A Lichnerowicz vanishing theorem for proper cocompact actions

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Abstract

We establish a Lichnerowicz type vanishing theorem for non-compact spin manifolds admitting proper cocompact actions, when the action group is unimodular.

1 Introduction

A well-known result of Lichnerowicz [2] states that if a compact spin manifold of dimension $4k$ admits a Riemannian metric of positive scalar curvature, then the index of the associated Dirac operator vanishes.

The purpose of this note is to establish an extension of this classical result to the case of non-compact manifolds admitting proper cocompact group actions. To be more precise, we will prove such a result, when the action group is unimodular, for the Mathai-Zhang index introduced in [3] for such actions. The main result is stated in Theorem 3.1.

This note is organized as follows. In Section 2, we recall the definition of the Mathai-Zhang index [3] for spin manifolds. In Section 3 we prove our main result mentioned above.

2 The Mathai-Zhang index on spin manifold

Let $M$ be a non-compact even dimensional spin manifold. Let $G$ be a locally compact group. We assume that $G$ acts on $M$ properly and cocompactly, where by proper we mean that the following map

$$G \times M \to M \times M, \quad (g, x) \mapsto (x, gx),$$

is proper (the pre-image of a compact subset is compact), while by cocompact we mean that the quotient space $M/G$ is compact.

We make the assumption that $G$ preserves the spin structure on $M$.
Let $g^{TM}$ be a Riemannian metric on the tangent vector bundle $TM$. Without loss of generality, we can and we will assume that $g^{TM}$ is $G$-invariant (cf. [3 (2.3)]).

Let $S(TM) = S_+(TM) \oplus S_-(TM)$ be the complex vector bundle of spinors associated to $(TM, g^{TM})$. Let $g^{S(TM)}$ and $\nabla^{S(TM)}$ be the canonically induced Hermitian metric and connection on $S(TM)$. Then the action of $G$ on $M$ lifts to an action on $S(TM)$, preserving $g^{S(TM)}$ and $\nabla^{S(TM)}$.

Let $E$ be a Hermitian vector bundle over $M$ such that it admits a $G$-action lifted from the action of $G$ on $M$. Let $g^E$ be a $G$-invariant Hermitian metric on $E$, and $\nabla^E$ be a $G$-invariant Hermitian connection on $E$.

For any tangent vector $X \in TM$, let the Clifford action $c(X)$ on $S(TM)$ extend to act on $S(TM) \otimes E$ by acting as identity on $E$, and we still denote this action by $c(X)$. The $G$-invariant metrics and connections on $S(TM)$ and $E$ induce canonically a $G$-invariant Hermitian metric $g^{S(TM) \otimes E}$ and a $G$-invariant connection $\nabla^{S(TM) \otimes E}$ on $S(TM) \otimes E$.

The twisted (by $E$) Dirac operator acting on $\Gamma(S(TM) \otimes E)$ is given by

$$D^E = \sum_{i=1}^{\dim M} c(e_i) \nabla^{S(TM) \otimes E}_{e_i} : \Gamma(S(TM) \otimes E) \rightarrow \Gamma(S(TM) \otimes E), \quad (2.1)$$

where $e_1, \ldots, e_{\dim M}$ is an oriented orthonormal basis of $TM$.

Let $\Gamma(S(TM) \otimes E)$ carry the natural inner product such that for any $s_1, s_2 \in \Gamma(S(TM) \otimes E)$ with compact support,

$$(s_1, s_2) = \int_M (s_1, s_2)_{S(TM) \otimes E} dv^{TM}. \quad (2.2)$$

Let $\| \cdot \|_0$ be the $L^2$-norm associated to the inner product (2.2), let $\| \cdot \|_1$ be a (fixed) $G$-invariant Sobolev 1-norm. Let $H^0(M, S(TM) \otimes E)$ be the completion of $\Gamma(S(TM) \otimes E)$ under $\| \cdot \|_0$.

Denote the space of $G$-invariant smooth sections of $S(TM) \otimes E$ by $\Gamma(S(TM) \otimes E)^G$.

Now recall that the compactness of $M/G$ guarantees the existence of a compact subset $Y$ of $M$ such that $G(Y) = M$ (cf. [4 Lemma 2.3]). Let $U, U'$ be two open subsets of $M$ such that $Y \subset U$ and that the closures $\overline{U}$ and $\overline{U'}$ are both compact in $M$, and that $\overline{U} \subset U'$. The existence of $U, U'$ is clear.

Following [3], let $f \in C_c^\infty(M)$ be a nonnegative function such that $f|_U = 1$ and $\text{Supp}(f) \subset U'$.

Let $H^0_f(M, S(TM) \otimes E)^G$ and $H^1_f(M, S(TM) \otimes E)^G$ be the completions of $\{ fs : s \in \Gamma(S(TM) \otimes E)^G \}$ under $\| \cdot \|_0$ and $\| \cdot \|_1$ respectively.

Let $P_f$ denote the orthogonal projection from $H^0(M, S(TM) \otimes E)$ to its subspace $H^0_f(M, S(TM) \otimes E)^G$.

Clearly, $P_f D^E$ maps $H^1_f(M, S(TM) \otimes E)^G$ into $H^0_f(M, S(TM) \otimes E)^G$.

We now recall a basic result from [3 Proposition 2.1].
Proposition 2.1 (Mathai-Zhang) The operator

\[ P_f D^E : H^1_f(M, S(TM) \otimes E)^G \to H^0_f(M, S(TM) \otimes E)^G \]

is a Fredholm operator.

