Continuous Time and Consistent Histories

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Abstract

We discuss the case of histories labelled by a continuous time parameter in the History Projection Operator consistent-histories quantum theory. In this approach—an extension to the generalised consistent histories theory—propositions about the history of the system are represented by projection operators on a Hilbert space. A continuous time parameter leads to a history algebra that is isomorphic to the canonical algebra of a quantum field theory. We describe how the appropriate representation of the history algebra may be chosen by requiring the existence of projection operators that represent propositions about time averages of the energy. We define the action operator for the consistent histories formalism, as the quantum analogue of the classical action functional, for the simple harmonic oscillator case. We show that the action operator is the generator of two types of time transformations that may be related to the two laws of time-evolution of the standard quantum theory: the state-vector reduction and the unitary time-evolution. We construct the corresponding classical histories and demonstrate the relevance with the quantum histories; we demonstrate how the requirement of the temporal logic structure of the theory is sufficient for the definition of classical histories. Furthermore, we show the relation of the action operator to the decoherence functional which describes the dynamics of the system. Finally, the discussion is extended to give a preliminary account of quantum field theory in this approach to the consistent histories formalism.
Chapter 1

Introduction

In classical Newtonian theory, time is introduced as an external parameter; and in all the existing approaches to quantum theory, the treatment of time is inherited from the classical theory. On the other hand, general relativity treats time as an internal parameter of the theory: in particular, it is one of the coordinates of the spacetime manifold. When we combine the two theories in quantum gravity, this essential difference in the treatment of time appears as a major problem—one of the aspects of what is known as the ‘Problem of Time’. One of the directions towards a solution of the problem is to construct ‘timeless’ quantum theories, i.e. theories where time is not a fundamental ingredient of the theory.

One such formalism is the consistent histories approach to quantum theory in which time appears as the label on a time-ordered sequence of projection operators which represents a ‘history’ of the system. In the original scheme by Gell-Mann and Hartle \[9, 10, 11\], the crucial object is the decoherence function written as \(d(\alpha, \beta) = \text{tr}(\hat{C}_\alpha \rho \hat{C}_\beta)\) where \(\rho\) is the initial density-matrix, and where the class operator \(\hat{C}_\alpha\) is defined in terms of the standard Schrödinger-picture projection operators \(\alpha_t\), as

\[
\hat{C}_\alpha := U(t_0, t_1)\alpha_{t_1} U(t_1, t_2)\alpha_{t_2} \ldots U(t_{n-1}, t_n)\alpha_{t_n} U(t_n, t_0)
\] (1.0.1)

where \(U(t, t') = e^{-i(t-t')H/\hbar}\) is the unitary time-evolution operator from time \(t\) to \(t'\). Each
projection operator \( \alpha_t \) represents a proposition about the system at time \( t \), and the class operator \( \tilde{C}_\alpha \) represents the composite history proposition “\( \alpha_t \) is true at time \( t_1 \), and then \( \alpha_{t_2} \) is true at time \( t_2 \), and then \( \ldots \), and then \( \alpha_{t_n} \) is true at time \( t_n \)”.

The motivation for the work that will be presented here, may be elucidated in several key points about the consistent histories theory construction.

1. The consistent histories approach allows the description of an approximately classical domain emerging from the macroscopic behaviour of a closed physical system, as well as its microscopic properties in terms of the conventional Copenhagen quantum mechanics. This is possible through the decoherence condition: the requirement for ‘decoherence’ (negligible interference between histories leads to the assignment of a probability measure) selects a consistent set of histories that can be represented on a classical (Boolean) logic lattice, thus having a classical logical structure. Hence in the consistent histories theory, emphasis must be given to the observation that, although in atomic scales a system is described by quantum mechanics, it may also be described by classical mechanics and ordinary logic. Therefore a more refined logical structure seems to be a necessary part of any consistent histories formalism. However, the Gell-Mann and Hartle approach lacks the logical structure of standard quantum mechanics in the sense that the fundamental entity (i.e. history) for the description of the system is not represented by a projector in the standard Hilbert space representation: as a product of (generically, non-commuting) projection operators, the class-operator \( \tilde{C}_\alpha \), representing a history, is not itself a projector.

2. This difference between the representation of propositions in standard quantum mechanics and in the history theory was resolved in the alternative approach of the ‘History Projection Operator’ (or HPO for short) theory [12, 13], in which the history proposition “\( \alpha_{t_1} \) is true at time \( t_1 \), and then \( \alpha_{t_2} \) is true at time \( t_2 \), and then \( \ldots \), and then \( \alpha_{t_n} \) is true at time \( t_n \)” is represented by the tensor product \( \alpha_{t_1} \otimes \alpha_{t_2} \otimes \cdots \otimes \alpha_{t_n} \) which, unlike \( \tilde{C}_\alpha \), is a genuine projection operator on the tensor product of copies of the standard Hilbert space \( \mathcal{V}_n = \mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2} \otimes \cdots \otimes \mathcal{H}_{t_n} \). Hence the ‘History Projection Operator’ formalism restores the quantum logic structure as it is in the case of single-time quantum theory.

3. However, the introduction of the tensor product \( \mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2} \otimes \cdots \otimes \mathcal{H}_{t_n} \) led to a
quantum theory where the notion of time appears mainly via its partial ordering property (quasi-temporal behaviour). In particular, we do not have a clear notion of time evolution in the sense that there is no natural way to express the time translations from one time slot—that refers to one copy of the Hilbert space $H_t$—to another one, that refers to another copy $H_{t'}$. As we shall see, the situation changes when the the continuous limit of such tensor products is introduced: henceforward, time appears uniformly in a continuous way.

4. One of the original problems in the development of the HPO theory was the lack of a clear physical meaning of the quantities involved. The introduction of the history group by Isham and Linden [8] made a significant step in this direction in the sense that, the spectral projectors of the history Lie algebra represent propositions about phase space observables of the system. Furthermore, it transpired that the history algebra for one-dimensional quantum mechanics is infinite dimensional—in fact, it is isomorphic to the canonical commutation algebra of a standard quantum field theory in one spatial dimension. This suggested that it would be profitable to study the history theory using tools that are normally employed in quantum field theory. We will use such tools extensively in what follows.

5. The choice of the continuous-time treatment introduced in the definition of the history algebra (history commutation relations at unequal times) by a delta-function, has a striking consequence: the physical observables of the theory are intrinsically time-averaged quantities; this means that the physical quantities cannot be defined at sharp moments in time. This is an important feature of the HPO theory. In this respect, it is closer to quantum field theory formalisms but with the essential difference that the time (spacetime) smearing does not appear only as a mathematical requirement but is also an intrinsic property of the fundamental elements of the theory.

This latter result, together with the preceding reasoning, was the starting point for the work that will be presented here. As we shall see, the introduction of the continuous-time treatment enables the definition of time transformations in the HPO theory, leading finally the notion of a time flow.

We will now give a brief description of the contents of this work.
In chapter 2, we summarise the generalised consistent histories theory in the form originally developed by Gell-Mann and Hartle. We also show how a history is represented in standard quantum mechanics, using the underlying logical structure of the theory. We then give a detailed presentation of the History Projection Operator theory based on the ideas of Isham \[12\]. In particular, we emphasise the logical structure of the theory, which is one of the key discriminating factors from previous consistent histories formalisms.

In chapter 3, we explain the choice of treating time as a continuous parameter. The construction of the history group is an important part of the HPO theory, therefore we present its definition and the original attempt to find the representation space of the history algebra for the example of a particle moving on a line $\mathcal{R}$, as presented in \[8\]. We comment on the observation that the history commutation relations are identical to the ones for the one spatial dimension quantum field theory. We then embark on a more physically motivated construction, based on the fact that the requirement for the existence of a Hamiltonian operator properly defined on the history space uniquely selects the history algebra representation space. In particular we examine the example of the simple harmonic oscillator in one dimension. As we explained previously, in HPO the interesting question that arises is how the Schrödinger-picture objects with different time labels—referring to the corresponding copies of the standard Hilbert space—are related. This work is the result of the collaboration between the authors of the article \[22\]. The work presented in the following chapters is published in \[1\].

In chapter 4 we will show that there is a transformation law ‘from one Hilbert space to another’. The generator of these time transformations is the ‘action operator’, a quantum analogue of the classical action functional. The definition of Heisenberg-picture operators will be used to demonstrate the time transformation law.

The main theme of chapter 5 is exactly the time transformation structure of HPO theory. In particular, we will show that there exist two types of time transformation, generated by the kinematical (Liouville operator) and the dynamical (Hamiltonian operator) part of the action operator. We will try to interpret the two-fold time law comparing it with the two types of time evolution of standard quantum theory, \textit{i.e.}, the state-vector reduction and the unitary time evolution.
It is interesting to examine how this novel structure with respect to the two time transformations in HPO appear in classical case. To this end, we define classical histories as an analogue of the quantum ones. Furthermore, we will show that, taking into account the temporal logic structure of the theory, classical histories can be defined without any reference to the quantum case.

The dynamics of the theory is described by the decoherence functional: it is natural to seek then the appearance of the action operator in its expression. We first present a summary of the use of coherent states for the definition of the decoherence functional as was originally presented in \cite{8}. Then we show that the operators involved have a functional relation with the action operator.

In chapter 6, we extend the discussion to the HPO theory of a free scalar field. In particular, starting from the quantum mechanics history group, we write a possible candidate for the quantum field theory history algebra. The question of external Lorentz invariance is examined. We comment on the intriguing result that in HPO two possible Poincaré groups appear, as a consequence of the two types of time transformation.
CHAPTER 1. INTRODUCTION
Chapter 2

A Review of the Consistent Histories

2.1 Introduction

The origins of the consistent histories theory lies in the attempt, introduced by Everett, to apply quantum mechanics to closed systems. The usual Copenhagen formulation of quantum mechanics is inadequate for quantum cosmology as it assumes a division of the universe into ‘observer’ and ‘observed’, and for the early universe it posits an external ‘classical domain’.

The post-Everett formulation of quantum mechanics stresses the consistency of probability sum rules as the primary criterion for determining which sets of histories may be assigned probabilities, and the *decoherence* (absence of interference between individual histories) as a sufficient condition for the consistency of probability sum rules. Such sets of histories are called ‘consistent’ or ‘decoherent’ and can be manipulated according to the rules of ordinary Boolean logic.

The consistent histories theory introduces a new treatment of the notion of time that
CHAPTER 2. A REVIEW OF THE CONSISTENT HISTORIES

opens up the possibility of eventually finding a different way to address the relevant problems in quantum gravity. In this sense, the Hamiltonian quantum mechanics is constructed by choosing one set of spacelike surfaces to define time. Hence it is restricted to a particular choice for the direction of time. Hartle partly resolved this problem by using the sum-over-histories formulation of quantum mechanics to bring histories in a spacetime form so that the theory does not require a privileged notion of time. In addition it works for a large variety of temporal coarse-grainings such as spacetime regions. However, it is restricted only to configuration space histories.

Isham suggested a refinement of the Gell-Mann and Hartle axioms for a generalized histories approach by constructing analogues of the lattice structure employed in standard quantum logic. Its quasitemporal structure is coded in a partial semigroup of temporal supports incorporated in the lattice of history propositions by the correspondence of a temporal support to each history proposition. This treatment of the notion of time is of great significance for quantum gravity and quantum field theories in curved spacetime.

2.2 The Formalism of Consistent Histories

The usual ‘Copenhagen’ formulation of quantum mechanics is inadequate for the description of possible histories for the universe because of the absence in this case of the external observer. There thereby arises a need to be able to assign probabilities about alternative histories of a subsystem without using the notion of a measurement procedure as a necessary ingredient. The idea that was developed in this direction was that the primary criterion for the assignment of probabilities is the consistency of probability sum rules, and the sufficient condition for this is the absence of quantum mechanical interference between individual histories, i.e., the notion of decoherence.

After the original attempt for a quantum description of the universe by Everett, there followed the construction of a generalized quantum mechanics that led to the formulation of the history theory approach.
2.2. THE FORMALISM OF CONSISTENT HISTORIES

2.2.1 Histories in Standard Quantum Theory

It is useful to summarise very briefly how ‘histories’ are understood in the conventional
interpretation of an open, Hamiltonian quantum system that is subject to measurements
by an external observer.

To this end, let \( U(t_1, t_0) \) denote the unitary time-evolution operator from time \( t_0 \) to
\( t_1 \); i.e., \( U(t_1, t_0) = e^{-i(t_1-t_0)H/\hbar} \). Then, in the Schrödinger picture, the density operator
state \( \rho(t_0) \) at time \( t_0 \) evolves in time \( t_1 - t_0 \) to \( \rho(t_1) \), where

\[
\rho(t_1) = U(t_1, t_0)\rho(t_0)U(t_1, t_0)^\dagger = U(t_1, t_0)\rho(t_0)U(t_1, t_0)^{-1}.
\]  

Suppose that a measurement is made at time \( t_1 \) of a property represented by a projection
operator \( P \). Then the probability that the property will be found is

\[
\text{Prob}(P = 1; \rho(t_1)) = \text{tr}(P\rho(t_1)) = \text{tr}(PU(t_1, t_0)\rho(t_0)U(t_1, t_0)^\dagger) = \text{tr}(P(t_1)\rho(t_0))
\]

(2.2.2)

where

\[
P(t_1) := U(t_1, t_0)^\dagger P(t_0)U(t_1, t_0)
\]

(2.2.3)
is the Heisenberg-picture operator defined with respect to the fiducial time \( t_0 \). If the result
of this measurement is kept then, according to the Von Neumann ‘reduction’ postulate,
the appropriate density matrix to use for any further calculations is

\[
\rho_{\text{red}}(t_1) := \frac{P(t_1)\rho(t_0)P(t_1)}{\text{tr}(P(t_1)\rho(t_0))}.
\]

(2.2.4)

Now suppose a measurement is performed of a second observable \( Q \) at time \( t_2 > t_1 \).
Then, according to the above, the conditional probability of getting \( Q = 1 \) at time \( t_2 \) given
that \( P = 1 \) was found at time \( t_1 \) (and that the original state was \( \rho(t_0) \)) is

\[
\text{Prob}(Q = 1|P = 1 \text{ at } t_1; \rho(t_0)) = \text{tr}(Q(t_2)\rho_{\text{red}}(t_1)) = \frac{\text{tr}(Q(t_2)P(t_1)\rho(t_0)P(t_1))}{\text{tr}(P(t_1)\rho(t_0))}
\]

(2.2.5)
The probability of getting \( P = 1 \) at \( t_1 \) and \( Q = 1 \) at \( t_2 \) is this conditional probability
multiplied by \( \text{Prob}(P = 1; \rho(t_1)) \), i.e.,

\[
\text{Prob}(P = 1 \text{ at } t_1 \text{ and } Q = 1 \text{ at } t_2; \rho(t_0)) = \text{tr}(Q(t_2)P(t_1)\rho(t_0)P(t_1)).
\]

(2.2.6)
Generalising to a sequence of measurements of propositions $\alpha_{t_1}, \alpha_{t_2}, \ldots, \alpha_{t_n}$ at times $t_1, t_2, \ldots, t_n$, the joint probability of finding all the associated properties is
\[
\text{Prob}(\alpha_{t_1} = 1 \text{ at } t_1 \text{ and } \alpha_{t_2} = 1 \text{ at } t_2 \text{ and } \ldots \alpha_{t_n} = 1 \text{ at } t_n; \rho(t_0)) = 
\text{tr} \left( \alpha_{t_n}(t_n) \ldots \alpha_{t_1}(t_1) \rho(t_0) \alpha_{t_1}(t_1) \ldots \alpha_{t_n}(t_n) \right) \quad (2.2.7)
\]

The conditions that must be satisfied for the probability assignments Eq. (2.2.7) to be consistent are presented below, in the context of the Gell-Mann and Hartle axioms.

### 2.2.2 The Gell-Mann and Hartle Generalised Consistent Histories Approach

The generalised consistent-histories approach to quantum theory can be formulated in several different ways. In the original scheme by Gell-Mann and Hartle [4, 10, 11], the main assumption of the consistent-histories interpretation of quantum theory is that, under appropriate conditions, a probability assignment is still meaningful for a closed system, with no external observers or associated measurement-induced state-vector reductions (thus signalling a move from ‘observables’ to ‘beables’). The satisfaction or otherwise of these conditions is determined by the behaviour of the \textit{decoherence functional} $d_{\rho,H}(\alpha, \beta)$ which, for the pair of sequences of projection operators $\alpha := (\alpha_{t_1}, \alpha_{t_2}, \ldots, \alpha_{t_n})$ and $\beta := (\beta_{t_1}, \beta_{t_2}, \ldots, \beta_{t_n})$ is defined as
\[
d_{\rho,H}(\alpha, \beta) = \text{tr}(\tilde{C}_\alpha^\dagger \rho \tilde{C}_\beta) \quad (2.2.8)
\]

where $\rho$ is the initial density-matrix, $H$ is the Hamiltonian, and where the \textit{class operator} $\tilde{C}_\alpha$ is defined in terms of the standard Schrödinger-picture projection operators $\alpha_{t_i}$ as
\[
\tilde{C}_\alpha := U(t_0, t_1)\alpha_{t_1}U(t_1, t_2)\alpha_{t_2} \ldots U(t_{n-1}, t_n)\alpha_{t_n}U(t_n, t_0), \quad (2.2.9)
\]

where $U(t, t') = e^{-i(t-t')H/\hbar}$ is the unitary time-evolution operator from time $t$ to $t'$. Each projection operator $\alpha_{t_i}$ represents a proposition about the system at time $t_i$, and the class operator $\tilde{C}_\alpha$ represents the composite history proposition “$\alpha_{t_1}$ is true at time $t_1$, and then $\alpha_{t_2}$ is true at time $t_2$, and then . . . , and then $\alpha_{t_n}$ is true at time $t_n$.”

At this point it is useful to gather together a few definitions that can be conveniently associated with these ideas.
2.2. THE FORMALISM OF CONSISTENT HISTORIES

• A homogeneous history is any time-ordered sequence \((\hat{\alpha}_{t_1}, \hat{\alpha}_{t_2}, \ldots, \hat{\alpha}_{t_n})\) of projection operators.

• A homogeneous history \(\beta := (\hat{\beta}_{t_1}, \hat{\beta}_{t_2}, \ldots, \hat{\beta}_{t_n})\) is coarser than another history \(\alpha := (\hat{\alpha}_{t_1}, \hat{\alpha}_{t_2}, \ldots, \hat{\alpha}_{t_n})\) if, for every \(t_i\), \(\hat{\alpha}_{t_i} \leq \hat{\beta}_{t_i}\) where \(\leq\) denotes the usual ordering operation on the space of projection operators, i.e., \(\hat{P} \leq \hat{Q}\) means that the range of \(\hat{P}\) is a subspace of the range of \(\hat{Q}\) (this includes the possibility that \(\hat{P} = \hat{Q}\) so that, in particular, every homogeneous history is trivially coarser than itself). This relation on the set of homogeneous histories is a partial ordering \(\leq\).

• Two homogeneous histories \(\alpha := (\hat{\alpha}_{t_1}, \hat{\alpha}_{t_2}, \ldots, \hat{\alpha}_{t_n})\) and \(\beta := (\hat{\beta}_{t_1}, \hat{\beta}_{t_2}, \ldots, \hat{\beta}_{t_n})\) are disjoint if, for at least one time point \(t_i\), \(\hat{\beta}_{t_i}\) is disjoint from \(\hat{\alpha}_{t_i}\), i.e., the ranges of these two projection operators are orthogonal subspaces of \(\mathcal{H}\).

• In calculating a decoherence functional it may be necessary to go outside the class of homogeneous histories to include inhomogeneous histories. A history of this type arises as a logical ‘or’ (denoted \(\lor\)) operation on a pair of disjoint homogeneous histories \(\alpha := (\hat{\alpha}_{t_1}, \hat{\alpha}_{t_2}, \ldots, \hat{\alpha}_{t_n})\) and \(\beta := (\hat{\beta}_{t_1}, \hat{\beta}_{t_2}, \ldots, \hat{\beta}_{t_n})\). Such a history \(\alpha \lor \beta\) is generally not itself a collection of projection operators (i.e., it is not homogeneous) but, when computing the decoherence functional, it is represented by the operator \(\hat{C}_{\alpha \lor \beta} := \hat{C}_\alpha + \hat{C}_\beta\). The coarse-graining relations \(\alpha \leq \alpha \lor \beta\) and \(\beta \leq \alpha \lor \beta\) are deemed to apply to this disjoint ‘or’ operation. The ‘negation’ operation \(\neg\) also usually turns a homogeneous history into an inhomogeneous history, with \(\hat{C}_{\neg \alpha} := 1 - \hat{C}_\alpha\).

A brief description of the elements of the theory follows.

