Atomic Josephson junction with two bosonic species

Giovanni Mazzarella\textsuperscript{1}, Marco Moratti\textsuperscript{1}, Luca Salasnich\textsuperscript{2}, Mario Salerno\textsuperscript{3} and Flavio Toigo\textsuperscript{1}

\textsuperscript{1} Dipartimento di Fisica ‘Galileo Galilei’ and CNISM, Università di Padova, Via Marzolo 8, 35131 Padova, Italy
\textsuperscript{2} CNR-INFM and CNISM, Unità di Padova, Via Marzolo 8, 35131 Padova, Italy
\textsuperscript{3} Dipartimento di Fisica ‘E R Caianiello’ and CNISM, Università di Salerno, Via Allende 1, 84081 Baronissi (SA), Italy

Received 25 February 2009, in final form 5 May 2009
Published 1 June 2009
Online at stacks.iop.org/JPhysB/42/125301

Abstract
We study an atomic Josephson junction (AJJ) in the presence of two interacting Bose–Einstein condensates (BECs) confined in a double-well trap. We assume that bosons of different species interact with each other. The macroscopic wavefunctions of the two components obey a system of two 3D coupled Gross–Pitaevskii equations (GPEs). We write the Lagrangian of the system, and from this we derive a system of coupled ordinary differential equations (ODEs), for which the coupled pendula represent the mechanical analogues. These differential equations control the dynamical behaviour of the fractional imbalance and of the relative phase of each bosonic component. We perform the stability analysis around the points which preserve the symmetry and get an analytical formula for the oscillation frequency around the stable points. Such a formula could be used as an indirect measure of the inter-species s-wave scattering length. We also study the oscillations of each fractional imbalance around zero and nonzero—the macroscopic quantum self-trapping (MQST)—time averaged values. For different values of the inter-species interaction amplitude, we carry out this study both by directly solving the two GPEs and by solving the corresponding coupled pendula equations. We show that, under certain conditions, the predictions of these two approaches are in good agreement. Moreover, we calculate the crossover value of the inter-species interaction amplitude which signals the onset of MQST.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The prediction [1] of Bose–Einstein condensation and the realization in the laboratory of BECs [2] paved the way for many important theoretical and experimental developments. One of these is the study of the atomic counterpart [3–6] of the Josephson effect which occurs in superconductor–oxide–superconductor junctions [7]. In [3–6], realization of the AJJ is taken into account from a theoretical point of view. A few years ago, Albiez et al [8] provided an experimental realization of the AJJ focusing on the data obtained experimentally with the predictions of a many-body two-mode model [10] and a mean-field description. Under certain conditions, a coherent transfer of matter consisting of condensate bosons flows across the junction. In the above references, the AJJ physics is explored in the presence of a single bosonic species. The possibility of managing via magnetic and optical Feshbach resonances the intra- and inter-species interactions [11, 12] makes BEC mixtures very promising candidates for successfully investigating quantum coherence and nonlinear phenomena such as the existence of self-trapped modes and intrinsically localized states. Localized states induced by the nonlinearity were shown to be quite generic for multicomponent systems in external trapping potentials. In particular, the emergence of coupled bright solitons from the modulational instability of binary mixtures of BECs in optical lattices was found numerically in [13]. More sophisticated coupled localized states of two-component condensates both in optical lattices and in parabolic...
traps were reported in [14]. The existence of dark–bright states of binary BEC mixtures was demonstrated in [15]. On the other hand, the existence of localized states of different symmetry types (mixed symmetry states) was numerically and analytically demonstrated in [16]. Properties of coupled gap solitons in binary BECs mixtures with repulsive interactions were also analysed in the multidimensional case [17] as well as for combined linear and nonlinear optical lattices [18]. Although gap-soliton breathers of multicomponent Gross–Pitaevskii equations (GPEs) involving periodic oscillations of the two components densities localized on adjacent sites of an optical lattice have been found [16] (in analogy with what was done for the single component case in [6], such states can also be seen as matter wave realizations of Josephson junctions), not much numerical and theoretical study has been done until now on AJJ of binary mixtures.

Recently, this has been considered in [19, 20] for the case of a bosonic binary mixture trapped in a double-well potential, for which a coupled pendula system of ODEs for the temporal evolution of the relative population and relative phase of each component was derived. Using this reduced system, the authors of [20] have predicted the analogues of the phase of each component was derived. Using this reduced potential, for which a coupled pendula system of ODEs for et al so that the question of the validity of such a prediction remains open. For single-component condensates, Salasnich et al [5] have shown that a good agreement exists between the results obtained from the GPE and those of the ODE. Similar agreement was obtained in [6] for AJJ realized with weakly interacting solitons localized in two adjacent wells of an optical lattice. However, the situation may be quite different for multicomponent condensates, due to the interplay of intra- and inter-species interactions which enlarge the number of achievable states (for instance, mixed symmetry states can exist only in the presence of the inter-species interaction) as well as their stability, making the system much more complicated.

