Hua’s Theorem with the Primes in Piatetski-Shapiro Prime Sets

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Abstract: In this paper, we study the hybrid problem of Hua’s theorem and the Piatetski-Shapiro prime number theorem, and obtain results in this direction of the nonhomogeneous case \( k = 3 \), which deepen the classical result of Hua.

Keywords: Piatetski-Shapiro prime; exponential sum; circle method; mixed power

1 Introduction and main result

In 1937, I. M. Vinogradov [27] solved the ternary Goldbach problem. He proved that, for sufficiently large odd integer \( N \), there holds

\[
\sum_{p_1 + p_2 + p_3 = N} 1 = \frac{1}{2} \mathcal{G}(N) \frac{N^2}{\log^3 N} + O\left(\frac{N^2}{\log^4 N}\right),
\]

where \( \mathcal{G}(N) \) is the singular series

\[
\mathcal{G}(N) = \prod_{p|N} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p|N} \left(1 + \frac{1}{(p-1)^2}\right).
\]

For any sufficiently large odd integer \( N \) and fixed positive integer \( k \), let \( \mathcal{R}(N, k) \) be the number of representations of \( N \) in the form

\[ N = p_1 + p_2 + p_3^k, \]

where \( p_1, p_2, p_3 \) are primes. In 1938, L. K. Hua [10] generalized the result of Vinogradov and proved that

\[
\mathcal{R}(N, k) = \frac{k^2}{k+1} \mathcal{G}(N, k) \frac{N^{1+1/k}}{\log^3 N} + O\left(\frac{N^{1+1/k}}{\log^4 N}\right),
\]

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where
\[ \mathcal{S}(N, k) = \prod_p \left(1 + \frac{B_p(N, k)}{(p-1)^3}\right), \]
\[ B_q(N, k) = \sum_{a=1}^{q} C_q(a, k) e\left(-\frac{aN}{q}\right), \quad C_q(a, k) = \sum_{\ell=1}^{q} e\left(\frac{\ell^k}{q}\right). \]

In 1986, Wirsing [28], motivated by the earlier work of Erdős and Nathanson [4] on sums of squares, considered the question of whether one could find thin subsets \( S \) of primes which were still sufficient to obtain all sufficiently large odd integers as sums of three of them. He obtained the very satisfactory answer that there exist such sets \( S \) with the property that \( \sum_{p \leq x, p \in S} 1 \ll (x \log x)^{1/3} \). This result was later rediscovered by Ruzsa. Wirsing’s result, which is obviously best possible apart from the logarithmic factor, is based on probabilistic considerations and does not lead to a subset of the primes which is constructive or recognizable.

We fix a real number \( c \) and consider the number of \( n \leq x \) such that the integer part \( [n^{1/c}] \) is a prime. In the case that \( 0 < c \leq 1 \) every prime \( \leq x^c \) occurs in this fashion and it is a simple consequence of the prime number theorem that we have the expected asymptotic formula
\[ \sum_{n \leq x} 1 = (1 + o(1)) \frac{x}{c \log x}. \] (1.1)

We let \( \gamma = 1/c \), so that the set of the Piatetski-Shapiro primes of type \( \gamma < 1 \)
\[ \mathcal{P}_\gamma = \{p : p = [n^{1/\gamma}] \text{ for some } n \in \mathbb{N}\} \]
is a well-known thin set of prime numbers. Piatetski-Shapiro [22] proved the much more difficult result that the asymptotic formula (1.1) still holds in the range \( 1 < c < 12/11 \). Since then, this range for \( c \) has been improved by a number of authors [1, 7, 12, 13, 15, 16, 18–20, 23]. The best results are given by [24] and [25], where it is proved that
\[ \pi_\gamma(x) \sim \frac{x^{\gamma}}{\log x} \]
for \( 2426/2817 < \gamma < 1 \), and
\[ \pi_\gamma(x) \gg \frac{x^{\gamma}}{\log x} \]
for \( 205/243 \leq \gamma < 1 \).

In 1992, A. Balog and J. P. Friedlander [2] considered the ternary Goldbach problem with variables restricted to Piatetski-Shapiro primes. They proved that, for \( 20/21 < \gamma \leq 1 \) fixed, any sufficiently large odd integer \( N \) can be written as three primes with each
prime of the form \([n^{1/\gamma}]\). Rivat [23] extended the range \(20/21 < \gamma \leq 1 \) to \(188/199 < \gamma \leq 1\); Kumchev [18] extended the range to \(50/53 < \gamma \leq 1\). Jia [14] used a sieve method to enlarge the range to \(15/16 < \gamma \leq 1\).

In 1998, Zhai [30] considered the hybrid problem of quadratic Waring-Goldbach problem with each prime variable restricted to Piatetski-Shapiro sets. To be specific, he proved that, for \(43/44 < \gamma \leq 1\) fixed, every sufficiently large integer \(N\) satisfying \(N \equiv 5 \pmod{24}\) can be written as five squares of primes with each prime of the form \([n^{1/\gamma}]\). Later, in 2005, Zhang and Zhai [29] improved the result of Zhai [30] and enlarge the range to \(249/256 < \gamma \leq 1\).

In 2004, Cui [3] studied the hybrid problem of Hua’s theorem \((k = 2)\) with each prime variable restricted to Piatetski-Shapiro sets. He proved that, for any \(104/105 < \gamma \leq 1\) fixed, every sufficiently large odd integer can be written as the sum of two primes and a prime square with all primes of the form \([n^{1/\gamma}]\).

In this paper, we consider the hybrid problem of Hua’s theorem \((k = 3)\) with each prime variable restricted to Piatetski-Shapiro sets and prove the following theorem.

**Theorem 1.1**  Let \(0 < \gamma_i \leq 1\) \((i = 1, 2, 3)\), \(0 < \delta_j < 1\) \((j = 1, 3)\) satisfying

\[
\begin{align*}
\frac{\gamma_1 + \gamma_2}{2} + \frac{\delta_1}{40} &> 1, \\
\frac{\gamma_1 + \gamma_2}{2} + \frac{\delta_3}{3} &> 1, \\
73(1 - \gamma_i) + 86\delta_1 &< 9 \quad (i = 1, 2), \\
1714(1 - \gamma_3) + 1725\delta_3 &< 46.
\end{align*}
\]

Then for sufficiently large odd integer \(N\), the equation

\[N = p_1 + p_2 + p_3^3, \quad p_i \in \mathcal{P}_{\gamma_i}, \quad i = 1, 2, 3\]

is solvable.

From Theorem 1.1, we know that one may require three summands to be Piatetski-Shapiro primes of different type. In particular, by choosing \(\gamma_1 = \gamma_2 = \gamma_3 = \gamma\), we obtain

**Corollary 1.2**  For any fixed \(2816/2825 < \gamma \leq 1\), every sufficiently large odd integer \(N\) can be written as the sum of two primes and a cube of prime with all primes of the form \([n^{1/\gamma}]\).

However, the above result is not the best one. Taking \(\gamma_1 = \gamma_2 = \gamma\) as in Corollary 1.2, we can enlarge the range of the value of \(\gamma_3\) and obtain...
Corollary 1.3 For any fixed $\frac{2816}{2825} < \gamma \leq 1$ and $\frac{3335}{199692} < \gamma_3 \leq 1$, every sufficiently large odd integer $N$ can be written as the sum of two primes of the form $\lfloor n^{1/\gamma} \rfloor$ and a cube of prime with the prime of the form $\lfloor n^{1/\gamma_3} \rfloor$.

If we take $\gamma_1 = \gamma_2 = 1$, then we can obtain

Corollary 1.4 For any fixed $\frac{1668}{1714} < \gamma \leq 1$, every sufficiently large odd integer $N$ can be written as the sum of two primes and a cube of prime with the last prime of the form $\lfloor n^{1/\gamma} \rfloor$.

Notation. Throughout this paper, $p, p_1, \cdots$ are primes; $N$ always denotes a sufficiently large natural number; $\varepsilon$ always denotes an arbitrary small positive constant, which may not be the same at different occurrences; $n \sim X$ means $X < n \leq 2X$. We use $[x], \{x\}$ and $\|x\|$ to denote the integral part of $x$, the fractional part of $x$ and the distance from $x$ to the nearest integer correspondingly. $\Lambda(n)$ denotes von Mangold’s function; $\mu(n)$ denotes Möbius function; $e(x) = e^{2\pi ix}$; $\mathcal{L} = \log N$; $\psi(x) = x - \lfloor x \rfloor - \frac{1}{2}$. $f(x) \ll g(x)$ means that $f = O(g(x))$; $f(x) \asymp g(x)$ means that $f(x) \ll g(x) \ll f(x)$.

We also define

$$P = N^{1/3}, \quad S_1(N, \alpha) = \sum_{p \leq N} e(\alpha p), \quad S_3(N, \alpha) = \sum_{p \leq P} e(\alpha p^3),$$

$$T_1(N, \alpha) = \frac{1}{\gamma} \sum_{p \leq N, p \in \mathcal{P}_\gamma} p^{1-\gamma} e(\alpha p), \quad T_3(N, \alpha) = \frac{1}{\gamma} \sum_{p \leq P, p \in \mathcal{P}_\gamma} p^{1-\gamma} e(\alpha p^3),$$

$$T_{1,i}(N, \alpha) = \frac{1}{\gamma_i} \sum_{p \leq N, p \in \mathcal{P}_{\gamma_i}} p^{1-\gamma_i} e(\alpha p), \quad (i = 1, 2).$$

2 Preliminary Lemmas

Lemma 2.1 For any real numbers $\alpha$ and $\tau \geq 1$, there must be integers $a$ and $q$, $(a, q) = 1$, $1 \leq q \leq \tau$, such that

$$\alpha = \frac{a}{q} + \lambda, \quad |\lambda| \leq \frac{1}{q^\tau}. \quad (2.1)$$

Proof. See C. D. Pan and C. B. Pan [21], Lemma 5.19. ⊠

Lemma 2.2 Let $\alpha$ be as in Lemma 2.1. Then

$$S_1(N, \alpha) \ll N \mathcal{L}^4 \left( \frac{1}{q^{1/2}} + \frac{1}{N^{1/6}} + \frac{q^{1/2}}{N^{1/2}} \right).$$
Proof. See Vaughan [26], Theorem 3.1. ■

Lemma 2.3 Let \( \alpha \) be as in Lemma 2.1. Then

\[
S_3(N, \alpha) \ll N^{1/3+\varepsilon} \left( \frac{1}{q} + \frac{1}{N^{1/6}} + \frac{q}{N} \right)^{1/16}.
\]

Proof. See Harman [6], Theorem 1. ■

Lemma 2.4 Let \( \gamma, \delta_1 \) satisfy \( 0 < \gamma \leq 1, \delta_1 > 0 \) and

\[
73(1-\gamma) + 86\delta_1 < 9.
\]

Then, uniformly in \( \alpha \), we have

\[
T_1(N, \alpha) = S_1(N, \alpha) + O\left(N^{1-\delta_1-\varepsilon}\right)
\]

where the implied constant depends only on \( \gamma \) and \( \delta_1 \).

