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The moduli space of the tropicalizations of Riemann surfaces

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Abstract. In this paper we study the moduli space of the tropicalizations of Riemann surfaces. We first tropicalize a smooth pointed Riemann surface by its pair of pants decomposition. Then we can construct the moduli space of tropicalizations based on a fixed regular tropicalization, and compactify it by adding strata parametrizing weighted contractions. This, in the covering level, is analogous to adding the frontier set, subordinate to a pants decomposition, to the Teichmüller space. We show that this compact moduli space is also Hausdorff. In the end, we compare this moduli space with the moduli space of Riemann surfaces, establishing a partial order-preserving correspondence between the stratifications of these two moduli spaces.

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1. Introduction

We wish to construct the moduli space of the tropicalizations of Riemann surfaces with marked points in this paper, so that a partial order-preserving correspondence can be established between the stratifications of this moduli space and the moduli space of Riemann surfaces.

The tropical methods have already been used to study the moduli theory of algebraic curves during the past decade, via the help of non-Archimedean geometry.

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In [7], Caporaso constructed the moduli space of tropical curves with marked points. And later in [3], Abramovich, Caporaso and Payne identified this space with the skeleton of the moduli stack of stable curves, via the analytification of its coarse moduli space. Hence they established an order-reversing correspondence between the stratifications of the two moduli spaces. For their purpose they tropicalize an algebraic curve by its dual graph and interpret the edge-length as the “complexity” of the node. These work are also summarized in the expository papers [2, 8]. Along the way, Chan, Galatius and Payne recently studied the topology of tropical moduli spaces to look into the cohomology of the moduli space of algebraic curves [9, 10]. On the other hand, Odaka compactified the moduli space of Riemann surfaces (without marked points) by attaching the moduli of tropical curves as boundary in [13]. Those tropical curves are obtained as the Gromov–Hausdorff collapse of Riemann surfaces by fixing their diameters with respect to the hyperbolic metric. It turns out that there also exists a bijective and order-reversing correspondence between the stratifications of this boundary and the Deligne–Mumford boundary.

However, we want to establish an order-preserving correspondence between stratifications of some moduli space in a tropical sense and the moduli space of Riemann surfaces (or complex algebraic curves). In order to realize that, we tropicalize a smooth pointed Riemann surface by the graph defined by its pair of pants decomposition, and the edge-length is accordingly given by the length of the geodesic representative of the corresponding boundary component of the (hyperbolic) pair of pants. We denote this tropicalization by \((G, \ell)\). The underlying graph \(G\) is always trivalent and we call such a topological tropicalization a regular tropicalization for a smooth pointed Riemann surface. Since a regular tropicalization is not unique, we always fix a regular tropicalization \(G\) to start with for constructing our moduli space. Now each tropicalization \((G, \ell)\) corresponds to a point in the open cone

\[ C(G) := \mathbb{R}_{>0}^{\vert E(G)\vert} \]

if to each edge we associate a coordinate. Of course, the automorphism group \(\text{Aut}(G)\) of the graph induces an action on this cone by permuting the coordinates. So we construct the moduli space of tropicalizations \((G, \ell)\) as the following quotient space:

\[ M_G := C(G)/\text{Aut}(G) = \mathbb{R}_{>0}^{\vert E(G)\vert}/\text{Aut}(G) = \mathbb{R}_{>0}^{3g-3+n}/\text{Aut}(G) \]

with the quotient topology.

Then a weighted contraction of \(G\), for which the genus of the graph is preserved, is introduced to represent the boundary point of \(C(G)\). In this sense the regular tropicalization is viewed as a \(0\)-weighted graph \((G, 0)\), and those dependent notions are thus modified accordingly. In particular, the above moduli space \(M_G\) will be rewritten as \(M^\text{tr}(G, 0)\). We denote by \((G', w') \preceq (G, w)\) if \((G', w')\) is a weighted contraction of \((G, w)\). Without surprise, a weighted contraction of \((G, 0)\) can be interpreted as the tropicalization of a nodal Riemann surface which is obtained by shrinking the corresponding geodesic representatives to a point. Hence we get a partially compactified space \(C(G, 0)^+ = \mathbb{R}_{\geq 0}^{\vert E(G)\vert}\) of the cone. This phenomenon looks like an analogy to adding the frontier set, subordinate to a pants decomposition, to the Teichmüller space.
However, this space is still not compact since the edge-length is allowed to go to arbitrarily large. So we need to see what it would like to be when the edge-length goes to infinity. In fact, if we let the edge-length go to infinity, the Riemann surface in question becomes the normalization of the nodal Riemann surface whose corresponding geodesic representatives goes to zero. This fact connects the two tropicalizations when the edge-length goes to infinity or zero respectively. But on the other hand, the normalized Riemann surface is not connected or of the same genus anymore for which we do not want to tropicalize. So that we can identify $\infty$ to 0 for each coordinate in order to construct our moduli space.

Therefore, we extend the action of $\text{Aut}(G, 0)$ to the compactified space $\overline{C}(G, 0) = (\mathbb{S}^1)^{|E(G)|}$ in such a way that the quotient space

$$
\overline{M}^{\text{tr}}(G, 0) := \overline{C}(G, 0)/\text{Aut}(G, 0) = (\mathbb{S}^1)^{|E(G)|}/\text{Aut}(G, 0)
$$

identifies isomorphic tropicalizations. This quotient space endowed with the quotient topology is our desired compactified moduli space for $M^{\text{tr}}(G, 0)$.

Then we arrive at the first main result of this paper: some properties of this moduli space.

**Theorem 1.1.** (Theorem 3.16). Let $(G, 0)$ be a regular tropicalization of a smooth $n$-pointed genus $g$ Riemann surface. Then we have

1. There is a stratification for $\overline{M}^{\text{tr}}(G, 0)$ as follows

$$
\overline{M}^{\text{tr}}(G, 0) = \bigcup_{(G', w') \leq (G, 0)} M^{\text{tr}}(G', w'),
$$

where $M^{\text{tr}}(G, 0)$ is open and dense in $\overline{M}^{\text{tr}}(G, 0)$.

2. $\overline{M}^{\text{tr}}(G, 0)$ is compact and Hausdorff as a topological space.

The tropicalization of a nodal Riemann surface can be realized as a weighted contraction of a regular tropicalization (i.e., a trivalent genus $g$ graph with $n$ leaves), via its normalization. So we have as well a stratification for the moduli space of Riemann surfaces $\overline{M}_{g,n}$ as follows:

$$
\overline{M}_{g,n} = \bigcup_{(G'_{\leq G}, w') \sim (G_{\leq G}, w)} M^{\text{rs}}(G'_{\leq G}, w'),
$$

where $M^{\text{rs}}(G'_{\leq G}, w')$ denotes the locus in $\overline{M}_{g,n}$ of those nodal Riemann surfaces obtained by shrinking those geodesic representatives, corresponding to contracted edges in $(G, 0)$ for getting $(G', w')$, to a point; and $(G'_{\leq G}, w') \sim (G''_{\leq G}, w'')$ if they support the Riemann surfaces of the same topological type.

Although the weighted contractions of a fixed regular tropicalization can not exhaust all the topological types of nodal Riemann surfaces, we still have the following partial partition analogy between $\overline{M}^{\text{tr}}(G, 0)$ and $\overline{M}_{g,n}$. 
Theorem 1.2. (Theorem 4.3). Fix a regular tropicalization \((G, \emptyset)\) of an \(n\)-pointed genus \(g\) Riemann surface. Let \((G', \mathbf{w}')\) be a weighted contraction of it. Then the association as below

\[
M^{tr}(G', \mathbf{w}') \mapsto M^{rs}(G'_{\leq G}, \mathbf{w}')
\]
gives a map from the stratification of \(\overline{M}^{tr}(G, \emptyset)\) to the stratification of \(\overline{M}_{g,n}\). And we have

1. \(\dim M^{tr}(G', \mathbf{w}') = \dim M^{rs}(G'_{\leq G}, \mathbf{w}') = |E(G')|\).
2. Suppose \((G'', \mathbf{w}'') \preceq (G', \mathbf{w}')\), then we have \(M^{tr}(G'', \mathbf{w}'') \subset M^{tr}(G', \mathbf{w}')\) and \(M^{rs}(G''_{\leq G}, \mathbf{w}'') \subset M^{rs}(G'_{\leq G}, \mathbf{w}')\).

Note that, unfortunately, the above association in the theorem is neither surjective nor injective (see Sect. 4 for the explanation).

Since our tropicalization is essentially a tropical curve, we largely make use of the techniques on the moduli of tropical curves (or equivalently, weighted metric graphs) which are well elaborated in [7,8], except for some suitable modifications for our situation.

The paper is organized as follows. In Sect. 2 we tropicalize a smooth \(n\)-pointed Riemann surface through its (hyperbolic) pair of pants decomposition. In Sect. 3 we construct our moduli space of tropicalizations based on a fixed regular tropicalization, and then compactify it by adding strata parametrizing weighted contractions. We can also show some basic properties of this space. In the last section, Sect. 4, we compare this space with the moduli space of Riemann surfaces, trying to establish some correspondence between stratifications of these two moduli spaces.

