Bootstrapped Pivots for Sample and Population Means and Distribution Functions

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Abstract

In this paper we introduce the concept of bootstrapped pivots for the sample and the population means. This is in contrast to the classical method of constructing bootstrapped confidence intervals for the population mean via estimating the cutoff points via drawing a number of bootstrap sub-samples. We show that this new method leads to constructing asymptotic confidence intervals with significantly smaller error in comparison to both of the traditional $t$-intervals and the classical bootstrapped confidence intervals. The approach taken in this paper relates naturally to super-population modeling, as well as to estimating empirical and theoretical distributions.

1 Introduction

Unless stated otherwise, $X, X_1, \ldots$ throughout are assumed to be independent random variables with a common distribution function $F$ (i.i.d. random variables), mean $\mu := E_X X$ and variance $0 < \sigma^2 := E_X (X - \mu)^2 < +\infty$. Based on $X_1, \ldots, X_n$, a random sample on $X$, for each integer $n \geq 1$, define

$$X_n := \frac{1}{n} \sum_{i=1}^{n} X_i$$
$$S_n^2 := \frac{1}{n} \sum_{i=1}^{n} (X_i - X_n)^2$$

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the sample mean and sample variance, respectively, and consider the classical Student $t$−statistic

$$
T_n(X) := \frac{\bar{X}_n}{S_n/\sqrt{n}} = \frac{\sum_{i=1}^{n} X_i}{S_n/\sqrt{n}}. \tag{1.1}
$$

that, in turn, on replacing $X_i$ by $X_i - \mu$, $1 \leq i \leq n$, yields

$$
T_n(X - \mu) := \frac{\bar{X}_n - \mu}{S_n/\sqrt{n}} = \frac{\sum_{i=1}^{n} (X_i - \mu)}{S_n/\sqrt{n}}, \tag{1.2}
$$

the classical Student pivot for the population mean $\mu$.

Define now $T_{mn}^*$ and $G_{mn}^*$, randomized versions of $T_n(X)$ and $T_n(X - \mu)$, respectively, as follows:

$$
T_{mn}^* := \frac{\sum_{i=1}^{n} \left( \frac{w_{(n)}^{(i)} - \frac{1}{n}}{m_n} \right) X_i}{S_n \sqrt{\sum_{i=1}^{n} \left( \frac{w_{(n)}^{(i)} - \frac{1}{n}}{m_n} \right)^2}} = \frac{\sum_{i=1}^{n} \left( \frac{w_{(n)}^{(i)} - \frac{1}{n}}{m_n} \right) (X_i - \mu)}{S_n \sqrt{\sum_{i=1}^{n} \left( \frac{w_{(n)}^{(i)} - \frac{1}{n}}{m_n} \right)^2}}, \tag{1.3}
$$

$$
G_{mn}^* := \frac{\sum_{i=1}^{n} \left| \frac{w_{(n)}^{(i)} - \frac{1}{n}}{m_n} \right| (X_i - \mu)}{S_n \sqrt{\sum_{i=1}^{n} \left( \frac{w_{(n)}^{(i)} - \frac{1}{n}}{m_n} \right)^2}}, \tag{1.4}
$$

where, the weights $(w_1^{(n)}, \ldots, w_n^{(n)})$ have a multinomial distribution of size $m_n := \sum_{i=1}^{n} w_i^{(n)}$ with respective probabilities $1/n$, i.e.,

$$(w_1^{(n)}, \ldots, w_n^{(n)}) \overset{d}{=} \text{multinomial}(m_n; \frac{1}{n}, \ldots, \frac{1}{n}).$$

The just introduced respective randomized $T_{mn}^*$ and $G_{mn}^*$ versions of $T_n(X)$ and $T_n(X - \mu)$ can be computed via re-sampling from the set of indices $\{1, \ldots, n\}$ of $X_1, \ldots, X_n$ with replacement $m_n$ times so that, for each $1 \leq i \leq n$, $w_i^{(n)}$ is the count of the number of times the index $i$ of $X_i$ is chosen in this re-sampling process.

**Remark 1.1.** In view of the preceding definition of $w_i^{(n)}$, $1 \leq i \leq n$, they form a row-wise independent triangular array of random variables such that $\sum_{i=1}^{n} w_i^{(n)} = m_n$ and, for each $n \geq 1$,

$$(w_1^{(n)}, \ldots, w_n^{(n)}) \overset{d}{=} \text{multinomial}(m_n; \frac{1}{n}, \ldots, \frac{1}{n}).$$
i.e., the weights have a multinomial distribution of size \( m_n \) with respective probabilities \( 1/n \). Clearly, for each \( n \), \( w_i^{(n)} \) are independent from the random sample \( X_i \), \( 1 \leq i \leq n \). Weights denoted by \( w_i^{(n)} \) will stand for triangular multinomial random variables in this context throughout.

We note that our approach to the bootstrap is to benefit (cf. Corollary 2.1 and Remark 2.1) from the refinement provided by the respective scalings \( \left| \frac{w_i^{(n)} - \frac{1}{n}}{S_n \sqrt{\sum_{i=1}^{n} \left( \frac{w_i^{(n)} - \frac{1}{n}}{m_n} \right)^2}} \right| \) of \( \left( X_i - \mu \right) \)'s and \( \left( \frac{w_i^{(n)} - \frac{1}{n}}{m_n} \right) \) of \( X_i \)'s in \( G_{m_n}^* \) and \( T_{m_n}^* \) which result from the re-sampling, as compared to \( 1/(S_n \sqrt{n}) \) of the classical central limit theorem (CLT) simultaneously for \( T_n(X) \) and \( T_n(X - \mu) \). Viewed as pivots, \( G_{m_n}^* \) and \( T_{m_n}^* \) are directly related to the respective parameters of interest, i.e., the population mean \( \mu = E_X X \) and the sample mean \( \bar{X}_n \), when, e.g., it is to be estimated as the sample mean of an imaginary random sample from an infinite super-population. This approach differs fundamentally from the classical use of the bootstrap in which re-sampling is used to capture the sampling distribution of the pivot \( T_n(X - \mu) \) for \( \mu \) using \( T_{m_n}^* \) that, by definition, is no longer related to \( \mu \) (cf. Section 4).

In the literature of the bootstrap, the asymptotic behavior of \( T_{m_n}^* \) is usually studied by conditioning on the observations on \( X \). In this paper we use the method of first conditioning on the weights \( w_i^{(n)} \) for both \( T_{m_n}^* \) and \( G_{m_n}^* \), and then conclude also their limit law in terms of the joint distribution of the observations and the weights. We use the same approach for studying the asymptotic laws of the sample and population distribution functions.

For similarities and differences between the two methods of conditioning, i.e., on the observations and on the weights, for studying the convergence in distribution of \( T_{m_n}^* \), we refer to Csörgő et al. [2].

**Notations.** Let \((\Omega_X, \mathcal{F}_X, P_X)\) denote the probability space of the random variables \( X, X_1, \ldots, \) and \((\Omega_w, \mathcal{F}_w, P_w)\) be the probability space on which the weights \((w_1^{(1)}, w_1^{(2)}, w_2^{(2)}), \ldots, (w_1^{(n)}, \ldots, w_n^{(n)}), \ldots)\) are defined. In view of the independence of these two sets of random variables, jointly they live on the direct product probability space \((\Omega_X \times \Omega_w, \mathcal{F}_X \otimes \mathcal{F}_w, P_{X,w} = P_X P_w)\). For each \( n \geq 1 \), we also let \( P_{1|w}(\cdot) \) and \( P_{1|X}(\cdot) \) stand for the conditional probabilities given \( \mathcal{F}_w := \sigma(w_1^{(n)}, \ldots, w_n^{(n)}) \) and \( \mathcal{F}_X := \sigma(X_1, \ldots, X_n) \), respectively, with corresponding conditional expected values \( E_{1|w}(\cdot) \) and \( E_{1|X}(\cdot) \).

We are to outline now our view of \( T_{m_n}^* \) and \( G_{m_n}^* \) in terms of their related roles in statistical inference as they are studied in this exposition.
To begin with, $T_{m_n}^*$ is one of the well studied bootstrapped versions of the Student $t$-statistic $T_n(X)$ for constructing bootstrap confidence intervals for $\mu$. Unlike $T_n(X)$ however, that can be transformed into $T_n(X - \mu)$, the Student pivot for $\mu$ as in (1.2), its bootstrapped version $T_{m_n}^*$ does not have this straightforward property, i.e., it does not yield a pivotal quantity for the population mean $\mu = E_XX$ by simply replacing each $X_i$ by $X_i - \mu$ in its definition. Indeed, it takes considerably more effort to construct and establish the validity of bootstrapped $t$-intervals for $\mu$ by means of approximating the sampling distribution of $T_n(X - \mu)$ via that of $T_{m_n}^*$. For references and recent contributions along these lines we refer to Csörgő et al. [2] and to Section 7 of this exposition.

Along with other results, Section 2 contains the main objective of this paper which is to show that when $E_X|X|^r < +\infty$, $r \geq 3$, then the confidence bound for the population mean $\mu$, $G_{m_n}^* \leq z_{1-\alpha}$, where $z_{1-\alpha}$ is the $(1 - \alpha)$th percentile of a standard normal distribution, converges to its nominal probability coverage $1 - \alpha$ at a significantly better rate of at most $O(n^{-1})$ (cf. Theorem 2.1 Corollary 2.1 and Remark 2.1) than that of the traditional $t$-interval $T_n(X - \mu) \leq z_{1-\alpha}$ which is of order $O(n^{-1/2})$. This improvement results from incorporating the additional randomness provided by the weights $w_i(n)$. In fact, it is shown in Section 2 that, when $E_X|X|^r < +\infty$, $r \geq 3$, the conditional distribution functions, given $w_i(n)$’s, of both $G_{m_n}^*$ and $T_{m_n}^*$ converge to the standard normal distribution at a rate that is at best $O(n^{-1})$. A numerical study of the results of Section 2 is also presented there.

In Section 3, $G_{m_n}^*$ is introduced as a natural direct asymptotic pivot for the population mean $\mu = E_XX$, while in Section 4, $T_{m_n}^*$ is studied on its own as a natural pivot for the sample mean $\bar{X}_n$. Furthermore, in Section 5, both $G_{m_n}^*$ and $T_{m_n}^*$ will be seen to relate naturally to sampling from a finite population that is viewed as an imaginary random sample of size $N$ from an infinite super-population (cf., e.g., Hartley and Sielken Jr. [8]), rather than to sampling from a finite population of real numbers of size $N$.

Confidence intervals (C.I.’s) are established in a similar vein for the empirical and theoretical distributions in Section 6.

In Section 7 we compare the rate at which the actual probability coverage of the C.I. $G_{m_n}^* \leq z_{1-\alpha}$ converges to its nominal level, to the classical bootstrap C.I. for $\mu$, in which $T_n(X - \mu)$ is the pivot and the cutoff points are estimated by the means of $T_{m_n}^*(1), \ldots, T_{m_n}^*(B)$ which are computed versions of $T_{m_n}^*$, based on $B$ independent bootstrap sub-samples of size $m_n$. In this
paper we specify the phrase “classical bootstrap C.I.” for the aforementioned C.I.’s based on bootstrapped cutoff points.

In Section 7 we also introduce a more accurate version of the classical bootstrap C.I. that is based on a fixed number of bootstrap sub-samples $B$. In other words, for our version of the classical bootstrap C.I. to converge to its nominal probability coverage, $B$ does not have to be particularly large. This result coincides with one of the results obtained by Hall [7] in this regard. Also, as it will be seen in Section 8 when $B = 9$ bootstrap sub-samples are drawn, in our version of the classical bootstrap, the precise nominal coverage probability 0.9000169 coincides with the one of level 0.9 proposed in Hall [7] for the same number of bootstrap replications $B = 9$. Our investigation of the classical bootstrap C.I.’s for $\mu$ in Section 7 shows that the rate at which their actual probability coverage approaches their nominal one is no better than that of the classical CLT, i.e., $O(n^{-1/2})$.

Section 8 is devoted to presenting a numerical comparison between the three methods of constructing C.I.’s for the population mean $\mu = E_X X$, namely the C.I. based on our bootstrapped pivot $G_{mn}^*$, the traditional C.I. based on the pivot $T_n (X - \mu)$ and the classical bootstrap C.I.. As it will be seen in Section 8 the use of the bootstrapped pivot $G_{mn}^*$ tends to significantly outperform the other two methods by generating values closer to the nominal coverage probability more often.

The proofs are given in Section 9.

2 The rate of convergence of the CLT’s for $G_{mn}^*$ and $T_{mn}^*$

The asymptotic behavior of the bootstrapped $t$-statistics has been extensively studied in the literature. A common feature of majority of the studies that precede paper [2] is that they were conducted by means of conditioning on a given random sample on $X$. In Csörgő et al. [2], the asymptotic normality of $T_{mn}^*$, when $0 < \sigma^2 := E_X (X - \mu)^2 < +\infty$, was investigated via introducing the approach of conditioning on the bootstrap weights $w_i^{(n)}$’s (cf. (4.4) in this exposition), as well as via revisiting the method of conditioning on the sample when $X$ is in the domain of attraction of the normal law, possibly with $E_X X^2 = +\infty$. Conditioning on the weights, the asymptotic normality of the direct pivot $G_{mn}^*$ for the population mean $\mu$, when $0 < \sigma^2 < +\infty$, can be proven by simply replacing $(\frac{w_i^{(n)}}{m_n} - \frac{1}{n})$ by $|\frac{w_i^{(n)}}{m_n} - \frac{1}{n}|$ in the proof of part (a) of Corollary 2.1 in paper [2], leading to concluding (3.5) as $n, m_n \to +\infty$
so that \( m_n = o(n^2) \).

One efficient tool to control the error when approximating the distribution function of a statistic with that of a standard normal random variable is provided by Berry-Esseen type inequalities (cf. for example Serfling [10]), which provides an upper bound for the error for any finite number of observations in hand. It is well known that, as the sample size \( n \) increases to infinity, the rate at which the Berry-Esseen upper bound vanishes is \( n^{-1/2} \).

Our Berry-Esseen type inequality for the respective conditional, given \( w_i^{(n)} \)'s, distributions of \( G_m^* \) and \( T_m^* \), as in [13] and [14] respectively, reads as follows.

**Theorem 2.1.** Assume that \( E_X|X|^3 < +\infty \) and let \( \Phi(.) \) be the standard normal distribution function. Also, for arbitrary positive numbers \( \delta, \varepsilon \), let \( \varepsilon_1, \varepsilon_2 > 0 \) be so that \( \delta > (\varepsilon_1/\varepsilon)^2 + P_X(|S_n - \sigma^2| > \varepsilon_1^2) + \varepsilon_2 > 0 \), where, for \( t \in \mathbb{R} \), \( \Phi(t - \varepsilon) - \Phi(t) > -\varepsilon_2 \) and \( \Phi(t + \varepsilon) - \Phi(t) < \varepsilon_2 \). Then, for all \( n, m_n \) we have

\[
(A) \quad P_w \left\{ \sup_{-\infty < t < +\infty} \left| P_{X|w}(G_m^* \leq t) - \Phi(t) \right| > \delta \right\} \\
\leq \delta^{-2}(1 - \varepsilon)^{-3}(1 - \frac{1}{n})^{-3}\left( \frac{n^2}{m_n^3} + \frac{n^2}{m_n^2} + \frac{25m_n^2}{n^2} \right) + \varepsilon^{-2} \frac{m_n^2}{(1 - \frac{1}{n})} \left\{ \frac{1 - \frac{1}{n}}{n^3m_n^3} + \frac{(1 - \frac{1}{n})^4}{m_n^3} + \frac{(m_n - 1)(1 - \frac{1}{n})^2}{nm_m^3} + \frac{4(n - 1)}{n^3m_n} + \frac{1}{m_n^2} \right\},
\]

and also

\[
(B) \quad P_w \left\{ \sup_{-\infty < t < +\infty} \left| P_{X|w}(T_m^* \leq t) - \Phi(t) \right| > \varepsilon \right\} \\
\leq \delta^{-2}(1 - \varepsilon)^{-3}(1 - \frac{1}{n})^{-3}\left( \frac{n^2}{m_n^3} + \frac{n^2}{m_n^2} + \frac{25m_n^2}{n^2} \right) + \varepsilon^{-2} \frac{m_n^2}{(1 - \frac{1}{n})} \left\{ \frac{1 - \frac{1}{n}}{n^3m_n^3} + \frac{(1 - \frac{1}{n})^4}{m_n^3} + \frac{(m_n - 1)(1 - \frac{1}{n})^2}{nm_m^3} + \frac{4(n - 1)}{n^3m_n} + \frac{1}{m_n^2} \right\},
\]
where
\[ \delta_n := \frac{\delta - (\varepsilon_1/\varepsilon)^2 - P_X(\left| S_n^2 \right| > \varepsilon)^2 + \varepsilon_2}{CE_X |X - \mu|^3/\sigma^3/2}, \]

with \( C \) being the Berry-Esseen universal constant.

The following result, a corollary to Theorem 2.1, gives the rate of convergence of the respective conditional CLT’s for \( G_{m_n}^* \) and \( T_{m_n}^* \).

