Measurement-induced spin squeezing in a cavity

Hiroki Saito and Masahito Ueda
Department of Physics, Tokyo Institute of Technology, Tokyo 152-8551, Japan
and CREST, Japan Science and Technology Corporation (JST), Saitama 332-0012, Japan
(Dated: October 15, 2018)

Spin squeezing of an ensemble of \( N \) atoms in a high-Q cavity is shown to be enhanced by continuous measurement of photons in the cavity. A degree of spin squeezing proportional to \( N^{-1/3} \) is attained as a quasi-steady state over a broad set of initial states of the atoms and photons.

PACS numbers: 42.50.Ct, 42.50.Lc, 03.65.Ud, 06.20.Dk

I. INTRODUCTION

Squeezed spin states (SSS) \(^1\), in which quantum uncertainty of the collective spin in a direction orthogonal to the mean spin vector is suppressed below the standard quantum limit, have recently been realized experimentally \(^2\) \(^3\) \(^4\). In the SSS of atomic systems, entanglement between atoms is crucial for suppressing fluctuations in collective atomic spin, and therefore such systems are of potential interest in quantum information processing as well as in atomic interferometry.

A number of methods to generate SSS in atomic systems have been proposed, and these can be classified into several categories. A few examples that are relevant to the subject of the present paper are: (i) Squeezing is transferred between photons and atoms via their interactions \(^2\) \(^3\) \(^4\) \(^5\) \(^6\) \(^7\). Using this property, SSS can be realized in an atomic system by copying the squeezing from the system of photons to the atoms. This scheme has been demonstrated experimentally \(^2\). (ii) A probe laser beam propagating through atoms establishes entanglement between the atoms, with measurement of the probe laser inducing state reduction in the atomic states, resulting in SSS \(^8\) \(^9\) \(^10\). This kind of measurement-induced squeezing scheme has also been employed in experiments \(^11\) \(^12\). (iii) The interaction between atoms and photons confined in a high-Q cavity entangles them, and states of both atoms and photons develop into squeezed states \(^13\) \(^14\) \(^15\).

In the method (iii) proposed by Wineland \textit{et al.} \(^12\) and by us \(^13\) \(^14\) \(^15\), photons prepared in a coherent state and atoms prepared in a coherent spin state undergo interactions in a high-Q cavity, and after a certain interaction time SSS is realized temporarily, but not in a steady state. Precise control over the initial state and interaction time is therefore required. In the present paper, a novel scheme to generate SSS is proposed that combines methods (iii) and (ii) to circumvent the stringent requirements of the precise control needed for method (iii). SSS is thus obtained as a “quasi-steady state”, regardless of the initial state, by performing continuous measurement of photons in the cavity. This drastically reduces experimental difficulties in producing SSS. Moreover, a degree of spin squeezing proportional to \( N^{-1/3} \), where \( N \) is the number of atoms, can be achieved using this method, in contrast to the \( N^{-1/4} \) attainable by method (iii) \(^15\).

This paper is organized as follows. Section II describes the system we consider and develops a theory of continuous measurement of photon number in the cavity that is being pumped and contains atoms to be squeezed. Section III shows that the atomic state in the cavity develops into an SSS by continuous measurement of photons in the cavity. Section IV discusses possible experimental schemes and concludes this paper.

II. CONTINUOUS MEASUREMENT OF PHOTON NUMBER IN A PUMPED CAVITY WITH ATOMS

A. Description of the system

An operational theory of continuous photon counting has been proposed by Srinivas and Davies \(^14\), and microscopic modeling of this theory has been reported in Ref. \(^15\). The nonunitary state evolution due to measurement back action associated with the Srinivas-Davies model was investigated in Ref. \(^10\). In these studies, continuous measurement of single-mode photons in a high-Q cavity was considered. We extend this model to the case of a pumped cavity containing \( N \) two-level atoms as illustrated in Fig. 1.

We first consider the system without photon counting, namely, without the probe atoms and detector shown in Fig. 1. The high-Q cavity is assumed to sustain single-mode photons of frequency \( \omega_c \) and be pumped by a coherent light source with an effective amplitude \( A \) and frequency \( \omega_p \) inside the cavity. The cavity contains two-level atoms with transition frequency \( \omega_a \), which couple to photons via the Jaynes-Cummings interaction \(^17\) with coupling constant \( g \). We assume that \( g \) is constant and the same for all the atoms, that is, we consider the Lamb-Dicke limit, where all atoms experience the same photon field. It is then convenient to introduce a set of collective
the Hamiltonian (3) is time-independent:

$$\hat{H}_0 = \frac{\hbar}{2} \sum_{j=1}^{N} \left( |e_j\rangle \langle e_j| - |g_j\rangle \langle g_j| \right),$$

where $N$ is the number of atoms, and $|g_j\rangle$ and $|e_j\rangle$ are the ground and excited states of the $j$th atom. We shall also use other collective spin operators defined by $\hat{S}_z \equiv (\hat{S}_+ + \hat{S}_-) / 2$ and $\hat{S}_y \equiv (\hat{S}_+ - \hat{S}_-) / (2i)$.

The Hamiltonian for the system is given by

$$\hat{H}_0 = \omega_n \hat{n} + \omega_a \hat{S}_z + A (e^{i \omega_a t} \hat{a} + e^{-i \omega_a t} \hat{a}^\dagger) + g (\hat{a} \hat{S}_+ + \hat{a}^\dagger \hat{S}_-),$$

where $\hat{a}$ ($\hat{a}^\dagger$) is the annihilation (creation) operator of the photon field, $\hat{n} \equiv \hat{a}^\dagger \hat{a}$ is the photon-number operator, and $A$ and $g$ are taken to be real and positive without loss of generality. In Eq. (3), the first and the second terms on the right-hand side (rhs) describe the free Hamiltonian for the photon field and the collective atomic system, respectively; the third term describes the effect of coherent pumping of the photon field, and the last term describes the interaction between the photon field and the atomic system. In the rotating frame of reference $e^{i \omega_a (\hat{n} + \hat{S}_z)} |\psi\rangle$, the Hamiltonian (3) is time-independent:

$$\hat{H}_0^{\text{rot}} = \delta_{\text{cp}} \hat{n} + \delta_{\text{ap}} \hat{S}_z + A (\hat{a} + \hat{a}^\dagger) + g (\hat{a} \hat{S}_+ + \hat{a}^\dagger \hat{S}_-),$$

where $\delta_{\text{cp}} \equiv \omega_c - \omega_p$ and $\delta_{\text{ap}} \equiv \omega_a - \omega_p$.

