DEFORMATION OF A PROJECTION IN THE MULTIPLIER ALGEBRA AND PROJECTION LIFTING FROM THE CORONA ALGEBRA

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Abstract. Let \( X \) be a unit interval or a unit circle and let \( B \) be a \( \sigma_p \)-unital, purely infinite, simple \( C^* \)-algebra such that its multiplier algebra \( M(B) \) has real rank zero. Then we determine necessary and sufficient conditions for a projection in the corona algebra of \( C(X) \otimes B \) to be liftable to a projection in the multiplier algebra. This generalizes a result proved by L. Brown and the author \([4]\). The main technical tools are divided into two parts. The first part is borrowed from the author’s result, \([13, \text{Theorem 3.3}]\). The second part is a proposition showing that we can produce a sub-projection, with an arbitrary rank which is prescribed as K-theoretical data, of a projection or a co-projection in the multiplier algebra of \( C(X) \otimes B \) under a suitable “infinite rank and co-rank” condition.

1. Introduction

Let \( X \) be a (finite dimensional) locally compact Hausdorff space and \( B \) a \( C^* \)-algebra. We are concerned with the projection lifting problem from the corona algebra to the multiplier algebra of a \( C^* \)-algebra of the form \( C(X) \otimes B \). An earlier result in this direction is that of W. Calkin \([6]\), who showed that a projection in the quotient algebra \( B(H)/K \) is liftable to a projection in \( B(H) \) where \( H \) is a separable infinite dimensional Hilbert space and \( K \) is the \( C^* \)-algebra of compact operators on \( H \); in our setting this corresponds to the case that \( X \) is a one point set and \( B \) is the \( C^* \)-algebra of compact operators. A generalization of this result was given by L. Brown and the author as follows.

Theorem 1.1. Let \( X \) be \([0,1]\), \((\infty,\infty)\), \([0,\infty)\), or a unit circle. A projection \( f \) in the corona algebra of \( C(X) \otimes K \), represented by finite
umber of projection valued functions \((f_0, \cdots, f_n)\) under a suitable partition \(\{x_1, \ldots, x_n\}\) of the interior of \(X\), is liftable to a projection in the multiplier algebra if and only if there exist \(l_0, \cdots, l_n\) satisfying the following conditions; suppose \(k_i\)'s are the essential codimensions of \(f_i(x_i)\) and \(f_{i-1}(x_i)\) for \(1 \leq i \leq n\).

1. \(l_i - l_{i-1} = -k_i\) for \(i > 0\) and \(l_0 - l_n = -k_0\) in the circle case, if for some \(x\) in \(X_i\), \(f_i(x)\) has finite rank, then
   \[
   l_i \geq -\text{rank}(f_i(x)),
   \]
2. if for some \(x\) in \(X_i\), \(1 - f_i(x)\) has finite rank, then
   \[
   l_i \leq \text{rank}(1 - f_i(x)),
   \]
3. if either end point of \(X_i\) is infinite, then
   \[
   l_i = 0.
   \]

We note that our result was obtained from the following interesting proposition on continuous fields of Hilbert spaces. This result says, roughly speaking, that a continuous field of Hilbert spaces such that each fiber’s rank is greater and equal to \(n\) has a trivial subfield of rank \(m\) for any \(m \leq n\). (See also [9, Proposition 3.2].)

**Proposition 1.2.** ([4, Corollary A.5]) If \(X\) is a separable metric space such that whose covering dimension is less than or equal to 1 and \(H\) is a continuous field of Hilbert spaces over \(X\) such that \(\dim(H_x) \geq n\) for every \(x \in X\), then \(H\) has a trivial subfield of rank \(n\). Equivalently, if \(p\) is a strongly continuous projection valued function on \(X\) such that \(\text{rank}(p(x)) \geq n\) for every \(x \in X\), then there is a norm continuous projection valued function \(q\) such that \(q \leq p\) and \(\text{rank}(q(x)) = n\) for every \(x \in X\).

We denote by \(M(B)\) the multiplier algebra of a stable \(C^*\)-algebra \(B\), and consider the projection valued map \(p : X \to M(B)\), which is continuous with respect to the strict topology. Note that the associated fiber \(p(x)H_B\) is a Hilbert submodule so that we have no proper notion of rank. But at least we can distinguish “finiteness” and “infiniteness” of a Hilbert (sub)module using finiteness and infiniteness of the corresponding fiberwise projection in \(M(B)\). We are going to show that:

**Lemma 1.3.** Let \(p\) be a continuous section from \(X\) to \(M(B)\) with respect to the strict topology on \(M(B)\) where \(B\) is a \(\sigma_p\)-unital, stable \(C^*\)-algebra of real rank zero such that \(M(B)\) contains a halving full projection. In addition, when we denote its image on \(x \in X\) by \(p_x\),
assume that $p_x$ is a full, properly infinite projection for each $x$. Then for any $\alpha \in K_0(B)$ there exists a norm continuous section $r$ from $X$ to $B$ such that $r_x \leq p_x$ such that $[r]_{K_0(B)} = \alpha$.

Then using the above lemma we can generalize Theorem 1.1 to the case $B$ is a $\sigma_p$-unital, purely infinite simple $C^*$-algebra provided that $K_0(B)$ is an ordered group.

**Theorem 1.4** (Theorem [3.11]). A projection $f$ in $C(C(X) \otimes B)$ represented by $(f_0, \ldots, f_n)$ under a suitable partition $\{x_1, \ldots, x_n\}$ of the interior of $X$ is liftable to a projection in $M(C(X) \otimes B)$ where $f_i(x)$'s are full and properly infinite projections for all $i$ and $x \in X$ if and only if there exist $l_0, \ldots, l_n$ in $K_0(B)$ satisfying the following conditions; suppose $k_i$'s are the (generalized) “essential codimensions” of $f_i(x_i)$ and $f_{i-1}(x_i)$ for $1 \leq i \leq n$.

(5) $l_i - l_{i-1} = -k_i$ for $i > 0$ and $l_0 - l_n = -k_0$ in the circle case,

if for some $x$ in $X_i$, $f_i(x)$ belongs to $B$, then

(6) $l_i \geq -[f_i(x)]_0$,

if for some $x$ in $X_i$, $1 - f_i(x)$ belongs to $B$, then

(7) $l_i \leq [1 - f_i(x)]_0$,

if either end point of $X_i$ is infinite, then

(8) $l_i = 0$.

In addition, we are going to show that two projections in the corona algebra are equivalent under suitable conditions on associated generalized essential codimensions.

2. THE ESSENTIAL CODIMENSION

As we observe in Theorem 1.4, the technical tool other than Lemma 1.3 is a K-theoretic notion which is called the generalized essential codimension. In fact, this generalizes the notion of classical essential codimension of Brown, Douglas, Fillmore. In Section 2 we give a careful treatment of this notion using rudiments of Kasparov’s KK-theory although it appeared in [13] without showing the connection to the classical essential codimension.