The aim of this paper is just to perform a systematic investigation of possible Josephson oscillations which can arise in binary BEC mixtures trapped in a double-well potential, as a function of the system parameters. In this regard, we derive the reduced coupled pendula system proceeding from a Lagrangian formulation and from the canonical equations of motion. We show that for certain conditions and range of parameters, there exists a good agreement between the solutions of the two GPEs and the predictions provided by the coupled pendula equations. We look for the stationary points that preserve the symmetry and study their stability; we get an analytical formula for the oscillation frequencies around the equilibrium points. This formula shows the possibility of determining the inter-species s-wave scattering length from the frequency.

We analyse the influence of the inter-species interaction on the temporal evolution of each relative population. In particular, by employing the coupled pendula equations we show the existence of MQST when the inter-species interaction amplitude is greater than a certain value, for which we are able to provide an analytical formula. As done by Satija et al [20], we calculate the values of the relative populations associated with the degenerate GPE states that break the symmetry of the fractional imbalances. In addition, we perform the stability analysis by explicitly calculating the associated oscillation frequencies. We, moreover, show that the MQST-like evolution obtained by solving the coupled pendula equations is close to that obtained by integrating the two coupled GPEs.

Proceeding from the works of Albiez et al [8] and Gati et al [9], we correlate our theoretical work with experiments. Finally, we draw our conclusions.

2. AJJ with two bosonic species: quasi-analytical approach

We consider two BECs of repulsively interacting bosons with different atomic species denoted below by 1 and 2. We suppose that the two BECs are confined in a double-well trap produced, for example, by a far off-resonance laser barrier that separates each trapped condensate into two parts, L (left) and R (right). We assume, moreover, that the two condensates interact with each other. In the mean-field approximation, the macroscopic wavefunctions $\Psi_i(\mathbf{r}, t)$, $(i = 1, 2)$, of the interacting BECs in a trapping potential $V_{trap}(\mathbf{r})$ at zero temperature satisfy the two coupled GPEs:

$$i\hbar \frac{\partial \Psi_i}{\partial t} = -\frac{\hbar^2}{2m_i} \nabla^2 \Psi_i + [V_{trap}(\mathbf{r}) + g_{ij} |\Psi_i|^2 + g_{ij} |\Psi_j|^2] \Psi_i.$$  

(1)

Here $\nabla^2$ denotes the 3D Laplacian and $\Psi_i(\mathbf{r}, t)$ is subject to the normalization condition

$$\iint d\mathbf{r} d\mathbf{y} d\mathbf{z} |\Psi_i(\mathbf{x}, y, z)|^2 = N_i,$$  

(2)

with $N_i$ being the number of bosons of the $i$th species. Similarly, $m_i$, $a_i$, and $g_{ij} = 4\pi\hbar^2 a_{ij}/m_i$ denote the atomic mass, the s-wave scattering length and the intra–species coupling constant of the $i$th species; $g_{ij} = 2\pi\hbar^2 a_{ij}/m_i$ being the inter–species coupling constant, with $m_i = m_i m_j/(m_i + m_j)$ being the reduced mass, and $a_{ij}$ is the associated s-wave scattering length. In the following, we shall consider both $g_{ij}$ and $a_{ij}$ as free parameters, due to the possibility of changing the scattering lengths $a_i$ and $a_{ij}$ at will by using the technique of Feshbach resonances. Here we take into account the case in which the two BECs interact attractively; see [11] and [21].

The trapping potential for both components is taken to be the superposition of a strong harmonic confinement in the radial $(x–y)$ plane and of a double-well (DW) potential in the axial $(z)$ direction. For the $i$th component, we model this trapping potential in the form

$$V_{trap}(\mathbf{r}) = \frac{m_i a_i^2}{2} (x^2 + y^2) + V_{DW}(z),$$  

(3)

where, for symmetric configurations in the z-direction, we model the DW potential as
\[ V_{\text{DW}}(z) = V_L(z) + V_R(z), \]
\[ V_L(z) = -V_0 \text{Sech}^2 \left( \frac{z + z_0}{a} \right), \]
\[ V_R(z) = -V_0 \text{Sech}^2 \left( \frac{z - z_0}{a} \right), \]
\[ V_0 = \hbar \omega_1 \left[ 1 + \text{Sech}^2 \left( \frac{2z_0}{a} \right) \right]^{-1}, \]
\[ i.e. \text{the combination of two Pöschl–Teller (PT) potentials, } V_L(z) \text{ and } V_R(z), \text{ separated by a potential barrier, the height of which can be changed by varying } a, \text{ centred around the points } -z_0 \text{ and } z_0 \text{ (see figure 1). Note that the usage of PT potentials is only for the benefit of improving accuracy in our numerical GPE calculations (see below), taking advantage of the integrability of the underlying linear system. We remark, however, that the obtained results are of generic validity also for more confining (e.g. not saturating to zero at large distances) double-well potentials.} \]

