Quasi Exactly Solvable $N \times N$-Matrix Schrödinger Operators.

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Abstract

New examples of matrix quasi exactly solvable Schrödinger operators are constructed. One of them constitutes a matrix generalization of the quasi exactly solvable anharmonic oscillator, the corresponding invariant vector space is constructed explicitly. Also investigated are matrix generalizations of the Lamé equation.

1 Introduction

The topic of Quasi Exactly Solvable (QES) operators [1, 2] has been developed in the last years. It consists of differential operators (mainly Schrödinger ones) which possess a finite dimensional invariant vector space, say $\mathcal{V}$, of functions. So, the restriction of the eigenvalue equation to the space $\mathcal{V}$ leads to an algebraic eigenvalue problem.
Although scalar QES operators have been classified in one \cite{3} and several \cite{4} variables, a classification of matrix QES operators is still missing. Yet, interesting examples of them have been obtained in relation with the stability analysis of soliton solutions occurring in some field theories \cite{3,4}.

This problem was first addressed in \cite{7} and further developed in \cite{5} and \cite{8}. More recently \cite{9,10}, interesting tools for the classification of $N \times N$ matrix QES operators in one spatial dimension have been constructed and applied to the case $N=2$ \cite{11} (although potentials of the Lamé-type are not included in this classification).

Here we consider a suitable class of finite dimensional vector spaces of $N$ polynomials in a real variable and we construct families of operators preserving sub-classes of these vector spaces. The corresponding QES equations respectively constitute “coupled channel” generalizations of the scalar QES equations.

Following the basic idea of QES operators \cite{1} we consider the finite dimensional vector space of $N$-tuples of polynomials of given degree $n_1, n_2, \ldots, n_N$ in a real variable $x$. We slightly generalize this vector space by setting

$$V = P \left( \mathcal{P}(n_1) \oplus \mathcal{P}(n_2) \oplus \cdots \oplus \mathcal{P}(n_N) \right) \quad (1)$$

where $\mathcal{P}(n_i), i = 1, \ldots, N$ denotes the set of real polynomials of degree at most $n_i$ in $x$ while $P$ is a fixed invertible $N \times N$ matrix operator; $P$ can be interpreted as a change of basis in the vector space $\mathcal{P}(n_1) \oplus \cdots \oplus \mathcal{P}(n_N)$. With such an interpretation, and if we assume $n_1 \geq n_2 \geq \cdots \geq n_N$, it is reasonable to choose the matrix $P$ as a lower triangular matrix with $P_{ii} = 1$.

### 2 QES Anharmonic matrix potentials

We consider an operator of the form

$$H(y) = -\frac{d^2}{dy^2} \Pi_N + M_3(y^2) \quad (2)$$
where $M_3(x)$ is a $N \times N$ hermitian matrix whose elements are polynomials of degree at most three in the argument $x$. After a standard "gauge transformation" of $H(y)$ with the factor

$$\phi(y) = \exp\left\{-\frac{P_2}{2}y^4 + p_1y^2\right\}$$

($p_1, p_2$ are arbitrary real parameters, $p_2 > 0$) and the change of variable $x = y^2$, the operator equivalent to (2) reads

$$\hat{H}(x) = -\left(4x \frac{d^2}{dx^2} + 2 \frac{d}{dx}\right)I_N + 8(p_2x + p_1)x \frac{d}{dx}I_N - \left(4p_2^2x^3 + 8p_1p_2x^2 + (4p_1^2 - 6p_2)x - 2p_1\right)I_N + M_3(x)$$

We now determine the form of the matrix $M_3(x)$ such that the operator $\hat{H}(x)$ possesses a finite dimensional invariant vector space of the type (1) for generic values of $p_1, p_2$. In this purpose, the differential operators in the second line of (4) (i.e. $8p_2x^2d/dx$ and $8p_1xd/dx$) have separately to be completed into operators which have this property.

