Conditions for genuine multipartite continuous-variable entanglement

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We derive a hierarchy of continuous-variable multipartite entanglement conditions in terms of second-order moments of position and momentum operators that extends some previous results. Each condition of the hierarchy corresponds to a convex optimization problem which, given the covariance matrix of the state, can be numerically solved in a straightforward way. Our approach can be used to get an analytical condition for genuine multipartite entanglement. We demonstrate that even a special case of our conditions can very efficiently detect entanglement. Using multi-objective optimization it is also possible to numerically verify genuine entanglement of some realistic states realized experimentally. As a by-product of our method, we get some matrix inequalities that are hard to obtain directly.

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Introduction. In multipartite case there are many different notions of entanglement, ranging from the most specific to the most general ones. A specific kind of entanglement means that we precisely specify the groups of modes that are separable from each other, i.e., we specify partition of the set of indices \( \{1, \ldots, n\} \). A partition is a disjoint set \( \{I_1, \ldots, I_k\} \) of nonempty subsets of indices whose union is equal to \( \{1, \ldots, n\} \). A partition with \( k = 2 \) is called a bipartition. From these specific kinds of separability we can construct more general kinds by considering mixtures of specific kinds according to some criteria. From example, a multipartite state is called \( k \)-separable if it is a mixture of states each of which has \( k \) separable groups of modes (these groups can be different for the different terms of the mixture). In the case of \( k = n \) equal to the number of modes we get the notion of full separability, which is the most specific notion of separability of all in the sense that the least number of separable states is fully separable. For \( k = 2 \) we get the notion of biseparability, which is the most general. Thus, a state is biseparable if it is a mixture of states each of which has two separable groups of modes. In the \( n \)-partite case there are \( 2^n - 1 \) possible bipartitions, so there are exponentially many ways for a state to be biseparable. Any state that is not biseparable may be referred to as genuine multipartite entangled.

Many second-order entanglement conditions for \( n \)-partite continuous-variable (CV) systems deal with lower bounds for the quantity \( \text{tr} (\mathbf{G}) \) for a given matrix \( \mathbf{G} \). To find the minimal value of \( \text{tr} (\mathbf{G}) \) we can use \( \mathbf{G} = \mathbf{G}(X, P) \) in case reads as \( \mathbf{G} = (\hat{x}^T \hat{x} + \hat{p}^T \hat{p}) \) instead of \( \hat{x} \) and \( \hat{p} \), since we can replace the wave function \( \psi(x) \) by \( \psi(x + x_0) e^{-i(x \cdot p_0)} \), where \( x_0 = (\hat{x}) \) and \( p_0 = (\hat{p}) \), we transform one quantity to the other. Lastly, by ignoring the off-diagonal blocks of the covariance matrix, \( \gamma_{xp} = \gamma_{xp} \), we still get a valid covariance matrix: If \( \gamma = \begin{pmatrix} \gamma_{xx} & \gamma_{xp} \\ \gamma_{px} & \gamma_{pp} \end{pmatrix} \) is a covariance matrix of a quantum state, then so is \( \tilde{\gamma} = \begin{pmatrix} \gamma_{xx} & 0 \\ 0 & \gamma_{pp} \end{pmatrix} \). In fact, \( \gamma \) is a covariance matrix iff it satisfies the inequality \( \gamma + (i/2) J \geq 0 \), where \( J = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \). The covariance matrix of the state with respect to \( (\hat{x}, \hat{p}) \) is \( \gamma' = \begin{pmatrix} \gamma_{xx} & -\gamma_{xp} \\ -\gamma_{px} & \gamma_{pp} \end{pmatrix} \) and thus it is also a valid covariance matrix. A formal proof can be obtained with the determinant identity \( \det \mathbf{G} = \det (\mathbf{G}) \). Since both \( \gamma \) and \( \gamma' \) are covariance matrices, then so is their average \( \bar{\gamma} = (1/2)(\gamma + \gamma') \). So, if we have only the diagonal blocks \( \gamma_{xx} \) and \( \gamma_{pp} \) of the covariance matrix, we can explicitly assume that we are working with a quantum state whose covariance matrix is block-diagonal, \( \gamma_{xp} = \gamma_{px} = 0 \).

Quantumness bound. To find the minimal value of \( \gamma \) we use the Williamson’s theorem [12]. This theorem states that there is a symplectic matrix \( \mathbf{S} \) such that \( \mathbf{S}^T \mathbf{M} \mathbf{S} = \begin{pmatrix} 0 & \Lambda \\ \Lambda & 0 \end{pmatrix} \), where \( \Lambda \) is a diagonal \( n \times n \) matrix. Since each symplectic transform is imple-
mentable as a unitary transformation, starting with some state \( \hat{\psi} \) we have \( \langle \hat{\psi} | M \hat{\psi} \rangle = (\hat{T}^\dagger (\hat{A}^\dagger \hat{A}) \hat{T})' \rangle \) for the appropriately transformed state \( \hat{\psi}' \). The minimum of \( G \) is achieved for \( \hat{\psi}' \) being the vacuum state, and in this case the equality \( G = \text{Tr} \Lambda \) takes place. We thus obtain the following tight inequality: \( G = \text{Tr}(M^2) \geq \text{Tr} \Lambda \).

