NON COMMUTATIVE SCALAR FIELDS FROM SYMPLECTIC DEFORMATION

M. Daoud$^a$ and A. Hamama$^b$

$^a$ Abdus Salam International Centre for Theoretical Physics, Strada Costiera 11, 34014 Trieste, Italy

$^b$High Energy Laboratory, Faculty of Sciences, University Mohamed V, P.O. Box 1014, Rabat, Morocco

Abstract

This paper is concerned with the quantum theory of noncommutative scalar fields in two dimensional space time. It is shown that the noncommutativity originates from the deformation of symplectic structures. The quantization is performed and the modes expansions of the fields, in presence of an electro-magnetic background, are derived. The Hamiltonian of the theory is given and the degeneracies lifting, induced by the deformation, is also discussed.

$^{1}$Faculté des Sciences, Département de Physique, Agadir, Morocco; email: m_daoud@hotmail.com
1 Introduction

In the last decade the noncommutative field theories have been extensively investigated (for a review see [1-2]) in connection with the low energy description of string theory with a constant magnetic background in the presence of $D$-branes [3]. It is important to note the construction of field theories in noncommutative space-time encounters many problems as the violation of Lorentz invariance, the ultraviolet divergences, and the breaking of unitarity and causality. However, despite such technical problems, the noncommutative geometry still considered as one promising candidate to provide a rigorous framework to investigate and to understand the noncommutative space time structure at Planck scale [4] in relation with quantum gravity.

It is also important to mention that motivated by the noncommutative field theories the quantum mechanical problems in noncommutative space have received a considerable attention [5-9]. More recently it was shown that the noncommutative geometry via the Moyal star product is relevant in many branches of physics and especially in condensed matter as for instance quantum Hall effect in different geometries [10-11], bosonization in higher dimensions [12] and braiding quantum statistics [13-19].

In this work, we shall be concerned with the quantum theory of noncommutative scalar fields where the fields and their conjugate momenta obey deformed equal time commutations rules. It is important to stress that the quantum theory of noncommutative fields is defined on a commutative space-time contrarily to the noncommutative field theories which arise from the non-commutativity of space-time. The interest for this new kind of quantum theory of noncommutative fields, introduced in [20-24], is essentially motivated by Lorentz invariance violation, the problem of neutrino oscillations and the asymmetry of the dispersion relation for particles and antiparticles.

In this paper we propose a noncommutative formulation of two dimensional scalar field theory underlying the noncommutative algebra with a constant commutator of fields and simultaneously with a constant commutator of momenta as well. This formulation is done from a symplectic point of view. Indeed we first suggest a modification of the symplectic structure associated to phase space associated to two massless scalar fields. Secondly, performing a dressing transformation, we write the deformed two-form in a canonical form. The Hamiltonian is then converted in a new one involving terms arising from the deformation. We show that the dynamic described by a deformed Hamiltonian and canonical two-form is equivalent to the description which uses undeformed Hamiltonian and deformed symplectic structure. Note that in a particular case, which will be specified below, the construction developed here agrees with the results derived in [24] and most importantly it start at classical level by deforming the symplectic structure. In quantizing the theory the deformation induces a modification of the commutation
rules as well as in the spectrum of the system. These results can be viewed as an extension of
the symplectic approach to non-commutative mechanics initiated in [25-27].

The arrangement of this paper is as follows. In the section 2, we first review some elements
concerning symplectic structures in deriving the evolution of classical fields and the canonical
quantization. This review, included here for completeness, is useful and necessary to understand
and to perform the approach developed in the next sections. The section 3 concerns the deforma-
tion from a symplectic point of view. We consider two classical scalar fields and we modify
the symplectic two-form to take into account the presence of an electro-magnetic background.
We define the corresponding Poisson brackets from which we get ones between fields and their
conjugate momenta. In section 4, following the procedure suggested in [20-24], we quantize the
theory, we propose the Hamiltonian associated to a quantum theory of two noncommutative
scalar fields and we discuss the degeneracies lifting induced by the deformation. Solving the
Heisenberg equations of motion, we give the modes expansion of the noncommutative fields. As
mentioned above, the approach essentially originates from the deformation of the symplectic
structures (symplectic two-form and Poisson brackets) in the phase space of the scalar fields.
We end section 4 with some comments concerning the dressing transformation, which play an
important role in the quantizing the model. Concluding remarks close this paper.

