LOCALIZATIONS OF MORAVA E-THEORY AND DEFORMATIONS OF FORMAL GROUPS

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Abstract. We study the relationship between the transchromatic localizations of Morava E-theory, \(L_{K(n-1)}E_n\), and formal groups. In particular, we show that the coefficient ring \(\pi_0L_{K(n-1)}E_n\) has a modular interpretation, representing deformations of formal groups with certain extra structure, and derive similar descriptions of the cooperations algebra and \(E_{n-1}\)-homology of this spectrum. As an application, we show that \(L_{K(1)}E_2\) has exotic \(E_\infty\) structures not obtained by \(K(1)\)-localizing the \(E_\infty\) ring \(E_2\).

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1. Introduction

Stable homotopy theory is concerned with the study and construction of cohomology theories. Many interesting cohomology theories are complex orientable, meaning that they can be equipped with a system of Chern classes for complex vector bundles; as it happens, this data is neatly encoded in a purely algebraic object known as a formal group. The relation between formal groups and stable homotopy theory in general has been an enduring theme over the last fifty years. It was first sounded by Quillen, who proved in [Qu69] that the complex cobordism theory \(MU\), which carries the universal theory of Chern classes, also carries the universal formal group law. The Landweber exact functor theorem [La76], in a deep baritone, showed that cohomology theories could be constructed out of formal groups, and this version of the theme has been trumpeted again by the modern theory of topological modular forms [TMF14] and its higher generalizations [BL10]. Others like Morava and Ravenel took up the call by relating the height filtration on formal groups to the \(K(n)\)-local filtration on the stable homotopy category [Mora85, Rav04], and we now know that the classification of formal groups is linked to deep structural properties of the stable homotopy category as a whole [DHS88, HSm98].

The theme has found its clearest expression in the study of Morava E-theory. This is a complex oriented cohomology theory constructed from the algebraic geometry of deformations of formal groups. In [LT66], Lubin and Tate proved that a height \(n\) formal group \(\Gamma\) over a perfect, characteristic \(p\) field \(k\) has a universal deformation which lives over the ring \(Wk[[u_1, \ldots, u_{n-1}]]\). The parameters \(u_i\) control the height of the deformation: for example, inverting \(u_{n-1}\) forces the height to be at most \(n - 1\). By the theorem of Goerss, Hopkins,
and Miller \cite{GH04}, there is an essentially unique complex oriented, $K(n)$-local, $\mathcal{E}_\infty$ ring spectrum, called Morava $E$-theory $E = E(k, \Gamma)$, with
\[ \pi_* E = Wk[[u_1, \ldots, u_{n-1}]]u_1^{\pm 1}, |u_1| = 2, |u_i| = 0, \]
and with formal group the universal deformation defined by Lubin and Tate.

This theorem suggests that the relationship between stable homotopy theory and the algebraic geometry of formal groups is extremely close when localized at a single prime and height. In particular, basically all topological facts about $E$-theory should be expressible in terms of formal groups. For example:

1. The profinite group $\mathbb{G}_n$ of automorphisms of the field $k$ and the formal group $\Gamma$ acts on the Lubin-Tate ring, and this extends to an action of $\mathbb{G}_n$ on $E_n$ by $\mathcal{E}_\infty$ maps. By a theorem of Devinatz and Hopkins \cite{DH04}, and see Theorem 3.7 in this document,
\[ \pi_* L_{K(n)}(E^{n(s+1)}) \simeq \text{Hom}_{cts}(\mathbb{G}_n^x, \pi_* E_n). \]
This means that the $K(n)$-local $E_n$-based Adams spectral sequence for the sphere takes the form
\[ E_2^{st} = H^s(\mathbb{G}_n, \pi_1 E_n) \Rightarrow \pi_{t-s} L_{K(n)} S. \]
Moreover, $L_{K(n)} S$ is the homotopy fixed points of $\mathbb{G}_n$ acting on $E_n$, in a sense described by \cite{DH04}. This also means that the $E_n$-comodule structure on the completed $E$-homology of a space or spectrum is just a continuous $\mathbb{G}_n$-action.

2. The completed $E$-homology of a $K(n)$-local $\mathcal{E}_\infty$ ring spectrum carries power operations, which are parametrized by isogenies of deformations of the formal group $\Gamma$ \cite{AHS04, Re09}.

3. The $E$-theory of Eilenberg-MacLane spaces for abelian groups can be described in terms of exterior powers of the universal deformation of $\Gamma$ \cite{Pe11, HL13}.

4. For nonabelian groups $G$, there is a character map from $E^0 BG$ with ‘image in a ring of ‘generalized class functions’ on conjugacy classes of $n$-tuples of commuting elements of $p$-power order in $G$, which becomes an isomorphism after base changing to a ring parametrizing certain level structures on formal groups \cite{HKR00, Sta13}.

Of the transchromatic objects straddling heights $n-1$ and $n$, one of the most basic is the $K(n-1)$-localization of a height $n$ $E$-theory, known in this document as $L_{K(n-1)} E_n$ or just $LE$. This is even periodic, with
\[ LE_0 = Wk[[u_1, \ldots, u_{n-1}]]u_1^{\pm 1}[u_{n-1}^{-1}](p, u_1, \ldots, u_{n-2}] \cong Wk((u_{n-1}))_p[u_1, \ldots, u_{n-2}], \]
a complete local ring with residue field $k((u_{n-1}^{-1}))$, over which there is a naturally defined height $n-1$ formal group $\mathbb{H} = \mathbb{G}_n \otimes k((u_{n-1}^{-1}))$. This looks very much like the height $n-1$ $E$-theory associated to $\mathbb{H}$ – the only problem being that the field $k((u_{n-1}))$ is not perfect, so that the Lubin-Tate theorem does not apply.

This article has two goals. The first is to clarify the relationship between $LE$ and formal groups. As we prove, the ring $LE_0$ classifies deformations of $\mathbb{H}$ together with certain extra structure, which can be briefly described as follows.

**Theorem 1.1** (Theorem 4.10). Continuous ring homomorphisms $LE_0 \to R$, for a complete local ring $R$, correspond to deformations of $\mathbb{H}$ over $R$, together with a choice of last Lubin-Tate coordinate,
\[ j : Wk((u_{n-1}))_p^{\pm} \to R. \]

We construct similar modular interpretations of other invariants of $LE$, namely the completed $E_{n-1}$-homology
\[ (E_{n-1})_* LE = \pi_* L_{K(n-1)}(E_{n-1} \wedge LE) = \pi_* L_{K(n-1)}(E_{n-1} \wedge E_n) \]
and the completed cooperations algebra

\[ \hat{L}E \wedge LE = \pi_\ast L_{K(n-1)}(E_n \wedge LE) = \pi_\ast L_{K(n-1)}(E_n \wedge E_n). \]

Like \( LE \), these rings are even periodic and it suffices to describe them in degree 0.

**Theorem 1.2** (Proposition 5.11, Theorem 5.4). Continuous maps \((E_n^{\wedge n-1})_0 LE \to R\), for complete local rings \( R \), represent pairs of a deformation of \( \mathbb{H} \) and a deformation of the height \( n-1 \) formal group \( \Gamma \) over \( R \), together with an isomorphism of these formal groups over \( R/\mathfrak{m} \), and a choice of last Lubin-Tate coordinate,

\[ j : Wk((u_{n-1}))_p^{\wedge} \to R. \]

Continuous maps \( LE_0^{\wedge} LE \to R \), for complete local rings \( R \), represent pairs of deformations of \( \mathbb{H} \) over \( R \), pairs of last Lubin-Tate coordinates

\[ j,j' : Wk((u_{n-1}))_p^{\wedge} \to R, \]

and an isomorphism of the deformations over \( R/\mathfrak{m} \).

Interestingly, the “last Lubin-Tate coordinate” appearing in these theorems exists mostly separately from the formal group deformation data, a fact which is useful in doing constructions with these objects.

The second goal is to study power operations and \( E_\infty \) structures on the spectra \( LE \). Here we restrict ourselves to \( n = 2 \) and \( n-1 = 1 \), so that we can take advantage of the obstruction theory defined in [GH05], which allows one to construct \( K(1) \)-local \( E_\infty \) ring spectra out of power operations data known as a \( \theta \)-algebra structure. We prove:

**Theorem 1.3** (Theorem 6.19, Corollary 6.20). There are \( E_\infty \) structures on \( L_{K(1)}E_2 \) not obtained by \( K(1) \)-localizing the unique \( E_\infty \) structure on \( E_2 \).

Besides serving as an instructive calculation in \( K(1) \)-local obstruction theory, this result, and the methods used to prove it, point the way towards studying the transchromatic behavior of highly structured ring spectra and their power operations. In particular, these exotic \( E_\infty \)-algebras are \( K(1) \)-localizations of \( K(2) \)-local spectra, and are \( E_\infty \)-algebras, but only one of them – the \( K(1) \)-localization of the canonical \( E_\infty \)-algebra \( E_2 \) – is a \( K(1) \)-localization of a \( K(2) \)-local \( E_\infty \)-algebra. It is likely that, as is the case here, \( K(n) \)-local power operations generally satisfy integrality conditions that can be distorted on their \( K(n-1) \)-localizations. In future work, the author hopes to prove the analogous statement at all heights \( n \).

1.1. **Other approaches.** Before outlining the paper, let’s highlight some of the difficulties afoot in this transchromatic situation, and compare the approach taken here with some of the other ones in the literature. Again, \( E_0 \) carries the universal deformation of a height \( n \) formal group \( \Gamma \) over a finite field \( k \), and roughly speaking, passing to the \( K(n-1) \)-localization should be equivalent to restricting to deformations which are height exactly \( n-1 \). But it’s unclear how exactly to think of these objects as deformations of \( \Gamma \), since \( \Gamma \) itself is not a deformation of this type. From an algebraic point of view, the ring map \( E_0 \to LE_0 \) is analogous to the map

\[ \mathbb{Z}_p[[x]] \to \mathbb{Z}_p((x))_p^{\wedge}. \]

This map is not continuous with respect to the maximal ideal topologies on the two rings, and so does not induce a map

\[ \text{Spf} \mathbb{Z}_p[[x]] \leftarrow \text{Spf} \mathbb{Z}_p((x))_p^{\wedge}, \]

one is somehow trying to take the complement of the unique point of \( \text{Spf} \mathbb{Z}_p[[x]] \).
One possible approach is to treat $\mathbb{Z}_p((x))^\wedge_p$ as a topological ring such that the above map is continuous; that is, a basis of neighborhoods of 0 is given by $(p,x)^r\mathbb{Z}_p[[x]]$. This is a topology with comparatively many open sets. The subring $\mathbb{Z}_p((x))$ is open, and a continuous map $\mathbb{Z}_p((x)) \to R$ is simply a continuous map $\mathbb{Z}_p[[x]] \to R$ sending $x$ to a unit; on the other hand, $\mathbb{Z}_p((x))_p$ is not dense in $\mathbb{Z}_p((x))^\wedge_p$, so that some choices with no simple description are needed to define maps $\mathbb{Z}_p((x))^\wedge_p \to R$. Torii uses this topology to study localizations of $E$-theory $[To11]$.

Another approach is to treat the maximal ideal topology and the topology coming from $\mathbb{Z}_p[[x]]$ as coexisting simultaneously on $\mathbb{Z}_p((x))^\wedge_p$. One way to do this is to treat $\mathbb{Z}_p((x))^\wedge_p$ as a pro-object which is the limit of the rings $\mathbb{Z}/p^n((x))$, each of which carries the $x$-adic topology. This idea is explored by [MGPS], who develop a theory of “pipe rings” based on ideas from the theory of higher local fields in number theory [Morr12, Ka00], and prove analogous theorems about $LE$ in this context. There are some serious difficulties in the algebraic geometry that have so far presented obstacles to more widespread adoption of this program.

We remark that some of the results here have pipe parallels: for example, just as $LE_0$ represents augmented deformations of the formal group $\mathbb{H}$ as a complete local ring, it also does as a pipe ring. Additionally, we will use some basic concepts from pipe theory to prove results about $\theta$-algebras in section 6.

Third, we should mention the recently introduced framework of condensed [CS] or py-knotic [H] mathematics, which replaces topological groups, rings, and so on with better-behaved categories of sheaves on the site of compact Hausdorff spaces. The author believes that this framework may present a way around some of the difficulties in the theory of pipe rings, but this has not been worked out.

In this paper we take the fourth and simplest approach, which is to think of $LE_0$ as a topological ring with its maximal ideal topology (for example, $\mathbb{Z}_p((x))_p^\wedge$ carries the $p$-adic topology). Thus a continuous map $\mathbb{Z}_p((x))_p^\wedge \to R$, for some complete local ring $R$, descends to a map $\mathbb{F}_p((x)) \to R/\mathfrak{m}$ which may not have any nice continuity properties; note that $\mathbb{F}_p((x))$ has infinite transcendence degree over $\mathbb{F}_p(x)$, so that such a map is not determined by the image of $x$. Of course, the multitude of such maps is part of the reason for the multitude of $\mathcal{E}_\infty$ structures found in Theorem 6.19.

Let us finally comment that one can also view $LE_0$ as carrying not a formal group but a $p$-divisible group. This is the result of base changing the $p$-power torsion subgroups of $G^u$ to $LE_0$; it fits into an exact sequence

$$0 \to G^{for} \to G \to G^{et} \to 0$$

where $G^{for}$ is the formal group of $LE$, and $G^{et}$ is a Galois twist of a constant group scheme of the form $\mathbb{Q}_p/\mathbb{Z}_p$. Homotopy theorists have long thought that this $p$-divisible group structure should manifest itself in transchromatic phenomena, in particular due to the use of $p$-divisible groups of abelian varieties in defining topological automorphic forms [Lu09, Lu18a, Lu18b, BL10]; the best work on this subject is still [Sta13], which uses $p$-divisible groups like these to produce character maps.

Again, we take a simpler approach here, forgetting about the étale part of this $p$-divisible group and only considering the deformation of the formal group $\mathbb{H}$.

1.2. Outline of the paper. There are two background sections: section 2 concerns topics from algebraic geometry (particularly concerning formal groups and deformation theory), and section 3 discusses Morava $E$-theory. For purposes of comparison, we review the Devinatz-Hopkins calculation (as Theorem 3.7) of the completed cooperations coalgebra $E^\wedge_4 E$. 

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This text discusses various approaches to understanding topological objects, particularly in the context of algebraic geometry and homotopy theory. It references several works and authors, indicating a deep exploration into the relationships between different mathematical structures. The text highlights the use of pipe rings and condensed mathematics as frameworks that may provide new insights into the study of $E_0$ and its deformations. It also touches on the connection between $p$-divisible groups and transchromatic phenomena, showing how these concepts might be used to understand the complex behavior of $E$-theory.
In section 4, we describe basic facts about $LE$, and prove (Theorem 4.10) that its coefficient ring represents so-called augmented deformations of a height $n - 1$ formal group.

In section 5, we describe the cooperations $LE^*LE$ (Proposition 5.11), and the height $n - 1$ $E$-theory $(E_{n-1})^*LE$ (Theorem 5.4).

