The Renormalized Thermal Mass with Non-Zero Charge Density

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The linear $\delta$ expansion is used to obtain corrections up to $O(\delta^2)$ to the self-energy for a complex scalar field theory with a $\lambda(\phi^*\phi)^2$ interaction at high temperature and non-zero charge density. The calculation is done in the imaginary-time formalism via the Hamiltonian form of the path integral. Nonperturbative results are generated by a systematic order by order variational procedure and the dependence of the critical temperature on the chemical potential $\mu$ is obtained.

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I. INTRODUCTION

It has been known for some time that conventional perturbation theory is inadequate for describing high-temperature field theory\footnote{arXiv:hep-th/0005069v1 8 May 2000}, with the perturbation expansion breaking down at some order in the coupling constant in the parameter regime where the temperature is very much greater than the bare mass. There have been several attempts to circumvent this problem (for example Refs.\footnote{H. F. Jones and Philip Parkin} use resummed perturbative expansions and systematically include all relevant diagrams). A temperature-dependent renormalization group approach has been used in Ref.\footnote{H. F. Jones and Philip Parkin}. The method we will employ in this paper is the linear $\delta$ expansion (see, for example\footnote{arXiv:hep-th/0005069v1 8 May 2000}), which contains elements of both of the above methods. It involves a mass shift which is determined order-by-order in the expansion by a non-perturbative criterion, but prior to the (crucial) optimization stage it merely uses low-order perturbation theory with modified propagators and vertices.

Pinto and Ramos\footnote{arXiv:hep-th/0005069v1 8 May 2000} have successfully applied the method to finite-temperature symmetry breaking in a scalar field theory with a $\phi^4$ interaction up to second order in $\delta$, which in particular involves evaluating the non-trivial “setting-sun” diagram. In the present paper we extend this approach to tackle the problem of a non-zero chemical potential; in so doing we consider a complex scalar field rather than a real scalar field. For $\mu = 0$, the effect is simply to alter the symmetry factors of the one- and two-loop diagrams, but for $\mu \neq 0$ there is a shift in the energy component of the Matsubara propagator, which makes the diagrams considerably more difficult to evaluate but presents no problem of principle.

To our knowledge there have been only a few attempts at this problem. Benson et al.\footnote{arXiv:hep-th/0005069v1 8 May 2000} performed a one-loop calculation, while Funakubo and Sakamoto\footnote{arXiv:hep-th/0005069v1 8 May 2000} used a renormalization group approach in the large-$N$ limit of an $O(N)$ theory (in which the setting-sun diagram does not appear). In the standard lattice Monte-Carlo approach it is impossible to incorporate a non-zero chemical potential, because the Euclidean action then becomes complex and cannot be used as a real, positive statistical weight.

We outline the linear $\delta$ expansion technique below and explain how it can generate nonperturbative results, while in Sec. II we present briefly the construction of the theory in the imaginary time formalism and include the chemical potential associated with a conserved charge. We then go on to calculate all diagrams contributing to the self-energy, up to second order in $\delta$. Sec. III includes results for the critical temperature and its behaviour as one allows $\mu$ to take non-zero values. The appendix contains a detailed calculation of the two-loop setting sun diagram in the high temperature expansion.

A. The Linear Delta Expansion

The linear $\delta$ expansion (LDE) is a technique that allows the use of an analytic approach to probe the nonperturbative sector of the field theory to which it is applied. It has been employed with success in a wide variety of areas\footnote{arXiv:hep-th/0005069v1 8 May 2000}, with convergence of the expansion rigorously demonstrated in some simple zero- and one-dimensional models (see, for example,\footnote{arXiv:hep-th/0005069v1 8 May 2000}). The method involves the introduction of an artificial expansion parameter, $\delta$, and the modification of the action of the theory under consideration via

$$S \rightarrow S^\delta = (1 - \delta)S_0(\{\eta_i\}) + \delta S,$$

(1)
where \( \{ \eta_i \} \) is some set of variational parameters and \( S_0 \) represents the action of some soluble theory. This trial action is not determined by the method; however, an appropriate choice is usually suggested by the form of the theory under consideration. The \( \delta \)-modified action can then be used to evaluate some desired physical quantity as a power series in \( \delta \) (truncated at some finite order, \( N \)), with \( \delta \) then set equal to 1 at the end of the calculation. We label this quantity \( P_N \), noting that it will, in general, have a residual dependence on the \( \{ \eta_i \} \). The variational element is introduced by fixing these parameters according to some specified criterion. We shall use the principle of minimal sensitivity (PMS) \([11]\) whereby the \( \{ \eta_i \} \) are chosen at a stationary point of the quantity \( P_N \), namely:

\[
\frac{\partial P_N}{\partial \eta_i} \bigg|_{\eta_i = \bar{\eta}_i} = 0.
\]

It is this order by order fixing that allows for non-perturbative behaviour to emerge (the optimized variational parameters becoming functions of the order of truncation, and so being more correctly labelled by \( \{ \eta_i(N) \} \)), and can also provide for convergence of the expansion, two particularly desirable features.

