Chapter

Microwave Heating of Low-Temperature Plasma and Its Application

Tetyana Frolova, Vyacheslav Buts, Gennadiy Churyumov, Eugene Odarenko and Vladimir Gerasimov

Abstract

In this chapter, the results of theoretical and experimental studies of the interaction of an electromagnetic field with a plasma (fundamental interaction of the wave-particle type) both in the regime of standing waves (in the case of a resonator) and in the case of traveling waves in a waveguide are presented. The results of computer modeling the distribution of a regular electromagnetic field for various designs of electrodynamic structures are considered. The most attractive designs of electrodynamic structures for practical application are determined. A brief review and analysis of some mechanisms of stochastic plasma heating are given as well as the conditions for the formation of dynamic chaos in such structures are determined. Comparison analysis of microwave plasma heating in a regular electromagnetic field (in a regime with dynamical chaos) with plasma heating by random fields is considered. It is shown, that stochastic heating of plasma is much more efficient in comparison with other mechanisms of plasma heating (including fundamental interaction of the wave-wave type). The results obtained in this work can be used to increase the efficiency of plasma heating as well as to develop promising new sources of electromagnetic radiation in the microwave and optical ranges.

Keywords: Low-temperature plasma, microwave heating, stochastic heating, electromagnetic wave, dynamic chaos, electrodynamic structure

1. Introduction

In recent years, the most promising method for creating low-temperature plasma is the method of microwave heating. As of now, gas-discharge plasma is effectively used in various fields, including microwave electronics, lighting technology, medicine, plasma chemistry, etc. The use of microwave heating to excite gases and to create plasma allows significantly to reduce the overall dimensions of devices. This is a very current task especially under creating small-size devices, including the annular laser gyroscopes, different types of lasers, electrodeless plasma lamps, etc. [1].

Created designs of a miniature helium-neon laser and a source of incoherent optical radiation based on an electrodeless sulfur lamp are shown in Figure 1 [2, 3]. The unifying factor of these structures is the method of microwave excitation (heating) of the gas mixture in the working volume of the devices. Considering the
significant difference in the frequencies of the exciting microwave field, the designs of electrodynamic systems, in the volume of which a regular electromagnetic field is formed, have significant differences. These features of the excitation elements must be taken into account when designing devices, the operation of which is based on the use of gas-discharge media (plasma).

The analysis shows that the main structural elements of these devices are the active medium and the electrodynamic system for the formation of a regular electromagnetic field. A gas or a gas mixture (sometimes with the addition of an impurity, for example, sulfur S8, as in the case of a sulfur lamp) is considered as an active medium with nonlinear properties. Under the action of an external microwave electromagnetic field, the gas mixture is ionized and plasma is formed. At this stage, it is important to understand the ongoing physical processes that underlie the

Figure 1.
Examples of structural elements of sources of electromagnetic radiation with microwave excitation. A ring laser gyroscope [2] (a) and an optical radiation source based on an electrodeless sulfur lamp (b) [2].
formation of plasma with the necessary quantitative characteristics. To solve such a problem, it is necessary to develop a mathematical model of the plasma, the use of which will make it possible to estimate the quantitative parameters of the gas (gas mixture) to provide the necessary plasma characteristics.

On the other hand, the process of ionization of the active medium (gas or gas mixture) occurs as a result of the action of a regular electromagnetic field. One of the conditions for effective plasma heating is the arrangement of an active medium into an area with the maximum intensity of the electric component of the regular electromagnetic field. To achieve this, it is necessary to know the distribution of the components of the electromagnetic field in the volume of the electrodynamic structure, which requires additional studies of its properties. Currently, there are two approaches to the formation of an electromagnetic field with the aim of its subsequent use for exciting or heating various media: resonance (or a regime of a standing wave) and interference, when two traveling waves add up to give a resultant standing wave [3]. In the first case, a regular electromagnetic field is excited in a resonator. The shape and dimensions of the given resonator are selected from the conditions of the frequency range used, taking into account the maximum intrinsic quality factor of the oscillatory system. The interference method of forming a standing wave involves the use of two traveling waves moving towards each other. To implement this approach, a regular waveguide is used, inside which a bulb with an active medium is placed.

Thus, in order to increase the efficiency of conversion of the energy of the electromagnetic field into the internal energy of the gas–discharge medium, it is necessary to a knowledge of the distribution of a regular electromagnetic field as well as an understanding of the physical processes in the active medium.

In the latter case, it is important to further develop the theory of plasma processes taking into account their chaotization, as well as to understand the conditions for the occurrence of regimes with dynamic chaos (conditions of stochastic heating of plasma).

In the given chapter the theoretical and experimental studies of microwave heating of plasma by an electromagnetic field are discussed. The conditions for increasing the efficiency of the microwave heating at the expense of enhancing the intensity of a regular electromagnetic field or using dynamic chaos mode, including the methods of its achieving (the cases of Cherenkov’s and cyclotron resonances) are determined. The computer modeling results of electrodynamic systems are presented and the process of formation of a standing electromagnetic wave in electrodynamic systems (resonator and waveguide) is investigated, conditions for a local increase of intensity of the regular field in the region of the active medium are determined.

2. Computer modeling of the electrodynamic systems

As mentioned above, the energy of the microwave electromagnetic field is used to excite and heat the plasma. Various designs of oscillatory structures (resonators) are used as electrodynamic structures that form fields with the required distribution of power lines, the type and shape of which depend on the frequency of the electromagnetic field used. For example, for the excitation (pumping) of the active medium in helium-neon lasers, electromagnetic oscillations are used, the frequency of which lies in the range of 200 ... 400 MHz with an average microwave power level of ~ 2 ... 5 W. In this case, a classical oscillatory circuit in the form of a flat capacitor is used as an oscillatory electrodynamic system. Figure 2 shows the design of such a capacitor, the results of simulation and experiment [4]. An analysis of the
results obtained showed that the pump parameters of the active medium (helium-neon mixture) depend on the dimensions of the flask (its inner diameter and length) containing the active medium and are selected taking into account the maximum efficiency of the pump energy transfer process (see Figure 1a).

