Article

Evaluation of Infinite Series by Integrals

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Abstract: We examine a large class of infinite triple series and establish a general summation formula. This is done by expressing the triple series in terms of definite integrals involving arctangent function that are evaluated in turn in closed forms. Numerous explicit formulae are tabulated for the triple series whose values result in elegant expressions as π, ln 2 and the Catalan constant G.

Keywords: infinite series; Catalan’s constant; integration by parts

MSC: 11B65; 65B10

1. Introduction and Outline

In mathematics and applied sciences, there exist numerous infinite series [1,2]. For example, the double series of Mordell [3,4] and Tornheim [5–8] play an important role in the number theory. In his collected works, Ramanujan [9] made explorations to several remarkable series (see also [10]).

From calculus, there are two simple but well-known series

\[ \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = \ln 2. \]

Their tensor product with zeta function and extensions lead to intensive investigations on Euler–Zagier sums and multiple zeta values (see for example [11–13]). These works suggest the authors to examine naturally, for \( \lambda \in \mathbb{Z} \) and \( \mu, \nu \in \mathbb{N}_0 \), the following triple series

\[ S_{\lambda}(\mu, \nu) = \sum_{n=0}^{\infty} \binom{-\lambda}{n} \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{2n+2i+2\mu-1} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{2n+2j+2\nu-1}. \quad (1) \]

The aim of this article is to evaluate explicitly this series. Firstly, it is trivial to see that there holds the following symmetry:

\[ S_{\lambda}(\mu, \nu) = S_{\lambda}(\nu, \mu). \quad (2) \]

Then for \( m \in \mathbb{N}_0 \) and \( n \in \mathbb{Z} \), recall the binomial identity

\[ \binom{-\lambda}{n} = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \binom{k-\lambda}{n+m}, \]
which can be explained by the finite differences (i.e., the $m$th difference of a polynomial of degree $m + n$). When $m \leq \min\{\mu, \nu\}$, the series $S_{\lambda}(\mu, \nu)$ can be manipulated as follows:

$$S_{\lambda}(\mu, \nu) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \sum_{n=m}^{\infty} \frac{(k - \lambda)}{n + m} \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{2n+2i+2\mu-1} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{2n+2j+2\nu-1}$$

where the lower limit $m$ of the sum with respect to $n$ is replaced by 0 because

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \binom{k - \lambda}{n} = 0 \quad \text{for} \quad 0 \leq n < m.$$

Hence, we get the following recurrence relation.

**Lemma 1** (Recurrence relation: $0 \leq m \leq \min\{\mu, \nu\}$).

$$S_{\lambda}(\mu, \nu) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} S_{\lambda-k}(\mu - m, \nu - m).$$

Particularly for $m = 1$, the following simplified recurrence holds:

$$S_{\lambda}(\mu, \nu) = S_{\lambda-1}(\mu - 1, \nu - 1) - S_{\lambda}(\mu - 1, \nu - 1). \quad (3)$$

By writing the two inner sums as definite integrals

$$\sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{2n+2i+2\mu-1} = \sum_{i=1}^{\infty} (-1)^{i-1} \int_{0}^{1} x^{2n+2i+2\mu-2} \, dx = \int_{0}^{1} x^{2n+2\mu} \left\{ \sum_{i=1}^{\infty} (-1)^{i-1} x^{2i-2} \right\} \, dx = \int_{0}^{1} x^{2n+2\mu} \frac{1}{1 + x^2} \, dx,$$

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{2n+2j+2\nu-1} = \sum_{j=1}^{\infty} (-1)^{j-1} \int_{0}^{1} y^{2n+2j+2\nu-2} \, dy = \int_{0}^{1} y^{2n+2\nu} \left\{ \sum_{j=1}^{\infty} (-1)^{j-1} y^{2j-2} \right\} \, dy = \int_{0}^{1} y^{2n+2\nu} \frac{1}{1 + y^2} \, dy;$$

where exchanging orders of summation and integration is justified by Lebesgue’s dominated convergence theorem ([14], §11.32), we can express $S_{\lambda}(\mu, \nu)$ as

$$S_{\lambda}(\mu, \nu) = \sum_{n=0}^{\infty} \binom{-\lambda}{n} \int_{0}^{1} x^{2n+2\mu} \frac{1}{1 + x^2} \, dx \int_{0}^{1} y^{2n+2\nu} \frac{1}{1 + y^2} \, dy = \int_{0}^{1} \int_{0}^{1} \frac{x^2y^2}{(1 + x^2)(1 + y^2)} \sum_{n=0}^{\infty} \binom{-\lambda}{n} x^{2n}y^{2\nu} \, dx \, dy.$$
Evaluating the inner sum by the binomial theorem

\[
\sum_{n=0}^{\infty} \binom{-\lambda}{n} x^{2n} y^{2n} = (1 + x^2 y^2)^{-\lambda},
\]

leads us to the following double integral representation.

**Lemma 2** (Integral representation: \(\lambda \in \mathbb{Z}\) and \(\mu, \nu \in \mathbb{N}_0\)).

\[
S_{\lambda}(\mu, \nu) = \int_0^1 \int_0^1 \frac{x^{2\nu y^{2\nu}}}{(1 + x^2)(1 + y^2)(1 + x^2 y^2)^\lambda} \, dx \, dy.
\]

Based on Lemmas 1 and 2, we deduce the following preliminary facts.

- \(\lambda, \mu, \nu\) with \(\lambda = 0\) In this case, the corresponding double integral becomes a product of two single integrals

\[
S_0(\mu, \nu) = \int_0^1 \int_0^1 \frac{x^{2\nu y^{2\nu}}}{(1 + x^2)(1 + y^2)} \, dx \, dy = \int_0^1 \frac{x^{2\mu}}{1 + x^2} \, dx \int_0^1 \frac{y^{2\nu}}{1 + y^2} \, dy.
\]

Both integrals are computable since their integrands are simple rational functions.