### II. The Charged Scalar Field at Finite Temperature

In Minkowski space, the Lagrangian density of a massive, complex scalar field \( \varphi(x) \) with a \( V(\varphi^* \varphi) \) interaction term is given by

\[
\mathcal{L} = (\partial_{\mu} \varphi^*)(\partial_{\mu} \varphi) - \frac{m^2}{2} \varphi^* \varphi - V(\varphi^* \varphi)
\]

(3)

This Lagrangian density has a well-known global \( U(1) \) gauge symmetry which leads to a conserved charge \( Q \) and an associated chemical potential \( \mu \), so that the grand partition function is

\[
Z(\beta, \mu) = \text{Tr} e^{-\beta (H - \mu Q)}
\]

(4)

Proceeding through the Hamiltonian form of the path integral for \( Z \), in the imaginary-time formalism of finite-temperature field theory \([12]\), we arrive at

\[
Z(\beta, \mu) = \int \mathcal{D}(\varphi^*, \varphi) e^{-S_{\text{E}}(\beta, \mu)}
\]

(5)

where the Euclideanized action is (dropping the suffix for future notational convenience)

\[
S(\beta, \mu) = S_F(\beta, \mu) + S_I(\beta, \mu) + S_C(\beta, \mu)
\]

(6)

with

\[
S_F(\beta, \mu) = \int_T d^4x \varphi^* \left[ -\frac{\partial^2}{\partial \tau^2} + 2\mu \frac{\partial}{\partial \tau} - \nabla^2 + m^2 - \mu^2 \right] \varphi,
\]

(7)

\[
S_I(\beta, \mu) = \int_T d^4x \mathcal{V}(\varphi^* \varphi),
\]

(8)

\[
S_C(\beta, \mu) = \int_T d^4x \varphi^* \left[ -A \left( \frac{\partial^2}{\partial \tau^2} - 2\mu \frac{\partial}{\partial \tau} + \nabla^2 + \mu^2 \right) + Bm^2 \right] \varphi + \int_T d^4x \mathcal{V}(\varphi^* \varphi),
\]

(9)

where \( \tau = it, x = (\tau, \mathbf{x}) \) and \( \int_T d^4x = \int_0^\beta d\tau \int d^3\mathbf{x} \). Here \( S_C(\beta, \mu) \) represents those counterterms required to render the model finite.

A sensible choice for our delta-modified action, in the presence of a \((\varphi^* \varphi)^2\) self-interaction, \( V(\varphi^* \varphi) = (\lambda/4)(\varphi^* \varphi)^2 \), would then be

\[
S^\delta(\beta, \mu) = S_F^\delta(\beta, \mu) + S_I^\delta(\beta, \mu) + S_C^\delta(\beta, \mu),
\]

(10)

with

\[
S_F^\delta(\beta, \mu) = \int_T d^4x \varphi^* \left[ -\frac{\partial^2}{\partial \tau^2} + 2\mu \frac{\partial}{\partial \tau} - \nabla^2 + \Omega^2 - \mu^2 \right] \varphi,
\]

(11)

\[
S_I^\delta(\beta, \mu) = \int_T d^4x \left[ \frac{\delta \lambda}{4}(\varphi^* \varphi)^2 - \delta \eta^2 \varphi^* \varphi \right],
\]

(12)

\[
S_C^\delta(\beta, \mu) = \int_T d^4x \varphi^* \left[ -A^\delta \left( \frac{\partial^2}{\partial \tau^2} - 2\mu \frac{\partial}{\partial \tau} + \nabla^2 + \mu^2 \right) + B^\delta \Omega^2 \right] \varphi + \int_T d^4x \left[ C^\delta \frac{\delta \lambda}{4}(\varphi^* \varphi)^2 - B^\delta \delta \eta^2 \varphi^* \varphi \right],
\]

(13)

2
introducing the variational parameter Ω, or equivalently η, given by η² = Ω² − m². In subsequent sections we will evaluate the thermal mass up to O(δ²), using dimensional regularization to renormalize the theory. It is an important point, raised in [6], that the arbitrary variational parameter η becomes a function of the bare parameters (which is how non-perturbative behaviour arises in the LDE), so we must eliminate any divergences before we apply the PMS optimization procedure. The renormalization procedure we use is identical to that of [6] and is based on [11]; we forgo any detailed discussion of the cancelling of temperature-dependent divergences and any systematic calculation of the coefficients of the counterterms Aδ, Bδ and Cδ, the details being essentially identical to those discussed in [6].

