Segal-Bargmann transform:  
the $q$-deformation

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March 23, 2017

Abstract

We give identifications of the $q$-deformed Segal-Bargmann transform and define the Segal-Bargmann transform on mixed $q$-Gaussian variables. We prove that, when defined on the random matrix model of Śniady for the $q$-Gaussian variable, the classical Segal-Bargmann transform converges to the $q$-deformed Segal-Bargmann transform in the large $N$ limit. We also show that the $q$-deformed Segal-Bargmann transform can be recovered as a limit of a mixture of classical and free Segal-Bargmann transform.

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1 Introduction

Let $H$ be a real finite-dimensional Hilbert space. Let $\gamma$ be the standard Gaussian measure on $H$, whose density with respect to the Lebesgue measure at $h \in H$ is $(2\pi)^{-d/2} \exp(-\|h\|_H^2/2)$. Let $\mu$ be the Gaussian measure on the complexification $H^C = H + iH$ of $H$ whose density with respect to the Lebesgue measure at $h \in H^C$ is $\pi^{-d} \exp(-\|h\|_{H^C}^2)$. For all $f \in L^2(H, \gamma)$, the map $z \mapsto \int_H f(z - x) \, d\gamma(x)$, admits an analytic continuation $\mathcal{S}(f)$ to $H^C$. Furthermore, the map $\mathcal{S}(f)$ is in the closed subspace of holomorphic functions of $L^2(H^C, \mu)$, hereafter denoted by $\mathcal{H}L^2(H^C, \mu)$. The resulting map

$$\mathcal{S} : L^2(H, \gamma) \to \mathcal{H}L^2(H^C, \mu) \quad (1.1)$$

is known as the Segal-Bargmann transform, introduced by Segal [21, 22] and Bargmann [1, 2] in early 1960s.

1.1 $q$-Deformed Segal-Bargmann Transform

In [28], Leeuwen and Massen considered a $q$-deformation of the Segal-Bargmann transform in the one-dimensional case. For all $0 \leq q < 1$, the measure replacing the Gaussian measure is the $q$-Gaussian measure $\nu_q$ on $\mathbb{R}$, whose density with respect to the Lebesgue measure is

$$\nu_q(dx) = \mathbb{1}_{|x| \leq 2/\sqrt{1-q}} \frac{1}{\pi} \sqrt{1-q} \sin \theta \prod_{n=1}^{\infty} (1 - q^n) |1 - q^n e^{2i\theta}|^2 \, dx,$$

where $\theta \in [0, \pi]$ is such that $x = 2 \cos(\theta)/\sqrt{1-q}$. The $q$-deformation of the Segal-Bargmann transform is then defined through the kernel

$$\Gamma_q(x, z) = \prod_{k=0}^{\infty} \frac{1}{1 - (1-q)q^k x z + (1-q)q^{2k} z^2}, \quad |x| \leq \frac{2}{\sqrt{1-q}}, \quad |z| < \frac{1}{\sqrt{1-q}}.$$

For all function $f \in L^2(\mathbb{R}, \nu_q)$, the function

$$\mathcal{S}_q(f) : z \mapsto \int_{\mathbb{R}} f(x) \Gamma_q(x, z) \, d\nu_q(x)$$

is defined on the unit disk of radius $1/\sqrt{1-q}$ and the map $\mathcal{S}_q$ is in fact an isomorphism of Hilbert space between $L^2(\mathbb{R}, \nu_q)$ and a reproducing kernel Hilbert space of analytic function on the unit disk of radius $1/\sqrt{1-q}$ which plays the role of the complexified version of $\nu_q$.

Let us remark that $\lim_{q \to 1} \nu_q(dx) = \exp(-x^2/2)/\sqrt{2\pi} \, dx$ and $\lim_{q \to 1} \Gamma_q(x, z) = \exp(xz - z^2/2)$, which suggest to denote the standard normal distribution by $\nu_1$ and the classical Segal-Bargmann transform $\mathcal{S}$ on $L^2(\mathbb{R}, \nu_1)$ by $\mathcal{S}_1$. The case $q = 0$, studied in [3], is of particular interest, since $\nu_0(dx) = \mathbb{1}_{|x| \leq \sqrt{4-\pi^2}} \frac{2}{2\pi} \, dx$ is the well-known semicircular law and the so-called free Segal-Bargmann transform $\mathcal{S}_0$ maps isometrically $L^2(\mathbb{R}, \nu_0)$ to the Hardy space of analytic functions on the unit disc. Although beyond the scope of this article, let us mention also the related work [5], where Blitvić and Kemp define a refinement of the $q$-deformed Segal-Bargmann transform.

1.2 Matrix Approximations

In [3], Biane proves that the free Segal-Bargmann transform $\mathcal{S}_0$ is the limit of the classical Segal-Bargmann transform on Hermitian matrices in the following sense: for all $N \geq 1$, let $\mathbb{M}_N$ be the space of complex matrices of size $N \times N$, let $\mathbb{H}_N$ be the subspace of Hermitian matrices $M = M^*$ of size $N \times N$, and let $\text{Tr}$ denote the usual trace. Let $\gamma_N$ be the standard Gaussian measure on $\mathbb{H}_N$ for the norm $\|M\|^2 = N\text{Tr}(MM^*)$, and $\mu_N$ be
the standard Gaussian measure on \( \mathbb{M}_N = \mathbb{H}_N + i\mathbb{H}_N \) for the norm \( \|M\|^2 = 2N \Re \text{Tr}(M^2) \). This way we can consider the Segal-Bargmann transform \( \mathcal{S} : L^2(\mathbb{H}_N, \gamma_N) \to \mathcal{H} L^2(\mathbb{M}_N, \mu_N) \) defined by \( \{1\} \). Biane extends the transform \( \mathcal{S} \) to act on \( \mathbb{M}_N \)-valued functions, by applying \( \mathcal{S} \) entrywise. More precisely, endowing \( \mathbb{M}_N \) with the norm \( \|M\|_{\mathbb{M}_N}^2 = \text{Tr}(MM^*)/N \), he considers the Hilbert space tensor products \( L^2(\mathbb{H}_N, \gamma_N; \mathbb{M}_N) = L^2(\mathbb{H}_N, \gamma_N) \otimes \mathbb{M}_N \) and \( \mathcal{H} L^2(\mathbb{M}_N, \mu_N; \mathbb{M}_N) = \mathcal{H} L^2(\mathbb{M}_N, \mu_N) \otimes \mathbb{M}_N \), as well as the boosted Segal-Bargmann transform

\[
\mathcal{S}_N = \mathcal{S} \otimes \text{Id}_{\mathbb{M}_N} : L^2(\mathbb{H}_N, \gamma_N; \mathbb{M}_N) \to \mathcal{H} L^2(\mathbb{M}_N, \mu_N; \mathbb{M}_N).
\]

Each polynomial can be seen as an element of \( L^2(\mathbb{H}_N, \gamma_N; \mathbb{M}_N) \) (or of \( \mathcal{H} L^2(\mathbb{M}_N, \mu_N; \mathbb{M}_N) \)) via the polynomial calculus, and Biane proved that, restricted to those polynomial functions, the Segal-Bargmann transform \( \mathcal{S}_N \) converges to the free Segal-Bargmann transform \( \mathcal{S}_0 \) in the following sense: for all polynomial \( P \),

\[
\lim_{N \to \infty} \|\mathcal{S}_N(P) - \mathcal{S}_0(P)\|_{\mathcal{H} L^2(\mathbb{M}_N, \mu_N; \mathbb{M}_N)} = 0.
\]

One of the motivation of this article is to prove that the \( q \)-deformed Segal-Bargmann transform \( \mathcal{S}_q \) can also be approximated by the classical one for \( 0 < q \leq 1 \). In the model of Biane, \( L^2(\mathbb{H}_N, \gamma_N; \mathbb{M}_N) \) is an approximation of \( L^2(\mathbb{R}, \nu_0) \) in the sense that, for all polynomial \( P \), \( \|P\|_{L^2(\mathbb{R}, \nu_0)} = \lim_{N \to \infty} \|P\|_{L^2(\mathbb{H}_N, \gamma_N; \mathbb{M}_N)} \). In the case of \( 0 < q \leq 1 \), we replace the previous model by a model of Śniady introduced in [23] in order to approximate \( L^2(\mathbb{R}, \nu_q) \). Let us briefly describe this model.

Let \( d \geq 0 \). We endow \( \mathbb{M}_d \) by the inner products, quotient if necessary, \( \langle A, B \rangle_1 = \frac{1}{d} \text{Tr}(AB^*) \) and \( \langle A, B \rangle_0 = \text{Tr}(A) \text{Tr}(B^*) \). For all \( S \subset \{1, \ldots, N\} \), we define the inner product \( \langle \cdot, \cdot \rangle_S \) on \( \mathbb{M}_d \) to be the inner product of the Hilbert space tensor product \( \bigotimes_{r=1}^N (\mathbb{M}_d, \langle A, B \rangle_{\mathbb{M}_r}) \). Let \( \sigma = (\sigma_S)_{S \subset \{1, \ldots, N\}} \) be a family of real numbers indexed by all subsets of \( \{1, \ldots, N\} \). It determines an averaged inner product \( \langle A, B \rangle_{\sigma} = \sum_{S \subset \{1, \ldots, N\}} \sigma_S \cdot \langle A, B \rangle_S \) on \( \mathbb{M}_d \). Let \( \gamma_{dN}^{\sigma} \) be the Gaussian measure on \( \mathbb{H}_d \) whose characteristic function is given by

\[
\int_{\mathbb{H}_d} \exp(i \text{Tr}(MX)) \, d\gamma_{dN}^{\sigma}(X) = \exp(-\|M\|_d^2/2)
\]

and \( \mu_{dN}^{\sigma} \) be the Gaussian measure on \( \mathbb{M}_d \) whose characteristic function is given by

\[
\int_{\mathbb{M}_d} \exp(i \text{Tr}(MX)) \, d\mu_{dN}^{\sigma}(X) = \exp(-\Re \|M\|_d^2/4).
\]

Denoting by \( \text{supp} \gamma_{dN}^{\sigma} \) the support of \( \gamma_{dN}^{\sigma} \), which is a linear subspace of \( \mathbb{H}_d \), we have \( \text{supp} \mu_{dN}^{\sigma} = \text{supp} \gamma_{dN}^{\sigma} + i\text{supp} \gamma_{dN}^{\sigma} \). The linear space \( \text{supp} \gamma_{dN}^{\sigma} \) can be endowed with a unique inner product such that \( \gamma_{dN}^{\sigma} \) is the standard Gaussian measure on \( \text{supp} \gamma_{dN}^{\sigma} \), and therefore the Segal-Bargmann transform

\[
\mathcal{S} : L^2(\mathbb{H}_d, \gamma_{dN}^{\sigma}) = L^2(\text{supp} \gamma_{dN}^{\sigma}, \gamma_{dN}^{\sigma}) \to \mathcal{H} L^2(\text{supp} \mu_{dN}^{\sigma}, \mu_{dN}^{\sigma}) = \mathcal{H} L^2(\mathbb{M}_d, \mu_{dN}^{\sigma})
\]

is well-defined as in \([1] \). Following the model of Biane, we consider the two following Hilbert space tensor products \( L^2(\mathbb{H}_d, \gamma_{dN}^{\sigma}; \mathbb{M}_d) = L^2(\mathbb{H}_d, \gamma_{dN}^{\sigma}) \otimes \mathbb{M}_d \) and \( \mathcal{H} L^2(\mathbb{M}_d, \mu_{dN}^{\sigma}; \mathbb{M}_d) = \mathcal{H} L^2(\mathbb{M}_d, \mu_{dN}^{\sigma}) \otimes \mathbb{M}_d \), where \( \mathbb{M}_d \) is endowed with the norm \( \|M\|_{\mathbb{M}_d}^2 = \text{Tr}(MM^*)/d^N \). Finally, we consider the boosted Segal-Bargmann transform

\[
\mathcal{S}_{dN} = \mathcal{S} \otimes \text{Id}_{\mathbb{M}_d} : L^2(\mathbb{H}_d, \gamma_{dN}^{\sigma}; \mathbb{M}_d) \to \mathcal{H} L^2(\mathbb{M}_d, \mu_{dN}^{\sigma}; \mathbb{M}_d).
\]

**Theorem 1.1** (see Theorem 3.14). Let \( 0 \leq q \leq 1 \). Under technical assumptions [H.1] [H.2] [H.3] and [H.4] on \( \sigma \) (see Section 2.3) which ensure that, for all polynomial \( P \),

\[
\lim_{N \to \infty} \|P\|_{L^2(\mathbb{H}_d, \gamma_{dN}^{\sigma}; \mathbb{M}_d)} = \|P\|_{L^2(\mathbb{R}, \nu_q)},
\]

4
the Segal-Bargmann transform $\mathcal{S}_d^N$ converges to the $q$-deformed Segal-Bargmann transform $\mathcal{S}_q$ in the following sense: for all polynomial $P$,
\[
\lim_{N \to \infty} \| \mathcal{S}_d^N (P) - \mathcal{S}_q (P) \|_{\mathcal{H}L^2(M_{dN}, \gamma_{\nu^2_q M_{dN}})} = 0.
\]

We are able to prove Theorem 1.1 in the two parameter setting and in the multidimensional case.

1.3 Two Parameter Case

A simple scaling of $\mathcal{S} : L^2(H, \gamma) \to \mathcal{H}L^2(H^C, \mu)$ gives us a unitary isomorphism $\mathcal{S}^t$ which depends on one parameter $t > 0$. It is also possible to consider one scaling for the space $L^2(H, \gamma)$ and another scaling for the transform $\mathcal{S}$. It yields to the two-parameter Segal-Bargmann transform $\mathcal{S}^{s,t}$, where $s$ and $t$ are two parameters with $s > \frac{1}{2} > 0$, which was defined by Driver and Hall in [11, 13]. In this article, all the definition and results are considered in this two-parameter setting. In particular, we shall generalize the transform $\mathcal{S}_q$ of Leeuwen and Massen to a $q$-deformed Segal-Bargmann transform (with $-1 < q < 1$) given by
\[
\mathcal{S}_q^{s,t}(f) : z \mapsto \int_{\mathbb{R}} f(x) \Gamma_q^{s,t}(x, z) \nu_q^s (dx)
\]
where $\Gamma_q^{s,t}$ is a generating function and $\nu_q^s$ is scaled from $\nu_q$ so that it has variance $s$. With this formula, we are able to compute the range of the Segal-Bargmann transform, which is a reproducing kernel Hilbert space of analytic functions in an ellipse. It allows us to prove Theorem 3.14 which is a version of Theorem 1.1 with two parameters $s$ and $t$.

1.4 Multidimensional Case

In [3], Biane extends the free Segal-Bargmann transform $\mathcal{S}_0$ to the multidimensional case, replacing $\mathbb{R}$ by an arbitrary real Hilbert space $H$. The space $L^2(H, \gamma)$ has to be replaced by a non-commutative generalization of a $L^2$-space. More precisely, in the classical case, $L^2(H, \gamma)$ can be viewed as the space of square-integrable random variables generated by the Gaussian field on $H$. If $-1 \leq q \leq 1$, it is possible to define some $q$-deformations of Gaussian field over $H$ (see Section 4.2). The free Segal-Bargmann transform $\mathcal{S}_0$ acts on the space of square-integrable random variables generated by a $0$-deformed Gaussian field on $H$ (called semicircular system in [3]).

In [16], Kemp generalizes Biane’s results and defines a $q$-deformed Segal-Bargmann transform $\mathcal{S}_q$ acting on the space of square-integrable random variables generated by a $q$-deformed Gaussian field on $H$. In [14], the second author defined the two-parameter free Segal-Bargmann transform $\mathcal{S}_q^{s,t}$ acting on the space of square-integrable random variables generated by a $0$-deformed Gaussian field on $H$. In this article, we will follow [3, 14, 16] and define the two-parameter $q$-deformed Segal-Bargmann transform $\mathcal{S}_q^{s,t}$ acting on the space of square-integrable random variables generated by a $q$-deformed Gaussian field on $H$. Of course, if we consider $H = \mathbb{C}$, the $q$-Segal-Bargmann transform $\mathcal{S}_q^{s,t}$ is equivalent to the integral transform $\mathcal{S}_q^{s,t}$ already defined in (1.2); that is to say the integral transform gives an explicit formula of the $q$-Segal-Bargmann transform in the one dimensional setting (see Corollary 4.7).