2.2.3 The Gell-Mann and Hartle axioms

The Gell-Mann and Hartle axioms postulate a new approach to quantum theory in which the notion of history has a fundamental role; i.e., a ‘history’ in this generalised sense can be an irreducible entity in its own right, not necessarily derived from time-ordered strings of single-time propositions. These axioms and definitions are essentially as follows:

\(^1\)A relation \(\leq\) on a set \(X\) is a partial ordering if it satisfies the conditions (i) for all \(x \in X\), \(x \leq x\); (ii) \(x \leq y\) and \(y \leq x\) implies \(x = y\); and (iii) \(x \leq y\) and \(y \leq z\) implies \(x \leq z\).
CHAPTER 2. A REVIEW OF THE CONSISTENT HISTORIES

1. The fundamental ingredients in the theory are a space of histories and a space of decoherence functionals which are complex-valued functions of pairs of histories. The value \(d(\alpha, \beta)\) of such a decoherence functional \(d\) is a measure of the extent to which the histories \(\alpha\) and \(\beta\) ‘interfere’ with each other.

2. The set of histories possesses a partial order \(\leq\). If \(\alpha \leq \beta\) then \(\beta\) is said to be coarser than \(\alpha\), or a coarse-graining of \(\alpha\); dually, \(\alpha\) is a finer than \(\beta\), or a fine-graining of \(\beta\). Heuristically this means that \(\alpha\) provides a more precise specification than \(\beta\).

3. A history \(\alpha\) is defined to be fine-grained if the only histories \(\beta\) for which \(\beta \leq \alpha\) are 0 or \(\alpha\) itself. In standard quantum theory, histories of this type arise as time-ordered sequences of projection operators whose ranges are all one-dimensional subspaces of the Hilbert space. In general, the fine-grained histories are the sets of exhaustive, alternative histories of a closed system which are the most refined description to which one can contemplate assigning probabilities.

4. The allowed coarse grainings. The operation of coarse graining partitions a set of fine-grained histories into an exhaustive set of exclusive classes \(\{c_{\alpha}\}\), its class being a coarse-grained history.

5. There is a notion of two histories \(\alpha, \beta\) being disjoint, written \(\alpha \perp \beta\). Heuristically, if \(\alpha \perp \beta\) then if either \(\alpha\) or \(\beta\) is ‘realised’ the other is automatically excluded.

6. There is a unit history 1 (heuristically, the history that is always realised) and a null history 0 (heuristically, the history that is never realised). For all histories \(\alpha\) we have \(0 \leq \alpha \leq 1\).

7. Two histories \(\alpha, \beta\) that are disjoint can be combined to form a new history \(\alpha \lor \beta\) (heuristically, the history ‘\(\alpha\) or \(\beta\)’) which is the least upper bound of \(\alpha\) and \(\beta\) with respect to the partial ordering \(\leq\).

8. A set of histories \(\alpha^1, \alpha^2, \ldots, \alpha^N\) is said to be exclusive if \(\alpha^i \perp \alpha^j\) for all \(i, j = 1, 2, \ldots, N\). The set is exhaustive (or complete) if it is exclusive and if \(\alpha^1 \lor \alpha^2 \lor \ldots \lor \alpha^N = 1\).

9. The decoherence functional measures interference between the members of a coarse-grained set of histories. The decoherence functional is a complex-valued functional,
2.2. THE FORMALISM OF CONSISTENT HISTORIES

\( d(\alpha', \alpha) \), defined for each pair of histories in a coarse-grained set \( \{\alpha\} \). It must satisfy the following conditions:

i) Hermiticity: \( d(\alpha', \alpha) = d^*(\alpha', \alpha) \)

ii) Positivity: \( d(\alpha, \alpha) \geq 0 \)

iii) Normalization: \( \sum_{\alpha' \in \mathcal{C}} d(\alpha', \alpha) = 1 \)

It is important to note that this axiomatic scheme is given a physical interpretation only in relation to consistent sets of histories. A complete set \( \mathcal{C} \) of histories is said to be (strongly) consistent with respect to a particular decoherence functional \( d \) if \( d(\alpha, \beta) = 0 \) for all \( \alpha, \beta \in \mathcal{C} \) such that \( \alpha \neq \beta \). Under these circumstances, \( d(\alpha, \alpha) \) is given the physical interpretation as the probability that the history \( \alpha \) will be ‘realised’. The Gell-Mann and Hartle axioms then guarantee that the usual Kolmogoroff probability sum rules will be satisfied.

When we consider Hamiltonian quantum mechanics the set of histories are represented by chains of projections onto exhaustive sets of orthogonal subspaces of a Hilbert space. Then the fine-grained histories correspond to the possible sequences of sets of projections onto a complete set of states, a set at every time. The set of coarse-grained histories consist of sequences of independent alternatives at definite moments of time so that every history can be represented as a chain of projections.

In order to develop history theory to the fully 4-dimensional spacetime form that does not need a privileged notion of time, Hartle generalised quantum mechanics by enlarging the set of alternatives to include spacetime ones, not necessarily defined on spacelike surfaces. He then incorporated the dynamics of the theory by using the spacetime path integrals in the decoherence functional, thus succeeding in a construction that allows plurality in the selection of different temporal coarse grainings.

The sum over histories approach is described with the familiar three elements of a history theory. The fine-grained histories are paths in a configuration space of generalized coordinates \( \{q^i\} \), expressed as single-valued functions of the physical time. The operation of coarse-graining can be made with an especially natural partition of the configuration
space into a mutually exclusive and exhaustive set of subsets at each moment of time. The
decoherence functional can be written in a path-integral form as
\[
d(\alpha', \alpha) = \int_{C_{\alpha'}} \delta q' \int_{C_\alpha} \delta q \delta (q_f' - q_f) e^{i/\hbar (S[q'_f(\tau)] - S[q(\tau)])} \rho(q'_0, q_0)
\]  
for an interval of time t=0 to t=T. The integrals are defined over paths that begin at a
point \(q_0\) at \(t=0\), end at a point \(q_f\) at \(t=T\) and lie in the class \(c_\alpha\), and \(\rho(q'_0, q_0)\) is a density
matrix. The integration with the primes is defined in analogy. For the case of the above
partition, coming with a fixed choice of time, the decoherence functional coincides with
that of the Hamiltonian quantum mechanics histories.

In heuristic sense, the sum-over-histories approach provides a purely covariant frame-
work for the treatment of field theories because the path integration is over fields defined
over spacetime. The use of the path integral method in this case has the ability to ac-
commodate various choices of temporal structures as encoded within the different spacetime
coarse grainings. On the other hand, the theory is restricted only for configuration space
histories. This means that fewer sets of coarse-grained histories are possible since there is
a unique set of fine grained histories.

### 2.3 The History Projection Operator Approach: the
Discrete-Time Case

As was illustrated in the previous paragraph, the consistent histories programme affords
the possibility of escaping the measurement problem and the concept of state-vector re-
duction induced by an external observer associated with the Copenhagen interpretation
of quantum theory. In addition, the fact that the notion of history can be used as a funda-
mental theoretical entity rather than a time-ordered string of events, enables a novel way
of addressing the problem of time in quantum gravity situations. The ‘quasi-temporal’
nature of the consistent histories formalism inspired Isham to further develop a consistent
histories approach in such a way that the theory is equipped with a generalisation of the
quantum logic structure of the standard quantum theory.

Starting from the ideas of Mittlestaedt and Stachow on the logic of sequential propo-
2.3. THE HISTORY PROJECTION OPERATOR APPROACH: THE DISCRETE-TIME CASE

Isham’s key idea was the observation that the statement that a certain universe (i.e., history) is ‘realised’ is itself a proposition, and therefore the set of all such histories might possess a lattice structure analogous to the lattice of single-time propositions in standard quantum logic. In particular, a (general) history proposition should be represented by a projection operator in some Hilbert space. In the Gell-Mann and Hartle approach this is exactly not the case since the C-representation of a history defined as \( \hat{C}_\alpha := \hat{a}_{t_n}(t_n)\hat{a}_{t_{n-1}}(t_{n-1})\ldots\hat{a}_{t_1}(t_1) \), the product of (Heisenberg picture) projection operators \( \hat{a}_{t_k}(t_k) \) is usually not itself a projection operator.

2.3.1 A History Version of Standard Quantum Theory

One of the main aims was to find candidates for the ‘history analogues’ of the lattice \( \mathcal{L} \) and the state-space \( \mathcal{R} \) of the standard Hamiltonian theory. These analogues of \( \mathcal{L} \) and \( \mathcal{R} \) is a space \( \mathcal{UP} \) of history-propositions and a space \( \mathcal{D} \) of decoherence functionals. The construction of such a general scheme was motivated by the special example of a history version of standard quantum logic. By this is meant a generalisation of the ideas in paragraph 2.2 in which strings of projection operators are replaced by strings of single-time propositions belonging to the lattice \( \mathcal{L} \) of some ‘standard’ quantum logic theory.

In the quantum-logic version of the history theory we consider a system with a lattice \( \mathcal{L} \) of single-time propositions, and define a history filter to be any finite collection \( (\alpha_{t_1}, \alpha_{t_2}, \ldots, \alpha_{t_n}) \) of single-time propositions \( \alpha_{t_i} \in \mathcal{L} \) which is time-ordered in the sense that \( t_1 < t_2 < \ldots < t_n \). Thus, in the special case where \( \mathcal{L} \) is identified with the lattice \( P(\mathcal{H}) \) of projection operators on a Hilbert space \( \mathcal{H} \), a history filter is what we called a homogeneous history above.

In the case of standard quantum logic, a history filter is a time-labelled version of what Mittelstaedt and Stachow call a sequential conjunction \[^3^, ^3^, ^7^\] i.e., it corresponds to the proposition ‘\( \alpha_{t_1} \) is true at time \( t_1 \), and then \( \alpha_{t_2} \) is true at time \( t_2 \), and then \ldots and then \( \alpha_{t_n} \) is true at time \( t_n \)’. The phrase ‘history filter’ is intended to capture the idea that each single-time proposition \( \alpha_{t_i} \) in the collection \( (\alpha_{t_1}, \alpha_{t_2}, \ldots, \alpha_{t_n}) \) serves to ‘filter out’ the properties of the system that are realised in this potential history of the universe.
2.12 CHAPTER 2. A REVIEW OF THE CONSISTENT HISTORIES

It is important to be able to manipulate history filters that are associated with different sets of time points. To this end, it is useful to think of a history filter as something that is defined at every time point but which is ‘active’ only at a finite subset of points. This can be realised mathematically by defining it to be equal to the trivial proposition at all but the active points. More precisely, in standard quantum logic we shall define a history filter \( \alpha \) to be an element of the space \( \mathcal{F}(\mathcal{T}, \mathcal{L}) \) of maps from the space of time points \( \mathcal{T} \) (in the present case, the real line \( \mathbb{R} \)) to the lattice \( \mathcal{L} \) with the property that each map is (i) equal nowhere to the null single-time proposition, and (ii) equal to the unit single-time proposition for all but a finite set of \( t \) values. It will be convenient to append to this space the null history filter which is defined to be the null single-time proposition at all points \( t \in \mathcal{T} \).

It follows that, in a standard quantum theory realised on a Hilbert space \( \mathcal{H} \), a history filter (i.e., a homogeneous history) is represented by an element \( \alpha \) of the space of functions \( \mathcal{F}(\mathcal{T}, \mathcal{P}(\mathcal{H})) \) where \( \hat{\alpha}_t \) (the value of the map \( \alpha \) at \( t \in \mathcal{T} \)) is equal to the unit operator for all but a finite set of time points \( t \in \mathcal{T} \). Objects of this type can be regarded as projection operators on the weak direct sum \( \mathcal{F}(\mathcal{T}, \mathcal{H}) \) of \( \mathcal{H} \)-valued functions on \( \mathcal{T} \).

In the context of quantum cosmology, a history filter is a possible ‘universe’ complete with whatever quasi-temporal attributes it may, or may not, possess. For this reason, the set of all history filters in the general theory will be denoted \( \mathcal{U} \); in the case of standard quantum logic we will write \( \mathcal{U}(\mathcal{L}) := \mathcal{F}(\mathcal{T}, \mathcal{L}) \) to indicate the underlying lattice \( \mathcal{L} \) of single-time propositions.

The temporal properties of a history filter \( \alpha \in \mathcal{U}(\mathcal{L}) \) are encoded in the finite set of time points at which it is active; i.e., the points \( t \in \mathcal{T} \) such that \( \alpha_t \neq 1 \). This motivates the following definitions:

1. The set of \( t \in \mathcal{T} \) for which \( \alpha_t \neq 1 \) is called the temporal support, or just support, of \( \alpha \in \mathcal{U}(\mathcal{L}) \), and is denoted \( \sigma(\alpha) \).

2. The set of all possible temporal supports will be denoted \( \mathcal{S} \); in the present case this is just the set of all ordered finite subsets of \( \mathcal{T} = \mathbb{R} \).

3. The support of the null history is defined to be the empty subset of \( \mathbb{R} \).
The fact that the essential temporal properties of the space of history filters $U(L)$ in standard Hamiltonian quantum theory is reflected in its set of temporal supports raised the possibility to construct a theory that is more general than the standard theory and in which the quasi-temporal structure is reflected in the structure of the support space. Furthermore, in the general case in order for the ‘or’ and ‘not’ operations to be included, $U$ is extended to a larger space $UP$ of all history propositions which has the structure of an orthocomplemented lattice.

2.3.2 The HPO-theory for Standard Quantum Theory

The next step was to construct an operator representation of standard quantum theory in which every history proposition is represented by a genuine projection operator, thus the whole $UP$ can be identified with the projection lattice of some new Hilbert space. To accomplish this Isham examined the special case where $L = P(H)$, i.e., a Hilbert space based quantum system rather than a general lattice $L$. Still, for a general history theory called an HPO theory (History Projection Operator theory), the space $UP$ is an ortho-complemented lattice that can be represented by projection operators.

As we have explained previously, $\hat{C}_a$ is not a projection operator, and therefore it is not part of the propositional lattice associated with the Hilbert space $H$ on which it is defined. This makes it difficult to know what is the $C$-representative of, for example, $\alpha \lor \beta$ when $\alpha$ and $\beta$ are not disjoint. Indeed, if $P$ and $Q$ are projection operators, the product $PQ$ generally fails to be so. However, the tensor product $P \otimes Q$ is a projection operator, and is hence a candidate to represent the two-time homogeneous history $(P, Q)$. More generally, if we consider the set $U\{t_1, \ldots, t_n\}$ of all homogeneous histories with (for the moment) a fixed support $\{t_1, t_2, \ldots, t_n\}$, we can represent any such $\alpha = (\hat{a}_{t_1}, \hat{a}_{t_2}, \ldots, \hat{a}_{t_n})$ with the tensor product

$$\theta(\hat{a}_{t_1}, \hat{a}_{t_2}, \ldots, \hat{a}_{t_n}) := \hat{a}_{t_1} \otimes \hat{a}_{t_2} \otimes \ldots \otimes \hat{a}_{t_n} \quad (2.3.1)$$

which acts on the tensor-product space $\otimes_{t \in \{t_1, \ldots, t_n\}} H_t$ of $n$ copies of $H$.

That the tensor product appears in a natural way can be seen from the following
Hence the map from $\alpha$ is that unlike the standard representation with in the histories approach to standard quantum theory. The important result from this homogeneous history ($\alpha$ vector space $W$ of vector spaces $V$ is multilinear operator. 

In constructing the decoherence functional, the map

$$ (\hat{\alpha}_t_1, \hat{\alpha}_t_2, \ldots, \hat{\alpha}_t_n) \mapsto \text{tr} (\hat{\alpha}_t_n(t_n)\hat{\alpha}_t_{n-1}(t_{n-1}) \ldots \hat{\alpha}_t_1(t_1)\hat{B}) $$

(2.3.2)

is multilinear with respect to the vector space structure of $\oplus_{t \in \{t_1, \ldots, t_n\}} B(H)_t$ for any $\hat{B} \in B(H)$. However, the fundamental property of the tensor product of a finite collection of vector spaces $V_1, V_2, \ldots, V_n$ is that any multilinear map $\mu : V_1 \times V_2 \times \ldots \times V_n \to W$ to a vector space $W$ factorises uniquely through the tensor product to give the chain of maps

$$ V_1 \times V_2 \times \ldots \times V_n \overset{\theta}{\rightarrow} V_1 \otimes V_2 \otimes \ldots \otimes V_n \overset{\theta'}{\rightarrow} W. $$

(2.3.3)

Hence the map from $\alpha = (\hat{\alpha}_t_1, \hat{\alpha}_t_2, \ldots, \hat{\alpha}_t_n)$ to $\hat{\alpha}_t_1 \otimes \hat{\alpha}_t_2 \otimes \ldots \otimes \hat{\alpha}_t_n$ arises naturally in the histories approach to standard quantum theory. The important result from this construction is that unlike the standard representation with $\hat{C}_\alpha$, no information about the homogeneous history $(\hat{\alpha}_t_1, \hat{\alpha}_t_2, \ldots, \hat{\alpha}_t_n)$ is lost by representing it with the tensor product $\hat{\alpha}_t_1 \otimes \hat{\alpha}_t_2 \otimes \ldots \otimes \hat{\alpha}_t_n$ and unlike $\hat{C}_\alpha$, the operator $\hat{\alpha}_t_1 \otimes \hat{\alpha}_t_2 \otimes \ldots \otimes \hat{\alpha}_t_n$ is a projection operator.

Hence, with the aid of the map $\theta$, the operator representation $\prod_{t \in \{t_1, \ldots, t_n\}} P(H)_t$ of the space of homogeneous histories $U_{\{t_1, \ldots, t_n\}}$ with temporal support $\{t_1, t_2, \ldots, t_n\}$ is embedded in the space $P(\otimes_{t \in \{t_1, \ldots, t_n\}} H_t)$ of projection operators on the Hilbert space $\otimes_{t \in \{t_1, \ldots, t_n\}} H_t$. The space $P(\otimes_{t \in \{t_1, \ldots, t_n\}} H_t)$ carries the usual lattice structure of projection operators and is therefore a natural model for the space of history propositions based on homogeneous histories with support $\{t_1, t_2, \ldots, t_n\}$. In this model, history filters/homogeneous histories are represented by homogeneous projectors, and a general history proposition is represented by an inhomogeneous projector. This explains why the collection $(\hat{\alpha}_t_1, \hat{\alpha}_t_2, \ldots, \hat{\alpha}_t_n)$ was referred to earlier as a ‘homogeneous’ history.

The decoherence functionals will be computed with the aid of the map $D : \otimes_{t \in \{t_1, \ldots, t_n\}} B(H) \to B(H)$ defined by

$$ D(\hat{A}_1 \otimes \hat{A}_2 \otimes \ldots \hat{A}_n) := \hat{A}_n(t_n)\hat{A}_{n-1}(t_{n-1}) \ldots \hat{A}_1(t_1) $$

(2.3.4)

on homogeneous operators and then extended by linearity. Thus, on a homogeneous history $\alpha \in U_{\{t_1, \ldots, t_n\}}$, the $C$-map is defined by

$$ \hat{C}_\alpha := D(\theta(\alpha)) $$

(2.3.5)
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and then extended by linearity to the appropriate set of inhomogeneous histories.

To incorporate arbitrary supports one needs to collect together the operator algebras \( \otimes_{t \in s} B(\mathcal{H})_t \) for all supports \( s \in \mathcal{S} \). The natural way of doing this is to use an infinite tensor product of copies of \( B(\mathcal{H}) \).

Let \( \Omega \) denote a family of unit vectors in the Cartesian product \( \prod_{t \in \mathcal{T}} \mathcal{H}_t \) of copies of \( \mathcal{H} \) labelled by the time values \( t \in \mathcal{T} \); i.e., \( t \mapsto \Omega_t \) is a map from \( \mathcal{T} \) to the unit sphere in \( \mathcal{H} \). Then an infinite tensor product \( \otimes_{t \in \mathcal{T}} \Omega_t \) of operator algebras \( B(\mathcal{H}) \) is naturally associated with this case. It is defined to be the weak closure (i.e., the closure in the weak operator topology) of the set of all finite sums of functions from \( \mathcal{T} \) to \( B(\mathcal{H}) \) that are equal to the unit operator for all but a finite set of \( t \)-values. This definition accommodates arbitrary temporal supports, and the set of all projection operators in \( \otimes_{t \in \mathcal{T}} B(\mathcal{H})_t \) can be taken as a model for the complete space \( \mathcal{UP} \) of history propositions in a standard Hilbert-space based, quantum theory.

2.3.3 The General Axioms for History Propositions

The axioms for the HPO approach to the consistent histories theory can be viewed as a more detailed version of the original Gell-Mann and Hartle axioms. The general axioms and definitions are as follows.

H1. The space of history filters. The fundamental ingredient in a theory of histories is a space \( \mathcal{U} \) of history filters, or possible universes. This space has the following structure.

1. \( \mathcal{U} \) is a partially-ordered set with a unit history filter 1 and a null history filter 0 such that \( 0 \leq \alpha \leq 1 \) for all \( \alpha \in \mathcal{U} \).

2. \( \mathcal{U} \) has a meet operation \( \wedge \) which combines with the partial order \( \leq \) to form a meet semi-lattice with unit 1 so that \( 1 \wedge \alpha = \alpha \) for all \( \alpha \in \mathcal{U} \). The null history is absorptive in the sense that \( 0 \wedge \alpha = 0 \) for all \( \alpha \in \mathcal{U} \).