A. Calculating the thermal mass

We recall that the Feynman rules in frequency space in the presence of an overall charge are as follows:

1. To every line of the diagram assign a factor ∆F(iωn,k) ≡ [−(iωn + μ)² + ωk²]⁻¹, where ωn = 2πn/β and ωk = √(k² + Ω²), and an arrow in the direction of momentum flow

2. Assign a factor −δλ to each vertex

3. Assign a factor δη² to each insertion

4. Integrate over every internal line with the measure T∑ₙ∫d³k/(2π)³

5. There must be conservation of charge at each vertex, i.e. the number of arrows entering a vertex must be the same as the number leaving

We are now in a position to estimate corrections to the thermal mass (which we do up to O(δ²)), defined by

\[ m^{2}_{T,μ} = \Omega^2 + Π(i\omega_n, p), \] (14)

evaluated on-shell, i.e. iωn = Ω − μ, p = 0, where Π is the thermal self-energy. This choice of four-momentum is discussed in the Appendix, where the setting-sun diagram is calculated, the only momentum-dependent contribution to Π up to O(δ²).

B. The thermal mass at first order

![Diagram](image)

FIG. 1. Diagrams contributing at O(δ)

To lowest order, the relevant contributions are

\[ Π_{T,μ}^{δ} = Π_{1}^{δ} + Π_{2}^{δ} \]

\[ = −δη² + δλT \sum_{n} \frac{d^{3}k}{(2π)^{3}} ∆F(iω_n, k). \]

(15)

The frequency sum in (15) can be treated in a particularly concise and efficient manner through the mixed representation \[ ∆F(τ, k) \]

of \[ ∆F :\]

\[ ∆F(τ, k) = T \sum_{n} e^{-iω_n τ} ∆F(iω_n, k) \]

\[ = \frac{1}{2ω_k} \left[ (1 + n^-_k) e^{−(ω_k−μ)τ} + n^+_k e^{(ω_k+μ)τ} \right], \]

(16)

where

\[ n^±_k = \frac{1}{e^{β(ω_k±μ)}−1} \]

(17)
is the Bose-Einstein distribution function in the presence of a non-zero chemical potential. The Matsubara propagator is recovered by Fourier transforming the mixed propagator with respect to the $\tau$ variable:

$$\Delta_F(i\omega_n, k) = \int_0^\beta d\tau e^{i\omega_n \tau} \Delta_F(\tau, k).$$

(18)

The contribution of $\Pi_2^\delta$ in (13) can then be calculated trivially to give

$$\Pi_2^\delta = \delta \lambda T \sum_n \int \frac{d^3k}{(2\pi)^3} \int_0^\beta d\tau e^{i\omega_n \tau} \Delta_F(\tau, k)$$

$$= \delta \lambda T \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} (1 + n^+_k + n^-_k).$$

(19)

Using dimensional regularization [14] one finds that the $O(\delta^2)$ contribution to the thermal mass is

$$m_{T,\mu}^2 = \Omega^2 - \mu^2 - \delta \eta^2 + \frac{\lambda}{16\pi^2} \Omega^2 \left[ -\frac{1}{e} + \ln \left( \frac{\Omega^2}{4\pi M^2} \right) + \gamma - 1 \right] + \delta \frac{\lambda T^2}{\pi^2} h_3(y, r),$$

(20)

where $M$ is a mass scale introduced by dimensional regularization and, in the notation of [15],

$$h_3(y, r) = \frac{1}{\Gamma(3)} \int_0^\infty dx \frac{x^2}{\sqrt{x^2 + y^2}} \left[ \frac{1}{\exp(\sqrt{x^2 + y^2} - ry) - 1} \right],$$

(21)

leading to the following expression for the renormalized thermal mass at first order:

$$m_{T,\mu}^2 = \Omega^2 - \delta \eta^2 + \delta \frac{\lambda T^2}{12} - \delta \frac{\lambda T \Omega}{4\pi} \sqrt{1 - r^2} + \delta \frac{\lambda \Omega^2}{16\pi^2} \left[ \ln \left( \frac{4\pi T^2}{M^2} \right) - \gamma - 2r^2 \right].$$

(25)

In this and all subsequent calculations we neglect terms of $O(1/T)$. 