2. Tropicalization by pants decompositions

This section is devoted to giving a brief introduction to the tropicalization of a Riemann surface with marked points. We tropicalize a pointed Riemann surface by using its pair of pants decomposition. In fact, a pants decomposition of a closed topological surface implies some important information on the moduli space of Riemann surfaces supported on it, e.g., its parameters, dimension and so on. A good reference on Riemann surfaces and their moduli is Looijenga’s notes [11], as well as a much bigger and comprehensive volume of Arbarello–Cornalba–Griffiths [4].

Let \(S\) be a closed connected topological surface of genus \(g\). It is well-known that a conformal structure with an orientation endowed on \(S\) turns \(S\) into a Riemann surface. Indeed, it coincides with the classical definition of a Riemann surface: the surface \(S\) admits an atlas of complex charts with holomorphic transition functions. That is because a conformal structure on \(S\) with an orientation defines a complex structure \(J_p\) on each tangent space \(T_p S\) of \(S\) at \(p\), and a Riemannian metric on a surface (hence on \(S\)) is locally conformal to a Euclidean metric so that the coordinate changes of this atlas are holomorphic. Therefore, it is not surprising that we can describe the variation of complex structures on \(S\) in terms of the variation of conformal structures on it, up to some equivalence encoded by a group action.
Let $P$ be a closed subset of $S$, not necessarily finite, but can be empty. Then all the diffeomorphisms of $S$ leaving $P$ pointwise fixed form a group, denoted by $\text{Diff}(S, P)$. We write its identity component as $\text{Diff}^0(S, P)$, which is a normal subgroup of $\text{Diff}(S, P)$. Likewise, those orientation preserving diffeomorphisms of $\text{Diff}(S, P)$ form a group as well, denoted by $\text{Diff}^+(S, P)$, also containing $\text{Diff}^0(S, P)$ as a normal subgroup. Then the mapping class group of $(S, P)$ is defined as the quotient $\text{Mod}(S, P) := \text{Diff}^+(S, P)/\text{Diff}^0(S, P)$, endowed with the discrete topology.

To an embedded circle $\alpha \subset S \setminus P$ one can associate an element of $\text{Mod}(S, P)$, called Dehn twist and denoted by $D_{\alpha}$, as follows. Embedding the circle $S^1$ into $\mathbb{C}$ such that $|s| = 1$ for any $s \in S^1$ and thus an orientation on $S^1$ is given accordingly, let $\phi : (-1, 1) \times S^1 \to S \setminus P$ be an orientation preserving open embedding such that $\phi(0, S^1) = \alpha$, let $\theta : (-1, 1) \to [0, 2\pi]$ be a smooth monotone function taking values constant 0 on $(-1, -\frac{1}{2})$ and constant $2\pi$ on $(\frac{1}{2}, 1)$. Define a mapping $h : S \to S$ such that $h(\phi(t, u)) = \phi(t, ue^{-\sqrt{-1}\theta(t)})$ and identity on $S \setminus \phi((-1, 1) \times S^1)$. It is obvious that $h$ is a differmorphism and one can check its equivalence class in $\text{Mod}(S, P)$ is only determined by the isotopy class of $\alpha$ relative to $P$.

Such an embedded circle $\alpha$ is called a nonseparating curve if its complement $S \setminus \alpha$ is connected. Then the complement becomes the interior of a compact connected surface of genus $g - 1$ with two boundary components. Otherwise it is called a separating curve. This circle divides $S$ into two connected components, written as $S'$ and $S''$, each of which is the interior of a compact surface with $\alpha$ as its boundary. If the genera of $S'$ and $S''$ are $g'$ and $g''$ respectively, then we have $g = g' + g''$. And the set $P$ of marked points is also divided into two subsets: $P' := P \cap S'$ and $P'' := P \cap S''$, lying on $S'$ and $S''$ respectively.

The following theorem is obtained by Dehn and Lickorish.

**Theorem 2.1.** (Dehn–Lickorish). Let $S$ and $P$ be given as above. Then the mapping class group $\text{Mod}(S, P)$ of $(S, P)$ is generated by finitely many Dehn twists.

Recall that a conformal structure on a 2-dimensional real vector space is an inner product on it given up to a scalar multiplication. Then the spaces $\text{Conf}(T_pS)$, for each $p \in S$ the set of conformal structures on the tangent space $T_pS$, form a disk bundle over $S$, denoted by $\text{Conf}(S)$. A smooth section of this bundle is called a conformal structure on $S$.

From now on we will always assume $P$ is finite or empty. The group $\text{Diff}^+(S, P)$ acts on $\text{Conf}(S)$. Then the Téichmüller space of $(S, P)$ is defined as the space of conformal structures on $S$ given up to isotopy relative to $P$: 

$$T(S, P) := \text{Diff}^0(S, P) \setminus \text{Conf}(S),$$

which can also be written as $T_g, P$ since now the genus is enough to characterize the underlying surface. Likewise, we can also define another orbit space, the space of conformal structures on $S$ given up to orientation-preserving differmorphisms relative to $P$:

$$M_{g, P} = M(S, P) := \text{Diff}^+(S, P) \setminus \text{Conf}(S) = \text{Mod}(S, P) \setminus T(S, P).$$
In fact, this is the moduli space of Riemann surfaces of genus \( g \) with \( P \)-marked. If the elements of \( P \) are ordered: \( P = \{p_1, p_2, \ldots, p_n\} \), then we can write \( T_{g,p} \) and \( M_{g,p} \) as \( T_{g,n} \) and \( M_{g,n} \) respectively.

A pair of pants is a compact surface with boundary of genus zero such that its boundary has three connected components. If it is endowed with a hyperbolic structure for which the three boundary components are geodesics, then it is called a hyperbolic pair of pants. Since it is homeomorphic to a sphere with three holes, its Euler characteristic is \( 2 - 2g - n = 2 - 0 - 3 = -1 \).

Let \( S^o := S \setminus P \). From now on we will assume \( 2 - 2g - n < 0 \) throughout, or equivalently, the Euler characteristic of \( S^o \) is negative. This means if a conformal structure \( J \) is given on \( S \), then its restriction to \( S^o \) underlies a complete hyperbolic metric, making any element of \( P \) to be a cusp. Therefore, \( S^o \) can be identified with a quotient of the upper half plane \( \mathbb{H} \), i.e., \( \Gamma \backslash \mathbb{H} \) with some \( \Gamma \subset \text{PSL}(2, \mathbb{R}) \).

Then a pants decomposition of \( S^o \) is given as follows.

**Definition 2.2.** A pants decomposition of \( S^o \) is a closed one dimensional submanifold \( A \subset S \) which does not meet \( P \) given up to isotopy relative to \( P \) (so every connected component of \( A \) is an embedded circle) such that every connected component of \( S^o \setminus A \) is differentiable to the interior of a pair of pants.

The topological feature of the pants decomposition actually can be characterized by a topological graph, or simply a graph, by which we mean a finite one dimensional CW complex attached with finitely many (can be empty) half-open (also half-closed) 1-cells, denoted by \( G \): we allow at most one of the two boundary components of each 1-cell not to be glued to any 0-cell. We denote by \( V(G) \) the set of vertices (or 0-cells) and by \( \text{Edge}(G) \) the set of its edges (or 1-cells). So to every \( e \in \text{Edge}(G) \) we associate the set of vertices, \( \{v, v'\} \) (if each boundary component of \( e \) is glued to a 0-cell) or \( \{v\} \) (if only one boundary component of \( e \) is glued to a 0-cell), which form the boundary of \( e \). We call \( v \) and \( v' \), or \( v \) only, the endpoints of \( e \), and we say that \( e \) is adjacent to \( v \) if \( v \) is an endpoint of \( e \). If \( v = v' \), then we say that the edge \( e \) is a loop based at \( v \). The degree of a vertex \( v \) is defined as the number of edges having \( v \) as an endpoint, with the convention that the loop based at \( v \) being counted twice. This suggests an associated notion half edge by which we mean a non-closed 1-cell emanating from one vertex. We will hence further distinguish the edge with two (possibly identical) endpoints from the edge with only one endpoint, in the next section, in order to simplify defining the morphism between graphs. The following application to our situation justifies the introduction of the above notions.

Now the boundary of \( S^o \setminus A \) is either a connected component of \( A \) or a singleton in \( P \). One can then easily encode the topological type of the triple \((S, P; A)\) by a connected graph \( G(S, P; A) \): the vertex set \( V(S, P; A) = \pi_0(S^o \setminus A) \), the edge set \( \text{Edge}(S, P; A) = \pi_0(A) \cup P \) with the obvious incidence relation: the edge \( e \) is adjacent to the vertex \( v \) if and only if \( e \) represents a boundary component of the pair of pants indexed by \( v \). Those edges indexed by \( \pi_0(A) \) are called interior edges: the separating curve defines an edge connecting two vertices, while the nonseparating curve defines a loop based at a vertex. Likewise, those edges indexed by \( P \) are
called the exterior edges: they look like ‘leaves’ each of which emanates from one vertex.

**Proposition 2.3.** The graph $G(S, P; A)$ is trivalent (i.e., any vertex is of degree 3), has $2g - 2 + n$ vertices and $3g - 3 + n$ interior edges.

**Proof.** Each vertex of the graph $G(S, P; A)$ is indexed by the interior of a pair of pants, which is homeomorphic to a thrice punctured sphere. Since it has three connected boundary components, it is clear that the graph is trivalent.