To compute the minimal value of \( \mathcal{G} \) we need to know the diagonal elements \( \lambda_i \) of \( M \). These numbers are referred to as symmetric spectrum of \( M \) and they can be directly obtained from the matrix \( M \) according to the fact that \( \pm \lambda_i \) are the eigenvalues of \( JM \). In our special case of block-diagonal \( M = \left( \begin{array}{cc} X & 0 \\ 0 & P \end{array} \right) \) we can get these numbers directly in terms of \( X \) and \( P \). In fact, we have the equality \( JM = \left( \begin{array}{cc} 0 & E \\ -E & 0 \end{array} \right) \left( \begin{array}{cc} X & 0 \\ 0 & P \end{array} \right) = \left( \begin{array}{cc} X & 0 \\ 0 & P \end{array} \right) \left( \begin{array}{cc} -X & 0 \\ 0 & P \end{array} \right) \). The characteristic equation of this matrix reads as \( \chi(\lambda) = \det(\chi^2 + EP) = 0 \). Since the diagonal blocks commute with the off-diagonal ones, according to (4) this equation can be simplified as \( \chi(\lambda) = \det(\lambda^2 E + X P) = 0 \). Substituting the eigenvalues \( \pm \lambda_i \) into this equation, we see that the diagonal elements \( \lambda_i \) satisfy the equation \( \det(XP - \lambda^2 E) = \det(\sqrt{X} \sqrt{P} - \lambda^2 E) = 0 \) and thus are eigenvalues of the symmetric matrix \( \sqrt{X} \sqrt{P} \). The matrices \( X \) and \( P \) can be swapped in this derivation. We finally get the following relation:

\[
\min_{\hat{\psi}} G = \text{Tr} \Lambda = \text{Tr} \sqrt{\sqrt{X} \sqrt{P} \sqrt{X}} = \text{Tr} \sqrt{\sqrt{X} \sqrt{P} \sqrt{X}}.
\] (1)

To put it in another way, we have the inequality
\[
\text{Tr}(X \gamma_{xx}) + \text{Tr}(P \gamma_{pp}) \geq \sqrt{\text{Tr}(X \gamma_{xx}) \text{Tr}(P \gamma_{pp})},
\]
which is valid for all positive definite matrices \( X \) and \( P \) and all correlation matrices \( \gamma_{xx} \) and \( \gamma_{pp} \).

If matrices \( X \) and \( P \) commute then the right-hand side of Eq. (1) reduces to \( \text{Tr} \sqrt{X} \sqrt{P} = \text{Tr} \sqrt{P} \sqrt{X} \). This expression, \( \text{Tr} \sqrt{X} \sqrt{P} \), is a lower bound for \( G \) independent of commutation properties of \( X \) and \( P \). In fact, in terms of wave functions the quantity \( G \) reads as

\[
G = \int \left( \langle \psi(x) \rangle^2 + (\nabla \psi^*)^T P(\nabla \psi) \right) dx.
\] (2)

If we take a general wave function of the form \( \psi(x) = f(x)e^{i\varphi(x)} \), where \( f(x) = |\psi(x)| \) is a real wave function and \( \varphi(x) \) is the phase, we will see that the first term in Eq. (2) does not depend on \( \varphi(x) \), while the other term is equal to \( \int (\nabla f)^T P(\nabla f) + \int (\nabla \varphi)^T P(\nabla \varphi) \) dx. It follows that \( \langle \psi(x) \rangle \geq G(f) \) and thus real wave functions are sufficient to minimize \( G \). For a real wave function \( f(x) \) we have the equality \( G = \int (|u|^2 - \langle \psi(x) \rangle)^2 + (|v|^2 - \langle \psi(x) \rangle)^2 \) dx, where the vector fields \( u(x) \) and \( v(x) \) are defined via \( u(x) = f(x)x \) and \( v(x) = \nabla f(x) \). We can write this equality in a more compact form as \( G \geq \int \left( \left| \langle \hat{u}(x) \rangle \right|^2 \right) dx \). Now we can estimate \( G \) as follows: \( G \geq 2 \left| \int \langle \hat{u}(x), \hat{v}(x) \rangle dx \right| \), or \( G \geq 2 \left| \int x^T \sqrt{X} \sqrt{P} \sqrt{X} \right| f(x) dx \). From the relation

\[
\int u_j(x)v_k(x) \langle \psi(x) \rangle = \int x_j f(x) \hat{\partial}_{x_k} f(x) dx = -\delta_{jk} / 2 \text{ we get the inequality } G \geq \text{Tr} \sqrt{X} \sqrt{P} \sqrt{X} \). We thus have two lower bounds for \( G \) — the tight one is given by the inequality (11) and the other one, not necessarily tight, have just been obtained with the help of Cauchy-Schwarz inequality. Since the tight bound is the best bound possible, we derive the following inequality for a pair of positive definite matrices \( X \) and \( P \):

\[
\text{Tr} \sqrt{\sqrt{X} \sqrt{P} \sqrt{X}} \geq \text{Tr} \sqrt{X} \sqrt{P} \sqrt{X} = \text{Tr} \sqrt{X} \sqrt{P} \sqrt{X} \sqrt{P} \sqrt{X}. \] (3)

This inequality is a special case of Araki-Lieb-Thirring trace inequalities [13,17], which also have quantum mechanical background and read as \( \text{Tr} \delta^{1/2} B \delta^{1/2} \geq \text{Tr} \delta'^{1/2} B' \delta'^{1/2} \), where \( A \) and \( B \) are arbitrary positive definite matrices, \( q \geq 0 \) and \( 0 \leq r \leq 1 \). The case of \( q = 1 \) and \( r = 1/2 \) corresponds to the inequality (3).

