Asymptotic Approximations to Truncation Errors of Series Representations for Special Functions

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Summary. Asymptotic approximations ($n \to \infty$) to the truncation errors $r_n = -\sum_{\nu=0}^{\infty} a_\nu$ of infinite series $\sum_{\nu=0}^{\infty} a_\nu$ for special functions are constructed by solving a system of linear equations. The linear equations follow from an approximative solution of the inhomogeneous difference equation $\Delta r_n = a_{n+1}$. In the case of the remainder of the Dirichlet series for the Riemann zeta function, the linear equations can be solved in closed form, reproducing the corresponding Euler-Maclaurin formula. In the case of the other series considered – the Gaussian hypergeometric series $_2F_1(a, b; c; z)$ and the divergent asymptotic inverse power series for the exponential integral $E_1(z)$ – the corresponding linear equations are solved symbolically with the help of Maple. The practical usefulness of the new formalism is demonstrated by some numerical examples.

1 Introduction

A large part of special function theory had been developed already in the 19th century. Thus, it is tempting to believe that our knowledge about special functions is essentially complete and that no significant new developments are to be expected. However, up to the middle of the 20th century, research on special functions had emphasized analytical results, whereas the efficient and reliable evaluation of most special functions had been – and to some extend still is – a more or less unsolved problem.

Due to the impact of computers on mathematics, the situation has changed substantially. We witness a revival of interest in special functions. The general availability of electronic computers in combination with the development of powerful computer algebra systems like Maple or Mathematica opened up many new applications, and it also created a great demand for efficient and reliable computational schemes (see for example [7, 11, 21] or [19, Section 13] and references therein).
Most special functions are defined via infinite series. Examples are the Dirichlet series for the Riemann zeta function,

$$\zeta(s) = \sum_{\nu=0}^{\infty} (\nu + 1)^{-s}. \quad (1)$$

which converges for \(\text{Re}(s) > 1\), or the Gaussian hypergeometric series

$$2F_1(a, b; c; z) = \sum_{\nu=0}^{\infty} \frac{(a)_\nu (b)_\nu}{(c)_\nu \nu!} z^\nu, \quad (2)$$

which converges for \(|z| < 1\).

The definition of special functions via infinite series is to some extent highly advantageous since it greatly facilitates analytical manipulations. However, from a purely numerical point of view, infinite series representations are at best a mixed blessing. For example, the Dirichlet series (1) converges for \(\text{Re}(s) > 1\), but is notorious for extremely slow convergence if \(\text{Re}(s)\) is only slightly larger than one. Similarly, the Gaussian hypergeometric series (2) converges only for \(|z| < 1\), but the corresponding Gaussian hypergeometric function is a multivalued function defined in the whole complex plane with branch points at \(z = 1\) and \(\infty\). A different computational problem occurs in the case of the asymptotic series for the exponential integral:

$$ze^z E_1(z) \sim \sum_{m=0}^{\infty} (-1/z)^m m! = 2F_0(1, 1; -1/z), \quad z \to \infty. \quad (3)$$

This series is probably the most simple example of a large class of series that diverge for every finite argument \(z\) and that are only asymptotic in the sense of Poincaré as \(z \to \infty\). In contrast, the exponential integral \(E_1(z)\), which has a cut along the negative real axis, is defined in the whole complex plane.

Problems with slow convergence or divergence were encountered already in the early days of calculus. Thus, numerical techniques for the acceleration of convergence or the summation of divergent series are almost as old as calculus. According to Knopp [9, p. 249], the first systematic work in this direction can be found in Stirling’s book [17], which was published already in 1730 (recently, Tweddle [20] published a new annotated translation), and in 1755 Euler [6] published the series transformation which now bears his name.

The convergence and divergence problems mentioned above can be formalized as follows: Let us assume that the partial sums \(s_n = \sum_{\nu=0}^{n} a_\nu\) of a convergent or divergent but summable series form a sequence \(\{s_n\}_{n=0}^{\infty}\) whose elements can be partitioned into a (generalized) limit \(s\) and a remainder or truncation error \(r_n\) according to

$$s_n = s + r_n, \quad n \in \mathbb{N}_0. \quad (4)$$

This implies
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\[ r_n = - \sum_{\nu=n+1}^{\infty} a_{\nu}, \quad n \in \mathbb{N}_0. \]  

(5)

At least in principle, a convergent infinite series can be evaluated by adding up the terms successively until the remainders become negligible. This approach has two obvious shortcomings. Firstly, convergence can be so slow that it is uneconomical or practically impossible to achieve sufficient accuracy. Secondly, this approach does not work in the case of a divergent but summable series because increasing the index \( n \) normally only aggravates divergence.

As a principal alternative, we can try to compute a sufficiently accurate approximation \( \tilde{r}_n \) to the truncation error \( r_n \). If this is possible, \( \tilde{r}_n \) can be eliminated from \( s_n \), yielding a (much) better approximation \( s_n - \tilde{r}_n \) to the (generalized) limit \( s \) than \( s_n \) itself.

This approach looks very appealing since it is in principle remarkably powerful. In addition, it can avoid the troublesome asymptotic regime of large indices \( n \), and it also works in the case of divergent but summable sequences and series. Unfortunately, it is by no means easy to obtain sufficiently accurate approximations \( \tilde{r}_n \) to truncation errors \( r_n \). The straightforward computation of \( r_n \) by adding up the terms does not gain anything.

The Euler-Maclaurin formula, which is discussed in Section 2, is a principal analytical tool that produces asymptotic approximations to truncation errors of monotone series in terms of integrals plus correction terms. Unfortunately, it is not always possible to apply the Euler-Maclaurin formula. Given a reasonably well behaved integrand, it is straightforward to compute a sum of integrand values plus derivatives of the integrand. But for a given series term \( a_n \), it may be prohibitively difficult to differentiate and integrate it with respect to the index \( n \).

In Section 3, an alternative approach for the construction of asymptotic approximations \( (n \to \infty) \) to the truncation errors \( r_n \) of infinite series is proposed that is based on the solution of a system of linear equations. The linear equations exist under very mild conditions: It is only necessary that the ratio \( a_{n+2}/a_{n+1} \) or similar ratios of series terms possesses an asymptotic expansion in terms of inverse powers \( 1/(n + \alpha) \) with \( \alpha > 0 \). Moreover, it is also fairly to solve these linear equations since they have a triangular structure.

The asymptotic nature of the approximants makes it difficult to use them also for small indices \( n \), although this would be highly desirable. In Section 4, it is mentioned briefly that factorial series and Padé approximants can be helpful in this respect since they can accomplish a numerical analytic continuation.

