Output Impedance Diffusion into Lossy Power Lines

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Abstract—Output impedances are inherent elements of power sources in the electrical grids. In this paper, we give an answer to the following question: What is the effect of output impedances on the inductivity of the power network? To address this question, we propose a measure to evaluate the inductivity of the power grid. By exploiting this measure together with the algebraic connectivity of the network topology, one can tune the output impedances in order to impose a desired level of inductivity on the power system. Results show that the more “connected” the network is, the more the output impedances diffuse into the network.

Index Terms—Microgrid, Power network, Output impedance, Graph theory, Laplacian matrix, Kron reduction

I. INTRODUCTION

OUTPUT impedance is an important and inevitable element of any power producing device, such as synchronous generators and inverters. Synchronous generators typically possess a highly inductive output impedance according to their large stator coils, and are prevalently modeled by a voltage source behind an inductance. Similarly, inverters have an inductive output impedance due to the low pass filter in the output, which is necessary to eliminate the high frequencies of the modulation signal.

There are motives to add an impedance to the inherent output impedance of the inverters, one of the most important of which is to enhance the performance of droop controllers in a lossy network. Droop controllers show a better performance in a dominantly inductive network (or analogously in dominantly resistive networks for the case of inverse-droop controllers) [1]–[6] (see Figure 1). The additional output impedance is also employed to correct the reactive power sharing error due to line mismatch [2], [7], [8], supply harmonics to nonlinear loads [9], [1], [10], share current among sources resilient to parameters mismatch and synchronization error [11], decrease sensitivity to line impedance unbalances [3], [12], reduce the circulating currents [13], limit output current during voltage sags [14], minimize circulating power [15], and damp the LC resonance in the output filter [6]. In most of these methods, to avoid the costs and large size of an additional physical element, a virtual output impedance is employed, where the electrical behavior of a desired output impedance is simulated by the inverter controller block.

Although an inductive output impedance, either resulting from the inherent output filter or the added output impedance, is considered as a means to regulate the inductive behavior of the resulting network, there is a lack of theoretical analysis to verify the feasibility of this method and to quantify the effect of the output impedances on the network inductivity/resistivity. Note that the output impedance cannot be chosen arbitrarily large, since a large impedance substantially boosts the voltage sensitivity to current fluctuations, and results in high frequency noise amplification [6]. Furthermore, there is the fundamental challenge of quantifying inductivity/resistivity of a network, which is nontrivial unless the overall network has uniform line characteristics (homogeneous). This is not the case here as the augmented network will be nonuniform (heterogeneous) even if the initial network is.

In this paper, we examine the effect of the output impedances on a homogeneous power distribution grid by proposing a quantitative measure for the inductivity of the resulting heterogeneous network. Similarly, a dual measure is defined for its resistivity. Based on these measures, we show that the network topology plays a major role in the diffusion of the output impedance into the network. Furthermore, we exploit the proposed measures to maximize the effect of the added output impedances on the network inductivity/resistivity. We demonstrate the validity and practicality of the proposed method on various examples and special cases.

The structure of the paper is as follows: In Section II the notions of Network Inductivity Ratio (Ψ_NIR) and Network Resistivity Ratio (Ψ_NRR) are proposed. In Section III the proposed measures are analytically computed for various cases of output inductors and resistors. In Section IV the proposed measure is evaluated with the Kron reduction in the phasor domain. Finally, Section V is devoted to conclusions.
Consider an electrical network with an arbitrary topology, where we assume that all the sources and loads are connected to the grid via power converter devices (inverters) [16]. The network of this grid is represented by a connected and weighted undirected graph \( G(V, E, \Gamma) \). The nodes \( V = \{1, \ldots, n\} \) represent the inverters, and the edge set \( E \) accounts for the distribution lines. The total number of edges is denoted by \( m \), i.e., \( |E| = m \). The edge weights are collected in the diagonal matrices with the line resistances and inductances on their diagonal, respectively. We have \( R \in \mathbb{R}^{n \times n} \) and \( L \in \mathbb{R}^{n \times n} \) be the diagonal matrices with the line resistances and inductances on their diagonal, respectively. We define

\[
b_{ik} = \begin{cases} 
+1 & \text{if } i \text{ is the tail of edge } k \\
-1 & \text{if } i \text{ is the head of edge } k \\
0 & \text{otherwise}
\end{cases}
\]

with \( b_{ik} \) being the \((i, k)\)th element of \( B \).

