Existence and Stability for a Nonlinear Coupled $p$-Laplacian System of Fractional Differential Equations

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1. Introduction

The theoretical development of fractional calculus and its applications is more important to model nonlinear complex problems with the arbitrary fractional order. The subject of fractional differential equations (FDEs) has become an important area in real life because of their ability to model a lot of physical phenomena associated with rapid and concise changes with their significance in science and engineering through the past three decades, such as chemistry, physics, biology, engineering, visco-elasticity, electrotechnical, signal processing, electrochemistry, and controllability (see the details, [1–9], and the reference therein). In the near time, the nonlinear fractional partial differential equations are the most applied research area in which most authors and scientists are focused for their investigation. In this case, the Caputo derivative plays a great role to analyze the specific application of nonlinear PDEs. In [10], the authors have studied the cancer treatment model based on Caputo–Fabrizio fractional derivative. After integrating the model into the Caputo–Fabrizio fractional derivative, they have analyzed the existence of the solution as well. The Caputo–Fabrizio fractional derivative is implemented in [11] for the modeling and characterizing of the alcoholism. By applying the fixed-point theorem, they have studied the existence and uniqueness of the alcoholism model. The spread of the SIQR model is investigated by [12] using the Caputo derivative. They have justified the stability and uniqueness of the nonvirus equilibrium and virus equilibrium point.

For this problem, different authors proposed different numerical solution techniques. The analysis with the non-linear time-fractional HIV/AIDS transmission model is considered in [13], in which the numerical solution is found using the fractional variational iteration method with convergence analysis. The nonlinear garden equation is studied in [14] based on the Atangana–Baleau Caputo derivative. He has highlighted the fixed-point theorem for proving the existence and uniqueness of the garden equation.

One of the main difficulties for the solution of the nonlinear fractional PDEs is to analyze the existence theory of solutions. Sufficient conditions for the existence and
uniqueness of solutions (EUS) have been obtained by using different nonlinear analysis techniques and fixed-point theorems (for more details, read [15–18]). Also, the boundary value problems with various boundary conditions for many ordinary differential equations are studied [19–23]. However, the theory of boundary value problems for nonlinear FDEs is still not discussed more, and many problems of this theory require to be explored. On the contrary, the investigation of coupled systems of the differential equations is also significant because systems of this kind appear in various applied nature problems (refer [24–28]).

The topological degree theory is a useful tool in nonlinear analysis with numerous applications to operatorial equations, optimization theory, fractal theory, and other topics. We will see the following consideration of topological degree theory with boundary conditions based on the Caputo fractional derivative by different authors. Isaila [29] applied the topological degree theory to establish sufficient conditions for the existence of a solution for the following nonlinear integral equations:

\[
\pi(w) = \mathcal{L}(w, \pi(w)) + \int_a^b \mathcal{Q}(w, \pi(\mathbf{3}))d\mathbf{3}, \quad w \in [a, b],
\]

where \( \mathcal{L} : [a, b] \times IR \rightarrow IR \) and \( \mathcal{Q} : [a, b] \times [a, b] \times IR \rightarrow IR \) are continuous functions. In their study [30], Wang et al. used the topological degree method to obtain some existence conditions of the solution for the following nonlocal Cauchy problem:

\[
\begin{align*}
&cD^\alpha w = \mathcal{L}(w, \pi(w)), \quad 0 \leq w \leq W, \\
&\pi(0) + h(\pi) = \pi_0,
\end{align*}
\]

where \( cD^\alpha \) denotes the Caputo fractional derivative with order \( \alpha \in (0, 1) \) and \( \mathcal{L} : C([0, W], IR) \rightarrow IR \) and \( \pi_0 \in IR \) are continuous. The nonlocal term \( h : C([0, W], IR) \rightarrow IR \) is a given function. Proceeding on the same fashion, Shah and Khan [31] proved the EUS for a coupled system under the fractional derivatives by using the technique of degree theory given as follows:

\[
\begin{align*}
&cD^\alpha \pi_1(w) = \mathcal{L}_1(w, \pi_1(w), \pi_2(w)), \quad w \in [0, 1], \\
&cD^\beta \pi_2(w) = \mathcal{L}_2(w, \pi_1(w), \pi_2(w)), \quad w \in [0, 1], \\
&a_1\pi_1(0) - \beta_1\pi_1(\theta) - \delta_1\pi_1(1) = \phi_1(\pi_1), \\
&a_2\pi_2(0) - \beta_2\pi_2(\theta) - \delta_2\pi_2(1) = \phi_2(\pi_2),
\end{align*}
\]

where \( a_1, a_2, \beta_1, \beta_2, \delta_1, \delta_2, \phi_1, \phi_2 : C((0, 1], IR) \rightarrow IR \) are continuous. Khan et al. [32] used the above-mentioned method to study the following coupled system in the sense of Caputo derivatives with \( p \)-Laplacian:

\[
\begin{align*}
&cD^\alpha \phi_p(D^{\rho_1}_{\theta_1} \pi_1(\omega)) + \mathcal{L}_1(w, \pi_1(w)), \\
&cD^\beta \phi_p(D^{\rho_2}_{\theta_2} \pi_2(\omega)) + \mathcal{L}_2(w, \pi_1(w)), \\
&\rho_1 \pi_1(0) = \phi_p(D^{\rho_1}_{\theta_1} \pi_1(\omega))\big|_{\omega=0} = \phi_p(D^{\rho_1}_{\theta_1} \pi_1(\omega))\big|_{\omega=\eta_1}, \\
&\pi_1(1) = \frac{\Gamma(2 - \delta_1)}{\eta_1^{\delta_1 - 1}} \phi_p\left(\frac{\rho_1}{\theta_1}, \mathcal{L}_1(w, \pi_1(w))\right)\big|_{\omega=0}, \\
&\rho_2 \pi_2(0) = \phi_p(D^{\rho_2}_{\theta_2} \pi_2(\omega))\big|_{\omega=0} = \phi_p(D^{\rho_2}_{\theta_2} \pi_2(\omega))\big|_{\omega=\eta_2}, \\
&\pi_2(1) = \frac{\Gamma(2 - \delta_2)}{\eta_2^{\delta_2 - 1}} \phi_p\left(\frac{\rho_2}{\theta_2}, \mathcal{L}_1(w, \pi_1(w))\right)\big|_{\omega=0},
\end{align*}
\]

where \( \rho_i, \theta_i \in (1, 2] \) and \( \delta_i, \eta_i \in (0, 1) \), for \( i = 1 \) and \( 2 \). The study of positive solutions to boundary value problems for fractional-order differential equations using the topological degree theory technique is rarely available in the literature, so this research field needs further elaboration. Most papers that dealt the topological degree theory with fractional orders belong to \( (0, 1) \) or \( (1, 2] \). For the uniqueness and existence analysis of nonlinear fractional differential equations, the case only Caputo fractional derivative is used frequently.