2 Symplectic structures and field quantization

We shall consider two real massless bosonic fields denoted by $\Phi^i(x) \equiv \Phi(x,t)$, $i = 1, 2$. The
corresponding action is

$$S = \int_{\Sigma} dtdx \mathcal{L} = \frac{1}{2l} \sum_{i=1,2} \int_{\Sigma} dtdx ((\partial_t \Phi^i)(\partial_t \Phi^i) - (\partial_x \Phi^i)(\partial_x \Phi^i)).$$

(1)

The space-time region $\Sigma$ will be considered to be of the form $[0,l] \times [0,T]$ where $[0,l]$ is line
segment is the spatial region. The equation of motion are given by the variational principle such
that the classical orbit of the fields which connects the initial and final configurations $\Phi^i(x,0)$
and $\Phi^i(x,T)$ at time $t = 0$ and $t = T$, minimizes the action. Hence, the minimization condition
$\delta S = 0$ gives the equations of motion (Euler-Lagrange equations) since the surface contribution
of $\delta S$:

$$\delta S_{surface} = \sum_{i=1,2} \int dtdx \frac{\partial}{\partial x^\mu} \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi^i)} \delta \Phi^i \right)$$

(2)

vanishes when we fix the initial and final fields configurations, i.e. $\delta \Phi^i = 0$ at time $t = 0$ and
$t = T$ and also assume that either $\delta \Phi^i = 0$ or $\delta \mathcal{L} / \partial (\partial_\mu \Phi^i)$ vanishes at $x = 0$ and $x = l$. In equation
(2) $\mu = 0, 1$ stands for time and space directions respectively. The summation over repeated
indices is understood. This is the standard way to get the trajectory of classical fields by means
of Euler-Lagrange equations. There exists another way based on the symplectic structure of the phase space to describe the classical evolution of the fields. Indeed, if one consider the variations of the fields with $\delta \Phi^i$ non-vanishing at time $t = 0$ and $t = T$, the integration of the surface contribution term gives

$$l \int dt dx \frac{\partial}{\partial x^\mu} \left( \frac{\partial L}{\partial (\partial_\mu \Phi^i)} \delta \Phi^i \right) = \mathcal{A}(T) - \mathcal{A}(0)$$

(3)

where

$$\mathcal{A}(t) = \sum_{i=1,2} \int dx \partial_t \Phi^i \delta \Phi^i.$$  

(4)

This is the canonical one form, i.e. a differential one form on the phase space. In equation (4), $\delta$ denotes the exterior derivative on the field space and the time derivatives of the fields must be treated as variables since $\mathcal{A}$ is defined at fixed time. The phase space whose variables (coordinates) are $\Phi^i(x, t)$ and the canonical momentum $\Pi^i(x, t) = \partial_t \Phi^i(x, t)$ is now equipped with a symplectic structure, a canonical two-form, defined by

$$\Omega_0 = \delta \mathcal{A}(t) = \sum_{i=1,2} \int dx \delta \Phi^i \wedge \delta \Pi^i$$

(5)

which can be seen as field strength corresponding to one-form $\mathcal{A}$ viewed as $U(1)$ gauge potential.

It can also be written as

$$\Omega_0 = \frac{1}{2} \sum_{I,J} \int dx dx' (\Omega_0)_{IJ}(x, x') \delta \xi^I(x) \wedge \delta \xi^J(x')$$

(6)

where we denote the phase space coordinates by $\xi^I(x) \equiv \xi^{ii'}(x) = \Phi^i(x)$ (resp. $\Pi^i(x)$) for $i' = 1$ (resp. for $i' = 2$) and

$$(\Omega_0)_{IJ}(x, x') \equiv (\Omega_0)_{IJ} \delta(x - x') = \delta_{ij} \epsilon_{i'j'} \delta(x - x').$$

(7)

It follows that the Poisson bracket of two functionals $\mathcal{F}$ and $\mathcal{G}$ is

$$\{ \mathcal{F}, \mathcal{G} \} = \sum_{I,J} \int dx (\Omega_0)^{-1}_{IJ} \frac{\delta \mathcal{F}}{\delta \xi^I} \frac{\delta \mathcal{G}}{\delta \xi^J}$$

(8)

where $\Omega_0^{-1}_{IJ}$ are the elements of the inverse matrix of $\Omega_{IJ}$. In particular the Poisson brackets for the coordinates on the phase space are the inverse of the symplectic two-form (treated as matrix):

$$\{ \xi^{ik}(x'), \xi^{jl}(x) \} = \delta^{ij} \epsilon^{kl} \delta(x - x'),$$

(9)

or alternatively

$$\{ \Phi^i(x, t), \Phi^j(x', t) \} = 0 \quad \{ \Phi^i(x, t), \Pi^j(x', t) \} = \delta^{ij} \delta(x - x') \quad \{ \Pi^i(x, t), \Pi^j(x', t) \} = 0.$$  

(10)
As we are concerned with two component scalar field theory in a compact domain, with the boundary condition we can expand the fields as

$$\Phi^i(x, t) = \frac{1}{\sqrt{l}} \sum_{n \in \mathbb{Z}} q_n^i \exp(-inx)$$

and

$$\Pi^i(x, t) = \frac{1}{\sqrt{l}} \sum_{n \in \mathbb{Z}} p_n^i \exp(-inx)$$

where the normal or Fourier modes $q_n$ and $p_n$ satisfy the conditions

$$q_{-n} = q_n^* \quad p_{-n} = p_n^*$$

required by the reality of fields and their time derivatives.

