COINCIDENCE THEOREMS FOR NONLINEAR HYBRID CONTRACTIONS

Y.J. CHO
Deptartment of Mathematics
Gyeongsang National University
Chinju 660-701, Korea

B. FISHER
Department of Mathematics
and Computer Science
University of Leicester
Leicester LE1 7RH, England

G.S. GENGA
Deptartment of Mathematics
Sunchon National University
Sunchon 540-742, Korea

(Received June 5, 1995 and in revised form December 12, 1995)

ABSTRACT. In this paper, we give some common fixed point theorems for single-valued mappings and multi-valued mappings satisfying a rational inequality. Our theorems generalize some results of B. Fisher, M. L. Diviccaro et al. and V. Popa

KEY WORDS AND PHRASES: Compatible mappings, weakly commuting mappings, coincidence points and fixed points.

1991 AMS SUBJECT CLASSIFICATION CODES: 54H25, 47H10.

1. INTRODUCTION

Let $(X,d)$ be a metric space and let $f$ and $g$ be mappings from $X$ into itself. In [1], Sessa defined $f$ and $g$ to be weakly commuting if

$$d(gfx,fgx) \leq d(gx,fx)$$

for all $x$ in $X$. It can be seen that two commuting mappings are weakly commuting, but the converse is false as shown in the Example of [2].

Recently, G. Jungck [3] extended the concept of weak commutativity in the following way.

DEFINITION 1.1. Let $f$ and $g$ be mappings from a metric space $(X,d)$ into itself. The mappings $f$ and $g$ are said to be compatible if

$$\lim_{n \to \infty} (fgx_n, gfx_n) = 0$$

whenever $(x_n)$ is a sequence in $X$ such that $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = z$ for some $z$ in $X$.

It is obvious that two weakly commuting mappings are compatible, but the converse is not true. Some examples for this fact can be found in [3].

Recently, H. Kaneko [4] and S. L. Singh et al. [5] extended the concepts of weak commutativity and compatibility [6] for single-valued mappings to the setting of single-valued and multi-valued mappings, respectively.

Let $(X,d)$ be a metric space and let $CB(X)$ denote the family of all nonempty closed and bounded subsets of $X$. Let $H$ be the Hausdorff metric on $CB(X)$ induced by the metric $d$, i.e.,

$$H(A,B) = \max\left\{ \sup_{x \in A} d(x,B), \sup_{y \in B} d(y,A) \right\}$$
for \( A, B \in CB(X) \), where \( d(x, A) = \inf_{y \in A} d(x, y) \).

It is well-known that \((CB(X), H)\) is a metric space, and if a metric space \((X, d)\) is complete, then \((CH(X), H)\) is also complete.

Let \( \delta(A, B) = \sup \{d(x, y) : x \in A \text{ and } y \in B\} \) for all \( A, B \in CB(X) \). If \( A \) consists of a single point \( a \), then we write \( \delta(A, B) = \delta(a, B) \). If \( \delta(A, B) = 0 \), then \( A = B = \{a\} \).

**Lemma 1.1** \([8]\). Let \( A, B \in CB(X) \) and \( k > 1 \). Then for each \( a \in A \), there exists a point \( b \in B \) such that \( d(a, b) \leq k \delta(A, B) \).

Let \((X, d)\) be a metric space and let \( f : X \to X \) and \( S : X \to CB(X) \) be single-valued and multi-valued mappings, respectively.

**Definition 1.2.** The mappings \( f \) and \( S \) are said to be weakly commuting if for all \( x \in X \), \( fSx \in CB(X) \) and

\[ H(Sfx, fSx) \leq d(fx, Sx), \]

where \( H \) is the Hausdorff metric defined on \( CB(X) \).

**Definition 1.3.** The mappings \( f \) and \( S \) are said to be compatible if

\[ \lim_{n \to \infty} d(fy_n, Sfx_n) = 0 \]

whenever \( \{x_n\} \) and \( \{y_n\} \) are sequences in \( X \) such that \( \lim_{n \to \infty} fx_n = \lim_{n \to \infty} y_n = z \) for some \( z \in X \), where \( y_n \in Sx_n \) for \( n = 1, 2, \ldots \).

