We study the performance of supcode—a family of dynamically correcting pulses designed to cancel simultaneously both Overhauser and charge noise for singlet-triplet spin qubits—adapted to silicon devices with electrostatic control. We consider both natural Si and isotope-enriched Si systems, and in each case we investigate the behavior of individual gates under static noise and perform randomized benchmarking to obtain the average gate error under realistic $1/f$ noise. We find that in most cases supcode pulses offer roughly an order of magnitude reduction in gate error, and especially in the case of isotope-enriched Si, supcode yields gate operations of very high fidelity. We also develop a version of supcode that cancels the charge noise only, \textit{\alpha}J-supcode, which offers a level of error reduction comparable to the original supcode while yielding gate times that are 30\% to 50\% shorter. Our results show that \textit{\alpha}J-supcode is particularly beneficial for isotope-enriched Si devices where charge noise dominates Overhauser noise, providing a fast, simple, and effective approach to error suppression, bringing gate errors well below the quantum error correction threshold in principle.

\vspace{10pt}

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Ever since Kane’s original proposal for a silicon-based quantum computer\cite{Kane98}, Si has been a prime candidate as a host material for solid-state quantum computing\cite{Cleland11}. A decisive advantage of Si is that it is available in an isotope-enriched form (\textsuperscript{28}Si) that has zero nuclear spin. This allows for the nearly complete removal of decoherence due to the hyperfine interaction between the qubit and surrounding nuclear spins (Overhauser noise), leading to remarkably long coherence times and high control fidelity\cite{Barthel06,Albright13}. Recent years have witnessed substantial experimental progress in the fabrication, initialization, readout and control of spins in both phosphorous donors and gate-defined quantum dots in Si systems\cite{Hanson10,Allan13}. For a P donor qubit on Si, both the donor electron spin\cite{Petta05} and the nuclear spin\cite{Hanson08} have been explored as potential qubits, exhibiting coherence times up to 30 seconds and control fidelities exceeding 99.99\% in recent experiments\cite{Petta10}, both of which are the highest achieved for any solid state qubit. In laterally defined Si quantum dot systems, coherent operation of a singlet-triplet spin qubit\cite{Jeffrey12,Jeffrey13} has also been demonstrated\cite{Matta12,Barnes13}.

While Si qubits have been shown repeatedly to possess long information storage (i.e. quantum memory) times, it remains an open question on how to extend this long-lived coherence to qubits undergoing quantum gate operations. Experiments have progressed to such a stage where high fidelity quantum gates are now within reach. While quantum devices built on isotope-enriched Si already enjoy many advantages, some noise remains. In particular, the residual \textsuperscript{29}Si impurities distort the wave function of the donor qubit or the lateral confinement potential of the quantum dot, which either broadens the ESR resonance in the former case\cite{Nemitz08,Resch10} or shifts the exchange interaction in the latter\cite{Hunten12}, both causing inaccuracies in control. More importantly, additional errors can arise through electrical fluctuations from either the leads or nearby charged impurities, and from variations in control parameters\cite{Barnes13,Allan13,Jeffrey12}. While a vast literature on dynamically corrected gates (DCGs) has existed long before the conception of spin qubits, it was only recently realized that the special constraints imposed by solid state spin systems often require brand-new approaches to developing DCGs\cite{Oh2013,Allan13,Jeffrey12}. While one of these approaches, known as supcode\cite{King07}, has already been experimentally demonstrated for nitrogen-vacancy centers in diamond\cite{Jelezko04} and is also applicable to single donor spins in Si, it is particularly suitable for singlet-triplet qubits subject to Overhauser and charge noise because it is designed to yield comparatively simple pulse sequences that cancel both while respecting all experimental constraints, including the restriction to positive, single-axis control. In the case of isotope-enriched Si, however, canceling Overhauser noise is no longer necessary, leaving open the possibility that robust quantum operations on Si devices might be realized with even simpler and faster control sequences.

