Strict power concavity of a convolution

Jun O’Hara1· Shigehiro Sakata2

Received: 5 June 2020 / Accepted: 13 October 2021 / Published online: 9 November 2021
© Fondazione Annali di Matematica Pura ed Applicata and Springer-Verlag GmbH Germany, part of Springer
Nature 2021

Abstract
We give a sufficient condition for the strict parabolic power concavity of the convolution in space variable of a function defined on \( \mathbb{R}^n \times (0, +\infty) \) and a function defined on \( \mathbb{R}^n \). Since the strict parabolic power concavity of a function defined on \( \mathbb{R}^n \times (0, +\infty) \) naturally implies the strict power concavity of a function defined on \( \mathbb{R}^n \), our sufficient condition implies the strict power concavity of the convolution of two functions defined on \( \mathbb{R}^n \). As applications, we show the strict parabolic power concavity and strict power concavity in space variable of the Gauss–Weierstrass integral and the Poisson integral for the upper half-space.

Keywords Strict power concavity · The Borell–Brascamp–Lieb inequality · Strict parabolic power concavity

Mathematics Subject Classification 26B25 · 26D15 · 90C25 · 52A41 · 52A40

1 Introduction

In this paper, we are interested in the strict power concavity of the convolution,

\[
 f \ast g(x) = \int_{\mathbb{R}^n} f(x-y)g(y) \, dy, \quad x \in \mathbb{R}^n,
\]

of two nonnegative measurable functions \( f \) and \( g \) defined on \( \mathbb{R}^n \).

The first-named author is partially supported by JSPS Kakenhi (No. 19K03462). The second-named author is partially supported by JSPS Kakenhi (No. 20K14320) and funds (No. 205004) from the Central Research Institute of Fukuoka University.

Jun O’Hara
ohara@math.s.chiba-u.ac.jp

Shigehiro Sakata
ssakata@fukuoka-u.ac.jp

1 Department of Mathematics and Informatics, Faculty of Science, Chiba University, 1-33 Yayoi-cho, Inage, Chiba 263-8522, Japan

2 Department of Applied Mathematics, Faculty of Science, Fukuoka University, 8-19-1 Nanakuma, Jonan, Fukuoka 814-0180, Japan
Let us recall the notion of power concavity and its basic properties. Let $A$ be a convex subset of $\mathbb{R}^n$, $f$ a nonnegative function defined on $A$, and $p \in \mathbb{R} \cup \{ \pm \infty \}$. $f$ is said to be $p$-concave on $A$ if, for any $x_0, x_1 \in A$ and $\lambda \in [0, 1]$, the inequality
\[
f((1 - \lambda)x_0 + \lambda x_1) \geq M_p(f(x_0), f(x_1); \lambda)
\] (1.2)
holds. Here, for $a, b \in [0, +\infty)$ and $\lambda \in [0, 1],
\[
M_p(a, b; \lambda) := \begin{cases} 
0 & (ab = 0), \\
((1 - \lambda)a^p + \lambda b^p)^{1/p} & (ab > 0, \ p \notin \{\pm \infty, 0\}), \\
\max\{a, b\} & (ab > 0, \ p = +\infty), \\
\min\{a, b\} & (ab > 0, \ p = -\infty)
\end{cases}
\] (1.3)
is called the $p$th mean of $a$ and $b$ of ratio $\lambda$. $f$ is said to be strictly $p$-concave on $A$ if the inequality (1.2) strictly holds for any distinct $x_0, x_1 \in A$ and $\lambda \in (0, 1)$.

When $f$ is positive on $A$ and $p \in \mathbb{R}$, $f$ is $p$-concave if and only if $x \mapsto f(x)^p$ is concave for $p \in (0, +\infty)$, $x \mapsto \log f(x)$ is concave for $p = 0$, and $x \mapsto f(x)^p$ is convex for $p \in (-\infty, 0)$ (see Subsect. 2.2 for the details). As a consequence of Jensen’s inequality, if $p \geq q$, then, for any $a, b \in [0, +\infty)$ and $\lambda \in [0, 1],
\[
M_p(a, b; \lambda) \geq M_q(a, b; \lambda)
\] (1.4)
holds (see, for example, [9, Sect. 2.9]). Thus, for any $p \in \mathbb{R} \cup \{ +\infty \}$, $p$-concave (resp. strictly $p$-concave) functions are $-\infty$-concave (resp. strictly $-\infty$-concave).

$-\infty$-concavity is also called quasi-concavity, and we use this terminology hereafter. It directly follows from definition that any strictly quasi-concave function has at most one global maximum point. Furthermore, if $f$ is strictly quasi-concave on $A$, then, for any convex subset $C$ of $A$, the restriction of $f$ to $C$ is strictly quasi-concave on $C$. Thanks to these properties, strict quasi-concavity plays an important role for optimization problems in, for example, economics. Namely, for a maximization problem with an objective function $f$, if $f$ is strictly quasi-concave, then we have at most one global optimal solution.

As Gardner explains in [6, Sect. 11], the power concavity of a convolution can be derived from the Borell–Brascamp–Lieb inequality (BBL-inequality, for short). The BBL-inequality is an integral inequality (see Theorem 2.9 of this paper for the precise statement). It was shown by Borell [1, Theorem 3.1] and by Brascamp and Lieb [3, Theorem 3.3], independently, around the same time. The proof of the BBL-inequality can be found in, for example, [5, Sect. 3.3], [8] and [16]. These references include probabilistic applications of the BBL-inequality.

Let us review the process of deriving the power concavity of (1.1) from the BBL-inequality according to [6, Sect.11] (see also [5, Sect. 3.3] and [21, Sect.2]). Hölder’s inequality implies that, for $a, b, c, d \in [0, +\infty)$, $p, q \in \mathbb{R} \cup \{ \pm \infty \}$ and $\lambda \in [0, 1]$, if $p + q \geq 0$, then
\[
M_p(a, b; \lambda)M_q(c, d; \lambda) \geq M_{p+q}(ac, bd; \lambda)
\] (1.5)
holds, where...
Strict power concavity of a convolution

\[ \ell' = \begin{cases} \frac{pq}{p + q} & (p + q \neq 0), \\ -\infty & (p + q = 0, (p, q) \neq (0, 0)), \\ 0 & (p, q) = (0, 0), \end{cases} \]  

(1.6)

and we understand \(+\infty + (-\infty) = -\infty + \infty = 0\) (see, for example, [6, Lemma 10.1]). It follows from (1.5) that if \(p + q \geq 0\), then, for any \(p\)-concave function \(f\) and \(q\)-concave function \(g\), the function

\[ \mathbb{R}^n \times \mathbb{R}^n \ni (x, y) \mapsto f(x - y)g(y) \]  

(1.7)

is \(\ell\)-concave on \(\mathbb{R}^n \times \mathbb{R}^n\). It follows from the BBL-inequality that if \(\ell' \geq -1/n\), then, for any \(\ell\)-concave function \(F\) defined on \(\mathbb{R}^m \times \mathbb{R}^n\) such that the integral

\[ G(x) = \int_{\mathbb{R}^n} F(x, y) \, dy \]  

(1.8)

exists for each \(x \in \mathbb{R}^m\), the function \(G\) is \(\ell/(1 + n\ell')\)-concave on \(\mathbb{R}^m\) (see \([1, Theorem 4.3]\) and \([3, Corollary 3.5]\)). Here, we understand that \(\ell/(1 + n\ell')\) is equal to \(-\infty\) when \(\ell = -1/n\) and to \(1/n\) when \(\ell = +\infty\). Using this application of the BBL-inequality with \(m = n\) and \(F\) in (1.7), we obtain the \(\ell/(1 + n\ell')\)-concavity of (1.1).

One of our results of this paper (Theorem 3.9) is the strict version of the above. We show that if the following conditions are satisfied, then (1.1) is strictly \(\ell/(1 + n\ell')\)-concave on \(\mathbb{R}^n\):

(i) \(f\) is strictly \(p\)-concave on \(\mathbb{R}^n\).

(ii) \(g\) is \(q\)-concave on \(\mathbb{R}^n\).

(iii) \(\mathbb{R}^n \setminus g^{-1}(0)\) is bounded, and its interior is not empty.

(iv) \(p + q \geq 0\) and \(\ell' \geq -1/n\).

Compared to the process of deriving (not necessarily strict) power concavity, to show the strict power concavity of (1.1), it is essentially sufficient to add two extra assumptions, the strictness of the power concavity of \(f\) and the boundedness of the support of \(g\).

Our result can be applied to the Gauss–Weierstrass integral,

\[ W_g(x, t) = \frac{1}{(4\pi t)^{n/2}} \exp \left( -\frac{|x|^2}{4t} \right) \ast g(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \exp \left( -\frac{|x-y|^2}{4t} \right) g(y) \, dy, \]  

(1.9)

where \(g\) is a bounded measurable function defined on \(\mathbb{R}^n\). It is well known that \(W_g\) satisfies the Cauchy problem for the heat equation

\[ \begin{cases} \left( \frac{\partial}{\partial t} - \Delta \right) W_g(x, t) = 0, & (x, t) \in \mathbb{R}^n \times (0, +\infty), \\ W_g(x, 0^+) = g(x), & x \in \mathbb{R}^n. \end{cases} \]  

(1.10)

Since the Gauss–Weierstrass kernel is strictly 0-concave on \(\mathbb{R}^n\) at any fixed \(t \in (0, +\infty)\), our result implies that, for any 0-concave function \(g\) such that its support is a convex body (compact convex set with non-empty interior), the function \(W_g(\cdot, t) : \mathbb{R}^n \to (0, +\infty)\) is strictly 0-concave on \(\mathbb{R}^n\) at any fixed \(t \in (0, +\infty)\) (Proposition 3.16). We refer to Brascamp...
and Lieb's investigation [3, Sect. 4] for the pioneering work on concavity properties of a solution of a partial differential equation (see also, for example, [11]–[14]).