**Remark 1.1.**

1. Definition 1.3 is slightly different from the Kaneko's definition \([6]\).
2. If \( S \) is a single-valued mapping on \( X \) in Definitions 1.2 and 1.3, then Definitions 1.2 and 1.3 become the definitions of weak commutativity and compatibility for single-valued mappings.
3. If the mappings \( f \) and \( S \) are weakly commuting, then they are compatible, but the converse is not true.

In fact, suppose that \( f \) and \( S \) are weakly commuting and let \( \{x_n\} \) and \( \{y_n\} \) be two sequences in \( X \) such that \( y_n \in Sx_n \) for \( n = 1, 2, \ldots \) and \( \lim_{n \to \infty} fx_n = \lim_{n \to \infty} y_n = z \) for some \( z \in X \). From \( d(fx_n, Sx_n) \leq d(fx_n, y_n) \), it follows that \( \lim_{n \to \infty} d(fx_n, Sx_n) = 0 \). Thus, since \( f \) and \( S \) are weakly commuting, we have

\[ \lim_{n \to \infty} H(Sfx_n, fSx_n) = 0. \]

On the other hand, since \( d(fy_n, Sfx_n) \leq H(fSx_n, Sfx_n) \), we have

\[ \lim_{n \to \infty} d(fy_n, Sfx_n) = 0, \]

which means that \( f \) and \( S \) are compatible.

**Example 1.1.** Let \( X = [1, \infty) \) be a set with the Euclidean metric \( d \) and define \( fx = 2x^4 - 1 \) and \( Sx = [1, x^2] \) for all \( x \geq 1 \). Note that \( f \) and \( S \) are continuous and \( S(X) = f(X) = X \). Let \( \{x_n\} \) and \( \{y_n\} \) be sequences in \( X \) defined by \( x_n = y_n = 1 \) for \( n = 1, 2, \ldots \). Thus we have

\[ \lim_{n \to \infty} fx_n = \lim_{n \to \infty} y_n = 1 \in X, \quad y_n \in Sx_n. \]

On the other hand, we can show that \( H(fSx_n, Sfx_n) = 2(x_n^4 - 1)^2 \to 0 \) if and only if \( x_n \to 1 \) as \( n \to \infty \) and so, since \( d(fy_n, Sfx_n) \leq H(fSx_n, Sfx_n) \), we have

\[ \lim_{n \to \infty} d(fy_n, Sfx_n) = 0. \]

Therefore, \( f \) and \( T \) are compatible, but \( f \) and \( T \) are not weakly commuting at \( x = 2 \).
We need the following lemmas for our main theorems, which is due to G. Jungck [2]

**LEMMA 1.2.** Let \( f \) and \( g \) be mappings from a metric space \((X, d)\) into itself. If \( f \) and \( g \) are compatible and \( fz = gz \) for some \( z \in X \), then
\[
fz = gz = gfz = ffz.
\]

**LEMMA 1.3.** Let \( f \) and \( g \) be mappings from a metric space \((X, d)\) into itself. If \( f \) and \( g \) are compatible and \( fz \to z \) for some \( z \in X \), then we have the following
\[
\begin{align*}
(1) & \quad \lim_{n \to \infty} gfz_n = fz \text{ if } f \text{ is continuous at } z, \\
(2) & \quad fgz = gfz \text{ and } fz = gz \text{ if } f \text{ and } g \text{ are continuous at } z
\end{align*}
\]

2. COINCIDENCE THEOREMS FOR NONLINEAR HYBRID CONTRACTIONS

In this section, we give some coincidence point theorems for nonlinear hybrid contractions, i.e., contractive conditions involving single-valued and multi-valued mappings. In the following Theorem 2.1, \( S(X) \) and \( T(X) \) mean \( S(X) = \bigcup_{x \in X} Sx \) and \( T(X) = \bigcup_{x \in X} Tx \), respectively.

