ON REAL PROJECTIVE CONNECTIONS, V.I. SMIRNOV’S APPROACH, AND BLACK HOLE TYPE SOLUTIONS OF THE LIOUVILLE EQUATION

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Dedicated to my teacher Ludwig Dmitrievich Faddeev on the occasion of his 80th birthday

Abstract. We consider real projective connections on Riemann surfaces and corresponding solutions of the Liouville equation. It is shown that these solutions have singularities of special type (of a black hole type) on a finite number of simple analytical contours. The case of the Riemann sphere with four real punctures, considered in V.I. Smirnov’s thesis (Petrograd, 1918), is analyzed in detail.

1. Introduction

One of a central problems of mathematics in the second half of the 19th century and at the beginning of the 20th century was the problem of uniformization of Riemann surfaces. The classics, Klein and Poincaré, associate it with the study of ordinary differential equations of the second order with regular singular points [1, 2]. Another approach to the uniformization problem was proposed by Poincaré [3]. It consists of finding a complete conformal metric of constant negative curvature, and it reduces to the global solvability of the Liouville equation, a special nonlinear partial differential equations of elliptic type on a Riemann surface.

In this paper we illustrate the relation between these two approaches and describe solutions of the Liouville equation, which correspond to ordinary differential equations of the second order with real monodromy group. In modern physics literature on the Liouville equation it is rather common to assume that for the Fuchsian uniformization of a Riemann surface it is sufficient to have an ordinary differential equation of the second order with a real monodromy group. However, already the classics knew that it is not the case, as they were analyzing in detail ordinary differential equations of the second order with real monodromy group on Riemann surfaces of genus 0 with punctures. Nonetheless, they did not consider the relation with the Liouville equation, and here we partially fill this gap.

Namely, in Section 2, following the lectures [4], we briefly describe the theory of projective connections on a Riemann surface — invariant method for defining a corresponding ordinary differential equation of the second order with regular singular points. Ibid, following [5, 6], we review the main results on the Fuchsian uniformization, Liouville equation and the complex
geometry of the moduli space. In Section 3, following the paper [7], we present modern classification of projective connections with real monodromy group, and review the results of V.I. Smirnov thesis [8] (Petrograd, 1918). This work, published in [9, 10], was the first where for the case of four real punctures a complete classification of equations with real monodromy group was given. In Section 3.2, we give a modern interpretation of the results of V.I. Smirnov. Finally, in Section 4 we describe solutions of the Liouville equation with a black hole type singularities associated with real projective connections. To the best of our knowledge, these solutions were not previously considered in the literature.

2. Projective connections, uniformization and the Liouville equation

2.1. Projective connections. Let $X_0$ be a compact Riemann surface of genus $g$ with marked points $x_1, \ldots, x_n$, where $2g + n - 2 > 0$, and let $\{U_\alpha, z_\alpha\}$ be a complex-analytic atlas with local coordinates $z_\alpha$ and transition functions $z_\alpha = g_{\alpha\beta}(z_\beta)$ on $U_\alpha \cap U_\beta$. Denote by $X = X_0 \setminus \{x_1, \ldots, x_n\}$ a corresponding Riemann surface of type $(g, n)$, the surface of genus $g$ with $n$ punctures. The collection $R = \{r_\alpha\}$, where $r_\alpha$ are holomorphic functions on $U_\alpha \cap X$, is called (holomorphic) projective connection on $X$, if on every intersection $U_\alpha \cap U_\beta \cap X$,

$$r_\beta = r_\alpha \circ g_{\alpha\beta}(g_{\alpha\beta}'^2) + \mathcal{S}(g_{\alpha\beta}),$$

where $\mathcal{S}(f)$ is the Schwarzian derivative of a holomorphic function $f$, 

$$\mathcal{S}(f) = \frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'}\right)^2.$$

In addition, it is assumed that if $x_i \in U_\alpha$ and $z_\alpha(x_i) = 0$, then

$$(2.1) \quad r_\alpha(z_\alpha) = \frac{1}{2z_\alpha^2} + O\left(\frac{1}{|z_\alpha|}\right), \quad z_\alpha \to 0.$$

