Abstract—We study the class of reach-avoid dynamic games in which multiple agents interact noncooperatively, and each wishes to satisfy a distinct target criterion while avoiding a failure criterion. Reach-avoid games are commonly used to express safety-critical optimal control problems found in mobile robot motion planning. Here, we focus on finding time-consistent solutions, in which future motion plans remain optimal even when a robot diverges from the plan early on due to, e.g., intrinsic dynamic uncertainty or extrinsic environment disturbances. Our main contribution is a computationally-efficient algorithm for multi-agent reach-avoid games which renders time-consistent solutions for all players. We demonstrate our approach in two- and three-player simulated driving scenarios, in which our method provides safe control strategies for all agents.

I. INTRODUCTION

Consider a canonical motion planning problem, in which a mobile robot must reach a pre-specified goal condition while avoiding obstacles or other hazards. There are a wide variety of approaches for solving these types of problems, including randomized graph planning, smooth (non)convex optimization, and dynamic programming. In many ways, such methods are interchangeable in that they produce an optimal robot trajectory which achieves the desired outcome without violating any safety constraints. This paper concentrates on a subtle distinction, however, which is intimately related to solution robustness. That is, we focus on the time consistency of solutions to these reach-avoid problems, as illustrated in Fig. 1.

Informally, a time-consistent solution is one which is still optimal despite variation early on in the robot’s trajectory. Time-inconsistent plans are thus highly susceptible to uncertainty in robot motion and environment structure, such as the location of obstacles. This susceptibility is well-known in the motion planning community, and necessitates rapid re-planning to account for the latest sensor observations.

Established methods for identifying time-consistent solutions to reach-avoid problems amount to solving a Partial Differential Equation (PDE) whose dimensionality corresponds to the dimension of the robot’s state space. Unfortunately, this requires exponential computation time and memory in general, relegating reach-avoid verification to offline operation. Recent progress establishes an efficient technique for computing real-time local solutions in the avoid-only case [1]. Unfortunately, however, this approach is time-inconsistent. In that context, the contributions of this paper are threefold:

1) Efficient reach-avoid algorithm. We propose a novel, computationally efficient technique for finding approximate solutions to reach-avoid problems.

2) Time consistency. We modify this approach such that it finds (locally) time-consistent control strategies, in which later control inputs are still optimal despite suboptimal behavior earlier in the trajectory.

3) Multi-agent, game-theoretic settings. Many modern robotic systems operate in the presence of other agents with potentially competing objectives (e.g., traffic). We derive and demonstrate our methods in the setting of multi-player reach-avoid games.

II. PRIOR WORK

Robot motion planning methods aim to guide a robot from an initial configuration to a desired one while avoiding undesired conditions such as collisions with obstacles. These two temporal logic properties are commonly referred to, respectively, as liveness (a desired condition will eventually take place) and safety (undesired conditions will never take place) [2]. Historically, such problems have been primarily addressed through two complementary lenses: search and optimization. Search-based methods [3]–[5] construct (potentially randomized) graphs of feasible robot motion, and identify optimal paths therein while enforcing safety during graph construction by rejecting any candidate nodes or edges.
that violate the state constraints. However, the performance of these methods suffers in high-dimensional problems, making them challenging to use in real-time applications and especially in safety-critical settings.

Continuous optimization-based methods exploit the differentiability of robot dynamics and planning objectives to solve certain motion planning problems more efficiently. However, since these methods rely upon derivative information, they find local solutions as opposed to the global solutions identified by graph search methods. Still, optimization-based methods [6, 7] are widely used in real-time motion planning and Model Predictive Control (MPC) [8].

Our method falls within the optimization-based category, but uses a level set method to enforce richer trajectory-wide properties in the form of temporal logic specifications, rather than state and control constraints at individual time instants. With level set methods, reach-avoid objectives can be constructed so that the sign of a scalar outcome determines whether the safety and liveness conditions have been met [10], [11]. This continuous scalar encoding induces a dynamic programming relation in the form of a Hamilton-Jacobi (HJ) PDE or variational inequality, whose viscosity solution [12] determines the outcome of the game when all players choose their control inputs optimally. This Hamilton-Jacobi equation is usually solved numerically [13]; however computation scales poorly with state dimension due to the “curse of dimensionality” [14].