In order to make use of the results of [9, 10] we conveniently rewrite $H$ according to

$$\hat{H}(x) = -\left(4x \frac{d^2}{dx^2} + 2 \frac{d}{dx}\right)I_N + 8p_2Q_+ + 8p_1Q_0 + 8p_2(W_3(x) + B)$$

where the operators $Q_+, Q_0$ are defined by

$$Q_+ = x^2 \frac{d}{dx} + 2xA - B , \quad Q_0 = x \frac{d}{dx} + A$$

and the constant matrices $A, B$ are chosen to obey $[A, B] = B$, in such a way that

$$[Q_+, Q_0] = Q_+ .$$

Without loosing generality [11], we can choose $A$ and $B$ in the form

$$A = \text{Diag}(0, 1, 2, \ldots, N - 1) - \frac{p}{2}I_N , \quad B_{a,b} = c_0\delta_{a,b+1}$$
The matrix $W_3(x)$ is defined by identification of (4) and (5); it is symmetric and does not contain derivative. In the following, we further assume this matrix to be irreducible by $x$-independent changes of basis; in particular, we exclude the cases where $W_3$ is diagonal.

In [9, 10], it was demonstrated that $Q_+, Q_0$ admit a finite-dimensional vector space; we now construct it explicitely. For this purpose, we define a family of vector spaces characterized by two integers $N$ and $p$

\[ V(N, p) = \mathcal{P}(p) \oplus \mathcal{P}(p - 2) \oplus \ldots \oplus \mathcal{P}(p - 2N + 2) \quad (9) \]

and an (invertible) $N \times N$ matrix operator $P$ with matrix elements

\[ P_{ij} = \begin{cases} 0 & \text{if } i < j \\ 1 & \text{if } i = j \\ \left( \prod_{k=0}^{i-j-1} \frac{c_{j+k}}{(p + 2 - 2j - k)} \right) \frac{1}{(i-j)!} \left( \frac{d}{dx} \right)^{i-j} & \text{if } i > j \end{cases} \quad (10) \]

The vector space $V' \equiv PV$ can be seen as a change of basis on the space $V$ (for brevity, we do not write the dependence on $p, N$ anylonger).

The following proposition, which can be checked after an algebra, provides the explicit form of the invariant vector space of $Q_+, Q_0$:

Proposition 1

\[ P^{-1}BP = P^{-1}(Q_+ + B)P - (Q_+ + B) \quad (11) \]
\[ P^{-1}Q_+ P = x^2 \frac{d}{dx} + 2xA \quad (12) \]
\[ P^{-1}Q_0 P = Q_0 \quad (13) \]

Because the operators on the right hand side of these equalities preserve $V$, it follows immediately that $Q_+$ and $Q_0$ preserve the space $V'$.

The requirement that the $(W_3 + B)$-part of $\hat{H}$ also preserves $V'$ is guaranteed by the Proposition 2 below. For later use we will note by $J_+, J_0, J_-$ the usual irreducible
representation of sl(2) by $N \times N$ matrices, i.e.

$$[J_0, J_{\pm}] = \pm J_{\pm}, \quad [J_+, J_-] = 2J_0. \quad (14)$$

In particular we can set $J_0 = A - \frac{1}{N} \text{Tr} A$.

**Proposition 2**

If $W_3(x)$ is symmetric and irreducible, then the condition

$$P^{-1}(W_3 + B)P\mathcal{V} \subseteq \mathcal{V} \quad (15)$$

is fulfilled if and only if

$$B = cJ_+ \quad , \quad W_3 = -c(J_+ + J_-) \quad (16)$$

The values of the different parameters $c_k$ entering in the matrix $B$ are therefore fixed up to the multiplicative factor $c$.

So far, we proved that the condition is necessary and sufficient for $N$ up to seven and we are confident that it is true for arbitrary $N$.

