Abstract. We analyse logic actions of Polish groups which arise in continuous logic. We extend the generalised model theory of H.Becker from [3] to the case of Polish $G$-spaces when $G$ is an arbitrary Polish group.

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0 Introduction

The paper is devoted to Polish group actions from the point of view of continuous logic. Let $(Y,d)$ be a Polish space and $Iso(Y)$ be the corresponding isometry group endowed with the pointwise convergence topology. Then $Iso(Y)$ is a Polish group.

It is worth noting that any Polish group $G$ can be realised as a closed subgroup of the isometry group $Iso(Y)$ of an appropriate Polish space $(Y,d)$. Moreover it is shown by J.Melleray in [26] (Theorem 6) that $G$ can be chosen as the automorphism group of a continuous metric structure on $(Y,d)$ which is approximately ultrahomogeneous.

For any countable continuous signature $L$ the set $Y_L$ of all continuous metric $L$-structures on $(Y,d)$ can be considered as a Polish $Iso(Y)$-space. We call this action

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logic and show that it is universal for Borel reducibility of orbit equivalence relations of Polish $G$-spaces with closed $G \leq Iso(Y)$.

Note that for any tuple $\bar{s} \in Y$ the map $g \to d(\bar{s}, g(\bar{s}))$ can be considered as a graded subgroup of $G$. For any continuous sentence $\phi$ we have a graded subset of $Y_L$ defined by $M \to \phi^M$.

Typical notions naturally arising for logic actions can be applied in the general case of a Polish $G$-space $X$ with $G$ as above. If we consider $G$ together with a family of graded subgroups as above, then distinguishing an appropriate family $B$ of graded subsets of $X$ we arrive at the situation very similar to the logic space $Y_L$. For example we can treat elements of $B$ as continuous formulas.

In the case of the logic action of closed subgroups of $S_\infty$ on the space of $X_L$ of discrete structures on $\omega$ this approach was realised by H.Becker in [2] and [3]. Immitating topologies generated by sets of the form $Mod(\phi, \bar{s}) = \{ M \in X_L : M \models \phi(\bar{s}) \}$, where $\bar{s} \in \omega$ and formulas $\phi$ are chosen from a countable fragment of $L_{\omega_1\omega}$, he introduces the concept of a nice topology. Many theorems of traditional model theory can be generalised to topological statements concerning spaces with nice topologies.

In fact we generalise this approach to the general case of Polish $G$-actions removing the Becker’s assumption that $G < S_\infty$. In particular we define graded Vaught transforms, graded nices bases and prove an existence theorem stating that under natural circumstances appropriately defined nice topologies can be always found (see Section 2).

When a family of graded subsets $B$ generates such a topology as a nice graded basis (a term which will be defined later) then treating elements of $B$ as continuous formulas, we may extend theorems of continuous model theory to some topological statements. In Section 4 we obtain topological versions of several theorems from logic, for example Ryll-Nardzewski theorem.

The construction of nice bases arises in the most natural form when one considers the case of continuous logic actions over $U$, the Urysohn space of diameter 1. Let $L$ be a countable continuous signature and $U_L$ be the $Iso(U)$-space of all continuous $L$-structures. Our main theorem of Section 3 roughly states that graded subsets associated with continuous $L$-formulas form a nice basis on $U_L$.

We may interpret this theorem that $U$ is the continuous counterpart of $\omega$ in moving from the case of discrete logic $S_\infty$-actions to the case of actions on spaces of continuous structures. This motivates some very basic questions. A theorem from [3] (Corollary 1.13) states that in the case of logic actions of $S_\infty$ on the space of countable structures each nice topology is defined by model sets $Mod(\phi)$ of formulas of a countable fragment of $L_{\omega_1\omega}$. Can this statement be extended to $U_L$?

This issue depends on the Lopez-Escobar theorem on invariant Borel subsets of logic spaces (one of the ingredients of Corollary 1.13 of [3]). Is there an appropriate continuous version of the Lopez-Escobar theorem? We find an example which shows some complications in possible generalisations of this material over $U$. On the other hand it is worth noting that very recently (a year after the first arXiv version of our paper) S.Coskey, M.Lupini in [13] and I.Ben Yaacov, A.Nies and T.Tsankov in [9] have proved very natural continuous versions of the Lopez-Escobar theorem.
Viewing the logic space $Y_L$ as a Polish space one may consider Borel/algorithmic complexity of interesting subsets of $Y_L$. This is the main concern of Section 5 (the last one). It demonstrates some new setting arising in the approach of the logic space of continuous structures.

We now give some preliminaries in detail.

**Polish group actions.** A Polish space (group) is a separable, completely metrizable topological space (group). Sometimes we extend the corresponding metric to tuples by

$$d((x_1, ..., x_n), (y_1, ..., y_n)) = \max(d(x_1, y_1), ..., d(x_n, y_n)).$$

If a Polish group $G$ continuously acts on a Polish space $X$, then we say that $X$ is a Polish $G$-space. We say that a subset of $X$ is invariant if it is $G$-invariant.

If $B$ is a subset of $X$ and $u$ is a non-empty open subset of $G$ then let

$$B^*u = \{x \in X : \{g \in u : gx \in B\} \text{ is comeagre in } u\},$$

$$B^\Delta u = \{x \in X : \{g \in u : gx \in B\} \text{ is not meagre in } u\}$$

These operations are called Vaught transforms. Their properties can be found in Section 5 of [4].

Let $(Y, d)$ be a Polish space and $Iso(Y)$ be the corresponding isometry group endowed with the pointwise convergence topology. Then $Iso(Y)$ is a Polish group. A compatible left-invariant metric can be obtained as follows: fix a countable dense set $S = \{s_i : i \in \{1, 2, ...\}\}$ and then define for two isometries $\alpha$ and $\beta$ of $Y$

$$\rho_S(\alpha, \beta) = \sum_{i=1}^{\infty} 2^{-i} \min(1, d(\alpha(s_i), \beta(s_i))).$$

The metric completion of $(Iso(Y, d), \rho_S)$ can be naturally considered as a semigroup of isometric embeddings of $(Y, d)$ into itself. Let $In(Y)$ be the semigroup of all isometric embeddings of this space.

We will study closed subgroups of $Iso(Y)$. We fix a dense countable set $\Upsilon \subset Iso(Y)$. In any closed subgroups of $Iso(Y, d)$ we distinguish the base consisting of all sets of the form $N_{\sigma, q} = \{\alpha : \rho_S(\alpha, \sigma) < q\}$, $\sigma \in \Upsilon$ and $q \in \mathbb{Q}$.

**Continuous structures.** We now fix a countable continuous signature

$$L = \{d, R_1, ..., R_k, ..., F_1, ..., F_l, ...\}.$$ 

Let us recall that a metric $L$-structure is a complete metric space $(M, d)$ with $d$ bounded by 1, along with a family of uniformly continuous operations on $M$ and a family of predicates $R_i$, i.e. uniformly continuous maps from appropriate $M^{k_i}$ to $[0, 1]$. It is usually assumed that to a predicate symbol $R_i$ a continuity modulus $\gamma_i$ is assigned so that when $d(x_j, x'_j) < \gamma_i(\varepsilon)$ with $1 \leq j \leq k_i$ the corresponding predicate of $M$ satisfies

$$|R_i(x_1, ..., x_j, ..., x_{k_i}) - R_i(x_1, ..., x'_j, ..., x_{k_i})| < \varepsilon.$$
It happens very often that $\gamma_i$ coincides with $id$. In this case we do not mention the appropriate modulus. We also fix continuity moduli for functional symbols.

Note that each countable structure can be considered as a complete metric structure with the discrete $(0,1)$-metric.

Atomic formulas are the expressions of the form $R_i(t_1,\ldots,t_r)$, $d(t_1,t_2)$, where $t_i$ are terms (built from functional $L$-symbols). In metric structures they can take any value from $[0,1]$. Statements concerning metric structures are usually formulated in the form

$$\phi = 0,$$

where $\phi$ is a formula, i.e. an expression built from 0,1 and atomic formulas by applications of the following functions:

$$x/2, \quad x-y = \max(x-y,0), \quad \min(x,y), \quad \max(x,y), \quad |x-y|,$$

$$\neg(x) = 1-x, \quad x+y = \min(x+y,1), \quad \sup x \text{ and } \inf x.$$  

Sometimes statements are called conditions; we will use both names. A theory is a set of statements without free variables (here $\sup x$ and $\inf x$ play the role of quantifiers).

We often extend the set of formulas by the application of dotted products by positive rational numbers. This means that when $q \cdot x$ is greater than 1, the dotted product of $q$ and $x$ is 1. Since the context is always clear, we preserve the same notation $q \cdot x$. The continuous logic after this extension does not differ from the basic case.

It is worth noting that for any continuous relational structure $M$, any formula is a $\gamma$-uniform continuous function from the appropriate power of $M$ to $[0,1]$, where $\gamma$ is bounded by a linear combination of the continuity moduli of $L$-symbols appearing in the formula.

It is observed in Appendix A of [11] that instead of continuity moduli one can consider inverse continuity moduli. Slightly modifying that place in [11] we define it as follows.

**Definition 0.1** A continuous monotone function $\delta : [0,1] \to [0,1]$ with $\delta(0) = 0$ is an inverse continuity modulus of a map $F(\bar{x}) : X^n \to [0,1]$ if for any $\bar{a}, \bar{b}$ from $X^n$,

$$|F(\bar{a}) - F(\bar{b})| \leq \delta(\max_{1 \leq i \leq n}(d(a_i,b_i))).$$

It is also worth noting that for any continuous relational structure $M$, where any $n$-ary relation has $n \cdot id$ as an inverse continuity modulus, any formula admits an inverse continuity modulus which is a linear function.

For a continuous structure $M$ defined on $(Y,d)$ let $Aut(M)$ be the subgroup of $Iso(Y,d)$ consisting of all isometries preserving the values of atomic formulas. It is easy to see that $Aut(M)$ is a closed subgroup with respect to the topology on $Iso(Y)$ defined above.

For every $c_1,\ldots,c_n \in M$ and $A \subseteq M$ we define the $n$-type $tp(\bar{c}/A)$ of $\bar{c}$ over $A$ as the set of all $\bar{x}$-conditions with parameters from $A$ which are satisfied by $\bar{c}$ in $M$. Let $S_n(T_A)$ be the set of all $n$-types over $A$ of the expansion of the theory $T$ by constants
from $A$. There are two natural topologies on this set. The logic topology is defined by the basis consisting of sets of types of the form $[\phi(\bar{x}) < \varepsilon]$, i.e. types containing some $\phi(\bar{x}) \leq \varepsilon'$ with $\varepsilon' < \varepsilon$. The logic topology is compact.

The $d$-topology is defined by the metric

$$d(p, q) = \inf \{ \max_{i \leq n} d(c_i, b_i) \mid \text{there is a model } M \models p(\bar{c}) \land q(\bar{b}) \}. $$

By Propositions 8.7 and 8.8 of [5] the $d$-topology is finer than the logic topology and $(S_n(T_A), d)$ is a complete space.

Definability in continuous structures is introduced as follows.

**Definition 0.2** Let $A \subseteq M$. A predicate $P : M^n \to [0, 1]$ is definable in $M$ over $A$ if there is a sequence $(\phi_k(x) : k \geq 1)$ of $L(A)$-formulas such that predicates interpreting $\phi_k(x)$ in $M$ converge to $P(x)$ uniformly in $M^n$.

A theory $T$ is separably categorical if any two separable models of $T$ are isomorphic. By Theorem 12.10 of [5] a complete theory $T$ is separably categorical if and only if for each $n > 0$, every $n$-type $p$ is principal. The latter means that for every model $M \models T$, the predicate $\text{dist}(x, p(M))$ is definable over $\emptyset$.

Another property equivalent to separable categoricity states that for each $n > 0$, the metric space $(S_n(T), d)$ is compact. In particular for every $n$ and every $\varepsilon$ there is a finite family of principal $n$-types $p_1, ..., p_m$ so that their $\varepsilon$-neighbourhoods cover $S_n(T)$.

In the classical first order logic a countable structure $M$ is $\omega$-categorical if and only if $\text{Aut}(M)$ is an oligomorphic permutation group, i.e. for every $n$, $\text{Aut}(M)$ has finitely many orbits on $M^n$. In continuous logic we have the following modification.

**Definition 0.3** An isometric action of a group $G$ on a metric space $(X, d)$ is said to be approximately oligomorphic if for every $n \geq 1$ and $\varepsilon > 0$ there is a finite set $F \subset X^n$ such that

$$G \cdot F = \{ g\bar{x} : g \in G \text{ and } \bar{x} \in F \}$$

is $\varepsilon$-dense in $(X^n, d)$.

Assuming that $G$ is the automorphism group of a non-compact separable continuous metric structure $M$, $G$ is approximately oligomorphic if and only if the structure $M$ is separably categorical (C. Ward Henson, see Theorem 4.25 in [29]). It is also known that separably categorical structures are approximately homogeneous in the following sense: if $n$-tuples $\bar{a}$ and $\bar{c}$ have the same types (i.e. the same values $\phi(\bar{a}) = \phi(\bar{b})$ for all $L$-formulas $\phi$) then for every $c_{n+1}$ and $\varepsilon > 0$ there is an tuple $b_1, ..., b_n, b_{n+1}$ of the same type with $\bar{c}, c_{n+1}$, so that $d(a_i, b_i) \leq \varepsilon$ for $i \leq n$. In fact for any $n$-tuples $\bar{a}$ and $\bar{b}$ there is an automorphism $\alpha$ of $M$ such that

$$d(\alpha(\bar{c}), \bar{a}) \leq d(tp(\bar{a}), tp(\bar{c})) + \varepsilon.$$  

(i.e $M$ is strongly $\omega$-near-homogeneous in the sense of Corollary 12.11 of [5]).
The following notion is helpful when we study some concrete examples, for example the Urysohn space. A relational continuous structure $M$ is approximately ultrahomogeneous if for any $n$-tuples $(a_1, ..., a_n)$ and $(b_1, ..., b_n)$ with the same quantifier-free type (i.e. with the same values of predicates for corresponding subtuples) and any $\varepsilon > 0$ there exists $g \in Aut(M)$ such that

$$\max\{d(g(a_j), b_j) : 1 \leq j \leq n\} \leq \varepsilon.$$ 

As we already mentioned any Polish group can be chosen as the automorphism group of a continuous metric structure which is approximately ultrahomogeneous.

The bounded Urysohn space $U$ (see Section 3) is ultrahomogeneous in the traditional sense: any partial isomorphism between two tuples extends to an automorphism of the structure $[31]$. Note that this obviously implies that $U$ is approximately ultrahomogeneous.

## 1 The space of metric structures.

In the first part of this section we introduce logic actions of isometry groups on spaces of continuous structures. In the second part we prove that these spaces are universal for Borel reducibility of orbits equivalence relations.

### 1.1 Logic action

We now fix a countable continuous signature

$$L = \{d, R_1, ..., R_k, ..., F_1, ..., F_l, ...\}$$

and a Polish space $(Y, d)$. Let $S$ be a dense countable subset of $Y$. Let $seq(S) = \{\bar{s}_i : i \in \omega\}$ be the set (and an enumeration) of all finite sequences (tuples) from $S$. Let us define the space of metric $L$-structures on $(Y, d)$. Using the recipe as in the case of $Iso(Y)$ we introduce a metric on the set of $L$-structures as follows. Enumerate all tuples of the form $(\varepsilon, j, \bar{s})$, where $\varepsilon \in \{0, 1\}$ and when $\varepsilon = 0$, $\bar{s}$ is a tuple from $seq(S)$ of the length of the arity of $R_j$, and for $\varepsilon = 1$, $\bar{s}$ is a tuple from $seq(S)$ of the length of the arity of $F_j$. For metric $L$-structures $M$ and $N$ let

$$\delta_{seq(S)}(M, N) = \sum_{i=1}^{\infty} \{2^{-i}|R_j^M(\bar{s}) - R_j^N(\bar{s})| : i \text{ is the number of } (\varepsilon, j, \bar{s})\}.$$ 

Since the predicates and functions are uniformly continuous (with respect to moduli of $L$) and $S$ is dense in $Y$, we see that $\delta_{seq(S)}$ is a complete metric. Moreover by an appropriate choice of rational values for $R_j(\bar{s})$ we find a countable dense subset of metric structures on $Y$, i.e. the space obtained is Polish. We denote it by $Y_L$. It is clear that $Iso(Y)$ acts on $Y_L$ continuously. Thus we consider $Y_L$ as an $Iso(Y)$-space and call it the space of the logic action on $Y$.

$^1$resp. $2^{-i}d(F_j^M(\bar{s}), F_j^N(\bar{s}))$ when $\varepsilon = 1$
It is convenient to consider the following basis of the topology of $Y_L$. Fix a finite sublanguage $L' \subset L$, a finite subset $S' \subset S$, a finite tuple $q_1, ..., q_t \in Q \cap [0,1]$ and a rational $\varepsilon \in [0,1]$ with $1 - \varepsilon < 1/2$. Consider a diagram $D$ of $L'$ on $S'$ of some inequalities of the form

$$d(F_j(\bar{s}), s') > \varepsilon, \quad d(F_j(\bar{s}), s') < 1 - \varepsilon,$$

(i.e. in the case of relations we consider negations of statements of the form: $|R_j(\bar{s}) - q_i| \leq \varepsilon$, $|R_j(\bar{s}) - q_i| \geq 1 - \varepsilon$). The set of metric $L$-structures realizing $D$ is an open set of the topology of $Y_L$ and the family of sets of this form is a basis of this topology. Compactness theorem for continuous logic (see [11]) shows that the topology is compact. We will call it logic too.

The following proposition is very helpful.

**Proposition 1.1** For any continuous formula $\phi(\bar{v}, \bar{w})$ of the language $L$ there is a natural number $n$ such that for any tuple $\bar{a} \in Y$ and $\varepsilon \in [0,1]$, the subset

$$\text{Mod}(\phi, \bar{a}, < \varepsilon) = \{(M, \bar{c}) : M \models \phi(\bar{a}, \bar{c}) < \varepsilon\}$$

( or $\text{Mod}(\phi, \bar{a}, > \varepsilon) = \{(M, \bar{c}) : M \models \phi(\bar{a}, \bar{c}) > \varepsilon\}$ )

of the corresponding space $Y_{L,e}$ of $\bar{c}$-expansions of $L$-structures, belongs to $\Sigma_n$.

**Proof.** The proof is by induction on the complexity of $\phi$. Assume that $\phi$ is atomic. A straightforward argument shows that if, for example, $\phi$ is $P(\bar{v}, \bar{c})$, then for every $\varepsilon$ the set of $L\bar{c}$-structures satisfying $\phi(\bar{a}, \bar{c}) < \varepsilon$ is open.

If $\phi$ is of the form $\neg \psi(\bar{v}, \bar{w})$, then $\text{Mod}(\phi, \bar{a}, < \varepsilon)$ is the set $\text{Mod}(\psi, \bar{a}, > 1 - \varepsilon)$, i.e. of the next $\Sigma_n$-class with respect to sets of the form $\text{Mod}(\psi, \bar{a}, < \varepsilon')$.

If $\phi$ is of the form $\max_{i \leq k} \psi_i(\bar{v}, \bar{w})$, $k \in \omega$, then $\text{Mod}(\phi, \bar{a}, < \varepsilon)$ is the intersection of the sets $\text{Mod}(\psi_i, \bar{a}, < \varepsilon)$, $i \leq k$. The cases of other Boolean connectives are similar.

When $\phi = \inf \psi(\bar{v}, u, \bar{w})$, the corresponding subset is the (countable) union

$$\bigcup \{\text{Mod}(\psi, \bar{a}, s_{k+1}, < \varepsilon) : s_{k+1} \in S\},$$

i.e. of the same $\Sigma_n$-class with $\psi$.

When $\phi = \sup \psi(\bar{v}, u, \bar{w})$, the subset $\text{Mod}(\phi, \bar{a}, < \varepsilon)$ is the (countable) union of all possible intersections

$$\bigcap \{\text{Mod}(\psi, \bar{a}, s_{k+1}, < \varepsilon') : s_{k+1} \in S\}, \quad \varepsilon' < \varepsilon \text{ and } \varepsilon' \in \mathbb{Q}.$$

$\Box$
1.2 Reduction

Let \((Y, d)\) be a Polish metric space with diameter not greater than 1 and \(G < Iso(Y, d)\) be a Polish group. The following theorem is the main result of Section 1.

**Theorem 1.2** There is a continuous relational signature \(L^*\) such that for any Polish \(G\)-space \(X\) there is a Borel 1-1-map \(M : X \rightarrow Y_L\) such that for any \(x, x' \in X\) structures \(M(x)\) and \(M(x')\) are isomorphic if and only if \(x\) and \(x'\) are in the same \(G\)-orbit.

In other words the map \(M\) is a Borel \(G\)-invariant 1-1-reduction of the \(G\)-orbit equivalence relation on \(X\) to the \(Iso(Y)\)-orbit equivalence relation on the space \(Y_{L^*}\).