Thus, our motivation to this study is developing a sufficient condition for the coupled nonlinear fractional derivative that is based on both Caputo and Riemann–Liouville derivatives. The fractional order in our study is expanded to
where \( q, \theta, \beta, \beta_1 \in (n - 1, n], n \geq 3, 1 < \lambda, \sigma \leq 2 \). \( ^{c}D^{\rho} \) and \( ^{c}D^{\beta} \) denote the Caputo fractional derivatives, \( ^{D}D^{\rho} \) and \( ^{D}D^{\beta} \) are the Riemann–Liouville fractional derivatives, and \( \mathcal{G}, \mathcal{Z}, [0, 1] \times IR \rightarrow IR \) are nonlinear functions, and the boundary functions \( \phi_\alpha \) and \( \psi_{\alpha} \) are defined on \( L[0,1] \). \( \phi_p \) represents the \( p \)-Laplacian operator such that \( \phi_p(v) = |v|^{p-2}v \), and \( \phi^\sigma\) denotes the inverse of \( p \)-Laplacian, where \( \frac{1}{\phi_p} + \frac{1}{\phi^\sigma} = 1 \). Since it is difficult to find the exact solution of the nonlinear differential equations, stability and uniqueness have played a great role to get the approximate solution for the given nonlinear problems. Therefore, scientists and researchers have given attention to study the various forms of stability to the nonlinear problems in the sense of Ulam and their multiple types in the last few decades. We observe that the concept of Hyers–Ulam stability is fundamental in realistic problems, such as numerical analysis, biology, and economics (see \([33–38]\)).

The remaining part of this manuscript is structured as follows. In Section 2, we have introduced some basic definitions and lemmas that we need to prove our main results. By using the topological degree theory, the results of existence and uniqueness for the solutions are obtained in Section 3. In Section 4, we investigate the stability of Hyers–Ulam to our proposed coupled system. The theoretical results are demonstrated by providing an example in Section 5, and finally, we have drawn the conclusion in Section 6.

### 2. Preliminaries

In this section, we introduce some basic notions, definitions, and important lemmas which are used in this article. Let \( \Omega = C([0, 1], IR) \) be a Banach space for all continuous functions \( \pi: [0, 1] \rightarrow IR \) with the norm \( ||\pi|| = sup ||\pi(\omega)||: 0 \leq \omega \leq 1 \).

Further, \( \Omega = \pi \times \Omega \) is also a Banach space under the norms \( ||(\pi_1, \pi_2)|| = ||\pi_1|| + ||\pi_2|| \) and \( ||(\pi_1, \pi_2)|| = \max (||\pi_1||, ||\pi_2||) \). The family of each bounded set of \( P(\Omega) \) symbolized by \( B \).

**Definition 1.** For \( \mathcal{X}(w): (0, +\infty) \rightarrow R \), the Caputo fractional derivative of noninteger order \( \rho > 0 \) is known by

\[
^{c}D^{\rho}\mathcal{X}(w) = \frac{1}{\Gamma(n-\rho)} \int_{0}^{w} (w-\xi)^{n-\rho-1} \mathcal{X}^{(n)}(\xi) d\xi,
\]

where \( n-1 < \rho < n \), the integral on the right side is pointwise defined on \((0, \infty)\), and \( \mathcal{X}(w) \) is a continuous function.

**Definition 2.** For \( \mathcal{X}(w): (0, +\infty) \rightarrow R \), the Riemann–Liouville fractional derivative of noninteger order \( \rho > 0 \) is known by

\[
^{R}D^{\rho}\mathcal{X}(w) = \frac{1}{\Gamma(n-\rho)} \left( \frac{d}{dw} \right)^{n-\rho} \int_{0}^{w} (w-\xi)^{n-\rho-1} \mathcal{X}(\xi) d\xi,
\]

where \( n-1 < \rho < n \), the integral on the right side is pointwise defined on \((0, \infty)\), and \( \mathcal{X}(w) \) is a continuous function.

**Definition 3.** For \( \mathcal{X}(w): (0, +\infty) \rightarrow R \), the Riemann–Liouville fractional integral of order \( \rho > 0 \) is defined by

\[
^{I}_{0}D^{-\rho}\mathcal{X}(w) = \frac{1}{\Gamma(\rho)} \int_{0}^{w} (w-\xi)^{\rho-1} \mathcal{X}(\xi) d\xi,
\]

where the integral on the right side is pointwise defined on \((0, +\infty)\) and \( \Gamma(\rho) \) indicates the Gamma function defined as

\[
\Gamma(\rho) = \int_{0}^{\infty} e^{-\xi} \xi^{\rho-1} d\xi.
\]

**Lemma 1** (see \([39]\)). Let \( \rho > 0 \) and \( \mathcal{X} \in C((0,1) \cap L^{1}(0,1) \cdot \mathcal{X}(w) = \pi(\omega) + c_0 + c_1 w + c_2 w^2 + \cdots + c_{n-1} w^{n-1} \),

for \( n = 0, 1, 2, \ldots, n - 1 \).

**Lemma 2** (see \([2, 8]\)). Let \( \theta \in (n - 1, n] , \mathcal{X} \in C^{\alpha-1}, \) and \( ^{c}D^{\theta} \) is the fractional derivative for Caputo, then

\[
^{I}D^{\rho}\mathcal{X}(w) = \mathcal{X}(w) + a_1 + a_2 w + a_3 w^2 + \cdots + a_n w^{n-1},
\]

for \( a_i \in \mathbb{R} \) and \( i = 1, 2, 3, 4, \ldots, n \).

**Lemma 3** (see \([2, 8]\)). Let \( \rho \in (n - 1, n] , \mathcal{X} \in C^{\alpha-1}, \) and \( ^{R}D^{\rho} \) is the fractional derivative for Riemann–Liouville, then
Lemma 4 (see [22]). For \( q, \varepsilon > 0 \), the following relations are satisfying:

\[
\begin{align*}
D^\varepsilon g^p &= \frac{\Gamma(\varepsilon + 1)}{\Gamma(1 + \varepsilon - \theta)} g^{p-\theta}, \\
I^\theta g^p &= \frac{\Gamma(\varepsilon + 1)}{\Gamma(1 + \varepsilon + \theta)} g^{p+\theta}.
\end{align*}
\]

Proposition 2 (see [31]). Let \( \mathcal{F}, \mathcal{G} : \Psi \rightarrow \Pi \) be \( \theta \)-Lipschitz operators with constants \( \eta_1 \) and \( \eta_2 \), respectively, then \( \mathcal{F} + \mathcal{G} : \Psi \rightarrow \Pi \) is \( \theta \)-Lipschitz with constants \( \eta_1 + \eta_2 \).

Lemma 5 (see [39]). Let \( \phi_p \) be a nonlinear \( p \)-Laplacian operator.

