Quantum chaos inside space-temporal Sinai billiards

Andrea Addazi

Dipartimento di Fisica, Università di L’Aquila, 67010 Coppito, AQ
LNGS, Laboratori Nazionali del Gran Sasso, 67010 Assergi AQ, Italy

Abstract

We discuss general aspects of non-relativistic quantum chaos theory of scattering of a quantum particle on a system of a large number of naked singularities. We define such a system space-temporal Sinai billiard. We discuss the problem in semiclassical approach. We show that in semiclassical regime the formation of trapped periodic semiclassical orbits inside the system is unavoidable. This leads to general expression of survival probabilities and scattering time delays, expanded to the chaotic Pollicott-Ruelle resonances. Finally, we comment on possible generalizations of these aspects to relativistic quantum field theory.

1 Introduction

The Quantum Chaos theory studies the systems in classical chaos and quantum mechanics regimes. Quantum Sinai Billiard is a well known example of quantum chaotic system [1, 2]. However, quantum chaotic scatterings in contest of general relativity were not studied as well in literature.

In this paper, we will discuss general aspects of quantum scatterings of wave functions on a complicated space-time topology composed of a large number of horizonless singularities, randomly oriented. We dub such a system space-temporal Sinai billiard. What we will expect is that the initial probability will be fractioned into two contributions. In fact, a part of the initial probability density will ”escape” by the system while a part will remain ”trapped” forever in the system because of back and fourth scatterings among the singular geometries. This can be easily understood by a classical chaotic mechanics point of view. In fact, the definition of a classical chaotic scatterings of a particle is the following: a classical mechanics scattering problem in which the incident particle can be trapped ideally forever in a class of classical orbits; but the periodic orbits are unstable saddle solutions and their number grows exponentially with time. Chaotic scatterings have a high sensitivity to the initial conditions manifesting...
itself in a fractal chaotic invariant set, which is also called chaotic saddle [3, 4]. Energy shells closed to the chaotic saddle energy shell will continue to be chaotic. In simpler chaotic systems, examples are Kolmogorov-Arnold-Moser (KAM) elliptic islands, that contain stable periodic orbits. KAM stable periodic orbits undergo to chaotic bifurcations, rupturing the smoothed topology of the invariant set [5, 6]. As usually happening for chaotic saddles, KAM islands are surrounded by a layer of chaotic trajectories. Another typical example is the hyperbolic set of hyperbolic unstable trajectories: solutions are exponentially growing or decreasing but the number of directions are constants of motion. In our case, periodic orbits will be forever trapped in back and forth scatterings among the the space-temporal Sinai billiard. As generically happening in classical chaotic scattering problems, these trajectories will necessary exist in the phase space of the system Our problem is nothing but a complication with respect to a simpler and well known example of classical chaotic scattering problem: a 2d classical elastic scattering of a particle on a system of N fixed disks of radius \( a \) [7, 8]. In this simple problem, kinetic energy is assumed to be conserved, i.e no any dissipations are considered. For one disk the problem is trivially un-chaotic: the differential cross section is just \( \frac{d\sigma}{d\theta} = \frac{\pi}{4} |\sin \frac{\theta}{2}| \) for \( \theta \) in the range \([-\pi, \pi]\); and no trapped periodic orbit are possible. However, with two disks, an unstable periodic orbit is the one bouncing back and forth forever among the two disks. With the increasing of the number of disks one can easily get that the number of trapped periodic orbit will exponentially increase. For example, as shown in [9, 10], in a three disks’ system, the number of unstable periodic orbits proliferate as \( 2^n \) where \( n \) is the number of bounces in unit of the period. If the radius is the distance among the next neighboring disk is \( R > 2.04822142 a \). From Classical chaotic scatterings we can get the main feature of the quantum semiclassical chaotic problem associated and about semiclassical periodic orbits. So, because of multiple diffractions and back and fourth scatterings, one will also expect that the resultant wave function is ”chaotized” by the system: the total wave function is a superposition of the initial one plus all the spherical ones coming from each ”scatterators”. A part of the initial infalling information will be trapped ”forever” in the system, i.e for all the system life-time. In order to describe the evolution of the infalling informations, a quantum mechanical approach based on wave functions is not useful, in this system. A wave functions approach can be substituted by a quantum statistical mechanics approach in terms of density matrices. From the point of view of a Quantum field theory, a S-matrix
approach is not useful in this case, even if "fundamentally true": in order to calculate $\langle \text{in}|S|\text{out} \rangle$ ($\text{in}$ is the in-going plane wave, where $\text{out}$ is the out-going result), we have to get unknown informations on the precise geometric configuration inside the system and about the trapped information state inside it. Such a system can emit a quasi thermalized mixed information state without losing any informations at fundamental level. In other words, we suggest that the space-time non-trivial topology prepares an entangled state as well as an experimental apparatus can prepare an entangled state by an initial pure state. Then considering also possible interaction terms among quantum particles, the chaotization effect will also be more dramatically efficient. In particular, in quantum field theory, interactions in the lagrangian density functional induce n-wave mixings inside the space-temporal Sinai billiard. Thinking about the ingoing state as a collection of coherent quantum fields, these will be scattered into the system and, they will meet each others inside "the trap", they will scatter each others, coupled by lagrangian interactions. A complicated cascade of hadronic and electromagnetic processes is expected. For example, these will produce a large amount of neutral pions, that will electromagnetically decay into two entangled photons $\pi^0 \rightarrow \gamma \gamma$ ($\tau \simeq 10^{-16}$ s in the rest frame). However, also from only one plane wave infalling in the system, the final state emitted by the system will be a mixed state: this is just an effect of the information losing inside the system because of trapped chaotic zones inside. This phenomena is a new form of quantum decoherence induced by the space-time topology. Usually, quantum decoherence is the effective losing of infalling informations in a complex system, like coherent light pumped in a non-linear crystal. In this case, the complex topology of space-time catalyzes the effective losing of information.

A possible applications of our result is in contest of theoretical cosmology. In particular, it was suggested the presence of topologically defects, as a net of cosmic strings, can affect the gaussianity of the CMB spectrum $[12, 13, 14, 15, 16]$. A critical cosmic string sources a conic naked singular geometry $[12, 13, 14, 15, 16]$. So that, a net of cosmic strings is thought as a complicate superposition of conic geometries.

This paper is organized as follows: in section 2 we will discuss the chaotic scattering problem on the space-temporal Sinai Billiard in classical (subsection 2.1) and semiclassical approaches (subsection 2.2), then we will discuss the problem with non-relativistic quantum scattering methods (subsection 2.3). At the end of section 2, we will comment on possible extension to the chaotic quantum field theory effects (subsection 2.4).
Finally we show our conclusions and outlooks in section 3.