C. The thermal mass at second order

![Diagrams contributing at $O(\delta^2)$](image)
At \( \mathcal{O}(\delta^2) \) there are five diagrams (shown in Fig. 2) which provide corrections to the self-energy. We begin by evaluating the first diagram, which in the high temperature limit is

\[
\Pi_1^{\delta^2} = \delta^2 \frac{\lambda^2 \Omega^2}{8 \pi \Omega} \frac{1}{\sqrt{1 - r^2}} - \delta^2 \frac{\lambda^2 \eta^2}{16 \pi^2} \left[ -\frac{1}{\epsilon} + \ln \left( \frac{4 \pi T^2}{M^2} \right) - \gamma \right].
\]  

(26)

The next two diagrams are constructed using mass and vertex counterterms to give \( \mathcal{O}(\delta^2) \) diagrams that eliminate the temperature-dependent divergences arising from \( \Pi_1^{\delta^2} \). Explicitly, we obtain in the high temperature limit

\[
\Pi_2^{\delta^2} = -\delta^2 \frac{\lambda^2 \Omega^2}{2(16\pi^2)^2 c^2} + \delta^2 \frac{\lambda^2}{16 \pi^2} \cdot \frac{\left[ T^2 - T \frac{\Omega^2}{4 \pi} \sqrt{1 - r^2} + \frac{\Omega^2}{8 \pi} \right]}{\sqrt{1 - r^2}} + \delta^2 \frac{\lambda^2}{16 \pi^2} \left[ \ln \left( \frac{4 \pi T^2}{M^2} \right) - \gamma - 2 r^2 \right]
\]

(27)

\[
\Pi_3^{\delta^2} = -\delta^2 \frac{5 \lambda^2 \Omega^2}{2(16 \pi^2)^2 c^2} + \delta^2 \frac{5 \lambda^2}{32 \pi^2} \left[ T^2 - 3 - T \frac{\Omega^2}{2 \pi} \sqrt{1 - r^2} + \frac{\Omega^2}{4 \pi^2} \right] \left[ \ln \left( \frac{4 \pi T^2}{M^2} \right) + (2 \gamma - 1) \ln \left( \frac{\Omega^2}{4 \pi M^2} \right) + 2.4 \right]
\]

(28)

The momentum-independent two-loop diagram is given by

\[
\Pi_4^{\delta^2} = \delta^2 \frac{\lambda^2 \Omega^2}{64 \pi^4} \frac{1}{\sqrt{1 - r^2}} - \delta^2 \frac{\lambda^2}{16 \pi^2} \left[ T^2 - T \frac{\Omega^2}{4 \pi} \sqrt{1 - r^2} + \frac{\Omega^2}{8 \pi} \right] + \delta^2 \frac{\lambda^2 \Omega^2}{16 \pi^2} \left[ \ln \left( \frac{4 \pi T^2}{M^2} \right) - \gamma + \frac{1}{2} \right] + \delta^2 \frac{\lambda^2 \Omega^2}{64 \pi^4} \left[ \ln \left( \frac{\Omega^2}{4 \pi M^2} \right) + (2 \gamma - 1) \ln \left( \frac{\Omega^2}{4 \pi M^2} \right) + 2.4 \right]
\]

(29)

Finally, we have the momentum-dependent setting sun diagram, which we evaluate with the Euclidean self-energy on shell (the details of the calculation are provided in Appendix A):

\[
\Pi_5^{\delta^2} = \delta^2 \frac{3 \lambda^2 \Omega^2}{8 \pi} \frac{1}{\sqrt{1 - r^2}} + \delta^2 \frac{3 \lambda^2 \Omega^2}{2(32 \pi^2)^2 c^2} + \delta^2 \frac{\lambda^2 p^2}{2 \pi} \left[ T^2 - T \frac{\Omega^2}{8 \pi} \sqrt{1 - r^2} + \frac{\Omega^2}{4 \pi^2} \right] + \delta^2 \frac{3 \lambda^2}{16 \pi^2} \left[ \ln \left( \frac{\Omega^2}{4 \pi T} \right) + \gamma - \frac{1}{2} + r^2 \right] + \delta^2 \frac{3 \lambda^2 \Omega^2}{128 \pi^4} \left[ \ln \left( \frac{\Omega^2}{T^2} \right) + 5.0669 + \frac{1}{2} \ln(1 - r^2) - \frac{2 \gamma - 1}{\pi^2} (2 \ln 2 - 1) - \frac{1}{\pi^2} \ln \left( \frac{1 + r}{1 - r} \right) \ln 2 \right].
\]

(30)

The divergent parts of these diagrams can all be eliminated by a suitable choice of the counterterm in Eq. (13). As mentioned above, we use the minimal subtraction prescription.

