \quad P_\Phi - a.s.,$$

Figure 1: Information patterns for variable-length block-coding schemes on a FSMC with causal feedback and CSI.
\[ \mathbb{P}_\Phi(S_{t+1} = s, Y_t = x \mid W, S_{t}^{t-1}, Y_{t}^{t-1}, X_{t}^{t}) = P(s, y \mid S_{t}, X_{t}) , \quad \mathbb{P}_\Phi - a.s. . \]

It is convenient to introduce the following notation for the information patterns available at the encoder and decoder side. For every \( t \) we define the sigma-fields \( \mathcal{E}_t := \sigma(S_{t}^{t-1}, Y_{t}^{t-1}) \), describing the feedback information available at the encoder side, and \( \mathcal{F}_t := \sigma(S_{t}^{t}, Y_{t}^{t}) \), describing the information available at the decoder. Clearly

\[
\{\emptyset, \Omega\} = \mathcal{E}_0 = \mathcal{F}_0 \subseteq \mathcal{E}_1 \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{A} .
\]

In particular we end up with two nested filtrations: \( \mathcal{F} := (\mathcal{F}_t)_{t \in \mathbb{Z}^+} \) and \( \mathcal{E} := (\mathcal{E}_t)_{t \in \mathbb{Z}^+} . \)

**Definition 4** A transmission time \( T \) is a stopping time for the filtration \( \mathcal{F} . \)

**Definition 5** Given a causal feedback encoder \( \Phi \) as in (14) and a transmission time \( T \), a sequential decoder for \( \Phi \) and \( T \) is a \( \mathcal{W} \)-valued \( \mathcal{F}_T \)-measurable random variable.

Notice that with Def.s 4 and 5 we are assuming that perfect causal state knowledge is available at the receiver. In particular the fact that the transmitter’s and the receiver’s information patterns are nested guarantees that encoder and decoder stay synchronized while using a variable-length scheme.

Given a causal feedback encoder \( \Phi \) as in Def. 3 a transmission time \( T \) and a sequential decoder \( \Psi \), we will call the triple \((\Phi, T, \Psi)\) a variable-length block-coding scheme and define its error probability as

\[ p_e(\Phi, T, \Psi) := \mathbb{P}_\Phi(\Psi \neq W) . \]

Following Burnashev’s approach we shall consider the expected decoding time \( E_{\Phi}[T] \) as a measure of the delay of the scheme \((\Phi, T, \Psi)\) and accordingly define its rate as

\[ R(\Phi, T, \Psi) := \frac{\log |\mathcal{W}|}{E_{\Phi}[T]} . \]

We are now ready to state our main result. It is formulated in an asymptotic setting, considering countable families of causal encoders and sequential decoders with asymptotic average rate below capacity and vanishing error probability.

**Theorem 1** For any \( R \) in \((0, C)\)

1. any family \((\Phi^n, T_n, \Psi^n)_{n \in \mathbb{N}}\) of variable-length block-coding schemes such that

\[ \lim_{n \to \infty} p_e(\Phi^n, T_n, \Psi^n) = 0 , \quad \liminf_{n \to \infty} R(\Phi^n, T_n, \Psi^n) \geq R , \]

satisfies

\[ \limsup_{n \to \infty} - \frac{1}{E_{\Phi^n[T_n]}} \log p_e(\Phi^n, T_n, \Psi^n) \leq E_B(R) . \]

2. there exists a family \((\Phi^n, T_n, \Psi^n)_{n \in \mathbb{N}}\) of variable-length block-coding schemes satisfying (16) and such that

- if \( D < +\infty \)

\[ \lim_{n \to \infty} - \frac{1}{E_{\Phi^n[T_n]}} \log p_e(\Phi^n, T_n, \Psi^n) = E_B(R) , \]

- if \( D = +\infty \)

\[ p_e(\Phi^n, T_n, \Psi^n) = 0 , \quad \forall n \in \mathbb{N} . \]

We observe that Burnashev’s original result [5] for memoryless channels can be recovered as a particular case of Theorem 1 when the state space is trivial, i.e. \(|S| = 1\).
3 An upper bound on the achievable error exponent

The aim of this section is to provide an upper bound on the error exponent of an arbitrary variable-length block coding scheme. A first observation is that, without any loss of generality, we can restrict ourselves to the case when the Burnashev coefficient $D$ is finite, since otherwise the claim (17) is trivially true. The main result of this section is contained in Theorem 7 whose proof will pass through a series of intermediate steps, contained in Lemmas 2, 3, and 5.

The main idea, taken from Burnashev’s original paper 5 (see also 17 and 2), is to obtain two different upper bounds for the error probability. Differently from [5] and [17], we will follow an approach similar to the one proposed in [2] and look at the behaviour of the a posteriori error probability, instead of that of the a posteriori entropy. The two bounds correspond to two distinct phases which can be recognized in any sequential transmission scheme and will be the content of Sections 3.1 and 3.2. The first one is provided in Lemma 3 whose proof is based on an application of Fano’s inequality combined with a martingale argument invoking Doob’s optional stopping theorem. The second bound is given by Theorem 5 whose proof combines the use of the log-sum inequality with another application of Doob’s optional stopping theorem. In Section 3.3 these two bounds will be combined obtaining Theorem 7 which is a generalization to our setting of Burnashev’s result [5].

3.1 A first bound on the error probability

Suppose we are given a causal feedback encoder $\Phi = (\mathcal{W}, (\phi_t))$ as in (14) and a transmission time $T$ as in Def. 4. Our goal is to find a lower bound for the error probability $p_e(\Phi, t, \Psi)$ where $\Psi$ is an arbitrary sequential decoder for $\Phi$ and $T$.

It will be convenient to define for every $t \geq 0$ the $\sigma$-algebra $\mathcal{G}_t := \mathcal{E}_{t+1}$ describing the encoder’s feedback information at time $t+1$. $\mathcal{G} := (\mathcal{G}_t)_{t \in \mathbb{Z}_+}$ will denote the corresponding filtration. We define the maximum a posteriori error probability conditioned on the $\sigma$-algebras $\mathcal{F}_t$ and $\mathcal{G}_t$ respectively by

$$P_{MAP}^\Phi(t) := 1 - \max_{w \in \mathcal{W}} \{ P_\Phi(W = w | \mathcal{F}_t) \}, \quad \tilde{P}_{MAP}^\Phi(t) := 1 - \max_{w \in \mathcal{W}} \{ P_\Phi(W = w | \mathcal{G}_t) \}.$$

Clearly $P_\Phi(t)$ is an $\mathcal{F}_t$-measurable random variable while $\tilde{P}_\Phi(t)$ is $\mathcal{G}_t$-measurable.

It is a well known fact that the decoder minimizing the error probability over the class of fixed-length decoders $\{ \Psi : \mathcal{S}^t \times \mathcal{Y}^t \rightarrow \mathcal{W} \}$ is the maximum a posteriori one, defined by

$$\Psi^t_{MAP}(s, y) = \arg\max_{w \in \mathcal{W}} \{ P_\Phi(W = w | S_1^t = s, Y_1^t = y) \}, \quad s \in \mathcal{S}^t, \ y \in \mathcal{Y}^t,$$

(with the convention for the operator argmax to arbitrarily assign one of the optimizing values in case of non-uniqueness). It will be convenient to consider the larger class of decoders $\{ \Psi : \mathcal{S}^{t+1} \times \mathcal{Y}^t \rightarrow \mathcal{W} \}$ (differing from the previous one because of the possible dependence on the state at time $t+1$); over such a class, the optimal decoder is given by

$$\tilde{\Psi}^t_{MAP}(s, y) = \arg\max_{w \in \mathcal{W}} \{ P_\Phi(W = w | S_1^{t+1} = s, Y_1^t = y) \}, \quad s \in \mathcal{S}^{t+1}, \ y \in \mathcal{Y}^t.$$

It follows that for any $\Psi : \mathcal{S}^t \times \mathcal{Y}^t \rightarrow \mathcal{W}$ we have

$$p_e(\Phi, t, \Psi) \geq p_e(\Phi, t, \Psi^t_{MAP}) \geq p_e(\Phi, t, \tilde{\Psi}^t_{MAP}) = \mathbb{E}_\Phi \left[ \tilde{P}_{MAP}^\Phi(t) \right].$$

The discussion above naturally generalizes from the fixed length setting to the sequential one. In particular, given a stopping time $T$ for the decoder filtration $\mathcal{F}$, we observe that, since
\(\mathcal{F}_t \subseteq \mathcal{G}_t\) for every \(t \geq 0\), \(T\) is also stopping time for the filtration \(\mathcal{G}\) and \(\mathcal{F}_T \subseteq \mathcal{G}_T\). It follows that the error probability of the scheme \((\Phi, T, \Psi)\), where \(\Psi\) is an arbitrary \(\mathcal{F}_T\)-measurable \(W\)-valued r.v., is lower bounded by that corresponding to the sequential improved MAP decoder \(\tilde{\Psi}^T_{\text{MAP}}\), defined by

\[
\tilde{\Psi}^T_{\text{MAP}} := \arg\max_{w \in \mathcal{W}} \{ \mathbb{P}_\Phi(W = w|\mathcal{G}_T) \}.
\]

Therefore we can conclude that

\[
p_e(\Phi, T, \Psi) \geq \mathbb{E}_\Phi \left[ \tilde{P}^\Phi_{\text{MAP}}(T) \right],
\]

for any \(\mathcal{F}_T\)-measurable \(W\)-valued random variable \(\Psi\).

In the sequel we will lower bound the righthand side of (20). In particular, since the random variable \(W\) is uniformly distributed over the message set \(\mathcal{W}\), and since \(S_1\) is independent of \(W\), we have that

\[
\mathbb{P}_\Phi(W = w|\mathcal{G}_0) = \mathbb{P}_\Phi(W = w) = \frac{1}{|\mathcal{W}|}, \quad w \in \mathcal{W},
\]

so that

\[
\tilde{P}^\Phi_{\text{MAP}}(0) = \frac{|\mathcal{W}| - 1}{|\mathcal{W}|}.
\]

Moreover we have the following recursive lower bound for \(\tilde{P}^\Phi_{\text{MAP}}(t)\) (see Proposition 2 in [2] for a similar result in the memoryless case).

**Lemma 2** Given any causal feedback encoder \(\Phi\), we have, for every \(t \in \mathbb{N}\),

\[
\tilde{P}^\Phi_{\text{MAP}}(t) \geq \lambda \tilde{P}^\Phi_{\text{MAP}}(t - 1) \quad \mathbb{P}_\Phi - \text{a.s.}
\]

**Proof** See Appendix A. \(\square\)

For every \(\delta\) in \((0, \frac{1}{2})\), we now consider the random variable

\[
\tau_\delta := \min \left\{ T, \inf \left\{ n \in \mathbb{N} : \tilde{P}^\Phi_{\text{MAP}}(t) \leq \delta \right\} \right\}.
\]

(21)

It is immediate to verify that \(\tau_\delta\) is a stopping time for the filtration \(\mathcal{G}\). Moreover the event \(\{\tilde{P}^\Phi_{\text{MAP}}(\tau_\delta) > \delta\}\) implies the event \(\{\tau_\delta = T\}\), so that an application of Markov inequality and (20) give us

\[
\mathbb{P}_\Phi \left( \tilde{P}^\Phi_{\text{MAP}}(\tau_\delta) > \delta \right) \leq \mathbb{P}_\Phi \left( \tilde{P}^\Phi_{\text{MAP}}(T) > \delta \right) \leq \frac{1}{\delta} \mathbb{E}_\Phi \left[ \tilde{P}^\Phi_{\text{MAP}}(T) \right] \leq \frac{1}{\delta} p_e(\Phi, T, \Psi).
\]

We introduce the following notation for the a posteriori entropy

\[
\Gamma_t := - \sum_{w \in \mathcal{W}} \mathbb{P}_\Phi(W = w|\mathcal{G}_t) \log \mathbb{P}_\Phi(W = w|\mathcal{G}_t), \quad t \in \mathbb{Z}_+.
\]

Observe that, since \(S_1\) is independent of the message \(W\), then

\[
\Gamma_0 = \log |\mathcal{W}|, \quad \mathbb{P}_\Phi - \text{a.s.}
\]
It is easy to verify that, whenever \( \hat{P}_{MAP}(\tau_\delta) \leq \delta \), we have
\[
\Gamma_{\tau_\delta} \leq H(\delta) + \delta \log |W|.
\]
Hence the expected value of \( \Gamma_{\tau_\delta} \) can be bounded from above as follows:
\[
\mathbb{E}_\Phi [\Gamma_{\tau_\delta}] = \mathbb{E}_\Phi \left[ \Gamma_{\tau_\delta} \big| \hat{P}_{MAP}(\tau_\delta) \leq \delta \right] \mathbb{P}_\Phi \left( \hat{P}_{MAP}(\tau_\delta) \leq \delta \right) + \mathbb{E}_\Phi \left[ \Gamma_{\tau_\delta} \big| \hat{P}_{MAP}(\tau_\delta) > \delta \right] \mathbb{P}_\Phi \left( \hat{P}_{MAP}(\tau_\delta) > \delta \right)
\leq (H(\delta) + \delta \log |W|) \mathbb{P}_\Phi \left( \hat{P}_{MAP}(\tau_\delta) \leq \delta \right) + \mathbb{P}_\Phi \left( \hat{P}_{MAP}(\tau_\delta) > \delta \right) \log |W|
\leq H(\delta) + (\delta + \frac{H}{\delta}) M(\Phi, T, \Psi) \log |W|.
\]

We now introduce, for every time \( t \) in \( \mathbb{N} \), a \( \mathcal{P}(\mathcal{X}) \)-valued random variable \( \Upsilon_{\Phi,t} \) describing the channel input distribution induced by the encoder \( \Phi \) at time \( t \):
\[
\Upsilon_{\Phi,t}(x) := \mathbb{P}(X_t = x|E_t) = \mathbb{P}(\phi_t(W, S^t_1, Y^{t-1}_1) = x|S^t_1, Y^{t-1}_1), \quad x \in \mathcal{X}, \quad (23)
\]
Notice that \( \Upsilon_{\Phi,t} \) is \( E_t \)-measurable, i.e. equivalently it is a function of the pair \( (S^t_1, Y^{t-1}_1) \). The subscript in \( \Upsilon_{\Phi,t} \) emphasizes the fact that this quantity depends on the encoder \( \Phi \), with no restriction on it but to be causal.