We now consider continuous photon counting for the system described by the Hamiltonian (3). Experimentally, photon counting may be performed using a photodetector placed in the immediate vicinity of the cavity, where photons leaking out of the cavity are detected. This kind of continuous photodetection can be simulated using a sequence of probe atoms that are injected through the cavity [12], as illustrated in Fig. 1. The probe atoms are prepared in the ground state and injected into the cavity, with internal states detected upon leaving the cavity. If a probe atom is found to be in an excited state, we can say that one photon has been detected (one-count process). If it is found to have remained in the ground state, we can say that no photon has been detected (no-count process) [12, 16]. In the remaining part of this section we discuss nonunitary state evolution of the system in the one-count process, the no-count process, and more general processes.

### B. State evolution in the one-count process

If the passage time of each probe atom through the cavity is much shorter than the time scale of the state evolution described by Eq. (3), the density operator of the entire system (i.e., cavity photons plus atoms excluding the probe atoms) immediately after the one-count process at time $t$ is given by (see Appendix A for the derivation)

$$\hat{\rho}(t) = \frac{\hat{a} \hat{\rho}(t) \hat{a}^\dagger}{\text{Tr}[\hat{a} \hat{\rho}(t) \hat{a}^\dagger]},$$

where $t$ denotes a time infinitesimally later than $t$ and Tr is the trace of the entire system. The denominator on the rhs of Eq. (5) multiplied by $\gamma \Delta t$ gives the probability that a photon is detected between $t$ and $t + \Delta t$, where $\gamma$ is the photodetection rate defined by Eq. (4). The expression (5) takes the same form as that of the Srinivas-Davies model [14]. Immediately after the one-count process (5), the expectation value of an arbitrary operator $\hat{O}$ is given by

$$\left\langle \frac{\hat{\rho}(t)}{\hat{\rho}(t+\Delta t)} \right\rangle = \left\langle \frac{\hat{a} \hat{\rho}(t) \hat{a}^\dagger}{\hat{a}^\dagger \hat{\rho}(t) \hat{a}} \right\rangle,$$

where $\hat{a} \hat{\rho}(t) \hat{a}^\dagger$ denotes a time infinitesimally later than $t$ and Tr is the trace of the entire system. The denominator on the rhs of Eq. (5) multiplied by $\gamma \Delta t$ gives the probability that a photon is detected between $t$ and $t + \Delta t$, where $\gamma$ is the photodetection rate defined by Eq. (4). The expression (5) takes the same form as that of the Srinivas-Davies model [14]. Immediately after the one-count process (5), the expectation value of an arbitrary operator $\hat{O}$ is given by

$$\left\langle \hat{O} \right\rangle = \left\langle \hat{O} \hat{\rho}(t) \hat{O}^\dagger \right\rangle,$$

where $\hat{O}$ is the photon-number operator $\hat{n}$, we obtain

$$\left\langle \hat{n} \right\rangle_t = \left\langle \hat{n} \right\rangle_t - \frac{(\Delta \hat{n})^2_t}{\langle \hat{n} \rangle_t}.$$

When $\hat{O}$ is some spin operator $\hat{S}$, which commutes with $\hat{a}$, Eq. (6) becomes

$$\langle \hat{S} \rangle_t = \langle \hat{S} \rangle_t + \frac{(\Delta \hat{S} \Delta \hat{n})_t}{\langle \hat{n} \rangle_t}.$$
This indicates that upon the detection of a photon, the expectation value of the collective spin operator changes by an amount that depends on the correlation between the photons and atoms. As an example, let us consider an entangled state between an atom and a photon, where the ground and excited states of the atom are denoted by $|g\rangle$ and $|e\rangle$, and the vacuum and single-photon state by $|0\rangle$ and $|1\rangle$. Suppose that the state of the system before measurement is given by $(|g\rangle|1\rangle + |e\rangle|0\rangle)/\sqrt{2}$. Then $\langle S_\gamma \rangle$ decreases from 1/2 to 0 by the one-count process. This is because the measurement reveals that the cavity had contained a photon, and hence that the state of the system after the one-count process is given by $|g\rangle|0\rangle$. Similarly, $\langle S_\gamma \rangle$ increases upon the one-count process from 1/2 to 1 for an entangled initial state $(|g\rangle|0\rangle + |e\rangle|1\rangle)/\sqrt{2}$. In contrast, when the initial state is separable, i.e., $\langle S_\gamma \rangle = \langle S_\gamma \rangle(t_\gamma)$, the atomic quantities do not change by the one-count process.

### C. State evolution in the no-count process

When no photon is detected between $t$ and $t+T$, the density operator is shown to evolve according to the following equation (see Appendix A for the derivation),

$$
\hat{\rho}(t+T) = \frac{e^{-i \hat{H}_{\text{nc}} T/\hbar} \hat{\rho}(t) e^{i \hat{H}_{\text{nc}} T/\hbar}}{\text{Tr}\left[e^{-i \hat{H}_{\text{nc}} T/\hbar} \hat{\rho}(t) e^{i \hat{H}_{\text{nc}} T/\hbar}\right]}.
$$

(9)

where $\hat{H}_{\text{nc}} = \hat{H}_{\text{nc}}^0 - i\hbar \gamma \hat{n}/2$ is the non-Hermitian operator that governs the state evolution during the no-count process. The denominator on the rhs of Eq. (9) gives the probability that no photon is detected between $t$ and $t+T$. It follows from Eq. (9) that the master equation for the no-count process is given by

$$
\frac{\partial \hat{\rho}(t)}{\partial t} = \frac{i}{\hbar} \left[\hat{\rho}(t) \hat{H}_{\text{nc}}^\dagger - \hat{H}_{\text{nc}} \hat{\rho}(t)\right] + \gamma \langle \hat{n} \rangle_{t} \hat{\rho}(t).
$$

(10)

In fact, it can be shown by direct substitution of Eq. (9) into Eq. (10) that Eq. (9) is the solution of Eq. (10).

