ISOTYPED ALGEBRAS

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ABSTRACT. The paper is essentially a continuation of [PZ], whose main notion is that of logic-geometrical equivalence of algebras (LG-equivalence of algebras). This equivalence of algebras is stronger than elementary equivalence. In the paper we introduce the notion of isotyped algebras and relate it to LG-equivalence. We show that these notions coincide. The idea of the type is one of the central ideas in Model Theory. The correspondence introduced in the paper stimulates a bunch of problems which connect universal algebraic geometry and Model Theory. We provide a new general view on the subject, arising "on the territory" of universal algebraic geometry. This insight yields also applications of algebraic logic in Model Theory. Application of algebraic logic in Model theory makes some approaches more transparent.

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1. General view

1.1. Introduction. The notion of a type of an algebra, like of any other algebraic system, came from Model Theory and turns to be one of its key notions (see, for example, [Ho], [Ma]).

The paper is devoted to isotyped algebras that is to the algebras with the same types. We consider algebras which belong to a fixed variety of algebras Θ. We approach to the notion of a type from the positions of universal algebraic geometry (UAG). On the one hand, universal algebraic geometry is an equational algebraic geometry in an arbitrary variety of algebras Θ. This means that
algebraic sets are defined by systems of equations in free algebras from $\Theta$. On the other hand, universal algebraic geometry spreads to First Order Logic (FOL) geometry in an arbitrary $\Theta$ \textit{(logical geometry)}. This means that algebraic sets are defined by arbitrary first order formulas, semantically compressed in the given $\Theta$. In the case of logical geometry algebraic sets are called elementary sets and arbitrary first order formulas replace equations (see [PZ] for details).

In the paper we proceed from the system of notions of algebraic logic. In principle, it is possible to translate this approach to the usual model theoretic language. However, we believe that application of algebraic logic makes the main ideas of the paper more transparent and consistent.

The bridge between logical geometry and model theory is provided via algebraic logic by the means of the algebra of formulas $\Phi = \Phi(X)$, $X$ is a finite set of variables. In fact, $\Phi(X)$ is \textit{a set of first order formulas over $X$} which is converted in a special way into an algebra of formulas. The precise definition of the algebra $\Phi = \Phi(X)$ is given in Section 5.5 (see also [P6]-[P8]).

Let $W = W(X)$ be the free algebra in $\Theta$ over $X$. An equality $w \equiv w'$, which is an element in the algebra $\Phi(X)$, corresponds to an equation $w = w'$ in $W(X)$. So, equalities are considered as nullary operations \textit{(constants)} in $\Phi(X)$. Boolean algebras with equalities of the form $w \equiv w'$, and with acting quantifiers $\exists x$ by all $x \in X$ are called \textit{extended boolean algebras} \textit{(for the list of identities see [PZ], Section 2.1 and Subsection 5.2 of this paper)}. The algebra $\Phi(X)$ is an example of an extended boolean algebra. However, in $\Phi(X)$ there are other operations $s_*$ (see below).

It was mentioned that universal algebraic geometry is an algebraic geometry associated to an arbitrary variety of algebras $\Theta$. If $\Theta = \text{Com} - P$ is the variety of commutative associative algebras with unit over the field $P$ then we arrive to classical algebraic
geometry. One of the principal problems related to universal algebraic geometry is to understand what a part of rich geometry of the variety $\Theta = \text{Com} - P$ (i.e. of the classical algebraic geometry) survives in other varieties $\Theta$. On the other hand, the new ideas related to UAG and, especially, to logical geometry appear in the classical situation of $\text{Com} - P$.

It is quite important to note that equational algebraic geometry (AG for short) is connected with the category $\Theta^0$ of the free in $\Theta$ algebras $W = W(X)$ with finite sets $X$. In order to easy the intuition one should mention that in classical case $\Theta^0$ is just the category of all polynomial algebras over a field $P$. The role of $\Theta^0$ in logical geometry (LG for short) plays the special category $\text{Hal}_\Theta^0$, whose objects are the algebras of formulas $\Phi(X)$ (we use the notation ”$\text{Hal}$” in order to remind the role played by P.Halmos in algebraic logic). The categories $\Theta^0$ and $\text{Hal}_\Theta^0$ are bounded by the covariant functor

$$\Theta^0 \rightarrow \text{Hal}_\Theta^0.$$

This functor attaches a morphism $s_* : \Phi(X) \rightarrow \Phi(Y)$ in $\text{Hal}_\Theta^0$ to each homomorphism-morphism $s : W(X) \rightarrow W(Y)$ in $\Theta^0$. Here $s_*$ is a boolean homomorphism which is in some sense compatible with quantifiers and equalities. The same $s_*$ can be treated as an operation in special multi-sorted Halmos algebras (see Section 5 for details). In particular, the operation $s_*$ can present in a record of elements from $\Phi(X)$.

Now let us make one more step towards the general theory. Denote by $\Phi^0 = \Phi^0(X)$ the subalgebra in $\Phi(X)$ generated by all equalities in the signature of the boolean operations and quantifiers. This is an extended boolean algebra which is a subalgebra in the extended boolean algebra $\Phi(X)$. This subalgebra is usually less than $\Phi(X)$ since the operations $s_*$ are not involved in the records of the elements from $\Phi^0$. We shall note here, that the
algebra $\Phi^0(X)$ cannot be defined independently from the algebra $\Phi(X)$. In its turn $\Phi(X)$ is defined through the means of algebraic logic.

Remark 1.1. Axioms of Halmos algebras (Subsection 5.3) imply that if we have $s : W(Y) \rightarrow W(X)$ and $v$ is an equality in $\Phi^0(Y)$ then $s_*(v)$ is an equality in $\Phi^0(X)$. However, in general, for the formula $v = \exists y v_0 \in \Phi^0(Y)$ the formula $s_*(v)$ can be not in $\Phi^0(X)$. (See, for example, Proposition 4.11).

Now we shall consider the origin of the morphisms and operations $s_*$. The reasons for introducing $s_*$ are as follows. It is well-known that the standard algebraic first order logic uses an infinite number of variables. Denote the infinite set of variables by $X^0$. Assuming the needs of logical geometry we shall deal with the system $\Gamma$ of all finite subsets $X$ in $X^0$. In algebraic logic this approach leads to Halmos categories and multi-sorted (i.e., $\Gamma$-sorted) Halmos algebras. Different $\Phi(X), X \in \Gamma$ should be somehow connected. This peculiarity requires introduction of the operations and morphisms of the type $s_*$ (see Section 5 for details).

We use some notions which can be found in [PZ]. For the sake of completeness many definitions from [PZ] are reproduced throughout the paper. Papers devoted to universal algebraic geometry ([BMR], [KMR], [MR], [P6], [P3], [P7], [P8], etc.) makes the material more friendly.

The paper is organized as follows. In Section 1 we introduce the notion of isotyped algebras and relate it to the known notions of universal algebraic geometry. Section 2 deals with noetherian properties. In Section 3 we introduce the notion of a logically separable algebra and give examples of isotyped but not isomorphic algebras. Section 4 is devoted to logically perfect algebras. We show that every group can be embedded into a logically perfect one. We also consider two examples of abelian groups and
study their behavior with respect to logical properties. Section 5 makes the paper self-complete. In this section we provide the reader with the notions of algebraic logic. In particular we define the algebra of formulas $\Phi(X)$ which plays a principal role in all considerations. In Section 5 there is also a list of open problems.

In this paper we have no difficult theorems. But we have some new insight and new problems in the field of mathematics which can be characterized as Logic in Algebra. Concerning the objects of investigation, in this paper an algebra $H \in \Theta$ takes a primary role and is considered from the perspective of its logical and geometric invariants.

1.2. Algebra $\text{Bool}(W(X), H)$. Let $H$ be an algebra in $\Theta$. Take a free in $\Theta$ algebra $W = W(X)$ with a finite $X$ and consider the set $\text{Hom}(W, H)$ as an affine space. Its points are homomorphisms $\mu : W \to H$. If $X = \{x_1, \ldots, x_n\}$, then $\text{Hom}(W, H)$ is isomorphic to $H^{(n)}$ and a point $\mu$ meets a tuple $\vec{a} = (a_1, \ldots, a_n) \in H^{(n)}$.

Denote by $\text{Bool}(W(X), H)$ the boolean algebra of all subsets of $\text{Hom}(W(X), H)$. Define the action of quantifiers $\exists x, x \in X$. Recall (see [H]) that if $B$ is a boolean algebra, then the mapping $\exists : B \to B$ is an existential quantifier if

1. $\exists(0) = 0$,
2. $\exists(a) \geq a$,
3. $\exists(a \land \exists b) = \exists a \land \exists b$.

A universal quantifier $\forall : B \to B$ is defined dually as $\forall(a) = \neg(\exists(\neg a))$.

Let now $A \in \text{Bool}(W(X), H)$ and $x \in X$. We set: $\mu \in \exists x A$ if there is $\nu \in A$ such that $\mu(x') = \nu(x')$ for every $x' \in X$, $x' \neq x$. The necessary conditions (1) – (3) hold true and the definition of existential quantifier perfectly agrees with intuition.

Let, further, $w \equiv w'$ be an equality in the algebra of formulas $\Phi(X)$. Define the corresponding element of the algebra $\text{Bool}(W(X), H)$ by $\text{Val}_H^X(w \equiv w') = \{\mu : W \to$
\( H | (w, w') \in Ker(\mu) \). These elements are considered as equalities in \( \text{Bool}(W(X), H) \).

Thus \( \text{Bool}(W(X), H) \) is defined as an extended Boolean algebra.

As we will see in Subsection 5.6, the correspondence \( w \equiv w' \rightarrow \text{Val}^X_H(w \equiv w') \) is naturally extended up to a homomorphism value of extended boolean algebras

\[
\text{Val}^X_H : \Phi(X) \rightarrow \text{Bool}(W(X), H).
\]  

We consider also the category \( \text{Hal}_\Theta(H) \) of all \( \text{Bool}(W(X), H) \) with the natural morphisms \( s_* \). The values and morphisms in \( \text{Hal}_\Theta^0 \) and \( \text{Hal}_\Theta(H) \) are connected by the following commutative diagram

\[
\begin{array}{ccc}
\Phi(X) & \xrightarrow{s_*} & \Phi(Y) \\
\text{Val}^X_H | & & \text{Val}^Y_H |
\end{array}
\]

\[
\begin{array}{ccc}
\text{Bool}(W(X), H) & \xrightarrow{s_*} & \text{Bool}(W(Y), H).
\end{array}
\]

**Remark 1.2.** The existence of homomorphisms \( \text{Val}^X_H \) for every algebra \( H \) in \( \Theta \) which satisfy the diagram above was a leading idea towards the definition of algebra of formulas \( \Phi(X) \) and the category of such algebras of formulas \( \text{Hal}_\Theta^0 \). This is a place we faced with advantages of application of algebraic logic.

More precisely, our aim is to define the system of algebras of formulas \( \Phi(X) \), where \( X \) are finite subsets of \( X^0 \) and to define the category \( \text{Hal}_\Theta^0 \) of these algebras in such a way that for every algebra \( H \in \Theta \) and every \( s : W(X) \rightarrow W(Y) \) the conditions (*) and (**) are fulfilled.

In fact, what we have to do is to define the upper level of the diagram (**) and the vertical arrows having already the lower level of the diagram. This leads to constructions and conditions which are realized in Section 5.
Such approach to the definition of the algebra $\Phi(X)$ and the category $\text{Hal}_\Theta^0$ is well coordinated with the approach based on application of the equivalence of Lindenbaum-Tarski.

1.3. **Logical kernel of a point, types, isotyped algebras.**

Define the notion of the *logical kernel* of a point $\mu : W(X) \to H$. So, along with $\text{Ker}(\mu)$ we will consider the logical kernel $L\text{Ker}(\mu)$. This logical kernel is an important logical invariant of a point.

**Definition 1.3.** Let $u \in \Phi(X)$ and $\mu : W(X) \to H$ be a point in $\text{Hom}(W(X), H)$. We set: $u \in L\text{Ker}(\mu)$, if $\mu \in \text{Val}_H^X(u)$.

In this case we say that $\text{Val}_H^X(u)$ is the value of a formula $u$ in $\text{Bool}(W(X), H)$ and a point $\mu$ is a solution of the "equation" $u$ in $\text{Hom}(W(X), H)$. This definition corresponds to the usual inductive definition of a point satisfying a formula (see [Ma]).

We have also $\text{Ker}(\mu) = L\text{Ker}(\mu) \cap M_X$, where $M_X$ is a set of all equalities $w \equiv w'$, $w, w' \in W(X)$.

Show that the kernel $L\text{Ker}(\mu)$ is an ultrafilter of the boolean algebra $\Phi(X)$. First prove that it is a filter. Let $u_1, u_2 \in L\text{Ker}(\mu)$. We have $\mu \in \text{Val}_H^X(u_1) \cap \text{Val}_H^X(u_2) = \text{Val}_H^X(u_1 \land u_2)$. Hence, $u_1 \land u_2 \in L\text{Ker}(\mu)$. Let, now, $u \in L\text{Ker}(\mu)$, $v \in \Phi(X)$. We have: $\mu \in \text{Val}_H^X(u) \cup \text{Val}_H^X(v) = \text{Val}_H^X(u \lor v)$. Thus, $u \lor v \in L\text{Ker}(\mu)$ and $L\text{Ker}(\mu)$ is a filter.