Theorem 1.1 is true in the multidimensional case. Indeed, Theorem 4.8 shows that the two-parameter $q$-deformed Segal-Bargmann transform $\mathcal{S}_q^{s,t}$ acting on the space of square-integrable random variables generated by a $q$-deformed Gaussian field on $H$ can be approximated by the classical Segal-Bargmann transform.

1.5 Mixture of $q$-Deformed Segal-Bargmann Transform

In fact, it is possible to deform a Gaussian field over $\mathbb{R}^n$ in a much more complicated way, where a $q_{ij}$-deformed Gaussian random variable is considered for each direction of the canonical basis of $\mathbb{C}^n$, and where the correlation relation between two different variables is determined by some factors $q_{ij}$ ($q_{ij} = 1$ yields the classical independence of random variables and $q_{ij} = 0$ yields the free independence of random variables).
This deformation, first considered by Speicher in [25], is known as mixed $q$-Gaussian variables, and is uniquely determined by a symmetric matrix $Q = (q_{ij})_{1 \leq i, j \leq n}$ with elements in $[-1, 1]$. The case of the previous section corresponds to the case where all the elements of $Q$ are equal to a single $-1 \leq q \leq 1$. It is also possible in this framework to define a $Q$-deformed Segal-Bargmann transform $\mathcal{S}_Q^{s,t}$, and restricted on the one-dimensional directions, $\mathcal{S}_Q^{s,t}$ yields to the already defined $\mathcal{S}_{q_{ii}}^{s,t}$. In particular, if all $q_{ii}$ are equal to 0, $\mathcal{S}_Q^{s,t}$ can be seen as a noncommutative mixture of the classical Segal-Bargmann transform $\mathcal{S}_q^{s,t}$ (see Remark 5.7).

In [24], Speicher proves the following central limit theorem: every $q$-deformed Gaussian random variable can be approximated by a normalised sum of mixed $q$-Gaussian variables for some appropriate choice of $Q$ with elements in $\{-1, 1\}$. Similarly, Młotkowski proves in [18] that the elements of $Q$ can be chosen in $\{0, 1\}$ in the central limit theorem of Speicher.

Our last result, summed up in Theorem 5.6, is the fact that the $q$-deformed Segal-Bargmann transform $\mathcal{S}_Q^{s,t}$ can be approximated by a noncommutative mixture $\mathcal{S}_Q^{s,t}$ of the classical Segal-Bargmann transform applied on normalised sum of mixed $q$-Gaussian variables (see Remark 5.7).

1.6 Organization of the Paper

A brief outline of the paper is as follows. In Section 2, we introduce the Segal-Bargmann transform $\mathcal{S}_q^{s,t}$, continue with a summary of the (mixed) $q$-random variables and end by a description of the random matrix model of Śniady. In Section 3, we introduce the two-parameter $q$-deformed Segal-Bargmann transform $\mathcal{S}_q^{s,t}$, and prove Theorem 1.1. In Section 4, we introduce the two-parameter $q$-deformed Segal-Bargmann transform in the multidimensional case, and prove Theorem 4.3, the analogue of Theorem 1.1 in this multidimensional setting. Finally, in Section 5, we introduce the mixture of $q$-deformed Segal-Bargmann transform, and prove Theorem 5.6.

2 Preliminaries

We begin by briefly introduce the already existing objects and results that will be useful for us: the two-parameter Segal-Bargmann transform, the $q$-deformation of the Gaussian measure, the $q$-deformation of independent Gaussian random variables and the model of random matrix of Śniady which allows to approximate those $q$-deformed Gaussian random variables.

2.1 Segal-Bargmann Transform

Let $H$ be a real finite-dimensional Hilbert space of dimension $d \geq 1$. For all $t > 0$, we define $\gamma_t$ to be a Gaussian measure on $H$ whose density with respect to the Lebesgue measure at $x \in H$ is $(2\pi t)^{-d/2} \exp(-\|x\|^2/2t)$. For all $r, s > 0$, we define $\gamma_{r,s}$ to be a Gaussian measure on the complexification $H^C = H + iH$ of $H$ whose density with respect to the Lebesgue measure at $x + iy \in H^C$ is $(2\pi \sqrt{rs})^{-d} \exp(-\|x\|^2/2r - \|y\|^2/2s)$. In other words, identifying $H^C = H + iH$ with $H \times H$, we have $\gamma_{r,s} = \gamma_r \otimes \gamma_s$: the parameters $r$ and $s$ define the respective scaling of the Gaussian measure on the real and the imaginary part of $H$.

In [11], Driver and Hall introduced a general version of the Segal-Bargmann transform which depends on two parameters $s$ and $t$. Let $s > t/2 > 0$. For all $f \in L^2(H, \gamma_s)$, the map

$$z \mapsto \int_H f(z - x) \, d\gamma_t(x),$$

has a unique analytic continuation $\mathcal{S}_q^{s,t}(f)$ to $H^C$. Furthermore, the map $\mathcal{S}_q^{s,t}(f)$ is in the closed subspace of holomorphic functions of $L^2(H^C, \gamma_{s-t/2,t/2})$, denoted in the following by $\mathcal{H}L^2(H^C, \gamma_{s-t/2,t/2})$. The two
parameter Segal-Bargmann transform is the isomorphism of Hilbert space

$$\mathcal{S}^{s,t} : L^2(H, \gamma_s) \rightarrow \mathcal{H}L^2(H^C, \gamma_{s-t/2,t/2})$$

(2.2)

The standard case considered by Segal and Bargmann corresponds to the case $s = t$, and the Segal-Bargmann $\mathcal{S}$ considered in the introduction corresponds to the case $s = t = 1$.

### 2.2 $q$-Gaussian Measure

In this section, we will review some facts about $q$-Gaussian measures and $q$-Hermite polynomials. More discussions can be found in [6, 27].

**Definition 2.1.** Let $-1 < q < 1$ and $t \geq 0$. The $q$-Gaussian measure $\nu_q$ of variance 1 is defined to be

$$\nu_q(dx) = \text{1}_{|x| \leq 2/\sqrt{1-q}} \frac{1}{\pi} \sqrt{1-q} \sin \theta \prod_{n=1}^{\infty} (1-q^n)|1 - q^n e^{2i\theta}|^2 \, dx$$

where $\theta \in [0, \pi]$ is such that $x = 2 \cos \theta / \sqrt{1-q}$. The $q$-Gaussian measure $\nu_q^t$ of variance $t$ is given by $\nu_q^t(dx) = \nu_q(dx / \sqrt{t})$.

Let $-1 < q < 1$. For all integer $n$, set $[n]_q = 1 + \cdots + q^{n-1}$. For all $t \neq 0$, the $q$-Hermite polynomials $H^{q,t}_n$ of parameter $t$ are defined by $H^{q,t}_0(x) = 1$, $H^{q,t}_1(x) = x$ and the recurrence relation

$$H^{q,t}_{n+1}(x) = xH^{q,t}_n(x) - t[n]_q H^{q,t}_{n-1}(x).$$

They form an orthogonal family with respect to $\nu_q$ with norm $[n]_q t^n$. Their generating function

$$\Gamma_q^t(x, z) := \sum_{k=0}^{\infty} \frac{x^k}{[k]_q^t} H^{q,t}_k(x) = \prod_{k=0}^{t} \frac{x}{t - (1-q)q^k z x + (1-q)q^{2k}z^2},$$

where $[n]_q = \prod_{j=1}^{n} [j]_q$, converges whenever $|x| \leq \frac{2\sqrt{t}}{\sqrt{1-q}}$ and $|z| < \sqrt{\frac{t}{1-q}}$.

### 2.3 $q$-Gaussian Variables and Wick Product

**Definition 2.2.** A non-commutative probability space $(\mathcal{A}, \tau)$ is a unital $\ast$-algebra with a linear functional $\tau : \mathcal{A} \rightarrow \mathbb{C}$ such that $\tau [1_A] = 1$ and $\tau [A^* A] \geq 0$ for all $A \in \mathcal{A}$. The element of $\mathcal{A}$ are called random variables.

If $\mathcal{X}$ is a subset of $\mathcal{A}$, we denote by $L^2(\mathcal{X}, \tau)$ the Hilbert space given by the completion of the (quotiented if necessary) space of the $\ast$-algebra generated by $\mathcal{X}$ with respect to the norm $||A||^2 = \tau [A^* A]$, and by $\mathcal{H}L^2(\mathcal{X}, \tau)$ the Hilbert space given by the completion of the (quotiented if necessary) space of the algebra generated by $\mathcal{X}$ with respect to the same norm.

The following definition of $q$-Gaussian variables can be considered as a $q$-deformation of the Wick formula of Gaussian variables (the classical case corresponds to $q = 1$). Let $P_2(n)$ be the set of pairing of $\{1, \ldots, n\}$. Let $\pi$ be a pairing of $\{1, \ldots, n\}$. A quadruplet $1 \leq i < j < k < l \leq n$ is called a crossing of $\pi$ if $\{i, k\} \in \pi$ and $\{j, l\} \in \pi$. The number of crossings of the pairing $\pi$ is denoted by $\text{cr}(\pi)$.

**Definition 2.3.** Let $-1 \leq q \leq 1$. A set $\mathcal{X}$ of self-adjoint and centred non-commutative random variables in a non-commutative probability space $(\mathcal{A}, \tau)$ is said to be jointly $q$-Gaussian if, for all $X_1, \ldots, X_n \in \mathcal{X}$, we have

$$\tau [X_1 \cdots X_n] = \sum_{\pi \in P_2(n)} q^{\text{cr}(\pi)} \prod_{\{i, j\} \in \pi} \tau [X_i X_j].$$

(2.3)
Two sets of jointly q-Gaussian variables $\mathcal{X}$ and $\mathcal{Y}$ are called q-independent if and only if $\mathcal{X} \cup \mathcal{Y}$ is jointly q-Gaussian and the elements of $\mathcal{X}$ are orthogonal with the elements of $\mathcal{Y}$ in $L^2(\mathcal{A}, \tau)$.

A set $\mathcal{Z}$ of non-commutative centred random variables in a non-commutative probability space $(\mathcal{A}, \tau)$ is said to be jointly q-Gaussian if $\{\mathcal{R}Z, \exists Z : Z \in \mathcal{Z}\}$ is jointly q-Gaussian. Moreover, if $\tau([\mathcal{R}Z]^2) = s$ and $\tau([\mathcal{Z}]^2) = t$, we say that $\mathcal{Z}$ is a $(s, t)$-elliptic q-Gaussian variable.

Let $\mathcal{X}$ be any set of jointly q-Gaussian variables (not necessarily self-adjoint). Then, by linearity, it follows that the linear span of $\{X, X^* : X \in \mathcal{X}\}$ is also jointly q-Gaussian. If we take $X_1 = \cdots = X_n = X = X^*$ in (2.3) and such that $\tau[X^2] = 1$, we obtain the formula for the moments of the q-Gaussian measure $\nu_q$:

$$\tau[X^n] = \sum_{\pi \in \mathcal{P}_2(n)} q^{\text{cr}(\pi)} = \int \prod x_i^n \nu_q(dx).$$

The q-Gaussian measure is called the distribution of the q-Gaussian variable $X$. 

**Remark 2.4.** A family of self-adjoint jointly q-Gaussian variables $(X_i)_{i \in I}$ is, up to isomorphism, a q-Gaussian process as defined in [6], with covariance $c : I \times I \rightarrow \mathbb{R}$ given by $c(i, j) = \tau[X_i X_j]$. We can view them as operators acting on a q-deformation of the Fock space over $\mathcal{H}$ (see Section 2.4). In the literature, for example in [6] [7] [10] [12] [16], the q-Gaussian variables have often been considered in this particular representation. Since our work only involves the non-commutative distribution of the q-Gaussian variables, we found more convenient to forget about the representation of a q-Gaussian variables and define it via its non-commutative distribution. This non-commutative distribution is implicitly given in [7] Proposition 2], or alternatively in [12] Corollary 2.1].

**Definition 2.5.** Let $-1 \leq q \leq 1$. Let $n \geq 0$. The Wick product of $n$ jointly q-Gaussian variables $X_1, \ldots, X_n$, denoted by $X_1 \odot \cdots \odot X_n$, is uniquely defined by the following recursion formula: the empty Wick product is 1 and

$$X_1 \odot \cdots \odot X_n = X_1 \cdot (X_2 \odot \cdots \odot X_n) - \sum_{i=2}^n q^{i-1} \tau[X_i X_1](X_1 \odot \cdots \odot \hat{X}_i \odot \cdots \odot X_n)$$

where the hat means that we omit the corresponding element in the product.

**Remark 2.6.** The Wick product has been considered in [6] and [12] with different notation. Considering a set of self-adjoint jointly q-Gaussian variables $(X_i)_{i \in I}$ acting on the q-deformation of the Fock space over $L^2(X_i, \tau)$, the Wick product $X_1 \odot \cdots \odot X_n$ coincides with the quantity denoted by $\Psi(X_1 \otimes \cdots \otimes X_n)$ in [6] Definition 2.5) (they satisfy the same recursion formula thanks to [6] Proof of Proposition 2.7]). The Wick product $X_1 \odot \cdots \odot X_n$ is denoted by $\mathcal{W}_{\mathcal{X}}$ in [12].

In [12] is given an explicit formula for the Wick product of jointly q-Gaussian variables which are self-adjoint that we will present now. By linearity, the formula is also valid for non-necessarily self-adjoint variables. A Feynman diagram $\gamma$ on $\{1, \ldots, n\}$ is a partition of $\{1, \ldots, n\}$ into one- and two-element sets. The set of Feynman diagrams on $\{1, \ldots, n\}$ is denoted by $\mathcal{F}(n)$, and we have $\mathcal{P}_2(n) \subset \mathcal{F}(n)$. We extend naturally the notion of crossing to $\mathcal{F}(n)$: a quadruplet $1 \leq i < j < k < l \leq n$ is called a crossing of $\gamma$ if $\{i, k\} \in \gamma$ and $\{j, l\} \in \gamma$. The number of crossings of a Feynman diagram $\gamma$ is denoted by $\text{cr}(\gamma)$. Similarly, a triplet $1 \leq i < j < k \leq n$ is called a gap of $\gamma$ if $\{i, k\} \in \gamma$ and $\{j\} \in \gamma$. The number of gaps of a Feynman diagram $\gamma$ is denoted by $\text{gap}(\gamma)$. Finally, the number of pairings of a Feynman diagram $\gamma$ is denoted by $2\gamma$. 

**Theorem 2.7** (Theorem 3.1 of [12]). The Wick product of jointly q-Gaussian variables $X_1, \ldots, X_n$ is given by

$$X_1 \odot \cdots \odot X_n = \sum_{\gamma \in \mathcal{F}(n)} (-1)^{2\gamma} q^{\text{gap}(\gamma) - \text{cr}(\gamma)} \prod_{\{a,b\} \in \gamma} \tau[X_a X_b] \prod_{\{c\} \in \gamma} X_c.$$

In the following proposition, we sum up some properties of the Wick product which can be found in [6] and in [12] for self-adjoint jointly q-Gaussian variables. The general case follows by linearity.
Proposition 2.8.  
1. The Wick product is multilinear on the linear span of jointly $q$-Gaussian variables.

2. If $X_1, \ldots, X_n$ is jointly $q$-Gaussian, $(X_1 \cdots X_n)^* = X_n^\ast \cdots X_1^\ast$.

3. If $X_1, \ldots, X_{n+m}$ is jointly $q$-Gaussian (with $n, m \geq 0$),
\[ \tau \left[ (X_1 \cdots X_n) \cdot (X_{n+1} \cdots X_{n+m}) \right] = \delta_{n,m} \sum_{\pi \in P_2(n,m)} q^{\text{cr} (\pi)} \prod_{\{i,j\} \in \pi} \tau [X_i X_j]. \]

4. If \( \{X_i\}_{i \in I} \) is a set of jointly $q$-Gaussian variables, the set of Wick products
\[ \{X_{i(1)} \cdots X_{i(n)}\}_{n \geq 0, i(1), \ldots, i(n) \in I} \]
is a spanning set of the algebra $C \langle X_i : i \in I \rangle$ generated by $\{X_i\}_{i \in I}$.