### III. RESULTS

Having obtained an expression for the renormalized thermal mass, we can set \( \delta = 1 \) and obtain numerical results for this mass, \( m_{T,\mu}^2(\tilde{\eta}) \), where \( \tilde{\eta} \) is determined via the PMS condition:

\[
\left. \frac{\partial m_{T,\mu}^2(\eta)}{\partial \eta} \right|_{\eta=\tilde{\eta}} = 0.
\]

(31)

At \( \mathcal{O}(\delta) \), \( \tilde{\eta} \) does not depend on the coupling and so does not generate non-perturbative information, but non-perturbative behaviour emerges at \( \mathcal{O}(\delta^2) \). Fig. 3 is a typical plot, showing a clear maximum in \( m_{T,\mu}^2(\eta) \).
It will be of particular interest to study the $\mu$ dependence of the critical temperature $T_c$, the signal for the phase transition being taken to be $m^2_{T_c,\mu} = \mu^2$, which leads to an implicit equation for $T_c = T_c(\mu)$ (with dependence on the renormalized bare mass and coupling of the theory suppressed in the notation). Fig. 4 shows a plot of the contours of $m^2_{T_c,\mu}$ for $m^2 = -M^2, \lambda = 0.5$ in a region of the $(T, \mu)$ plane, with the thicker line indicating the critical temperature. Below this line the LDE breaks down - the thermal mass being a monotonically decreasing function of $\eta$ and thus lacking extrema.

In Fig. 5 we present a comparison of $T_c(\mu)$ as calculated by the LDE with the first-order estimate

$$m^2 + \frac{\lambda T^2}{12} = \mu^2$$

provided by perturbation theory in the high temperature limit. 

FIG. 3. Dependence of the thermal mass squared on the variational parameter $\eta$. All masses measured in units of $M$.

FIG. 4. Contours of the thermal mass squared in intervals of $\frac{1}{2}M^2$ in the $(T, \mu)$ plane for $\lambda = 0.5$ with $m^2 = -M^2$. Both axes are in units of $M$. 

In Fig. 5 we present a comparison of $T_c(\mu)$ as calculated by the LDE with the first-order estimate
FIG. 5. Dependence of the critical temperature on $\mu$ at $O(\delta^2)$ for $m^2 = -M^2$, $\lambda = 0.5$, with both axes in units of $M$. The dotted line indicates the first-order approximation given by Eq. (32).

The two curves are quite similar in shape, and converge at large $T_c$. The reason for this behaviour is that in the high $T$ limit, it so happens that $\bar{\eta}^2 \to \mu^2 - m^2$, which is equivalent to $r^2 \to 1$. Examining those contributions from $\Pi_4$ in (29) we see that if this is the case, we require the coefficients of the $1/\sqrt{(1 - r^2)}$ terms to disappear, which is exactly the same condition as (32).

Fig. 6 illustrates the high-$T$ behaviour of the second-order approximation to the thermal mass, with $\mu/T_c$ approaching a constant as $T_c$ increases; the constant being determined from (32) to be $\sqrt{(\lambda/12)}$.

FIG. 6. The variation of $\mu/T_c$ with $T_c$ for $m^2 = -M^2$, $\lambda = 0.5$. The dashed line indicates the constant value $\sqrt{(\lambda/12)}$. $T_c$ is in units of $M$.

IV. CONCLUSIONS

In this paper we have shown how the problem of finite chemical potential for a charged scalar theory can be formulated in the context of the linear delta expansion. The graphs to be evaluated are essentially those of ordinary
perturbation theory, with modified mass and coupling parameters. The chemical potential appears explicitly in the 
Lagrangian, and also in the Bose-Einstein factors occurring in the momentum integrals. Renormalization has been 
implemented on the lines of Ref. [6], where the importance of renormalizing before applying the variational aspect of 
the method was emphasized. We obtain unambiguous PMS points, as illustrated in Fig. 3.

The results have been plotted in Fig. 4 as a contour plot of the thermal mass in the (T, µ) plane, and in Fig. 5 as 
Tc versus µ for given values of m², λ. This latter curve approaches the result of resummed perturbation theory at 
high temperature, but differs significantly from it at lower temperatures. The reason for the convergence at higher 
temperatures can be understood in terms of the properties of the stationary points in the variational parameter η.