We start our analysis with the voltages across the edges of the graph \( G \). Let \( R_e \in \mathbb{R}^{m \times m} \) and \( L_e \in \mathbb{R}^{m \times m} \) be the diagonal matrices with the line resistances and inductances on their diagonal, respectively. We have

\[
R_e I_e + L_e \dot{I}_e = B^T V, 
\]

where \( I_e \in \mathbb{R}^m \) denotes the current flowing through the edges. The orientation of the currents is taken in agreement with their diagonal, respectively. We have \( \Gamma = \text{diag}(\gamma) := \text{diag}(\tau_1^{-1}, \tau_2^{-1}, \ldots, \tau_m^{-1}) \).

We can rewrite (1) as [17]

\[
\dot{I}_e = \Gamma B^T V.
\]

Hence, \( r B I_e + \ell \dot{I}_e = B \Gamma B^T V \), and

\[
\dot{I} + \ell I = \mathcal{L} V, 
\]

where \( I := B I_e \) is the vector of nodal current injections. The matrix \( \mathcal{L} = B \Gamma B^T \) is the Laplacian matrix of the graph \( G(V, E, \Gamma) \) with the weight matrix \( \Gamma \).

Note that, as the network (3) is homogeneous, its inductivity behavior is simply determined by the ratio \( \frac{r}{\ell} \). However, clearly, network homogeneity will be lost once the output impedances are augmented to the network. This makes the problem of determining network inductivity nontrivial and challenging. To cope with the heterogeneity resulting from the addition of the output impedances, we need to depart from the homogeneous form (3), and develop new means to assess the network inductivity. To this end, we consider the more general representation

\[
R I + \dot{I} = \mathcal{L} V_o, 
\]

where \( V_o \in \mathbb{R}^n \) is the vector of voltages of the augmented nodes (the dark green nodes in Figure 1), and \( R \in \mathbb{R}^{n \times n} \) and \( L \in \mathbb{R}^{n \times n} \) are matrices associated closely with the resistances and inductances of the lines, respectively. We will show that the overall network after the addition of the output impedances, can be described by (4). Note that this description cannot necessarily be realized with passive RL elements. Therefore, while the inductivity behavior of the homogeneous network (3) is simply determined by the ratio \( \frac{r}{\ell} \), the one of (4) cannot be trivially quantified.

The idea here is to promote the rate of convergence as a suitable metric quantifying the inductivity/resistivity of the network. For the network dynamics in (3), the rate of convergence of the solutions is determined by the ratio \( \frac{r}{\ell} \). The more inductive the lines are, the slower the rate of convergence is. Now, we seek for a similar property in (4). Notice that the solutions of (3) are damped with corresponding eigenvalues of \( L^{-1} R \). Throughout the paper, we assume the following property:

**Assumption 1** The eigenvalues of the matrix \( L^{-1} R \) are all positive and real.

It will be shown that Assumption 1 is satisfied for all the cases considered in this paper.

Figure 2 sketches the behavior of homogeneous solutions of (4). Among all the solutions, we choose the fastest one as our measure for inductivity, and the slowest one for resistivity of the network. Opting for these worst case scenarios allows us to guarantee a prescribed inductivity or resistivity ratio by proper design of output impedances. These choices are formalized in the following definitions.

**Definition 1** Let \( I(t, I_0) \) denote the homogeneous solution of (4) for an initial condition \( I_0 \in \text{im } B \). Let the set \( M_L \subseteq \mathbb{R}^+ \) be given by

\[
M_L := \{ \sigma \in \mathbb{R}^+ \mid \exists \mu \text{ s.t. } \|I(t, I_0)\| \geq \mu e^{-\sigma t} \|I_0\|, \forall t \in \mathbb{R}^+, \forall I_0 \in \text{im } B\}.
\]

Then we define the Network Inductivity Ratio (NIR) as

\[
\Psi_{\text{NIR}} := \frac{1}{\inf(M_L)}.
\]
\[ M_R := \{ \sigma \in \mathbb{R}^+ \mid \exists \mu \text{ s.t.} ||I(t, \sigma)|| \leq \mu e^{-\sigma t} ||I_0||, \forall t \in \mathbb{R}^+, \forall I_0 \in \text{im} B \}. \]

We define the Network Resistivity Ratio (NRR) as
\[
\Psi_{\text{NRR}} := \sup(M_R).
\]

Note that the set \( M_L \) is bounded from below and \( M_R \) is bounded from above by definition and Assumption 1. Interestingly, in case of the homogeneous network (3), \( e \) without output impedances, we have \( \Psi_{\text{NIR}} = \frac{\ell}{r} \) and \( \Psi_{\text{NRR}} = \frac{\ell}{r} \), which are natural measures to reflect the inductivity and resistivity of an RL homogeneous network.