2.4 Mixed $q$-Gaussian Variables

Let $Q = (q_{ij})_{i,j \in I}$ be a symmetric matrix with elements in $[-1, 1]$. We recall now the construction of the mixed $q$-Gaussian variables operators $X_i = c_i + c_i^\ast$, where $c_i$ satisfy the commutation relations of the form
\[ c_i^\ast e_j = q_{ij} c_j^\ast + \delta_{ij} 1. \] (2.4)

We consider a complex Hilbert space $K$ with an orthonormal basis $\{e_i\}_{i \in I}$, and the algebraic full Fock space
\[ \mathcal{F}(K) = \mathbb{C} \Omega + \bigoplus_{n=1}^{\infty} (K)^{\otimes n} \]
where $\Omega$ is a unit vector called the vacuum. The set of permutations of $\{1, \ldots, n\}$ is denoted by $\mathfrak{S}_n$, and a pair $1 \leq a < b < l \leq n$ is called an inversion of a permutation $\pi \in \mathfrak{S}_n$ if $\pi(a) \geq \pi(b)$. We define the Hermitian form $\langle \cdot, \cdot \rangle_Q$ to be the conjugate-linear extension of
\[ \langle \Omega, \Omega \rangle_Q = 1 \]
\[ \langle e_{i(1)} \otimes \cdots \otimes e_{i(k)}, e_{j(1)} \otimes \cdots \otimes e_{j(l)} \rangle_Q = \delta_{k\ell} \sum_{\pi \in \mathfrak{S}_k} \prod_{i = j \circ \pi} q_{i(a)i(b)}. \]

The $Q$-Fock space $\mathcal{F}_Q(K)$ is the completion of the quotient of $\mathcal{F}(K)$ by the kernel of $\langle \cdot, \cdot \rangle_Q$. For any $i \in I$, define the left creation operator $c_i$ on $\mathcal{F}_Q(K)$ to extend
\[ c_i(\Omega) = e_i \]
\[ c_i(e_{i(1)} \otimes \cdots \otimes e_{i(k)}) = e_i \otimes e_{i(1)} \otimes \cdots \otimes e_{i(k)}. \]

The annihilation operator is its adjoint, which can be computed as
\[ c_i^\ast(\Omega) = 0 \] (2.5)
\[ c_i^\ast(e_{i(1)} \otimes \cdots \otimes e_{i(k)}) = \sum_{\ell=1}^{k} \delta_{ii(\ell)} q_{ii(1)} \cdots q_{ii(\ell-1)} \cdot e_{i(1)} \otimes \cdots \otimes e_{i(\ell-1)} \otimes e_{i(\ell+1)} \otimes \cdots \otimes e_{i(k)}. \] (2.6)

Finally, we define the mixed $q$-Gaussian variables $X_i$ to be $c_i + c_i^\ast$. We can compute explicitly the mixed moment of those variables with respect to the vector state $\tau[.] = \langle \Omega, \Omega \rangle_Q$. 

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Proposition 2.9 (Proof of Theorem 4.4 of [8]). We have

\[ \tau[X_{i(1)} \cdots X_{i(n)}] = \sum_{\pi \in P_2(n)} \prod_{\{a,b\} \in \text{cr} (\pi)} q_{i(a)i(b)} \prod_{\{a,b\} \in \pi} \delta_{i(a)i(b)}. \]  

(2.7)

As a consequence, the distribution of the variable \( X_i \) is the \( q_{ii} \)-Gaussian measure. Let us remark that if all the \( q_{ij} \) are equal to a single \( q \), the set \( \{X_i\}_{i \in I} \) is jointly \( q \)-Gaussian. Finally, let us mention that it is also possible to define some Wick product for mixed \( q \)-Gaussian variables: see [15, 17].

2.5 Random Matrix Model of Śniady

Let \( d \geq 0 \). We endow \( \mathbb{M}_d \) by the inner products \( \langle A, B \rangle_1 = \frac{1}{d} \text{Tr}(AB^*) \) and \( \langle A, B \rangle_0 = \text{Tr}(A)\text{Tr}(B^*) \). For all \( S \subset \{1, \ldots, N\} \), we define the inner product \( \langle \cdot, \cdot \rangle_S \) on \( \mathbb{M}_d^N \approx \bigotimes_{r=1}^N \mathbb{M}_d \) to be the inner product of the Hilbert space tensor product \( \bigotimes_{r=1}^N (\mathbb{M}_d, \langle A, B \rangle_{1S(r)}) \). Let \( \sigma = (\sigma_S)_{S \subset \{1, \ldots, N\}} \) be a family of real numbers indexed by all subsets of \( \{1, \ldots, N\} \). We define the inner product on \( \mathbb{M}_d^N \) given by

\[ \langle A, B \rangle_\sigma = \sum_{S \subset \{1, \ldots, n\}} \sigma_S^2 \cdot \langle A, B \rangle_S. \]

In order to be concrete, let us compute the inner product of elementary matrices. Setting

\[ T_{ij,kl}^S = \langle E_{j,i}, E_{k,l} \rangle_{1S(r)} = \begin{cases} \delta_{i,l} \delta_{j,k} & \text{if } r \in S \vspace{1mm} \\ \delta_{i,j} \delta_{k,l} & \text{if } r \notin S \end{cases}, \]

and, for all \( i = (i_1, \ldots, i_N) \), \( j, k, l \in \{1, \ldots, d\}^N \),

\[ T_{ij,kl}^S = \langle E_{j,i}, E_{k,l} \rangle_S = \prod_{r=1}^N T_{ij,kl}^{S_{rj,kl}}, \]

we have, for all \( i = (i_1, \ldots, i_N) \), \( j, k, l \in \{1, \ldots, d\}^N \),

\[ \langle E_{j,i}, E_{k,l} \rangle_\sigma = \sum_{S \subset \{1, \ldots, N\}} \sigma_S^2 T_{ij,kl}^S. \]  

(2.8)

Theorem 2.10 (Theorem 1 of [23]). Let \( \mathcal{X} = \{X_t\}_{t \in T} \) be a set of self-adjoint variables which are jointly \( q \)-Gaussian.

For each \( N \geq 0 \), let \( \sigma^{(N)} = (\sigma_S^{(N)})_{S \subset \{1, \ldots, N\}} \) be a family of real numbers, and let \( \mathcal{X}^{(N)} = \{X^{(N)}_t\}_{t \in T} \) be the Gaussian stochastic process on \( \mathbb{H}_d^N \) (indexed by \( T \)), uniquely defined by the following covariance: for all \( M, N \in \mathbb{H}_d^N \) and all \( s, t \in T \), one has

\[ \mathbb{E} \left[ \text{Tr}(MX_t^{(N)})\text{Tr}(NX_s^{(N)}) \right] = \tau[X_tX_s]\langle M, N \rangle_{\sigma^{(N)}}. \]

In other words, the entries of the matrices in \( \mathcal{X}^{(N)} \) are centered Gaussian variables with the following covariance: for all \( i, j, k, l \in \{1, \ldots, d\}^N \) and all \( s, t \in T \), one has

\[ \mathbb{E} \left[ \text{Tr}(E_{j,i}X_t^{(N)})\text{Tr}(E_{k,l}X_s^{(N)}) \right] = \mathbb{E} \left[ \text{Tr}(E_{j,i}X_t^{(N)})\text{Tr}(E_{k,l}X_s^{(N)}) \right] = \tau[X_tX_s] \sum_{S \subset \{1, \ldots, N\}} (\sigma_S^{(N)})^2 T_{ij,kl}^S. \]

Under the technical assumptions \( [H.1] [H.2] [H.3] \) and \( [H.4] \), \( \mathcal{X}^{(N)} \) converges to \( \mathcal{X} \) in noncommutative distribution in the following sense: for all \( t_1, \ldots, t_n \), we have

\[ \lim_{N \to \infty} \mathbb{E} \left[ \frac{1}{d^N} \text{Tr}(X_{t_1}^{(N)} \cdots X_{t_n}^{(N)}) \right] = \tau[X_{t_1} \cdots X_{t_n}]. \]
Before presenting the technical assumptions \[ H.1 \ H.2 \ H.3 \] and \[ H.4 \] let us present two simple examples of family of real numbers \( \sigma^{(N)} = (\sigma_S^{(N)})_{S \subset \{1, \ldots, N\}} \) (for \( N \geq 0 \)) fulfilling all assumptions. Those examples are taken from [23, Proposition 1 and 2]: if \( q \) can be written as \( q = \exp(c^2/d^2 - c^2) \) for a real number \( c > 0 \), the sequence of functions defined by

\[
(\sigma_S^{(N)})^2 = \left( \frac{c}{\sqrt{N}} \right)^{|S|} \left( 1 - \frac{c}{\sqrt{N}} \right)^{N-|S|}
\]

fulfils the assumptions of Theorem [2.10] if \( q \) can be written as \( q = \exp(c^2/d^2 - c^2) \) for a real number \( c > 0 \), the sequence of functions defined for \( N \) sufficiently large by

\[
(\sigma_S^{(N)})^2 = \begin{cases} 
\frac{1}{|c\sqrt{N}|} & \text{if } |S| = |c\sqrt{N}| \\
0 & \text{otherwise}
\end{cases}
\]

fulfils the assumptions of Theorem [2.10]

**Definition 2.11.** For each \( N \geq 0 \), let \( \sigma^{(N)} = (\sigma_S^{(N)})_{S \subset \{1, \ldots, N\}} \) be a family of real numbers. The assumptions \[ H.1 \ H.2 \ H.3 \] and \[ H.4 \] are given as follow:

\[ H.1 \] for each \( N \in \mathbb{N} \),

\[
\sum_{S \subset \{1, \ldots, N\}} (\sigma_S^{(N)})^2 = 1,
\]

\[ H.2 \] we have

\[
\lim_{N \to \infty} \sum_{S_1, S_2, S_3 \subset \{1, \ldots, N\}} (\sigma_{S_1}^{(N)})^2 (\sigma_{S_2}^{(N)})^2 (\sigma_{S_3}^{(N)})^2 = 0,
\]

\[ H.3 \] there exists a sequence \( (p_i)_{i \geq 0} \) of nonnegative real numbers such that \( \sum_{i \geq 0} p_i = 1 \), \( \sum_{i \geq 0} p_i/d^2i = q \), and such that, for any \( k \in \mathbb{N} \) and any nonnegative integers numbers \( (n_{ij})_{1 \leq i < j \leq k} \), we have

\[
\lim_{N \to \infty} \sum_{S_1, \ldots, S_k \subset \{1, \ldots, N\}} (\sigma_{S_1}^{(N)})^2 (\sigma_{S_1}^{(N)})^2 = \prod_{1 \leq i < j \leq k} p_{n_{ij}},
\]

\[ H.4 \] for each \( k \in \mathbb{N} \),

\[
\lim_{N \to \infty} \sum_{S_1, \ldots, S_n \subset \{1, \ldots, N\}} \frac{(\sigma_{S_1}^{(N)})^2 \cdots (\sigma_{S_k}^{(N)})^2}{N^{|A_1 \setminus (A_2 \cup \cdots \cup A_n)|}} = 0.
\]

### 3 The Two-Parameter \( q \)-Deformed Segal-Bargmann Transform

In this section, we define the \( q \)-deformed Segal-Bargmann transform \( \mathcal{J}^s_t \) with parameters \( s > t/2 > 0 \) and prove Theorem [3.14] which reduces to Theorem [1.1] when \( s = t = 1 \).

#### 3.1 An Integral Representation

The integral representation for the one-parameter and the two-parameter cases are similar. The two-parameter case is in fact a generalization of the one-parameter; we separate here simply to make the presentation of the computations clearer.
One-Parameter Case

**Definition 3.1.** Let $-1 < q < 1$ and $t \geq 0$. We define the $q$-deformed Segal-Bargmann transform $\mathcal{S}_q^t$ by

$$\mathcal{S}_q^t f(z) = \int f(x) \Gamma_q^t(x, z) \nu_q^t(\, dx).$$

Observe that $\mathcal{S}_q^t H^t_n(z) = z^n$ and $\mathcal{S}_q^t$ is injective (by looking at the Fourier expansion of $L^2(\nu_q^t)$).

**Remark 3.2.** When $t = 1$, the transform $\mathcal{S}_q^1$ coincides with the the transform $W$ from [28]. The method is different; while van Leeuwen and Maassen discovered the integral kernel by solving an eigenvalue equation [28, Equation (8)], we make use of the generating function directly to match the result from the Fock space. The method we present here will give us a two-parameter generalization in later sections.

**Theorem 3.3.** The transform $\mathcal{S}_q^t$ is a unitary isomorphism between $L^2(\nu_q^t)$ and the reproducing kernel Hilbert space $\mathcal{H}_q^t$ of analytic functions on the disk $B \left(0, \sqrt{\frac{1}{1-q}}\right)$ generated by the positive-definite sesqui-analytic kernel

$$K_q^t(z, \zeta) = \int \Gamma_q^t(x, z) \Gamma_q^t(x, \zeta) \nu_q^t(\, dx) = \sum_{k=0}^{\infty} \frac{1}{\Gamma_q^t(k+1)} \left(\frac{z\zeta}{t}\right)^k.$$

**Proof.** Let us denote $\Gamma_q^t(x, z)$ by $\Gamma_q(x)$ for each $z \in B \left(0, \sqrt{\frac{1}{1-q}}\right)$ and $x \in \mathbb{R}$. We also write $K_q^t(z) = K_q^t(\zeta)$ as an analytic function on $B \left(0, \sqrt{\frac{t}{1-q}}\right)$. Observe that

$$\mathcal{S}_q^t f(\zeta) = \langle f, \bar{\Gamma}_q \rangle_{L^2(\nu_q^t)}.$$

Define $\mathcal{H}_q^t \equiv \mathcal{S}_q^t(L^2(\nu_q^t))$ equipped with the inner product

$$\langle F, G \rangle_{\mathcal{H}_q^t} := \langle (\mathcal{S}_q^t)^{-1} F, (\mathcal{S}_q^t)^{-1} G \rangle_{L^2(\nu_q^t)}$$

which is well-defined since $\mathcal{S}_q^t$ is injective on $L^2(\nu_q^t)$. By construction $\mathcal{S}_q^t$ is a unitary isomorphism between $L^2(\nu_q^t)$ and $\mathcal{H}_q^t$. Finally, we see that $K_q^t(z) = \mathcal{S}_q^t \bar{\Gamma}_q(z)$ and, for any $F \in \mathcal{H}_q^t$,

$$\langle F, K_q^t \rangle_{\mathcal{H}_q^t} = \langle (\mathcal{S}_q^t)^{-1} F, (\mathcal{S}_q^t)^{-1} K_q^t \rangle_{L^2(\nu_q^t)} = \langle (\mathcal{S}_q^t)^{-1} F, \bar{K}_q \rangle_{L^2(\nu_q^t)} = \mathcal{S}_q^t((\mathcal{S}_q^t)^{-1} F)(\zeta) = F(\zeta),$$

which shows that $K_q^t$ is a reproducing kernel for $\mathcal{H}_q^t$. \qed

**Remark 3.4.** Since $\mathcal{S}_q^t$ coincides with $W$ from [28], the reproducing kernel Hilbert space $\mathcal{H}_q^t$ actually is equal to the space $H^2(D_q, \mu_q)$ considered in [28].

**Analytic Continuation of a Generating Function** In this subsection, we study the analytic continuation on $y$ to the following generating function

$$A(r, x, y) = \sum_{k=0}^{\infty} h^q_k(x) h^q_k(y) \frac{r^n}{(q)_n} = \prod_{k=0}^{\infty} (1 - 4rq^kxy + 2r^2q^{2k}(-1 + 2x + 2y) - 4r^3q^{3k}xy + r^4q^{4k}),$$

where $x, y \in [-1, 1]$, $0 < |r| < 1$ which is either real or purely imaginary, and $h^q_k(x) = H^q_k \left(\frac{x}{2}\sqrt{1-q}\right)$.