Let now $u \in \Phi(X), u \notin L\text{Ker}(\mu)$, i.e., $\mu \notin \text{Val}_H^X(u), \mu \in \neg \text{Val}_H^X(u) = \text{Val}_H^X(\neg u)$. Then $\neg u \in L\text{Ker}(\mu)$ and, hence, $L\text{Ker}(\mu)$ is an ultrafilter.

**Definition 1.4.** Every ultrafilter in $\Phi(X)$ we call $X$-type. A type $T$ we call $X$-type of the algebra $H \in \Theta$, if $T = L\text{Ker}(\mu)$ for some $\mu : W(X) \to H$.

**Remark 1.5.** Compare with the definition of a type from [Ma].
We say that $T$ is realized in the algebra $H$ if $T$ is an $X$-type of $H$. Denote by $S^X(H)$ the system of all $X$-types of the algebra $H$. This is an important logical invariant of the algebra $H$.

**Definition 1.6.** Algebras $H_1$ and $H_2$ in $\Theta$ are called isotyped, if for any finite $X$ every $X$-type of the algebra $H_1$ is an $X$-type of the algebra $H_2$ and vice versa.

Thus, the algebras $H_1$ and $H_2$ are *isotyped* if

$$S^X(H_1) = S^X(H_2).$$

for any $X$. Note, further, that the logical kernel $L\text{Ker}(\mu)$ of $\mu : W(X) \to H$ contains the elementary $X$-theory of the algebra $H$. Indeed, if $u \in Th^X(H)$ then $Val^H_{X}(u) = Hom(W(X), H)$. In particular, $\mu \in Val^X_H(u)$ and $u \in L\text{Ker}(\mu)$. Thus $Th^X(H) \subset L\text{Ker}(\mu)$.

It is clear now that if $H_1$ and $H_2$ are isotyped then

$$Th^X(H_1) = Th^X(H_2),$$

where $Th^X(H)$ is the elementary $X$-theory of $H$.

**Definition 1.7.** We say that an algebra $H$ is *saturated*, if for every $X$ any ultrafilter $T$ in $\Phi(X)$ which contains $Th^X(H)$ is realizable in $H$.

**Remark 1.8.** Observe that the definition of a saturated algebra which is used in Model theory operates also with constants. In our definition constants are already incorporated in the signature of the variety $\Theta$.

Now we can state that if the saturated algebras $H_1$ and $H_2$ are elementary equivalent i.e., $Th(H_1) = Th(H_2)$ then they are isotyped. Thus, the saturated algebras are elementary equivalent if and only if they are isotyped. As we noticed above the isotyped algebras are always elementary equivalent. However, in general the notion to be isotyped is more strong than to be elementary.
equivalent. In fact the relation on algebras "to be isotyped" can be treated as a generalization of the idea of saturated algebras.

In what follows we will consider logically noetherian algebras. The statement for such algebras is also two sided: logically noetherian algebras $H_1$ and $H_2$ are elementary equivalent if and only if they are isotyped.

We consider the definition of isotyped algebras from the logical geometry perspective, but first we treat (equational) algebraic geometry. Note that in algebraic geometry it is also useful to speak of atomic kernel. It is all equalities $w \equiv w'$ and inequalities $w \not\equiv w'$ belonging to $LKe\mu$. Denote this atomic kernel by $AtKer(\mu)$. If $w \equiv w' \notin AtKer(\mu)$, then $w \not\equiv w' \in AtKer(\mu)$. In fact, $AtKer(\mu)$ is the kernel of $\mu$ represented in the algebra of formulas $\Phi(X)$.

We consider also a special logical kernel $LKer^0(\mu)$ defined by

$$LKe^0(\mu) = LKe(\mu) \cap \Phi^0(X).$$

This kernel is an ultrafilter in the algebra $\Phi^0(X)$.

1.4. The main Galois correspondences in algebraic geometry and logical geometry. $AG$- and $LG$-equivalence of algebras. Consider the Galois correspondence between subsets $A$ in $Hom(W(X), H)$ and systems of equations $T$ in $W = W(X)$. These $T$ can be viewed as binary relations in $W$. For each $T$ we set:

$$T'_H = A = \{\mu : W \to H | T \subset Ker(\mu)\}.$$ 

For an arbitrary $A$ we set

$$A'_H = T = \bigcap_{\mu \in A} Ker(\mu).$$

We have here the Galois correspondence and we can speak of the Galois closures $A''_H$, $T''_H$. Every set $A \subset Hom(W(X), H)$ of the form $A = T''_H$ is closed, and we call it an algebraic set.
Every system of equations $T$ of the form $T = A'_H$ is an $H$-closed congruence in $W$.

Let us do the same for the logical geometry, substituting $Ker(\mu)$ by $LKer(\mu)$. Here $T$ is an arbitrary subset in $\Phi = \Phi(X)$. We set:

$$T^L_H = A = \{\mu : W \to H | T \subset LKer(\mu)\},$$

$$A^L_H = T = \bigcap_{\mu \in A} LKer(\mu).$$

The corresponding closures are $T^{LL}_H$ and $A^{LL}_H$. Each $A$ of the form $A = T^L_H$ is called an elementary set (it can be defined for an infinite $T$ as well). Each $T = A^L_H$ is an $H$-closed filter in the boolean algebra $\Phi = \Phi(X)$. We have also

$$T^L_H = A = \bigcap_{u \in T} Val^X_H(u),$$

$$T = A^L_H = \{u \in \Phi(X) | A \subset Val^X_H(u)\}.$$

Recall that algebras $H_1$ and $H_2$ in $\Theta$ are called geometrically (AG-equivalent), if for any finite $X$ and any $T$ in $W(X)$ we have

$$T''_{H_1} = T''_{H_2}$$

(see [P6], [P7]), and $H_1$ and $H_2$ are called LG-equivalent if always

$$T^{LL}_{H_1} = T^{LL}_{H_2}$$

for $T \subset \Phi(X)$ (see [PZ]).

Note that starting from the logical kernel $LKer^0(\mu)$ one can also establish the Galois correspondence between subsets $T$ in $\Phi^0(X)$ and the sets of points $A$ in $Hom(W(X), H)$.

1.5. **LG-equivalence and isotyped algebras.** We interrelate LG-equivalence and isotypeness of algebras. Let $A$ be a subset in $Hom(W(X), H)$, consisting of a single point $\mu : W(X) \to H$. We have

$$T = A^L_H = \{\mu\}^L_H = LKer(\mu).$$
Hence, $L\text{Ker}(\mu) = \{\mu\}_H^L$ and $L\text{Ker}(\mu)$ is an $H$-closed ultrafilter. Let $T$ be an ultrafilter, and $T = L\text{Ker}(\mu)$. Take $T_H^L = A_0$ and let $\nu \in A_0$. We have $\{\nu\}_H^L = L\text{Ker}(\nu) \supset A_0_H^L = T = L\text{Ker}(\mu)$. We see that any two points in $A_0$ have the same logical kernel. Besides, we can note that if $T$ is an ultrafilter in $\Phi(X)$, then this $T$ is $X$-type for $H$ if and only if the set $A_0 = T_H^L$ is not empty.

**Definition 1.9.** Define the equivalence $\rho = \rho_H^X$ on the set $\text{Hom}(W(X), H)$ setting $\mu \rho \nu$ if and only if $L\text{Ker}(\mu) = L\text{Ker}(\nu)$.

Consider the quotient set $\text{Hom}(W(X), H)/\rho = \overline{\text{Hom}}(W(X), H)$. Hence, there is a bijection $\overline{\text{Hom}}(W(X), H) \to S^X(H)$.

Note also that every coset of the equivalence $\rho$ is an elementary set, defined by the type $L\text{Ker}(\mu) = T$, where $\mu$ belongs to the coset. According to [P7], each elementary set is invariant under the action of the group of automorphisms $\text{Aut}(H)$. Hence, if $\mu$ and $\nu$ are conjugated by an automorphism $\sigma \in \text{Aut}(H)$, then $\mu \rho \nu$. Recall that the action of a group $\text{Aut}(H)$ in $\text{Hom}(W, H)$ is defined by a transition $\mu \to \mu \sigma$. Under certain conditions the opposite is true as well. In these cases cosets of the equivalence $\rho$ are exactly the orbits of the action of the group $\text{Aut}(H)$.

The following theorem is the main one.

**Theorem 1.10.** Algebras $H_1$ and $H_2$ are $LG$-equivalent if and only if they are isotyped.

Proof. Let $H_1$ and $H_2$ be $LG$-equivalent algebras. This means that for any finite $X$ and any set $T$ of formulas from $\Phi(X)$ this $T$ is $H_1$-closed if and only if $T$ is $H_2$-closed.

Take now an ultrafilter $T$ in $\Phi(X)$ and let $T$ be an $X$-type over $H_1$. Such $T$ is $H_1$-closed and, consequently, $H_2$-closed. Here $T_H^L$ is not empty and, hence, $T$ is an $X$-type over $H_2$. The transition from $H_2$ to $H_1$ works in a similar way. Therefore, $H_1$ and $H_2$ are isotyped.
Let, further, $H_1$ and $H_2$ be isotyped. This means, in particular, that if $T = LKer(\mu)$ for $\mu : W(X) \to H_1$, then $T = LKer(\nu)$ for some $\nu : W(X) \to H_2$ as well. The same is true for the transition from $H_2$ to $H_1$. Thus $T$ is simultaneously $H_1$- and $H_2$-closed.

Let, now, $T$ be an $H_1$-closed filter in $\Phi(X)$. We want to check that $T$ is $H_2$-closed. Take $A = T^L_{H_1}$. Then $T = A^L_{H_2} = \bigcap_{\mu \in A} LKer(\mu)$. This also means that $u \in T$ if and only if $A \subset Val^X_{H_1}(u)$.

We will see that the intersection of $H$-closed filters is also $H$-closed. As we know every $H_1$-closed filter of the form $LKer(\mu)$ is $H_2$-closed. Thus $T = \bigcap_{\mu \in A} LKer(\mu)$ is an $H_2$-closed filter. Similarly, $H_2$-closedness of $T$ implies its $H_1$-closedness. This is true for any $X$. Hence, $H_1$ and $H_2$ are LG-equivalent.

It remains to check that the intersection $T = \bigcap_{\alpha} T_\alpha$ with all $H$-closed $T_\alpha, \alpha \in I$ is also $H$-closed.

Take $T_\alpha = (A_\alpha)^L_H$ and check that $\bigcap_{\alpha} (A_\alpha)^L_H = (\bigcup_{\alpha} A_\alpha)^L_H$. Let $u \in \bigcap_{\alpha} T_\alpha = T$. The inclusions $u \in T_\alpha$ and $A_\alpha \subset Val^X_H(u)$ always hold true. So, $\bigcup_{\alpha} A_\alpha \subset Val^X_H(u)$ and, hence $u \in (\bigcup_{\alpha} A_\alpha)^L_H$.

Let $u \in (\bigcup_{\alpha} A_\alpha)^L_H$. Then $\bigcup_{\alpha} A_\alpha \subset Val^X_H(u)$. The inclusions $A_\alpha \subset Val^X_H(u)$ and $u \in (A_\alpha)^L_H = T_\alpha$ always hold true. Hence, $u \in T$. \qed

From this theorem follows that if the algebras $H_1$ and $H_2$ are $LG$-equivalent, then they are elementary equivalent (see [PZ]). Besides, if $H_1$ and $H_2$ are isotyped, then the categories of elementary sets $LK_{\Theta}(H_1)$ and $LK_{\Theta}(H_2)$ ([PZ]) are isomorphic (see also Section 5).

**Remark 1.11.** Our definition of a type corresponds to the notion of a complete type in Model theory. A complete type is an $H$-closed ultrafilter defined by a single point. However, an arbitrary $H$-closed set is defined by a set of points. In this case also there are relations with the model theoretic general theory of
types. In particular, the $H-$closure of an arbitrary type is always an intersection of complete types.

1.6. **Infinitary logic.** Let us make some remarks on the relations with infinitary logic. We start from quasiidentities. Take a binary relation $T$ in $W = W(X)$. Consider a formula (quasiidentity)

$$\left(\bigwedge_{(w, w') \in T} (w \equiv w')\right) \rightarrow (w_0 \equiv w_0').$$

(\ast)

If $T$ is infinite, it is an infinitary quasiidentity.

Let, further, $H$ be an algebra in $\Theta$ and consider $T''_H$. It is proved in \[P5\] that $(w_0, w_0') \in T''_H$ if and only if the quasiidentity (\ast) holds in $H$.

Proceed now from $T \subset \Phi(X)$ and consider a formula

$$\left(\bigwedge_{u \in T} u\right) \rightarrow v,$$

(\ast\ast)

where $v \in \Phi(X)$. The formula (\ast\ast) is infinitary if $T$ is an infinite set. We write for short $T \rightarrow v$. This formula holds in $H$ if and only if $v \in T''_H \llbracket P8 \rrbracket$. Denote by $Im(Th)(H)$ the set of all formulas of the form $T \rightarrow v$, holding in $H$. It is the *implicative theory* of the algebra $H$. We may say that the algebras $H_1$ and $H_2$ are $LG$-equivalent (isotyped), if and only if their implicative theories coincide, i.e., $Im(Th)(H_1) = Im(Th)(H_2)$ (see \[PZ\]). The presence here of infinitary formulas is naturally stipulated by the aims of universal algebraic geometry. Such formulas are not new also in Model Theory. In particular, they participate in the theory of abstract elementary classes ($AEC$-classes) of models \[G\].