### III. Calculating the Network Inductivity/Resistivity Measure (\( \Psi_{\text{NIR}}/\Psi_{\text{NRR}} \))

In this section, based on Definitions 1 and 2, we compute the network inductivity/resistivity ratio for both cases of uniform and nonuniform output impedances.

#### A. Uniform Output Impedances

In most cases of practical interest, the output impedance consists of both inductive and resistive elements. We investigate the effect of the addition of such output impedances on the network inductivity ratio. The change in network resistivity ratio can be studied similarly, and thus is omitted here. Consider the uniform output impedances with the inductive part \( \ell \) and the resistive component \( r \) (in series), added to the network (3). Note that the injected currents \( I \) now pass through the output impedances, as shown in Figure 3.

Clearly, we have
\[
V = V_o - r_o I - \ell_o \dot{I}.
\]

Having (3) and (5), the overall network can be described as
\[
(r_o \mathcal{L} + r \mathcal{I}) I + (\ell_o \mathcal{L} + \ell \mathcal{I}) \dot{I} = \mathcal{L} V_o,
\]
where \( \mathcal{I} \in \mathbb{R}^{n \times n} \) denotes the identity matrix, and \( \mathcal{L} \) is the Laplacian matrix of \( \mathcal{G} \) as before. In view of equation (4), the matrices \( R \) and \( L \) are given by \( R = r_o \mathcal{L} + r \mathcal{I} \) and \( L = \ell_o \mathcal{L} + \ell \mathcal{I} \), respectively. As both matrices are positive definite, the eigenvalues of the product \( L^{-1}R \) are all positive and real, see [18, Ch. 7]. Hence, Assumption 1 is satisfied. To calculate the measure \( \Psi_{\text{NIR}} \) for the inductivity of the resulting network, we investigate the convergence rates of the homogeneous solution of (6). This brings us to the following theorem:

**Theorem 1** Consider a homogeneous network (3) with the resistance per length unit \( r \) and inductance per length unit \( \ell \). Suppose that an output resistance \( r_o \) and an output inductance \( \ell_o \) are attached in series to each node. Assume that \( \frac{r_o}{\ell_o} < \frac{r}{\ell} \).

Then the network inductivity ratio is given by
\[
\Psi_{\text{NIR}} = \frac{\ell_o \lambda_2 + \ell}{r_o \lambda_2 + r},
\]
where \( \lambda_2 \) is the algebraic connectivity of the graph \( \mathcal{G}(\mathcal{V}, \mathcal{E}, \Gamma) \).

**Proof:** The homogeneous solution is
\[
I(t) = e^{-r_o L + r \mathcal{I}}(I_0 \mathcal{L} + 1^{T} \mathcal{I})^{-1} I_0.
\]

The Laplacian matrix can be decomposed as \( \mathcal{L} = \mathcal{U}^T \Lambda \mathcal{U} \). Here, \( \mathcal{U} \) is the matrix of eigenvectors and \( \Lambda = \text{diag} \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) where \( \lambda_1 < \lambda_2 < \cdots < \lambda_n \) are the eigenvalues of the matrix \( \mathcal{L} \). Note that \( \lambda_1 = 0 \). We have
\[
I(t) = \mathcal{U} e^{-r_o L + r \mathcal{I}}(I_0 \mathcal{L} + 1^{T} \mathcal{I})^{-1} \mathcal{U}^T I_0
\]

where \( \Lambda = \text{diag} \{\frac{\lambda_2 + r}{\ell_o \lambda_2 + r}, \ldots, \frac{\lambda_2 + r}{\ell_o \lambda_2 + r}\} \). Noting that \( \mathcal{U} \) is unitary and by the Kirchhoff Law, \( \mathcal{U}^T I_0 = 0 \), we have
\[
I(t) = \tilde{\mathcal{U}} e^{-\lambda_t \tilde{\mathcal{L}}} \tilde{\mathcal{U}}^T I_0 = \sum_{i=1}^{n-1} e^{-\lambda_i t} \tilde{U}_i \tilde{U}_i^T (\sum_{i=1}^{n-1} \alpha_i \tilde{U}_i)
\]

where \( \tilde{U}_i \) denotes the \( i \) th column of \( \tilde{U} \), and we used again \( \mathcal{U}^T I_0 = 0 \) to write \( I_0 \) as the linear combination
\[
I_0 = \sum_{i=1}^{n-1} \alpha_i \tilde{U}_i.
\]

Hence
\[
||I(t)||^2 = \sum_{i=1}^{n-1} \alpha_i^2 e^{-2\lambda_i t}.
\]