This formula is known as the $q$-Mehler formula and has been studied analytically and combinatorially; see e.g. [6, Theorem 1.10] or [19, Equation (24)]. By a standard theorem (see [20, Theorem 15.4]), the analytic
continuation on the parameter \( y \) of \( \Lambda \) is to solve, for a single \( 0 < |r| < 1 \) and all \( x \in [-1,1] \), what \( y \) make
\[
1 - 4rq^kxy + 2r^2q^{2k}(-1 + 2x + 2y) - 4r^3q^{3k}xy + r^4q^{4k} = 0.
\]
The equation
\[
4r^2q^{2k}y^2 - 4tq^kx(1 + r^2q^{2k})y + r^4q^{4k} + 1 + 2r^2q^{2k}(2x^2 - 1) = 0
\]
has solution
\[
y = \frac{1}{2} \left( \left( \frac{1}{rq^k} + rq^k \right) x \pm i \left( \frac{1}{rq^k} - rq^k \right) \sqrt{1 - x^2} \right).
\]
It follows that precisely when
\[
y = \begin{cases}
\frac{1}{2} \left( \left( \frac{1}{|r||q|^k} + |r||q|^k \right) x \pm i \left( \frac{1}{|r||q|^k} - |r||q|^k \right) \sqrt{1 - x^2} \right) & \text{if } r \in \mathbb{R} \\
\frac{1}{2} \left( \pm \left( \frac{1}{|r||q|^k} + |r||q|^k \right) \sqrt{1 - x^2} \right) + \left( \frac{1}{|r||q|^k} - |r||q|^k \right) ix & \text{if } r \in i\mathbb{R}
\end{cases}
\]
(3.1)
for some \( x \in [-1,1] \), \( \Lambda(r, x, y) \) has a zero for the particular \( y \). Denote \( \Omega_{k,r} \) the bounded component, which contains 0, of the complement of the ellipse. Let
\[
\varphi_1(u) = \frac{1}{|r||q|^u} + |r||q|^u
\]
and
\[
\varphi_2(u) = \frac{1}{|r||q|^u} - |r||q|^u.
\]
The derivative \( \varphi'_1(u) = (-\log |q|) \left( \frac{1}{|r||q|^u} - |r||q|^u \right) > 0 \) implies \( \varphi_1 \) is increasing. Obviously \( \varphi_2 \) is increasing. Therefore \( \Omega_{k,r} \) is increasing. Whence the \( y \) parameter in \( \Lambda(r, x, y) \) can be analytically continued to the ellipse \( \Omega_{0,r} \).

**Proposition 3.5.** The generating function \( \Lambda(r, x, z) \) can be analytically continued to \( x \in [-1,1] \) and \( z \in \Omega_{0,r} \) which is an ellipse with major axis \([-\frac{1}{2}(1/|r| + |r|), \frac{1}{2}(1/|r| + |r|)]\) and minor axis \( i[-\frac{1}{2}(1/|r| - |r|), \frac{1}{2}(1/|r| - |r|)]\).

**Two-Parameter Case** We intend to define the integral \( q \)-Segal-Bargmann transform \( \mathcal{S}_q^{s,t} \) by
\[
\mathcal{S}_q^{s,t} f(z) = \int f(x) \Gamma_q^{s,t}(x, z) \nu_q^s \, (dx)
\]
where
\[
\Gamma_q^{s,t}(x, z) = \sum_{k=0}^{\infty} \frac{H_k^{q,s-t}(z) H_k^{q,s}(x)}{s^k [n]_q!}
\]
\[
= \sum_{k=0}^{\infty} (s - t)^{k/2} s^{k/2} \frac{H_k^{q,s-t}(z) H_k^{q,s}(x)}{s^k [n]_q!}
\]
\[
= \sum_{k=0}^{\infty} \left( 1 - \frac{t}{s} \right)^{k/2} \frac{H_k^{q,s-t}(z) H_k^{q,s}(x)}{[n]_q!}.
\]
By [6] Theorem 1.10], this series converges for \( |x|, |z| \leq 2\sqrt{1/\sqrt{1-q}} \) for real \( x, z \).

**Case** \( s > t \):
It is easy to see that
\[ \Gamma_{q}^{s,t}(x,z) = \Lambda \left( \sqrt{1 - \frac{t}{s}}, \frac{x\sqrt{1-q}}{2\sqrt{s}}, \frac{z\sqrt{1-q}}{2\sqrt{s-t}} \right). \]

By proposition 3.5, \( \Gamma_{q}^{s,t}(x,z) \) is defined as an analytic function on the ellipse \( E_{s,t} \) with major axis
\[ \frac{2\sqrt{s-t}}{\sqrt{1-q}} \left( \left(1 - \frac{t}{s}\right) - \left(1 - \frac{t}{s}\right) \right) = \frac{2(2s-t)}{\sqrt{s}\sqrt{1-q}} \]
and minor axis
\[ \frac{2\sqrt{s-t}}{\sqrt{1-q}} \left( \left(1 - \frac{t}{s}\right) - \left(1 - \frac{t}{s}\right) \right) = \frac{2t}{\sqrt{s}\sqrt{1-q}}. \]

Case \( s < t \):

Similarly,
\[ \Gamma_{q}^{s,t}(x,z) = \Lambda \left( i\sqrt{\frac{t}{s}-1}, \frac{x\sqrt{1-q}}{2\sqrt{s}}, \frac{z\sqrt{1-q}}{2\sqrt{s-t}} \right). \]

By proposition 3.5, \( \Gamma_{q}^{s,t}(x,z) \) is defined as an analytic function on the ellipse \( E_{s,t} \) with major axis on the purely imaginary axis of length
\[ \frac{2\sqrt{t-s}}{\sqrt{1-q}} \left( \left(\frac{t}{s} - 1\right) - \left(\frac{t}{s} - 1\right) \right) = \frac{2(2s-t)}{\sqrt{s}\sqrt{1-q}} \]
and minor axis on the real axis of length
\[ \frac{2\sqrt{t-s}}{\sqrt{1-q}} \left( \left(\frac{t}{s} - 1\right) - \left(\frac{t}{s} - 1\right) \right) = \frac{2t}{\sqrt{s}\sqrt{1-q}}. \]

Remark 3.6. When \( q = 0 \), the ellipse coincides with the ellipse where the Brown measure of an elliptic element is distributed; see [4].

3.2 The Integral Transform

Definition 3.7. Let \(-1 < q < 1\) and \(s > t/2 > 0\). We define the \(q\)-deformed Segal-Bargmann transform \( S_{q}^{s,t} \) by
\[ S_{q}^{s,t} f(z) = \int f(x) \Gamma_{q}^{s,t}(x,z) \nu_{q}^{s,t}(dx) \]
for all \( f \in L^{2}(\nu_{q}^{s}) \). \( S_{q}^{s,t} f \) is an analytic function on the ellipse \( E_{s,t} \).

Observe that \( S_{q}^{s,t} H_{n}^{s,t}(z) = H_{n}^{s,t}(z) \). The two-parameter analogue of Theorem 3.3 holds:

Theorem 3.8. The transform \( S_{q}^{s,t} \) is a unitary isomorphism between \( L^{2}(\nu_{q}^{s}) \) and the reproducing kernel Hilbert space \( H_{q}^{s,t} \) of analytic functions on the ellipse \( E_{s,t} \) generated by the positive-definite sesqui-analytic kernel
\[ K_{q}^{s,t}(z,\zeta) = \int \Gamma_{q}^{s,t}(x,z) \Gamma_{q}^{s,t}(x,\zeta) \nu_{q}^{s}(dx). \]
3.3 Segal-Bargmann Transform and Conditional Expectation

The goal of this section is to prove Corollary 3.13 showing that the $q$-deformed Segal-Bargmann transform can be written as the action of a "$q$-deformed heat kernel". This result is already known for $q = 0$, thanks to [9, Theorem 3.1].

Recall that the Wick product $X_1 \cdots X_n$ is orthogonal in $L^2(\mathcal{A}, \tau)$ to all products in $X_1, \ldots, X_n$ of degree strictly less than $n$. Since $X_1 \cdots X_n - X_1 \cdots X_n$ is in the span of the products in $X_1, \ldots, X_n$ of degree strictly less than $n$, $X_1 \cdots X_n - X_1 \cdots X_n$ can be seen as the orthogonal projection of $X_1 \cdots X_n$ onto the span of the products in $X_1, \ldots, X_n$ of degree strictly less than $n$. Because the Wick product can be seen as some orthogonal projection, the link with the conditional expectation is not surprising.

**Definition 3.9.** Let $\mathcal{X}$ be a subset of a non-commutative space $(\mathcal{A}, \tau)$. The conditional expectation

$$\tau[\cdot | \mathcal{X}] : L^2(\mathcal{A}, \tau) \to L^2(\mathcal{X}, \tau)$$

is the orthogonal projection of $L^2(\mathcal{A}, \tau)$ onto $L^2(\mathcal{X}, \tau)$.

**Remark 3.10.** If $(\mathcal{A}, \tau)$ is a $W^*$-probability space, that is to say a von Neumann algebra with an appropriate $\tau$, the conditional expectation $\tau[\cdot | \mathcal{X}]$ maps $\mathcal{X}$ into the von Neumann algebra $W^*(\mathcal{X})$ generated by $\mathcal{X}$.

**Proposition 3.11.** Let $\mathcal{X}$ and $\mathcal{Y}$ be two sets of jointly $q$-Gaussian variables which are $q$-independent. Let $X_1, \ldots, X_n \in \mathcal{X} \cup \mathcal{Y}$. We have

$$\tau \left[ X_1 \cdots X_n \big| \mathcal{X} \right] = 0$$

if one of the $X_i$s belongs to $\mathcal{Y}$, and $X_1 \cdots X_n$ if all $X_i$s are in $\mathcal{X}$.

**Proof.** If all $X_i$s are in $\mathcal{X}$, $X_1 \cdots X_n$ is in $L^2(\mathcal{X}, \tau)$ and the conditional expectation does not affect $X_1 \cdots X_n$. If one of the $X_i$s belongs to $\mathcal{Y}$, it is sufficient to verify that $X_1 \cdots X_n$ is orthogonal to $L^2(\mathcal{X}, \tau)$, and it is an immediate consequence of the following fact: for all $X_{n+1}, \ldots, X_{n+m} \in \mathcal{X}$,

$$\tau \left[ (X_1 \cdots X_n) \cdot (X_{n+1} \cdots X_{n+m})^* \right] = 0.$$

Indeed, using of Proposition 2.8 the computation of the trace always involves a factor $\tau[X_iX_i^*]$ between a $X_i \in \mathcal{Y}$ and a $X_j \in \mathcal{X}$, which vanishes. \hfill \Box

**Corollary 3.12.** Let $\mathcal{X} = \{X_i\}_{i \in I}$, $\mathcal{Y} = \{Y_j\}_{j \in J}$ and $\mathcal{Z} = \{Z_j\}_{j \in J}$ be three sets of jointly $q$-Gaussian variables which are $q$-independent. The conditional expectations $\tau[\cdot | \mathcal{X} \cup \mathcal{Z}] : L^2(\mathcal{A}, \tau) \to L^2(\mathcal{X} \cup \mathcal{Z}, \tau)$ and $\tau[\cdot | \mathcal{X}] : L^2(\mathcal{A}, \tau) \to L^2(\mathcal{X}, \tau)$ coincide on $L^2(\mathcal{X} \cup \mathcal{Y}, \tau)$.

**Proof.** Thanks to Proposition 3.11 the two conditional expectations coincide on the Wick products of elements in $\mathcal{X} \cup \mathcal{Y}$ which is a dense subset of $L^2(\mathcal{X} \cup \mathcal{Y}, \tau)$ (see Proposition 2.8). \hfill \Box

**Corollary 3.13.** Let $Z$ be a $(s,t)$-elliptic $q$-Gaussian variable in $(\mathcal{A}, \tau)$. If $Y$ is a $(t,0)$-elliptic $q$-Gaussian variable which is $q$-independent from $Z$, we have, for all polynomial $P$,

$$\mathcal{S}_q^{s,t} P(Z) = \tau[P(Y + Z) | Z].$$

**Proof.** It suffices to prove the theorem for the Hermite polynomials $\{H_n^{q,s}\}_{n \geq 0}$. Because $\mathcal{S}_q^{s,t} H_n^{q,s} = H_n^{q,s-t}$ for all $n \geq 0$, we need to prove that, for all $n \geq 0$,

$$\tau[H_n^{q,s}(Y + Z) | Z] = H_n^{q,s-t}(Z).$$
We compute 
\[ \tau[Z^2] = s - t \text{ and } \tau[(Y + Z)^2] = \tau[Z^2] + \tau[Y^2] = (s - t) + t = s, \]
and we deduce the following equalities by induction:
\[ H_n^{q,s-t}(Z) = Z^{on} \text{ and } H_n^{q,s}(Y + Z) = (Y + Z)^{on}. \]
Let us conclude by the following computation where we use Proposition 3.11:
\[ \tau[H_n^{q,s}(Y + Z)|Z] = \tau[(Y + Z)^{on}|Z] = Z^{on} = H_n^{q,s-t}(Z). \]
\[ \square \]

### 3.4 Random Matrix Model

Let \( \gamma_{d^N}^{\sigma,t} \) be the Gaussian measure on \( \mathbb{H}_{d^N} \) whose characteristic function is given by
\[
\int_{\mathbb{H}_{d^N}} \exp(i\text{Tr}(MX)) \, d\gamma_{d^N}^\sigma(X) = \exp(-t\|M\|_{\sigma}/2). 
\]
The measure \( \gamma_{d^N}^\sigma \) is supported on the following vector subspace
\[ K_{\sigma} = \{ X \in \mathbb{H}_{d^N} : \text{Tr}(MX) = 0 \text{ for all } M \in \mathbb{H}_{d^N} \text{ such that } \|M\|_{\sigma}^2 = 0 \}. \]
In particular, if \( \| \cdot \|_{\sigma}^2 \) is not faithful, \( \gamma_{d^N}^{\sigma,t} \) is not absolutely continuous with respect to the Lebesgue measure on \( \mathbb{H}_{d^N} \). However, \( \gamma_{d^N}^{\sigma,t} \) is absolutely continuous with respect to the Lebesgue measure on the vector space \( K_{\sigma} \). More precisely, using the Riesz representation theorem, let us define the linear map \( K : K_{\sigma} \rightarrow K_{\sigma} \) to be the unique linear map such that, for all \( x, y \in K_{\sigma}, \text{Tr}(xy) = \langle \phi(x), y \rangle_{\sigma} \). With respect to the Lebesgue measure on the vector space \( K_{\sigma} \), the measure \( \gamma_{d^N}^{\sigma,t} \) has density proportional to
\[ \exp \left( -\frac{1}{2t} \text{Tr}(x\phi(x)) \right) = \exp \left( -\frac{1}{2t} \|\phi(x)\|_{\sigma}^2 \right). \]
The quantity \( \|\phi(x)\|_{\sigma} \) is known as the Mahalanobis distance from \( x \) to \( 0 \), and it is the norm of \( K_{\sigma} \) for which \( \gamma_{d^N}^{\sigma,t} \) is the standard Gaussian measure.

We follows now Section 2.1 in order to define the Segal-Bargmann transform \( \mathcal{S}^{s,t} \) on \( L^2(K_{\sigma}, \gamma_{d^N}^{\sigma,s}) \). First, we consider the Gaussian measure \( \mu_{d^N}^{\sigma,r,s} \) on \( K_{\sigma} + iK_{\sigma} \) which is given by \( \gamma_{d^N}^{\sigma,t} \otimes \gamma_{d^N}^{\sigma,s} \) when identifying \( K_{\sigma} + iK_{\sigma} \) with \( K_{\sigma} \times K_{\sigma} \). A short computation shows that \( \mu_{d^N}^{\sigma,r,s} \) is the Gaussian measure on \( \mathbb{M}_{d^N} \) whose characteristic function is given by
\[
\int_{\mathbb{M}_{d^N}} \exp(i\text{Tr}(MX^*)) \, d\mu_{d^N}^{\sigma}(X) = \exp(-r\|\Re M\|_{\sigma}^2/2 - s\|\Im M\|_{\sigma}^2/2). 
\]
The Segal-Bargmann transform
\[ \mathcal{S}^{s,t} : L^2(\mathbb{H}_{d^N}, \gamma_{d^N}^{\sigma,s}) = L^2(K_{\sigma}, \gamma_{d^N}^{\sigma,s}) \rightarrow \mathcal{H}L^2(K_{\sigma} + iK_{\sigma}, \mu_{d^N}^{\sigma,s-t/2,t/2}) = \mathcal{H}L^2(\mathbb{M}_{d^N}, \mu_{d^N}^{\sigma,s-t/2,t/2}) \]
is well-defined as in (2.2).