With the increasing frequency, an electrodynamic system usually modified its shape and dimensions. For frequencies in the range of more than 1000 MHz, cavity resonators are usually used (see Figure 1b). In our case, we used a cylindrical resonator having the following geometrical dimensions: a diameter \( \approx 172 \) mm and a height \( \approx 120 \) mm. As a source of electromagnetic oscillation, there was used a magnetron generator, possessing a frequency of generation \( \approx 2.45 \) GHz and the output power \( \approx 800 \) W.

Figure 3 illustrates the results of a computer modeling of the two modes of oscillations excited into the cylindrical resonator, namely, the \( H_{111} \) and \( H_{011} \) modes, correspondingly. The dependences of the reflection coefficient modulus from frequency and the spatial distributions of components of an electric field of the given oscillation modes have been shown.

As is seen, the different modes of oscillations are excited in the cylindrical resonators. As a rule, the given modes have distinct frequencies and possessing different distributions of the electric component of an electromagnetic field. The interest is the oscillations with the frequencies close to the frequency of the magnetron, i.e., to the frequency of 2.45 GHz. On the other hand, it is significant that the selected mode of oscillation had a maximum of the electric field in the area corresponding to the location where must be the bulb with the gaseous mixture. This permits to make an excitation process more effective.

In addition to the resonance excitation method of the plasma for forming a standing wave, we may use the interference of two coherent waves propagating towards each other in a waveguide [3]. Let us consider this approach using an
example of the waveguide with a cross-section (72x34) mm. This approach may be realized using the waveguide structure that is schematically shown in Figures 4 and 5. An electromagnetic wave $E$ from the output of the magnetron originates to the input of the waveguide structure at $x = L_0$ (see Figure 5). Next at $x = 0$ and $z = 0$, the wave is divided into two waves and that enters the inputs 1 and 2 of a waveguide and propagates towards each other (see Figures 4 and 5). As a result of the interference of the two coherent waves in the waveguide, the standing wave is formed as it may be seen from Figure 6.

For increasing the intensity of the electric component of a total electromagnetic field and enhancing the efficiency of exciting plasma are of interest to view a case of contraction of the narrow size $b$. The main results of computer modeling the propagation of the waves in the space of the waveguide $L_0 - L_A$ taking into account a change of the size of the narrow wall in the waveguide are shown in Figure 7.

As indicated in Figure 7, the general regularity of changing the intensity of the total electromagnetic field in the waveguide $E = E_1 + E_2$ is nonlinear and satisfies the condition $E \sim 1/b$, where $b$ – the size of the narrow wall of the waveguide. By varying the high of the narrow wall of the waveguide we may choose a necessary value of the intensity of electric components and thus increasing the efficiency of controlling by a process of plasma heating. Such an approach of increasing the efficiency of the excitation process of the plasma mixture can be used for the choice of an optimum design of the electrodynamic structure. Among possible electrodynamic structures having an enhanced concentration of the electric component of electromagnetic field one can select both the resonant species of such structures and non-resonant. To the first group, we can relate toroid and coaxial resonators. The
second group of such structures can be presented by the single- and double-ridged waveguides. Thus, an application of the above-mentioned electrodynamic structures enables improving the process of exciting plasma and enhancing the efficiency of transformation of energy.

Figure 4.
Schematic image of the waveguide structure with a metal insert having a height $h$.

Figure 5.
Image of the curve $L_0$ – $L_4$ along which the electric field value is calculated.
3. Mathematical models and modeling of stochastic processes in plasma

3.1 Mathematical description of the state of the plasma and its model

Due to the variety of processes taking place in a spatially inhomogeneous plasma, an analytical description of a real plasma in the general case is very difficult. Therefore, simplified plasma models are usually considered, stipulating the conditions under which a real plasma can be close to its accepted model.

The state of a real plasma at an arbitrary pressure is determined by a) the concentration of particles of all kinds $N$ (the number of particles per unit volume);
b) their speed distribution functions $N_i(n)$; c) the population of the excited levels $N_k$ (the number of particles per unit volume, excited in the state $k$); d) the spatial distribution of these quantities.

It is extremely difficult to obtain information about all the listed characteristics since theoretical studies of the state of plasma-like media require the compilation and solution of a system of equations connecting the indicated quantities with external conditions.

The basic equations describing the nonlinear states of plasma have limitations, primarily related to the possibility of obtaining their solutions. Too simple mathematical plasma models also have limited capabilities due to their inability to adequately reflect the behavior of real plasma. The strongest difference between the real state of the plasma and its mathematical description is observed in the so-called boundary zones, where the plasma passes from one physical state to another (for example, from a state with a low degree of ionization to a state with a high degree of ionization). In this case, the plasma cannot be described using simple smooth functions and a probabilistic approach is required to describe it. Effects such as a spontaneous change in the state of plasma are a consequence of the complex nonlinear interaction of charged particles that make up it. Therefore, to describe plasma, models of the state of the plasma are used, which relate the values of its main parameters, and, therefore, determine its basic properties and behavior.

3.1.1 Local thermodynamic equilibrium (LTE) model

To describe the low-temperature plasma in the bulb of an electrodeless sulfur lamp, which is formed under the action of an electromagnetic field, one can use the LTE model. This model makes it possible to qualitatively and quantitatively describe the continuous emission spectrum of the lamp, as well as the distribution of the main physical quantities of sulfuric plasma.

According to the LTE model, the temperature in different elements of the volume of the medium is different, there is a radiation flux outward (the radiation field is anisotropic), but for each element of the volume of the medium, the Boltzmann and Maxwell distributions, as well as the Saha formula, are valid. Moreover, all of them for a given volume include the same local temperature value, which is the same for all types of particles.

Basic equations describing the LTE model:

– the number of atoms or ions in an arbitrary excited state $k$ (population of the state $k$) is determined by the Boltzmann formula

$$N_k = N_0 \frac{g_k}{g_0} \exp \left( \frac{-E_k}{kT} \right) = N_0 \frac{g_k}{U} \exp \left( \frac{-E_k}{kT} \right),$$  \hspace{1cm} (1)

where $N_0$ – the population of the ground state; $g_0$ – the statistical weight of this state; $g_k$ – statistical weight of the excited state; $E_k$ – the energy of the excited state, measured from the ground level.