**Corollary 2.1.** Assume that \( E_X |X|^r < +\infty, r \geq 3 \). If \( n, m_n \to +\infty \) in such a way that \( m_n = o(n^2) \), then, for arbitrary \( \delta > 0 \), we have

(A) \[ P_{\omega_n} \left\{ \sup_{-\infty < t < +\infty} \left| P_{X|\omega} (G_{m_n}^* \leq t) - \Phi(t) \right| > \delta \right\} = O \left( \max \left\{ \frac{m_n}{n^2}, \frac{1}{m_n} \right\} \right), \]

(B) \[ P_{\omega_n} \left\{ \sup_{-\infty < t < +\infty} \left| P_{X|\omega} (T_{m_n}^* \leq t) - \Phi(t) \right| > \delta \right\} = O \left( \max \left\{ \frac{m_n}{n^2}, \frac{1}{m_n} \right\} \right). \]

Moreover, if \( E_X X^4 < +\infty \), if \( n, m_n \to +\infty \) in such a way that \( m_n = o(n^2) \) and \( n = o(m_n^2) \) then, for \( \delta > 0 \), we also have

(C) \[ P_{\omega_n} \left\{ \sup_{-\infty < t < +\infty} \left| P_{X|\omega} (G_{m_n}^{**} \leq t) - \Phi(t) \right| > \delta \right\} = O \left( \max \left\{ \frac{m_n}{n^2}, \frac{1}{m_n}, \frac{n}{m_n^2} \right\} \right), \]

(D) \[ P_{\omega_n} \left\{ \sup_{-\infty < t < +\infty} \left| P_{X|\omega} (T_{m_n}^{**} \leq t) - \Phi(t) \right| > \delta \right\} = O \left( \max \left\{ \frac{m_n}{n^2}, \frac{1}{m_n}, \frac{n}{m_n^2} \right\} \right), \]

where

\[ G_{m_n}^{**} := \frac{\sum_{i=1}^{n} \left| \frac{w^{(n)}_{i}}{m_n} - \frac{1}{n} \right| (X_i - \mu)}{S_{m_n}^* \sqrt{\sum_{i=1}^{n} \left( \frac{w^{(n)}_{i}}{m_n} - \frac{1}{n} \right)^2}}, \]

\[ T_{m_n}^{**} := \frac{\sum_{j=1}^{n} \left( \frac{w^{(n)}_{i}}{m_n} - \frac{1}{n} \right) X_i}{S_{m_n}^* \sqrt{\sum_{i=1}^{n} \left( \frac{w^{(n)}_{i}}{m_n} - \frac{1}{n} \right)^2}}, \]

and \( S_{m_n}^2 \) is the bootstrapped sample variance defined as

\[ S_{m_n}^2 := \frac{\sum_{i=1}^{n} \left( \frac{w^{(n)}_{i}}{m_n} - \frac{1}{n} \right)^2}{m_n}, \]

where

\[ \bar{X}_{m_n} := \sum_{i=1}^{n} \frac{w^{(n)}_{i}}{m_n} X_i / m_n. \]
Then we compute the following empirical values.

$$X \sum_{n=1}^{500} \text{distribution for each } n$$ then generating 500 sets of $X$.

$m$ to the CLT for where, (2.5) and (2.6) hold true when $E X^4 < +\infty$.

Remark 2.2. When $E X^4 < +\infty$, the extra term of $n/m^2$ which appears in the rate of convergence of $G_m^*$ and $T_m^*$ in (C) and (D) of Corollary 2.1 is the rate at which $P_w \{ P_{X,w} | S_{m}^2 - S_{n}^2 | > \varepsilon_1 \} > \varepsilon_2 \}$ approaches zero as $n, m \to +\infty$, where $\varepsilon_1$ and $\varepsilon_2$ are arbitrary positive numbers.

Furthermore, in view of Lemma 1.2 of S. Csörgő and Rosalsky [4], the conditional CLT’s resulting from (A), (B), (C) and (D) of Corollary 2.1 imply respective unconditional CLT’s in terms of $P_{X,w}$. Moreover, when $m_n = n \to +\infty$, we conclude

$$\sup_{-\infty < t < +\infty} | P_{X,w}(G_{m_n}^* \leq t) - \Phi(t) | = O(1/n), \quad (2.5)$$
$$\sup_{-\infty < t < +\infty} | P_{X,w}(T_{m_n}^* \leq t) - \Phi(t) | = O(1/n), \quad (2.6)$$
$$\sup_{-\infty < t < +\infty} | P_{X,w}(G_{m_n}^* \leq t) - \Phi(t) | = O(1/n), \quad (2.7)$$
$$\sup_{-\infty < t < +\infty} | P_{X,w}(T_{m_n}^* \leq t) - \Phi(t) | = O(1/n), \quad (2.8)$$

where, (2.5) and (2.6) hold true when $E X^3 < +\infty$ and (2.7) and (2.8) hold true when $E X^4 < +\infty$.

In the following Table 1 we use the software R to present some numerical illustrations of our Theorem 2.1 and its Corollary 2.1 for $m_n$ as compared to the CLT for $T_n(X - \mu)$.

The conditional probability of $G_{m_n}^*$, with $m_n = n$, given $w_i^{(n)}$'s, is approximated by its empirical counterpart. On taking $m_n = n$, the procedure for each $n$ is generating a realization of the weights $(w_1^{(n)}, \ldots, w_n^{(n)})$ first, and then generating 500 sets of $X_1, \ldots, X_n$ to compute the conditional empirical distribution $\sum_{t=1}^{500} \mathbb{1}(G_{m_n}^* \leq 1.644854)/500$. Simultaneously, and for the same generated 500 sets of $X_1, \ldots, X_n$, we also compute the empirical distribution $\sum_{t=1}^{500} \mathbb{1}(T_n(X - \mu) \leq 1.644854)/500$, where, here and throughout this paper, $\mathbb{1}(.)$ stands for the indicator function.

This cycle is repeated 500 times, i.e., for 500 realizations $(w_1^{(n)(s)}, \ldots, w_n^{(n)(s)})$, $1 \leq s \leq 500$. For each $s$, 500 sets $X_1^{(s)}, \ldots, X_n^{(s)}$, $1 \leq t \leq 500$ are generated. Then we compute the following empirical values.
\[ emp_x |_{w} G^* := \frac{\sum_{s=1}^{500} \mathbf{1} \left( \frac{\sum_{t=1}^{500} \mathbf{1}( G_{m_{n_s}}^{(s,t)} \leq 1.644854) }{500} - 0.95 \right) \leq 0.01 }{500}, \]

\[ emp_x T_n(X - \mu) := \frac{\sum_{s=1}^{500} \mathbf{1} \left( \frac{\sum_{t=1}^{500} \mathbf{1}( T_{n_s}^{(s,t)}(X - \mu) \leq 1.644854) }{500} - 0.95 \right) \leq 0.01 }{500}, \]

where, \( G_{m_{n_s}}^{(s,t)} \) stands for the computed value of \( G_{m_{n_s}}^* \), with \( m_{n_s} = n_s \), based on \( (w_{n_s}^{(n_s)}(s), \ldots, w_{n_s}^{(n_s)}) \) and \( X_1^{(s,t)}, \ldots, X_n^{(s,t)} \) and \( T_{n_s}^{(s,t)}(X - \mu) \) stands for the computed value of \( T_n(X - \mu) \) based on \( X_1^{(s,t)}, \ldots, X_n^{(s,t)} \).

The numerical results, which are presented in the following table, indicate a significantly better performance of \( G_{m_{n_s}}^* \), with \( m_{n_s} = n_s \), in comparison to that of the Student \( t \)-statistic \( T_n(X - \mu) \).

**Table 1: Comparing \( G_{m_{n_s}}^* \), with \( m_{n_s} = n_s \), to \( T_n(X - \mu) \)**

| Distribution  | \( n \) | \( emp_x |_{w} G^* \) | \( emp_x T_n(X - \mu) \) |
|---------------|--------|-----------------|-----------------|
| Poisson(1)    | 20     | 0.552           | 0.322           |
|               | 30     | 0.554           | 0.376           |
|               | 40     | 0.560           | 0.364           |
| Lognormal(0,1)| 20     | 0.142           | 0.000           |
|               | 30     | 0.168           | 0.000           |
|               | 40     | 0.196           | 0.000           |
| Exponential(1)| 20     | 0.308           | 0.016           |
|               | 30     | 0.338           | 0.020           |
|               | 50     | 0.470           | 0.094           |

In view of (2.5)-(2.8), we also compare the rate of convergence of actual coverage probability of the confidence bounds \( G_{m_{n_s}}^* \leq z_{1-\alpha} \), with \( m_{n_s} = n_s \) and \( 0 < \alpha < 1 \), to those of the traditional confidence bounds \( T_n(X - \mu) \leq z_{1-\alpha} \) and the classical bootstrap confidence bounds of size \( 1 - \alpha \) (cf. Table 2 in Section 8).
3 Bootstrapped asymptotic pivots for the population mean \( \mu \)

We are now to present \( G_{m_n}^* \) of (1.4), and some further versions of it, as direct asymptotic bootstrap pivots for the population mean \( \mu = E_X X \) when 0 < \( \sigma^2 := E_X (X - \mu)^2 < +\infty \).

We note that for the numerator term of \( G_{m_n}^* \) we have
\[
E_{X|w}(\sum_{i=1}^n \frac{w_i^{(n)}}{m_n} - \frac{1}{n}(X_i - \mu)) = 0.
\]
Furthermore, given \( w_i^{(n)} \)'s, for the bootstrapped weighted average
\[
\sum_{i=1}^n \frac{w_i^{(n)}}{m_n} - \frac{1}{n}(X_i - \mu) =: \bar{X}_{m_n}^*(\mu),
\]
we have
\[
E_{X|w}(\sum_{i=1}^n \frac{w_i^{(n)}}{m_n} - \frac{1}{n}(X_i - \mu)) = 0.
\]

and the same holds true if \( n \to +\infty \) as well.

In view of (3.1)
\[
\frac{\sum_{i=1}^n |w_i^{(n)} - \frac{1}{n}|X_i}{\sum_{j=1}^n |w_j^{(n)} - \frac{1}{n}|} =: \bar{X}_{n,m_n}^*.
\]
is an unbiased estimator for \( \mu \) with respect to \( P_{X|w} \).

It can be shown that when \( E_X X^2 < +\infty \), as \( n, m_n \to +\infty \) such that \( m_n = o(n^2) \), \( \bar{X}_{n,m_n}^* \) is a consistent estimator for the population mean \( \mu \) in terms of \( P_{X|w} \), i.e.,
\[
\bar{X}_{n,m_n}^* \to \mu \quad \text{in probability} - P_{X|w}, \quad (3.4)
\]
In Appendix 1 we give a direct proof for (3.4) for the important case when \( m_n = n \), for which the CLT’s in Corollary 2.1 hold true at the rate \( n^{-1} \).

As to \( G_{m_n}^* \) of (1.4), on replacing \( \left( \frac{w_i^{(n)}}{m_n} - \frac{1}{n} \right) \) by \( \left| \frac{w_i^{(n)}}{m_n} - \frac{1}{n} \right| \) in the proof of (a) of Corollary 2.1 of Csörgő et al. [2], as \( n, m_n \to +\infty \) so that \( m_n = o(n^2) \) we arrive at
\[
P_{X|w}(G_{m_n}^* \leq t) \to P(Z \leq t) \quad \text{in probability} - P_w \quad \text{for all} \ t \in \mathbb{R}, \quad (3.5)
\]
and, via Lemma 1.2 in S. Csörgő and Rosalsky [4], we conclude also the unconditional CLT

\[ P_{X,w}(G_{m_n}^* \leq t) \to P(Z \leq t) \text{ for all } t \in \mathbb{R}. \]  

(3.6)

where, and also throughout, Z stands for a standard normal random variable.

For studying \( G_{m_n}^* \) possibly in terms of weights other than \( w_i^{(n)} \), we refer to Appendix 4.

When \( E_X X^2 < +\infty \), in Appendix 1 we show that when \( n \) is fixed and \( m := m_n \to +\infty \), the bootstrapped sample variance \( S_{m_n}^2 \), as defined in (2.3), converges in probability-\( P_{X,w} \) to the sample variance \( S_n^2 \), i.e., as \( m := m_n \to +\infty \) and \( n \) is fixed, we have (cf. (10.4) in Appendix 1 or Remark 2.1 of Csörgő et al. [2])

\[ S_m^2 \to S_n^2 \text{ in probability} - P_{X,w}. \]  

(3.7)

For related results along these lines in terms of \( u \)- and \( v \)-statistics, we refer to Csörgő and Nasari [3], where in a more general setup, we establish in probability and almost sure consistencies of bootstrapped \( u \)- and \( v \)-statistics.

In Appendix 1 we also show that, when \( E_X X^4 < +\infty \), if \( n, m_n \to +\infty \) so that \( n = o(m_n) \), then we have (cf. (10.4) in Appendix 1)

\[ (S_{m_n}^2 - S_n^2) \to 0 \text{ in probability} - P_{X,w}. \]  

(3.8)

When \( E_X X^4 < +\infty \), the preceding convergence also holds true when \( n = o(m_n^2) \) (cf. the proof of Corollary 2.1).

On combining (3.8) with the CLT in (3.6), when \( E_X X^2 < +\infty \), as \( n, m_n \to +\infty \) so that \( m_n = o(n^2) \) and \( n = o(m_n) \), the following unconditional CLT holds true as well in terms of \( P_{X,w} \)

\[ G_{m_n}^{**} \to Z, \]  

(3.9)

where, and also throughout, \( \to \) stands for convergence in distribution and \( G_{m_n}^{**} \) is as defined in (2.1).

Furthermore, when \( E_X X^2 < +\infty \), under the same conditions, i.e., on assuming that \( n, m_n \to +\infty \) so that \( m_n = o(n^2) \) and \( n = o(m_n) \), mutatis mutandis, via (b) of Lemma 2.1 of Csörgő et al. [2], equivalently to (3.9), in terms of \( P_{X,w} \), we also arrive at

\[ \tilde{G}_{m_n}^{**} := \frac{\sum_{i=1}^n |w_i^{(n)}| m_n^{-1} (X_i - \mu)}{S_{m_n}^* / \sqrt{m_n}} \to Z. \]  

(3.10)
Thus, without more ado, we may now conclude that the unconditional
CLT’s as in (3.6), (3.9) and (3.10), respectively for $G^*_m$, $G^{**}_m$ and $\tilde{G}^{**}_m$,
the bootstrapped versions of the traditional Student pivot $T_n(X - \mu)$ for the
population mean $\mu$ as in (1.2), can be used to construct exact size asymptotic
C.I.’s for the population mean $\mu = E_X X$ when $0 < \sigma^2 := E_X (X - \mu)^2 < +\infty$,
if $n, m_n \to +\infty$ so that $m_n = o(n^2)$ in case of (3.6), and, in case of (3.9)
and (3.10), so that $n = o(m_n)$ as well. Moreover, when $E_X X^4 < +\infty$, if
$n, m_n \to +\infty$ so that $m_n = o(n^2)$ and $n = o(m_n^2)$ (cf. the proof of Corollary
2.1), then (3.9) and (3.10) continue to hold true.

We spell out the one based on $G^*_m$ as in (3.6). Thus, when
$0 < \sigma^2 := E_X (X - \mu)^2 < +\infty$ and $m_n = o(n^2)$, then, for any $\alpha \in (0, 1)$, we conclude (cf.
also (3.1)-(3.4) in comparison) a $1 - \alpha$ size asymptotic C.I. for the population
mean $\mu = E_X X$, which is also valid in terms of the conditional distribution
$P_{X|w}$ as follows

$$
\bar{X}_{n,m_n}^* - z_{\alpha/2} \frac{S_n}{\sum_{j=1}^{n} \left( \frac{n_j}{m_n} - \frac{1}{n} \right)} \leq \mu \leq \bar{X}_{n,m_n}^* + z_{\alpha/2} \frac{S_n}{\sum_{j=1}^{n} \left( \frac{n_j}{m_n} - \frac{1}{n} \right)}
$$

(3.11)

where $z_{\alpha/2}$ satisfies $P(Z \geq z_{\alpha/2}) = \alpha/2$ and $\sqrt{\cdot} := \sqrt{\sum_{j=1}^{n} \left( \frac{n_j}{m_n} - \frac{1}{n} \right)^2}$.

When $E_X X^2 < +\infty$ on assuming $n = o(m_n)$ and when $E_X X^4 < +\infty$ on
assuming $n = o(m_n^2)$ (cf. the proof of Corollary 2.1) as well, as $n, m_n \to +\infty$,
then we can replace $S_n$ by $S^*_m$ in (3.11), i.e., the thus obtained $1 - \alpha$ size asymptotic C.I. for the population mean $\mu$ is then based on the CLT for $G^*_m$
as in (3.9). A similar $1 - \alpha$ size asymptotic C.I. for the population mean $\mu$
can be based on the CLT for $G^{**}_m$ as in (3.10).

4 Bootstrapped asymptotic pivots for the sample mean

We are now to present $T^*_m$, and further versions of it, as asymptotic boot-
strapped pivots for the sample mean $\bar{X}_n$ when $0 < \sigma^2 = E_X (X - \mu)^2 < +\infty$.
In view of its definition, when studying $T^*_m$, we may, without loss of generality, forget about what the numerical value of the population mean $\mu = E_X X$
could possibly be.

To begin with, we consider the associated numerator term of $T^*_m$, and write
\[
\sum_{i=1}^{n} \left( \frac{w_i^{(n)}}{m_n} - \frac{1}{n} \right) X_i = \frac{1}{m_n} \sum_{i=1}^{n} w_i^{(n)} X_i - \frac{1}{n} \sum_{i=1}^{n} X_i = \bar{X}_{m_n}^* - \bar{X}_n. \tag{4.1}
\]

The term \( \bar{X}_{m_n}^* \) is known as the bootstrapped sample mean in the literature.