Separability bounds. We now show that Eq. (11) can be used to get a hierarchy of necessary conditions for multipartite separability. If a pure state with real wave function is separable and if modes \( i \) and \( j \) are separable from each other then \( \langle \hat{p}_i \hat{p}_j \rangle = 0 \). In fact, separability for pure states means that the wave function is factorizable, i.e., \( f(x) = g(x')h(x'') \), where, without loss of generality, we can assume that \( x' = (x_1, \ldots, x_k) \), \( x'' = (x_{k+1}, \ldots, x_n) \), \( 1 \leq i \leq k \) and \( k + 1 \leq j \leq n \). We have \( \langle \hat{p}_i \hat{p}_j \rangle = \int \langle \hat{p}_i \hat{p}_j \rangle f(x) \hat{\partial}_{x_k} f(x) dx = \int \hat{\partial}_{x_k} g(x')h(x'')g(x')h(x'') d'x'' = 0 \). The fact that \( f(x) \) is real is important for the validity of the last step. The seeming asymmetry in position and momentum operators is superficial — if we worked in momentum representation and dealt with real wave function in that representation we would have \( \langle \hat{p}_i \hat{p}_j \rangle = 0 \). While \( \langle \hat{p}_i \hat{p}_j \rangle = 0 \) if modes \( i \) and \( j \) are separable, for the position correlations we can say only that they factorize: \( \langle \hat{p}_i \hat{p}_j \rangle = \langle \hat{p}_i \rangle \langle \hat{p}_j \rangle \). Taking the wave function \( f_0(x) = f(x + x_0) \), where \( x_0 = (x) \) is the vector of averages computed for the wave function \( f(x) \), we get a new wave function with \( \langle x \rangle = 0 \) and \( G_0 = G - \langle x \rangle^2 \). We see that to minimize \( G \) we can also assume that \( \langle x \rangle = 0 \) and thus \( \langle \hat{p}_i \hat{p}_j \rangle = 0 \) if \( i \) and \( j \) are separable.

So, for a real factorizable wave function to minimize \( G \) some of the elements of the corresponding correlation matrices \( \gamma_{xx} \) and \( \gamma_{pp} \) must be equal to zero. More precisely, for \( \{I_1, \ldots, I_k\} \)-factorizable states all submatrices of the form \( A[I_1 I_2] \), \( i \neq j \), are zero. The notation \( A[I J] \), where \( I \) and \( J \) are sets of indices, is used to denote the submatrix of \( A \) formed by the intersection of rows with indices in \( I \) and columns with indices in \( J \). We thus have the equality
\[
\text{Tr}(X \gamma_{xx}) + \text{Tr}(P \gamma_{pp}) = \text{Tr}(X' \gamma_{xx}) + \text{Tr}(P' \gamma_{pp}) ,
\]
where \( X' \) is obtained from \( X = (X_{ij})_{i,j=1}^n \) by replacing its elements corresponding to zero elements of \( \gamma_{xx} \) by arbitrary real numbers \( u_{ij} \) subject to the condition that \( X' \) is symmetric and positive definite and the same procedure is applied to \( P' \) to produce \( P' \). In other words, we can replace all elements of
the submatrices of the form \(X[I_i|I_j]\) and \(P[I_i|I_j]\), \(i \neq j\), by arbitrary real numbers in such a way that the resulting matrices are again symmetric and positive definite. This construction is better illustrated by an example. For

\[ n = 4, \] consider 2|134- and 1|234-factorizable states. In the former case we have \(I_1 = \{2\}, I_2 = \{1, 3, 4\}\), and in the latter case we have \(I_1 = \{1\}, I_2 = \{2\}, I_3 = \{3, 4\}\). The matrices \(X' = X'(u)\) corresponding to these two cases look as

\[
\begin{pmatrix}
X_{11} & u_1 & u_2 & u_3 \\
u_1 & X_{22} & u_4 & u_5 \\
u_2 & u_4 & X_{33} & u_6 \\
u_3 & u_5 & u_6 & X_{44}
\end{pmatrix},
\begin{pmatrix}
X_{11} & u_1 & u_2 & u_3 \\
u_1 & X_{22} & u_4 & u_5 \\
u_2 & u_4 & X_{33} & u_6 \\
u_3 & u_5 & u_6 & X_{44}
\end{pmatrix}.
\tag{4}
\]

In the first case we replace elements of the submatrices \(X[I_1|I_2]\) and \(X[I_2|I_1]\) by arbitrary numbers, and in the second case we replace submatrices \(X[I_1|I_2]\), \(X[I_2|I_1]\), \(X[I_1|I_3]\), \(X[I_3|I_1]\), \(X[I_2|I_3]\), \(X[I_3|I_2]\). Different submatrices are marked by different colors (symmetric parts are marked by the same color). The more factorizable parts the state has the more elements can be replaced by arbitrary numbers. For a completely factorizable state we can freely choose all the off-diagonal entries. Not to overload the notation, we fix some kind of separability, i.e., some decomposition \(\{I_1, \ldots, I_k\}\) of the indices, and use it in all considerations below. Applying the inequality \(
\sum_{k=1}^{n} \frac{1}{k} \geq \ln n
\), we get that for a factorizable state of any kind with real wave functions and, thus, for all separable states of the same kind, we have

\[
\mathcal{E}(X, P) = \mathcal{G}(X, P) - \max_{(u, v) \in \Omega_x \times \Omega_p} F_{X, P}(u, v) \geq 0,
\tag{5}
\]

where \(F_{X, P}(u) = \text{Tr} \sqrt{X'(u)P'(v)}\sqrt{X'(u)}\) and \(\Omega_x, \Omega_p\) are the sets of all points \(u\) and \(v\) respectively such that \(X'(u)\) and \(P'(v)\) are positive definite. For example, for a bipartition \(\{I, J\}\) with \(|I| = k, |J| = n - k\) the vectors \(u\) and \(v\) have \(k(n-k)\) components, so in this case \(\Omega_x, \Omega_p \subset \mathbb{R}^{k(n-k)}\). For full separability \(u\) and \(v\) have \(n(n-1)/2\) components, so \(\Omega_x, \Omega_p \subset \mathbb{R}^{n(n-1)/2}\). To each kind of separability corresponds its own maximization problem over its own sets \(\Omega_x, \Omega_p\) of its own dimension. There are \(2^{n-1} - 1\) different partitions of the indices of an \(n\)-partite state and many more partitions into three or more parts. If for a given state there is a pair of matrices \(X\) and \(P\) such that an inequality of the form \(\mathcal{E}(X, P) > 0\) is violated, then the state is not separable of the corresponding kind. If there are \(X\) and \(P\) such that the inequalities \(\mathcal{E}(X, P) \geq 0\) are violated for all partitions simultaneously, then the state is genuine multipartite entangled.

