TRIANGULAR BASES IN QUANTUM CLUSTER ALGEBRAS AND MONOIDAL CATEGORIFICATION CONJECTURES

FAN QIN

To the memory of Professor Andrei Zelevinsky

ABSTRACT. We consider the quantum cluster algebras which are injective-reachable and introduce a triangular basis in every seed. We prove that, given some initial conditions, there exists a unique common triangular basis for all seeds. This basis is parametrized by tropical points as expected in the Fock-Goncharov conjecture.

As an application, we prove the existence of the common triangular bases for the quantum cluster algebras arising from representations of quantum affine algebras and partially for those arising from quantum unipotent subgroups. This result implies monoidal categorification conjectures of Hernandez-Leclerc or Fomin-Zelevinsky in the corresponding cases.

CONTENTS

1. Introduction 1
2. Preliminaries on additive categorifications 6
3. Dominance order and pointed sets 12
4. Correction technique 17
5. Injective pointed sets 20
6. Triangular bases 28
7. Graded quiver varieties and minuscule modules 44
8. Facts and conjectures about monoidal categorifications 54
9. Monoidal categorification conjectures 65
References 77

1. INTRODUCTION

1.1. Background. Cluster algebras were invented by Sergey Fomin and Andrei Zelevinsky around the year 2000 in their seminal work [FZ02]. These are commutative algebras with generators defined recursively called cluster variables. The quantization was later introduced in [BZ05]. Fomin and Zelevinsky aimed to develop a combinatorial approach to the canonical bases in quantum groups (discovered by Lusztig
[Lus90] and Kashiwara [Kas90] independently) and the theory of total positivity in algebraic groups (by Lusztig [Lus93][Lus94]). They conjectured that the cluster structure should serve as an algebra framework for the study of the “dual canonical bases” in various coordinate rings and their $q$-deformations. In particular, they proposed the following conjecture.

**Conjecture.** _All monomials in the variables of any given cluster (the cluster monomials) belong to the dual canonical basis._

This claim would imply the positivity conjecture of the Laurent phenomenon of skew-symmetric cluster algebras, which was recently proved by Lee and Schiffler [LS13] by elementary algebraic computation.

It soon turns out that (quantum) cluster algebras are related to many other areas. For example, by using cluster algebras, Bernhard Keller has proved the periodicity conjecture of $Y$-systems in physics, cf. [Kel13]. We refer the reader to B. Keller’s introductory survey [Kel12] for a review of applications of cluster algebras.

However, despite the success in other areas, the original motivation to study cluster algebras remains largely open. In previous literature, the following bases have been constructed and shown to contain the (quantum) cluster monomials.

(i) By using preprojective algebras, cf. [GLS11], Geiss, Leclerc, and Schröer have shown that, if $G$ is a semi-simple complex algebraic group and $N \subset G$ a maximal nilpotent subgroup, then the coordinate algebra $\mathbb{C}[N]$ admits a canonical cluster algebra structure. They further established the generic basis of this algebra, which contains the cluster monomials. As an important consequence, they identified the generic basis with Lusztig’s dual semi-canonical basis, which is a variant of the dual canonical basis, cf. [Lus00]. By [Pla13], this basis is parametrized by tropical points as expected by the Fock-Goncharov conjecture [FG06].

(ii) For commutative cluster algebras arising from triangulated surfaces, Musiker, Schiffler and Williams constructed various bases, cf. [MSW13]. Their positivity were discussed in [Thu13]. Fock and Goncharov also constructed “canonical bases” in [FG06].

(iii) For (quantum) cluster algebras of rank 2, Kyungyong Lee, Li Li, Dylan Rupel, A. Zelevinsky introduced (quantum) greedy bases, cf. [LLZ14] [LLRZ14a][LLRZ14b].

As a new approach to cluster algebras, David Hernandez and Bernard Leclerc found a cluster algebra structure on the Grothendieck ring of finite dimensional representations of quantum affine algebras in [HL10]. They proposed the following monoidal categorification conjecture.

**Conjecture.** _All cluster monomials belong to the basis of the simple modules._
Partial results in type $A$ and $D$ were due to [HL10] [HL13b]. Inspired by the work of Hernandez and Leclerc, Hiraku Nakajima used graded quiver varieties to study cluster algebras [Nak11] and verified the monoidal categorification conjecture for cluster algebras arising from bipartite quivers. His work was later generalized to all acyclic quivers by Yoshiyuki Kimura and the author [KQ12] (cf. [Lee13] for a different and quick proof).

By the work of Khovanov, Lauda [KL09], and Rouquier [Rou08], a quantum group $U_q(n)$ admits a categorification by the modules of quiver Hecke algebras, in which the dual canonical basis vectors correspond to the finite dimensional simple modules. Therefore, Fomin and Zelevinsky’s conjecture can also be viewed a monoidal categorification conjecture of the corresponding (quantum) cluster algebras.

1.2. Construction, results and comments. We shall consider quantum cluster algebras $\mathcal{A}$ that are injective-reachable (Assumption 2), by which we mean, for any principal quiver $Q(t)$ (with a generic potential) of a seed $t$, we can obtain the “tilting object” formed by the injective right-modules of its Jacobian algebra via a mutation sequence $\Sigma$, cf. Section 5.1. Starting from the new seed $\Sigma t$, also denoted by $t[1]$, we can again consider the corresponding injective modules and obtain a new seed $t[2]$. Repeating this process, we obtain the notion of an injective-reachable chain of seeds $\{t[d]\}_{d \in \mathbb{Z}}$.

Fix any seed $t$, we define the “pointed set” $\Pi^t$ consisting of normalized products of quantum cluster variables in $t$ and $t[1]$. The triangular basis $L^t$ for the seed $t$, if it exists, is defined to be the unique bar-invariant basis such that every element of $\Pi^t$ has a $\prec_t$-unitriangular decomposition into $L^t$, where $\prec_t$ is certain order and $m = q^{-\frac{1}{2}} \mathbb{Z}[q^{-\frac{1}{2}}]$, cf. Section 3.1 5.3.

Our existence theorem 6.4.1 states that if there exists a common triangular basis in the seeds $\{t[d]\}_{d \in \mathbb{Z}}$, such that it has positive structure constants, satisfies certain local conditions, and its parametrization by the tropical points on these seeds are compatible with the tropical transformations, then this basis lifts to the unique common triangular basis $L$ with the same properties for all seeds.

By construction, the common triangular basis $L$ contains all the quantum cluster monomials. Therefore, if we can show that the basis of the simple modules (or the dual canonical basis) produces the common triangular basis, the monoidal categorification conjecture is verified.

Notice that the seeds $t[d]$ for all $d$ have isomorphic principal quivers. Consequently, we have the reduced existence theorem 6.4.7 as one possible reduction of the existence theorem 6.4.1, whose crucial part is the finite criterion that the cluster variables along the mutation sequence from $t$ to $t[1]$ and $t[-1]$ are contained in the triangular basis $L^t$ of $t$. 
As applications, we prove the existence of the common triangular basis in the following cases, cf. Theorem 9.4.1 9.4.2 9.4.3.

(i) Assume that $\mathcal{A}$ arises from the quantum unipotent subgroup $A_q(n(w))$ associated with a reduced word $w$ (called type (i)). If the Cartan matrix is of Dynkin type $ADE$, or if the word $w$ is inferior to an adaptable word under the left or right weak order, then the dual canonical basis gives the common triangular basis after normalization and localization at the frozen variables.

(ii) Assume that $\mathcal{A}$ arises from representations of quantum affine algebras (called type (ii)). Then, after localization at the frozen variables, the basis of the simple modules gives the common triangular basis.

We also find an easy proof of Conjecture 8.3.3 about cluster structures on all integral quantum unipotent subgroups, cf. Proposition 9.1.3.

Remark 1.2.1 (Key points in the proofs). The existence theorem 6.4.1 is established by elementary algebraic computation and the combinatorics of cluster algebras. We control the triangular bases by the positivity of their structure constants as well as their expected compatibility with the tropical transformations. On the one hand, the expected compatibility implies that the new quantum cluster variables obtained from one-step mutations are contained in the triangular basis of the original seed, cf. Example 6.2.4 Proposition 6.2.3. On the other hand, if these new quantum cluster variables are contained in the original triangular basis, we construct the triangular bases in the new seeds which are compatible with the original one, cf. Proposition 6.3.5.

When we consider applications to a quantum cluster algebra $\mathcal{A}$ of type (i) or (ii), most initial conditions are obtained by induction based on the properties of the standard basis (or dual PBW basis) and the $T$-systems. By the reduced existence theorem 6.4.7, it remains to verify that finite many quantum cluster variables are contained in the initial triangular basis $L^0$. Some of them are known to be Kirillov-Reshetikhin modules, cf. [Nak03][GLS13][HL13b]. We compare the rest with the characters of some variants of Kirillov-Reshetikhin modules in the language of graded quiver varieties, cf. Section 7.4. We simplify our treatment by freely changing coefficient pattern with the help of the correction technique introduced in [Qin13], cf. Section 4.

Remark 1.2.2. The idea of constructing the common triangular basis of a quantum cluster algebra first appeared in the work of Berenstein and Zelevinsky [BZ12], whose construction was over acyclic quivers. For general acyclic quivers, it is not clear if their construction agrees with the triangular basis in this paper or not.
Remark 1.2.3. The existence theorem is stated for quantum cluster algebras associated with skew-symmetrizable matrix under several assumptions, which essentially demand good properties of the combinatorics of cluster algebras, cf. Assumption 1.3.4.

Such properties are well known for the quiver case (the skew-symmetric case) thanks to the categorification by cluster categories. The recent paper [GHKK14] found lots of results about combinatorics of the skew-symmetrizable case, which might be useful in considering the corresponding triangular bases.

The recent paper by Gross, Hacking, Keel and Kontsevich [GHKK14] provided another positive basis of commutative cluster algebras parametrized by tropical points and a more precise form of the Fock-Goncharov dual basis conjecture was verified. Their work also implied the positivity of the Laurent phenomenon of commutative cluster algebras.

When the author was preparing the present article, Seok-Jin Kang, Masaki Kashiwara, Myungho Kim and Se-Jin Oh established an approach to cluster algebras on quantum unipotent subgroups by studying simple modules of quiver Hecke algebras, which reduced the verification of the corresponding monoidal categorification conjecture to one-step mutations in one seed [KKKO14]. This result allows them to verify the conjecture for quantum cluster algebras arising from quantum unipotent subgroups in a forthcoming paper.

1.3. Contents. In the first part of Section 2, we recall basic definitions and properties of quantum cluster algebras $\mathcal{A}$ and its categorifications (including the Calabi-Yau reduction). This part serves as a background for the construction of the common triangular basis.

In Section 3, we consider the lattice $D(t)$ of leading degrees for every seed $t$ of a cluster algebra together with its dominance order $\prec_t$. We then define and study pointed elements in the quantum torus $T(t)$, whose name is borrowed from [LLZ14]. We recall the tropical transformation expected by [FG09]. We end the section by a natural discussion of dominant degrees.

In Section 4, we recall and reformulate the correction technique introduced in [Qin13], which is easy but useful to keep track of equations involving pointed elements when the coefficients and quantization change.

In Section 5, we construct and study the injective pointed set $\mathcal{I}'$ consisting of normalized products of the cluster variables and the injectives in any given seed $t$.

In section 6, we introduce the triangular bases $L'_t$ of a quantum cluster algebra such that they are unitriangular to the injective pointed sets $\mathcal{I}'$ respectively. We present local conditions on $t$, such as positivity and unitriangularity of the basis $L'_t$ (Section 6.1). As a crucial step, we study one step mutations (Section 6.2). Then we discuss when
the triangular basis in one seed becomes the triangular basis in a new seed (Section 6.3). We conclude the section by proving the existence theorem 6.4.1 for the common triangular basis.

We devote the rest of the paper to the existence and applications of the common triangular bases.

In Section 7, we review the theory of graded quiver varieties for acyclic quivers and the associated Kirillov-Reshetikhin modules. As the main result, we compute $qt$-characters of a special class of simple modules. These slightly generalize Nakajima’s notations and results [Nak04][Nak03]. A reader unfamiliar with this topic might skip this section and refer to Nakajima’s works.

In Section 8, we review the notion of monoidal categorifications, the quantum cluster algebras $\mathcal{A}(i)$ associated with a word $i$, and the monoidal categorification conjecture in type $(i)$. We then introduce the quantum cluster structure $\mathcal{A}(i)$ on a subring $\mathcal{R}(i,a)$ of the graded Grothendieck ring over graded quiver varieties, where the word $i$ and grading $a$ are adaptable. We present the corresponding monoidal categorification conjecture.

In section 9, we show that the basis of the simple modules produces the initial triangular bases of the corresponding quantum cluster algebra $\mathcal{A}(i)$ after localization. We verify local conditions in the initial seed. In order to show that it lifts to the common triangular basis, it remains to check that certain cluster variables are contained in this basis. We mainly work in the case when $i$ is the power $c^{N+1}$ of an acyclic Coxeter word $c$. Consequences are summarized in the end.

ACKNOWLEDGEMENTS

Special thanks are due to Bernhard Keller and Christof Geiss for their encouragements. The author is grateful to Hiraku Nakajima and Yoshiyuki Kimura for important discussion and comments. He thanks David Hernandez, Pierre-Guy Plamondon, and Pavel Tumarkin for interesting discussion on quantum affine algebras, cluster algebras and Coxeter groups. He also thanks Andrei Zelevinsky and Dylan Rupel for suggesting the name “pointed element”. He thanks Yoshiyuki Kimura, David Hernandez and Bernhard Keller for remarks on a previous version of this paper.

The author thanks the Mathematical Sciences Research Institute at Berkeley for the financial support, where part of this work was done during his stay in 2012.

2. PRELIMINARIES ON ADDITIVE CATEGORIFICATIONS

2.1. Quantum cluster algebras. We review the definition of quantum cluster algebras introduced by [BZ05]. Our notations will be similar to those in [Qin12][KQ12].
Let $m \geq n$ be two non-negative integers. Let $\tilde{B}$ be an $m \times n$ integer matrix. Its entry in position $(i, j)$ is denoted by $b_{ij}$. We define the principal part and the coefficient pattern of the matrix $\tilde{B}$ to be its upper $n \times n$ submatrix $B$ and lower $(m - n) \times n$ submatrix respectively. We shall always assume the matrix $\tilde{B}$ to be of full rank $n$ and its principal part matrix skew-symmetrizable.

A quantization matrix $\Lambda$ is an $m \times m$ skew-symmetric integer matrix.

**Definition 2.1.1 (Compatible pair).** The pair $(\Lambda, \tilde{B})$ is called compatible, if their product satisfies

$$(1) \quad \Lambda \cdot (\tilde{B}) = \begin{pmatrix} D & 0 \\ 0 & \end{pmatrix}$$

for some $n \times n$ diagonal matrix $D = \text{Diag}(d_1, d_2, \ldots, d_n)$ whose diagonal entries are strictly positive integers.

By [BZ05], the product $DB$ is skew symmetric.

The quantization matrix $\Lambda$ gives the bilinear form $\Lambda(\cdot, \cdot)$ on the lattice $\mathbb{Z}^m$ such that $\Lambda(g, h) = g^T \Lambda h$ for $g, h \in \mathbb{Z}^m$. Here we use $(\cdot)^T$ to denote the matrix transposition.

Let $q$ be an indeterminate. Define $q^{\frac{1}{2}}$ to be its square root such that $(q^{\frac{1}{2}})^2 = q$. The quantization matrix $\Lambda$ makes the usual Laurent polynomial ring into a quantum torus.

**Definition 2.1.2.** The quantum torus $T$ is the Laurent polynomial ring $\mathbb{Z}[q^{\pm \frac{1}{2}}][X_1^{\pm 1}, \ldots, X_m^{\pm 1}]$ endowed with the twisted product $\ast$ such that

$$X^g \ast X^h = q^{\frac{1}{2} \Lambda(g, h)} X^{g+h},$$

for any $g, h \in \mathbb{Z}^m$.

We denote the usual product of $T$ by $\cdot$. The notation $X^g$ denotes the monomial $\prod_{1 \leq i \leq m} X_i^{g_i}$ of the $X$-variables $X_i$ with respect to the usual product. Recall that the $Y$-variables $Y_k$, $1 \leq k \leq n$, are defined as $X^{\tilde{B}e_k}$, where $e_k$ is the $k$-th unit vector in the lattice $\mathbb{Z}^m$. The monomials $Y^v$, $v \in \mathbb{N}^n$, are similarly defined as $\prod_k Y_k^{v_k}$.

By [BZ05], the quantum torus $T$ is contained in its skew-field of fractions $\mathcal{F}$.

The bar involution $\overline{\cdot}$ on $T$ is the anti-automorphism of $T$ sending $q^{\frac{1}{2}}$ to $q^{-\frac{1}{2}}$ and $X^g$ to $X^g$.

A sign $\epsilon$ is an element in $\{-1, +1\}$. For any $1 \leq k \leq n$ and sign $\epsilon$, we associate to $\tilde{B}$ the $m \times m$ matrix $E_\epsilon$ whose entry at position $(i, j)$ is

$$e_{ij} = \begin{cases} -1 & \text{if } i = j = k \\ \max(0, -\epsilon b_{ik}) & \text{if } i \neq k, j = k \\ \delta_{ij} & \text{if } j \neq k \end{cases}.$$
and the \( n \times n \) matrix \( F \), whose entry at position \((i, j)\) is

\[
f_{ij} = \begin{cases} 
-1 & \text{if } i = j = k \\
\max(0, \epsilon b_{kj}) & \text{if } i = k, j \neq k \\
\delta_{ij} & \text{if } i \neq k
\end{cases}.
\]

We only consider triples \((\Lambda, \tilde{B}, X)\) such that \((\Lambda, \tilde{B})\) is a compatible pair and \(X = \{X_1, \ldots, X_m\}\) the \(X\)-variables in the corresponding quantum torus \(\mathcal{T}\).

Let \([\ ]_+\) denote the function \(\max\{ , 0\}\).

**Definition 2.1.3** (Mutation [FZ02] [BZ05]). Fix a sign \(\epsilon\) and an integer \(1 \leq k \leq n\). The mutation \(\mu_k\) on a triple \((\Lambda, \tilde{B}, X)\) is the operation that generates the new triple \((\Lambda', \tilde{B}', X')\):

\[
\tilde{B}' = E_i \tilde{B} E_i, \\
\Lambda' = E_i^T \Lambda E_i, \\
\begin{align*}
X_i' &= X_i, \quad \forall 1 \leq i \leq m, i \neq k, \\
X_k' &= X_k^{-1} \cdot \left( \prod_{1 \leq i \leq m} X_i^{[b_{ki}]_+} + \prod_{1 \leq j \leq m} X_j^{[b_{kj}]_+} \right).
\end{align*}
\]

The last equation above is called the *exchange relation* at the vertex \(k\). By this definition, the quantum torus \(\mathcal{T}\) and the new quantum torus \(\mathcal{T}'\) associated with the new triple share the same skew-field of fractions.

Let \((\Lambda, \tilde{B}, X)\) be a given triple. Let \(\mathbb{T}_n\) denote an \(n\)-regular tree with root \(t_0\) whose edges will be labeled by \(\{1, \ldots, n\}\). There exists a unique way of associate a triple \((\Lambda(t), \tilde{B}(t), X(t))\) with each vertex \(t\) of \(\mathbb{T}_n\) such that

(i) \((\Lambda(t_0), \tilde{B}(t_0), X(t_0)) = (\Lambda, \tilde{B}, X)\).

(ii) If two vertices \(t\) and \(t'\) are related by an edge labeled \(k\), then the triple \((\Lambda(t'), \tilde{B}(t'), X(t'))\) is obtained from \((\Lambda(t), \tilde{B}(t), X(t))\) via the mutation \(\mu_k\).

We call the triples \((\Lambda(t), \tilde{B}(t), X(t))\) the quantum seeds. The \(X\)-variables \(X_i(t), 1 \leq i \leq m\), are called quantum cluster variables. The quantum cluster monomials are the monomials of quantum cluster variables from the same seed. Notice that, when \(j > n\), the cluster variables \(X_j(t)\) remain the same for all seeds \(t\). Consequently, we call them the frozen variables or coefficients and simply denote them by \(X_j\). The vertices \(1, 2, \ldots, n\) are said to be exchangeable and form the set \(\mathbb{E}\). Let \(\mathbb{Z}[q^{ \frac{1}{\epsilon} \pm}]\) denote the coefficient ring \(\mathbb{Z}[q^{ \frac{1}{\epsilon} \pm}]X_j^{\pm}X_{n<j \leq m}\). The frozen variables commute with any quantum cluster variable up to to a power of \(q^{\frac{1}{\epsilon}}\), which situation is called q-commute or quasi-commute.

Given any quantum seed parametrized by a vertex \(t\), we call it the seed \(t\) for simplicity. We also use \(\tilde{B}^t\) and \(\Lambda^t\) to denote \(\tilde{B}(t)\) and \(\Lambda(t)\) respectively.
Definition 2.1.4. The quantum cluster algebra $\mathcal{A}$ is the $\mathbb{Z}[q^{\pm \frac{1}{2}}]$-subalgebra of $\mathcal{F}$ generated by the quantum cluster variables and the inverses of the frozen variables, where we use the twisted product.

The specialization $\mathcal{A}|_{q^{\frac{1}{2}}=1}$ gives us the commutative cluster algebra, which is denoted by $\mathcal{A}_{\mathbb{Z}}$. We also define the quantum cluster algebra $\mathcal{A}^t$ as the subalgebra of $\mathcal{A}$ generated by the quantum cluster variables $X_i(t)$, $1 \leq i \leq m$.

The quantum cluster algebra is contained in the quantum torus $\mathcal{T}(t)$ for any seed $t$, cf. the quantum Laurent phenomenon [FZ02][BZ05].

Fix the initial seed $t_0$. For any given quantum cluster variable, we can define its extended $g$-vector $\tilde{g}_i(t)$ in $\mathbb{Z}^m$, cf. [Tra11] [FZ07]. Its restriction to the first $n$ components is called the corresponding $g$-vector, which is the grading of the cluster variable $X_i(t)$ when the $B$-matrix is of principal coefficients type, namely, when

$$\tilde{B}(t_0) = B^p(t_0) = \begin{pmatrix} B(t_0) \\ 1_n \end{pmatrix}.$$ 

We refer the reader to the survey [Kel12] for more details on the combinatorics of cluster algebras.

2.2. Quivers and cluster categories. We review the necessary notions on the additive categorification of a cluster algebra by cluster categories, more details could be found in [Kel08] [Pla11b].

Let $\tilde{B}$ be the $m \times n$ matrix introduced before. Assume that its principal part $B$ is skew-symmetric.

Then we can construct the quiver $\tilde{Q} = \tilde{Q}(\tilde{B})$. This is an oriented graph with the vertices $1, 2, \ldots, m$ and $[b_{ij}]_+$ arrows from $i$ to $j$, where $1 \leq i, j \leq m$. We call $\tilde{Q}$ an ice quiver, its vertices $1, \ldots, n$ the exchangeable vertices, and its vertices $n+1, \ldots, m$ the frozen vertices. Its principal part is defined to be the full subquiver $Q = Q(B)$ on the vertices $1, 2, \ldots, n$. For any given arrow $h$ in a quiver, we let $s(h)$ and $t(h)$ to denote its start and terminal respectively.

The quiver $Q$ is called acyclic if it contains no oriented cycles, and bipartite if any of its vertices is either a source point or a sink point.

Example 2.2.1. Figure 1 provides an example of an ice quiver, where we use the diamond nodes to denote the frozen vertices.

We fix the base field to be the complex field $\mathbb{C}$. Choose a generic potential $\tilde{W}$ for the ice quiver $\tilde{Q}$ in the sense of [DWZ08]. Associate the Ginzburg algebra $\Gamma(Q, \tilde{W})$ to the quiver with potential $(\tilde{Q}, \tilde{W})$. The restriction of $(\tilde{Q}, \tilde{W})$ to the vertices $[1, n]$ is called the principal part and denoted by $(Q, W)$.
Claire Amiot introduced the generalized cluster category $\mathcal{C}$ in [Ami09] as the quotient category

$$\mathcal{C} = \text{per } \Gamma / \mathcal{D}_{fd} \Gamma,$$

where $\text{per } \Gamma$ is the perfect derived category of dg-modules $\Gamma$ and $\mathcal{D}_{fd} \Gamma$ its full subcategory whose objects have finite dimensional homology.

The natural functor $\pi : \text{per } \Gamma \rightarrow \mathcal{C}$ gives us

$$T_i = \pi(e_i \Gamma), \quad \forall 1 \leq i \leq m,$$

$$T = \oplus T_i.$$

By Pierre-Guy Plamondon’s work [Pla11b], the presentable cluster category $\mathcal{D}$ is defined as the full subcategory of $\mathcal{C}$ which consists of objects $M$ such that there exist triangles

$$M_1 \rightarrow M_0 \rightarrow M \rightarrow SM_1,$$

$$M^0 \rightarrow M^1 \rightarrow S^{-1}M \rightarrow SM^0,$$

with $M_1, M_0, M^0, M^1$ in $\text{add } T$, and $S$ is the shift functor in $\mathcal{C}$. Following [Pal08], we define the index and coindex of $M$ as

$$\text{Ind}^T M = [M_0 : T] - [M_1 : T],$$

$$\text{Coind}^T M = [M^0 : T] - [M^1 : T],$$

where $[ : T]$ denote the multiplicities of an object in $\text{add } T$ with respect to the indecomposables $T_i$, $1 \leq i \leq m$. Notice that $\text{Coind}^T M = -\text{Ind}^T S^{-1}M$.

Let $F$ denote\(^1\) the functor $\text{Ext}^1_C(T, \ )$ from $\mathcal{D}$ to the category of right modules of $\text{End}_C T$. Then, similar to [Pal08, Lemma 2.1 2.3], [Pla11b, Lemma 3.6, Notation 3.7], we have

$$\text{Ind}^{S^{-1}T} S^{-1} M + \text{Ind}^T S^{-1} M = -\tilde{B} \cdot \dim F(S^{-1}M). \quad (2)$$

An object $M$ in $\mathcal{C}$ is said to be coefficient-free if

(i) it contains no direct summand $T_j$ for $j > n$, and 
(ii) the module $FM$ is supported at the principal quiver $Q$.

\(^1\)Our functor $F$ is the composition of $S$ with the functor $F$ in [Pla11b].
By [DWZ10] [Pla11b], we define the cluster characters for coefficient-free objects \( M \) in \( \mathcal{D} \) as
\[
x_M = x^{\text{Ind} M}(\sum_{e \in \mathbb{N}^n} \chi(\text{Gr}_e FM) Y^e),
\]
where \( \text{Gr}_e FM \) denotes the variety of the \( e \)-dimensional submodules of \( FM \) and \( \chi \) the Euler-Poincaré characteristic.

By [Pla11b], there exists a unique way of associating an object \( T(t) = \bigoplus_{1 \leq i \leq m} T_i(t) \) in \( \mathcal{D} \) with each seed \( t \) such that

1. \( T(h_0) = T \),
2. if two vertices \( t \) and \( t' \) are related by an edge labeled \( k \), then the object \( T(t') \) is obtained from \( T(t) \) by replacing one summand in a unique way.

Theorem 2.2.2 ([DWZ10][Pla11b][Nag13]). The commutative cluster variables \( X_i(t)|_{q \rightarrow 1} \) agree with the cluster characters \( x_{T_i(t)} \).

2.3. Calabi-Yau reduction. In this subsection, we assume that that the Jacobian algebra of the quiver with potential \((\tilde{Q}, \tilde{W})\) is finite dimensional. Consider the full subcategory of \( \mathcal{U} \) consisting of the coefficient-free objects. Take the ideal \( \Gamma_F \) consisting of the morphisms factoring through some \( T_j, j > n \). Then, by [IY08][Pla13], \( \mathcal{U} = \mathcal{U}/(\Gamma_F) \) is equivalent to the generalized cluster category of \((Q, W)\).

For any coefficient-free object \( X \), denote its image in \( \mathcal{U} \) by \( \overline{X} \). Let \( \Sigma \) denote the shift functor in \( \mathcal{U}/(\Gamma_F) \). Any given \( \Sigma T_k, 1 \leq k \leq n \), has the lift \( \Sigma T_k \) by [IY08]. Define \( \Sigma T = \bigoplus_{k=1}^n \Sigma T_k \) and \( \bigoplus_{n<j \leq m} T_j \).

For any given \( 1 \leq k \leq n \), by taking \( S^{-1}M = \Sigma^{-1}T_k \) in (2), we get
\[
\text{Ind}^{S^{-1}T} \Sigma^{-1}T_k + \text{Ind}^T(\Sigma^{-1}T_k) = -\tilde{B}(T) \cdot \text{Ext}^1_{C}(T, \Sigma^{-1}T_k).
\]

Lemma 2.3.1. We have \( \text{Ind}^{S^{-1}T} \Sigma^{-1}T_k = -\text{Ind}^{\Sigma^{-1}T} T_k \).

Proof. \( T_k \) fits into a triangle of the following type ([Pla13, Lemma 3.15]):
\[
\Sigma^{-1}T_k \rightarrow T_F \rightarrow T_k \rightarrow S\Sigma^{-1}T_k,
\]
where \( T_F \) is some direct sum of \( T_j, j > n \). Rewrite it as the triangle
\[
S^{-1}T_F \rightarrow S^{-1}T_k \rightarrow \Sigma^{-1}T_k \rightarrow T_F.
\]
The claim follows by definition. \( \square \)

The following equality follows as a consequence
\[
\text{Ind}^T(\Sigma^{-1}T_k) + \tilde{B}(T) \cdot \text{Ext}^1_{C}(T, \Sigma^{-1}T_k) = \text{Ind}^{\Sigma^{-1}T} T_k.
\]

Notice that \( \text{Ext}^1_{C}(T, \Sigma^{-1}T_k) \) is the \( k \)-th projective right-module of the Jacobian algebra principal of \((Q, W)\). We will later discuss this equality in Example 5.1.6 with more details.
3. Dominance order and pointed sets

3.1. Constructions. Let $t$ be any vertex of the $n$-regular tree $T_n$.

Define the degree lattice $D(t)$ to be $\mathbb{Z}^m$.

**Definition 3.1.1** (Dominance order). For any given $\tilde{g}', \tilde{g} \in D(t)$, we say $\tilde{g}' \prec_t \tilde{g}$ (or $\tilde{g}$ dominates $\tilde{g}'$) if we have $\tilde{g}' = \tilde{g} + B(t)v$ for some $0 \neq v \in \mathbb{N}^n$. We denote $\tilde{g}' \preceq_t \tilde{g}$ if $\tilde{g}' \prec_t \tilde{g}$ or $\tilde{g}' = \tilde{g}$.

**Lemma 3.1.2.** For any $\tilde{g}_1 \prec_t \tilde{g}_0$, there exists finitely many $\tilde{g}$ such that $\tilde{g}_1 \prec_t \tilde{g} \prec_t \tilde{g}_0$.

**Proof.** Notice that $\tilde{B}(t)$ is of full rank. The claim follows from definition. □

Recall that the quantum torus $\mathcal{T}(t)$ is the Laurent polynomial ring generated by $X_i(t)$, $1 \leq i \leq m$.

**Definition 3.1.3** (Leading term). For any Laurent polynomial $Z \in \mathcal{T}(t)$, the leading terms are those Laurent monomials of $Z$ such that, among all the Laurent monomials of $Z$ with non-zero coefficients, they have maximal multidegrees with respect to the order $\prec_t$.

**Definition 3.1.4** (Pointed element). If a Laurent polynomial $Z \in \mathcal{T}(t)$ has a unique leading term whose degree is $\eta$, we say its leading (or maximal) degree is $\deg^t Z = \eta$. If its leading degree has coefficient $q^\alpha$ for some $\alpha \in \frac{1}{2} \mathbb{Z}$, we define the normalization of $Z$ in the quantum torus $\mathcal{T}(t)$ to be $[Z]^t = q^{-\alpha}Z$.