In this Rapid Communication, we show that supcode pulses can substantially improve gate fidelities in silicon singlet-triplet spin qubits. We begin by quantifying the extent to which charge noise corrupts the application of individual quantum gates and demonstrate the cancelation of these errors effected by supcode pulses. We then provide a more systematic analysis of the performance of supcode by studying its effectiveness for long sequences of quantum gates subject to $1/f$ noise, which describes both Overhauser and charge noise\cite{Cleland11}. Through randomized benchmarking\cite{Knill08}, we extract the average error per gate for the set of single-qubit Clifford gates, and we find that supcode pulses remain robust against noise in this
We also develop a version of SUPCODE that works particularly well for isotope-enriched Si while being about 30% to 50% faster, and we demonstrate that it offers a level of error reduction comparable to the original version. These results illustrate that Si quantum devices can achieve long coherence times not only when the qubit is idle but even while it is undergoing operations.

We start with the model Hamiltonian governing a singlet-triplet qubit
\[ H(t) = \frac{\hbar}{2} \sigma_x + J \left[ \frac{\epsilon(t)}{2} \right] \sigma_z. \] 
The computational bases are \( |0\rangle = |T\rangle = (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2} \) and \( |1\rangle = |S\rangle = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2} \), where \( |\downarrow\rangle = c_{\downarrow}^\dagger c_{\uparrow}^\dagger |\text{vacuum}\rangle \) with \( c_{\sigma}^\dagger \) creating an electron with spin \( \sigma \) in the \( j \)th dot. As can be seen from Eq. (1), rotations around the \( z \)-axis are performed with a magnetic field gradient across the double-dot system, which in energy units reads \( h = g \mu_B \Delta B_z \). This magnetic field gradient can be generated by a micromagnet, as demonstrated in Si/SiGe quantum dot systems, where \( h \) is approximately 30–60 neV. Rotations around the \( z \)-axis are done using the exchange interaction \( J \), the energy level splitting between \( |S\rangle \) and \( |T\rangle \). The magnitude of \( J \) is controlled by the detuning \( \epsilon \), namely the tilt of the double-well confinement potential, which in turn can be controlled efficiently by changing the gate voltages. Experimentally, \( J \) can be tuned from close to zero up to tens of \( eV \). One therefore has fast, all-electrical control over the rotation rate around the \( z \)-axis. On the other hand, the magnetic field gradient \( h \) cannot be changed efficiently during a gate operation, and we will take it to be constant throughout the computation.

In general, decoherence can arise through both \( h \) and \( J \). Fluctuations arising from Overhauser noise add a small, but unknown error term \( \delta h \) to the Hamiltonian:
\[ h \rightarrow h + \delta h. \] 
For natural Si, \( \delta h \) is on the scale of neV, corresponding to a few percent of \( h \). However, for isotope-enriched silicon, \( \delta h \) can be reduced by up to three orders of magnitude, down to \( \delta h/h \sim 10^{-5} \). On the other hand, impurities on the substrate lead to deformations of the confinement potential and consequently the energy level structure, thereby causing noise [21–26]. This noise produces fluctuations in the exchange energy, \( \delta J \), through fluctuations in the detuning \( \epsilon \). In this work, we assume a phenomenological form of \( J = J_0 \exp(\epsilon/\epsilon_0) \), implying \( \delta J = J_0 \delta \epsilon \), but our method applies to other forms also. Estimates of \( \delta \epsilon/\epsilon_0 \) vary, ranging from 10% to 50% (cf. Ref. 59). We study a range of \( \delta \epsilon \) values in this work.