Proof. This is, all in essentials, deduced from the process of the proof of Kumchev [17] Theorem 2. ■

Lemma 2.5 We have

\[
\int_0^1 \left| T_1(N, \alpha) \right|^2 d\alpha \ll N^{2-\gamma}.
\]

Proof. See Cui [3], Lemma 6. ■

Lemma 2.6 Suppose that \( f(x) : [a, b] \to \mathbb{R} \) has continuous derivatives of order up to 2 on \([a, b]\), where \( 1 \leq a < b \leq 2a \). Suppose further that

\[
0 < c_1 \lambda_1 \leq |f'(x)| \leq c_2 \lambda_1, \quad c_3 \lambda_1 a^{-1} \leq |f''(x)| \leq c_4 \lambda_1 a^{-1}, \quad x \in [a, b],
\]

where \( c_j (j = 1, 2, 3, 4) \) are absolute constants. Then

\[
\sum_{a<n \leq b} e\left(f(n)\right) \ll a^{1/2} \lambda_1^{1/2} + \lambda_1^{-1}. \tag{2.2}
\]

If \( c_2 \lambda_1 \leq 1/2 \), then we have

\[
\sum_{a<n \leq b} e\left(f(n)\right) \ll \lambda_1^{-1}. \tag{2.3}
\]

Proof. See Jia [11], Lemma 1. ■
Lemma 2.7 Let \( k \geq 3 \) be an integer, and suppose that \( f(x) : [a, b] \to \mathbb{R} \) has continuous derivatives of order up to \( k \) on \( [a, b] \), where \( 1 \leq a < b \leq 2a \). Suppose further that
\[
0 < \lambda_k \leq |f^{(k)}(x)| \leq A\lambda_k, \quad x \in [a, b].
\]
Then
\[
\sum_{a < n < b} e\left(f(n)\right) \ll_{A,k,\varepsilon} N^{1+\varepsilon} \left(\lambda_k^{1/k-1} + N^{-1/k(k-1)} + N^{-2/k(k-1)} \lambda_k^{-2/k^2(k-1)}\right).
\]
Proof. See Heath-Brown [9], Theorem 1. ■

Lemma 2.8 Let \( \mathcal{I} \) be a subinterval of \( (Y, 2Y] \) and let \( J \) be a positive integer. Then
\[
\left|\sum_{n \in \mathcal{I}} z_n\right|^2 \leq \left(1 + \frac{Y}{J}\right) \sum_{|j| \leq J} \left(1 - \frac{|j|}{J}\right) \sum_{n, n+j \in \mathcal{I}} z_{n+j} z_n.
\]
Proof. See Heath-Brown [7], Lemma 5. ■

Lemma 2.9 Suppose that \( 1/2 < \alpha < 1 \), \( H \geq 1 \), \( N \geq 1 \), \( \Delta > 0 \). Let \( S(H, N, \Delta, \gamma) \) denote the number of solutions of the inequality
\[
|h_1 n_1^\alpha - h_2 n_2^\alpha| \leq \Delta, \quad h_1, h_2 \sim H, n_1, n_2 \sim N.
\]
Then we have
\[
S(H, N, \Delta, \gamma) \ll HN \log 2HN + \Delta N^{2-\alpha}.
\]
Proof. See the discussion on pp. 256-257 of Heath-Brown [7]. ■

Lemma 2.10 For any \( H \geq 1 \), we have
\[
\psi(\theta) = -\sum_{0 < |h| \leq H} \frac{e(\theta h)}{2\pi i h} + O(g(\theta, H)),
\]
where
\[
g(\theta, H) = \min\left(1, \frac{1}{H\|\theta\|}\right) = \sum_{h = -\infty}^{\infty} a_h e(\theta h)
\]
and
\[
a_h \ll \min\left(\frac{\log 2H}{H}, \frac{1}{|h|}, \frac{H}{|h|^2}\right).
\]
Proof. See pp. 245 of Heath-Brown [7]. ■

Lemma 2.11 Let \( z \geq 1 \) and \( k \geq 1 \). Then, for any \( n \leq 2z^k \),
\[
\Lambda(n) = \sum_{j=1}^{k} (-1)^{j-1} {k \choose j} \sum_{n_1 n_2 \cdots n_j = n} \mu(n_1+1) \cdots \mu(n_j). 
\]
Proof. See pp. 1366-1367 of Heath-Brown \[8\].

**Lemma 2.12** Suppose that
\[
L(H) = \sum_{i=1}^{m} A_i H^{a_i} + \sum_{j=1}^{n} B_j H^{-b_j},
\]
where \(A_i, B_j, a_i\) and \(b_j\) are positive. Assume that \(H_1 \leq H_2\). Then there is some \(H\) with \(H_1 \leq H \leq H_2\) and
\[
L(H) \ll \sum_{i=1}^{m} A_i H^{a_i} + \sum_{j=1}^{n} B_j H^{-b_j} + \sum_{i=1}^{m} \sum_{j=1}^{n} (A_i B_j)^{1/(a_i+b_j)}.
\]
The implied constant depends only on \(m\) and \(n\).

Proof. See Graham and Kolesnik \[5\], Lemma 2.4.

**Lemma 2.13** Suppose that 
\(f(x) \ll B, f'(x) \gg \Delta\) for \(x \sim N\). Then we have
\[
\sum_{n \sim N} \min \left( D, \frac{1}{\|f(n)\|} \right) \ll (B+1) \left( D + \frac{1}{\Delta} \right) \log \left( 2 + \frac{1}{\Delta} \right).
\]

Proof. See Jia \[11\], Lemma 3.

**Lemma 2.14** Suppose \(f(x)\) and \(g(x)\) are algebraic function in \([a, b]\) and 
\[
\frac{1}{R} \leq |f''(x)| \ll \frac{1}{R}, \quad |f'''(x)| \ll \frac{1}{RU} (U \geq 1),
\]
\[
|g(x)| \leq G, \quad |g'(x)| \ll U^{-1} G.
\]
Let \([\alpha, \beta]\) be the image of \([a, b]\) under the mapping \(y = f'(x)\). Then we have
\[
\sum_{\alpha < n \leq \beta} g(n)e(f(n)) = \sum_{\alpha < u \leq \beta} \frac{g(n_u)}{\sqrt{f''(n_u)}} e\left( f(n_u) - un_u + \frac{1}{8} \right)
+ O(G \log(\beta - \alpha + 2) + U^{-1} G(b - a + R))
+ O\left( G \min \left( \sqrt{R}, \frac{1}{\|\alpha\|} \right) + G \min \left( \sqrt{R}, \frac{1}{\|\beta\|} \right) \right),
\]
where \(n_u\) is the solution of \(f'(n) = u\).

Proof. See Jia \[11\], Lemma 5.

For the sum of the form 
\[
\min \left( 1, \frac{H_1}{H} \right) \sum_{h \sim H} \left| \sum_{m \sim M} \sum_{n \sim N} a_m b_n e\left( \alpha m^3 n^3 + h(mn + u)\gamma \right) \right|
\]
with
\[
MN \sim x, \ a_m \ll x^\varepsilon, \ b_n \ll x^\varepsilon
\]
for every fixed \(\varepsilon\), it is usually called a “Type I” sum, denoted by \(S_I(M, N)\), if \(b_n = 1\) or \(b_n = \log n\); otherwise it is called a “Type II” sum, denoted by \(S_{II}(M, N)\).
Lemma 2.15 Suppose that $48(1 - \gamma) + 48\delta < 1$, $|a_m| \ll 1$, $|b_n| \ll 1$, $MN \asymp x$. Then, for

$$x^{24(1-\gamma)+24\delta+\varepsilon} \ll M \ll x^{\gamma-2\delta-\varepsilon},$$

(2.7)

we have

$$S_{11}(M, N) \ll x^{1-\delta-\varepsilon}.$$  

(2.8)

Proof. Let $Q$ be a positive integer satisfying $1 \leq Q \leq HN \log^{-1} x$. For each $q$ ($1 \leq q \leq Q$), define

$$w_q := \{(n, h) : 4HN^\gamma(q-1)Q^{-1} < hn^\gamma \ll 4HN^\gamma qQ^{-1}, h \sim H, n \sim N\}.$$ 

Then we have

$$S := \sum_{h \sim H} \left| \sum_{m \sim M} \sum_{n \sim N} a_m b_n e(\alpha m^3 n^3 + h(mn + u)^\gamma) \right|$$

$$= \sum_{h \sim H} \sum_{m \sim M} \sum_{n \sim N} a_m b_n c_h e(\alpha m^3 n^3 + h(mn + u)^\gamma)$$

$$= \sum_{m \sim M} a_m \sum_{h \sim H n \sim N} b_n c_h e(\alpha m^3 n^3 + h(mn + u)^\gamma)$$

$$= \sum_{m \sim M} a_m \sum_{q=1}^{Q} \sum_{(n, q) \in w_q} b_n c_h e(\alpha m^3 n^3 + h(mn + u)^\gamma),$$

where $|c_h| = 1$, $h \sim H$. By Cauchy’s inequality, we obtain

$$|S|^2 \ll \left( \sum_{m \sim M} |a_m|^2 \right) \left( \sum_{m \sim M} \left| \sum_{q=1}^{Q} \sum_{(n, q) \in w_q} b_n c_h e(\alpha m^3 n^3 + h(mn + u)^\gamma) \right|^2 \right)$$

$$\ll MQ \sum_{m \sim M} \sum_{q=1}^{Q} \left| \sum_{(n, q) \in w_q} b_n c_h e(\alpha m^3 n^3 + h(mn + u)^\gamma) \right|^2$$

$$\ll MQ \sum_{q=1}^{Q} \sum_{(n_1, h_1) \in w_q} \sum_{(n_2, h_2) \in w_q} e(\alpha m^3 (n_1^3 - n_2^3) + h_1 (mn_1 + u)^\gamma - h_2 (mn_2 + u)^\gamma)$$

$$=: MQ \sum_{m \sim M} \left| \sum_{m \sim M} e(f(m)) \right|,$$

(2.9)

where $f(m) = \alpha m^3 (n_1^3 - n_2^3) + h_1 (mn_1 + u)^\gamma - h_2 (mn_2 + u)^\gamma$. The outer sum runs over all the quadruples $(h_1, n_1, h_2, n_2)$ with $(h_1, n_1), (h_2, n_2) \in w_q$.

