Random-field-induced order in a bosonic t-J model

Yoshihito Kuno, Takamasa Mori and Ikuo Ichinose

Department of Applied Physics, Nagoya Institute of Technology, Nagoya, 466–8555, Japan
E-mail: ikuo@nitech.ac.jp

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Abstract
In this paper, we study the effect of a random quenched external field for spin order and also multiple Bose–Einstein condensation (BEC). This system is realized by the cold atomic gases in an optical lattice. In particular, we are interested in the strong-repulsion region of the two-component gases for which the bosonic t-J model is a good effective model. In the bosonic t-J model, a long-range order of the pseudo-spin and also BEC of atoms appear quite naturally as in the fermion t-J model for the high-temperature superconducting materials. Random Raman scattering between two internal states of a single atom plays a role of the random external field, and we study its effects on the pseudo-spin order and the BEC by means of quantum Monte-Carlo simulations. The random external field breaks a continuous U(1) symmetry existing in the original bosonic t-J model and it induces new orders, named random-field-induced order (RFIO). We show a phase diagram of the bosonic t-J model with the random external magnetic field and study the robustness of the RFIO states. We also study topological excitations like vortices and the domain wall in the RFIO state. Finally, we point out the possibility of a quantum bit by the RFIO.

Keywords: quantum gases, optical lattice, two internal states, Bose–Einstein condensation, random external field, Monte-Carlo simulation, replica methods
1. Introduction

Quenched disorder plays a very important role in condensed matter physics. A prominent example is the Anderson localization that predicts all quantum states are localized in one and two spatial dimensions if interactions between particles can be neglected [1]. It has been proved rigorously that quenched disorder destroys ordered states and rounds the singularity of phase transitions [2]. However, recently, a counter-intuitive possibility was pointed out and examined. That is, a quenched disorder generates a new ordered state, which is different from the original one, if that quenched disorder breaks the continuous symmetry of the original system without the quenched disorder. This possibility was first studied in a classical XY spin model coupled with a random external magnetic field [3] and then the resultant order is called random-field-induced order (RFIO). Shortly after the proposal, it was shown that such phenomena of the RFIO can be observed by experiments on systems of ultra-cold bosonic atoms of multiple-internal states [4]. In Bose atomic gases, a Bose–Einstein condensed (BEC) state is the genuine ordered state, and a random Raman scattering of the internal states of the atom plays the role of a quenched random external magnetic field. Then, it is expected that the RFIO can be observed in the ultra-cold atomic gas systems. There are other interesting works on the RFIO [5].

In this paper, we shall investigate the RFIO in detail for the two-component cold bosonic gases in a square optical lattice (OL). In particular, we consider the bosonic t-J model, which is a low-energy effective model for the Bose–Hubbard model in the strong-repulsion limit [6, 7]. This model exhibits both the pseudo-spin order and the BECs and is therefore suitable for study of the RFIO. We employ quantum Monte-Carlo (MC) simulations that take into account all of the fluctuations. Thus, the present study is in sharp contrast to the previous ones that used an estimation of the classical energy of the XY spin configurations for observing a possible RFIO [3] and a mean-field theory with Gross-Pitaevskii equations for the BEC of the RFIO [4]. Furthermore, we will investigate the behavior of low-energy topological excitations such as vortices and domain walls in the RFIO states, and reveal their interesting properties. A finite-temperature ($T$) phase diagram is also obtained, which is useful for a discussion on the robustness of the RFIO states.

The present paper is organized as follows. In section 2, we introduce and explain the bosonic t-J model with a random external field. Path-integral quantization using the slave-particle representation is explained. Effective field theory for the pseudo-spin and BECs is derived by integrating out the amplitude degrees of freedom of the slave-particle field variables. In section 3, a replica mean-field theory is applied to the effective field theory and effects of the random field are studied. This study clearly shows how the RFIO appears as a result of the random field with a moderate fluctuation. In section 4, the results of the numerical simulations are given. Phase diagrams at vanishing $T$ as well as finite-$T$ are obtained. Various correlation functions, which are used for the identification of the orders, are shown. In section 5, topological excitations like vortices and the domain wall are investigated numerically. Properties of these excitations are discussed from the view point of the RFIO. Section 6 is devoted to the conclusion.

2. Models and numerical methods

2.1. Bosonic t-J model with random Rabi coupling

The bosonic t-J model, which describes the dynamics of two internal states of a boson, which we call $a$- and $b$-boson for simplicity, in a square OL, is defined by the following Hamiltonian [7],
$H_{\text{ElJ}} = H_{\text{I}} + H_{V}$,  

$$H_{\text{I}} = -\sum_{\langle i,j \rangle}(t_a a_i^\dagger a_j + t_b b_i^\dagger b_j + \text{h.c.}) - J_{xy} \sum_{\langle i,j \rangle}(S_i^x S_j^x + S_i^y S_j^y) + J_z \sum_{\langle i,j \rangle} S_i^z S_j^z - \mu \sum_i(1 - a_i^\dagger a_i - b_i^\dagger b_i),$$

$$H_{V} = \frac{V_0}{4} \sum_i \left( (a_i^\dagger a_i - \bar{\rho}_{ai})^2 + (b_i^\dagger b_i - \bar{\rho}_{bi})^2 \right),$$

where $a_i^\dagger (a_i)$ and $b_i^\dagger (b_i)$ are boson creation (destruction) operators at site $i$ of the square lattice and $t_a$ and $t_b$ are the hopping amplitude between the nearest-neighbor (NN) sites. Pseudo-spin operator $\vec{S}_i$ is given as $\vec{S}_i = \frac{1}{2}B_i^\dagger \vec{\sigma} B_i$ with $B_i = (a_i, b_i)^\dagger$, and $\vec{\sigma}$ is the Pauli spin matrix. In the t-J model, the doubly-occupied state is excluded at each site and the density of atoms is controlled by the chemical potential $\mu$. It was proven that the Hamiltonian $H_{\text{I}}$ is derived from the Bose–Hubbard model in the strong one-site repulsion limit by integrating out multiple-particle states, and the exchange couplings $J_{xy}$ and $J_z$ are related with the intra and inter-repulsions between atoms. In the present study, however, we shall treat these parameters as free ones because the system $H_{\text{I}}$ might be derived from a Bose-gas system on the Lieb lattice that is a bosonic counterpart of the d-p model for the strongly-correlated electron systems.

