Transition to zero cosmological constant and phantom dark energy as solutions involving change of orientation of spacetime manifold

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Abstract
The main conclusion of long-standing discussions concerning the role of solutions with degenerate metric \((g \equiv \det(g_{\mu\nu}) = 0)\) was that in the first-order formalism they are physically acceptable and must be included in the path integral. In particular, they may describe topology changes and reduction of the ‘metrical dimension’ of spacetime. The latter implies disappearance of the volume element \(\sqrt{-g}d^4x\) of a 4D spacetime in a neighborhood of the point with \(g = 0\). We pay attention to the fact that besides \(\sqrt{-g}\), the 4D spacetime differentiable manifold also possesses a ‘manifold volume measure’ (MVM) described by a 4-form which is sign indefinite and generically independent of the metric. The first-order formalism proceeds with an originally independent connection and metric structures of the spacetime manifold. In this paper we bring up the question of whether the first-order formalism should be supplemented with degrees of freedom of the spacetime differentiable manifold itself, e.g. by means of the MVM. It turns out that adding the MVM degrees of freedom to the action principle in the first-order formalism one can realize very interesting dynamics. Such a two measures field theory (TMT) enables radically new approaches to the resolution of the cosmological constant problem. We show that fine tuning free solutions describing a transition to the \(\Lambda = 0\) state involve oscillations of \(g_{\mu\nu}\) and MVM around zero. The latter can be treated as a dynamics involving changes of orientation of the spacetime manifold. As we have shown earlier, in realistic scale invariant models (SIM), solutions formulated in the Einstein frame satisfy all existing tests of general relativity (GR). Here we reveal surprisingly that in SIM, all ground-state solutions with \(\Lambda \neq 0\) appear to be degenerate either in \(g_{00}\) or in MVM. Sign indefiniteness of MVM in a natural way yields a dynamical realization of a phantom cosmology \((w < -1)\). It is very important that for all solutions, the metric tensor rewritten in the Einstein frame has regularity properties exactly as in GR. We discuss new physical effects which arise from
1. Introduction: degenerate metric, manifold volume measure and orientation of the spacetime manifold

Solutions with degenerate metric were the subject of long-standing discussions starting probably with the paper by Einstein and Rosen [1]. In spite of some difficulty interpreting solutions with degenerate metric in the classical theory of gravitation, the prevailing view was that they have physical meaning and must be included in the path integral [2–4]. As shown in [2, 5], in the first-order formulation of an appropriately extended general relativity, solutions with \( g(x) = \det(g_{\mu\nu}) = 0 \) can describe changes of the spacetime topology. A similar idea is also realized in Ashtekar’s variables [6, 7]. There are also known classical solutions [8–14] with the change of the signature of the metric tensor.

The spacetime regions with \( g(x) = 0 \) can be treated as having ‘metrical dimension’ \( D < 4 \) (using terminology by Tseytlin [4]).

The simplest solution with \( g(x) = 0 \) is \( g_{\mu\nu} = 0 \) while the affine connection is arbitrary (or, in the Einstein–Cartan formulation, the vierbein \( e^a_\mu = 0 \) and \( \omega^{ab}_\mu \) is arbitrary). Such solutions have been studied by D’Auria and Regge [3], Tseytlin [4], Witten [15], Horowitz [5], Giddings [16] and Bañados [17]; it has been suggested that \( g_{\mu\nu} = 0 \) should be interpreted as essentially a non-classical phase in which diffeomorphism invariance is unbroken and is realized at high temperature and curvature.

Now we would like to bring up a question: whether the equality \( g(x) = 0 \) really with a necessity means that the dimension of the spacetime manifold in a small neighborhood of the point \( x \) may become \( D < 4 \)? At first sight it should be so because the volume element is

\[
dV_{\text{metrical}} = \sqrt{-g} \, d^4x.
\]

Note that the latter is the ‘metrical’ volume element, and the possibility of describing the volume of the spacetime manifold in this way appears after the four-dimensional differentiable manifold \( M_4 \) is equipped with the metric structure. For a solution with \( g_{\mu\nu} = 0 \), the situation with the description of the spacetime becomes even worse. However, in spite of the lack of the metric, the manifold \( M_4 \) may still possess a nonzero volume element and have the dimension \( D = 4 \). The well known way to realize it consists of the construction of a differential 4-form build for example by means of four differential 1-forms \( d\varphi_a, \) \( a = 1, 2, 3, 4 \):

\[
d\varphi_1 \wedge d\varphi_2 \wedge d\varphi_3 \wedge d\varphi_4.
\]

Each of the 1-forms \( d\varphi_a \) may be defined by a scalar field \( \varphi_a(x) \). The appropriate volume element of the four-dimensional differentiable manifold \( M_4 \) can be represented in the following way:

\[
dV_{\text{manifold}} = 4! \, d\varphi_1 \wedge d\varphi_2 \wedge d\varphi_3 \wedge d\varphi_4 \equiv \Phi \, d^4x
\]

where

\[
\Phi \equiv \varepsilon_{abcd} \varepsilon^{\mu\nu\lambda\sigma} \left( \partial_\mu \varphi_a \right) \left( \partial_\nu \varphi_b \right) \left( \partial_\lambda \varphi_c \right) \left( \partial_\sigma \varphi_d \right)
\]

is the volume measure independent of \( g_{\mu\nu} \) as opposed to the case of the metrical volume measure \( \sqrt{-g} \). In order to emphasize the fact that the volume element (2) is metric independent we will call it a manifold volume element and the measure \( \Phi \)—a manifold volume measure.
If $\Phi(x) \neq 0$ one can think of four scalar fields $\varphi_a(x)$ as describing a homomorphism of an open neighborhood of the point $x$ on the four-dimensional Euclidean space $R^4$. However, if one allows a dynamical mechanism of metrical dimensional reduction of the spacetime by means of degeneracy of the metrical volume measure $\sqrt{-g}$, there is no reason to ignore a possibility of a similar effect permitting degenerate manifold volume measure $\Phi$. The possibility of such a (or even stronger, with a sign change of $\Phi$) dynamical effect seems to be more natural since the manifold volume measure $\Phi$ is sign indefinite (in measure theory, sign indefinite measures are known as signed measures [18]). Note that the metrical and manifold volume measures are not obliged generically to be simultaneously nonzero.

The original idea to use differential forms as describing dynamical degrees of freedom of the spacetime differentiable manifold has been developed by Taylor in his attempt [19] to quantize the gravity. Taylor argued that quantum mechanics is not compatible with a Riemannian metric spacetime; moreover, in the quantum regime spacetime is not even an affine manifold. Only in the classical limit do the metric and connection emerge, which allows one to then construct a traditional spacetime description. Of course, the transition to the classical limit is described in [19] rather in the form of a general prescription. Thereupon we would like to pay attention to the additional possibility which was ignored in [19]. Namely, in the classical limit not only do the metric and connection emerge but also some of the differential forms could keep (or restore) certain dynamical effects in the classical limit. In such a case, the traditional spacetime description may occur to be incomplete. Our key idea is that one of these lost differential forms, the 4-form (2) survives in the classical limit as describing dynamical degrees of freedom of the volume measure of the spacetime manifold, and hence can affect the gravity theory on the classical level.

If we add four scalar fields $\varphi_a(x)$ as new variables to a set of usual variables (like metric, connection and matter degrees of freedom) which undergo variations in the action principle, then one can expect an effect of gravity and matter on the manifold volume measure $\Phi$ and vice versa. We will see later in this paper that in fact such effects exist and in particular classical cosmology solutions of a significant interest exist where $\Phi$ vanishes and changes sign.

As is well known, the four-dimensional differentiable manifold is orientable if it possesses a differential form of degree 4 which is nonzero at every point on the manifold. Therefore two possible signs of the manifold volume measure (3) are associated with two possible orientations of the spacetime manifold. The latter means that besides a dimensional reduction and topology changes on the level of the differentiable manifold, the incorporation of the manifold volume measure $\Phi$ allows us to realize solutions describing dynamical change of the orientation of the spacetime manifold.

In the light of the existence of two volume measures, the simplest way to take into account this fact in the action principle consists of the modification of the action which should now consist of two terms, one with the usual measure $\sqrt{-g}$ and another—with the measure $\Phi$,

$$S_{\text{mod}} = \int (\Phi L_1 + \sqrt{-g} L_2) \, d^4x,$$

where two Lagrangians $L_1$ and $L_2$ coupled with the manifold and metrical volume measures appear, respectively. According to our previous experience [22–34] in two measures field theory (TMT) we will proceed with an additional basic assumption that, at least on the classical level, the Lagrangians $L_1$ and $L_2$ are independent of the scalar fields $\varphi_a(x)$, i.e. the manifold volume measure degrees of freedom enter into TMT only through the manifold

1 An opposite view on the role of the volume element has been studied by Wilczek [20]. Another idea of modified volume element was studied in [21].