We say $Z$ is pointed at $\eta$ (or $\eta$-pointed) if $Z$ has a unique leading term, and this term has degree $\eta$ with coefficient 1.

For simplicity, we often denote $\deg^t Z$ and $[Z]^t$ by $\deg Z$ and $[Z]$ respectively when the context is clear.

**Definition 3.1.5** (Pointed set). Let $W$ be a subset of $D(t)$. A subset $L$ of $\mathcal{T}(t)$ is said to be pointed at $W$ (or $W$-pointed) if it consists of pointed elements, and, if the map sending $Z \in L$ to $\deg^t Z \in D(t)$ is a bijection from $L$ to $W$.

We also use $L^t$ to denote the $D(t)$-pointed set $L$. For the element of $L$ with leading degree $\tilde{g} \in W$, denote it by $L(\tilde{g}; t)$, $L^t(\tilde{g})$, or simply $L(\tilde{g})$.

It immediately follows from the definition that any pointed set is $\mathbb{Z}[q^{\frac{1}{2}}]$-linearly independent.

**Example 3.1.6.** By [Pla13], for any commutative cluster algebra studied in [GLS11], the dual semicanonical basis, after localization at the frozen variables, is a $D(t)$-pointed set for any seed $t$.

Assume that we are given a collection of elements $Z_i$ in the quantum torus $\mathcal{T}(t)$ such that, at any Laurent monomial $m$, there are finitely
many $Z_i$ with non-zero coefficient $z_{i,m}$. Then we let the sum $\sum_i Z_i$ denote the element $\sum_m (\sum_i z_{i,m}) m$ in the completion of the quantum torus $\mathcal{T}(t)$.

**Definition 3.1.7** (Unitriangularity). Let $\mathbb{L}$ be a given $W$-pointed set, $W \subset D(t)$. Any pointed element $Z \in \mathcal{T}(t)$ is said to be $\prec_t$-unirangular to $\mathbb{L}$, if it has the following (possibly infinite) expansion into $\mathbb{L}$:

$$Z = \mathbb{L}(\deg Z; t) + \sum_{\vec{g} \prec \deg Z} b_{\vec{g}} \mathbb{L}(\vec{g}; t), \ b_{\vec{g}} \in \mathbb{Z}[q^{\pm \frac{1}{2}}].$$

If the coefficients $b_{\vec{g}}$ are contained in $\mathfrak{m} = q^{-\frac{1}{2}} \mathbb{Z}[q^{-\frac{1}{2}}]$, we say $Z$ is $(\prec_t, \mathfrak{m})$-unitriangular to $\mathbb{L}$.

A pointed subset of $\mathcal{T}(t)$ is said to be $\prec_t$-unitriangular (resp. $(\prec_t, \mathfrak{m})$-unitriangular) to $\mathbb{L}$ if all its elements have this property.

We might write the notation $\prec_t$ as $\prec$ or simply omit it, when this order is clear from the context.

**Remark 3.1.8.** It follows from Lemma 3.1.2 that the coefficients $b_{\vec{g}}$ in (4) are uniquely determined by $Z$. In fact, denote the coefficient of the Laurent expansion of $Z$ at any given degree $\vec{g}'' \prec_t \deg' Z$ by $\tilde{Z}_{\vec{g}''}$ and that of $\mathbb{L}(\vec{g})'$ by $a_{\vec{g}', \vec{g}''}$. We must have the following equation:

$$\tilde{Z}_{\vec{g}''} = 1 \cdot a_{\deg' Z, \vec{g}''} + \sum_{\vec{g}' \prec \vec{g}'' \prec \deg' Z} b_{\vec{g}'} a_{\vec{g}'', \vec{g}''} + b_{\vec{g}''} \cdot 1.$$

It follows that the coefficient $b_{\vec{g}''}$ is determined by those $b_{\vec{g}'}$ appearing. By Lemma 3.1.2, this is a finite algorithm for any given $\vec{g}''$.

In general, there could exist many infinite expansion of $Z$ into $\mathbb{L}$ if we don’t require them to take the unitriangular form (4).

**Lemma 3.1.9** (Expansion properties). Let $\mathbb{L}$ be any $D(t)$-pointed subset of $\mathcal{T}(t)$. Let $Z$ be any pointed element in $\mathcal{T}(t)$. Then the following statements are true.

(i) $Z$ is $\prec_t$-unitriangular to $\mathbb{L}$.

(ii) If $Z$ and $\mathbb{L}$ are bar-invariant and $Z$ is $(\prec_t, \mathfrak{m})$-unitriangular to $\mathbb{L}$, then $Z$ equals $\mathbb{L}(\deg Z; t)$.

(iii) If $Z$ a $\mathbb{Z}[q^{\pm \frac{1}{2}}]$-linear combination of finitely many elements of $\mathbb{L}$, then such a combination is unique and takes the unitriangular form (4).

**Proof.** (i)(ii) The statements are obvious.

(iii) Suppose that some $\mathbb{L}(\eta)$, $\eta \in D(t)$, appears in the expansion of $Z$ into $\mathbb{L}$ with non-zero coefficient $b(\eta)$, such that $\eta \neq_t \deg^t Z$. Because the expansion is finite, we can always find a degree $\eta$ such that it is maximal among those having this property. Then the coefficient of $X(t)^\eta$ of $Z$ equals $b(\eta) \neq 0$. But the coefficient of this Laurent monomial in the pointed element $Z$ must vanish. This contradiction shows that such $\eta$ cannot exist. \(\square\)
Lemma 3.1.10 (Inverse transition). Let $M$, $L$ be any two $(D(t))$-pointed subsets of $T(t)$. If $M$ is $(<_t, m)$-unitriangular to $L$, then $L$ is $(<_t, m)$-unitriangular to $M$ as well.

Proof. For any $\tilde{g} \in D(t)$, let us denote

$$M(\tilde{g}; t) = L(\tilde{g}; t) + \sum_{\tilde{g}^1 <_{_t} \tilde{g}} c_{\tilde{g}, \tilde{g}^1} L(\tilde{g}^1; t), \quad c_{\tilde{g}, \tilde{g}^1} \in m,$$

$$L(\tilde{g}; t) = M(\tilde{g}; t) + \sum_{\tilde{g}^1 <_{_t} \tilde{g}} b_{\tilde{g}, \tilde{g}^1} M(\tilde{g}^1; t).$$

Expand RHS of the second equation by using the first equation. Then the coefficient of any factor $L(\tilde{g}^2; t), \tilde{g}^2 <_{_t} \tilde{g}$, appearing in RHS is the following three terms’ sum

$$c_{\tilde{g}, \tilde{g}^2} + b_{\tilde{g}, \tilde{g}^2} + \sum_{\tilde{g}^2 <_{_t} \tilde{g}^1 <_{_t} \tilde{g}} b_{\tilde{g}, \tilde{g}^1} c_{\tilde{g}^1, \tilde{g}^2},$$

which must vanish.

We verify the claim by induction on the order of the degrees $\tilde{g}^2$ such that $\tilde{g}^2 <_{_t} \tilde{g}$. If $\tilde{g}^2$ is maximal among such degrees, the last term vanishes and we have $b_{\tilde{g}, \tilde{g}^2} \in m$. Assume $b_{\tilde{g}, (\tilde{g}^2)'} \in m$ for any $(\tilde{g}^2)'$ such that $\tilde{g}^2 <_{_t} (\tilde{g}^2)'$. Then the first and last term belong to $m$. Consequently, $b_{\tilde{g}, \tilde{g}^2} \in m$ belong to $m$ as well. \hfill \square

3.2. Tropical transformation.

Definition 3.2.1 (Tropical transformation, [FG09][FZ07, (7.18)]). For any $1 \leq k \leq n$, we consider the piecewise linear map $\phi_{\mu_k t}: D(t) \to D(\mu_k t)$, such that any given degree $\tilde{g} = (\tilde{g}_j)_{1 \leq j \leq m} \in D(t)$ has the image $\tilde{g}' = (\tilde{g}'_j)_{1 \leq j \leq m} = \phi_{\mu_k t}(\tilde{g})$ with the following components

$$\tilde{g}_k = -\tilde{g}_k,$$

$$\tilde{g}'_i = \tilde{g}_i + b_{ik}(t)[\tilde{g}_k]_+, \text{ if } 1 \leq i \leq m, \ i \neq k, \ b_{ik} \geq 0,$$

$$\tilde{g}'_j = \tilde{g}_j - b_{kj}(t)[-\tilde{g}_k]_+, \text{ if } 1 \leq j \leq m, \ j \neq k, \ b_{kj} \geq 0.$$

Remark 3.2.2. The piecewise linear map (5) is the tropicalization of the rule [FG09, (13)]. It describes the mutation of the tropical $\mathbb{Z}$-points $\mathcal{X}'(\mathbb{Z}^n) = D(t)$ of the $\mathcal{X}$-variety (the variety of the tropical $Y$-variables for the cluster algebra defined via $B(t)^T$). Fock and Goncharov conjectured that these tropical points parametrize a “canonical” basis, [FG06][FG09, Section 5].

On the cluster algebra side, Fomin and Zelevinsky wrote this transformation in [FZ07, (7.18)] and conjectured that it describes the transformation of the $g$-vectors of cluster variables. When the cluster algebra admits a categorification by a cluster category, this formula (5) can be interpreted as the transformation rule of the indices of object in the cluster category, cf. [DK07][KY11][Pla13]. We refer the reader to [FZ07, Remark 7.15] [Pla13, Section 3.5] [FG09] for more details.
The following observation is obvious.

**Lemma 3.2.3** (Sign coherent transformation). Assume that two vectors \( \eta^1, \eta^2 \) in \( D(t) \) are sign coherent at the \( k \)-th component, namely, we have simultaneously \( \eta^1_k, \eta^2_k \geq 0 \) or \( \eta^1_k, \eta^2_k \leq 0 \). The tropical transformation (5) is additive on them:

\[
\phi_{\mu \lambda} \eta^1 + \phi_{\mu \lambda} \eta^2 = \phi_{\mu \lambda} (\eta^1 + \eta^2).
\]

We shall keep the following assumption throughout this paper.

**Assumption 1** (Categorification). The combinatorics of the quantum cluster algebra \( A \) can be categorified. More precisely, there is a category \( C \), such that

(i) the cluster variables \( X_i(t), 1 \leq i \leq m, t \in T_n \), correspond to objects \( M_i(t) \) in \( C \).

(ii) Given any two seeds \( t^1, t^2 \) and any mutation sequence from \( t^1 \) to \( t^2 \), we define \( \phi_{t^2, t^1} \) to be the composition of the transformations (5) along this sequence. Then \( \phi_{t^2, t^1} \) is independent of the choice of the sequence.

Moreover, there exists an index map \( \text{Ind}^t \) from \( M_i(t) \) to \( \mathbb{Z}^m \) for every seed \( t \), such that the indices of the objects \( M_i(t), t \in T_n \), are related by the map \( \phi_{t^2, t^1} \):

\[
\text{Ind}^{t^2}(M_i(t)) = \phi_{t^2, t^1} \text{Ind}^{t^1}(M_i(t)).
\]

(iii) In any quantum torus \( T(t_0) \), the cluster variables \( X_i(t) \) has the following Laurent expansion

\[
X_i(t) = X(t_0)^{\text{Ind}^{\text{int}}M_i(t)}(1 + \sum_{0 \neq v \in \mathbb{N}^n} c_v(q^{\frac{1}{2}})Y(t_0)^v),
\]

where the coefficients \( c_v(q^{\frac{1}{2}}) \in \mathbb{Z}[q^{\pm \frac{1}{2}}] \). In particular, \( \text{Ind}^{\text{int}}M_i(t) \) equals \( \text{deg}^{t_0} X_i(t) \).

(iv) The quantization matrices satisfy, for any \( t, t_0 \in T_n \),

\[
\Lambda_{ij}^t = \Lambda_{ij}^{t_0}(\text{Ind}^{t_0}M_i(t), \text{Ind}^{t_0}M_j(t)).
\]

**Remark 3.2.4.** When \( B(t) \) is skew-symmetric, Assumption 1 is known to hold when the categorification is provided by the (generalized) cluster category in Section 2.2. The verification of most conditions could be found in Plamondon’s works [Pla11b][Pla11a][Pla13]. By translating (5) into multiplication by \( E_\epsilon \) in Definition 2.1.3, we see that the last condition can be deduced from the sign coherence of \( g \)-vectors. This condition implies that the quantization matrices \( \Lambda(t) \) lift to a bilinear form \( \lambda \) on categories, cf. [Qin12].

Let \( \text{pr}_n : \mathbb{Z}^m \rightarrow \mathbb{Z}^n \) denote the projection onto the first \( n \)-components. For any \( \tilde{g} \) in \( D(t) \), we denote the projection \( \text{pr}_n \tilde{g} \) by \( g \). Define \( \text{pr}_f : \mathbb{Z}^m \rightarrow \mathbb{Z}^m \) to be the projection onto the last \( m - n \) components.
As the coefficient-free version of the piecewise linear map $\phi_{t',t}$, we have the following restriction map.

**Lemma 3.2.5.** The restriction $\text{pr}_n \phi_{t',t}|_{\mathbb{Z}^n} : \mathbb{Z}^n \to \mathbb{Z}^n$ is a bijection of sets.

**Proof.** It suffices to verify the statement for the case $t' = \mu_k t$, where we have

$$(\text{pr}_n \phi_{t',t})(\text{pr}_n \phi_{t,t'}) = 1.$$ 

□

We are interested in the pointed sets whose leading degree parametrization is compatible with the tropical transformation rules.

**Definition 3.2.6** (Compatible pointed sets). Let $t_1, t_2$ be two seeds. Let $\mathbb{L}^1$ be a $D(t_1)$-pointed set and $\mathbb{L}^2$ a $D(t_2)$-pointed set. We say $\mathbb{L}^1$ is compatible with $\mathbb{L}^2$ if, for any $\eta \in D(t_1)$, we have

$$\mathbb{L}(\phi_{t_2,t_1} \eta; t_2) = \mathbb{L}(\eta; t_1).$$

By Assumption 1(ii), this compatibility relation is reflexive and transitive.

3.3. Dominant degrees.

**Definition 3.3.1** (Dominant degree cone). For any given seed $t$, define the dominant degree cone $D^\dagger(t) \subset D(t)$ to be the cone generated by the leading degrees of all cluster variables $\deg_t(X_i(t'))$, for any $1 \leq i \leq m$ and $t' \in \mathbb{T}_n$. Its elements are called the dominant degrees of the lattice $D(t)$.

By (6) in Assumption 1, we have $D^\dagger(t) = \phi_{t,t_0} D^\dagger(t_0)$, $\forall t, t_0 \in \mathbb{T}_n$.

The following observation is obvious.

**Lemma 3.3.2.** Let $Z$ be any element in $\mathbb{A}^\dagger$. Then, for any seed $t$, the $\prec_t$-maximal degrees of the Laurent expansion of $Z$ in $T(t)$ are contained in $D^\dagger(t)$.

**Definition 3.3.3** (Bounded from below). We say that $(\prec_t, D^\dagger(t))$ is bounded from below if for any $\vec{g} \in D^\dagger(t)$, there exists finitely many $\vec{g}' \in D^\dagger(t)$ such that $\vec{g}' \prec_t \vec{g}$.

**Remark 3.3.4.** When $\mathbb{A}$ is of type (i) or (ii) in the sense of Section 8, take the canonical initial seed $t_0$. In Section 9.1, we shall see that $(\prec_{t_0}, D^\dagger(t_0))$ is bounded from below.

The following question arises naturally.

**Question 3.3.5.** If $(\prec_{t_0}, D^\dagger(t_0))$ is bounded from below, is it true that $(\prec_t, D^\dagger(t))$ is bounded from below for all $t \in \mathbb{T}_n$?
4. Correction technique

The correction technique was introduced in [Qin13] in order to keep track of bases when the quantization and the coefficient pattern of a quantum cluster algebra change. In this section, we recall and reformulate this correction technique. It will help us simplify many arguments by coefficient changes.

4.1. Similarity and variation. Let \( \var^* \) be any isomorphism of finite \( n \)-elements sets \( \var^* : \text{ex}^{(2)} \to \text{ex}^{(1)} \). We associate to it an isomorphism \( \var \) of the lattices \( \var : \mathbb{Z}^{\text{ex}^{(3)}} \to \mathbb{Z}^{\text{ex}^{(2)}} \) such that we have \( \var(v)_k = v_{\var^*(k)} \), \( \forall v = (v_k) \in \mathbb{Z}^{\text{ex}^{(1)}} \).

Let \( \tilde{B}^{(i)} = (b_{jk}^{(i)}) \) be an \( I^{(i)} \times \text{ex}^{(i)} \) matrix, \( i = 1, 2 \), where \( I^{(i)} \) has \( m^{(i)} \) elements and \( \text{ex}^{(i)} \subset I^{(i)} \) has \( n \) elements. The matrices \( \tilde{B}^{(1)} \) and \( \tilde{B}^{(2)} \) are called similar, if there exists an isomorphism \( \var^* : \text{ex}^{(2)} \to \text{ex}^{(1)} \) such that the matrix entries satisfy \( b_{jk}^{(2)} = b_{\var^*(j) \var^*(k)}^{(1)} \), \( \forall j, k \in \text{ex}^{(2)} \). In this case, we denote the principal part \( B^{(2)} \) as \( \var B^{(1)} \).

Example 4.1.1. Take \( I^{(i)} = \text{ex}^{(i)} = \{1, 2, 3\} \), where \( i = 1, 2 \). Define the map \( \var^* : \text{ex}^{(2)} \to \text{ex}^{(1)} \) as the permutation \( \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \). Define the matrices \( B^{(1)} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \) and \( B^{(2)} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \).

Then we have \( B^{(2)} = \var B^{(1)} \).

We denote by \( \text{pr}_{\text{ex}^{(i)}} : \mathbb{Z}^{I^{(i)}} \to \mathbb{Z}^{\text{ex}^{(i)}} \) the natural projection onto the coordinates at \( \text{ex}^{(i)} \).

For any given full rank matrices \( \tilde{B}^{(i)} \), \( i = 1, 2 \), consider the Laurent polynomial rings \( \mathcal{T}^{(i)} = \mathbb{Z}[X_j \, | \, j \in I^{(i)}] \). For any \( k \in \text{ex}^{(i)} \), define the \( k \)-th \( Y \)-variable \( Y_k^{(i)} = \prod_{j \in I^{(i)}} (X_j b_{jk}^{(i)}). \)

Consider the pointed elements \( M^{(i)} = X^{\eta^{(i)}} (\sum_{v \in \text{ex}^{(i)}} c_v^{(i)} (Y_v^{(i)})^v) \), \( i = 1, 2, \eta^{(i)} \in \mathbb{Z}^{m^{(i)}}, c_0^{(i)} = 1, c_v^{(i)} \in \mathbb{Z} \). \( M^{(1)} \) and \( M^{(2)} \) are called similar, if we have

\[
\text{pr}_{\text{ex}^{(2)}} \eta^{(2)} = \var(\text{pr}_{\text{ex}^{(1)}} \eta^{(1)}), \\
c_v^{(2)} = c_{\var^{-1}(v)}^{(1)}, \forall v \in \mathbb{N}^{\text{ex}^{(2)}}.
\]

Assume that there is an embedding from \( I^{(1)} \) to \( I^{(2)} \) such that its restriction induces an isomorphism \( \var^* : \text{ex}^{(2)} \to \text{ex}^{(1)} \). Then we can naturally embed \( \mathbb{Z}^{I^{(1)}} \) into \( \mathbb{Z}^{I^{(2)}} \) by adding 0 on the extra coordinate. Denote this embedding by \( \var \).

By abuse of notation, define the variation map \( \var \) to be the map from the set of pointed elements in \( \mathcal{T}^{(1)} \) to that of \( \mathcal{T}^{(2)} \) such that, for any pointed element \( M^{(1)} \), its image is the unique pointed element \( M^{(2)} \).
which is similar to $M^{(1)}$ and satisfies
\[ \var \eta^{(1)} = \eta^{(2)}. \]

Now, let us take into account of quantization.

**Definition 4.1.2 (Similar compatible pair).** For $i = 1, 2$, let $(\tilde{B}^{(i)}, \Lambda^{(i)})$ be compatible pairs such that $(\tilde{B}^{(i)})^T \Lambda^{(i)} = D^{(i)} \oplus 0$, where $D^{(i)}$ is a diagonal matrix $\text{Diag}(d_{k}^{(i)})_{k \in \text{ex}^{(i)}}$ with diagonal entries in $\mathbb{Z}_{>0}$. We say these two compatible pairs are similar, if there exists an isomorphism $\var^{*} : \text{ex}^{(2)} \rightarrow \text{ex}^{(1)}$, such that the principal $B$-matrices $B^{(2)} = \var B^{(1)}$, and, in addition, there exists a positive integer $\delta$ such that $D^{(2)} = \delta \var D^{(1)}$.

For $i = 1, 2$, consider the quantum torus $\mathcal{T}^{(i)} = \mathbb{Z}[q_{\pm \frac{1}{2}}][X_{j}^\pm]_{j \in \text{I}^{(i)}}$ associated with similar compatible pairs $(\tilde{B}^{(i)}, \Lambda^{(i)})$.

**Definition 4.1.3 (Similar pointed elements).** We say the pointed elements $M^{(i)} = X^{v^{(i)}}(\sum_{e \in \mathbb{N}^{(i)}} c_{e}^{(i)} q_{\var_{v}^{(i)}}^{(1)}(Y_{1}^{(i)})^{v})$, $c_{0}^{(i)} = 1$, $c_{0}^{(i)}(q_{\frac{1}{2}}^{(1)} \in \mathbb{Z}[q_{\pm \frac{1}{2}}]$, in $\mathcal{T}^{(i)}$ are similar if
\[ \var_{\text{pr}^{(2)}}(\eta^{(2)}) = \var(\var_{\text{pr}^{(1)}}(\eta^{(1)})), \]
\[ c_{e}^{(2)}(q_{\frac{1}{2}}^{(1)} = c_{e}^{(1)}(q_{\var_{v}^{(1)}}^{(1)}), \quad \forall v \in \mathbb{N}^{(2)}. \]

**Definition 4.1.4 (variation map).** Let the embedding $I^{(1)} \rightarrow I^{(2)}$ be given as before. The variation map $\var$ is the injective application from the set of the pointed elements of $\mathcal{T}^{(1)}$ to that of $\mathcal{T}^{(2)}$ sending any pointed element $M^{(1)}$ to the unique similar pointed element $M^{(2)}$ subject to
\[ \var(\eta^{(1)}) = \eta^{(2)}. \]

**Example 4.1.5.** Let us take two quantum seeds $t_{0}^{(1)}$, $t_{0}^{(2)}$ with the following two compatible pairs respectively
\[
(B^{(1)}, \Lambda^{(1)}) = \left( \begin{array}{ccc}
0 & -2 & 0 \\
2 & 0 & 1 \\
1 & 0 & 1
\end{array} \right), \quad \left( \begin{array}{ccc}
0 & 0 & -1 \\
0 & 0 & 2 \\
0 & 1 & -2
\end{array} \right)
\]
\[
(B^{(2)}, \Lambda^{(2)}) = \left( \begin{array}{ccc}
0 & -2 & 0 \\
2 & 0 & 1 \\
-1 & 2 & 0
\end{array} \right), \quad \left( \begin{array}{ccc}
0 & 0 & -1 \\
0 & 0 & 2 \\
-2 & 1 & -2
\end{array} \right)
\]

These compatible pairs are similar via the identification of vertices $\var^{*}(i) = (i)$, $i = 1, 2$. The following quantum cluster variables obtained by the same mutation sequence are similar:
\[
X_{2}(\mu_{2} \mu_{1} t_{0}^{(1)}) = X^{-c_{2}}(1 + Y_{2} + (q^{\frac{1}{2}} + q^{-\frac{1}{2}})Y_{1}Y_{2} + Y_{1}^{2}Y_{2}) \in \mathcal{T}(t^{(1)})
\]
\[
X_{2}(\mu_{2} \mu_{1} t_{0}^{(2)}) = X^{-c_{2} + c_{4}}(1 + Y_{2} + (q^{\frac{1}{2}} + q^{-\frac{1}{2}})Y_{1}Y_{2} + Y_{1}^{2}Y_{2}) \in \mathcal{T}(t^{(2)})
\]
The variation map \( \var \) would send \( X_2(\mu_2 \mu_1 t_0^{(1)}) \) to the similar element 
\[ X^{-c_2}(1 + Y_2 + (q^{1/2} + q^{-1/2})Y_1Y_2 + Y_1^2 Y_2) \in \mathcal{T}(t^{(2)}). \]

4.2. Correction. We consider algebraic equations involving the pointed elements. Fix two compatible pairs \((\tilde{B}^{(i)}, \Lambda^{(i)})\), \(i = 1, 2\). As in the previous discussion, assume that there is an embedding from \( I^{(1)} \) to \( I^{(2)} \) whose restriction induces an isomorphism \( \var^* : \text{ex}^{(2)} \rightarrow \text{ex}^{(1)} \), such that 
\[ B^{(2)} = \var B^{(1)} \text{ and } D^{(2)} = \delta \var D^{(1)}. \]

Let \([\cdot]^{(i)}\) denote the normalization in the quantum torus \( \mathcal{T}^{(i)} \) and \( \deg^{(i)}(\cdot) \) denote the leading degree of a pointed element in \( \mathcal{T}^{(i)} \). Then [Qin13, Thm 9.2] can be reformulated as follows.

**Theorem 4.2.1 ([Qin13]).** Fix an integer \( s \geq 1 \). Assume that we have equations

\[
\begin{align*}
    P &= [M_1 * M_2 * \cdots * M_s]^{(1)}, \\
    Q &= \sum_{j \geq 0} c_j (q^{1/2}) N_j,
\end{align*}
\]

for pointed elements \( P, Q, M_i \) and (possibly infinitely many) \( N_j \) in the quantum torus \( \mathcal{T}^{(1)} \), such that \( c_0 = 1, c_j(q^{1/2}) \in \mathbb{Z}[q^{1/2}] \), and

\[ \deg^{(1)} N_j = \deg^{(1)} Q + \tilde{B}^{(1)} u_j, \text{ where } u_j \in \mathbb{N}_{\text{ex}^{(1)}}, u_0 = 0. \]

Then, for any \( P^{(2)}, Q^{(2)}, M_i^{(2)}, N_j^{(2)} \) in \( \mathcal{T}^{(2)} \) similar to \( P, Q, M_i, N_j \), we have the following equations in \( \mathcal{T}^{(2)} \):

\[
\begin{align*}
    P^{(2)} &= f_M [M_1^{(2)} * M_2^{(2)} * \cdots * M_s^{(2)}]^{(2)}, \\
    Q^{(2)} &= \sum_{j \geq 0} c_j (q^{1/2}) f_j N_j^{(2)}
\end{align*}
\]

where \( f_M, f_j \) are the Laurent monomials in the frozen variables \( X_i, n < i \leq m^{(2)} \), given by

\[
\begin{align*}
    f_M &= \deg^{(2)} P^{(2)} - \sum_{i=1}^s \deg^{(2)} M_i^{(2)}, \\
    f_j &= X^{\deg^{(2)} Q^{(2)} + \tilde{B}^{(2)} \var(u_j) - \deg^{(2)} N_j^{(2)}}.
\end{align*}
\]

In particular, if \( P^{(2)}, Q^{(2)}, M_i^{(2)}, N_j^{(2)} \) are given by image of the the variation map in Definition 4.1.4, we have

\[
\begin{align*}
    f_M &= 1, \\
    f_j &= X^{\tilde{B}^{(2)} \var(u_j) - \var(\tilde{B}^{(1)} u_j)}.
\end{align*}
\]

**Proof.** We consider Laurent expansions of (7) (8) and compare their coefficients term by term. Details are given in [Qin13, Theorem 9.2].

The following observations are immediate consequences of the correction technique 4.2.1 and the existence of (quantum) \(F\)-polynomials [Tra11][FZ07].

**Lemma 4.2.2.** Let \(t, t'\) be two given seeds (probably associated with different quantum cluster algebras). Assume that \((\tilde{B}(t), \Lambda(t))\) is similar to \((\tilde{B}(t'), \lambda(t'))\) via the variation map \(\var\). Let \(\tilde{\mu} = \mu_{\var^i_1} \cdots \mu_{\var^i_r}\), \(r \in \mathbb{N}\), be any given sequence of mutations on \(\text{ex}^{(1)}\) and \(\var \tilde{\mu} = \mu_{i_1} \cdots \mu_{i_1}\) the corresponding sequence on \(\text{ex}^{(2)}\) Then we have the following results.

(i) \((\tilde{B}(\tilde{\mu} t), \Lambda(\tilde{\mu} t))\) is similar to \((\tilde{B}(\var(\tilde{\mu} t')), \lambda((\var(\tilde{\mu} t'))))\) via \(\var\).

(ii) Assume that \(B(t)\) is skew-symmetric. Let \(\var''\) denote the new variation maps from \(\var(T(\tilde{\mu} t))\) to \(\var(T(\var(\tilde{\mu} t')))\). Then the quantum cluster variable \(\var'' X_{\var'' i_r}(\tilde{\mu} t)\) and \(X_{i_r}(\var(\tilde{\mu} t'))\) are similar when they are viewed as Laurent polynomials in \(T(t)\) and \(T(t')\) respectively.

(iii) Assume that \(B(t)\) is skew-symmetric. Let \(Z\) be any given pointed element in the quantum torus \(T(t)\). Assume that it has a (possibly infinite) unitriangular expansion

\[
Z = \sum_{j \geq 0} c_j N_j,
\]

where \(N_j\) are normalized twisted products of quantum cluster variables of \(A(t)\), \(c_0 = 1\), \(\deg^i N_0 = \det^i Z\), and \(c_j \in \mathbb{m}, \deg^j N_j <_t \deg^i N_0\) if \(j > 0\). Then, for any element \(Z't \in T(t')\) similar to \(Z\), we have

\[
Z' = \sum_{j \geq 0} c_j f_j N'_j,
\]

where \(N'_j\) are normalized products of the quantum cluster variables of \(A(t')\) associated with the same mutation sequences, the coefficients \(f_j\) are given by Theorem 4.2.1.

5. **Injective pointed sets**

5.1. **Injective-reachable.** A vertex \(t\) is said to be injective-reachable via \((\Sigma, \sigma)\), for some sequence of mutations \(\Sigma\) and a permutation \(\sigma\) of \(\{1, \ldots, n\}\), if we have the following equalities

\[
pr_n \deg^i (X_{\sigma(i)}(\Sigma t)) = -e_i, \quad 1 \leq i \leq n,
\]

\[
b_{\sigma(i) \sigma(j)}(\Sigma t) = b_{ij}(t), \quad \forall 1 \leq i, j \leq n.
\]

Let us denote \(X_{\sigma(i)}(\Sigma t)\) by \(I_i(t)\) and call it an injective quantum cluster variable for \(t\) (or simply an injective for \(t\)). The meaning of “injective” will be explained in Remark 5.1.5.

We trivially extend the action of \(\sigma\) to a permutation of \(\{1, \ldots, m\}\). Let \(\sigma\) act naturally on \(\mathbb{Z}^m\) by pull back such that \(\sigma e_i = e_{\sigma^{-1} i}\).

We make the following assumptions for the rest of this paper.
Assumption 2. The cluster algebra contains an injective-reachable seed.

Assumption 3. If \( t \) is injective-reachable via \((\Sigma, \sigma)\), then for any \( 1 \leq k \leq n \), the vertex \( t' = \mu_k t \) is injective-reachable via \((\mu_{\sigma(k)} \Sigma \mu_k, \sigma)\).