1.7. **Galois correspondence and morphisms in \(Hal^0_{\check{\Theta}}\).** Let us show the relation between the Galois correspondence and morphisms in the category $Hal^0_{\check{\Theta}}$. Let $s : W(X) \rightarrow W(Y)$ and
$s_* : \Phi(X) \rightarrow \Phi(Y)$ be given. For every $u \in \Phi(X)$ we have

$$Val_H^Y(s_* u) = s_* Val_H^X(u),$$

Here, $\mu \in s_* Val_H^X(u)$ if $\mu s \in Val_H^X(u)$.

For $T \subset \Phi(X)$ denote by $s_* T$ the subset in $\Phi(Y)$ which consists of $s_* u$, $u \in T$. We have

$$(s_* T)_H^L = s_* T_H^L,$$

(see [P7]). Thus, if $A = T_H^L$ is an elementary set in $Hom(W(X), H)$ then $s_* A$ is an elementary set in $Hom(W(Y), H)$. As usual $\mu \in s_* A$ if $\mu s \in A$.

Connect now $X$ and $Y$—types over $H$ for the given $s : W(X) \rightarrow W(Y)$ and $s_* : \Phi(X) \rightarrow \Phi(Y)$. For each point $\nu : W(X) \rightarrow H$ we denote by $L Ker^X(\nu)$ its logical kernel, having in mind that this kernel is calculated in $\Phi(X)$. Let us check that

$$s_* L Ker^X(\mu s) \subset L Ker^Y(\mu)$$

for $\mu : W(Y) \rightarrow H$.

Let $u \in \Phi(X)$ and let $u \in L Ker^X(\mu s)$. This gives $\mu s \in Val_H^X(u)$. Thus $\mu \in s_* Val_H^X(u) = Val_H^Y(s_* u)$, i.e., $s_* u \in L Ker^Y(\mu)$. The inclusion is checked.

Apply once again the $L$-transition. We have:

$$(L Ker^Y(\mu))_H^L = B \subset (s_* L Ker^X(\mu s))_H^L = s_* (L Ker^X(\mu s))_H^L = s_* A.$$

Here, $B$ is the closure of the point $\mu$, $A$ is the closure of the point $\mu s$ and $B \subset Hom(W(Y), H)$, $A \subset Hom(W(X), H)$, $s_* A \subset Hom(W(Y), H)$, and $B \subset s_* A$. $B$ and $A$ are minimal elementary sets (i.e., they do not contain other elementary sets), while $s_* A$ is not necessarily a minimal set.

1.8. Relations $\rho$ and $\tau$. Along with the relation $\rho$ consider relations $\rho_0$ and $\tau$ on the given set $Hom(W(X), H)$. The relation $\rho_0$ is determined by the decomposition of $Hom(W(X), H)$ into
orbits of the group $Aut(H)$. The inclusion $\rho_0 \subset \rho$ always holds, but we are interested in the situation of the equality $\rho_0 = \rho$.

Let $X = \{x_1, \ldots, x_n\}$ and let

$$(\mu(x_1), \ldots, \mu(x_n)) = \bar{a} = (a_1, \ldots, a_n),$$

$$(\nu(x_1), \ldots, \nu(x_n)) = \bar{b} = (b_1, \ldots, b_n)$$

for the points $\mu$ and $\nu$. Denote by $A$ and $B$ subalgebras in $H$, generated by all $a_1, \ldots, a_n$ and all $b_1, \ldots, b_n$ respectively. We set: $\mu \tau \nu$ if and only if the transitions $a_i \rightarrow b_i$ determine the isomorphism $\eta : A \rightarrow B$. We have $\rho_0 \subset \tau$. Actually the following theorem takes place:

**Theorem 1.12.** The condition $\mu \tau \nu$ holds true if and only if $At\text{Ker}(\mu) = At\text{Ker}(\nu)$ in $\Phi(X)$ or what is the same $\text{Ker}(\mu) = \text{Ker}(\nu)$ in $W(X)$.

**Proof.** Let $\mu \tau \nu$ hold and the corresponding isomorphism $\eta : A \rightarrow B$ be given. Take $w \equiv w' \in At\text{Ker}(\mu)$. We have $w^\mu = w'^\mu$ in $A$ and $w'^{\mu\eta} = w''^{\mu\eta}$ in $B$. The equality $w'' = w''$ holds in $B$ and, hence, $w \equiv w' \in At\text{Ker}(\nu)$. Similarly, if $w \not\equiv w' \in At\text{Ker}(\mu)$, then $w \not\equiv w' \in At\text{Ker}(\nu)$. Thus $At\text{Ker}(\mu) \subset At\text{Ker}(\nu)$. The second direction is similar.

Let us check the converse statement. Consider homomorphisms $\alpha : W(X) \rightarrow A$ and $\beta : W(X) \rightarrow B$, where $X = \{x_1, \ldots, x_n\}$, $\alpha(x_i) = a_i = \mu(x_i)$, $\beta(x_i) = b_i = \nu(x_i)$. Let $w(x_1, \ldots, x_n) = w'(x_1, \ldots, x_n)$ lie in the kernel $\text{Ker}(\alpha)$. Then $w \equiv w' \in At\text{Ker}(\mu) = At\text{Ker}(\nu)$. Therefore, $w \equiv w' \in \text{Ker}(\beta)$. More precisely, $w \equiv w' \in \text{Ker}(\alpha)$ if and only if $w \equiv w' \in \text{Ker}(\beta)$, i.e., $\text{Ker}(\alpha) = \text{Ker}(\beta)$. We get an isomorphism $A \rightarrow B$ induced by $a_i \rightarrow b_i$. \hfill \Box

It is clear that $L\text{Ker}(\mu) = L\text{Ker}(\nu)$ implies $At\text{Ker}(\mu) = At\text{Ker}(\nu)$ which also means that $\rho \subset \tau$. This gives us $\rho_0 \subset \rho \subset \tau$ and if $\tau = \rho_0$ for the given $X$ and $H$ then $\rho = \rho_0$. 


Let us show that every coset of the relation \( \tau \) is an elementary set. Let \( T \) be an arbitrary congruence on the algebra \( W = W(X) \). Consider the set of formulas in \( \Phi(X) \) defined by \( T' = T_1' \cup T_2' \), where \( T_1' = \{ w \equiv w' | (w, w') \in T \} \) and \( T_2' = \{ w \not\equiv w' | (w, w') \not\in T \} \). It is easy to understand that a point \( \mu : W(X) \to H \) satisfies the set \( T' \), that is \( \mu \in (T')^L_H \), if and only if \( \text{Ker}(\mu) = T \). Hence, the coset of the relation \( \tau \) containing the point \( \mu \) is an elementary set defined by the set of formulas \( T' \) with \( \text{Ker}(\mu) = T \). We will always use this remark dealing with the relation \( \tau \).

Note that the main future problem in this paper is to find the conditions on algebra \( H \) which provide isomorphism \( \eta : A \to B \) to be realized by some automorphism of the algebra \( H \). In this case we have \( \tau = \rho_0 \).

1.9. Isotypeness of points. Along with the notion of isotypeness of algebras we introduce also the notion of isotyped points over algebras. We say that two \( X \)-points \( \mu, \nu : W(X) \to H \) are isotyped if \( L\text{Ker}(\mu) = L\text{Ker}(\nu) \) which means that \( \mu \rho \nu \) holds true.

This notion lead to the definition of logical isotypeness also for elements of the algebra \( H \). We say that the elements \( a_1 \) and \( a_2 \) of an algebra \( H \) are isotyped if the corresponding points \( \mu, \nu \in \text{Hom}(W(x), H) \) with \( \mu(x) = a_1 \) and \( \nu(x) = a_2 \) are isotyped. Actually the notion of isotyped elements of \( H \) is related to the following idea.

We proceed from a formula or a set of formulas \( T \). Let the elements \( a_1 \) and \( a_2 \) from \( H \) with the corresponding points \( \mu, \nu \in \text{Hom}(W(x), H) \) be isotyped. Then the points \( \mu \) and \( \nu \) are isotyped and let \( T \) belong to \( L\text{Ker}(\mu) \). Then \( T \) belongs to \( L\text{Ker}(\nu) \) as well. So, both points satisfy \( T \) and let formulas from \( T \) describe some algebraic property of elements in \( H \). Isotypeness of \( a_1 \) and \( a_2 \) means that these elements both satisfy this algebraic
property. For example, if $g$ and $g'$ are isotyped and $g$ has a finite order $n$, then $g'$ has the same order. Here $T$ consists of the formula $x^n \equiv 1$.

Consider another example of engel elements and nil-elements in groups. Recall that an element $g \in H$ is an $n$-engel element, if there is $n = n(g)$ such that $[a, g, \ldots, g] = 1$ for any $a \in H$. Here $[a, g, \ldots, g]$ stands for a composite commutator, where $g$ is repeated $n$ times. Thus, the point $\mu$ with $\mu(x) = a$, $\mu(y) = g$ satisfies the formula $\forall x([x, y, \ldots, y] = 1)$ if and only if $g$ is an $n$-engel element. This formula is $T$.

Proceed further from $\mu, \nu : W(x, y) \to H$, $\mu(x) = a$, $\mu(y) = g$, $\nu(x) = a'$, $\nu(y) = g'$. Let $\mu$ and $\nu$ be isotyped (i.e., $g$ and $g'$ are isotyped) and the element $g$ be an $n$-engel one. Then $g'$ is engel as well. These remarks imply that if $H$ is noetherian group and $g$ belongs to its nilpotent radical, then $g'$ belongs to it as well. See [Ba] and [P2]. The similar fact is true for the solvable radical of a noetherian group for the corresponding $T$ and isotyped $g$ and $g'$ (see [P10]).

Let us pass to nil-elements. An element $g$ of a group $G$ is a nil-element, if for every $a \in G$ there exists $n = n(a, g)$ such that $[a, g, \ldots, g] = 1$, where $g$ is taken $n$ times. The property of being nil-element is not expressed as a formula, since the definition uses a quantifier of the type $\exists n$ for natural $n$.

We can improve the situation in the following way. Fix two elements $g$ and $g'$ in the group $G$ and consider all possible pairs $(a, g)$ and $(a', g')$ where $a$ and $a'$ are one-to-one related. Suppose that the corresponding $\mu, \nu : W(x, y) \to H$ are isotyped. Assume now that $g$ is a nil-element. Then the pair $(a, g)$ satisfies an equation $[x, y, \ldots, y] = 1$. Isotypeness of $\mu, \nu$ implies that the pair $(a', g')$ satisfies the same equation. This holds for every appropriate $a$ and $a'$, and, hence, $g'$ is also nil-element.

Let now $G$ be a solvable group (or, more generally, radical group ([P1])), and $g$ and $g'$ be its isotyped elements. Then, if one of them
belongs to a locally nilpotent radical, then the second one does. It follows from the previous remarks and from [P1]. We can also consider various other radicals in other groups. A lot of natural problems arise in this way.

2. Logical noetherianity

2.1. Definitions and problems. Let $H$ be an algebra from $\Theta$. Consider three conditions of noetherianity for $H$.

1. An algebra $H$ is logically noetherian, if for any finite $X$ an arbitrary elementary set $A \subset Hom(W(X), H)$ is finitely definable. This means that if $A = T_H^L$ then there exists a finite set $T_0$ (not necessarily a subset of $T$) such that $A = (T_0)_H^L$.

2. An algebra $H$ is strictly logically noetherian, if for any finite $X$ and arbitrary elementary set $A = T_H^L \subset Hom(W(X), H)$ there exists a finite set $T_0 \subset T$ such that $A = (T_0)_H^L$.

3. An algebra $H$ is weakly logically noetherian if for any $T \rightarrow v$ which holds in $H$ there exists a finite subset $T_0 \subset T$ (possibly depending on $v$) such that $T_0 \rightarrow v$ holds in $H$.

For each of these noetherian conditions isotypeness of the algebras $H_1$ and $H_2$ is equivalent to their elementary equivalence.

Clearly, every finite algebra is logically noetherian in any sense. In Model Theory there exist examples of infinite logically noetherian algebras (see also Proposition 2.4).

We will use now the following remark. All elementary sets in the given $Hom(W(X), H)$ constitute a sub-lattice in the lattice of all subsets. Dually, we have a lattice of all $H$-closed filters in $\Phi(X)$. Strict noethrianiy of $H$ is equivalent to the noetherianity condition of the corresponding lattice of filters and, what is the same, to the condition of artinianity of the lattice of elementary sets for a finite $X$.

The principal problem related to Model Theory is as follows:
**Problem 2.1.** To develop a general approach for constructing logically noetherian algebras $H$ in different varieties $\Theta$.

Here $\Theta$ can be the variety of groups, the variety of abelian groups, the variety of commutative associative algebras. For example, we have a problem to describe the logically noetherian abelian groups.

Let us make some remarks on algebras $H$ with the finite set of types $S^X(H)$ for every $X$. It follows from the definitions that if the set $S^X(H)$ is finite, then we have also a finite number of different $H$-closed filters in $\Phi(X)$. We have also a finite number of elementary sets in $Hom(W(X), H)$. This gives artinianity and noetherianity of the corresponding lattices and strict noetherianity of the algebra $H$. It is clear that if we have a finite number of $Aut(H)$-orbits in $Hom(W(X), H)$, then the set $S^X(H)$ is finite as well.

2.2. **Automorphic finitarity of algebras.** In this subsection we consider the question raised in Problem 2.1.

**Definition 2.2.** Algebra $H$ is called automorphically ($Aut(H)$)-finitary, if there is a finite number of $Aut(H)$-orbits in $Hom(W(X), H)$ for any finite $X$.