The term $H_{V}$ in equation (3) controls the density fluctuations of atoms at each site from the mean value $\bar{\rho}_{ai}$ and $\bar{\rho}_{bi}$. This term is expected to appear naturally for describing the practical phenomena in experiments at low energies and therefore, we explicitly added it to the Hamiltonian.

In the present study, we add the following terms that describe the quenched random external fields,

$$H_{R} = H_{\text{ElJ}} + \sum_i (J_i^x S_i^x + J_i^y S_i^y),$$

where $J_i^x$ and $J_i^y$ take random real variables with the vanishing mean value. For the practical numerical study, we use the following distribution function $P(J_i)$,

$$P(J_i^x) = \frac{1}{\sigma_x \sqrt{\pi}} \exp\left[-\left(J_i^x / \sigma_x\right)^2\right], \quad P(J_i^y) = \frac{1}{\sigma_y \sqrt{\pi}} \exp\left[-\left(J_i^y / \sigma_y\right)^2\right],$$

where $\sigma_x(\sigma_y)$ are positive parameters. In the cold atomic systems, the above terms are realized by the Rabi oscillation with the Raman laser of a random complex amplitude $\Omega_i = \Omega_i^R + i\Omega_i^I$, and
then $J_t^x(J_t^y) \propto \Omega_t^x(\Omega_t^y)$ as $S_t^x = a_t^+ b_t + b_t^+ a_t$ and $S_t^y = \frac{1}{i}(a_t^+ b_t - b_t^+ a_t)$. It is expected that the complex Raman amplitude $\Omega$ is realized experimentally using speckle laser light [8].

It should be noted that the $U(1) \times U(1)$ symmetry of $H_{\text{EdJ}}$ in equation (1), i.e., $(a_i, b_i) \rightarrow (e^{i\alpha}a_i, e^{i\beta}b_i)$ with arbitrary constants $\alpha$ and $\beta$, is preserved only for the case $\sigma_x = \sigma_y$, otherwise the quenched disorder, $J_t^x$ and $J_t^y$, explicitly breaks the symmetry as $U(1) \times U(1) \rightarrow U(1) \times Z_2$.

2.2. Numerical methods: path-integral Monte-Carlo simulations

In order to study the model $H_T$ by means of quantum MC simulations, we use the path-integral method with the slave particle description for the local constraint of the t-J model. The boson creation operators are expressed by the slave particle operators $\phi_{ai}$, $\phi_{bi}$ and $\phi_{hi}$ as follows,

$$a_i^+ = \phi_{ai}^+ \phi_{hi}, \quad b_i^+ = \phi_{bi}^+ \phi_{hi},$$  

and the physical state of the slave particle $\langle \text{Phys} |$ must satisfy

$$\langle \phi_{ai}^+ \phi_{ai}^+ \phi_{hi} \phi_{hi}^+ | \text{Phys} \rangle = | \text{Phys} \rangle.$$

Then the partition function for the system $H_T$ is given by

$$Z = \int [d\phi_a d\phi_b d\phi_h] \exp \left[ - \int d\tau \left( \sum_{\alpha=a,b,h} (\bar{\phi}_{\alpha i} \phi_{\alpha i}) + H_T \right) \right],$$

where $\tau$ is the imaginary time, $\bar{\phi}_{ai} = \frac{d\phi_{ai}}{d\tau}$ and $H_T$ is expressed in terms of the slave particles by using equation (6). For the path integral in equation (8), the local constraint $\bar{\phi}_{ai} \phi_{ai} + \bar{\phi}_{bi} \phi_{bi} + \bar{\phi}_{hi} \phi_{hi} = 1$ can be imposed by using a Lagrange multiplier field $\lambda_i(\tau)$,

$$\prod_\tau \delta(\bar{\phi}_{ai} \phi_{ai} + \bar{\phi}_{bi} \phi_{bi} + \bar{\phi}_{hi} \phi_{hi} - 1) = \int [d\lambda] e^{i\int d\tau(\bar{\phi}_{ai} \phi_{ai} + \bar{\phi}_{bi} \phi_{bi} + \bar{\phi}_{hi} \phi_{hi} - 1)\lambda_i}.$$

To obtain a positive-definite action for carrying out the path-integral MC simulation, we parameterize the fields as $\phi_{ai} = \sqrt{\rho_{ai}} e^{i\omega_{ai}} (\alpha = a, b, h)$, and analytically calculate the integral over the amplitudes $\rho_{ai}$. By the term $H_V$ in $H_T$, the integration can be carried out in powers of the density fluctuations, $\delta\rho_{ai} = \rho_{ai} - \bar{\rho}_{ai}$. As a result, the Berry phase $\sum_\alpha \bar{\phi}_{ai} \phi_{ai}$ generates terms like $\frac{1}{V_0} \sum_\alpha (\omega_{ai} + \lambda_i)^2$ in the action.