3. \( \mathcal{U} \) is a partial semi-group with composition law denoted \( \circ \). If \( \alpha, \beta \in \mathcal{U} \) can be combined to give \( \alpha \circ \beta \in \mathcal{U} \) we say that \( \beta \) follows \( \alpha \), or \( \alpha \) precedes \( \beta \), and write \( \alpha \triangleleft \beta \).
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The ◦ and ∧ laws are compatible in the sense that if α ◦ β is defined then it is equal to α ∧ β.

4. The null and unit histories can always be combined with any history filter α to give

\[
\alpha \circ 1 = 1 \circ \alpha = \alpha \quad (2.3.6) \\
\alpha \circ 0 = 0 \circ \alpha = 0. \quad (2.3.7)
\]

H2. \textit{The space of temporal supports.} Any quasi-temporal properties of the system are encoded in a partial semi-group \(\mathcal{S}\) of supports with unit *. The support space has the following properties.

1. There is a homomorphism \(\sigma: \mathcal{U} \to \mathcal{S}\) of partial semi-groups that assigns a support to each history filter. The support of 0 and 1 is defined to be * \(\in \mathcal{S}\).

2. A history filter \(\alpha\) is \textit{nuclear} if it has no non-trivial decomposition of the form \(\alpha = \beta \circ \gamma\) with \(\beta, \gamma \in \mathcal{U}\); a temporal support \(s\) is \textit{nuclear} if it has no non-trivial decomposition of the form \(s = s_1 \circ s_2\) with \(s_1, s_2 \in \mathcal{S}\). Nuclear supports are the analogues of points of time; nuclear history filters are the analogues of single-time propositions.

A decomposition of \(\alpha \in \mathcal{U}\) as \(\alpha = \alpha^1 \circ \alpha^2 \circ \ldots \circ \alpha^N\) is \textit{irreducible} if the constituent history filters \(\alpha^i \in \mathcal{U}, i = 1, 2, \ldots, N\) are all nuclear.

A \textit{resolution} of the semi-group homomorphism \(\sigma: \mathcal{U} \to \mathcal{S}\) is a chain of semigroups \(\mathcal{U}_i\) and semi-group homomorphisms \(\sigma_i\) so that \(\sigma\) factors as the composition

\[
\mathcal{U} \xrightarrow{\sigma} \mathcal{U}_1 \xrightarrow{\sigma_1} \mathcal{U}_2 \xrightarrow{\sigma_2} \ldots \xrightarrow{\sigma_{k-1}} \mathcal{U}_k \xrightarrow{\sigma_k} \mathcal{S}. \quad (2.3.8)
\]

H3. \textit{The space of history propositions.} The space \(\mathcal{U}\) of history filters is embedded in a larger space \(\mathcal{U}\mathcal{P}\) of \textit{history propositions}. This space is an ortho-complemented lattice with a structure that is consistent with the semi-lattice structure on the subspace \(\mathcal{U}\). One may also require the lattice to be countably complete, or even complete, depending on its cardinality. In addition:

1. The space \(\mathcal{U}\mathcal{P}\) can be generated from \(\mathcal{U}\) by the application of a finite (or, perhaps, countably infinite) number of \(\neg, \lor\) and \(\land\) lattice operations. This captures the idea
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that elements of \( \mathcal{UP} \) represent propositions ‘about’ history filters (i.e., about possible universes).

2. An important role may be played by representations of the partial semi-group \( \mathcal{S} \) in the automorphism group of the lattice \( \mathcal{UP} \).

Any representation of the ortho-complemented lattice \( \mathcal{UP} \) by projection operators on a Hilbert space is called an \textit{HPO quantisation} of the system.

Two history propositions \( \alpha \) and \( \beta \) are said to be \textit{disjoint}, denoted \( \alpha \perp \beta \), if \( \alpha \leq \neg \beta \).

A set of history propositions \( \{\alpha^1, \alpha^2, \ldots, \alpha^N\} \) is \textit{exclusive} if its elements are pairwise disjoint. It is \textit{exhaustive (or complete)} if \( \alpha^1 \lor \alpha^2 \lor \ldots \lor \alpha^N = 1 \). Countable sets (i.e., with \( N = \infty \)) are permitted where appropriate.

It should be noted that the definitions above are the direct HPO analogues of the corresponding ideas introduced earlier in the context of the Gell-Mann and Hartle axioms for consistent histories.

H4. \textit{The space of decoherence functionals.} A \textit{decoherence functional} is a complex-valued map of pairs of history propositions; the set of all such maps is denoted \( \mathcal{D} \). There may be a natural topology on \( \mathcal{UP} \) such that each decoherence functional \( d \in \mathcal{D} \) is a \textit{continuous} function of its arguments. Any decoherence functional has the following properties:

1. The ‘inner-product’ type conditions:
   - \textit{Hermiticity}: \( d(\alpha, \beta) = d(\beta, \alpha)^* \) for all \( \alpha, \beta \in \mathcal{UP} \).
   - \textit{Positivity}: \( d(\alpha, \alpha) \geq 0 \) for all \( \alpha \in \mathcal{UP} \).
   - \textit{Null triviality}: \( d(0, \alpha) = 0 \) for all \( \alpha \in \mathcal{UP} \).

2. Conditions related to the potential probabilistic interpretation:
   - \textit{Additivity}: if \( \alpha \perp \beta \) are general history propositions then, for all \( \gamma \in \mathcal{UP} \), \( d(\alpha \lor \beta, \gamma) = d(\alpha, \gamma) + d(\beta, \gamma) \).
   - \textit{Normalisation}: \( d(1, 1) = 1 \).

\footnote{Such a condition holds in standard quantum theory because \( \hat{A} \mapsto \text{tr}(\hat{A}\hat{B}) \) is a weakly continuous function on bounded subsets of \( \mathcal{B}(\mathcal{H}) \) for each trace-class operator \( \hat{B} \).}
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H5. The (tentative) physical interpretation of these axioms is the same as that of the Gell-Mann and Hartle axioms, \textit{i.e.}, the diagonal element $d(\alpha, \alpha)$ is interpreted as the probability of the history proposition $\alpha$ being ‘true’ when $\alpha$ is part of a consistent set. If this is not the case, no direct physical meaning is ascribed to the real number $d(\alpha, \alpha)$.

To summarise, in the HPO theory every history proposition is represented as a projection operator on a certain Hilbert space. This provides valuable clues about the possible lattice structure on $\mathcal{UP}$ in the general case and suggests the existence of novel concepts. The collection $\mathcal{UP}$ of all history propositions in a general history theory can be equipped with a lattice structure that is similar in some respects to the lattice of propositions in standard quantum logic. Any quasi-temporal properties of the theory are coded in the space $\mathcal{S}$ of supports associated with the subspace $\mathcal{U}$ of history filters. A Boolean lattice would correspond to a history version of a classical theory, and quantum-mechanical superselection rules would arise in the usual way via the existence of a non-trivial center for the lattice $\mathcal{UP}$. 
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Chapter 3

Continuous Time in the History Projection Operator Theory

3.1 Introduction

The introduction of a continuous time clearly poses difficulties for any approach to the consistent-history theory: in the class-operator scheme one has to define continuous products of projection operators. In the HPO approach, the difficulty is to define a continuous tensor product of projection operators.

In the original construction of the continuous-time histories by Isham and Linden \cite{8} the problem was resolved by exploiting the existence of continuous tensor products of coherent states. However, several interesting issues were sidestepped in the process. For example, the projectors onto coherent states do not have a clear physical interpretation.

In what follows, we will re-address the question of continuous time in the HPO theory, inclining towards a more physically-motivated construction. As in \cite{8}, the starting point is the history group: a history-analogue of the canonical group used in standard quantum mechanics. The key idea is that a unitary representation of the history group
leads to a self-adjoint representation of its Lie algebra, the spectral projectors of which are to be interpreted as propositions about the histories of the system. Thus we employ a history group whose associated projection operators represent propositions about continuous-time histories. As we shall see, it will transpire that the history algebra for one-dimensional quantum mechanics is infinite dimensional—in fact, it is isomorphic to the canonical commutation algebra of a standard quantum field theory in one spatial dimension. This suggests that it might be profitable to study the history theory using tools that are normally employed in quantum field theory. In [22], we showed that the physically appropriate representation of the history algebra can be selected by requiring the existence of operators that represent propositions about the time-averaged values of the energy. The Fock space thus constructed can be related to the notion of a continuous tensor product as used in [8], thus establishing the link with the idea of continuous temporal logic.

3.2 The Choice of Time as a Continuous Parameter

Most discussions of the consistent-histories formalism have involved histories defined at a finite set of discrete time points. However, it is important to extend this to include a continuous time variable, especially for potential applications to quantum field theory and quantum gravity.

As was mentioned in the Introduction, temporal logic is a structure that ought to be of particular importance in any approach to consistent-histories theory. Indeed, this is one of the key features of the HPO theory. In normal, single-time quantum mechanics, a statement about a physical quantity is represented by a projector on the standard Hilbert space. Likewise in HPO, a history (a temporal statement about properties of the system), is represented by a genuine projection operator (a tensor product of projectors), on the tensor product of copies of the standard Hilbert space. The one effect of introducing the temporal logic by using tensor products, is the fact that, there is no natural way of generating a time translation from one time slot to another. As we shall see, this situation changes when we consider the the continuous limit of such tensor products, in which time now appears uniformly in a continuous way.
3.3. THE HISTORY SPACE

However, even after having made this step, one is still involved in the use of quantities (specifically, certain projectors in the history quantum space) that do not have a clear physical meaning. For this, Isham and Linden \[8\] introduced the history group for discrete-time histories, and hence made a significant connection in which the spectral projectors of the history Lie algebra represent propositions about phase space observables of the system. The next crucial step is to introduce a continuous time variable by introducing a delta function in the description of its history commutation relations (at unequal times).

As an immediate consequence, an intriguing feature of the HPO theory appears; that all interesting history propositions are about time-averaged physical quantities. In other words, the physical quantities in HPO are time averaged and they cannot be defined at sharp moments in time. Whether or not one considers this to be a more natural way to identify physical observables is, to a certain extent, a matter of opinion. However, it is worth emphasising that, in quantum field theories, only after proper spacetime averaging (the analogue situation of the history quantum mechanics time averaging) do the operator-valued distributions correspond to physical observables.

Hence, one can argue that a theory in which the fundamental elements are time-averaged quantities by construction—without at the same time contradicting the standard quantum theory treatment—generates an interest that should be exploited.

Finally, it is worth adding that, to some extent, it is a matter of personal opinion, whether or not time should be regarded as a continuum. But then, following a similar reasoning, it is natural to expect the physical quantities not to be defined at sharp moments of time. Hence the time-averaged, or (field theoretic) spacetime-averaged observables emerge from the physical interpretation of the theory, rather than purely from mathematical necessity.

3.3 The History Space
3.3.1 The History Group

We start by considering the HPO version of the quantum theory of a particle moving on the real line $\mathbb{R}$. As explained before, the history proposition “$\alpha_{t_1}$ is true at time $t_1$, and then $\alpha_{t_2}$ is true at time $t_2$, and then . . . , and then $\alpha_{t_n}$ is true at time $t_n$” is represented by the projection operator $\alpha_{t_1} \otimes \alpha_{t_2} \otimes \cdots \otimes \alpha_{t_n}$, on the $n$-fold tensor product $V_n = \mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2} \otimes \cdots \otimes \mathcal{H}_{t_n}$, of $n$-copies of the Hilbert-space $\mathcal{H}$, of the canonical theory. Since $\mathcal{H}$ carries a representation of the Heisenberg-Weyl group with Lie algebra

$$[x, p] = i\hbar,$$

the Hilbert space $V_n$ is expected to carry a unitary representation of the $n$-fold product group whose generators satisfy

$$[x_k, x_m] = 0 \quad (3.3.2)$$
$$[p_k, p_m] = 0 \quad (3.3.3)$$
$$[x_k, p_m] = i\hbar \delta_{km} \quad (3.3.4)$$

with $k, m = 1, 2, \ldots, n$. Thus the Hilbert space $V_n$ carries a representation of the ‘history group’ whose Lie algebra is defined to be that of Eqs. (3.3.2)–(3.3.4). However, we can also turn the argument around and define the history version of $n$-time quantum mechanics by starting with Eqs. (3.3.2)–(3.3.4). In this approach, $V_n$ arises as a representation space for Eqs. (3.3.2)–(3.3.4), and tensor products $\alpha_{t_1} \otimes \alpha_{t_2} \otimes \cdots \otimes \alpha_{t_n}$ that correspond to sequential histories about the values of position or momentum (or linear combinations of them) are then elements of the spectral representations of this Lie algebra.

We shall employ this approach to discuss continuous-time histories. Thus, motivated by Eqs. (3.3.2)–(3.3.4), we start with the history-group whose Lie algebra (referred to in what follows as the ‘history algebra’, or HA for short) is

$$[x_{t_1}, x_{t_2}] = 0 \quad (3.3.5)$$
$$[p_{t_1}, p_{t_2}] = 0 \quad (3.3.6)$$
$$[x_{t_1}, p_{t_2}] = i\hbar \tau \delta(t_1 - t_2) \quad (3.3.7)$$

where $-\infty \leq t_1, t_2 \leq \infty$; the constant $\tau$ has dimensions of time [31]. Note that these
operators are in the Schrödinger picture: they must not be confused with the Heisenberg-picture operators $x(t), p(t)$ of normal quantum theory.

The choice of the Dirac delta function in the right hand side of Eq. (3.3.7), instead of the Kronecker delta function that seems more natural in dealing with Schrödinger picture operators, is closely associated with the requirement for treating time as a continuous variable. As emphasised earlier, one consequence is the fact that the observables cannot be defined at sharp moments of time but rather as time-averaged quantities.

An important observation is that Eqs. (3.3.5)–(3.3.7) are mathematically the same as the canonical commutation relations of a quantum field theory in one space dimension:

\begin{align}
[\phi(x_1), \phi(x_2)] &= 0 \tag{3.3.8} \\
[\pi(x_1), \pi(x_2)] &= 0 \tag{3.3.9} \\
[\phi(x_1), \pi(x_2)] &= i\hbar\delta(x_1 - x_2). \tag{3.3.10}
\end{align}

This analogy will be exploited fully in this chapter. For example, the following two issues arise immediately. Firstly—to be mathematically well-defined—equations of the type Eqs. (3.3.5)–(3.3.7) must be smeared with test functions to give

\begin{align}
[x_f, x_g] &= 0 \tag{3.3.11} \\
[p_f, p_g] &= 0 \tag{3.3.12} \\
[x_f, p_g] &= i\hbar \tau \int_{-\infty}^{\infty} f(t) g(t) \, dt, \tag{3.3.13}
\end{align}

which leads at once to the question of which class $s$ of test functions to use. The minimal requirement for the right hand side of Eq. (3.3.13) to make sense is that $s$ must be a linear subspace of the space $L^2(\mathbb{R}, dt)$ of square integrable functions on $\mathbb{R}$. For the moment we shall leave $s$ unspecified beyond this.

The second issue is concerned with finding the physically appropriate representation of the HA Eqs. (3.3.11)–(3.3.13), bearing in mind that infinitely many unitarily inequivalent representations are known to exist in the analogous case of Eqs. (3.3.29)–(3.3.31). Note that this problem does not arise in standard quantum mechanics, or in the history version of quantum mechanics with propositions defined at a finite number of times, since—by
CHAPTER 3. CONTINUOUS TIME IN THE HISTORY PROJECTION OPERATOR THEORY

the Stone-von Neumann theorem—there is a unique representation of the corresponding algebra up to unitarily equivalence.

Of course, the physically appropriate Fock representation in the histories formalism is expected to involve some type of continuous tensor product; this was the path followed in \[8\]. On the other hand, according to a famous paper by Araki \[15\], in standard quantum field theory, the requirement that the Hamiltonian exists as a proper self-adjoint operator is sufficient to select a unique representation; for example, the representations appropriate for a free boson field with different masses are unitarily inequivalent. In our case, this suggests that the appropriate representation of the algebra Eqs. (3.3.11)–(3.3.13) should be chosen by requiring the existence of operators that represent history propositions about (time-averaged) values of the energy. As we shall see, this is indeed the case.

Before we exploit this statement further, it is useful to briefly review the original work for finding the representation space of the HA \[8\].

3.3.2 The History Group Representation on \( \mathcal{V}_{cts} \)

From the perspective of the history theory the physically appropriate representation space is expected to involve some type of continuous tensor product. Indeed in \[8\], Isham and Linden showed how the requirement of representing the history group on a Hilbert space (which we will denote \( \mathcal{V}_{cts} \)) leads to a continuous tensor product of copies of the standard Hilbert space \( L^2(\mathbb{R}, dx) \).

The starting point is the fact that the representation space of the single-time Weyl group Eq. (3.3.1) can be written as a Fock space \( \exp T \) where \( T \simeq \mathbb{C} \); hence in order to find the representation of the history group Eqs. (3.3.11)–(3.3.13), one proceeds by using Fock construction techniques. This involves taking a complexification of the space of real test functions \( L^2(\mathbb{R}, dx) \) used to smear the generators of the history algebra Eqs. (3.3.11)–(3.3.13)

\[
L^2(\mathbb{R}, dx) \cong L^2(\mathbb{R}, dx) \oplus L^2(\mathbb{R}, dx) \\
L^2(\mathbb{R}, dx) \cong \mathbb{C}
\]

(3.3.14)

where we have chosen the complexified space to be \( T = \mathbb{C} \).
3.3. **THE HISTORY SPACE**

Each Fock space carries a special class of states, the normalised coherent states defined as

\[ |z\rangle \overset{\text{def}}{=} e^{-\frac{1}{2}|z|^2} |z\rangle + z a^\dagger |0\rangle, \quad (3.3.15) \]

and satisfy \( \langle z | w \rangle = e^{-\frac{1}{2}|z|^2 - \frac{1}{2}|w|^2 + z^* w} \). The idea of coherent states is a necessary mathematical tool in the construction of continuous tensor products. They are related to the elements \( \exp z \rangle \) of the Fock space \( \exp \mathbb{C} \) by

\[ \exp z \rangle = e^{\frac{1}{2}|z|^2} |z\rangle, \]

and the Hilbert space \( \exp \mathbb{C} \) is isomorphic to \( L^2(\mathbb{C}) \) via

\[ \exp z \rangle \mapsto \langle x | \exp z \rangle = (2\pi)^{-\frac{1}{4}} e^{\frac{1}{2}xz - \frac{1}{4}|z|^2 - \frac{1}{4}x^2}. \]

On the other hand, we consider the continuous tensor product of a one-parameter family of \( t \mapsto \mathcal{H}_t \) of standard Hilbert spaces. In general, one may try to define the inner product as

\[ \langle \otimes_t u_t | \otimes_t v_t \rangle \overset{\text{def}}{=} e^{\int_{-\infty}^{\infty} \log \langle u_t, v_t \rangle_{\mathcal{H}_t} dt} \]

(3.3.17)

This is intended to be the continuous analogue of the inner product between discrete tensor products of vectors

\[ \langle u_1 \otimes u_2 \otimes \cdots \otimes u_n | v_1 \otimes v_2 \otimes \cdots \otimes v_n \rangle \overset{\text{def}}{=} \prod_{i=1}^{n} \langle u_i, v_i \rangle = e^{\sum_{i=1}^{n} \log \langle u_i, v_i \rangle}. \]

(3.3.18)

If \( \mathcal{H}_t \) is an exponential Hilbert space \( \mathcal{H}_t = \exp K_t \), then the construction works since

\[ \langle \exp \phi_t | \exp \psi_t \rangle_{\mathcal{K}_t} = e^{(\phi_t, \psi_t)_{\mathcal{K}_t}} \]

(3.3.19)

and so the definition of the scalar product on the continuous tensor product of copies of \( \exp K_t \) as

\[ \langle \otimes_t \exp \phi_t | \otimes_t \exp \psi_t \rangle_{\otimes_t \mathcal{K}_t} \overset{\text{def}}{=} e^{\int_{-\infty}^{\infty} (\phi_t, \psi_t)_{\mathcal{K}_t} dt} \]

(3.3.20)

is well-defined.

Furthermore, the scalar product \( \int_{-\infty}^{\infty} (\phi_t, \psi_t)_{\mathcal{K}_t} dt \) is the inner product on the direct integral Hilbert space \( \int \mathcal{K}_t \), hence we can write

\[ \langle \otimes_t \exp \phi_t | \otimes_t \exp \psi_t \rangle_{\otimes_t \mathcal{K}_t} = \langle \exp \phi(\cdot) | \exp \psi(\cdot) \rangle_{\exp \int \mathcal{K}_t} \]

(3.3.21)
In fact, there exists the useful isomorphism
\[ \otimes_t \exp \mathcal{K}_t \simeq \exp \int \mathcal{K}_t \]  
\[ \otimes_t |\exp \phi_t \rangle \mapsto |\exp (\cdot)\rangle \]  
(3.3.22)

The existence of yet another isomorphism between the Hilbert space \( L^2(\mathbb{R}) \) and the direct integral \( \int \oplus C_t dt \) via
\[ \int \oplus C_t dt \simeq L^2(\mathbb{R}, dt) \]  
(3.3.23)
\[ \int \oplus w_t dt \mapsto w(\cdot) \]  
(3.3.24)
links together the previous results with the Fock space \( \exp C_t \simeq L^2(\mathbb{R}) \) that represents the history group, i.e. the continuous-time Weyl group.