Statistical sums over all energy levels $E_n$ of the corresponding ions (atoms) is equal

$$U = \sum_n g_n \exp \left( \frac{-E_n}{kT} \right).$$  \hspace{1cm} (2)

In the case of a single ionization of a gas, the concentrations of atoms, ions, and electrons are related to each other by the Saha formula
\[
\frac{N_e N_i}{N_a} = 2 \frac{(2p m_e)^{3/2}}{h^3} (kT)^{3/2} U_e \frac{U}{U_a(T)} \exp \left( -\frac{E_i}{kT} \right),
\]

where \(m_e\) – the electron mass; \(E_i\) – ionization energy; \(U_i(T)\) and \(U_a(T)\) are the sums over the states of ions and atoms; \(g = 2\) – the statistical weight of electrons.

In plasma of a complex chemical composition, equation (3) is valid for the ions of each chemical element; chemical reactions can occur there, dissociation and recombination of molecules can occur. All these reactions obey the law of mass action with the same temperature \(T\).

The distribution of particles of any kind \(i\) by velocity \(\nu\) is expressed by the Maxwell function

\[
N_i(n) = 4pN_i \left( \frac{M_i}{2\pi kT} \right)^{3/2} \exp \left( -\frac{M_i n^2}{2kT} \right),
\]

where \(M_i\) – the mass of particles; \(N_i(n)\) – the number of particles (concentration) with velocities ranging from \(n\) to \(n + dn\); \(N_i\) – concentration equal to

\[
N_i = \int_0^\infty N_i(\nu) d\nu.
\]

The pressure \(p\) in the plasma is found from the equation of state

\[
p = \sum_i N_i kT.
\]

Local thermodynamic equilibrium is a state of a plasma in which all distribution functions are in equilibrium, except for one concerning radiation: there is no equilibrium of optical processes, as a result of which the Planck formula turns out to be unsuitable.

LTE is typical for most stationary plasmas obtained under laboratory conditions. Under conditions of LTE plasma, the detailed equilibrium with respect to optical transitions is violated; therefore, it is advisable to consider radiation and absorption separately.

Plasma, in which radiation of a given wavelength is practically not absorbed, is optically thin for this radiation. The radiation intensity \(J_{ki}\) of an optically thin plasma in the LTE state within the spectral line with the following frequency \(n_{ki}\) may be written as

\[
J_{ki} = N_k A_{ki} h n_{ki} = N_0 A_{ki} h n_{ki} \exp \left( -\frac{E_k}{kT} \right).
\]

LTE plasma, described by a single parameter \(T\), can exist in a limited pressure range.

The numerical model of the optical radiation source can be a spherical or cylindrical flask made of transparent anhydrous quartz glass filled with a metered amount of sulfur \(\sim 1 \ldots 3\) mg (and it is also possible to introduce impurities, for example, CaBr\(_2\) or indium iodide InI, etc.) and a buffer gas (argon, neon, krypton) under the pressure of \(\sim 45 \ldots 170\) torrs. By changing the composition of the lamp bulb filling, it is possible to carry out theoretical studies of the output spectral characteristics of optical radiation and their dependence in the optical wavelength.
range on the microwave pump power, the temperature distribution inside the bulb, plasma electrical conductivity, etc.

Buffer gas (argon) – serves for initial ionization and obtaining a glow discharge (gas pressure is set at the initial stage). We obtain the dependence of the dynamics of changes in pressure in the flask on temperature.

3.2 Stochastic plasma heating

3.2.1 Introduction

By stochastic heating, we mean a process in which, as a result of nonlinear dynamics, plasma particles move chaotically in the fields of regular electromagnetic waves. Their dynamics differ little from the dynamics of particles in random fields. Below, such regimes we will call regimes with dynamic chaos. The conditions for the occurrence of such modes will be the condition of overlapping nonlinear resonances (Chirikov’s criterion). It is known that in the vicinity of resonances the dynamics of particles are described by equations of nonlinear oscillators, in particular, by the equation of a mathematical pendulum. Therefore, the algorithm for finding the conditions for the emergence of regimes with dynamic chaos (conditions of stochastic heating) can be described as follows:

1. The conditions for resonant interaction of waves with particles are found. There should be several such resonances.

2. Equations of nonlinear oscillators are determined, which describe the dynamics of particles in the vicinity of resonances.

3. The conditions for the overlap of nonlinear resonances of these oscillators are found.

These conditions will be the conditions of stochastic heating.

Below, this algorithm is used for the case of Cherenkov resonances, as well as for cyclotron resonances.

3.2.2 The case of Cherenkov resonances

3.2.2.1 Heating of particles in the field of several transverse electromagnetic waves

Consider the dynamics of motion of charged particles in the field of several electromagnetic waves. Expressions for the electric and magnetic fields of these waves can be represented in this form

\[
\begin{align*}
\bar{E} &= \sum_n \bar{E}_n, \\
\bar{H} &= \sum_n \bar{H}_n, \\
\bar{E}_n &= \text{Re} (\mathcal{E}_n e^{i\psi_n}), \\
\bar{H}_n &= \frac{c}{\omega_n} \left[ k_n \bar{E}_n \right],
\end{align*}
\]

(8)

where \(\psi_n = k_n r - \omega_n t\).
These fields satisfy Maxwell’s equations. The equations of motion of a charged particle in fields (8) have the traditional form

\[
\frac{d\vec{P}}{dt} = e\vec{E} + \frac{e}{c}[\vec{v}\vec{H}].
\]  

(9)

It is convenient to write these equations in the following dimensionless variables for both dependent and independent variables:

\[
\begin{align*}
\omega_n &= \omega_n/\omega_0, \\
\vec{P} &= \vec{P}/mc, \\
\vec{v} &= \vec{v}/c, \\
k_n &\equiv k_n c/\omega_0, \\
\vec{r} &= \omega_0 \vec{r}/c, \\
\vec{E}_n &\equiv e\vec{E}_n/mc\omega_n - \text{is the wave force parameter.}
\end{align*}
\]

Eq. (9) can be conveniently supplemented with the energy equation

\[
\dot{\gamma} = \frac{\vec{P}}{\gamma} \frac{e\vec{E}}{mc\omega_0}.
\]  

(10)

Substituting fields (8) into Eqs. (9) and (10) and using these dimensionless variables, we can obtain the following, convenient for further analysis, equations

\[
\begin{align*}
\vec{P} &= \sum_n \vec{E}_n \left(\omega_n - k_n \vec{r}\right) + \sum_n k_n \left(\vec{v} \vec{E}_n\right), \\
\dot{\gamma} &= \frac{\vec{P}}{\gamma} \sum_n \omega_n \vec{E}_n,
\end{align*}
\]  

(11)

where \(\vec{E}_n = \text{Re} \left(\vec{E}_n e^{i\psi_n}\right)\); \(\psi_n \equiv k_n \vec{r} - \omega_n t\).