We always read mutation sequences \( \mu_{i_n} \cdots \mu_{i_2} \mu_{i_1} \) from right to left in this paper, which we also denote by \((i_s, \ldots, i_2, i_1)\) for simplicity.

Lemma 5.1.1. If \( t \) is injective-reachable via \((\Sigma, \sigma)\), then for any sequence of mutations \( \mu_k = \mu_{k_s} \cdots \mu_{k_2} \mu_{k_1}, 1 \leq k_1, \ldots, k_s \leq n \), the vertex \( t' = \mu_k t \) is injective-reachable via \((\mu_{\sigma(k)} \Sigma \mu_k^{-1}, \sigma)\), where \( \mu_{\sigma(k)} \Sigma \mu_k^{-1} \) is defined to be \( \mu_{\sigma(k_1)} \cdots \mu_{\sigma(k_2)} \mu_{\sigma(k_1)} \).

Definition 5.1.2 (Injective-reachable cluster algebra). A cluster algebra is called injective-reachable if all its seeds are injective-reachable.

Proposition 5.1.3. Assumption 3 is equivalent to the claim that, for any \( 1 \leq k \leq n \), we have

\[
\deg^i I_k(t') = - \deg^i I_k(t) + \sum_{1 \leq i \leq n} [b_{ik}(t)]_+ \deg^i I_i(t) + \sum_{0 < j \leq m-n} [b_{n+j, \sigma k}(\Sigma t)]_+ e_{n+j}
\]

Proof. First assume that (9) holds. Omitting the symbol \((t)\), we can write

\[
\Pr_n \deg^i I_k(t') = e_k + \sum_{1 \leq i : n,b_{ik} \geq 0} (-b_{ik})e_i.
\]

And Assumption 3 follows from the following straightforward calculation:

\[
\Pr_n deg^\mu_k I_k(t') = \Pr_n \phi_{\mu_k,t} \deg^i I_k(t')
= \Pr_n \phi_{\mu_k,t} \Pr_n deg^i I_k(t')
= \Pr_n (\phi_{\mu_k,t} e_k + \phi_{\mu_k,t} \sum_{1 \leq i : n,b_{ik} \geq 0} (-b_{ik})e_i)
= \Pr_n (e_k + \sum_{1 \leq i : n,b_{ik} \geq 0} b_{ik} e_i + \sum_{1 \leq i : n,b_{ik} \geq 0} (-b_{ik})e_i)
= - e_k.
\]

Conversely, assume that Assumption 3 is true. Because the map \( \Pr_n \phi_{\mu_k,t} \) is invertible on \( \mathbb{Z}^n \), the above calculation implies we must have

\[
(10) \quad \Pr_n deg^i I_k(t') = \Pr_n (\deg^i I_k(t) + \sum_{1 \leq i \leq n} [b_{ik}]_+ \deg^i I_i(t)).
\]

On the other hand, consider the exchange rule in the seed \( \Sigma t \):

\[
X_{\sigma k}(\mu_{\sigma k} \Sigma t) = \frac{\prod_{1 \leq i \leq m} X_i(\Sigma t)^{[b_{ik}, \sigma k(\Sigma t)]_+} + \prod_{1 \leq j \leq m} X_j(\Sigma t)^{[b_{jk}, \sigma k(\Sigma t)]_+}}{X_{\sigma k}(\Sigma t)}.
\]
View this equation in $\mathcal{T}(t)$. Because $X_{\sigma k}(\mu_{\sigma k}\Sigma t)$ is a pointed element whose unique leading term has coefficient 1, its leading degree must be chosen from either
\[
\deg^t \left( \prod_{1 \leq i \leq m} X_i(\Sigma t)^{[b_{i,\sigma k}(\Sigma t)]_+} \right) - \deg^t X_{\sigma k}(\Sigma t)
\]
or
\[
\deg^t \left( \prod_{1 \leq j \leq m} X_j(\Sigma t)^{[b_{\sigma k,j}(\Sigma t)]_+} \right) - \deg^t X_{\sigma k}(\Sigma t).
\]
Their images in $\mathbb{Z}^n$ under the projection $\text{pr}_n$ are different except the trivial case when $\sigma k$ is an isolated point in $Q(\Sigma t)$. (10) implies that we should choose the former one. This choice gives us (9). □

**Remark 5.1.4.** Notice that the definition of injective-reachable is not affected by quantization or frozen variables.

Assumption 3 holds true, if the cluster algebra defined on the principal part $B(t)$ admits an appropriate categorification. When $B(t)$ is skew-symmetric, such a categorification has been established by (generalized) cluster categories [Ami09][Pla11b], cf. Section 2.2. For cluster algebras of type (i), a different categorification via Frobenius category is available, and a verification of Assumption 3 is written in [GLS11, Proposition 13.4].

Let us impose the principal coefficients on $t$, namely
\[
\tilde{B}(t) = \begin{pmatrix} B(t) \\ \text{Id}_n \end{pmatrix}
\]
cf. [FZ07]. We can calculate the leading degrees of the cluster variables appearing in the assumption, following the mutation rule of (extended) $g$-vectors [FZ07, (6.12)] [Tra11, (3.32)]. As a consequence, Assumption 3 is equivalent to the claim that the matrix $\tilde{B}(\Sigma t)$ satisfies
\[
b_{n+i,\sigma(i),\sigma(j)}(\Sigma t) = -\delta_{ij}, \ 1 \leq i, j \leq n.
\]

We denote $X_i(t)$ by $P_{\sigma i}(\Sigma t)$ and call it a projective quantum cluster variable for $\Sigma t$ (or simply a projective for $\Sigma t$). We make the following assumption on the Laurent expansions of $I_i(t)$ and $P_{\sigma i}(\Sigma t)$ for the rest of this paper. Its meaning will be clear in Remark 5.1.5.

Recall that $\sigma^{-1}e_i = e_{\sigma i}, 1 \leq i \leq m$.

**Assumption 4 (Cluster expansions).** Let $t$ be any given seed. When $q^{\frac{1}{\delta}}$ specializes to 1, the Laurent expansion of $I_i(t)|_{q^{\frac{1}{\delta}} \rightarrow 1}$ in $\mathcal{T}(t)$ takes the following form:
\[
I_i(t)|_{q^{\frac{1}{\delta}} \rightarrow 1} = X_{\sigma i}(\Sigma t)|_{q^{\frac{1}{\delta}} \rightarrow 1}
= X(t)^{\deg^t I_i(t)}(1 + Y(t)^{e_i} + \sum_{d \in \mathbb{N}^n; d > e_i} \alpha_d Y(t)^d),
\]
(11)
and the Laurent expansion of $P_i(t)|_{q^\frac{1}{2} \to 1}$ in $T(t)$ takes the following form:

\[
(12) \quad P_i(\Sigma t)|_{q^\frac{1}{2} \to 1} = X_{\sigma_i^{-1}}(t)|_{q^\frac{1}{2} \to 1} = X(\Sigma t)^{\deg_{\Sigma T} P_i(\Sigma t)} Y(\Sigma t)^{p(i, \Sigma t)} \cdot \left( \sum_{d \in \mathbb{N}^n : e_i < d \leq p(i, \Sigma t)} \beta_d Y(\Sigma t)^{-d} + Y(\Sigma t)^{-e_i} + 1 \right),
\]

for some dimension vector $p(i, \Sigma t) \in \mathbb{N}^n$, coefficients $\alpha_d, \beta_d \in \mathbb{Z}$.

Moreover, the degrees satisfy

\[
(13) \quad \deg_{\Sigma T} (P_i(\Sigma t) Y(t)^{p(i, \Sigma t)}) = \sigma^{-1} \deg^t I_{\sigma_i^{-1}}(t).
\]

**Remark 5.1.5 (Injectives and projectives).** Assume that the cluster algebra admits the categorification in Section 2. We rewrite (3) as

\[
\text{Ind}^\Sigma T_i + \widetilde{B}(\Sigma T) \cdot \text{Ext}_{C}(\Sigma T, T_i) = \text{Ind}^T \Sigma T_i.
\]

Recall that $T_i$ correspond to the quantum cluster variable $X_i(t)$. Thanks to the Calabi-Yau reduction, $\Sigma T_i$ corresponds to the $i$-th injective right module of the Jacobian algebra associated with the principal quiver $Q(t)$ and its generic potential. So $\Sigma T_i$ corresponds to the quantum cluster variable $I_i(t) = X_{\sigma_i}(t[1])$ on the vertex $\sigma_i$ in $Q(t[1])$. On the other hand, thanks to Calabi-Yau reduction, $T_i$ correspond to the indecomposable projective of the Jacobian algebra associated with $Q(t[1])$ and its generic potential at this vertex.

The above equation holds when we replace $\Sigma T$ and $T$ by their direct summands $\Sigma T_i$ and $T_j$ respectively, $1 \leq j \leq m$. Notice that the multiplicity of $\Sigma T_j$ contributes the $\sigma_j$-th component in the degree lattice $D(t[1]) = \mathbb{Z}^m$. Also recall that $\sigma e_{\sigma_j} = e_j$. We obtain

\[
\sigma(\deg^t P_{\sigma_i} + \widetilde{B}(t[1]) \cdot p(\sigma_i, t[1])) = \deg^t I_i(t),
\]

where $p(\sigma_i, t[1])$ is the dimension of the $\sigma_i$-th projective right module of the Jacobian algebra associated with the quiver $Q(t[1])$ and its generic potential. Equation (13) follows.

**Example 5.1.6 (Compare injectives and projectives).** Consider a seed $t$ whose ice quiver $\widetilde{Q}(t)$ is given in Figure 2. It is of principal coefficients
with the canonical quantization matrix

\[
\Lambda(t) = \begin{pmatrix}
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & -1 & 0 & 0
\end{pmatrix}.
\]

Take the sequence \( \Sigma_4 = \mu_1 \mu_3 \mu_2 \mu_4 \mu_1 \) and the permutation

\[
\sigma = \begin{pmatrix}
1 & 2 & 3 & 4 \\
4 & 2 & 3 & 1
\end{pmatrix}.
\]

The seed \( t[−1] \) is obtained from \( t \) by applying \((\sigma \Sigma_4)^{-1} = (4, 1, 2, 3, 4)\). Its ice quiver is given in Figure 3. \( t[−1] \) is injective-reachable via \((\sigma \Sigma_4, \sigma) = ((4, 3, 2, 1, 4), \sigma)\).

In the quantum torus \( T(t[−1]) \), we have

\[
\begin{align*}
I_1(t[−1]) &= X_4(t) = X^{-e_1+e_5+e_8}(1 + Y_1 + Y_1Y_2), \\
I_2(t[−1]) &= X_2(t) = X^{-e_2+e_6+e_7+e_8}(1 + Y_2 + Y_2Y_4), \\
I_3(t[−1]) &= X_3(t) = X^{-e_3+e_7}(1 + Y_3 + Y_2Y_3 + Y_2Y_3Y_4), \\
I_4(t[−1]) &= X_1(t) = X^{-e_4+e_5+e_6+e_7}(1 + Y_4 + Y_1Y_4).
\end{align*}
\]

It is straightforward to check that the projective right modules of the Jacobian algebra of \((Q(t), W(t))\) have dimensions

\[
\begin{align*}
p(1, t) &= e_1 + e_2 + e_3, \\
p(2, t) &= e_2 + e_3 + e_4, \\
p(3, t) &= e_3, \\
p(4, t) &= e_1 + e_4.
\end{align*}
\]

The \( Y \)-variables of the seed \( t \) in the quantum torus \( T(t) \) are given by

\[
\begin{align*}
Y_1 &= X^{e_2-e_4+e_5} \\
Y_2 &= X^{-e_1+e_3+e_4+e_6} \\
Y_3 &= X^{-e_2+e_7} \\
Y_4 &= X^{e_1-e_2+e_8}.
\end{align*}
\]
The projectives for $t$, obtained by applying the sequence $(\sigma \Sigma)^{-1}$, take the following form in the quantum torus $T(t)$:

- $P_1(t) = X_4(t[-1]) = X^{-e_3}(1 + Y_3 + Y_2Y_3 + Y_1Y_2Y_3)$
  \[= X^{-e_1+e_5+e_6+e_7}(Y^{-e_1-e_2-e_3} + Y^{-e_1-e_2} + Y^{-e_1} + 1),\]

- $P_2(t) = X_2(t[-1]) = X^{e_2-e_3-e_4}(1 + Y_3 + Y_4 + Y_3Y_4 + Y_2Y_3Y_4)$
  \[= X^{-e_2+e_6+e_7+e_8}(Y^{-e_2-e_3-e_4} + Y^{-e_2-e_4} + Y^{-e_2-e_3} + Y^{-e_2} + 1),\]

- $P_3(t) = X_3(t[-1]) = X^{e_2-e_3}(1 + Y_3)$
  \[= X^{-e_3+e_7}(Y^{-1}_3 + 1),\]

- $P_4(t) = X_1(t[-1]) = X^{-e_1}(1 + Y_1 + Y_1Y_4)$
  \[= X^{-e_4+e_5+e_8}(Y^{-e_1-e_4} + Y^{-e_4} + 1).\]

This example meets the expectation of Assumption 4.

5.2. Injective-reachable chain. Let $t$ be any seed which is injective-reachable via an associated pair $(\Sigma_t, \sigma_t)$. We denote the seed $t$ by $t[0]$.

For any integer $d \geq 0$, by Lemma 5.1.1, we recursively obtain that the seed $t[d]$ is injective reachable via the pair $(\Sigma_{t[d]}, \sigma_{t[d]}) = (\sigma^d \Sigma_t, \sigma_t)$, and we denote $t[d+1] = \Sigma_{t[d]}t[d]$.

Let $\text{ord } \sigma_t$ denote the order of the permutation $\sigma_t$. Then $t[\text{ord } \sigma_t - 1]$ is injective-reachable via

\[(\Sigma_{t[\text{ord } \sigma_t - 1]}, \sigma_{t[\text{ord } \sigma_t - 1]}) = (\sigma_t^{\text{ord } \sigma_t - 1} \Sigma_t, \sigma_t) = (\sigma_t^{-1} \Sigma_t, \sigma_t).\]
Moreover, the principal part of the quiver of the seed \( t[\ord \sigma_t] \) is the same as that of \( t \). Since injective-reachable is independent of the coefficient part, the seed \( t[-1] = \sigma_t \Sigma_t^{-1} t \) is injective-reachable via \( (\sigma_t \Sigma_t^{-1}, \sigma_t) \). Recursively, we obtain that the seed \( t[-d] = \sigma_t^d \Sigma_t^{-1} t[-d + 1] \) is injective-reachable via \( (\sigma_t^{-d} \Sigma_t, \sigma_t) \), for any integer \( d \geq 1 \).

**Definition 5.2.1** (Injective-reachable chain). The chain of seeds \( (t[d])_{d \in \mathbb{Z}} \) is called an injective-reachable chain.

### 5.3. Injective pointed set.

For any \( g \in \mathbb{Z}^n \), denote \( g_+ = (\max\{((g)_i, 0)\})_{1 \leq i \leq n} \) and \( g_- = (\max\{((-g)_i, 0)\})_{1 \leq i \leq n}. \) We will also view \( g, g_+ \in \mathbb{Z}^n \) as elements of \( \mathbb{Z}^m \) by putting zero at the last \( m-n \) components.

**Definition 5.3.1** (Injective pointed set). For any \( (f, d_X, d_I) \in \mathbb{Z}^{m-n} \oplus \mathbb{N}^n \oplus \mathbb{N}^n \), we define the following pointed element of \( T(t) \)

\[
(14) \quad I(f, d_X, d_I; t) = \prod_{1 \leq i \leq m-n} X_{n+i}^{f_i} \ast X(t)^{d_X} \ast I(t)^{d_I}.
\]

Define the injective pointed set \( I_t \) to be \( \{I(\tilde{g}; t), \tilde{g} \in D(t)\} \) such that for any \( \tilde{g} \in D(t) \), \( I(\tilde{g}; t) = I(g_f, g_+, g_-; t) \), where \( g_f \) is defined to be \( \tilde{g} - g_+ - \deg f(I(t)^{g_-}). \)

We also view \( g_f \) as an element in \( \mathbb{Z}^{m-n} \) by removing its first \( n \) components.

By definition, \( I_t \) is a \( D(t) \)-pointed \( \mathbb{Z}[q^\frac{1}{2}] \)-linearly independent subset of \( T(t) \). The following lemma suggests that its elements behave well under one-step mutations.

**Lemma 5.3.2** (Neighboring injective pointed set). For any given \( t \in \mathbb{T}_n \) and \( 1 \leq k \leq n \), define \( t' \) to be \( \mu_k t \). Then the set \( I_t \) is \( D(t) \)-pointed in \( T(t) \) with respect to the order \( \prec \). Furthermore, for any \( \tilde{g} \in D(t') \), we have

\[
\deg I_t(\tilde{g}; t') = \phi_{t,t'}(\tilde{g}).
\]

**Proof.** The degrees in \( D(t') \) change by the piecewise linear formula (5) which depends on the sign of the \( k \)-th components. In our situation, the degrees of the factors of \( I(\tilde{g}; t') \) are \( \deg_{t'} X(t')^{g_f}, \deg_{t'} X(t')^{g_+}, \deg_{t'} I(t')^{g_-}. \) They are sign coherent at the \( k \)-th component. Therefore, we can use Lemma 3.2.3 and deduce that

\[
\phi_{t,t'}(\tilde{g}) = \phi_{t,t'} \deg_{t'} (X(t')^{g_f} \ast X(t')^{g_+} \ast I(t')^{g_-}) = \phi_{t,t'} \deg_{t'} (X(t')^{g_f}) + \phi_{t,t'} \deg_{t'} (X(t')^{g_+}) + \phi_{t,t'} \deg_{t'} (I(t')^{g_-}) = \deg_{t'} (X(t')^{g_f}) + \deg_{t'} (X(t')^{g_+}) + \deg (I(t')^{g_-}) \quad \text{(Assumption 1)}
\]

\[
= \deg_{t} (X(t')^{g_f} \ast X(t')^{g_+} \ast I(t')^{g_-}) = \deg_{t} (I(\tilde{g}; t')).
\]
It remains to check that the coefficient of the leading term remains to be 1. We have to verify the following equality:

\[ \Lambda(t)(\phi_* \deg^t X(t')^{g_j}, \phi_* X(t')^{g_+}) = \Lambda(t)(\deg^t X(t')^{g_j}, \deg^t X(t')^{g_+}), \]

\[ \Lambda(t)(\phi_* \deg^t X(t')^{g_j}, \phi_* I(t')^{g_-}) = \Lambda(t)(\deg^t X(t')^{g_j}, \deg^t I(t')^{g_-}), \]

\[ \Lambda(t)(\phi_* \deg^t X(t')^{g_j}, \phi_* I(t')^{g_-}) = \Lambda(t)(\deg^t X(t')^{g_+}, \deg^t I(t')^{g_-}) \]

Decompose the degrees appearing above into sum of unit vectors by Lemma 3.2.3. Then it suffices to verify the following equations for any pair of unit vectors \((e_i, e_j), 1 \leq i, j \leq m\), sign-coherent at the \(k\)-th component:

\[ \Lambda(t)(\phi_* e_i, \phi_* e_j) = \Lambda(t)(e_i, e_j), \]

These equations follow from the mutation rule of quantization matrices, \(\forall i, j \neq k:\)

\[ \Lambda(t)_{ij} = \Lambda(t')_{ij}, \]

\[ \Lambda(t)(e_i, -e_k + [b_{jk}(t')]_+ e_j) = e^T_1 (E_{-1}(t')^T \Lambda(t') E_{-1}(t')) E_{-1}(t') e_k = e^T_1 \Lambda(t') e_k \]

\[ \Lambda(t)(e_i, e_k - [-b_{jk}(t')]_+ e_j) = e^T_1 (E_1(t')^T \Lambda(t') E_1(t')) E_1(t') (-e_k) = e^T_1 \Lambda(t') (-e_k). \]

\[ \square \]

**Remark 5.3.3.** The notion of leading terms depends on the choice of seed and so does the normalization factor. Different pointed elements \(Z_t \in \mathcal{T}(t^1)\) might have identical leading degrees in the quantum torus \(\mathcal{T}(t^2)\) of the new seed \(t_2\), and a pointed element \(Z\) in \(\mathcal{T}(t^1)\) might not be pointed in the \(\mathcal{T}(t^2)\). The normalization factor of the twisted product \(Z_1 \ast Z_2\) might also change if they are not quantum cluster variables in the same cluster.

**Example 5.3.4.** We consider the initial acyclic quiver \(Q = Q(t_0)\) of type \(A_2\) in Figure 4. Then \(\tilde{B}(t_0) = B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\). Take the quantization matrix \(\Lambda(t_0)\) to be \(-B^{-1} = B\).

Then \(t_0\) is injective-reachable via \((\Sigma, \sigma) = (\mu_2 \mu_1, 1)\). We have

\[ X_i(t_0) = X_i, \ i = 1, 2, \]

\[ I_1(t_0) = X_1^{-1} + X_1^{-1} X_2 = X^{-e_1} (1 + Y_1) \]

\[ I_2(t_0) = X_1^{-1} X_2^{-1} + X_2^{-1} + X_1^{-1} = X^{-e_2} (1 + Y_2 + Y_1 Y_2) \]

The set \(I^0\) consists of the normalized ordered products of \(X_i(t_0)\) and \(I_j(t_0)\) such that \(X_i(t_0)\) and \(I_i(t_0)\) do not appear simultaneously. It is apparently \(D(t_0) = \mathbb{Z}^2\) pointed.
Consider the neighboring seed $\mu_1 t_0$, its quantization matrix is $\Lambda(\mu_1 t_0) = -B$. In the quantum torus $T(\mu_1 t_0)$, we have the following Laurent expansions

\[\begin{align*}
X_1(t_0) &= X^{-e_1+e_2}(1 + Y_1) \\
X_2(t_0) &= X_2 \\
I_1(t_0) &= X_1 \\
I_2(t_0) &= X^{-e_2}(1 + Y_2).
\end{align*}\]

It is straightforward to check that $I^{t_0}$ is a $D(\mu_1 t_0)$-pointed set.

If we proceed to the seed $\mu_2 \mu_1 t_0 = t_0 \{1\}$. The quantization matrix $\Lambda(t_0 [1]) = B$. In the quantum torus $T(t_0 [1])$, we have the following Laurent expansions

\[\begin{align*}
X_1(t_0) &= P_1(t_0 [1]) = X^{-e_2}(1 + Y_2 + Y_1 Y_2) \\
X_2(t_0) &= P_2(t_0 [1]) = X^{e_1 - e_2}(1 + Y_2) \\
I_1(t_0) &= X_1 \\
I_2(t_0) &= X_2.
\end{align*}\]

It is straightforward to check that the set of leading degrees of $I^{t_0}$ in this seed is $\mathbb{N} \times \mathbb{Z}$. In fact, the different pointed elements $[X_1(t_0) * I_2(t_0)]^{t_0}$ and $1$ in $T(t_0)$ will have the same leading degree $0$ in $T(t_0 [1])$. The normalization factors of $X_1 * I_2$ are also different in $T(t_0)$ and $T(t_0 [1])$.

\[\begin{align*}
&\text{Figure 4. A quiver } Q \text{ of type } A_2. \\
&\end{align*}\]

6. TRIANGULAR BASES

In this section, we consider the triangular basis $L^t$ with respect to the injective pointed set $I^t$. In the existence theorem 6.4.1, we shall prove that a common triangular basis for all seeds exists, provided some initial local conditions are satisfied.

6.1. Local conditions. For any $1 \leq k \leq n$, let us denote $X_k(\mu_k t)$ and $X_{\sigma(k)}(\mu_{\sigma(k)} t)$ by $X'_k(t)$ and $I'_k(t)$ respectively.

**Definition 6.1.1** (Local conditions). For any given vertex $t \in T_n$ and $1 \leq k \leq n$, we consider the following local conditions on $t$:
L0(t) There exists a bar-invariant \( D(t) \)-pointed basis \( L^t \) in \( \mathbb{A} \), which satisfies \( L_0(0;t) = 1 \) and factors through the frozen variables:

\[
(15) \quad \prod_{i=1}^{m-n} X_{n+i}(t)^{i_1} * L(\tilde{g}; t) \in L^t, \quad f \in \mathbb{Z}^{m-n}.
\]

L1(t) The structure constants of \( L^t \) belong to \( \mathbb{N}[q^{\pm \frac{1}{2}}] \).

L2(t) \( \mathbf{I}^t \) is \((\prec_{t}, m)\)-unitriangular to \( L^t \).

L3(k; t) For any \( S \in L^t \), the normalized twisted product \([X_k(t) \ast S]^t \) is \((\prec_{t}, m)\)-unitriangular to \( L^t \) in the quantum torus \( T(t) \).

We also consider the following auxiliary conditions:

L4(k; t) For any integer \( d_k > 0 \), \( X_k(t)^{d_k} \in L^t \).

L5(k; t) For any integer \( d_k > 0 \), \( I_k(t)^{d_k} \in L^t \).

L3'(k; t) \( \mathbf{I}^t \) is \((\prec_{t}, m)\)-unitriangular to the \( D(t) \)-pointed set

\[
[X_k(t) \ast \mathbf{I}^t(\tilde{g}; t)]^t = \{ [X_k(t) \ast \mathbf{I}^t(\tilde{g}; t)]^t | \tilde{g} \in D(t) \}.
\]

L3''(t) For any cluster monomials \( \prod_{i=1}^{m-n} X_{n+i}(t)^{i_1} \), \( X(t)^{d_x}, I(t)^{d_t} \), their normalized twisted product \( \mathbf{I}(f, d_X, d_I; t) \) is \((\prec_{t}, m)\)-unitriangular to \( \mathbf{I}^t \):

\[
\mathbf{I}(f, d_X, d_I; t) = \mathbf{I}(\tilde{g}; t) + \sum_{\tilde{g} \prec \tilde{g}'} c_{\tilde{g}} \mathbf{I}(\tilde{g}'; t),
\]

where \( c_{\tilde{g}} \in m, \tilde{g} = \deg^t \mathbf{I}(f, d_X, d_I; t) \).

L4'(k; t) \( (X_k(t))^{d_k} = \mathbf{I}(\tilde{g}; t) + \sum_{\tilde{g} \prec \tilde{g}'} u_{\tilde{g}} \mathbf{I}(\tilde{g}'; t), u_{\tilde{g}} \in m, \forall 1 \leq k \leq n, \)

\( d_k \in \mathbb{N} \), where \( \tilde{g} = \deg^t ((X_k(t))^{d_k}) \).

L5'(k; t) \( (I_k(t))^{d_k} = \mathbf{I}(\tilde{g}; t) + \sum_{\tilde{g} \prec \tilde{g}'} v_{\tilde{g}} \mathbf{I}(\tilde{g}'; t), v_{\tilde{g}} \in m, \forall 1 \leq k \leq n, \)

\( d_k \in \mathbb{N} \), where \( \tilde{g} = \deg^t ((I_k(t))^{d_k}) \).

For simplicity, we often omit the notation \( t \) and, if a condition L2? (k) is satisfied by \( k \) from a set of vertices \( E \), we denote this property by \( L?(E) \);

Remark 6.1.2. Notice that we allow infinite expansions in L3', L3'', L4', L5'. Also, Condition L0 implies that the Laurent monomials of the frozen variables are contained in \( L^t \).

Remark 6.1.3. (15) is an analogue of the factorization property of the dual canonical bases of quantum unipotent subgroups, cf. [Kim12, Section 6.3].

We propose Condition L2 as an analog of the transition property between dual PBW bases and dual canonical bases and use Condition L3 to strengthen L2. If the seed \( t \) has acyclic principal part with appropriate coefficient pattern, then \( \mathbf{I}^t \) is the dual PBW basis, and we can take \( L^t \) to be the dual canonical basis. The conditions L0(t) L1(t) L2(t) are well known results and, moreover, \( L^t \) contains all quantum cluster monomials by [KQ12].
where $b_L$ is the coefficient of each basis element

Lemma 6.1.5. Assume that a vertex $x$ satisfies $L_0 L_2$. For any $g, g' \in \mathbb{Z}^m$, $1 \leq i \leq n$, $1 \leq k \leq n$, if $g' \prec_t g$, we have

$$(16) \quad \Lambda(t)(\deg^t X_i(t), \deg^t Y_k(t)) = \Lambda(t)(\deg^t Y_k(t), \deg^t I_i(t)) = -\delta_{ik}d,$$

$$(17) \quad \Lambda(t)(\deg^t X_k(t), \tilde{g}) \leq \Lambda(t)(\deg^t X_k(t), \tilde{g'}),$$

$$(18) \quad \Lambda(t)(\tilde{g'}, \deg^t I_k(t)) \leq \Lambda(t)(\tilde{g}, \deg^t I_k(t)).$$

Proof. The first equation follows from the definition of compatible pairs, and it implies the remaining equations. \hfill \Box

Lemma 6.1.6 (Positive expansion). For any given vertex $t \in \mathbb{T}_n$, if we have $L_0$, $L_1$, and if the cluster monomials of $t$ are contained in $L'$, then the Laurent expansions of the elements of $L'$ in $\mathcal{T}(t)$ have non-negative coefficients.

Proof. The expansion coefficients can be interpreted as $q^{\frac{1}{2}}$-shifted of the structure constants, cf. [HL10, Proposition 2.2]. \hfill \Box

Lemma 6.1.7. Assume that a vertex $t \in \mathbb{T}_n$ satisfies $L_0 L_2$.

(i) If $t$ satisfies $L_0 L_2$ for another bar-invariant $D(t)$-pointed bases $L'$, then we have $L' = L'$.

(ii) The cluster monomials $X(t)^d$, $I(t)^d$, $d \in \mathbb{N}^n$, are contained in $L'$.

(iii) If $t$ further satisfies $L_1$, then $L_4(k)$ and $L_5(k)$ are equivalent to the claim that $X_k(t), I_k(t)$ are contained in $L'$ respectively.

Proof. The statements (i)(ii) follow from the bar-invariance of the triangular basis. The statement (iii) follows from the positive Laurent expansions of $L'$ in the seeds $t$ and $\Sigma t$ respectively. We give a proof that $X_k(t) \in L'$ implies $L_4(k)$ here. The proof for $L_5(k)$ is same.

For simplicity, we specialize $q^{\frac{1}{2}}$ to 1 and omit the symbol $(t)$. Consider the following product of basis elements in $L'$

$$(X_k')^{\delta_k} = \sum_{\tilde{g} \leq \deg^t X_k^{\delta_k}} b(\tilde{g})L(\tilde{g}; t),$$

where $b(\tilde{g}) \in \mathbb{N}$, $b(\deg^t X_k^{\delta_k}) = 1$. The Laurent expansion of LHS is $X_j^{\delta_{[b_k]}} + X_k^{\delta_k}(1 + Y_k)^{\delta_k}$. By Lemma 6.1.5, the positive Laurent expansion of each basis element $L(\tilde{g}; t)$ appearing must take the form

$$(X_j^{\delta_{[b_k]}} + X_k^{\delta_k} F_{\tilde{g}}(Y_k),$$

where $F_{\tilde{g}}(Y_k)$ is a polynomial in $Y_k$ whose coefficients are no larger than those of $(1 + Y_k)^{\delta_k}$ at all degrees. Let $F_{\tilde{g}}(Y_k)$ denote the polynomial $F_{\tilde{g}}(Y_k^{-1})Y_k^{\delta_k}$. Then its coefficients are also no larger than those of $(1 + Y_k)^{\delta_k}$. \hfill \Box
Consider the Laurent expansion of $\mathbf{L}(\widetilde{g}; t)$ in $\mathcal{T}(\mu_k t)$. We obtain
\[
\mathbf{L}(\widetilde{g}; t) = X_k^{\ast d_k} \cdot \frac{1}{(1 + \gamma_k)^{d_k}} F_g(Y_k) = X_k^{\ast d_k} \cdot \frac{y_k^{\ast d_k} F'_g(Y'_k)}{(1 + y_k)^{d_k}}.
\]
This is an Laurent polynomial in $\mathcal{T}(\mu_k t)$ if and only if $\frac{F'_g(Y'_k)}{(1 + y_k)^{d_k}}$ is a Laurent polynomial in $\mathcal{T}(\mu_k t)$. But $\frac{F'_g(Y'_k)}{(1 + y_k)^{d_k}}$ has coefficients no larger than those of $(1 + Y'_k)^{d_k}$. It follows that $F'_g(Y'_k) = (1 + Y'_k)^{d_k}$ and, consequently, $(X_k^*)^{d_k} = \mathbf{L}(\widetilde{g}; t)$.

