A Unified Framework for Causal Inference with Multiple Imputation Using Martingale

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Summary. Multiple imputation is widely used to handle confounders missing at random in causal inference. Although Rubin’s combining rule is simple, it is not clear whether or not the standard multiple imputation inference is consistent when coupled with the commonly-used average causal effect (ACE) estimators. This article establishes a unified martingale representation for the average causal effect (ACE) estimators after multiple imputation. This representation invokes the wild bootstrap inference to provide consistent variance estimation. Our framework applies to asymptotically normal ACE estimators, including the regression imputation, weighting, and matching estimators. We extend to the scenarios when both outcome and confounders are subject to missingness and when the data are missing not at random.

Keywords: Causality; Congeniality; Martingale representation; Weighted bootstrap

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1 Introduction

Causal inference is a central goal in many disciplines, such as medicine, econometrics, political and social sciences. When all confounders that influence both treatment and outcome are observed, the average causal effect (ACE) of the treatment is identifiable. The literature has proposed many ACE estimators, such as regression imputation (Hahn, 1998, Heckman et al., 1997), (augmented) propensity score weighting (Horvitz and Thompson, 1952, Rosenbaum and Rubin, 1983, Robins et al., 1994, Bang and Robins, 2005, Cao et al., 2009) and matching (Rosenbaum, 1989, Heckman et al., 1997, Hirano et al., 2003, Hansen, 2004, Rubin, 2006, Abadie and Imbens, 2006, Stuart, 2010, Abadie and Imbens, 2016) to adjust for confounders (e.g., Imbens and Rubin, 2015).

However, it is ubiquitous that confounders are only partially observed in practice. A widely-used approach to handle incomplete/missing data is multiple imputation (MI). The National Research Council has recommended MI as one of its preferred approaches of addressing missing data in 2010 (National Research Council, 2010). The idea of MI is to fill the missing values multiple times by sampling from the posterior predictive distribution of the missing values given the observed values. Then, full sample analyses can be applied straightforwardly to the imputed data sets, and these multiple results are summarized by an easy-to-implement combining rule for inference (Rubin, 1987). Previous works have used MI for causal inference with partially observed confounders, e.g., Qu and Lipkovich (2009), Crowe et al. (2010), Mitra and Reiter (2011), and Seaman and White (2014).

MI can provide valid frequentist inferences in various applications (e.g., Clogg et al., 1991). On the other hand, many authors have found that Rubin’s variance estimator is not always consistent (e.g., Fay, 1992, Kott, 1995, Fay, 1996, Binder and Sun, 1996, Wang and Robins, 1998, Robins and Wang, 2000, Nielsen, 2003 and Kim et al., 2006). To ensure the validity of Rubin’s variance estimation, imputations must be proper (Rubin, 1987). A sufficient condition for this is the congeniality condition of Meng (1994), imposed on both the imputation model and the subsequent full sample analysis. Even with a correctly specified imputation model, Yang and Kim (2016) showed that MI is not necessarily congenial for the method of moments estimation, so common statistical procedures can be incompatible with MI. This phenomenon becomes pronounced for causal inference because many full sample estimators are available for estimating the ACE. The validity of Rubin’s variance estimator using these full sample estimators for causal inference is largely unexplored and questionable. Given the popularity of MI in practice, it is important to develop a valid inference procedure for utilizing MI in causal inference.

In this article, we establish a novel martingale representation of the MI estimator of the
ACE. Our key insight is that the MI estimator is intrinsically created in a sequential manner: first, the posterior samples of parameters are drawn from the posterior distribution, which is asymptotically equivalent to the sampling distribution of the maximum likelihood estimator based on the Bernstein-von Mises theorem (van der Vaart, 2000; Chapter 10); second, the predictive posterior samples of the missing data are drawn conditioned on the observed data. This conceptualization leads to an asymptotically linear expression of the MI estimator in terms of a sequence of random variables which have conditional mean zero given the sigma algebra generated from the preceding variables (i.e., a martingale representation). The martingale representation invokes the wild/weighted bootstrap procedure (Wu, 1986, Liu, 1988) that provides valid variance estimation and inference regardless of which full sample estimator is adopted in MI.

We show the asymptotic validity of our proposed bootstrap inference method for the MI estimators using the martingale central limit theory (Hall and Heyde, 1980) and the asymptotic property of weighted sampling of martingale difference arrays (Pauly et al., 2011). Although the validity of the proposed method is based on the asymptotic results as the sample size goes to infinity, the simulations results demonstrate that it performs well for finite samples. It is worthwhile to compare the proposed method with the improper MI approach proposed by Wang and Robins (1998), Robins and Wang (2000). The idea of improper MI is to use Monte Carlo imputation as a tool to compute the maximum likelihood estimator and therefore, it requires the imputation size \( m \) to be large in order to reduce the Monte Carlo error. In contrast, our proposed method allows the imputation size \( m \) to be fixed at a small value. This property is appealing for releasing multiply imputed datasets for public usage. Moreover, improper MI can only deal with regular estimators but not non-regular estimators such as the matching estimators. The proposed method can be applied to a wide range of the ACE estimators adopted in MI, including the outcome regression, weighting and matching estimators. Indeed, the simulation studies indicate that Rubin’s variance estimator overestimates the variance for the IPW and matching estimators because these two estimators are not self-efficient (Meng, 1994, Xie and Meng, 2017), while the proposed variance estimation procedure is consistent for all types of estimators.

Importantly, our framework can easily accommodate the scenarios when both outcome and confounders have missing values and when the missing data are missing not at random. In the former case we only need to add the imputation step for the missing outcomes. In the latter case, we only need to modify the imputation model by further considering the missing data probability model in the data likelihood function. Our research is likely to bridge the advantages of MI and its wide applications in causal inference.

The rest of the paper is organized as follows. Section 2 introduces the background
information and basic setup. Section 3 presents the martingale representation for the MI ACE estimators and the wild bootstrap inference procedure and establishes its validity. Section 4 extends the proposed method to the scenario with other causal estimands, the scenario where both outcome and the confounders have missing values and the scenario where the confounders are missing not at random. In Section 5, we evaluate the finite sample performance of the proposed method using simulation studies. In section 6, we apply the proposed wild bootstrap inference method to a U.S National Health and Nutrition Examination Survey data. Section 7 concludes.

2 Background and setup

2.1 Potential outcomes framework

Following Neyman (1923) and Rubin (1974), we use the potential outcomes framework to formulate the causal problem. Denote $X$ to be a vector of $p$-dimensional confounders. Suppose that the treatment is a binary variable $A \in \{0, 1\}$, with 0 and 1 being the labels for control and active treatments, respectively. Under the Stable Unit Treatment Value assumption (Rubin, 1980), for each level of treatment $a$, there exists a potential outcome $Y(a)$, representing the outcome had the unit, possibly contrary to the fact, been given treatment $a$. We make the causal consistency assumption that links the observed outcome with the potential outcomes; i.e., the observed outcome $Y$ is the potential outcome $Y(A)$ under the actual treatment.

We focus on the average causal effect (ACE) $\tau = E\{Y(1) - Y(0)\}$. Our methodology applies to a broader class of causal estimands in Li et al. (2018); we discuss the extension to other causal estimands in Section 4.1. The fundamental problem in estimating the ACE is that for each unit, we observe at most one of the two potential outcomes $Y(0)$ and $Y(1)$.

Throughout we make the following assumptions that are common in the causal inference literature.

**Assumption 1 (Ignorability)** $\{Y(0), Y(1)\} \perp \perp A \mid X$.

**Assumption 2 (Overlap)** There exist constants $c_1$ and $c_2$ such that $0 < c_1 \leq e(X) \leq c_2 < 1$ almost surely, where $e(X) = P(A = 1 \mid X)$ is called the propensity score.
2.2 Common estimators for the ACE

It is well known that under Assumptions 1 and 2, the ACE can be identified and estimated through many different approaches including outcome regression, augmented/inverse probability weighting (AIPW/IPW), or matching. See Imbens (2004) and Rosenbaum (2002) for surveys of these estimators.

Define $\mu_a(X) = E\{Y(a) \mid X\}$. Under Assumption 1, $\mu_a(X) = E(Y \mid A = a, X)$. In practice, the outcome distribution and the propensity score are often unknown and therefore have to be modeled and estimated.

