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Elementary formulae
for social distancing scenarios:
Application to COVID-19 mitigation
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Abstract
Social distancing has been enacted in order to mitigate the spread of COVID-19. Like many authors, we adopt the classic epidemic SIR model, where the infection rate is the control variable. Its differential flatness property yields elementary closed-form formulae for open-loop social distancing scenarios, where, for instance, the increase of the number of uninfected people may be taken into account. Those formulae might therefore be useful to decision makers. A feedback loop stemming from model-free control leads to a remarkable robustness with respect to severe uncertainties of various kinds. Although an identification procedure is presented, a good knowledge of the recovery rate is not necessary for our control strategy. Several convincing computer experiments are displayed.

Index Terms
Biomedical control, COVID-19, social distancing, SIR model, nonlinear feedback control, flatness-based control, model-free control, robustness, identifiability, algebraic differentiator

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I. INTRODUCTION

In less than two years an abundant mathematically oriented literature has been devoted to the worldwide COVID-19 pandemic. Some of the corresponding calculations had even a significant political impact (see, e.g., [1]). A novel control technique for improving the social distancing is presented here. This fundamental topic has already been tackled by many authors: see, e.g., [2], [3], [6], [7], [8], [10], [12], [13], [14], [15], [21], [25], [30], [38], [37], [38], [41], [43], [44], [46], [57]. Most of those papers are based on the famous SIR (Susceptible-Infected-Recovered/Removed) model, which goes back to [27] in 1927, or on slight modifications. This communication is also using the SIR model:

- When, like in several papers, the infection rate is the control variable, the SIR model is (differentially) flat ([20]). Remember that flatness-based control is one of the most popular model-based control setting, especially with respect to concrete applications: see, e.g., [3], [9], [31], [32], [35], [45], [47], [49], [50], [51], [53], [54], [55], [63] for some recent publications. Note that flatness has already been utilized in [23] for studying COVID-19 but with other purposes.
- There are severe uncertainties: model mismatch, poorly known initial conditions, …We therefore close the loop around the reference trajectory via model-free control, or MFC, in the sense of [16], [17]. MFC, which is easy to implement, has already been illustrated in a number of practical situations. Some new contributions are listed here: [22], [26], [29], [33], [39], [40], [48], [52], [58], [59], [60], [61], [64], [65]. Let us single out here the excellent work by [56] on ventilators, which are obviously related to COVID-19.

In order to be more specific consider a flat system with a single input $u$ and a single output $y$. Assume that $y$ is a flat output. Our strategy may be summarized as follows:

1) To any output reference trajectory $y^*$ corresponds at once thanks to flatness an open-loop control $u^*$.
2) Let $z$ be some measured output. Write $z^*$ the corresponding reference trajectory. Set $u = u^* + \Delta u$, where $\Delta u$ is the control of an ultra-local local model [16]. Its output $\Delta z = z - z^*$ is the tracking error. Closing the loop via an intelligent controller [16] permits to ensure local stability around $z^*$ in spite of severe mismatches and disturbances.

Our paper is organized as follows.

Section III shows that
- the SIR model, where the infection rate is the control variable, is flat and the population of recovered/removed individuals is a flat output;
- the recovery rate is identifiable in the sense of [19].

Section III is devoted to a flatness-based control strategy, i.e., to a feedforward approach. Elementary closed-form of the control and state variables may be easily derived. Various scenarios, where for instance the number of uninfected persons is increased, may thus be easily suggested to decision makers.

Closing the loop via an intelligent proportional regulator, stemming from model-free control, is the subject of Section IV. Computer simulations confirm an excellent robustness with respect to severe uncertainties.

A time-varying recovery rate is estimated in Section V via algebraic estimation methods ([19]). Techniques from Section IV show however good performances if this rate is wrongly assumed to be constant.

Some concluding remarks may be found in Section VI.

II. MODELING ISSUES

A. The SIR model

The SIR model (see, e.g., [62] for a most pleasant introduction) reads:

$$\begin{align*}
\dot{S} &= -\beta IS \\
\dot{I} &= \beta IS - \gamma I \\
\dot{R} &= \gamma I
\end{align*}$$

(1)

$S$, $I$ and $R$, which are non-negative quantities, correspond respectively to the fractions of susceptible, infected and recovered/removed individuals in the population. We may set therefore

$$S + I + R = 1$$

(2)

$\beta$, $0 < \beta \leq \beta \leq \beta$, which is here the control variable\footnote{Softening social distancing implies increasing $\beta(t)$.}, and the constant parameter $\gamma > 0$ are the infection and recovery rates.