This result is slightly connected with the conjecture of G. Hjorth, that for any Borel equivalence relation \(E\) Borel reducible to an orbit equivalence relation of a Polish \(G\)-space there is a Polish \(G\)-space \(X\) such that the orbit equivalence relation \(E^X_G\) is Borel and \(E \leq_B E^X_G\). Since any Polish \(G\) can be realised as an isometry group, we may always assume that \(E\) is Borel reducible to the orbit equivalence relation of an \textit{logic} action. We hope that this observation may bring some model-theoretic tools for the Hjorth’s conjecture.

Let us start with some preliminaries. Let \(⟨X, τ, d⟩\) be a Polish \(G\)-space with a basis \(A = \{A_l : l ∈ ω\}\). To describe a reduction of the \(G\)-space \(X\) to an \(Iso(Y)\)-space of countinuous structures on \((Y, d)\) we use \([26]\) and some standard ideas already applied for closed subgroups of \(S_\infty\) (see Section 6.1 of \([18]\)).

We will assume that \(d\) and \(d^\tau\) have values from \([0, 1]\) (if necessary we may replace them by \(\frac{d(x, y)}{1 + d(x, y)}\)). We fix some countable dense set \(S ⊂ Y\) and enumerate \(S = \{s_1, s_2, ...\}\) and all orbits of \(G\) of finite tuples of \(S\) (i.e. of \(Seq(S)\)). For the closure of such an \(n\)-orbit \(C\) define a predicate \(R_C\) on \((Y, d)\) by

\[
R_C(y_1, ..., y_n) = d((y_1, ..., y_n), \bar{c}) \quad \text{(i.e. } \inf \{d(\bar{y}, \bar{c}) : \bar{c} ∈ C\}\).
\]

It is proved in Theorem 6 of \([26]\) that the continuous structure \(M\) of all these predicates on \(Y\) is approximately ultrahomogeneous and \(G\) is its automorphism group.

Let \(L\) be the language of \(M\). For every pair of natural numbers \(k > 0\) and \(l\) we add to \(L\) a predicate \(R_{k,l}\) of arity \(k\). The extended language will be denoted by \(L^*\). Then to every \(x ∈ X\) we assign an \(L^*\)-expansion of \(M\) where the predicates \(R_{k,l}(y_1, ..., y_k)\) are interpreted as follows:

\[
\inf\{\max(d((h(y_1), ..., h(y_k)), (s_1, ..., s_k)), d^\tau(hx, x')) : x' ∈ A_l \text{ and } h ∈ G\}.
\]

It is easily seen, that these predicates are uniformly continuous with respect to the continuous modulus \(id\) for each variable. Let \(M(x)\) denote this expansion. Let \(M\) denote the map \(x → M(x)\). The following proposition implies the theorem above.

**Proposition 1.3** The map \(M\) is a Borel \(G\)-invariant 1-1-reduction of the \(G\)-orbit equivalence relation on \(X\) to the \(Iso(Y)\)-orbit equivalence relation on the space \(Y_{L^*}\) of all \(L^*\)-structures. Moreover the \(M\)-preimage of any open subset of \(Y_{L^*}\) belongs to \(Δ_3\) and the \(M\)-image of any open subset of \(X\) belongs to \(F_σ\).
Proof. To see $G$-invariantness note that the condition $gx = x'$ with $g \in G$, implies the property that for every $k, l \in \omega$ and $y_1, \ldots, y_k \in Y$

$$R_{k,l}^{M(x')}(y_1, \ldots, y_k) = R_{k,l}^{M(x)}(g^{-1}(y_1), \ldots, g^{-1}(y_k))$$

(i.e. $g$ maps $M(x)$ to $M(x')$). This follows from the fact that for any $h \in G$

$$\max(d((h(y_1), \ldots, h(y_k)), (s_1, \ldots, s_k)), d^\tau(hx', A_l)) =$$

$$\max(d((hg^{-1}(y_1)), \ldots, hg^{-1}(y_k)), (s_1, \ldots, s_k)), d^\tau(hgx, A_l)).$$

Let us check that the map $x \rightarrow M(x)$ is injective. Assume $x, x' \in X$ and $x \neq x'$. Then there are basic open sets $A_l$ and $A_m$ such that $d(A_l, A_m) > 0$, $A_l \cap \{x, x'\} = \{x\}$ and $A_m \cap \{x, x'\} = \{x'\}$. Since the $G$-action is continuous, there is an open set $V \subset G$ containing the identity such that $Vx \subseteq A_l$ and $Vx' \subseteq A_m$. We may think that $V$ consists of all $h \in G$ such that for some sufficiently small $\varepsilon$ and a natural $k$

$$\sum_{i=1}^{k} 2^{-i} \min(1, d(h(c_i), c_i)) < \varepsilon.$$ 

This obviously means that $R_{k,l}^{M(x)}(c_1, \ldots, c_k)$ (which is 0) cannot be equal to $R_{k,l}^{M(x')} (c_1, \ldots, c_k)$, i.e. $M(x) \neq M(x')$.

We can now see that if the structures $M(x)$ and $M(x')$ are isomorphic then $x$ and $x'$ belong to the same $G$-orbit. Indeed, by the choice of relations in $M(x)$ such an isomorphism can be realized by an element (say $g$) of $G$. Then we see that $M(x') = M(gx)$ and thus by the definition of relations $R_{k,l}$ (in particular for tuples of the form $c_1, \ldots, c_k$), $x' = gx$.

Let us prove the last statement of the proposition. Fix a finite sublanguage $L' \subseteq L^*$, a finite subset $S' \subseteq S$, a finite tuple $q_1, \ldots, q_l \in Q \cap [0, 1]$ and a small rational $\varepsilon \in [0, 1]$. Consider a diagram $D$ of $L'$ on $S'$ consisting of atomic statements and negations (in the standard sense) of atomic statements of the following form:

$$|R(\bar{s}) - q_i| \leq \varepsilon, |R(\bar{s}) - q_i| \geq \varepsilon.$$

The set of metric $L^*$-structures realizing $D$ is an $F_\sigma$-set of the logic topology. Since each $M(x)$ belongs to the closed subset of all expansions of $M$ we will assume that $D$ is consistent with the elementary diagram of $M$. As a result $D$ is determined by formulas of the form (with or without $\neg$ in its standard meaning)

$$|R_{k,l}(s_{i_1}, \ldots, s_{i_k}) - q_i| \leq \varepsilon, |R_{k,l}(s_{i_1}, \ldots, s_{i_k}) - q_i| \geq \varepsilon, s_{i_j} \in C'.$$

Thus the set of $x$ with $M(x) \models D$ is a finite intersection of sets of the form

$$\{x : \inf\{\max(d((h(s_{i_1}), \ldots, h(s_{i_k})), (s_1, \ldots, s_k)), d^\tau(hx, x')) : x' \in A_l \text{ and } h \in G\} < \varepsilon\},$$

$$\{x : \inf\{\max(d((h(s_{i_1}), \ldots, h(s_{i_k})), (s_1, \ldots, s_k)), d^\tau(hx, x')) : x' \in A_l \text{ and } h \in G\} \leq \varepsilon\}.$$
(or their complements), where the first one is open and thus the second one is an $F_\sigma$-set. From this we conclude that the preimage of an open set belongs to $\Delta_3$.

Now consider the case of the $\mathcal{M}$-image of the basic open set $A_i$. Note that for any point $a$ of $A_i$ there is some $A_m$ with the closure satisfying $a \in \overline{A_m} \subseteq A_i$. On the other hand it is easy to see that the $\mathcal{M}$-image of the closure $\overline{A_m}$ consists of all $M(x)$ such that for every $k$, $R_{k,m}(s_1,\ldots,s_k) = 0$. It is clear that this is a closed set. This means that the $\mathcal{M}$-image of $A_i$ is an $F_\sigma$-set. □

2 Graded subsets and nice bases

In this section we introduce the main notions of the paper and develop the basic theory of nice topologies. In particular in the second part of the section we prove an existence theorem for nice topologies. The first part develops necessary techniques.

2.1 Graded subsets

Proposition 1.1 naturally fits to the notion of graded subsets, introduced in [8]. Let us recall this notion.

A function $\phi$ from a space $X$ to $[-\infty, +\infty]$ is upper (lower) semi-continuous if the set $\phi_{<r}$ (resp. $\phi_{\geq r}$) is open for all $r \in \mathbb{R}$ (here $\phi_{<r} = \{z \in X : \phi(z) < r\}$, a cone).

A graded subset of $X$, denoted $\phi \subseteq X$, is a function $X \to [0, \infty]$. It is open (closed), $\phi \subseteq_o X$ (resp. $\phi \subseteq_c X$), if it is upper (lower) semi-continuous. We also write $\phi \in \Sigma_1$ when $\phi \subseteq_o X$ and we write $\phi \in \Pi_1$ when $\phi \subseteq_c X$. We will assume below that values of a graded subset belong to $[0, 1]$.

By Lemma 2.3 of [8] if $\Phi$ is a family of upper semi-continuous functions, then $\inf \Phi : x \to \inf \{f(x) : f \in \Phi\}$ is upper semi-continuous as well. If additionally $X$ admits a countable base then there exists a countable subfamily $\Phi_0 \subseteq \Phi$ such that $\inf \Phi_0 = \inf \Phi$.

When $G$ is a Polish group, then a graded subset $H \subseteq G$ is called a graded subgroup if

$$H(1) = 0 \ , \ \forall g \in G(H(g) = H(g^{-1})) \text{ and } \forall g, g' \in G(H(gg') \leq H(g) + H(g')).$$

By induction we define Borel classes $\Sigma_\alpha$, $\Pi_\alpha$ with $\Pi_\alpha = \{\neg \phi : \phi \in \Sigma_\alpha\}$. We will say that a graded subset $\phi$ is $\Sigma_\alpha$ if $\phi = \inf \Phi$ for some countable family of graded subsets from $\bigcup\{\Pi_\gamma : \gamma < \alpha\}$. Note that in this case $\phi_{<r} \in \Sigma^X_\alpha$ for all $r \in [0, 1]$. On the other hand if $\phi \in \Pi_\alpha$, then for any $r > 0$, $\phi_{\leq r} \in \Pi^X_\alpha$. In this paper we consider merely Borel graded subsets.

By standard inductive argument one can prove that for any Borel $A \subseteq X$, its characteristic function $O_A$ defined by $O_A(x) = \begin{cases} 0 & \text{if } x \in A; \\ 1 & \text{if } x \notin A \end{cases}$ is a Borel graded subset. Moreover $O_A$ is a $\Sigma_\alpha(\Pi_\alpha)$-graded subset if and only if $A \in \Sigma^X_\alpha$ (resp. $A \in \Pi^X_\alpha$). Sometimes we will identify subsets with their characteristic graded subsets. In particular $G$ may stand for both $G$ and $O_G$. 


It is clear that for every continuous structure $M$ (defined on $Y$) any continuous formula $\phi(x)$ defines a clopen graded subset of $M^{[T]}$. Moreover note that when $\phi(x, \bar{c})$ is a continuous formula with parameters $\bar{c} \in M$ and $\delta$ is a linear inverse continuous modulus for $\phi(x, y)$ (see Definition 0.1), then $\phi$ is invariant with respect to the open graded subgroup $H_{\delta, \bar{c}} \subseteq \text{Aut}(M)$ defined by

$$H_{\delta, \bar{c}}(g) = \delta(\max(d(c_1, g(c_1)), \ldots, d(c_n, g(c_n))))$$

where $g \in \text{Aut}(M)$, in the following sense:

$$\phi(g(a), \bar{c}) \leq \phi(a, \bar{c}) + H_{\delta, \bar{c}}(g).$$

The fact that $H_{\delta, \bar{c}}$ is a graded subgroup follows from the condition that the action is isometric and $\delta$ is linear. The invariantness is a consequence of

$$\phi(g(\bar{a}), g(\bar{c})) = \phi(\bar{a}, \bar{c})$$

together with uniform $\delta$-continuity of $\phi(g(x), g(y))$. We arrive at the following definition.

**Definition 2.1** Let $X$ be a continuous $G$-space. A graded subset $\phi \subseteq X$ is called invariant with respect to a graded subgroup $H \subseteq G$ if for any $g \in G$ and $x \in X$ we have $\phi(g(x)) \leq \phi(x) + H(g)$.

Since $H(g) = H(g^{-1})$, the inequality from the definition is equivalent to $\phi(x) \leq \phi(g(x)) + H(g)$.

In fact Proposition 1.1 says that any continuous sentence $\phi(\bar{c})$ defines a graded subset of $Y_L$ which belongs to $\Sigma_n$ for some $n$:

$$\phi(\bar{c}) \text{ takes } M \text{ to the value } \phi^M(\bar{c}).$$

We extend this as follows.

**Lemma 2.2** Let $\delta$ be an inverse continuity modulus for $\phi(x)$, which is linear. The graded subset defined by $\phi(\bar{c}) \subseteq Y_L$ is invariant with respect to the graded stabiliser $H_{\delta, \bar{c}} \subseteq \text{Iso}(Y)$ defined as above, i.e.

$$H_{\delta, \bar{c}}(g) = \delta(\max(d(c_1, g(c_1)), \ldots, d(c_n, g(c_n))))$$

where $g \in \text{Iso}(Y)$.

**Proof.** Since $\phi^M(\bar{c}) = \phi^M(\bar{c})$ and $\phi^M(\bar{x})$ is a $\delta$-uniformly continuous map $Y^{[T]} \to [0, 1]$, we have

$$\phi^M(\bar{c}) \leq \phi^M(\bar{c}) + H_{\delta, \bar{c}}(g).$$

□

A graded subset $\phi \subseteq X$ is called meagre in an open $\psi \subseteq X$ if there is $r > 0$ such that $\phi_{<s}$ is meagre in $\psi_{<s}$ for all $s \leq r$ (this extends Definition 3.5 of [S]).

We now define two Vaught transforms as follows. Let $G$ continuously act on $X$. For any non-empty open $J \subseteq G$ let

$$\phi^J(x) = \inf\{r + s : \{h : \phi(h(x)) < r \} \text{ is not meagre in } J_{<s}\},$$
\[ \phi^u_J(x) = \sup \{ r - s : h : \phi(h(x)) \leq r \} \] is not comeagre in \( J_{<s} \).

Using the original topological Vaught transforms one can rewrite the above definitions as follows:

\[ \phi^\Delta_J(x) = \inf \{ r + s : x \in (\phi_{<r})^\Delta(J_{<s}) \}, \]
\[ \phi^{\ast}_J(x) = \sup \{ r - s : x \notin (\phi_{\leq r})^\ast(J_{<s}) \}. \]

Note that since \( J \) is an open graded subset, if \( \{ h : \phi(h(x)) < r \} \) is not meagre in \( J_{<s} \), then it is not meagre in any \( J_{<s'} \) with \( s < s' \).

The lemma below expresses some basic relationships between usual topological Vaught transforms and their graded versions introduced above.

**Lemma 2.3** Let \( A \subseteq X \) be a Borel set and \( \phi \subseteq X \) be a Borel graded set. Let \( u \subseteq G \) be an open set and \( J \subseteq G \) be an open graded set. Then for any \( r \in (0, 1] \cap \mathbb{Q} \) we have the following equalities:

1. \( (\phi_{<r})^\Delta = (\phi^\Delta_O)_{<r} \) and \( (\phi_{\leq r})^\ast = (\phi^\ast_O)_{\leq r} \);
2. \( A^{\Delta(J_{<r})} = (O_{A^\Delta})_{<r} \) and \( A^{\ast(J_{<r})} = (O_{A^\ast})_{\leq r} \).

**Proof.** (1) For any \( s \in (0, 1] \) we have \( (O_u)_{<s} = u \) and so \( (\phi_{<r})^\Delta = (\phi_{<r})^\Delta(O_u)_{<s} \) and \( (\phi_{\leq r})^\ast = (\phi_{\leq r})^\ast(O_u)_{<s} \). Using this we argue as follows:

\[ x \in (\phi_{<r})^\Delta \]
\[ \Downarrow \]
\[ x \in (\bigcup \{ \phi_{<r'} : r' < r \})^\Delta \]
\[ \Downarrow \]
\[ (\exists r' < r)(x \in (\phi_{<r'})^\Delta) \]
\[ \Downarrow \]
\[ (\exists r' < r)(\forall s > 0)(x \in (\phi_{<r'})^\Delta(O_u)_{<s}) \]
\[ \Downarrow \]
\[ (\exists r' < r)(\inf \{ r'' + s : x \in (\phi_{<r''})^\Delta(O_u)_{<s} \} \leq r') \]
\[ \Downarrow \]
\[ x \in (\phi^\Delta_O)_{<r} \]

Since the set of rational numbers is dense in \([0, 1]\) we may assume that \( \bigcup \) and \( \bigcap \) above and below are applied to countable families. The next equality can be derived in a similar way.

\[ x \in (\phi_{\leq r})^\ast \]
\[ \Downarrow \]
\[ x \in (\bigcap \{ \phi_{\leq r'} : r' > r \})^\ast \]
\[ \Downarrow \]
\[ (\forall r' > r)(x \in (\phi_{\leq r'})^\ast) \]
\[ \Downarrow \]
\[ (\forall r' > r)(\forall s > 0)(x \in (\phi_{\leq r'})^\ast(O_u)_{<s}) \]
\[ \Downarrow \]
\[ (\forall r' > r)(\sup \{ r'' - s : x \notin (\phi_{<r''})^\ast(O_u)_{<s} \} < r') \]
\[ \Downarrow \]
\[ x \in (\phi^\ast_O)_{\leq r} \]
Lemma 2.4 Let $J \subseteq G$ be an open graded subset. Then:

1. $\phi^J = 1 - (1 - \phi)^\Delta J$, i.e. $\phi^J(x) = 1 - (1 - \phi)^\Delta J(x)$ for all $x \in X$.
2. $\phi^\Delta J \leq \phi^* J$, i.e. $\phi^\Delta J(x) \leq \phi^* J(x)$ for all $x \in X$.
3. If $\phi$ is a graded $\Sigma_\alpha$-subset, then $\phi^\Delta J$ is also $\Sigma_\alpha$. If $\phi$ is a graded $\Pi_\alpha$-subset, then $\phi^* J(x)$ is also $\Pi_\alpha$.
4. Vaught transforms of Borel graded subsets are Borel.

Proof. First observe that for any $r, s \in [0, 1]$ we have $1 - (r-s) = (1-r) + s$. Now statement (1) is straightforward:

\[
1 - (\phi)^* J(x) = 1 - \inf \{ (1-r) + s : x \in X \setminus (\phi_{\leq r})^{* J(\leq s)} \} = \\
\inf \{ (1-r) + s : x \in ((1 - \phi^{< r})^{\Delta J(\leq s)}) = \\
\inf \{ (1-r) + s : x \in ((1 - \phi^{< s})^{\Delta J(\leq s)}) = \\
(1 - \phi)^\Delta J(x).
\]

To prove (2) take any $x \in X$ and $r \in [0, 1]$ such that $\phi^\Delta J(x) > r$. Then for some $\varepsilon > 0$ we have $\phi^\Delta J(x) > r + 4\varepsilon$, and so $x \notin (\phi_{\leq r + 3\varepsilon})^{\Delta J(\leq s)}$. Then by the properties of the original topological Vaught transforms we see that $x \notin (\phi_{\leq r + 3\varepsilon})^{(J(\leq s))}$. Hence $\phi^{* J}(x) \geq (r + 3\varepsilon) - \varepsilon$, i.e. $\phi^{* J}(x) > r$.

Statement (3) follows by induction starting with the case when $\phi$ is open. In this case for any $x \in X$ and any open $U \subseteq G$ the set $\{ h \in U : \phi(h(x)) < r' \}$ is open too. Thus the set

\[
\{ x : \{ h \in U : \phi(h(x)) < r' \} \text{ is not meagre in } J_{< s} \}
\]

coincides with

\[
\{ x : \{ h \in U : \phi(h(x)) < r' \} \cap J_{< s} \neq \emptyset \}.
\]

Since the action of $G$ is continuous and both $\phi$ and $J$ are open, the latter set is open too. Now note that $(\phi^\Delta J)_{< r}$ is a union of sets of this form (taking $U = G$ above).
When $\phi = \inf \Psi$ where $\Psi$ is a countable subfamily of $\bigcup_{\gamma < \alpha} \Pi_\gamma$, then

$$\phi^J(x) = \inf \{ r + s : \{ h : \inf \Psi(h(x)) < r \} \text{ is not meagre in } J_{<s} \}.$$ 

Since $\{ h : \inf \Psi(h(x)) < r \}$ is not meagre in $J_{<s}$ if and only if one of $\{ h : \psi(h(x)) < r \}$, $\psi \in \Psi$, is not meagre in $J_{<s}$, we see that $\phi^J = \inf \{ \psi^J : \psi \in \Psi \}$, i.e. belongs to $\Sigma_\alpha$.

