UNIVERSAL DEFORMATION RINGS OF STRINGS MODULES OVER A CERTAIN SYMMETRIC TAME ALGEBRA

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Abstract. Let \( k \) be an algebraically closed field, let \( \Lambda \) be a finite dimensional \( k \)-algebra and let \( V \) be a \( \Lambda \)-module with stable endomorphism ring isomorphic to \( k \). If \( \Lambda \) is self-injective then \( V \) has a universal deformation ring \( R(\Lambda, V) \), which is a complete local commutative Noetherian \( k \)-algebra with residue field \( k \). Moreover, if \( \Lambda \) is also a Frobenius \( k \)-algebra then \( R(\Lambda, V) \) is stable under syzygies. We use these facts to determine the universal deformation rings of string \( \Lambda_{r} \)-modules whose stable endomorphism ring isomorphic to \( k \), where \( \Lambda_{r} \) is a symmetric special biserial \( k \)-algebra that has quiver with relations depending on the four parameters \( r = (r_{0}, r_{1}, r_{2}, k) \) with \( r_{0}, r_{1}, r_{2} \geq 2 \) and \( k \geq 1 \). Universal deformation rings and Frobenius algebras and Stable endomorphism rings and Special biserial algebras [2000]16G10 and 16G20 and 20C20

1. Introduction

Let \( k \) be a field of arbitrary characteristic, and let denote by \( \hat{\mathcal{C}} \) the category of all complete local commutative Noetherian \( k \)-algebras with residue field \( k \). Suppose that \( \Lambda \) is a fixed finite dimensional \( k \)-algebra and let \( V \) be a finitely generated \( \Lambda \)-module. Let \( R \) be an arbitrary object in \( \hat{\mathcal{C}} \). A lift \((M, \phi)\) of \( V \) over \( R \) is a finitely generated \( R \otimes_k \Lambda \)-module \( M \) that is free over \( R \) together with an isomorphism of \( \Lambda \)-modules \( \phi : k \otimes_R M \to V \). If \( \Lambda \) is self-injective and the stable endomorphism ring of \( V \) is isomorphic to \( k \), then there exists a particular object \( R(\Lambda, V) \) in \( \hat{\mathcal{C}} \) and a lift \((U(\Lambda, V), \phi_{U(\Lambda, V)})\) of \( V \) over \( R(\Lambda, V) \), which is universal with respect to all isomorphism classes of lifts of \( V \) over such \( k \)-algebras \( R \) (see [10] and §2). The ring \( R(\Lambda, V) \) and the isomorphism class of the lift \((U(\Lambda, V), \phi_{U(\Lambda, V)})\) are respectively called the universal deformation ring and the universal deformation of \( V \). Traditionally, universal deformations rings are studied when \( \Lambda \) is equal to a group algebra \( kG \), where \( G \) is a finite group and \( k \) has positive characteristic \( p \) (see e.g., [3, 4, 5, 6, 7, 8, 9]). In particular, it was proved by F. M. BLEHER and T. CHINBURG in [4] that if \( V \) is a finitely generated \( kG \)-module whose stable endomorphism ring is isomorphic to \( k \), then \( V \) has a universal deformation ring \( R(G, V) \). Observe that \( kG \) is an example of a self-injective \( k \)-algebra (see e.g., [2, Prop. 3.1.2]). This approach has recently led to the solution of various open problems, e.g., the construction of representations whose universal deformation rings are not local complete intersections (see [3, 5, 6]). Universal deformation rings of modules over more general finite dimensional algebras have been studied by many authors in different contexts (see e.g., [14, 18] and their references). The main motivation of this article is that sophisticated results from representation theory of finite dimensional algebras, such as Auslander-Reiten quivers, stable equivalences, and combinatorial description of modules can be used to arrive at a deeper understanding of universal deformation rings.

In this article, we assume that \( k \) is algebraically closed and consider the basic \( k \)-algebra

\[
(1) \quad \Lambda_{\bar{r}} = kQ/I_{\bar{r}}
\]

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where \( \bar{r} = (r_0, r_1, r_2, k) \) with \( r_0, r_1, r_2 \geq 2, k \geq 1 \), \( Q \) is the quiver

\[
Q = \begin{array}{ccc}
0 & \tau_0 & 1 \\
\tau_2 & 1 & \tau_1 \\
& \tau_1 & \tau_2 \\
\end{array}
\]

and \( I_r \) is the ideal of the path algebra \( kQ \) generated by the relations

\[
\{ \tau_0 \zeta_0, \zeta_1 \tau_1, \tau_1 \zeta_1, \tau_2 \zeta_2, \zeta_0 \tau_2, \zeta_0 \tau_0 - (\tau_2 \tau_1 \tau_0)^k, \zeta_1^3 - (\tau_2 \tau_2 \tau_1)^k, \zeta_2^2 - (\tau_1 \tau_2 \tau_1)^k \}.
\]

The algebra \( \Lambda_\bar{r} \) is among the class of algebras of dihedral type, which were introduced by K. Erdmann in [13] to classify all tame blocks of group algebras of finite groups with dihedral defect groups up to Morita equivalence. However, \( \Lambda_\bar{r} \) is not Morita equivalent to a block of a group algebra (see [13, Lemma IX.5.4]). Since \( \Lambda_\bar{r} \) is a special biserial algebra, all the non-projective indecomposable \( \Lambda_\bar{r} \)-modules can be described combinatorially as so-called strings and bands modules as introduced in [11] (see also §3.1). We denote by \( \Gamma_\bar{r}(\Lambda_\bar{r}) \) the stable Auslander-Reiten quiver of \( \Lambda_\bar{r} \). The components of \( \Gamma_\bar{r}(\Lambda_\bar{r}) \) consisting in string modules are two 3-tubes and infinitely many components of type \( \mathbb{Z} \mathbb{A}_\infty^\infty \). The components consisting of band modules are infinitely many 1-tubes.

In [10], the particular case \( \bar{r} = (2, 2, 2, 1) \) has been considered. In particular, there are exactly three components \( \mathcal{C} \) of \( \Gamma_\bar{r}(\Lambda_\bar{r}) \) of type \( \mathbb{Z} \mathbb{A}_\infty^\infty \), each contain a simple \( \Lambda_\bar{r} \)-module. If \( \mathcal{C} \) is such a component then \( \Omega(\mathcal{C}) = \mathcal{C} \) and there are exactly three \( \Omega \)-orbits of \( \Lambda(2,2,2,1) \)-modules in \( \mathcal{C} \) whose stable endomorphism ring is isomorphic to \( k \); the universal deformation rings are either isomorphic to \( k \), or to \( k[[t]]/(t^2) \), or to \( k[[t]] \) (see [10, Prop. 3.9]). Moreover, if \( \mathcal{T} \) is one 3-tubes of \( \Gamma_\bar{r}(\Lambda(2,2,2,1)) \) then \( \Omega(\mathcal{T}) \) is the other 3-tube and there are exactly three \( \Omega \)-orbits of \( \Lambda(2,2,2,1) \)-modules in \( \mathcal{T} \) whose stable endomorphism ring is isomorphic to \( k \); the universal deformation rings are either isomorphic to \( k \) or to \( k[[t]] \) (see [10, Prop. 3.11]).

In this article, we let \( \bar{r} = (r_0, r_1, r_2, k) \) with \( r_0, r_1, r_2 \geq 2 \) and \( k \geq 1 \) be arbitrary. We study the two 3-tubes and the components \( \mathcal{C} \) of \( \Gamma_r(\Lambda_r) \) of type \( \mathbb{Z} \mathbb{A}_\infty^\infty \) containing a module whose endomorphism ring is isomorphic to \( k \). Our goal is to investigate how universal deformation rings change when inflating modules from \( \Lambda_{(r_0, r_1, r_2, k)} \) to \( \Lambda_{(r_0', r_1', r_2', k') \in } \), where \( \Lambda_{(r_0', r_1', r_2', k') \in } \) surjects onto \( \Lambda_{(r_0, r_1, r_2, k)} \) when \( r_0' \geq r_0, r_1' \geq r_1, r_2' \geq r_2, k' \geq k \).

If \( M \) and \( N \) are two indecomposable \( \Lambda_r \)-modules belonging to the same component of \( \Gamma_r(\Lambda_r) \), we say that \( N \) is a successor of \( M \) provided that there exists an irreducible homomorphism \( M \rightarrow N \). Throughout this article, we identify the vertices of the quiver \( Q \) with elements of the cyclic group with three elements \( \mathbb{Z}/3 \).

A summary of the main results concerning \( \Lambda_r = kQ/I_r \) is as follows (cf. [10, Prop. 3.9, Prop. 3.11]); for more precise statements, see Propositions 4.1, 4.4, 4.5 and 4.6.

**Theorem 1.1.** Let \( \Lambda_r \) be as in (1) where \( \bar{r} = (r_0, r_1, r_2, k) \) with \( r_0, r_1, r_2 \geq 2 \) and \( k \geq 1 \), and let \( \Gamma_r(\Lambda_r) \) denote the stable Auslander- Reiten quiver of \( \Lambda_r \).