**THEOREM 2.1.** Let \((X, d)\) be a complete metric space. Let \( f, g : X \to X \) be continuous mappings and \( S, T : X \to CB(X) \) be \( H \)-continuous multi-valued mappings such that
\[
T(X) \subset f(X) \quad \text{and} \quad S(X) \subset g(X),
\]
the pairs \( f, S \) and \( g, T \) are compatible mappings,
\[
H_p(Sx, Ty) \leq \frac{cd(fx, Sx)d^p(gy, Ty) + bd(fx, Ty)d^p(gy, Sx)}{\delta(fx, Sx) + \delta(gy, Ty)}
\]
for all \( x, y \in X \) for which \( \delta(fx, Sx) + \delta(gy, Ty) \neq 0 \), where \( p \geq 1, b \geq 0 \) and \( 1 < c < 2 \). Then there exists a point \( z \in X \) such that \( fz \in Sz \) and \( gz \in Tz \), i.e., \( z \) is a coincidence point of \( f, S \) and \( g, T \).

**PROOF.** Choose a real number \( k \) such that \( 1 < k < \left( \frac{2}{c} \right)^\frac{1}{p} \) and let \( x_0 \) be an arbitrary point in \( X \). Since \( Sz_0 \subset g(X) \), there exists a point \( x_1 \in X \) such that \( gx_1 \in Sz_0 \) and so there exists a point \( y \in Tx_1 \) such that
\[
d(gx_1, y) \leq kH(Sx_0, Tx_1),
\]
which is possibly by Lemma 1.1. Since \( Tx_1 \subset f(X) \), there exists a point \( x_2 \in X \) such that \( y = fx_2 \) and so we have
\[
d(gx_1, fx_2) \leq kH(Sx_0, Tx_1).
\]
Similarly, there exists a point \( x_3 \in X \) such that \( gx_3 \in Sz_2 \) and
\[
d(gx_3, fx_2) \leq kH(Sx_2, Tx_1).
\]
Inductively, we can obtain a sequence \( \{x_n\} \) in \( X \) such that
\[
f(x_{2n}) \in Tx_{2n-1}, \quad n \in N,
\]
\[
gx_{2n+1} \in Sz_{2n}, \quad n \in N_0 = N \cup \{0\},
\]
\[
d(gx_{2n+1}, fx_{2n}) \leq kH(Sx_{2n}, Tx_{2n-1}), \quad n \in N,
\]
\[
d(gx_{2n+1}, fx_{2n+2}) \leq kH(Sx_{2n}, Tx_{2n+1}), \quad n \in N_0,
\]
where \( N \) denotes the set of positive integers.

First, suppose that for some \( n \in N \)
\[
\delta(fx_{2n}, Sz_{2n}) + \delta(gx_{2n+1}, Tx_{2n+1}) = 0.
\]
Then $f_{2n} \in S_{2n}$ and $g_{2n+1} \in T_{2n+1}$ and so $x_{2n}$ is a coincidence point of $f$ and $S$ and $x_{2n+1}$ is a coincidence point of $g$ and $T$.

Similarly, $\delta(f_{2n+2}, S_{2n+2}) + \delta(g_{2n+1}, T_{2n+1}) = 0$ for some $n \in N$ implies that $x_{2n+1}$ is a coincidence point of $g$ and $T$ and $x_{2n+2}$ is a coincidence point of $f$ and $S$.

Now, suppose that $\delta(f_{2n+1}, S_{2n+1}) + \delta(g_{2n+1}, T_{2n+1}) \neq 0$ for $n \in N_0$. Then, by (2.3), we have

$$d^p(g_{2n+1}, f_{2n+2}) \leq k^p H^p(S_{2n}, T_{2n+1})$$

$$\leq k^p \frac{cd(f_{2n}, S_{2n})d^p(g_{2n+1}, T_{2n+1}) + bd(f_{2n+1}, T_{2n+1})d^p(g_{2n+1}, S_{2n})}{\delta(f_{2n}, S_{2n}) + \delta(g_{2n+1}, T_{2n+1})}$$

$$\leq k^p \frac{cd(f_{2n}, g_{2n+1})d^p(g_{2n+1}, f_{2n+2}) + bd(f_{2n+1}, f_{2n+2})d^p(g_{2n+1}, g_{2n+1})}{\delta(f_{2n}, S_{2n}) + \delta(g_{2n+1}, T_{2n+1})}$$

$$\leq k^p \frac{cd(f_{2n}, g_{2n+1})d^p(g_{2n+1}, f_{2n+2})}{d(f_{2n}, g_{2n+1}) + d(g_{2n+1}, f_{2n+2})}.$$