2 Chaotic space-temporal Sinai Billiard

In this section, we will discuss the non-relativistic quantum scattering problem of a wave function on a system of naked singular geometries, which we dub space-temporal Sinai Billiard. In particular, we are interested to conic singular geometries disposed in an idealized 3D box. We will discuss the general aspects of the problem in classical and semiclassical approach. Then, we will discuss what happen in a 3D box of N conic singularities, in the non-relativistic quantum scattering approach. Finally, we will comment on a generalization to quantum field theory and quantum particle interactions.

2.1 Classical chaotic scattering on a Space-temporal Sinai Billiard

The classical chaotic scattering of a particle on a Space-temporal Sinai Billiard is characterized by a classical Hamiltonian system $\dot{r} = \partial H / \partial p$ and $\dot{p} = -\partial H / \partial r$ with an initial condition $x_0 = (r_0, p_0)$ in the space of phase. In particular, Let us consider the case of a conic singularity: supposed disposed along the z-axis, the conic metric is

$$ds^2 = -dt^2 + dr^2 + \left(1 - \frac{\Psi}{2\pi}\right)^2 r^2 d\psi^2 + dz^2$$

where $\Psi$ is the deficit angle, related to the opening angle as $\Theta = 2\pi - \Psi$. To consider a generic field function on a conic surface is equivalent to consider these on a Euclidean plane with an extra periodicity condition

$$\phi(\theta) = \phi(\theta + \Theta)$$

where $\Theta$ is the open angle of the cone and $\theta$ is a new angle variable defined so that

$$ds_E^2 = d\tau^2 + dr^2 + r^2 d\theta^2 + dz^2$$

Such an Euclidean background has a topology $S^1 \times S^1 \times R^2$ or $T^2 \times R^2$, i.e. is a cylinder with a torus as its base.

In a box of cones, we consider N conic singularities with random orientation of their axis. In particular, we can define N Hamiltonian systems for each cones, describing the motion of the particle on each of N cones. Clearly, one can obtain similar systems by the geodesic equations $\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = 0$ of the particles in each conic metrics, i.e
the propagation of the particle on the conic hypersurfaces. We can easily show that
the effective Hamiltonian obtained for the propagation on one cone is
\[ H_I = \frac{1}{2m} p_i g^{ij} p_j = \frac{p^2}{2m}, \quad x_t(\theta + \Theta) = x_t(\theta) \] (4)
where we have used (2,3) (in Minkowskian form). This Hamiltonian is written choosing
the reference frame as the z-axis oriented along the cone’s axis. As a consequence,
also with one cone, we have a class of trapped trajectories infinitely going around
the \( \theta \)-direction. Clearly, for a system of \( N \) cones randomly oriented, extra angles
parameterizing their axis directions with respect to the chosen z-axis will enter in
the definition (4). The solution of such a system will be determined by a trajectory
\( x_t = \phi^t(x_0) \) solving the Cauchy problem of classical mechanics. In this case, we will
expect a proliferation of trapped periodic unstable trajectories, as anticipated in the
introduction, because of an infinite back and forth scattering among the \( N \) cones.

Let us define the action of the classical problem:
\[ S(E) = -\int_{\Sigma} r \cdot p \] (5)
where \( \Sigma \) is the energy shell \( H = E \) where a scattering orbit is sited. The time delay is
defined as
\[ T(E) = \frac{\partial S}{\partial E} \] (6)
If the impact parameters of the initial orbits \( \rho \) has a probability density \( w(\rho) \), the
probability density conditioned by energy \( E \) of the corresponding time delays is
\[ P(\tau|E) = \int d\rho w(\rho) \delta(\tau + T(\rho|E)) \] (7)
where the condition ”corresponding time delays” is encoded in the integral though the
Dirac’s delta. (7) is useful to describe the escape of the particle from the trapped
orbits’ zone. Inspired by \( N \) disks problems studied in literature [9, 10], an hyperbolic
invariant set is expected to occur. In this case the decays’ distribution rate is expected
to exponentially decrease, i.e
\[ \lim_{t \to \infty} \frac{P(\tau|E)}{t} = -\gamma(E) \] (8)
On the other hand, for non-hyperbolic sets, like KAM elliptic islands, power low decays
are generically expected \( P(t|E) \sim 1/t^\alpha \), where \( \alpha \) depends by the articular density of
trapped orbits.
Now, let us discuss the time delays in our system of cones. If unstable periodic orbit exists in our scattering problem, eq. (6) will have $\rho$-poles, i.e., it becomes infinite for precise initial impact parameters $\rho$. Let starts with the simplest case of a scattering on a simple cone. In this case, the integral (6) has only a couple of asymptotic divergent direction along the path $\mathbf{x}_t^\theta = (\theta, p\theta)$. The particle will be infinitely trapped in this path if and only if its initial incident direction is parallel to open angle $\theta$ of the cone. This condition correspond to all trajectories with a $z$ value in the range of the cone height.

Now let us complicate the problem considering two cones. In these case the number of divergent asymptotes of $T$ correspond to three couples: i) cycles around the first cone, ii) cycles around the second cone, iii) trapped back and forth trajectories between the two cones.

One can easily get that for a $N$ number of conic singularities the number of the divergent asymptotes for the time-delay function will proliferate. These divergent asymptotes are connected to the fractal character of the invariant set. A geometric way to see the problem is the following: we can consider a $2\nu - 2$ Poincaré surface with section in the Hamiltonian flown on a fixed energy surface, where $\nu$ is the number of degree of freedom of the system. In our case, we consider a 4d Poincaré surface. The time-delay of the orbit necessary to go-out from the cones at large enough distances is $T_{\pm}(\rho|E)$, for every initial Chauchy condition in the Poincaré section. $T_{+}(\rho|E) \to \infty$ for stable surfaces of orbits trapped forever. On the other hand, $T_{-} \to \infty$ on unstable manifolds of orbits.

In other words, $|T_{-}(\rho|E)| + |T_{+}(\rho|E)|$ is a localizator functions for the fractal set trapped trajectories.