Using the equations (5) and (11-12), the symplectic two-form $\Omega_0$ rewrites

$$\Omega_0 = \sum_{i=1,2} \sum_{n \in \mathbb{Z}} \delta q_n^i \wedge \delta p_{-n}^i$$

and the Poisson brackets take the simple form

$$\{F, G\} = \sum_{i=1,2} \sum_{n \in \mathbb{Z}} \frac{\partial F}{\partial q_n^i} \frac{\partial G}{\partial p_{-n}^i} - \frac{\partial F}{\partial p_{-n}^i} \frac{\partial G}{\partial q_n^i}.$$  

In particular, the Poisson brackets corresponding to the canonical coordinates of the phase space generated by the Fourier modes $q_n^i$ and $p_n^i$ are given by

$$\{q_n^i, q_m^j\} = 0 \quad \{p_n^i, p_m^j\} = 0 \quad \{q_n^i, p_m^j\} = \delta_{i,j} \delta_{m+n,0}.$$  

The lagrangian is given by

$$L = \int dx L = \frac{1}{2} \sum_{n} (p_n^i p_{-n}^i - n^2 q_n^i q_{-n}^i)$$

in terms of the Fourier modes. The canonical Hamiltonian reads

$$H_0 = \sum_{i=1,2} \int dx \Pi^i \partial_t \Phi^i - \int dx L = \frac{1}{2} \sum_{n} (p_n^i p_{-n}^i + n^2 q_n^i q_{-n}^i).$$

Since we defined $p_n^i = \dot{q}_n^i$, the canonical momenta conjugate to $q_n^i$ is defined by

$$\pi_n^i = \frac{\partial L}{\partial \dot{q}_n^i} = \dot{q}_{-n}^i = p_{-n}^i.$$  

The Hamiltonian is the generator of time translation. The time evolution of a function $F$ is given by the Hamilton’s equation of motion $\dot{F} = \{F, H_0\}$ that gives

$$\frac{dq_n^i}{dt} = \{q_n^i, H_0\} = p_n^i \quad \frac{dp_n^i}{dt} = \{p_n^i, H_0\} = -n^2 q_{-n}^i.$$  

To pass over the quantum theory in the Heisenberg picture, all canonical variables become Heisenberg operators satisfying commutation relations corresponding to Poisson brackets as...
(Poisson Bracket) $\rightarrow -i$ (commutator)

The equations (20) are easily solvable and one recover the evolution of the fields which can also be derived from the Euler-Lagrange equations. However, the symplectic prescription discussed has the merit to be more appropriate to introduce deformed target space in the spirit of the recent results obtained in [20-23] and [24]. This issue is the purpose of the next section.

3 Deformed symplectic structure

Hereafter, we consider two components abelian field theory in two dimensional space-time. The generalization to arbitrary number of components is straightforward. As mentioned above the dynamic of the system is governed by the Hamiltonian $H_0$ (18). We now assume that the symplectic structure of the phase space is modified due to the interaction between the field components $\Phi^1$ and $\Phi^2$ in presence of an electromagnetic background. This can be formulated by replacing the canonical two-form $\Omega_0$ by a closed new one as follows

$$\Omega = \Omega_0 + \frac{1}{2} \sum_{ij} E_{ij} \int \delta x \delta x' (x) \wedge \delta x'^2 (x) - \frac{1}{2} B_{ij} \int \delta x \delta x'^1 (x) \wedge \delta x'^1 (x)$$

which is non degenerate ($\det \Omega \neq 0$) when the antisymmetric tensors $E_{ij}$ and $B_{ij}$ satisfy the condition $\det(1_{2 \times 2} - EB) \neq 0$. This is easily seen in writing $\Omega$ in a matrix form. We assume in this work that such a condition is satisfied. To find the classical equations of motion and to establish the connection between the classical and quantum theory, it is necessary to define the Poisson brackets associated with the new phase space geometry in a consistent way. Indeed, recalling that the Poisson brackets for the coordinates on the phase space are the inverse of the symplectic form as matrix, we have

$$\{F, G\} = \sum_{I,J} \int \delta x \Omega^{IJ} \frac{\delta F}{\delta \xi^I} \frac{\delta G}{\delta \xi^J}$$

where $\Omega^{IJ}$ is the inverse matrix of $\Omega_{IJ}$ (24) and $\mathcal{F}$ and $\mathcal{G}$ are two functionals on the phase space. As we are interested by scalar field theory, we use the equations (11-12) and (22) to write the symplectic form as

$$\Omega = \delta q^i_n \wedge \delta p^i_{-n} + \frac{1}{2} E_{ij} \delta p^i_n \wedge \delta p^j_{-n} - \frac{1}{2} B_{ij} \delta q^i_n \wedge \delta q^j_{-n}$$
and the Poisson brackets take the simple form