(2.4)

If $d(g_{2n+1}, f_{2n+2}) = 0$ and $d(f_{2n}, g_{2n+1}) \neq 0$ in (2.4), then $g_{2n+1} = f_{2n+2} \in T_{2n+1}$ and so $x_{2n+1}$ is a coincidence point of $g$ and $T$. But the case of $d(f_{2n}, g_{2n+1}) = 0$ and $d(g_{2n+1}, f_{2n+2}) \neq 0$ in (2.4) cannot occur.

In fact, if $d(f_{2n}, g_{2n+1}) = 0$ and $d(g_{2n+1}, f_{2n+2}) \neq 0$ in (2.4), then we have $d(g_{2n+1}, f_{2n+2}) = 0$, which is impossible. From (2.4), we have

$$d^p(g_{2n+1}, f_{2n+2})[d(f_{2n}, g_{2n+1}) + d(g_{2n+1}, f_{2n+2})]$$

$$\leq k^p cd(f_{2n}, g_{2n+1})d^p(g_{2n+1}, f_{2n+2}),$$

which implies that

$$d(g_{2n+1}, f_{2n+2}) \leq (k^p - 1)d(f_{2n}, g_{2n+1}).$$

On the other hand, from (2.3), we have

$$d^p(g_{2n+3}, f_{2n+2})$$

$$\leq k^p H^p(S_{2n+2}, T_{2n+1})$$

$$\leq k^p \frac{cd(f_{2n+2}, S_{2n+2})d^p(g_{2n+1}, T_{2n+1}) + bd(f_{2n+1}, T_{2n+1})d^p(g_{2n+1}, S_{2n+2})}{\delta(f_{2n+2}, S_{2n+2}) + \delta(g_{2n+1}, T_{2n+1})}$$

$$\leq k^p \frac{cd(f_{2n+2}, g_{2n+3})d^p(g_{2n+1}, f_{2n+2}) + bd(f_{2n+1}, f_{2n+2})d^p(g_{2n+1}, g_{2n+1})}{\delta(f_{2n+2}, g_{2n+3}) + d(g_{2n+1}, f_{2n+2})},$$

which implies that, if $\alpha = d(x_{2n+3}, f_{2n+2})/d(f_{2n+2}, g_{2n+1})$, then $\alpha^p + \alpha^{p-1} \leq k^p \alpha$ Thus $\alpha < 1$ and we have

$$d(g_{2n+3}, f_{2n+2}) \leq d(f_{2n+2}, g_{2n+1}).$$

Repeating the above argument, since $0 \leq k^p - 1 < 1$, it follows that $\{g_{2n}, f_{2n}, g_{2n+1}, f_{2n+2}, \ldots, g_{2n-1}, g_{2n}, g_{2n+1}, \ldots\}$ is a Cauchy sequence in $X$. Since $(X, d)$ is a complete metric space, let $\lim_{n \to \infty} g_{2n+1} = \lim_{n \to \infty} f_{2n} = z$.

Now, we will prove that $fz \in Sz$, that is, $z$ is a coincidence point of $f$ and $S$. For every $n \in N$, we have

$$d(fgx_{2n+1}, Sz) \leq d(fg_{2n+1}, Sf_{2n}) + H(Sf_{2n}, Sz).$$

(2.5)

It follows from the $H$-continuity of $S$ that

$$\lim_{n \to \infty} H(Sf_{2n}, Sz) = 0$$

(2.6)

since $f_{2n} \to z$ as $n \to \infty$. Since $f$ and $S$ are compatible mappings and $\lim_{n \to \infty} f_{2n} = \lim_{n \to \infty} y_n = z$, where $y_n = g_{2n+1} \in S_{2n}$ and $z_n = x_{2n}$, we have
\[ \lim_{n \to \infty} d(fy_n, Sz_n) = \lim_{n \to \infty} d(fgx_{2n+1}, Sz_{2n}) = 0. \]  