Let $\lambda = h_1 n_1^\gamma - h_2 n_2^\gamma$. Then we have $|\lambda| \ll 4HN^\gamma Q^{-1}$. It is easy to verify that

$$f^{(4)}(m) = \gamma(\gamma - 1)(\gamma - 2)(\gamma - 3) \left( h_1 n_1^4 (mn_1 + u)^{\gamma - 4} - h_2 n_2^4 (mn_2 + u)^{\gamma - 4} \right)$$

$$= \gamma(\gamma - 1)(\gamma - 2)(\gamma - 3) \left( \lambda m^{\gamma - 4} + O\left( \frac{H}{H_1 M^4} \right) \right).$$
Thus, there exists a constant $C(\lambda) > 0$ such that $f^{(4)}(m) \asymp |\lambda|M^{-4}$ for $|\lambda| \geq C(\lambda)M^{-\gamma}HH^{-1}$. By Lemma 2.7 with $k = 4$, the estimate of the inner sum in (2.9) is
\[ \sum_{m \sim M} e(f(m)) \ll M^{2/3 + \gamma/12 + \varepsilon} |\lambda|^{1/12} + M^{11/12 + \varepsilon} + M^{1 - \gamma/24 + \varepsilon} |\lambda|^{-1/24}. \tag{2.10} \]

If $|\lambda| < C(\lambda)M^{-\gamma}(H \leq H_1)$ or $|\lambda| < C(\lambda)M^{-\gamma}HH^{-1}(H > H_1)$, we use the trivial bound $M$ to estimate the inner sum.

By Lemma 2.9, the contributions of $M$ to $|S|^2$ are (with $H \leq H_1$)
\[ \ll M^2Q(HN \log 2HN + M^{-\gamma}HN^{2-\gamma}) \ll M^2QHN \log 2HN \tag{2.11} \]
and (with $H > H_1$)
\[ \ll M^2Q(HN \log 2HN + HH^{-1}M^{-\gamma}HN^{2-\gamma}) \ll M^2QHN \log 2HN. \tag{2.12} \]

By noting that $|\lambda| \ll HN^{-\gamma}Q^{-1}$, then the contribution of $M^{11/12 + \varepsilon}$ to $|S|^2$ is
\[ \ll MQ \cdot M^{11/12 + \varepsilon} \cdot S(H, N, AH \cdot N^{-\gamma}Q^{-1}, \gamma) \ll M^{23/12 + \varepsilon}Q \cdot (HN \log 2HN + HN^{-\gamma}Q^{-1} \cdot HN^{2-\gamma}) \ll M^{23/12 + \varepsilon}H^2N^2 \ll M^{-1/12 + \varepsilon}H^2x^2. \tag{2.13} \]

Similarly, the contribution of $M^{2/3 + \gamma/12 + \varepsilon} |\lambda|^{1/12}$ to $|S|^2$ is
\[ \ll MQ \cdot M^{2/3 + \gamma/12 + \varepsilon} \cdot H^{1/12}N^{-\gamma/12}Q^{-1/12} \cdot S(H, N, AH \cdot N^{-\gamma}Q^{-1}, \gamma) \ll M^{5/3 + \gamma/12 + \varepsilon}Q^{11/12}H^{1/12}N^{-\gamma/12} \cdot (HN \log 2HN + HN^{-\gamma}Q^{-1} \cdot HN^{2-\gamma}) \ll M^{-1/3 + \varepsilon}Q^{-1/12}H^{25/12}x^{2+\gamma/12}. \tag{2.14} \]

By a splitting argument and Lemma 2.9, the contributions of $M^{1 - \gamma/24 + \varepsilon} |\lambda|^{-1/24}$ to $|S|^2$ are (with $H \leq H_1$ and $|\lambda| > C(\lambda)M^{-\gamma}$)
\[ \ll M^{2-\gamma/24 + \varepsilon}Q(\log x) \times \max_{M^{-\gamma} \ll U \ll HN^{\gamma}Q^{-1}} \sum_{U < |\lambda| \leq 2U} |\lambda|^{-1/24} \ll M^{2-\gamma/24 + \varepsilon}Q(\log x) \times \max_{M^{-\gamma} \ll U \ll HN^{\gamma}Q^{-1}} U^{-1/24} \cdot S(H, N, U) \ll M^{2-\gamma/24 + \varepsilon}Q(\log x) \times \max_{M^{-\gamma} \ll U \ll HN^{\gamma}Q^{-1}} (U^{-1/24}HN \log 2HN + U^{23/24}HN^{2-\gamma}) \ll M^{2-\gamma/24 + \varepsilon}Q(\log x) \cdot (M^{\gamma/24}HN \log 2HN + H^{23/24}N^{23\gamma/24}Q^{-23/24} \cdot HN^{2-\gamma}) \ll M^{2+\varepsilon}QHN \log^2 x + Q^{1/24}H^{47/24}M^{2-\gamma/24}x^{2-\gamma/24} \log x \tag{2.15} \]
and (with $H > H_1$ and $|\lambda| > C(\lambda)H H_1^{-1}$)

$$
\ll M^{2-\gamma/24+\varepsilon} Q(\log x) \times \max_{M^{-\gamma} H H_1^{-1} \leq U \leq H N^{\gamma} Q^{-1}} \sum_{U \leq |\lambda| < 2U} |\lambda|^{-1/24}

\ll M^{2-\gamma/24+\varepsilon} Q(\log x) \times \max_{M^{-\gamma} H H_1^{-1} \leq U \leq H N^{\gamma} Q^{-1}} (U^{-1/24} \cdot S(H, N, U, \gamma))

\ll M^{2-\gamma/24+\varepsilon} Q(\log x) \times \max_{M^{-\gamma} H H_1^{-1} \leq U \leq H N^{\gamma} Q^{-1}} (U^{-1/24} H N \log 2HN + U^{23/24} H N^{2-\gamma})

\ll M^{2-\gamma/24+\varepsilon} Q(\log x) (M^{\gamma/24}(H_1 H^{-1})^{1/24} HN \log 2HN

+ H^{23/24} N^{23\gamma/24} Q^{-23/24} \cdot H N^{2-\gamma})

\ll M^{2+\varepsilon} QHN \log^2 x + Q^{1/24} H^{47/24} M^x x^{2-\gamma/24} \log x.

(2.16)

From (2.11)-(2.16), we can get

$$(\log x)^{-2} |S|^2 \ll M^{-1/12+\varepsilon} H^2 x^2 + M^{2+\varepsilon} H N Q + M^{\varepsilon} H^{47/24} x^{2-\gamma/24} Q^{1/24}

+ M^{-1/3+\varepsilon} H^{25/12} x^{2+\gamma/12} Q^{-1/12}.

(2.17)$$

By Lemma 2.12, we can choose an optimal $Q \in [1, H N \log^{-1} x]$ such that

$$(\log x)^{-3} |S|^2 \ll M^{-1/12+\varepsilon} H^2 x^2 + M^{2+\varepsilon} H N Q + M^{\varepsilon} H^{47/24} x^{2-\gamma/24} + M^{-1/9+\varepsilon} H^2 x^2

+ M^{-1/4+\varepsilon} H^2 x^{(23+\gamma)/12} + M^{-3/13+\varepsilon} H^2 x^{(25+\gamma)/13}.

(2.18)$$

Therefore, we have

$$(\log x)^{-2} S_{11}(M, N) \ll M^{-1/24+\varepsilon} H_1 x + M^{1/2+\varepsilon} H_1^{1/2} x^{1/2}

+ M^{-1/8+\varepsilon} H_1 x^{(23+\gamma)/24} + M^{-3/26+\varepsilon} H_1 x^{(25+\gamma)/26}

+ M^{\varepsilon} H_1^{47/48} x^{1-\gamma/48} + M^{-1/18+\varepsilon} H_1 x

(2.19)$$

From (2.19) we know that, under the condition (2.7), the result of Lemma 2.15 follows. ■

Lemma 2.16 Suppose that $16(1-\gamma)+16\delta < 1$, $|a_m| \ll 1$, $b_n = 1$ or $b_n = \log n$, $MN \asymp x$. Let

$$
\begin{align*}
& a_1 = \frac{3}{2} - 19(1-\gamma) - 19\delta, & a_2 = \frac{12}{11} - \frac{144}{11} (1-\gamma) - \frac{144}{11} \delta, \\
& a_3 = 1 - \frac{35}{3} (1-\gamma) - \frac{35}{3} \delta, & a_4 = \frac{18}{17} - \frac{192}{17} (1-\gamma) - \frac{192}{17} \delta, \\
& a_5 = \frac{13}{11} - \frac{118}{11} (1-\gamma) - \frac{118}{11} \delta, & a_6 = \frac{24}{23} - \frac{216}{23} (1-\gamma) - \frac{216}{23} \delta, \\
& a_7 = \frac{26}{29} - \frac{194}{29} (1-\gamma) - \frac{201}{29} \delta, & a_8 = \frac{24}{29} - \frac{180}{29} (1-\gamma) - \frac{186}{29} \delta, \\
& a_9 = \frac{46}{57} - \frac{346}{57} (1-\gamma) - \frac{357}{57} \delta, & a = \min (a_1, a_2, \cdots, a_9) - \varepsilon.
\end{align*}
$$