We are interested in studying the dynamical oscillations of the populations of each condensate between the left (L) and right (R) wells when the barrier is large enough so that the link is weak. To exploit the strong harmonic confinement in the \((x,y)\) plane and get the effective 1D equations describing the dynamics in the \(z\)-direction, we write the Lagrangian associated with the GPEs in (1) as
\[ L = \int \text{d}^3r \left[ \sum_{i=1,2} \dot{\Psi}_i \left( i\hbar \frac{\partial}{\partial t} - \frac{\hbar^2}{2m_i} \nabla^2 + \right) \right] \Psi_i - \frac{V_{\text{trap}}(r)}{2} |\Psi_i|^2 - \frac{g_i}{2} |\Psi_i|^4 - \frac{g_{ij}}{2} |\Psi_i|^2 |\Psi_j|^2, \]
where \(\Psi_i\) denotes the complex conjugate of \(\Psi_i\), and \(i \neq j\) and adopt the ansatz [22]
\[ \Psi_i(x, y, z, t) = \frac{1}{\sqrt{\pi a_{\perp, i}}} \exp \left[ -\frac{x^2 + y^2}{2a_{\perp, i}^2} \right] f_i(z, t), \]
where \(a_{\perp, i} = \sqrt{\frac{\hbar}{m_i \omega_i}}\) and \(f_i(z, t)\) obey \(\int_{-\infty}^{\infty} \text{d}z |f_i(z, t)|^2 = N_i\), so that the normalization condition, equation (2), is satisfied. By inserting this ansatz (6) in (5) and performing the integration in the radial plane, we obtain the effective 1D Lagrangian for the fields \(f_i(z, t)\)
\[ \mathcal{L} = \int \text{d}z \left[ \sum_{i=1,2} \dot{f}_i \left( i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m_i} \nabla^2 \right) f_i - (\epsilon_i + V_{\text{DW}}(z)) |f_i|^2 - \frac{\hbar}{2} |f_i|^4 - \frac{g_{ij}}{2} |f_i|^2 |f_j|^2 \right]. \]
where the effective parameters for the 1D dynamics are given in terms of the original ones by \(\epsilon_i = \frac{\hbar^2}{2m_i} \left( \frac{m_i \omega_i^2 a_{\perp, i}^2}{2} \right) \), \(\tilde{g}_i = \frac{\hbar}{2m_i} \) and \(\tilde{g}_{ij} = \frac{\hbar g_{ij}}{\pi (a_{\perp, i} a_{\perp, j})} \). By varying \(L\) with respect to \(f_i\), we obtain the 1D GPE for the field \(f_i\)
\[ i\hbar \frac{\partial f_i}{\partial t} = -\frac{\hbar^2}{2m_i} \frac{\partial^2 f_i}{\partial z^2} + (\epsilon_i + V_{\text{DW}}(z)) f_i + \tilde{g}_i |f_i|^2 f_i + \tilde{g}_{ij} |f_j|^2 f_i. \]

It is possible to study the AJJ dynamics described by equation (8) by using the two-mode approximation discussed by Milburn et al in [10]. In particular, we assume, for each \(f_i\), the following time-dependent wavefunction decomposition:
\[ f_i(z, t) = \psi^L_i(t) \phi^L_i(z) + \psi^R_i(t) \phi^R_i(z), \]
where
\[ \psi_i^\alpha(t) = \sqrt{N_i^\alpha} e^{i \theta_i^\alpha(t)}, \]
with \(\alpha = L, R\), and a constant total number of particles given by \(N_i^L + N_i^R = \left| \psi_i^L(t) \right|^2 + \left| \psi_i^R(t) \right|^2 = N_i\), with \(\int_{-\infty}^{+\infty} \text{d}z |\phi^\alpha_i(z)|^2 = 1\) and \(\int_{-\infty}^{+\infty} \text{d}z \phi_i^L(z) \phi_i^R(z) = 0\). Neglecting terms of order greater than 2 in the overlaps of \(\phi^\alpha_i\), we can write the Lagrangian (7) in terms of \(N_i^\alpha\) and \(\theta_i^\alpha\) as
\[ \mathcal{L} = \sum_{i=1,2} \left[ -\hbar \dot{\theta}_i^L N_i^L - \hbar \dot{\theta}_i^R N_i^R - E_i^L N_i^L - E_i^R N_i^R \right. \]
\[ + 2K_i \sqrt{N_i^L N_i^R} \cos (\theta_i^L - \theta_i^R) \]
\[ - \left( \frac{U_i^L}{2} \left( N_i^L \right)^2 + \frac{U_i^R}{2} \left( N_i^R \right)^2 \right) \]
\[ - U_{12}^L N_i^L N_i^R - U_{12}^R N_i^R N_i^L, \]
where
\[ E_i^\alpha = \int \text{d}z \left[ \frac{\hbar^2}{2m_i} \left( \frac{d\phi_i^\alpha}{dz} \right)^2 \right. \]
\[ + \left( V_{\text{DW}} + \frac{\hbar^2}{2m_i} \frac{m_i \omega_i^2 a_{\perp, i}^2}{2} \right) \left( \phi_i^\alpha \right)^2 \right], \]
\[ U_i^\alpha = \tilde{g}_i \int \text{d}z (\phi_i^\alpha)^4. \]
\[ K_i = - \int \text{d}z \left[ \frac{\hbar^2}{2m_i} \frac{d^2\phi_i^\alpha}{dz^2} + V_{\text{DW}} \phi_i^\alpha \phi_i^\alpha \right]. \]