First we note that \( E_{w|X}(\bar{X}_{m_n}^*) = \bar{X}_n \), i.e., given \( X \), the Efron bootstrap mean \( \bar{X}_{m_n}^* \) is an unbiased estimator of the sample mean \( \bar{X}_n \) and, consequently, \( \text{(4.1)} \) exhibits the bias of \( \bar{X}_{m_n}^* \) vis-à-vis \( \bar{X}_n \), its conditional mean, given the data \( X \). Moreover, when the original sample size \( n \) is assumed to be fixed, then on taking only one large bootstrap sub-sample of size \( m := m_n \), as \( m \to +\infty \), we have

\[
\bar{X}_{m}^* \to \bar{X}_n \text{ in probability} - P_{X,w} \tag{4.2}
\]

(cf. \( \text{(10.3)} \) of Appendix 1, where it is also shown that the bootstrap estimator of the sample mean that is based on taking a large number, \( B \), of independent bootstrap sub-samples of size \( n \) is equivalent to taking only one large bootstrap sub-sample).

Further to \( \text{(4.2)} \), as \( n, m_n \to +\infty \), then (cf. \( \text{(10.3)} \) in Appendix 1)

\[
(\bar{X}_{m_n}^* - \bar{X}_n) \to 0 \text{ in probability} - P_{X,w}. \tag{4.3}
\]

Back to \( T_{m_n}^* \) and further to \( \text{(4.3)} \), we have that \( E_{X|w}(\sum_{i=1}^{n} (\frac{w_i^{(n)}}{m_n} - \frac{1}{n}) X_i) = 0 \) and, if \( n, m_n \to +\infty \) so that \( m_n = o(n^2) \), then (cf. part (a) of Corollary 2.1 of Csörgő et al. [2])

\[
P_{X|w}(T_{m_n}^* \leq t) \to P(Z \leq t) \text{ in probability} - P_w \text{ for all } t \in \mathbb{R}, \tag{4.4}
\]

Consequently, as \( n, m_n \to +\infty \) so that \( m_n = o(n^2) \), we arrive at (cf., e.g., Lemma 1.2 in S. Csörgő and Rosalsky [4])

\[
P_{X,w}(T_{m_n}^* \leq t) \to P(Z \leq t) \text{ for all } t \in \mathbb{R}, \tag{4.5}
\]

an unconditional CLT.

Moreover, in view of the latter CLT and \( \text{(3.8)} \), as \( n, m_n \to +\infty \) so that \( m_n = o(n^2) \) and \( n = o(m_n) \), in terms of probability-\( P_{X,w} \) we conclude the unconditional CLT

\[
T_{m_n}^{**} \overset{d}{\longrightarrow} Z, \tag{4.6}
\]

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where, $T_{m_n}^{**}$ is as defined in (2.2). In turn we note also that, under the same conditions, i.e., on assuming that $n, m_n \to +\infty$ so that $m_n = o(n^2)$ and $n = o(m_n)$, via (b) of Corollary 2.1 of Csörgő et al [2], equivalently to (4.6) in terms of probability-$P_{X,w}$ we also arrive at the unconditional CLT

$$
\tilde{T}_{m_n}^{**} := \frac{\sum_{i=1}^{n} \left( \frac{w^{(i)}}{m_n} - \frac{1}{n} \right) X_i}{S_n^*/\sqrt{m_n}} \xrightarrow{d} Z.
$$

(4.7)

Thus, when conditioning on the weights, via (4.4) we arrive at the unconditional CLTs as in (4.5), (4.6) and (4.7), respectively for $T_{m_n}^*$ and $\tilde{T}_{m_n}^{**}$ and $T_{m_n}^{**}$. These CLT’s in hand for the latter bootstrapped versions of the classical Student $t$–statistic $T_n(X)$ (cf. (1.1)) can be used to construct exact size asymptotic C.I.’s for the sample mean $\bar{X}_n$, treated as a random variable jointly with its bootstrapped version $\bar{X}_m^*$.

We spell out the one based on $T_{m_n}^*$ as in (4.5). Accordingly, when $E_{X}X^2 < +\infty$ and $m_n, n \to +\infty$ so that $m_n = o(n^2)$, then for any $\alpha \in (0, 1)$, we conclude (cf. also (4.1)) a $1 - \alpha$ size asymptotic confidence set, which is also valid in terms of the conditional distribution $P_{X|w}$, that covers $\bar{X}_n$ as follows

$$
\bar{X}_m^* - z_{\alpha/2} S_n \sqrt{\cdot} \leq \bar{X}_n \leq \bar{X}_{m_n}^* + z_{\alpha/2} S_n \sqrt{\cdot},
$$

(4.8)

where $z_{\alpha/2}$ is as in (3.11), and $\sqrt{\cdot} := \sqrt{\sum_{j=1}^{n} \left( \frac{w^{(j)}}{m_n} - \frac{1}{n} \right)^2}$.

When $E_{X}X^2 < +\infty$, on assuming $n = o(m_n)$ and when $E_{X}X^4 < +\infty$ on assuming $n = o(m_n^2)$ as well as $m_n, n \to +\infty$, then we can replace $S_n$ by $S_{m_n}^*$ in (4.8), i.e., then the thus obtained $1 - \alpha$ asymptotic confidence set that covers $\bar{X}_n$ is based on the CLT for $T_{m_n}^{**}$ as in (4.6). The equivalent CLT in (4.7) can similarly be used to cover $\bar{X}_n$ when $E_{X}X^2 < +\infty$ and also when $E_{X}X^4 < +\infty$.
5 Bootstrapping a finite population when it is viewed as a random sample of size \( N \) from an infinite super-population

As before, let \( X, X_1, \ldots \) be independent random variables with a common distribution function \( F \), mean \( \mu = E_X X \) and, unless stated otherwise, variance \( 0 < \sigma^2 := E_X(X - \mu)^2 < +\infty \). A super-population outlook regards a finite population as an imaginary large random sample of size \( N \) from a hypothetical infinite population. In our present context, we consider \( \{X_1, \ldots, X_N\} \) to be a concrete or imaginary random sample of size \( N \geq 1 \) on \( X \), and study it as a finite population of \( N \) independent and identically distributed random variables with respective mean and variance

\[
\bar{X}_N := \frac{1}{N} \sum_{i=1}^{N} X_i \quad \text{and} \quad S_N^2 := \frac{1}{N} \sum_{i=1}^{N} (X_i - \bar{X}_N)^2
\] (5.1)

that are to be estimated via taking samples from it.

Our approach in this regard is to assume that we can view an imaginary random sample \( \{X_1, \ldots, X_N\} \) as a finite population of real valued random variables with \( N \) labeled units, and sample its set of indices \( \{1, \ldots, N\} \) with replacement \( m_N \) times so that for each \( 1 \leq i \leq N \), \( w_i^{(N)} \) is the count of the number of times the index \( i \) of \( X_i \) is chosen in this re-sampling process. Consequently, as to the weights \( w_i^{(N)}, 1 \leq i \leq N \), mutatis mutandis, Remark 1.1 obtains with \( m_N = \sum_{i=1}^{N} w_i^{(N)} \) and, for each \( N \geq 1 \), the multinomial weights

\[
(w_1^{(N)}, \ldots, w_N^{(N)}) \overset{d}{=} \text{multinomial}(m_N; \frac{1}{N}, \ldots, \frac{1}{N})
\]

are independent from the finite population of the \( N \) labeled units \( \{X_1, \ldots, X_N\} \).

In the just described context, mutatis mutandis, under their respective conditions the respective results as in (3.7) and (4.2) are applicable to consistently estimate \( \bar{X}_N \) and \( S_N^2 \) of (5.1) respectively with \( N \) fixed and \( m := m_N \to +\infty \), while the respective CLT’s in (4.6) and (4.7) can be adapted to construct exact size asymptotic confidence sets for covering \( \bar{X}_N \). For illustrating this, \( \text{à la } T_{m_n} \) in (4.6), we let
\[ T_{mN}^{**} := \frac{\sum_{i=1}^{N} \left( \frac{w_i^{(N)}}{m_N} - \frac{1}{N} \right) X_i}{S_{mN}^* \sqrt{\sum_{j=1}^{N} \left( \frac{w_j^{(N)}}{m_N} - \frac{1}{N} \right)^2}} \]

\[ =: \frac{\bar{X}_m - \bar{X}_N}{S_{mN}^* \sqrt{\sum_{j=1}^{N} \left( \frac{w_j^{(N)}}{m_N} - \frac{1}{N} \right)^2}}, \quad (5.2) \]

where \( \bar{X}_{mN} := \sum_{i=1}^{N} X_i w_i^{(N)}/m_N \), the bootstrapped finite population mean and, à la \( S_{m_n}^* \) in (2.3),

\[ S_{mN}^{*2} := \sum_{i=1}^{N} \left( w_i^{(N)} \right) (X_i - \bar{X}_{mN})^2 / m_N \quad (5.3) \]

is the bootstrapped finite population variance.

With \( N \) fixed and \( m = m_N \to +\infty \), when \( X \) has a finite variance, à la (4.2) and (3.7) respectively, we arrive at

\[ \bar{X}_m \to \bar{X}_N \text{ in probability} - P_{X,w} \quad (5.4) \]

and

\[ S_{m}^{*2} \to S_N^{2} \text{ in probability} - P_{X,w}, \quad (5.5) \]

i.e., \( \bar{X}_m \) and \( S_{m}^{*2} \) are consistent estimators of the finite population mean \( \bar{X}_N \) and variance \( S_N^{2} \) respectively. Also, à la (4.3), as \( N, m_N \to +\infty \), we conclude

\[ (\bar{X}_{mN} - \bar{X}_N) \to 0 \text{ in probability} - P_{X,w} \quad (5.6) \]

and, if \( N, m_N \to +\infty \) so that \( N = o(m_N) \), then à la (3.8), we also have

\[ (S_{mN}^{*2} - S_N^{2}) \to 0 \text{ in probability} - P_{X,w}. \quad (5.7) \]

Furthermore, when studying \( T_{mN}^{**} \) as in (5.2) and, à la \( \tilde{T}_{m_N}^{**} \) as in (4.7),

\[ \tilde{T}_{m_N}^{**} := \frac{\sum_{i=1}^{N} \left( \frac{w_i^{(N)}}{m_N} - \frac{1}{N} \right) X_i}{S_{mN}^* / \sqrt{m_N}}, \quad (5.8) \]

we may, without loss of generality, forget about what the value of the super-population mean \( \mu = E_X X \) would be like. On assuming that \( 0 < E_X X^2 < \)
$+\infty$, if $N, m_N \to +\infty$ so that $m_N = o(N^2)$ and $N = o(m_N)$, then in view of the respective equivalent statements of (4.6) and (4.7), in terms of probability-$P_{X,w}$, we arrive at having the unconditional CLT’s

$$T_{m_N}^{**} \xrightarrow{d} Z$$

and

$$\tilde{T}_{m_N}^{**} \xrightarrow{d} Z.$$  

(5.9)

(5.10)

Back to $T_{m_N}^{**}$ as in (5.2), via the CLT of (5.10), for any $\alpha \in (0, 1)$, we conclude a $1 - \alpha$ size asymptotic confidence set, which is also valid in terms of the conditional distribution $P_{X|w}$, that covers $\bar{X}_N$ as follows

$$\bar{X}_{m_N}^* - z_{\alpha/2} S_{m_N}^* \sqrt{\cdot} \leq \bar{X}_N \leq \bar{X}_{m_N}^* + z_{\alpha/2} S_{m_N}^* \sqrt{\cdot}.$$  

(5.11)

with $z_{\alpha/2}$ as in (3.11) and $\sqrt{\cdot} := \sqrt{\sum_{j=1}^{N} \left(\frac{w_i^{(N)}}{m_N} - \frac{1}{N}\right)^2}$.

In the present context of taking samples with replacement from a finite population with $N$ labeled units, $\{X_1, \ldots, X_N\}$, that is viewed as an imaginary random sample of size $N$ from an infinite super-population with mean $\mu = E_X X$ and variance $0 < \sigma^2 = E_X (X - \mu)^2 < +\infty$, we can also estimate the population mean $\mu$ via adapting the results of of our Section 3 to fit this setting.

To begin with, the bootstrapped weighted average (cf. (3.1)) in this context becomes

$$\bar{X}_{m_N}^*(\mu) := \sum_{i=1}^{N} \left[ \frac{w_i^{(N)}}{m_N} - \frac{1}{N} \right] (X_i - \mu)$$  

(5.12)

and, given the weights, we have

$$E_{X|w}(\bar{X}_{m_N}^*(\mu)) = 0.$$  

(5.13)

Furthermore, with the finite population $N$ fixed and $m := m_N$, as $m \to +\infty$, we also conclude (cf. (3.2))

$$\bar{X}_{m_N}^*(\mu) = \bar{X}_m^*(\mu) \to 0 \text{ in probability} - P_{X,w},$$  

(5.14)

and the same holds true if $N \to +\infty$ as well.
A consistent estimator of the super population mean \( \mu \), similarly to (3.3), can be defined as

\[
\sum_{i=1}^{N} \left| \frac{w_{i}^{(N)}}{m_{N}} - \frac{1}{N} \right| X_{i} =: \bar{X}_{N,m_{N}}^{*},
\]

(5.15)

Thus, similarly to (3.4), it can be shown that, as \( N, m_{N} \to +\infty \) such that \( m_{N} = o(N^{2}) \), \( \bar{X}_{N,m_{N}}^{*} \) is a consistent estimator for the super-population mean \( \mu \), in terms of \( P_{X,w} \), i.e.,

\[
\bar{X}_{N,m_{N}}^{*} \to \mu \text{ in probability } - P_{X,w}.
\]

(5.16)

Now, with the bootstrapped finite population variance \( S_{m_{N}}^{*2} \) as in (5.3) (cf. also (5.7)), along the lines of arguing the conclusion of (3.9), as \( N, m_{N} \to +\infty \) so that \( m_{N} = o(N^{2}) \) and \( N = o(m_{N}) \), we conclude the following unconditional CLT as well

\[
\tilde{G}_{m_{N}}^{*} := \frac{\sum_{i=1}^{N} \left| \frac{w_{i}^{(N)}}{m_{N}} - \frac{1}{N} \right| (X_{i} - \mu)}{S_{m_{N}}^{*} \sqrt{\sum_{j=1}^{N} \left( \frac{w_{j}^{(N)}}{m_{N}} - \frac{1}{N} \right)^{2}}} \quad d \to Z.
\]

(5.17)

Furthermore, under the same conditions, equivalently to (5.17) \( \tilde{a} \text{ la } (3.10) \), we also have the unconditional CLT in parallel to that of (5.17)

\[
G_{m_{N}}^{*} := \frac{\sum_{i=1}^{N} \left| \frac{w_{i}^{(N)}}{m_{N}} - \frac{1}{N} \right| (X_{i} - \mu)}{S_{m_{N}}^{*} / \sqrt{m_{N}}} \quad d \to Z.
\]

(5.18)

Both (5.17) and (5.18) can be used to construct a \( 1 - \alpha \) size asymptotic C.I., with \( \alpha \in (0,1) \), for the unknown super-population mean \( \mu \). We spell out the one based on the CLT of (5.17). Accordingly, as \( N, m_{N} \to +\infty \) so that \( m_{N} = o(N^{2}) \) and \( N = o(m_{N}) \), we arrive at the following \( 1 - \alpha \) size asymptotic C.I., which is valid in terms of the conditional distribution \( P_{X|w} \), as well as in terms of the joint distribution \( P_{X,w} \), for the unknown super-population mean \( \mu \)

\[
\bar{X}_{N,m_{N}}^{*} - \frac{z_{\alpha/2} S_{m_{N}}^{*}}{\sqrt{\sum_{j=1}^{N} \left| \frac{w_{j}^{(N)}}{m_{N}} - \frac{1}{N} \right|}} \leq \mu \leq \bar{X}_{N,m_{N}}^{*} + \frac{z_{\alpha/2} S_{m_{N}}^{*}}{\sqrt{\sum_{j=1}^{N} \left| \frac{w_{j}^{(N)}}{m_{N}} - \frac{1}{N} \right|}}
\]

(5.19)
with $z_{\alpha/2}$ as in (3.11) and $\sqrt{\sum_{j=1}^{N} \left( \frac{w_{j}^{(N)}}{m_{N}} - \frac{1}{N} \right)^{2}}$, a companion of the $1-\alpha$ size asymptotic confidence set of (5.11) that, in the same super-population setting, covers the unknown finite population mean $\bar{X}_{N}$ under the same conditions.

6 Bootstrapped CLT’s and C.I.’s for the empirical and theoretical distributions

Let $X, X_1, X_2, \ldots$ be independent real valued random variables with a common distribution function $F$ as before, but without assuming the existence of any finite moments for $X$. Consider $\{X_1, \ldots, X_N\}$ a concrete or imaginary random sample of size $N \geq 1$ on $X$ of a hypothetical infinite population, and defined their empirical distribution function

$$F_N(x) := \sum_{i=1}^{N} \mathbb{1}(X_i \leq x)/N, \quad x \in \mathbb{R},$$

(6.1)

and the sample variance of the indicator variable $\mathbb{1}(X_i \leq x)$ by

$$S_N^2 := \frac{1}{N} \sum_{i=1}^{N} \left( \mathbb{1}(X_i \leq x) - F_N(x) \right)^2 = F_N(x)(1 - F_N(x)), \quad x \in \mathbb{R},$$

(6.2)

that, together with the theoretical distribution function $F$, are to be estimated via taking samples from $\{X_1, \ldots, X_N\}$ as in our previous sections in general, and as in Section 5 in particular. On replacing $N$ by $n$ in case of a real sample of size $n$, our present general formualtion of the problems in hand, as well as the results thus concluded, mutatis mutandis, continue to hold true when interpreted in the context of Sections 3 and 4 that deal with a concrete random sample $\{X_1, \ldots, X_n\}$ of size $n \geq 1$.