In Section 5, the formalism proposed in this article is applied to the truncation error of the Dirichlet series for the Riemann zeta function. It is shown that the linear equations can in this case be reduced to a well known recurrence formula of the Bernoulli numbers. Accordingly, the terms of the corresponding Euler-Maclaurin formula are exactly reproduced.

In Section 6, the Gaussian hypergeometric series \( _2F_1(a, b; c; z) \) is treated. Since the terms of this series depend on three parameters and one argument, a
closed form solution of the linear equations seems to be out of reach. Instead, approximations are computed symbolically with the help of the computer algebra system Maple. The practical usefulness of these approximations is demonstrated by some numerical examples.

In Section 7, the divergent asymptotic inverse power series for the exponential integral $E_1(z)$ is treated. Again, the linear equations are solved symbolically with the help of Maple, and the practical usefulness of these solutions is demonstrated by some numerical examples.

2 The Euler-Maclaurin Formula

The derivation of the Euler-Maclaurin formula is based on the assumption that $g(x)$ is a smooth and slowly varying function. Then, $\int_M^N g(x) dx$ with $M, N \in \mathbb{Z}$ can be approximated by the finite sum $\frac{1}{2}g(M) + g(M + 1) + \cdots + g(N - 1) + \frac{1}{2}g(N)$. This finite sum can also be interpreted as a trapezoidal quadrature rule. In the years between 1730 and 1740, Euler and Maclaurin derived independently correction terms to this quadrature rule, which ultimately yielded what we now call the Euler-Maclaurin formula (see for example [19, Eq. (1.20)])

$$\sum_{\nu = M}^{N} g(\nu) = \int_{M}^{N} g(x) \, dx + \frac{1}{2} [g(M) + g(N)]$$

$$+ \sum_{j=1}^{k} \frac{B_{2j}}{(2j)!} \left[ g^{(2j-1)}(N) - g^{(2j-1)}(M) \right] + R_k(g), \quad (6a)$$

$$R_k(g) = -\frac{1}{(2k)!} \int_{M}^{N} B_{2k}(x - \lfloor x \rfloor) g^{(2k)}(x) \, dx. \quad (6b)$$

Here, $g^{(m)}(x)$ is the $m$-th derivative, $\lfloor x \rfloor$ is the integral part of $x$, $B_m(x)$ is a Bernoulli polynomial defined by the generating function $te^{xt}/(e^t - 1) = \sum_{n=0}^{\infty} B_n(x) t^n/n!$, and $B_m = B_m(0)$ is a Bernoulli number.

It is not a priori clear whether the integral $R_k(g)$ in (6) vanishes as $k \to \infty$ for a given function $g(x)$. Thus, the Euler-Maclaurin formula may lead to an asymptotic expansion that ultimately diverges. In this article, it is always assumed that the Euler-Maclaurin formula and related expansions are only asymptotic in the sense of Poincaré.

Although originally used to express the in the early 18th century still unfamiliar integral in terms more elementary quantities, the Euler-Maclaurin formula is now often used to approximate the truncation error $r_n = -\sum_{\nu=0}^{n+1} a_{\nu}$ of a slowly convergent monotone series by an integral plus correction terms. The power and the usefulness of this approach can be demonstrated convincingly via the Dirichlet series (1) for the Riemann zeta function.
The terms \((\nu + 1)^{-s}\) of the Dirichlet series are obviously smooth and slowly varying functions of the index \(\nu\), and they can be differentiated and integrated easily. Thus, the application of the Euler-Maclaurin formula \(\text{(6)}\) with \(M = n + 1\) and \(N = \infty\) to the truncation error of the Dirichlet series yields:

\[
- \sum_{\nu=n+1}^{\infty} (\nu + 1)^{-s} = - \frac{(n+2)^{1-s}}{s-1} - \frac{1}{2} (n+2)^{-s} - \sum_{j=1}^{k} \frac{(s)_{2j-1}}{(2j)!} B_{2j} (n+2)^{-s-2j+1} + R_k(n, s), \quad (7a)
\]

\[
R_k(n, s) = \frac{(s)_{2k}}{(2k)!} \int_{n+1}^{\infty} \frac{B_{2k}(x - \lfloor x \rfloor)}{(x+1)^{s+2k}} \, dx. \quad (7b)
\]

Here, \((s)_m = s(s+1) \cdots (s+m-1) = \Gamma(s+m)/\Gamma(s)\) with \(s \in \mathbb{C}\) and \(m \in \mathbb{N}\) is a Pochhammer symbol.

In \([3, \text{Tables 8.7 and 8.8, p. 380}]\) and in \([25, \text{Section 2}]\) it was shown that a few terms of the sum in \(7a\) suffice for a convenient and reliable computation of \(\zeta(s)\) with \(s = 1.1\) and \(s = 1.01\), respectively. For these arguments, the Dirichlet series for \(\zeta(s)\) converges so slowly that it is practically impossible to evaluate it by adding up its terms.

In order to understand better its nature, the Euler-Maclaurin formula \(\text{(6)}\) is rewritten in a more suggestive form. Let us set \(M = n + 1\) and \(N = \infty\), and let us also assume \(\lim_{N \to \infty} g(N) = \lim_{N \to \infty} g'(N) = \lim_{N \to \infty} g''(N) = \cdots = 0\). With the help of \(B_0 = 1, B_1 = -1/2, \text{ and } B_{2n+1} = 0\) with \(n \in \mathbb{N}\) (see for example \([19, \text{p. 3}]\)), we obtain:

\[
- \sum_{\nu=n+1}^{\infty} g(\nu) = -B_0 \int_{n+1}^{\infty} g(x) \, dx + \sum_{\mu=1}^{m} \frac{(-1)^{\mu-1} B_\mu}{\mu!} g^{(\mu-1)}(\nu) + R_m(g), \quad (8a)
\]

\[
R_m(g) = \frac{(-1)^m}{(m)!} \int_{n+1}^{\infty} B_m(x - \lfloor x \rfloor) g^{(m)}(x) \, dx, \quad m \in \mathbb{N}. \quad (8b)
\]

In the same way, we obtain for the Euler-Maclaurin approximation \(\text{(7)}\) to the truncation error of the Dirichlet series:

\[
- \sum_{\nu=n+1}^{\infty} (\nu + 1)^{-s} = \sum_{\mu=0}^{m} \frac{(-1)^{\mu-1} (s)_{\mu-1} B_\mu}{\mu!} (n+2)^{1-s-\mu} + R_m(n, s), \quad (9a)
\]

\[
R_m(n, s) = \frac{(-1)^m (s)_m}{(m)!} \int_{n+1}^{\infty} \frac{B_m(x - \lfloor x \rfloor)}{(1+x)^{s+m}} \, dx, \quad m \in \mathbb{N}. \quad (9b)
\]