The final step is to check whether the operator in the first line of (5) also preserves $\mathcal{V}'$. In this respect, it is sufficient to study the condition $A \frac{d}{dx} \mathcal{V}' \subset \mathcal{V}'$. From the structure of the vector space $\mathcal{V}'$, it is quickly seen that this holds only for $N=2$. This shows that the generalization of the QES anharmonic oscillator to matrices is possible only for two-dimensional matrices and the 3-parameter family of operators of [9] is recovered.

Our results thus lead to a family of 3 parameters matrix quasi exactly solvable anharmonic oscillators $(p_1, p_2, c)$. The parameters $p_1, p_2$ determine the anharmonic frequencies of the $y^6$- and $y^4$-parts of the potential $M_3$, while the constant $c$ ensures a non-trivial coupling between the different oscillators.
3 Clebsch-Gordan coefficients

The results presented in the previous section contain a formulation of the Clebsch-
Gordan matrix of the tensor product of two representations of sl(2) in terms of dif-
ferential operators. In this section we would like to further analyze some aspects of
this realization.

We consider on the one hand the standard representation of the algebra sl(2)
by \( N \times N \)-matrix generators noted \( J_+, J_0, J_− \) and on the other hand the realization
expressed in terms of differential operators:

\[
j_+ = x^2 \frac{d}{dx} + 2\mu x \quad \text{,} \quad j_0 = x \frac{d}{dx} + \mu \quad \text{,} \quad j_- = \frac{d}{dx}
\]

(17)

With the choice \( \mu = -n/2 \) (\( n \) integer), this realization preserves the vector space
\( \mathcal{P}(n) \) and constitutes the corner stone of QES equations [1].

Then, we consider the tensor product of these representations, the generators of
which are

\[
\tilde{Q}_{\pm,0} = J_{\pm,0} + j_{\pm,0} \quad \text{.}
\]

(18)

Again, if \( \mu = -n/2 \), it is finite dimensional and preserves, in a reducible way, the
vector space

\[
\mathcal{V}_0 = \mathcal{P}(n) \oplus \mathcal{P}(n) \ldots \oplus \mathcal{P}(n) \quad , \quad N \text{ times}
\]

(19)

The explicit decomposition of the representation (18), (19) into irreducible ones is
achieved in two steps. We define a first transformation by means of

\[
Q_{\pm,0} = e^{xJ_−} \tilde{Q}_{\pm,0} e^{-xJ_−} \quad \text{,}
\]

(20)

leading to

\[
Q_+ = x^2 \frac{d}{dx} + 2x(J_0 + \mu \mathbb{I}) + J_+ \quad , \quad Q_0 = x \frac{d}{dx} + J_0 + \mu \mathbb{I} \quad , \quad Q_- = \frac{d}{dx}
\]

(21)

which can be identified with the operators (8) if \( A = J_0 + \mu \mathbb{I} \), i.e. if \( \mu = \frac{N−p−1}{2} \).

In other words the two irreducible representations \( J_\epsilon, j_\epsilon \) under investigation have
dimensions \( N \) and \( p − N + 2 \), respectively.
The second step leading to the desired decomposition reads

$$\overline{Q}_{\pm,0} = P^{-1}Q_{\pm,0}P$$

where the operator $P$ is defined in eq.(10). The form of the operators $\overline{Q}$ is available from Proposition 1:

$$\overline{Q}_+ = x^2 \frac{d}{dx} + 2xA , \quad \overline{Q}_0 = x \frac{d}{dx} + A , \quad \overline{Q}_- = \frac{d}{dx}$$

It clearly reveals that the vector space (14) is preserved. As a consequence the matrix $P_{cg} = \exp(-xJ_-)P$ constitutes the Clebsch-Gordan matrix of the decomposition.

It should be noticed that the operators $J_-, j_-$ preserve separately the vector space (19). Then, since the transformation (20) commutes with $J_-$, it results that both $J_-$, $j_-$ also preserve $V'$. This statement in fact provides the proof that the condition (16) of the Proposition 2 above is indeed sufficient.