Let us remind the definition of sensitivity to initial conditions, defined by the Lyapunov exponents

$$\lambda(x_0|\delta x_0) = \lim_{t \to \infty} \frac{1}{t} \frac{1}{\delta x_0} \frac{\delta x_t}{\delta x_0}$$

(9)

where $\delta x_{0,t}$ are infinitesimal perturbation of the initial condition $x_0$ and the resultant orbit $x_t$. In general, the Lyapunov exponents depend on the initial perturbation and on the orbit perturbation. However, $\lambda$ becomes un-sensible by the orbit in ergodic invariant sets. These sets are characterized by the following hierarchy of Lyapunov exponents in a system with $\nu$-degrees of freedom:

$$0 = \lambda_{\nu} \leq \lambda_{\nu-1} \leq \ldots \leq \lambda_2 \leq \lambda_1$$

(10)
while

\[ 0 = \lambda_{\nu+1} \geq \ldots \geq \lambda_{2\nu} \quad (11) \]

In a Hamiltonian system, the symplectic flows of the Hamiltonian operator implies that

\[ \sum_{k=0}^{2\nu} \lambda_k = 0 \]

and

\[ \lambda_{2\nu-k+1} = -\lambda_k \]

where \( k = 1, 2, \ldots, 2\nu \). In our case, the number of degree of freedom is \( \nu = 3 \), so that the number of independent Lyapunov’s exponents characterizing the chaotic scattering is three.

The exponentially growing number of unstable periodic trajectories inside the invariant set is characterized by a topological number

\[ h = \lim_{t \to \infty} \frac{1}{t} \ln(N\{\tau_o \geq t\}) \quad (12) \]

where \( N \) is the number of periodic orbits of period minor than \( t \), \( \tau_o \) is the periodic orbit time. Such a number is the so called topological entropy \( h > 0 \) if the system is chaotic while \( h = 0 \) if non-chaotic. For a system like a large box of cones, this number will be infinite. Such a number will diverge just with only three cones as happen just in a system of three 2d-disks.

In our system, as for disks, a hyperbolic invariant set or something of similar is expected. For this set \( \delta V \) small volumes are exponentially stretched by

\[ g_\omega = \exp\{\sum_{\lambda_k > 0} \lambda_k t_\omega\} > 1 \quad (13) \]

because of its unstable orbits; where \( t_\omega \) is the time interval associated to the periodic orbit of period \( n \), i.e. to the symbolic dynamic \( \omega = \omega_1\ldots\omega_n \) corresponding to all the nonperiodic and periodic orbits remaining closed in a \( \delta V \) for a time \( t_\omega \). Using (13), we can weight the probabilities for trapped orbits as

\[ \mu_\alpha(\omega) = \frac{|g_\omega|^{-\alpha}}{\sum_\omega |g_\omega|^{-\alpha}} \quad (14) \]

This definition is intuitively understood: a highly unstable trajectory with \( g_\omega >> 1 \) is weighted as \( \mu_\alpha \simeq 0 \). The definition (14) is normalized \( \sum_\omega \mu_\alpha(\omega) = 1 \). With \( \alpha = 1 \) we recover the ergodic definition for the Hamiltonian system.
An intriguing question will be if we can determine the Hausdorff dimension of the fractal sets for our box of cones. In principle, the answer is yes, but in practice the problem seems really hard to solve. In order to get the problem let us define the Ruelle topological pressure

\[ P(\alpha) = \lim_{t \to \infty} \frac{1}{t} \ln \sum_{\omega, t < t < t + \Delta t} |g_\omega|^{-\alpha} \] (15)

Ruelle topological pressure is practically independent by \( t, \Delta t \) for a large \( \Delta t \). The Ruelle topological pressure has a series of useful relations:

1) \( P(\alpha_1 + \alpha_2) \leq P(\alpha_1) + P(\alpha_2) \)

2) \( P(0) = h \), \( i.e \) for \( \alpha = 0 \) the Ruelle topological pressure is just equal to the topological entropy.

3) \( P(1) = -\gamma \), \( i.e \) for \( \alpha = 1 \) the Ruelle topological pressure is just equal to the escape rate.

4) The Ruelle topological pressure is connected to Lyapunov’s exponents as

\[ \frac{dP}{d\beta}(1) = -\lim_{t \to \infty} \sum_{\omega, t < t < t + \Delta t} \mu_1(\omega) \ln |g_\omega| = - \sum_{\lambda_k > 0} \lambda_k \]

The last relation is the one connecting the Ruelle topological pressure with the Hausdorff dimension \( d_H \): 5) \( P(d_H) = 0 \). The Hausdorff dimension of a system with \( \nu \) d.o.f is bounded as \( 0 \leq d_H \leq \nu - 1 \) for the subspace of unstable directions, while a corresponding set of stable directions has exactly the same dimension of the previous one. Let us note that for a system with \( \nu = 1 \) the Hausdorff dimension will collapse to \( d_H = 0 \), \( i.e \) no chaotic dynamics. In our case, \( 0 \leq d_H \leq 2 \) and in principle it can be founded as a root of the Ruelle topological pressure.

2.2 Semiclassical chaotic scattering on a Space-time Sinai Biliard

In Semiclassical approach, the main aspects of fully classical limit are not jeopardized by quantization: trapped periodic orbits, invariant sets and so on. In semiclassical approach we can generalize the classical notion of time delay for a semiclassical quantum system.

Let us remind, just to fix our conventions, that \( \psi_t(r) \) is obtained by an initial \( \psi_0(r_0) \) by the unitary evolution

\[ \psi_t(r) = \int d\mathbf{r}_0 K(r, \mathbf{r}_0, t) \psi_0(\mathbf{r}_0) \] (16)
where $K$ is the propagator, represented as a non-relativistic Feynman path integral as

$$K(r, r_0, t) = \int D\tau e^{i\tau}$$

where

$$I = \int_0^t dt L(r, \dot{r})$$

$I$ the action and $L$ the lagrangian of the particle. The semiclassical limit is obtained in the limit

$$I = \int_0^t [p \cdot dr - H dr] \gg \hbar$$

so that the leading contribution to the path integral is just given by classical orbits. The corresponding WKB propagator has a form

$$K_{WKB}(r, r_0, t) \simeq \sum_n A_n(r, r_0, t)e^{\frac{iI_n}{\hbar}}$$

where we sum on all over the classical orbits of the system. The amplitudes $A_n$ are

$$A_n(r, r_0, t) = \frac{1}{(2\pi i\hbar)^{\nu/2}} \sqrt{|\text{det}[\partial r_0/\partial r_0 I_n(r, r_0, t)]|} e^{-\frac{i\pi h_n}{2}}$$

($h_n$ counts the number of conjugate points along the n-th orbit).

The probability amplitude is related to Lyapunov exponents as

$$|A_n| \sim \exp \left(-\frac{1}{2} \sum_{\lambda_k > 0} \lambda_k t \right)$$

along unstable orbits. On the other hand,

$$|A_n| \sim |t|^{-\nu/2}$$

along stable orbits.