The reformulated Euler-Maclaurin approximation \(\text{(9)}\) looks suspiciously like a truncated expansion of the truncation error in terms of the asymptotic
sequence \( \{(n + 2)^{-\mu}\}_{\mu=0}^\infty \) of inverse powers. An analogous interpretation of the reformulated Euler-Maclaurin formula is possible if we assume that the quantities \( \int_{n+1}^\infty g(x)dx, g(n), g'(n), g''(n), \ldots \) form an asymptotic sequence \( \{G_\mu(n)\}_{\mu=0}^\infty \) for \( n \to \infty \) according to

\[
G_0(n) = \int_{n+1}^\infty g(x)dx, \quad (10a)
\]

\[
G_\mu(n) = g^{(\mu-1)}(n), \quad \mu \in \mathbb{N}. \quad (10b)
\]

The expansion of the truncation error \( -\sum_{\nu=n+1}^\infty g(\nu) \) in terms of the asymptotic sequence \( \{G_\mu(n)\}_{\mu=0}^\infty \) according to (8) has the undeniable advantage that the expansion coefficients do not depend on the terms \( g(\nu) \) and are explicitly known. The only remaining computational problem is the determination of the leading elements of the asymptotic sequence \( \{G_\mu(n)\}_{\mu=0}^\infty \). In the case of the Dirichlet series (1), this is trivially simple. Unfortunately, the terms of most series expansions for special functions are (much) more complicated than the terms of the Dirichlet series (1). In those less fortunate cases, it can be extremely difficult to do the necessary differentiations and integrations. Thus, the construction of the asymptotic sequence \( \{G_\mu(n)\}_{\mu=0}^\infty \) may turn out to be an unsurmountable problem.

### 3 Asymptotic Approximations To Truncation Errors

Let us assume that we want to construct an asymptotic expansion of a special function \( f(z) \) as \( z \to \infty \). First, we have to find a suitable asymptotic sequence \( \{\varphi_j(z)\}_{j=0}^\infty \). Obviously, \( \{\varphi_j(z)\}_{j=0}^\infty \) must be able to model the essential features of \( f(z) \) as \( z \to \infty \). On the other hand, \( \{\varphi_j(z)\}_{j=0}^\infty \) should also be sufficiently simple in order to facilitate the necessary analytical manipulations. In that respect, the most convenient asymptotic sequence is the sequence \( \{z^{-j}\}_{j=0}^\infty \) of inverse powers, and it is also the one which is used almost exclusively in special function theory. An obvious example is the asymptotic series (3).

The behavior of most special functions as \( z \to \infty \) is incompatible with an expansion in terms of inverse powers. Therefore, an indirect approach has to be pursued: For a given \( f(z) \), one has to find some \( g(z) \) such that the ratio \( f(z)/g(z) \) admits an asymptotic expansion in terms of inverse powers:

\[
f(z)/g(z) \sim \sum_{j=0}^\infty c_j z^{-j}, \quad z \to \infty. \quad (11)
\]

Although \( f(z) \) cannot be expanded in terms of inverse powers \( \{z^{-j}\}_{j=0}^\infty \), it can be expanded in terms of the asymptotic sequence \( \{g(z)/z^j\}_{j=0}^\infty \). The asymptotic series (3) is of the form of (11) with \( f(z) = E_1(z) \) and \( g(z) = \exp(-z)/z \).
It is the central hypothesis of this article that such an indirect approach is useful for the construction of asymptotic approximations to remainders
\( r_n = -\sum_{\nu=n+1}^{\infty} a_\nu \) of infinite series as \( n \to \infty \). Thus, instead of trying to use the technically difficult Euler-Maclaurin formula (6), we should try to find some \( \rho_n \) such that the ratio \( r_n/\rho_n \) admits an asymptotic expansion as \( n \to \infty \) in terms of inverse powers \( \{ (n+\alpha)^{-j} \}_{j=0}^{\infty} \) with \( \alpha > 0 \).

A natural candidate for \( \rho_n \) is the first term \( a_{n+1} \) neglected in the partial sum \( s_n = \sum_{\nu=0}^{n} a_\nu \), but in some cases it is better to choose instead \( \rho_n = a_n \) or \( \rho_n = (n+\alpha)a_{n+1} \) with \( \alpha > 0 \). Moreover, the terms \( a_{n+1} \) and the remainders \( r_n \) of an infinite series are connected by the inhomogeneous difference equation
\[
\Delta r_n = r_{n+1} - r_n = a_{n+1}, \quad n \in \mathbb{N}_0. \tag{12}
\]

In Jagerman’s book [8, Chapter 3 and 4], solutions to difference equations of that kind are called Nörlund sums.

If we knew how to solve (12) efficiently and reliably for essentially arbitrary inhomogeneities \( a_{n+1} \), all problems related to the evaluation of infinite series would in principle be solved. Unfortunately, this is not the case. Nevertheless, we can use (12) to construct the leading terms of an asymptotic expansion of \( r_n/a_{n+1} \) or of related expressions in terms of inverse powers.

For that purpose, we make the following ansatz:
\[
r^{(m)}_n = -a_{n+1} \sum_{\mu=0}^{m} \gamma^{(m)}_\mu \frac{1}{(n+\alpha)^\mu}, \quad n \in \mathbb{N}_0, \quad m \in \mathbb{N}, \quad \alpha > 0. \tag{13}
\]

This ansatz, which is inspired by the theory of converging factors [1, 13] and by a truncation error estimate for Levin’s sequence transformation [10] proposed by Smith and Ford [16, Eq. (2.5)] (see also [22, Section 7.3] or [24, Section IV]), is not completely general and has to be modified slightly both in the case of the Dirichlet series (1) for the Riemann zeta function, which is discussed in Section 5, and in the case of the divergent asymptotic series (3) for the exponential integral, which is discussed in Section 7. Moreover, the ansatz (13) does not cover the series expansions of all special functions of interest. For example, in [23] a power series expansion for the digamma function \( \psi(z) \) was analyzed whose truncation errors cannot be approximated by a truncated power series of the type of (13). Nevertheless, the examples considered in this article should suffice to convince even a sceptical reader that the ansatz (13) is indeed computationally useful.