Let us introduce some auxiliary characteristic of the particle, which we will further call the partial energy of the particle, which satisfies the following equation

\[
\dot{\gamma}_n = \omega_n \left(\vec{v} \vec{E}_n\right).
\]  

(12)

From the definition of this partial energy, it follows that it determines the value of the energy that a particle would have if it moved only in the field of one \(n\)-th electromagnetic wave. Using the definition of this partial energy, we obtain from Eqs. (11) and (12) the following integral of motion

\[
\vec{P} - \sum_n \text{Re} \left(i \vec{E}_n e^{i\psi_n}\right) - \sum_n k_n \gamma_n = \vec{C}.
\]  

(13)

A possible dispersion diagram of three waves interacting with particles is shown in Figure 8. This figure shows both the dispersion characteristics of the waves themselves (\(\omega_0, \omega_1, \omega_2; k_0, k_1, k_2\)) and the dispersion characteristics of the combination waves with which the Cherenkov resonance of plasma particles (\(v_{ph_1}, v_{ph_2}\)) occurs. In the general case, Eqs. (11) and (12) together with the integral (13) can be studied only by numerical methods. To obtain analytical results, we will assume that the force parameter of each of the waves acting on the particle is small. In this case, all the characteristics of a particle (its energy, momentum, coordinate, velocity) can be represented as a sum of slowly varying and rapidly changing quantities

\[
\begin{align*}
\vec{P} &= \vec{P} + \vec{P}', \\
\gamma_n &= \gamma_n + \gamma_n',
\end{align*}
\]  

(14)
In this case, we can get the following expressions and equations that relate fast and slow variables:

\[
\tilde{P} = \sum_n \frac{k_n \gamma_n}{\omega_n} + C,
\]

\[
\tilde{P} = \sum_n \text{Re} \left( i \tilde{E}_n e^{i \psi_n} \right) + \sum_n \frac{k_n \gamma_n}{\omega_n},
\]

\[
\gamma_n = \omega_n \nu \tilde{E}_n = \omega_n \nu \text{Re} \left( \tilde{E}_n e^{i \psi_n} \right),
\]

\[
\tilde{\gamma}_n = \omega_n \nu \tilde{E}_n,
\]

\[
\tilde{\gamma}_n = \text{Re} \left( \Gamma_n e^{i \psi_n} \right),
\]

where \( \Gamma_n = -i \omega_n \nu \tilde{E}_n e^{i \psi_n} \).

The equations for fast variables can be integrated

\[
\tilde{\gamma}_n = \text{Re} \left[ i \omega_n \left( \nu \tilde{E}_n \right) e^{i \psi_n} / \omega_n - k_n \nu \right],
\]

\[
\tilde{P} = \sum_n \text{Re} \left\{ i e^{i \psi_n} \left[ \tilde{E}_n + k_n \left( \nu \tilde{E}_n \right) / \omega_n \right] \right\}.
\]

The equations for the slow variables take the following form:

\[
\tilde{P} = \sum_{m,n} \frac{1}{\gamma} \left[ \text{Re} \left( i \tilde{E}_m e^{i \psi_m} \right) \left[ \text{Re} \left( \tilde{E}_n e^{i \psi_n} \right) \right] \right]
\]
and

\[ \dot{\gamma} = \frac{1}{\gamma} \sum_{m,n} \text{Re} \left( i \vec{E}_m e^{i\psi_m} \right) \omega_n \text{Re} \left( \vec{E}_n e^{i\psi_n} \right) = \]

\[ = \sum_{m,n} \frac{1}{2\gamma} \omega_n \vec{E}_m \left[ \cos (\psi_m + \psi_n + \pi/2) + \cos (\psi_m - \psi_n + \pi/2) \right]. \]

(18)

Below, we use the obtained equations and integrals to analyze the dynamics of some physical systems that are of considerable interest.

3.2.2.2 Resonances

In accordance with the algorithm described above, we will find resonances. In addition, we find equations that describe the dynamics of particles in the vicinity of resonances. All waves (1) are transverse and fast. In the original formulation of the problem, there is no mechanism for the resonant interaction of such waves separately with plasma particles. However, plasma particles can have a Cherenkov resonance with a beating wave (with a virtual wave; a combination wave). Indeed, let there be only two fast transverse waves (numbered 1 and 2) among those waves that act on a particle. The beats of these waves form a slow combination wave, the phase velocity of which can be close to the average particle velocity. In this case, the dynamics of particles can be described by the dynamics of a nonlinear pendulum (mathematical pendulum). Let’s show it. Let us denote the phase difference of these waves through \( \theta \), i.e. \( \theta = \psi_1 - \psi_2 \). For this phase difference, we obtain the following differential equation

\[ \frac{d\theta}{dt} = \chi v - \Omega = \Delta(\gamma), \]

(19)

where \( \chi = k_1 - k_2 \), \( \Omega = \omega_1 - \omega_2 \).

In this case, we assume that the parameters are close to the conditions of the Cherenkov resonance with the combination wave (\( \Omega / \chi \equiv v \)). The second equation of system (11), taking into account the dynamics of slow and fast variables, can be rewritten as

\[ \frac{d\gamma}{d\tau} = \frac{1}{\gamma} E \cdot \Omega \cdot \cos \theta, \]

(20)

where \( E = \vec{E}_1 \vec{E}_2 \).

