FRACTIONAL SUPERSYMMETRY ALGEBRA AND LACUNARY HERMITE POLYNOMIALS

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Abstract. We consider a realization of fractional supersymmetry of quantum mechanics of order \( r \), where the Hamiltonian and the supercharges involves reflection operators. It is shown that the Hamiltonian has a \( r \)-fold degenerate spectrum and the eigenvalues of the hermitian supercharges are the zeros of the associated Hermite polynomials of Askey and Wimp and its eigenfunctions involve lacunary Hermite polynomials.

Intertwining operator, Special functions, Dunkl operator, Integral transforms. [2010] Primary 33C60; Secondary 26A33, 43A32

1. Introduction

In [10, 11, 12], the authors formulated a supersymmetry quantum mechanics for one-dimensional systems by using difference-differential operators known in the literature as Dunkl operators [18, 6]. One of its characteristic features is that both a supersymmetric Hamiltonian and a supercharge component involve reflection operators. In addition their related wave functions are expressed in terms of the Hermite orthogonal polynomials, see [10]. The fractional supersymmetric quantum mechanics of order \( r \) is an extension of the ordinary supersymmetric quantum mechanics, introduced by Witten [16]. An ordinary supersymmetric quantum-mechanical system may be generated by three operators \( Q, Q^\dagger \) and \( H \) satisfying

\[
Q^2 = 0 \quad \text{and} \quad QQ^\dagger + Q^\dagger Q = H.
\]

The operator \( H \) is a self-adjoint operator and is referred to as the Hamiltonian operator. The operator \( Q \) gives rise to two dependent supercharges \( Q_- = Q \) and \( Q_+ = Q^\dagger \), which are connected via Hermitian conjugation. They are nilpotent operators of order 2 and commute with the Hamiltonian \( H \).

The framework of the fractional supersymmetric mechanics has been shown to be quite fruitful. Among many works, we may quote the deformed Heisenberg algebra introduced in connection with parafermionic and parabosonic systems [5], the \( C_\lambda \)-extended oscillator algebra developed in the framework of parasupersymmetric quantum mechanics [13], and the generalized Weyl-Heisenberg algebra \( W_k \) related to \( \mathbb{Z}_k \)-graded supersymmetric quantum mechanics [5].

Due to the fact that the case \( r = 2 \) corresponds to ordinary supersymmetry, it is interesting to study what kind of supersymmetric Hamiltonian involving reflection operators admit exact eigenfunctions which are expressible in terms of classical orthogonal polynomial system.

The Hermite polynomials \( H_n(x) \) have the following generating function

\[
\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = e^{2xt-t^2}.
\]
The lacunary Hermite polynomials may be defined by the following generating function [7]

\[ G(r, s) = \sum_{n=0}^{\infty} H_{nr+s}(x) \frac{t^{nr+s}}{(nr+s)!} \]

for arbitrary integers \(2 \leq r\) and \(0 \leq s \leq r - 1\), it is only the specific generating functions

\[ G(1, 0) = e^{-t^2+2xt}, \]
\[ G(2, 0) = e^{-t^2} \cosh(2xt), \]
\[ G(2, 1) = e^{-t^2} \sinh(2xt). \]

They appeared in a number of circumstances, including for example the treatment of Cauchy problems in partial differential equations [9, 2] and in calculations involving coherent and squeezed states [2].

The purpose of the present paper is to study some mathematical properties of the fractional supersymmetry quantum mechanics and investigate such a possibility of realizing \(r = 2\) supersymmetry for the case of fractional supersymmetry can be derived from supercharges involving reflections. In Section 2, we review the definition of the ordinary supersymmetric algebra, and we give some of their realizations. In section 3, we present a realization of the fractional supersymmetric quantum mechanics. Finally, in section 4 we construct a basis involving the lacunary Hermite polynomials that diagonalize simultaneously the Hamiltonian and the supercharge and we show that the eigenvalues of the supercharge are the zeros of the associated Hermite polynomials.