1. If \( 1 < p \leq 2, j_1, j_2 > 0 \), and \( |j_1|, |j_2| \geq \rho > 0 \), then

\[
|\phi_p(j_1) - \phi_p(j_2)| \leq (p - 1)\rho^{p-2}|j_1 - j_2|.
\]

2. If \( p > 2 \) and \( |j_1|, |j_2| \leq \rho^* \), then

\[
|\phi_p(j_1) - \phi_p(j_2)| \leq (p - 1)\rho^{*-p}|j_1 - j_2|.
\]

Theorem 1 (see [29]). Let \( \mathcal{F} : \Omega \rightarrow \Omega \) be a \( \theta \)-contraction, and \( \Xi = \{ \omega \in \Omega : \exists 0 \leq \rho \leq 1 \} \). Then, for \( \theta \in \mathbb{R} \), \( \mathcal{F} \) is a \( \theta \)-Lipschitz operator with constant \( \eta \).

Thus, \( \mathcal{F} \) has at least one fixed point, and the set of the fixed points of \( \mathcal{F} \) lies in \( \mathcal{B}_{\theta}(0) \).

The above theorem that we mentioned plays a substantial role in obtaining our main results.

3. Main Results

In the current section, we establish some appropriate conditions for proposed coupled system (5).

Theorem 2. Let \( \mathcal{L} : [0, 1] \rightarrow \mathbb{R} \) be a \( q \) times’ integrable function. Then, for \( q \in (3, n] \) and positive integer \( n \geq 4 \), the solution of the boundary value problem is as follows:

\[
\begin{align*}
\epsilon D^\theta \left( \phi_p \left( \mathcal{R}^\theta D^{\theta j} \pi_1 (w) \right) \right) &= \mathcal{L}_j (w, \pi_2 (w)), \quad w \in [0, 1], \\
\left( \phi_p \left( \mathcal{R}^\theta D^{\theta j} \pi_1 (w) \right) \right)_{|w=0}^{i} &= 0, \quad i = 0, 1, 2, \ldots, n - 1, \\
I^{k-\theta} \pi_1 (w)_{|w=0}^{k} &= 0, \quad k = 1, 2, 4, \ldots, n, \\
D^j \pi_1 (w)_{|w=1}^{j} &= \frac{1}{\Gamma (j)} \int_{0}^{1} (W - 3)^{j-1} \varphi (\pi_1 (3)) d\mathfrak{S},
\end{align*}
\]

(16)

is given by

\[
\begin{align*}
\pi_1 (w) &= \frac{1}{\Gamma (q_1 - 2 - l)} \int_{0}^{1} (W - 3)^{l-1} \varphi (\pi_1 (3)) d\mathfrak{S} \\
&+ \int_{0}^{1} \mathcal{G}^\vartheta (w, \mathfrak{S}) \varphi \left( \frac{1}{\Gamma (\theta)} \int_{0}^{1} (3 - r)^{\theta-1} \mathcal{L}_j (r, \pi_2 (r)) d\mathfrak{S} \right) d\mathfrak{S},
\end{align*}
\]

(17)

where \( \mathcal{G}^\vartheta (w, 3) \) is the Green’s function provided by
Proof. Applying the integral operator \( l^q \) and using Lemma 2 on (16), we get
\[
\phi_p \left[ R \mathcal{D}^q \pi_1 (w) \right] = l^q \mathcal{I}_x (w, \pi_2 (w)) + a_1 + a_2 w + a_3 \omega^2 + a_4 \omega^3 + \cdots + a_n \omega^{n-1}.
\] (19)

Using the condition \( \big( \phi_p \left[ R \mathcal{D}^q \pi_1 (w) \right] \big) \big|_{w=0} = 0 \) for \( i = 0, 1, 2, \ldots, n-1 \), in (19), we obtain \( a_1 = a_2 = a_3 = \cdots = a_n = 0 \), and then, we get
\[
\phi_p \left[ R \mathcal{D}^q \pi_1 (w) \right] = l^q \mathcal{I}_1 (w, \pi_2 (w)).
\] (20)

From (20), we have
\[
R \mathcal{D}^q \pi_1 (w) = \phi_q \left( l^q \mathcal{I}_1 (w, \pi_2 (w)) \right).
\] (21)

Applying the operator \( l^q \) and using Lemma 3 in (21), we get
\[
\pi_1 (w) = \frac{1}{\Gamma (q)} \int_0^w \left( w - \mathcal{M} \right)^{q-1} \phi_q \left( \frac{1}{\Gamma (q)} \int_0^\mathcal{M} \left( \mathcal{M} - r \right)^{q-1} \mathcal{I}_1 (\mathcal{M}, \pi_2 (\mathcal{M})) dr \right) d \mathcal{M}
+ \frac{G^q (w, \mathcal{M}) \phi_q \left( \frac{1}{\Gamma (q)} \int_0^\mathcal{M} \left( \mathcal{M} - r \right)^{q-1} \mathcal{I}_1 (\mathcal{M}, \pi_2 (\mathcal{M})) dr \right) d \mathcal{M},
\] (25)

which can be written after rearranging as follows:
\[
\pi_1 (w) = \frac{G^q (w, \mathcal{M}) \phi_q \left( \frac{1}{\Gamma (q)} \int_0^\mathcal{M} \left( \mathcal{M} - r \right)^{q-1} \mathcal{I}_1 (\mathcal{M}, \pi_2 (\mathcal{M})) dr \right) d \mathcal{M},
\] (26)

where \( G^q (w, \mathcal{M}) \) is the Green's function defined in (18). \( \square \)

In view of Theorem 2, the identical coupled system of Hammerstein-kind integral equations to fractional differential equation coupled system (5) is given as follows:
\[
\pi_1 (w) = \frac{G^q (w, \mathcal{M}) \phi_q \left( \frac{1}{\Gamma (q)} \int_0^\mathcal{M} \left( \mathcal{M} - r \right)^{q-1} \mathcal{I}_1 (\mathcal{M}, \pi_2 (\mathcal{M})) dr \right) d \mathcal{M},
\] (27)

where \( G^q (w, \mathcal{M}) \) is the Green's function provided by
From \( G^{(2)} \) and \( G^{(3)} \) obviously,

\[
\max_{w \in [0, 1]} |G^{(2)}(w, \mathfrak{t})| = \frac{\Gamma(q_1 - \sigma - 2)(1 - \mathfrak{t})^{q_1 - \sigma - 1}}{\Gamma(q_1 - 2)\Gamma(q_1 - \sigma)}, \quad \mathfrak{t} \in [0, 1].
\]

We define the operators \( \mathcal{F}_1: \Pi_1 \to \Pi_1 \) and \( \mathcal{F}_2: \Pi_2 \to \Pi_2 \) as

\[
\mathcal{F}_1(\pi_1)(w) = \int_0^1 G^{(2)}(w, \mathfrak{t})\varphi_1\left(\frac{1}{\Gamma(\theta_2)}\right) \int_0^\mathfrak{t} (1 - \mathfrak{t})^{\theta_1} \mathcal{L}_1(\mathfrak{t}, \pi_2(\mathfrak{t}))d\mathfrak{t} d\mathfrak{t},
\]

\[
\mathcal{F}_2(\pi_1)(w) = \int_0^1 G^{(3)}(w, \mathfrak{t})\varphi_2\left(\frac{1}{\Gamma(\theta_2)}\right) \int_0^\mathfrak{t} (1 - \mathfrak{t})^{\theta_1} \mathcal{L}_2(\mathfrak{t}, \pi_2(\mathfrak{t}))d\mathfrak{t} d\mathfrak{t}.
\]