Accordingly, we view a concrete or imaginary random sample $\{X_1, \ldots, X_N\}$ as a finite population of real valued random variables with $N$ label units, and sample its set of indices $\{1, \ldots, N\}$ with replacement $m_N$ times so that for $1 \leq i \leq N$, $w_i^{(N)}$ is the count of the number of times the index $i$ of $X_i$ is chosen in this re-sampling process and, as in Section 5 $m_N = \sum_{i=1}^{N} w_i^{(N)}$ and, for each $N \geq 1$, the multinomial weights

$$\left( w_1^{(N)}, \ldots, w_N^{(N)} \right) \overset{d}{=} \text{multinomial}\left( m_N; \frac{1}{N}, \ldots, \frac{1}{N} \right)$$

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are independent from the finite population of the $N$ labeled units $\{X_1, \ldots, X_N\}$.

Define the standardized bootstrapped empirical process

$$
\alpha_{mN,N}(x) := \frac{\sum_{i=1}^{N} \left( \frac{w_i^{(N)}}{mN} - \frac{1}{N} \right) 1(X_i \leq x)}{\sqrt{F(x)(1 - F(x))} \sqrt{\sum_{j=1}^{N} \left( \frac{w_j^{(N)}}{mN} - \frac{1}{N} \right)^2}} 
$$

$$
= \frac{\sum_{i=1}^{N} \frac{w_i^{(N)}}{mN} 1(X_i \leq x) - F_N(x)}{\sqrt{F(x)(1 - F(x))} \sqrt{\sum_{j=1}^{N} \left( \frac{w_j^{(N)}}{mN} - \frac{1}{N} \right)^2}}, \ x \in \mathbb{R} 
$$

(6.3)

where

$$
F_{mN,N}(x) := \sum_{i=1}^{N} \frac{w_i^{(N)}}{mN} 1(X_i \leq x), \ x \in \mathbb{R}, 
$$

(6.4)

is the bootstrapped empirical process.

We note that

$$
E_{X \mid W}(F_{mN,N}(x)) = F(x), \ E_{W \mid X}(F_{mN,N}(x)) = F_N(x) \text{ and } E_{X \mid W}(F_{mN,N}(x)) = F(x). 
$$

(6.5)

Define also the bootstrapped finite population variance of the indicator random variable $1(X_i \leq x)$ by putting

$$
S_{mN,N}^2(x) := \sum_{i=1}^{N} \frac{w_i^{(N)}}{mN} [1(X_i \leq x) - F_{mN,N}(x)]^2/mN 
$$

(6.6)

$$
= F_{mN,N}(x)(1 - F_{mN,N}(x)), \ x \in \mathbb{R}. 
$$

With $N$ fixed and $m = m_N \to +\infty$, along the lines of (6.4) we arrive at

$$
F_{mN,N}(x) \to F_N(x) \text{ in probability } - P_{X \mid W} \text{ pointwise in } x \in \mathbb{R}, 
$$

(6.7)

and, consequently, point-wise in $x \in \mathbb{R}$, as $m = m_N \to +\infty$,

$$
S_{mN,N}^2(x) \to F_N(x)(1 - F_N(x)) = S_N^2(x) \text{ in probability } - P_{X \mid W}. 
$$

(6.8)
Furthermore, à la [5.6], as \( N,m_N \to +\infty \), pointwise in \( x \in \mathbb{R} \), we conclude
\[
(F_{m_N,N}^*(x) - F_N(x)) \to 0 \text{ in probability} - P_{X,w}, \quad (6.9)
\]
that, in turn, pointwise in \( x \in \mathbb{R} \), as \( N,m_N \to +\infty \), implies
\[
(S_{m_N,N}^{*2} - S_N^2(x)) \to 0 \text{ in probability} - P_{X,w}. \quad (6.10)
\]

We wish to note and emphasize that, unlike in (5.7), for concluding (6.10), we do not assume that \( N = o(m_N) \) as \( N,m_N \to +\infty \).

Further to the standardized bootstrapped empirical process \( a_{N,m_N}^{(1)}(x) \), we now define the following Studentized/self-normalized bootstrapped versions of this process:

\[
\hat{a}_{m_N,N}^{(1)}(x) := \frac{\sum_{i=1}^{N}(\frac{w_i}{m_N} - \frac{1}{N})1(X_i \leq x)}{\sqrt{F_N(x)(1 - F_N(x))}\sum_{j=1}^{N}(\frac{w_j}{m_N} - \frac{1}{N})^2} \quad (6.11)
\]

\[
\hat{a}_{m_N,N}^{(1)}(x) := \frac{\sum_{i=1}^{N}(\frac{w_i}{m_N} - \frac{1}{N})1(X_i \leq x)}{\sqrt{F_{m_N,N}^*(x)(1 - F_{m_N,N}^*(x))}\sum_{j=1}^{N}(\frac{w_j}{m_N} - \frac{1}{N})^2} \quad (6.12)
\]

\[
\hat{a}_{m_N,N}^{(2)}(x) := \frac{\sum_{i=1}^{N}|\frac{w_i}{m_N} - \frac{1}{N}|(1(X_i \leq x) - F(x))}{\sqrt{F_N(x)(1 - F_N(x))}\sum_{j=1}^{N}(\frac{w_j}{m_N} - \frac{1}{N})^2} \quad (6.13)
\]

\[
\hat{a}_{m_N,N}^{(2)}(x) := \frac{\sum_{i=1}^{N}|\frac{w_i}{m_N} - \frac{1}{N}|(1(X_i \leq x) - F(x))}{\sqrt{F_{m_N,N}^*(x)(1 - F_{m_N,N}^*(x))}\sum_{j=1}^{N}(\frac{w_j}{m_N} - \frac{1}{N})^2}. \quad (6.14)
\]

Clearly, on replacing \( X_i \) by \( 1(X_i \leq x) \) and \( \mu \) by \( F(x) \) in the formula in (5.12), we arrive at the respective statements of (5.13) and (5.14) in this context. Also, replacing \( X_i \) by \( 1(X_i \leq x) \) in the formula as in (5.15), we conclude the statement of (5.16) with \( \mu \) replaced \( F(x) \).

In Lemma 5.2 of Csörgő et al. [2] it is shown that, if \( m_N,N \to +\infty \) so that \( m_N = o(N^2) \), then
\[
M_N := \frac{\max_{1 \leq i \leq N}(\frac{w_i}{N} - \frac{1}{N})^2}{\sum_{j=1}^{N}(\frac{w_j}{N} - \frac{1}{N})^2} \to 0 \text{ in probability} - P_w. \quad (6.15)
\]
This, mutatis mutandis, combined with (a) of Corollary 2.1 of Csörgö et al. [2], as \( N, m_N \to +\infty \) so that \( m_N = o(N^2) \), yields

\[
P_{X,w}(\hat{\alpha}_{m,N}^{(s)}(x) \leq t) \to P(Z \leq t) \text{ in probability}, \quad \text{for all } x, t \in \mathbb{R},
\]

(6.16)

with \( s = 1 \) and also for \( s = 2 \), and via Lemma 1.2 in S. Csörgö and Rosalsky [4], this results in having also the unconditional CLT

\[
P_{X,w}(\hat{\alpha}_{m,N}^{(s)}(x) \leq t) \to P(Z \leq t) \text{ for all } x, t \in \mathbb{R},
\]

(6.17)

with \( s = 1 \) and also for \( s = 2 \).

On combining (6.17) and (6.10), as \( N, m \to +\infty \) so that \( m_N = o(N^2) \), when \( s = 1 \) in (6.17), we conclude

\[
\hat{\alpha}_{m,N}^{(1)}(x) \overset{d}{\to} Z
\]

(6.18)

and, when \( s = 2 \) in (6.17), we arrive at

\[
\hat{\alpha}_{m,N}^{(2)}(x) \overset{d}{\to} Z
\]

(6.19)

for all \( x \in \mathbb{R} \).

Furthermore, as \( m_N, N \to +\infty \) so that \( m_N = o(N^2) \), in addition to (6.15), we also have (cf. (9.8))

\[
\left| \sum_{i=1}^{N} \left( \frac{w_i^{(N)}}{N} - \frac{1}{N} \right)^2 - \frac{1}{m_N} \right| = o_{P,w}(1).
\]

(6.20)

Consequently, in the CLT’s of (6.16)–(6.19), the term \( \sqrt{\sum_{i=1}^{N} \left( \frac{w_i^{(N)}}{N} - \frac{1}{N} \right)^2} \) in the respective denominators of \( \hat{\alpha}_{m,N}^{(s)}(x) \) and \( \hat{\alpha}_{m,N}^{(s)}(x) \), \( s = 1, 2 \), can be replaced by \( 1/\sqrt{m_N} \).

In case of a \textit{concerted random samples} of size \( N \geq 1 \) on \( X \), the CLT’s for \( \hat{\alpha}_{m,N}^{(1)}(x) \) and \( \hat{\alpha}_{m,N}^{(1)}(x) \) can both be used to construct a \( 1 - \alpha, \alpha \in (0, 1) \), size asymptotic confidence sets for covering the empirical distribution function \( F_N(x) \), point-wise in \( x \in \mathbb{R} \), as \( N, m_N \to +\infty \) so that \( m_N = o(N^2) \), while in case of an \textit{imaginary random sample} of size \( N \geq 1 \) on \( X \) of a hypothetical infinite population, the CLT for \( \hat{\alpha}_{m,N}^{(1)}(x) \) works also similarly estimating the, in this case, unknown empirical distribution function \( F_N(x) \). The respective CLT’s for \( \hat{\alpha}_{m,N}^{(2)}(x) \) and \( \hat{\alpha}_{m,N}^{(2)}(x) \) work in a similar way for
point-wise estimating the unknown distribution function $F(x)$ of a hypothetical infinite population in case of a concrete and imaginary random sample of size $N \geq 1$ on $X$. Furthermore, mutatis mutandis, the Berry-Ésseen type inequality (A) of our Theorem 2.1 continues to hold true for both $\hat{\alpha}_{m_N,N}^{(2)}(x)$ and $\hat{\alpha}_{m_N,N}^{(2)}(x)$, and so does also (B) of Theorem 2.1 for both $\hat{\alpha}_{m_N,N}^{(1)}(x)$ and $\hat{\alpha}_{m_N,N}^{(1)}(x)$, without the assumption $E_X|X|^3 < +\infty$, for the indicator random variable $1(X \leq x)$ requires no moments assumptions.

**Remark 6.1.** In the context of this section, (A) and (B) of Corollary 2.1 read as follows: As $N, m_N \to +\infty$ in such a way that $m_N = o(N^2)$, then, mutatis mutandis, (A) and (B) hold true for $\hat{\alpha}_{m_N,N}^{(1)}(x)$ and $\hat{\alpha}_{m_N,N}^{(2)}(x)$, with $O(\max\{m_N/N^2, 1/m_N\})$ in both. Consequently, on taking $M_N = N$, we immediately obtain the optimal $O(N^{-1})$ rate conclusion of Remark 2.1 in this context, i.e., uniformly in $t \in \mathbb{R}$ and point-wise in $x \in \mathbb{R}$ for $\hat{\alpha}_{m_N,N}^{(1)}(x)$ and $\hat{\alpha}_{m_N,N}^{(2)}(x)$.

Also, for a rate of convergence of the respective CLT’s via (C) and (D) of Corollary 2.1 for $\hat{\alpha}_{m_N,N}^{(1)}(x)$ and $\hat{\alpha}_{m_N,N}^{(2)}(x)$, as $N, m_N \to +\infty$ in such away that $m_N = O(N^2)$, we obtain the rate $O(\max\{m_N/N^2, 1/m_N\})$. The latter means that observing a sub-sample of size $m_N = N^{1/2}$ will result in the rate of convergence of $N^{-1/2}$ for $\hat{\alpha}_{m_N,N}^{(1)}(x)$ and $\hat{\alpha}_{m_N,N}^{(2)}(x)$, uniformly in $t \in \mathbb{R}$ and point-wise in $x \in \mathbb{R}$.

In the context of this section, asymptotic $1 - \alpha$ size confidence sets, in terms of the joint distribution $P_{X,w}$ which are also valid in terms of $P_{X|i}$ for the empirical and the population distribution functions $F_N(x)$ and $F(x)$, for each $x \in \mathbb{R}$, are in fact of the forms (5.11) and (5.19), respectively, on replacing $X_i$ by $1(X_i \leq x)$ which are spelled out as follows

\[
F_{m_N,N}^*(x) - z_{\alpha/2} S^*_{m_N} \sqrt{\frac{1}{m_N}} \leq F_N(x) \leq F_{m_N,N}^*(x) + z_{\alpha/2} S^*_{m_N} \sqrt{\frac{1}{m_N}}
\]

\[
\frac{F_{m_N,N}^*(x) - z_{\alpha/2} S^*_{m_N} \sqrt{\frac{1}{m_N}}}{\sum_{j=1}^{N} |w_{(j)}^{N}/m_N - 1/N|} \leq F(x) \leq \frac{F_{m_N,N}^*(x) + z_{\alpha/2} S^*_{m_N} \sqrt{\frac{1}{m_N}}}{\sum_{j=1}^{N} |w_{(j)}^{N}/m_N - 1/N|}
\]

with $z_{\alpha/2}$ as in (3.11), $\sqrt{\gamma} := \sqrt{\frac{1}{N} \sum_{j=1}^{N} (w_{(j)}^{N}/m_N - 1/N)^2}$ and $S^*_{m_N} = F_{m_N,N}^*(x)(1 - F_{m_N,N}^*(x))$.
In this section we show that, unlike our direct pivot $G^*_{m_n}$ of (1.4) for $\mu$, the indirect use of $T^*_{m_n}$ of (1.3) along the lines of the classical method of constructing a bootstrap C.I. for the population mean $\mu$ does not lead to a better error rate than that of the classical CLT for $T_n(X - \mu)$, which is $n^{-1/2}$ under the conditions of Corollary 2.1.

A bootstrap estimator for a quantile $\eta_{n,1-\alpha}$, $0 < \alpha < 1$, of $T_n(X - \mu)$ is the solution of the inequality

$$P(T^*_{m_n} \leq x|X_1, \ldots, X_n) := P_{w|X}(T^*_{m_n} \leq x) \geq 1 - \alpha,$$

i.e., the smallest value of $x = \hat{\eta}_{n,1-\alpha}$ that satisfies (7.1), where $T^*_{m_n}$ is as defined in (1.3). Since the latter bootstrap quantiles should be close to the true quantiles $\eta_{n,1-\alpha}$ of $T_n(X - \mu)$, in view of (7.1), it should be true that

$$P_{w|X}(T_n(X - \mu) \leq \hat{\eta}_{n,1-\alpha}) \approx 1 - \alpha. \quad (7.2)$$

In practice, the value of $\hat{\eta}_{n,\alpha}$ is usually estimated by simulation (cf. Efron and Tibshirani [5], Hall [7]) via producing $B \geq 2$ independent copies of $T^*_{m_n}$, usually with $m_n = n$, given $X_1, \ldots, X_n$.

A classical bootstrap C.I. of level $1 - \alpha$, $0 < \alpha < 1$, for $\mu$ is constructed by using $T_n(X - \mu)$ as a pivot and estimating the cutoff point $\hat{\eta}_{n,1-\alpha}$ using the $(B + 1). (1 - \alpha)$th largest value of $T^*_{m_n}(b)$, $1 \leq b \leq B$, where, each $T^*_{m_n}(b)$, is computed based on the $b$-th bootstrap sub-sample. We note that the preceding method of constructing a classical bootstrap C.I. at level $1 - \alpha$ is for the case when $(B + 1). (1 - \alpha)$ is an integer already.

For the sake of comparison of our main results, namely Theorem 2.1 and Corollary 2.1, to the classical bootstrap C.I.’s which were also investigated by Hall [7], here we study the rate of convergence of classical bootstrap C.I.’s.

As it will be seen below, our investigations agree with those of Hall [7] in concluding that the number of bootstrap sub-samples $B$ does not have to be particularly large for a classical bootstrap C.I. to reach its nominal probability coverage.

In this section, we also show that the rate at which the probability of the event that the conditional probability, given $w^{(n)}_i$’s, of a classical bootstrap C.I. for $\mu$ deviating from its nominal probability coverage by any given positive number, vanishes at a rate that can, at best, be $O(n^{-1/2})$, as $n \rightarrow +\infty$ (cf. Theorem 7.1). This is in contrast to our Corollary 2.1 and Remark 2.1.
It is also noteworthy that the rate of convergence of a joint distribution is essentially the same as those of its conditional versions to the same limiting distribution. Therefore, the preceding rate of, at best, $O(n^{-1/2})$ for the conditional, given $w_i^{(n)}$'s, probability coverage of classical bootstrap C.I.'s is inherited by their probability coverage in terms of the joint distribution.

In what follows $B \geq 2$ is a fixed positive integer, that stands for the number of bootstrap sub-samples of size $m_n$, generated via $B$ times independently re-sampling with replacement from \{\(X_i, 1 \leq i \leq n\)\}. We let $T_{m_n}^*(1), \ldots, T_{m_n}^*(B)$ be the versions of computed $T_{m_n}^*$ based on the drawn $B$ bootstrap sub-samples. We now state a multivariate CLT for the $B$ dimensional vector

\[
\left( \frac{T_n(X - \mu) - T_{m_n}^*(1)}{\sqrt{2}}, \ldots, \frac{T_n(X - \mu) - T_{m_n}^*(B)}{\sqrt{2}} \right)^T.
\]

The reason for investigating the asymptotic distribution of the latter vector has to do with computing the actual probability coverage of the classical bootstrap C.I.'s as in (7.4) below. Furthermore, in Section 7.2, we compute the actual probability coverage of the classical bootstrap C.I.'s as in (7.4), and show that for a properly chosen finite $B$, the nominal probability coverage of size $1 - \alpha$ will be achieved, as $n$ approaches $+\infty$. We then use the result of the following Theorem 7.1 to show that the rate at which the actual probability coverage, of a classical bootstrap C.I. constructed using a finite number of bootstrap sub-samples, approaches its nominal coverage probability, $1 - \alpha$, is no faster than $n^{-1/2}$.