For the practical numerical calculation, we introduce a lattice for the imaginary time direction, and we denote the site of three-dimensional (3D) cubic space-time lattice $r$. With the resultant action on the lattice $A_{Lxy}$, the partition function is given by

$$Z = \int [d\omega_{ai} d\lambda_r] e^{-A_{Lxy}},$$

with

$$A_{Lxy} = A_{Lt} + A_L(e^{i\theta_x}, e^{-i\theta_y}) + A_q,$$
where

\[ A_{Lr} = -c_r \sum_r \sum_{\alpha=a,b,h} \cos \left( \omega_{\alpha,r+\tau} - \omega_{\alpha r} + \lambda_r \right), \]  

(12)

\[ A_L \left( e^{i \theta_r}, e^{-i \theta_r} \right) = -\sum_{\langle r,r' \rangle} \left( C_3^a \cos \left( \theta_{ar} - \theta_{ar'} \right) + C_3^b \cos \left( \theta_{br} - \theta_{br'} \right) \right) 
+ C_1 \cos \left( \theta_{sr} - \theta_{sr'} \right), \]  

(13)

and

\[ A_q = -\sum_{\langle r,r' \rangle} \left( \tilde{J}_i^x \cos \left( \theta_{sr} - \theta_{sr'} \right) + \tilde{J}_i^y \sin \left( \theta_{sr} - \theta_{sr'} \right) \right). \]  

(14)

In equations (11) \sim (14), dynamical variables are

\[ \theta_{sr} = \omega_{ar} - \omega_{br}, \quad \theta_{ar} = \omega_{ar} - \omega_{hr}, \quad \theta_{br} = \omega_{br} - \omega_{hr}, \]

and parameters are related to the original ones as,

\[ c_r = \frac{1}{V_0 \Delta \tau}, \]

\[ C_1 = 4J\bar{\rho}_a^2 \bar{\rho}_b^2 \Delta \tau \propto J/(c_r V_0), \]

\[ C_3^a = \frac{t_a}{2\Delta \tau} \left( 1 - \bar{\rho}_a - \bar{\rho}_b \right) \Delta \tau \propto t_a/(c_r V_0), \]

\[ C_3^b = \frac{t_b}{2\Delta \tau} \left( 1 - \bar{\rho}_a - \bar{\rho}_b \right) \Delta \tau \propto t_b/(c_r V_0), \]

\[ \tilde{J}_i^{x(y)} = J^{x(y)} \Delta \tau = J^{x(y)}/(c_r V_0), \]  

(15)

where \( \Delta \tau \) is the lattice spacing of the imaginary time. Note that \( \bar{\rho}_a, \ldots, \tilde{J}_i^{x(y)} \) are all dimensionless. (We have put \( \hbar = 1 \).) Please notice that the quenched disorder variables \( \tilde{J}_i^{x(y)} \) are independent of the imaginary time \( \tau \).

There are comments on the derivation of \( A_{Lxy} \) and advantages of the MC simulation on it. On performing the path integral of \( \rho_{ai} \), the higher-order terms of the fluctuations \( \delta \rho_{ai} \) are ignored, e.g., in the hopping term,

\[ a_i^\dagger a_j \rightarrow \sqrt{\rho_{ai} \rho_{aj}} \exp \left[ i \left( -\omega_{ai} + \omega_{hi} + \omega_{aj} - \omega_{hj} \right) \right]. \]  

(16)

The above approximation is legitimate for \( \delta \rho/\bar{\rho} \ll 1 \) as in the experiments of large \( \bar{\rho} \) [9] or small \( \delta \rho \). In the previous paper [10], we studied the non-random case in detail and verified that the results obtained by the MC simulations on \( A_{Lxy} \) are in good agreement with those obtained by the Gross-Pitaevskii theory. Furthermore in the previous paper [11], we studied the phase diagram of \( H_{EdJ} \) with a finite \( J_z \) by using \( A_{Lxy} \) and found that the supersolid state forms in a certain parameter region. The obtained phase diagram is in agreement with that of the two-component Bose–Hubbard model, which was obtained by using the MC simulation with the worm algorithm [12], although the case of commensurate filling factors was studied there. One advantage of the present MC method for studying the bosonic t-J model is its rapid convergence, and therefore, the large-scale MC simulation is possible. Furthermore, various correlation functions as well as the density of topological excitations can be calculated accurately, as we show in sections 4 and 5.
3. Replica mean-field theory

Before going into the numerical calculations, we briefly study the model given by equation (10) by means of the replica methods. In particular, we are interested in the case of the single-component random field like \( \tilde{J}^x \neq 0 \) and \( \tilde{J}^y = 0 \), and see how the order of \( S^y \) shows up whereas that of \( S^x \) does not. For simplicity, we shall consider the case of the total filling factor = 1, i.e., the filling factor of each particle is \( \frac{1}{2} \), and focus on the pseudo-spin symmetry, though the extension to the case with a finite hole density is rather straightforward.

In the replica method studying the effects of quenched random variables, a replica index \( \nu = 1, 2, \ldots, n \) is introduced for each dynamical variable. In the present system, \( \omega_{ai} \rightarrow \omega_{ai}^\nu (\alpha = a, b, h) \), and the partition function of the replica system \( Z_n \) is given by,

\[
Z_n = \int \left\{ \prod_i (d\tilde{J}^x) P(\tilde{J}^x_i) \right\} \prod_{i, \nu} \left( d\omega_{ai}^\nu d\omega_{bi}^\nu \right) \exp \left[ -\sum_{\nu} \left( A_{S}^\nu + \int d\tau S_{i}^{x\nu} \right) \right],
\]

\[
A_{S}^\nu = \int d\tau \left[ \frac{1}{V_0} \sum_{i, \nu} (\dot{\omega}_{ai}^\nu + \dot{\omega}_{bi}^\nu) - C_1 \sum_{i, \mu} \left( S_{i}^{x\nu} S_{i}^{y\mu} + S_{i}^{y\nu} S_{i}^{y\mu} \right) \right],
\]

where \( \dot{\omega}_{ai}^\nu = \frac{d\omega_{ai}^\nu}{d\tau} \), etc, and \([\ldots]\) denotes the average over the random variables \( \tilde{J}^x_i \) with \( P(\tilde{J}^x_i) \).

After calculating \( Z_n \), the limit \( n \rightarrow 0 \) is taken to obtain \( \log Z \).

In \([Z^n]\) in equation (17), the integration over \( \tilde{J}^x_i \) can be carried out readily to obtain,

\[
[Z^n] = \int \left\{ \prod_{i, \nu} (d\omega_{ai}^\nu d\omega_{bi}^\nu) \right\} \exp \left[ -\sum_{\nu} A_{S}^\nu \right.
\]

\[+ \frac{\sigma_c^2}{4} \sum_{i} \left( \int d\tau \sum_{\nu} S_{i}^{x\nu} (\tau) \right) \left( \int d\tau' \sum_{\nu'} S_{i}^{x\nu'} (\tau') \right) \].

