Role of quasiparticles in the growth of a trapped Bose-Einstein condensate

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A major extension of the model of condensate growth introduced by us is made to take account of the evolution of the occupations of lower trap levels (quasiparticles) by scattering processes, and of the full Bose-Einstein formula for the occupations of higher trap levels, which are assumed to have a time independent occupation. The principal effect is a speedup of the growth rate by somewhat less than an order of magnitude, the precise value depending on the the assumptions made on scattering and transition rates for the quasiparticle levels.

In a previous paper [1] we introduced a formula for the growth of a Bose-Einstein condensate, in which growth resulted exclusively from stimulated collisions of atoms where one of the atoms is left in the condensate. This gave a very simple growth formula which predicts a rate of growth of the order of magnitude of that observed in current experiments [2–4]. The direct stimulated effect must eventually be very important, once a significant amount of condensate has formed, but in the initial stages there will of course also be a significant number of transitions to excited states of the condensate (quasiparticles), whose populations will then also grow. As well as this, there will be interactions between the condensate, the quasiparticles and the atomic vapor from which the condensate forms. This paper extends the description of the condensate growth to include these factors.

All of these effects are encompassed by the description given in [1,2]. However, the practical extension of this description to take account of the additional effects would involve a calculation of all the eigenfunctions for the trapped condensate, and the detailed summation over all processes involving these. Since the number of levels involved is of the order of tens of thousands, this could be a formidable task. However some quite reasonable estimates can be made for the overall effects of these processes, and from these we can derive a set of easily solvable differential equations for the populations of the condensate and the lower energy quasiparticles.

As in our previous work, we divide the states in the potential into the condensate band, $R_C$, which consists of the energy levels significantly affected by the presence of a condensate in the ground state, and the non-condensate band, $R_{NC}$, which contains all the remaining energy levels above the condensate band. The division between the two bands is taken to be at the value, $E_R$.

The picture we shall use assumes that $R_{NC}$ consists of a large “bath” of atomic vapor, whose distribution function is given, for the energy levels greater than a value which we shall call $E_{\text{max}}$ (with $E_{\text{max}} > E_R$), by a time-independent equilibrium Bose-Einstein distribution $\{\exp[(E - \mu)/k_B T] - 1\}^{-1}$. The value of $E_{\text{max}}$ will be assumed to be small enough for the majority of atoms to have energies higher than $E_{\text{max}}$, so that this part of the bath can be treated as being essentially undepleted by the process of condensate growth.

The energy levels between $E_R$ and $E_{\text{max}}$ are taken to have a time dependent population, since the continuation of the equilibrium Bose-Einstein formula to lower energies eventually leads to an unrealistic situation in which the populations and transition rates become too large for the populations to be considered to be constant—this demonstrates that the initial condition in which the vapor is at a positive chemical potential cannot apply for all energies. The choice of $E_{\text{max}}$ is thus determined as a lower limit to the equilibrium distribution, with the distribution in the range between $E_{\text{max}}$ and $E_R$ treated as time dependent, and computed as part of the growth kinetics.

The value $E_R$, above which the energy levels are taken as unperturbed, was fixed at $2\mu$. This value and the ground state energy level—the chemical potential
\[ \mu(N) - \mu \] put bounds on the energy levels of the states in between. As a simple expression of this fact, the levels between \( \mu(N) \) and \( E_R \) are determined by interpolating linearly between the two extremes, using a density of states \( \mathcal{N}[E - \mu(N)]^2 \), where \( \mathcal{N} \) is a normalization chosen so that the cumulative number of states matches the corresponding cumulative value for an unperturbed three-dimensional harmonic oscillator when \( E = E_R \). As an illustration of the effect of this, we show in Fig.1 the cumulative occupation \( \mathcal{N}(E) \), and the occupation per unit energy interval \( f(E) = \frac{d\mathcal{N}(E)}{dE} \), when the system is in equilibrium. (The condensate population itself is not shown.)

Fig.2: The transitions being considered: Left—scattering; Right—Condensate growth.

The dynamics we will consider will arise from two kinds of process as illustrated in Fig.2.