**Theorem 7.1.** Assume that $\mathbb{E}|X|^3 < +\infty$. Consider a positive integer $B \geq 2$ and let $\mathcal{H}$ be the class of all half space subsets of $\mathbb{R}^B$. Define

\[
\mathbb{H}_n := \left( \frac{T_n(X - \mu) - T_{m_n}^*(1)}{\sqrt{2}}, \ldots, \frac{T_n(X - \mu) - T_{m_n}^*(B)}{\sqrt{2}} \right)^T,
\]

and let $\mathcal{Y} := (Z_1, \ldots, Z_B)^T$ be a $B$-dimensional Gaussian vector with mean $(0, \ldots, 0)^T_{1 \times B}$ and covariance matrix

\[
\begin{pmatrix}
1 & 1/2 & \cdots & 1/2 \\
1/2 & 1 & \cdots & 1/2 \\
\vdots & \vdots & \ddots & \vdots \\
1/2 & 1/2 & \cdots & 1
\end{pmatrix}_{B \times B}.
\]
Then, as \( n, m \to \infty \) in such a way that \( m_n = o(n^2) \), for \( \varepsilon > 0 \), the speed at which

\[
\left\{ \sup_{A \in \mathcal{H}} \left( \bigotimes_{b=1}^{B} P_{X|w(b)}(B_n \in A) - P(Y \in A) \right) > \varepsilon \right\}
\]

approaches zero is at best \( n^{-1/2} \), where \( \bigotimes_{b=1}^{B} P_{X|w(b)}(.) \) is the conditional probability, given \( (w_1^{(n)}(1), \ldots, w_n^{(n)}(1)), \ldots, (w_1^{(n)}(B), \ldots, w_n^{(n)}(B)) \).

We note in passing that, on account of \( m_n = o(n^2) \), Theorem 7.1 also holds true when \( m_n = n \), i.e., when \( B \) bootstrap sub-samples of size \( n \) are drawn from the original sample of size \( n \). Also, by virtue of Lemma 2.1 of S. Csörgő and Rosalsky [4], the unconditional version of the conditional CLT of Theorem 7.1 continues to hold true under the same conditions as those of the conditional one, with the same rate of convergence that is \( n^{-1/2} \) at best.

### 7.1 The classical bootstrap C.I.’s

The classical method of establishing an asymptotic \( 1 - \alpha \) size bootstrap C.I. for \( \mu \), as mentioned before and formulated here, is based on the \( B \geq 2 \) ordered bootstrap readings \( T_n^*[1] \leq \ldots \leq T_n^*[B] \), resulting from \( B \) times independently re-sampling bootstrap sub-samples of size \( m_n \), usually with \( m_n = n \), from the original sample with replacement by setting (cf., e.g., Efron and Tibshirani [3])

\[
T_n(X - \mu) \leq T_{m_n}^*[\nu],
\]

where \( T_{m_n}^*[\nu] \) is the \( \nu = (B + 1)(1 - \alpha) \)th order statistic of the \( T_{m_n}^*(b) \), \( 1 \leq b \leq B \), a bootstrap approximation to \( \hat{\eta}_{n,\alpha} \) as in (7.1). For simplicity, we assume here that \( \nu \) is an integer already.

The so-called ideal bootstrap C.I. for \( \mu \) is obtained when \( B \to +\infty \). The validity of ideal bootstrap C.I.’s was established by Csörgő et al. [2] in terms of sub-samples of size \( m_n \) when \( E_XX^2 < +\infty \) and it was also studied previously by Hall [7] when \( E_X|X|^{4+\delta} < +\infty \), with \( \delta > 0 \). The common feature of the results in the just mentioned two papers is that they both require that \( n, m_n, B \to +\infty \), with \( m_n = n \) in Hall [7].

In view of Theorem 7.1, however, we establish another form of a bootstrap C.I. for \( \mu \) of level \( 1 - \alpha \) when \( B \) is fixed, as formulated in our next section.
7.2 A precise version of the classical bootstrap C.I. for \( \mu \) with fixed \( B \)

We first consider the counting random variable \( Y \) that counts the number of negative (or positive) components \( Z_t, 1 \leq t \leq B \), in the Gaussian vector \( \mathbb{Y} := (Z_1, \ldots, Z_B)^T \). The distribution of \( Y \) is

\[
P(Y \leq y) = \sum_{\ell=0}^{y} \binom{B}{\ell} P(Z_1 < 0, \ldots, Z_\ell < 0, Z_{\ell+1} > 0, \ldots, Z_B > 0),
\]

where, \( y = 0, \ldots, B \) and \( \mathbb{Y} := (Z_1, \ldots, Z_B)^T \) has \( B \)-dimensional normal distribution with mean \((0, \ldots, 0)^T_{1 \times B}\) and covariance matrix as in (7.3).

Let \( y_{1-\alpha} \) be the \((1-\alpha)\)th percentile of \( Y \), in other words,

\[
P(Y \leq y_{1-\alpha} - 1) < 1 - \alpha \quad \text{and} \quad P(Y \leq y_{1-\alpha}) \geq 1 - \alpha.
\]

We establish a more accurate version of \(1 - \alpha\) level classical bootstrap C.I. based on a finite number of, \(B\)-times, re-sampling by setting

\[
T_n(X - \mu) \leq T_{m_n}^*[y_{1-\alpha}],
\]

(7.5)

where, \( T_{m_n}^*[y_{1-\alpha}] \) is the \(y_{1-\alpha}\)th largest order statistic of the \(B\) bootstrap versions \( T_{m_n}^*(1), \ldots, T_{m_n}^*(B) \) of \( T_{m_n}^* \), constituting a new method for the bootstrap estimation of \( \hat{\eta}_{n,\alpha} \) as in (7.2), as compared to that of (7.4).

To show that as \( n, m_n \to +\infty \), the probability coverage of the bootstrap C.I. (7.5) approaches its nominal probability of size \( 1 - \alpha \), we first let

\[
\bigotimes_{b=1}^{B} \Omega_{X, w(b)}, \bigotimes_{b=1}^{B} \mathcal{X}_{X, w(b)}, \bigotimes_{b=1}^{B} P_{X, w(b)}, 1 \leq b \leq B,
\]

be the joint probability space of the \(X\)’s and the preceding array of the weights \((w_1(n)(b), \ldots, w_n(n)(b))\), \(1 \leq b \leq B\), as in Theorem (7.31) i.e., the probability space generated by \(B\) times, independently, re-sampling from the original sample \(X_1, \ldots, X_n\). Employing now the definition of order statistics, we can compute the actual coverage probability of (7.5) as follows:
\[ \alpha_n(B) := \bigotimes_{b=1}^{B} P_{X,b(w)} \{ T_n \leq T_{m_n}^*[y_{1-\alpha}] \} \]

\[ = \sum_{\ell=0}^{y_{1-\alpha}} \left( \begin{array}{c} B \\ \ell \end{array} \right) \prod_{b=1}^{B} P_{X,b(w)} \{ T_n(X - \mu) - T_{m_n}^*(1) \leq 0, \ldots, T_n(X - \mu) - T_{m_n}^*(\ell) \leq 0, \\
T_n(X - \mu) - T_{m_n}^*(\ell + 1) > 0, \ldots, T_n(X - \mu) - T_{m_n}^*(B) > 0 \} \]

\[ \rightarrow \sum_{\ell=0}^{y_{1-\alpha}} \left( \begin{array}{c} B \\ \ell \end{array} \right) P(Z_1 < 0, \ldots, Z_{\ell} < 0, Z_{\ell+1} > 0, \ldots, Z_B > 0) = P(Y \leq y_{1-\alpha}) \]

as \( n, m_n \rightarrow +\infty \) such that \( m_n = o(n^2) \).

The preceding convergence results from Theorem 7.1 on assuming the same conditions as those of the latter theorem. It is noteworthy that the conditional version of the preceding convergence, in view of Theorem 7.1, holds also true in probability-P\(_w\) when one replaces the therein joint probability \( \prod_{b=1}^{B} P_{X,b(w)} \) by the conditional one \( \prod_{b=1}^{B} P_{X|w(b)} \).

Recalling that \( y_{1-\alpha} \) is the \((1-\alpha)\)th percentile of \( Y \), we now conclude that

\[ \alpha_n(B) \rightarrow \beta \geq 1 - \alpha. \] (7.6)

This means that the new bootstrap C.I. (7.5) is valid.

**Remark 7.1.** The choice of the cutoff point \( T_{m_n}^*[\nu] \) in the classical bootstrap C.I. (7.4) is done in a blindfold way in comparison to the more informed choice of \( T_{m_n}^*[y_{1-\alpha}] \) in (7.5). We also note that increasing the value of \( B \) to \( B + 1 \) for the same confidence level \( 1 - \alpha \), only results in a different cutoff point, which is a consequence of a change in the distribution of \( Y \) which is now based on a \((B + 1)\)-dimensional Gaussian vector.

**8 Numerical comparisons of the three C.I.'s for the population mean \( \mu = E_X X \)**

The numerical study below shows that, in terms of the joint distribution \( P_{X,w} \), our confidence bound of level 0.9000169 using the bootstrapped pivot \( G_{m_n}^* \), with \( m_n = n \), for \( \mu \) outperforms the traditional confidence bound of level 0.9000169 with the pivot \( T_n(X - \mu) \) and also the classical bootstrapped confidence bound of the same level.
We note that the classical bootstrap confidence bound, as in (7.4), coincides with our improved version of it (7.5) when \( B = 9 \).

In order to numerically compare the performance of the three confidence bounds for the population mean \( \mu \) in terms of the joint distribution \( P_{X,w} \), we let \( m_n = n \) and take a similar approach to the one used to illustrate Theorem 2.1 and its Corollary 2.1. The only difference is that here we generate the weights \( w_i^{(n)} \)'s and the data \( X_1, \ldots, X_n \) simultaneously. More precisely, here we generate 500 sets of the weights \((w_1^{(n)}(b), \ldots, w_n^{(n)}(b))\) for \( b = 1, \ldots, 9 \) and \( X_1, \ldots, X_n \) at the same time. We then compute the empirical distributions. The cycle is repeated 500 times. At the end, we obtain the relative frequency of the empirical distributions that did not deviate from the nominal limiting probability \( \Phi(1.281648) = 0.9000169 \) by more than 0.01. This procedure is formulated as follows.

\[
\begin{align*}
\text{emp}_X T_n(X - \mu) & := \frac{1}{500} \sum_{s=1}^{500} \mathbb{1}\{0.9000169 - \frac{\sum_{t=1}^{500} \mathbb{1}(T_{n}^{(s,t)}(X-\mu) \leq 1.281648)}{500} \leq 0.01\}, \\
\text{emp}_{X,w} G^* & := \frac{1}{500} \sum_{s=1}^{500} \mathbb{1}\{0.9000169 - \frac{\sum_{t=1}^{500} \mathbb{1}(G_{m_n}^{(s,t)} \leq 1.281648)}{500} \leq 0.01\}, \\
\text{emp}_{X,w} \text{Boot} & := \frac{1}{500} \sum_{s=1}^{500} \mathbb{1}\{0.9000169 - \frac{\sum_{t=1}^{500} \mathbb{1}(T_{m_n}^{(s,t)}(X-\mu) \leq \max_{1 \leq b \leq 9} T_{m_n}^{(s,t)}(b))}{500} \leq 0.01\},
\end{align*}
\]

where, for each \( 1 \leq s \leq 500 \), \( T_n^{(s,t)}(X - \mu) \) and \( G_{m_n}^{(s,t)} \), with \( m_n = n \), stand for the respective values of \( T_n(X - \mu) \) and \( G_{m_n}^{*} \), with \( m_n = n \), which are computed, using 500 sets of \( (X_1^{(s,t)}, \ldots, X_n^{(s,t)}) \) and \( (w_1^{(n)(s,t)}, \ldots, w_n^{(n)(s,t)}) \), \( 1 \leq t \leq 500 \).

In a similar vein, for each \( 1 \leq b \leq 9 \) and each \( 1 \leq s \leq 500 \), \( T_{m_n}^{(s,t)}(b) \) represents the value of \( T_{m_n}^{(s,t)} \) which are computed, using the 500 simultaneously generated samples \( (X_1^{(s,t)}, \ldots, X_n^{(s,t)}) \) and \( (w_1^{(n)(s,t)}(b), \ldots, w_n^{(n)(s,t)}(b)) \) for \( t = 1, \ldots, 500 \).

The number 0.9000169 is the precise nominal probability coverage of the interval \( T_n(X - \mu) \leq \max_{1 \leq b \leq 9} T_{m_n}^{*}(b) \) which, in view of our Theorem 7.1, is \( P(1 \leq Y \leq 9) = 1 - P(Z_1 > 0, \ldots, Z_9 > 0) \), where \( Y \) is number of negative \( Z_i \)'s, \( 1 \leq i \leq 9 \), in the 9-dimensional Gaussian vector \((Z_1, \ldots, Z_9)^T\). To compute the probability \( P(Z_1 > 0, \ldots, Z_9 > 0) \), we use the Genz algorithm (cf. Genz [6]), which is provided in the software R.
It is noteworthy that the examined classical bootstrap confidence bound for $\mu$, which is based on our modified version of the classical bootstrap confidence bound in (7.5) with $B = 9$ bootstrap sub-samples, coincides with the 0.9 level classical bootstrap confidence bound for $\mu$, of the form (7.4), that was constructed by Hall [7], based on the same number of bootstrap sub-samples.

We use the statistical software R to conduct the latter numerical study and present the results in the following table.

| Distribution     | $n$ | $emp_{X,w}G^*$ | $emp_{X,T_n}(X - \mu)$ | $emp_{X,w,Boot}$ |
|------------------|----|----------------|------------------------|------------------|
| Poisson(1)       | 20 | 0.48           | 0.302                  | 0.248            |
|                  | 30 | 0.494          | 0.300                  | 0.33             |
|                  | 40 | 0.496          | 0.350                  | 0.316            |
| Lognormal(0, 1)  | 20 | 0.028          | 0.000                  | 0.000            |
|                  | 30 | 0.048          | 0.000                  | 0.004            |
|                  | 40 | 0.058          | 0.000                  | 0.002            |
| Exponential(1)   | 20 | 0.280          | 0.026                  | 0.058            |
|                  | 30 | 0.276          | 0.026                  | 0.084            |
|                  | 40 | 0.332          | 0.048                  | 0.108            |

**9 Proofs**

**Proof of Theorem 2.1**

Due to similarity of the two cases we only give the proof of part (A) of this theorem. The proof relies on the fact that via conditioning on the weights $w_i^{(n)}$'s, $\sum_{i=1}^{n} \left| \frac{w_i^{(n)}}{m_n} - \frac{1}{n} \right| (X_i - \mu)$ as a sum of independent and non-identically distributed random variables. This in turn enables us to use a Berry-Esseen type inequality for self-normalized sums of independents and non-identically distributed random variables. Also, some of the ideas in the proof are similar to those of Slutsky’s theorem.
We now write
\[ G_{m_n}^* = \frac{\sum_{i=1}^{n} |X_i - \mu|}{\sqrt{\sum_{i=1}^{n} \left( \frac{w_i}{m_n} - \frac{1}{n} \right)^2}} + \frac{\sum_{i=1}^{n} |w_i/(m_n) - 1/n|}{\sqrt{\sum_{i=1}^{n} \left( \frac{w_i}{m_n} - \frac{1}{n} \right)^2}} \left( \frac{\sigma}{\sqrt{S_n}} - 1 \right) \]
\[ =: Z_{m_n} + Y_{m_n}. \]  

(9.1)

In view of the above setup, for \( t \in \mathbb{R} \) and \( \varepsilon_1 > 0 \), we have
\[ - P_{X|w}(|Y_{m_n}| > \varepsilon) + P_{X|w}(Z_{m_n} \leq t - \varepsilon) \]
\[ \leq P_{X|w}(G_{m_n}^* \leq t) \]
\[ \leq P_{X|w}(Z_{m_n} \leq t + \varepsilon) + P_{X|w}(|Y_{m_n}| > \varepsilon). \]  

(9.2)

Observe now that for \( \varepsilon_2 > 0 \) we have
\[ P_{X|w}(|Y_{m_n}| > \varepsilon) \leq P_{X|w}(|Z_{m_n}| > \frac{\varepsilon}{\varepsilon_1}) + P_X \left( |S_n^2 - \sigma^2| > \varepsilon_1^2 \right). \]  

(9.3)

One can readily see that
\[ P_{X|w}(|Z_{m_n}| > \frac{\varepsilon}{\varepsilon_1}) \leq \left( \frac{\varepsilon}{\varepsilon_1} \right)^2 \frac{\sum_{i=1}^{n} \left( \frac{w_i}{m_n} - \frac{1}{n} \right)^2 \sqrt{X(x - \mu)^2}}{\sigma^2 \sum_{i=1}^{n} \left( \frac{w_i}{m_n} - \frac{1}{n} \right)^2} \]
\[ = \left( \frac{\varepsilon_1}{\varepsilon} \right)^2. \]

