Scaling of the elastic energy of small balls for maps between manifolds with different curvature tensors

Milan Krömer and Stefan Müller

December 12, 2022

Contents

1 Introduction 1
2 Preliminaries 3
3 A new notion of convergence for blow-ups 4
4 Compactness and \(\Gamma\)-convergence 8
5 Convergence of the energy 13

1 Introduction

Motivated by experiments and formal asymptotic expansions in the physics literature \[1\], Maor and Shachar \[9\] studied the behaviour of a model elastic energy of maps between manifolds with incompatible metrics. For thin objects they analysed the scaling of the minimal elastic energy as a function of the thickness. In particular, they established the following result.

Theorem 1.1 (\[9\], Thm 1.1). Let \((M, g)\) be an oriented \(n\)-dimensional Riemannian manifold. Let \(p \in M\) and consider a small ball \(B_h(p)\) around \(p\). For a map \(u\) in the Sobolev space \(W^{1,2}(B_h(p); \mathbb{R}^n)\) define the energy

\[
E_{B_h(p)}(u) = \int_{B_h(p)} \text{dist}^2(du, \text{SO}(g,e)) \, d\text{Vol}_g
\]

where \(\text{SO}(g,e)(p')\) denotes the set of orientation preserving isometries from \(T_{p'}M\) to \(\mathbb{R}^n\) (equipped with the Euclidean metric \(e\) and the standard orientation) and where the distance is taken with respect to the Frobenius norm for tensors in \(\mathbb{R}^n \otimes T^*_pM\), see (2.4) and (2.5) below for explicit formulae. For a measure \(\nu\) the average with respect to \(\nu\) is denoted by \(\int_E f \, d\nu = (\nu(E))^{-1} \int f \, d\nu\).

For a tensor \(A \in T_pM \otimes (T^*_pM)^{\otimes 3}\) define a map \(B : T_pM \supset B_1(0) \to T_pM \otimes T^*_pM\) by \(B(X)(Y) = A(X,Y,X)\) and an energy

\[
\mathcal{I}_A := \min_{f \in W^{1,2}(B_1(0);T_pM)} \int_{B_1(0)} |\text{sym} \, df - \frac{1}{6} B|^2 \, d\text{Vol}_{g(p)}.
\]

Then

\[
\lim_{h \to 0} \frac{1}{h^2} \inf E_{B_h(p)} = \mathcal{I}_{\mathcal{R}(p)},
\]

where \(\mathcal{R}(p)\) is the Riemann curvature tensor at \(p\).

In (1.2) the norm is the Frobenius norm of tensors in \(T_pM \otimes T^*_pM\) and the symmetric part of a linear map \(L : T_pM \to T_pM\) is defined by \(\text{sym} \, L = \frac{1}{2}(L + L^T)\) where \(L^T\) is the adjoint map given by \(g(p)(L^T X, Y) = g(p)(X, LY)\).
In [9] it is shown that the quadratic quantity \( I_{\mathcal{R}(p)} \) is actually induced by a scalar product and in particular \( I_{\mathcal{R}(p)} = 0 \) if and only if \( \mathcal{R}(p) = 0 \). Recall that by Gauss’ theorema egregium, a small ball \( B_h(p) \) in \( \mathcal{M} \) can be mapped into \( \mathbb{R}^n \) with zero energy \( E_{B_h(p)} \) if and only if \( \mathcal{R} \equiv 0 \) on \( B_h(p) \).

In local coordinates \( \mathcal{I}_A \) is given as follows. Let \( e_1, \ldots, e_n \) be any \( g(p) \)-orthonormal basis of \( T_p\mathcal{M} \). Then

\[
\mathcal{I}_A = \min_{f \in W^{1,2}(B_1(0);\mathbb{R}^n)} \int_{B_1(0)} \frac{1}{2} \left( \frac{\partial \tilde{f}^i}{\partial x^j} + \frac{\partial \tilde{f}^j}{\partial x^i} \right)^2 - \frac{1}{6} \sum_{j,l=1}^n A_{ijkl} x^j x^l \, dx
\]

where now \( B_1(0) \) is the unit ball in \( \mathbb{R}^n \) and

\[
A_{ijkl} = g(p)(e_i, A(e_j, e_k, e_l)).
\]

The functions \( f \) and \( \tilde{f} \) are related by the identity \( \tilde{f}^i(x) = g(p)(e_i, f(\sum_{j=1}^n x^j e_j)) \).

Based on Theorem 1.1 and heuristic reasoning in the physics literature, Maor and Shachar raise the question whether Theorem 1.1 can be generalized to non-flat targets with \( \mathcal{R} \) replaced by the difference of the curvature tensors in the target and the domain [9]. Here we show that this is true if the difference of the curvature tensors is properly interpreted.

**Theorem 1.2.** Let \( (\mathcal{M}, g) \) and \( (\tilde{\mathcal{M}}, \tilde{g}) \) be smooth oriented Riemannian manifolds and suppose that \( \tilde{\mathcal{M}} \) is compact. For \( p \in \mathcal{M}, h > 0 \) and a map \( u \) in the Sobolev space \( W^{1,2}(B_h(p);\mathcal{M}) \) define the energy

\[
E_{B_h(p)}(u) = \int_{B_h(p)} \text{dist}^2(du, SO(g, \tilde{g})) \, d\text{Vol}_g
\]

where \( \text{dist}(du, SO(g, \tilde{g}))(p') \) denotes the Fronenius distance in \( T_u(p', \tilde{\mathcal{M}} \otimes T_p^*\mathcal{M}) \) of \( du(p') \) from the set of orientation preserving isometries from \( T_{p'}\tilde{\mathcal{M}} \) to \( T_u(p', \mathcal{M}) \). Then

\[
\lim_{h \to 0} \frac{1}{h^4} \inf \min_{q \in \mathcal{M}} \min_{Q \in SO(T_p\mathcal{M}, T_p\tilde{\mathcal{M}})} \mathcal{I}_{\mathcal{R}(p) - RQ},
\]

where \( \mathcal{R}^Q \) is the pullback of the the Riemann curvature tensor \( \mathcal{R}(q) \) under \( Q \), i.e.,

\[
\mathcal{R}^Q(X, Y, Z) = Q^{-1} \mathcal{R}(q)(QX, QY, QZ)
\]

and where \( SO(T_p\mathcal{M}, T_p\tilde{\mathcal{M}}) \) denotes the set of orientation preserving isometries from \( T_p\mathcal{M} \) (equipped with the metric \( g(p) \)) and \( T_q(\tilde{\mathcal{M}}) \) (equipped with the metric \( \tilde{g}(q) \)).

The result can be extended to noncompact targets \( \tilde{\mathcal{M}} \), if \( \tilde{\mathcal{M}} \) satisfies a uniform regularity condition near infinity and if the minimum over \( q \) is replaced by an infimum, see Corollary 5.2 below. In particular the result holds for the hyperbolic space \( \mathbb{H}^n_K \) of constant curvature \( K < 0 \), and we recover Theorem 1.1 if we take \( \tilde{\mathcal{M}} = \mathbb{R}^n \).

The heuristic argument for the validity of both theorems is simple. In normal coordinates (i.e., those induced by the exponential map) in a neighbourhood of \( p \in \mathcal{M} \) and \( q = u(p) \in \tilde{\mathcal{M}} \) the metrics behave like \( g(v) = \text{Id} + q(v) + \mathcal{O}(|v|^3) \) and \( \tilde{g}(v) = \text{Id} + \tilde{q}(v) + \mathcal{O}(|v|^3) \) where \( q \) and \( \tilde{q} \) are homogeneous of degree 2 and determined by the Riemann curvature tensors at \( p \) and \( q \), respectively, see [2] below. This suggests to look for approximate minimizers of the elastic energy of the form

\[
u(\exp_p X) = \exp_q(Q(X + h^3 f(X/h)))
\]

with \( Q \in SO(T_p\mathcal{M}, T_q\tilde{\mathcal{M}}) \) and \( f : T_p\mathcal{M} \to T_q\mathcal{M} \). Then \( d(\exp_q^{-1} \circ u \circ \exp_p) = \text{Id} + h^2 df \) and optimization over \( f \) and \( Q \) should yield the asymptotically optimal behaviour of the energy.