(2.20)
If there holds
\[ M \ll x^6, \quad (2.21) \]
then we have
\[ S_I(M, N) \ll x^{1-\delta-\varepsilon}. \quad (2.22) \]

**Proof.** Applying partial summation to the inner sum, we have
\[
\left| \sum_{m \sim M} \sum_{n \sim N} a_m b_n e(\alpha m^3 n^3 + h(mn + u)^\gamma) \right| \\
\leq \sum_{m \sim M} \left| \sum_{n \sim N} b_n e(\alpha m^3 n^3 + h(mn + u)^\gamma) \right| \\
\leq (\log x) \sum_{m \sim M} \left| \sum_{n \sim N} e(\alpha m^3 n^3 + h(mn + u)^\gamma) \right| =: (\log x) \cdot K_h.
\]
Thus, we obtain
\[
(\log x)^{-1} \cdot S_I(M, N) \ll \min \left( 1, \frac{H_1}{H} \right) \sum_{h \sim H} K_h,
\]
where
\[
K_h = \sum_{m \sim M} \left| \sum_{n \sim N} e(\alpha m^3 n^3 + h(mn + u)^\gamma) \right|.
\]
By Hölder’s inequality, we have
\[
K_h^8 \ll M^7 \sum_{m \sim M} \left| \sum_{n \sim N} e(\alpha m^3 n^3 + h(mn + u)^\gamma) \right|^8. \quad (2.23)
\]
Suppose \( z_n = z_n(m, u, \alpha) = \alpha m^3 n^3 + h(mn + u)^\gamma \). Let \( Q, J, L \) be three positive integers, which satisfy \( 1 \leq Q \leq N \log^{-1} x, 1 \leq J \leq N \log^{-1} x, 1 \leq L \leq N \log^{-1} x \).

Applying Lemma 2.8 to the inner sum of (2.23), we get
\[
\left| \sum_{n \sim N} e(z_n) \right|^2 \ll \frac{N^2}{Q} \sum_{|q| \leq Q} \left( 1 - \frac{|q|}{Q} \right) \sum_{n \sim N} e(z_{n+q} - z_n) \\
\ll \frac{N^2}{Q} + \frac{N}{Q} \sum_{1 \leq q \leq Q} \left( 1 - \frac{q}{Q} \right) \sum_{n \sim N} e(z_{n+q} - z_n). 
\]
Therefore, by Cauchy’s inequality, we have
\[
\left| \sum_{n \sim N} e(z_n) \right|^4 \ll \frac{N^4}{Q^2} + \frac{N^2}{Q^2} \left( \sum_{1 \leq q \leq Q} \left( 1 - \frac{q}{Q} \right)^2 \right) \left( \sum_{1 \leq q \leq Q} \sum_{n \sim N} e(z_{n+q} - z_n) \right)^2 \\
\ll \frac{N^4}{Q^2} + \frac{N^2}{Q} \sum_{1 \leq q \leq Q} \sum_{n \sim N} e(z_{n+q} - z_n)^2. \quad (2.24)
\]
Applying Lemma 2.8 to the inner sum of (2.24), we have

\[
\left| \sum_{n \sim N, n+q \sim N} e(z_{n+q} - z_n) \right|^2
\leq \frac{N^2}{J} \sum_{|j| \leq J} \left( 1 - \frac{|j|}{J} \right) \sum_{n \sim N, n+q \sim N} e(z_{n+q+j} - z_{n+j} - z_{n+q} + z_n)
\leq \frac{N^2}{J} + \frac{N}{J} \left( \sum_{n \sim N, n+q \sim N} e(z_{n+q+j} - z_{n+j} - z_{n+q} + z_n) \right)^2.
\]

(2.25)

Putting (2.25) into (2.24), we have

\[
\left| \sum_{n \sim N} e(z_n) \right|^4 \leq \frac{N^4}{Q^2} + \frac{N^3}{JQ} \sum_{1 \leq q \leq Q} \sum_{1 \leq j \leq J} \left| \sum_{N < n < 2N-q-j} e(z_{n+q+j} - z_{n+j} - z_{n+q} + z_n) \right|^2.
\]

(2.26)

Therefore, by Cauchy’s inequality, we have

\[
\left| \sum_{n \sim N} e(z_n) \right|^8
\leq \frac{N^8}{Q^4} + \frac{N^6}{J^2Q^2} \left( \sum_{1 \leq q \leq Q} \sum_{1 \leq j \leq J} \left| \sum_{N < n < 2N-q-j} e(z_{n+q+j} - z_{n+j} - z_{n+q} + z_n) \right| \right)^2.
\]

(2.27)

Set \( y_n = y_n(q,j) = z_{n+q+j} - z_{n+j} - z_{n+q} + z_n \). Applying Lemma 2.8 to the inner sum of (2.27), we have

\[
\left| \sum_{N < n < 2N-q-j} e(y_n) \right|^2
\leq \frac{N}{L} \sum_{|\ell| \leq L} \left( 1 - \frac{|\ell|}{L} \right) \sum_{N < n \leq 2N-q-\ell} e(y_{n+\ell} - y_n)
\leq \frac{N^2}{L} + \frac{N}{L} \sum_{1 \leq |\ell| \leq L} \left( 1 - \frac{|\ell|}{L} \right) \sum_{N < n \leq 2N-q-\ell} e(y_{n+\ell} - y_n).
\]

(2.28)
Putting (2.28) into (2.27), we have

\[
\left| \sum_{n \sim N} e(z_n) \right|^8 \leq \frac{N^8}{Q^4} + \frac{N^8}{J^2} + \frac{N^8}{L} + \frac{N^7}{LJQ} \sum_{q=1}^Q \sum_{j=1}^J \sum_{\ell=1}^L \left( 1 - \frac{\ell}{L} \right) \sum_{N<n<2N-q-j-\ell} e(y_{n+\ell} - y_n).
\]

(2.29)

Put (2.29) into (2.23), we obtain

\[
K_h^8 \leq \frac{x^8}{Q^4} + \frac{x^8}{J^2} + \frac{x^8}{L} + \frac{x^7}{LJQ} \times \sum_{q=1}^Q \sum_{j=1}^J \sum_{\ell=1}^L \left| \sum_{m \sim M} \sum_{N<n<2N-q-j-\ell} e(y_{n+\ell} - y_n) \right| =: \frac{x^8}{Q^4} + \frac{x^8}{J^2} + \frac{x^8}{L} + \frac{x^7}{LJQ} \sum_{q=1}^Q \sum_{j=1}^J \sum_{\ell=1}^L |E_{q,j,\ell}|,
\]

(2.30)

where

\[
E_{q,j,\ell} = \sum_{m \sim M} \sum_{N<n<2N-q-j-\ell} e(y_{n+\ell} - y_n).
\]

(2.31)

Let

\[
\Delta(n^\gamma; q, j, \ell) = (n+q+j+\ell)^\gamma - (n+q+j)^\gamma - (n+q+\ell)^\gamma - (n+j+\ell)^\gamma + (n+q)^\gamma + (n+j)^\gamma + (n+\ell)^\gamma - n^\gamma.
\]

Then we have

\[
y_{n+\ell} - y_n = \left| z_{n+q+j+\ell} - z_{n+q+j} - z_{n+q+\ell} - z_{n+j+\ell} + z_{n+q} + z_{n+j} + z_{n+\ell} - z_n \right| \]

\[
= 6a q j m^3 + \left( h\left( m(n+q+j+\ell) + u \right)^\gamma - h\left( m(n+q+j) + u \right)^\gamma \right) - \left( h\left( m(n+q+\ell) + u \right)^\gamma - h\left( m(n+q) + u \right)^\gamma \right) - \left( h\left( m(n+j+\ell) + u \right)^\gamma - h\left( m(n+j) + u \right)^\gamma \right) + \left( h\left( m(n+\ell) + u \right)^\gamma - h(mn + u)^\gamma \right)
\]

\[
= 6a q j m^3 + h m^\gamma \Delta(n^\gamma; q, j, \ell)
\]

\[
+ \gamma h \int_0^a \left( (m(n+q+j+\ell) + t)^{\gamma-1} - (m(n+q+j) + t)^{\gamma-1} \right) dt
\]

\[
- \gamma h \int_0^a \left( (m(n+q+\ell) + t)^{\gamma-1} - (m(n+q) + t)^{\gamma-1} \right) dt
\]

\[
- \gamma h \int_0^a \left( (m(n+j+\ell) + t)^{\gamma-1} - (m(n+j) + t)^{\gamma-1} \right) dt
\]

\[
+ \gamma h \int_0^a \left( (m(n+\ell) + t)^{\gamma-1} - (mn + t)^{\gamma-1} \right) dt
\]