Let $E$ be a (right) Hilbert $B$-module. We denote $\mathcal{L}(E, F)$ by the $C^*$-algebra of adjointable, bounded operators from $E$ to $F$. The ideal of ‘compact’ operators from $E$ to $F$ is denoted by $\mathcal{K}(E, F)$. When $E = F$, we write $\mathcal{L}(E)$ and $\mathcal{K}(E)$ instead of $\mathcal{L}(E, E)$ and $\mathcal{K}(E, E)$. Throughout the paper, $A$ is a separable $C^*$-algebra, and all Hilbert
modules are assumed to be countably generated over a separable $C^*$-algebra. We use the term representation for a $\ast$-homomorphism from $A$ to $\mathcal{L}(E)$. We let $H_B$ be the standard Hilbert module over $B$ which is $H \otimes B$ where $H$ is a separable infinite dimensional Hilbert space. We denote $M(B)$ by the multiplier algebra of $B$. It is well-known that $\mathcal{L}(H_B) = M(B \otimes K)$ and $\mathcal{K}(H_B) = B \otimes K$ where $K$ is the $C^*$-algebra of the compact operators on $H$ [11].

Let us recall the definition of Kasparov group $KK(A, B)$. We refer the reader to [12] for the general introduction of the subject. A KK-cycle is a triple $(\phi, \psi, u)$, where $\phi : A \to \mathcal{L}(E_i)$ are representations and $u \in \mathcal{L}(E_0, E_1)$ satisfies that

(i) $u\phi_0(a) - \phi_1(a)u \in \mathcal{K}(E_0, E_1),$
(ii) $\phi_0(a)(u^*u - 1) \in \mathcal{K}(E_1), \phi_1(a)(uu^* - 1) \in \mathcal{K}(E_1)$. The set of all $KK$-cycles will be denoted by $\mathbb{E}(A, B)$. A cycle is degenerate if

$$u\phi_0(a) - \phi_1(a)u = 0, \quad \phi_0(a)(u^*u - 1) = 0, \quad \phi_1(a)(uu^* - 1) = 0.$$ 

An operator homotopy through $KK$-cycles is a homotopy $(\phi_0, \phi_1, u_t)$, where the map $t \to u_t$ is norm continuous. The equivalence relation $\sim_{\text{oh}}$ is generated by operator homotopy and addition of degenerate cycles up to unitary equivalence. Then $KK(A, B)$ is defined as the quotient of $\mathbb{E}(A, B)$ by $\sim$. When we consider non-trivially graded $C^*$-algebras, we define a triple $(E, \phi, F)$, where $\phi : A \to \mathcal{L}(E)$ is a graded representation, and $F \in \mathcal{L}(E)$ is of odd degree such that $F\phi(a) - \phi(a)F$, $(F^2 - 1)\phi(a)$, and $(F - F^*)\phi(a)$ are all in $\mathcal{K}(E)$ and call it a Kasparov $(A, B)$-module. Other definitions like degenerate cycle and operator homotopy are defined in similar ways.

In the above, we introduced the Fredholm picture of $KK$-group. There is an alternative way to describe the element of $KK$-group. The Cuntz picture is described by a pair of representations $\phi, \psi : A \to \mathcal{L}(H_B) = M(B \otimes K)$ such that $\phi(a) - \psi(a) \in \mathcal{K}(H_B) = B \otimes K$. Such a pair is called a Cuntz pair. They form a set denoted by $\mathbb{E}_h(A, B)$. A homotopy of Cuntz pairs consists of a Cuntz pair $(\Phi, \Psi) : A \to M(\mathcal{C}([0, 1]) \otimes (B \otimes K))$. The quotient of $\mathbb{E}_h(A, B)$ by homotopy equivalence is a group $KK_h(A, B)$ which is isomorphic to $KK(A, B)$ via the mapping sending $[\phi, \psi]$ to $[\phi, \psi, 1]$.

**Definition 2.1** (the generalized essential codimension). Given two projections $p, q \in M(B \otimes K)$ such that $p - q \in B \otimes K$, we consider representations $\phi, \psi$ from $\mathbb{C}$ to $M(B \otimes K)$ such that $\phi(1) = p, \psi(1) = q$. Then $(\phi, \psi)$ is a Cuntz pair so that we define $[p : q]$ as the class
\([\phi, \psi] \in KK_h(\mathbb{C}, B) \cong K(B)\) and call the (generalized) essential codimension of \(p\) and \(q\).

**Remark 2.2.** We recall that BDF’s original definition of the essential codimension of \(p\) and \(q\) in \(B(H)\) is given by the Fredholm index of \(V^*W\) where \(V\) and \(W\) are isometries such that \(VV^* = q\) and \(W^*W = p\) \([3]\). This looks different with the above definition. But in the case \(B = K\) or \(C\) \([p : q]\) is mapped to \([\phi, \psi, 1]\) in \(KK(\mathbb{C}, \mathbb{C})\). Then the map from \(KK(\mathbb{C}, \mathbb{C})\) to \(\mathbb{Z}\) sends \([\phi, \psi, 1]\) to the Fredholm index of \(qp\) viewing \(qp\) as the operator from \(pH\) to \(qH\). Thus it equals to the Fredholm index of \(V^*W\).

The following demonstrate that the generalized essential codimension behaves like the original essential codimension (see \([1, \text{Section 1}]\)).

**Lemma 2.3.** \([:]\) has the following properties.

1. \([p_1 : p_2] = [p_1]_0 - [p_2]_0\) if either \(p_1\) or \(p_2\) belongs to \(B\), where \([p_i]_0\) is the \(K_0\)-class of a projection \(p_i\).
2. \([p_1 : p_2] = -[p_2 : p_1]\).
3. \([p_1 : p_3] = [p_1 : p_2] + [p_2 : p_3]\), when sensible.
4. \([p_1 + q_1 : p_2 + q_2] = [p_1 : p_2] + [q_1 : q_2]\), when sensible.