Lemma 6.1.6(iii) is not essential to this paper. It simplifies Proposition 6.2.3 such that the statement for the case $d_k = 1$ is sufficient.

**Lemma 6.1.7.** Let us consider any $t \in \mathbb{T}_n$. Assume that it verifies $L_0, L_2$, then the following claims are true.

(i) $L_3(k)$ and $L_3'(k)$ are equivalent, for any $1 \leq k \leq n$. $L_3([1, n])$ and $L_3''$ are equivalent.

(ii) $L_4(k)$ and $L_4'(k)$ are equivalent.

(iii) $L_5(k)$ and $L_5'(k)$ are equivalent.

**Proof.** (ii)(iii) are obvious by $L_2$ and bar-invariance. We give a proof for (i).

Assume that we have $L_3(k; t)$. By $L_2$, it implies that $[X_k * \mathbf{I}]^t$ is $\mathbf{m}$-unitriangular to $\mathbf{L}'$ and, consequently, $\mathbf{m}$-unitriangular to $\mathbf{I}'$. Therefore, $\mathbf{I}'$ is $\mathbf{m}$-unitriangular to $[X_k * \mathbf{I}]^t$.

Assume that we have $L_3'(k; t)$. Let $S$ be any element in $\mathbf{L}'$. Then $S$ is $\mathbf{m}$-unitriangular to $\mathbf{I}'$:

\[
S = \sum_{\widetilde{g} \prec \deg S} b(\widetilde{g}) \mathbf{I}(\widetilde{g}; t),
\]

where $b(\deg S) = 1$, $b(\widetilde{g}) \in \mathbf{m}$ if $\widetilde{g} \neq \deg S$. The normalized product $[X_k * S]^t$ is $\mathbf{m}$-unitriangular to $[X_k * \mathbf{I}]^t$:

\[
[X_k * S]^t = \sum_{\widetilde{g} \prec \deg S} b(\widetilde{g}) q^\frac{1}{2} \Lambda(e_k, \deg \widetilde{g} - \deg S)[X_k * \mathbf{I}(\widetilde{g}; t)]^t.
\]

Using $L_3'(k)$, we see that $X_k * S$ is $\mathbf{m}$-unitriangular to $\mathbf{I}'$ and, consequently, to $\mathbf{L}'$.

If we have $L_3''$. For any given $\widetilde{g} \in D(t)$ and $1 \leq k \leq n$, take $(f, d_X, d_I)$ such that the normalized product $\mathbf{I}(f, d_X, d_I; t)$ equals $[X_k * \mathbf{I}(\widetilde{g}; t)]^t$, then it is $\langle \iota, \mathbf{m} \rangle$-unitriangular to $\mathbf{I}'$. For the given $k$, $L_3''(k)$ follows by taking the inverse transition and it implies $L_3(k)$.

If, conversely, $t$ satisfies $L_3([1, n])$, we obtain $L_3''$ by repeatedly left multiplying the $X$-variables to the triangular basis element $\mathbf{I}(f, 0, d_I; t) \in \mathbf{L}'$. 
\[\square\]
Lemma 6.1.8. Assume that $B(t)$ is skew-symmetric.

(i) Let $t$ and $t'$ be two similar seeds such that $(\tilde{B}(t), \Lambda(t))$ is similar to $(\tilde{B}(t'), \Lambda(t'))$ via the variation map $\text{var}$. Assume we have a basis $L'$ of $A(t)$ so that $t$ satisfies $L0(t)$ and $L2(t)$. Let $L''$ denote the set of the pointed elements in $T(t')$ similar to the elements of $L'$. Then it is the basis of $A(t')$, such that $t'$ satisfies $L0(t')$ and $L2(t')$.

(ii) If $t$ also satisfies $L1$ or $L3$, the same holds for $t'$.

(iii) Furthermore, for some given mutation sequence $\mu = \mu_{\text{var}^{-i_2}} \cdots \mu_{\text{var}^{-i_1}}$, $r \in \mathbb{N}$, assume that the the quantum cluster variable $X_{\text{var}^{-i_2}}(\mu t)$ is contained in $L'$, then the quantum cluster variable $X_{i_r}(\text{var}^{-1} \mu t')$ is contained in $L''$.

Proof. (i) $L'$ is $D(t')$-pointed and therefore linearly independent. It factors through the frozen variables by construction. Any given basis element $b \in L'$ is a polynomial of quantum cluster variables of $A(t)$ with coefficients in $\mathbb{Z}[q^{\pm \frac{1}{2}}]P$. Then any element $b' \in L''$ similar to $b$ is a polynomial of quantum cluster variables of $A(t')$ with coefficients in $\mathbb{Z}[q^{\pm \frac{1}{2}}]P'$ by Lemma 3.1.9(iii) and 4.2.2(iii). Therefore, $b'$ is contained in $A(t')$. We see that $L'$ spans $A(t')$ by Theorem 4.2.1 and the condition $L2(t')$ by Lemma 4.2.2(iii).

(ii) The conditions follow from Theorem 4.2.1 and Lemma 4.2.2(iii) respectively.

(iii) Recall that $X_{i_r}(\text{var}^{-1} \mu t')$ is similar to $X_{\text{var}^{-i_2}}(\mu t)$ by Lemma 4.2.2. The claim follows from the construction of $L''$. \qed

6.2. One step mutations. We shall show in Proposition 6.2.3 that the expected transformation rule (5) for the leading degrees of triangular basis elements implies the important conditions $L4$, $L5$: quantum cluster variables obtained by one step mutations are contained in the triangular bases.

We will control the Laurent expansions of the related triangular basis elements with the help of the positivity of the structure constants of the basis $L'$ and Assumption 4. To warm up, we start with a straightforward calculation.

Proposition 6.2.1. For any given $t$ in $\mathbb{T}_n$ and $1 \leq k \leq n$, assume that we have $L0$, $L1$, $L2$, $L3([1,n])$, then the Laurent expansion of $L'(\text{deg}^t X_k'(t))$ in $T(t)$ contains that of $X_k'(t)$.

Proof. Omit the notation $t$ for simplicity.

Denote $e_+ = \sum_j [b_{kj}]_+ e_j$, $X_+ = X^{e_+}$. Then we have $\text{deg}^t X_k' = e_+ - e_k$ and $X_k' = X_k^{-1} X_+(1 + Y_k)$.

We consider the normalized twisted product $[X^{-\text{deg}^t I_k - e_k} * X_+ * I_k]$, where the first factor is a Laurent monomial of the frozen variables. By Lemma 6.1.6(ii), $I_k$ and the cluster monomials in $t$ belong to the basis $L'$. By Lemma 6.1.5, the Laurent expansion of $I_k$ has non-negative
coefficients. Further applying Assumption 4, we see that the Laurent expansion of $I_k$ must take the form

$$I_k = X^{\deg I_k} (1 + Y_k + \sum_{d \in \mathbb{N}^n; d > e_k} \alpha_d Y^d), \quad \alpha_d \in \mathbb{N}[q^{\pm \frac{1}{2}}].$$

Therefore, the Laurent expansion of $[X^{\deg I_k} I_k - e_k * X_+ * I_k]$ takes the form

$$(19)\quad [X^{\deg I_k} I_k - e_k * X_+ * I_k] = X^{\deg I_k} I_k - e_k X_+ X^{\deg I_k} I_k (1 + Y_k + \sum_{d > e_k} \alpha_d q^{\frac{1}{2} \Lambda (e_+, \deg Y^d) Y^d})$$

$$= X^{-1} X_+ + X^{-1} X_+ Y_k + \sum_{d > e_k} \alpha_d q^{\frac{1}{2} \Lambda (e_+, \deg Y^d)} X^{-1} X_+ Y^d.$$

On the other hand, by L3, we have

$$(20)\quad [X^{\deg I_k} I_k - e_k * X_+ * I_k] = L'(e_+ - e_k) + \sum_{\eta < e_+ - e_k} c_{\eta} L'(\eta),$$

where the coefficients $c_{\eta} \in \mathbb{N}[q^{\pm \frac{1}{2}}] \cap \mathfrak{m}$. Comparing (19) with the Laurent expansion of (20). They all have non-negative coefficients in $\mathbb{N}[q^{\pm \frac{1}{2}}]$. By analyzing these coefficients, we easily deduce that the monomial $X_+ Y_k$ can only be the contribution of $L'(e_+ - e_k)$.

Therefore, the Laurent expansion of $L'(e_+ - e_k)$ takes the form

$$L'(e_+ - e_k) = X^{-1} X_+ + X^{-1} X_+ Y_k + \sum_{d \in \mathbb{N}^n; d > e_k} u_d X^{-1} X_+ Y^d.$$

□

Remark 6.2.2. In the proof of Proposition 6.2.1, the calculation of the desired basis element $L'^{(\deg X'_k(t))}$ is based on the following two steps:

- (i) put the desired basis element $L'^{(\deg X'_k(t))}$ in the product of cluster variables contained in $L'$ (20), and
- (ii) analyze its Laurent expansion by using various positivity properties, as well as partially known cluster expansions in Assumption 4.

However, Proposition 6.2.1 can not precisely control $L'^{(\deg X'_k(t))}$, because the desired basis element appears as the leading term in (20). To fix this, in Proposition 6.2.3, we put it as the last term in a similar but less intuitive equation.

Proposition 6.2.3 (One step mutation). Let any $t \in \mathbb{T}_n$ and $k \in [1, n]$ be given. Assume we have conditions $L_0(t), L_1(t), L_2(t)$ and $L_0(t[-1]), L_3([1, n]; t[-1])$. Furthermore, assume the basis $L'$ agrees with the basis $L'[-1]$, and $L'^{(\deg X'_k(t))} = L'[-1] (\deg X'_k(t))$. Then we have $X'_k(t) = L'(\deg X'_k(t))$. 

Proof. The quantum cluster variable $X'_k(t)$ is given by the exchange relation

$$X'_k(t) = X_k(t)^{-1}X(t)^{\hat{z}^t f_1} + X_k(t)^{-1}X(t)^{\hat{z}^t f_2},$$

where

$$i = \sum_{1 \leq i \leq n} [b_{ik}(t)]_i e_i,$$

$$f_1 = \sum_{1 \leq i \leq m-n} [b_{n+i,k}(t)]_{i+n} e_{i+n},$$

$$j = \sum_{1 \leq j \leq n} [b_{kj}(t)]_j e_j,$$

$$f_2 = \sum_{1 \leq j \leq m-n} [b_{k,j+n}(t)]_{j+n} e_{j+n}.$$

Recall that $\sigma e_i = e_{\sigma^{-1}i}$. In $T(t[-1])$, the above equation becomes

$$X'_k(t) = I_{\sigma^{-1}k}(t[-1])^{-1}X(t[-1])^{f_1} I(t[-1])^{\sigma_2} + I_{\sigma^{-1}k}(t[-1])^{-1}X(t[-1])^{f_2} I(t[-1])^{\sigma_2},$$

By (9), we have

$$\text{deg}^t_{[-1]} X'_k(t) = -\text{deg}^t_{[-1]} I_{\sigma^{-1}k}(t[-1]) + f_1 + \text{deg}^t_{[-1]} I(t[-1])^{\sigma_2}.$$

Let us denote

$$\text{deg}^t_{[-1]} I_{\sigma^{-1}k}(t[-1]) = -e_{\sigma^{-1}k} + f_3.$$

Correspondingly, we have the following expansion of a product of elements in $L^{t[-1]}$:

$$q^{-\alpha}X(t[-1])^{e_{\sigma^{-1}k} - f_3} \ast X(t[-1])^{f_1} I(t[-1])^{\sigma_2} = L_{[-1]}^{t}(\text{deg}^t_{[-1]} X'_k(t)) + \sum_{\eta} p_{\eta} L^{t[-1]}(\eta).$$

where the non-negative coefficients $p_{\eta}$ belong to $m$ by $L3([1, n]; t[-1])$, and the degree $\alpha$ given by

$$\alpha = \frac{1}{2} \Lambda(t[-1]) (e_{\sigma^{-1}k} - f_3, f_1 + \text{deg}^t_{[-1]} I(t[-1])^{\sigma_2}).$$
Notice that, by using Assumption (iv), $\alpha$ can also be written as
\[
\alpha = \frac{1}{2} \Lambda(t[-1])(-\deg^{t[-1]} I_{\sigma-1k}(t[-1]), f_1 + \deg^{t[-1]} I(t[-1])^{\sigma k}) \]
\[
= -\frac{1}{2} \Lambda(t[-1])(\deg^{t[-1]} I_{\sigma-1k}(t[-1]), f_1 + \deg^{t[-1]} I(t[-1])^{\sigma k}) \]
\[
= -\frac{1}{2} \Lambda(t)(\deg^t X_k(t), f_1 + \ell) \]
\[
= \frac{1}{2} \Lambda(t)(-\varepsilon_k, f_1 + \ell) \]
\[
= \frac{1}{2} \Lambda(t)(\deg^t P_k(t)Y(t)^{p(k,t)} - f_3, \deg^t X(t)^{f_1+\ell}),
\]
where we use (13) in the last equality.

We now view equation (22) in the quantum torus $T(t)$, which becomes
\[
q^{-\alpha} P_k(t)X(t[-1])^{-f_3} \ast X(t)^{f_1+\ell} = L^{t[-1]}(\deg^{t[-1]} X_k'(t)) + \sum_{\eta} p_{\eta} L^{t[-1]}(\eta)
\]
\[
= L^{t}(\deg^t X_k'(t)) + \sum_{\eta} p_{\eta} L^{t[-1]}(\eta).
\]
By (13), we also have
\[
\deg^t X_k'(t) = \ell + f_2 - \varepsilon_k
\]
\[
= -\varepsilon_k + f_3 - (\ell + f_1 - \ell + f_2) + \ell + f_1 - f_3
\]
\[
= \deg^{t[-1]} I_{\sigma-1k}(t[-1]) - \deg^t Y_k(t) + \ell + f_1 - f_3
\]
\[
= \deg^t P_k(t)Y(t)^{p(k,t)} - \varepsilon_k X(t)^{\ell + f_1 - f_3}.
\]
By (12) and L1(t), the Laurent expansion of $P_k(t)$ takes the form
\[
P_k(t) = X(t)^{\deg^t P_k(t)Y(t)^{p(k,t)} \cdot (\sum_{e_k < d \leq p(k,t) \beta_d Y(t)^{-d} + Y_k(t)^{-1} + 1)}),
\]
where $\beta_d \in \mathbb{N}[q^{\pm \frac{1}{2}}$]. Also, the coefficients $X(t)^{-f_3} = X(t[-1])^{-f_3}$. Consequently, the Laurent expansion of LHS of (23) becomes
\[
q^{-\alpha} P_k(t)X(t)^{-f_3} \ast X(t)^{f_1+\ell}
\]
\[
= q^{-\alpha} (X(t)^{\deg^t P_k(t) - f_3} Y(t)^{p(k,t)} \cdot (\sum_{e_k < d \leq p(k,t)} \beta_d Y(t)^{-d} + Y_k(t)^{-1} + 1)) \ast X(t)^{f_1+\ell}
\]
\[
= \sum_{e_k < d < p(k,t)} q^{\alpha d} \beta_d X(t)^{\deg^t P_k(t) + \ell - f_3 + f_1} Y(t)^{p(k,t) - d}
\]
\[
+ X(t)^{\deg^t P_k(t) + \ell - f_3 + f_1} Y(t)^{p(k,t) - e_k} + X(t)^{\deg^t P_k(t) + \ell - f_3 + f_1} Y(t)^{p(k,t)}
\]
\[
= \sum_{e_k < d < p(k,t)} q^{\alpha d} \beta_d X(t)^{\deg^t X_k'(t) Y(t)^{e_k - d} + X(t)^{\deg^t X_k'(t) + X(t)^{\deg^t X_k'(t) Y_k(t)}}
\]
where \( \alpha_d = \frac{1}{2} \Lambda^t(\deg^t Y(t)^{-d}, \hat{\ell}) \leq 0 \) for any \( d > e_k \).

Notice that the last two terms sum to \( X_k'(t) \). Similar to the discussion in Proposition 6.2.1, we compare this Laurent expansion to the Laurent expansion of the RHS of (23), which has non-negative coefficients by Lemma 6.1.5.

First, \( L^t(\deg X_k'(t)) \) contains \( X^{\deg X_k'(t)} \) because it needs to have the leading degree \( \deg^t X_k'(t) \). Notice that the Laurent expansion of \( L^t(\deg X_k'(t)) \) is contained in (24), and the only term with degree dominated by \( \deg^t X_k'(t) \) is \( X(t)^{\deg^t X_k'(t)} Y_k(t) \). Moreover, if the term \( X^{\deg^t X_k'(t)} Y_k(t) \) is contained in some other basis elements \( L^{t[-1]}(\eta) \) in (23), this will contradicts to the properties that \( p_\eta \in \mathfrak{m} \) and bar-invariance of \( L^{t[-1]}(\eta) \). Therefore, the desired basis element \( L^t(\deg X_k'(t)) \) must be \( X^{\deg^t X_k'(t)} Y_k(t) + X^{\deg^t X_k'(t)} = X_k'(t) \).

**Example 6.2.4** (Control Laurent expansion). We continue Example 5.1.6. Take \( k = 4 \). We want to control the Laurent expansion of \( L^t(\deg X_4'(t); t) \) by locating it as the last few terms of a Laurent polynomial in \( \mathcal{T}(t) \) and deduce that it is the triangular basis element \( L^t(\deg X_4'(t)) \).

We have the following quantum cluster variables in \( \mathcal{T}(t) \):

\[
X_4'(t) = X^{e_2-e_4}(1 + Y_4) = X^{e_1-e_4+e_8}(Y_4^{-1} + 1), \\
P_4(t) = X^{-e_4+e_5+e_8} (Y^{-e_1-e_4} + Y^{-e_4} + 1).
\]

Denote the coefficient part of the degree \( \deg^{t[-1]} \mathcal{L}_{\sigma^{-1}}(t[-1]) = -e_1 + e_5 + e_8 \) by \( f_3 = e_5 + e_8 \). Denote \( \hat{\ell} = \sum_{1 \leq i \leq n}[b_k(t)]_i e_i = e_1, f_1 = \sum_{i\geq n}[b_k(t)]_i e_i = e_8. \) Then \( \sigma \hat{\ell} = e_4. \)

The quantization matrix \( \Lambda(t[-1]) \) is given by

\[
\Lambda(t[-1]) = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
0 & 1 & 1 & 0 & -1 & 0 & 1 & 0 \\
0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0
\end{pmatrix}.
\]

Assume that \( L^t = L^{t[-1]} \) with positive structure constants and \( L^t(\deg^t X_4'(t)) = L^{t[-1]}(\deg^{t[-1]} X_4'(t)) \). Compute the degree

\[
\deg^{t[-1]} X_4'(t) = e_1 - e_4 + e_6 + e_7 \\
= e_{\sigma^{-1}4} - f_3 + f_1 + \deg^{t[-1]} I(t[-1]) \sigma \hat{\ell}.
\]

Then the desired basis element \( L^t(\deg^t X_4'(t)) \) appears in the \( (\prec t[-1], \mathfrak{m}) \)-unitriangular expansion of the following normalized product in
\(\mathcal{T}(t[-1]):\)

\[
q^{-a} X(t[-1])^{e_{a-1} - f_3} * X(t[-1])^{f_1} I(t[-1])^{e_2} = q^{-a} P_k(t) X(t[-1])^{-f_3} * X(t)^{i + f_1}.
\]

Compute the quantization degree

\[
\alpha = \frac{1}{2} \Lambda(t[-1])(e_1 - f_3, f_1 + \deg t[-1] I(t[-1])^{e_2})
\]

\[
= \frac{1}{2} \Lambda(t[-1])(e_1 - e_5 - e_8, -e_4 + e_5 + e_7 + e_8)
\]

\[
= \frac{1}{2}
\]

\[
= \frac{1}{2} \Lambda(t)(-e_4, e_1 + e_8)
\]

\[
= \frac{1}{2} \Lambda(t)(\deg t P_4(t) Y_1 Y_4 - f_3, f_1 + i).
\]

Now, view the twisted product (25) in \(\mathcal{T}(t)\), we obtain

\[
q^{-a} P_4(t) X^{-f_3} * X^{f_1 + i} = q^{-\frac{1}{a}} X^{e_1 - e_4 + i} Y_1^{a-1} Y_4^{-1} + X^{e_1 - e_4 + e_8}(Y_4^{-1} + 1)
\]

\[
= q^{-\frac{1}{a}} X^{-e_5 + X^{i}}.
\]

Recall that the expansion of the normalized product into \(\mathcal{L}^t[-1]\) takes the form

\[
q^{-a} P_4(t) X(t)^{-f_3} * X(t)^{f_1 + i} = \mathcal{L}^t[-1](\deg t[-1] X_4'(t)) + \sum_{\eta < t[-1]\deg t[-1]} p_\eta \mathcal{L}^t[-1](\eta)
\]

\[
= \mathcal{L}^t(\deg t X_4'(t)) + \sum_{\eta < t[-1]\deg t[-1]} p_\eta \mathcal{L}^t[-1](\eta)
\]

with non-negative coefficients \(p_\eta \in \mathfrak{m}\).

It follows that \(X_4'(t)\) must agree with \(\mathcal{L}^t(\deg t X_4'(t))\).

6.3. Triangular bases for new seeds. We shall show in Proposition 6.3.2 that, if the quantum cluster variables \(X'_k(t), I'_k(t)\) are contained in the triangular basis \(\mathcal{L}'\), then \(\mathcal{L}'\) implies the existence of the triangular basis \(\mathcal{L}'\) for the new seed \(t' = \mu_k t\) and they are compatible. We further verify the local conditions for the new triangular basis in Proposition 6.3.5.

**Lemma 6.3.1.** Let \(t, t'\) be two given seeds. Assume that we have \(\mathcal{L}0(t)\) and \(\mathcal{L}0(t')\) with \(\mathcal{L}' = \mathcal{L}'\) given by \(\mathcal{L}'(\bar{g}; t') = \mathcal{L}(\phi_{t,t'}(\bar{g}); t), \forall \bar{g} \in D(t')\). Let \(Z \in A\) be any element which has a unique leading degree in each of \(\mathcal{T}(t)\) and \(\mathcal{T}(t')\), such that we have \(\deg t Z = \eta Z, \deg t' Z = \bar{g} Z = \phi_{t,t'}(\eta Z)\), for some \(\eta \in D(t)\).

If \([Z]\) is \((\prec_t, \mathfrak{m})\)-unitriangular to \(\mathcal{L}'\), then it is also pointed in \(\mathcal{T}(t')\) and \((\prec_{t'}, \mathfrak{m})\)-unitriangular to \(\mathcal{L}'\) as well.
Proof. By the assumption, in \( \mathcal{T}(t) \), we have
\[
[Z]_t^t = L(\eta_Z; t) + \sum_{\eta' \prec \eta_Z} c_{\eta'} L(\eta'; t), \quad c_{\eta'} \in \mathfrak{m}.
\]
Because \( Z \in \mathcal{A} \), RHS is a finite sum. View this equation in \( \mathcal{T}(t') \) and rewrite RHS in terms of \( L' \), we obtain
\[
[Z]_t^t = L(\tilde{g}_Z; t') + \sum_{\eta' \prec \eta_Z} c_{\eta'} L(\phi_{t,t}(\eta'); t'), \quad c_{\eta'} \in \mathfrak{m}.
\]
Notice that \( \phi_{t,t}\eta' \neq \phi_{t,t}\eta_Z = \tilde{g}_Z \). By this equation, \( [Z]_t^t \) has coefficient 1 at its leading degree \( \tilde{g}_Z \). In particular, \( [Z]_t^t \) is pointed at \( \tilde{g}_Z \) in \( \mathcal{T}(t') \). Moreover, this expansion is a finite sum. By applying Lemma 3.1.9(i)(iii), we see it must take the form
\[
[Z]_t^t = L(\tilde{g}_Z; t') + \sum_{\tilde{g'} \prec \nu \tilde{g}_Z} c_{\tilde{g'}} L(\tilde{g'}; t), \quad c_{\tilde{g'}} \in \mathfrak{m}.
\]
\[\square\]

**Proposition 6.3.2** (triangular basis in new seed). Let \( t, t' \) be seeds related by a mutation at some vertex \( k \). Assume that we have \( L_0(t) \)
\( L_1(t) \) \( L_2(t) \) \( L_3([1,n]; t) \) as well as \( L_4(k;t) \) and \( L_5(k;t) \). Furthermore, assume that, in the quantum torus \( \mathcal{T}(t), \mathfrak{I}' \) is \((\prec_t, \mathfrak{m})\)-unitriangular to \( \mathfrak{I}' \). Then the following claims are true.

(i) The quantum cluster monomials in \( t' \) belong to \( \mathfrak{L}' \).

(ii) \( \mathfrak{L}' \) is \( \mathfrak{D}(t') \)-pointed in \( \mathcal{T}(t') \), such that
\[
\deg'(L(\eta; t)) = \phi_{t,t}(\eta), \quad \eta \in \mathfrak{D}(t).
\]

In other words, we have the bar-invariant \( \mathfrak{D}(t') \)-pointed basis \( \mathfrak{L}' \)
defined by \( L(\tilde{g}; t') = L(\phi_{t,t}\tilde{g}; t) \) for any \( \tilde{g} \in \mathfrak{D}(t) \).

(iii) \( \mathfrak{I}' \) is \((\prec_{t'}, \mathfrak{m})\)-unitriangular to \( \mathfrak{L}' \) in \( \mathcal{T}(t') \).

(iv) \( \mathfrak{I}' \) is \((\prec_{t'}, \mathfrak{m})\)-unitriangular to \( \mathfrak{I}' \) in \( \mathcal{T}(t') \).

**Proof.** (i) By \( L_3([1,n]; t) \), the normalized product \([X_d x] = (X_d)^{d_1} t), d_X \in \mathbb{N} \), \( d_k \) \( \in \mathbb{N} \), is \((\prec_t, \mathfrak{m})\)-unitriangular to \( \mathfrak{L}' \). The statement follows by taking those \( d_X \) with the \( k \)-th coordinate vanishing.

(ii) It follows from Lemma 5.3.2 that \( \mathfrak{I}(\tilde{g}; t') \) is pointed at degree \( \phi_{t',t}\tilde{g} \) in \( \mathcal{T}(t) \). By the hypothesis in the Proposition, it is \((\prec_t, \mathfrak{m})\)-unitriangular to \( \mathfrak{I}' \) and, consequently, to \( \mathfrak{L}' \). So we have
\[
\mathfrak{I}(\tilde{g}; t') = L(\eta; t) + \sum_{\eta' \prec \eta} c_{\eta\eta'} L(\eta'; t), \quad \eta, \eta' \in \mathfrak{D}(t), \quad \tilde{g} = \phi_{t,t}(\eta), \quad c_{\eta\eta'} \in \mathfrak{m}.
\]

View (27) in \( \mathcal{T}(t') \) from now on. Since \( \mathfrak{I}(\tilde{g}; t') \) belong to \( \mathcal{A} \), this linear combination is a finite sum. Furthermore, because \( X_k(t') \) and \( I_k(t') \) belong to \( \mathfrak{L}' \) by \( L_4(k;t) \) \( L_5(k;t) \), the coefficients \( c_{\eta\eta'} \) are structure constants of the basis \( \mathfrak{L}' \) and thus non-negative by \( L_1(t) \). In addition, by the same proof of Lemma 6.1.5, \( L_4(k;t) \) and \( L_5(k;t) \) imply that the
Laurent expansion of each term of RHS in the quantum torus $\mathcal{T}(t')$ must have non-negative coefficients. Finally, notice that LHS has a unique leading term of degree $\tilde{g}$ with coefficient 1. Therefore, there exists a unique leading term of degree $\tilde{g}$ in RHS with coefficient 1, which must be the contribution from $L(\eta; t)$. Consequently, $L(\eta; t)$ is pointed at $\tilde{g}$ in $\mathcal{T}(t')$.

(iii) We obtain the claim by using (27) for each element $I'(\tilde{g}; t')$.

(iv) $I'$ is $\leq_t, \mathfrak{m}$)-unitriangular to $L'$ by Lemma 5.3.2 and Lemma 6.3.1. $Z'$ is $\leq_t, \mathfrak{m}$)-unitriangular to $I'$ by (ii) and Lemma 3.1.10. The claim follows.

\begin{remark}
The positivity condition $L_1(t)$ in Proposition 6.3.2 is only used for proving the claim (ii).
\end{remark}

We enhance Proposition 6.3.2 by verifying the local conditions for the new seed $t'$ equipped with the new triangular basis $L'$. 

\begin{lemma}
Let $t$ be any given seed. Assume that we have $L_0(t)$ $L_2(t)$ $L_3([1, n]; t)$. Let $Z$ be a element in $\mathcal{T}(t)$ with a unique maximal degree.

If the normalization $[Z]^t$ in $\mathcal{T}(t)$ is $\leq_t, \mathfrak{m}$)-unitriangular to $I'$, then for any $f \in \mathbb{Z}_{m-n}$, $d_x, d_I \in \mathbb{N}^n$, the normalized product $[\prod_i X_{n+i}(t)^{f_i} \times X(t)^{d_x} \times Z \times I(t)^{d_I}]^t$ is $\leq_t, \mathfrak{m}$)-unitriangular to $I'$ as well.

\begin{proof}
Notice that we have $L_3''(t)$ by Lemma 6.1.7. Substitute the factor $[Z]^t$ by its expansion in $I'$. By using Lemma 6.1.4 and $L_3''(t)$, we obtain the desired result.
\end{proof}
\end{lemma}

\begin{proposition}[local conditions for new seeds]
Let $t, t'$ be seeds related by a mutation at some given vertex $k$.

(i) The conditions $L_0(t')$, $L_1(t')$, $L_2(t')$, $L_3([1, n]; t')$, $L_4(k; t)$, $L_5(k; t)$ imply $L_0(t')$, $L_1(t')$, $L_2(t')$ with $L(t\tilde{g}; t') = L(\phi_{t,v}(\tilde{g}); t)$.

(ii) The conditions $L_0(t')$, $L_2(t')$, $L_3([1, n]; t)$ and conditions $L_0(t')$, $L_2(t')$ with $L(t\tilde{g}; t') = L(\phi_{t,v}(\tilde{g}); t)$ imply $L_3'([1, n]; t')$.

\begin{proof}
For simplicity, we omit the notation $(t)$ inside $X(t)^d$, $X_k(t)$, $I(t)^{d'}, I'_k(t)$, $d, d' \in \mathbb{N}^n$ in the proof.

Proof of claim (i):

By Definition 5.3.1, for any given $\tilde{g} \in D(t')$, we have, in $\mathcal{T}(t)$,

$I(t\tilde{g}; t') = [X^{g_k} X^{[g_k]} \times (X_k')^{[g_k]} \times (I'_k)^{-g_k}]^t$,

where $g_k \in \mathbb{Z}^n$ is obtained from $g$ by imposing $g_k = 0$. By Lemma 5.3.2, it is normalized in $\mathcal{T}(t)$ as well:

$I(\tilde{g}; t') = [X^{g_k} X^{[g_k]} \times (X_k')^{[g_k]} \times (I'_k)^{-g_k}]^t$.

