WEAK HOPF ALGEBRA SYMMETRIES OF $C^*$-ALGEBRA INCLUSIONS

KORNÉL SZLACHÁNYI

In 2-dimensional conformal field theory or in 1-dimensional quantum lattice systems the superselection sectors (charges) $p, q, \ldots$ often have composition laws $p \otimes q = \sum_r N_{pq}^r \cdot r$ such that the non-negative integer multiplicities $N_{pq}^r$ do not admit an integer solution for the dimensions $d_r$ of the equation $\sum_r N_{pq}^r d_r = d_p d_q$. This implies that it is impossible to find a semisimple quantum group, more precisely a semisimple Hopf algebra, $H$ the irreducible representations of which would be in one-to-one correspondence with the sectors $p, q, \ldots$ and the tensor product of irreducible representations of $H$ would produce precisely the numbers $N_{pq}^r$ as their branching numbers.

A solution for this problem has been found in [1] where we proposed to replace $C^*$-Hopf algebras with $C^*$-weak Hopf algebras the comultiplication of which is coassociative but not unit preserving. This allows the dimension of a tensor product of representations be smaller than the product of the dimensions of these representations,

$$\dim(D \otimes D') \leq \dim D \cdot \dim D'.$$

$C^*$-weak Hopf algebras extended earlier generalizations by T. Yamanouchi [21] and Y. Hayashi [8] and can be considered as an axiomatic approach to the depth 2 paragoups of A. Ocneanu. A detailed analysis of the structure of weak Hopf algebras has been carried out in [2, 16, 12, 3]. For a recent review see [15].

Independently of the above development the motivation to axiomatize a fairly non-commutative Poisson groupoid has lead J-H. Lu to the introduction of Hopf algebroids in [11]. Later it was shown by P. Etingof and D. Nikshych [4] that weak Hopf algebras are just the finite dimensional versions of Hopf algebroids. See also [18].

The definition of a Hopf algebroid $A$ contains the data of a base algebra $B$ and two algebra maps $s: B^{op} \to A$ and $t: B \to A$, called the source and the target, respectively. They are non-commutative versions of the algebra of functions over the space of units of a groupoid. In a weak Hopf algebra $A$ the images of these maps has been called $A^R$ and $A^L$, respectively, although they are not part of the data but have been "discovered" in studying the properties of the coproduct $\Delta: A \to A \otimes A$.

$C^*$-weak Hopf algebras can be applied to the characterization of finite index depth 2 inclusions $N \subset M$ of von Neumann algebras. If the centers of $M$ and $N$ are finite dimensional there exists a $C^*$-weak Hopf algebra $A$ acting regularly on $M$ such that $N$ is the invariant subalgebra. Such an approach has been initiated in [17] and worked out for type II$_1$ factors by D. Nikshych and L. Vainerman in [13, 14]. One should mention here an other approach [6] by M. Enock and J.-M. Vallin.

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which uses Hopf bimodules to characterize depth 2 inclusions with arbitrary (non-finite) index.

The paper is organized as follows. In Section 1 the axioms of weak bialgebras and weak Hopf algebras are presented with a brief description of the properties of $A^L$, $A^R$, representation theory, the $C^*$-structure, and the Haar measure. In Section 2 we summarize the requirements for a "regular" action of a $C^*$-weak Hopf algebra on a unital $C^*$-algebra. The first two sections are based on earlier papers on the subject while Section 3 contains new results. In Section 3 we outline the proof of a reconstruction theorem stating that if a certain inclusion $N \subset M$ of $C^*$-algebras is given then there is a $C^*$-weak Hopf algebra $A$ and a regular action of $A$ on $M$ such that $N$ is the invariant subalgebra. We actually work with abstract 2-categorical "inclusions".

1. Weak Hopf Algebras

1.1. Weak bialgebras.

**Definition 1.1.** A weak bialgebra over a field $K$ consists of the data $\langle A, m, u, \Delta, \varepsilon \rangle$ where

1. $\langle A, m, u \rangle$ is an algebra, i.e., the multiplication $m: A \otimes A \to A$ and the counit $u: K \to A$ satisfy
   (a) associativity: $m \circ (m \otimes id) = m \circ (id \otimes m)$ and
   (b) unit properties: $m \circ (u \otimes id) = id = m \circ (id \otimes u)$

2. $\langle A, \Delta, \varepsilon \rangle$ is a coalgebra, i.e., the comultiplication $\Delta: A \to A \otimes A$ and the counit $\varepsilon: A \to K$ satisfy
   (a) coassociativity: $(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$ and
   (b) counit properties: $(\varepsilon \otimes id) \circ \Delta = id = (id \otimes \varepsilon) \circ \Delta$

3. The algebra and coalgebra structures satisfy the compatibility conditions
   (a) $\Delta$ is multiplicative / $m$ is comultiplicative, i.e., as maps $A \otimes A \to A \otimes A$,
   \[ \Delta \circ m = (m \otimes m) \circ (id \otimes \tau \otimes id) \circ (\Delta \otimes \Delta) \]
   where $\tau: A \otimes A \to A \otimes A$ denotes the flip map $x \otimes y \mapsto y \otimes x$, $\tau$
   (b) $\varepsilon$ is weakly multiplicative, i.e., as maps $A \otimes A \otimes A \to K$,
   \[ (\varepsilon \otimes \varepsilon) \circ (m \otimes m) \circ (id \otimes \Delta \otimes id) = \varepsilon \circ m \circ (m \otimes id) \]
   \[ (\varepsilon \otimes \varepsilon) \circ (m \otimes m) \circ (id \otimes \Delta^{op} \otimes id) = \varepsilon \circ m \circ (m \otimes id) \]
   where $\Delta^{op} := \tau \circ \Delta$ is opposite comultiplication,
   (c) $u$ is weakly comultiplicative, i.e., as maps $K \to A \otimes A \otimes A$,
   \[ (id \otimes m \otimes id) \circ (\Delta \otimes \Delta) \circ (u \otimes u) = (\Delta \otimes id) \circ \Delta \circ u \]
   \[ (id \otimes m^{op} \otimes id) \circ (\Delta \otimes \Delta) \circ (u \otimes u) = (\Delta \otimes id) \circ \Delta \circ u \]
   where $m^{op} := m \circ \tau$ is opposite multiplication.

Usually one uses elements of $A$ to express identities and this is what we will do in most of the cases. So we shall write $xy$ instead of $m(x \otimes y)$ and use the unit element 1 instead of $u: k \mapsto 1k$. Then axiom (3.a) could then be simply written as $\Delta(ab) = \Delta(a)\Delta(b)$. Or, using Sweedler’s notation $\Delta(a) = a_{(1)} \otimes a_{(2)}$ which suppresses a possible summation, (3.a) takes the form
\[ (ab)_{(1)} \otimes (ab)_{(2)} = a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}, \]
well, the beauty of which is not very convincing. However, the weak multiplicativity properties under (3.b) can be rather nicely written using Sweedler’s notation as
\[ \varepsilon(ab_{(1)})\varepsilon(b_{(2)}c) = \varepsilon(abc) = \varepsilon(ab_{(2)})\varepsilon(b_{(1)}c). \]
Anyhow, the above set of axioms show manifestly the self-duality of the structure. Considering the axioms as commutative diagrams in the category $\text{Vec}_K$ of vector spaces then the axioms stay invariant if we reverse the directions of all the arrows and change simultaneously $\Delta \leftrightarrow m$ and $\varepsilon \leftrightarrow u$.