Projective connections form an affine space $\mathcal{P}(X)$ over the vector space $\mathcal{Q}(X)$ of holomorphic quadratic differentials on $X$; elements of $\mathcal{Q}(X)$ are collections $Q = \{q_\alpha\}$ with the transformation law

$$q_\beta = q_\alpha \circ g_{\alpha\beta}(g_{\alpha\beta}')^2$$

and additional condition that $q_\alpha(z_\alpha) = O(|z_\alpha|^{-1})$ as $z_\alpha \to 0$, if $x_i \in U_\alpha$ and $z_\alpha(x_i) = 0$. The vector space $\mathcal{Q}(X)$ has complex dimension $3g - 3 + n$. (For more details on projective connections and quadratic differentials see [4] and the references therein).

A projective connection $R$ naturally determines a linear differential equation of the second order on the Riemann surface $X$, the Fuchsian differential equation

$$(2.2) \quad \frac{d^2 u_\alpha}{dz_\alpha^2} + \frac{1}{2} r_\alpha u_\alpha = 0,$$
where \( U = \{ u_\alpha \} \) is understood and as a multi-valued differential of order \(-\frac{1}{2}\) on \( X \). Equation (2.2) determines the monodromy group, a representation of the fundamental group \( \pi_1(X, x_0) \) of the Riemann surface \( X \) with the marked point \( x_0 \) in \( \text{PSL}(2, \mathbb{C}) = \text{SL}(2, \mathbb{C})/\{ \pm I \} \). Condition (2.1) implies that under the monodromy representation standard generators of \( \pi_1(X, x_0) \), which correspond to the loops around the punctures \( x_i \), are mapped to parabolic elements in \( \text{PSL}(2, \mathbb{C}) \).

2.2. Uniformization. According to the uniformization theorem

\[ X \cong \Gamma \backslash \mathbb{H}, \]

where \( \mathbb{H} = \{ \tau \in \mathbb{C} : \text{Im } \tau > 0 \} \) is the Poincaré model of the Lobachevsky plane, and \( \Gamma \subset \text{PSL}(2, \mathbb{R}) \) is a Fuchsian group of type \((g, n)\), which acts on \( \mathbb{H} \) by linear fractional transformations. In other words, there exists a complex-analytic covering \( J : \mathbb{H} \to X \) whose automorphism group is \( \Gamma \). The inverse function to \( J \) — a multi-valued analytic function \( J^{-1} : X \to \mathbb{H} \) — is locally univalent linear polymorphic function on \( X \) (the latter means that its branches are connected by linear fractional transformations in \( \Gamma \)). The Schwarzian derivatives of \( J^{-1} \) with respect to \( z_\alpha \) are well-defined on \( U_\alpha \) and determine the Fuchsian projective connection \( R_F = \{ \mathcal{S}_{z_\alpha}(J^{-1}) \} \) on \( X \), and multi-valued functions \( \frac{1}{\sqrt{(J^{-1})'}} \) and \( \frac{J^{-1}}{\sqrt{(J^{-1})'}} \) satisfy Fuchsian differential equation (2.2) with \( R = R_F \). The monodromy group of this equation, up to the conjugation in \( \text{PSL}(2, \mathbb{R}) \), is the Fuchsian group \( \Gamma \).

Klein [1] and Poincaré [2] were solving the uniformization problem of a Riemann surface \( X \) by choosing such a projective connection in Fuchsian equation (2.2) that its monodromy group is a Fuchsian group \( \Gamma \) with the property that (2.3) holds. However, a direct proof of the existence of the Fuchsian projective connection \( R_F \) on \( X \) turned out to be a very difficult problem which has not been completely solved to this day (see [11, 12]). In the case of Riemann surfaces of type \((0, n)\), to which we further restrict ourselves, this problem is formulated as follows.