The final section 6 concerns power operations and $E_\infty$ structures, and works specifically with $L_{K(1)}E_2$. After a detour into the theory of $\theta$-algebras, the existence of exotic $E_\infty$ structures on this spectrum is proved in Theorem 6.19.

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1.4. Notation. For the sake of reference, here is some notation introduced elsewhere in the paper:

- $k$ is a field of characteristic $p$, generally perfect, often even finite. $W$ denotes the Witt vectors functor. $A$ is the completed Laurent series ring $Wk((u_{n-1}))^\wedge_p$.
- $\text{CLN}$ is the category of complete local noetherian rings and continuous ring homomorphisms – for $\text{CLN}_A$, see Definition 2.1. $\text{Gpd}$ is the category of groupoids. Other categories are written in sans-serif, but typically identifiable from their names.
- $\Gamma_n$ (sometimes just $\Gamma$) is a formal group over $k$ of finite height $n$.
- $E = E_n = E(k, \Gamma)$ is the Morava $E$-theory for $(k, \Gamma)$; this is even periodic, with periodicity class $u$ in degree 2 and $\pi_0E$ non-canonically isomorphic to $Wk[[u_1,\ldots,u_{n-1}]]$. $LE$ is the localization $L_{K(n-1)}E_n$. At a certain point, $F$ is used for $E_{n-1}$, and $K$ for $E_1$.
- $BP$ is the Brown-Peterson spectrum with $BP_*=\mathbb{Z}(p)[v_1,v_2,\ldots]$, where $|v_1|=2(p^i-1)$; the map $BP \to E$ sends $v_n \mapsto u^p, v_i \mapsto 0$ for $i > n$, and $v_i \mapsto u^{p^i-1}u_i$ for $i < n$. $I_n$ is the ideal $(p,v_1,\ldots,v_{n-1})$, or its image in any $BP_*$-module.
- Note that like $E^*_E$ denotes completed homology $\pi_*L_{K(n)}(E \wedge E)$. Which $n$ is intended varies, but is generally clear from context.

- $G^u$ is the universal deformation formal group of $\Gamma$, defined over $E_0$. In particular, there is a canonical isomorphism $G^u \otimes E_0 k \cong \Gamma$. We write $H$ for the height $n-1$ formal group $G^u \otimes k((u_{n-1}))$. $G$ will generally be used for other formal groups. If $f: R_1 \to R_2$ is a ring map and $G$ is a formal group over $R_1$, then $f^*G$ is its base change over $R_2$ – the upper star because this is the pullback of $G \to \text{Spec} R_1$ along $\text{Spec} R_2 \to \text{Spec} R_1$.
- We will write $\text{Def}_G$ for the deformation functor of $\Gamma$, and $\text{Def}_{G}^{\text{aug}}$ for the functor of deformations of $H$ augmented with $\Lambda$-algebra structure; both are defined more explicitly below.

In section 6, $\psi^p$ and $\theta$ are certain operations on $\theta$-algebras.
2. Background on deformation theory

In this introductory section, we review some facts from deformation theory, in particular the Lubin-Tate theorem on deformations of formal groups.

2.1. Witt vectors and Cohen rings. We will begin by presenting a deformation-theoretic point of view on the Witt vectors of fields.

Definition 2.1. Let CLN be the category of complete noetherian local rings and continuous maps, where a ring $R$ with maximal ideal $m$ is equipped with its $m$-adic topology. More generally, if $A$ is a ring equipped with a chosen maximal ideal $I$, let $\text{CLN}_A$ be the category whose objects are complete noetherian local $A$-algebras $R$ such that the structure map $A \to R$ sends $I$ into the maximal ideal of $R$, and whose morphisms are continuous $A$-algebra maps.

We will not define the Witt vectors, but urge the reader to consult [Hes08, Rab14, Se79] for more information. Let’s recall that if $R$ is a ring, its Witt vectors $W(R)$ are another ring equipped with:

- a Frobenius map $F : W(R) \to W(R)$, which is a ring homomorphism;
- a Verschiebung map $V : W(R) \to W(R)$, which is additive and satisfies $V(xF(y)) = V(x)y$ and $FV = p$;
- a multiplicative lift, which is a group homomorphism $R^\times \to W(R)^\times$ written $x \mapsto [x]$.

We are mainly concerned with the Witt vectors of perfect fields, which are described by the following statement.

Proposition 2.2 (cf. [Rab14]). If $k$ is a perfect field of characteristic $p$, then $W(k)$ is a $p$-torsion-free complete local ring with maximal ideal $VW(k) = pW(k)$ and residue field $k$. Every element of $W(k)$ has a unique expansion of the form $\sum_{n=0}^{\infty} p^n [b_n]$, with $b_n \in R$.

The Witt vectors of a perfect characteristic $p$ field enjoy a universal property, which we can describe as follows.

Theorem 2.3. Let $k$ be a perfect field of characteristic $p$ and let $R$ be a ring which is complete with respect to an ideal $I$ that contains $p$. Then for any map $i : k \to R/I$, there exists a unique continuous map completing any diagram of the form

$\begin{array}{c}
Wk \xrightarrow{\alpha} R \\
k \xrightarrow{i} R/I.
\end{array}$

Proof. This proof originally goes back to Cartier, and one should consult [Se79, II.4-6]. Recall that elements of $Wk$ for $k$ perfect can be uniquely written in the form $\sum [a_n] p^n$, where $[a_n]$ is the Teichmüller lift of $a_n \in k$. The idea is that there is a unique multiplicative lift $\tau : k^\times \to R^\times$. One is then forced to send $\sum [a_n] p^n$ to $\sum \tau(a_n) p^n$, which converges by completeness of $R$.

Regard $k$ as a subring of $R/I$. For each $a \in k$, define

$U_n(a) = \{ x^{p^n} : x \in R, \ x \equiv a^{p^{-n}} \pmod{I} \}.$

Here $a^{p^{-n}}$ is the unique $p^n$th root of $a$ in $k$. We have $U_{n+1}(a) \subseteq U_n(a)$. Moreover, if $x^{p^n}$ and $y^{p^n}$ are elements in $U_n(a)$, then $x \equiv y \pmod{I}$, and thus $x^{p^n} \equiv y^{p^n} \pmod{P^{n+1}}$ using the binomial theorem and the fact that $p \in I$. By completeness of $R$, there is a unique element in $\bigcap_{n \geq 0} U_n(a)$. Call this $\tau(a)$. 

One now observes that $\tau(a^p) = \tau(a)^p$, and that $\tau$ is the unique section $k^\times \to R^\times$ with this property. Indeed, if $\tau'$ also has this property, then

$$\tau'(a) = \tau'(a^{p^{-n}}) \in U_n(a)$$

for all $n$, so $\tau'(a) = \tau(a)$. Thus, there is at most one multiplicative section. But $\tau$ is also multiplicative, because $U_n(a) \cdot U_n(b) \subseteq U_n(ab)$. □

Remark 2.4. Taking $I$ to be the maximal ideal $m$ of a complete local noetherian ring $R$, one can see this as a trivial case of deformation theory. Indeed, define the functor of deformations of nothing, $\text{Def} : \text{CLN} \to \text{Sets}$, by

$$\text{Def}(R) = \{i : k \to R/m\}.$$  

Then we have proved that $\text{Def}(R) \cong \text{Hom}_{\text{cts}}(Wk, R)$.

However, the above theorem proves slightly more. A map $k \to R/I$ gives deformations of nothing over numerous complete local rings, namely the completions of the localizations of $R$ at maximal ideals containing $I$. The theorem implies that these deformations of nothing assemble over $\text{Spf } R$ to give a unique deformation of nothing over everything, which the reader must admit is really something.

If $k$ is a non-perfect characteristic $p$ field, its ring of Witt vectors is worse-behaved. There is still a surjection $w_0 : W(k) \to k$ with kernel $VW(k)$, but this ideal need not be principal, and $V(1) \neq p$. In addition, the universal property of Theorem 2.3 is not satisfied. Instead, one can construct rings having a weak version of this universal property. (Topologists wishing to know more should also consult the last section of [AMS98].)

Definition 2.5. A Cohen ring for a characteristic $p$ field $k$ is a complete discrete valuation ring with residue field $k$ and uniformizer $p$.

Example 2.6. The Witt vectors of a perfect field $k$ are a Cohen ring for $k$. For an imperfect field, we have $px = V(1)x = V(F(x))$, so the set of multiples of $p$ is in general a proper subset of the maximal ideal $VWk$. Thus, the Witt vectors are not a Cohen ring in this case.

Example 2.7. If $k$ is perfect, the completed Laurent series ring $\Lambda = Wk((x))_p$ is a Cohen ring for the Laurent series field $k((x))$.

Theorem 2.8 ([Stacks, Tag 0323]). Every characteristic $p$ field $k$ has a Cohen ring. If $C$ is a Cohen ring for the characteristic $p$ field $k$, then for every $n$, $\mathbb{Z}/p^n\mathbb{Z} \to C/p^nC$ is formally smooth.

Corollary 2.9. If $C$ is a Cohen ring for $k$, $R$ is a ring which is complete with respect to an ideal $I$ containing $p$, and $i : k \to R/I$ is an inclusion, then there exists a map completing the diagram

$$
\begin{array}{ccc}
C & \longrightarrow & R \\
\downarrow & & \downarrow \\
k & \longrightarrow & R/I.
\end{array}
$$

Proof. Starting with the map $C/p = k \to R/I$, we use formal smoothness to inductively construct lifts

$$
\begin{array}{ccc}
\mathbb{Z}/p^n\mathbb{Z} & \longrightarrow & R/I^n \\
\downarrow & & \downarrow \\
C/p^nC & \longrightarrow & R/I^{n-1}
\end{array}
$$
along the square-zero extensions \( R/I^n \to R/I^{n-1} \). By completeness, these assemble to a map \( C \to R \) with the desired property. \( \square \)

**Corollary 2.10.** Any two Cohen rings for \( k \) are (non-uniquely) isomorphic.

**Proof.** Let \( C_1 \) and \( C_2 \) be Cohen rings for \( k \). By the previous corollary, there exists a map \( f : C_1 \to C_2 \) reducing to the identity on \( k \). If \( f(x) = 0 \), then \( x \) must be divisible by \( p \), and writing \( x = px_1 \) and proceeding inductively, we see that \( x \) is divisible by all powers of \( p \), so is zero by completeness. Thus, \( f \) is injective. If \( y \in C_2 \), then there is an \( x_0 \in C_2 \) such that \( f(x_0) - y \) is divisible by \( p \), so is equal to some \( p y_1 \). Proceeding inductively again and using completeness, we see that \( f \) is surjective. \( \square \)

2.2. Formal groups and their deformations.

**Definition 2.11.** By a formal group over a ring \( R \), we mean a connected one-dimensional smooth commutative formal group scheme over \( R \), which admits a global coordinate. If \( R \) has a topology (for example, if \( R \) is a complete local ring), then the formal scheme structure on any formal group is taken to be compatible with the topology. Thus, the underlying formal scheme of any formal group over \( R \) is non-canonically isomorphic to \( \text{Spf} \ R[[x]] \).

Given a ring homomorphism \( R \to R' \) (continuous, if the rings involved have topologies), and a formal group \( \Gamma \) over \( R \), we write \( \Gamma \otimes_R R' \) for the base change of \( \Gamma \) to \( R' \).

Formal groups over a field of characteristic \( p \) are stratified by an invariant known as height, which can be briefly defined as follows. Choosing a coordinate \( x \) for the formal group \( \Gamma \) and writing \( [p] \Gamma(x) \) for the power series that expresses formal multiplication by \( p \), one finds that

\[
[p] \Gamma(x) = ux^p + \cdots
\]

for some \( 0 < n \leq \infty \) independent of the choice of \( x \). This \( n \) is the **height** of \( \Gamma \). We will only be concerned with formal groups of finite height (i.e., where \( [p] \Gamma(x) \) is not identically 0) in this paper.

We write \( \mathcal{M}_{fg} \) for the moduli of formal groups, which is an algebraic stack. For further background on formal groups, see [Rav04, Appendix 2] and [Go08].

**Definition 2.12.** Let \( \Gamma \) be a formal group over \( k \) and \( R \) a complete local noetherian ring with maximal ideal \( m \). A **deformation** of \( \Gamma \) over \( R \) is a triple

\[
(\mathcal{G}, \overline{\iota}, \alpha),
\]

where

- \( \mathcal{G} \) is a formal group over \( R \),
- \( \overline{\iota} \) is an inclusion \( k \to R/m \),
- and \( \alpha \) is an isomorphism \( \Gamma \otimes_k R/m \to \mathcal{G} \otimes_R R/m \).

An **isomorphism** of deformations of \( \Gamma \) over \( R \), \( \phi : (\mathcal{G}_1, \overline{\iota}_1, \alpha_1) \to (\mathcal{G}_2, \overline{\iota}_2, \alpha_2) \) is

- the condition that \( \overline{\iota}_1 = \overline{\iota}_2 \),
- and an isomorphism \( \phi : \mathcal{G}_1 \to \mathcal{G}_2 \) of formal groups over \( R \),
- such that the square

\[
\begin{array}{ccc}
\Gamma \otimes_k R/m & \xrightarrow{\alpha_1} & \mathcal{G}_1 \otimes_R R/m \\
1 & \downarrow & \phi \\
\Gamma \otimes_k R/m & \xrightarrow{\alpha_2} & \mathcal{G}_2 \otimes_R R/m
\end{array}
\]

commutes.
If $\Gamma$ is a formal group over $k$, then let
\[
\text{Def}_\Gamma : \text{CLN} \to \text{Gpd}
\]
be the functor that sends $R$ to the groupoid of deformations of $\Gamma$ over $R$ and their isomorphisms. More generally, if $A$ is a ring with a maximal ideal $I$ such that $A/I = k$, and $\Gamma$ is a formal group over $k$, then let
\[
\text{Def}_\Gamma^A : \text{CLN}_A \to \text{Gpd}
\]
be the functor that sends an $A$-algebra $R$ to the groupoid of deformations $(\mathbb{G}, \tilde{i}, \alpha)$ of $\Gamma$ over $R$ such that $\tilde{i} : A/I \to R/m_R$ is the reduction of the $A$-algebra structure map, and their isomorphisms.

The Lubin-Tate theorem is originally stated in the following form \[LT66\]:

**Theorem 2.13** (Lubin-Tate). Let $\Gamma$ be a formal group of finite height $n$ over a field $k$ of characteristic $p > 0$, let $A$ be a ring with an ideal $I$ such that $A/I \cong k$, and suppose given a diagram
\[
\begin{array}{ccc}
A & \xrightarrow{i} & R \\
\downarrow & & \downarrow \\
k & \xrightarrow{\tilde{i}} & R/m
\end{array}
\]

where $R$ is a complete local noetherian ring with maximal ideal $m$. Then there is a formal group $\mathbb{G}^u$ over $A[[u_1, \ldots, u_{n-1}]]$ such that, for any formal group $\mathbb{G}$ over $R$ with an isomorphism
\[
\alpha : \Gamma \otimes_k R/m \xrightarrow{\sim} \mathbb{G} \otimes_R R/m,
\]
there is a unique continuous $A$-algebra map $f : A[[u_1, \ldots, u_{n-1}]] \to R$ and a unique isomorphism
\[
\mathbb{G}^u \otimes A[[u_1, \ldots, u_{n-1}]] \xrightarrow{\sim} \mathbb{G}
\]
that reduces to $\alpha$ over $R/m$.