The level density of bounded quantum states is related to the trace of the propagator. In semiclassical limit, the trace over the propagator is peaked on around the periodic orbits and stationary saddles. This allows to quantize à la Bohr-Sommerfeld semiclassical unstable periodic orbits, that are densely sited in the invariant set. The semiclassical quantum time delay is

$$\mathcal{T} = \int \frac{d\Gamma_{ph}}{(2\pi \hbar)^{\nu-1}} [\delta(E-H_0+V) - \delta(E-H_0)] + O(\hbar^{2-\nu}) + 2 \sum_p \sum_p \tau_{a=1}^{\infty} \tau_p \cos \left( \frac{a S_p}{\pi} - \frac{\pi p m_p}{2} \right) \sqrt{|\text{det}(M_p^0)|} + O(\hbar)$$
where \(d\Gamma_{ph} = dpdr\). The sum is on all over the periodic orbits, where primary periodic orbits are labelled as \(p\) and the number of their repetitions are labelled as \(a\); \(S_p(E) = \int p \cdot dr\), \(\tau_p = \int_E S_p(E)\), \(m_p\) is the Maslov index, and \(\mathcal{M}\) is the \((2\nu - 2) \times (2\nu - 2)\) Poincaré map matrix defined in the neighborhood of the \(a\)-orbit.

Now, let us consider a simplified problem with only \(\nu = 2\) d.o.f, in order to more easily get analytical important properties of semiclassical chaotic scatterings and their features. Let us consider a generic projection of our box of cones to a 2d plane. Now, we study the dynamics in this plane, ignoring the existence of a third dimension. However, we can be so general in our consideration to be practically valid for every chosen projection! Clearly, we remark that we know well how this problem can be only a different simplified problem with respect the 3d one. In this case, the matrix \(\mathcal{M}\) has two eigenvalues: \(\{g_p, g_p^{-1}\}\), where \(g_p\) is the classical factor \(|g_p| = \exp(\lambda_p \tau_p)\). The complicate equation (22) for the time delay is just reduced to

\[
\mathcal{T}(E) = \mathcal{T}_0(E) - 2\hbar Im \frac{d \ln Z(E)}{dE} + O(\hbar) \tag{23}
\]

where \(\mathcal{T}_0(E)\) is the analytical part of the time-delay function given by the first integral in (22), while \(Z(E)\) is the Zeta function

\[
Z(E) = \prod_p \prod_{a=0}^\infty \left(1 - e^{i\alpha_p} \frac{1}{g_p^a \sqrt{|g_p|}}\right) \tag{24}
\]

where

\[
\phi_p = \frac{1}{\hbar} S_p - \frac{\pi}{2} m_p
\]

From (23) and (24) one could get, as an application of the Mittag-Leffler theorem, that the pole of the resolvent operators exactly corresponds to the zeros of the Zeta function. In complex energies’ plane, the contribution of periodic orbits to the trace of the resolvent operator is related to the Z function by the simple relation

\[
tr_\frac{1}{z - H} = \left|_p \frac{d}{dz} \ln Z(z) = \frac{1}{i\hbar} \sum_p \sum_a \tau_a e^{i\alpha_p} \frac{1}{|g_p|^{a/2}} \right.	ag{25}
\]

(we omit extra higher inverse powers of \(|g_p|\)). But the poles of the resolvent operator and the zeros of the Zeta function are nothing but scattering resonances:

\[
Z(E_a = E_a - i\Gamma_a/2) = 0
\]

Let us comment that if the invariant set contains a single orbit, resonances \(E_a\) satisfy the Bohr-Sommerfeld quantization condition

\[
S_p(E_a) = 2\pi \hbar \left(a + \frac{1}{4} m_p\right) + O(\hbar^2)
\]
while widths satisfy 

\[ \Gamma_a = \frac{\hbar}{\tau_p} \ln|g_p(\mathcal{E}_r)| + O(\hbar) \]

This last relation is intuitively understood: for a large instability of the periodic orbit \( g_p \gg 1 \), the resonances’ lifetime \( \tau_a = \hbar/\Gamma_a \ll 1 \).

Let us return on our general problem, from 2d to 3d. The resonances will not always dominate the time evolution of a wavepacket. In fact, in a system like our one, one could expect so many resonances that after the first decays the system will proceed to an average distribution over these resonances’ peaks. Considering a wavepacket \( \psi_t(\mathbf{r}) \) over many resonances in a region \( W \) in the \( \nu \)-dimensional space, the quantum survival probability is

\[ P(t) = \int_W |\psi_t(\mathbf{r})|^2 d\mathbf{r} \quad (26) \]

that can be also rewritten in terms of the initial density operator \( \rho_0 = |\psi_0\rangle\langle \psi_0| \) as

\[ P(t) = \text{tr} \mathcal{I}_D(\mathbf{r}) e^{-\frac{\mu t}{\hbar}} \rho_0 e^{+\frac{\mu t}{\hbar}} \quad (27) \]

where \( \mathcal{I}_D \) is a distribution equal to 1 for \( \mathbf{r} \) into \( D \) while is zero out of the region \( D \). As done for the time-delay equal to 1 for \( \mathbf{r} \) into \( D \) while is zero out of the region \( D \).

\[ P(t) \approx \int \frac{d\Gamma_{ph}}{(2\pi \hbar)^\nu} \mathcal{I}_D e^{\mathbf{L}_{cl} t} \tilde{\rho}_0 + O(\hbar^{1+\nu}) + \frac{1}{\pi \hbar} \int dE \sum_p \sum_a \cos \left( \frac{a S_p}{\hbar} - \frac{\pi}{2} \mathbf{m}_p \right) \left| \sqrt{\det(\mathbf{m}_p - 1)} \right| \int_p \mathcal{I}_D e^{\mathbf{L}_{cl}^\dagger \tilde{\rho}_0 dt} + O(\hbar^0) \quad (28) \]

where \( \mathbf{L}_{cl} \) is the classical Liouvillian operator, defined in terms of classical Poisson brackets as \( \mathbf{L}_{cl} = \{ H_{cl}, \ldots \} \text{Poisson} \); \( \tilde{\rho}_0 \) is the Wigner transform of the initial density state.

