Nonabelian Cut Diagrams and Their Applications*  

C.S. Lam†  

Department of Physics, McGill University, 
3600 University St., Montreal, P.Q., Canada H3A 2T8  

Abstract  

A new kind of cut diagram is introduced to sum Feynman diagrams with nonabelian vertices. Unlike the Cutkosky diagrams which compute the discontinuity of single Feynman diagrams, the nonabelian cut diagrams represent a resummation of both the real and the imaginary parts of Feynman diagrams related by permutations. Several applications of the technique are reported, including a resolution of the apparent inconsistency of the baryon problem in large-$N_c$ QCD, a simplified calculation of high-energy low-order QCD diagrams, and progress made with this technique on the unitarization of the BFKL equation.  

1 Nonabelian cut diagrams  

This note serves to introduce the recently developed nonabelian cut diagrams [1, 2] and to summarize their applications to date [2, 3, 4].  

A tree diagram of the type shown in Fig. 1, with bosons of relatively small momenta $q_i$ absorbed or emitted from a particle of large momentum $p$, has an amplitude given approximately by  

$$-2\pi i\delta \left( \sum_{j=1}^{n} \omega_j \right) \left( \prod_{i=1}^{n-1} \frac{1}{\omega_j + i\epsilon} \right) \cdot t_1 t_2 \cdots t_n \cdot V \equiv a[12 \cdots n] \cdot t[12 \cdots n] \cdot V ,$$  

where $\omega_j = 2p \cdot q_j$, $t[12 \cdots n] \equiv t_1 t_2 \cdots t_n$, with $t_i$ being some nonabelian vertices. $V$ contains other objects such as coupling constants, etc., independent of the ordering of the bosons. For easy reference we shall assume $p$ to be the momentum of a fermion, though the kinematical and combinatorial formulas discussed below remain unaltered whatever that particle is.  

The momenta $q_i$ have been assumed in (1.1) to be much smaller than the momentum $p$, so that the approximation $(p + \sum q_j)^2 - M^2 \simeq 2p \cdot \sum q_j$ can be used to compute the denominators of propagators. This would be so in the presence of a very large fermion mass, as is the case for baryons in large-$N_c$ QCD. It is also valid when the fermion is taken from the incoming beam of a high energy process like quark-quark scattering. These two applications will be discussed separately in the next two sections.
Our aim is to compute the sum of the nonabelian amplitude (1.1) over the $n!$ permuted tree diagrams, with the help of two exact combinatorial formulas involving $a[\cdots]$ and $t[\cdots]$. Since the momenta $q_i$ are allowed to be offshell, these formulas are equally valid when the tree in Fig. 1 is part of a much larger loop diagram, so the formalism can be used just as well to compute sums of loop diagrams with the boson lines so permuted.

To state these formulas we must first introduce suitable notations to describe and to manipulate these tree diagrams. We will use $[\sigma] = [\sigma_1 \sigma_2 \cdots \sigma_n]$ to denote a tree diagram whose bosons are numbered in the order $\sigma_1, \sigma_2, \cdots, \sigma_n$, from left to right. Hence Fig. 1 is $[12 \cdots n]$. If $[T_i]$ are tree diagrams, then $[T_1 T_2 \cdots T_A]$ is taken to mean the tree diagram obtained by merging these $A$ trees together in that order. For example, if $[T_1] = [123]$ and $[T_2] = [45]$, then $[T_1 T_2] = [12345]$. The notation $\{T_1; T_2; \cdots; T_A\}$, on the other hand, is used to denote the set of all tree diagrams obtained by interleaving the trees $T_1, T_2, \cdots, T_A$ in all possible ways. This set contains $(\sum_n n_a)!/\prod_n n_a!$ trees if $n_a$ is the number of boson lines of tree $T_a$. In the example above, $\{T_1; T_2\}$ contains the $5!/3!2! = 10$ trees $[12345], [12435], [12453], [14235], [14253], [14523], [41235], [41253], [41523],$ and $[45123]$. Correspondingly, $a\{T_1; T_2; \cdots; T_A\}$ is defined to be the sum of the amplitudes $a[T]$ for every tree $T$ in this set.