2 For a more detailed discussion of the role of scalars $\varphi_a(x)$ in the TMT dynamics, see the end of section 2.
volume measure $\Phi$. In such a case, the action (4) possesses an infinite dimensional symmetry
\[ \phi_a \rightarrow \phi_a + f_a(L_1), \tag{5} \]
where $f_a(L_1)$ are arbitrary functions of $L_1$ (see details in [24]). One can hope that this
symmetry should prevent the emergence of the scalar fields $\phi_a(x)$ dependence in $L_1$ and $L_2$
after quantum effects are taken into account.

Note that equation (4) is just a convenient way to present the theory in a general form. In
concrete models studied in the present paper, we will see that the action (4) can be always
rewritten in an equivalent form where each term in the action has its own total volume measure
and the latter is a linear combination of $\Phi$ and $\sqrt{-g}$.

In the following section it will be shown that the spacetime geometry described in terms
of the original metric and connection of the underlying action (4) is not generically Riemannian.
However by making use a change of variables to the Einstein frame one can represent the
resulting equations of motion in the Riemannian (or Einstein–Cartan) spacetime.

In our previous investigations we have shown that TMT enables radically new approaches
to the resolution of the cosmological constant (CC) problem [24, 25, 33] (for an alternative
approach see [35]). Intrinsic features of TMT allow us to realize a scalar field dark energy
model where all dependence of the scalar field appears as a result of the spontaneous breakdown
of the dilatation symmetry. Solutions of this model formulated in the Einstein frame satisfy
all existing tests of general relativity (GR) [30, 31, 34]. A new sort of dynamical protection
from the initial singularity of the curvature becomes possible [33]. It also allows us to realize a
phantom dark energy in the late time universe without introducing an explicit phantom scalar
field [33].

In contrast to all our previous investigations of TMT, the purpose of the present paper
is to study the dynamics of the metric $g_{\mu\nu}$ and the manifold volume measure $\Phi$ (used in
the underlying action (4)) in a number of TMT models. Attention is mainly concentrated on the
analysis of the amazing features of the ‘irregularity’ of $g_{\mu\nu}$ and $\Phi$ (involving a change of
orientation of the spacetime manifold) in the course of transitions to a ground state and in the
phantom dark energy. It is very important to note immediately that in the Einstein frame the
metric tensor has regularity properties exactly as in GR. The organization of the paper is as
follows. In section 2 we discuss general features of classical dynamics in TMT. In section 3 we
consider the pure gravity model. In sections 4 and 5, in the framework of a simple scalar field
model I, we analyze in detail the behavior of two volume measures in the course of transition
to the ground state with zero CC. In section 6 we study a (generically broken) intrinsic TMT
symmetry which is restored in the ground states; the relation of this symmetry restoration
to the old CC problem [36] is also analyzed; a discussion of this effect is continued in
section 8. In section 7 we briefly present the scalar field model II with spontaneously broken
global scale invariance [25], studied in detail in [33]. In the framework of such a class of
models, an interesting dynamics of the metric and the manifold volume measure in the course
of transition to ground states is analyzed in section 8. In section 9 we reveal that the possibility
of realizing a phantom dark energy without explicitly introducing a phantom scalar field
(demonstrated in [33]) has its origin in a sign indefiniteness of the manifold volume measure
(3). Finally, since one cannot check directly whether a tiny/zero cosmological constant is
fine-tuned or not, in section 10 we discuss the new physical effects which arise from this
theory and in particular a strong gravity effect in high energy physics experiments.

2. Classical equations of motion

Varying the measure fields $\phi_a$, we get $B^{\mu a}_{\nu} \partial_\nu L_1 = 0$ where $B^{\mu a}_{\nu} = \varepsilon^{\mu \nu \alpha \beta} \varepsilon_{\alpha \beta \gamma \delta} \partial_\gamma \phi_b \partial_\delta \phi_c \partial_\gamma \phi_d$. Since $\text{Det}(B^{\mu a}_{\nu}) = \frac{-1}{\Phi^4} \Phi^3$ it follows that if $\Phi \neq 0$ the constraint
\[ L_1 = sM^4 = \text{const.} \]  

must be satisfied, where \( s = \pm 1 \) and \( M \) is a constant of integration with the dimension of mass. Variation of the metric \( g^{\mu\nu} \) gives

\[
\zeta \frac{\partial L_1}{\partial g^{\mu\nu}} + \frac{\partial L_2}{\partial g^{\mu\nu}} - \frac{1}{2} g^{\mu\nu} L_2 = 0, \tag{7}
\]

where

\[ \zeta = \frac{\Phi}{\sqrt{-g}}, \tag{8} \]

is the scalar field build of the scalar densities \( \Phi \) and \( \sqrt{-g} \).

We study models with the Lagrangians of the form

\[
L_1 = -\frac{1}{b_g} R(\Gamma, g) + L_1^m, \quad L_2 = -\frac{1}{\kappa} R(\Gamma, g) + L_2^m, \tag{9}
\]

where \( \Gamma \) stands for affine connection, \( R(\Gamma, g) = g^{\mu\nu} R_{\mu\nu}(\Gamma) \), \( R_{\mu\nu}(\Gamma) = R^\lambda_{\mu\nu\lambda}(\Gamma) \) and

\[
R^\lambda_{\mu\nu\sigma}(\Gamma) = \Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\nu\sigma} g_{\mu\nu} - \sigma,\mu g_{\mu\nu} \tag{10}
\]

\[
\sigma,\mu \equiv \frac{1}{\zeta + b_g} \zeta,\mu. \tag{11}
\]

If \( \zeta \neq \text{const.} \) the covariant derivative of \( g_{\mu\nu} \) with this connection is nonzero (nonmetricity) and consequently the geometry of the spacetime with the metric \( g_{\mu\nu} \) is generically non-Riemannian. The gravity and matter field equations obtained by means of the first-order formalism contain both \( \zeta \) and its gradient as well. It turns out that at least at the classical level, the measure fields \( \phi_a \) affect the theory only through the scalar field \( \zeta \).

For the class of models (9), the consistency of the constraint (6) and the gravitational equations (7) have the form of the following constraint:

\[
(\zeta - b_g)(sM^4 - L_1^m) + g^{\mu\nu} \left( \frac{\partial L_1^m}{\partial g^{\mu\nu}} + \frac{\partial L_2^m}{\partial g^{\mu\nu}} \right) - 2L_2^m = 0, \tag{12}
\]

which determines \( \zeta(x) \) (up to the chosen value of the integration constant \( sM^4 \)) as a local function of matter fields and metric. Note that the geometrical object \( \zeta(x) \) does not have its own dynamical equation of motion and its spacetime behavior is totally determined by the metric and matter fields dynamics via the constraint (12). Together with this, since \( \zeta \) enters into all equations of motion, it generically has straightforward effects on the dynamics of the matter and gravity through the forms of potentials, variable fermion masses and selfinteractions [22–34].

For understanding the structure of TMT it is important to note that TMT (where, as we suppose, the scalar fields \( \phi_a \) enter only via the measure \( \Phi \)) is a constrained dynamical system. In fact, the volume measure \( \Phi \) depends only upon the first derivatives of fields \( \phi_a \) and this dependence is linear. The fields \( \phi_a \) do not have their own dynamical equations: they are auxiliary fields. All of their dynamical effect is displayed only in the following two ways: (a)
in generating the constraint (6) (or (12)); (b) in the appearance of the scalar field $\zeta$ and its gradient in all equations of motion.

3. Pure gravity TMT model

Let us start from the simplest TMT model with action (4) where

$$L_1 = -\frac{1}{b_\kappa^2} R(\Gamma, g) - \Lambda_1,$$
$$L_2 = -\frac{1}{\kappa} R(\Gamma, g) - \Lambda_2$$

and $\Lambda_1, \Lambda_2$ are constants. Note that $\Lambda_1 = \text{const.}$ cannot have a physical contribution to the field equations (in this model—only gravitational) because $\Phi_1/\Lambda_1$ is a total derivative. Nevertheless, we keep $\Lambda_1$ to see explicitly how $\Lambda_1$ appears in the result. $\Lambda_2/2$ would have a sense of the cosmological constant in the regular, non-TMT, theory (i.e. with the only measure $\sqrt{-g}$).

Following the procedure described in section 2 we obtain the gravitational equations (7) and the constraint (12) in the following form:

$$R_{\mu\nu}(\Gamma) = \frac{\kappa}{b_\kappa^2 \Lambda_2} \varepsilon_{\mu\nu}$$
$$\zeta = b_\kappa - \frac{2\Lambda_2}{sM^4 + \Lambda_1} = \text{const.},$$

where $sM^4$ is the constant of integration that appears in equation (6) and we have assumed that the total volume measure in the gravitational term of the action is nonzero, that is $\Phi/b_\kappa + \sqrt{-g} \neq 0$.

