AMENABILITY PROPERTIES OF THE CENTRES OF GROUP ALGEBRAS

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Abstract. Let $G$ be a locally compact group, and $\text{ZL}^1(G)$ be the centre of its group algebra. We show that when $G$ is compact $\text{ZL}^1(G)$ is not amenable when $G$ is either nonabelian and connected, or is a product of infinitely many finite nonabelian groups. We also, study, for some non-compact groups $G$, some conditions which imply amenability and hyper-Tauberian property, for $\text{ZL}^1(G)$.

Let $G$ be a locally compact group and $L^1(G)$ denote the group algebra, i.e. the subalgebra of the measure algebra $\mathcal{M}(G)$ consisting of measures which are absolutely continuous with respect to the left Haar measure. We let

$$\text{ZL}^1(G) = \{ f \in L^1(G) : f * g = g * f \text{ for all } g \in L^1(G) \}$$

be the centre of $L^1(G)$. Our goal is to study amenability and weak amenability for $\text{ZL}^1(G)$.

We show that when $G$ is compact, $\text{ZL}^1(G)$ is generally not amenable. In fact, it fails to be amenable whenever $G$ is either non-abelian and connected (Section 1.4), or when $G$ is a product of infinitely many non-abelian finite groups (Section 1.5). These results make substantial use of some discoveries from the intensive study of central idempotent measures on compact groups of D. Rider [24]. It is mentioned in [26] that it is known to B. E. Johnson that $\text{ZL}^1(G)$ fails to be amenable for some compact group $G$. However no further information is provided. Our results, and techniques therein, lead us towards the following.

Conjecture 0.1. If $G$ is compact, then $\text{ZL}^1(G)$ is amenable if and only if $G$ admits an open abelian subgroup.

We note that by [19], $G$ admits an open abelian subgroup if and only if the set of degrees of its irreducible representations is bounded. We address the
above conjecture with an illustrative example (Section 1.6). As a complement to many of the methods used in the prior sections, we illustrate two examples using hypergroup techniques (Section 1.7).

We close the article with a study of some non-compact groups. We use results of R.D. Mosak and J. Liukkonen [20, 18, 21] extensively. When the commutator of $G$ with the open subgroup, which supports all elements of $ZL^1(G)$, is finite, then $ZL^1(G)$ is amenable (Section 2.2). When $G$ has relatively compact conjugacy classes, then $ZL^1(G)$ is hyper-Tauberian (Section 2.3). We outline the basic theory of hyper-Tauberian algebras, below.

0.1. Amenability. If $A$ is a Banach algebra, we let $A \hat{\otimes} A$ denote the projective tensor product of $A$ with itself. Following B. E. Johnson [14], we say $A$ is amenable if it admits a bounded approximate diagonal (b.a.d.): a bounded net $(\mu_\alpha) \subset A \hat{\otimes} A$ which satisfies

$$m(\mu_\alpha)a, am(\mu_\alpha) \to a \text{ and } a \cdot \mu_\alpha - \mu_\alpha \cdot a \to 0$$

for $a$ in $A$, where $m : A \hat{\otimes} A \to A$ is the multiplication map, and the module actions of $A$ on $A \hat{\otimes} A$ are given on elementary tensors by $a \cdot (b \otimes c) = (ab) \otimes c$ and $(b \otimes c) \cdot a = b \otimes (ca)$. As shown in [14], amenability is equivalent to the existence of a virtual diagonal: an element $M$ in $(A \hat{\otimes} A)^{**}$ such that

$$a \cdot M = M \cdot a \text{ and } am^{**}(M) = m^{**}(M)a = a$$

for $a$ in $A$, where the module actions of $A$ on $(A \hat{\otimes} A)^{**}$ and $A^{**}$, are the second adjoints of the module actions of $A$ on $A \hat{\otimes} A$ and $A$, respectively, and $m^{**}$ is the second adjoint of the multiplication map.

We can quantify amenability via the amenability constant, which was defined in [15]. Let

$$\text{AM}(A) = \inf \left\{ \sup_\alpha \| \mu_\alpha \| : (\mu_\alpha) \text{ is a b.a.d. for } A \right\}$$

where we allow the infemum of an empty set be $\infty$.

The above definition is equivalent to a cohomological one: $A$ is amenable if every derivation into a dual Banach $A$-bimodule is inner; see [13] for more on this. We say $A$ is weakly amenable if every bounded derivation into $A^*$ is inner. If $A$ is commutative, this is equivalent to having every bounded derivation into any symmetric bimodule be inner; see [2]. We will not directly conduct any computations with derivations. We note the important fact that $L^1(G)$ is amenable exactly when $G$ is an amenable group [13].

0.2. The hyper-Tauberian property. Let $A$ be a commutative semisimple Banach algebra. Suppose $A$ is regular on its spectrum $\mathcal{X}$; we regard $A$ as an algebra of functions on $\mathcal{X}$. If $\varphi \in A^*$ we define

$$\text{supp}\varphi = \left\{ \chi \in \mathcal{X} : \text{ for every neighbourhood } U \text{ of } \chi \text{ there is } f \in A \text{ such that } \text{supp}f \subset U \text{ and } \varphi(f) \neq 0 \right\}.$$
A linear operator $T : \mathcal{A} \to \mathcal{A}^*$ is said to be local if

$$\text{supp} Tf \subset \text{supp} f$$

for every $f$ in $\mathcal{A}$. We say $\mathcal{A}$ is hyper-Tauberian if every bounded local operator $T : \mathcal{A} \to \mathcal{A}^*$ is an $\mathcal{A}$-module map. This concept was developed by the second named author [25] to study the reflexivity of the (completely bounded) derivation space of $\mathcal{A}$. However, it has nice applications to weak amenability and spectral synthesis problems, which we summarise.

**Theorem 0.2.** If $\mathcal{A}$ is hyper-Tauberian then

(i) $\mathcal{A}$ is weakly amenable;

(ii) finite subsets of $\mathcal{X}$ are sets of spectral synthesis; and

(iii) if $\mathcal{A} \hat{\otimes} \mathcal{A}$ is semi-simple, then $\{(\chi, \chi) : \chi \in \mathcal{X}\}$ is a set of local synthesis for that algebra, and hence is a set of spectral synthesis when $\mathcal{A}$ has a bounded approximate identity.

See [25] Theorem 5, Corollary 8 and Theorem 6, for the proof.

1. Compact groups

1.1. **Notation.** In this section we let $G$ denote a compact group. Let $\hat{G}$ denote the set of equivalence classes of irreducible representations of $G$. By standard abuse of notation, we will use $\hat{G}$ to denote a set or representatives, one from each equivalence class. We let $d_\pi$ denote the dimension of $\pi$. We let $\text{ZM}(G)$ denote the centre of the measure algebra.

Let for $\pi$ in $\hat{G}$,

$$\chi_\pi = \text{Tr}\pi(\cdot) \text{ and } \psi_\pi = \frac{1}{d_\pi} \chi_\pi$$

so $\chi_\pi$ is the character of $\pi$ and $\psi_\pi$ the normalised character with $\psi_\pi(e) = 1$. If $\mu \in \text{M}(G)$ we let

$$\hat{\mu}(\pi) = \int_G \pi(s)d\mu(s) \in \mathcal{B}(\mathcal{H}_\pi).$$

If $\mu \in \text{ZM}(G)$ then it is well-known, and straightforward to compute that

$$\hat{\mu}(\pi) = \int_G \tilde{\psi}_\pi d\mu \cdot I_{\mathcal{H}_\pi}$$

and we then let

$$(1.1) \quad \hat{\mu}(\pi) = \int_G \tilde{\psi}_\pi d\mu.$$

We note that $f \mapsto \tilde{f}$ is the Gelfand transform on $\text{ZL}^1(G)$. 
1.2. Some functorial properties of the centre of the group algebra.

We recall, as observed in [20, Prop. 1.5], that the map

\[ P = P_G : L^1(G) \to ZL^1(G), \quad Pf(s) = \int_G f(ts^{-1})dt \]

is a surjective quotient map.