The Sturm-Liouville problem associated to \( \mathbf{L}_{cl} \) defines the Pollicott-Ruelle resonances

\[ \mathbf{L}_{cl} \phi_n = \{ H_{cl}, \phi_n \} \text{Poisson} = \lambda_n \phi_n \quad (29) \]

The eigenstates \( \phi_n \) are Gelfald-Schwartz distributions. They are the ones with unstable manifolds in the invariant set. On the other hand, the adjoint problem

\[ \mathbf{L}_{cl}^\dagger \tilde{\phi}_n = \tilde{\lambda}_n \tilde{\phi}_n \quad (30) \]

has eigenstates associated to stable manifolds. The eigenvalues \( \lambda_n \) are in general complex. They have a real part \( Re(\lambda_n) \leq 0 \) because of they are associated to an ensemble
bounded periodic orbits. On the other hand $Im(\lambda_n)$ describe the decays of the statistical ensambles. As shown in

one can expand the survival probability over the Pollicot-Ruelle resonances as

$$P(t) \simeq \int \sum_n (\mathcal{L}_D|\phi_n(E)\rangle \langle \tilde{\phi}_n(E)|e^{\lambda_n(E)t}|\phi_n(E)\rangle \langle \tilde{\phi}_n(E)|\tilde{\rho}_0) \tag{31}$$

From this expansion, one can consider the 0-th leading order: it will be just proportional to an exponential $e^{\lambda_0(E)t}$. The long-time decay of the system is expected to be related to the classical escape rate $\gamma(E)$. So that we conclude that the survival probability goes as $P(t) \sim e^{-\gamma(E)t}$, i.e $s_0 = -\gamma(E)$.

As a consequence, the cross sections from A to B $\sigma_{AB} = |S_{AB}|^2$ are dramatically controlled by the Pollicott-Ruelle resonances. Let us consider cross sections’ autocorrelations

$$C_E(\bar{E}) = \langle \sigma_{BA}(E - \bar{E})\sigma_{AB}(E + \bar{E}) \rangle - |\langle \sigma_{BA}(E) \rangle|^2 \tag{32}$$

with $E$ labelling the energy shell considered. Let us perform the Fourier transform

$$\tilde{C}_E(t) = \int_{-\infty}^{+\infty} C_E(\bar{E})e^{-i\bar{E}t}d\bar{E} \tag{33}$$

As done for the survival probability, we expand (33) all over the Pollicott-Ruelle spectrum so that we obtain

$$\tilde{C}_E(t) \simeq \sum_n \tilde{C}_n e^{x(-\text{Re}\lambda_n(E)t)\cos Im\lambda_n(E)t} \tag{34}$$

where $\tilde{C}_n$ are coefficients of this expansion. In particular the leading order of (34) is related to (31) for $Im\lambda_0 = 0$:

$$\tilde{C}_E(t) \simeq e^{\gamma(E)t} \tag{35}$$

corresponding to the main Lorentzian peak

$$C_E(\bar{E}) \sim \frac{1}{E^2 + (h\gamma(E))^2} \tag{36}$$

while (34) corresponds to a spectral correlation

$$C_E(\bar{E}) \simeq \sum_n \left\{ \frac{C_n}{(E - hIm\lambda_n)^2 + (h\text{Re}\lambda_n)^2} + \frac{C_n}{(-E - hIm\lambda_n)^2 + (h\text{Re}\lambda_n)^2} \right\} \tag{37}$$

We conclude resuming that a semiclassical quantum chaotic scattering approach leads to following conclusions about the box of cones problem: i) the existence of chaotic
regions of trapped trajectories has to be a consequence of our scattering problem; ii) the qualitative behavior of survival probability and correlation function is qualitatively understood as a decreasing function in time with an exponent determined by classical chaos scattering considerations.

2.3 Non-Relativistic Quantum Scattering

Let us consider the Schrödinger equation for a particle, in a cone geometry

\[ i \frac{\partial}{\partial t} \psi(x) = -\frac{\Delta_c}{2m} + A \frac{\delta(r - \bar{r})}{r} \]  

(38)

where \( \Delta_c \) is the Laplacian in the conical geometry. For simplicity, we have considered a cone with its axis coincident with the z-axis. In fact, the radius of the cone boundary is \( r = \bar{r} \), and it can be encoded in the equation as a \( \delta \)-potential, while \( A \) is the dimensional "coupling" of the potential.

As usually done for this type of problem, we can separate the variables as

\[ \psi(t, x) \sim e^{-i\omega t} \phi_n(r) (\sin n\nu \theta, \cos n\nu \theta)^T, \quad n = 0, 1, 2, ... \]  

(39)

and defining the adimensional parameter \( a = 2mA \) and substituting (39) to (38) we obtain

\[ \frac{d^2 \phi_n(r)}{dr^2} + \frac{1}{r} \frac{d \phi_n(r)}{dr} + \left[ k_z^2 - \frac{n^2\nu^2}{r^2} - \frac{a}{r} \delta(r - \bar{r}) \right] \phi_n(r) = 0 \]  

(40)

We demand as contour conditions

\[ \phi_n(a + o^+) - \phi_n(a + o^-) = 0 \]  

(41)

so that we can map such a problem to another free-like equation

\[ \frac{d^2 \phi_n(r)}{dr^2} + \frac{k_z^2 - \frac{n^2\nu^2}{r^2}}{r^2} f_n(r) = 0 \]  

(42)

This equation can be also rewritten as

\[ \frac{d^2 u_n(r)}{dr^2} + \left( k_z^2 - \frac{n^2\nu^2}{r^2} \right) u_n(r) = 0 \]  

(43)

where \( u_n = r\phi_n \) and \( k_z \).

The solution (regular) corresponding to the continuous part of the spectrum is

\[ \phi_n(r) = c_n^0 J_{n\nu}(k_z r), \quad r < \bar{r} \]  

(44)

Perhaps this problem could be found in standard test of advanced quantum mechanics and non-relativistic quantum scattering theory. I have not found any useful references about this particular problem of quantum scattering, so that I have just decided to repeat the exercise in all the details.
\[ \phi_n(r) = c_n^-(k_z)H_{nv}^-(k_z r) - c_n^+(k_z)H_{nv}^+(k_z r), \quad r > \bar{r} \]  \hspace{1cm} (45)

These solutions are valid for all values of \( a \) in the \( \delta \)-potential. Our problem has two matching conditions

\[ c_n^0(k_z)J_{nv}(k_z \bar{r}) = c_n^-(k_z)H_{nv}^-(k_z \bar{r}) - c_n^+(k_z)H_{nv}^+(k_z \bar{r}) \] \hspace{1cm} (46)

\[ c_n^0(k_z)\left[ \frac{a}{k_z \bar{r}} J_{nv}(k_z \bar{r}) + J'_{nv}(k_z \bar{r}) \right] = c_n^-(k_z)H_{nv}^-(k_z \bar{r}) - c_n^+(k_z)H_{nv}^+(k_z \bar{r}) + 2i/\pi \] \hspace{1cm} (47)

(prime is the differentiation with respect to the adimensional variable \( k_z r \)).