We cannot expect that the ansatz (13) satisfies the inhomogeneous difference equation (12) exactly. However, we can choose the unspecified coefficients \( \gamma^{(m)}_\mu \) in (13) in such a way that only a higher order error remains:
\[
\frac{r_n^{(m)} - r_n^{(m)}}{a_{n+1}} = \sum_{\mu=0}^{m} \gamma^{(m)}_\mu \frac{a_{n+1}}{(n+\alpha)^\mu} - \frac{a_{n+2}}{a_{n+1}} \sum_{\mu=0}^{m} \gamma^{(m)}_\mu \frac{(n+\alpha+1)}{(n+\alpha+1)^\mu} \tag{14}
\]
\[
= 1 + O(n^{-m-1}), \quad n \to \infty. \tag{15}
\]
The approach of this article depends crucially on the assumption that the ratio $a_{n+2}/a_{n+1}$ can be expressed as an (asymptotic) power series in $1/(n+\alpha)$. If this is the case, then the right-hand side of (14) can be expanded in powers of $1/(n+\alpha)$ and we obtain:

$$r_{n+1} - r_n$$

$$= \sum_{\mu=0}^{m} \frac{C_{\mu}^{(m)}}{(n+\alpha)^{\mu}} + \mathcal{O}(n^{-m-1}), \quad n \to \infty. \quad (16)$$

Now, (16) implies that we have solve the following system of linear equations:

$$C_{\mu}^{(m)} = \delta_{\mu 0}, \quad 0 \leq \mu \leq m. \quad (17)$$

Since $C_{\mu}^{(m)}$ with $0 \leq \mu \leq m$ contains only the unspecified coefficients $\gamma_0^{(m)}, \ldots, \gamma_{m}^{(m)}$, but not $\gamma_{m+1}^{(m)}, \ldots, \gamma_{m}^{(m)}$, the linear system (17) has a triangular structure and the unspecified coefficients $\gamma_0^{(m)}, \ldots, \gamma_{m}^{(m)}$ can be determined by solving successively the equations $C_0^{(m)} = 1, C_1^{(m)} = 0, \ldots, C_m^{(m)} = 0$.

Another important aspect is that the linear equations (17) do not depend explicitly on $m$, which implies that the coefficients $\gamma_{\mu}^{(m)}$ in (13) also do not depend explicitly on $m$. Accordingly, the superscript $m$ of both $C_{\mu}^{(m)}$ and $\gamma_{\mu}^{(m)}$ is superfluous and will be dropped in the following Sections.

4 Numerical Analytic Continuation

Divergent asymptotic series of the type of (11) can be extremely useful computationally: For sufficiently large arguments $z$, truncated expansions of that kind are able to provide (very) accurate approximations to the corresponding special functions, in particular if the series is truncated in the vicinity of the minimal term. If, however, the argument $z$ is small, truncated expansions of that kind produce only relatively poor or even completely nonsensical results.

We can expect that our asymptotic expansions in powers of $1/(n+\alpha)$ have similar properties. Thus, we can be confident that they produce (very) good results for sufficiently large indices $n$, but it would be overly optimistic to assume that these expressions necessarily produce good results in the nonasymptotic regime of moderately large or even small indices $n$.

Asymptotic approximants can often be constructed (much) more easily than other approximants that are valid in a wider domain. Thus, it is desirable to use asymptotic approximants also outside the asymptotic domain. This means that we would like to use our asymptotic approximants also for small indices $n$ in order to avoid the computationally problematic asymptotic regime of large indices. Obviously, this is intrinsically contradictory. We also must find a way of extracting additional information from the terms of a truncated divergent inverse power series expansion.
Often, this can be accomplished at low computational costa by converting an inverse power series \( \sum_{n=0}^{\infty} c_n / z^n \) to a factorial series \( \sum_{n=0}^{\infty} \tilde{c}_n / (z)^n \). Factorial series, which had already been known to Stirling [20, p. 6], frequently have superior convergence properties. An example is the incomplete gamma function \( \Gamma(a, z) \), which possesses a divergent asymptotic series of the type of (11) [5, Eq. (6) on p. 135] and also a convergent factorial series [5, Eq. (1) on p. 139]. Accordingly, the otherwise so convenient inverse powers are not necessarily the computationally most effective asymptotic sequence.

The transformation of an inverse power series to a factorial series can be accomplished with the help of the Stirling numbers of the first kind which are normally defined via the expansion \((z - n + 1)_n = \sum_{\nu=0}^{n} S^{(1)}(n, \nu) z^\nu \) of a Pochhammer symbol in terms of powers. As already known to Stirling (see for example [20, p. 29] or [14, Eq. (6) on p. 78]), the Stirling numbers of the first kind occur also in the factorial series expansion of an inverse power:

\[
\frac{1}{z^{k+1}} = \sum_{k=0}^{\infty} \frac{(-1)^\kappa S^{(1)}(k+\kappa, k)}{(z)^{k+\kappa+1}}, \quad k \in \mathbb{N}_0 .
\] (18)

This infinite generating function can also be derived by exploiting the well known recurrence relationships of the Stirling numbers.

With the help of (18), the following transformation formula can be derived easily:

\[
\sum_{n=0}^{\infty} \frac{c_n}{z^n} = c_0 + \frac{c_1}{(z)_1} + \sum_{k=2}^{\infty} \frac{(-1)^k}{(z)_k} \sum_{\kappa=1}^{k} (-1)^\kappa S^{(1)}(k-1, \kappa-1) c_\kappa .
\] (19)

Let us now assume that the coefficients \( \gamma_\mu \) with \( 0 \leq \mu \leq m \) of a truncated expansion of \( r_n / a_{n+1} \) in powers of \( 1/(n+\alpha) \) according to (13) are known. Then, (19) implies that we can use the transformation scheme

\[
\tilde{\gamma}_\mu = \begin{cases} 
\gamma_\mu, & \mu = 0, 1, \\
\sum_{\nu=1}^{\mu} (-1)^{\mu+\nu} S^{(1)}(\mu-1, \nu-1) \gamma_\nu, & \mu \geq 2,
\end{cases}
\] (20)

to obtain instead of (13) the truncated factorial series

\[
\tilde{r}^{(m)}_n = -a_{n+1} \left[ \sum_{\mu=0}^{m} \frac{\tilde{\gamma}_\mu}{(n+\alpha)_\mu} + O(n^{-m-1}) \right], \quad n \to \infty .
\] (21)

Padé approximants, which convert the partial sums of a formal power series to a doubly indexed sequence of rational functions, can also be quite helpful. They are now used almost routinely in applied mathematics and theoretical physics to overcome convergence problems with power series (see for example the monograph by Baker and Graves-Morris [2] and references therein). The
ansatz (13) produces a truncated series expansion of \( r_n/a_{n+1} \) in powers of \( 1/(n + \alpha) \), which can be converted to a Padé approximant, i.e., to a rational function in \( 1/(n + \alpha) \).

The numerical results presented in Sections 6 and 7 that the conversion to factorial series and Padé approximants improves the accuracy of our approximants, in particular for small indices \( n \).