Recently, however, differential dynamic programming methods [7], such as the Iterative Linear-Quadratic Regulator (ILQR) [15] have proven extremely successful in trajectory optimization problems, and have been extended to multi-agent settings [16], [17]. These ideas have motivated the investigation of an efficient level set formulation for the safety (avoid-only) case [1]. As we prove in [16], this approach unfortunately results in a time-inconsistent control strategy. In this work, we develop a novel approach which retains the computational advantages of prior work while yielding a more robust, time-consistent solution.

III. PROBLEM FORMULATION

A. Reach-Avoid Problem

We begin by considering a general reach-avoid optimal control problem, in which we seek to drive the state \( x_t \in \mathcal{X} \subseteq \mathbb{R}^n \) of a dynamical system into a (possibly dynamic) target region \( T \subseteq \mathcal{X} \) at some future time \( t \) while avoiding a (possibly time-varying) failure region \( \mathcal{F} \subseteq \mathcal{X} \) at all previous times. Let the discrete-time dynamics of the system be given by \( x_{t+1} = f_t(x_t, u_t) \), with \( u_t \in \mathcal{U} \subseteq \mathbb{R}^m \) denoting the control input at time \( t \).

The reach-avoid condition for a generic state trajectory \( x := (x_0, \ldots, x_T) \) can be expressed as:

\[
\exists t \in \{0, \ldots, T\} : x_t \in T_t \land \forall \tau \in \{0, \ldots, t\} : x_\tau \notin \mathcal{F}_\tau \quad (1)
\]

We only assume that \( T \) is closed and \( \mathcal{F} \) is open, following the convention in the literature [10], [11]; no other assumptions (convexity, connectedness, etc.) are required. We can then characterize the target and failure sets using two (possibly time-varying) Lipschitz continuous margin functions, \( \ell_t, g_t : \mathcal{X} \rightarrow \mathbb{R} \), as follows:

\[
\ell_t(x_t) \leq 0 \iff x_t \in T_t, \quad g_t(x_t) > 0 \iff x_t \in \mathcal{F}_t. \quad (2)
\]

This characterization is the basis of the level set method, which allows us to encode the reach-avoid condition [1] through the following scalar objective, which we seek to minimize by our choice of control strategy:

\[
J_s(x) = \min_{t \in \{s, \ldots, T\}} \min_{\tau \in \{s, \ldots, t\}} \{ \ell_t(x_t), \max_{\tau \in \{s, \ldots, t\}} g_t(x_\tau) \}. \quad (3)
\]

Note that we have expressed this objective-to-go along the time horizon \( \{s, \ldots, T\} \) for an arbitrary \( s \in \{0, \ldots, T\} \): states visited by trajectory \( x \) prior to \( s \) are neglected by \( J_s \). In general, we seek solutions for \( s = 0 \), but as we shall see, subproblems for other \( s > 0 \) will allow us to define and derive more robust time-consistent solutions.

B. Time Consistency

Time consistency expresses the desirable property that an optimal control strategy remains optimal for sub-problems beginning at intermediate times along its execution despite earlier suboptimal actions. In other words, the robot has no incentive to deviate from a time-consistent strategy.

The traditional definitions of time consistency in the optimal control and dynamic game theory literature (cf. [18, Ch. 5]) are specific to problems with time-additive costs. Here, we extend these definitions to reach-avoid problems in which margin function values at early times can dominate overall cost. The extensions below preserve the spirit of the time-additive definitions, and continue to encode the rationality of later control decisions regardless of state history.

The notion of time consistency can be applied to a variety of control strategy classes, such as state feedback policies and open-loop control signals. Let \( \gamma \in \Gamma \) be the allowed control strategies along the time horizon \( \{0, \ldots, T\} \), and denote by \( \gamma_s \in \Gamma_s \) their truncation to the time horizon \( \{s, \ldots, T\} \) for \( s \in \{0, \ldots, T\} \).