Convex optimization problem. The sets \(\Omega_x\) and \(\Omega_p\) are nonempty and convex. It is enough to prove this for one of the sets. The point \(u_0\) whose components are the elements of the original matrix \(X\) that they replaced is clearly in \(\Omega_x\) since \(X \geq 0\), so \(\Omega_x\) is nonempty. The set \(\Omega_x\) is, in fact, the intersection of the convex cone of positive definite matrices and some affine plane and thus \(\Omega_x\) is convex. The set \(\Omega_x \times \Omega_p\) is also convex as the Cartesian product of two convex sets.

The function \(F_{X, P}(u, v)\) in Eq. 4 is a concave function of \((u, v) \in \Omega_x \times \Omega_p\) for fixed \(X\) and \(P\). Since \(F_{X, P}(u, v)\) is the restriction of the function \(F(X, P) = \text{Tr} \sqrt{X P} \sqrt{X}\) to the product of some affine planes, it is enough to prove joint concavity of \(F(X, P)\) itself. Due to the equality \(F(X, P) = \min_{\tau_{XZ}, \tau_{YP}} (\text{Tr}(X \tau_{XZ}) + \text{Tr}(P \tau_{YP}))\), it immediately follows the \(F(X, P)\) is concave as the minimum of concave (in fact, linear) functions. Thus, \(-F_{X, P}(u, v)\) is convex and the left-hand side of Eq. 5 is convex optimization problem.

To demonstrate entanglement of some kind we need to find a pair of matrices \(X\) and \(P\) such that \(\mathcal{E}(X, P) < 0\). In practice, however, we should take into account the errors in the measurement of the matrix elements of \(\gamma_{xx}\) and \(\gamma_{pp}\). Assuming that the errors in the individual elements of \(\gamma_{xx}\) and \(\gamma_{pp}\) are independent, the standard deviation of \(G\) is given by the expression

\[
\sqrt{\|X \circ \sigma_{xx}\|^2 + \|P \circ \sigma_{pp}\|^2},
\]

where \(\sigma_{xx} = \sigma_{x1}^{(u_j)}, \sigma_{pp} = \sigma_{p1}^{(v_k)}\) are the matrices of standard deviations of individual elements of \(\gamma_{xx}\) and \(\gamma_{pp}\) respectively and \(A \circ B\) is the Hadamard (entry-wise) product of matrices. Since it is, in fact, Euclidean norm, it is a convex function of \((X, P)\). So, to be on the safe side the right function to be minimized reads as

\[
\mathcal{E}(X, P) = G(X, P) + s \sigma(X, P) - \max_{(u, v)} F_{X, P}(u, v),
\tag{6}
\]

where \(s\) is the level of certainty with which we can claim that the state is entangled. Usually, the "three-sigma rule", \(s = 3\) is used \([13]\), but the larger \(s\) the better.

The convexity plays a central role in the separability characterization, so it is no surprising that convexity appears here again at a larger scale. We claim that the function \(\mathcal{E}(X, P)\) is convex on the set of all pairs of semidefinite matrices \((X, P)\). This set is obviously convex. The matrices \(X\) and \(P\) have totally \(n(n+1)/2\) independent elements, so we have another convex optimization problem over some convex subset of \(\mathbb{R}^{n(n+1)}\). To prove the convexity of \(\mathcal{E}(X, P)\) we have to prove the concavity of \(\max_{(u, v)} F_{X, P}(u, v)\). The key element of this proof is the fact that \(F_{X, P}(u, v)\) is jointly concave with respect to all four variables \(X, P, u, v\). It follows from the concavity of \(F(X, P)\) and the relation \(\theta(X_1 + (1-\theta)X_2) = \theta X_1' + (1-\theta)X_2'\), \(0 \leq \theta \leq 1\), and similar relation for \(P\). Here we have three sets — the set \(\Omega_{x1} \times \Omega_{p1}\) of points \((u, v)\) where \(X_1'\) and \(P_1'\) are positive definite, the similar set \(\Omega_{x2} \times \Omega_{p2}\) for \(X_2\) and \(P_2\), and the set \(\Omega_x \times \Omega_p\) for \(X = \theta X_1 + (1-\theta)X_2, P = \theta P_1 + (1-\theta)P_2\). In general, these are distinct sets, but one can easily see that \(\theta \Omega_{x1} \times \Omega_{p1} + (1-\theta)\Omega_{x2} \times \Omega_{p2} \subseteq \Omega_x \times \Omega_p\). The argument given in Ref. \([14]\) can be applied here to conclude that the maximum \(\max_{(u, v)} F_{X, P}(u, v)\) over a convex set of a jointly concave function is also concave. This finished
the proof of the convexity of the function $\mathcal{E}(X, P)$ defined by Eq. (6).

Since $\mathcal{E}(X, P)$ is homogeneous, $\mathcal{E}(\lambda X, \lambda P) = \lambda \mathcal{E}(X, P)$ for $\lambda \geq 0$, it makes sense to put some condition on the matrices $X$ and $P$. The simplest is a linear condition, for example, the condition $\text{Tr}(X_{\gamma xx} + P_{\gamma pp}) = C$, where $C$ is an arbitrary fixed positive constant. We thus arrive to the following entanglement condition for any partition of the index set:

$$
\min_{u, v} \text{Tr}(X_{\gamma xx} + P_{\gamma pp}) = C (s \sigma(X, P) - \max_{u, v} F_{X, P}(u, v)) < -C.
$$

(7)

If, for given $\gamma_{xx}$, $\gamma_{pp}$, $\sigma_{xx}$ and $\sigma_{pp}$, this minimum drops below $-C$ then the state under study is not separable of the corresponding kind. If these inequalities are valid for all bipartitions, then the state is genuine multipartite entangled.