5 The Dirichlet Series for the Riemann Zeta Function

In this Section, an asymptotic approximation to the truncation error of the Dirichlet series for the Riemann zeta function is constructed by suitably adapting the approach described in Section 3. In the case of the Dirichlet series (1), we have:

\[
\begin{align*}
  s_n &= \sum_{\nu=0}^{n} (\nu + 1)^{-s}, \\
  r_n &= -\sum_{\nu=n+1}^{\infty} (\nu + 1)^{-s} = -(n + 2)^{-s} \sum_{\nu=0}^{\infty} \left(1 + \frac{\nu}{n + 2}\right)^{-s}, \\
  \Delta r_n &= (n + 2)^{-s}.
\end{align*}
\] (23)

It is an obvious idea to express the infinite series on the right-hand side of (23) as a power series in \( 1/(n + 2) \). If \( \nu < n + 2 \), we can use the binomial series

\[
(1 + z)^a = \sum_{m=0}^{\infty} \binom{a}{m} z^m
\]

which converges for \( |z| < 1 \). We thus obtain

\[
[1 + \nu/(n + 2)]^{-s} = \sum_{m=0}^{\infty} \frac{(s)_m}{m!} \left[-\nu/(n + 2)\right]^m.
\] (25)

The infinite series converges if \( \nu/(n + 2) < 1 \). Thus, an expansion of the right-hand side of (23) in powers of \( 1/(n + 2) \) can only be asymptotic as \( n \to \infty \). Nevertheless, a suitably truncated expansion suffices for our purposes.

In the case of the Dirichlet series for the Riemann zeta function, we cannot use ansatz (13). This follows at once from the relationship

\[
\Delta n^\alpha = (n + 1)^\alpha - n^\alpha = \alpha n^{\alpha-1} + O(n^{\alpha-2}), \quad n \to \infty.
\] (26)

Thus, we make the following ansatz, which takes into account the specific features of the Dirichlet series (1):

\[
r_n^{(m)} = (n + 2)^{-s} \sum_{\mu=0}^{m} \frac{\gamma_\mu}{(n + 2)^\mu}, \quad m \in \mathbb{N}, \quad n \in \mathbb{N}_0.
\] (27)

This ansatz is inspired by the truncation error estimate for Levin’s \( u \) transformation (10) (see also (22) Section 7.3 or (24) Section IV)).
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As in Section 3, the unspecified coefficients $\gamma_\mu$ are chosen in such a way that only a higher order error remains:

$$\Delta r_n^{(m)} = (n + 2)^{-s} \left[ 1 + O \left( n^{-m-1} \right) \right], \quad n \to \infty. \quad (28)$$

For that purpose, we write:

$$r_n^{(m)} - r_n^{(m)} = \sum_{\mu=0}^{m} \frac{\gamma_\mu}{(n + 2)^\mu} - \left[ \frac{n + 3}{n + 2} \right]^{1-s} \sum_{\mu=0}^{m} \frac{\gamma_\mu}{(n + 3)^\mu} \quad (29)$$

$$= \sum_{\mu=0}^{m} \frac{\gamma_\mu}{(n + 2)^\mu} \left\{ 1 - \left[ 1 + 1/(n + 2) \right]^{1-s-\mu} \right\}. \quad (30)$$

With the help of the binomial series [12, p. 38], we obtain:

$$1 - \left[ 1 + 1/(n + 2) \right]^{1-s-\mu} = \sum_{\lambda=0}^{\infty} \frac{(s + \mu - 1)_{\lambda+1}}{(\lambda + 1)!} \frac{(-1)^\lambda}{(n + 2)^{\lambda+1}}. \quad (31)$$

Inserting (31) into (30) yields:

$$\sum_{\mu=0}^{m} \frac{\gamma_\mu}{(n + 2)^\mu} \left\{ 1 - \left[ 1 + 1/(n + 2) \right]^{1-s-\mu} \right\} = \sum_{\mu=0}^{m} \frac{\gamma_\mu}{(n + 2)^\mu} \sum_{\lambda=0}^{\infty} \frac{(s + \mu - 1)_{\lambda+1}}{(\lambda + 1)!} \frac{(-1)^\lambda}{(n + 2)^{\lambda+1}} \quad (32)$$

$$= - \sum_{\nu=0}^{\infty} (n + 2)^{-\nu-1} \sum_{\lambda=0}^{\min(\nu,m)} \frac{(1 - s - \nu)_{\lambda+1} \gamma_{\nu-\lambda}}{\lambda + 1)!}. \quad (33)$$

Thus, we obtain the following truncated asymptotic expansion:

$$\frac{r_{n+1}^{(m)} - r_n^{(m)}}{(n + 2)^{-s}} = - \sum_{\mu=0}^{m} (n + 2)^{-\mu}$$

$$\times \sum_{\lambda=0}^{\mu} \frac{(1 - s - \mu)_{\lambda+1} \gamma_{\mu-\lambda}}{(\lambda + 1)!} + O \left( n^{-m-1} \right), \quad n \to \infty. \quad (34)$$

The unspecified coefficients $\gamma_\mu$ have to be determined by solving the following system of linear equations, whose triangular structure is obvious:

$$\sum_{\lambda=0}^{\mu} \frac{(1 - s - \mu)_{\lambda+1} \gamma_{\mu-\lambda}}{(\lambda + 1)!} = \delta_{\mu 0}, \quad 0 \leq \mu \leq m. \quad (35)$$

For a more detailed analysis of the linear system (35), let us define $\beta_\mu$ via

$$\gamma_\mu = (-1)^\mu \frac{(s)_{\mu-1}}{\mu!} \beta_\mu, \quad \mu \in \mathbb{N}_0. \quad (36)$$
Inserting (36) into (35) yields:

\[
\begin{align*}
\sum_{\lambda=0}^{\mu} \frac{(1-s-\mu)_{\lambda+1}}{(\lambda+1)!} & \frac{(-1)^{\mu-\lambda}(s)_{\mu-\lambda-1}}{(\mu-\lambda)!} \beta_{\mu-\lambda} = \delta_{\mu0}, \quad \mu \in \mathbb{N}_0. \\
\end{align*}
\]

(37)

Next, we use \((1-s-\mu)_{\lambda+1} = (-1)^{\lambda+1}(s)_{\mu}\) and obtain

\[
\begin{align*}
\sum_{\lambda=0}^{\mu} \frac{(1-s-\mu)_{\lambda+1} \gamma_{\mu-\lambda}}{(\lambda+1)!} &= (-1)^{\mu+1}(s)_{\mu} \sum_{\lambda=0}^{\mu} \frac{\beta_{\mu-\lambda}}{((\lambda+1)!(\mu-\lambda)!)} \\
&= (-1)^{\mu+1}(s)_{\mu} \sum_{\sigma=0}^{\mu} \frac{(\mu+1)!}{(\mu-\sigma+1)!\sigma!} \beta_{\sigma} \\
&= (-1)^{\mu+1}(s)_{\mu} \sum_{\sigma=0}^{\mu} \frac{\beta_{\sigma}}{\sigma!} = \delta_{\mu0}, \quad \mu \in \mathbb{N}_0.
\end{align*}
\]