Following the model of Biane, we consider the two following Hilbert space tensor products
\[ L^2(\mathbb{H}_{d^N}, \gamma_{d^N}^{\sigma,s}; \mathbb{M}_{d^N}) = L^2(\mathbb{H}_{d^N}, \gamma_{d^N}^{\sigma,s}) \otimes \mathbb{M}_{d^N} \]
and
\[ \mathcal{H} L^2(\mathbb{M}_{dN} ; \mu_{dN}^{\sigma,s-t/2,t/2} ; \mathbb{M}_{dN}) = \mathcal{H} L^2(\mathbb{M}_{dN} ; \mu_{dN}^{\sigma,s-t/2,t/2}) \otimes \mathbb{M}_{dN}, \]
where \( \mathbb{M}_{dN} \) is endowed with the norm \( \| M \|_{\mathbb{M}_{dN}}^2 = \text{Tr}(MM^*)/d^N \). Finally, we consider the boosted Segal-Bargmann transform
\[ \mathcal{S}_d^{s,t} = \mathcal{S}_d^{s,t} \otimes \text{Id}_{\mathbb{M}_{dN}} : L^2(\mathbb{H}_{dN} ; \gamma_{dN}^\sigma ; \mathbb{M}_{dN}) \to \mathcal{H} L^2(\mathbb{M}_{dN} ; N_{\mu_{dN}^{\sigma,s-t/2,t/2}}^{\sigma,s-t/2,t/2} ; \mathbb{M}_{dN}). \]

**Theorem 3.14.** Let \( 0 \leq q < 1 \). Assuming (H.1), (H.2), (H.3) and (H.4) on \( \sigma \) ensures that the Segal-Bargmann transform \( \mathcal{S}_d^{q,s,t} \) converges to the \( q \)-deformed Segal-Bargmann transform \( \mathcal{S}_q^{s,t} \) in the following sense: for all polynomial \( P \), we have
\[ \lim_{N \to \infty} \left\| \mathcal{S}_d^{s,t}(P) - \mathcal{S}_q^{s,t} P \right\|_{\mathcal{H} L^2(\mathbb{M}_{dN} ; N_{\mu_{dN}^{\sigma,s-t/2,t/2}}^{\sigma,s-t/2,t/2} ; \mathbb{M}_{dN})} = 0. \]

**Proof.** Let us denote by \( Q \) the polynomial \( \mathcal{S}_q^{s,t} P \).

For all \( z \in \mathbb{H}_{dN} \), we have
\[ (\mathcal{S}_d^{s,t}(P))(z) = \int_{\mathbb{H}_{dN}} P(z - x) \, d\gamma_{dN}^{\sigma,s}(x). \]

Because both side are analytic in \( z \), the equality is valid for all \( z \in \mathbb{M}_{dN} \). Thus we can compute
\[ \left\| \mathcal{S}_d^{s,t}(P) - Q \right\|_{\mathcal{H} L^2(\mathbb{M}_{dN} ; N_{\mu_{dN}^{\sigma,s-t/2,t/2}}^{\sigma,s-t/2,t/2} ; \mathbb{M}_{dN})} = \int_{\mathbb{M}_{dN}} \int_{\mathbb{H}_{dN}} \int_{\mathbb{H}_{dN}} \left( P(z - x) - Q(z) \right) \left( P(z - y) - Q(z) \right)^* \, d\gamma_{dN}^{\sigma,s}(x) \, d\gamma_{dN}^{\sigma,s}(y) \, d\mu_{dN}^{\sigma,s-t/2,t/2}(z). \]

Considering three independent random matrices \( X^{(N)}, Y^{(N)} \) and \( Z^{(N)} \) of respective distribution \( \gamma_{dN}^{\sigma,s}, \gamma_{dN}^{\sigma,t} \) and \( N_{\mu_{dN}^{\sigma,s-t/2,t/2}}^{\sigma,s-t/2,t/2} \), we can rewrite
\[ \left\| \mathcal{S}_d^{s,t}(P) - Q \right\|_{\mathcal{H} L^2(\mathbb{M}_{dN} ; N_{\mu_{dN}^{\sigma,s-t/2,t/2}}^{\sigma,s-t/2,t/2} ; \mathbb{M}_{dN})} = \mathbb{E} \left[ (P(Z^{(N)} + X^{(N)}) - Q(Z^{(N)}))(P(Z^{(N)} + Y^{(N)}) - Q(Z^{(N)}))^* \right]. \]

Let \( X, Y \) be two \((t,0)\)-elliptic \( q \)-Gaussian random variables and \( Z \) be a \((s-t/2, t/2)\)-elliptic \( q \)-Gaussian random variable such that \( X, Y \) and \( Z \) are \( q \)-independent. Remark that, for any random Hermitian matrix \( X^{(N)} \) distributed according to \( \gamma_{dN}^{\sigma,s,t} \), for all \( M, N \in \mathbb{H}_{dN} \), one has
\[ \mathbb{E} \left[ \text{Tr}(MX^{(N)}\text{Tr}(NX^{(N)})) \right] = t(M, N)_\sigma. \]

Moreover, for any random matrix \( Z \) distributed according to \( \mu_{dN}^{\sigma,s-t/2,t/2} \), \( \Re Z \) and \( \Im Z \) are two independent Hermitian random matrices distributed according to \( \gamma_{dN}^{\sigma,s-t/2} \) and \( \gamma_{dN}^{\sigma,t/2} \). Thus, we can apply Theorem 2.10 which says that the Hermitian random matrices \( X^{(N)}, Y^{(N)} \), \( \Re Z^{(N)} \) and \( \Im Z^{(N)} \) converge in noncommutative distribution to \( X, Y, \Re Z \) and \( \Im Z \). In particular, we have the following convergence:
\[ \lim_{N \to \infty} \mathbb{E} \left[ (P(Z^{(N)} + X^{(N)}) - Q(Z^{(N)}))(P(Z^{(N)} + Y^{(N)}) - Q(Z^{(N)}))^* \right] = \tau \left[ (P(Z + X) - Q(Z))(P(Z + Y) - Q(Z))^* \right]. \]
Thus the limit \( \tau [ (P(Z + X) - Q(Z))(P(Z + Y) - Q(Z))^*] \) of \( \left\| \mathcal{S}^{s,t}_q(P) - Q \right\|_{\mathcal{H}L^2(M_{dN}, \mu_{dN}^{s-t/4,t/2}; \mathbb{M}_{dN})} \) vanishes:

\[
\tau [ (P(Z + X) - Q(Z))(P(Z + Y) - Q(Z))^*] = \tau [ (P(Z + X) - \tau[P(Z + X)|Z,Y])(P(Z + Y) - Q(Z))^*] = \tau [ (P(Z + X) - P(Z + X))(P(Z + Y) - Q(Z))^*] = 0.
\]

\[\square\]

4 Multidimensional \( q \)-Segal-Bargmann Transform

In this section, we will extend the definition of the \( q \)-Segal-Bargmann transform \( \mathcal{S}^{s,t}_q \) to a multidimensional setting, and prove Theorem 4.8 which says that Theorem 3.14 is also true in this new setting. In order to understand the multidimensional case for \( -1 \leq q \leq 1 \), we decide first to explain the infinite-dimensional case for the classical Segal-Bargmann transform.

4.1 Classical Segal-Bargmann Transform in the Infinite-Dimensional Case

The content of this section is entirely expository. In Section 4.1 we shall define a version of the Segal-Bargmann transform in a probabilistic framework which allows to consider infinite-dimensional Hilbert spaces. In Section 4.2 and 4.1 we give two alternative descriptions of the Segal-Bargmann transform which are adapted to consider \( q \)-deformations.

In a probabilistic framework In order to consider the \( q \)-deformation of this Segal-Bargmann transform, it is convenient to have a version of the \( L^2 \)-spaces with more probabilistic flavor. Let \( h \in H \). The continuous linear functional \( (\cdot, h) \in H^* \) can be considered as a random variable defined on the probability space \((H, \mathcal{B}, \gamma_s)\) (where \( \mathcal{B} \) is the Borel \( \sigma \)-field of \( H \)). Let us denote by \( X(h) \) the linear functional \( x \mapsto \langle x, h \rangle \) defined on \( H \) and by \( Z(h) \) the linear functional \( z \mapsto \langle z, h \rangle \) defined on \( H^C \). Because \( H \) is finite-dimensional, the \( \sigma \)-field generated by the random variables \( (X(h))_{h \in H} \) is the Borel \( \sigma \)-field \( \mathcal{B} \) of \( H \). Denoting by \( L^2(X) \) the random variables of \( L^2(H, \mathcal{B}, \gamma_s) \) which are measurable with respect to the \( \sigma \)-field generated by the random variables \( (X(h))_{h \in H} \), we have \( L^2(X) = L^2(H, \mathcal{B}, \gamma_s) \). Furthermore, it is well-known that the density in \( L^2(H, \mathcal{B}, \gamma_s) \) of polynomial variable follows from Hölder inequality. Finally, the three following Hilbert spaces are identical:

\[
L^2(X) = \mathbb{C}[X(h) : h \in H]^{L^2(H, \mathcal{B}, \gamma_s)} = L^2(H, \mathcal{B}, \gamma_s).
\]

In the same way, denoting by \( \mathcal{H}L^2(Z) \) the completion of the algebra of random variables \( \mathbb{C}[Z(h) : h \in H] \) in \( L^2(H^C, \mathcal{B}, \gamma_{s-t/2,t/2}) \) we have the equality between the three following Hilbert spaces (where the first equality is a definition):

\[
\mathcal{H}L^2(Z) = \mathbb{C}[Z(h) : h \in H]^{L^2(H^C, \mathcal{B}, \gamma_{s-t/2,t/2})} = \mathcal{H}L^2(H^C, \mathcal{B}, \gamma_{s-t/2,t/2}).
\]

The Segal-Bargmann map \( 2.2 \) can now be seen as an isomorphism between two spaces of random variables

\[
\mathcal{S}^{s,t}_q : L^2(X) \rightarrow \mathcal{H}L^2(Z).
\]
From the definition 2.1, the action of $\mathcal{S}_{s,t}$ on $\mathbb{C}[X(h) : h \in H]$ is easily described in the following way. The Hermite polynomials of parameter $s$ are defined by $H_{0}^{s}(x) = 1$, $H_{1}^{s}(x) = x$ and the recurrence relation $xH_{n}^{s}(x) = H_{n+1}^{s}(x) + nH_{n-1}^{s}(x)$. If $h_{1}, \ldots, h_{k}$ is an orthonormal family of $H$, the Hermite polynomials $H_{n_{1}}^{s}(X(h_{1})) \cdots H_{n_{k}}^{s}(X(h_{k}))$ form an orthonormal family of $L^{2}(X)$ and the action of $\mathcal{S}_{s,t}$ on this basis is

$$\mathcal{S}_{s,t} : H_{n_{1}}^{s}(X(h_{1})) \cdots H_{n_{k}}^{s}(X(h_{k})) \mapsto H_{n_{1}}^{s-t}(Z(h_{1})) \cdots H_{n_{k}}^{s-t}(Z(h_{k})).$$

(4.1)

The formula (4.1) determines $\mathcal{S}_{s,t}$ on $\mathbb{C}[X(h) : h \in H]$ by linearity, and thus (4.1) determines uniquely $\mathcal{S}_{s,t}$ on $L^{2}(X)$ by continuity.

**In the infinite dimensional case** The first approach of Section 2.1 can not extend directly to the infinite-dimensional setting because the Gaussian measures $\gamma_{s}$ do not make sense as measures on an infinite-dimensional Hilbert space. The dual point of view of Section 4.1 allows to define the Segal-Bargmann transform on infinite-dimensional Hilbert spaces. Indeed, $X$ and $Z$ of last section are particular cases of what we will called Gaussian fields. One has just to replace the underlying probability space $(H, B, \gamma_{s})$, which is not well-defined, by a sufficiently big one $(\Omega, F, \mathbb{P})$. In the following, the underlying probability space $(\Omega, F, \mathbb{P})$ will be completely arbitrary, but in concrete cases, the measure of reference $\mathbb{P}$ is often supported on a space $\Omega$ bigger than $H$. For example, in [11], the measure of reference $\mathbb{P}$ is a Wiener measure on a Wiener space whose Cameron-Martin space is $H$.

Let us fix an underlying probability space $(\Omega, F, \mathbb{P})$ and call random variables the measurable functions on $\Omega$. For all real Hilbert space, a linear map $X$ from $H$ to the space of real random variables is called a Gaussian field on $H$ if, for all $h \in H$, $X(h)$ is centered Gaussian with variance $\mathbb{E}[(X(h))^{2}] = \|h\|^{2}$. For all $r, s \geq 0$, a linear map $Z$ from $H$ to the space of complex random variables is called an $(r, s)$-elliptic Gaussian field if it has the same distribution as $\sqrt{r}Z_{1} + i\sqrt{s}Z_{2}$, where $Z_{1}$ and $Z_{2}$ are two Gaussian fields on $H$ which are independent (in particular, an $(r, 0)$-elliptic Gaussian field is real-valued and an $(0, s)$-elliptic Gaussian field is purely imaginary-valued). Let $r, s \geq 0$, and let $Z$ be an $(r, s)$-elliptic Gaussian field. Following the last section, we define $\mathcal{H}L^{2}(Z)$ to be the completion of the algebra of random variables $\mathbb{C}[Z(h) : h \in H]$ in $L^{2}(\Omega, F, \mathbb{P})$. When $s = 0$ or $r = 0$, $\mathcal{H}L^{2}(Z)$ coincide with the random variables of $L^{2}(\Omega, F, \mathbb{P})$ which are measurable with respect to the $\sigma$-field generated by the random variables $(Z(h))_{h \in H}$, and we will simply write $L^{2}(Z)$ instead of $\mathcal{H}L^{2}(Z)$.

Let $s > t/2 \geq 0$. In Section 4.1, $X$ was an $(s, 0)$-elliptic Gaussian field and $Z$ was an $(s - t/2, t/2)$-elliptic Gaussian field on a finite-dimensional Hilbert space $H$. Thanks to Section 4.1, we have the following proposition.

**Proposition 4.1.** Let $H$ be a (possibly infinite-dimensional) Hilbert space, $X$ be an $(s, 0)$-elliptic Gaussian field on $H$ and $Z$ be an $(s - t/2, t/2)$-elliptic Gaussian field on $H$. The map given, for all orthonormal family $h_{1}, \ldots, h_{k}$ of $H$, by

$$\mathcal{S}_{s,t} : H_{n_{1}}^{s}(X(h_{1})) \cdots H_{n_{k}}^{s}(X(h_{k})) \mapsto H_{n_{1}}^{s-t}(Z(h_{1})) \cdots H_{n_{k}}^{s-t}(Z(h_{k})),$$

(4.2)

is a well-defined isometry from $\mathbb{C}[X(h) : h \in H]$ to $\mathbb{C}[Z(h) : h \in H]$ which extends uniquely to an isomorphism of Hilbert space $\mathcal{S}_{s,t} : L^{2}(X) \rightarrow \mathcal{H}L^{2}(Z)$, called in the following the (two-parameter) Segal-Bargmann transform.

**Segal-Bargmann transform and Wick products** In order to define $q$-deformation of the Segal-Bargmann transform, we give here a second description of the Gaussian fields and of the Segal-Bargmann transform defined in Section 4.1.

