Axisymmetric contact between a rigid punch and a coated foundation with rough surfaces

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Abstract. Axisymmetric contact problem for a coated foundation and a rigid punch is considered. It is assumed that the shape of the thin upper layer (coating) and the shape of the punch surface can be described by different rapidly changing functions. We obtain basic integral equation for this problem and additional condition. A projection method is developed for this problem. It allows one to obtain the solution of the equation with high accuracy that cannot be done by known methods.

1. Introduction
Some plane and axisymmetric contact problems for bodies with coatings were considered in [1–5]. It were some investigations of plane and axisymmetric problems for conformal contact between layered foundations and punches, contact problems for nonuniform foundations and rough punches. This paper devoted to the contact problem for foundation with homogeneous coating of variable thickness and annular punch with complex base forms for a first time.

2. Statement of the contact problem
We assume that a viscoelastic aging layer of an arbitrary thickness $H$ with a thin coating made of another viscoelastic aging material of an thickness $h(r)$ lies on a rigid basis. We denote the moments of coating and lower layer production by $\tau_1$ and $\tau_2$, respectively. We also assume that the coating rigidity is less than the rigidity of the lower layer or they are of the same order of magnitude. There is smooth contact or perfect contact between layers and between the lower layer and the rigid base.

At time $\tau_0$, the axial force $P(t)$ starts to indent a rigid annular punch of inner radius $a$ and outer radius $b$ into the surface of such a foundation. The function $g(r)$ describes distance between contact surfaces in nondeformable state and called backlash function. If the function $f(r)$ describes the form of the punch then $g(r) = f(r) - h(r) - h_0$, where $h_0 = \min_{r \in [a,b]} |f(r) - h(r)|$. In the case of conformal contact $g(r) \equiv 0$. Contact area is constant: it is a ring of inner radius $a$ and outer radius $b$. The coating is assumed to be thin compared with the width of contact area, i.e., its thickness satisfies the condition $h(r) \ll (b - a)$.  

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Figure 1. Axisymmetric contact interaction

As a result of this interaction, the punch is immersed in the base by the quantity $\delta(t)$. The contact interaction scheme is shown in figure 1.

The integral equations and additional condition for described problem has a form [6, 7]

$$\theta \left[ \frac{q(r,t)h(r)}{E_1(t-\tau_1)} - \int_{\tau_0}^t \frac{q(r,\tau)h(r)}{E_1(\tau-\tau_1)} K_1(t-\tau_1,\tau-\tau_1) d\tau \right]$$

$$+ \frac{2(1-\nu_2^2)}{H} \left[ \int_a^b k_{ax} \left( \frac{r}{H}, \frac{\rho}{H} \right) \frac{q(\rho,t)}{E_2(t-\tau_2)} \rho d\rho \right]$$

$$- \int_{\tau_0}^t K_2(t-\tau_2,\tau-\tau_2) \int_a^b k_{ax} \left( \frac{r}{H}, \frac{\rho}{H} \right) \frac{q(\rho,\tau)}{E_2(\tau-\tau_2)} \rho d\rho d\tau \right] = \delta(t) - g(r), \quad (1)$$

$$2\pi \int_a^b q(\rho,t)\rho d\rho = P(t), \quad K_k(t,\tau) = E_k(\tau) \frac{\partial}{\partial \tau} \left[ \frac{1}{E_k(\tau)} + C_k(t,\tau) \right], \quad r \in [a,b], \quad t \geq \tau_0, \quad (2)$$

where $\delta(t)$ is the punch settlement; $E_k(t)$ are the Young moduli of the coating ($k = 1$) and the lower layer ($k = 2$) and $\nu_2$ is Poisson’s ratio of the lower layer; $I$ is the identity operator; $K^{(k)}(t,\tau)$ ($k = 1, 2$) are tensile creep kernels; $C^{(k)}(t,\tau)$ ($k = 1, 2$) are the tensile creep functions; $\theta$ is a dimensionless coefficient depending on the contact conditions between coating and lower layer; in the case of a smooth coating-layer contact, we have $\theta = 1 - \nu_1^2$, and in the case of a perfect contact, $\theta = (1 - \nu_1 - 2\nu_2^2)/(1 - \nu_1)$, where $\nu_1$ is Poisson’s ratio of the coating; $k_{ax}(r/H,\rho/H)$ is known kernel of the plane contact problem, which has the form