Therefore, we have \( \mathcal{F}(\pi_1, \pi_2) = (\mathcal{F}_1, \mathcal{F}_2)(\pi_1, \pi_2) \), \( \mathcal{F}(\pi_1, \pi_2) = (\mathcal{F}_1, \mathcal{F}_2)(\pi_1, \pi_2) \) and \( \mathcal{F}(\pi_1, \pi_2) = \mathcal{F}(\pi_1, \pi_2) + \mathcal{F}(\pi_1, \pi_2) \). Thus, the equivalent operator equation for the topped system of Hammerstein-kind integral equations (27) is provided by

\[
(\pi_1, \pi_2) = \mathcal{F}(\pi_1, \pi_2) = \mathcal{F}(\pi_1, \pi_2) + \mathcal{F}(\pi_1, \pi_2).
\]

Consequently, the solutions of system (27) are the fixed points of operator equation (32).

Now, we need to list the following assumptions to complete our results.

\( (H_1) \) For \( h, \pi_1, \ell, \pi_2 \in IR \), the nonlinear functions \( \varphi \) and \( \omega \) satisfy \( \|\varphi(h) - \varphi(\pi_1)\| \leq K_{\varphi}\|h - \pi_1\| \) and \( \|\omega(\ell) - \omega(\pi_2)\| \leq K_{\omega}\|\ell - \pi_2\| \) such that \( K_{\varphi}, K_{\omega} \in [0, 1) \).

\( (H_2) \) With the positive constants given \( C_\varphi, C_\omega, N_\pi, N_\omega, \) and \( q_1 \in [0, 1) \), the nonlinear functions \( \varphi \) and \( \omega \) for \( \pi_1, \pi_2 \in IR \) satisfy the following growth conditions: \( |\varphi(\pi_1)| \leq C_\varphi|\pi_1|^{q_1} + N_\varphi \) and \( |\omega(\pi_2)| \leq C_\omega|\pi_2|^{q_1} + N_\omega \).

\( (H_3) \) With the presence of constants \( g, h, N_{\mathcal{L}_1}, N_{\mathcal{L}_2}, \) and \( q_2 \in [0, 1) \), the nonlinear functions \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) for \( \pi_1, \pi_2 \in IR \) satisfy the following growth conditions:

\[
|\mathcal{L}_1(\pi_1, \pi_2)| \leq \varphi_1(|\pi_1|^{q_2} + N_{\mathcal{L}_1}),
\]

\[
|\mathcal{L}_2(\pi_1, \pi_2)| \leq \varphi_2(|\pi_2|^{q_2} + N_{\mathcal{L}_2}).
\]

\( (H_4) \) For \( h, \pi_1, \ell, \pi_2 \in IR \), there exists positive constants \( L_{\mathcal{L}_1} \) and \( L_{\mathcal{L}_2} \) such that

\[
|\mathcal{L}_1(\pi_1, \pi_2) - \mathcal{L}_1(\pi_1, \pi_2)| \leq L_{\mathcal{L}_1}|\ell - \pi_2|,
\]

\[
|\mathcal{L}_2(\pi_1, \pi_2) - \mathcal{L}_2(\pi_1, \pi_2)| \leq L_{\mathcal{L}_2}|h - \pi_2|.
\]

**Theorem 3.** Assume that \( (H_1) \) and \( (H_2) \) hold true. Then, the operator \( \mathcal{F} \) is Lipschitz and satisfies the following growth condition:

\[
\left\|\mathcal{F}(\pi_1, \pi_2)\right\| \leq C_\mathcal{L}\left\|(\pi_1, \pi_2)\right\|^{q_2} + N_{\mathcal{L}}, \quad \forall (\pi_1, \pi_2) \in \Omega.
\]
Proof. By assumption \((H_1)\), we get

\[
|\mathcal{F}_1(\pi_1)(\omega) - \mathcal{F}_1(\pi_1)(\omega)| = \left| \frac{\Gamma((\pi_1 - \lambda - 2)}{\Gamma((\pi_1 - 2)\Gamma(\lambda)} \int_0^W (W - 3)^{k_{1}}|\varphi(\pi_1) - \varphi(\pi_1)|d\mathcal{H} \right|
\]

which yields

\[
\|\mathcal{F}_1(\pi_1) - \mathcal{F}_1(\pi_1)\| \leq \mathcal{K}_\pi\|\pi_1 - \pi_1\|, \tag{37}
\]

where

\[
\mathcal{K}_\pi = \frac{K_\pi\Gamma((\pi_1 - \lambda - 2)\Gamma(\lambda + 1)}{\Gamma((\pi_1 - 2)\Gamma(\lambda + 1)} \in [0, 1]. \tag{38}
\]

To get the growth condition, consider

\[
|\mathcal{F}_2(\pi_1)(\omega)| = \left| \frac{\Gamma((\pi_1 - \lambda - 2)}{\Gamma((\pi_1 - 2)\Gamma(\lambda)} \int_0^W (W - 3)^{k_{1}}|\varphi(\pi_1)(\pi_1)|d\mathcal{H} \right|
\]

which means that

\[
\|\mathcal{F}_2(\pi_1)(\pi_1)\| \leq \mathcal{K}_\pi\|\pi_1 - \pi_1\| \tag{39}
\]

which implies that

\[
\|\mathcal{F}_2(\pi_1)(\pi_1)\| \leq \left| \frac{\Gamma((\pi_1 - \lambda - 2)\Gamma(\lambda + 1)}{\Gamma((\pi_1 - 2)\Gamma(\lambda + 1)} \left[ C_\pi\|\pi_1\| + N_\pi \right] \right|
\]

Thus,

\[
\|\mathcal{F}_2(\pi_1, \pi_2)\| \leq C_\pi\|\pi_1\| + C_\pi\|\pi_2\| + N_\pi \tag{44}
\]

where
Theorem 4. Suppose that \((H_3)\) is satisfied. Then, the operator \(G\) is continuous and satisfies the following growth condition:
\[
\|G(\pi_1, \pi_2)\| \leq \Lambda \|\pi_1, \pi_2\|^{\sigma_2} + \Theta, \quad \text{for all} \ (\pi_1, \pi_2) \in \Omega,
\]
where \(\Lambda = \gamma(g + h)\) and \(\Theta = \gamma(N_{\mathcal{X}_1} + N_{\mathcal{X}_2})\) such that

\[
\gamma = \max \left\{ \left( \frac{1}{\Gamma(\theta_1 + 1)} + \frac{\Gamma(\theta_1 - \lambda - 2)}{\Gamma(\theta_1 - 2) \Gamma(\theta_1 - \lambda + 1)} \right) \right\}^{\sigma_1 - 1},
\]
and
\[
\gamma \left( \frac{1}{\Gamma(\theta_2 + 1)} + \frac{\Gamma(\theta_2 - \sigma - 2)}{\Gamma(\theta_2 - 2) \Gamma(\theta_2 - \sigma + 1)} \right) \right\}^{\sigma_2 - 1}.
\]