2. ORDINARY SUPERSYMMETRIC QUANTUM MECHANICS

Let us remember the standard algebra of harmonic oscillator:

\[ [a_-, a_+] = 1, \]
\[ [H, a_-] = -a_-, \quad [H, a_+] = a_+ . \]

In the coordinates representation the operators \(H, a_-\) and \(a_+\) are given by

\[ a_- = \frac{1}{\sqrt{2}} \left( \frac{d}{dx} + x \right), \quad a_+ = \frac{1}{\sqrt{2}} \left( -\frac{d}{dx} + x \right), \]

and

\[ H = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} x^2 \]

The eigenvalue problem corresponding to the harmonic oscillator is the differential equation for \(\psi(x)\)

\[ -\frac{1}{2} \frac{d^2\psi(x)}{dx^2} + \frac{1}{2} x^2 \psi(x) = \lambda \psi(x). \]

The wave functions corresponding to the well-known eigenvalue

\[ \lambda_n = n + \frac{1}{2}, \quad n = 0, 1, 2, \ldots \]

are given by

\[ \psi_n(x) = (\sqrt{\pi} n! 2^n)^{-1/2} e^{-x^2/2} H_n(x). \]
Here $H_n(x)$ is the Hermite polynomials of order $n$.

Recall that the ordinary supersymmetric quantum mechanics algebra with two generators $Q$ and $Q^\dagger$, as introduced by Witten, reads [16]

\begin{align*}
Q^2 &= Q^\dagger 2 = 0 \\
[H, Q] &= [H, Q^\dagger] = 0 \\
\{Q, Q^\dagger\} &= H.
\end{align*}

A realization of supersymmetric quantum mechanics is formulated by taking the following supercharges

\begin{align*}
Q_i^- &= \frac{1}{\sqrt{2}} \left( \frac{d}{dx} + U(x) \right) \Pi_i, \quad Q_i^+ &= \frac{1}{\sqrt{2}} \Pi_i \left( -\frac{d}{dx} + U(x) \right), \quad i = 0, 1
\end{align*}

where $U(x)$ is an odd function $U(-x) = -U(x)$, and $\Pi_0$ and $\Pi_1$ are the orthogonal projection operators

\begin{align*}
\Pi_0 &= \frac{1}{2} (1 + R), \quad \Pi_0 = \frac{1}{2} (1 - R).
\end{align*}

Here $R$ is the reflection operator

\begin{align*}
(Rf)(x) = f(-x).
\end{align*}

It is clear that

\begin{align*}
Q_{i-}^\dagger = Q_{i+}
\end{align*}

A straightforward computation shows that

\begin{align*}
Q_i^2 = Q_i^{2} = 0, \quad i = 0, 1,
\end{align*}

and the Hamiltonian $H_i$ defined by (2.10) takes the form

\begin{align*}
H_0 &= -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} U^2 - \frac{1}{2} (\Pi_0 - \Pi_1) \left[ \frac{d}{dx}, U \right], \\
H_1 &= -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} U^2 + \frac{1}{2} (\Pi_0 - \Pi_1) \left[ \frac{d}{dx}, U \right].
\end{align*}

In the particular case when $U(x) = x$, the supercharges $Q_i^-$ and $Q_i^+$ are expressed as

\begin{align*}
Q_i^- &= \frac{1}{\sqrt{2}} \left( \frac{d}{dx} + x \right) (1 - \Pi_i) = a_-(1 - \Pi_i), \\
Q_i^+ &= \frac{1}{\sqrt{2}} (1 - \Pi_i) \left( -\frac{d}{dx} + x \right) = (1 - \Pi_i) a_+.
\end{align*}

Substituting (2.17) and (2.18) into (2.3), we obtain

\begin{align*}
H_0 &= -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} x^2 + \frac{1}{2} (\Pi_0 - \Pi_1), \\
H_1 &= -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} x^2 - \frac{1}{2} (\Pi_0 - \Pi_1).
\end{align*}

It follows

\begin{align*}
H_0|n\rangle &= (n + \frac{1 - (-1)^n}{2}) |n\rangle, \\
H_1|n\rangle &= (n + \frac{1 + (-1)^n}{2}) |n\rangle.
\end{align*}
The spectrum of $H_0$ consist only for the even number starting with zero. Each level is degenerate except for the ground states which is unique and the spectrum of $H_1$ consists only for the odd number and each level is degenerate.