**Proof.** Let \(\phi_i\)'s and \(\psi_i\)'s be elements in \(\text{Hom}(\mathbb{C}, M(B))\) such that \(\phi_i(1) = p_i\) and \(\psi_i(1) = q_i\) respectively. Without loss of generality we let \(B\) be a stable \(C^*\)-algebra and \(\Theta_B : M_2(B) \to B\) be an inner isomorphism. Then

1. In general, the isomorphism

   
   \[KK_h(A, B) \to KK(A, B)\]

   maps \([\phi_1, \phi_2]\) to a cycle \([\phi_1, \phi_2, 1]\). Note that, when \(A = \mathbb{C}\),

   \[
   (\phi_1, \phi_2, 1) = \begin{pmatrix}
   H_B \oplus H_B, \\
   \phi_1 \\
   \phi_2
   \end{pmatrix}, \begin{pmatrix}
   0 & 1 \\
   1 & 0
   \end{pmatrix}
   \]

   is a compact perturbation of

   \[
   \begin{pmatrix}
   H_B \oplus H_B, \\
   \begin{pmatrix}
   p_1 \\
   p_2
   \end{pmatrix}, \\
   \begin{pmatrix}
   0 & p_1p_2 \\
   p_2p_1 & 0
   \end{pmatrix}
   \end{pmatrix}.
   \]

   The latter is decomposed to \(((1-p_1)(H_B) \oplus (1-p_2)(H_B), 0, 0) \oplus (p_1(H_B) \oplus p_2(H_B), \begin{pmatrix}
   p_1 \\
   p_2
   \end{pmatrix}, \begin{pmatrix}
   0 & p_1p_2 \\
   p_2p_1 & 0
   \end{pmatrix})\)

   so that \((\phi_1, \phi_2, 1)\) is represented as

   \[
   \begin{pmatrix}
   p_1(H_B) \oplus p_2(H_B), \\
   \begin{pmatrix}
   p_1 \\
   p_2
   \end{pmatrix}, \\
   \begin{pmatrix}
   0 & p_1p_2 \\
   p_2p_1 & 0
   \end{pmatrix}
   \end{pmatrix}.
   \]
Now we view $p_2 p_1$ as an essential unitary operator from $p_1(H_B)$ to $p_2(H_B)$, thus its generalized Fredholm index is given by $[p_1]_0 - [p_2]_0$ (see [18]). In fact, it is realized as the following diagram

\[
p_1(H_B) \oplus (1 - p_1)(H_B) \xrightarrow{I \oplus U^*} p_2(H_B) \oplus (1 - p_2)(H_B)
\]

\[
p_1(H_B) \oplus H_B \xrightarrow{p_2 p_1 \oplus I} p_2(H_B) \oplus H_B
\]

where $U, W \in M(B) = \mathcal{L}(H_B)$ are isometries such that $UU^* = 1 - p_1, WW^* = 1 - p_2$.

(2) \[
\begin{pmatrix}
\cos(\frac{\pi}{2}t) & \sin(\frac{\pi}{2}t) \\
-\sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t)
\end{pmatrix}
\]
defines a homotopy from \[
\begin{pmatrix}
\phi_2 & 0 \\
0 & \phi_1
\end{pmatrix}
\]
to \[
\begin{pmatrix}
\phi_1 & 0 \\
0 & \phi_2
\end{pmatrix}
\]. Thus

\[
[\phi_1, \phi_2] + [\phi_2, \phi_1] = \Theta_B \circ \begin{pmatrix}
\phi_1 & 0 \\
0 & \phi_2
\end{pmatrix}, \Theta_B \circ \begin{pmatrix}
\phi_2 & 0 \\
0 & \phi_1
\end{pmatrix}
\]

\[
= \Theta_B \circ \begin{pmatrix}
\phi_1 & 0 \\
0 & \phi_2
\end{pmatrix}, \Theta_B \circ \begin{pmatrix}
\phi_1 & 0 \\
0 & \phi_2
\end{pmatrix}
\]

\[
= 0.
\]

(3) Similarly,

\[
[\phi_1, \phi_2] + [\phi_2, \phi_3] = \Theta_B \circ \begin{pmatrix}
\phi_1 & 0 \\
0 & \phi_2
\end{pmatrix}, \Theta_B \circ \begin{pmatrix}
\phi_2 & 0 \\
0 & \phi_3
\end{pmatrix}
\]

\[
= \Theta_B \circ \begin{pmatrix}
\phi_1 & 0 \\
0 & \phi_2
\end{pmatrix}, \Theta_B \circ \begin{pmatrix}
\phi_2 & 0 \\
0 & \phi_3
\end{pmatrix}
\]

\[
= [\phi_2, \phi_2] + [\phi_1, \phi_3]
\]

\[
= [\phi_1, \phi_3] .
\]

(4) Since $\phi_i$ and $\psi_i$ are orthogonal, i.e. $\phi_i \psi_i = 0$, $\phi_i + \psi_i$ is a homomorphism. Note that \[
\begin{pmatrix}
\phi_i & 0 \\
0 & \psi_i
\end{pmatrix}
\]
is homotopic to \[
\begin{pmatrix}
\phi_i + \psi_i & 0 \\
0 & 0
\end{pmatrix}
\]
up to stability.

\[
[\phi_1, \phi_2] + [\psi_1, \psi_2] = \Theta_B \circ \begin{pmatrix}
\phi_1 & 0 \\
0 & \psi_1
\end{pmatrix}, \Theta_B \circ \begin{pmatrix}
\phi_2 & 0 \\
0 & \psi_2
\end{pmatrix}
\]

\[
= \Theta_B \circ \begin{pmatrix}
\phi_1 + \psi_1 & 0 \\
0 & 0
\end{pmatrix}, \Theta_B \circ \begin{pmatrix}
\phi_2 + \psi_2 & 0 \\
0 & 0
\end{pmatrix}
\]

\[
= [\phi_1 + \psi_1, \phi_2 + \psi_2].
\]

□
Lemma 2.4. Let $p$ and $q$ be projections in $M(B)$ such that $p - q \in B$. If there is a unitary $U \in 1 + B$ such that $UpU^* = q$, then $[p : q] = 0$. In particular, if $\|p - q\| < 1$, then $[p : q] = 0$.

Proof. Since $(\phi_1, \phi_2, 1)$ is unitarily equivalent to $(\phi_1, \phi_1, 1)$ which is a degenerate element, it follows that $[p : q] = 0$ in $\text{KK}(\mathbb{C}, B)$.

If $\|p - q\| < 1$, we can take $a = (1 - q)(1 - p) + qp \in 1 + B$. Since $aa^* = a^*a = 1 - (p - q)^2 \in 1 + B$,

$$\|a^*a - 1\| = \|p - q\|^2 < 1, \quad aa^* - 1 = \|p - q\|^2 < 1.$$ Moreover, it follows that

$$ap = qp = qa.$$ Hence, $a$ is invertible element and $U = a(a^*a)^{-\frac{1}{2}} \in 1 + B$ is a unitary such that $UpU^* = q$. \hfill \Box

Proposition 2.5. Let $p, q$ be projections in $M(B)$ such that $p - q \in B$. Suppose that $(K_0(B), K_0(B)^+)$ is an ordered group where the positive cone $K_0(B)^+ = \{[p]_0 \mid p \in P_\infty(B)\}$.