Proof. Let \(\mathcal{B} = \{(\pi_1, \pi_2) \in \Omega : \|\pi_1, \pi_2\| \leq r\}\) be a bounded set with a sequence \((\pi_{1n}, \pi_{2n})\) converging to \((\pi_1, \pi_2)\) in \(\mathcal{B}\). In order to show that \(G\) is continuous, we have to prove that
\[
\|G(\pi_{1n}, \pi_{2n}) - G(\pi_1, \pi_2)\| \to 0 \quad \text{as} \quad n \to \infty.
\]
Let us choose the following:

\[
\|G(\pi_{1n}, \pi_{2n}) - G(\pi_1, \pi_2)\| \to 0 \quad \text{as} \quad n \to \infty.
\]

The continuity of \(\mathcal{L}_1\) implies that
\[
|\mathcal{L}_1(\tau, \pi_{1n}(\tau)) - \mathcal{L}_1(\tau, \pi_{2n}(\tau))| \to 0 \quad \text{as} \quad n \to \infty,
\]
and then
\[
\|G(\pi_{1n}) - G(\pi_1)\| \to 0, \quad \text{as} \quad n \to \infty.
\]

and also, we can in the same way prove that
\[
\|G(\pi_{2n}) - G(\pi_2)\| \to 0, \quad \text{as} \quad n \to \infty.
\]
The continuity of $\mathcal{L}_2$ implies that $|\mathcal{L}_2(\tau, \pi_\nu(\tau) - \mathcal{L}_2(\tau, \pi_\nu(\tau))| \to 0$ as $n \to \infty$, and then,

$$\|\mathcal{G}_2(\pi_\nu) - \mathcal{G}_2(\pi_\nu)\| \to 0, \quad \text{as} \quad n \to \infty. \quad (51)$$

Thus, from (49) and (51), we have

$$\|\mathcal{G}(\pi_\nu, \pi_\nu) - \mathcal{G}(\pi_\nu, \pi_\nu)\| \leq (q + 1)\rho_1^q \int_0^1 \mathcal{G}(w, \mathfrak{A}) \left[ \frac{1}{\Gamma(\theta_1)} \int_0^3 (\mathfrak{A} - r)^{\theta_1 - 1}\mathcal{L}_1(\tau, \pi_\nu(\tau)) \right] d\mathfrak{A} - \mathcal{L}_1(\tau, \pi_\nu(\tau))|d\mathfrak{A}

+ (q + 1)\rho_2^q \int_0^1 \mathcal{G}(w, \mathfrak{A}) \left[ \frac{1}{\Gamma(\theta_2)} \int_0^3 (\mathfrak{A} - r)^{\theta_2 - 1}\mathcal{L}_2(\tau, \pi_\nu(\tau)) \right] d\mathfrak{A}.

(52)

From the continuity of $\mathcal{L}_1$ and $\mathcal{L}_2$ and (52), we have

$$\|\mathcal{G}(\pi_\nu, \pi_\nu) - \mathcal{G}(\pi_\nu, \pi_\nu)\| \to 0 \quad \text{as} \quad n \to \infty. \quad (53)$$

To calculate (46) for $\mathcal{G}$, using assumption $(H_2)$ and (29), we obtain

$$|\mathcal{G}_1(\pi_\nu)(w)| = \int_0^1 \mathcal{G}(w, \mathfrak{A}) \phi_q\left( \frac{1}{\Gamma(\theta_1)} \int_0^3 (\mathfrak{A} - r)^{\theta_1 - 1}\mathcal{L}_1(\tau, \pi_\nu(\tau)) \right) d\mathfrak{A}

\leq \int_0^1 \mathcal{G}(w, \mathfrak{A}) \phi_q\left( \frac{1}{\Gamma(\theta_1)} \int_0^3 (\mathfrak{A} - r)^{\theta_1 - 1}\mathcal{L}_1(\tau, \pi_\nu(\tau)) \right) d\mathfrak{A}

\leq \Gamma(\theta_1 - \lambda - 2) \left[ \frac{1}{\Gamma(\theta_1 + 1)} \right] ^q \left[ h|\pi_\nu|^{q_2} + N_{\mathcal{A}} \right].$$

(54)

From assumption $(H_2)$ and (29), we get

$$|\mathcal{G}_2(\pi_\nu)(w)| = \int_0^1 \mathcal{G}(w, \mathfrak{A}) \phi_q\left( \frac{1}{\Gamma(\theta_2)} \int_0^3 (\mathfrak{A} - r)^{\theta_2 - 1}\mathcal{L}_2(\tau, \pi_\nu(\tau)) \right) d\mathfrak{A}

\leq \int_0^1 \mathcal{G}(w, \mathfrak{A}) \phi_q\left( \frac{1}{\Gamma(\theta_2)} \int_0^3 (\mathfrak{A} - r)^{\theta_2 - 1}\mathcal{L}_2(\tau, \pi_\nu(\tau)) \right) d\mathfrak{A}

\leq \Gamma(\theta_2 - \sigma - 2) \left[ \frac{1}{\Gamma(\theta_2 + 1)} \right] ^q \left[ h|\pi_\nu|^{q_2} + N_{\mathcal{A}} \right].$$

(55)

Then,

$$|\mathcal{G}_2(\pi_\nu)(w)| \leq \int_0^1 \mathcal{G}(w, \mathfrak{A}) \phi_q\left( \frac{1}{\Gamma(\theta_2)} \int_0^3 (\mathfrak{A} - r)^{\theta_2 - 1}\mathcal{L}_2(\tau, \pi_\nu(\tau)) \right) d\mathfrak{A}

\leq \left[ \frac{1}{\Gamma(\theta_2 + 1)} \right] ^q \left[ h|\pi_\nu|^{q_2} + N_{\mathcal{A}} \right].$$

(56)
By the help of (54) and (55), we have obtained
\[
\|\mathcal{G} (\pi_1, \pi_2)\| = \|\mathcal{G}_1 (\pi_2)\| + \|\mathcal{G}_2 (\pi_1)\|
\leq \gamma (g\|\pi_2\|^{q_1} + N_{x_1}) + \gamma (h\|\pi_1\|^{q_1} + N_{x_2})
\leq \gamma (g + h)\|\pi_2\|^{q_1} + \|\pi_1\|^{q_1} + \gamma (N_{x_1} + N_{x_2})
= \Lambda\|\pi_1, \pi_2\|^{q_1} + \Theta.
\]
(57)

Hence, it follows that
\[
\|\mathcal{G}_1 \pi_{1_n} (w_1) - \mathcal{G}_1 \pi_{1_n} (w_2)\|
\leq \left[ (w_1^{q_1} - w_2^{q_1}) \frac{1}{\Gamma (\theta_1 + 1)} + \left( w_1^{q_1 - 3} - w_2^{q_1 - 3} \right) \frac{1}{\Gamma (\theta_1 - 2)\Gamma (\theta_1 - \lambda + 1)} \right]^{q_1 - 1}
\times (g\|\pi_2\|^{q_1} + N_{x_1}).
\]
(60)