The optimization problem (7) is unlikely to have a compact analytical solution. We now show that in some cases one can get useful results even without solving this problem. First, we reproduce the results of Ref. [2]. Consider the rank-one matrices $X = h h^T$ and $P = g g^T$. The square root of a rank-one matrix $A = a a^T$ is given by $\sqrt{A} = A/\|a\|$, so we have $\text{Tr} \sqrt{X P X} = \|h, g\|$. As a concrete example let us consider four-partite case and 1/2/34-separable states. We are free to change some elements of the matrices $X$ and $P$. Let us just change the sign of the $P$'s elements that are marked in Eq. (1):

$$
P' = \begin{pmatrix} g_1^2 & \pm g_1 g_2 & \pm g_1 g_3 & \pm g_1 g_4 \\ \pm g_1 g_2 & g_2^2 & \pm g_2 g_3 & \pm g_2 g_4 \\ \pm g_1 g_3 & \pm g_2 g_3 & g_3^2 & \pm g_3 g_4 \\ \pm g_1 g_4 & \pm g_2 g_4 & \pm g_3 g_4 & g_4^2 \end{pmatrix}.
$$

(8)

For appropriate combinations of signs we can get the equalities $P' = g g^T$, where the new vector reads $g = (\pm g_1, \pm g_2, g_3, g_4)$ and thus $\text{Tr} \sqrt{X P X} = \|h, g\| = \|h_1 g_1 + h_2 g_2 + h_3 g_3 + h_4 g_4\|$. The maximum of these four expressions is $|h_1 g_1| + |h_2 g_2| + |h_3 g_3| + |h_4 g_4|$. This results can be extended to all $n$ and arbitrary kind of separability and coincide with the results obtained in Ref. [2].

**Analytical solution.** The convex optimization problem (7) allows one to quickly test states for entanglement of some kind, but the number of these kinds grows extremely fast with the number of parts. We now derive an analytical condition for genuine multipartite entanglement. It is a single condition that does not require testing exponentially many bipartitions, but it does not provide the best possible bound. Consider the following quantity:

$$
G_n = \sum_{1 \leq i < j \leq n} ((\hat{x}_i + \hat{x}_j)^2 + (\hat{p}_i - \hat{p}_j)^2).
$$

(9)

It is the general quantity $G$, where all diagonal elements of $X$ and $P$ are $n - 1$ and off-diagonal elements of $X$ are 1 and those of $P$ are $-1$. The minimal value of $G_n$ is easy to compute and is equal to $(n - 2)^2/2$. Since matrices $X$ and $P$ are completely symmetric with respect to different parts, it is enough to consider only bipartitions of the form $\{1, \ldots, k\} \cup \{k + 1, \ldots, n\}$ for $k \leq n/2$. We do not change the elements of $X$, and in the matrix $P$ we set all $v_i$ to 1 (so we change the matrix elements from $-1$ to 1). Denote the resulting matrix by $P'_k$. The matrices $X$ and $P'_k$ commute, so that $\text{Tr}(\sqrt{X P'_k X})^{1/2}$ is easy to compute: $\text{Tr} \sqrt{X P'_k X} = -(n - 1)/(\sqrt{n(n - 2)} + \sqrt{n - 2})$. The minimum of this expression over $1 \leq k \leq n/2$ is attained for $k = 1$. We thus have that any biseparable state must satisfy the inequality

$$
G_n \geq (n - 1)/(\sqrt{n(n - 2)} + \sqrt{n - 2}) + 4(n - 1)/\sqrt{n(\sqrt{2n - 2} + \sqrt{n - 2})}.
$$

(10)

If this inequality is violated, then the state is genuine multipartite entangled. Table I summarizes the lower bounds of $G_n$ for some $n$ obtained with the analytical condition (10) and computed numerically from Eq. (6).

| n  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|----|----|----|----|----|----|----|----|
| a  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
| b  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
| f  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |

**TABLE 1.** Lower bounds for $G_n$ for different kinds of $n$-partite states. The $q$ row is quantumness bound, $(n - 1)/\sqrt{n(n - 2)}$. The $a$ row is the biseparability bound given by the analytical expression (10). The $b$ row is the true biseparability bound given by the solution of the optimization problem (7). The last row, $f$, is the full separability bound, $n(n - 1)$.

A similar genuine entanglement condition has been obtained in Ref. [2] in terms of rank-one matrices. The gap between quantum bound and biseparability bound there decreases as $O(1/n)$, where $n$ is the number of parts [21]. In our condition this gap is $O(1)$ and so it does not tend to zero for large $n$.

**Conclusion.** We have developed a method to test multipartite states for arbitrary kinds of entanglement. Our approach allows both numerical and analytical treatment. Numerically, it reduces to a convex optimization problem, which allows fast and accurate solution. We have shown that it is very efficient at detecting ordinary entanglement and can detect genuine multiparticle entanglement in a reasonable amount of time. Analytically, it allows to reproduce (and thus generalize) some known results as well as to obtain an analytical genuine multiparticle entanglement condition. With our approach we can easily obtain some results from matrix theory, like joint concavity of a matrix function of two matrix arguments or reproduce a trace-class inequality, which are difficult to get in a direct way.
In practice, however, we need to take into account possible errors of measurements of the covariance matrix. These errors are given in the form of the standard deviations of the individual matrix elements, from which we can easily compute the standard deviation of the left-hand side of Eq. (11) and we should take this value into account, as it has been done in [3]. To understand what values of $s$ are sufficient to guarantee that our results are correct we need to know the probability that the result of a measurement lies outside $s$ sigma interval. For a Gaussian probability distribution with the mean $\mu$ and the standard deviation $\sigma$ this probability is given by the expression

$$P(s) = 1 - \int_{\mu-s\sigma}^{\mu+s\sigma} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1 - \text{erf}\left(\frac{s}{\sqrt{2}}\right),$$

which depends only on $s$. This probability decreases very quickly as $s$ growth. The order of values of $P(s)$ for some $s$ are shown in the table below.