Let $X$ be a Gaussian field on $H$. The Wick product is the result of the Gram-Schmidt process for the basis of $L^{2}(X)$ given by monomials. More precisely, for all $n \geq 0$ and $h_{1}, \ldots, h_{n} \in H$, we define the Wick product $X(h_{1}) \circ \cdots \circ X(h_{n})$ of $X(h_{1}), \ldots, X(h_{n})$ as the unique element of

$$X(h_{1}) \cdots X(h_{n}) + \text{Span}\{X(k_{1}) \cdots X(k_{m}) : m < n, k_{1}, \ldots, k_{m} \in H\}$$
which is orthogonal to $\text{Span}\{X(k_1) \cdots X(k_m) : m \leq n, k_1, \ldots, k_m \in H\}$, or equivalently, such that
\[
\mathbb{E}[(X(h_1) \diamond \cdots \diamond X(h_n)) \cdot X(k_1) \cdots X(k_m)] = 0
\]
for all $m < n, k_1, \ldots, k_m \in H$. In certain cases, the Wick product can be computed explicitly. For all $n \geq 0, m_1, \ldots, m_n \geq 1$ and $h_1, \ldots, h_n$ an orthonormal family of $H$, we have
\[
X(h_1)^{m_1} \diamond \cdots \diamond X(h_n)^{m_n} = H_{m_1}^1(X(h_1)) \cdots H_{m_n}^1(X(h_n)).
\]

Let $Z$ be a Gaussian $(s, t)$-elliptic system on $H$. In the same way, for all $n \geq 0$ and $h_1, \ldots, h_n \in H$, we define the Wick product $Z(h_1) \diamond \cdots \diamond Z(h_n)$ of $Z(h_1), \ldots, Z(h_n)$ as the unique element of
\[
Z(h_1) \cdots X(h_n) + \text{Span}\{Z(k_1) \cdots Z(k_m) : m < n, k_1, \ldots, k_m \in H\}
\]
which is orthogonal to $\text{Span}\{Z(k_1) \cdots Z(k_m) : m \leq n, k_1, \ldots, k_m \in H\}$, or equivalently, such that
\[
\mathbb{E}[(Z(h_1) \diamond \cdots \diamond Z(h_n)) \cdot Z(k_1) \cdots Z(k_m)] = 0
\]
for all $m < n, k_1, \ldots, k_m \in H$. By multilinearity and the discussion above, for all $n \geq 0, m_1, \ldots, m_n \geq 1$ and $h_1, \ldots, h_n$ an orthonormal family of $H$, we have
\[
Z(h_1)^{m_1} \diamond \cdots \diamond Z(h_n)^{m_n} = H_{m_1}^{s-t}(Z(h_1)) \cdots H_{m_n}^{s-t}(Z(h_n)).
\]

We are now able to give an alternative description of the Segal-Bargmann transform. Let $X$ be a $(s, 0)$-elliptic Gaussian system, and $Z$ be a Gaussian $(s, t)$-elliptic system on $H$. From (4.2), we deduce that, for all orthonormal family $h_1, \ldots, h_k$ of $H$, we have
\[
\mathcal{S}^{s,t}(X(h_1)^{m_1} \cdots X(h_n)^{m_n}) = Z(h_1)^{m_1} \cdots Z(h_n)^{m_n}
\]
which can be generalized by multilinearity to the following.

**Proposition 4.2.** Let $X$ be a $(s, 0)$-elliptic Gaussian system, and $Z$ be a Gaussian $(s, t)$-elliptic system on $H$. For all $n \geq 0$ and $h_1, \ldots, h_n \in H$, we have
\[
\mathcal{S}^{s,t}(X(h_1) \diamond \cdots \diamond X(h_n)) = Z(h_1) \diamond \cdots \diamond Z(h_n). \tag{4.3}
\]

**Segal-Bargmann transform and conditional expectations** In the proof of Theorem 4.8, we will need a third description of the Segal-Bargmann transform, which follows directly from the definition. Let $X$ be a $(s, 0)$-elliptic Gaussian system, and $Z$ be a Gaussian $(s, t)$-elliptic system on $H$. If $Y$ is a $(t, 0)$-elliptic Gaussian system which is independent from $Z$, we have, for all $P \in \mathbb{C}[x_h : h \in H]$,
\[
\mathcal{S}^{s,t}(P(X(h) : h \in H)) = \mathbb{E} [P(Z(h) + Y(h) : h \in H)|Z(h) : h \in H]. \tag{4.4}
\]

Because the formula only involves finitely many variables $h$ for each $P \in \mathbb{C}[x_h : h \in H]$, it is enough to prove the formula for finite-dimensional Hilbert spaces $H$. For convenience, we take the particular case of Section 4.1. $X(h)$ is the linear functional $x \mapsto \langle x, h \rangle$ defined on $(H, \mathcal{B}, \gamma_s)$ and $Z(h)$ the linear functional $z \mapsto \langle z, h \rangle$ defined on $(H^C, \mathcal{B}, \gamma_{s-t/2,t/2})$. Let $P \in \mathbb{C}[x_h : h \in H]$. For all $z \in H$,
\[
\mathcal{S}^{s,t}(P(X(h) : h \in H))(z) = \int_{H} P(X(h) : h \in H)(z - x) \, d\gamma_t(x)
\]
\[
= \int_{H} P((z - x, h) : h \in H) \, d\gamma_t(x)
\]
\[
\mathcal{S}^{s,t}(P(X(h) : h \in H))(z) = \int_{H} P((Z(h))(z) - (X(h))(x) : h \in H) \, d\gamma_t(x).
\]

The last line is also valid for all $z \in H^C$, since each side is analytic. We recognize the conditioning of two independent set of variables: by enlarging the underlying probability space, we assume that there exists a $(t, 0)$-elliptic Gaussian system $Y$ independent from $Z$ and rewrite the last equality as follows.
Proposition 4.3. Let $X$ be a $(s, 0)$-elliptic Gaussian system, and $Z$ be a Gaussian $(s, t)$-elliptic system on $H$. Let us assume that there exists a $(t, 0)$-elliptic Gaussian system $Y$ independent from $Z$. For all $P \in \mathbb{C}[x_h : h \in H]$, we have

$$\mathcal{S}^{s,t}(P(X(h) : h \in H)) = \mathbb{E}\left[P(Z(h) + Y(h) : h \in H)\big| Z(h) : h \in H\right].$$

(4.5)

4.2 The $q$-Deformation of the Segal-Bargmann Transform

Definition 4.4. Let $-1 \leq q \leq 1$. A $q$-Gaussian field $X_q$ on $H$ is a linear map from $H$ to a non-commutative probability space $(\mathcal{A}, \tau)$ which is an isometry for the $L^2$-norm and such that $(X_q(h))_{h \in H}$ is jointly $q$-Gaussian.

A $(r, s)$-elliptic $q$-Gaussian field $Z_q$ is a linear map from $H$ to a non-commutative probability space $(\mathcal{A}, \tau)$ which can be decomposed as $\sqrt{r}X_q + i\sqrt{s}Y_q$, where $X_q$ and $Y_q$ are two $q$-Gaussian field which are $q$-independent. Elliptic $q$-Gaussian fields are $q$-independent if the previous decomposition holds simultaneously with $q$-Gaussian fields which are all $q$-independent.

The following definition of the Segal-Bargmann transform in the infinite-dimensional case coincide with the classical Segal-Bargmann transform if $q = 1$, with the definition of Kemp in [16] if $s = t$, and with the definition of the second author in [14] if $q = 0$.

Proposition / Definition 4.5. Let $X_q$ be a $q$-Gaussian $(s, 0)$-elliptic system, and $Z_q$ be a $q$-Gaussian $(s - t/2, t/2)$-elliptic system from $H$ to $\mathcal{A}$. The (q-deformed) Segal-Bargmann transform $\mathcal{S}_q^{s,t}$ is the unique unitary isomorphism from $L^2(X_q, \tau)$ to $\mathcal{H}L^2(Z_q, \tau)$ such that, for all $h_1, \ldots, h_n \in H$,

$$\mathcal{S}_q^{s,t}(X_q(h_1) \cdots \diamond X_q(h_n)) = Z_q(h_1) \cdots \diamond Z_q(h_n).$$

(4.6)

We will see in Corollary 4.7 that this transform is indeed a generalization of Definition 3.7.

Proof. The unicity is clear. It remains to prove the existence and the unitarity. Let us first remark that, for all $h, k \in H$, we have

$$\tau[Z_q(h)Z_q(k)^*] = \tau[\Re Z_q(h)^*] + 0 + \tau[\Im Z_q(h)\Im Z_q(h)] = (s - t/2)\langle h, k \rangle_H + (t/2)\langle h, k \rangle_H = s\langle h, k \rangle_H = \tau[X_q(h)X_q(k)^*].$$

Combined with Proposition 2.8, it follows that, for all $h_1, \ldots, h_n \in H$ and $k_1, \ldots, k_m \in H$,

$$\langle X_q(h_1) \cdots \diamond X_q(h_n), X_q(k_1) \cdots \diamond X_q(k_m) \rangle_{L^2(X_q, \tau)} = \langle Z_q(h_1) \cdots \diamond Z_q(h_n), Z_q(k_1) \cdots \diamond Z_q(k_m) \rangle_{\mathcal{H}L^2(Z_q, \tau)}.$$

We deduce the existence of the unitary linear map $\mathcal{S}_q^{s,t}$ from $C(X_q(h) : h \in H)$ to $C(Z_q(h) : h \in H)$ given by (4.6), and we extend this map to $\mathcal{S}_q^{s,t} : L^2(X_q, \tau) \to \mathcal{H}L^2(Z_q, \tau)$ by density. \hfill \Box

Here again, the $q$-deformed Segal-Bargmann transform can be seen as the action of a ”$q$-deformed heat kernel”, a result which extends [9] Theorem 3.1 to $-1 \leq q \leq 1$.

Theorem 4.6. Let $X_q$ be a $q$-Gaussian $(s, 0)$-elliptic system, and $Z_q$ be a $q$-Gaussian $(s, t)$-elliptic system from $H$ to $\mathcal{A}$. If $Y_q$ is a $q$-Gaussian $(t, 0)$-elliptic system which is $q$-independent from $Z_q$, we have, for all noncommutative polynomial $P \in \mathbb{C}[x_h : h \in H]$,

$$\mathcal{S}_q^{s,t}(P(X_q(h) : h \in H)) = \tau[P(Y_q(h) + Z_q(h) : h \in H)|Z_q].$$
Proof. For all \( h_1, \ldots, h_n \), we define a polynomial \( P_{h_1, \ldots, h_n} \in \mathbb{C}(x_h : h \in H) \) by the following recursion formula: \( P_{\emptyset} = 1 \) and
\[
P_{h_1, \ldots, h_n} = x_{h_1} \cdot P_{h_2, \ldots, h_n} - \sum_{i=2}^{n} q^{i-1} s(h_1, h_i)_H \cdot P_{h_1, \ldots, \hat{h}_i, \ldots, h_n}
\]
where the hat means that we omit the corresponding element in the product. Since \( \{P_{h_1, \ldots, h_n}\}_{n \geq 0, h_1, \ldots, h_n \in H} \) is a spanning set of \( \mathbb{C}(x_h : h \in H) \), it suffices to prove the theorem for those polynomials. Remark that, for all \( h, k \in H \), \( \tau[X_q(h)X_q(k)] = s(h, k)_H \). Consequently, the variables \( P_{h_1, \ldots, h_n}(X_q(h) : h \in H) \) and \( X_q(h_1) \circ \cdots \circ X_q(h_n) \) satisfies the same recursion formula, and we have
\[
P_{h_1, \ldots, h_n}(X_q(h) : h \in H) = X_q(h_1) \circ \cdots \circ X_q(h_n).
\]

Similarly, we compute
\[
\tau[(Y_q + Z_q)(h) \cdot (Y_q + Z_q)(k)] = \tau[Z_q(h)Z_q(k)] + \tau[Y_q(h)Y_q(h)] = (s - t) \langle h, k \rangle_H + t \langle h, k \rangle_H = s \langle h, k \rangle_H,
\]
and we deduce the following equality by induction:
\[
P_{h_1, \ldots, h_n}(Y_q(h) + Z_q(h) : h \in H) = (Y_q + Z_q)(h_1) \circ \cdots \circ (Y_q + Z_q)(h_n).
\]

Let us conclude by the following computation where we use Proposition \[3.11\] to compute the conditional expectation:
\[
\tau[P_{h_1, \ldots, h_n}(Y_q(h) + Z_q(h) : h \in H)|Z_q] = \tau[(Y_q + Z_q)(h_1) \circ \cdots \circ (Y_q + Z_q)(h_n)|Z_q]
\]
\[
= Z_q(h_1) \circ \cdots \circ Z_q(h_n)
\]
\[
= \mathcal{S}^{s,t}_q(X_q(h_1) \circ \cdots \circ X_q(h_n))
\]
\[
= P_{h_1, \ldots, h_n}(X_q(h) : h \in H).
\]

\[\square\]

Combining Theorem \[4.6\] with Corollary \[3.13\] and Definitions \[4.5\] and \[4.7\] of \( \mathcal{S}^{s,t}_q(P(X_q(h))) \) for one polynomial \( P \).

Corollary 4.7. Let \(-1 < q < 1\). For a unit vector \( h \) and a polynomial \( P \), we have
\[
\mathcal{S}^{s,t}_q(P(X_q(h))) = \mathcal{S}^{s,t}_q(P(Z_q(h))).
\]

4.3 Large N Limit

Let us construct a boosted version of the Gaussian \((s, 0)\)-elliptic system on a Hilbert space \( H \). Let us consider the tensor product Hilbert space \( H \otimes_{\mathbb{R}} \mathbb{H}_{dN} \) of \( H \) with \( \mathbb{H}_{dN}, ||\cdot||_\sigma \). Let \( X : H \otimes_{\mathbb{R}} \mathbb{H}_{dN} \rightarrow L^2(\mathbb{X}) \subset L^2(\Omega, \mathcal{F}, \mathbb{P}) \) be a Gaussian \((s, 0)\)-elliptic system. We define
\[
X^{(N)} : H \rightarrow L^2(\mathbb{X}) \otimes_{\mathbb{R}} \mathbb{H}_{dN} \simeq L^2(\mathbb{X}) \otimes_\mathbb{C} \mathbb{M}_{dN} \subset L^2(\Omega, \mathbb{P}; \mathbb{M}_{dN})
\]
by duality as the unique linear map \( X^{(N)} \) from \( H \) to the random variables with value in \( \mathbb{M}_{dN} \) such that \( X(h \otimes M) = \text{Tr}(MX^{(N)}(h)) \). Each variable \( \text{Tr}(MX^{(N)}(h)) \) is Gaussian with the covariance given by
\[
\mathbb{E} \left[ \text{Tr}(MX^{(N)}(h)) \text{Tr}(NX^{(N)}(k)) \right] = s \langle h, k \rangle_H \langle M, N \rangle_\sigma.
\]

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In other words, if the norm of \( h \) is 1, the distribution of the random matrix \( X^{(N)}(h) \) is the Gaussian distribution \( \gamma_{d^N}^{\sigma \tau} \) of Section 3.4 and if \( k \) is another vector orthogonal to \( h \), the random matrices \( X^{(N)}(h) \) and \( X^{(N)}(k) \) are independent.