(i) If \( \mathcal{T} \) is one of two the 3-tubes then \( \Omega(\mathcal{T}) \) is the other 3-tube. There are exactly three \( \Omega \)-orbits of modules in \( \mathcal{T} \cup \Omega(\mathcal{T}) \) whose stable endomorphism ring is isomorphic to \( k \). If \( X_0 \) is a module that belongs to the boundary of \( \mathcal{T} \), then these three \( \Omega \)-orbits are represented by \( X_0 \), by a successor \( X_1 \) of \( X_0 \), and by a successor \( X_2 \) of \( X_1 \) that does not lie in the \( \Omega \)-orbit of \( X_0 \). The universal deformation rings are

\[
R(\Lambda_r, X_0) \cong k, \quad R(\Lambda_r, X_1) \cong k, \quad R(\Lambda_r, X_2) \cong k[[t]].
\]

(ii) There are exactly three distinct components \( \mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2 \) of \( \Gamma_r(\Lambda_r) \) of type \( \mathbb{Z} \mathbb{A}_\infty^\infty \), which each contain exactly one simple \( \Lambda_r \)-module. For all \( i \in \{0, 1, 2\} \mod 3 \), the component \( \mathfrak{A}_i \) is \( \Omega \)-stable if and only if \( r_i = 2 \) and there are exactly three \( \Omega \)-orbits of modules in \( \mathfrak{A}_i \cup \Omega(\mathfrak{A}_i) \) whose stable endomorphism ring is isomorphic to \( k \). If for all \( i \in \{0, 1, 2\} \mod 3 \), \( U_{i,0} \) denotes the unique simple module lying in \( \mathfrak{A}_i \), then these three \( \Omega \)-orbits are represented by \( U_{i,0} \), by a successor \( U_{i,1} \)
of $U_{i,0}$, and by a successor $U_{i,2}$ of $U_{i,1}$ that does not lie in the $\Omega$-orbit of $U_{i,0}$. The universal deformation rings are

$$R(\Lambda, U_i, 0) \cong k[[t]]/(t^i), \quad R(\Lambda, U_i, 1) \cong k, \quad R(\Lambda, U_i, 2) \cong k[[t]].$$

(iii) There are three distinct components $\mathcal{B}_0, \mathcal{B}_1$ and $\mathcal{B}_2$ of $\Gamma_s(\Lambda_\infty)$ of type $\mathcal{Z}A_{\infty}$ that contain exactly a module of length 1 whose endomorphism ring is isomorphic to $k$. Let $i \in \{0, 1, 2\} \mod 3$ and let $V_i, 0$ be a module of minimal length in $\mathcal{B}_i$. If $k = 1$ then $\mathcal{B}_i = \Omega(\mathcal{A}_{i+2})$, where $\mathcal{A}_{i+2}$ is as in (ii). In particular, $\mathcal{B}_i$ is $\Omega$-stable if and only if $k = 1$ and $i + 2 = 2$. There are exactly three $\Omega$-orbits of modules in $\mathcal{B}_i \cup \Omega(\mathcal{B}_i)$ whose stable endomorphism ring is isomorphic to $k$. These three $\Omega$-orbits are represented by $V_i, 0$, by a successor $V_{i, 1}$ of $V_i, 0$, and by a successor $V_{i, -1}$ of $V_i, 0$ that does not lie in the $\Omega$-orbit of $V_{i, 1}$. If $k = 1$ then the universal deformation rings are

$$R(\Lambda, V_i, 0) \cong k, \quad R(\Lambda, V_i, 1) \cong k[[t]]/(t^{i+2}), \quad R(\Lambda, V_i, -1) \cong k[[t]].$$

If $k \geq 2$ then the universal deformation rings are

$$R(\Lambda, V_i, 0) \cong k, \quad R(\Lambda, V_i, 1) \cong k[[t]], \quad R(\Lambda, V_i, -1) \cong k[[t]].$$

(iv) If $k \geq 2$ then there are three distinct components $\mathcal{C}_0, \mathcal{C}_1$ and $\mathcal{C}_2$ of $\Gamma_s(\Lambda_\infty)$ of type $\mathcal{Z}A_{\infty}$, which each contain a module of length 2 whose endomorphism rings is isomorphic to $k$. Let $i \in \{0, 1, 2\} \mod 3$ and let $W_i, 0$ be a module of minimal length in $\mathcal{B}_i$. For all $i \in \{0, 1, 2\} \mod 3$, the component $\mathcal{C}_i$ is $\Omega$-stable if and only if $k = 2$, and there are exactly three $\Omega$-orbits of modules in $\mathcal{C}_i$ whose stable endomorphism ring is isomorphic to $k$. These three $\Omega$-orbits are represented by $W_i, 0$, by a successor $W_{i, -1}$ of $W_i, 0$, and by a successor $W_{i, -2}$ of $W_{i, -1}$ that does not lie in the $\Omega$-orbit of $W_{i, 0}$. The universal deformation rings are

$$R(\Lambda, W_i, 0) \cong k[[t]]/(t^i), \quad R(\Lambda, W_i, -1) \cong k, \quad R(\Lambda, W_i, -2) \cong k[[t]].$$

This article is organized as follows. In §2, we recall the definitions of deformations and universal deformation rings and summarize some of their properties. In §3, we give a precise description of string modules for $\Lambda$, describe the components of $\Gamma_s(\Lambda_\infty)$ that contain string modules using hooks and co-hooks (see [11]), and give a description of the homomorphisms between string modules as determined in [15]. Moreover, we describe the indecomposable projective modules for $\Lambda$ and classify all $\Lambda_\infty$-modules with endomorphism ring isomorphic to $k$ (see Proposition 3.1). In §4, we prove Theorem 1.1.

See e.g., [1, 2, 13] for further information about basic concepts from representation theory of finite dimensional algebras, such as the definition and properties of the syzygy functor $\Omega$ and the definition of the Auslander-Reiten quiver of an arbitrary Artinian algebra $\Lambda$.

## 2. Universal Deformation Rings

Let $k$ be a field of arbitrary characteristic and denote by $\hat{C}$ the category of all complete local commutative Noetherian $k$-algebras with residue field $k$. Note that the morphisms in $\hat{C}$ are continuous $k$-algebra homomorphisms that induce the identity map on $k$. Suppose that $\Lambda$ is a finite dimensional $k$-algebra and $V$ is a fixed finitely generated $\Lambda$-module. We denote by $\text{End}_\Lambda(V)$ (respectively, by $\text{End}_\Lambda(V)$) the endomorphism ring (respectively, the stable endomorphism ring) of $V$. Let $R$ be an arbitrary object in $\hat{C}$. A lift $(M, \phi)$ of $V$ over $R$ is a finitely generated $R \otimes_k \Lambda$-module $M$ that is free over $R$ together with an isomorphism of $\Lambda$-modules $\phi : k \otimes_R M \to V$. Two lifts $(M, \phi)$ and $(M', \phi')$ over $R$ are isomorphic if there exists an $R \otimes_k \Lambda$-module isomorphism $f : M \to M'$ such that $\phi' \circ (\text{id}_k \otimes f) = \phi$, where $\text{id}_k$ denotes the identity map on $k$. If $(M, \phi)$ is a lift of $V$ over $R$ we denote by $[M, \phi]$ its isomorphism class and say that $[M, \phi]$ is a deformation of $V$ over $R$. Let us define $\text{Def}_\Lambda(V, R)$ the set of all deformations of $V$ over $R$. The deformation functor over $V$ is the covariant functor $\hat{F}_V : \hat{C} \to \text{Sets}$ defined as follows: for all objects $R$ in $\hat{C}$ define $\hat{F}_V(R) = \text{Def}_\Lambda(V, R)$ and for all morphisms $\alpha : R \to R'$ in $\hat{C}$ let $\hat{F}_V(\alpha) : \text{Def}_\Lambda(V, R) \to \text{Def}_\Lambda(V, R')$ be defined as $\hat{F}_V(\alpha)([M, \phi]) = [R' \otimes_{R, \alpha} M, \phi_\alpha]$, where $\phi_\alpha : k \otimes_{R'} (R' \otimes_{R, \alpha} M) \to V$ is the composition of $\Lambda$-module isomorphisms

$$k \otimes_{R'} (R' \otimes_{R, \alpha} M) \cong k \otimes_R M \overset{\phi}{\to} V.$$
Suppose there exists an object $R(\Lambda, V)$ in $\mathcal{C}$ and a deformation $[U(\Lambda, V), \phi_{U(\Lambda, V)}]$ of $V$ over $R(\Lambda, V)$ with the following property. For each $R$ in $\mathcal{C}$ and for all lifts $M$ of $V$ over $R$ there exists a morphism $v : R(\Lambda, V) \to R$ in $\mathcal{C}$ such that
\[
\hat{F}_V(v)[U(\Lambda, V), \phi_{U(\Lambda, V)}] = [M, \phi],
\]
and moreover $v$ is unique if $R$ is the ring of dual numbers $k[[t]]/(t^2)$. Then $R(\Lambda, V)$ and $[U(\Lambda, V), \phi_{U(\Lambda, V)}]$ are respectively called the \textit{versal deformation ring} and \textit{versal deformation} of $V$. If the morphism $v$ is unique for all $R$ in $\mathcal{C}$ and lifts $(\Lambda, \phi)$ of $V$ over $R$, then $R(\Lambda, V)$ and $[U(\Lambda, V), \phi_{U(\Lambda, V)}]$ are respectively called the \textit{universal deformation ring} and the \textit{universal deformation} of $V$. In other words, the universal deformation ring $R(\Lambda, V)$ represents the deformation functor $\hat{F}_V$ in the sense that $\hat{F}_V$ is naturally isomorphic to the Hom functor $\text{Hom}_{\Lambda}(R(\Lambda, V), -)$. Using Schlessinger’s criteria [17, Thm. 2.11] and using methods similar to those in [16], it is straightforward to prove that the deformation functor $\hat{F}_V$ is continuous, that every finitely generated $\Lambda$-module $V$ has a versal deformation ring and that this versal deformation is universal provided that the endomorphism ring $\text{End}_{\Lambda}(V)$ is isomorphic to $k$ (see [10, Prop. 2.1]).