Put $Q_i = \frac{1}{\sqrt{2}}(Q_i - Q_i^+)$, ($i = 0, 1$). From equation (2.14) and (2.9) the hermitian supercharges $Q_i$ shares the property $Q_i^* = Q_i$ and commutes with the supersymmetric Hamiltonian $H_i$, so it can be diagonalized simultaneously with $H_i$.

The supercharges $Q_i$ acts on the wave function as

\begin{align}
Q_i \psi_{2n+i}(x) &= \sqrt{(2n+2+i)/2} \psi_{2n+2+i}(x) \\
Q_i \psi_{2n+i+1}(x) &= \sqrt{(2n+1+i)/2} \psi_{2n+1+i}(x).
\end{align}

It is readily seen that the states

\begin{equation}
\psi_{n \varepsilon}^{(i)}(x) = \frac{1}{2}(\psi_{2n+i}(x) + \varepsilon \psi_{2n+i+1}(x)), \quad \varepsilon = \pm 1.
\end{equation}

obey

\begin{equation}
Q_i \psi_{n \varepsilon}^{(i)}(x) = \varepsilon \sqrt{(2n + 2 + i)/2} \psi_{2n+i\varepsilon}(x).
\end{equation}

The eigenfunctions $\psi_{n \varepsilon}^{(i)}(x)$ of the supercharges $Q_i$ which is defined in equation (2.26) involve the lacunary Hermite polynomials $H_{2n}(x)$ and $H_{2n+1}(x)$.

### 2.1. Associated Laguerre and Hermite polynomials

The Hermite polynomials satisfy the three terms recursion relation

\begin{align}
2xH_n(x) &= H_{n+1}(x) + 2(n+1)H_{n-1}(x) \quad (n > 0), \\
H_{-1}(x) &= 0 \quad H_0(x) = 1.
\end{align}

The associated Hermite polynomials $H_n(x,c)$ are defined by the following three terms recurrence relation

\begin{align}
2xH_n(x,c) &= H_{n+1}(x,c) + 2(n+1 + c)H_{n-1}(x,c) \quad (n > 0), \\
H_{-1}(x,c) &= 0 \quad H_0(x,c) = 1,
\end{align}

They were introduced by Askey and Wimp in [1], where their weight functions and explicit formulas were also found. When $c > -1$, they satisfy the orthogonality relations

\begin{equation}
\int_{-\infty}^{\infty} \frac{H_n(x,c)H_m(x,c)}{|D_{-c}(ix\sqrt{2})|^2} dx = 2^n \sqrt{\pi} \Gamma(n+c+1) \delta_{nm}.
\end{equation}

From the three terms recurrence relations (2.28) and (2.27) we see that the first $r$ associated Hermite polynomial $H_0(x,nr)$, ..., $H_{r-1}(x,nr)$ are related to the lacunary Hermite polynomials by

\begin{equation}
H_s(x,nr) = H_{nr+s}(x), \quad s = 0, 1, \ldots r - 1.
\end{equation}

Note that the polynomials $H_n(x,c)$ can be expressed in terms of the associated Laguerre polynomials $L_n^\nu(x,c)$ and $L_n^\nu(c,x)$, which are defined by the three terms recurrence relations [14]

\begin{align}
(2n + 2c + \nu + 1 - x)L_n^\nu(x,c) &= (n+c+1)L_{n+1}^\nu(x,c) + (n+c+\nu)L_{n-1}^\nu(x,c) \\
L_0^\nu(x,c) &= 1, \quad L_1^\nu(x,c) = \frac{2c + \nu + 1 - x}{c + 1}
\end{align}
and
\[(2.33) \quad (2n + 2c + \nu + 1 - x)L_n^\nu(x, c) = (n + c + 1)L_{n+1}^\nu(x, c) + (n + c + \nu)L_{n-1}^\nu(x, c)\]

\[(2.34) \quad L_0^\nu(x, c) = 1, \quad L_1^\nu(x, c) = \frac{2c + \nu + 1 - x}{c + 1}\]