- \(\lambda, \mu, \nu\) with \(\lambda < 0\) According to the binomial theorem, by expanding \((1 + x^2 y^2)^{-\lambda}\) in Lemma 2, we can express \(S_{\lambda}(\mu, \nu)\) in terms of \(S_0(\mu', \nu')\).

- \(\lambda, \mu, \nu\) with \(\lambda > 0\) In view of Lemma 1, we can write \(S_{\lambda}(\mu, \nu)\) in terms of \(S_{\lambda}(\mu', 0)\) or \(S_{\lambda'}(0, \nu')\). Taking account of symmetry, we only need to evaluate \(S_{\lambda}(\mu, 0)\) for \(\lambda \in \mathbb{Z}\) and \(\mu \in \mathbb{N}_0\).

By making use of the algebraic relation

\[
x^{2\mu} = (-1)^\mu + (1 + x^2) \sum_{k=1}^{\mu} (-1)^{\mu-k} x^{2k-2},
\]

we can express

\[
S_{\lambda}(\mu, 0) = (-1)^\mu S_{\lambda}(0, 0) + \sum_{k=1}^{\mu} (-1)^{\mu-k} T_{\lambda}(k - 1),
\]

where

\[
T_{\lambda}(\mu) = \int_0^1 \int_0^1 \frac{x^{2\mu} y^{2\mu}}{(1 + x^2)(1 + x^2 y^2)^\lambda} \, dx \, dy.
\]

Therefore, the evaluation of series \(S_{\lambda}(\mu, \nu)\) is simplified to analyzing series \(S_{\lambda}(0, 0)\) and \(T_{\lambda}(\mu)\). They will be treated separately in the next two sections. Finally, the paper will end in Section 4, where a conclusive theorem will be presented together with several tabulated sample formulae.

Throughout the paper, we shall utilize the following notations. For an indeterminate \(x\) and \(n \in \mathbb{N}_0\), the rising and falling factorials are defined by \((x)_0 = (\lambda)_0 = 1\) and

\[
(x)_n = x(x + 1) \cdots (x + n - 1) \quad \text{and} \quad (x)_n = x(x - 1) \cdots (x - n + 1) \quad \text{for} \quad n \in \mathbb{N}.
\]

Analogous to the harmonic numbers and odd harmonic numbers [13,15–18]

\[
H_n = \sum_{k=1}^{\infty} \frac{1}{k} \quad \text{and} \quad O_n = \sum_{k=1}^{\infty} \frac{1}{2k - 1}.
\]
we shall employ their “skewed” variants (cf. [19–22]):
\[
\bar{H}_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \quad \text{and} \quad \bar{O}_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1}.
\] (5)

Most of our summation formulae are expressed in terms of \( \pi \), \( \ln 2 \) and Catalan’s constant (cf. [23–25])
\[
G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \approx 0.915965594.
\] (6)

In order to assure the accuracy of computations, numerical tests for all the equations have been made by appropriately devised Mathematica commands.

2. Evaluation of \( S_\lambda(0, 0) \)

In view of the integral expression in Lemma 2, consider the difference
\[
2S_{1+\lambda}(0, 0) - S_\lambda(0, 0) = \int_0^1 \int_0^1 \frac{(1 - x^2 y^2)}{(1 + x^2)(1 + y^2)(1 + x^2 y^2)^{\lambda+1}} \, dx \, dy.
\]

By applying the equation
\[
1 - x^2 y^2 = (1 + x^2) + (1 + y^2) - (1 + x^2)(1 + y^2)
\]
and then the symmetry, we can manipulate the double integral
\[
2S_{1+\lambda}(0, 0) - S_\lambda(0, 0) = \int_0^1 \int_0^1 \frac{dx \, dy}{(1 + x^2)(1 + x^2 y^2)^{\lambda+1}}
+ \int_0^1 \int_0^1 \frac{dx \, dy}{(1 + y^2)(1 + x^2 y^2)^{\lambda+1}} - \int_0^1 \int_0^1 \frac{dx \, dy}{(1 + x^2 y^2)^{\lambda+1}}
= 2 \int_0^1 \int_0^1 \frac{dx \, dy}{(1 + x^2)(1 + x^2 y^2)^{\lambda+1}} - \int_0^1 \int_0^1 \frac{dx \, dy}{(1 + x^2 y^2)^{\lambda+1}},
\]
which yields the expression
\[
2S_{1+\lambda}(0, 0) - S_\lambda(0, 0) = \int_0^1 \int_0^1 \frac{(1 - x^2)}{(1 + x^2)(1 + x^2 y^2)^{\lambda+1}} \, dx \, dy
= \int_0^1 \frac{1 - x^2}{1 + x^2} dx \int_0^1 \frac{dy}{(1 + x^2 y^2)^{\lambda+1}}
= \int_0^1 \frac{1 - x^2}{1 + x^2} \Theta_{\lambda+1}(x) dx.
\]

Henceforth, \( \Theta_\lambda(x) \) is defined by the parametric integral
\[
\Theta_\lambda(x) = \int_0^1 \frac{dy}{(1 + x^2 y^2)^{\lambda}} \quad \text{for} \quad \lambda \in \mathbb{N}_0.
\] (7)

By employing the integration by parts
\[
\Theta_\lambda(x) = \Theta_{\lambda-1}(x) - \int_0^1 \frac{(xy)^2 dy}{(1 + x^2 y^2)^{\lambda}}
= \Theta_{\lambda-1}(x) - \frac{\Theta_{\lambda-1}(x)}{2(\lambda - 1)} + \frac{1}{2(\lambda - 1)(1 + x^2)^{\lambda-1}},
\]
we get the following recurrence relation

\[ \Theta_\lambda(x) = \frac{2\lambda - 3}{2\lambda - 2} \Theta_{\lambda-1}(x) + \frac{1}{2(\lambda - 1)(1 + x^2)^{\lambda-1}}. \]