Recall that the $k$-algebra $\Lambda$ is said to be self-injective if the regular left $\Lambda$-module $\Lambda$ is injective and that $\Lambda$ is called a Frobenius algebra provided that the right $\Lambda$-modules $\Lambda^\ast = \text{Hom}_k(\Lambda, k)$ are isomorphic. Recall also that $\Lambda$ is said to be a symmetric algebra provided that $\Lambda$ is a Frobenius algebra and there exists a non-degenerate associative bilinear form $\theta : \Lambda \times \Lambda \to k$ with $\theta(ab, c) = \theta(b, ac)$ for all $a, b, c \in \Lambda$. By [12, Prop. 9.9], every Frobenius algebra is self-injective.

Remark 2.1. If $\Lambda$ is self-injective and $(\Lambda, \phi)$ is a lift of $\Lambda$ over an object $R$ in $\mathcal{C}$ with $\text{End}_\Lambda(V) \cong k$, then the deformation $[M, \phi]$ does not depend on the particular choice of the $\Lambda$-module isomorphism. More precisely, if $f : M \to M'$ is an $R \otimes_k \Lambda$-module isomorphism with $(M', \phi')$ a lift of $V$ over $R$, then there exists an $R \otimes_k \Lambda$-module isomorphism $\hat{f} : M \to M'$ such that $\phi' \circ (\text{id}_k \otimes R \hat{f}) = \phi$. In other words, $[M, \phi] = [M', \phi']$ in $\hat{F}_V(R) = \text{Def}_\Lambda(V, R)$ (see [10, Thm. 2.6]).

We denote the first syzygy of $V$ by $\Omega V$, i.e., $\Omega V$ is the kernel of a projective cover $P_V \to V$, (see e.g., [1, pp. 124-126]).

Example 2.2. Let $G$ be a finite group and consider the group algebra $kG$, which is a self-injective $k$-algebra (see e.g., [2, Prop. 3.1.2] and [12, Prop. 9.6]). It was proved in [4] that if $V$ is a finitely generated $kG$-module whose stable endomorphism ring is isomorphic to $k$ then $V$ has a universal deformation ring $R(kG, V)$. Moreover, the stable endomorphism ring of $\Omega V$ is also isomorphic to $k$ and the universal deformation rings $R(kG, V)$ and $R(kG, \Omega V)$ of $V$ and $\Omega V$, respectively, are isomorphic.

The following result generalizes the properties of universal deformation rings mentioned in Example 2.2 to arbitrary Frobenius $k$-algebras (see [10, Thm. 2.6]).

Theorem 2.3. Let $\Lambda$ be a finite dimensional self-injective $k$-algebra, and suppose that $V$ is a finitely generated $\Lambda$-module whose stable endomorphism ring $\text{End}_\Lambda(V)$ is isomorphic to $k$.

(i) The module $V$ has a universal deformation ring $R(\Lambda, V)$.

(ii) If $P$ is a finitely generated projective $\Lambda$-module, then $\text{End}_\Lambda(V \oplus P) \cong k$ and $R(\Lambda, V) \cong R(\Lambda, V \oplus P)$.

(iii) If $\Lambda$ is also a Frobenius algebra, then $\text{End}_\Lambda(\Omega V) \cong k$ and $R(\Lambda, V) \cong R(\Lambda, \Omega V)$.

3. Some Remarks about the Representation Theory of $\Lambda_\varphi$ and Classification of $\Lambda_\varphi$-modules whose Endomorphism Ring is Isomorphic to $k$

For the remainder of this article, let $k$ be an algebraically closed field of arbitrary characteristic and let $\Lambda_\varphi = k Q / I_\varphi$ as in (1). We identify the vertices of $Q$ with elements of $\mathbb{Z} / 3$ (the cyclic group of three elements).

The algebra $\Lambda_\varphi$ is one of the algebras of dihedral type studied by K. Erdmann in [13]. In particular, $\Lambda_\varphi$ is a symmetric $k$-algebra. However, by [13, Lemma IX.5.4], $\Lambda_\varphi$ is not Morita equivalent to a block of a group algebra. Since $\Lambda_\varphi$ is a special biserial algebra, all the non-projective indecomposable $\Lambda_\varphi$-modules can be described combinatorially as so-called strings and bands modules (see [11]). In this article, we are only concerned about these string modules, which are described as follows.
3.1. String modules for $\Lambda_f$. Given each arrow $\zeta_0, \zeta_1, \tau_1, \zeta_2, \tau_2$ of $Q$, we define a formal inverse by $\zeta_0^{-1}, \zeta_1^{-1}, \tau_1^{-1}, \zeta_2^{-1}, \tau_2^{-1}$, respectively. Let $s(\zeta_0) = 0 = s(\zeta_1^{-1}), s(\tau_0) = 0 = s(\tau_2^{-1}), s(\zeta_1) = 1 = s(\zeta_2^{-1}), s(\tau_1) = 1 = s(\zeta_0^{-1}), s(\zeta_2) = 2 = s(\zeta_1^{-1})$ and $s(\tau_2) = 2 = s(\tau_1^{-1})$. Let $e(\zeta_0) = 0 = e(\zeta_1^{-1}), e(\tau_0) = 1 = e(\zeta_2^{-1}), e(\zeta_1) = 1 = e(\zeta_1^{-1}), e(\tau_1) = 2 = e(\tau_2^{-1}), e(\zeta_2) = 2 = e(\zeta_2^{-1})$ and $e(\tau_2) = 0 = e(\tau_1^{-1})$. By a word of length $n \geq 1$ we mean a sequence $w_n \cdots w_1$, where the $w_j$ is either an arrow or a formal inverse of an arrow and where $s(w_j+1) = e(w_j)$ for $1 \leq j \leq n - 1$. We define $(w_n \cdots w_1)^{-1} = w_1^{-1} \cdots w_n^{-1}, s(w_n \cdots w_1) = s(w_1)$ and $e(w_n \cdots w_1) = e(w_n)$. If $i \in \{0, 1, 2\}$ mod 3 is a vertex of $Q$, we define an empty word $1_i$ of length zero with $e(1_i) = i = s(1_i)$ and $(1_i)^{-1} = 1_i$. For all $i \in \{0, 1, 2\}$ mod 3 there exists a path in $Q$ of length 3 starting and ending at $i$, namely

$$L_i = \tau_i + 2\tau_{i+1}\tau_i.$$  

Denote by $W$ the set of all words and let

$$J = \{\zeta_0^0, \zeta_1^2, \zeta_2^3, \tau_0\zeta_0, \zeta_1\tau_0, \tau_1\zeta_1, \tau_2\zeta_2, \zeta_0\tau_2, \zeta_2^k, \zeta_1^k, \tau_2^k\}.$$  

Let $\sim$ be the equivalence relation on $W$ defined by $w \sim w'$ if and only if $w = w'$ or $w^{-1} = w'$. A string is a representative $C$ of an equivalence class under the relation $\sim$ where either $C = 1_i$ for some vertex $i$ of $Q$, or $C = w_n \cdots w_1$ with $n \geq 1$ and $w_j \neq w_{j+1}^{-1}$ for $1 \leq j \leq n - 1$ and no sub-word of $C$ or its formal inverse belong to $J$. If $C$ is a string such that $s(C) = e(C)$, then we let $C^0 = 1_{s(C)}$. If $C = w_n \cdots w_1$ and $D = v_m \cdots v_1$ are strings of length $n, m \geq 1$, respectively, we say that the composition of $C$ and $D$ is defined provided $w_n \cdots w_1v_m \cdots v_1$ is a string, and write $CD = w_n \cdots w_1v_m \cdots v_1$; we say that the composition of $C$ with $1_i$ is defined provided $s(C) = i$ (respectively, $e(C) = i$), and in this case we have $C1_i \sim C$ (respectively, $1_iC \sim C$). In particular, if $C = w_n \cdots w_1$ is a string of length $n \geq 1$ then $C \sim w_n \cdots w_{j+1}1_{e(w_j)}w_j^{-1} \cdots w_1$ for all $1 \leq j \leq n - 1$. If $C = w_n \cdots w_1$ is a string of length $n \geq 1$ then there exists and indecomposable $\Lambda_f$-module $M[C]$, called the string module corresponding to the string representative $C$, which can be described as follows. There is an ordered $k$-basis $\{z_0, z_1, \ldots, z_n\}$ of $M[C]$ such that the action of $\Lambda_f$ on $M[C]$ is given by the following representation $\varphi_C : \Lambda_f \to \text{Mat}(n + 1, k)$. Let $v(j) = e(w_j)$ for $0 \leq j \leq n - 1$ and $v(n) = s(w_n)$. Then for each vertex $i \in \{0, 1, 2\}$ mod 3 and for each arrow $\zeta \in \{\zeta_0, \zeta_1, \tau_1, \zeta_2, \tau_2\}$ in $Q$ and for all $0 \leq j \leq n$ define

$$(3) \quad \varphi_C(i)(z_j) = \begin{cases} z_j, & \text{if } v(j) = i \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \varphi_C(\zeta)(z_j) = \begin{cases} z_{j-1}, & \text{if } w_j = \zeta \\ z_{j+1}, & \text{if } w_{j+1} = \zeta^{-1} \\ 0, & \text{otherwise} \end{cases}$$

We call $\varphi_C$ the canonical representation and $\{z_0, z_1, \ldots, z_n\}$ a canonical $k$-basis for $M[C]$ relative to the string representative $C$. Note that $M[C] \cong M[C^{-1}]$. If $C = 1_i$ with $i \in \{0, 1, 2\}$ mod 3 then $M[C]$ is the simple $\Lambda_f$-module corresponding to vertex $i$. We denote the simple $\Lambda_f$-modules corresponding to the vertices 0, 1 and 2 of $Q$ by $M[1_0], M[1_1]$ and $M[1_2]$, respectively.