4 Lamé-type operators.

In the following the Jacobi elliptic functions

$$\text{sn}(z,k) , \quad \text{cn}(z,k) , \quad \text{dn}(z,k) .$$

of argument $z$ and modulus $k$ [12] are abbreviated respectively by sn, cn, dn. These functions are periodic with period $4K(k), 4K(k), 2K(k)$ respectively ($K(k)$ is the complete elliptic integral of the first type).

It is well known that in order to reveal the algebraic properties of the Lamé equation

$$\left[-\frac{d^2}{dz^2} + k^2 N(N + 1) \text{sn}^2(z)\right] \psi(z) = E \psi(z) ,$$

the relevant change of variable consists in posing $x = \text{sn}^2(z,k)$. In particular the second derivative operator is transformed into

$$\frac{d^2}{dz^2} = 4x(1 - x)(1 - k^2x)\frac{d^2}{dx^2} + 2(3k^2x^2 - 2(1 + k^2)x + 1)\frac{d}{dx}$$
and Eq. (25) becomes a Fushs equation with four regular-singular points (at \(x = 0, 1/k^2, 1, \infty\)).

To our knowledge, attempts to construct (and classify) Schrödinger QES matrix operators with this type of change of variable have not been attempted so far. Particular cases are emphasized in [5, 13]. The natural choice is to consider \(N \times N\) Schrödinger matrix operators with potentials depending on the Jacobi elliptic functions and which possess algebraic properties similar to Eq. (25). More specifically, we consider operators of the form

\[
H(z) = -\frac{d^2}{dz^2} I_N + V_D(z) + V_I(z)
\]  

(27)

with

\[
V_D = \text{sn}^2 \text{diag}(a_1, a_2, \ldots, a_N) + \text{diag}(b_1, b_2, \ldots, b_N)
\]

(28)

where \(a_j, b_j\) denote real constants (without losing generality we assume \(\sum_{j=1}^{N} b_j = 0\)) and \(V_I\) is a symmetric off-diagonal matrix of the form

\[
(V_I)_{ij} = \begin{cases} 
\theta_{ij} \text{sn}^{\alpha_{1ij}} \text{cn}^{\alpha_{2ij}} \text{dn}^{\alpha_{3ij}} & \text{if } i \neq j \\
0 & \text{if } i = j
\end{cases}
\]

Owing the periodicity of the Jacobi elliptic functions, the family hamiltonian above is to be considered on the Hilbert space of periodic functions on \([0, 4K(k)]\).

Unfortunately, it has not been possible to classify the QES operators of the form (27), here we will describe the cases that we were able to treat. The difficulty of achieving the classification will appear from these few examples. The properties of the Jacobi functions that are useful to make the calculations are listed in the appendix.

4.1 case \(N=1\)

This case corresponds to the Lamé equation which was discussed lengthly e.g. in [14].
4.2 Case N=2

The non-diagonal potential reduces to only one component

\[ V_I = \theta \, \text{sn}^{\alpha_1} \text{cn}^{\alpha_2} \text{dn}^{\alpha_3} \sigma_1 \]  

(29)

Using a similarity transformation of the form

\[ \hat{H}(x) = U^{-1}(z)H(z)U(z) , \quad U(z) = \text{diag}(\text{sn}^{\beta_1} \text{cn}^{\beta_2} \text{dn}^{\beta_3}, \text{sn}^{\gamma_1} \text{cn}^{\gamma_2} \text{dn}^{\gamma_3}) \]  

(30)

sets the operator (27) into a form with polynomial coefficients in the variable \( x = \text{sn}^2 \) provided (see Appendix)

- \( \beta_j, \gamma_j = 0 \) or \( 1 \) , \( j = 1, 2, 3 \)
- \( \alpha_j \pm (\beta_j - \gamma_j) = \) non-negative even integer , \( j = 1, 2, 3. \)