We already know that the Euler characteristic of $S^\circ$, a pair of pants and a circle is $2 - 2g - n, -1$ and $0$ respectively. So by the additive property of Euler characteristic we can deduce that there are $2g - 2 + n$ connected components of $S^\circ\setminus A$ if $A$ induces a pants decomposition of $S^\circ$, i.e., there are $2g - 2 + n$ vertices in $G(S, P; A)$. Since the graph is trivalent, there are $3(2g - 2 + n) = 6g - 6 + 3n$ half edges where the interior edges are counted twice and the exterior edges are counted once. The number of the exterior edges is just $n$, i.e., the number of the elements of $P$. So there are $(6g - 6 + 3n - n)/2 = 3g - 3 + n$ interior edges in $G(S, P; A)$. 

**Remark 2.4.** It should be noted that the pants decomposition for a given $S^\circ$ is not unique, which can be seen from Fig. 1.

Let us assume that a conformal structure $J$ is given on $S^\circ$ and a pants decomposition induced by $A$ is fixed. As discussed before, it underlies a natural hyperbolic structure on $S^\circ$. In fact, there is a unique closed geodesic representing each connected component $\alpha$ of $A$, called a geodesic representative. And these representative geodesics are disjoint. Therefore, a length is defined by the conformal structure $J$ on the geodesic representative of $\alpha$, denoted by $\ell_\alpha(J)$. Since the length is invariant up to the isotopy class of the conformal structure we have a well-defined function as follows

$$\ell_\alpha : T(S, P) \to \mathbb{R}_{>0}; [J] \mapsto \ell_\alpha(J).$$

We know from hyperbolic geometry that a hyperbolic pair of pants up to isometry is determined by the lengths of its three boundary components. So if a pants decomposition for $S^\circ$ is fixed and the lengths of those geodesic representatives are given, then each hyperbolic pair of pants is determined accordingly up to isometry. However, from these data one still can not recover $(S, J)$. That is because even though we know that which pair of boundary components should be welded onto each other from the decomposition, there still exists a rotation over an angle for
two $S^1$ to be welded onto each other. This causes a so-called geodesic shearing action along the geodesic representative of the connected component $\alpha$ of $A$ so that two conformal structures may differ by a geodesic shearing along the geodesic representative. We denote by $\theta_\alpha$ this angle of the geodesic shearing action along $\alpha$:

$$\theta_\alpha : T(S, P) \to \mathbb{R}; \ [J] \mapsto \theta_\alpha(J).$$

Therefore, the above discussion essentially shows that $T(S, P)$ locally looks like $\mathbb{R}^{6g-6+2n}$ as a topological space (length function resp. geodesic shearing provides a coordinate for each geodesic representative, and there are $3g-3+n$ interior edges for the decomposition).

Now we are ready to present the so-called Fenchel–Nielsen parametrization for the Teichmüller space $T(S, P)$.

**Theorem 2.5.** (Fenchel–Nielsen parametrization). The Fenchel–Nielsen parametrization, given as above by the length function resp. geodesic shearing for each geodesic representative, defines a global homeomorphism from $(\mathbb{R}^0 > 0 \times \mathbb{R})^{\pi_0(A)}$ to $T(S, P)$. In fact, the action of geodesic shears $\mathbb{R}^{\pi_0(A)}$ gives rise to a principal fibration for $T(S, P)$ which can be illustrated by $\ell : T(S, P) \to \mathbb{R}_{>0}^{\pi_0(A)}$.

Moreover, the Dehn twists around the connected components of $A$ generate a free abelian subgroup of $\text{Mod}(S, P)$ of rank $|\pi_0(A)|$.

**Remark 2.6.** We note that the Fenchel–Nielsen parametrization a priori depends on the pants decomposition. But the Fenchel–Nielsen parametrizations associated to different pants decompositions are differentiable in terms of each other. So the (smooth) manifold structure is independent of the pants decompositions.

In particular, $T(S, P)$ even has a complex manifold structure: if $[J] \in T(S, P)$ is represented by a Riemann surface $R$, then one can show (see [11, Section 8] for a heuristic approach) that the tangent space at $[J]$ can be identified with $H^1(R, \mathcal{V}_R(-P))$, where $\mathcal{V}_R(-P)$ is the sheaf of holomorphic vector fields on $R$ which vanish in $P$.

Inspired by multiple pants decompositions existing for a given $S^0$, we consider the collection $C(S^0)$ of isotopy classes of embedded circles in $S^0$ that do not bound a disk in $S$ which meets $P$ in at most one point. We make this collection the vertex set of a simplicial complex, called the curve complex, such that: a nonempty finite subset $d \subset C(S^0)$ spans a simplex if and only if its elements can be represented by mutually disjoint embedded circles. So the simplex $d$ of $C(S^0)$ can be thought of as an isotopy class of closed one dimensional submanifold $B \subset S^0$ such that every connected component of $S^0 \setminus B$ has negative Euler characteristic. We then notice that a pants decomposition $A$ of $S^0$ defines a maximal simplex of $C(S^0)$ and that every maximal simplex is of this form. Observe that the mapping class group $\text{Mod}(S, P)$ acts on this complex.

We readily notice that if we allow the length $\ell_\alpha$ associated to $\alpha$ to go to zero, it would lead to the appearance of a node with the corresponding angle $\theta_\alpha$ unspecified. It is indeed a (nodal) Riemann surface degenerated from the smooth ones. This suggests that a bordification of the Teichmüller space can be constructed by enlarging the range of the Fenchel–Nielsen parameters. Now for each $\alpha$ the parameter space
for \((\ell_\alpha, \theta_\alpha)\) can be identified with the space \(\mathbb{R}_{\geq 0} \times \mathbb{R}/\{0\} \times \mathbb{R}\). Then for the pants decomposition \(A\) a frontier set \(F_A\) is added to the Teichmüller space to get a bigger space by letting \(\ell_\alpha\) include the value 0, and with \(\theta_\alpha\) undefined when \(\ell_\alpha = 0\), for each \(\alpha \in A\). It is clear that the points of \(F_A\) parameterize the degenerate Riemann surfaces with each \(\ell_\alpha = 0\) specifying a node. In particular, for a simplex \(d \subset A\) the \(d\)-null stratum is defined as \(S(d) := \{J \in F_A \mid \ell_\alpha(J) = 0 \text{ iff } \alpha \in d\}\). So the frontier set \(F_A\) is the union of the \(d\)-null stratum running over the subsimplices of \(A\), i.e.,

\[ F_A = \bigcup_{d \subset A} S(d). \]

In fact, the strata \(S(d)\) are the products of lower dimensional Teichmüller spaces. The neighborhood basis for points of \(T(S, P) \cup F_A\) is prescribed by the requirement that for each simplex \(d \subset A\), the projection \(((\ell_\beta, \theta_\beta), \ell_\alpha) : T(S, P) \cup F_A \rightarrow \prod_{\beta \notin d}(\mathbb{R}_{\geq 0} \times \mathbb{R}) \times \prod_{\alpha \in d} \mathbb{R}_{\geq 0}\) is continuous.

We further consider an even bigger space realized by taking over the variation of the pants decompositions. It is not hard to see that for a simplex \(d\) contained in both pants decompositions \(A\) and \(A'\), the specified neighborhood systems for \(T(S, P) \cup S(d)\) are equivalent. Thus we can define the augmented Teichmüller space \(\overline{T}(S, P)\) as

\[ \overline{T}(S, P) := T(S, P) \cup_{d \subset C(S')} S(d), \]

which is clearly a stratified topological space. This was first studied by Abikoff [1] so that we sometimes also call this augmented Teichmüller space the Abikoff bordification. The space \(\overline{T}(S, P)\) is not locally compact since no point of the frontier has a relatively compact neighborhood, and the neighborhood bases are unrestricted in the \(\theta_\alpha\) parameters for \(\alpha\) a \(d\)-null. Note that the action of \(\text{Mod}(S, P)\) on \(T(S, P)\) extends naturally to an action by homeomorphisms on \(\overline{T}(S, P)\), although the action is not properly discontinuous anymore. Abikoff [1] and Bers [5] respectively showed that the quotient \(\overline{T}(S, P)/\text{Mod}(S, P)\) is topologically the Deligne–Mumford compactification of the moduli space of stable curves. For more descriptions on the augmented Teichmüller space, interested readers can also consult Wolpert [14].

Now for each smooth Riemann surface supported on \(S\) with \(P\)-marked, we associate to each interior edge of \(G(S, P; A)\) the length \(\ell_\alpha(J)\) of its corresponding geodesic representative, to each exterior edge of \(G(S, P; A)\) a length \(\infty\). By this operation the graph \(G\) is made to be a metrized graph, denoted by \((G, \ell)\). We say that the metric graph \((G, \ell)\) gives a tropicalization of the smooth Riemann surface with \(P\)-marked. Sometimes we call the graph \(G(S, P; A)\) a topological tropicalization of \(S\) if we do not want to consider its metric structure.

Conversely, it is clear that for a given interior edge of \(G\), i.e., corresponding to a geodesic representative, any length taking value in \(\mathbb{R}_{\geq 0}\) can be realized by a Riemann surface supported on \((S, P)\) leaving other edge-lengths fixed. This means that the graph \(G\) endowed with an arbitrary length function \(\ell\) on its interior edges can always be tropicalized from a Riemann surface.