Thus, from (2.5), (2.6) and (2.7), we have \( \lim_{n \to \infty} d(fgx_{2n+1}, Sz) = 0 \) and so, from

\[ d(fz, Sz) \leq d(fz, fgx_{2n+1}) + d(fgx_{2n+1}, Sz) \]

and the continuity of \( f \), it follows that \( d(fz, Sz) = 0 \), which implies that \( fz \in Sz \) since \( Sz \) is a closed subset of \( X \). Similarly, we can prove that \( gz \in Tz \), that is, \( z \) is a coincidence point of \( g \) and \( T \). This completes the proof.

If we put \( f = g = i_x \) (the identity mapping on \( X \)) in Theorem 2.1, we have the following

**COROLLARY 2.2** ([1]) Let \( (X, d) \) be a complete metric space and let \( S, T : X \to CB(X) \) be \( H \)-continuous multi-valued mappings such that

\[ H^p(Sx, Ty) \leq \frac{cd(x, Sz)d^p(y, Ty) + bd(x, Ty)d^p(y, Sz)}{\delta(x, Sz) + \delta(y, Ty)} \]  

for all \( x, y \in X \) for which \( \delta(x, Sz) + \delta(y, Ty) \neq 0 \), where \( p \geq 1, b \geq 0 \) and \( 1 < c < 2 \). Then \( S \) and \( T \) have a common fixed point in \( X \), that is, \( z \in Sz \) and \( z \in Tz \).

Assuming that \( f = g = i_x \) on \( X \) in Theorem 2.1, we have the following

**COROLLARY 2.3.** Let \( (X, d) \) be a complete metric space and let \( f : X \to X \) be a continuous single-valued mapping and \( S : X \to CB(X) \) be an \( H \)-continuous multi-valued mapping such that

\[ S(X) \subset f(X), \]

\[ f \text{ and } S \text{ are continuous mappings}, \]

\[ H^p(Sx, Ty) \leq \frac{cd(fx, Sz)d^p(fy, Sy) + bd(fx, Ty)d^p(fy, Sz)}{\delta(fx, Sz) + \delta(fy, Sy)} \]

for all \( x, y \in X \) for which \( \delta(fx, Sz) + \delta(fy, Sy) \neq 0 \), where \( p \geq 1, b \geq 0 \) and \( 1 < c < 2 \). Then there exists a point \( z \in X \) such that \( fz \in Sz \), i.e., \( z \) is a coincidence point of \( f \) and \( S \).

**REMARK 2.1.** If we put \( p = 1 \) in Theorem 2.1, Corollaries 2.2 and 2.3, we can obtain further corollaries.

3. FIXED POINT THEOREMS FOR SINGLE-VALUED MAPPINGS

In this section, using Theorem 2.1, we can obtain some fixed point theorems for single-valued mappings in a metric space.

If \( S \) and \( T \) are single-valued mappings from a metric space \( (X, d) \) into itself in Theorem 2.1, we have the following

**THEOREM 3.1.** Let \( (X, d) \) be a complete metric space. Let \( f, g, S \) and \( T \) be continuous mappings from \( X \) into itself such that

\[ S(X) \subset g(X) \text{ and } T(X) \subset f(X), \]

the pairs \( f, S \) and \( g, T \) are compatible mappings,

\[ \text{either (i) } d^p(Sx, Ty) \leq \frac{cd(fx, Sz)d^p(gy, Ty) + bd(fx, Ty)d^p(gy, Sz)}{d(fx, Sz) + d(gy, Ty)}, \]

\[ \text{if } d(fx, Sz) + d(gy, Ty) \neq 0 \text{ for all } x, y \in X, \text{ where } p \geq 1, b \geq 0 \text{ and } 1 < c < 2, \text{ or} \]

\[ \text{(ii) } d(Sz, Ty) = 0 \text{ if } d(fx, Sz) + d(gy, Ty) = 0. \]
Then \( f, g, S \) and \( T \) have a unique common fixed point \( z \) in \( X \). Further, \( z \) is the unique common fixed point of \( f, S \) and of \( g, T \).