The case of $\phi^\ast$ with $\phi \in \Pi_\alpha$ follows from the $\Delta J$-case and statement (1).

Statement (4) follows from statement (3). $\square$

The argument of statement (3) is also applied in the following lemma.

**Lemma 2.5** Let $H$ be an open graded subgroup of $G$.

(1) If $\phi$ is an open graded subset then $\phi^H \leq \phi$;

(2) For any graded subset $\phi$ both $\phi^H(x)$ and $\phi^\Delta H(x)$ are $H$-invariant:

$$|\phi^H(h(x)) - \phi^H(x)| \leq H(h) \text{ and }$$

$$|\phi^\Delta H(h(x)) - \phi^\Delta H(x)| \leq H(h).$$

Moreover if $\phi$ is $H$-invariant, then

$$\phi^H(x) = \phi(x) = \phi^\Delta H(x).$$

**Proof.** (1) Let $\phi(x) = r$. Then for any $\varepsilon > 0$ the set $\{ h : \phi(h(x)) < r + \varepsilon \}$ is open and intersects open $H_{<\varepsilon}$ (they contain the neutral element). Thus $\{ h : \phi(h(x)) < r + \varepsilon \}$ is not meagre in $H_{<\varepsilon}$ and $\phi^H(x) \leq r + 2\varepsilon$.

(2) Let $H(h) = t$. Since $H$ is a graded subgroup, $H(h^{-1}) = t$ and $H(gh), H(gh^{-1}) \leq H(g) + t$, for all $g \in G$. Note that for any $r, s \in (0, 1]$ we have

$$\{ g : \phi(g(h(x))) \leq r \} = \{ g : \phi(g(x)) \leq r \} h^{-1} \text{ and } H_{<h^{-1}} \subseteq H_{<s+t}.$$

Thus if the set $\{ g : \phi(g(x)) \leq r \}$ is not comeagre in $H_{<s}$, then the set $\{ g : \phi(g(h(x))) \leq r \}$ is not comeagre in $H_{<s+t}$. Hence

$$\{ r^{-}(s+t) : x \notin (\phi_{<r})^{(H_{<s})} \} \subseteq \{ r^{-} s' : h(x) \notin (\phi_{<r})^{(H_{<s})} \}.$$ 

Since for every $u, w, v \in [0, 1]$ we have $u^{-}(w+v) = (u-w)^{-}v$, then the latter implies $\phi^H(x)^{-}t \leq \phi^H(h(x))$. Replacing $x$ by $h(x)$ and $h$ by $h^{-1}$, we obtain $\phi^H(h(x)) \leq \phi^H(x)^{+}t$.

A similar argument works for $\phi^\Delta H$.

To see the last statement let $\phi(x) = r$. Take any $\varepsilon > 0$ and consider the set $\{ h : r^{-}\varepsilon < \phi(h(x)) < r+\varepsilon \}$. Since this set is comeagre in (coincides with) the open $H_{<\varepsilon}$, we see that

$$|\phi^H(x) - \phi(x)| \varepsilon \text{ and } |\phi^\Delta H(x) - \phi(x)| \varepsilon.$$ 

$\square$
If \( H \) is a graded subgroup, then for every \( g \in G \) we define the graded coset \( Hg \) and the graded conjugate \( H^g \) as follows:
\[
Hg(h) = H(hg^{-1}) \\
H^g(h) = H(ghg^{-1}).
\]
Observe that if \( H \) is open, then \( Hg \) is an open graded subset and \( H^g \) is an open graded subgroup.

**Lemma 2.6** Let \( H \) be an open graded subgroup, \( g \in G \) and \( \rho = Hg \). Let \( \phi \subseteq X \) be a Borel graded subset. Then both \( \phi^{\Delta \rho} \) and \( \phi^*\rho \) are \( H^g \)-invariant.

**Proof.** Let \( H^g(h) = t \). For every \( f \in G \) we have
\[
\rho(fh^{-1}) = H(fh^{-1}g^{-1}) = H((fg^{-1})(gh^{-1}g^{-1})) \leq \rho(f) + H^g(h^{-1}) 
\]
i.e. \( \rho(fh^{-1}) \leq \rho(f) + t \).

Hence for every \( s \in (0, 1] \) we have \( \rho_{< s} h^{-1} \subseteq H_{< s+t} \). Using this and
\[
\{ f : \phi(f(h(x))) < r \} = \{ f : \phi(f(x)) < r \} h^{-1}, \text{ where } r \in (0, 1],
\]
we see that if the set \( \{ f : \phi(f(x)) < r \} \) is not meagre in \( \rho_{< s} \), then the set \( \{ f : \phi(fh(x)) < r \} \) is not meagre in \( \rho_{< s+t} \), i.e. if \( x \in (\phi_{< r})^{\Delta(\rho_{< s})} \) then \( h(x) \in (\phi_{< r})^{\Delta(\rho_{< s+t})} \).

Therefore \( \{ r + s' : h(x) \in (\phi_{< r})^{\Delta(\rho_{< s'})} \} \supseteq \{ r + (s+t) : x \in (\phi_{< r})^{\Delta(\rho_{< s})} \} \). This implies \( \phi^{\Delta \rho}(hx) \leq \phi^{\Delta \rho}(x) + H^g(h) \).

In a similar way we proceed with \( \phi^*\rho \). \( \Box \)

The following lemma will be used in Section 4. In the proof we develope the arguments from Lemmas 2.4(3) and 2.5(1).

**Lemma 2.7** Let \( H \) be an open graded subgroup and \( \phi \) and \( \psi \) be open graded subsets. If for some \( \varepsilon > 0 \), \( \phi_{< r} \) is contained in the closure of \( H_{< \varepsilon} \psi_{< t} \) then \( (\phi^{\Delta H})_{< r} \) is contained in the closure of \( H_{< \varepsilon}(\psi^{\Delta H})_{< t+\varepsilon} \).

**Proof.** Take any \( x \in (\phi^{\Delta H})_{< r} \). Find \( \sigma < r \) and \( g_1 \in \{ h : \phi(h(x)) < r - \sigma \} \cap H_{< \sigma} \). We can choose \( g_2 \in H_{< \varepsilon} \) and \( y \) with \( \psi(y) = t' < t \) and \( g_2^{-1}(y) \) sufficiently close to \( g_1(x) \). Thus for some \((\text{any})~ \varepsilon' < \frac{t-t'}{2} \), the open set \( H_{< \varepsilon} \) has a non-empty intersection with \( \{ h : \psi(hg_2^{-1}(y)) < t' + \varepsilon' \} \), i.e. \( g_2^{-1}(y) \in (\psi^{\Delta H})_{< t+\varepsilon} \) and \( g_1^{-1}g_2^{-1}(y) \in H_{< \varepsilon}(\psi^{\Delta H})_{< t+\varepsilon} \). In particular \( x \) belongs to the closure of \( H_{< \varepsilon}(\psi^{\Delta H})_{< t+\varepsilon} \). \( \Box \)

The following statement follows by the same proof (even with some simplifications):

\[
\text{if under circumstances above, } \phi_{< r} \text{ is contained in the closure } \overline{\psi_{< t}} \text{ then } (\phi^{\Delta H})_{< r} \text{ is contained in } \overline{H_{< \varepsilon}(\psi^{\Delta H})_{< t}}.
\]
2.2 Nice bases

We now consider $G$ together with a distinguished countable family of clopen graded subsets $\mathcal{R}$. We will assume that a countable family of sets of the form $\rho<q$, for $\rho \in \mathcal{R}$ and real $r$, forms a basis of the topology of $G$. In fact we usually assume it for $\{\rho<q : \rho \in \mathcal{R} \text{ and } q \in \mathbb{Q}^+\}$.

We also assume that $\mathcal{R}$ consists of graded cosets, i.e. for such $\rho \in \mathcal{R}$ there is a graded subgroup $H \in \mathcal{R}$ and an element $g_0 \in G$ so that for any $g \in G$, $\rho(g) = H(gg_0^{-1})$.

**Remark 2.8** For every Polish group $G$ there is a countable family of open graded subsets $\mathcal{R}$ as above.

Indeed, fix an arbitrary compatible right-invariant metric $d \leq 1$ on $G$. Put $H(g) = d(1,g)$. It is easily seen that $H(x)$ is of an open graded subgroup such that $\{H<q : r \in [0,1] \cap \mathbb{Q}^+\}$ is a basis of open neighbourhoods of the unity $1$. Now take a dense countable subgroup $G_0 \subset G$ and let $\mathcal{R}$ consist of all conjugates of $H$ by elements from $G_0$ and $G_0$-cosets of these $H^q$.

As a result we have a countable dense subgroup $G_0 < G$ so that $\mathcal{R}$ is closed under $G_0$-conjugacy and consists of all $G_0$-cosets of graded subgroups from $\mathcal{R}$. It is easy to see that we may additionally assume that the set of graded subgroups from $\mathcal{R}$ is closed under max and dotted multiplication by positive rational numbers (i.e. the product is also bounded by 1).

We will see below that if the space $(Y,d)$ is good enough (for example the Urysohn space), then the family $\mathcal{R}$ of graded subsets of $G = \text{Iso}(Y)$ can be chosen among graded cosets of the form

$$\rho(g) = q(d(\bar{b},g(\bar{a}))) , \text{ where } q \in \mathbb{Q}^+ \text{ and } \bar{a}\bar{b} \text{ is an appropriate tuple from } S_Y,$$

If the metric is bounded by 1 we mean the dotted multiplication by $q$ in the formula above.

When we consider a $(G,G_0,\mathcal{R})$-space $X$ we usually distinguish a similar family too: we choose a countable family $\mathcal{U}$ of open (clopen) graded subsets of $X$ so that a countable family of sets of the form $\sigma<q$, for $\sigma \in \mathcal{U}$ and real $r$, forms a basis of the topology of $X$. To formalize this let us introduce the following notion.

**Definition 2.9** A family $\mathcal{U}$ of open graded subsets of $X$ is called a graded basis of the topology $\tau$ if the family $\{\phi<q : \phi \in \mathcal{U}, q \in \mathbb{Q} \cap (0,1)\}$ is a basis of $\tau$.

By Proposition 2.C.2 of [2] there exists a unique partition of $X$, $X = \bigcup \{Y_t : t \in T\}$ into invariant $G_δ$-sets $Y_t$ such that every $G$-orbit from $Y_t$ is dense in $Y_t$. To construct this partition we define for any $t \in 2^\mathbb{N}$ the set

$$Y_t = (\bigcap \{GA_j : t(j) = 1\}) \cap (\bigcap \{X \setminus GA_j : t(j) = 0\})$$

where $A_j$ are taken from the corresponding basis of sets the form $\sigma<q$. Now let $T = \{t \in 2^\mathbb{N} : Y_t \neq \emptyset\}$. This partition is called canonical.
Remark 2.10 When $A$ is open, $GA = A^G$. Thus each $Y_t$ as above is an intersection of sets of the form $(\sigma_r)^G$ or their complements. By Lemma 2.3 each piece of the canonical partition can be constructed as an intersection of sets of the form $(\sigma^G)_{<r}$ (with $\sigma \in \mathcal{U}$) or their complements. Here we may assume that $r$ can take only countably many values.

Along with the $d$-topology $\tau$ we shall consider some special topology on $X$. For this purpose we apply the idea from [3] of extension of our $\mathcal{U}$ to a nice basis.

Definition 2.11 Let $\mathcal{R}$ be a graded basis of $G$ consisting of cosets of open graded subgroups of $G$ which also belong to $\mathcal{R}$. Assume that the subfamily of $\mathcal{R}$ of all open graded subgroups is closed under max and dotted multiplication by numbers from $\mathbb{Q}^+$. We say that a family $\mathcal{B}$ of Borel graded subsets of the $G$-space $(X, \tau)$ is a nice basis with respect to $\mathcal{R}$ if:

(i) $\mathcal{B}$ is countable and generates the topology finer than $\tau$;
(ii) for all $\phi_1, \phi_2 \in \mathcal{B}$, the functions $\neg \phi_1, \min(\phi_1, \phi_2), \max(\phi_1, \phi_2), |\phi_1 - \phi_2|, \phi_1 \cdot \phi_2$ belong to $\mathcal{B}$;
(iii) for all $\phi \in \mathcal{B}$ and $q \in \mathbb{Q}^+$ the dotted product $q \cdot \phi$ belongs to $\mathcal{B}$;
(iv) for all $\phi \in \mathcal{B}$ and open graded subsets $\rho \in \mathcal{R}$ we have $\phi^{*\rho}, \phi^{\Delta \rho} \in \mathcal{B}$;
(v) for any $\phi \in \mathcal{B}$ there exists an open graded subgroup $H \in \mathcal{R}$ such that $\phi$ is $H$-invariant.

It will be usually assumed that the constant function $1$ belongs to $\mathcal{B}$, i.e. in particular all constant functions $q, q \in \mathbb{Q} \cap [0, 1]$ are in $\mathcal{B}$.

Definition 2.12 A topology $t$ on $X$ is $\mathcal{R}$-nice for the $G$-space $(X, \tau)$ if the following conditions are satisfied.

(a) The topology $t$ is Polish, $t$ is finer than $\tau$ and the $G$-action remains continuous with respect to $t$.

(b) There exists a graded basis $\mathcal{B}$ of $t$ which is nice with respect to $\mathcal{R}$.

Theorem 2.13 Let $G$ be a Polish group and $\mathcal{R}$ be a countable graded basis satisfying the assumptions of Definition 2.11 and the following closure property:

$$H^g \in \mathcal{R}.$$  

Let $(X, \tau)$ be a $G$-space and $\mathcal{F}$ be a countable family of Borel graded subsets of $X$ generating a topology finer than $\tau$ such that each $\phi \in \mathcal{F}$ is invariant with respect to some graded subgroup $H \in \mathcal{R}$.

Then there is an $\mathcal{R}$-nice topology for $(X, \tau, G)$ such that $\mathcal{F}$ consists of open graded subsets.

Proof. First we shall construct an increasing sequence $(S_n)_{n \in \omega}$ of countable families of Borel graded subsets of $X$ along with an increasing sequence $(A_n)_{n \in \omega}$ of countable

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2the conditions of Remark 2.8 imply these assumptions
bases of Polish topologies on $X$. We proceed by induction. We put $S_0 = \mathcal{F}$. Suppose that we have already constructed $S_n$. Then $\{\phi_{< r} : \phi \in S_n, r \in (0, 1] \cap \mathbb{Q}^+\}$ is a countable family of Borel subsets of $X$. Accordingly to Remark 5.1.4 in [1] it can be extended to a countable basis of a Polish topology on $X$. We define $\mathcal{A}_n$ to be such a basis. Then we define $S_{n+1}$ to be the least family of graded subsets extending $S_n \cup \{O_B : B \in \mathcal{A}_n\}$ which is closed under conditions (ii)-(iv) of Definition 2.11. We see that $S_{n+1}$ is countable and by Lemma 2.4 it consists of Borel graded subsets.

Having defined sequences $(S_n)$ and $(\mathcal{A}_n)$ we put $S = \bigcup\{S_n : n \in \omega\}$ and $\mathcal{A} = \bigcup\{\mathcal{A}_n : n \in \omega\}$. Observe that for every $n \in \omega$ we have

$$(*) \quad \mathcal{A}_n \subseteq \{\phi_{< r} : \phi \in S_{n+1}, r \in (0, 1] \cap \mathbb{Q}^+\} \subseteq \mathcal{A}_{n+1}.$$  

Thus we see that $\mathcal{A} = \{\phi_{< r} : \phi \in S, r \in (0, 1] \cap \mathbb{Q}^+\}$. By Remark 5.1.4 in [1] $\mathcal{A}$ is a basis of a Polish topology on $X$ finer then the topology generated by $\mathcal{F}$ (say $t'$), so $S$ is its graded basis. Although the topology defined by graded basis $S$ is Polish, in general it may not preserve continuity of the action. So we have to look for some smaller topology.

Obviously $S$ satisfies closure conditions (ii)-(iv) of Definition 2.11. This results in the following properties of $\mathcal{A}$.

**Claim 1.** $\mathcal{A}$ is a Boolean algebra of subsets of $X$ closed under both (standard) Vaught transforms.

*Proof.* Supposing that $A, B \in \mathcal{A}$, we get $O_A, O_B \in S$. Then by closure properties of $S$ we have $1 - O_A \in S$ and $\max\{O_A, O_B\} \in S$. Since $X \setminus A = (1 - O_A)_{< 1}$ and $A \cap B = (\max\{O_A, O_B\})_{< 1}$, thus $X \setminus A$ and $A \cap B$ are elements of $\mathcal{A}$. Next by Lemma 2.3(3), for any $\rho \in \mathcal{R}$ and $r \in (0, 1] \cap \mathbb{Q}$ we have $A^{\Delta(\rho_{< r})} = ((O_A)^{\Delta \rho})_{< r}$, thus $A^{\Delta(\rho_{< r})} \in \mathcal{A}$.

Now consider the family $B = \{(\phi^{\Delta \rho}) : \phi \in S, \rho \in \mathcal{R}\}$. We claim that it has the following properties.

**Claim 2.** (1) If $\psi \in S$ is $H$-invariant for some open graded subgroup $H \in \mathcal{R}$, then $\psi \in B$.

(2) $B$ satisfies conditions (ii)-(v) of Definition 2.11.

(3) $B = \{(\phi^{\rho}) : \phi \in S, \rho \in \mathcal{R}\}$.

(4) The family $\{\phi_{< r} : \phi \in B, r \in (0, 1] \cap \mathbb{Q}^+\}$ is closed under intersection.

*Proof.* (1) Accordingly to Lemma 2.5 we have $\psi^{H} = \psi$ and $\psi^{*H} = \psi$.

(2) First we examine condition (v). Take an arbitrary $\phi \in S$ and $\rho \in \mathcal{R}$. By the hypothesis of the theorem, there are $g \in G$ and an open graded subgroup $H \in \mathcal{R}$ such that $\rho = H g$. Then $\phi^{\Delta \rho}$ is $H^g$-invariant by Lemma 2.6.

Now take any $\psi_1, \psi_2 \in B$ and open graded subgroup $H_1, H_2 \in \mathcal{R}$ such that $\psi_i$ is $H_i$-invariant, for $i = 1, 2$. Put $H = \max(H_1, H_2)$, then $\psi_1, \psi_2$ are also $H$-invariant. Now it is easy to see that both $\min(\psi_1, \psi_2)$ and $\max(\psi_1, \psi_2)$ are $H$-invariant and each
of the graded subsets $\psi_1 - \psi_2, \psi_1 + \psi_2, |\psi_1 - \psi_2|$ is $2 \cdot H$-invariant. By Lemma 2.4 $\neg \psi_1$ is $H_1$-invariant and $r \cdot \psi_1$ is $r \cdot H_1$-invariant. Now the rest follows from point (1).

Since $B$ is closed under both graded transforms, thus (iv) follows from the construction of $B$. Point (3) is an immediate consequence of (1).

(4) Take any $A, B \in \{ \phi_{<r}: \phi \in B, r \in (0, 1] \cap \mathbb{Q}^+ \}$. There are $\phi_A, \phi_B \in \mathcal{S}$ and $s, t \in (0, 1] \cap \mathbb{Q}^+$ such that $A = (\phi_A)_t$ and $B = (\phi_B)_s$. Then $s \cdot \phi_A, t \cdot \phi_B \in B$, $A = (s \cdot \phi_A)_s, B = (t \cdot \phi_B)_t$ and $A \cap B = (\max(s \cdot \phi_A, t \cdot \phi_B))_{st}$. Hence $A \cap B \in \{ \phi_{<r}: \phi \in B, r \in (0, 1] \cap \mathbb{Q}^+ \}$.

**Claim 3.** The family $\{ \phi_{<r}: \phi \in B, r \in (0, 1] \cap \mathbb{Q}^+ \}$ is a basis of the topology generated by the subbasis $\{ A^{\Delta u}: A \in \mathcal{A}, u \in \{ \rho_{<s}: \rho \in \mathcal{R}, s \in (0, 1] \cap \mathbb{Q}^+ \} \}$.

**Proof.** Since the family $\{ \phi_{<r}: \phi \in B, r \in (0, 1] \cap \mathbb{Q}^+ \}$ is closed under intersection, it suffices to show that each of its elements is a union of elements of the family $\{ A^{\Delta u}: A \in \mathcal{A}, u \in \{ \rho_{<s}: \rho \in \mathcal{R}, s \in (0, 1] \cap \mathbb{Q}^+ \} \}$ and vice versa. It follows directly from the definition of a graded $\Delta$-transform, that every element from $\{ \phi_{<r}: \phi \in B, r \in (0, 1] \cap \mathbb{Q}^+ \}$ is a union of elements from $\{ A^{\Delta u}: A \in \mathcal{A}, u \in \{ \rho_{<s}: \rho \in \mathcal{R}, s \in (0, 1] \cap \mathbb{Q}^+ \} \}$. On the other hand if $A \in \mathcal{A}$, then $O_A \in \mathcal{S}$ and $A^{\Delta (\rho_{<s})} = (O_A)^{\Delta (\rho_{<s})}$. Hence

$$\{ A^{\Delta u}: A \in \mathcal{A}, u \in \{ \rho_{<s}: \rho \in \mathcal{R}, s \in (0, 1] \cap \mathbb{Q}^+ \} \} \subseteq \{ \phi_{<r}: \phi \in B, r \in (0, 1] \cap \mathbb{Q}^+ \}.$$

To close the argument we have to recall Theorem 5.2.1 from [4]. Actually, the main part of its proof consists of justification of the following statement.