Iterating this relation \( \lambda - 1 \) times (under the condition \( \lambda \geq 1 \)), we find that

\[ \Theta_\lambda(x) = \frac{\lambda - 3}{\lambda - 1} \Theta_1(x) + \frac{1}{2(\lambda - 1)} \sum_{k=1}^{\lambda-1} \frac{\lambda - 1}{k(1 + x^2)^{\lambda-k}}, \]

where \( \Theta_0(x) \) and \( \Theta_1(x) \) are evaluated explicitly below

\[ \Theta_0(x) = 1 \quad \text{and} \quad \Theta_1(x) = \int_0^1 \frac{dy}{1 + x^2 y^2} = \arctan \frac{x}{y}. \]

**Lemma 3** (\( \lambda \in \mathbb{N} \)).

\[ \Theta_\lambda(x) = \frac{\lambda - 3}{\lambda - 1} \frac{\arctan x}{x} + \frac{1}{2} \sum_{k=1}^{\lambda-1} \frac{\lambda - 3}{k(1 + x^2)^{\lambda-k}}, \quad (8) \]

\[ \Theta_\lambda(1) = \frac{\lambda - 3}{\lambda - 1} \frac{\pi}{4} + \sum_{k=1}^{\lambda-1} \frac{\lambda - 3}{(2\lambda + 1)(\lambda)_{k}} \int_0^1 \frac{1 - x^2}{(1 + x^2)^{2+\lambda-k}} dx. \quad (9) \]

By substitution, we can proceed further

\[ 2S_{2+\lambda}(0,0) - S_{\lambda}(0,0) = \int_0^1 \frac{1}{1 + x^2} \Theta_{\lambda+1}(x) dx \]

\[ = \left( \frac{\lambda - 3}{\lambda - 1} \right) \int_0^1 \frac{1 - x^2}{x(1 + x^2)} \arctan x \]

\[ + \frac{\lambda}{2} \frac{\lambda - 3}{(2\lambda + 1)(\lambda)_{k}} \int_0^1 \frac{1 - x^2}{(1 + x^2)^{2+\lambda-k}} dx. \]

Evaluate the two integrals

\[ \int_0^1 \frac{1 - x^2}{x(1 + x^2)} \arctan x \]

\[ \int_0^1 \frac{1 - x^2}{(1 + x^2)^{2+\lambda-k}} dx = 2\Theta_{2+\lambda-k}(1) - \Theta_{1+\lambda-k}(1); \]

where the first one is done as follows:

\[ \diamondsuit = \int_0^1 \frac{1 - x^2}{x(1 + x^2)} \arctan x \]

\[ = \int_0^\frac{\pi}{2} 2y \cot(2y) dy \]

\[ = - \int_0^\frac{\pi}{2} \ln \sin(2y) dy = - \frac{1}{2} \int_0^\frac{\pi}{2} \ln \sin y dy \]

\[ = - \frac{1}{2} \int_0^\frac{\pi}{2} \ln \cos y dy = - \frac{1}{4} \int_0^\frac{\pi}{2} \ln \sin(2y) dy \]

\[ = \frac{\pi}{8} \ln 2 - \frac{1}{8} \int_0^\frac{\pi}{2} \ln \sin x dx = \frac{\pi}{8} \ln 2 - \frac{1}{4} \int_0^\frac{\pi}{2} \ln \sin x dx \]

\[ = \frac{\pi}{8} \ln 2 + \frac{\diamondsuit}{2} = \frac{\pi}{4} \ln 2. \]
Then we can express the difference
\[ 2S_{t+\lambda}(0,0) - S_{\lambda}(0,0) = \left( \frac{\lambda - \frac{1}{2}}{\lambda} \right) \frac{\pi}{4} \ln 2 \]
\[ + \sum_{k=1}^{\lambda} \frac{(\lambda + \frac{1}{2})^k}{(2\lambda + 1)(\lambda k)} \left\{ 2\Theta_{2+\lambda-k}(1) - \Theta_{1+\lambda-k}(1) \right\}. \]

Denote the last sum by \( \nabla \), we can manipulate it as follows:
\[ \nabla = \sum_{k=1}^{\lambda} \frac{(\lambda + \frac{1}{2})^k}{(2\lambda + 1)(\lambda k)} \left\{ 2\Theta_{2+\lambda-k}(1) - \Theta_{1+\lambda-k}(1) \right\} \]
\[ = \sum_{k=0}^{\lambda-1} \frac{2(\lambda + \frac{1}{2})^{k+1}}{(2\lambda + 1)(\lambda k+1)} - \sum_{k=1}^{\lambda} \frac{(\lambda + \frac{1}{2})^k}{(2\lambda + 1)(\lambda k)} \Theta_{1+\lambda-k}(1) \]
\[ = \Theta_{\lambda+1}(1) \frac{(\frac{1}{2}\lambda)!}{\lambda!} \Theta_1(1) + \sum_{k=1}^{\lambda-1} \frac{(1+\lambda-k)(\lambda + \frac{1}{2})^{k}}{(2\lambda + 1)(\lambda k+1)} \Theta_{1+\lambda-k}(1). \]

Therefore, we have established the following recurrence relation.