Since $\zeta = \text{const.}$ the connection $\Gamma_{\mu\nu}^\lambda$, equation (10), coincides with the Christoffel’s connection coefficients of the metric $g_{\mu\nu}$. Therefore in the model under consideration, the spacetime with the metric $g_{\mu\nu}$ is (pseudo) Riemannian. It follows from equations (14) and (15) the resulting Einstein equations:

$$G_{\mu\nu}(g) = \frac{\kappa}{2} \Lambda g_{\mu\nu}; \quad \Lambda = \frac{b_\kappa}{2} (sM^4 + \Lambda_1).$$

Constancy of $\zeta(x)$ on the mass shell, equation (15), means that for the described solution the manifold and metrical volume measures coincide up to a normalization factor. However, this is true only on the mass shell; if we try to start from this assumption in the underlying action the resulting solution would be completely different.

The model possesses a few interesting features in what concerns the CC:

1. The effective CC $\Lambda$ appears as a constant of integration (as we noted above, the parameter $\Lambda_1$ has no physical meaning and it can be absorbed by the constant of integration). The effective regular, non-TMT, gravity theory provides the same equations if the cosmological constant is added explicitly.

2. The effective cosmological constant $\Lambda$ does not depend at all on the CC-like parameter $\Lambda_2$ (which should describe a total vacuum energy density including vacuum fluctuations of all matter fields). The latter resembles the situation in the unimodular theory [37, 38].

3. Note that $\Lambda$ becomes very small if the integration constant is chosen such that $sM^4 + \Lambda_1$ is very small. The latter is equivalent to a solution with $\Phi/b_\kappa \gg \sqrt{-g}$. In the limit where the metrical volume measure $\sqrt{-g} \rightarrow 0$ while the manifold volume measure $\Phi$ remains nonzero, we get $\Lambda \rightarrow 0$. Thus a $\Lambda = 0$ state is realized for a solution which involves a reduction of the metrical dimension to $D^{(g)} < 4$ and at the same time the dimension of the spacetime as a differentiable manifold remains $D^{(m)} = 4$. 
In the limit where the free parameter $b_g \to \infty$, the gravitational term in the underlying action (equations (4) and (13)) with coupling to the manifold measure $/Phi_1$ approaches zero; then TMT takes the form of a regular (non-TMT) field theory, but the effective cosmological constant $/Lambda_1$ becomes infinite. If we wish to reach a very small value of $/Lambda_1$ keeping the integration constant arbitrary, one should take the opposite limit where $b_g \ll 1$. Then in the underlying action, the weight of the gravitational term with coupling to the manifold volume measure $/Phi_1$ increases with respect to the regular one with coupling to the metrical measure $\sqrt{-g}$.

The above speculations can be regarded as a strong indication that TMT possesses a potential for resolution of the CC problem. In the following sections we will study this issue in more realistic models.

4. Scalar field model I

Let us now study a model including gravity as in equations (9) and a scalar field $\phi$. The action has the same structure as in equation (4) but it is more convenient to write it in the following form:

$$S^{(1)}_{mod} = \frac{1}{b_g} \int d^4x \left[ -\frac{1}{\kappa} (\Phi + b_g \sqrt{-g}) R(\Gamma, g) ight. $$

$$ \left. + (\Phi + b_g \sqrt{-g}) \frac{1}{2} g^{\mu\nu} \phi,\mu \phi,\nu - \Phi V_1(\phi) - \sqrt{-g} V_2(\phi) \right]. $$

The appearance of the dimensionless factor $b_\phi$ is explained by the fact that without fine tuning it is impossible in general to provide the same coupling of the $\phi$ kinetic term to the measures $\Phi$ and $\sqrt{-g}$. $V_1(\phi)$ and $V_2(\phi)$ are potential-like functions; we will see below that the physical potential of the scalar $\phi$ is a complicated function of $V_1(\phi)$ and $V_2(\phi)$.

The constraint (12) reads now

$$ (\zeta - b_g) [s M^4 + V_1(\phi)] + 2 V_2(\phi) + b_g \frac{\delta}{2} g^{\alpha\beta} \phi,\alpha \phi,\beta = 0, $$

where

$$ \delta = b_g - b_\phi. $$

Since $\zeta \neq \text{const.}$, the connection (10) differs from the connection of the metric $g_{\mu\nu}$. Therefore the spacetime with the metric $g_{\mu\nu}$ is non-Riemannian. To see the physical meaning of the model we perform a transition to a new metric $\tilde{g}_{\mu\nu}$

$$ \tilde{g}_{\mu\nu} = (\zeta + b_g) g_{\mu\nu}, $$

where the connection $\Gamma^\lambda_{\mu\nu}$ becomes equal to the Christoffel connection coefficients of the metric $\tilde{g}_{\mu\nu}$ and the spacetime turns into (pseudo) Riemannian. This is why we call the set of dynamical variables using the metric $\tilde{g}_{\mu\nu}$ the Einstein frame. One should point out that the transformation (20) is not a conformal one since $(\zeta + b_g)$ is sign indefinite. But $\tilde{g}_{\mu\nu}$ is a regular pseudo-Riemannian metric. For the action (17), gravitational equations (7) in the Einstein frame take a canonical GR form with the same $\kappa = 16\pi G$

$$ G_{\mu\nu}(\tilde{g}_{\alpha\beta}) = \frac{\kappa}{2} T_{\mu\nu}^{eff}. $$

Here $G_{\mu\nu}(\tilde{g}_{\alpha\beta})$ is the Einstein tensor in the Riemannian spacetime with the metric $\tilde{g}_{\mu\nu}$ and the energy–momentum tensor reads

$$ T_{\mu\nu}^{eff} = \frac{\zeta + b_g}{\zeta + b_\phi} (\phi,\mu \phi,\nu - \tilde{g}_{\mu\nu} X) - \tilde{g}_{\mu\nu} \frac{b_g - b_\phi}{(\zeta + b_g)} X + \tilde{g}_{\mu\nu} V_{\text{eff}}(\phi; \zeta, M). $$
where
\[ X \equiv \frac{1}{2} \tilde{g}^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta}. \]
and the function \( V_{\text{eff}}(\phi; \zeta, M) \) is defined as follows:
\[ V_{\text{eff}}(\phi; \zeta, M) = \frac{b_g [sM^4 + V_1(\phi)] - V_2(\phi)}{(\zeta + b_g)^2}. \] (24)

The scalar \( \phi \) field equation following from equation (17) and rewritten in the Einstein frame reads
\[ \frac{1}{\sqrt{-\tilde{g}}} \left[ \frac{\zeta + b_g}{\zeta + b_g} \sqrt{-\tilde{g}} \tilde{g}^{\mu\nu} \partial_\mu \phi \right] + \frac{\zeta V'_1 + V'_2}{(\zeta + b_g)^2} = 0. \] (25)

The scalar field \( \zeta \) in equations (22)–(25) is determined by means of the consistency equation (12) which in the Einstein frame (20) takes the form
\[ (\zeta - b_g) [sM^4 + V_1(\phi)] + 2V_2(\phi) + \delta \cdot b_g (\zeta + b_g)X = 0. \] (26)

5. Manifold measure and the old cosmological constant problem: cosmological dynamics with \( |\Phi|/\sqrt{-g} \rightarrow \infty \)

It is interesting to see the role of the manifold volume measure in the resolution of the CC problem. We accomplish this now in the framework of the scalar field model I of the previous section. The \( \zeta \)-dependence of \( V_{\text{eff}}(\phi; \zeta, M) \), equation (24), in the form of inverse square like \((\zeta + b_g)^{-2}\) has a key role in the resolution of the old CC problem in TMT. One can show that if quantum corrections to the underlying action generate nonminimal coupling like \( \propto R(\Gamma, g) \phi^2 \) in both \( L_1 \) and \( L_2 \), the general form of the \( \zeta \)-dependence of \( V_{\text{eff}} \) remains similar: \( V_{\text{eff}} \propto (\zeta + f(\phi))^{-2} \), where \( f(\phi) \) is a function. The fact that only such a type of \( \zeta \)-dependence emerges in \( V_{\text{eff}}(\phi; \zeta, M) \), and a \( \zeta \)-dependence is absent for example in the numerator of \( V_{\text{eff}}(\phi; \zeta, M) \), is a direct result of our basic assumption that \( L_1 \) and \( L_2 \) in action (4) are independent of the manifold measure fields \( \phi_a \).

Generically, in action (17), \( b_\phi \neq b_\zeta \) which yields a nonlinear kinetic term (i.e. the \( k \)-essence type dynamics) in the Einstein frame3. But for purposes of this section it is enough to take a simplified model with \( b_\phi = b_\zeta \) (which is in fact a fine tuning) since the nonlinear kinetic term has no qualitative effect on the zero CC problem. In such a case solving the constraint (26) for \( \zeta \) and substituting into equations (22)–(25) we obtain equations for the scalar-gravity system which can be described by the regular GR effective action with the scalar field potential
\[ V_{\text{eff}}(\phi) = \frac{(sM^4 + V_1(\phi))^2}{4[b_g (sM^4 + V_1(\phi)) - V_2(\phi)]}. \] (27)
For an arbitrary nonconstant function \( V_1(\phi) \) there exist infinitely many values of the integration constant \( sM^4 \) such that \( V_{\text{eff}}(\phi) \) has the absolute minimum at some \( \phi = \phi_0 \) with \( V_{\text{eff}}(\phi_0) = 0 \) (provided \( b_g [sM^4 + V_1(\phi_0)] - V_2(\phi_0) > 0 \). This effect takes place as \( sM^4 + V_1(\phi_0) = 0 \) without fine tuning of the parameters and initial conditions. Note that the choice of the scalar field potential in the GR effective action in a form proportional to a perfect square-like emerging in equation (27) would mean a fine tuning.