If $H$ is $Aut(H)$-finitary, then there is a finite number of types on $H$ and hence a finite number of elementary sets. Thus, $H$ is strictly logically noetherian. So, there arise a question whether an algebra $H$ is automorphically finitary.

We shall specify the problem above for the case of groups:

**Problem 2.3.** To develop general methods for constructing infinite automorphically finitary groups.

Consider an example.

**Proposition 2.4.** Let $H$ be an infinite abelian group with the identity $x^p \equiv 1$ with prime $p$. Such a group is $Aut(H)$-finitary.
Remark 2.5. This proposition is not unexpected in view of the well known model theoretic result by Ryll-Nardzewski (see [CK], [Ma]) which basically states that a complete theory is $\aleph_0$-categorical if and only it has a finite number of types. However, for the sake of completeness we present an independent simple proof of the proposition 2.4.

Proof. Proceed from the variety $\Theta$ of all abelian groups with the identity $x^p \equiv 1$, $p$ is prime. Fix $X = \{x_1, \ldots, x_n\}$ and let $W = W(X)$ be a free in $\Theta$ group over $X$. This group is finite. Let $T$ be a subgroup of $W$.

Take a formula $u = (\bigwedge_{u_i \in T}(u_i \equiv 0)) \land (\bigwedge_{v_i \not\in T}(v_i \neq 0))$. Denote $Val_H^X(u) = V$. Obviously, a point $\mu : W \to H$ belongs to $V \subset Hom(W(X), H)$ if and only if $Ker(\mu) = T$.

Let now $\mu$ and $\nu$ be two points in $V$, $A$ and $B$ their images in $H$. Then $A = \langle a_1, \ldots, a_n \rangle$, $B = \langle b_1, \ldots, b_n \rangle$. Since $Ker(\mu) = Ker(\nu)$, we have a commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\sigma} & B \\
\mu \downarrow & & \downarrow \nu \\
W(X) & & \\
\end{array}
$$

Here $\sigma$ is an isomorphism, $\sigma(a_i) = b_i$ and $\mu \sigma = \nu$. Subgroups $A$ and $B$ have complements in $H$, i.e., $A \bigoplus A' = B \bigoplus B' = H$. Since $H$ is infinite, $A'$ and $B'$ are infinite and isomorphic. This means that the isomorphism $\sigma : A \to B$ can be extended up to an automorphism $\sigma \in Aut(H)$. Therefore $\mu$ and $\nu$ are conjugated by this automorphism. Besides, sequences $(a_1, \ldots, a_n)$, $(b_1, \ldots, b_n)$ are also conjugated by the automorphism $\sigma$. This means that the set $V$ belongs to $Aut(H)$-orbit, determined by each of the points $\mu$ and $\nu$. On the other hand, since $V$ is an elementary set, the pointed orbit is contained in $V$. Therefore, $V$ is equal to the orbit containing $\mu$. 

Every orbit, determined by a point \( \mu : W \to H \) is of this form. Indeed, take \( T = \text{Ker}(\mu) \) and a formula \( u \) constructed by \( T \). Then \( V = \text{Val}_H^X(u) \) is an \( \text{Aut}(H) \)-orbit, determined by the point \( \mu \).

Since there are finite sets of different \( T \) and \( u \), we have a finite set of orbits. Moreover, all of them are one-defined elementary sets. The group \( H \) is an infinite strictly logically noetherian group. 

\[ \square \]

Note that we use in this proof a representation of the point \( \mu \in H^{(n)} \) as a point - homomorphism \( \mu : W \to H \).

A similar situation holds in finitely dimension vector spaces over arbitrary finite field. If the field is infinite, then the description of orbits is the same, but the number of orbits is infinite. Various other examples of infinite automorphically finitary (and thus strictly logically noetherian) groups were suggested to me by A.Olshansky (algebraic approach) and B.Zilber (model theoretic approach). In their constructions the groups are far from being abelian.

In this concern there is a problem whether there exist infinite noetherian groups which are also logically noetherian. Noetherian polycyclic groups and their finite extensions are not the case. Infinite cyclic group is also not logically noetherian.

Let us conclude this section with one useful remark.

Suppose that \( H_1 \) and \( H_2 \) are isotyped and \( H_1 \) is strictly logically noetherian. Then \( H_2 \) is strictly logically noetherian too.

Indeed, let \( H_1 \) and \( H_2 \) be isotyped and \( H_1 \) be strictly logically noetherian. Then for \( T \subset \Phi(X) \) and some finite part \( T_0 \subset T \) we have \( T_{H_2}^{LL} = T_{H_1}^{LL} = T_{0H_1}^{LL} = T_{0H_2}^{LL} \) and, hence, \( T_{H_2}^{LL} = T_{0H_2}^{LL} \). Therefore, \( H_2 \) is strictly logically noetherian.
3. ISOTYPENESS AND ISOMORPHISM

3.1. Separable algebras. Another general problem is to study relations between isotypeness property and isomorphism of algebras. First of all, there are various examples of non-isomorphic isotyped algebras. Even for the case of fields there are examples of such kind. We will present examples of isotyped but not isomorphic algebras in Subsection 3.2.

Definition 3.1. An algebra $H \in \Theta$ is called separable in $\Theta$ if each $H' \in \Theta$ isotyped to $H$ is isomorphic to $H$.

Remark 3.2. Sometimes it is worth to modify Definition 3.1 and to consider only finitely-generated $H'$.

Problem 3.3. For which $\Theta$ every free in $\Theta$ algebra $W = W(X)$ with finite $X$ is separable?

In other words this problem asks when every free in $\Theta$ algebra $W = W(X)$ with finite $X$ can be distinguished in $\Theta$ by means of logic of types (i.e., LG-logic).

The problem is stated for an arbitrary variety $\Theta$, but most of all we are interested in the variety of groups $\Theta = Grp$ and the variety $\Theta = Com - P$ of commutative and associative algebras over a field $P$.

Z.Sela (unpublished) showed that free noncommutative groups $F(X)$ and $F(Y)$ are isotyped if and only if they are isomorphic.

3.2. Examples of non-isomorphic isotyped algebras.

Proposition 3.4. Let algebras $H_1$ and $H_2$ be infinitely dimension vector spaces over a field $P$. Then they are isotyped.

Proof. We use a method which can be applied in other cases as well. Proceed from a variety $\Theta$ of vector spaces over $P$. Take $X = \{x_1, \ldots, x_n\}$ and let $W = W(X)$ be the corresponding free object, i.e., an arbitrary linear space of dimension $n$. Consider
points $\mu : W \to H_1$ and $\nu : W(X) \to H_2$. A sequence $\bar{a} = (a_1, \ldots, a_n)$, $a_i = \mu(x_i)$ corresponds to the point $\mu$. Similarly, we have $\bar{b} = (b_1, \ldots, b_n)$ for $\nu$. Denote $A = < a_1, \ldots, a_n >$ and $B = < b_1, \ldots, b_n >$. The points $\mu$ and $\nu$ we call isomorphic, if there is an isomorphism $\alpha : A \to B$ with $\alpha(a_i) = b_i$. We write $\alpha \mu = \nu$. We have also $\mu = \alpha^{-1} \nu$. If $H_1 = H_2$ then isomorphism of the points $\mu$ and $\nu$ means that $\mu$ and $\nu$ satisfy the relation $\tau$.

Along with the free algebra $W = W(X)$ consider the algebra of formulas $\Phi = \Phi(X)$. A formula $u \in \Phi$ is called correct, if for isomorphic $\mu$ and $\nu$ the inclusion $\mu \in Val_{H_1}(u)$, i.e., $u \in LKer(\mu)$, holds if and only if $u \in LKer(\nu)$, i.e., $\nu \in Val_{H_2}(u)$.

We intend to check that in our case every formula $u$ is correct. It is easy to see that for an arbitrary $\Theta$ all the equalities $w \equiv w'$ are correct, if $u$ is correct, then its negation $\neg u$ is correct, and if $u_1$ and $u_2$ are correct, then $u_1 \lor u_2$ and $u_1 \land u_2$ are correct as well.

However the implication if $u$ is correct, then so is $\exists x_i u$ for every $x_i$ is not probably true for arbitrary $\Theta$. Here there arises a question to find an example when the correctness condition is not fulfilled. We shall check that in the situation under consideration the implication is valid.

Without loss of generality take $x_i = x_1$. So, let $\mu$ and $\nu$ be isomorphic, $\nu = \alpha \mu$, $\mu \in Val_{H_1}(\exists x_1 u) = \exists x_1 Val_{H_1}(u)$. There exists $\mu_1 \in Val_{H_1}(u)$ with $\mu(y) = \mu_1(y)$ for each $y \neq x_1, y \in X$.

Take $a = \mu_1(x_1)$. For $\mu_1$ we have a sequence $(a, a_2, \ldots, a_n)$. Recall that $\bar{a} = (a_1, a_2, \ldots, a_n)$ and $\bar{b} = (b_1, b_2, \ldots, b_n)$, $b_i = \alpha'(a_i)$, for $\mu$ and $\nu$ respectively. Take $A_1 = < a, a_2, \ldots, a_n > = < a, < a_2, \ldots, a_n >>$. We want to investigate an isomorphism $\alpha_1 : A_1 \to B_1$ with $B_1 = < b, < b_2, \ldots, b_n >>$, where $b_1 = \alpha_1(a)$, $b_i = \alpha_1(a_i) = \alpha(a_i)$, $i = 2, \ldots, n$. The isomorphism $\alpha : A \to B$ induces an isomorphism $\alpha' : < a_2, \ldots, a_n > \to < b_2, \ldots, b_n >$. Suppose first that $a \in < a_2, \ldots, a_n >$. In this case we have an isomorphism $\alpha_1 : A_1 \to B_1$ with $\alpha_1(a) = \alpha'(a) = b$. Let $a \notin < a_2, \ldots, a_n >$. Then $A_1$ is a vector space with the dimension
greater by one than the dimension of the space \(< a_2, \ldots, a_n >\).

Take an arbitrary \(b \in H_2\) which does not lie in \(< b_2, \ldots, b_n >\). We have a vector space \(B_1 = < b, < b_2, \ldots, b_n > >\) of the same dimension as \(A_1\). Assuming \(\alpha_1(a) = b\) we determine an isomorphism \(\alpha_1 : A_1 \to B_1\), extending the isomorphism \(\alpha'\).

Take further \(\nu_1 : W \to H_2\) defined by the rule \(\nu_1(x_i) = \alpha_1 \mu_1(x_i)\), \(i = 1, \ldots, n\). Here \(\mu_1\) and \(\nu_1\) are isomorphic. Since \(\mu_1 \in Val_{H_1}(u)\), then \(\nu_1 \in Val_{H_1}(u)\) due to correctness of the formula \(u\). The points \(\nu\) and \(\nu_1\) coincide on the variables \(x_2, \ldots, x_n\) by the construction. Hence, \(\nu \in Val_{H_2}(\exists x_1 u) = \exists x_1 Val_{H_2}(u)\). Similarly, if \(\nu \in \exists x_1 Val_{H_2}(u)\), then \(\mu = \alpha^{-1}\nu \in Val_{H_1}(\exists x_1 u)\).

Show that if \(u \in \Phi(Y)\) is correct then the formula \(s_* u \in \Phi(X)\) where \(s : W(Y) \to W(X)\) and \(s_* : \Phi(Y) \to \Phi(X)\) (see Section 5 for the definition of the mapping \(s_*\)) is correct as well. Let \(\mu\) and \(\nu\) be isomorphic by the isomorphism \(\alpha\), and \(\mu \in Val_{H_1}(s_* u) = s_* Val_{H_1}(u)\). Here \(\mu s \in Val_{H_1}(u)\). Apply the isomorphism \(\alpha\) with \(\nu = \alpha \mu\). This gives the isomorphism \((\alpha \mu)s = \nu s = \alpha(\mu s)\), and \(\nu s\) and \(\mu s\) are isomorphic.

Suppose \(u\) is correct. Then \(\nu s \in Val_{H_2}(u)\) and \(\nu \in s_* Val_{H_2}(u) = Val_{H_2}(s_* u)\). The formula \(s_* u\) is also correct. Using the definition of the algebra \(\Phi(X)\) and the fact that all the equalities are correct, we may conclude that all \(u \in \Phi(X)\) are correct. If \(\mu\) and \(\nu\) are isomorphic, then for any \(u \in \Phi(X)\) we have \(\mu \in Val_{H_1}(u)\) if and only if \(\nu \in Val_{H_2}(u)\).

Let now \(\mu\) and \(\nu\) be isomorphic. Since \(u\) is correct we have \(u \in LKer(\mu)\) if and only if \(u \in LKer(\nu)\). Thus \(LKer(\mu) = LKer(\nu)\). If the dimensions of \(H_1\) and \(H_2\) are infinite, for any point \(\mu\) we can construct a point \(\nu\) isomorphic to it. Hence, for each point \(\mu \in Hom(W(X), H_1)\) there is \(\nu \in Hom(W(X), H_2)\) with \(LKer(\mu) = LKer(\nu)\).

The opposite is also true. This means that \(H_1\) and \(H_2\) are isotyped. It is clear that they are not necessarily isomorphic. □
The method from the proposition above can be used also in the case when $H_1$ and $H_2$ are infinite abelian groups of the finite exponent $p$. From the other hand, what can be said if $H_1$ and $H_2$ are free abelian groups of infinite range, or free noncommutative groups of infinite range? Using considerations similar to those from Proposition 3.4 we may, in particular, study locally cyclic torsion-free groups to see if they are isotyped. This also should give examples of isotyped but not isomorphic algebras.