Let $G$ be a Polish group, $X$ be a Borel $G$-space and $P$ be a countable Boolean algebra of Borel sets generating the Borel structure of $X$ and closed under Vaught transforms. Then the family

$$\{ B^{\Delta u}: B \in P, u \text{- basic open subset of } G \}$$

is a subbasis of a Polish topology which makes the action continuous.

Now applying Claim 1 we see that $\{ A^{\Delta u}: A \in \mathcal{A}, u \in \{ \rho_{<s}: \rho \in \mathcal{R}, s \in (0, 1] \cap \mathbb{Q}^+ \} \}$ is a subbasis of a Polish topology $t$ such that the $G$-action on $X$ is continuous with respect to $t$. Accordingly to Claim 3, $B$ is its graded basis. It follows from the assumptions of the theorem and Claim 2 (1) that $t$ is finer than $t'$. Then by Claim 2 (2) $t$ is a nice topology and $B$ is its nice graded basis. \( \square \)

The following observation shows that the assumptions of the theorem above are natural.

**Proposition 2.14** Let $G$ be a Polish group and let $\mathcal{R}$ be a countable basis consisting of open graded cosets. Let $(X, \tau)$ be a $G$-space. Then there is a graded basis $\mathcal{F}$ for $((X, \tau), G)$ such that each $\phi \in \mathcal{F}$ is invariant with respect to some open graded subgroup $H \in \mathcal{R}$.
Proof. Take an arbitrary graded basis \( U \). Then \( A = \{ \phi_r : \phi \in U, r \in \mathbb{Q} \cap (0, 1) \} \) is a basis for \( \tau \). We claim that the family

\[
\{ O^H_B : B \in A, H \in \mathcal{R} \text{ a graded subgroup} \}
\]

is a graded basis for \( \tau \). To see this consider any \( A \in A \) and \( x_0 \in A \). By continuity of the action there are an open graded subgroup \( H \in \mathcal{R}, s \in \mathbb{Q} \cap (0, 1) \) and \( B \in A \) such that \( x_0 \in B \subseteq A \) and for every \( x \in B \) and \( h \in H_{<s} \) the element \( hx \) belongs to \( A \). On the one hand since \( x_0 \in B \) and \( O_B \) is open, thus by Lemma 2.5 (1) we have \( O^H_B(x_0) = 0 \). On the other hand if \( x \in (O^H_B)_{<s} \) then there is \( h \in H_{<s} \) with \( hx \in B \) and by \( H(h^{-1}) = H(h) \) we have \( x = h^{-1}hx \in A \). Hence \( x_0 \in (O^H_B)_{<s} \subseteq A \). □

3 Logic action over the Urysohn space

The construction of nice bases naturally arises when one considers the case of continuous logic actions over \( U \), the Urysohn space of diameter 1. This is the unique Polish metric space which is universal and ultrahomogeneous, i.e. every isometry between finite subsets of \( U \) extends to an isometry of \( U \). The space \( U \) is considered in the continuous signature \( \langle d \rangle \).

Let \( L \) be a countable continuous signature and \( U_L \) be the \( Iso(U) \)-space of all \( L \)-structures defined as in Section 1.1. In the present section we will consider nice topologies on \( U_L \). Our main theorem states that under some natural circumstances graded subsets associated with continuous formulas form a nice basis on \( U_L \). We call it ‘logical’ and consider it as the most natural example of nice bases.

A theorem from [3] (Corollary 1.13) states that in the case of logic actions of \( S_\infty \) on the space of countable structures each nice topology is defined by model sets \( Mod(\phi) \) of formulas of a countable fragment of \( L_{\omega_1\omega} \). Can this statement be extended to \( U_L \)?

We find an example which shows some complications in possible generalisations. Since this issue depends on the Lopez-Escobar theorem on invariant Borel subsets of logic spaces, we also discuss possible continuous versions of this theorem.

3.1 When is a nice topology logical?

The countable counterpart of \( U \) is the rational Urysohn space of diameter 1 \( QU \), which is both ultrahomogeneous and universal for countable metric spaces with rational distances and diameter \( \leq 1 \). It is shown in Section 5.2 of [8] that there is an embedding of \( QU \) into \( U \) so that:

(i) \( QU \) is dense in \( U \);
(ii) any isometry of \( QU \) extends to an isometry of \( U \) and \( Iso(QU) \) is dense in \( Iso(U) \);
(iii) for any \( \varepsilon > 0 \), any partial isometry \( h \) of \( QU \) with domain \( \{ a_1, \ldots, a_n \} \) and any isometry \( g \) of \( U \) such that \( d(g(a_i), h(a_i)) < \varepsilon \) for all \( i \), there is an isometry \( \hat{h} \) of \( QU \) that extends \( h \) and is such that for all \( x \in U \), \( d(\hat{h}(x), g(x)) < \varepsilon \).
We now define a family of clopen graded subgroups of $Iso(U)$ which satisfies the conditions of Theorem 2.13. Let $G_0$ be a dense countable subgroup of $Iso(\mathbb{Q}U)$.

**Family $\mathcal{R}^U$.** Let $\mathcal{R}_0$ be the family of all clopen graded subgroups of $Iso(U)$ of the (dotted) form

$H_{q,\bar{s}} : g \to q \cdot d(g(\bar{s}), \bar{s})$, where $\bar{s} \subset \mathbb{Q}U$, and $q \in \mathbb{Q}^+$.

It is clear that $\mathcal{R}_0$ is closed under conjugacy by elements of $G_0$. Consider the closure of $\mathcal{R}_0$ under the function $\text{max}$ and define $\mathcal{R}^U$ to be the family of all $G_0$-cosets of graded subgroups from $\text{max}(\mathcal{R}_0)$. Then $\mathcal{R}^U$ is countable and the family of all $(H_{q,\bar{s}})_1$ where $H \in \mathcal{R}_0$ and $l \in \mathbb{Q}$, generates the topology of $Iso(U)$. Moreover it is easy to see that $G_0$ and $\mathcal{R}^U$ satisfy all the conditions of Remark 2.8 and in particular $\mathcal{R}^U$ satisfies the conditions of Theorem 2.13.

**Remark 3.1** It is worth noting that any graded subgroup of the form $\text{max}(H_1, ..., H_k)$ where

$H_i : g \to q_i \cdot d(g(\bar{s}_i), \bar{s}_i)$, where $\bar{s}_i \subset \mathbb{Q}U$, and $q_i \in \mathbb{Q}$,

has a graded subgroup from $\mathcal{R}_0$. Indeed, let $\bar{s}$ consist of all elements appearing in $\bar{s}_i$, $i \leq k$, and $q = \text{max}(q_1, ..., q_k)$. Then

$H_{q,\bar{s}} : g \to q \cdot d(g(\bar{s}), \bar{s})$

is a graded subgroup of $\text{max}(H_1, ..., H_k)$.

Let $L$ be a relational language of a continuous signature with inverse continuity moduli $\leq n \cdot \text{id}$ for $n$-ary relations. Let $\mathcal{B}_0$ be the family of all graded subsets defined by continuous $L$-sentences as follows

$\phi(\bar{s}) : M \to \phi^M(\bar{s})$, where $\bar{s} \in \mathbb{Q}U$.

It is easy to see that for any continuous sentence $\phi(\bar{s})$ there is a number $q \in \mathbb{Q}$ (depending on the continuity modulus of $\phi$) such that the graded subset as above is $H_{q,\bar{s}}$-invariant. We will prove below (in Theorem 3.3) that $\mathcal{B}_0$ is an $\mathcal{R}^U$-nice basis. This can be considered as a version of Theorem 1.10 of [3] which states that in the discrete case of the $S_\infty$-space of countable $L$-structures all formulas as above already form a nice basis.

To prove this theorem we need some preliminary work. We start with some property of the Urysohn space which is interesting by itself. In fact in the following lemma we evaluate the distance between some types in $S_n(Th(U))$. In the proof of Theorem 3.3 this lemma will be applied in the situation when $q = 0$. The case $q > 0$ can be considered as a new amalgamation property of the Urysohn space, see Section 3.2 for some applications of this case.

**Lemma 3.2** Let $a_1, ..., a_n \in U$, $0 \leq q < n$ and $\varepsilon > 0$ satisfy all inequalities of the form

$(2\left(\begin{array}{c} n-q \\varepsilon \end{array}\right) + 1)\varepsilon < d(a_i, a_j)$ for $i < j \leq n$ and
\[
((2^{n-q}) + 1)\varepsilon < d(a_i, a_j) + d(a_i, a_k) - d(a_j, a_k) \text{ for triples } a_i, a_j, a_k \text{ with } \\
i, j, k \leq n, |\{i, j, k\}| = 3 \text{ and } d(a_i, a_j) + d(a_i, a_k) \neq d(a_j, a_k).
\]

Assume that there is no geodesic triple \(a_i, a_j, a_k\) with \(k \leq q < i, j\) and \(a_j\) between \(a_i\) and \(a_k\).

Let \(B\) be an \(n\)-element metric space consisting of elements \(b_i\) so that for each pair \(i < j \leq n, |d(b_i, b_j) - d(a_i, a_j)| \leq \varepsilon\). We assume that \(a_1 = b_1, ..., a_q = b_q\) and the metric defined on \(\{b_1, ..., b_q\}\) in the space \(B\) coincides with the metric defined on \(\{a_1, ..., a_q\}\) in the space \(\mathbb{U}\).

Then the space \(B\) embeds into \(\mathbb{U}\) over \(\{a_1, ..., a_q\}\) so that for each \(q < i \leq n, d(a_i, b_i) = (2^{(n-q)} + 1)\varepsilon\).

**Proof.** To see this note that by universality of \(\mathbb{U}\) it is enough to build a metric space on \(\{a_1, ..., a_n, b_1, ..., b_n\}\) so that \(d(a_i, b_i) = (2^{(n-q)} + 1)\varepsilon\) for all \(q < i \leq n\). Let \(l_1 \leq l_2 \leq ... \leq l_{\binom{n-q}{2}}\) be all values of all \(\min(d(a_i, a_j), d(b_i, b_j))\) with \(q < i\) and \(q < j\). We enumerate all pairs \((i, j)\) so that \(q < i, j\) and for \((i, j)\) having number \(m\) the equality \(\min(d(a_i, a_j), d(b_i, b_j)) = l_m\) is satisfied. Let us define \(d(a_i, b_j)\) and \(d(b_i, a_j)\) by induction according this enumeration. At every step \(m\) we choose \(\varepsilon_m \leq m\varepsilon\) so that \(\frac{\varepsilon}{2} \leq \varepsilon_m\) and

\[
d(a_i, b_j) = d(b_i, a_j) = \min(d(a_i, a_j), d(b_i, b_j)) - \varepsilon_m.
\]

If \(\min(d(a_i, a_j), d(b_i, b_j)) = l_1\), let

\[
d(a_i, b_j) = d(b_i, a_j) = \min(d(a_i, a_j), d(b_i, b_j)) - \frac{\varepsilon}{2}.
\]

Our assumptions guarantee any possible metric triangle condition for distances defined in the formulation (with \(d(a_i, b_i)\)) together with these \(d(a_i, b_j)\) and \(d(b_i, a_j)\). We now assume that all \(d(a_i, b_j)\) and \(d(b_i, a_j)\) defined for all pairs \((i, j)\) numbered before \(m\) (together with all \(d(a_i, b_i) = (2^{(n-q)} + 1)\varepsilon\) and given \(d(a_i, a_j), d(b_i, b_j)\)) satisfy any possible metric triangle condition for distances.

At step \(m\) define the corresponding \(d(a_i, b_j)\) and \(d(b_i, a_j)\) as follows. Assume that the definition

\[
d(a_i, b_j) = d(b_i, a_j) = \min(d(a_i, a_j), d(b_i, b_j)) - \frac{\varepsilon}{2}
\]

does not imply any inequality contradicting the metric condition for triangles of any set \(\{a_i, b_i, a_j, b_j, a_k, b_k\}\) with \(k \leq n\) so that all remaining distances in \(\{a_i, b_i, a_j, b_j, a_k, b_k\}\) are already defined. Then we take this definition.

Consider the contrary case. If the definition above does not define a metric in \(\{a_i, b_i, a_j, b_j, a_k, b_k\}\), where distances for pairs \((i, k), (j, k)\) were defined before, then we may assume that \(|\{a_i, b_i, a_j, b_j, a_k, b_k\}| = 6\) (by the assumptions on geodesic tripls in the formulation), \(d(a_i, a_k) + d(a_j, a_k) = d(a_i, a_j)\) and

\[
d(a_i, b_k) + d(b_j, b_k) < \min(d(a_i, a_j), d(b_i, b_j)) - \frac{\varepsilon}{2}.
\]
Indeed, the inequalities of the form
\[ d(b_j, b_k) + \min(d(a_i, a_j), d(b_i, b_j)) - \frac{\varepsilon}{2} < d(a_i, b_k), \]
\[ d(a_i, b_k) + \min(d(a_i, a_j), d(b_i, b_j)) - \frac{\varepsilon}{2} < d(b_j, b_k) \]
(with possible interchange \(a \leftrightarrow b\)) cannot happen by assumptions on \(\varepsilon\), distances in \(\{a_1, \ldots, a_n\} \cup B\) and the assumption
\[ \max(\min(d(a_i, a_k), d(b_i, b_k)), \min(d(a_j, a_k), d(b_j, b_k))) \leq \min(d(a_i, a_j), d(b_i, b_j)). \]
When \(d(a_i, a_k) + d(a_j, a_k) \neq d(a_i, a_j)\) we have
\[ (2\left(\frac{n - q}{2}\right) + 1)\varepsilon < d(a_k, a_j) + d(a_i, a_k) - d(a_j, a_i) \]
which easily implies
\[ d(a_i, b_k) + d(b_j, b_k) \geq \min(d(a_i, a_j), d(b_i, b_j)) - \frac{\varepsilon}{2}. \]

Let us consider the case \(d(a_i, a_k) + d(a_j, a_k) = d(a_i, a_j)\). We choose such a \(k\) with the maximal \(\varepsilon_m\) appearing in the definition of \(d(a_i, b_k)\). Then let \(\varepsilon_m\) be \(\varepsilon_m\) for the chosen \(k\). We obviously have the inequality
\[ d(a_i, b_k) + d(b_j, b_k) \geq \min(d(a_i, a_j), d(b_i, b_j)) - \varepsilon_m. \]
The inequalities
\[ d(a_i, b_k) \leq d(b_j, b_k) + \min(d(a_i, a_j), d(b_i, b_j)) - \varepsilon_m \]
and
\[ d(b_j, b_k) \leq d(a_i, b_k) + \min(d(a_i, a_j), d(b_i, b_j)) - \varepsilon_m. \]
follow from the fact that our procedure guarantees
\[ \min(d(a_i, a_j), d(b_i, b_j)) \geq \max(\min(d(a_i, a_k), d(b_i, b_k)), \min(d(a_k, a_j), d(b_k, b_j))) \geq \varepsilon_m. \]
Since the latter inequalities work for any \(k\) with distances for pairs \((i, k), (j, k)\) defined before (not only for \(k\) chosen as above) we see that our definition of \(d(a_i, b_j) = d(a_j, b_i)\) does not destroy the metric triangle condition for triples were \(d(a_i, b_j)\) or \(d(a_j, b_i)\) appear. \(\Box\)

**Theorem 3.3** The family \(\mathcal{B}_0\) is an \(R^U\)-nice basis.

**Proof.** To get this statement we use the strategy of Theorem 1.10 of [3]. Conditions (i) - (iii) and (v) of Definition 2.11 are easily seen in this case. For example to see condition (i) it suffices to note that \(\tau\) is generated by open sets of the form \(\{M : |\psi^M(s) - q_1| < q_2\}\) where \(\psi\) is an atomic formula, \(s \in Q\cup\) and \(q_1, q_2 \in Q \cap (0, 1)\). As in [3] we concentrate on condition (iv) of the definition of a nice basis. The following claim is the main point of the proof.
Let $B \in \mathcal{B}_0$ be the graded subset of $\mathbb{U}_L$ and $H \in \mathcal{R}_0$ be a graded coset of $Iso(\mathbb{U})$ corresponding to a graded subgroup of the form $H_{q,\delta}$. Then $B^{+H}$ belongs to $\mathcal{B}_0$.

We fix $B \in \mathcal{B}_0$ and $H \in \mathcal{R}^U$, and find a continuous formula $\phi$, pairwise distinct $a_0, \ldots, a_{l-1}, b_0, \ldots, b_{m-1}, c_0, \ldots, c_{n-1} \in QU$ and pairwise distinct $b_0', \ldots, b_{m-1}', c_0', \ldots, c_{n-1}' \in QU$ so that the following three conditions are satisfied:

1. the type of $b_0, \ldots, b_{m-1}, c_0, \ldots, c_{n-1}$ in the pure structure $(\mathbb{U}, d)$ coincides with the type of $b_0', \ldots, b_{m-1}', c_0', \ldots, c_{n-1}'$;
2. $H(g) = q \cdot \max \{d(g(b_0), b_0), \ldots, d(g(b_{m-1}), b_{m-1}), d(g(c_0), c_0), \ldots, d(g(c_{n-1}), c_{n-1})\}$ for an appropriate $q \in (0, 1] \cap \mathbb{Q}$;
3. $\phi = \phi(b_0, \ldots, b_{m-1}, a_0, \ldots, a_{l-1})$ is a continuous $L$-formula with parameters so that the graded subset $B$ of $\mathbb{U}_L$ is defined by

$$M \to B(M) = \phi^M(b_0, \ldots, b_{m-1}, a_0, \ldots, a_{l-1}).$$

Let $\rho : [0, 1] \to [0, 1]$ be the inverse continuity modulus for $\phi$. By our assumptions on the continuous signature $L$, $\rho$ is a linear function. We want to show that $B^{+H}$ belongs to $\mathcal{B}_0$.

Let $q_0$ be a rational number such that

$$(l + m + n)^2 + 1)q_0 < d(s_i, s_j) \text{ for } s_1 \neq s_2 \in \{b_0, \ldots, b_{m-1}, c_0, \ldots, c_{n-1}, a_0, \ldots, a_{l-1}\}$$

and

$$(l + m + n)^2 + 1)q_0 < d(s_i, s_j) + d(s_i, s_k) - d(s_j, s_k)$$

for triples $s_i, s_j, s_k \in \{b_0, \ldots, b_{m-1}, c_0, \ldots, c_{n-1}, a_0, \ldots, a_{l-1}\}$ with $i, j, k \leq n, |\{i, j, k\}| = 3$ and $d(s_i, s_j) + d(s_i, s_k) \neq d(s_j, s_k)$.

We also choose $m_0$ so that $((l + n + m)^2 + 1) < m_0$ and $m_0 \cdot q_0 \geq 1$.