Similar to the reasoning in [9], the proof of Theorem 1.2 relies on a corresponding \( \Gamma \)-convergence result where the notion of convergence of sequences of maps \( u_h : B_h(p) \to \tilde{\mathcal{M}} \) incorporates a blow-up which reveals the map \( f \). One key additional difficulty for non-flat targets is that maps
$u_h$ with small energy need not be continuous. Thus $u_h(B_h(p))$ may not be contained in a single chart of $\tilde{\mathcal{M}}$ and we cannot rely on Taylor expansion in exponential coordinates in the target.

To overcome this difficulty, we define a new notion of convergence of the maps $u_h$ which is based on Lipschitz approximations and exploits the fact that Sobolev maps agree with Lipschitz maps on a large subset. The idea to use Lipschitz approximation to treat manifold-valued maps has already been used in \cite{1}, pp. 390–391. The use of Lipschitz approximations to define a suitable notion of convergence after blow-up seems, however, to be new. We believe that this approach might be useful for other problem involving manifold-valued maps, too.

The remainder of this paper is organized as follows. In Section 2 we introduce the relevant notation and definitions, in particular the definition of Sobolev maps with values in a Riemannian manifold. In Section 3 we introduce a new notion of convergence based on blow-ups of Lipschitz approximations and show that the limit is well-defined, and in particular does not depend on which Lipschitz approximation is used. Based on this convergence notion we establish compactness and $\Gamma$-convergence results in Section 4. Finally, in Section 5 we deduce Theorem 1.2, i.e. convergence of the rescaled energy, in the usual way from compactness and $\Gamma$-convergence.

2 Preliminaries

Here we recall three facts: the notion of Sobolev spaces of maps with values in a Riemannian manifold, the expression of $\text{dist}(du, SO(g, \hat{g}))$ in local coordinates, and the expansion of the metric near the origin in normal coordinates.

For the rest of this paper $(\mathcal{M}, g)$, $(\tilde{\mathcal{M}}, \tilde{g})$ will always denote smooth oriented Riemannian $n$-dimensional manifolds. We often drop $g$ or $\hat{g}$ in the notation. We denote by $d_g$ the inner metric of $\mathcal{M}$, i.e. $d_g(p, p')$ is given by the infimum of the length of curves connecting $p$ and $p'$.

The Sobolev spaces $W^{1,p}(\mathcal{M})$ of functions $u : \mathcal{M} \to \mathbb{R}$ are defined by using a partition of unity and local charts. The definition of Sobolev maps with values in $\tilde{\mathcal{M}}$ is more subtle, since Sobolev maps need not be continuous and hence the image of a small ball in $\mathcal{M}$ may not be contained in a single chart of $\tilde{\mathcal{M}}$. To overcome this difficulty, we use the fact that $\tilde{\mathcal{M}}$ can be isometrically embedded in some $\mathbb{R}^s$ if $s$ is chosen sufficiently large. We thus may assume that $\tilde{\mathcal{M}} \subset \mathbb{R}^s$ and for an open subset $U \subset \mathcal{M}$ we define

$$W^{1,p}(U; \tilde{\mathcal{M}}) = \left\{ u \in W^{1,p}(\mathcal{M}; \mathbb{R}^s) : u(x) \in \tilde{\mathcal{M}} \text{ for a.e. } x \in U \right\}. \quad (2.1)$$

It is easy to check that for a map $u \in W^{1,p}(U; \tilde{\mathcal{M}})$ the weak differential $du$ (obtained by viewing $u$ as a map with values in $\mathbb{R}^s$) satisfies $\text{range}(du(x)) \subset T_{u(x)}\tilde{\mathcal{M}}$ for a.e. $x \in U$.

Equivalently, one can define the Sobolev space $W^{1,p}(U; \tilde{\mathcal{M}})$ by viewing $\tilde{\mathcal{M}}$ as a metric space with the inner metric $d_{\tilde{g}}$ and use the theory of Sobolev spaces with values in a metric space, see, for example, \cite{5} and \cite{11}. Alternatively, one can use the intrinsic definition Sobolev maps with values in manifolds, introduced by Convent and van Schaftingen [2].

We denote by $\mathbb{R}^{n \times n}$ the space of real $n \times n$ matrices and by $O(n) = \left\{ A \in \mathbb{R}^{n \times n} : A^T A = \text{Id} \right\}$ and $SO(n) = \left\{ A \in O(n) : \det A = 1 \right\}$ the orthogonal and special orthogonal group. On $\mathbb{R}^{n \times n}$ we use the Frobenius norm given by

$$|A|^2 = \text{tr} A^T A = \sum_{i,j=1}^{n} A_{ij}^2. \quad (2.2)$$

This norm is invariant under the left and right action of $O(n)$:

$$|RAQ| = |A| \quad \forall R, Q \in O(n). \quad (2.3)$$
For a (weakly) differentiable map $u$ from an open subset of $(\mathcal{M}, g)$ to $(\hat{\mathcal{M}}, \hat{g})$ we define $\text{dist}(du, SO(g, \hat{g}))$ as follows. For $p \in \mathcal{M}$ let $V = (V_1, \ldots, V_n)$ be a positively oriented orthonormal basis of $(T_p\mathcal{M}, g(p))$, let $\hat{V}$ be a positively oriented orthonormal basis of $T_{u(p)}\hat{\mathcal{M}}$ and let $A = (du)_V \hat{V}$ be the matrix representation of $du(p)$ in these bases, i.e., $du(p)V_j = \sum_{i=1}^n A_{ij} \hat{V}_i$. Then

$$\text{dist}(du, SO(g, \hat{g})) := \min_{Q \in SO(n)} |(du)_V \hat{V} - Q|. \quad (2.4)$$

In view of (2.3), the right hand side does not depend on the choice of (positively oriented) orthonormal bases. If $\tilde{X}$ and $\tilde{\hat{X}}$ are general positively oriented bases and if we define matrices $(g_{\tilde{X}})_{ij} = g(p)(X_i, X_j)$ and $(\hat{g}_{\tilde{\hat{X}}})_{ij} = \hat{g}(u(p))(\hat{X}_i, \hat{X}_j)$ then $V_i = \sum_{j=1}^n (g_{\tilde{X}})_{ij}^{-1/2} X_j$ and $\hat{V}_i = \sum_{j=1}^n (\hat{g}_{\tilde{\hat{X}}})_{ij}^{-1/2} \hat{X}_j$ define orthonormal bases. Thus, if $(du)_X \tilde{X}$ is the matrix representation with respect to $X$ and $\tilde{X}$ we get

$$\text{dist}(du, SO(g, \hat{g})) = \min_{Q \in SO(n)} |\hat{g}^{1/2}(du)_X \tilde{X} \hat{g}^{-1/2} - Q|. \quad (2.5)$$

In particular we see that $\text{dist}(du, SO(g, \hat{g}))$ behaves natural under pullback. More precisely, if $N$ and $\hat{N}$ are oriented $n$-dimensional manifolds and $\varphi : N \to \mathcal{M}$, $\psi : \hat{N} \to \hat{\mathcal{M}}$ are smooth orientation-preserving diffeomorphisms then

$$\text{dist}(du, SO(g, \hat{g})) = \text{dist}(d(\psi^{-1} \circ u \circ \varphi), SO(\varphi^*g, \psi^*\hat{g})) \quad (2.6)$$

where $\varphi^*g$ denotes the pullback metric given by $\varphi^*g(x)(X, Y) = g(\varphi(x))(d\varphi X, d\varphi Y)$ and $\psi^*\hat{g}$ is given by the analogous expression.

Finally we recall the expansion of the metric in local coordinates. Let $p \in \mathcal{M}$, let $V = (V_1, \ldots, V_n)$ be an orthonormal basis of $(T_p\mathcal{M}, g(p))$, let $n_V : \mathbb{R}^n \to T_p\mathcal{M}$ be given by $n_V(x) = \sum_{j=1}^n x^j V_j$, and let $(e_1, \ldots, e_n)$ denote the standard basis of $\mathbb{R}^n$. Then

$$\left( (\exp_p \circ n_V)^* \hat{g} \right)_{ik} (x) := \left( (\exp_p \circ n_V)^* g \right)(x)(e_i, e_k) = \delta_{ik} + \frac{1}{3} \mathcal{R}^i_{jkl}(p)x^j x^l + \mathcal{O}(|x|^3). \quad (2.7)$$

where $\mathcal{R}$ is the Riemann curvature tensor, i.e.,

$$\mathcal{R}(U, V, W) = \nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U, V]} W \quad (2.8)$$

and

$$\mathcal{R}^i_{jkl}(p) = g(p)(V_i, \mathcal{R}(p)(V_j, V_k, V_l)). \quad (2.9)$$

### 3 A new notion of convergence for blow-ups

In this section we introduce a notion of convergence of blow-ups of a sequence of maps $u_{h_k} : B_{h_k}(p) \to \mathcal{M}$ which is based on a suitable approximation by Lipschitz maps. We show in particular that this notion of convergence does not depend on the precise choice of the approximation.