**Remark 3.5.** In fact, the proof of Proposition 3.4 follows already from the freeness of vector spaces. We gave a detailed proof having in mind applications to other situations.

### 4. Logically perfect algebras

#### 4.1. Embedding of algebras

Remind that we defined equivalence $\rho$ on the set (affine space) $\text{Hom}(W(X), H)$. If $\mu$ and $\nu$ are two points $W(X) \to H$, then $\mu\rho\nu$ if and only if $L\text{Ker}(\mu) = L\text{Ker}(\nu)$ (see 1.5). These logical kernels are calculated in the algebra of formulas $\Phi(X)$. A group $\text{Aut}(H)$ acts on the set $\text{Hom}(W(X), H)$ by the rule $\mu \to \mu\sigma$, $\sigma \in \text{Aut}(H)$. Here each elementary set in $\text{Hom}(W(X), H)$ is invariant under the action of the group $\text{Aut}(H)$. This implies that $\mu\rho(\mu\sigma)$ always hold true.

**Definition 4.1.** An algebra $H \in \Theta$ is called *logically perfect* if for any $X$ the relation $\mu\rho\nu$ holds if and only if the points $\mu$ and $\nu$ are conjugated by an automorphism of the given $H$.

This condition means that for any $X$ every coset of the relation $\rho$ is an orbit of the group $\text{Aut}(H)$. Hence, for logically perfect groups every orbit is an elementary set.

Let us formulate the following problem:

**Problem 4.2.** Consider conditions when a given algebra $H \in \Theta$ can be embedded into a perfect algebra $H' \in \Theta$. 
This problem is closely related to the following well-known results of Model Theory [Ma]:

1. For every finite set $X$, algebra $H \in \Theta$, and points $\mu, \nu : W(X) \rightarrow H$ the condition $LKer(\mu) = LKer(\nu)$ is equivalent to the following one: for an elementary embedding $H \rightarrow G \in \Theta$ there is $\sigma \in Aut(G)$ with $\mu \sigma = \nu$.

2. There exists a large algebra $H \in \Theta$, such that for every $X$ and points $\mu, \nu : W(X) \rightarrow H$ we have $LKer(\mu) = LKer(\nu)$ if and only if $\mu \sigma = \nu$ for some $\sigma \in Aut(H)$.

Both these theorems are theorems of existence and rely on the compactness theorem. So they are highly non-constructive. Our interest is a bit different. We look at the specific varieties $\Theta$. The question is how to realize the transition from $H$ to $H'$ by constructions in $\Theta$ (see, for example, Proposition 4.2, where $\Theta$ is the variety of groups).

So, we want to find out how logically perfect algebras look like. The first question is

**Problem 4.3.** For which algebras $H \in \Theta$ every $Aut(H)$-orbit in $Hom(W(X), H)$ is an elementary set for any finite $X$?

It was noted earlier that if $H$ is logically perfect then $Aut(H)$ orbits are the elementary sets. In fact, the opposite statement is true as well (Proposition 4.4). This fact explains the importance of Problem 4.3.

**Proposition 4.4.** An algebra $H$ is logically perfect if and only if every $Aut(H)$-orbit in $Hom(W(X), H)$ is an elementary set for every $X$.

*Proof.* Let us fix a finite $X$ and take a point $\mu : W(X) \rightarrow H$. Let $A$ be an $Aut(H)$-orbit defined by $\mu$, $\mu \in A$. Suppose that $A$ is an elementary set. Then $A^{LL}_H = A$. We have $A^{LL}_H \subseteq \{\mu\}^{LL}_H = LKer(\mu)$. Then $\{\mu\}^{LL}_H = LKer(\mu)^L_H \subseteq A^{LL}_H = A$. The point $\mu$ belongs to the elementary set $\{\mu\}^{LL}_H$ and thus the whole orbit $A$
lies in \{\mu\}^{LL}_H. We get the equality \{\mu\}^{LL}_H = A. We have also
\{\mu\}^{LLL}_H = A_LH = \{\mu\}^L_H = L\text{Ker}(\mu).

Thus, for every orbit A we have \(A_LH = L\text{Ker}(\mu)\), where \(\mu\) is a point from A. We used that all orbits A are elementary sets. Different types correspond to different orbits over \(H\) and such correspondence exhausts all types related to \(H\).

Let now \(\mu \rho \nu\) and let \(A\) be an \(\text{Aut}(H)\)-orbit over \(\mu\), \(B\) an \(\text{Aut}(H)\)-orbit over \(\nu\). We have \(L\text{Ker}(\mu) = L\text{Ker}(\nu)\) i.e., \(A^L_H = B^L_H\), and thus \(A = B\) since \(A\) and \(B\) are elementary sets. This means that the points \(\mu\) and \(\nu\) belong to a common orbit. This gives \(\mu \rho_0 \nu\). Hence \(\rho = \rho_0\) and thus the algebra \(H\) is perfect. □

Consider the notion of strictly logically perfect algebras.

**Definition 4.5.** An algebra \(H \in \Theta\) is called strictly logically perfect if for any \(X\) and points \(\mu : W(X) \to H\) and \(\nu : W(X) \to H\) the condition \(L\text{Ker}^0(\mu) = L\text{Ker}^0(\nu)\) implies \(\mu = \nu \sigma\) for some \(\sigma \in \text{Aut}(H)\).

This notion is indeed more strict than being logically perfect in the usual sense. As it was done before one can prove that an algebra \(H\) is strictly logically perfect if for any \(X\) every \(\text{Aut}(H)\)-orbit in \(\text{Hom}(W(X), H)\) is an elementary set defined by some set of formulas \(T \subset \Phi^0(X)\). In this case we say that every orbit is a strictly elementary set.

4.2. **Method of HNN-extensions.** This subsection describes the case \(\Theta = \text{Grp}\). We use here the method of HNN-extensions (see \cite{HNN}).

Let \(a_i, b_i, i \in I\) be two sets of elements in the group \(H\), \(A\) and \(B\) be subgroups in \(H\), generated by all \(a_i\) and \(b_i\) respectively. Consider a system of equations \(ta_i t^{-1} = b_i, i \in I\). It is proved (Theorem 1 from \cite{HNN}) that such a system has a solution for
some $t$ belonging to a group $G$ containing $H$, if and only if the subgroups $A$ and $B$ are isomorphic under transition $a_i \to b_i$.

This theorem has a lot of applications. In particular, it implies that for every torsion free group $H$ there exists a torsion free group $G$ containing $H$, such that there is only one non-trivial class of conjugated elements in $G$. This means also the following. Let $G$ be such a group and $W = W(x)$ a free cyclic group. Consider the affine space $\text{Hom}(W, G)$ and an elementary set $A$, defined by a single "equation" $x \neq 1$. As usual, it is invariant under the action of the group $\text{Aut}(G)$. On the other hand, any two elements $\mu$ and $\nu$ in the given elementary set are conjugated by an element in $\text{Aut}(G)$. Hence, $A$ is the unique nontrivial orbit of the group $\text{Aut}(G)$. We would like to build a group $G$ in which something similar holds in the affine space $\text{Hom}(W(X), G)$, for each finite $X$. Namely, we would like to construct a group $G$ by given group $H$ such that for any $X$ in the affine space $\text{Hom}(W(X), G)$ would be a finite number $\text{Aut}(G)$–orbits and each of the orbit should be an elementary set.

Let us start from the situation when $H$ is a set without operations.

Let us use the scheme from the proof of Proposition 2.4. Given an equivalence $\tau$ on the set $X = \{x_1, \ldots, x_n\}$, consider formulas $x_i \equiv x_j$ if $x_i \tau x_j$, and $x_i \not\equiv x_j$ otherwise. Let $u = u_\tau$ be a conjunction of all such equalities and inequalities. Pass to $V = \text{Val}_H^X(u)$. A point $\mu : X \to H$ belongs to $V$ if and only if $\text{Ker}(\mu) = \tau$. If $A$ is an image of the point $\mu$, then we have a bijection $X/\tau \to A$. If $\mu$ and $\nu$ are two points in $V$, then $\text{Ker}(\mu) = \text{Ker}(\nu) = \tau$, which gives a commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\sigma} & B \\
\mu \downarrow & & \nu \downarrow \\
X & & 
\end{array}
$$
Here $\sigma$ is a bijection and $B$ is the image of the point $\nu$. The sets $A$ and $B$ are finite, while the set $H$ we regard as infinite. We have $A \cup A' = H = B \cup B'$ where $A'$ and $B'$ are of the same cardinality. This leads to the extension of the bijection $\sigma$ up to a permutation $\sigma \in S_H$. According to diagram $\nu = \mu \sigma$ and $\sigma(a_i) = b_i$, where $a_i = \mu(x_i)$ and $b_i = \nu(x_i)$. Hence, every $V_\tau \tau = Val^X_H(u_\tau)$ is an orbit of the permutation group $S_H$, and these are all the orbits. Thus we obtained a finite number of finitely defined orbits.

4.3. Embedding theorem. In the proof of the embedding theorem we will use Theorem 2 from [HNN]:

Let $\eta_i : A_i \to B_i, i \in I$, be isomorphisms of subgroups of the group $H$. There exists a group $G = \langle H, F \rangle$, where $F$ is freely generated by elements $t_i$ so that $\eta_i(a) = t_i at_i^{-1}$ for each $a \in A_i$.

Our next goal is the following:

**Theorem 4.6.** Each group can be embedded into a logically perfect group.

This theorem seems to be a particular case of a similar model theoretic result. We present here an independent group theoretic proof, using HNN-theory.

**Proof.** Let $\bar{a} = (a_1, \ldots, a_n)$ and $\bar{b} = (b_1, \ldots, b_n)$ be two sequences of elements in $H$, and let $A$ and $B$ be subgroups, generated by the elements $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, respectively. According to Theorem 1 from [HNN], there exists a group $G = G_{(\bar{a}, \bar{b})}$ containing $H$ and such that some automorphism of the group $G$ takes $\bar{a}$ into $\bar{b}$ if and only if this determines an isomorphism $\eta : A \to B$.

Having this in mind, consider an equivalence $\tau$ defined in $H^{(n)}$, $n$ is fixed. Define $\bar{a} \tau \bar{b}$ if there is an isomorphism $\eta = \eta_{(\bar{a}, \bar{b})} : A \to B$, extending correspondence $a_i \to b_i$. If $\bar{a} \tau \bar{b}$ holds true, then we have a group $G = G_{(\bar{a}, \bar{b})}$ with the element $t = t_{(\bar{a}, \bar{b})}$, such that $t$ determines an inner automorphism of the group $G$, taking $\bar{a}$ to $\bar{b}$. The group $G$ can be represented as $G = \langle H, t \rangle$. The equivalence
\(\tau\) is automatically extended to the space \(\text{Hom}(W(X), H)\), \(X = \{x_1, \ldots, x_n\}\).

Consider further the relations \(\tau_n\) for every natural \(n\). The relations \(\tau_n\) are defined on all sequences \(\bar{a} = (a_1, \ldots, a_n)\) of the length \(n\). A subgroup \(A\) in \(H\), generated by the elements \(a_1, \ldots, a_n\), corresponds to every sequence. We write \(A = A(\bar{a})\). As before, \(\bar{a}\tau_n \bar{b}\) if the transitions \(a_i \rightarrow b_i\) determine the isomorphism \(\eta(\bar{a}, \bar{b}): A(\bar{a}) \rightarrow B(\bar{b})\). Consider elements \(t(\bar{a}, \bar{b})\), which freely generate the group \(F_n\). According to [HNN], we have \(G_n = \langle H, F_n \rangle\) so that every \(t(\bar{a}, \bar{b})\) determines an inner automorphism of the group \(G_n\) which, like \(\eta(\bar{a}, \bar{b})\), transforms \(a_i\) into \(b_i\).

On the next step we vary \(n\) and consider the relations \(\tau_n\) for different \(n\). Take different \(t(\bar{a}, \bar{b})\) with \(\bar{a}\tau_n \bar{b}\) for all \(n\). Generate by these \(t(\bar{a}, \bar{b})\) the free group \(F\). Then we have a group \(G = \langle H, F \rangle\).

If \(\bar{a}\tau_n \bar{b}\), then \(t(\bar{a}, \bar{b})\) induces an isomorphism \(\eta(\bar{a}, \bar{b}): A(\bar{a}) \rightarrow B(\bar{b})\). All this is valid due to Theorem 2 from [HNN].

Denote the group \(G = \langle H, F \rangle\) by \(H'\). Iterating the transition \(H \rightarrow H'\) we get the increasing sequence of groups \(H, H', H'', \ldots\), etc. Denote by \(H^0\) the union of all these groups. Consider sequences \(\bar{a} = (a_1, \ldots, a_n)\), where \(a_i \in H^0\). For every \(n\) if \(\bar{a}\tau_n \bar{b}\), then there exists an element \(t \in H^0\), such that the inner automorphism \(\hat{t}\) transforms \(\bar{a}\) into \(\bar{b}\).

Show now that for every \(X = \{x_1, \ldots, x_n\}\) in \(\text{Hom}(W(X), H^0)\) the equality \(\tau = \tau_n = \rho_0\) holds true and every coset of the relation \(\tau\) is \(\text{Aut}(H^0)\)-orbit.