1. If $q \in B$, then $[p : q] \geq [q]_0$.
2. If $1 - q \in \tilde{B}$, then $[p : q] \leq [1 - q]_0$.

Proof. (1) Note that $[p : q] = [p]_0 - [q]_0$ by Lemma 2.3-(1). Hence $[p : q] \geq -[q]_0$.

(2) Notice that $[1 - p]_0 - [1 - q]_0 \in K_0(B)$. By Lemma 2.3-(4) $[p : q] = -[1 - p : 1 - q] = [1 - q]_0 - [1 - p]_0 \leq [1 - q]_0$. \hfill \Box

Lemma 2.6. Suppose projections $p_t, q_t \in M(B)$ are defined for each $t$ in a set of $\mathbb{R}$. Then $[p_t : q_t]$ is constant if $t \rightarrow p_t - q_t$ is norm continuous in $B$.

Proof. Straightforward. \hfill \Box

Remark 2.7. Originally, the proof of this lemma for $B = K$ and $M(B) = B(H)$ was nontrivial (see [4, Corollary 2.6]). However, it is now built in the definition of $\text{KK}_h$.

The next theorem, which we do not claim its originality, was proved in [13] exhibits the most important property of the essential codimension (see [4, Theorem 2.7]).

Theorem 2.8. Let $B$ be a non-unital ($\sigma$-unital) purely infinite simple $C^*$-algebra such that $M(B \otimes K)$ has real rank zero. Suppose two projections $p$ and $q$ in $M(B \otimes K) = L(H_B)$ such that $p - q \in B \otimes K$ and
neither of them is in \( B \otimes K \). If \([p : q] \in K_0(B)\) vanishes, then there is a unitary \( u \) in \( 1 + B \otimes K \) such that \( upu^* = q \).

**Remark 2.9.**

(i) As it was pointed out in [8] and [13], the crucial point of Theorem 2.8 is that the implementing unitary \( u \) has the form “identity + compact”. This requirement to obtain the reasonable generalization of BDF’s original statement has been very useful in KK-theory (see [8] and [13]).

(ii) Without restrictions on \( C^*\)-algebra \( B \), but on \( p \in M(B \otimes K) \) being halving projection, any compact perturbation of \( p \) is of the form \( upu^* \) where a unitary \( u \) is in \( 1 + B \otimes K \) [25].

3. Deformation of a projection and its applications

Let \( B \) be a simple stable \( C^* \)-algebra such that the multiplier algebra \( M(B) \) has real rank zero. Let \( X \) be \([0, 1], [0, \infty), (-\infty, \infty) \) or \( \mathbb{T} = [0, 1]/\{0, 1\} \). When \( X \) is compact, let \( I = C(X) \otimes B \) which is the \( C^* \)-algebra of (norm continuous) functions from \( X \) to \( B \). When \( X \) is not compact, let \( I = C_0(X) \otimes B \) which is the \( C^* \)-algebra of continuous functions from \( X \) to \( B \) vanishing at infinity. Then \( M(I) \) is given by \( C_0(X, M(B)_s) \), which is the space of bounded functions from \( X \) to \( M(B) \), where \( M(B) \) is given the strict topology. Let \( C(I) = M(I)/I \) be the corona algebra of \( I \) and also let \( \pi : M(I) \to C(I) \) be the natural quotient map. Then an element \( f \) of the corona algebra can be represented as follows: Consider a finite partition of \( X \), or \( X \setminus \{0, 1\} \) when \( X = \mathbb{T} \) given by partition points \( x_1 < x_2 < \cdots < x_n \) all of which are in the interior of \( X \) and divide \( X \) into \( n + 1 \) (closed) subintervals \( X_0, X_1, \ldots, X_n \). We can take \( f_i \in C_0(X_i, M(B)_s) \) such that \( f_i(x_i) - f_{i-1}(x_i) \in B \) for \( i = 1, 2, \ldots, n \) and \( f_0(x_0) - f_n(x_0) \in B \) where \( x_0 = 0 = 1 \) if \( X \) is \( \mathbb{T} \). The following lemma and the statement after it were shown in [13].

**Lemma 3.1.** The coset in \( C(I) \) represented by \((f_0, \cdots, f_n)\) consists of functions \( f \) in \( M(I) \) such that \( f - f_i \in C(X_i) \otimes B \) for every \( i \) and \( f - f_i \) vanishes (in norm) at any infinite end point of \( X_i \).

Similarly \((f_0, \cdots, f_n)\) and \((g_0, \cdots, g_n)\) define the same element of \( C(I) \) if and only if \( f_i - g_i \in C(X_i) \otimes B \) for \( i = 0, \cdots, n \) if \( X \) is compact. \((f_0, \cdots, f_n)\) and \((g_0, \cdots, g_n)\) define the same element of \( C(I) \) if and only if \( f_i - g_i \in C(X_i) \otimes B \) for \( i = 0, \cdots, n - 1 \), \( f_n - g_n \in C_0([x_n, \infty)) \otimes B \) if \( X \) is \([0, \infty)\). \((f_0, \cdots, f_n)\) and \((g_0, \cdots, g_n)\) define the same element of \( C(I) \) if and only if \( f_i - g_i \in C(X_i) \otimes B \) for \( i = 1, \cdots, n - 1 \), \( f_n - g_n \in C_0([x_n, \infty)) \otimes B \), \( f_0 - g_0 \in C_0((-\infty, x_1]) \otimes B \) if \( X = (-\infty, \infty) \).
A virtue of the above unnecessarily complicated description of an element in \( \mathcal{C}(I) \) is the following theorem which says a projection in \( \mathcal{C}(I) \) is locally liftable.

**Theorem 3.2.** \([13, \text{Theorem } 3.2]\) Let \( I \) be \( \mathcal{C}(X) \otimes B \) or \( \mathcal{C}_0(X) \otimes B \) where \( B \) is a stable \( \mathcal{C}^* \)-algebra such that \( M(B) \) has real rank zero. Then a projection \( f \) in \( M(I)/I \) can be represented by \( (f_0, f_1, \ldots, f_n) \) as above where \( f_i \) is a projection valued function in \( \mathcal{C}(X_i) \otimes M(B) \), for each \( i \).

**Remark 3.3.** The theorem says that any projection \( f \) in the corona algebra of \( \mathcal{C}(X) \otimes B \) for some \( \mathcal{C}^* \)-algebras \( B \) can be viewed as a “locally trivial fiber bundle” with the Hilbert modules as fibers in the sense of Dixmier and Duady \([10]\).

The following theorem will be used as one of our technical tools.