This problem can be viewed as a scattering one. The corresponding solution for the S-matrix is

\[ S_n(k_z) = \frac{a J_{nv}(k_z \bar{r})H_{nv}^-(k_z \bar{r}) + 2i/\pi}{a J_{nv}(k_z \bar{r})H_{nv}^+(k_z \bar{r}) - 2i/\pi} \] \hspace{1cm} (48)

related to \( f_n \) as usual:

\[ S_n = 1 + 2ik_zf_n \]

so that

\[ |S_n| = 1 \rightarrow S_n = e^{2i\delta_n} \]

We also remind as \( f_n \) is related to this phase \( \delta_n \):

\[ f_n = e^{2i\delta} - 1 \]

\[ \frac{2ik_z}{2i/\pi} = \frac{e^{i\delta_n} \sin \delta_n}{k_z} \]

Let us remind that, as usual, the asymptotic expansion of the radial part of the wave function can be written as the sum of the incident plane-wave on the conic geometry and the spherical one as

\[ \frac{1}{(2\pi)^{3/2}} \left[ e^{ik_z z} + f(\theta, \phi)e^{ikr} \frac{\sin \delta}{r} \right] \]

Now, Let us consider a series of scatterings on a large number of N cones, disposed with a uniform random distribution of axis. Let us suppose a box of \( n \times m \times p \) cones, \( n \) in the x-axis, \( m \) in y-axis, \( p \) in z-axis (not necessary disposed as a regular lattice). Let us call \( N_1, N_2 \) the sides sited in the xy-planes, \( M_{1,2} \) in xz-planes, \( P_{1,2} \) in zy-planes, edges of the box of cones. Suppose an incident plane wave \( \psi_0 \) on the 2D surface \( N_1 \), with \( n \times m \) cones: \( n \times m \) conic singularities will diffract the incident wave in \( n \times m \)-components. We want to evaluate the S-matrix from the in-state 0 to the out-the box one. One will expect that a fraction of initial probability density will escape from the box by the sides \( N_{1,2}M_{1,2}, P_{1,2} \), another fraction will be trapped ”forever” (for a time-life equal to the one of the system) inside the box. As a consequence, we
have to consider all possible diffraction stories/paths. We also have to consider more complicated diffraction paths: the initial wave can scatter back and forth in the system before going-out.

We can consider the problem as a superposition of the initial wave function, assumed as a wave plane, and the diffracted wave functions for each conic singularities. In this system, we can label the position of all the conic singularities as $(i, j, k)$, where $i = 1, ..., n$, $j = 1, ..., m$, $k = 1, ..., p$. The total wave function can be written as

$$\phi_0 + f(n_0, n_{111})\frac{e^{ikr_{111}}}{r_{111}} + f(n_0, n_{121})\frac{e^{ikr_{121}}}{r_{121}} + ... + f(n_0, n_{1N1})\frac{e^{ikr_{1N1}}}{r_{1N1}}$$

(49)

$$+ f(n_{111}, n_{121})\frac{e^{ikr_{221}}}{r_{221}} + ... + f(n_{111}, n_{1N1})\frac{e^{ikr_{2N1}}}{r_{2N1}}$$

$$+ f(n_{111}, n_{211})\frac{e^{ikr_{321}}}{r_{321}} + f(n_{111}, n_{221})\frac{e^{ikr_{421}}}{r_{421}} + ... + f(n_{111}, n_{212})\frac{e^{ikr_{321}}}{r_{321}} + ... + f(n_{111}, n_{21P})\frac{e^{ikr_{321}}}{r_{321P}} + ... + f(n_{111}, n_{22P})\frac{e^{ikr_{321}}}{r_{321P}} + ...$$

where $n_0$ is the wave versor of the incident plane wave, $n_{ijk}$ are wave versors of the scattered waves from the conic singularities in positions $ijk$, $r_{ijk}$ are radii from positions $ijk$.

Under this approximation, we can use the transition amplitudes of the one scattering problem considered in the previous section.

The resultant wave function will be a superposition of an infinite series of waves. As a consequence, the total wave function will be highly chaotized by the superposition of all the scattered waves.

An S-matrix for one possible diffraction path is

$$\langle in | S_{111}^{th - short} | out \rangle = S_{0-111} S_{111-222} S_{222-333} \cdots S_{(n-1)(m-1)(p-1)-(nmp)}$$

(50)

where $S_{111-222}$ represents the S-matrix for a process from in-state (after a scattering on) 111 and with an out-state (after a scattering on) 222. This formulation can be consider if and only if the interdistances among singularities are much higher than the cones’ sizes.

We can write a generic S-matrix for one diffraction path as

$$\langle in | S^{Kth} | out \rangle = S_{0-1jk} S_{ijk} S_{i'j'k'} \cdots S_{(n-1)j'(m-1)k'(p-1)-(n'j'm'kp)}$$

(51)

with conditions

$$i \leq i' \leq i + 1$$

(52)
\begin{align*}
j &\leq j' \leq j + 1 \quad (53) \\
k &\leq k' \leq k + 1 \quad (54) \\
... \\
i^{n-1} &\leq i^n \leq i^{n-1} + 1 \quad (55) \\
j^{m-1} &\leq j^m \leq j^{m-1} + 1 \quad (56) \\
k^{p-1} &\leq k^p \leq k^{p-1} + 1 \quad (57)
\end{align*}

represent a class of paths similar to (50).

These class of paths are "minimal" ones: there are not back-transitions. "Minimal paths" are \( n \times m \times p \times (n - 1) \); while the number of non-minimal paths will diverge.

The total S-matrix is the (infinite) sum on all diffraction paths

\[
\langle in \mid S_{n}^{OUT} \mid out \rangle = \sum_{\text{paths}} \langle in \mid S_{n}^{K-th} \mid out \rangle \quad (58)
\]

The S-matrix for one diffraction path can be written as

\[
(S^{Kth})_n = \prod_{j=first}^{last} \frac{a_j J_{\nu}(k_j \vec{r}_j) H^{*}_{\nu}(k_j \vec{r}_j) + \frac{2i}{\pi}}{a_j J_{\nu}(k_j \vec{r}_j) H^{*}_{\nu}(k_j \vec{r}_j) - \frac{2i}{\pi}} \quad (59)
\]

where the product is performed from the first scattering to the last one, and \( a_j, \vec{r}_j, k_j \) depend by the particular j-th conic singularity (\( k_j \) depends on the direction of the conic axis).

### 2.4 Quantum field theories

In this section we will formally discuss the problem of scattering from a QFT point of view. If one considers the path integral behavior in the UV energy regime, the fields’ configurations start to "feel" the effect of the non-trivial topology and naked cones’ singularities. Information is chaotically mixed in this limit. In fact fields start to be randomly diffused by presence of randomly oriented cones. A part of the fields’ energy density will be trapped in the irregularities.