**Proposition 1.1.** \( ZL^1(G) \hat{\otimes} ZL^1(G) \cong ZL^1(G \times G) \).

**Proof.** This follows from the fact that \( P_G \) is a surjective quotient map and that in the identification \( L^1(G) \hat{\otimes} L^1(G) \cong L^1(G \times G) \), we have that \( P_G \otimes P_G = P_{G \times G} \). \( \square \)

If \( N \) is a closed normal subgroup of \( G \), we have a map

\[ T_N : C(G) \to C(G/N), \quad T_N f(sN) = \int_N f(sn)dn \]

for every \( sN \) in \( G/N \). This map extends to a surjective quotient map from \( L^1(G) \) to \( L^1(G/N) \) which we again denote \( T_N \). See [23, Thm. 3.5.4].

**Proposition 1.2.** \( T_N(ZL^1(G)) = ZL^1(G/N) \) and \( T_N : ZL^1(G) \to ZL^1(G/N) \) is a surjective quotient map.

**Proof.** It is sufficient to verify that

\[ T_N \circ P_G = P_{G/N} \circ T_N \]

since each are surjective quotient maps. For \( f \in C(G) \) we have for \( s \) in \( G \), using Weyl’s integral formula, that

\[
T_N \circ P_G f(sN) = \int_N \int_G f(ts^{-1})dt\,dn \\
= \int_N \int_{G/N} \int_N f(tn's(tn')^{-1})dn'\,dt\,N\,dn \\
= \int_{G/N} \int_N \int_N f(tn's(tn')^{-1})dn'\,dn\,dt\,N \\
= \int_{G/N} T_N f(ts^{-1})dt\,N = P_{G/N} \circ T_N f(sN).
\]

Since \( C(G) \) is dense in \( L^1(G) \) we are done. \( \square \)

**Corollary 1.3.** If \( N \) is a closed normal subgroup of \( G \) then \( AM(ZL^1(G)) \geq AM(ZL^1(G/N)) \). In particular, if \( ZL^1(G) \) is amenable, then \( ZL^1(G/N) \) is amenable.

**Proof.** If \( (\mu_\alpha) \subset ZL^1(G) \hat{\otimes} ZL^1(G) \) is an approximate diagonal for \( ZL^1(G) \), then it is a standard fact that \( (T_N \otimes T_N(\mu_\alpha)) \) is an approximate diagonal for \( ZL^1(G/N) \). \( \square \)
Let \( \tilde{m} : M(G \times G) \to M(G) \) be given by
\[
\int_G u \, d\tilde{m}(\mu) = \int_{G \times G} u(st) \, d\mu(s, t) \text{ for } u \in C(G).
\]
Then \( \tilde{m}(\mu \otimes \nu) = \mu \ast \nu \) and, \( \tilde{m} \) is the weak* continuous extension of the multiplication map \( m : L^1(G) \otimes L^1(G) \cong L^1(G \times G) \to L^1(G) \).

**Proposition 1.4.**

(i) \( ZM(G) \) is weak* closed in \( M(G) \) and \( ZL^1(G) \) is weak* dense in \( ZM(G) \).

(ii) \( \tilde{m}(ZM(G \times G)) = ZM(G) \) and \( \tilde{m} : ZM(G \times G) \to ZM(G) \) is a homomorphism.

**Proof.**

(i) The product on \( M(G) \) is well-known to be weak* continuous in each variable. Hence if \( (\mu_s) \) is a net contained in \( ZM(G) \) with weak* limit point \( \mu \), then for each \( \nu \) in \( M(G) \) we have
\[
\nu \ast \mu = w^* \lim_{\alpha} \nu \ast \mu_s = w^* \lim_{\alpha} \mu_s \ast \nu = \nu \ast \mu
\]
so \( \mu \in ZM(G) \). The map \( \tilde{P} : M(G) \to ZM(G) \) given by
\[
\int_G u \, d\tilde{P}(\mu) = \int_G \left( \int_G u(s^{-1}ts) \, d\mu(t) \right) \, ds
\]
for \( u \in C(G) \), is a weak* continuous extension of the map \( P = P_G \) in (1.2). Moreover, \( \tilde{P} \) is a quotient map onto \( ZM(G) \), being the adjoint of the injection \( ZC(G) \hookrightarrow C(G) \), where \( ZC(G) \) is the convolutive centre of \( C(G) \). Hence if \( \mu \in ZM(G) \) and \( (f_\alpha) \) is a net from \( L^1(G) \) with \( w^* \lim_\alpha f_\alpha = \mu \), then \( w^* \lim_\alpha P(f_\alpha) = P(\mu) = \mu \).

(ii) Suppose that \( \mu \in ZM(G \times G) \). Then for any \( u \) in \( C(G) \) and \( s \) in \( G \) we have
\[
\int_G u \, d(\delta_s \ast \tilde{m}(\mu) \ast \delta_{s^{-1}}) = \int_G u(s^{-1}ts) \, d\tilde{m}(\mu)(t) = \int_G u(s^{-1}xys) \, d\mu(x, y)
\]
\[
= \int_{G \times G} u(s^{-1}xss^{-1}ys) \, d\mu(x, y) = \int_{G \times G} u(xy) \, d\mu(x, y) = \int_G u \, d\tilde{m}(\mu)
\]
so \( \tilde{m}(\mu) \in ZM(G) \). Hence \( \tilde{m}(ZM(G \times G)) \subseteq ZM(G) \). Now if \( \mu \in ZM(G \times G) \) and \( \nu \in M(G \times G) \) then for \( u \) in \( C(G) \) we have
\[
\int_G u \, d\tilde{m}(\mu \ast \nu) = \int_{G \times G} u(st) \, d(\mu \ast \nu)(s, t)
\]
\[
= \int_{G \times G} \int_{G \times G} u(sst') \, d\mu(s, t) \, d\nu(s', t')
\]
\[
= \int_{G \times G} \int_{G \times G} u(sst') \, d\mu(s, t) \, d\nu(s', t'),
\]
since \( \delta_{(e, s')} \ast \mu \ast \delta_{(e, s'^{-1})} = \mu \) for any \( s' \)
\[
= \int_G \int_G u(xy) \, d\tilde{m}(\mu)(x) \, d\tilde{m}(\nu)(y) = \int_G u \, d(\tilde{m}(\mu) \ast \tilde{m}(\nu)).
\]
Observe that we actually proved that $\tilde{m}$ is a (left) $\text{ZM}(G \times G)$-module map. Since $\tilde{m}(\mu \otimes \delta_e) = \mu$, it follows that $\tilde{m}(\text{ZM}(G \times G)) = \text{ZM}(G)$. □

1.3. Approximate diagonals for centres of compact group algebras.

We note that $\text{ZTrig}(G) = \text{span}\{\chi_\pi : \pi \in \hat{G}\}$ is dense in $\text{ZL}^1(G)$. To see this, we first recall that the set $\text{Trig}(G) = \{\pi_{ij} : i, j = 1, \ldots, d, \pi \in \hat{G}\}$ of matrix coefficients is dense in $L^1(G)$. It is easily checked that $P\pi_{ij} = \psi_\pi = d^{-1}\chi_\pi$ for each $\pi_{ij}$ where $P$ is the map defined in (1.2). Then if $(u_n) \subset \text{Trig}(G)$ converges to $f$ in $\text{ZL}^1(G)$, we have $\lim_n Pu_n = Pf = f$.

**Lemma 1.5.** There exists a net $(f_\beta)$ in $\text{ZTrig}(G)$ such that $(f_\beta)$ is a bounded approximate identity for $\text{L}^1(G)$. Moreover, if for each $\beta$ we have $a_\beta = \sum_{\pi \in \hat{G}} a_\beta^\pi \chi_\pi$ where $a_\beta^\pi = 0$ except for finitely many elements $\pi$, then for each $\pi$ in $\hat{G}$ we have

$$\lim_\beta a_\beta^\pi = d_\pi.$$

**Proof.** Let $(U)$ be a base of neighbourhoods of the identity in $G$, each invariant for inner automorphisms. Then $(e_U) = \left(\frac{1}{\lambda(U)}1_U\right)$ is a central approximate identity for $L^1(G)$. Since $\text{ZTrig}(G)$ is dense in $\text{ZL}^1(G)$ we can find for each $\varepsilon > 0$ and $U$ as above, $f_\varepsilon, U \in \text{ZTrig}(G)$ such that $\|f_\varepsilon, U - e_U\|_1 < \varepsilon$. Then $(f_\beta) = (f_\varepsilon, U)$ is the desired bounded approximate identity. Since for each $\pi$ in $\hat{G}$ we have

$$a_\beta^\pi = \frac{\partial}{d_\pi} \chi_\pi = f_\beta^* \chi_\pi \xrightarrow{\beta} \chi_\pi,$$

it follows that $\lim_\beta a_\beta^\pi = d_\pi$. □

We recall from [10, (27.43)] that $\hat{G} \times \hat{G} = \{\pi \times \sigma : \pi, \sigma \in \hat{G}\}$.