3.2.2.3 Particle dynamics near resonance

We will assume that the initial energy of a particle exactly corresponds to the Cherenkov resonance of a particle with a combination wave. It means that \( \Delta(\gamma_0) = 0 \). In addition, we will take into account that as a result of the interaction of waves with particles, the energy of the particle has not changed much. In this case, the resonance detuning \( \Delta(\gamma) \) can be expanded into a Taylor series:

\[ \Delta(\gamma) = \Delta(\gamma_0) + \delta \frac{\partial \Delta}{\partial \gamma} |_{\gamma_0} \gamma_0. \]

(21)
Then Eqs. (19) and (20) will be completely closed and take the following form
\[
\frac{d\theta}{d\tau} = \delta\gamma \left( \frac{\partial \Delta}{\partial \gamma} \right)_{\gamma_0},
\]
\[
\frac{d\delta\gamma}{d\tau} = \frac{\varepsilon \Omega}{\gamma_0} \cos \theta.
\] (22)

The system of equations (22) is equivalent to the equation of the mathematical pendulum
\[
\ddot{\theta} = \left( \frac{\partial \Delta}{\partial \gamma} \right)_{\gamma_0} \frac{\varepsilon}{\gamma_0} \cos \theta.
\] (23)

The half-width of the nonlinear resonance of the pendulum (23) is 
\[\sqrt{\left( \frac{\partial \Delta}{\partial \gamma} \right)_{\gamma_0}} = \frac{\varepsilon \Omega}{\gamma_0}.\] If there are many waves (three or more), then each pair of waves can organize a combination wave. The phase velocities of these waves can be easily selected in the required way for efficient particle heating. So if the distance between the phase velocities of the nearest combination waves turns out to be less than the sum of the half-widths of nonlinear resonances, then the dynamics of particles in the field of these waves will be chaotic.

It is enough for us to consider the dynamics of particles in the field of three waves. Two of these waves propagate in the same direction, the third propagates towards them. The condition for overlapping nonlinear resonances can be written as
\[
\left| v_{\phi h} - v_{\phi k} \right| \leq \sqrt{\left( \frac{\partial \Delta}{\partial \gamma} \right)_{\gamma_0}} \left[ \sqrt{\varepsilon_i \Delta \omega_i} + \sqrt{\varepsilon_{i+1} \Delta \omega_{i+1}} \right],
\] (24)

where \( v_{\phi h} = \Delta \omega_i / (k_0 + k_i) \), \( i = \{1, 2, \ldots n\} \), \( \Delta \omega_i \equiv 1 - \omega_i \).

The left side of inequality (24) describes the distance between nonlinear resonances. The right side represents itself the sum of half-widths of two adjacent nonlinear resonances. If inequality (24) is satisfied, then the dynamics of particles become chaotic. This fact is confirmed by both analytical and numerical studies.

Let us briefly describe the results of a numerical study of the original system of Eq. (11) for the case of interaction of particles with three waves (see Figure 8). The dynamics of particles was investigated in a field of small and identical field strengths \( \varepsilon_i = 0.03 \) and with large \( -\varepsilon_i = 0.3 \). In Figure 9 shows the dependence of the change in energy on time for particles with an initial velocity equal to zero. The wave vectors of the waves were equal: \( k_1 = -0.8 \), \( k_2 = -1 \), \( k_0 = 1.2 \). In Figure 10 shows the temporal dynamics of particle energy with large field strength \( \varepsilon_i = 0.3 \).

Figures 9 and 10 it is seen that at low strengths of the electromagnetic field of the waves, the particle performs regular oscillations, being in one nonlinear Cherenkov resonance with one of the combination waves. With an increase in the field strength under the action of the fields, the particle transitions from resonance to resonance, the dynamics of particle motion is irregular with significant changes in the particle energy.

In this section, it will be shown that using regimes with dynamic chaos, it is possible to propose rather simple and efficient schemes for heating solid-state plasma up to temperatures required for nuclear fusion. Moreover, the heating process proceeds extremely quickly, so that all known plasma instabilities do not have time to develop. To prove the possibility of such heating, we will use all the results obtained above. We will assume that the frequency of the laser radiation that acts on a solid target is much higher than the plasma frequency. Then the results
obtained above can be used in the first approximation. This means that we can assume that condition (24) of overlap of nonlinear Cherenkov resonances is satisfied in the field of laser radiation. If these conditions are met, we can assume that the dynamics of particles is chaotic. Then, by averaging over random phases and random positions of particles, we can find the following expression for the mean square of the change in the energy of particles

\[
\langle (\Delta \gamma)^2 \rangle \approx \mathcal{E}^4 (\Delta \omega)^2 \cdot \tau / (4 \gamma_0^2).
\]  

(25)
Here, the angle brackets denote averaging over phases and positions of particles

\[ \langle L \rangle \equiv \frac{1}{2\pi} \int_0^{2\pi} d\left( \frac{\vec{k} \cdot \vec{r}}{\pi} \right) \cdot \lim_{T \to 0} \frac{1}{T} \int_{-T}^{T} L \cdot dt, \]  

\[ \bar{E}_n = \bar{E}_n e^{-i\omega_n}. \]  

(26)

In deriving (25), we assumed that \( \Delta \omega_0 \approx \Delta \omega_2 \equiv \Delta \omega \approx \varepsilon_1 \approx \varepsilon_2 \equiv \varepsilon \) and that the averaging time is much longer than the decoupling time of correlations of particle motion \( (\tau > \tau_k) \). The decoupling time of the correlation can be estimated by the value \( \tau_k \approx 1/\omega \cdot \ln K \). Here \( K \) is the ratio of the width of nonlinear resonances to the distance between them. At \( K > 1 \), the decoupling time of the correlation is commensurate with the period of the HF field.

A similar analysis of the particle dynamics can be carried out for the case of a large number of waves interacting with particles. The analytical analysis practically does not differ from the one carried out above. Numerical calculations were carried out as well. Let us note the most important results of these studies. The growth rate of the average energy of an ensemble of particles and its maximum energy depends both on the strength of the electromagnetic waves, on the number of combination waves participating in the interaction, as well as on the distance between their nonlinear resonances. Thus, the maximum energy that particles can accumulate in the case of overlap of all Cherenkov resonances from \( N \) combination waves is determined by the sum of the distances between these resonances

\[ \sum_{i=0}^{N-1} \left( v_{ph,i} - v_{ph} \right) = v_{ph,N} - v_{ph,0}. \]  

(27)

3.2.2.4 Comparison of heating efficiency

It is of interest to compare the efficiency of plasma heating by fields of regular electromagnetic waves (in a regime with dynamic chaos) with plasma heating by random fields. In random fields, we can write the following equation for the particle energy

\[ \frac{d\gamma}{d\tau} = \left( \frac{\vec{v}}{\vec{E}_n} \right). \]  

(28)

Here \( \vec{E}_n \) – the field strength of the random wave.