For illustrative purposes let us consider the model with
\[ V_1(\phi) = \frac{1}{2} \mu_1^2 \phi^2, \quad V_2(\phi) = V_2^{(0)} + \frac{1}{2} \mu_2^2 \phi^2. \] (28)

3 See also [33] where we study in detail a model with dilatation symmetry which also results in the \( k \)-essence type dynamics.
Recall that adding a constant to $V_1$ does not affect the equations of motion, while $V_2^{(0)}$ absorbs the bare CC and all possible vacuum contributions. We take negative integration constant, i.e. $s = -1$, and the only restriction on the values of the integration constant $M$ and the parameters is that the denominator in (27) is positive.$^4$

Consider spatially flat FRW universe with the metric in the Einstein frame
\[
\tilde{g}_{\mu\nu} = \text{diag}(1, -a^2, -a^2, -a^2),
\]
where $a = a(t)$ is the scale factor. Each cosmological solution ends with the transition to a $\Lambda = 0$ state via damping oscillations of the scalar field $\phi$ toward its absolute minimum $\phi_0$. The appropriate oscillatory regime in the phase plane is presented in figure 1.

It follows from the constraint (26) (where we took $\delta = 0$) that $|\zeta| \to \infty$ as $\phi \to \phi_0$. More exactly, oscillations of $sM^4 + V_1$ around zero are accompanied with a singular behavior of $\zeta$ each time $\phi$ crosses $\phi_0$:
\[
\frac{1}{\zeta} \sim sM^4 + V_1(\phi) \to 0 \quad \text{as} \quad \phi \to \phi_0
\]
and $\zeta^{-1}$ oscillates around zero together with $sM^4 + V_1(\phi)$. Taking into account that the metric in the Einstein frame $\tilde{g}_{\mu\nu}$, equation (29), is regular we deduce from equation (20) that the metric $g_{\mu\nu}$ used in the underlying action (17) becomes degenerate each time $\phi$ crosses $\phi_0$:
\[
g_{00} = \frac{\tilde{g}_{00}}{\zeta + b_0} \sim \frac{1}{\zeta} \to 0; \quad g_{ii} = \frac{\tilde{g}_{ii}}{\zeta + b_i} \sim -\frac{1}{\zeta} \to 0 \quad \text{as} \quad \phi \to \phi_0,
\]
where we have taken into account that the energy density approaches zero and therefore for this cosmological solution the scale factor $a(t)$ remains finite in all times $t$. Therefore
\[
\sqrt{-g} \sim \frac{1}{\zeta} \to 0 \quad \text{and} \quad \Phi = \zeta^{1/2} \sqrt{-g} \sim \frac{1}{\zeta} \to 0 \quad \text{as} \quad \phi \to \phi_0.
\]

The detailed behavior of $\zeta$, the manifold measure $\Phi$ and $g_{\mu\nu}$—components$^5$ are shown in figure 2.

$^4$ For the case $s = +1$ and the ground state with nonzero cosmological constant see appendix A.

$^5$ Since the metric in the Einstein frame $\tilde{g}_{\mu\nu}$ is diagonal, equation (29), it is clear from the transformation (20) that $g_{\mu\nu}$ is also diagonal.
Recall that the manifold volume measure $\Phi$ is a signed measure [18] and therefore it is not a surprise that it can change sign. But TMT shows that including the manifold degrees of freedom into the dynamics of the scalar-gravity system we discover an interesting dynamical effect: a transition to zero vacuum energy is accompanied by oscillations of $\Phi$ around zero. Similar oscillations\(^6\) simultaneously occur with all components of the metric $g_{\mu\nu}$ used in the underlying action (17).

The measure $\Phi$ and the metric $g_{\mu\nu}$ pass zero only in a discrete set of moments in the course of transition to the $\Lambda = 0$ state. Therefore there is no problem with the condition $\Phi \neq 0$ used for the solution (6). Also there is no problem with singularity of $g_{\mu\nu}$ in the underlying action since

$$\lim_{\phi \to \phi_0} \Phi g^{\mu\nu} = \text{finite} \quad \text{and} \quad \sqrt{-g} g^{\mu\nu} \sim \frac{1}{\zeta} \to 0 \quad \text{as} \quad \phi \to \phi_0.$$  

(33)

The metric in the Einstein frame $\tilde{g}_{\mu\nu}$ is always regular because degeneracy of $g_{\mu\nu}$ is compensated in equation (20) by singularity of the ratio of two measures $\zeta \equiv \Phi / \sqrt{-g}$.

\section{6. Restoration of intrinsic TMT symmetry in the course of transition to zero cosmological constant state}

Let us now turn to intrinsic symmetry of TMT which can reveal itself in a model with only the manifold volume measure $\Phi$. Indeed, if in equation (9) we take the limit\(^7\) $b_g \to 0$ and $L_2^m \to 0$ then equation (12) reads

$$L_1^m - g^{\mu\nu} \frac{\partial L_1^m}{\partial g_{\mu\nu}} = sM^4, \quad \text{if} \quad \zeta \neq 0.$$  

(34)

If in addition $L_1^m$ is homogeneous of degree 1 in $g^{\mu\nu}$ then the integration constant $M$ must be zero. The simplest example of a model for $L_1^m$ satisfying this property is the massless scalar

\(^6\) Note however that these oscillations do not affect the sign of the metrical volume measure $g = \det(g_{\mu\nu})$ used in the underlying action (17). This notion is useful when comparing our model with an approach developed in [42–44].

\(^7\) For the model of section 4 it means that in equation (17) we take the limit $b_g \to 0, b_\phi \to 0$ and $V_2 \to 0$. 

Figure 2. Oscillations of the measure $\Phi$, the original metric $g_{\mu\nu}$ and the rhs of equation (41) during the transition to $\Lambda = 0$ state.
field. In such a case the theory is invariant under transformations in the space of the scalar fields $\phi_a$

$$\phi_a \rightarrow \phi'_a = \phi'_a(\phi_b)$$

resulting in the transformation of the manifold volume measure $\Phi$

$$\Phi(x) \rightarrow \Phi'(x) = J(x)\Phi(x), \quad J(x) = \text{Det} \left( \frac{\partial \phi'_a}{\partial \phi_b} \right)$$

simultaneously with the local transformation of the metric

$$g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x) = J(x)g_{\mu\nu}(x).$$

This symmetry was studied in earlier publications [22] where we called it the local Einstein symmetry (LES).

Consider now linear transformations in the space of the scalar fields $\phi_a$

$$\phi_a \rightarrow \phi'_a = A^b_a\phi_b + C_b, \quad a, b = 1, 2, 3, 4$$

where $A^b_a = \text{constants}, C_b = \text{constants}$. Then LES (35)–(37) is reduced to transformations of the global Einstein symmetry (GES) with $J = \text{det}(A^b_a) = \text{const}$. Note that the Einstein symmetry contains a $\mathbb{Z}_2$ subgroup of the sign inversions when $J = -1$

$$\Phi \rightarrow -\Phi, \quad g_{\mu\nu} \rightarrow -g_{\mu\nu}.$$ (39)

LES as well as GES appear to be explicitly broken if $L_1^m$ is not a homogeneous function of degree 1 in $g^{\mu\nu}$, for example as in the model where $L_1^m$ describes a scalar field with a nontrivial potential. The Lagrangian $L_1^m$ generically breaks the Einstein symmetry too. The transformation of GES originated by the infinitesimal linear transformations $\phi_a(x) \rightarrow \phi'_a(x) = (1 + \epsilon/4)\phi_a(x), \epsilon = \text{const},.$ yields the following variation of the action (4) written in the form $S = \int L d^4x$ where $L = \Phi L_1^m + \sqrt{-g}L_2$

$$\delta S = \int \left[ -\frac{\partial L}{\partial g^{\mu\nu}}g^{\mu\nu} + L_1 \frac{\partial \Phi}{\partial \phi_a,\mu} \phi_a,\mu \right] \epsilon \, d^4x.$$ (40)

The first term in (40) equals zero on the mass shell giving the gravitational equation (7); recall that we proceed in the first-order formalism. Integrating the second term by part, using equation (6) and the definition (3) of the measure $\Phi$, we reduce the variation (40) to

$$\delta S = \epsilon \int \partial_\mu j^\mu d^4x$$

where $\partial_\mu j^\mu = sM^4\Phi$ and $j^\mu = sM^4B^a_\mu \phi_a$. In the presence of topological defects with $\Phi = 0$, equation (6) does not hold anymore all over spacetime, and one should keep $L_1$ in the definition of the current: $j^\mu = L_1^m B^a_\mu \phi_a$. In subsection 8.4 we will see how such a situation may be realized.