and have the orthogonality relations
\[(2.35) \quad \int_0^\infty L_n^\nu(x, c)L_m^\nu(x, c) x^{\nu-1} e^{-x} \frac{|\psi(c, 1 - \nu; xe^{-i\pi}|)^2}{\Gamma(c + 1)\Gamma(\nu + c + 1)} = \frac{(\nu + c + 1)_n}{(\nu + 1)_n} \delta_{nm}\]

\[(2.36) \quad \int_0^\infty L_n^\nu(x, c)L_m^\nu(x, c) x^{\nu-1} e^{-x} \frac{|\psi(-\nu, xe^{-i\pi}|)^2}{\Gamma(c + 1)\Gamma(\nu + c + 1)} = \frac{(\nu + c + 1)_n}{(\nu + 1)_n} \delta_{nm}\]

Namely, we have [14]
\[(2.37) \quad H_{2n}^\nu(x, c) = (-4)^n(1 + c/2)_nL_n^{1/2}(x^2, c/2),\]
\[(2.38) \quad H_{2n+1}^\nu(x, c) = 2x(-4)^n(1 + c/2)_nL_n^{-1/2}(x^2, c/2).\]

The first few polynomials are
\[(2.39) \quad H_0(x, c) = 1, \quad H_1(x, c) = 2x,\]
\[(2.40) \quad H_2(x, c) = 4x^2 - 2(c + 1), \quad H_3(x, c) = 8x^3 - 4(2c + 3)x.\]

Furthermore,
\[(2.41) \quad H_n(x, c) = \sum_{k=0}^{[n/2]} \frac{(-2)^k(c)_k(n - k)!}{k!(n - 2k)!} H_{n-2k}(x).\]

The associated Laguerre and Hermite polynomials are also birth and death process polynomials studied by Ismail, Letessier and Valent [14].

3. Realization of fractional supersymmetric quantum mechanics

First of all, we review the fractional supersymmetry algebra of order \(r\). Let \(H\) be a Hamiltonian; it said to be supersymmetric of order \(r\), if there are supercharges \(Q_-\) and \(Q_+\) such that the superalgebra relations
\[(3.1) \quad Q_- = Q_+ = 0, \quad [H, Q_-] = [H, Q_+] = 0, \quad Q_+^r = Q_+,
\[(3.2) \quad Q_-^r + Q_+^{-1}Q_+Q_- + Q_+ + \cdots + Q_-Q_+Q_-^{-1} + Q_+Q_- = (r - 1)Q_-^{-1}H.\]

Note that the operators \(a_-\) and \(a_+\) satisfying (2.1) may be realized in \(\mathcal{F}\) as
\[(3.3) \quad a_-|n> = \sqrt{n}|n - 1> \quad \text{and} \quad a_+|n> = \sqrt{n+1}|n + 1>,\]
where \(\mathcal{F}\) is the Fock space, which is generated by the complete set of orthonormal vectors \(\{|n>\}\), constructed over the vacuum state defined by the relations:
\[(3.4) \quad a|0> = 0.\]
The number operator $N$, which counts the number of particles, or objects in general in a system, is defined by $N = a^\dagger a$ and is represented as

$$N |n\rangle = n |n\rangle.$$  

It satisfies the commutation relations

$$[a_-, N] = a_-, \quad [a_+, N] = -a_+.$$

Now, let $r$ be a fixed integer such that $r \geq 2$ and $K$ be an operator defined by the relations

$$K^r = 1, \quad a_- K = \varepsilon_r K a_-, \quad a_+ K = \varepsilon_r^{-1} K a_+, \quad K^* = K^{-1}.$$  

where $\varepsilon_r = e^{2\pi i/r}$ is a primitive root of unity. The operator $K$ can be realized by using the number operator $N$ in the form

$$K = e^{2\pi i r N / r},$$

and acts on the space $\mathcal{F}$ as

$$K |n\rangle = \varepsilon_r^n |n\rangle,$$

and so introduces $\mathbb{Z}_r$-grading structure on the Fock space $\mathcal{F}$ as