Similarly, we have
\[
\|\mathcal{G}_2 \pi_{1_n} (w_1) - \mathcal{G}_2 \pi_{1_n} (w_2)\|
\leq \left[ (w_1^{q_1} - w_2^{q_1}) \frac{1}{\Gamma (\theta_2 + 1)} + \left( w_1^{q_1 - 3} - w_2^{q_1 - 3} \right) \frac{1}{\Gamma (\theta_2 - 2)\Gamma (\theta_2 - \sigma + 1)} \right]^{q_1 - 1}
\times (\|\pi_2\|^{q_1} + N_{x_2}).
\]
(61)

Both the right sides of (55) and (61) tend to be zero as \(w_1 \to w_2\). Therefore, the operators \(\mathcal{G}_1\) and \(\mathcal{G}_2\) are equicontinuous, and hence, \(\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2)\) is equicontinuous on \(E\). Thus, \(\mathcal{G} (E)\) is compact by the theorem of Arzela–Ascoli. Moreover, through Proposition 3, \(\mathcal{G}\) is \(\theta\)-Lipschitz with constant zero.

**Theorem 5.** The operator \(\mathcal{G}: \Omega \to \Omega\) is \(\theta\)-Lipschitz with constant zero and is compact.

**Proof.** Take a bounded set \(E\) and a sequence \((\pi_{1_n}, \pi_{2_n})\) such that \(E \subset \mathcal{B}_E \subset \Omega\). Then, using (46), we have
\[
\left\| \mathcal{G} (\pi_{1_n}, \pi_{2_n}) \right\| \leq \Lambda\|\pi_{1_n}, \pi_{2_n}\|^{q_1} + \Theta, \quad \text{for all } (\pi_{1_n}, \pi_{2_n}) \in \Omega,
\]
(58)

which means that \(\mathcal{G}\) is bounded. Now, for all \((\pi_{1_n}, \pi_{2_n}) \in E\), we have, for \(0 \leq w_1 < w_2 \leq 1\),
\[
\left\| \mathcal{G}_1 \pi_{1_n} (w_1) - \mathcal{G}_1 \pi_{1_n} (w_2) \right\|
= \int_0^1 G^{\phi_1} (w_1, \mathfrak{M}) \phi_{\theta_1} \left( \frac{1}{\Gamma (\theta_1)} \right) \int_0^\mathfrak{M} (\mathfrak{M} - r)^{\theta_1 - 1} \mathcal{I}_1 ((r, \pi_2 (r)) \, dr \, d\mathfrak{M})
- \int_0^1 G^{\phi_1} (w_2, \mathfrak{M}) \phi_{\theta_1} \left( \frac{1}{\Gamma (\theta_1)} \right) \int_0^\mathfrak{M} (\mathfrak{M} - r)^{\theta_1 - 1} \mathcal{I}_1 ((r, \pi_2 (r)) \, dr \, d\mathfrak{M})
\leq \int_0^1 G^{\phi_1} (w_1, \mathfrak{M}) - G^{\phi_1} (w_2, \mathfrak{M}) \phi_{\theta_1} \left( \frac{1}{\Gamma (\theta_1)} \right) \int_0^\mathfrak{M} (\mathfrak{M} - r)^{\theta_1 - 1} \phi_{\tau} (g\|\pi_2\|^{q_1} + N_{x_1}) \, dr \, d\mathfrak{M}.
\]
(59)

Obviously, \(\|\pi_{1_n}, \pi_{2_n}\|\) is bounded. If not correct, take \(\|\pi_{1_n}, \pi_{2_n}\| = \delta\) such that \(\delta \to \infty\) and \(q_3 \in (0, 1)\). Consequently,
\[
\begin{align*}
1 & \leq (C_{x_2} + \Lambda) \frac{\|\pi_{1_n}, \pi_{2_n}\|^{q_1}}{\|\pi_{1_n}, \pi_{2_n}\|^{q_1}} + \frac{N_{x_2} + \Theta}{\|\pi_{1_n}, \pi_{2_n}\|^{q_1}} \\
1 & \leq \frac{(C_{x_2} + \Lambda) \delta^{q_1}}{\delta^{q_1}} + \frac{N_{x_2} + \Theta}{\delta^{q_1}} \\
1 & \leq \frac{(C_{x_2} + \Lambda) \delta^{q_1}}{\delta^{q_1-1}} + \frac{N_{x_2} + \Theta}{\delta^{q_1-1}} \to 0 \quad \text{as } \delta \to \infty,
\end{align*}
\]
(63)

which is a contradiction. So, \(\mathcal{B}_E\) is bounded. Thus, the boundedness of \(\mathcal{G}\) follows by Theorem 7, we conclude that \(\mathcal{G}\) has at least one fixed point and that \(\|\pi_{1_n}, \pi_{2_n}\|\) is a solution of system (5), and the set of solutions is bounded in \(\Omega\).}

**Theorem 6.** Suppose that \((H_1)-(H_3)\) are satisfied with \(\Lambda + C_{x_2} \leq 1\). Then, the toppled system (5) has at least one solution \((\pi_{1_n}, \pi_{2_n}) \in \Omega\). Furthermore, the set of solutions of (5) is bounded in \(\Omega\).
Theorem 7. Suppose that \((H_1)-(H_4)\) hold and \(\chi < 1\), where

\[
\chi = \frac{K_\varphi \Gamma(\theta_1 - \lambda - 2) W^\lambda}{\Gamma(\theta_1 - 2) \Gamma(\lambda + 1)} + \frac{K_\sigma \Gamma(\theta_2 - \sigma - 2) W^\sigma}{\Gamma(\theta_2 - 2) \Gamma(\sigma + 1)} \\
+ (q - 1) \rho_1^{q-2} L_{\mathcal{F}_1}\left(\frac{1}{\Gamma(\theta_1 + 1)} + \frac{\Gamma(\theta_1 - \lambda - 2)}{\Gamma(\theta_1 - 2) \Gamma(\theta_1 - \lambda + 1)}\left(\frac{1}{\Gamma(\theta_1 + 1)}\right)\right) \\
+ (q - 1) \rho_2^{q-2} L_{\mathcal{F}_2}\left(\frac{1}{\Gamma(\theta_2 + 1)} + \frac{\Gamma(\theta_2 - \sigma - 2)}{\Gamma(\theta_2 - 2) \Gamma(\theta_2 - \sigma + 1)}\left(\frac{1}{\Gamma(\theta_2 + 1)}\right)\right)
\]

Then, toppled system (5) has a unique solution.