By Lemma 3.2 for any subspace $\{b_0', \ldots, b_{m-1}', c_0', \ldots, c_{n-1}', a_0', \ldots, a_{l-1}'\} \subset \mathbb{U}$ and any $\varepsilon < q_0$ so that for each pair $s_1, s_2 \in \{b_0, \ldots, b_{m-1}, c_0, \ldots, c_{n-1}, a_0, \ldots, a_{l-1}\}$ and the corresponding $s_1', s_2' \in \{b_0', \ldots, b_{m-1}', c_0', \ldots, c_{n-1}', a_0', \ldots, a_{l-1}'\}$ we have $|d(s_1, s_2) - d(s_1', s_2')| \leq \varepsilon$, there is an embedding $\alpha$ of the space $\{b_0, \ldots, b_{m-1}, c_0, \ldots, c_{n-1}, a_0, \ldots, a_{l-1}\}$ into $\mathbb{U}$ so that for each $s_i \in \{b_0, \ldots, b_{m-1}, c_0, \ldots, c_{n-1}, a_0, \ldots, a_{l-1}\}$,

$$d(\alpha(s_i), s_i') \leq ((l + n + m)^2 + 1)\varepsilon.$$

For such $\varepsilon$ and $\alpha$ viewing

$$\phi^M(b_0', \ldots, b_{m-1}', a_0', \ldots, a_{l-1}')(q \cdot \max \{d(b_0, b_0'), \ldots, d(b_{m-1}, b_{m-1}'), d(c_0, c_0'), \ldots, d(c_{n-1}, c_{n-1}')\})$$

as

$$[\phi^M(b_0', a_0') - \phi^M(\alpha(b), \alpha(a)) + \phi^M(\alpha(b), \alpha(a))] -$$

$$[q \cdot \max \{d(b, b'), d(c, c')\} - q \cdot \max \{d(b, \alpha(b)), d(c, \alpha(c))\} +$$

$$q \cdot \max \{d(b, \alpha(b)), d(c, \alpha(c))\}],$$

we have the required formula and embedding.
we see (by the triangle condition) that it is not greater than

\[ \phi^M(\alpha(b), \alpha(a)) - q \cdot \max(d(b, \alpha(b)), d(c, \alpha(c))) + p((l+n+m)^2 + 1) \varepsilon + q((l+n+m)^2 + 1) \varepsilon. \]

Let \( \psi(u_0', ..., u_{m-1}', v_0', ..., v_{n-1}') \) be the following formula:

\[
sup_{\bar{a}, \bar{u}} |\phi(\bar{u}, \bar{w}) - q \cdot \max(d(u_0', u_0), ..., d(u_{m-1}', u_{m-1}), d(v_0', v_0), ..., d(v_{n-1}', v_{n-1}))| = (\rho + q \cdot id)(m_0 \cdot \max(|d(z_1, z_2) - d(s_1, s_2)| : \text{the place of } z_1, z_2 \text{ in } \{u_0, ..., u_{m-1}, v_0, ..., v_{n-1}, w_0, ..., w_l\}) \text{ corresponds to the place of } s_1, s_2 \text{ in } \{b_0, ..., b_{m-1}, c_0, ..., c_{n-1}, a_0, ..., a_{l-1}\}).
\]

To see that \( B^{sH} \) is determined by \( \psi(b_0', ..., b_{m-1}', c_0', ..., c_{n-1}') \) note that for a tuple \( a''_0, ..., a''_{m-1}, b''_0, ..., b''_{m-1}, c''_0, ..., c''_{n-1} \) isomorphic to \( a_0, ..., a_{l-1}, b_0, ..., b_{m-1}, c_0, ..., c_{n-1} \) and for any \( L \)-expansion \( M \) of \( (\mathbb{U}, d) \) we have

\[ \phi^M(\bar{b}'', \bar{a}'') - q \cdot \max(d(b_0', b_0''), ..., d(b_{m-1}', b_{m-1}''), d(c_0', c_0''), ..., d(c_{n-1}', c_{n-1}'')) \leq \phi^M(\bar{b}', \bar{a}'). \]

In particular for any \( r \in [0, 1] \) greater than \( \psi^M(b_0', ..., b_{m-1}', c_0', ..., c_{n-1}') \) if \( g \) maps each \( b_i' \) to \( b_i \) and each \( c_i' \) to \( c_i \) (i.e. \( H(g) = q \cdot d(b'c', b''c'') \)), then \( g \) maps \( M \) to \( B_{< r + H(g)} \) (by \( \phi^M(\bar{b}, \bar{a}) = \phi^M(\bar{b}'', \bar{a}'') \)). We see that for any decomposition \( r = r' - r'' \) all isometries from \( H_{< r''} \) take \( M \) to \( B_{< r'} \).

On the other hand if the expansion \( M \) does not satisfy \( \psi(b_0', ..., b_{m-1}', c_0', ..., c_{n-1}') < r \), then for any \( \delta > 0 \) there is a tuple \( \bar{a}''_0b''c'' \) such that \( r - \delta \) is less than

\[ [\phi^M(\bar{b}'', \bar{a}'') - q \cdot \max(d(b_0', b_0''), ..., d(b_{m-1}', b_{m-1}''), d(c_0', c_0''), ..., d(c_{n-1}', c_{n-1}''))] - (\rho(m_0q_1) + q \cdot m_0q_1) \]

where

\[ q_1 = \max(|d(z_1, z_2) - d(s_1, s_2)| : \text{the place of } z_1, z_2 \text{ in } \{b_0', ..., b_{m-1}', c_0', ..., c_{n-1}', a_0'', ..., a_{l-1}'\}) \]

corresponds to the place of \( s_1, s_2 \) in \( \{b_0, ..., b_{m-1}, c_0, ..., c_{n-1}, a_0, ..., a_{l-1}\} \).

If the value of the formula above is positive then \( q_1 \leq q_0 \). Then as we already know by Lemma 3.2 there is a map \( \alpha \) taking \( \bar{b}'c'\bar{a}' \) to a tuple in \( \mathbb{U} \), say \( \alpha(\bar{b}'c'\bar{a}') \), which is at distance \( \leq ((l + m + n)^2 + 1)q_1 \) from \( \bar{b}'c'\bar{a}'' \). In particular,

\[ \phi(\bar{b}'', \bar{a}'') - q \cdot \max(d(b_0', b_0''), ..., d(b_{m-1}', b_{m-1}''), d(c_0', c_0''), ..., d(c_{n-1}', c_{n-1}'')) \]

does not differ from

\[ \phi(\alpha(\bar{b}'), \alpha(\bar{a}')) - q \cdot \max(d(b_0', \alpha(b_0')), ..., d(b_{m-1}', \alpha(b_{m-1}')), d(c_0', \alpha(c_0')), ..., d(c_{n-1}', \alpha(c_{n-1}'))) \]

more than \( \rho((l + m + n)^2 + 1) \cdot q_1 + q((l + m + n)^2 + 1) \cdot q_1 \). Since

\[ \rho((l + m + n)^2 + 1) \cdot q_1 + q((l + m + n)^2 + 1) \cdot q_1 \leq \rho(m_0q_1) + qm_0q_1 \]
we may replace $\bar{b}''e''a''$ by $\alpha(\bar{b}''e''a'')$ keeping the value of the formula above greater than $r - \delta$ (substituting $\alpha(\bar{b}''e''a'')$ we see that the final part of the formula disappears).
As a result we may assume that $\bar{b}''e''a''$ and $\bar{b}c\bar{a}$ are in the same orbit under $Iso(\mathbb{U})$.

This argument shows that the value of $\psi(b'_0, ..., b''_{m-1}, c'_0, ..., c''_{n-1})$ becomes $\text{sup}$ of the corresponding subformula with respect to subspaces $a'_0, ..., a'_{l-1}, b'_0, ..., b''_{m-1}, c'_0, ..., c''_{n-1}$ isomorphic to $a_0, ..., a_{l-1}, b_0, ..., b_{m-1}, c_0, ..., c_{n-1}$. Since any isometry $h$ taking $a''_0, ..., a''_{l-1}, b''_0, ..., b''_{m-1}, c''_0, ..., c''_{n-1}$ as above to $a_0, ..., a_{l-1}, b_0, ..., b_{m-1}, c_0, ..., c_{n-1}$ maps $M$ to $B_{>r+H(h)}\delta$, there is a presentation $r - \delta = r'' - r''$ such that for sufficiently small $\varepsilon$ the basic open set of all isometries of $\mathbb{U}$ taking $b''_0, ..., b''_{m-1}, c''_0, ..., c''_{n-1}, a''_0, ..., a''_{l-1}$ to the $\varepsilon$-neighborhood of $b_0, ..., b_{m-1}, c_0, ..., c_{n-1}, a_0, ..., a_{l-1}$ is contained in $H_{<r''}$ but does not contain an element taking $M$ to $B_{<r''-\varepsilon}$. In particular, $M \notin B^H_{<r''}$. Thus $M \notin B^H_{<r''}$.

To finish the proof of the theorem note that the proof given above can be easily generalised to the case of a graded coset of the form $\max(H_1, ..., H_k)$ where each $H_i$ is defined as $H$ above. The modification concerns the form of the formula $\psi$: after the $\phi$-part we should apply $\max$ to an appropriate linear functions. □

Family $\mathcal{R}^\vee$. We now define an extension of $\mathcal{R}^U$. Let $\mathcal{R}_0^\vee$ be the extension of $\mathcal{R}_0$ by graded subsets of the form

$$H_{q,s}^\vee: g \to q \cdot \sqrt{d(g(s), s)}, \text{ where } s \subset \mathbb{Q}U, \text{ and } q \in \mathbb{Q}^+.$$ 

To see that graded sets of this form are graded subgroups take $g_1, g_2, g_3 \in Iso(\mathbb{U})$ with $g_1 \cdot g_2 = g_3$. Since all $g_i$ are isometries, $d(g_1g_2(s), s) \leq d(g_1(s), s) + d(g_2(s), s)$. Thus $\sqrt{d(g_1g_2(s), s)} \leq \sqrt{d(g_1(s), s)} + \sqrt{d(g_2(s), s)}$ which implies the required inequality. When we apply $\max$ to a finite family of graded subgroups we obviously obtain a graded subgroup too. Let $\mathcal{R}^\vee$ be the family of all $G_0$-cosets of graded subgroups from $\max(\mathcal{R}_0^\vee)$. It is clear that it satisfies the corresponding assumptions of Theorem 2.13. It is also clear that Remark 3.1 holds for subgroups from $\mathcal{R}_0^\vee$.

We will now prove that $\mathcal{B}_0$ is not an $\mathcal{R}^\vee$-nice basis. Thus there are cases forbidden by logical bases. Note that by Theorem 2.13 the family $\mathcal{B}_0$ can be also extended to an $\mathcal{R}^\vee$-nice basis of the $G$-space $\mathbb{U}_L$. Let $\mathcal{B}^\vee$ be a basis of this form. We view it as the best example of a 'non-logical' basis.

**Proposition 3.4** Let $L$ be the language corresponding to the continuous signature $\langle d, c \rangle$, where $c$ is a constant symbol. Then the family $\mathcal{B}_0$ of graded subsets of $\mathbb{U}_L$ defined above does not form an $\mathcal{R}^\vee$-nice basis. In particular $\mathcal{B}_0 \neq \mathcal{B}^\vee$.

**Proof.** We fix $u_0 \in \mathbb{Q}U$ and consider the graded subset $\psi$ of $\mathbb{U}_L$ defined by the formula $10 \cdot d(u_0, c)$ (with dot-product). Let $H$ be the graded subgroup of $Iso(\mathbb{U})$ defined by

$$g \to \sqrt{d(g(u_0), u_0)},$$

and let $\theta = \psi^{\Delta H}$.

**Claim.** If a structure $M \in \mathbb{U}_L$ satisfies $\frac{1}{10} < d(u_0, c) \leq \frac{1}{2}$ then $\theta(M) = \sqrt{d(u_0, c)}$.

Indeed let $q = d(u_0, c)$. Consider an isometric copy of $[0, 1]$ in $\mathbb{U}$ so that $0$ is identified with $c$ and the number $q$ with $u_0$. Take any $a$ between $u_0$ and $c$. Let
part of graded subgroups of the form note that by Remark 3.1 it suffices to verify the formulation for property holds: \( H \) there is an property for graded subgroups if for any graded subgroup \( \text{In this subsection we apply Lemma 3.2 to some useful property of the space } \mathbb{U}_L. \) We formulate it in the most general form.

**Definition 3.5** Let \((G, \mathcal{R})\) and \((X, \mathcal{U})\) satisfy the assumptions of Remark 2.8 and Definition 2.9. We say that the \((G, \mathcal{R})\)-space \((X, \mathcal{U})\) has the **approximation property for graded subgroups** if for any graded subgroup \( H \in \mathcal{R} \) and for any \( \varepsilon > 0 \) there is an \( H \)-invariant \( \phi \in \mathcal{U} \) and \( \delta > 0 \) such that for any \( c \) and \( c' \in X \) the following property holds:

\[
|\phi(c) - \phi(c')| \leq \delta, \text{ then there is } g \in G \text{ with } H(g) < \varepsilon \text{ and } d(g(c'), c) < \varepsilon.
\]

Let us consider the \(G\)-space \( \mathbb{U}_L \) under any \( \mathcal{R}^U \)-nice basis \( \mathcal{B}_0 \) containing the family \( \mathcal{B}_0 \). We consider \( \mathbb{U}_L \) under the metric \( \delta_{\text{seq}(\mathcal{Q})} \) defined as in Section 1.1 with respect to some enumeration of \( \text{seq}(\mathcal{Q}\mathbb{U}) \). To see that \( \mathbb{U}_L \) satisfies the approximation property for graded subgroups note that by Remark 3.1 it suffices to verify the formulation for graded subgroups of the form

\[
H_{q, \bar{s}} : g \rightarrow q \cdot d(g(\bar{s}), \bar{s}), \text{ where } \bar{s} \subset \mathcal{Q}\mathbb{U}, \text{ and } q \in \mathbb{Q}.
\]
By the definition of the space $U_L$ to guarantee that two $L$-structures $M$ and $N$ are distant $\leq \epsilon$ it suffices to find an appropriate tuple $\bar{a} \subset \mathbb{Q}U$ and appropriate $\epsilon' > 0$ so that the values of the corresponding atomic $L$-formulas (say $\psi_i, i \in I$) on subtuples of $\bar{a}$ do not differ in $M$ and $N$ more than $2\epsilon'$.

On the one hand $\mathbb{Q}U$ is universal homogeneous and on the other hand it is dense in $U$. Thus the tuple $\bar{a}$ can be replaced by another one where there is no triples of its elements which lie on geodesic lines (when we find such a triple we may replace the middle element by a sufficiently close element outside the line). Moreover we may assume that there is no geodesic triple intersecting both $\bar{a}$ and $\bar{a} \setminus \bar{s}$.

By the values of $\psi_i$'s in $M$ we build $\inf$-formulas over $\bar{s}$ so that if they have values in $M$ and $N$ which are close enough (i.e. as in the first part of the implication of Definition 3.5 for $c = M$ and $c' = N$), then there is $\bar{a}'$ such that the value in $M$ of any $\psi_i(\bar{s}a)$ does not differ from the value in $N$ of $\psi_i(\bar{s}a')$ more than $\epsilon'$. (These $\inf$-formulas just express the existence of $\bar{a}'$ so that $|\psi_i(\bar{s}a) - r| \leq \frac{\epsilon'}{4}$ where $r$ is a rational number with $|\psi_i(\bar{s}a) - r| \leq \frac{\epsilon'}{4}$.)

Now we use Lemma 3.2. It in particular says that there is sufficiently small $\delta > 0$ such that if distances between pairs in $\bar{s}a$ and the corresponding distances in $\bar{s}a'$ do not differ more than $\delta$, then there is an isometry $g \in Iso(U)$ (apply ultrahomogenity) such that $d(g(\bar{s}a'), \bar{s}a) \leq \epsilon'$. Choosing such $g$ we obtain that the values of appropriate atomic $L$-formulas in $g(N)$ on $\bar{s}a$ does not differ from the corresponding values in $M$ more than $2\epsilon'$.

As a result we obtain the following proposition.

**Proposition 3.6** The approximation property for graded subgroups holds in the space $U_L$ under any nice basis $B$ containing the family of graded subsets defined by all continuous $L$-sentences over parameters from $\mathbb{Q}U$.

### 3.3 Impossible versions of Lopez-Escobar theorem

The Lopez-Escobar theorem states that in the (discrete) $S_\infty$-space of countable structures any invariant Borel subset is defined by a formula of $L_{\omega_1\omega}$. This theorem is crucial for Corollary 1.13 of [3] that any nice topology is defined by a countable fragment of $L_{\omega_1\omega}$. In this subsection we discuss some versions of this theorem in the space of continuous structures.

Let us fix some continuous language $L$ and the corresponding space $U_L$ of $L$-structures on $U$. Consider the family $R^U$ of open graded subgroups of $Iso(U)$ defined in Section 3.1. To adapt the argument of Corollary 1.13 of [3] we need the following version of the Lopez-Escobar theorem.

Let $H$ be a graded subgroup from $R^U$ and $\lambda$ be a Borel graded subset of $U_L$ which is $H$-invariant.

Find an $L_{\omega_1\omega}$-formula $\phi$ over $U$ which defines $\lambda$ by the map $M \rightarrow \phi^M$.

We remind the reader that continuous $L_{\omega_1\omega}$-formulas are defined by the standard procedure applied to countable conjunctions and disjunctions (see [7]). Each continuous infinite formula depends on finitely many free variables. The main demand
is the existence of continuity moduli of such formulas. It is usually assumed that a continuity modulus $\delta_{\phi,x}$ satisfies the equality

$$\delta_{\phi,x}(\varepsilon) = \sup \{ \delta_{\phi,x}(\varepsilon') : 0 < \varepsilon' < \varepsilon \}$$

and

$$\delta_{\wedge,\phi,x}(\varepsilon) = \sup \{ \delta'_{\wedge,\phi,x}(\varepsilon') : 0 < \varepsilon' < \varepsilon \}, \text{ where } \delta'_{\wedge,\phi,x} = \inf \{ \delta_{\phi,x} : \phi \in \Phi \}.$$  

S.Coskey, M.Lupini, I.Ben Yaacov, A.Nies and T.Tsankov have proved in [9] and [13] that in the case when $H = Iso(\mathbb{U})$ the formula $\phi$ realising $\lambda$ as above, can be found. On the other hand the case of an arbitrary $H$ still looks open. It is clear that when $H$ depends on some parameters from $\mathbb{U}$ (for example when $H$ is a graded stabiliser of a finite subset from $\mathbb{U}$) the formula $\phi$ must be of the form $\phi'(\bar{a})$, where $\bar{a}$ is a tuple from $\mathbb{U}$. Note that (contrary to the classical case) it can happen that the corresponding formula $\phi'(\bar{x})$ with variable $\bar{x}$ instead of $\bar{a}$ is not an $L_{\omega_1\omega}$-formula. This suggests the following considerations.

Assuming that $L$ has constant symbols let $L_{\omega_1\omega}^{tame}$ consist of all formulas of the form $\phi(\bar{z}) = \phi'(\bar{c}, \bar{z})$, where $\bar{c}$ is a tuple of $L$-constants and $\phi'(\bar{x}, \bar{z})$ is a continuous $L_{\omega_1\omega}$-formula without constants. It is clear that $L_{\omega_1\omega}^{tame}$ is a fragment of $L_{\omega_1\omega}$ which is closed under finitary iterations of basic continuous connectives. Note that when $L$ is discrete, $L_{\omega_1\omega}^{tame} = L_{\omega_1\omega}$.

We now give an example which shows that the Lopez-Escobar theorem does not hold for $L_{\omega_1\omega}^{tame}$.

Let $L = \langle d, R^1, c \rangle$ be a continuous signature where $c$ is a constant symbol and $R$ is a symbol of a predicate with continuity modulus $id$. Consider the logic space $\mathbb{U}_L$ of all continuous $L$-structures on the Urysohn space $\mathbb{U}$. To each continuous $L$-structure $M \in \mathbb{U}_L$ we associate a real number $r_M$ defined as follows:

$$M \rightarrow r_M = \sin(\frac{1}{R(c)^M}).$$

We assume that $r_M = 0$ for $R(c)^M = 0$. Since $R(c)$ is a formula and $\sin(\frac{1}{x})$ is continuous in $[0, 1]$, the function $M \rightarrow r_M$ is Borel. It is obviously $Iso(\mathbb{U})$-invariant.

We want to prove that this function is not defined by a continuous $L_{\omega_1\omega}^{tame}$-formula.

**Proposition 3.7** There is no $L_{\omega_1\omega}^{tame}$-sentence $\psi$ so that for any $M \in \mathbb{U}_L$ the numbers $r_M$ and $\psi^M$ are the same.

**Proof.** If the constant symbol $c$ does not appear in a continuous $L_{\omega_1\omega}$-formula $\psi$, then for any $\langle L \setminus \{c\} \rangle$-structure on $\mathbb{U}$ of the form $R(x) = d(x, u_0)$ with some fixed $u_0 \in \mathbb{U}$ the value $\psi^M$ coincides with $\psi^{M'}$ where $M = \langle \mathbb{U}, d, R, c \rangle$ with $c = u_0$ and $M' = \langle \mathbb{U}, d, R, c' \rangle$ with $d(c', u_0) = \frac{2}{\pi}$. This means that $\psi$ cannot define the function $M \rightarrow r_M$.

Suppose that $\phi(x)$ is a continuous $L_{\omega_1\omega}$-formula without $c$ so that for any $M \in \mathbb{U}_L$, $r_M = \phi(c)^M$. Let $u_0 \in \mathbb{U}$ and $M_0$ be the structure of the signature $\langle d, R \rangle$ on $\mathbb{U}$ where $R(x) = d(x, u_0)$. 

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Let $\delta_{\phi,x}$ be a continuity modulus of $\phi$. Then for any two $L$-expansions of $M_0$ (say by $c$ and $c'$)
\[ d(c, c') \leq \delta_{\phi,x}(\varepsilon) \Rightarrow |r_{(M_0,c)} - r_{(M_0,c')}| < \varepsilon. \]
Choosing $c, c'$ in an isometric copy of $[0, 1]$ in $U$ (identifying $u_0$ with 0) we obtain a continuity modulus for $\sin(\frac{1}{x})$ in $(0, 1]$. This is a contradiction. □

4 Canonical pieces which are $G$-orbits

We now consider canonical pieces (see Section 2.2) which are $G$-orbits. We take the assumptions of Remark 2.8, Definition 2.11 and Definition 2.12, i.e. in particular $G$ is a Polish group with a dense countable subgroup $G_0$ and a graded basis $\mathcal{R}$ consisting of graded $G_0$-cosets. In this section we additionally assume that that all graded subgroups from $\mathcal{R}$ are clopen (as in the case of Theorem 3.3). By $\mathcal{B}$ and $t$ we denote a nice basis and the corresponding nice topology of $(X, \tau)$, which is a Polish $G$-space.

