Entangled states in a Josephson charge qubit coupled to a superconducting resonator

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Abstract

We study the dynamics of a quantum superconducting circuit which is the analogue of an atom in a high-Q cavity. The circuit consists of a Josephson charge qubit coupled to a superconducting resonator. The charge qubit can be treated as a two level quantum system whose energy separation is split by the Josephson energy $E_j$. The superconducting resonator in our proposal is the analogue of a photon box and is described by a quantum harmonic oscillator with characteristic frequency $\omega_r$. The coupling between the charge qubit and the resonator is realized by a coupling capacitance $C_c$. We have studied the eigenstates as well as the dynamics of the quantum circuit. Interesting phenomena occur when the Josephson energy equals the oscillator frequency, $E_j = h\omega_r$. Then the quantum circuit is described by entangled states. We have deduced the time evolution of these states in the limit of weak coupling between the charge qubit and the resonator. We found Rabi oscillations of the excited charge qubit eigenstate. This effect is explained by the spontaneous emission and re-absorption of a single photon in the superconducting resonator.

I. INTRODUCTION

Recently, a substantial interest in the theory of quantum information and computing has revived the physical research on quantum systems. The elementary unit of quantum information is a two-state system, usually referred to as a quantum bit (qubit). Basic operations are realized by preparation and manipulation of, as well as a measurement on, entangled states in systems which consist of several coupled qubits. However, the fabrication of physical systems which would enable the actual implementation of quantum algorithms is far from being realized in the near future and a substantial amount of fundamental research is still needed.

During the past five years, great progress has been made in the manipulation of entangled states in systems consisting of up to four qubits based on ion traps and atoms in a high-Q cavity. These two experiments demonstrate clearly and unambiguously the possibility to coherently control the entangled states of a limited number of qubits, as well as to perform
a quantum measurement on them. In spite of this success, it seems quite difficult to realize circuits consisting of the large number of ion traps or atoms in a cavity necessary for quantum computation.

It has been suggested that small solid state devices fabricated using nanolithography technologies are promising for quantum circuit integration. However, the coherent manipulation of entangled states as well as the realization of quantum measurements remain fundamental issues to be investigated. One of the main challenges is to gain control over all possible sources of decoherence. At present the best candidates for the implementation of quantum gates based on solid state devices are circuits using small Josephson junctions. In the superconducting state, such circuits contain less intrinsic sources of decoherence. Indeed it has been experimentally demonstrated that a single Cooper pair box is a macroscopic two level system which can be coherently controlled. At about the same time, theoretical works have proposed the use of the Cooper pair box as a qubit (the so-called Josephson charge qubit) in the context of quantum computers. In particular, systems consisting of several charge qubits with controlled couplings have been discussed, the quantum measurement problem has been addressed and the decoherence time has been estimated. More recently, qubits based on superconducting loops containing small Josephson junctions have been proposed (Josephson phase qubits) and are currently studied. But up to now, the existence of entangled states, which are at the heart of quantum information processing, has been demonstrated neither for charge qubits nor for phase qubits.

In this article we propose to study one of the simplest Josephson circuits in which entangled states can be realized. It consists of a charge qubit coupled to a superconducting resonator and can be described theoretically by a two level system coupled to a quantum harmonic oscillator. After a description of this quantum circuit in the next section, the Hamiltonian describing it will be derived in Sec. III. In Sec. IV the time evolution of the eigenstates is obtained and we demonstrate the existence of entanglement. In the last Section, we discuss the dynamics of the quantum circuit for typical experimental values of the system parameters.

II. QUANTUM CIRCUIT

The Josephson circuit we study hereafter is depicted in Fig. 1. It consists of three different elementary circuits: a Cooper pair box, an LC-resonator and a ”coupling” capacitor.