To present the current conservation in the generally coordinate invariant form one has to use the covariant divergence. However when doing this using the original metric $g_{\mu\nu}$ we encounter the non-metricity. It is much more transparent to use the Einstein frame (20) where the spacetime becomes pseudo-Riemannian and the covariant derivative of the metric $\tilde{g}_{\mu\nu}$ equals zero identically. Thus with the definition $j^\mu = \sqrt{-g}J^\mu$, using the definition of $\zeta$ in equation (7) and the transformation to the Einstein frame (20) we obtain

$$\tilde{\nabla}_\mu J^\mu \equiv \frac{1}{\sqrt{-\tilde{g}}} \partial_\mu(\sqrt{-\tilde{g}}J^\mu) = sM^4 \zeta \left( \zeta + b_\zeta \right)^2.$$ (41)

As one should expect, when $L_2 \equiv 0$ and $L_1^m$ is homogeneous of degree 1 in $g^{\mu\nu}$, i.e. in the case of unbroken GES, the current is conserved because in this case the integration constant $M = 0$.

---

8 Note that the pure gravity model of section 3 is invariant both under the LES and the GES.
As we have seen in the framework of the scalar field model of section 5, the dynamical evolution pushes $|\zeta| \equiv |\Phi_1|/\sqrt{-g} \to \infty$ as the gravity + scalar field $\phi$-system approaches (without fine tuning) the $\Lambda = 0$ ground state $\phi = \phi_0$. Therefore according to equation (41),

$$\nabla_\mu J^\mu \to 0 \quad \text{as} \quad \phi \to \phi_0.$$  

For the model of section 5, the damping oscillations of the rhs of equation (41) around zero are shown in figure 2. Thus, the GES explicitly broken in the underlying action emerges in the vacuum which, as it turns out, has zero energy density. And vice versa, emergence of GES due to $|\zeta| \to \infty$ implies, according to equation (24), a transition to a $\Lambda = 0$ ground state.

Another way to reach the same conclusion is to look at the underlying action (17). In virtue of equation (33), it is evident that in the course of transition to the ground state, the terms in (17) coupled to the metric volume measure $\sqrt{-g}$ become negligible in comparison with the corresponding terms coupled to the manifold volume measure $\Phi_1$; besides the term $-\int V_1(\phi) \Phi_1 d^4x$ (which also breaks the GES) disappears as $\phi \to \phi_0$. The only terms surviving in the transition to the $\Lambda = 0$ ground state are the following

$$\frac{1}{\kappa} \int \Phi d^4x \left[ -\frac{1}{\kappa} R(\Gamma_1, g) + \frac{1}{2} g^{\mu\nu} \phi_\mu \phi_\nu \right]$$

and they respect the GES.

One should note however that one can regard the GES as the symmetry responsible for a zero CC only if TMT is taken in the strict framework formulated in section 1. In fact, let us consider for example a modified TMT model where the manifold volume measure degrees of freedom enter in the Lagrangian $L_1$ in contrast to our additional basic assumption made in section 1 (after equation (4)). Namely let us assume that the Lagrangian $L_1$ in equation (4) involves a term proportional to $\Phi_1/\sqrt{-g}$ that explicitly breaks the infinite dimensional symmetry (5). To be more concrete we consider a model with the action

$$S = S^{(1)}_{\text{mod}} - \lambda \int \frac{\Phi^2}{\sqrt{-g}} d^4x,$$

where $S^{(1)}_{\text{mod}}$ is the action defined in equation (17). Such an addition to action (17) respects the GES but it is easy to see that it affects the theory in such a way that without fine tuning it is impossible generically to reach a zero CC (see appendix B).

7. Scalar field model II. Global scale invariance

Let us now turn to analyzing the results of the TMT model possessing a global scale invariance studied early in detail [25–27, 29–33]. The scalar field $\phi$ playing the role of a model of dark energy appears here as a dilaton, and a spontaneous breakdown of the scale symmetry results directly from the presence of the manifold volume measure $\Phi$. In other words, this SSB is an intrinsic feature of TMT.

In the context of the present paper this model is of significant interest because cosmological solutions of the FRW universe exhibit two unexpected results: (a) the ground state as well as the asymptotic of quintessence-like evolution (in co-moving frame) possess certain degeneracies in $\Phi$ or $g_{\mu\nu}$; (b) superaccelerating expansion of the universe (phantom cosmology) appears as the direct dynamical effect when $\Phi < 0$, i.e. as the orientation of the spacetime manifold is opposite to the regular one. In this section we present the model and some of its relevant results. Regimes (a) and (b) will be analyzed in the following two sections.
The action of the model reads
\[ S = \frac{1}{b_g} \int d^4x \, e^{\alpha \phi/M_p} \left[ -\frac{1}{\kappa} R(\Gamma, g) (\Phi + b_\phi \sqrt{-g}) 
+ (\Phi + b_\phi \sqrt{-g}) \frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - e^{\alpha \phi/M_p} (\Phi V_1 + \sqrt{-g} V_2) \right] \] (45)
and it is invariant under the global scale transformations \( (\theta = \text{const.}) \)
\[ g_{\mu\nu} \rightarrow e^2 \theta g_{\mu\nu}, \quad \Gamma_{\mu\beta}^{\gamma} \rightarrow \Gamma_{\mu\beta}^{\gamma}, \quad q_\mu \rightarrow \lambda_{ab} q_\mu \quad \text{where} \]
\[ \det(\lambda_{ab}) = e^{2 \theta}, \quad \phi \rightarrow \phi - \frac{M_p}{\alpha} \theta. \] (46)

The appearance of the dimensionless parameters \( b_g \) and \( b_\phi \) is explained by the same reasons we mentioned after equations (9) and (17). In contrast to the model of section 4, we now deal with exponential (pre-) potentials where \( V_1 \) and \( V_2 \) are constant dimensionfull parameters.

The remarkable feature of this TMT model is that equation (6), being the solution of the equation of motion resulting from variation of the manifold volume measure degrees of freedom, spontaneously breaks the scale symmetry (46): this happens due to the appearance of a dimensionfull integration constant \( s M^4 \) in equation (6). One can show [33] that in the case of the negative integration constant \( s = -1 \) and \( V_1 > 0 \), the ground state appears to be again (as it was in the scalar field model I of section 4) a zero CC state without fine tuning of the parameters and initial conditions. The behavior of \( \Phi \) and \( g_{\mu\nu} \) in the course of transition to the \( \Lambda = 0 \) state is qualitatively the same as we observed in section 5 for the scalar field model I. Therefore in the present paper studying the model (45) we restrict ourselves with the choice \( s = +1 \) and \( V_1 > 0 \).

Similar to the model of section 4, equations of motion resulting from the action (45) are noncanonical and the spacetime is non-Riemannian when using the original set of variables. This is because all the equations of motion and the solution for the connection coefficients include terms proportional to \( \partial_\mu \zeta \). However, when working with the new metric (\( \phi \) remains the same)
\[ \tilde{g}_{\mu\nu} = e^{\alpha \phi/M_p} (\zeta + b_g) g_{\mu\nu}, \] (47)
which we call the Einstein frame, the connection becomes Riemannian and the general form of all equations becomes canonical. Since \( \tilde{g}_{\mu\nu} \) is invariant under the scale transformations (46), spontaneous breaking of the scale symmetry is reduced in the Einstein frame to the spontaneous breakdown of the shift symmetry
\[ \phi \rightarrow \phi + \text{const.} \] (48)

After the change of variables to the Einstein frame (47) the gravitational equation takes the standard GR form with the same Newton constant as in the action (45)
\[ G_{\mu\nu}(\tilde{g}_{\alpha\beta}) = \frac{\kappa}{2} T_{\mu\nu}^{\text{eff}}, \] (49)
where \( G_{\mu\nu}(\tilde{g}_{\alpha\beta}) \) is the Einstein tensor in the Riemannian spacetime with the metric \( \tilde{g}_{\mu\nu} \).

The energy–momentum tensor \( T_{\mu\nu}^{\text{eff}} \) reads
\[ T_{\mu\nu}^{\text{eff}} = \frac{\zeta + b_\phi}{\zeta + b_g} \left( \rho_{,\mu} \phi_{,\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{g}^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} \right) - \tilde{g}_{\mu\nu} \left( \frac{b_g - b_\phi}{2(\zeta + b_g)} \tilde{g}^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} + \tilde{g}_{\mu\nu} V_{\text{eff}}(\phi, \zeta; M) \right), \] (50)
where the function \( V_{\text{eff}}(\phi, \zeta; M) \) is defined as follows:
\[ V_{\text{eff}}(\phi, \zeta; M) = \frac{b_g [M^4 e^{-2\alpha \phi/M_p} + V_1] - V_2}{(\zeta + b_g)^2}. \] (51)
Note that the $\zeta$-dependence of $V_{\text{eff}}(\phi, \zeta; M)$ is the same as in equation (24) of the model of section 4.