**Proof.** The existence of the point \( w \) with \( f w = S w \) and \( g w = T w \) follows from Theorem 2.1. From (ii) of (3.3), since \( d(fw, Sw) + d(gw, Tw) = 0 \), it follows that \( d(Sw, Tw) = 0 \) and so \( Sw = f w = gw = Tw \). By Lemma 1.2, since \( f \) and \( S \) are compatible mappings and \( f w = Sw \), we have

\[
Sw = S Sw = f Sw = f w,
\]

which implies that \( d(fSw, SSw) + d(gw, Tw) = 0 \) and, using the condition (ii) of (3.3), we have

\[
Sw = SSw = Tw = gw = f w
\]

and so \( f w = z \) is a fixed point of \( S \). Further, (3.4) and (3.5) implies that

\[
Sz = f Sw = SSw = f z = z.
\]

Similarly, since \( g \) and \( T \) are compatible mappings, we have \( T z = g z = z \). Using (ii) of (3.3), since \( d(fz, Sz) + d(gz, Tz) = 0 \), it follows that \( d(Sz, Tz) = 0 \) and so \( Sz = Tz \). Therefore, the point \( z \) is a common fixed point of \( f, g, S \) and \( T \).

Next, we will show the uniqueness of the common fixed point \( z \). Let \( z' \) be another common fixed point of \( f \) and \( S \). Using the condition (ii) of (3.3), since \( d(fz', Sz') + d(gz, Tz) = 0 \), it follows that \( dz, z' = d(Tz, Sz') = 0 \) and so \( z = z' \). This completes the proof.

Now, we give an example of Theorem 3.1 with \( p = 1 \) and \( f, g \)

**Example 3.1.** Let \( X = \{1, 2, 3, 4\} \) be a finite set with the metric \( d \) defined by

\[
\begin{align*}
d(1, 3) &= d(1, 4) = d(2, 3) = d(2, 4) = 1, \\
d(1, 2) &= d(3, 4) = 2.
\end{align*}
\]

Define mappings \( f, S, T : X \rightarrow X \) by

\[
\begin{align*}
f(1) &= 1, \quad f(2) = 2, \quad f(3) = 4, \quad f(4) = 3, \\
S(1) &= S(2) = S(4) = 2, \quad S(3) = 3, \\
T(1) &= T(2) = T(3) = T(4) = 2.
\end{align*}
\]

From

\[
\begin{align*}
S f(1) &= S(1) = 2 = f(2) = fS(1), \\
S f(2) &= S(2) = 2 = f(2) = fS(2), \\
d(S f(3), f S(3)) &= d(S(4), f(3)) = d(2, 4) < 2 = d(3, 4) = d(S(3), f(3))
\end{align*}
\]

and

\[
\begin{align*}
d(S f(4), f S(4)) &= d(S(3), f(2)) = d(3, 2) = 1 = d(2, 3) = d(S(4), f(4)).
\end{align*}
\]

it follows that \( f \) and \( S \) are weakly commuting mappings and so they are compatible. Clearly, \( f, S \) and \( T \) are continuous and

\[
S(x) = \{2, 3\} \subset X = f(X), \quad T(X) = \{2\} \subset X = f(X).
\]

Further, we can show that the inequality (i) of (3.3) holds with \( c = \frac{3}{2} \) and \( b = 2 \) and the condition (ii) of (3.3) holds only for the point 2. Therefore, all the conditions of Theorem 3.1 are satisfied and the point 2 is a unique common fixed point of \( f, S \) and \( T \).