**Assumption 3 (Outcome model)** The parametric model $\mu_a(X; \beta_a)$ is a correct specification for $\mu_a(X)$, for $a = 0, 1$; i.e., $\mu_a(X) = \mu_a(X; \beta^*_a)$, where $\beta^*_a$ is the true model parameter.

**Assumption 4 (Propensity score model)** The parametric model $e(X; \alpha)$ is a correct specification for $e(X)$; i.e., $e(X) = e(X; \alpha^*)$, where $\alpha^*$ is the true model parameter.

Under Assumption 3 let $\hat{\beta}_a$ be a consistent estimator of $\beta^*_a$ for $a = 0, 1$. Under Assumption 4 let $\hat{\alpha}$ be a consistent estimator of $\alpha^*$.

We now review common estimators for $\tau$. The outcome regression estimator of $\tau$ is

$$\hat{\tau}_{n,\text{reg}} = \frac{1}{n} \sum_{i=1}^{n} \left\{ \mu_1(X_i; \hat{\beta}_1) - \mu_0(X_i; \hat{\beta}_0) \right\}. \quad (1)$$

The IPW estimator is

$$\hat{\tau}_{n,\text{IPW}} = \frac{1}{n} \sum_{i=1}^{n} \left\{ \frac{A_i Y_i}{e(X_i; \hat{\alpha})} - \frac{(1 - A_i)Y_i}{1 - e(X_i; \hat{\alpha})} \right\}. \quad (2)$$

The AIPW estimator is

$$\hat{\tau}_{n,\text{AIPW}} = \frac{1}{n} \sum_{i=1}^{n} \left\{ \frac{A_i Y_i}{e(X_i; \hat{\alpha})} + \left( 1 - \frac{A_i}{e(X_i; \hat{\alpha})} \right) \mu_1(X_i; \hat{\beta}_1) - \frac{(1 - A_i) Y_i}{1 - e(X_i; \hat{\alpha})} - \left( 1 - \frac{1 - A_i}{1 - e(X_i; \hat{\alpha})} \right) \mu_0(X_i; \hat{\beta}_0) \right\}. \quad (3)$$

The AIPW estimator is doubly robust, in the sense that it is consistent if either Assumption 3 or 4 holds, and it is locally efficient if both Assumptions hold (Rotnitzky and Vansteelandt, 2015).
Another commonly used estimator for $\tau$ is the matching estimator. To fix ideas, we consider the case of matching with replacement and the number of matches to be fixed as $M$ ($M \geq 1$). Throughout, we use the Euclidean distance for matching, although our discussion applies to other distance measures. Let $J_X(i)$ be the index set of the nearest $M$ neighbors for unit $i$ in its opposite treatment group based on the matching variable $X$. For unit $i$, the potential outcome $Y_i(A_i)$ is observed, and the potential outcome $Y_i(1 - A_i)$ is approximated by the average of the observed outcomes from the $M$ matched units in the opposite treatment group; i.e.,

$$
\hat{Y}_i(1) = \begin{cases} 
\frac{1}{M} \sum_{j \in J_X(i)} Y_j & \text{if } A_i = 0, \\
Y_i & \text{if } A_i = 1,
\end{cases} \quad \hat{Y}_i(0) = \begin{cases} 
Y_i & \text{if } A_i = 0, \\
\frac{1}{M} \sum_{j \in J_X(i)} Y_j & \text{if } A_i = 1.
\end{cases}
$$

The matching estimator of $\tau$ is

$$\hat{\tau}_{n,\text{mat}} = \frac{1}{n} \sum_{i=1}^{n} \{ \hat{Y}_i(1) - \hat{Y}_i(0) \}. \quad (4)$$

### 2.3 Asymptotically linear characterizations of the ACE estimators

To establish a unified framework, it is important to note that the above estimators are asymptotically linear. Let $\hat{\tau}_n$ denote a generic estimator of $\tau$. Under mild regularity conditions,

$$\hat{\tau}_n = \frac{1}{n} \sum_{i=1}^{n} \psi(A_i, X_i, Y_i) + o_p(1). \quad (5)$$

Under Assumption 3 for the outcome model, let

$$S_a(A, X, Y; \beta_a) = \frac{\partial \mu_a(X; \beta_a)}{\partial \beta_a} \{ Y - \mu_a(X; \beta_a) \}$$

be the estimating function for $\beta_a^*$ for $a = 0, 1$. Under Assumption 4 for the propensity score model, let

$$S(A, X; \alpha) = \frac{A - e(X; \alpha)}{e(X; \alpha)\{1 - e(X; \alpha)\}} \frac{\partial e(X; \alpha)}{\partial \alpha}$$

be the score function for $\alpha^*$, and let

$$\Sigma_\alpha = \mathbb{E} \left[ \frac{1}{e(X; \alpha)\{1 - e(X; \alpha)\}} \left\{ \frac{\partial e(X; \alpha)}{\partial \alpha} \right\}^2 \right]$$

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be the Fisher information matrix for $\alpha$.

Under Assumption 4, the asymptotically linear form of $\hat{\tau}_S$ where $\hat{e}_i = e(X_i; \alpha)$, $\hat{e}_i = \partial e(X_i; \alpha)/\partial \alpha^T$, $S_i^* = S(A_i, X_i; \alpha)$, $\mu_{ai} = \mu_a(X_i; \beta_a^*)$, $\hat{\mu}_{ai} = \partial \mu_a(X_i; \beta_a^*)/\partial \beta_a^T$, $S_{ai}^* = S_a(A_i, X_i, Y_i; \beta_a^*)$, $\hat{S}_{ai}^* = \partial S_a(A_i, X_i, Y_i; \beta_a^*)/\partial \beta_a^T$ for $a = 0, 1$.

Under Assumption 3 or Assumption 4, the asymptotically linear form of $\hat{\tau}_S$ where $\hat{e}_i = e(X_i; \alpha)$, $\hat{e}_i = \partial e(X_i; \alpha)/\partial \alpha^T$, $S_i^* = S(A_i, X_i; \alpha)$, $\mu_{ai} = \mu_a(X_i; \beta_a^*)$, $\hat{\mu}_{ai} = \partial \mu_a(X_i; \beta_a^*)/\partial \beta_a^T$, $S_{ai}^* = S_a(A_i, X_i, Y_i; \beta_a^*)$, $\hat{S}_{ai}^* = \partial S_a(A_i, X_i, Y_i; \beta_a^*)/\partial \beta_a^T$ for $a = 0, 1$.

Under Assumption 4, the asymptotically linear form of $\hat{\tau}_{n, \text{reg}}$ is

$$\psi_{\text{reg}}(A_i, X_i, Y_i) = \mu_{1i}^* - \mu_{0i} - E(\hat{\mu}_1) \left\{ E(\hat{S}_1^*) \right\}^{-1} S_{1i}^* + E(\hat{\mu}_0) \left\{ E(\hat{S}_0^*) \right\}^{-1} S_{0i}^*.$$

Under Assumption 4, the asymptotically linear form of $\hat{\tau}_{n, \text{IPW}}$ is

$$\psi_{\text{IPW}}(A_i, X_i, Y_i) = \frac{A_i Y_i}{e_i^*} - \frac{(1 - A_i) Y_i}{1 - e_i^*} - E \left\{ \frac{A Y}{(e^*)^2} + \frac{(1 - A) Y}{(1 - e^*)^2} \right\} e^* \left( \mu_i - \frac{(1 - A_i) Y_i}{1 - e_i^*} \right) \mu_{0i}^* + E \left\{ \frac{A Y - \mu_i^*}{(e^*)^2} + \frac{(1 - A) Y - \mu_0^*}{(1 - e^*)^2} \right\} e^* \left( \mu_i - \frac{(1 - A_i) Y_i}{1 - e_i^*} \right) \mu_{0i}^* - E \left\{ \frac{(1 - A) Y - \mu_i^*}{(1 - e^*)^2} \right\} e^* \left( \mu_i - \frac{(1 - A_i) Y_i}{1 - e_i^*} \right) \mu_{0i}^* \left\{ E(\hat{S}_1) \right\}^{-1} S_{1i}^* + E \left\{ \frac{1 - A}{1 - e^*} \right\} \mu_0^* \left\{ E(\hat{S}_0) \right\}^{-1} S_{0i}^*.$$