Let $p \in \mathcal{M}$. We set $B(h) = \{ p' \in \mathcal{M} : d_g(p, p') < h \}$ where $d_g$ is the inner metric induced by the Riemannian metric $g$ on $\mathcal{M}$. In $T_p\mathcal{M}$ we consider the balls $B_r(0) = \{ X \in T_p\mathcal{M} : g(p)(X, X) < r^2 \}$. Let $\text{inj}(p)$ denote the injectivity radius, i.e., the supremum of all $r > 0$ such that the exponential map $\exp_p$ is injective on $B_r(0)$. Then for $h < \text{inj}(p)$ the exponential map is a smooth diffeomorphism from $B_h(0) \subset T_p\mathcal{M}$ to $B_{h}(p) \subset \mathcal{M}$.

Using a positively oriented orthonormal frame $V = (V_1, \ldots, V_n)$ of $T_p\mathcal{M}$ and the corresponding map $n_V : \mathbb{R}^n \to T_p\mathcal{M}$ given by $n_V(x) = \sum_{j=1}^n x^j V_j$ we can identify maps $f : B_1(0) \subset T_p\mathcal{M} \to T_p\mathcal{M}$

\footnote{Some authors define the Riemann curvature tensor by $\mathcal{R}'(U, V, W) = \mathcal{R}(U, V, W)$ where $\mathcal{R}(U, V, W)$ is given by (2.3). Then $\mathcal{R}'(X, Y, X) = \mathcal{R}(Y, X, X) = -\mathcal{R}(X, Y, X)$ and thus $\left( (\exp_p \circ n_V)^* \hat{g} \right)_{ik} (x) = \delta_{ik} - \frac{1}{3} \mathcal{R}^i_{jkl}(p)x^j x^l + \mathcal{O}(|x|^3)$.}
with maps \( \tilde{f} : B_1(0) \subset \mathbb{R}^n \to \mathbb{R}^n \) by setting \( \tilde{f} = \psi^{-1} \circ f \circ \psi \). In this way we can define the Sobolev space \( W^{1,2}(B_1(0), \mathbb{R}^n) \) with \( B_1(0) \subset T_pM \) and we introduce the following equivalence relation on that space

\[
\tag{3.1}
f \sim g \quad \text{if } f - g \text{ is affine and } D(f - g) \text{ is skew-symmetric.}
\]

Here symmetry of \( Df \) is defined using the scalar product \( g(p) \). Equivalently, \( Df \) is symmetric if and only if \( D\bar{f} \) is symmetric as a map from \( \mathbb{R}^n \) to \( \mathbb{R}^n \) with respect to the standard Euclidean metric.

For \( p \in M \) and \( q' \in \hat{M} \) we denote by \( SO(T_pM, T_q\hat{M}) \) the set of orientation preserving linear isometries from \( T_pM \) to \( T_q\hat{M} \) (equipped with the metrics \( g(p) \) and \( \hat{g}(q') \), respectively). By \( \mu \) we denote the standard measure on \( M \): \( \mu(E) = \int_E d\text{Vol}_g \). Recall that for a measure \( \nu \) we denote the average with respect to \( \nu \) by \( \int_E f d\nu = (\nu(E))^{-1} \int_E f d\nu \).

**Definition 3.1.** Let \( h_k > 0 \) with \( \lim_{k \to \infty} h_k = 0 \), let \( p \in M \), and let \( u_k \) be a sequence of maps in \( W^{1,2}(B_{h_k}(p); M) \). Let \( q \in M \), \( Q \in SO(T_pM, T_q\hat{M}) \), and \( f \in W^{1,2}(B_1(0), T_pM) \sim \) where \( B_1(0) \subset T_pM \).

We say that \( u_k \) converges to the triple \( (q, Q, f) \), if the following three conditions hold.

(i) \( u_k \) converges to the constant map \( q \) in measure, i.e.,

\[
\lim_{k \to \infty} \frac{1}{\mu(B_{h_k}(p))} \mu \left( \{ x \in B_{h_k}(p) : d_{q}(u_k(x), q) \geq \delta \} \right) = 0 \quad \text{for every } \delta > 0;
\]

(ii) there exist Lipschitz maps \( w_k : B_{h_k}(p) \to \hat{M} \) such that

\[
\sup_k \text{Lip } w_k < \infty,
\]

\[
\sup_k \frac{1}{h_k} \frac{1}{\mu(B_{h_k}(p))} \mu \left( \{ x \in B_{h_k}(p) : w_k(x) \neq u_k(x) \} \right) < \infty;
\]

(iii) Set

\[
q_k := \exp_q \left( \int_{B_1(0)} (\exp_{q^{-1}} \circ w_k \circ \exp_p)(h_k X) d\text{Vol}_{g(p)}(X) \right).
\]

Then there exist \( Q_k \in SO(T_pM, T_q\hat{M}) \), \( c_k \in \mathbb{R}^n \), and an element \( \tilde{f} \) of the equivalence class \( f \) such that \( Q_k \to Q \) and the maps \( f_k : B_1(0) \subset T_pM \to T_pM \) defined by

\[
f_k(X) := \frac{1}{h_k} \left\{ Q_k^{-1} \left( \exp_{q^{-1}} \circ w_k \circ \exp_p \right)(h_k X) - X - c_k \right\}
\]

satisfy

\[
f_k \to \tilde{f} \quad \text{in } W^{1,2}(B_1(0), T_pM)
\]

We denote this convergence by \( u_k \to (q, Q, f) \).

**Remark 3.2.** 1. To see that the right hand sides of (3.5) and (3.6) are well defined for sufficiently large \( k \) note that it follows from (3.2), (3.3), and (3.4) that

\[
\lim_{k \to \infty} \sup_{p' \in B_{h_k}(p)} d_g(w_k(p'), q) = 0.
\]

Hence, for large enough \( k \), the set \( w_k(B_{h_k}(x)) \) is contained in a ball around \( q \) on which \( \exp_q^{-1} \) is defined and a diffeomorphism. Moreover (3.8) implies that

\[
\lim_{k \to \infty} d_g(q_k, q) = 0
\]

and thus \( \exp_{q_k^{-1}} \circ w_k \) is also well-defined for \( k \) large enough.
2. The linear maps $Q_k$ have different target spaces. To define the convergence $Q_k \to Q$ one uses a local trivialization of the tangent bundle $T\mathcal{M}$. More explicitly, one can check convergence by expressing $Q_k$ in a smooth local frame, see the proof of Lemma 3.3 below.

3. The reader might wonder why we introduce the points $q_k$ rather than defining $f_k$ simply by using $\exp_q^{-1}$. The point is that the Lipschitz estimate on $w_k$ ensures that the image $w_k(B_{h_k}(p))$ is contained in a ball of radius $C h_k$ around $q_k$. Thus in normal coordinates around $q_k$, one can obtain estimates like (2.7) with error terms of order $O(h_k^2)$. Normal coordinates around $q$ give only weaker estimates since we know $d_{\bar{g}}(q_k,q) \to 0$, but in general there is no rate of convergence in terms of $h_k$.

4. Instead of the points $q_k$ one can use in (3.6) a more intrinsically defined Riemannian centre of mass which depends only on the maps $w_k$ and not on $q$. Indeed, the Lipschitz condition on $w_k$ and the fact that the images of the maps $w_k$ stays in a bounded set of $\mathcal{M}$ imply that, for sufficiently large $k$, there exists a unique point $\tilde{q}_k$ which minimizes the quantity $D(q') = \int_{B_{h_k}(p)} d^2_{\bar{g}}(w_k(q'),d\text{Vol}_q$, see [6, Def. 1.3]. We have opted for the more pedestrian definition (3.5) because it is simpler and is sufficient for our purposes.

We show next that if $u_k \to (q,Q,f)$, then $Q$ and $f$ are uniquely determined by the sequence $u_k$. In particular, they do not depend on the choices of $w_k, Q_k,$ and $c_k$. Note that $q$ is determined by $u_k$ in view of (3.2). We also show that $c_k$ is of order $h_k$.