The following result relates three relevant quantities characterizing the performances of any causal encoder sequential decoder pair: the cardinality of the message set \( W \), the error probability of the encoder decoder pair, and the the mutual information cost \( c(5) \) up to the stopping time \( \tau_\delta \):
\[
C_\delta(\Phi, T) := \mathbb{E}_\Phi \left[ \sum_{t=1}^{\tau_\delta} c(S_t, \Upsilon_{\Phi,t}) \right]. \quad (24)
\]

**Lemma 3** For any variable-length block-coding scheme \((\Phi, T, \Psi)\) and any \( 0 < \delta < \frac{1}{2} \), we have
\[
C_\delta(\Phi, T) \geq \left( 1 - \delta - \frac{P_e(\Phi, T, \Psi)}{\delta} \right) \log |W| - H(\delta).
\]

**Proof** See Appendix A. \( \Box \)

### 3.2 A lower bound to the error probability of a composite binary hypothesis test

We now consider a particular binary hypothesis testing problem which will arise while proving the main result. Suppose we are given a causal feedback encoder \( \Phi = (W, (\phi_t)) \). Consider a nontrivial binary partition of the message set
\[
W = W_0 \cup W_1, \quad W_0 \cap W_1 = \emptyset, \quad W_0, W_1 \neq \emptyset, \quad (26)
\]
and a sequential binary hypothesis test \( \hat{\Psi} = (T, \hat{\Psi}) \) (where \( T \) is stopping time for the filtration \( \mathcal{G} \), and \( \hat{\Psi} \) is a \( \mathcal{G}_T \)-measurable \( \{0, 1\} \)-valued random variable) between the hypothesis \( \{W \in W_0\} \) and the hypothesis \( \{W \in W_1\} \). Following the standard statistical terminology we call \( \hat{\Psi} \) a composite test since it must decide between two classes of probability laws for the process \((S, Y)\) rather than between two single laws. For every \( t \), we define the \( \mathcal{P}(\mathcal{X}) \)-valued random variables \( \Upsilon^0_{\Phi,t} \) and \( \Upsilon^1_{\Phi,t} \) by
\[
\Upsilon^i_{\Phi,t}(x) = \mathbb{P}_\Phi(X_t = x|W \in W_i, E_t), \quad x \in \mathcal{X}, \quad i = 0, 1.
\]
The r.v. \( Y_{\Phi,t}^0 \) (respectively \( Y_{\Phi,t}^1 \)) represents the channel input distribution at time \( t \) induced by the encoder \( \Phi \) when restricted to the message subset \( W_0 \) (resp. \( W_1 \)). Notice that

\[
Y_{\Phi,t} = P(\Phi(W \in W_0 | \mathcal{E}_t)Y_{\Phi,t}^0 + P(\Phi(W \in W_1 | \mathcal{E}_t)Y_{\Phi,t}^1.
\]

Let \( \tau \) be another stopping time for the filtration \( \mathcal{G} \), such that \( \tau \leq T \). Let us consider the conditional expectation terms

\[
L_i := E_\Phi \left[ \log \frac{P_\Phi \left( S_{\tau+2}^T, Y_{\tau+1}^T | W \in W_i, \mathcal{G}_\tau \right)}{P_\Phi \left( S_{\tau+2}^T, Y_{\tau+1}^T | W \notin W_i, \mathcal{G}_\tau \right)} \middle| W \in W_i, \mathcal{G}_\tau \right], \quad i = 0, 1.
\]

Both \( L_0 \) and \( L_1 \) are \( \mathcal{G}_\tau \)-measurable random variables. In particular \( L_0 \) equals the Kullback-Leibler information divergence between the \( \mathcal{G}_\tau \)-conditioned probability distributions of the pair \( (S_{\tau+2}^T, Y_{\tau+1}^T) \) respectively given \( \{W \in W_0\} \) and \( \{W \in W_1\} \); an analogous interpretation is possible, mutatis mutandis, for \( L_1 \).

In the special case when both \( \tau \) and \( T \) are deterministic constants, an application of the log-sum inequality would show that, for \( i = 0, 1 \), \( L_i \) can be upperbounded by the \( \mathcal{G}_\tau \)-conditional expected value of the sum of the information divergence costs \( d \left( Y_{\Phi,t}^i, S_i \right) \) from time \( \tau + 1 \) to \( T \), and analogously for \( L_1 \), with the terms \( d \left( S_t, Y_{\Phi,t}^i \right) \). It turns out that the same is true in our setting where both \( \tau \) and \( T \) are stopping times for the filtration \( \mathcal{G} \), as stated in the following lemma, whose proof requires, besides an application of the log-sum inequality, a martingale argument invoking Doob's optional stopping theorem.

**Lemma 4** Let \( \tau \) and \( T \) be stopping times for the filtration \( \mathcal{G} \) such that \( \tau \leq T \), and consider a partition of the message set as in (26). Then

\[
L_i \leq E_\Phi \left[ \sum_{t=\tau+1}^{T} d \left( Y_{\Phi,t}, S_t \right) \middle| W \in W_i, \mathcal{G}_\tau \right], \quad P_\Phi - a.s., \quad i = 0, 1. \tag{27}
\]

**Proof** See Appendix A.

Suppose now that \( W_1 \) is a \( \mathcal{G}_\tau \)-measurable random variable taking values in \( 2^W \setminus \{\emptyset, W\} \), the class of nontrivial proper subsets of the message set \( W \). In other words, we are assuming that \( W_1 \) is a random subset of the message set \( W \), deterministic function of the pair \( (S_1^{T+1}, Y_1^T) \).

The following result gives a lower bound on the error probability of the binary test \( \tilde{\Psi} \) conditioned on the \( \sigma \)-algebra \( \mathcal{G}_\tau \) in terms of the information divergence terms

\[
E_\Phi \left[ \sum_{t=\tau+1}^{T} d \left( S_t, Y_{\Phi,t}^i \right) \middle| \mathcal{G}_\tau \right], \quad i = 0, 1.
\]

**Lemma 5** Let \( \Phi \) be any causal encoder, and \( \tau \) and \( T \) be stopping times for the filtration \( \mathcal{G} \) such that \( \tau \leq T \). Then, for every \( 2^W \setminus \{\emptyset, W\} \)-valued \( \mathcal{G}_\tau \)-measurable r.v. \( W_1 \), we have that \( P_\Phi - a.s.

\[
E_\Phi \left[ \sum_{t=\tau+1}^{T} d \left( S_t, Y_{\Phi,t}^i (W \in W_1) \right) \middle| \mathcal{G}_\tau \right) \geq \log \frac{Z}{4} - \log P \left( \tilde{\Psi} \neq 1 | \{W \in W_1\} \right) \tag{28}
\]

where

\[
Z := \min \left\{ P_\Phi \left( W \in W_0 \middle| \mathcal{G}_\tau \right), P_\Phi \left( W \in W_1 \middle| \mathcal{G}_\tau \right) \right\}.
\]

**Proof** See Appendix A.
3.3 Burnashev bound for Markov channels

**Lemma 6** Let $\Phi$ be a causal feedback encoder and $T$ a transmission time for $\Phi$. Then, for every $0 < \delta < 1/2$ there exists a $\mathcal{G}_{\tau_3}$-measurable random subset $W_1$ of the message set $W$, whose a posteriori error probabilities satisfy

$$1 - \lambda \delta \geq \mathbb{P}(W \in W_1 | \mathcal{G}_{\tau_3}) \geq \lambda \delta.$$  \hfill (29)

**Proof** See Appendix A.

To a causal encoder $\Phi$ and a transmission time $T$, for every $0 < \delta < 1/2$ we define the quantity $D_{\delta}(\Phi, T) := \max_{W_1, \mathcal{G}_{\tau_3}} \mathbb{E} \left[ \sum_{t=\tau_3+1}^T d(S_t, Y_{\Phi,t}^{\{W \in W_1\}}) \right]$ \hfill (30)

The quantity $D_{\delta}(\Phi, T)$ equals the maximum, over all possible choices of a nontrivial partition of the message set $W$ as a function of the joint channel state output process $(S_1^{\tau_3+1}, Y_1^{\tau_3})$ stopped at the intermediate time $\tau_3$, of the averaged sum of the information divergence costs $d(S_t, Y_{\Phi,t}^{\{W \in W_1\}})$ incurred between times $\tau_3 + 1$ and $T$. Intuitively $D_{\delta}(\Phi, T)$ measures the maximum error exponent achievable by the encoder $\Phi$ when transmitting a binary message between times $\tau_3$ and $T$.

Based on Lemma 3 and Lemma 5 we will now prove the main result of this section, consisting in an upper bound on the largest error exponent achievable by variable-length block-coding schemes with perfect causal state knowledge and output feedback.

**Theorem 7** Consider a variable-length block-coding scheme $(\Phi, T, \Psi)$. Then, for every $\delta \in (0, 1/2)$,

$$- \log p_e(\Phi, T, \Psi) \leq \frac{D}{C} C_{\delta}(\Phi, T) + D_{\delta}(\Phi, T) - \frac{D}{C} \log |W| (1 - \alpha) - \beta,$$  \hfill (31)

where

$$\alpha := \delta + \frac{p_e(\Phi, T, \Psi)}{\delta}, \quad \beta := \log \frac{\lambda \delta}{4} - \frac{D}{C} H(\delta).$$

**Proof** Let $W_1$ be a $\mathcal{G}_{\tau_3}$-measurable subset of the message space $W$ satisfying (29). We define the binary sequential decoder

$$\tilde{\Psi}_{\delta} := 1_{W_1}(\Psi).$$

Notice that the definition above is consistent in the sense that $\tilde{\Psi}$ is $\mathcal{G}_T$-measurable, since $\Psi$ is $\mathcal{G}_T$-measurable, while $W_1$ is $\mathcal{G}_{\tau_3}$-measurable and $\mathcal{G}_{\tau_3} \subseteq \mathcal{G}_T$.

We can lower bound the error probability of the composite hypothesis test $\tilde{\Psi}_{\delta}$ conditioned on $\mathcal{G}_{\tau_3}$ using Lemma 5 and (29), obtaining

$$- \log \mathbb{P}_\Phi \left( \tilde{\Psi}_{\delta} \neq 1_{W_1}(W) \mid \mathcal{G}_{\tau_3} \right) + \log \frac{\lambda \delta}{4} \leq \mathbb{E}_\Phi \left[ \sum_{t=\tau_3+1}^T d(S_t, Y_{\Phi,t}^{\{W \in W_1\}}) \mid \mathcal{G}_T \right].$$

It is clear that the error event of the pair $\Psi$ is implied by the error event of $\tilde{\Psi}_{\delta}$. It follows
that

\[-\log p_e (\Phi, T, \Psi) + \log \frac{\lambda \delta}{4} = - \log E_\Phi \left[ \mathbb{P} \left( \Psi \neq W \mid G_{\tau_3} \right) \right] + \log \frac{\lambda \delta}{4} \]

\[
\leq - \log E_\Phi \left[ \mathbb{P} \left( \tilde{\Psi}_\delta \neq 1_{W_1} (W) \mid G_{\tau_3} \right) \right] + \log \frac{\lambda \delta}{4}
\]

\[
= - \log E_\Phi \left[ \mathbb{P} \left( \tilde{\Psi}_\delta \neq 1_{W_1} (W) \mid G_{\tau_3} \right) \right] + \log \frac{\lambda \delta}{4}
\]

\[
\leq E_\Phi \left[ - \log \mathbb{P}_\Phi \left( \tilde{\Psi}_\delta \neq 1_{W_1} (W) \mid G_{\tau_3} \right) \right] + \log \frac{\lambda \delta}{4}
\]

\[
\leq E_\Phi \left[ \sum_{t=\tau_3+1}^{T} d \left( S_t, \Upsilon_{\phi,t}^{1_{W \in W_1}} \right) \mid G_{\tau_3} \right]
\]

\[
\leq D_\delta (\Phi, T),
\]

the second inequality in (32) following from Chebychev inequality. The claim now follows by taking a linear combination of (32) and (25).

In the memoryless case, i.e. when the state space is trivial (|S| = 1), Burnashev’s original result (see (4.1) in [5], see also (12) in [2]) can be recovered from (31) by optimizing over the channel input distributions \( \Upsilon_{\phi,t}, \Upsilon_{\phi,t}^0, \) and \( \Upsilon_{\phi,t}^1. \)

In order to prove Part 1 of Theorem 1 it remains to consider countable families of variable-length coding schemes with vanishing error probability and to show that asymptotically the upper bound in (31) reduces to the Burnashev exponent \( E_B (R). \) This involves new technical challenges which will be the object of next section.

4 Markov decision problems with stopping time horizons

In this section we shall recall some concepts about Markov decision processes which will allow us to asymptotically estimate the terms \( C_\delta (\Phi, T) \) and \( D_\delta (\Phi, T) \) respectively in terms of the capacity \( C \) defined in (6) and the Burnashev coefficient \( D \) of the FSMC.

The main idea is to interpret the maximization of \( C_\delta (\Phi, T) \) and \( D_\delta (\Phi, T) \) as stochastic control problems with average cost criterion [1]. The control is the channel input distribution chosen as a function of the available feedback information and the controller is identified with the encoder. The main novelty these problems have with respect to those traditionally addressed by Markov decision theory consists in the fact that, as a consequence of considering variable-length coding schemes, we shall need to deal with the situation when the horizon is neither finite (in the sense of being a deterministic constant) nor infinite (in the sense of being concerned with the asymptotic normalized average running cost), but rather it is allowed to be a random stopping time. In order to handle this case we adopt the convex analytical approach, a technique first introduced by Manne in [18] (see also [9]) for the finite state finite action setting, and later developed in great generality by Borkar [4].