**Theorem 1.6.** Let $G$ be a compact group and $(f_\beta)$ be as in Lemma 1.5 above. For each $\beta$ define

$$\mu_\beta = \sum_{\pi \in \hat{G}} (a_\beta^\pi)^2 \chi_\pi \in \text{ZL}^1(G) \hat{\otimes} \text{ZL}^1(G).$$

Then $(\mu_\beta)$ is an approximate diagonal for $\text{ZL}^1(G)$. Moreover, the following are equivalent

(i) $(\mu_\beta)$ is bounded;

(ii) $\text{ZL}^1(G)$ is amenable; and

(iii) there is a measure $\mu$ in $\text{ZM}(G \times G)$ which satisfies

(1.3) $\tilde{\mu}(\pi \times \sigma) = \delta_{\pi, \sigma}$

where $\delta$, in this context, is the Kronecker symbol. For such $\mu$ we have that $\tilde{m}(\mu) = \delta_e$ and $(f \otimes \delta_e)^* \mu = \mu^*(\delta_e \otimes f)$ for $f$ in $\text{ZL}^1(G)$. 

Note that we thus have that $ZL^1(G)$ is pseudo-amenable, in the sense defined in [5].

**Proof.** It is clear that for each $\pi$ in $\widehat{G}$ and each $\beta$ we have $\chi_\pi \cdot \mu_\beta = \mu_\beta \cdot \chi_\pi$. Since $Z \text{Trig}(G)$ is dense in $ZL^1(G)$, it follows that $f \cdot \mu_\beta = \mu_\beta \cdot f$ for each $f$ in $ZL^1(G)$ too. Also

$$m(\mu_\beta) = \sum_{\pi \in \widehat{G}} \frac{(a_\pi^\beta)^2}{d_\pi} \chi_\pi = f_\beta \ast f_\beta$$

so $(m(\mu_\beta))$ is a bounded approximate identity. Thus $(\mu_\beta)$ is an approximate diagonal for $ZL^1(G)$. It is immediate that $(i) \implies (ii)$.

$(ii) \implies (i)$. If we suppose that $ZL^1(G)$ is amenable, it admits a bounded approximate diagonal $(\mu'_\gamma)$. We may assume $(\mu'_\gamma)$ is weakly Cauchy, i.e. it converges to a virtual diagonal $M$ in $(ZL^1(G) \otimes ZL^1(G))^\ast$. With $(f_\beta)$ as in the lemma above, let for each $\beta$, $F_\beta = \{ \pi \in \widehat{G} : a_\pi^\beta \neq 0 \}$ and $A_\beta = \text{span}\{ \chi_\pi : \pi \in F_\beta \}$. Then $A_\beta \otimes A_\beta$ is a finite dimensional ideal in $ZL^1(G) \otimes ZL^1(G)$ which contains $f_\beta \otimes f_\beta$. Then $((f_\beta \otimes f_\beta) \ast \mu'_\gamma)$ is a bounded net in $A_\beta \otimes A_\beta$ with limit point $\mu'_\beta$. Write

$$\mu'_\beta = \sum_{\pi, \sigma \in F_\beta} c_{\pi, \sigma}^\beta \chi_\pi \otimes \chi_\sigma.$$ 

Then for any $\pi \in F_\beta$, using that $f \ast \chi_\pi = \chi_\pi \ast f$, we have

$$\chi_\pi \cdot \mu'_\beta = (f_\beta \otimes f_\beta) \ast \left[ \lim_{\gamma} \chi_\pi \cdot \mu'_\gamma \right] = (f_\beta \otimes f_\beta) \ast \left[ \lim_{\gamma} \mu'_\gamma \cdot \chi_\pi \right] = \mu'_\beta \cdot \chi_\pi$$

and thus

$$\sum_{\sigma \in F_\beta} \frac{c_{\pi, \sigma}^\beta}{d_\pi} \chi_\pi \otimes \chi_\sigma = \sum_{\sigma \in F_\beta} \frac{c_{\sigma, \pi}^\beta}{d_\pi} \chi_\sigma \otimes \chi_\pi.$$ 

It follows from the orthogonality relations of the characters that $c_{\pi, \sigma}^\beta = 0$ if $\sigma \neq \pi$ and hence

$$\mu'_\beta = \sum_{\pi \in F_\beta} \frac{c_{\pi, \pi}^\beta}{d_\pi} \chi_\pi \otimes \chi_\pi.$$ 

Since $m(\mu'_\beta) = m(f_\beta \otimes f_\beta) \ast \lim_{\gamma} m(\mu'_\gamma) = f_\beta \ast f_\beta$ we obtain

$$\sum_{\pi \in F_\beta} \frac{c_{\pi, \pi}^\beta}{d_\pi} \chi_\pi = \sum_{\pi \in F_\beta} \frac{(a_\pi^\beta)^2}{d_\pi} \chi_\pi$$

and thus $c_{\pi, \pi}^\beta = (a_\pi^\beta)^2$ for each $\pi$ in $F_\beta$. Then for each $\beta$ we have $\mu_\beta = \mu'_\beta$, so

$$\|\mu_\beta\| = \|\mu'_\beta\| \leq \|f_\beta\|_1 \sup_{\gamma} \|\mu'_\gamma\|$$

and, since $(f_\beta)$ is bounded, $(\mu_\beta)$ is bounded too.
(i) ⇒ (iii). Using Proposition 1.1 we identify \((\mu_\beta)\) as a bounded net \(M(G \times G)\). It thus has a weak* cluster point \(\mu\). We note that \(\mu\) is, in fact, a limit point. Indeed, \(\text{Trig}(G \times G)\) is uniformly dense in \(\mathcal{C}(G \times G)\), and if \(u \in \text{Trig}(G \times G)\) it is clear that

\[
\sum_{\pi \in \hat{G}} d_{\pi}^2 \int_{G \times G} u(s, t) \chi_\pi(s) \chi_\pi(t) d(s, t) = \lim_{\beta} \sum_{\pi \in \hat{G}} (a_{\beta})^2 \int_{G \times G} u(s, t) \chi_\pi(s) \chi_\pi(t) d(s, t) = \lim_{\beta} \langle u, \mu_\beta \rangle
\]

as all sums in the expression are finite. Moreover, the above expression must be \(\int_{G \times G} u(s, t) d\mu(s, t)\). By Proposition 1.4, \(\mu \in ZM(G \times G)\). By (1.1) we find

\[
\hat{\mu}(\pi \times \sigma) = \frac{1}{d_\pi d_\sigma} \int_{G \times G} \chi_\pi(s) \chi_\sigma(t) d\mu(s, t) = \delta_{\pi, \sigma}.
\]

Let \(R : \mathcal{C}(\hat{G} \times \hat{G}) \to \mathcal{C}(\hat{G})\) be the map of restriction to the diagonal: \(Ru(\pi) = u(\pi, \pi)\). Note that for any \(\nu \in ZM(G \times G)\), \((\hat{\nu}(\nu)) = R\hat{\nu}\). Thus we have for any \(\pi \in \hat{G}\)

\[
(\hat{\nu}(\nu))^{\sim}(\pi) = R\hat{\mu}(\pi) = 1 = \hat{\delta}_e(\pi)
\]

so \(\hat{\nu}(\nu) = \delta_e\). Also, if \(f \in ZL^1(G)\) and \(\pi, \sigma \in \hat{G}\) then, \((f \otimes \delta_e)^{\sim}(\pi \times \sigma) = \hat{f}(\pi)\) while \((\delta_e \otimes f)^{\sim}(\pi \times \sigma) = \hat{f}(\sigma)\). It follows that

\[
(f \otimes \delta_e)^{\sim}\hat{\mu}(\pi \times \sigma) = f(\pi)\delta_{\pi, \sigma} = \hat{\mu}(\delta_e \otimes f)^{\sim}(\pi \times \sigma)
\]

so it follows that \((f \otimes \delta_e) * \mu = \mu * (\delta_e \otimes f)\).

