Abstract. This paper gives a self-contained group-theoretic proof of a dual version of a theorem of Ore on distributive intervals of finite groups. We deduce a bridge between combinatorics and representations in finite group theory.

1. Introduction

Oystein Ore proved in 1938 that a finite group is cyclic if and only if its subgroup lattice is distributive, and he extended one side as follows, where $[H, G]$ will be an interval in the subgroup lattice of the group $G$ (idem throughout the paper).

**Theorem 1.1** ([4]). Let $[H, G]$ be a distributive interval of finite groups. Then there is $g \in G$ such that $\langle Hg \rangle = G$.

This paper first recalls our short proof of Theorem 1.1 and then gives a self-contained group-theoretic proof of the following dual version, where $G_{(V^H)}$ will be the pointwise stabilizer subgroup of $G$ for the fixed-point subspace $V^H$ (see Definition 3.1).

**Theorem 1.2.** Let $[H, G]$ be a distributive interval of finite groups. Then there exists an irreducible complex representation $V$ of $G$ such that $G_{(V^H)} = H$.

We deduce a bridge between combinatorics and representations:

**Corollary 1.3.** The minimal number of irreducible components for a faithful complex representation of a finite group $G$ is at most the minimal length $\ell$ for an ordered chain of subgroups

$$\{e\} = H_0 < H_1 < \cdots < H_\ell = G$$

such that $[H_i, H_{i+1}]$ is distributive (or better, bottom Boolean).

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It is a non-trivial upper bound involving the subgroup lattice only. These results were first proved by the author as applications to finite group theory of results on planar algebras [5, Corollaries 6.10, 6.11]. For the convenience of the reader and for being self-contained, this paper reproduces some preliminaries of [1] and [5].

2. Ore’s theorem on distributive intervals

2.1. Basics in lattice theory. We refer to [7] for the notions of finite lattice $L$, meet $\land$, join $\lor$, subgroup lattice $\mathcal{L}(G)$, sublattice $L' \subseteq L$, interval $[a, b] \subseteq L$, minimum $\hat{0}$, maximum $\hat{1}$, atom, coatom, distributive lattice, Boolean lattice $\mathcal{B}_n$ (of rank $n$) and complement $b^\complement$ (with $b \in \mathcal{B}_n$). The top interval of a finite lattice $L$ is the interval $[t, \hat{1}]$, with $t$ the meet of all the coatoms. The bottom interval of a finite lattice $L$ is the interval $[\hat{0}, b]$, with $b$ the join of all the atoms. A lattice with a Boolean top interval will be called top Boolean; idem for bottom Boolean.

Lemma 2.1. A finite distributive lattice is top and bottom Boolean.

Proof. See [7] items a-i p254-255 which uses Birkhoff’s representation theorem (a finite lattice is distributive if and only if it embeds into some $\mathcal{B}_n$). □

2.2. The proof. Øystein Ore proved the following result in [4, Theorem 4, p267].

Theorem 2.2. A finite group $G$ is cyclic if and only if its subgroup lattice $\mathcal{L}(G)$ is distributive.

Theorem [1.1] is an extension by Ore of one side of Theorem 2.2 to any distributive interval of finite groups [4, Theorem 7, p269].

Definition 2.3. An interval of finite groups $[H, G]$ is said to be H-cyclic if there is $g \in G$ such that $\langle H, g \rangle = G$. Note that $\langle H, g \rangle = \langle Hg \rangle$.

We will give our short alternative proof of Theorem 1.1 by extending it to any top Boolean interval (see Lemma 2.1) as follows:

Theorem 2.4. A top Boolean interval $[H, G]$ is H-cyclic.

Proof. The proof follows from the claims below.

Claim: Let $M$ be a maximal subgroup of $G$. Then $[M, G]$ is $M$-cyclic.