In Section 4.1 we shall first reformulate the problem of optimizing the terms \( C_\delta (\Phi, T) \) and \( D_\delta (\Phi, T) \) with respect to the causal encoder \( \Phi. \) Then, we present a brief review of the convex analytical approach to Markov decision problems in Section 4.2 presenting the main ideas and definitions. In Section 4.3 we will prove a uniform convergence theorem for the empirical measure process and use this result to treat the asymptotic case of the average cost problem with stopping time horizon. The main result of this section is contained in Theorem 11 which is then applied in Section 4.4 together with Theorem 7 in order to prove Part 1 of Theorem 1.
4.1 Markov decision problems with stopping time horizons

We shall consider a controlled Markov chain over \( S \), with compact control space \( U := P(X) \), the space of channel input distributions. Let \( g : S \times U \to \mathbb{R} \) be a continuous (and thus bounded) cost function; in our application \( g \) will coincide either with the mutual information cost \( c \) defined in (5) or with the information divergence cost \( d \) defined in (9). We prefer to consider the general case in order to deal with both problems at once.

The evolution of the system is described by a state sequence \( S = (S_t) \), an output sequence \( Y = (Y_t) \) and a control sequence \( U = (U_t) \). If at time \( t \) the system is in state \( S_t = s \) in \( S \), and a control \( U_t = u \) in \( U \) is chosen according to some policy, then a cost \( g(s, u) \) is incurred and the system produces the output \( Y_t = y \) in \( Y \) and moves to next state \( S_{t+1} = s_\pi \) in \( S \) according to the stochastic kernel \( Q(s, y|s, u) \), defined in (3). Once the transition into next state has occurred, a new action is chosen and the process is repeated.

At time \( t \), the control \( U_t \) is allowed to be an \( \mathcal{E}_t \)-measurable random variable, where \( \mathcal{E}_t = \sigma(S_1^t, Y_1^{t-1}) \) is the encoder’s feedback information pattern at time \( t \); in other words we are assuming that \( U_t = \pi_t(S_1^t, Y_1^{t-1}) \) for some map

\[
\pi_t : S^t \times Y^{t-1} \to U.
\]

We define a feasible policy \( \pi \) as a sequence \((\pi_t)_{t \in \mathbb{N}}\) of such maps. Once a feasible policy \( \pi \) has been chosen, a joint probability distribution \( \mathbb{P}_\pi \) for state, control and output sequences is well defined; we will denote by \( \mathbb{E}_\pi \) the corresponding expectation operator.

Let \( \tau \) be a stopping time for the filtration \( \mathcal{G} = (\mathcal{G}_t) \) (recall that \( \mathcal{G}_t = \mathcal{E}_{t+1} \) describes the encoder’s feedback and state information at time \( t + 1 \)), and consider the following optimization problem: maximize

\[
\frac{1}{\mathbb{E}_\pi[\tau]} \mathbb{E}_\pi \left[ \sum_{t=1}^\tau g(S_t, \pi_t(S_1^t, Y_1^{t-1})) \right]
\]

over all feasible policies \( \pi = (\pi_t) \) such that \( \mathbb{E}_\pi[\tau] \) is finite.

Clearly, in the special case when \( \tau \) is a constant \( \tau = \mathbb{N} \) reduces to the standard finite-horizon problem which is usually solved with dynamic programming tools. Another special case is when \( \tau \) is geometrically distributed and independent from the processes \( S, U \) and \( Y \). In this case (33) reduces to the so-called discounted problem which has been widely studied in the stochastic control literature [1]. However, what makes the problem nonstandard is that in (33) \( \tau \) is allowed to be an arbitrary stopping time for the filtration \( \mathcal{F} \), generally correlated with the processes \( S, U \) and \( Y \).

4.2 The convex analytical approach

We review some of the ideas of the convex-analytical approach following [4].

A feasible policy \( \pi \) is said to be stationary if the current control depends on the current state only and is independent of the past state and output history and of the time, i.e. there exists a map \( \pi : S \to U \) such that \( \pi_t(s_1^t, y_1^{t-1}) = \pi(s_t) \) for all \( t \). We will identify a stationary policy as above with the map \( \pi : S \to U \). It has already been noted in Section 2.1 that, for every stationary policy \( \pi \), the stochastic matrix \( Q_\pi \) as defined in (4) is irreducible, so that existence and uniqueness of an invariant measure \( \mu_\pi \) in \( P(S) \) are guaranteed. It follows that, if a stationary policy \( \pi \) is used, then the normalized running cost \( \frac{1}{\pi} \sum_{t=1}^\infty g(S_t, \pi(S_t)) \) converges...
Figure 2: A schematic representation of the optimization problem (40). The large triangular space is the infinite dimensional Prohorov space $P(S \times U)$. Its gray-shaped subset represents the close convex set $K$ of all occupation measures. The set of extreme points of $K$ is $K_e$ and corresponds to the set of all occupation measures associated to stationary deterministic policies. The optimal value of the linear functional $\eta \mapsto \langle \eta, g \rangle$ happens to be achieved on $K_e$ and thus corresponds to the occupation measure $\eta^*_\pi$ associated to an optimal deterministic stationary policy $\pi^*: S \rightarrow P(X)$.

$P_\pi$-almost surely to $\sum_{s \in S} \mu_\pi(s)g(s, \pi(s))$. Define

$$G := \max_{\pi: S \rightarrow U} \sum_{s \in S} \mu_\pi(s)g(s, \pi(s)).$$

(34)

Observe that the optimization in the righthand side of (34) has the same form of those in the definitions (5) and (12) of the capacity and the Burnashev coefficient of an ergodic FSMC given in Section 2. Notice that compactness of the space $U^S$ of all stationary policies and continuity of the cost $g(s, \pi(s))$ and of the invariant measure $\mu_\pi$ as functions of the stationary policy $\pi$ guarantee the existence of an optimal value in the above maximization.

We now consider stationary randomized policies. These are defined as maps $\tilde{\pi}: S \rightarrow P(U)$, where $P(U)$ denotes the space of probability measures on $U$, equipped with its Prohorov topology. To any stationary randomized policy $\tilde{\pi}$ the following control strategy is associated: if at time $t$ the state is $S_t = s$, then the control $U_t$ is randomly chosen in the control space $U$ with conditional distribution given the available information $\mathcal{E}_t = \sigma(S_1, Y_{t-1}^t)$ equal to $\pi(s)$. Observe that there are two levels of randomization. The control space itself has already been defined as the space of channel input probability distributions $P(X)$, while the strategy associated to the stationary randomized policy $\tilde{\pi}$ chooses a control at random with conditional distribution $\tilde{\pi}(S_t)$ in $P(U) = P(P(X))$. Of course randomized stationary policies are a generalization of deterministic stationary policies, since to any deterministic stationary policy $\pi: S \rightarrow U$ it is possible to associate the randomized policy $\tilde{\pi}(s) = \delta_{\pi(s)}$. To any randomized stationary policy $\tilde{\pi}: S \rightarrow P(U)$ we associate the stochastic matrix describing the associated state transition probabilities

$$(Q_{\tilde{\pi}}(s_+|s))_{s, s_+ \in S} ; \quad Q_{\tilde{\pi}}(s_+|s) := \int_U Q(s_+|s, u)[\tilde{\pi}](s) (du).$$

(35)

Similarly to the case of stationary deterministic policies, it is not difficult to conclude that, since $Q_{\tilde{\pi}}$ can be written as a convex combination of a finite number of stochastic matrices $Q_f$, with $f: S \rightarrow X$, all of which are irreducible, then $Q_{\tilde{\pi}}$ itself is irreducible and thus admits a unique state ergodic measure $\mu_{\tilde{\pi}}$ in $P(S)$. This motivates the following definition.
**Definition 6**  For every stationary (randomized) policy \( \pi : S \rightarrow \mathcal{P}(U) \) the occupation measure of \( \pi \) is \( \eta_\pi \) in \( \mathcal{P}(S \times U) \) defined by

\[
\langle \eta_\pi, h \rangle = \int_{S \times U} h(s,u) d\eta_\pi(s,u) = \sum_{s \in S} \mu_\pi(s) \int_U h(s,u) d\pi(s), \quad \forall h \in \mathcal{C}_b(S \times U),
\]

where \( \mu_\pi \) in \( \mathcal{P}(S) \) is the invariant measure of the stochastic matrix \( Q_\pi \), while \( \mathcal{C}_b(S \times U) \) is the space of bounded continuous maps from \( S \times U \) to \( \mathbb{R} \).

The occupation measure \( \eta_\pi \) can be viewed as the long-time empirical frequency of the joint state-control process governed by the stationary (randomized) policy \( \pi \). In fact, for every time \( n \in \mathbb{N} \), we can associate to the controlled Markov process the empirical measure \( \nu_n \) which is a \( \mathcal{P}(S \times U) \)-valued random variable sample-path-wise defined by

\[
\langle \nu_n, h \rangle := \frac{1}{n} \sum_{t=1}^n h(S_t, U_t), \quad \forall h \in \mathcal{C}_b(S \times U). \tag{36}
\]

Observe that \( \nu_n \) is a probability measure on the product space \( S \times U \), and is itself a random variable since it is defined as a function of the joint state control random process \( (S_t, U_t)^t \).

Then, it can be verified that, if the process is controlled by a stationary (randomized) policy \( \pi \), then

\[
\lim_{n \to \infty} \nu_n = \eta_\pi \quad \mathbb{P}_\pi - \text{a.s.} \tag{37}
\]

We will denote by \( K \) the set of the occupation measures associated to all the stationary randomized policies, i.e.

\[
K := \{ \eta_\pi \mid \pi : S \rightarrow \mathcal{P}(U) \} \subseteq \mathcal{P}(S \times U), \tag{38}
\]

and by \( K_e \) the set of all occupation measures associated to stationary deterministic policies

\[
K_e := \{ \eta_\pi \mid \pi : S \rightarrow U \} \subseteq \mathcal{P}(S \times U).
\]

Well known results (see [4]) show that both \( K \) and \( K_e \) are closed subsets of \( \mathcal{P}(S \times U) \). Moreover \( K \) is convex and \( K_e \) coincides with the set of extreme points of \( K \). Furthermore it is possible to characterize \( K \) as the the set of zeros of the continuous linear functional

\[
F : \mathcal{P}(S \times U) \rightarrow [0,1]^S, \quad F_s(\eta) := \eta(\{s\}, U) - \int_{S \times U} Q_S(s \mid j, u) d\eta(j, u),
\]

i.e.

\[
K = \{ \eta \in \mathcal{P}(S \times U) : F(\eta) = 0 \}. \tag{39}
\]

In fact it is possible to think of \( ||F(\eta)|| \) (here and throughout the paper \( ||x|| := \max_i |x_i| \) will denote the \( L^\infty \)-norm of a vector \( x \)) as a measure of how far the \( S \)-marginal of a measure \( \eta \) in \( \mathcal{P}(S \times U) \) is from being invariant for the state process.

If one were interested in optimizing the infinite-horizon running average cost

\[
\liminf_{n \in \mathbb{N}} \frac{1}{n} \mathbb{E}_\pi \left[ \sum_{t=1}^n g(S_t, U_t) \right] = \liminf_{n \in \mathbb{N}} \mathbb{E}_\pi \left[ \langle \nu_n, g \rangle \right]
\]

over all (randomized) stationary policies \( \pi \), then (37) and (38) would immediately lead to the following convex optimization problem:

\[
\max_{\eta \in K} \langle \eta, g \rangle. \tag{40}
\]
In fact, using (39), (40) can be rewritten as an infinite dimensional linear programming problem

$$\max_{\eta \in \mathcal{P}(S \times U): F(\eta) = 0} \langle \eta, c \rangle$$

We notice that, since $\mathcal{U}$ is compact and $S$ is finite, $\mathcal{P}(S \times U)$ is compact. Thus both $K$ and $K_e$ are compact. It follows that, since the map

$$\mathcal{P}(S \times U) \ni \eta \mapsto \langle \eta, g \rangle \in \mathbb{R}$$

is continuous, it achieves its maxima both on $K$ and $K_e$; moreover, the same map is linear so that these maxima do coincide, i.e. the maximum over $K$ is achieved in an extreme point. Thus we have the following chain of equalities

$$G = \max_{\pi: S \rightarrow U} \sum_{s \in S} \mu_\pi(s) g(s, \pi(s))$$

$$= \max_{\pi: S \rightarrow U} \langle \eta_\pi, g \rangle$$

$$= \max_{\eta \in K_e} \langle \eta, g \rangle$$

$$= \max_{\eta \in K} \langle \eta, g \rangle$$

$$= \max_{\eta \in \mathcal{P}(S \times U): F(\eta) = 0} \langle \eta, c \rangle$$

(42)

We observe that the last term in (42) both the constraints and the object functionals are linear. This indicates (infinite dimensional) linear programming as a possible approach for computing $G$, alternative to the dynamic programming ones based on policy or value iteration techniques [1], [4]. Moreover, it shows an easy way to generalize the theory taking into account average cost constraints (see [17] where the Burnashev exponent of DMCs with average cost constraints is studied). In fact, in the convex analytical approach these constraints merely translate into additional constraints for the linear program.

4.3 An asymptotic solution to Markov decision problems with a stopping time horizon

It is known that, under the ergodicity and continuity assumptions we have made, $G$ defined in (34) is the sample-path optimal value for the infinite horizon problem with cost $g$ not only over the set of all stationary policies, but also over the larger set of all feasible policies (actually over all admissible policies, see [3]). This means that, for every feasible policy $\pi = (\pi_t)$,

$$\limsup_{n \in \mathbb{N}} \frac{1}{n} \sum_{t=1}^{n} g(S_t, \pi_t(S_{t-1}^t, Y_{t-1}^t)) \leq G$$

$$\mathbb{P}_\pi - a.s.$$ (43)

Moreover, it is a known fact that for an arbitrary sequence of policies $(\pi^n)$ we have

$$\limsup_{n \in \mathbb{N}} \frac{1}{n} \mathbb{E}_{\pi^n} \left[ \sum_{t=1}^{n} g(S_t, U_t) \right] = \limsup_{n \in \mathbb{N}} \frac{1}{n} \mathbb{E}_{\pi^n} \left[ \sum_{t=1}^{n} g(S_t, \pi^n_t(S_{t-1}^t, Y_{t-1}^t)) \right] \leq G$$

(44)

i.e. the limit of the optimal values of finite horizon problems coincides with infinite horizon optimal value. (44) can be proved by using dynamic programming arguments based on Bellman principle of optimality. As shown in [27], (44) is useful in characterizing the capacity
of channels with memory and feedback with fixed-length codes. Actually, a much more general result than (44) can be proved, as explained in the sequel.