(iii) ⇒ (ii). Let \((f_\alpha)\) be any bounded approximate identity in \(ZL^1(G \times G)\). We will show that any weak* cluster point \(M\) of \((\mu * f_\alpha)\) in \(ZL^1(G \times G)\)** is a virtual diagonal. We may assume \(M\) is a limit point. First, if \(f \in ZL^1(G)\) we have

\[
f \cdot M = \lim_{\alpha} (f \otimes \delta_e) * \mu * f_\alpha = \lim_{\alpha} \mu * (\delta_e \otimes f) * f_\alpha = \lim_{\alpha} \mu * f_\alpha * (\delta_e \otimes f) = M \cdot f.
\]

Second, we note it follows from Proposition 1.4 that \((m(f_\alpha))\) is a bounded approximate identity for \(ZL^1(G)\). We let \(E\) be any weak* cluster point of \((m(f_\alpha))\), which we may consider to be a limit point. We then have, again by Proposition 1.4, and using \(\hat{m}(\mu) = \delta_e\), that

\[
m^{**}(M) = \lim_{\alpha} m(\mu * f_\alpha) = \lim_{\alpha} \hat{m}(\mu) * m(f_\alpha) = \lim_{\alpha} m(f_\alpha) = E.
\]

It is clear that \(f \cdot E = E \cdot f = f\) for \(f \in ZL^1(G)\). Thus \(M\) is a virtual diagonal.

Note that if \(G\) is abelian, then \(\mu\) is the Haar measure of the anti-diagonal subgroup \(A = \{(s, s^{-1}) : s \in G\}\). Indeed, if we denote the latter by \(\lambda_A\) then
we have for \( \chi, \psi \) in \( \hat{G} \)
\[
\hat{\lambda}_A(\chi \times \psi) = \int_G \overline{\chi}(s)\psi(s^{-1})ds = \int_G \overline{\chi}(s)\psi(s)ds = \delta_{\chi, \psi} = \hat{\mu}(\chi \times \psi)
\]
and hence \( \mu = \lambda_A \). Though the definition of \( \lambda_A \), as above, makes sense for any compact group, it forms a central measure only when \( G \) is abelian.

Suppose \( d_G = \sup_{\pi \in \hat{G}} d_{\pi} < \infty \). Then for \( u, v \) in Trig(\( G \)) we use the Cauchy-Schwarz inequality and Bessel’s inequality on the orthonormal set \( \{ \chi_{\pi} : \pi \in \hat{G} \} \) to see that for the approximate diagonal \((\mu_\beta)\) in the theorem above we have
\[
\left| \lim_{\beta} \int_{G \times G} u(s)v(t)\mu_\beta(s, t)dsdt \right| = \left| \sum_{\pi \in \hat{G}} d_\pi^2 \int_G u(s)\overline{\chi}_{\pi}(s)ds \int_G v(t)\chi_{\pi}(t)dt \right|
\leq d_G^2 \sum_{\pi \in \hat{G}} |\langle u|\chi_{\pi} \rangle||\langle v|\chi_{\pi} \rangle|
\leq d_G^2 \|u\|_2 \|v\|_2 \leq d_G^2 \|u\|_\infty \|v\|_\infty.
\]

Since Trig(\( G \)) is dense in \( \mathcal{C}(G) \), it follows that \((\mu_\beta)\) converges to a bimeasure in the terminology of [2], i.e. an element \( \mu \) of \( (\mathcal{C}(G) \widehat{\otimes} \mathcal{C}(G))^* \). Conjecture 0.1 if true, would further imply that \( \mu \in M(G \times G) \).

1.4. Connected groups.

**Theorem 1.7.** If \( G \) is a non-abelian connected compact group, then \( ZL^1(G) \) is not amenable.

**Proof.** There is a family \( \{G_i\}_{i \in I} \) of compact connected Lie groups, at least one of which is simple (in the sense of Lie groups) with finite centre, such that
\[
G \cong \left( \prod_{i \in I} G_i \right)/A
\]
where \( A \) is a central subgroup of \( P = \prod_{i \in I} G_i \). See [22, 6.5.6], for example. Hence \( G \) admits, as a quotient
\[
\prod_{i \in I} G_i/Z(G_i) \cong P/Z(P) \cong (P/A)/(Z(P)/A).
\]

Let \( i_0 \) be so \( G_{i_0} \) is simple with finite centre. Then \( G_{i_0}/Z(G_{i_0}) \) is simple with trivial centre. Hence there is a closed normal subgroup \( N \) of \( G \) such that \( G/N \) is a simple Lie group with trivial centre. By [24 Lem. 9.1] we obtain “Condition I” on \( G/N \), which is the property that
\[
\lim_{d_{\pi} \to \infty} \psi_{\pi}(sN) = 0 \text{ for } sN \in G/N \setminus \{eN\}.
\]

Hence, there is a sequence \( \{\pi_n\}_{n=1}^\infty \subset \hat{G} \) such that
\[
(1.4) \quad \psi_n(s) = 1 \text{ for } s \in N \text{ and } \lim_{n \to \infty} \psi_n(s) = 0 \text{ for } s \in G \setminus N
\]
where $\psi_n = \psi_{\pi_n}$. Indeed, choose any sequence of representations $\{\tilde{\pi}_n\}_{n=1}^{\infty} \subset \hat{G}/N$ where $\lim_{n \to \infty} d_{\tilde{\pi}_n} = \infty$, and let $\pi_n = \tilde{\pi}_n \circ q$ where $q : G \to G/N$ is the quotient map.

If it were the case that $\text{ZL}_1(G)$ were amenable, then we would obtain $\mu$ in $\text{ZM}(G \times G)$ as in (1.3). Let us see that the existence of such $\mu$ gives a contradiction. Let $N$ and $(\psi_n)$ be as in (1.4). Define two sequences $(u_n)$ and $(v_n)$ of functions on $G \times G$ by

$$u_n = \psi_n \otimes \psi_n \quad \text{and} \quad v_n = \psi_n \otimes \psi_{n+1}.$$ 

Then $(u_n)$ and $(v_n)$ are bounded sequences with

$$\lim_{n \to \infty} u_n(s, t) = \lim_{n \to \infty} v_n(s, t) = \begin{cases} 1 & \text{if } (s, t) \in N \times N \\ 0 & \text{if } (s, t) \not\in N \times N. \end{cases}$$

Hence it follows from the Lebesgue dominated convergence theorem that

(1.5) $$\lim_{n \to \infty} \int_{G \times G} u_n \, d\mu = \mu(N \times N) = \lim_{n \to \infty} \int_{G \times G} v_n \, d\mu.$$ 

However, by (1.3) we have that

$$\int_{G \times G} u_n \, d\mu = \bar{\mu}(\tilde{\pi}_n \times \tilde{\pi}_n) = 1 \quad \text{while} \quad \int_{G \times G} v_n \, d\mu = \bar{\mu}(\tilde{\pi}_n \times \tilde{\pi}_{n+1}) = 0$$

which contradicts (1.5).

1.5. **Products of finite groups.** Let $G$ be a finite group. We will treat $G$ as a compact group so we have normalised Haar integral: $\int_G f = \frac{1}{|G|} \sum_{s \in G} f(s)$. Then it is well known that

(1.6) $$\text{ZL}_1(G) = \text{span}\{\chi_\pi : \pi \in \hat{G}\}$$

Moreover, if we let for any $x$ in $G$, $C_x = \{sx s^{-1} : s \in G\}$ denote the conjugacy class, and $\text{Conj}(G) = \{C_x : x \in G\}$, then since elements of $\text{ZL}_1(G)$ are constant on conjugacy classes we have

(1.7) $$\text{ZL}_1(G) = \text{span}\{1_C : C \in \text{Conj}(G)\}$$

where $1_C$ is the indicator function of $C$. We will let $f(C) = f(x)$ where $C = C_x$, for $f \in \text{ZL}_1(G)$.