B. Flatness

Equations (1)-(2) show that System (1) is flat and that $R$ is a flat output [20]. The other system variables may be expressed as differential rational functions of $R$, i.e., as rational functions of $R$ and its derivatives up to some finite order:

$$I = \frac{\dot{R}}{\gamma}$$

(3)

$$S = 1 - R - \frac{\dot{R}}{\gamma}$$

(4)

$$\beta = \frac{-\dot{S}}{IS} = \frac{1}{S} \left( \frac{\dot{I}}{I} + \gamma \right)$$

(5)

Remark 1: If $\gamma$ is not constant, but a differentiable function of time, Equations (3)-(4)-(5) remain valid: System (1) is still flat and $R$ is still a flat output. Equation (3) shows however that $\dot{r}$ is needed.
C. Identifiability of the recovery rate

Equation (5) yields

$$\gamma = \beta S - \frac{I}{I}$$

$$\gamma$$ is a differential rational function of $$R$$ and $$\beta$$; it is thus rationally identifiable [19].

Remark 2: The above equation does not work for an identifiability purpose if $$\gamma$$ is time-varying; $$\dot{\gamma}$$ is sitting on its right hand-side. If we assume that $$I$$ and $$S$$ are measured, Equation (4) yields

$$\gamma = \frac{I - \beta IS}{I}$$

(6)

$$\gamma$$ is still rationally identifiable with respect to $$I$$, $$S$$, $$\beta$$. It will be useful in Section V.

III. FLATNESS-BASED CONTROL

A. Preparatory calculations

Set

$$I_{\text{reference}}(t) = I_0 e^{-\lambda t}$$

where $$t \geq 0$$, $$0 \leq I_0 \leq 1$$, and $$\lambda \geq 0$$ is some constant parameter.

Remark 3: The reproduction number (see, e.g., [24], [62]) is thus set to

$$\exp(-\lambda) < 1$$

If we set $$R(0) = 0$$, it yields

$$R_{\text{reference}}(t) = \frac{\gamma I_0}{\lambda} (1 - e^{-\lambda t}) - I_0 e^{-\lambda t}$$

and the open-loop control

$$\beta_{\text{flat}}(t) = \frac{\gamma - \lambda}{1 - \frac{2I_0}{\lambda} (1 - e^{-\lambda t}) - I_0 e^{-\lambda t}}$$

Thus

$$\lim_{t \to +\infty} \beta_{\text{flat}}(t) = \frac{\lambda(\gamma - \lambda)}{\lambda - \gamma I_0}$$

(7)

The following inequalities are straightforward:

$$\gamma I_0 < \lambda < \gamma$$

(8)

$$\lambda < \gamma$$ follows from $$\beta > 0$$; $$\gamma I_0 < \lambda$$ follows from

$$\lim_{t \to +\infty} S(t) = 1 - \frac{\gamma I_0}{\lambda} = S(\infty) > 0$$

(9)

Introduce the more or less precise quantity $$\beta_{\text{accept}}$$, where $$\beta < \beta_{\text{accept}} < \frac{\gamma}{\lambda}$$. It stands for the “harshest” social distancing protocols which are “acceptable” in the long run. Equation (7) yields therefore

$$\lambda(\gamma - \lambda) \lambda - \gamma I_0 = \beta_{\text{accept}}$$

The positive root of the corresponding quadratic algebraic equation

$$\lambda^2 + (\beta_{\text{accept}} - \gamma)\lambda - \gamma I_0\beta_{\text{accept}} = 0$$

is

$$\lambda_{\text{accept}} = \frac{\gamma - \beta_{\text{accept}} + \sqrt{\Delta_{\text{accept}}}}{2}$$

where $$\Delta_{\text{accept}} = (\gamma - \beta_{\text{accept}})^2 + 4\gamma I_0\beta_{\text{accept}} \geq 0$$. The fundamental inequality

$$\gamma I_0 < \lambda_{\text{accept}} < \gamma$$

follows from

$$\lim_{\lambda \to \gamma I_0} \frac{\lambda(\gamma - \lambda)}{\lambda - \gamma I_0} = +\infty, \quad \lim_{\lambda \to \gamma I_0} \frac{\lambda(\gamma - \lambda)}{\lambda - \gamma I_0} = 0$$

Equation (7) leads to the notation

$$S_{\text{accept}}(\infty) = 1 - \frac{\gamma I_0}{\lambda_{\text{accept}}}$$

The inequality

$$S(\infty) < S_{\text{accept}}(\infty) \quad \text{if} \quad \lambda < \lambda_{\text{accept}}$$

demonstrates that the proportion of uninfected people decreases if the social distancing obligations are relaxed.
B. Two computer experiments

Set $\gamma = 0.1$, $\beta_{\text{accept}} = 0.22$. Figure 1 displays the open-loop evolutions of $\beta$, $I$, $S$ when $\lambda = \lambda_{\text{accept}}$. Those behaviors are quite satisfactory.