Indeed, for the special case \( \mathcal{K}_t = C_t \), we summarise the previous isomorphisms as
\[ \exp C_t \simeq L^2(\mathbb{R}) \]  
(3.3.25)
\[ \otimes_t \exp C_t \simeq \exp \int \oplus C_t dt \]  
(3.3.26)
\[ \exp(L^2(\mathbb{R})) \simeq \exp \int \oplus C_t dt \]  
(3.3.27)

To conclude that
\[ \mathcal{V}_{\text{cts}} \overset{\text{def}}{=} \otimes_t (L^2(\mathbb{R})) \simeq \exp \left(L^2(\mathbb{R}, dt)\right) . \]  
(3.3.28)

Hence, the Fock space \( \exp(L^2(\mathbb{R})) \) on which the history algebra is naturally represented, is isomorphic to the space \( \exp \int \oplus C_t dt \), the continuous tensor product of copies of the standard Hilbert space.

### 3.3.3 The Hamiltonian Algebra

Having found the representation space of the history group from a mathematical perspective, we now turn to explore a more physically meaningful way of uniquely selecting the representation space of the history algebra.
3.3. THE HISTORY SPACE

We return to the crucial observation—for the construction of the theory—that there exists a strong resemblance between the history algebra Eqs. (3.3.2)–(3.3.4), and the canonical commutation relations of a quantum field theory in one space dimension

\[
\begin{align*}
[\phi(x_1), \phi(x_2)] &= 0 \\
[\pi(x_1), \pi(x_2)] &= 0 \\
[\phi(x_1), \pi(x_2)] &= i\hbar \delta(x_1 - x_2).
\end{align*}
\]

We start with the ubiquitous example of the one-dimensional, simple harmonic oscillator with Hamiltonian

\[H = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2.\]  
(3.3.32)

As we have seen, the naïve idea behind the HPO theory is that to each time \(t\) there is associated a Hilbert space \(\mathcal{H}_t\) that carries propositions appropriate to that time (the ‘naïvety’ refers to the fact that, in a continuous tensor product \(\otimes_{t \in \mathbb{R}} \mathcal{H}_t\), the individual Hilbert spaces \(\mathcal{H}_t\) do not strictly exist as subspaces; this is related to the need to smear operators). Thus we expect to have a one-parameter family of operators

\[H_t := \frac{p_t^2}{2m} + \frac{m\omega^2}{2}x_t^2\]  
(3.3.33)

that represent the energy at time \(t\).

As it stands, the right hand side of Eq. (3.3.33) is not well-defined, just as in normal canonical quantum field theory it is not possible to define products of field operators at the same spatial point. However, the commutators of \(H_t\) with the generators of the HA can be computed formally as

\[
\begin{align*}
[H_t, x_s] &= -i\hbar \frac{\delta(t-s)}{m}p_s \\
[H_t, p_s] &= i\hbar m\omega^2 \delta(t-s)x_s \\
[H_t, H_s] &= 0
\end{align*}
\]

and are the continuous-time, history analogues of the familiar result in standard quantum theory:

\[
\begin{align*}
[H, x] &= -i\hbar \frac{\delta(t-s)}{m}p \\
[H, p] &= i\hbar m\omega^2 x.
\end{align*}
\]
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In standard quantum theory, the spectrum of the Hamiltonian operator can be computed directly from the algebra of Eqs. (3.3.37)–(3.3.38) augmented with the requirement that the underlying representation of the canonical commutation relations Eq. (3.3.1) is irreducible. This suggests that we try to define the history theory by requiring the existence of a family of operators $H_t$ that satisfy the relations Eqs. (3.3.34)–(3.3.36) and where the representation of the canonical history algebra Eqs. (3.3.5)–(3.3.7) is irreducible. More precisely, we augment the HA with the algebra (in semi-smeared form)

$$
\left[ H(\chi), p_t \right] = i\hbar \omega x_t
$$

where $H(\chi)$ is the history energy-operator, time averaged with the function $\chi$; heuristically, $H(\chi) = \int_{-\infty}^{\infty} dt \chi(t) H_t$.

It is useful to integrate these equations in the following sense. If self-adjoint operators $H(\chi)$ exist satisfying Eqs. (3.3.39)–(3.3.41), we can form the unitary operators $e^{iH(\chi)/\hbar}$, and these satisfy

$$
e^{iH(\chi)/\hbar} x_t e^{-iH(\chi)/\hbar} = \cos [\omega \chi(t)] x_t + \frac{1}{m\omega} \sin [\omega \chi(t)] p_t
$$

$$
e^{iH(\chi)/\hbar} p_t e^{-iH(\chi)/\hbar} = -m\omega \sin [\omega \chi(t)] x_t + \cos [\omega \chi(t)] p_t.
$$

However, it is clear that the right hand side of Eqs. (3.3.42)–(3.3.43) defines an automorphism of the canonical history algebra Eqs. (3.3.5)–(3.3.7). Thus the task in hand can be rephrased as that of finding an irreducible representation of the HA in which these automorphisms are unitarily implementable: the self-adjoint generators of the corresponding unitary operators will then be the desired time-averaged energy operators $H(\chi)$ [strictly speaking, weak continuity is also necessary, but this poses no additional problems in the cases of interest here].

3.3.4 The Fock Representation

It is natural to contemplate the use of a Fock representation of the HA since this plays such a central role in the analogue of a free quantum field in one spatial dimension. To
3.3. THE HISTORY SPACE

this end, we start by defining the ‘annihilation operator’

\[ \hat{b}_t := \sqrt{\frac{\hbar}{2m \omega}} x_t + i \sqrt{\frac{1}{2m \omega \hbar}} p_t \]  

(3.3.44)

in terms of which the HA (3.3.5)–(3.3.7) becomes

\[ [\hat{b}_t, \hat{b}_s] = 0 \]  

(3.3.45)

\[ [\hat{b}_t, \hat{b}_s^\dagger] = \delta(t - s). \]  

(3.3.46)

Note that

\[ \hbar \omega \hat{b}_t^\dagger \hat{b}_s = \frac{1}{2m} p_t p_s + \frac{m \omega^2}{2} x_t x_s - \frac{\hbar \omega}{2} \delta(t - s) \]  

(3.3.47)

which suggests that there exists an additively renormalised version of the operator \( \hat{H}_t \) in Eq. (3.3.33) of the form \( \hbar \omega \hat{b}_t^\dagger \hat{b}_t \). In turn, this suggests strongly that a Fock space based on Eq. (3.3.44) should provide the operators we seek.

To make this explicit we recall that the bosonic Fock space \( \mathcal{F}[\mathcal{H}] \) associated with a Hilbert space \( \mathcal{H} \) is defined as

\[ \mathcal{F}[\mathcal{H}] := \mathbb{C} \oplus \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H}) \oplus \cdots \]  

(3.3.48)

where \( \mathcal{H} \otimes \mathcal{H} \) denotes the symmetrised tensor product of \( \mathcal{H} \) with itself. Any unitary operator \( U \) on the ‘one-particle’ space \( \mathcal{H} \) gives a unitary operator \( \Gamma(U) \) on \( \mathcal{F}[\mathcal{H}] \) defined by

\[ \Gamma(U) := 1 \oplus U \oplus (U \otimes U) \oplus \cdots \]  

(3.3.49)

Furthermore, if \( U = e^{iA} \) for some self-adjoint operator \( A \) on \( \mathcal{H} \), then \( \Gamma(U) = e^{i\Gamma(A)} \) where

\[ d\Gamma(A) := 0 \oplus A \oplus (A \otimes 1 + 1 \otimes A) \oplus \cdots. \]  

(3.3.50)

The implications for us of these well-known constructions are as follows. Consider the Fock space \( \mathcal{F}[L^2(\mathbb{R}, dt)] \) that is associated with the Hilbert space \( L^2(\mathbb{R}, dt) \) via the annihilation operator \( \hat{b}_t \) defined in Eq. (3.3.44); i.e., the space built by acting with (suitably smeared) operators \( \hat{b}_t^\dagger \) on the ‘vacuum state’ \( |0\rangle \) that satisfies \( \hat{b}_t |0\rangle = 0 \) for all \( t \in \mathbb{R} \). The equations Eq. (3.3.42)–(3.3.43) show that, if it exists, the operator \( e^{i\hat{H}(\chi)/\hbar} \) acts on the putative annihilation operator \( \hat{b}_t \) as

\[ e^{i\hat{H}(\chi)/\hbar} \hat{b}_t e^{-i\hat{H}(\chi)/\hbar} = e^{-i\omega \chi(t)} \hat{b}_t. \]  

(3.3.51)
However, thought of as an action on $L^2(\mathbb{R},dt)$, the operator $U(\chi)$ defined by

$$
(U(\chi)\psi)(t) := e^{-i\omega \chi(t)}\psi(t)
$$

is unitary for any measurable function $\chi$. Hence, using the result mentioned above, it follows that in this particular Fock representation of the HA the automorphism on the right hand side of Eq. (3.3.51) is unitarily implementable, and hence the desired self-adjoint operators exist. Note that $H(\chi) = \hbar \omega d\Gamma(\hat{\chi})$, where the self-adjoint operator $\hat{\chi}$ is defined on $L^2(\mathbb{R},dt)$ as

$$
(\hat{\chi} \psi)(t) := \chi(t)\psi(t).
$$

In summary, we have shown that the Fock representation of the HA Eqs. (3.3.5)–(3.3.7) associated with the annihilation operator $b_t$ of Eq. (3.3.44) is such that there exists a family of self-adjoint operators $H(\chi)$ for which the algebra Eqs. (3.3.39)–(3.3.41) is satisfied. This Fock space is the desired carrier of the history propositions in our theory. Note that, in this case, the natural choice for the test function space $\phi \subseteq L^2(\mathbb{R},dt)$ used in Eqs. (3.3.11)–(3.3.13) is simply $L^2(\mathbb{R},dt)$ itself.

The position history-variable $x_t$ can be written in terms of $b_t$ and $b^\dagger_t$ as

$$
x_t = \sqrt{\frac{\hbar}{2m\omega}} \left( b_t + b^\dagger_t \right)
$$

and has the correlation function

$$
\langle 0| x_t x_s |0 \rangle = \frac{\hbar}{2m\omega} \delta(t-s).
$$

Thus the carrier space of our history theory is Gaussian white noise.

### 3.4 The n-particle History Propositions

The Fock-space construction produces a natural collection of history propositions: namely, those represented by the projection operators onto what, in a normal quantum field theory, would be called the ‘$n$-particle states’. To see what these correspond to physically in our case we note first that a $\delta$-function normalised basis for $\mathcal{F}[L^2(\mathbb{R},dt)]$ is given by the vectors $|0\rangle$, $|t_1\rangle$, $|t_1, t_2\rangle$, $\ldots$ where $|t_1\rangle := b_t^\dagger |0\rangle$, $|t_1, t_2\rangle := b_t^\dagger b_{t_2}^\dagger |0\rangle$, etc (of course, properly
normalised vectors are of the form $|\phi\rangle \equiv b^\dagger_\phi |0\rangle$, etc for suitable smearing function $\phi$. The physical meaning of the projection operators of the form $|t\rangle\langle t|$ (or, more rigorously, $|\phi\rangle\langle \phi|$), $|t_1, t_2\rangle\langle t_1, t_2|$, etc, can be seen by studying the equations

$$H(\chi)|0\rangle = 0$$  \hspace{1cm} (3.4.1)
$$H(\chi)|t\rangle = \hbar \omega \chi(t)|t\rangle$$  \hspace{1cm} (3.4.2)
$$H(\chi)|t_1, t_2\rangle = \hbar \omega [\chi(t_1) + \chi(t_2)]|t_1, t_2\rangle$$  \hspace{1cm} (3.4.3)

or, in totally unsmeared form,

$$H_\chi|0\rangle = 0$$  \hspace{1cm} (3.4.4)
$$H_\chi|t_1\rangle = \hbar \omega \delta(t - t_1)|t_1\rangle$$  \hspace{1cm} (3.4.5)
$$H_\chi|t_1, t_2\rangle = \hbar \omega [\delta(t - t_1) + \delta(t - t_2)]|t_1, t_2\rangle.$$  \hspace{1cm} (3.4.6)

It is clear from the above that, for example, the projector $|t_1, t_2\rangle\langle t_1, t_2|$ represents the proposition that there is a unit of energy $\hbar \omega$ concentrated at the time point $t_1$ and another unit concentrated at the time point $t_2$. Note that $H(\chi)|t, t\rangle = 2\hbar \omega \chi(t)|t, t\rangle$, and hence $|t, t\rangle\langle t, t|$ represents the proposition that there are two units of energy concentrated at the single time point $t$ (thus exploiting the Bose-structure of the canonical history algebra!). This interpretation of projectors like $|t_1, t_2\rangle\langle t_1, t_2|$ is substantiated by noting that the time-averaged energy obtained by choosing the averaging function $\chi$ to be 1 acts on these vectors as

$$\int_{-\infty}^{\infty} ds \ H_s |t\rangle = \hbar \omega |t\rangle$$  \hspace{1cm} (3.4.7)
$$\int_{-\infty}^{\infty} ds \ H_s |t_1, t_2\rangle = 2\hbar \omega |t_1, t_2\rangle$$  \hspace{1cm} (3.4.8)

and so on. This is the way in which the HPO account of the simple harmonic oscillator recovers the integer-spaced energy spectrum of standard quantum theory.

Finally, we note in passing that

$$\frac{1}{\hbar \omega} \int_{-\infty}^{\infty} ds \ s H_s |t_1, t_2, \ldots, t_n\rangle = (t_1 + t_2 + \cdots + t_n)|t_1, t_2, \ldots, t_n\rangle$$  \hspace{1cm} (3.4.9)

so that $\frac{1}{\hbar \omega} \int_{-\infty}^{\infty} ds \ s H_s$ acts as a ‘total time’ or ‘center-of-time’ operator.
3.5 The Extension to Three Dimensions

The extension of the formalism above to a particle moving in three spatial dimensions appears at first sight to be unproblematic. The analogue of the history algebra Eqs. (3.3.5)–(3.3.7) is

\[
\begin{align*}
[x_{i_1}, x_{i_2}] & = 0 \quad (3.5.1) \\
[p_{i_1}, p_{i_2}] & = 0 \quad (3.5.2) \\
[x_{i_1}, p_{i_2}] & = i\hbar \delta^{ij} \delta(t_1 - t_2) \quad (3.5.3)
\end{align*}
\]

\(i, j = 1, 2, 3\); while the formal expression Eq. (3.3.33) for the energy at time \(t\) becomes

\[
H_t := \frac{\mathbf{p}_t \cdot \mathbf{p}_t}{2m} + \frac{m\omega^2}{2} \mathbf{x}_t \cdot \mathbf{x}_t. \tag{3.5.4}
\]

It is straightforward to generalise the discussion above to this situation and, in particular, to find a Fock representation of Eqs. (3.5.1)–(3.5.3) in which the rigorous analogues of Eq. (3.5.4) exist as properly defined self-adjoint operators. However, an interesting issue then arises that has no analogue in one-dimensional quantum theory. Namely, we expect to have angular-momentum operators whose formal expression is

\[
L^i_t := \epsilon^{ijk} x^j_{i_1} p^k_{i_2} \tag{3.5.5}
\]

and whose commutators can be computed heuristically as

\[
[L^i_t, L^j_s] = i\hbar \epsilon^{ijk} \delta(t - s) L^k_t. \tag{3.5.6}
\]

Such operators \(L^i_t\) can be constructed rigorously using, for example, the method employed for the energy operators \(H_t\): viz., compute the automorphisms of the canonical history algebra that are formally induced by the angular-momentum operators and then see if these automorphism can be unitarily implemented in the given Fock representation. However, the interesting observation is that, even if this can be done (which is the case, see below), this does not guarantee in advance that the commutators in Eq. (3.5.6) will be reproduced: in particular, it is necessary to check directly if a \(c\)-number central extension is present since we know from other branches of theoretical physics that algebras of the type in Eq. (3.5.6) are prone to such anomalies.
3.5. THE EXTENSION TO THREE DIMENSIONS

An obvious technique for evaluating such a commutator would be to define the angular momentum operators by point-splitting in the form

\[ L^i_{t, \epsilon} := i\hbar \epsilon^{jk}(b^j_{t})^\dagger b^k_{t+\epsilon} \]  \hspace{1cm} (3.5.7)

so that the commutator in Eq. (3.5.6) is the analogue of an equal-time commutator in standard quantum field theory, and the point-splitting is the analogue of spatial point splitting. It is then straightforward to compute the commutators of these point-split operators and take the limit \( \epsilon \to 0 \). The result is the anticipated algebra Eq. (3.5.6).

However, in standard quantum field theory it is known that the limit of the commutator has to be considered at unequal times (i.e., using Heisenberg-picture operators), and that there is a subtle relation between the two limits of the times becoming equal and the spatial point splitting tending to zero \(^{17}\). Therefore, in order to calculate correctly the commutator in our case it seems appropriate to consider the analogue of an unequal time commutator, namely

\[ [L^i_{\chi, t, \epsilon}, L^j_{0, s, \epsilon}] \]  \hspace{1cm} (3.5.8)

where

\[ L^i_{\chi, t, \epsilon} := i\hbar \epsilon^{jk}(b^j_{\chi, t})^\dagger b^k_{\chi, t+\epsilon} \]  \hspace{1cm} (3.5.9)

and where

\[ b^k_{\chi, t} := e^{i\chi(t)}b^k_{t}e^{-i\chi(t)} = e^{-i\omega\chi(t)}b^k_{t} \]  \hspace{1cm} (3.5.10)

is a time-averaged Heisenberg picture operator of the type defined earlier.

It is not difficult to show that

\[ [L^i_{\chi, t, \epsilon}, L^j_{0, s, \epsilon}] = -\hbar^2 e^{i\omega(\chi(t)-\chi(t+\epsilon))} \times \left[ \delta(t - s + \epsilon) \left( (b^i_{t})^\dagger b^j_{t+2\epsilon} - \delta^{ij} (b^m_{t})^\dagger b^m_{t+2\epsilon} \right) 
\quad - \delta(t - s - \epsilon) \left( (b^i_{t})^\dagger b^j_{t+\epsilon} - \delta^{ij} (b^m_{t})^\dagger b^m_{t+\epsilon} \right) \right] \]  \hspace{1cm} (3.5.11)

and then, by evaluating the matrix element of the commutator in the vacuum state, one sees that there is no central extension in this case. Furthermore, by considering the matrix element of the commutator in general coherent states, one can check that the limits of \( \epsilon \to 0 \) and \( \chi \to 0 \) are straightforward, and that as long as the test functions are smooth, the angular momentum generators do indeed satisfy the heuristic commutator Eq. (3.5.6) in the limit.
CHAPTER 3. CONTINUOUS TIME IN THE HISTORY PROJECTION OPERATOR THEORY
Chapter 4

The Action Operator

4.1 Introduction

In the previous chapter we showed that, for the example of a simple harmonic oscillator in one dimension, the requirement of the existence of the Hamiltonian operator—which represents propositions about the time-averaged values of the energy of the system—together with the explicit relation between the Hamiltonian and the creation and annihilation operators, uniquely selects a particular Fock space as the representation space of the history algebra \([1.2-1.4]\) on the history space \(V_{cts}\).

The history algebra generators \(x_t\) and \(p_t\) can be seen heuristically as operators, (actually they are operator-valued distributions on \(V_{cts}\), that for each time label \(t\), are defined on the Hilbert space \(H_t\). The question then arises if, and how, these Schrödinger-picture objects with different time labels are related: in particular, is there a transformation law ‘from one Hilbert space to another’? One anticipates that the analogue of this question in the context of a histories treatment of a relativistic quantum field theory would be crucial to showing the Poincaré invariance of the system. The main goal of the present chapter is to enhance the theory so as to have a clearer view of the time transformation issue. This will ultimately allow us to address the problem of the Poincaré covariance of a history version of quantum field theory \([32]\).
In the Hamilton-Jacobi formulation of Classical Mechanics, it is the action functional that plays the role of the generator of a canonical transformation of the system from one time to another. Indeed, the Hamilton-Jacobi functional $S$, evaluated on the realised path of the system—i.e., for a solution of the classical equations of motion, under some initial conditions—is the generating function of a canonical transformation which transforms the system variables, for example position $x$ and momentum $p$, from an initial time $t = 0$ to another time $t$. It is therefore natural to investigate whether a quantum analogue of the action functional exists for the HPO theory and what role it plays in regard to the concept of time evolution in the theory.