We start with a proposition which is a version of Lindstrom’s model completeness theorem (that an $\forall\exists$-axiomatizable $\aleph_0$-categorical theory is model complete).

Proposition 4.1 Let $X = Gx_0$ for some (any) $x_0 \in X$ and $X$ be a $G_\delta$-subset of $X$. Then both topologies $\tau$ and $t$ are equal on $X$.

Proof. We have to check that every element of $\mathcal{B}$ is a $\tau$-open graded subset on $X$. Let $\phi \in \mathcal{B}$, $x_1 \in X$ and $\phi(x_1) = r_1 < r$. Take an open graded subgroup $H$ of $G$ from $\mathcal{R}$ such that $\phi$ is $H$-invariant. Let $r' < r - r_1$. Since by the Effros’ theorem on $G_\delta$-orbits, the canonical map $g \to gx_1$ is an open map $G \to X$, we see that $H_{<r'}x_1$ is a $\tau$-open subset of $\phi_{<r} \cap X$ containing $x_1$. □

In the following definition we introduce farther counterparts of model theoretic notions.

Definition 4.2 Let $t$ be an $\mathcal{R}$-nice topology for the $G$-space $(X, \tau)$. Let $H$ be an open graded subgroup from $\mathcal{R}$ and $X$ be an invariant $G_\delta$-subset of $X$ with respect to the $t$-topology.

1. A family $\mathcal{F}$ of subsets of the form $\phi_{<r}$ with $H$-invariant $\phi \in \mathcal{B}$ is called an $H$-type in $X$, if it is maximal with respect to the condition that $X \cap \bigcap \mathcal{F} \neq \emptyset$.

2. An $H$-type $\mathcal{F}$ is called principal if for every $\varepsilon > 0$ there is an $H$-invariant graded basic set $\phi \in \mathcal{B}$ and there is $r$ such that $\phi_{<r} \in \mathcal{F}$ and for each $B = \psi_{<t} \in \mathcal{F}$, the set $\psi_{<t+\varepsilon}$ contains $\phi_{<r} \cap X$. Then we say that $\phi_{<r}$ $\varepsilon$-defines $\mathcal{F}$.

Each type $\mathcal{F}$ is determined by any element from $X \cap \bigcap \mathcal{F}$. It is worth noting that when $x$ and $z$ determine different $H$-types, then there is an $H$-invariant $\psi$ and two rational numbers $r_1$ and $r_2 \leq 1$ so that $\psi(x) < r_1$, $1 - \psi(z) < r_2$ and $r_1 < 1 - r_2$.

Lemma 4.3 Assume that $c$ defines a principal $H$-type $\mathcal{F}$ and $\phi_{<r} \frac{c}{2}$-defines $\mathcal{F}$ (with an $H$-invariant graded basic set $\phi \in \mathcal{B}$ and $r > 0$).
Then for any $H$-invariant $\psi$ and any $c'$ and $c''$ with $\max(\phi(c'), \phi(c'')) < r$, we have $|\psi(c') - \psi(c'')| < \varepsilon$. In particular $\phi_{<r}$ $\varepsilon$-defines the $H$-type of any element of $\phi_{<r}$.

Proof. Note that $|\psi(c') - \psi(c)| < \frac{\varepsilon}{2}$. Indeed, if $\psi(c) < \psi(c')$, then applying the definition of principal types, $\psi(c') < \psi(c) + \frac{\varepsilon}{2}$. If $\psi(c') < \psi(c)$, then repeat this argument for $1 - \psi(x)$. The rest is clear. □

Note that the following lemma is related to omitting types theorems from logic.

**Lemma 4.4** Let $G$, $\mathcal{R}$, $\mathcal{B}$ and $t$ satisfy the assumptions of the section and $X$ be an invariant $t$-$G_\delta$-subset. Let $H \in \mathcal{R}$. Then for any non-principal $H$-type $\mathcal{F}$ the set

$$\bigcap_{g \in G} \left( g \left( \bigcup_{B \in \mathcal{F}} (X \setminus B) \right) \right)$$

is nonempty and $G$-invariant. In particular if $X$ is a $G$-orbit then any $H$-type of $X$ is principal.

Proof. Suppose $\mathcal{F}$ is non-principal. Find $\varepsilon > 0$ so that no $\phi_{<t}$ $\varepsilon$-defines $\mathcal{F}$, i.e. by Lemma 4.3 there is no $\phi_{<t}$ in $\mathcal{F}$ so that for any $H$-invariant $\psi$ and any $c'$ and $c''$ with $\max(\phi(c'), \phi(c'')) < r$, we have $|\psi(c') - \psi(c'')| < \varepsilon$. We claim that $\bigcap \{ H_{<\frac{\varepsilon}{2}} B : B \in \mathcal{F} \}$ is meager in $(X, t)$. Indeed, otherwise it would be comeager in $X \cap B_{\mathcal{F}}$ for some non-empty $B_{\mathcal{F}}$ of the form $\phi_{<r}$ with $\phi \in \mathcal{B}$. We fix these $\phi$ and $r$. Since $\mathcal{B}$ contains constant functions we may assume that $r < \frac{\varepsilon}{2}$. Consider $\phi^{\Delta H}$. It is an $H$-invariant graded set from $\mathcal{B}$. Since $\phi$ is open, $\phi^{\Delta H}$ is open too. If $B \in \mathcal{F}$ with $B = c_{<t}$, then the closure of $H_{<\frac{\varepsilon}{2}} B \cap X$ contains $B_{\mathcal{F}} \cap X$. Thus by Lemma 2.7 $(\phi^{\Delta H})_{<t}$ is contained in $H_{<t}(\psi^{\Delta H})_{<t+\frac{\varepsilon}{2}}$ and $H$-invariantness of $\psi$ (i.e. in particular $(\psi^{\Delta H})_{<t} = c_{<t}$ and $H_{<\varepsilon} \psi_{<t} \subseteq \psi_{<t+\varepsilon}$) implies that $(\phi^{\Delta H})_{<r}$ is contained in $\psi_{\leq t+r+\frac{\varepsilon}{2}}$, i.e. in $\psi_{<t+\varepsilon}$. This is a contradiction with the assumption that $\mathcal{F}$ is non-principal.

Now $X \cap \bigcup_{B \in \mathcal{F}} (X \setminus H_{<\frac{\varepsilon}{2}} B)$ is comeagre in $X$. Since this set is contained in

$$\bigcap \{ X \cap h(\bigcup_{B \in \mathcal{F}} (X \setminus B)) : h \in H_{<\frac{\varepsilon}{2}} \},$$

for every $g \in G$, the set

$$g(\bigcap \{ X \cap h(\bigcup_{B \in \mathcal{F}} (X \setminus B)) : h \in H_{<\frac{\varepsilon}{2}} \}),$$

is comeagre in $X$.

The graded group $H$ is of countable index in $G$ (by $H \subseteq G$), i.e. for every $\varepsilon'$ the group $G$ is covered by countably many left translates of $H_{<\varepsilon'}$. In particular we may choose a countable subgroup $G_{cd} \subset G$ such that for every $g \in G$ there is $g' \in G_{cd} \cap gH_{<\frac{\varepsilon}{2}}$. It is clear that

$$\bigcap_{g \in G_{cd}} g(\bigcap \{ X \cap h(\bigcup_{B \in \mathcal{F}} (X \setminus B)) : h \in H_{<\frac{\varepsilon}{2}} \})$$

is comeagre in $X$. Thus by Lemma 4.3 there is no $\phi_{<t}$ in $\mathcal{F}$ so that for any $H$-invariant $\psi$ and any $c'$ and $c''$ with $\max(\phi(c'), \phi(c'')) < r$, we have $|\psi(c') - \psi(c'')| < \varepsilon$. We claim that $\bigcap \{ H_{<\frac{\varepsilon}{2}} B : B \in \mathcal{F} \}$ is meager in $(X, t)$.
is comeagre in $X$. Note that
\[ \bigcap_{g \in G_{cd}} \{ X \cap h(\bigcup_{B \in F} (X \setminus B)) : h \in H_{< \frac{\epsilon}{2}} \} = \bigcap_{g \in G} \left( g(\bigcup_{B \in F} (X \setminus B)) \right). \]
Indeed if $x \in \bigcap_{g \in G_{cd}} \{ X \cap h(\bigcup_{B \in F} (X \setminus B)) : h \in H_{< \frac{\epsilon}{2}} \}$ and $g \in G$ then find $g' \in G_{cd} \cap gH_{< \frac{\epsilon}{2}}$ with $g = g'h'$, $h' \in H_{< \frac{\epsilon}{2}}$. Thus
\[ (g')^{-1}(x) \in \bigcap_{B \in F} \{ X \cap h(\bigcup_{B \in F} (X \setminus B)) : h \in H_{< \frac{\epsilon}{2}} \} \]
and $(h')^{-1}(g')^{-1}(x) \in X \cap \bigcup_{B \in F} (X \setminus B)$, i.e. $x \in X \cap g(\bigcup_{B \in F} (X \setminus B))$. The rest is clear.
We see that the intersection $\bigcap_{g \in G} \{ g(\bigcup_{B \in F} (X \setminus B)) \}$ is $G$-invariant and comeagre.

To prove the remaining part suppose that $F$ is a non-principal $H$-type of $X$ and $X$ is a $G$-orbit. Then, by the previous statement, $X \subseteq \bigcup_{B \in F} (X \setminus B)$. This contradicts to the definition of a type. \(\square\)

The following statement is a version of Ryll-Nardzewski’s theorem. We remind the reader that we assume that graded cosets from $\mathcal{R}$ are clopen and are represented by elements of $G_0 <_{\text{dense}} G$. The group $G$ is considered under a left-invariant metric (for example defined as $\rho_S$ in Section 0).

**Theorem 4.5** Let $(X, \mathcal{U})$ be a Polish $(G, \mathcal{R})$-space and $\mathfrak{t}$ be a nice topology of $X$ with the nice basis $\mathcal{B}$. A piece $X$ of the canonical partition with respect to the topology $\mathfrak{t}$ is a $G$-orbit if and only if for any basic clopen graded subgroup $H \in \mathcal{R}$ any $H$-type of $X$ is principal.

**Proof.** By Lemma 4.4 we have the necessity of the theorem.

For sufficiency we use the back-and-forth argument. Let $x, y \in X$. We build a set $\Gamma$ of tuples $(H_i, \phi_i, r_i, \varepsilon_i, H'_i, \phi'_i, g_i)$, $i \in \omega$, with the following properties:

(a) each $H_i$ (and $H'_i$) is a clopen graded subgroup from $\mathcal{R}$ and $g_i$ belongs to $G_0$;
(b) each $\phi_i$ is an $H_i$-invariant basic $\mathfrak{t}$-clopen graded subset with $x \in (\phi_i)_{< r_i}$ and each $\phi'_i$ is an $H'_i$-invariant basic $\mathfrak{t}$-clopen graded subset with $y \in (\phi'_i)_{< r_i}$;
(c) for each even $i > 0$, $H_{i+1} \subseteq \max(H_i, H_{i-1})$, $H'_{i+1} = g_{i+1}H_{i+1}g_{i+1}^{-1}$ (i.e. $= (g_{i+1}H_{i+1})$), $\phi_{i+1} \subseteq \max(\phi_i, \phi_{i-1})$, $\phi'_{i+1} = g_{i+1}\phi_{i+1}$ and the $H_{i+1}$-type of $x$ is $\frac{\varepsilon_i + 1}{2}$-defined by $(\phi_{i+1})_{< r_{i+1}}$ as a principal type;
(d) for each odd $i > 0$, $H'_{i+1} \subseteq \max(H'_i, H'_{i-1})$, $H_{i+1} = g_{i+1}H_{i+1}g_{i+1}$, $\phi'_{i+1} \subseteq \max(\phi'_i, \phi'_{i-1})$, $\phi_{i+1} = g_{i+1}\phi'_{i+1}$ and the $H'_{i+1}$-type of $y$ is $\frac{\varepsilon_i + 1}{2}$-defined by $(\phi'_{i+1})_{< r_{i+1}}$ as a principal type;

\[ \max(\text{diam}((\phi_i)_{\leq r_i}), \text{diam}((H_i)_{\leq \varepsilon_i}), \text{diam}((\phi'_i)_{\leq r_i}), \text{diam}((H'_i)_{\leq \varepsilon_i})) < 2^{-i}, \]

\(\text{by } H^g \text{ we denote the graded group } g^{-1}hg \to H(h) \text{ and by } ^gH \text{ we denote the graded group } g^h \to H(h)\)
\[ H_i(g_i^{-1}g_{i+1}) \leq \varepsilon_i \text{ and } H'_i(g_{i+1}g_i^{-1}) \leq \varepsilon_i \text{ for all } i. \]

It is worth noting here that since \( G \) is considered under a left-invariant metric, 
\[ \text{diam}((g_i H_i)_{\leq \varepsilon_i}) = \text{diam}((H_i)_{\leq \varepsilon_i}) < 2^{-i}. \]

At Step 0 let \( H_0 = H'_0 = H_{-1} = H'_{-1} = G \), \( g_0 = id \), \( \varepsilon_0 = 1 \) and \( \phi_0 = \phi'_0 \) be a G-invariant basic graded set so that for some \( r_0 \), \( X \subset (\phi_0)_{< r_0} \).

At step \( i + 1 \) (assuming that \( i \) is even) take any basic \( C \subset (\phi_i)_{< r_i} \) of the form \( \psi_{< r} \) where \( \psi \in \mathcal{B} \) is a \((t \text{-clop})\) graded subset of \( \max(\phi_i, \phi_{i-1}) \), with \( x \in C \) and \( r \) so small that \( \text{diam}(C) < 2^{-(i+1)} \). Let \( H_{i+1} \) be a basic clopen graded subgroup of \( \max(H_i, H_{i-1}) \) such that \( \psi \) is \( H_{i+1}\)-invariant. We may choose \( H_{i+1} \) so that for some \( \varepsilon < 2^{-(i+1)} \), \( \text{diam}((H_{i+1})_{\leq \varepsilon}) < 2^{-(i+1)} \).

We denote this \( \varepsilon \) by \( \varepsilon_{i+1} \). Let \( \phi_{i+1} (\phi_i)_{< r_{i+1}} \subset C \frac{\varepsilon_{i+1}}{2} \text{-define the (principal) } H_{i+1} \text{-type of } x. \) We may assume that \( \phi_{i+1} \subseteq \psi \) and \( r_{i+1} \leq r \) for \( \psi \) and \( r \) above.

Let \( g_{i+1} \) be any element of \( G_0 \) which maps the clopen set \( (\phi_{i+1})_{< r_{i+1}} \) to a set containing \( y \). The existence of such \( g_{i+1} \) follows from the assumption that \( x \) and \( y \) belong to the same canonical piece. On the other hand by the inductive assumptions, \( g_i \phi_{i+1} \) is an \( g_i H_{i+1} \)-invariant clopen graded subset of \( \phi'_i \). Let us compute \( H'_i(g_i g_i^{-1}) \).

By Lemma 4.3 for any \( H'_i \)-invariant graded subset \( \theta \subseteq \phi'_i \) the values \( \theta(g_i g_i^{-1}) \text{ and } \theta(y) \) do not differ more than \( \varepsilon_i \). Since \( \theta \) is \( H'_i \)-invariant, \( H'_i(g_i g_i^{-1}) < \varepsilon_i \). Since \( H_i = g_i^{-1} H'_i g_i \), \( H_i(g_i^{-1} g_i) = H'_i(g_i g_i^{-1} g_i g_i^{-1}) < \varepsilon_i \), i.e. the corresponding part of condition (d) is satisfied.

Let \( H'_{i+1} = (g_{i+1} H_{i+1}) \) and \( \phi'_{i+1} = g_{i+1} \phi_{i+1} \). It is clear that properties (a)-(d) are satisfied for \( i \).

The case of odd \( i \) is symmetric.

As a result we have a Cauchy sequence \( g_i, i \in \omega \). Let \( g \in G \) be the limit of this sequence. Note that for each \( i \) the element \( g_i \) maps \( (\phi_i)_{< r_i} \) to \( (\phi'_{i})_{< r_i} \) and

\[ \{x\} = \bigcap \{(\phi_i)_{< r_i} : i \in \omega\} \text{ and } \{y\} = \bigcap \{(\phi'_i)_{< r_i} : i \in \omega\}. \]

Thus \( g \) maps a Cauchy sequence converging to \( x \) to a Cauchy sequence converging to \( y \). Thus \( g \) maps \( x \) to \( y \). \( \square \)

5 Complexity of some subsets of the logic space

Viewing the logic space \( Y_L \) as a Polish space one may consider Borel/algorithmic complexity of interesting subsets of \( Y_L \). In this section we fix a countable dense subset \( S_Y \) of \( Y \) and study subsets of \( Y_L \) which are invariant with respect to isometries stabilising \( S_Y \) setwise. The best example of this situation is the logic space \( U_L \) over the bounded Urysohn metric space \( U \) where distinguishing the countable counterpart \( \mathbb{Q}U \) of \( U \) (see Section 3) we study \( \text{Iso}(\mathbb{Q}U) \)-invariant subsets of \( U_L \).

This approach corresponds to considering a structure on \( Y \) (say \( M \)) together with its presentation over \( S_Y \), i.e. the set

\[ \text{Diag}(M, S_Y) = \{(\phi, q) : M \models \phi < q, \text{ where } q \in [0, 1] \cap \mathbb{Q} \text{ and } \phi \text{ is a continuous sentence with parameters from } S_Y\}. \]
In this section we examine separable categoricity and ultrahomogeneity. In particular in Section 5.1 we find a Borel subset $SC$ of $Y_L$ which is $Iso(Y_L)$-invariant and each separably categorical structure on $Y$ is homeomorphic to a structure from $SC$.

Since any Polish group can be realised as the automorphism group of an approximately ultrahomogeneous structures, it makes sense in order to characterise some special properties of Polish groups to study the corresponding subclasses of approximately ultrahomogeneous structures and then to study the complexity of these subclasses. In Section 5.2 we consider two opposite subsets of approximately ultrahomogeneous structures from $Y_L$ which correspond to natural properties of automorphism groups: separable oligomorphicity and admitting of complete left invariant metrics.

In Section 5.3 we study complexity of the index set of computable members of $SC$. We believe that these ideas can be applied to investigation of complexity of other topological and model theoretic notions. In fact our intention in this section is to demonstrate some new settings arising in the approach of the logic space of continuous structures.

Our special attention to separable categoricity is motivated by ubiquity of it in this paper: the Urysohn space is separably categorical, the material of Section 4 is an abstract form of Ryll-Nardzewski’s theorem, the approximation property for graded subgroups for logic spaces over the Urysohn space (Section 3.2) is a consequence of categoricity.

5.1 Separable categoricity

We preserve all the assumptions of Section 1 on the space $(Y, d)$. For simplicity we assume that all $L$-symbols are of continuity modulus $id$. We reformulate separable categoricity as follows.

**Proposition 5.1** Let $M$ be a non-compact, separable, continuous, metric structure on $(Y, d)$. The structure $M$ is separably categorical if and only if for any $n$ and $\varepsilon$ there are finitely many conditions $\phi_i(\bar{x}) \leq \delta_i$, $i \in I$, so that any $n$-tuple of $M$ satisfies one of these conditions and the following property holds:

for any $i \in I$, any $a_1, ..., a_n \in M$ realising $\phi_i(\bar{x}) \leq \delta_i$ and any finite set of formulas $\Delta(x_1, ..., x_n, x_{n+1})$ realised in $M$ and containing $\phi_i(\bar{x}) \leq \delta_i$, there is a tuple $b_1, ..., b_n, b_{n+1}$ realising $\Delta$ such that $\max_{i \leq n} d(a_i, b_i) < \varepsilon$.

To prove this proposition we start with the following observation.

**Lemma 5.2** Let $M$ be a non-compact, separable, continuous, metric structure on $(Y, d)$. The structure $M$ is separably categorical if and only if for any $n$ and $\varepsilon$ there are finitely many conditions $\phi_i(\bar{x}) \leq \delta_i$, $i \in I$, so that any $n$-tuple of $M$ satisfies one of these conditions and for any $i \in I$, any $a_1, ..., a_n \in M$ realising $\phi_i(\bar{x}) \leq \delta_i$ and any type $p(x_1, ..., x_n, x_{n+1})$ realised in $M$ and containing $\phi_i(\bar{x}) \leq \delta_i$, there is a tuple $b_1, ..., b_n, b_{n+1}$ realising $p$ such that $\max_{i \leq n} d(a_i, b_i) < \varepsilon$. 