$$k_{ax}(r,\rho) = \int_0^\infty L(u)J_0(\rho u)J_0(\rho u) du,$$

and, in the case of a smooth contact between the lower layer and the rigid base, $L(u) = (\cosh 2u - 1)/\sinh 2u + 2u)$, and in the case of a perfect contact, $L(u) = (2\kappa \sinh 2u - 4u)/(2\kappa \cosh 2u + 4u^2 + 1 + \kappa^2)$ ($\kappa = 3 - 4\nu_2$) (see, for example, [8]).
If we make a change of variables according to the formulas
\[(r^*)^2 = \frac{r^2 - a^2}{b^2 - a^2}, \quad (\rho^*)^2 = \frac{\rho^2 - a^2}{b^2 - a^2}, \quad t^* = \frac{t}{\tau_0}, \quad \tau^* = \frac{\tau_k}{\tau_0}, \quad \lambda = \frac{H}{b - a}, \quad \eta = \frac{a}{b - a},\]
\[\zeta^2 = \frac{b + a}{b - a}, \quad \delta^* (t^*) = \frac{\delta(t)}{b - a}, \quad g^*(\rho^*) = \frac{g(\rho)}{b - a}, \quad m^*(\rho^*) = \frac{\theta}{1 - \nu^2} 2(b - a),\]
\[c^*(t^*) = \frac{E_2(t - \tau_2)}{E_1(t - \tau_1)}, \quad q^*(\rho^*, t^*) = \frac{2q(r, t)(1 - \nu^2)}{E_2(t - \tau_2)}, \quad P^*(t^*) = \frac{P(t)(1 - \nu^2)}{\pi E_2(t - \tau_2)(b^2 - a^2)^2}.\]
(3)

then we obtain the following mixed integral equation with the additional condition
\[c(t)m(r)(I - V_1)q(r, t) + (I - V_2)Fq(r, t) = \delta(t) - g(r),\]
\[\int_0^1 q(\rho, t)\rho\,d\rho = P(t), \quad r \in [0, 1], \quad t \geq 1.\]
(4)

Hereinafter we will omit the asterisks in formulas with dimensionless variables.

It easy to see that there exist two versions of the substitution: 1) the settlement of the punch is known (i.e., the right-hand side of the integral equation is given); 2) the indenter force is known. We will construct the solution for the case with known force \(P(t)\) and unknown settlement \(\delta(t)\) and contact pressures \(q(r, t)\). To this end we will use projection method described in [9].

3. Solution for version with known indenting force

We will find the solution in the form
\[q(r, t) = \frac{Q(r, t)}{\sqrt{m(r)}} - \frac{g(r)}{m(r)}(I - V_1)^{-1} \frac{1}{c(t)},\]
(5)

where \(Q(r, t)\) is new unknown function. Then the integral equation and auxiliary condition (4) can be reduced to the following integral equation with the Hilbert–Schmidt kernel \(k(x, \xi)\) [10, 11]:
\[c(t)(I - V_1)Q(r, t) + (I - V_2)AQ(r, t) = \frac{\delta(t)}{\sqrt{m(r)}} + \frac{\tilde{c}(t)\tilde{g}(r)}{\sqrt{m(r)}},\]
\[\int_0^1 \frac{Q(\rho, t)}{\sqrt{m(\rho)}}\rho\,d\rho = \tilde{P}(t), \quad r \in [0, 1], \quad t \geq 1,\]
(6)

where
\[A f(r) = \int_0^1 k(r, \rho)f(\rho)\rho\,d\rho, \quad k(r, \rho) = \frac{k_{ax}(r, \rho)}{\sqrt{m(r)m(\rho)}}, \quad \tilde{g}(r) = \int_0^1 \frac{k_{ax}(r, \rho)g(\rho)}{m(\rho)}\rho\,d\rho,\]
\[\tilde{c}(t) = (I - V_2)(I - V_1)^{-1} \frac{1}{c(t)}, \quad \tilde{P}(t) = P(t) + (I - V_1)^{-1} \frac{1}{c(t)} \int_0^1 \frac{g(\rho)}{m(\rho)}\rho\,d\rho.\]