As emphasized in the introduction, the problem of a non-zero chemical potential is not amenable to treatment by 
the usual Monte-Carlo lattice method. A lattice version of the present calculation is in progress.

An extension of the present work which we intend to pursue in the near future is to free it from the dependence 
on the high temperature expansion, thereby enabling us to consider a system with non-zero chemical potential at low 
temperature.

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APPENDIX A:

![Diagram](image)

FIG. 7. The two-loop momentum-dependent diagram

In this appendix we will derive in some detail the expression for the two-loop setting sun diagram. We begin 
by mimicking the approach of Parwani [2] to split up the diagram into three parts; containing zero, one and two 
Bose-Einstein factors respectively. Explicitly,

$$\Pi_5^{\delta^2} = -\frac{\delta^2 \lambda^2}{2} M^{\delta^2} T^2 \sum_{l,m} \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{d^{d-1}q}{(2\pi)^{d-1}} \Delta_F(i\omega_l, k) \Delta_F(i\omega_m, q) \Delta_F(i\omega_{r}, -r),$$

(A1)

where d = 4 − 2ε is the space-time dimension and r = p − k − q. If we now use the mixed representation of the 
Matsubara propagator defined in [8], the frequency sums become trivial and eventually we have

$$\Pi_5^{\delta^2}(i\omega_n, p) = -\frac{\delta^2 \lambda^2}{2} \left(G_0(i\omega_n, p) + G_1(i\omega_n, p) + G_2(i\omega_n, p)\right),$$

(A2)

where

$$G_0(i\omega_n, p) = \int d[k, q] S_{\mu}(\omega_k, \omega_q, \omega_r),$$

(A3)

$$G_1(i\omega_n, p) = \int d[k, q] \left[n_k^+ \left( S_{\mu}(\omega_k, \omega_q, \omega_r) + S_{\mu}(-\omega_k, \omega_q, \omega_r) + S_{\mu}^{+}(\omega_k, \omega_q, \omega_r) + S_{\mu}^{-}(-\omega_k, \omega_q, \omega_r) \right) + n_k^- \left( S_{\mu}(\omega_k, \omega_q, \omega_r) + S_{\mu}(-\omega_k, \omega_q, \omega_r) + S_{\mu}^{+}(\omega_k, \omega_q, \omega_r) + S_{\mu}^{-}(-\omega_k, \omega_q, \omega_r) \right) \right],$$

(A4)
\[ G_2(i\omega_n, \mathbf{p}) = \int d[k, q] \left[ n_k^+ n_q^- (S^+_{\mu}(\omega_k, \omega_q, \omega_r) + S_\mu(-\omega_k, \omega_q, \omega_r) + S_\mu(\omega_k, -\omega_q, \omega_r) - S^+_{\mu}(\omega_k, \omega_q, -\omega_r)) \right. \]
\[ + n_k^+ n_q^- (S^+_{\mu}(\omega_k, \omega_q, \omega_r) + S^-_{\mu}(\omega_k, -\omega_q, \omega_r)) \right. \]
\[ + n_k^- n_q^+ (S^-_{\mu}(\omega_k, \omega_q, \omega_r) + S^+_{\mu}(\omega_k, -\omega_q, \omega_r)) \right. \]
\[ + n_k^- n_q^- (S^-_{\mu}(\omega_k, \omega_q, \omega_r) + S^-_{\mu}(\omega_k, \omega_q, -\omega_r)) \right] \] with the definitions
\[ S^+_{\mu}(\omega_k, \omega_q, \omega_r) = \frac{1}{\pm(i\omega_n + \mu) + \omega_k + \omega_q + \omega_r}, \]
\[ S_{\mu}(\omega_k, \omega_q, \omega_r) = S^+_{\mu}(\omega_k, \omega_q, \omega_r) + S^-_{\mu}(\omega_k, \omega_q, \omega_r), \]
\[ d[k, q] = M^4 \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{d^{d-1}q}{(2\pi)^{d-1}} \frac{1}{8\omega_k \omega_q \omega_r}. \]