The nonlocal terms in equation (18) can be reduced to local ones by using a Hubbard–Storatonovich transformation with auxiliary fields \( m_i(\tau) \) as

\[
\exp \left[ \frac{1}{4\sigma_c} \sum_{i} \left( \int d\tau \sum_{\nu} S_{i}^{x\nu} (\tau) \right) \left( \int d\tau' \sum_{\nu'} S_{i}^{x\nu'} (\tau') \right) \right]
\]

\[= \int [dm_i] \exp \left[ -\frac{1}{\sigma_c} \sum_{i} m_i^2 + \sum_{i} m_i \left( \int d\tau \sum_{\nu} S_{i}^{x\nu} (\tau) \right) \right]. \]

We also apply a mean-field theory (MFT) for the spin part of \( A_{S}^\nu \) in equation (17) as

\[
\sum_{i, \mu} \left( S_{i}^{x\nu} S_{i}^{y\mu} + S_{i}^{y\nu} S_{i}^{y\mu} \right) \rightarrow \sum_i \left( 4 \langle S^{x\nu} \rangle S_{i}^{x\nu} - 2 \langle S^{x\nu} \rangle^2 + 4 \langle S^{y\nu} \rangle S_{i}^{y\nu} - 2 \langle S^{y\nu} \rangle^2 \right). \]
In this MFT, the partition function of the replica system \([Z^n]_{\text{MFT}}\) is given as
\[
[Z^n]_{\text{MFT}} = \int [dm_i] [d\omega_{\nu}] [d\omega_{\sigma}] \exp \left[ -\frac{1}{\sigma_x^2} \sum_i m_i^2 + \int d\tau \sum_{i,\nu} \left( m_i + 4C_1 \langle S^{\nu} \rangle \right) S^{\nu}_i \right. \\
+ \int d\tau \sum_{i,\nu} \left\{ -2C_1 \langle S^{\nu} \rangle^2 + 4C_1 \langle S^{\nu} \rangle S^{\nu}_i - 2C_1 \langle S^{\nu} \rangle^2 \right\} \\
- \frac{1}{V_0} \int d\tau \sum_{i,\nu} \left[ \left( \dot{\omega}_{\nu, i}^v \right)^2 + \left( \dot{\omega}_{\sigma, i}^v \right)^2 \right] \right].
\tag{21}
\]

In equation (21), the integration of \(\omega_{\nu, i}^v\) can be carried out to obtain a Ginzburg–Landau (GL) theory for the pseudo-spin order. To this end, we use the following on-site Green functions as we consider the system at a sufficiently low temperature,
\[
\left\langle e^{i\omega_{\nu, i}^v(t')} \exp{e^{-i\omega_{\nu, i}^v(t')}} \right\rangle = \delta_{\nu,\nu'} \delta_{ij} e^{-Vj \tau - \tau'^{-1}}, \quad (\alpha = a, b).
\tag{22}
\]

Then
\[
[Z^n]_{\text{MFT}} = \int [dm_i] \exp \left[ -\frac{1}{\sigma_x^2} \sum_i m_i^2 - 2C_1 \int d\tau \sum_{\nu} \left( \langle S^{\nu} \rangle^2 + \langle S^{\nu} \rangle \right) \right.
\\
+ \int d\tau \sum_{i,\nu} \left\{ \frac{\chi_i^v}{4V_0} + \frac{4C_1^2}{V_0} \langle S^{\nu} \rangle \right\} \right],
\tag{23}
\]
\[
\chi_i^v = m_i + 4C_1 \langle S^{\nu} \rangle. \quad \text{We are interested in the replica-symmetric solution and set} \quad \sum_{\nu} \langle S^{\nu} \rangle = n \langle S^x \rangle, \quad \text{etc. We also introduce a cutoff} \quad \beta = 1/(k_B T) \quad \text{for the integral of the imaginary time} \quad \tau. \quad \text{Then the integration over} \quad m_i \quad \text{can be done to obtain}
\\
\int [dm_i] \exp \left[ -\frac{1}{\sigma_x^2} \sum_i m_i^2 + \frac{\beta m}{V_0} \left( m_i + 4C_1 \langle S^x \rangle \right) \right]
\\
= \exp \left[ \frac{1}{\sigma_x^2} - \frac{\beta m}{V_0} C_1 \langle S^x \rangle \right] + \frac{4C_1^2 \beta m}{V_0} \langle S^x \rangle^2
\tag{25}
\]
\[
\text{Finally we obtain the effective potential,} \quad V_{\text{Rep}}, \quad \text{by taking the limit} \quad n \to 0,
\]
\[
V_{\text{Rep}} \equiv \lim_{n \to 0} \left( \frac{1 - [Z^n]_{\text{MFT}}}{\beta m} \right)
\\
= \left( 2C_1 - \frac{4C_1^2}{V_0} \right) \left( \langle S^x \rangle^2 + \langle S^y \rangle^2 \right) \quad - \frac{\beta m}{V_0} \langle C_1 \rangle \langle S^x \rangle \frac{1}{\sigma_x^2} \left( \frac{1}{\sigma_x^2} - \frac{\beta m}{V_0} \right) \langle S^x \rangle^2.
\tag{26}
\]

It is obvious that two limits, \(\beta \to \infty\) and \(n \to 0\), are not interchangeable in \(V_{\text{Rep}}\) given by equation (26). For a finite \(\sigma_x\) and at finite temperature, the last term in \(V_{\text{Rep}}\) (26) vanishes for
\( n \to 0 \) and both \( \langle S^z \rangle \) and \( \langle S^y \rangle \) can have a nonvanishing value for \( C_1 > \frac{1}{V_0} \), i.e., in the case in which the spin interaction \( J \) dominates the suppression of the density fluctuations \( V_0 \). On the other hand, for the case of a large fluctuation of the random field \( \sigma_x \to \infty \), the \( \langle S^x \rangle \)-terms in \( V_{\text{Rep}} \),

\[
\left( 2C_1 - \frac{4C_1^2}{V_0} \right) \langle S^y \rangle^2 - \frac{\beta n}{\sigma^2} \frac{C_1}{V_0} \left( \frac{C_1}{V_0} \right)^2 \to 2C_1 \langle S^x \rangle^2, \quad \sigma_x \to \infty, \tag{27}
\]

and then \( S^x \) does not condense whereas \( S^y \) does. Similarly for \( T \to 0 \), the last term of \( V_{\text{Rep}} \) in equation (26) gives a finite contribution for any nonvanishing \( \sigma_x \), and \( S^x \) does not condense.