Proof: For $g \in G$ with $g \notin M$, we have $\langle M, g \rangle = G$ by maximality. ■

Claim: A Boolean interval $[H, G]$ is H-cyclic.
Proof: Let $M$ be a coatom in $[H, G]$, and $M^\complement$ be its complement. By the previous claim and induction on the rank of the Boolean lattice, we can assume $[H, M]$ and $[H, M^\complement]$ both to be $H$-cyclic, i.e. there are $a, b \in G$ such that $\langle H, a \rangle = M$ and $\langle H, b \rangle = M^\complement$. For $g = ab$, $a = gb^{-1}$ and $b = a^{-1}g$, so $\langle H, a, g \rangle = \langle H, g, b \rangle = \langle H, a, b \rangle = M \lor M^\complement = G$. Now, $\langle H, g \rangle = \langle H, g \rangle \lor H = \langle H, g \rangle \lor (M \land M^\complement)$ but by distributivity $\langle H, g \rangle \lor (M \land M^\complement) = (\langle H, g \rangle \lor M) \land (\langle H, g \rangle \lor M^\complement)$. So $\langle H, g \rangle = \langle H, a, g \rangle \land \langle H, g, b \rangle = G$. The result follows. ■ □

Claim: $[H, G]$ is $H$-cyclic if its top interval $[K, G]$ is $K$-cyclic.

Proof: Consider $g \in G$ with $\langle K, g \rangle = G$. For any coatom $M \in [H, G]$, we have $K \subseteq M$ by definition, and so $g \notin M$, then a fortiori $\langle H, g \rangle \subsetneq M$. It follows that $\langle H, g \rangle = G$. ■ □

The converse is false because $\langle S_2, (1234) \rangle = S_4$ whereas $[S_2, S_4]$ is not top Boolean.

3. Dual Ore’s theorem on distributive intervals

3.1. Basics in Galois connections.

Definition 3.1. Let $W$ be a representation of a group $G$, $K$ a subgroup of $G$, and $X$ a subspace of $W$. We define the fixed-point subspace

$$W^K := \{w \in W \mid kw = w, \forall k \in K\},$$

and the pointwise stabilizer subgroup

$$G_{(X)} := \{g \in G \mid gx = x, \forall x \in X\}.$$ 

Lemma 3.2. Let $G$ be a finite group, $H, K$ two subgroups, $V$ a representation of $G$ and $X, Y$ two subspaces of $V$. Then

1. $H \subseteq K \Rightarrow V^K \subseteq V^H$,
2. $X \subseteq Y \Rightarrow G_{(Y)} \subseteq G_{(X)}$,
3. $V^{H \lor K} = V^H \lor V^K$,
4. $H \subseteq G_{(V^H)}$,
5. $V^{G_{(V^H)}} = V^H$,
6. $[H \subseteq K$ and $V^K \subseteq V^H] \Rightarrow K \not\subseteq G_{(V^H)}$.

Proof. (1) and (2) are immediate.

(3) First $H, K \subseteq H \lor K$, so $V^{H \lor K}$ is included in $V^H$ and $V^K$, so in $V^H \lor V^K$. Now take $v \in V^H \lor V^K$, then $\forall h \in H$ and $\forall k \in K$, $hv = kv = v$, but any element $g \in H \lor K$ is of the form $h_1 k_1 h_2 k_2 \cdots h_r k_r$ with $h_i \in H$ and $k_i \in K$, it follows that $gv = v$ and so $V^H \lor V^K \subseteq V^{H \lor K}$.

(4) Take $h \in H$ and $v \in V^H$. Then by definition $hv = v$, so $H \subseteq G_{(V^H)}$.

(5) From (1) and (4) we deduce that $V^{G_{(V^H)}} \subseteq V^H$. Now take $v \in V^H$
and \( g \in G_{(V^G)} \), by definition \( g v = v \), so \( V^G \subseteq V^{G_{(V^G)}} \) also.