In the convex analytical approach, the key point in proof of (43) consists in showing that, under any -generally non stationary- feasible policy \( \pi \), the empirical measure process \((\nu_n)\) as defined in (36) converges \( P_{\pi} \)-almost surely to the set \( K \). The way this is usually proven is by using a martingale central limit theorem in order to show that the finite-dimensional process \( F(\nu_n) \) converges to 0 almost surely. The following is a stronger result, providing an exponential upper bound on the tails of the random sequence \( ||F(\nu_n)||_n \), this bound being uniform with respect to the choice of the policy \( \pi \) in \( \Pi \).

**Lemma 8** For every \( \varepsilon > 0 \), and for every feasible policy \( \pi \)

\[
P_{\pi} \left( ||F(\nu_n)|| \geq \varepsilon + \frac{1}{n} \right) \leq 2|S| \exp \left( -n\varepsilon^2 / 2 \right).
\]  

**(Proof)** See Appendix B. \( \Box \)

We emphasize the fact that the bound (45) is uniform with respect to the choice of the feasible policy \( \pi \). It is now possible to drive conclusions on the tails of the running average cost \( \frac{1}{n} \sum_{t=1}^{n} g(S_t, U_t) \) based on (45). The core idea is the following. By the definition of the empirical measure \( \nu_n \), we can rewrite the normalized running cost as

\[
\frac{1}{n} \sum_{t=1}^{n} g(S_t, U_t) = \langle \nu_n, g \rangle.
\]

Since the map \( \eta \mapsto \langle \eta, g \rangle \) is continuous over \( \mathcal{P}(S \times U) \), and \( G = \max \{ \langle \eta, g \rangle \mid \eta \in K \} \), we have that, whenever \( \nu_n \) is close to the set \( K \), \( \langle \nu_n, g \rangle \) cannot be much larger than \( G \). It follows that, if with high probability \( \nu_n \) is close enough to \( K \), then with high probability \( \langle \nu_n, g \rangle \) cannot be much larger than \( G \). In order to show that with high probability \( \nu_n \) is close to \( K \), we want to use (45). In fact, if for some \( x \) in \( \mathcal{P}(S \times U) \) the quantity \( ||F(x)|| \) is very small, then \( x \) is necessarily close to \( G \). More precisely, we define the function

\[
\gamma : \mathbb{R}^+ \rightarrow \mathbb{R}, \quad \gamma(x) := \sup \{ \langle \eta, g \rangle \mid \eta \in \mathcal{P}(S \times U) : ||F(\eta)|| \leq x \}.
\]

Clearly \( \gamma \) is nondecreasing and \( \gamma(0) = G \). Moreover we have the following result.

**Lemma 9** The map \( \gamma \) is upper semicontinuous. (i.e. \( x_n \to x \Rightarrow \limsup_n \gamma(x_n) \leq \gamma(x) \))

**(Proof)** See Appendix B. \( \Box \)

For every \( k \) in \( \mathbb{N} \) we now introduce the random process \( (G^k_n) \)

\[
G^k_n := \sup_{i \geq n} \langle \nu_i, g \rangle, \quad n \in \mathbb{N}.
\]

Clearly the process \( (G^k_n) \) is samplepathwise non increasing in \( n \).

**Lemma 10** Let \( (\tau_k) \) be a sequence of stopping times for the filtration \( \mathcal{F} \) and \( (\pi^k) \) be a sequence of feasible policies such that \( E_{\pi^k}[\tau_k] < \infty \) for every \( k \) and

\[
\lim_{k \in \mathbb{N}} \mathbb{P}_{\pi^k} (\tau_k \leq M) = 0, \quad \forall M \in \mathbb{N}.
\]  

Then

\[
\lim_{k \in \mathbb{N}} \mathbb{P}_{\pi^k} \left( G^k_{\tau_k} > \gamma(\varepsilon) \right) = 0, \quad \forall \varepsilon > 0.
\]  

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The following result can be considered as an asymptotic estimate of (33). It consists in a generalization of (44) from a deterministic increasing sequence of time horizons to a sequence of stopping times satisfying the ‘probabilistic divergence’ requirement (46).

**Theorem 11** Let \((\tau_k)\) be a sequence of stopping times for the filtration \(\mathcal{F}\) and \((\pi^k)\) be a sequence of feasible policies such that \(E_{\pi^k}[\tau_k] < \infty\) for every \(k\), and (46) holds true. Then

\[
\limsup_{k \in \mathbb{N}} \frac{1}{E_{\pi^k}[\tau_k]} E_{\pi^k} \left[ \sum_{t=1}^{\tau_k} g(S_t, U_t) \right] \leq G. \tag{48}
\]

**Proof** Let us fix an arbitrary \(\varepsilon > 0\). By applying Lemma 10, we obtain

\[
\frac{E_{\pi^k} \left[ \sum_{t=1}^{\tau_k} g(S_t, U_t) \right]}{E_{\pi^k}[\tau_k]} = \frac{E_{\pi^k}[\tau_k(v_{\tau_k}, g)]}{E_{\pi^k}[\tau_k]} = \frac{E_{\pi^k}[\tau_k(v_{\tau_k}, g) \{G_{\tau_k} \leq \gamma(\varepsilon)\} + G_{\tau_k} > \gamma(\varepsilon)\] - \frac{g_{\max} P_{\pi^k}(G_{\tau_k} > \gamma(\varepsilon))}{\varepsilon \to 0^+} \gamma(\varepsilon) = G. \tag{49}
\]

Therefore (48) follows from the arbitrariness of \(\varepsilon > 0\), and the fact that, as a consequence of Lemma 9, we have

\[
\lim_{\varepsilon \to 0^+} \gamma(\varepsilon) = G.
\]

**4.4 Proof of Part 1 of Theorem 1**

We are now ready to step back to the problem of upperbounding the error exponent of variable-length block-coding schemes over FSMCs. We want to combine the result in Theorem 7 with that in Theorem 11 in order to finally prove Part 1 of Theorem 1.

Let \((\Phi^k, T_k, \Psi^k)\) be a sequence of variable-length block coding schemes satisfying (16). Our goal is to prove that

\[
\limsup_{k \in \mathbb{N}} \frac{-\log p_e(\Phi^k, T_k, \Psi^k)}{E_{\Phi^k}[T_k]} \leq D \left(1 - \frac{R}{C}\right). \tag{49}
\]

A first simple conclusion that can be drawn from Theorem 7 using the crude bounds

\[
c(\Upsilon_{\Phi^k,t}, S_t) \leq \log |\mathcal{X}|, \quad d(S_t, \Upsilon_{\Phi^k,t}^i) \leq d_{\max}, \quad i = 0, 1,
\]

is that

\[
\limsup_{k \in \mathbb{N}} \frac{-\log p_e(\Phi^k, T_k, \Psi^k)}{E_{\Phi^k}[T_k]} \leq D \frac{\log |\mathcal{X}| + d_{\max} - R(1 - \delta) - \log \frac{\lambda \delta}{4} + D}{C} \mathcal{H}(\delta) < +\infty. \tag{50}
\]

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Thus the error probability does not decay to zero faster than exponentially with the expected transmission time $\mathbb{E}_{\Phi^k}[T_k]$.

The core idea to prove (49) consists in introducing a real sequence $(\delta_k)$ and showing that both

$$
\tau_k := \min \left\{ T_k, \inf \left\{ t \in \mathbb{N} \mid P_{MAP}^k(t) \leq \delta_k \right\} \right\},
$$

and $T_k - \tau_k$ diverge in the sense of satisfying (46). The sequence $(\delta_k)$ needs to be carefully chosen: we want it to be asymptotically vanishing in order to guarantee that $\tau_k$ diverges, but not too fast since otherwise $T_k - \tau_k$ would not diverge. It turns out that one possible good choice is

$$
\delta_k := \frac{-1}{\log p_e (\Phi^k, T_k, \Psi^k)}.
$$

It is immediate to verify that

$$
\lim_{k \in \mathbb{N}} p_e (\Phi^k, T_k, \Psi^k) = 0
$$

implies

$$
\lim_{k \in \mathbb{N}} \delta_k = 0, \quad \lim_{k \in \mathbb{N}} \frac{p_e (\Phi^k, T_k, \Psi^k)}{\delta_k} = 0. \quad (51)
$$

**Lemma 12** In the previous setting, for every fixed $M$ in $\mathbb{N}$, we have

$$
\lim_{k \in \mathbb{N}} \mathbb{P}_{\Phi^k} (\tau_k \leq M) = 0, \quad \lim_{k \in \mathbb{N}} \mathbb{P}_{\Phi^k} (T_k - \tau_k \leq M) = 0. \quad (52)
$$

**Proof** See Appendix B.

Lemma 12 allows us to apply Theorem 11 first to the mutual information cost $c$ obtaining

$$
\limsup_{k \in \mathbb{N}} \frac{C_{\delta_k}(\Phi^k, T_k)}{\mathbb{E}_{\Phi^k}[\tau_k]} = \limsup_{k \in \mathbb{N}} \frac{\mathbb{E}_{\Phi^k} \left[ \sum_{t=1}^{\tau_k} c(S_t, Y_{\Phi,t}) \right]}{\mathbb{E}_{\Phi^k}[\tau_k]} \leq C
$$

and then to the information divergence cost $d$ obtaining

$$
\limsup_{k \in \mathbb{N}} \frac{D_{\delta_k}(\Phi^k, T_k)}{\mathbb{E}_{\Phi^k}[T_k - \tau_k]} \leq D.
$$

Therefore, by applying Theorem 7 we get

$$
D \geq \limsup_{k \in \mathbb{N}} \frac{1}{\mathbb{E}_{\Phi^k}[T_k]} \left( \frac{D}{C} C_{\delta_k}(\Phi^k, T_k) + D_{\delta_k}(\Phi^k, T_k) \right)
$$

$$
\geq \limsup_{k \in \mathbb{N}} \frac{-\log p_e (\Phi^k, T_k, \Psi^k)}{\mathbb{E}_{\Phi^k}[T_k]} + \frac{D}{C} \frac{\log |W_k|}{\mathbb{E}_{\Phi^k}[T_k]} (1 - \alpha_k) + \frac{\beta_k}{\mathbb{E}_{\Phi^k}[T_k]}
$$

$$
= \frac{D}{C} R + \limsup_{k \in \mathbb{N}} \frac{-\log p_e (\Phi^k, T_k, \Psi^k)}{\mathbb{E}_{\Phi^k}[T_k]}
$$

thus proving (17).
An asymptotically optimal scheme

In this section we propose and analyze a family of causal coding schemes with feedback asymptotically achieving the Burnashev exponent $E_B(R)$, thus proving Part 2 of Theorem 1.

The scheme we propose can be viewed as a generalization of the one introduced by Yamamoto and Itoh in [33] and consists of a sequence of epochs. Each epoch is made up of two distinct fixed-length transmission phases, respectively named communication and confirmation phase. In the communication phase the message to be sent is encoded in a block code and transmitted over the channel. At the end of this phase the decoder makes a tentative decision about the message sent based on the observation of the channel outputs and of the state sequence. As perfect causal feedback is available at the encoder, the result of this decision is known at the encoder. In the confirmation phase a binary acknowledge message, confirming the decoder’s estimation if it is correct, or denying it when it is wrong, is sent by the encoder through a fixed-length repetition code function. The decoder performs a binary hypothesis test in order to decide whether a deny or a confirmation message has been sent. If an acknowledge is detected the transmission halts, while if a deny is detected the system restarts transmitting the same message with the same protocol. Again because of perfect feedback availability at the encoder, there are no synchronization problems.

More precisely we design our scheme as follows. Given a design rate $R$ in $(0, C)$, let us fix an arbitrary $\gamma$ in $(\frac{R}{C}, 1)$. For every $n$ in $\mathbb{N}$, consider a message set $W_n$ of cardinality $|W_n| = \exp([nR])$ and two blocklengths $\hat{n}$ and $\tilde{n}$ respectively defined as $\hat{n} = \lceil n\gamma \rceil$, $\tilde{n} := n - \hat{n}$.