### D. State evolution in the referring and non-referring measurement processes

Time evolution under continuous photodetection is thus a stochastic process, in which the state evolution associated with one-count and no-count processes are described by Eqs. (9) and (10), respectively. The probability of the one-count process occurring between $t$ and $t+dt$ is $\gamma \langle \hat{n} \rangle_{t} dt$ and that of the no-count process is $1 - \gamma \langle \hat{n} \rangle_{t} dt$. The number of photons detected and the times at which they are detected, therefore, differ run by run even for the same initial state. The measurement process in which we read the outcome of the measurement at every instant of time is referred to as referring measurement process [10]. When the photons are detected at $t_1, t_2, \cdots, t_n$ from $t = 0$ to $t = T$, the time evolution of the density operator is given by

$$
\hat{\rho}(T) = \frac{\hat{P}(T; t_1, \cdots, t_n) \hat{\rho}(0) \hat{P}^\dagger(T; t_1, \cdots, t_n)}{\text{Tr}[\hat{P}(T; t_1, \cdots, t_n) \hat{\rho}(0) \hat{P}^\dagger(T; t_1, \cdots, t_n)]}.
$$

(11)

where

$$
\hat{P}(T; t_1, \cdots, t_n) = e^{-i \hat{H}_{\text{nc}} (T - t_n)/\hbar} \hat{a} e^{-i \hat{H}_{\text{nc}} (t_n - t_{n-1})/\hbar} \hat{a}^\dagger \cdots \hat{a} e^{-i \hat{H}_{\text{nc}} (t_2 - t_1)/\hbar} \hat{a} e^{-i \hat{H}_{\text{nc}} t_1}/\hbar.
$$

(12)

If the detector is switched on but we do not read out the measurement results, the state evolves into a statistical mixture of an ensemble of all possible outcomes, which is referred to as non-referring measurement process [10]. The master equation describing the state evolution in the non-referring measurement is obtained as follows. In the non-referring measurement process, $\hat{\rho}(t+dt)$ is a statistical mixture of the density operator $\rho_1(t+dt)$ of the one-count process and the density operator $\rho_0(t+dt)$ of the no-count process:

$$
\hat{\rho}(t+dt) = \gamma \langle \hat{n} \rangle_{t} \hat{\rho}_1(t+dt) dt + (1 - \gamma \langle \hat{n} \rangle_{t}) \hat{\rho}_0(t+dt).
$$

(13)

Substituting $\hat{\rho}_1(t+dt)$ for $\hat{\rho}(t+)\$ in Eq. (10) gives

$$
\gamma \langle \hat{n} \rangle_{t} \hat{\rho}_1(t+dt) dt = \gamma \hat{a} \hat{\rho}(t) \hat{a}^\dagger dt.
$$

(14)

Setting $T = dt$ in Eq. (10) gives

$$
\hat{\rho}_0(t+dt) = \hat{\rho}(t) + \frac{i}{\hbar} \left[\hat{\rho}(t) \hat{H}_{\text{nc}}^\dagger - \hat{H}_{\text{nc}} \hat{\rho}(t)\right] dt + \gamma \langle \hat{n} \rangle_{t} \hat{\rho}(t) dt + O(dt^2).
$$

(15)

Substituting Eqs. (14) and (15) into Eq. (13) gives the master equation for the non-referring measurement:

$$
\frac{\partial \hat{\rho}(t)}{\partial t} = \frac{i}{\hbar} \left[\hat{\rho}(t) \hat{H}_{\text{nc}}^\dagger - \hat{H}_{\text{nc}} \hat{\rho}(t)\right] + \frac{\gamma}{2} \left[\hat{a} \hat{\rho}(t) \hat{a}^\dagger - \hat{\rho}(t) \hat{\rho}(t) - \hat{\rho}(t) \hat{n}\right].
$$

(16)

When the cavity contains no atoms or when $g = 0$, we can show that the state of the photon field eventually evolves into a coherent state $|\alpha\rangle$ with $\alpha = -A/(\delta_\text{cp} - i\gamma/2)$ in both referring and non-referring measurement processes, unless the initial state is orthogonal with $|\alpha^*\rangle$ (see Appendix B). This fact is interesting in that the photon number never increases for the steady state $|\alpha\rangle$ despite the fact that the cavity is being pumped and that no photons are detected. This can be understood from the equation describing the time evolution of the mean photon number in the no-count process

$$
\frac{d \langle \hat{n} \rangle_{t}}{dt} = iA (\hat{a} - \hat{a}^\dagger)_{t} - \gamma (\langle \Delta \hat{n} \rangle^2)_{t},
$$

(17)

which can be obtained from Eq. (10). For the steady state $|\alpha\rangle$ with $\alpha = -A/(\delta_\text{cp} - i\gamma/2)$, the pumping term [the first term of the rhs of Eq. (17)] balances with the last term in Eq. (17) which describes the decrease in the number of photons due to the state reduction by the no-count process.
III. GENERATION OF SQUEEZED SPIN STATE BY CONTINUOUS PHOTODETECTION

A. Squeezed spin state

Because the spin operators $\hat{S}_x, \hat{S}_y, \hat{S}_z$ obey the commutation relation $[\hat{S}_x, \hat{S}_y] = i\hat{S}_z$, and the corresponding cyclic permutations, the spin vector fluctuates around the mean value $(\langle \hat{S}_x \rangle, \langle \hat{S}_y \rangle, \langle \hat{S}_z \rangle)$. For example, let us consider the lowest eigenstate of $\hat{S}_z$, $|S, -S\rangle$, where $S(S+1)$ is the eigenvalue of $\hat{S}^2 \equiv \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2$. The mean spin vector of this state is given by $\langle \hat{S} \rangle = (0, 0, -S)$. Quantum fluctuations around this are isotropic and given by $\langle (\Delta \hat{S}_x)^2 \rangle = \langle (\Delta \hat{S}_y)^2 \rangle = S/2$, satisfying the minimum uncertainty relation $\langle (\Delta \hat{S}_x)^2 \rangle \langle (\Delta \hat{S}_y)^2 \rangle + \langle (\Delta \hat{S}_z)^2 \rangle = (S^2/4)$. The SSS in atomic systems is discussed in Ref. [7].