\textbf{Remark 2.7.} It is not surprising that the process of tropicalization discards some information, in this case, on the geodesic shears.
If we let some edges of \((G, \ell)\) go to zero, it would naturally lead to a contracted graph which in the first place could naively be viewed as the boundary point to the moduli of the tropicalizations \((G, \ell)\). But we readily notice that the genus of the contracted graph (see the definition below) would not be preserved. Then we introduce a weighted version of the graph in order to capture the genus. It happens that this weighted contraction could be viewed as the tropicalization of a nodal Riemann surface obtained by shrinking the corresponding geodesic representatives to a point. In fact, this nodal Riemann surface is parametrized by some point of the frontier set \(\mathcal{F}_A\) to the Teichmüller space. We will look into this phenomenon in more detail in the next section.

Conversely, given a nodal Riemann surface, we will as well see in Sect. 4 that it could always be tropicalized to a weighted graph via its normalization.

### 3. Moduli of tropicalizations

In this section we first construct the moduli space of the tropicalizations of smooth \(n\)-pointed genus \(g\) Riemann surfaces, as a topological space. And then we compactify it by adding strata which can be interpreted as parametrizing the tropicalizations of nodal \(n\)-pointed Riemann surfaces. We then show that the compactified space is also Hausdorff.

#### 3.1. The moduli space of tropicalizations

Now let us fix a pants decomposition \(A\) for \(S^\circ\). We are thus given the tropicalization, \((G, \ell)\), of a smooth \(n\)-pointed Riemann surface supported on \(S\). From now on we will assume that the elements in \(P\) are ordered.

The genus of a graph is defined as its first Betti number
\[
g(G) := b_1(G) = \dim_{\mathbb{Z}} H_1(G, \mathbb{Z}) = |E(G)| - |V(G)| + |\text{connected components}|
\]
where \(E(G)\) does not count leaves. This suggests us to distinguish the interior edge from the exterior edge, so we will call an interior edge just an edge, an exterior edge a leaf in what follows. The sets of interior edges and exterior edges are denoted by \(E(G)\) and \(L(G)\) respectively.

Then we can show that the genus of \(G\) is just \(g\), i.e., the genus of the smooth Riemann surface from which it is tropicalized.

**Proposition 3.1.** The genus of the tropicalization \((G, \ell)\) of the smooth \(n\)-pointed Riemann surface supported on \(S\) is just \(g\).

**Proof.** We directly apply the above formula.
\[
g(G) = |E(G)| - |V(G)| + |\text{connected components}|
= (3g - 3 + n) - (2g - 2 + n) + 1
= g.
\]
The numbers of vertices and edges of \(G\) are already obtained in Proposition 2.3, and the number of connected components is just 1 since \(G\) is connected. \(\square\)
Conversely, if we are given a trivalent genus $g$ connected graph with $n$ leaves, we can recover a genus $g$ surface with $n$ marked points as follows. We first associate to each vertex a pair of pants and to a half edge or a leaf which is adjacent to that vertex a boundary component, then we weld the corresponding boundary components if there is an edge connecting two (possibly identical) vertices. In the meantime those boundary components corresponding to leaves are shrunk to a marked point. A topological surface is thus recovered. That the surface is still of genus $g$ is due to the above proposition because the given graph is a tropicalization of the recovered surface. It is clear that the surface has $n$ marked points since they are recovered from the $n$ leaves.

In order to construct the moduli space of those tropicalizations of $n$-pointed smooth Riemann surfaces supported on $S$, denoted by $M_G$, it is not surprising to introduce the open cone $$ \mathcal{C}(G) := \mathbb{R}_{\geq 0}^{3g-3+n}$$ endowed with the usual topology. To every point $(\ell_1, \ldots, \ell_{|E(G)|})$ in the cone there corresponds a unique metric graph $(G, \ell)$ for which the length of its $i$-th edge is $\ell_i$.

But we notice that the symmetry of the graph would cause that some distinct graphs, in the sense of corresponding to distinct points in the cone, are “isomorphic” to each other. So before we can go further, we need to introduce the automorphism group of a graph first. For that we define a multivalued so-called endpoint map $$ \epsilon : E(G) \cup L(G) \to V(G)$$ which assigns an edge (or a leaf) its endpoints. The images of distinct elements of $E(G) \cup L(G)$ under this map should thus be thought of as equal if and only if they are identical as a set (of singleton or two elements).

**Definition 3.2.** A map $f : V(G) \cup E(G) \cup L(G) \to V(G') \cup E(G') \cup L(G')$ is called a morphism from graph $G$ to graph $G'$ if we have $f(V(G)) \subset V(G')$, $f(E(G)) \subset V(G') \cup E(G')$, $f(L(G)) \subset L(G')$, and the diagram below is commutative.

\[
\begin{array}{ccc}
V(G) \cup E(G) \cup L(G) & \xrightarrow{f} & V(G') \cup E(G') \cup L(G') \\
\downarrow (id_V, \epsilon) & & \downarrow (id_{V'}, \epsilon') \\
V(G) \cup E(G) \cup L(G) & \xrightarrow{f} & V(G') \cup E(G') \cup L(G')
\end{array}
\]

The morphism $f$ is an isomorphism if it induces, by restriction, three bijections $f_V : V(G) \to V(G')$, $f_E : E(G) \to E(G')$, $f_L : L(G) \to L(G')$ and $f_L$ preserves the ordering if the elements of $L(G)$ and $L(G')$ are numbered. Then an automorphism of $G$ is an isomorphism between $G$ and itself.
Besides the notion of an automorphism of a graph, there is also a common type of graph morphism, called (edge) contraction, playing an important role in this paper. Let \( Q \subset E(G) \) be a set of edges, we denote by \( G/\overline{Q} \) the graph removing all the edges in \( Q \) and identifying the endpoints of each edge in \( Q \). Then we have a resulting map \( f : G \rightarrow G/\overline{Q} \) by fixing anything outside of \( Q \), called (edge) contraction. On the other hand, let \( E_1 := E(G) \setminus Q \), we can also obtain a graph \( G - E_1 \) by just removing all the edges of \( E_1 \). Then every connected component of \( G - E_1 \) can be contracted to a vertex of \( G/\overline{Q} \); conversely, the preimage \( f^{-1}(v') \subset G \) for every vertex \( v' \in V(G/\overline{Q}) \) is a connected component of \( G - E_1 \). In particular, we have

\[
b_1(G - E_1) = \sum_{v' \in V(G/\overline{Q})} b_1(f^{-1}(v')).
\]  

(3.1)

By the additivity of \( b_1 \) we have that

\[
b_1(G) = b_1(G/\overline{Q}) + b_1(G - E_1).
\]  

(3.2)

It is clear that all the automorphisms of a graph \( G \) form a group, denoted by \( \text{Aut}(G) \). By passing from \( f \) to \( f_E \) it induces a homomorphism from \( \text{Aut}(G) \) to the symmetry group on \( |E(G)| \) edges. Hence \( \text{Aut}(G) \) acts on the open cone \( \mathbb{R}^{|E|} \) by permuting the coordinates. Therefore, we construct the moduli space of tropicalizations \( (G, \ell) \) with \( G \) underlying the given pants decomposition:

\[
M_G := \mathcal{C}(G)/\text{Aut}(G) = \mathbb{R}^{|E(G)|}_{>0}/\text{Aut}(G) = \mathbb{R}^{3g-3+n}_{>0}/\text{Aut}(G)
\]

with the quotient topology.

**Remark 3.3.** By passing from \( f \) to \( f_E \) one may lose some non-trivial elements while fixing the edge set \( E(G) \), for instance, an element which is not identity on \( V(G) \) but fixing the edge set (see Fig. 2). That means \( \text{Aut}(G) \) may contain non-trivial elements acting trivially on \( \mathcal{C}(G) \).

### 3.2. Weighted contractions

Now we want to add the boundary strata to the open cone \( \mathcal{C}(G) \) in order to get the closed cone \( \mathcal{C}(G)^+ = \mathbb{R}^{|E(G)|}_{\geq 0} \). That means some edges of the metric graph \( (G, \ell) \) would go to zero, leading to the contraction of \( G \). Since the length of each edge of the graph is defined as the length of its corresponding geodesic representative of \( \alpha \in A \), that the length going to zero amounts to shrinking the corresponding geodesic representative until we get a node. By doing this we get a nodal Riemann surface, and we say that the contracted graph \( G' \) is a tropicalization of this nodal Riemann surface.
Remark 3.4. Shrinking a closed curve α in a smooth Riemann surface leads to a nodal Riemann surface. But the tropicalization of the nodal Riemann surface a priori depends on the tropicalization G of the smooth one, hence it is not unique either. On the other hand, if we start from a tropicalization G for a smooth one, the various contractions of G unfortunately would not exhaust all the nodal Riemann surfaces of arithmetic genus g. This point of view will become more clear in the next section when we compare this moduli space with the moduli space of Riemann surfaces.

We know that those nodal Riemann surfaces, obtained by shrinking some α’s from a smooth one, still have arithmetic genus g. For a graph, however, after contracting an edge, its genus might decrease. So we introduce a weighted version for these notions, provided by Brannetti–Melo–Viviani [6], so as to remedy this problem.

Definition 3.5. A graph G is called a weighted graph if it is endowed with a weight function on the vertices $w : V(G) \to \mathbb{Z}_{\geq 0}$, denoted by $(G, w)$. We can think of an unweighted graph as a 0-weighted graph $(G, 0)$.

A weighted metric graph $(G, w, \ell)$ is defined likewise if we start from a metric graph $(G, \ell)$.