**Growth:** A collision between a pair of atoms initially in the bath of atomic vapor results in one of the atoms having a final energy less than \( E_{\text{max}} \).

**Scattering:** A collision between an atom initially in an energy level below \( E_{\text{max}} \) and a bath atom transfers the first atom to another energy level below \( E_{\text{max}} \).

Our treatment therefore omits any scattering between atoms which both have energies less than \( E_{\text{max}} \), which is reasonable if the number of atoms in the bath is almost 100% of the total number of atoms.

The mechanism of condensate growth, as in Fig.2, under appropriate approximations, gives rise to equations of motion for the number of atoms in the condensate band \( N \), and the number of quasiparticle excitations \( n_m \) (with energies \( e_m < E_{\text{max}} \)) in the condensate, which can be written as follows. First define

\[
\dot{n}_m^+ \equiv 2W_{m+}^+(N) \left( 1 - e^{(\mu(N) - \mu + e_m)/kBT} \right) n_m + 1, \quad (1)
\]

\[
\dot{n}_m^- \equiv 2W_{m-}^+(N) \left( 1 - e^{-(\mu(N) + \mu + e_m)/kBT} \right) n_m + 1. \quad (2)
\]

Then the multilevel growth equations are

\[
\dot{n}_m = \dot{n}_m^+ + \dot{n}_m^-, \quad (3)
\]

\[
\dot{N} = 2W_0^+(N) \left( 1 - e^{(\mu(N) - \mu)/kBT} \right) N + 1.
\]

Here \( 2(N+1)W^+(N) \) is the transition rate for an atom to enter the condensate level as a result of a collision between two atoms in the “bath” of atomic vapor—the factor \( N+1 \), takes account of both the “spontaneous” term, and the “stimulated” term induced by the presence of the condensate. The reversed process occurs with the overall rate \( 2NW^+(N)e^{(\mu(N)-\mu)/kBT} \)—that is with no “spontaneous” term, and with a factor dependent on the difference of the chemical potential \( \mu \) of the vapor, and that \( \mu(N) \) of the condensate. As a result, equilibrium in the large \( N \) limit occurs at equality of the two chemical potentials.

During the process of BEC formation, the spectrum of eigenvalues makes a transition from the unperturbed spectrum of trap levels to the case where the spectrum is strongly affected by the condensate in the ground state. The Bogoliubov spectrum of a condensed gas is valid in the case where the number of particles in the condensate, \( n_0 \), is so large that it is valid to write \( n_0 \approx N \). Thus, during the initial stages of condensate formation, where this is not true, one must use another formalism. In this paper we will consider the situation in which the interaction between the particles is very weak, as is in practice the case. This means that we will be able to use the unperturbed spectrum for the initial stages of condensation, and only use the Bogoliubov description once enough condensate has formed to make the effective interaction rather stronger.

The basic formalism of Fig.2 can still be carried out in this case, and the modification that is found is rather minor—essentially, we make the substitution \( N \rightarrow n_0 \) in the chemical potential and the \( W^+(N), W^{++}(N) \) functions, and set \( W_m^+ \rightarrow 0 \), since this term comes from the mixing of creation and annihilation operators which arises from the Bogoliubov method. In order to simplify the equations we also group the levels in narrow bands of energy with \( g_k \) levels per group, and for simplicity use the same notation \( n_k \) now for the number of particles in the energy band with mean energy \( \epsilon_k \). (This corresponds to the ergodic assumption used in Fig.2.) We then deduce

\[
\dot{n}_m |_{\text{growth}} = 2W_m^+(n_0) \left\{ 1 - e^{\frac{\mu(n_0) - \mu + e_m}{kBT}} \right\} n_m + g_m \}, \quad (5)
\]

\[
\dot{n}_0 |_{\text{growth}} = 2W_0^+(n_0) \left\{ 1 - e^{\frac{\mu(n_0)}{kBT}} \right\} n_0 + 1. \quad (6)
\]