In the language defined above, we can restate this as follows.

**Corollary 2.14** (Lubin-Tate theorem, equivalent form). Let $\Gamma$ be a formal group of finite height $n$ over a field $k$ of characteristic $p > 0$. Then the functor $\text{Def}_\Gamma^A$ on $\text{CLN}_A$ is represented by $A[[u_1, \ldots, u_{n-1}]]$. In other words, for $R \in \text{CLN}_A$, the groupoid $\text{Def}_\Gamma^A(R)$ is naturally equivalent to the discrete groupoid $\text{Maps}_{A,cts}(A[[u_1, \ldots, u_{n-1}]], R)$.

Typically, one is interested in representing the functor $\text{Def}_\Gamma$ on the category $\text{CLN}$, not on the categories $\text{CLN}_A$. Using Theorem 2.13 this version of the theorem can be proved if the field $k$ is perfect (and most often, this refinement is the one used by homotopy theorists).

**Corollary 2.15** (Lubin-Tate theorem for perfect fields). Let $\Gamma$ be a formal group of finite height $n$ over a perfect field $k$ of characteristic $p > 0$. Then the functor $\text{Def}_\Gamma$ on $\text{CLN}$ is represented by $Wk[[u_1, \ldots, u_{n-1}]]$.

**Proof.** Given $(\mathbb{G}, \tilde{i}, \alpha) \in \text{Def}_\Gamma(R)$, let $i : Wk \to R$ be the unique continuous lift over $\tilde{i}$ guaranteed by Theorem 2.13. The map $i$ makes $R$ an object of $\text{CLN}_{Wk}$, and $(\mathbb{G}, i, \alpha)$ an object of $\text{Def}_\Gamma^{Wk}(R)$. This proves that the inclusion map $\text{Def}_\Gamma^{Wk}(R) \to \text{Def}_\Gamma(R)$ is essentially surjective, but $\text{Def}_\Gamma^{Wk}(R)$ is a full subgroupoid of $\text{Def}_\Gamma(R)$ by definition. So the inclusion is an equivalence, and on the other hand we have
\[
\text{Maps}_{Wk,cts}(Wk[[u_1, \ldots, u_{n-1}]], R) = \text{Maps}_{cts}(Wk[[u_1, \ldots, u_{n-1}]], R),
\]
again by Theorem 2.13. \qed
Remark 2.16. The ring \( Wk[[u_1, \ldots, u_{n-1}]] \) is called the **Lubin-Tate ring** for \((k, \Gamma)\). As a result of Corollary 2.15, it carries a deformation \((\G^u, 1, \alpha^u)\) which is a universal deformation of \(\Gamma\), in the sense that any other deformation \((\G, i, \alpha)\) over \(R \in \text{CLN}\) is uniquely isomorphic to the base change of \(\G^u\) along a unique map \(Wk[[u_1, \ldots, u_{n-1}]] \rightarrow R\).

In fact, this can be made fairly explicit. If \(\Gamma\) has the Honda formal group law over \(k\), with \(p\)-series \([p] \Gamma(x) = x p^n\) (for some chosen coordinate \(x\)), then we can choose a coordinate on \(G_u\) such that \([p] G_u(x) = p x + G_u u_1 x^p + G_u u_2 x^{p^2} + \cdots + G_u u_{n-1-1} x^{p^{n-1}} + G_u x^{p^n}\).

The isomorphism \(\alpha^u : \Gamma \sim \Gamma \otimes k\) matches these two coordinates.

Remark 2.18. There is a left action of \(\text{Aut}(k, \Gamma)\) on \(\text{Def}_\Gamma\), defined as follows. Given \((\G, i, \alpha) \in \text{Def}_\Gamma(R)\) and \((\tau : k \rightarrow k, g : \Gamma \rightarrow \tau^* \Gamma) \in \text{Aut}(k, \Gamma)\) (see Definition 2.17), define

\[
(\tau, g)(\G, i, \alpha) = (\G, i \circ \tau, \alpha g^{-1} : \Gamma \otimes_k R / \mathfrak{m} \xrightarrow{\alpha^{-1}} \Gamma \otimes_k R / \mathfrak{m} \xrightarrow{\alpha^i} \G \otimes_R R / \mathfrak{m}).
\]

Remark 2.19. Typically, in homotopy theory, \(k\) is taken to be a finite field containing \(\mathbb{F}_{p^n}\). Then all automorphisms of \(\Gamma\) are defined over \(k\), i.e., \(\text{Aut}_k(\Gamma) = \text{Aut}_{\mathbb{F}_{p^n}}(\Gamma)\). It follows that

\[\text{Aut}(k, \Gamma) = \text{Gal}(k / \mathbb{F}_p) \rtimes \text{Aut}_k(\Gamma)\].

Moreover, \(\text{Aut}_k(\Gamma)\) is a profinite group, isomorphic to the group of units in a maximal order of the division algebra of invariant \(1/n\) over \(Wk[1/p]\). One commonly writes \(S_n\) for \(\text{Aut}_k(\Gamma)\) and \(G_n\) for \(\text{Aut}(k, \Gamma)\) in this case.

2.3. Automorphisms of formal groups.

**Definition 2.17.** Let \(\Gamma\) be a height \(n\) formal group over a field \(k\). The **Morava stabilizer group** \(\text{Aut}(k, \Gamma)\) is the group of pairs \((\tau, g)\), where \(\tau : k \rightarrow k\) is an automorphism, and \(g\) is an isomorphism of formal groups over \(k\), \(g : \Gamma \rightarrow \tau^* \Gamma\).

Equivalently, one can write \((\tau, g) \in \text{Aut}(k, \Gamma)\) as a commutative square

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{\sim} & \Gamma \\
\downarrow & & \downarrow \\
\text{Spec } k & \xrightarrow{\tau^*} & \text{Spec } k \\
\end{array}
\]

where the horizontal maps are isomorphisms, the top one a (non-\(k\)-linear) isomorphism of formal groups. Then \(g : \Gamma \rightarrow \tau^* \Gamma\) is the map induced by the universal property of the pullback.

The composition law can be written

\[
(\tau_2, g_2)(\tau_1, g_1) = (\tau_2 \tau_1, \tau_2^* (g_1) g_2).
\]

The odd variance of this formula is a result of our choice to write \(\tau\) as a map of fields (that is, rings) and \(g\) as a map of formal groups (that is, formal schemes).

**Remark 2.18.** There is a left action of \(\text{Aut}(k, \Gamma)\) on \(\text{Def}_\Gamma\), defined as follows. Given \((\G, i, \alpha) \in \text{Def}_\Gamma(R)\) and \((\tau : k \rightarrow k, g : \Gamma \rightarrow \tau^* \Gamma) \in \text{Aut}(k, \Gamma)\) (see Definition 2.17), define

\[
(\tau, g)(\G, i, \alpha) = (\G, i \circ \tau, \alpha g^{-1} : \Gamma \otimes_k R / \mathfrak{m} \xrightarrow{\alpha^{-1}} \Gamma \otimes_k R / \mathfrak{m} \xrightarrow{\alpha^i} \G \otimes_R R / \mathfrak{m}).
\]

Remark 2.19. Typically, in homotopy theory, \(k\) is taken to be a finite field containing \(\mathbb{F}_{p^n}\). Then all automorphisms of \(\Gamma\) are defined over \(k\), i.e., \(\text{Aut}_k(\Gamma) = \text{Aut}_{\mathbb{F}_{p^n}}(\Gamma)\). It follows that

\[\text{Aut}(k, \Gamma) = \text{Gal}(k / \mathbb{F}_p) \rtimes \text{Aut}_k(\Gamma)\].

Moreover, \(\text{Aut}_k(\Gamma)\) is a profinite group, isomorphic to the group of units in a maximal order of the division algebra of invariant \(1/n\) over \(Wk[1/p]\). One commonly writes \(S_n\) for \(\text{Aut}_k(\Gamma)\) and \(G_n\) for \(\text{Aut}(k, \Gamma)\) in this case.
Now that we have described the automorphisms of a formal group over a field, we describe how they deform. While deformations of formal groups are controlled by Lubin-Tate parameters, isomorphisms of formal groups deform uniquely.

**Definition 2.20.** Let $G_1$ and $G_2$ be two formal groups over rings a ring $A$. The **moduli of isomorphisms** from $G_1$ to $G_2$ is the functor $\text{Iso}(G_1, G_2): \text{Alg}_A \to \text{Sets}$ given by

$$\text{Iso}(G_1, G_2)(B) = \{ \phi : G_1 \otimes_A B \to G_2 \otimes_A B \text{ an isomorphism of formal groups} \}.$$ 

**Theorem 2.21** ([Goerss-Hopkins-Miller]) If $\Gamma_1$ and $\Gamma_2$ are formal groups over a ring $A$, then $\text{Iso}(\Gamma_1, \Gamma_2)$ is an affine $A$-scheme. Moreover, if $G_1$ and $G_2$ are height $n$ formal groups over a field $k$ of characteristic $p$, then $\text{Iso}(G_1, G_2)$ is pro-étale over $k$.

We will use the following easy corollary repeatedly.

**Corollary 2.22.** Let $G_1$ and $G_2$ be height $n$ formal groups over a field $k$ of characteristic $p$, and let $R$ be a a $k$-algebra which is complete with respect to an ideal $I$. Then $\text{Iso}_R(G_1, G_2) \to \text{Iso}_{R/I}(G_1, G_2)$ is an isomorphism.

**Proof.** If $I$ is nilpotent, this is just the infinitesimal criterion of étaleness. In general, we write $R = \lim R/I^n$ and note that the moduli of isomorphisms commutes with limits. \qed

## 3. Background on homotopy theory

### 3.1. Morava $E$-theory

The Lubin-Tate rings can be realized in homotopy theory in an interesting way, as the coefficient rings of cohomology theories known as **Morava $E$-theory**.

**Theorem 3.1.** Let $\Gamma$ be a height $n$ formal group over a perfect field $k$ of characteristic $p$. There is a complex orientable, even periodic ring spectrum $E = E(k, \Gamma)$ such that $n_0 E = Wk[[u_1, \ldots, u_{n-1}]]$ and the formal group of $E$ (rescaled to degree zero) is the universal deformation of $\Gamma$.

One way to prove this is using the Landweber exact functor theorem, which applies to the Lubin-Tate rings because they are flat over the moduli of formal groups [Goerss-Hopkins-Miller].

There are also more homotopical constructions available. For instance, the Brown-Peterson spectrum $BP$ is a summand of $p$-local complex cobordism, with coefficient ring

$$BP_* = \mathbb{Z}(\rho)[v_1, v_2, \ldots], \quad |v_i| = 2(p^i - 1).$$

One can define a ring spectrum $BP(n)$ as the quotient of $BP$ by the ideal $(v_{n+1}, v_{n+2}, \ldots)$ [EKMM]. The Johnson-Wilson $E$-theory $E(n)$ is then $BP(n)[v_n^{-1}]$. This admits an étale extension in ring spectra, $\widehat{E(n)}$, with

$$\pi_*\widehat{E(n)} = \pi_*E(n)[u]/(u(p^n - 1) - v_n) = \mathbb{Z}(\rho)[u_1, \ldots, u_{n-1}][u^\pm 1]$$

where $|u| = 2$ and $|u_i| = 0$, and $u_i = v_i u_1^{1-p^i}$. Finally, the $K(n)$-localization of $\widehat{E(n)}$ is a Morava $E$-theory. Specifically, it is the Morava $E$-theory for the Honda formal group of height $n$ over $\mathbb{F}_p$, with

$$[p]_\Gamma(x) = x^{p^n}.$$ 

Both constructions lack a certain something. Both the Landweber exact functor theorem, and the process of quotienting ring spectra by ideals, only produce Morava $E$-theory as a homotopy commutative ring spectrum. More advanced obstruction theory techniques imply the following, for which one should see [GH04] and [Re98].

**Theorem 3.2** ([Goerss-Hopkins-Miller]). There is a unique $E_\infty$ ring spectrum, up to $E_\infty$ homotopy equivalence, whose underlying ring spectrum is $E(k, \Gamma)$. Moreover, the space of $E_\infty$ endomorphisms of this spectrum is homotopy equivalent to the discrete group $\text{Aut}(k, \Gamma)$. 

In other words, by insisting on more structure – the structure of an $E_\infty$ ring spectrum – we are able to realize the Lubin-Tate formal groups in stable homotopy theory in an essentially unique way. Thus, there is an unexpectedly tight correspondence between chromatic stable homotopy theory and the algebraic geometry of formal groups, so long as one works locally on both sides: Morava $E$-theories are $K(n)$-local, while the Lubin-Tate formal groups are universal deformations of formal groups over fields. In turn, this suggests that, as a general principle, all phenomena having to do with Morava $E$-theory can be described in terms of formal groups.

3.2. L-completeness and completed $E$-homology. The $E$-theory of a point – the Lubin-Tate ring – is a complete local ring. The $E$-theory spectrum itself also satisfies a completeness property, in that it is $K(n)$-local: this implies that, letting $I = (i_0, \ldots, i_{n-1})$ range over a cofinal sequence of $n$-tuples of integers such that the generalized Moore spectrum $S/(p^0, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$ is defined, the natural map

$$E \to \text{holim}_I E \wedge S/(p^0, \ldots, v_{n-1}^{i_{n-1}})$$

is an equivalence. One expects to be able to deduce completeness properties for the $E$-homology and $E$-cohomology functors as well. For example, for any spectrum $X$,

$$E^*X = \pi_* F(X, E) = \pi_* \text{holim}_I F(X, E \wedge S/(p^0, \ldots, v_{n-1}^{i_{n-1}})) = \pi_* \text{holim}_I F(X, E) \wedge S/(p^0, \ldots, v_{n-1}^{i_{n-1}}).$$

This is an $E_*$-module with a derived completeness property to be explained momentarily.

The $E$-homology groups $E_*X = \pi_*(E \wedge X)$ do not generally have this completeness property, because the smash product does not distribute over the homotopy limit. Instead, we work with completed $E$-homology,

$$E^\wedge_*X = \pi_* L_{K(n)}(E \wedge X).$$

This is not a homology theory in the Eilenberg-Steenrod sense. Nevertheless, it is often better behaved than uncompleted $E$-homology: for example, while $E_*E$ is quite complicated, $E^\wedge_*E$ is pro-free over $E$ and has the simple Hopf algebroid description $\text{Maps}_{cl}(G_n, E_*)$.

We now describe the derived completeness property of $E$-cohomology and completed $E$-homology, known as L-completeness. More information, and proofs of the results mentioned here, can be found in [GM92], [HSt99 Appendix A], and [BF15]. In practice, the distinction between L-completeness and classical completeness will not matter much here, as any L-complete module we will ever consider will be classically complete.