Having \( \frac{r_o}{\ell_o} < \frac{r}{\ell} \), it is straightforward to see that
\[
\frac{r_o \lambda_2 + r}{\ell_o \lambda_2 + \ell} \geq \frac{r_o \lambda_i + r}{\ell_o \lambda_i + \ell}, \quad \forall i.
\]

and bearing in mind that \( ||I_0||^2 = \sum_{i=1}^{n-1} \alpha_i^2 \), we conclude that
\[
||I(t)|| \geq e^{-\frac{r_o \lambda_2 + r}{\ell_o \lambda_2 + r} t} ||I_0||,
\]
which yields \( \Psi_{\text{NIR}} = \frac{\ell_o \lambda_2 + \ell}{r_o \lambda_2 + r} \). Note that (9) holds with equality in case \( I_0 \) belongs to the span of the corresponding
eigenvector of the second smallest eigenvalue of the Laplacian matrix \( L \). This completes the proof.

Theorem 1 provides a compact and easily computable expression which quantifies the network inductivity behavior. Moreover, the expression (7) is an easy-to-use measure that can be exploited to choose the output impedances in order to impose a desired degree of inductivity on the network. The only information required is the line parameters \( r, \ell \), and the algebraic connectivity of the network; see [19] and [20] for more details on algebraic connectivity and its lower and upper bounds in various graphs.

**Remark 1** Algebraic connectivity is a measure of connectivity of the weighted graph \( G \), which depends on both the density of the edges and the weights (inverse of the lines lengths). Hence, Theorem 1 reveals the fact that: “The more connected the network is, the more the output impedance diffuses into the network.”

**Remark 2** In case \( \frac{r_o}{\ell_o} > \frac{r}{\ell} \), the network inductivity ratio will be given by

\[
\Psi_{\text{NIR}} = \frac{\ell_o \lambda_{\text{max}} + \ell}{r_o \lambda_{\text{max}} + r}
\]

where \( \lambda_{\text{max}} \) is the largest eigenvalue of the Laplacian matrix of \( G \). Furthermore, if \( \frac{r_o}{\ell_o} = \frac{r}{\ell} \), then \( \lambda = \frac{r}{\ell} L \) and \( \Psi_{\text{NIR}} = \frac{r}{\ell} \). However, the condition \( \frac{r_o}{\ell_o} < \frac{r}{\ell} \) assumed in Theorem 1 is more relevant since the resistance \( r_o \) of the inductive output impedance is typically small.

**Remark 3** In case, the resistance part of the output impedance is negligible, i.e. \( r_o = 0 \), the network inductivity ratio reduces to

\[
\Psi_{\text{NIR}} = \frac{\ell_o \lambda_2 + \ell}{r}
\]

As mentioned in Section [21] in low-voltage microgrids where the lines are dominantly resistive, the inverse-droop method is employed. In this case, a purely resistive output impedance is of advantage [21].

**Corollary 1** Consider a homogeneous distribution network with the resistance per length \( r \), inductance per length unit \( \ell \), and output inductors \( \ell_o \). Then the network resistivity ratio is given by

\[
\Psi_{\text{NRR}} = \frac{r_o \lambda_2 + r}{\ell}
\]

where \( \lambda_2 \) is the algebraic connectivity of the graph \( G(V, E, \Gamma) \).

**Proof:** The proof can be constructed in an analogous way to the proof of Theorem 1 and is therefore omitted.

**Remark 4** The algebraic connectivity of the network can be estimated through distributed methods [22], [23]. Furthermore, line parameters (resistance and inductance) can be identified through PMUs (Phase Measurement Units) [24], [25], [26]. Therefore, our proposed measure can be calculated in a distributed manner.

1) **Case Study: Identical Line Lengths**

Recall that the notion of network inductivity ratio allows us to quantify the inductivity behavior of the network, while the model [6], in general, cannot be synthesized with \( RL \) elements only. A notable special case where the model [6] can be realized with \( RL \) elements is a complete graph with identical line lengths. Although such case is improbable in practice, it provides an example to assess the validity and credibility of the introduced measures. Interestingly, \( \Psi_{\text{NIR}} \) matches precisely the inductance to resistance ratio of the lines of the synthesized network in this case:

**Theorem 2** Consider a network with a uniform complete graph where all the edges have the length \( \tau \). Suppose that the lines have inductance \( \ell_o \in \mathbb{R} \) and resistance \( r_o \in \mathbb{R} \). Attach an output inductance \( \ell_o \) in series with a resistance \( r_o \) to each node. Then the model of the augmented graph can be equivalently synthesized by a new \( RL \) network with identical lines, each with inductance \( \ell := n\ell_o + \ell_o \) and resistance \( r := nr_o + r_o \), where \( n \) denotes the number of nodes. Furthermore, the resulting network inductivity ratio \( \Psi_{\text{NIR}} \) is equal to \( \frac{\ell_o}{r_o} \).