We previously developed a family of composite pulses (SUPCODE) that cancels both Overhauser noise and charge noise. The basic idea is as follows: assuming the noise is quasi-static, one expands the evolution operator to first order in both \( \delta h \) and \( \delta \epsilon \). In order to cancel these error terms, the original rotation is supplemented by an (imperfect) identity operation, carefully designed such that the error arising from performing this identity operation exactly cancels that of the target rotation. This formalism reduces to a multi-dimensional optimization problem involving six coupled nonlinear equations (corresponding to error terms for \( \sigma_x, \sigma_y \), and \( \sigma_z \), each of which has \( \delta \epsilon \) and \( \delta h \) components), from which the pulse parameters (the strength of the exchange interaction and the duration of each piece) can be straightforwardly obtained. A detailed description of the theoretical framework is presented in Ref. [22] and we shall refer to these sequences as “full SUPCODE” sequences in this work because they cancel both Overhauser noise and charge noise.

While undoubtedly full SUPCODE works for the problem at hand, further optimizations are possible for an isotope-enriched Si system. Because the Overhauser noise is negligible in this case, one can completely ignore the \( \delta h \) component of the error, leading to a simplification of the problem since only three coupled nonlinear equations need to be solved, and the resulting sequences are much shorter and simpler. In Fig. 1 we show pulse profiles comparing the full SUPCODE and \( \delta J - \text{SUPCODE} \) for a Hadamard gate. Detailed sequences and pulse parameters for each of the single-qubit Clifford gates are given in the Supplemental Material. In obtaining these results, we restricted the strength of the exchange to \( J \leq 10 \) eV so that the resulting sequences are more feasible for experimental implementation. In the Supplemental Material, we also show that \( \delta J - \text{SUPCODE} \) is about 30% to 50% shorter than the full SUPCODE, and that the number of pieces as well as the total angle swept around the Bloch sphere are also substantially reduced (with the exception of \( \pi \)-rotations, which are the same as naive pulses since no charge noise occurs if \( J \) is not pulsed on at all). We shall also show, in the remainder of this paper, that for parameter regimes of interest, the error reduction of \( \delta J - \text{SUPCODE} \) is comparable to that of full SUPCODE for isotope-enriched Si, making it particularly suitable for experimental realization in these systems.

Figure 2 shows the infidelity (cf. Ref. 11) of the Hadamard gate versus quasi-static charge noise strength. We model natural Si as having \( \delta h/h = 3 \% \), while isotope-enriched Si has \( \delta h/h = 2 \times 10^{-5} \), consistent with experimental values. Fig. 2(a) shows the result for natural Si. While the curve for the naive pulse saturates at an infidelity of about 10−3, the full SUPCODE has a residual error of only 10−7 for charge noise of less than 1%, and if the quantum error correction threshold is set at 10−4, the range of charge noise that can be tolerated is as large as 5%. This is as expected because full SUPCODE cancels both \( \delta h \) and \( \delta \epsilon \) errors, and should offer reduction in error even when \( \delta h \) is held at a non-vanishing value. Turning to the curve for \( \delta J - \text{SUPCODE} \), we see that for most of the \( \delta \epsilon \) range (\( \delta \epsilon \lesssim 5 \% \)), its performance is worse than the naive pulse. This is understandable because \( \delta J - \text{SUPCODE} \) completely ignores the Overhauser noise, and this is precisely the range where Overhauser noise
is dominant. Remarkably, for the range where charge noise is considerably larger than the Overhauser noise ($\delta \epsilon / \epsilon_0 \gtrsim 10\%$), $\delta J$-SUPCODE reduces the error even more than full SUPCODE. This effect presumably comes from the reduced length and complexity of the pulse sequence during which less error accumulates.

Figure 2(b) shows the case of isotope-enriched Si. From the different scale of the y-axis compared with Fig. 2(a) one immediately notices that the gate error is substantially reduced for all three curves as expected for negligible Overhauser noise. The exceedingly low errors achieved by the full SUPCODE are consistent with the remarkably long coherence times measured in experiments. At a typical quantum error correction threshold of $10^{-4}$, the naive pulse tolerates charge noise up to 1%, whereas the full SUPCODE tolerates as much as 7%. While this range does not differ from the case of natural Si, the gate error for charge noise less than $10^{-2}$ is substantially smaller due to the absence of Overhauser noise. Note that for the isotope-enriched Si, $\delta J$-SUPCODE has low error comparable with full SUPCODE for a wide range of noise amplitudes. Even in the regime where it is not as effective as full SUPCODE ($\delta \epsilon / \epsilon_0 \lesssim 1\%$), the error is still greatly reduced compared to that of the naive pulse.