Under Assumption 3 or Assumption 4, the asymptotically linear form of $\hat{\tau}_{n, \text{AIPW}}$ is

$$\psi_{\text{AIPW}}(A_i, X_i, Y_i) = \frac{A_i Y_i}{e_i^*} + \left( 1 - A_i \right) Y_i - E \left\{ \frac{A Y}{(e^*)^2} + \frac{(1 - A) Y}{(1 - e^*)^2} \right\} e^* \left( \mu_i - \frac{(1 - A_i) Y_i}{1 - e_i^*} \right) \mu_{0i}^* + E \left\{ \frac{A Y - \mu_i^*}{(e^*)^2} + \frac{(1 - A) Y - \mu_0^*}{(1 - e^*)^2} \right\} e^* \left( \mu_i - \frac{(1 - A_i) Y_i}{1 - e_i^*} \right) \mu_{0i}^* - E \left\{ \frac{(1 - A) Y - \mu_i^*}{(1 - e^*)^2} \right\} e^* \left( \mu_i - \frac{(1 - A_i) Y_i}{1 - e_i^*} \right) \mu_{0i}^* \left\{ E(\hat{S}_1) \right\}^{-1} S_{1i}^* + E \left\{ \frac{1 - A}{1 - e^*} \right\} \mu_0^* \left\{ E(\hat{S}_0) \right\}^{-1} S_{0i}^*.$$

Following Abadie and Imbens (2006), the asymptotically linear form of $\hat{\tau}_{n, \text{mat}}$ is

$$\psi_{\text{mat}}(A_i, X_i, Y_i) = (2A_i - 1) \left[ \{ Y_i - \mu_{1-A_i}(X_i) \} + M^{-1} K_M(i) \{ Y_i - \mu_{A_i}(X_i) \} \right],$$

where $K_M(i) = \sum_{l=1}^n \mathbb{I}\{ i \in J_X(l) \}$ is the number of times that unit $i$ is used as a match, where $\mathbb{I}(\cdot)$ is an indicator function; i.e., for an event $\mathcal{E}$, $\mathbb{I}(\mathcal{E}) = 1$ if $\mathcal{E}$ is true and $\mathbb{I}(\mathcal{E}) = 0$ otherwise. Due to the non-smoothness of the matching estimator caused by the fixed number of matches in Abadie and Imbens (2006), the naive bootstrap, i.e., resampling the individuals and obtaining the replicates of the matching estimator by repeating the matching procedure on each bootstrap resamples, fails to provide asymptotically valid inference. Therefore, the naive nonparametric bootstrap can not be used to build a unified framework for causal inference after MI.
2.4 MI in the presence of missing confounders

We consider the case where \( X = (X_1, \ldots, X_p) \) contains missing values. Let \( R = (R_1, \ldots, R_p) \) be the vector of missing indicators such that \( R_j = 1 \) if the \( j \)-th component \( X_j \) is observed and 0 if it is missing. Also, let \( 1_p \) denote the \( p \)-vector of 1’s. We write \( X = (X_R, X_{\overline{R}}) \), where \( X_R \) and \( X_{\overline{R}} \) represent the observed and missing parts of \( X \), respectively. This notation depends on the missingness pattern; e.g., if \( R_1 = 1 \) and \( R_j = 0 \) for \( j = 2, \ldots, p \), then \( X_R = X_1 \) and \( X_{\overline{R}} = (X_2, \ldots, X_p) \). With missing values in \( X \), the aforementioned full sample estimators (1)–(4) are not feasible.

To facilitate applying full sample estimators, MI creates multiple complete data sets by filling in missing values. Let units be indexed by \( i = 1, \ldots, n \). Assume unit \( i \) has the complete data \( Z_i = (A_i, X_i, Y_i, R_i) \) and the observed data \( Z_{\text{obs},i} = (A_i, X_{R,i}, Y_i, R_i) \). Denote \( Z = (Z_1, \ldots, Z_n) \) and \( Z_{\text{obs}} = (Z_{\text{obs},1}, \ldots, Z_{\text{obs},n}) \). Assume that the observed data likelihood is \( f(Z_{\text{obs}}; \theta) \) with the true parameter value \( \theta_0 \). The MI procedure proceeds as follows.

**Step MI-1.** Create \( m \) complete data sets by filling in missing values with imputed values generated from the posterior predictive distribution. Specifically, to create the \( j \)-th imputed data set, first generate \( \theta^* (j) \) from the posterior distribution \( p(\theta | Z_{\text{obs}}) \), and then generate \( X^*_{R,i} \) from \( f(X_{\overline{R},i} | A_i, X_{R,i}, Y_i, R_i; \theta^*(j)) \) for each missing \( X_{\overline{R},i} \).

**Step MI-2.** Apply a full sample estimator of \( \tau \) to each imputed data set. Let \( \hat{\tau} (j) \) be the estimator applied to the \( j \)-th imputed data set, and \( \hat{V} (j) \) be the full sample variance estimator for \( \hat{\tau} (j) \).

**Step MI-3.** Use Rubin’s combining rule to summarize the results from the multiple imputed data sets. The MI estimator of \( \tau \) is \( \hat{\tau}_\text{MI} = m^{-1} \sum_{j=1}^m \hat{\tau} (j) \), and Rubin’s variance estimator is

\[
\hat{V}_\text{MI}(\hat{\tau}_\text{MI}) = W_m + (1 + m^{-1}) B_m, \tag{6}
\]

where \( W_m = m^{-1} \sum_{j=1}^m \hat{V} (j) \) and \( B_m = (m - 1)^{-1} \sum_{j=1}^m (\hat{\tau} (j) - \hat{\tau}_\text{MI})^2 \).

We first elucidate our method by assuming the confounders are missing at random (MAR) in the sense of Rubin (1976).

**Assumption 5 (Missingness at random)** We have \( X_{\overline{R}} \indep R | Z_{\text{obs}} \).
Under Assumption 5, \( f(A_i, X_i, Y_i, R_i; \theta) = f(A_i, X_{R_i,i}, Y_i, R_i; \theta) f(X_{R_i,i} | A_i, X_{R_i,i}, Y_i, R_i = 1_p; \theta) \) is identifiable, which justifies the likelihood-based or Bayesian inference. Moreover, the posterior distribution of the missing data can be decomposed to

\[
f(X_{R_i,i} | A_i, X_{R_i,i}, Y_i, R_i; \theta^{(j)}) \propto f(Y_i | X_{R_i,i}, X_{R_i,i}, A_i; \theta^{(j)})
\times f(A_i | X_{R_i,i}, X_{R_i,i}; \theta^{(j)}) f(X_{R_i,i} | X_{R_i,i}; \theta^{(j)}),
\]

which does not depend on the missingness pattern probability for \( R_i \).

The variance of the MI estimator can be decomposed to

\[
\text{var}(\hat{\tau}_{\text{MI}}) = \text{var}(\hat{\tau}_n) + \text{var}(\hat{\tau}_{\text{MI}} - \hat{\tau}_n) + 2\text{cov}(\hat{\tau}_{\text{MI}} - \hat{\tau}_n, \hat{\tau}_n),
\]

In Rubin’s variance estimator (6), \( W_m \) estimates the within-imputation variance \( \text{var}(\hat{\tau}_n) \), and \( (1 + m^{-1})B_m \) estimates the between-imputation variance \( \text{var}(\hat{\tau}_{\text{MI}} - \hat{\tau}_n) \). However, it ignores the covariance between \( \hat{\tau}_{\text{MI}} - \hat{\tau}_n \) and \( \hat{\tau}_n \). Rubin’s variance estimator is asymptotically unbiased only under the congeniality condition (Meng, 1994), i.e., \( \text{cov}(\hat{\tau}_{\text{MI}} - \hat{\tau}_n, \hat{\tau}_n) = o(1) \). Therefore, Rubin’s variance estimator using the different full sample estimator \( \hat{\tau}_n \) may be inconsistent.