Then the genus of a weighted graph is modified by adding weights of vertices to the first Betti number:

$$g(G, w) := b_1(G) + \sum_{v \in V(G)} w(v).$$

Therefore, the contraction of a weighted graph can be naturally modified by setting the weight of each vertex of the contracted graph as the genus of its weighted preimage.

Definition 3.6. Let $Q \subset E(G)$ be a set of edges of a weighted graph $(G, w)$. A contraction of $Q$ in $(G, w)$ is called a weighted contraction if the weight function $w_Q$ on the contracted graph $G_Q$ is defined as follows:

$$w_Q(v') := b_1(f^{-1}(v')) + \sum_{v \in f^{-1}(v')} w(v)$$

for each vertex $v' \in V(G_Q)$.

We immediately notice that the weighted contraction would not change the genus of a weighted graph.

Proposition 3.7. Using the notations above, we have

$$g(G_Q, w_Q) = g(G, w).$$
Proof. Let \( Q \) and \( E_1 \) be defined as above. Then we have

\[
g(G, w) = b_1(G) + \sum_{v \in V(G)} w(v) = b_1(G/Q) + b_1(G - E_1) + \sum_{v \in V(G)} w(v)
\]

by (3.2)

\[
= b_1(G/Q) + \sum_{v' \in V(G/Q)} b_1(f^{-1}(v')) + \sum_{v \in V(G)} w(v)
\]

by (3.1)

\[
= b_1(G/Q) + \sum_{v' \in V(G/Q)} b_1(f^{-1}(v')) + \sum_{v' \in V(G/Q)} \sum_{v \in f^{-1}_v(v')} w(v)
\]

\[
= b_1(G/Q) + \sum_{v' \in V(G/Q)} \left( b_1(f^{-1}(v')) + \sum_{v \in f^{-1}_v(v')} w(v) \right)
\]

by Definition 3.6

\[
= b_1(G/Q) + \sum_{v' \in V(G/Q)} w/Q(v')
\]

\[
= g(G/Q, w/Q).
\]

\( \square \)

If we regard the tropicalization \( G \) for our smooth Riemann surfaces as a 0-weighted graph, then by this proposition we can see that all the weighted contractions of this weighted graph \((G, 0)\) still stay in the category of genus \(g\) weighted graph (with \(n\) leaves). And we denote that \((G', w') \preceq (G, w)\) if \((G', w')\) is a weighted contraction of \((G, w)\). In what follows we will call such a \((G, 0)\), tropicalized from a smooth Riemann surface, a regular tropicalization. It is clear that a regular tropicalization is always trivalent and 0-weighted.

We can define the automorphism group \( Aut(G, w) \) of a weighted graph \((G, w)\) as the subgroup of \( Aut(G) \) preserving the weights on vertices. Then if a tropicalization of a (singular) Riemann surface is \((G, w)\), we can also define the moduli space of tropicalizations based on this weighted graph \((G, w)\) as follows:

\[
M^w_G := \mathcal{C}(G, w) / Aut(G, w) = \mathbb{R}_{>0}^{\vert E(G) \vert} / Aut(G, w).
\]

Hence \( M^w_G \) can be rewritten as \( M^w(G, 0) \), and \( Aut(G) \) as \( Aut(G, 0) \). Then the space \( \mathcal{C}(G, w)^+ \) can be decomposed as follows

\[
\mathcal{C}(G, w)^+ = \bigsqcup_{Q \subset E(G)} \mathcal{C}(G/Q, w/Q).
\]

Remark 3.8. This decomposition is exactly an analogue of the stratification of the union of the Teichmüller space \( T(S, P) \) and the frontier set \( F_A \) subordinate to the pants decomposition \( A \), which is decomposed as

\[
T(S, P) \cup F_A = T(S, P) \cup_{d \in A} S(d).
\]
In fact, the open cone $\mathcal{C}(G/Q, w/Q)$ parametrizes those tropicalizations coming from those nodal Riemann surfaces parametrized by the stratum $\mathcal{S}(d)$ if the simplex $d$ is spanned by the embedded circles representing the edges in $Q$.

Now let us see how the action of the group $\text{Aut}(G, w)$ extends to the closed cone $\mathcal{C}(G, w)^+$, in order to define the partially compactified moduli space $M^\text{tr}(G, w)^+$. For an open face of $\mathcal{C}(G, w)^+$, i.e., the open cone of some weighted contraction $(G', w')$ of $(G, w)$, there is a natural action of $\text{Aut}(G', w')$ so that we can similarly define $M^\text{tr}(G', w') = \mathbb{R}_{\geq 0}^{E(G')} / \text{Aut}(G', w')$. But it may also happen that there are two distinct contractions from $(G, w)$ being isomorphic, i.e., there is an isomorphism $f : (G', w') \cong (G'', w'')$. Then we have an isometry $\phi_f : \mathcal{C}(G', w') \cong \mathcal{C}(G'', w'')$ induced by $f$. We also want to identify these isomorphic tropicalizations since the surfaces supported on these tropicalizations are of the same topological type. Therefore, we extend the action of $\text{Aut}(G, w)$ to the closed cone $\mathcal{C}(G, w)^+$ in such a manner that the quotient space

$$M^\text{tr}(G, w)^+ := \mathcal{C}(G, w)^+ / \text{Aut}(G, w) = \mathbb{R}_{\geq 0}^{E(G)} / \text{Aut}(G, w)$$

identifies isomorphic tropicalizations. And this quotient space endowed with the quotient topology is our desired partially compactified moduli space for $M^\text{tr}(G, w)$.

**Lemma 3.9.** The quotient map $\pi^+ : \mathcal{C}(G, w)^+ \to M^\text{tr}(G, w)^+$ has finite fibers.

**Proof.** Since our $(G, w)$ is a finite graph, its automorphism group $\text{Aut}(G, w)$ is finite. So is any weighted contraction of it. On the other hand it has only finitely many weighted contractions, so for a given weighted contraction, there can only be finitely many other weighted contractions being isomorphic to it. Therefore the preimage of a point in $M^\text{tr}(G, w)^+$ contains only finitely many points in $\mathcal{C}(G, w)^+$.

□

**Lemma 3.10.** $M^\text{tr}(G', w') \subset M^\text{tr}(G, w)^+$ if and only if $(G', w') \preceq (G, w)$.

**Proof.** This is a direct consequence of the fact that $\mathcal{C}(G', w') \subset \mathcal{C}(G, w)^+$ if and only if $(G', w') \preceq (G, w)$.

□

**Remark 3.11.** By the argument in Lemma 3.9 as well as the preceding discussion, we know that the extension of the action of $\text{Aut}(G, w)$ to the boundary of $\mathcal{C}(G, w)$ consists of two types: (1) the action of $\text{Aut}(G/Q, w/Q)$ on the open face $\mathcal{C}(G/Q, w/Q)$, and (2) the isometry $\phi_f$ between faces induced by the isomorphism $f$ between weighted contractions.

### 3.3. Extended tropicalizations

Nevertheless, one can notice that the space $M^\text{tr}(G, w)^+$ is still not compact. This can easily be seen by allowing the edge-lengths to go to arbitrarily large. In order to solve this problem, we introduce the so-called extended length function for
edges by allowing their lengths to be infinity. Namely, now we have an “extended” length function
\[ \ell : E(G) \to \mathbb{R}_\geq \cup \{\infty\}, \]
then the tropicalization \((G, w, \ell)\) is called an extended tropicalization.

In fact, letting the length go to infinity amounts to enlarging the corresponding geodesic representative until the Riemann surface is cut open from there, and each of the two boundary components becomes a marked point. In other words, the edge can be understood as being broken up into two leaves. By this operation, depending on the geodesic representative is a separating curve or not, we get (1) either a new Riemann surfaces with two connected components for which the sum of their genera is \(g\) and there is one new marked point on each component, (2) or a new Riemann surface with genus being 1 less and having two new marked points on it. Then we say that the graph \((G, w, \ell)\) is a tropicalization of this (possibly disconnected) Riemann surface.

The Riemann surfaces in these two cases are still smooth but there is also some defect in each case: in the first case, the Riemann surface is not connected anymore; while in the latter one the Riemann surface is no longer of genus \(g\). So these are not the Riemann surfaces that we want to tropicalize for our moduli space.

We immediately, however, notice that the Riemann surface corresponding to edge-length going to infinity is just the normalization of the nodal Riemann surface when the length of the same edge goes to zero. This fact happens to connect the two tropicalizations when edge goes to \(\infty\) and 0 respectively: the Riemann surface from which the former tropicalization comes is just the normalization of the latter one, although the normalized one is no longer a connected genus \(g\) Riemann surface.

It suggests that we can identify 0 and \(\infty\) for each edge so that \((\mathbb{R}_\geq \cup \{\infty\})/\sim\) is homeomorphic to \(S^1\) and hence is compact. The explicit homeomorphism between them can be given by
\[ h : (\mathbb{R}_\geq \cup \{\infty\})/\sim \to S^1; \quad x \mapsto \exp \left(2\pi \sqrt{-1} \frac{x}{x+1}\right), \]
where \(0 \sim \infty\). And one possible distance function on \((\mathbb{R}_\geq \cup \{\infty\})/\sim\) compatible with this topology can be given by
\[ d(x, y) := 2\pi \min \left( \left| \frac{x}{x+1} - \frac{y}{y+1} \right|, 1 - \left| \frac{x}{x+1} - \frac{y}{y+1} \right| \right) \]
when a metric structure is needed.