Let \( X_0 = \mathbb{P}^1 \) be the Riemann sphere and \( X \) be a Riemann surface of genus 0 with \( n \) punctures \( z_1, \ldots, z_n \). Without the loss of generality it can be assumed that \( z_{n-2} = 0, z_{n-1} = 1, z_n = \infty \) and \( X = \mathbb{C} \setminus \{ z_1, \ldots, z_{n-3}, 0, 1 \} \). Equation (2.2) takes the form

\[ \frac{d^2 u}{dz^2} + \frac{1}{2} \sum_{i=1}^{n-1} \left( \frac{1}{2(z - z_i)^2} + \frac{c_i}{z - z_i} \right) u = 0, \]

where \( z \) is the global complex coordinate on \( X \). The complex parameters \( c_1, \ldots, c_{n-1} \) satisfy two conditions

\[ \sum_{i=1}^{n-1} c_i = 0, \quad \sum_{i=1}^{n-1} z_i c_i = 1 - \frac{n}{2}, \]
which allow to express $c_{n-2}$ and $c_{n-1}$ explicitly in terms of $z_1, \ldots, z_{n-3}$ and the remaining $n - 3$ parameters $c_1, \ldots, c_{n-3}$.

In the classical approach of Klein and Poincaré to the uniformization problem it was required, given the singular points $z_1, \ldots, z_{n-3}, 0, 1, \infty$, to choose the parameters $c_1, \ldots, c_{n-3}$ in such a way that the monodromy group of equation (2.4) is a Fuchsian group isomorphic to the fundamental group of the Riemann surface $X$. Then a ratio of two linear independent solutions of equation (2.4) would be, up to a linear fractional transformation, the desired multi-valued mapping $J^{-1}: X \to \mathbb{H}$ realizing the uniformization of the Riemann surface $X$. Corresponding $\Gamma$-automorphic function $J: \mathbb{H} \to \mathbb{C}$ is called Klein’s Hauptmodul (Hauptfunktion). The complex numbers $c_1, \ldots, c_{n-3}$, the accessory parameters of the Fuchsian uniformization of the surface $X$, are uniquely determined by the singular points $z_1, \ldots, z_{n-3}$. Moreover

\[ (2.6) \quad \mathcal{J}(J^{-1})(z) = \sum_{i=1}^{n-1} \left( \frac{1}{2(z-z_i)^2} + \frac{c_i}{z-z_i} \right). \]

To prove the existence of accessory parameters, Poincaré in [2] proposed the so-called “continuity method”. However, using this method it was not possible to obtain a rigorous solution of the uniformization problem, and subjected to criticism, it was soon abandoned. The ultimate solution of the uniformization problem was obtained by Koebe and Poincaré in 1907 by using quite different methods, in particular, by using potential theory (for modern exposition, see e.g. [13]).

### 2.3. The Liouville equation

Projection on $X$ of the Poincaré metric $(\text{Im } \tau)^{-2}|d\tau|^2$ on $\mathbb{H}$ is a complete conformal metric on $X$ of constant negative curvature $-1$. It has the form $e^{\varphi(z)}|dz|^2$, where

\[ (2.7) \quad e^{\varphi(z)} = \frac{|(J^{-1})'(z)|^2}{(\text{Im } J^{-1}(z))^2}. \]

The smooth function $\varphi$ on $X$ satisfies the Liouville equation

\[ (2.8) \quad \varphi_{zz} = \frac{1}{2}e^{\varphi} \]

and has the following asymptotics

\[ (2.9) \quad \varphi(z) = \begin{cases} -2 \log |z - z_i| - 2 \log | \log |z - z_i|| + o(1), & z \to z_i, \ i \neq n, \\ -2 \log |z| - 2 \log \log |z| + o(1), & z \to \infty. \end{cases} \]

In [3] Poincaré proposed an approach to the uniformization problem based on the Liouville equation. Namely, he proved that in the class of smooth real-valued functions on $X$ with asymptotics (2.9) Liouville equation (2.8) is uniquely solvable. From here it follows that $T_\varphi = \varphi_{zz} - \frac{1}{2}\varphi^2$ is a rational function of the form (2.6) and the differential equation

\[ \frac{d^2u}{dz^2} + \frac{1}{2}T_\varphi u = 0 \]
has a Fuchsian monodromy group which uniformizes the Riemann surface $X$.