For small enough junction capacitance $C_j$, gate capacitance $C_g$, and coupling capacitance $C_c$, the charging energy of the box is large compared to thermal fluctuations and the excess charge of the box is quantized. On the other hand, we assume the charging energy to be smaller than the superconducting gap $\Delta$, such that no quasiparticles are present in the box. Thus the excess charge is entirely due to the presence of Cooper pairs and charge quantization occurs in units of $2e$. The gate voltage $V_g$ is used as an external control parameter. When the gate charge $N_g = -C_g V_g / e$ is equal to unity, the Cooper pair box can be viewed as a macroscopic two-level quantum system whose energy separation is split by the Josephson energy $E_j$. The two eigenstates $|\rangle$ and $|+\rangle$ correspond to a coherent superposition of the two different charge states of the box. When $N_g \approx 1$, the Cooper pair box will be referred to as a Josephson charge qubit.
The LC-resonator system can be described by a quantum harmonic oscillator whose characteristic frequency is given by \( \omega_r \). This system is the analogue of a high-Q cavity.

The capacitance \( C_c \) plays a crucial role in our proposed circuit since it couples the charge qubit and the resonator to each other. These two circuits are no longer independent and the system must be considered in its totality. Thus the proposed quantum circuit of Fig. 2 realizes the simple situation in which a two level system is coupled to a harmonic oscillator. In spite of its simplicity, such a system describes a great variety of interesting situations.

III. HAMILTONIAN

The circuit depicted in Fig. 1 can be characterized mechanically by two generalized coordinates, \( \phi_j \) and \( \phi_r \). These coordinates are associated with the voltage drop \( \delta V_j \) over the junction and \( \delta V_r \) over the resonator, respectively, according to the Josephson relation
\[
\phi_i = 2e\delta V_i t / \hbar \quad (i = j, r).
\]
We seek the Lagrangian \( \mathcal{L}(\phi_j, \phi_r, \dot{\phi}_j, \dot{\phi}_r) = T - V \) describing the dynamics of these variables. The potential energy \( V \) is a function of the coordinates only,
\[
V(\phi_j, \phi_r) = -E_j \cos \phi_j + \frac{E_r}{2} \phi_r^2, \tag{1}
\]
where \( E_r = (1/L_r)(\hbar/2e)^2 \) is the energy associated with the inductance \( L_r \) of the resonator. The kinetic energy \( T \) is quadratic in the velocities \( \dot{\phi}_j \) and \( \dot{\phi}_r \). It is just the free electrostatic energy stored in the capacitators present in the circuit. This free energy can be written as
\[
T = \frac{1}{2} \left[ C_{\Sigma j} (\hbar \dot{\phi}_j/2e)^2 + C_{\Sigma r} (\hbar \dot{\phi}_r/2e)^2 + 2C_{\Sigma C} (\hbar/2e)^2 \dot{\phi}_j \dot{\phi}_r \right]. \tag{2}
\]
Here we introduced the capacitances \( C_{\Sigma C} = C_c + C_g, \quad C_{\Sigma j} = C_j + C_{\Sigma C}, \quad C_{\Sigma r} = C_r + C_{\Sigma C}. \) Note that the Lagrangian contains an interaction between the resonator and the junction:
\[
\mathcal{L}_{int} = C_{\Sigma C} (\hbar/2e)^2 \dot{\phi}_j \dot{\phi}_r. \]
The effective coupling between these two parts of the circuit is determined by the sum of the gate capacitance and the coupling capacitance. We also note that the equations of motion \( d(\partial \mathcal{L}/\partial \dot{\phi}_i)/dt + \partial \mathcal{L}/\partial \phi_i = 0 \) express current conservation in the circuit.

Through the Josephson junction, Cooper pairs can tunnel from or onto the island. The number of excess Cooper pairs on the island, \( n \), depends on the gate voltage. Charge neutrality leads us to relation
\[
2ne = C_{\Sigma j} (\hbar \dot{\phi}_j/2e) + C_{\Sigma C} (\hbar \dot{\phi}_r/2e) + N_g e. \tag{3}
\]
It is always possible to add a term to the Lagrangian which is a total time derivative. Let us add the term \( \hbar \dot{\phi}_j N_g / 2 \). As a result, the momenta conjugate to \( \phi_j \) and \( \phi_r \) are
\[
p_j = \partial \mathcal{L} / \partial \dot{\phi}_j = (\hbar/2e)[C_{\Sigma j} (\hbar \dot{\phi}_j/2e) + C_{\Sigma C} (\hbar \dot{\phi}_r/2e) + N_g e] = \hbar n,
p_r = \partial \mathcal{L} / \partial \dot{\phi}_r = C_{\Sigma r} (\hbar/2e)^2 \dot{\phi}_r + C_{\Sigma C} (\hbar/2e)^2 \dot{\phi}_j.
\]
Note in particular that the momentum \( p_j \) is proportional to \( n \).