**Theorem 3.4.** \([13, \text{Theorem } 3.3]\) Let \( I \) be \( \mathcal{C}(X) \otimes B \) where \( B \) is a \( \sigma \)-unital, non-unital, purely infinite simple \( \mathcal{C}^* \)-algebra such that \( M(B) \) has real rank zero or \( K_1(B) = 0 \) (See \([23]\)). Let a projection \( f \) in \( \mathcal{M}(I)/I \) be represented by \( (f_1, f_2, \ldots, f_n) \), where \( f_i \) is a projection valued function in \( \mathcal{C}(X_i) \otimes M(B), \) for each \( i \), as in Theorem 3.2. If \( k_i = [f_i(x_i) : f_{i-1}(x_i)] = 0 \) for all \( i \), then the projection \( f \) in \( \mathcal{M}(I)/I \) lifts.

Now we want to observe the necessary conditions for a projection in \( \mathcal{C}(I) \) to lift. If \( f \) is liftable to a projection \( g \) in \( \mathcal{C}(I) \), we can use the same partition of \( X \) so that \((g_0, \ldots, g_n)\) and \((f_0, \ldots, f_n)\) define the same element \( f \) where \( g_i \) is the restriction of \( g \) on \( X_i \). Then, for each \( i \), \([g_i(x) : f_i(x)]\) is defined for all \( x \). By Lemma 2.6 this function must be constant on \( X_i \) since \( g_i - f_i \) is norm continuous. So we can let \( l_i = [g_i(x) : f_i(x)] \).

Since \( g_i(x_i) = g_{i-1}(x_i) \), we have \([g_i(x_i) : f_i(x_i)] + [f_i(x_i) : f_{i-1}(x_i)] = [g_{i-1}(x_i) : f_{i-1}(x_i)]\) by Lemma 2.2-(3). In other words,

\[
 l_i - l_{i-1} = -k_i \quad \text{for } \quad i > 0 \quad \text{and} \quad l_0 - l_n = -k_0 \quad \text{in the circle case}.
\]

Moreover, if \((K_0(B), K_0(B)^+)\) is an ordered group with the positive cone \( K_0(B)^+ = \{[p]\mid p \in \mathcal{P}_\infty(B)\} \), we apply Proposition 2.5 and Lemma 2.4 to projections \( g_i(x) \) and \( f_i(x) \), and we have the following restrictions on \( l_i \).

(i) If for some \( x \) in \( X_i \), \( f_i(x) \) belongs to \( B \), then
\[
 l_i \geq [f_i(x)]_0,
\]

(ii) If for some \( x \) in \( X_i \), \( 1 - f_i(x) \) belongs to \( B \), then
\[
 l_i \leq [1 - f_i(x)]_0,
\]

where \([p]_0 = \{x \mid x \in p\} \cup \{x \mid x \in p^*\} \), \( 0 \leq \cdot \leq 1 \), and \( 0 = 1 \).
(iii) If either end point of $X_i$ is infinite, then

$$l_i = 0.$$  

Are these necessary conditions sufficient? Our strategy of showing the converse is to perturb the $f_i$'s so that $f_i(x_i) : f_{i-1}(x_i)$ vanishes for all $i$. To do this we need to deform a field of orthogonal projections by embedding a field a projections with arbitrary “rank” information which is given by a K-theoretical term. This implies that a field of Hilbert modules defined by a field of projections should be ambient so that any field of submodules with arbitrary rank can be embedded. Unlike a Hilbert space there is no proper algebraic notion of rank of a multiplier projection whose image can be regarded as a Hilbert submodule of the standard Hilbert module $H_B$. However, at least we want to distinguish submodules in terms of finiteness and infiniteness of corresponding projections. Recall that a projection $p$ in a unital C*-algebra $A$ is called a halving projection if both $1 - p$ and $p$ are Murray-von Neumann equivalent to the unit in $A$ and a projection $p$ in $A$ is called properly infinite if there are mutually orthogonal projections $e, f$ in $A$ such that $e \leq p, f \leq p$, and $e \sim f \sim p$. Then it is easy to check that a projection in the multiplier algebra of a stable C*-algebra is Murray-von Neumann equivalent to 1 if and only if it is full and properly infinite. We denote by $\mathfrak{P}$ the set of all full and properly infinite projections in $M(B)$. We assume $M(B)$ has a halving projection and fix the halving projection $H$.

For the moment we allow $X$ to be any finite dimensional topological space. Let $p$ be a section which is a continuous map from $X$ to $M(B)$ with respect to the strict topology on $M(B)$. We denote its image on $x \in X$ by $p_x$. By the observation so far, it is natural to put the condition that $p_x$ is full and properly infinite for every $x \in X$ to have an analogue of an infinite dimensional Hilbert bundle or a continuous field of separable infinite dimensional Hilbert spaces.

We need the Michael selection theorem as the important step to get the main result Lemma 3.6.

**Theorem 3.5.** (See [17]) Let $X$ be a paracompact finite dimensional topological space. Let $Y$ be a complete metric space and $S$ be a set-valued lower semicontinuous map from $X$ to closed subsets of $Y$, i.e. for each open subset $U$ of $Y$ the set $\{x \in X \mid S(x) \cap U \neq \emptyset\}$ is open. Let $R$ be the range $\{S(x) : x \in X\}$ of $S$. Then, if for some $m > \dim X$ we have

(i) $S(x)$ is $m$-connected;


(ii) each \( R \in \mathcal{R} \) has the property that every point \( x \in R \) has an arbitrarily small neighborhood \( V(x) \) such that \( \pi_m(R' \cap V(x)) = \{e\} \) for every \( R' \in \mathcal{R} \);

then there exists a continuous map \( s \) from \( X \) to \( Y \) such that \( s(x) \) is in \( S(x) \) for all \( x \in X \).

Here, \( \pi_m \) is the \( m \)th homotopy group, defined by \( \pi_m(Z) = [S^m, Z] \).

The following lemma was actually proved in [16] under the assumption that \( p_x \) is halving for every \( x \), but this assumption can be weakened keeping the same proof [19].

**Lemma 3.6.** Let \( B \) be a \( \sigma_p \)-unital, stable \( C^* \)-algebra and \( p \) be a section which is a map from a compact, finite dimensional topological space \( X \) to \( M(B) \) with respect to the strict topology on \( M(B) \). Suppose \( p_x \) is a full, properly infinite projection in \( M(B) \) for each \( x \). Then there exists a continuous map \( u : X \to \mathfrak{M} \) such that \( u_x^*u_x = p_x, u_xu_x^* = H \in M(B) \), where \( \mathfrak{M} \) is the set of all partial isometries with the initial projection in \( \mathfrak{P} \) and range projection \( H \).