**Theorem 1.8.** If $G$ is a finite group, then $\text{ZL}_1(G)$ has unique diagonal and we have

$$\text{AM}(\text{ZL}_1(G)) = \frac{1}{|G|^2} \sum_{C, C' \in \text{Conj}(G)} \left| \sum_{\pi \in \hat{G}} d_{\pi}^2 \overline{\chi_\pi(C) \chi_\pi(C')} \right| |C||C'|.$$ 

**Proof.** That

$$\mu = \sum_{\pi \in \hat{G}} d_{\pi}^2 \chi_\pi \otimes \chi_\pi$$

and
is the unique diagonal for \( ZL^1(G) \) follows from the proof of Theorem 1.6. However, using the relations \( \chi_{\pi} \ast \chi_{\sigma} = \delta_{\pi, \sigma} d_{\pi}^{-1} \chi_{\pi} \) for \( \pi, \sigma \) in \( \hat{G} \), that \( \mu \) is a diagonal is easily verified manually using (1.6). The uniqueness of the diagonal in any amenable finite dimensional commutative algebra has been observed in [6, Prop. 0.2].

If \( C \in \text{Conj}(G) \) with \( C = C_x \), we let \( \overline{C} = C_{x^{-1}} \). The operation \( C \mapsto \overline{C} \) is an involution on \( \text{Conj}(G) \). If \( \pi \in \hat{G} \) then \( \chi_{\pi}(C) = \overline{\chi_{\pi}(C)} \). We appeal to (1.7) to obtain

\[
\mu = \sum_{\pi \in \hat{G}} d_{\pi}^2 \left( \sum_{C \in \text{Conj}(G)} \chi_{\pi}(C) 1_C \right) \otimes \left( \sum_{C' \in \text{Conj}(G)} \chi_{\pi}(C') 1_{C'} \right)
\]

\[
= \sum_{\pi \in \hat{G}} d_{\pi}^2 \left( \sum_{C \in \text{Conj}(G)} \chi_{\pi}(C) 1_C \right) \otimes \left( \sum_{C' \in \text{Conj}(G)} \chi_{\pi}(C') 1_{C'} \right)
\]

\[
= \sum_{C, C' \in \text{Conj}(G)} \left( \sum_{\pi \in \hat{G}} d_{\pi}^2 \chi_{\pi}(C) \chi_{\pi}(C') \right) 1_C \otimes 1_{C'}.
\]

We then compute \( \text{AM}(ZL^1(G)) = \|\mu\|_1 \) to finish. \( \square \)

**Corollary 1.9.** If \( G \) is a non-abelian finite group, then \( \text{AM}(ZL^1(G)) > 1 \).

**Proof.** Letting \( C' = C \) we obtain lower bound

\[
\text{AM}(ZL^1(G)) \geq \frac{1}{|G|^2} \sum_{C \in \text{Conj}(G)} \sum_{\pi \in \hat{G}} d_{\pi}^2 |\chi_{\pi}(C)|^2 |C|^2
\]

\[
= \frac{1}{|G|} \sum_{\pi \in \hat{G}} d_{\pi}^2 \sum_{C \in \text{Conj}(G)} |C||\chi_{\pi}(C)|^2 \frac{|C|}{|G|}.
\]

Since \( G \) is nonabelian we have \( |C| > 1 \) for some conjugacy class \( C \). Moreover, there is some \( \pi \) so \( \chi_{\pi}(C) \neq 0 \). Thus we find

\[
\text{AM}(ZL^1(G)) > \frac{1}{|G|} \sum_{\pi \in \hat{G}} d_{\pi}^2 \sum_{C \in \text{Conj}(G)} |\chi_{\pi}(C)|^2 \frac{|C|}{|G|}
\]

\[
= \frac{1}{|G|} \sum_{\pi \in \hat{G}} d_{\pi}^2 \|\chi_{\pi}\|_2^2 = 1
\]

since \( \|\chi_{\pi}\|_2 = 1 \) and \( \sum_{\pi \in \hat{G}} d_{\pi}^2 = |G| \). \( \square \)

Let us take a second look at the proof of the above corollary. The Schur orthogonality relations tell us that the \( \hat{G} \times \text{Conj}(G) \) matrix

\[
U = \begin{bmatrix} |C|^{1/2} \chi_{\pi}(C) \end{bmatrix}
\]

is the unique diagonal for \( ZL^2(G) \). The above holds for any amenable finite dimensional commutative algebra. The unique diagonal in any amenable finite dimensional commutative algebra has been observed in [6, Prop. 0.2].
is unitary. Letting $C' = C$ we obtain lower bound

$$\text{AM}(ZL^1(G)) \geq \frac{1}{|G|} \sum_{C \in \text{Conj}(G)} \sum_{\pi \in \hat{G}} d_{\pi}^2 |\chi_\pi(C)|^2 \left| \frac{C}{|G|} \right|$$

where $\| \cdot \|_2$ denotes the Hilbert-Schmidt norm. Is it possible to get a lower estimate in terms of $\max_{\pi \in bG} d_\pi$? If so, Conjecture 0.1 may be shown to hold for compact totally disconnected groups which do not admit an open abelian subgroup.

**Theorem 1.10.** If $G = \prod_{i=1}^\infty G_i$ where each $G_i$ is a nonabelian finite group, then $ZL^1(G)$ is not amenable.

**Proof.** For each $i$, the diagonal $\mu_i$ for $ZL^1(G_i)$ promised by Theorem 1.8 is an idempotent in $ZL^1(G \times G)$. Hence by [24, Thm. 5.3], there is a constant $\delta > 0$ – in fact $\delta \geq 1/700$ – for which

$$\text{AM}(ZL^1(G_i)) \geq 1 + \delta$$

for each $i$. Since $G$ admits, for each $n$, $G(n) = \prod_{i=1}^n G_i$ as a quotient, we have that

$$\text{AM}(ZL^1(G)) \geq \text{AM}(ZL^1(G(n))) = \prod_{i=1}^n \text{AM}(ZL^1(G_i)) \geq (1 + \delta)^n.$$

Hence we have that $\text{AM}(ZL^1(G)) = \infty$ and $ZL^1(G)$ is not amenable. □

1.6. **An amenable example.** The following example further illustrates Conjecture 0.1

Let $G = \mathbb{T} \times \mathbb{Z}_2$ where $\mathbb{T} = \{ s \in \mathbb{C} : |s| = 1 \}$ and $\mathbb{Z}_2 = \{-1, 1\}$. The group law and inverse are given by

$$(s, a)(t, b) = (st^a, ab) \quad \text{and} \quad (s, a)^{-1} = (s^{-a}, a)$$

for $(s, a), (t, b)$ in $G$. An application of the “Mackey machine”, see [4, Sec. 6.6] for example, gives us $\hat{G} = \{ 1, \sigma, \pi_n : n \in \mathbb{N} \}$ where

$$1(s, a) = 1, \quad \sigma(s, a) = a \quad \text{and} \quad \pi_n(s, a) = \begin{bmatrix} s^n & 0 \\ 0 & s^{-n} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{(1-a)/2}$$

for $(s, a)$ in $G$. It follows that we have normalised characters $1, \sigma$ and

$$\psi_{\pi_n}(s, a) = \begin{cases} \frac{1}{2}(s^n + s^{-n}) & \text{if } a = 1 \\ 0 & \text{if } a = -1 \end{cases}$$

for $(s, a)$ in $G$. We note that all of the calculations thus far, and hence the next proposition, also hold if $\mathbb{T}$ is replaced by any compact abelian group $T$ admitting only 1 as a real character, i.e. for $\chi$ in $\hat{T}$, $\chi = \chi$ implies $\chi = 1$. For sake of concreteness, we will continue with $T = \mathbb{T}$. 

Theorem 1.11. For $G = \mathbb{T} \ltimes \mathbb{Z}_2$, $\text{ZL}_1(G)$ is amenable.