Under the same assumptions under which formula (25) was obtained, we find

\[ \left\langle (\Delta \gamma)^2 \right\rangle = v^2 \varepsilon_n^2 \tau. \]  

(29)

Let us assume that the energy in the field of the noise wave is equal to the energy of the field of coherent radiation. In this case \( \varepsilon_n^2 \cdot \Delta \omega_n = \varepsilon^2 \cdot \Delta \omega \). Here \( \Delta \omega < < \Delta \omega_n \) is the width of the spectrum of the noise field, \( \Delta \omega = \omega/Q \) is the width of the spectrum of coherent radiation, \( Q \) is the \( Q \)-factor of the optical resonator \( (Q \sim 10^6 - 10^7) \).

\[ K \equiv \frac{\left\langle (\Delta \gamma)^2 \right\rangle}{\left\langle (\Delta \gamma)^2 \right\rangle_n} > \frac{\varepsilon^2 (\Delta \omega_0)^2 Q}{4\gamma_0^2 v_0^2}. \]  

(30)

In the vast majority of cases \( K > > 1 \).
It should be noted that many other heating mechanisms are also less efficient than heating in the dynamic chaos regime. In particular, one can point to the well-known turbulent heating. In turbulent heating schemes, radiation incident on plasma as a result of nonlinear processes excites random fluctuations of fields in the plasma. It is these random fluctuations that heat the plasma particles. As we saw above, this mechanism is less efficient than dynamic heating. In addition, the transformation of regular fields incident on the plasma into random fields requires a significant time.

The closest to the one considered is the scenario of plasma heating, which is associated with collisions of particles of dense (solid-state) plasma. The collision frequency, as is known, is proportional to the plasma density \( n = 10^{22} \text{cm}^{-3} \) and at a temperature \( T = 7 \text{ keV} \) is \( v = 10^{12} \text{s}^{-1} \). If the frequency of the laser radiation \( \omega = 5 \times 10^{15} \text{s}^{-1} \) and the amplitude of the laser wave \( E = 0.1 \), then the heating of the plasma to a temperature of 7 keV occurs in a time \( \Delta t_H = 2 \times 10^{-14} \text{s} \), i.e. in a time significantly shorter than the time of collision between particles. Thus, there is a range of laser radiation and plasma parameters at which dynamic heating is much more efficient than other heating mechanisms.

Let us estimate the possibility of using dynamic heating of solid-state plasma to thermonuclear temperatures. In this case, we need to heat the plasma ions. In this case, direct dynamic heating of ions is ineffective. Indeed, as follows from formula (25), this time is proportional to the fourth power of the mass \( (r_H \sim m_i^4) \).

In this case, the ion heating scheme may look as follows: the laser field \( E = 0.1, \omega = 5 \cdot 10^{15} \text{s}^{-1} \) heats plasma electrons \( n = 10^{22} \text{cm}^{-3} \) to a temperature of 7 keV. This heating takes place over time \( t < 10^{-13} \text{s} \). During the time \( t \sim 10^{-9} \text{s} \), the heated electrons transfer their energy to the ions. This time is rather short. During this time, a solid-state target of radius \( r = 0.1 \) will not increase its size too much. Note that the rapid heating of electrons and the rapid transfer of energy from electrons to ions make it possible to avoid the development of plasma instabilities.

### 3.2.3 Plasma heating in an external magnetic field

We saw above that the dynamics of charged particles in the field of a combination wave in the vicinity of the Cherenkov resonance of particles with a combination wave is described by the equation of a mathematical pendulum. If there are several combination waves (we saw above that three transverse electromagnetic waves can generate two combination waves), then to describe the dynamics of particles, it is necessary to analyze a model that contains two equations of a mathematical pendulum. As we saw above, stochastic instability developed when the nonlinear resonances of these mathematical pendulums crossed (see [5, 6]). The particle dynamics became random. Thus, in this model of the interaction of charged particles with electromagnetic waves, the result turned out to be analogous to the motion of particles in a random field. Above, using the example of plasma heating by three laser waves, an expression was obtained that characterizes the efficiency of plasma heating in the field of three regular laser waves. Another common scheme for realizing plasma heating is that the plasma is placed in an external constant magnetic field. To analyze the appearance of conditions for effective plasma heating in such installations based on regimes with dynamic chaos, we note that the presence of an external magnetic field leads to the fact that regimes with dynamic chaos can be realized even when the plasma is exposed to only one external electromagnetic wave. It turns out that the role of a large number of waves, in this case, is played by resonances (cyclotron resonances), and also that the dynamics of particles in the vicinity of cyclotron resonances is described by the model of a mathematical pendulum. Overlapping of nonlinear cyclotron resonances leads, as above,
to the development of local instability (stochastic instability). As a result, we get method for effective plasma heating [7–11].

Consider a charged particle (electron) that moves in an external constant magnetic field \( H_0 \) of magnitude directed along the axis \( z \) and in the field of an electromagnetic wave of arbitrary polarization. The components of the electric and magnetic fields of such a wave can be represented as

\[
\begin{align*}
\vec{E} &= \text{Re} \left( E_0 e^{i\psi} \right), \\
\vec{H} &= \text{Re} \left( \frac{1}{k_0} \left[ k \vec{E} \right] \right),
\end{align*}
\]

where \( \psi \equiv \omega t - \vec{k} \cdot \vec{r} \), \( \vec{E}_0 = \vec{a} E_0 \); \( \vec{a} = \{a_x, ia_y, a_z\} \) – wave polarization vector; \( k_0 = \omega / c \); \( \omega, \vec{k} \) – frequency and wave vector of the wave. We introduce the following dimensionless variables:

\[
\begin{align*}
p_1 &= \frac{p}{m c}, \\
k_1 &= \frac{k}{k_0}, \\
\tau &= \omega t, \\
\vec{v}_1 &= k_0 \vec{v}, \\
\vec{e} &= e \vec{E}_0 / m c \omega, \quad \vec{v}_{ph1} = \vec{v}_{ph} / c = \omega / k c.
\end{align*}
\]