We choose to evaluate the self-energy on shell ($\mathbf{p} = 0, i\omega_n = \Omega - \mu$). The reason we evaluate at this point rather than at $\mathbf{p} = 0, i\omega_n = \Omega$ is best explained as follows [16]. In setting up the theory, we are dealing with an effective Hamiltonian, $\hat{H}_{\text{eff}} = \hat{H} - \mu \hat{Q}$ rather than the real Hamiltonian, $\hat{H}$. We would naturally choose to evaluate the self-energy at $i\omega_n = \Omega$ for the real Hamiltonian, which means evaluating at $i\omega_n = \Omega - \mu$ in the effective theory. Following this prescription, we find that
\[ \Re\{G_0(\Omega - \mu, 0)\} = -\frac{3\Omega^2}{2(16\pi)^2} \left[ \frac{1}{\epsilon^2} + \frac{3 - 2\gamma}{\epsilon} - \frac{2}{\epsilon} \ln \left( \frac{\Omega^2}{4\pi M^2} \right) \right] - \frac{\mu^2}{(32\pi^2)^2} \frac{1}{\epsilon} \]
\[ - \frac{3\Omega^2}{(16\pi)^2} \left[ \ln^2 \left( \frac{\Omega^2}{4\pi M^2} \right) + (2\gamma - \frac{17}{6}) \ln \left( \frac{\Omega^2}{4\pi M^2} \right) + 1.9785 \right]. \] (A6)

We now calculate the contribution from $G_1$. Writing
\[ G_1(i\omega_n, \mathbf{p}) = G^+_1(i\omega_n, \mathbf{p}) + G^-_1(i\omega_n, \mathbf{p}), \] (A7)
where $G^\pm_1$ represents that part of $G_1$ with an associated BE factor of $n^\pm_k$ respectively, and decomposing this factor into a UV divergent part and a UV finite part in the manner of the $\mu = 0$ case one obtains
\[ \Re\{G^\pm_1(\Omega - \mu, 0)\} = F^\pm_0 + F^\pm_1 + F^\pm_2(\Omega^2). \] (A8)

Here
\[ F^\pm_0 = \frac{3}{32\pi^2} \frac{T^2}{\pi^2} h_3(y, \pm r) \frac{1}{\epsilon}, \] (A9)
\[ F^\pm_1 = \frac{3}{32\pi^2} \frac{T^2}{\pi^2} h_3(y, \pm r) \left[ \ln \left( \frac{4\pi M^2}{\Omega^2} \right) + 2 - \gamma \right], \] (A10)
and
\[ F^\pm_2(\Omega^2) = \frac{1}{4(2\pi)^4} \int_0^\infty dk \int_0^\infty dq \left[ q \ln \left( \frac{X^\pm}{X^-} \right) - 6k \right], \] (A11)
with
\[ X^\pm = \left[ \Omega^2 - (\omega_k + \omega_q + \omega_{k\pm q})^2 \right] \left[ \Omega^2 - (-\omega_k + \omega_q + \omega_{k\pm q})^2 \right] \left[ \Omega \pm \omega_{k\pm q} \right]^2, \] (A12)
where the subscript $\pm$ refers to the $\omega_{k\pm q}$ term. Following a similar procedure to that of Ref. [2] we find that in the high-$T$ limit
\[ F^\pm_2(\Omega^2) \sim \frac{T^2}{128\pi^2} \left[ \ln \left( \frac{\Omega}{T} \right) - 0.54597 \right] \] (A13)
Thus, collecting all the terms together,
\[ \Re[G_1(\Omega - \mu, 0)] = F_0 + F_1 + F_2(\Omega^2), \] (A14)

with
\[ F_0 = \frac{3T^2}{16\pi^4} h_5^*(y, r) \frac{1}{e}, \] (A15)
\[ F_1 = \frac{3T^2}{16\pi^4} h_4^*(y, r) \left[ -\ln \left( \frac{\Omega^2}{4\pi M^2} \right) + 2 - \gamma \right], \] (A16)

and
\[ F_2(\Omega^2) \sim \frac{T^2}{64\pi^2} \left[ \ln \left( \frac{\Omega}{7} \right) - 0.54597 \right]. \] (A17)

Finally, we consider the contribution from \( G_2 \), which contains a BE factor for each loop and so is UV finite. We write
\[ \Re[G_2(\Omega - \mu, 0)] = H^{++}(\Omega^2) + H^{-+}(\Omega^2) + H^{-+}(\Omega^2) + H^{--}(\Omega^2) \] (A18)

with
\[ H^{\pm\pm}(\Omega^2) = \frac{1}{4(2\pi)^2} \int_0^\infty dk \frac{k n_k^\pm}{\omega_k} \int_0^\infty dq \frac{q n_q^\pm}{\omega_q} \ln \left| \frac{\omega_+^\pm}{\omega_-^\pm} \right|, \] (A19)