We observe similar behavior for all single-qubit Clifford gates (not shown here) with the exception of $x$-rotations, where $\delta J$-SUPCODE has a constant error because charge noise does not come into play. The results suggest that while isotope-enriched Si already allows for much more precise gate operation compared with natural Si, the implementation of carefully designed composite pulses further reduce the error, offering orders-of-magnitude-longer coherence times. $\delta J$-SUPCODE is particularly suitable for isotope-enriched Si because it provides almost the same level of error reduction with faster gate times and less pulse sequence complexity.

While individual gate errors such as those shown in Fig. 2 are strongly indicative of the improvements afforded by SUPCODE, a more complete characterization of this improvement is obtained by studying the error over long sequences of gates through randomized benchmarking. We have so far assumed a static noise model, which is a good approximation for single gates, but a more realistic model taking into account temporal variations of the noise is needed for longer gate sequences. To include these effects, we employ a $1/f$ noise model in which both the Overhauser and charge noise have a power spectrum of the form $S_k(\omega) = \beta_k/\omega^\gamma$, where $k \in \{h, J\}$ denotes Overhauser ($h$) and charge noise ($J$), and $\beta_h$ and $\beta_J$ denote the noise amplitudes. We choose the exponent to be $\gamma = 1.5$, which is consistent with the upper bound estimated from experiments. Previous work suggests that for $\gamma < 1$ composite pulses would not be expected to improve the gate fidelity since the noise is similar to white noise. However for $\gamma = 1.5$, the noise is more concentrated at low frequencies, and we expect SUPCODE to have a more significant effect. We implement randomized benchmarking by averaging the fidelity over random sequences of single-qubit Clifford gates and over different noise realizations for a varying number of
FIG. 3: Decay of the gate fidelity via randomized benchmarking under 1/f noise for charge noise with amplitude $\beta_J/\epsilon_0 = 1\%$ for (a) natural Si and (b) isotope-enriched Si. The naïve pulse, full supcode and $\delta J$-supcode are shown as black, red and blue lines respectively, and the thin black dot-dashed, red dashed and blue dotted lines are fits of the corresponding data to $\frac{1}{2}(1 + e^{-\gamma n})$. Note the different scales of both the $x$ and $y$ axes.

FIG. 4: Average error per gate vs. amplitude of the charge noise $\beta_J/\epsilon_0$ for (a) natural Si and (b) isotope-enriched Si. The gate error is shown for the naïve pulses (black solid), full supcode (red dashed), and $\delta J$-supcode (blue dotted). Note the different scales of the $y$-axis.

gates in the sequences. The fidelity of a sequence behaves as $\frac{1}{2}(1 + (1 - d)^n)$, where $d$ is the average error per gate, and $n$ is the number of Clifford gates applied. We fit our results to the exponentially decaying function $\frac{1}{2}(1 + e^{-\gamma n})$, where the average error per gate is related to the fitted exponent via $d = 1 - e^{-\gamma}$. The realizations of 1/f noise are generated using a weighted sum of random telegraph signals.

Figure 3 shows the randomized benchmarking results for charge noise with $\beta_J/h = 1\%$ (note that one should not directly compare this result to that of constant noise with $\delta \epsilon/\epsilon_0 = 1\%$ since the noise spectra are completely different). Here, natural Si is modeled as $\beta_n/h = 3\%$ and isotope-enriched Si as $\beta_n/h = 2 \times 10^{-5}$. Fig. 3(a) shows the result for natural Si. These results are generated using the average of 200 realizations of noise/gate sequences, each of which contains up to 300 Clifford gates. All curves asymptote to fidelity 1/2 as expected, but the naïve pulse drops to 1/2 much more steeply, reaching the asymptotic value after less than 20 gate operations.