We substitute the factors $(X_k')^{[g_k]}$ or $(I'_k)^{-g_k}$ appearing by using the equations in $L_4'(k; t)$, $L_5'(k; t)$. Because at most one of them could

appear, by using Lemma 6.3.4, we obtain that, for any \( \tilde{g} \in D(t') \),

\[
I(\tilde{g}; t') \in I(\eta; t) + \sum_{\eta' \prec \eta} mI(\eta'; t).
\]

where \( \eta = \phi_{t,t'}(\tilde{g}) \). Namely, \( I' \) is \((\prec_t, m)\)-unitriangular to \( I \). By Proposition 6.3.2, \( t' \) satisfies L0 L1 L2, such that \( L(\tilde{g}; t') = L(\phi_{t,t'} \tilde{g}; t) \).

**Proof of claim (ii):** Take any \( S \in L' \).

First, take any \( i \neq k \). We have \( X_i(t') = X_i(t) \). By L3(i; t), the normalized twisted product \( [X_i(t') * S]^t = [X_i(t) * S]^t \) is \((\prec_t, m)\)-unitriangular to \( L' \).

In \( T(t') \), the twisted product \( [X_i(t) * S]^t \) has the unique maximal degree \( \deg^t(X_i(t)) + \deg^t S \). This maximal degree turns out to be \( \phi_{\nu t} \deg^t X_i(t) + \phi_{\nu t} \deg^t S = \phi_{\nu t} \deg^t(X_i(t) * S) \), thanks to the sign coherence at the \( k \)-th components. Therefore, we can apply Lemma 6.3.1 and deduce that \( [X_i(t') * S]^t \) is pointed and \((\prec_{t'}, m)\)-unitriangular to \( L' \) in \( T(t') \).

It remains to prove that the normalized twisted product \( [X_k' * S]^{t'} \) is \((\prec_{t'}, m)\)-unitriangular to \( L' \).

Denote \( \deg^t S = \tilde{g} \). By Proposition 6.3.2(ii)(iii)(iv), \( S \) is \((\prec_{t'}, m)\)-unitriangular to \( I' \). Therefore, by applying Lemma 6.1.4 in \( T(t') \), we have

\[
[X_k' * S]^{t'} = [X_k' * \sum_{\phi_{\nu', t^\prime} \eta^\prime \leq \nu, \tilde{g}} c_{\eta^\prime} I(\eta; t)]^{t'}
\]

where \( c_{\eta} = 1, \alpha_{\eta} = 0 \), if \( \phi_{\nu, t^\prime} \eta = \tilde{g} \), and \( c_{\eta} \in m, \alpha_{\eta} \leq 0 \) if \( \phi_{\nu, t^\prime} \eta \prec_{t'} \tilde{g} \). It suffices to show that each term \( [X_k' * I(\eta; t)]^{t'} \) is \((\prec_{t'}, m)\)-unitriangular to \( L' \).

Let us write

\[
I(\eta; t) = [X^{\eta_f} * X^{[\eta_k]_+} * X_k^{[\eta_k]_+} * I_k^{-[\eta_k]_+} * I^{[-\eta_k]_+}]^t
\]

where \( \eta_k \) is obtained from \( \pr_n \eta \) by setting the \( k \)-th component to 0, \( \eta_f \) is the degree of coefficients such that the leading terms of both side agree. This element is also normalized in \( T(t') \) by Lemma 5.3.2:

\[
I(\eta; t) = [X^{\eta_f} * X^{[\eta_k]_+} * X_k^{[\eta_k]_+} * I_k^{-[\eta_k]_+} * I^{[-\eta_k]_+}]^{t'}.
\]

(Case ii-a) If \( \eta_k \leq 0 \), then \( I(\eta; t) \) does not contain any factor \( X_k \). Consider the following normalized product in \( T(t) \)

\[
[X_k' * I(\eta; t)]^t = [X^{\eta_f} * X^{[\eta_k]_+} * X_k' * X_k^{[\eta_k]_+} * I_k^{-[\eta_k]_+} * I^{[-\eta_k]_+}]^t.
\]

The leading degrees \( \deg^t X_k' \) and \( \deg^t I(\eta; t) \) are sign coherent at the \( k \)-th components. It follows that

\[
\deg^{t'} (X_k' * I(\eta; t)) = \phi_{t', t} \deg^t (X_k' * I(\eta; t)).
\]
In addition, $X'_k \in L^{t'} = L^t$ is $(\prec_t, m)$-unitriangular to $I^t$ and, by Lemma 6.3.4, $X'_k \star I(\eta; t)$ is $(\prec_t, m)$-unitriangular to $I^t$ after normalization in $T(t)$. Therefore, we can change the seed from $t$ to $t'$ by applying Lemma 6.3.1 and obtain that $[X'_k \star I(\eta; t)]^{t'} = [X'_k \star I(\eta; t)]^{t'}$ and that $[X'_k \star I(\eta; t)]^{t'}$ is $(\prec_{t'}, m)$-unitriangular to $L^{t'}$.

(Case ii-b) If $\eta_k > 0$, we have the following normalized product

\[(28)\quad [X'_k \star I(\eta; t)]^{t'} = [X^{\eta_k} X^{[\eta_k]} + X'_k X_k^\eta \star I^{-[\eta_k]}]^{t'}
  = [X^{\eta_k} X^{[\eta_k]} + (X'_k X_k) X_k^{\eta-1} \star I^{-[\eta_k]}]^{t'}.
\]

By the exchange relation at vertex $k$, we have, in the quantum torus $T(t')$,

\[\ [X'_k \star X_k]^{t'} = X^\hat{k} + q^{-\frac{d_k}{2}} X^\hat{k} \]

\[= X^\hat{k} + q^{-\frac{d_k}{2}} X^\hat{k}Y_k^{-1}(t),\]

where $\hat{i} = \sum_i [h_{ik}(t)] + e_i$ and $\hat{j} = \sum_j [h_{kj}(t)] + e_j$.

Notice that the leading term of the pointed element $[X'_k \star X_k]^{t'}$ in $T(t')$ is $X^\hat{k}$. By the definition of normalization, (28) becomes

\[ [X'_k \star I(\eta; t)]^{t'} = [X^{\eta_k} X^{[\eta_k]} + (X^\hat{k} + q^{-\frac{d_k}{2}} X^\hat{k}Y_k^{-1}) \star X_k^{\eta-1} \star I^{-[\eta_k]}]^{t'}
  = [X^{\eta_k} X^{[\eta_k]} + X^\hat{k}X_k^{\eta-1} \star I^{-[\eta_k]}]^{t'}
  + q^{-\frac{d_k}{2}} q^{\alpha_k} [X^{\eta_k} X^{[\eta_k]} + X^\hat{k}Y_k^{-1}X_k^{\eta-1} \star I^{-[\eta_k]}]^{t'},\]

where, by the sign coherence at the $k$-th components and the mutation rule of quantization matrices, we have

\[\alpha_k = \frac{1}{2} \Lambda(t')(\deg Y_k^{-1}, (\eta_k - 1) \deg X_k) = \frac{1}{2} \Lambda(t)(\deg Y_k^{-1}, (\eta_k - 1) \deg X_k) = - (\eta_k - 1) \frac{d_k}{2} \leq 0.\]

Now, in the quantum torus $T(t)$, the leading degrees of the factors in both terms are sign coherent at the $k$-th components. It follows that

\[\deg X^\eta X^{[\eta_k]} + X^\hat{k}X_k^{\eta-1} \star I^{-[\eta_k]} = \phi_{t'}, \deg X^{\eta_k} X^{[\eta_k]} + X^\hat{k}X_k^{\eta-1} \star I^{-[\eta_k]} +
\deg X^\eta X^{[\eta_k]} + X^\hat{k}Y_k^{-1}X_k^{\eta-1} \star I^{-[\eta_k]} = \phi_{t'}, \deg X^{\eta_k} X^{[\eta_k]} + X^\hat{k}Y_k^{-1}X_k^{\eta-1} \star I^{-[\eta_k]} +\]

By $L^{t''}(t)$, these terms, when normalized in $T(t)$, are $(\prec_t, m)$-unitriangular to $L^t$. Applying Lemma 6.3.1, their normalization in $T(t')$ remain the same, and, when normalized in $T(t')$, they become $(\prec_{t'}, m)$-unitriangular to $L^{t'}$. \qed
6.4. Existence theorem. We present the existence theorem about the common triangular basis. The criterion can be reduced to a finite problem, as we shall see in the reduced existence theorem 6.4.7.

**Theorem 6.4.1 (Existence theorem)**. Assume that Assumption 1 2 3 4 hold. Further assume that there exists some injective-reachable chain of seeds \((t[d])_{d \in \mathbb{Z}}\), which satisfies the following local conditions:

- \(L_0(t[d]):\) there exists a bar-invariant \(D(t[d])\)-pointed basis \(L^{[d]}\) in \(A\), which factors through the frozen variables and satisfies \(L_0(t[d]) = 1\),
- \(L_1(t[d]):\) the structure constants of \(L^{[d]}\) belong to \(\mathbb{N}[q^\pm 1]\),
- \(L_2(t[d]):\) the injective pointed set \(I^{[d]}\) is \((<t[d], m)\)-unitriangular to \(L^{[d]}\),
- \(L_3([1, n]; t[d]):\) for any \(s \in L^{[d]}\) and \(k \in [1, n]\), the normalized twisted product \([X_k(t[d]) * S^{[d]}]\) is \((<t[d], m)\)-unitriangular to \(L^{[d]}\) in the quantum torus \(T(t[d])\).
- Compatibility with tropical transformations:
  - \(L(\eta; t[d]) = L(\phi_{t[d+1], t[d]}(\eta); t[d + 1])\)
  for any \(d \in \mathbb{Z}\), \(\eta \in D(t[d])\).

Then, for any vertex \(t' \in T_n\), the local conditions \(L_0(t'), L_1(t'), L_2(t'), L_3([1, n]; t')\) are satisfied with the basis \(L'\) defined by \(L(\tilde{g}; t') = L(\phi_t, \nu(\tilde{g}); t), \forall \tilde{g} \in D(t')\).

**Proof of Theorem 6.4.1.** For any \(1 \leq k \leq n\), denote \(t = \mu_k t\).

Observe that, by Proposition 6.2.3 and Lemma 6.1.6 (iii), if we have \(L_0(t[d]), L_1(t[d]), L_2(t[d]), L_3([1, n]; t[d])\), together with \(L(\eta; t[d]) = L(\phi_{t[d-1], t[d]}(\eta); t[d - 1])\) for all \(d \in \mathbb{Z}\), then \(L_4(\sigma^k t[d])\) and, equivalently, \(L_5(\sigma^d k; t[d - 1])\), hold true for all \(d \in \mathbb{Z}\). Further applying Proposition 6.3.5, we obtain triangular bases in the seeds \(t'[d], d \in \mathbb{Z}\), which satisfy the compatibility conditions

\[
L(\tilde{g}; t'[d]) = L(\phi_{t[d-1], t[d]}(\eta); t'[d - 1]),
\]

and

\[
L(\tilde{g}; t'[d]) = L(\phi_{t[d], t[d]}(\eta); t[d]),
\]

By Proposition 6.3.5, we obtain that for \(t' = \mu_k t\), the chain \((t[d])_{d \in \mathbb{Z}}\) verifies all the local conditions in the theorem. Moreover, the common triangular basis on it is compatible with the common triangular basis on \((t[d])\).

We verify the theorem by repeatedly applying the above arguments for all seeds along any mutation sequence.

**Definition 6.4.2.** Under the assumption of the Theorem 6.4.1, the common injective-triangular basis \(L'\), \(t \in T_n\), is called the common triangular basis of \(A\), which we can denote by \(L\) without confusion.

**Remark 6.4.3.** By Theorem 6.4.1, the elements of the common triangular basis \(L\) have distinct unique leading terms in every quantum torus,
and their degrees change under the piecewise linear formula (5). Therefore, \( L \) possesses the property of the basis expected by Fock-Goncharov conjecture [FG06].

For the rest of this subsection, we require that the cluster algebra is associated with skew-symmetric matrices, so that the quantum \( F \)-polynomials are known to exist.

A change of the coefficient pattern or quantization will not affect the existence of the common triangular basis.

**Corollary 6.4.4.** Assume \( B(t_0) \) is skew-symmetric. Let \( t_0, t'_0 \) be two initial seeds with similar compatible pairs in the sense of Section 4 and \( \mathcal{A}(t_0), \mathcal{A}(t'_0) \) the corresponding quantum cluster algebras respectively. Assume that \( \mathcal{A}(t_0) \) verifies all the conditions in Theorem 6.4.1.

Let \( L^{t_0} \) denote the initial triangular basis of \( \mathcal{A}(t'_0) \) constructed as similar elements of the initial basis \( L^{t_0} \subset \mathcal{A}(t_0) \), cf. 6.1.8. Then \( \mathcal{A}(t_0) \) verifies all the conditions in Theorem 6.4.1 such that \( L^{t_0} \) lifts to the common triangular basis of \( \mathcal{A}(t'_0) \).

**Proof.** We can prove it based on Theorem 6.4.1 and the correction technique in Section 4.

Alternatively, we can deduce it from Theorem 6.4.7 and the existence of quantum \( F \)-polynomials. □

Next, we use the correction technique in Section 4 to reduce the existence theorem 6.4.1.

**Proposition 6.4.5.** Assume that some seed \( t \) satisfies \( L_0(t), L_1(t), L_2(t), L_3(t) \). Then, for any \( d \in \mathbb{Z} \), \( t[d] \) satisfies the conditions \( L_0(t[d]), L_1(t[d]), L_2(t[d]), L_3(t[d]) \) as well.

**Proof.** We know these conditions are satisfied in \( t[0] = t \). Any given seed \( t[d], d \in \mathbb{Z} \), is similar to \( t[0] \). So we obtain its triangular basis as similar elements to the elements of \( L^{t[0]} \) by Lemma 6.1.8. We deduce that the local conditions are satisfied the correction technique in Section 4. □

**Lemma 6.4.6.** Let \( \mu \) be any mutation sequence and \( (t[d])_{d \in \mathbb{Z}} \) a chosen injective-reachable chain of seeds on which \( L_0 \) and \( L_2 \) hold. If for some \( 1 \leq k \leq n \), the cluster variable \( X_k(\mu t) \) belongs to the basis \( L^t \), then for any \( d \in \mathbb{Z} \), the cluster variable \( X_{\sigma^d k}(\sigma^d \mu(t[d])) \) belongs to \( L^{t[d]} \).

**Proof.** The compatible pairs of the seeds \( t[d], d \in \mathbb{Z} \) are similar. Because \( X_k(\mu t) \) is unitriangular to \( I^t \), we have a similar unitriangular expansion of \( X_{\sigma^d k}(\sigma^d \mu(t[d])) \) in \( I^{t[d]} \) by Theorem 4.2.1, where the coefficients \( f_j \) in the Theorem are absorbed into basis elements of \( I^{t[d]} \). The claim follows. □

The following theorem is a possible reduction of the existence theorem 6.4.1, which reduces the criterion to a finite problem.
Recall that Assumption 134 are satisfied by skew-symmetric (quantum) cluster algebras.

**Theorem 6.4.7** (Reduced existence theorem). Assume that \( t \) is a seed injective-reachable via \((\Sigma, \sigma)\) and \( B(t) \) a skew-symmetric matrix. Assume that \( t \) verifies the local conditions \( L_0(t), L_1(t), L_2(t), L_3([1, n]; t), \) and, furthermore, its triangular basis \( L^t \) contains the quantum cluster variables obtained along the mutation sequences \( \Sigma \) and \( \sigma \Sigma^{-1} \) starting from \( t \).

Then, for any vertex \( t' \in T_n \), the local conditions \( L_0(t'), L_1(t'), L_2(t'), L_3([1, n]; t') \) are satisfied with the basis \( L^{t'} \) defined by \( L(\tilde{g}; t') = L(\phi_{t,t'}(\tilde{g}); t), \forall \tilde{g} \in D(t'). \)

**Proof.** By Proposition 6.4.5 and the correction technique 4.2.1, the existence of \( L^t \) implies the existence of the triangular basis \( L^{l[d]}, d \in \mathbb{Z} \), which satisfies \( L_0, L_1, L_2, L_3 \) as well. It remains to show that these bases are compatible.

Repeatedly using Proposition 6.3.5, we can construct the triangular bases \( L^{l'} \) for the seeds \( t' \) obtained along the mutation sequence \( \sigma \Sigma^{-1} \) from \( t \) to \( t[-1] \). These bases are compatible with \( L^t \). In particular, \( L^{l[-1]} \) and \( L^t \) are compatible. By using Lemma 6.4.6, we deduce the similar result for \( L^{l[d]}, d \in \mathbb{Z} \).

Now, we have found all conditions demanded by Theorem 6.4.1. The claim follows.

---

7. **Graded quiver varieties and minuscule modules**

Fix a symmetric generalized Cartan matrix \( C \). As a preparation for the statement of monoidal categorification in Section 8.4 and the proof of theorem 9.4.1, we review graded quiver quivers, working in a framework slightly generalized than that of \cite{Nak03}. Following the arguments of \cite{Nak03}, we obtain characters of some simple modules as our main result Proposition 7.4.3(i)(ii).

A reader unfamiliar with graded quiver varieties, quantum affine algebras, or only interested in type ADE might admit the construction in \cite{Nak03}\cite{Nak04} and skip this section, whose main result will only be used in the proof of Proposition 9.2.4.

**7.1. A review of graded quiver varieties.** We start with a review of the graded quiver varieties used in \cite{KQ12}\cite{Qin13}, whose construction follows from that of \cite{Nak01}\cite{Nak04}. We choose its grading as in \cite{Qin13}\cite{KQ12}.

**Graded quiver varieties.** Given any generalized symmetric Cartan matrix, we associate the diagram \( \Gamma \) to the Cartan matrix such that its set of vertices is \( I \) and has \(-C_{ij}\) edges between any two different vertices \( i \) and \( j \). We then choose an orientation \( \Omega \) of its edges.
Fix an acyclic quiver \((\Gamma, \Omega)\) associated with \(C\). For any arrow \(h\) in \(\Omega\), we denote its opposite arrow by \(\overline{h}\). Let \(\overline{\Omega}\) denote the set of arrows opposite to the arrows in \(\Omega\). Define the \(H\) to be the union of \(\Omega\) and \(\overline{\Omega}\).

We follow the torus grading in [Qin13, Section 4.3][KQ12]. Choose a map \(\xi\) from \(I\) to \(\{1, 2, \ldots, r\}\), whose images are denoted by \(\xi_i\) for \(i \in I\), such that \(\xi_i > \xi_j\) whenever there exists a nontrivial path from \(i\) to \(j\) in the acyclic quiver \((\Gamma, \Omega)\). Define \(\epsilon_{ij} \in \{\pm 1\}\) to be the sign of \(\xi_i - \xi_j\) and

\[
\epsilon_{ij} = \frac{\xi_i - \xi_j}{r}.
\]

It follows that \(\epsilon_{s(h)\overline{(h)}} = 1\) if \(h \in \Omega\) and \(\epsilon_{s(h)\overline{(h)}} = -1\) if \(h \in \overline{\Omega}\).

**Example 7.1.1.** Consider the acyclic quiver \((\Gamma, \Omega)\) in Figure 5. We have \(\epsilon_{21} = \epsilon_{23} = \epsilon_{34} = \epsilon_{45} = 1\), \(\epsilon_{12} = \epsilon_{32} = \epsilon_{43} = \epsilon_{54} = -1\).

![Figure 5. An acyclic quiver \((\Gamma, \Omega)\) of Cartan type \(A_5\)](image)

Define the set of vertices

\[
\mathbb{W} = I \times 2\mathbb{Z}
\]
\[
\mathbb{V} = I \times (1 + 2\mathbb{Z}).
\]

The \(q\)-analog of Cartan matrix \(C_q\) is defined to be the linear map from \(\mathbb{Z}^{I\times\mathbb{Z}}\) to \(\mathbb{Z}^{I\times\mathbb{Z}}\), such that any \(\eta = (\eta_{i,b})\) has the image given by

\[
(C_q\eta)_{i,a} = \eta_{i,a-1} + \eta_{i,a+1} + \sum_{j \neq i} C_{ij}\eta_{j,a+\epsilon_{ij}}.
\]

For any finitely supported \(v \in \mathbb{N}^\mathbb{V}\), \(w \in \mathbb{N}^\mathbb{W}\), we say the pair \((v, w)\) is \(l\)-dominant if \(w - C_q v \geq 0\).

For any finite supported \(w\), define \(C_q^{-1}w\) to be the unique vector such that \((C_q^{-1}w)_{i,b} = 0\) when \(b \ll 0\) and \(C_q(C_q^{-1}w) = w\). We refer the reader to [HL11, Example 2.2] for an example, where a comparison of notations is presented in Remark 7.2.1.
Example 7.1.2. Consider the acyclic quiver \((\Gamma, \Omega)\) in Figure 6. Then we can only choose \(\xi_1 = 3, \xi_2 = 2, \xi_3 = 1\). We have \(\epsilon_{12} = \epsilon_{23} = \epsilon_{13} = 1, \epsilon_{21} = \epsilon_{32} = \epsilon_{31} = -1\), and the following equation

\[
(C_q \eta)_{2,a} = \eta_{2,a-1} + \eta_{2,a+1} - \eta_{1,a-1} - \eta_{3,a+1}.
\]

The component \(\eta_{2,a+1}\) contributes to the components of degree \((2, a), (2, a + 2), (1, a + 1 + \epsilon_{21}), (3, a + 1 + \epsilon_{23})\) via the map \(C_q\).

Figure 6. An acyclic quiver \((\Gamma, \Omega)\) of Cartan type \(A_2^{(1)}\)

For any \(d \in \mathbb{Z}\), define the shift operator \([d]\) to be the linear map from \(\mathbb{Z}^{I \times \mathbb{Z}}\) to \(\mathbb{Z}^{I \times \mathbb{Z}}\) such that, for any \(\eta \in \mathbb{Z}^{I \times \mathbb{Z}}\), we have

\[
(\eta[d])_{i,a} = \eta_{i,a+d}.
\]

Take any finitely supported dimension vector \(v, v'\), in \(\mathbb{N}^V\) and \(w\) in \(\mathbb{N}^W\). Denote the associated graded \(\mathbb{C}\)-vector by \(V, V'\) and \(W\). We define the following vector spaces

\[
L(w, v) = \oplus_{(i,a) \in W} \text{Hom}(W_{i,a}, V_{i,a-1}),
\]

\[
L(v, w) = \oplus_{(i,b) \in V} \text{Hom}(V_{i,b}, W_{i,b-1}),
\]

\[
E(\Omega; v, v') = \bigoplus_{h \in H, b \in 1+2\mathbb{Z}} \text{Hom}(V_{s(h), b}, V'_{t(h), b-1+\epsilon_{s(h), t(h)}}) = \bigoplus_{h \in \Omega, b \in 1+2\mathbb{Z}} \text{Hom}(V_{s(h), b}, V'_{t(h), b})
\]

\[
\oplus \bigoplus_{\tau \in \Omega, h \in 1+2\mathbb{Z}} \text{Hom}(V_{s(\tau), b}, V'_{t(\tau), b-2}).
\]

Define the vector space \(\text{Rep}(\Omega; v, w) = E(\Omega; v, v) \oplus L(w, v) \oplus L(v, w)\), whose coordinates will be denoted by

\[
(B, t, j) = ((B_h)_{h \in H, t, j}) = ((B_h)_{h \in \Omega}, (B_{\tau})_{\tau \in \tau}, (t_i), (j_i)) = ((\oplus_b B_{h, b}), (\oplus_{\tau} B_{\tau, b}), (\oplus_{a_i} t_i), (\oplus_{b_{j_i}} b)).
\]

Notice that \(\text{Rep}(\Omega; v, w)\) can be naturally viewed as the vector space of \((v, w)\)-dimensional representations of a quiver with vertices \(V \sqcup W\). We denote this quiver by \(\tilde{\Gamma}_\Omega\) can call it the framed repetition quiver associated with the acyclic orientation \(\Omega\).

Different acyclic orientations produce isomorphic framed repetition quivers. Choose and fix such a quiver \(\tilde{\Gamma}_\Omega\) from now on.
The analog of the moment map $\mu$ is the linear map from $\text{Rep}(\Omega; v, w)$ to $L(v, v[-1])$ given by

$$
\mu(B, i, j) = \bigoplus_{(i, b) \in V} \mu(B, i, j),
$$

where

$$
\mu(B, i, j) = \bigoplus_{h \in \Omega, t(h) = i} b_{h, 1} b_{h, 2} - \sum_{h \in \Omega, s(h) = i} b_{h, 1} b_{h, 2} + (i, b-1, j, b).
$$

Example 7.1.3. Let the acyclic quiver $(\Gamma, \Omega)$ be given by Figure 5. Part of the the framed repetition quiver $\tilde{\Gamma}_\Omega$ is shown in Figure 7.

![Figure 7](image)

The variety $\mu^{-1}(0)$ has the natural action by the reductive algebraic group $G_v = \prod_{(i, b) \in V} GL(V, b)$ such that, for any given $g = (g_{i, b})$, the action is given by

$$
g(i) = g_i,
$$

$$
g(j) = g^{-1},
$$

$$
g(b_h) = g_{t(h)} b_h g_{s(h)}^{-1}, \quad \forall h \in H.
$$

Fix the character $\chi_v$ of $G_v$ given by $\chi_v(g) = \prod_{i, b}(\det g_{i, b})^{-1}$.

Define $M^\bullet(v, w)$ to be the geometric invariant quotient of the $G_v$-variety $\mu^{-1}(0)$ associated with $\chi_v$ in the sense of Mumford, cf. [MFK94]. Denote the categorical quotient by $M_0^\bullet(v, w)$ and the natural projective morphism from $M^\bullet(v, w)$ to $M_0^\bullet(v, w)$ by $\pi$. Denote the fiber
\[ \pi^{-1}(0) \text{ by } \mathcal{L}^\bullet(v, w). \] The varieties \( \mathcal{M}^\bullet(v, w), \mathcal{M}_0^\bullet(v, w) \text{ and } \mathcal{L}^\bullet(v, w) \) are called the graded quiver varieties.

For any \( v' < v \), there is a natural embedding of \( \mathcal{M}_0^\bullet(v', w) \) into \( \mathcal{M}_0^\bullet(v, w) \). Moreover, by our construction, \( \mathcal{M}_0^\bullet(v, w) \) stabilizes when \( v \) is large enough. Denote the union \( \cup_v \mathcal{M}_0^\bullet(v, w) \) by \( \mathcal{M}_0^\bullet(w) \).

**Grothendieck ring and characters.** We have intersection cohomology sheaves \( IC(v, w) \), where \( w - C_q v \geq 0 \), for closed subvarieties \( \mathcal{M}_0^\bullet(v, w) \) of \( \mathcal{M}_0^\bullet(w) \). Define the operation of \( \mathbb{Z}[t^\pm] \) on the Grothendieck ring of the derived category of constructible sheaves over \( \mathbb{C} \)-vector spaces on \( \mathcal{M}_0^\bullet(w) \) such that \( t \) acts as the shift functor. Let \( K_w \) denote the free abelian group generated by the classes of the sheaves \( IC(v, w) \) over \( \mathbb{Z}[t^\pm] \) inside the Grothendieck group.

Let \( R_w = \text{Hom}_{\mathbb{Z}[t^\pm]}(K_w, \mathbb{Z}[t^\pm]) \) denote the dual of \( K_w \). Define \( R_t \) to be the subspace of \( \prod_w R_w \) which consists of the basis \( \{ S(w) \} \), such that the value of \( S(w) \) on \( IC(v, w') \) is \( \delta_{w, w' - C_q v} \).

Define the quantum torus \( \mathcal{Y} \) to be the Laurent polynomial ring \( \mathbb{Z}[t^\pm][Y_{i,a}]_{(i,a) \in W} \) equipped with the twisted product \( \ast \) defined by\(^2\)

\[
Y^{w_1} \ast Y^{w_2} = t^{-\mathcal{E}(m_1^1, m_2^1)} Y^{w_1 + w_2},
\]

\[
\mathcal{E}(m_1, m_2) = -w^1[1] \cdot C_q^{-1} w^2 + w^2[1] \cdot C_q^{-1} w^1,
\]

for any \( w^1, w^2 \in \mathbb{N}^W \), cf. [KQ12][Qin13, (40)].

We let \([\ ]\) to denote the normalization by \( t \)-factors in \( \mathcal{Y} \).

There exists a natural multiplication on \( R_t \), which is derived from the geometric restriction functor and the Euler twist \( \mathcal{E}(\ , \ ) \), cf. [Nak11]. Let \( R_t \) denote the Grothendieck ring equipped with this multiplication. Furthermore, the structure constants of its natural basis \( \{ S(w) \} \) are given by

\[
S(w^1) \otimes S(w^2) = \sum_{v \geq 0, w^1 + w^2 = C_q v} a_v(t) S(v^1 + w^2 - C_q v),
\]

such that \( a_v(t) \in \mathbb{N}[t^\pm], a_0 = 1 \). Let \( R_{t=1} \) denote the specialization of \( R_t \) by taking \( t = 1 \).

Let \( \chi_{t,q} \) denote the \( t \)-analog of the \( q \)-characters (\( qt \)-character for short)

\[
\chi_{t,q} : R_t \to \mathcal{Y} = \mathbb{Z}[Y_{i,a}]_{(i,a) \in I \times \mathbb{Z}}.
\]

\( \chi_{t,q} \) is an injective ring homomorphism with respect to the twisted product \( \ast \), cf. [Nak04][VV03][Her04]. We refer the reader to [Qin13] for its precise definition in our convention. In particular, define the Laurent monomial

\[
A_{i,b} = Y_{i,b-1} Y_{i,b+1} \prod_{j \neq i} Y_{j,b+\epsilon_j}^{C_{ij}}.
\]

\(^2\)Our \( t \) is inverse to that of Nakajima [Nak03] by this definition.
Then the character of any $S(w)$ is a formal series

$$\chi_{t,q}S(w) = Y^w (1 + \sum_{\substack{0 \neq v}} b_v(t) A^v),$$

such that $b_v(t) \in \mathbb{N}[t^\pm]$. Any monomial $Y^w A^v$ is said to be $l$-dominant if the pair $(v, w)$ is. We say the leading term of $\chi_{t,q}S(w)$ is $Y^w$. The simple module $S(w)$ is called *minuscule* if $Y^w$ is the only one $l$-dominant monomial appearing in its character.

**Truncated characters.** We consider truncated $qt$-characters which were first introduced in [HL10].

For any function $c \in (2\mathbb{Z})^I$. Denote its value at $i \in I$ by $c_i$. When the function $c$ takes a constant value $c \in 2\mathbb{Z}$, we simply denote it by $c$.

We say $c$ is $\Omega'$-adaptable (or adaptable for short), if the full subquiver of the fixed framed repetition quiver $\widetilde{\Gamma}_\Omega$ on the vertices $(i, c_i - 1)$, $i \in I$, is the acyclic quiver $(\Gamma, \Omega')$. We shall always assume $c$ to be adaptable for some $\Omega'$ from now on.

Let $\chi_{t,q} \leq \Sigma$ denote the truncation of $\chi_{t,q}$ at the subring $\mathbb{Z}^\pm t^\pm | Y_{i,a}^{\pm} |_{i,a} \in I \times \mathbb{Z}, a \leq c_i$.

### 7.2. A review of Kirillov-Reshetikhin modules. In this section, we review Kirillov-Reshetikhin modules in the language of graded quiver varieties, where we use the arguments of [Nak03] in our setting.