(38)

(39)

Thus, the linear system (38) is equivalent to the well known recurrence formula

\[
\sum_{\nu=0}^{n} \binom{n+1}{\nu} B_{\nu} = 0, \quad n \in \mathbb{N},
\]

(41)

of the Bernoulli numbers (see for example [19, Eq. (1.11)]) together with the initial condition \(B_0 = 1\). Thus, the ansatz \(24\) reproduces the finite sum \(20\) or \(22\) of the Euler-Maclaurin formula for the truncation error of the Dirichlet series, which is not really surprising since asymptotic series are unique, if they exist. Only the integral \(7b\) or \(9b\) cannot be reproduced in this way.

### 6 The Gaussian Hypergeometric Series

The simplicity of the terms of the Dirichlet series facilitates the derivation of explicit asymptotic approximations to truncation errors by solving a system of linear equations in closed form. A much more demanding test for the feasibility of the new formalism is the Gaussian hypergeometric series, which depends on three parameters \(a, b,\) and \(c\), and one argument \(z\).

Due to the complexity of the terms of the Gaussian hypergeometric series, there is little hope in obtaining explicit analytical solutions to the linear equations. From a pragmatist’s point of view, it is therefore recommendable to use computer algebra systems like Maple and Mathematica and let the computer do the work.

In the case of a nonterminating Gaussian hypergeometric series, we have:
Asymptotic Approximations to Truncation Errors

\[ s_n(z) = \sum_{\nu=0}^{n} \frac{(a)_\nu (b)_\nu}{(c)_\nu} z^\nu, \quad (42) \]

\[ r_n(z) = -\sum_{\nu=n+1}^{\infty} \frac{(a)_\nu (b)_\nu}{(c)_\nu} z^\nu \]

\[ = -\frac{(a)_{n+1} (b)_{n+1}}{(c)_{n+1}(n+1)!} z^{n+1} \sum_{\nu=0}^{\infty} \frac{(a+n+1)\nu (b+n+1)\nu}{(c+n+1)\nu (n+2)\nu} z^\nu, \quad (43) \]

\[ \Delta r_n(z) = \frac{(a)_{n+1} (b)_{n+1}}{(c)_{n+1}(n+1)!} z^{n+1}. \quad (44) \]

Since \([a+n+1]_{\nu} (b+n+1)_{\nu}/(c+n+1)_{\nu} (n+2)_{\nu}\) can be expressed as a power series in \(1/(n+1)\), the following ansatz make sense:

\[ r^{(m)}_n(z) = -\frac{(a)_{n+1} (b)_{n+1}}{(c)_{n+1}(n+1)!} z^{n+1} \sum_{\nu=0}^{m} \frac{\gamma_{\mu}}{(n+1)^\mu}, \quad m \in \mathbb{N}, \quad n \in \mathbb{N}_0, \quad |z| < 1. \quad (45) \]

Again, we choose the unspecified coefficients \(\gamma_{\mu}\) in (45) in such a way that only a higher order error error remains:

\[ \Delta r^{(m)}_n(z) = r^{(m)}_{n+1}(z) - r^{(m)}_n(z) \]

\[ = \frac{(a)_{n+1} (b)_{n+1}}{(c)_{n+1}(n+1)!} z^{n+1} [1 + O(n^{-m-1})], \quad n \to \infty. \quad (46) \]

This convergence condition can be reformulated as follows:

\[ \frac{r^{(m)}_{n+1}(z) - r^{(m)}_n(z)}{[a+n+1]_{\nu} (b+n+1)_{\nu}/(c+n+1)_{\nu} (n+1)!}] \]

\[ = \sum_{\mu=0}^{m} \frac{\gamma_{\mu}}{(n+1)^\mu} - \frac{(a+n+1)(b+n+1)}{(c+n+1)(n+2)} z \sum_{\mu=0}^{m} \frac{\gamma_{\mu}}{(n+2)^\mu} \quad (47) \]

\[ = 1 + O(n^{-m-1}), \quad n \to \infty. \quad (48) \]

Now, we only have to do an asymptotic expansion of (47) in terms of the asymptotic sequence \(\{1/(n+1)^{\mu}\}_{j=0}^{\infty}\) as \(n \to \infty\). This yields:

\[ \frac{r^{(m)}_{n+1}(z) - r^{(m)}_n(z)}{[a+n+1]_{\nu} (b+n+1)_{\nu}/(c+n+1)_{\nu} (n+1)!}] \]

\[ = \sum_{\mu=0}^{m} \frac{C_{\mu}}{(n+1)^\mu} + O(n^{-m-1}), \quad n \to \infty. \quad (49) \]

We then obtain the following system of coupled linear equations in the unspecified coefficient \(\gamma_{\mu}\) with \(0 \leq \mu \leq m:\
As discussed in Section 3, a coefficient $C_\mu$ with $0 \leq \mu \leq m$ contains only the unspecified coefficients $\gamma_0, \ldots, \gamma_\mu$ but not $\gamma_{\mu+1}, \ldots, \gamma_m$. Thus, the symbolic solution of these linear equations for a Gaussian hypergeometric function $_2F_1(a, b; c; z)$ with unspecified parameters $a$, $b$, and $c$ and unspecified argument $z$ is not particularly difficult for a computer algebra system, since the unspecified coefficient $\gamma_\mu$ can be determined successively. The following linear equations were constructed with the help of Maple 8:

\begin{align}
C_0 &= (1 - z) \gamma_0 = 1, \\
C_1 &= (c - a - b + 1) z \gamma_0 + (1 - z) \gamma_1 = 0, \\
C_2 &= [(c - b + 1) a + (c + 1) b - 1 - c - c^2] z \gamma_0 \\
&\quad+ (c + 2 - b - a) z \gamma_1 + (1 - z) \gamma_2 = 0, \\
C_3 &= \{(c + 1) b - 1 - c - c^2\} a - (1 + c + c^2) b + c^2 + c + 1 \} z \gamma_0 \\
&\quad+ [(c + 2 - b) a + (c + 2) b - 3 - c^2 - 2 c] z \gamma_1 \\
&\quad+ (3 - b - a + c) z \gamma_2 + (1 - z) \gamma_3 = 0.
\end{align}