Our result can also be applied to the Poisson integral for the upper half-space,

\[
P_g(x, t) = \frac{2t}{\sigma_n(S^n)} \left( |r|^2 + r^2 \right)^{-(n+1)/2} g(x)
= \frac{2t}{\sigma_n(S^n)} \int_{\mathbb{R}^n} (|x - y|^2 + r^2)^{-(n+1)/2} g(y) \, dy,
\]

where \(S^n\) denotes the \(n\)-dimensional unit sphere, \(\sigma_n\) denotes the \(n\)-dimensional spherical Lebesgue measure, and \(g\) is a bounded measurable function defined on \(\mathbb{R}^n\). As an analytic property, \(P_g\) satisfies the Cauchy problem for the 1/2-heat equation

\[
\begin{cases}
\quad \left( \frac{\partial}{\partial t} + \sqrt{-\Delta} \right) P_g(x, t) = 0, \ (x, t) \in \mathbb{R}^n \times (0, +\infty), \\
P_g(x, 0^+) = g(x), \quad x \in \mathbb{R}^n,
\end{cases}
\]

which is equivalent to the boundary value problem for the Laplace equation

\[
\begin{cases}
\quad \left( \Delta + \frac{\partial^2}{\partial t^2} \right) P_g(x, t) = 0, \ (x, t) \in \mathbb{R}^n \times (0, +\infty), \\
P_g(x, 0^+) = g(x), \quad x \in \mathbb{R}^n.
\end{cases}
\]

As a geometric property, when \(g\) is the characteristic function \(x_D\) of a body (the closure of a bounded open set) \(D\) in \(\mathbb{R}^n\), \(P_{x_D}(x, t)\) is proportional to the solid angle of \(D\) at \((x, t)\). Namely,

\[
P_{x_D}(x, t) = \frac{2\sigma_n((D \ast (x, t)) \cap (S^n + (x, t))))}{\sigma_n(S^n)}
\]

where \(D \ast (x, t)\) denotes the cone of base \(D\) and vertex \((x, t)\) (see [18, p. 2157]).

It was shown in [19, Proposition 3.7 (1)] that if \(\Omega\) is a convex body in \(\mathbb{R}^n\), then \(P_{x_\Omega}(\cdot, t) : \mathbb{R}^n \to (0, +\infty)\) is strictly \(-1\)-concave on \(\mathbb{R}^n\) at any fixed \(t \in (0, +\infty)\). This fact is generalized by our result since the Poisson kernel is strictly \(-1/(n + 1)\)-concave on \(\mathbb{R}^n\) at any fixed \(t \in (0, +\infty)\). Namely, if \(q \geq 1\), then, for any \(q\)-concave function \(g\) such that its support is a convex body, the function \(P_g(\cdot, t) : \mathbb{R}^n \to (0, +\infty)\) is strictly \(q/(1 - q)\)-concave on \(\mathbb{R}^n\) (Proposition 3.18). We remark that the characteristic function of a convex body is \(+\infty\)-concave on \(\mathbb{R}^n\) and \(q/(1 - q)\) = \(-1\) for \(q = +\infty\).

Since \(W_g\) and \(P_g\) are solutions of evolution equations, it is natural to investigate those concavity properties involving the space and the time variables jointly. In order to investigate such a concavity property, the notion of parabolic power concavity of a function defined on a parabolically convex set in \(\mathbb{R}^n \times (0, +\infty)\) was introduced by Ishige and Salani [11], and exciting concavity properties of solutions of parabolic problems were shown. As in the case of a strictly power concave function on a convex set, the strict parabolic power concavity of a function on a parabolically convex set guarantees the uniqueness of a global maximum point (see Subsections 2.3 and 2.4 for the precise definitions). This is the reason why we are also interested in the strict parabolic power concavity of the convolution in space variable,
$\Gamma(x, t) = \varphi(\cdot, t) \ast \psi(x) = \int_{\mathbb{R}^n} \varphi(x - y, t) \psi(y) \, dy, \ (x, t) \in \mathbb{R}^n \times (0, +\infty), \quad (1.15)$
of two measurable functions $\varphi : \mathbb{R}^n \times (0, +\infty) \to [0, +\infty)$ and $\psi : \mathbb{R}^n \to [0, +\infty)$.

The argument to show the power concavity of (1.1) also works for the parabolic power concavity of $\Gamma$ in (1.15), that is, it is derived from the parabolic power concavity of $\varphi$ and the power concavity of $\psi$ through the BBL-inequality. In the main theorem (Theorem 3.6), we give a sufficient condition for the strict parabolic power concavity of $\Gamma$ in (1.15). Compared to the process of deriving (not necessarily strict) parabolic power concavity, to show the strict parabolic power concavity of $\Gamma$ in (1.15), it is essentially sufficient to add two extra assumptions, the "almost-strictness" of the parabolic power concavity of $\varphi$ and the boundedness of the support of $\psi$. Note that the strict parabolic power concavity of $\Gamma$ in (1.15) implies the strict power concavity of (1.1). To be precise, if $\Gamma$ in (1.15) is parabolically $p$-concave (resp. strictly parabolically $p$-concave), then, at any fixed $t$, the function $\Gamma(\cdot, t) : \mathbb{R}^n \to [0, +\infty)$ is $p$-concave (resp. strictly $p$-concave). Therefore, the strict parabolic power concavity of $\Gamma$ in (1.15) is the most important subject in this paper.

The strict parabolic power concavity of $W_{X\Omega}$ and $P_{X\Omega}$ can be derived from our main theorem, where $\Omega$ is a bounded convex subset of $\mathbb{H}^n$ with non-empty interior. In particular, the strict parabolic power concavity of $P_{X\Omega}$ is the usual strict quasi-concavity on $\mathbb{R}^n \times (0, +\infty)$. Recalling the application to optimization problems and the geometric interpretation of $P_{X\Omega}$, our result states that, at an art museum, when we look at a convex picture $\Omega$ on the wall from an area $E \subset \mathbb{R}^2 \times (0, +\infty)$, if $E$ is compact and convex, then there is a unique point with the maximum viewing solid angle. Thus, we obtain the uniqueness of an optimal solution to a generalization of Regiomontanus’ angle maximization problem. We refer to [15, Sect. 3.1] for this kind of issue.

## 2 Preliminaries

In this section, after setting our notation, we introduce the notions of power concave functions, parabolically convex sets and parabolically power concave functions. We also show some of their fundamental properties.

### 2.1 Notation

For a subset $X$ of $\mathbb{H}^n$, we denote by $\text{int} \, X$, $\text{cl} \, X$ and $\chi_X$ the interior, closure and characteristic function of $X$, respectively. For $x \in \mathbb{R}^n$ and $\rho \in (0, +\infty)$, we denote by $B(x, \rho)$ the open ball centered at $x$ of radius $\rho$. Let $S^{n-1}$ be the boundary of $B(0, 1)$. For $\mu, \nu \in \mathbb{R}$, $X \subset \mathbb{R}^n$, we use the Minkowski addition

$$\mu X + \nu Y = \{ \mu x + \nu y \mid x \in X, \ y \in Y \}. \quad (2.1)$$

In particular, when $Y$ is a singleton $\{ y \}$ in (2.1), we write

$$\mu X + \nu y = \mu X + \nu \{ y \} = \{ \mu x + \nu y \mid x \in X \}. \quad (2.2)$$

For $p \in \mathbb{R} \cup \{ +\infty \}$, $a, b \in [0, +\infty)$ and $\lambda \in [0, 1]$, $M_p(a, b; \lambda)$ is defined in (1.3). The convex combination of two points $x_0$ and $x_1$ of ratio $\lambda \in [0, 1]$ is denoted by
A compact convex set with non-empty interior in $\mathbb{R}^n$ is called a **convex body**. For a convex body $K$ in $\mathbb{R}^n$, we denote by $h_K$ the **support function** of $K$, that is,

$$h_K(u) = \max\{x \cdot u \mid x \in K\}, \quad u \in S^{n-1}. \quad (2.4)$$

Put

$$H^-(h, u) = \{y \in \mathbb{R}^n \mid y \cdot u \leq h\}, \quad (h, u) \in \mathbb{R} \times S^{n-1}. \quad (2.5)$$

### 2.2 Power concave functions

Let us recall the definition of power concavity of a function.

**Definition 2.1** Let $A$ be a convex set in $\mathbb{R}^n$, $f$ a nonnegative function defined on $A$, and $p \in \mathbb{R} \cup \{\pm \infty\}$. $f$ is said to be $p$-**concave** on $A$ if, for any $x_0, x_1 \in A$ and $\lambda \in [0, 1]$, the inequality

$$f(\lambda x_0 + (1 - \lambda) x_1) \geq M_p(f(x_0), f(x_1); \lambda) \quad (2.6)$$

holds. 0-concavity and $-\infty$-concavity are also called **log-concavity** and **quasi-concavity**, respectively. $f$ is said to be strictly $p$-**concave** on $A$ if both of the following conditions hold:

(i) $f$ is $p$-concave on $A$.

(ii) Equality in (2.6) holds if and only if any of the conditions $x_0 = x_1$, $\lambda = 0$ or $\lambda = 1$ holds.

For $p \in \mathbb{R} \cup \{\pm \infty\}$, the compositions of a $p$-concave function with a homothety and with a translation are $p$-concave. To be precise:

**Remark 2.2** Let $A, f$ and $p$ be as in Definition 2.1. Suppose that $f$ is $p$-concave (resp. strictly $p$-concave) on $A$. Then, the following statements hold:

(1) Let $s \in \mathbb{R} \setminus \{0\}$. The function $(1/s)A \ni x \mapsto f(sx)$ is $p$-concave (resp. strictly $p$-concave) on $(1/s)A$.

(2) Let $\xi \in \mathbb{R}^n$. The function $A - \xi \ni x \mapsto f(x + \xi)$ is $p$-concave (resp. strictly $p$-concave) on $A - \xi$.

It directly follows from definition that:

**Remark 2.3** ([3, p. 373]) Let $A$ and $f$ be as in Definition 2.1. $f$ is quasi-concave on $A$ if and only if, for any $a \in [0, +\infty)$, the super-level set $\{x \in A \mid f(x) > a\}$ is convex (or empty).

As mentioned in Introduction, for any $p \in \mathbb{R} \cup \{\pm \infty\}$, $p$-concave functions are quasi-concave. Thus, Remark 2.3 implies:
Remark 2.4 Let $A$, $f$ and $p$ be as in Definition 2.1. If $f$ is $p$-concave on $A$, then, for any $a \in [0, +\infty)$, the super-level set $\{x \in A \mid f(x) > a\}$ is convex (or empty). In particular, if $f$ is $p$-concave on $A$, then $A \setminus f^{-1}(0)$ is convex (or empty).