One may get a good approximation for functions \(\phi_i^L(z)\) and \(\phi_i^R(z)\) when the double-well potential \(V_{\text{DW}}(z)\) is such that the two lowest energy eigenvalues of the corresponding Schrödinger equation constitute a closely spaced doublet well separated from the higher excited levels, and \(\tilde{g}_i\)’s are not too large (see, for example, [10]). If the real symmetric function \(\phi_i^L(z)\) and the real antisymmetric function \(\phi_i^R(z)\) are
the wavefunctions of the ground state and the first excited state, respectively, then \( \phi_i^L(z) \) and \( \phi_i^R(z) \) may be chosen as

\[
\phi_i^L(z) = \frac{\phi_i^z(z) + \phi_i^A(z)}{\sqrt{2}}, \quad \phi_i^R(z) = \frac{\phi_i^z(z) - \phi_i^A(z)}{\sqrt{2}}.
\]  

(13)

Remember that \( \phi_i^z(z) \) and \( \phi_i^A(z) \) satisfy the relations: \( \int_{-\infty}^{\infty} dz |\phi_i^z(z)|^2 = 1 \) and \( \int_{-\infty}^{\infty} dz x \phi_i^z(z) \phi_i^A(z) = 0 \). Having chosen \( V_{ijw} \) as the sum of two Pöschl–Teller (PT) wells (see equation (4)), the functions \( \phi_i^L(z) \) and \( \phi_i^R(z) \) may be analytically calculated following a perturbative approach. Let us consider the eigenvalue problem corresponding to equation (8) with \( \hat{g}_i = \hat{g}_{jj} = 0 \). We know exactly the wavefunctions for this eigenvalue problem when the potential is given by a single \( V_a(z) \) (\( a = L, R \)), for example \( V_L(z) \). The wavefunction of the ground state is [23]

\[
\phi_i^{L,PT}(z) = A \left[ 1 - \text{Tanh}^2 \left( \frac{z + z_0}{a} \right) \right]^{B/2}.
\]

(14)

In equation (14) \( A \), equal for both sides, ensures the normalization of the wavefunction. Since we assume that the two lowest energetic levels are well separated from the higher ones, when the potential is perturbed by the presence of \( V_a(z) \), we look for the eigenstates in the form of a linear superposition of \( \phi_i^{L,PT}(z) \) and \( \phi_i^{R,PT}(z) \). For each component, to the first-order of such a perturbative theory, the ground-state wavefunction \( \phi_i^L(z) \) and the first excited state wavefunction \( \phi_i^R(z) \) read

\[
\phi_i^L(z) = M_S (\phi_i^{L,PT}(z) + \phi_i^{R,PT}(z))
\]

\[
\phi_i^A(z) = M_A (\phi_i^{L,PT}(z) - \phi_i^{R,PT}(z)).
\]

(15)

Here \( M_S \) and \( M_A \) ensure the normalization of \( \phi_i^L(z) \) and \( \phi_i^A(z) \), respectively. Note that \( K_i \) is equal to \( (E_i^S - E_i^L)/2 \), with \( E_i^S \) and \( E_i^L \), also perturbatively calculated, the energies associated with \( \phi_i^L(z) \) and \( \phi_i^A(z) \), respectively. The quantity

\[
\omega = \left( \frac{E_i^A - E_i^S}{\hbar} \right)
\]

(16)

is the Rabi frequency. This frequency characterizes the oscillations of a particle between the states \( \phi_i^L \) and \( \phi_i^R \). By using equation (4) in the decomposition (13), we are able to write the functions \( \phi_i^L(z) \) and \( \phi_i^R(z) \) in terms of \( \phi_i^{L,PT}(z) \) and \( \phi_i^{R,PT}(z) \) in the following way:

\[
\phi_i^L = \frac{(M_S + M_A) \phi_i^{L,PT}(z) + (M_S - M_A) \phi_i^{R,PT}(z)}{\sqrt{2}}.
\]

(17)

Note that \( \phi_i^L(z) \) and \( \phi_i^A(z) \), and the associated energies, may be numerically found as the wavefunctions of the two lowest states of the eigenvalue problem corresponding to equation (8) in the absence of interactions. Then, by using the decomposition (13), one calculates the functions \( \phi_i^L(z) \) and \( \phi_i^R(z) \). We have verified that the perturbative theory provides practically the same results as the numerical approach.

Let us, now, focus on the Lagrangian (11). The conjugate momenta of the generalized coordinates \( N_{i\alpha} \) and \( \theta_{i\alpha} \) are given by

\[
p_{N_{i\alpha}} = \frac{\partial L}{\partial \dot{N}_{i\alpha}} = 0, \quad p_{\theta_{i\alpha}} = \frac{1}{\hbar} \frac{\partial L}{\partial \dot{\theta}_{i\alpha}} = -N_{i\alpha}.
\]

(18)

The Hamiltonian of the system is

\[
H = -\sum_{i=1,2} \left[ p_{i\alpha} E_i^L + p_{i\beta} E_i^R \right] - \sum_{i=1,2} 2K_i \sqrt{p_{i\alpha} \cos (\theta_i^L - \theta_i^R)} + \sum_{i=1,2} \left( \frac{U_i^L}{2} p_{i\alpha}^2 + \frac{U_i^R}{2} p_{i\beta}^2 \right) + U_{12} p_{1\alpha} p_{2\beta} + U_{12} p_{1\beta} p_{2\alpha}.
\]