Similarly, let \( Z : H \otimes_{\mathbb{R}} \mathbb{H}_d \rightarrow L^2(\mathcal{X}) \subset L^2(\Omega, \mathcal{F}, \mathbb{P}) \) be a Gaussian \((s-t/2, t/2)\)-elliptic system. We define
\[
Z^{(N)} : H \rightarrow L^2(\mathcal{X}) \otimes_{\mathbb{R}} \mathbb{H}_d \simeq L^2(\mathcal{X}) \otimes_{\mathbb{C}} \mathbb{M}_d \subset L^2(\Omega, \mathbb{P} ; \mathbb{M}_d)
\]
by duality as the unique linear map \( Z^{(N)}(h) \) from the random variables with value in \( \mathbb{M}_d \) such that \( Z(h \otimes M) = \text{Tr}(MZ^{(N)}(h)) \). The Segal-Bargmann transform \( S^{s,t}_d : L^2(\mathcal{X}) \rightarrow \mathcal{H}L^2(\mathbb{Z}) \) is well-defined as in \( \ref{4.1} \). Finally, we consider the following boosted Segal-Bargmann transform
\[
S^{s,t}_{d^N} = S^{s,t} \otimes Id_{\mathbb{M}_d} : L^2(\mathcal{X}) \otimes \mathbb{M}_d \rightarrow \mathcal{H}L^2(\mathbb{Z}) \otimes \mathbb{M}_d.
\]

Theorem 4.8. Let \( 0 \leq q \leq 1 \). Assuming \( (H.1), (H.2), (H.3) \) and \( (H.4) \) on \( \sigma \) ensures that the Segal-Bargmann transform \( S^{s,t}_{d^N} \) converges to the \( q \)-deformed Segal-Bargmann transform \( S^{q,s}_q \) : \( L^2(X_q, \tau) \rightarrow \mathcal{H}L^2(\mathbb{Z}, \tau) \) in the following sense: for all polynomial \( P \) and \( q \in \mathbb{C}[x_h : h \in H] \) such that
\[
S^{q,s}_q(P(X_q(h)) : h \in H)) = Q(Z_q(h) : h \in H),
\]
the norm \( \| P(X^{(N)}(h)) : h \in H) \|_{L^2(\mathcal{X})} = \| S^{q,s}_q(P(X^{(N)}(h)) : h \in H)) \|_{H^2(\mathbb{Z})} \) converges, as \( N \) tends to \( \infty \), to \( \| P(X_q(h) : h \in H) \|_{L^2(\mathcal{X}, \tau)} = \| Q(Z_q(h) : h \in H) \|_{H^2(\mathbb{Z}, \tau)} \) and
\[
\lim_{N \rightarrow \infty} \| S^{q,s}_q(P(X^{(N)}(h)) : h \in H)) - Q(Z^{(N)}(h) : h \in H) \|_{H^2(\mathbb{Z})} = 0.
\]

Proof. Remark that, for all \( M, N \in \mathbb{H}_d \), and all \( h, k \in H \), we have
\[
\mathbb{E} \left[ \text{Tr}(MX^{(N)}(h))\text{Tr}(NX^{(N)}(k)) \right] = \tau[X_q(h)X_q(k)](M, N)_{\sigma}.
\]
We can apply Theorem 2.10 which says that the random matrices \( \{ X^{(N)}(h) : h \in H \} \) converge in noncommutative distribution to \( \{ X_q(h) : h \in H \} \). In particular, we have the following convergences:
\[
\lim_{N \rightarrow \infty} \| P(X^{(N)}(h)) : h \in H) \|_{L^2(\mathcal{X})} = \| P(X_q(h) : h \in H) \|_{L^2(\mathcal{X}, \tau)}
\]

The proof of the second limit uses the following lemma. Let \( Y : H \otimes_{\mathbb{R}} \mathbb{H}_d \rightarrow L^2(\mathcal{Y}) \subset L^2(\Omega, \mathcal{F}, \mathbb{P}) \) be a Gaussian \((t, 0)\)-elliptic system independent from \( (Z^{(N)}(h))_{h \in H} \), and define
\[
Y^{(N)} : H \rightarrow L^2(\mathcal{Y}) \otimes_{\mathbb{R}} \mathbb{H}_d \simeq L^2(\mathcal{Y}) \otimes_{\mathbb{C}} \mathbb{M}_d \subset L^2(\Omega, \mathbb{P} ; \mathbb{M}_d)
\]
by duality as the unique linear map \( Y^{(N)}(h) \) from \( H \) to the random variables with value in \( \mathbb{M}_d \) such that \( Y(h \otimes M) = \text{Tr}(MY^{(N)}(h)) \).

Lemma 4.9. For all \( P \in \mathbb{C}[x_h : h \in H] \), we have
\[
S^{d^N}_{q,s}(P(X^{(N)}(h)) : h \in H)) = \mathbb{E}\left[ P(Z^{(N)}(h) + Y^{(N)}(h)) : h \in H) \right] Z^{(N)}(h) : h \in H.
\]

Proof of Lemma 4.9. One can apply \( \ref{4.3} \) for each coordinate of \( P(X^{(N)}(h)) : h \in H) \) in any basis of \( \mathbb{M}_d \). Alternatively, one can reason as follows.
of Section 4.1 X(h ⊗ M) is the linear functional x ⊗ N → ⟨x, h⟩_H Tr(NM) defined on (H ⊗ ℌdN, B, γs), Y(h ⊗ M) is the linear functional x ⊗ N → ⟨x, h⟩_H Tr(NM) defined on (H ⊗ ℌdN, B, γt) and Z(h ⊗ M) the linear functional z ⊗ N → ⟨z, h⟩_Tr(N^*M) defined on (H_C ⊗ ℌdN, B, γs-t/2, t/2). We consider then the matrix-valued random variables X^{(N)}(h) : x ⊗ M → ⟨x, h⟩_H · M, Y^{(N)}(h) : x ⊗ M → ⟨x, h⟩_H · M and Z^{(N)}(h) : z ⊗ M → ⟨z, h⟩_H_C · M.

Let P ∈ ℂ[X_h : h ∈ H]. We use here the definition 2.1 of the Segal-Bargmann transform $S^{s,t}$, which is also valid for $S^{s,t}$ by linearity: for all z ⊗ M ∈ H ⊗ ℌdN,

$$S^{s,t}(P(X^{(N)}(h) : h ∈ H))(z ⊗ M) = \int_{H ⊗ ℌdN} P(X^{(N)}(h) : h ∈ H)(z ⊗ M - x ⊗ N) dγ_t(x ⊗ N)$$

$$= \int_{H ⊗ ℌdN} P(⟨z, h⟩_M - ⟨x, h⟩_N : h ∈ H) dγ_t(x ⊗ N)$$

$$= \int_{H ⊗ ℌdN} P(⟨Z^{(N)}(h)(z ⊗ M) - Y^{(N)}(h)(x ⊗ N) : h ∈ H) dγ_t(x ⊗ N).$$

The last term is also valid for all z ⊗ M ∈ H_C ⊗ ℌdN, since each side is analytic in z ⊗ M. We recognize the wanted conditioning of Lemma 4.9.

Let us consider an independent copy W^{(N)} of Y^{(N)}. We consider also two q-Gaussian (t, 0)-elliptic system W_q and Y_q which are q-independent from each others and from Z_q. Remark that, for all M, N ∈ ℌdN, and all h, k ∈ H, we have

$$\mathbb{E} \left[ Tr(M U^{(N)}(h))Tr(N V^{(N)}(k)) \right] = τ[U_q(h)V_q(k)](M, N)_σ,$$

where the symbols U and V can be replaced by any from the symbols W_q, Y_q, ℍZ_q and ℍZ_q. Thus, we can apply Theorem 2.10 which says that the random matrices \{W^{(N)}(h), Y^{(N)}(h), ℍZ^{(N)}(h), ℍZ^{(N)}(h) : h ∈ H\} converge in noncommutative distribution to \{W_q(h), Y_q(h), ℍZ_q(h), ℍZ_q(h) : h ∈ H\}. In particular, we have the following convergence:

$$\lim_{N → ∞} \left\| S^{s,t}_{dN}(P(X^{(N)}(h) : h ∈ H)) - Q(Z^{(N)}(h) : h ∈ H) \right\|_{HL^2(Z ⊗ ℌdN)}$$

$$= \lim_{N → ∞} \left\| \mathbb{E} \left[ P(Z^{(N)}(h) + Y^{(N)}(h) : h ∈ H) | Z \right] - Q(Z^{(N)}(h) : h ∈ H) \right\|_{HL^2(Z ⊗ ℌdN)}$$

$$= \lim_{N → ∞} \left\| \mathbb{E} \left[ P(Z^{(N)}(h) + Y^{(N)}(h) : h ∈ H) - Q(Z^{(N)}(h) : h ∈ H) | Z \right] \right\|_{HL^2(Z ⊗ ℌdN)}$$

$$= \lim_{N → ∞} \mathbb{E} \left[ \frac{1}{dN} Tr \left( \left( P(Z^{(N)}(h) + Y^{(N)}(h) : h ∈ H) - Q(Z^{(N)}(h) : h ∈ H) \right)^* \right) \right]$$

$$= \mathbb{E} \left[ Tr \left( (P(Z_q(h) + Y_q(h) : h ∈ H) - Q(Z_q(h) : h ∈ H))^* \right) \right]$$

The last quantity vanishes because Theorem 4.6 tells us that

$$Q(Z_q(h) : h ∈ H) = S^{s,t}_{dN}(P(X_q(h))) = τ[P(Z_q(h) + Y_q(h) : h ∈ H)]Z_q.$$

□
5 Mixture of Classical and Free Segal-Bargmann Transform

In this section, we shall define the Segal-Bargmann transform for a mixture of classical and free random variables and then we recover the \( q \)-Segal Bargmann transform in the limit.

5.1 The Mixed \( q \)-Deformed Segal-Bargmann Transform

Let \( Q = (q_{ij})_{i,j \in I} \) be a symmetric matrix with elements in \([-1, 1]\). We consider a complex Hilbert space \( K \) with an orthonormal basis \( \{e_i\}_{i \in I} \), the Fock space \( F_Q(K) \), and the set of mixed \( q \)-Gaussian variables \( \{X_i\}_{i \in I} \) acting on \( F_Q(K) \) as defined in Section 2.4. The set \( \{\sqrt{s}X_i\}_{i \in I} \) are the mixed \( q \)-Gaussian variables of variance \( s \). Remark that the map \( A \mapsto A(\Omega) \) extend to a unitary isomorphism from \( L^2(\{\sqrt{s}X_i\}_{i \in I}, \tau) \) to \( F_Q(K) \).

As in Section 2.3, we will define the mixed \( q \)-Gaussian variables as a set of variables \( \{Z_i\}_{i \in I} \) indexed by \( I \) such that \( \{\Re Z_i, \Im Z_i\}_{i \in I} \) are a set of mixed \( q \)-Gaussian variables with prescribed variance. The first step is to replace the index set \( I \) by the index set \( I \times \{1, 2\} \), and the matrix \( Q \) by the matrix

\[
\tilde{Q} = \begin{pmatrix} Q & Q \\ Q & \bar{Q} \end{pmatrix} = Q \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.
\]

We consider the complex Hilbert space \( K^2 \) with an orthonormal basis \( \{e_{i,1}, e_{i,2}\}_{i \in I} \). Considering the Fock space \( F_Q(K^2) \), we define the set of mixed \( q \)-Gaussian variables \( \{X_{i,1}, X_{i,2}\}_{i \in I} \) acting on \( F_Q(K^2) \) as defined in Section 2.4. Finally, we set the mixed \( q \)-Gaussian \( \sigma - t/2, t/2 \)-elliptic variables

\[
Z_i = \sqrt{s-t/2}X_{i,1} + i\sqrt{t/2}X_{i,2}.
\]

Remark that the map \( A \mapsto A(\Omega) \) extend to a unitary isomorphism from \( \mathcal{H}L^2(\{Z_i\}_{i \in I}, \tau) \) to \( F_Q(K^2) \).

We are ready to define the mixed \( q \)-deformed Segal-Bargmann transform.

**Definition 5.1.** The mixed \( q \)-deformed Segal-Bargmann transform \( \mathcal{S}^{q,t}_Q \) is the unitary isomorphism so that the following diagram commute:

\[
\begin{array}{ccc}
F_Q(K) & \xrightarrow{\delta^t_Q} & F_Q(K^2) \\
A \mapsto A\Omega & \quad \quad \quad & A \mapsto A\Omega \\
L^2(\{X_i\}_{i \in I}, \tau) & \xrightarrow{\mathcal{S}^{q,t}_Q} & \mathcal{H}L^2(\{Z_i\}_{i \in I}, \tau)
\end{array}
\]

where \( \delta^t_Q \) is the Fock extension of \( \delta^t_Q(\sqrt{s}e_i) = \sqrt{s-t/2}e_{i,1} + i\sqrt{t/2}e_{i,2} \), meaning that

\[
\delta^t_Q(h_1 \otimes \ldots \otimes h_k) = \delta^t_Q(h_1) \otimes \ldots \otimes \delta^t_Q(h_k).
\]

For all \( i \in I \), we have

\[
H_{q_{ii},s}^{\sqrt{s}X_i}\Omega = (\sqrt{s}e_i)^{\otimes n}.
\]

Indeed, the definition of the Hermite polynomials is adjusted with the definition (2.6) of the annihilation operator \( c_{i(1)}^s \) in such a way that, by a direct induction, for all \( n \geq 2 \), we have

\[
H_{q_{ii},s}^{\sqrt{s}X_i}\Omega = \sqrt{s}(c_i + c_i^*)(\sqrt{s}e_i)^{\otimes n-1} - s \sum_{k=2}^n q^{n-2} (\sqrt{s}e_i)^{\otimes n-2}
\]

\[
= \sqrt{s}e_i(\sqrt{s}e_i)^{\otimes n-1}
\]

\[
= (\sqrt{s}e_i)^{\otimes n}.
\]

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Similarly, we have

\[ H_{n}^{q_{ii},s^{-1}}(Z_{i})\Omega = (\sqrt{s - t/2}e_{(i,1)} + i\sqrt{t/2}e_{(i,2)})^{\otimes n}. \]

We deduce the following result, which says that restricted on the different \( L^{2}(X_{i}, \tau) \), the mixed \( q \)-deformed Segal-Bargmann transform \( \mathcal{S}_{Q}^{s,t} \) coincides with the \( q \)-deformed Segal-Bargmann transform \( \mathcal{S}_{q_{ii}}^{s,t} \).

**Proposition 5.2.** For all \( i \in I \) and all polynomial \( P \), we have

\[ \mathcal{S}_{Q}^{s,t}(P(\sqrt{s}X_{i})) = (\mathcal{S}_{q_{ii}}^{s,t}P)(Z_{i}). \]

### 5.2 The \( q \)-Segal-Bargmann Transform in the Limit

Set \( I = \mathbb{N} \). We choose \( Q = (q_{ij} = q_{ji})_{i,j \in I} \) randomly in \( \{-1, +1\} \) or in \( \{0, +1\} \), as independent random variables, identically distributed, with \( \mathbb{E}[q_{ij}] = q \). We consider the mixed \( q \)-Gaussian variables \( \{\sqrt{s}X_{i}\}_{i \in I} \) of variance \( s \) as defined in the previous section.

**Remark 5.3.** Let us recall first that \( q_{ii} = -1, 0 \) or 1 means respectively that \( X_{i} \) is a Bernoulli variable, a semi-circular variable or a Gaussian variable. Secondly, \( q_{ii} = 0 \) or 1 means respectively that \( X_{i} \) and \( X_{j} \) are freely independent or classically independent.

Let us consider the sum

\[ X^{(n)} := \frac{\sqrt{s}X_{1} + \ldots + \sqrt{s}X_{n}}{\sqrt{n}}. \]

These variables define an approximation of a \( q \)-Gaussian variable. Speicher’s central limit theorem ([24 Theorem 1]) makes this statement precise whenever \( q_{ij} \in \{-1, +1\} \). If \( q_{ij} \in \{0, +1\} \), it is not complicated (using for example the characterisation with cumulants of [26]) to prove that we fall in the framework of \( \Lambda \)-freeness of Młotkowski. More precisely, if we define \( \Lambda \) to be the set of \( \{i, j\} \) such that \( q_{ij} = 1 \) and \( i \neq j \), the algebras generated by the different \( X_{i} \) are \( \Lambda \)-free. We can use Młotkowski’s central limit theorem ([18 Theorem 4]) and we get the following result.

**Theorem 5.4** (Theorem 1 of [24] and Theorem 4 of [18]). Almost surely, the variable \( X^{(n)} \) converges to a \( q \)-Gaussian variable \( X \) of variance \( s \) in noncommutative distribution in the following sense: for all polynomial \( P \), we have

\[ \lim_{N \to \infty} \tau \left[ P(X_{(n)}) \right] = \tau[P(X)]. \]

We consider now the mixed \( q \)-Gaussian \( (s - t/2, t/2) \)-elliptic variables \( \{Z_{i}\}_{i \in I} \), where the relations are governed by the matrix \( \tilde{Q} \), and we set

\[ Z^{(n)} := \frac{Z_{1} + \ldots + Z_{n}}{\sqrt{n}}. \]

The entries of \( \tilde{Q} \) are not any more independent but only block-independent. Nevertheless, as used in [16 Section 4.2], a straightforward modification of Speicher’s proof and of Młotkowski’s proof generalizes the theorem to this case.