**Proof.** We define a map \( F_H : \mathfrak{M} \to \mathfrak{P} \) by \( F_H(v) = v^*Hv \). Then it can be shown that \( F_H^{-1}(p) \) is closed and contractible, and \( F_H \) is an open mapping as in [16]. Let \( Y \) be norm closed ball in \( M(B) \) of elements with the norm less than equal to 2. If we define a set-valued map \( S : X \to 2^Y \) by \( S(x) = F_H^{-1}(p_x) \), it can be shown that this map is lower semi-continuous using the openness of the map \( F_H \) (see also [16]). Then we apply the Michael selection theorem to the map \( S \) so that there is a cross section \( s : X \to Y \) such that \( s(x) \in S(x) \). Then we define \( u_x = Hs(x) \). Since \( F_H(s(x)) = p_x \), it follows that \( u_xu_x^* = H \) and \( u_x^*u_x = p_x \). \( \square \)

Recall that a closed submodule \( E \) of the Hilbert module \( F \) over \( B \) is complementable if and only if there is a submodule \( G \) orthogonal to \( E \) such that \( E \oplus G = F \). The Kasparov stabilization theorem says that a countably generated closed submodule of \( H_B \) is complementable whence it is the image of a projection in \( \mathcal{L}(H_B) \). Let \( \mathfrak{F} = ((F_x)_{x \in X}, \Gamma) \) be a continuous field of Hilbert modules. A continuous field of Hilbert modules \( ((E_x)_{x \in X}, \Gamma') \) is said to be complementable to \( \mathfrak{F} \) when \( E_x \) is a complementable submodule of \( F_x \) for each \( x \in X \). Then the following is a geometrical interpretation of Lemma 3.6.

**Proposition 3.7.** A complementable subfield \( ((E_x)_{x \in X}, \Gamma') \) of the constant module \( ((H_B)_{x \in X}, \Gamma) \), where \( \Gamma \) consists of the (norm) continuous section from \( X \) to \( H_B \), is in one to one correspondence to a continuous projection-valued map \( p : X \to \mathcal{L}(H_B) \), where the latter
is equipped with the $*$-strong topology (In general, we say that $\{T_i\}$ in $\mathcal{L}(X)$ converges to $T$ $*$-strongly if and only if both $T_i(x) \to T(x)$ and $T_i^*(x) \to T^*(x)$ in $X$ for all $x \in X$).

Proof. Since $E_x$ is a complementable submodule of $H_B$, it is an image of a projection $p_x \in \mathcal{L}(H_B)$. This it defines a map $p : X \to \mathcal{L}(H_B)$. Note that in general when $H$ is a continuous field of Hilbert modules and $((E_x)_{x \in X}, \Gamma')$ is a complemented subfield of $\mathcal{H}$, $x \mapsto p_x(\gamma(x))$ is continuous if and only if $x \mapsto \|p_x(\gamma(x))\|$ is continuous for $\gamma \in \Gamma$. In addition, $\Gamma' = \{x \mapsto p_x(\gamma(x)) \mid \gamma \in \Gamma\}$. So if $\mathcal{H}$ is a trivial field, the map $x \mapsto p_x(\xi)$ for $\xi \in H_B$ is in $\Gamma'$, and thus continuous. This implies that the map $x \mapsto p_x$ is strongly continuous. Conversely, suppose that we are given a strongly continuous map $x \mapsto p_x \in \mathcal{L}(H_B)$. Let $E_x = p_x(H_B)$, and define a section $\gamma_x : x \mapsto p_x(\xi)$ for each $\xi \in H_B$. Let $\Lambda = \{\gamma_x \in \prod_{x \in X} E_x \mid \xi \in H_B\}$ and $\Gamma'' = \{\gamma \in \prod_x E_x \mid \gamma$ satisfies $(\ast)$\}.

$(\ast)$ For any $x \in X$ and $\varepsilon > 0$, there exists $\gamma' \in \Lambda$ such that $\|\gamma(x) - \gamma'(x)\| \leq \varepsilon$ in a neighborhood of $x$.

Then we can check that $((E_x)_{x \in X}, \Gamma'')$ is a complemented subfield of a trivial field.

Thus we have an analogue of well-known Dixmier’s triviality theorem on continuous fields of Hilbert spaces in the Hilbert module setting. We note that the strict topology on $M(B \otimes K) \simeq \mathcal{L}(H_B)$ coincides with the $*$-strong topology on bounded sets (see [20, Proposition C.7]).

Corollary 3.8. Let $B$ be a stable $C^*$-algebra and $X$ a finite dimensional compact Hausdorff space. Then a complementable subfield of Hilbert modules associated with a projection-valued map $p : X \to M(B)_s$ is isomorphic to a trivial field provided that each $p_x$ is a full, properly infinite projection in $M(B)$.

Proof. Lemma 3.6 says that $p \in M(C(X) \otimes B)$ is globally full, and properly infinite if $p_x$ is full and properly infinite for each $x$ in $X$. Thus $p$ is Murray-von Neumann equivalent to $1_{M(C(X) \otimes B)}$.

Lemma 3.9. Under the same hypothesis on $B$ and $X$ as in Lemma 3.8 let $p$ be a section which is a map from $X$ to $M(B)$ with respect to the strict topology on $M(B)$. Suppose that there exist a continuous map $u : X \to \mathfrak{M}$ such that $u_x^*u_x = p_x$, $u_xu_x^* = H \in M(B)$. In addition, given an $\alpha \in K_0(B)$ we suppose that there exist a projection $q \in H B H \subset B$
such that \([q] = \alpha \in K_0(B)\). Then there exists a norm continuous section \(r\) from \(X\) to \(B\) such that \(r_x \leq p_x\) such that \([r]_{K_0(B)} = \alpha\).

**Proof.** Since \(q \in B\), \(x \to qu_x \in B\) is norm continuous so that \(x \to r_x = (qu_x)^*qu_x = u_x^*qu_x\) is norm continuous. Note that \((qu_x)^*qu_x = qu_xu_x^*q = qHq = q\). Thus \([r]_{K_0(B)} = [r_x] = [q] = \alpha\). □

In summary, we state what we need as a final form.

**Lemma 3.10.** Let \(p\) be a section from \(X\) to \(M(B)\) with respect to the strict topology on \(M(B)\) where \(B\) is a \(\sigma_p\)-unital, stable \(C^*\)-algebra of real rank zero such that \(M(B)\) contains a halving full projection. In addition, assume that \(p_x\) is a full, properly infinite projection for each \(x\). Then for any \(\alpha \in K_0(B)\) there exists a norm continuous section \(r\) from \(X\) to \(B\) such that \(r_x \leq p_x\) such that \([r]_{K_0(B)} = \alpha\).