It follows from the definition of $M_{+\infty}$ that:

**Remark 2.5** ([3, p. 373]) Let $A$ and $f$ be as in Definition 2.1. Then, the following statements hold:

1. Let $\Omega \subset A$ be a convex set, and $c \in [0, +\infty)$. The function $c\chi_{\Omega}$ is $+\infty$-concave on $A$.
2. Let $\Omega = A \setminus f^{-1}(0)$, and fix an arbitrary $\xi \in \Omega$. If $f$ is $+\infty$-concave on $A$, then $f = f(\xi)\chi_{\Omega}$.

By definition, positive power concave functions are described as follows.

**Remark 2.6** Let $A$ and $f$ be as in Definition 2.1. Suppose that $f$ is positive on $A$. Then, the following statements hold:

1. Let $p \in (0, +\infty)$. $f$ is $p$-concave if and only if $f^p : A \ni x \mapsto f(x)^p \in (0, +\infty)$ is concave.
2. $f$ is log-concave if and only if $\log f : A \ni x \mapsto \log f(x) \in \mathbb{R}$ is concave.
3. Let $p \in (-\infty, 0)$. $f$ is $p$-concave if and only if $f^p : A \ni x \mapsto f(x)^p \in (0, +\infty)$ is convex.

Positivity and continuity are fundamental properties of strictly power concave functions.

**Lemma 2.7** Let $A$, $f$ and $p$ be as in Definition 2.1. Suppose that $f$ is $p$-concave on $A$. Then, the following statements hold:

1. Let $p \in \mathbb{R} \cup \{+\infty\}$. If $f$ is positive on $\text{int} A$, then $f$ is continuous on $\text{int} A$.
2. Let $p \in \mathbb{R} \cup \{-\infty\}$. If $f$ is strictly $p$-concave on $\text{int} A$, then $f$ has to be positive on $\text{int} A$.
3. Let $p \in \mathbb{R}$. If $f$ is strictly $p$-concave on $\text{int} A$, then $f$ is continuous on $\text{int} A$.

**Proof**

1. The statement for $p = +\infty$ follows from Remark 2.5. Since $f = (f^p)^{1/p}$ for $p \in \mathbb{R} \setminus \{0\}$ and $f = \exp \log f$, the statement for $p \in \mathbb{R}$ follows from the well-known theorem in convex analysis: any concave/convex function is continuous on the interior of its domain (see [20, Theorem 1.5.3] or [17, Theorem 10.1]).

2. Fix an arbitrary $x \in \text{int} A$. We take a small enough $\varepsilon \in (0, +\infty)$ such that $B(x, \varepsilon) \subset A$. Let $v \in S^{n-1}$. Then, we have

$$f(x) = f\left(\frac{1}{2}\left(x + \frac{\varepsilon}{2}v\right) + \frac{1}{2}\left(x - \frac{\varepsilon}{2}v\right)\right) > M_p\left(f\left(x + \frac{\varepsilon}{2}v\right), f\left(x - \frac{\varepsilon}{2}v\right), \frac{1}{2}\right) \geq 0.$$  

3. follows from (1) and (2).

There exists a discontinuous strictly quasi-concave function.
Lemma 2.8 Let \( k \) be a positive function defined on \([0, +\infty)\). Put \( k^\circ = k(| \cdot |) \). Suppose that \( k \) is strictly decreasing on \([0, +\infty)\). Then, the following statements hold:

1. \( k \) is strictly quasi-concave on \([0, +\infty)\).
2. \( k^\circ \) is strictly quasi-concave on \( \mathbb{R}^n \).

Proof

1. Let \( r_0, r_1 \in [0, +\infty) \), and \( \lambda \in (0, 1) \). Suppose \( r_0 < r_1 \). Then, we have \( r_0 < r_\lambda < r_1 \). Since \( k \) is strictly decreasing, we have \( k(r_\lambda) > k(r_1) = M_{-\infty}(k(r_0), k(r_1); \lambda) \).
2. Let \( x_0, x_1 \in \mathbb{R}^n \), and \( \lambda \in (0, 1) \). Suppose \( x_0 \neq x_1 \). Put \( r_0 = |x_0| \) and \( r_1 = |x_1| \). Since \( k \) is strictly decreasing and \( |x_\lambda| \leq r_\lambda \), we have \( k^\circ(x_\lambda) = k(|x_\lambda|) \geq k(r_\lambda) \). Equality holds if and only if there exists a positive \( s \) such that \( x_0 = sx_1 \). Thus, the strict quasi-concavity of \( k \) shown in (1) completes the proof.

At the end of this subsection, let us review the precise statement of the BBL-inequality with our notation. See also [1, Theorem 3.1], [4, Theorem 3.1] and [7, Theorem 10.1].

Theorem 2.9 [3, Theorem 3.3] Let \( f_0 \) and \( f_1 \) be nonnegative integrable functions defined on \( \mathbb{R}^n \). Suppose that the \( L^1 \)-norms of \( f_0 \) and \( f_1 \) are both positive. Let \( \ell' \in [-1/n, +\infty) \), and

\[
S(y) = \text{ess sup} \left \{ M_{\ell'}(f_0(y_0), f_1(y_1); \lambda) \ \middle | \ (y_0, y_1) \in \mathbb{R}^n \times \mathbb{R}^n, y_\lambda = y \right \}, \ y \in \mathbb{R}^n.
\]

Then, we have

\[
\int_{\mathbb{R}^n} S(y) \ dy \geq M_{\ell'/(1+n\ell')} \left( \int_{\mathbb{R}^n} f_0(y) \ dy, \int_{\mathbb{R}^n} f_1(y) \ dy; \lambda \right).
\]

2.3 Parabolically convex sets

The notion of \( \alpha \)-parabolic convexity of a subset of \( \mathbb{R}^n \times (0, +\infty) \) was introduced in [10]. It is an extension of the usual parabolic convexity introduced in [2]. We show some basic properties of parabolically convex sets.

Definition 2.10 ([10, Definition 3.5]) Let \( E \) be a subset of \( \mathbb{R}^n \times (0, +\infty) \), and \( \alpha \in \mathbb{R} \). \( E \) is said to be \( \alpha \)-parabolically convex if, for any \( (x_0, t_0), (x_1, t_1) \in E \) and \( \lambda \in [0, 1) \), \( (x_\lambda, M_\alpha(t_0, t_1; \lambda)) \in E \) holds.

We remark that the original parabolic convexity [2] corresponds to the case where \( \alpha = 1/2 \).

Example 2.11 Let \( 0 \leq a < b \leq +\infty \), \( I = (a, b) \), and \( \alpha \in \mathbb{R} \). Put
\[
E = \begin{cases} 
\{(x, t) | |x| < t^\alpha, \ t \in I\} & (\alpha \neq 0), \\
\{(x, t) | |x| < \log t, \ t \in I\} & (\alpha = 0).
\end{cases}
\]

Then, \(E\) is \(\alpha\)-parabolically convex.

We show that convex sets in \(\mathbb{R}^n\) can generate parabolically convex sets in \(\mathbb{R}^n \times (0, +\infty)\).

**Proposition 2.12** Let \(A\) be a convex set in \(\mathbb{R}^n\), and \(\alpha \in \mathbb{R} \setminus \{0\}\). Put

\[
\widehat{A}_\alpha = \left\{ (x, t) \in \mathbb{R}^n \times (0, +\infty) \left| \frac{x}{t^\alpha} \in A \right. \right\}.
\]

Then, \(\widehat{A}_\alpha\) is \(\alpha\)-parabolically convex.

The proof is directly completed by the convex combination

\[
\frac{x_j}{M_\alpha(t_0, t_1; \lambda)^\alpha} = \frac{(1 - \lambda)t_0^\alpha}{M_\alpha(t_0, t_1; \lambda)^\alpha t_0^\alpha} \frac{x_0}{t_0^\alpha} + \frac{\lambda t_1^\alpha}{M_\alpha(t_0, t_1; \lambda)^\alpha t_1^\alpha} \frac{x_1}{t_1^\alpha},
\]

(2.7)

This is sometimes used throughout this paper.

When we connect the two cases where \(\alpha \neq 0\) and where \(\alpha = 0\), we use the following relations:

\[
\log M_0(t_0, t_1; \lambda) = M_1(\log t_0, \log t_1; \lambda), \quad (t_0, t_1, \lambda) \in (1, +\infty) \times (1, +\infty) \times [0, 1]; 
\]

(2.8)

\[
\exp M_1(t_0, t_1; \lambda) = M_0(e^{t_0}, e^{t_1}; \lambda), \quad (t_0, t_1, \lambda) \in (0, +\infty) \times (0, +\infty) \times [0, 1].
\]

(2.9)

**Corollary 2.13** Let \(A\) be a convex set in \(\mathbb{R}^n\). Put

\[
\widehat{A}_0 = \left\{ (x, t) \in \mathbb{R}^n \times (1, +\infty) \left| \frac{x}{\log t} \in A \right. \right\}.
\]

Then, \(\widehat{A}_0\) is 0-parabolically convex.

**Proof** Let \(\widehat{A}_1\) be as in Proposition 2.12 with \(\alpha = 1\). We remark

\[
\widehat{A}_0 = \left\{ (x, t) \in \mathbb{R}^n \times (1, +\infty) \left| (x, \log t) \in \widehat{A}_1 \right. \right\}.
\]

Thanks to the relation (2.8), Proposition 2.12 with \(\alpha = 1\) completes the proof.

\(\widehat{A}_\alpha\) in Proposition 2.12 or in Corollary 2.13 is concretely given when \(A\) is a *convex cone*, that is, \(A\) additionally has the property that, for any \((x, s) \in \mathbb{A} \times (0, +\infty), sx \in A\) holds.

**Proposition 2.14** Let \(A, \alpha\) and \(\widehat{A}_\alpha\) be as in Proposition 2.12. \(A\) is a convex cone if and only if \(\widehat{A}_\alpha = A \times (0, +\infty)\).
Proof Suppose that $A$ is a convex cone. Let $(x, t) \in \hat{A}_\alpha$. By the definition, we have $x/t^\alpha \in A$. Since $A$ is a convex cone, we have $x = t^\alpha(x/t^\alpha) \in A$. Thus, $(x, t) \in A \times (0, +\infty)$. On the other hand, let $(x, t) \in A \times (0, +\infty)$. Since $A$ is a convex cone, we have $x/t^\alpha \in A$, that is, $(x, t) \in \hat{A}_\alpha$.