Proof. Let \((\pi_1, \pi_2)\) and \((\bar{\pi}_1, \bar{\pi}_2) \in \Omega\) are two solutions, then

\[
|\mathcal{F}(\pi_1, \pi_2) - \mathcal{F}(\bar{\pi}_1, \bar{\pi}_2)| = |[\mathcal{F}(\pi_1, \pi_2) + \mathcal{G}(\pi_1, \pi_2)] - [\mathcal{F}(\bar{\pi}_1, \bar{\pi}_2) + \mathcal{G}(\bar{\pi}_1, \bar{\pi}_2)]| \\
\leq |\mathcal{F}(\pi_1, \pi_2) - \mathcal{F}(\bar{\pi}_1, \bar{\pi}_2)| + |\mathcal{G}(\pi_1, \pi_2) - \mathcal{G}(\bar{\pi}_1, \bar{\pi}_2)|.
\]

and after simplification, we obtain

\[
\left\|\mathcal{F}(\pi_1, \pi_2) - \mathcal{F}(\bar{\pi}_1, \bar{\pi}_2)\right\| \\
\leq \left(\frac{K_\varphi \Gamma(\theta_1 - \lambda - 2) W^\lambda}{\Gamma(\theta_1 - 2) \Gamma(\lambda + 1)} + \frac{K_\sigma \Gamma(\theta_2 - \sigma - 2) W^\sigma}{\Gamma(\theta_2 - 2) \Gamma(\sigma + 1)} \\
+ (q - 1) \rho_1^{q-2} L_{\mathcal{F}_1}\left(\frac{1}{\Gamma(\theta_1 + 1)} + \frac{\Gamma(\theta_1 - \lambda - 2)}{\Gamma(\theta_1 - 2) \Gamma(\theta_1 - \lambda + 1)}\left(\frac{1}{\Gamma(\theta_1 + 1)}\right)\right) \\
+ (q - 1) \rho_2^{q-2} L_{\mathcal{F}_2}\left(\frac{1}{\Gamma(\theta_2 + 1)} + \frac{\Gamma(\theta_2 - \sigma - 2)}{\Gamma(\theta_2 - 2) \Gamma(\theta_2 - \sigma + 1)}\left(\frac{1}{\Gamma(\theta_2 + 1)}\right)\right)\right\|\right\| \leq \chi \|\pi_1, \pi_2\| - (\bar{\pi}_1, \bar{\pi}_2) \|.
\]

which implies that

\[
\left\|\mathcal{F}(\pi_1, \pi_2) - \mathcal{F}(\bar{\pi}_1, \bar{\pi}_2)\right\| \leq \chi \|\pi_1, \pi_2\| - (\bar{\pi}_1, \bar{\pi}_2) \|.
\]

Thus, the operator \(\mathcal{F}\) is a contraction as \(\chi < 1\), and by the Banach fixed-point theorem, \(\mathcal{F}\) has a unique fixed point, and then, considered toppled system (5) has a unique solution. □

4. Hyers–Ulam Stability

In this section, we investigate the stability of Hyers–Ulam for the suggested toppled system.

Definition 7. We say that the toppled system of Hammerstein-kind integral equations (27) is Hyers–Ulam stable if
there exists positive constants $a, b, c, \text{ and } d$ such that, for each $\xi_1, \xi_2 > 0$ and any solution $(\pi_1', \pi_2')$ of the system

$$
\begin{aligned}
\left\{&\right.
\left\{\begin{array}{l}
\pi_1'(w) - \frac{\Gamma(q_1 - \lambda - 2)w^{\rho_1-3}}{\Gamma(q_1 - 2)\Gamma(\lambda)} \int_0^W (W - \mathcal{F})^{\lambda-1} \varphi(\pi_1(\mathcal{F}))d\mathcal{F} \\
\quad \quad + \int_0^1 G^{\rho_1}(w, \mathcal{F}) \varphi_1 \left( \frac{1}{\Gamma(\theta_1)} \right) \int_0^\mathcal{F} (\mathcal{F} - \tau)^{\theta_1-1} \mathcal{Z}_1(\tau, \pi_2'(\tau))d\tau \right) d\mathcal{F} \\
\pi_2'(w) - \frac{\Gamma(q_2 - \sigma - 2)w^{\rho_2-3}}{\Gamma(q_2 - 2)\Gamma(\sigma)} \int_0^W (W - \mathcal{F})^{\sigma-1} \omega(\pi_2(\mathcal{F}))d\mathcal{F} \\
\quad \quad + \int_0^1 G^{\rho_2}(w, \mathcal{F}) \varphi_2 \left( \frac{1}{\Gamma(\theta_2)} \right) \int_0^\mathcal{F} (\mathcal{F} - \tau)^{\theta_2-1} \mathcal{Z}_2(\tau, \pi_2'(\tau))d\tau \right) d\mathcal{F} \leq \xi_1,
\end{array}
\right.
\end{aligned}
$$

there exists $(\pi_1, \pi_2)$ which is the unique solution of (27) satisfying that

$$
\begin{align}
|\pi_1(w) - \pi_1'(w)| &\leq a\xi_1 + b\xi_2, \\
|\pi_2(w) - \pi_2'(w)| &\leq c\xi_1 + d\xi_2.
\end{align}
$$

Theorem 8. The toppled system (5) is Hyers–Ulam stable under hypotheses $(H_1)\text{--}(H_4)$.

Proof. With the help of Definition 7 and Theorem 7, suppose that $(\pi_1, \pi_2)$ to be the correct solution and the pair $(\pi_1^*, \pi_2^*)$ be the other solution of system (27). Then, we have, from the first equation of (27),

$$
\begin{aligned}
|\pi_1(w) - \pi_1'(w)| &= \frac{\Gamma(q_1 - \lambda - 2)w^{\rho_1-3}}{\Gamma(q_1 - 2)\Gamma(\lambda)} \int_0^W (W - \mathcal{F})^{\lambda-1} \varphi(\pi_1(\mathcal{F}))d\mathcal{F} \\
&\quad + \int_0^1 G^{\rho_1}(w, \mathcal{F}) \varphi_1 \left( \frac{1}{\Gamma(\theta_1)} \right) \int_0^\mathcal{F} (\mathcal{F} - \tau)^{\theta_1-1} \mathcal{Z}_1(\tau, \pi_2'(\tau))d\tau \right) d\mathcal{F} \\
&\quad - \frac{\Gamma(q_1 - \lambda - 2)}{\Gamma(q_1 - 2)\Gamma(\lambda)} \int_0^W (W - \mathcal{F})^{\lambda-1} \varphi(\pi_1^*(\mathcal{F}))d\mathcal{F} \\
&\quad - \int_0^1 G^{\rho_1}(w, \mathcal{F}) \varphi_1 \left( \frac{1}{\Gamma(\theta_1)} \right) \int_0^\mathcal{F} (\mathcal{F} - \tau)^{\theta_1-1} \mathcal{Z}_1(\tau, \pi_2'(\tau))d\tau \right) d\mathcal{F} \\
&\leq \frac{\Gamma(q_1 - \lambda - 2)}{\Gamma(q_1 - 2)\Gamma(\lambda)} \int_0^W (W - \mathcal{F})^{\lambda-1} (\varphi(\pi_1) - \varphi(\pi_1^*))d\mathcal{F} \\
&\quad + \int_0^1 \|G^{\rho_1}(w, \mathcal{F})\| \phi_1 \left[ \frac{1}{\Gamma(\theta_1)} \right] \int_0^\mathcal{F} (\mathcal{F} - \tau)^{\theta_1-1} \mathcal{Z}_1(\tau, \pi_2'(\tau))d\tau \right] d\mathcal{F} \\
&\quad - \phi_1 \left[ \frac{1}{\Gamma(\theta_1)} \right] \int_0^\mathcal{F} (\mathcal{F} - \tau)^{\theta_1-1} \mathcal{Z}_1(\tau, \pi_2'(\tau))d\tau \right] d\mathcal{F} \\
&\leq \frac{\Gamma(q_1 - \lambda - 2)}{\Gamma(q_1 - 2)\Gamma(\lambda)} \int_0^W (W - \mathcal{F})^{\lambda-1} (\varphi(\pi_1) - \varphi(\pi_1^*))d\mathcal{F} \\
&\quad + (q - 1)\rho_1^{\theta_1-2} \int_0^1 \|G^{\rho_1}(w, \mathcal{F})\| \left( \frac{1}{\Gamma(\theta_1)} \right) \int_0^\mathcal{F} (\mathcal{F} - \tau)^{\theta_1-1} \mathcal{Z}_1(\tau, \pi_2'(\tau))d\tau \right] d\mathcal{F}.
\end{aligned}
$$
Then,