$$\mathcal{F} = \bigoplus_{j=0}^{r-1} \mathcal{F}_j,$$

where

$$\mathcal{F}_j = \{| nr + j \rangle : n = 0, 1, \ldots \}.$$  

For $j = 0, 1, \ldots, r - 1$, we denote by $\Pi_j$, the orthogonal projection from $\mathcal{F}$ onto its subspace $\mathcal{F}_j$, which can be represented as

$$\Pi_j = \frac{1}{r} \sum_{l=0}^{r-1} \varepsilon^{-lj}_r K^l.$$  

Equivalently, the reflections $K^l$ are expressed in terms of $\Pi_j$ by

$$K^l = \sum_{j=0}^{r-1} \varepsilon^{-lj}_r \Pi_j, \quad l = 0, \ldots, r - 1.$$  

It is clear that they form a system of resolution of the identity

$$\Pi_0 + \Pi_1 + \cdots + \Pi_{r-1} = 1 \quad \text{and} \quad \Pi_i \Pi_j = \delta_{ij} \Pi_i.$$  

The action of $\Pi_j$ on $\mathcal{F}$ can be taken to be

$$\Pi_j |kr + l\rangle = \delta_{jl} |kr + l\rangle.$$  

3.1. Supercharges. Let $r$ and $j$ be two fixed integers such that $r \geq 2$ and $0 \leq j \leq r - 1$. Following [5], one can construct two supercharges $Q_{j-}$ and $Q_{j+}$ by means of the orthogonal projections $\Pi_k$ and the creation $a_+$ and annihilation $a_-$ operators introduced before as follows

$$Q_{j-} = a_- (1 - \Pi_j) = (1 - \Pi_{j-1}) a_-, \quad Q_{j+} = a_+ (1 - \Pi_{j-1}) = (1 - \Pi_j) a_+.$$  

Obviously, the operators $a_-, a_+, Q_{j-}$ and $Q_{j+}$ satisfy the intertwining relations

$$\Pi_{j-1} a_- = a_- \Pi_j \quad \text{and} \quad a_+ \Pi_{j-1} = \Pi_j a_+$$

$$Q_{j-} a_- = a_- Q_{j+} \quad \text{and} \quad a_+ Q_{j+} = Q_{j+1} a_+.$$  

"
It is easy to see that the supercharges $Q_{j-}$ and $Q_{j+}$ have the Hermitian conjugation relations

\begin{equation}
Q_{j-}^* = Q_{j+},
\end{equation}

Furthermore, the operators $a_-$ and $a_+$ can be developed as

\begin{equation}
a_- = \frac{1}{r-1} \sum_{j=0}^{r-1} Q_{j-} \quad \text{and} \quad a_+ = \frac{1}{r-1} \sum_{j=0}^{r-1} Q_{j+}.
\end{equation}

Now, by making use of the relations (3.14) and (3.15), we easily obtain

\begin{equation}
Q_j^k = \begin{cases} 
    a_k (1 - \Pi_j - \ldots - \Pi_j k) & \text{if } k \leq r - 1 - j \\
    a_k (1 - \Pi_j - \ldots - \Pi_{N-1} - \cdots - \Pi_{k+j-r-1}) & \text{if } r - j \leq k \leq r - 1.
\end{cases}
\end{equation}

In particular for $k = r$, we get

\begin{equation}
Q_j^r = a^r (1 - \Pi_j - \ldots - \Pi_{r-1} - \cdots - \Pi_{j-1}) = 0.
\end{equation}

Similarly, we have \( Q_j^r = 0 \).

Hence, the supercharge operators $Q_{j+}$ are nilpotent operators of order $r$.

3.2. Supersymmetric Hamiltonian of order $r$. We can use the operators $a_-$, $a_+$ and the orthogonal projections $\Pi_0$, $\Pi_1$, $\ldots$, $\Pi_{r-1}$ introduced before for obtaining a realization of supersymmetric Hamiltonian. We first summarize the key properties

\begin{equation}
Q_j a_- = a_- a_j \Pi_{j-2},
\end{equation}

\begin{equation}
Q_j^r Q_j^r = a^r a_j \Pi_{r-1},
\end{equation}

\begin{equation}
Q_j^{r-1-k} Q_j^k = a^{r-1-k} a_j \Pi_{r-2} + (1 + \frac{r}{2}) \Pi_{j-1}.
\end{equation}