For illustration, we conduct a numerical experiment to assess the congeniality condition for the outcome regression, IPW, AIPW and matching estimators of the ACE. The data generating mechanism is described in scenario (a) in Section 5. For each simulated data set, we compute the full sample point estimators \( \hat{\tau}_n \) assuming the confounders are fully observed and the multiple imputation point estimators \( \hat{\tau}_{\text{MI}} \). Table 1 presents the simulations results of the variances of the full sample point estimators and the MI point estimators and the covariance between \( \hat{\tau}_{\text{MI}} - \hat{\tau}_n \) and \( \hat{\tau}_n \). The covariance is significantly negative for the IPW estimator and the matching estimator. Rubin’s variance estimator overestimates the variances of the IPW estimator and matching estimator. As consequence, MI is not congenial for the IPW and matching estimators. Thus, the congeniality condition required for MI can be quite restrictive for general ACE estimation.
Table 1: Simulation results of the full sample point estimators and MI point estimators based on 5,000 simulated data sets

| Method     | $\hat{\tau}_n$ | var($\hat{\tau}_n$) | var($\hat{\tau}_{MI}$) | var($\hat{\tau}_{MI} - \hat{\tau}_n$) | cov($\hat{\tau}_{MI} - \hat{\tau}_n$, $\hat{\tau}_n$) |
|------------|-----------------|----------------------|------------------------|----------------------------------------|--------------------------------------------------|
| Regression | 33              | 51                   | 18                     | 0                                      | 0                                                |
| IPW        | 110             | 115                  | 33                     | -14                                    | -13                                              |
| AIPW       | 35              | 53                   | 20                     | -7                                     | -2                                               |
| matching   | 43              | 55                   | 26                     | -7                                     | -2                                               |

3 A Martingale Representation of the MI estimators of causal effects

3.1 A novel martingale representation

Based on the unified linear form of the full sample estimator as in (5), we will express the MI estimator in a general form as

$$\hat{\tau}_{MI} - \tau = \frac{1}{m} \sum_{j=1}^{m} (\hat{\tau}(j) - \tau) = \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} \left\{ \psi(A_i, X_{i}^{*}(j), Y_i) - \tau \right\} + o_p(1),$$

where $X_{i}^{*}(j) = (X_{R_i,i}, X_{R_i,i}^{*})$ and $o_p(1)$ is due to (5).

To express (7) further, it is important to understand the properties of the posterior distribution and the imputed values $X_{i}^{*}(j)$. Using the Bernstein-von Mises theorem [van der Vaart, 2000, Chapter 10], under certain regularity conditions, conditioned on the observed data, the posterior distribution $p(\theta | Z_{obs})$ converges to a normal distribution with mean $\hat{\theta}$ and variance $n^{-1} I^{-1}_{\text{obs}}$ almost surely, where $\hat{\theta}$ is the maximum likelihood estimator (MLE) of $\theta_0$ and $I^{-1}_{\text{obs}}$ is the inverse of the Fisher information matrix. Let $S(\theta; A, X, Y, R)$ be the score function of $\theta$. In the presence of missing data, define the mean score function $\bar{S}(\theta_0; Z_{obs,i}) = E\{S(\theta_0; A_i, X_i, Y_i, R_i) | Z_{obs,i}, \theta_0\}$. The MLE $\hat{\theta}$ can be viewed as the solution to the mean score equation $\sum_{i=1}^{n} \bar{S}(\hat{\theta}; Z_{obs,i}) = 0$. Under certain regularity conditions, we
can then express $\hat{\theta} - \theta_0 = n^{-1}T^{-1}_{\text{obs}} \sum_{i=1}^n \tilde{S}(\theta_0; Z_{\text{obs},i}) + o_p(1)$. It is insightful to write (7) as

$$\hat{\tau}_{\text{MI}} - \tau = \frac{1}{nm} \sum_{i=1}^n \sum_{j=1}^m \left[ \psi(A_i, X_i^{(j)}, Y_i) - E\{\psi(A_i, X_i, Y_i) \mid Z_{\text{obs},i}, \hat{\theta} \} \right]$$

$$+ \frac{1}{nm} \sum_{i=1}^n \sum_{j=1}^m \left[ E\{\psi(A_i, X_i, Y_i) \mid Z_{\text{obs},i}, \theta_0 \} - \tau + \Gamma T^{-1}_{\text{obs}} S(\theta_0; Z_{\text{obs},i}) \right] + o_p(1).$$

Now, by a Taylor expansion of $E\{\psi(A_i, X_i, Y_i) \mid Z_{\text{obs},i}, \hat{\theta} \}$ around the true value $\theta_0$,

$$\hat{\tau}_{\text{MI}} - \tau = \frac{1}{nm} \sum_{i=1}^n \sum_{j=1}^m \left[ \psi(A_i, X_i^{(j)}, Y_i) - E\{\psi(A_i, X_i, Y_i) \mid Z_{\text{obs},i}, \hat{\theta} \} \right]$$

$$+ \frac{1}{nm} \sum_{i=1}^n \sum_{j=1}^m \left[ E\{\psi(A_i, X_i, Y_i) \mid Z_{\text{obs},i}, \theta_0 \} - \tau + \Gamma T^{-1}_{\text{obs}} S(\theta_0; Z_{\text{obs},i}) \right] + o_p(1),$$

where

$$\Gamma = E\left[E\{\psi(A_i, X_i, Y_i) S(\theta_0; A_i, X_i, Y_i) \mid Z_{\text{obs},i}, \theta_0 \} - E\{\psi(A_i, X_i, Y_i) \mid Z_{\text{obs},i}, \theta_0 \} S(\theta_0; Z_{\text{obs},i})\right]^T.$$

Based on (9), we can write

$$n^{1/2}(\hat{\tau}_{\text{MI}} - \tau) = \sum_{k=1}^{n+nm} \xi_{n,k} + o_p(1),$$

where

$$\xi_{n,k} = \begin{cases} \frac{1}{n^{1/2}} \left[ E\{\psi(A_i, X_i, Y_i) \mid Z_{\text{obs},i}, \theta_0 \} - \tau + \Gamma T^{-1}_{\text{obs}} S(\theta_0; Z_{\text{obs},i}) \right], & \text{if } k = i, 1 \leq i \leq n, \\
\frac{1}{n^{1/2}} \left[ \psi(A_i, X_i^{(j)}, Y_i) - E\{\psi(A_i, X_i, Y_i) \mid Z_{\text{obs},i}, \hat{\theta} \} \right], & \text{if } k = n + (i - 1)m + j. \end{cases}$$

Based on the decomposition in (10), the first $n$ terms of $\xi_{n,k}$ contribute to the variability of $\hat{\tau}_{\text{MI}}$ because of the unknown parameters, and the rest $nm$ terms of $\xi_{n,k}$ contribute to the variability of $\hat{\tau}_{\text{MI}}$ because of the imputations given the parameter values, reflecting the sequential MI procedure.

Consider the $\sigma$-fields

$$\mathcal{F}_{n,k} = \begin{cases} \sigma\{Z_{\text{obs},1}, \ldots, Z_{\text{obs},k}\}, & \text{if } 1 \leq k \leq n, \\
\sigma\{Z_{\text{obs},1}, \ldots, Z_{\text{obs},n}, X^{(1)}_1, \ldots, X^{(j)}_i\}, & \text{if } k = n + (i - 1)m + j. \end{cases}$$
Obviously, $E(\xi_1) = 0$ and $E(\xi_{n,k} \mid Z_{\text{obs}},1,\ldots,Z_{\text{obs}},k-1) = E(\xi_{n,k}) = 0$ for $1 < k \leq n$. Using the Bernstein-von Mises theorem (van der Vaart, 2000, Chapter 10), under certain regularity conditions, $E(\xi_{n,k} \mid F_{n,k-1}) = 0$ for $k = n + (i-1)m + j$, where $i = 1, \ldots, n$, and $j = 1, \ldots, m$. Therefore,

$$ \left\{ \sum_{i=1}^{k} \xi_{n,i} F_{n,k}, 1 \leq k \leq n(1+m) \right\} $$

is a martingale for each $n \geq 1$.

### 3.2 Wild bootstrap for the MI estimators of the ACE

Based on the martingale representation, we propose the wild bootstrap procedure to estimate the variance of $\hat{\tau}_{\text{MI}}$.

**Step 1.** Sample $u_k$, for $k = 1, \ldots, n+nm$, to satisfy that $E(u_k \mid Z_{\text{obs}}) = 0$, $E(u_k^2 \mid Z_{\text{obs}}) = 1$ and $E(u_k^4 \mid Z_{\text{obs}}) < \infty$.