With these notations, we can proceed to state the two combinatorial theorems. The factorization formula [1] asserts that the sum of amplitudes on the left of the following equation factorizes into the expression on the right:

$$a\{T_1; T_2; \cdots; T_A\} = \prod_{a=1}^A a[T_a].$$  \hspace{1cm} (1.2)

In the special case when each $[T_i] = [i]$ is just a vertex, the set $\{1; 2; \cdots; A\}$ consists of the $A!$ permuted trees of $[12 \cdots n]$, and the factorization formula is simply the well-known eikonal
formula (5)

\[ a\{1;2;\cdots; A\} = \prod_{a=1}^{A} \left[ -2\pi i \delta(\omega_a) \right]. \quad (1.3) \]

The factorization formula is used to show reggeized factorization [4]. It is also applied to derive the multiple commutator formula [1] used to sum up the \( n! \) permuted trees of the nonabelian amplitude (1.1):

\[ \sum_{\sigma \in S_{n}} a[\sigma] t[\sigma] = \sum_{\sigma \in S_{n}} a[\sigma] t[\sigma']' \quad (1.4) \]

The symbols and meaning of this equation will be explained below.

To each Feynman tree diagram \( [\sigma] = [\sigma_1 \sigma_2 \cdots \sigma_n] \) we associate a cut diagram \( [\sigma]_c \) by putting cuts on specific fermion propagators as follows. Proceeding from left to right, a cut is put after a number if and only if a smaller number does not occur to its right. An external line is considered to be equivalent to a cut so there is never an explicit cut at the end of the tree.

Designating a cut by a vertical bar behind a number, here are some examples of where cuts are to be put into Feynman trees: \( [1234]_c = [1|2|3|4] \), \( [3241]_c = [3241] \), and \( [2134]_c = [21|3|4] \).

The complementary cut diagram \( [\sigma]'_c \) is one where lines cut in \( [\sigma]_c \) are not cut in \( [\sigma]'_c \), and vice versa. Thus \( [1234]'_c = [1234] \), \( [3241]'_c = [3|2|4|1] \), and \( [2134]'_c = [2|134] \).

The cut amplitude \( a[\sigma]_c \) is the Feynman amplitude \( a[\sigma] \) with the propagator at a cut taken to be the Cutkosky cut propagator \( -2\pi i \delta(\sum_j \omega_j) \) instead of the Feynman propagator \( (\sum_j \omega_j + i\epsilon)^{-1} \). However, cuts are placed here only on high speed fermion lines, and as (1.4) indicates, these nonabelian cut diagrams represent a resummation and not a discontinuity as is the case in Cutkosky cut diagrams.

The nonabelian quantum number \( t[\sigma]'_c \) is determined from the complementary cut diagram \( [\sigma]'_c \) by replacing the product of nonabelian vertices separated by cuts with their commutators. For example, \( t[1234]'_c = t[1234] = t_1 t_2 t_3 t_4 \), \( t[3214]'_c = t[3214] = t_3 t_2 t_4 t_1 \), and \( t[2134]'_c = t[2134] = t_2 t_1 t_3 t_4 \).

This completes the description and the explanation of eq. (1.4). Eq. (1.4) reduces to (1.3) when all the \( t_a \) commute, so it can be considered as a nonabelian generalization of the eikonal formula.

Physically we may think of the eikonal formula as exhibiting a very interesting interference phenomenon. According to (1.3), as a result of the interference of the \( A! \) amplitudes, a very strong \( A \)-dimensional peak is found at all \( \omega_a = 0 \). At any other energy the amplitude vanishes by destructive interference. The nonabelian version (1.4) is more subtle because it does different things to different nonabelian channels. The peak at \( \omega_a = 0 \) is generally weaker, being only of \( B \leq A \) dimensions, but this is compensated by having \( A - B \) commutators of the nonabelian vertices. The exact physical significance of the commutators depends somewhat on the application and will be discussed separately in the next two sections.
2 The baryon problem in large-$N_c$

Suppose Fig. 1 represents meson-baryon inelastic scattering in large-$N_c$ QCD. The Yukawa coupling constant $g$ at each vertex is known to be proportional to $\sqrt{N_c}$, hence the tree diagram will grow like $N_c^{n/2}$. On the other hand, the complete tree amplitude after summing over the the $n!$ permuted diagrams is known to behave like $N_c^{1-n/2}$, so there is a discrepancy of $n - 1$ powers of $N_c$ between individual Feynman diagrams and their sum. For the meson-baryon inelastic scattering problem at large $N_c$ to be consistent, a strong cancellation must occur between the individual diagrams, and more so for larger $n$.