By taking the homeomorphism \(h\) on each coordinate we can have a homeomorphism from \((\mathbb{R}_\geq \cup \{\infty\})/\sim)^{|E(G)|}\) to \((S^1)^{|E(G)|}\), denoted by \(h\) as well. Thanks to this homeomorphism, we will not distinguish the open cone \(\mathcal{C}(G, w)\) with its image under \(h\) in the topological sense in what follows. Similarly, a metric structure on \((\mathbb{R}_\geq \cup \{\infty\})/\sim)^{|E(G)|}\) compatible with the topology is just the product of the metric on each coordinate given above.

Now we can define a compactification of the open cone \(\mathcal{C}(G, w)\), denoted by \(\overline{\mathcal{C}(G, w)}\), as follows:
\[ \overline{\mathcal{C}(G, w)} := (S^1)^{|E(G)|}. \]
Remark 3.12. We can immediately notice that there is a bijection between the closed cone $\mathcal{C}(G, w)^+$ and the space $\overline{\mathcal{C}(G, w)}$, although they are endowed with a different topology.

Likewise the space $\overline{\mathcal{C}(G, w)}$ can be decomposed as follows

$$\overline{\mathcal{C}(G, w)} = \bigsqcup_{Q \subseteq E(G)} \mathcal{C}(G/Q, w/Q). \quad (3.3)$$

As same as we did for $\mathcal{C}(G, w)^+$, we extend the action of $\text{Aut}(G, w)$ to the compactified space $\overline{\mathcal{C}(G, w)}$ in such a manner that the quotient space

$$\overline{M^\text{tr}}(G, w) := \overline{\mathcal{C}(G, w)}/\text{Aut}(G, w) = (S^1)^{|E(G)|}/\text{Aut}(G, w)$$

identifies isomorphic tropicalizations. This quotient space endowed with the quotient topology is our desired compactified moduli space for $\overline{M^\text{tr}}(G, w)$.

Remark 3.13. We only consider weighted contractions rather than extended tropicalizations in our space $\overline{\mathcal{C}(G, w)}$, although we identify 0 and $\infty$ for each edge. That is because we want our tropicalizations to come from connected $n$-pointed genus $g$ Riemann surfaces.

By the same argument for Lemmas 3.9 and 3.10 we readily have the following similar properties for $\overline{M^\text{tr}}(G, w)$.

**Lemma 3.14.** (1) The quotient map $\overline{\pi} : \overline{\mathcal{C}(G, w)} \to \overline{M^\text{tr}}(G, w)$ has finite fibers.
(2) $\overline{M^\text{tr}}(G', w') \subset \overline{M^\text{tr}}(G, w)$ if and only if $(G', w') \preceq (G, w)$.

Now we are ready to see what this space $\overline{M^\text{tr}}(G, w)$ looks like.

**Proposition 3.15.** Let $(G, w)$ be some weighted contraction coming from a regular tropicalization of a smooth pointed Riemann surface. Then we have

(1) There is a stratification for $\overline{M^\text{tr}}(G, w)$ as follows

$$\overline{M^\text{tr}}(G, w) = \bigsqcup_{(G', w') \preceq (G, w)} \overline{M^\text{tr}}(G', w'),$$

where $\overline{M^\text{tr}}(G, w)$ is open and dense in $\overline{M^\text{tr}}(G, w)$.
(2) $\overline{M^\text{tr}}(G, w)$ is compact and Hausdorff as a topological space.

**Proof.** The stratification for $\overline{M^\text{tr}}(G, w)$ comes directly from the decomposition (3.3) for $\overline{\mathcal{C}(G, w)}$ and the definition of $\overline{M^\text{tr}}(G, w)$.

Since $\overline{M^\text{tr}}(G, w)$ is defined as $\overline{\mathcal{C}(G, w)}/\text{Aut}(G, w)$ and $\overline{\mathcal{C}(G, w)}$ is open and dense in $\overline{\mathcal{C}(G, w)}$, by $\overline{\pi}^{-1}(\overline{M^\text{tr}}(G, w)) = \overline{\mathcal{C}(G, w)}$ we also have $\overline{M^\text{tr}}(G, w)$ is open and dense in $\overline{M^\text{tr}}(G, w)$. This completes the proof of the statement (1).

Since $\overline{\mathcal{C}(G, w)}$ is compact, it is clear that $\overline{M^\text{tr}}(G, w)$ is compact as well by definition.

Now we show the Hausdorffness of $\overline{M^\text{tr}}(G, w)$. For the simplicity of notations, we denote the open sectional face $\overline{\mathcal{C}(G/Q, w/Q)}$ (they are not open faces anymore in $\overline{\mathcal{C}(G, w)}$ due to different topology) by $F_Q$ for any $Q \subset E(G)$, and its closure in
\( \mathcal{U}(G, w) \) by \( F_Q \). As discussed in Remark 3.11, there are two types of maps leading to identifying a point in \( \overline{\mathcal{U}(G, w)} \) with other points in order to get \( \overline{\mathcal{M}^G(G, w)} \):

1. An element \( g \in \text{Aut}(G/\mathcal{Q}, w/\mathcal{Q}) \) identifying \( p \) with \( gp \) for any \( p \in F_Q \).
2. An isometry \( \phi_f : F_Q \to F_{Q'} \), induced by an isomorphism \( f : (G/\mathcal{Q}, w/\mathcal{Q}) \to (G_{/\mathcal{Q}'}, w_{/\mathcal{Q}'}) \), identifying \( p \) with \( \phi_f(p) \) for any \( p \in F_Q \).

Let \( \overline{u} \) and \( \overline{v} \) be two distinct points in \( \overline{\mathcal{M}^G(G, w)} \). Write their preimages as \( \overline{\pi}^{-1}(\overline{u}) = \{ u_1, \ldots, u_s \} \) and \( \overline{\pi}^{-1}(\overline{v}) = \{ v_1, \ldots, v_t \} \) respectively. Then there exists a sufficiently small \( \varepsilon > 0 \) (using the metric structure on \( \overline{\mathcal{C}(G, w)} \)) such that the following holds.

(a) For every \( i, j \) the open balls \( B(u_i, \varepsilon) \) and \( B(v_j, \varepsilon) \) do not intersect each other.
(b) If \( \overline{F_Q} \cap B(u_i, \varepsilon) \neq \emptyset \) then \( u_i \in F_Q \); likewise, if \( \overline{F_Q} \cap B(v_j, \varepsilon) \neq \emptyset \) then \( v_j \in F_Q \).

Let \( U := \bigcup_{i=1}^{s} B(u_i, \varepsilon) \) and \( V := \bigcup_{j=1}^{t} B(v_j, \varepsilon) \). It is clear that \( U \) and \( V \) are open subsets of \( \overline{\mathcal{C}(G, w)} \), and they are disjoint by (a) above.

We claim that
\[
\overline{\pi}^{-1}(\overline{\pi}(U)) = U \quad \text{and} \quad \overline{\pi}^{-1}(\overline{\pi}(V)) = V.
\]

We extend the preceding action (i) and map (ii) to \( \overline{\mathcal{C}(G, w)} \) by fixing anything outside of \( \overline{\mathcal{F}_Q} \). Namely, we set \( gp = p \) and \( \phi_f(p) = p \) for any \( p \in \overline{\mathcal{C}(G, w)} \setminus \overline{\mathcal{F}_Q} \), every \( g \in \text{Aut}(G/\mathcal{Q}, w/\mathcal{Q}) \) and every \( \phi_f \) as above.

Now in order to prove the claim, it suffices to prove that the open subset \( U \) is invariant under the extended action (i) and the extended map (ii).

For that we pick \( Q \) and let \( g \in \text{Aut}(G/\mathcal{Q}, w/\mathcal{Q}) \). As \( g \) acts trivially outside \( \overline{\mathcal{F}_Q} \) we can assume that \( \overline{\mathcal{F}_Q} \cap U \neq \emptyset \). That means there exists a \( u_i \) such that \( \overline{\mathcal{F}_Q} \cap B(u_i, \varepsilon) \neq \emptyset \). This implies that \( u_i \in \overline{\mathcal{F}_Q} \) by (b) above. Without loss of generality, we can assume that \( u_i \in F_Q \) for otherwise if \( u_i \in \overline{\mathcal{F}_Q} \setminus F_Q \) we can always find a \( Q' \supset Q \) such that \( u_i \in F_{Q'} \). Hence we have \( gu_i \in \overline{\pi}^{-1}(\overline{u}) \) and thus
\[
(\overline{\mathcal{F}_Q} \cap B(u_i, \varepsilon))^{gu_i} = \overline{\mathcal{F}_Q} \cap B(gu_i, \varepsilon) \subset U.
\]

This proves the invariance of \( U \) under the extended action of (i).