**Step 2.** Compute the bootstrap replicate as $T^* = n^{-1/2} \sum_{k=1}^{n+nm} \hat{\xi}_{n,k} u_k$, where

$$ \hat{\xi}_{n,k} = \begin{cases} \frac{1}{n^{1/2}} \left[ E\{\psi(A_i, X_i, Y_i) \mid Z_{\text{obs},i}, \hat{\theta}\} - \hat{\tau} + \hat{\Gamma}_{\text{obs}}^{-1} S(\hat{\theta}; Z_{\text{obs},i}) \right], & \text{if } k = i, 1 \leq i \leq n, \\ \frac{1}{n^{1/2}m} \left[ \psi(A_i, X_{i}^{*(j)}, Y_i) - E\{\psi(A_i, X_i, Y_i) \mid Z_{\text{obs},i}, \hat{\theta}\} \right], & \text{if } k = n + (i-1)m + j. \end{cases} $$

**Step 3.** Repeat Step 1–Step 2 $B$ times, and estimate the variance of $\hat{\tau}_{\text{MI}}$ by the sample variance of the $B$ copies of $T^*$.

**Remark 1** There are many choices for generating $u_k$, such as the the standard normal distribution, Mammen’s (1993) two point distribution

$$ \mu_k = \begin{cases} \frac{1-5^{1/2}}{2}, & \text{with probability } \frac{1+5^{-1/2}}{2}, \\ \frac{5^{1/2}+1}{2}, & \text{with probability } \frac{1-5^{-1/2}}{2}, \end{cases} $$

or a simpler distribution with probability 0.5 of being 1 and probability 0.5 of being −1. The wild bootstrap procedure is not sensitive to the choice of the sampling distribution of $\mu_k$. In particular, one can also use the nonparametric bootstrap weights; that is, let $u_k = (nm + n)^{-1/2}(W_k - \overline{W})$, where $\{W_k : k = 1, \ldots, n(m+1)\}$ follows a multinomial distribution with $n(m+1)$ draws on $n(m+1)$ cells with equal probability, and $\overline{W} = (nm + n)^{-1} \sum_{k=1}^{n(m+1)} W_k$.

Several authors have used the nonparametric bootstrap to estimate the variance of the MI estimators. Schomaker and Heumann (2018) combined MI with bootstrap to do inference
for the quantity of interest. However, their discussions restrict to the maximum likelihood estimators of model parameters and require bootstrap on top of MI, which is computationally intensive. Moreover, in the causal inference literature in absence of missing data, Abadie and Imbens (2008) has demonstrated that nonparametric bootstrap can not provide consistent variance estimation for the matching estimators of the ACE due to the non-smooth nature of the matching procedure. It is important to note that the proposed wild bootstrap procedure with the nonparametric bootstrap weights is different from the naive bootstrap. The martingale representation and the wild bootstrap procedure work for the asymptotically linear ACE estimators including the matching estimator.

We show the asymptotic validity of the above bootstrap inference method by the following theorem.

**Theorem 1** Under Assumptions 1, 2, and Assumptions ??-?? in the supplementary material,

\[
\sup_r \left| \Pr(\frac{1}{2}T^* \leq r|Z_{obs}) - \Pr\left(\frac{1}{2}(\hat{\tau}_{MI} - \tau) \leq r\right) \right| \xrightarrow{p} 0,
\]

as \( n \to \infty \).

We provide the proof of Theorem 1 in the supplementary material, which draws on the martingale central limit theory (Hall and Heyde, 1980) and the asymptotic property of weighted sampling of martingale difference arrays (Pauly et al., 2011). Theorem 1 indicates that the distribution of the wild bootstrap statistic consistently estimates the distribution of the MI estimator.

### 4 Extension

#### 4.1 Different causal estimands

Our inference framework extends to a wide class of causal estimands, as long as the estimand admits an asymptotically linear full sample estimator as in (3). For example, we can consider the average causal effects over a subset of population (Crump et al., 2006, Li et al., 2018), including the average causal effect on the treated. We can also consider nonlinear causal estimands. For example, for a binary outcome, the log of the causal risk ratio is

\[
\log \text{CRR} = \log \frac{P(Y(1) = 1)}{P(Y(0) = 1)} = \log \frac{E\{Y(1)\}}{E\{Y(0)\}},
\]
and the log of the causal odds ratio is
\[
\log \text{COR} = \log \frac{P\{Y(1) = 1\}/P\{Y(1) = 0\}}{P\{Y(0) = 1\}/P\{Y(0) = 0\}} = \log \frac{E\{Y(1)\}/[1 - E\{Y(1)\}]}{E\{Y(0)\}/[1 - E\{Y(0)\}]].
\]

The key insight is that under Assumptions 1 and 2 we can estimate \(E\{Y(a)\}\) with commonly-used estimators, denoted by \(\hat{E}\{Y(a)\}\), for \(a = 0, 1\). We can then obtain an estimator for the log CRR as \(\log[\hat{E}\{Y(1)\}/\hat{E}\{Y(0)\}]\). By the Taylor expansion, we can linearize these estimators and establish a similar linear form as (5), which serves as the basis to construct the weighted bootstrap inference.

### 4.2 Missingness not at random

If Assumption 5 fails, the missing pattern also depends on the missing values themselves even after controlling for the observed data, a scenario known as missing not at random (MNAR). In our motivating example discussed in Section 6, the family poverty ratio is likely to be missing not at random because subjects with higher income may be less likely to disclose their income information (Davern et al., 2005). In general, MNAR occurs frequently for sensitive questions regarding e.g. alcohol consumption, income, etc.

Causal inference with data missing not at random is more challenging because the full data distribution and therefore the ACE are not identifiable in general. To utilize MI in causal inference with confounders MNAR, we require identification conditions that ensure the full data distribution is identifiable. For example, Wang et al. (2014) introduced a nonresponse instrument as a sufficient condition for the identifiability of the observed likelihood. Miao et al. (2016) investigated identifiability of normal and normal mixture models with nonignorable missing data. Yang et al. (2019) proposed an outcome-independence missingness mechanism under which the missing data mechanism is independent of the outcome given the treatment and confounders and establish general identification conditions.

Our proposed method can easily extend to the scenario where the confounders are MNAR when additional assumptions are made for identifiability of the full data distribution. After the identification check, we only need to modify the posterior predictive distribution of \(X_{R,i}^{(j)}\). For example, following Yang et al. (2019), we assume that the missingness pattern \(R\) is independent of the outcome given the treatment and confounders.

**Assumption 6 (Outcome-independent missingness)** We have \(Y \perp \perp R \mid (A, X_R, X_{\bar{R}})\).

Under certain regularity conditions, the full data distribution \(f(A, X, Y, R)\) is identifiable (Yang et al., 2019). Then in Step MI-1, the posterior distribution of \(X_{R,i}^{(j)}\) can
be decomposed to \( f(X_{R_i} \mid A_i, X_{R_i}, Y_i, R_i; \theta^*(j)) \propto f(Y_i \mid X_{R_i}, X_{R_i}, A_i; \theta^*(j)) f(R_i \mid X_{R_i}, X_{R_i}, A_i; \theta^*(j)) f(A_i \mid X_{R_i}, X_{R_i}, Y_i; \theta^*(j)) f(R_i \mid X_{R_i}, A_i; \theta^*(j)) \).

After imputation, the wild bootstrap steps remain exactly the same.

4.3 Partially observed outcome and confounders

In some cases, both the outcome and the confounders are subject to missingness. Our framework can easily accommodate this scenario by adding an outcome imputation step in the MI procedure.