\[ \{ \mathcal{F}, \mathcal{G} \} = \sum_{ikn} (\omega_1^{-1})_{ik} \frac{\delta \mathcal{F}}{\delta q_i^n} \left[ \frac{\delta \mathcal{G}}{\delta p_{i-n}^n} \right] - (\omega_2^{-1})_{ik} \frac{\delta \mathcal{F}}{\delta q_i^n} \left[ \frac{\delta \mathcal{G}}{\delta p_{i-n}^n} \right] \]  

where the functionals \( \mathcal{F} \) and \( \mathcal{G} \) are now expressed in terms of \( q_i^n \) and \( p_i^n \) generating the phase space. The matrix elements of \( \omega_1 \) and \( \omega_2 \), occurring in (27), are defined by

\[(\omega_1)_{ij} = \delta_{ij} - \mathcal{E}_{ik} \mathcal{B}_{kj}, \]  
\[(\omega_2)_{ij} = \delta_{ij} - \mathcal{B}_{ik} \mathcal{E}_{kj}. \]

The last equations can be read in matrices form as \( \omega_1 = 1 - \mathcal{E} \mathcal{B} \) and \( \omega_2 = 1 - \mathcal{B} \mathcal{E} \), respectively.

It follows that the modified canonical Poisson brackets are

\[ \{ q_i^n, q_j^m \} = - \sum_k (\omega_1^{-1})_{ik} \mathcal{E}_{kj} \delta_{m+n,0}, \]  
\[ \{ p_i^n, p_j^m \} = \sum_k (\omega_2^{-1})_{ik} \mathcal{B}_{kj} \delta_{m+n,0}, \]  
\[ \{ q_i^n, p_j^{-m} \} = (\omega_1^{-1})_{ij} \delta_{n,m} = (\omega_2^{-1})_{ji} \delta_{n,m}. \]

According to the modification of the symplectic structure of the phase space, we introduce the vector fields \( X_{\mathcal{F}} \) associated to a given functional \( \mathcal{F}(q_i^n, p_i^n) \)

\[ X_{\mathcal{F}} = \sum_n X_i^n \frac{\delta}{\delta q_i^n} + Y_i^n \frac{\delta}{\delta p_i^n} \]  

such that the interior contraction of \( \Omega \) with \( X_{\mathcal{F}} \) gives

\[ i(X_{\mathcal{F}}) \Omega = \delta \mathcal{F}. \]

A straightforward calculation leads to

\[ X_i^n = \sum_j (\omega_1^{-1})_{ij} \left( \frac{\delta \mathcal{F}}{\delta q_j^n} - \sum_k \mathcal{E}_{jk} \frac{\delta \mathcal{F}}{\delta q_k^n} \right) \]  
\[ Y_i^n = - \sum_j (\omega_2^{-1})_{ij} \left( \frac{\delta \mathcal{F}}{\delta q_j^n} - \sum_k \mathcal{B}_{jk} \frac{\delta \mathcal{F}}{\delta p_k^n} \right) \]

and one can check that

\[ i(X_{\mathcal{F}}) i(X_{\mathcal{G}}) \Omega = \{ \mathcal{F}, \mathcal{G} \}. \]

Thus in the deformed case the fields and their conjugate momentum satisfy the following Poisson algebra

\[ \{ \Phi^i(x, t), \Phi^j(x', t) \} = - \sum_k (\omega_1^{-1})_{ik} \mathcal{E}_{kj} \delta(x - x'). \]
\{\Phi^i(x, t), \Pi^j(x', t)\} = (\omega_2^{-1})_{ij} \delta(x - x'), 
(39)
\}
\{\Pi^i(x, t), \Pi^j(x', t)\} = \sum_k (\omega_2^{-1})_{ik} B_{kj} \delta(x - x'). 
(40)

Clearly, in the limiting case \( E = 0 \) and \( B = 0 \), we recover the canonical Poisson brackets (10).

To simply our purpose, let us now set
\begin{align*}
E_{ij} &= \theta \epsilon_{ij} \\
B_{ij} &= \bar{\theta} \epsilon_{ij}
\end{align*}
(41)
where \( \epsilon_{ij} \) is the usual antisymmetric tensor \( (\epsilon_{12} = -\epsilon_{21} = 1) \). With this choice, the Poisson brackets (30-32) read simply
\begin{align*}
\{q^i_n, q^j_m\} &= -\frac{\theta}{1 + \theta \bar{\theta}} \epsilon_{ij} \delta_{m+n,0} \\
\{p^i_n, p^j_m\} &= \frac{\bar{\theta}}{1 + \theta \bar{\theta}} \epsilon_{ij} \delta_{m+n,0} \\
\{q^i_n, p^j_{-m}\} &= \frac{1}{1 + \theta \bar{\theta}} \delta_{ij} \delta_{n,m}
\end{align*}
(42-44)
reflecting a deviation from the canonical brackets. At this stage, it is remarkable that the symplectic form (26) and the corresponding Poisson brackets (27) can be converted in the canonical forms by means of the so-called dressing transformation which furnishes a simply way to quantize the theory. This is the main task of the next section.