Suppose interdistances much higher than cones’ dimensions. This case is a simplified one with respect to the realistic problem. In this case, we can define a transition amplitude for each cone. Let us suppose to be interested to calculate the transition amplitude for a field configuration \( \phi_0 \) to a field configuration \( \phi_N \). \( \phi_0 \) is the initial field configuration defined on a \( t_0 \), before entering in the system, while \( \phi_N \) is a field
configuration of a time $t_N$, corresponding to an out-going state from the system. For simplicity, we can formalize the simplified problem as a 4D-box, with $n \times m \times p$ conic singularities in 3D, $n$ in the x-axis, $m$ in y-axis, $p$ in z-axis (not necessarily disposed as a regular lattice). Let us call $\mathcal{N}_1, \mathcal{N}_2$ the sides sited in the xy-planes, $\mathcal{M}_{1,2}$ in xz-planes, $\mathcal{P}_{1,2}$ in yz-planes, delimiting the 3D-space-box. Let us consider an incident field $\phi_0$ on the 2D plane $\mathcal{N}_1$, with $n \times m$ conic singularities. Then the $n \times m$ conic singularities will scatter the incident field in $n \times m$-waves. From each diffractions, the out-waves will scatter on a successive cones, penetrating in the box, or to the other nodes in the same plane $\mathcal{N}_1$, and so on. Our problem is to evaluate the S-matrix from the in-state 0 to the out-the box state. One will expect that a fraction of initial probability density will escape from the 3D box by the sides $\mathcal{N}_1 \mathcal{M}_{1,2}, \mathcal{P}_{1,2}$, another fraction will be trapped ”forever” (for a time-life equal to the one of the system) inside the box. As a consequence, one has to consider all possible diffraction stories or diffraction paths. Clearly, one has also to consider paths in which the initial wave goes back and forth in the system before going-out.

One example of propagation Path $0 - 111 - 222 - 333 - ... - nmp - N$

$$\langle \phi_0, t_0 | \phi_{111, \text{in}}, t_{111, \text{in}} \rangle \langle \phi_{111, \text{in}}, t_{111, \text{in}} | \phi_{111, \text{out}}, t_{111, \text{out}} \rangle \langle \phi_{111, \text{out}}, t_{111, \text{out}} | \phi_{222, \text{in}}, t_{222, \text{in}} \rangle \times \langle \phi_{222, \text{in}}, t_{222, \text{in}} | \phi_{222, \text{out}}, t_{222, \text{out}} \rangle \cdots \langle \phi_{(n-1,m-1,p-1)}, t_{(n-1),(m-1),(p-1)} | \phi_{nmp}, t_{nmp} \rangle \langle \phi_{n,m,p}, t_{n,m,p} | \phi_N, t_N \rangle$$

where $| \phi_{ijk, \text{in}}, t_{ijk, \text{in}} \rangle$ and $| \phi_{ijk, \text{out}}, t_{ijk, \text{out}} \rangle$ are states before and after entering in the conic geometry $ijk$. In order to evaluate $\langle \phi_0, t_0 | \phi_{nmp}, t_{nmp} \rangle$ one has to consider all the possible propagation paths from the initial position to the $nmp$-th conic singularity.

We define these amplitudes as

$$\langle \phi_{ijk}, t_{ijk} | \phi_{i'j'k', \text{in}}, t_{i'j'k', \text{in}} \rangle = \int_{\mathcal{M}_0} D\phi e^{i\Pi[\phi]}$$

while

$$\langle \phi_{ijk, \text{in}}, t_{ijk, \text{in}} | \phi_{ijk, \text{out}}, t_{ijk, \text{out}} \rangle = \int_{\mathcal{M}_{ijk}} D\phi e^{i\Pi[\phi]}$$

where $\mathcal{M}_0$ is the Minkowski space-time, while $\mathcal{M}_{ijk}$ is the ijk-cone space-time. Again one can easily get that for a large system of naked conic singularities, it will exist a class of propagators’ paths, reaching the out state $| \phi_N, t_N \rangle$ only for a time $t_N \to \infty$.

A simple example can be the propagator paths

$$| \langle \phi_{ijk, \text{in}}, t_{ijk} | \phi_{i'j'k'}, t_{i'j'k'} \rangle |^2 | \langle \phi_{ijk, \text{in}}, t_{ijk}^{(1)} | \phi_{i'j'k'}, t_{i'j'k'}^{(1)} \rangle |^2 \cdots | \langle \phi_{ijk, \text{in}}, t_{ijk}^{(\infty)} | \phi_{i'j'k'}, t_{i'j'k'}^{(\infty)} \rangle |^2$$

17
where \( t^\infty_{ijk} > \ldots > t^{(1)}_{ijk} > t_{ijk} \) and \( t^\infty_{ijk'} > \ldots > t^{(1)}_{ijk'} > t_{ijk'} \). This amplitude is non-vanishing in such a system as an infinite sample of other ones. We can formally group these propagators in a \( \langle BOX|BOX \rangle \) propagator, evaluating the probability that a field will remain in the box of cones after a time larger than the system life-time. On the other hand, let call \( \langle BOX|OUT \rangle \) and \( \langle OUT|OUT \rangle \) the other processes.

Considering interactions, one will also use S-matrices. We can write a generic S-matrix for one diffraction path as

\[
\langle in|S^{K_{th}}|out \rangle = S_0 - 1_{jk}S_{ijk}S_{i'j'k'}\ldots S_{(in-1jm-1kp-1)\ldots(injmkp)}
\] (64)

A class of paths like the one in (50), are like (64) with conditions

\[
i \leq i' \leq i + 1
\] (65)

\[
j \leq j' \leq j + 1
\] (66)

\[
k \leq k' \leq k + 1
\] (67)

\[
\ldots
\]

\[
i^{n-1} \leq i^n \leq i^{n-1} + 1
\] (68)

\[
j^{m-1} \leq j^m \leq j^{m-1} + 1
\] (69)

\[
k^{p-1} \leq k^p \leq k^{p-1} + 1
\] (70)

We call these class of paths ”minimal paths”. In fact, in these paths there are not back-transitions. The total number of ”minimal paths” is is \( n \times m \times p \times (n - 1) \). On the other hand, the number of paths with back and forth scatterings will diverge.

As a consequence, the total S-matrix is the sum over all possible infinite diffraction paths

\[
\langle in|S^{OUT}_n|out \rangle = \sum_{\text{paths}} \langle in|S^{K_{th}}_n|out \rangle
\] (71)

accounting for all the paths leading from the in-state to the out-of-box state.