Definition 1 (Weak time consistency): An optimal control strategy \( \gamma^* \in \Gamma \) for the reach-avoid objective \( J_0 \) is weakly time-consistent if its truncation to \( \{s, \ldots, T\} \) remains optimal for the truncated reach-avoid problem with objective \( J_s \) that results after applying \( \gamma^* \) on \( \{0, \ldots, s-1\} \). That is, given the optimal strategy \( \gamma^* \) (which, by assumption, achieves the minimum \( J_0 \) from all initial conditions \( x_0 \in X_0 \)), consider any alternative strategy \( \gamma \in \Gamma \) that is identical to \( \gamma^* \) at times \( t < s \) and may differ thereafter. Then the following inequality must hold for every such \( \gamma \):

\[
J_s(x^\gamma_0) \leq J_s(x^\gamma_0) \quad \forall s \in \{0, \ldots, T\}, \forall x_0 \in X_0,
\]

with \( x^\gamma_0 \) denoting the state trajectory resulting from applying strategy \( \gamma \) from initial conditions \( x_0 \) at time 0. A solution \( \gamma^* \) that is not weakly time-consistent is time-inconsistent.

Definition 2 (Strong time consistency): An optimal control strategy \( \gamma^* \in \Gamma \), defined for times \( t \in \{0, \ldots, T\} \) is strongly time-consistent if it remains optimal for the
truncated reach-avoid problem when restricted to begin at an intermediate time \( s \) from an arbitrary state \( \tilde{x} \). That is, consider any other strategy \( \gamma \) defined at subsequent times \( t \in \{s, \ldots, T\} \). Then the following inequality must hold for every such \( \gamma \):

\[
J_s(x_{s,x}^\gamma) \leq J_s(x_{s,x}^\gamma), \forall s \in \{0, \ldots, T\}, \forall \tilde{x} \in X.
\]

A strongly time-consistent solution \( \gamma^* \) is also weakly time-consistent; the converse is not in general true.

**Remark 1:** In some cases, we may only be interested in the reach-avoid problem from a known initial state, in which case \( x_0 := \{x_0\} \). This is commonly the case for online trajectory optimization schemes, like the one proposed here.

C. General-Sum Multi-Player Reach-Avoid Games

For the remainder of this paper, we consider the more general setting of an \( N \)-player game in which each player’s objective, denoted \( J^i \) (with target and failure margin functions \( \ell^i \) and \( g^i \)), is of the form (3), and the overall state of the system \( x \) evolves jointly with all players’ inputs \( u^i, i \in \{1, \ldots, N\} \), i.e., \( x_{t+1} = f_t(x_t, u^1_t, \ldots, u^N_t) \). In this multi-player setting, time-consistency is defined analogously to [Def. 1 and 2] where “solution” is understood to refer to the desired game-theoretic equilibrium concept, as in [18, Ch. 5, footnote 44]—in our case, a local Nash equilibrium.

**Definition 3:** Local Nash Equilibrium [19, Def. 1] A tuple of player strategies \( \gamma^* := (\gamma^*_i)^{N+1}_{i=1} \) is a local Nash equilibrium if there exists a neighborhood of strategies \( \tilde{\Gamma}^i \subset \Gamma^i \) for each player (with \( \gamma^* \in \tilde{\Gamma}^i \)) within which player has no incentive to unilaterally deviate from strategy \( \gamma^* \).

That is, the following must hold for every player \( i \):

\[
\exists \tilde{\Gamma}^i \subset \Gamma^i : \gamma^* \in \tilde{\Gamma}^i \land J_0^i(x_{0 x}^{\gamma^*}) \leq J_0^i(x_{0 x}^{(\gamma^*_0, \gamma^*)}), \forall \gamma^* \in \tilde{\Gamma}^i, \forall x_0 \in X_0,
\]

where \( (\gamma^* \in \tilde{\Gamma}^i) \) represents a tuple of strategies where all players except for \( i \) follow \( \gamma^* \) and player \( i \) uses \( \gamma^i \).