(19)

The evolution equations for the fractional imbalance \( z_i = (N_i^L - N_i^R)/N_i \) and for the relative phase \( \theta_i = \theta_i^R - \theta_i^L \) for each component are derived from the canonical equations associated with the Hamiltonian (19):

\[
p_{\theta_{i\alpha}} = \frac{1}{\hbar} \frac{\partial H}{\partial \dot{\theta}_{i\alpha}}, \quad \theta_{i\alpha} = \frac{1}{\hbar} \frac{\partial H}{\partial p_{\theta_{i\alpha}}}.
\]

(20)

By subtracting the equation for \( p_{\theta_{i\alpha}} \) from that for \( p_{\theta_{i\beta}} \), we obtain the equation for the temporal evolution of \( z_i \). Similar arguments lead to the equation for \( \theta_i(t) \). In the following, we shall assume two wells to be symmetric, i.e. \( E_i^L = E_i^R, U_i^L = U_i^R \equiv U_i, U_{12} = U_{12} \equiv U_{12} \). The fractional imbalance and the relative phase for each component vary in time according to the following (coupled pendula) equations:

\[
z_i = -\frac{2}{\hbar} K_i \sin \theta_i \frac{\sqrt{1 - z_i^2}}{h} \frac{\cos \theta_i}{h} + \frac{U_{12} N_i z_i}{h}.
\]

(21)

Note that when \( U_{12} = 0 \), equations (21) reduce to the usual equations for a single component obtained in [4]. At this point, we observe that it is possible to obtain equation (21) proceeding from the following equations:

\[
N_i \dot{z}_i = -\frac{1}{\hbar} \frac{\partial \tilde{H}}{\partial \theta_i}, \quad N_i \dot{\theta}_i = \frac{1}{\hbar} \frac{\partial \tilde{H}}{\partial \dot{z}_i},
\]

(22)

where \( \tilde{H} \) is

\[
\tilde{H} = -\sum_{i=1,2} 2K_i N_i \left[ \sqrt{(1 - z_i^2)} \cos \theta_i \right] + \sum_{i=1,2} \frac{U_i}{2} (N_i^2 z_i^2 + U_{12} N_i N_j z_i z_j).
\]

(23)

To fix ideas, let us consider, as done in [12], a mixture of two bosonic isotopes of the same atom, so that one may have different intra-species and inter-species interactions. For simplicity, for the time being we will neglect the mass difference between the two species. We compare the temporal
Figure 2. Fractional imbalances of the two bosonic species versus time. Here $N_1 = 100$ and $N_2 = 150$. In each plot of $z_i(t)$, from top to bottom: $U_1 = U_2 = U_{12} = 0$; $U_1 = 0.001$, $U_{12} = 0$; $U_1 = U_2 = U = 0.001$, $U_{12} = -U/2$. The dashed line represents data from integration of equation (8), the dot-dashed line represents solution of equation (21) with $K_1 = K_2 = K = 0.0148$ and the continuous line, in the second and third panels, represents solution of equation (21) with the best-fit $K$’s, say $K_{ij}$. For each $z_i(t)$, $K_{ij} = 0.0155$ ($U_1 = U_2 = 0.001$, $U_{12} = 0$) and $K_{ij} = 0.0151$ ($U_1 = U_2 = U = 0.001$, $U_{12} = -U/2$). We used the initial conditions $z_i(0) = 0.1$, $z_i(0) = 0.15$ and $\theta_i(0) = 0$. Time is measured in units of $(\omega_1)^{-1} = (\omega_2)^{-1} \equiv \omega^{-1}$, lengths are measured in units of $a_{1,2} = a_1 = a_2$ and energies in units of $\hbar \omega$. 

The small amplitude oscillation frequency, say $\omega_\pm$, around the point (i) is

$$\omega_\pm = \frac{1}{\hbar} \left(K_1 (2K_1 + U_1 N_1) + K_2 (2K_2 + U_2 N_2) \pm \Delta \right)^{1/2},$$

$$\Delta = [-4K_1 K_2 (4K_1 K_2 + 2K_1 U_2 N_2 + 2K_2 U_1 N_1) - U_1^2 N_1 N_2 + U_2^2 N_1 N_2] + (K_1 (2K_1 + U_1 N_1) + K_2 (2K_2 + U_2 N_2))^2]^{1/2}$$

with and – corresponding to the normal modes of the linearized system associated with equation (21). When $z_i(0) = \mp z_i(0) < 1$, $U_1 = U_2$, $K_1 = K_2$ and $N_1 = N_2$, the fractional imbalances $z_i$ oscillate around the point (i) according to the law

$$z_i(t) = z_i(0) \cos (\omega_\pm)^1 t.$$
oscillation frequency associated with class (iv). We have verified that for the stationary points of type (ii), under certain conditions (analytically achievable but very complicated), the eigenvalues of the Jacobian matrix are all of the form \( i \omega \). The small amplitude oscillation frequency, say \( \omega^{(2)} \), around the point (ii) is

\[
\omega^{(2)} = \frac{1}{\hbar} \left( K_1 (2 K_1 + U_1 N_1) + K_2 (2 K_2 - U_2 N_2) \pm \Delta \right)^{1/2},
\]

where \( \Delta = \left[ -4 K_1 K_2 (K_1 N_2 - K_2 N_1) + U_1^2 N_1 N_2 - U_2 (U_2 N_2 N_1) + (K_1 (2 K_1 + U_1 N_1) + K_2 (2 K_2 - U_2 N_2))^2 \right]^{1/2}. \tag{28}
\]