The Liouville equation is the Euler-Lagrange equation for the functional

$$ S(\psi) = \lim_{\varepsilon \to 0} \left( \int_{X_\varepsilon} (|\psi_z|^2 + e^\psi) d^2z + 2\pi \varepsilon + 4\pi n \log |\log \varepsilon| \right), $$

where $d^2z$ is the Lebesgue measure on $\mathbb{C}$,

$$ X_\varepsilon = X \setminus \left( \bigcup_{i=1}^{n-1} \{|z - z_i| < \varepsilon\} \bigcup \{|z| > 1/\varepsilon\} \right), $$

and $\psi$ belongs to the class of smooth functions $X$ with asymptotics (2.9).

The quantity $T_\psi = \psi_{zz} - \frac{1}{4} \psi_z^2$ plays the role of $(2,0)$-component of the stress-energy tensor in the classical Liouville theory, and

$$ T_\varphi = \mathcal{F}(J^{-1}). $$

Denote by

$$ \mathcal{M}_{0,n} = \{(z_1, \ldots, z_{n-3}) \in \mathbb{C}^{n-3} : z_i \neq z_j, i \neq j, z_i \neq 0, 1\} $$

the moduli space of Riemann surfaces of genus 0 with $n$ ordered punctures (rational curves with $n$ marked points). Critical values of the Liouville action functional $S(\psi)$ (the values on the extrema $\varphi$ for all surfaces $X$) determine a smooth function $S : \mathcal{M}_{0,n} \to \mathbb{R}$, the classical action of the Liouville equation. As was proved in [5, 6], the classical action of the Liouville equation plays a fundamental role in the complex geometry of the moduli space $\mathcal{M}_{0,n}$. Namely, $S$ is a common antiderivative for the accessory parameters

$$ c_i = -\frac{1}{2\pi} \frac{\partial S}{\partial z_i}, \quad i = 1, \ldots, n-3, $$

as well as a Kähler potential for the Weil-Petersson metric on $\mathcal{M}_{0,n}$,

$$ -\frac{\partial^2 S}{\partial z_i \partial z_j} = \left\langle \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right\rangle_{WP}, \quad i, j = 1, \ldots, n-3. $$

The statement that for surfaces of genus 0 the classical action of the Liouville equation is a common antiderivative for accessory parameters was conjectured by Polyakov\footnote{Lecture at Leningrad’s branch of V.A. Steklov Mathematical Institute, 1982, unpublished.} on the basis of the semiclassical analysis of the conformal Ward identities of the quantum Liouville theory (see [14]).
3. Real projective connections and V.I. Smirnov thesis

3.1. General case. Let \( X \) be a Riemann surface of type \((g,n)\). Projective connection \( R \) on \( X \) is called real (correspondingly, Fuchsian), if its monodromy group, up to a conjugation in \( \text{PSL}(2, \mathbb{C}) \), is a subgroup in \( \text{PSL}(2, \mathbb{R}) \) (correspondingly, is a Fuchsian group). By the uniformization theorem, Fuchsian projective connection \( R_F \) is uniquely characterized by the condition that its monodromy group is precisely the Fuchsian group \( \Gamma \) which uniformizes the Riemann surface \( X \). It is rather natural to ask whether it is possible to characterize a projective connection \( R_F \) on \( X \) by simpler conditions like being real (see [15, p. 224]) or Fuchsian? The answer to both these statements is negative.

Namely, when \( X = \Gamma \backslash \mathbb{H} \) is a compact Riemann surface of genus \( g > 1 \), Goldman [7] has shown that to every integral Thurston’s measurable geodesic lamination \( \mu = \sum i m_i \gamma_i \) (disjoint union), where \( \gamma_i \) are simple closed geodesics in the hyperbolic metric on \( X \), and \( m_i \) are non-negative integers, there corresponds a Riemann surface \( Gr_\mu(X) \) of genus \( g \) with a projective connection \( R(\mu) \) having the monodromy group \( \Gamma \). Riemann surfaces \( Gr_\mu(X) \) are obtained from \( X \) by the so-called “grafting” procedure that generalizes classic examples of Maskit-Hejhal and Sullivan-Thurston (see [7]). Moreover, the set of all Fuchsian projective connections on all Riemann surfaces of genus \( g \) is isomorphic to the direct product of the Teichmüller space \( T_g \) and the set of integral Thurston’s measurable geodesic laminations on a topological surface of genus \( g \). In [16] it is proved that on each Riemann surface \( X \) of genus \( g > 1 \) there are infinitely many Fuchsian projective connections.