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Proof. By Theorem 12.10 of [5] a complete theory $T$ is separably categorical if and only if for each $n > 0$, every $n$-type is principal. An equivalent condition states that for each $n > 0$, the metric space $(S_n(T), d)$ is compact. In particular for every $n$ and every $\varepsilon$ there is a finite family of principal $n$-types $p_1, ..., p_m$ so that their $\varepsilon/2$-neighbourhoods cover $S_n(T)$.

Thus when $M$ is separably categorical, given $n$ and $\varepsilon$, we find appropriate $p_i$, $i \in I$, define $P_i(\bar{x}) = \text{dist}(\bar{x}, p_i(M))$, the corresponding definable predicates and $n$-conditions $\phi_i(\bar{x}) \leq \delta_i$ describing the corresponding $\varepsilon/2$-neighbourhoods of $p_i$. The rest follows by strong $\omega$-near-homogeneity.

To see the converse assume that $M$ satisfies the property from the formulation. To see that $G = \text{Aut}(M)$ is approximately oligomorphic take any $n$ and $\varepsilon$ and find finitely many conditions $\phi_i(\bar{x}) \leq \delta_i$, $i \in I$, satisfying the property from the formulation for $n$ and $\varepsilon/4$. Choose $\bar{a}_i$ with $\phi_i(\bar{a}_i) \leq \delta_i$ and let $F = \{\bar{a}_i : i \in I\}$. To see that $G \cdot F$ is $\varepsilon$-dense we only need to show that if $\bar{a}$ satisfies $\phi_i(\bar{x}) \leq \delta_i$, then there is an automorphism which takes $\bar{a}$ to the $\varepsilon$-neighbourhood of $\bar{a}_i$. This is verified by "back-and-forth" as follows. Let $(\varepsilon_k)$ be an infinite sequence of positive real numbers whose sum is less than $\varepsilon/4$. At every step $l$ (assuming that $l \geq n$) we build a finite elementary map $\alpha_l$ and $l$-tuples $\bar{c}_l$ and $\bar{d}_l$ so that

- $\bar{c}_n = \bar{a}$ and $\bar{d}_n = \bar{a}_i$;
- for $l > n$, $\alpha_l$ takes $\bar{c}_l$ to $\bar{d}_l$;
- for $l > n + 1$, the first $l - 1$ coordinates of $\bar{c}_l$ (resp. $\bar{d}_l$) are at distance less than $\varepsilon_l$ away from the corresponding coordinates of $\bar{c}_{l-1}$ (resp. $\bar{d}_{l-1}$);
- the sets $\bigcup \{\bar{c}_l : l \in \omega\}$ and $\bigcup \{\bar{d}_l : l \in \omega\}$ are dense in $M$.

In fact we additionally arrange that for even $l$, $\bar{c}_{l+1}$ extends $\bar{c}_l$ and for odd $l$ $\bar{d}_{l+1}$ extends $\bar{d}_l$. In particular the type of $\bar{c}_{l+1}$ always extends the type of $\bar{c}_l$. At the $(n+1)$-th step we find finitely many conditions $\phi_j(\bar{x}) \leq \delta_j$, $j \in J$, so that any $(n+1)$-tuple of $M$ satisfies one of these conditions and for any $j \in J$, any $a'_1, ..., a'_{n+1} \in M$ realising $\phi_j(\bar{x}) \leq \delta_j$ and any type $p(x_1, ..., x_{n+1}, x_{n+2})$ realised in $M$ and containing $\phi_j(\bar{x}) \leq \delta_j$, there is a tuple $b_1, ..., b_{n+1}, b_{n+2}$ realising $p$ such that $\max_{t \leq n+1} d(a'_t, b_t) < \varepsilon_{n+1}$. Now by the choice of $i$ for any extension of $\bar{a} = \bar{c}_n$ to an $(n+1)$-tuple $\bar{c}_{n+1}$ we can find a tuple $\bar{d}_{n+1}$ realising $tp(\bar{c}_{n+1})$ so that the first $n$ coordinates of $\bar{d}_{n+1}$ are at distance less than $\varepsilon/4$ away from the corresponding coordinates of $\bar{d}_n = \bar{a}_i$. If $n$ is even we choose such $\bar{c}_{n+1}$ and $\bar{d}_{n+1}$; if $n$ is odd we replace the roles of $\bar{c}_{n+1}$ and $\bar{d}_{n+1}$. For the next step we fix the condition $\phi_j(\bar{x}) \leq \delta_j$ satisfied by $\bar{c}_{n+1}$ and $\bar{d}_{n+1}$.

The $(l+1)$-th step is as follows. Assume that $l$ is even (the odd case is symmetric). Extend $\bar{c}_l$ to an appropriate $\bar{c}_{l+1}$ (aiming to density of $\bigcup \{\bar{c}_l : l \in \omega\}$). There are finitely many conditions $\phi_k(\bar{x}) \leq \delta_k$, $k \in K$, so that any $(l+1)$-tuple of $M$ satisfies one of these conditions and for any $k \in K$, any $a'_1, ..., a'_{l+1} \in M$ realising $\phi_k(\bar{x}) \leq \delta_k$ and any type $p(x_1, ..., x_{l+1}, x_{l+2})$ realised in $M$ and containing $\phi_k(\bar{x}) \leq \delta_k$, there is a tuple $b_1, ..., b_{l+1}, b_{l+2}$ realising $p$ such that $\max_{t \leq l+1} d(a'_t, b_t) < \varepsilon_{l+1}$. We find the condition
satisfied by $\bar{c}_{i+1}$ and a tuple $\bar{d}_{i+1}$ realising $tp(\bar{c}_{i+1})$ so that the first $l$ coordinates of $\bar{d}_{i+1}$ are at distance less than $\epsilon_1$ away from the corresponding coordinates of $\bar{d}_i$.

As a result for every $k$ we obtain Cauchy sequences of $k$-restrictions of $\bar{c}_i$-s and $\bar{d}_i$-s. For $k = n$ their limits are not distant from $\bar{a}$ and $\bar{a}_i$ more than $\epsilon/2$. Moreover the limits $\lim\{\bar{c}_i\}$ and $\lim\{\bar{d}_i\}$ are dense subsets of $Y$ and realise the same type. This defines the required automorphism of $M$. □

Proof of Proposition 5.1. It suffices to show that the condition of the formulation implies the corresponding condition of Lemma 5.2. Given $n$ and $\epsilon$ take the family $\phi_i(\bar{x}) \leq \delta_i$, $i \in I$, satisfying the condition of the proposition for $n$ and $\epsilon/2$. Let $p(\bar{x}, x_{n+1})$ be a type with $\phi_i(\bar{x}) \leq \delta_i$ and $a_1, ..., a_n$ be as in the formulation.

Let $(\epsilon_k)$ be an infinite sequence of positive real numbers whose sum is less than $\epsilon/2$. Now apply the condition of the formulation of the proposition to $n+1$ and $\epsilon_1/2$ and find an appropriate finite family of inequalities such that one of them, say $\psi(\bar{x}, x_{n+1}) \leq \tau$, belongs to $p$ and for any $c_1, ..., c_n, c_{n+1} \in M$ realising $\psi(\bar{x}, x_{n+1}) \leq \tau$, and any finite subset $\Delta \subset p$ containing $\psi(\bar{x}, x_{n+1}) \leq \tau$ there is a tuple $c_1', ..., c_n', c_{n+1}'$ realising $\Delta$, such that $\max_{i \leq n+1} d(c_i, c'_i) \leq \epsilon_1/2$. Then let $b_1', ..., b_n', b_{n+1}'$ be a tuple realising $\phi_i(\bar{x}) \leq \delta_i$ and $\psi(\bar{x}, x_{n+1}) \leq \tau$ such that $\max_{i \leq n} d(a_i, b'_i) < \epsilon/2$.

For $n+1$ and $\epsilon_2/2$ find an appropriate condition $\psi'(\bar{x}, x_{n+1}) \leq \tau'$ from $p$ so that any $c_1, ..., c_n, c_{n+1} \in M$ realising $\psi'(\bar{x}, x_{n+1}) \leq \tau'$, and any finite subset $\Delta \subset p$ containing $\psi'(\bar{x}, x_{n+1}) \leq \tau'$ there is a tuple $c_1', ..., c_n', c_{n+1}'$ realising $\Delta$, such that $\max_{i \leq n+1} d(c_i, c'_i) < \epsilon_2/2$. Let $b_1', ..., b_n', b_{n+1}'$ be a tuple realising $\phi_i(\bar{x}) \leq \delta_i$, $\psi(\bar{x}, x_{n+1}) \leq \tau$ and $\psi'(\bar{x}, x_{n+1}) \leq \tau'$ such that $\max_{i \leq n+1} d(b_i', b'_i) < \epsilon_1/2$. Note that $\max_{i \leq n} d(a_i, b_i') < \epsilon/2 + \epsilon_1/2$.

Continuing this procedure we obtain a Cauchy sequence of $(n+1)$-tuples so that its limit satisfies $p$ and is not distant from $\bar{a}$ more than $\epsilon$. □

We now apply this proposition in order to prove the following theorem.

**Theorem 5.3** Let $S = S_Y$ be a dense countable subset of $Y$. There is an $Iso(S)$-invariant Borel subset $SC_S \subset Y_L$ consisting of separably categorical continuous structures on $(Y, d)$ such that any separably categorical continuous $L$-structure on $Y$ is homeomorphic to a structure from $SC_S$.

**Proof.** Let $SC_S$ be the set of all $L$-structures $M$ on $Y$ so that for every $n$ and rational $\epsilon$ there is a finite set $F$ of tuples $\bar{a}_i$ from $S$ together with conditions $\phi_i(\bar{x}) \leq \delta_i$, $(i \in I$ and all $\delta_i$ are rational) with $\phi_i^M(\bar{a}_i) \leq \delta_i$, $i \in I$, and the following property

any $n$-tuple $\bar{a}$ from $S$ satisfies in $M$ one of these $\phi_i(\bar{x}) \leq \delta_i$ and

when $\phi_i^M(\bar{a}) \leq \delta_i$ and $\bar{c}$ is an $(n+1)$-tuple from $S$ with $c_1, ..., c_n$ satisfying $\phi_i(\bar{x}) \leq \delta_i$ in $M$, for any finite set $\Delta$ of $L$-formulas $\phi(\bar{y})$, $|\bar{y}| = n+1$ with $\phi^M(\bar{c}) = 0$ there is an $(n+1)$-tuple $\bar{b} \in S$ so that $\max_{j \leq n} d(a_j, b_j) \leq \epsilon/2$ and $\phi^M(\bar{b}) = 0$ for all formulas $\phi \in \Delta$.

To see that $SC_S$ is a Borel subsets of $Y_L$ it suffices to note that given rational $\epsilon > 0$, finitely many formulas $\phi_i(\bar{x})$, $i \in I$, with $|\bar{x}| = n+1$, and an $n$-tuple $\bar{a}$ from $S$ the set of $L$-structures $M$ on $Y$ with the property that
there is an \((n+1)\)-tuple \(\bar{b} \in S\) so that \(max_{j \leq n}(d(a_j, b_j)) \leq \varepsilon\) and \(\phi_i^M(\bar{b}) = 0\) for all \(i \in I\),

is a Borel subset of \(\mathbf{Y}_L\). The latter follows from Lemma 1.1, which in particular says that any set of \(L\)-structures of the form

\[
\{ M : M \models max(max_{j \leq n}(d(a_j, b_j)) - \varepsilon), max_{i \in I}(\phi_i(\bar{b})) = 0 \}
\]

is a Borel subset of \(\mathbf{Y}_L\).

Note that the density of the set of all tuples from \(S\) in all \(\mathbf{Y}^n\) implies that any continuous structure \(M\) from \(\mathcal{SC}_S\) satisfies the following property for every \(n\) and rational \(\varepsilon\) there is a finite set \(F_{n,\varepsilon}\) of \(n\)-tuples \(\bar{a}_i\) from \(S\) together with conditions \(\phi_i(\bar{x}) \leq \delta_i\), \(i \in I\), (all \(\delta_i\) are rational) with \(\phi_i^M(\bar{a}_i) \leq \delta_i\), \(i \in I\), so that any \(n\)-tuple \(\bar{a}\) from \(\mathbf{Y}\) satisfies one of these \(\phi_i(\bar{x}) \leq \delta_i\) and when \(\phi_i(\bar{a}) \leq \delta_i\) and \(\bar{c}\) is an \((n+1)\)-tuple from \(\mathbf{Y}\) with \(c_1, ..., c_n\) satisfying \(\phi_i(\bar{x}) \leq \delta_i\), for any finite set \(\Delta\) of \(L\)-formulas \(\phi(\bar{y}), |\bar{y}| = n + 1\), with \(\phi(\bar{c}) = 0\) there is an \((n+1)\)-tuple \(\bar{b} \in S\) so that \(max_{j \leq n}(d(a_j, b_j)) \leq \varepsilon/2\) and \(\phi(\bar{b}) = 0\) for all formulas \(\phi \in \Delta\).

It is now clear that any \(M \in \mathcal{SC}_S\) satisfies the condition of Proposition 5.1, i.e. \(M\) is separably categorical.

To finish the proof note that Proposition 5.1 also implies that if \(M\) is a separably categorical structure on \(\mathbf{Y}\), there is a dense set \(S' \subseteq \mathbf{Y}\) so that \(M\) belongs to the corresponding Borel set of \(L\)-structures \(\mathcal{SC}_{S'}\). To see this we just extend \(S_\mathbf{Y}\) to some countable \(S'\) which satisfies the property of Proposition 5.1 in which we additionally require that \(a_1, ..., a_n \in S'\). It is clear any homeomorphism taking \(S'\) onto \(S\) takes \(M\) into \(\mathcal{SC}_S\).

The proof above demonstrates that \(\mathcal{SC}_S\) is of Borel level \(\omega\).

### 5.2 Complexity of sets of approximately ultrahomogeneous structures

Since any Polish group can be realised as the automorphism group of an approximately ultrahomogeneous structures it makes sense to consider the subset of \(\mathbf{Y}_L\) of all approximately ultrahomogeneous structures. Then one can try to characterise properties of Polish groups by description of the corresponding classes of approximately ultrahomogeneous structures and then to study the complexity of these classes. In this subsection we consider two opposite properties: separable oligomorphicity and admitting of complete left invariant metrics. Our results are not complete. For example we do not know if the class of approximately ultrahomogeneous \(L\)-structures on \(\mathbf{Y}\) is a Borel subset of \(\mathbf{Y}_L\).

We use the following characterisation of approximate ultrahomogeneity from Section 6.1 of [20].
Let $M$ be a separable continuous relational structure. Then $M$ is approximately ultrahomogeneous if and only if for any $\varepsilon$, for any quantifier free type $p(x_1, ..., x_n, x_{n+1})$ realised in $M$ and any $a_1, ..., a_n \in M$ realising the restriction of $p$ to $x_1, ..., x_n$, there is a tuple $b_1, ..., b_n, b_{n+1}$ realising $p$ such that $\max_{i \leq n} d(a_i, b_i) < \varepsilon$.

**Theorem 5.4** Let $S = S_Y$ be a dense countable subset of $Y$. There is an $\text{Iso}(S)$-invariant Borel subset $\text{SCU}_S \subset Y_L$ consisting of separably categorical approximately ultrahomogeneous $L$-structures on $(Y, d)$ such that any separably categorical approximately ultrahomogeneous $L$-structure on $Y$ is homeomorphic to a structure from $\text{SCU}_S$.

**Proof.** We start with the claim that when $M$ is a non-compact, separable, continuous, metric structure on $(Y, d)$ the following properties are equivalent:

1. The structure $M$ is separably categorical and approximately ultrahomogeneous;
2. For any $n$ and $\varepsilon$ there are finitely many quantifier free conditions $\phi_i(x) \leq \delta_i$, $i \in I$, so that any $n$-tuple of $M$ satisfies one of these conditions and for any $i \in I$, any $a_1, ..., a_n \in M$ realising $\phi_i(x) \leq \delta_i$ and any quantifier free type $p(x_1, ..., x_n, x_{n+1})$ realised in $M$ and containing $\phi_i(x) \leq \delta_i$, there is a tuple $b_1, ..., b_n, b_{n+1}$ realising $p$ such that $\max_{i \leq n} d(a_i, b_i) < \varepsilon$.
3. For any $n$ and $\varepsilon$ there are finitely many quantifier free conditions $\phi_i(x) \leq \delta_i$, $i \in I$, so that any $n$-tuple of $M$ satisfies one of these conditions and the following property holds:

   for any $i \in I$, any $a_1, ..., a_n \in M$ realising $\phi_i(x) \leq \delta_i$ and any finite set of quantifier free formulas $\Delta(x_1, ..., x_n, x_{n+1})$ realised in $M$ and containing $\phi_i(x) \leq \delta_i$, there is a tuple $b_1, ..., b_n, b_{n+1}$ realising $\Delta$ such that $\max_{i \leq n} d(a_i, b_i) < \varepsilon$.

Indeed, to see the implication $1 \rightarrow 2$ we apply the proof of Lemma 5.2 as follows. Since the theory $T = Th(M)$ is separably categorical for each $n > 0$, every $n$-type is principal and the metric space $(S_n(T), d)$ is compact. In particular for every $n$ and every $\varepsilon$ there is a finite family of principal $n$-types $p_1, ..., p_m$ so that their $\varepsilon/2$-neighbourhoods cover $S_n(T)$.

Note that by approximate ultrahomogeneity for every $i \leq m$,

$$\text{dist}(x, p_i(M)) = \text{dist}(x, p^{qe}_i(M)),$$

where $p^{qe}_i$ is the quantifier free part of $p_i$. Thus when $M$ is separably categorical, given $n$ and $\varepsilon$, we find appropriate $p_i$, $i \in I$, define $P_i(x) = \text{dist}(x, p^{qe}_i(M))$, the corresponding definable predicates and quantifier free $n$-conditions $\phi_i(x) \leq \delta_i$ describing the corresponding $\varepsilon/2$-neighbourhoods of $p_i$. The rest follows by strong $\omega$-near-homogeneity and approximate ultrahomogeneity.

Since by the reformulation of approximate ultrahomogeneity above condition 2 obviously implies approximate ultrahomogeneity, the implication $2 \rightarrow 1$ follows easily from Lemma 5.2.
The equivalence $2 \iff 3$ follows by arguments of the proof of Proposition 5.1.

We now repeat the proof of Theorem 5.3 to show that the following property defines a required Borel subset $SCU_C$ of $Y_L$: for every $n$ and rational $\varepsilon$ there is a finite set $F$ of $n$-tuples $\bar{a}_i$ from $S$ together with quantifier free conditions $\phi_i(\bar{x}) \leq \delta_i$, $i \in I$ (all $\delta_i$ are rational), with $\phi_i(\bar{a}_i) \leq \delta_i$, $i \in I$, so that any $n$-tuple $\bar{a}$ from $S$ satisfies one of these $\phi_i(\bar{x}) \leq \delta_i$ and when $\phi_i(\bar{a}) \leq \delta_i$ and $\bar{c}$ is an $(n+1)$-tuple from $S$ with $c_1, \ldots, c_n$ satisfying $\phi_i(\bar{x}) \leq \delta_i$, for any finite set $\Delta$ of quantifier free $L$-formulas $\phi(\bar{y})$, $|\bar{y}| = n + 1$, with $\phi(\bar{c}) = 0$ there is an $(n+1)$-tuple $\bar{b} \in S$ so that $\max_{j \leq n}(d(a_j, b_j)) \leq \varepsilon/2$ and $\phi(\bar{b}) = 0$ for all formulas $\phi \in \Delta$.

□

The following observation shows that the automorphism group of a non-compact separably categorical approximately ultrahomogeneous structure does not admit a compatible complete left-invariant metric. Indeed, it is an easy exercise that a non-compact separably categorical structure properly embeds into itself. Thus condition (c) below supports our claim.

**Proposition 5.5** Let $G$ be the automorphism group of an approximately ultrahomogeneous continuous $L$-structure $M$ on the space $(Y, d)$. The following conditions are equivalent:
(a) the group $G$ admits a compatible complete left-invariant metric;
(b) the group $G$ is closed in $In(Y, d)$;
(c) there is no proper embedding of $M$ into itself.

**Proof.** Let $\rho_S$ be the standard metric of $Iso(Y)$ ($S = S_Y$, see Introduction). To see that (b) implies (a) note that closedness of $G$ in $In(Y, d)$ guarantees that any Cauchy $\rho_S$-sequence of elements from $G$ has a limit in $G$, i.e. $\rho_S$ is a compatible complete left-invariant metric of $G$.

For the converse note that by Lemma 2.1 of [19] when $G$ has a compatible complete left-invariant metric, any compatible left-invariant metric is complete. Thus the metric $\rho_S$ is complete and $G$ is closed in $In(Y, d)$.