Considering first the cases where \( V_{12} \) is a constant (\( \alpha_{1,2,3} = 0 \)) or a single linear factor in one of the Jacobi functions, we were able to show that the only possible QES operators are available for \( \theta = 0. \)

For the three cases a) \( V_{12} = \theta \, \text{sn} \, \text{cn}, \) b) \( V_{12} = \theta \, \text{sn} \, \text{dn} \) and c) \( V_{12} = \theta \, \text{cn} \, \text{dn}, \) the construction of non-decoupled QES operators is possible. One remarkable feature is that in each case, two sets of values of the coupling constants \( a_1, a_2, b_1, \theta \) lead to four different algebraizations of the corresponding operator. We now discuss it in detail (posing \( b \equiv b_1 = -b_2, \))

4.2.1 Case \( V_{12} = \theta \, \text{sn} \, \text{cn} \)

The two sets of values of the coupling constants leading to QES operators are noted Type 1 and Type 2.

**Type 1**

\[ a_1 = k^2(4m^2 + 2m + 1) - 2b \]
\[ a_2 = k^2(4m^2 + 2m + 1) + 2b \]
\[ \theta^2 = 4b^2 - k^4(1 + 4m)^2 \]
The parameters $b$ and $k$ remain free, $m$ is an integer. Four invariant spaces are available:

$$V_1 = \begin{pmatrix} \text{sn} & 0 \\ 0 & \text{cn} \end{pmatrix} \begin{pmatrix} 1 & \kappa_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{P}(m-1) \\ \mathcal{P}(m) \end{pmatrix}, \quad \kappa_1 = \frac{-\theta}{2b + k^2(1 + 4m)} \quad (31)$$

$$V_2 = \begin{pmatrix} \text{cn} & 0 \\ 0 & \text{sn} \end{pmatrix} \begin{pmatrix} 1 & \kappa_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{P}(m-1) \\ \mathcal{P}(m) \end{pmatrix}, \quad \kappa_2 = \frac{\theta}{2b - k^2(1 + 4m)} \quad (32)$$

$$V_3 = \begin{pmatrix} \text{dn} & 0 \\ 0 & \text{sn} \end{pmatrix} \begin{pmatrix} 1 & \kappa_3 \times \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{P}(m-1) \\ \mathcal{P}(m-1) \end{pmatrix}, \quad \kappa_3 = -\frac{1}{\kappa_2} \quad (33)$$

$$V_4 = \begin{pmatrix} \text{sn} \text{cn} \text{dn} & 0 \\ 0 & \text{dn} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \kappa_4 \times & 1 \end{pmatrix} \begin{pmatrix} \mathcal{P}(m-1) \\ \mathcal{P}(m-1) \end{pmatrix}, \quad \kappa_4 = -\frac{1}{\kappa_1} \quad (34)$$

**Type 2**

$$a_1 = k^2(4m^2 + 6m + 1) - 2b$$

$$a_2 = k^2(4m^2 + 6m + 1) + 2b$$

$$\theta^2 = 4b^2 - k^4(3 + 4m)^2$$

The corresponding invariant vector spaces read

$$V_5 = \begin{pmatrix} 1 & 0 \\ 0 & \text{sn} \text{cn} \end{pmatrix} \begin{pmatrix} 1 & \kappa_5 \times \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{P}(m) \\ \mathcal{P}(m) \end{pmatrix}, \quad \kappa_5 = \frac{k^2(4m+3) - 2b}{\theta} \quad (35)$$

$$V_6 = \begin{pmatrix} \text{sn} \text{cn} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \kappa_6 \times & 1 \end{pmatrix} \begin{pmatrix} \mathcal{P}(m) \\ \mathcal{P}(m) \end{pmatrix}, \quad \kappa_6 = \frac{k^2(4m+3) - 2b}{\theta} \quad (36)$$

$$V_7 = \begin{pmatrix} \text{sn} \text{dn} & 0 \\ 0 & \text{cn} \text{dn} \end{pmatrix} \begin{pmatrix} 1 & \kappa_7 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{P}(m-1) \\ \mathcal{P}(m) \end{pmatrix}, \quad \kappa_7 = \frac{2b - k^2(4m+3)}{\theta} \quad (37)$$

$$V_8 = \begin{pmatrix} \text{cn} \text{dn} & 0 \\ 0 & \text{sn} \text{dn} \end{pmatrix} \begin{pmatrix} 1 & \kappa_8 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{P}(m-1) \\ \mathcal{P}(m) \end{pmatrix}, \quad \kappa_8 = \frac{2b + k^2(4m+3)}{\theta} \quad (38)$$