The above results seem interesting but they are obtained by the MFT. For example, the assumption of the constant mean field \( \langle S^x \rangle \) is not correct for \( \sigma_x \to \infty \). Therefore, more reliable studies are welcome. The numerical calculations in the subsequent sections give reliable results and reveal detailed properties of the RFIO.

4. Numerical results

4.1. Phase diagrams at low temperature

In this and subsequent sections, we shall show the results obtained by means of the numerical MC simulations. The model is defined by equations (10) \( \sim \) (14), and the local-update MC simulation was used for calculation of the physical quantities for fixed random variables \( \{ \tilde{J}^z \} \) \( \{ \tilde{J}^y \} \). The standard Metropolis algorithm [13] was used for the local update. For the local update of the angle variables \( \theta_{\alpha i} \), random variables \( \Delta \theta \) used for generating a candidate of a new variable \( \theta_{\text{new}} = \theta_{\text{old}} + \Delta \theta \) was chosen in the range \( |\Delta \theta| < \frac{\pi}{3} \). The typical sweep for the thermalization is 100,000 and for the measurement is \( (40,000) \times (10 \text{ samples}) \). The typical acceptance ratio is \( 40\% \sim 50\% \), and errors were estimated from 10 samples by the jackknife methods [14].

We first show the phase diagram of the system without the random field, which was obtained in the previous study [7]. For the case of \( t_x = t_y \) and at \( T = 0 \), there are three phases, the phase with no long-range order (LRO), the FM phase and the phase of double BECs, which we often denote 2SF, accompanying the FM order. See figure 1. In particular, in the states with the FM order, the pseudo-spin \( (S^x, S^y) \) has a LRO in an arbitrary direction, i.e., \( \langle S^x \rangle / \langle S^y \rangle = \tan \theta_s \) with an arbitrary angle \( \theta_s \). This is observed by calculating the correlation functions \( G_S^x (r) \) and \( G_S^y (r) \) defined by

\[
G_S^{x(y)} (r) = \frac{1}{L^3} \sum_{n_0} \left\langle S^{x(y)} (r + n_0) S^{x(y)} (n_0) \right\rangle, \quad G_S (r) = G_S^x (r) + G_S^y (r), \tag{28}
\]

where \( L \) is the linear size of the 3D lattice, and sites \( r_0 \) and \( n_0 + r \) are located in the same spatial 2D lattice, i.e., \( G_S^{x(y)} (r) \) is an equal-time correlator. We put the lattice size of the imaginary-time direction \( N_t = \) the lattice size of the spatial direction \( L \). The angle \( \theta_s \) takes various values depending on the initial configurations and random variables used local updates in the MC simulations. This result comes from the U(1) symmetry of the pseudo-spin rotation in the system of the action \( A_{Lr} + A_L \).

1 More precisely in the MC simulations, the temperature of the system \( T \) is given as \( k_B T = \frac{E_t}{N_t} \), where \( N_t \) is the lattice size in the imaginary-time direction. Then the system at \( T \to 0 \) is realized as \( N_t \to \infty \).
In this paper, we study the system in the random fields \( A^\alpha_q \) in equation (14). We first consider the case \( J^0 \). In this single-component external system, the U(1) spin symmetry is reduced to the \( Z_2 \) symmetry of the Ising type \( \langle S^x, S^y \rangle \rightarrow -\langle S^x, S^y \rangle \). This fact implies that there might exist a preferred direction in the pseudo-spin order. This is actually the case, as we show below.

In numerical studies, we first determine the random variables \( \tilde{J}^x_i \) according to the distribution \( P(\tilde{J}_i) \) in equation (5). To this end, we used the box-Muller methods. Then the MC simulation is carried out for the system with the fixed \( \tilde{J}^x_i \) by the local update of \( \omega_\alpha \) and \( \lambda_r \). The final physical quantities are obtained by averaging calculated quantities for each sample over \( \sim 5 \times 10^5 \) \( \tilde{J}^x_i \) samples. The phase boundary is determined by calculating the internal energy \( E \) and the specific heat \( C \), which are defined as

\[
E = \left\langle A_{Lxy} \right\rangle / L^3, \quad C = \left\langle \left( A_{Lxy} - E \right)^2 \right\rangle / L^3,
\]

where the mean value \( \left\langle \cdot \right\rangle \) includes the average over samples of the random fields, as we explained above. In order to identify the physical properties of each phase, we also calculate the boson correlation functions besides the pseudo-spin one in equation (28),

\[
G_6(r) = \frac{1}{L^3} \sum_{r_0} \left\langle e^{i\theta_{\alpha r_0}} e^{-i\theta_{\alpha r_0+r}} \right\rangle,
\]

\[
G_6(r) = \frac{1}{L^3} \sum_{r_0} \left\langle e^{i\theta_{\alpha r_0}} e^{-i\theta_{\alpha r_0+r}} \right\rangle,
\]

where, as in \( G_6^{\alpha}(r) \), sites \( r_0 \) and \( r_0 + r \) are located in the same spatial 2D lattice.