The Hamiltonian is obtained with the help of the Legendre transform \( H = p_j \dot{\phi}_j + p_r \dot{\phi}_r - \mathcal{L} \). We find \( H = H_j + H_r + H_c \), where

\[
3
\[ H_j = E_{C,j}(2n - N_g)^2 - E_j \cos \phi_j, \]  
\[ H_r = E_{C,r}(2p_r/\hbar)^2 + E_r \phi_r^2/2, \]  
\[ H_c = -E_{C,c}(2n - N_g)(2p_r/\hbar). \]  

The charging energies \( E_{C,i} (i = j, r, c) \) appearing here are given by \( E_{C,i} = e^2/2C_{i,\text{eff}}, \) with \( C_{j,\text{eff}} = C_j + (1/C_{\Sigma_c} + 1/C_r)^{-1}, \) \( C_{r,\text{eff}} = C_r + (1/C_{\Sigma_c} + 1/C_j)^{-1}, \) and \( C_{c,\text{eff}} = (C_{\Sigma_c} + (1/C_j + 1/C_r)^{-1})[(C_j + C_r)/2C_{\Sigma_c}]. \) The Hamiltonian equations of motion, \( \dot{p}_i = -\partial H/\partial \phi_i, \) \( \dot{\phi}_i = \partial H/\partial p_i \) lead us again to current conservation.

In order to obtain the quantum mechanical Hamiltonian \( \hat{H} \), we replace \( p_i, \phi_i \) by the corresponding operators, with \([p_k, \phi_m] = (\hbar/i)\delta_{k,m}. \) In particular, in \( \phi \)-representation we have \( p_k = (\hbar/i)\partial/\partial \phi_k. \) Note also that \([n, \phi_j] = -i\) and \( n = -i\partial/\partial \phi_j. \) Below we discuss the various contributions to \( \hat{H} \) in some detail.

\textit{Josephson junction.} The commutation relation \([n, \phi_j] = -i\) implies \([n, e^{i\phi_j}] = e^{i\phi_j}. \) Using the basis states \( |n\rangle, \) where \( n \) corresponds to the number of excess Cooper pairs on the island, we thus have \( e^{i\phi_j}|n\rangle = |n + 1\rangle. \) Similarly, \( e^{-i\phi_j}|n\rangle = |n - 1\rangle. \) Therefore we can write \( \hat{H}_j \) as

\[ \hat{H}_j = E_{C,j} \sum_n (2n - N_g)^2|n\rangle\langle n| - \frac{E_j}{2} \sum_n (|n + 1\rangle\langle n| + |n - 1\rangle\langle n|). \]  

(7)

If the gate-voltage is such that \( N_g \approx 1, \) the states with \( n = 0 \) and \( n = 1 \) are almost degenerate. At low temperatures, the Hamiltonian \( \hat{H}_j \) involves only these two states, and thus can be presented as a matrix

\[ \hat{H}_j \approx \begin{pmatrix} E_{C,j}N_g^2 & -E_j/2 \\ -E_j/2 & E_{C,j}(2 - N_g)^2 \end{pmatrix}. \]  

(8)

This matrix can be diagonalized. The eigenvalues are

\[ E_\pm = E_{C,j}[1 + (\delta N_g)^2] \mp \frac{1}{2}\sqrt{(\delta E_g)^2 + E_j^2}, \]  

(9)

where \( \delta N_g = N_g - 1 \) and \( \delta E_g = -4E_{C,j}\delta N_g. \) The corresponding eigenstates are

\[ |-\rangle = \alpha|0\rangle + \beta|1\rangle \]  

(10)

\[ |+\rangle = \beta|0\rangle - \alpha|1\rangle \]  

(11)

where \( \alpha^2 = 1 - \beta^2 = [1 + \delta E_g/\sqrt{(\delta E_g)^2 + E_j^2}] / 2. \)

\textit{Resonator.} Since the LC-circuit constitutes just a harmonic oscillator with a characteristic frequency \( \omega_r = \sqrt{1/L_rC_{r,\text{eff}}}, \) the Hamiltonian \( \hat{H}_r \) can be written in the standard way

\[ \hat{H}_r = \hbar \omega_r (a^\dagger a + 1/2), \]  