3.2. The stable Auslander-Reiten quiver of $\Lambda_f$. We denote by $\Gamma_s(\Lambda_f)$ the stable Auslander-Reiten quiver of $\Lambda_f$ (see [1, VII]). For all $i \in \{0, 1, 2\}$ mod 3 there exists a path in $Q$ of length 3 starting and ending at $i$, namely

$$L_i = \tau_i + 2\tau_{i+1}\tau_i.$$  

The components $\mathfrak{C}$ of $\Gamma_s(\Lambda_f)$ consisting of string $\Lambda_f$-modules are two 3-tubes and infinitely many non-periodic components of type $\mathbb{ZA}_\infty$. In the following, we describe the irreducible morphism between string $\Lambda_f$-modules.

Assume $C = w_nw_{n-1}\cdots w_1$ with $n \geq 1$ is a string. We say that $C$ is directed if all $w_j$ are arrows and we say that $C$ is a maximal directed string if $C$ is directed and if for any arrow $\zeta$ in $Q$, $\zeta C \in J$. Let $\mathcal{M}$ be the set of all maximal directed strings, i.e.,

$$\mathcal{M} = \{\zeta_0^{-1}, \zeta_1^{-1}, \zeta_2^{-1}, \tau_2^{-1}\tau_1, \zeta_0^{-1}, \zeta_2^{-1}\tau_1, \zeta_0^{-1}\tau_0\tau_2, \zeta_2^{-1}\tau_1\tau_0\}.$$  

Let $C$ be a string. We say that $C$ starts on a peak (respectively, starts in a deep) provided that there is no arrow $\zeta$ in $Q$ such that $C\zeta$ (respectively, $C\zeta^{-1}$) is a string; we also say that $C$ ends in a peak (respectively,
ends in a deep) provided that there is no arrow $\gamma$ in $Q$ such that $\gamma^{-1}C$ (respectively, $\gamma C$) is a string. If $C$ is a string not starting on a peak (respectively, not starting in a deep), say $C\zeta$ (respectively, $C\zeta^{-1}$) is a string for some arrow $\zeta$ then there is a unique directed string $D \in \mathcal{M}$ such that $C_b = C\zeta D^{-1}$ (respectively, $C_c = C\zeta^{-1} D$) is a string. We say $C_b$ (respectively, $C_c$) is obtained from $C$ by adding a hook (respectively, a co-hook) on the right side. Dually, if $C$ is a string not ending on a peak (respectively, not ending in a deep), say $\gamma^{-1}C$ (respectively, $\gamma C$) is a string for some arrow $\gamma$ in $Q$ then there is a unique directed string $E \in \mathcal{M}$ such that $h C = E\gamma^{-1}C$ (respectively, $c C = E^{-1}\gamma D$) is a string. We say $h C$ (respectively, $c C$) is obtained from $C$ by adding a hook (respectively, a co-hook) on the right side. By [11], all irreducible morphisms between string modules are either canonical injections $M[C] \to M[C_b]$, $M[C] \to M[h C]$, or canonical projections $M[C_c] \to M[C]$, $M[c C] \to M[C]$. Suppose $M[C]$ is a string module of minimal length such that $M[C]$ belongs to a component $\mathcal{E}$ of $\Gamma_s(\Lambda_F)$ of type $\mathbb{Z}_3^\infty$. Since none of the projective $\Lambda_F$-modules is uniserial then

![Figure 1. The stable Auslander-Reiten component near $M[C]$.](image)

near $M[C]$ the component $\mathcal{E}$ looks as in Figure 1.

### 3.3. Homomorphisms between string modules for $\Lambda_F$

Let $S$ and $T$ be strings for $\Lambda_F$. Suppose $C$ is a substring of both $S$ and $T$ such that the following conditions (i) and (ii) are satisfied.

(i) $S \sim BCD$, where $B$ is a substring which is either of length zero or $B = B'\zeta$ for an arrow $\zeta$, and $D$ is a substring which is either of length zero or $D = \gamma^{-1}D'$ for an arrow $\gamma$, i.e., $S \sim B' \xleftarrow{\zeta} C \xrightarrow{\gamma} D'$.

(ii) $T \sim ECF$, where $E$ is a substring which is either of length zero or $E = E'\epsilon^{-1}$ for an arrow $\epsilon$, and $F$ is a substring which is either of length zero or $F = \mu F'$ for an arrow $\mu$, i.e., $T \sim E' \xrightarrow{\epsilon} C \xleftarrow{\mu} F'$.

Then by [15] there exists a composition of $\Lambda_F$-module homomorphisms

\[(5) \quad \sigma_C : M[S] \to M[C] \to M[T].\]

We call $\sigma_C$ a canonical homomorphism from $M[S]$ to $M[T]$ that factors through $M[C]$. It follows from [15] that each $\Lambda_F$-module homomorphism from $M[S]$ to $M[T]$ can be written uniquely as a $k$-linear combination of canonical $\Lambda_F$-module homomorphisms as in (5). In particular, if $M[S] = M[T]$ then the canonical endomorphisms generate $\text{End}_{\Lambda_F}(M[S])$.

### 3.4. Projective Indecomposable $\Lambda_F$-modules and modules whose endomorphism ring is isomorphic to $k$

For all $i \in \{0, 1, 2\}$ mod $3$ vertex of $Q$, the radical series of the projective indecomposable
\(\Lambda_\ell\)-module \(P_i\) can be described as in the following figure.

\[
\begin{array}{c}
P_i = \begin{array}{c}
\mathcal{L}^{k-1} \\
\varepsilon \rightarrow \\
M[\mathbb{I}_i] \\
\downarrow \\
M[\mathbb{I}_{i+2}] \\
\downarrow \\
M[\mathbb{I}_{i+1}] \\
\downarrow \\
M[\mathbb{I}_i] \\
\end{array}
\end{array}
\begin{array}{c}
\tau_i \\
\tau_{i+1} \\
\tau_{i+2} \\
\tau_i \\
\varepsilon \\
\end{array}
\begin{array}{c}
\mathcal{L}^{k-1} = \\
\varepsilon \rightarrow \\
M[\mathbb{I}_i] \\
\downarrow \\
M[\mathbb{I}_{i+2}] \\
\downarrow \\
M[\mathbb{I}_{i+1}] \\
\downarrow \\
M[\mathbb{I}_i] \\
\end{array}
\begin{array}{c}
\zeta_i \rightarrow \\
\zeta_i \rightarrow \\
\zeta_i \rightarrow \\
\zeta_i \rightarrow \\
\zeta_i \rightarrow \\
\end{array}
\begin{array}{c}
\zeta_i \\
\zeta_i \\
\zeta_i \\
\zeta_i \\
\zeta_i \\
\end{array}
\end{array}
\]

The following result provides a classification of all \(\Lambda_\ell\)-modules whose endomorphism ring is isomorphic to \(k\).

**Proposition 3.1.** Let \(M[S]\) be a string \(\Lambda_\ell\)-module, where \(\bar{r} = (r_0, r_1, r_2, k)\) and \(r_0, r_1, r_2 \geq 2, k \geq 1\). Then \(M[S]\) has endomorphism ring isomorphic to \(k\) if and only if for some \(i \in \{0, 1, 2\} \mod 3\) the string representative \(S\) is equivalent either to \(\mathbb{I}_i\), or to \(\tau_i\), or to \(\tau_{i+1}\).