\[
=: 6a q j m^3 + h m^\gamma \Delta(n^\gamma; q, j, \ell) + I_1 - I_2 - I_3 + I_4.
\]
By noting that
\[ I_1 \asymp h m \ell \int_0^u (m(n + q + j) + t)^{\gamma-2} dt, \quad I_2 \asymp h m \ell \int_0^u (m(n + q) + t)^{\gamma-2} dt, \]
\[ I_3 \asymp h m \ell \int_0^u (m(n + j) + t)^{\gamma-2} dt, \quad I_4 \asymp h m \ell \int_0^u (m n + t)^{\gamma-2} dt, \]
we obtain
\[ I_1 - I_2 - I_3 + I_4 = (I_1 - I_2) - (I_3 - I_4) \]
\[ \asymp h m \ell \int_0^u \left( (m(n + q + j) + t)^{\gamma-2} - (m(n + q) + t)^{\gamma-2} \right) dt \]
\[ - h m \ell \int_0^u \left( (m(n + j) + t)^{\gamma-2} - (m n + t)^{\gamma-2} \right) dt \]
\[ \asymp h m^2 j \ell \int_0^u \left( (m(n + q) + t)^{\gamma-3} - (m n + t)^{\gamma-3} \right) dt \]
\[ \asymp h m^3 q j \ell \int_0^u (m n + t)^{\gamma-4} dt \asymp h q j \ell M^3 x^{\gamma-4}. \]
Thus, we get
\[ y_{n+\ell} - y_n = 6 a q j \ell m^3 + h m^2 \Delta(n; q, j, \ell) + O(h q j \ell M^3 x^{\gamma-4}) \]
\[ =: G(m, n) + O(h q j \ell M^3 x^{\gamma-4}). \tag{2.32} \]
Putting (2.32) into (2.31), we have
\[ E_{q, j, \ell} = \sum_{m \sim M} \sum_{N < n \leq 2N - q - j - \ell} e \left( G(m, n) + O(h q j \ell M^3 x^{\gamma-4}) \right) \]
\[ = \sum_{m \sim M} \sum_{N < n \leq 2N - q - j - \ell} e(G(m, n)) \left( 1 + O(h q j \ell M^3 x^{\gamma-4}) \right) \]
\[ = \sum_{m \sim M} \sum_{N < n \leq 2N - q - j - \ell} e(G(m, n)) + O(h q j \ell M^3 x^{\gamma-3}). \tag{2.33} \]
For any \( t \neq 0, 1 \), we have
\[ \Delta(n^t; q, j, \ell) = t \int_0^\ell ((n + q + j + \tau)^{t-1} - (n + q + \tau)^{t-1}) d\tau \]
\[ \quad - t \int_0^\ell ((n + j + \tau)^{t-1} - (n + \tau)^{t-1}) d\tau \]
\[ \asymp t(t-1) \int_0^\ell ((n + q + \tau)^{t-2} - (n + \tau)^{t-2}) d\tau \]
\[ \asymp t(t-1)(t-2) q j \int_0^\ell (n + \tau)^{t-3} d\tau \]
\[ = t(t-1)(t-2) q j \ell n^{t-3} \]
\[ + t(t-1)(t-2)(t-3) q j \int_0^\tau \int_0^\tau (n + \xi)^{t-4} d\xi d\tau \]
\[ = t(t-1)(t-2) q j \ell n^{t-3} + O(N^{t-4} q j \ell^2). \tag{2.34} \]
Similarly, we also have
\begin{align*}
\Delta(n^t; q, j, \ell) &= t(t - 1)(t - 2)qj\ell n^{t-3} + O(N^{t-4}qj^2\ell), \\
\Delta(n^t; q, j, \ell) &= t(t - 1)(t - 2)qj\ell n^{t-3} + O(N^{t-4}q^2j\ell).
\end{align*}

Combining (2.34), (2.35) and (2.36), we obtain
\begin{align*}
\Delta(n^t; q, j, \ell) &= t(t - 1)(t - 2)qj\ell n^{t-3} + O(N^{t-4}qj\ell(q + j + \ell)) \\
&= t(t - 1)(t - 2)qj\ell n^{t-3} \left(1 + \frac{q + j + \ell}{N}\right).
\end{align*}

Therefore, it is easy to get
\begin{align*}
\frac{\partial G}{\partial n} &= \gamma hm^\gamma \Delta(n^{\gamma - 1}; q, j, \ell) \\
&= \gamma(\gamma - 1)(\gamma - 2)(\gamma - 3)hqj\ell m^\gamma n^{\gamma - 4} \left(1 + O\left(\frac{q + j + \ell}{N}\right)\right)
\end{align*}

and
\begin{align*}
\frac{\partial^2 G}{\partial n^2} &= \gamma(\gamma - 1)hm^\gamma \Delta(n^{\gamma - 2}; q, j, \ell) \\
&= \gamma(\gamma - 1)(\gamma - 2)(\gamma - 3)(\gamma - 4)hqj\ell m^\gamma n^{\gamma - 5} \left(1 + O\left(\frac{q + j + \ell}{N}\right)\right).
\end{align*}

If \(|\gamma(\gamma - 1)(\gamma - 2)(\gamma - 3)hqj\ell m^\gamma n^{\gamma - 4}| \leq 1/500\), then from (2.3) of Lemma 2.6 we have
\begin{align*}
\sum_{m \sim M} \sum_{N < n \leq 2N - q - j - \ell} e(G(m, n)) &\ll MN^4(hqj\ell M^\gamma N^\gamma)^{-1} \asymp MN^4(hqj\ell x^\gamma)^{-1}.
\end{align*}

In the rest of this Lemma, we always suppose that \(|\gamma(\gamma - 1)(\gamma - 2)(\gamma - 3)hqj\ell m^\gamma n^{\gamma - 4}| > 1/500\). By Lemma 2.14, we have
\begin{align*}
\sum_{N < n \leq 2N - q - j - \ell} e(G(m, n)) \\
&= e\left(\frac{1}{8}\right) \sum_{\alpha \leq \nu \leq \beta} \left(\frac{\partial^2 G}{\partial n^2}(m, n_{\nu})\right)^{-1/2} e(G(m, n_{\nu}) - \nu n_{\nu}) + R_1(m, q, j, \ell),
\end{align*}

where
\begin{align*}
\frac{\partial G}{\partial n}(m, n_{\nu}) &= \gamma hm^\gamma \Delta(n_{\nu}^{\gamma - 1}; q, j, \ell) = \nu, \\
\alpha &= \frac{\partial G}{\partial n}(m, N), \quad \beta = \frac{\partial G}{\partial n}(m, 2N - q - j - \ell), \\
R &= N^5(hqj\ell x^\gamma)^{-1}, \quad \nu = \frac{\partial G}{\partial n}(m, n_{\nu}) \asymp hqj\ell m^\gamma N^{\gamma - 4},
\end{align*}

\begin{align*}
R_1(m, q, j, \ell) &\ll \log x + RN^{-1} + \min\left(\sqrt{R}, \max\left(\frac{1}{\|\alpha\|}, \frac{1}{\|\beta\|}\right)\right).
\end{align*}
Now, we only need to estimate the exponential sum

\[ \sum_{m \sim M} \min \left( \sqrt{R}, \frac{1}{\| \alpha \|} \right) + \sum_{m \sim M} \min \left( \sqrt{R}, \frac{1}{\| \beta \|} \right) \]

\[ \ll M \log x + M R N^{-1} + \sum_{m \sim M} \left( \frac{\partial^2 G}{\partial n^2} (m, n) \right)^{-1/2} e(G(m, n) - \nu n) \]

\[ \ll M \log x + x^{4-\gamma} (hqj \ell M^3)^{-1} + (hqj \ell)^{1/2} M^{3/2} x^{(\gamma - 3)/2} \log x. \]  

(2.46)

Now, we only need to estimate the exponential sum

\[ \sum_{m \sim M} \sum_{\alpha < \nu \leq \beta} \left( \frac{\partial^2 G}{\partial n^2} (m, n) \right)^{-1/2} e(G(m, n) - \nu n) \]

\[ = \sum_{\nu} \sum_{m \in \mathcal{I}_\nu} \left( \frac{\partial^2 G}{\partial n^2} (m, n) \right)^{-1/2} e(G(m, n) - \nu n). \]  

(2.47)

where \( \mathcal{I}_\nu \) is a subinterval of \((M, 2M]\).

For fixed \( \nu \), define \( \Delta = \Delta(n_\nu; q, j, \ell) \), where \( \lambda \) is arbitrary real number. Taking derivative of \( m \) on both sides of the equation (2.42), we have

\[ n_\nu' = -\frac{\gamma \Delta_{\gamma-1}}{(\gamma - 1)m \Delta_{\gamma-2}}. \]  

(2.48)

Combining (2.34) and (2.39), we get

\[ \frac{d}{dm} \left( \frac{\partial^2 G}{\partial n^2} (m, n) \right) \]

\[ = \frac{\gamma^2 hm^{\gamma-1}}{\Delta_{\gamma-2}} \left( \gamma - 1 \right) \Delta_{\gamma-2} - (\gamma - 2) \Delta_{\gamma-1} \Delta_{\gamma-3} \]

\[ = \frac{\gamma^2 (\gamma - 1)(\gamma - 2) (\gamma - 3) hm^{\gamma-1} n_\nu^{-5}}{\Delta_{\gamma-2}} \left( 1 + O \left( \frac{g + j + \ell}{N} \right) \right), \]  

(2.49)

so that \( \left( \frac{\partial^2 G}{\partial n^2} (m, n) \right)^{-1/2} \) is monotonic in \( m \).

Let \( g(m) = G(m, n_\nu(m)) - \nu n_\nu(m) \). By a series of simple calculation, we obtain

\[ g'(m) = 180 q j \ell m^2 + \gamma hm^{\gamma-1} \Delta, \]

\[ g''(m) = 360 q j \ell m + \frac{\gamma h}{\gamma - 1} \frac{(\gamma - 1)^2 \Delta_{\gamma-2} \Delta_{\gamma-2} - \gamma^2 \Delta_{\gamma-1}^2}{m^{2-\gamma} \Delta_{\gamma-2}} \]

\[ =: 360 q j \ell m + \frac{\gamma h}{\gamma - 1} \frac{g_1(m) - g_2(m)}{g_0(m)}, \]  

(2.50)

where

\[ g_1(m) = (\gamma - 1)^2 \Delta_{\gamma-2}, \quad g_2(m) = \gamma^2 \Delta_{\gamma-1}^2, \quad g_0(m) = m^{2-\gamma} \Delta_{\gamma-2}. \]

Hence

\[ g'''(m) = 360 q j \ell + \frac{\gamma h}{\gamma - 1} \frac{(g_1'(m) - g_2'(m))g_0(m) - g_0'(m)(g_1(m) - g_2(m))}{g_0''(m)}. \]  

(2.51)
Putting \( g'(m) \), we get

\[
g''(m) = 36\alpha q\ell + \frac{\gamma h}{(\gamma - 1)^2} \frac{g_3(m) - g_4(m)}{g_5(m)}, \tag{2.53}
\]

where

\[
g_3(m) = 3\gamma^2 (\gamma - 1)^2 (\gamma - 2)^2 \Delta_{\gamma-1}^2 \Delta_{\gamma-2} + (\gamma - 1)^3 (\gamma - 2) \Delta_{\gamma-1}^3 \Delta_{\gamma-2},
\]

\[
g_4(m) = \gamma^2 (\gamma - 2)^2 \Delta_{\gamma-1}^2 \Delta_{\gamma-3}, \quad g_5(m) = m^{3-\gamma} \Delta_{\gamma-2}^3.
\]