| $s$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ | $10$ |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $P(s)$ | $0.32$ | $0.05$ | $10^{-3}$ | $10^{-4}$ | $10^{-6}$ | $10^{-9}$ | $10^{-12}$ | $10^{-23}$ |

In many cases, the value of $s = 3$ is sufficient (the three-sigma rule). For $s \geq 5$ this probability is negligible, and for $s \geq 6$ it is practically zero. Even if the real probability distribution is not perfectly Gaussian it is unlikely to have long tail, so Eq. (11) gives a reasonable estimate. Even if this estimate is wrong by several orders of magnitude, provided that we have verified violation with $s \geq 6$ we are still on the safe side. The larger $s$ we set the larger the probability of the correct result is, but the more difficult it will be to find a violation with such $s$. From this table we can conclude that we should search a violation with $s$ not smaller than 3 and not larger than 6 — the event of getting the right result outside of six sigma interval is practically impossible.

It happens that rank-one matrices work surprisingly well. Consider the four-partite state that was analyzed in Ref. [1]. It has the following covariance matrix:

$$\gamma_{xx} = \begin{pmatrix}
1.09921 & 0.16092 & -0.17609 & -0.84831 \\
0.16092 & 0.40938 & -0.16060 & -0.18963 \\
-0.17609 & -0.16060 & 0.46060 & 0.04319 \\
-0.84831 & -0.18963 & 0.04319 & 1.06419
\end{pmatrix},$$

$$\gamma_{pp} = \begin{pmatrix}
1.09921 & 0.35533 & 0.36439 & 0.91386 \\
0.35533 & 0.92282 & 0.57440 & 0.43388 \\
0.36439 & 0.57440 & 1.04339 & 0.34868 \\
0.91386 & 0.43388 & 0.34868 & 1.06419
\end{pmatrix}$$

where $h_I$ is a $|I_j|$-vector with elements whose indicies belong to $I_j$. For example, in the four-partite case for a bipartition $\{1, 4\} \cup \{2, 3\}$ the inequality (11) reads as

$$h^T \gamma_{xx} h + g^T \gamma_{pp} g - \sum_{j=1}^{k} |h^T_{I_j} g_{I_j}| \geq 0,$$
The standard deviation matrix reads as

\[
\sigma_{xx} = \begin{pmatrix}
0.00327 & 0.01041 & 0.00894 & 0.00647 \\
0.01041 & 0.00822 & 0.01848 & 0.01899 \\
0.00894 & 0.01848 & 0.00861 & 0.01345 \\
0.00647 & 0.01899 & 0.01345 & 0.00549
\end{pmatrix},
\]

(15)

\[
\sigma_{pp} = \begin{pmatrix}
0.00458 & 0.01009 & 0.02767 & 0.04289 \\
0.01009 & 0.01023 & 0.02101 & 0.02085 \\
0.02767 & 0.02101 & 0.01466 & 0.01955 \\
0.04289 & 0.02085 & 0.01955 & 0.00455
\end{pmatrix}
\]

A very simple way to search for violation of the condition (11) is to randomly generate 4-vectors \(h\) and \(g\) and check whether this condition is violated or not, and if it is, how strong the violation is. Then just record the maximal observed violation. As a measure of violation we use the quantity

\[
s = \frac{\text{rhs} - \text{lhs}}{\sigma},
\]

(16)

where rhs and lhs are the right-hand side and left-hand side of the corresponding inequality. For the inequality (6) these are the functions of the matrices \(X\) and \(P\), and for the inequality (11) they are functions of the vectors \(h\) and \(g\). In the case of inequality (11) this approach requires only simple matrix algebra manipulations, which can be done very efficiently with tools like Intel Math Kernel Library. A simple parallel Fortran program has been written and run on a low-end 4-core desktop PC. The total time to test all seven possible bipartitions in this four-partite case is 4 minutes (using all four cores available). Table II compares our results with those obtained in Ref. [6]. We see that for the state under study our approach is superior to that of Ref. [6] (which uses a genetic algorithm to find the best violation), since it is simpler and gives better results, though, as we have mentioned before, from practical point of view all violations larger than 6 are of the same value.

We now apply our technique to the six-partite state also considered in Ref. [6]. We have performed two runs of our program on the same hardware as in the previous case, one with a smaller number of random trials and the other with 200 times more trials. The first run takes approximately 4 minutes to perform all 31 tests, the other one takes around 12 hours. As Table III demonstrates, in this case the optimization based on a genetic algorithm gives somewhat larger violations. On the other hand, we do not know what computational resources were used to perform that optimization and how much time it took.

As we have already said, all violation larger than 6 are of the same practical value and our method produced much better violations in just a few minutes on a low-end PC.

The last state considered in Ref. [6] is a ten-partite state. It has been reported that the smallest violation of 1.1 was obtained for the bipartition 1, 10/23456789. The corresponding probability to get wrong result is

\[
P(1.1) \approx 0.27, \text{ and it is not small enough to conclude that the state under study is not } 1, 10/23456789\text{-separable}.
\]

Randomly generating vectors \(h\) and \(g\), we have found that the inequality (11) for this kind of separability can be violated with \(s = 3.65\). The corresponding probability

\[
P(3.65) < 3 \times 10^{-4}
\]

is much smaller and gives a strong confidence that the state is 1, 10/23456789-entangled. The vectors are

\[
h = (-0.65, -1.22, -0.21, 0.01, 0.365, -0.32, 0.25, 0.28, -0.88, -0.94),
\]

(17)

\[
g = (0.22, -1.04, -0.12, -0.04, 0.42, -0.34, 0.22, 0.30, -0.56, 0.73).
\]

The violations of other kinds of biseparability are all larger than 3, so the standard three-sigma test is passed for all bipartitions. Note that the conditions (13) and their special case (14) form a natural hierarchy — the violation of a finer kind of separability cannot be smaller than the violation of a more coarse one, so the violation for the full separability must be the largest. The violations reported in Ref. [6] show some strange behavior — the violation for full separability is smaller than violations of some more coarse kinds. But this may be an artifact of an implementation of the genetic optimization algorithm.