The scalar field $\zeta$ is determined by means of the constraint similar to equation (26) of section 4:

$$(b_g - \zeta)[4e^{-2a\phi/M_p} + V_1] - 2V_2 - \delta \cdot b_g(\zeta + b_g)X = 0,$$

where

$$X \equiv \frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} \quad \text{and} \quad \delta = \frac{b_g - b_\phi}{b_g}.$$  \hfill (52)

The dilaton $\phi$ field equation in the Einstein frame is reduced to the following:

$$\frac{1}{\sqrt{-g}} g_{\mu} \left[ \frac{\zeta + b_\phi}{\zeta + b_g} \sqrt{-g} g^{\mu\nu} \partial_{\nu} \phi \right] - \frac{2\alpha}{(\zeta + b_g)^2 M_p} M^4 e^{-2a\phi/M_p} = 0,$$

where again $\zeta$ is a solution of the constraint (52). Note that the dilaton $\phi$ dependence in all equations of motion in the Einstein frame appears only in the form $M^4 e^{-2a\phi/M_p}$, i.e. it results only from the spontaneous breakdown of the scale symmetry (46).

The effective energy–momentum tensor (50) can be represented in a form of that of a perfect fluid

$$T_{\mu\nu}^{\text{eff}} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu}, \quad \text{where} \quad u_\mu = \frac{\phi_{,\mu}}{(2X)^{1/2}}$$

with the following energy and pressure densities resulting from equations (50) and (51) after inserting the solution $\zeta = \zeta(\phi, X; M)$ of equation (52)

$$\rho(\phi, X; M) = X + \frac{(M^4 e^{-2a\phi/M_p} + V_1)^2 - 2\delta b_g(M^4 e^{-2a\phi/M_p} + V_1)X - 3\delta^2 b_g^2 X^2}{4[b_g(M^4 e^{-2a\phi/M_p} + V_1) - V_2]},$$

$$p(\phi, X; M) = X - \frac{(M^4 e^{-2a\phi/M_p} + V_1 + \delta b_gX)^2}{4[b_g(M^4 e^{-2a\phi/M_p} + V_1) - V_2]}.$$  \hfill (56)

In a spatially flat FRW universe with the metric $\tilde{g}_{\mu\nu} = \text{diag}(1, -a^2, -a^2, -a^2)$ filled with the homogeneous scalar field $\phi(t)$, the $\phi$ field equation of motion takes the form

$$Q_1 \ddot{\phi} + 3H Q_2 \dot{\phi} - \frac{\alpha}{M_p} Q_3 M^4 e^{-2a\phi/M_p} = 0,$$

where $H$ is the Hubble parameter and we have used the following notations:

$$\dot{\phi} \equiv \frac{d\phi}{dt} \quad \text{(59)}$$

$$Q_1 = 2[b_g(M^4 e^{-2a\phi/M_p} + V_1) - V_2]\rho_X = (b_g + b_\phi)(M^4 e^{-2a\phi/M_p} + V_1) - 2V_2 - 3\delta^2 b_g^2 X$$

$$Q_2 = 2[b_g(M^4 e^{-2a\phi/M_p} + V_1) - V_2]\rho_X = (b_g + b_\phi)(M^4 e^{-2a\phi/M_p} + V_1) - 2V_2 - \delta^2 b_g^2 X$$

$$Q_3 = \frac{1}{[b_g(M^4 e^{-2a\phi/M_p} + V_1) - V_2]}
\left[(M^4 e^{-2a\phi/M_p} + V_1)[b_g(M^4 e^{-2a\phi/M_p} + V_1) - 2V_2] + 2\delta b_g V_2 X + 3\delta^2 b_g^2 X^2 \right]$$  \hfill (60)
The nonlinear $X$-dependence appears here in the framework of the fundamental theory without exotic terms in the Lagrangians $L_1$ and $L_2$. This effect follows just from the fact that there are no reasons to choose the parameters $b_g$ and $b_\phi$ in action (45) to be equal in general; on the contrary, the choice $b_g = b_\phi$ would be a fine tuning. Thus the above equations represent an explicit example of $k$-essence [39] resulting from first principles. The system of equations (21), (56)–(58) accompanied with the functions (60)–(62) and written in the metric $\tilde{g}_{\mu\nu} = \text{diag}(1, -a^2, -a^2, -a^2)$ can be obtained from the $k$-essence type effective action

$$S_{\text{eff}} = \int \sqrt{-\tilde{g}} \, d^4x \left[ -\frac{1}{\kappa} R(\tilde{g}) + p(\phi, X; M) \right],$$

where $p(\phi, X; M)$ is given by equation (57). In contrast to the simplified models studied in literature [39], it is impossible here to represent $p(\phi, X; M)$ in a factorizable form like $\tilde{K}(\phi)\tilde{p}(X)$. The scalar field effective Lagrangian, equation (57), can be represented in the form

$$p(\phi, X; M) = K(\phi)X + L(\phi)X^2 - U(\phi),$$

where the potential

$$U(\phi) = \left[ V_1 + M^4 e^{-2\alpha\phi/M_p} \right]^2 4\left[ b_g(V_1 + M^4 e^{-2\alpha\phi/M_p}) - V_2 \right]$$

and $K(\phi)$ and $L(\phi)$ depend on $\phi$ only via $M^4 e^{-2\alpha\phi/M_p}$. Note that $U(\phi) > 0$ for any $\phi$ provided $b_g > 0$, $V_1 > 0$ and $b_g V_1 \geq V_2$, (66)

which we will assume in what follows. Note that besides the presence of the effective potential $U(\phi)$, the Lagrangian $p(\phi, X; M)$ differs from that of [40] by the sign of $L(\phi)$: in our case $L(\phi) < 0$ provided the conditions (66). This result cannot be removed by a choice of the parameters of the underlying action (45) while in [40] the positivity of $L(\phi)$ was an essential assumption. This difference plays a crucial role for the possibility of a dynamical protection from the initial singularity of the curvature studied in detail in [33].

The model allows a power law inflation (where the dilaton $\phi$ plays the role of the inflaton) with a graceful exit to a zero or tiny cosmological constant state. In what it concerns to primordial perturbations of $\phi$ and their evolution, there are no differences with the usual (i.e. one-measure) model with the action (63)–(65).

8. Degeneracies of $g_{00}$ and $\Phi$ in $\Lambda \neq 0$ ground states

8.1. Fine-tuned $\delta = 0$ models

We are now going to analyze some of the cosmological solutions for the late universe in the framework of the scale invariant model of the previous section. These solutions surprisingly exhibit that asymptotically, as $t \to \infty$, either $g_{00} \to 0$ or $\Phi \to 0$.

In the late universe, the kinetic energy $X \to 0$. Therefore in many cases the role of the nonlinear $X$ dependence becomes qualitatively unessential. This is why, for simplicity, in this
section we can restrict ourselves with the fine tuned model with $\delta = 0$. In such a case the constraint (52) yields
\[
\zeta = \frac{b_g V_1 - 2 V_2 + b_g M^4 e^{-\alpha \phi / M_p}}{V_1 + M^4 e^{-\alpha \phi / M_p}}.
\] (68)

The energy density and pressure take then the form
\[
\rho(0) = \rho|_{\delta=0} = \frac{1}{2} \dot{\phi}^2 + U(\phi); \quad p(0) = p|_{\delta=0} = \frac{1}{2} \dot{\phi}^2 - U(\phi),
\] (69)

where $U(\phi)$ is determined by equation (65). The $\phi$-equation (58) is reduced to
\[
\ddot{\phi} + 3H \dot{\phi} + \frac{dU(\phi)}{d\phi} = 0.
\] (70)

Applying this model to of the late time cosmology of the spatially flat universe and assuming that the scalar field $\phi \to \infty$ as $t \to \infty$, it is convenient to rewrite the potential $U(\phi)$ in the form
\[
U(\phi) = \Lambda + V(\phi),
\] (71)

where
\[
\Lambda = \frac{V_1^2}{4(b_g V_1 - V_2)},
\] (72)

is the positive cosmological constant and
\[
V(\phi) = \left(\frac{b_g V_1 - 2 V_2}{b_g V_1 - V_2}\right) V_1 M^4 e^{-2 \alpha \phi / M_p} + \left(\frac{b_g V_1 - V_2}{b_g V_1 + M^4 e^{-2 \alpha \phi / M_p} - V_2}\right) M^8 e^{-4 \alpha \phi / M_p}.
\] (73)

It is evident that if $b_g V_1 > 2 V_2$ or $b_g V_1 = 2 V_2$ then $V(\phi)$ is a sort of a quintessence-like potential and therefore quintessence-like scenarios can be realized. This means that the dynamics of the late time universe is governed by the dark energy which consists of both the cosmological constant and the potential slow decaying to zero as $\phi \to \infty$. In the opposite case, $b_g V_1 < 2 V_2$, the potential $V(\phi)$, and also $U(\phi)$, has an absolute minimum at some finite value of $\phi$, and therefore the cosmological scenario is different from the quintessence-like scenario. Details of the cosmological evolution starting from the early inflation and up to the late time universe governed by the potential $U(\phi)$ have been studied in [33] for each of these three cases. Here we want to analyze what kind of degeneracy appears in the ground state depending on the region in the parameter space.