For any $i \in I$, $k \in \mathbb{N}$, $a \in 2\mathbb{Z}$, define the dimension vector

$$w_{k,a}^{(i)} = e_{i,a} + e_{i,a+2} + \ldots + e_{i,a+2(k-1)}.$$

Inspired by [Nak03], we call the simple modules $S(w_{k,a}^{(i)})$ the Kirillov-Reshetikhin modules and denote it by $W_{k,a}^{(i)}$. They were introduced in [KR90] for quantum affine algebras.

**Remark 7.2.1.** When taking the quiver $(\Gamma, \Omega)$ to be bipartite and choosing any $j \in I$, we can identify our graded framed quiver with the corresponding quiver in [Nak03] by identifying our vertex $(i, s - \xi_{ij})$ with the vertex $(i, s)$ in [Nak03]. We can then translate the results of [Nak03] to Section 7.2.

For any Laurent monomial $m \neq 1$ in $\mathcal{Y}$ and $j \in I$, define

$$r_j(m) = \max \{ s \in \mathbb{Q} | m \text{ contains the factor } Y_{i,s+\xi_{ij}} \text{ for some } i \in I \}.$$

Here we use $\xi_{ij}$ to distinguish the grading of the vertices in $(\Gamma, \Omega)$. It follows that $r_j(m) + \xi_{ij} = r_i(m)$, for any $i, j$, and that the non-positive criterion in the following definition is independent of the choice of $j$.

**Definition 7.2.2 (right negative).** The monomial $m$ is said to be right negative if, for some $j \in I$, the factors $Y_{i, r_j(m) + \xi_{ij}}$ has non-positive powers for any $i \in I$. 

Example 7.2.3. Let us take the quiver $(\Gamma, \Omega)$ in Figure 6. Then the monomial $Y_{w_{k,0}^{(2)}}^{(2)} = Y_{2,0}Y_{2,2} \cdots Y_{2,2k-2}$ is $l$-dominant with $r_2(Y_{w_{k,0}^{(2)}}^{(2)}) = 2k - 2$, $r_1(Y_{w_{k,0}^{(2)}}^{(2)}) = 2k - 2 - \frac{1}{3}$, $r_3(Y_{w_{k,0}^{(2)}}^{(2)}) = 2k - 2 + \frac{1}{3}$.

For any $1 \leq s \leq k$, the monomial $m_s = Y_{w_{k,0}^{(2)}}^{(2)}A_{2,2k-2s+1}^{-1} \cdots A_{2,2k-3}^{-1}A_{2,2k-1}^{-2}$ is right negative with $r_2(m_s) = 2k$.

Notice that the $l$-dominant monomials are not right negative. The product of two right negative monomials is still right negative.

In [Nak03], Nakajima computed truncated $qt$-characters of Kirillov-Reshetikhin modules by studying right negative monomials and using combinatorial properties of $qt$-characters defined over graded quiver varieties. His arguments remain effective for our graded quiver varieties and imply the following results.

Theorem 7.2.4 ([Nak03]). (i) All monomials in $\chi_{t,q}W_{k,a}^{(i)}$ are right negative except its leading term $Y_{w_{k,a}^{(i)}}^{(i)}$.

(ii) Let $m$ be a right negative monomial appearing in the truncated $qt$-character $\chi_{t,q,\leq2k}W_{k,a}^{(i)}$, then we have

\[
m = \prod_{t=0}^{k-1} Y_{i,a+2t} \prod_{t=s+1}^{k} A_{i,a+2t-1}^{-1}
= \prod_{t=0}^{s-1} Y_{i,a+2t} \prod_{t=s+1}^{k} (Y_{i,a+2t}^{-1} \prod_{j \neq i} Y_{j,a+2t-1+\epsilon_{ij}}^{-C_{ij}})
= Y_{i,a} \cdots Y_{i,a+2(s-1)}^{-1} Y_{i,a+2(s+1)}^{-1} \cdots Y_{i,a+2k}^{-1} \prod_{j \neq i} (Y_{j,a+2s+1+\epsilon_{ij}} \cdots Y_{j,a+2k-1+\epsilon_{ij}})^{-C_{ij}},
\]

where $0 \leq s \leq k - 1$.

(iii) We have the following equality in the quantum torus $\mathcal{Y}$;

\[
[\chi_{t,q}W_{1,a+2k}^{(i)} \ast \chi_{t,q}W_{k,a}^{(i)}] = \chi_{t,q}W_{k+1,a}^{(i)} + t^{-1} \chi_{t,q}S(w_{k-1,a}^{(i)} - \sum_{j \neq i} C_{ij} w_{1,a+2k-1+\epsilon_{ij}}^{(j)}).
\]

Recall that $[\cdot]$ denote the normalization in the quantum torus such that the leading term has coefficient 1.

Notice that $m$ becomes the leading term $Y_{w_{k,a}^{(i)}}^{(i)}$ of $\chi_{t,q}W_{k,a}^{(i)}$ if we take $s = k$. The following Lemma was used in the proof of Theorem 7.2.4.

Lemma 7.2.5 ([Nak03]). Take monomials $m$ and $m'$ from $\chi_{t,q}W_{k,a}^{(i)}$ and $\chi_{t,q}W_{1,a+2k}^{(i)}$ respectively. If the product $mm'$ is not right negative, then the following claims are true.
Moreover, introduced in \((38)\) rings with quantum groups, we will need a different 7.3. A different deformation.

\(w\) for any \((39)\) \(R\) with the corresponding extended Grothendieck ring \(\begin{equation}
\text{Theorem 7.2.6 (\cite{Nak03}). The following equation holds in the quantum torus } Y.t:\n\left[\chi_{t,q}W^{(i)}_{k,a+2} \ast \chi_{t,q}W^{(i)}_{k,a}\right] = \left[\chi_{t,q}W^{(i)}_{k-1,a+2} \ast \chi_{t,q}W^{(i)}_{k+1,a}\right] + t^{-1}\prod_{j \neq i}(\chi_{t,q}W^{(j)}_{k,a+1+\epsilon_{ij}}) - C_{ij}].
\end{equation}\)

Moreover, \(W^{(i)}_{k-1,a+2} \otimes W^{(i)}_{k+1,a}\) agrees with the simple module \(S(w^{(i)}_{k+1,a} + w^{(i)}_{k-1,a+2})\) up to \(t\)-power in \(\mathcal{R}_t\).

### 7.3. A different deformation.

In order to compare Grothendieck rings with quantum groups, we will need a different \(t\)-deformation (introduced in \cite{Her04} as an algebraic approach to \(qt\)-characters) and work with the corresponding extended Grothendieck ring \(\mathcal{R} = \mathcal{R}_t \otimes \mathbb{Z}[t^{\pm \frac{1}{2}}]\) as in \cite{HL11}. To be more precise, as in \cite[Section 6.1]{KQ12}, we define the quantum torus \(Y^H_{t_2}\) as the Laurent polynomial ring \(\mathbb{Z}[t^{\pm \frac{1}{2}}][Y_{t,a}(i,a) \in \mathcal{W}]\) equipped with the twisted product \(\ast\) defined by

\(Y^{w_1} \ast Y^{w_2} = t^{\frac{1}{2}\mathcal{N}(m_1, m_2)} Y^{w_1+w_2},\)

\(\mathcal{N}(m_1, m_2) = w_1[1] \cdot C_q^{-1} w_2 - w_2[1] \cdot C_q^{-1} w_1 - w_1[-1] \cdot C_q^{-1} w_2 + w_2[-1] \cdot C_q^{-1} w_1,\)

for any \(w_1, w_2 \in \mathbb{N}^{\mathcal{W}},\) cf. \cite[Section 6]{KQ12}. Replacing the Euler form \(-\mathcal{E}(,\ )\) in the construction of the twisted multiplication in \(\mathcal{R}_{t_2} = \mathcal{R}_t \otimes \mathbb{Z}[t^{\pm \frac{1}{2}}]\) by \(\frac{1}{2}\mathcal{N}(,\ ),\) we obtain the extended Grothendieck ring \(\mathcal{R}_t^H\), which will be simply denoted by \(\mathcal{R}\). Denote the corresponding \(qt\)-character from \(\mathcal{R}\) to \(Y^H_{t_2}\) by \(\chi_{t,q}^H\).

### Remark 7.3.1.

The construction of \(\mathcal{N}(,\ )\) in \(39)\) as a variant of \(\mathcal{E}(,\ )\) is inspired by \cite[(6)]{HL11}.

Assume that the quiver \((\Gamma, \Omega)\) is of bipartite orientation. Choose and fix some \(j \in I\). Then our vertices \((i, b - \epsilon_{ij})\) are labeled by \((i, b)\) in \cite{HL11}, and \(C_q^{-1}\) is translated as follows:

\(C_q^{-1} e_{j,a} \cdot e_{i,b-\epsilon_{ij}} = C_{ij}(b-a).\)
Consequently, (39) is translated into
\[
\mathcal{N}(e_i,p-e_{ij}, e_j,s) = \tilde{C}_{ij}(p-1-s) - \tilde{C}_{ji}(s-1-p) - \tilde{C}_{ij}(p+1-s) + \tilde{C}_{ji}(s+1-p).
\]
Then it identifies with [HL11, (6)] because the matrix $\tilde{C}$ is symmetric.

Lemma 7.3.2. For any $c \in 2\mathbb{N}$, $d, d' \in \mathbb{N}$, and $i \in I$, we have
\[
-\mathcal{E}(Y_{i,c-2d} \cdots Y_{i,c-2} Y_{i,c}, A_{j,c-1-2d}^{-1}) = \begin{cases} -1 & \text{if } i = j, d' = d \\ 0 & \text{else} \end{cases}
\]
\[
\mathcal{N}(Y_{i,c-2d} \cdots Y_{i,c-2} Y_{i,c}, A_{j,c-1-2d}^{-1}) = \begin{cases} -2 & \text{if } i = j, d' = d \\ 0 & \text{else} \end{cases}.
\]

Proof. With the help of [Qin13, Lemma 4.3], for any $(i, a) \in W$ and $(j, b) \in V$, we make the following computation.
\[
-\mathcal{E}(e_i,a, -C_q e_{j,b}) = -e_i,a[1] \cdot C_q^{-1} C_q e_{j,b} + C_q e_{j,b}[1] \cdot C_q^{-1} e_i,a
\]
\[
= -e_i,a[1] \cdot e_{j,b} + e_{j,b}[1] \cdot C_q^{-1} e_i,a
\]
\[
= -e_{i,a-1} \cdot e_{j,b} + e_{j,b-1} \cdot e_{i,a}
\]
(40)
\[
\begin{cases} 1 & i = j, b = a + 1 \\ -1 & i = j, b = a - 1 \\ 0 & \text{else} \end{cases}
\]

(41)
\[
\mathcal{N}(e_i,a, -C_q e_{j,b}) = -\mathcal{E}(e_i,a, -C_q e_{j,b})
\]
\[
- e_i,a[-1] \cdot C_q^{-1} (-C_q e_{j,b}) + (-C_q e_{j,b})[-1] \cdot C_q^{-1} e_i,a
\]
\[
= -\mathcal{E}(e_i,a, -C_q e_{j,b}) + e_{i,a+1} \cdot e_{j,b} - e_{j,b+1} \cdot e_{i,a}
\]
\[
= \begin{cases} 2 & i = j, b = a + 1 \\ -2 & i = j, b = a - 1 \\ 0 & \text{else} \end{cases}
\]

The desired claim follows. \qed

7.4. Computation of characters. We compute $q_t$-characters of some simple modules which will be useful in the proof of Theorem 9.4.1.

Recall that we have calculated the Euler form $\mathcal{E}$ in Lemma 7.3.2. Inspired by Lemma 7.2.5, we consider the following product. Results of this type was found in [FH14].

Lemma 7.4.1. Fix integers $h, k \in \mathbb{N}$ such that $k \geq 2$, $k > h \geq 1$. Take monomials $m$ and $m'$ from $\chi_t q W^{(i)}_{k,a}$ and $\chi_t q W^{(i)}_{h,a+2(k-h+1)}$ respectively. If the product $mm'$ is not right negative, then the following claims are true.

\( (i) \) The monomials take the form

\[
(42) \quad m' = Y_{i,a+2(k-h+1)} \cdots Y_{i,a+2k}
\]

\[
(43) \quad m = Y_{i,a} \cdots Y_{i,a+2(k-1)} \cdot \prod_{t=s+1}^{k} (A_{i,a+2t-1}^{-1}), \quad 0 \leq s \leq k.
\]

In particular, \( mm' \) is \( l \)-dominant if and only if \((k-h) \leq s \leq k\).

\( (ii) \) The coefficient of \( mm' \) in \([\chi_{t,q} W_{k,a}^{(i)}] \star \chi_{t,q} W_{k,a}^{(i)} \) is 1 if \((k-h) < s \leq k \) and \( t^{-1} \) if \( 0 \leq s \leq (k-h) \).

Notice that, when \( s = k \), \( m \) is the leading term in \( \chi_{t,q} W_{k,a}^{(i)} \).

**Example 7.4.2.** We consider the vectors \( w \in \mathbb{N}^W \) with support on Figure 8.

Take \( k = 3, h = 2, i = 2 \). Consider the following dimension vectors

\[
w_{k,a}^{(2)} = e_{2,a} + e_{2,a+2} + e_{2,a+4},
\]

\[
w_{h,a+2(k-h+1)}^{(2)} = e_{2,a+4} + e_{2,a+6},
\]

\[
w_{k+1,a}^{(2)} = e_{2,a} + e_{2,a+2} + e_{2,a+4} + e_{2,a+6},
\]

\[
w_{k-h,a+2(k-h+1)}^{(2)} = e_{2,a+4},
\]

\[
w_{h,a+2(k-h)+1+\epsilon_{21}}^{(1)} = e_{1,a+4} + e_{1,a+6},
\]

\[
w_{h,a+2(k-h)+1+\epsilon_{23}}^{(3)} = e_{3,a+4} + e_{3,a+6}.
\]

We compute the truncated \( qt \)-characters of the Kirillov-Reshetikhin modules

\[
\chi_{t,q \leq a+6} W_{k,a}^{(2)} = Y_{2,a} Y_{2,a+2} Y_{2,a+4} (1 + A_{2,a+5}^{-1} + A_{2,a+3}^{-1} A_{2,a+5}^{-1} + A_{2,a+1}^{-1} A_{2,a+3}^{-1} A_{2,a+5}^{-1}),
\]

\[
\chi_{t,q \leq a+6} W_{h,a+2(k-h+1)}^{(2)} = Y_{2,a+4} Y_{2,a+6},
\]

\[
\chi_{t,q \leq a+6} W_{k+1,a}^{(2)} = Y_{2,a} Y_{2,a+2} Y_{2,a+4} Y_{2,a+6},
\]

\[
\chi_{t,q \leq a+6} W_{k-h,a+2(k-h+1)}^{(2)} = Y_{2,a+4} (1 + A_{2,a+5}^{-1}).
\]

For simplicity, specialize \( t \) to 1. It is easy to compute the following difference

\[
\chi_{t,q \leq a+6} W_{k,a}^{(2)} \star \chi_{t,q \leq a+6} W_{h,a+2(k-h+1)}^{(2)} - \chi_{t,q \leq a+6} W_{k+1,a}^{(2)} \star \chi_{t,q \leq a+6} W_{k-h,a+2(k-h+1)}^{(2)} = Y_{2,a} Y_{1,a+4} Y_{1,a+6} Y_{3,a+4} Y_{3,a+6} (1 + A_{2,a+1}^{-1}^{-1}).
\]

We observe that the difference has only one \( l \)-dominant monomial. In fact, we will show it is the truncated character of the minuscule module \( W_{k-h,h,a}^{(2)} \) in Proposition 7.4.3.
For $0 \leq h \leq k$, we define:

$$w_{k-h,h,a}^{(i)} = w_{k-h,a}^{(i)} - \sum_{j \neq i} C_{ij} w_{h,a+2(k-h)+1+e_{ij}}^{(j)}$$

(44)

$$W_{k-h,h,a}^{(i)} = S(w_{k-h,h,a}^{(i)}).$$

Then $W_{k,0,a}^{(i)} = W_{k,a}^{(i)}$.

Similar to Theorem 7.2.4, by using Lemma 7.4.1, we have the following consequences on the modules $W_{k-h,h,a}^{(i)}$. The claim (iv) follows from an analogous proof of [Nak03, Lemma 4.1] or from Proposition 9.1.5.

**Proposition 7.4.3.** (i) All monomials in $\chi_{t,q} W_{k-h,h,a}^{(i)}$ are right negative except its leading term $Y^{w_{k-h,h,a}^{(i)}}$.

(ii) Let $m$ be a right negative monomial appearing in the truncated $q\ell$-character $\chi_{t,q} \leq 2k W_{k-h,h,a}^{(i)}$, then we have

$$m = Y^{w_{k-h,h,a}^{(i)}} \prod_{t=s+1}^{k-h} A_{t,a+2t-1}^{-1}$$

(45)

where $0 \leq s < k - h$.

(iii) We have the following equality in the quantum torus $Y$;

$$[\chi_{t,q} W_{h,a+2(k-h+1)}^{(i)} * \chi_{t,q} W_{k,a}^{(i)}]$$

$$= [\chi_{t,q} W_{k-h,a+2(k-h+1)}^{(i)} * \chi_{t,q} W_{k+1,a}^{(i)}] + t^{-1} \chi_{t,q} W_{k-h,h,a}^{(i)}$$

(46)

where $[]$ denote the normalization by $t$-factors.

(iv) In the Grothendieck ring $\mathcal{R}_t$, $W_{k-h,a+2(k-h+1)}^{(i)} \otimes W_{k+1,a}^{(i)}$ agrees with $S(w_{k+1,a}^{(i)} + w_{k-h,a+2(k-h+1)}^{(i)})$ up to normalization.

8. Facts and conjectures about monoidal categorifications

8.1. Monoidal categorification.
Definitions. We refer the reader to [HL10] and [Nak01] for more details.

Let $\mathcal{A}_Z$ be any given commutative cluster algebra. We give the following definition following [HL10].

**Definition 8.1.1.** We say that $\mathcal{A}_Z$ admits a monoidal categorification if there exists a tensor category $(\mathcal{U}, \otimes)$, such that

(i) There exists a ring isomorphism $\kappa$ from $\mathcal{A}_Z$ to the Grothendieck ring $\mathcal{R}_{t=1}$ of $\mathcal{U}$.

(ii) $\kappa$ sends the cluster variables (resp. the cluster monomials) to classes of prime real simple objects (resp. classes of real simple objects).

Assume that the set of simple objects (simples for short) of $\mathcal{U}$ are parametrized as $\{S(w)\}$ where $w$ are some parameters. By abuse of notation, we denote an object and its class in the Grothendieck ring $\mathcal{R}_{t=1}$ by the same symbol. Denote the multiplication in the Grothendieck ring $\mathcal{R}_{t=1}$ by $\otimes$.

Next, let a quantum cluster algebra $\mathcal{A}$ be given. Let $(\mathcal{U}, \otimes)$ be a monoidal category and $\mathcal{R}$ its graded Grothendieck ring equipped with a twisted product $\ast$ and a bar-involution $(\overline{\cdot})$ such that $\overline{t} = t^{-1}$. Assume that $\mathcal{R}$ has a $\mathbb{Z}[t^\pm]$-basis $\{S(w)\}$, called the simple basis, where $S(w)$ are classes of simple objects and bar-invariant. Here the $t$-grading in $\mathcal{R}$ comes from the natural grading of objects in $\mathcal{U}$ or from choosing a $t$-deformation of the previous Grothendieck ring $\mathcal{R}_{t=1}$. We assume that the simple basis in the graded Grothendieck ring has positive structure constants with respect to the twisted product $\ast$:

$$S(w^1) \ast S(w^2) = \sum_{w^3} a_{w^1,w^2}^{w^3} S(w^3), \quad a_{w^1,w^2}^{w^3} \in \mathbb{N}[t^\pm], \quad \forall w^1, w^2.$$  

(47)

Fix an integer $d \geq 1$. Let $\mathcal{R}$ denote the natural extension $\mathcal{R}_t \otimes_{\mathbb{Z}[t^\pm]} \mathbb{Z}[t^\pm d]$. By the convention of this article, we will take $d = 2$.

**Definition 8.1.2.** We say that $\mathcal{U}$ provides a monoidal categorification of $\mathcal{A}$ if

(i) There exists a ring isomorphism $\kappa$ from $\mathcal{A}$ to the graded Grothendieck ring $\mathcal{R}$ of $\mathcal{U}$ such that $\kappa(q^\frac{1}{d}) = t^{\frac{1}{d}}$ and it commutes with the bar-involutions.

(ii) $\kappa$ sends the quantum cluster variables (resp. the quantum cluster monomials) to classes of prime real simple objects (resp. classes of real simple objects).

**Standard basis and properties.** In this article, we shall work with the case that the set of parameters $w$ is $\bigoplus_{k=1}^l \mathbb{N} \beta_k$, whose basis vector $\beta_k$ is called the $k$-th root vectors. The object $S(\beta_k)$ is called the $k$-th fundamental object.

Endow $\mathbb{N}^l$ with the natural lexicographical order $\prec_w$ such that for any $w = (w_k)$, $w' = (w'_k)$, $w \prec_w w'$ if and only if there exists some
Let Proposition 8.1.3. A by $S$ where $\alpha$ is some special integer. Denote $m = t^{-\frac{1}{2}} \mathbb{Z}[t^{-\frac{1}{2}}]$. We shall further assume that the set of classes $\{M(w)\}$ decomposes $(\preceq_w, m)$-unitaly into the basis $\{S(w)\}$ in the sense of Definition 3.1.7. Then it is also a $\mathbb{Z}[t^{\pm \frac{1}{2}}]$-basis of $R$, which is called the standard basis.

Assume $\kappa$ is given and we fix the initial seed $t_0$. If the elements $\kappa^{-1}S(w)$ have distinct leading degrees $\deg S(w) \in D(t_0) = \mathbb{Z}^m$ in the quantum torus $\mathcal{T}(t_0)$ (Definition 3.1.3), we define the map $\theta^{-1}$ sending $w$ to $\deg \kappa^{-1}S(w)$. Denote the elements $\kappa^{-1}S(w)$ and $\kappa^{-1}M(w)$ in $A^U$ by $S(w)$ and $M(w)$ respectively.

The following observation states that a monoidal categorification of $A_{\mathbb{Z}}$ implies that of $A$.

**Proposition 8.1.3.** Let $\mathcal{U}$ be a monoidal category which satisfies (47) with its graded Grothendieck ring $R$ isomorphic to a quantum cluster algebra $A^U$ by identifying some simples with the initial quantum cluster variables. Assume that $\mathcal{U}$ has the commutative Grothendieck ring $\mathcal{R}_{t=1}$ as the specialization of $R$ at $t^\frac{1}{2} = 1$ and, moreover, $\mathcal{U}$ provides a monoidal categorification of the commutative cluster algebra $A^U_{\mathbb{Z}}$. Then $\mathcal{U}$ provides a monoidal categorification of $A^U$ as well.

**Proof.** Let $t$ be any seed and $\{S_i(t)\}$ the collection of simples in $\mathcal{R}_{t=1}$ identified with the commutative cluster variables $\{x_i(t)\}$. Then, for any index $k$, there exists some simple module $S_k(t)^*$ such that we have the following exchange relation in $\mathcal{R}_{t=1}$:

$$S_k(t) \otimes S_k(t)^* = S_+ + S_-,$$

where $S_+$ and $S_-$ are monomials in $\{S_i(t)\}$. In particular, $S_+$ and $S_-$ are simples.

Now, assume that after the quantization, $\{S_i(t)\}$ is identified with $\{X_i(t)\}$ the set of the quantum cluster variables in $t$. The twisted product takes the form

$$S_k(t) \otimes S_k(t)^* = f_+S_+ + f_-S_-,$$

where the coefficient $f_+, f_- \in \mathbb{N}[t^{\pm \frac{1}{2}}]$ by (47). It follows that $f_+$ and $f_-$ must be powers of $t^{\pm \frac{1}{2}}$ so that they specialize to 1.

Next, left multiplying $S_k(t)^{-1}$, by the bar-invariance of $S_k(t)^*$, we must get

$$S_k(t)^* = [S_k(t)^{-1} * S_+]^t + [S_k(t)^{-1} * S_-]^t,$$
where \([\cdot]\) denote normalization in the quantum torus \(\mathbb{Z}[t^{\pm\frac{1}{2}}] [S_i(t)^\pm]\), cf. Definition 3.1.4. Therefore, \(S_k(t)^*\) is identified with the quantum cluster variable \(X_k(\mu_k t)\).

Starting from the initial seed \(t_0\) and repeating the above argument, we obtain the desired claim. \(\square\)

8.2. **Quantum cluster algebras associated with words.** We shall recall various quantum cluster algebras associated with words, which are expected to admit monoidal categorifications. More details could be found in [GLS11].

**Ice quiver of type (i)(ii).** Fix an non-empty vertex set \(I = \{1, \ldots, r\}\) and a generalized Cartan matrix \(C = (C_{i,j})_{i,j \in I}\). For any \(i \in I\), let \(s_i\) denote the simple reflection at the \(i\)-th simple root. By choosing an acyclic orientation \(\Omega\) on the diagram \(\Gamma\) associated with \(C\). We obtain an acyclic quiver \((\Gamma, \Omega)\).

Let \(i = (i_l, i_{l-1}, \ldots, i_2, i_1)\) be a non-empty word of length \(l\) with elements in \(I\).

We say the word \(i\) is **reduced**, if the Weyl group element \(w(i) = s_{i_l} \cdots s_{i_2} s_{i_1}\) has length \(l\).

We say the word \(i\) is **\(\Omega\)**-adaptable (or adaptable for short), if there exists an acyclic quiver \((\Gamma, \Omega)\) associated with the Cartan matrix \(C\), such that \((i_l, \ldots, i_2, i_1)\) is a sink sequence of the quiver when read from right to left, cf. [GLS07, Section 3.7] for more details.

An adaptable word \((i_r, \ldots, i_2, i_1)\) which contains every \(i \in I\) exactly once is called an **acyclic Coxeter word**.

We consider the following types of the pair \((C, i)\):

(i) The generalized Cartan matrix \(C\) is symmetric, and \(i\) is reduced.

(ii) The generalized Cartan matrix \(C\) is of type \(ADE\), and the word \(i\) is \(\Omega\)-adaptable for some acyclic quiver \((\Gamma, \Omega)\) associated \(C\).

Without loss of generality, we can assume that the diagram \(\Gamma\) associated with \(C\) is connected.

For any \(1 \leq k \leq l\), \(i \in I\), we define the following notations (cf. [GLS11, Section 9.8] [BFZ05])

\[
\begin{align*}
k^{\text{max}} &= \max \{ s \in [1, l] | i_s = i_k \}, \\
k^{\text{min}} &= \min \{ s \in [1, l] | i_s = i_k \}, \\
k^+ &= \min \{ l + 1, s \in [k + 1, l] | i_s = i_k \}, \\
k^- &= \max \{ 0, s \in [1, k - 1] | i_s = i_k \}, \\
k^+(i) &= \min \{ l + 1, s \in [k + 1, l] | i_s = i \}, \\
k^-(i) &= \max \{ 0, s \in [1, k - 1] | i_s = i \}, \\
\max_i &= \max \{ k \in [1, l] | i_k = i \}, \\
\min_i &= \min \{ k \in [1, l] | i_k = i \}.
\end{align*}
\]
For any interval \([a, b] = \{ k \in \mathbb{Z} | a \leq k \leq b \}\) in \([1, l]\), define the multiplicities

\[
m(i, [a, b]) = | \{ k \in [a, b], i_k = i \} |, \\
m(i) = m(i, [1, l]), \\
m_k^+ = m(i_k, [k, l]) - 1, \\
m_k^- = m(i_k, [1, k]) - 1.
\]

We define \(k[0] = k\), and for any integer \(1 \leq d \leq m_k^+\), we recursively define the \(d\)-th offset \(k[d]\) of \(k\) to be \(k[d-1]^+\). The integer \(m(i_k, [k^{\min}, k])\) is called the \textit{level} of the vertex \(k\).

If we define the following subset of \([a, b]\):

\[
[a, b](i) = \{ k | k \in [a, b], i_k = i \}.
\]

Then the cardinality of \([1, l](i)\) is just \(m(i_k)\). The subset \([1, l](i)\) is naturally a sequence (read from right to left) whose elements are ordered as the following:

\[
\text{min} i[m(i) - 1], \text{min} i[m(i) - 2], \ldots, \text{min} i[1], \text{min} i[0].
\]

Following \cite{GLS11} [GLS11, Section 2.4] \cite{BFZ05}, we associate the quiver \(\Gamma_i\) to \(i\) as the following:

- The vertices of \(\Gamma_i\) are \(1, 2, \ldots, l\).
- For \(1 \leq s, t \leq l\), there are \(-C_{i,s,i_t}\) arrows from \(t\) to \(s\) if \(t^+ \geq s^+ > t > s\). These are called the \textit{ordinary arrows} of \(\Gamma_i\).
- For any \(1 \leq s \leq l\), there is an arrow from \(s\) to \(s^+\) if \(s^+ \leq l\). These are called the \textit{horizontal arrows} of \(\Gamma_i\).

By default, we consider words satisfying the following assumption.

**Assumption 5.** We assume that every element of \(I\) appears in the word \(i\) at least once.

**Example 8.2.1** (type \(A\)). The quiver in Figure 9 is the quiver \(\Gamma_i\) associated with the adaptable word \(i = (1, 2, 1, 3, 2, 1, 4, 3, 2, 1)\) and the type \(A_4\) Cartan matrix given by

\[
C = \begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 2
\end{pmatrix}.
\]

The pair \((C, i)\) is of type both (i) and (ii).

**Example 8.2.2** (\cite{GLS11, Example 13.2}). The quiver in Figure 10 is the quiver associated with the non-adaptable word \(i = (2, 3, 2, 1, 2, 1, 3, 1, 2, 1)\)

\footnote{In our convention, the quiver \(\Gamma_i\) is opposite to that of \cite{GLS11}.}
and the Cartan matrix \( C = \begin{pmatrix} 2 & -3 & -2 \\ -3 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix} \). The pair \((C, i)\) is of type \((i)\). Here, we put a number \(s\) on an arrow to indicate that there are \(s\)-many arrows drawn here.

\[ \text{Mutation sequences.} \] Following [GLS11, Section 13.1], for any \(1 \leq k \leq l\), we define the following mutation sequence (read from right to left):

\[ \overleftarrow{\mu}_k = \mu_{k_{\min}[m(i_k,[k,l]) - 2]} \cdots \mu_{k_{\min}[1]} \cdots \mu_{k_{\min}}. \]  

(48)

Notice that, when \(m(i_k,[k,l]) - 2 < 0\), this sequence is trivial.

Define the mutation sequence (read from right to left)

\[ \Sigma_1 = \overleftarrow{\mu}_1 \cdots \overleftarrow{\mu}_2 \overleftarrow{\mu}_1. \]  

(49)
We also define a permutation $\sigma_i$ of $[1, l]$ such that, for any $k \in [1, l]$, 
$0 \leq d \leq m(i, [1, l]) - 2$, we have

$$\sigma_i(k_{\text{max}}) = k_{\text{max}},$$

$$\sigma_i(k_{\text{min}}[d]) = k_{\text{min}}[m(i, [1, l]) - 2 - d].$$

We always view the quiver $\Gamma_i$ as an ice quiver by freezing the vertices $\max_i$, $i \in I$. The number of exchangeable vertices is $n = l - r$. We associate the initial $l \times n$ matrix $\overline{B}(t_0)$ to the quiver $\Gamma_i$ such that, at position $(i, j)$, its entry $b_{ij}$ equals the difference between the number of arrows from $i$ to $j$ and the number of arrows from $j$ to $i$. Let $A_\mathcal{Z}(i) = A_\mathcal{Z}(t_0)$ denote the commutative cluster algebra arising from this initial matrix $\overline{B}(t_0)$.

By [GLS11, Proposition 13.4], $t_0$ is injective-reachable via $(\Sigma_i, \sigma_i)$ in the sense of Section 5.1.