SUPCODE pulses, on the other hand, maintain the fidelity much longer, with 70% fidelity after 60 gate operations. This clearly demonstrates that SUPCODE pulses play a crucial role in error reduction. It is also remarkable that the performance of $\delta J$-SUPCODE is almost the same as that of the full SUPCODE. Turning to Fig. 3(b), the results for isotope-enriched Si, we see a huge reduction in error. Here we average over 100 realizations of the noise and gate sequences and increase the number of gates in each sequence to 500. We see that even with the naïve pulses, the fidelity remains larger than 80% after 400 gate operations. Most remarkably, when SUPCODE pulses are used, the fidelity remains greater than 95% even after 500 gate operations. This fact reinforces the main point of this paper: while isotope-enriched Si is already superior to natural Si, application of SUPCODE reduces the error much further. We see again that $\delta J$-SUPCODE has an error reduction ability comparable with the full SUPCODE, so it is particularly useful for isotope-enriched Si since it is typically shorter, less complicated, and works almost perfectly when charge noise plays a much more significant role than Overhauser noise.
We have also performed randomized benchmarking for different magnitudes of the charge noise, and these results are summarized in Fig. 3, where we show the average error per gate, $d$, obtained from the fitted decaying exponent. We note from Fig. 3(a) that, for charge noise below 3% in natural Si, the naïve pulse has a gate error of approximately 20-30%. While this is a large error, it can be reduced by roughly one order of magnitude by the application of SUPCODE pulses. Fig. 3(b) shows the result for isotope-enriched Si. The gate errors are again much lower than those of natural Si: on a log-log plot, the gate error of the naïve pulses reduces linearly from around 1% for 3% charge noise, to $10^{-5}$ for a charge noise of 0.1%. In this range, the gate error is further reduced by an order of magnitude when SUPCODE pulses are used.

In conclusion, we studied the performance of SUPCODE for both static noise and $1/f$ noise in Si (natural and isotope-enriched). While isotope-enriched Si already has much lower gate error compared to natural Si, application of SUPCODE further reduces the error in a substantial way. We also demonstrated that δ-J-SUPCODE performs comparably to the full SUPCODE, so it is particularly useful for isotope-enriched Si since it typically generates simpler and faster sequences. We note that the same protocol applies to two-qubit gates for single-spin qubits as well.

Our results clearly show that the extremely long coherence times of Si qubits pertain not only to information storage but also to information processing, if SUPCODE is applied. Thus, Si spin qubits may now be close to quantum error corrected fault tolerant quantum computing applications.

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In this Supplementary Material we provide details of the pulse parameter of the single-qubit Clifford gate for $\delta J$-supcode, which is obtained from solving the three coupled nonlinear equation (involving $\sigma_x$, $\sigma_y$, and $\sigma_z$ respectively) relevant to the charge noise only. Note that $x$-rotations $R(\hat{x}, -\pi/2)$, $R(\hat{x}, \pi/2)$, and $R(\hat{x}, \pi)$ do not need to turn on $J$ at all (therefore having no $\delta J$ error) and when charge noise is the only error source, the naive pulses suffice.