If $A$ is a finite dimensional bialgebra then $\hat{A} := \text{Hom}(A, K)$ is also a bialgebra if we define the structure maps $\hat{m}, \hat{\Delta}, \hat{u}, \hat{\varepsilon}$ by means of the canonical pairing $\langle \cdot, \cdot \rangle: \hat{A} \times A \to K$,

\begin{align*}
(1.1) \quad \langle \varphi \psi, a \rangle & := \langle \varphi \otimes \psi, \Delta(a) \rangle \\
(1.2) \quad \langle \hat{\Delta}(\varphi), a \otimes b \rangle & := \langle \varphi, ab \rangle \\
(1.3) \quad \langle \hat{1}, a \rangle & := \varepsilon(a) \\
(1.4) \quad \hat{\varepsilon}(\varphi) & := \langle \varphi, 1 \rangle,
\end{align*}

where $\varphi, \psi \in \hat{A}, a, b \in A$, and we used the notation $\langle \cdot, \cdot \rangle$ also for the pairing of $A \otimes A$ with $\hat{A} \otimes \hat{A} \cong \text{Hom}(A \otimes A, K)$. This is exactly how ordinary bialgebras behave. A weak bialgebra $A$ becomes a bialgebra in case either of the following conditions hold:

1. $\Delta$ is unital, $\Delta(1) = 1 \otimes 1$.
2. $\varepsilon$ is multiplicative, $\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$ for all $a, b \in A$.

The basic feature in weak bialgebras (WBA’s for short) is the presence of two canonical subalgebras $A_L \subset A$ and $A_R \subset A$ both of which reduce to $K$ in case of $A$ is an ordinary bialgebra. The following are equivalent definitions of $A_L$ and $A_R$.

**Lemma & Definition 1.2.** The following conditions on elements $l$, respectively $r$, of a WBA $A$ are equivalent

1. $\Delta(l) = (l \otimes 1)\Delta(1) = \Delta(1)(l \otimes 1)$
2. $\varepsilon(l) = (1 \otimes r)\Delta(1) = \Delta(1)(1 \otimes r)$
3. $\exists f: A \to K$ such that $(f \otimes \text{id}) \circ \Delta(1) = l$
4. $\exists f: A \to K$ such that $(\text{id} \otimes f) \circ \Delta(1) = r$

The set of such $l$’s and $r$’s will be denoted by $A_L$ and $A_R$, respectively.

It not difficult to see from the first of these three properties that $A_L$ and $A_R$ are subalgebras of $A$ and that they commute with each other. $\Delta(1)$ is an idempotent which not only belongs to $A_R \otimes A_L$ by property (2) but also spans $A_L$ and $A_R$ in the sense of property (3). For the proof of these properties and many more we refer to [2, 18].

1.2. **Representations of WBA’s.** The meaning of $A_L, A_R$ can be better understood in terms of the representation theory of the weak bialgebra. Let $\mathcal{A}M$ denote the category of left $A$-modules. We define a monoidal structure on this category using the coalgebra structure of $A$. The monoidal product $V \Box W$ of two $A$-modules is defined as the subspace of the $K$-module tensor product $V \otimes W$ which is the image of the idempotent $\Delta(1)$. The monoidal product of intertwiners $t: V \to V'$ and $s: W \to W'$ is the restriction of $t \otimes s: V \otimes W \to V' \otimes W'$ to the subspace $V \Box W = 1_{(1)} \cdot V \otimes 1_{(2)} \cdot W$.

This procedure is similar to how representation theory of a bialgebra or Hopf algebra is related to the representation theory of the underlying field or ring $K$. 
The "only" difference is the need of projecting out a subspace by acting with $\Delta(1)$. There is another, more elegant way of formulating the monoidal structure of $A$ which uses the fact that every bimodule category has a very natural monoidal structure.

Let $L$ denote the ring $A^L$ and its injection to $A$ be $t: L \to A$. The ring $A^R$ is isomorphic to $L$ with opposite multiplication via the map

$$s: L^{op} \to A, \quad s(l) = 1(1)\varepsilon(1(2)l)$$

Since the images of $s$ and $t$ commute, we can make $A$ into an $L$-$L$-bimodule with the definitions

$$l \in L, \ a \in A \mapsto l \cdot a := t(l)a, \quad a \cdot l := s(l)a.$$  

Then we have the forgetful functor $\phi: _A\mathcal{M} \to L\mathcal{M}_L$ from the category of $A$-modules to the monoidal category of $L$-$L$-bimodules. Thus a monoidal structure on $A\mathcal{M}$ can be introduced by requiring that the forgetful functor $\phi$ be strictly monoidal.

This means precisely that we define the underlying space of the $A$-module $V \Box W$ as the bimodule tensor product $V \otimes_L W$, which is indeed smaller then the $K$-tensor product $V \otimes W$ and can be shown to be isomorphic to the subspace projected out by $\Delta(1)$, as above. The proof uses the following identity shown in [2], (2.31a),

$$a_{(1)}s(l) \otimes a_{(2)} = a_{(1)} \otimes a_{(2)}t(l), \quad l \in L, \ a \in A.$$  

Now it is not surprising that the monoidal unit $U$ of $A\mathcal{M}$, also called the trivial representation is not one-dimensional but is a representation on the space $L = A^L$.

So we have a left action of $A$ on $L$, but also, we had a right action of $L$ on $A$, so the notation $a \cdot l$ would be ambiguous. Instead we use the special notation

$$a \triangleright l := \varepsilon(1(1)al)1(2).$$

In this way $L$ becomes a left $A$-module denoted $U$ for which there exist natural isomorphisms $U \Box V \cong V \cong U \Box V$ for all $A$-modules $V$. Of course, they are the natural isomorphisms $L \otimes_L V \cong V \cong V \otimes_L L$ of the bimodule category $L\mathcal{M}_L$. The very fact that these natural isomorphisms (and also the associativity natural isomorphism) for the two categories $A\mathcal{M}$ and $L\mathcal{M}_L$ coincide, means that the forgetful functor $A\mathcal{M} \to L\mathcal{M}_L$ is strictly monoidal.

The description of weak bialgebras through the bimodule category over $L$ yields a direct connection to the bialgebroids defined by Lu in [11]. A short review of the connection of these concepts can be found in [13].