\section{Dressing transformation and Quantization}

To begin note that under the following transformation
\begin{align*}
Q_n^i &= a q_n^i + \frac{1}{2} b \theta \sum_k \epsilon_{ki} p_n^k \\
P_n^i &= c p_n^i + \frac{1}{2} d \theta \sum_k \epsilon_{ki} q_n^k
\end{align*}
(45-46)

the Poisson brackets (42-44) give the canonical ones
\begin{align*}
\{Q_n^i, Q_m^j\} &= 0 \\
\{P_n^i, P_m^j\} &= 0 \\
\{Q_n^i, P_{-m}^j\} &= \delta_{ij} \delta_{n,m}
\end{align*}
(47)
when the scalars \( a, b, c \) and \( d \) satisfy the following constraints
\begin{align*}
4a^2 - 4ab - \theta \bar{\theta} b^2 &= 0 \\
4c^2 - 4cd - \theta \bar{\theta} d^2 &= 0 \\
4ac + 2\theta \bar{\theta} (ad + bc) - \theta \bar{\theta} bd &= 4(1 + \theta \bar{\theta}).
\end{align*}
The transformation (45-46) generalizes the so-called dressing transformation introduced in [24]. As simple solution of the above equations, we consider

\[ a = c = \frac{1}{b} = \frac{1}{d} = \frac{1}{\sqrt{2}} \sqrt{1 + \sqrt{1 + \theta \bar{\theta}}}. \]  

(48)

With these new dynamical variables, \( \Omega \) takes a canonical form

\[ \Omega = \sum \delta Q^i_n \wedge \delta P^i_{-n}. \]  

(49)

Inverting the above dressing transformation (45-46)

\[ q^i_n = \frac{a}{\sqrt{1 + \theta \bar{\theta}}} \left[ Q^i_n + \frac{\theta}{2a^2} \sum_k \epsilon_{ik} P^k_n \right], \]

(50)

\[ p^i_n = \frac{a}{\sqrt{1 + \theta \bar{\theta}}} \left[ P^i_n + \frac{\bar{\theta}}{2a^2} \sum_k \epsilon_{ik} Q^k_n \right], \]  

(51)

the Hamiltonian (18), denoted in what follows by \( H \) to avoid any confusion, becomes

\[ H = \frac{a^2}{2(1 + \theta \bar{\theta})} \left[ \sum_n (1 + \frac{\theta^2}{4a^4} n^2) P^i_n P^i_{-n} + (\frac{\bar{\theta}^2}{4a^4} + n^2) Q^i_n Q^i_{-n} + (\frac{\theta}{a^2} n^2 + \frac{\bar{\theta}}{a^2}) \sum_j \epsilon_{ij} P^j_{-n} Q^j_n \right]. \]  

(52)

It is important to note that thanks to the dressing transformation (45-46), the dynamic described by the Hamiltonian \( H_0 \) on the deformed symplectic space is converted to one described by underformed (canonical) symplectic structure (49) and the Hamiltonian (52) expressed in terms of the new dynamical modes of the theory. Evidently the \((\theta, \bar{\theta})\)-dependent terms in \( H \) arise from the deformation of the symplectic structure. This suggests that the fields \( \Phi^1 \) and \( \Phi^2 \) interacts with a given potential \( V_{int} \) to have \( H = H_0 + V_{int} \) where \( H_0 \) is the free Hamiltonian given by (18) modulo the substitution \((q, p) \rightarrow (Q, P)\). In this respect the deformation of the symplectic structure can be seen as a perturbation reflecting the action of some external potential on the system. This feature is very similar to the Landau problem in quantum mechanics.

We now come to the quantization of the model. Since, as mentioned early, the two descriptions are equivalent, we shall quantize the system using the canonical prescription. Thus, we replace the phase space variables by operators satisfying commutation rules where the commutators are given by \( i \) times the associated Poisson brackets. Consequently, we have

\[ [Q^i_n, Q^j_m] = 0 \quad [P^i_n, P^j_m] = 0 \quad [Q^i_n, P^j_{-m}] = i \delta_{m+n,0} \delta_{i,j}. \]  

(53)

This gives

\[ [q^i_n, q^j_m] = -i \frac{\theta}{1 + \theta \bar{\theta}} \epsilon_{ij} \delta_{m+n,0} \]  

(54)

\[ [p^i_n, p^j_m] = i \frac{\bar{\theta}}{1 + \theta \bar{\theta}} \epsilon_{ij} \delta_{m+n,0} \]  