IV. TECHNICAL APPROACH

Recent work [1] develops an efficient method for solving reachability (reach-only or avoid-only) optimal control problems to a locally optimal feedback policy, as well as multi-player reachability games to a local feedback Nash equilibrium. This approach is based upon earlier work for computing local Nash equilibria in general-sum dynamic games with time-additive costs using ILQ approximations [16], theoretical analysis of which is given in [20]. In the reachability setting of [1], each player’s objective is of the form \( J_s(x) = \min_{\ell \in \{u, \ldots, T\}} \ell_\ell(x_i) \), which can be seen as a special case of (3) for \( g_i(\cdot) \equiv -\infty \).

Each Linear-Quadratic (LQ) approximation of the game is formed by first finding the time \( \tau \) at which the above reachability objective attains its minimum for the current iteration’s state trajectory, and then minimizing (liveness) the value of \( \ell_\ell(x_\tau) \) (alternatively maximizing for safety problems). The resulting LQ game is in the standard time-additive cost form, with only one nonzero state cost term along the entire horizon (namely, at \( t = \tau \)), since the remaining time steps do not locally contribute to \( J_s(x) \) within a neighborhood of the current trajectory iterate. To reflect this structure, we refer to this approach as the “pinch-point method.”

A. Pinch-Point Reach-Avoid

Our first contribution is to extend the pinch-point method to reach-avoid problems with player objectives in the form of (3). The extension is conceptually straightforward, but the more complex form of the reach-avoid objective (3) requires carefully keeping track of both \( \ell \) and \( g \) to resolve which margin function, and at what time, is determining the value of \( J_s(x) \) along the current trajectory iterate \( x \).

The proposed extension is summarized in Alg. 2, which forms a subroutine of the iterative LQ game procedure of Alg. 1 from [16] and [1].

In Alg. 1 we iteratively refine the set of strategies for all players, \( \gamma = (\gamma^1, \ldots, \gamma^N) \) by solving LQ games which locally approximate the underlying reach-avoid game where each player’s objective is of the form (3). The reach-avoid pinch-point method in Alg. 2 (extending that of [1]), forms a standard, time-additive LQ approximation of the type considered in standard references such as [18, Ch. 6]. In particular, we find a linear approximation to the dynamics about the current trajectory iterate \( x \) (iteration index suppressed for
B. Time-Consistent Reach-Avoid

Our second, and more substantial, contribution is a locally strongly time-consistent algorithm for reach-avoid problems. This approach shares the ILQ structure of [Alg. 1] but we derive a new time-consistent solution for the LQ case, given in [Alg. 3]. As before, we begin by identifying times at which the reach-avoid objective $J^*$ takes the value of either the target margin function $f^*_t$ or failure margin function $g^*$.

However, where [Alg. 2] records only a single so-called pinch point which determines the value of $J_0^*$, here we record a set of critical times $\{\tau_j^i\}$ and associated margin functions $m^i_{\tau_j,j^i}()$—which are either $f^i_{\tau_j,j^i}()$ or $g^i_{\tau_j,j^i}()$—such that

$$J^i_t(\cdot) = m^i_{\tau_j,j^i}(\cdot), \forall \tau_{j-1} < t \leq \tau_j,$$

presuming that the critical times $\tau_j$ are indexed by $j \in \mathbb{N}$.

After computing the set of critical times for each player $i$, [Alg. 3] solves the reach-avoid LQ game in which each player’s cost-to-go at time $t$ is a quadratic approximation of the associated margin function $m^i_{\tau_j,j^i}()$ for the subsequent critical time $\tau_j$. As in [Alg. 4], the quadratic approximation $J^i_{\cdot}$ has the form

$$J^i_{\cdot} = \frac{1}{2} \left( x^T_{\tau_j} Q^i_{\tau_j} + 2q^i_{\tau_j}^T x_{\tau_j} + \sum_{s=\tau_j}^{\tau_j\tau_j+1} \left( u^T_{s\tau_j} R^i_s + r^i_s u^T_{s\tau_j} \right) u_{s\tau_j} \right),$$

with $\forall \tau_{j-1} < t \leq \tau_j$. Here, $Q^i_{\tau_j}$ and $q^i_{\tau_j}$ constitute a quadratic approximation of time $t$’s active margin function $m^i_{\tau_j,j^i}$ about the current trajectory iterate $x$, and we incorporate the same control regularization term as in [Alg. 4] in the terms $R^i_s$ and $r^i_s$.