The physical meaning of the SSS in atomic systems is such that $\gamma \gg S$, and hence we derive an approximate master equation of the no-count process for large $\gamma$. In the following derivation, we assume $\gamma \gg A$ and $A \sim gS$. Defining $\hat{\rho}'(t) \equiv e^{i\gamma t/2} \hat{\rho}(t) e^{-i\gamma t/2}$, we rewrite the master equation from the no-count process as

$$\hat{H}_{nc}/\hbar = A(\hat{a} + \hat{a}^\dagger) + g(\hat{a}\hat{S}_+ + \hat{a}^\dagger\hat{S}_-) - \frac{i\gamma}{2}\hbar,$$

(20)

where we define $\hat{\sigma} \equiv g\hat{S}_x + A$.

As will be shown in Sec. III.C, the generation of SSS is effective for $\gamma/g \gtrsim S$, and hence we derive an approximate master equation of the no-count process for large $\gamma$. The no-count process (9) is then given by

$$\frac{\partial \hat{\rho}'(t)}{\partial t} = i\hat{\rho}'(t) \left( e^{\frac{\gamma t}{2}}\hat{a}\hat{a}^\dagger + e^{-\frac{\gamma t}{2}}\hat{a}^\dagger\hat{a} \right)$$

$$- i\left( e^{-\frac{\gamma t}{2}}\hat{a}\hat{a}^\dagger + e^{\frac{\gamma t}{2}}\hat{a}^\dagger\hat{a} \right) \hat{\rho}'(t) + \gamma \langle \hat{n} \rangle \hat{\rho}'(t).$$

(21)

Since $\gamma \gg A$, the photon field is almost always in the vacuum state, which is justified below. We may therefore approximate $\hat{\rho}'(t)$ as $\hat{\rho}'(t) \simeq \langle 0 | \otimes \hat{\rho}_s(t)$, where $\hat{\rho}_s(t)$ denotes the reduced density operator for the spin. Integrating Eq. (21) iteratively gives

$$\hat{\rho}'(t + \Delta t) = \langle 0 | \otimes \left\{ \hat{\rho}_s(t) + \gamma \int_t^{t+\Delta t} dt_1 \langle \hat{n} \rangle_{t_1} \hat{\rho}_s(t) \right\}$$

$$- \int_t^{t+\Delta t} dt_2 \int_t^{t_2} dt_1 \left\{ e^{\gamma(t_1 - t_2)} (\hat{\sigma}^\dagger \hat{\sigma} \hat{\rho}_s(t) + \hat{\rho}_s(t) \hat{\sigma}^\dagger \hat{\sigma}) + \gamma^2 \langle \hat{n} \rangle_{t_1} \langle \hat{n} \rangle_{t_2} \right\}$$

$$+ |1 \rangle \langle 1 | \otimes \int_t^{t+\Delta t} dt_2 \int_t^{t_2} dt_1 2 e^{\gamma(t_1 + t_2)} \hat{\sigma} \hat{\rho}_s(t) \hat{\sigma}^\dagger + \cdots,$$

(22)

where we have kept only relevant terms. We take $\Delta t$ such that $\gamma \Delta t \gg 1$ and $\gamma \Delta t A^2/\gamma^2 \ll 1$ are satisfied. From the last term of Eq. (22), we find that the mean photon number is given by $\langle \hat{n} \rangle_{t_1} \approx 4 \gamma^2 \langle \hat{\sigma}^\dagger \hat{\sigma} \rangle_{t_1}$, which is $O(A^2/\gamma^2)$. The last term and the term including $\gamma^2 \langle \hat{n} \rangle_{t_1} \langle \hat{n} \rangle_{t_2}$ in Eq. (22) are therefore $O(A^2/\gamma^2)$ and can be ignored. Thus,
Eq. (22) can be approximated as
\[
\dot{\hat{\rho}}(t + \Delta t) \simeq |0\rangle\langle 0| \otimes \left\{ \hat{\rho}_s(t) - \frac{2\Delta t}{\gamma} [\hat{\sigma}^\dagger \hat{\sigma} \hat{\rho}_s(t) + \hat{\rho}_s(t) \hat{\sigma}^\dagger \hat{\sigma}] + \frac{4\Delta t}{\gamma} (\hat{\sigma}^\dagger \hat{\sigma})_t \hat{\rho}_s(t) \right\}.
\] (23)

Since, the last two terms in the curly brackets in Eq. (23) are \(O(\gamma \Delta t^2 / \gamma^2)\), which is assumed to be small, we obtain an effective master equation of the spin state for the no-count process:
\[
\frac{\partial \hat{\rho}_s(t)}{\partial t} = -\frac{2}{\gamma} [\hat{\sigma}^\dagger \hat{\sigma} \hat{\rho}_s(t) + \hat{\rho}_s(t) \hat{\sigma}^\dagger \hat{\sigma}] + \frac{4}{\gamma} (\hat{\sigma}^\dagger \hat{\sigma})_t \hat{\rho}_s(t).
\] (24)