(55)
\[ [q^i_n, p^j_{-m}] = i \frac{1}{1 + \theta \bar{\theta}} \delta_{i,j} \delta_{n,m}, \quad (56) \]

from which we obtain the equal time commutation relations for the fields \( \Phi^1 \), \( \Phi^2 \) and their conjugate momenta:

\[ [\Phi^i(x, t), \Phi^j(x', t)] = -i \frac{\theta}{1 + \theta \bar{\theta}} \epsilon_{ij} \delta(x - x') \quad (57) \]

\[ [\Pi^i(x, t), \Pi^j(x', t)] = i \frac{\bar{\theta}}{1 + \theta \bar{\theta}} \epsilon_{ij} \delta(x - x') \quad (58) \]

\[ [\Phi^i(x, t), \Pi^j(x', t)] = -i \frac{1}{1 + \theta \bar{\theta}} \delta_{ij} \delta(x - x'). \quad (59) \]

At this stage, we introduce the operators

\[
\begin{align*}
a^i_n &= \sqrt{\frac{\Delta_n}{2}} (Q_n + i \frac{P_n^i}{\Delta_n}) \\
a^{i+}_n &= \sqrt{\frac{\Delta_n}{2}} (Q_n - i \frac{P_n^i}{\Delta_n})
\end{align*} \quad (60)
\]

where

\[
\Delta_n = \sqrt{\frac{\theta^2 + 4a^4_n n^2}{4a^4 + \theta^2 n^2}}.
\quad (61)
\]

They satisfy the usual Heisenberg commutation relations \([a^i_n, a^j_m] = \delta_{i,j} \delta_{n,m} \). The Hamiltonian \( H \) can be written as the sum of two contributions:

\[
H - H_L = \frac{1}{4 \frac{1}{1 + \theta \bar{\theta}}} \sum_{in} \sqrt{\bar{\theta}^2 + 4a^4_n n^2} (4a^4 + \theta^2 n^2)(1 + 2a^{i+}_n a^{i}_n) \quad (62)
\]

and

\[
H_L = \frac{i}{2 \frac{1}{1 + \theta \bar{\theta}}} \sum_{ijn} (\bar{\theta} + \theta n^2) \epsilon_{ijn} a^{i+}_n a^{j}_n. \quad (63)
\]

The Hamiltonian \( H \) can be diagonalized by considering the operators

\[
A^1_n = \frac{1}{\sqrt{2}} (a^1_n - ia^2_n) \quad A^2_n = \frac{1}{\sqrt{2}} (a^1_n + ia^2_n). \quad (64)
\]

Indeed, substituting (64) in (62-63), we obtain

\[
H = \sum_{n} [\omega_n + (\omega_n - \bar{\omega}_n) A^{1+}_n A^1_n + (\omega_n + \bar{\omega}_n) A^{2+}_n A^2_n] \quad (65)
\]

where

\[
\omega_n = \frac{1}{2 \frac{1}{1 + \theta \bar{\theta}}} \sqrt{\theta^2 + 4a^4_n n^2} (4a^4 + \theta^2 n^2) \quad (66)
\]

and

\[
\bar{\omega}_n = \frac{1}{2(1 + \theta \bar{\theta})} (\bar{\theta} + \theta n^2). \quad (67)
\]
From the last three equations, it is easily seen that the deformation induces a lifting of the degeneracies of the spectrum. The Hamiltonian is a superposition of two independents one dimensional oscillators unlike to undeformed case where the Hamiltonian is a sum of two-dimensional oscillators. The dynamics of this system is described by the Heisenberg equations:

$$\frac{dA_n^1}{dt} = -i[A_n^1, H] = -i\omega_n^- A_n^1$$

$$\frac{dA_n^2}{dt} = -i[A_n^2, H] = -i\omega_n^+ A_n^2$$

(68)

where \( \omega_n^\pm = \omega_n \pm \bar{\omega}_n \). Thus, we have:

$$A_n^1(t) = \hat{A}_n^1 \exp(-i\omega_n^- t) \quad A_n^2(t) = \hat{A}_n^2 \exp(-i\omega_n^+ t)$$

(69)

where the operators \( \hat{A}_n^1 \) and \( \hat{A}_n^2 \) are time-independents. Consequently, using the equations (50), (60), (64) and (69) we obtain the normal modes of the model as

$$q_n^1(t) = \frac{1}{2}[\Lambda_n^+(\hat{A}_n^1 \exp(-i\omega_n^- t) + \hat{A}_n^1+ \exp(+i\omega_n^- t)) + \Lambda_n^-(\hat{A}_n^2 \exp(-i\omega_n^+ t) + \hat{A}_n^2+ \exp(+i\omega_n^+ t))]$$

(70)

and

$$q_n^2(t) = i\frac{1}{2}[\Lambda_n^+(\hat{A}_n^1 \exp(-i\omega_n^- t) - \hat{A}_n^1+ \exp(+i\omega_n^- t)) + \Lambda_n^-(\hat{A}_n^2 \exp(+i\omega_n^+ t) - \hat{A}_n^2+ \exp(-i\omega_n^+ t))]$$