The identity operation and the Hadamard gate can be made resistant to charge noise by using the following pulse sequence:

$$U(J, \pi + \frac{\phi}{2}) U(j_3, \pi) U(j_2, \pi) U(j_1, \pi) U(j_0, 2\pi) U(j_1, \pi) U(j_2, \pi) U(j_3, \pi) U(J, \pi + \frac{\phi}{2})$$

(S-1)

with parameters provided in Table I.

| $\delta J$-supcode | $J_0$ | $J_1$ | $J_2$ | $J_3$ |
|-------------------|------|------|------|------|
| $J$               | 1    | 0.099998 | 0.9849 | 0.12167 |
| $R(\hat{x} + \hat{z}, \pi)$ | 1    | $\pi$ | 1.3604 | 10.000 |

$R(\hat{x}, -\pi/2)$ is done using the sequence

$$U(J = 1, \pi) U(j_3 = 0, \pi + \frac{\phi}{2}) U(j_4, \pi) U(j_2, \pi) U(j_3, \pi) U(j_1 = 0, \pi) U(j_0, 2\pi)$$

(S-2)

with parameters given in Table II.

$R(\hat{x}, \pi/2)$ is used as the sequence

$$U(J = 1, \pi) U(j_3 = 0, \pi + \frac{\phi}{2}) U(j_4, \pi) U(j_2, \pi) U(j_3, \pi) U(j_1 = 0, \pi) U(j_0, 2\pi)$$

(S-3)

with parameters given in Table III.

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**Table I: Parameters of the $\delta J$-supcode sequence, Eq. (S-1), appropriate for the identity operation and the Hadamard gate.**

Strengths of the exchange interactions are given in units of $\hbar$.

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**Supplementary material**

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TABLE II: Parameters of the $\delta J$-supcode sequence, Eq. (S-2), appropriate for Clifford $z$-rotations. Strengths of the exchange interactions are given in units of $\hbar$.

| $\phi$ | $j_0$ | $j_1$ | $j_2$ | $j_3$ |
|--------|-------|-------|-------|-------|
| $R(\hat{z}, \pi/2)$ | $\pi/2$ | 0.39727 | 0.9998 | 0.12151 | 0.9998 |
| $R(\hat{z}, \pi)$ | $\pi$ | 0.94156 | 10.000 | 0.12518 | 9.9994 |

TABLE III: Parameters of the $\delta J$-supcode sequence, Eq. (S-3), appropriate for Clifford $R(\hat{z}, -\pi/2)$ rotation. Strengths of the exchange interactions are given in units of $\hbar$.

| $\phi$ | $j_0$ | $j_1$ | $j_2$ | $j_3$ | $j_4$ |
|--------|-------|-------|-------|-------|-------|
| $R(\hat{z}, -\pi/2)$ | $-\pi/2$ | 0.9090 | 0.26697 | 0.1544 | 0.68434 | 10.000 |

A general rotation around any arbitrary axis can be achieved by the following sequence

$$U(J = 0, \phi_a)U(J = 1, \pi)U(j_4 = 1, \pi + \theta_4)U(j_3, \pi)U(j_2, \pi)U(j_1, \pi)U(j_0, 2\pi) \times U(j_1, \pi)U(j_2, \pi)U(j_3, \pi)U(j_4 = 1, \pi - \theta_4)U(J = 0, \phi_b)U(J = 1, \pi)U(J = 0, \phi_c).$$

(S-4)

with parameters given in Table IV.

TABLE IV: Parameters of the $\delta J$-supcode sequence, Eq. (S-4), appropriate for all remaining Clifford gates. Strengths of the exchange interactions are given in units of $\hbar$.