**Proof:** The nodal injected currents satisfy \( rI + \ell \dot{I} = \mathcal{L}V \). In this network, \( r = \frac{2\ell_o}{\ell}, \ell = \frac{\ell_o}{\ell}, \) and \( \mathcal{L} = \frac{2}{\ell} \Pi \) where \( \Pi := \mathbb{1} - \frac{1}{n} \mathbb{1}^{T}. \) Hence,

\[
r_o I + \ell_o \dot{I} = n\Pi V.
\]

By appending the output impedance we have \( V = V_o - r_o I - \ell_o \dot{I} \). Hence (10) modifies to

\[
(nr_o \Pi + \ell_o \mathcal{L})I + (n\ell_o \Pi + \ell \mathcal{L}) \dot{I} = n\Pi V_o,
\]

which results in

\[
(n\ell_o \Pi + \ell_o \mathcal{L})^{-1}(nr_o \Pi + \ell_o \mathcal{L})I + \dot{I} = n(n\ell_o \Pi + \ell \mathcal{L})^{-1} \Pi V_o.
\]

Since \( (n\ell_o \Pi + \ell \mathcal{L})^{-1} = \frac{1}{\ell_o + n\ell_o} \mathcal{L} + \frac{\ell_o}{\ell_o + n\ell_o} \Pi^{T} \), we obtain

\[
(nr_o \Pi + \ell_o \mathcal{L})I + (n\ell_o + \ell) \dot{I} = n\Pi V_o,
\]

where we used \( \Pi^{T} \mathcal{L} = 0 \) and \( \Pi^{T} \Pi = 0 \). Similarly we have

\[
I + \ell_o (nr_o \Pi + \ell_o \mathcal{L})^{-1} \dot{I} = n(nr_o \Pi + \ell_o \mathcal{L})^{-1} \Pi V_o,
\]

and hence \( r_o I + \ell_o \dot{I} = n\Pi V_o \). This equation is analogous to (10) and corresponds to a uniform complete graph with identical line resistance \( r_c = nr_o + r_o \) and inductance \( \ell_c = n\ell_o + \ell_o \).

Note that the algebraic connectivity of the weighted Laplacian \( L \) is \( \frac{2}{\ell} \). By Theorem 1 the inductivity ratio is then computed as

\[
\Psi_{\text{NIR}} = \frac{2\ell_o + \ell}{\frac{2}{\ell} r_o + r} = \frac{\ell_o}{r_o}.
\]
2) Case Study: Constant Current Loads

So far, we have considered loads which are connected via power converters. The same definitions and results can be extended to the case of loads modeled with constant current sinks. Consider the graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \Gamma)$ divided into source (S) and load nodes (L), and decompose the Laplacian matrix accordingly as

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_{SS} & \mathcal{L}_{SL} \\ \mathcal{L}_{LS} & \mathcal{L}_{LL} \end{bmatrix}.$$ 

We have

$$rI_S + \ell I_S = \mathcal{L}_{SS}V_S + \mathcal{L}_{SL}V_L$$  \hspace{1cm} (12) \hspace{1cm} rI_L + \ell I_L = \mathcal{L}_{LS}V_S + \mathcal{L}_{LL}V_L \hspace{1cm} (13)$$

Suppose that the load nodes are attached to constant current loads $I_L = -I_o^*$. Then from (13) we obtain

$$-rI_o^* = \mathcal{L}_{LS}V_S + \mathcal{L}_{LL}V_L,$$

and therefore

$$-r\mathcal{L}_{LL}^{-1}I_o^* - \mathcal{L}_{LS}^{-1}L_v = V_L.$$  \hspace{1cm} (14)

Substituting (14) into (12) yields

$$rI_S + \ell I_S = \mathcal{L}_{red}V_o - r\mathcal{L}_{LS}^{-1}I_o^*.$$  

Here the Schur complement $\mathcal{L}_{red} = \mathcal{L}_{SS} - \mathcal{L}_{SL}\mathcal{L}_{LL}^{-1}\mathcal{L}_{LS}$ is again a Laplacian matrix known as the Kron-reduced Laplacian [27], [28]. Bearing in mind that $V_G = V_o - \ell_o I_S - r_o I_S$, the system becomes

$$\left(r\mathcal{I} + r_o \mathcal{L}_{red}\right)I_S + \left(\ell\mathcal{I} + \ell_o \mathcal{L}_{red}\right)I_S = \mathcal{L}_{red}V_o - r\mathcal{L}_{LS}^{-1}I_o^*,$$  \hspace{1cm} (15)

and one can repeat the same analysis as above working with $\mathcal{L}_{red}$ instead of $\mathcal{L}$. Note that (15) matches the model (4) with the difference of a constant. As this constant term does not affect the homogeneous solution, the network inductivity and resistivity ratios are obtained analogously as before, where the algebraic connectivity is computed based on the Kron reduced Laplacian. \qed

B. Non-uniform Output Impedances

In this section we investigate the case where output impedances with different magnitudes are connected to the network, and we quantify the network inductivity ratio $\Psi_{NIR}$ under this non-uniform assignment. The case with non-uniform resistances can be treated in an analogous manner.