Hence

\[
g^{(4)}(m) = \frac{\gamma h}{(\gamma - 1)^2} \frac{(g_3(m) - g_4(m))g_5(m) - g_5(m)(g_3(m) - g_4(m))}{g_5^3(m)}, \tag{2.54}
\]

where

\[
g_3'(m) = \left( (\gamma - 1)^2 (\gamma + 1)(\gamma + 2) \Delta_{\gamma-1} \Delta_{\gamma-2}^3 + 6\gamma^2 (\gamma - 1)(\gamma - 2) \Delta_{\gamma-1}^2 \Delta_{\gamma-2} \Delta_{\gamma-3} - \right.
\]

\[
3(\gamma - 1)^3 (\gamma - 2) \Delta_{\gamma-1} \Delta_{\gamma-2} \Delta_{\gamma-3} \right) n'_\nu(m),
\]

\[
g_4'(m) = \left( 3\gamma^3 (\gamma - 1)(\gamma - 2) \Delta_{\gamma-1}^2 \Delta_{\gamma-2} \Delta_{\gamma-3} + \gamma^3 (\gamma - 2)(\gamma - 3) \Delta_{\gamma-1}^3 \Delta_{\gamma-4} \right) n'_\nu(m),
\]

\[
g_5'(m) = \frac{m^{2-\gamma}}{\gamma - 1} \left( (\gamma - 1)(3-\gamma) \Delta_{\gamma-2}^3 - 3\gamma(\gamma - 2) \Delta_{\gamma-1} \Delta_{\gamma-2} \Delta_{\gamma-3} \right). \tag{2.55}
\]

Put (2.55) into (2.54), we obtain

\[
g^{(4)}(m) = \frac{\gamma h}{(\gamma - 1)^2} \frac{1}{m^{3-\gamma} \Delta_{\gamma-2}^5} \left( \gamma^2 (\gamma - 1)^2 (\gamma - 2)^2 \Delta_{\gamma-1}^2 \Delta_{\gamma-2}^4 - 
\]

\[
-2\gamma^3 (\gamma - 1)(\gamma - 2)(\gamma + 2) \Delta_{\gamma-2}^2 \Delta_{\gamma-3} - 
\]

\[
r^4 (\gamma - 2)(\gamma - 3) \Delta_{\gamma-1} \Delta_{\gamma-2} \Delta_{\gamma-4} - 
\]

\[
-(\gamma - 1)^4 (\gamma - 2)(\gamma - 3) \Delta_{\gamma} \Delta_{\gamma-2}^5 + 3\gamma^4 (\gamma - 2)^2 \Delta_{\gamma-2}^2 \Delta_{\gamma-3} \right). \tag{2.56}
\]

Combining (2.37), we have

\[
g^{(4)}(m) = c_0(\gamma) h q j \ell m^{\gamma-4} n_{\nu}^{\gamma-3} \left( 1 + \left( \frac{q + j + \ell}{N} \right) \right) \times h q j \ell M^{-1} x^{\gamma-3}, \tag{2.57}
\]
where \( c_0(\gamma) = -8\gamma^2(\gamma - 1)(\gamma - 2)^2(\gamma - 3)(\gamma - 4)^{-3}(3\gamma - 8) \).

By partial summation and Lemma 2.7 with parameter \( k = 4 \), we obtain

\[
\sum_{\nu} \sum_{m \in \mathbb{Z}_+} \left( \frac{\partial^2 G}{\partial n^2} (m, n_\nu) \right)^{-1/2} e(G(m, n_\nu) - vn_\nu) \\
\ll M^{1+\epsilon} \left( (hqj\ell M^{-1} x^{\gamma-3})^{1/12} + M^{-1/12} + M^{-1/6} (haj\ell M^{-1} x^{\gamma-3})^{-1/24} \right) \\
\times (hqj\ell M^7 N^{\gamma-5})^{-1/2} \cdot haj\ell M^7 N^{\gamma-4} \\
\ll (hqj\ell)^{7/12} M^{29/12+\epsilon} x^{(\gamma-3)/12} + (hqj\ell)^{1/2} M^{29/12+\epsilon} x^{(\gamma-3)/2} \\
+ (hqj\ell)^{11/24} M^{19/8+\epsilon} x^{11(\gamma-3)/24}. \tag{2.58}
\]

From (2.33), (2.40), (2.46) and (2.58), we get

\[
(\log x)^{-1} \cdot E_{q,j,\ell} \\
\ll haj\ell M^3 x^{\gamma-3} + M + (hqj\ell M^3)^{-1} x^{4-\gamma} + (hqj\ell)^{1/2} M^{29/12+\epsilon} x^{(\gamma-3)/2} \\
+ (hqj\ell)^{11/24} M^{19/8+\epsilon} x^{11(\gamma-3)/24}. \tag{2.59}
\]

Putting (2.59) into (2.30), we get

\[
(\log x)^{-4} \cdot K_h^8 \ll x^8 Q^{-4} + x^8 J^{-2} + x^8 L^{-1} + M x^7 + (hQJLM^3)^{-1} x^{11-\gamma} \\
+ hQJLM^3 x^{\gamma+4} + (hQJL)^{7/12} M^{29/12+\epsilon} x^{(\gamma+9)/12} \\
+ (hQJL)^{1/2} M^{29/12+\epsilon} x^{(\gamma+11)/2} \\
+ (hQJL)^{11/24} M^{19/8+\epsilon} x^{11(\gamma+17)/24+45/8}. \tag{2.60}
\]

Next, we apply Lemma 2.12 to (2.60) in \( Q, J, L \) one step at a time. First, for fixed \( Q \) and \( J \), we choose an optimal \( L \in [1, N \log^{-1} x] \) and obtain

\[
(\log x)^{-4} \cdot K_h^8 \ll x^{15/2} + M x^7 \log x + M^{8/19+\epsilon} x^{140/19} + M^{11/18+\epsilon} x^{22/3} \\
+ M^{24/35+\epsilon} x^{256/35} + x^8 Q^{-4} + (hQJ)^{-1} M^{-2} x^{10-\gamma} \log x \quad x^{8} J^{-2} \\
+ hQJLM^3 x^{\gamma+4} + (hQJ)^{7/12} M^{29/12+\epsilon} x^{(\gamma+9)/12} \\
+ (hQJ)^{1/2} M^{29/12+\epsilon} x^{(\gamma+11)/2} + (hQJ)^{11/24} M^{19/8+\epsilon} x^{11(\gamma+24+45/8)} \\
+ (hQJ)^{11/35} M^{29/18+\epsilon} x^{(\gamma+19)/3} + (hQJ)^{11/35} M^{57/35+\epsilon} x^{(11(\gamma+23)/35} \\
+ (hQJ)^{11/24} M^{3/2} x^{7/2+6} + (hQJ)^{7/19} M^{29/19+\epsilon} x^{7(\gamma+17)/19}. \tag{2.61}
\]

Second, for fixed \( Q \), we choose an optimal \( J \in [1, N \log^{-1} x] \) and obtain

\[
(\log x)^{-6} \cdot K_h^8 \ll x^{15/2} + M^{1+\epsilon} x^7 + M^{8/19+\epsilon} x^{140/19} + M^{11/18+\epsilon} x^{22/3} \\
+ M^{24/35+\epsilon} x^{256/35} + M^{15/26+\epsilon} x^{189/26} + M^{17/24+\epsilon} x^{29/4} \\
+ M^{35/46+\epsilon} x^{333/46} + h^{-1} M^{-1} x^{9-\gamma} Q^{-1} + x^8 Q^{-4}
\]

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Finally, we choose an optimal $Q \in [1, N \log^{-1} x]$ and obtain

$$
(\log x)^{-2} \cdot K_h
\ll
x^{15/16} + M^{1/8} + M^{1/9} + x^{35/38} + M^{11/14} + x^{11/12}
+ M^{3/5} + M^{17/12} + x^{29/32} + M^{35/36} + x^{333/368}
+ M^{1/2} + M^{23/140} + x^{117/140} + M^{23/162} + x^{139/162}
+ M^{1/4} + x^{25/28} + M^{11/18} + x^{105/18} + M^{23/216} + x^{8/9}
+ M^{23/206} + x^{183/206} + x^{1-\gamma/8} + h^{1/8} + M^{3/8} + x^{(\gamma+4)/8}
+ h^{7/96} M^{29/96} + x^{(\gamma+9)/96} + h^{1/16} M^{29/96} + x^{(\gamma+11)/16}
+ h^{11/192} M^{19/64} + x^{11/192} + 45/64 + h^{1/16} M^{3/16} + x^{(\gamma+12)/16}
+ h^{7/152} M^{29/152} + x^{(\gamma+17)/152} + h^{1/24} M^{29/144} + x^{(\gamma+19)/24}
+ h^{11/280} M^{57/280} + x^{(11\gamma+223)/280} + h^{1/12} M^{1/4} + x^{(\gamma+8)/12}
+ h^{7/124} M^{29/124} + x^{(\gamma+13)/124} + h^{1/20} M^{29/120} + x^{(\gamma+15)/20}
+ h^{11/236} M^{57/236} + x^{(11\gamma+179)/236} + h^{1/20} M^{3/20} + x^{(\gamma+16)/20}
+ h^{7/180} M^{29/180} + x^{(\gamma+21)/180} + h^{1/28} M^{29/168} + x^{(\gamma+23)/28}
+ h^{11/324} M^{19/108} + x^{(11\gamma+267)/324} + h^{1/10} M^{3/10} + x^{(\gamma+6)/10}
+ h^{7/110} M^{29/110} + x^{(\gamma+11)/110} + h^{1/18} M^{29/108} + x^{(\gamma+13)/18}
+ h^{11/214} M^{57/214} + x^{(11\gamma+157)/214} + h^{1/18} M^{1/6} + x^{(\gamma+14)/18}
+ h^{7/166} M^{29/166} + x^{(\gamma+19)/166} + h^{1/26} M^{29/156} + x^{(\gamma+21)/26}
+ h^{11/302} M^{57/302} + x^{(11\gamma+245)/302} + h^{1/14} M^{3/14} + x^{(\gamma+10)/14}
+ h^{7/138} M^{29/138} + x^{(\gamma+15)/138} + h^{1/22} M^{29/132} + x^{(\gamma+17)/22}
+ h^{11/258} M^{57/258} + x^{(11\gamma+201)/258} + h^{1/22} M^{3/22} + x^{(\gamma+18)/22}
+ h^{7/194} M^{29/194} + x^{(\gamma+23)/194} + h^{1/30} M^{29/180} + x^{(\gamma+25)/30}
+ h^{11/346} M^{57/346} + x^{(11\gamma+289)/346}.
\tag{2.63}
\]
From (2.63) we know that, under the condition (2.20), the result of Lemma 2.16 follows.