At this point, it is worth commenting on the fact that there is an interesting relation between the definition of the action operator in HPO, and the well known work by Dirac, on the Langrangian theory for quantum mechanics [26]. Motivated by the fact that—contrary to the Hamiltonian—the Langrangian method can be expressed relativistically on account of the action function being a relativistic invariant, Dirac tried to take over the ideas of the classical Langrangian theory, albeit not the equations of the Langrangian theory per se [24]. In doing so, he showed that the transformation function $\langle q|Q \rangle$ that connects the two position representations—in which $q$ (the position at time $t$) and $Q$ (the position at another time $t'$) are multiplicative operators—is the quantum analogue of $e^{i\frac{\hbar}{S}}$ where $S$ is the classical action functional. He also obtained the contact transformations of the classical action functional in quantum mechanics; from this work the path integral approach to quantum theory was eventually developed. We will show that in HPO, the quantum analogue of the action functional acts in a similar way: it is the generator of time transformations in the sense that it relates the position and the momentum observables of the system at one time, with the position and momentum observables at another time, (as mentioned above, it resembles the canonical transformations generated by the Hamilton-Jacobi action functional).

In what follows, we prove the existence of the action operator $S_\kappa$, using the same type of quantum field theory methods that were used earlier to prove the existence of the Hamiltonian operator $H_\kappa$. 
4.2 The Definition of the Action Operator

In the Hamiltonian formalism for a classical system, the action functional is defined as

\[ S_{\text{cl}} := \int^{+\infty}_{-\infty} (p \dot{q} - H) \, dt \]  \hspace{1cm} (4.2.1)

where \( q \) is the position, \( p \) is the momentum and \( H \) the Hamiltonian of the system. Following the same line of thought as we the one we used in the definition of the Hamiltonian algebra, we want to find a representation of the history algebra in which their exists a one-parameter family of operators \( S_t \)—or better, their smeared form \( S_{\lambda, \kappa} \). Heuristically, we have

\[ S_t := (p_t \dot{x}_t - H_t) \]  \hspace{1cm} (4.2.2)

\[ S_{\lambda, \kappa} := \int^{+\infty}_{-\infty} (\lambda(t)p_t \dot{x}_t - \kappa(t)H_t) \, dt \]  \hspace{1cm} (4.2.3)

where \( S_{\lambda, \kappa} \) is the smeared action operator with smearing functions \( \lambda(t), \kappa(t) \). In order to discuss the existence of an operator \( S_{\lambda, \kappa} \) we note that, if this operator exists, the Hamiltonian algebra eqs. (3.3.39—3.3.41), would be augmented in the form

\[
\begin{align*}
[S_{\lambda, \kappa}, x_f] &= i\hbar (x \frac{d}{dt} \lambda_f) + \frac{P_f}{m} \frac{d}{dt} \lambda_f, \\
[S_{\lambda, \kappa}, p_f] &= i\hbar (p \frac{d}{dt} \lambda_f) + m\omega x_f \frac{d}{dt} \lambda_f, \\
[S_{\lambda, \kappa}, H_{\kappa'}] &= i\hbar H \frac{d}{dt} (\kappa' \frac{d}{dt} \lambda(t)p_t^2) \, dt \\
[S_{\lambda, \kappa}, S'_{\lambda, \kappa}] &= i\hbar H \frac{d}{dt} (\kappa' \frac{d}{dt} \lambda(t)) - i\hbar H \frac{d}{dt} (\lambda(t)) - i\hbar \int^{\infty}_{-\infty} \left( (\kappa(t) \frac{d}{dt} \lambda(t)) \frac{d}{dt} \lambda(t) + \kappa'(t) \frac{d}{dt} \lambda(t) \right) \, dt
\end{align*}
\]  \hspace{1cm} (4.2.4—4.2.7)

Although we have defined the action operator in a general smeared form, in what follows we will mainly employ only the case \( \lambda(t) = 1 \) and \( \kappa(t) = 1 \) that accords with the expression for the classical action functional. This choice of smearing functions poses no technical problems, provided we keep to the requirement that the smearing functions for the position and momentum operators are square-integrable functions. In particular, the products of the smearing functions \( f \) and \( g \) in eqs (4.2.4—4.2.7) with the test functions \( \lambda(t) = 1 \) and \( \kappa(t) = 1 \) are still square-integrable.
The Existence of the Action Operator in HPO. We now examine whether the action operator actually exists in the Fock representation of the history algebra employed in our earlier discussion in Chapter 3. Henceforward we choose $\lambda(t) = 1$. Then the formal commutation relations are

$$S_\kappa := \int_{-\infty}^{+\infty} (p_t \dot{x}_t - \kappa(t) H_t) dt$$  \hspace{1cm} (4.2.8)

$$[S_\kappa, x_f] = i\hbar (x_f + \frac{p_f}{m})$$  \hspace{1cm} (4.2.9)

$$[S_\kappa, p_f] = i\hbar (p_f + m\omega x_{\kappa f})$$  \hspace{1cm} (4.2.10)

$$[S_\kappa, H_{\kappa'}] = i\hbar H_{\kappa'}$$  \hspace{1cm} (4.2.11)

$$[S_\kappa, S_{\kappa'}] = i\hbar H_{\kappa'} - i\hbar H_{\kappa}$$  \hspace{1cm} (4.2.12)

A key observation is that if the operators $e^{i\kappa S_\kappa}$ existed they would produce the history algebra automorphism

$$e^{i\kappa S_\kappa} b_i e^{-i\kappa S_\kappa} = e^{i\omega \int_{t}^{t+s} \kappa(t+s') ds'} e^{i \frac{\omega}{\pi t} b_i}$$  \hspace{1cm} (4.2.13)

or, in the more rigorous smeared form

$$e^{i\kappa S_\kappa} b_f e^{-i\kappa S_\kappa} = b_{\Sigma f}$$  \hspace{1cm} (4.2.14)

where the unitary operator $\Sigma_\kappa$ is defined on $L^2(\mathbb{R})$ by

$$(\Sigma_\kappa \psi)(t) := e^{-i\omega \int_{t}^{t+s} \kappa(t+s') ds'} \psi(t+s).$$  \hspace{1cm} (4.2.15)

As we explained in the previous chapter, an important property of the Fock construction states that if $e^{i\kappa A}$ is a unitary operator on the Hilbert space $L^2(\mathbb{R})$, there exists a unitary operator $\Gamma(e^{i\kappa A})$ that acts on the exponential Fock space $\mathcal{F}(L^2(\mathbb{R}))$ in such a way that

$$\Gamma(e^{i\kappa A}) b_i \Gamma(e^{i\kappa A})^{-1} = b_{e^{i\kappa A} f}.$$  \hspace{1cm} (4.2.16)

In addition, there exists a self-adjoint operator $d\Gamma(A)$ on $\mathcal{F}(L^2(\mathbb{R}))$ such that

$$\Gamma(e^{i\kappa A}) = e^{i\kappa d\Gamma(A)}$$  \hspace{1cm} (4.2.17)

in terms of the self-adjoint operator $A$ that acts on $L^2(\mathbb{R})$. In particular, it follows that the representation of the history algebra on the Fock space $\mathcal{F}(L^2(\mathbb{R}))$ carries a (weakly
4.3. The Definition of the Liouville Operator

The first term of the action operator eq. (4.2.19) is identical to the kinematical part of the classical action functional eq. (4.2.1). For reasons that will become apparent later, we write $S_\kappa$ as the difference between two operators: the ‘Liouville’ operator and the Hamiltonian operator. The Liouville operator is formally written as

$$V := \int_{-\infty}^{+\infty} (p_t \dot{x}_t) dt$$

(4.3.1)

where

$$S_\kappa = V - H_\kappa.$$  

(4.3.2)

We prove the existence of $V$ on $\mathcal{F}(L^2(\mathbb{R}))$ using the same technique as before. Namely, we can see at once that the history algebra automorphism

$$e^{sV} b_f e^{-sV} = b_{B_s f}$$

(4.3.3)

is unitarily implementable. Here, the unitary operator $B_s$, $s \in \mathbb{R}$, acting on $L^2(\mathbb{R})$ is defined by

$$(B_s f)(t) := e^{sD} f(t) = e^{isD} f(t) = f(t + s)$$

(4.3.4)

where $D := -i \frac{d}{dt}$. The Liouville operator $V$ has the following commutation relations with the generators of the history algebra:

$$[V, x_f] = -i\hbar x_f$$

(4.3.5)
CHAPTER 4. THE ACTION OPERATOR

\[ [V, p_f] = -i\hbar p_f \]  \hspace{1cm} (4.3.6)
\[ [V, H_\kappa] = -i\hbar H_\kappa \]  \hspace{1cm} (4.3.7)
\[ [V, S_\kappa] = i\hbar H_\kappa \]  \hspace{1cm} (4.3.8)
\[ [V, H] = 0 \]  \hspace{1cm} (4.3.9)
\[ [V, S] = 0 \]  \hspace{1cm} (4.3.10)
\[ [H, S] = 0 \]  \hspace{1cm} (4.3.11)

where we have defined \( H := \int_{-\infty}^{\infty} H_t dt \). As we shall see, these commutators will play an important role in the physical interpretation of the Liouville operator.

We notice that \( V \) transforms, for example, \( b_t \) from one time \( t \) —that refers to the Hilbert space \( \mathcal{H}_t \)—to another time \( t + s \), that refers to \( \mathcal{H}_{t+s} \). More precisely, \( V \) transforms the support of the operator-valued distribution \( b_t \) from \( t \) to \( t + s \):

\[ e^{i\hbar sV} b_f e^{-i\hbar sV} = b_{fs} \]  \hspace{1cm} (4.3.12)

where \( f_s(t) := f(s + t) \). We shall return to the significance of this later.

4.4 The Fourier-transformed n-particle History Propositions

We shall now briefly consider the eigenvectors of the action operator \( S \)—as we shall see later, these play a significant part in understanding the role in the quantum history theory of the classical solutions in the equations of motion. An interesting family of history propositions emerged from the representation space \( \mathcal{F}[L^2(\mathbb{R}, dt)] \), acting on the \( \delta \)-function normalised basis of states \( |0\rangle, |t_1\rangle := b_{t_1}^\dagger |0\rangle, |t_1, t_2\rangle := b_{t_1}^\dagger b_{t_2}^\dagger |0\rangle \ etc \); or, in smeared form, \( |\phi\rangle := b_{\phi}^\dagger |0\rangle \ etc \). As noted in Chapter 3, the projection operator \( |t\rangle \langle t| \) corresponds to the history proposition ‘there is a unit energy \( \hbar \omega \) concentrated at the time point \( t \’ \). The physical interpretation for this family of propositions, was deduced from the action of the Hamiltonian operator on the family of \( |t\rangle \) states.

To study the behaviour of the \( S \) operator, a particularly useful basis for \( \mathcal{F}[L^2(\mathbb{R}, dt)] \) is
4.5. THE VELOCITY OPERATOR

The Fourier-transforms of the $|t\rangle$-states. Indeed, if we consider the Fourier transformations

$$
|\nu\rangle = \int_{-\infty}^{+\infty} e^{i\nu t} b^{\dagger} t |0\rangle dt (4.4.1)
$$

$$
|\nu_1, \nu_2\rangle = \int_{-\infty}^{+\infty} e^{i\nu_1 t_1} e^{i\nu_2 t_2} b^{\dagger}_{t_1} b^{\dagger}_{t_2} |0\rangle dt_1 dt_2 (4.4.2)
$$

$$
b_{\nu} = \int_{-\infty}^{+\infty} e^{i\nu t} b t dt (4.4.3)
$$

$$
b^{\dagger}_{\nu} = \int_{-\infty}^{+\infty} e^{-i\nu t} b^{\dagger} t dt (4.4.4)
$$

the Fourier transformed $|\nu\rangle$- states are defined by $|\nu\rangle := b^{\dagger}_{\nu} |0\rangle$, $|\nu_1, \nu_2\rangle := b^{\dagger}_{\nu_1} b^{\dagger}_{\nu_2} |0\rangle$ etc. These states are eigenvectors of the operator $S$:

$$
S |0\rangle = 0 (4.4.5)
$$

$$
S |\nu\rangle = \hbar (\nu - \omega) |\nu\rangle (4.4.6)
$$

$$
S |\nu_1, \nu_2\rangle = \hbar [(\nu_1 - \omega) + (\nu_2 - \omega)] |\nu_1, \nu_2\rangle (4.4.7)
$$

$$
\vdots
$$

and we note in particular that $e^{\pm \nu S} |0\rangle = |0\rangle$.

The $|\nu\rangle$-states are also eigenstates of the time-averaged history Hamiltonian operator:

$$
H |0\rangle = 0 (4.4.8)
$$

$$
H |\nu\rangle = \hbar \omega |\nu\rangle (4.4.9)
$$

$$
H |\nu_1, \nu_2\rangle = 2\hbar \omega |\nu_1, \nu_2\rangle (4.4.10)
$$

$$
\vdots
$$

which is consistent with these states being eigenstates of $S = V - H$, since $[S, H] = 0$ from Eq. (4.3.11). Again, we see how the integer-spaced spectrum of the standard quantum theory of the simple harmonic oscillator appears in the HPO theory.

4.5 The Velocity Operator

The HPO approach to the consistent-histories theory has the striking feature that, formally, there exists an operator that corresponds to propositions about the velocity of the
system: namely, \( \dot{x}_t := \frac{d}{dt} x_t \). More rigorously, we can adopt the procedure familiar from standard quantum field theory and define

\[
\dot{x}_f := -x_f'
\] (4.5.1)

which is meaningful provided that (i) the test-function \( f \) is differentiable; and (ii) \( f \) ‘vanishes at infinity’ so that the implicit integration by parts used in Eq. (4.5.1) is allowed; i.e., heuristically, \( x_f = \int_{-\infty}^{\infty} dt x_t f(t) \).

The rigorous existence of \( \dot{x}_t \) depends on the precise choice of test-function space used in the smeared form of the HA in Eqs. (3.3.11)–(3.3.13). In the analogous situation in normal quantum field theory, the test-functions are chosen so that the spatial derivatives of the quantum field exist, this being necessary to define the Hamiltonian operator. In our case, the situation is somewhat different since the energy operator \( H_t \) [see Eq. (3.3.3)] does not depend on \( \dot{x}_t \) and hence there is no \textit{a priori} requirement for \( \dot{x}_t \) to exist. However, what \textit{is} clear from Eq. (3.3.3) is that if \( \dot{x}_t \) exists then

\[
[x_t, \dot{x}_s] = 0
\] (4.5.2)

and hence our theory allows for history propositions that include assertions about the position of the particle and its velocity at the same time; in particular, the velocity \( \dot{x}_t \) and momentum \( p_t \) are not related. In this context it should be emphasised once more that \( x_t, t \in \mathbb{R}, \) is a one-parameter family of Schrödinger-picture operators—it is \textit{not} a Heisenberg-picture operator, and the equations of motion do not enter at this level.

There is an interesting approach to understanding Eq. (4.5.2) that comes from the underlying continuous time structure of the theory. As we explained in the previous chapter, because of the history group construction the histories are defined continuously with respect to the time; as a result, we can ask questions about the position \( x_t \) of the system at \textit{any} time. Hence, using ideas drawn from standard quantum theory, we can produce an operational procedure for defining the velocity \( \dot{x}_t \) by using the same \textit{measurement apparatus} as the one we could have used for evaluating the position of the system. In general, the information that we require by asking a question about the system, is determined by the choice of the smearing function for the \( x_t \) and \( \dot{x}_t \) operators.

The existence of a velocity operator that commutes with position is a striking property
of the HPO approach to consistent histories and raises some intriguing questions. For example, a classic paper by Park and Margenau contains an interesting discussion of the uncertainty relations, including a claim that it is possible to measure position and momentum simultaneously provided the latter is defined using time-of-flight measurements. The existence in our formalism of the vanishing commutator Eq. (4.5.2) throws some new light on this old discussion. Also relevant in this respect is Hartle’s discussion of the operational meaning of momentum in a history theory. In particular, he emphasises that an accurate measurement of momentum requires a long time-of-flight, whereas—on the other hand—our definition of velocity as the time-derivative of the history variable clearly involves a vanishingly small time interval. Presumably this is the operational difference between momentum and velocity in the HPO approach to consistent histories.

The existence of the Liouville operator in the HPO scheme, allows an interesting comparison between the velocity operator and the momentum operator: namely, the latter is defined by the history commutation relation of the position with the Hamiltonian, while we can define the velocity operator from the history commutation relation of the position with the Liouville operator:

\[
[x_f, H] = i\hbar \frac{p_f}{m} \quad (4.5.3)
\]

\[
[x_f, V] = i\hbar \dot{x}_f \quad (4.5.4)
\]

These relations illustrate the different nature of the momentum and the velocity: in particular, the behaviour of the momentum is fundamentally dynamical (as shown by the relation to the Hamiltonian operator), whereas the velocity is fundamentally kinematical (as shown by the relation with the Liouville operator).

### 4.6 The Heisenberg Picture

At this point in our discussion, it is useful to investigate the analogue of the Heisenberg picture in our continuous-time HPO theory. This would help to clarify the relation between momentum and velocity; it will also be a central feature in our discussion in Chapter 6 of the history version of relativistic quantum field theory.
In standard quantum theory, the Heisenberg-picture version of an operator $A$ is defined with respect to a time origin $t = 0$ as

$$A_H(s) := e^{isH/\hbar} A e^{-isH/\hbar}. \quad (4.6.1)$$

In particular, for the simple harmonic oscillator we have

$$x(s) = \cos[\omega s] x + \frac{1}{m\omega} \sin[\omega s] p \quad (4.6.2)$$
$$p(s) = -m\omega \sin[\omega s] x + \cos[\omega s] p. \quad (4.6.3)$$

The Heisenberg-picture operator $x(s)$ satisfies the classical equation of motion

$$\frac{d^2 x(s)}{ds^2} + \omega^2 x(s) = 0, \quad (4.6.4)$$

and the commutator of these operators is

$$[x(s_1), x(s_2)] = \frac{i\hbar}{m\omega} \sin[\omega(s_1 - s_2)] \quad (4.6.5)$$

which, on using the Heisenberg-picture equation of motion

$$p := m \frac{dx(s)}{ds} \bigg|_{s=0}, \quad (4.6.6)$$
reproduces the familiar canonical commutation relation Eq. (3.3.1).

In trying to repeat this construction for the history theory we might be tempted to define the Heisenberg-picture analogue of, say, $x_t$ as

$$x_{H,t}(s) := e^{iH_{t,\kappa}/\hbar} x_t e^{-iH_{t,\kappa}/\hbar}. \quad (4.6.7)$$

However, this expression is not well-defined since it corresponds to choosing the test-function in Eq. (3.3.42) as $\chi(t') := s\delta(t - t')$, which leads to ill-defined products of $\delta(t - t')$.

What is naturally suggested instead is to define ‘time-averaged’ Heisenberg quantities

$$x_{\kappa,t} := e^{iH_{\kappa}/\hbar} x_t e^{-iH_{\kappa}/\hbar} = \cos[\omega\kappa(t)] x_t + \frac{1}{m\omega} \sin[\omega\kappa(t)] p_t \quad (4.6.8)$$

for suitable test functions $\kappa$. The analogue of the equation of motion Eq. (4.6.4) is the functional differential equation

$$\frac{\delta^2 x_{\kappa,t}}{\delta\kappa(s_1)\delta\kappa(s_2)} + \delta(t - s_1)\delta(t - s_2)\omega^2 x_{\kappa,t} = 0, \quad (4.6.9)$$
4.6. THE HEISENBERG PICTURE

while the history analogue of Eq. (4.6.6) is

\[ \delta(t - s)p_t = m \frac{\delta x_{\kappa,t}}{\delta \kappa(s)} \bigg|_{\kappa=0} \]  

(4.6.10)

and the analogue of the ‘covariant commutator’ Eq. (4.6.5) is

\[ [x_{\kappa_1,t_1}, x_{\kappa_2,t_2}] = i\hbar \delta(t_1 - t_2) \sin[\omega(\kappa_1(t_1) - \kappa_2(t_2))] \]  

(4.6.11)

which correctly reproduces the canonical history algebra. However, the use of the expression Eq. (4.6.8) to define a ‘Heisenberg-picture’ operator, lacks the analogy with the classical equations of motion Eq. (4.6.4).

Before developing this point further however, it is worth noting that any definition of a Heisenberg-picture in the HPO theory will involve two time labels: an ‘external’ label \( t \)—that specifies the time the proposition is asserted—and an ‘internal’ label \( s \) that, for a fixed time \( t \), is the time parameter of the Heisenberg picture associated with the copy \( \mathcal{H}_t \) of the standard Hilbert space. Using the results from the history algebra automorphisms Eqs. (3.3.51, 4.3, 4.2.13) for the definition of the \( H, V \) and \( S \) operators, it can be seen that the two labels appear naturally in the final version of the Heisenberg picture: they are related to the groups that produce the two types of time transformations. In addition, the analogy with the classical expressions is regained.