In figure 1, we exhibit the obtained phase diagram for \( \sigma_x = 0.3 \) and \( \sigma_y = 0 \), which corresponds to \( J^y_0 = 0 \). Typical behaviors of \( E \) and \( C \) of various system sizes, which are used for identification of the phase boundaries, are also shown in figure 2. The calculations indicate that all phase transitions are of the second order. The locations of the phase boundaries are almost the same as the ones of the original bosonic t-J model. However, the FM order is replaced by the RFIO, as the pseudo-spin correlation function in figure 3 indicates that only the y-
component of the pseudo-spin has a LRO and the correlation of the \( x \)-component vanishes quite rapidly as a function of \( r \). This result means that the relative phase of the condensations of the \( a \)- and \( b \)-boson operators has definite values \( \theta \approx \pm \frac{\pi}{2} \). To verify this, we measured the relative phase at each site and the result is shown in figure 4. The magnetization in the phase of RFIO is slightly smaller than that in the original t-J model. The above observation is in good agreement

**Figure 2.** Internal energy \( E \) and specific heat \( C \) for various system sizes. Results indicate the existence of second-order phase transitions. \( C^q_3 = C^p_3 \) and \( \sigma = 0.3 \).

**Figure 3.** Correlation functions, which are used to identify the various phases. \( G_S(r) \), \( G_a(r) \), \( G_b(r) \), \( G^{x}_S(r) \), and \( G^{y}_S(r) \). \( C^q_3 = C^p_3 \) and \( \sigma = 0.3 \).
with the previous studies of the related systems such as the classical XY spin model and two component BEC of the cold atoms [3, 4].

It is interesting to see how the phase diagram is changed when both components of the random field are turned on. It is not so difficult to show that the U(1) symmetry is restored for the case $\sigma = \sigma_y$, and then the relative phase $\theta = \theta_a - \theta_b$ takes an arbitrary value. From this observation, one may expect that the range of $\theta$ is expanded for nonvanishing $\sigma_x$ and $\sigma_y$ depending on the ratio $\sigma_x/\sigma_y$. However, this is not the case. We studied the cases with various values of $\sigma_x/\sigma_y$, and found that $\theta$ takes $\pm \pi/2$ ($0$ or $\pi$) for $\sigma_x/\sigma_y > 1$ ($<1$). For a typical example, see figure 5.

Let us perform the ‘Gedanken experiment’ in which the parameter $V_0$ is varied with the other parameters remaining fixed. For larger $V_0$, fluctuations of the densities of atoms in each site are suppressed and then fluctuations of the phase degrees of freedom of the boson operators are enhanced. In fact, from the action in equations (12) and (15), it is seen that the phases $\omega_{ai}$ vary rapidly in the $\tau$-direction for small $c_\tau$, even though their orders are generated in the spatial direction for a sufficiently large hopping amplitude and a spin-exchange coupling.

We show the results of the numerical study in figure 6. The phase diagram is given in the $(1/\sqrt{V_0} - \sigma_i)$ plane for $C_1 = 2.0$ and $C_3 = C_3^0 = 0.2$. In the 3D region of smaller $V_0$, phase

![Figure 4](image-url)  
Figure 4. Histogram of the relative phase of the condensation of the $a$- and $b$-atoms at constant $\tau$. The results clearly indicate that the relative phase $\theta = \theta_a - \theta_b \approx \pi/2$. $C_1 = 2.0$, $C_3 = C_3^0 = 0.2$, $c_\tau = 2.0$ and $\sigma = 0.3$.

![Figure 5](image-url)  
Figure 5. Pseudo-spin correlation function for various $\sigma_x$ and $\sigma_y$. $G_{S_x}(r)$, $G_{S_y}(r)$ and $G_{S_z}(r)$. Orientation of the magnetization of the pseudo-spin is determined by the relative magnitude of $\sigma_x$ and $\sigma_y$. 

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transition from the 3D XY-spin ordered state to the 3D RFIO takes place as $\sigma_x$ is increased. Both states have their own LROs. On the other hand, for larger $V_0$, the system has a quasi-LRO for smaller $\sigma_x$, and the state turns to that of the RFIO with the genuine $Z_2$ LRO as $\sigma_x$ is increased. In the limit $\frac{1}{V_0} \to 0$, the system can be regarded as a classical 2D system. Therefore, the study of this limit reproduces the result of the previous study on the classical XY model in 2D with the random external field [3]. There the RFIO forms simply as properties of the lowest energy state.

4.2. Robustness of a finite RFIO state

The phase diagram obtained in the previous section, figure 6, shows that the state of the quasi-LRO changes to the state with the genuine Ising-type RFIO as $\sigma_x$ increases for a large $V_0$. This result indicates the robustness of the RFIO state. In order to verify this fact, we study how finite-size systems of the quasi-LRO and also the RFIO change under the updates of the MC simulations. The result of $\langle S^x \rangle$, $\langle S^y \rangle$ for fixed $\tau$ is shown in figure 7. Parameters are
$\tau = 2.0$, $C_1 = 2.0$ and $C_C = 0.2$. For this case, the direction of $\langle S^z \rangle$ changes as a result of quantum fluctuations.

On the other hand for $\tau = 0.3$, the orientation of $\langle S^z \rangle$ changes as a result of quantum fluctuations.

From figure 7, it is obvious that, in the ordinary system without the random external field, the average of the pseudo-spin (magnetization) is nonvanishing but unstable under the MC update, i.e., the orientation of the magnetization fluctuates strongly because of the finiteness (smallness) of the system. On the other hand, in the random case, the average is quite stable and stays $\langle S^z \rangle = \pm 1$, $\langle S_i^z \rangle = 0$. Then one may wonder that the spin behaves as a classical spin and has lost its quantum properties. In order to study it, we measured the behavior of $\langle S_i^z \rangle$ as a function of $\tau$. See figure 8. The results in figure 8 indicate the following fact. For a small $V_0$ (i.e., large $c_\tau$), the phase degrees of freedom of the boson operators $a_i$ and $b_i$ have small quantum fluctuations as their boson densities fluctuate rather largely. Then, the spin behaves as a classical spin. On the other hand, for a large $V_0$ (small $c_\tau$), the spin behaves as a quantum spin and a superposition of the $\uparrow$ spin and $\downarrow$ spin is possible. This result indicates that the 2D RFIO state can be used as a quantum qubit in the quantum information device. In figure 9, we show a method to make a superposed state $|\uparrow\rangle + |\downarrow\rangle$ and also an entangled state of two qubits. The study of quantum superpositions of macroscopically distinct states has a long history [15]. The
above study indicates the possibility that a mesoscopic RFIO state is a candidate for the quantum mesoscopic superposed state.