(12)

where
\[ \phi_r = 2 \sqrt{\frac{E_{C,r}}{\hbar \omega_r}} (a^\dagger + a), \quad (13) \]
\[ p_r = \frac{i \hbar}{4} \sqrt{\frac{\hbar \omega_r}{E_{C,r}}} (a^\dagger - a). \quad (14) \]

**Coupling term.** The coupling term can also be written using the operators \( a, a^\dagger \):
\[ \hat{H}_c = -i \frac{E_{C,c}}{2} \sqrt{\frac{\hbar \omega_r}{E_{C,r}}} (2n - N_g) \left( a^\dagger - a \right). \quad (15) \]

Note that the characteristic coupling energy is \( E_c = \sqrt{\hbar \omega_r/E_{C,r}E_{C,c}}/2 \).

A general analysis of the Hamiltonian \( \hat{H} \) is beyond the scope of the present paper and will be presented elsewhere. In the next section we will discuss an explicit matrix form of \( \hat{H} \), which can be obtained under certain simplifying conditions which are nevertheless experimentally relevant.

**IV. EIGENSTATES AND ENTANGLEMENT**

Throughout this section we will work in the zero-temperature limit. We are interested in the situation \( N_g \approx 1 \), such that we have to consider the charge qubit states \( |-\rangle \) and \( |+\rangle \) only. Furthermore, as far as the resonator is concerned, we will consider \( \hbar \omega_r = E_j \) and work only with the ground state \( |0\rangle \) and the first excited state \( |1\rangle \). In the limit of weak coupling, \( E_c \ll \hbar \omega_r \), the coupled system can be characterized by the four basis states \(|-0\rangle, |-1\rangle, |+0\rangle, |+1\rangle \). The Hamiltonian matrix for this low-energy subspace reads
\[ \hat{H} = \begin{pmatrix}
E_0 & iE_\beta & 0 & -2i\alpha \beta E_c \\
-iE_\beta & E_1 & 2i\alpha \beta E_c & 0 \\
0 & -2i\alpha \beta E_c & E_2 & iE_\alpha \\
2i\alpha \beta E_c & 0 & -iE_\alpha & E_3
\end{pmatrix}, \quad (16) \]

which is a hermitian matrix describing the two-level system coupled to the lowest states of the resonator. Here, \( E_0 = E_{-} + \hbar \omega_r/2, E_1 = E_{-} + 3\hbar \omega_r/2, E_2 = E_{+} + \hbar \omega_r/2, E_3 = E_{+} + 3\hbar \omega_r/2, E_\alpha = E_c(2\alpha^2 - N_g) \), and \( E_\beta = E_c(2\beta^2 - N_g) \).

Suppose that the system has been prepared in the state \( |\psi(t = 0)\rangle = |+, 0\rangle \) at time \( t = 0 \). This situation can be achieved by a suitable manipulation of the gate voltage \( V_g \) at times prior to \( t = 0 \). At times \( t > 0 \), the time evolution of \( |\psi(t)\rangle \) describing the system is governed by the Hamiltonian \( (16) \). We keep \( V_g \) fixed such that \( N_g = 1 \) at \( t > 0 \). Thus we have \( \alpha^2 = \beta^2 = 1/2 \), and hence \( E_\alpha = E_\beta = 0 \). Moreover, as \( \hbar \omega_r = E_j \), we have \( E_1 = E_2 = E_{C,j} + E_j \equiv \bar{E} \): without coupling, the state \( |+, 0\rangle \) would be degenerate with the state \( |-1\rangle \). Thus the Hamiltonian takes the simple form
\[ \hat{H} = \begin{pmatrix}
E_0 & 0 & 0 & -iE_c \\
0 & \bar{E} & iE_c & 0 \\
0 & -iE_c & \bar{E} & 0 \\
iE_c & 0 & 0 & E_3
\end{pmatrix}. \quad (17) \]
Note in particular that the state $|+, 0\rangle$ couples to the state $|-, 1\rangle$; as a result, the degeneracy between them is lifted and the states become entangled. The precise form of the entanglement is governed by the central $2 \times 2$ block of the matrix (17). The eigenstates of this block are

$$
|\chi_1\rangle = \frac{|-, 1\rangle + i|+, 0\rangle}{\sqrt{2}}, \\
|\chi_2\rangle = \frac{|-, 1\rangle - i|+, 0\rangle}{\sqrt{2}},
$$
corresponding to the eigen energies $\bar{E} - E_c$ and $\bar{E} + E_c$, respectively. These two excited eigenstates thus correspond to a maximum entanglement of charge qubit and resonator states, induced by the capacitive coupling between them.