An important example for a left $A$-module is the linear space of the dual bialgebra $\hat{A}$ endowed with the left action $a \triangleright \varphi := \varphi_{(1)}\langle \varphi_{(2)}, a \rangle$, where $a \in A$ and $\varphi \in \hat{A}$. Similarly, a right $A$-module structure on $\hat{A}$ can be defined by the right action $\varphi \leftarrow a := \langle \varphi_{(1)}, a \rangle\varphi_{(2)}$. For showing that these are indeed actions one can use the identities

$$\langle b \triangleright \varphi, a \rangle = \langle \varphi, ab \rangle = \langle \varphi \leftarrow a, b \rangle.$$  

1.3. **Weak Hopf algebras.** A weak Hopf algebra (WHA for short) is a weak bialgebra $A$ for which there exists an antipode, i.e., a linear map $S: A \to A$ satisfying
the antipode axioms

\begin{align*}
\text{(1.10)} & \quad a_{(1)} S(a_{(2)}) = \varepsilon(1) a 1 \\
\text{(1.11)} & \quad S(a_{(1)}) a_{(2)} = 1 \varepsilon(1) a _{1(2)} \\
\text{(1.12)} & \quad S(a_{(1)}) a_{(2)} S(a_{(3)}) = a
\end{align*}

for all \( a \in A \). Denoting by \( \pi^L \) and \( \pi^R \) the right hand sides of axioms \( \text{(1.10)} \) and \( \text{(1.11)} \), respectively, we can make the following observations. In any WBA \( \pi^L \) and \( \pi^R \) are idempotents in \( \text{End} A \) that project onto \( A^L \) and \( A^R \), respectively. But they are idempotents also in the convolution sense. The convolution product is the associative binary operation on \( \text{End} A \) defined by

\begin{equation}
\text{(1.13)} \quad f * g := m \circ (f \otimes g) \circ \Delta
\end{equation}

which has \( I = u \circ \varepsilon \) as its unit. One obtains the following identities in any WBA:

\begin{align*}
\text{(1.14)} & \quad \pi^L * \text{id}_A = \text{id}_A, \quad \pi^L * \pi^L = \pi^L, \\
\text{(1.15)} & \quad \text{id}_A * \pi^R = \text{id}_A, \quad \pi^R * \pi^R = \pi^R.
\end{align*}

In Hopf algebra theory one defines the antipode as the convolution inverse of \( \text{id}_A \). Assume that \( \sigma * \text{id}_A = I \). Then

\[
I = \sigma * \text{id}_A = \sigma * \text{id}_A * \pi^R = I * \pi^R = \pi^R
\]

but the image of \( \pi^R \) is \( A^R \) while the image of \( I \) is \( K \). So if the WBA is not a bialgebra then \( \text{id}_A \) is never convolution invertible. Equations \( \text{(1.14)} \) and \( \text{(1.15)} \) suggest that a convolution inverse can exist in a groupoid sense because \( \pi^L \) is the target and \( \pi^R \) is the source projection of \( \text{id}_A \). Therefore we require for a convolution inverse \( S \) that

\[
\text{(1.16)} \quad \text{id}_A * S = \pi^L, \quad S * \text{id}_A = \pi^R
\]

which are precisely the first two antipode axioms. The third one is then equivalent to either one of the equations

\[
\text{(1.17)} \quad S * \pi^L = S, \quad \pi^R * S = S
\]

that formulate the requirement that the source and target of \( S \) be the target and source of \( \text{id}_A \), respectively.

**Proposition 1.3** (see Thm 2.10 of \([2]\)). The antipode \( S \) of a WHA \( A \) satisfies the following properties.

1. \( S(ab) = S(b)S(a) \) for \( a, b \in A \), i.e., \( S \) is antimultiplicative,
2. \( S(a_{(1)}) \otimes S(a_{(2)}) = S(a_{(2)}) \otimes S(a_{(1)}) \) for \( a \in A \), i.e., \( S \) is anticomultiplicative,
3. \( S(1) = 1, \varepsilon \circ S = \varepsilon \),
4. \( S(A^L) = A^R \) and \( S(A^R) = A^L \),
5. if \( \dim A < \infty \) then \( S \) is invertible.

**Example 1.4.** Let \( A = KG \) be the groupoid algebra of a finite groupoid \( G \). Define comultiplication \( \Delta \) as the linear extension of the diagonal map \( g \mapsto g \otimes g \), \( g \in G \). This map is not unital

\[
\text{(1.18)} \quad \Delta(1) = \sum_{u \in O} u \otimes u \neq \sum_{u \in O} \sum_{v \in O} u \otimes v = 1 \otimes 1
\]
where $O \subset G$ denotes the set of units. One can easily check that $\Delta$ makes $KG$ into a WHA with

$$\varepsilon(g) = 1, \quad S(g) = g^{-1}, \quad g \in G.$$  

In this WHA $A^L = A^R$ and they coincide with the groupoid algebra $KO$ of the (totally disconnected) subgroupoid of units of $G$. Thus $A^L$ is Abelian and spanned by the pairwise orthogonal idempotents $u \in O$.

### 1.4. Weak $C^*$-Hopf algebras.

**Definition 1.5.** A $C^*$-WHA is a weak Hopf algebra $\langle A, \Delta, \varepsilon, S \rangle$ over the complex field $\mathbb{C}$ in which $A$ is a finite dimensional $C^*$-algebra and $\Delta$ is a $^*$-algebra map.

Since the counit and the antipode are uniquely determined by $\Delta$, one can immediately see that

$$\varepsilon(a^*) = \varepsilon(a), \quad S(a^*)^* = S^{-1}(a), \quad a \in A.$$  

The identity $1^*_1 \otimes 1^*_2 \equiv \Delta(1)^* = \Delta(1^*) \equiv \Delta(1)$ together with (3) implies that $A^L$ and $A^R$ are closed under the $^*$-operation, hence they are $C^*$-subalgebras of $A$.

The dual WHA $\hat{A}$ can be given a $^*$-operation by setting

$$\langle \phi^*, a \rangle := \langle \phi, S(a)^* \rangle.$$  

In this way $\hat{A}$ becomes a $^*$-algebra and $\hat{\Delta}$ a $^*$-homomorphism. However, it is not obvious whether $\hat{A}$ is also a $C^*$-algebra, inspite of its finite dimension. What is missing is to show that $\hat{A}$ possesses a faithful $^*$-representation. What helps here is the existence of a Haar measure.

**Theorem 1.6.** Let $\hat{A}$ be a $C^*$-WHA. Then there exists a unique $h \in A$, called the Haar integral (or Haar measure), such that

$$ah = \pi^L(a)h, \quad ha = h\pi^R(a) \quad \forall a \in A,$$

$$\pi^L(h) = 1 = \pi^R(h).$$

The Haar integral satisfies

$$h(1) \otimes ah(2) = S(a)h(1) \otimes h(2) \quad \forall a \in A$$

$$h(1)a \otimes h(2) = h(1) \otimes h(2)S(a) \quad \forall a \in A$$

$$h \mapsto \hat{A} = A^L$$

$$\hat{A} \leftarrow h = \hat{A}^R$$

$$h = h^2 = h^* = S(h)$$

$$\langle \phi(\psi \leftarrow a), h \rangle = \langle \phi \leftarrow S(a)\psi), h \rangle, \quad \forall \phi, \psi \in \hat{A}, \quad \forall a \in A$$

$$\langle (a \rightarrow \psi), h \rangle = \langle \phi(S(a) \rightarrow \psi), h \rangle, \quad \forall \phi, \psi \in \hat{A}, \quad \forall a \in A$$

Moreover, the sesquilinear form on $\hat{A}$ defined by

$$\langle \phi, \psi \rangle := \langle \phi^* \psi, h \rangle$$

is non-degenerate and positive, hence defines a Hilbert space structure on $\hat{A}$. It makes the left regular representation of $\hat{A}$ into a faithful $^*$-representation, in this way proving that $\hat{A}$ is a $C^*$-WHA, too.
Since \( \text{dim} A \) is finite, the Haar measure above is analogous to the Haar measure on a finite group, i.e., the counting measure normalized to give a total mass 1.