Real projective connections on Riemann surfaces of type \((g,n)\) were also studied by Faltings [15]. As shown in [7] for compact Riemann surfaces, to each half-integral Thurston’s measurable lamination \( \mu \) there is a Riemann surface \( Gr_\mu(X) \) of genus \( g \) with the real projective connection.

3.2. Surfaces of type \((0,4)\) and V.I. Smirnov thesis. Classics have associated the uniformization problem of Riemann surfaces with the differential equations. As a basic example, they considered the case of Riemann surfaces of type \((0,4)\); the corresponding problem was to find an accessory parameter in equation \((2.4)\) such that its monodromy group was Fuchsian or Kleinian group (a discrete subgroup of \( \text{PSL}(2, \mathbb{C}) \)). For the special case of real singular points Klein [17] has proposed an approach that uses Sturm’s oscillation theorem, and Hilb has proved [18] that equation \((2.4)\) has a Fuchsian monodromy group for infinitely many values of the accessory parameter. Hilbert [19, Kap. XX] reduced this problem to the study of a certain integral equation.

The problem of real monodromy group of equation \((2.4)\) with four real singular points has been completely solved by V.I. Smirnov in his thesis [8], published in Petrograd in 1918 (its main content was presented in [9, 10]). Namely, consider equation \((2.4)\) with singular points \( z_1 = 0, z_2 = a, z_3 = 0 \).
$z_4 = \infty$, where $0 < a < 1$. Writing general solution of equations (2.5) in the form

$$c_1 = 1 + \frac{1 + 2\lambda}{a}, \quad c_2 = \frac{1 + 2\lambda}{a(a - 1)}, \quad c_3 = -\frac{a + 2\lambda}{a - 1},$$

where $\lambda$ is the accessory parameter, and performing the change of the dependent variable $y = \sqrt{z(z - a)(z - 1)}u$, we arrive at the equation

$$\begin{array}{c}
\frac{d}{dz} \left( p(z) \frac{dy}{dz} \right) + (z + \lambda) y = 0, \\
p(z) = z(z - a)(z - 1).
\end{array}$$

Denote by $(y^{(1)}_i, y^{(2)}_i)$ the standard basis in the solution space (3.1), which in the neighborhood a singular point $z_i$ consists of normalized holomorphic solutions

$$y^{(1)}_i(z, \lambda) = 1 + \sum_{k=1}^{\infty} a_{ik}(z - z_i)^k, \quad i = 1, 2, 3,$$

$$y^{(2)}_i(z, \lambda) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{a_{ik}}{z^k},$$

and the solutions

$$y^{(2)}_i(z, \lambda) = y^{(1)}_i(z, \lambda) \log(z - z_i) + \tilde{y}_i(z, \lambda), \quad i = 1, 2, 3,$$

$$y^{(2)}_4(z, \lambda) = y^{(1)}_4(z, \lambda) \log \frac{1}{z} + \tilde{y}_4(z, \lambda),$$

where $\tilde{y}_i(z, \lambda)$ are holomorphic in the neighborhood of $z_i$. For real $\lambda$ the power series $y^{(1,2)}_i(z, \lambda)$ and $\tilde{y}_i(z, \lambda)$ have real coefficients.

To determine real $\lambda$ for which the monodromy group of equation (3.1) is real, classics have used the notion of real continuation. Namely (see [8]), if in the neighborhood of a singular point $a$ for real $z < a$ we have

$$y(z) = c \log(a - z) + f(z),$$

where $\log 1 = 0$ and the function $f(z)$ is holomorphic in a neighborhood of $a$, then the real continuation of $y$ to the domain $z > a$ is defined as

$$y(z) = c \log(z - a) + f(z).$$

The following statement holds.