**Proof.** If \(S\) is equivalent either to one of the strings \(\mathbb{I}_0, \mathbb{I}_1\), or to \(\mathbb{I}_2\), then it follows from Schur’s Lemma that \(\text{End}_{\Lambda_\ell}(M[S]) \cong k\). If \(S\) is equivalent to one of the strings \(\tau_0, \tau_1, \tau_2, \tau_1\tau_0, \tau_2\tau_1, \text{ or } \tau_0\tau_2\) then the only canonical endomorphism in \(\text{End}_{\Lambda_\ell}(M[S])\) is the identity homomorphism, which implies that \(\text{End}_{\Lambda_\ell}(M[S])\) is one-dimensional over \(k\). Next assume that \(M[S]\) is a string \(\Lambda_\ell\)-module with endomorphism ring isomorphic to \(k\). Let denote by \(n\) the length of \(S\). If \(n = 0\) then \(S\) is equivalent either to \(\mathbb{I}_0\), or to \(\mathbb{I}_1\), or to \(\mathbb{I}_2\). If \(n = 1\) then \(S\) is equivalent to an arrow. By hypothesis, \(S\) is equivalent neither to \(\zeta_0\), nor to \(\zeta_1\), nor to \(\zeta_2\), for otherwise \(\text{dim}_k \text{End}_{\Lambda_\ell}(M[S]) \geq 2\). This implies that \(S\) is equivalent either to \(\tau_0\), or to \(\tau_1\), or to \(\tau_2\). For the remainder of the proof, assume that \(n \geq 2\) and let \(m\) be maximal such that the string representative \(S\) contains a substring equivalent to \(\zeta_i^{-m}\) for some \(i \in \{0, 1, 2\} \mod 3\), and put \(m = 0\) provided that \(S\) does not contain as substring any of the strings \(\zeta_0, \zeta_1, \zeta_2\) or any of their formal inverses. If \(m > 0\) then there exist suitable strings \(D\) and \(D'\) such that \(S \sim D\zeta_i^{-m}D'\). It follows from the maximality of \(m\) that the string \(\zeta_i^{-m}\) starts in a deep and ends on a peak. Therefore, there exists a non-trivial canonical endomorphism of \(M[S]\) factoring through \(M[\mathbb{I}_i]\) implying that \(\text{dim}_k \text{End}_{\Lambda_\ell}(M[S]) \geq 2\), which contradicts our hypothesis. Thus \(m = 0\), implying that \(S\) does not contain as substrings the arrows \(\zeta_0, \zeta_1, \text{ or } \zeta_2\) or any of their formal inverses. Thus, there exist \(i \in \{0, 1, 2\}\) mod 3 and an integer \(l \in \{0, \ldots, k-1\}\) such that either \(S \sim \mathcal{L}^{l\tau_i} \zeta_i^{-1}\) or \(S \sim \mathcal{L}^{l\tau_i+2}\). If \(l = 0\) then \(S\) is equivalent either to \(\tau_1\tau_0\), or to \(\tau_2\tau_1\), or to \(\tau_0\tau_2\). Assume then that \(l > 0\). If \(S \sim \mathcal{L}^{l\tau_i+2}\) (respectively, \(S \sim \mathcal{L}^{l\tau_i+2}\tau_i\tau_i+1\)) then there exists a non-trivial canonical endomorphism of \(M[S]\) factoring through \(M[\tau_{i+2}]\) (respectively, through \(M[\tau_{i+2}\tau_i+1]\)) implying that \(\text{dim}_k \text{End}_{\Lambda_\ell}(M[S]) \geq 2\), contradicting again our hypothesis. This finishes the proof of Proposition 3.1. \(\square\)

4. COMPONENTS OF \(\Gamma_s(\Lambda_\ell)\) OF TYPE \(\mathbb{Z}A^\infty\) CONTAINING A MODULE WHOSE ENDOMORPHISM RING IS ISOMORPHIC TO \(k\) AND 3-TUBES

For all \(i \in \{0, 1, 2\} \mod 3\) we define:

\[
\zeta_i = \tau_i \zeta_i^{-r_i+1} \\
\bar{\zeta}_i = \mathcal{L}^{l_i+2} \tau_i + \tau_i \zeta_i^{-1}
\]

4.1. Components of \(\Gamma_s(\Lambda_\ell)\) of type \(\mathbb{Z}A^\infty\) containing a module whose endomorphism ring is isomorphic to \(k\).
Proposition 4.1. For \( i \in \{0, 1, 2\} \) mod 3, let \( \mathfrak{A}_i \) be the component of the stable Auslander-Reiten quiver of \( \Lambda_r \) containing the simple \( \Lambda_r \)-module \( M[\mathfrak{1}_i] \), where \( \mathfrak{1}_i = (r_0, r_1, r_2, k) \) and \( r_0, r_1, r_2 \geq 2, k \geq 1 \). Define
\[
(1_i)_h = a_i + 2 \\
(1_i)hh = a_i + 2a_i + 1.
\]
The component \( \mathfrak{A}_i \) is \( \Omega \)-stable if and only if for \( r_i = 2 \). If \( k = 1 \) then the module \( M[r_{i+1}] \) lies in \( \Omega(\mathfrak{A}_i) \). The modules in \( \mathfrak{A}_i \cup \Omega(\mathfrak{A}_i) \) whose stable endomorphism rings are isomorphic to \( k \) are precisely the modules in \( \Omega \)-orbits of the modules \( U_0 = M[\mathfrak{1}_i] \), \( U_1 = M[(1_i)_h] \) and \( U_2 = M[(1_i)hh] \). Their universal deformation rings are
\[
R(\Lambda_r, U_0) \cong k[[t]]/(t^{r_i}), \\
R(\Lambda_r, U_1) \cong k, \\
R(\Lambda_r, U_2) \cong k[[t]].
\]

Proof. Let \( i \in \{0, 1, 2\} \) mod 3 be fixed. Using hooks and co-hooks (see §3.2), we see that all \( \Lambda_r \)-modules in \( \mathfrak{A}_i \cup \Omega(\mathfrak{A}_i) \) lie in the \( \Omega \)-orbit of either
\[
A_{0,0} = M[(a_i + 2a_i + 1)^q], \\
A_{0,1} = M[(a_i + 2a_i + 2)^q], \\
A_{0,2} = M[(a_i + 2a_i + 2)^q]^2, \\
B_{0,0} = M[(b_i + 1)^q], \\
B_{0,1} = M[(b_i + 1)^q]^2, \\
B_{0,2} = M[(b_i + 1)^q]^2.
\]
for some \( q \geq 0 \). Note for example that \( A_{0,0} = M[\mathfrak{1}_i] = B_{0,0}, A_{0,1} = M[(1_i)_h], A_{0,2} = M[(1_i)hh], B_{0,1} = M[(\mathfrak{1}_i)] \) and \( B_{0,2} = M[\mathfrak{1}_i] \). Since \( \Omega M[\mathfrak{1}_i] = M[\zeta^{-r_i} \mathfrak{1}_i^{k-1} r_i + 1] \) then \( \mathfrak{A}_i = \Omega(\mathfrak{A}_i) \).

Using §3.3 and the description of the projective indecomposable \( \Lambda_r \)-module \( P_i \) in (6), it is straightforward to show that the stable endomorphism ring of \( A_{0,j} \) is isomorphic to \( k \) for \( j \in \{0, 1, 2\} \) and that \( \text{Ext}^1_{\Lambda_r}(A_{0,j}, A_{0,j}) \) is isomorphic to \( k \) for \( j \in \{0, 2\} \) and zero for \( j = 1 \). On the other hand, for \( q \geq 1 \) and for \( j \in \{0, 1, 2\} \), the \( \Lambda_r \)-module \( A_{0,j} \) has a non-zero endomorphism which factors through \( M[\mathfrak{1}_i] \) and which does not factor through a projective \( \Lambda_r \)-module. Assume that \( r_i = 2 \). Since in this case \( \mathfrak{A}_i \) is \( \Omega \)-stable, then for all \( j \in \{0, 1, 2\} \) mod 3 and for all \( q \geq 0 \), the \( \Lambda_r \)-module \( B_{0,j} \) lies in the \( \Omega \)-orbit of \( A_{0,j} \) for some \( j' \in \{0, 1, 2\} \) and \( q' \geq 0 \). In particular, \( B_{0,1} = \Omega^{-1} A_{0,0}, B_{0,2} = \Omega^{-1} A_{0,1} \) and \( B_{1,0} = \Omega^{-1} A_{0,2} \). If \( r_i \geq 3 \) then each of the modules \( B_{0,1}, B_{0,2} \) and \( B_{0,0} \) with \( j \in \{0, 1, 2\} \) and \( q \geq 1 \) have a non-zero endomorphism factoring through \( M[\mathfrak{1}_i] \) and which does not factor through a projective \( \Lambda_r \)-module. Therefore, for all \( r_i \geq 2 \), the modules in \( \mathfrak{A}_i \cup \Omega(\mathfrak{A}_i) \) whose stable endomorphism rings are isomorphic to \( k \) are precisely the modules in \( \Omega \)-orbits of the modules \( A_{0,0}, A_{0,1} \) and \( A_{0,2} \).

Since \( \text{Ext}^1_{\Lambda_r}(A_{0,1}, A_{0,0}) = 0 \), it follows that \( R(\Lambda_r, A_{0,1}) \cong k \). Since \( \text{Ext}^1_{\Lambda_r}(A_{0,2}, A_{0,0}) \) is isomorphic to \( k \) for \( j \in \{0, 2\} \), it follows that \( R(\Lambda_r, A_{0,j}) \) is a quotient of \( k[[t]] \) for \( j \in \{0, 2\} \).

Let the \( \Lambda_r \)-module \( A_{0,0} = M[\mathfrak{1}_i] \).

Claim 4.2. The universal deformation ring \( R(\Lambda_r, A_{0,0}) \) of \( A_{0,0} \) is isomorphic to \( k[[t]]/(t^{r_i}) \).

Proof of Claim. For all \( l \in \{0, \ldots, r_i - 1\} \) let \( S_l = \zeta^{-l} \). Then for all \( l \in \{1, \ldots, r_i - 1\} \) there exists a non-trivial canonical endomorphism \( \sigma_l \) of the \( \Lambda_r \)-module \( M[S_l] \) which factors through \( M[S_{l-1}] \), namely
\[
\sigma_l : M[S_l] \to M[S_{l-1}] \to M[S_l].
\]
Observe that the kernel of \( \sigma_l \) and the image of \( \sigma_l^{-1} \) are isomorphic to \( A_{0,0} \), and that \( \sigma_l \) is a quotient of \( M[S_l] \) and \( M[S_{l-1}] \). Thus, for all \( l \in \{0, \ldots, r_i - 1\} \), the \( \Lambda_r \)-module \( M[S_l] \) is naturally a \( k[[t]]/(t^{l+1}) \otimes_k \Lambda_r \)-module where the action of \( t \) over \( m \in M[S_l] \) is given as \( t \cdot m = \sigma_l(m) \). In particular, \( t M[S_l] \cong M[S_{l-1}] \) for all \( l \in \{1, \ldots, r_i - 1\} \).