**Lemma 3.3.** Suppose that $u_k, w_k, Q_k, c_k, q, Q, f,$ and $\tilde{f}$ are as in Definition 3.1 and in particular conditions (3.2)–(3.7) hold. Suppose that there exist $w_k', Q_k', c_k', f_k', Q', f'$ and $\tilde{f}'$ such that conditions (3.3)–(3.7) hold for the primed quantities. Then $Q' = Q$ and $f' = f$ (as equivalence classes).

Moreover, if conditions (i)–(iii) in Definition 3.1 are satisfied, then

$$\sup_k \frac{|c_k|}{h_k} < \infty. \quad (3.10)$$

**Proof.** Step 1: Estimate for $d_{\bar{g}}(q_k,q_k')$.

Let $\tilde{w}_k(X) = w_k(\exp_p h_k X)$, $\tilde{w}_k'(X) = w_k'(\exp_p h_k X)$. Then, by (3.3),

$$\text{Lip} \tilde{w}_k + \text{Lip} \tilde{w}_k' \leq Ch_k,$$

and, by (3.4),

$$\mu(\{X \in B_{1}(0) : \tilde{w}_k(X) \neq \tilde{w}_k'(X)\}) \leq Ch_k^4.$$  

Thus for each $X \in B_{1}(0)$ there exists $Y \in B_{1}(0)$ such that $|Y - X| \leq Ch_k^{4/n}$ and $\tilde{w}_k(Y) = \tilde{w}_k'(Y)$. It follows that

$$\sup_X |\tilde{w}_k(X) - \tilde{w}_k'(X)| \leq Ch_k^{1+4/n},$$

and

$$\sup_{x \in B_{h_k}(p)} |\exp_q^{-1} w_k(x) - \exp_q^{-1} w_k'(x)| \leq Ch_k^{1+4/n}.$$  

Since

$$\frac{1}{\mu(B_{h_k}(p))} \mu(\{x : \exp_q^{-1} w_k(x) \neq \exp_q^{-1} w_k'(x)\}) \leq Ch_k^4,$$

we get

$$d_{\bar{g}}(q_k,q_k') \leq \frac{C}{h_k^{n}} (Ch_k^{4+n} h_k^{1+4/n}) \leq Ch_k^{5+4/n}.$$
Step 2: Comparison of $\exp^{-1}_{q_k}$ and $\exp^{-1}_{q_k'}$.

Here and in the rest of the argument it is convenient to work in local coordinates. Thus let $\hat{V} = (\hat{V}_1, \ldots, \hat{V}_n)$ be a smooth, positively oriented, orthonormal frame defined in an open neighbourhood of $q$. For $q'$ in that neighbourhood consider the isometries $V_{q'(q')}: \mathbb{R}^n \to TM_{q'}$ given by $V_{q'(q')} := \sum_{j=1}^n y_j \hat{V}_j(q')$. Similarly, fix a positively oriented orthonormal basis $\tilde{V}$ of $T_p\mathcal{M}$ and define $\tilde{V}$ in the same way.

Recall that $\text{inj}(q)$ denotes the injectivity radius of $\exp_q$. Thus there exists a $\rho > 0$ such that for $\tilde{q}, \tilde{q}' \in B_\rho(q)$ and $x \in B_{\text{inj}(q)/2}(0) \subset \mathbb{R}^n$ the expression

$$v_{\tilde{q}, \tilde{q}'}(x) = (\tilde{r}_{-1}^{\tilde{q}} \circ \exp_{\tilde{q}}^{-1} \circ \exp_q \circ \tilde{r}_q)(x)$$

is well defined and smooth as a map from $B_\rho(q) \times B_\rho(q) \times B_{\text{inj}(q)/2}(0)$ to $\mathbb{R}^n$. Moreover $v_{\tilde{q}, \tilde{q}'} \equiv \text{Id}$. Thus

$$\|dv_{\tilde{q}, \tilde{q}'}(x) - \text{Id}\| \leq C d\tilde{q} \circ \tilde{q}' \quad \forall \tilde{q}, \tilde{q}' \in B_\rho/2(q), \quad \forall x \in B_{\text{inj}(q)/4}(0). \quad (3.11)$$

It follows from (3.9) and Step 1 that the maps $\tilde{v}_k$ given by

$$\tilde{v}_k(x) = \frac{1}{h_k} \left( \tilde{r}_{-1}^{\tilde{q}_k} \circ \exp_{\tilde{q}_k}^{-1} \circ \exp_{q_k} \circ \tilde{r}_q \right)(h_k x) \quad (3.12)$$

are well-defined for sufficiently large $k$ and $x \in B_{\text{inj}(q)/2h_k}$ and satisfy

$$|d\tilde{v}_k(x) - \text{Id}| \leq C h_k^{5+4/n} \quad \forall x \in B_{\text{inj}(q)/4h_k}. \quad (3.13)$$

Step 3: Uniqueness of $Q$ and $f$.

Using the frames introduced in Step 2, we define maps $\tilde{f}_k: B_1(0) \subset \mathbb{R}^n \to \mathbb{R}^n$ and linear maps $\tilde{Q}_k: \mathbb{R}^n \to \mathbb{R}^n$ by

$$\tilde{Q}_k = \tilde{r}_k^{-1} \circ Q_k \circ \tilde{r}_q, \quad (3.14)$$

$$\tilde{f}_k = \tilde{r}_k^{-1} \circ f_k \circ \tilde{r}_q. \quad (3.15)$$

and similarly for the primed quantities. We use the analogous definitions for the limits $Q$ and $\hat{f}$ (with $q_k$ replaced by $q$). Then $\tilde{Q}_k, \tilde{Q}_k' \in SO(n)$ and $Q_k \to Q$ if and only if $\tilde{Q}_k \to \hat{Q}$. Similarly $f_k \to \hat{f}$ in $W^{1,2}$ if and only if $\tilde{f}_k \to \tilde{f}$ in $W^{1,2}$.

We also define the following maps from $B_1(0) \subset \mathbb{R}^n$ to $\mathbb{R}^n$:

$$\tilde{w}_k(x) = \frac{1}{h_k} \left( \tilde{r}_{-1}^{\tilde{q}_k} \circ \exp_{\tilde{q}_k}^{-1} \circ \exp_q \circ \tilde{r}_q \right)(h_k x), \quad (3.16)$$

$$\tilde{w}_k'(x) = \frac{1}{h_k} \left( \tilde{r}_{-1}^{\tilde{q}_k'} \circ \exp_{\tilde{q}_k'}^{-1} \circ \exp_q \circ \tilde{r}_q \right)(h_k x), \quad (3.17)$$

$$\tilde{w}_k''(x) = \frac{1}{h_k} \left( \tilde{r}_{-1}^{\tilde{q}_k} \circ \exp_{\tilde{q}_k}^{-1} \circ \exp_q \circ \tilde{r}_q \right)(h_k x). \quad (3.18)$$

Then

$$\tilde{w}_k' = \tilde{v}_k \circ \tilde{w}_k',$$

where $\tilde{v}_k$ is given by (3.12), and

$$\text{Lip} \tilde{w}_k + \text{Lip} \tilde{w}_k' \leq C, \quad \mathcal{L}^n(\{|\tilde{w}_k \neq \tilde{w}_k'\}) \leq C h_k^4. \quad (3.19)$$

It follows from the definitions of $f_k$ and $f_k'$, as well as the definition of $\tilde{v}_k$ in (3.12) that

$$df_k = \frac{1}{h_k^2} \left( (\tilde{Q}_k)^{-1} d\tilde{w}_k - \text{Id} \right), \quad (3.20)$$

$$d\tilde{f}_k' = \frac{1}{h_k^2} \left( (\tilde{Q}_k')^{-1} d\tilde{v}_k \circ \tilde{w}_k' - \text{Id} \right). \quad (3.21)$$
Now we first exploit the second estimate in (3.19) and the estimate (3.13) for $d\bar{\nu}_k - \text{Id}$ to show that $Q_k$ and $\bar{Q}_k$ have the same limit. Let $E_k = \{ \bar{\nu}_k \neq \bar{\nu}'_k \}$. Then $d\bar{\nu}_k = d\bar{\nu}'_k$ a.e. in $B_1(0) \setminus E_k$. Thus, by (3.13) and the estimates of the Lipschitz constants in (3.19), we get

$$|d(\bar{\nu}_k \circ \bar{\nu}'_k) - d\bar{\nu}_k)| \leq C h_k^{5+4/n} \quad \text{a.e. in } B_1(0) \setminus E_k. \tag{3.22}$$