Proof. Let $\mu = 1 \otimes 1 + \sigma \otimes \sigma - 2(1 + \sigma) \otimes (1 + \sigma) + \nu$, where $\nu = \lambda_D + \lambda_A$, the sum of the Haar measures on the subgroups of $G \times G$ given by $D = \{((t, 1), (t, 1)) : t \in \mathbb{T}\}$ and $A = \{((t, 1), (t^{-1}, 1)) : t \in \mathbb{T}\}$, each normalised to have total mass 1. We note that $\nu \in \text{ZM}(G)$ since for $(s, a)$ in $G$ we have

$$\begin{align*}
\delta_{(s,a),(t,b)} \ast \nu \ast \delta_{(s^{-a},a),(t^{-b},b)} = &\delta_{(1,a),(1,b)} \ast \delta_{(s^a,1),(t^b,1)} \ast \nu \ast \delta_{(s^{-a},1),(t^{-b},1)} \ast \delta_{(1,a),(1,b)} = \nu.
\end{align*}$$

Thus $\mu \in \text{ZM}(G)$. Now for $\pi, \rho$ in $\hat{G}$ we have

$$\begin{align*}
\hat{\mu}(\pi \times \rho) = &\int_{G \times G} (1 \otimes 1 + \sigma \otimes \sigma - 2(1 + \sigma) \otimes (1 + \sigma)) \cdot \psi_\pi \otimes \psi_\rho \\
&+ \int_{\mathbb{T}} (\psi_\pi(s,1)\psi_\rho(s,1) + \psi_\pi(s,1)\psi_\rho(s^{-1},1))ds \\
= &\ (1) + (2)
\end{align*}$$

where

$$\begin{align*}
(1) = &\left\{ \begin{array}{ll}
-1 & \text{if } (\pi, \rho) = (1, 1), (\sigma, \sigma) \\
-2 & \text{if } (\pi, \rho) = (1, \sigma), (\sigma, 1) \\
0 & \text{if } (\pi, \rho) = (1, \pi_n), (\sigma, \pi_n), (\sigma, \pi_n), (\pi_n, \sigma) \\
\end{array} \right. \\
&\text{and}
\end{align*}$$

$$\begin{align*}
(2) = &\ 2 \int_{\mathbb{T}} \psi_\pi(s,1)\psi_\rho(s,1) ds = \left\{ \begin{array}{ll}
2 & \text{if } (\pi, \rho) = (1, 1), (\sigma, \sigma), (1, \sigma), (\sigma, 1) \\
1 & \text{if } (\pi, \rho) = (\pi_n, \pi_n), n \in \mathbb{N} \\
0 & \text{if } (\pi, \rho) = (1, \pi_n), (\pi_n, 1), (\sigma, \pi_n), (\pi_n, \sigma) \\
\end{array} \right. \\
&\text{and } (\pi_n, \pi_m), n \neq m, n, m \in \mathbb{N}
\end{align*}$$

Thus it follows that $\mu$ satisfies (1.3).

We remark that the measure $\mu$ corresponds to the (formal) Fourier series

$$\begin{align*}
1 \otimes 1 + \sigma \otimes \sigma - 2(1 + \sigma) \otimes (1 + \sigma) + 4 \left[ \frac{1}{2} (1 + \sigma) \otimes (1 + \sigma) + \sum_{n=1}^{\infty} \chi_{\pi_n} \otimes \chi_{\pi_n} \right]
\end{align*}$$

as suggested by Theorem 1.6. The coefficient 4, in the second term, is $1/\lambda(G_{(1,1)} \ltimes G_{(1,1)})$, where $G_{(1,1)} \cong \mathbb{T}$ is the connected component of the identity in $G$. The second term corresponds to the Fourier series for $\lambda_A + \lambda_D$ on $\mathbb{T} \times \mathbb{T}$, as may be revealed by a simple computation which we leave to the reader.

Let us make a few observations about $G = \mathbb{T} \ltimes \mathbb{Z}_2$. First we compute, for $s$ in $\mathbb{T}$ and $(t, b)$ in $G$

$$(t, b)(s, 1)(t^{-b}, b) = (s^b, 1) \text{ and } (t, b)(s, -1)(t^{-b}, b) = (t^2 s^b, -1).$$

Hence we deduce that

$$\text{Conj}(G) = \{(1, 1), (-1, 1), (s, 1), (s^{-1}, 1)\}_{\text{Im} s > 0}, G_{(1, -1)}$$
where $G_{(1,-1)}$ is the connected component of $(1,-1)$. Moreover we compute commutators

$$\left([t,b],(s,a)\right) = (t,b)(s,a)(t^{-b},b)(s^{-a},a) = (t^{1-a}s^{1-b},1).$$

Letting $a = 1$, $b = -1$ and $s,t$ be arbitrary in $\mathbb{T}$ we find, in the notation of Section 2.2 that $G' = [G, G_0] = G_{(1,1)}$. In particular, notice that the assumptions of Theorem 2.2, below, are not necessary for $\mathrm{ZL}^1(G)$ to be amenable.

Let us close by noting the following decomposition

$$\mathrm{ZL}^1(\mathbb{T} \rtimes \mathbb{Z}_2) = \mathbb{Z}_2 \mathbb{L}^1(\mathbb{T}) \oplus \mathbb{C}(1-\sigma)$$

where $\mathbb{Z}_2 \mathbb{L}^1(\mathbb{T}) = \{ f \in \mathbb{L}^1(\mathbb{T}) : \tilde{f} = f \}$, $\tilde{f}(s) = f(s^{-1})$. We note that both of the components of this decomposition are closed subalgebras, but neither is an ideal. Hence it is not apparent that $\mathbb{Z}_2 \mathbb{L}^1(\mathbb{T})$ is amenable. We show this fact in the next section.

1.7. **The hypergroup approach.** We indicate, by way of two examples, how the problem of amenability for $\mathrm{ZL}^1(G)$ can be treated by using hypergroups. We refer to [3] for the definition of a hypergroup $K$ and its left Haar measure $\lambda_K$, or to [12], where a hypergroup is referred to as a “convos”. If $G$ is a compact group, then $K = \text{Conj}(G)$ is a hypergroup [12, 8.4]. Since $K$ is compact and commutative, it admits a Haar measure. Moreover we have $\mathrm{ZL}^1(G) \cong \mathbb{L}^1(K)$, where $\mathbb{L}^1(K)$ is the hypergroup algebra. Such $K$ is a **strong hypergroup** in the sense that its character set $\hat{K}$ is a (discrete) hypergroup under pointwise multiplication. In fact $\hat{K}$ identifies naturally with $\{\psi_\pi\}_{\pi \in \hat{G}}$.

We first consider $G = \text{SU}(2)$. By [12, 15.4], $\text{Conj}({\text{SU}}(2))$ identifies naturally with a hypergroup whose underlying set is $K = [-1,1]$. We will not explicitly need the convolution formula on $K$, but we will require the formula for the Haar measure

$$\int_K f d\lambda_K = \frac{2}{\pi} \int_{-1}^1 f(x)(1-x^2)^{1/2} dx = \frac{2}{\pi} \int_0^\pi f(\cos \theta) \sin^2 \theta d\theta$$

where $dx, d\theta$ each denote integration with respect to Lebesgue measure, and the (non-normalised) characters are given by

$$\{\chi_k\}_{k=0}^\infty \quad \text{where} \quad \chi_k(\cos \theta) = \frac{\sin(k+1)\theta}{\sin \theta}.$$  

Note that $\chi_k$ is, up to identification, the character of the unique representation of $\text{SU}(2)$ of dimension $k+1$.

**Theorem 1.12.** $\mathrm{ZL}^1(\text{SU}(2))$ is not amenable.

**Proof.** We first note that by Lemma 1.5 there is a bounded approximate identity for $\mathbb{L}^1(K)$, $(e_n) \subset \text{Trig}(K) = \text{span}\{\psi_k : k \in \mathbb{N}_0\}$ where $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. This bounded approximate identity may be taken to be a sequence, $(e_n)$,
and we have for each \( n \), \( e_n = \sum_{k=0}^{\infty} a_k^{(n)} \chi_k \) where \( a_k^{(n)} = 0 \) for all but finitely many indices \( k \). We obtain, again from Lemma 1.5, that

\[
(1.9) \quad \lim_{n \to \infty} a_k^{(n)} = k + 1
\]