Interpreting the pairing \( \langle \psi, h \rangle \) as the integral \( \int \psi \) of the function \( \psi \) w.r.t. the Haar measure and the Sweedler arrows \( a \mapsto \cdot a \) and \( \cdot a \mapsto a \) as left and right translations, respectively, properties (1.29, 1.30) become the right and left invariance of the Haar measure, respectively.

The maps \( \varphi \mapsto (h \mapsto \varphi) \) and \( \varphi \mapsto (\varphi \mapsto h) \) are conditional expectations from \( \hat{A} \) onto \( \hat{A}_L \) and \( \hat{A}_R \), respectively. They tell us that \( \hat{A}_L \) is the space of functions invariant under left translations by all \( a \in A \). This space is, in general, different from the space \( \hat{A}_R \) that consists of the functions invariant under right translations.

The Haar state \( \langle \cdot, h \rangle \) on \( \hat{A} \) is not a trace, in general. In order to compute its modular automorphism we have to acquaint with the canonical grouplike element.

**Proposition 1.7.** Let \( A \) be a \( C^* \)-WHA and \( h \) its Haar measure and let \( \hat{h} \) be the Haar measure of the dual WHA \( \hat{A} \). Then \( \hat{h} \mapsto h \) and \( h \mapsto \hat{h} \) are positive and invertible and we can define

\[
g := g_L g_R^{-1} \quad \text{where} \quad g_L = (\hat{h} \mapsto h)^{1/2}, \quad g_R = (h \mapsto \hat{h})^{1/2},
\]

and call it the canonical grouplike element of \( A \). It is the unique \( g \in A \) for which

\[
g \geq 0 \quad \text{and invertible}
\]

\[
gag^{-1} = S^2(a), \quad a \in A
\]

\[
\text{tr}_r g = \text{tr}_r g^{-1} \quad \text{for all irrep} \ r \ \text{of} \ A.
\]

The name ”grouplike” refers to its invertibility and to the special form of its coproduct

\[
\Delta(g) = (g \otimes g)\Delta(1) = \Delta(1)(g \otimes g).
\]

In Hopf algebras grouplike elements can be very sparse. The fact that \( g \) exists in any \( C^* \)-WHA is related to that \( g \) belongs to the ”trivial” subalgebra \( A_L A_R \), which consists of the scalars in case of Hopf algebras. Notice also that \( g \) is not as grouplike as it could be, namely \( g \) is not unitary but positive instead.

The modular automorphism of the Haar state on \( A \) can now be expressed in terms of \( g_L \) and \( g_R \) as

\[
\langle \hat{h}, ab \rangle = \langle \hat{h}, b(g_L g_R a(g_L g_R)^{-1}) \rangle, \quad a, b \in A.
\]

One can prove that \( \hat{h} \) is a trace precisely if \( S^2 = \text{id}_A \). If this happens the WHA is called a weak Kac algebra.

**1.5. Soliton sectors of \( C^* \)-WHA’s.** The category \( \text{rep} A \) of finite dimensional \( * \)-representations of a \( C^* \)-WHA \( A \) has been studied in detail in [3]. Here I would like to emphasize only one aspect, the groupoid like vacuum structure of \( \text{rep} A \). This property is independent of the fact that WHA’s are quantum groupoids, it reflects rather a ”quantum 2-groupoid” feature.

The monoidal unit of the monoidal category \( \text{rep} A \) is the GNS representation associated to the positive linear functional \( \varepsilon: A \rightarrow \mathbb{C} \). This is the representation mentioned in Subsection 1.3 endowed with the scalar product \( \langle l_1, l_2 \rangle = \varepsilon(l_1^* l_2) \), \( l_1, l_2 \in A_L \) which makes the left action (1.8) a \( * \)-representation of \( A \). The point is that this ”trivial representation” may be reducible. This happens precisely when the inclusion \( A_L \subset A \) is not connected. The reason is that \( \text{End} U \cong Z^L \equiv A^L \cap \)
Center \( A \) according to Proposition 2.15 of [2]. The irreducibles \( V_\mu \) occurring in the decomposition

\[ U = \bigoplus_\mu V_\mu \]

are called vacuum representations, the name borrowed from the theory of solitons in 1+1-dimensional quantum field theory. Each of them is selfconjugate and occurs with multiplicity 1 in \( U \). WHA’s with irreducible trivial representation are called \textit{pure} (since \( \varepsilon \) is pure) or \textit{connected} (since the Bratteli diagram of \( A^L \subset A \) is connected).

If \( V_r \) is an arbitrary irreducible representation from the equivalence class (or sector, for short) \( r \) then there exists one and only one vacuum sector, called \( r^L \), for which

\[ V_r \cong U \otimes V_r \cong \bigoplus_\mu V_\mu \otimes V_r \cong V_{r^L} \otimes V_r . \]

Similarly, there exists one and only one vacuum sector \( r^R \) for which \( V_r \otimes V_{r^R} \cong V_r \).

The vacua \( r^L \) and \( r^R \) of the sector \( r \) behave very much like the source and target of a groupoid.

- The monoidal product of sectors \( p \otimes q = \{0\} \) if \( p^R \neq q^L \).
- If \( p^R = q^L \) and \( r \) is a sector contained in \( p \otimes q \) then \( r^L = p^L \) and \( r^R = q^R \).
- If \( \bar{r} \) denotes the class of the conjugate then \( \bar{r}^L = r^R \) and \( \bar{r}^R = r^L \).
- If there is a sector with left vacuum \( \mu \) and right vacuum \( \nu \) and there is one with left vacuum \( \lambda \) and right vacuum \( \mu \) then there is a sector with left vacuum \( \lambda \) and right vacuum \( \nu \).

2. \textbf{Regular actions of \( C^* \)-weak Hopf algebras}

2.1. \textbf{Module algebras}. Categorically speaking a module algebra \( M \) over a WHA or WBA \( A \) is a monoid in the category of left \( A \)-modules. More explicitly,

1. \( M \) is a left \( A \)-module, the action of \( a \) on \( m \) is denoted by \( a \triangleright m \), thus
   \[
   a \triangleright (b \triangleright m) = ab \triangleright m , \quad a,b \in A, \ m \in M
   \]
   \[
   1 \triangleright m = m \quad m \in M
   \]
2. \( M \) is an algebra with unit \( 1_M \).
3. multiplication of \( M \) is an \( A \)-module map, i.e.,
   \[
   a \triangleright (mm') = (a(1) \triangleright m)(a(2) \triangleright m') , \quad a \in A, \ m,m' \in M
   \]
4. the unit of \( M \) is an \( A \)-module map, i.e.,
   \[
   a \triangleright 1_M = \pi^L(a) \triangleright 1_M , \quad a \in A
   \]
If \( A \) is a \( C^* \)-WHA one adds the requirements

5. \( M \) is a \( C^* \)-algebra.
6. \( m \mapsto a \triangleright m \) is continuous for all \( a \in A \).
7. \( (a \triangleright m)^* = S(a)^* \triangleright m^* \) for \( a \in A, \ m \in M \).
2.2. The invariant subalgebra. The invariants of an A-module algebra M are those elements of M that transform the same way under the action of A as the identity 1_M does. So we define

\[(2.1) \quad M^A := \{ n \in M \mid a \triangleright n = \pi^L(a) \triangleright n, \ a \in A \} .\]

This subspace of M is actually a sub-C^*-algebra. It is easy to see that the Haar measure h \in A provides a conditional expectation onto the invariant subalgebra, h \triangleright M = M^A.