4.3. Finite temperature phase diagram

In this subsection, we shall study the finite-$T$ phase diagram of the random system. In particular, we are interested in how the states with the RFIO evolve as $T$ is increased. This study is closely related with the stability of the RFIO states investigated in the previous subsection.

In the present MC simulation, the temperature is given as $k_B T = 1/(N_c \Delta \tau)$. Then, the system at low $T$ is realized for sufficiently large $N_c$. The temperature of the system is increased by decreasing $\Delta \tau$ for fixed $N_c$, and therefore the parameters in the action vary as indicated in equations (15). It is obvious that the original 3D system tends to be quasi-1D as $\Delta \tau \to$ small, and then the LROs disappear, which is nothing but a finite-$T$ phase transition.

It is interesting how the ordered states evolve as $T$ is increased. In order to identify the finite-$T$ phase diagram, we measured the internal energy $E$ and the specific heat $C$ as the investigation of the quantum phase transition in the previous section. We also calculated the various correlation functions to identify each phase transition. We show the obtained results and the phase diagram in figure 10. In figure 10, for the states of the 2SF for moderate hopping amplitude $C_3^a = C_3^b$, the specific heat $C$ exhibits two sharp peaks that indicate a second-order
phase transition. The correlation functions show that the 2SF state first loses the properties of the BECs and then the pseudo-spin LRO as $T$ is increased. On the other hand, for the deep 2SF state, a first-order phase transition takes place from the 2SF to the disordered state directly. This disordered state should be distinguished from the PM state in the low $T$ phase diagram. The latter appears as a result of the competition between the $V_0$-term and the hopping, whereas the present one comes from the effect of the thermal fluctuations.

5. Topological excitations in the RFIO state

It is interesting to compare topological excitations in the genuine FM state and RFIO state. There are two topological objects that play an important role near the phase boundary:

(i) vortices of the $a$ and $b$-atoms and their bound state;
(ii) domain wall of the relative phase of the BECs of the $a$ and $b$-atoms.

It is well known that the $a$ and $b$-vortices proliferate in the PM state, whereas, in the FM state, the spatial overlap of these two kinds of vortices increases as a result of the coherent condensation of the spin operator $S^i_x$ and/or $S^i_y$. Furthermore, in a constant external magnetic field $\mathbf{h}$, the Zeeman coupling $\mathbf{h} \cdot \mathbf{S}_i$ generates a linear-potential between the $a$ and $b$-vortices and ‘confinement of vortices’ takes place. For example $\mathbf{h} = (h, 0)$, the Zeeman coupling is given as $\hbar S^i_x = h \cos (\theta_{ai} - \theta_{bi})$, where $\theta_{ai}$ ($\theta_{bi}$) is the phase of the $a$($b$)-atom, and then for a configuration of the vortex pair, an extra energy is generated that is proportional to $\hbar \cdot (\text{distance between two vortices in a pair})$. In two-gap superconductors, a mixing of the two Cooper pairs gives a similar effect to the Zeeman coupling in the FM, and therefore it generates a confinement of vortex [16].

In the system in a random external magnetic field, the Zeeman coupling such as $\mathbf{J}_i^x \cos (\theta_{ai} - \theta_{bi})$ shows up. As seen in the previous section, configurations like $(\theta_{ai} - \theta_{bi}) \sim \pm \pi$ dominate because of the random Zeeman coupling. Therefore, it is expected that the interaction between a pair of an $a$-vortex and a $b$-vortex in the RFIO is quantitatively different from that in a constant magnetic field, i.e., a constant Rabi oscillation that prefers configurations with $(\theta_{ai} - \theta_{bi}) \sim 0$. We investigate this problem in this section. The expected configuration of a vortex pair in the RFIO state is shown in figure 11.
We calculate the local density of vortices $V_r^a$ and $V_r^b$, which is defined as

$$V_r^A \equiv \begin{cases} v_{Ar}, & |v_{Ar}| \geq 1/2 \\ 0, & |v_{Ar}| < 1/2 \end{cases}$$

(31)

with the vorticity $v_{Ar}$ at site $r$ of the 3D space-time lattice,

$$v_{Ar} \equiv \frac{1}{4} \left[ \sin \left( \theta_{A,r+\hat{x}} - \theta_{A,r} \right) + \sin \left( \theta_{A,r+\hat{y}} - \theta_{A,r+\hat{x}} \right) - \sin \left( \theta_{A,r+\hat{y}} + \theta_{A,r+\hat{x}} \right) - \sin \left( \theta_{A,r+\hat{x}} + \theta_{A,r+\hat{y}} \right) \right],$$

(32)

where $A = a, b$. Here we have introduce a cutoff and set $V_r^A = 0$ if $|v_{Ar}|$ is smaller than $1/2$. This cutoff is useful for clarifying the locations of vortices. From the local vortex density $V_r^A$ in equation (32), we measure the overlap of the vortex configurations of the $a$- and $b$-atoms by calculating $dV$ in each time slice, which is defined as

$$dV = \frac{1}{N_v} \sum_{r \in \{r: \text{fixed}\}} \left( V_r^a - V_r^b \right)^2,$$

(33)

where $N_v$ is the total number of vortex $N_v = N_v^a - N_v^b \simeq N_v^b - N_v^a$, $N_v^{A+} = \sum_{r \in \{r: \text{fixed}\}} V_r^A \theta(V_r^A)$, and $N_v^{A-} = \sum_{r \in \{r: \text{fixed}\}} V_r^A \theta(-V_r^A)$ ($A = a, b$) with the Heaviside $\theta$-function, $\theta(\chi)$. We show the calculation of $dV$ and also the total number of vortices and antivortices, $(N_v^{A+}, N_v^{A-})$ in figure 12. Larger $dV$ means a smaller overlap of the $a$- and $b$-vortices.