**Fixed-length block-coding for the transmission phase**

It is known from previous works (see [27] for instance) that the capacity $C$ of the stationary Markov channel we are considering is achievable by fixed-length block-coding schemes. Thus, since the rate of the first transmission phase is below capacity,

$$\hat{R} := \lim_{n \in \mathbb{N}} \frac{\log |W_n|}{\hat{n}} = \frac{R}{\gamma} < C,$$

there exists a family of causal encoders $(\hat{\Phi}^n)$ parametrized by an index $n$ in $\mathbb{N}$

$$\hat{\Phi}^n = \left( W_n, (\hat{\phi}_t^n) \right), \quad \hat{\phi}_t^n : W_n \times S^t \times Y^{t-1} \to X,$$

and a corresponding family $(\hat{\Psi}^n)$ of decoders of fixed length $\hat{n}$ (notice that $n$ is the index while $\hat{n}$ is the blocklength)

$$\hat{\Psi}^n : S^{\hat{n}} \times Y^{\hat{n}} \to W_n,$$
with error probability asymptotically vanishing in \( n \). More precisely, since the state space \( \mathcal{S} \) is finite, the pair \((\hat{\Phi}^n, \hat{\Psi}^n)\) can be designed in such a way that the probability \( \mathbb{P}_{\hat{\Phi}^n}(\hat{\Psi}^n \neq W | W = w, S_1 = s) \) of error conditioned on the transmission of any message \( w \) in \( \mathcal{W}_n \) and of an initial state \( s \) approaches zero uniformly with respect both to \( w \) and \( s \), i.e.

\[
p(n) := \max_{w \in \mathcal{W}_n} \max_{s \in \mathcal{S}} \mathbb{P}_{\hat{\Phi}^n}(\hat{\Psi}^n \neq W | W = w, S_1 = s) \xrightarrow{n \to \infty} 0.
\]

The triple \((\hat{\Phi}^n, \hat{n}, \hat{\Psi}^n)\) will be used in the first phase of each epoch of our iterative transmission scheme. \( \square \)

**Binary hypothesis test for the confirmation phase**

For the second phase, instead, we consider a causal binary input encoder \( \tilde{\Phi}^n \) based on the optimal stationary policies in the maximization problem \([13]\). More specifically, we define \( \tilde{\Phi}^n \) by

\[
\tilde{\phi}_t^n : \{0,1\} \times \mathcal{S}^t \to \mathcal{X}, \quad \tilde{\phi}_t^n(m, s) = f_{m}^*(s_t), \quad m = 0, 1, \quad t = 1, \ldots, \tilde{n},
\]

where \( f_0^*, f_1^* : \mathcal{S} \to \mathcal{X} \) are such that

\[
D = \sum_{s \in \mathcal{S}} \mu_{f_0}(s) D\left( P(\cdot, \cdot | s, f_0(s)) || P(\cdot, \cdot | s, f_1(s)) \right)
\]

Suppose that an acknowledge message \( m = 0 \) is sent. Then it is easy to verify that the pair sequence \((S_{t+1}, Y_{t})_{t=1}^{\tilde{n}}\) forms a Markov chain over the space of the achievable channel state output pairs

\[
\mathcal{Z} := \{ (s_+, y) \in \mathcal{S} \times \mathcal{Y} \text{ s.t. } \exists s \in \mathcal{S}, \exists x \in \mathcal{X} : P(s_+, y | s, x) > 0 \}
\]

with transition probability matrix

\[
P_0 = \left( P_0(s_+, y | s, y_-) := P(s_+, y | s, f_0^*(s)) \right).
\]

Analogously, if a deny message \( m = 1 \) has been sent, then \((S_{t+1}, Y_{t})_{t=1}^{\tilde{n}}\) forms a Markov chain with transition probability matrix

\[
P_1 = \left( P_1(s_+, y | s, y_-) := P(s_+, y | s, f_1^*(s)) \right).
\]

It follows that a decoder for \( \hat{\Phi}^n \) is a binary hypothesis test between two Markov chain hypothesis. Notice that for both chains the transition probabilities \( P_0(s_+, y | s, y_-) \) and \( P_1(s_+, y | s, y_-) \) respectively do not depend on the second component \( y_- \) of the past state only, but only on its first component \( s \) as well as on the full future state \((s_+, y)\).

When the coefficient \( D \) is finite, as a consequence of Assumption \([2]\) and \([11]\), we have that both the stochastic matrices \( P_0 \) and \( P_1 \) are both irreducible over \( \mathcal{Z} \), with the invariant measure of \( P_i \) given by

\[
\tilde{\mu}_i \in \mathcal{P}(\mathcal{Z}), \quad \tilde{\mu}_i(s_+, y) := \sum_{s \in \mathcal{S}} \mu_{f_i}(s) P(s_+, y | s, f_i(s)), \quad i = 0, 1.
\]

Using binary hypothesis test results for irreducible Markov chains (see \([20]\) and \([8]\, pagg.72-82]) it is possible to show that a decoder

\[
\hat{\Psi}^n : (\mathcal{S} \times \mathcal{Y})^{\tilde{n}-1} \to \{0,1\}
\]
can be chosen in such a way that, asymptotically in \( n \), its type 1 error probability achieves the exponent (recall (13))
\[
\sum_{z,z_+ \in \mathcal{Z}} \tilde{\mu}_0(z) P_0(z_+ | z) \log \frac{\tilde{\mu}_0(z) P_0(z_+ | z)}{\mu_0(z) P_1(z_+ | z)} = \sum_{s,s_+ \in \mathcal{S}} \tilde{\mu}_0(s,y_-) P(s_+, y | s, f_0^s(s)) \log \frac{P(s_+, y | s, f_0^s(s))}{P(s_+, y | s, f_1^s(s))} \\
= \sum_{s \in \mathcal{S}} \mu_0(s) D \left( P(\cdot, \cdot | s, f_0^s(s)) \parallel P(\cdot, \cdot | s, f_1^s(s)) \right) = \frac{D}{n}
\]
while its type one error probability is vanishing. More specifically, since the state space is finite, we have that, defining \( p_i(n) \) as the maximum over all possible initial states of the error probability of the pair \((\tilde{\Psi}^n, \Psi^n)\) conditioned on the transmission of a \('i'\) confirmation message, i.e.
\[
p_i(n) := \max_{s \in \mathcal{S}} P_{\tilde{\Phi}} \left( \tilde{\Psi}^n \left( S_2^n, Y_1^{n-1} \right) \neq i \mid W = i, S_1 = s \right), \quad i = 0, 1,
\]
we have
\[
\lim_{n \to \infty} p_0(n) = 0, \quad \lim_{n \to \infty} \frac{- \log p_1(n)}{n} = D. \quad (55)
\]
When the coefficient \( D \) is infinite, then the stochastic matrix \( P_0 \) is irreducible over \( \mathcal{Z} \), while there exists at least a pair \( z, z_+ \in \mathcal{Z} \) such that \( P_0(z_+ | z_-) > 0 \) while \( P_1(z_+ | z_-) = 0 \). It follows that a sequence of binary tests \((\tilde{\Psi}^n)\), with \( \tilde{\Psi}^n : (\mathcal{S} \times \mathcal{Y})^{\tilde{n}-1} \to \{0, 1\} \), can be designed such that
\[
\lim_{n \to \infty} p_0(n) = 0, \quad p_1(n) = 0, \quad n \in \mathbb{N}. \quad (56)
\]
Such a family of tests is given for instance by allowing \( \tilde{\Psi}^n(z) \) to equal 0 if and only if the \((\tilde{n} - 1)\)-tuple \( z \) contains a symbol \( z_- \) followed by a \( z_+ \).

Once fixed \( \tilde{\Phi}^n, \tilde{\Psi}^n, \Phi^n \) and \( \tilde{\Psi}^n \), the iterative protocol described above defines a variable-length block-coding scheme \((\Phi^n, T_n, \Psi^n)\). As mentioned above the scheme consists of a sequence of epochs, each of fixed length \( n \); in particular we have
\[
T_n = n \tau_n,
\]
where
\[
\tau_n := \inf \left\{ k \in \mathbb{N} : \tilde{\Psi}^n(S_{(k-1)n+\tilde{n}+1}^{kn}, Y_{(k-1)n+\tilde{n}+1}^{kn}) = 0 \right\},
\]
is a positive integers valued random variable describing the number of epochs occurred until transmission halts.

The following result characterizes the asymptotic performances of the family of schemes \((\Phi^n, T_n, \Psi^n)\).

**Proposition 13** For every design rate \( R \) in \((0, C)\), and \( \gamma \) in \((0, C)\), we have
\[
\lim_{n \to \infty} \frac{\log \left| \mathcal{W}_n \right|}{\mathbb{E}_{\Phi^n}[T_n]} = R \quad (57)
\]
and
- if \( D < +\infty \)
\[
\lim_{n \to \infty} \frac{- \log p_e(\Phi^n, T_n, \Psi^n)}{\mathbb{E}_{\Phi^n}[T_n]} = D(1 - \gamma), \quad (58)
\]
• if $D = +\infty$

$$p_e(\Phi^n, T_n, \Psi_n) = 0, \quad n \in \mathbb{N}. \quad (59)$$

**Proof** We introduce the following notation. First, for every $k \in \mathbb{N}$:

- $\hat{e}_k := \{\tilde{\Psi}(S_{(k-1)n+1}^{(k-1)n}, \tilde{Y}_{(k-1)n+1}^{(k-1)n}) \neq W\}$ is the error event of the first transmission phase of the $k$-th epoch;

- $\check{e}_k := \{\tilde{\Psi}(S_{(k-1)n+1}^{(k-1)n+1}, \tilde{Y}_{(k-1)n+1}^{(k-1)n+1}) \neq \check{1}_{e_k}\}$ is the error event of the second transmission phase of the $k$-th epoch;

Clearly we have

$$\mathbb{P}_{\Phi^n}(\hat{e}_k | \mathcal{F}_{(k-1)n}) \leq p(n), \quad \mathbb{P}_{\Phi^n}(\check{e}_k | \mathcal{F}_{(k-1)n+1}) \leq p_{1_{e_k}}(n).$$

The transmission halts the first time a confirmation is detected at the end of the second phase, i.e. the first time either a correct transmission in the first phase is followed by a successful transmission of an acknowledge message in the second phase, or an incorrect transmission in the first phase is followed by a misreceived transmission of a deny message in the second phase. It follows that we can rewrite $\tau_n$ as

$$\tau_n = \inf \{ k \in \mathbb{N} \text{ s.t. } (e_k \cap \check{e}_k) \cup ((e_k)^c \cap (\check{e}_k)^c) \}. \quad (61)$$

We claim that

$$\mathbb{P}_{\Phi^n}(\tau_n \geq k) \leq (p(n) + p_0(n))^{k-1}. \quad (60)$$

Indeed (60) can be shown by induction. It is clearly true for $k = 1$. Suppose it is true for some $k$ in $\mathbb{N}$; then

$$\mathbb{P}_{\Phi^n}(\tau_n \geq k + 1) = \mathbb{P}_{\Phi^n}(\tau_n \geq k + 1 | \tau_n \geq k) \mathbb{P}_{\Phi^n}(\tau_n \geq k)$$

$$= (\mathbb{P}_{\Phi^n}(e_{k+1}) (1 - \mathbb{P}_{\Phi^n}(\hat{e}_{k+1} | e_k)) + (1 - \mathbb{P}_{\Phi^n}(e_{k+1})) \mathbb{P}_{\Phi^n}(\check{e}_{k+1} | (e_k)^c) \mathbb{P}_{\Phi^n}(\tau_n \geq k)$$

$$\leq (p(n) + p_0(n)) \mathbb{P}_{\Phi^n}(\tau_n \geq k)$$

$$\leq (p(n) + p_0(n))^k.$$

Thus $\tau_n$ is stochastically dominated by the sum of a constant 1 plus a r.v. with geometric distribution of parameter $p(n) + p_0(n)$. It follows that its expected value can be bounded as follows

$$1 \leq \mathbb{E}_{\Phi^n}[\tau_n] = \sum_{t \geq 1} \mathbb{P}_{\Phi^n}(\tau_n \geq t) \leq \sum_{t \geq 1} (p(n) + p_0(n))^{t-1} \leq \frac{1}{1 - (p(n) + p_0(n))}.$$

Hence, from (53) and (55) we have

$$\lim_{n \in \mathbb{N}} \mathbb{E}_{\Phi^n}[\tau_n] = 1. \quad (61)$$

From (61) it immediately follows that

$$\lim_{n \in \mathbb{N}} \frac{\log |\mathcal{W}_n|}{\mathbb{E}_{\Phi^n}[T_n]} = \lim_{n \in \mathbb{N}} \frac{\log (\exp([nR]))}{n \mathbb{E}_{\Phi^n}[\tau_n]} = R.$$
Moreover, transmission ends with an error if and only if an error happens in the first transmission phase followed by a type-1 error in the second phase, so that, the error probability of the overall scheme \( (\Phi^n, T_n, \Psi^n) \) can be bounded as follows

\[
p_e(\Phi^n, T_n, \Psi^n) = \mathbb{P}_{\Phi^n}(e_{\tau_n} \cap \hat{e}_{\tau_n}) = \sum_{t \geq 1} \mathbb{P}_{\Phi^n}(e_t \cap \hat{e}_t \cap \{\tau_n = t\}) = \sum_{t \geq 1} \mathbb{P}_{\Phi^n}(e_t \cap \hat{e}_t \cap \{\tau_n \geq t\}) \leq p(n)p_1(n) \sum_{t \geq 1} \mathbb{P}_{\Phi^n}(\tau_n \geq t) \leq \frac{p(n)p_1(n)}{1 - p(n)p_0(n)}.
\]

When \( D \) is infinite, (62) directly implies (59). When \( D \) is finite from (53), (55), (61) and (62) it follows that

\[
\liminf_{n \in \mathbb{N}} \frac{-\log p_e(\Phi^n, T_n, \Psi^n)}{\mathbb{E}_{\Phi^n}[T_n]} = \liminf_{n \in \mathbb{N}} \frac{-\log p_e(\Phi^n, T_n, \Psi^n)}{n \mathbb{E}_{\Phi^n}[\tau_n]} \geq \liminf_{n \in \mathbb{N}} (1 - p(n)p_0(n)) \left( -\log p_1(n) + \frac{1}{n} \log (1 - p(n)p_0(n)) \right) = D (1 - \gamma),
\]

which proves (58).

It is clear that (18) follows from (58) and the arbitrariness of \( \gamma \) in \((\frac{R}{C}, 1)\), so that Part 2 of Theorem 1 is proved.

We end this section with the following observation. It follows from (60) that the probability that the proposed transmission scheme halts after more than one epoch is bounded by \( p(n) + p_0(n) \), a term which is vanishing asymptotically with \( n \). Then, even if the transmission time is variable, it is constant with high probability. As also observed in [17] in the memoryless case, this is a desirable property from a practical point of view.

6 An example

We consider a FSMC as in Fig.4, with state space \( S = \{G, B\} \), input and output spaces \( X = Y = \{0, 1\} \) and stochastic kernel given by:

\[
P_S(s_+, y|s, x) = P_S(s_+, x|s)P_Y(y|x, s), \quad s, s_+ \in S, \ x, y \in \{0, 1\},
\]

\[
P_S(G|0) = \alpha_0 \quad P_S(G|B, 0) = \beta_0 \quad P_S(G|B, 1) = \beta_1,
\]

\[
P_Y(1|0, G) = P_Y(0, G|1) = p_G, \quad P_Y(1|B, 0) = P_Y(0, B|1) = p_B,
\]

where \( 0 < p_G < p_B < \frac{1}{2}, \) and \( \alpha_0, \alpha_1, \beta_0, \beta_1 \in (0, 1) \). For any stationary policy \( \pi : S \rightarrow \mathcal{P}(\{0, 1\}) \), the state invariant measure associated to \( \pi \) can be made explicit:

\[
\mu_\pi(B) = \frac{\alpha_0[\pi(G)](0) + \alpha_1[\pi(G)](1)}{\alpha_0[\pi(G)](0) + \alpha_1[\pi(G)](1) + \beta_0[\pi(B)](0) + \beta_1[\pi(B)](1)}, \quad \mu_\pi(G) = 1 - \mu_\pi(B).
\]
Figure 4: A simple FSMC with binary state space $\mathcal{S} = \{G, B\}$ and binary input/output space $\mathcal{X} = \mathcal{Y} = \{0, 1\}$: notice that the state transition probabilities are allowed to depend on the current input (ISI).