To see this explicitly, we define a Heisenberg-picture analogue of \( x_t \) as

\[ x_{\kappa,t,s} : = e^{\frac{i}{\hbar} \hat{S}_H \kappa} x_t e^{-\frac{i}{\hbar} \hat{S}_H \kappa} \]  

(4.6.12)

\[ = \cos[\omega \kappa(t)]x_t + \frac{1}{m\omega} \sin[\omega \kappa(t)]p_t \]

\[ p_{\kappa,t,s} : = e^{\frac{i}{\hbar} \hat{S}_H \kappa} p_t e^{-\frac{i}{\hbar} \hat{S}_H \kappa} \]  

(4.6.13)

\[ = -m\omega \sin[\omega \kappa(t)]x_t + \cos[\omega \kappa(t)]p_t \]

where \( \kappa(t) \) is now a ‘fixed’ function. The commutation relations for these operators are

\[ [x_{\kappa,t}(s), x_{\kappa',t'}(s')] = \frac{i\hbar}{m\omega} \sin[\omega \kappa(s' - s)] \delta(t - t') \]  

(4.6.14)

\[ [x_{\kappa,t}(s), S_{\kappa'}] = i\hbar \left[ \cos[\omega \kappa(t)] \dot{x}_t + \frac{1}{m\omega} \sin[\omega \kappa(t)] \dot{p}_t - \kappa' \frac{p_t}{m} \right] \]  

(4.6.15)

\[ [p_{\kappa,t}(s), S_{\kappa'}] = i\hbar \left[ \cos[\omega \kappa(t)] \dot{p}_t - m\omega \sin[\omega \kappa(t)] \dot{x}_t + \kappa'(t)x_{\kappa,t,s} \right] \]  

(4.6.16)

which are clearly compatible with the HPO analogue of the equations of motion

\[ \frac{d^2}{ds^2} x_{\kappa,t,s} + \omega^2 \kappa(t)^2 x_{\kappa,t,s} = 0 \]  

(4.6.17)
which follow directly from the definition of the operator $x_{\kappa, t, s}$ in Eq. (4.6.13). We notice the strong resemblance with standard quantum theory; for the case $\kappa(t) = 1$, the classical expressions are fully recovered.
Chapter 5

Time Transformations in the HPO

5.1 Introduction

One exciting feature of the HPO theory is the way that the time transformations appear in the formalism.

In what follows, we will show that, constructed as a quantum analogue of the classical action functional, $S_\kappa$ does indeed act as a generator of time-transformations in the HPO theory. Furthermore, we will argue that $S_\kappa$ is related to the two laws of time-evolution in standard quantum theory namely, state-vector reduction, and unitary time-evolution between measurements.

A comparison with the classical theory seems appropriate at this point; thus, we present a classical analogue of the HPO where the continuous-time classical histories can be seen as analogues of the continuous-time quantum histories.

We further exploit the above analogy to discuss the ‘classical’ behaviour of the history quantum scheme. In particular, we expect the action operator to be involved in some way...
with the dynamics of the theory. To this end, we show how the action operator appears in the expression for the decoherence functional, with operators acting on coherent states, as used by Isham and Linden [8].

5.2 The Two Types of Time Transformation

In standard classical mechanics, the Hamiltonian $H$ is the generator of time transformations. In terms of Poisson brackets, the generalised equation of motion for an arbitrary function $u$ is given by

$$\frac{du}{dt} = \{u, H\} + \frac{\partial u}{\partial t}. \quad (5.2.1)$$

In a HPO theory, the Hamiltonian operator $H_t$ produces phase changes in time, preserving the time label $t$ of the Hilbert space on which, at least formally, $H_t$ is defined. On the other it is the Liouville operator $V$ that assigns, analogous to the classical case, history commutation relations, and produces time transformations ‘from one Hilbert space to another’. The action operator generates a combination of these two types of time-transformation. If we use the notation $x_f(s)$ for the history Heisenberg-picture operators smeared with respect to the time label $t$ Eq. (4.6.13) (with $\kappa = 1$), we observe that they behave as standard Heisenberg-picture operators, with time parameter $s$. A novel result however is the observation that, the operator $x_f(s)$ for instance, changes with respect to time parameter $t$ also, in the sense that at a later time $t' = t + \alpha$ the operator valued distribution $x_t$ is smeared by the function $f'(t') = f(t + \alpha)$. Furthermore, their history commutation relations are

$$[x_f(s), V] = i\hbar \dot{x}_f(s) \quad (5.2.2)$$
$$[x_f(s), H] = \frac{i\hbar}{m} p_f(s) \quad (5.2.3)$$
$$[x_f(s), S] = i\hbar (\dot{x}_f(s) - \frac{1}{m} p_f(s)) \quad (5.2.4)$$

which, as we shall see, strongly resemble the corresponding expressions in the classical history theory which we shall develop.

We now define a one-parameter group of transformations $T_V(\tau)$, with elements $e^{\tau V}$, $\tau \in \mathbb{R}$ where $V$ is the Liouville operator Eq. (4.3.1), and we consider its action on the
5.2. THE TWO TYPES OF TIME TRANSFORMATION

Heisenberg-picture operator $b_{t,s}$; for simplicity we write the unsmeared expressions

$$e^{i\bar{\hbar}\tau V}b_{t,s}e^{-i\bar{\hbar}\tau V} = b_{t+\tau,s},$$

which makes particularly clear the sense in which the Liouville operator is the generator of transformations of the time parameter $t$ labelling the Hilbert spaces $\mathcal{H}_t$.

Next we define a one-parameter group of transformations $T_H(\tau)$, with elements $e^{i\bar{\hbar}\tau H}$, where $H$ is the time-averaged Hamiltonian operator

$$e^{i\bar{\hbar}\tau H}b_{t,s}e^{-i\bar{\hbar}\tau H} = b_{t,s+\tau}.$$  \hspace{1cm} (5.2.6)

Thus the Hamiltonian operator is the generator of phase changes of the time parameter $s$, produced only on one Hilbert space $\mathcal{H}_t$, for a fixed value of the ‘external’ time parameter $t$.

Finally, we define the one-parameter group of transformations $T_S(\tau)$, with elements $e^{i\bar{\hbar}\tau S}$, where $S$ is the action operator, which acts as

$$e^{i\bar{\hbar}\tau S}b_{t,s}e^{-i\bar{\hbar}\tau S} = b_{t+s+\tau}.$$  \hspace{1cm} (5.2.7)

We see that the action operator generates both types of time transformations—a feature that appears only in the HPO scheme.

In Fig.1a,b, we represent the tensor product of Hilbert spaces as a sequence of planes (each one representing a copy of the standard Hilbert planes), and a quantum continuous-time history as a curve in that space. Each plane is labeled by the time label $t$ that the corresponding Hilbert space $\mathcal{H}_t$ carries. Thus a history is depicted as a curve along an $n$-fold sequence of ‘Hilbert planes’ $\mathcal{H}_{ti}$. As we will explain later, in analogy to this, we can represent a classical history as a curve along an $n$-fold sequence of planes corresponding to copies of the standard phase-space $\Gamma_{ti}$, as we will explain later. The time transformations generated by the Liouville operator, shift the path in the direction of the ‘Hilbert planes’. On the other hand, the Hamiltonian operator generates time transformations that move the history curve in the direction of the path, as represented on one ‘Hilbert plane’.
Figure 5.1: Quantum and classical history curves. In Figure 1.a the transformation of the history curves generated by $V$ is represented by the dashed line, while the transformation generated by $H$ are represented by the dotted line. The curves drawn on each ‘Hilbert plane’ correspond to the Hamiltonian transformations as effected on the corresponding Hilbert space. In Figure 1.b the classical history remains invariant under the corresponding time transformations.

5.2.1 The Two-fold Time Interpretation

In standard quantum theory, time-evolution is described by two different laws: the state-vector reduction that occurs when a measurement is made, and the unitary time-evolution that takes place between measurements. Thus, according to von Neumann, one has to
augment the Schrödinger equation with a collapse of the state vector associated with a measurement \[27\].

I would like to claim that the two types of time-transformations observed in the HPO theory are associated in some way with the two dynamical processes in standard quantum theory: the time transformations generated by the Liouville operator \( V \) are related to the state-vector reduction (more precisely, the time ordering implied by the state-vector reduction), while the time transformations produced by the Hamiltonian operator \( H \) are related to the unitary time-evolution between measurements.

The argument in support of this assertion is as follows. Keeping in mind the description of the history space as a tensor product of single-time Hilbert spaces \( \mathcal{H}_t \), the \( V \) operator acts on the Schrödinger-picture projection operators by translating them in time from one Hilbert space to another. These time-ordered projectors appear in the expression for the decoherence functional that defines probabilities. In history theory, the expression for probabilities in a consistent set, is the same as that derived in standard quantum theory using the projection postulate on a time-ordered sequence of measurements \[20, 11\]. It is this that suggests a relation of the Liouville operator to ‘state-vector reduction’.

Indeed, the class operator in Eq. (5.4.9) that represents a history as a time-ordered sequence of Schrödinger-picture operators interleaved with the unitary-time operator, was constructed in a way that imitates the state-vector reduction (time evolution) of the standard quantum theory. In the HPO theory, for a set of consistent histories, the Liouville operator is the generator of time transformations that takes a Schrödinger-picture operator at some time \( t \) (corresponding to the Hilbert space copy with the same t-time label, \( \mathcal{H}_t \)) to another time \( t' \) (corresponding to the Hilbert space copy with the same t-time label \( \mathcal{H}_{t'} \)). Following the description for the class operator in generalised consistent histories, we argue that in a similar way, the action of the Liouville operator can be related to the process of state-vector reduction.

To strengthen this claim, in what follows we will show the analogy of \( V \) with the \( S_{cts} \) operator (an approximation of the derivative operator that we shall define shortly) that appears in the decoherence function in the HPO formalism, and is implicitly related to the state-vector reduction by specifying the time-ordering of the action of the single-time
5.8 CHAPTER 5. TIME TRANSFORMATIONS IN THE HPO

projectors. The action of $V$ as a generator of time translations depends on the partial (in fact, total) ordering of the time parameter treated as the causal structure in the underlying spacetime. Hence, the $V$-time translations illustrate the purely kinematical function of the Liouville operator.

The Hamiltonian operator producing transformations, via a type of Heisenberg time-evolution, appears as the ‘clock’ of the theory. As such, it depends on the particular physical system that the Hamiltonian describes. Indeed, we would expect the definition of a ‘clock’ for the evolution in time of a physical system to be connected with the dynamics of the system concerned. We note that the smearing function $\kappa(t)$ used in the definition of the Hamiltonian operator can be interpreted as a mechanism of implementing the idea of reparametrizing time—a concept that plays a key role in quantum gravity; in the present context however, $\kappa$ is kept fixed for a particular physical system.

The coexistence of the two types of time-evolution, as reflected in the action operator identified as the generator of such time transformations, is a striking result. In particular, its definition is in accord with its classical analogue, namely the Hamilton action functional. In classical theory, a distinction between a kinematical and a dynamical part of the action functional also arises in the sense that the first part corresponds to the symplectic structure and the second to the Hamiltonian.

5.3 The Classical Imprint of the HPO

Let us now consider more closely the relation of the classical and the quantum histories. We have shown above how the action operator generates time translations from one Hilbert space to another, through the Liouville operator; and on each labeled Hilbert space $\mathcal{H}_t$, through the Hamiltonian operator. We now wish to discuss in more detail the analogue of these transformations in the classical case.

We recall that a history is a time-ordered sequence of propositions about the system. The continuous-time quantum history in the HPO system, makes assertions about the values of the position or the momentum of the system, or a linear combination of them,
at each moment of time, and is represented by a projection operator on the continuous tensor product of copies of the standard Hilbert space.

This raises the intriguing question of the extent to which one expects a continuous-time classical history theory should reflect the underlying temporal logic of the situation. Thus the assertions about the position and the momentum of the system at each moment of time should be represented on an analogous history space: as we shall see, this can be achieved by using the Cartesian product of a continuous family (labelled by the time $t$) of copies of the standard classical state space.

In classical mechanics, a (fine-grained) classical history is represented by a path in the state space. Indeed, a path $\gamma$ is defined as a map from the real line into the standard phase-space $\Gamma$:

$$
\gamma : \mathbb{R} \rightarrow \Gamma \\
\quad \quad \quad \quad t \mapsto (q(\gamma(t)), p(\gamma(t)))
$$

where $q^i(\gamma(t))$ and $p_j(\gamma(t))$, $i, j = 1, 2, \ldots, N$ (where the dimension of $\Gamma$ is $2N$) are the position and momentum coordinates of the path $\gamma$, at the time $t$. For our purposes, we shall consider the path $t \mapsto \gamma(t)$ to be defined for $t$ in some finite time interval $[t_1, t_2]$. We shall denote by $\Pi$ the set of all such $C^\infty$ paths.

The key idea of this new approach to classical histories is contained in the symplectic structure of the theory: the choice of the Poisson bracket must be such that it includes entries at different moments of time. Thus we suppose that the space of functions on $\Pi$ is equipped with the ‘history Poisson bracket’ defined by

$$
\{ q^i_t, p_{j,t'} \} = \delta^i_j \delta(t - t')
$$

where we defined the functions $q_t$ on $\Pi$ as

$$
q^i_t : \Pi \rightarrow \mathbb{R} \\
\gamma \mapsto q^i_t(\gamma) := q(\gamma(t))^i
$$

and similarly for $p_t$.

$^1$The notation here is somewhat cryptic but hopefully the intention will be apparent.
The Temporal Logic and the HPO Classical Histories Proposal

Before proceeding any further with this construction, it is worth mentioning that the temporal logic of classical history theory can be defined without any reference to the quantum case.

We start by recalling that there exists a correspondence between single-time history propositions $P$ and subsets $\mathcal{C}$ of the phase space $\Gamma$: a proposition is represented by a characteristic function $\chi_C(s)$, defined as

$$\chi_C(s) \overset{\text{def}}{=} \begin{cases} 1 & \text{if } s \in C, \\ 0 & \text{if } s \not\in C. \end{cases}$$

(5.3.3)

where $C$ is a subset of the state space $\Gamma$.

Let us suppose that the proposition $P$ corresponds to the subset $C_P \subset \Gamma$, and the proposition $Q$ corresponds to the subset $C_Q \subset \Gamma$. Then, it is obvious that the proposition “$P$ and $Q$” corresponds to the subset $C_P \cap C_Q \subset \Gamma$.

Next, we aim to define temporal propositions: for instance “$P$ at time $t_1$ and then $Q$ at time $t_2$”. For a state $s \in \Gamma$, the proposition is true if $s \in C_P$ and $s \in \tau_{(t_1,t_2)}(C_Q)$, where

$$\tau_{(t,t')} : \Gamma \mapsto \Gamma$$

$$s_t \mapsto \tau_{(t,t')}(s_t) := s_{t'}.$$  

(5.3.4)

denotes the time-development map that evolves a state $s \in \Gamma$ at time $t$, to the corresponding state $\tau_{(t,t')}(s)$ at time $t'$.

Hence, from this perspective the temporal proposition “$P$ at time $t_1$ and then $Q$ at time $t_2$” corresponds to the subset

$$C_P \cap \tau_{(t_1,t_2)}^{-1}(C_Q)$$

(5.3.5)

This can be regarded as a classical analogue of a quantum class operator.

Furthermore, let us suppose that, the proposition $Q$ is in fact, the proposition $\tau_{(t_1,t_2)}(P)$ to be true at time $t_2$. Then the corresponding subset of the temporal proposition “$P$ at t_2. Then the corresponding subset of the temporal proposition “$P$ at
5.3. **THE CLASSICAL IMPRINT OF THE HPO**

Time \( t_1 \) and then Q at time \( t_2 \) is

\[
C_P \cap \tau_{(t_1,t_2)}^{-1}(\tau_{(t_1,t_2)}(C_P)) = C_P \cap C_P = C_P
\]

(5.3.6)

Hence, the problem that arises is how to discriminate the above temporal proposition from the proposition “P is true at time \( t_1 \)”. From another perspective, what we seek is another representation of temporal logic that is independent of the specific dynamics of a particular system.

Actually, the natural solution to this, is to take the Cartesian product of copies of the phase space. More precisely, we employ a mathematical model in which each single-time proposition corresponds to a subset of a particular copy of the phase space \( \Gamma_t \), labeled by the time parameter. The proposition “\( P_{t_1} \) and then \( P_{t_2} \) and then...and then \( P_{t_n} \)” corresponds to the subset

\[
C_{t_1} \times C_{t_2} \times \cdots \times C_{t_n}
\]

(5.3.7)

of the Cartesian product \( \Gamma_{t_1} \times \Gamma_{t_2} \times \cdots \times \Gamma_{t_n} \) of copies of the standard phase space.

We know that a natural way to proceed with the quantisation algorithm for such formalism, is to take the tensor product of copies of Hilbert spaces \( \mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2} \otimes \cdots \otimes \mathcal{H}_{t_n} \). This is an important result because it provides another, rather novel justification of the original quantum construction of the HPO theory. Furthermore it shows that, in general, one may well start by defining classical HPO histories, and then proceed with their quantum analogues.

**The Classical Analogue of the HPO Theory** We now define the history action functional \( S_h(\gamma) \) on \( \Pi \) as

\[
S_h(\gamma) := \int_{t_1}^{t_2} [p_t \dot{q}_t - H_t(p_t,q_t)](\gamma) \, dt
\]

(5.3.8)

where \( q_t(\gamma) \) is the position coordinate at the time point \( t \in [t_1,t_2] \) of the path \( \gamma \), and \( \dot{q}_t(\gamma) \) is the velocity coordinate at the time point \( t \in [t_1,t_2] \) of the path \( \gamma \).

\(^3\)This is a compact expression, where the indices \( i,j \) used previously are implicit.
We also define the history classical analogues for the Liouville and time-averaged Hamiltonian operators as

\[ V_h(\gamma) := \int_{t_1}^{t_2} [p_t \dot{q}_t](\gamma) \, dt \quad (5.3.9) \]

\[ H_h(\gamma) := \int_{t_1}^{t_2} [H_t(p_t, q_t)](\gamma) \, dt \quad (5.3.10) \]

\[ S_h(\gamma) = V_h(\gamma) - H_h(\gamma) \quad (5.3.11) \]

In classical mechanics, the least action principle states that there exists a functional

\[ S(\gamma) = \int_{t_1}^{t_2} [p \dot{q} - H(p, q)](\gamma) \, dt \]

such that the physically realised path is a curve in state space, \( \gamma_0 \), with respect to which the condition \( \delta S(\gamma_0) = 0 \) holds, when we consider variations around this curve. From this, the Hamilton equations of motion are deduced to be

\[ \dot{q} = \{ q, H \} \quad (5.3.12) \]

\[ \dot{p} = \{ p, H \} \quad (5.3.13) \]

where \( q \) and \( p \)—the coordinates of the realised path \( \gamma_0 \)—are the solutions of the classical equations of motion. For any function \( F(q, p) \) of the classical solutions it is also true that

\[ \{ F, H \} = \dot{F} \quad (5.3.14) \]

In the case of classical continuous-time histories, one can formulate the above variational principal in terms of the Hamilton equations with the statement: A classical history \( \gamma_{cl} \) is the realised path of the system—i.e. a solution of the equations of motion of the system—if it satisfies the equations

\[ \{ q_t, V_h \}(\gamma_{cl}) = \{ q_t, H_h \}(\gamma_{cl}) \quad (5.3.15) \]

\[ \{ p_t, V_h \}(\gamma_{cl}) = \{ p_t, H_h \}(\gamma_{cl}) \quad (5.3.16) \]

where \( \gamma_{cl} = t \mapsto (q_t(\gamma_{cl}), p_t(\gamma_{cl})) \), and \( q_t(\gamma_{cl}) \) is the position coordinate of the realised path \( \gamma_{cl} \) at the time point \( t \). The eqs. (5.3.15) and (5.3.16) are the history equivalent of the Hamilton equations of motion. Indeed, for the case of the simple harmonic oscillator in one dimension the eqs. (5.3.15) and (5.3.16) become

\[ \dot{q}_t(\gamma_{cl}) = \frac{p_t}{m}(\gamma_{cl}) \quad (5.3.17) \]

\[ p_t(\gamma_{cl}) = -m\omega^2 q_t(\gamma_{cl}) \quad (5.3.18) \]
where $\dot{q}_t(\gamma_{cl}) = \dot{q}(\gamma_{cl}(t))$ is the value of the velocity of the system at time $t$. One would have expected the result in eqs. (5.3.15–5.3.16) for the classical analogue of the histories formalism, as it shows that the classical analogue of the two types of time-transformation in the quantum theory coincide.

From the eqs. (5.3.15–5.3.16) we also conclude that the canonical transformation generated by the history action functional $S_h(\gamma_{cl})$, leaves invariant the paths that are classical solutions of the system:

\[
\{q_t, S_h\}(\gamma_{cl}) = 0 \quad (5.3.19)
\]
\[
\{p_t, S_h\}(\gamma_{cl}) = 0 \quad (5.3.20)
\]

It also holds that any function $F$ on $\Pi$ satisfies the equation

\[
\{F, S_h\}(\gamma_{cl}) = 0 \quad (5.3.21)
\]

Some of these statements are implicit in previous work by C. Anastopoulos [24]; an interesting application of a similar extended Poisson bracket using a different formulation has been done by I. Kouletsis [25].

### 5.3.1 Classical Coherent States for the Simple Harmonic Oscillator

In the case of the simple harmonic oscillator, the relation between the classical and the quantum history theories can be further exemplified by using coherent states. This special class of states was used in [8] to represent certain continuous-time history propositions in the history space. Coherent states are particularly useful for this purpose since they form a natural (over-complete) base for the Fock space representation of the history algebra.