The structure of general WHA-actions is at least as complicated as the structure of general actions of finite groups. What we are interested in is rather the formulation of conditions for a "nice" action (being outer and Galois, e.g.) which makes the WHA together with its action uniquely determined by the inclusion of the invariant subalgebra N = M^A in M.

2.3. The relative commutant. When M is a module algebra over an ordinary C^*-Hopf algebra the relative commutant (M^A)' \cap M can be arbitrary small. So the "natural thing" is to consider irreducible inclusions for which (M^A)' \cap M \cong C^1. There can be situations, however, when an inclusion N \subset M is reducible and we want to describe N as an invariant subalgebra. In this case WHA’s are useful for the following reason. For module algebras M over a C^*-WHA there is a non-trivial lower bound for the relative commutant. Namely, for faithful actions (i.e., a \triangleright A = 0 \Rightarrow a = 0) the map \[(2.2) \quad A^L \ni l \mapsto l \triangleright 1_M \in N' \cap M \]

is an injective C^*-algebra map. So the "natural thing" is to consider module algebras for which (M^A)' \cap M \cong A^L.

2.4. The crossed product and Galois actions. In order to formulate our next condition we need the notion of the crossed product algebra. The crossed product algebra M \rtimes A of an A-module algebra M with A is equal, as a linear space, to the A^L-module tensor product M \otimes_{A^L} A, where the right A^L-module structure of M is defined by \( m \cdot a = m(1) a (2) \), and the *-algebra structure on M \otimes_{A^L} A is given by

\[(2.3) \quad (m \rtimes a)(m' \rtimes a') = m(a(1) \triangleright m') \rtimes a(2) a' \]
\[(2.4) \quad (m \rtimes a)^* = (a^*_1 \triangleright m^*) \rtimes a^*_2 . \]

The crossed product contains M and A as C^*-subalgebras in the form \{ m \rtimes 1 \mid m \in M \} and \{ 1_M \rtimes a \mid a \in A \}, respectively. In the sequel we identify M and A with these subalgebras. In this sense we have, for example, the following relation in the crossed product

\[ hmh = (h(1) \triangleright m) h(2) h = (h(1) \triangleright m) h(2) S(h(3)) h = (1(1) h \triangleright m) 1(2) h \]
\[ = (h \triangleright m) h \]

proving that the conditional expectation h \triangleright : M \rightarrow M^A is implemented by the projection h \in A \subset M \rtimes A. Therefore the basic construction hMh is a subalgebra of the crossed product.

Example 2.1. If A is a C^*-WHA and ˆA its dual then A is a ˆA-module algebra via the Sweedler arrow, ˆϕ\triangleright a = ˆϕ \triangleleft a, ˆϕ \in ˆA, a \in A. The invariant subalgebra is
The crossed product $C^*$-algebra $A \rtimes \hat{A}$ is called the Weyl algebra or Heisenberg double since it is the algebra generated by $A$ (the "momenta") and $\hat{A}$ (the "coordinates") satisfying the generalized Weyl commutation relations
\begin{equation}
\phi a = a_1 \langle \phi (1), a_2 \rangle \phi (2).
\end{equation}
It has been shown in [3] that the basic construction for the inclusion $A_L \subset A$ is precisely the Weyl algebra.

For a Galois action of $A$ on $M$ one requires that the crossed product be equal to the basic construction for $M_A \subset M$. That is to say $M \rtimes A$ is generated as an algebra by $M$ and by the Haar element $h$, being the Jones projection in this case. Thus the above example is a Galois action.

**Example 2.2.** Let $M = A_L$ and let $A$ act on $M$ as the trivial representation: $a \triangleright l = \pi (al)$. Then $A_L$ is a module algebra in the $C^*$-sense. The invariant subalgebra is precisely $Z_L$. Now it is clear that the action of $A$ on $A_L$ is not Galois in general. The crossed product $A_L \rtimes A$ coincides with the $C^*$-algebra $A$. Hence $A_L$ is Galois precisely if $Z_L \subset A_L \subset A$ is a basic construction. Such special WHA’s occur as symmetries of depth 1 inclusions.

### 2.5. Regular actions.

The next definition summarizes our requirements on the $A$-module algebra $M$.

**Definition 2.3.** A module algebra $M$ over the $C^*$-Hopf algebra $A$ is called regular if
\begin{enumerate}
\item $M$ is minimal, i.e., $(M_A)' \cap M = A_L$,
\item $M_A \subset M \subset M \rtimes A$ is a basic construction in the sense of Jones [8],
\item the conditional expectation $h \triangleright \cdot : M \to M_A$ is of finite index in the sense of Watatani [20].
\end{enumerate}

For a regular module algebra $M$ one has complete control over the other relative commutants, too.

\begin{align}
&M' \cap (M \rtimes A) = A_R \\
&(M_A)' \cap (M \rtimes A) = A \\
&\text{Center } M_A = A_L \cap \text{Center } A \\
&\text{Center } M = A_L \cap A_R \\
&\text{Center } (M \rtimes A) = A_R \cap \text{Center } A
\end{align}

The dual WHA $\hat{A}$ acts on the crossed product via the Sweedler arrow on $A$, i.e.,
\begin{equation}
\phi \triangleright (m \times a) := m \times (\phi \rightarrow a), \quad \phi \in \hat{A}, \ m \in M, \ a \in A
\end{equation}
is a left action of $\hat{A}$ on $M \rtimes A$ with the invariant subalgebra being just $M$. The crossed product algebra $(M \rtimes A) \rtimes A$ contains the algebra $(1_M \times A) \rtimes \hat{A}$ isomorphic to the Weyl algebra. It is precisely the relative commutant $(M_A)' \cap ((M \rtimes A) \times \hat{A})$.

### 3. The reconstruction theorem

In this Section we would like to investigate the problem of whether an inclusion $N \subset M$ of unital $C^*$-algebras is isomorphic to the inclusion $M_A \subset M$ of the
invariant subalgebra with respect to a regular action of an appropriate $C^*$-WHA $A$.

If such a WHA-action exists then there must be a conditional expectation $E : M \to N$ of finite index type. Furthermore, $A$ must be isomorphic, as a $C^*$-algebra, to $N' \cap M_2$ where $M_2$ is the basic construction for $N \subset M$. For obtaining information about the coproduct of $A$ one may look at the next member of the Jones tower, $M_3$, because it contains $\hat{A}$ as the relative commutant $M \cap M_3$. The derived tower of the Jones tower over $N \subset M$ is therefore completely known,

\begin{align*}
N &\subset M \subset M_2 \subset M_3 \subset \ldots \\
\bigcup &\bigcup \bigcup \bigcup \\
N' \cap N &\subset N' \cap M \subset N' \cap M_2 \subset N' \cap M_3 \subset \ldots \\
Z^L &\subset A^L \subset A \subset A \rtimes \hat{A}
\end{align*}

The derived tower is again a Jones tower starting from the second item $A^L$ due to the fact that the Weyl algebra is a Jones extension of $A^L \subset A$ [3]. This means, by definition, that $N \subset M$ is of depth 2.