Suppose $\hat{A}_\alpha = A \times (0, +\infty)$. Let $(x, s) \in A \times (0, +\infty)$. Since $(x, s^{-1/\alpha}) \in A \times (0, +\infty) = \hat{A}_\alpha$, we obtain $sx = x/(s^{-1/\alpha})^\alpha \in A$. □

Remark 2.15 Let $A$ and $\hat{A}_0$ be as in Corollary 2.13. Let $\hat{A}_1$ be as in Proposition 2.12 with $\alpha = 1$. Then, the following statements hold:

1. $\hat{A}_0 = \{(x, t) \in \mathbb{R}^n \times (1, +\infty) \mid (x, \log t) \in \hat{A}_1\}$ (which was mentioned in the proof of Corollary 2.13).
2. $\hat{A}_1 = \{(x, t) \in \mathbb{R}^n \times (0, +\infty) \mid (x, e^t) \in \hat{A}_0\}$.
3. $\hat{A}_0 = A \times (1, +\infty)$ if and only if $\hat{A}_1 = A \times (0, +\infty)$.

Corollary 2.16 Let $A$ and $\hat{A}_0$ be as in Corollary 2.13. $A$ is a convex cone if and only if $\hat{A}_0 = A \times (1, +\infty)$.

Proof Thanks to Remark 2.15 (3), Proposition 2.14 with $\alpha = 1$ completes the proof. □

Conversely, parabolically convex sets in $\mathbb{R}^n \times (0, +\infty)$ naturally generate convex sets in $\mathbb{R}^n$ since $t = M_\alpha(t, t; \lambda)$.

Remark 2.17 Let $E$ be a subset of $\mathbb{R}^n \times (0, +\infty)$, and $\alpha \in \mathbb{R}$. Put

$$\hat{E}(t) = \{x \in \mathbb{R}^n \mid (x, t) \in E\}, \quad t \in (0, +\infty).$$

If $E$ is $\alpha$-parabolically convex, then, for each $t \in (0, +\infty)$, $\hat{E}(t)$ is convex (or empty).

For each $\alpha \in \mathbb{R}$, $\alpha$-parabolically convex sets have the same basic properties as in [2, Sections 1 and 2] (which corresponds to the case where $\alpha = 1/2$). The properties are not used for the proof of our main theorem, but we show them here, which might be of help in understanding the shape of an $\alpha$-parabolically convex set. The proofs are slightly different from [2].

Remark 2.18 Let $E$ be a subset of $\mathbb{R}^n \times (0, +\infty)$, and $\alpha \in \mathbb{R}$. $E$ is $\alpha$-parabolically convex if and only if, for any $(x_0, t_0), (x_1, t_1) \in E$, both of the following two hold:

(i) If $t_0 \neq t_1$, then, for any $\theta \in [0, 1]$,

$$E \ni \begin{cases} \left(\frac{\alpha^\alpha - \alpha^\theta}{\alpha^\theta - 0} x_0 + \frac{\alpha^\alpha - \alpha^\theta}{\alpha^\theta - 0} x_1, t_\theta\right) & (\alpha \neq 0), \\
\left(\log t_1 - \log t_0 x_0 + \log t_1 - \log t_0 x_1, t_\theta\right) & (\alpha = 0). \end{cases}$$

(ii) If $t_0 = t_1$, then, for any $\theta \in [0, 1], (x_\theta, t_0) \in E$.

Proposition 2.19 Let $E$ be a subset of $\mathbb{R}^n \times (0, +\infty)$, and $\alpha \in \mathbb{R} \setminus \{0\}$. Put
\[ \mathcal{E}_a(s;E) = \left\{ y \left| \left( \frac{y}{s}, s^{-1/a} \right) \in E \right. \right\}, \ s \in (0, +\infty); \]
\[ \omega_a(x,t) = \left( \frac{x}{t^a}, \frac{1}{t^a} \right), \ (x,t) \in \mathbb{R}^n \times (0, +\infty). \]

The following statements are equivalent:

(i) \( E \) is \( a \)-parabolically convex.
(ii) For any \( s_0, s_1 \in (0, +\infty) \) and \( \theta \in [0, 1] \), \( (1 - \theta)\mathcal{E}_a(s_0; E) + \theta\mathcal{E}_a(s_1; E) \subseteq \mathcal{E}_a(s; E) \) holds.
(iii) \( \omega_a(E) \) is convex.

**Proof** (ii) \( \Longleftrightarrow \) (iii) follows from
\[ \omega_a(E) = \left\{ (y,s) \in \mathbb{R}^n \times (0, +\infty) \mid y \in \mathcal{E}_a(s; E) \right\}. \]

(i) \( \Longleftrightarrow \) (ii): Let \( y_0 \in \mathcal{E}_a(s_0; E) \), \( y_1 \in \mathcal{E}_a(s_1; E) \), and \( \theta \in [0, 1] \). Put \( \lambda = \theta s_1/s_0 \in [0, 1] \). Since we have \( (y_0/s_0, s_0^{-1/a}) \in E \) and \( (y_1/s_1, s_1^{-1/a}) \in E \), we obtain
\[ \left( \frac{y_\theta}{s_\theta}, s_\theta^{-1/a} \right) = \left( (1 - \lambda) \frac{y_0}{s_0} + \lambda \frac{y_1}{s_1}, M_a \left( s_0^{-1/a}, s_1^{-1/a}; \lambda \right) \right) \in E. \]

(ii) \( \Longleftrightarrow \) (i): Let \( (x_0, t_0), (x_1, t_1) \in E \), and \( \lambda \in [0, 1] \). Put \( \theta = \lambda t_1^a/M_a(t_0, t_1; \lambda)^a \in [0, 1] \). Since we have \( t_0^{-a}x_0 \in \mathcal{E}_a(t_0^{-a}; E) \) and \( t_1^{-a}x_1 \in \mathcal{E}_a(t_1^{-a}; E) \), we have \( (1 - \theta)t_0^{-a}x_0 + \theta t_1^{-a}x_1 \in \mathcal{E}_a((1 - \theta)t_0^{-a} + \theta t_1^{-a}, E) \). Hence, we obtain
\[ (x_\lambda, M_a(t_0, t_1; \lambda)) = \left( \frac{1 - \theta) t_0^{-a}x_0 + \theta t_1^{-a}x_1}{(1 - \theta) t_0^{-a} + \theta t_1^{-a}}, \left( (1 - \theta)t_0^{-a} + \theta t_1^{-a} \right)^{-1/a} \right) \in E. \]

\( \square \)

**Remark 2.20** Let \( E \) be a subset of \( \mathbb{R}^n \times (1, +\infty) \), and \( \widetilde{E} = \{ (x, \log t) \mid (x,t) \in E \} \). Put
\[ \mathcal{E}_0(s; E) = \left\{ y \left| \left( \frac{y}{s}, e^{1/s} \right) \in E \right. \right\}, \ s \in (0, +\infty); \]
\[ \omega_0(x,t) = \left( \frac{x}{\log t}, \frac{1}{\log t} \right), \ (x,t) \in \mathbb{R}^n \times (1, +\infty). \]

Let \( \mathcal{E}_1 \) and \( \omega_1 \) be as in Proposition 2.19 with \( \alpha = 1 \). Then, the following statements hold:

(1) \( E \) is 0-parabolically convex if and only if \( \widetilde{E} \) is 1-parabolically convex.
(2) \( \mathcal{E}_0(s; E) = \mathcal{E}_1(s; \widetilde{E}) \) for any \( s \in (0, +\infty) \).
(3) \( \omega_0(E) = \omega_1(\widetilde{E}) \).

**Corollary 2.21** Let \( E, \mathcal{E}_0 \) and \( \omega_0 \) be as in Remark 2.20. The following statements are equivalent:

[Springer]
(i) $E$ is 0-parabolically convex.
(ii) For any $s_0, s_1 \in (0, +\infty)$ and $\theta \in [0, 1]$, $(1 - \theta)E_0(s_0; E) + \theta E_0(s_1; E) \subset E_0(s_0; E)$ holds.
(iii) $\omega_0(E)$ is convex.

**Proof** Thanks to Remark 2.20, Proposition 2.19 with $\alpha = 1$ completes the proof. \(\square\)

The set $\tilde{E}$ in Remark 2.20 is concretely given when $E$ is a convex cylinder.

**Remark 2.22** Let $A$ be a convex subset of $\mathbb{R}^n$, and $I$ an interval in $(1, +\infty)$. Let $\tilde{\cdot}$ be the operator as in Remark 2.20. Then, $\tilde{A} \times I = A \times \log I$.

**Proposition 2.23** Let $A$ be a subset of $\mathbb{R}^n$, $I$ an interval in $(0, +\infty)$, and $\alpha \in \mathbb{R}$. $A \times I$ is $\alpha$-parabolically convex if and only if $A$ is convex.

**Proof** The “only if” part follows from Remark 2.17. The “if” part follows from definition. \(\square\)

### 2.4 Parabolically power concave functions

The notion of *parabolic power concavity* of a function was introduced in [11] (see also [10]). In this subsection, we slightly extend the notion and show several basic properties of parabolically power concave functions.

**Definition 2.24** Let $\alpha \in \mathbb{R}$, $E$ an $\alpha$-parabolically convex set in $\mathbb{R}^n \times (0, +\infty)$, $\varphi$ a nonnegative function defined on $E$, and $p \in \mathbb{R} \cup \{\pm \infty\}$. $\varphi$ is said to be $\alpha$-parabolically $p$-concave on $E$ if, for any $(x_0, t_0), (x_1, t_1) \in E$ and $\lambda \in [0, 1]$, the inequality

$$\varphi(x_\lambda, M_\alpha(t_0, t_1; \lambda)) \geq M_p(\varphi(x_0, t_0), \varphi(x_1, t_1); \lambda)$$

holds. $\varphi$ is said to be *strictly* $\alpha$-parabolically $p$-concave on $E$ if both of the following conditions hold:

(i) $\varphi$ is $\alpha$-parabolically $p$-concave on $E$.
(ii) Equality in (2.10) holds if and only if any of the conditions $(x_0, t_0) = (x_1, t_0)$, $\lambda = 0$ or $\lambda = 1$ holds.

When $\alpha \neq 0$, $\varphi$ is said to be *almost-strictly* $\alpha$-parabolically $p$-concave on $E$ if both of the following conditions hold:

(i) $\varphi$ is $\alpha$-parabolically $p$-concave on $E$.
(ii) Equality in (2.10) holds if and only if any of the conditions $x_0/t_0^\alpha = x_1/t_1^\alpha$, $\lambda = 0$ or $\lambda = 1$ holds.