We find a time-consistent solution of this LQ game by modifying the standard backward recursion used to solve time-additive LQ games [18 Cor. 6.1]. That is, we represent the cost-to-go from a given state with the matrix-vector pair $(Z^i_t, z^i_t)$, where $J^i_t(x_t) = \frac{1}{2} (x^T_t Z^i_t + z^T_t) x_t$. Whenever $t$ is a critical time $\tau_j$, we fix $(Z^i_t, z^i_t) = (Q^i_{\tau_j}, q^i_{\tau_j})$, and at intervening times we follow the standard time-additive coupled Riccati recursion [18] setting $Q^i_t$ and $q^i_t$ to zero since no margin functions are active during this interval.

This procedure explicitly considers the sequence of LQ games whose objectives encode the subsequent critical time.
and associated margin function. Since feedback Nash solutions to LQ games are strongly time-consistent [18, Chapter 6], our approach identifies time consistent solutions as well. To make this precise, we define local time consistency for a single agent, in line with Def. 1 and Def. 2.

Definition 4 (Local strong time consistency): A locally optimal control strategy \( \gamma \in \Gamma \), defined for times \( t \in \{0, \ldots, T\} \) is locally strongly time-consistent if it remains near-optimal for the truncated reach-avoid problem restricted to begin at an intermediate time \( s \in (0, T) \) from an arbitrary state \( \tilde{x} \) within a neighborhood of the optimal trajectory at that time. That is, for any \( \epsilon > 0 \), there exist \( \delta_x, \delta_\gamma > 0 \) such that, for each time step \( s \in \{0, \ldots, T\} \),

\[
J_s(x_{s,\tilde{x}}) \leq J_s(x_{s,\tilde{x}}^{\gamma^*}) + \epsilon, \quad \forall \tilde{x} \in B(x_{s,\tilde{x}}^{\gamma^*}, \delta_x), \forall \gamma \in B(\gamma^*, \delta_\gamma).
\]

As before, the definition extends to multi-player decision problems by interpreting “optimality” as desired by the game solution concept (e.g. feedback Nash equilibrium). We now state our main theorem regarding the time-consistency of [Alg. 1] and [Alg. 3].

Theorem 2: Let the solution \( \gamma^* \) returned by [Alg. 1] using the subroutine provided in [Alg. 3] be locally optimal under the feedback Nash equilibrium concept. Then, \( \gamma^* \) exhibits (feedback Nash) local strong time consistency.

Proof: By construction, the affine feedback policies produced by [Alg. 3] have constitute a feedback Nash equilibrium to the LQ game defined by the derivatives of players’ reach-avoid objectives \( J_{f,0} \). Further, the stage-wise feedback strategies \((K_t, k_t)_{t=1}^{N} \) at each time \( t \) are optimal for the objective \( J_{f,t} \), as given by [5]. Therefore, these strategies are globally strong-time consistent for the LQ game that locally approximates the overall dynamic game. By hypothesis, \( \gamma^* \) is a local feedback Nash equilibrium, indicating that there exists \( \delta_{\gamma} > 0 \) such that, at each time \( t \in \{0, \ldots, T\} \),

\[
J_t (x_{t,\tilde{x}}) \leq J_t (x_{t,\tilde{x}}^{\gamma^*}), \forall \tilde{x} \in B(x_{t,\tilde{x}}^{\gamma^*}, \delta_{\gamma}).
\]

Therefore, due to the smoothness of players’ costs and the continuous dependence of trajectories \( \gamma^* \) upon initial conditions (cf. [21, Thm. 3.23]), for any \( \epsilon > 0 \) there exists a \( \delta_\gamma > 0 \) such that, for all \( \tilde{x} \in B(x_{t,\tilde{x}}^{\gamma^*}, \delta_{\gamma}) \), the inequality of Def. 4 holds.