(6) Suppose that \( K \subseteq G_{(V^K)} \), then \( V^K \supseteq V^{G_{(V^K)}} = V^K \) by (1) and (5). Hence \( V^K = V^H \) by (1), contradiction with \( V^K \not\subseteq V^H \). \( \square \)

3.2. Induced representation.

**Definition 3.3.** Let \( G \) be a finite group and \( H \) a subgroup. Consider the set \( G/H = \{g_1H, \ldots, g_sH\} \), with \( g_1 = e \). Let \( V \) be a complex representation of \( H \). The induced representation \( \text{Ind}_{H}^{G}(V) \) is a space \( \bigoplus_i g_iV \) (we identify \( eV \) with \( V \)) on which \( G \) acts as follows:

\[
g \cdot \left( \sum_i g_i v_i \right) = \sum_i g_{\tau(i,g)}(h_{i,g} \cdot v_i)
\]

with \( gg_i = g_{\tau(i,g)}h_{i,g}, \tau(i,g) \in \{1, \ldots, s\} \) and \( h_{i,g} \in H \).

Let \( \langle \cdot, \cdot \rangle \) be the usual normalized inner product of finite dimensional complex representations (up to equivalence) of a finite group \( G \).

**Lemma 3.4** (Frobenius reciprocity, [3] p62). Let \( G \) be a finite group and \( H \) a subgroup. Let \( V \) (resp. \( W \)) be a finite dimensional complex representation of \( G \) (resp. of \( H \)). Let \( \text{Ind}(W) \) be the induction to \( G \) and \( \text{Res}(V) \) the restriction to \( H \), then \( \langle V, \text{Ind}(W) \rangle_G = \langle \text{Res}(V), W \rangle_H \).

**Lemma 3.5.** Let \([H, G]\) be an interval of finite groups. Let \( V_1, \ldots, V_r \) be the irreducible complex representations of \( G \) (up to equivalence). Then

\[
|G : H| = \sum_{i=1}^r \dim(V_i) \dim(V_i^H).
\]

**Proof.** The following proof is due to Tobias Kildetoft. Let \( 1^{G_H} \) be the trivial representation of \( H \) induced to \( G \). On one hand, it has dimension \( |G : H| \), and on the other hand, this dimension is also

\[
\sum_i \dim(V_i)\langle V_i, 1^{G_H} \rangle_G = \sum_i \dim(V_i)\langle V_i, 1^H \rangle_H = \sum_i \dim(V_i) \dim(V_i^H).
\]

The first equality follows from Frobenius reciprocity. \( \square \)

**Lemma 3.6.** Let \([H, G]\) be an interval of finite groups with \( H \neq G \). Then there is a non-trivial irreducible complex representation \( V \) of \( G \) such that \( V^H \neq 0 \).

**Proof.** By Lemma 3.5 \( \sum_i \dim(V_i) \dim(V_i^H) = |G : H| \geq 2 \) because \( H \neq G \). The result follows. \( \square \)
Lemma 3.7. Let $G$ be a finite group, $K$ a subgroup and $U$ a finite dimensional complex representation of $K$. For any irreducible component $V$ of the induction $W = \text{Ind}_K^G(U)$, there exists $\tilde{U} \subseteq V$ equivalent to $U$ as a representation of $K$.

Proof. Direct by Frobenius reciprocity because $\langle V, W \rangle_G = \langle V, U \rangle_K$. □

Lemma 3.8. Let $[H, G]$ be an interval of finite groups, $K \in [H, G]$ and $U$ a finite dimensional complex representation of $K$ such that $U^H \neq 0$. Let $W$ be the induction $\text{Ind}_K^G(U)$. Let $V$ be an irreducible component of $W$. Then

1. $G(W^H) \subseteq K(U^H)$,
2. $V^H \neq 0$ and $K(V^H) \subseteq K(U^H)$,
3. if $K(U^H) = H$ then $G(W^H) = K(V^H) = H$.