For the oscillations of \( z_i \) around the point (ii), one may use arguments analogous to those employed for class (i). If we start, now, from the points of class (ii), and replace \( U_1 \) with \(-U_1\), we get the stability regions and the oscillation frequency for the points of type (iii). Let us focus, to fix the ideas, on the frequency \( (26) \) and on the formula of equation \( (12) \) which gives the inter-species interaction amplitude \( U_{12} \). We note that \( \bar{z}_{12} \) is directly related to the inter-species s-wave scattering length. This quantity, then, can be determined from the oscillation frequency \( (26) \) once one keeps fixed \( K_1, U_1 \) and \( N_1 \). We will discuss this point in more detail in the following.

At this point, it is worth observing that, because of the nonlinearity associated with the inter- and intra-species interactions, there is a class of degenerate GPE eigenstates that breaks the \( z_i \) symmetry. Let us assume that \( U_1 = U_2 = U, K_1 = K_2 = K \) and \( N_1 = N_2 = N \). In correspondence with \( \theta_i = \pi \), we have looked for nonzero stationary solutions of the system \( (21) \). We have found four classes of fractional imbalances corresponding to the \( z_i \) broken symmetry; we have verified that two of these classes do not correspond to a stable equilibrium. Let us consider the two classes describing stable equilibrium, say I and II. For class I, we have

\[
\begin{align*}
&z_{1,SB}^{(I)} = \pm \sqrt{1 - \frac{2 K}{N (U + U_{12})}}, \\
z_{2,SB}^{(I)} = z_{1,SB}^{(I)},
\end{align*}
\]

provided that \( |(U + U_{12})| > 2K/N \). When \( 0 < U < 2K/N \), the solution \( (29) \) is always stable, and the corresponding oscillation frequency is

\[
\omega^{(I)}_A = \frac{1}{\hbar} \sqrt{(N (U + U_{12}))^2 - 4 K^2}. \tag{30}
\]

For \( U > 2K/N \), the solution \( (29) \) is stable when

\[
U_{12} > \bar{U}_{12}^{(I)} = \frac{2K}{N} \left( -U N/2K \right) + \frac{3^{1/3} - (9UN/2K + \sqrt{3 + 81(NUN/2K)^2})^{2/3}}{3^{2/3}(9UN/2K - \frac{\sqrt{3 + 81(NUN/2K)^2})^{1/3}}}. \tag{31}
\]

The corresponding oscillation frequency is

\[
\omega^{(I)}_A = \frac{1}{\hbar} \sqrt{(N (U + U_{12}))^2 + 4K^2 \left( \frac{U_{12} - U}{U + U_{12}} \right)}. \tag{32}
\]

It is possible to determine the crossover value, say \( U_{12}^{(cr)} \), of the inter-species interaction strength signing the onset of the self-trapping. We start by evaluating the Hamiltonian \( (23) \) in \( z_i = z_{1,SB}^{(I)} \) and \( \theta_i = \pi \). Let us denote the energy obtained in this way by \( E^{(I)} \). This energy reads

\[
E^{(I)} = \frac{4K^2}{U + U_{12}} + N^2 (U + U_{12}). \tag{33}
\]

Then, we evaluate the Hamiltonian \( (23) \) at \( t = 0 \), i.e. \( \bar{H}(z_i(0), \theta_i(0)) \). We require that

\[
\bar{H}(z_i(0), \theta_i(0)) > 4KN \tag{34}
\]

with \( 4KN \) the value got by \( E^{(I)} \) when \( z_{1,SB}^{(I)} = 0 \), i.e. when \( 2K = N(U + U_{12}) \). Then, we get

\[
U_{12}^{(ICR)} = \frac{K}{N z_i(0) z_0(0)} \left[ 4 - \frac{U N}{2K} \sum_{i=1,2} z_i(0)^2 \right] + 2 \sum_{i=1,2} \sqrt{1 - z_i(0)^2} \cos \theta_i(0). \tag{35}
\]

When the condition \( U_{12} > U_{12}^{(ICR)} \) is satisfied, the system will be self-trapped. For the solution of type II, we have

\[
\begin{align*}
&z_{1,SB}^{(II)} = \pm \sqrt{1 - \frac{2K}{N (U - U_{12})}}, \\
z_{2,SB}^{(II)} = -z_{1,SB}^{(II)},
\end{align*}
\]

provided that \( |(U - U_{12})| > 2K/N \). When \( 0 < U < 2K/N \), this solution is always stable and is characterized by the oscillation frequency

\[
\omega^{(II)}_A = \frac{1}{\hbar} \sqrt{(N (U - U_{12}))^2 - 4K^2}. \tag{36}
\]