Let $\bar{R}_k := \bar{Q}_k^{-1} \bar{Q}'_k$, multiply (3.21) by $-\bar{R}_k$, add (3.20), and multiply the resulting equation by $h_k^2 (1 - 1_{E_k})$. This yields

$$h_k^2 (d\bar{f}_k - \bar{R}_k d\bar{f}'_k) (1 - 1_{E_k}) = O(h_k^{5+4/n}) + (\bar{R}_k - \text{Id}) (1 - 1_{E_k}). \tag{3.23}$$

Since $\bar{f}_k$ and $\bar{f}'_k$ converge weakly in $L^2$, $\bar{R}_k \in SO(n)$, and $\mathcal{L}^n(\bar{E}_k) \to 0$, it follows that $|\bar{R}_k - \text{Id}| \leq C h_k^5$. In particular, $\bar{R}_k \to \text{Id}$ as $h_k \to 0$ and hence $\bar{Q} = \bar{Q}'$. To show that $\bar{f} \sim \bar{f}'$, we note that there exists a subsequence $k_j \to \infty$ such that the limit

$$A := \lim_{j \to \infty} \frac{\bar{R}_{k_j} - \text{Id}}{h_{k_j}^2}$$

exists. Since $\bar{R}_k \in SO(n)$, it follows that $A$ is skewsymmetric. Dividing (3.23) by $h_k^2$ and passing to the limit along the subsequence $k_j$, we get $d\bar{f} - d\bar{f}' = A$. Thus $\bar{f} \sim \bar{f}'$. This is equivalent to $\bar{f} \sim \bar{f}'$ or $f = f'$ (as equivalence classes).

Step 4: Proof of (3.10).

It follows from the definition of $q_k$ and the Lipschitz bound on $w_k$ that $w_k(B_{h_k}(p))$ is contained in a ball $B_{Ch_k}(q_k)$. Thus Taylor expansion of $\bar{\nu}_k = \exp^{-1}_{q_k} \circ \exp_q$ around $Z_k = \exp^{-1}_q(q_k)$ yields

$$\exp^{-1}_{q_k} \circ w_k = \bar{\nu}_k \circ \exp^{-1}_q \circ w_k = 0 + d\bar{\nu}_k(Z_k)[\exp^{-1}_q \circ w_k - Z_k] + O(h_k^2).$$

Hence

$$\int_{B_1(0)} (\exp^{-1}_{q_k} \circ w_k \circ \exp_p)(h_k X) \, d\text{Vol}_{g(p)}(X)$$

$$= d\bar{\nu}_k(Z_k) \left[ \int_{B_1(0)} \left( (\exp^{-1}_q \circ w_k \circ \exp_p)(h_k X) - Z_k \right) \, d\text{Vol}_{g(p)}(X) \right] + O(h_k^2)$$

$$= O(h_k^2) \tag{3.24}$$

where we used the definition (3.5) of $q_k$ for the last identity. Since $f_k$ is bounded in $L^2$, equation (3.10) now follows by integrating (3.24) over $X \in B_1(0)$ and using (3.24).

\[ \square \]

4 Compactness and $\Gamma$-convergence

For $u_h \in W^{1,2}(B_h(p); \hat{\mathcal{M}})$ define the energy of $u_h$ by

$$E_{B_h(p)}(u_h) := \int_{B_h(p)} \text{dist}^2(du_h, SO(g, \hat{g})) \, d\text{Vol}_{g}.$$

For points $p \in \mathcal{M}$ and $q \in \hat{\mathcal{M}}$, an orientation preserving isometry $Q \in SO(T_p \mathcal{M}, T_q \hat{\mathcal{M}})$, and the unit ball $B_1(0)$ in $T_p \mathcal{M}$ we define a functional $T^Q : W^{1,2}(B_1(0); T_p \mathcal{M}) \to \mathbb{R}$ by

$$T^Q(u) = \int_{B_1(0)} |\text{sym} \, df(X) - B(X)|^2 \, d\text{Vol}_{g(p)}(X), \tag{4.1}$$
where $|\cdot|$ denotes the Frobenius norm on $T_p\mathcal{M} \otimes T_p^*\mathcal{M}$ and $\mathcal{B}(X)$ is the element of $T_p\mathcal{M} \otimes T_p^*\mathcal{M}$ given by

$$
\mathcal{B}(X)(Y) = \frac{1}{6} \left( \mathcal{R}(p)(X,Y,X) - \tilde{\mathcal{R}}^Q(X,Y,X) \right)
$$

with

$$
\tilde{\mathcal{R}}^Q(X,Y,X) := Q^{-1}\mathcal{R}(q)(QX,QY,QX).
$$

It follows directly from the definition that $\mathcal{T}^{q,Q}$ depends only on the equivalence class of $f$ (where the equivalence relation is given by $\mathcal{A}$). We will thus view $\mathcal{T}^{q,Q}$ also as a functional on the space $W^{1,2}(B_1(0); T_p\mathcal{M})/\sim$ without change of notation.

Our main result is the following compactness and $\Gamma$-convergence result.

**Theorem 4.1.** Let $(\mathcal{M}, g)$ and $(\tilde{\mathcal{M}}, \tilde{g})$ be smooth, oriented, $n$-dimensional Riemannian manifolds. Then the following assertions hold:

(i) Compactness: Assume in addition that $\tilde{\mathcal{M}}$ is compact. Let $h_k \to 0$ and assume that there exists a constant $C > 0$ such the maps $u_k : B_{h_k}(p) \to \tilde{\mathcal{M}}$ satisfy $E_{h_k}(u_k) \leq Ch_k^4$. Then there exists a subsequence $h_{k_j} \to 0$ such that

$$u_{k_j} \longrightarrow (q, Q, f)$$

in the sense of Definition 6.1.

(ii) $\Gamma - \lim \inf$ inequality: if $h_k \to 0$ and $u_k \rightarrow (q, Q, f)$, then

$$
\lim_{k \to \infty} \frac{1}{h_k^4} E_{h_k}(u_k) \geq \mathcal{T}^{q,Q}(f).
$$

(iii) Recovery sequence: Given a triple $(q, Q, f)$ and $h_k \to 0$, there exists $u_k$ such that $u_k \rightarrow (q, Q, f)$ and

$$
\lim_{k \to \infty} \frac{1}{h_k^4} E_{h_k}(u_k) = \mathcal{T}^{q,Q}(f).
$$

The combination of properties (ii) and (iii) can be stated concisely as the fact that $\frac{1}{h_k^4} E_{h_k} \Gamma$-converges (with respect to the convergence in Definition 6.1) to $\mathcal{T}$ with $\mathcal{T}(q, Q, f) = \mathcal{T}^{q,Q}(f)$.

To prove compactness, we use the following result on Lipschitz approximation of $\mathbb{R}^s$-valued Sobolev maps. This is a minor variation of the classical result by Liu [8, Thm. 1], see also [3, Section 6.6.3, Thm. 3].

**Lemma 4.2** ([3], Prop. A.1). Let $s, n \geq 1$ and $1 \leq p < \infty$ and suppose $U \subset \mathbb{R}^n$ is a bounded Lipschitz domain. Then there exists a constant $C = C(U, n, s, p)$ with the following property:

For each $u \in W^{1,p}(U, \mathbb{R}^s)$ and each $\lambda > 0$ there exists $v : U \to \mathbb{R}^s$ such that

(i) $\text{Lip} v \leq C \lambda$,

(ii) $L^n \left( \{ x \in U : u(x) \neq v(x) \} \right) \leq \frac{C}{\lambda^p} \int_{\{ x \in U : |u(x)| > \lambda \} \cap \{ x \in U : |v(x)| > \lambda \}} |du|^p_e \, dx$.

Here $|\cdot|_e$ denotes the Frobenius norm with the respect to the standard scalar products on $\mathbb{R}^n$ and $\mathbb{R}^s$.

**Remark 4.3.** It is easy to see that the constant $C(U, n, s, p)$ can be chosen invariant under dilations of $U$, i.e., $C(rU, n, s, p) = C(U, n, s, p)$. Indeed, given $u \in W^{1,p}(rU, \mathbb{R}^s)$ apply the lemma to the rescaled function $\tilde{u} : U \to \mathbb{R}^s$ given by $\tilde{u}(x) = r^{-1}u(rx)$, obtain a Lipschitz approximation $\tilde{v} : U \to \mathbb{R}^s$ and define $v$ by $v(y) = r\tilde{v}(y/r)$.
Proof of Theorem 4.1 (compactness). We proceed in two steps. First we show that there exists a good Lipschitz approximation \( w_k \) of \( u_k \) and then deduce compactness by expressing \( \text{dist}(dw_k, SO(g, \tilde{g})) \) in terms of normal coordinates in \( M \) and \( \tilde{M} \).