**Proposition 1** (\( \lambda \in \mathbb{Z} \)).

\[
\begin{align*}
\lambda < 0 & \quad S_{\lambda}(0,0) = \sum_{k=0}^{\lambda} \left( -\lambda \right)^k \left\{ \frac{\pi}{4} - \Theta_k \right\}^2, \\
\lambda = 0 & \quad S_0(0,0) = \frac{\pi^2}{16}, \\
\lambda \geq 1 & \quad S_1(0,0) = \frac{\pi^2}{32} + \frac{\pi}{8} \ln 2, \\
2S_{t+\lambda}(0,0) - S_{\lambda}(0,0) & = \frac{\Theta_{\lambda+1}(1)}{\lambda} \frac{\pi}{4} (1 - \ln 2) \\
& + \sum_{k=1}^{\lambda-1} \frac{(1+\lambda-k)(\lambda + \frac{1}{2})^{k}}{(2\lambda + 1)(\lambda k+1)} \Theta_{1+\lambda-k}(1). \]
\]

**Remark 1.** According to this proposition, it can be seen that \( S_{\lambda}(0,0) \in \mathbb{Q}(1, \pi, \pi^2) \) for \( \lambda \leq 0 \) and \( S_{\lambda}(0,0) \in \mathbb{Q}(1, \pi, \pi^2, \pi \ln 2) \) for \( \lambda > 0 \); where \( \mathbb{Q}(\Lambda) \) denotes the \( \mathbb{Q} \)-linear space spanned by \( \Lambda \subset \mathbb{R} \). Furthermore, by iterating the recursion \( \lambda - 1 \) times, we derive, for \( \lambda \geq 1 \), the following explicit formula
\[ S_{\lambda}(0,0) = \frac{\pi \ln 2}{2^{\lambda+2}} + \frac{\pi^2}{2^{\lambda+4}} + \sum_{i=2}^{\lambda} \left\{ \frac{2^{i-1-\lambda}}{i-1} \Theta_i(1) - \frac{\pi(1-\ln 2)}{2^{i+\lambda-i}} \frac{(i - \frac{3}{2})}{(i - 1)} \right\} \\
+ 2^{i-1-\lambda} \sum_{k=1}^{\lambda-2} \frac{(i-k)(\lambda + \frac{1}{2})^{k}}{(2i-1)(i-1)k+1} \Theta_{1+\lambda-k}(1). \]

**Proof.** The initial values corresponding to \( \lambda = 0 \) and \( \lambda = 1 \) are determined by
\[ S_0(0,0) = \int_0^1 \int_0^1 \frac{dx dy}{(1+x^2)(1+y^2)} = \frac{\pi^2}{16}, \]
\[ 2S_1(0,0) = S_0(0,0) + \int_0^1 \frac{1-x^2}{1+x^2} \Theta_1(x) dx \]
\[ = \frac{\pi^2}{16} + \int_0^1 \frac{(1-x^2) \arctan x}{x(1+x^2)} dx = \frac{\pi^2}{16} + \frac{\pi}{4} \ln 2. \]
When $\lambda < 0$, the integral can be evaluated directly

$$S_\lambda(0,0) = \int_0^1 \int_0^1 \frac{(1 + x^2 y^2)^{-\lambda}}{(1 + x^2)(1 + y^2)} \, dx \, dy$$

$$= \sum_{k=0}^{\infty} \binom{-\lambda}{k} \int_0^1 \frac{x^{2k}}{1 + x^2} \, dx \int_0^1 \frac{y^{2k}}{1 + y^2} \, dy$$

$$= \sum_{k=0}^{\infty} \binom{-\lambda}{k} \left\{ \frac{\pi}{4} - O_k \right\}^2,$$

where the last line is justified by

$$\int_0^1 \frac{x^{2k}}{1 + x^2} \, dx - \frac{\pi}{4} (-1)^k = (-1)^{k+1} \int_0^1 \frac{1 - (-x^2)^k}{1 + x^2} \, dx$$

$$= (-1)^{k+1} \sum_{i=1}^{k} \int_0^1 (-x^2)^{i-1} \, dx$$

$$= (-1)^{k+1} \sum_{i=1}^{k} \frac{(-1)^{i-1}}{2i - 1}.$$

We highlight from Proposition 1 the following infinite series identities.

$$S_1(0,0) = \frac{\pi^2}{32} + \frac{\pi}{8} \ln 2,$$

$$S_2(0,0) = \frac{1}{8} + \frac{\pi^2}{64} + \frac{\pi}{8} \ln 2,$$

$$S_3(0,0) = \frac{3}{16} + \frac{\pi}{128} + \frac{\pi^2}{128} + \frac{7\pi}{64} \ln 2,$$

$$S_4(0,0) = \frac{31}{144} + \frac{\pi}{64} + \frac{\pi^2}{256} + \frac{3\pi}{32} \ln 2,$$

$$S_5(0,0) = \frac{65}{288} + \frac{89\pi}{4096} + \frac{\pi^2}{512} + \frac{83\pi}{1024} \ln 2.$$

3. Evaluation of $T_\lambda(\mu)$

Now we are going to evaluate

$$T_\lambda(\mu) = \int_0^1 \int_0^1 \frac{x^{2\mu} \, dx \, dy}{(1 + y^2)(1 + x^2 y^2)^{\lambda + \mu}}.$$