The mutual information costs are given by

$$c(G,u) = H(u(1)\alpha_1 + u(0)\alpha_0) + H(u(1)p_G + u(0)(1-p_G)) - H(p_G) - (u_G H(\alpha_1) + u(0) H(\alpha_0)), \quad c(B,u) = H(u(1)\beta_1 + u(0)\beta_0) + H(u(1)p_B + u(0)(1-p_B)) - H(p_B) - (u(1) H(\beta_1) + u(0) H(\beta_0)),$$

$H$ denoting the binary entropy function. The information divergence costs instead are given by

$$d(G,\delta_{f_0(G)}) = D(p_G\|1-p_G) + D(\alpha_{f_0(G)}\|\alpha_{f_1(G)}), \quad d(B,\delta_{x^G_G}) = D(p_B\|1-p_B) + D(\alpha_{f_0(G)}\|\alpha_{f_1(G)}),$$

where, for $x, y$ in $[0, 1]$, $D(x||y) := x \log \frac{x}{y} + (1-x) \log \frac{1-x}{1-y}$.

In Fig. 5 and Fig. 6 the special case when $p_G = 0.001$, $p_B = 0.1$, $\alpha_0 = 1 - \beta_0 = 0.7$, $\alpha_1 = 1 - \beta_1 = \gamma$, is studied as a function of $\gamma$ in $(0, 1)$. In particular in Fig 5 the capacity and the optimal policy $\pi^* : \{G, B\} \rightarrow \mathcal{P}(\{0, 1\})$ is plotted as a function of $\gamma$ in $(0, 1)$.

In Fig. 5 and Fig. 6 the special case when $p_G = 0.001$, $p_B = 0.1$, $\alpha_0 = 1 - \beta_0 = 0.7$ and $\alpha_1 = 1 - \beta_1 = \gamma$ is studied as a function of the parameter $\gamma$ in $(0, 1)$. In particular in Fig 5 the capacity and the optimal policy $\pi : \mathcal{S} \rightarrow \mathcal{X}$ are plotted as a function of $\gamma$. Notice that for $\gamma = 0.7$ the channel has no ISI and actually coincides with a memoryless Gilbert-Elliot channel: for that value the optimal policy chooses the uniform distribution both in the good state $G$ as well as in the bad state $B$. For values of $\gamma$ below 0.7 (resp. beyond 0.7), instead, the optimal policy puts more mass on the input symbol 1 (resp. the symbol 0) both in state
Figure 6: The thick solid line is a plot of the Burnashev coefficient $D$ (evaluated with natural log base) of the FSMC of Fig.4 for the same values of the parameters as in Fig.5.

$G$ and state $B$, and it is more unbalanced in state $B$. In Fig.6 the Burnashev coefficient of the channel is plotted as a function of the parameter $\gamma$, as well as the the values of the ergodic Kullback-Leibler cost corresponding to the four possible policies $f_0: \{G, B\} \to \{0, 1\}$. Observe as the minimum value of $D$ is achieved for $\gamma = 0.7$; in that case all the four non trivial policies $f_0, f_1$ give the same value of the Kullback-Leibler cost.

Finally it is worth to consider the simple non-ISI case when $\alpha_0 = \alpha_1 = \beta_0 = \beta_1$. In this case the state ergodic measure is the uniform one on $\{G, B\}$. Notice by a basic convexity argument we get that its capacity $C$ and Burnashev coefficient $D$ satisfy

$$C = 1 - \frac{1}{2} \mathcal{H}(p_G) - \frac{1}{2} \mathcal{H}(p_B) > 1 - \mathcal{H}(\frac{1}{2} p_G + \frac{1}{2} p_B) =: \tilde{C},$$

$$D = \frac{1}{2} \mathcal{D}(p_G || 1 - p_G) + \frac{1}{2} \mathcal{D}(p_B || 1 - p_B) \geq \mathcal{D}(\frac{1}{2} p_G + \frac{1}{2} p_B || 1 - \frac{1}{2} p_B - \frac{1}{2} p_G) =: \tilde{D}. \quad (63)$$

In the (63) and (64) $\tilde{C}$ and $\tilde{D}$ correspond respectively to the capacity and the Burnashev coefficient of memoryless binary symmetric channel with crossover probability equal to the ergodic average of the crossover probabilities $p_B$ and $p_G$. Such a channel is introduced in practice when channel interleavers are used in order to apply to FSMCs coding techniques designed for DMCs. While this approach reduces the decoding complexity, it is well known that it reduces the achievable capacity (63) (see [13]). Inequality (64) shows that this approach causes also a loss in the Burnashev coefficient of the channel.

### 7 Conclusions

In this paper we studied the error exponent of FSMCs with feedback. We have proved an exact single-letter characterization of the reliability function for variable-length block-coding schemes with perfect causal output feedback, generalizing the result obtained by Burnashev [7] for memoryless channels. Our assumptions are that the channel state is causally observable both at the encoder and the decoder and the stochatic kernel describing the channel satisfies some mild ergodicity properties.
As a first topic for future research, we would like to extend our result to the case when the state is either observable at the encoder only or it is not observable at neither side. We believe that the techniques used in [27] in order to characterize the capacity of FSMCs with state not observable may be adopted to handle our problem as well. The main idea consists in studying a partially observable Markov decision process and reduce it to a fully observable one with a larger state space. However some technical issues may appear since, in order to deal with average cost problems, we used finiteness of the state space in our proofs in Section 4. Finally, it would be interesting to consider the problem of finding universal schemes which do not require exact knowledge of the channel statistics but use feedback in order to estimate them.

A Proofs for Section 3

For the reader’s convenience all statements are repeated before their proof.

Lemma 2 Given any causal feedback encoder $\Phi$, we have, for every $t$ in $\mathbb{N}$,

$$\hat{P}^\Phi_{MAP}(t) \geq \lambda \hat{P}^\Phi_{MAP}(t-1) \quad \mathbb{P}_\Phi - a.s.$$  

Proof A first observation is that

$$\mathbb{P}_\Phi \left( \bigcap_{x \in \mathcal{X}} \{ P(S_{t+1}, Y_t | S_t, x) = 0 \} \right) = 0, \quad \forall t \in \mathbb{N}.$$  

It follows that, $\mathbb{P}_\Phi$-almost surely, for every $t$ in $\mathbb{N}$

$$P(S_{t+1}, Y_t | S_t, X_t) \geq \lambda S_t \geq \lambda.$$  

Let us fix an arbitrary message $w$ in $\mathcal{W}$. We have

$$\mathbb{P}_\Phi (W = w | G_t) \geq \mathbb{P}_\Phi (W = w | G_t) \mathbb{P}_\Phi (S_{t+1}, Y_t | G_t)$$

$$= \mathbb{P}_\Phi (W = w | G_{t-1}) \mathbb{P}_\Phi (S_{t+1}, Y_t | W = w, G_{t-1})$$

$$= \mathbb{P}_\Phi (W = w | G_{t-1}) P(S_{t+1}, Y_t | S_t, X_t)$$

$$\geq \lambda \mathbb{P}_\Phi (W = w | G_{t-1}).$$

It follows that

$$\hat{P}^\Phi_{MAP}(t) = \mathbb{P}_\Phi \left( \tilde{\Psi}^t_{MAP} \neq W | G_t \right)$$

$$= \sum_{w \in \mathcal{W}} \mathbb{P}_\Phi (W = w | G_t)$$

$$\geq \sum_{w \in \mathcal{W}} \lambda \mathbb{P}_\Phi (W = w | G_{t-1})$$

$$\geq \lambda \hat{P}^\Phi_{MAP}(t-1).$$

Lemma 3 For any variable-length block-coding scheme $(\Phi, T, \Psi)$ and any $0 < \delta < \frac{1}{2}$, we have

$$C_\delta(\Phi, T) \geq \left( 1 - \frac{p_e(\Phi, T, \Psi)}{\delta} \right) \log |\mathcal{W}| - H(\delta).$$
Proof For every \( n \) we introduce a random variable \( \Gamma_n \) describing the conditional message entropy given the information \( \mathcal{G}_n \) available at the encoder at time \( n + 1 \). Consider the real valued random variable \( V_n \) defined by

\[
V_n := \Gamma_n + \sum_{t=1}^{n} c(S_t, \Phi_{s,t}) , \quad n \in \mathbb{Z}^+ ,
\]

We claim that \( (V_n, \mathcal{G}_n)_{n \in \mathbb{Z}^+} \) is a submartingale. Indeed, for every \( n \) in \( \mathbb{Z}^+ \), \( V_n \) is \( \mathcal{G}_n \)-measurable, since \( \Gamma_n \) is, and so do both \( S_t \) and \( \Phi_{s,t} \) for every \( 1 \leq t \leq n \). Moreover we have

\[
E_{\Phi} \left[ \Gamma_{n-1} - \Gamma_n \mid \mathcal{G}_{n-1} \right] = E_{\Phi} \left[ \log \frac{P_{\Phi}(W \mid \mathcal{G}_n)}{P_{\Phi}(W \mid \mathcal{G}_{n-1})} \mid \mathcal{G}_{n-1} \right] \]

\[
= E_{\Phi} \left[ \log \frac{P_{\Phi}(S_{n+1}, Y_n \mid W, \mathcal{G}_{n-1})}{P_{\Phi}(S_{n+1}, Y_n \mid \mathcal{G}_{n-1})} \mid \mathcal{G}_{n-1} \right] \]

\[
\leq E_{\Phi} \left[ \log \frac{P_{\Phi}(S_{n+1}, Y_n \mid X_n, \mathcal{G}_{n-1})}{P_{\Phi}(S_{n+1}, Y_n \mid \mathcal{G}_{n-1})} \mid \mathcal{G}_{n-1} \right] \]

\[
= \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} \sum_{s, x} \Phi_{s,n}(x) P(s_+, y \mid s, x) \log \frac{P(s_+, y \mid s, x)}{\sum_{z \in \mathcal{X}} \Phi_{s,n}(z) P(s_+, y \mid s, z)} \]

\[
= c(\Phi_{s,n}, S_n) ,
\]

the inequality in the formula above following from the data processing inequality once noted that, because of the causality of the encoder and the Markovian structure of the channel,

\[
(W, S^n_1, Y^{n-1}_1) - (X_n, S_n) - (Y_n, S_{n+1})
\]

forms a Markov chain. It follows that

\[
E_{\Phi} \left[ V_n - V_{n-1} \mid \mathcal{G}_{n-1} \right] = E_{\Phi} \left[ \Gamma_n - \Gamma_{n-1} + c(\Phi_{s,t}, S_t) \mid \mathcal{G}_{n-1} \right] \geq 0 .
\]

Moreover, \( (V_n) \) has uniformly bounded increments since

\[
|V_n - V_{n-1}| \leq |c(\Phi_{s,t}, S_t)| + |\Gamma_n - \Gamma_{n-1}| \leq \log |\mathcal{X}| + 2 \log |\mathcal{W}| < +\infty .
\]

Doob’s optional sampling theorem can thus be applied to the submartingale \( (V_n, \mathcal{G}_n)_{n \in \mathbb{Z}^+} \) and the stopping time \( \tau_\delta \), concluding that

\[
\log |\mathcal{W}| = E_{\Phi} [\Gamma_0 \mid \mathcal{G}_0] = E_{\Phi} [V_0 \mid \mathcal{G}_0] \leq E_{\Phi} [V_{\tau_\delta}] = E_{\Phi} [\Gamma_{\tau_\delta}] + E_{\Phi} \left[ \sum_{t=1}^{\tau_\delta} c(S_t, \Phi_{s,t}) \right] . \quad (65)
\]

Finally, combining (65) with (22), we obtain

\[
C_\delta(\Phi, T) = E_{\Phi} \left[ \sum_{t=1}^{\tau_\delta} c(S_t, \Phi_{s,t}) \right] \geq \left( 1 - \frac{P_{\Phi}(\Phi_{s,t}, \Psi)}{\delta} \right) \log |\mathcal{W}| - H(\delta) .
\]

Lemma 4 Let \( \tau \) and \( T \) be stopping times for the filtration \( \mathcal{G} \) such that \( \tau \leq T \), and consider a partition of the message set as in (26). Then

\[
L_i \leq E_{\Phi} \left[ \sum_{t=\tau+1}^{T} d(\Phi_{s,t}, S_t) \mid W \in \mathcal{W}_i, \mathcal{G}_\tau \right] , \quad P_{\Phi} - a.s. , \quad i = 0, 1 .
\]
Proof We will prove the claim for $i = 0$. Define for $t \geq 0$

$$Z_t := \log \frac{\sum_{x \in \mathcal{X}} P(S_{t+1}, Y_t | S_t, x) \Upsilon^0_{\Phi,t}(x)}{\sum_{x \in \mathcal{X}} P(S_{t+1}, Y_t | S_t, x) \Upsilon^1_{\Phi,t}(x)},$$

with the agreement $\log \frac{0}{0} = 0$. We have that

$$|Z_t| \leq 2 \log \frac{1}{\lambda}. \quad (66)$$

Indeed if $P(S_{t+1}, Y_t | S_t, x) = 0$ for every $x$ in $\mathcal{X}$, then $Z_t = 0$ by definition. If instead there exists $x$ in $\mathcal{X}$ such that $P(S_{t+1}, Y_t | S_t, x) > 0$, then

$$|Z_t| = \left| \log \frac{\sum_{x \in \mathcal{X}} P(S_{t+1}, Y_t | S_t, x) \Upsilon^0_{\Phi,t}(x)}{\sum_{x \in \mathcal{X}} P(S_{t+1}, Y_t | S_t, x) \Upsilon^1_{\Phi,t}(x)} \right| \leq 2 \log \left( \inf_{x \in \mathcal{X}} \{P(S_{t+1}, Y_t | S_t, x)\} \right)^{-1} = 2 \log \frac{1}{\lambda_{S_t}} \leq 2 \log \frac{1}{\lambda}$$