Let \( l \in \{1, \ldots, r_i - 1\} \) be fixed and let \( \{b_l\} \) be a \( k \)-basis of \( A_{0,0} \). Using the isomorphism \( M[S_l]/t M[S_l] \cong A_{0,0}, \) we can lift \( b_l \) to an element \( b_l \in M[S_l] \). It follows that \( \{b_1\} \) is linearly independent over \( k \) and that \( \{t^a b_1 \mid 0 \leq a \leq l\} \) is a \( k \)-basis of \( t M[S_l] \cong M[S_{l-1}] \). Therefore, \( \{b_l\} \) is a \( k[[t]]/(t^{l+1}) \)-basis of \( M[S_l] \), which means that \( M[S_l] \) is free over \( k[[t]]/(t^{l+1}) \). Moreover, \( M[S_l] \) lies in a short exact sequences of \( \Lambda_r \)-modules
\[
0 \to t M[S_l] \to M[S_l] \to k \otimes k[[t]]/(t^{l+1}) M[S_l] \to 0.
\]
Consequently, there exists an isomorphism of $\Lambda$-modules $\phi_\ell : \k[t]\otimes_{\k[[t]]}(t^{\ell+1})M[S_1] \to A_{0,0}$, which implies that $(M[S_1], \phi_\ell)$ is a lift of $A_{0,0}$ over $\k[t]/(t^{\ell+1})$. Consider the lift $(M[S_{r,-1}], \phi_{r,-1})$ of $A_{0,0}$ over $\k[t]/(t^{r})$. Since $\text{End}_{\Lambda}(A_{0,0}) \cong \k$ then by Theorem 2.3(i), there exists a unique morphism $\alpha : R(\Lambda, A_{0,0}) \to \k[t]/(t^{r})$ in $\hat{C}$ such that $M[S_{r,-1}] \cong \k[t]/(t^{r}) \otimes _{R(\Lambda, A_{0,0}),\alpha} U(\Lambda, A_{0,0})$, where $R(\Lambda, A_{0,0})$ and $U(\Lambda, A_{0,0})$ are respectively the universal deformation ring and the universal deformation of the $\Lambda$-module $A_{0,0}$. Since $(M[S_1], \phi_1)$ is not the trivial lift of $A_{0,0}$ over $\k[t]/(t^2)$, it follows that there exists a unique surjective morphism $\alpha' : R(\Lambda, A_{0,0}) \to \k[t]/(t^{2})$ in $\hat{C}$ such that $M[S_1] \cong \k[t]/(t^{2}) \otimes _{R(\Lambda, A_{0,0}),\alpha'} U(\Lambda, A_{0,0})$. By considering the natural projection $\pi_{r,-2} : \k[t]/(t^{2}) \to \k[t]/(t)$ and the lift $(U', \phi_{U'})$ of $A_{0,0}$ over $\k[t]/(t^2)$ corresponding to the morphism $\pi_{r,-2} \circ \alpha$, we obtain

\[
U' \cong \k[t]/(t^2) \otimes _{R(\Lambda, A_{0,0}),\pi_{r,-2} \circ \alpha} U(\Lambda, A_{0,0}) \\
\cong \k[t]/(t^{2}) \otimes _{\k\Lambda_{\alpha}} \k[t]/(t^{r}) \otimes _{R(\Lambda, A_{0,0}),\alpha} U(\Lambda, A_{0,0}) \\
\cong \k[t]/(t^{2}) \otimes _{\k\Lambda_{\alpha}} \k[t]/(t^{r}) \otimes _{R(\Lambda, A_{0,0}),\alpha} U(\Lambda, A_{0,0}) \\
\cong M[S_{r,-1}] \otimes _{\k} \Lambda_{\alpha} \cong M[S_1].
\]

It follows from Remark 2.1 that $[U', \phi_{U'}] = [M[S_1], \phi_1]$ in $\hat{F}_{A_{0,0}}(\k[t]/(t^2))$. The uniqueness of $\alpha'$ implies $\alpha' = \pi_{r,-2} \circ \alpha$. Since $\alpha'$ is surjective, it follows that $\alpha$ is also surjective. We want to prove that $\alpha$ is an isomorphism. Suppose this is false. Then there exists a surjective $\k$- algebra homomorphism $\alpha_0 : R(\Lambda, A_{0,0}) \to \k[t]/(t^{r+1})$ in $\hat{C}$ such that $\pi_{r+1,-1} \circ \alpha_0 = \alpha$, where $\pi_{r+1,-1} : \k[t]/(t^{r+1}) \to \k[t]/(t^{r})$ is the natural projection. Let $M_0$ be a $\k[t]/(t^{r+1}) \otimes _{\k\Lambda_{\alpha}} \Lambda_{\alpha}$-module which defines a lift of $A_{0,0}$ over $\k[t]/(t^{r+1})$ corresponding to $\alpha_0$. Since the kernel of $\pi_{r+1,-1}$ is $(t^{r})/(t^{r+1})$, then $M_0/t^r M_0 \cong M[S_{r,-1}, 1]$ . Consider the $\k[t]/(t^{r+1}) \otimes _{\k\Lambda_{\alpha}} \Lambda_{\alpha}$-module homomorphism $g : M_0 \to t^r M_0$ defined by $g(m) = t^r m$ for all $m \in M_0$. Since $M_0$ is free over $\k[t]/(t^{r+1})$, if follows that the kernel of $g$ is isomorphic to $tM_0$. Since $g$ is a surjection, it follows that $M_0/tM_0 \cong t^r M_0$, which implies that $t^r M_0 \cong A_{0,0}$. Hence, there exists a non-split short exact sequence of $\k[t]/(t^{r+1}) \otimes _{\k} \Lambda_{\alpha}$-modules

\[
0 \to A_{0,0} \to M_0 \to M[S_{r,-1}, 1] \to 0.
\]

Since $\Omega M[S_{r,-1}, 1] = \Omega M[S_{r,-1}] \cong M[S_{r,-1}] = M[k^{-1}r+2r_{r+1}]$, then

\[
\text{Ext}^1_{\Lambda}(M[S_{r,-1}], A_{0,0}) = \text{Hom}_{\Lambda}(M[S_{r,-1}], A_{0,0}) = 0.
\]

It follows that the sequence (9) splits as a sequence of $\Lambda$-modules. Hence $M_0 = A_{0,0} \oplus M[S_{r,-1}]$ as $\Lambda$-modules. Identifying the elements of $M_0$ as $(a, m)$ with $a \in A_{0,0}$ and $m \in M[S_{r,-1}]$ we see that the $t$ acts on $(a, m) \in M_0$ at $t \cdot (a, m) = (\mu(m), \sigma_{r,-1}(m))$, where $\mu : M[S_{r,-1}] \to A_{0,0}$ is a surjective $\Lambda$-module homomorphism and $\sigma_{r,-1}$ is as in (8). Since the canonical homomorphism $e : M[S_{r,-1}] \to M[1]$, $M[1]$ generates $\text{Hom}_{\Lambda}(M[S_{r,-1}], A_{0,0})$, then there exists $c \in \k$ such that $\mu = ec$, which implies that the kernel of $\mu$ is $tM[S_{r,-1}]$. Therefore $t^r(a, m) = (e(t^{r-1}m), \sigma_{r,-1}(m)) = (0, 0)$ for all $a \in A_{0,0}$ and $m \in M[S_{r,-1}]$, which contradicts the fact that $t^r M_0 \cong A_{0,0}$. Thus $\alpha : R(\Lambda, A_{0,0}) \to \k[t]/(t^{r+1})$ is an isomorphism and $R(\Lambda, A_{0,0}) \cong \k[t]/(t^{r+1})$. This finishes the proof of Claim 4.2.