Applying now the preceding in (9.3), in view of (9.2) can be replaced by
\[ - \left( \frac{\varepsilon_1}{\varepsilon} \right)^2 - P_X \left( |S_n^2 - \sigma^2| > \varepsilon_1^2 \right) + P_{X|w}(Z_{m_n} \leq t - \varepsilon) \]
\[ \leq P_{X|w}(G_{m_n}^* \leq t) \]
\[ \leq \left( \frac{\varepsilon_1}{\varepsilon} \right)^2 + P_X \left( |S_n^2 - \sigma^2| > \varepsilon_1^2 \right) + P_{X|w}(Z_{m_n} \leq t + \varepsilon). \]  

(9.4)

Now, the continuity of the normal distribution \( \Phi \) allows us to choose \( \varepsilon_3 > 0 \) so that so that \( \Phi(t + \varepsilon) - \Phi(t) < \varepsilon_2 \) and \( \Phi(t - \varepsilon) - \Phi(t) > -\varepsilon_2 \). This combined with (9.4) imply that
\[ - \left( \frac{\varepsilon_1}{\varepsilon} \right)^2 - P_X \left( |S_n^2 - \sigma^2| > \varepsilon_1^2 \right) + P_{X|w}(Z_{m_n} \leq t - \varepsilon) - \Phi(t - \varepsilon) - \varepsilon_2 \]
\[ \leq P_{X|w}(G_{m_n}^* \leq t) - \Phi(t) \]
\[ \leq \left( \frac{\varepsilon_1}{\varepsilon} \right)^2 + P_X \left( |S_n^2 - \sigma^2| > \varepsilon_1^2 \right) + P_{X|w}(Z_{m_n} \leq t + \varepsilon) - \Phi(t + \varepsilon) + \varepsilon_2. \]  

(9.5)
We now use the Berry-Esseen inequality for independent and not identically distributed random variables (cf., e.g., Serfling [10]) to write

\[ P_X|w(Z_{m_n} \leq t + \varepsilon_1) - \Phi(t + \varepsilon_1) \leq \frac{C E_X|X - \mu|^3}{\sigma^{3/2}} \cdot \frac{\sum_{i=1}^{n} |w^{(n)}(m_i) - \frac{1}{n}|^3}{(\sum_{i=1}^{n} (w^{(n)}(m_i) - \frac{1}{n})^2)^{3/2}} \]

and

\[ P_X|w(Z_{m_n} \leq t - \varepsilon_1) - \Phi(t - \varepsilon_1) \geq -\frac{C E_X|X - \mu|^3}{\sigma^{3/2}} \cdot \frac{\sum_{i=1}^{n} |w^{(n)}(m_i) - \frac{1}{n}|^3}{(\sum_{i=1}^{n} (w^{(n)}(m_i) - \frac{1}{n})^2)^{3/2}} \]

where \( C \) is the universal constant of Berry-Esseen inequality.

Incorporating these approximations into (9.5) we arrive at

\[
\sup_{-\infty < t < +\infty} \left| P_X|w(G^*_{m_n} \leq t) - \Phi(t) \right| \\
\leq \left( \frac{\varepsilon_1}{\varepsilon} \right)^2 \cdot P_X(|S^2_n - \sigma^2| > \varepsilon_1^2) + \frac{C E_X|X - \mu|^3}{\sigma^{3/2}} \cdot \frac{\sum_{i=1}^{n} |w^{(n)}(m_i) - \frac{1}{n}|^3}{(\sum_{i=1}^{n} (w^{(n)}(m_i) - \frac{1}{n})^2)^{3/2}} + \varepsilon_1.
\]

From the preceding relation we conclude that

\[
P_w(\sup_{-\infty < t < +\infty} \left| P_X|w(G^*_{m_n} \leq t) - \Phi(t) \right| > \delta) \leq P_w\left( \frac{\sum_{i=1}^{n} |w^{(n)}(m_i) - \frac{1}{n}|^3}{(\sum_{i=1}^{n} (w^{(n)}(m_i) - \frac{1}{n})^2)^{3/2}} > \delta_n \right)
\]

(9.6)

with \( \delta_n \) as defined in the statement of Theorem 2.1.

For \( \varepsilon > 0 \), the right hand side of (9.6) is bounded above by

\[
P_w \left\{ \sum_{i=1}^{n} \left| \frac{w^{(n)}(m_i)}{m_n} - \frac{1}{n} \right|^3 > \frac{\delta_n (1 - \varepsilon)^{3/2} (1 - \frac{1}{n})^{3/2}}{m_n^{3/2}} \right\} \\
+ P_w \left( \left| \frac{m_n - 1}{n - 1} \sum_{i=1}^{n} \left( \frac{w^{(n)}(m_i)}{m_n} - \frac{1}{n} \right)^2 - 1 > \varepsilon \right) \right)
=: \Pi_1(n) + \Pi_2(n).
\]
We bound $\Pi_1(n)$ above by

$$\delta_n^{-2}(1 - \varepsilon)^{-3}(1 - \frac{1}{n})^{-3}m_n^{-3}(n + n^2)E_w(w_1^{(n)} - \frac{m_n}{n})^6$$

$$= \delta_n^{-2}(1 - \varepsilon)^{-3}(1 - \frac{1}{n})^{-3}m_n^{-3}(n + n^2)\left\{\frac{15m_n^3}{n^3} + \frac{25m_n^2}{n^2} + \frac{m_n}{n}\right\}. \quad (9.7)$$

As for $\Pi_2(n)$, recalling that $E_w\left(\sum_{i=1}^{n}(\frac{w_i^{(n)}}{m_n} - \frac{1}{n})^2\right) = (1 - \frac{1}{n})^2$, an application of Chebyshev’s inequality yields

$$\Pi_2(n) \leq \frac{m_n^2}{\varepsilon^2(1 - \frac{1}{n})^2}E_w\left(\sum_{i=1}^{n}\left(\frac{w_i^{(n)}}{m_n} - \frac{1}{n}\right)^2 - (1 - \frac{1}{n})^2\right)^2$$

$$= \frac{m_n^2}{\varepsilon^2(1 - \frac{1}{n})^2} \left\{nE_w\left(\frac{w_1^{(n)}}{m_n} - \frac{1}{n}\right)^4 + n(n-1)E_w\left[(\frac{w_1^{(n)}}{m_n} - \frac{1}{n})^2(\frac{w_2^{(n)}}{m_n} - \frac{1}{n})^2\right] - (1 - \frac{1}{n})^2\right\}. \quad (9.8)$$

We now use the fact that $w^{(n)}$’s are multinomially distributed to compute the preceding relation. After some algebra it turns out that it can be bounded above by

$$\frac{m_n^2}{\varepsilon^2(1 - \frac{1}{n})^2}\left\{\frac{1 - \frac{1}{n}}{n^3m_n^3} + \frac{(1 - \frac{1}{n})^4}{m_n^3} + \frac{(m_n - 1)(1 - \frac{1}{n})^2}{nm_n^3} + \frac{4(n - 1)}{n^3m_n} + \frac{1}{m_n^2} \right.\right.$$

$$\left.\left. - \frac{1}{nm_n^2} + \frac{n - 1}{n^3m_n^3} + \frac{4(n - 1)}{n^2m_n^3} - \frac{(1 - \frac{1}{n})^2}{m_n^2}\right\}. \quad (9.9)$$

Incorporating (9.7) and (9.9) into (9.6) completes the proof of part (A) of Theorem 2.1. \Box

**Proof of Corollary 2.1**

The proof parts (A) and (B) of this corollary is an immediate consequence of Theorem 2.1.

To prove parts (C) and (D) of this corollary, in view of Theorem 2.1 it suffices to show that, for arbitrary $\varepsilon_1, \varepsilon_2 > 0$, as $n, m_n \to +\infty$,

$$P_w(P_{X|w}(|S_m^*-S_n^2| > \varepsilon_1) > \varepsilon_2) = O\left(\frac{n}{m_n^2}\right). \quad (9.10)$$
To prove the preceding result we first note that

\[
S_{m_n}^2 - S_n^2 = \sum_{1 \leq i \neq j \leq n} \left( \frac{w^{(n)}_i w^{(n)}_j}{m_n(m_n - 1)} - \frac{1}{n(n-1)} \right) \frac{(X_i - X_j)^2}{2} = \sum_{1 \leq i \neq j \leq n} \left( \frac{w^{(n)}_i w^{(n)}_j}{m_n(m_n - 1)} - \frac{1}{n(n-1)} \right) \left( \frac{(X_i - X_j)^2}{2} - \sigma^2 \right)
\]

By virtue of the preceding observation, we proceed with the proof of (9.10) by first letting \( d_{i,j}^{(n)} := \frac{w^{(n)}_i w^{(n)}_j}{m_n(m_n - 1)} - \frac{1}{n(n-1)} \) and writing

\[
P_w \{ P_{X|w} \left( \sum_{1 \leq i \neq j \leq n} d_{i,j}^{(n)} \left( \frac{(X_i - X_j)^2}{2} - \sigma^2 \right) \right) > \varepsilon_1 \varepsilon_2 \} \leq P_w \{ E_{X|w} \left( \sum_{1 \leq i \neq j \leq n} d_{i,j}^{(n)} \left( \frac{(X_i - X_j)^2}{2} - \sigma^2 \right)^2 \right) > \varepsilon_1^2 \varepsilon_2^2 \}. \tag{9.11}
\]

Observe now that

\[
E_{X|w} \left( \sum_{1 \leq i \neq j \leq n} d_{i,j}^{(n)} \left( \frac{(X_i - X_j)^2}{2} - \sigma^2 \right)^2 \right) = E \left( \frac{(X_1 - X_2)^2}{2} - \sigma^2 \right)^2 \sum_{1 \leq i \neq j \leq n} (d_{i,j}^{(n)})^2
\]

\[
+ \sum_{1 \leq i, j, k \leq n} d_{i,j}^{(n)} d_{i,k}^{(n)} E_X \left( \left( \frac{(X_i - X_j)^2}{2} - \sigma^2 \right) \left( \frac{(X_i - X_k)^2}{2} - \sigma^2 \right) \right)
\]

\[
+ \sum_{1 \leq i, j, k, l \leq n} d_{i,j}^{(n)} d_{k,l}^{(n)} E_X \left( \left( \frac{(X_i - X_j)^2}{2} - \sigma^2 \right) \left( \frac{(X_k - X_l)^2}{2} - \sigma^2 \right) \right) \tag{9.12}
\]

We note that in the preceding relation, since \( i, j, k \) are distinct, we have that

\[
E_X \left( \left( \frac{(X_i - X_j)^2}{2} - \sigma^2 \right) \left( \frac{(X_i - X_k)^2}{2} - \sigma^2 \right) \right) = E \{ E \left( \frac{(X_i - X_j)^2}{2} - \sigma^2 \mid X_i \right) E \left( \frac{(X_i - X_k)^2}{2} - \sigma^2 \mid X_i \right) \} = E_X \left( \frac{X_i^2}{4} - \sigma^2 \right).
\]

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Also, since \( i, j, k, l \) are distinct, we have that
\[
E_X \left( \frac{(X_i - X_j)^2}{2} - \sigma^2 \right) \left( \frac{(X_k - X_l)^2}{2} - \sigma^2 \right) = E_X \left( \frac{(X_i - X_j)^2}{2} - \sigma^2 \right) = 0.
\]

Therefore, in view of (9.12) and (9.11), the proof of (9.10) follows if we show that
\[
\sum_{1 \leq i \neq j \leq n} (d_{i,j}^{(n)})^2 = O_P \left( \frac{1}{m_n^2} \right) \tag{9.13}
\]
and
\[
\sum_{1 \leq i, j, k \leq n, i, j, k \text{ are distinct}} d_{i,j}^{(n)} d_{i,k}^{(n)} = O_P \left( \frac{n}{m_n^2} \right). \tag{9.14}
\]

Noting that, as \( n, m_n \to +\infty \),
\[
E_w \left\{ \sum_{1 \leq i \neq j \leq n} (d_{i,j}^{(n)})^2 \right\} \sim \frac{1}{m_n^2}
\]
and
\[
E_w \left| \sum_{1 \leq i, j, k \leq n, i, j, k \text{ are distinct}} d_{i,j}^{(n)} d_{i,k}^{(n)} \right| \leq n^3 E_w (d_{1,2}^{(n)})^2 \sim \frac{n}{m_n^2}.
\]
The preceding two conclusions imply (9.13) and (9.14), respectively. Now the proof of Corollary 2.1 is complete. \( \square \)

**Proof of Theorem 7.1**

Once again this proof will be done by the means of conditioning on the weights \( w^{(n)} \)'s and this allows us to think of them as constant coefficients for the \( X \)'s.

When using the bootstrap to approximate the cutoff points, via repeated re-sampling, like in the C.I. (7.5), the procedure can be described as *vectorizing* each centered observation. More precisely, as a result of drawing \( B \) bootstrap sub-samples and each time computing the value of \( T^*_m(b), 1 \leq b \leq B, \)

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for each univariate random variable \((X_i - \mu)\), \(1 \leq i \leq n\), we have the following transformation.

\[
(X_i - \mu) \mapsto \left[ \frac{(w_i^{(n)(1)/m_n - 1/n})}{S_n \sqrt{\sum_{1 \leq j \leq n}(w_j^{(n)(1)/m_n - 1/n})^2}} (X_i - \mu) \right]_{B \times 1}
\]

Viewing the problem from this perspective and replacing the sample variance \(S_n^2\) by \(\sigma^2\) result in having

\[
\left[ S_n(T_n(X - \mu) - T_{m_n}(1))/(\sigma \sqrt{2}) \right] \quad : \quad \left[ S_n(T_n(X - \mu) - T_{m_n}(B))/(\sigma \sqrt{2}) \right]_{B \times 1}
\]

\[
= \sum_{i=1}^{n} \left[ \left( \frac{1}{\sigma \sqrt{n}} - \frac{(w_i^{(n)(1)/m_n - 1/n})}{\sigma \sqrt{\sum_{1 \leq j \leq n}(w_j^{(n)(1)/m_n - 1/n})^2}} \right) (X_i - \mu) \right]_{B \times 1}
\]

Conditioning on \(w_i^{(n)}\)s, the preceding representation is viewed as a sum of \(n\) independent but not identically distributed \(B\)-dimensional random vectors. This, in turn, enables us to use Theorem 1.1 of Bentkus [1] to derive the rate of the conditional CLT in Theorem 7.1. For the sake of simplicity, we give the proof of this theorem only for \(B = 2\), as the proof essentially remains the same for \(B \geq 2\). Also, in the proof we will consider half spaces of the form \(A = \{(x, y) \in \mathbb{R}^2 : x \leq a, \ y > b\}\), where \(a, b \in \mathbb{R}\), as other forms of half spaces can be treated in the same vein. Moreover, in what will follow we let \(\| \cdot \|\) stand for the Euclidean norm on \(\mathbb{R}^2\).

We now continue the proof by an application of Theorem 1.1 of Bentkus [1] as follows.
\[
\begin{align*}
&\sup_{(x_1, x_2) \in \mathbb{R}^2} \left| \bigotimes_{b=1}^2 P_{X|\omega(b)} \begin{bmatrix} S_n(T_n(X - \mu) - T_{m_n}^*(1))/(\sigma \sqrt{2}) \leq x_1 \\ S_n(T_n(X - \mu) - T_{m_n}^*(2))/(\sigma \sqrt{2}) > x_2 \end{bmatrix} \right| \quad - P(Z_1 \leq x_1, Z_2 > x_2) \leq \frac{c}{4} \sum_{i=1}^n E_X \left| C_n^{-1/2} \begin{bmatrix} \frac{1}{\sqrt{n}} - a_{i,n}(1) \left( \frac{X_i - \mu}{\sigma \sqrt{2}} \right) \\ \vdots \\ \frac{1}{\sqrt{n}} - a_{i,n}(B) \left( \frac{X_i - \mu}{\sigma \sqrt{2}} \right) \end{bmatrix} \right|^3, \quad (9.15)
\end{align*}
\]

where \(c\) is an absolute constant and
\[
C_n^{-1/2} = \begin{pmatrix} A_n & -B_n \\ -B_n & A_n \end{pmatrix}
\]

with
\[
A_n = \frac{1 + \sqrt{1 - A_n^2(1, 2)}}{\sqrt{2 + 2\sqrt{1 - A_n^2(1, 2)}(2 - A_n^2(1, 2) + \sqrt{1 - A_n^2(1, 2)})}}
\]
\[
B_n = \frac{A_n^2(1, 2)(1 + \sqrt{1 - A_n^2(1, 2)})}{\sqrt{2 + 2\sqrt{1 - A_n^2(1, 2)}(2 - A_n^2(1, 2) + \sqrt{1 - A_n^2(1, 2)})}}.
\]

where \(A_n(1, 2) := 1/2(\sum_{i=1}^n a_{i,n}(1)a_{i,n}(2) + 1)\).