**Proof.** If we denote a halving (strictly) full projection by \(H\), \(HBH\) is a full hereditary subalgebra of \(B\) so that it is stably isomorphic to \(B\) by [2, Corollary 2.6]. Hence \(K_0(HBH) = K_0(B)\). Since \(B\) is a \(C^*\)-algebra of real rank zero, so is \(HBH\) by [5] Corollary 2.8. This it satisfies the strong \(K_0\)-surjectivity, i.e. there exists a projection \(q\) in \(HBH\) such that \([q]_0 = \alpha\). Then the conclusion follows from Lemma 3.9 and Lemma 3.10. □

Now we restrict ourselves to the case \(I = C(X) \otimes B\) where \(X\) is \([0, 1]\), \([0, \infty)\), \((-, \infty)\), or \([0, 1]/\{0, 1\}\), and \(B\) is a \(\sigma_p\)-unital, purely infinite simple \(C^*\)-algebra such that \(M(B)\) has real rank zero and has a full halving projection \(H\). From now on we assume that \(K_0(B)\) is an ordered abelian group and drop 0 in the expression of an element in \(K_0(B)\). Note that \(B\) is a stable \(C^*\)-algebra by Zhang’s dichotomy [23]. Also, it satisfies the strong \(K_0\)-surjectivity [15]. Thus we can apply Lemma 3.10 to (one dimensional) closed intervals \(X_i\)'s, which come from a partition of \(X\) associated with a local representation of a projection in the corona algebra of \(I\).

**Theorem 3.11.** A projection \(f\) in \(C(I)\) represented by \((f_0, \cdots, f_n)\) is liftable to a projection in \(M(I)\) where \(f_i(x)\)'s are halving projections for all \(i\) and \(x \in X\) if and only if there exist \(l_0, \cdots, l_n\) satisfying above conditions (9), (10), (11), (12).

**Proof.** Given \(l_i\)'s satisfying (9), (10), (11), (12), we will show there exist \(g_0, \cdots, g_n\) such that \([g_i(x_i) : g_{i-1}(x_i)] = 0\) for \(i > 0\) and \([g_0(x_0) : g_n(x_0)] = 0\) in the circle case.

First observe that if we have \(g_i\)'s such that \(l_i = [g_i(x_i) : f_i(x_i)]\), we have \([g_i(x_i) : g_{i-1}(x_i)] = 0\) by (9). Thus it is enough to show that there exist \(g_0, \cdots, g_n\) such that \([g_i(x_i) : f_i(x_i)] = l_i\).
\( l_i = 0 \): Take \( g_i = f_i \).

\( l_i > 0 \): By Lemma 3.10 the continuous field determined by \( 1 - f_i \) has a trivial subfield which is given by a projection valued function \( q \leq 1 - f_i \) such that \([q(x)]_0 = l_i\). So we take \( g_i = f_i + q \).

\( l_i < 0 \): Similarly, the continuous field determined by \( f_i \) has a trivial subfield which is given by a projection valued function \( q' \leq f_i \) such that \([q'(x)]_0 = -l_i\). So we take \( g_i = f_i - q' \).

Then the conclusion follows from Theorem 3.4.

Then we want to investigate some equivalence relations for projections using above arguments. As before, let \( p \) and \( q \) be two projections in \( \mathcal{C}(I) \).

**Lemma 3.12.** Let \((p_0, \ldots, p_n)\) and \((q_0, \ldots, q_n)\) be local liftings of \( p \) and \( q \) such that \( q_i(x) \) is a halving projection for each \( x \) in \( X_i \).

If \( \sum_{i=1}^{n}[p_i(x_i) : p_{i-1}(x_i)] = \sum_{i=1}^{n}[q_i(x_i) : q_{i-1}(x_i)] \), or \( \sum_{i=1}^{n}[p_i(x_i) : p_{i-1}(x_i)] + [p_0(x_0) : p_n(x_0)] = \sum_{i=1}^{n}[q_i(x_i) : q_{i-1}(x_i)] + [q_0(x_0) : q_n(x_0)] \) in the circle case, then we can find a perturbation \((q'_0, \ldots, q'_n)\) of \( q \) such that \([p_i(x_i) : p_{i-1}(x_i)] = [q'_i(x_i) : q'_{i-1}(x_i)] \) for \( i = 1, \ldots, n \) or \([p_i(x_i) : p_{i-1}(x_i)] = [q'_i(x_i) : q'_{i-1}(x_i)] \) for \( i = 1, \ldots, n + 1 \) modulo \( n + 1 \).

**Proof.** Let \([p_i(x_i) : p_{i-1}(x_i)] = k_i, [q_i(x_i) : q_{i-1}(x_i)] = l_i \). If \( d_i = k_i - l_i \), note that

\[
\sum [p_i(x_i) : p_{i-1}(x_i)] = \sum [q_i(x_i) : q_{i-1}(x_i)] \quad \text{if and only if} \quad \sum d_i = 0.
\]

Let \( q'_0 = q_0 \). Suppose that we have constructed \( q'_0, \ldots, q'_i \) such that \([p_j(x_j) : p_{j-1}(x_j)] = [q'_j(x_j) : q'_{j-1}(x_j)] \) for \( j = 1, \ldots, i \) and \([q_{i+1}(x_{i+1}) : q'_{i+1}(x_{i+1})] = l_{i+1} - \sum_{k=1}^{i} d_k \).

Let \( r \) be a projection valued (norm continuous) function on \( X_{i+1} \) such that \( r \leq 1 - q_{i+1} \) and \([q]_{K_0(B)} = d_{i+1} + \sum_{k=1}^{i} d_k \). Then take \( q'_{i+1} = q + q_{i+1} \). Then

\[
[q'_{i+1}(x_{i+1}) : q'_i(x_{i+1})] = [q_{i+1}(x_{i+1}) : q'_i(x_{i+1})] + [q(x_{i+1}) : 0]
\]

\[
= l_{i+1} - \sum_{k=1}^{i} d_k + d_{i+1} + \sum_{k=1}^{i} d_k
\]

\[
= l_{i+1} + k_{i+1} - l_{i+1}
\]

\[
= k_{i+1}
\]
\[ [q_{i+2}(x_{i+2}) : q'_{i+1}(x_{i+2})] = [q_{i+2}(x_{i+2}) : q_{i+1}(x_{i+2})] + [0 : q(x_{i+2})] \]
\[ = l_{i+2} - (d_{i+1} + \sum_{k=1}^{i} d_k) \]
\[ = l_{i+2} - \sum_{k=1}^{i+1} d_k \]

By induction, we can get \( q_0, \ldots, q_{n-1} \) such that \([p_j(x_j) : p_{j-1}(x_j)] = [q'_j(x_j) : q'_{j-1}(x_j)]\) for \( j = 1, \ldots, n-1 \) as we want. Finally, since we also have \([q_n(x_n) : q'_{n-1}(x_n)] = l_n - \sum_{k=1}^{n-1} d_k = l_n + d_n = k_n \) from \( \sum_{k=1}^{n-1} d_k + d_n = 0 \), we take \( q'_n = q_n \).