Without loss of generality, we can assume that the vector \( \vec{k} \) has only two nonzero components \( k_x \) and \( k_z \). The equations of motion of a particle can be reduced to the form

\[
\begin{align*}
\dot{\vec{p}} &= \left( 1 - \frac{k \vec{p}}{\gamma} \right) \text{Re} \left( \vec{e} \vec{e}^w \right) + \frac{k}{\gamma} \text{Re} \left( \vec{p} \vec{e} \right) \vec{e}^w + \frac{\omega H}{\gamma} \left[ \vec{p} \vec{e} \right], \\
\dot{\vec{r}} &= \vec{p} / \gamma, \\
\dot{\psi} &= k \vec{p} / \gamma - 1,
\end{align*}
\]

where \( \tau \equiv \omega t, \vec{e} / H_0 / H_0; \omega_H = e H_0 / m c \omega; \psi = \vec{k} \cdot \vec{r} - \tau \) [7].

The last term on the right-hand side of the first vector equation describes the Lorentz force that acts on a particle in a constant external field. Multiplying the first of equations (32) by \( \vec{p} \) and taking into account that \( p^2 = \gamma^2 - 1 \), we obtain the following equation for changing the particle energy

\[
\dot{\gamma} = \text{Re} \left( \vec{v} \vec{e} \right) e^w.
\]

Using this equation, from equations (32) we find the following integral of motion

\[
\vec{p} - \text{Re} \left( i \vec{e} e^w \right) + \omega_H \left[ \vec{v} \vec{e} \right] - k \vec{e} = \text{const.}
\]

For what follows, it is convenient to pass to new variables \( p_\perp, p_\|, \theta, \xi \) and \( \eta \), which are related to the old following ratios

\[
\begin{align*}
p_x &= p_\perp \cos \theta, \\
p_y &= p_\perp \sin \theta, \\
p_z &= p_\|, \\
x &= \xi - \frac{p_\perp}{\omega_H} \sin \theta, \\
y &= \eta + \frac{p_\perp}{\omega_H} \cos \theta.
\end{align*}
\]
In these variables, Eqs. (32) taking into account the integral (34) take the form

\[
\dot{p}_\perp = (1 - k_z v_x) \sum_n \left( \epsilon_x n / \mu J_n - \epsilon_z J_n \right) \cos \theta_n + k_z v_x \sum_n n / \mu J_n \cos \theta_n,
\]

\[
\dot{\theta} = -\frac{\omega_H}{\gamma} + (1 - k_z v_x) \sum_n \left( \epsilon_x J_n' - \epsilon_z J_n \right) \sin \theta_n + k_z v_x \sum n / \mu J_n \sin \theta_n + k_z v_x \sum_n J_n' \cdot \sin \theta_n,
\]

\[
\dot{p}_z = \sum_n \cos \theta_n \left[ \epsilon_x J_n \left( n / \mu + v_z \epsilon_z \right) - \epsilon_z J_n \right],
\]

\[
\dot{\gamma} = \frac{e_0}{\gamma} W_s \cdot \cos \theta_s,
\]

\[
\theta_n = k_z z + k_z \xi - n \theta - \tau.
\]

(36)

In obtaining (36), we used the expansion

\[
\cos(x - \mu \sin \theta) = \sum_{n=-\infty}^{\infty} J_n(\mu) \cos(x - n \theta),
\]

(37)

where \( \mu = k_z p_\perp / \omega_H \).

At \( (e_0 << 1) \) the effective interaction of the particle with the wave occurs when one of the resonance conditions is satisfied

\[
\Delta_s(\gamma_0) \equiv k_z v_x + s \frac{\omega_H}{\gamma_0} - 1 = 0.
\]

Assuming condition (38) had performed and introducing the resonant phase \( \theta_s = s \theta - \tau \) from the system of equations (36), after averaging, we obtain the following equations of motion

\[
\dot{p}_\perp = \frac{1}{p_\perp} (1 - k_z v_x) W_s \cdot e_0 \cos \theta_s,
\]

\[
\dot{\theta} = \frac{1}{\gamma} k_z W_s e_0 \cos \theta_s,
\]

\[
\dot{\gamma} = e_0 \frac{\omega_H}{\gamma} W_s \cdots \cos \theta_s,
\]

where \( W_s \equiv \alpha_s p_\perp J_n + \epsilon_z p_\perp J_n' + \epsilon_z p_\perp J_n \).

(39)

3.2.4 The condition for the appearance of dynamic chaos (the condition of stochastic heating)

We will assume that the particle energy changes little as a result of the interaction with the electromagnetic wave \( (\gamma = \gamma_0 + \tilde{\gamma}), \tilde{\gamma} << \gamma_0 \), and the resonance condition (38) is exactly satisfied for a particle with energy \( \gamma_0 \). Then, doing decomposition \( \Delta_s(\gamma) \) near \( \gamma_0 \), the last two equations of the system (36), we obtain a closed system of two equations for determining \( \tilde{\gamma} \) and \( \theta_s \).
\[
\frac{d\gamma}{d\tau} = \frac{\varepsilon_0}{\gamma_0} W_s \cos \theta,
\]
\[
\frac{d\theta}{d\tau} = \frac{k_z^2 - 1}{\gamma_0} \frac{d\gamma}{d\tau}.
\]

Equations (40) represent the equation of a mathematical pendulum. Of them, we find the width of the nonlinear resonance
\[
\Delta \theta_s = 4\sqrt{\frac{\varepsilon_0}{\gamma_0} (k_z^2 - 1) \cdot W_s / \gamma_0^2},
\]
\[
\Delta \gamma_s = 4\sqrt{\frac{\varepsilon_0}{\gamma_0} W_s / (k_z^2 - 1)}.
\]

To find the distance between resonances, we write the resonance conditions (38) and the averaged conservation law (34) for two adjacent resonances (see [6, 7, 11])
\[
k_z p_{s+1} + (s + 1)\omega_H - \gamma_{s+1} = 0,
\]
\[
\gamma_{s+1} - p_{s+1}/k_z = C,
\]
\[
k_z p_s + s\omega_H - \gamma_s = 0,
\]
\[
\gamma_s - p_s/k_z = C.
\]