To prove the invariance of \( U \) under the extended map (ii). Let \( p \in \overline{\mathcal{F}_Q} \setminus U \), then there exists a \( u_i \) such that \( p \in \overline{\mathcal{F}_Q} \cap B(u_i, \varepsilon) \). Let \( f \) be an isomorphism from \( (G/\mathcal{Q}, w/\mathcal{Q}) \) to another contraction \( (G_{/\mathcal{Q}'}, w_{/\mathcal{Q}'}) \) and \( \phi_f \) be the induced isometry from \( \overline{\mathcal{F}_Q} \) to \( \overline{\mathcal{F}}_{Q'} \). Then we have
\[
\phi_f(\overline{\mathcal{F}_Q} \cap B(u_i, \varepsilon)) = \overline{\mathcal{F}}_{Q'} \cap B(\phi_f(u_i), \varepsilon) \subset U
\]
since we know that \( \phi_f(u_i) \in \overline{\pi}^{-1}(\overline{u}) \) by (ii) above.

Now the claim is proved. This yields that \( \overline{\pi}(U) \) and \( \overline{\pi}(V) \) are open and disjoint in \( \overline{\mathcal{M}^G(G, w)} \). Since \( \overline{u} \in \overline{\pi}(U) \) and \( \overline{v} \in \overline{\pi}(V) \), the Hausdorffness of \( \overline{\mathcal{M}^G(G, w)} \) is proved.

By applying this proposition to a regular tropicalization of smooth \( n \)-pointed Riemann surfaces, we immediately have
Theorem 3.16. Let \((G, 0)\) be a regular tropicalization of a smooth \(n\)-pointed Riemann surface. Then we have

1. There is a stratification for \(\overline{M}^{ir}(G, 0)\) as follows

\[
\overline{M}^{ir}(G, 0) = \bigsqcup_{(G', w') \preceq (G, 0)} \overline{M}^{ir}(G', w'),
\]

where \(\overline{M}^{ir}(G, 0)\) is open and dense in \(\overline{M}^{ir}(G, 0)\).

2. \(\overline{M}^{ir}(G, 0)\) is compact and Hausdorff as a topological space.

Remark 3.17. If we choose another regular tropicalization \((G_1, 0)\) for this smooth \(n\)-pointed Riemann surface, we still have the above theorem for \(\overline{M}^{ir}(G_1, 0)\). But these two moduli spaces as disjoint union of orbifolds are different because the automorphism group of the tropicalizations may not be the same.

Question 3.18. Can we connect these two moduli spaces through the moduli space of Riemann surfaces or corresponding Teichmüller space? Since \((G, 0)\) and \((G_1, 0)\) are both tropicalized from a smooth \(n\)-pointed genus \(g\) Riemann surface.

4. Comparing with other moduli spaces

In this section we compare our moduli space of tropicalizations with the moduli space of Riemann surfaces (or equivalently, complex algebraic curves), since this is the main goal for us to introduce this moduli space, for which we hope it could provide a new angle to understand the tropicalization of the moduli space of algebraic curves. Next we also compare it with the moduli space of tropical curves in the sense of [7], since a tropicalization is essentially a tropical curve.

In fact, in [7], Caporaso established the so-called moduli space of \(n\)-pointed genus \(g\) tropical curves, \(\overline{M}_{g, n}^{trop}\). And in [3], Abramovich, Caporaso and Payne identified this space with the skeleton \(\Sigma(\overline{M}_{g, n})\) of the moduli stack of curves \(\overline{M}_{g, n}\), via the analytification of its coarse moduli space, \(\overline{M}_{g, n}^{an}\). This can be illustrated by the following commutative diagram.

\[
\overline{M}_{g, n}^{an} \xrightarrow{\text{Retraction}} \Sigma(\overline{M}_{g, n}) \xrightarrow{\sim} \overline{M}_{g, n}^{trop}
\]

There is also a stratification on the moduli stack \(\overline{M}_{g, n}\) which can be described in a combinatorial way:

\[
\overline{M}_{g, n} = \bigsqcup_{(G, w) \text{ stable, genus } g, \text{ } n \text{ leaves}} \mathcal{M}^{alg}(G, w),
\]
where \( \mathcal{M}^{\text{alg}}(G, w) \) parametrizes those curves whose dual graph is isomorphic to \((G, w)\). Here by stable graphs we mean the degree of the vertex is required to be \( \geq 3 \) (resp. \( \geq 1 \)) if \( w(v) = 0 \) (resp. \( w(v) = 1 \)). The codimension of \( \mathcal{M}^{\text{alg}}(G, w) \) in \( \overline{M}_{g,n} \) is just the number of edges of \((G, w)\), and \( \mathcal{M}^{\text{alg}}(G', w') \subset \mathcal{M}^{\text{alg}}(G, w) \) if and only if \((G, w) \preceq (G', w')\).

Therefore, we see that there is an order-reversing correspondence between the moduli space of algebraic curves and the moduli space of tropical curves. We will, however, show that there is an order-preserving correspondence, in a partial way though, between our moduli space of tropicalizations and the moduli space of Riemann surfaces.

4.1. Comparing with the moduli space of Riemann surfaces

Since nodal Riemann surfaces are parametrized by the boundary strata of the Deligne–Mumford space \( \overline{M}_{g,n} \), it is necessary for us to see how to tropicalize a nodal Riemann surface before establishing the correspondence between the two moduli spaces.

If we are given a nodal Riemann surface in \( \overline{M}_{g,n} \), its normalization gives rise to a (possibly disconnected) smooth Riemann surface. Every node marks a point on each branch on which it lies. By this operation each component becomes a pointed smooth Riemann surface and we tropicalize it as before. Thus we get a tropicalization for each smooth component and we join each pair of leaves, coming from the same node, to get an edge. Hence we get a connected trivalent graph of genus \( g \): joining leaves would not change the degree of vertices; separating nodes play no role for the genus; the normalization of a nonseparating node reduces the genus of the resulting Riemann surface (and hence of its tropicalization) by 1, while joining leaves corresponding to that node increases the genus of the tropicalization by 1 so that the genus has not been altered. Then we contract those newly formed edges so as to get a weighted contraction. We say this weighted contraction is a tropicalization of the nodal Riemann surface. Note that the tropicalization is not unique since the tropicalization of a smooth Riemann surface is not unique.

In the meantime, we also know that \( (G', w') \) underlies some nodal Riemann surface by letting those geodesic representatives corresponding to contracted edges go to zero. From this point of view, we denote by \( M^{\text{rs}}(G', w') \) the locus in \( \overline{M}_g \) of those nodal Riemann surfaces obtained by shrinking those geodesic representatives, corresponding to contracted edges in \((G, 0)\), to a point. Here the superscript “rs” stands for “Riemann surface”.

Remark 4.1. The space \( M^{\text{rs}}(G', w') \), unfortunately, also depends on the regular tropicalization \((G, 0)\). The graph \((G', w')\) may be contracted from some other regular tropicalization, say \((G_1, 0)\). By shrinking corresponding geodesic representatives we may get a nodal Riemann surface of distinct topological type. This can be seen from Fig. 3.
Fig. 3. Isomorphic graphs, contracted from different regular tropicalizations, support surfaces of distinct topological types

On the other hand, suppose \((G'', w'')\) is another weighted contraction of \((G, 0)\) which is not isomorphic to \((G', w')\), the moduli space \(M^{rs}(G'' \leq_G, w'')\) may be the same as \(M^{rs}(G' \leq_G, w')\), since they may support surfaces of the same topological type. This is illustrated by an example in Fig. 4.

Lemma 4.2. Fix a regular tropicalization \((G, 0)\) of a smooth \(n\)-pointed genus \(g\) Riemann surface. Let \((G', w')\) be a weighted contraction of it. We have that \(M^{rs}(G' \leq_G, w')\) is an irreducible quasi-projective variety and its dimension is equal to \(|E(G')|\).

Proof. Let \((C, p) \in M^{rs}(G' \leq_G, w')\) where \(p\) is an ordered form of \(P\). Then \(d := 3g - 3 + n - |E(G')|\) is the number of its nodes by construction. Denote by \(C_1, \ldots, C_k\) its irreducible components, and by \(n_i\) the number of marked points on \(C_i\). It is clear that \(\sum_{i=1}^k n_i = n\).

We take the normalization of \(C\) and write it as \(v : \bigsqcup_{i=1}^k C_i^v \to C\), where each \(C_i^v\) is a smooth genus \(g_i\) Riemann surface with \(n_i + d_i\) marked points. Here \(d_i := |v^{-1}(C_{\text{sing}}) \cap C_i^v|\). We notice that \((C_i^v, p^{(i)})\) falls into \(M_{g_i, n_i+d_i}\) since the normalization preserves the stability of Riemann surfaces. It is clear that by definition \(M^{rs}(G' \leq_G, w')\) is a coarse moduli space of a Deligne–Mumford stack and there is a gluing morphism defined naturally on the level of stacks. Then the gluing morphism on the stacks descends to such a following surjective morphism on the coarse moduli spaces:
Fig. 4. Nonisomorphic weighted contractions supporting surfaces of the same topological type

\[ M_{g_1,n_1+d_1} \times \cdots \times M_{g_k,n_k+d_k} \to M_{rs}(G'_\leq G, w'). \]

This morphism can be naturally extended to the following morphism:

\[ \overline{M}_{g_1,n_1+d_1} \times \cdots \times \overline{M}_{g_k,n_k+d_k} \to \overline{M}_{g,n}. \]

Since \( M_{rs}(G'_\leq G, w') \) is open in its image, it is quasi-projective. It is well-known that \( \overline{M}_{g_i,n_i+d_i} \) is irreducible of dimension \( 3g_i - 3 + n_i + d_i \) for all \( i = 1, \ldots, k \). Hence \( M_{rs}(G'_\leq G, w') \) is irreducible as well.