A class of coherent states in the relevant Fock space is generated by unitary transformations on the cyclic vacuum state:

\[
|f, h\rangle := U[f, h]|0\rangle \quad (5.3.22)
\]

where $U[f, h]$ is the Weyl operator defined as

\[
U[f, h] := e^{\frac{i}{\hbar}(xf - ph)}, \quad (5.3.23)
\]
where \( f \) and \( h \) are test functions in \( L^2(\mathbb{R}) \). The Weyl generator

\[
\alpha(f, h) := x(f) - p(h)
\]  

(5.3.24)

can alternatively be written as

\[
\alpha(f, h) = \frac{\hbar}{i}(b^{\dagger}(w) - b(w^*))
\]  

(5.3.25)

where \( w := f + ih \).

Suppose now that, for a pair of functions \((f, h)\), the operator \(\alpha(f, h)\) commutes with the action operator \(S\):

\[
[S, \alpha(f, h)] = 0.
\]  

(5.3.26)

Then any pair \((f, h)\) satisfying this equation is necessarily a solution of the system of differential equations:

\[
\dot{f} + m\omega^2 h = 0
\]  

(5.3.27)

\[
\dot{h} - \frac{f}{m} = 0
\]  

(5.3.28)

We see that if we identify \( f \) with the classical momentum \( p_{cl} \) and \( h \) with the classical position \( x_{cl} \), then the eqs. (5.3.27–5.3.28) are precisely the classical equations of motion for the simple harmonic oscillator:

\[
x_{cl}'' + \omega^2 x_{cl} = 0.
\]  

(5.3.29)

The classical solutions \((f, h)\) distinguish a special class of Weyl operators \(\alpha_{cl}(f, h)\), and hence a special class of coherent states:

\[
| \exp z_{cl} \rangle := U_{\alpha_{cl}(f, h)} | 0 \rangle
\]  

(5.3.30)

where \( z_{cl} := f + ih \).

These classical-like features stem from the following relation with \(S\)

\[
[S, U_{\alpha_{cl}}] = 0
\]  

(5.3.31)

\[
[S, P_{\exp z_{cl}}] = 0
\]  

(5.3.32)
5.4. THE DECOHERENCE FUNCTION ARGUMENT

where \( P_{\exp z_{cl}} \) is the projection operator onto the (non-normalised) coherent state \( |\exp z_{cl}\rangle \):

\[
P_{\exp z_{cl}} := \frac{|\exp z_{cl}\rangle\langle \exp z_{cl}|}{\langle \exp z_{cl}| \exp z_{cl}\rangle} \quad (5.3.33)
\]

We note that there exists an analogy between eqs. (5.3.19–5.3.20) and eq. (5.3.32), if we consider \((f,h)\) to be the classical solution: \( t \mapsto (q_t, p_t)(\gamma_{cl}) \). In classical histories, the canonical transformation eqs. (5.3.19–5.3.20) generated by the history action functional vanishes on a solution to the equations of motion. On the other hand, when we deal with quantum histories, the action operator produces the classical equations of motion eqs. (5.3.27–5.3.28) when we require that it commutes with the projector (as in eq. (5.3.33)) which corresponds to a classical solution \((f,h)\) of the system. However, this is not directly related to the actual classical limit of the theory: to make any such physical predictions we must involve the decoherence functional and the coarse graining operation.

Notice that many facets of the construction above hold for a generic potential, as long as there exists a representation on \( \mathcal{V}_{cts} \) of the history algebra on which the action operator is defined. However, it is a subject for future research to uncover the analogue of the coherent-states construction in such situations.

5.4 The Decoherence Function Argument

In the consistent histories quantum theory, the dynamics of a system is described by the decoherence functional. In a classical theory it is the action functional that plays a similar role in regard to the dynamics of the system. It is only natural then, to seek for the appearance of the action operator in the decoherence functional. The aim in this section is to write the HPO expression for the decoherence functional with respect to an operator that includes \( S \), and to compare this operator \((i.e., its matrix elements)\), with the operator \( S_{cts}U \) that appears in the decoherence functional \([8]\). Hence, it is useful to present beforehand the definition of the history propositions and of the decoherence functional for the special case of the coherent-states, as presented in \([8]\).
5.4.1 The Coherent States History Propositions

In the construction of the history space \( \mathcal{V}_{cts} = \otimes_{t \in \mathbb{R}} \mathcal{H}_t \) in section 3.3, the use of the coherent states played an important role in demonstrating the isomorphism

\[
\mathcal{V}_{cts} \overset{\text{def}}{=} \otimes_{t \in \mathbb{R}} \left( L^2_t(\mathbb{R}) \right) \simeq \exp \left( L^2(\mathbb{R}, dt) \right). \tag{5.4.1}
\]

The (non normalised) exponential state \(| \exp \phi_t \rangle \in L^2_t(\mathbb{R})\) can be written in terms of the normalised coherent states \(| \mu(t) \rangle\) as

\[
| \exp \mu(t) \rangle = e^{\frac{1}{2}|\mu|^2} | \mu(t) \rangle. \tag{5.4.2}
\]

The usual normalised coherent states \(| \mu(t) \rangle\) are defined as

\[
| \mu(t) \rangle = e^{\frac{1}{2}|\mu|^2} + \mu(t)^{a\dagger} | 0 \rangle. \tag{5.4.3}
\]

with inner product

\[
\langle \lambda | \mu(t) \rangle = e^{\lambda(t)^* \mu(t) - \frac{1}{2} |\lambda(t)|^2 - \frac{1}{2} |\mu(t)|^2}. \tag{5.4.4}
\]

The isomorphism Eq. (5.4.1) allows one to identify the projector \( P_{\otimes_t | \exp \lambda(t) \rangle} \) onto the vector \( \otimes_t | \exp \lambda(t) \rangle \) in \( \otimes_t L^2_t(\mathbb{R}) \), with the projector \( P_{| \exp \lambda \rangle} = e^{-\langle \lambda, \lambda \rangle} | \exp \lambda \rangle \langle \exp \lambda | \) onto the vector \( | \exp \lambda \rangle \) in \( \exp L^2_t(\mathbb{R}, dt) \). The action of the latter is

\[
P_{| \exp \lambda \rangle} | \exp \mu \rangle = e^{\langle \lambda, \mu - \lambda \rangle} | \exp \lambda \rangle. \tag{5.4.5}
\]

Furthermore, the projector \( P_{[a,b] | \exp \lambda \rangle} \), corresponding to history propositions that involve a finite time integral \([a,b]\) was defined in such a way that it is “active” in the region \([a,b]\), otherwise it is equal to the unit operator

\[
P_{[a,b] | \exp \lambda \rangle} | \exp \mu \rangle \overset{\text{def}}{=} \exp \left( \int_a^b \lambda^*(t) (\mu(t) - \lambda(t)) \, dt \right) | \exp \lambda \ast \mu \rangle. \tag{5.4.6}
\]

where

\[
(\lambda \ast \mu)(t) \overset{\text{def}}{=} \begin{cases} 
\lambda(t) & \text{if } t \in [a,b], \\
\mu(t) & \text{otherwise.}
\end{cases} \tag{5.4.7}
\]
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5.4.2 The Coherent States Decoherence Functional

As we have discussed earlier, in the Gell-Mann and Hartle generalised histories [11, 20], the decoherence functional \( d(\alpha, \beta) \) is a complex-valued function of pairs of homogeneous histories \( (\alpha_{t_1}, \alpha_{t_2}, \ldots, \alpha_{t_n}) \) and \( (\beta_{t'_1}, \beta_{t'_2}, \ldots, \beta_{t'_m}) \), defined as

\[
d(\alpha, \beta) = \text{tr}(\tilde{C}_\alpha^\dagger \rho \tilde{C}_\beta)
\]

(5.4.8)

where \( \rho \) is the initial density-matrix, and where the class operator \( \tilde{C}_\alpha \) is defined in terms of the standard Schrödinger-picture projection operators \( \alpha_{t_i} \) as

\[
\tilde{C}_\alpha := U(t_0, t_1) \alpha_{t_1} U(t_1, t_2) \alpha_{t_2} \ldots U(t_{n-1}, t_n) \alpha_{t_n} U(t_n, t_0),
\]

(5.4.9)

where \( U(t, t') = e^{-i(t-t')H/\hbar} \) is the unitary time-evolution operator from time \( t \) to \( t' \).

As shown in section 2.3, for the case of discrete time histories on the discrete tensor product of the standard Hilbert space \( \mathcal{V} \) [14], the decoherence functional can be written as

\[
d(\alpha, \beta) = \text{tr}(\otimes_{n} H) \otimes (\otimes_{m} H) (\alpha \otimes \beta X)
\]

(5.4.10)

where \( X \) is independent of \( \alpha \) defined as \( \alpha = \alpha_{t_1} \otimes \alpha_{t_2} \otimes \cdots \otimes \alpha_{t_n} \) and \( \beta \) defined as \( \beta = \beta_{t'_1} \otimes \beta_{t'_2} \otimes \cdots \otimes \beta_{t'_m} \).

In the HPO formalism, the decoherence functional \( d \) has been constructed for the special case of continuous-time projection operators corresponding to coherent states [8], as discussed above. The decoherence functional \( d(\mu, \nu) \) for two such continuous-time histories is denoted by

\[
d(\mu, \nu) = \text{tr}_{\mathcal{V}_{cts}} (P_{\exp(\mu)} \otimes P_{\exp(\nu)} X)
\]

(5.4.11)

where

\[
X := \langle 0 | P_{-\infty} | 0 \rangle (\mathcal{S}_{cts} \mathcal{U})^\dagger \otimes (\mathcal{S}_{cts} \mathcal{U}).
\]

(5.4.12)

and the two continuous time projectors \( P_{\exp(\mu)} \) and \( P_{\exp(\nu)} \) correspond to the two continuous-time histories \( \mu \) and \( \nu \) respectively. The appearance of the operator \( \mathcal{S}_{cts} \mathcal{U} \) in the expression of the decoherence functional is not restricted to coherent state propositions, but it arises in the decoherence functional [33].

We will now demonstrate certain relations between \( \mathcal{S}_{cts} \mathcal{U} \) and the three crucial operators \( H, V, S \) of the HPO theory, in order to emphasise that the appearance of such
operators in the description of the dynamics is not just a matter of the mathematical formulation used, but it is also a consequence of the physical interpretation of the theory.

5.4.3 The Appearance of the Action Operator in the Decoherence Functional

The operator $S_{cts}$ that appears in the expression for the $d(\mu, \nu)$ was defined in [22] as an approximation of the derivative operator in the sense that

$$S_{cts}\exp(\nu(\cdot)) = |\exp(\nu(\cdot) + \dot\nu(\cdot))\rangle,$$

while the dynamics was introduced by the operator $U$, defined in such way that the notion of time evolution is encoded in the expression

$$e^{i(\lambda, \dot\lambda)}e^{\frac{\pi}{\hbar}H[\lambda]} = \text{tr}_{\nu_{cts}}(S_{cts}U|\exp(\lambda(\cdot))\rangle)$$  \hspace{1cm} (5.4.14)

We expect $V$ and $H$ to play a similar role to that of $S_{cts}$ and $U$ respectively, inside an expression for the decoherence functional. To demonstrate this we will use the type of Fock space construction given in eqs. (4.2.16—4.2.17). In particular, we use the property

$$\Gamma(A)|\exp(\nu(\cdot))\rangle = |\exp(A\nu(\cdot))\rangle$$  \hspace{1cm} (5.4.15)

where $A$ is an operator that acts on the elements $\nu(\cdot)$ of the base Hilbert space $\mathcal{H}$, while the operator $\Gamma(A)$, defined by eq. (4.2.16), acts on the coherent states $|\exp\nu(\cdot)\rangle$ of the Fock space $e^{\mathcal{H}}$.

We notice that $U$ is related to the unitary time-evolution eq. (5.4.14) in a similar way to that of the Hamiltonian operator $H$

$$e^{isH}|\exp(\nu(\cdot))\rangle = \Gamma(e^{is\omega I})|\exp(\nu(\cdot))\rangle = |\exp(e^{is\omega}\nu(\cdot))\rangle$$  \hspace{1cm} (5.4.16)

where $I$ is the unit operator. We also notice that the action of the operator $e^{isH}$ produces phase changes, as reflected in the right hand side of eq. (5.4.16) (which has been calculated for the special case of the simple harmonic oscillator). Furthermore, when the operator $S_{cts}$ acts on a coherent state eq. (5.4.13), it transforms it to another coherent state which
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involves the addition to the defining function $\nu(\cdot)$ in a way that involves the time derivative of $\nu$; and it is noteworthy that the Liouville operator $V$ acts in a similar way:

$$e^{isV}|\exp\nu(\cdot)\rangle = \Gamma(e^{isD})|\exp\nu(\cdot)\rangle = |\exp(e^{isD}\nu(\cdot))\rangle$$

(5.4.17)

where

$$(e^{isD}\nu)(t) = \nu(t + s)$$

(5.4.18)

where $D := -\frac{d}{dt}$. The operator $e^{isD}$ acts on the base Hilbert space, and corresponds to the operator $e^{isV}$ under the $\Gamma$-construction on the Fock space; that is, it acts on the vector $\nu(t)$ and transforms it to another one $\nu(t + s)$, which, for each time $t$ is translation by the time interval $s$.

This suggests that we define the operator $A_s := e^{isS}$, where $S := \int_{-\infty}^{\infty}(p_1\dot{x}_1 - H_t)dt$ is the action operator for the simple harmonic oscillator, which one expects to be related to the operator $S_{cts}U$. For this reason, we write the matrix elements of both operators and compare them.

The general formula for the matrix elements of an arbitrary operator $T$ with respect to the coherent states basis in the history space in $\mathbb{R}$ is

$$\langle \exp\mu(\cdot)|T|\exp\nu(\cdot)\rangle = e^{((\mu|\tau_{\mu\nu})+(\tau_{\mu\nu}|\tau_{\mu\nu}))}\langle \exp\lambda(\cdot)|T|\exp\lambda(\cdot)\rangle\Big|_{\lambda = \bar{\lambda} = 0}$$

(5.4.19)

hence we need only compare the diagonal matrix elements of the two operators $S_{cts}U$ and $A_s$. Thus we have

$$\langle \exp(\lambda(\cdot)|S_{cts}U|\exp(\lambda(\cdot)) = e^{(\lambda,\lambda + \hat{\lambda})}e^{\frac{1}{2}H[\lambda]}$$

(5.4.20)

where $H[\lambda] := \int_{-\infty}^{\infty}H(\lambda(t))dt$ and $H(\lambda) := H(\lambda, \lambda) = \langle\lambda|H|\lambda\rangle/\langle\lambda|\lambda\rangle$; and

$$\langle \exp\lambda(\cdot)|A_s|\exp\lambda(\cdot)\rangle = e^{(\lambda,e^{is(\omega I + D)}\lambda)}$$

(5.4.21)

with

$$(e^{is(\omega I + D)}\lambda)(t) = e^{is\omega}(t + s)$$

(5.4.22)

We also write the diagonal matrix elements of the Liouville, the Hamiltonian and the action operators

$$\langle \exp\lambda(\cdot)|V|\exp\lambda(\cdot)\rangle = \langle \lambda, D\lambda \rangle e^{(\lambda,\lambda)}$$

(5.4.23)

An important property of the Fock construction states that when there exists a unitary operator $e^{isA}$ acting on $L^2(\mathbb{R})$, there exists a unitary operator $\Gamma(e^{isA})$ that acts on the exponential Fock space.
5.4. THE DECOHERENCE FUNCTION ARGUMENT

\[ \langle \exp \lambda(\cdot) | H | \exp \lambda(\cdot) \rangle = \langle \lambda, \omega I \lambda \rangle e^{(\lambda, \lambda)} \]  \hspace{1cm} (5.4.24)

\[ \langle \exp \lambda(\cdot) | S | \exp \lambda(\cdot) \rangle = \langle \lambda, (\omega I + D) \lambda \rangle e^{(\lambda, \lambda)} \]  \hspace{1cm} (5.4.25)

\[ \langle \exp \lambda(\cdot) | S | \exp \lambda(\cdot) \rangle = \langle \lambda, (\omega I + D) \lambda \rangle e^{(\lambda, \lambda)} \]  \hspace{1cm} (5.4.26)

We can also write both of the above operators on the history space \( \mathcal{F}(L^2(\mathbb{R})) \) using their corresponding operators on the Hilbert space \( L^2(\mathbb{R}) \). The \( \Gamma \) construction shows that

\[ S_{cts} = \Gamma(1 + iD) \]  \hspace{1cm} (5.4.28)

\[ S_{cts}U = \Gamma(1 + i\sigma) \]  \hspace{1cm} (5.4.29)

\[ A_s = \Gamma(e^{i\sigma}) = e^{isd\Gamma(\sigma)} \]  \hspace{1cm} (5.4.30)

\[ V = d\Gamma(D) \]  \hspace{1cm} (5.4.31)

\[ S = d\Gamma(\omega I + D) \]  \hspace{1cm} (5.4.32)

\[ H = d\Gamma(I) \]  \hspace{1cm} (5.4.33)

where \( \sigma = \omega I + D \), and \( I \) is the unit operator. As expressions of the same function \( \sigma \), the operators \( S_{cts}U \) and \( A_s \) commute. However, we cannot readily compute their common spectrum because the operator \( S_{cts}U \) is not self-adjoint.

We might speculate that the value of the decoherence functional is maximised for a continuous-time projector that corresponds to a coarse graining around the classical path. Indeed, if we take such a generic projection operator \( P \), we expect that it should commute with the operator \( S_{cts}U \). In this context, we noticed earlier that the projection operator which corresponds to a classical solution \((f, h)\) commutes with the action operator

\[ [S_{cts}U, P_{(f, h)}] = 0. \]  \hspace{1cm} (5.4.34)

Finally, this argument should be compared with the similar condition for classical histories:

\[ \{S_h, F_C\}(\gamma_{id}) = 0. \]  \hspace{1cm} (5.4.35)
Chapter 6

A Study of a Free Relativistic Quantum Field

6.1 Introduction

We wish now to extend the discussion to the HPO theory of a free scalar field. Hartle [20] proposed a consistent histories approach to quantum field theory based on path integrals, and Blencowe [21] gave a careful analysis of the use of class operators. However, almost nothing has been said about the HPO scheme in this context, and we shall now briefly present the necessary developments. The resemblance of the history version of quantum mechanics (‘field theory in zero spatial dimensions’) to a canonical field theory in one spatial dimension suggests that the history version of quantum field theory in three spatial dimensions should resemble canonical quantum field theory in four spatial dimensions. We shall see that this expectation is fully justified.
6.2 The Canonical History Algebra

The first step in constructing an HPO version of quantum field theory is to foliate four-dimensional Minkowski space-time with the aid of a time-like vector \( n^\mu \) that is normalised by \( \eta_{\mu \nu} n^\mu n^\nu = 1 \), where the signature of the Minkowski metric \( \eta_{\mu \nu} \) has been chosen as \((+,-,-,-)\). The canonical commutation relations for a standard bosonic quantum field theory (the analogue of Eq. (3.3.1)) in three spatial dimensions are

\[
\begin{align*}
\left[ \phi(x_1), \phi(x_2) \right] &= 0 \\
\left[ \pi(x_1), \pi(x_2) \right] &= 0 \\
\left[ \phi(x_1), \pi(x_2) \right] &= i\bar{\hbar} \delta^3(x_1 - x_2).
\end{align*}
\] (6.2.1 - 6.2.3)

In constructing the associated HPO theory we shall assume that the passage from the canonical algebra Eq. (3.3.1) to the history algebra Eqs. (3.3.5)–(3.3.7) is reflected in the field theory case by passing from Eqs. (6.2.1)–(6.2.3) to

\[
\begin{align*}
\left[ \phi_{t_1}(x_1), \phi_{t_2}(x_2) \right] &= 0 \\
\left[ \pi_{t_1}(x_1), \pi_{t_2}(x_2) \right] &= 0 \\
\left[ \phi_{t_1}(x_1), \pi_{t_2}(x_2) \right] &= i\bar{\hbar} \delta^3(t_1 - t_2) \delta^3(x_1 - x_2)
\end{align*}
\] (6.2.4 - 6.2.6)

where, for each \( t \in \mathbb{R} \), the fields \( \phi_t(x) \) and \( \pi_t(x) \) are associated with the spacelike hypersurface \((n,t)\) whose normal vector is \( n \) and whose foliation parameter is \( t \); in particular, the three-vector \( x \) in \( \phi_t(x) \) or \( \pi_t(x) \) denotes a vector in this space.