Consider \(\text{Hom}(W(X), H^0)\). Take two points \(\mu, \nu : W(X) \rightarrow H^0\). The condition \(\mu \tau \nu\) means that \(\mu \tau_n \nu\) for some \(n\), and if \((\mu(x_1), \ldots, \mu(x_n))) = \bar{a} = (a_1, \ldots, a_n)\) and \((\nu(x_1), \ldots, \nu(x_n))) = \bar{b} = (b_1, \ldots, b_n)\), then \(\bar{a}\tau_n \bar{b}\). We have an inner automorphism \(\sigma = \hat{t}\), transforming \(\bar{a}\) into \(\bar{b}\) and, simultaneously, \(\mu\) into \(\nu\). So, \(\mu \tau \nu\) implies \(\mu \sigma = \nu\) for some \(\sigma \in \text{Aut}(H^0)\), \(\tau \subset \rho_0\). Besides,
\( \rho_0 \subset \tau \) and \( \tau = \rho_0 \). We have also \( \tau = \rho \). Finitely, \( \rho = \rho_0 \) and the group \( H^0 \) is perfect.

\[ \square \]

**Remark 4.7.** All \( \text{Aut}H^0 \)-orbits in \( \text{Hom}(W(X), H^0) \) are elementary sets for every finite \( X \). However the number of orbits is infinite and equals to the number of different \( \text{Ker}(\alpha) \) for the points \( \alpha : W(X) \to H^0 \). This group is not good enough for generalization of the example of a group with a single non-trivial conjugacy class \( [\text{HNN}] \), \( [\text{Ku}] \).

On the other hand, in Proposition 2.4 we constructed a group \( H \) with finite number \( \text{Aut}H \)-orbits and each of them is an elementary set. The group \( H \) is logically perfect group.

Note that a group \( G \) is called **homogeneous** if every isomorphism \( \eta : A \to B \) of its finitely generated subgroups is realized by an inner automorphism of \( G \). Such groups were constructed by Ph. Hall, O.Kegel, B.Neumann (see \( [\text{Ha}] \), \( [\text{Ke}] \), \( [\text{Ne}] \)). It is easy to see that every group of such type is logically perfect since here we have \( \tau = \rho_0 \) and \( \rho = \rho_0 \). Homogeneity property is considered in model theory with respect to an arbitrary algebraic system. In this case the automorphism of a system is not assumed to be inner. The notion of logically perfect algebra is close to the notion of homogeneous algebra. It is easy to see that the algebra \( H \) is homogeneous if and only if for every \( X \) we have \( \tau = \rho_0 \). We keep the term ”perfectness” having in mind the observation that perfectness in groups is provided, usually, by inner automorphisms (like in Theorem 4.6).

In general, a logically perfect algebra is not necessarily homogeneous: it can happen that \( \rho = \rho_0 \) but \( \tau \neq \rho \). However, the following theorem takes place:

**Theorem 4.8.** (G.I.Zhitomiskii) An algebra \( H \) in \( \Theta \) is homogeneous if and only if \( H \) is strictly logically perfect.
Proof. We use the notion of a correct formula (see Proposition 3.4). Recall this notion with respect to algebra \( \Phi^0 \). A formula \( u \in \Phi^0 \) is called correct, if for any isomorphic \( \mu \) and \( \nu \) the inclusion \( \mu \in Val_H(u) \), i.e., \( u \in LKer^0(\mu) \), holds if and only if \( u \in LKer^0(\nu) \), i.e., \( \nu \in Val_H(u) \).

If every formula \( u \in \Phi^0(X) \) is correct then \( LKer^0(\mu) = LKer^0(\nu) \), for the isomorphic \( \mu, \nu \).

Let now \( H \) be strictly logically perfect. Using induction by cardinality of the set \( X \) we will prove that every formula \( u \in \Phi^0(X) \) is correct. Let \( X = \{x\} \). Then every boolean formula over equalities is correct. The formula \( \exists xu \) has 0 and 1 as values. So for one-element \( X \) correctness takes place. Let \( X = \{x_1, \ldots, x_n\} \) and let for any \( X \) such that \( |X| < n \) the property is proved. First of all note that the correctness property is preserved under application of the boolean operations. Let \( u \in \Phi^0(X) \) be correct. Show that \( \exists x_nu \) is correct. Let the points \( \mu, \nu : W(X) \to H \) be given, let \( \mu_0, \nu_0 \) be their restrictions on \( X_0 = \{x_1, \ldots, x_{n-1}\} \). Let \( A, B \) be the images of the points \( \mu, \nu \), respectively, in \( H \) and \( A_0, B_0 \) be the corresponding images for \( \mu_0, \nu_0 \). Let \( \alpha \) be an isomorphism \( A \to B \). Denote by \( \alpha_0 \) the isomorphism \( A_0 \to B_0 \) induced by \( \alpha \). Take \( \alpha \mu = \nu \) and \( \alpha_0 \mu_0 = \nu_0 \). Since every formula in \( \Phi^0(X_0) \) is correct we have \( LKer^0(\mu_0) = LKer^0(\nu_0) \) for isomorphic \( \mu_0, \nu_0 \). Now, since \( H \) is strictly logically perfect we have an automorphism \( \sigma \in Aut_H \) such that \( \mu_0 \sigma = \nu_0 \). Take \( \mu \sigma \) and \( \nu \). Their restrictions to \( X_0 \) coincide.

Let now \( \mu \in Val_H^X(\exists x_nu) = \exists x_n Val_H^X(u) \). Then there is \( \mu' \in Val_H^X(u) \), such that \( \mu \) and \( \mu' \) coincide on \( X_0 \). Then \( \mu \sigma \) and \( \mu' \sigma \) coincide on \( X_0 \). But \( \mu \sigma \) and \( \nu \) coincide on \( X_0 \). Then \( \mu' \sigma = \nu' \) and \( \nu \) coincide on \( X_0 \). Besides, along with \( \mu' \) the point \( \mu' \sigma = \nu' \) belongs to \( Val_H^X(u) \). Then \( \nu \in Val_H^X(\exists x_nu) \). Hence \( \mu \in Val_H^X(\exists x_nu) \) implies \( \nu \in Val_H^X(\exists x_nu) \). In a similar way one can see that \( \nu \in Val_H^X(\exists x_nu) \) implies \( \mu \in Val_H^X(\exists x_nu) \). Thus the formula \( \exists x_nu \) is also correct. Therefore all formulas
from $\Phi^0(X)$ are correct. So, if $\mu$ and $\nu$ are isomorphic then $L\text{Ker}^0(\mu) = L\text{Ker}^0(\nu)$. Using once again the condition of strict logical perfectness we have $\mu\sigma = \nu$ for some $\sigma \in \text{Aut}H$ which extends $\alpha : A \to B$. This means that algebra $A$ is homogeneous.

Conversely, let $H$ be homogeneous. Then $\mu\tau \nu$ means that $\mu\sigma = \nu$ for some $\sigma \in \text{Aut}H$. Hence, a coset of $\tau$ is an orbit. However, a coset for $\tau$ is defined by a set $T \subset \Phi^0(X)$. This means that the algebra $H$ is strictly logically perfect. \hfill \Box

**Remark 4.9.** In fact, Theorem 4.8 can be deduced from the criterion $\tau = \rho_0$. However, we presented here the original edifying proof.

**Remark 4.10.** Note that from the proof of Theorem 4.6 could be seen that every group is embedded into a homogeneous one. This fact can also be deduced from the method of HNN-extensions.

Now we give an example of logically perfect but not a homogeneous group. This example also belongs to G.I. Zhitomirskii.

**Proposition 4.11.** The infinite cyclic group $\mathbb{Z}$ is logically perfect.

*Proof.* Consider the infinite cyclic group which is represented as the additive group of integers $\mathbb{Z}$. This group is also a $\mathbb{Z}$-module. Each subgroup of $\mathbb{Z}$ is a submodule. We take the variety of all abelian groups or the variety of $\mathbb{Z}$-modules as $\Theta$. It is clear that the group $\mathbb{Z}$ is not homogeneous. Let us check that $\mathbb{Z}$ is logically perfect.

Take a set $X = \{x_1, \ldots, x_n\}$. Pick a variable $y$ and let $Y = \{y, X\}$. Define the map $s : Y \to X$ by the rule $s(x_i) = x_i$, $s(y) = 0$. So we have morphisms $s : W(Y) \to W(X)$ and $s_* : \Phi(Y) \to \Phi(X)$. For each formula $v \in \Phi(Y)$ there is the formula $u = s_*v$, $u \in \Phi(X)$ which does not necessarily belong to $\Phi^0(X)$.

Let $\mu : W(X) \to \mathbb{Z}$ be a point. We have $\mu \in \text{Val}^X_\mathbb{Z}(u) = s_*\text{Val}^Y_\mathbb{Z}(v)$ if and only if $\mu s \in \text{Val}^Y_\mathbb{Z}(v)$. 
Take two points $\mu, \nu : W(X) \to \mathbb{Z}$ and let $LKer(\mu) = LKer(\nu)$. Let us show that $\mu$ and $\nu$ are conjugated by an automorphism of $\mathbb{Z}$. This group has only the identity automorphism and the automorphism which takes an element to the inverse one. We shall check that $\mu = \nu$ or $\mu = -\nu$.

Let $\mu(x_i) = a_i, \nu(x_i) = b_i, i = 1, \ldots, n$. If all $a_i$ are equal to zero, then $\mu = \nu$. So we can assume that $a_1 \neq 0$.

For the point $\mu$ we are going to construct a special test formula $v \in \Phi(Y)$ such that $\mu \in Val^{X}(u), u = s_*v$. This will imply $\nu \in Val^{X}(u)$. First, we shall define $v_0 \in \Phi(Y)$. Then $v$ is constructed as $v = \exists y v_0$.

Let $v_0$ be the formula

$$(x_1 \equiv |a_1|y) \land (x_2 \equiv sgn(a_2a_1)|a_2|y) \land \ldots \land (x_n \equiv sgn(a_na_1)|a_n|y).$$

Here, $|a|$ stands for the absolute value of $a$ and $sgn(a)$ is a sign of $a$.

Define a point $\gamma : W(Y) \to \mathbb{Z}$ by $\gamma(x_i) = \mu(x_i) = a_i$ and $\gamma(y) = sgn(a_1)1$. The point $\gamma$ satisfies $v_0$. Since $\mu s$ and $\gamma$ coincide on $X$ then the point $\mu s$ satisfies the formula $\exists y v_0 \equiv v$. In other words $\mu \in Val^{X}(u)$, where $u = s_*v$. We have also $\nu s \in Val^{X}(v)$. From this follows that for some values $c$ of $y$ and $b_i$ of $x_i$ we have

$$b_1 = |a_1|c, \ldots, b_n = sgn(a_na_1)|a_n|c.$$  

This gives that $b_i = a_i c$ if $a_1 > 0$, $b_i = -a_i c$ if $a_1 < 0$. Interchanging $\mu$ and $\nu$ we have $d$ such that $a_i = b_i d$ if $b_1 > 0$ and $a_i = -b_id$ if $b_1 < 0$. In such a way we arrive to $c = 1$ or $c = -1$. In the first case we have $\mu = \nu$, and in the second one $\mu = -\nu$. The proof of the example is finished. \qed

The group $\mathbb{Z}$ is not strictly logically perfect since $\mathbb{Z}$ is not homogeneous. It would be interesting to get other examples which work in different $\Theta$. 
Let us make one more remark. Let us take $\mu : W(X) \to H$ and consider its closure $A = \{ \mu \}_{L^L_H} = (L\ker(\mu))_{L^L_H}$. This $A$ is a minimal elementary set and every elementary set in $Hom(W(X), H)$ is the union of such disjoint $A$. If $H$ is logically perfect then these minimal elementary sets coincide with the orbits of $Aut(H)$. This is a property of the lattice of elementary sets in $Hom(W(X), H)$. It is clear that the minimal elementary sets one to one correspond to types.

Now we give two simple examples of strictly logically perfect abelian groups.

4.4. **Example.** Let us study a concrete example. Let a group $H$ be a discrete direct product of all simple cyclic groups of different prime orders. We show that for a given $H$ all $Aut(H)$-orbits in $Hom(W, H)$ are elementary sets.

Proceed from the variable $x \in W = W(X)$ and write down a formula $u = u(x)$ of the form $x \neq 1 \land x^m = 1 \land u_0$, where $u_0$ is the conjunction of all $x^{m_i} \neq 1$ by all divisors $m_i$ of the number $m$.

Only elements $g$ of the order $m$ satisfy the formula $u = u(x)$. There is a finite number of such elements, namely $(p_1 - 1) \ldots (p_k - 1)$, if $m = p_1 \ldots p_k$.

Take a finite $X$ and consider $Hom(W(X), H)$. Let $(\mu(x_1), \ldots, \mu(x_n)) = \bar{g} = (g^1, \ldots, g^n)$ for a point $\mu : W(X) \to H$. Represent each $g^i$ as $g^i = g^i_1 \ldots g^i_{k_i}$. The factors are organized by increasing of their prime orders. The order of $g^i$ is some $m^i$.

Proceed further from the set of formulas $u(x_1), \ldots, u(x_n)$ for the orders $m_1, \ldots, m_n$ and let $u$ be their conjunction. Pass to the elementary set $Val^X_H(u) = A$. All the points $\mu : W(X) \to H$ of the type described above are included in this set.

**Proposition 4.12.** Every $A = Val^X_H(u)$ is an orbit of the group $Aut(H)$ and every orbit has such form.
Proof. Let \( \mu : W(X) \to H \) be a point in \( A \). For every automorphism \( \sigma \in \text{Aut}(H) \) we have the inclusion \( \mu \sigma \in A \). We should check whether \( \nu = \mu \sigma \) for any other point \( \nu : W(X) \to H \) from \( A \) and some automorphism \( \sigma \).