We note that \(C_n^{-1/2}\) is the inverse of a positive square root of the covariance matrix of the vector
\[
\left( S_n(T_n(X - \mu) - T_{m_n}^*(1))/(\sigma \sqrt{2}), S_n(T_n(X - \mu) - T_{m_n}^*(2))/(\sigma \sqrt{2}) \right),
\]
which is
\[
\begin{pmatrix} 1 & A_n(1, 2) \\ A_n(1, 2) & 1 \end{pmatrix}.
\]

Some algebra shows that the R.H.S. of (9.15) is equal to
\[
c^{2^{-3/2}}E_X[X_1 - \mu]^3 \sum_{i=1}^n \left\{ \frac{2(A_n - B_n)^2}{n} + \frac{(A_n^2 + B_n^2)(a_{i,n}^2(1) + a_{i,n}^2(2))}{\sqrt{n}} - 2\frac{(A_n - B_n)^2(a_{i,n}(1) + a_{i,n}(2))}{\sqrt{n}} - 4A_nB_na_{i,n}(1)a_{i,n}(2) \right\}^{3/2}.
\]
In summary, so far, we have shown that

\[
\sup_{(x_1, x_2) \in \mathbb{R}^2} \left| \bigotimes_{b=1}^{2} P_{X|w(b)} \left[ S_n(T_n(X - \mu) - T_{m_n}^*(1))/\sigma \leq x_1 \right] \right.
\]

\[-P(Z_1 \leq x_1, Z_2 > x_2)\]

\[\leq c2^{-3/2}E_{X} |X_1 - \mu|^3 \sum_{i=1}^{n} \left\{ \frac{2(A_n - B_n)^2}{n} + \left( A_n^2 + B_n^2 \right) \left( a_{i,n}^2(1) + a_{i,n}^2(2) \right) \right\}
\]

\[-2 \frac{(A_n - B_n)^2 (a_{i,n}(1) + a_{i,n}(2))}{n} - 4A_nB_n a_{i,n}(1)a_{i,n}(2) \right\} \frac{3}{2}
\]

\[=: R(n). \quad (9.16)\]

To investigate the speed at which \(P_w(R(n) > \varepsilon)\) approaches zero, as \(n, m_n \to +\infty\), we first note that there is no cancelation of terms in the general term of the sum in \(R(n)\) for each fixed \(n, m_n\). The other important observation concerns \((A_n - B_n)^2\), which is the coefficient of the term 1/n in \(R(n)\). It can be shown that as \(n, m_n \to +\infty\) in such a way that \(m_n = o(n^2)\),

\[\sum_{i=1}^{n} a_{i,n}(1)a_{i,n}(2) \to 0 \text{ in probability} - P_w \quad (9.17)\]

(cf. Appendix 2 for details). The latter means that, as \(n, m_n \to +\infty\) in such a way that \(m_n = o(n^2)\), the following in probability-\(P_w\) statement holds true.

\[(A_n - B_n)^2 \to \left( \left( (1 + \sqrt{3/4})^2 - 1/2(1 + \sqrt{3/4}) \right)/ \left( \sqrt{2 + 2\sqrt{3/4}(7/4 + \sqrt{34})} \right) \right)^2
\]

\[=: D > 0 \quad (9.18)\]

The preceding shows that \(A_n\) and \(B_n\) do not contribute to the speed at which \(P_w(R(n) > \varepsilon)\) approaches zero at a rate no faster than \(n^{-1/2}\). To further elaborate on the latter conclusion we employ Markov’s inequality followed by an application of Jensen’s inequality to write

\[P_w(R(n) > \varepsilon) \leq \varepsilon^{-1} c_2 2^{-3/2} n E_w^{1/2} \left[ \frac{2(A_n - B_n)^2}{n} + \left( A_n^2 + B_n^2 \right) \left( a_{i,n}^2(1) + a_{i,n}^2(2) \right) \right]
\]

\[-2 \frac{(A_n - B_n)^2 (a_{i,n}(1) + a_{i,n}(2))}{n} - 4A_nB_n a_{i,n}(1)a_{i,n}(2) \right\} \frac{3}{2}
\]

\[\leq \varepsilon^{-1} c n^{-1/2} E_w^{1/2} |A_n - B_n|^3 + L(n, c, \varepsilon). \quad (9.19)\]
It is easy to check that $A_n - B_n$ is uniformly bounded in $n$. By this, and in view of (9.18), the dominated convergence theorem implies that, as $n, m_n \to +\infty$,

$$E_{\omega}^{1/2} |A_n - B_n|^3 \to D^{3/2}.$$  

Now, it is clear that the R.H.S. of (9.19) approaches zero no faster than $n^{-1/2}$.

To complete the proof of this theorem, for $\varepsilon_1, \varepsilon_2 > 0$, we use a Slutsky type argument to arrive at the following approximation.

$$-P_{X|w}(\left| \frac{T_n(X - \mu) - G^*_{m_n}(1)}{\sigma \sqrt{2}}(\sigma / S_n - 1) \right| > \varepsilon_1)$$

$$-P_{X|w}(\left| \frac{T_n(X - \mu) - G^*_{m_n}(2)}{\sigma \sqrt{2}}(\sigma / S_n - 1) \right| > \varepsilon_2)$$

$$+P_{X|w}(\left| \frac{T_n(X - \mu) - G^*_{m_n}(1)}{\sigma \sqrt{2}} \leq x_1 - \varepsilon_1 \right| \frac{T_n(X - \mu) - G^*_{m_n}(2)}{\sigma \sqrt{2}} > x_2 + \varepsilon_2) - \Phi(x_1 - \varepsilon_1, x_2 + \varepsilon_2)$$

$$+\Phi(x_1 - \varepsilon_1, x_2 + \varepsilon_2) - \Phi(x_1, x_2)$$

$$\leq P_{X|w}(\left| \frac{T_n(X - \mu) - G^*_{m_n}(1)}{\sigma \sqrt{2}} \leq x_1 \right| \frac{T_n(X - \mu) - G^*_{m_n}(2)}{\sigma \sqrt{2}} > x_2) - \Phi(x_1, x_2)$$

$$\leq P_{X|w}(\left| \frac{T_n(X - \mu) - G^*_{m_n}(1)}{\sigma \sqrt{2}}(\sigma / S_n - 1) \right| > \varepsilon_1)$$

$$+P_{X|w}(\left| \frac{T_n(X - \mu) - G^*_{m_n}(2)}{\sigma \sqrt{2}}(\sigma / S_n - 1) \right| > \varepsilon_2)$$

$$+P_{X|w}(\left| \frac{T_n(X - \mu) - G^*_{m_n}(1)}{\sigma \sqrt{2}} \leq x_1 + \varepsilon_1 \right| \frac{T_n(X - \mu) - G^*_{m_n}(2)}{\sigma \sqrt{2}} > x_2 - \varepsilon_2) - \Phi(x_1 + \varepsilon_1, x_2 - \varepsilon_2)$$

$$+\Phi(x_1 + \varepsilon_1, x_2 - \varepsilon_2) - \Phi(x_1, x_2).$$

By virtue of (9.16) and also by continuity of the bivariate normal distribution function, for some $\varepsilon_3 > 0$, we can replace the preceding approximations by
\[ P_{x|w} \left( \frac{T_n(X-\mu)-G_{mn}^*(1)}{\sqrt{2}} \leq x_1 \right) - \Phi(x_1, x_2) \]
\[ \leq P_{x|w} \left( \left| \frac{S_n(T_n(X-\mu)-G_{mn}^*(1))}{\sigma \sqrt{2}} \right| > \varepsilon_1 \right) \]
\[ + P_{x|w} \left( \left| \frac{S_n(T_n(X-\mu)-G_{mn}^*(2))}{\sigma \sqrt{2}} \right| > \varepsilon_2 \right) \]
\[ + R(n) + \varepsilon_3. \]

One can show that, as \( n, m_n \to +\infty \) so that \( m_n = o(n^2) \), \( P_{x|w}(S_n(T_n(X-\mu)-G_{mn}^*)/\sigma \sqrt{2} \leq t) \to \Phi(t) \) in probability-\( P_w \) (cf. Appendix 3 for details).

The latter implies that the first two terms in the R.H.S. of (9.20) approach zero, as \( n, m_n \to +\infty \). As we have already noted, \( R(n) \) goes to zero with a rate that as best is \( n^{-1/2} \) in probability-\( P_w \). By sending \( \varepsilon_1, \varepsilon_2 \to 0 \), \( \varepsilon_3 \) goes to zero too and this finishes the proof of Theorem 7.1. \( \square \)

10  Appendix 1

Consider the original sample \((X_1, \ldots, X_n)\) and assume that the sample size \( n \geq 1 \) is fixed. It is known that the bootstrap estimator of the mean based on \( B \) independent sub-samples \((X_1^*(b), \ldots, X_n^*(b))\), \( 1 \leq b \leq B \) can be computed as

\[ \hat{X}_{nB}^* = \frac{\sum_{b=1}^B \hat{X}_n^*(b)}{B} \]
\[ \quad = \frac{1}{nB} \sum_{i=1}^n \left( \sum_{b=1}^B w_i^{(n)}(b) \right) X_i, \]

where \( w_i^{(n)}(b) \) is the # of times the index \( i \), i.e., \( X_i \) is chosen in the \( b \)th bootstrap sub-sample \((X_1^*(b), \ldots, X_n^*(b))\). Observe now that \( \sum_{b=1}^B w_i^{(n)}(b) \) counts the total # of times \( X_i \) has appeared in the \( m := nB \) bootstrap sub-samples. Also, observe that for fixed \( n \), \( B \to +\infty \) is equivalent to \( m \to +\infty \). Therefore, in view of (10.1) we can write

\[ \hat{X}_{nB}^* = X_m^*. \]

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This means that taking a large number, \( B \), of independent bootstrap subsamples is equivalent to taking only one large bootstrap sub-sample.

We are now to show that when \( n \) is fixed, as \( m \to +\infty \), we have \( \bar{X}_{*m} \to \bar{X}_n \) in probability \( P_{X,w} \). To do so, without loss of generality we assume that \( \mu = 0 \), let \( \varepsilon_1, \varepsilon_2 > 0 \) and write

\[
P_w \{ P_{X,w}( | \bar{X}_{*m} - \bar{X}_n | > \varepsilon_1 ) > \varepsilon_2 \} \leq P_w \{ E_{X,w}( \sum_{i=1}^{n} \left( \frac{w_i(n)}{m} - \frac{1}{n} \right) X_i )^2 > \varepsilon_1^2 \varepsilon_2 \}
\]

\[
= P_w \{ \sum_{i=1}^{n} \left( \frac{w_i(n)}{m} - \frac{1}{n} \right)^2 > \sigma^{-2} \varepsilon_1 \varepsilon_2 \}
\]

\[
\leq \sigma^{-2} \varepsilon_1^{-2} \varepsilon_2^{-1} n E_w \left( \frac{w_i(n)}{m} - \frac{1}{n} \right)^2
\]

\[
\leq \sigma^{-2} \varepsilon_1^{-2} \varepsilon_2^{-1} \left( \frac{1 - \frac{1}{n}}{m} \right) \to 0, \text{ as } m \to \infty.
\]

(10.3)

The preceding means that \( P_{X,w}( | \bar{X}_{*m} - \bar{X}_n | > \varepsilon_1 ) \to 0 \) in probability-\( P_w \), hence from the dominated convergence theorem we conclude that \( \bar{X}_{*m} \to \bar{X}_n \) in probability \( P_{X,w} \). □

We are now to show that the bootstrap sample variance which we denote by \( S_{m}^{*2} \) is a in probability consistent estimator of the ordinary sample variance \( S_n^2 \) for each fixed \( n \), when \( m \to +\infty \). To do so, we denote the bootstrap sub-sample of size \( m \) by \( (X_1^*, \ldots, X_m^*) \). By this we have that \( S_{m}^{*2} = \sum_{k=1}^{m} (X_k^* - \bar{X}_m^*)^2 / (m - 1) \). Employing now the \( u \)-statistic representation of the sample variance enables us to write

\[
S_{m}^{*2} = \frac{\sum_{1 \leq k \leq l \leq m} (X_k^* - X_l^*)^2}{2m(m-1)}
\]

\[
= \frac{\sum_{1 \leq i \leq j \leq n} w_i^{(n)} w_j^{(n)} (X_i - X_j)^2}{2m(m-1)}.
\]

The preceding relation is the weighted form of \( S_{m}^{*2} \) and it is based on the fact that the terms \( (X_k^* - X_l^*)^2 \), \( 1 \leq k \neq l \leq m \), are replications of \( (X_i - X_j)^2 \), \( 1 \leq i \neq j \leq n \). Therefore, the deviation \( S_{m}^{*2} - S_n^2 \) can be written as follows.

\[
\sum_{1 \leq i \neq j \leq n} \left( \frac{1}{2m(m-1)} - \frac{1}{2n(n-1)} \right)(X_i^2 - X_j^2).
\]

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Now for $\varepsilon_1, \varepsilon_2 > 0$ we write

\[
P_w\left(P_{X|w}\left(|S_m^2 - S_n^2| > \varepsilon_1\right) > \varepsilon_2\right)
\]

\[
= P_w\left(P_{X|w}\left(|\sum_{1 \leq i \neq j \leq n}\left(\frac{w_i^{(n)}w_j^{(n)}}{m(m-1)} - \frac{1}{n(n-1)}\right)(X_i - X_j)^2| > 2\varepsilon_1\right) > \varepsilon_2\right)
\]

\[
\leq P_w\left(\sum_{1 \leq i \neq j \leq n}\left|\frac{w_i^{(n)}w_j^{(n)}}{m(m-1)} - \frac{1}{n(n-1)}\right|E_X(X_i - X_j)^2 > 2\varepsilon_1\varepsilon_2\right)
\]

\[
\leq P_w\left(\sum_{1 \leq i \neq j \leq n}\left|\frac{w_i^{(n)}w_j^{(n)}}{m(m-1)} - \frac{1}{n(n-1)}\right| > \varepsilon_1\varepsilon_2\sigma^{-2}\right). \quad (10.4)
\]

The preceding relation can be bounded above by:

\[
\varepsilon_1^{-2}\varepsilon_2^{-2}\sigma^4\left\{n(n-1)E_w\left(\left|\frac{w_i^{(n)}w_j^{(n)}}{m(m-1)} - \frac{1}{n(n-1)}\right|^2\right)\right.
\]

\[+ n(n-1)(n-2)E_w\left(\left|\frac{w_i^{(n)}w_j^{(n)}}{m(m-1)} - \frac{1}{n(n-1)}\right|\left|\frac{w_i^{(n)}w_j^{(n)} - w_i^{(n)}w_j^{(n)}}{m(m-1)} - \frac{1}{n(n-1)}\right|\right)
\]

\[+ n(n-1)(n-2)(n-3)E_w\left(\left|\frac{w_i^{(n)}w_j^{(n)}}{m(m-1)} - \frac{1}{n(n-1)}\right|\left|\frac{w_i^{(n)}w_j^{(n)} - w_i^{(n)}w_j^{(n)}}{m(m-1)} - \frac{1}{n(n-1)}\right|\right)\left\}\right.
\]

\[\leq \varepsilon_1^{-2}\varepsilon_2^{-2}\sigma^4\left\{n(n-1)E_w\left(\left|\frac{w_i^{(n)}w_j^{(n)}}{m(m-1)} - \frac{1}{n(n-1)}\right|^2\right)\right.
\]

\[+ n(n-1)(n-2)E_w\left(\left|\frac{w_i^{(n)}w_j^{(n)}}{m(m-1)} - \frac{1}{n(n-1)}\right|^2\right)
\]

\[+ n(n-1)(n-2)(n-3)E_w\left(\left|\frac{w_i^{(n)}w_j^{(n)}}{m(m-1)} - \frac{1}{n(n-1)}\right|^2\right)\right\}
\]

\[= \varepsilon_1^{-2}\varepsilon_2^{-2}\sigma^4\left\{n(n-1)\left(\frac{1}{n^4m^2} + \frac{n}{n^4m^2} + \frac{n^2}{n^4m^2}\right)\right\}.
\]

Clearly, the preceding term approaches zero when $m \to +\infty$, for each fixed $n$. By this we have shown that $S_m^2 \to S_n^2$ in probability-$P_{X,w}$, when $n$ is fixed and only $m \to +\infty$. □
Consistency of $X_{n,m_n}$ in (3.3)

We give the proof of (5.16) for $m_n = n$, noting that the below proof remains the same for $m_n \leq n$ and it can be adjusted for the case $m_n = kn$, where $k$ is a positive integer. In order establish (5.16) when $m_n = n$, we first note that

$$E_{X|w} \left( \sum_{i=1}^{n} \left| \frac{w_i^{(n)}}{n} - \frac{1}{n} \right| \right) = 2(1 - \frac{1}{n})^n$$

and for $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$, we proceed as follows.

$$P_w \{ P_{X|w} (|X_{n,m_n} - \mu| > \varepsilon_1) > \varepsilon_2 \} \leq P_w \{ P_{X|w} (|X_{n,m_n} - \mu| > \varepsilon_1) > \varepsilon_2, \left| \sum_{i=1}^{n} \frac{w_i^{(n)}}{n} - \frac{1}{n} \right| - 2(1 - \frac{1}{n})^n \leq \varepsilon_3 \}$$

$$+ P_w \{ \left| \sum_{j=1}^{n} \frac{w_j^{(n)}}{n} - \frac{1}{n} \right| - 2(1 - \frac{1}{n})^n > \varepsilon_3 \}$$

$$\leq P_w \{ P_{X|w} \left( \sum_{i=1}^{n} \left| \frac{w_i^{(n)}}{n} - \frac{1}{n} \right| - \varepsilon_1 (1 - \frac{1}{n})^n - \varepsilon_3 \right) > \varepsilon_2 \}$$

$$+ \varepsilon_3^{-2} E_w \left( \sum_{j=1}^{n} \left| \frac{w_j^{(n)}}{n} - \frac{1}{n} \right| - 2(1 - \frac{1}{n})^n \right)^2$$

$$\leq P_w \{ \sum_{i=1}^{n} \left( \frac{w_i^{(n)}}{n} - \frac{1}{n} \right)^2 > \sigma^{-2} (2(1 - \frac{1}{n})^n - \varepsilon_3)^2 \varepsilon_2 \}$$

$$+ \varepsilon_3^{-2} \{ nE_w (\frac{w_i^{(n)}}{n} - \frac{1}{n})^2 + n(n-1)E_w (\frac{w_i^{(n)}}{n} - \frac{1}{n}) (\frac{w_2^{(n)}}{n} - \frac{1}{n}) - 4(1 - \frac{1}{n})^{2n} \}$$

$$=: K_1(n) + K_2(n).$$

A similar argument to that in (10.3) implies that, as $n \to +\infty$, and then $\varepsilon_3 \to 0, K_1(n) \to 0$. As for $K_2(n)$, we note that

$$E_w (\frac{w_i^{(n)}}{n} - \frac{1}{n})^2 = n^{-2}(1 - \frac{1}{n})$$

$$E_w \left( \left| \frac{w_i^{(n)}}{n} - \frac{1}{n} \right| \left| \frac{w_2^{(n)}}{n} - \frac{1}{n} \right| \right) = -n^{-3} + 4n^{-2}(1 - \frac{1}{n})^n (1 - \frac{1}{n - 1})^n.$$