We now introduce another missingness indicator \( R_Y \) for \( Y \); i.e., \( R_Y = 1 \) if \( Y \) is observed and \( R_Y = 0 \) otherwise. In Step MI-1, we first generate \( \theta^*(j) \) from the posterior distribution \( p(\theta \mid Z_{\text{obs}}) \). Then for unit \( i \) with \( R_Y = 1 \), generate \( X^*_i \) from \( f(X_{R_i} \mid A_i, X_{R_i}, Y_i, R_i; \hat{\theta}) \), for unit \( i \) with \( R_Y = 0 \), generate \( X^*_i \) and \( Y^*_i \) from \( f(X_{R_i}, Y_i \mid A_i, X_{R_i}, R_i, R_Y = 0; \hat{\theta}^*(j)) \) to create the \( j \)th the imputed data set. Then the MI estimator can be written in a general form with both imputed outcome and confounders as

\[
\hat{\tau}_{\text{MI}} - \tau = \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} \left\{ \psi(A_i, X^*_i, Y^*_i) - \tau \right\} + o_p(1).
\]

Accordingly, the martingale difference arrays in the wild bootstrap procedure can be written as

\[
\hat{\xi}_{n,k} = \begin{cases} 
\frac{1}{n^{1/2}} \left[ E\left\{ \psi(A_i, X_i, Y_i) \mid Z_{\text{obs}}, \hat{\theta} \right\} - \hat{\tau} + \hat{\tau} \hat{Z}_{\text{obs}}^{-1} \hat{S}(\hat{\theta} ; Z_{\text{obs}}) \right], & \text{if } 1 \leq k = i \leq n, \\
\frac{1}{n^{1/2}m} \left[ \psi(A_i, X^*_i, Y^*_i) - E\left\{ \psi(A_i, X_i, Y_i) \mid Z_{\text{obs}}, \hat{\theta} \right\} \right], & \text{if } k = n + (i - 1)m + j.
\end{cases}
\]

Other steps in the MI and wild bootstrap procedures remain the same as described for the scenario when only confounders have missing values.

5 Simulation

We conduct simulation studies to evaluate the finite sample performance of the proposed inference when MI adopts different full sample estimators including the outcome regression, IPW, AIPW and matching estimators.

For each sample, the confounder \( X = (X_{[1]}, X_{[2]}) \) are sampled from a multivariate normal distribution with mean \((0, 0)\), variance \((1, 1)\) and a correlation coefficient 0.3. The potential
outcomes follow \( Y(0) = 1 + 3X_{[1]} + 1.5X_{[2]} + \epsilon(0) \) and \( Y(1) = 2X_{[1]} + X_{[2]} + \epsilon(1) \), where \( \epsilon(0) \sim \mathcal{N}(0, \sigma^2_0) \), \( \epsilon(1) \sim \mathcal{N}(0, \sigma^2_1) \) with \( \sigma_0 = \sigma_1 = 1 \), and \( \epsilon(0) \) and \( \epsilon(1) \) are independent. So the true value of ACE is \( \tau = -1 \). We generate the treatment indicator \( A \) from Bernoulli\( \{ \pi_A(X) \} \) and \( \pi_A(X) = P(A = 1 \mid X) = \Phi(-0.2 + 0.1A + 0.6X_{[1]} + 0.3X_{[2]}) \), where \( \Phi(\cdot) \) is the cumulative density function for the standard normal distribution. In the sample, we assume \( A \) and \( X_{[1]} \) are fully observed, but \( X_{[2]} \) and \( Y \) can be partially observed with the missing indicators \( R_{[2]} \) and \( R_Y \), respectively. We consider four scenarios:

(a) \( X_{[2]} \) is missing at random; i.e., its missingness depends only on the observed data. Let \( R_{[2]} \sim \text{Bernoulli}\{ \pi_{R}(A, X_{[1]}, Y) \} \), where \( \pi_{R}(A, X_{[1]}, Y) = \Phi(-0.2 + 0.1A + 0.6X_{[1]} + 0.3Y) \) with the missing rate being about 50%. Moreover, the inference procedure assumes the correct missingness mechanism;

(b) \( X_{[2]} \) is missing not at random; i.e., its missingness depends on unobserved data. Let \( R_{[2]} \sim \text{Bernoulli}\{ \pi_{R}(X_{[2]}) \} \), where \( \pi_{R}(X_{[2]}) = \Phi(0.2 + 1X_{[2]}) \) with the missing rate being about 45%. Moreover, the inference procedure assumes the correct missingness mechanism;

(c) \( X_{[2]} \) is missing not at random as in scenario (b); but the inference procedure assumes an incorrect missingness at random mechanism;

(d) both \( X_{[2]} \) and \( Y \) are missing not at random, with the missingness indicators \( R_{[2]} \) and \( R_Y \), respectively. Let \( R_{[2]} \sim \text{Bernoulli}\{ \pi_{R}(X_{[2]}) \} \), where \( \pi_{R}(X_{[2]}) = \Phi(0.8 + 1X_{[2]}) \) with the missing rate being about 30%. Let \( R_Y \sim \text{Bernoulli}\{ \pi_{Y}(A, X) \} \), where \( \pi_{Y}(A, X) = \Phi(1 + 0.2A + 0.5X_{[1]} + 0.5X_{[2]}) \) with the missing rate being about 20%.

We generate 5,000 Monte Carlo samples with size \( n = 2000 \) for each scenario. In MI, the missing data mechanism is specified according to the above scenarios and other components of the distribution are correctly specified. We use non-informative priors for parameters. Suppose that the prior distribution for each coefficient in the outcome model, the propensity score model and the missing indicator model is \( \mathcal{N}(0, 100) \); the prior distribution for the variance parameters \( \sigma_0 \) and \( \sigma_1 \) in the outcome regression model is Gamma(0.01, 0.01); the prior distribution for the mean of \( X \) is \( (0, 0) \); the prior distribution for the variance covariance matrix of \( X \) is \( I_2 \), where \( I_2 \) is the 2-dimensional identity matrix. We consider three sizes of multiple imputation with \( m = 5, 10 \) or 100. To generate the posterior samples of the missing values \( X_{[R]}^{*j} \), we use Gibbs sampling with 5,000 iterations, discard first 2,000 burn-in samples, and randomly choose \( m \) posterior samples from the remaining 3,000 draws. For each imputed data set, we calculate the full sample
point estimators and variance estimators of ACE using outcome regression, IPW, AIPW and matching, and then use Rubin’s method to get the corresponding MI estimators $\hat{\tau}_{\text{MI}}$ and Rubin’s variance estimators $\hat{V}_{\text{MI}}$. For the matching estimator, we set the number of matches as $M = 1$.

We compare the standard MI inference and the proposed bootstrap inference. For the standard MI inference, the 100$(1 - \alpha)$% confidence intervals are calculated as $(\hat{\tau}_{\text{MI}} - t_{\nu,1-\alpha/2}\hat{V}^{1/2}_{\text{MI}}, \hat{\tau}_{\text{MI}} + t_{\nu,1-\alpha/2}\hat{V}^{1/2}_{\text{MI}})$, where $t_{\nu,1-\alpha/2}$ is the 100$(1 - \alpha/2)$% quantile of the $t$ distribution with degree of freedom $\nu = (m-1)\lambda^{-2}$ with $\lambda = (1 + m^{-1})B_m / \{W_m + (1 + m^{-1}B_m)\}$. For the proposed bootstrap procedure, we use $B = 1,000$, sample the weights $\mu_k$ from the Mammen’s two point distribution suggested in Remark 1 and calculate the variance estimate $\hat{V}_{\text{BS}}$. The corresponding 100$(1 - \alpha)$% confidence interval are estimated using two different methods: (i) quantile-based confidence interval $(\hat{\tau}_{\text{MI}} - q_{\alpha/2}^{\nu} \hat{V}^{1/2}_{\text{MI}}, \hat{\tau}_{\text{MI}} + q_{\alpha/2}^{\nu} \hat{V}^{1/2}_{\text{MI}})$, where $q_{\alpha/2}^{\nu}$ and $q_{\alpha^*}^{\nu}$ are the $(1 - \alpha/2)$th and $(\alpha/2)$th quantiles of $T^*$; (ii) the Wald-type confidence interval $(\hat{\tau}_{\text{MI}} - z_{1-\alpha/2} \hat{V}^{1/2}_{\text{BS}}, \hat{\tau}_{\text{MI}} + z_{1-\alpha/2} \hat{V}^{1/2}_{\text{BS}})$, where $z_{1-\alpha/2}$ is the $(1 - \alpha/2)$th quantile of the standard normal distribution.