By the definition, we have \( \mu \sigma(x) = \mu(x) \circ \sigma = g \circ \sigma \) for every \( x \in X \). Let \( g = g_1 \ldots g_k \), where the order of \( g_i \) is \( p_i \). Let, besides, \( \nu(x) = g' = g'_1 \ldots g'_k \), where the order of \( g'_i \) is also \( p_i \) and \( g_i \) and \( g'_i \) are generators of the cyclic group of the order \( p_i \). Therefore the transition \( g_i \to g'_i \) determines automorphisms of these cyclic subgroups. This holds for each \( x \in X \) and, hence, we have an automorphism \( \sigma \), taking \( g \) into \( g' \). We also have \( \mu \sigma = \nu \).

Prove now that every orbit is some set \( A \). Take an arbitrary \( X = \{x_1, \ldots, x_n\} \) and a point \( \mu : W(X) \to H \). A sequence \( \bar{g} = (g^1, \ldots, g^n) \) corresponds to this point. A specific \( A \) containing \( \mu \) corresponds to \( \mu \), and this is valid for each \( A \). Applying to \( \bar{g} \) arbitrary automorphisms \( \sigma \in \text{Aut}(H) \) we get the whole orbit \( A \).

There is an infinite number of different orbits, and thus an infinite number of elementary sets and types. The interesting point here is that every orbit is finitely defined. It is easy to understand that the group \( H \) satisfies none of the noetherianity conditions and, besides, if some \( H' \) is isotyped with \( H \), then \( H \) and \( H' \) are isomorphic. Actually this result is derived from elementary equivalence of \( H \) and \( H' \).

Let us prove this fact and then make some remarks on noetherianity.

Let \( H \) be a group from the 4.12 and \( H' \) a group isotyped to \( H \). Then they are elementary equivalent and they have the same identities. Thus, \( H' \) is abelian. The formula \( \exists x (x^p \equiv 1) \) in \( H \) means that also in \( H' \) there exist elements of the order \( p \). The formula \( x_1^p \equiv x_2^p \equiv 1 \to x_1 \equiv x_2^{m_1} \lor \ldots \lor x_1 \equiv x_2^{m_k} \) with \( m_i < p \) implies that \( H' \) contains only one cyclic subgroup of the order \( p \).
The formula $x^{p^n} \equiv 1 \rightarrow x^p \equiv 1$ means that all $p$-elements are of the order $p$.

All this holds in $H'$. Therefore, $H'$ is a direct product of all cyclic subgroups of all prime orders, that leads to isomorphism.

Let us show that $H$ is not logically noetherian. Take an infinite subset $M$ in the set of primes with the infinite complement $M'$. Consider a formula $\neg(x^p \equiv 1 \rightarrow x \equiv 1)$ by all $p \in M$. This set is not reduced to a finite one since $M'$ is infinite.

4.5. **Additive group of rational numbers.** Proceed now from the additive group $H$ of rational numbers.

First of all we show that this group is homogeneous. Fix a set $X = \{x_1, \ldots, x_n\}$. Let $W(X)$ be the free abelian group over $X$. Take the points $\mu, \nu : W(X) \rightarrow H$. Let $A$ and $B$ be the images of $\mu$ and $\nu$, respectively, and $\alpha : A \rightarrow B$ be an isomorphism. The groups $A$ and $B$ are cyclic and thus $A = \{a\}$, $B = \{b\}$. We assume that $\alpha(a) = b$. Let the sequence $a_1, \ldots, a_n$, where $a_i = m_i a \in A$ corresponds to $\mu$. Correspondingly, we have $(b_1, \ldots, b_n)$ for $\nu$. Here, $\alpha(a_i) = b_i = m_i b$, where all $m_i$ are integers. Take $s = b/a$, $sa = b$. Multiplication of the element $h \in H$ by $s$ is an automorphism of the group $H$. This automorphism extends the initial $\alpha$.

If, further, $A$ is the orbit of the point $\mu$ then this orbit is defined by a set of formulas $T$ determined by the kernel of the point $\mu$.

The same orbit one can describe using another approach. Let $x$ be an auxiliary variable and consider the formula

$$u = u(m_1, \ldots, m_n) = \exists x(x^{m_1} = x_1 \land \ldots \land x^{m_n} = x_n).$$

If we restrict the value of this formula on the initial set $X$ then we come to the orbit we are looking for.

In the next definition and Theorem 4.14 a possible use of the auxiliary variable $x$ is taken into account.
Definition 4.13. An algebra $H$ is called logically locally noetherian if for each finite set $X$ the closure of a finite set of points $\mu_i : W(X) \to H$ is finitely defined.

Theorem 4.14. The locally cyclic group $H$ is logically locally noetherian.

Proof. First of all we show that for every point $\mu : W(X) \to H$ we have $L\ker(\mu) = \{u\}_H^L$ for some formula $u$ of the form $u = u(m_1, \ldots, m_n)$. Besides, $\{\mu\}_H^L$ is a finitely definable closure of the point $\mu$.

Let, $u = u(m_1, \ldots, m_n)$. Then $Val_X^X(u) = \{u\}_H^L$ and $(Val_X^X(u))_H^L = \{u\}_H^{LL}$. Let now $\mu \in A_0 = Val_X^X(u)$. Then $\{\mu\}_H^L \supset A_0 = \{u\}_H^{LL}$ and $L\ker(\mu) \supset \{u\}_H^{LL}$. By the definition of the operator $L$ we have $A_0 = \cap_{\nu \in A_0} L\ker(\nu)$. It follows from the previous considerations that $\mu$ and $\nu$ are isotyped. Hence, $A_0 = L\ker(\mu) = \{u\}_H^{LL}$. We have also $(L\ker(\mu))_H^L = A_0$ for arbitrary $\mu \in A_0$.

We will need now a remark on the lattice of all elementary sets in the given $\hom(W(X), H)$. Let $A$ and $B$ be two such sets, $A_H^L = T_1$, $B_H^L = T_2$. Let $T_1^0$ be a subset in $T_1$ with $(T_1^0)_H^L = T_1$. Similarly, take $T_2^0$ in $T_2$ with $(T_2^0)_H^L = T_2$. Denote by $T_1^0 \lor T_2^0$ a set of all $u \lor v$, $u \in T_1^0$, $v \in T_2^0$. Check that $(T_1^0 \lor T_2^0)_H^L = A \cup B$. Indeed, $(T_1^0 \lor T_2^0)_H^L = \cap_{u \lor v} Val_X^X(u \lor v) = \cap_{u \lor v} (Val_X^X(u) \cup Val_X^X(v)) = (\cap_{u \in T_1^0} Val_X^X(u)) \cup (\cap_{v \in T_2^0} Val_X^X(v)) = A \cup B$.

Apply this to the group $H$. Let $u_1 = u(m_1, \ldots, m_n)$, $u_2 = u(m'_1, \ldots, m'_n)$, $A = Val_X^X(u_1)$, $B = Val_X^X(u_2)$. Then $A \cup B = Val_X^X(u_1 \lor u_2)$. This implies that the union of any finite number of the sets of the type $A$ is an elementary finitely defined set.

Let, further, $A_0$ be a finite set of points $\mu : W(X) \to H$. Every $\mu \in A_0$ belongs to some $Val_X^X(u)$, $u = u(m_1, \ldots, m_n)$. Hence, $A_0$ is contained in a finite union of the sets of the type $Val_X^X(u)$.
Let $A_0$ be a subset in some $A = (L\text{Ker}(\mu))^L_H$. Then $A^L_{0H} = \bigcap_{\nu \in A_0} L\text{Ker}(\nu) = L\text{Ker}(\mu)$ for a point $\mu \in A_0$. The closure is $A^{LL}_{0H} = A$.

Take now a finite set of points $A_0 = \{\mu_1, \ldots, \mu_k\}$, $\mu_i \in L\text{Ker}(\mu_i)^L_H$ and show that its closure is finitely defined. Here we have $A^L_{0H} = \bigcap_{\mu_i \in A_0} L\text{Ker}(\mu_i)$.

It follows from the remarks on elementary sets that $A^{LL}_{0H} = \bigcup_{\mu_i} L\text{Ker}(\mu_i)^L_H = \bigcup_{\mu_i} A_i$, $A_i = L\text{Ker}(\mu_i)^L_H$. For the group of rational numbers $H$ all $A_i$ are finitely defined. Then $\bigcup_{\mu_i} A_i$ is finitely defined as well. Therefore, the closure $A^{LL}_{0H}$ is finitely defined.

□

Now we might compare local logical noetherianity to other noetherianity conditions. We do not study this problem here. Just note that every logically noetherian algebra $H$ is locally logically noetherian. However, if every finitely generated subgroup in $H$ is logically noetherian, it does not mean that $H$ is logically locally noetherian.

5. SOME FACTS FROM ALGEBRAIC LOGIC. APPENDIX

5.1. **Introduction.** For every given variety of algebras $\Theta$ we distinguish an ordinary pure logic in $\Theta$ and algebraic logic in $\Theta$. Formulas of algebraic logic are formulas of the pure logic, compressed by the semantic relation in the given $\Theta$. Polyadic Halmos algebras and cylindric Tarski algebras are the main general structures of the algebraic logic. The essential characteristic property of these structures is that they admit an infinite set of variables. Denote this set by $X^0$. (See [H], [HMT]).

In our case we are forced to consider a system of all finite subsets $X$ in $X^0$ instead of such big $X^0$. Denote this system by $\Gamma$. Then we pass to a multi-sorted algebra with a system of sorts $\Gamma$. 

We come, in particular, to multi-sorted Halmos algebras and to Halmos categories.

5.2. **A category - algebra** $Hal_\Theta(H)$. Let us start with an important example, namely, Halmos category $Hal_\Theta(H)$. Here $H$ is an algebra in $\Theta$. Objects of this category are extended boolean algebras $Bool(W(X), H)$. Define morphisms $s_* : Bool(W(X), H) \rightarrow Bool(W(Y), H)$. Denote by $\Theta^0$ the category of free in $\Theta$ algebras $W = W(X)$, $X \in \Gamma^0$. We have also a category $K^0_\Theta(H)$ of all affine spaces over $H$. Its morphisms are mappings $\tilde{s} : Hom(W(Y), H) \rightarrow Hom(W(X), H)$, where $s : W(X) \rightarrow W(Y)$ is a morphism in $\Theta^0$ and $\tilde{s}(\nu) = \nu s : W(Y) \rightarrow H$ for $\nu : W(X) \rightarrow H$. For each $A \subset Hom(W(X), H)$ we set $s_*A = \tilde{s}^{-1}A$. This determines a morphism in $Hal_\Theta(H)$. Every $s_*$ is also a homomorphism of Boolean algebras and it is correlated with quantifiers and equalities (see also Subsection 5.4).

Passing to general definitions, let us refine the notion of extended Boolean algebras.

Recall that in Algebraic Logic (AL) quantifiers are treated as operations on Boolean algebras. Let $B$ be a Boolean algebra. Its **existential quantifier** is a mapping $\exists : B \rightarrow B$ with the conditions:

1. $\exists 0 = 0$,
2. $\exists a > a$,
3. $\exists(a \land \exists b) = \exists a \land \exists b$.

The **universal quantifier** $\forall : B \rightarrow B$ is defined dually:

1. $\forall 1 = 1$,
2. $\forall a < a$,
3. $\forall(a \lor \forall b) = \forall a \lor \forall b$.

Here 0 and 1 are zero and unit of the algebra $B$ and $a, b$ are arbitrary elements of $B$. The quantifiers $\exists$ and $\forall$ are coordinated in the usual way: $\exists a = \forall \overline{a}$, $\forall a = \exists \overline{a}$. 
Let $\Theta$ and $W = W(X) \in \Theta$ be fixed and $B$ be a Boolean algebra. We call $B$ an extended Boolean algebra in $\Theta$ over $W(X)$, if

1. There are defined quantifiers $\exists x$ for all $x \in X$ in $B$ with $\exists x \exists y = \exists y \exists x$ for all $x, y \in X$.
2. To every formula $w \equiv w'$, $w, w' \in W$ it corresponds a constant in $B$, denoted also by $w \equiv w'$. Here,
   2.1. $w \equiv w$ is the unit of the algebra $B$.
   2.2. For every $n$-ary operation $\omega \in \Omega$, where $\Omega$ is a signature of the variety $\Theta$, we have
   $$w_1 \equiv w_1' \land \ldots \land w_n \equiv w_n' < w_1 \ldots w_n \omega \equiv w_1' \ldots w_n' \omega.$$

We can consider the variety of such algebras for the given $\Theta$ and $W = W(X)$.

5.3. Halmos category. A general definition.

**Definition 5.1.** A category $\Upsilon$ is a Halmos category if:

1. Every its object has the form $\Upsilon(X)$, and this object is an extended Boolean algebra in $\Theta$ over $W(X)$.
2. Morphisms are of the form $s_* : \Upsilon(X) \to \Upsilon(Y)$, where $s : W(X) \to W(Y)$ are morphisms in $\Theta^0$, $s_*$ are homomorphisms of Boolean algebras and the transition $s \to s_*$ is given by a covariant functor $\Theta^0 \to \Upsilon$.
3. There are identities controlling the interaction of morphisms with quantifiers and equalities. The coordination with the quantifiers is as follows:
   3.1. $s_1 \exists x a = s_2 \exists x a$, $a \in \Upsilon(X)$, if $s_1 y = s_2 y$ for every $y \in X$, $y \neq x$.
   3.2. $s_* \exists x a = \exists (sx)(s_* a)$ if $sx = y \in Y$ and $y = sx$ is not in the support of $sx'$, $x' \in X$, $x' \neq x$.
4. The following conditions describe coordination with equalities:
4.1. \( s_\ast (w \equiv w') = (sw \equiv sw') \) for \( s : W(X) \to W(Y) \), \( w, w' \in W(X) \).