It follows from Eq. (24) that the system evolves into the lowest eigenstate of the operator
\[
\hat{\sigma}^\dagger \hat{\sigma} = g^2 \hat{S}_+ \hat{S}_- + g A (\hat{S}_+ + \hat{S}_-) + A^2,
\] (25)
if the initial spin state is not orthogonal with the lowest eigenstate. To see this, let us expand the density operator as
\[
\hat{\rho}(t) = \sum_{n,m} c_{nm} |\phi_n\rangle \langle \phi_m|,
\] (26)
where \(|\phi_n\rangle\) is the eigenstate of the operator with eigenvalue \(\varepsilon_n\), i.e., \(\hat{\sigma}^\dagger \hat{\sigma} |\phi_n\rangle = \varepsilon_n |\phi_n\rangle\). The master equation (24) is rewritten as
\[
c_{nm}(t) = \frac{2}{\gamma} (2 \langle \hat{\sigma}^\dagger \hat{\sigma} \rangle_t - \varepsilon_n - \varepsilon_m) c_{nm}(t),
\] (27)
which can be solved to give
\[
c_{nm}(t) = \exp \left[ \frac{4}{\gamma} \int dt' \langle \hat{\sigma}^\dagger \hat{\sigma} \rangle_t - \frac{2}{\gamma} (\varepsilon_n + \varepsilon_m) t \right] c_{nm}(0).
\] (28)
Thus, the system develops into the state that minimizes \(\varepsilon_n + \varepsilon_m\), that is, the lowest eigenstate of the operator, if the corresponding initial coefficient \(c_{nm}(0)\) is not zero. We note that Eq. (25) is independent of \(\gamma\), and therefore small experimental fluctuations in \(\gamma\) do not affect the steady state. Since the relaxation coefficient in Eq. (24) is proportional to \(\gamma^{-1}\), the relaxation becomes slower for larger \(\gamma\). This is because the number of photons in the cavity is smaller for larger \(\gamma\), and therefore the state reduction caused by the no-count process becomes less effective.

The steady state of the master equation (24) is the SSS, as will be numerically shown in Sec. III C. We evaluate perturbatively the degree of spin squeezing of the steady state. When \(A \gg g\), the first-order perturbation with respect to \(g / A\) yields (Appendix C)
\[
\zeta = 1 - \frac{g}{4A} (N - 1) < 1,
\] (29)
showing that the state is an SSS. When \(A \ll g\), the second-order perturbation with respect to \(A / g\) yields (Appendix C)
\[
\zeta = 1 - \frac{g}{4A} (N - 1) < 1,
\] (30)
showing that the state is again an SSS.

### C. Exact numerical results

We numerically simulate the stochastic process of the continuous measurement described by Eqs. (5) and (10) with \(\hat{H}_{ac}\) given by Eq. (22). We divide the time into a small interval \(\Delta t\) such that \(\gamma (\hat{n}) \Delta t\) is always much smaller than unity. We then generate a random number \(r\) distributed uniformly between 0 and 1 for each time interval. If \(0 < r \leq \gamma (\hat{n}) \Delta t\), we assume that the one-count process occurs between \(t\) and \(t + \Delta t\) and calculate the density operator at \(t + \Delta t\) according to Eq. (11). If \(\gamma (\hat{n}) \Delta t < r \leq 1\), we assume that the no-count process occurs during \(t\) and \(t + \Delta t\) and calculate the density operator at \(t + \Delta t\) according to Eq. (10).

Figure 3 shows an example of the stochastic time evolution of various quantities in the referring measurement process with \(A / g = 5\), \(\gamma / g = 10\), and \(N = 10\), where the photon field is initially in the vacuum state and the atoms are in \(|S, -S\rangle\). The impulses that can be seen at the bottom of Fig. 2(c) indicate the times at which photons are detected. They are clearly bunched, reflecting the fact that \(g^{(2)} \equiv \langle \hat{a}^\dagger \hat{a}^2 \rangle / \langle \hat{a}^\dagger \hat{a} \rangle^2\) in the cavity is larger than unity most of the time as shown in Fig. 2(a). The bunching nature of photons makes the duration of each no-count process longer, helping the system reach the no-count steady state (shown as plateaus in Fig. 2). We note that in the no-count steady state, the SSS is generated with \(\zeta \simeq 0.56\) [solid curve in Fig. 2(c)], where \(\zeta\) is defined in Eq. (18). In the non-referring measurement process, in contrast, no SSS is obtained [dashed line in Fig. 2(c)], obtained by numerically integrating Eq. (10), indicating that the information concerning the detection times of photons is crucial for obtaining the SSS. It should be emphasized that in such a referring measurement process \(\zeta\) is smaller than unity most of the time, and therefore the SSS is obtained as a quasi-steady state.

Figure 3(a) shows the quasi-probability distribution (QPD) corresponding to the spin state at \(t = 15\), when the no-count steady state is reached. We can see that the spin fluctuation is squeezed in the azimuthal direction. However, at \(t = 16\) the spin state is disturbed by the photon counts and the corresponding QPD shown in Fig. 3(b) exhibits a pattern similar to a rotated Dicke state.
The degree of spin squeezing, $\zeta$, depends on the parameters $A$, $\gamma$, and $N$. Figure 2 shows the dependence of $\zeta_{st}$ on $A$ and $\gamma$ for $N = 2$ and 10, where $\zeta_{st}$ is the steady-state value of $\zeta$ in the no-count process. We see that the SSS is obtained for a wide range of parameters. We find that in both cases of $N = 2$ and $N = 10$ the maximum squeezing is obtained for $A/g \approx N/2 = S$, and we have numerically confirmed that this finding holds true for other values of $N$. For $\gamma \gg A$, $\zeta_{st}$ depends little on $\gamma$, in agreement with the analytic results of Eqs. (29), (30), and (31).

Figure 3 shows the dependence of $\zeta_{st}$ and the angle of the spin vector $\theta_{st}$ on $N$, where the squares are obtained by solving the master equation (10) and the circles by diagonalizing Eq. (29). Both plots are in excellent agreement with the analytic results of Eqs. (25), (29), and (31). The solid curves show examples of time evolutions of $\langle \hat{n} \rangle$, $g^{(2)} \equiv \langle \hat{a}^{\dagger 2}\hat{a}^2 \rangle/\langle \hat{n} \rangle$, $\langle \hat{S}_{z} \rangle$, and $\zeta$ as defined in Eq. (18) under continuous photodetection with a steady-state value of $\zeta = 10$. The initial state of the photon field is $|0\rangle_a$, the vacuum state, and that of the atoms is $|S, S_z = -S \rangle$. The impulses in (c) indicate the times at which photons are detected. The dashed lines show the values for the steady state in the no-referring measurement.