We use $L(k_{\text{min}}, k)$ to denote the initial cluster variables $X_k(t_0)$, $1 \leq k \leq l$. When $\overline{\mu}_k$ is nontrivial, applying it to the seed $\overline{\mu}_1 \cdots \overline{\mu}_k t_0$, we obtain the new cluster variables

$$X_{k_{\text{min}}[d]}(\overline{\mu}_k \cdots \overline{\mu}_2 \overline{\mu}_1 t_0), \ 0 \leq d \leq m(i, [k, l]) - 2,$$

which we will denote by $L(k^+, k[d]^+)$. 

**Example 8.2.3.** In Example 8.2.1, The cluster variables in the canonical initial seed $t_0$ are denoted by

$$L(1, 10), L(1, 8), L(1, 5), L(1, 1),$$

$$L(2, 9), L(2, 6), L(2, 2),$$

$$L(3, 7), L(3, 3), L(4, 4).$$

The mutation sequence $\Sigma_4$ is the composition of the following mutation sequences (read from right to left):

$$\overline{\mu}_1 = \mu_8 \mu_5 \mu_3 \overline{\mu}_2 = \mu_6 \mu_2 \overline{\mu}_3 = \mu_3 \overline{\mu}_4 = 1 \overline{\mu}_5 = \mu_5 \mu_1$$

$$\overline{\mu}_6 = \mu_2 \overline{\mu}_7 = 1 \overline{\mu}_8 = \mu_1 \overline{\mu}_9 = 1 \overline{\mu}_{10} = 1$$

By applying the mutation sequence $\overline{\mu}_1$ to $t_0$, we obtain the new cluster variables (from right to left):

$$L(5, 10), L(5, 8), L(5, 5).$$

We continue with the sequence $\overline{\mu}_2$ to obtain $L(6, 9), L(6, 6)$. After applying all factors of $\Sigma_4$, we obtain the cluster variables of $\Sigma_4 t_0$ denoted by

$$L(1, 10), L(5, 10), L(8, 10), L(10, 10),$$

$$L(2, 9), L(6, 9), L(9, 9),$$

$$L(3, 7), L(7, 7), L(4, 4).$$
In the convention of Section 5.1, the cluster variables will be denoted as follows:

\[
X_{10}(t_0), I_1(t_0), I_5(t_0), I_8(t_0),
X_9(t_0), I_2(t_0), I_6(t_0),
X_7(t_0), I_3(t_0),
X_4(t_0).
\]

The permutation \( \sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 8 & 6 & 3 & 4 & 5 & 2 & 7 & 1 & 9 & 10 \end{pmatrix} \).

### 8.3. Monoidal categorification conjecture: type (i)

We refer the reader to [Kim12] and [GLS13] for precise definitions in this subsection.

**Quantum cluster algebra structure.** When the pair \((C, i)\) is of type (i), \(A^1_J(i)\) is isomorphic to the coordinate ring of the unipotent subgroup \(N(w_1)\), cf. [BFZ05][GLS11]. Moreover, the canonical initial seed \(t_0\) has a natural quantization matrix \(A(t_0)\) by [GLS13]. We define the rational quantum cluster algebra to be \(A^1_Q(i) = A^1(t_0) \otimes \mathbb{Q}(q^{\frac{1}{2}})\).

The quantum unipotent subgroup \(A_Q(n(w_1))\) is an subalgebra of \(U_Q(n)\). It contains its integral form \(A_{Z[\mathbb{Z}^{\pm}]}(n(w_1))\). It has the dual PBW basis and the dual canonical basis as the restrictions from that of \(A_{Z[\mathbb{Z}^{\pm}]}(n)\). We refer the reader to [Kim12] for more details. Denote \(A_Q(v^{\frac{1}{2}})(n(w_1))\) is of type (i). Then

**Theorem 8.3.1 ([GLS11][GLS13]).** Assume \((C, i)\) is of type (i). Then there is an algebra isomorphism\(^4\) \(\kappa\) from the quantum cluster algebra \(A^1_Q(i)\) to the quantum unipotent subgroup \(A_Q(v^{\frac{1}{2}})(n(w_1))\) such that

\[
\kappa(q^{\frac{1}{2}}) = v^{\frac{l}{2}}
\]

and the images of the quantum cluster variables \(\kappa L(a, b)\), where \(1 \leq a \leq b \leq l\) such that \(i_a = i_b\), are \(v^{\frac{l}{2}}\)-shifts of the unipotent quantum minors

\[
D(a, b) = D_{s_{i_1} s_{i_2} \cdots s_{i_a}(w_{i_a}), s_{i_1} s_{i_2} \cdots s_{i_b}(w_{i_b})}.
\]

Moreover, \(\kappa^{-1}A_{Z[\mathbb{Z}^{\pm}]}(n(w_1))\) is contained in \(A^1(i)\).

Let \(B^*(w_1) = \{b(w) | w \in \mathbb{N}^l\}\) denote the dual canonical basis of the quantum unipotent subgroup \(A_Q(n(w_1))\). The map \(\kappa\) transports the bar-involution on \(T(t_0)\) to a bar-involution on \(A_Q(n(w_1))\). By multiplying \(v^{\frac{l}{2}}\)-factors, we obtain a bar-invariant basis \(\{S(w) | w \in \mathbb{N}^l\}\).

Then, the rational quantum cluster algebra has the bar-invariant basis \(\{S(w)\} = \{\kappa^{-1}S(w)\}\).

**Conjecture 8.3.2 ([FZ02][GLS07]).** Assume \((C, i)\) is of type (i). Then the quantum cluster monomials of \(A(i)\) are contained in \(\{S(w)\}\).

\(^4\)In [GLS13], the square roots \(q^{\frac{1}{2}}\) and \(v^{\frac{l}{2}}\) did not appear because the quantum cluster algebra is rescaled and its bar-involution is defined differently.
By the work of Khovanov, Lauda [KL09] and Rouquier [Rou08], the quantum group $U(n)$ can be viewed as the Grothendieck ring of modules of the corresponding quiver Hecke algebra, such that $v$-shifts of the dual canonical basis corresponds to the classes of finite dimensional simple modules [VV11][Rou12]. It follows that the integral form $A_{\mathbb{Z}[v^\pm]}(n(w_1))$ is isomorphic to the graded Grothendieck ring $\mathcal{R}(w(1))$ of certain monoidal subcategory $U(w_1)$, and each $S(w)$ represent the class of a simple module.

The category $U(w_1)$ satisfies the assumptions in Section 8.1. In particular, $\mathcal{R}$ has the basis $\{M(w)\}$ with expected properties, which corresponds to the normalization of the dual PBW basis. Here, we use Kashiwara’s bilinear form following the convention of the [GLS13].

**Conjecture 8.3.3** (Integral form, [GLS13, Conjecture 12.7]). The isomorphism $\kappa$ restricts to an isomorphism from $A^\dagger$ to $A_{\mathbb{Z}[v^\pm]}(n(w_1))$.

We shall give a proof of Conjecture 8.3.3 in Proposition 9.1.3.

**Lemma 8.3.4.** Conjecture 8.3.3 implies Conjecture 8.3.3.

**Proof.** Notice the algebra $A^\dagger$ is generated by the quantum cluster variables over $\mathbb{Z}[q^{\frac{1}{2}}]$. Therefore, $\kappa A^\dagger$ is generated by elements in $\{S(w)\}$ and, consequently, contained in the integral form $A_{\mathbb{Z}[v^\pm]}(n(w_1))$. But the integral form is generated by the fundamental modules $M(\beta_k)$ over $\mathbb{Z}[v^\pm]$, which are images of quantum cluster variables. Therefore, $\kappa A^\dagger$ is the integral form. □

**Embeddings of Grothendieck rings.** Let $i_1, \ldots, i_k$ denote the word $(i_1, \ldots, i_k)$ whose associated root vectors are denoted as $\beta_1, \beta_2, \ldots, \beta_k$. Similarly, let $i_1, \ldots, i_k$ denote the word $(i_1, \ldots, i_k)$ whose associated root vectors are denoted as $\beta_1, \beta_2, \ldots, \beta_k$, and $i_1, \ldots, i_k$ the word $(i_1, \ldots, i_k)$ with root vectors denoted by $\beta_1, \beta_2, \ldots, \beta_k$. Then we have the natural embedding of $\mathcal{R}(w_{l,i})$ into $\mathcal{R}(w_i)$ sending $S(\beta_k)$ to $S(\beta_k)$, $1 \leq k \leq l - 1$, and the simple basis into the simple basis.

Moreover, the Lusztig’s braid group symmetry $T_{i_1}$ at vertex $i_1$ maps $D_{s_{i_1} s_{i_2} \cdots s_{i_k}} \omega_{i_1, s_{i_2} \cdots s_{i_k}} \omega_{i_k}$ to $(1 - v^2)^{\langle h_{i_1}, \xi_k \rangle} D_{s_{i_1} s_{i_2} \cdots s_{i_k}} \omega_{i_1, s_{i_2} \cdots s_{i_k}} \omega_{i_k}$, where $\langle h_{i_1}, \xi_k \rangle$ linearly depends on the homogeneous grading $\xi_k$ of $D_{s_{i_1} s_{i_2} \cdots s_{i_k}} \omega_{i_1, s_{i_2} \cdots s_{i_k}} \omega_{i_k}$.

And it will send the dual canonical basis of $A_{Q(v^{\frac{1}{2}})}(n(w_{l-1}))$ into that of $A_{Q(v^{\frac{1}{2}})}(n(w_1))$ modulo these factors depending on homogeneous gradings, cf. [Kim12]. Consequently, after rescaling, we have the embedding from $\mathcal{R}(w_{l-1})$ into $\mathcal{R}(w_1)$ sending $S(\beta_k)$ to $S(\beta_k)$ and the simple basis into the simple basis. Notice that, if we work with Lusztig’s bilinear form instead, then the above factor does not appear, cf. [Lus93, 38.2.1].

Therefore, there is an natural embedding from $\mathcal{R}(w_{l-1})$ to $\mathcal{R}(w_1)$. Denote the images of the above embeddings by $\mathcal{R}(w_1)_{\hat{1}}, \mathcal{R}(w_1)_{\hat{2}}, \mathcal{R}(w_1)_{\hat{1}, \hat{2}}$ respectively.
8.4. Monoidal categorification conjecture: adaptable word. When the pair \((C, i)\) is of type (ii), we refer the reader to \([HL10]\) for the original monoidal categorification conjecture of \(A_Z(i)\) in terms of finite dimensional representation of quantum affine algebras. In this section, we shall present a quantized and slightly generalized version of this conjecture in terms of graded quiver varieties, where \(i\) is adaptable and \(C\) any generalized symmetric Cartan matrix.

*Subring via adaptable embedding.* Fix the framed repetition quiver \(\tilde{\Gamma}_\Omega\) associated with an acyclic quiver \((\Gamma, \Omega)\), cf. Section 7.

Let \(i\) be any given \(\overline{\Omega'}\)-adaptable word for an acyclic quiver \((\Gamma, \Omega)\). Fix a multidegree \(a = (a_k)_{1 \leq k \leq l} \in (2\mathbb{Z})^l\) such that \(a_k[d+1] = a_k[d] - 2\). Define the corresponding map \(\iota_a\) which sends the vertices \(1 \leq k \leq l\) of \(\tilde{\Gamma}_i\) to \((i_k, a_k)\). It follows that \(\iota_a(\min_i[d]) = (i, a_{\min_i} - 2d), \forall i \in I, 0 \leq d \leq m(i) - 1\). (50)

We further assume the multidegree \(a\) to be \(\Omega'\)-adaptable (or adaptable for simplicity) such that the full subquiver of \(\tilde{\Gamma}_\Omega\) on the vertices \((\min_i, a_{\min_i} - 1)\), where \(i \in I\), is isomorphic to \((\Gamma, \Omega')\). Because the word \(i\) is adaptable, an adaptable embedding multidegree \(a\) always exists. The associated map \(\iota_a\) is called an adaptable embedding.

**Example 8.4.1.** We take the framed repetition quiver \(\tilde{\Gamma}_\Omega\) associated with \((\Gamma, \Omega)\) in Figure 5. Part of this quiver is drawn in 7.

Take the word \(i = (2132143215432)\) (read from right to left). It is \(\overline{\Omega}\)-adaptable with respect to the orientation \(\overline{\Omega}\). The associated ice quiver \(\Gamma_i\) is drawn in 11. Recall that we have \(i_1 = i_6 = i_{10} = i_{13} = 2, 1[0] = 1, 1[1] = 6, 1[2] = 10, 1[3] = 13, 13^{\min} = 10^{\min} = 6^{\min} = 1^{\min} = 1, m_6 = 1, m_6 = 2\).

For any chosen \(a \in 2\mathbb{Z}\), we have an \(\Omega\)-adaptable multidegree \(a\) and the associated adaptable embedding \(\iota_a\) such that \(a_k = a + 6\) for \(1 \leq k \leq 5\), cf. Figure 7.

Let us take another adaptable word \(i' = (21354)\). It is \(\overline{\Omega'}\)-adaptable where the acyclic quiver \((\Gamma, \Omega')\) is given in Figure 12. A possible adaptable adaptable grading \(\iota_a\) could have the image

\[
\{(1, a + 4), (2, a + 2), (3, a + 4), (4, a + 6), (5, a + 6)\}
\]

in the set of vertices of the framed repetition quiver \(\tilde{\Gamma}_\Omega\).

Fix an adaptable embedding \(\iota_a\). Define \(R(i, a)\) to be the subring of the extended graded Grothendieck ring \(R\) generated by the simples \(W_{i, a_k}, i \in I, 1 \leq k \leq l\). It has the simple basis and standard basis as expected in Section 8.1.
Figure 11. The ice quiver $\Gamma_1$ associated with an adaptable word $i$ and $A_5$ Cartan matrix.

Figure 12. An acyclic quiver $(\Gamma, \Omega')$ of Cartan type $A_5$

Quantum cluster algebra structure. We denote by $\vartheta_\Delta: \mathcal{T}(t_0) \to \mathcal{Y}^H_{t_2}$ the embedding of the Laurent polynomial rings such that

\[
\vartheta_\Delta(q^\frac{1}{2}) = t^\frac{1}{2},
\]

\[
\vartheta_\Delta(X_k) = Y_{i_k,a_k}Y_{i_k,a_k-1}\cdots Y_{i_k,a_{\min}}, \quad 1 \leq k \leq l.
\]

It follows that we have

\[
\vartheta_\Delta(Y_k) = \vartheta_\Delta(X^{\tilde{B}(t_0)e_k}) = A^{-1}_{i_k,a_k-1} = A^{-1}_{i_k,a_k-2m_k-1}.
\]

The twisted multiplication on $\mathcal{Y}^H_{t_2}$ then induces the twisted multiplication on $\mathcal{T}(t_0)$ via the morphism $\vartheta_\Delta$ such that

\[
\Lambda(t_0)(e_k, e_s) = \mathcal{N}(w^{(i_k)}_{m_k+1,a_k}, w^{(i_s)}_{m_s+1,a_s}).
\]

By Lemma 7.3.2, it is compatible with the matrix $\tilde{B}$ such that

\[
\Lambda(t_0) \cdot (-\tilde{B}) = \begin{pmatrix} -21 \cr 0 \end{pmatrix}.
\]
We denote the quantum cluster algebra obtained via this quantization by $\mathcal{A}(i)$.

By comparing $T$-systems with specific exchange relations, we obtain the following result, whose proof goes the same as that of [GLS13, Proposition 12.1].

**Theorem 8.4.2.** Assume that the word $i$ and the multidegree $\underline{a}$ are adaptable. Then there exists an injective algebra homomorphism $\kappa^{-1}$ from the Grothendieck ring $R(i, \underline{a})$ to the quantum cluster algebra $\mathcal{A}^i(i)$, such that $\kappa^{-1}(t^T) = q^{\hat{T}}$ and, for any $1 \leq b \leq s \leq l$ such that $i_b = i_s$, $\kappa^{-1}W^{(i)}_{m(i_b, [b, s], \underline{a})}$ agrees with the quantum cluster variables $L(b, s)$.

**Remark 8.4.3.** When the word $i$ is reduced adaptable, $R(i, \underline{a}) = R^H(i, \underline{a})$ is also isomorphic to the quantum unipotent subgroup $A_\mathbb{Q}(\underline{a})$, cf. the proofs of [GLS13, Theorem 12.3][HL11, Theorem 6.1(a)] based on $T$-systems. In this case, $S(w)$ represents a simple module of the quiver Hecke algebra.

If the Cartan matrix $C$ is of Dynkin type $ADE$, $R(i, \underline{a})$ and $R(i, \underline{a}) = R^H(i, \underline{a})$ are $t$-deformations of the Grothendieck ring of the monoidal subcategory $U(i, \underline{a})$ of finite dimensional representations of the quantum affine algebras $U_q(\mathfrak{g})$, where each $S(w)$ represents a simple module as constructed by [Nak01].

The corresponding monoidal categorification conjecture takes the following form.

**Conjecture 8.4.4.** Assume that $i$ is adaptable. Then the quantum cluster monomials of $\mathcal{A}^i(i)$ are contained in $\kappa^{-1}\{S(w)\}_{w \in \mathbb{W}}$.

**Theorem 8.4.5** ([HL13a, Theorem 5.1], [GLS11, Theorem 3.1]). $\kappa^{-1}$ induces an isomorphism from the extended Grothendieck ring $R(i, \underline{a}) \otimes \mathbb{Q}$ to $\mathcal{A}_\mathbb{Z} \otimes \mathbb{Q}$.

**Embeddings of Grothendieck rings.** Let $\underline{a}_\Gamma$ to be the multidegree obtained from $\underline{a}$ by deleting the component $a_1$. Similarly, construct $\underline{a}_\Gamma^+$ and $\underline{a}_\Gamma^*$ by deleting $a_1$ and $a_1, a_l$ from $\underline{a}$ respectively. By construction, we have natural embeddings of $R(i-i_1, \underline{a}_1), R(i-i_l, \underline{a}_l), R(i-i_1-i_l, \underline{a}_{1,l})$ into $R(i, \underline{a})$, which send $S(\beta_k)$ to $S(\beta_k)$ and the simple bases into the simple basis. As before, we denote the images by $R(i, \underline{a})_{\Gamma^+}, R(i, \underline{a})_{\check{\Gamma}}, R(i, \underline{a})_{\hat{\Gamma}}$ respectively.

9. **Monoidal categorification conjectures**

We refer the reader to Section 2.8 for the necessary notations and definitions used in this section.

Fix a pair $(C, i)$ of type (i) or (ii). We want to know if the quantum cluster algebra $\mathcal{A} = \mathcal{A}(i)$ has the common triangular basis or not. We assume that every $i \in I$ appears at least once in $i$ (Assumption 5).
Recall that we have the corresponding extended Grothendieck ring $\mathcal{R}(i)$ which can be taken as an extension of a quantum unipotent subgroup if $i$ is reduced or a subring $\mathcal{R}(i,a)$ for some adaptable multidegree $a$ if $i$ is adaptable.

9.1. Local conditions in the initial seed. We work in the canonical initial seed $t_0$ associated with $(C,i)$ in this section.

Define the injective linear map $\theta^{-1}$ from $\mathbb{N}^d = \oplus_{1 \leq k \leq \ell} \mathbb{N}\beta_k$ to $D^d(t_0)$ such that, for any $i \in I$, we have
\[
\theta^{-1}\beta_{i[d]} = e_{i[d]} - e_{i[d-1]}, \quad 1 \leq d \leq m(i) - 1,
\]
\[
\theta^{-1}\beta_{\min_i} = \epsilon_{\min_i}.
\]

Lemma 9.1.1. $\theta^{-1}$ is an isomorphism.

Proof. Recall that there exists an isomorphism $\kappa^{-1}$ from $\mathcal{A}^d_{\mathbb{Z}}(i) \otimes \mathbb{Q}$ to $\mathcal{R}_{t=1}(i) \otimes \mathbb{Q}$. It sends the standard basis element $M(w)$ to a pointed element $\kappa^{-1}M(w)$ with leading degree $\theta^{-1}w$. Notice that the simple basis element $\kappa^{-1}S(w)$ is also pointed at $\theta^{-1}w$.

Any element $Z$ in $\mathcal{A}^d_{\mathbb{Z}}(i) \otimes \mathbb{Q}$ has the expansion into the simple basis elements $\kappa^{-1}S(w)$. Consequently, the $<_{t_0}$-maximal degrees of $Z$ take the form $\theta^{-1}w$, $w \in \mathbb{N}^d$. It follows by definition that the dominant degree lattice $D^d(t_0)$ equals $\theta^{-1}\mathbb{N}^d$. \[
\square\]

Notice that, for any $\tilde{g}^1, \tilde{g}^2 \in D^d(t_0)$, the condition $\tilde{g}^1 <_{t_0} \tilde{g}^2$ implies that $g^1 \prec \tilde{w} g^2$ with respect to the lexicographical order $\prec_w$ on $\oplus \mathbb{N}\beta_k$. We have the following useful consequence.

Lemma 9.1.2. $(<_{t_0}, D^d(t_0))$ is bounded from below.

With the help of Lemma 9.1.1 9.1.2, we obtain the following proposition, which verifies Conjecture 8.3.3 for type (i) quantum cluster algebras. It was already known for type (ii) quantum cluster algebras by [HL13a, Theorem 5.1].

Proposition 9.1.3. The injective homomorphism $\kappa^{-1}$ from $\mathcal{R}(i)$ to $\mathcal{A}^d(i)$ is an isomorphism. In particular, $\mathcal{S}(w)$ is a basis of $\mathcal{A}^d(i)$.

Proof. Work in the initial quantum torus $\mathcal{T}(t_0)$. Observe that any element of $\mathcal{A}^d(i)$ has maximal degrees in $D^d(t_0)$. For any given element $Z$ of $\mathcal{A}^d(i)$, combining this observation and the expansion algorithm in Remark 3.1.8, we obtain a unique $<_{t_0}$-untriangular expansion of $Z$ into the $D^d(t_0)$-pointed set $\{\mathcal{S}(w)\}$:
\[
Z = \mathcal{S}(\theta \deg^{t_0} Z) + \sum_{\theta^{-1}w <_{t_0} \deg^{t_0} Z} b_w \mathcal{S}(w), \quad b_w \in \mathbb{Z}[q^{\pm \frac{1}{2}}].
\]

Then this expansion is finite because $(<_{t_0}, D^d(t_0))$ is bounded from below. Therefore, $\{\mathcal{S}(w)\}$ is a basis of $\mathcal{A}^d(i)$. The claim follows. \[
\square\]
For any $1 \leq a \leq b \leq l$ with $i_a = i_b$, use $S(a, b)$ to denote the simple module $S(\beta_a + \beta_{a[1]} + \ldots + \beta_b)$. Then we have the quantum cluster variables $X_k(t_0) = S(\theta e_k) = S(k_{\min}, k)$ and, for any $k \neq k_{\max}$, $I_k(t_0) = S(k[1], k_{\max})$.

**Remark 9.1.4.** When the word $i$ is adaptable, we can construct the graded Grothendieck ring $\mathcal{R}(i) = \mathcal{R}(i, \alpha)$ as in Section 7 for some choice of adaptable multidegree $\alpha = (a_k) \in \mathbb{Z}^l$, such that the root vectors $\beta_k$ corresponds to the unit vectors $e_{i_k, a_k}$. In this situation, the map $\theta$ identify the $X$-variables $X_k$ and $Y$-variables $Y_k = \prod_{1 \leq s \leq l} X_{s}^{b_{s k}(t_0)}$ with the monomials $Y_{i_k, a_k} Y_{i_k, a_k+2} \cdots Y_{i_k, a_k+\min}$ and $A_{i_k, a_k}^{-1}$ respectively in the convention of Section 7, cf. Section 8.4.

For any $w$, the basis element $S(w) = \kappa^{-1}S(w)$ has the leading degree $\tilde{g} = \theta^{-1}w$ and we denote it by $L(\tilde{g})$. Denote the basis $\{S(w)\}$ by $L^t_{\kappa t_0}$.

Recall that we have another basis formed by the elements

$$M(w) = [S(w_1) \ast S(w_2) \cdots \ast S(w_l)]^t_{\kappa t_0}. 
$$

It is $(\prec_{w}, m)$-unitriangular to the basis $L^t_{\kappa t_0}$. By Lemma 3.1.9(iii), it is $(\prec_{l_0}, m)$-unitriangular to $L^l_{\kappa t_0}$ as well. From now on, by default, we shall only consider the dominance orders $\prec_{t}$ associated with a seed $t$ (or denoted by $\prec$ when there is no confusion). Notice that $N^t = \oplus_k N^t_k \beta_k$ inherits the order $\prec_{t_0}$ via the isomorphism $\theta$.

**Proposition 9.1.5** (Factorization property). The simple basis $\{S(w)\}$ of $\mathcal{R}$ factors through the simples $S(\theta e_s)$, for any $1 \leq s \leq l$ such that $s = s_{\max}$ (longest Kirillov-Reshetikhin modules):

$$[S(w) \ast S(\theta e_s)] = [S(\theta e_s) \ast S(w)] = S(w + \theta e_s), \forall w \in N^t.
$$

In type (i), this property was proved in [Kim12]. With the help of the standard basis and $T$-systems, we will give a proof by induction in the end of this subsection.

**Corollary 9.1.6.** By taking into account of the inverses of $S(\theta e_s)$, where $s_{\max} = s$, the simple basis $\{S(w)\}$ generates a basis of the quantum cluster algebra $\mathcal{A}(i)$, which we denote by $L^i_{t}$. Its structure constants are contained in $\mathbb{N}[q^{\frac{1}{2}}]$.

We use $t_0(l^-)$ to denote the seed from $t_0$ by freezing $l^-$. Because the only exchangeable vertex connected to $l$ is $l^-$, the frozen vertex $l$ does not connect to any exchangeable vertex in the corresponding ice quiver $\hat{Q}(t_0(l^-))$. Consequently, the quantum cluster algebra $\mathcal{A}^i(t_0(l^-))$ is generated by $\mathcal{A}^i(t_0(l^-))$ and the frozen variable $X_l$. It follows that $\mathcal{A}^i(t_0(l^-))$ has the following basis

$$L^l(t_0(l^-)) = \{X_l^d L(\eta)L(\eta) \in L^i(t_0(i - i)), d \in \mathbb{N}\}.
$$

As a consequence, we obtain the following Lemma.
Lemma 9.1.7. The restriction of \( L^{t_0} \) on the subalgebra \( A^l(t_0) \) gives its basis \( L^{t_0} \).

Proposition 9.1.8. (triangular product) The condition \( L \mathcal{A}(k;t_0) \) is satisfied by \( L^{t_0} \) for any \( 1 \leq k \leq l \).

Proof. By proposition 9.1.5, it suffices to verify the claim for \( X_k, k \neq l \).

We prove the claim by induction on the length \( l \) of the word \( \mathbf{i} \). The case \( l = |I| \) is trivial.

Now assume the claim has been verified for the words of length \( l - 1 \).

The quantum cluster variable \( X_k \) is the basis element \( L(e_k) = S(\theta e_k) = S(\beta_k + \beta_{[k]} + \ldots + \beta_{k_{\min}}) \). By Proposition 9.1.5, it suffices to verify the claim for \( k \neq l \).

Let \( S(w) \) be any basis elements in \( L^{t_0} \), where \( w \in \mathbb{N}^l \). Because the simple basis is unitriangular to the standard basis, we have the following \((\prec, \mathbf{m})\)-unitriangular decomposition

\[
S(w) = \sum_{w^1 \leq w} c(w, w^1) M(w^1)
\]

where \( w^1 \beta_i \) is the \( l \)-th component of \( w^1 \) and \( w_i^1 = w^1 - w_i^1 \beta_i \). \([\quad] \) is the normalization.

Now left multiplying the initial cluster variable \( X_k \), by Lemma 6.1.4, we still have a \((\prec, \mathbf{m})\)-unitriangular decomposition

\[
[X_k * S(w)] = \sum_{w^1 \leq w} c(w, w^1) [X_k * M(w^1) * M(w_i^1 \beta_i)]
\]

As discussed before, the quantum cluster algebra \( A^l(i - i) \) associated with the length \( l - 1 \) word \( i - i \) is a subalgebra of \( A^l(i) \) and it contains the standard basis element \( M(w_i^1) \). It also contains \( X_k \) because \( k \neq l \). By induction hypothesis, the normalized product \([X_k * M(w^1)]\) is unitriangular to the simple basis of \( A^l(i - i) \), and, consequently, to its standard basis. Namely, we have \((\prec, \mathbf{m})\)-unitriangular decompositions

\[
[X_k * M(w_i^1)] = \sum_{w^2 \leq \theta e_k + w_i^1} d(w^2, \theta e_k + w_i^1) M(w^2).
\]

Reusing Lemma 6.1.4, we now obtain the \((\prec, \mathbf{m})\)-unitriangular decomposition

\[
[X_k * S(w)] = \sum_{w^1 \leq w, w^2 \leq \theta e_k + w_i^1} e(w, w^1, w^2) [M(w^2) * M(w_i^1 \beta_i)],
\]
where \((w^2)_l = 0\), \(e(w, w, w^2)\) is contained in \(\mathfrak{m}\) if \(w^l_1 + w^2 \neq w + \theta e_k\) and equals to 1 if \(w^l_1 + w^2 = w + \theta e_k\).

Recall that the standard basis of \(A^{t, l}_n\) is unitriangular to its simple basis. The claim follows. \(\square\)

**Corollary 9.1.9.** (triangular basis) The basis \(L^{t, l}_n\) verifies Condition L2(\(t_0\)). Namely, it is the unique triangular basis for the seed \(t_0\).

We finish this subsection by a proof of Proposition 9.1.5.

**Lemma 9.1.10.** Take any \(Z\) in \(\mathcal{R}\). If \(Z\) is \((\prec, \mathfrak{m})\)-unitriangular to the standard basis, then for any \(w_1, w_l \in \mathbb{N}\), the normalized product \([Z \ast S(w_l \beta_l)]\) and \([S(w_1 \beta_1) \ast Z]\) are \((\prec, \mathfrak{m})\)-unitriangular to the standard basis.

**Proof.** Notice that we can replace \(Z\) by a unitriangular decomposition into the standard basis. The claim follows from Lemma 6.1.4. \(\square\)

**Lemma 9.1.11.** Take any \(Z = \sum a_i S(w^i)\) in \(\mathcal{R}\) such that \(a_i \in \mathbb{N}[t^{\pm \frac{1}{2}}]\). If there exists some simple module \(S(w)\) such that \(Z \ast S(w)\) or \(S(w) \ast Z\) is a simple module up to \(t^{\pm \frac{1}{2}}\)-shift, then \(Z\) is a simple module.

**Proof.** The claim follows by the positivity of the structure constants of the simple basis. \(\square\)

**Proof of Proposition 9.1.5.** We prove the claim by induction.

If the adaptable word has length \(l = |I|\), the statement is trivial.

Assume that the Proposition has been verified for any adaptable word of length less than \(l\). For any given word \(i\) of length \(l\), \(w = \sum_{k=1}^n w_k \beta_k\) with \(w_k \in \mathbb{N}\), and any \(1 \leq s \leq l\) such that \(s = s_{\text{max}}\), we want to show the normalized product \([S(\theta e_s) \ast S(w)]\) is a simple module. By its bar-invariance, it suffices to show that this product is \((\prec, \mathfrak{m})\)-unitriangular to the simple basis, or, equivalently, to the standard basis.

Let \(w_1, w_l, w_{1l}\) denote the dimension vector obtained from \(w\) by subtracting \(w_1 \beta_1, w_l \beta_l, \) and \(w_1 \beta_1 + w_l \beta_l\) respectively.

(i) Assume \(s \neq l\).