The multiple commutator formula (1.4) can be used to demonstrate that this indeed happens [3, 6], so the large-$N_c$ baryon problem is indeed self consistent. Essentially, what happens is that each time a commutator appears, the $N_c$ power is reduced by 1. So in the term where the complementary cut diagram has a cut in every baryon propagator, a reduction by a factor of $N_c^n$ occurs, which is precisely what is required to achieve the cancellation needed for the consistency. Note that in this case the corresponding cut diagram contains only Feynman propagators. All the other terms in (1.4) have at least one cut propagator in the cut diagram, so the amplitude $a[\sigma]_c$ would contain at least one $\delta(\omega_a)$ which is zero in the generic energy configuration where all the meson energies are non-vanishing. By analytic continuation, we can now assert that the cancellation always occurs and the baryon problem is self-consistent at all energies.

3 High-energy QCD scattering

Consider quark-quark elastic scattering in an $SU(N_c)$ theory where the ‘quarks’ are allowed to carry any color. In this problem $N_c$ could be 3 and is not necessarily large. We are interested in the situation where the energy $\sqrt{s}$ is much larger than the momentum transfer $\sqrt{-t}$. Since high-energy cross-sections are spin independent, what is being discussed below applies to gluon-gluon scattering as well.

Unlike the situation of last section where terms containing a $\delta(\omega_a)$ may be discarded, the destructive interference exhibited here is much more subtle. In a loop amplitude, the contribution from the $\delta$-function must be retained, but it turns out that the amplitude would be a factor of $\ln s$ down in the presence of each $\delta(\omega_a)$ [2, 4].

The role of the nonabelian quantum numbers $t_a$, in this case color, is also different. The commutators now specify in which color channels these interference effects are to be seen. For example, the cut diagram without any cut has no interference suppression in its amplitude. The quantum number of that term is given by a multiple commutator of $t_a$, corresponding to the adjoint color channel, so in $SU(3_c)$ the octet channel remains intact. The amplitude in any other color channel has at least one $\delta(\omega_a)$ and will be suppressed.

It can be shown [4] that such automatic suppression in the nonabelian cut diagrams does not occur in Feynman diagrams. Individual Feynman diagrams would bear a larger power of $\ln s$ in these channels, and the suppression would occur only when the individual Feynman diagrams are added together [2, 4, 7]. This means that contributions to such
color channels can be obtained only when individual Feynman diagrams are computed to subleading orders—a very difficult task in general. In contrast, this is not be a problem in nonabelian cut diagrams so they may be computed just to leading-log accuracies. This is a great advantage especially for high-order calculations.

The gluon in QCD scattering is known to be reggeized, and the Low-Nussinov Pomeron appears in the two-reggeon-exchange amplitude. A leading-log computation of this Pomeron leads to a total cross-section growing like a positive power of $s$, violating the Froissart bound. Subleading logarithms are therefore needed to restore unitarity to the BFKL equation, and these may come from multi-reggeon exchanges. What is lacking is the proof that the sum of Feynman diagrams indeed factorizes into multi-reggeon amplitudes. To be sure this factorization has been verified in low-order calculations, completely to the 6th order and partially to the 8th and 10th orders. An attempt to extend this to all orders in the usual approach would be extremely difficult, on account of the delicate cancellations discussed in the last paragraph, and because it is difficult to see why the sum of the complicated diagrams with all the criss-crossing of lines should factorize. Nonabelian cut diagrams are potentially capable of solving both of these difficulties, for delicate cancellations plaguing sums of Feynman diagrams do not occur here, and the factorization formula (1.2) which led to their derivation can also be used to obtain reggeon factorization. Accordingly we have initiated a program to use the nonabelian cut diagrams to study the validity of this multi-reggeon factorization. We are now able to prove it to be true for $s$-channel-ladder diagrams to all orders. More work is required for more complicated diagrams.

4 Acknowledgements

I wish to thank my collaborators Y.J. Feng, O. Hamidi-Ravari, and K.F. Liu. This research is supported in part by the by the Natural Science and Engineering Research Council of Canada, and the Fonds pour la Formation de Chercheurs et l’Aide à la Recherche de Québec.

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[*] Contributed paper to the 28th International Conference on High Energy Physics in Warsaw, July 1996.

[†] Electronic address: lam@physics.mcgill.ca

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