Thus the inclusion $N \subset M$ can be the inclusion of the invariant subalgebra $N = M^A$ w.r.t. a regular $C^*$-WHA action only if it is a finite index depth 2 inclusion of unital $C^*$-algebras with finite dimensional centers. The letter condition comes partly from the first derived tower, saying that Center $N = Z^L \equiv A^L \cap $ Center $A$, partly from the second derived tower, saying that Center $M = A^L \cap A^R \equiv \hat{A}^L \cap $ Center $A$. The above conditions are not only necessary but also sufficient.

**Theorem 3.1.** Let $N \subset M$ be an inclusion of unital $C^*$-algebras. Then the conditions listed below are necessary and sufficient for the existence of a $C^*$-WHA $A$ and a regular action of $A$ on $M$ such that $M^A = N$.

1. There exists a conditional expectation $E : M \to N$ of finite index type [20].
2. $N \subset M$ is of depth 2.
3. Center $N$ (and/or Center $M$) is finite dimensional.

The WHA $A$ is uniquely determined by the inclusion if we require that the restriction of the square of the antipode onto $A^L$ be the identity.

The proof of this Theorem has not been published yet although it was implicitly present in [17]. Meanwhile a proof of an important special case has been appeared in [13] where D. Nikshych and L. Vainerman considered type $\text{II}_1$ von Neumann algebra factors. In [13] they went beyond the depth 2 case and have constructed a Galois correspondence for finite depth inclusions $N \subset M$ of $\text{II}_1$ factors. As I will try to explain below the restriction to factors is not really essential and it just hides the interesting feature of non-trivial vacuum structure.

In this section we shall outline a proof of the above Theorem in the hope of that a detailed proof will be available in the near future.

3.1. **2-categorical generalization.** An inclusion $N \subset M$ of unital $C^*$-algebras is just a special case of of a unit preserving *-homomorphism $N \to M$. Unital $C^*$-algebras are the objects (=0-cells) of a $C^*$-2-category the arrows (=1-cells) of which
are the unit preserving $*$-algebra maps and for parallel arrows $\alpha, \beta: N \to M$ the intertwiners (= 2-cells) from $\alpha$ to $\beta$ are the elements $t \in M$ satisfying the intertwiner relation $\tau t(n) = \beta(n) t$ for all $n \in N$. For example the selfintertwiners of the 0-cell $M$, considered as a special 1-cell $\text{id}_M: M \to M$, are the elements $c$ of $M$ for which $cm = mc$ for all $m \in M$, i.e., $\text{End} M = \text{Center} M$ as a $C^*$-algebra.

In general a 2-category $\mathcal{C}$ consists of 0, 1, and 2-dimensional cells $\mathcal{C}^0 \subset \mathcal{C}^1 \subset \mathcal{C}^2$ and there are two partially defined associative composition laws for 2-cells $s, t \in \mathcal{C}^2$.

- The horizontal composition $s \times t$ is defined for 2-cells $s: \alpha \to \alpha'$ where $\alpha, \alpha': M \to L$ and $t: \beta \to \beta'$ where $\beta, \beta': N \to M$.
- The vertical composition $s \circ t$ is defined for $s: \beta \to \gamma$ and $t: \alpha \to \beta$ where $\alpha, \beta, \gamma: N \to M$ are parallel arrows. The result is $s \circ t: \alpha \to \gamma$.

They obey the interchange law
\[
(s \circ t) \times (s' \circ t') = (s \times s') \circ (t \times t')
\]
for all 2-cells for which the left hand side is defined. The usual convention is to identify the 0-cells $M$ with the arrows $\text{id}_M: M \to M$ serving as a (partial) unit for $\times$ and the 1-cell $\alpha: N \to M$ with the identity intertwiner $1_\alpha: \alpha \to \alpha$ which serves as the (partial) unit for $\circ$. So all units for $\times$ are units for $\circ$, too.

The basic example of a 2-category is $\mathbf{Cat}$, the 2-category of small categories. Its 0-cells are the small categories, the 1-cells are the functors, and the 2-cells are the natural transformations.

A 2-category is called a $C^*$-2-category if the sets $\text{Hom}(\alpha, \beta)$ of 2-cells from $\alpha$ to $\beta$ are Banach spaces for all pairs of parallel 1-cells $\beta, \beta': N \to M$, and if the vertical composition makes $\text{Hom}(\alpha, \alpha)$ into a $C^*$-algebra for each 1-cell $\alpha$. For a precise definition we refer to [11].

The basic example of a $C^*$-2-category is $C^*$-$\mathbf{Alg}$, the category of unital $C^*$-algebras. Its 0-cells are the small unital $C^*$-algebras, the 1-cells are the unit preserving $*$-algebra maps, and the 2-cells are the intertwiners as it has been defined above.

For our purposes the category $C^*$-$\mathbf{Alg}$ may turn out to be too small in the following sense. If $N \subset M$ is an inclusion possessing a conditional expectation $E: M \to N$ of finite index, this means that $E$ has a quasibasis, i.e., a finite set \{\(m_i\)\} of elements of $M$ such that
\[
\sum_{i=1}^n m_i E(m_i^* m) = m, \quad \forall m \in M.
\]
Then one may construct a (2-sided) dual $\iota: M \to N$ of the inclusion map $\iota: N \to M$ as a $*$-algebra map from $M$ to a finite amplification $N \otimes M_n$ of $N$ by the formula
\[
\iota(m) = \sum_{i,j} E(m_i^* mm_j) \otimes e_{ij}
\]
where \{\(e_{ij}\)\} is a set of matrix units of $M_n$. However, $\iota$ belongs to $C^*$-$\mathbf{Alg}$ only for $n = 1$, otherwise it is an arrow from $M$ to $N$ in the larger $C^*$-2-category $C^*$-$\text{amp}$ of ”amplimorphisms”. This category is basically the same as the $C^*$-2-category of finitely generated projective Hilbert bimodules over unital $C^*$-algebras.

We do not need the details about these categories here because we want to switch to a general $C^*$-2-category $\mathcal{C}$ and lift the conditions (1-2-3) of Theorem 3.4 as assumptions on an arrow $\iota$ of $\mathcal{C}$. Then we arrive to the following
Theorem 3.2. Let $\mathcal{C}$ be a $C^*$-2-category and $\iota: N \to M$ be an arrow satisfying the following assumptions.

1. $\iota$ has a dual (or conjugate) $\bar{\iota}: M \to N$ with intertwiners $\bar{\text{R}}: M \to \iota \times \bar{\iota}$ and $\text{R}: N \to \bar{\iota} \times \iota$ satisfying the conjugacy equations
   \[
   (\bar{\text{R}}^* \times \iota) \circ (\iota \times \text{R}) = \iota, \\
   (\bar{\iota} \times \bar{\text{R}}^*) \circ (\bar{\text{R}} \times \iota) = \bar{\iota}.
   \]

2. $\iota$ is of depth 2, i.e., $\iota \times \bar{\iota} \times \iota$ is a direct summand of a finite multiple of $\iota$.

3. $\text{End } N$ is a finite dimensional $C^*$-algebra.

Then $A := \text{End } \iota \times \bar{\iota}$ and $B := \text{End } \bar{\iota} \times \iota$ are finite dimensional $C^*$-algebras and there exists a non-degenerate bilinear form $\langle \cdot, \cdot \rangle: A \times B \to \mathbb{C}$ which makes $A$ and $B$ into $C^*$-WHA’s in duality.