When \( U > 2K/N \), the solution \( (36) \) is stable if

\[
U_{12} < \bar{U}_{12}^{(II)} = \frac{2K}{N} \left( \frac{U N/2K}{3^{1/3} - (9UN/2K + \sqrt{3 + 81(NUN/2K)^2})^{2/3}} \right), \tag{37}
\]

and the corresponding oscillation frequency reads

\[
\omega^{(II)}_A = \frac{1}{\hbar} \sqrt{(N (U - U_{12}))^2 + 4K^2 \left( \frac{U + U_{12}}{U_{12} - U} \right)}. \tag{38}
\]

Also for solution II, it is possible to determine the inter-species interaction amplitude which signals the self-trapping onset, say \( U_{12}^{(ICR)} \), by using the same argument employed for class I. Also in this case, we proceed by evaluating the Hamiltonian \( (23) \) in \( z_i = z_{1,SB}^{(II)} \) and \( \theta_i = \pi \). Let us denote the energy obtained in this way by \( E^{(II)} \). This energy reads

\[
E^{(II)} = \frac{4K^2}{U - U_{12}} + N^2 (U - U_{12}). \tag{39}
\]

The condition to find \( U_{12}^{(ICR)} \) is equation \( (34) \), and we obtain that \( U_{12}^{(ICR)} \) coincides with \( U_{12}^{(ICR)} \).

An interesting task, now, is to analyse the influence of the interaction between the two BECs on the temporal evolution of the fractional imbalances. We have fixed \( U_1 \) and \( U_2 \) and analysed \( z_1(t) \) and \( z_2(t) \) for different values of the inter-species interaction amplitude \( U_{12} \). The greater the absolute value of \( U_{12} \), the greater the deformation of
θi(zi(t)) upper panels of each right). We used the initial conditions z(t) for different values of time, as displayed in the figure. The oscillations around ⟨z(t)⟩ = 0, as shown in the two upper panels (from left to right) and in the first lower panel (from left) of each zi(t) of figure 3. Note that as long as the oscillations are harmonic (see the first panel of each zi(t)), the time evolution of the fractional imbalances may be described in terms of a carrier wave of frequency ωc, given by equation (26) with U12 = 0, modulated by a wave of frequency ωm. The frequency ωc is much greater than ωm. We found that there exists a value of the inter-species interaction amplitude for which the relative population in each trap oscillates around a nonzero time averaged value, ⟨zi(t)⟩ ≠ 0, which corresponds to the macroscopic quantum self-trapping (MQST) as discussed in [4] for a single component. To support this interpretation, we have studied the behaviour of the density profiles of the two species as a function of z and for different values of time. In particular, to find |f1(z, t)|2, we have numerically solved the two coupled GPEs (8) for those values of the interaction amplitudes for which the self-trapped is predicted to occur by the coupled pendula equations. We have summarized the results of this analysis in the last two panels of figure 3. Finally, it is interesting to observe that there is a good agreement between MQST predicted by the coupled pendula equations (21) and that obtained by numerically solving the two 1D GPEs (8). This comparison is displayed in figure 4, where we can observe oscillation around a nonzero time averaged value of fractional imbalances. We followed the same fitting procedure adopted in obtaining figure 2.

At this point, we observe that the question of the AJJ with two bosonic species could also be addressed from the experimental point of view. This could be done considering, for example, BEC binary mixtures of two bosonic isotopes of the same alkali atom [12]. The works referenced in [8] and [9] provide the ideal guide lines for these kind of experiments. By engineering a double-well potential as suggested, for instance, by Gati et al [9], the measured fractional imbalances zi(t) could be compared with those obtained both by solving equation (8)(see [8] and [9]) and the ODE system (21). It could be possible to measure, moreover, the inter-species Σ-wave scattering length a12 by using, to fix the ideas, the frequency (26). The mixture could be prepared in such a way that all the

Figure 3. Fractional imbalances of the first and second bosonic species versus time. Here N1 = 100 and N2 = 150, K1 = K2 = 0.0148 and U1 = U2 = 0.1K1. Plots are for different values of U12. U12 = −U/20 and U12 = −U/2 (the two upper panels of each zi(t), from left to right); U12 = −U and U12 = −1.2U (the two lower panels of each zi(t), from left to right). We used the initial conditions z1(0) = 0.1, z2(0) = 0.15 and θ1(0) = 0. Here, we report the profiles |f1|2 and |f2|2 as functions of z and for different values of time, as displayed in the figure. The quantities |f1|2 are obtained by integrating equation (8) when U12 = −1.2U. Units are as in figures 1 and 2.