It is easy to check by induction that for every $n \geq 0$

$$\log \frac{\mathbb{P}_\Phi(S^{n+1}_1, Y^n_1 | W \in W_0)}{\mathbb{P}_\Phi(S^{n+1}_1, Y^n_1 | W \in W_1)} = \sum_{t=1}^{n} Z_t. \quad (67)$$

Indeed (67) holds true for $n = 0$, since $S_1$ is independent from $W$ (with the agreement for an empty summation to equal zero). Moreover, suppose (67) holds true for some $n$. Then

$$\log \frac{\mathbb{P}_\Phi(S^{n+2}_1, Y^{n+1}_1 | W \in W_0)}{\mathbb{P}_\Phi(S^{n+2}_1, Y^{n+1}_1 | W \in W_1)} = \log \frac{\mathbb{P}_\Phi(S^{n+1}_1, Y^n_1 | W \in W_0) \mathbb{P}_\Phi(S_{n+2}, Y_{n+1} | W \in W_0, \mathcal{E}_{n+1})}{\mathbb{P}_\Phi(S^{n+1}_1, Y^n_1 | W \in W_1) \mathbb{P}_\Phi(S_{n+2}, Y_{n+1} | W \in W_1, \mathcal{E}_{n+1})}$$

$$= \log \frac{\mathbb{P}_\Phi(S^{n+1}_1, Y^n_1 | W \in W_0) \sum_{x \in \mathcal{X}} P(S_{n+2}, Y_{n+1} | S_{n+1}, x) \Upsilon^0_{\Phi,n+1}(x)}{\mathbb{P}_\Phi(S^{n+1}_1, Y^n_1 | W \in W_1) \sum_{x \in \mathcal{X}} P(S_{n+2}, Y_{n+1} | S_{n+1}, x) \Upsilon^1_{\Phi,n+1}(x)}$$

$$= \log \frac{\mathbb{P}_\Phi(S^{n+1}_1, Y^n_1 | W \in W_0) + Z_{n+1}}{\mathbb{P}_\Phi(S^{n+1}_1, Y^n_1 | W \in W_1)} + Z_{n+1} = \sum_{t=1}^{n+1} Z_t.$$

Now, by applying the log-sum inequality and recalling the definition (9) of the cost $d$, we have, for ever $t \geq 1$,

$$\mathbb{E}_\Phi [Z_t | W \in W_0, \mathcal{E}_t] = \mathbb{E}_\Phi \left[ \log \frac{\sum_{x \in \mathcal{X}} P(S_{t+1}, Y_t | S_t, x) \Upsilon^0_{\Phi,t}(x)}{\sum_{x \in \mathcal{X}} P(S_{t+1}, Y_t | S_t, x) \Upsilon^1_{\Phi,t}(x)} \right]_{W \in W_0, \mathcal{E}_t}$$

$$= \sum_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} \left( \sum_{x \in \mathcal{X}} \Upsilon^0_{\Phi,t}(x) P(s, y | S_t, x) \right) \log \frac{\sum_{x \in \mathcal{X}} P(s, y | S_t, x) \Upsilon^0_{\Phi,t}(x)}{\sum_{x \in \mathcal{X}} P(s, y | S_t, x) \Upsilon^1_{\Phi,t}(x)}$$

$$\leq \sum_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} \left( \sum_{x \in \mathcal{X}} \Upsilon^0_{\Phi,t}(x) P(s, y | S_t, x) \right) \log \frac{P(s, y | S_t, x) \Upsilon^0_{\Phi,t}(x)}{P(s, y | S_t, x) \Upsilon^1_{\Phi,t}(x)}$$

$$= d(\Upsilon^0_{\Phi,t}, S_t). \quad (68)$$
From (67) and (68) it follows that, if we define
\[ V_n := \log \frac{\mathbb{P}_\Phi \left( S_{n+1}^0, Y_{n+1}^0 \mid W \in W_0 \right)}{\mathbb{P}_\Phi \left( S_n^0, Y_n^0 \mid W \in W_1 \right)} - \sum_{t=1}^{n} d(S_t, Y_{\Phi,t}^0), \quad n \geq 0, \]
then \((V_n, G_n)_{n \geq 0}\) is a submartingale with respect to the conditioned probability measure \(\mathbb{P}_\Phi \cdot \mid W \in W_0\). Moreover it follows from (65) (recall that we are assuming \(\lambda > 0\) and that this is equivalent to the boundedness of \(K\)) that \((V_n)\) has uniformly bounded increments:
\[ |V_{n+1} - V_n| \leq |Z_n+1| + |d(S_t, Y_{\Phi,n+1}^0)| \leq \log \frac{1}{\lambda} + d_{\text{max}} < +\infty. \]
Thus, since \(\tau \leq T\), Doob’s optional stopping theorem can be applied yielding
\[ \mathbb{E}_\Phi \left[ V_T - V_\tau \mid W \in W_0, G_\tau \right] \leq 0. \] (69)
Then the claim follows from (69), after noticing that
\[ V_T - V_\tau = \log \frac{\mathbb{P}_\Phi \left( S_{T+1}^0, Y_{T+1}^0 \mid W \in W_0, G_\tau \right)}{\mathbb{P}_\Phi \left( S_T^0, Y_T^0 \mid W \in W_1, G_\tau \right)} - \sum_{t=\tau+1}^{T} d(S_t, Y_{\Phi,t}^0), \quad \mathbb{P}_\Phi - \text{a.s.}. \]

**Lemma 5** Let \(\Phi\) be any causal encoder, and \(\tau\) and \(T\) be stopping times for the filtration \(\mathcal{G}\) such that \(\tau \leq T\). Then, for every \(2^{|W|}\)-valued \(\mathcal{G}_\tau\)-measurable r.v. \(W_1\), we have \(\mathbb{P}_\Phi\)-a.s.
\[ \mathbb{E}_\Phi \left[ \sum_{t=\tau+1}^{T} d(S_t, Y_{\Phi,t}^{W \in W_1}) \mid \mathcal{G}_\tau \right] \geq \log \frac{Z}{4} - \log \mathbb{P} \left( \bar{\Psi} \neq 1 \mid W_1 \right) \mathcal{G}_\tau, \]
where
\[ Z := \min \left\{ \mathbb{P}_\Phi \left( W \in W_0 \mid G_\tau \right), \mathbb{P}_\Phi \left( W \in W_1 \mid G_\tau \right) \right\}. \]

**Proof** First we will prove the statement when \(W_1\) is a fixed, non-trivial subset of the message set \(W\). From the log-sum inequality it follows that
\[ L_0 = \mathbb{E}_\Phi \left[ \log \frac{\mathbb{P}_\Phi \left( S_{T+1}^0, Y_{T+1}^0 \mid W \in W_0, G_\tau \right)}{\mathbb{P}_\Phi \left( S_T^0, Y_T^0 \mid W \in W_1, G_\tau \right)} \mid W \in W_0, G_\tau \right] \]
\[ \geq \sum_{i=0,1} \mathbb{P}_\Phi \left( \bar{\Psi} = i \mid W \in W_i, G_\tau \right) \log \frac{\mathbb{P}_\Phi \left( \bar{\Psi} = i \mid W \in W_i, G_\tau \right)}{\mathbb{P}_\Phi \left( \bar{\Psi} = i \mid W \in W_i, G_\tau \right)} \]
\[ \geq -H \left( \mathbb{P}_\Phi \left( \bar{\Psi} = 1 \mid W \in W_0, G_\tau \right) \right) - \mathbb{P}_\Phi \left( \bar{\Psi} = 0 \mid W \in W_0, G_\tau \right) \log \mathbb{P}_\Phi \left( \bar{\Psi} = 1 \mid W \in W_0, G_\tau \right) \]
\[ \geq -\log 2 - \mathbb{P}_\Phi \left( \bar{\Psi} = 0 \mid W \in W_0, G_\tau \right) \log \mathbb{P}_\Phi \left( \bar{\Psi} = 1 \mid W \in W_0, G_\tau \right). \]
We now consider the error probability of \(\bar{\Psi}\) conditioned on the sigma-field \(G_\tau\):
\[ \mathbb{P}_\Phi \left( \bar{\Psi} \neq 1 \mid W_1 \right) \mathcal{G}_\tau = \mathbb{P}(W \in W_0 \mid G_\tau) \mathbb{P}_\Phi(\bar{\Psi} = 1 \mid W \in W_0, G_\tau) \]
\[ + \mathbb{P}(W \in W_1 \mid G_\tau) \mathbb{P}_\Phi(\bar{\Psi} = 0 \mid W \in W_1, G_\tau) \]
\[ \geq \min_{i=0,1} \left\{ \mathbb{P}_\Phi(W \in W_i \mid G_\tau) \right\} \mathbb{P}_\Phi(\bar{\Psi} = 1 \mid W \in W_0, G_\tau) \]
\[ = Z \mathbb{P}_\Phi(\bar{\Psi} = 1 \mid W \in W_0, G_\tau). \]
From Lemma 4 it follows that

\[
\mathbb{E}_\Phi \left[ \sum_{t=\tau+1}^{T} d \left( S_t, Y_{\Phi,t}^0 \right) \left| W \in \mathcal{W}_0, \mathcal{G}_\tau \right. \right] \\
\geq \mathbb{E}_\Phi \left[ \log \frac{\mathbb{P}_\Phi \left( S_{\tau+1}^T, Y_{\tau+1}^T \mid W \in \mathcal{W}_0, \mathcal{G}_\tau \right)}{\mathbb{P}_\Phi \left( S_{\tau+1}^T, Y_{\tau+1}^T \mid W \in \mathcal{W}_1, \mathcal{G}_\tau \right)} \left| W \in \mathcal{W}_0, \mathcal{G}_\tau \right. \right] \\
\geq - \log 2 - \mathbb{P}_\Phi \left( \tilde{\Psi} = 0 \mid W \in \mathcal{W}_0, \mathcal{G}_\tau \right) \log \mathbb{P}_\Phi \left( \tilde{\Psi} = 1 \mid W \in \mathcal{W}_0, \mathcal{G}_\tau \right) \\
\geq - \log 2 - \mathbb{P}_\Phi \left( \tilde{\Psi} = 0 \mid W \in \mathcal{W}_0, \mathcal{G}_\tau \right) \log \left( \frac{1}{Z} \mathbb{P}_\Phi \left( \tilde{\Psi} \neq 1 \mid W_1(W) \mid \mathcal{G}_\tau \right) \right). \tag{70}
\]

An analogous derivation leads to

\[
\mathbb{E}_\Phi \left[ \sum_{t=\tau+1}^{T} d \left( S_t, Y_{\Phi,t}^1 \right) \left| W \in \mathcal{W}_1, \mathcal{G}_\tau \right. \right] \\
\geq - \log 2 - \mathbb{P}_\Phi \left( \tilde{\Psi} = 1 \mid W \in \mathcal{W}_1, \mathcal{G}_\tau \right) \log \left( \frac{1}{Z} \mathbb{P}_\Phi \left( \tilde{\Psi} \neq 1 \mid W_1(W) \mid \mathcal{G}_\tau \right) \right). \tag{71}
\]

If we now average (70) and (71) with respect to the posterior distribution of \( W \) given \( \mathcal{G}_\tau \), we obtain (27). Finally, since the claim holds true for every choice of \( \mathcal{W}_0 \) in \( 2^\mathcal{W} \setminus \{ \emptyset, \mathcal{W} \} \), then it continues to hold true also when \( \mathcal{W}_0 \) is a \( 2^\mathcal{W} \setminus \{ \emptyset, \mathcal{W} \} \)-valued \( \mathcal{G}_\tau \)-measurable random variable.