Proof. (1) $U^H \subseteq W^H$, so by Lemma 3.2(2), $G(W^H) \subseteq G(U^H)$. Now, $U^H \neq 0$, so by definition of the induction, $g \cdot U^H \subseteq U$ if and only if $gK = K$, if and only if $g \in K$. Thus $G(U^H) \subseteq K(U^H)$, and so $G(W^H) \subseteq K(U^H)$.

(2) Take $\tilde{U}$ as for Lemma 3.7. Then $0 \neq \tilde{U}^H \subseteq V^H$ and $K(V^H) \subseteq K(U^H) = K(V^H)$.

(3) By (1) and Lemma 3.2(4),

$$H \subseteq G(W^H) \subseteq K(U^H) = H,$$

so $G(W^H) = H$. Idem, by (2) and Lemma 3.2(4), $K(V^H) = H$. □

3.3. The proof.

Definition 3.9. The group $G$ is called linearly primitive if it admits an irreducible complex representation $V$ which is faithful, i.e. $G(V) = \{e\}$.

Definition 3.10. The interval $[H, G]$ is called linearly primitive if there is an irreducible complex representation $V$ of $G$ such that $G(V^H) = H$.

Lemma 3.11. A maximal interval $[H, G]$ is linearly primitive.

Proof. By Lemma 3.6, there is a non-trivial irreducible complex representation $V$ of $G$ with $V^H \neq 0$. By Lemma 3.2(4), $H \subseteq G(V^H)$. If $G(V^H) = G$ then $V$ must be trivial (by irreducibility), so by maximality $G(V^H) = H$. □

Lemma 3.12. The interval $[H, G]$ is linearly primitive if its bottom interval $[H, K]$ is so.
Proof. Let \([H, K]\) be the bottom interval of \([H, G]\), i.e. \(K = \bigvee_i K_i\) with \(K_1, \ldots, K_n\) the atoms of \([H, G]\). By assumption, there is an irreducible complex representation \(U\) of \(K\) such that \(K(U) = H\). Let \(V\) be an irreducible component of \(\text{Ind}_K^G(U)\). By Lemma 3.8(3), \(K(V) = H\). Now if \(\exists i\) such that \(V \subseteq K_i \subseteq K(V) = H\), contradiction with \(H \not\subseteq K_i\). So \(\forall i \ V \not\subseteq V^{K_i} \subseteq V\). By Lemma 3.2(6), we deduce that \(K_i \not\subseteq G(V) \ \forall i\), so by minimality \(G(V) = H\). □

A dual version of Theorem 2.4 is the following:

**Theorem 3.13.** A bottom Boolean interval \([H, G]\) is linearly primitive.

**Proof.** By Lemma 3.12, we can reduced to Boolean intervals. We make an induction on the rank of the Boolean lattice. The rank one case is handled in Lemma 3.11. Assume that it is true at rank \(n < n\). We will write a proof at rank \(n \geq 2\). Let \(K\) be a coatom of \([H, G]\). Then \([H, K]\) is Boolean of rank \(n - 1\), so by assumption, it is linearly primitive, thus there is an irreducible complex representation \(U\) of \(K\) such that \(K(U) = H\). For any irreducible component \(V\) of \(W = \text{Ind}_K^G(U)\), we have

\[K \cap G(V) = K(V) = H\]

by Lemma 3.8(3). Thus, by the Boolean structure, \(G(V) \leq K\) because

\[K \cap = K \cap V = K \cap V \cap (K \cap G(V)) = (K \cap V) \cap (K \cap G(V)) = K \cap G(V).

But \(K\) is a coatom of \([H, G]\), so \(K\) is an atom and then

\[G(V) \in \{H, K\}\]  

Assume that every irreducible component \(V\) of \(W\) satisfies \(G(V) = K\). There are irreducible complex representations \(V_1, \ldots, V_r\) of \(G\) such that \(W = \bigoplus_i V_i\), then by Lemma 3.8(3)

\[H = G(W) = \bigwedge_i G(V_i) = K,\]

thus \(H = K\), contradiction. So there is an irreducible component \(V\) of \(W\) such that \(G(V) = H\), and the result follows. □

Theorem 1.2 follows directly from Theorem 3.13 and Lemma 2.1.
4. A BRIDGE BETWEEN COMBINATORICS AND REPRESENTATIONS

We restate Corollary 1.3 by using the notation $H^i$ instead of $H_i$ because it will be more convenient for the proof.