When $E \subset \mathbb{R}^n \times (1, +\infty)$, $\varphi$ is said to be *almost-strictly* 0-parabolically $p$-concave on $E$ if both of the following conditions hold:
(i) \( \varphi \) is \( \alpha \)-parabolically \( p \)-concave on \( E \).

(ii) Equality in (2.10) holds if and only if any of the conditions \( x_0 / \log t_0 = x_1 / \log t_1 \), \( \lambda = 0 \) or \( \lambda = 1 \) holds.

Similarly to Definition 2.1, \( \alpha \)-parabolic 0-concavity and \( \alpha \)-parabolic \( -\infty \)-concavity are also called \( \alpha \)-parabolic log-concavity and \( \alpha \)-parabolic quasi-concavity, respectively.

For \( \alpha \in \mathbb{R} \) and \( p \in \mathbb{R} \cup \{\pm \infty\} \), the composition of an \( \alpha \)-parabolically \( p \)-concave function with a homothety is \( \alpha \)-parabolically \( p \)-concave. To be precise:

**Remark 2.25** Let \( \alpha, E, \varphi \) and \( p \) be as in Definition 2.24. Let \( s \in \mathbb{R} \setminus \{0\} \), and \( \tau \in (0, +\infty) \). Put \( E_{s, \tau} = \{(x, t) \in \mathbb{R}^n \times (0, +\infty) \mid (sx, \tau t) \in E\} \). Then, the following statements hold:

1. If \( \varphi \) is \( \alpha \)-parabolically \( p \)-concave (resp. strictly \( \alpha \)-parabolically \( p \)-concave) on \( E \), then the function \( E_{s, \tau} \ni (x, t) \mapsto \varphi(sx, \tau t) \) is \( \alpha \)-parabolically \( p \)-concave (resp. strictly \( \alpha \)-parabolically \( p \)-concave) on \( E_{s, \tau} \).

2. If \( \alpha \neq 0 \) and \( \varphi \) is almost-strictly \( \alpha \)-parabolically \( p \)-concave on \( E \), then the function \( E_{s, \tau} \ni (x, t) \mapsto \varphi(sx, \tau t) \) is almost-strictly \( \alpha \)-parabolically \( p \)-concave on \( E_{s, \tau} \).

As we see in the next proposition, 0-parabolically \( p \)-concave functions defined on a 0-parabolically convex set generate 1-parabolically \( p \)-concave functions defined on a 1-parabolically convex set, and vice versa.

**Proposition 2.26** Let \( E \) be a 0-parabolically convex subset of \( \mathbb{R}^n \times (1, +\infty) \), \( \varphi \) a nonnegative function defined on \( E \), and \( p \in \mathbb{R} \cup \{\pm \infty\} \). Let \( \tilde{E} \) be as in Remark 2.20. Put

\[
\tilde{\varphi}(x, t) = \varphi(x, e^t) \quad (x, t) \in \tilde{E}.
\]

Then, \( \varphi \) is 0-parabolically \( p \)-concave (resp. almost-strictly/strictly 0-parabolically \( p \)-concave) on \( E \) if and only if \( \tilde{\varphi} \) is 1-parabolically \( p \)-concave (resp. almost-strictly/strictly 1-parabolically \( p \)-concave) on \( \tilde{E} \).

**Proof** By Remark 2.20 (1), \( \tilde{E} \) is 1-parabolically convex. The relations (2.8) and (2.9) complete the proof.

We show that \( p \)-concave functions can generate \( \alpha \)-parabolically \( p \)-concave functions.

**Proposition 2.27** Let \( A \) be a convex set in \( \mathbb{R}^n \), \( f \) a nonnegative function defined on \( A \), \( \alpha \in \mathbb{R} \setminus \{0\} \), and \( p \in \mathbb{R} \cup \{\pm \infty\} \). Let \( \tilde{A}_\alpha \) be as in Proposition 2.12. Put

\[
\tilde{f}_{p, \alpha}(x, t) = \begin{cases} 
\frac{r^\alpha}{\alpha!} f \left( \frac{x}{r^\alpha} \right) (p \neq 0), \\
\exp \left( t^\alpha \log f \left( \frac{x}{t^\alpha} \right) \right) (p = 0), 
\end{cases} 
(x, t) \in \tilde{A}_\alpha.
\]

If \( f \) is \( p \)-concave (resp. strictly \( p \)-concave) on \( A \), then \( \tilde{f}_{p, \alpha} \) is \( \alpha \)-parabolically \( p \)-concave (resp. almost-strictly \( \alpha \)-parabolically \( p \)-concave) on \( \tilde{A}_\alpha \).

**Proof** We give a proof for the case where \( p \neq 0 \). The argument in the case where \( p = 0 \) goes parallel.
Let \((x_0, t_0), (x_1, t_1) \in \hat{A}_a\), and \(\lambda \in [0, 1]\). Using the convex combination \((2.7)\), the \(p\)-concavity of \(f\) implies
\[
\hat{f}_{p,a}(x, M_a(t_0, t_1; \lambda)) = M_a(t_0, t_1; \lambda)^{a/p} f\left(\frac{(1 - \lambda)t_0^a}{M_a(t_0, t_1; \lambda)^a} x_0 + \frac{\lambda t_1^a}{M_a(t_0, t_1; \lambda)^a} x_1\right) \\
\geq M_a(t_0, t_1; \lambda)^{a/p} M_p\left(f\left(\frac{x_0}{t_0^a}\right), f\left(\frac{x_1}{t_1^a}\right)\right) \frac{\lambda t_1^a}{M_a(t_0, t_1; \lambda)^a} \\
= M_p\left(\hat{f}_{p,a}(x_0, t_0), \hat{f}_{p,a}(x_1, t_1); \lambda\right).
\]
\[\square\]

**Corollary 2.28** Let \(A, f\) and \(p\) be as in Proposition 2.27. Let \(\hat{f}_{p,1}\) be as in Proposition 2.27 with \(a = 1\). Let \(\hat{A}_0\) be as in Corollary 2.13. Put
\[
\hat{f}_{p,0}(x, t) = \hat{f}_{p,1}(x, \log t), \ (x, t) \in \hat{A}_0.
\]
If \(f\) is \(p\)-concave (resp. strictly \(p\)-concave) on \(A\), then \(\hat{f}_{p,0}\) is \(0\)-parabolically \(p\)-concave (resp. almost-strictly \(0\)-parabolically \(p\)-concave) on \(\hat{A}_0\).

**Proof** Propositions 2.27 with \(a = 1\) and 2.26 complete the proof (see also Remark 2.15).
\[\square\]

We show that Proposition 2.27 constructs radially symmetric parabolically power concave functions.

**Proposition 2.29** Let \(\kappa\) be a nonnegative function defined on \([0, +\infty) \times (0, +\infty)\), \(\alpha \in \mathbb{R} \setminus \{0\}\), \(p \in \mathbb{R} \cup \{\pm \infty\}\), and \(\tau \in (0, +\infty)\). We consider the following conditions for \(\kappa\):

(i) For any \((r, t) \in [0, +\infty) \times (0, +\infty)\), we have
\[
\kappa(r, t) = \begin{cases} 
  t^{\alpha/p} \kappa\left(\frac{r}{t^\alpha}, \tau\right) & (p \neq 0), \\
  \exp\left(t^\alpha \log \kappa\left(\frac{r}{t^\alpha}, \tau\right)\right) & (p = 0).
\end{cases}
\]

(ii) \(\kappa(\cdot, \tau)\) is \(p\)-concave (resp. strictly \(p\)-concave) on \([0, +\infty)\).

(iii) For each \(t \in (0, +\infty)\), \(\kappa(\cdot, t)\) is decreasing (resp. strictly decreasing) on \([0, +\infty)\).

Put
\[\kappa^\circ(x, t) = \kappa(|x|, t), \ (x, t) \in \mathbb{R}^n \times (0, +\infty).
\]

Then, the following statements hold:
(1) If (i) and (ii) are satisfied, then \( \kappa \) is \( \alpha \)-parabolically \( p \)-concave (resp. almost-strictly \( \alpha \)-parabolically \( p \)-concave) on \([0, +\infty) \times (0, +\infty)\).

(2) If (i), (ii) and (iii) are satisfied, then \( \kappa^{\circ} \) is \( \alpha \)-parabolically \( p \)-concave (resp. almost-strictly \( \alpha \)-parabolically \( p \)-concave) on \( \mathbb{R}^n \times (0, +\infty) \).

**Proof**

(1) In Proposition 2.27, put \( n = 1 \), \( A = [0, +\infty) \), and \( f = \kappa(\cdot, \tau) \). Since \( A \) is a convex cone in \( \mathbb{R} \), by Proposition 2.14, we have \( \hat{A}_a = A \times (0, +\infty) \). Thus, Proposition 2.27 completes the proof.

(2) Let \((x_0, t_0), (x_1, t_1) \in \mathbb{R}^n \times (0, +\infty)\), and \( \lambda \in [0, 1] \). Put \( r_0 = |x_0| \) and \( r_1 = |x_1| \). By the condition (iii) and \( |x_\lambda| \leq r_\lambda \), we have

\[
\kappa^{\circ}(x_\lambda, M_a(t_0, t_1; \lambda)) = \kappa(|x_\lambda|, M_a(t_0, t_1; \lambda)) \geq \kappa(r_\lambda, M_a(t_0, t_1; \lambda)).
\]

When the condition (iii) is strictly satisfied, equality holds if and only if there exists a positive \( s \) such that \( x_0 = sx_1 \). Thus, the \( \alpha \)-parabolic \( p \)-concavity of \( \kappa \) shown in (1) completes the proof.

**Corollary 2.30** Let \( \kappa, p, \tau \) and \( \circ \) be as in Proposition 2.29. Put

\[
\kappa_0(r, t) = \kappa(r, \log t), \quad (r, t) \in [0, +\infty) \times (1, +\infty).
\]

Then, the following statements hold:

(1) If (i) and (ii) in Proposition 2.29 with \( \alpha = 1 \) are satisfied, then \( \kappa_0 \) is 0-parabolically \( p \)-concave (resp. almost-strictly 0-parabolically \( p \)-concave) on \([0, +\infty) \times (1, +\infty)\).