Under assumptions (1) and (2) assumption (3) is equivalent to assuming $\text{End } M$ is finite dimensional.

3.2. Outline of the proof.

3.2.1. Standard rigidity intertwiners. Given a conjugate $\bar{\iota}$ of $\iota$ there is some freedom in choosing the intertwiners $\bar{\text{R}}$ and $\text{R}$. A special class of choices are termed standard. They have been defined in case of $C^*$-categories with irreducible unit [10]. It can be generalized easily to $C^*$-categories with reducible unit $U$ if $U$ decomposes into finitely many irreducibles [3]. The $C^*$-2-category generalization is straightforward.

Let
\[
\iota = \bigoplus_a \bigoplus_{i=1}^{m_a} \iota_{a i}
\]
be the decomposition of $\iota$ into pairwise inequivalent irreducibles $\iota_a: N \to M$, each of them with some multiplicity $m_a$. Let $\bar{\iota}_a$ be a conjugate of $\iota_a$ with rigidity intertwiners $\bar{R}_a: M \to \iota_a \times \bar{\iota}_a$ and $R_a: N \to \bar{\iota}_a \times \iota_a$. Then $\bar{R}_a^* \circ \bar{R}_a$ is a selfintertwiner of $M$ and $R_a^* \circ R_a$ is a selfintertwiner of $N$. They must be proportional to a minimal projection of the Abelian algebras $\text{End } M$ and $\text{End } N$, respectively, since $\iota_a$ is irreducible. Therefore we can multiply $R_a$ and $\bar{R}_a$ with numbers, if necessary, to obtain a choice for which
\[
\bar{R}_a^* \circ \bar{R}_a = d_a \cdot P_{a L}, \quad R_a^* \circ R_a = d_a \cdot P_{a R}
\]
with the same positive number $d_a$ in both equations. Here $P_{a L}$ refers to a minimal (central) projection of $\text{End } M$ or $\text{End } N$ depending on whether the the $\mu$ is a left or a right "vacuum" of $a$. (Later $\text{End } M$ will become the $Z^L$ of the WHA $A$ and $\text{End } N$ that of $B$, so this explains the "vacuum sector" terminology.)

Having been chosen rigidity intertwiners for each $\iota_a$ we are ready to write down the standard rigidity intertwiners for $\iota$. Let
\[
w_{ai}: \iota_a \to \iota, \quad \bar{w}_{ai}: \bar{\iota}_a \to \bar{\iota} \quad i = 1, \ldots m_a
\]
be intertwiners chosen in such a way that
\[
w_{ai}^* \circ w_{bj} = \delta_{ab} \delta_{ij} \quad i = 1, \ldots m_a
\]
be intertwiners chosen in such a way that
\[
\sum_a \sum_{i=1}^{m_a} w_{ai} \circ w_{ai}^* = \iota.
\]
I.e., the $\mathcal{w}$ intertwiners provide a direct sum diagram for (3.5). Then the intertwiners

$$\mathcal{R} = \sum_{a} \sum_{i} (w_{ai} \times \bar{w}_{ai}) \circ \bar{R}_{a} : M \rightarrow \iota \times \bar{\iota}$$

$$R = \sum_{a} \sum_{i} (\bar{w}_{ai} \times w_{ai}) \circ R_{a} : N \rightarrow \bar{\iota} \times \iota$$

are rigidity intertwiners for $\iota$. They will be called the standard rigidity intertwiners. Although they are not unique, depend on the choice of the direct sum diagram, the maps

$$\Psi_{\iota} : \text{End} \iota \rightarrow \text{End} M , \quad \Psi(l) := \bar{R}^{*} \circ (l \times \bar{\iota}) \circ \bar{R}$$

$$\Phi_{\iota} : \text{End} \iota \rightarrow \text{End} N , \quad \Phi(l) := R^{*} \circ (\bar{\iota} \times \iota) \circ R$$

are uniquely determined faithful positive traces, called the standard traces. These traces are of finite index type, i.e., have a quasibasis. Ind $\Psi_{\iota} = \text{Ind} \Phi_{\iota}$. Furthermore,

$$\text{tr}_{M}(\Psi_{\iota}(l)) = \text{tr}_{N}(\Phi_{\iota}(l)) \quad l \in \text{End} \iota$$

where $\text{tr}_{M}$ is the trace on $\text{End} M$ which takes the value 1 on each minimal projector and $\text{tr}_{N}$ is the analogue trace on $\text{End} N$.

In a similar fashion one defines the standard trace $\Psi_{\bar{\iota}} : \text{End} \bar{\iota} \rightarrow \text{End} N$, for example, using the standard rigidity intertwiner $R$. Also we can construct the standard trace $\Psi_{\iota \times \bar{\iota}}$ using the standard rigidity intertwiner

$$\bar{R}_{\iota \times \bar{\iota}} = (\iota \times R \times \bar{\iota}) \circ \bar{R}.$$ 

Finally we will need the standard trace $\Psi_{\iota \times \bar{\iota} \times \iota} : \text{End}(\iota \times \bar{\iota} \times \iota) \rightarrow \text{End} M$. The following abbreviations will be used

$$\Psi_{1} = \Psi_{\iota} , \quad \Psi_{12} := \Psi_{\iota \times \bar{\iota}} , \quad \Psi_{123} := \Psi_{\iota \times \bar{\iota} \times \iota}.$$ 

3.2.2. The pairing. We define the $C^{*}$-algebras $A := \text{End}(\iota \times \bar{\iota})$ and $B := \text{End}(\bar{\iota} \times \iota)$. They are finite dimensional, as all intertwiner spaces are in a $C^{*}$-category the monoidal unit of which has finite dimensional endomorphism algebra. Motivated by the pairing formula of Theorem 4.11 in [8] we make the following Ansatz

$$\langle a, b \rangle = \text{tr}_{M} \circ \Psi_{123}(U_{23}U_{12}b_{3}a_{2}z_{3}), \quad a \in A, b \in B$$

where $U_{12} := \bar{R} \circ \bar{R}^{*}$ and $U_{23} := R \circ R^{*}$ are meant to be embedded into $\text{End}(\iota \times \bar{\iota} \times \iota)$ in the obvious way, just like $a$ and $b$. The pairing also contains the yet undetermined element $z \in \text{End} \iota$ in the form of $z_{1} := z \times \bar{\iota} \times \iota$ and $z_{2} := \iota \times \bar{\iota} \times z$.