Recalling the partial fraction decomposition

$$\frac{1}{(1+y)(1+xy)^\lambda} = \frac{1}{(1-x)^{\lambda}(1+y)} - \sum_{k=1}^{\lambda} \frac{x}{(1-x)^{1+\lambda-k}(1+xy)^k},$$
we can integrate
\[
\int_0^1 \frac{dy}{(1 + y^2)(1 + x^2y^2)^\lambda} = \int_0^1 \frac{1}{(1 - x^2)^\lambda(1 + y^2)}dy
- \sum_{k=1}^{\lambda} \int_0^1 \frac{x^2}{(1 - x^2)^{1+\lambda-k}(1 + x^2y^2)^\lambda}dy
= \frac{\pi}{4(1 - x^2)^\lambda} - \sum_{k=1}^{\lambda} \frac{x^2\Theta_k(x)}{(1 - x^2)^{1+\lambda-k}}.
\]
This reduces \(T_\lambda(\mu)\) to a single integral
\[
T_\lambda(\mu) = \int_0^1 R(\lambda, \mu; x)dx,
\]
where the integrand is given by
\[
R(\lambda, \mu; x) = \frac{\pi x^{2\mu}}{4(1 - x^2)^\lambda} - \sum_{k=1}^{\lambda} \frac{x^{2+2\mu}\Theta_k(x)}{(1 - x^2)^{1+\lambda-k}}.
\]
Observing that
\[
R(\lambda, \mu; x) - R(\lambda, \mu + 1; x) = R(\lambda - 1, \mu; x) - x^{2+2\mu}\Theta_1(x),
\]
we find the recurrence relation
\[
T_\lambda(1 + \mu) = T_\lambda(\mu) - T_{\lambda-1}(\mu) + \Delta(\lambda, \mu),
\]
where the non–homogeneous term \(\Delta(\lambda, \mu)\) reads explicitly as
\[
\Delta(\lambda, \mu) = \int_0^1 x^{2+2\mu}\Theta_1(x)dx
= \left(\frac{\lambda - \frac{3}{2}}{\lambda - 1}\right) \int_0^1 x^{1+2\mu} \arctan x\,dx
+ \sum_{k=1}^{\lambda-1} \frac{\langle \lambda - \frac{3}{2} \rangle_{k-1}}{2(\lambda - 1)_k} \int_0^1 \frac{x^{2+2\mu}}{(1 + x^2)^{\lambda-k}}\,dx.
\]
The integral \(\Delta(\lambda, \mu)\) will be evaluated in Lemma 6. According to the recurrence (11), we infer that as long as \(T_\lambda(0)\) are known, we can deduce all \(T_\lambda(\mu)\) for \(\mu > 0\). Analogously, we claim that as long as \(T_\lambda(\mu)\) are known, we can deduce all \(T_\lambda(\mu)\) for \(\lambda \leq 0\).

3.1. \(\Delta(\lambda, \mu)\)

In order to evaluate \(\Delta(\lambda, \mu)\) explicitly, we have to do that for the above two integrals, that are treated in the next two separate lemmas.

Firstly, it is not hard to evaluate the arctan-integral below.

Lemma 4 \((m \in \mathbb{N}_0)\). Let \(\chi\) be the logical function defined by \(\chi(\text{true}) = 1\) and \(\chi(\text{false}) = 0\). Then we have the following integral formulae:

\[
I(2m) = \int_0^1 x^{2m} \arctan x\,dx = \frac{\pi}{8m + 4} - \frac{(-1)^m \ln 2}{4m + 2} + \frac{(-1)^m H_m}{4m + 2},
\]
\[
I(2m + 1) = \int_0^1 x^{2m+1} \arctan x\,dx = \frac{\pi \chi(m\text{-even})}{4m + 4} - \frac{(-1)^m \bar{H}_{m+1}}{2m + 2},
\]
Then for another integral of the rational function
\[ J(m, n) = \int_0^1 \frac{x^m}{(1 + x^2)^n} \, dx \] where \( m \in \mathbb{N}_0 \) and \( n \in \mathbb{N} \),

by means of the integration by parts, we have
\[ J(m, n) = \frac{m}{2(n-1)} \int_0^1 \frac{x^{m-2}}{(1 + x^2)^{n-1}} \, dx - \frac{m}{2(n-1)(1 + x^2)^{n-1}} \bigg|_0^1 \]

which results in the following recurrence relation
\[ J(m, n) = \frac{m - 1}{2(n-1)} J(m - 2, n - 1) - \frac{1}{2^n(n-1)}. \]

Under the replacement \( m \to \delta + 2m \) with \( \delta \in \{0, 1\} \), the above equation can be restated as
\[ J(\delta + 2m, n) = \frac{\delta - 1 + 2m}{2n - 2} J(\delta + 2m - 2, n - 1) - \frac{1}{2^n(n-1)}. \]

Iterating \( \ell \) times this equation gives that
\[ J(\delta + 2m, n) = \left( \frac{\delta - 1 + m}{n - 1} \right)_\ell J(\delta + 2m - 2\ell, n - \ell) - \sum_{k=1}^{\ell} \frac{\left( \frac{\delta - 1 + m}{n - 1} \right)_k}{2^{1+n-k}(n-1)_k}. \]

By assigning \( \ell = m \) and \( \ell = n - 1 \) for \( n > m \) and \( n \leq m \), respectively, we get from the above equation the following recurrent formulae.

Lemma 5 \( (m \in \mathbb{N}_0, n \in \mathbb{N} \text{ and } \delta \in \{0, 1\}). \]

\[
\begin{align*}
\text{n > m:} \quad J(\delta + 2m, n) &= \left( \frac{1 + \delta}{n - 1} \right)_m J(\delta, n - m) - \sum_{k=1}^{m} \frac{\left( \frac{\delta - 1 + m}{n - 1} \right)_k}{2^{1+n-k}(n-1)_k}, \\
\text{n \leq m:} \quad J(\delta + 2m, n) &= \left( \frac{\delta - 1 + m}{n - 1} \right)_J(2 + \delta + 2m - 2n, 1) - \sum_{k=1}^{n-1} \frac{\left( \frac{\delta - 1 + m}{n - 1} \right)_k}{2^{1+n-k}(n-1)_k}.
\end{align*}
\]