(2) If (i), (ii) and (iii) in Proposition 2.29 with \( \alpha = 1 \) are satisfied, then \( \kappa_0^{\circ} \) is 0-parabolically \( p \)-concave (resp. almost-strictly 0-parabolically \( p \)-concave) on \( \mathbb{R}^n \times (1, +\infty) \).

**Proof** Propositions 2.29 with \( \alpha = 1 \) and 2.26 complete the proof (see also Remarks 2.15 and 2.22).

Conversely, \( \alpha \)-parabolically \( p \)-concave functions naturally generate \( p \)-concave functions since \( \tau = M_a(\tau, r; \lambda) \).

**Remark 2.31** Let \( \alpha \in \mathbb{R} \), \( E \) an \( \alpha \)-parabolically convex set in \( \mathbb{R}^n \times (0, +\infty) \), \( \varphi \) a nonnegative function defined on \( E \), \( p \in \mathbb{R} \cup \{ \pm \infty \} \), and \( \tau \in (0, +\infty) \). Let \( \tilde{E} \) be as in Remark 2.17. Suppose \( \tilde{E}(\tau) \neq \emptyset \). Put

\[
\tilde{\varphi}_\tau(x) = \varphi(x, \tau), \quad x \in \tilde{E}(\tau).
\]

If \( \varphi \) is \( \alpha \)-parabolically \( p \)-concave (resp. strictly/almost-strictly \( \alpha \)-parabolically \( p \)-concave) on \( E \), then \( \tilde{\varphi}_\tau \) is \( p \)-concave (resp. strictly \( p \)-concave) on \( \tilde{E}(\tau) \).
3 Main theorem and its applications

3.1 Lemmas for the main theorem

Lemma 3.1 Let $I$ be an interval in $(0, +\infty)$, and $\varphi$ a nonnegative function defined on $\mathbb{R}^n \times I$, $\alpha \in \mathbb{R}$, and $p \in \mathbb{R} \cup \{\pm \infty\}$. Put

$$
\Phi(x, y, t) = \varphi(x - y, t), \quad (x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times I.
$$

If $\varphi$ is $\alpha$-parabolically $p$-concave on $\mathbb{R}^n \times I$, then $\Phi$ is $\alpha$-parabolically $p$-concave on $\mathbb{R}^n \times \mathbb{R}^n \times I$.

Proof Let $(x_0, y_0, t_0), (x_1, y_1, t_1) \in \mathbb{R}^n \times \mathbb{R}^n \times I$, and $\lambda \in [0, 1]$. Since $\varphi$ is $\alpha$-parabolically $p$-concave on $\mathbb{R}^n \times I$, we have

$$
\Phi(x_1, y_1, M_{\alpha}(t_0, t_1; \lambda)) = \varphi((1 - \lambda)(x_0 - y_0) + \lambda(x_1 - y_1), M_{\alpha}(t_0, t_1; \lambda))
$$

$$
\geq M_p(\varphi(x_0 - y_0, t_0), \varphi(x_1 - y_1, t_1); \lambda)
$$

$$
= M_p(\Phi(x_0, y_0, t_0), \Phi(x_1, y_1, t_1); \lambda).
$$

\[\square\]

Remark 3.2 Let $I$, $\varphi$, $\alpha$, $p$ and $\Phi$ be as in Lemma 3.1. Suppose that $\varphi$ is almost-strictly $\alpha$-parabolically $p$-concave on $\mathbb{R}^n \times I$. Then, the following statements hold:

(1) We have

$$
\Phi(x_1, y_1, M_{\alpha}(t_0, t_1; \lambda)) = M_p(\Phi(x_0, y_0, t_0), \Phi(x_1, y_1, t_1); \lambda)
$$

$$
\iff \begin{cases}
\frac{x_0 - y_0}{t_0} = \frac{x_1 - y_1}{t_1} \quad (\alpha \neq 0), \\
\frac{x_0 - y_0}{\log t_0} = \frac{x_1 - y_1}{\log t_1} \quad (\alpha = 0).
\end{cases}
$$

(2) For each $(x, t) \in \mathbb{R}^n \times I$, $\Phi(x, \cdot, t)$ is strictly $p$-concave on $\mathbb{R}^n$ (see also Remarks 2.2 and 2.31).

Lemma 3.3 Let $\Phi_0, \Phi_1 \in [0, +\infty)$, and $\psi$ a nonnegative function defined on $\mathbb{R}^n$. Let $p$ and $q \in \mathbb{R} \cup \{\pm \infty\}$. Let $\ell$ be as in (1.6). Suppose that $\psi$ is $q$-concave on $\mathbb{R}^n$, and that $p + q \geq 0$. Then, for any $y_0, y_1 \in \mathbb{R}^n$ and $\lambda \in [0, 1]$, we have

$$
M_p(\Phi_0, \Phi_1; \lambda)\psi(y_\lambda) \geq M_\ell(\Phi_0 \psi(y_0), \Phi_1 \psi(y_1); \lambda).
$$

Proof (1.5) with $(a, b, c, d) = (\Phi_0, \Phi_1, \psi(y_0), \psi(y_1))$ completes the proof. \[\square\]

Remark 3.4 Let $\Omega$ be a convex set in $\mathbb{R}^n$ with non-empty interior, and $x \in \text{cl} \Omega$. Then, $x \in \text{cl} \text{ int} \Omega$. This follows from [20, Theorem 1.1.15 (b)].
Proposition 3.5 Let \( \Omega \) be a bounded convex set in \( \mathbb{R}^n \) with non-empty interior, \( K = \text{cl} \Omega \), \( s \in (0, 1) \), \( \mu \in [0, +\infty) \), and \( v \in S^{n-1} \). Suppose \((s, \mu) \neq (1, 0)\). Then, \( \Omega \setminus (sK - \mu v) \) has an interior point.

Proof Let us first show the statement under the assumption \( h_K(v) > h_{sK-\mu v}(v) \). Since we have \( sK - \mu v \subset H^{-}(h_{sK-\mu v}(v), v) \), it is sufficient to show that \( \Omega \setminus H^{-}(h_{sK-\mu v}(v), v) \) has an interior point.

Let \( x \in K \) be such that \( x \cdot v = h_K(v) \). By Remark 3.4, we have

\[
B\left( x, \frac{h_K(v) - h_{sK-\mu v}(v)}{2} \right) \cap \text{int} \Omega \neq \emptyset.
\]

We take a point \( y \) from the above intersection. We remark

\[
y \cdot v = x \cdot v + (y - x) \cdot v \geq h_K(v) - |y - x| > h_K(v) - \frac{h_K(v) - h_{sK-\mu v}(v)}{2} > h_{sK-\mu v}(v),
\]

that is, \( y \not\in H^{-}(h_{sK-\mu v}(v), v) \). Let us show that \( y \) is an interior point of \( \Omega \setminus H^{-}(h_{sK-\mu v}(v), v) \).

Since \( y \in \text{int} \Omega \), there exists a positive \( \delta \) such that \( B(y, \delta) \subset \Omega \). Let

\[
\varepsilon = \min \left\{ \frac{h_K(v) - h_{sK-\mu v}(v)}{2}, \delta \right\} > 0.
\]

Fix an arbitrary \( z \in B(y, \varepsilon) \). By the definition of \( \varepsilon \), we have \( z \in \Omega \). Since we have

\[
|z - x| \leq |z - y| + |y - x| < \varepsilon + \frac{h_K(v) - h_{sK-\mu v}(v)}{2} \leq h_K(v) - h_{sK-\mu v}(v),
\]

we have

\[
z \cdot v = x \cdot v + (z - x) \cdot v \geq h_K(v) - |z - x| > h_{sK-\mu v}(v).
\]

Thus, \( z \not\in H^{-}(h_{sK-\mu v}(v), v) \).

Next, we show the statement under the assumption \( h_K(v) \leq h_{sK-\mu v}(v) \). Since \( h_{sK-\mu v}(v) = sh_K(v) - \mu \), the assumption implies \( \mu \leq (s - 1)h_K(v) \) and \( s < 1 \). Since we have \( sK - \mu v \subset H^{-}(sK-\mu v, -v), -v \), it is sufficient to show that \( \Omega \setminus H^{-}(sK-\mu v, -v), -v \) has an interior point.

Since \( \Omega \) has an interior point, we have the positivity of the width of \( K \), that is, \( h_K(v) + h_K(-v) > 0 \). Thus, we obtain

\[
h_K(-v) - h_{sK-\mu v}(-v) = h_K(-v) - (sh_K(-v) + \mu) \geq (1 - s)(h_K(v) + h_K(-v)) > 0.
\]

Let \( x \in K \) be such that \( h_K(-v) = x \cdot (-v) \). By Remark 3.4, we have

\[
B\left( x, \frac{h_K(-v) - h_{sK-\mu v}(-v)}{2} \right) \cap \text{int} \Omega \neq \emptyset.
\]

We take a point \( y \) from the above intersection. In the same manner as above, it is shown that \( y \) is an interior point of \( \Omega \setminus (sK - \mu v) \). \( \square \)
3.2 Main theorem

Theorem 3.6 Let I be an interval in \((0, +\infty)\), \(\varphi\) a nonnegative measurable function defined on \(\mathbb{R}^n \times I\), \(\psi\) a nonnegative measurable function defined on \(\mathbb{R}^n\), \(\alpha \in \mathbb{R}\), \(p \in \mathbb{R}\), and \(q \in \mathbb{R} \cup \{+\infty\}\). Let \(\ell^0\) be as in (1.6). Assume that the following conditions are satisfied:

(i) \(\varphi\) is almost-strictly \(\alpha\)-parabolically \(p\)-concave on \(\mathbb{R}^n \times I\).
(ii) \(\psi\) is \(q\)-concave on \(\mathbb{R}^n\).
(iii) \(\mathbb{R}^n \setminus \psi^{-1}(0)\) is bounded, and its interior is not empty.
(iv) \(p + q \geq 0\) and \(\ell^0 \geq -1/n\).

Then, the function
\[
\Gamma(x, t) = \varphi(x, t) \ast \psi(x) = \int_{\mathbb{R}^n} \varphi(x - y, t)\psi(y) \, dy, \quad (x, t) \in \mathbb{R}^n \times I,
\]
is strictly \(\alpha\)-parabolically \(\ell^0/(1 + n\ell^0)\)-concave on \(\mathbb{R}^n \times I\).