**Definition 3.3.** Let $R$ be a Noetherian ring and let $I$ be an ideal in $R$, generated by a regular sequence of length $n$. We write $M^\wedge$ for the $I$-adic completion of an $R$-module $M$, and $L_0M$ for the $s$th derived functor of $I$-adic completion applied to $M$. There are natural maps

$$M \to L_0M \to M^\wedge.$$

A module $M$ is **L-complete** if the natural map $M \to L_0M$ is an isomorphism, and (classically) **complete** if the map $M \to M^\wedge$ is an isomorphism.

Write $\text{Mod}^\wedge_R$ for the category of L-complete $R$-modules.

In practice, we will usually take $R = E_*$, $I = I_n = (p, u_1, \ldots, u_{n-1})$, and work with the degreewise versions of these properties for graded rings. The hypotheses on $R$ are less general than those in [GM92] (which weakens the Noetherian condition) and more general than those in [HSt99] and [BF15] (which require $R$ to be a local ring and $I$ its maximal ideal). However, the proofs in [HSt99] and [BF15] do not rely on $I$ being maximal.
The following proposition summarizes results from [GM92] and [HSt99].

**Proposition 3.4.**  
(1) \( \text{Mod}^\wedge_R \) is an exact subcategory of \( \text{Mod}_R \), which is closed under extensions, limits, and \( \lim^1 \) of towers.  
(2) There is an adjunction \( L_0: \text{Mod}_R \Rightarrow \text{Mod}^\wedge_R : i \), where \( i \) is the inclusion.  
(3) If \( M \) is free, or finitely generated, then the natural map \( M^\wedge \to L_0 M \) is an isomorphism.  
(4) If \( M \) is classically complete, it is \( L \)-complete.  
(5) In general, \( L_s M = 0 \) for \( s \geq n + 1 \), and there are natural exact sequences  
\[
0 \to \lim_{k}^1 \text{Tor}^R_{s+1}(R/I^k, M) \to L_s M \to \lim_{k} \text{Tor}^R_s(R/I^k, M) \to 0.
\]

**Proposition 3.5** ([HSt99] Corollary 2.3, Proposition 8.4). The functors \( E_*^\wedge \) and \( E_*^\wedge \) are naturally valued in the category \( \text{Mod}_{E_*}^\wedge \) of graded \( E_* \)-modules which are \( L \)-complete with respect to the ideal \( I_n = (p, u_1, \ldots, u_{n-1}) \).

**Proposition 3.6** ([HSt99] Proposition 8.4). If \( X \) is finite, then \( E_*^\wedge X = E_*X \). If \( E_*X \) is \( E_* \)-free, then \( E_*^\wedge X \) is its \( I_n \)-adic completion.

*Proof.* We repeat the proof from [HSt99] because it will be relevant later on. If \( X \) is finite, then \( E \wedge X \) is an iterated cofiber of suspensions of \( E \), so is already \( K(n) \)-local. For general \( X \), \( v_n \) is invertible on \( E \wedge X \), so we have  
\[
L_{K(n)}(E \wedge X) = \text{holim}_I E \wedge X \wedge S/(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}}).
\]
If \( E_*X \) is \( E_* \)-flat, then  
\[
\pi_*(E \wedge X \wedge S/(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}})) = E_*X/(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}}),
\]
and the transition maps in the inverse system of homotopy groups are surjective. Therefore,  
\[
\pi_*(\text{holim}_I E \wedge X \wedge S/(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}})) = \lim_{I} E_*X/(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}}) = (E^\wedge X)^{\wedge}.
\]

\qed

### 3.3. Cooperations for \( E \)-theory.

The completed \( E \)-homology \( E_*^\wedge X = \pi_*L_{K(n)}(E \wedge X) \) of a space or spectrum \( X \) is naturally a complete comodule for a coalgebra of cooperations \( E_*^\wedge X \). This has a surprisingly simple form. In this section, we write \( \mathbb{G}_n = \text{Aut}(k, \Gamma) \).

**Theorem 3.7** ([DHI04]). If \( E = E(k, \Gamma) \) where \( \Gamma \) is the height \( n \) Honda formal group over a perfect field \( k \) of characteristic \( p \), then there is a Hopf algebroid isomorphism  
\[
E_*^\wedge E \cong \text{Maps}_{cts}(\mathbb{G}_n, E_*),
\]
where \( \text{Maps}_{cts}(\mathbb{G}_n, E_*) \) is the \( E_* \)-algebra of continuous set maps \( \mathbb{G}_n \to E_* \).

As an immediate application, if \( X \) is a spectrum such that \( E_*^\wedge X \) is honestly \( I_n \)-complete, then the \( E_*^\wedge E \)-comodule structure on \( E_*^\wedge X \) is equivalently given by a continuous \( \mathbb{G}_n \) action. (If \( E_*^\wedge X \) is merely \( L \)-complete, then a comodule structure should instead be taken as the definition of “continuous action”.) Thus in the \( K(n) \)-local \( E \)-based Adams spectral sequence  
\[
E_2 = \text{Ext}^E_{E_*^\wedge}(E_*, E_*^\wedge X) \Rightarrow \pi_*L_{K(n)}X,
\]
we can rewrite the \( E_2 \) page as group cohomology \( H^*_c(\mathbb{G}_n, E_*^\wedge X) \).

For the sake of inspiring the arguments below, we give a proof of this theorem. Let \( \text{Alg}_{E_*} \) be the category of even periodic, \( I_n \)-adically complete \( E_* \)-algebras, and let \( \text{Alg}_{E_*}^\wedge \) be the category of \( I_n \)-adically complete \( E_0 \)-algebras. The proof will show that \( E_*^\wedge E \) and \( \text{Maps}_{cts}(\mathbb{G}_n, E_*) \) represent the same functor on \( \text{Alg}_{E_*}^\wedge \).

**Lemma 3.8.** The ring \( E_*^\wedge E \) is an object in \( \text{Alg}_{E_*}^\wedge \).
Proof. Since $E$ is Landweber exact,

$$E_* E = \pi_*(E \wedge E) = E_* \otimes_{BP_*} BP_* BP \otimes_{BP_*} E_*$$

is even periodic and flat as a left $E_*$-module. The theory $E \wedge E$ is $L_n$-local, which means that

$$L_{K(n)}(E \wedge E) = \lim_I E \wedge E \wedge S/I = \lim_I E \wedge E/I$$

where $I$ ranges over ideals $(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$ such that the associated generalized Moore spectrum exists. Again, the homotopy groups of the objects in the limit diagram are

$$E_*(E/I) = E_* \otimes_{BP_*} BP_* BP \otimes_{BP_*} E_*/I.$$ 

Thus, the maps in the diagram are surjective on homotopy groups, so that

$$\pi_* L_{K(n)}(E \wedge E) = \lim_I E_*(E/I) = (E_* \otimes_{BP_*} BP_* BP \otimes_{BP_*} E_*)_{I_n} \wedge *.$$

As $I_n$ and its powers are images of invariant ideals in $BP_* BP$, it doesn’t matter whether we complete with respect to the $I_n$ coming from the left or right $E_*$-module structure. Clearly, this is an object of $\text{Alg}_{E_*}^\wedge$.

\[ \square \]

Lemma 3.9. For $R_* \in \text{Alg}_{E_*}^\wedge$, pre-composition with the completion map $E_* E \to E_*^\wedge E$ induces an isomorphism

$$\text{Hom}_{\text{Alg}_{E_*}^\wedge}(E_*^\wedge E, R_*) \cong \text{Hom}_{E_*}(E_* E, R_*).$$

Moreover, for any map $E_*^\wedge E \to R_*$ in $\text{Alg}_{E_*}^\wedge$, the composition

$$E_* \xrightarrow{\eta} E_* E \to R_*$$

is also continuous.

Proof. Let $R_* \in \text{Alg}_{E_*}^\wedge$ and give $R_*$ the $I_n$-adic topology. As we saw in the proof of the previous lemma, $I_n$ is the image of an invariant ideal in $BP_* BP$. Thus, any map

$$f : E_* \otimes_{BP_*} BP_* BP \otimes_{BP_*} E_* \to R_*$$

extending the given map $E_* \to R_*$ has $f(I_n \cdot E_* E) \subseteq I_n R_*$. In particular, $R_*$ is also complete with respect to $I_n \cdot E_* E$, so that $f$ factors uniquely through the completion $E_*^\wedge E$. Moreover, $f(\eta_R(I_n \cdot E_*))$ is also in $I_n R_*$, so that the map $E_* \to R_*$ coming from the right unit is also continuous.

\[ \square \]

Let $\text{Alg}_{E_*}^\wedge, \text{loc}$ be the full subcategory of $R \in \text{Alg}_{E_*}^\wedge$ such that $R_0$ is complete local. Then the map $E_0 \to R_0$ classifies an object $(G, i, \alpha) \in \text{Def}_\Gamma(R_0)$.

Proposition 3.10. Let $R_* \in \text{Alg}_{E_*}^\wedge, \text{loc}$, and let $(G, i, \alpha)$ be the deformation of $\Gamma$ classified by $E_0 \to R_0$. Then the set of maps $\text{Hom}_{\text{Alg}_{E_*}^\wedge}(E_*^\wedge E, R_*)$ is naturally isomorphic to the set of pairs $(j, \gamma)$, where $j$ is a map $k \to R_0/\mathfrak{m}$, and $\gamma$ is an isomorphism of formal groups over $R_0/\mathfrak{m}$.

Proof. By Lemma 3.9

$$\text{Hom}_{\text{Alg}_{E_*}^\wedge}(E_*^\wedge E, R_*) \cong \text{Hom}_{E_*}(E_* E, R_*) \cong \text{Hom}_{E_0}(E_0 E, R_0).$$

Write $BPP$ for 2-periodic $BP$. Then we also have

$$E_0 E = E_0 \otimes_{BPP_0} BPP_0 BPP \otimes_{BPP_0} E_0.$$
The Hopf algebroid \((BPP_0, BPP_0 BPP)\) presents the moduli of \(p\)-local formal groups, so there is a pullback square of stacks

\[
\begin{array}{c}
\text{Spec } E_0 E \\
\downarrow j \\
\text{Spec } E_0 \\
\downarrow \\
\text{M}_{fg}.
\end{array}
\]

Again, Lemma 3.5 implies that, for any \(E_0\)-map \(E_0 E \to R_0\), the map \(E_0 \to R_0\) coming from the right unit is also continuous, and thus classifies a deformation. So we have a homotopy pullback of groupoids,

\[
\text{Hom}_{\text{Alg}_{E_*}^h}(E_*^h E, R_*) \longrightarrow \text{Def}_\Gamma(R_0) \longrightarrow \text{M}_{fg}(R_0).
\]

An object in the pullback is given by another deformation \((G', j, \beta) \in \text{Def}_\Gamma(R_0)\) and an isomorphism \(\phi : G \to G'\). An isomorphism in the pullback is an isomorphism \(\psi : (G', j, \beta) \to (G'', j, \delta)\) such that the obvious triangle involving \(\phi\) commutes. Now, there is an isomorphism of deformations \(\phi^{-1} : (G', j, \beta) \to (G, j, \beta \circ \phi^{-1})\); this is the only isomorphism from \((G', j, \beta)\) to a deformation whose underlying formal group is exactly \(G\). It follows that the connected components of the pullback groupoid are contractible, as expected, and correspond to pairs

\[(j, \beta : \Gamma \otimes^j R_0/m \xrightarrow{\sim} G \otimes R_0/m).
\]

Equivalently, they correspond to pairs

\[(j : k \to R_0/m, \gamma = \beta^{-1}\alpha : \Gamma \otimes^j R_0/m \xrightarrow{\sim} \Gamma \otimes^j R_0/m).
\]

\[\square\]

**Example 3.11.** The group \(\text{Aut}(k, \Gamma)\) acts on this set by pre-composing with the map \(j\) and post-composing with the isomorphism \(\gamma\). However, this action need not be transitive. For example, let \(\Gamma\) be the height 1 formal group over the perfect field \(K = \overline{\mathbb{F}_p((u^{1/p^\infty}))}\) with \(p\)-series

\[\lbrack p \rbrack_\Gamma(x) = ux^p.
\]

Let \(L = K \otimes_{\mathbb{F}_p} \mathbb{F}_p((u^{1/p^\infty}, v^{1/p^\infty}))\), and let \(R_0\) be the algebraic closure of \(L\). There are two maps \(j_1, j_2 : K \to R_0\), respectively sending \(u\) to \(u\) and to \(v\). The base changes of \(\Gamma\) along these maps are isomorphic over \(R_0\) via

\[\phi(x) = (u/v)^{1/(p-1)} x.
\]

This isomorphism is not induced by an element of \(\text{Aut}(k, \Gamma)\).

We now specialize to the case where \(\Gamma\) is the height \(n\) Honda formal group over a finite field \(k\) containing \(\mathbb{F}_p\), with \([p]_\Gamma(x) = xp^n\). In this case, the formal group is algebraic enough to prevent the above subtlety from occurring. (In fact, the following argument works in slightly more generality: one can take \([p]_\Gamma(x) = ux^{p^n}\) for \(u \in k^\times\), which at least implies that \(\text{Frob}_\Gamma\) is central in \(\text{End}_{E}(\Gamma)\).) The author thanks Paul Goerss for pointing out this subtlety and the following method of addressing it.

**Proof of Theorem 3.7.** First, we need to construct a continuous \(E_*\)-algebra map \(f : E_*^h E \to \text{Maps}_{ct}(G_n, E_*)\). Such a map is adjoint to a continuous map

\[G_n \to \text{Hom}_{\text{Alg}_{E_*}^h}(E_*^h E, E_*)\].

Let $(G^n, 1, \alpha^n)$ be the universal deformation over $E_*$. The $E_*$-algebra structure map $E_* \to E_*$ is just the identity map, which classifies this deformation. By Proposition 3.10
\[ \text{Hom}_{\text{Alg}_{E_*}}(E_*, E) \cong \{(j : k \to k, \gamma : \Gamma \xrightarrow{\sim} \Gamma \otimes j k)\}. \]

Since $k$ is a finite field, any map $k \to k$ is an isomorphism. So the right-hand side is exactly $\mathbb{G}_n$, defining the desired map.

For simplicity’s sake, we now restrict everything to degree zero. To show that the map $f : E_0^* E \to \text{Maps}_{\text{cts}}(G_n, E_0)$ is an isomorphism, it suffices, since both sides are flat and complete $E_0$-algebras, that it induces an isomorphism mod $I_n$. Now, $I_n$ is an invariant ideal in $BP$, so
\[ E_0^* E/I_n = k \otimes_{BP_0} BP_0 BP \otimes_{BP_0} k. \]

A map from this into a ring $R$ is the same as a pair of maps $i, j : k \to R$ and an isomorphism $\gamma : \Gamma \otimes^i_k R \to \Gamma \otimes^j_k R$ of formal groups over $R$. Now, if $\Gamma$ is the Honda formal group over $k \supseteq \mathbb{F}_{p^n}$, we have a coordinate $x$ for $\Gamma$ with $[p]_{\Gamma}(x) = x^{p^n}$, and this must commute in the obvious way with any isomorphism $\gamma$. It follows that the coefficients of $\gamma$, viewed as a power series in $x$, are fixed by the $n$th power of Frobenius. Since $R$ is an $\mathbb{F}_{p^n}$-algebra via $i$, the subring of elements of $R$ fixed by the $n$th power of Frobenius is a product of copies of $\mathbb{F}_{p^n}$.