Up to now it has been shown that the four-partite state with the covariance matrix (11) is not separable for any fixed kind of separability. We demonstrate that this state is genuine entangled. To do this we need to find a pair of vectors \(h\) and \(g\) or a pair of matrices \(X\) and \(P\) that simultaneously violate the inequalities (11) or (5) for all bipartitions. First, we have tried to violate the inequalities Eq. (11). It took nearly one day, but we were able

| Bipart. | Violation | h | g |
|--------|-----------|---|---|
| Ref. [6], 4m | | | |
| 1|234 | 20.93 | 26.48 | (1.97, −0.01, 0.49, 1.88) |
| 2|134 | 13.17 | 18.08 | (−0.40, −1.99, −1.48, −0.74) |
| 3|124 | 11.21 | 16.10 | (0.31, 1.93, 1.62, 0.32) |
| 4|123 | 21.06 | 26.57 | (0.23, 1.65, −1.46, −0.19) |
| 12|34 | 24.34 | 27.79 | (1.91, 0.42, 0.75, 1.90) |
| 13|24 | 23.52 | 26.17 | (−1.52, −0.17, −0.54, −1.58) |
| 14|23 | 4.66 | 9.72 | (0.22, 1.65, 0.18, 0.77) |

The total time to perform all seven tests is 4 minutes.
| Bipartition | Violation |
|------------|-----------|
|            | Ref. [6]  | 12h | 4m |
| 1[23456]   | 40.086    | 40.111 | 37.500 |
| 2[13456]   | 36.185    | 34.097 | 33.967 |
| 3[12456]   | 29.274    | 19.715 | 17.683 |
| 4[12356]   | 20.010    | 18.869 | 16.935 |
| 5[12346]   | 27.146    | 26.680 | 22.979 |
| 6[12345]   | 49.220    | 48.077 | 44.187 |
| 12[3456]   | 53.541    | 53.085 | 50.142 |
| 13[2456]   | 45.569    | 45.958 | 41.684 |
| 14[2356]   | 44.789    | 42.163 | 42.509 |
| 15[2346]   | 45.282    | 40.410 | 38.565 |
| 16[2345]   | 31.177    | 29.995 | 25.686 |
| 23[1456]   | 40.158    | 38.636 | 37.462 |
| 24[1356]   | 37.698    | 36.633 | 31.716 |
| 25[1346]   | 35.256    | 31.106 | 29.877 |
| 26[1345]   | 47.016    | 42.269 | 40.199 |
| 34[1256]   | 24.833    | 22.592 | 20.694 |
| 35[1246]   | 28.794    | 25.927 | 23.390 |
| 36[1245]   | 50.193    | 48.021 | 44.561 |
| 45[1236]   | 30.629    | 28.950 | 27.092 |
| 46[1235]   | 51.500    | 50.153 | 47.510 |
| 56[1234]   | 56.080    | 55.390 | 51.521 |
| 123[456]   | 56.661    | 55.474 | 54.345 |
| 124[356]   | 54.402    | 52.666 | 52.044 |
| 125[346]   | 50.653    | 48.953 | 47.993 |
| 126[345]   | 28.957    | 26.470 | 22.905 |
| 134[256]   | 47.675    | 46.956 | 46.154 |
| 135[246]   | 47.279    | 43.109 | 40.016 |
| 136[245]   | 34.237    | 29.400 | 26.527 |
| 145[236]   | 47.331    | 42.957 | 41.199 |
| 146[235]   | 35.340    | 34.072 | 30.970 |
| 156[234]   | 39.487    | 38.426 | 36.880 |

TABLE III. The comparison of the violation of the separability condition for the six-party state considered in Ref. [6]. Provided that the separability condition can be so strongly violated for all bipartitions it is absolutely unnecessary to test other kinds of separability, i.e. kinds with partitions of modes into three or more groups.

For these matrices we have
\[
\text{Tr}(X_{\gamma_{xx}}) + \text{Tr}(P_{\gamma_{pp}}) = 1.47484, \tag{19}
\]
and
\[
\sqrt{\|X \sigma_{xx}\|^2 + \|P \sigma_{pp}\|^2} = 0.01947. \tag{20}
\]

The maximum \(\max_{(u,v)} f_{X',P'}(u,v)\) for different bipartitions is presented below. The elements of the matrices that were optimized over are highlighted. For bipartition \(\{1\} \cup \{2,3,4\}\) the maximum is attained at
\[
X' = \begin{pmatrix}
0.39234 & -0.10873 & 0.158136 & 0.116524 \\
-0.10873 & 0.88526 & 0.09450 & 0.09080 \\
0.158136 & 0.09450 & 0.58391 & 0.20795 \\
0.116524 & 0.09080 & 0.20795 & 0.39504
\end{pmatrix}
\]
\[
P' = \begin{pmatrix}
-0.22992 & -0.11914 & 0.113758 & 0.083761 \\
-0.11914 & 0.52598 & -0.32316 & -0.16699 \\
0.113758 & -0.32316 & 0.39949 & 0.06971 \\
0.083761 & -0.16699 & 0.06971 & 0.31242
\end{pmatrix},
\]
and is equal to
\[
\max_{(u,v)} f_{X',P'}(u,v) = 1.65474. \tag{21}
\]
For bipartition \( \{2\} \cup \{1, 3, 4\} \)