For the sake of simplicity, throughout this subsection, we consider the case where the resistive parts of the output impedances are negligible (see Remark 5 for relaxing this assumption). Let $D = \text{diag}(\ell_{o1}, \ell_{o2}, \cdots, \ell_{on})$, where $\ell_{oi}$ is the (nonzero) output inductance connected to the node $i$. We have

$$rI + \ell I = LV; \quad V = V_o - D\dot{I},$$

and hence

$$rI + (\ell I + LD)\dot{I} = LV_o.$$  \hspace{1cm} (16)

Note that $LD$ is similar to $D^{1/2}LD^{1/2}$ and therefore has nonnegative real eigenvalues. In view of equation (4), here $R = rI$ and $L = \ell I + LD$. Hence, the matrix $L^{-1}R$ possesses positive real eigenvalues, and Assumption 1 holds.

The matrix $LD$ is also similar to $DL$, which can be interpreted as the (asymmetric) Laplacian matrix of a directed connected graph noted by $\hat{\mathcal{G}}(\hat{\mathcal{V}}, \hat{\mathcal{E}}, \hat{\Gamma})$ with the same nodes as the original graph $\mathcal{V} = \{1, \ldots, n\}$, but with directed edges $\hat{\mathcal{E}} \subset \mathcal{V} \times \mathcal{V}$. As shown in Figure 4, in this representation, for any $(i, j) \in \hat{\mathcal{E}}$, there exists a directed edge from node $i$ to node $j$ with the weight $\ell \tau_{ij}^{-1}$ (recall that $\tau_{ij}$ is the weight of the edge $(i, j) \in \mathcal{E}$ of the original graph $\mathcal{G}$). Hence, the weight matrix $\hat{\mathcal{L}} \in \mathbb{R}^{2m \times 2m}$ is the diagonal matrix with the weights $\ell \tau_{ij}^{-1}$ on its diagonal. Note that the edge set $\hat{\mathcal{E}}$ is symmetric in the sense that $(i, j) \in \hat{\mathcal{E}} \iff (j, i) \in \hat{\mathcal{E}}$, and its cardinality is equal to $2m$. We take advantage of this graph to obtain the network inductivity ratio $\Psi_{NIR}$, as formalized in the following theorem.

**Theorem 3** Consider a homogeneous network with the resistance per length unit $r$, inductance per length unit $\ell$, edge lengths $\tau_1, \ldots, \tau_n$, and output inductors $\ell_{o1}, \ell_{o2}, \ldots, \ell_{on}$. Then the network inductivity ratio is given by

$$\Psi_{NIR} = \frac{\lambda_2 + \ell}{r},$$

where $\lambda_2$ is the algebraic connectivity of the graph $\hat{\mathcal{G}}(\hat{\mathcal{V}}, \hat{\mathcal{E}}, \hat{\Gamma})$ defined above.

**Proof:** Let $L' = D^{1/2}LD^{1/2}$. The homogeneous solution to (16) is

$$I(t) = e^{-r(\ell I + LD)^{-1}t}I_0 = D^{-1/2}e^{-r\ell D^{-1/2}(\ell I + LD)^{-1}D^{1/2}t}D^{1/2}I_0 = D^{-1/2}e^{-r(\ell I + L')^{-1}t}D^{1/2}I_0.$$  