From the following well-known relation $a^r a_j + a_+ a_- k a^{k-1}$ and relations (3.31), (3.20) and (3.21) lead to

\begin{equation}
\sum_{k=0}^{r-1} Q_j^{r-1-k} Q_j^k = (r-1) Q_j^{r-2} \left[ a_+ a_- + (2 - \frac{r}{2}) \Pi_{j-2} (r) + (1 - \frac{r}{2}) \Pi_{j-1} \right].
\end{equation}

On the other hand from (3.10), we see that

\begin{equation}
Q_j^{r-2} \sum_{k=2}^{r-1} \left( 1 + k - \frac{r}{2} \right) \Pi_{r+j-k-1} = 0
\end{equation}

This yields

\begin{equation}
\sum_{k=0}^{r-1} Q_j^{r-1-k} Q_j^k = (r-1) Q_j^{r-2} H_j,
\end{equation}

where, the operator $H_j$ is given explicitly by

\begin{equation}
H_j = a_+ a_- + \sum_{k=0}^{r-1} \left( 1 + k - \frac{r}{2} \right) \Pi_{r+j-k-1}.
\end{equation}

It remains to prove that

\begin{equation}
[H, Q_{j-}] = [H, Q_{j+}] = 0.
\end{equation}
Indeed, from (3.21), we have

\[(3.27) \quad Q_j - H = a_- a_+ a_- (1 - \Pi_j) + \sum_{k=0}^{r-2} (1 + k - \frac{r}{2}) a_- (1 - \Pi_j) \Pi_{r+j-k-1} \]

\[(3.28) = a_+ a_- a_- (1 - \Pi_j) + \sum_{k=0}^{r-2} (2 + k - \frac{r}{2}) \Pi_{r+j-k-2} (1 - \Pi_{j-1}) a_- \]

\[(3.29) = a_+ a_- a_- (1 - \Pi_j) + \sum_{k=1}^{r-1} (1 + k - \frac{r}{2}) \Pi_{r+j-k-1} (1 - \Pi_{j-1}) a_- \]

\[(3.30) = HQ_{j-}. \]

Thus

\[(3.31) \quad [H_j, Q_j - ] = 0 \]

Since the orthogonal projection \(\Pi_k\) is hermitian conjugate \(\Pi_k^* = \Pi_k\), we easily check that the operators \(H_j\) share that property

\[(3.32) \quad H_j^* = H_j. \]

Combining this with (3.31), we get

\[(3.33) \quad [H_j, Q_j^+] = 0 \]

Given the representation (3.31), \(Q_j^-\) and \(Q_j^+\) take the value

\[(3.34) \quad Q_j^- |nr + j + s\rangle = \sqrt{nr + j + s} |nr + j + s - 1\rangle, \quad s = 1, \ldots, r - 1, \]

\[(3.35) \quad Q_j^+ |nr + j + s\rangle = \sqrt{nr + j + s + 1} |nr + j + s + 1\rangle, \quad s = 0, \ldots, r - 2, \]

\[(3.36) \quad Q_j^- |nr + j\rangle = 0, \quad Q_j^+ |(n+1)r + j - 1\rangle = 0. \]

and

\[(3.37) \quad H_j | nr + j + s\rangle = (rn + r/2 + j) | nr + j + s\rangle. \]

These show that the Hamiltonian \(H_j\) has a \(r\)-fold degenerate spectrum.

4. SIMULTANEOUS DIGITALIZATION

Notice that the hermitian supercharge operator \(Q_j\) defined by

\[(4.1) \quad Q_j = \frac{1}{\sqrt{2}} (Q_j^- + Q_j^+). \]

We have

\[(4.2) \quad Q_j |nr + j + s\rangle = a_s(r, n, j) |nr + j + s - 1\rangle + a_{s+1}(r, n, j) |nr + j + s + 1\rangle, \]

\[(4.3) \quad Q_j |nr + j\rangle = a_1(r, n, j) |nr + j + 1\rangle, \]

\[(4.4) \quad Q_j |(n+1)r + j - 1\rangle = a_{r-1}(r, n, j) |(n+1)r - 2\rangle, \]

where

\[a_s(r, n, j) := \sqrt{(nr + j + s)/2}, \quad s = 1, \ldots, r - 1. \]