**Theorem 1** (Klein, Hilbert). Equation (3.1) has a real monodromy group if $\lambda$ is real and one of the following conditions holds.

I) The solution $y^{(1)}_0(z, \lambda)$ is holomorphic in a neighborhood of the singular point $z_2 = a$.

II) The solution $y^{(1)}_1(z, \lambda)$ is holomorphic in a neighborhood of the singular point $z_3 = 1$.

III) Under the real continuation through $z = a$ the solution $y^{(1)}_0(z, \lambda)$ is holomorphic in a neighborhood of the singular point $z_3 = 1$. 

Wherein in the case I) the ratio $\eta = \sqrt{-1} y_3^{(1)}/y_1^{(1)}$ of linear independent solutions of equation $= (3.1)$, when going around the singular points 0, $a$ and 1, is transformed by real linear fractional transformations; in the case II) analogous role is played by the ratio $\eta = \sqrt{-1} y_2^{(1)}/y_4^{(1)}$, and in the case III) — by the ratio $\eta = \sqrt{-1} y_1^{(1)}/y_2^{(1)}$ (see [8] for details).

Conditions I)-III) determine the following three Sturm-Liouville type spectral problems for equation (3.1). Namely, one needs to determine such values of $\lambda$ that

1) on the interval $[0,a]$ there is a solution which is regular at 0 and $a$;
2) on the interval $[a,1]$ there is a solution which is regular at $a$ and 1;
3) there is a solution regular at 0 such that under the real continuation through $a$ it is regular at 1.

Using classical Sturm’s method, V.I. Smirnov [8] proved the following result.

**Theorem 2** (V.I. Smirnov). Each of Sturm-Liouville problems 1) – 3) has simple unbounded discrete spectrum. Namely, the following statements hold.

i) The spectral problem 1) has infinitely many eigenvalues $\mu_k$, $k \in \mathbb{N}$, accumulating to $\infty$ and satisfying the inequalities

$$-a < \mu_1 < \mu_2 < \ldots$$

ii) The spectral problem 2) has infinitely many eigenvalues $\mu_{-k}$, $k \in \mathbb{N}$, accumulating to $-\infty$ and satisfying the inequalities

$$-a > \mu_{-1} > \mu_{-2} > \ldots$$

iii) The spectral problem 3) has infinitely many eigenvalues $\lambda_k$, $k \in \mathbb{Z}$, accumulating to $\pm \infty$ and satisfying the inequalities

$$\cdots < \mu_{-2} < \lambda_{-1} < \mu_{-1} < \lambda_0 < \mu_1 < \lambda_1 < \mu_2 < \ldots$$

The case $\lambda = \lambda_0$ corresponds to the Fuchsian uniformization of the Riemann surface $X = \mathbb{C} \setminus \{0,a,1\}$ and the ratio $\eta = \sqrt{-1} y_1^{(1)}/y_2^{(1)}$ maps one-to-one the upper half-plane of the variable $z$ to the interior of a circular rectangle with zero angles and sides orthogonal to $\mathbb{R} \cup \{\infty\}$. Normalizing $\eta$ by a real linear fractional transformation such that the images of all singular points 0, $a$, $1$ and $\infty$ are finite, we get a rectangle on Fig. 1 (cf. Fig. 9 in [17]).

![Fig. 1](image-url)
Analytically continuing $\eta(z)$ to the lower half-plane of the variable $z$ we obtain multi-valued linear polymorphic function $\eta : X \to \mathbb{H}$ with Fuchsian group $\Gamma$ such that $J = \eta^{-1}$ determines the isomorphism (2.3).

Corresponding ratio $\eta$ is a one-to-one function on the upper half-plane of variable $z$ also in cases $\lambda = \mu_{\pm 1}$. Thus when $\lambda = \mu_1$ we have $\eta(0) = \eta(a) = \infty$, $\eta(1) = \eta(\infty) = 0$. Normalizing $\eta = \sqrt{-1} y_3^{(1)} / y_1^{(1)}$ such that the images of the singular points are finite, we obtain a one-to-one map $\eta$ of the upper half-plane of the variable $z$ onto the interior of a degenerate circular rectangle on Fig. 2. Corresponding monodromy groups will be Schottky groups.