Lemma 3.7 If Theorem 3.6 is true for \(\alpha = 1\), then it is true for \(\alpha = 0\).

Proof We assume \(I \subset (1, +\infty)\) when we discuss almost-strict 0-parabolic power concavity of a function (see Definition 2.24). By Remark 2.22, \(\mathbb{R}^n \times I = \mathbb{R}^n \log I \subset \mathbb{R}^n \times (0, +\infty)\). Let \(\widetilde{\varphi}\) be as in Proposition 2.26. By the condition (i) with \(\alpha = 0\) and Proposition 2.26, \(\widetilde{\varphi}\) is almost-strictly 1-parabolically \(p\)-concave on \(\mathbb{R}^n \times \log I\). Thus, by Theorem 3.6 with \(\alpha = 1\), the function
\[
\widetilde{\Gamma}(x, t) = \widetilde{\varphi}(x, t) \ast \psi(x) = \int_{\mathbb{R}^n} \widetilde{\varphi}(x - y, t)\psi(y) \, dy, \quad (x, t) \in \mathbb{R}^n \times \log I,
\]
is strictly 1-parabolically \(\ell^0/(1 + n\ell^0)\)-concave on \(\mathbb{R}^n \times \log I\). Since \(\Gamma(x, t) = \widetilde{\Gamma}(x, \log t)\) for any \((x, t) \in \mathbb{R}^n \times I\), Proposition 2.26 completes the proof.

Proof of Theorem 3.6 Due to Lemma 3.7, we give a proof in the case where \(\alpha \neq 0\).

Let \((x_0, t_0), (x_1, t_1) \in \mathbb{R}^n \times I\), and \(\lambda \in (0, 1)\). Suppose \((x_0, t_0) \neq (x_1, t_1)\). Put
\[
\Phi(x, y, t) = \varphi(x - y, t), \quad \Phi_p(x, y, t) = \Phi(x, y, t)\psi(y), \quad (x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times I,
\]
\[
S(y) = \operatorname{ess sup} \left\{ M_{\ell^0} \left( \Phi_p(x_0, y_0, t_0), \Phi_p(x_1, y_1, t_1); \lambda \right) \mid (y_0, y_1) \in \mathbb{R}^n \times \mathbb{R}^n, \, y_1 = y, \, y \in \mathbb{R}^n \right\}.
\]

Let \(\Omega = \mathbb{R}^n \setminus \psi^{-1}(0)\). By Remark 3.2 (2) and Lemma 2.7 (2), for each \((x, t) \in \mathbb{R}^n \times I\), \(\Phi_p(x, \cdot, t)\) is positive on \(\operatorname{int} \Omega\). By Theorem 2.9 with \(f_0 = \Phi_p(x_0, \cdot, t_0)\) and \(f_1 = \Phi_p(x_1, \cdot, t_1)\),
\[
\int_{\mathbb{R}^n} S(y) \, dy \geq M_{\ell^0/(1+n\ell^0)}(\Gamma(x_0, t_0), \Gamma(x_1, t_1); \lambda).
\]

Thus, it is sufficient to show
Strict power concavity of a convolution

\[ \int_{\mathbb{R}^n} S(y) \, dy < \Gamma(x, M_a(t_0, t_1; \lambda)) = \int_{\mathbb{R}^n} \Phi(x, y, M_a(t_0, t_1; \lambda)) \, dy. \]

Since \( \Phi(x, y, M_a(t_0, t_1, \lambda)) = 0 \) for any \( y \in \mathbb{R}^n \setminus \Omega \), we have

\[ \Gamma(x, M_a(t_0, t_1, \lambda)) = \int_\Omega \Phi(x, y, M_a(t_0, t_1, \lambda)) \, dy. \]

Since \( \Omega \) is convex (see Remark 2.4), if \( y \in \mathbb{R}^n \setminus \Omega \), then, for any \( (y_0, y_1) \in \mathbb{R}^n \times \mathbb{R}^n \) with \( y_\lambda = y \), we have \( (y_0, y_1) \notin \Omega \times \Omega \). From this property, we have \( S(y) = 0 \) for any \( y \in \mathbb{R}^n \setminus \Omega \), which implies

\[ \int_{\mathbb{R}^n} S(y) \, dy = \int_\Omega S(y) \, dy. \]

Thus, our aim is to show

\[ \int_{\Omega} S(y) \, dy < \int_\Omega \Phi(x, y, M_a(t_0, t_1, \lambda)) \, dy. \]

We construct a subset \( \Omega' \) of \( \Omega \) such that \( \Omega' \) has non-empty interior, and that \( S(y) < \Phi(x, y, M_a(t_0, t_1, \lambda)) \) for any \( y \in \Omega' \). Let \( K = \text{cl} \Omega \). Since \( M_\rho(\Phi(x_0, y_0, t_0), \Phi(x_1, y_1, t_1); \lambda) = 0 \) for any \( (y_0, y_1) \in K \times K \), we have

\[ S(y) = \text{ess sup} \left\{ M_\rho(\Phi(x_0, y_0, t_0), \Phi(x_1, y_1, t_1); \lambda) \left| (y_0, y_1) \in K \times K, \ y_\lambda = y \right. \right\}. \]

By Lemma 3.3 with \( \Phi_0 = \Phi(x_0, y_0, t_0) \) and \( \Phi_1 = \Phi(x_1, y_1, t_1) \), we have

\[ S(y) \leq \text{ess sup} \left\{ M_\rho(\Phi(x_0, y_0, t_0), \Phi(x_1, y_1, t_1); \lambda) \left| (y_0, y_1) \in K \times K, \ y_\lambda = y \right. \right\} \psi(y). \]

By the continuity of \( \Phi(x, y, t) \) (see Lemma 2.7 and Remark 3.2 (2)) and the compactness of \( K \), there exists a pair \( (\eta_0, \eta_1) \in \mathbb{R}^n \times \mathbb{R}^n \) such that \( (\eta_0, \eta_1) \in K \times K, \ \eta_\lambda = y \), and

\[ \text{ess sup} \left\{ M_\rho(\Phi(x_0, \eta_0, t_0), \Phi(x_1, \eta_1, t_1); \lambda) \left| (y_0, y_1) \in K \times K, \ y_\lambda = y \right. \right\} = M_\rho(\Phi(x_0, \eta_0, t_0), \Phi(x_1, \eta_1, t_1); \lambda). \]

Let

\[ K' = r_1 \left( \left( \frac{\lambda}{r_0} + \frac{1 - \lambda}{r_1} \right) K - \lambda \left( \frac{x_0}{r_0} - \frac{x_1}{r_1} \right) \right) \cap r_0 \left( \left( \frac{\lambda}{r_0} + \frac{1 - \lambda}{r_1} \right) K + (1 - \lambda) \left( \frac{x_0}{r_0} - \frac{x_1}{r_1} \right) \right), \]

and \( \Omega' = \Omega \setminus K' \). Proposition 3.5 guarantees that \( \Omega' \) has non-empty interior.

It is directly shown that \( y \in K' \) if and only if there exists a pair \( (y_0, y_1) \in \mathbb{R}^n \times \mathbb{R}^n \) such that

\[ (y_0, y_1) \in K \times K, \quad (3.1) \]

\[ y_\lambda = y, \quad (3.2) \]
\[ \frac{x_0 - y_0}{r_0^a} = \frac{x_1 - y_1}{r_1^a}. \] (3.3)

If \( y \in \Omega' \) is expressed by (3.1) and (3.2) for \((y_0, y_1) = (\eta_1, \eta_2)\), then (3.3) does not hold for \((y_0, y_1) = (\eta_1, \eta_2)\). Thus, by Remark 3.2 (1), we have

\[ M_p(\Phi(x_0, \eta_0, t_0), \Phi(x_1, \eta_1, t_1); \lambda) < \Phi(x_\lambda, \eta_\lambda, M_a(t_0, t_1); \lambda) = \Phi(x_\lambda, y, M_a(t_0, t_1); \lambda) \]

for any \( y \in \Omega' \). Hence, we obtain

\[ S(y) < \Phi(x_\lambda, y, M_a(t_0, t_1); \lambda) \psi(y) = \Phi_\psi(x_\lambda, y, M_a(t_0, t_1); \lambda) \]

for any \( y \in \Omega' \), and the proof is completed. \( \square \)

**Corollary 3.8** Let \( I, q, \psi, \alpha, p, q \) and \( \Gamma' \) be as in Theorem 3.6. If all the conditions (i)–(iv) in Theorem 3.6 are satisfied, then \( \Gamma' \) has at most one maximum point in \( \mathbb{R}^n \times I \).

Theorem 3.6 improves [19, Theorem 3.4].

**Theorem 3.9** Let \( f \) and \( g \) be nonnegative measurable functions defined on \( \mathbb{R}^n \). Let \( p \in \mathbb{R} \), and \( q \in \mathbb{R} \cup \{+\infty\} \). Let \( \ell \) be as in (1.6). Assume that the following conditions are satisfied:

(i) \( f \) is strictly \( p \)-concave on \( \mathbb{R}^n \).

(ii) \( g \) is \( q \)-concave on \( \mathbb{R}^n \).

(iii) \( \mathbb{R}^n \setminus g^{-1}(0) \) is bounded, and its interior is not empty.

(iv) \( p + q \geq 0 \) and \( \ell \geq -1/n \).

Then, the function

\[ G(x) = f \ast g(x) = \int_{\mathbb{R}^n} f(x - y)g(y) \, dy, \quad x \in \mathbb{R}^n, \]

is strictly \( \ell/(1 + n\ell) \)-concave on \( \mathbb{R}^n \).