Step 1: Lipschitz approximation: There exists a constant \( C > 0 \) and Lipschitz maps \( w_k : B_{h_k}(p) \to \tilde{M} \) such that, for all sufficiently large \( k \),

\[
\text{Lip} \ w_k \leq C, \quad \frac{1}{\mu(B_{h_k}(p))}\mu(\{u_k \neq w_k\}) \leq Ch_k^4. \tag{4.4, 4.5}
\]

The construction of the maps \( w_k \) is very similar to the construction in \([7]\) pp. 390–391]. We include the details for the convenience of the reader. To construct \( w_k \), we recall that in view of the Nash imbedding theorem \([10]\) Theorem 3), we can view \( \tilde{M} \) as a subset of \( \mathbb{R}^s \) for large \( s \), with the metric on the tangent space of \( \tilde{M} \) induced by the Euclidean metric of \( \mathbb{R}^s \). Let \( \mathcal{V} = (V_1, \ldots, V_n) \) be a positively oriented, orthonormal basis of \( T_pM \) and define \( \hat{u}_k : B_{h_k}(0) \subset \mathbb{R}^n \to \tilde{M} \subset \mathbb{R}^s \) by

\[
\hat{u}_k = u_k \circ \exp_p \circ \mathcal{V}_k(x) = \sum_{j=1}^n x^jV_j. \quad \text{Let } (\tilde{g})_{ij} = ((\exp_p \circ \mathcal{V}_k)^*g)(e_i, e_j) \text{ be the coefficients of the pullback metric in the standard Euclidean basis. Then by (2.7) that}
\]

\[
|\tilde{g}_{ij} - \delta_{ij}| \leq Ch_k^2 \quad \text{on } B_{h_k}(0). \tag{4.6}
\]

Since the Frobenius norm of a map in \( SO(n) \) is \( \sqrt{n} \) and since \( \tilde{M} \) is isometrically imbedded into \( \mathbb{R}^s \) it follows from (4.6) that

\[
|d\hat{u}_k|_e \leq (1 + Ch_k^2)(\sqrt{n} + \text{dist}(du_k, SO(g, \tilde{g}))) \tag{4.7}
\]

In particular for sufficiently large \( k \) we have

\[
|d\hat{u}_k|_e \geq 4\sqrt{n} \quad \implies \quad \text{dist}(du_k, SO(g, \tilde{g})) \geq \frac{1}{2}|d\hat{u}_k|_e \geq 2\sqrt{n}. \tag{4.8}
\]

Now apply Lemma 4.2 and Remark 4.3 with \( u = u_k, U = B_{h_k}(0) \) and \( \lambda = 4\sqrt{n} \). Denote the corresponding Lipschitz approximation by \( \tilde{u}_k \) and set \( E_k^2 = \{x \in B_{h_k}(0) : \tilde{u}_k \neq \hat{u}_k\} \). Then

\[
\text{Lip} \ \tilde{u}_k \leq C. \tag{4.9}
\]

Using that, in addition, \( \det \tilde{g}(x) \geq (1 + Ch_k^2)^{-1} \geq \frac{1}{2} \) we get

\[
\mathcal{L}^n(E_k^2) = \frac{C}{\lambda^2} \int_{\{x \in B_{h_k}(0) : \|d\hat{u}_k|_e \geq \lambda\}} |d\hat{u}_k|_e^2 \ dx \leq \frac{C}{\lambda^2} \int_{B_{h_k}(0)} \text{dist}^2(du_k, SO(g, \tilde{g})) \ Vol_g \ \leq \ C\mu(B_{h_k}(p))h_k^4. \tag{4.10}
\]

In general, the map \( \tilde{u}_k \) takes values in \( \mathbb{R}^s \) rather than in \( \tilde{M} \). This difficulty can be easily overcome by projecting back to \( M \). Indeed, since \( M \) is compact, there exists a \( \rho > 0 \) and a smooth projection \( \pi_{\tilde{M}} \) from a \( \rho \)-neighbourhood of \( \tilde{M} \) in \( \mathbb{R}^s \) to \( M \). Now by (4.10), there exists an \( x' \in B_{h_k}(0) \) such that \( \tilde{u}_k(x') = \hat{u}_k(x') \in \tilde{M} \). Since the distance function is 1-Lipschitz we deduce that \( \text{dist}(\tilde{u}_k(x), M) \leq C|x - x'| \leq Ch_k \) for all \( x \in B_{h_k}(0) \). Then \( \hat{u}_k := \pi_{\tilde{M}} \circ \tilde{u}_k \) is well-defined for sufficiently large \( k \) and satisfies \( \text{Lip} \hat{u}_k \leq C \). Since \( \pi|_{\tilde{M}} = \text{id} \) we have \( \{\hat{u}_k \neq \tilde{u}_k\} \subset \{\tilde{u}_k \neq \hat{u}_k\} \). Finally, using that \( \exp_p \circ \mathcal{V}_k \), is Bilipschitz in a neighbourhood of 0, we see that \( w_k := \hat{u}_k \circ (\exp_p \circ \mathcal{V}_k)^{-1} \) satisfies (4.4) and (4.5).
Step 2: Compactness
The estimate $\text{Lip } w_k \leq C$ implies that the image of $w_k$ is contained in the ball $B(w_k(p), Ch_k)$. Since $\mathcal{M}$ is compact, there exists a subsequence $k_j \to \infty$ and $q \in \mathcal{M}$ such that $w_{k_j}(p) \to q$ as $j \to \infty$. Hence $\lim_{j \to \infty} \sup_{B_{h_{k_j}}} d_g(w_{k_j}, q) = 0$ and in view of (1.5) we get, for all $\delta > 0$,

$$\lim_{j \to \infty} \frac{1}{\mu(B_{h_{k_j}}(p))} \mu \left( \left\{ p' \in B_{h_{k_j}}(p) : d_g(w_{k_j}(p'), q) \geq \delta \right\} \right) = 0.$$  

Thus condition (i) in Definition 3.1 is satisfied for the subsequence $k_j$. Condition (ii) in Definition 3.1 is equivalent to (4.4) and (4.5).

To verify condition (iii) in Definition 3.1 consider the points $q_{k_j}$ defined by

$$q_{k_j} := \exp_q \left( \int_{B_1(0)} (\exp^{-1}_q \circ w_{k_j} \circ \exp_p)(h_{k_j} x) d \text{Vol}_g(X) \right).$$

Since $\exp_q$ and $\exp_p$ are Bilipschitz with Bilipschitz constant close to one in a small neighbourhood of the origin, it follows that $q_{k_j} \to q$ as $j \to \infty$ and that the image of $w_{k_j}$ is contained in $B_{2Ch_{k_j}}(q_{k_j})$ for $j$ sufficiently large.

Note also that the approximation properties (4.4) and (4.5) in combination with the hypothesis $E_{B_{h_{k_j}}}(u_k) \leq Ch_k^4$ imply that

$$\int_{B_{h_{k_j}}(p)} \text{dist}^2(dw_k, SO(g, \tilde{g})) d \text{Vol}_g \leq Ch_k^4. \quad (4.11)$$

Now it is convenient to work in local coordinates, as in the proof of Lemma 3.3. To simplify the notation, we write $w_k$ instead of $w_{k_j}$. Consider again the maps $\tilde{w}_k : B_1(0) \subset \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\tilde{w}_k(x) = \frac{1}{h_k} (i^{-1}_{\text{Vol}(q_k)} \circ \exp^{-1}_q \circ w_k \circ \exp_p \circ \text{Vol}(h_k)) (h_k x). \quad (4.12)$$

We now apply first the formula (2.10) for $\text{dist}(du, SO(g, \tilde{g}))$ with $\varphi_k(x) = \exp_p \circ \text{Vol}(h_k) (h_k x)$ and $\psi_k(x) = \exp_q \circ \text{Vol}(q_k) (h_k x)$ and then (2.25). This yields

$$\text{dist}(dw_k(\varphi(x)), SO(g, \tilde{g})) = \text{dist} \left( (\tilde{g}^{(k)} \circ \tilde{w}_k(x))^{1/2} dw_k (\tilde{g}^{(k)})^{-1/2}(x), SO(n) \right), \quad (4.13)$$

where $\tilde{g}^{(k)}$ is the metric (expressed in the standard basis of $\mathbb{R}^n$) obtained from the metric $g$ on $\mathcal{M}$ by pullback under $\varphi_k$ and similarly for $\tilde{g}^{(k)}$.