In the circle case, we perturb \( q_0 \) to \( q'_0 \) such that \([q'_n(x_n) : q'_{n-1}(x_n)] = k_n \) and \([q_0(x_0) : q'(x_0)] = l_0 - \sum_{k=1}^{n} d_k = l_0 + d_0 = k_0. \)

Next is an analogous result that is more symmetrical.

**Lemma 3.13.** Let \((p_0, \ldots, p_n)\) and \((q_0, \ldots, q_n)\) be local liftings of \( p \) and \( q \) such that \( p_i(x) \) and \( q_i(x) \) are full, properly infinite projections for each \( x \) in \( X_i \).

If \( \sum_{i} [p_i(x_i) : p_{i-1}(x_i)] = \sum_{i} [q_i(x_i) : q_{i-1}(x_i)] \), or \( [p_i(x_i) : p_{i-1}(x_i)] = [q'_i(x_i) : q'_{i-1}(x_i)] \) for \( i = 1, \ldots, n + 1 \) modulo \( n + 1 \), then we can find perturbations \((q'_0, \ldots, q'_n)\) of \( q \) and \((p'_0, \ldots, p'_n)\) of \( p \) such that \([p'_i(x_i) : p'_{i-1}(x_i)] = [q'_i(x_i) : q'_{i-1}(x_i)]\) for all \( i \).

**Proof.** The proof proceeds as above with one exception: If \( d_{i+1} + \sum_{k=1}^{i} d_k \geq 0 \), we make \( p'_i \leq p_i \) rather than making \( q'_i \geq q_i \). \( \square \)

An operator on \( H_B \) is called a Fredholm operator when it is invertible modulo \( K(H_B) \) the ideal of compact operators. In fact, a generalized Atkinson theorem says that an operator \( F \) for which there exists a compact \( K \in K(H_B) \) such that \( \text{Ker}(F + K) \) and \( \text{Ker}(F + K)^* \) are finitely generated and \( \text{Im} F + K \) is closed is a Fredholm operator and vice versa by [18]. Thus we can define an index of a Fredholm operator in \( K_0(B) \) as the difference of two classes of finitely generated modules. Let us denote its index by \( \text{Ind} \). For more details, we refer the reader to [26, 18].

**Proposition 3.14.** Suppose \( p \) and \( q \) are given by projection valued functions \((p_0, p_1, \ldots, p_n)\) and \((q_0, q_1, \ldots, q_n)\), where both \( p_i(x) \) and \( q_i(x) \) are full and properly infinite projections for each \( x \) in \( X_i \). If \( \sum_i k_i = \sum_i l_i \), then \( p \sim q \).

**Proof.** By Lemma 3.13 and the assumption we may arrange that \( k_i = l_i \) for each \( i \). Since \( p_i(x) \) and \( q_i(x) \) are full, properly infinite projections for
each $x$ in $X_i$, there is a (double) strongly continuous function $u_i$ on each $X_i$ such that $u_i^*u_i = p_i, u_iu_i^* = q_i$ by Corollary 3.8. Note that $u_{i-1}(x)$ is a unitary from $p_{i-1}(x)H$ onto $q_{i-1}(x)H$ so that $\text{Ind}(u_{i-1}(x)) = 0$. Then $k_i = li$ implies that

$$\text{Ind}(q_i(x)u_{i-1}(x)p_i(x)) = -l_i + \text{Ind}(u_{i-1}(x)) + k_i = 0,$$

where the first index is for maps from $p_i(x)H_B$ to $q_i(x)H_B$, and, for example, the index of $p_{i-1}(x)p_i(x)$ as a map from $p_i(x)H_B$ to $p_{i-1}(x)H_B$ is $k_i$. Also

$$q_i(x)u_{i-1}(x)p_i(x) - u_{i-1}(x) \in B.$$

There is a compact perturbation $v_i$ of $q_i(x)u_{i-1}(x)p_i(x)$ such that $v_i^*v_i = p_i(x), v_i^*v_i = q_i(x)$, and $v_i - u_{i-1}(x) \in B$.

By the triviality of the continuous field of Hilbert modules determined by $p_i$ and the path connectedness of the unitary group of $M(B)$ [18], there is a path $\{v(t) : t \in [x_i, x]\}$ such that $v(t)^*v(t) = v(t)v(t)^* = p_i(t), v(x_i) = u_i(x_i)^*v_i$, and $v(x) = p_i(x)$ for some $x \in X_i$. Then we let $w_i = u_i^*v$ on $[x_i, x]$ so that

$$w_i(x_i) - u_{i-1}(x) = v_i - u_{i-1}(x) \in B,$$

$$w_i^*w_i = v^*u_i^*u_i v = v^*p_i v = p_i,$$

$$w_i w_i^* = u_i^*v v^*u_i^* = u_i p_i u_i^* = q_i.$$

Finally, we define

$$u_i' = \begin{cases} w_i, & \text{on } [x_i, x], \\ u_i, & \text{on } [x, x_{i+1}]. \end{cases}$$

In the $(-\infty, \infty)$-case we do the above for $i = 1, \ldots, n$ and let $u_0' = u_0$.

In the circle case we do it for $i = 0, \ldots, n$. \hfill \square

**Corollary 3.15.** Suppose $p$ and $q$ are given by projection valued functions $(p_0, p_1, \ldots, p_n)$ and $(q_0, q_1, \ldots, q_n)$, where both $p_i(x)$ and $q_i(x)$ are halving projections for each $x$ in $X_i$. If $\sum_i k_i = \sum_i l_i$, then $p \sim u q$.

**Corollary 3.16.** Suppose $p$ and $q$ are given by projection valued functions $(p_0, p_1, \ldots, p_n)$ and $(q_0, q_1, \ldots, q_n)$. If $\sum_i k_i = \sum_i l_i$, then $[p] \sim [q]$ in $K_0$.

**Proof.** Since $p_i(x)$ and $q_i(x)$ are halving, we apply Proposition 3.14 to $1 - p$ and $1 - q$, and obtain that $1 - p \sim 1 - q$. It follows that $p \sim u q$. \hfill \square

**Proof.** We replace $p$ with $p \oplus 1 \oplus 0$ and $q$ with $q \oplus 1 \oplus 0$, still being equal in $K_0$. Also, note that $k_i$’s and $l_i$’s are not changed. Then by Kasparov’s absorption theorem $p_x(H_B) \oplus H_B \simeq H_B$ and so is $q_x(H_B)$.
Thus $p_x \oplus 1$ and $q_x \oplus 1$ are Murray-von Neumann equivalent. This implies that $p_x \oplus 1$ and $q_x \oplus 1$ are full, properly infinite projections in $M(B)$. The conclusion follows from Proposition 3.14.

4. Acknowledgements

The author wishes to thank S. Zhang for pointing out his previous result (2.9) (ii) during a conference. He also wishes to thank P.W. Ng for confirming Lemma 3.6.

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