In the above lemma, the two integrals on the right are evaluated explicitly below:

\[
\begin{align*}
J(0, n) &= \int_0^1 \frac{dx}{(1 + x^2)^n} = \Theta_n(1) = \frac{\pi}{4(n-1)_{n-1}} + \sum_{k=1}^{n-1} \frac{(n-3)_{k-1}}{(n-1)_k 2^{1+n-k} k!}, \\
J(1, n) &= \int_0^1 \frac{x}{(1 + x^2)^n} \, dx = \begin{cases} \frac{\ln 2}{2}, & n = 1; \\
\frac{1 - 2^{1-n}}{2n - 2}, & n > 1; \end{cases} \\
J(m, 0) &= \frac{1}{m + 1}, \quad J(2m, 1) = \int_0^1 x^{2m} \, dx = (-1)^m \frac{\pi}{4} - (-1)^m \Theta_m; \\
J(2m + 1, 1) &= \int_0^1 x^{2m+1} \, dx = (-1)^m \frac{\ln 2}{2} - (-1)^m \bar{H}_m.
\end{align*}
\]

Finally, we find the following expression for the non-homogeneous term.
Lemma 6 ($\lambda, \mu \in \mathbb{N}_0$).

\[
\Delta(\lambda, \mu) = \int_0^1 x^{2+2\mu} \Theta_\lambda(x) \, dx = \left(\frac{\lambda - \frac{3}{2}}{\lambda - 1}\right) I(2\mu + 1) + \sum_{k=1}^{\lambda-1} \frac{(\lambda - \frac{3}{2})^{k-1}}{2(\lambda - 1)^k} I(2\mu + 2, \lambda - k).
\]

3.2. $T_\lambda(0)$ with $\lambda \geq 1$

For $\lambda = 1$, it is not difficult to check that

\[
T_1(0) = \int_0^1 \int_0^1 \frac{dxdy}{(1+y^2)(1+x^2y^2)} = \int_0^1 \arctan y \, dy = \frac{G}{2} + \frac{\pi}{8} \ln 2.
\]

This is justified by combining two integrals

\[
\int_0^1 \arctan \frac{y}{1+y^2} \, dy - \frac{G}{2} = \int_0^1 \arctan \frac{y}{1+y^2} \, dy - \int_0^1 \frac{\arctan y \, dy}{2y} = \int_0^1 \frac{(1-y^2) \arctan y}{2y(1+y^2)} \, dy = \frac{\pi}{8} \ln 2.
\]

Interestingly, there is a counterpart integral evaluation

\[
\int_0^1 \frac{y \arctan \frac{y}{1+y^2} \, dy}{(1+y^2)} - \frac{G}{2} = \int_0^1 \frac{y \arctan \frac{y}{1+y^2} \, dy}{(1+y^2)} - \int_0^1 \frac{\arctan y \, dy}{2y} = \int_0^1 \frac{(y^2-1) \arctan y}{2y(1+y^2)} \, dy = -\frac{\pi}{8} \ln 2.
\]

In general for $\lambda > 1$, integrating with respect to $x$ gives

\[
T_\lambda(0) = \int_0^1 \int_0^1 \frac{dxdy}{(1+y^2)(1+x^2y^2)} = \int_0^1 \frac{\Theta_\lambda(y)}{1+y^2} \, dy.
\]

Recalling Lemma 3, we can compute the rightmost integral

\[
\int_0^1 \frac{\Theta_\lambda(y)}{1+y^2} \, dy = \left(\frac{\lambda - \frac{3}{2}}{\lambda - 1}\right) \int_0^1 \frac{\arctan \frac{y}{1+y^2} \, dy}{y(1+y^2)} + \sum_{k=1}^{\lambda-1} \frac{(\lambda - \frac{1}{2})^{k-1}}{(\lambda - 1)^k} \int_0^1 \frac{dy}{(1+y^2)^{1+\lambda-k}} = \left(\frac{\lambda - \frac{3}{2}}{\lambda - 1}\right) \left\{ \frac{G}{2} + \frac{\pi}{8} \ln 2 \right\} + \frac{1}{2\lambda - 1} \sum_{k=1}^{\lambda-1} \frac{(\lambda - \frac{1}{2})^{k-1}}{(\lambda - 1)^k} \Theta_{1+\lambda-k}(1).
\]

This leads us to the general formula

\[
T_\lambda(0) = \left(\frac{\lambda - \frac{3}{2}}{\lambda - 1}\right) \left\{ \frac{G}{2} + \frac{\pi}{8} \ln 2 \right\} + \frac{1}{2\lambda - 1} \sum_{k=1}^{\lambda-1} \frac{(\lambda - \frac{1}{2})^{k-1}}{(\lambda - 1)^k} \Theta_{1+\lambda-k}(1).
\]

3.3. $T_\lambda(\mu)$ with $\lambda \leq 0$

When $\lambda \leq 0$, it is almost routine to proceed with
\[
T_\lambda(\mu) = \int_0^1 \int_0^1 \frac{x^{2\mu} dxdy}{(1 + y^2)(1 + x^2 y^2)^\lambda}
= \sum_{k=0}^\lambda \left( -\lambda \right) \left[ \frac{(-1)^k}{2\mu + 2k + 1} \left\{ \frac{\pi}{4} - \bar{O}_k \right\} \right].
\]

Summing up, we have proved the following general result.