Using the expansion (2.7) of the metric in normal coordinates and Proposition 4.4 below we deduce that

$$\text{dist}(d\tilde{w}_k, SO(n))(x) \leq (1 + Ch_k^2) \text{dist}(dw_k, SO(g, \tilde{g})) (\exp_p \circ \text{Vol}(h_k x)) + Ch_k^2. \quad (4.14)$$

In view of (4.11) this implies that

$$\int_{B_1(0)} \text{dist}^2 (d\tilde{w}_k, SO(n)) dx \leq Ch_k^4. \quad (4.15)$$

By the rigidity estimate in [4] Thm. 3.1 there exists a constant rotation $\tilde{Q}_k \in SO(n)$ such that

$$\int_{B(0,1)} \left| \tilde{Q}_k^{-1} d\tilde{w}_k - \text{Id} \right|^2 dx \leq Ch_k^4. \quad (4.16)$$

Thus there exists $\tilde{c}_k \in \mathbb{R}^n$ such that the functions

$$\tilde{f}_k = \frac{1}{h_k^2} (\tilde{Q}_k^{-1} w_k - \text{Id} - \tilde{c}_k)$$

are uniformly Lipschitz in $\mathbb{R}^n$. This completes the proof of Claim 3.1.
are bounded in $W^{1,2}(B_1(0);\mathbb{R}^n)$ and hence a subsequence converges weakly in $W^{1,2}(B_1(0);\mathbb{R}^n)$. Unwinding definitions, we see that condition (iii) in Definition 3.1 is satisfied.

\[ \square \]

**Proposition 4.4.** Let $A, B, F \in \mathbb{R}^{n\times n}$ and assume that $A$ and $B$ are invertible. Then

\[
\text{dist}(F, SO(n)) \\
\leq (1 + |A^{-1} - \text{Id}|)(1 + |B^{-1} - \text{Id}|) \text{dist}(AFB, SO(n)) \\
+ |A^{-1} - \text{Id}| + |B^{-1} - \text{Id}| + |A^{-1} - \text{Id}||B^{-1} - \text{Id}|.
\]

(4.17)

For $A = \text{diag}(a^{-1}, 1, \ldots, 1)$, $B = \text{diag}(b^{-1}, 1, \ldots, 1)$, $F = \text{diag}(abc, 1, \ldots, 1)$, with $a, b, c > 1$ the equality holds.

**Proof.** There exist $Q \in SO(n)$ such that $\text{dist}(AFB, SO(n)) = |AFB - Q|$. Set $A_Q = Q^{-1}AQ$ and $\tilde{F}_Q = Q^{-1}F$. Then $\text{dist}(AFB, SO(n)) = |A_Q\tilde{F}_QB - \text{Id}|$ and

\[
|F - Q| = |\tilde{F}_Q - \text{Id}| \leq |\tilde{F}_Q - A_Q^{-1}B^{-1}| + |A_Q^{-1}B^{-1} - \text{Id}|
\]

(4.18)

Now expand $B^{-1}$ and $A_Q^{-1}$ as $B^{-1} = \text{Id}+(B^{-1}-\text{Id})$ and $A_Q^{-1} = Q^{-1}A_Q^{-1}Q = \text{Id}+Q^{-1}(A^{-1}-\text{Id})Q$ and use that $|XY| \leq |X| |Y|$ and $|Q^{-1}(A^{-1} - \text{Id})Q| = |A^{-1} - \text{Id}|$.

**Proof of Theorem 4.4** (T - $\lim\inf$ inequality). Let $V = (V_1, \ldots, V_n)$ be a positively oriented orthonormal basis of $T_pM$ and set $\tilde{V}_k = (Q_k V_1, \ldots, Q_k V_n)$. Then $\tilde{V}_k$ is a positively oriented orthonormal basis of $T_{q_k}M$. Set $\varphi_k(x) = (\exp_{q_k} \circ \tilde{V}_k)(h_kx)$ and $\psi_k(x) = (\exp_{q_k} \circ \tilde{V}_k)(h_kx)$. Let $w_k$ be as in Definition 4.1 and define

\[
\tilde{w}_k := \psi_k^{-1} \circ w_k \circ \varphi_k, \quad \tilde{E}_k := \{ x : w_k \circ \varphi_k(x) \neq \text{Id} \circ \varphi_k(x) \}.
\]

(4.19)

Then $\mathcal{L}^n(\tilde{E}_k) \leq Ch_k^4$ and

\[
E_{h_k}(w_k) \geq \int_{B_1(0) \setminus \tilde{E}_k(x)} 1_{B_1(0)}(x) \text{dist}(dw_k(\varphi_k(x)), SO(g, \tilde{g})) d\text{Vol}_{\tilde{g}}(x).
\]

(4.20)

Since the functions $w_k$ satisfy a uniform Lipschitz bound, we can obtain the lower bound by expressing $\text{dist}(dw_k(\varphi_k(x)), SO(g, \tilde{g}))$ in normal coordinates at $p$ and $q_k$ and using Taylor expansion on the large set where $dw_k$ is close to $SO(g, \tilde{g})$. Specifically, using (4.13) we get

\[
\text{dist}(dw_k(\varphi(x)), SO(g, \tilde{g})) = \text{dist} \left( (\tilde{g}^{(k)})^{1/2}(\tilde{w}_k(x)) \right. \left. d\tilde{w}_k \left( \tilde{g}^{(k)} \right)^{-1/2}(x), SO(n) \right),
\]

(4.21)

where $\tilde{g}^{(k)}$ is the metric obtained from $g$ by pullback under $\varphi_k$ and similarly for $\tilde{g}$. The expansion (2.7) of the metric in normal coordinates yields

\[
\tilde{g}^{(k)}_{im}(x) = h_k^2 \left( \delta_{im} - \frac{1}{3} \sum_{j,l=1}^n g(p)(V_i, R(p)(V_j, V_m, V_l)) h_k^2 x^j x^l + O(h_k^3 |x|^3) \right),
\]

(4.22)

\[
\tilde{g}^{(k)}_{im}(y) = h_k^2 \left( \delta_{im} - \frac{1}{3} \sum_{j,l=1}^n g(q_k)(Q_k V_i, \tilde{R}(q_k)(Q_k V_j, Q_k V_m, Q_k V_l)) h_k^2 y^j y^l \right. \left. + O(h_k^3 |y|^3) \right).
\]

(4.23)

Moreover, it follows from the definition of $\tilde{w}_k$ and $f_k$ that

\[
\tilde{w}_k = \text{id} + t_k \tilde{w}^{-1}_k \circ f_k \circ \tilde{t}_k + t_k^{-1} e_k.
\]
Now by (3.10) we have \( c_k \to 0 \). Since \( f_k \) is bounded in \( L^2 \) it follows that \( \tilde{w}_k \to \text{id} \) in \( L^2 \). In view of the uniform Lipschitz bound on \( \tilde{w}_k \) we see that \( \tilde{w}_k \to \text{id} \) uniformly. Thus

\[
G_k := \left( \ell(\ell(k))^{1/2} \circ \tilde{w}_k \right) / h^2_k \to G \quad \text{in} \quad L^2(B_1(0); \mathbb{R}^{n \times n})
\]

(4.24)

with

\[
G_{\text{im}}(x) = d(\ell \circ \tilde{f} \circ \nu)(x) - \sum_{j,l=1}^n A_{jml} x^j x^l, \quad \text{and} \quad A_{jml} = \frac{1}{6} g(p)(V_i, (\mathcal{R}(p) - \mathcal{R}Q)(V_j, V_m, V_l)).
\]

(4.25)

Now set \( F_k := \{ x \in B_1(0) : |h_G^2 G_k| > h_k \} \) and for \( x \notin F_k \) use the Taylor expansion

\[
\text{dist}^2(\text{Id} + h_G^2 G_k, SO(n)) = |\text{sym} \ h_G^2 G_k|^2 + \mathcal{O}(h_k)|h_G^2 G_k|^2.
\]

By (2.7) we have \( d\text{Vol}_{g(x)} = h^n(1 + \mathcal{O}(h_k^n))\mathcal{L}^n \). Using that \( \mathcal{L}^n(\bar{E}_k \cup F_k) \to 0 \) and that positive semidefinite quadratic forms are weakly lower semi-continuous, we deduce that

\[
\begin{align*}
\liminf_{k \to \infty} \frac{1}{h_k^2} \int_{B_1(0)} \left( 1_{B_1(0)} \text{sym} \ h_G^2 G_k \right) d\text{Vol}_{g(x)} &
\geq \liminf_{k \to \infty} \int_{B_1(0)} |1_{B_1(0)} \text{sym} \ G_k|^2 \ dx \\
&\geq \int_{B_1(0)} |\text{sym} \ G_k|^2 \ dx.
\end{align*}
\]

(4.26)

Now the assertion follows from (4.20), (4.21), (4.24), (4.25) and (4.26).