**Remark 3.1.** Theorem 3.1 assures that \( f, g, S \) and \( T \) have a unique common fixed point in \( X \). However, either \( f \) or \( g \) or \( S \) or \( T \) can have other fixed points. Indeed, in Example 3.1, \( f \) and \( S \) have two fixed points.
REMARK 3.2. From the proof of Theorem 3.1, it follows that if the condition (ii) of (3.3) is omitted in the hypothesis of Theorem 3.1, then $f, g, S$ and $T$ have a coincidence point $w$, i.e., $fw = gw = Sw = Tw$

If we put $f = g = i_x$ in Theorem 3.1, we have the following.

COROLLARY 3.2. Let $(X, d)$ be a complete metric space and let $S, T : X \to X$ be continuous mappings such that

\begin{equation}
\text{either (i) } d^p(Sx, Ty) \leq \frac{cd(x, Sx)d^p(y, Ty) + bd(x, Ty)d^p(y, Sx)}{d(x, Sx) + d(y, Ty)} \quad (3.5)
\end{equation}

for all $x, y \in X$ if $d(x, Sx) + d(y, Ty) \neq 0$, where $p \geq 1$, $b \geq 0$ and $1 < c < 2$, or

\begin{equation}
\text{(ii) } d(Sx, Ty) = 0 \quad \text{if } d(x, Sx) + d(y, Ty) = 0.
\end{equation}

Then $S$ and $T$ have a unique common fixed point $w$ in $X$.

Assuming that $f = g$ and $S = T$ on $X$ in Theorem 3.1, we have the following.

COROLLARY 3.3. Let $(X, d)$ be a complete metric space and let $f, S : X \to X$ be continuous mappings such that

\begin{equation}
S(X) \subseteq f(X), \quad (3.6)
\end{equation}

$f$ and $S$ are compatible mappings, \quad (3.7)

\begin{equation}
\text{either (i) } d^p(Sx, Ty) \leq \frac{cd(x, Sx)d^p(fy, Sy) + bd(fx, Sy)d^p(fy, Sx)}{d(fx, Sx) + d(fy, Sy)} \quad (3.8)
\end{equation}

for all $x, y \in X$ if $d(fx, Sx) + d(fy, Sy) \neq 0$, where $p \geq 1$, $b \geq 0$ and $1 < c < 2$, or

\begin{equation}
\text{(ii) } d(Sx, Sy) = 0 \quad \text{if } d(fx, Sx) + d(fy, Sy) = 0.
\end{equation}

Then $f$ and $S$ have a unique common fixed point $w$ in $X$.

REMARK 3.3. (1) If $p = 1$ in Corollary 3.2, we obtain the result of B. Fisher [9].

(2) Theorem 3.1 is an extension of the results of M.L. Diviccaro, S. Sessa and B. Fisher [10].

REMARK 3.4. Conditions (3.6) and (3.7) are necessary in Corollary 3.3 (and so Theorem 3.1) [3].

EXAMPLE 3.1. Let $X = [0, 1]$ with the Euclidean metric $d(x, y) = |x - y|$ and define two mappings $f, S : X \to X$ by

\[
Sx = \frac{1}{4} \quad \text{and} \quad fx = \frac{1}{2}x
\]

for all $x \in X$. Note that $f$ and $S$ are continuous and $S(X) = \{\frac{1}{4}\} \subseteq [0, \frac{1}{2}] = f(X)$.

Since $d(Sx, Sy) = 0$ for all $x, y \in X$, all the conditions of Corollary 3.3 are satisfied except the compatibility of $f$ and $S$. In fact, let $\{x_n\}$ be a sequence in $X$ defined by $x_n = \frac{1}{2}$ for $n = 1, 2, \ldots$. Then we have

\[
\lim_{n \to \infty} f x_n = \lim_{n \to \infty} \frac{1}{2} x_n = \frac{1}{4}, \quad \lim_{n \to \infty} S x_n = \lim_{n \to \infty} \frac{1}{4} = \frac{1}{4}
\]

but

\[
\lim_{n \to \infty} d(S f x_n, f S x_n) = \lim_{n \to \infty} \left| \frac{1}{4} - \frac{1}{8} \right| = \frac{1}{8}.
\]

Thus $f$ and $S$ are not compatible mappings. But $f$ and $S$ have no common fixed points in $X$. 
ACKNOWLEDGMENT. The first author was supported in part by the Basic Science Research Institute Program, Ministry of Education, Korea, 1995, Project No. BSRI-95-1405

REFERENCES

[1] POPA, V., Theorems on multifunctions satisfying a rational inequality, *Comment. Math. Univ. Carolinae*, 24 (4) (1983), 673-680.