**Lemma 6** Let \( \Phi \) be a causal feedback encoder and \( T \) a transmission time for \( \Phi \). Then, for every \( 0 < \delta < 1/2 \) there exists a \( \mathcal{G}_{\tau_3} \)-measurable random subset \( \mathcal{W}_1 \) of the message set \( \mathcal{W} \), whose a posteriori error probabilities satisfy

\[ 1 - \lambda \delta \geq \mathbb{P} \left( W \in \mathcal{W}_1 \mid \mathcal{G}_{\tau_3} \right) \geq \lambda \delta, \quad i = 0, 1. \]

**Proof** Suppose first that \( \hat{P}_{MAP}(\tau_3) \leq \delta \). Then, since clearly \( \hat{P}_{MAP}(\tau_3 - 1) \geq \delta \), by Lemma 2 we have

\[ \hat{P}_{MAP}(\tau_3) \geq \lambda \hat{P}_{MAP}(\tau_3 - 1) \geq \lambda \delta \]

It follows that if we define \( \mathcal{W}_1 := \{ \Psi_{MAP}(\tau_3) \} \), we have

\[ \mathbb{P}_\Phi(W \in \mathcal{W}_1 \mid \mathcal{G}_{\tau_3}) = 1 - P_{MAP}(\tau_3) \geq 1 - \delta \geq \lambda \delta, \quad \mathbb{P}_\Phi(W \notin \mathcal{W}_1 \mid \mathcal{G}_{\tau_3}) = P_{MAP}(\tau_3) \geq \lambda \delta. \]

If instead \( \hat{P}_{MAP}(\tau_3) > \delta \), the a posteriori probability of any message \( w \) in \( \mathcal{W} \) at time \( \tau_3 \) satisfies \( \mathbb{P}_\Phi(W = w \mid \mathcal{G}_{\tau_3}) \leq 1 - \delta \). Then it is possible to construct \( \mathcal{W}_1 \) in the following way. Introduce an arbitrary labelling of \( \mathcal{W} = \{ w_1, w_2, \ldots, w_{|\mathcal{W}|} \} \). For any \( 1 \leq i \leq |\mathcal{W}| \), define \( \mathcal{W}_{(i)} = \{ w_1, \ldots, w_i \} \). Set \( k := \inf \{ 1 \leq i \leq |\mathcal{W}| : \mathbb{P}_\Phi \left( W \in \mathcal{W}_{(i)} \mid \mathcal{G}_t \right) \geq \lambda \delta \} \), and define \( \mathcal{W}_1 = \mathcal{W}_{(k)} \). Then clearly \( \mathbb{P}_\Phi(W \in \mathcal{W}_1 \mid \mathcal{G}_t) \geq \lambda \delta \), while

\[
\mathbb{P}_\Phi(W \notin \mathcal{W}_1 \mid \mathcal{G}_t) = 1 - \mathbb{P}_\Phi(W \in \mathcal{W}_{(k)} \mid \mathcal{G}_t) \\
= 1 - \mathbb{P}_\Phi(W \in \mathcal{W}_{(k-1)} \mid \mathcal{G}_t) - \mathbb{P}_\Phi(W = w_k \mid \mathcal{G}_t) \\
\geq 1 - \lambda \delta - (1 - \delta) \geq \lambda \delta.
\]
B Proofs for Section 4

Lemma 8 For every \( \varepsilon > 0 \), and for every feasible policy \( \pi \)
\[ \mathbb{P}_\pi \left( \| F(\nu_n) \| \geq \varepsilon + \frac{1}{n} \right) \leq 2|\mathcal{S}| \exp \left( -\frac{n\varepsilon^2}{2} \right). \]

Proof Let us fix an arbitrary admissible policy \( \pi \) in \( \Pi \). For every \( s \) in \( \mathcal{S} \) consider the following random process:
\[ Z_0^s := 0 , \quad Z_1^s := 0 , \quad Z_n^s := (n-1)F_s(\nu_{n-1}) + \mathbb{1}_{\{s_n=s\}} - \mathbb{1}_{\{s_1=s\}} , \quad n \geq 2 . \]
We have
\[ Z_n^s = (n-1)F_s(\nu_{n-1}) + \mathbb{1}_{\{s_n=s\}} - \mathbb{1}_{\{s_1=s\}} \]
\[ = (n-1)\nu_{n-1}(\{s\}, U) + \mathbb{1}_{\{s_n=s\}} - \mathbb{1}_{\{s_1=s\}} - (n-1) \int_{\mathcal{S} \times \mathcal{U}} Q_s(s \mid j, u) d\nu_{n-1}(j, u) \]
\[ = \sum_{t=2}^{n} \mathbb{1}_{\{s_t=s\}} - \sum_{t=2}^{n} Q(s \mid S_{t-1}, U_{t-1}) \]
\[ = \sum_{t=2}^{n} \left( \mathbb{1}_{\{s_t=s\}} - \mathbb{E}_\pi \left[ \mathbb{1}_{\{s_t=s\}} | \mathcal{E}_{t-1} \right] \right) . \]
It is immediate to check that \( Z_n^s \) is \( \mathcal{E}_n \)-measurable. Moreover
\[ \mathbb{E}_\pi [Z_{n+1}^s | \mathcal{E}_n] = Z_n^s , \forall n \geq 0 , \]
so that \( (Z_n^s, \mathcal{E}_n, \mathbb{P}_\pi)_{n \geq 0} \) is a martingale. Moreover, \( (Z_n^s) \) has uniformly bounded increments since \( |Z_1^s - Z_0^s| \leq a_1 := 0 , \) while
\[ |Z_{n+1}^s - Z_n^s| = |\mathbb{1}_{\{s_{n+1}=s\}} - \mathbb{E}_\pi \left[ \mathbb{1}_{\{s_{n+1}=s\}} | \mathcal{E}_n \right] | \leq a_{n+1} := 1 , \quad n \geq 1 . \]
It follows that we can apply Hoeffding-Azuma inequality [19], obtaining
\[ \mathbb{P}_\pi (|Z_{n+1}^s| \geq \varepsilon n) \leq 2 \exp \left( -\frac{\varepsilon^2 n^2}{2 \sum_{k=1}^{m+1} a_k} \right) = 2 \exp \left( -\frac{\varepsilon^2 n}{2} \right) . \]
By simply applying a union bound, we can conclude that
\[ \mathbb{P}_\pi (\| F(\nu_n) \| \geq \varepsilon + \frac{1}{n}) = \mathbb{P}_\pi \left( \max_{s \in \mathcal{S}} |Z_{n+1}^s + \mathbb{1}_{\{s_1=s\}} - \mathbb{1}_{\{s_{n+1}=s\}}| \geq \varepsilon n + 1 \right) \]
\[ \leq \mathbb{P}_\pi \left( \bigcup_{s \in \mathcal{S}} \{ |Z_{n+1}^s| \geq \varepsilon n \} \right) \]
\[ \leq \sum_{s \in \mathcal{S}} \mathbb{P}_\pi (|Z_{n+1}^s| \geq \varepsilon n) \leq 2|\mathcal{S}| \exp \left( -\frac{\varepsilon^2 n}{2} \right) . \]

Lemma 9 The map \( \gamma \) is upper semicontinuous. (i.e. \( x_n \to x \Rightarrow \limsup_n \gamma(x_n) \leq \gamma(x) \))

Proof Possibly up to a subsequence, with no loss of generality we can assume that
\[ \gamma(x_n) \to \limsup_n \gamma(x_n) . \]
Since $S \times U$ is compact, the Prohorov space $P(S \times U)$ is compact as well \[3\]. Thus, since the map $\eta \mapsto ||F(\eta)||$ is continuous, the sublevel $\{||F(\eta)|| \leq x\}$ is compact. It follows that for every $n$ there exists $\eta_n$ in $P(S \times U)$ such that

$$\gamma(x_n) = \sup \{ \langle \eta, g \rangle | \eta \in P(S \times U) : ||F(\eta)|| \leq x_n \} = \langle \eta_n, g \rangle , \quad ||F(\eta_n)|| \leq x_n .$$

Since $P(S \times U)$ is compact we can extract a converging subsequence $(\eta_{n_k})$; define $\overline{\eta} := \lim_k \eta_{n_k}$. Clearly

$$||F(\overline{\eta})|| = \lim_k ||F(\eta_{n_k})|| \leq x .$$

It follows that

$$\gamma(x) = \sup \{ \langle \eta, g \rangle | \eta \in P(S \times U) : ||F(\eta)|| \leq x \} \geq \langle \overline{\eta}, g \rangle = \lim_k \langle \eta_{n_k}, g \rangle = \limsup_n \gamma(x_n) .$$

Lemma 10 Let $(\tau_k)$ be a sequence of stopping times for the filtration $\mathcal{F}$ and $(\pi^k)$ be a sequence of feasible policies such that $E_{\pi^k}[\tau_k] < \infty$ for every $k$ and \[4\] holds true. Then

$$\lim_{k \in \mathbb{N}} P_{\pi^k} (G_{\tau_k}^k > \gamma(\varepsilon)) = 0 , \quad \forall \varepsilon > 0 .$$

Proof For every $m$ in $\mathbb{Z}_+$ such that $P_{\pi^k} (\tau_k \geq m) > 0$ we have

$$P_{\pi^k} (G_{\tau_k}^k > \gamma(\varepsilon) \mid \tau_k \geq m) = \sum_{t \geq m} P_{\pi^k} (\tau_k = t) \cdot P_{\pi^k} (G_{\tau_k}^k > \gamma(\varepsilon) \mid \tau_k = t) \leq \sum_{t \geq m} P_{\pi^k} (\tau_k = t) \cdot P_{\pi^k} (G_{\tau_k}^k > \gamma(\varepsilon) \mid \tau_k = t) \leq \sum_{t \geq 0} P_{\pi^k} (\tau_k = t) \cdot P_{\pi^k} (G_{\tau_k}^k > \gamma(\varepsilon) \mid \tau_k = t) = P_{\pi^k} (G_{\tau_k}^k > \gamma(\varepsilon)) .$$

An application of the Bayes rule thus gives us

$$P_{\pi^k} (\tau_k \geq m) \geq P_{\pi^k} (\tau_k \geq m \mid G_{\tau_k}^k > \gamma(\varepsilon)) , \quad \forall k \text{ s.t. } P_{\pi^k} (G_{\tau_k}^k > \gamma(\varepsilon)) > 0 ,$$

which in turns implies

$$E_{\pi^k} [\tau_k] = \sum_{m \geq 0} P_{\pi^k} (\tau_k \geq m) \geq \sum_{m \geq 0} P_{\pi^k} (\tau_k \geq m \mid G_{\tau_k}^k > \gamma(\varepsilon)) = E_{\pi^k} [\tau_k \mid G_{\tau_k}^k > \gamma(\varepsilon)] . \quad (72)$$

On the other hand, for every $\varepsilon > 0$, using a union bound estimation and \[45\] we get,

$$P_{\pi^k} (G_{\tau_k}^k > \gamma(\varepsilon + \frac{1}{n})) = P_{\pi^k} \left( \bigcup_{t \geq n} \{ \langle \theta_t, c \rangle > \gamma(\varepsilon + \frac{1}{n}) \} \right) \leq \sum_{t \geq n} P_{\pi^k} (\langle \theta_t, c \rangle > \gamma(\varepsilon + \frac{1}{n})) \leq 2 |S| \sum_{t \geq n} \exp (-t \varepsilon^2 / 2) \leq 2 |S| \exp \left( -n \varepsilon^2 / 2 \right) \frac{1 - \exp (-\varepsilon^2 / 2)}{(1 - \exp (-\varepsilon^2 / 2))} \quad (73)$$
It follows that for every $M$ in $\mathbb{N}$ we have
\[
\mathbb{P}_{\pi^k}(G^k_{\tau_k} > \gamma (\varepsilon + \frac{1}{M})) = \mathbb{P}_{\pi^k}(\{G^k_{\tau_k} > \gamma (\varepsilon + \frac{1}{M})\} \cap \{\tau_k \geq M\}) + \mathbb{P}_{\pi^k}(\{G^k_{\tau_k} > \gamma (\varepsilon + \frac{1}{M})\} \cap \{\tau_k < M\})
\leq \sum_{t \geq M} \mathbb{P}_{\pi^k}(\{G^k_{\tau_k} > \gamma (\varepsilon + \frac{1}{M})\} \cap \{\tau_k = t\}) + \mathbb{P}_{\pi^k}(\tau_k < M)
\leq \sum_{t \geq M} \mathbb{P}_{\pi^k}(G^k_t > \gamma (\varepsilon + \frac{1}{M})) + \mathbb{P}_{\pi^k}(\tau_k < M)
\leq \sum_{t \geq M} \frac{2|S|}{(1 - \exp(-\varepsilon^2/2))^2} \exp(-M\varepsilon^2/2) + \mathbb{P}_{\pi^k}(\tau_k < M)
= \frac{2|S|}{(1 - \exp(-\varepsilon^2/2))^2} \exp(-M\varepsilon^2/2) + \mathbb{P}_{\pi^k}(\tau_k < M),
\]
so that it follows from (46)
\[
\limsup_{k \in \mathbb{N}} \mathbb{P}_{\pi^k}(G^k_{\tau_k} > \gamma (\varepsilon + \frac{1}{M})) \leq \frac{2|S|}{(1 - \exp(-\varepsilon^2/2))^2} \exp(-M\varepsilon^2/2) + \limsup_{k \in \mathbb{N}} \mathbb{P}_{\pi^k}(\tau_k \leq M)
\leq \frac{2|S|}{(1 - \exp(-\varepsilon^2/2|S|^2))^2} \exp(-M\varepsilon^2/2),
\]
and by the arbitrariness of $M$ in $\mathbb{N}$ we get the claim. \hfill \blacksquare

Lemma 12 In the previous setting, for every fixed $M$ in $\mathbb{N}$, we have
\[
\lim_{k \in \mathbb{N}} \mathbb{P}_{\Phi^k}(\tau_k \leq M) = 0, \quad \lim_{k \in \mathbb{N}} \mathbb{P}_{\Phi^k}(T_k - \tau_k \leq M) = 0.
\]

Proof From Lemma 2 we have
\[
\mathbb{P}_{\Phi^k} \left( T_k \geq \mathbb{P}_{\Phi^k}(T_k) \lambda^{T_k - \tau_k} \geq \lambda \delta \lambda^{T_k - \tau_k}. \right.
\]
This implies that, for every $M$ in $\mathbb{N}$,
\[
p_c \left( \Phi^k, T_k, \Psi^k \right) = \mathbb{E}_{\Phi^k} \left[ \mathbb{P}_{\Phi^k}(T_k) \right] \geq \mathbb{E}_{\Phi^k} \left[ \mathbb{P}_{\Phi^k}(T_k | T_k - \tau_k \leq M) \right] \mathbb{P}_{\Phi^k}(T_k - \tau_k \leq M)
\geq \lambda \delta \lambda^M \mathbb{P}_{\Phi^k}(T_k - \tau_k \leq M).
\]
It follows that
\[
\mathbb{P}_{\Phi^k}(T_k - \tau_k \leq M) \leq \lambda^{M-1} p_c \left( \Phi^k, T_k, \Psi^k \right) \frac{k \to \infty}{\delta_k} 0.
\]
In order to show the first part of the claim, suppose that $\mathbb{P}_{\Phi^k}(T_k) \leq \delta_k$. Then
\[
\frac{|W_k| - 1}{|W_k|} \lambda^{\tau_k} \leq \mathbb{P}_{\Phi^k}(T_k) \leq \delta_k.
\]
It follows that we have
\[
\mathbb{P}_{\Phi^k}(\tau_k \leq M) \leq \mathbb{P}_{\Phi^k}(\{\tau_k \leq M\} \cap \{\mathbb{P}_{\Phi^k}(T_k) \leq \delta_k\}) + \mathbb{P}_{\Phi^k}(\mathbb{P}_{\Phi^k}(T_k) > \delta_k)
\leq \mathbb{P}_{\Phi^k}(\frac{|W_k| - 1}{|W_k|} \lambda^M \leq \delta_k) + \mathbb{P}_{\Phi^k}(\tau_k = T_k) \frac{k \to \infty}{0}.
\]

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