By Theorem [2.21] the isomorphism $\gamma$ is defined over $\mathbb{F}_{p^n}$.

Thus, the data $(i, j, \gamma)$ is always base changed from data of the form
\[ (1 : k \to k, j : k \to k, \gamma : \Gamma \xrightarrow{\sim} j^* \Gamma). \]

Since $k$ is finite, $j$ is an isomorphism. This is precisely an element of the Morava stabilizer group $G_n = \text{Aut}(k, \Gamma)$ (cf. Definition [2.17]). Thus, the map $E_0^* E/I_n \to \text{Maps}_{\text{cts}}(G_n, k)$ is an isomorphism.

\[ \square \]

4. Localized $E$-theory and augmented deformations

Our primary concern in this paper is the spectrum $L_{K(n-1)}E_n$, where $n \geq 2$. We abbreviate this spectrum by $LE$.

4.1. Coefficients. The coefficient ring $LE_*$ is even periodic, with
\[ LE_0 = Wk[[u_1, \ldots, u_{n-1}]](u_1^{n-1})/(p, u_1, \ldots, u_{n-2}). \]

Proof. By [Rav84], $BP$ satisfies the telescope conjecture, in the sense that there is an equality of Bousfield classes $(BP \wedge T(n-1)) = (BP \wedge K(n-1))$, where $T(n)$ is a $v_{n-1}$-telescope of a finite type $n-1$ spectrum. As $E_n$ is a $BP$-module, it also satisfies the telescope conjecture. By [HSt99] Proposition 7.10, we then have
\[ L_{K(n-1)}E_n = \text{holim} S/(p^{i_0}, v_1^{i_1}, \ldots, v_{n-2}^{i_{n-2}}) \wedge v_{n-1}^{-1} E_n, \]

where the limit is over type $(n-1)$ generalized Moore spectra. We observe that
\[ (v_{n-1}^{-1} E_n)_0 S/(p^{i_0}, v_1^{i_1}, \ldots, v_{n-2}^{i_{n-2}}) = E_*[u_{n-1}^{-1}]/(p^{i_0}, u_1^{i_1}, \ldots, u_{n-2}^{i_{n-2}}), \]

which is even periodic with
\[ (v_{n-1}^{-1} E_n)_0 S/(p^{i_0}, v_1^{i_1}, \ldots, v_{n-2}^{i_{n-2}}) = Wk[[u_1, \ldots, u_{n-1}]](u_1^{n-1})/(p^{i_0}, u_1^{i_1}, \ldots, u_{n-2}^{i_{n-2}}). \]

The transition maps in the diagram are surjective, so there is no $\lim^1$ and the result is still even periodic. The limit on $\pi_0$ is the completion $Wk[[u_1, \ldots, u_{n-1}]](u_1^{n-1})/(p, u_1, \ldots, u_{n-2})$, as desired. \[ \square \]
Proposition 4.2. We have
\[ LE_0 = Wk((u_{n-1}))_{p}^\wedge[[u_1, \ldots, u_{n-2}]]. \]

Proof. Elements of both rings can be identified as certain possibly infinite formal sums
\[ \sum \{ a_I u_1^{i_1} \cdots u_{n-2}^{i_{n-2}} u_{n-1}^{i_{n-1}} : a_I \in Wk, i_j \in \mathbb{N} \text{ for } 1 \leq j \leq n-2, i_{n-1} \in \mathbb{Z}. \} \]
Such a sum is in \( LE_0 \) if and only if its reduction modulo each power of \((p, u_1, \ldots, u_{n-2})\) is in \( k((u_{n-1})) \). In other words, the exponents \( i_{n-1} \) appearing in all nonzero terms with \( i_0, \ldots, i_{n-2} \) less than some fixed \( i \) are bounded below. On the other hand, such a sum is in \( Wk((u_{n-1}))_{p}^\wedge[[u_1, \ldots, u_{n-2}]] \) if and only if the terms with each fixed \( i_1, \ldots, i_{n-2} \) add up to an element of \( u_1^{i_{n-1}} \cdots u_{n-2}^{i_{n-2}} Wk((u_{n-1}))_{p}^\wedge \). That is, the exponents \( i_{n-1} \) appearing in the nonzero terms with fixed \( i_1, \ldots, i_{n-2} \), and with \( i_0 \) less than some fixed \( i \), are bounded below. Since there are only finitely many choices of \( i_1, \ldots, i_{n-2} \) less than any fixed \( i \), the two conditions are in fact equivalent. \( \square \)

Forgetting about the ring structure, the spectrum \( LE \) also has a simple description.

Proposition 4.3 (\cite[Theorem 3.10]{AMS98}). The spectrum \( LE \) splits as a \( K(n-1) \)-local coproduct of copies of \( E_{n-1} \).

4.2. Completed homology.

Definition 4.4. We write \( \text{Mod}_{LE}^\wedge \) for the category of graded \( LE_\ast \)-modules which are (degreewise) L-complete with respect to the ideal \( I_{n-1} = (p, \ldots, u_{n-2}) \).

The spectrum \( LE \) defines completed homology and cohomology theories:
\[ LE^\wedge X = \pi_\ast F(X, LE), \]
\[ LE^\wedge_\ast X = \pi_\ast L_{K(n-1)}(LE \wedge X) = \pi_\ast L_{K(n-1)}(E \wedge X). \]

Proposition 4.5. The functors \( LE^\ast \) and \( LE^\wedge_\ast \) from \( \text{HoTop} \) to \( \text{Mod}_{LE}^\wedge \) naturally factor through \( \text{Mod}_{LE}^\wedge_{E_1} \).

Proof. The homology of the sphere is complete, and thus L-complete. Since \( \text{Mod}_{LE}^\wedge \) is an abelian category closed under extensions, the same follows for any finite complex. Now let \( X \) be an arbitrary spectrum and write \( X \) as a filtered colimit of its finite subcomplexes \( X_\alpha \). Then \( LE^\wedge X = \lim LE^\ast X_\alpha \), which is also L-complete.

Finally, the completed homology of \( X \) is
\[ LE^\wedge_\ast X = \pi_\ast L_{K(n-1)}(E_\ast \wedge L_{n-1}X) = \pi_\ast \text{holim}(E[u_{n-1}]^{-1}/(p_0^i, \ldots, u_{n-2}^{i_{n-2}}) \wedge L_{n-1}X). \]

There is a Milnor exact sequence
\[ 0 \to \text{lim}^1 \pi_{k+1}(E[u_{n-1}]^{-1}/(p_0^i, \ldots, u_{n-2}^{i_{n-2}}) \wedge L_{n-1}X) \to \]
\[ LE^\wedge_k X \to \text{lim} \pi_k(E[u_{n-1}]^{-1}/(p_0^i, \ldots, u_{n-2}^{i_{n-2}}) \wedge L_{n-1}X) \to 0. \]
Each term in the limit diagram is torsion to a power of \( I_{n-1} \) and thus L-complete. Since L-complete modules are closed under extensions, limits and \text{lim}^1, \( LE^\wedge_k X \) is L-complete for each \( k \). \( \square \)

Proposition 4.6. If \( X \) is finite, then \( LE^\wedge_\ast X = LE^\ast X \), which is complete in the ordinary sense.

Proof. If \( X \) is finite, then \( LE \wedge X \) is in the thick subcategory generated by \( LE \). In particular, it is \( K(n-1) \)-local. It follows that \( LE^\ast X = LE^\wedge_\ast X \), a finite L-complete \( LE_\ast \)-module. Since it is finitely generated, it is also complete in the ordinary sense \cite[GM92]{GM92}. \( \square \)
Proposition 4.7. If $LE_sX$ is free over $LE_s$, then $LE^\wedge_sX$ is its $I_{n-1}$-completion.

Proof. In the Milnor exact sequence for $LE^\wedge_sX$, we have $\pi_* (LE/I \wedge X) \cong \pi_* (LE \wedge X) / I$, where $I$ is an ideal in $LE_0$. Thus, the transition maps in the towers are surjective, the $\lim^1$ term vanishes, and the $\lim$ term is the ordinary completion of $LE_sX$. □

4.3. Augmented deformations.

Definition 4.8. Write $\Lambda$ for the coefficient ring of completed Laurent series, $\mathbb{W}((u_{n-1}))^\wedge$. This is a complete local ring with residue field $k((u_{n-1}))$. As we have shown, $LE_0 = \mathbb{W}((u_{n-1}))^\wedge [[u_1, \ldots, u_{n-2}]]$.

We will also write $\mathbb{H}^n$ for the base change of the universal deformation formal group $\mathbb{G}^n$ over $E_0$ to $LE_0$, and $\mathbb{H}$ for its base change to the residue field $k((u_{n-1}))$. By the discussion in Remark 2.16, if we started with the Honda formal group law with $p$-series $[p]_\Gamma(x) = x^{p^n}$, then $\mathbb{H}$ has a coordinate with $p$-series $[p]_H(x) = u_{n-1}^{p^{n-1}} + \mathbb{H} x^{p^n}$.

In particular, its height is $n - 1$.

Definition 4.9. Let $\mathbb{H}$ be a formal group over $k((u_{n-1}))$. An augmented deformation of $\mathbb{H}$ over $(R, m) \in \text{CLN}$ is a triple $(G, i, \alpha)$, where:

- $G$ is a formal group over $R$,
- $i: \Lambda \to R$ is a local ring homomorphism (that is, continuous for the maximal ideal topology), inducing a map $\tilde{i}: k((u_{n-1})) \to R/m$,
- and $\alpha: \mathbb{H} \otimes_{k((u_{n-1}))} R/m \sim G \otimes_R R/m$ is an isomorphism of formal groups over $R/m$.

An isomorphism of augmented deformations of $\mathbb{H}$ over $R$, $\phi: (G_1, i_1, \alpha_1) \to (G_2, i_2, \alpha_2)$, is

- the condition that $i_1 = i_2$,
- and a map $\phi: G_1 \to G_2$ of formal groups over $R$,
- such that the square

$$
\begin{array}{ccc}
\Gamma \otimes_{k((u_{n-1}))} R/m & \xrightarrow{\alpha_1} & G_1 \otimes_R R/m \\
\downarrow & & \downarrow \phi \\
\Gamma \otimes_{k((u_{n-1}))} R/m & \xrightarrow{\alpha_2} & G_2 \otimes_R R/m 
\end{array}
$$

commutes.

Let

$$
\text{Def}^{\text{aug}}_\mathbb{H}: \text{CLN} \to \text{Gpd}
$$

be the functor that sends $R$ to the groupoid of augmented deformations of $\mathbb{H}$ over $R$ and their isomorphisms.

Theorem 4.10. The functor $\text{Def}^{\text{aug}}_\mathbb{H}$ is represented by $LE_0$. That is, for $R$ a complete local ring, the groupoid $\text{Def}^{\text{aug}}_\mathbb{H}(R)$ is naturally equivalent to the discrete groupoid $\text{Maps}_{\text{cts}}(LE_0, R)$ of continuous maps with respect to the maximal ideal topology on both rings.
Proof. This is more or less an immediate consequence of the Lubin-Tate theorem, in the form of Theorem 2.13. Consider the following deformation in $\text{Def}_{\text{aug}}^H(LE_0)$:

$$(\mathbb{H}^u, 1 : \Lambda \to \Lambda, \alpha^u : \mathbb{H}^u \sim \mathbb{H}^u \otimes k((u_{n-1})))$$

where $\alpha^u$ is the canonical isomorphism given by the definition of $\mathbb{H}$. We will show that this deformation is universal.

Given a local ring homomorphism $f : LE_0 \to R$, we obtain an augmented deformation

$$(H^u \otimes_{LE_0} R, f|_\Lambda, \alpha^u \otimes f|_{LE_0} R) \in \text{Def}_{\text{aug}}^H(R)$$

by base change. Note that the map $f|_\Lambda$ is local because $f$ is.

On the other hand, suppose given $(G, i, \alpha) \in \text{Def}_{\text{aug}}^H(R)$. We must exhibit a unique continuous map $f : LE_0 \to R$ and a unique isomorphism between $(G, i, \alpha)$ and an augmented deformation of the above form. Since $i$ is local, we may regard $R$ as a $\Lambda$-algebra of the form given in Theorem 2.13, and $(G, i, \alpha)$ as an object of $\text{Def}_{\text{aug}}^\Lambda(R)$. Theorem 2.13 now implies that $(G, i, \alpha)$ is uniquely isomorphic, as an object of $\text{Def}_{\text{aug}}^\Lambda(R)$, to a base change of the universal deformation along a unique local ring map

$$\Lambda[[u_1, \ldots, u_{n-2}]] \to R$$

compatible with $i$. Equivalently, it is uniquely isomorphic, as an object of $\text{Def}_{\text{aug}}^H(R)$, to a base change of $(\mathbb{H}^u, 1, \alpha^u)$ along a unique local ring map

$$\Lambda[[u_1, \ldots, u_{n-2}]] \to R.$$  

□

5. The $E$-theory of $E$-theory

In this section, we let $k$ be a finite field containing $\mathbb{F}_{p^n}$ and $\mathbb{F}_{p^{n-1}}$. We let $E$ be the $E$-theory associated to a height $n$ formal group $\Gamma$ over $k$, and $F$ the $E$-theory associated to the height $n - 1$ Honda formal group $\Gamma_{n-1}$ over $k$. We have just described the homotopy groups of $LE$; as $LE$ is a $K(n-1)$-local spectrum, it is natural to study it in terms of its completed $F$-homology, which we do in this section. We will also obtain a description of the coalgebra of cooperations, $LE^\wedge$.  

5.1. The completed $E_{n-1}$-homology of $E_n$.

Proposition 5.1. $F^\wedge E = F^\wedge_{E^\wedge} LE$.

Proof. The map $E \to LE$ is a $K(n-1)$-local equivalence, so it remains so after smashing with $F$. □

Proposition 5.2. $F^\wedge_{E^\wedge}$ is even periodic and flat over $F_{E^\wedge}$.