[2] SESSA, S., On a weak commutativity conditions of mappings in fixed point considerations, *Publ. Inst. Math. (Beograd)*, 32 (46) (1982), 146-153.

[3] JUNGCK, G., Compatible mappings and common fixed point, *Internat. J. Math. Math. Sci.* 9 (4) (1986), 771-779.

[4] KANEKO, H., A Common fixed point of weakly commuting multi-valued mappings, *Math. Japon.*, 33 (5) (1988), 741-744.

[5] SINGH, S.L., HA, K.S and CHO, Y.J., Coincidence and fixed points of nonlinear hybrid contractions, *Internat. J. Math. Math. Sci.*, 12 (2) (1989), 247-256.

[6] KANEKO, H. and SESSA, S., Fixed point theorems for compatible multi-valued and single-valued mappings, *Internat. J. Math. Math. Sci.*, 12 (1989), 257-262.

[7] RUS, I.A., Fixed point theorems for multi-valued mappings in a complete metric space, *Math. Japon.*, 20 (1975), 21-24.

[8] NADLER, S.B., Multi-valued contraction mappings, *Pacific J. Math.*, 20 (2) (1969), 475-488.

[9] FISHER, B., Theorems on mappings satisfying a rational inequality, *Comment. Math. Univ. Carolinae*, 19 (1978), 37-44.

[10] DIVICCARO, M.L., SESSA, S. and FISHER, B., Common fixed point theorems with a rational inequality, *Bull. of Institute of Math.*, Academia Sinica, 14 (3) (1986), 277-285.
Space dynamics is a very general title that can accommodate a long list of activities. This kind of research started with the study of the motion of the stars and the planets back to the origin of astronomy, and nowadays it has a large list of topics. It is possible to make a division in two main categories: astronomy and astrodynamics. By astronomy, we can relate topics that deal with the motion of the planets, natural satellites, comets, and so forth. Many important topics of research nowadays are related to those subjects. By astrodynamics, we mean topics related to spaceflight dynamics.

It means topics where a satellite, a rocket, or any kind of man-made object is travelling in space governed by the gravitational forces of celestial bodies and/or forces generated by propulsion systems that are available in those objects. Many topics are related to orbit determination, propagation, and orbital maneuvers related to those spacecrafts. Several other topics that are related to this subject are numerical methods, nonlinear dynamics, chaos, and control.

The main objective of this Special Issue is to publish topics that are under study in one of those lines. The idea is to get the most recent researches and published them in a very short time, so we can give a step in order to help scientists and engineers that work in this field to be aware of actual research. All the published papers have to be peer reviewed, but in a fast and accurate way so that the topics are not outdated by the large speed that the information flows nowadays.

Before submission authors should carefully read over the journal's Author Guidelines, which are located at http://www.hindawi.com/journals/mpe/guidelines.html. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

| Manuscript Due       | July 1, 2009 |
|----------------------|-------------|
| First Round of Reviews| October 1, 2009 |
| Publication Date     | January 1, 2010 |

Lead Guest Editor

Antonio F. Bertachini A. Prado, Instituto Nacional de Pesquisas Espaciais (INPE), São José dos Campos, 12227-010 São Paulo, Brazil; prado@dem.inpe.br

Guest Editors

Maria Cecilia Zanardi, São Paulo State University (UNESP), Guaratinguetá, 12516-410 São Paulo, Brazil; cecilia@feg.unesp.br

Tadashi Yokoyama, Universidade Estadual Paulista (UNESP), Rio Claro, 13506-900 São Paulo, Brazil; tadashi@rc.unesp.br

Silvia Maria Giulia Winter, São Paulo State University (UNESP), Guaratinguetá, 12516-410 São Paulo, Brazil; silvia@feg.unesp.br