**Proof** Let

\[ \varphi(x, t) = t^{1/p}f \left( \frac{x}{t} \right), \quad (x, t) \in \mathbb{R}^n \times (0, +\infty). \]

Proposition 2.27 guarantees that \( \varphi \) is almost-strictly 1-parabolically \( p \)-concave on \( \mathbb{R}^n \times (0, +\infty) \) (see also Proposition 2.14). By Theorem 3.6, the function

\[ \Gamma(x, t) = \varphi(\cdot, t) \ast g(x) = \int_{\mathbb{R}^n} \varphi(x - y, t)g(y) \, dy, \quad (x, t) \in \mathbb{R}^n \times (0, +\infty), \]

is strictly 1-parabolically \( \ell/(1 + n\ell) \)-concave on \( \mathbb{R}^n \times (0, +\infty) \). Since \( G = \Gamma(\cdot, 1) \), Remark 2.31 completes the proof. \( \square \)

**Corollary 3.10** Let \( f, g, p, q \) and \( G \) be as in Theorem 3.9. If all the conditions (i)–(iv) in Theorem 3.9 are satisfied, then \( G \) has at most one maximum point in \( \mathbb{R}^n \).
3.3 Applications to concrete convolutions

In this subsection, we show the strict parabolic power concavity and strict power concavity in space variable of the Gauss–Weierstrass integral (1.9) and the Poisson integral (1.11). As applications of Theorem 3.6, the strict $1/2$-parabolic quasi-concavity of the Gauss–Weierstrass integral and the strict 1-parabolic quasi-concavity of the Poisson integral are given.

**Example 3.11** Let $a \in \mathbb{R} \setminus \{0\}$, $b \in [1, +\infty)$, and $c \in \mathbb{R}$. Suppose $c/a < 0$. Put

$$
\kappa(r, t) = r^a \exp \left( -\frac{\rho^b}{t^p} \right), \ (r, t) \in [0, +\infty) \times (0, +\infty).
$$

Then, $\kappa$ satisfies the conditions (i)–(iii) in Proposition 2.29 with $a = c/b$, $p = c/(ab)$ and $r = 1$. Thus, the function

$$
\kappa^a(x, t) = \kappa(|x|, t), \ (x, t) \in \mathbb{R}^n \times (0, +\infty),
$$

is almost-strictly $c/b$-parabolically $c/(ab)$-concave on $\mathbb{R}^n \times (0, +\infty)$. In particular, applying this investigation with $a = -n/2$, $b = 2$ and $c = 1$, Remark 2.25 (2) guarantees that the Gauss–Weierstrass kernel

$$
\mathbb{R}^n \times (0, +\infty) \ni (x, t) \mapsto \frac{1}{(4\pi t)^{n/2}} \exp \left( -\frac{|x|^2}{4t} \right)
$$

is almost-strictly $1/2$-parabolically $-1/n$-concave on $\mathbb{R}^n \times (0, +\infty)$.

**Proposition 3.12** Let $\Omega$ be a bounded convex set in $\mathbb{R}^n$ with non-empty interior. Let $W$ be as in (1.9). $W_{X\Omega}$ is strictly $1/2$-parabolically quasi-concave on $\mathbb{R}^n \times (0, +\infty)$.

**Example 3.13** Let $a \in [0, +\infty)$, $b \in [1, +\infty)$, and $c \in (-\infty, 0)$. Suppose $(a, b) \neq (0, 1)$ and $c < -a$. Put

$$
\kappa(r, t) = r^a \left( r^b + t^b \right)^{c/b}, \ (r, t) \in [0, +\infty) \times (0, +\infty).
$$

Then, $\kappa$ satisfies the conditions (i)–(iii) in Proposition 2.29 with $a = 1$, $p = 1/(a + c)$ and $r = 1$. Thus, the function

$$
\kappa^a(x, t) = \kappa(|x|, t), \ (x, t) \in \mathbb{R}^n \times (0, +\infty),
$$

is almost-strictly $1/(a + c)$-concave on $\mathbb{R}^n \times (0, +\infty)$. In particular, applying this investigation with $a = 1$, $b = 2$ and $c = -(n+1)$, the Poisson kernel

$$
\mathbb{R}^n \times (0, +\infty) \ni (x, t) \mapsto \frac{2t}{\sigma_n(S^\alpha)} \left( |x|^2 + t^2 \right)^{-(n+1)/2}
$$

is almost-strictly $1$-parabolically $-1/n$-concave on $\mathbb{R}^n \times (0, +\infty)$.

**Proposition 3.14** Let $\Omega$ be a bounded convex set in $\mathbb{R}^n$ with non-empty interior. Let $P$ be as in (1.11). $P_{X\Omega}$ is strictly $1$-parabolically quasi-concave on $\mathbb{R}^n \times (0, +\infty)$. 

\[ \text{Springer} \]
As applications of Theorem 3.9, the strict log-concavity in space variable of the Gauss–Weierstrass integral (1.9) and the strict power concavity in space variable of the Poisson integral (1.11) are given.

**Example 3.15** Let \( t \in (0, +\infty), b \in (1, +\infty) \) and \( c \in \mathbb{R} \). Put

\[
k_t(r) = \exp \left( -\frac{r^b}{r^c} \right), \quad r \in [0, +\infty).
\]

Then, \( k_t \) is strictly log-concave on \([0, +\infty)\) and strictly decreasing on \([0, +\infty)\). Thus, the function

\[
 k_t^0(x) = k_t(|x|), \quad x \in \mathbb{R}^n,
\]
is strictly log-concave on \( \mathbb{R}^n \). In particular, applying this investigation with \( b = 2 \) and \( c = 1 \), the function

\[
 \mathbb{R}^n \ni x \mapsto \frac{1}{(4\pi t)^{n/2}} \exp \left( -\frac{|x|^2}{4t} \right)
\]
is strictly log-concave on \( \mathbb{R}^n \).

**Proposition 3.16** Let \( g \) be a nonnegative function defined on \( \mathbb{R}^n \), and \( q \in \mathbb{R} \cup \{ +\infty \} \). Assume that the following conditions are satisfied:

(i) \( g \) is \( q \)-concave on \( \mathbb{R}^n \).
(ii) \( \mathbb{R}^n \setminus g^{-1}(0) \) is bounded, and its interior is not empty.
(iii) \( q \geq 0 \).

Let \( W \) be as in (1.9). For any \( t \in (0, +\infty) \), the function \( Wg(\cdot, t) : \mathbb{R}^n \to (0, +\infty) \) is strictly log-concave on \( \mathbb{R}^n \).

**Example 3.17** Let \( t \in (0, +\infty), b \in (1, +\infty), \) and \( c \in (-\infty, 0) \). Put

\[
k_t(r) = \left( r^b + t^b \right)^{c/b}, \quad r \in [0, +\infty).
\]

Then, \( k_t \) is strictly \( 1/c \)-concave on \([0, +\infty)\) and strictly decreasing on \([0, +\infty)\). Thus, the function

\[
 k_t^0(x) = k_t(|x|), \quad x \in \mathbb{R}^n,
\]
is strictly \( 1/c \)-concave on \( \mathbb{R}^n \). In particular, applying this investigation with \( b = 2 \) and \( c = -(n + 1) \), the function

\[
 \mathbb{R}^n \ni x \mapsto \frac{2t}{\sigma_n(S^n)} \left( |x|^2 + t^2 \right)^{-(n+1)/2}
\]
is strictly \(-1/(n + 1)\)-concave on \( \mathbb{B}^n \).
Proposition 3.18 Let $g$ be a nonnegative function defined on $\mathbb{R}^n$, and $q \in \mathbb{R} \cup \{+\infty\}$. Assume that the following conditions are satisfied:

(i) $g$ is $q$-concave on $\mathbb{R}^n$.
(ii) $\mathbb{R}^n \setminus g^{-1}(0)$ is bounded, and its interior is not empty.
(iii) $q \geq 1$.

Let $P$ be as in (1.11). For any $t \in (0, +\infty)$, the function $Pg(\cdot, t) : \mathbb{R}^n \to (0, +\infty)$ is strictly $q/(1-q)$-concave on $\mathbb{R}^n$.

Acknowledgements The authors would like to express their deep gratitude to the anonymous reviewer(s) for careful reading of this paper and helpful suggestions.

References

1. Borell, C.: Convex set functions in $d$-space. Period. Math. Hungar. 6(2), 111–136 (1975)
2. Borell, C.: A note on parabolic convexity and heat conduction. Ann. Inst. H. Poincaré Probab. Statist. 32, 387–393 (1996)
3. Brascamp, H.J., Lieb, E.H.: On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation. J. Funct. Anal. 22, 366–389 (1976)
4. Dancs, S., Uhrin, B.: On a class of integral inequalities and their measure-theoretic consequences. J. Math. Anal. Appl. 74, 388–400 (1980)
5. Dharmadhikari, S., Joag-Dev, K.: Unimodality, Convexity, and Applications. Academic Press, Cambridge (1988)
6. Gardner, R.J.: An early version of G2002, available at http://faculty.wwu.edu/gardner/research.html
7. Gardner, R.J.: The Brunn-Minkowski inequality. Bull. Amer. Math. Soc. 39(3), 355–405 (2002)
8. Das Gupta, S.: Brunn–Minkowski and its aftermath, J. Multivariate Analysis 10 (1988), 296–318
9. Hardy, G.H., Littlewood, J.E., Pólya, G.: Inequalities. Cambridge Univ, Press (1934)
10. Ishige, K., Salani, P.: On a new kind of convexity for solutions of parabolic problems. Discrete Contin. Dyn. Syst. Ser. S 4, 851–864 (2011)
11. Ishige, K., Salani, P.: Parabolic power concavity and parabolic boundary value problems. Math. Ann. 238, 1091–1117 (2014)
12. Kawohl, B.: Rearrangement and Convexity of Level Sets in PDE. Lecture Notes in Math, vol. 1150. Springer, Berlin (1985)
13. Kennington, A.U.: Power concavity and boundary value problems. Indiana Univ. Math. J. 34, 687–704 (1985)
14. Korevaar, N.J.: Convex solutions to nonlinear elliptic and parabolic boundary value problems. Indiana Univ. Math. J. 32, 603–614 (1983)
15. Nahin, P.J.: When Least Is Best: How Mathematicians Discovered Many Clever Ways to Make Things as Small (or as Large) as Possible. Princeton Univ, Press (2011)
16. Rinott, Y.: On convexity of measures. Ann. Probab. 4, 1020–1026 (1976)
17. Rockafellar, R.T.: Convex Analysis. Princeton Univ. Press, Princeton, NJ (1970)
18. Sakata, S.: Geometric estimation of a potential and cone conditions of a body. J. Geom. Anal. 27(3), 2155–2189 (2017)
19. Sakata, S.: Every convex body has a unique illuminating center. J. Geom. 108(2), 655–662 (2017)
20. Schneider, R.: Convex Bodies: the Brunn-Minkowski Theory, second, expanded Cambridge Univ Press, Cambridge (2014)
21. Uhrin, B.: Some remarks about the convolution of unimodal functions. Ann. Probab. 12(2), 640–645 (1984)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.