Note that $L'$ is positive semi-definite and thus $\ell I + L'$ is invertible. Bearing in mind that 0 is an eigenvalue of the matrix $L'$ with the corresponding normalized eigenvector...
\[ \mathcal{U}_1 = (I^T D^{-1} I) \cdot 2 \cdot D^{1/2} I, \]
and by the spectral decomposition
\[ \mathcal{L}' = \mathcal{U} \mathcal{M} \mathcal{T}, \]
we find that
\[ I(t) = D^{1/2} e^{-r(t \mathcal{L} + \Lambda)} I_0 = D^{1/2} \mathcal{U} e^{-r(t \mathcal{L} + \Lambda)} I_0 \]
\[ = D^{1/2} \left[ \mathcal{U}_1 \tilde{U} \right] e^{-r \left[ \begin{array}{c} \frac{1}{2} \\ 0 \end{array} \right] \cdot \left( \begin{array}{c} 0 \\ 1 \end{array} \right)} \mathcal{T} \left[ \begin{array}{c} \mathcal{U}_1^T \\ \mathcal{U}_0^T \end{array} \right] D^{1/2} I_0 , \]
where \( \tilde{\Lambda} = \text{diag} \left\{ \frac{1}{\lambda_2 + r}, \ldots, \frac{1}{\lambda_n + r} \right\} \) and \( 0 < \lambda_2 < \lambda_3 < \cdots < \lambda_n \) are nonzero eigenvalues of the matrix \( \mathcal{L}' \). Let \( \tilde{I}(t) = D^{1/2} I(t) \). Noting that by the Kirchhoff Law, \( I \mathcal{T} I_0 = 0 \), we have
\[ \tilde{I}(t) = \tilde{U} e^{-r \tilde{\Lambda} t} \tilde{U}^T \tilde{I}_0 . \]

Since \( \mathcal{U}_1^T \tilde{I}_0 = 0 \) we can write \( \tilde{I}_0 \) as the linear combination
\[ \tilde{I}_0 = \tilde{U} X, \quad X \in \mathbb{R}^{(n-1) \times 1} . \]
Now we have
\[ \tilde{I}(t) = \tilde{U} e^{-r \tilde{\Lambda} t} X, \quad \| \tilde{I}(t) \|^2 = X^T e^{-2r \tilde{\Lambda} t} X . \]

Hence
\[ \| \tilde{I}(t) \|^2 \geq e^{-2r \tilde{\Lambda} t} \| \tilde{I}_0 \|^2, \]
\[ I^T(t) D I(t) \geq e^{-2r \tilde{\Lambda} t} I^T_0 D I_0, \]
\[ \| I(t) \| \geq \mu e^{-2r \tilde{\Lambda} t} \| I_0 \|, \]
where
\[ \mu := \sqrt{\text{min}(\ell_n) \over \text{max}(\ell_n)} . \]

This yields \( \Psi_{\text{NIR}} = \frac{\lambda_2 + \ell}{\lambda_1 + \ell} \). Note that the eigenvalues of \( \mathcal{L}' \) and \( \mathcal{L} \) are the same. This completes the proof. \( \square \)

**Remark 5** The results of Theorem 3 can be generalized to the case of non-uniform output impedances, each containing a nonzero resistor \( r_o \) and a nonzero inductor \( \ell_o \) in series. In this case, the network can be modeled by
\[ (r \mathcal{I} + \mathcal{L} D_t \mathcal{I}) + (\mathcal{L} + \mathcal{L} D_r \mathcal{I}) \mathcal{I} = \mathcal{L} \mathcal{V}_o , \]
where \( D_r := \text{diag} \{ r_{o_1}, r_{o_2}, \ldots, r_{o_n} \} \) and \( D_t := \text{diag} \{ \ell_{o_1}, \ell_{o_2}, \ldots, \ell_{o_n} \} \). Analogously to the proof of Theorem 3 it can be shown that the network inductivity ratio is calculated as
\[ \Psi_{\text{NIR}} = \min_{\ell \in \{2,3,\ldots,n\}} \frac{\lambda_i + \ell}{\lambda_i + r_i} , \]
where \( \lambda_i \) denotes the \( i \)th eigenvalue of the matrix \( \mathcal{L} D_t \), and \( \lambda_{r_i} = 0 \). Similarly, \( \lambda_{r_i} \) denotes the \( i \)th eigenvalue of the matrix \( \mathcal{L} D_r \), and \( \lambda_{r_i} = 0 \). \( \square \)

**Example 1** Consider the graph \( \mathcal{G} \) with the Laplacian \( \mathcal{L} \), consisting of four nodes in a star topology. The line lengths are 5mH, 7mH, and 9mH, as depicted in Figure 5. Note that here again, the weights of the edges are the inverse of the distances. Attach the output impedances \( D = \text{diag} \{ \ell_{o_1}, \ell_{o_2}, \ell_{o_3}, \ell_{o_4} \} \) to each inverter, and assume that we have limited resources of inductors, namely \( \ell \)
\[ \sum_i \ell_{o_i} = c, \quad c \in \mathbb{R}^+. \]

Figure 5 shows different values of the second smallest eigenvalue of the matrix \( \mathcal{L} \mathcal{D} \mathcal{L} \). We obtain that the maximal algebraic connectivity is achieved when no output impedance is used (wasted) for the node in the middle. Interestingly, the optimal value of output inductor for each node is proportional to its distance to the middle node.