For a completeness of our discussion, let us reformulate the non-relativistic quantum problem in a non-relativistic path integral formulation. We will use here the bracket-notation, in which the propagator from \((x_0, t_0)\) to \((x_1, t_1)\) is

\[
K(x_0, t_0; x, t_1) = \langle x_0, t_0|x, t_1 \rangle
\]
This will be equivalent to wave functions’ formulation considered in section 3.2. In this case, a problem of \(\langle OUT|OUT\rangle\) is reformulated not with propagators in the fields’ space but in the same space-time points. \(\langle OUT|OUT\rangle\) will account for all possible paths leading to an in-coming state \(|x_0, t_0\rangle\) to another state out of the box. Again, such a problem is chaotized by the fact that one has to consider the interference of all possible paths passing for all possible conic geometries. A simple example of a path inside the OUT-OUT class of paths is

\[
\langle x_0, t_0|x_{111,in}, t_{111,in}\rangle \langle x_{111,in}, t_{111,in}|x_{111,out}, t_{111,out}\rangle \langle x_{111,out}, t_{111,out}|x_{222,in}, t_{222,in}\rangle
\]

\[
\times \langle x_{222,in}, t_{222,in}|x_{222,out}, t_{222,out}\rangle \cdots \langle x_{(n-1,m-1,p-1), (n-1),(m-1),(p-1)}|x_{nmp}, t_{nmp}\rangle
\]

where \(|x_{ijk,in}, t_{ijk,in}\rangle\) and \(|x_{ijk, out}, t_{ijk, out}\rangle\) are states before and after entering in the conic geometry \(ijk\).

One can find trapped propagators like

\[
|\langle x_{ij}, t_{ij}|x'_{ij'k'}, t'_{ij'k'}\rangle|^2 |\langle x_{ij}, t^{(1)}_{ij}|x'_{ij'k'}, t^{(1)}_{ij'k'}\rangle|^2 \cdots |\langle x_{ij}, t^{(\infty)}_{ij}|x'_{ij'k'}, t^{(\infty)}_{ij'k'}\rangle|^2
\]

where \(t^{(\infty)}_{ij} > \cdots > t^{(1)}_{ij} > t_{ij}\) and \(t^{(\infty)}_{ij'k'} > \cdots > t^{(1)}_{ij'k'} > t_{ij'k'}\). A class of paths from OUT to BOX state will be attracted in these trapped paths. This is a reformulation of what we have concluded in Section 3.2.

Now, let us return to QFT formulation. We are against a chaotic quantum field theory problem. In a chaotic quantum field theory, there are not trapped trajectories in space-time but there trapped configurations in the infinite dimensional space of fields! In analogy to semiclassical chaotic non-relativistic quantum mechanics, one can consider a semiclassical approximation in a regime in which the fields’ action is much higher than \(\hbar\): \(I >> \hbar\). In this approximation, we have a formal understanding of the chaotic quantum field theory problem. The corresponding WKB propagator for a quantum field has a form

\[
\langle \phi_0, t_0|\phi_1, t_1\rangle \simeq \sum_n A_n(\phi_0, t_0|\phi_1, t_1)e^{i\frac{\sqrt{I_n}}{\hbar}}
\]

where we are summing on all over the classical orbits in the fields’ configurations’ space, while amplitudes \(A_n\) are

\[
A_n(\phi_0, t_0|\phi_1, t_1) = \frac{1}{(2\pi i\hbar)^{n/2}} \sqrt{|\text{det}[\partial_{\phi_0} \partial_{\phi_0} I_n[r, r_0, t]]|} e^{-\frac{\hbar h_n}{2}}
\]

where \(h_n\) counts the number of conjugate points along the \(n\)-th orbits.
In my knowledge, I will expect that all rigorous results obtained in literature of classical chaotic scatterings, about the existence of invariant set with their topological robust proprieties discussed in part above, are not rigorously extended for an infinite dimensional space of fields and a complete theory regarding these aspects in QFT is not known to me. Nevertheless let us intuitively think something similar happen in space of fields, even if more complicated. The presence of chaotic zones of trapped periodic fields’ configurations in a subregion of the configurations’ space, corresponding to the one confined into our system, is expected for our problem. Also for fields, chaotic unstable trajectories in the fields’ space are expected, as well as a large number of fields’ resonances in QFT S-matrices, generalizing Pollicot-Ruelle ones. The survival probability for a field are expected to exponentially decrease as in semiclassical quantum mechanical case.

On the other hand, a general space of different fields, the presence of interaction terms in the lagrangian leads to tree-level transitions’ processes that has to be considered as leading orders in the semiclassical saddle point perturbative expansion. As a consequence, chaotic fields’ trapped trajectories have to be thought as a multifields’ ones. The result can be imagined as a chaotic cascade of processes among fields, in which a part of different fields are trapped in the system continuously interacting and scatterings and decaying each others. For example, let us imagine one pure electromagnetic wave entering inside the box of cones. This starts to be diffracted into different direction, so that initial coherent photons will start to re-meet each other in a different state. Of course, if their energy is enough, they can produce couples of $e^+ e^-$, $q\bar{q}$ and so on. Then, these fields will interact each other through electromagnetic, strong and weak interactions. The final system will be full of new fields, and it will have highly chaotic trapped zone.

3 Conclusions and outlooks

In this paper, we have discussed aspects of quantum chaos in a problem of scattering of quantum particles on a system of a large number of horizonless naked singularities, randomly oriented. We have assumed that the scattered particle has a too small mass to induce a relevant back-reaction to the space-time metric. The general results that we obtained can be resume as follows:

- In semiclassical regime, chaotic trapped zone of semiclassical periodic orbits will be
inevitably formed inside the system. The initial information state transits to a highly chaotized final state. The final state is fractioned in a forever trapped part inside the system -or at least trapped until the life-time of the space-time configuration.

- The transition or survival probabilities calculated inside the system are dominated by the presence of a peculiar spectrum of resonances known in literature as Pollicott-Ruelle eigenvalues.

We conclude that the problem considered can be only an example of a new interesting physics regime in which chaotic effects, quantum mechanics and general relativity cannot be neglected at the same time. In fact, the space-temporal Sinai billiard cannot be considered as a classical (gravitational) newtonian system while it induces chaotic behaviors to quantum particles scattered on it. So that, the space-temporal Sinai billiard is an example of a quantum chaos problem in General Relativity. Finally, we mention that in recent papers, we suggested that this regime can lead to a reinterpretation of the quantum black hole nature and its information paradoxes. In particular, we suggested that black holes are a superposition of a large number of horizonless naked singularities [17, 18, 19]. In other words, the Penrose diagram of a black hole is an approximate superposition of a large number of Penrose diagrams of singular geometries. But we have to admit that our reinterpretation of black holes remains speculative and not enough quantitative. We hope that future progresses on quantum chaos in general relativity could be helpful in order to understand black hole physics.

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