Observing now that, as $n \to +\infty$,

$$n(n-1)E_w (\left| \frac{w_i^{(n)}}{n} - \frac{1}{n} \right| \left| \frac{w_2^{(n)}}{n} - \frac{1}{n} \right|) - 4(1 - \frac{1}{n})^{2n} \to 0,$$
we imply that, as \( n \to +\infty \), \( K_2(n) \to 0 \). By this we have concluded the consistency of \( \bar{X}_{n,m_n}^* \) for the population mean \( \mu \), when \( m_n = n \). □

11 Appendix 2

We are now to show that, as \( n, m \to +\infty \) in such a way that \( m_n/n^2 \to 0 \), as in Theorem 7.1, (9.17) holds true. In order to do so, we let \( \varepsilon_1, \varepsilon_2 \) and \( \varepsilon_3 \) be arbitrary positive numbers and write

\[
P\left( \left| \sum_{i=1}^{n} a_{i,n}(1) a_{i,n}(2) \right| > \varepsilon_1 \right)
\]

\[
\leq P\left( \frac{m_n}{1 - \frac{1}{n}} \left| \sum_{i=1}^{n} \left( \frac{w_i^{(n)}(1)}{m_n} - \frac{1}{n} \right) \frac{w_i^{(n)}(2)}{m_n} - \frac{1}{n} \right| > \varepsilon_1 \right) + P\left( \left| \sum_{i=1}^{n} \left( \frac{w_i^{(n)}(1)}{m_n} - \frac{1}{n} \right)^2 - 1 \right| > \varepsilon_2 \right) + P\left( \left| \sum_{i=1}^{n} \left( \frac{w_i^{(n)}(2)}{m_n} - \frac{1}{n} \right)^2 - 1 \right| > \varepsilon_3 \right)
\]

=: \( \pi_1(n) + \pi_2(n) + \pi_3(n) \).

The last two terms in the preceding relation have already been shown to approach zero as \( m_n^2/n \to 0 \) (cf. (9.8)). We now show that the first term approaches zero as well in view of the following argument which relies on the facts that \( w_i^{(n)} \)'s, \( 1 \leq i \leq n \) are multinomially distributed and that for each \( 1 \leq i, j \leq n \), \( w_i^{(n)}(1) \) and \( w_j^{(n)}(2) \) are i.i.d. (in terms of \( P_w \)).

To show that \( \pi_1(n) = o(1) \), as \( n, m_n \to +\infty \) such that \( m_n/n^2 \to 0 \), in what will follow, we put \( \varepsilon_4 := \varepsilon_1(1 - \varepsilon_2)(1 - \varepsilon_3) \) to write

\[
\pi_1(n) \leq \varepsilon_4^2 \frac{m_n^2}{(1 - \frac{1}{n})^2} \left\{ nE_w^2 \left( \frac{w_i^{(n)}(1)}{m_n} - \frac{1}{n} \right)^2 + n(n-1)E_w^2 \left[ \left( \frac{w_i^{(n)}(1)}{m_n} - \frac{1}{n} \right) \left( \frac{w_i^{(n)}(2)}{m_n} - \frac{1}{n} \right) \right] \right\}
\]

\[
= \varepsilon_4^2 \frac{m_n^2}{(1 - \frac{1}{n})^2} \left\{ n \left( \frac{1 - \frac{1}{n}}{m_n} \right)^2 + n(n-1) \left( \frac{-1}{m_n n^2} \right)^2 \right\}
\]

\[
\leq \varepsilon_4^2 \left( \frac{1}{n} + \frac{1}{n^2(1 - \frac{1}{n})^2} \right) \to 0.
\]
The preceding result completes the proof of (9.17). □

12 Appendix 3

Noting that the expression \( S_n(T_n(X - \mu) - G_{m_n}^*)/\sigma\sqrt{2} \) can be written as

\[
S_n(T_n(X - \mu) - G_{m_n}^*)/\sigma\sqrt{2} = \sum_{i=1}^{n} \left( \frac{1}{\sqrt{n}} - \frac{(w_i^{(n)} - \frac{1}{n})}{\sqrt{\sum_{j=1}^{n} (w_j^{(n)} - \frac{1}{n})^2}} \right) (X_i - \mu)
\]

makes it clear that, in view of Lindeberge-Feller CLT, in order to have \( P_{X_{w}}(S_n(T_n(X - \mu) - G_{m_n}^*)/\sigma\sqrt{2} \leq t) \rightarrow \Phi(t) \) in probability-\( P_{w} \), it suffices to show that, as \( n, m_n \rightarrow +\infty \) so that \( m_n = o(n^2) \),

\[
\max_{1 \leq i \leq n} \left| \frac{(w_i^{(n)} - \frac{1}{n})}{\sqrt{\sum_{j=1}^{n} (w_j^{(n)} - \frac{1}{n})^2}} - \frac{1}{\sqrt{n}} \right| = o_{P_{w}}(1).
\]

(12.1)

Considering that the L.H.S. of (12.1) is bounded above by

\[
\max_{1 \leq i \leq n} \frac{|w_i^{(n)} - \frac{1}{n}|}{\sqrt{\sum_{j=1}^{n} (w_j^{(n)} - \frac{1}{n})^2}} + \frac{1}{\sqrt{n}},
\]

(12.1) follows if one shows that, as \( n, m_n \rightarrow +\infty \) so that \( m_n = o(n^2) \),

\[
\max_{1 \leq i \leq n} \frac{|w_i^{(n)} - \frac{1}{n}|}{\sqrt{\sum_{j=1}^{n} (w_j^{(n)} - \frac{1}{n})^2}} = o_{P_{w}}(1).
\]

(12.2)

In order to establish the latter, for \( \varepsilon, \varepsilon' > 0 \), we write:

\[
P_{w}\left( \frac{\max_{1 \leq i \leq n} (w_i^{(n)} - \frac{1}{n})^2}{\sum_{i=1}^{n} (w_i^{(n)} - \frac{1}{n})^2} > \varepsilon \right)
\]

\[
\leq P_{w}\left( \frac{\max_{1 \leq i \leq n} (w_i^{(n)} - \frac{1}{n})^2}{\sum_{i=1}^{n} (w_i^{(n)} - \frac{1}{n})^2} > \varepsilon, \left| m_n - \frac{1}{n} \sum_{i=1}^{n} (w_i^{(n)} - \frac{1}{n})^2 - 1 \right| \leq \varepsilon' \right)
\]

\[
+ P_{w}\left( \left| \frac{1}{1 - \frac{1}{n}} \sum_{i=1}^{n} (w_i^{(n)} - \frac{1}{n})^2 - 1 \right| > \varepsilon' \right)
\]

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An upper bound for \( P_w(\max_{1 \leq i \leq n} (w_i^{(n)} - \frac{1}{n})^2 > \frac{\varepsilon(1 - \varepsilon')(1 - \frac{1}{n})}{m_n}) \) is

\[
nP_w \left( \frac{w_1^{(n)}}{m_n} - \frac{1}{n} \right) > \sqrt{\frac{\varepsilon(1 - \varepsilon')(1 - \frac{1}{n})}{m_n}} 
\]

\[
\leq n \exp \left\{-\frac{\varepsilon(1 - \varepsilon')(1 - \frac{1}{n})}{2 \left( \frac{m_n}{n} + \sqrt{\varepsilon(1 - \varepsilon')(1 - \frac{1}{n})} \right)} \right\}.
\]

The preceding relation, which is due to Bernstein’s inequality, is a general term of a finite series when \( m_n = o(n^2) \).

\( P_w(\sum_{i=1}^{n} (w_i^{(n)} - \frac{1}{n})^2 - \frac{(1 - \frac{1}{n})}{m_n}) > \frac{\varepsilon(1 - \varepsilon')(1 - \frac{1}{n})}{m_n} \) was already shown, (cf. (9.8)), to approach zero as \( n, m_n \to +\infty \) such that \( m_n = o(n^2) \).

13 Appendix 4

Viewing the bootstrapped mean as a randomly weighted partial sum, allows one to think about randomly weighted partial sums of the general form

\[
\sum_{i=1}^{n} v_i^{(n)} X_i,
\]  

where, \( X_i \)'s are i.i.d. random variables and \( v_i^{(n)} \)'s are random weights which are independent from \( X_i \)'s and posses the properties that \( E_v(v_i^{(n)}/m_n) = 1/n \) and \( \sum_{i=1}^{n} v_i^{(n)} = m_n \). The motivation behind studying the latter sums is that, in addition to bootstrapping, they allows considering the problem of stochastically re-weighing (designing) the observations using a random weights \( v_i^{(n)} \) whose distribution is usually known. Naturally, in this setup, on taking \( v_i^{(n)} := w_i^{(n)} \) for each \( 1 \leq i \leq n \), \( n \geq 1 \), i.e., on assuming the random weights to be multinomially distributed, the randomly weighted partial sum in (13.1)
coincides with the bootstrapped mean as in (2.4). This idea first appeared in the area of Bayesian bootstrap (cf. for example Rubin [9]). The convergence in distribution of the partial sums of (13.1) were also studied by Csörgő et al. [2] via conditioning on the weights (cf. Theorem 2.1 and Corollary 2.2 therein). Noting that in the latter results only randomly weighted statistics similar to $T^*_m$ which are natural pivots for the sample variance $\bar{X}_n$ were studied. In view of the fact that $G^*_m$, as defined by (1.4), was seen to be a natural pivot for the population mean $\mu := E_X X$, in a similar fashion and to Theorem 2.1 and its Corollary 2.2 of Csörgő et al. [2], here we state a more general conditional CLT, given the weights $v^{(n)}_i$’s, for the partial sums $\sum_{i=1}^n |v^{(n)}_i m - \frac{1}{n}| (X_i - \mu)$, noting that the proofs of these results almost identical to those of Theorem 2.1 and its Corollary 2.2 of Csörgő et al. [2] in view of the more general setup on notations in the latter paper. In order to estate the results we first generalize the definition of $G^*_{m_n}$ and $G^{**}_{m_n}$, as defined in (1.4) and (2.1), respectively, to become

\[ G^*_m := \frac{\sum_{i=1}^n |v^{(n)}_i m - \frac{1}{n}| (X_i - \mu)}{S_n \sqrt{\sum_{i=1}^n (v^{(n)}_i m - \frac{1}{n})^2}}, \]  

\[ G^{**}_{m_n} := \frac{\sum_{i=1}^n |v^{(n)}_i m - \frac{1}{n}| (X_i - \mu)}{S^{**}_{m_n} \sqrt{\sum_{i=1}^n (v^{(n)}_i m - \frac{1}{n})^2}}, \]  

where, $m_n = \sum_{i=1}^n v^{(n)}_i$.

**Theorem 13.1.** Let $X, X_1, X_2, \ldots$ be real valued i.i.d. random variables with mean 0 and variance $\sigma^2$, and assume that $0 < \sigma^2 < \infty$. Put $V_{i,n} := |(v^{(n)}_i m_n - \frac{1}{n})X_i|$, $1 \leq i \leq n$, $V_n^2 := \sum_{i=1}^n (v^{(n)}_i m_n - \frac{1}{n})^2$, $M_n := \frac{\max_{1 \leq i \leq n} (v^{(n)}_i m_n - \frac{1}{n})^2}{\sum_{i=1}^n (v^{(n)}_i m_n - \frac{1}{n})^2}$, and let $Z$ be a standard normal random variable throughout. Then as, $n, m_n \to \infty$, having
\[ M_n = o(1) \text{ a.s.} - P_v \]

is equivalent to concluding the respective statements of
\[(13.5) \text{ and } (13.6) \]
simultaneously as follows
\[ P_{X|\nu}(\mathcal{G}_{m_n}^* \leq t) \longrightarrow P(Z \leq t) \text{ a.s.} - P_v \text{ for all } t \in \mathbb{R} \] (13.5)

and
\[ \max_{1 \leq i \leq n} P_{X|\nu}(V_{i,n}/(S_n V_n) > \varepsilon) = o(1) \text{ a.s.} - P_v, \text{ for all } \varepsilon > 0, \] (13.6)

and, in a similar vein, having
\[ M_n = o_{P_v}(1) \]

is equivalent to concluding the respective statements of
\[(13.8) \text{ and } (13.9) \]
as below simultaneously
\[ P_{X|\nu}(\mathcal{G}_{m_n}^* \leq t) \longrightarrow P(Z \leq t) \text{ in probability} - P_v \text{ for all } t \in \mathbb{R} \] (13.8)

and
\[ \max_{1 \leq i \leq n} P_{X|\nu}(V_{i,n}/(S_n V_n) > \varepsilon) = o_{P_v}(1), \text{ for all } \varepsilon > 0. \] (13.9)

Moreover, assume that, as \( n, m_n \to \infty \), we have for any \( \varepsilon > 0 \),
\[ P_{X|\nu}\left( \left| \frac{S_{m_n}^2 / m_n}{\sigma^2 \sum_{i=1}^n \left( \frac{v_n(i)}{m_n} - \frac{1}{n} \right)^2} - 1 \right| > \varepsilon \right) = \begin{cases} o(1) \text{ a.s.} - P_v, & (13.10) \\ o_{P_v}(1). & (13.11) \end{cases} \]

Then, as \( n, m_n \to \infty \), via (13.10), the statement of (13.4) is also equivalent to having (13.12) and (13.13) simultaneously as below
\[ P_{X|\nu}(\mathcal{G}_{m_n}^{**} \leq t) \longrightarrow P(Z \leq t) \text{ a.s.} - P_v \text{ for all } t \in \mathbb{R} \] (13.12)

and
\[ \max_{1 \leq i \leq n} P_{X|\nu}(V_{i,n}/(S_{m_n}^*/\sqrt{m_n}) > \varepsilon) = o(1) \text{ a.s.} - P_v, \text{ for all } \varepsilon > 0, \] (13.13)

and, in a similar vein, via (13.11), the statement (13.7) is also equivalent to having (13.14) and (13.15) simultaneously as below

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\[ P_{X|\nu} \left( T_{m_n}^{**} \leq t \right) \rightarrow P(Z \leq t) \text{ in probability} - P_{\nu} \text{ for all } t \in \mathbb{R} \]  
(13.14)

and

\[ \max_{1 \leq i \leq n} P_{X|\nu}(V_{i,n}/(S_{m_n}^{*}/\sqrt{m_n}) > \varepsilon) = o_{P_{\nu}}(1), \text{ for all } \varepsilon > 0. \]  
(13.15)

For verifying the technical conditions (13.10) and (13.11) as above, one does not need to know the actual finite value of \( \sigma^2 \).

Now suppose that \( v_i^{(n)} = \zeta_i, 1 \leq i \leq n \), where \( \zeta_i \) are positive i.i.d. random variables. In this case, noting that the bootstrapped \( t \)-statistic \( G_{m_n}^{*} \) defined by (13.2) is of the form:

\[ G_{m_n}^{*} = \frac{\sum_{i=1}^{n} \left( \zeta_i - \frac{1}{n} \right) (X_i - \mu)}{S_n \sqrt{\sum_{i=1}^{n} \left( \zeta_i - \frac{1}{n} \right)^2}}, \]  
(13.16)

where \( m_n = \sum_{i=1}^{n} \zeta_i \).

The following Corollary 13.1 to Theorem 13.1 establishes the validity of this scheme of bootstrap for \( G_{m_n}^{*} \), as defined by (13.16), via conditioning on the weights \( \zeta_i \)'s.

**Corollary 13.1.** Assume that \( 0 < \sigma^2 = \text{var}(X) < \infty \), and let \( \zeta_1, \zeta_2, \ldots \) be a sequence of positive i.i.d. random variables which are independent of \( X_1, X_2, \ldots \). Then, as \( n \rightarrow \infty \),

(a) if \( E_\zeta(\zeta_1^4) < \infty \), then, mutatis mutandis, (13.4) is equivalent to having (13.5) and (13.6) simultaneously and, spelling out only (13.6), in this context it reads

\[ P_{X|\zeta}(G_{m_n}^{*} \leq t) \rightarrow P(Z \leq t) \text{ a.s.} - P_\zeta, \text{ for all } t \in \mathbb{R}, \]  
(13.17)

(b) if \( E_\zeta(\zeta_1^2) < \infty \), then, mutatis mutandis, (13.7) is equivalent (13.8) and (13.9) simultaneously, and spelling out only (13.8), in this context it reads

\[ P_{X|\zeta}(G_{m_n}^{*} \leq t) \rightarrow P(Z \leq t) \text{ in probability} - P_\zeta, \text{ for all } t \in \mathbb{R}, \]  
(13.18)

where \( Z \) is a standard normal random variable.
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