Now let \( \mu_n = \sum_{k=0}^{\infty} (a_k^{(n)})^2 \chi_k \otimes \chi_k \), so \( (\mu_n)_{n=1}^{\infty} \) is the approximate identity from Theorem 1.6 and we are done once we establish \( (\mu_n) \) is not bounded. The using the fact that \( L^1(K) \otimes L^1(K) \cong L^1(K \times K) \) and (1.8) we have

\[
\left( \frac{\pi}{2} \right)^2 \|\mu_n\|_1 = \int_0^{\pi} \int_0^{\pi} |\mu_n(\cos \theta, \cos \theta')| \sin^2 \theta \sin^2 \theta' d\theta d\theta'
\]

\[
\geq \int_0^{\pi/2} \int_0^{\pi/2} \left| \sum_{k=0}^{\infty} (a_k^{(n)})^2 \frac{\sin(k+1)\theta \sin(k+1)\theta'}{\sin \theta' \sin \theta} \right| \sin^2 \theta \sin^2 \theta' d\theta d\theta'
\]

\[
\geq \left| \sum_{k=0}^{\infty} (a_k^{(n)})^2 \sin(k+1)\theta \sin(k+1)\theta' \sin \theta \sin \theta' \right|
\]

\[
= \sum_{k=0}^{\infty} (a_k^{(n)})^2 \left( \int_0^{\pi/2} \sin(k+1)\theta \sin \theta d\theta \right)^2
\]

\[
= \sum_{k=0}^{\infty} (a_k^{(n)})^2 \left( \frac{1}{2} \int_0^{\pi/2} (\cos k\theta - \cos(k+2)\theta) d\theta \right)^2
\]

\[
= \sum_{k=0}^{\infty} \left( \frac{a_k^{(n)}(k+1)}{k(k+2)} \sin k\pi \right)^2 = \sum_{j=0}^{\infty} \left( \frac{a_{2j+1}^{(n)}(2j+2)}{(2j+1)(2j+3)} \right)^2
\]

Let \( f_n = \left( \frac{a_{2j+1}^{(n)}(2j+2)}{(2j+1)(2j+3)} \right)^{\infty}_{j=0} \). If \( (\mu_n) \) is bounded, then \( (f_n) \) is bounded in \( \ell^2(\mathbb{N}_0) \), in which cases the latter sequence has a cluster point \( f \). We have, by (1.9), that \( f(j) = \frac{(2j+2)^2}{(2j+1)(2j+3)} \), which means that \( f \) cannot be an element of \( \ell^2(\mathbb{N}_0) \). Thus \( (\mu_n) \) must not be bounded. \( \square \)

Let us now turn out attention to \( \mathbb{Z}_2 L^1(\mathbb{T}) \), from the last section. We let

\[
\psi_0 = \frac{1}{2} (1 + \sigma) \quad \text{and} \quad \psi_n = \psi_{\pi n} \quad \text{for} \quad n \in \mathbb{N}.
\]

Then the family of all \( \mathbb{Z}_2 \)-invariant characters of \( \mathbb{Z}_2 L^1(\mathbb{T}) \) is \( \mathcal{X}_{\mathbb{Z}_2}(\mathbb{T}) = \{ \psi_n \}_{n \in \mathbb{N}_0} \). Observe, under pointwise multiplication, that \( \mathcal{X}_{\mathbb{Z}_2}(\mathbb{T}) \) satisfies the same multiplication rules as the cosine functions \( \{ \cos(m \cdot) \}_{m \in \mathbb{N}_0} \), and hence is isomorphic to the Chebychev polynomial hypergroup of the first kind \( \mathcal{C} \).

There is a commutative hypergroup \( K = [-1, 1] \), which is isomorphic to the double conjugacy class hypergroup \( \mathbb{T} / \mathbb{Z}_2 \), such that \( \mathbb{Z}_2 L^1(\mathbb{T}) \cong L^1(K) \).
The Haar measure on $K$ is given by
\[ \int_K f \, d\lambda_K = \frac{1}{\pi} \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} \, dx = \frac{1}{\pi} \int_{0}^{\pi} f(\cos \theta) \, d\theta \]
and the characters, in the present identification, are given by $\psi_n(\cos \theta) = \cos n\theta$ for $n \in \mathbb{N}_0$.

**Theorem 1.13.** $Z_2 \mathcal{L}^1(\mathbb{T})$ is amenable.

**Proof.** We let $K_n : [0, \pi] \to \mathbb{R}^{\geq 0}$ for $n \in \mathbb{N}_0$ denote the well-known Fejer kernel (see [16, 2.5], for example), so
\[ K_n = \sum_{k=0}^{n} \left( 1 - \frac{2k}{n+1} \right) \psi_{k} \circ \cos. \]
Then let $\mu_n = \sum_{k=0}^{n} \left( 1 - \frac{2k}{n+1} \right) \psi_{k} \otimes \psi_{k}$. We have that $(\mu_n)$ is an approximate diagonal by Theorem 1.6. Moreover, $(\mu_n)$ is bounded since
\[ \|\mu_n\|_1 = \frac{1}{\pi^2} \int_{0}^{\pi} \int_{0}^{\pi} \left| \sum_{k=0}^{n} \left( 1 - \frac{2k}{n+1} \right) \cos k\theta \cos k\theta' \right| \, d\theta \, d\theta' = \frac{1}{2\pi^2} \int_{0}^{\pi} \int_{0}^{\pi} \left( K_n(\theta + \theta') + K_n(\theta - \theta') \right) \, d\theta \, d\theta' = 1. \]
Thus $Z_2 \mathcal{L}^1(\mathbb{T}) \cong \mathcal{L}^1(K)$ is amenable. \qed

2. Some non-compact groups

2.1. Preliminaries and Notation. If $G$ is a locally compact group, then $\mathcal{L}^1(G) \neq \{0\}$ if and only if $G$ has a relatively compact neighbourhood which is invariant under inner automorphisms, i.e. $G$ is an $[IN]$-group; see [21, Prop. 1]. In fact, it is shown in [18, Cor. 1.5] that $\mathcal{L}^1(G)$ is related to certain centers of $[FIA]_B^{-}$-groups, which we define below.

Let $\text{Aut}(G)$ denote the space of continuous automorphism of $G$ which can be endowed with a Hausdorff topology [9 (26.5)]. We let $\text{Inn}(G) = \{ s \mapsto t s t^{-1} : t \in G \}$ denote the group of inner automorphisms in $\text{Aut}(G)$. We say $G$ has relatively compact inner automorphisms if $\text{Inn}(G)$ is relatively compact in $\text{Aut}(G)$. More generally, if there is a relatively compact subgroup $B$ of $\text{Aut}(G)$ such that $B \supset \text{Inn}(G)$ we say $G$ is of class $[FIA]_B^{-}$. We let for $\beta \in B$ and $f \in \mathcal{L}^1(G)$, \( f \circ \beta(s) = f(\beta(s)) \) for almost every $s \in G$. We then let
\[ Z_B \mathcal{L}^1(G) = \{ f \in \mathcal{L}^1(G) : f \circ \beta = f \text{ for all } \beta \text{ in } B \}. \]
This is a subalgebra of $\mathcal{L}^1(G)$. The result [18, Cor. 1.5], to which we alluded, above, is that for an $[IN]$-group $G$, there is open normal subgroup $G_0$ of $G$ generated by all elements with relatively compact conjugacy classes, and
a closed normal subgroup of $G_0$, $N$, which is the intersection if all $\text{Inn}(G)/G_0$-invariant neighbourhoods of $e$, so that group $B = \{ sN \mapsto t^{-1}stN : t \in G_0 \}$ is relatively compact in $\text{Aut}(H)$ where $H = G_0/N$, and

\[(2.1) \quad ZL^1(G) \cong ZBL^1(H).\]

We let $\mathcal{X}_B(G)$ denote the Gelfand spectrum of $Z_B L^1(G)$, and let $\mathcal{X}(G) = \mathcal{X}_{\text{Inn}(G)}(G)$. The identification (2.1) gives a natural identification $\mathcal{X}(G) \cong \mathcal{X}_B(H)$. It follows from [20, 4.12] (see [21, 4.2]) that $\mathcal{X}_B(G)$ may be identified with a certain family of continuous positive definite functions on $G$.

We record the following important structural result, which will be key to many of the results which follow. It summarises results from [18, Prop. 2.3] and [21, Lem. 1]. See the summary presented in [26].