8.2. The case $b_g V_1 > 2 V_2$

Let us consider the case when the relation between the parameters $V_1$ and $V_2$ satisfies the condition $b_g V_1 > 2 V_2$. It follows from equation (68) that
\[
\zeta \to \frac{b_g V_1 - 2 V_2}{V_1} = \text{const} > 0 \quad \text{as} \quad \phi \to \infty.
\] (74)

By making use the $(00)$ component of equation (47), we see that
\[
g_{00} = \frac{e^{-\alpha \phi / M_p}}{\zeta + b_g} \to 0.
\] (75)

In order to get the asymptotic time dependence of $g_{00}$ and the spatial components of the metric
\[
g_{ii} = \frac{e^{-\alpha \phi / M_p}}{\zeta + b_g} a(t)^2, \quad i = 1, 2, 3
\] (76)
as \( t \to \infty \), we have to know a solution \( a = a(t), \phi = \phi(t) \). We can find analytically the asymptotic (as \( \phi \to \infty \)) behavior of a cosmological solution for a particular value of the parameter \( \alpha = \sqrt{\frac{3}{8}} \). In such a case, keeping only the leading contribution of the \( \phi \)-exponent in equation (73), we deal with the following system of equations:

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{1}{3M_p^2} \rho^{(0)} \tag{77}
\]

\[
\phi + 3 \frac{\dot{a}}{a} \dot{\phi} - \frac{2\alpha V_1(b_2 V_1 - 2V_2)}{M_p^2} \frac{4(b_2 V_1 - V_2)^2 M^4 e^{-2\alpha \phi/M_p}}{4(b_2 V_1 - V_2)^2} = 0, \tag{78}
\]

where \( \rho^{(0)} \) is determined by equation (69). The exact analytic solution for these equations is as follows [24]:

\[
\phi(t) = \text{const.} + \frac{M_p}{2\alpha} \ln(M_p t), \quad a(t) \propto t^{1/3} e^{\lambda t}, \quad \lambda = \frac{1}{M_p} \sqrt{\frac{\Lambda}{3}}, \tag{79}
\]

where \( \Lambda \) is determined by equation (72). Therefore we obtain for the asymptotic cosmic time behavior of the components of the metric \( g_{\mu\nu} \)

\[
g_{00} \sim \frac{1}{t^{1/2}} \to 0; \quad g_{ii} \sim -t^{1/6} e^{2\lambda t} \quad \text{as} \quad t \to \infty. \tag{80}
\]

So in the course of the expansion of the very late universe, only \( g_{00} \) asymptotically vanishes while the space components \( g_{ii} \) behave qualitatively in the same manner as the space components of the metric in the Einstein frame \( \tilde{g}_{ii} \). Respectively, the asymptotic behavior of the volume measures is as follows:

\[
\Phi \approx \frac{b_2 V_1 - 2V_2}{V_1} \sqrt{-g} \sim e^{3\lambda t} \quad \text{as} \quad t \to \infty. \tag{81}
\]

The GES is asymptotically restored that can be seen from the asymptotic time behavior of the conservation law (67)

\[
\nabla_{\mu} J^\mu \sim \text{const.} e^{-2\alpha \phi/M_p} \sim \frac{1}{t}. \tag{82}
\]

### 8.3. The case \( b_2 V_1 = 2V_2 \)

In this case the asymptotic form of \( V(\phi) \) is

\[
V(\phi) \approx \frac{M^8}{2b_2 V_1} e^{-4\alpha \phi/M_p} \tag{83}
\]

and \( \zeta \) asymptotically approaches zero according to

\[
\zeta = \frac{b_2 M^4 e^{-2\alpha \phi/M_p}}{V_1 + M^4 e^{-2\alpha \phi/M_p}} \to 0 \quad \text{as} \quad \phi \to \infty. \tag{84}
\]

Similar to the previous subsection, the analytic form of the asymptotic (as \( \phi \gg M_p \)) cosmological solution exists for a particular value of the parameter \( \alpha = \sqrt{3/32} \)

\[
\phi(t) = \text{const.} + \frac{M_p}{4\alpha} \ln(M_p t), \quad a(t) \propto t^{1/3} e^{\lambda t}, \quad \lambda = \frac{1}{M_p} \sqrt{\frac{\Lambda}{3}}, \tag{85}
\]

where now

\[
\Lambda = \frac{V_1}{2b_2}. \tag{86}
\]
For this solution we obtain the following asymptotic cosmic time behavior for the components of the metric $g_{\mu\nu}$ and volume measures

$$g_{00} \sim \frac{1}{t^{1/4}} \to 0; \quad g_{ii} \sim -t^{5/12} e^{2\lambda t} \quad \text{as} \quad t \to \infty,$$

$$\sqrt{-g} \sim \sqrt{t} e^{3\lambda t}, \quad \Phi \sim e^{3\lambda t} \quad \text{as} \quad t \to \infty.$$  \hfill (87)

The asymptotic time behavior of the conservation law describing the asymptotic restoration of the GES is the same as in equation (82).

8.4. The case $0 < b_g V_1 < 2V_2$

In this case the potential $U(\phi)$, equation (65), has an absolute minimum

$$\Lambda = U(\phi_{\text{min}}) = \frac{V_2}{b_g} \quad \text{at} \quad \phi = \phi_{\text{min}} = -\frac{M_p^2}{2\alpha} \ln \left( \frac{2V_2 - b_g V_1}{b_g M_p^4} \right).$$  \hfill (89)

The spatially flat universe described in the Einstein frame with the metric $\tilde{g}_{\mu\nu} = \text{diag}(1, -a^2, -a^2, -a^2)$, in a finite time [23] reaches this ground state where it expands exponentially

$$a \propto e^{\lambda t}, \quad \lambda = M_p^{-1} \sqrt{\Lambda/3}$$  \hfill (90)

and $\Lambda$ is given by equation (89). A surprising feature of this case is that $\zeta$, equation (68), disappears in the minimum

$$\zeta(\phi_{\text{min}}) = 0.$$  \hfill (91)

The components of the metric $g_{\mu\nu}$ in the ground state are as follows:

$$g_{00}\big|_{\text{ground state}} = \left( \frac{2V_2 - b_g V_1}{b_g M_p^4} \right)^{1/2} = \text{const}, \quad g_{ii}\big|_{\text{ground state}} = -g_{00}\big|_{\text{ground state}} \cdot e^{2\lambda t}$$  \hfill (92)

with the respective behavior of the metrical volume measure $\sqrt{-g} \propto \exp(3\lambda t)$. Hence the manifold volume measure in the ground state disappears

$$\Phi\big|_{\text{ground state}} = 0.$$  \hfill (93)

in view of equation (91).

Disappearance of the manifold volume measure $\Phi$ in the ground state may not allow us to get equation (6) by varying the $\varphi_0$ fields in action (45). Therefore in the conservation law (67) one should use the current in the form $j^\mu = L_1 B_\mu^\alpha \varphi_\alpha$ as we have noted after equation (40). Recall that $L_1$ is constituted by the terms of the Lagrangian in (45) coupled to the measure $\Phi$. However, after using the gravitational equation obtained by varying $g_{\mu\nu}$ in (45) and substituting the ground state value $\phi = \phi_{\text{min}}$ into $L_1$, we obtain $L_1 = M_p^4$. Hence, the conservation law (67) in the ground state reads just

$$\nabla_\mu J^\mu\big|_{\text{ground state}} = 0.$$  \hfill (94)
9. Sign indefiniteness of the manifold volume measure as the origin of a phantom dark energy

We turn now to the non-fine-tuned case of the model of section 7 applied to the spatially flat universe. We start from a short review of our recent results [33] concerning qualitative structure of the appropriate dynamical system which consists of equation (58) and the equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3M_p^2} \rho,$$

(95)

where the energy density $\rho$ is defined by equation (56). The case of interest of this section is realized when the parameters of the model satisfy the condition

$$ (b_g + b_\phi)V_1 - 2V_2 < 0. $$

(96)

In this case the phase plane has a very interesting structure presented in figure 3. Recall that the functions $Q_1, Q_2, Q_3$ are defined by equations (60)–(62).