We seek the solution \(Q(r, t)\) in the class of functions continuous in time \(t\) in the Hilbert space \(L_2(0, 1)\). To this end, we at first construct an orthonormal system.
of functions \( \{p_0(r), p_1(r), p_2(r), \ldots \} \) in \( L^2(0,1) \) which contains \( 1/\sqrt{m(r)} \) and remaining basis functions can be written as the products of functions depending on \( r \) and weight function \( 1/\sqrt{m(r)} \). So we will orthonormalize the system of linearly independent functions \( \{1/\sqrt{m(r)}, r/\sqrt{m(r)}, r^2/\sqrt{m(r)}, \ldots \} \) in \( L^2(0,1) \).

The Hilbert space \( L^2(0,1) \) can be represented as the direct sum of orthogonal subspaces \( L^2(0,1) = L^2_{(0)}(0,1) \oplus L^2_{(1)}(0,1) \), where \( L^2_{(0)}(0,1) \) is the Euclidean space with basis \( \{p_0(r)\} \) and \( L^2_{(1)}(0,1) \) is the Hilbert space with basis \( \{p_1(r), p_2(r), \ldots \} \). The integrand and the right-hand side of (6) can also be represented as the algebraic sum of functions continuous in time \( t \) and ranging in \( L^2_{(1)}(0,1) \) and \( L^2_{(2)}(0,1) \), respectively, i.e.,

\[
Q(r,t) = Q_0(r,t) + Q_1(r,t), \quad \delta(t) + \tilde{c}(t)\tilde{g}(r) = \Delta_0(r,t) + \Delta_1(r,t),
\]

where \( Q_0(r,t), \Delta_0(r,t) \in L^2_{(0)}(0,1), Q_1(r,t), \Delta_1(r,t) \in L^2_{(1)}(0,1) \) and

\[
Q_0(r,t) = z_0(t)p_0(r), \quad \Delta_0(r,t) = \left[\sqrt{g_0} \delta(t) + \tilde{g}_0 \tilde{c}(t)\right]p_0(r), \quad \Delta_1(r,t) = \tilde{c}(t)\tilde{g}_1(r),
\]

\[
\tilde{g}_0 = \sum_{k=0}^{\infty} K_{0k} \int_0^1 p_1(\rho)g(\rho)d\rho, \quad \tilde{g}_1(r) = \tilde{g}(r) - \tilde{g}_0 p_0(r),
\]

\[
K_{ml} = \int_0^1 \int_0^1 k(r,\rho)p_m(\rho)p_1(\rho)r\rho d\rho dr, \quad k(r,\rho) = \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} K_{ml}p_m(\rho)p_l(\rho).
\]  

The term \( Q_0(r,t) \in L^2_{(0)}(0,1) \) is determined by the additional condition (6):

\[
Q_0(r,t) = z_0(t)p_0(r), \quad z_0(t) = \sqrt{2P(t)},
\]  

and the term \( \Delta_1(r,t) \in L^2_{(1)}(0,1) \) is known.