(71)

where the \((\theta, \bar{\theta})\)-dependent constants \( \Lambda_n^\pm \) are defined by

$$\Lambda_n^\pm = \left[\frac{1}{\sqrt{\Delta_n}} \pm \frac{\theta}{2a^2} \sqrt{\Delta_n}\right].$$

(72)

It is clear that in the limiting case \( \theta = 0 \) and \( \bar{\theta} = 0 \), one recovers the usual results arising from the equations (20). Note also that for \( \bar{\theta} = 0 \), our results agree with ones obtained in [24]. We end up this section by some remarks related the dressing transformation (45-46). In this sense, we will show that the deformed Poisson algebra Eqs.(42-44) is un-equivalent to un-deformed one (47). The un-equivalency occurs if the two algebras can not be transformed to each other by a unitary transformation. This means that the dressing transformation (45-46) should be not unitary or more precisely not orthogonal since its elements are reals. In fact, it is easy to check that the orthogonality requires the following conditions

$$a^2 + \frac{1}{4} \theta^2 b^2 = 1 \quad c^2 + \frac{1}{4} \theta^2 d^2 = 1 \quad bc\theta = ad\bar{\theta}.$$  

(73)

Setting

$$a = \cos \varphi_1 \quad c = \cos \varphi_2 \quad b\theta = 2\sin \varphi_1 \quad d\bar{\theta} = 2\sin \varphi_2$$

(74)

one can see that last equality in (73) is satisfied if

$$\varphi_1 = \varphi_2 + n\pi \quad n \in \mathbb{N}$$

(75)

which implies

$$a = \pm c \quad b\theta = \pm d\bar{\theta}$$

(76)
Consistency with (48) gives $\theta = \bar{\theta}$. This shows that the dressing transformation is orthogonal when the strengths of magnetic and electric backgrounds are equal. Finally, we stress that the transformation (45-46) is similar to Darboux coordinates transformation (see for instance [26-27]).

5  Concluding remarks

We have clarified a procedure generating the noncommutative scalar field theories. The key point of this procedure is the deformation of the symplectic structure of the phase space of classical fields. Having constructed the Poisson brackets, we quantized the model under consideration following the standard canonical scheme thanks to the so called dressing transformation (45-46). An interesting feature of this approach lies on the deformation of the symplectic structure which is introduced before the quantization process. The present results can be extended in various directions. In particular, we believe that this approach can be adapted to the theory of noncommutative chiral fields in two dimension in relation with fractional quantum Hall effect. Indeed, it is well established that for an incompressible quantum Hall droplet, the edges excitations are described by a chiral scalar field. In this respect, the quantum theory of noncommutative chiral fields can provides us with an unified scheme to classify different Hall hierarchies and can bring new fractional filling factors with interesting physical consequences. We hope to report on this issue in a forthcoming work.

Acknowledgments

MD would like to thank the hospitality and kindness of Condensed Matter and Statistical Physics section of Abdus Salam International Centre for Theoretical Physics (AS-ICTP) where this work was done. The authors are indebted to the referee for his constructive comment.

References

[1] M. R. Douglas and N. A. Neskrasov, Noncommutative field theory, Rev. Mod. Phys. 73 (2001) 977 [hep-th/0106048].

[2] R. J. Szabo, Quantum field theory on noncommutative spaces, Phys. Rep. 378 (2003) 207, [hep-th/0109162].

[3] N. Seiberg and E. Witten, String theory and noncommutative geometry, JHEP 9909 (1999) 032 [hep-th/9908142].
[4] S. Doplicher, K. Fredenhagen, and J. E. Roberts, *The quantum structure of space-time at the planck scale and quantum fields*, Commun. Math. Phys. **172** (1995) 187–220, [hep-th/0303037].

[5] V. P. Nair, *Quantum mechanics on a noncommutative brane in m(atrix) theory*, Phys. Lett. **B505** (2001) 249–254, [hep-th/0008027].

[6] V. P. Nair and A. P. Polychronakos, *Quantum mechanics on the noncommutative plane and sphere*, Phys. Lett. **B505** (2001) 267, [hep-th/0011172].

[7] R. Jackiw, V. P. Nair, *Anyon spin and the exotic central extension of the planar Galilei group*, Phys. Lett. **B480** (2000) 237, [hep-th/0003130].

[8] C. Duval, P. A. Horvathy, *Anyons with anomalous gyromagnetic ratio and the Hall effect*, Phys.Lett. **B594** (2004) 402, [hep-th/0402191]; *Non-commuting coordinates, exotic particles, anomalous anyons in the Hall effect*, Theor.Math.Phys. **144** (2005) 899, [hep-th/0407010].