**Theorem 5.5** (Theorem 1 of [24] and Theorem 4 of [18]). Almost surely, the variable \( Z^{(n)} \) converges to a \( q \)-Gaussian \( (s - t/2, t/2) \)-elliptic variable \( Z \) in noncommutative distribution in the following sense: for all polynomial \( P \), we have

\[ \lim_{N \to \infty} \tau \left[ P(Z^{(n)}) \right] = \tau[P(Z)]. \]

The following theorem says that the mixed \( q \)-Segal-Bargmann transform is also an approximation of the \( q \)-deformed case.
Theorem 5.6. Set $I = \mathbb{N}$. We choose $Q = (q_{ij} = q_{ji})_{i,j \in I}$ randomly in $\{-1, +1\}$ or in $\{0, +1\}$, as independent random variables, and identically distributed with $\mathbb{E}[q_{ij}] = q$ for $i > j$. We consider the mixed $q$-Gaussian variables $\{\sqrt{s}X_i\}_{i \in I}$ of variance $s$, the mixed $q$-Gaussian $(s - t/2, t/2)$-elliptic variables $\{Z_i\}_{i \in I}$ and the mixed $q$-Segal-Bargmann transform $\mathcal{J}^{s,t}_Q : L^2(\{X_i\}_{i \in I}, \tau) \to \mathcal{H}L^2(\{Z_i\}_{i \in I}, \tau)$.

Almost surely, the Segal-Bargmann transform $\mathcal{J}^{s,t}_Q$ converges to the $q$-deformed Segal-Bargmann transform $\mathcal{J}^{s,t}_q$ in the following sense: considering the sums

$$X^{(n)} := \frac{\sqrt{s}X_1 + \ldots + \sqrt{s}X_n}{\sqrt{n}} \quad \text{and} \quad Z^{(n)} := \frac{Z_1 + \ldots + Z_n}{\sqrt{n}},$$

for all polynomial $P$, we have $\lim_{n \to \infty} \left\| \mathcal{J}^{s,t}_Q (P(X^{(n)})) - \mathcal{J}^{s,t}_q P(Z^{(n)}) \right\|_{\mathcal{H}L^2(\{Z_i\}_{i \in I}, \tau)} = 0$.

Remark 5.7. • We can choose $q_{ii}$ arbitrarily. For example, if we choose $q_{ii} = 1$, Proposition 5.2 tells us that $\mathcal{J}^{s,t}_Q$ restricted to $L^2(\sqrt{s}X_i, \tau)$ is the classical Segal-Bargmann transform $\mathcal{J}^{s,t}_{q_{ii}}$.

• Now, assume that $q_{ij} \in \{0, +1\}$. We define $\Lambda$ to be the (random) set of $\{i, j\}$ such that $q_{ij} = 1$ and $i \neq j$. The algebras generated by the different $X_i$ are $\Lambda$-free in the sense of [18]. Decomposing $L^2(\sqrt{s}X_i, \tau) = L^0_i \oplus \mathbb{C}$ (with $L^0_i$ composed of the operators $A$ such that $\tau[A] = 0$), and $L^2(Z_i, \tau) = \mathcal{H}L^0_i \oplus \mathbb{C}$ decomposed similarly, we have the $\Lambda$-free products observed in [18]:

$$L^2(\{X_i\}_{i \in I}, \tau) = \bigoplus_{(i(1), \ldots, i(m)) \in S(I, \Lambda)} L^0_i(1) \otimes \ldots \otimes L^0_i(m),$$

$$\mathcal{H}L^2(\{Z_i\}_{i \in I}, \tau) = \bigoplus_{(i(1), \ldots, i(m)) \in S(I, \Lambda)} \mathcal{H}L^0_i(1) \otimes \ldots \otimes \mathcal{H}L^0_i(m),$$

where $S(I, \Lambda)$ is the set of reduced words over $I$ modulo the relations $(\ldots, i, j, \ldots) \simeq (\ldots, j, i, \ldots)$ if $\{i, j\} \in \Lambda$ and $(\ldots, i, i, \ldots) \simeq (\ldots, i, \ldots)$, or, more specifically, a set of representatives of minimal length. Finally, $\mathcal{J}^{s,t}_Q : L^2(\{X_i\}_{i \in I}, \tau) \to \mathcal{H}L^2(\{Z_i\}_{i \in I}, \tau)$ can be decomposed as

$$\bigoplus_{(i(1), \ldots, i(m)) \in S(I, \Lambda)} \mathcal{J}^{s,t}_{q_{i(1)}} \otimes \ldots \otimes \mathcal{J}^{s,t}_{q_{i(n)}},$$

or as a $\Lambda$-free product of classical Segal-Bargmann transform whenever $q_{ii} = 1$ for all $i \in I$.

Proof. We consider the index set $I \times \{1, 2, 3, 4\}$, and the matrix

$$R = \begin{pmatrix}
Q & Q & Q & Q \\
Q & Q & Q & Q \\
Q & Q & Q & Q \\
Q & Q & Q & Q
\end{pmatrix} = Q \otimes \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{pmatrix}.$$

We consider the complex Hilbert space $K^4 \supseteq K^2$ with an orthonormal basis $\{e_{i(1)}, e_{i(2)}, e_{i(3)}, e_{i(4)}\}_{i \in I}$ and the Fock space $\mathcal{F}_R(K^4)$. We have the canonical inclusion $\mathcal{F}_Q(K^2) \subseteq \mathcal{F}_R(K^4)$ given by the natural extension of $K^2 \subseteq K^4$. We define the set of mixed $q$-Gaussian variables $\{X_{i(1)}, X_{i(2)}, X_{i(3)}, X_{i(4)}\}_{i \in I}$ acting on $\mathcal{F}_Q(K^4)$ as defined in Section 2.4, which is an extension of the already defined action of $\{X_{i(1)}, X_{i(2)}\}_{i \in I}$ on $\mathcal{F}_Q(K^2)$. The action of the mixed $q$-Gaussian $(s - t/2, t/2)$-elliptic variables $Z_i$ extends to $\mathcal{F}_R(K^4)$ by

$$Z_i = \sqrt{s - t/2}X_{i(1)} + i\sqrt{t/2}X_{i(2)},$$

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and the action of $Z^{(n)}$ extends to $\mathcal{F}_R(K^4)$ by

$$Z^{(n)} := \frac{Z_1 + \ldots + Z_n}{\sqrt{n}}.$$  

Finally, we define also

$$Y_i = \sqrt{i}X_{(i,3)}, \quad Y^{(n)} := \frac{Y_1 + \ldots + Y_n}{\sqrt{n}},$$  

and

$$W_i = \sqrt{i}X_{(i,4)}, \quad W^{(n)} := \frac{W_1 + \ldots + W_n}{\sqrt{n}}.$$  

**Lemma 5.8.** For all polynomial $Q$, we have

$$\left\| \mathcal{G}_{\tau}^s \left( P(X^{(n)}) \right) - Q(Z^{(n)}) \right\|_{\mathcal{H}(\mathcal{L}(\{Z_i \mid i \in I, \tau\}))} = \tau \left[ \left( P(Z^{(n)} + Y^{(n)}) + Q(Z^{(n)}) \right) \ast \left( P(Z^{(n)} + W^{(n)}) + Q(Z^{(n)}) \right) \right].$$

**Proof of Lemma.** For all $i(1), \ldots, i(n) \in I$, we define a polynomial $P_{i(1),\ldots,i(n)} \in \mathbb{C}[x_i : i \in I]$ by the following recursion formula:

$$P_{i(1),\ldots,i(n)} = x_{i(1)} \cdot P_{i(2),\ldots,i(n)} - \sum_{k=2}^{n} \delta_i (i(k)) q_i(i(2) \cdots q_i(i(1))i(k-1) \cdot P_{i(1),\ldots,i(k),\ldots,i(n)}.$$

where the hat means that we omit the corresponding index. Since $\{P_{i(1),\ldots,i(n)}\}_{n \geq 0, i(1),\ldots,i(n) \in I}$ is a spanning set of $\mathbb{C}[x_i : i \in I]$, it suffices to prove the theorem for those polynomials.

We have

$$P_{i(1),\ldots,i(n)}(\sqrt{s}X_i : i \in I) \Omega = \sqrt{s}e_{i(1)} \otimes \cdots \otimes \sqrt{s}e_{i(n)}.$$  

Indeed, the definition of the polynomials $\{P_{i(1),\ldots,i(n)}\}_{n \geq 0, i(1),\ldots,i(n) \in I}$ is adjusted with the definition (2.6) of the annihilation operator $c_{i(1)}^*$ in such a way that, by a direct induction, for all $i(1), \ldots, i(n) \in I$, we have

$$P_{i(1),\ldots,i(n)}(X_i : i \in I) \Omega = \sqrt{s} (c_{i(1)} + c_{i(1)}^*) \cdot \sqrt{s}e_{i(1)} \otimes \cdots \otimes \sqrt{s}e_{i(n)}$$  

$$- s \sum_{k=2}^{n} \delta_i (i(k)) q_i(i(2) \cdots q_i(i(1))i(k-1) \cdot \sqrt{s}e_{i(1)} \otimes \cdots \otimes \sqrt{s}e_{i(1)}$$  

$$= \sqrt{s}e_{i(1)} \otimes \cdots \otimes \sqrt{s}e_{i(n)}$$  

Similarly, setting $h_i = \sqrt{s - t/2}e_{i(n),1} + i\sqrt{t/2}e_{i(n),2} + \sqrt{s}e_{i(n),3}$, it follows from the definition of the polynomials $\{P_{i(1),\ldots,i(n)}\}_{n \geq 0, i(1),\ldots,i(n) \in I}$ that

$$P_{i(1),\ldots,i(n)}(Z_i + Y_i : i \in I) \Omega = h_{i(1)} \otimes \cdots \otimes h_{i(n)}.$$  

Indeed, for all $i(1), \ldots, i(n) \in I$, we have

$$(\sqrt{s - t/2}e_{i(1),1}^* + i\sqrt{s - t/2}e_{i(1),2}^* + \sqrt{t}e_{i(2),3}^*) \cdot h_{i(2)} \otimes \cdots \otimes h_{i(n)}$$  

$$= (s - t/2 - t/2 + t) \sum_{k=2}^{n} \delta_i (i(k)) q_i(i(2) \cdots q_i(i(1))i(k-1) \cdot h_{i(1)} \otimes \cdots \otimes h_{i(k)} \otimes \cdots \otimes h_{i(n)}$$  

$$= s \sum_{k=2}^{n} \delta_i (i(k)) q_i(i(2) \cdots q_i(i(1))i(k-1) \cdot h_{i(1)} \otimes \cdots \otimes h_{i(k)} \otimes \cdots \otimes h_{i(n)}$$
which allows to write the induction step

\[ P_{i(1),\ldots,i(n)}(Z_i + Y_i : i \in I) \Omega = (\sqrt{s-t/2}c_{i(1),1} + i \sqrt{s-t/2}c_{i(1),2} + \sqrt{t}c_{i(2),3}) \cdot h_i(2) \odot \cdots \odot h_{i(n)} \]

\[ + (\sqrt{s-t/2}c_{i(1),1} + i \sqrt{s-t/2}c_{i(1),2} + \sqrt{t}c_{i(2),3}) \cdot h_i(2) \odot \cdots \odot h_{i(n)} \]

\[ - s \sum_{k=2}^{n} \delta_{i(1)(k)}q_{i(1)(2)} \cdots q_{i(1)(k-1)} \cdot h_i(2) \odot \cdots \odot h_i(k) \odot \cdots \odot h_{i(n)} \]

\[ = (\sqrt{s-t/2}c_{i(1),1} + i \sqrt{s-t/2}c_{i(1),2} + \sqrt{t}c_{i(2),3}) \cdot h_i(2) \odot \cdots \odot h_{i(n)} \]

Setting \( k_i = \sqrt{s-t/2}c_{i(n),1} + i \sqrt{t/2}e_{i(n),2} + \sqrt{s}e_{i(n),4}, \) the same computation yields

\[ P_{i(1),\ldots,i(n)}(Z_i + W_i : i \in I) \Omega = k_i(1) \odot \cdots \odot k_i(n). \]

Finally, we have

\[ \tau \left[ (P_{i(1),\ldots,i(n)}(Z_i + Y_i : i \in I) + Q(Z_i : i \in I))^* \cdot (P_{i(1),\ldots,i(n)}(Z_i + W_i : i \in I) + Q(Z_i : i \in I)) \right] \]

\[ = \langle k_i(1) \otimes \cdots \otimes k_i(n) - Q(Z_i : i \in I) \rangle_h, h_i(1) \otimes \cdots \otimes h_i(n) - Q(Z_i : i \in I) \rangle_{R}. \]

Because the \( e_{i(3)} \) occur only in \( h_i(1) \otimes \cdots \otimes h_i(n), \) and the \( e_{i(4)} \) occur only in \( k_i(1) \otimes \cdots \otimes k_i(n), \) they do not contribute to the scalar product, and we can replace \( h_i(1) \otimes \cdots \otimes h_i(n) \) and \( k_i(1) \otimes \cdots \otimes k_i(n) \) by

\[ (\sqrt{s-t/2}e_{i(n),1} + i \sqrt{t/2}e_{i(n),2}) \otimes \cdots \otimes (\sqrt{s-t/2}e_{i(n),1} + i \sqrt{t/2}e_{i(n),2}) = \delta_Q^{s,t}(\sqrt{s}e_{i(1)} \otimes \cdots \otimes \sqrt{s}e_{i(n)}), \]

which yields

\[ \tau \left[ \langle P_{i(1),\ldots,i(n)}(Z_i + Y_i : i \in I) + Q(Z_i : i \in I))^* \cdot (P_{i(1),\ldots,i(n)}(Z_i + W_i : i \in I) + Q(Z_i : i \in I)) \rangle_h, h_i(1) \otimes \cdots \otimes h_i(n) - Q(Z_i : i \in I) \rangle_{R}. \]

Let us denote by \( Q \) the polynomial \( S_{q}^{s,t}P. \) Let \( Y, W \) be two \((0,0)\)-elliptic \( q \)-Gaussian random variables and \( Z \) be a \((s-t/2,t/2)\)-elliptic \( q \)-Gaussian random variable such that \( Y, W \) and \( Z \) are \( q \)-independent. Thanks to the discussion before Theorem 5.3.15.4 we know that we can apply 24.1 Theorem 1 in the case of \( q_{ij} \in \{-1,+1\} \) (or 18.1 Theorem 4) in the case \( q_{ij} \in \{0,+1\} \) which says that the mixed \( q \)-Gaussian random variables \( Y^{(n)}, W^{(n)}, RZ^{(n)} \) and \( \mathbb{Z}Z^{(n)} \) converge in noncommutative distribution to the \( q \)-Gaussian random variables \( Y, W, RZ \) and \( \mathbb{Z}Z. \) In particular, we have the following convergence:

\[ \lim_{n \to \infty} \tau \left[ (P(Z^{(n)} + Y^{(n)}) - Q(Z^{(n)}))^* (P(Z^{(n)} + W^{(n)}) - Q(Z^{(n)})) \right] = \tau \left[ (P(Z + Y) - Q(Z))^* (P(Z + W) - Q(Z)) \right]. \]

From Corollary 3.11 and Corollary 3.13, we know that

\[ Q(Z) = S_{q}^{s,t}P(Z) = \tau[P(Z + Y)|Z] = \tau[P(Z + Y)|Z,W]. \]
Thus the limit $\tau [(P(Z + Y) - Q(Z))^*(P(Z + W) - Q(Z))]$ of $\|F^{*,t}_Q (P(X^{(n)})) - Q(Z^{(n)})\|_{H^2(\mathbb{L}, \tau)}$ vanishes:

$$\tau [(P(Z + Y) - Q(Z))^*(P(Z + W) - Q(Z))] = \tau [(P(Z + Y) - \tau [P(Z + Y)|Z, W])^*(P(Z + W) - Q(Z))]$$

$$= \tau [(P(Z + Y) - P(Z + Y))^*(P(Z + W) - Q(Z))]$$

$$= 0.$$ 

**Acknowledgements**

The first author was partially funded by the ERC Advanced Grant “NCDFP” held by Roland Speicher. The second author was funded by the same ERC Advanced Grant “Non-commutative distributions in free probability” (grant no. 339760). The second author would like to thank Roland Speicher for allowing his stay in Saarbrücken, Germany so the authors had a chance to collaborate.

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