IV. MODEL-FREE CONTROL

A. Ultra-local model

Set $\Delta I(t) = I(t) - I_{\text{reference}}(t)$, $\beta(t) = \beta_{\text{flat}}(t) + \Delta \beta(t)$. In order to take into account the various uncertainties, introduce the ultra-local model ([16])

$$\frac{d}{dt} \Delta I = F + \alpha \Delta \beta$$ (10)

- The function $F$, which is data-driven, subsumes the poorly known structures and disturbances.
- The parameter $\alpha$, which does not need to be precisely determined, is chosen such that the three terms in Equation (10) are of the same magnitude.
- $F_{\text{est}} = -\frac{2}{\tau} \int_{t-\tau}^{t} ((t - 2\tau) \Delta I(\sigma) + \alpha \sigma(\tau - \sigma) \Delta \beta(\sigma)) d\sigma$, where $\tau > 0$ is “small”, gives a real-time estimate, which in practice is implemented via a digital filter.

B. Intelligent proportional controller

Introduce ([16]) the intelligent proportional controller, or iP,

$$\Delta \beta = -\frac{F_{\text{est}} + K_P \Delta I}{\alpha}$$ (11)

where $K_P$ is a tuning gain. Equations (10) and (11) yield

$$\frac{d}{dt} \Delta I + K_P \Delta I = F - F_{\text{est}}$$

Set $K_P > 0$. Then $\lim_{t \to +\infty} \Delta I(t) = 0$ if the estimate $F_{\text{est}}$ is “good,” i.e., if $F - F_{\text{est}}$ is “small.” Local stability is ensured.

Remark 4: When compared to classic PIs and PIDs (see, e.g., [4]), the gain tuning of the iP is straightforward.

C. Computer experiments

The sampling time interval is 2 hours. In Equations (10) and (11), $\alpha = 0.1$, $K_P = 1$. Figure 2 displays excellent results in spite of
- errors on initial conditions;
- the fuzzy character of any measurement of the social distancing. It is expressed by an additive corrupting white Gaussian noise $N(0, 5.10^{-3})$ on $\beta$.

V. ON THE RECOVERY RATE $\gamma$

Assume now that $\gamma$ is a differentiable time function. Equation (6) yields the algebraic estimator

$$\gamma_{\text{est}} = \left[ \dot{I} \right]_{\text{est}} - \beta IS$$ (12)

where $[\dot{I}]_{\text{est}}$ is an estimate of $\dot{I}$ obtained along the lines developed in [34], [42] for algebraic differentiators. Figure 3-c displays excellent results. The flatness-based computer experiments is achieved as in Section [II-B] i.e., $\gamma = 0.1$ is assumed to be constant. Lack of space prevents us from displaying our convincing simulations in the more realistic situation with noise corruption.

Closing the loop via model-free control yields as demonstrated in Figures 3-a-b a rather satisfactory behavior. Should we deduce that the exact knowledge of the recovery rate is unimportant?

VI. CONCLUSION

The relevance and usefulness of such control-theoretic considerations for non-pharmaceutical mitigation policies against COVID-19 are questioned in [11]. We certainly do not claim to set aside those objections in this preliminary short study. The combination however of flatness-based and model-free controls, like in [18] for in silico cancer treatments, presents perhaps some major advantages:
- Flatness-based control allows to present in a straightforward way a wealth of reference trajectories in order to take into account various constraints.
- Closing the loop via model-free control permits a remarkable robustness with respect to many severe uncertainties. Those features should of course be confirmed by further investigations.
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Fig. 1: Open loop: $I_0 = 0.05$ (–) and $I_0 = 0.1$ (- -)

Fig. 2: Error on initial conditions and fuzzy $\beta$ – blue(- -): reference trajectory
(a) \( \beta \) – blue(- -): reference trajectory  
(b) \( I \) – blue(- -): reference trajectory  
(c) \( S \) – blue(- -): reference trajectory  

\( \gamma \) (- -) and \( \gamma_{test} \) (blue –) 

Fig. 3: Variable recovery rate \( \gamma \)