Proof of Theorem 4.1 (recovery sequence). Let \( q \in \mathcal{M} \), \( Q \in SO(n)(T_p M, T_p \mathcal{M}) \) and let \( \tilde{f} \in W^{1,2}(B_1(0), T_p M) \) be a representative of \( f \). There exists Lipschitz maps \( \tilde{f}_k \) such that \( \tilde{f}_k \to \tilde{f} \) in \( W^{1,2} \) and \( \text{Lip} \tilde{f}_k \leq h_k^{-1} \). Set \( c_k = -h_k^2 \int_{B_1(0)} \tilde{f}_k \) and define

\[
u_k(\exp_p(h_k X)) := w_k(\exp_p(h_k X)) := \exp_q(h_k Q(X + h_k^2 \tilde{f}_k(X) + c_k)).
\]

(4.27)

Then (3.2)–(3.4) hold, and the definition (3.5) of \( q_k \) in combination with the definition of \( c_k \) implies that \( q_k = q \). The definition (3.6) of \( f_k \) with the choice \( Q_k = Q \) yields \( f_k = \tilde{f}_k \). Thus \( u_k \to (q, Q, f) \).

To show convergence of the rescaled energy, we define \( G_k \) as in (4.24) and (4.19), with the frame \( \tilde{Y}_k = (QV_1, \ldots, QV_n) \) in the target space (recall that \( q_k = q \) and \( Q_k = Q \)). Then \( G_k \to G \) in \( L^2 \) (strongly), with \( G \) given by (4.24). Since \( \text{dist}^2(F, SO(n)) \leq C |F - \text{Id}|^2 \) and \( \int_{|G_k| \geq h_k^{-1}} |G_k|^2 \ dx \to 0 \), Taylor expansion shows that

\[
\lim_{k \to \infty} \frac{1}{h_k^2} \int_{B_1(0)} \text{dist}(\text{Id} + h_G^2 G_k, SO(n)) \ dx = \int_{B_1(0)} |\text{sym} \ G|^2 \ dx.
\]

In view of (4.24) and the choice \( u_k = w_k \), we get the desired assertion.

5 Convergence of the energy

It is easy to see that the quadratic functional \( f \to T^{q, Q}(f) \) attains its minimum in \( W^{1,2}(B_1(0), T_p M) \). Set

\[
m^{q, Q} := \min_{f \in W^{1,2}(B_1(0), T_p M)} T^{q, Q}(f).
\]

(5.1)

Theorem 5.1. Let \( \mathcal{M} \) be compact. Then

\[
\lim_{k \to 0} \frac{1}{h_k^2} \min_{u \in W^{1,2}(B_k(p); \mathcal{M})} E_{B_k(p)}(u) = \bar{m} := \min_{q \in \mathcal{M}} \min_{Q \in SO(T_p M, T_q \mathcal{M})} m^{q, Q}.
\]

(5.2)
Proof. This is a standard consequence of Theorem 4.1. We include the details for the convenience of the reader.

It is easy to see that the map \( q, Q \mapsto m^{q, Q} \) is continuous as a map from the subbundle \( SO(T_p M, T\bar{M}) \subset T M \otimes T^*_p M \) to \( \mathbb{R} \). Since \( \bar{M} \) is compact, so is \( SO(T_p M, T\bar{M}) \). Thus the minimum on the right hand side of (5.2) exists.

Upper bound: set \( L^+ = \limsup_{h \to 0} h^{-4} \inf_{u \in W^{1,2}(B_h(p), \bar{M})} E_{B_h(p)}(u) \) and let \( h_k \to 0 \) be a subsequence along which the limit superior is realised. Let \( q \in \mathcal{M} \), \( Q \in SO(T_p M, T_q \bar{M}) \), and let \( f \) be a minimiser of \( T^{q, Q} \). It follows from Theorem 4.1 (iii) that \( L^+ \leq m^{q, Q} \). Optimising over \( Q \) and \( q \), we get \( L^+ \leq \tilde{m} \).

Lower bound: set \( L^- = \liminf_{h \to 0} h^{-4} \inf_{u \in W^{1,2}(B_h(p), \bar{M})} E_{B_h(p)}(u) \) and let \( h_k \to 0 \) be a subsequence which realises the limit inferior. Then there exist maps \( u_k \) such that

\[
\lim_{k \to \infty} \frac{1}{h_k} E_{B_{h_k}(p)}(u_k) = L^-.
\]

By Theorem 4.1 (iii) there exists a subsequence \( u_{k_j} \) which converges to \((q, Q, f)\) in the sense of Definition 3.1. Thus Theorem 4.1 (iii) implies that \( L^- \geq T^{q, Q}(f) \geq m^{q, Q} \geq \tilde{m} \).

A slight modification of the arguments in the proof of Theorem 4.1 yields the following extension for non-compact targets.

**Corollary 5.2.** Suppose that \( \bar{M} \) is complete and satisfies the following uniform regularity condition: there exists a \( \rho > 0 \) such that the injectivity radius satisfies \( \text{inj}(q) \geq \rho \) for all \( q \in \mathcal{M} \) and the the pullback metrics \( \exp_q^* g \) are uniformly bounded in \( C^3(B_\rho(0)) \). Then

\[
\lim_{h \to 0} \frac{1}{h} \inf_{u \in W^{1,2}(B_h(p), \bar{M})} E_{B_h(p)}(u) = \inf_{q \in \mathcal{M}} \min_{Q \in SO(T_p M, T_q \bar{M})} m^{q, Q}.
\]

**Acknowledgements**

The authors thank Cy Maor for very helpful suggestions and for pointing out reference [7]. This work is an extension of the first author’s B.Sc. thesis at the University of Bonn. In that thesis a recovery sequence is constructed, and compactness and the \( \Gamma - \lim \inf \) inequality are shown under the additional hypothesis that the original sequence \( u_k \) satisfies a uniform Lipschitz bound and \( u_k(p) \) is fixed. The second author has been supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) through the Hausdorff Center for Mathematics (GZ EXC 59 and 2047/1, Projekt-ID 390685813) and the collaborative research centre *The mathematics of emerging effects* (CRC 1060, Projekt-ID 211504053).

**References**

[1] H. Aharoni, J.M. Kolinski, M. Moshe, I. Meirzada, I., and E. Sharon, Internal stresses lead to net forces and torques on extended elastic bodies, Physical Review Letters 117 (2016), 124101.

[2] A. Convent and J. Van Schaftingen, Intrinsic co-local weak derivatives and Sobolev spaces between manifolds, Ann. Sc. Norm. Super. Pisa Cl. Sci. 16 (5) (2016), 97–128.

[3] L.C. Evans and R.F. Gariepy, Measure theory and fine properties of functions, CRC Press, 1992.

[4] G. Friesecke, R.D. James, and S. Müller, A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity, Comm. Pure Appl. Math. 55 (2002), 1461–1506.
[5] J. Heinonen, P. Koskela, N. Shanmugalingam, and J.T. Tyson, Sobolev spaces on metric measure spaces – an approach based on upper gradient, Cambridge Univ. Press, 2015.

[6] H. Karcher, Riemannian center of mass and mollifier smoothing, Comm. Pure Appl. Math. 30 (1977), 509–541.

[7] R. Kupferman, C. Maor, A. Shachar, Reshetnyak rigidity for Riemannian manifolds, Arch. Rat. Mech. Anal. 231 (2019), 367–408.

[8] F.-C. Liu, A Luzin type property of Sobolev functions, Indiana Univ. Math. J. 26 (1977), 645–651.

[9] C. Maor and A. Shachar, On the role of curvature in the elastic energy of non-Euclidean thin bodies. J. Elasticity, 134 (2019), 149–173.

[10] J. Nash, The imbedding problem for Riemannian manifolds, Ann. Math. 63 (1956), 20–63.

[11] Y.G. Reshetnyak, Y. G., Sobolev classes of functions with values in a metric space, Sib. Math. Journal 38 (1997), 567–583.