Next consider the string $\Lambda$-module $A_{0,2} = M[(1_1)_{h k}]$

\[\text{Claim 4.3.} \text{ The universal deformation ring } R(\Lambda, A_{0,2}) \text{ of } A_{0,2} \text{ is isomorphic to } \k[t].\]

\[\text{Proof of Claim.} \text{ Let } T_0 = (1_1)_{h k} \text{ and for all } l \geq 1, \text{ let } T_l = T_{l-1}r_1(1_1)_{h k}. \text{ Thus, for all } l \geq 1 \text{ and by using similar arguments as those in the proof of Claim 4.2, we get lifts } (M[T_i], \varphi_i) \text{ of } A_{0,2} \text{ over } \k[t]/(t^{l+1}), \text{ where for each } l \geq 1, \text{ t acts on } m \in M[T_1] \text{ at } t \cdot m = \delta_l(m), \text{ where } \delta_l \text{ is the non-trivial canonical endomorphism of } M[T_l] \text{ that factors through } M[T_{l-1}], \text{ namely}\]

\[\delta_l : M[T_1] \to M[T_{l-1}] \to M[T_l].\]

\[\text{Note that for all } l \geq 1, \text{ we have natural projections } \pi_{l,-1} : M[T_l] \to M[T_{l-1}]. \text{ Let } N_0 = \lim_{l \to \infty} M[T_l] \text{ and let } t \text{ act on } N_0 \text{ as } \lim_{l \to \infty} \pi_{l,-1}. \text{ In particular, } \k[t]/(t^{l+1}) N_0 \cong N_0/t N_0 \cong A_{0,2}, \text{ which implies that there exists an isomorphism of } \Lambda \text{-modules } \varphi_0 : \k[t]/(t^{l+1}) N_0 \to A_{0,2}. \text{ Let } n = \dim_k A_{0,2} \text{ and let } \{B_j\}_{1 \leq j \leq n} \text{ be a } k\text{-basis of} \]

For all $1 \leq j \leq n$, we are able to lift these elements $\bar{B}_j$ in $N_0/tN_0$ to elements $B_j$ of $N_0$ such that $\{B_j\}_{1 \leq j \leq n}$ is a generating set of the $k[[t]] \otimes_k \Lambda_1$-module $N_0$. It follows that $\{B_j\}_{1 \leq j \leq n}$ is a $k[[t]]$-basis of $N_0$, which implies that $N_0$ is free over $k[[t]]$. Therefore, $(N_0, \varphi_0)$ is a lift of $A_{0,2}$ over $k[[t]]$ and there exists a unique $k$-algebra homomorphism $\beta : R(\Lambda_r, A_{0,2}) \to k[[t]]$ in $\mathcal{C}$ corresponding to the deformation defined by $(N_0, \varphi_0)$, where $R(\Lambda_r, A_{0,2})$ is the universal deformation ring of $A_{0,2}$. Since $N_0/t^2N_0 \cong M[T_1]$ as $\Lambda_r$-modules, we can see as in the proof of Claim 4.2 that since $N_0/t^2N_0$ defines a non-trivial lift of $A_{0,2}$ over $k[[t]]/(t^2)$, then $\beta$ is a surjection. Since $R(\Lambda_r, A_{0,2})$ is a quotient of $k[[t]]$, it follows that $\beta$ is an isomorphism. Hence $R(\Lambda_r, A_{0,2}) \cong k[[t]]$. This finishes the proof of Claim 4.3, which finishes the proof of Proposition 4.1.

\[ \square \]

**Proposition 4.4.** For $i \in \{0, 1, 2\}$ mod 3, let $\mathfrak{B}_i$ be the component of $\Gamma_s(\Lambda_r)$ containing the $\Lambda_r$-module $M[\tau_i]$, where $\bar{r} = (r_0, r_1, r_2, k)$ and $r_0, r_1, r_2 \geq 3$, $k \geq 1$. Define

\[(\tau_i)_h = \tau_i \mathcal{W}_{i+2} \quad \text{and} \quad h(\tau_i) = \mathcal{W}_{i+1} \tau_i.\]

If $k = 1$ then $\mathfrak{B}_i = \Omega(\mathfrak{A}_{i+2})$, where $\mathfrak{A}_{i+2}$ is as in Proposition 4.1. Thus, $\mathfrak{B}_i = \Omega(\mathfrak{B}_i)$ if and only if $k = 1$ and $r_{i+2} = 2$. The modules in $\mathfrak{B}_i \cup \Omega(\mathfrak{B}_i)$ whose stable endomorphism rings are isomorphic to $k$ are precisely the modules in the $\Omega$-orbits of the modules $V_0 = M[\tau_i]$, $V_1 = M[(\tau_i)_h]$ and $V_{-1} = M[h(\tau_i)]$. If $k = 1$ then the universal deformation rings are

\[R(\Lambda_r, V_0) \cong k, \quad R(\Lambda_r, V_1) \cong k[[t]]/(t^{r_{i+2}}), \quad \text{and} \quad R(\Lambda_r, V_{-1}) \cong k[[t]].\]

If $k \geq 2$ then the universal deformation rings are

\[R(\Lambda_r, V_0) \cong k, \quad R(\Lambda_r, V_1) \cong k[[t]], \quad \text{and} \quad R(\Lambda_r, V_{-1}) \cong k[[t]].\]

**Proof.** Let $i \in \{0, 1, 2\}$ mod 3 be fixed. Using hooks and co-hooks (see §3.2) we see that all $\Lambda_r$-modules in $\mathfrak{B}_i$ lie in the $\Omega$-orbit of either

\[C_{q,0} = M[\tau_i (a_{i+2} b_{i+2} + a_{i+1})^q], \quad \text{or} \quad C_{q,1} = M[\tau_i (a_{i+2} b_{i+1} + a_{i+1})^q a_{i+2}], \quad \text{or} \quad C_{q,2} = M[\tau_i (a_{i+2} b_{i+1} + a_{i+1})^q a_{i+2} a_{i+1}], \quad \text{or} \quad D_{q,0} = M[(b_{i+2} b_{i+1})^q \tau_i], \quad \text{or} \quad D_{q,1} = M[b_{i+2} b_{i+1} (b_{i+2} b_{i+1})^q \tau_i], \quad \text{or} \quad D_{q,2} = M[b_{i+2} (b_{i+2} b_{i+1})^q \tau_i], \]

for some $q \geq 0$.

Note that $C_{0,0} = M[\tau_i] = D_{0,0}$, $C_{0,1} = M[(\tau_i)_h]$, $C_{0,2} = M[h(\tau_i)]$, $D_{0,1} = M[h(\tau_i)]$ and $D_{0,2} = M[\tau_i]$. By Proposition 4.1, it follows that if $k = 1$ then $M[\tau_i]$ lies in $\Omega(\mathfrak{A}_{i+2})$, which implies that $\mathfrak{B}_i$ is $\Omega$-stable if and only if $r_{i+2} = 2$ and $k = 1$.

Using §3.3 and the description of the projective indecomposable $\Lambda_r$-module $P_i$ in (6), it is straightforward to show that the stable endomorphism rings of $C_{0,j}$ and $D_{0,j}$ are isomorphic to $k$ for $j \in \{0, 1\}$, that $\text{Ext}_{}^1_{\Lambda_r}(C_{0,1}, C_{0,1})$ and $\text{Ext}_{}^1_{\Lambda_r}(D_{0,1}, D_{0,1})$ are isomorphic to $k$, and that $\text{Ext}_{}^1_{\Lambda_r}(C_{0,0}, C_{0,0}) = 0$. Moreover, $D_{q,0}$ and $D_{q,j}$ with $q \geq 1$ and $j \in \{0, 1, 2\}$ have a non-zero canonical endomorphism factoring through $M[\tau_i]$ that does not factor through a projective $\Lambda_r$-module. If $k \geq 2$ or $r_{i+2} \geq 3$ then $C_{0,2}$ and $C_{q,j}$ with $q \geq 1$ and $j \in \{0, 1, 2\}$ have a non-zero canonical endomorphism which factors through $M[\mathbb{A}_{i+1}]$ and which does not factor through a projective $\Lambda_r$-module. If $k = 1$ and $r_{i+2} = 2$ then $C_{0,2} = \Omega^{-1} C_{0,1}$, $C_{1,0} = \Omega^{-3} C_{0,0}$, $C_{1,1} = \Omega^{-3} D_{0,1}$, and the modules $C_{1,2}$ and $C_{q,j}$ with $q \geq 2$, $j \in \{0, 1, 2\}$ have a non-trivial canonical endomorphism factoring through $M[\mathbb{A}_{i+1}]$ that does not factor through a projective $\Lambda_r$-module. Therefore, for all $r_0, r_1, r_2 \geq 2$ or $k \geq 1$, the modules in $\mathfrak{B}_i \cup \Omega(\mathfrak{B}_i)$ whose stable endomorphism rings are isomorphic to $k$ are precisely the modules in the $\Omega$-orbits of $C_{0,0}, C_{0,1}$ and $D_{0,1}$.

Since $\text{Ext}_{}^1_{\Lambda_r}(C_{0,0}, C_{0,0}) = 0$, it follows that $R(\Lambda_r, C_{0,0}) \cong k$. Since $\text{Ext}_{}^1_{\Lambda_r}(C_{0,1}, C_{0,1})$ and $\text{Ext}_{}^1_{\Lambda_r}(D_{0,1}, D_{0,1})$ are both isomorphic to $k$ then $R(\Lambda_r, C_{0,1})$ and $R(\Lambda_r, C_{0,1})$ are quotients of $k[[t]]$. Assume that $k = 1$. Then
by Theorem 2.3 and Proposition 4.1, it follows that
\[ R(\Lambda_r, C_0, 1) \cong R(\Lambda_r, M^2 \mathbb{Z}_{i+2}) \cong k[[t]]/(t^{r+2}), \]
and
\[ R(\Lambda_r, D_0, 1) \cong R(\Lambda_r, \Omega M([\mathbb{Z}_{i+2}]_{hh})) \cong k[[t]]. \]

Next assume that \( k \geq 2 \). Let \( S_0 = (\tau_i)_h \), \( T_0 = h(\tau_i) \) and for all \( l \geq 1 \), let \( S_l = S_{l-1} \tau_{i+1}(\tau_i)_h \) and \( T_l = T_{l-1} \zeta_i^{-1} h(\tau_i) \). Then by using similar arguments as in proof of Claim 4.3 within the proof of Proposition 4.1, we obtain that \( R(\Lambda_r, C_0, 1) \cong k[[t]] \cong R(\Lambda_r, D_0, 1) \). This finishes the proof of Proposition 4.4. \( \square \)

Let \( \xi_i \) be the component of \( \Gamma_s(\Lambda_r) \) containing the string module \( M[\tau_{i+1} \tau_i] \) for some \( i \in \{0, 1, 2\} \mod 3 \). Observe that if \( k = 1 \) then \( \xi_i \) is one of the 3-tubes, otherwise \( \xi_i \) is a component of type \( ZA_{\infty} \). In Proposition 4.6, we determine the universal deformation rings of modules whose stable endomorphism ring is isomorphic to \( k \) lying in the 3-tubes (see Proposition 4.6). In the following result, we assume that \( k \geq 2 \).