![FIG. 2: The solid curves show examples of time evolutions of $\langle \hat{n} \rangle$, $g^{(2)} \equiv \langle \hat{a}^{\dagger 2}\hat{a}^2 \rangle/\langle \hat{n} \rangle$, $\langle \hat{S}_{z} \rangle$, and $\zeta$ as defined in Eq. (18) under continuous photodetection with a steady-state value of $\zeta = 10$. The initial state of the photon field is $|0\rangle_a$, the vacuum state, and that of the atoms is $|S, S_z = -S \rangle$. The impulses in (c) indicate the times at which photons are detected. The dashed lines show the values for the steady state in the no-referring measurement.](image)

![FIG. 3: Gray-scale images of the quasi-probability distributions of the spin state $\langle \theta, \phi | \hat{r}_+ | \theta, \phi \rangle$, where $\hat{r}_+$ is the reduced density operator of spin and $| \theta, \phi \rangle$ is the coherent spin state defined in Eq. (19). (a) At $t = 15$ in Fig. 2 (no-count steady state) and (b) at $t = 16$ (just after a few photons were counted).](image)

tablze T I: Power laws of the optimal squeezing parameter $\zeta$ with respect to the number of atoms $N$ for various squeezing schemes.

| Scheme                        | $\zeta$     |
|-------------------------------|-------------|
| two-axis twisting [1]         | $N^{-1}$    |
| one-axis twisting [1]         | $N^{-2/3}$  |
| Jaynes-Cummings (with measurement) | $N^{-1/3}$ |
| Jaynes-Cummings (without measurement) [2] | $N^{-1/4}$ |

The dependence of $\zeta$ on $N$ in these schemes.

**IV. DISCUSSIONS AND CONCLUSIONS**

When the number of excited atoms is much smaller than the total number of atoms $N$, the Holstein-Primakoff transformation [18] can be approximated by $\hat{S}_- \equiv (2S)^{1/2} \hat{b}$ and $\hat{S}_+ \equiv (2S)^{1/2} \hat{b}^\dagger$, where $\hat{b}$ and $\hat{b}^\dagger$ are the boson operators satisfying $[\hat{b}, \hat{b}^\dagger] = 1$. The non-Hermitian operator describing the no-count process (20) then becomes

$$\hat{H}_{nc}/\hbar = A(\hat{a} + \hat{a}^\dagger) + G(\hat{a}^\dagger \hat{b} + \hat{a} \hat{b}^\dagger) - i\gamma/2 \hat{n}, \quad (31)$$

where $G \equiv (2S)^{1/2} g$. In a similar manner to the derivation given in Appendix [18] one can show that during the no-count process the system develops into $|0\rangle_a |\beta\rangle_b$, where $|0\rangle_a$ is the vacuum state of the photon field and $|\beta\rangle_b = e^{\beta(\hat{a}^\dagger - \hat{a})} |0\rangle_b$ is the coherent state of the boson with $\beta = -A/G$. Therefore, the bosonization of spin eliminates the squeezing properties of atoms, and thus the spin algebra, which plays a role in nonlinear saturation, is crucial for obtaining squeezing in the present scheme.

Experimentally, the model presented may be implemented in various systems, such as a microwave cavity.
with Rydberg atoms [19], an optical cavity with cold atoms [21], and trapped ions coupled with their center-of-mass motion [12]. For example, let us consider a system in which Rb atoms in the circular Rydberg state pass through a superconducting microwave cavity [19].

The coupling constant between the atoms and the cavity photons can be varied from 0 to 24 kHz [19]. If we take \( \kappa/(2\pi) = 2 \text{ kHz} \), \( g/(2\pi) = 100 \text{ Hz} \), \( \tau_p = 10 \mu s \), and \( N = 10 \), the conditions (A3) are satisfied as \( \kappa \tau_p \simeq 0.1 \) and \( gS \tau_p \simeq A \tau_p \simeq 0.03 \), where \( \kappa \) is the coupling constant between probe atoms and photons, and \( \tau_p \) is the passage time of a probe atom through the cavity (see Appendix A). These parameters give \( \gamma/g = \kappa^2 \tau_p/g \simeq 2.5 \), and the lifetime of the Rydberg state \( \simeq 30 \text{ ms} \) imposes the constraint \( gt < 19 \). The cavity relaxation time now reaches 0.3 s [21]. Thus the scheme is experimentally feasible.

In conclusion, we have studied the system in which \( N \) two-level atoms interact with photons in a pumped cavity, and have shown that the atomic state develops into an SSS by continuous measurement of photon number in the cavity. In particular, the SSS is obtained as a quasi-steady state during the no-count process. The present method does not require fine tuning of the initial state and the time and strength of interaction, and therefore significantly facilitates experimental realization of the SSS.

Acknowledgments

This work was supported by the Special Coordination Funds for Promoting Science and Technology and a Grant-in-Aid for Scientific Research (Grant No. 11216204) by the Ministry of Education, Science, Sports, and Culture of Japan, by the Yamada Science Foundation, and by the Toray Science Foundation.

APPENDIX A: A MODEL OF THE CONTINUOUS MEASUREMENT OF PHOTON NUMBER

For continuous measurement of photon number, the Srinivas-Davies “ansatz” [14] is usually employed. However, it is by no means obvious whether it is valid in our system, and therefore, we explicitly derive nonunitary time evolutions (5) and (9) using the continuous photodetection model of Ref. [15].

We consider a situation in which probe atoms prepared in the ground state \( |g\rangle \) pass through the cavity successively, and the internal states of the atoms coming out of the cavity are detected by, e.g., an ionization detector (Fig. 1). When a probe atom is detected to be in the excited state \( |e\rangle \), we regard this event as a one-count process. When the atom coming out of the cavity remains in the ground state, we regard this event as a no-count process in the cavity.
The time evolution as the probe atom passes through the cavity is described by the Hamiltonian
\[ \hat{H}_p = \hat{H}_{0}^{\text{rot}} + \hbar \kappa (\hat{a}^\dagger \hat{\sigma}_+ + \hat{a} \hat{\sigma}_-), \]
where \( \hat{H}_0^{\text{rot}} \) is given in Eq. \( \ref{A1} \), \( \hat{\sigma}_+ \) and \( \hat{\sigma}_- \) are the raising and lowering operators of the probe atom, and \( \kappa \) is a coupling constant which we assume to be real. We assume that the transition frequency of the probe atom is resonant with the cavity mode. When the probe atom enters the cavity at time \( t \) and comes out of it at \( t + \tau_p \), the density operator \( \hat{\rho}_{sp} \) for the system and the probe atom evolves as
\[ \hat{\rho}_{sp}(t + \tau_p) = \hat{\rho}(t) \otimes |g\rangle \langle g| - \frac{i\tau_p}{\hbar} [\hat{H}_p, \hat{\rho}(t) \otimes |g\rangle \langle g|] \
- \frac{\tau_p^2}{2\hbar^2} [\hat{H}_p, [\hat{H}_p, \hat{\rho}(t) \otimes |g\rangle \langle g|]] + \cdots \] (A2)
where \( \hat{\rho}(t) \) is the density operator for the system (excluding the probe atom).