where
\[ Y_{++} = \Omega^2 - (-\omega_k + \omega_q + \omega_{k+q})^2[\Omega^2 - (\omega_k - \omega_q + \omega_{k+q})^2][\Omega + \omega_k + \omega_q]^2 - \omega_{k+q}^2, \] (A20)
\[ Y_{+-} = \Omega^2 - (\omega_k + \omega_q + \omega_{k+q})^2[\Omega^2 - (\omega_k + \omega_q - \omega_{k+q})^2][\Omega + \omega_k - \omega_q]^2 - \omega_{k+q}^2, \] (A21)
\[ Y_{-+} = \Omega^2 - (\omega_k + \omega_q + \omega_{k+q})^2[\Omega^2 - (\omega_k + \omega_q - \omega_{k+q})^2][\Omega - \omega_k + \omega_q]^2 - \omega_{k+q}^2, \] (A22)
\[ Y_{--} = \Omega^2 - (-\omega_k + \omega_q + \omega_{k+q})^2[\Omega^2 - (-\omega_k - \omega_q + \omega_{k+q})^2][\Omega - \omega_k - \omega_q]^2 - \omega_{k+q}^2. \] (A23)

Each of the \( H^{\pm\pm} \) pieces has a logarithmic IR divergence as \( \Omega \to 0 \). To deal with this, we extract the leading order behaviour of \( \ln |Y_{++}^{\pm}/Y_{--}^{\pm}| \) in this limit, separating into three terms as follows:
\[ \ln \left| \frac{Y_{++}^{\pm}}{Y_{--}^{\pm}} \right| \sim \pm \ln 2 \pm \ln \left| \frac{kq}{\Omega \sqrt{k^2 - q^2}} \right| + \frac{2}{3} \ln \left| \frac{k + q}{k - q} \right|. \] (A24)

Collecting those terms involving the \( \ln 2 \) piece alone, which we can take outside the integral, we calculate the coefficient of this term in the high-temperature limit \( a \equiv \beta / \Omega \ll 1 \), which we call \( C_{\ln 2} \), to be
\[ C_{\ln 2} = \frac{T^2}{(8\pi^2)^2} \ln^2 \left( \frac{1 + r}{1 - r} \right). \] (A25)

The remainder of \( \Re[G_2(\Omega - \mu, 0)] \) splits into two pieces which we define by
\[ H^1(\Omega^2) = \frac{1}{4(2\pi)^2} \int_0^\infty \frac{k dk}{\omega_k} (n_k^+ - n_k^-) \int_0^\infty \frac{q dq}{\omega_q} (n_q^+ - n_q^-) \ln \left| \frac{kq}{\Omega \sqrt{k^2 - q^2}} \right|, \] (A26)
\[ H^2(\Omega^2) = \frac{3}{8(2\pi)^2} \int_0^\infty \frac{k dk}{\omega_k} (n_k^+ + n_k^-) \int_0^\infty \frac{q dq}{\omega_q} (n_q^+ + n_q^-) \ln \left| \frac{k + q}{k - q} \right|. \] (A27)

Extracting the high-temperature limit of \( H^1(\Omega^2) \) gives, after using the symmetry of the integrand under \( k \leftrightarrow q \) to restrict the range of the \( q \) integration and making the changes of variable \( \Omega \xi = \omega_k, \Omega \eta = \omega_q \),
\[ H^1(\Omega^2) = \frac{\Omega^2}{4(2\pi)^2} \int_1^\infty d\xi (n_\xi^+ - n_\xi^-) \int_1^\xi d\eta (n_\eta^+ - n_\eta^-) \ln \left| \frac{(\xi^2 - 1)(\eta^2 - 1)}{\xi^2 - \eta^2} \right|, \] (A28)
where \( n^\pm_\xi = 1/(e^{a(\xi \pm r)} - 1) \). After making the successive approximations \(^1\)

\[
n^+_\xi - n^-_\xi \approx -2ra \frac{e^{a\xi}}{(e^{a\xi} - 1)^2} \approx -\frac{2r}{a\xi^2}
\]  

(A29)

in the integrand, we have, for \( a \ll 1 \),

\[
H^1(\Omega^2) = \frac{T^2 r^2}{(2\pi)^3}(\ln 2 - \frac{1}{2}).
\]  

(A30)

The calculation of \( H^2(\Omega^2) \) parallels the case for \( \mu = 0 \), leading to

\[
H^2(\Omega^2) = \frac{3T^2}{64\pi^2} \left[ -\frac{1}{2} \ln \left( \frac{\Omega^2 - \mu^2}{T^2} \right) - 1.50699 \right].
\]  

(A31)

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\(^1\)The second approximation in (A29) is strictly valid only for \( |a\xi| \ll 1 \); however, its validity is assured by the convergence of the resulting integral and has been checked numerically.