If $z \in \text{End} \iota$ is invertible then the bilinear form (3.16) is non-degenerate. This can be shown by proving that the Fourier transform

$$\mathcal{F} : B \rightarrow A , \quad \mathcal{F}(b) := \Psi_{3}(U_{23}U_{12}a_{2}z_{3})$$

is invertible and that the pairing can be written as

$$\langle a, b \rangle = \text{tr}_{M} \circ \Psi_{12}(\mathcal{F}(b)a).$$

By means of the pairing one defines coalgebra structures on $A$ and $B$.

$$\Delta_{A} : A \rightarrow A \otimes A , \quad \langle \Delta(a), b \otimes b' \rangle = \langle a, bb' \rangle$$

$$\varepsilon_{A} : A \rightarrow \mathbb{C} , \quad \varepsilon_{A}(a) = \langle a, 1_{B} \rangle$$

and similar expressions for $B$. Antipodes can be introduced by

$$S_{A}(a) := (a_{\ast})^{*} , \quad S_{B}(b) := (b_{\ast})^{*}.$$
where the lower star operation is the transpose of the upper one,

\begin{equation}
\langle a^*, b \rangle = \overline{\langle a, b^* \rangle}, \quad \langle a, b^* \rangle = \overline{\langle a^*, b \rangle}.
\end{equation}

Explicitely

\begin{align}
\varepsilon_A(a) &= \text{tr}_M(\bar{R}^* \circ (z \times \bar{i}) \circ a \circ (z \times i) \circ R) \\
S_A(b) &= (z \times z^{-1}) \circ \bar{a} \circ (z^{-1} \times z)
\end{align}

where \( a \mapsto \bar{a} \) is the action of the conjugation (left=right duality) functor w.r.t. the standard conjugacy intertwiners. The explicit form of the coproduct can be written only in terms of some quasibasis.

The difficult part will be to prove (vii) using the depth 2 property and an appropriate choice of \( z \), the remaining ones will hold true without any further assumptions.

The formula for \( \varepsilon_A \) shows that \( z \) must be Hermitean in order for \( \varepsilon_A \) to be positive. Assuming this the following properties can now be verified easily:

\begin{enumerate}
\item \( (\Delta_A \otimes \text{id}_A) \circ \Delta_A = (\text{id}_A \otimes \Delta_A) \circ \Delta_A \)
\item \( (\varepsilon_A \otimes \text{id}_A) \circ \Delta_A = \text{id}_A = (\text{id}_A \otimes \varepsilon_A) \circ \Delta_A \)
\item \( S_A(aa') = S_A(a')S_A(a) \)
\item \( \Delta_A \circ S_A = (S_A \otimes S_A) \circ \Delta_A^{op} \)
\item \( * \circ S_A \circ * \circ S_A = \text{id}_A \)
\item \( \Delta_A(a^*) = \Delta_A(a)^* \)
\end{enumerate}

and analogue statements for \( B \).

Comparing these properties with the original C*-WHA axioms of \cite{1} we see that what is missing for \( A \) with \( \Delta_A \) to be a C*-WHA, and \( B \) with \( \Delta_B \) its dual, is the verification of three more axioms:

\begin{enumerate}
\item[(vii)] \( \Delta_A(a)\Delta_A(a') = \Delta_A(aa') \)
\item[(viii)] \( a_{(1)} \otimes a_{(2)}S_A(a_{(3)}) = 1_{(1)}a \otimes 1_{(2)} \)
\item[(ix)] \( \varepsilon_A(aa') = \varepsilon_A(a_{(1)})\varepsilon_A(a_{(2)}) \)
\end{enumerate}

The difficult part will be to prove (vii) using the depth 2 property and an appropriate choice of \( z \), the remaining ones will hold true without any further assumptions.

3.2.3. The depth 2 condition. If \( \iota \times \bar{\iota} \times \iota \) is a direct summand of a finite multiple of \( \iota \) then there exists a finite set of intertwiners \( V_i : \iota \to \iota \times \bar{\iota} \times \iota \) such that \( \sum_i V_i \circ V_i^* \) is the identity at \( \iota \times \bar{\iota} \times \iota \). Introducing

\begin{equation}
v_{i} := (\iota \times \bar{\iota} \times \bar{R}^*) \circ (V_i \times \bar{i}) \in A
\end{equation}

we obtain the relation

\begin{equation}
\sum_i (v_i \times \iota) \circ U_{23} \circ (v_i^* \times \iota) = \iota \times \bar{\iota} \times \iota.
\end{equation}

Hence \( \{v_{i}\} \) is a quasibasis for \( \Psi_2 : \text{End}(\iota \times \bar{\iota}) \to \text{End} \iota \) and \( \text{End}(\iota \times \bar{\iota} \times \iota) \) is the basic construction for \( \text{End} \iota \subseteq \text{End}(\iota \times \bar{\iota}) \).

3.2.4. Multiplicativity of the coproduct. Given a Hermitean invertible \( z \) in the pairing of \( A \) and \( B \) we have the

**Proposition 3.3.** Under the conditions of Theorem 3.3 the following statements are equivalent:

1. \( \Delta_A \) is multiplicative
2. \( \Delta_B \) is multiplicative
3. \( b \mapsto (aa') = (b_{(1)} \mapsto a)(b_{(2)} \mapsto a') \)
4. \( (\iota \times b) \circ (a \times \iota) = ((b_{(1)} \mapsto a) \times \iota) \circ (\iota \times b_{(2)}) \)
(5) if \( f : \bar{\iota} \times \iota \times \bar{\iota} \times \iota \to \bar{\iota} \times \iota \) denotes the "fork" \( \bar{\iota} \times (\bar{\iota}^* \circ (z^{-1} \times \iota)) \times \iota \) then

\[
b \circ f = f \circ (b_{(1)} \times b_{(2)}).
\]

Let \( \{u_i\} \) be an orthonormal basis of \( A \) w.r.t. \( \text{tr} \circ \Psi_{12} \), i.e.,

\[
\text{tr} \circ \Psi_{12}(u_i^* u_j) = \delta_{ij}.
\]

Then \( u^i := F^{-1}(u_i^*) \in B \) is the dual basis of \( \{u_i\} \) w.r.t. the pairing. Hence

\[
\Delta_B(b) = \sum_i (u_i \cdot b) \otimes u^i.
\]

Lemma 3.4. The fork rule (3.27) holds if and only if \( v_i := u_i^* \circ (z^{-1} \times \bar{i}) \) is a quasibasis for \( \Psi_2 : A \to \text{End} \iota \), i.e., if

\[
\sum_i u_i^* z^{-2} U_{23} u_i = 1
\]

holds in \( \text{End} \iota \times \bar{i} \times \iota \).

The \( z' \)s that satisfy all conditions we have are precisely the Hermitean invertible \( z \in \text{End} \iota \) for which

\[
\text{tr}_a z^{-2} = d_a
\]

for all sector \( a \) contained in \( \iota \). If we add the somewhat ad hoc requirement that \( z \) be central in \( \text{End} \iota \) then \( z \) is unique up to a sign in each sector. The \( \pm \) signs can be reabsorbed into the freedom of choosing the standard rigidity intertwiners \( R, \bar{R} \) but non-central \( z \) cannot be made central by this method.

We remark that the \( z \) we use here is related to the canonical grouplike element of the resulting WHA by the formula \( z^2 = \epsilon L \) where the \( g' L \) has been defined in [3].

Although any \( C^* \)-WHA can occur as the symmetry of an inclusion \( N \subset M \) satisfying the conditions of Theorem 3.1, we do not need all of them. Namely, every such inclusion has a uniquely determined \( C^* \)-WHA with the additional property \( S^2 \vert_{A^L} = \text{id}_{A^L} \). Therefore the freedom of having non-trivial \( S^2 \vert_{A^L} \), or equivalently, non-tracial \( \epsilon \vert_{A^L} \) in a WHA is a property which is not utilized in applications to depth 2 inclusions. Thus the meaning of this degree of freedom of WHA’s is still waiting for explanation.

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Research Institute for Particle and Nuclear Physics, Budapest, H-1525 Budapest, P. O. Box 49, Hungary
E-mail address: szlach@rmki.kfki.hu