In using this algebra, we have in mind a representation that is some type of continuous tensor product \( \otimes_{t \in \mathbb{R}} \mathcal{H}_t \) where each \( \mathcal{H}_t \) carries a representation of the standard canonical commutation relations Eqs. (6.2.1)–(6.2.3) for a scalar field theory associated with the given spacetime foliation. However, to emphasise the underlying spacetime picture it is convenient to rewrite Eqs. (6.2.4)–(6.2.6) in terms of four-vectors \( X \) and \( Y \) as

\[
\begin{align*}
\left[ \phi(X), \phi(Y) \right] &= 0 \\
\left[ \pi(X), \pi(Y) \right] &= 0 \\
\left[ \phi(X), \pi(Y) \right] &= i\bar{\hbar} \delta^4(X - Y).
\end{align*}
\] (6.2.7 - 6.2.9)

In relating these expressions to those in Eqs. (6.2.4)–(6.2.6) the three-vector \( x \) may be equated with a four-vector \( x_n \) that satisfies \( n \cdot x_n = 0 \) (the dot product is taken with
6.3. THE HAMILTONIAN ALGEBRA

respect to the Minkowski metric $\eta_{\mu\nu}$ so that the pair $(t, x) \in \mathbb{R} \times \mathbb{R}^3$ is associated with the spacetime point $X = tn + x_n$ (in particular, $t = n \cdot X$). Note, however, that the covariant-looking nature of these expressions is deceptive and it is not correct to assume a priori that the fields $\phi(X)$ and $\pi(Y)$ transform as spacetime scalars under the action of some ‘external’ spacetime Poincaré group that acts on the $X$ and $Y$ labels—as things stand there is an implicit $n$ label on both $\phi$ and $\pi$. We shall return to this question later.

6.3 The Hamiltonian Algebra

The key idea of our HPO approach to quantum field theory is that the physically-relevant representation of the canonical history algebra Eqs. (6.2.4)–(6.2.6) [or, equivalently, Eqs. (6.2.7)–(6.2.9)] is to be selected by requiring the existence of operators that represent history propositions about temporal averages of the energy defined with respect to the chosen spacetime foliation. Thus, for a fixed foliation vector $n$, we seek a family of ‘internal’ Hamiltonians $H_{n,t}, t \in \mathbb{R}$, whose explicit formal form (i.e., the analogue of Eq. (3.3.33)) can be deduced from the standard quantum field theory expression to be

$$H_{n,t} := \frac{1}{2} \int d^4X \left\{ \pi(X)^2 + (n^\mu n^\nu - \eta^{\mu\nu}) \partial_\mu \phi(X) \partial_\nu \phi(X) + m^2 \phi(X)^2 \right\} \delta(t-n \cdot X). \quad (6.3.1)$$

The analogous, temporally-averaged object is

$$H_n(\chi) := \int_{-\infty}^{\infty} dt \chi(t) H_{n,t} \quad (6.3.2)$$

$$= \frac{1}{2} \int d^4X \left\{ \pi(X)^2 + (n^\mu n^\nu - \eta^{\mu\nu}) \partial_\mu \phi(X) \partial_\nu \phi(X) + m^2 \phi(X)^2 \right\} \chi(n \cdot X)$$

where $\chi$ is a real-valued test function.

As in the discussion above of the simple harmonic oscillator, the next step is to consider the commutator algebra that would be satisfied by the operators $H_n(\chi)$ if they existed. These field-theoretic analogues of Eqs. (3.3.39)–(3.3.41) are readily computed as

$$[H_n(\chi), \phi(X)] = -i\hbar \chi(n \cdot X) \pi(X) \quad (6.3.3)$$

$$[H_n(\chi), \pi(X)] = i\hbar \chi(n \cdot X) K_n \phi(X) \quad (6.3.4)$$

$$[H_n(\chi_1), H_n(\chi_2)] = 0 \quad (6.3.5)$$
where $K_n$ denotes the partial differential operator

\[
(K_n f)(X) := 
[\eta^{\mu\nu} - n^\mu n^\nu] \partial_\mu \partial_\nu + m^2 \] f(X). \tag{6.3.6}
\]

The exponentiated form of Eqs. (6.3.3)–(6.3.4) is

\[
e^{iH_n(x)/\hbar} \phi(X) e^{-iH_n(x)/\hbar} = 
\cos \left[ \chi(n \cdot X) \sqrt{K_n} \right] \phi(X) + \frac{1}{\sqrt{K_n}} \sin \left[ \chi(n \cdot X) \sqrt{K_n} \right] \pi(X) \tag{6.3.7}
\]

\[
e^{iH_n(x)/\hbar} \pi(X) e^{-iH_n(x)/\hbar} = 
-\sqrt{K_n} \sin \left[ \chi(n \cdot X) \sqrt{K_n} \right] \phi(X) + \cos \left[ \chi(n \cdot X) \sqrt{K_n} \right] \pi(X) \tag{6.3.8}
\]

where the square-root operator $\sqrt{K_n}$, and functions thereof, can be defined rigorously using the spectral theory of the self-adjoint, partial differential operator $K_n$ on the Hilbert space $L^2(\mathbb{R}^4, d^4X)$. Note that the expression $\chi(n \cdot X) \sqrt{K_n}$ is unambiguous since, viewed as an operator on $L^2(\mathbb{R}^4, d^4X)$, multiplication by $\chi(n \cdot X)$ commutes with $K_n$.

### 6.4 The Fock Space Representation

The right hand side of Eqs. (6.3.7)–(6.3.9) defines an automorphism of the CHA Eqs. (6.2.7)–(6.2.9) and the task is to find a representation of the latter in which these automorphisms are unitarily implemented. To this end, define new operators

\[
q(X) := K_n^{1/4} \phi(X) \tag{6.4.1}
\]

\[
p(X) := K_n^{-1/4} \pi(X) \tag{6.4.2}
\]

and

\[
b(X) := \frac{1}{\sqrt{2}} \left( q(X) + ip(X) \right) = \frac{1}{\sqrt{2}} \left( K_n^{1/4} \phi(X) + iK_n^{-1/4} \pi(X) \right) \tag{6.4.3}
\]

which satisfy

\[
[b(X), b(Y)] = 0 \tag{6.4.4}
\]

\[
[b^\dagger(X), b^\dagger(Y)] = 0 \tag{6.4.5}
\]

\[
[b(X), b^\dagger(Y)] = \hbar \delta^4(X - Y). \tag{6.4.6}
\]
6.5. **THE QUESTION OF EXTERNAL LORENTZ INVARIANCE**

Then

\[ e^{iH_n/\hbar} q(X) e^{-iH_n/\hbar} = \cos \left[ \chi(n \cdot X) \sqrt{K_n} \right] q(X) + \sin \left[ \chi(n \cdot X) \sqrt{K_n} \right] p(X) \]  

(6.4.7)

\[ e^{iH_n/\hbar} p(X) e^{-iH_n/\hbar} = -\sin \left[ \chi(n \cdot X) \sqrt{K_n} \right] q(X) + \cos \left[ \chi(n \cdot X) \sqrt{K_n} \right] p(X) \]  

(6.4.8)

and so

\[ e^{iH_n/\hbar} b(X) e^{-iH_n/\hbar} = e^{-i\chi(n \cdot X)\sqrt{K_n}} b(X). \]  

(6.4.10)

However, the operator defined on \( L^2(\mathbb{R}^4) \) by

\[ (U(\chi)\psi)(X) := e^{-i\chi(n \cdot X)\sqrt{K_n}} \psi(X) \]  

(6.4.11)

is unitary, and hence—using the same type of argument invoked earlier for the simple harmonic oscillator—we conclude that the desired quantities \( H_n(\chi) \) exist as self-adjoint operators on the Fock space \( \mathcal{F}[L^2(\mathbb{R}^4, d^4X)] \) associated with the creation and annihilation operators \( b^\dagger(X) \) and \( b(X) \). The spectral projectors of these operators then represent propositions about the time-averaged value of the energy in the spacetime foliation determined by \( n \).

6.5 **The Question of External Lorentz Invariance**

An important part of standard quantum field theory is a proof of invariance under the Poincaré group—something that, in the canonical formalism, is not totally trivial since the Schrödinger-picture fields depend on the reference frame (i.e., the spacetime foliation). The key ingredient is a construction of the generators of the Poincaré group as explicit functions of the canonical field variables; in practice, the first step is often to construct the Heisenberg-picture fields with the aid of the Hamiltonian, and then to demonstrate manifest Poincaré covariance within that framework. The canonical fields associated with any spacelike surface in a particular Lorentz frame can then be obtained by restricting the Heisenberg fields (and their normal derivatives) to the surface.
CHAPTER 6. A STUDY OF A FREE RELATIVISTIC QUANTUM FIELD

When considering the role of the Poincaré group in the HPO picture of consistent histories, the starting point is the observation that, heuristically speaking, for a given foliation vector $n$—and for each value of the associated time $t$—there will be a Hilbert space $H_t$ carrying an independent copy of the standard quantum field theory. In particular, therefore, for fixed $n$, there will be a representation of the Poincaré group associated with each spacelike slice $(n, t)$, $t \in \mathbb{R}$. Thus if $A_a$, $a = 1, 2, \ldots, 10$ denote the generators of the Poincaré group, there should exist a family of operators $A^a_t$ which, for each $t \in \mathbb{R}$, generate the ‘internal’ Poincaré group $P_{n,t}$ associated with the slice $(n, t)$. These operators will satisfy a ‘temporally gauged’ version of the Poincaré algebra. More precisely, if $C^{abc}$ are the structure constants of the Poincaré group, so that

$$[A^a, A^b] = iC^{abc}A^c,$$  \hfill (6.5.1)

then the algebra satisfied by the history theory operators $A^a_t$ is

$$[A^a_t, A^b_s] = i\delta(t - s)C^{abc}A^c_t,$$  \hfill (6.5.2)

which, of course, reflects the way in which the canonical commutation relations Eqs. (6.2.1–6.2.3) are replaced by Eqs. (6.2.4–6.2.6) in the history theory.

As always in quantum theory, the energy operator is of particular importance, and in the present case we have a family of Hamiltonian operators $H_{n,t}$, $t \in \mathbb{R}$, which are related to the generators $P^\mu_{n,t}$ of translations for the quantum field theory associated with the hypersurface $(n, t)$ by

$$H_{n,t} = n_\mu P^\mu_{n,t}. \hfill (6.5.3)$$

In fact, it is straightforward to show that

$$P^\mu_{n,t} = n^\mu H_{n,t} + \int d^4X \delta(t - n \cdot X)(n^\mu n \cdot \partial \phi - \partial^\mu \phi)\pi \hfill (6.5.4)$$

which suggests that, as would be expected, the components of $P^\mu_{n,t}$ normal to $n$ act are the generators of spatial translations in the hypersurface $(n, t)$. Indeed, Eq. (6.3.3) generalises to

$$[P^\mu_n(\chi), \phi(X)] = -i\hbar \chi(n \cdot X)\{n_\mu \pi(X) + (\partial_\mu \phi(X) - n_\mu n \cdot \partial \phi(X))\}. \hfill (6.5.5)$$

Similarly, the ‘temporally gauged’ Lorentz generators satisfy

$$[J^\mu_{n,t}, \phi(X)] = \hfill (6.5.6)$$

$$i\hbar \delta(t - n \cdot X)\{X^\mu (\partial^\nu \phi - n^\nu n \cdot \partial \phi) - X^\nu (\partial^\mu \phi - n^\mu n \cdot \partial \phi) - (X^\mu n^\nu - X^\nu n^\mu)\pi\}. \hfill (6.5.6)$$
6.5. THE QUESTION OF EXTERNAL LORENTZ INVARIANCE

As emphasised above, each generator of the group $P_{n,t}$ acts ‘internally’ in the Hilbert space $H_t$; in particular, this is true of the Hamiltonian, which (modulo the need to smear in $t$) generates translations along an ‘internal’ time label $s$ that is to be associated with each leaf $(n, t)$ of the foliation. It is important to note that $H_{n,t}$ does not generate translations along the ‘external’ time parameter $t$ that appears in the CHA Eqs. (6.2.4–6.2.6) and which labels the spacelike surface (of course, there is an analogous statement for the Hamiltonians $H_t$ in the HPO model of the simple harmonic oscillator considered earlier). The existence of these internal Poincaré groups is sufficient to guarantee covariance of physical quantities, such as transition amplitudes, that can be calculated in the class operator version of the theory.

However, the HPO formalism admits an additional type of Poincaré group—what we shall call the ‘external’ Poincaré group—which is defined to act on the pair of labels $(x, t)$ that appear in the CHA Eqs. (6.2.4–6.2.6). Thus these labels include the ‘external’ time parameter $t$ that specifies the leaf $(n, t)$ of the foliation associated with the timelike vector $n$. In the context of the covariant-looking version Eqs. (6.2.7–6.2.9) of the CHA, the main question is whether the fields $\phi(X)$ and $\pi(X)$ transform in a covariant way under this external group.

As far as the field $\phi(X)$ is concerned it seems reasonable to consider the possibility that this may an external scalar in the sense that there exists a unitary representation $U(\Lambda)$ of the external Lorentz group $U(\Lambda)$ such that

$$U(\Lambda)\phi(X)U(\Lambda)^{-1} = \phi(\Lambda X).$$

(6.5.7)

The spectral projectors of the (suitably smeared) operators $\phi(X)$ then represent propositions about the values of the spacetime field in a covariant way.

However, the situation for the field momentum $\pi(X)$ is different since this is intrinsically associated with the timelike vector $n$. Indeed, the natural thing would be to require the existence of a family of operators $\pi_n(X)$ where $n$ lies in the hyperboloid of all timelike (future-pointing) vectors, and such that

$$U(\Lambda)\pi_n(X)U(\Lambda)^{-1} = \pi_{\Lambda n}(\Lambda X).$$

(6.5.8)

The next step in demonstrating external Poincaré covariance would be to extend the
algebra (6.2.7–6.2.9) to include the \( n \) parameter on the \( \pi \) field; in particular, one would need to specify the commutator \([\pi_n(X), \pi_m(Y)]\), but it is not obvious \textit{a priori} what this should be.

Another possibility would be to try to combine the Heisenberg picture—and its associated ‘internal’ time \( s \)—with the external time parameter \( t \) of the spacetime foliation to give some scheme that was manifestly covariant in the context of a five-dimensional space with signature \((+++,--)\) associated with the variables \((x, t, s)\). However, we do not know if this is possible and the demonstration of external Poincaré covariance, if it exists, remains the subject for future research.
6.2 CHAPTER 6. A STUDY OF A FREE RELATIVISTIC QUANTUM FIELD
Chapter 7

Conclusions

We have discussed the introduction of continuous-time histories within the ‘HPO’ version of the consistent-histories formalism in which propositions about histories of the system are represented by projection operators on a ‘history’ Hilbert space. The history algebra (whose representations specify this space) for a particle moving in one dimension is isomorphic to the canonical commutation relations for a one-dimensional quantum field theory, thus allowing the history theory to be studied using techniques drawn from quantum field theory. In particular, we have shown how the problem of the existence of infinitely many inequivalent representations of the history algebra can be solved by requiring the existence of operators whose spectral projectors represent propositions about time-averages of the energy.

We have examined the example of the simple harmonic oscillator, in one dimension, within the History Projection Operator formulation of the consistent-histories scheme. We defined the action operator as the quantum analogue of the classical Hamilton action functional and we have proved its existence by finding a representation on the $\mathcal{F}(L^2(\mathbb{R}))$ space of the history algebra. We have shown that the action operator is the generator of two types of time transformations: translations in time from one Hilbert space $\mathcal{H}_t$, labeled by the time parameter $t$, to another Hilbert space with a different label $t$, and phase changes in time with respect to the time parameter $s$ of the standard Heisenberg-time
evolution that acts in each individual Hilbert space $\mathcal{H}_t$. We have expressed the action operator in terms of the Liouville and Hamiltonian operators—which are the generators of the two types of time transformation—and which correspond to the kinematics and the dynamics of the theory respectively.

We have constructed continuous-time classical histories defined on the continuous Cartesian product of copies of the phase space and demonstrated an analogous expression to the classical Hamilton’s equations.

We have shown that the action operator commutes with the defining operator of the decoherence functional, thus appearing in the expression for the dynamics of the theory, as would have been expected.

Finally, we have shown how the HPO scheme can be extended to the history version of canonical quantum field theory. We discussed the difference between the ‘internal’ and ‘external’ Poincaré groups and indicated how the former are implemented in the formalism. A major challenge for future research is to construct an HPO quantum field theory which is manifestly covariant under this external symmetry group.

One of the major reasons for undertaking this study was to provide new tools for tackling the recalcitrant problem of constructing a manifestly covariant quantum field theory in the consistent histories formalism. Work on this problem is now in progress with the expectation that the Hamiltonian and Liouville operators will play a central role in the proof of explicit Poincaré invariance of the theory.
Bibliography

[1] K. Savvidou. Preprint gr-qc/9811078.

[2] V.P. Belavkin. Reconstruction theorem for a quantum stochastic process. *Teor. Mat. Fizika*, 62:409–431, 1983.

[3] R.B. Griffiths. Consistent histories and the interpretation of quantum mechanics. *J. Stat. Phys.*, 36:219–272, 1984.

[4] R. Omnès. Logical reformulation of quantum mechanics. I. Foundations. *J. Stat. Phys.*, 53:893–932, 1988.

[5] P. Mittelstaedt. Time dependent propositions and quantum logic. *Jour. Phil. Logic*, 6:463–472, 1977.

[6] E.-W. Stachow. Logical foundations of quantum mechanics. *Int. J. Theor. Phys.*, 19:251–304, 1980.

[7] E.-W. Stachow. Sequential quantum logic. In E.G. Beltrametti and B.C. van Fraassen, editors, *Current Issues in Quantum Logic*, pages 173–191. Plenum Press, New York, 1981.

[8] C.J. Isham and N. Linden. Continuous histories and the history group in generalised quantum theory. *J. Math. Phys.* 36: 5392–5408, 1995

[9]

[10] R. Omnès. Logical reformulation of quantum mechanics. I. Foundations. *J. Stat. Phys.*, 53:893–932, 1988.
[11] M. Gell-Mann and J. Hartle. Quantum mechanics in the light of quantum cosmology. In W. Zurek, editor, *Complexity, Entropy and the Physics of Information, SFI Studies in the Science of Complexity, Vol. VIII*, pages 425–458. Addison-Wesley, Reading, 1990.

[12] C.J. Isham. Quantum logic and the histories approach to quantum theory. *J. Math. Phys.* 35:2157–2185, 1994.

[13] C.J. Isham and N. Linden. Quantum temporal logic and decoherence functionals in the histories approach to generalised quantum theory. *J. Math. Phys.* 35:5452–5476, 1994.

[14] C.J. Isham, N. Linden and S. Schreckenberg. The classification of decoherence functionals: an analogue of Gleason’s theorem. *J. Math. Phys.* 35:6360–6370, 1994.

[15] H. Araki. Hamiltonian formalism and the canonical commutation relations in quantum field theory. *J. Math. Phys.* 1:492–504, 1960.

[16] B. Simon. The $P(\phi)^2$ Euclidean (Quantum) Field Theory. Princeton University Press, Princeton, 1974.

[17] R. Jackiw Field theoretic investigations in current algebra. In *Current Algebra and Anomalies*, S.B. Treiman, R. Jackiw, B. Zumino and E. Witten, eds, World Scientific, Singapore, 1985.

[18] J. Park and H. Margenau. Simultaneous measurement in quantum theory. *Int. J. Theor. Phys.* 1:211–283, 1968.

[19] J. Hartle. Spacetime grainings in nonrelativistic quantum mechanics. *Phys. Rev.*, D44:3173–3195, 1991.

[20] J. Hartle. Spacetime quantum mechanics and the quantum mechanics of spacetime. In *Proceedings on the 1992 Les Houches School, Gravitation and Quantisation*. 1993.

[21] M. Blencowe. The consistent histories interpretation of quantum fields in curved spacetime. *Ann. Phys. New York* 211:87–111, 1991.

[22] C. Isham, N. Linden, K. Savvidou and S. Schreckenberg. “Continuous time and consistent histories”, *J. Math. Phys.* 37, 2261 (1998).
[23] C.J.Isham. “Quantum logic and the histories approach to quantum theory”, *J. Math. Phys.* **23**, 2157 1994.

[24] C.Anastopoulos. “On the selection of preferred consistent sets”, *Int.Jour. Theor. Phys.* **37**, 2261 (1998).

[25] I. Kouletsis. “A classical history theory: geometrodynamics and general field dynamics regained”, gr-qc 9801019.

[26] P.A.M. Dirac. “The Lagrangian in quantum mechanics”, In *Selected papers on quantum electrodynamics*, edited by J.Schwinger. Dover Publications, Inc. New York (1958).

[27] H.D.Zeh. *The physical basis of the direction of time*, Springer-Verlag Berlin Heidelberg 1992.

[28] F.A.Berezin. *The method of second quantization*, Academic Press New York and London 1966.

[29] H.Goldstein. *Classical Mechanics*, Addison-Wesley Publishing Company 1980.

[30] J.J.Halliwell *Notes on quantum cosmology lectures*, Imperial College, M.Sc. course in quantum fields and fundamental forces 1995.

[31] From a private communication of Prof. Tulsi Dass with
Prof. Chris Isham.

[32] K.Savvidou The continuous-time histories approach to quantum field theory: the Poincare symmetry, Work in preparation.

[33] C.Anastopoulos. “Quantum Fields in Nonstatic background: A Histories Perspective”, gr-qc 9903026.