For all other values of $\lambda_k$ and $\mu_k$ corresponding map $\eta$ is no longer one-to-one mapping on the upper half-plane of the variable $z$. Thus when $\lambda = \lambda_1$, the upper half-plane is mapped onto the interior of an annulus on Fig. 3 (cf. Fig. 10 in [17]). Here the function $\eta$ takes twice the values from the marked darker domain, which corresponds to the rectangle in Fig. 1. When $\lambda = \lambda_k$, this rectangle is wrapped over itself $2|k|$ times.

Similarly, when $\lambda = \mu_2$ the upper half-plane of the variable $z$ maps onto the interior of the annulus on Fig. 4. Here the function $\eta$ takes twice the values from the marked darker domain, which corresponds to the degenerate
rectangle in Fig. 2. When $\lambda = \mu_k$, this rectangle is wrapped over itself $|k|$ times.

It is instructive to compare these results of V.I. Smirnov with Goldman’s classification of Fuchsian and real projective connections on Riemann surfaces, generalized for the surfaces of type $(g, n)$. The Fuchsian series $\lambda = \lambda_k$ correspond to integral laminations in $[7]$, while series $\lambda = \mu_k$ corresponds to half-integral laminations.

4. BLACK HOLE TYPE SOLUTIONS OF THE LIOUVILLE EQUATION

Fuchsian uniformization of a Riemann surface $X$ determines a solution of the Liouville equation: a smooth function $\varphi$ on $X$, satisfying equation (2.8) and having asymptotics (2.9) (see Section 2.3). The function $\varphi$ is obtained from the ratio $J^{-1}$ of linear independent solutions of equation (2.4) by formula (2.7). This formula is well-defined because of the condition that the monodromy group $\Gamma$ of equation (2.4) is real; the smoothness of $\varphi$ is quarantined by the condition that $\Gamma$ uniformizes the Riemann surface $X$ and its image under the multi-valued map $J^{-1}$ is the upper half-plane $\mathbb{H}$.

Similarly, with each equation (2.4) with the real monodromy group one associates a solution of the Liouville equation. Put

\begin{equation}
\varphi(z) = \frac{|\eta'(z)|^2}{(\text{Im} \eta(z))^2},
\end{equation}

where $\eta$ is the ratio of linear independent solutions of equation (2.4), which transform by fractional linear transformations when going around the singular points (in the Fuchsian case $\eta = J^{-1}$). The function $\varphi$ is well-defined because the monodromy group is real, and has asymptotics (2.9). The latter follows from the theory of Fuchsian equations with equal exponents. However, solution (4.1) is no longer smooth: the image of $X$ under the multi-valued map $\eta$ has a nontrivial intersection with the real axis and the function $\varphi$ is singular on $\eta^{-1}(\mathbb{R})$.

Namely, it follows from results in [15, §6] that the inverse image $\eta^{-1}(\mathbb{R})$ is a disjoint union of finitely many simple closed analytic curves on $X$. Let $C$
be one of such curves. There is a branch of the multi-valued function \( \eta \) which maps \( C \) one-to-one onto the circle, so that \( C = \{ z = \eta^{-1}(t), t \in [\alpha, \beta] \} \). It is convenient to introduce the Schwarz function \( S \) of the analytic contour \( C \) by the formula
\[
S = \eta^{-1} \circ \eta,
\]
where \( \eta^{-1}(z) = \eta^{-1}(\bar{z}) \). The Schwarz function is defined in some neighborhood of the contour \( C \) and determines it by the equation \( \bar{z} = S(z) \) (see [20]).