**4.2.2 Case** $V_{12} = \theta \text{ sn dn}$

In the following cases we just write the values of the coupling constants leading to QES operators; the corresponding invariant vector spaces are similar to the ones
above, their explicit form can be obtained in a straightforward way.

**Type 1**

\[
a_1 = k^2(4m^2 + 2m + 1 - 2b)
\]

\[
a_2 = k^2(4m^2 + 2m + 1 + 2b)
\]

\[
\theta^2 = 4k^2b^2 - k^2(1 + 4m)^2
\]

**Type 2**

\[
a_1 = k^2(4m^2 + 6m + 1 - 2b)
\]

\[
a_2 = k^2(4m^2 + 6m + 1 + 2b)
\]

\[
\theta^2 = 4k^2b^2 - k^2(3 + 4m)^2
\]

**4.2.3 Case** \(V_{12} = \theta \, \text{cn} \, \text{dn}\)

**Type 1**

\[
a_1 = k^2(4m^2 + 2m + 1) - 2b \frac{k^2}{1 + k^2}
\]

\[
a_2 = k^2(4m^2 + 2m + 1) + 2b \frac{k^2}{1 + k^2}
\]

\[
\theta^2 = k^2(1 + 4m)^2 - 4b^2 \frac{k^2}{1 + k^2}
\]

**Type 2**

\[
a_1 = k^2(4m^2 + 6m + 1) - 2b \frac{k^2}{1 + k^2}
\]

\[
a_2 = k^2(4m^2 + 6m + 1) + 2b \frac{k^2}{1 + k^2}
\]

\[
\theta^2 = k^2(1 + 4m)^2 - 4b^2 \frac{k^2}{1 + k^2}
\]

Concerning the operators, we want to point out the following things:

- The extraction of the prefactor is done in two steps. After factorizing the appropriate products of Jacobi functions, the non-diagonal part of the potential
takes the form
\[
\begin{pmatrix}
0 & x^{2-\epsilon} \\
x^\epsilon & 0
\end{pmatrix}, \quad \epsilon = 2 \text{ or } 1 \text{ or } 0
\] (39)
which is clearly incompatible with an operator preserving a vector space of the form \(\mathcal{P}(n) \oplus \mathcal{P}(m)\) for integers \(m, n\). The setting of the operator in a form preserving such a vector space is realized by the second prefactor (the triangular matrix one). As an example if \(\epsilon = 0\) in (39), then use is made of the relations
\[
\begin{pmatrix}
1 & -A \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
1 & A \\
0 & 1
\end{pmatrix} =
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}
\] (40)
\[
\begin{pmatrix}
1 & -A \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & A \\
0 & 1
\end{pmatrix} =
\begin{pmatrix}
-A & A^2 \\
1 & A
\end{pmatrix}
\] (41)
with \(A \div x\) in order to eliminate the quadratic power of \(x\) occuring in \(V_{12}\).

- For \(V_{12} = \theta \sin \cos\), the limit \(k \to 0\) is non trivial and leads to a potential of the form
\[
\begin{pmatrix}
\cos^2(x) - \frac{1}{2} & \cos(x)\sin(x) \\
\cos(x)\sin(x) & \sin^2(x) - \frac{1}{2}
\end{pmatrix}
\] (42)
In particular, the dependence on \(m\) disappears, so this integer can be choosen arbitrarily large and an infinite set of algebraic eigenvectors occurs in this limit. This confirms the fact that the corresponding Schrödinger equation is completely solvable and is related to the stability of some soliton solutions occuring in the Goldstone model in 1+1 dimensions with a periodic condition for the space coordinate.