3 Proof of Theorem 1.1

In order to prove Theorem 1.1, it is sufficient for us to prove the following proposition.

Proposition 3.1 Suppose that \(0 < \gamma \leq 1, \delta > 0\) and

\[
1714(1 - \gamma) + 1725\delta < 46.
\]

Then, uniformly in \(\alpha\), we have

\[
T_3(N, \alpha) = S_3(N, \alpha) + O(P^{1-\delta-\varepsilon}),
\]

where the implied constant depends only on \(\gamma\) and \(\delta\).

3.1 Proof of Proposition 3.1

We have

\[
\frac{1}{\gamma} \sum_{p \leq P, p \in \mathcal{P}_\gamma} p^{1-\gamma} e(\alpha p^3) = \frac{1}{\gamma} \sum_{p \leq P} p^{1-\gamma} e(\alpha p^3) \left([-p^\gamma] - [-(p+1)^\gamma]\right)
\]

\[
= \sum_{p \leq P} e(\alpha p^3) + \frac{1}{\gamma} \sum_{p \leq P} p^{1-\gamma} e(\alpha p^3) (\psi(-(p+1)^\gamma) - \psi(-p^\gamma)) + O(\log P).
\]

By noting that, for \(p \sim x\) satisfying \(x \leq P^{1/2}\), we have

\[
\sum_{p \sim x} p^{1-\gamma} e(\alpha p^3) (\psi(-(p+1)^\gamma) - \psi(-p^\gamma)) \ll \sum_{p \sim x} p^{1-\gamma} \ll x^{2-\gamma} \ll P^{1-\gamma/2} \ll P^{1-\delta-\varepsilon}.
\]

Therefore, in order to prove Proposition 3.1, it is sufficient for us to prove that, for any \(x\) satisfying \(P^{1/2} < x \leq P\), there holds

\[
\sum_{p \sim x} p^{1-\gamma} e(\alpha p^3) (\psi(-(p+1)^\gamma) - \psi(-p^\gamma)) \ll x^{1-\delta-\varepsilon}.
\]

By partial summation, we have

\[
\sum_{p \sim x} p^{1-\gamma} e(\alpha p^3) (\psi(-(p+1)^\gamma) - \psi(-p^\gamma))
\ll (\log x)^{-1} \left| \sum_{n \sim x} \Lambda(n) n^{1-\gamma} e(\alpha n^3) (\psi(-(n+1)^\gamma) - \psi(-n^\gamma)) \right| + x^{3/2-\gamma} \log x.
\]

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Hence, we only need to show that
\[ \sum_{n \sim x} \Lambda(n)n^{1-\gamma}e\left(\alpha n^3\right)\left(\psi(-(n+1)\gamma) - \psi(-n\gamma)\right) \ll x^{1-\delta-\varepsilon}. \] (3.1)

Applying Lemma 2.10 to (3.1) with the parameter \( H = H_0 \), the contribution of the error term in (2.4) is
\[
\ll \sum_{n \sim x} \Lambda(n)n^{1-\gamma} \min\left(1, \frac{1}{H_0||n\gamma||}\right)
\ll x^{1-\gamma}\log x \sum_{n \sim x} \min\left(1, \frac{1}{H_0||n\gamma||}\right)
\ll x^{1-\gamma}\log x \sum_{n \sim x} a_h e(h\gamma)
\ll x^{1-\gamma}\log x \sum_{h=-\infty}^{\infty} |a_h| \left| \sum_{n \sim x} e(h\gamma) \right|.
\]

For \( h \neq 0 \), applying (2.2) to the inner sum, we get
\[
\left| \sum_{n \sim x} e(h\gamma) \right| \ll h^{1/2}x^{\gamma/2} + h^{-1}x^{1-\gamma}.
\]

Therefore, the contribution of the error term is
\[
\ll x^{1-\gamma}\log x \left( \frac{x \log H_0}{H_0} + \sum_{h=-\infty}^{\infty} |a_h| \left( |h|^{1/2}x^{\gamma/2} + |h|^{-1}x^{1-\gamma} \right) \right)
\ll x^{1-\gamma}\log x \left( \frac{x \log H_0}{H_0} + \sum_{0<h\leq H_0} \frac{1}{h} \left( h^{1/2}x^{\gamma/2} + h^{-1}x^{1-\gamma} \right) \right.
\left. + \sum_{h>H_0} \frac{H_0}{h^2} \left( h^{1/2}x^{\gamma/2} + h^{-1}x^{1-\gamma} \right) \right)
\ll x^{1-\gamma}\log^2 x \left( xH_0^{-1} + x^{1-\gamma} + H_0^{1/2}x^{\gamma/2} \right). \] (3.2)

Taking \( H_0 = x^{1-\gamma+\delta+\varepsilon} \), then (3.2) is \( \ll x^{1-\delta-\varepsilon} \).

From (3.1) and (3.2), it is easy to see that, in order to prove Proposition 3.1, we only need to prove
\[
\sum_{h \sim H} \frac{1}{h} \left| \sum_{n \sim x} \Lambda(n)n^{1-\gamma}e\left(\alpha n^3\right)\left(e\left(h(n+1)\gamma\right) - e(h\gamma)\right) \right| \ll x^{1-\delta-\varepsilon}. \] (3.3)

Set \( H_1 = x^{1-\gamma} \). If \( H \leq H_1 \), we write
\[
e(e(h(n+1)\gamma) - e(hn\gamma)) = 2\pi i h \gamma \int_0^1 (n + u)^{\gamma-1}e(h(n + u)\gamma)du. \] (3.4)
Putting (3.4) into the left hand side of (3.3) and combining partial summation, we can see that the left hand of (3.3) is

\[ \ll \sum_{h \sim H} \left| \sum_{n \sim x} \Lambda(n)e\left(\alpha n^3 + h(n + u)^\gamma\right) \right|. \tag{3.5} \]

If \( H > H_1 \), we divide the left hand side of (3.3) into two parts and treat them separately. Applying partial summation to the inner sum of the left hand side of (3.3), we can see that the left hand of (3.3) is

\[ \ll \frac{H_1}{H} \sum_{h \sim H} \left| \sum_{n \sim x} \Lambda(n)e\left(\alpha n^3 + h(n + u)^\gamma\right) \right|. \tag{3.6} \]

Combining (3.3), (3.5) and (3.6), it is sufficient to show that

\[ \min\left(1, \frac{H_1}{H}\right) \sum_{h \sim H} \left| \sum_{n \sim x} \Lambda(n)e\left(\alpha n^3 + h(n + u)^\gamma\right) \right| \ll x^{1 - \delta - \varepsilon}. \]

Take parameters \( a_1, \cdots, a_9 \) as condition (2.20) in Lemma 2.16. Let

\[ a = \min(a_1, \cdots, a_9) - \varepsilon, \quad b = 24(1 - \gamma) + 24\delta + \varepsilon, \quad c = \gamma - 2\delta - \varepsilon. \]

Obviously, it is easy to check that

\[ b < 2/3, \quad b < a, \quad 1 - c < c - b. \]

By Lemma 2.11 with \( k = 3 \), one can see that the exponential sum

\[ \min\left(1, \frac{H_1}{H}\right) \sum_{h \sim H} \left| \sum_{n \sim x} \Lambda(n)e\left(\alpha n^3 + h(n + u)^\gamma\right) \right| \]

can be written as linear combination of \( O \left( \log^6 x \right) \) sums of the form

\[ T = \min\left(1, \frac{H_1}{H}\right) \sum_{h \sim H} \left| \sum_{n_1 \sim N_1} \cdots \sum_{n_6 \sim N_6} (\log n_1)\mu(n_1)\mu(n_5)\mu(n_6) \right. \]

\[ \times e\left(\alpha(n_1 \cdots n_6)^3 + h(n_1 \cdots n_6 + u)^\gamma\right), \tag{3.7} \]

where \( N_1 \cdots N_6 \sim x; 2N_i \leq (2x)^{1/3}, i = 4, 5, 6 \) and some \( n_i \) may only take value 1. Therefore, it is sufficient for us to prove that, for each \( T \) defined as (3.7), there holds

\[ T \ll x^{1 - \delta - \varepsilon}. \]

Next, we will consider three cases.

**Case 1** If there exists an \( N_j \) such that \( N_j \geq x^{1 - b} \), then we must have \( j \leq 3 \) for the fact that \( 1 - b > 1/3 \). Let \( m = \prod_{i \neq j} n_i, n = n_j, M = \prod_{i \neq j} N_i, N = N_j \). In this case,
we can see that $\mathcal{T}$ is a sum of “Type I” satisfying $M \ll x^b \ll x^a$. By Lemma 2.16, the result follows.

**Case 2** If there exists an $N_j$ such that $x^{1-c} \leq N_j < x^{1-b}$, then we take $m = \prod_{i \neq j} n_i$, $n = n_j$, $M = \prod_{i \neq j} N_i$, $N = N_j$. Thus, $\mathcal{T}$ is a sum of “Type II” satisfying $x^b \ll M \ll x^a$. By Lemma 2.15, the result follows.