The above surjection \( M_{g_1,n_1+d_1} \times \cdots \times M_{g_k,n_k+d_k} \to M_{rs}(G'_\leq G, w') \) is clearly a finite map, we thus have

\[
\dim M_{rs}(G'_\leq G, w') = \sum_{i=1}^{k} (3g_i - 3 + n_i + d_i) = 3 \sum_{i=1}^{k} g_i - 3k + n + 2d
\]

since \( \sum_{i=1}^{k} d_i = 2d \). Applying \( g = \sum_{i=1}^{k} g_i + d - k + 1 \), we get

\[
\dim M_{rs}(G'_\leq G, w') = 3g - 3d + 3k - 3 - 3k + n + 2d = 3g - 3 + n - d = |E(G')|.
\]

\( \square \)
By the discussion in the beginning of this subsection, the tropicalization of a nodal Riemann surface is contracted from a trivalent genus \( g \) graph with \( n \) leaves (via the tropicalization of its normalization), so we can reexpress the stratification of \( \overline{M}_{g,n} \) by our notations \( M^{\text{tr}}(G', w') \) for the moduli strata as follows:

\[
\overline{M}_{g,n} = \bigsqcup \left( M^{\text{tr}}(G', w'), \text{G trivalent, genus } g, \text{ } n \text{ leaves} \right) / \sim
\]

where \((G', w') \sim (G''_G, w'')\) if they support the Riemann surfaces of the same topological type.

Then we have the following partition analogy between \( M^{\text{tr}}(G, Q) \) and \( \overline{M}_{g,n} \).

**Theorem 4.3.** Fix a regular tropicalization \((G, Q)\) of a smooth \( n \)-pointed genus \( g \) Riemann surface. Let \((G', w')\) be a weighted contraction of it. Then the association as below

\[
M^{\text{tr}}(G', w') \mapsto M^{\text{tr}}(G'_G, w')
\]

gives a map from the stratification of \( M^{\text{tr}}(G, Q) \) to the stratification of \( \overline{M}_{g,n} \). And we have

1. \( \dim M^{\text{tr}}(G', w') = \dim M^{\text{tr}}(G'_G, w') = |E(G')| \).
2. Suppose \((G'', w'') \preceq (G', w')\), then we have \( M^{\text{tr}}(G'', w'') \subseteq M^{\text{tr}}(G', w') \) and \( M^{\text{tr}}(G''_G, w'') \subseteq M^{\text{tr}}(G'_G, w') \).

**Proof.** That \( \dim M^{\text{tr}}(G'_G, w') = |E(G')| \) follows from Lemma 4.2. The dimension of \( M^{\text{tr}}(G', w') \) comes from the dimension of the cone \( C(G', w') \) which is equal to \( |E(G')| \). This completes the proof of the statement (1).

For (2), suppose \((G'', w'') \preceq (G', w')\), then we have \( M^{\text{tr}}(G'', w'') \subseteq M^{\text{tr}}(G', w') \) by Proposition 3.15. Now we will show that we also have \( M^{\text{tr}}(G''_G, w'') \subseteq M^{\text{tr}}(G'_G, w') \).

Let \((C, p) \in M^{\text{tr}}(G''_G, w'')\) be a pointed Riemann surface. We know that a neighborhood \( U \) of each node on \( C \) can be locally modeled as \(((\mathbb{C}^2, 0), z_1 z_2 = 0)\). We can take the neighborhood \( U \) as small as we like, then a deformation of the node is modeled by a map germ \( t : ((\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0); (z_1, z_2) \mapsto z_1 z_2 \), regarding the fiber \( U_t := t^{-1}(t) \) for small \( t \neq 0 \) as a ‘deformed \( U' \).

Now let \((G'', w'') = (G'/Q, w'/Q)\). We can deform all the nodes corresponding to the edges in \( Q \) at one time, so that we have a family of pointed Riemann surfaces supported on \((G'_G, w')\) specializing to \((C, p)\). Namely, there exists such a family, \( \pi : \mathcal{C} \rightarrow B \), over a one dimensional complex base manifold \((B, o), o\), with \( n \) pairwise disjoint holomorphic sections \((\sigma_b : B \rightarrow \mathcal{C})\) indicating the marked points, such that the fiber \((C_b := \pi^{-1}(b), \sigma(b))\) for every \( b \neq o \) can be tropicalized to \((G'_G, w')\) and the central fiber \((C_o, \sigma(o)) = (C, p)\). Hence we have \( M^{\text{tr}}(G''_G, w'') \subseteq M^{\text{tr}}(G'_G, w') \).

This theorem shows that there exists an order-preserving partial correspondence between the stratification of our moduli space \( M^{\text{tr}}(G, Q) \) and the stratification of \( \overline{M}_{g,n} \), not in a bijective manner though (see Remark 4.1 for the explanation why they are not in bijection).
4.2. Comparing with the moduli space of tropical curves

For the moduli space of tropical curves, we only consider stable tropical curves.

**Definition 4.4.** A stable $n$-pointed weighted tropical curve of genus $g$ is a triple $(G, w, \ell)$ where $(G, w)$ is a stable weighted graph with a length function $\ell : E(G) \cup L(G) \to \mathbb{R}_{>0} \cup \{\infty\}$ such that $\ell(x) = \infty$ if and only if $x \in L(G)$.

One readily notice that our tropicalizations for nodal Riemann surfaces with $2 - 2g - n < 0$ are always stable tropical curves. So we will omit “stable” in what follows if no confusion would arise.

In order to construct the moduli space of tropical curves, in [7], Caporaso first constructed the moduli space of tropical curves with fixed combinatorial type: $M_{\text{trop}}(G, w)^+ := C(G, w)^+ / \sim$, where $p_1 \sim p_2$ if the associated weighted metric graphs $(G_1, w_1, \ell_1)$ and $(G_2, w_2, \ell_2)$ are isomorphic to each other. Then the moduli space of $n$-pointed genus $g$ tropical curves is defined as

$$M_{g,n}^{\text{trop}} := \left( \bigsqcup_{G \text{ trivalent, genus } g, n \text{ leaves}} M_{g,n}^{\text{trop}}(G, 0)^+ \right) / \cong$$

where $\cong$ denotes the isomorphism of tropical curves. Finally this space can be compactified to

$$\overline{M}_{g,n}^{\text{trop}} := \left( \bigsqcup_{G \text{ trivalent, genus } g, n \text{ leaves}} \overline{M}_{g,n}^{\text{trop}}(G, 0)^+ \right) / \cong$$

by allowing the edge-length to go to infinity (so that the tropical curve becomes a so-called extended tropical curve), where $\cong$ denotes the isomorphism of extended tropical curves.

It can be shown that the topological space $\overline{M}_{g,n}^{\text{trop}}$ is compact and Hausdorff and of pure dimension $3g - 3 + n$. Moreover, it is connected through codimension one, which is a typical property of a tropical variety (see the Structure Theorem in [12, Section 3.3]), although $\overline{M}_{g,n}^{\text{trop}}$ is not a tropical variety (associated to a prime ideal in the sense of [12, Section 3.2]) in general.

Without surprise, one may notice that the construction for our moduli space of tropicalizations $\overline{M}_{g,n}^{\text{trop}}(G, 0)$ is also largely based on the techniques of the moduli of weighted metric graphs. However, there are still at least two main differences between $\overline{M}_{g,n}^{\text{trop}}(G, 0)$ and $\overline{M}_{g,n}^{\text{trop}}$.

1. The topology is different: for $\overline{M}_{g,n}^{\text{trop}}(G, 0)$ we also allow the edge-length to go to infinity, but we identify the edge-length $\infty$ to $0$ so as to get a topological space $(\mathbb{R}_{>0} \cup \{\infty\}) / \sim$, homeomorphic to $S^1$, for each coordinate to start with. That is because we want to regard our graphs as a tropicalization of some connected genus $g$ Riemann surface. When the length of some edge goes to infinity, the Riemann surface supported on it becomes the normalization of the nodal Riemann surface supported on the graph whose corresponding edge-length goes to zero. Thus the
Riemann surface becomes disconnected or of less genus, which is not the one we want to tropicalize.

While for $\overline{M_{g,n}^{\text{trop}}}$, one endows the set $\mathbb{R}_{\geq 0} \cup \{ \infty \}$ with the subspace topology of the one-point compactification $\mathbb{R} \cup \{ \infty \}$ of $\mathbb{R}$ to get a topological space, for each coordinate to start with.

(2) The “irreducibility” is different: the regular tropicalization for a smooth $n$-pointed Riemann surface is not unique, and on the other hand, a weighted graph may support Riemann surfaces of distinct topological types if it is contracted from different regular tropicalizations, so we fix a regular tropicalization $(G, 0)$ for constructing our space $\overline{M_{g,n}^{\text{trop}}}(G, 0)$.

While for $\overline{M_{g,n}^{\text{trop}}}$, it is glued by the moduli spaces of all the possible combinatorial types $\overline{M_{g,n}^{\text{trop}}}(G, 0)^+$ along the codimension one strata, so that each $\overline{M_{g,n}^{\text{trop}}}(G, 0)^+$ plays a role like an “irreducible component” in $\overline{M_{g,n}^{\text{trop}}}$. Hence in this sense, our $\overline{M_{g,n}^{\text{trop}}}(G, 0)$ is just “irreducible”.

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Declarations

Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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