| $\phi$ | $j_0$ | $j_1$ | $j_2$ | $j_3$ | $\theta_4$ | $\phi_a$ | $\phi_b$ | $\phi_c$ |
|--------|-------|-------|-------|-------|------------|----------|----------|----------|
| $R(\hat{y}, -\pi/2)$ | 10.000 | 0.41333 | 10.000 | 0.24570 | $\arctan[(4 + \sqrt{2\pi})/(-2\sqrt{2} + \pi)]$ | $3\pi/2$ | $3\pi/2$ | $\pi/2$ |
| $R(\hat{y}, \pi/2)$ | 10.000 | 0.41333 | 10.000 | 0.24570 | $\arctan[(4 + \sqrt{2\pi})/(-2\sqrt{2} + \pi)]$ | $5\pi/2$ | $3\pi/2$ | $3\pi/2$ |
| $R(\hat{y}, \pi)$ | 1.2144 | 10.000 | 0.26486 | 10.000 | $-2\arctan[(\pi + \sqrt{16 + \pi^2})/4]$ | $3\pi/2$ | $\pi$ | $\pi/2$ |
| $R(\hat{x} - \hat{z}, \pi)$ | 10.000 | 0.41333 | 10.000 | 0.24570 | $\arctan[(4 + \sqrt{2\pi})/(-2\sqrt{2} + \pi)]$ | $\pi/2$ | $3\pi/2$ | $\pi/2$ |
| $R(\hat{x} + \hat{y}, \pi)$ | 9.9957 | 0.30460 | 9.9964 | 0.27306 | $\arctan[(4 - \sqrt{2\pi})/(2\sqrt{2} + \pi)]$ | 0 | $\pi/2$ | 3$\pi$ |
| $R(\hat{x} - \hat{y}, \pi)$ | 9.9957 | 0.30460 | 9.9964 | 0.27306 | $\arctan[(4 - \sqrt{2\pi})/(2\sqrt{2} + \pi)]$ | $\pi$ | $5\pi/2$ | $2\pi$ |
| $R(\hat{y} + \hat{z}, \pi)$ | 10.000 | 0.33692 | 9.9951 | 0.34373 | $\arctan(4/\pi)$ | 7$\pi/2$ | $2\pi$ | $2\pi$ |
| $R(\hat{y} - \hat{z}, \pi)$ | 10.000 | 0.33692 | 9.9951 | 0.34373 | $\arctan(4/\pi)$ | $\pi/2$ | $\pi$ | 0 |
| $R(\hat{x} + \hat{y} + \hat{z}, 2\pi/3)$ | 8.5686 | 0.29793 | 10.000 | 0.27073 | $\arctan[(4 - \sqrt{2\pi})/(2\sqrt{2} + \pi)]$ | 0 | $\pi/2$ | $\pi/2$ |
| $R(\hat{x} + \hat{y} + \hat{z}, 4\pi/3)$ | 10.000 | 0.41333 | 10.000 | 0.24570 | $\arctan[\sqrt{2} + 8(2\sqrt{2} + \pi)/(\pi^2 - 8)]$ | 7$\pi/2$ | 7$\pi/2$ | 2$\pi$ |
| $R(\hat{x} + \hat{y} - \hat{z}, 2\pi/3)$ | 10.000 | 0.41333 | 10.000 | 0.24570 | $\arctan[(4 + \sqrt{2\pi})/(-2\sqrt{2} + \pi)]$ | 5$\pi/2$ | 3$\pi/2$ | 4$\pi$ |
| $R(\hat{x} + \hat{y} - \hat{z}, 4\pi/3)$ | 9.9957 | 0.30460 | 9.9964 | 0.27306 | $\arctan[(-8\sqrt{2} + \pi(8 - \sqrt{2\pi})]/(\pi^2 - 8)]$ | 0 | $\pi/2$ | 3$\pi/2$ |
| $R(\hat{x} - \hat{y} + \hat{z}, 2\pi/3)$ | 8.7287 | 0.29878 | 10.000 | 0.27104 | $\arctan[(-8\sqrt{2} + \pi(8 - \sqrt{2\pi})]/(\pi^2 - 8)]$ | $\pi/2$ | 5$\pi/2$ | 2$\pi$ |
| $R(\hat{x} - \hat{y} + \hat{z}, 4\pi/3)$ | 10.000 | 0.41333 | 10.000 | 0.24570 | $\arctan[(4 + \sqrt{2\pi})/(2\sqrt{2} + \pi)]$ | 2$\pi$ | 7$\pi/2$ | 3$\pi/2$ |
| $R(-\hat{x} + \hat{y} + \hat{z}, 2\pi/3)$ | 9.9957 | 0.30460 | 9.9964 | 0.27306 | $\arctan[(4 - \sqrt{2\pi})/(2\sqrt{2} + \pi)]$ | 3$\pi/2$ | $2\pi$ | 2$\pi$ |
| $R(-\hat{x} + \hat{y} + \hat{z}, 4\pi/3)$ | 10.000 | 0.41333 | 10.000 | 0.24570 | $\arctan[\sqrt{2} + 8(2\sqrt{2} + \pi)/(\pi^2 - 8)]$ | 4$\pi$ | 7$\pi/2$ | 5$\pi/2$ |
In Table V we compare the pulse length of the naïve pulse ($L_{naiv}$), full supcode ($L_{full}$) and $\delta$J-supcode ($L_{\delta J}$) in unit of ($h^{-1}$). From the ratio $L_{\delta J}/L_{full}$ we clearly see that $\delta$J-supcode is approximately 30% ~ 50% shorter than the full supcode (except the $x$-rotations), demonstrating its advantage to be implemented in isotope-enriched Si.