4.2. \( s^x_{w}a \land (w \equiv w') < s^x_{w'}a \) for an arbitrary \( a \in \Upsilon(X), x \in X \), where \( w, w' \in W(X) \), and \( s^x_{w} : W(X) \to W(X) \) is defined by the rule: \( s^x_{w}(x) = w, s^x_{w}(y) = y, y \in X, \ y \neq x. \)

The category \( Hal_{\Theta}(H) \) is an example of the Halmos category. Another important example is the category of formulas \( Hal_{\Theta}^0 \) of the algebras of formulas \( Hal_{\Theta}^0(X) = \Phi(X) \). This category plays in logical geometry the same role as the category \( \Theta^0 \) plays in AG.

5.4. Halmos algebras. We deal with multi-sorted Halmos algebras, associated with Halmos categories. Describe first the signature \( L_X \). Take \( L_X = \{ \lor, \land, -, \exists x, x \in X, M_X \} \) for every \( X \). Here \( M_X \) is a set of all equalities over the algebra \( W = W(X) \). We add all \( s : W(X) \to W(Y) \) to all \( L_X \), treating them as symbols of unary operations. Denote the new signature by \( L_{\Theta} \).

Consider further algebras \( \Upsilon = (\Upsilon_X, X \in \Gamma^0) \). Every \( \Upsilon_X \) is an algebra in the signature \( L_X \) and an unary operation (mapping) \( s_\ast : \Upsilon_X \to \Upsilon_Y \) corresponds to every \( s : W(X) \to W(Y) \).

**Definition 5.2.** We call an algebra \( \Upsilon \) in the signature \( L_{\Theta} \) a Halmos algebra, if

1. Every \( \Upsilon_X \) is an extended Boolean algebra in the signature \( L_X \).
2. Every mapping \( s_\ast : \Upsilon_X \to \Upsilon_Y \) is a homomorphism of Boolean algebras.
3. The identities, controlling interaction of operations \( s_\ast \) with quantifiers and equalities are the same as in the definition of Halmos categories.

It is clear now that each Halmos category \( \Upsilon \) can be viewed as a Halmos algebra and vice versa. In particular, this relates to \( Hal_{\Theta}(H) \). Recall also that homomorphisms of multi-sorted algebras work componentwise.
5.5. **Categories and algebras of formulas.** Denote by \( M = (M_X, X \in \Gamma) \) a multi-sorted set with the components \( M_X \).

Take the *absolutely free algebra* \( \Upsilon^0 = (\Upsilon^0_X, X \in \Gamma) \) over \( M \) in the signature \( L_\Theta \). Elements of each \( \Upsilon^0_X \) are First Order Logic (FOL) formulas which are inductively constructed from the equalities using the signature \( L_\Theta \). So, \( \Upsilon^0 \) is a multi-sorted algebra of pure FOL formulas over equalities.

Denote by \( Hal_\Theta \) the variety of \( \Gamma \)-sorted Halmos algebras in the signature \( L_\Theta \). Denote by \( Hal^0_\Theta \) the free algebra of this variety over the multi-sorted set of equalities \( M = (M_X, X \in \Gamma) \).

The same \( M \) determines the homomorphism \( \pi = (\pi_X, X \in \Gamma) : \Upsilon^0 \to Hal^0_\Theta \). If \( u \in \Upsilon^0_X \), then the image \( u^{\pi_X} = \bar{u} \) in \( Hal^0_\Theta(X) \) is viewed as a compressed formula.

*Setting* \( Hal^0_\Theta(X) = \Phi(X) = (\Upsilon^0_X)^{\pi_X} \) *we get the wanted algebra of compressed formulas which we used throughout the paper.* This is an extended Boolean algebra with additional operations of type \( s_* \).

Recall that the Halmos algebra of formulas \( Hal^0_\Theta \) is also a Halmos category. We have a covariant functor \( \Theta^0 \to Hal^0_\Theta \).

5.6. **Value of a formula.** The value \( Val^X_H(w \equiv w') \) corresponds to each equality \( w \equiv w', w, w' \in W(X) \). This determines a mapping \( Val_H : M \to Hal_\Theta(H) \) which is uniquely extended up to homomorphisms

\[
Val^0_H : \Upsilon^0 \to Hal_\Theta(H),
\]

and

\[
Val_H : Hal^0_\Theta \to Hal_\Theta(H).
\]

For every \( X \in \Gamma \) we have a commutative diagram

\[
\begin{array}{ccc}
\Upsilon^0_X & \xrightarrow{Val^0_X_H} & Bool(W(X), H) \\
\pi_X \downarrow & & \downarrow Val^X_H \\
\Phi(X) & \xrightarrow{Val^X_H} &
\end{array}
\]
Thus, for every $u \in \Upsilon_X^0$ and the corresponding $\bar{u} \in \Phi(X)$ we have the values $Val^X_H(u) = Val^X_H(\bar{u})$.

Let us make a remark on the kernel of the homomorphism $Val_H$. We have

$$\text{Ker}(Val_H) = Th(H) = (Th^X(H), X \in \Gamma).$$

Here $Th(H) = (Th^X(H), X \in \Gamma)$, is the elementary theory of the algebra $H$, i.e., the set of formulas $u \in Th^X(H)$ such that $Val^X_H(u) = \text{Hom}(W(X), H)$. It is clear also that the image $\text{Im} Val_H$ is a subalgebra in $Hal_\Theta(H)$ which consists of one-defined elementary sets.

5.7. The main theorem.

**Theorem 5.3.** [P7] The variety $Hal_\Theta$ is generated by all algebras $Hal_\Theta(H), H \in \Theta$.

This means that identities of all $Hal_\Theta(H)$ determine the variety of Halmos algebras $Hal_\Theta$.

If $\Theta_1$ is a subvariety in $\Theta$, then the variety $Hal_{\Theta_1}$ in $Hal_\Theta$, generated by all $Hal_{\Theta_1}(H), H \in \Theta_1$, corresponds to $\Theta_1$. Therefore, if $H_1$ and $H_2$ are algebras from $\Theta_1$, then they are isotyped in $\Theta_1$ if and only if they are isotyped in $\Theta$.

Note also the following general observation. It is clear that there is a canonical homomorphism of multi-sorted algebras

$$\pi_H : \Upsilon^0 \rightarrow Hal_\Theta(H),$$

where $\pi_H = (\pi^X_H, x \in \Gamma)$, and $\pi^X_H$ are homomorphisms $\pi^X_H : \Upsilon^0_X \rightarrow \text{Bool}(W(X), H))$. This homomorphism is unique because it takes equalities to the corresponding equalities. Hence the kernel $\text{Ker}(\pi_H)$ is also the congruence of identities of the algebra $Hal_\Theta(H)$. Consider $\tilde{\pi} = \bigcap_H \text{Ker}(\pi_H)$. This is the congruence of all identities of the variety generated by all $Hal_\Theta(H)$.

Theorem 5.3 now means that there is the equality

$$Hal^0_\Theta = \Upsilon^0 / \tilde{\pi}; \quad \Phi(X) = \Upsilon^0_X / \tilde{\pi}^X,$$
where $\tilde{\pi}^X = \pi_X$. It can be shown (see [P9], [P7]) that the congruence $\tilde{\pi}$ corresponds to Lindenbaum-Tarski congruence.

We gave the necessary information from algebraic logic. The conditions from Subsection 1.2 are now realized.

5.8. **Category of elementary sets.** First consider the category $Set_\Theta(H)$ of affine sets over an algebra $H$. Its objects are of the form $(X, A)$, where $A$ is an arbitrary subset in the affine space $\text{Hom}(W(X), H)$. The morphisms are

$$\left[ s \right]: (X, A) \to (Y, B).$$

Here $s: W(Y) \to W(X)$ is a morphism in $\Theta^0$. The corresponding $\tilde{s}: \text{Hom}(W(X), H) \to \text{Hom}(W(Y), H)$ should be coordinated with $A$ and $B$ by the condition: if $\nu \in A \subset \text{Hom}(W(X), H)$, then $\tilde{s}(\nu) \in B \subset \text{Hom}(W(Y), H)$. Then the induced mapping $\left[ s \right]: A \to B$ we consider as a morphism $(X, A) \to (Y, B)$.

Now we define the category of algebraic sets $K_\Theta(H)$ and the category of elementary sets $LK_\Theta(H)$. Both these categories are full subcategories in $Set_\Theta(H)$ and are viewed as important invariants of the algebra $H$. We call them AG- and LG-invariants of $H$.

The objects of $K_\Theta(H)$ are of the form $(X, A)$, where $A$ is an algebraic set in $\text{Hom}(W(X), H)$. If we take for $A$ the elementary sets, then we are getting the category of elementary sets $LK_\Theta(H)$. The category $K_\Theta(H)$ is a full subcategory in $LK_\Theta(H)$.

As it was mentioned, if algebras $H_1$ and $H_2$ are isotyped, then the categories $LK_\Theta(H_1)$ and $LK_\Theta(H_2)$ are isomorphic.

5.9. **Model Theory and algebraic logic.** Let us recall some known facts. Along with an algebra of formulas $\Phi = \Phi(X)$ we consider the algebra of pure formulas $\Upsilon^0_X$. Here formulas are identified with their records. Variables from the set $X$ take part in the records. A variable may be either bound by a quantifier,
or free. A formula $u$ is called closed (or a proposition) if all its variables are bound. If $u$ is a closed formula, then its value $\text{Val}_H^X(u)$ is a unit or zero of the algebra $\text{Bool}(W(X), H)$ (the whole space $\text{Hom}(W(X), H)$ or the empty set of points). The set of closed formulas $T$ in $\Upsilon^0$ or in $\text{Hal}_0^\Theta$ is called a theory. A theory is satisfiable if it has a model. A theory $T$ is called complete if every closed formula $u$ belongs to $T$ or $\neg u$ belongs to $T$. The theory $Th(H)$ is always complete. A theory $T$ is called categorical in the given cardinal $\alpha$ if any two of its models of the cardinality $\alpha$ are isomorphic.

This paper is one in the series of papers related to universal algebraic geometry [KMR], [BMR], [MR], [P6], [P2], [P4], [P8], etc. As we have mentioned, in the frameworks of the considered theory there arise various problems close to algebra and Model Theory. They seem to be new, although some of them may look simple to specialists (I mean specialists in Model Theory).

The following remarks will concern pure logic and algebraic logic in Model Theory. Model Theory is a combination of syntax and semantics. Syntax (languages) play an essential role comparable to that of semantics (models). One can speak of syntactic structure related to languages and of theories in languages. All this is applicable to concrete mathematical problems.

Algebraic logic is not just a syntax, but nevertheless it works in Model Theory. It is helpful to treat homomorphisms $\text{Val}_H^X : \Phi(X) \rightarrow \text{Bool}(W(X), H)$ interrelating formula and its value. Elementary theory can be presented as a kernel of such homomorphism. In concern to theory of types one can speak of logical kernel of a point. This kernel $\text{LKer}(\mu)$ automatically turns to be ultrafilter in the algebra of formulas $\Phi = \Phi(X)$. There are a lot of other reasons to apply algebraic logic in Model Theory.

Let us once more name open problems:
**Problem 5.4.** Consider infinite logically noetherian algebras in different varieties $\Theta$.

**Problem 5.5.** Consider separable algebras in different varieties $\Theta$.

**Problem 5.6.** What are the varieties $\Theta$ such that free algebras in $\Theta$ are logically separable.

**Problem 5.7.** Construct automorphically finitary algebras $H$ in different varieties $\Theta$. In other words, we want to get algebras $H$ in $\Theta$ such that for every finite $X$ there is a finite number of $\text{Aut}(H)$-orbits in the space $\text{Hom}(W(X), H)$.

**Problem 5.8.** A general problem: types and isotypeness in multi-sorted algebras. Consider this problem for the variety of representations of groups $\text{Rep} - K$.

Let us note that in [PZ] there was Problem 1.23 concerning non-isomorphic isotyped abelian groups. We see now that this problem can be easily solved (Proposition 3.4). In fact, the following problem is actual for abelian groups:

**Problem 5.9.** Find conditions when two abelian groups are elementary equivalent but not isotyped.

Finally,

**Problem 5.10.** Which noetherian groups are not logically noetherian.

In all the cases we mean special $\Theta$ and $H \in \Theta$. For instance, $\Theta$ may be modules, vector spaces, linear algebras, Lie, associative algebras (finitely dimension or not), etc. The concrete always illuminates the general.
5.10. **Acknowledgements.** The author is pleased to thank Zlil Sela who noticed that the logical kernel of a point is a type, and Alexei Miasnikov who payed my attention on the fact that the logic-geometrical equivalence of algebras is the same as isotypeness of algebras. I am also grateful to my colleagues Yu.Ershov, E.Katsov, V.Remeslennikov, E.Rips, A.Olshansky, G.Zhitomirski, B.Zilber and others for the constant support.

The paper was written in Jurmala, Latvia. I had perfect working conditions thanks to my good friends Anna Efimenko and Dima Koval, and Alla and Igor Duman. The manuscript was typed and prepared by E.& T.Plotkin.

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