The growth equations can be modified to include the terms derived in Fig.2 corresponding to the scattering of particles in the condensate band by the vapor particles. This is equivalent to scattering of particles by a heat bath, which leads to a rate equation for scattering of the
form (Where $\tilde{N}_{km} = 1/(\exp[(e_k - e_m)/k_B T] - 1)$, and by $k > m$ we mean $e_k > e_m$)

$$\gamma_{mk} = \sum_{k<m} \gamma_{mk}\{\tilde{N}_{mk}n_k(n_m + g_m) - (\tilde{N}_{mk} + 1)(n_k + g_k)n_m\}$$

$$+ \sum_{k>m} \gamma_{km}\{(\tilde{N}_{km} + 1)n_k(n_m + g_m) - \tilde{N}_{km}(n_k + g_k)n_m\}.$$ (7)

The formulae of QKIII give precise methods for computing the coefficients $\gamma_{km}$, but we can simplify their computation by adapting the kinetic equation of Holland et al. [6]. This methodology is based on a model in which the trap levels are all treated as being unaffected by the presence of the condensate, which should suffice as a first approximation. To apply it to this situation, we assume all the levels with energies greater than $E_{\text{max}}$ are thermalized, and sum out over these levels. The working is essentially straightforward, and yields an equation for the $n_m$ variables in the form (with $M$ the mass of the atom and $a$ the scattering length)

$$\dot{n}_m|_{\text{scatt}} = \frac{8Ma^2\omega^2}{\pi\hbar}e^{\mu/k_B T}\Gamma(T) \times$$

$$\left\{\sum_{k<m} \frac{1}{g_m}\left[n_k(g_m + n_m)e^{-\hbar\omega_{mk}/k_B T} - n_m(g_k + n_k)\right]\right\}$$

$$+ \sum_{k>m} \frac{1}{g_k}\left[n_k(g_m + n_m) - n_m(g_k + n_k)e^{-\hbar\omega_{km}/k_B T}\right].$$ (8)

where $\Gamma(T) \equiv \sum_{e_m>e_{\text{max}}} e^{-e_m/k_B T}$ has a value which depends on the spectrum of energies. For an isotropic 3-dimensional harmonic oscillator with frequency $\omega$, the energy levels above the zero point are $e_n = n\hbar\omega$, so that we find

$$\Gamma(T) = \frac{e^{-E_{\text{max}}/k_B T}}{1 - e^{-\hbar\omega/k_B T}}.$$ (9)

This corresponds to essentially to (7) when one makes the correspondences

$$\tilde{N}_{km} \rightarrow e^{-(e_k - e_m)/k_B T}, \quad 1 + \tilde{N}_{km} \rightarrow 1,$$ (10)

$$\gamma_{km} \rightarrow \gamma_{mk} \rightarrow 8Ma^2\omega^2e^{\mu/k_B T}\Gamma(T)/g_k \quad \text{with} \ k > m.$$ (11)

The equation for both growth and scattering is now given by adding (8) to (10)

$$\dot{n}_m = \dot{n}_m|_{\text{growth}} + \dot{n}_m|_{\text{scatt}},$$ (12)

where, for $m = 0$, we use (8) instead of (10).

The overall evolution of the system can now be found from the numerical solutions to (12). The parameters used were chosen so as to be in approximate agreement with the experimental work being conducted at MIT, where the growth of Bose-Einstein condensates of $^{23}$Na is being studied. In contrast to the estimate in (7) in which the bath distribution was approximated by a Maxwell-Boltzmann distribution, in this computation we use the full Bose-Einstein distribution, truncated at a lower energy of $E_{\text{max}}$, since lower energies are described by the $n_m$ variables.

In applying the theory two major approximations are made. Firstly, the $W_{m+}^{+}(N)$ functions were assumed to be equal to the $W_+^{+}(N)$ function, since the actual $W_{m}^{+}(N)$ terms are not easily calculated. The justification for this is that the $W_{m}^{+}(N)$ terms represent an averaging over all the levels contained in the $m$th group, and as such they may be expected to be of the same order of magnitude as $W_{+}^{+}(N)$. As a validity check, it was found that the effect on the condensate growth rate was small when the $W_{m}^{+}(N)$ terms were altered by a factor in the range $0.5 - 2$.