|            | $L_{naiv}$ | $L_{full}$ | $L_{\delta J}$ | $L_{\delta J}/L_{full}$ | $L_{naiv}$ | $L_{full}$ | $L_{\delta J}$ | $L_{\delta J}/L_{full}$ |
|------------|------------|------------|----------------|--------------------------|------------|------------|----------------|--------------------------|
| $I$        | 4.443      | 30.92      | 16.37          | 53.0%                    | 9.155      | 57.34      | 33.20          | 57.9%                    |
| $R(\hat{x}, -\pi/2)$ | 4.712      | 29.84      | 4.712          | 15.8%                    | 9.155      | 63.72      | 39.49          | 62.0%                    |
| $R(\hat{x}, \pi/2)$   | 1.571      | 42.07      | 1.571          | 3.73%                    | 12.30      | 66.33      | 42.45          | 64.0%                    |
| $R(\hat{x}, \pi)$     | 3.142      | 28.10      | 3.142          | 11.2%                    | 9.155      | 53.15      | 26.74          | 50.3%                    |
| $R(\hat{y}, -\pi/2)$  | 15.44      | 56.34      | 33.04          | 58.6%                    | 7.584      | 50.68      | 25.47          | 50.2%                    |
| $R(\hat{y}, \pi/2)$   | 12.30      | 60.89      | 39.32          | 64.6%                    | 13.87      | 71.13      | 50.32          | 70.7%                    |
| $R(\hat{y}, \pi)$     | 13.87      | 56.62      | 29.63          | 52.3%                    | 10.73      | 72.95      | 47.18          | 64.7%                    |
| $R(\hat{z}, -\pi/2)$  | 9.155      | 38.77      | 22.35          | 57.6%                    | 10.73      | 53.40      | 28.49          | 53.4%                    |
| $R(\hat{z}, \pi/2)$   | 6.014      | 47.66      | 25.62          | 53.8%                    | 7.584      | 62.24      | 38.02          | 61.1%                    |
| $R(\hat{z}, \pi)$     | 7.584      | 50.96      | 25.93          | 50.9%                    | 13.87      | 65.48      | 44.04          | 67.3%                    |
| $R(\hat{x} + \hat{z}, \pi)$ | 2.221      | 28.76      | 17.81          | 61.9%                    | 10.73      | 59.16      | 34.77          | 58.8%                    |
| $R(\hat{x} - \hat{z}, \pi)$ | 12.30      | 53.50      | 29.90          | 55.9%                    | 10.73      | 71.91      | 53.46          | 74.3%                    |

TABLE V: Pulse length of the naïve pulse ($L_{naiv}$), full supcode ($L_{full}$) and $\delta$ supcode ($L_{\delta J}$) in unit of ($h^{-1}$). The ratio $L_{\delta J}/L_{full}$, showing the reduction in length of $\delta$J-supcode from the full one, is also presented.