We assess the performance in terms of the relative bias of the variance estimator and the coverage rate of confidence intervals. The relative bias of the variance estimators are calculated as \(\frac{E(\hat{V}_{\text{MI}}) - \text{var}(\hat{\tau}_{\text{MI}})}{\text{var}(\hat{\tau}_{\text{MI}})}\times100\%\) and \(\frac{E(\hat{V}_{\text{BS}}) - \text{var}(\hat{\tau}_{\text{MI}})}{\text{var}(\hat{\tau}_{\text{MI}})}\times100\%\) correspondingly. The coverage rate of the 100$(1 - \alpha)$% confidence intervals is estimated by the percentage of the Monte Carlo samples for which the confidence intervals contain the true value.

Tables 2–5 present the simulation results for the four scenarios. When the imputation model is correctly specified as in scenarios (a), (b) and (d), the MI point estimator has small biases for all full sample estimators. Rubin’s variance estimator is unbiased for the outcome regression estimator and the AIPW estimator; however, it overestimates the variances of the IPW estimator and the matching estimator e.g. by as high as 34.2% and 23.2% in scenario (a). Due to variance overestimation, the coverage rate of Rubin’s method is above nominal level for the IPW and Matching estimators. In contrast, our proposed wild bootstrap procedure for variance estimation is unbiased for all four ACE estimators, and therefore the coverage rate of the confidence intervals based on our proposed wild bootstrap method is close to the nominal level. Moreover, the proposed method is not sensitive to the number of imputations $m$. However, in scenario (c) when the true missing data mechanism is missingness not at random while the inference procedure assumes missingness at random, the MI point estimator has large biases and all the confidence intervals have poor coverage rates; see Table 4.
Table 2: Simulation results: point estimate, true variance, relative bias of the variance estimator, coverage and mean width of interval estimate using Rubin’s method and the proposed wild bootstrap method under scenario (a) with missingness at random

| Method | $\tau_n$ | $m$ | $\hat{\tau}_n$ ($\times10^2$) | Point est ($\times10^4$) | True var ($\times10^4$) | Relative Bias (%) | Coverage (%) | Mean width ($\times10^2$) for 95% CI | Mean width ($\times10^2$) for 95% CI |
|--------|----------|-----|-------------------------------|--------------------------|--------------------------|-------------------|-------------|--------------------------------------|--------------------------------------|
|        |          |     | Rubin BS | Rubin BS | Wald | Rubin BS | Wald |                                           |                                         |
| Regression | 5       | -100.1 | 52.9 | -1.9 | 1.7 | 94.7 | 95.1 | 95.2 | 29.4 | 28.6 | 28.7 |
|          | 10      | -100.1 | 51.4 | -2.1 | 3.4 | 94.7 | 95.4 | 95.4 | 28.2 | 28.5 | 28.6 |
|          | 100     | -100.1 | 49.7 | -1.8 | 5.7 | 94.9 | 95.6 | 95.6 | 27.4 | 28.3 | 28.4 |
| IPW     | 5       | -99.9 | 118.7 | 29.4 | -0.1 | 95.9 | 94.3 | 94.4 | 45.8 | 39.8 | 40.3 |
|          | 10      | -99.9 | 115.2 | 34.2 | 1.8 | 96.0 | 94.4 | 94.5 | 44.9 | 39.4 | 40.0 |
|          | 100     | -99.9 | 111.8 | 32.6 | 2.1 | 96.3 | 94.4 | 94.8 | 44.3 | 39.0 | 39.6 |
| AIPW    | 5       | -100.1 | 54.5 | -4.0 | 95.2 | 94.3 | 94.5 | 30.8 | 28.2 | 28.3 |
|          | 10      | -100.1 | 52.9 | -2.6 | 95.2 | 94.3 | 94.4 | 29.6 | 27.9 | 28.1 |
|          | 100     | -100.1 | 51.1 | -0.8 | 95.3 | 94.6 | 94.8 | 28.7 | 27.7 | 27.9 |
| Matching | 5       | -100.1 | 57.4 | 21.7 | -4.5 | 96.7 | 94.2 | 94.4 | 34.2 | 28.9 | 29.0 |
|          | 10      | -100.1 | 55.4 | 21.8 | -3.4 | 96.7 | 94.2 | 94.3 | 32.7 | 28.5 | 28.7 |
|          | 100     | -100.1 | 53.1 | 23.2 | -1.6 | 97.1 | 94.4 | 94.7 | 31.7 | 28.2 | 28.3 |

Table 3: Simulation results: point estimate, true variance, relative bias of the variance estimator, coverage and mean width of interval estimate using Rubin’s method and the proposed wild bootstrap method under scenario (b) with missingness not at random

| Method | $\tau_n$ | $m$ | $\hat{\tau}_n$ ($\times10^2$) | Point est ($\times10^4$) | True var ($\times10^4$) | Relative Bias (%) | Coverage (%) | Mean width ($\times10^2$) for 95% CI | Mean width ($\times10^2$) for 95% CI |
|--------|----------|-----|-------------------------------|--------------------------|--------------------------|-------------------|-------------|--------------------------------------|--------------------------------------|
|        |          |     | Rubin BS | Rubin BS | Wald | Rubin BS | Wald |                                           |                                         |
| Regression | 5       | -100.0 | 46.5 | -3.4 | -2.1 | 95.0 | 94.9 | 95.0 | 26.9 | 26.3 | 26.4 |
|          | 10      | -100.0 | 45.4 | -3.4 | -0.8 | 94.8 | 94.9 | 95.2 | 26.2 | 26.2 | 26.3 |
|          | 100     | -100.0 | 44.4 | -3.3 | 0.4  | 94.6 | 95.0 | 95.0 | 25.7 | 26.0 | 26.1 |
| IPW     | 5       | -99.8 | 112.3 | 29.1 | 0.8  | 95.8 | 94.2 | 94.6 | 43.9 | 38.7 | 39.3 |
|          | 10      | -99.8 | 109.3 | 29.2 | 1.6  | 95.8 | 94.2 | 94.4 | 43.2 | 38.4 | 39.0 |
|          | 100     | -99.8 | 107.2 | 29.0 | 2.0  | 95.9 | 94.1 | 94.5 | 42.8 | 38.0 | 38.6 |
| AIPW    | 5       | -100.0 | 48.6 | 2.5  | -5.0 | 95.6 | 94.6 | 94.6 | 28.3 | 26.5 | 26.6 |
|          | 10      | -100.0 | 47.3 | 2.5  | -3.8 | 95.5 | 94.7 | 94.9 | 27.5 | 26.3 | 26.4 |
|          | 100     | -100.0 | 46.1 | 2.7  | -2.5 | 95.4 | 94.7 | 95.0 | 26.9 | 26.2 | 26.1 |
| Matching | 5       | -100.0 | 51.4 | 21.1 | -5.0 | 96.6 | 94.5 | 94.5 | 31.9 | 27.2 | 27.4 |
|          | 10      | -100.0 | 49.7 | 21.6 | -3.8 | 96.7 | 94.6 | 94.8 | 30.8 | 27.0 | 27.1 |
|          | 100     | -100.0 | 48.2 | 22.0 | -3.0 | 97.0 | 94.6 | 94.8 | 30.1 | 26.7 | 26.8 |

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Table 4: Simulation results: point estimate, true variance, relative bias of the variance estimator, coverage and mean width of interval estimate using Rubin’s method and the proposed wild bootstrap method under scenario (c) when the true missing mechanism is missing not at random but missingness at random is assumed