- The case \(V_{12} = \theta \cos \sin\) with \(b = (1 + k^2)/2\) was studied lengthly in. The corresponding Schrödinger equation is related to the stability analysis of the sphaleron solution in the U(1)-Abelian Higgs model in 1+1 dimensions (again with a periodic condition for the space coordinate).
4.3 Case N=3

We attempted to construct 3×3 matrix operators with some choice for the constants \( \alpha_{aij} \) \((a, i, j = 1, 2, 3)\) inspired from the results above. Several trials were unsuccessful (namely with \( V_{12} ÷ V_{23} ÷ \text{sn cn} \)); however the choice

\[
V_I = \begin{pmatrix}
0 & \theta_1 \text{cn} \text{dn} & V_{23} \\
\theta_1 \text{cn} \text{dn} & 0 & \theta_2 \text{cn} \text{dn} \\
V_{23} & \theta_2 \text{cn} \text{dn} & 0
\end{pmatrix}
\] (43)

leads to the wanted form of equations and preserves a vector space of the form

\[
V = \begin{pmatrix}
\text{cn} & 0 & 0 \\
0 & \text{dn} & 0 \\
0 & 0 & \text{cn}
\end{pmatrix} \cdot \begin{pmatrix}
1 & \alpha & \beta \\
0 & 1 & \gamma \\
0 & 0 & 1
\end{pmatrix} \cdot \begin{pmatrix}
\mathcal{P}(n-2) \\
\mathcal{P}(n-1) \\
\mathcal{P}(n)
\end{pmatrix}
\] (44)

The condition

\[
H(z)V \subseteq V
\] (45)

leads to ten equations for the parameters \( \alpha, \beta, \gamma, k, a_i, b_i \) and \( \theta_a \). These equations are compatible with each other only if \( k = 1 \) and \( V_{23} = \theta_3 \text{cn}^2 \). We found it convenient to leave the parameters \( \alpha, \beta, \gamma \) free and to express the coupling constants \( a_i, b_i \) and \( \theta_a \) in terms of them. We refrain to write these tedious expressions in general but we mention that they have the polynomial

\[
(1 + \gamma^2 + \beta^2)(1 + \beta^2 + \gamma^2 + \alpha^2 \gamma^2 - 2\alpha \beta \gamma)
\] (46)

as common denominator. Also they are such that

\[
a_1 + a_2 + a_3 = 2(6n^2 - 3n + 4)
\] (47)

For the case with e.g. \( \alpha = \beta = \gamma = 1 \) we find

\[
a_1 = a_2 = \frac{12n^2 - 10n + 11}{3}, \quad a_3 = \frac{2}{3}(6n^2 + n + 1)
\] (48)

\[
b_1 = b_2 = \frac{4n - 3}{3}, \quad b_3 = -2b_1
\] (49)
\[
\begin{align*}
\theta_1 &= \frac{7 - 2n}{6}, \quad \theta_2 = -\frac{4n + 1}{3}, \quad \theta_3 = \frac{2(4n + 1)}{3} \quad (50) \\
as another example, if \alpha = -\beta = 1, \quad \gamma = 0, \text{ we have} \\
a_1 &= \frac{2(6n^2 - 7n + 3)}{3}, \quad a_2 = a_3 = \frac{1}{3}(12n^2 - 2n + 9) \quad (51) \\
b_1 &= \frac{2(4n + 1)}{3}, \quad b_2 = -\frac{4n + 1}{3}, \quad b_3 = -b_1 - b_2 \quad (52) \\
\theta_1 &= \frac{3 - 4n}{3}, \quad \theta_2 = \frac{20n - 3}{6}, \quad \theta_3 = \frac{2(4n - 1)}{3} \quad (53)
\end{align*}
\]

Remarkably, replacing the first factor in (44) by \( \text{diag}(\text{dn}, \text{cn}, \text{dn}) \), we obtain the same solution for \( a_1, a_2, \ldots \). So the operator (43) possesses at least a double algebraization, but we have not attempted to construct the other possibly existing ones yet.