\[
X' = \begin{pmatrix}
  0.39234 & -0.20267 & 0.24691 & 0.30527 \\
  -0.20267 & 0.88526 & -0.01671 & 0.09080 \\
  0.22149 & -0.01671 & 0.58391 & 0.24154 \\
  0.30527 & 0.09080 & 0.24154 & 0.39504 \\
-0.22992 & -0.13140 & 0.02836 & -0.11723 \\
-0.13140 & 0.52598 & -0.07629 & -0.16699 \\
  0.02836 & -0.07629 & 0.39949 & 0.12842 \\
-0.11723 & -0.16699 & 0.12842 & 0.31242
\end{pmatrix},
\]

and is equal to

\[
\max_{(u,v)} F_{X,P}(u,v) = 1.66193. \tag{22}
\]

For bipartition \( \{3\} \cup \{1, 2, 4\} \)

\[
X' = \begin{pmatrix}
  0.39234 & -0.20267 & 0.22149 & 0.30527 \\
-0.20267 & 0.88526 & -0.1671 & 0.09080 \\
  0.22149 & -0.1671 & 0.58391 & 0.24154 \\
  0.30527 & 0.09080 & 0.24154 & 0.39504 \\
  0.22992 & -0.13140 & 0.02836 & -0.11723 \\
-0.13140 & 0.52598 & -0.07629 & -0.16699 \\
  0.02836 & -0.07629 & 0.39949 & 0.12842 \\
-0.11723 & -0.16699 & 0.12842 & 0.31242
\end{pmatrix},
\]

and is equal to

\[
\max_{(u,v)} F_{X,P}(u,v) = 1.56935. \tag{23}
\]

For bipartition \( \{4\} \cup \{1, 2, 3\} \)

\[
X' = \begin{pmatrix}
  0.39234 & -0.20267 & 0.24691 & 0.15483 \\
-0.20267 & 0.88526 & 0.09450 & 0.04047 \\
  0.24691 & 0.09450 & 0.58391 & 0.23966 \\
  0.15483 & 0.04047 & 0.23966 & 0.39504 \\
  0.22992 & -0.13140 & -0.00477 & 0.05094 \\
-0.13140 & 0.52598 & -0.32316 & -0.05608 \\
-0.00477 & -0.32316 & 0.39949 & 0.12953 \\
  0.05094 & -0.05608 & 0.12953 & 0.31242
\end{pmatrix},
\]

and is equal to

\[
\max_{(u,v)} F_{X,P}(u,v) = 1.63974. \tag{24}
\]

For bipartition \( \{1, 2\} \cup \{3, 4\} \)

\[
X' = \begin{pmatrix}
  0.39234 & 0.02067 & 0.24691 & 0.19766 & 0.11649 \\
-0.20267 & 0.88526 & -0.01671 & 0.09080 & 0.12997 \\
  0.22149 & -0.1671 & 0.58391 & 0.24154 & 0.32316 \\
  0.30527 & 0.09080 & 0.24154 & 0.39504 & 0.22992 \\
  0.22992 & -0.13140 & 0.02836 & -0.11723 & -0.00477 \\
-0.13140 & 0.52598 & -0.07629 & -0.16699 & -0.00477 \\
  0.02836 & -0.07629 & 0.39949 & 0.12842 & 0.02836 \\
-0.11723 & -0.16699 & 0.12842 & 0.31242 & -0.11723
\end{pmatrix},
\]

and is equal to

\[
\max_{(u,v)} F_{X,P}(u,v) = 1.81056. \tag{25}
\]

For bipartition \( \{1, 3\} \cup \{2, 4\} \)

\[
X' = \begin{pmatrix}
  0.39234 & -0.10013 & 0.24691 & 0.12997 \\
-0.10013 & 0.88526 & 0.03695 & 0.09080 \\
  0.24691 & -0.03695 & 0.58391 & 0.25307 \\
  0.12997 & 0.09080 & 0.25307 & 0.39504 \\
  0.22992 & -0.07273 & -0.00477 & 0.06225 \\
-0.07273 & 0.52598 & -0.05209 & -0.16699 \\
-0.00477 & -0.05209 & 0.39949 & 0.17734 \\
  0.06225 & -0.16699 & 0.17734 & 0.31242
\end{pmatrix},
\]

and is equal to

\[
\max_{(u,v)} F_{X,P}(u,v) = 1.74993. \tag{26}
\]

For bipartition \( \{1, 4\} \cup \{2, 3\} \)

\[
X' = \begin{pmatrix}
  0.39234 & -0.11435 & 0.18571 & 0.30527 \\
-0.11435 & 0.88526 & 0.09450 & -0.02360 \\
  0.18571 & 0.09450 & 0.58391 & 0.25307 \\
  0.30527 & -0.02360 & 0.25307 & 0.39504 \\
  0.22992 & -0.09331 & 0.05497 & -0.11723 \\
-0.09331 & 0.52598 & -0.32316 & -0.02171 \\
  0.05497 & -0.32316 & 0.39949 & 0.12643 \\
-0.11723 & -0.02171 & 0.12643 & 0.31242
\end{pmatrix},
\]

and is equal to

\[
\max_{(u,v)} F_{X,P}(u,v) = 1.56114. \tag{27}
\]

The smallest number among these maximums is the last one, 1.56114, so we have

\[
\frac{\max_{(u,v)} F_{X,P}(u,v) - \text{Tr}(X\gamma_{xx} + P\gamma_{pp})}{\sigma(X,P)} > s = 4.43199
\]

for all bipartitions simultaneously. The corresponding probability is \( P(4.43199) < 10^{-5} \), which is almost two orders of magnitude smaller than for the vectors \( h \) and \( g \) we found before, so one can be pretty sure that the state under study is genuine entangled.