**Proposition 2** \((\lambda \in \mathbb{Z} \text{ and } \mu \in \mathbb{N}_0)\). Assuming \(I(\mu)\) and \(J(\lambda, \mu)\) as in Lemmas 4 and 5, we have the following formulae:

\[
\begin{align*}
\lambda \leq 0 & \quad T_\lambda(\mu) = \sum_{k=0}^\lambda \left( -\lambda \right) \left[ \frac{(-1)^k}{2\mu + 2k + 1} \left\{ \frac{\pi}{4} - \bar{O}_k \right\} \right]; \\
\lambda \geq 1 & \quad T_\lambda(0) = \left( \frac{\lambda - 3}{\lambda - 1} \right) \left\{ \frac{G}{2} + \frac{\pi}{8} \ln 2 \right\} + \frac{1}{2\lambda - 1} \sum_{k=1}^{\lambda-1} \frac{\lambda - 3}{\lambda - 1} \Theta_{1+\lambda-k}(1); \\
T_\lambda(\mu + 1) & = T_\lambda(\mu) - T_{\lambda-1}(\mu) - \Delta(\lambda, \mu) + \left( \frac{\lambda - 3}{\lambda - 1} \right) I(2\mu + 1) + \sum_{k=1}^{\lambda-1} \frac{\lambda - 3}{\lambda - 1} J(2\mu + 2, \lambda - k).
\end{align*}
\]

**Remark 2.** This proposition clearly implies the remarkable fact that \(T_\lambda(\mu) \in \mathbb{Q}(1, \pi)\) for \(\lambda \leq 0\) and \(T_\lambda(\mu) \in \mathbb{Q}(1, G, \pi, \pi \ln 2)\) for \(\lambda > 0\).

According to Proposition 2, it is possible to compute \(T_\lambda(\mu)\) when \(\lambda \in \mathbb{Z}\) and \(\mu \in \mathbb{N}_0\) are specified by small values. For \(-3 \leq \lambda \leq 3\) and \(0 \leq \mu \leq 3\), the corresponding values of \(T_\lambda(\mu)\) are recorded in the following table.

| \(\lambda\) | \(\mu\) | 0 | 1 | 2 | 3 |
|-----------|--------|---|---|---|---|
| \(-3\)    |        | 76/105 + 4\pi/35 | 388/945 + 4\pi/315 | 988/3465 + 4\pi/1115 | 125/55 + 4\pi/1003 |
| \(-2\)    |        | 8/15 + 2\pi/15   | 32/105 + 2\pi/315   | 40/189 + 2\pi/315    | 16/99 + 2\pi/693   |
| \(-1\)    |        | 1/5 + \pi/50     | 1/5 + \pi/50       | 1/7 + \pi/70        | 1/9 + \pi/125     |
| 0         |        | \pi/4             | \pi/4              | \pi/20              | \pi/28             |
| 1         |        | 2\pi/3 + \pi ln 2 | 2\pi/3 + \pi ln 2   | 2\pi/3 + \pi ln 2   | 2\pi/3 + \pi ln 2  |
| 2         |        | 4\pi/9 + \pi ln 2 | 4\pi/9 + \pi ln 2   | 4\pi/9 + \pi ln 2   | 4\pi/9 + \pi ln 2  |
| 3         |        | 5\pi/12 + \pi ln 2| 5\pi/12 + \pi ln 2  | 5\pi/12 + \pi ln 2  | 5\pi/12 + \pi ln 2 |

4. Evaluation of \(S_{\lambda}(\mu, \nu)\)

Recalling the integral representation in Lemma 2, \(S_{\lambda}(\mu, \nu)\) is symmetric in \(\mu\) and \(\nu\). Suppose that \(\lambda \in \mathbb{Z}\) and \(\mu, \nu \in \mathbb{N}_0\) with \(\mu \geq \nu\). By Lemma 1, we can write

\[
S_{\lambda}(\mu, \nu) = \sum_{k=0}^{\nu} (-1)^{\nu-k} \binom{\nu}{k} S_{\lambda-k}(\mu - \nu, 0).
\]

In view of the algebraic relation
Theorem 1

Theorem 1 \((\lambda \in \mathbb{Z} \text{ and } \mu, \nu \in \mathbb{N}_0)\). For any triplet integers \(\lambda, \mu, \nu\) with \(\lambda \in \mathbb{Z}\) and \(\mu, \nu \in \mathbb{N}_0\), the corresponding \(S_\lambda(\mu, \nu)\) always has the value in \(\mathbb{Q}(1, \pi, \pi^2)\) for \(\lambda \leq 0\) and \(\mathbb{Q}(1, G, \pi, \pi^2, \pi \ln 2)\) for \(\lambda > 0\), where \(\mathbb{Q}(\Lambda)\) denotes the \(\mathbb{Q}\)-linear space spanned by \(\Lambda \subseteq \mathbb{R}\). More precisely, assuming \(S_\lambda(0,0)\) and \(T_\lambda(\mu)\) as in Proposition 1 and Proposition 2, respectively, the following infinite series identity holds:

\[
S_\lambda(\mu, \nu) = \sum_{k=0}^{\nu} (-1)^{\mu-k} \binom{\nu}{k} S_{\lambda-k}(0,0) + \sum_{k=0}^{\nu} \binom{\nu}{k} \sum_{j=1}^{\mu-\nu} (-1)^{\mu-\nu-j} T_{\lambda-k}(j-1).
\]

According to this theorem, we tabulate the summation formulae for triple series \(S_\lambda(\mu, \nu)\) with \(-4 \leq \lambda \leq 5\) and \(0 \leq \mu, \nu \leq 3\) below.