Let \( D^E_{\pm} : \Gamma(S_{\pm}(TM) \otimes E) \to \Gamma(S_{\pm}(TM) \otimes E) \) be the restrictions of \( D^E \) on \( \Gamma(S_{\pm}(TM) \otimes E) \) respectively. It has been shown in [3] that \( \text{ind} \ (P_f D^E_{\pm}) \) is independent of the choices of the cut-off function \( f \), as well as the \( G \)-invariant metrics and connections involved. Following [3, Definition 2.4], we denote \( \text{ind} \ (P_f D^E_{\pm}) \) by \( \text{ind}_G(D^E_{\pm}) \).

3 A Lichnerowicz vanishing theorem for Mathai-Zhang index

In this section, we extend the Lichnerowicz vanishing theorem to the case of \( \text{ind}_G(D^E_{\pm}) \) when the action group \( G \) is unimodular.

Let \( k^TM \) be the scalar curvature of \( g^TM \). Let \( R^E \) be the curvature of \( \nabla^E \).

Let \( \{e_i\} \) be a local orthonormal basis of \( TM \). Let \( c(R^E) \), which acts on \( \Gamma(S(TM) \otimes E) \), be defined by

\[
c(R^E) = \frac{1}{2} \sum_{i,j=1}^{\dim M} c(e_i)c(e_j) R^E(e_i, e_j).
\]

(3.1)

Recall the famous Lichnerowicz formula [2] (cf. [1, Chapter II, Theorem 8.17]), which states that,

\[
(D^E)^2 = -\Delta^E + \frac{k^TM}{4} + c(R^E),
\]

(3.2)

where \( \Delta^E = \sum_{i=1}^{\dim M} \left( \left( \nabla^E e_i \otimes e_i \right)^2 - \nabla^E S(TM) \otimes e_i \right) \) is the Bochner Laplacian.

We can now state the main result this note as follows.

**Theorem 3.1** Let \( M \) be a non-compact spin manifold carrying a proper cocompact action by a locally compact unimodular group \( G \), such that the \( G \)-action preserves the spin structure on \( M \). Let \( g^TM \) be a \( G \)-invariant metric on the tangent bundle \( TM \). Let \( E \) be a Hermitian vector bundle over \( M \), carrying a \( G \)-action lifted from that on \( M \), as well as a \( G \)-invariant Hermitian metric and a \( G \)-invariant Hermitian connection \( \nabla^E \). If one assumes that \( k^TM + 4c(R^E) > 0 \) holds pointwise over \( M \), then the Mathai-Zhang index \( \text{ind}_G(D^E_{\pm}) \) vanishes.
Proof. Since $G$ is unimodular, by a result of Mathai-Zhang \cite[Theorem 2.7]{3}, one has that
\[ \text{ind}_G (D^E_+) = \dim \left( (\text{Ker} \ D^E_+)^G \right) - \dim \left( (\text{Ker} \ D^E)^G \right). \quad (3.3) \]

Set $\mathcal{R} = \frac{L^M}{4} + c(R^E)$. For any $s \in (\text{Ker} \ D^E)^G$ and $\varphi \in C^\infty_c(M)$, by (3.2), one gets that
\[ 0 = \int_M \left( (D^E)^2 s, \varphi s \right) dv_{g^M} = \int_M \left( -\Delta^E s, \varphi s \right) dv_{g^M}. \quad (3.4) \]

By (3.4), one has, where we denote $\nabla^{S(TM) \otimes E}$ by $\nabla$ for simplicity,
\[ 0 = \int_M \left( \frac{1}{2} \sum_{i=1}^{\dim M} e_i \langle \nabla e_i s, \nabla e_i (\varphi s) \rangle + \varphi \langle \mathcal{R}s, s \rangle \right) dv_{g^M} \]
\[ = \int_M \left( \frac{1}{2} \sum_{i=1}^{\dim M} e_i \langle e_i (|s|^2), \varphi \nabla s \rangle + \varphi \langle \mathcal{R}s, s \rangle \right) dv_{g^M} \]
\[ = \int_M \left( \frac{1}{2} \sum_{i=1}^{\dim M} e_i \langle e_i (|s|^2), \nabla s \rangle + \frac{1}{2} \sum_{i=1}^{\dim M} \varphi e_i (e_i (|s|^2)) + \varphi \langle \mathcal{R}s, s \rangle \right) dv_{g^M} \]
\[ = \int_M \varphi \left( -\frac{1}{2} \Delta (|s|^2) + \nabla s \langle \nabla s + \langle \mathcal{R}s, s \rangle \right) dv_{g^M}. \quad (3.5) \]

Since $\varphi$ is arbitrary, one sees from (3.5) that for any $s \in (\text{Ker} \ D^E)^G$, there holds,
\[ \frac{1}{2} \Delta (|s|^2) = |\nabla s|^2 + \langle \mathcal{R}s, s \rangle \geq \langle \mathcal{R}s, s \rangle. \quad (3.6) \]

Since the $G$-action on $M$ is cocompact, and that $|s|^2$ is clearly $G$-invariant, $|s|^2$ attains its maximum at certain point $p \in M$, on which one has, by the standard maximum principle, that
\[ \Delta (|s|^2) \leq 0. \]

Combining with (3.6), one sees that under the assumption that $\mathcal{R} > 0$ over $M$,
\[ |s(p)|^2 = 0, \quad (3.7) \]
which implies that $s \equiv 0$ on $M$.

Combining with (3.3), one completes the proof of Theorem 3.1. \qed
Corollary 3.2 Let $M$ be a non-compact spin manifold carrying a proper cocompact action by a locally compact unimodular group $G$, such that the $G$-action preserves the spin structure on $M$. If $M$ admits a $G$-invariant Riemannian metric with positive scalar curvature, then the Mathai-Zhang index $\text{ind}_G(D_+)$ vanishes.

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