We see that the operator \(Q_j\) leaves the subspace generated by the states \(|nr + j + s\rangle, s = 0, 1, \ldots, r - 1\) invariant. Hence, \(Q_j\) can be represented by the following \(r \times r\)-tridiagonal
Jacobi matrix $A_r$

\[
A_r = \begin{pmatrix}
0 & a_1(r, n, j) & 0 & \cdots & 0 \\
a_1(r, n, j) & 0 & a_2(r, n, j) & \cdots & 0 \\
0 & a_2(r, n, j) & 0 & \cdots & a_{r-1}(r, n, j) \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & 0 \\
\end{pmatrix}.
\]

It is well known that if the coefficients of the subdiagonal of some Jacobi Matrix are different from zero, then all the eigenvalues of this matrix are real and nondegenerate. In what follows we will assume that the eigenvalues are ordered as follows from the smallest to the largest:

\[(4.5) \quad x_0(r, n, j) < x_1(r, n, j) < \cdots < x_{r-1}(r, n, j).\]

Introduce the normalized eigenvectors $| n, j, s \rangle$ of the supercharge $Q_j$

\[(4.6) \quad Q_j | n, j, s \rangle = x_s(r, n, j) | n, j, s \rangle, \quad s = 0, \ldots, r - 1.\]

The vector $| n, j, s \rangle$ can be expanded in the basis $| nr + j + s \rangle$, $s = 0, 1, \ldots, r - 1$, as

\[(4.7) \quad | n, j, s \rangle = \sum_{k=0}^{r-1} \sqrt{w_s P_k(x_s(r, n, j))} | nr + j + k \rangle,
\]

with $P_0(x) = 1$.

It follows from (4.7) and (4.2) that the coefficients $P_k(x)$ obey the three term recurrence relation

\[(4.8) \quad a_k(r, n, j) P_{k-1}(x) + a_{k+1}(r, n, j) P_{k+1}(x) = x P_k(x)\]

and hence are orthogonal polynomials with the initial condition $P_{-1}(x) = 0$ added. The monic orthogonal polynomials $\tilde{P}_k(x)$ related to $P_k(x)$ satisfy the three term recurrence relation

\[(4.9) \quad x \tilde{P}_k(x) = \tilde{P}_{k+1}(x) + 2(nr + j + k) \tilde{P}_{k-1}(x),
\]

\[(4.10) \quad \tilde{P}_{-1}(x) = 0, \quad \tilde{P}_0(x) = 1.\]

The polynomials $\tilde{P}_k(x)$ satisfying (4.9) are known to be the associated Hermite polynomials $H_k(x/2, c)$ with $c = nr + j$. Namely we have

\[(4.11) \quad \tilde{P}_k(x) = H_k(x/2, nr + j), \quad k = 0, 1, \ldots, r - 1.\]

It is well-known from the theory of orthogonal polynomials [3] that the eigenvalues $x_s(r, n, j)$ of the supercharge $Q_j$ are the roots of the characteristic polynomial $H_{(n+1)r+j}(x/2, nr + j)$

\[(4.12) \quad H_{(n+1)r+j}(x/2, nr + j) = (x - x_0(r, n, j)) \cdots (x - x_{r-1}(r, n, j))\]

and the discrete weights $w_k$ can be expressed as

\[(4.13) \quad w_k = \frac{2(2(nr + j + 1), c)^{1/2}}{2r^2 H_{r-1}(x_k(r, n, j)/2, nr + j) H_r(x_k(r, n, j)/2, nr + j)}\]

Using this, we get

\[(4.14) \quad \langle x | n, s \rangle = e^{-x^2/2} \sum_{k=0}^{r-1} \sqrt{w_k} H_k(x_s(r, n, j)/2, nr + j) H_{n+N+k}(x),\]
where

\[
\varrho_k = \sqrt{\frac{\pi}{(2^k (nr + j + 1)_k)^{1/2}}}
\]

\[
(4.15)
\]

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