To see that (b) implies (c) assume that there is a proper embedding of $M$ into itself (say $h$). Then for each sequence $s_1, \ldots, s_n \in S$ the quantifier free types of this sequence coincides with the quantifier free type of $h(s_1), \ldots, h(s_n)$. Since $M$ is approximately ultrahomogeneous for any $\varepsilon$ there is an automorphism $g_{n,\varepsilon}$ taking every $s_i$ to the $\varepsilon$-ball of $h(s_i)$. This produces a Cauchy $\rho_S$-sequence from $G$ with the limit $h$, contradicting the closedness of $G$ in $In(Y, d)$.

The negation of (b) implies the negation of (c) by an obvious reason. □

What is the complexity of the class of approximately ultrahomogeneous structures from this proposition? The following statement gives a partial answer. It somehow
corresponds to the result of M. Malicki [25] that the set of all Polish groups admitting compatible complete left-invariant metrics is coanalytic non-Borel as a subset of a standard Borel space of Polish groups.

**Corollary 5.6** The subset of $Y_L$ consisting of approximately ultrahomogeneous structures $M$ such that $\text{Aut}(M)$ admits a compatible complete left-invariant metric, is coanalytic in any Borel subset of approximately ultrahomogeneous structures.

**Proof.** It is enough to show that the subset of $L$-structures $M$ admitting proper embeddings into itself is analytic. To see this consider the extension of $L$ by a unary function $f$. All expansions of $L$-structures satisfying the property that $f$ is an isometry which preserves $L$-relations, form a closed subset of the (Polish) space of all $L \cup \{f\}$-structures.

If $s \in S = S_Y$ and $\varepsilon \in Q \cap [0, 1]$ then the condition that $f(S)$ does not intersect the $\varepsilon$-ball of $s$ is open. Thus the set of $L \cup \{f\}$-structures with a proper embedding $f$ into itself, is Borel. The rest is easy. □

### 5.3 Computable presentations

Consider the situation of Section 2.2. Let $G$ be a Polish group and $\mathcal{R}$ be a distinguished countable family of clopen graded cosets so that the family $\{\rho_{<q} : \rho \in \mathcal{R} \text{ and } q \in Q^+ \cap [0, 1]\}$ forms a basis of the topology of $G$. Let $G_0$ be a countable dense subgroup of $G$ so that $\mathcal{R}$ is closed under $G_0$-conjugacy and consists of all $G_0$-cosets of graded subgroups from $\mathcal{R}$. We may also assume that the set of graded subgroups from $\mathcal{R}$ is closed under $\text{max}$ and dotted multiplication by positive rational numbers.

If we assume that there is a 1-1-enumeration of the family

$$\{\rho_{<q} : \rho \in \mathcal{R} \text{ and } q \in Q^+ \cap [0, 1]\} \cup \{\rho_{>q} : \rho \in \mathcal{R} \text{ and } q \in Q^+ \cap [0, 1]\}$$

so that the relation of inclusion between members of this family is computable we arrive at the case that $G$ is a computably presented $\omega$-continuous domain, [16] and [17].

When we consider a $(G, G_0, \mathcal{R})$-space $(X, \tau)$ together with a distinguished countable $G_0$-invariant graded basis $U$ (see Definition 2.9) of clopen graded subsets so that the relation of inclusion between sets of the form

$$\sigma_{<r} \text{ or } \sigma_{>r} \text{ for } \sigma \in U \text{ and } r \in Q^+ \cap [0, 1],$$

is computable (under an appropriate coding) we also obtain a computably presented $\omega$-continuous domain. We denote it by $U_Q$. Note that in the discrete case these circumstances are standard and in particular arise when one studies computability in $S_\infty$-spaces of logic actions.

Let

$$U_Q^+ = \{\sigma_{<r} : \sigma \in U \text{ and } r \in Q^+ \cap [0, 1]\},$$

$$\mathcal{R}_Q^+ = \{\rho_{<q} : \rho \in \mathcal{R} \text{ and } q \in Q^+ \cap [0, 1]\}.$$

Below we will restrict ourselves by only $+$-parts of domains above.
Remark 5.7 It is worth noting that when we have a recursively presented Polish space in the sense of the book of Moschovakis [27] (Section 3), then a basis of the form $U_{Q}$ as above (in fact $U_{Q}^{+}$) can be naturally defined. Indeed, let us recall that a recursive presentation of a Polish space $(X, d)$ is any sequence $S_{X} = \{ x_{i} : i \in \omega \}$ which is a dense subset of $X$ satisfying the condition that $(i, j, m, k)$-relations

$$d(x_{i}, x_{j}) \leq \frac{m}{k + 1} \text{ and } d(x_{i}, x_{j}) < \frac{m}{k + 1}$$

are recursive. If in this case for all $i$ we define graded subsets $\phi_{i}(x) = d(x, x_{i})$, then all balls $(\phi_{i})_{<r}$, $r \in Q$, form a basis $U_{Q}^{+} = \{ B_{i} : i \in \omega \}$ of $X$ which under appropriate enumeration (together with co-balls $(\phi_{i})_{>r}$) satisfies our requirements above. When $G$ is a Polish group with a left-invariant metric $d$, then for any $q_{1}, \ldots, q_{k} \in Q$ and any tuple $h_{1}, \ldots, h_{k} \in G$ the graded subset $\phi_{q, \bar{h}}(x) = max_{i \leq k}(q_{i} \cdot d(h_{i}, xh_{i}))$ is a graded subgroup. If $G$ is a recursively presented space with respect to a dense countable subgroup $G_{0}$ and the multiplications is recursive, then let $V$ consist of all $\phi_{q, \bar{h}}$ with $\bar{h} \in G_{0}$ and let $\mathcal{R}$ consist of all $G_{0}$-cosets of these graded subgroups. The structure (domain) $(\mathcal{R}_{Q}, \subseteq)$ is computably presented.

If $G$ isometrically acts on $X$ and $x_{1}, \ldots, x_{k}$ is a finite subset of the recursive presentation $S_{X}$ then as we already know the function $\psi_{q, \bar{x}}(g) = max_{i \leq k}(q_{i} \cdot d(x_{i}, g(x_{i})))$ also defines a graded subgroup. When $G$ has a recursive multiplication and a recursive action on $X$ (see Section 3 of [27]) so that the recursive presentation $S_{X}$ is $G_{0}$-invariant, then let $V$ consist of all these subgroups and $\mathcal{R}$ consist of the $G_{0}$-cosets. Then the structure $(\mathcal{R}_{Q}, \subseteq)$ is computably presented.

We always assume that $U$ and the subfamily of all graded subgroups from $\mathcal{R}$ are closed under $max$ and dotted multiplying by positive rational numbers. Families $U_{Q}^{+}$ and $\mathcal{R}_{Q}^{+}$ are bases of the corresponding spaces.

We now take some computability assumptions. As we will see below they are satisfied in the majority of interesting cases.

A1. We assume that the sets of indices (under our enumeration) of $U_{Q}^{+}$, $\mathcal{R}_{Q}^{+}$ and the set of rational cones $H_{<r}$, $H_{>r}$ for basic subgroups from $\mathcal{R}$ are distinguished by computable relations on $\omega$. We denote the latter by $V_{Q}$.

A2. We assume that under our 1-1-enumerations of the families $\mathcal{R}_{Q}$ and $U_{Q}$ the binary relation to be in the pair $\sigma_{<r}$, $\sigma_{>r}$ for $\sigma \in \mathcal{R}$ or $\sigma \in U$ is computable.

A3. We also assume that the following relation is computable:

$$inv(V, U) \iff (V \in V_{Q}^{+}) \land (U \in U_{Q}^{+}) \land (U \text{ is } V\text{-invariant})$$

(recall the latter means that when $\phi(x) < r$ and $H(g) < s$ we have $\phi(g(x)) < r + s$).

A4. We assume that there is an algorithm deciding the problem whether for a natural number $i$ and for a basic set of the form $\sigma_{<r}$, for $\sigma$ from $U$ or $\mathcal{R}$ and $r \in Q$, the diameter of $\sigma_{<r}$ is less than $2^{-i}$.

---

4 apply $d(h_{i}, xyh_{i}) \leq d(h_{i}, xh_{i}) + d(xh_{i}, xyh_{i}) = d(h_{i}, xh_{i}) + d(h_{i}, yh_{i})$ together with the fact that max applied to graded subgroups gives graded subgroups again
In [27] it is defined that a point \( x \in X \) is recursive if the set \( \{ s : x \in B_s, B_s \in U_Q \} \) is computable. We immitate it in the following definition.

**Definition 5.8** We say that an element \( x \in X \) is computable if the relation
\[
Sat_x(U) \Leftrightarrow (U \in U_Q) \land (x \in U)
\]
is decidable.

In the case of the logic action of \( S_{\infty} \), when \( x \) is a structure on \( \omega \) and all \( H \) and \( \phi \) are two-valued, this notion is obviously equivalent to the notion of a computable structure.

We will denote by \( Sat_x(U_Q) \) the set \( \{ C : C \in U_Q \text{ and } Sat_x(C) \text{ holds } \} \).

The following lemma follows from the assumption that \( U \) is a graded basis and satisfies \( A4 \).

**Lemma 5.9** If \( x \in X \) is computable then there is a computable function \( \kappa : \omega \to U_Q^+ \) such that for all natural numbers \( n \), \( x \in \kappa(n) \) and \( \text{diam}(\kappa(n)) \leq 2^{-n} \).

We also say that an element \( g \in G \) is computable if the relation \(( N \in R_Q ) \land (g \in N) \) is computable. Then there is a computable function realising the same property as \( \kappa \) above but already in the case of the basis \( R_Q \).

In the following lemma we use standard indexations of the set of computable functions and of the set of all finite subsets of \( \omega \).

**Lemma 5.10** The following relations belong to \( \Pi^0_2 \):

1. \( \{ e : \text{the function } \varphi_e \text{ is a characteristic function of a subset of } U_Q \} \); 
2. \( \{(e, e') : \text{there is a computable element } x \in X \text{ such that the function } \varphi_e \text{ is a characteristic function of the set } Sat_x(U_Q) \text{ and the function } \varphi_{e'} \text{ realizes the corresponding function } \kappa \text{ defined in Lemma 5.9} \} \); 
3. \( \{(e, e') : \text{there is an element } g \in G \text{ such that the function } \varphi_e \text{ is a characteristic function of the subset } \{ N \in R_Q : g \in N \} \text{ and the function } \varphi_{e'} \text{ realizes the corresponding function } \kappa \text{ defined as in Lemma 5.9} \} \).

**Proof.** (1) Obviuos. Here and below we use the fact that a function is computable if and only if its graph is computably enumerable.

(2) The corresponding definition can be described as follows:

\[\forall n) ((\varphi_{e'}(n) \in U_Q^+) \land (\varphi_{e'}(n) \neq \emptyset) \land (\varphi_e(\varphi_{e'}(n)) = 1) \land (\text{diam}(\varphi_{e'}(n)) < 2^{-n})) \land \]
\[\forall d)(\exists n)(( \text{"every element } U' \text{ of the finite subset of } U_Q \text{ with the canonical index } d \text{ satisfies } \varphi_e(U') = 1") \leftrightarrow ( \text{"} \varphi_e(n) \text{ is contained in any element } U' \text{ of the finite subset of } U_Q \text{ with the canonical index } d" ))\].
The last part of the conjunction ensures that the intersection of any finite subfamily of \( \mathcal{U}_Q \) of cones \( U' \) with \( \varphi_e(U') = 1 \) contains a closed cone of the form \( \phi_{\leq r} \) of sufficiently small diameter. Now the existence of the corresponding \( x \) follows by Cantor’s theorem. (3) is similar to (2). □

As a result we see that the set of indices of computable elements of \( X \) belongs to \( \Sigma^0_3 \). If \( X \) is of the form \( Y_L \) then it makes sense to study complexity of sets of indices of computable structures of natural model-theoretic classes. In the case of first order structures this approach is traditional, see [1], [20]. We now illustrate it in the case of \( \mathbb{U}_L \) for relational \( L \) (for simplicity). In fact we give effective versions of results of Sections 5.1 and 5.2. We start with the following theorem. We will not directly apply its statement, but the arguments of the proof will be very helpful below.

**Theorem 5.11** The structure \( (\mathbb{U}, s)_{s \in \mathcal{Q}_U} \) of the expansion of the Urysohn space by constants from \( \mathcal{Q}_U \) has a decidable continuous theory: for every continuous sentence of the form \( \phi(\bar{s}) \) where \( \bar{s} \in \mathcal{Q}_U \) the value of \( \phi(\bar{s}) \) in \( \mathbb{U} \) is a computable real number.

**Proof.** We remind the reader that a real number \( r \geq 0 \) is computable if there is an algorithm which for any natural number \( n \) finds a natural number \( k \) such that
\[
\frac{k - 1}{n} \leq r \leq \frac{k + 1}{n}.
\]
To prove the theorem we use the main result of [10]. Let \( T_{\mathcal{Q}_U} \) consist of the standard axioms of \( \mathbb{U} \) (with rational \( \varepsilon \) and \( \delta \), see Section 5 in [31]) together with all quantifier free axioms describing distances between constants from \( \mathcal{Q}_U \). We claim that \( T_{\mathcal{Q}_U} \) is decidable. Since the set of all standard axioms of \( \mathbb{U} \) is decidable (see [31]), it suffices to check that the elementary (not continuous) theory of the structure \( \mathcal{Q}_U \) in the language of binary relations
\[
d(x, y) = q , \text{where } q \in \mathbb{Q} \cap [0, 1],
\]
is decidable. The latter is straightforward.

Note that \( T_{\mathcal{Q}_U} \) is complete, i.e. it axiomatises the continuous theory of some continuous structure. Indeed, otherwise there is a separable continuous structure \( M \models T_{\mathcal{Q}_U} \) such that for some tuple \( \bar{s} \in \mathcal{Q}_U \) the structures \( (\mathbb{U}, \bar{s}) \) and the reduct of \( M \), say \( M' \), to the signature \( (d, \bar{s}) \), do not satisfy the same inequalities of the form
\[
\phi(\bar{s}) \leq (<)q \text{ or } \phi(\bar{s}) \geq (>q \text{ where } q \in \mathbb{Q} \cap [0, 1]).
\]
On the other hand since \( \mathbb{U} \) is separably categorical and ultrahomogeneous, the structures \( M' \) and \( (\mathbb{U}, \bar{s}) \) are isomorphic, contradicting the previous sentence.

By Corollaries 9.8 and 9.11 of [10] there is an algorithm which for every continuous sentence \( \phi(\bar{s}) \) computes its value in \( \mathbb{U} \). □

Let \( L \) be a relational language. Let us consider the space \( \mathbb{U}_L \), the family of graded cosets \( \mathcal{R}^U \) and the nice basis \( B_0 \) defined in Section 3. The subfamily \( B_{qf} \) of graded
subsets from \( \mathcal{B}_0 \) corresponding to quantifier free \( L \)-formulas is considered as the the graded basis \( \mathcal{U} \) above.

To check that the \( Iso(\mathcal{U}) \)-space \( \mathcal{U}_L \) satisfies the computability conditions above (in particular \( A1 - A4 \)), note that \( QU \) under the language of binary relations

\[
d(x, y) = q, \text{ where } q \in \mathbb{Q} \cap [0, 1],
\]

has a presentation on \( \omega \) so that all relations first-order definable in \( QU \), are decidable. This follows from the fact that the elementary theory of the structure \( QU \) in the language expanded by all constants has quantifier elimination and is computably axiomatisable (i.e. the corresponding theory is decidable). We fix such a presentation.

Then we can code a \( q' \)-cone of a graded coset

\[
I_{q, \bar{s}, \bar{s}'} : g \rightarrow q \cdot d(g(\bar{s}'), \bar{s}), \text{ where } \bar{s} \bar{s}' \subset \mathbb{Q}U, \text{ and } q \in \mathbb{Q}^+.
\]

(i.e. the coset \( H_{q, \bar{s}}g_0 \) of the graded subgroup

\[
H_{q, \bar{s}} : g \rightarrow q \cdot d(g(\bar{s}), \bar{s}), \text{ where } \bar{s} \subset \mathbb{Q}U, \text{ and } q \in \mathbb{Q}^+.
\]

with respect to \( g_0 \) taking \( \bar{s}' \) to \( \bar{s} \))

by the number of the tuple \( (q, \bar{s}, \bar{s}', \bar{s'}, q, *) \), where * corresponds to one of the symbols \(<, \leq, >, \geq \). It is known that for any \( \bar{t} \in \mathcal{U} \) the algebraic closure of \( \bar{t} \) in \( \mathcal{U} \) coincides with \( \bar{t} \) (Fact 5.3 of [15]). Now using decidability of the elementary diagram of \( QU \) we see that the relation of inclusion between cones of this form is decidable. Cones of graded subgroups (i.e. the set \( \mathcal{V}_0 \)) are distinguished by the computable subset of tuples as above with \( \bar{s} = \bar{s}' \).

Since we interpret elements of \( \mathcal{B}_0 \) by \( L \)-formulas with parameters from \( \mathbb{Q}U \) and without free variables, it is obvious that both \( \mathcal{B}_0 \) and \( \mathcal{B}_{qf} \) can be coded in \( \omega \) so that the operations of connectives are defined by computable functions. Moreover \( \mathcal{B}_{qf} \) is a decidable subset of \( \mathcal{B}_0 \). Now all cones of the form \( \sigma_{<q}, \sigma_{\geq q}, \sigma_{\leq q}, \sigma_{\geq q} \) can be enumerated so that all natural relations between them (in particular relations from \( A2 \)) are computable. To satisfy \( A3 \) we define \( Inv(V, U) \) as follows.

\[
Inv(V, U) \Leftrightarrow \text{"}U \text{ is of the form } \sigma_{<l} \text{ for } \sigma \in \mathcal{B}_{qf}, \ V \text{ is of the form } H_{<k} \text{ for } H \in \mathcal{V} \text{ and the tuple of parameners of } \sigma_{<l} \text{ is contained in the tuple of elements of } \mathbb{Q}U \text{ which appears in the code of } H_{<k} \text{"}.
\]

This relations is obviously decidable.

Let \( \phi(\bar{s}) \) be a quantifier-free formula defining an element \( A \in \mathcal{B}_{qf} \). To compute \( diam(A) \) consider the definition of the metric \( \delta_{seq(QU)} \) of the space \( \mathcal{U}_L \) with respect to \( sec(QU) \) in the beginning of Section 1.1. Assuming (for simplicity) that \( \phi \) is a conjunction of atomic inequalities find all numbers \( i \) of tuples \( (j, \bar{s}') \) such that \( R_j(\bar{s}') \) appears in \( \phi(\bar{s}) \). We may assume that appearance of such subformulas forces inequalities of the form \( q_i' \leq R_j(\bar{s}') \leq q_i \) for rational \( 0 \leq q_i', q_i \leq 1 \). Let \( I \) be the (finite) subset of such \( i \). Then \( diam(A) \) is computed by

\[
\sum_{i=1}^{\infty} \{2^{-i} : i \notin I\} + \sum_{i \in I} 2^{-i} |q_i - q_i'|.
\]
The case of basic clopen sets of $\mathcal{R}^U$ is similar.

The following theorem is an effective version of main results of Sections 5.1 and 5.2.

**Theorem 5.12** Let $\mathcal{SC}_{QU}$ be the $\text{Iso}(\mathcal{QU})$-invariant Borel subset of $\mathbb{U}_L$ defined as in Theorem 5.3 (in particular, consisting of separably categorical continuous structures on $(\mathbb{U},d)$ such that any separably categorical continuous $L$-structure on $\mathbb{U}$ is homeomorphic to a structure from $\mathcal{SC}_{QU}$).

Let $\mathcal{SCU}_{QU}$ be the $\text{Iso}(\mathcal{QU})$-invariant Borel subset of $\mathbb{U}_L$ defined as in Theorem 5.4 (in particular consisting of separably categorical approximately ultrahomogeneous $L$-structures on $(\mathbb{U},d)$ such that any separably categorical approximately ultrahomogeneous $L$-structure on $\mathbb{U}$ is homeomorphic to a structure from $\mathcal{SCU}_{QU}$).

Then the subsets of indices of computable structures from $\mathcal{SC}_{QU}$ and $\mathcal{SCU}_{QU}$ respectively are hyperarithmetic.

**Proof.** We use the following observation.

The set of all pairs $(i,j)$ where $j$ is an index of a cone from $(\mathcal{B}_0)_Q$ and $i$ is an index of a computable structure from this cone, is hyperarithmetical of level $\omega$.

This is an effective version of Proposition 1.1. It follows from Lemma 5.10 by standard arguments. Note that as we have shown above (using decidability of the elementary diagram of $(\mathcal{QU},s)_{s \in \mathcal{QU}}$) all assumptions of Lemma 5.10 are satisfied under the circumstances of our theorem.

It remains to verify that definitions of sets $\mathcal{SC}_{QU}$ and $\mathcal{SCU}_{QU}$ from the proofs of Theorems 5.3 and 5.4 define hyperarithmetic subsets of indices of computable structures. This is straightforward. □

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