| \(\lambda\) | \(\mu\) | 0 | 1 | 2 | 3 |
|----------|----------|----|----|----|----|
| 0        | 0        | \(\frac{1}{\pi}\) | \(-\frac{2}{\pi}\) | \(-\frac{4}{\pi}\) | \(-\frac{8}{\pi}\) |
| 1        | 0        | \(\frac{1}{\pi}\) | \(-\frac{2}{\pi}\) | \(-\frac{4}{\pi}\) | \(-\frac{8}{\pi}\) |
| 2        | 0        | \(\frac{1}{\pi}\) | \(-\frac{2}{\pi}\) | \(-\frac{4}{\pi}\) | \(-\frac{8}{\pi}\) |
| 3        | 0        | \(\frac{1}{\pi}\) | \(-\frac{2}{\pi}\) | \(-\frac{4}{\pi}\) | \(-\frac{8}{\pi}\) |
| 4        | 0        | \(\frac{1}{\pi}\) | \(-\frac{2}{\pi}\) | \(-\frac{4}{\pi}\) | \(-\frac{8}{\pi}\) |
| 5        | 0        | \(\frac{1}{\pi}\) | \(-\frac{2}{\pi}\) | \(-\frac{4}{\pi}\) | \(-\frac{8}{\pi}\) |
| $\lambda \times \nu$ | 0       | 1       | 2       | 3       |
|-----------------|---------|---------|---------|---------|
| $\lambda = 1$   |         |         |         |         |
| $\nu = 1$       | $\frac{1}{16} + \frac{1}{8} \ln 2$ | $\frac{1}{16} + \frac{1}{8} \ln 2$ | $\frac{1}{16} + \frac{1}{8} \ln 2$ | $\frac{1}{16} + \frac{1}{8} \ln 2$ |
| $\nu = 2$       | $\frac{1}{16} + \frac{1}{8} \ln 2$ | $\frac{1}{16} + \frac{1}{8} \ln 2$ | $\frac{1}{16} + \frac{1}{8} \ln 2$ | $\frac{1}{16} + \frac{1}{8} \ln 2$ |
| $\nu = 3$       | $\frac{1}{16} + \frac{1}{8} \ln 2$ | $\frac{1}{16} + \frac{1}{8} \ln 2$ | $\frac{1}{16} + \frac{1}{8} \ln 2$ | $\frac{1}{16} + \frac{1}{8} \ln 2$ |

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References
1. Bromwich, T.J. An Introduction to the Theory of Infinite Series; MacMillan & Co. Limited: London, UK, 1908.
2. Knopp, K. Theory and Applications of Infinite Series; Blackie & Son Limited: London, UK, 1928.
3. Mordell, L.J. On the evaluation of some multiple series. J. Lond. Math. Soc. 1958, 33, 368–371.
4. Subbarao, M.V.; Sitaramachandrarao, R. On some infinite series of L. J. Mordell and their analogues. Pac. J. Math. 1985, 119, 245–255. [CrossRef]
5. Aliev, I.A.; Dil, A. Tornheim-like series, harmonic numbers and zeta values. arXiv 2020, arXiv:2008.02488v1.
6. Tornheim, L. Harmonic double series. Am. J. Math. 1950, 72, 303–314. [CrossRef]
7. Kadota, S.Y.; Okamoto, T.; Tasaka, K. Evaluation of Tornheim’s type of double series. Illinois J. Math. 2017, 61, 171–186. [CrossRef]
8. Tsumura, H. Evaluation formulas for Tornheim’s type of alternating double series. Math. Comp. 2003, 73, 251–258. [CrossRef]
9. Ramanujan, S. Collected Papers of Srinivasa Ramanujan; Hardy, G.H., Seshu Aiyar, P.V., Wilson, B.M., Eds.; Cambridge University Press: Cambridge, UK, 1927.
10. Chagas, J.Q.; Tenreiro Machado, J.A.; Lopes, A.M. Revisiting the formula for the Ramanujan constant of a series. Mathematics 2022, 10, 1539. [CrossRef]
11. Borwein, D.; Borwein, J.M. On an intriguing integral and some series related to zeta(4). Proc. Am. Math. Soc. 1995, 123, 1191–1198.
12. Borwein, D.; Borwein, J.M.; Girgensohn, R. Explicit evaluation of Euler sums. Proc. Edinb. Math. Soc. 1995, 38, 277–294. [CrossRef]
13. Chu, W. Hypergeometric series and the Riemann zeta function. Acta Arith. 1997, 82, 103–118. [CrossRef]
14. Rudin, W. Principles of Mathematical Analysis, 3rd ed.; McGraw–Hill, Inc.: New York, NY, USA, 1976.
15. Chen, H. Interesting series associated with central binomial coefficients, Catalan numbers and harmonic numbers. J. Integer Seq. 2016, 19, 16.1.5.
16. Chu, W. A binomial coefficient identity associated with Beukers’ conjecture on Apéry numbers. *Electron. J. Combin.* **2004**, 11, N15. [CrossRef]

17. Chu, W. Infinite series identities on harmonic numbers. *Results Math.* **2012**, 61, 209–221. [CrossRef]

18. Chu, W.; Wang, X.Y. Binomial series identities involving generalized harmonic numbers. *Integers* **2020**, 20, 98.

19. Batir, N. Finite binomial sum identities with harmonic numbers. *J. Integer Seq.* **2021**, 24, 21.4.3.

20. Boyadzhiev, K.N. Power series with skew-harmonic numbers, dilogarithms, and double integrals. *Tatra Mt. Mat. Publ.* **2013**, 56, 93–108. [CrossRef]

21. Frontczak, R. Binomial sums with skew-harmonic numbers. *Palest. J. Math.* **2021**, 10, 756–763.

22. Kargin, L.; Can, M. Harmonic number identities via polynomials with \( r \)-Lah coefficients. *C. R. Math. Acad. Sci. Paris* **2020**, 358, 535–550. [CrossRef]

23. Bradley, D.M. Representations of Catalan’s Constant. 2001. Available online: [www.researchgate.net/publication/2325473](http://www.researchgate.net/publication/2325473) (accessed on 2 February 2001).

24. Jamerson, G.; Lord, N. Integrals evaluated in terms of Catalan’s constant. *Math. Gaz.* **2017**, 101, 38–49. [CrossRef]

25. Stewart, S.M. A Catalan constant inspired integral odyssey. *Math. Gaz.* **2020**, 104, 449–459. [CrossRef]