Notice that the module \(S(w)\) is \((\prec, \mathfrak{m})\)-unitriangular to the standard basis:

\[
S(w) = \sum_{w^l_1 \leq w} c(w, w^1) M(w^1) = \sum_{w^l_1 \leq w} c(w, w^1)[M(w^1_1) \ast M(w^l_1 \beta_l)].
\]
Therefore, we have the unitriangular decomposition
\[
[S(\theta e_s) * S(w)] = \sum_{w^1 \leq w} c(w, w^1)[S(\theta e_s) * M(w^1)]
\]
\[
= \sum_{w^1 \leq w} c(w, w^1)[S(\theta e_s) * M(w^1)] M(w^1) \beta_i]
\]

Notice that \(M(w^1)\) is unitriangular to the basis \(\{S(w^2)\}\) of \(R(i)\). The above is written as
\[
[S(\theta e_s) * S(w)] = \sum_{w^1 \leq w, w^2 \leq w^1} d(w, w^1, w^2)[S(\theta e_s) * S(w^2)] M(w^1) \beta_i] \]
\[
= \sum_{w^1 \leq w, w^2 \leq w^1} d(w, w^1, w^2)[S(\theta e_s + w^2) M(w^1) \beta_i],
\]

where we used induction hypothesis on \(R(i) \sim R(i - i_l)\) in the last equality. Here, the coefficients are contained in \(m\) except the leading term with \(w^1 = w, w^2 = w^1\) equals to 1.

By Lemma 9.1.10, each factor is \((\prec, m)\)-unitriangular to the standard basis.

(ii) Assume \(s = l\). We divide the proof into two cases.

(ii-1) If that \(i_l \neq i_1\), the proof is similar to that of (i).

Consider \((\prec, m)\)-unitriangular decomposition of \(S(w)\):
\[
S(w) = \sum_{w^1 \leq w} c(w, w^1) M(w^1) \]
\[
= \sum_{w^1 \leq w} c(w, w^1) M(w^1) \beta_1] M(w^1).
\]

Since \(M(w^1) \beta_1\) commutes with \(S(\theta e_l)\) up to a \(t^2\) power, we have the following unitriangular decomposition
\[
[S(\theta e_l) * S(w)] = \sum_{w^1 \leq w} c(w, w^1) M(w^1) \beta_1] S(\theta e_l) M(w^1)])
\]
\[
= \sum_{w^1 \leq w} c(w, w^1) M(w^1) \beta_1] [S(\theta e_l) M(w^1)]
\]

Decomposing the terms \(M(w^1)\) into simple basis elements, the above is written as
\[
[S(\theta e_l) * S(w)] = \sum_{w^1 \leq w, w^2 \leq w^1} d(w, w^1, w^2)[M(w^1) \beta_1] S(\theta e_l) S(w^2)]
\]
\[
= \sum_{w^1 \leq w, w^2 \leq w^1} d(w, w^1, w^2)[M(w^1) \beta_1] S(\theta e_l + w^2)],
\]
where we have used induction hypothesis $\mathcal{R}_{\tilde{1}} \to \mathcal{R}(i - i_1)$ in the last equality. Here, the coefficients are contained in $\mathfrak{m}$ except the leading term with $w^1 = w$, $w^2 = w_1^1$ equals to 1.

By Lemma 9.1.10, the above expression is $(\prec, \mathfrak{m})$-unitriangular to the standard basis.

(ii-2) Assume that $i_l = i_1$.

Consider the exchange relation at vertex $l[-1]$ in the seed $\mu_l[-2] \cdots \mu_1[1] \mu_1 t_0$ of the quantum cluster algebra $\mathcal{A}(i)$ (cf. [GLS11] for its ice quiver). Via the isomorphism $\kappa^{-1}$, it corresponds to the following $T$-system in $\mathcal{R}$:

$$(53) \quad [S(\theta_{e_l[-1]}) \ast S(1[1], l)] = [S(\theta_{e_l}) \ast S(1[1], l[-1])] + t^{-1}[\prod_{1\leq t < l, t = t_{\text{max}}} S(\theta_{e_t})^{-C_{i,t}}].$$

Denote the last simple module $\prod_{1\leq t < l, t = t_{\text{max}}} S(\theta_{e_t})^{-C_{i,t}}$ by $S(q)$.

Let us take any $w'$ such that $w'_1 = w'_1 = 0$. Notice that, in the quantum torus $\mathcal{T}(t_0)$, the leading monomials of $[S(\theta_{e_l[-1]}) \ast S(1[1], l)]$ and $S(q)$ differ by $Y_{l[-1]} \cdots Y_1$, which commutes with the monomial $X^{\theta_{l[-1]} w'}$ by the definition of compatible pair. If we deal with the type $(ii)$, this commutativity also follows from (40)(41). Therefore, by using the above $T$-system, we obtain

$$[S(\theta_{e_l}) \ast S(1[1], l[-1])] \ast S(w') = [[S(\theta_{e_l[-1]}) \ast S(1[1], l)] \ast S(w')] - t^{-1}[S(q) \ast S(w')].$$

By the induction hypothesis on $\mathcal{R}_{\tilde{1}}$, $[S(1[1], l) \ast S(w')]$ is a simple, which we denote by $S(p)$. By the induction hypothesis on either $\mathcal{R}_{\tilde{1}}$ or $\mathcal{R}_{\tilde{1}}$, $[S(q) \ast S(w')]$ is a simple, which we denote by $S(p')$. So we have

$$[S(\theta_{e_l}) \ast S(1[1], l[-1])] \ast S(w') = [[S(\theta_{e_l[-1]}) \ast S(p)] - t^{-1}S(p')]$$

By Lemma 9.1.10, RHS is unitriangular to the standard basis. By induction hypothesis, on $\mathcal{R}_{\tilde{1}} \to \mathcal{R}(i - i_1 - i_1)$, $S(1[1], l[-1])$ quasi-commutes with $S(w')$. Also, $S(\theta_{e_l})$ quasi-commutes with $S(w')$ as it is a frozen variable in the corresponding quantum cluster algebra. Therefore, the normalized product $[S(\theta_{e_l}) \ast S(1[1], l[-1]) \ast S(w')]$ is bar-invariant. Consequently, it is a simple module. By Lemma 9.1.11, $[S(\theta_{e_l}) \ast S(w')]$ is a simple module.

Consider the $(\prec, \mathfrak{m})$-unitriangular decompositions of the module $S(w)$ as

$$S(w) = \sum_{w^1 \leq w} b(w, w^1)[M(w_1^1 \beta_1) \ast M(w_2^1 \beta_1) \ast M(w_1^1 \beta_1)]$$

$$= \sum_{w^1 \leq w, w^2 \leq w_1^1 \beta_1} c(w, w^1, w^2)[M(w_1^1 \beta_1) \ast S(w^1) \ast M(w_1^1 \beta_1)]$$
It follows that we have the \((\prec, \mathbf{m})\)-unitriangular decomposition

\[
[S(\theta e_i) \ast S(w)] = \sum d(w, w', w'') [S(\theta e_i) \ast M(w'_1 b_1) \ast S(w') \ast M(w''_1 b_1)]
\]

\[
= \sum d(w, w', w'') [M(w'_1 b_1) \ast [S(\theta e_i) \ast S(w')] \ast M(w''_1 b_1)].
\]

Because \([S(\theta e_i) \ast S(w')]\) is a simple module, the above expansion of \(S(\theta e_i) \ast S(w)\) is \((\prec, \mathbf{m})\)-unitriangular to the standard basis by Lemma 9.1.10. The desired claim follows by its bar-invariance. \(\square\)

9.2. **Compatibility.** By Theorem 6.4.7, in order to apply the existence theorem to the quantum cluster algebra \(\mathcal{A}(i)\), it remains to check the following property.

**Conjecture 9.2.1.** The quantum cluster variables obtained along the sequence \((\sigma \Sigma_i)^{-1}\) from \(t_0\) to \(t_0[-1]\) are contained in \(L^0\).

In Section 9.3, we will give an example as a different quick proof of this property for type \(A_4\) based on the level-rank duality.

In this section, we verify that the triangular bases in \(t_0\) and \(t_0[-1]\) are compatible when \(i = c^{N+1}\) for some acyclic Coxeter word \(c\) and \(N \in \mathbb{N}\).

Instead of working with the previously introduced mutation sequence \(\Sigma_i\), we define the following sequences (read from right to left)

\[
\tilde{\mu}^k = \mu_{max}^{-1} \cdots \mu_{max}^{-2} \mu_{max}^{-1}, \quad 1 \leq k \leq l,
\]

\[
\sigma \tilde{\mu}^k = \mu_{min}^{-1} \cdots \mu_{min}^{-1} \mu_{min}, \quad 1 \leq k \leq l,
\]

\[
\Sigma^i = (\sigma \tilde{\mu}^{-1})^{-1} \cdots \tilde{\mu}^{-2} \tilde{\mu}^{-1}.
\]

Notice that \(\sigma\) is an involution. We have

\[
\sigma(\Sigma^i)^{-1} = (\sigma \Sigma^i)^{-1} = \tilde{\mu}^{-1} \cdots \tilde{\mu}^{-2} \tilde{\mu}^{-1}.
\]

**Example 9.2.2.** Let us continue Example 8.2.1. The sequences \(\tilde{\mu}^{-10} = \mu_1 \mu_5 \mu_3, \tilde{\mu}^{-5} = \mu_8,\) and \(\tilde{\mu}^{-1}\) is trivial. The sequence \(\Sigma^i\) and \((\sigma \Sigma^i)^{-1}\) (read from right to left) are given by

\[
\Sigma^i = (1, 5, 8, 2, 6, 1, 5, 3, 2, 1)
\]

\[
(\sigma \Sigma^i)^{-1} = (8, 6, 3, 5, 8, 2, 6, 1, 5, 8).
\]

Let \(i - i_l\) denote the word obtained from \(i\) by removing the last element \(i_l\) and \(\Sigma^i - i_l\) the corresponding sequence. Then we have \(\Sigma^i = (\sigma \tilde{\mu}^{-1})^{-1} \Sigma^i - i_l = \tilde{\mu}^{-1} \Sigma^i - i_l\). This observation inductively implies the following result.

**Lemma 9.2.3.** For any \(d \in \mathbb{Z}\), the seed \(l[d]\) is injective-reachable via either \((\sigma^d \Sigma^i, \sigma)\) or \((\sigma^d \Sigma^i, \sigma)\). Moreover, the sets of the quantum cluster variables obtained along both sequences starting from \(l[d]\) are the same.
Proof. The statement for \( d = 0 \) is obvious by induction, from which we deduce the general statement by noticing that, up to the permutation \( \sigma \) of vertices, all the seeds involving have the same principal part. \( \square \)

Recall that when \( i \) is adaptable, we can choose an adaptable multidegree \( \underline{a} = (a_k) \in (2\mathbb{Z})^I \), and identify the root vector \( \beta_k \) with the unit vector \( e_{i_k,a_k} \) on graded quiver varieties by the embedding \( \iota_\underline{a} \). The graded Grothendieck ring \( \mathcal{R} \) can be realized as \( \mathcal{R}(i, \underline{a}) \) cf. Section 2.7.

**Proposition 9.2.4.** Assume \( i = c^{N+1} \). Then, for any \( 1 \leq k < l \) such that \( i_k = i_l \), the quantum cluster variables \( X_i(\mu_k \mu_{k[i]} \cdots \mu_l t_0) \) is the simple module \( \kappa^i W^{(i)}_{\mu_k, \mu_{k[i]} + 1, a_k} \).

**Proof.** It is easy to compute the Laurent expansions of these quantum cluster variables. By Proposition 7.4.3(i)(ii), the Laurent expansions of the simple modules \( \kappa^i W^{(i)}_{\mu_k, \mu_{k[i]} + 1, a_k} \) take the same values. \( \square \)

**Lemma 9.2.5.** Assume \( i = c^{N+1} \). Then all the quantum cluster variables appearing along the sequence \( (\sigma \Sigma^i)^{-1} \) from \( t_0 \) to \( t_0^{-1} \) also appear along the sequence \( (\mu^c)^N \), where \( \mu^c \) denote the sequence \( \mu^c l^{-r+1} \cdots \mu^c l^{-1} \mu^c \) starting from \( t_0 \).

**Proof.** The claim follows by tracking the exchange relations involved. Recall that the quantum cluster variables \( X_i(t) \), as well as their specialization \( x_i(t) = X_i(t)|_{q_1^{\sigma^{-1}}} \), are in bijection with the object \( T_i(t) \) in Section 2.2. By using Calabi-Yau reduction, it suffices to remove the frozen vertices and verify that the commutative cluster variables \( x_i(t) \) appearing on the sequence \( (\sigma \Sigma^i)^{-1} \) starting from quiver \( Q(t_0) \) are contained in those appearing on the sequence \( (\mu^c)^N \).

For any given \( 0 \leq d \leq N - 1 \), let \( \mu^c \Sigma_d \) denote the mutation sequence obtained from \( \mu^c \) by removing the factors \( \mu_{\min[d]} \) with \( i \in I, d' < d \), and \( \mu^c \Sigma_d \) the sequence obtained from \( \mu^c \) by removing the factors \( \mu_{\min[i]} \) with \( i \in I, d' \geq d \). Then, by tracking the exchange relations involved, cf. the arguments in [GLS11, Section 13.1], we see that \( \mu^c \) and \( \mu^c \Sigma_d \mu^c \Sigma_d \) give identical collection of cluster variables. Because \( (\sigma \Sigma^i)^{-1} = \mu^c \Sigma_{N-1} \cdots \mu^c \Sigma_{1} \mu^c \Sigma_{0} \), it suffices to verify that, starting from the quiver \( Q(t_0) \), the mutation sequences

\[
(\mu^c_{<N-1} \cdots \mu^c_{<1} \mu^c_{\geq 1})(\mu^c_{<0} \mu^c_{\geq 0})
\]

and

\[
(\mu^c_{<N-1} \cdots \mu^c_{<1} \mu^c_{<0})(\mu^c_{\geq N-1} \cdots \mu^c_{\geq 1} \mu^c_{\geq 0}),
\]

give identical collection of cluster variables.

Let us denote \( A = \mu^c_{\geq d+1} \) and \( B = \mu^c_{\leq d} \cdots \mu^c_{<1} \mu^c_{<0} \). The quiver \( Q' = \mu^c_{\geq d} \cdots \mu^c_{<0} Q(t_0) \), the set of vertices acted by \( A \) and that acted by \( B \) are separated by the full subquiver on \( \{\min[i] \mid i \in I\} \) of \( Q' \). By
definition of mutations, this property is preserved by the quivers appearing along $AB$ or $BA$. Therefore, the mutation sequences $AB$ and $BA$ give identical set of vertices. The claim follows by recursively swapping $A$ and $B$ for all $d$. □

Example 9.2.6. Let the Cartan matrix be of type $A_3$, $c = (321)$ and $N = 4$. Take $d = 2$, then the quiver $Q' = \mu_{\geq d} \cdots \mu_{\geq 0} Q(t_0)$ is drawn in Figure 13. Take $A = \mu_{\geq d+1}^{c} = \mu_{10}\mu_{11}\mu_{12}$ and $B = \mu_{\geq 2}^{c} \mu_{<1} = \mu_{13}\mu_{4}\mu_{5}\mu_{6}\mu_{1}\mu_{2}\mu_{3}$. The sets $\{1, 2, 3, 4, 5, 6\}$ and $\{10, 11, 12\}$ are always separated by the full subquiver on $\{7, 8, 9\}$ of the quivers appearing along mutation sequence $AB$ or $BA$.

Moreover, any such quiver appearing is a union of small blocks in Figure 14 for $i \neq j \in I$, $0 \leq d' \leq N - 2$, with at most one diagonal edge of multiplicity $-C_{ij}$.

Figure 13. A quiver $Q'$

\[ i[d' + 1] \quad \cdots \quad i[d'] \]

\[ -C_{ij} \quad \cdots \quad -C_{ij} \]

\[ j[d' + 1] \quad \cdots \quad j[d'] \]

Figure 14. small block

Recall that the set $I$ denote $\{1, 2, \ldots, r\}$.

Proposition 9.2.7. When $i = c^{N+1}$, the triangular bases in the seeds $t_0$ and $t_0[-1]$ are compatible. Moreover, they contain all the quantum cluster variables appearing along the sequence $(\sigma \Sigma^i)^{-1}$ from $t_0$ to $t_0[-1]$.

Proof. We have shown that the triangular basis $L^{t_0}$ in $t_0$ satisfies the conditions L0, L1, L2, L3. Observe that the all quantum cluster variables along the mutation sequence $\Sigma^i$ from $t_0$ to $t_0[1]$ are known to be Kirillov-Reshetikhin modules, and, in particular, contained in the
initial triangular basis $L'_0$. Combining this observation with Proposition 9.2.4 and Proposition 6.3.5(i)(ii), we see that, the triangular bases exists on all seeds along the sequence $\mu^i$ from $t_0$ to $\mu^i t_0$ and they are compatible with each other.

Notice that the principal quiver $Q(\mu^i t_0)$ is the same as that of the ice quiver associated with the word $\mathbf{i}' = (i_{t-1}, i_{t-2}, \ldots, i_1, i_0)$, which takes the form $(i_{r-1}, \ldots, i_1, i_0)^{N+1}$ for the acyclic Coxeter word $(i_{r-1}, \ldots, i_1, i_0)$. Applying Proposition 9.2.4 on the ice quiver $\Gamma'_i$ and using the correction technique in Lemma 6.1.8, we see that the quantum cluster variables obtained along the sequence $\mu^i t_0$ starting from $\mu^i t_0$ is contained in the canonical basis $L^\mu t_0$ of $\mu^i t_0$. Repeating the previous argument, we obtain that the triangular bases exists on all seeds along the sequence $\mu^i t_0$ to $\mu^i t_0$ and they are compatible with each other.

Repeating this process, we conclude that the triangular bases exists on all seeds along the sequence $(\mu^c)^N$ from $t_0$ to $(\mu^c)^N t_0$ and they are compatible with each other. Moreover, the initial triangular basis $L^0$ contains all the quantum cluster variables along this sequence. By Lemma 9.2.5, it already contains the quantum cluster variables along the sequence $(\sigma \Sigma^i)^{-1}$ starting from $t_0$. Repeating the previous argument, we obtain that the triangular bases in $t_0$ and $t_0[-1]$ are compatible.

9.3. Compatibility in type $A$ via level-rank duality. In this section, we consider the special case when $C$ is of type $A_r$ and $i = (1, 2, 1, 3, 2, 1, \ldots, r-1, \ldots, 2, 1)$. Then $w_i$ is the longest element in the Weyl group. The pair $(C, i)$ is of both type (i) and (ii).

In the following example, we check Conjecture 9.2.1 for the special case $A_4$, which, together with Proposition 6.4.5, will imply the type $A_4$ case in Theorem 9.4.1 9.4.2 9.4.3. This approach seems to be effective for general $r$, but we don’t pursue the generalization in this paper.

Example 9.3.1. Consider Figure 9. The exchange relations along the mutation sequence $\Sigma^i = (1, 5, 8, 2, 6, 1, 5, 3, 2, 1)$ are $T$-system relations. Consider the trivial permutation $\var^*$ of $[1, l]$. It induces an isomorphism from principal quiver $Q(t'_0)$ (Figure 15) to the principal quiver $Q(t_0)$. We have a variation map $\var$ from $\mathcal{T}(t)$ to $\mathcal{T}(t')$. The mutation sequence (read from right to left) $(\sigma \Sigma^i)^{-1} = (8, 6, 3, 5, 8, 2, 6, 1, 5, 8)$ becomes a new sequence on $Q(t'_0)$. It is straightforward to check that the quantum cluster variables appearing along this sequence correspond to Kirillov-Reshetikhin modules for the seed $t'_0$, which is again associated with the longest word. In particular, they are contained in the triangular basis $L^6$. By the correction technique in Lemma 4.4.2, similar quantum cluster variables obtained along $(\sigma \Sigma^i)^{-1}$ are contained in $L^6$.

9.4. Consequences. We summarize the consequences of the previous discussion.
Figure 15. Ice quiver $\tilde{Q}(t'_0)$ obtained by rotating the quiver in Figure 9 and changing coefficient pattern.

We have seen that the simple basis $\{S(w)\}$ provides the initial triangular basis in the seed $t_0$ after localization. Combining Proposition 9.2.7, we have verified all the conditions in Theorem 6.4.7 for the word $c^{N+1}$. Notice that, to any adaptable word $i$, we can associate an initial subword $i'$ of some $c^{N+1}$, such that $w_i = w_i'$ and, consequently, $A(i)$ agrees with $A(i')$. Notice that $i'$ is adaptable as well. By working with type (ii) quantum cluster algebras, we have the embedding from $R(i', a')$ to $R(c^{N+1}, a)$ and the following result.

**Theorem 9.4.1.** Assume that $C$ is symmetric and $i$ an adaptable word. Then the simple basis $\{S(w)\}$, after localization at the frozen variables, gives the common triangular basis for all seeds of the quantum cluster algebra $A(i)$ in the sense that it verifies all the conditions in Theorem 6.4.1. In particular, it is parametrized by tropical points as expected by Fock-Goncharov, and it contains all quantum cluster monomials.

The following result is a direct consequence.

**Theorem 9.4.2.** Conjecture 8.4.4 holds true. In particular, the quantum cluster monomials in $A$ of type (ii) are simple modules of quantum affine algebras.

For quantum cluster algebras arising from quantum unipotent subgroups, we have verified Conjecture 8.3.3 in Proposition 9.1.3. Moreover, Theorem 9.4.1 implies that the quantum cluster monomials of $A$ of type (i) are contained in the dual canonical basis for the following cases.

**Theorem 9.4.3.** Conjecture 8.3.2 holds true for any reduced word $w$ which is inferior than some adaptable word with respect to the left or right weak order. In particular, it holds true for any reduced word $w$ when the Cartan type is $ADE$.

**Proof.** The first claim follows from Theorem 9.4.1 and compositions of the embeddings of Grothendieck ring in the end of Section 8.3.

For type $ADE$, the claim holds for the longest word because it is adaptable. Any reduced word $w$ is inferior than the longest word with
respect to the left and right weak order, cf. [BB05]. The second claim follows.

**Remark 9.4.4.** When the quantum cluster algebra is of type (i). The claim that all seeds $t$ satisfy condition $L3(t)$ might be compared with the following weaker form of Leclerc’s conjecture [Lec03, Section 3.1].

Let $b_1, b_2$ be elements of the dual canonical basis $B^*$. Suppose that $b_1$ corresponds to a cluster variable. Then the expansion of $b_1, b_2$ on $B^*$ is of the following form:

$$b_1 b_2 = q^s b'' + \sum_{c \neq b''} \gamma_{b_1 b_2}(q)c,$$

where $s \in \mathbb{Z}$, $\gamma_{b_1 b_2} \in q^{s-1}\mathbb{Z}[q^{-1}]$.

Leclerc’s conjecture was recently proved in [KKKO14].

**REFERENCES**

[Ami09] Claire Amiot, *Cluster categories for algebras of global dimension 2 and quivers with potential*, Annales de l’institut Fourier 59 (2009), no. 6, 2525–2590, arXiv:0805.1035.

[BB05] Anders Bjorner and Francesco Brenti, *Combinatorics of coxeter groups*, Springer, 2005.

[BFZ05] Arkady Berenstein, Sergey Fomin, and Andrei Zelevinsky, *Cluster algebras. III. Upper bounds and double Bruhat cells*, Duke Math. J. 126 (2005), no. 1, 1–52.

[BZ05] Arkady Berenstein and Andrei Zelevinsky, *Quantum cluster algebras*, Adv. Math. 195 (2005), no. 2, 405–455, arXiv:math/0404446v2.

[BZ12] ______, *Triangular bases in quantum cluster algebras*, 2012, arXiv:1206.3586.

[DK07] Raika Dehy and Bernhard Keller, *On the combinatorics of rigid objects in 2-Calabi-Yau categories*, arXiv:0709.0882.

[DWZ08] Harm Derksen, Jerzy Weyman, and Andrei Zelevinsky, *Quivers with potentials and their representations I: Mutations*, Selecta Mathematica 14 (2008), no. 1, 59–119.

[DWZ10] ______, *Quivers with potentials and their representations II: Applications to cluster algebras*, J. Amer. Math. Soc. 23 (2010), no. 3, 749–790, arXiv:0904.0676v2.

[FZ02] Sergey Fomin and Andrei Zelevinsky, *Cluster algebras. I. Foundations*, J. Amer. Math. Soc. 15 (2002), no. 2, 497–529 (electronic), arXiv:math/0104151v1.

[FZ07] ______, *Cluster algebras IV: Coefficients*, Compositio Mathematica 143 (2007), 112–164, arXiv:math/0602259v3.
B. Leclerc, *Imaginary vectors in the dual canonical basis of $U_q(n)$*, Transform. Groups 8 (2003), no. 1, 95–104.

Kyungyong Lee, *Every finite acyclic quiver is a full subquiver of a quiver mutation equivalent to a bipartite quiver*, 2013, arXiv:1311.0711.

Kyungyong Lee, Li Li, Dylan Rupel, and Andrei Zelevinsky, *The existence of greedy bases in rank 2 quantum cluster algebras*, 2014, arXiv:1405.2414.

Kyungyong Lee, Li Li, Dylan Rupel, and Andrei Zelevinsky, *The existence of greedy bases in rank 2 quantum cluster algebras*, 2014, arXiv:1405.2311.

Kyungyong Lee, Li Li, and Andrei Zelevinsky, *Greedy elements in rank 2 cluster algebras*, 2014, arXiv:1405.2415.

Kyungyong Lee, Li Li, and Andrei Zelevinsky, *Greedy bases in rank 2 quantum cluster algebras*, 2014, arXiv:1405.2311.

Kyungyong Lee, Li Li, and Andrei Zelevinsky, *Greedy elements in rank 2 cluster algebras*, 2014, arXiv:1405.2414.

Kyungyong Lee, Li Li, and Andrei Zelevinsky, *Greedy elements in rank 2 cluster algebras*, 2014, arXiv:1405.2311.

Kyungyong Lee, Li Li, and Andrei Zelevinsky, *Greedy bases in rank 2 quantum cluster algebras*, 2014, arXiv:1405.2311.

Kyungyong Lee, Li Li, and Andrei Zelevinsky, *Greedy bases in rank 2 quantum cluster algebras*, 2014, arXiv:1405.2311.

Kyungyong Lee, Li Li, and Andrei Zelevinsky, *Greedy bases in rank 2 quantum cluster algebras*, 2014, arXiv:1405.2311.

Kyungyong Lee, Li Li, and Andrei Zelevinsky, *Greedy elements in rank 2 cluster algebras*, 2014, arXiv:1405.2414.

Kyungyong Lee, Li Li, and Andrei Zelevinsky, *Greedy bases in rank 2 quantum cluster algebras*, 2014, arXiv:1405.2311.

Kyungyong Lee, Li Li, and Andrei Zelevinsky, *Greedy elements in rank 2 cluster algebras*, 2014, arXiv:1405.2414.

Kyungyong Lee, Li Li, and Andrei Zelevinsky, *Greedy bases in rank 2 quantum cluster algebras*, 2014, arXiv:1405.2311.

Kyungyong Lee, Li Li, and Andrei Zelevinsky, *Greedy elements in rank 2 cluster algebras*, 2014, arXiv:1405.2414.

Kyungyong Lee, Li Li, and Andrei Zelevinsky, *Greedy bases in rank 2 quantum cluster algebras*, 2014, arXiv:1405.2311.

Kyungyong Lee, Li Li, and Andrei Zelevinsky, *Greedy elements in rank 2 cluster algebras*, 2014, arXiv:1405.2414.

Kyungyong Lee, Li Li, and Andrei Zelevinsky, *Greedy bases in rank 2 quantum cluster algebras*, 2014, arXiv:1405.2311.

Kyungyong Lee, Li Li, and Andrei Zelevinsky, *Greedy elements in rank 2 cluster algebras*, 2014, arXiv:1405.2414.

Kyungyong Lee, Li Li, and Andrei Zelevinsky, *Greedy bases in rank 2 quantum cluster algebras*, 2014, arXiv:1405.2311.

Kyungyong Lee, Li Li, and Andrei Zelevinsky, *Greedy elements in rank 2 cluster algebras*, 2014, arXiv:1405.2414.

Kyungyong Lee, Li Li, and Andrei Zelevinsky, *Greedy bases in rank 2 quantum cluster algebras*, 2014, arXiv:1405.2311.

Kyungyong Lee, Li Li, and Andrei Zelevinsky, *Greedy elements in rank 2 cluster algebras*, 2014, arXiv:1405.2414.

Kyungyong Lee, Li Li, and Andrei Zelevinsky, *Greedy bases in rank 2 quantum cluster algebras*, 2014, arXiv:1405.2311.

Kyungyong Lee, Li Li, and Andrei Zelevinsky, *Greedy elements in rank 2 cluster algebras*, 2014, arXiv:1405.2414.

Kyungyong Lee, Li Li, and Andrei Zelevinsky, *Greedy bases in rank 2 quantum cluster algebras*, 2014, arXiv:1405.2311.

Kyungyong Lee, Li Li, and Andrei Zelevinsky, *Greedy elements in rank 2 cluster algebras*, 2014, arXiv:1405.2414.

Kyungyong Lee, Li Li, and Andrei Zelevinsky, *Greedy bases in rank 2 quantum cluster algebras*, 2014, arXiv:1405.2311.

Kyungyong Lee, Li Li, and Andrei Zelevinsky, *Greedy elements in rank 2 cluster algebras*, 2014, arXiv:1405.2414.

Kyungyong Lee, Li Li, and Andrei Zelevinsky, *Greedy bases in rank 2 quantum cluster algebras*, 2014, arXiv:1405.2311.

Kyungyong Lee, Li Li, and Andrei Zelevinsky, *Greedy elements in rank 2 cluster algebras*, 2014, arXiv:1405.2414.

Kyungyong Lee, Li Li, and Andrei Zelevinsky, *Greedy bases in rank 2 quantum cluster algebras*, 2014, arXiv:1405.2311.

Kyungyong Lee, Li Li, and Andrei Zelevinsky, *Greedy elements in rank 2 cluster algebras*, 2014, arXiv:1405.2414.
[Qin12] Fan Qin, *Quantum cluster variables via Serre polynomials*, J. Reine Angew. Math. 2012 (2012), no. 668, 149–190, with an appendix by Bernhard Keller, arXiv:1004.4171, doi:10.1515/CRELLE.2011.129.

[Qin13] Fan Qin, *t-analog of q-characters, bases of quantum cluster algebras, and a correction technique*, International Mathematics Research Notices (2013), arXiv:1207.6604.

[Rou08] Raphaël Rouquier, *2-Kac-Moody algebras*, 2008, arXiv:0812.5023.

[Rou12] Raphaël Rouquier, *Quiver hecke algebras and 2-Lie algebras*, Algebra colloquium, vol. 19, World Scientific, 2012, pp. 359–410.

[Thu13] Dylan P Thurston, *A positive basis for surface skein algebras*, 2013, arXiv:1310.1959.

[Tra11] Thao Tran, *F-polynomials in quantum cluster algebras*, Algebr. Represent. Theory 14 (2011), no. 6, 1025–1061, arXiv:0904.3291v1.

[VV03] M. Varagnolo and E. Vasserot, *Perverse sheaves and quantum Grothendieck rings*, Studies in memory of Issai Schur (Chevaleret/Rehovot, 2000), Progr. Math., vol. 210, Birkhäuser Boston, Boston, MA, 2003, pp. 345–365, arXiv:math/0103182v3. MR MR1985732 (2004d:17023)

[VV11] M. Varagnolo and E. Vasserot, *Canonical bases and KLR-algebras*, J. Reine Angew. Math. 2011 (2011), no. 659, 67–100, arXiv:0901.3992, doi:10.1515/crelle.2011.068.

E-mail address: qin.fan.math@gmail.com