Here we assume that \( \tau_p \) is sufficiently small so that third and higher-order terms in Eq. \( \ref{A2} \) can be ignored. This approximation is valid if the change in the state of the system is small during \( \tau_p \). This condition is satisfied if the following conditions are met:

\[ A\tau_p^{-1} \ll 1, \quad gS\tau_p^{-1} \ll 1, \quad \delta_{cp}\tau_p^{-1} \ll 1, \quad \delta_{sp}\tau_p^{-1} \ll 1. \] (A3)

When the probe atom is detected to be in the excited state \( |e\rangle \), the density operator of the system is projected into
\[ \text{Tr}_{\text{probe}}[|\alpha\rangle \langle \alpha| \hat{\rho}_{sp}(t + \tau_p)] = \kappa^2 \tau_p^2 \hat{\rho}(t) \hat{n} \hat{1}. \] (A4)

The normalized form of Eq. \( \ref{A4} \) gives the density operator immediately after the no-count process:
\[ \hat{\rho}(t+) = \frac{\hat{n} \hat{1}}{\text{Tr}[\hat{n} \hat{1}]}, \] (A5)
where \( t+ \) denotes a time infinitesimally later than \( t \). The probability that the one-count process \( \ref{A4} \) occurs is given by the trace of Eq. \( \ref{A4} \), i.e., \( \gamma \tau_p \langle \hat{n} \rangle_t \), where
\[ \gamma \equiv \kappa^2 \tau_p, \]

can be interpreted as the rate of photodetection per one photon.

When the probe atom is detected in its ground state \( |g\rangle \), the density operator of the system is projected into
\[ \text{Tr}_{\text{probe}}[|g\rangle \langle g| \hat{\rho}_{sp}(t + \tau_p)] \approx \hat{\rho}(t) - \frac{i\tau_p}{\hbar} [\hat{H}_0^{\text{rot}}, \hat{\rho}(t)] - \frac{\tau_p^2}{2\hbar^2} [\hat{H}_0^{\text{rot}}, [\hat{H}_0^{\text{rot}}, \hat{\rho}(t)]] \
- \frac{\tau_p^2 \kappa^2}{2} [\hat{n}, \hat{\rho}(t) + \hat{\rho}(t) \hat{n}] \
\approx e^{-\tau_p (\hat{H}_0^{\text{rot}} - \hat{n})} \hat{\rho}(t) e^{\tau_p (\hat{H}_0^{\text{rot}} - \hat{n})}. \] (A7)

When successive probe atoms are found to be in the ground state over a time interval \( T \), the density operator of the system after such a no-count process is given by Eq. \( \ref{A7} \).

Probe atoms were introduced to explicitly derive Eqs. \( \ref{A4} \) and \( \ref{A5} \). However, these expressions may also describe a more general system under continuous photodetection, such as the situation in which photons leaking out of the cavity are detected by a photodetector.

**APPENDIX B: STEADY STATE IN A PUMPED CAVITY UNDER CONTINUOUS PHOTODETECTION**

We show that the state of the system described by the Hamiltonian \( \ref{A1} \) eventually evolves into a coherent state in both referring and non-referring measurements.

The non-Hermitian operator that describes the no-count process \( \ref{A1} \) is rewritten in the rotating frame by
\[ \hat{H}_{nc}/\hbar = A(\hat{a} + \hat{a}^\dagger) + (\delta_{cp} - i\gamma/2)\hat{n} \]
\[ = \Omega \left( \hat{a} + \frac{A}{\Omega} \right) \left( \hat{a} + \frac{A}{\Omega} \right) - \frac{A^2}{\Omega} \]
\[ = \Omega e^{\frac{i}{2}(\hat{a} + \hat{a}^\dagger)} \hat{n} e^{-\frac{i}{2}(\hat{a} + \hat{a}^\dagger)} - \frac{A^2}{\Omega}, \] (B1)
where we define \( \Omega \equiv \delta_{cp} - i\gamma/2 \). We then obtain
\[ e^{-i\hat{H}_{nc}t/\hbar} \theta e^{-i\hat{H}_{nc}t/\hbar} \theta = \sum_{n=0}^{\infty} e^{-i\delta_{cp}nt - n\gamma t/2} e^{\frac{i}{2}(\hat{a} + \hat{a}^\dagger)} |n\rangle \langle n| e^{-\frac{i}{2}(\hat{a} + \hat{a}^\dagger)}, \] (B2)

For \( \gamma t \gg 1 \), the term corresponding to \( n = 0 \), i.e., \( e^{\frac{i}{2}(\hat{a} + \hat{a}^\dagger)} |0\rangle \langle 0| e^{-\frac{i}{2}(\hat{a} + \hat{a}^\dagger)} \theta \) becomes dominant in the summation, where \( |\alpha\rangle \) is the coherent state with \( \alpha = -A/\Omega \). We thus obtain
\[ \lim_{t \to \infty} \frac{e^{-i\hat{H}_{nc}t/\hbar} |\psi\rangle}{|e^{-i\hat{H}_{nc}t/\hbar} |\psi\rangle|} = |\alpha\rangle \] (B3)
for \( |\alpha\rangle |\psi\rangle \neq 0 \). Therefore, during the no-count process \( \ref{A1} \) the state develops into \( |\alpha\rangle \) unless the initial state is orthogonal to \( |\alpha^*\rangle \). If \( \langle \alpha^* | \psi \rangle = 0 \), the term of \( n = 1 \) in the summation of Eq. \( \ref{B2} \) becomes dominant, and the state develops into a displaced number state.