[9] J. M. Carmona, J. L. Cortes, J. Gamboa, and F. Mendez, *Noncommutativity in field space and lorentz invariance violation*, Phys. Lett. **B565** (2003) 222–228, [hep-th/0207158]; *Quantum theory of noncommutative fields*, JHEP **03** (2003) 058, [hep-th/0301248].

[10] D. Karabali and V. P. Nair, *Edge States for quantum Hall droplets in higher dimensions and a generalized WZW model*, Nucl.Phys. **B697** (2004) 513 [hep-th/0403111]; *The effective action for edge states in higher dimensional quantum Hall systems*, Nucl.Phys. **B679** (2004) 427, [hep-th/0307281].

[11] M. Daoud and A. Jellal, *Effective Wess-Zumino-Witten Action for Edge States of Quantum Hall Systems on Bergman Ball*, Nucl.Phys. **B764** (2007) 109 [hep-th/0605289]; *Quantum Hall Droplets on Disc and Effective Weiss-Zumino-Witten Action for Edge States*, Int. Jour. Geom. Meth. Mod. Phys **4** (2007) 1187 [hep-th/0605290]; *Quantum Hall Effect on the Flag Manifold F_2*, [hep-th/0610157].

[12] A. P. Polychronakos, *Chiral actions from phase space (quantum Hall) droplets*, Nucl. Phys. **B705** (2005) 457, [hep-th/0408194]; *Kac-Moody theories for colored phase space (quantum Hall) droplets*, Nucl. Phys. **B711** (2005) 505, [hep-th/0411065].

[13] R. Oeckl, *Braided quantum field theory*, Commun. Math. Phys. **217** (2001) 451, [hep-th/9906225]; *Untwisting noncommutative \( \mathbb{R}^d \) and the equivalence of quantum field theories*, Nucl. Phys. **B581** (2000) 559, [hep-th/0003018].

[14] A. P. Balachandran, G. Mangano, A. Pinzul, and S. Vaidya, *Spin and statistics on the Groenenwold-Moyal plane: Pauli-forbidden levels and transitions*, Int. J. Mod. Phys. **A21** (2006) 3111–3126, [hep-th/0508002].
[15] A. P. Balachandran, A. Pinzul, and B. A. Qureshi, *UV-IR mixing in non-commutative plane*, Phys. Lett. B634 (2006) 434, [hep-th/0508151].

[16] A. P. Balachandran, T. R. Govindarajan, G. Mangano, A. Pinzul, B. A. Qureshi, and S. Vaidya, *Statistics and UV-IR mixing with twisted Poincaré invariance*, Phys. Rev. D75 (2007) 045009, [hep-th/0608179].

[17] A. Pinzul, *Twisted Poincaré group and spin-statistics*, Int. J. Mod. Phys. A20 (2005) 6268–6277.

[18] G. Fiore and J. Wess, *On ‘full twisted poincaré’ symmetry and qft on moyaal-weyl spaces*, [hep-th/0701078].

[19] M. Chaichian, P. P. Kulish, K. Nishijima, and A. Tureanu, *On a Lorentz-invariant interpretation of noncommutative space-time and its implications on noncommutative QFT*, Phys. Lett. B604 (2004) 98–102, [hep-th/0408069].

[20] J. M. Carmona, J. L. Cortes, J. Gamboa, F. Mendez, *Noncommutativity in Field Space and Lorentz Invariance Violation*, Phys.Lett. B565 (2003) 222, [hep-th/0207158]; *Quantum Theory of Noncommutative Fields*, JHEP 0303 (2003) 058, [hep-th/0301248 ].

[21] P. Arias, A. Das, J. Gamboa, J. Lopez-Sarrion, F. Mendez *CPT/Lorentz Invariance Violation and Neutrino Oscillation*, [hep-ph/0608007].

[22] J. M. Carmona, J. L. Cortes, A. Das, J. Gamboa, F. Mendez *Matter-antimatter asymmetry without departure from thermal equilibrium* Mod. Phys. Lett. A21 (2006) 883, [hep-th/0410143].

[23] A. Das, J. Gamboa, F. Mendez, J. Lopez-Sarrion *Chiral bosonization for non-commutative fields* JHEP 0405 (2004) 022, [hep-th/0402001].

[24] A. P. Balachandran, A. R. Queiroz, A. M. Marques, P. Teotonio-Sobrinho, *Quantum Fields with Noncommutative Target Spaces*, [arXiv:0706.0021].

[25] C. Duval and P. A. Horvathy, *The exotic Galilei group and the ”Peierls subsitution”*, Phys. Lett. B 479 (2000) 284, [hep-th/0002233].

[26] C. Duval and P. A. Horvathy, *Exotic galilean symmetry in the non-commutative plane, and the Hall effect*, J. Phys. A 34 (2001) 10097, [hep-th/0106089].

[27] P. A. Horvathy, *The non-commutative Landau problem*, Ann. Phys. (N. Y) 299 (2002) 128, [hep-th/0201007].