**Corollary 4.1.** The minimal number of irreducible components for a faithful complex representation of a finite group $G$ is at most the minimal length $\ell$ for an ordered chain of subgroups $\{e\} = H^0 < H^1 < \cdots < H^\ell = G$

such that $[H^i, H^{i+1}]$ is distributive (or better, bottom Boolean).

**Proof.** By Theorem 3.13 and Lemma 3.8(3), there are irreducible complex representations $V_1, \ldots, V_\ell$ of $G$ such that $H^i_{(V_i H^{i-1})} = H^{i-1}$. Take $W = \bigoplus_{i=1}^\ell V_i$. Then

$$\ker(\pi_W) = \bigwedge_{i=1}^\ell \ker(\pi_{V_i}) = \bigwedge_{i=1}^\ell G(V_i) \leq \bigwedge_{i=1}^\ell G(V_i H^{i-1}) = (\bigwedge_{i=1}^{\ell-1} G(V_i H^{i-1})) \wedge H^{\ell-1}$$

$$= \bigwedge_{i=1}^{\ell-1} H^{\ell-1}_{(V_i H^{i-1})} = \cdots = \bigwedge_{i=1}^{\ell-s} H^{\ell-s}_{(V_i H^{i-1})} = \cdots = H^0 = \{e\}.$$ 

So $W$ is faithful with $\ell$ irreducible components. The result follows. \square

Note that this upper bound involves the subgroup lattice only.

**Remark 4.2.** The modular maximal-cyclic group $M_4(2)$ and the abelian group $C_8 \times C_2$ have the same subgroup lattice, but the first is linearly primitive whereas the second is not. So the minimal number in Corollary 4.1 cannot be determined by the subgroup lattice only.

**Lemma 4.3.** For $H$ core-free, $G$ is linearly primitive if $[H, G]$ is so.

**Proof.** Let $V$ be an irreducible complex representation of $G$ such that $G_{(V H)} = H$. Now, $V^H \subseteq V$ so $G_{(V)} \subseteq G_{(V H)}$, but $\ker(\pi_V) = G_{(V)}$, it follows that $\ker(\pi_V) \subseteq H$; but $H$ is a core-free subgroup of $G$, and $\ker(\pi_V)$ a normal subgroup of $G$, so $\ker(\pi_V) = \{e\}$. \square

By Lemma 4.3 we can improve the bound of Corollary 4.1 by taking for $H^0$ any core-free subgroup of $H^1$ (instead of just $\{e\}$), and we can wonder whether it is the right answer in general, in particular:

**Question 4.4.** Is a finite group $G$ linearly primitive if and only if there is a core-free subgroup $H$ with $[H, G]$ bottom Boolean?
It is true for any finite simple group $S$, because any proper subgroup $M$ is core-free, and by choosing it maximal, $[M,S]$ is Boolean of rank one. Moreover, we have checked by GAP [2] that it is also true for any finite group $G$ of order less than 512.

**Remark 4.5.** A normal subgroup $N \trianglelefteq G$ is a modular element in $\mathcal{L}(G)$, see [6, p43]. If $H$ is a subgroup of $G$ such that $\forall K \in (1,H]$, $K$ is not modular in $\mathcal{L}(G)$, then $H$ is core-free. It follows that we can also improve the bound of Corollary 4.1 completely combinatorially. Nevertheless, by Remark 4.2, it cannot be the right answer in general.

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Institute of Mathematical Sciences, Chennai, India

E-mail address: sebastienpalcoux@gmail.com