This example shows that the complexity of the coefficients $C_\mu$ in (50) increases so rapidly with increasing index $\mu$ that a solution of the linear equations (50) becomes soon unmanageable for humans. This is also confirmed by the following solutions of (50) obtained symbolically with the help of Maple 8:

\begin{align}
(z - 1) \gamma_0 &= 1, \\
(z - 1)^2 \gamma_1 &= z (a + b - c - 1), \\
(z - 1)^3 \gamma_2 &= z \{(a^2 + (b - c - 2) a + b^2 - (c + 2) b + 1 + 2 c) z \\
&\quad+ (b - c - 1) a - (c + 1) b + 1 + c + c^2\}, \\
(z - 1)^4 \gamma_3 &= z \{(a^3 + (b - c - 3) a^2 + (b^2 - (c + 3) b + 3 + 3 c) a \\
&\quad+ b^3 - (c + 3) b^2 + (3 + 3 c) b - 1 + 3 c) z^2 \\
&\quad+ [(2 b - 2 c - 3) a^2 + (2 b^2 - (4 c + 8) b + 2 c^2 + 7 + 8 c) a \\
&\quad- (2 c + 3) b^2 + (2 c^2 + 7 + 8 c) b - 4 - 5 c^2 - 7 c] z \\
&\quad+ [- (c + 1) b + 1 + c^2 + c] a + (1 + c^2 + c) b \\
&\quad- 1 - c^2 - c - c^3\}.
\end{align}

The solutions (52), which are rational in $z$, demonstrate quite clearly a principal weakness of symbolic computing. Typically, the results are complicated and poorly structured algebraic expressions, and it is normally very difficult to gain further insight from them. Nevertheless, symbolic solutions of the linear equations (50) are computationally very useful.

For the Gaussian hypergeometric series with $a = 1/3$, $b = 7/5$, $c = 9/2$, and $z = -0.85$, Maple 8 produced for $m = 8$ and $n = 1$ the following results:
\[ r_1 = -0.016412471, \quad (53a) \]
\[ a_2 [4/4] = -0.016410482, \quad (53b) \]
\[ a_2 r_1^{(8)} = -0.016414203, \quad (53c) \]
\[ a_2 r_1^{(8)} = -0.004008195. \quad (53d) \]

It is in my opinion quite remarkable that for \( n = 1 \), which is very far away from the asymptotic regime, at least the Padé approximant \( a_2 [4/4] \) and the truncated factorial series \( a_2 r_1^{(8)} \) agree remarkably well with the “exact” truncation error \( r_1 \). In contrast, the truncated inverse power series \( a_2 r_1^{(8)} \) produces a relatively poor result. For \( n = 10 \), which possibly already belongs to the asymptotic regime, Maple 8 produced the following results:

\[ r_{10} = 0.000031925482, \quad (54a) \]
\[ a_{11} [4/4] = 0.000031925482, \quad (54b) \]
\[ a_{11} r_{10}^{(8)} = 0.000031925483, \quad (54c) \]
\[ a_{11} r_{10}^{(8)} = 0.000031925471. \quad (54d) \]

Finally, let me emphasize that the formalism of this article is not limited to a Gaussian hypergeometric series \( 2 \), but works just as well in the case of a generalized hypergeometric series \( p+1F_p(a_1, \ldots, a_{p+1}; \beta_1, \ldots, \beta_p; z) \).

7 The Asymptotic Series for the Exponential Integral

The divergent asymptotic series \( 3 \) for the exponential integral \( E_1(z) \) is probably the most simple model for many other factorially divergent asymptotic inverse power series occurring in special function theory. Well known examples are the asymptotic series for the modified Bessel function \( K_\nu(z) \), the complementary error function \( \text{erfc}(z) \), the incomplete gamma function \( \Gamma(a, z) \), or the Whittaker function \( W_{\kappa, \mu}(z) \). Moreover, factorial divergence is also the rule rather than the exception among the perturbation expansions of quantum physics (see \( 22 \) for a condensed review of the relevant literature).

The exponential integral \( E_1(z) \) can also be expressed as a Stieltjes integral:

\[ z e^z E_1(z) = \int_0^\infty \frac{e^{-t}dt}{1+t/z}. \quad (55) \]

If \( z < 0 \), this integral has to be interpreted as a principal value integral.

In the case of a factorially divergent inverse power series, it is of little use to represent the truncation error \( r_n(z) \) by a power series as in \( 14 \). If, however, we use \( \sum_{\nu=0}^n x^\nu = \vert 1-x^{n+1} \vert / \vert 1-x \vert \) in \( 55 \), we immediately obtain:
\[ s_n(z) = \sum_{\nu=0}^{n} (-1/z)^\nu \nu! , \quad (56) \]
\[ r_n(z) = (-z)^{-n-1} \int_0^\infty \frac{t^{n+1} e^{-t} dt}{1 + t/z} , \quad (57) \]
\[ \Delta r_n(z) = (-1/z)^{n+1} (n+1)! . \quad (58) \]

Because of the factorial growth of the coefficients in (3), it is advantageous to use instead of (13) the following ansatz:
\[ r^{(m)}_n(z) = (-1/z)^n n! \sum_{\mu=0}^{m} \frac{\gamma_\mu}{(n+1)^{\mu}} , \quad m \in \mathbb{N} , \quad n \in \mathbb{N}_0 . \quad (59) \]

Again, we choose the unspecified coefficients \( \gamma_\mu \) in (59) in such a way that only a higher order error remains:
\[ \Delta r^{(m)}_n(z) = (-1/z)^{n+1} (n+1)! [1 + O(n^{-m-1})] , \quad n \to \infty . \quad (60) \]

This convergence condition can be reformulated as follows:
\[ \frac{r^{(m)}_{n+1}(z) - r^{(m)}_n(z)}{(-1/z)^{n+1}(n+1)!} = \frac{-z}{n+1} \sum_{\mu=0}^{m} \frac{\gamma_\mu}{(n+1)^{\mu}} - \sum_{\mu=0}^{m} \frac{\gamma_\mu}{(n+2)^{\mu}} \]
\[ = 1 + O(n^{-m-1}) , \quad n \to \infty . \quad (61) \]

Next, we do an asymptotic expansion of the right-hand side of (61) in terms of the asymptotic sequence \( \{1/(n+1)^j\}_{j=0}^\infty \) as \( n \to \infty \). This yields:
\[ \frac{r^{(m)}_{n+1}(z) - r^{(m)}_n(z)}{(-1/z)^{n+1}(n+1)!} = \sum_{\mu=0}^{m} \frac{C_\mu}{(n+1)^{\mu}} + O(n^{-m-1}) , \quad n \to \infty . \quad (62) \]

Again, we have to solve the following system of linear equations:
\[ C_\mu = \delta_{\mu0} , \quad 0 \leq \mu \leq m . \quad (64) \]