The 3×3-matrix QES potential (43) strongly contrasts with the 2×2 ones obtained above. While the former depends on three free parameters but exists only on the full line (since \( k = 1 \)), the latter can have an arbitrary period (equal to \( 4K(k) \)) but has only one free parameter (for instance noted by \( b \)) for a fixed \( k \). The fact that the operator \( H(z) \) for a 3×3-matrix QES potential is quasi exactly solvable only for \( k^2 = 1 \) was unexpected to us. Accordingly, this operator corresponds to a 3×3 matrix version of the Pöschl-Teller operator.

5 Concluding remarks

In the first part of this paper we obtained the matrix generalization of the celebrated sextic QES anharmonic oscillator. The examples of operators presented in the second part reflect the difficulty to classify the coupled-channel (or matrix) QES Schrödinger equations when the change of variable involves several singular points like in (24). We hope that this note will motivate further investigation of this problem.
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6 Appendix A

For \( k = 0 \) we have \( K(0) = \frac{\pi}{2} \) and the Jacobi functions reduce to standard trigonometric functions: \( \text{sn}(z, 0) = \sin(z) \), \( \text{cn}(z, 0) = \cos(z) \), \( \text{dn}(z, 0) = 1 \). For \( k = 1 \), we have \( K(1) = \infty \) and the Jacobi functions reduce to elementary functions:

\[
\text{sn}(z, 1) = \tanh(z) \quad (54)
\]
\[
\text{cn}(z, 1) = \text{dn}(z, 1) = \frac{1}{\cosh(z)} \quad (55)
\]

In this limit, the Lamé equation becomes a Poschl-Teller equation. For generic values of \( k \), the Jacobi functions obey the following relations:

\[
\text{cn}^2 + \text{sn}^2 = 1 \quad , \quad \text{dn}^2 + k^2 \text{sn}^2 = 1 \quad (56)
\]
\[
\frac{d}{dz} \text{sn} = \text{cn} \text{ dn} \quad , \quad \frac{d}{dz} \text{cn} = -\text{sn} \text{ dn} \quad , \quad \frac{d}{dz} \text{dn} = -k^2 \text{sn} \text{ cn} \quad (57)
\]

These identities as well as the following ones are useful to establish the equations in the variable \( x = \text{sn}^2 \) after the prefactor including the Jacobi functions has been extracted:

| \( f \) | \( f''/f \) | \( (\text{sn} \text{ cn} \text{ dn}) f'/f \) |
|---|---|---|
| 1 | 0 | 0 |
| \text{sn} | \( 2k^2x - (1 + k^2) \) | \( k^2x^2 - (1 + k^2)x + 1 \) |
| \text{cn} | \( 2k^2x - 1 \) | \( k^2x^2 - x \) |
| \text{dn} | \( 2k^2x - k^2 \) | \( k^2x^2 - k^2x \) |
| \text{cn} \text{ dn} | \( 6k^2x - (1 + k^2) \) | \( 2k^2x^2 - (1 + k^2)x \) |
| \text{sn} \text{ dn} | \( 6k^2x - (1 + 4k^2) \) | \( 2k^2x^2 - (1 + 2k^2)x + 1 \) |
| \text{sn} \text{ cn} | \( 6k^2x - (4 + k^2) \) | \( 2k^2x^2 - (2 + k^2)x + 1 \) |
| \text{sn} \text{ cn} \text{ dn} | \( 12k^2x - 4(1 + k^2) \) | \( 3k^2x^2 - 2(1 + k^2)x + 1 \) |

No similar identities are available (to our knowledge) with different choices of the function \( f \).