Proof. As in the proof of Lemma 3.8, $F$ and $LE$ are both even periodic and Landweber exact, so

$$F_{E^\wedge} LE = F_{E^\wedge} \otimes_{BP} BP \otimes_{BP} LE_{E^\wedge},$$

which is even periodic and flat over $F_{E^\wedge}$, since $F \wedge E$ is $L_{n-1}$-local. The $K(n-1)$-localization satisfies

$$F_{E^\wedge}^\wedge LE = \pi_* \lim_{I}(F \wedge LE/I),$$

where $I$ ranges over a cofinal set of ideals of the form $(p^{i_0}, \ldots, p^{i_{n-1}})$. The objects in the limit diagram are

$$F_{E^\wedge} \otimes_{BP} BP \otimes_{BP} LE_{E^\wedge}/I,$$

and the maps in the diagram are surjective. Therefore, $F^\wedge_{E^\wedge}$ is also even periodic.
It remains to show that \( F_*^\wedge \) is \( F_* \)-flat. The above implies that
\[
F_*^\wedge LE = (F_* LE)_{i-1}^\wedge,
\]
(the degreewise completion), and therefore that
\[
F_*^\wedge LE / I = (F_* LE) / I = F_* / I \otimes_{BP_*} BP_* BP \otimes_{BP_*} LE_* / I,
\]
for any power \( I \) of \( I_{i-1} \). Since each of these is a flat \( F_* / I \)-module and the maps in the limit diagram computing the completion are surjective, [Stacks, Tag 0912] implies that \( F_*^\wedge LE \) is a flat \( (F_*)^\wedge I = F_* \)-module. \( \square \)

We now describe the functor represented by \( F_0^\wedge E \).

**Lemma 5.3.** For any complete local ring \( R \), pre-composition with the completion map \( F_0E \to F_0^\wedge E \) induces an isomorphism
\[
\text{Hom}_{cts}(F_0^\wedge E, R) \cong \text{Hom}_{cts}(F_0 E, R).
\]
Moreover, for any continuous map \( F_0^\wedge E \to F_0 \), the composition
\[
LE_0 \xrightarrow{\eta_R} F_0E \to R
\]
is also continuous.

**Proof.** This is just as in Lemma 5.2. Since \( I_{i-1} \) is an invariant ideal in \( BP_* \), any complete local ring \( R \) with a map \( f : F_0E \to R \) such that the restriction to \( F_0 \) is continuous must be complete with respect to \( I_{i-1} : F_0E \), so that \( f \) factors uniquely through the completion \( F_0^\wedge E \). Moreover, \( f \) also sends \( \eta_R(I_{i-1} \cdot E_0) \) into \( I_{i-1}R \), so that the map \( E_0 \to R \) coming from the right unit is also continuous. \( \square \)

**Theorem 5.4.** Let \( R \) be a complete local \( F_0 \)-algebra. There is a natural isomorphism between continuous \( F_0 \)-algebra maps \( F_0^\wedge E \to R \) and pairs \((j, \gamma)\), where \( j : \Lambda \to R \) is a \( p \)-adic continuous map and \( \gamma \) is an isomorphism of formal groups over \( R/m \), \( \gamma : \Gamma_{n-1} \otimes_k R/m \xrightarrow{\sim} \mathbb{H} \otimes_k^\wedge \rho((u_{n-1})) R/m \).

**Proof.** As before, we have
\[
F_0 LE = \pi_0(F_* \otimes_{BP_*} BP_* BP \otimes_{BP_*} LE_*) = F_0 \otimes_{BP_0} BP_0 BP \otimes_{BP_0} LE_0.
\]
An \( F_0 \)-algebra map \( F_0 LE \to R \) is equivalent to a map \( LE_0 \to R \) and an isomorphism over \( R \) between the base changes of the formal groups of \( F_0 \) and \( LE_0 \).

If \( R \) is complete local, then the previous lemma tells us that \( \text{Hom}_{cts}(F_0^\wedge E, R) = \text{Hom}_{cts}(F_0 E, R) \), and that the map \( LE_0 \to R \) is continuous. Hence, the map \( LE_0 \to R \) represents an object of \( \text{Def}^{\text{aug}}_{\mathbb{H}}(R) \). Likewise, the structure map \( F_0 \to R \) represents an object of \( \text{Def}_{R_{i-1}}(R) \), say \((G, i, \alpha)\). Thus, we have a pullback of groupoids:
\[
\begin{array}{ccc}
\text{Hom}_{cts}(F_0^\wedge E, R) & \to & \text{Def}^{\text{aug}}_{\mathbb{H}}(R) \\
\downarrow & & \downarrow \\
\{*\} & \to & M_{\text{fg}}(R).
\end{array}
\]
In other words, a map \( f : F_0E \to R \) corresponds to the data:
\[
(G', j : \Lambda \to R, \beta : \mathbb{H} \otimes_{k((u_{n-1})} R/m \xrightarrow{\sim} G' \otimes_R R/m) \in \text{Def}^{\text{aug}}_{\mathbb{H}}(R);
\]
\[
\phi : G \xrightarrow{\sim} G'.
\]
There is a unique isomorphism in the pullback groupoid which restricts to \( \phi^{-1} : G' \to G \) on formal groups. Composing with this isomorphism, one gets a unique object in the pullback
groupoid isomorphic to $f$ whose underlying formal group is $G$ and whose underlying isomorphism of formal groups is the identity. It follows that the groupoid is locally contractible, as expected. The rest of the data is given by $j$ and $\beta$, or equivalently by $j$ and $\gamma = \beta^{-1} \alpha : \Gamma_{n-1} \otimes_k R/\mathfrak{m} \xrightarrow{\sim} \mathbb{H} \otimes_{k((u_{n-1}))} R/\mathfrak{m}$. 

Now let’s see what this functorial description of $F_*^\wedge E$ can tell us about it algebraically.

**Proposition 5.5.** $F_0E/I_{n-1}$ is of the form $\text{Hom}(\text{Gal}(k/\mathbb{F}_p), L)$, where $L$ is a field. Therefore, $F_0^\wedge E$ is a finite product of complete local rings.

**Proof.** Armed with Theorem 5.4, this is essentially a reinterpretation of a result of Torii, [Toi11, Theorem 2.7], which in turn reinterprets a result from [Gr79]. For $R$ a complete local $k = F_0/I_{n-1}$-algebra,

$$\text{Hom}_k(F_0E/I_{n-1}, R) = \{ (\gamma : k((u_{n-1})) \to R, \gamma : \Gamma_{n-1} \otimes_k R/\mathfrak{m} \xrightarrow{\sim} \mathbb{H} \otimes_{k((u_{n-1}))} R/\mathfrak{m}) \}.$$ 

The étaleness of isomorphisms, Theorem [221], says that we can equivalently define $\gamma$ as an isomorphism between $\Gamma_{n-1}$ and $\mathbb{H}$ over $R$. There is a smallest extension $L$ of $k((u_{n-1}))$ over which $\Gamma_{n-1}$ and $\mathbb{H}$ become isomorphic, given by adjoining the coefficients of an isomorphism between any choice of formal group laws for $\Gamma$ as expected. The rest of the data is given by a local isomorphism of formal groups is the identity. It follows that the groupoid is locally contractible, as expected. The rest of the data is given by $j$ and $\beta$, or equivalently by $j$ and $\gamma = \beta^{-1} \alpha : \Gamma_{n-1} \otimes_k R/\mathfrak{m} \xrightarrow{\sim} \mathbb{H} \otimes_{k((u_{n-1}))} R/\mathfrak{m}$.

Since $k$ is finite, this is a finite product of fields. The corresponding splitting for $F_0^\wedge E$ itself follows from Hensel’s lemma. 

**Remark 5.6.** It’s possible to be slightly more explicit, using the formula

$$F_*^\wedge E/I_{n-1} = F_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} LE_*/I_{n-1}.$$ 

For $x \in BP_*$, write $x$ for $\eta_L(x)$ and $\overline{x}$ for $\eta_R(x)$, elements of $BP_*BP$. To avoid confusion, let $u_i$ and $u$ be the Lubin-Tate generators of $E_*$, and write $u_i^F$ and $u^F$ for the corresponding generators in $F_*$. The map $BP_* \to F_*$ sends

$$v_i \mapsto (u^F)^{\rho^{i-1}} u_i^F, \quad i \leq n-2,$$

$$v_{n-1} \mapsto (u^F)^{\rho^{n-1}-1},$$

$$v_i \mapsto 0, \quad i \geq n.$$ 

Likewise, the map $BP_* \to LE_*$ sends

$$\overline{v}_i \mapsto u^{\rho^{i-1}} u_i, \quad i \leq n-1,$$

$$\overline{v}_n \mapsto u^{\rho^{n-1}-1},$$

$$\overline{v}_i \mapsto 0, \quad i \geq n+1.$$ 

We can thus write

$$F_*^\wedge E = \left( Wk[[u_1^F, \ldots, u_{n-2}^F]][(u^F)^{\pm 1}][t_1, t_2, \ldots] \otimes_{\mathbb{F}_p} Wk((u_{n-1}))[u_1, \ldots, u_{n-2}][u^\pm 1] \right)^\wedge_{I_{n-1}}.$$
In degree zero, let \( s_i = t_i u^{1-p^i} \) and \( w = u^F / u \). Note that the ideal \( I_{n-1} \) contains \( p \) and all \( u_i \) and \( u_i^F \) (and thus all \( v_i \) and \( \bar{v}_j \)) for \( 1 \leq i \leq n-2 \). Therefore,

\[
F_0^\wedge E/I_{n-1} = (k \otimes_{F_p} k)((u_{n-1}))[s_1, s_2, \ldots, w^{\pm 1}]/(u^{1-p^{n-1}} v_{n-1} - u_{n-1}, u^{1-p^n} v_n - 1, v_{n+1}, \ldots).
\]

Now, by [Rav04, 4.3.1],

\[
(1) \quad v_{n-1+i} \equiv v_{n-1+i} + v_{n-1} t_i^{p^{n-1}} - v_i^{p^i} \quad (\text{mod } (p, v_1, \ldots, v_{n-2}, t_1, \ldots, t_{i-1})).
\]

Scaling to degree zero by multiplying by appropriate powers of \( u \), we get relations in \( F_0^\wedge E/I_{n-1} \):

\[
\begin{align*}
  u_{n-1} &= w^{p^{n-1}-1}, \\
  1 &= w^{p^{n-1}-1} s_1^{p^{n-1}} - w^{p(p^{n-1}-1)} s_1, \\
  0 &\equiv w^{p^{n-1}-1} s_i^{p^{n-1}} - w^{p(p^{n-1}-1)} s_i + f_i(s_1, \ldots, s_{i-1}) \quad \text{for } i \geq 2.
\end{align*}
\]

The first relation gives an embedding of \( k((w)) \) into \( F_0^\wedge E/I_{n-1} \), as a tamely ramified extension of \( k((u_{n-1})) \) (recall that \( k \) contains \( F_{p^{n-1}} \)). The remaining relations inductively define \( s_i \) as solutions to higher Artin-Schreier equations over the ring generated by \( k((w)) \) and \( s_1, \ldots, s_{i-1} \). In particular, \( F_0^\wedge E/I_{n-1} \) is ind-étale over \( k((u_{n-1})) \).

\text{Remark 5.7.} One should compare this result with the computation of \( E_0^\wedge E \). In both cases, the object calculated is a flat extension of a Lubin-Tate ring, it represents an isomorphism of deformations of formal groups, and its reduction mod \( I_{n-1} \) carries the action of a Morava stabilizer group. In the case of \( E_0^\wedge E \), this action splits, and in fact \( E_0^\wedge E \) is a profinite group algebra for the Morava stabilizer group. In the case of \( F_0^\wedge E \), the action is in a certain sense as complicated as possible, so that the only part of the action that splits is the Galois group. By Torii’s theorem, the other part \( \text{Aut}_k(\Gamma_{n-1}) \) of the Morava stabilizer group instead becomes the Galois group of a residue field extension \( L/k((u_{n-1})) \). In particular, \( \mathbb{H} \) only becomes isomorphic to the Honda formal group \( \Gamma_{n-1} \) after base change to \( L \). This is one way of saying that the formal group of \( LE_0 \) is as complicated a height \( n-1 \) formal group as possible.

The formal étaleness of \( F_0^\wedge E_0 \) over \( E_0 \) is measured by the vanishing of its completed cotangent complex. (To be precise, this is the complex representing derived functors of derivations of \( E_0 \)-algebras which are \( L \)-complete with respect to \( I_n \)). \( F_0^\wedge E \) is not formally étale over \( F_0 \), but it is formally smooth, as shown by the following calculation. (The cotangent complex in question is likewise defined using the category of \( F_0 \)-algebras which are \( L \)-complete with respect to \( I_{n-1} \).)

\textbf{Proposition 5.8.} \textit{The completed cotangent complex }\( \mathbb{L}_{F_0^\wedge E/F_0} \textit{ is concentrated in degree zero and free of infinite rank (equal to the transcendence degree of } k((u_{n-1}))/k \textit{).}

\textit{Proof.} In Theorem 5.3, we showed that \( F_0^\wedge E \) represents data of the following form on complete \( F_0 \)-algebras:

\[
(j : \Lambda \to R, \gamma \text{ an isomorphism of formal groups over } R/m).
\]
But the $\gamma$ part of the data is insensitive to infinitesimal thickenings. Thus, if $R$ is a complete local ring and $I$ a square-zero ideal, constructing a lift in the diagram

$$
\begin{array}{c}
LE_0 \longrightarrow R \\
\downarrow \\
\Lambda = Wk((u_{n-1}))^\wedge \longrightarrow R/I.
\end{array}
$$

reduces to constructing a lift in the diagram

$$
\begin{array}{c}
Wk \longrightarrow R \\
\downarrow \\
\Lambda = Wk((u_{n-1}))^\wedge \longrightarrow R/I.
\end{array}
$$

In other words, the complete cotangent complex $\mathbb{L}_{LE_0^\wedge LE/LE_0}$ is a base change of $\mathbb{L}_{\Lambda/Wk}$. As a consequence of Theorem 2.8, the map $Wk \to \Lambda$ is formally smooth if $k$ is perfect, meaning that its completed cotangent complex is $\Lambda$-free and concentrated in degree zero, and

$$\text{rank}_\Lambda \mathbb{L}_{\Lambda/Wk} = \text{rank}_{k((u_{n-1}))} \mathbb{L}_{k((u_{n-1}))}/k$$

which is equal to the transcendence degree of $k((u_{n-1}))/k$. (This is always infinite, but there is an easy cardinality argument if $k$ is finite: $k((u_{n-1}))$ is uncountable, while any field of finite transcendence degree over $k$ is countable.)

Finally, we make a comment about the $K(n-1)$-local Adams-Novikov spectral sequence for $LE$. Let $G_{n-1} = \text{Aut}(k, \Gamma_{n-1})$ be the height $n-1$ Morava stabilizer group. By Theorem 3.7 and the fact that $F_s^\wedge E$ is honestly complete, we can regard its $F_s^\wedge F$-comodule structure as a continuous $G_{n-1}$-module structure.

**Lemma 5.9.** The $G_{n-1}$-action on $F_s^\wedge E$ is extended. In other words, there is a $G_{n-1}$-module isomorphism:

$$F_s^\wedge E \cong \text{Maps}_{cts}(G_{n-1}, LE_s).$$

**Proof.** This is true for $F$, and $LE$ is a $K(n-1)$-local sum of copies of $F$, by Proposition 4.3.

**Corollary 5.10.** The $K(n-1)$-local Adams-Novikov spectral sequence

$$E_2 = H^p_{cts}(G_{n-1}, F_s^\wedge E) \Rightarrow \pi_s LE$$

is concentrated on the 0-line and collapses at $E_2$:

$$\pi_s LE = (F_s^\wedge E)^{\otimes_{G_{n-1}}}.$$

5.2. **Cooperations for $LE$.** The same arguments apply to the completed cooperations algebra $LE_0^\wedge LE$. Let us merely state the corresponding results without proof:

**Proposition 5.11.**

(a) $LE_0^\wedge LE$ is even periodic and flat over $LE_s$.