After introducing the orthogonal projection operators \( P_0 \) and \( P_1 \), mapping the space \( L^2(0,1) \) onto \( L^2_{(0)}(0,1) \) and \( L^2_{(1)}(0,1) \), respectively, we apply the operator \( P_1 \) to integral equation (6). As a result, we obtain the equation in \( L^2_{(1)}(0,1) \) with a known right-hand side for determining \( Q_1(r,t) \):

\[
c(t)(I - V_1)Q_1(r,t) + (I - V_2)P_1FQ_1(r,t) = -(I - V_2)P_1FQ_0(r,t) + \tilde{c}(t)\tilde{g}_1(r).
\]  

It is necessary to construct its solution in the form of a series in the eigenfunctions of the operator \( P_1F \):

\[
Q_1(r,t) = \sum_{k=1}^{\infty} z_k(t)\varphi_k(r).
\]  

The system of eigenfunctions \( \{\varphi_k(r)\}_{k=1,2,3,\ldots} \) of such an operator is a basis in the space \( L^2_{(1)}(0,1) \). The spectral problem for the operator \( P_1F \) can be written in the form

\[
P_1F\varphi_k(r) = \gamma_k\varphi_k(r), \quad \varphi_k(r) = \sum_{m=1}^{\infty} \psi_{km}p_m(r), \quad k = 1, 2, 3, \ldots
\]  

Using formulas (7)–(10) we can obtain expressions for determining expansion functions \( z_k(t) \)
(k = 1, 2, 3, . . .):
\[ z_k(t) = \left( I + W_k \right) \frac{(I - V_2)(g_k(I - V_1)^{-1}c^{-1}(t) - K_kz_0(t))}{c(t) + \gamma_k}, \]
\[ K_k = \sum_{m=1}^{\infty} K_{0m}\psi_{km}, \quad g_k = \sum_{m=1}^{\infty} \psi_{km} \sum_{l=0}^{\infty} K_{ml} \int_0^1 p_l(\rho)g(\rho)\rho d\rho, \]
\[ W_kf(t) = \int_1^t R_k^*(t, \tau)f(\tau) d\tau, \quad k = 2, 3, \ldots, \]
where \( R_k^*(t, \tau) (k = 1, 2, 3, \ldots) \) are resolvents of the kernel \( K_k^*(t, \tau) = [c(t)K^1(t, \tau) + \gamma_kK^2(t, \tau)]/[c(t) + \gamma_k] \).

The resulting solution has the following structure
\[ g(r, t) = \frac{1}{m(r)} \left\{ z_0(t)P_0(r) + \sum_{m=1}^{\infty} \left[ \sum_{k=1}^{\infty} \psi_{km}z_k(t) \right] P_m(r) \right\} - \frac{g(r)}{m(r)} \left( I - V_1 \right)^{-1} \frac{1}{c(t)}, \]
where the expression in braces is smooth (\( P_i(r) \) is polynomial of degree \( i \), \( p_i(r) = P_i(r)/\sqrt{m(r)} \)). One can explicitly write out the weight functions \( m(r) \) and \( g(r) \) in the solution. Note that the coating thickness is related to \( m(r) \) and backlash function is related to \( g(r) \). The formulas obtained permit obtaining efficient analytic solutions for the layers with rough coatings which can be described by complicated and rapidly oscillating functions. Such a result can hardly be done by other known methods.

To find the unknown settlement, we must apply the operator \( P_0 \) to integral equation (6). Then the formula for the settlement \( \delta(t) \) has a form:
\[ \delta(t) = \sqrt{2} \left\{ c(t)(I - V_1)z_0(t) + (I - V_2) \left[ K_{00}z_0(t) + \sum_{k=1}^{\infty} K_kz_k(t) - g_0(I - V_1)^{-1} \frac{1}{c(t)} \right] \right\}. \]

Conclusions

- We pose and solve axisymmetric problems of contact between viscoelastic aging foundations with rough coatings and rigid punches in the case, where the punch base surface and coating width are described by different rapidly changing functions. To this end we develop general projection method.
- The solution of the problem is obtained analytically, and, in the expressions for the contact stresses, the coating width and backlash function is distinguished explicitly, which allows one to perform computations for actual shapes of the coating surface, which are described by rapidly changing functions.
- The explicit formulas has been obtained for the punch settlement.
- Other known methods for the solution of this problem diverge with an increase in the time parameter or yield an error of up to 100% in determining contact stresses.

Acknowledgments
This work was financially supported by the Russian Science Foundation under Project No. 17-19-01257.
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