Figure 4. Fractional imbalances of the first and second bosonic species versus time. The dashed line represents data from integration of equation (8), the dot-dashed line represents solution of equation (21) with K1 = K2 ≡ K = 0.0148 and the continuous line represents solution of equation (21) with the best-fit K equal to 0.0151. Here is U1 = U2 = 0.1K, U12 = −0.14K, N1 = N2 = 100. We used the initial conditions z1(0) = 0.4 and θ1(0) = 0. The units are as in figures 1 and 2.
conditions \( z_1(0) = z_2(0), K_1 = K_2, U_1 = U_2, N_1 = N_2 \)— see the discussion about the four classes (i)–(iv) of stationary points—are verified and have small amplitude oscillations around the stationary point (i). Each \( z_i \) oscillates according to the law \( z_i(0) \cos(\omega_i(1) t) \) (see equation (27)). Let us suppose to fix both the characteristic quantities (in our case, they are \( \omega_i, a, z_0 \)) of the trapping potential—the group of Heidelberg displayed how this is possible for a given class of double-well traps [8], [9]—and the intra-species s-wave scattering length \( a_i \). Then, the functions \( \phi_i^{\mu} \) are known. Then, equations (12) provide the parameters \( K_i \) and \( U_i^{\mu} \). The measure of the period of \( z_i(0) \cos(\omega_i(1) t) \) leads to the corresponding frequency \( \omega_i(1) \), i.e. the left-hand side of equation (26). The solution of this equation gives the parameter \( U_{12} \). By means of the second line of equation (12), one gets \( \tilde{g}_{12} \) and, then \( a_{12} \).

3. Conclusions

We have analysed the atomic Josephson effect in the presence of a binary mixture of BECs. We have written the Lagrangian of the system, from which we have derived a system of coupled differential equations which governs the dynamical behaviour of the fractional imbalance and of the relative phase of each component. We have analysed the stable points that preserve the symmetry, and we have obtained an analytical formula for the frequency oscillations around these equilibrium points. In this regard, one of the most interesting features is the possibility of knowing the inter-species s-wave scattering length from these frequencies. We have shown that in correspondence with precise values of the inter-species interaction amplitude, the relative populations oscillate around a nonzero time averaged value. This behaviour corresponds to MQST, a well-known phenomenon when only one component is taken into account. We have compared the predictions of GPEs with those of the coupled pendula equations. We have performed this comparison in the case of total absence of interaction, in the case in which only the intra-species interaction is present, and in the case in which also the inter-species interaction is involved. We have found that, under certain conditions, the predictions of GPEs agree with those of the coupled pendula equations. We have shown that, under certain hypothesis, it is possible to obtain analytical expressions for the inter-species interaction amplitudes which signal the onset of the self-trapping. Finally, we have commented about the possibility of correlating our theoretical work with the experiments proceeding from the works of the group of Heidelberg; see [8] and [9].

Acknowledgments

This work has been partially supported by Fondazione CARIPARO through the Project 2006: ‘Guided solitons in matter waves and optical waves with normal and anomalous dispersion’.

References

[1] Bose S N 1924 Z. Phys. 26 178
[2] Anderson M H, Matthews M R, Wieman C E and Cornell E A 1995 Science 269 198
[3] Davis K B, Mewes M O, Andrews M R, van Druten N J, Durfee D S, Kurn D M and Ketterle W 1995 Phys. Rev. Lett. 75 3969
[4] Gati R and Oberthaler M K 2007 Phys. Rev. Lett. 95 053610
[5] Salasnich L, Parola A and Reatto L 2002 J. Phys. B: At. Mol. Opt. Phys. 35 5205–16
[6] Salerno M 2005 Laser Phys. 4 620–5
[7] Barone A and Paternò G 1982 Phys. Rev. Lett. 68 1374
[8] Albiez M, Gati R, Fölling J, Hunsmann S, Cristiani M and Oberthaler M K 2005 Phys. Rev. Lett. 95 010402
[9] Gati R and Oberthaler M K 2007 J. Phys. B: At. Mol. Opt. Phys. 40 R61–89
[10] Milburn C J, Corney J, Wright E M and Walls D F 1997 Phys. Rev. A 55 4318
[11] Thalhammer G, Barontini G, De Sarlo L, Catani J, Minardi F and Inguscio M 2008 Phys. Rev. Lett. 100 210402
[12] Papp S B and Wieman C E 2006 Phys. Rev. Lett. 97 100404
[13] Kostov N A, Enol’skii V Z, Gerdjikov V S, Konotop V V and Salerno M 2004 Phys. Rev. E 70 056617
[14] Kasamatsu K and Tsubota M 2006 Phys. Rev. A 74 013617
[15] Kevrekidis P G, Nistazakis H E, Frantzeskakis D J, Malomed B A and Carretero-Gonzalez R 2004 Eur. Phys. J. D 28 181
[16] Ostrovskaya E A and Kivshar Yu S 2004 Phys. Rev. Lett. 92 180405
[17] Cruz H A, Brazhnyi V A, Konotop V V, Alimov G L and Salerno M 2007 Phys. Rev. A 76 013603
[18] Gubeskys A, Malomed B A and Merhasin I M 2006 Phys. Rev. A 73 023607
[19] Abdullaev F Kh, Gammal A, Salerno M and Tomio L 2008 Phys. Rev. A 77 023615
[20] Xu X, Lu L and Li Y 2008 Phys. Rev. A 78 043609
[21] Satija I I, Naudus P, Balakrishnan R, Heward J, Edwards M and Clark C W 2009 Phys. Rev. A 79 033616
[22] Malomed B A and Carretero-Gonzalez R 2004 Eur. Phys. J. D 28 181
[23] Landau L and Lifshitz L 1959 Course in theoretical physics Quantum Mechanics: Non-Relativistic Theory vol 3 (New York: Pergamon)