**Case 3** If $N_j < x^{1-c} (j = 1, 2, 3, 4, 5, 6)$, without loss of generality, we assume that $N_1 \geq N_2 \geq \cdots \geq N_6$. Let $\ell$ denote the natural number $j$ such that

\[ N_1 N_2 \cdots N_{j-1} < x^{1-c}, \quad N_1 N_2 \cdots N_{j} \geq x^{1-c}. \]

Since $N_1 < x^{1-c}$ and $N_6 < x^{1-c}$, then $2 \leq \ell \leq 5$. Thus, we have

\[ x^{1-c} \leq N_1 N_2 \cdots N_{\ell} = (N_1 \cdots N_{\ell-1}) \cdot N_{\ell} < x^{1-c} \cdot x^{1-c} < x^{1-b}. \]

Let $m = \prod_{i=\ell+1}^{6} n_i$, $n = \prod_{i=1}^{\ell} n_i$, $M = \prod_{i=\ell+1}^{6} N_i$, $N = \prod_{i=1}^{\ell} N_i$. At this time, $\mathcal{T}$ is a sum of “Type II” satisfying $x^b \ll M \ll x^a$. By Lemma 2.15, the result follows.

Combining the above three cases, we can assert that Proposition 3.1 holds.

### 3.2 Proof of Theorem 1.1

Take parameters as follows:

\[ Q = N^{\sigma}, \quad \tau = N^{1-\sigma}, \]

where $\sigma$ satisfies $0 < \sigma \leq 1/6$ to be determined later. When $1 \leq a \leq q \leq Q$ and $(a, q) = 1$, define major arcs and minor arcs as following:

\[ \mathcal{M}(a, q) = \left\lbrack \frac{a}{q} - \frac{1}{q \tau}, \frac{a}{q} + \frac{1}{q \tau} \right\rbrack \]

and

\[ \mathcal{M} = \bigcup_{q \in \mathcal{Q}} \bigcup_{1 \leq a \leq q \atop (a, q) = 1} \mathcal{M}(a, q), \quad m = \left\lbrack \frac{1}{\tau}, 1 + \frac{1}{\tau} \right\rbrack \setminus \mathcal{M}. \]

It is easy to find that Theorem 1.1 is a direct corollary of the following theorem.

**Theorem 3.2** Under the condition of Theorem 1.1, we have

\[
\int_{\frac{1}{\tau}}^{1+\frac{1}{\tau}} T_{1,1}(N, \alpha) T_{1,2}(N, \alpha) T_{3}(N, \alpha) e(-N\alpha) d\alpha \\
= \int_{\frac{1}{\tau}}^{1+\frac{1}{\tau}} S_{1}^{2}(N, \alpha) S_{3}(N, \alpha) e(-N\alpha) d\alpha + O(N^{4/3} L^{-B}), \tag{3.8}
\]

where $B > 0$ is arbitrary.
3.3 Proof of Theorem 3.2

First, take \( \gamma = \gamma_3, \delta = \delta_3 \) in Proposition 3.1. By Lemma 2.5 and Cauchy’s inequality, we have

\[
\begin{align*}
\int_0^{1+\frac{1}{L}} T_{1,1}(N, \alpha)T_{1,2}(N, \alpha)\left(T_3(N, \alpha) - S_3(N, \alpha)\right)e(-N\alpha)d\alpha \\
\ll \max_{\alpha \in [\frac{1}{L}, \frac{1}{L} + \frac{1}{N}]} |T_3(N, \alpha) - S_3(N, \alpha)| \times \int_0^{1} |T_{1,1}(N, \alpha)T_{1,2}(N, \alpha)|d\alpha \\
\ll N^{(1-\delta_3)/3} \left( \int_0^{1} |T_{1,1}(N, \alpha)|^2d\alpha \right)^{\frac{1}{2}} \left( \int_0^{1} |T_{1,2}(N, \alpha)|^2d\alpha \right)^{\frac{1}{2}} \\
\ll N^{(1-\delta_3)/3} \cdot N^{1-\gamma_1/2} \cdot N^{1-\gamma_2/2} \\
\ll N^{7/3-(\gamma_1+\gamma_2)/2-\delta_3/3}.
\end{align*}
\] (3.9)

Second, we divide the integral into the major arcs and the minor arcs.

\[
\begin{align*}
\int_0^{1+\frac{1}{L}} T_{1,1}(N, \alpha)T_{1,2}(N, \alpha)S_3(N, \alpha)e(-N\alpha)d\alpha \\
= \left\{ \int_{\mathbb{M}} + \int_{\mathfrak{m}} \right\} T_{1,1}(N, \alpha)T_{1,2}(N, \alpha)S_3(N, \alpha)e(-N\alpha)d\alpha.
\end{align*}
\]

For

\[
Q < q \leq \tau, \quad (a, q) = 1, \quad 1 \leq a \leq q.
\]

Thus, we have

\[
\sup_{\alpha \in \mathfrak{m}} |S_3(N, \alpha)| \ll N^{\frac{1}{3}}+\epsilon \left( \frac{1}{Q} + \frac{1}{N^{1/\sigma}} + \frac{\tau}{N} \right)^{\frac{1}{10}} \\
\ll N^{\frac{1}{3}}+\epsilon \left( N^{-\sigma} + N^{-\frac{1}{6}} \right)^{\frac{1}{10}} \\
\ll N^{\frac{1}{3} - \frac{1}{10}+\epsilon}.
\]

Using Lemma 2.5 and Cauchy’s inequality, we have

\[
\begin{align*}
\int_{\mathfrak{m}} T_{1,1}(N, \alpha)T_{1,2}(N, \alpha)S_3(N, \alpha)e(-N\alpha)d\alpha \\
\ll \sup_{\alpha \in \mathfrak{m}} |S_3(N, \alpha)| \times \int_{\mathfrak{m}} |T_{1,1}(N, \alpha)T_{1,2}(N, \alpha)|d\alpha \\
\ll \sup_{\alpha \in \mathfrak{m}} |S_3(N, \alpha)| \times \left( \int_0^{1} |T_{1,1}(N, \alpha)|^2d\alpha \right)^{\frac{1}{2}} \left( \int_0^{1} |T_{1,2}(N, \alpha)|^2d\alpha \right)^{\frac{1}{2}} \\
\ll N^{\frac{1}{3} - \frac{1}{10}+\epsilon}.N^{1-\gamma_1/2}N^{1-\gamma_2/2} \\
\ll N^{\frac{7}{3}-(\gamma_1+\gamma_2)/2}N^{\frac{1}{3}-\frac{1}{10}+\epsilon}.
\end{align*}
\] (3.10)
For $\alpha \in \mathfrak{M}$, taking $\gamma = \gamma_i$ ($i = 1, 2$) in Lemma 2.2, we have
\[
\int_{\mathfrak{M}} T_{1,1}(N, \alpha)T_{1,2}(N, \alpha)S_3(N, \alpha)e(-N\alpha)d\alpha \\
\ll \int_{\mathfrak{M}} S_1^2(N, \alpha)S_3(N, \alpha)e(-N\alpha)d\alpha + \int_{\mathfrak{M}} S_1(N, \alpha)S_3(N, \alpha) \cdot O(N^{1-\delta_1}) \cdot e(-N\alpha)d\alpha \\
+ \int_{\mathfrak{M}} (O(N^{1-\delta_1}))^2 S_3(N, \alpha)e(-N\alpha)d\alpha \\
=: I + II + III,
\] (3.11)
say. For I, noting that
\[
\int_{\mathfrak{M}} S_1^2(N, \alpha)S_3(N, \alpha)e(-N\alpha)d\alpha \\
\ll \sup_{\alpha \in \mathfrak{M}} \left| S_3(N, \alpha) \right| \times \int_0^1 \left| S_1(N, \alpha) \right|^2d\alpha \\
\ll N^{\frac{4}{3} - \frac{\delta}{3} + \varepsilon}, \quad N \ll N^{\frac{1}{4} - \frac{\delta}{3} + \varepsilon},
\]
we have
\[
I = \int_1^{1+\frac{1}{\tau}} S_1^2(N, \alpha)S_3(N, \alpha)e(-N\alpha)d\alpha - \int_m S_1^2(N, \alpha)S_3(N, \alpha)e(-N\alpha)d\alpha \\
= \int_1^{1+\frac{1}{\tau}} S_1^2(N, \alpha)S_3(N, \alpha)e(-N\alpha)d\alpha + O(N^{\frac{4}{3} - \frac{\delta}{3} + \varepsilon}).
\] (3.12)
It is easy to see that the measure of $\mathfrak{M}$ is
\[
\ll \sum_{q \in Q} \sum_{a=1}^q \frac{1}{q^\tau} \ll \frac{Q}{\tau} \ll N^{2\sigma-1}.
\]
Combining the trivial bound $S_1(N, \alpha) \ll N$, $S_3(N, \alpha) \ll N^{1/3}$, we have
\[
\ll N^{1-\delta_1} \int_{\mathfrak{M}} \left| S_1(N, \alpha)S_3(N, \alpha) \right|d\alpha \\
\ll N^{1-\delta_1} \cdot N \cdot \mathfrak{M}^{\frac{1}{3}} \cdot \text{meas}(\mathfrak{M}) \ll N^{\frac{4}{3} + 2\sigma - \delta_1}
\] (3.13)
and
\[
\ll N^{2-2\delta_1} \int_{\mathfrak{M}} \left| S_3(N, \alpha) \right|d\alpha \\
\ll N^{2-2\delta_1} \cdot N^{\frac{4}{3}} \cdot \text{meas}(\mathfrak{M}) \ll N^{\frac{4}{3} + 2(\sigma - \delta_1)}.
\] (3.14)
Collecting the above formulas (3.9)-(3.14), under the conditions
\[
\frac{\gamma_1 + \gamma_2}{2} + \frac{\delta_3}{3} > 1, \quad \frac{\sigma}{16} + \frac{\gamma_1 + \gamma_2}{2} > 1, \quad 2\sigma - \delta_1 < 0,
\]
\[
73(1 - \gamma_i) + 86\delta_1 < 9 \quad (i = 1, 2), \quad 1714(1 - \gamma_3) + 1725\delta_3 < 46,
\]
25
i.e.
\[
\frac{\gamma_1 + \gamma_2}{2} + \frac{\delta_3}{3} > 1, \quad \frac{\gamma_1 + \gamma_2}{2} + \frac{\delta_1}{32} > 1,
\]
\[
73(1 - \gamma_i) + 86\delta_1 < 9 \quad (i = 1, 2), \quad 1714(1 - \gamma_3) + 1725\delta_3 < 46,
\]
the equation (3.8) holds.

Acknowledgement
The authors would like to express the most and the greatest sincere gratitude to Professor Wenguang Zhai for his valuable advice and constant encouragement.

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