**Lemma 2.1.** Let $G$ be an $[FIA]_{\mathcal{B}}$-group and suppose there exists a compact $B$-invariant subgroup $K$ such that each “$\beta$-commutator” $s^{-1}\beta(s) \in K$, where $\beta \in B$ and $s \in G$ (thus $G/K$ is abelian). Define an equivalence relation on $\mathcal{X}_B(G)$ by

$$\chi \sim \omega \iff \chi|_K = \omega|_K.$$ Let $[\chi]$ denote the equivalence class of $\chi$. Then

(i) there is a family of ideals $\{ J(\chi) : [\chi] \in \mathcal{X}_B(G)/\sim \}$ such that

$$J(\chi) \cap J(\omega) = \{0\} \text{ if } \chi \neq \omega, \quad Z_B L^1(G) = \bigoplus_{[\chi] \in \mathcal{X}_B(G)/\sim} J(\chi)$$

and each $J(\chi)$ is isomorphic to $L^1(G(\chi))$, where $G(\chi)$ is an abelian group, isomorphic to a quotient of an open subgroup of $G$ by $K$; and

(ii) $\{ \chi|_K : [\chi] \in \mathcal{X}_B(G)/\sim \}$ is an orthogonal family in $L^2(K)$.

Note that for such a compact subgroup as $K$ to exist, it is necessary and sufficient that the closed subgroup generated by $B$-commutators be compact. In this case $G$ is said to be an $[FD]_{\mathcal{B}}$-group. Note that if $G$ is compact we may take $K = G$ and we obtain, for each $\pi \in \widehat{G} \cong \mathcal{X}(G)$, $J(\chi_\pi) = \mathbb{C}\chi_\pi \cong L^1(G)/G$.

2.2. **Some amenable centres.** If $A, B$ are any pair of subgroups of $G$, we let $[A, B]$ denote the closed subgroup generated by commutators $\{ aba^{-1}b^{-1} : a \in A, b \in B \}$. The derived subgroup is given by $G' = [G, G]$.

The following result is a generalisation of [26, Thm. 1]. We recall that if $G$ is an $[IN]$-group, then the subgroup $G_0$, of all elements with relatively compact conjugacy classes is an open normal subgroup.

**Theorem 2.2.** If $[G, G_0]$ is finite, then $ZL^1(G)$ is amenable.

**Proof.** We may suppose that $ZL^1(G) \neq \{0\}$, so $G$ has an invariant neighbourhood. Let $B$ and $H = G_0/N$ be as in (2.1) and $K = [G, G_0]/N$. Then it is straightforward to check that $K$ is $B$-invariant and that it is generated by $B$-commutators. Since $K$ is finite, the orthogonality relations given in Lemma [21 (ii) imply that there are only finitely many ideals
\( \{ J(\chi) : [\chi] \in \mathcal{X}_B(G)/\sim \}. \) It then follows from Lemma 2.1 (i), [13] Prop. 5.2, and the fact that each \( L^1(G(\chi)) \) is amenable, that \( ZL^1(G) \cong ZB^1_1(H) \) is amenable.

Observe that condition of the theorem above holds when \( G' \) is finite. It also holds when \( G_0 = \{ e \} \), in which case \( G \) is called an infinite conjugacy class group.

2.3. Some hyper-Tauberian centres. We direct the reader to Section 0.2 for the definition and consequences of the hyper-Tauberian property.

**Proposition 2.3.** Suppose \( G, B \) and \( K \) are as in the hypotheses of Lemma 2.1. Then \( Z_B^1(G) \) is hyper-Tauberian.

**Proof.** By Lemma 2.1 (i) we may write

\[
Z_B^1(G) = \bigoplus_{[\chi] \in \mathcal{X}_B(G)/\sim} J(\chi) \cong \bigoplus_{[\chi] \in \mathcal{X}_B(G)/\sim} L^1(G(\chi)).
\]

Hence it follows from [25, Cor. 13] that \( Z_B^1(G) \) is hyper-Tauberian. □

We say that \( G \) is an \([FC]^-\)-group if each conjugacy class in \( G \) is relatively compact. In the notation of Section 2.1 this is the same as having \( G = G_0 \).

**Theorem 2.4.** If \( G \) is an \([FC]^-\)-group, then \( ZL^1(G) \) is hyper-Tauberian.

**Proof.** In the notation of (2.1) we have that \( H = G/N \) and \( B = \text{Inn}(H) \). Thus \( ZL^1(G) \cong ZL^1(H) \), and we may assume \( G \), itself, is an \([FIA]^-\)-group.

If \( G \) is compactly generated, then [8] (3.20) guarantees that the derived group \( K = G' \) is compact. Hence we can apply Proposition 2.3 and we are done.

If \( G \) is not compactly generated, we must localise our argument to a compactly generated subgroup. We first wish to see that \( Z\mathcal{C}_c(G) \), the space of all of compactly supported continuous elements of \( ZL^1(G) \), is dense in \( ZL^1(G) \). We note that \( P : L^1(G) \to ZL^1(G) \), given for almost every \( s \) in \( G \) by \( Pf(s) = \int_{\text{lim}(G)} f(\beta(s))d\beta \), defines a surjective quotient map. Hence if \( f \in ZL^1(G) \) and \( \langle u_n \rangle \subset \mathcal{C}_c(G) \) is a sequence with \( \lim_n u_n = f \), then \( \lim_n Pu_n = Pf = f \).

Now let \( T : ZL^1(G) \to ZL^1(G)^* \) be a local operator. To see that \( T \) is a \( ZL^1(G) \)-module map, it suffices to show that

\[
\langle T(u* v), w \rangle = \langle u*T(v), w \rangle
\]

for any \( u, v, w \) in \( Z\mathcal{C}_c(G) \). The set \( U = \{ s \in G : |u(s)| + |v(s)| + |w(s)| > 0 \} \) is \( \text{Inn}(G) \)-invariant, open and relatively compact. Hence \( U \) generates a normal open subgroup \( F \) of \( G \). We let \( B = \text{Inn}(G)|_F \) and note that \( F \) is an \([FIA]_B^-\)-group.

We have that the closed subgroup \( K \) generated by \( B \)-commutators in \( F \) is compact. This is noted in [17], though does follow obviously from [8] (3.20). Let us show how this can be proved from [8]. It is shown in
that $G'$ consists of periodic elements, elements which individually generate relatively compact subgroups of $G$. Hence $K = [F, G] \subset G'$ consists of periodic elements. Since $F$ is compactly generated and an $[FIA]_B$-group, it is clear that $K$ is compactly generated. Then by [8, (3.17)], $K$ is compact. Clearly $K$ is $B$-invariant. Thus $Z_B L^1(F)$ is hyper-Tauberian by Proposition 2.3. We note that $Z_B L^1(F)$ is the closed subalgebra of all elements of $ZL^1(G)$ which vanish almost everywhere off of $F$. Moreover, the mapping $\chi \mapsto \chi|_{Z_B L^1(F)}$ maps $\mathcal{X}(G)$ continuously onto $\mathcal{X}_B(F)$, by [18] Prop. 2.9. Let $\iota : Z_B L^1(F) \to ZL^1(G)$ be the injection map, so $\iota^* \circ T_\iota : Z_B L^1(F) \to Z_B L^1(F)^*$ is a local map. Then $\iota^* \circ T_\iota$ is a $Z_B L^1(F)$-module map. Since $u,v,w \in Z_B L^1$, we see that (2.2) holds.

We note that there are non-$[FC]$-groups for which the above result fails. Let $n \geq 3$ and $G_n = \mathbb{R}^n \rtimes SO(n)_d$, the semi-direct product of $\mathbb{R}^n$ with the discrete special orthogonal group. We have for odd $n$ that $ZL^1(G_n) \cong Z_{SO(n)} L^1(\mathbb{R}^n)$; for $n = 3$ this was observed in [18, p. 162]. (Note that for even $n$ we have $Z(SO(n)) = \{1, -1\} = \mathbb{Z}_2$ and we have $ZL^1(G_n) \cong Z_{SO(n)} L^1(\mathbb{R}^n \rtimes \mathbb{Z}_2)$.) It is proved in [23] Prop. 2.6.8 (see also [1] Thm. 5.5) that for $n \geq 3$, $Z_{SO(n)} L^1(\mathbb{R}^n)$ admits non-zero point derivations. Hence this algebra cannot even be weakly amenable, neverless hyper-Tauberian, as noted in Theorem 0.2. Moreover, for $n \geq 3$, it is shown [23, 2.6.10] that except for the augmentation character, no singleton in $\mathcal{X}_{SO(n)}(\mathbb{R}^n)$ is a set of spectral synthesis.

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