(b) Let $R$ be a complete local $LE_0$-algebra, with $LE_0 \to R$ classifying $(G, i, \alpha) \in \text{Def}_{\mathbf{H}}^{\text{aug}}(R)$. There is a natural isomorphism between continuous $LE_0$-algebra maps $LE_0^\wedge LE \to R$ and pairs $(j, \gamma)$, where $j : \Lambda \to R$ is a $p$-adically continuous map and $\gamma$ is an isomorphism of formal groups over $R/m$.

$$\mathbb{H} \otimes_{k((u_{n-1}))}^J R/m \to \mathbb{H} \otimes_{k((u_{n-1}))}^J R/m.$$ 

(c) The completed cotangent complex $\mathbb{L}_{LE_0^\wedge LE/LE_0}$ is concentrated in degree zero and free of infinite rank (equal to the transcendence degree of $k((u_{n-1}))/k$.)
6. Power operations and \( \mathcal{E}_\infty \) structures

6.1. Background on \( \theta \)-algebras. This section specializes to the case \( n = 2 \) and \( n - 1 = 1 \). We write \( K = E_1 \) and \( E = E_2 \), both over a finite field \( k \) containing \( \mathbb{F}_p^2 \). We write \( x \) for \( u_1 \), so that \( LE_0 \cong Wk((x))_p^\wedge \).

**Definition 6.1.** A \( \theta \)-algebra is a \( \mathbb{Z}_p \)-algebra \( R \) equipped with an operation \( \theta : R \to R \) such that

\[
\begin{align*}
\theta(x + y) &= \theta(x) + \theta(y) - \sum_{i=1}^{p-1} \frac{1}{p^i} x^i y^{p^i-1}, \\
\theta(xy) &= \theta(x)y^p + x^p \theta(y) + p\theta(x)\theta(y), \\
\theta(0) &= \theta(1) = 0.
\end{align*}
\]

These properties imply that the operation \( \psi^p : R \to R \) defined by

\[
\psi^p(x) = x^p + p\theta(x)
\]
is a ring homomorphism; in other words, any \( \theta \)-algebra \( R \) is equipped with a lift of the Frobenius on \( R/p \). If \( R \) is \( p \)-torsion-free, then the Frobenius lift \( \psi^p \) uniquely determines \( \theta \), but in general, \( \theta \) is strictly more data than \( \psi^p \).

**Definition 6.2.** A \( \theta \)-comodule algebra is a \( \theta \)-algebra \( R \) together with a continuous action of \( \mathbb{Z}_p^\times \) on \( R \) that commutes with \( \theta \). We write \( \text{ComodAlg}_{\mathfrak{g}_\theta} \) for the category of \( \theta \)-comodule algebras.

In \( K(1) \)-local homotopy theory, \( \theta \)-algebras arise from power operations on \( \mathcal{E}_\infty \) ring spectra. More precisely:

- If \( X \) is a \( K(1) \)-local \( \mathcal{E}_\infty \) ring spectrum, then \( K_0^\wedge X \) is naturally a \( \theta \)-comodule algebra [GH05, Theorem 2.2.4].
- If \( X \) is a \( K(1) \)-local \( \mathcal{E}_\infty \) ring spectrum, then \( \pi_0 X \) is naturally a \( \theta \)-algebra [Hop].

In the remainder of this section, we consider the case \( X = LE \). In section 6.2, we show that there are nonisomorphic \( \theta \)-algebra structures on \( LE_0 \). In section 6.3, we extend these structures to \( \theta \)-comodule algebra structures on \( K_0^\wedge E \). In section 6.4, we use an obstruction theory argument to show that any \( \theta \)-algebra structure on \( LE_0 \) is realized by some \( \mathcal{E}_\infty \) structure on \( LE \).

6.2. Constructing non-isomorphic \( \theta \)-algebra structures. Let \( X \) and \( X' \) be two \( \mathcal{E}_\infty \)-algebras abstractly equivalent to \( LE \). An equivalence \( X \to X' \) induces a \( p \)-adically continuous isomorphism of \( \theta \)-algebras \( \pi_0 X \to \pi_0 X' \). Thus, the question of identifying \( \mathcal{E}_\infty \) structures on \( LE \) up to equivalence is related to the question of identifying \( \theta \)-algebra structures on \( LE_0 = Wk((x))_p^\wedge \) up to isomorphism.

I thank Dominik Absmeier for catching a mistake in the original version of this argument.

**Lemma 6.3.** Given any \( \theta \)-algebra structure on \( Wk((x))_p^\wedge \), the operation \( \theta \) descends to a map \( \theta : W_2k((x)) \to k((x)) \).

**Proof.** We have to show that \( \theta(f) \mod p \) only depends on the class of \( f \mod p^2 \). First, because \( \psi \) is a ring homomorphism, we must have

\[
\psi(n) = n \quad \text{for } n \in \mathbb{Z};
\]

this forces

\[
\theta(n) = \frac{n - np}{p} \quad \text{for } n \in \mathbb{Z}.
\]

In particular, if \( v_p(n) = i \geq 1 \), then \( v_p(\theta(n)) = i - 1 \).
Now calculate
\[
\theta(f + p^2 g) = \theta(f) + \theta(p^2 g) - \sum_{i=1}^{p-1} \frac{1}{p^i} f^i p^{2(p-i)} g^{2(p-i)} = \theta(f) + p^2 \theta(g) + \theta(p^2 g) + p \theta(p^2 g) - O(p^2) = \theta(f) + O(p)
\]
because \( \theta(p^2) \) is divisible by \( p \).

We will show that there are two \( \theta \)-algebra structures, \( \theta_0 \) and \( \theta \), such that there is no \( p \)-adically continuous automorphism of \( W_k((x))_p^\wedge \) making the diagram

\[
\begin{array}{ccc}
W_k((x))_p^\wedge & \xrightarrow{\theta_0} & W_k((x))_p^\wedge \\
\downarrow f & & \downarrow f \\
W_k((x))_p^\wedge & \xrightarrow{\theta} & W_k((x))_p^\wedge
\end{array}
\]

commute.

To do this, it helps to introduce an alternate notion of continuity, besides the \( p \)-adic one we have been using to this point. We first give some motivation, recalling Section 1.1 from the introduction. The ring \( W_k((x))_p^\wedge = W_k((u_{n-1}))_p^\wedge \) arose in the first place via inverting an element in the maximal ideal of a complete local ring and then completing with respect to a smaller ideal — these algebraic operations being related to the topological one of \( K(n-1) \)-localizing a \( K(n) \)-local object. Treating \( W_k((x))_p^\wedge \) as a \( p \)-adically complete local ring, equipped with its \( p \)-adic topology, means forgetting about the topology that was previously on the ring \( W_k[[x]] \). This gives us a lot of freedom in specifying continuous maps out of \( W_k((x))_p^\wedge \) — just as there are many more discontinuous than continuous maps out of the ring \( k[[x]] \) — but also makes them harder to specify — just as a continuous ring map out of \( k[[x]] \) is determined by the image of \( x \), but a discontinuous one is not.

Instead, one could keep track of both the \( p \)-adic topology on \( W_k((x))_p^\wedge \) and the maximal ideal topology on \( W_k[[x]] \). This is formalized in the notion of pipe structure due to [MGPS], drawing on work on higher local fields in number theory [Ka00, Morr12]. Rather than give the full definition, we will describe what it means for a map from \( W_k((x))_p^\wedge \) to itself to be continuous for the pipe structure, as such maps are all we care about here.

**Definition 6.4.** Let \( \phi : W_k((x))_p^\wedge \to W_k((x))_p^\wedge \) be a map of sets (which is not necessarily a ring homomorphism). We say that \( \phi \) is pipe-continuous if it is the limit of an inverse system of maps

\[
\phi_i : W_{n_i}k((x)) \to W_{m_i}k((x)),
\]

where each \( W_{n_i}k((x))_p^\wedge \) has the \( x \)-adic topology.

**Remark 6.5.** The \( x \)-adic topology on \( W_nk((x)) \) is the same as the topology induced by the maximal ideal topology on \( W_nk[[x]] \). Indeed,

\[
(p, x)^{n+r}W_nk[[x]] \subseteq x^rW_nk[[x]] \subseteq (p, x)^rW_nk[[x]].
\]

To demonstrate the usefulness of this idea, we note that pipe-continuous ring homomorphisms are very easy to describe.

**Proposition 6.6.** A pipe-continuous ring homomorphism \( \phi : W_k((x))_p^\wedge \to W_k((x))_p^\wedge \) is completely determined by the image of \( x \), which can be any completed Laurent series that reduces mod \( p \) to a nonzero element of \( xk[[x]] \).
Proof. First, it is easy to see that a pipe-continuous ring homomorphism $\phi$ can be written as the limit of an inverse system

$$\phi_i : W_n k((x)) \to W_n k((x)),$$

and must in particular be $p$-adically continuous. Now, any continuous map $W_n k((x)) \to W_n k((x))$ is determined by the image of $x$: namely, $x$ can go to any invertible, topologically nilpotent element. Let $y \in W_n k((x))$ be invertible and topologically nilpotent. Then $y$ is not divisible by $p$, so writing

$$y = \sum_{i \geq -N} a_i x^i,$$

there is a least $i$, say $i = d$, such that $a_d$ is invertible in $W_n k$. If $d \leq 0$, then for any $r$, $y^{p^r} = \overline{a_d}^{p^r} x^{p^r d} + \cdots \mod p$, so $y$ is not topologically nilpotent.

Conversely, suppose that $y = \sum_{i \geq -n} a_i x^i$, and that $y$ reduces mod $p$ to $\overline{a_d} x^d + \cdots$ with $\overline{a_d} \neq 0$ and $d > 0$. Then mod $p$,

$$a_d^{-1} x^{-d} y \equiv 1 + \cdots \in k((x))^\times,$$

so that $a_d^{-1} x^{-d} y$ is invertible by Hensel’s lemma, and so $y$ itself is invertible. Now let $y_+$ be the sum of the terms of $y$ of degree $\geq d$, so that we can write $y = y_+ + py_-$. Then

$$y^{p^r} = y_+^{p^r} + \sum_{i=1}^{p^r} \binom{p^r}{i} y_-^i y_+^{p^r-i}.$$

By Kummer’s theorem on valuations of binomial coefficients,

$$v_p \left( \binom{p^r}{i} \right) = i + r - v_p(i) \geq r.$$

Thus, for $r$ sufficiently large, $y^{p^r} = y_+^{p^r}$ in $W_n k((x))$. Thus, $y$ is topologically nilpotent. □

**Proposition 6.7.** Every field automorphism of $k((x))$ is continuous.

Proof. Suppose that $k = \mathbb{F}_q$. Let $S$ be the set of power series $1 + a_1 x + \cdots$. Then $S$ is multiplicatively closed, and any $f \in S$ has a $(q-1)$th root $g \in S$, by the binomial theorem. On the other hand, an $f \not\in S$ either has $v_\omega(f) = 0$ but constant term not equal to 1, in which case it does not have a $(q-1)$th root at all, or $v_\omega(f) \neq 0$, in which case it has at most a $(q-1)^m$th root for some maximal $m$.

It follows that $S$ is exactly the set of elements of $k((x))$ which have a $(q-1)^m$th root for all $m$. Thus, any automorphism of $k((x))$ preserves $S$. Subtracting 1, we see that any automorphism of $k((x))$ preserves the set $x k[[x]]$, and thus that it preserves $x^r k[[x]]$ for every $r$. Thus, any automorphism is continuous. □

**Corollary 6.8.** If $\psi^p$ is any pipe-continuous Frobenius lift of $W k((x))^\wedge$, then the associated $\theta$ induces an $(x$-adically) continuous map

$$\theta : W_2 k((x)) \to k((x)).$$

Proof. We can recover this reduction of $\theta$ from $\psi^p$ mod $p^2$, using the formula $\psi^p(f) = f^p + p \theta(f)$ inside $W_2 k((x))$. In other words, $\theta(f) = \frac{1}{k}(\psi^p(f) - p \theta(f))$, where division by $p$ is the obvious isomorphism from the $p$-torsion of $W_2 k((x))$ to $k((x))$. Since $\psi^p(f)$ and $f \mapsto f^p$ are both continuous on $W_2 k((x))$, it follows that $\theta$ is also continuous. □

**Proposition 6.9.** There exist non-isomorphic $\theta$-algebra structures on $W k((x))^\wedge$. 
Proof. Let \( \psi_0^p \) and \( \psi^p \) be the unique pipe-continuous endomorphisms satisfying \( \psi_0^p(x) = x^p \) and \( \psi^p(x) = x^p + p \). Thus, \( \theta_0(x) = 0 \) and \( \theta(x) = 1 \). Suppose that \( f \) is an automorphism of \( Wk((x))^\wedge_p \) such that \( f\theta_0 = \theta f \). By Lemma 6.3, there is a commutative diagram

\[
\begin{array}{ccc}
W_2k((x)) & \xrightarrow{\theta_0} & k((x)) \\
\downarrow f & & \downarrow \overline{f} \\
W_2k((x)) & \xrightarrow{\theta} & k((x))
\end{array}
\]

where we have written \( f \) for the reduction of \( f \) mod \( p^2 \), and \( \overline{f} \) for the reduction of \( f \) mod \( p \). In particular, note that

\[ \theta(f(x)) = \overline{f}(\theta_0(x)) = 0. \]

We will show that this is not possible for any automorphism \( f \).

By Proposition 6.7, \( \overline{f} \) is a continuous automorphism of \( k((x)) \). This means that, mod \( p^2 \), \( f \) must be of the form

\[ f(x) \equiv ax + g(x) + ph(x^{-1}) \pmod{p^2}, \]

where \( a \in W_2k^\times \), \( g \in x^2W_2k[[x]] \), and \( h \) is a polynomial.

Now let us calculate \( \theta(f) \) mod \( p \).

\[
\theta(f(x)) \equiv \theta(ax) + \theta(g + ph) - \sum \frac{1}{p^i} x^i(g + ph)^{p-i}
\]

\[
\equiv a^p\theta(x) + \theta(a)x^p + \theta(g + ph) - \sum \frac{1}{p^i} x^i g^{p-i}
\]

\[
\equiv a^p + \theta(a)x^p + \theta(g) + \theta(ph) - \sum \frac{1}{p^i} x^i g^{p-i}
\]

\[
\equiv a^p + \theta(a)x^p + \theta(g) + \theta(p)h(x^{-1})^p - \sum \frac{1}{p^i} x^i g^{p-i}. \]

Since the only terms of negative degree in \( x \) come from \( \theta(p)h(x^{-1})^p \), we must have \( h \equiv 0 \) mod \( p \), so that the term \( \theta(p)h(x^{-1})^p \) disappears.