In current BEC experiments the confining harmonic potential is normally anisotropic, whereas (8) was restricted to an isotropic trap. The second approximation is therefore that the scattering rate factor $\Gamma(T)$ is equal to that for an isotropic 3D harmonic oscillator with frequency equal to the geometric mean frequency of the anisotropic trap. The precise value of this factor was found to have little effect on the solutions, so long as it was greater than about one tenth of the value given by (9).

The condensate rate of growth depends on the number of groups of levels considered in the model—modeling more groups in the condensate band increases the rate of growth, which approaches a limiting value. The number of groups modeled was therefore chosen as large as possible, but it was required that there were at least 4 levels in the first group of levels above the condensate level.

Fig. 3: Condensate growth for Sodium: Dotted line—Growth of the condensate for the uncorrected model of (8); Dashed line—scattering is neglected, but $W_{+}^{+}$ is given by using the Bose-Einstein correction; Solid line—with scattering and the Bose-Einstein correction. In all cases the initial amount of condensate at $t = 0$ was 500 atoms.

In all cases the initial amount of condensate at $t = 0$ was 500 atoms. The initial populations for the groups in the condensate band were generated by a method which models the experimental procedure. We start at some initial $t \ll 0$ with the bath at a chemical potential $\mu \approx 0$, and evolve
the equations of motion until equilibrium is reached; at \( t = 0 \) we then change \( \mu \) to a positive value such that \( \mu = \mu(N_{\text{final}}) \), where \( N_{\text{final}} \) is the final number of atoms in the condensate. Changing the populations at \( t = 0 \) merely moves the growth curve slightly forward or backward in time, with no effect on its basic shape. We present a sample of the results obtained in Fig.3. The trap parameters, \( \nu_x = 18.5\,\text{Hz}, \nu_y = \nu_z = 250\,\text{Hz}, \) temperatures and final condensate number are chosen in the range presently being investigated for sodium.

The typical behavior of the noncondensate levels is shown in Fig.4, for the case where the initial occupation of all groups below \( E_R \) is chosen to be zero, in order to show the speed of the relaxation process. The initial growth is in the population of the noncondensate levels—their occupations can become very large, but this is because the numbers of levels in each group are very large, so that there is little degeneracy, i.e., the number of atoms per level does not significantly exceed one. Even in the lowest group the degeneracy is no more that about 100. The moment the condensate achieves a significant degeneracy, the stimulated process takes over, and immediately draws the excess population of the noncondensate groups into the condensate, well before the full condensate occupation is achieved. Thus, apart from the initial transient, the condensate growth occurs essentially by the same mechanism as in [1], with the modification that the distribution over the noncondensate levels changes slowly in response to the change of the condensate chemical potential.

If scattering is entirely neglected the populations of the lowest noncondensate levels become several times larger than those of the the condensate before settling to their very much lower equilibrium values. However, the inclusion of even as little as 0.1% of the strength of scattering used here eliminates that effect almost entirely.

![Fig.4: Growth of noncondensate levels—the condensate number is the almost vertical black line.](image)

In contrast to our work, in which explicit use is made of trap eigenfunctions, other descriptions [7–9] of condensate growth either do not treat the case of a trapping potential, or consider only the case where trapped situation is not essentially different from the untrapped situation. Nevertheless, the equations we use have a strong connection with those of Stoof [9], but their interpretation is different, since Stoof’s Fokker-Planck equation treats the untrapped case, where the low-lying levels are extremely closely spaced, and relative coherences between these levels become important. The current experiments are probably closer to the situation of our model, since the growth rate is somewhat slower than the lowest trap frequency.

We can conclude from the results of this model that the main effect of the inclusion of the scattering and the computation of \( W^+ \) using the full Bose-Einstein formula is to speed up the condensate growth by up to one order of magnitude, the precise speedup depending on the condensate size and temperature. Even though we have only estimated the scattering and transition rates for the quasiparticle levels, the results are reasonably predictive, since it would be hard to credit the scattering or the \( W^+ \) factors as being very different from the values assumed, and also because the numerical predictions are not very sensitive to these precise values. Precise predictions will involve the detailed computation of these effects, rather than their estimation.

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