**IV. KRON REDUCTION IN PHASOR DOMAIN AND THE NETWORK INDUCTIVITY RATIO**

By leveraging Kron reduction and using the phasor domain, it is sometimes possible to synthesize an \( RL \) circuit for the augmented network model (4) (see 30 for more details). Note that the budget constraint (18) can also be used to reflect any disadvantage resulting from a large output impedance, e.g. the voltage drop.
As depicted in Figure 6 in the Kron reduced graph some of the edges coincide with the lines of the original network into which the output impedances were diffused, while others (dotted green) are created as a result of the Kron reduction. We refer to the former as *physical* and to the latter as *virtual* lines. To derive the Kron-reduced model, we first write the nodal currents as

\[
[I_0] = \begin{bmatrix}
y_0 I & -y_0 I \\
y_0 I + y_0 L & V_o
\end{bmatrix}
\begin{bmatrix}
V \end{bmatrix},
\]

where

\[
y_o = \frac{1}{j\omega L}, \quad y_r = \frac{1}{r+j\omega L}.
\]

The Kron-reduced model is then obtained as

\[
\mathcal{Y}_{red} = y_o[I - (I + y_r L)^{-1}].
\]

(19)

Since every path between the outer nodes of the graph passes only through internal nodes (see Figure 6), the resulting Kron-reduced network is a complete graph \([31]\). In the following example, we compare the line phase angles

\[
\theta_{ij} := \arctan(\frac{\text{Im}(1/\mathcal{Y}_{red ij})}{\text{Re}(1/\mathcal{Y}_{red ij})})
\]

for the line \(\{i, j\}\), to the phase angles suggested by \(\Psi_{NIR}\), namely

\[
\theta_{NIR} := \arctan(\omega \Psi_{NIR}).
\]

(20)

Note that the term \(\omega\) in the above is included to obtain reactance to resistance ratio from inductance to resistance ratio.

**Example 2** Line angles of a Kron-reduced Graph with uniform output impedances.

(a) Consider a 4-node complete graph with different distribution line lengths. As shown in Figure 7a, the proposed measure matches with the overall behavior of the line angles as the added output inductance increases. (b) Consider a 4-node uniform path graph. Figure 7b shows that the least inductivity behavior is observed for the virtual lines. Hence the less virtual lines the reduced graph contains, the more output impedance diffuses in the network, which is consistent with Remark 1. Also note that the inductance and resistance possess negative values at some edges for certain values of the output impedance. Therefore, it is difficult to extract a reasonable inductivity ratio for those edges from the Kron reduced phasor model. On the contrary, the proposed inductivity measure remains within the physically valid interval \(\arctan(\omega \Psi_{NIR}) \in [0 \pi/2]\).

Example 2 shows that \(\theta_{NIR}\) can be used as a measure that estimates the phase of the lines of the overall network. A desired amount of change in this measure can be optimized by appropriate choices of output impedances. The following example illustrates this case.

**Example 3** Output impedance optimization on the IEEE 13 node test feeder.

Figure 8 depicts the graph of the islanded IEEE 13 node test feeder. Here, all the distribution lines are assumed to have the same reactance per length equal to \(\omega L = 1.2 \Omega/\text{mile}\) and resistance per length equal to \(r = 0.7 \Omega/\text{mile}\), derived from the configuration 602 [32]. We consider the case where three inverters are connected to the nodes 1, 3, and 7. Example 2 shows that \(\theta_{NIR}\) can be used as a measure that estimates the phase of the lines of the overall network. A desired amount of change in this measure can be optimized by appropriate choices of output impedances. The following example illustrates this case.
since the typical values for inverter output filter inductance and implemented output virtual inductance range from 0.5mH to 50mH [23, 24, 25, 26, 27, 28, 29, 30]. However, note that the total inductance used in the (optimal) non-uniform case is considerably smaller than the one used in the uniform scheme.

V. CONCLUSION

In this paper, the influence of the output impedance on the inductivity and resistivity of the distribution lines has been investigated. Two measures, network inductivity ratio and network resistivity ratio, were proposed and analyzed without relying on the ideal sinusoidal signals assumption (phasors). The analysis revealed the fact that the more connected the graph is, the more output impedance diffuses into the network and the larger its effect will be. We have provided examples on how the impact of inductive output impedances on the network can be maximized in specific network topologies. We compared the proposed measure to the phase angles of the lines in a phasor-based Kron reduced network. Results confirm the validity and the effectiveness of the proposed metrics. Future works include investigating analytical solutions on maximizing the network inductivity/resistivity ratios, and incorporating different load models in the proposed scheme.

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