In terms of the Schwarz function it is easy to show that the solution \( \varphi \) has the same singularities on \( C \) as the function
\[
-\frac{4S'(z)}{(z - S(z))^2}.
\]

Namely, as \( z \to z_0 \in C \) along any non-tangential to \( C \) direction,
\[
e^{\varphi(z)} = -\frac{4S'(z_0)}{\left(\bar{z} - \bar{z}_0 - S'(z_0)(z - z_0)\right)^2}\left(1 + O(|z - z_0|)\right).
\]

Note that due to the condition \( S(S(z)) = z \) the function in the right hand side of (4.3) is real and positive. The singularities of the type (4.2)–(4.3) on a contour \( C \) are similar to the singularity on \( \mathbb{R} \) of the Poincaré metric on \( \mathbb{C} \setminus \mathbb{R} \), which corresponds to the Schwarz function \( S(z) = z \).

Thus we can state the following problem. On the Riemann surface \( X = \mathbb{C} \setminus \{ z_1, \ldots, z_{n-3}, 0, 1 \} \) find simple analytic contours \( C_1, \ldots, C_k \) and a function \( \varphi \) such that on \( X \setminus \bigcup_{j=1}^k C_j \) the function \( \varphi \) satisfy the Liouville equation (2.8), has asymptotics (2.9) at the punctures \( z_j \) and singularities of the type (4.2)–(4.3) on the contours \( C_j \). On each of the connected components of \( X \setminus \bigcup_{j=1}^k C_j \) \( e^{\varphi(z)}|dz|^2 \) determines a complete metric of constant negative curvature \(-1\).

From Goldman’s classification of real projective connections [7] it follows that there exists a family of such solutions parameterized by the “integral lattice” of integral and half-integral measurable Thurston’s laminations, implicitly defined by the grafting procedure.

From results in V.I. Smirnov thesis one gets rather explicit description of black hole type solutions for the case of four real singular points. Namely, from Theorem 2 we obtain the following result.

**Theorem 3.** All black hole type solutions of the Liouville equation with four real punctures 0, a, 1 and \( \infty \) are described as follows.

1) **Solutions of the Fuchsian type**, which correspond to the values of accessory parameter \( \lambda = \lambda_k \), with integer \( k \), and having \( 2|k| \) contours \( C_j \). When \( k > 0 \) these contours go over the points 0 and a, and when \( k < 0 \) — over the points a and 1.

2) **Solutions of the Schottky type**, which correspond to the values of accessory parameter \( \lambda = \mu_k \), with integer \( k \neq 0 \), and having \( 2|k| - 1 \)
contours $C_j$. When $k > 0$ these contours go over the points $0$ and $a$, and when $k < 0$ — over the points $a$ and $1$.

In the general case it is convenient to make a substitution $\chi(z) = e^{-\varphi(z)/2}$, which transform the Liouville equation (2.8) to

$$-(\chi_{zz} + |\chi_z|^2) = \frac{1}{4},$$

and asymptotics (2.9) — to

$$\chi(z) = \begin{cases} |z - z_i||\log|z - z_i||(1 + o(1)), & z \to z_i, \ i \neq n, \\ |z|\log|z|(1 + o(1)), & z \to \infty. \end{cases}$$

Singularities (4.2) transform into the vanishing condition on the contour $C$:

$$\chi(z) \sim \frac{z - S(z)}{2\sqrt{-S'(z)}},$$

so that under the Schwarz reflection $z^* = S(z)$ through $C$ real-valued function $\chi(z)$ changes sign. Elliptic partial differential equation (4.4) with asymptotics (4.5) and vanishing conditions (4.6) on contours $C_j$ is a boundary value problem with free boundary. For its solution it would interesting to use the continuation with respect to a parameter method together with the a priori estimates, as it was done in [3] for the Liouville equation (2.8) with asymptotics (2.9).

In conclusion, we note that the function $\chi$ plays an important role in the theory of the Liouville equation. Namely, it is a bilinear form in solutions of equation (2.4) and their complex conjugates and it equation (2.4)

$$\chi_{zz} + \frac{1}{2}T_\varphi \chi = 0$$

and the complex conjugate equation. In the quantum Liouville theory the field $\chi = e^{-\varphi(z)/2}$ describes a degenerate at the level 2 vector in the Verma module for the Virasoro algebra. For the black hole type solutions the function $\chi$ still satisfies equation (4.7). It would be interesting to find out what role it plays in the quantum Liouville theory.

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