| Method  | \( \hat{\tau}_n \) | \( m \) | Point est \((\times 10^2)\) | True var \((\times 10^4)\) | Relative Bias | Coverage (%) | Mean width \((\times 10^2)\) |
|---------|-------------------|------|-----------------|-----------------|--------------|--------------|-----------------|
|         |       |       |     |     |        | (Rubin) | (BS) | (Rubin) | (BS) | (Rubin) | (BS) | (Rubin) | (BS) |
| Regression | 5  | -106.7 | 48.6 | -2.1 | 2.3 | 83.9 | 84.0 | 84.4 | 27.6 | 27.5 | 27.6 |
|          | 10 | -106.7 | 47.2 | -1.6 | 4.0 | 83.2 | 83.9 | 84.3 | 26.9 | 27.3 | 27.4 |
|          | 100 | -106.7 | 46.2 | -1.5 | 5.0 | 82.6 | 83.9 | 84.0 | 26.4 | 27.2 | 27.3 |
| IPW      | 5  | -107.6 | 141.6 | 32.9 | 1.2 | 97.0 | 94.3 | 93.7 | 49.9 | 42.7 | 43.4 |
|          | 10 | -107.5 | 140.1 | 40.8 | 1.3 | 97.0 | 94.4 | 93.8 | 49.4 | 42.3 | 43.1 |
|          | 100 | -107.5 | 136.4 | 38.1 | 1.1 | 97.4 | 94.3 | 93.9 | 49.0 | 41.8 | 42.6 |
| AIPW     | 5  | -106.7 | 50.9 | 1.9  | -3.5 | 85.0 | 82.8 | 83.4 | 28.8 | 27.3 | 27.4 |
|          | 10 | -106.7 | 49.5 | 2.4  | -2.1 | 84.1 | 82.3 | 82.9 | 28.0 | 27.1 | 27.2 |
|          | 100 | -106.7 | 48.5 | 2.5  | -1.5 | 83.9 | 82.6 | 82.7 | 27.6 | 26.9 | 27.1 |
| Matching | 5  | -106.7 | 54.0 | 23.5 | -3.9 | 89.7 | 83.3 | 83.7 | 32.9 | 28.1 | 28.2 |
|          | 10 | -106.7 | 52.0 | 24.5 | -2.7 | 89.7 | 83.3 | 83.6 | 31.9 | 27.7 | 27.9 |
|          | 100 | -106.7 | 50.5 | 25.2 | -2.0 | 89.3 | 83.1 | 83.5 | 31.2 | 27.4 | 27.5 |

Table 5: Simulation results: point estimate, true variance, relative bias of the variance estimator, coverage and mean width of interval estimate using Rubin’s method and the proposed wild bootstrap method under scenario (d) where both the outcome and confounders are missing and missing not at random is assumed

| Method  | \( \hat{\tau}_n \) | \( m \) | Point est \((\times 10^2)\) | True var \((\times 10^4)\) | Relative Bias | Coverage (%) | Mean width \((\times 10^2)\) |
|---------|-------------------|------|-----------------|-----------------|--------------|--------------|-----------------|
|         |       |       |     |     |        | (Rubin) | (BS) | (Rubin) | (BS) | (Rubin) | (BS) | (Rubin) | (BS) |
| Regression | 5  | -100.0 | 47.5 | -1.1 | -1.5 | 95.2 | 94.2 | 95.3 | 27.6 | 26.7 | 26.8 |
|          | 10 | -100.0 | 46.2 | -1.2 | -0.2 | 95.1 | 94.9 | 95.2 | 26.8 | 26.5 | 26.6 |
|          | 100 | -100.0 | 45.2 | -1.3 | 0.5 | 95.1 | 95.1 | 95.2 | 26.2 | 26.3 | 26.4 |
| IPW      | 5  | -99.8  | 113.2 | 41.6 | -0.8 | 96.5 | 94.1 | 94.3 | 46.2 | 38.8 | 39.3 |
|          | 10 | -99.8  | 108.7 | 41.4 | 0.3 | 96.6 | 94.2 | 94.4 | 42.2 | 38.3 | 38.8 |
|          | 100 | -99.8  | 106.1 | 44.3 | -0.2 | 96.8 | 94.1 | 94.5 | 44.8 | 37.8 | 38.3 |
| AIPW     | 5  | -100.1 | 49.6 | 10.1 | -3.6 | 96.3 | 94.8 | 95.0 | 30.0 | 27.0 | 27.1 |
|          | 10 | -100.1 | 47.7 | 10.8 | -1.9 | 96.0 | 94.8 | 94.9 | 28.8 | 26.7 | 26.8 |
|          | 100 | -100.0 | 46.3 | 11.4 | -0.7 | 96.1 | 94.6 | 95.0 | 28.1 | 26.4 | 26.6 |
| Matching | 5  | -100.0 | 54.0 | 20.8 | -4.9 | 96.8 | 94.5 | 94.6 | 32.8 | 28.0 | 28.1 |
|          | 10 | -100.1 | 52.0 | 21.7 | -3.8 | 97.2 | 94.3 | 94.7 | 31.6 | 27.6 | 27.7 |
|          | 100 | -100.0 | 50.2 | 22.7 | -2.6 | 96.9 | 94.5 | 94.5 | 30.8 | 27.3 | 27.4 |
6 Application

We apply our method to a dataset from 2015-2016 U.S. National Health and Nutrition Examination Survey to estimate the ACE of education on general health satisfaction. The general health satisfaction outcome \( Y \) is fully observed with a lower value indicating better satisfaction.

A sample of 4,845 individuals is divided into two groups: one (76%) with at least high school education, denoted as \( A = 1 \), and the other one (24%) with education level lower than high school, denoted as \( A = 0 \). The covariates \( X \) consist of five categorical variables including age, race, gender, marital status, an indicator of ever having pre-diabetes risk, and one continuous variable family poverty ratio. The pre-diabetes risk indicator has about 18% missing values, and the family poverty ratio has about 10% missing values. The other four covariates are fully observed.

To facilitate imputation and estimation, we assume the outcome follows a linear regression model, i.e., \( Y(a) = X^T \beta_a + \epsilon(a) \), where \( \epsilon(a) \sim N(0, \sigma_a^2) \) for \( a = 0, 1 \). The treatment indicator follows Bernoulli\{\( \pi_A(X) \)} with \( \pi_A(X) = \Phi(X^T \alpha) \). The missing indicator follows Bernoulli\{\( \pi_R(X, A) \)} with \( \pi_R(X, A) = \Phi((X, A)^T \gamma) \), under which the missingness of the pre-diabetes risk indicator and the family poverty ratio probably depend on the missing values themselves but not the outcome variable (i.e., Assumption 6).

Table 6 shows that education has a significantly positive effect on the general health satisfaction. The variance estimation for the IPW estimator is slightly larger when estimated using Rubin’s method compared with estimated using our proposed method, which might indicate the possible overestimation of Rubin’s method. For the other point estimators, the two methods give very similar results, which might due to the fact that there are not many missing values in this survey dataset.

7 Conclusion

This paper establishes a unified martingale representation of the MI estimators of the ACE which invokes the wild bootstrap inference for consistent variance estimation. The simulation results indicate the good finite sample performance of the proposed method when MI adopts different full sample estimators including the outcome regression, IPW, AIPW and matching estimators. Our framework works well when the missing mechanism is either MAR or MNAR.

Our framework can also be extended in the following directions. First, multiple imputation was originated for survey data, which often contain design weights (or sample weights)
Table 6: Result for the ACE od education on general health satisfaction: point estimates, the variance of point estimators, and 95% confidence interval estimated using Rubin’s method and proposed wild bootstrap method.

| Method   | Point est | Var est ($\times 10^4$) | 95% CI     | Var est ($\times 10^4$) | 95% CI     |
|----------|-----------|------------------------|------------|------------------------|------------|
| Regression | -0.37     | 18                     | (-0.45,-0.29) | 17                     | (-0.45,-0.29) |
| IPW      | -0.24     | 44                     | (-0.37,-0.11) | 39                     | (-0.36,-0.11) |
| AIPW     | -0.31     | 21                     | (-0.40,-0.22) | 20                     | (-0.40,-0.22) |
| Matching | -0.27     | 34                     | (-0.38,-0.15) | 34                     | (-0.38,-0.15) |

to account for sample selection. If sampling weights are non-informative, the sample data follow the population model and therefore the imputation can be done by ignoring sampling weights; whereas, if sampling weights are informative, the sample data distribution is different from the population model and therefore imputation must take into account sampling weights. The full Bayesian imputation is difficult (if not impossible) to implement in this case. To mitigate this problem, Kim and Yang (2017) proposed an approximate Bayesian computation technique, which can be used for multiple imputation in complex sampling. It would be interesting to extend the martingale representation to this setting in our future work. Second, in the current work, we assume that the imputer’s model and the analyst’s model are the same and are correctly specified. Xie and Meng (2017) argued that uncongeniality of the imputer’s model and the analyst’s model is the rule but not an exception. Their findings suggest that even both models are correctly specified, if the imputation model is more saturate than the analysis model, the standard MI inference may be invalid. In future work, we will extend our framework to this setting for consistent inference allowing uncongeniality.

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Supplementary materials

The online supplementary material contains technical assumptions and proofs, and the R code that implements the proposed method is available at https://github.com/qianguan/miATE.

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