The time evolution of $|\psi(t)\rangle$ is given by

$$
|\psi(t)\rangle = \frac{1}{\sqrt{2i}} \left[ e^{-i(\bar{E} - E_c)t/\hbar} |\chi_1\rangle - e^{-i(\bar{E} + E_c)t/\hbar} |\chi_2\rangle \right].
$$

(18)

We deduce that the state $|\psi(t)\rangle$ oscillates coherently between $|-, 1\rangle$ and $|+, 0\rangle$. In fact, these so-called quantum Rabi oscillations can be interpreted as the spontaneous emission and re-absorption of one excitation quantum by the resonator. An interesting quantity is the probability $P_1(t)$ to find the harmonic oscillator in the state $|1\rangle$ after a certain time $t$. This probability shows Rabi oscillations as a function of $t$ with frequency $2E_c/\hbar$,

$$
P_1(t) = |\langle 1, -|\psi(t)\rangle|^2 = \frac{1}{2} \left[ 1 - \cos(2E_c t/\hbar) \right].
$$

(19)

Since these Rabi oscillations are characteristic for the entanglement realized in the system, their measurement would provide direct evidence of the presence of the entangled states $|\chi_1\rangle$ and $|\chi_2\rangle$. We will discuss the feasibility of such a measurement in the next Section.

V. DISCUSSION

For the numerical estimates presented below we will consider parameters related to a typical aluminium superconducting circuit. For the Josephson charge qubit we have chosen the following characteristics: $E_j = 26.1 \mu eV$, $E_{C,j} = 70 \mu eV$, $\Delta = 240 \mu eV$. As for the resonator, we take $L_r = 90 pF$ and $E_{C,r} = 12 neV$, as a result $\hbar \omega_r = 26.1 \mu eV$. Finally, the coupling capacitance is chosen to be of the same order of $C_j$, $C_c = 0.5 fF$, yielding $E_c = 256 neV$. Note that the coupling energy is indeed much smaller than the Josephson energy, which in turn is equal to the excitation energy of the resonator.

Using the above parameters, we have plotted $P_1(t)$, Eq. (19), as a function of time in Fig. 2. We clearly see the Rabi oscillations with periodicity $T_{\text{Rabi}} \approx 8$ ns.

In order to be able to observe these oscillations, we need to satisfy various conditions. First of all, in order to avoid the presence of quasiparticles, the temperature must be much lower than the gap $\Delta$. Secondly, it is necessary to have a decoherence time which is longer than $T_{\text{Rabi}}$. In our system, the decoherence time will be the shorter of the lifetime $\tau_r$ of the excited state of the resonator and the decoherence time $\tau_{\text{qubit}}$ of the charge qubit. A Q-factor of about 500 is quite realistic for a superconducting LC resonator. This yields a lifetime $\tau_r > 10$ ns. As for the qubit, the experiment by Nakamura has indicated that
\(\tau_{\text{qubit}} > 2\text{ns.} \) This lower bound is essentially the decoherence time associated with the coupling to the measuring probe. In principle, this time can be improved upon by modifications of the experimental set-up. Theoretical estimates\(^8\) show that a time \(\tau_{\text{qubit}} \sim 100 \text{ ns} \) should be feasible. Finally, a measurement of the number of excitations should be performed on the resonator. This can be done, \textit{e.g.}, along the lines of Ref.\(^{17}\), where the discrete, oscillator-like energy levels of an underdamped Josephson junction were measured.

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FIG. 1. The quantum circuit.

FIG. 2. Probability $P_1(t)$ to find the system, prepared in the state $|+, 0\rangle$ at $t = 0$, in the state $|-1\rangle$ after a time $t$. 