A transition from the PM state to the RFIO state takes place at $C_1 \approx 0.62$. The density of vortices and anti-vortices does not change substantially in the PM and RFIO phases, but the vortex overlap $dV$ changes drastically at the phase boundary. See figure 12. We also show similar quantities for the nonrandom case with constant $\hat{J}^x = 0.3$. The result indicates that the energy of the vortex pair in the RF system is smaller than that in the system of constant $\hat{J}^x = 0.3$. From the above result, it is also expected that the shape of the brick wall between the $a$- and $b$-vortex pair in the RFIO state is different from that in the constant field, although it is not so easy to observe it by snapshots of the MC simulations. See figure 11.

Let us turn to the domain wall of the relative phase $(\theta_{ar} - \theta_{br})$. This domain wall is closely related to the ‘string’ connecting $a$-atom and $b$-atom vortices. See figure 11. As we explained above, the calculation of $dV$ indicates that the energy of the domain wall is getting smaller as the randomness of $\hat{J}^x$ is getting larger from $\sigma_x = 0$. Then in the case of a large $\sigma_x$, the pseudo-spin loses its order as a result of a large spatial fluctuation of $\hat{J}^x$.

To verify that the above expectation is correct, we investigate configurations generated by the boundary condition such that the spins $\vec{S}_r$ on the left spatial boundary have $(\theta_{ar} - \theta_{br}) = -\pi/2$, whereas on the right spatial boundary $(\theta_{ar} - \theta_{br}) = \pi/2$. In figure 13, we show the expectation value of spin $\langle \vec{S}_r \rangle$ for various $\sigma_x$. For the case $C_1 = 3.0$ and $C_3^a = C_3^b = 2.0$, the result obviously indicates that, from $\sigma_x = 0$ to $\sigma_x = 1.1$, the stiffness of spins is getting stronger as a result of the larger fluctuation of the random variable $\hat{J}^x$. On the other hand, in the case of $\sigma_x = 3.0$, the pseudo-spins fluctuate rather strongly. Similar behavior is observed in the other cases in figure 13. This indicates that the strongly fluctuating random-field destroys the spin order. Then, it is an interesting problem to determine a critical
randomness \(\sigma_c\) for the order-disorder phase transition observed in the present numerical simulations.

6. Conclusion

In this paper, we studied the effect of a ‘random external field’ on the phase diagram of the bosonic t-J model, the properties of the states and the low-energy excitations in the RFIO state. This external field is realized by a random Rabi oscillation between two internal states in an atom induced by a random Raman laser. In the phase diagram of the bosonic t-J model without the random field, there exist ordered states such as the pseudo-spin FM state and the 2SF. We first investigated how the phase diagram is changed as a result of the random field and found that the ordered states move to the states with the RFIO. In the RFIO states, the original U(1) symmetry reduces the \(Z_2\)-Ising type, and therefore low-energy excitations in the RFIO states have different properties from those in the original ordered states of the t-J model.

By the replica-MFT, we first studied the low-energy properties of the quantum spin system in a random external field, and found that, for a sufficiently strong randomness, there appear the preferred directions of the spin order, which are perpendicular to the applied field. Then, by using the MC simulations, we studied the phase diagram of the effective field theory of the t-J model in applied random external fields, \((\tilde{J}^x, \tilde{J}^y)\). We found that the direction of the spin order is determined by which component of the applied field, \((\tilde{J}^x, \tilde{J}^y)\), is larger. We also studied the
finite-$T$ phase diagram and found that the RFIO of the spin survives at intermediate temperatures although the SF is destroyed by the thermal fluctuations.

Finally, physical properties of topological excitations such as the vortex and domain wall were studied. Binding energy of the vortex pair of the $a$- and $b$- bosons is smaller in the RFIO compared to that in the genuine t-J model. This means that the average distance between $a$- and $b$-vortices in a single vortex pair gets longer as $\sigma_x$ increases. Similarly, the width of the domain

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**Figure 13.** (Top panel) Expectation value of spin $\vec{S}_r$, $\langle \vec{S}_r \rangle$, for various $\sigma_x$ with a boundary condition such as $\uparrow (\downarrow)$ on the right (left) boundary. The measured length of $\langle \vec{S}_r \rangle$ at each site is almost unity. In the region of kinks, the length of $\langle \vec{S}_r \rangle$ is modified to show the direction of $\langle \vec{S}_r \rangle$ clearly. For larger $\sigma_x$, the random variable $\tilde{J}_x$ fluctuates more strongly. The result obviously shows that in the region of the moderate fluctuation of $\tilde{J}_x$, $\sigma_x = 0.3 \sim 1.1$, the domain wall is rather thin compared to the case of $\tilde{J}_x = 0$ ($\sigma_x = 0$). However, in the case of $\sigma_x = 3.0$, the spins fluctuate rather strongly as a result of a large fluctuation of $\tilde{J}_x$. The system size is $L_x = 50$, $L_y = 10$ and $L_t = 10$. $C_1 = 3.0$ and $C_3 = 2.0$. (Middle and bottom panels) Similar behavior is observed near the phase boundary for $(C_1, C_3) = (0.8, 0.5)$ and $(C_1, C_3) = (1.5, 0.1)$. 

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finite-$T$ phase diagram and found that the RFIO of the spin survives at intermediate temperatures although the SF is destroyed by the thermal fluctuations.
wall is thinner in the RFIO state. We hope that the above findings are observed by experiments on cold atomic gases. In the near future, we shall report studies on the behavior of vortex lattices that form as a result of the coupling to an artificial external vector potential.

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