\[
\begin{aligned}
\left\| \pi_1(w) - \pi_1^*(w) \right\| & \leq \frac{\Gamma(\theta_1 - \lambda - 2)W^{\lambda}K_\rho}{\Gamma(\theta_1 - 2)\Gamma(\lambda + 1)} \left\| \pi_1(w) - \pi_1^*(w) \right\| \\
& + (q - 1)\rho_1^Q \| \mathcal{L}_{\mathcal{F}_1} \left( \frac{1}{\Gamma(\theta_1 + 1)} + \frac{\Gamma(\theta_1 - \lambda - 2)}{\Gamma(\theta_1 - 2)\Gamma(\theta_1 - \lambda + 1)} \right) \left( \frac{1}{\Gamma(\theta_1 + 1)} \right) \\
& \times \left\| \pi_2(w) - \pi_2^*(w) \right\| \leq a\xi_1 + b\xi_2,
\end{aligned}
\]  

(71)

where

\[
\begin{aligned}
a &= \frac{\Gamma(\theta_1 - \lambda - 2)W^{\lambda}K_\rho}{\Gamma(\theta_1 - 2)\Gamma(\lambda + 1)}, \\
b &= (q - 1)\rho_1^Q \mathcal{L}_{\mathcal{F}_1} \left( \frac{1}{\Gamma(\theta_1 + 1)} + \frac{\Gamma(\theta_1 - \lambda - 2)}{\Gamma(\theta_1 - 2)\Gamma(\theta_1 - \lambda + 1)} \right) \left( \frac{1}{\Gamma(\theta_1 + 1)} \right).
\end{aligned}
\]

(72)

Similarly, we get

\[
\left\| \pi_2(w) - \pi_2^*(w) \right\| \leq c\xi_1 + d\xi_2,
\]  

(73)

where

\[
\begin{aligned}
c &= \frac{\Gamma(\theta_2 - \sigma - 2)W^{\sigma}K_\rho}{\Gamma(\theta_2 - 2)\Gamma(\sigma + 1)}, \\
d &= (q - 1)\rho_2^Q \mathcal{L}_{\mathcal{F}_1} \left( \frac{1}{\Gamma(\theta_2 + 1)} + \frac{\Gamma(\theta_2 - \sigma - 2)}{\Gamma(\theta_2 - 2)\Gamma(\theta_2 - \sigma + 1)} \right) \left( \frac{1}{\Gamma(\theta_2 + 1)} \right).
\end{aligned}
\]

(74)

Hence, by (71) and (73), integral equations’ toppled system (27) is Hyers–Ulam stable. Thus, proposed toppled system (5) is Hyers–Ulam stable.

\[ \Box \]

5. Illustrative Example

In this section, we introduce an application of our results, which were proved in Sections 3 and 4.

Example 1. Consider the following toppled fractional system with the \( p \)-Laplacian operator and integral boundary conditions for \( n = 5 \):

\[
\begin{cases}
\mathcal{D}^{14/3} \left( \phi_{\mathcal{D}}^{(R)D^{13/3}} \pi_1(w) \right) = -\frac{21w}{12} + \frac{1}{10} \cos(\pi_1(w)), & w \in [0, 1], \\
\mathcal{D}^{14/3} \left( \phi_{\mathcal{D}}^{(R)D^{13/3}} \pi_2(w) \right) = \frac{32}{15} + \frac{1}{10} \sin(\pi_1(w)), & w \in [0, 1], \\
\phi_{\mathcal{D}}^{(R)D^{13/3}} \pi_1(0) = 0, & i = 0, 1, 2, 3, 4, \\
\phi_{\mathcal{D}}^{(R)D^{13/3}} \pi_2(0) = 0, & j = 0, 1, 2, 3, 4, \\
\mathcal{D}^{3/2} \pi_1(w)|_{w=0} = \frac{1}{\Gamma(3/2)} \int_0^1 (1 - 3)^{(3/2)} \cos(\pi_1) \, d\mathfrak{S}, \\
\mathcal{D}^{3/2} \pi_2(w)|_{w=0} = \frac{1}{\Gamma(3/2)} \int_0^1 (1 - 3)^{(3/2)} \cos(\pi_2) \, d\mathfrak{S}.
\end{cases}
\]

(75)
where \( \theta_1 = \frac{14}{3}, \theta_2 = (14/3), p = 4, p_1 = p_2 = (13/3), \) and \( \lambda = \sigma = (3/2). \) Then, we obtain \( K_p = K_\omega = (1/6) \) and \( L_{x_1} = L_{x_2} = (1/10). \) Via simple calculation and taking \( \rho_1 = \rho_2 = (1/2), \) we get \( x = 0.1767 < 1. \) Hence, by Theorem 7, toppled system (75) has a unique solution. With comparable fashion, it is easy to verify the fulfillment of the conditions of Theorem 6. Likewise, the conditions of Theorem 8 can be easily confirmed, and consequently, the solution of system (75) is Hyers–Ulam stable.

### 6. Conclusion

In this study, we analyzed the stability and uniqueness solution of Caputo and Riemann–Liouville fractional derivatives with fractional orders \( n - 1 < \theta_1, \theta_2, p_1, p_2 \leq n, \) and \( n \geq 3. \) By using the topological degree theory, we have proved sufficient conditions for the EUS of the coupled system of fractional differential equations with integral boundary conditions involving the \( p \)-Laplacian operator. Also, we have found appropriate conditions for Hyers–Ulam stability of the solution for the considered system. At the end, we have provided an example that supported our results as we have done in Section 5 to confirm the theoretical analysis.

### Data Availability

No data were used to support this study.

### Conflicts of Interest

The authors declare that they have no conflicts of interest.

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