If the photons are detected at times \( t_1, t_2, \ldots, t_n \), the time development operator \( \ref{B1} \) for the referring measurement takes the form
\[ \hat{P}(T; t_1, \ldots, t_n) = e^{-i\hat{H}_{nc}T/\hbar} \prod_{k=1}^{n} \left[ e^{-i\delta_{hk}t_k} + \frac{A}{\Omega} (e^{-i\delta_{hk}t_k} - 1) \right], \] (B4)

From Eqs. \( \ref{B3} \) and \( \ref{B4} \) we obtain
\[ \lim_{T \to \infty} \hat{P}(T; t_1, \ldots, t_n) |\psi\rangle \propto |\alpha\rangle. \] (B5)
Thus, the state develops to $|\alpha\rangle$ in the referring measurement. This result is also true for non-referring measurement, since Eq. (20) depends neither on the number of detected photons nor on the times at which photons are detected.

APPENDIX C: PROOF OF EQS. (20) AND (30)

We calculate perturbatively the squeezing parameter for the lowest eigenstate of the operator $\tilde{\sigma}$. When $g/A \ll 1$, the first term on the rhs of Eq. (20) can be treated as a perturbation. For convenience, we rotate in the spin space by $\pi/2$ with respect to the $S_y$ axis. The operator $\tilde{\sigma}$ then becomes

$$\tilde{\sigma}^{\dagger} \tilde{\sigma} = A^2 \left[ 1 + \frac{2g}{A} \hat{S}_z + \frac{g^2}{A^2} (\hat{S}_z^2 - \hat{S}_x^2 - \hat{S}_y) \right].$$  \hspace{1cm} (C1)

The unperturbed ground state is then taken to be the eigenstate of $\hat{S}_z$ with eigenvalue $-S$, which we write as $|S, -S\rangle$. Performing first-order perturbation with respect to the last term of Eq. (C1), we obtain the ground state as

$$|S, -S\rangle + \frac{\sqrt{2Sg}}{4A} \left( |S, 1 - S\rangle + \frac{1}{4} \sqrt{2(2S - 1)} |S, 2 - S\rangle \right),$$  \hspace{1cm} (C2)

with the squeezing parameter calculated to be

$$\zeta = 1 - \frac{g}{4A} (2S - 1).$$  \hspace{1cm} (C3)

When $A \ll g$, the second term on the rhs of Eq. (20) may be treated as a perturbation. In this case, we must perform second-order perturbation to obtain spin squeezing. The ground state is then given by

$$\left( 1 - \frac{A^2}{4Sg^2} |S, -S\rangle - \frac{A}{\sqrt{2Sg}} |S, 1 - S\rangle + \frac{A^2}{g^2} \frac{1}{\sqrt{4S(2S - 1)}} |S, 2 - S\rangle \right),$$  \hspace{1cm} (C4)

with the squeezing parameter calculated to be

$$\zeta = 1 - \frac{A^2}{2S^2g^2}.$$  \hspace{1cm} (C5)

[1] M. Kitagawa and M. Ueda, Phys. Rev. Lett. 67, 1852 (1991); Phys. Rev. A 47, 5138 (1993).
[2] J. Hald, J. L. Sørensen, C. Schori, and E. S. Polzik, Phys. Rev. Lett. 83, 1319 (1999).
[3] A. Kuzmich, L. Mandel, and N. P. Bigelow, Phys. Rev. Lett. 85, 1594 (2000).
[4] B. Julsgaard, A. Kozhekin, and E. S. Polzik, Nature 413, 400 (2001).
[5] H. Saito and M. Ueda, Phys. Rev. Lett. 79, 3869 (1997).
[6] A. Kuzmich, K. Molmer, and E. S. Polzik, Phys. Rev. Lett. 79, 4782 (1997).
[7] H. Saito and M. Ueda, Phys. Rev. A 59, 3959 (1999).
[8] U. V. Poulsen and K. Mølmer, Phys. Rev. Lett. 87, 123601 (2001).
[9] A. Kuzmich, N. P. Bigelow, and L. Mandel, Europhys. Lett. 42, 481 (1998).
[10] Y. Takahashi, K. Honda, N. Tanaka, K. Toyoda, K. Ishikawa, and T. Yabuzaki, Phys. Rev. A 60, 4974 (1999).
[11] L. M. Duan, J. I. Cirac, P. Zoller, and E. S. Polzik, Phys. Rev. Lett. 85, 5643 (2000).
[12] D. J. Wineland, J. J. Bollinger, W. M. Itano, F. L. Moore, and D. J. Heinzen, Phys. Rev. A 46, R6797 (1992); D. J. Wineland, J. J. Bollinger, W. M. Itano, and D. J. Heinzen, ibid. 50, 67 (1994).
[13] M. Ueda, T. Wakahayashi, and M. Kuwata-Gonokami, Phys. Rev. Lett. 76, 2045 (1996).
[14] M. D. Srinivas and E. B. Davies, Opt. Acta 28, 981 (1981); 29, 235 (1982).
[15] N. Imoto, M. Ueda, and T. Ogawa, Phys. Rev. A 41, 4127 (1990).
[16] M. Ueda, Quantum Opt. 1, 131 (1989); M. Ueda, N. Imoto, and T. Ogawa, Phys. Rev. A 41, 3891 (1990); M. Ueda, Phys. Rev. A 41, 3875 (1990).
[17] E. Jaynes and F. Cummings, Proc. IEEE 51, 89 (1963).
[18] T. Holstein and H. Primakoff, Phys. Rev. 58, 1098 (1940).
[19] J. M. Raimond, M. Brune, and S. Haroche, Rev. Mod. Phys. 73, 565 (2001), and references therein.
[20] C. J. Hood, M. S. Chapman, T. W. Lynn, and H. J. Kimble, Phys. Rev. Lett. 80, 4157 (1998); C. J. Hood, T. W. Lynn, A. C. Doherty, A. S. Parkins, and H. J. Kimble, Science 287, 1447 (2000).
[21] S. Brattke, B. T. H. Varcoe, and H. Walther, Phys. Rev. Lett. 86, 3534 (2001).
[22] For example, J. J. Sakurai, Modern Quantum Mechanics, (Addison-Wesley, Reading, MA, 1994), Chap. 5.