The following linear equations were constructed with the help of Maple 8:
\[ C_0 = -\gamma_0 = 1 , \quad (65a) \]
\[ C_1 = -z \gamma_0 - \gamma_1 = 0 , \quad (65b) \]
\[ C_2 = (1-z) \gamma_1 - \gamma_2 = 0 , \quad (65c) \]
\[ C_3 = -\gamma_1 + (2-z) \gamma_2 - \gamma_3 = 0 , \quad (65d) \]
\[ C_4 = \gamma_1 - 3 \gamma_2 + (3-z) \gamma_3 - \gamma_4 = 0 . \quad (65e) \]

If we compare the complexity of the equations (65) with those of (51), we see that in the case of the asymptotic series (3) for the exponential integral...
there may be a chance of finding explicit expressions for the coefficients $\gamma_\mu$. At least, the solutions of the linear system \(65\) obtained symbolically with the help of Maple 8 look comparatively simple:

\[
\begin{align*}
\gamma_0 &= -1, & (66a) \\
\gamma_1 &= z, & (66b) \\
\gamma_2 &= -(z-1)z, & (66c) \\
\gamma_3 &= (z^2 - 3z + 1)z, & (66d) \\
\gamma_4 &= -(z^3 - 6z^2 + 7z - 1)z. & (66e)
\end{align*}
\]

Of course, this requires further investigations.

The relative simplicity of the coefficients in \(66\) offers other perspectives. For example, the \([2/2]\) Padé approximant to the truncated power series in \(59\) is compact enough to be printed without problems:

\[
[2/2] = \frac{-1 + \frac{3 - z}{n + 1} + \frac{2}{(n+1)^2}}{1 + \frac{2z - 3}{n+1} + \frac{z^2 - 2z + 2}{(n+1)^2}} = \frac{n^2 - n + zn + z}{n^2 - n + 2zn + z^2}. \quad (67)
\]

Padé approximants to the truncated power series in \(59\) seem to be a new class of approximants that are rational in both $n$ and $z$.

For the asymptotic series \(3\) for $E_1(z)$ with $z = 5$, Maple 8 produced for $m = 16$ and $n = 2$ the following results:

\[
\begin{align*}
\gamma_{2} &= 0.027\,889, & (68a) \\
\gamma_{2}[8/8] &= 0.027\,965, & (68b) \\
\gamma_{2}^{(16)} &= 0.028\,358, & (68c) \\
\gamma_{2}^{(16)} &= -177.788. & (68d)
\end{align*}
\]

The Padé approximant $\gamma_{2}[8/8]$ and the truncated factorial series $\gamma_{2}^{(16)}$ agree well with the “exact” truncation error $\gamma_{2}$, but the truncated inverse power series $\gamma_{2}^{(16)}$ is way off. For $n = 10$, all results agree reasonably well

\[
\begin{align*}
\gamma_{10} &= 0.250\,470\,879, & (69a) \\
\gamma_{10}[8/8] &= 0.250\,470\,882, & (69b) \\
\gamma_{10}^{(16)} &= 0.250\,470\,902, & (69c) \\
\gamma_{10}^{(16)} &= 0.250\,470\,221. & (69d)
\end{align*}
\]

### 8 Conclusions and Outlook

A new formalism is proposed that permits the construction of asymptotic approximations to truncation errors $\gamma_{n} = -\sum_{\nu=n+1}^{\infty} a_{\nu}$ of infinite series for...
special functions by solving a system of linear equations. Approximations to
truncation errors of monotone series can be obtained via the Euler-Maclaurin
formula. The formalism proposed here is, however, based on different assump-
tions and can be applied even if the terms of the series have a comparatively
complicated structure. In addition, the new formalism works also in the case
of alternating and even divergent series.

Structurally, the asymptotic approximations of this article resemble the
asymptotic inverse power series for special functions as $z \to \infty$, since they are
not expansions of $r_n$, but rather expansions of ratios like $r_n/a_{n+1}$, $r_n/a_n$, or
$r_n/[(n+\alpha)a_{n+1}]$ with $\alpha > 0$. This is consequential, because it makes it possible
to use the convenient asymptotic sequence $\{1/(n+\alpha)^j\}_{j=0}^\infty$ of inverse powers.
This greatly facilitate the necessary analytical manipulations and ultimately
leads to comparatively simple systems of linear equations.

As shown in Section 5, the new formalism reproduces in the case of th e
Dirichlet series (1) for the Riemann zeta function the expressions (7a) or (9a)
that follow from the Euler-Maclaurin formula. The linear equations (35) are
equivalent to the recurrence formula (41) of the Bernoulli numbers. Thus, only
the integral (7b) or (9b) cannot be obtained in this way.

Much more demanding is the Gaussian hypergeometric series (2), wh ich
is discussed in Section 6. The terms of this series depend on three in general
complex parameters $a$, $b$, and $c$ and one argument $z$. Accordingly, there is little
hope that we might succeed in finding an explicit solution to the linear equa-
tions. However, all linear equations considered in this article have a triangular
structure. Consequently, it is relatively easy to construct solutions symboli-
cally with the help of a computer algebra system like Maple. The numerical
results presented in Section 6 also indicate that the formalism proposed in
this article is indeed computationally useful.

As a further example, the divergent asymptotic series (3) for the exponen-
tial integral $E_1(z)$ is considered in Section 7. The linear equations are again
solved symbolically by Maple. Numerical results are also presented. This ex-
ample is important since it shows that the new formalism works also in the
case of factorially divergent series. The Euler-Maclaurin formula can only
handle convergent monotone series.

Although the preliminary results look encouraging, a definite assess-
ment of the usefulness of the new formalism for the computation of special functions
is not yet possible. This requires much more data. Consequently, the new
formalism should be be applied to other series expansions for special functions
and the performance of the resulting approximations should be analyzed and
compared with other computational approaches.

I suspect that in most cases it will be necessary to solve the linear equa-
tions symbolically with the help of a computer algebra system like Maple. Nev ertheless, it cannot be ruled out that at least for some special functions
with sufficiently simple series expansions explicit analytical solutions to the
linear equations can be found.
Effective numerical analytic continuation methods are of considerable relevance for the new formalism which produces asymptotic approximations. We cannot tacitly assume that these approximations provide good results outside the asymptotic regime, although it would be highly desirable to use them also for small indices. In Section 4, only factorial series and Padé approximants are mentioned, although many other numerical techniques are known that can accomplish such an analytic continuation. Good candidates are sequence transformations which are often more effective than the better known Padé approximants. Details can be found in books by Brezinski and Redivo Zaglia [4], Sidi [15], or Wimp [26], or in a review by the present author [22].

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