**Proposition 4.5.** For \( i \in \{0, 1, 2\} \mod 3 \), let \( \xi_i \) be the component of \( \Gamma_s(\Lambda_r) \) containing the \( \Lambda_r \)-module \( M[\tau_{i+1} \tau_i] \), where \( \vec{r} = (r_0, r_1, r_2, k) \) and \( r_0, r_1, r_2 \geq 2, k \geq 2 \). Define
\[ h(\tau_{i+1} \tau_i) = b_{i+2} \tau_{i+1} \tau_i \quad \text{and} \quad hh(\tau_{i+1} \tau_i) = b_{i+2} b_{i+2} \tau_{i+1} \tau_i. \]
The component \( \xi_i \) is \( \Omega \)-stable if and only if \( k = 2 \). The modules in \( \xi_i \) whose stable endomorphism ring is isomorphic to \( k \) are precisely the modules in the \( \Omega \)-orbits of the modules \( W_0 = M[\tau_{i+1} \tau_i], W_{-1} = M[h(\tau_{i+1} \tau_i)] \) and \( W_{-2} = M[h_h(\tau_{i+1} \tau_i)] \). Their universal deformation rings are
\[ R(\Lambda_r, W_0) \cong k[[t]]/(t^k), \quad R(\Lambda_r, W_{-1}) \cong k, \quad R(\Lambda_r, W_{-2}) \cong k[[t]]. \]

**Proof.** Let \( i \in \{0, 1, 2\} \mod 3 \) be fixed. Using hooks and co-hooks (see §3.2) we see that all \( \Lambda_r \)-modules in \( \xi_i \) lie in the \( \Omega \)-orbit of either
\[ E_{q,0} = M[\tau_{i+1} \tau_i(t_{i+2}^{q} \tau_{i+1}^{q})], \quad \text{or} \]
\[ E_{q,1} = M[\tau_{i+1} \tau_i(t_{i+2}^{q} \tau_{i+1}^{q})^{q} \tau_{i+1}^{q+2}], \quad \text{or} \]
\[ E_{q,2} = M[\tau_{i+1} \tau_i(t_{i+2}^{q} \tau_{i+1}^{q})^{q} \tau_{i+1}^{q+2}], \quad \text{or} \]
\[ F_{q,0} = M[b_{i+2} b_{i+2}^{q} \tau_{i+1}^{q+1}], \quad \text{or} \]
\[ F_{q,1} = M[b_{i+2} b_{i+2}^{q} \tau_{i+1}^{q+1}], \quad \text{or} \]
\[ F_{q,2} = M[b_{i+2} b_{i+2}^{q} \tau_{i+1}^{q+1}], \]
for some \( q \geq 0 \). Note that \( E_{0,0} = M[\tau_{i+1} \tau_i] = F_{0,0}, E_{0,1} = M[(\tau_{i+1} \tau_i)_h], E_{0,2} = M[(\tau_{i+1} \tau_i)_{hh}], F_{0,1} = M[h(\tau_{i+1} \tau_i)] \) and \( F_{0,2} = M[h_h(\tau_{i+1} \tau_i)] \). Since \( \Omega F_{0,0} = M[k(\tau_{i+1} \tau_i)^{k-2}] \), then \( \xi_i \) is \( \Omega \)-stable if and only if \( k = 2 \).

By using §3.3 and the description of the projective indecomposable \( \Lambda_r \)-module \( P_i \) in (6), it is straightforward to show that for all \( j \in \{0, 1, 2\} \), the stable endomorphism ring of \( F_{0,j} \) is isomorphic to \( k \) and for \( q \geq 1 \), the module \( F_{q,j} \) has a non-trivial canonical endomorphism which factors through \( M[\tau_{i+1} \tau_i] \) and which does not factor through a projective \( \Lambda_r \)-module.

Assume first that \( k = 2 \). Since in this case \( \xi_i \) is \( \Omega \)-stable, then for all \( j \in \{0, 1, 2\} \mod 3 \) and for all \( q \geq 0 \) the \( \Lambda_r \)-module \( E_{q,j} \) lies in the \( \Omega \)-orbit of \( F_{q,j} \) for some \( j' \in \{0, 1, 2\} \) and \( q' \geq 0 \). In particular, \( E_{0,0} = \Omega^{-1} F_{0,0}, E_{0,2} = \Omega^{-1} F_{0,1} \) and \( E_{1,0} = \Omega^{-1} F_{0,2} \). Next assume that \( k \geq 3 \). Then for all \( q \geq 0 \) and \( j \in \{0, 1, 2\} \) the module \( E_{q,j} \) has a non-trivial canonical endomorphism, which factors through \( M[\mathbb{Z}_{i+2}] \) and which does not factor through a projective \( \Lambda_r \)-module. Therefore for all \( k \geq 2 \), the modules in \( \xi_i \) whose stable endomorphism ring is isomorphic to \( k \) are precisely the modules in the \( \Omega \)-orbits of the modules \( F_{0,0}, F_{0,1} \) and \( F_{0,2} \).

Since \( \operatorname{Ext}^1_{\Lambda_r}(F_{0,1}, F_{0,1}) = 0 \), it follows that \( R(\Lambda_r, F_{0,1}) \cong k \). Since \( \operatorname{Ext}^1_{\Lambda_r}(F_{0,0}, F_{0,0}) \) and \( \operatorname{Ext}^1_{\Lambda_r}(F_{0,2}, F_{0,2}) \) are isomorphic to \( k \) then \( R(\Lambda_r, F_{0,0}) \) and \( R(\Lambda_r, F_{0,2}) \) are quotients of \( k[[t]] \). Let \( T_0 = h(\tau_{i+1} \tau_i) \) and for all \( 0 \leq j \leq k-1 \) and \( l \geq 1 \), let \( S_j = t_{i+2}^{l-1} \tau_{i+1} \) and \( T_j = h(\tau_{i+1} \tau_i) t_{i+1}^{-1} T_{j-1} \). By using similar arguments as those in the proof of Proposition 4.1, we obtain that \( R(\Lambda_r, F_{0,0}) \cong k[[t]]/(t^k) \) and \( R(\Lambda_r, F_{0,2}) \cong k[[t]] \). This finishes the proof of Proposition 4.5. \( \square \)
4.2. 3-tubes.

**Proposition 4.6.** Let $\mathfrak{T}_1$ and $\mathfrak{T}_2$ be the two 3-tubes of $\Gamma_s(\Lambda_r)$, with $\vec{r} = (r_0, r_1, r_2, k)$ and $r_0, r_1, r_2 \geq 2$ and $k \geq 1$. Then $\Omega(\mathfrak{T}_1) = \mathfrak{T}_2$. Let $T = \zeta_0^{-r_0+1}$ and define

$$T_h = \zeta_0^{-r_0+1} r_2 \zeta_2^{r_0+1} \quad \text{and} \quad T_{hh} = \zeta_0^{-r_h+1} r_1 \zeta_1^{r_1+1}.$$ 

The modules in $\mathfrak{T}_1 \cup \mathfrak{T}_2$ whose stable endomorphism rings are isomorphic to $k$ are precisely the modules in the $\Omega$-orbit of $X_0 = M[T]$, $X_1 = M[T_h]$ and $X_2 = M[T_{hh}]$. Their universal deformation rings are

$$R(\Lambda_r, X_0) \cong k, \quad R(\Lambda_r, X_1) \cong k, \quad R(\Lambda_r, X_2) \cong k[[t]].$$

**Proof.** Using the description of the projective indecomposable $\Lambda_r$-modules in (6), we see that $\Omega(\mathfrak{T}_1) = \mathfrak{T}_2$. Using §3.3 and the description of the projective indecomposable $\Lambda_r$-module $P_h$ in (6), it is straightforward to show that the only $\Lambda_r$-modules in $\mathfrak{T}_1 \cup \mathfrak{T}_2$ whose stable endomorphism rings are isomorphic to $k$ lie in the $\Omega$-orbit of either $X_0 = M[T]$, $X_1 = M[T_h]$ or $X_2 = M[T_{hh}]$. Since $\text{Ext}^1_{\Lambda_r}(X_j, X_j) = 0$ for $j \in \{0, 1\}$, we have that $R(\Lambda_r, X_j) \cong k$ for $j \in \{0, 1\}$. Since $\text{Ext}^1_{\Lambda_r}(X_2, X_2)$ is isomorphic to $k$, it follows that $R(\Lambda_r, X_2)$ is a quotient of $k[[t]]$. Let $S_0 = T_{hh}$ and for all $l \geq 1$, let $S_l = S_{l-1} \tau_l T_{hh}$. By using similar arguments as those in the proof of Claim 4.3 within the proof of Proposition 4.1, we obtain that $R(\Lambda_r, X_2) \cong k[[t]]$, which proves Proposition 4.6.

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