We showcase this result in Fig. 1, which compares the trajectories computed by the ILQ reach-avoid scheme (Alg. 1) using the pinch-point (Alg. 2) and time-consistent (Alg. 3) subroutines. In the pinch-point formulation of Sec. IV-A, all trajectories that passed through the target while avoiding the obstacle was indistinguishably optimal. Here, however, any trajectory which terminates at the target while always avoiding the obstacle is indistinguishable. This set of trajectories is precisely that which satisfies the reach-avoid condition \( \Pi \) on subsets of the time horizon \( t \in \{s, \ldots, T\} \) and is hence time-consistent.

C. Computational Efficiency and Optimality of Solutions

While the proposed time-consistent approach introduces an additional Boolean check in Line 5 of [Alg. 3] both time-(in)consistent subroutines have asymptotic complexity commensurate with that of the standard coupled Riccati solution to time-additive LQ games, which scales with the cube of

| PP | target reached (#) | mean (max) # iters | safe after target (#) |
|----|-------------------|-------------------|---------------------|
| TC | 93                | 18.6 (77)         | 78                  |

TABLE 1: Summary statistics of pinch-point (PP) and time-consistent (TC) reach-avoid solvers from 100 trajectories shown in Fig. 1, starting from the same 100 randomly sampled initial states. Both methods were terminated as soon as the condition \( J_0 \leq 0 \) was met (target reached without previous constraint violations; if allowed to iterate further, TC finds 84 (instead of 78) trajectories that remain safe for all time.

the number of players and the state dimension, and is linear in the time horizon [16, 18]. This polynomial scaling is equivalent to that of standard second-order solution methods such as Newton’s method or ILQR in the single-player setting, making the family of ILQ game solution methods amenable to real-time implementation. Indeed, experimental evidence in Table 1 suggests that our time-consistent method further improves the iteration complexity of [Alg. 1] while also improving safety performance.

V. APPLICATION TO SAFE AUTONOMOUS DRIVING

We showcase our approach in two case studies motivated by autonomous driving. The code is available online [5].

A. Two-Player Defensive Driving

Reach-avoid problems are widely relevant to a variety of domains. To demonstrate the theoretical and algorithmic contributions of Sec. IV, we first showcase them in a defensive driving scenario based upon that of [22], illustrated in Fig. 2. Here, we see an “ego” vehicle passing an oncoming car on a two-lane street. The ego vehicle wishes to reach the end of its lane while avoiding collision with the oncoming car. Following [22], we encode a “defensive driving” mentality for the ego vehicle in a noncooperative dynamic game where, for some part of the horizon \( t \in \{0, \ldots, t_{\text{react}} \} \) the oncoming vehicle wishes to collide, and thereafter its driver “reacts” and seeks to avoid collision. We make several key changes to the formulation of [22]. First, our formulation is a reach-avoid game, rather than a time-additive game. Second, we allow the ego player to choose both vehicles’ control inputs after \( t_{\text{react}} \). This allows the oncoming vehicle to behave truly adversarially until \( t_{\text{react}} \), and rewards it for causing an inevitable collision at any point in the horizon, even after \( t_{\text{react}} \).

We formulate the game as follows. Each vehicle is modeled as a kinematic bicycle [21] which presumes that tires do not slip. Here, the state vector \( x := (p_i^x, p_i^y, \theta_i, v_i^x, v_i^y) \) is comprised of center rear-axle position, heading, front wheel angle, and speed. The overall game state is then \( x = (x_{\text{ego}}, x_{\text{onc}}) \) and each vehicle’s states evolve as

\[
x_{i+1} - x_i \quad \Delta t = \frac{v_i \cos \theta_i, v_i \sin \theta_i, \frac{v_i \tan \phi_i}{L_i}, \omega_i, a_i}{},
\]

where \( \omega_i \) and \( a_i \) are the front wheel rate and longitudinal acceleration of car \( i \), \( L_i \) is the wheelbase (4 m here) and \( \Delta t \) is the discrete time step (0.1 s). The ego’s control input is then

\[
u_{i}^{\text{ego}} = \begin{cases} \omega_i^{\text{ego}}, a_i^{\text{ego}}, & 0 \leq t < t_{\text{react}} \\ (\omega_i^{\text{ego, onc}}, a_i^{\text{ego, onc}}), & t_{\text{react}} < t \leq T \end{cases}
\]

[github.com/SafeRoboticsLab/Reach-Avoid-Games]
and the oncoming vehicle’s input affects only the first $t_{\text{react}}$ steps, i.e. $\nu_t^{\text{onc}} = (\omega_t^{\text{onc}}, a_t^{\text{onc}})$ for only $0 \leq t < t_{\text{react}}$.