As for the other terms, write \( v_x \) for the \( x \)-adic valuation on \( k((x)) \). The term \( a^p \) has \( v_x = 0 \) – thus, for \( \theta(f) \) to equal zero, it is necessary for some other term to also have \( x \)-adic valuation 0. However,

\[
v_x(a^p) = 0,
\]

\[
v_x(x\theta(a)) = 1,
\]

\[
v_x(x^i g^{p-i}) = i + (p - i)v_x(g) \geq p + 1 \geq 3.
\]

As for \( g \), still working mod \( p \),

\[
\theta(x^n) = x^n \theta(x^{n-1}) + x^p(x^{n-1}) \theta(x) = n x^p(x^{n-1}),
\]

by induction on \( n \). In particular,

\[
v_x(\theta(x^n)) \geq 2 \text{ when } n \geq 2.
\]

By the multiplication formula, the same is true for \( \theta(bx^n) \), when \( b \in W_2k \). Thus, a monomial \( g \) with \( v(g) \geq 2 \) has \( v(\theta(g)) \geq 2 \) as well. By induction and using the theta sum formula, the same is true for polynomials. By continuity of \( \theta \) (Corollary 6.8, the same is true for power series.
It follows that \( v_x(\theta(g)) \geq 2 \). Thus, it’s impossible for \( \theta(f(x)) \) to equal 0, which is what we needed to prove.

6.3. **The \( K \)-theory of \( E_2 \).** The ring \( K^\wedge E \) is a \( K^\wedge K \)-comodule. By Theorem 3.7 (and the fact that \( K^\wedge E \) is classically complete), it is equivalently a continuous \( \mathbb{Z}_p^\wedge \)-module, and there’s a homotopy fixed points spectral sequence

\[
E_2 = H^*_ct(\mathbb{Z}_p^\wedge, K^\wedge E) \Rightarrow \pi_* LE.
\]

**Proposition 6.10.** The cohomology of \( \mathbb{Z}_p^\wedge \) acting on \( K^\wedge E \) is concentrated in degree zero, so that the above spectral sequence collapses at \( E_2 \).

For the next proposition, write \( \sigma \) for the Frobenius map of any ring or scheme of characteristic \( p \). If \( G \) is a formal group over a ring \( R \) of characteristic \( p \), we define the formal group \( G^{(p)} \) and the **relative Frobenius** \( \text{Frob} : G \to G^{(p)} \) by the following pullback square:

\[
\begin{array}{ccc}
\mathbb{G} & \xrightarrow{\sigma} & \mathbb{G} \\
\downarrow & & \downarrow \\
\text{Spec } R & \xrightarrow{\sigma} & \text{Spec } R \\
\end{array}
\]

**Proposition 6.11.** Every \( \theta \)-algebra structure on \( LE_0 \) extends to a \( \theta \)-comodule algebra structure on \( K_0^\wedge E \).

**Proof.** Fix a \( \theta \)-algebra structure on \( LE_0 \). Since \( K_0^\wedge E \) is torsion-free, it suffices to extend \( \psi^p \) over \( K_0^\wedge E \). Note that \( \psi^p \) should not be \( K_0 \)-linear in the most obvious sense: rather, \( K_0 = Wk \) carries a unique Frobenius lift (which we will also write \( \sigma \)), and \( \psi^p : K_0^\wedge E \to K_0^\wedge E \) should be \( K_0 \)-linear where \( K_0 \) acts on the target via \( \sigma \).

By Theorem 5.3 to define \( \psi^p : K_0^\wedge E \to K_0^\wedge E \), it suffices to define a \( j : LE_0 \to K_0^\wedge E \) and an isomorphism of formal groups,

\[
\gamma : (\Gamma_1 \otimes_k K_0^\wedge E/p)^{(p)} = \Gamma_1 \otimes_i^p K_0^\wedge E/p \xrightarrow{\sim} \mathbb{H} \otimes_k^\wedge (k((x))) K_0^\wedge E/p.
\]

Note that the twisted \( K_0 \)-algebra structure on \( K_0^\wedge E/p \) also forces a Frobenius twist on the map \( i \) used to identify \( \Gamma_1 \) with the formal group of \( K_0^\wedge E/p \).

Define

\[
j : LE_0 \xrightarrow{\psi^p} LE_0 \hookrightarrow K_0^\wedge E.
\]

Observe that the reduction of \( j \) mod \( p \) is

\[
\overline{j} : k((x)) \xrightarrow{\sigma} k((x)) \hookrightarrow K_0^\wedge E/p,
\]

so that

\[
\mathbb{H} \otimes_{k((x))} K_0^\wedge E/p = (\mathbb{H} \otimes_{k((x))} K_0^\wedge E/p)^{(p)},
\]

where we have written \( \text{can} \) for the canonical morphism \( LE_0 \hookrightarrow K_0^\wedge E \).

Now \( K_0^\wedge E \) carries an isomorphism

\[
\gamma_{\text{can}} : \Gamma \otimes_k K_0^\wedge E/p \xrightarrow{\sim} \mathbb{H} \otimes^{\text{can}}_{k((x))} K_0^\wedge E/p;
\]

we define \( \gamma \) to be the base change of \( \text{can} \) along the Frobenius,

\[
\gamma = \gamma_{\text{can}}^{(p)} : \Gamma_1 \otimes_k^p K_0^\wedge E/p \xrightarrow{\sim} \mathbb{H} \otimes_{k((x))}^\wedge K_0^\wedge E/p.
\]
The given \((j, \gamma)\) define a map \(\psi^p : K_0^\wedge E \to K_0^\wedge E\), which we have to show lifts the Frobenius. In other words, we must show that the square

\[
\begin{array}{ccc}
K_0^\wedge E & \xrightarrow{\psi^p} & K_0^\wedge E \\
\downarrow & & \downarrow \\
K_0^\wedge E/p & \xrightarrow{\sigma} & K_0^\wedge E/p
\end{array}
\]

commutes. Both compositions are \(K_0\)-algebra maps, when \(K_0\) acts on the target \(K_0^\wedge E/p\) via the Frobenius lift \(\sigma\), so we can check that they are equal using Theorem 5.4. The top right composition represents the pair

\[
(\bar{f} = \psi^p = \sigma_{k((x))}, \gamma = \gamma_{\text{can}}^{(p)}).
\]

Applying a Frobenius to \(K_0^\wedge E/p\) clearly twists its subring \(k((x))\) by a Frobenius. Finally, \(K_0^\wedge E\) carries the coefficients of a power series which is the universal isomorphism \(\gamma_{\text{can}}\) between \(\Gamma_1\) and \(\mathbb{F}\); applying the Frobenius to the coefficients of these power series is the same as pulling back this isomorphism along the Frobenius.

Finally, to show that we have defined a \(\theta\)-comodule algebra structure on \(K_0^\wedge E\), we have to show that the \(\psi^p\) just defined commutes with the action of the Morava stabilizer group. In general, a group element \((\tau, g) \in \text{Aut}(k, \Gamma_1)\) acts on a point \((j, \gamma) \in \text{Hom}_{cts}(F_0^\wedge E, R)\) by precomposition with \(\gamma\):

\[
(\tau, g)(j, \gamma) = (j, \gamma g^{-1}_1 : \Gamma_1 \otimes_k^\Gamma R/m \xrightarrow{g^{-1}} \Gamma_1 \otimes_k^\Gamma R/m \xrightarrow{\gamma} R/m).
\]

This operation commutes with pullback of \(\gamma\) along the Frobenius. \(\square\)

6.4. Obstruction theory. In [GH04, GH05], Goerss and Hopkins construct an obstruction theory for realizing a \(\theta\)-comodule algebra as the completed \(K\)-homology of a \(K(1)\)-local \(E_{\infty}\) ring spectrum. Before describing the obstruction theory, we must make some preliminary definitions. Note that our situation is slightly simplified by the fact that \(K_0^\wedge E\) is classically \(p\)-complete (rather than just \(L\)-complete) and concentrated in even degrees; the definitions below need to be modified to handle more general cases.

**Definition 6.12** ([GH05, Definition 2.2.7]). Let \(A\) be a \(p\)-complete \(\theta\)-comodule algebra. An \(A\)-\(\theta\)-module is a \(p\)-complete \(A\)-module \(M\) equipped with a continuous action by \(\psi^k, k \in \mathbb{Z}_p\), and an operation \(\theta : M \to M\), such that

- for \(a \in A\) and \(m \in M\), \(\psi^k(am) = \psi^k(a)\psi^k(m)\);
- for \(a \in A\) and \(m \in M\), \(\theta(am) = a^p\theta(m) + p\theta(a)\theta(m)\).

Write \(\text{Mod}_{\theta}^A\) for the category of \(A\)-\(\theta\)-modules.

**Remark 6.13.** There is an equivalence between \(\text{Mod}_{\theta}^A\) and the category of abelian group objects in \((\text{ComodAlg}_{\theta}/A)/A\), given by sending

\[
[B \to A] \in \text{Ab}((\text{ComodAlg}_{\theta}/A) \leftrightarrow \ker(B \to A).
\]

Indeed, as a ring, \(B\) is a split square-zero extension of \(A\), of the form \(A \oplus M \xrightarrow{\text{proj}} A\); the structure on \(M\) induced by the \(\theta\)-comodule algebra structure on \(A\) is precisely the \(A\)-\(\theta\)-module structure defined above.

**Definition 6.14.** Let \(A\) be a \(\theta\)-comodule algebra, and \(M\) an \(A\)-\(\theta\)-module. A **derivation** from \(A\) into \(M\) is a map \(A \to A \oplus M\) of \(\theta\)-algebras augmented over \(A\), where \(A \oplus M\) is the split square-zero extension defined in Remark 6.13. We write \(\text{Der}_{\text{ComodAlg}_{\theta}}(A, M)\) for the abelian group of derivations from \(A\) into \(M\).
Definition 6.15. The André-Quillen cohomology groups of the \( \theta \)-algebra \( A \), with coefficients in \( A \)-\( \theta \)-modules \( M \), are the left derived functors
\[
D^s_{\ComodAlg}(A, \cdot)
\]
of \( \text{Der}_{\ComodAlg}(A, M) \).

Remark 6.16. Making sense of these derived functors requires defining a well-behaved model structure on simplicial \( \theta \)-algebras, which is done in [GH05, Proposition 2.3.1].

Remark 6.17. Completely analogously but forgetting about the \( \mathbb{Z}_p^\times \)-action everywhere, we can define \( \theta \)-modules over a \( \theta \)-algebras, derivations, and André-Quillen cohomology.

We can now state the obstruction theory. This is a special case of [GH05, Theorem 3.3.7] as well as the results on \( \theta \)-algebras in [GH05]; see [Szy] for a more precise statement.

Theorem 6.18. Let \( A \) be an \( p \)-complete \( \theta \)-comodule algebra. Then there are successively defined obstructions to realizing \( A \) as \( K_0^E X \), where \( X \) is an \( \mathcal{E}_\infty \) algebra such that \( K_0^E X \) is concentrated in even degrees, in the André-Quillen cohomology groups
\[
D^{s+2}_{\ComodAlg}(A, \Omega^s A), \ s \geq 1.
\]

There are successively defined obstructions to the uniqueness of this realization in
\[
D^{s+1}_{\ComodAlg}(A, \Omega^s A), \ s \geq 1.
\]

We now prove that any \( \theta \)-algebra structure on \( \Lambda \) is induced by an \( \mathcal{E}_\infty \)-algebra structure on \( LE \), using the argument of [GH05, Section 2.4.3]. Thanks again to Dominik Absmeier for calling my attention to a mistake in the earlier version of this statement.

Theorem 6.19. For any \( p \)-complete \( \theta \)-comodule algebra \( A \) such that \( A \) is isomorphic to \( K_0^E \) as a ring with \( \mathbb{Z}_p^\times \)-action, there is an even periodic \( \mathcal{E}_\infty \)-algebra \( X \) with \( K_0^E X = A \) as \( \theta \)-comodule algebras, and with \( X \cong L_{K(1)} E_2 \) as homotopy commutative ring spectra.

Proof. We want to show that the obstruction groups
\[
D^{s+2}_{\ComodAlg}(A, \Omega^s A)
\]
vanish for \( s \geq 1 \). Since \( \Omega^s A \cong K_0^E \Omega^s E \) is an extended \( \mathbb{Z}_p^\times \)-module by Lemma 5.9, these reduce [GH05, Proposition 2.4.7] to André-Quillen cohomology of \( \theta \)-algebras without \( \mathbb{Z}_p^\times \)-action:
\[
D^{s+2}_{\ComodAlg}(A, \Omega^s A) \cong D^{s+2}_{\Alg}(A, \Omega^s LE_0).
\]
The complete cotangent complex of \( A \) over \( Wk \) is a \( \theta \)-module, and there is a composite functor spectral sequence
\[
\text{Ext}^q_{\text{Mod}_{\theta,A}}(\pi_q L_{A/Wk}, \Omega^s LE_0) \Rightarrow D^{p+q}(A, \Omega^s LE_0).
\]
But \( Wk = K_0 \to K_0^E = A \) is formally smooth by Theorem 5.4 and Proposition 5.8. Thus, \( L_{A/Wk} \) is just the Kähler differentials \( \Omega_{A/Wk} \) concentrated in degree zero, and these are a \( p \)-completion of a free module. Finally, there is a resolution \( \Omega_{A/Wk} \) by free \( \theta \)-modules over \( A \),
\[
0 \to A[\theta] \otimes_A \Omega_{A/Wk} \xrightarrow{\theta} A[\theta] \otimes_A \Omega_{A/Wk} \to \Omega_{A/Wk}.
\]
For any complete \( \theta \)-module \( M \) over \( A \),
\[
\text{Ext}^q_{\text{Mod}_{\theta,A}}(A[\theta] \otimes_A \Omega_{A/Wk}, M) = \text{Ext}^q_{\text{Mod}_{A}}(\Omega_{A/Wk}, M),
\]
which is concentrated in degree zero because \( A \) is pro-free. Thus, the André-Quillen cohomology groups are concentrated in cohomological degrees 0 and 1, and in particular, those that can contain obstructions vanish.
This produces a $K(1)$-local $\mathcal{E}_\infty$-algebra $X$ with $K^1_0X = A$. Since $K^1_0X \cong K^1_0 E$ as $\mathbb{Z}_p^\wedge$-modules, we also have $K^1_t X \cong K^1_t E$ as $\mathbb{Z}_p^\wedge$-modules for all $t$. In particular, these are extended and $\mathbb{Z}_p$-free, and so the Adams-Novikov spectral sequence

$$E_2 = \text{Ext}^*_\mathbb{Z}_p(K^1_0 E, K^1_t X) \Rightarrow [E, X]^*,$$

is concentrated on the 0-line. Thus the isomorphism $K^1_t E \to K^1_t X$ of comodule algebras lifts to an equivalence $E \to X$ of homotopy commutative ring spectra. \hfill \Box

**Corollary 6.20.** There exist $\mathcal{E}_\infty$-algebras with underlying ring spectrum $L_{K(1)} E_2$, which are not equivalent to the $K(1)$-localization of the $\mathcal{E}_\infty$-algebra $E_2$.

**Proof.** This follows from Theorem 6.19 and Proposition 6.3. \hfill \Box

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