From these conditions, we find the following value of the distance between resonances
\[
\delta \gamma = \omega_H / (1 - k_z^2).
\]

From expressions (42) and (43) it follows that in carrying out the inequality
\[
\varepsilon_0 > \frac{\omega_H^2}{4} \left[ \sqrt{W_i} + \sqrt{W_{i+1}} \right]^2 (1 - k_z^2),
\]
nonlinear resonances overlap. A regime of stochastic instability sets in, and inequality (44) is the condition for stochastic heating of plasma particles. Note that the width of the nonlinear resonance as well as the distance between resonances must be calculated along with the integrals of motion (see Figure 11). In this figure, the dotted lines show the boundaries of nonlinear resonances (the position of the separatrices), the solid lines are the cyclotron resonances themselves, and the bold arrow denotes the integral. In all cases, the particle dynamics run according to integrals. When the integral line coincides with the resonance line, the autoresonance condition occurs. Under autoresonance conditions, particles can resonantly acquire unlimited energy.

3.2.5 Experimental studies of stochastic heating of plasma in a constant magnetic field

After theoretical work, a large series of experimental studies of stochastic plasma heating was carried out. At the same time, theoretical estimates showed that for stochastic heating of the plasma by the field of one external electromagnetic wave, the field strength of this wave should be sufficiently high (~ 10^6 V/cm) [8]. This result, for example, follows from an analysis of conditions (44). Additional numerical studies have shown that if several waves are excited in the experimental setup, for example, two waves with the same frequencies but different wave vectors, then
for the occurrence of conditions for stochastic heating, the field strengths significantly decrease (≈ 10^4 V/cm) [10, 11].

The main experiments were carried out on a setup, the scheme of which is described in detail in [9]. The main element of this setup is a cylindrical resonator, the general view of which is shown in Figure 12. The resonator is made of a copper tube with an inner diameter of 16 cm and a length of 66 cm. The modes $H_{10}$ and $H_{20}$ are excited in this resonator. The central axis of the resonator coincided with the direction of the external inhomogeneous magnetic field. This field formed a magnetic trap. The mirror ratio was chosen equal to 1.2 ... 2. The length of the uniform part of the magnetic field of the trap in the cavity was varied from 25 to 66 cm. A loop probe was located in the central part of the cavity. Plasma in the cavity was generated by an electron beam with an energy of 400–600 eV and a current of 60–100 mA due to a beam–plasma discharge [9]. The pressure in the resonator could be regulated in the range of $10^{-4}$ ... $10^{-6}$ mm Hg. In the main series of experiments, the plasma density was within $\approx 10^9$ cm$^{-3}$.

The sequence of working of the equipment in time is as follows. An electron beam is injected into the cavity. As a result of the beam–plasma discharge, a plasma is formed with a density of up to $\approx 10^{11}$ cm$^{-3}$. The resonator was excited at a frequency of $\approx 2.7$ GHz, for which the cyclotron resonance was performed at a magnetic induction at the minimum of the trap equal to $\approx 0.1$ T. The length of the uniform part of the magnetic field of the trap in the resonator was varied from 25 to 66 cm. The oscillation power of the magnetron could be varied in the range 0.1–1.0 MW in a pulse with a duration of 1.8 μs. and was fed into the resonator through a waveguide with a cross-section of 72x34 mm. By varying the delay time between the electron beam pulse and the high-frequency power pulse, the required plasma density was selected in the range $10^7$–$10^9$ cm$^{-3}$ at a pressure of $10^{-5}$–$10^{-4}$ mm Hg. Argon was used as a plasma-forming gas.

The experiments investigated the fluxes of microwave, optical and X–ray radiation. Simultaneously, using a set of foil plates (up to 15 layers of aluminum foil), electron fluxes with energies up to 1 MeV were recorded. The results are shown in Figure 13. Estimation of the electron energy at the maximums of the X–ray radiation intensity showed that at $t = 2$ μsec the electron energy reaches 100–150 keV, while at $t = 1$ μsec the electron energy is 8–10 times higher ($\approx 1$ MeV).
4. Conclusions

In this chapter, there were considered different approaches to exciting plasma by a regular electromagnetic field. As a microwave source, there was used a magnetron generator as well as two types of electrodynamic structures: resonators (the cylindrical resonator) and waveguides (the rectangular waveguide). An application of the given electrodynamic structures allowed the formation of an electromagnetic field needed for effective exciting plasma in the area of location of an active medium (for example, a bulb with gases mixture). The carried out investigations have pointed to distinct aspects of forming a regular electromagnetic field and its excitation as well as the features of plasma heating. It is significant that for exciting
plasma and enhancing efficiency in its heating it is necessary to optimize not only the shape and special distribution of the electric field in the cavity but also its intensity. In this regard, a preference is given to the resonant electrodynamic structures having a concentrated capacity as a parameter. Among such structures is the coaxial resonator loaded on a capacity as well as the toroidal resonator. In the case of an application of the interference method for forming an electromagnetic field and enhancing the effectiveness of plasma heating a great interest is using the single- and double-ridged waveguides instead of the regular waveguides. On the other hand, a comparison of the theoretical and experimental data showed that the most effective is heating of any plasma when the interaction of plasma particles with regular electromagnetic waves occurs in the dynamic chaos regime. Note that the described mechanisms relate to the interaction of the wave-particle type.

Another fundamental interaction (the interaction of wave-wave type) can also be used to heat the plasma. But in that event, such a heating mechanism (turbulent heating) contains two stages. At the first stage, the energy of regular waves is transformed into the energy of less efficient noise vibrations.

The experimental implementation of the conditions for stochastic heating of plasma by the field of regular electromagnetic waves with a high rate of energy transfer from electromagnetic waves to the energy of thermal motion of plasma electrons has been carried out. It is shown that the average energy of plasma electrons reached values ~ 1 MeV in times less than 1 μsec.

Also, it is necessary to note that in the experiment stochastic heating and, accordingly, X-ray radiation from the plasma was observed only when several spatial modes were excited in the resonator or when the resonator was excited by two frequencies. This fact is in full agreement with the results of the analysis of theoretical models.
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