The ego’s objective is of the form $\{1\}$. The failure set $\mathcal{F}_t^{\text{ego}}$ contains all joint states in which (a) the ego vehicle and the oncoming vehicle are in collision or violating road boundaries, (b) the ego vehicle’s steering angle exceeds a fixed range (here, $\pm 30^\circ$) and (c) the oncoming vehicle violates its own analogous constraints for $t > t_{\text{react}}$. The failure margin function $g_t^{\text{ego}}(\cdot)$ encodes signed distance to the above-defined set with the positive-inside convention. Similarly, the target function $\ell_t^{\text{ego}}(\cdot)$ encodes the ego vehicle’s signed distance to a goal region in the plane, illustrated in Fig. 2 in this case with the negative-inside convention.

The oncoming car seeks to collide with the ego vehicle or force it to drive off the road before the oncoming vehicle leaves the road. Its target margin function $\ell_t^{\text{onc}}(\cdot)$ encodes the negative of the maximum between (a) the signed distance between the vehicles, and (b) the signed distance between the ego car and the road boundaries (with safety corresponding to positive values). The failure margin function $g_t^{\text{onc}}(\cdot)$ encodes signed distance to the road boundaries.

We display the results of Alg. 1 with the time-consistent reach-avoid LQ subroutine from Alg. 3 in Fig. 2 for several different values of $t_{\text{react}}$. For low values of $t_{\text{react}}$, the ego vehicle has ample time to avoid the adversarial oncoming vehicle since adversarial control is only active over a short time interval. At larger values of $t_{\text{react}}$, however, the ego vehicle cannot reach the target set without avoiding the failure set, and therefore fails to satisfy the reach-avoid condition of $\{1\}$. By varying $t_{\text{react}}$, we are thus able to generate a range of increasingly strong defensive maneuvers.

**B. Three-Player Intersection**

We extend the two-player driving scenario above by modeling one pedestrian and two cars who wish to cross a T-intersection modeled on that of $\{1\}$, $\{16\}$. As illustrated in Fig. 3, one car wishes to drive straight through the intersection, while another wishes to make a left turn. Both are modeled as in $\{7\}$. Meanwhile, a pedestrian with bounded instantaneous velocity control $(v'_x, v'_y)$ wishes to cross the intersection at a crosswalk. Each agent has a reach-avoid objective of the form $\{3\}$, where the failure set contains collisions and, in the case of vehicles, leaving the allowed lanes, and the target set contains states with position greater than a desired distance ahead. Our proposed time-consistent algorithm finds safe trajectories for all agents.

**VI. Conclusion**

We study local algorithms for finding solutions to reach-avoid problems. Our three core contributions are: (a) an efficient algorithm for reach-avoid problems which extends the reachability-only case of $\{1\}$, (b) a time-consistent variation upon this algorithm whose output is robust to suboptimal or unanticipated behavior, and (c) derivations of our methods in the multi-player game-theoretic setting. In particular, we prove both the time-inconsistency of the algorithm based upon $\{1\}$ and the (local) time consistency of our core algorithm. We also demonstrate our approach in a simulated robust motion planning example which models a range of defensive driving behavior.

While we are motivated by the clear applications of reach-avoid problems to safety assurance in motion planning for autonomous vehicles, we also believe it is highly relevant to a broader range of robust control practitioners working in robotic grasping, surgical robotics, aerial collision avoidance, etc. Future work will be investigate these applications and seek to validate the reliability and computational efficiency of our work in these varied physical systems.

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