We are interested in the equation of state $w = p/\rho < -1$, where pressure $p$ and energy density $\rho$ are given by equations (56) and (57). The line indicated in figure 3 as 'line $w = -1$' coincides with the line $Q_2(\phi, X) = 0$ because

$$ w + 1 = \frac{X}{\rho} \frac{Q_2}{[b_g(M^4 e^{-2\phi/M_p} + V_1) - V_2]}. $$

(97)

Phase curves in zone 3 correspond to the cosmological solutions with the equation of state $w < -1$. In zone 2, $w > -1$ but this zone has no physical meaning since the squared sound speed of perturbations

$$ c_s^2 = \frac{Q_2}{Q_1} $$

(98)

is negative in zone 3. But in zone 2, $c_s^2 > 0$. Some details of numerical solutions describing the cross of the phantom divide $w = -1$ and the super-accelerating expansion of the universe are presented in figures 4 and 5.

Note that the superaccelerating cosmological expansion is obtained here without introducing an explicit phantom scalar field into the underlying action (45). In [33] we have discussed this effect from the point of view of the effective k-essence model realized in the Einstein frame when starting from action (45). A deeper analysis of the same effect yields the conclusion that the true and profound origin of the appearance of an effective phantom dynamics in our model is sign-indefiniteness of the manifold volume measure $\Phi$. In fact, using the constraint (52), equations (97) and (47) it is easy to show that

$$ \Phi + b_\phi \sqrt{-g} = (w + 1) \frac{X}{4X} \frac{b_g(M^4 e^{-2\phi/M_p} + V_1 - \delta \cdot b_g X)}{[b_g(M^4 e^{-2\phi/M_p} + V_1) - V_2]^{\frac{3}{2}}} $$

(99)

where $a$ is the scale factor. The expression in the lhs of this equation is the total volume measure of the $\phi$ kinetic term in the underlying action (45)

$$ \int d^4x (\Phi + b_\phi \sqrt{-g}) \frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu}. $$

(100)

The sign of this volume measure coincides with the sign of $w + 1$ as well as with the sign of the function $Q_2$ (see equation (97)). In figure 5 we present the result of the numerical solution for the scale factor dependence of $w$ and $(\Phi + b_\phi \sqrt{-g})/a^3$. Thus crossing the phantom divide occurs when the total volume measure of the $\phi$ kinetic term in the underlying action changes sign from positive to negative for dynamical reasons. This dynamical effect appears here as a dynamically well-founded alternative to the usually postulated phantom kinetic term of a scalar field Lagrangian [41].
Figure 3. The phase portrait (in the phase plane $(\phi, \dot{\phi})$) for the model with $\alpha = 0.2$, $\delta = 0.1$, $V_1 = 10M^4$ and $V_2 = 9.996M^4$. The region with $\rho > 0$ is divided into two dynamically disconnected regions by the line $Q_1(\phi, \dot{\phi}) = 0$. To the left of this line $Q_1 > 0$ (the appropriate zone we call zone 1) and to the right $Q_1 < 0$. The $\rho > 0$ region to the right of the line $Q_2(\phi, \dot{\phi}) = 0$ is divided into two zones (zone 2 and zone 3) by the line $Q_2 = 0$ (the latter coincides with the line where $w = -1$). In zone 2 $w > -1$ but $c_2 > 0$. In zone 3 $w < -1$ and $c_2 > 0$. Phase curves started in zone 2 cross the line $w = -1$. All phase curves in zone 3 exhibit processes with super-accelerating expansion of the universe. Besides all the phase curves in zone 3 demonstrate dynamical attractor behavior to the line which asymptotically, as $\phi \to \infty$, approaches the straight line $\dot{\phi} = 0$.

10. Summary and discussion

Introducing the spacetime manifold volume element (2) and adding the appropriate degrees of freedom to a set of traditional variables (metric, connection, matter fields) we reveal that such a two measures theory (TMT) takes up a special position between alternative theories. First, the equations of motion can be rewritten in the Einstein frame (where the spacetime becomes Riemannian) with the same Newtonian constant as in the underlying action (where the spacetime is generically non-Riemannian). Second, the theory possesses remarkable features in what it concerns the CC problem. Third, the TMT model with spontaneously broken dilatation symmetry satisfies all existing tests of GR. There are other interesting results, for example a possibility of a dynamical protection from the initial singularity of the curvature.

In this paper we have studied the behavior of the manifold volume measure $\Phi$ and the metric tensor $g_{\mu\nu}$ (used in the underlying TMT action) in cosmological solutions for a number of scalar field models of dark energy. We have made a special accent on the sign indefiniteness of the manifold volume measure $\Phi$ that may yield interesting physical effects. An example of
such a type of effects we have seen in section 9: the total volume measure of the dilaton scalar field kinetic term in the underlying action can change sign from positive to negative in the course of dynamical evolution of the late time universe. In the Einstein frame, this transition corresponds to the crossing of the phantom divide of the dark energy.

We have found out that in all studied models, the transition to the ground state is always accompanied by a certain degeneracy either in the metric (e.g., in $g_{00}$ or in all components) or in the manifold volume measure $\Phi$, or even in both of them. This result differs sharply from what was expected, e.g. in [2–4] where degenerate metric solutions have been associated with...
high curvature and temperature phases. One should only take into account that degeneracy
of $g_{\mu \nu}$ and/or $\Phi$ in the (transition to) ground state takes place only when one works with
the set of variables of the underlying TMT action. In the Einstein frame, we deal with the
effective picture where the measure $\Phi$ does not present at all and the metric tensor $\tilde{g}_{\mu \nu}$ (see
equations (20) or (47)) has the same regularity properties as in GR. The regularity of $\tilde{g}_{\mu \nu}$ results
from the singularity of the transformations (20) or (47)): degeneracy of $g_{\mu \nu}$ in a discrete set
of moments is compensated by a singularity of $\zeta$.

10.1. The CC problem

TMT provides two different possibilities for resolution of the CC problem: one which
guarantees zero CC without fine tuning (see however the end of section 6 and appendix
B); another which allows an unexpected way to reach a tiny CC. Which of these possibilities
is realized depends on the sign of the integration constant $s M^4$, $s = \pm 1$. We are now going to
discuss these two issues.

10.1.1. The case $\Lambda = 0$ in TMT. This case is of special interest for two reasons. First, as it
was shown earlier [33], the conditions of Weinberg’s no-go theorem [36] fail and a transition
to a zero CC state in TMT can be realized without fine tuning. This becomes possible for
example if $V_1(\phi) > 0$ and the integration constant $s M^4 < 0$. Second, as we have shown in
section 5, in the course of transition to a zero CC state, $g_{\mu \nu}$ and $\Phi$ oscillate synchronously
around zero and they cross zero each time $t_i$ ($i = 1, 2, 3, \ldots$) when the scalar field $\phi$ crosses
the (zero) absolute minimum of the potential (27) (or of the potential (65) for the model of
section 7 with $V_1 < 0$, see [33]).

One should recall that $\zeta(x)$ does not have its own dynamics: its values at the spacetime
point $x$ are determined directly and immediately by the local configuration of the matter fields
and gravity through the algebraic constraint, which is nothing but a consistency condition of
the equations of motion. $\zeta(x)$ does not possess inertia and therefore it changes together and
synchronously with changing matter and gravity fields. This notion is very important when
trying to answer the natural question: can oscillations of $\zeta(x)$ be a source for particle creation?
The answer is—no, it cannot. In fact, there is a coupling of $\zeta$ with fermions. But the structure
of this coupling in the Einstein frame has very surprising features which we will briefly review
in the following subsection. Here we are only formulating the conclusion: emergence of even
a tiny amount of fermionic matter immediately yields a rearrangement of the vacuum9 in such
a way that $\zeta$ instantly ceases the regime of oscillations and rapidly enters into a regime of
monotonous approach to a nonzero constant. It is interesting to note that the latter effect may
explain why the present day cosmological constant is most likely tiny but nonzero, in spite
of the existence of a fine tuning free classical solution described by a transition to the $\Lambda = 0$
state.

An overall change of sign of $g_{\mu \nu}$ in the course of these oscillations means a change of the
signature from $(+−−−)$ to $(−+++)$ and vice versa, while oscillations of the sign of $\Phi$
describe the change of orientation of the spacetime manifold. The latter means that the
arena of the gravitational dynamics should contain two spacetime manifolds with opposite
orientations. The discrete set of changes of the orientations happens in the form of a smooth
dynamical process in the course of which the spacetime passes the ‘degenerate’ phase where
both the metrical structure and the total 4D-volume measure disappear. The latter means
also that the term ‘orientation of the spacetime manifold’ loses any sense at moments $t_i$

9 The possibility of a vacuum deformation in a different approach has been shown by MacKenzie, Wilczek and Zee
[46].
We conclude therefore that two 4D differentiable manifolds with opposite orientations (described by means of a sign indefinite volume 4-form) equipped with connection and metrical structure still are not enough to describe the arena of the gravitational dynamics: the complete description of the spacetime dynamics also requires the mentioned degenerate phase. This situation is somewhat similar to that discussed in the introduction: first-order formulation of GR where the degenerate phase with $g_{\mu\nu} = 0$ should also be added [2–4]; see, e.g. the recent discussion by Bañados [17] where the limiting process $g_{\mu\nu} \rightarrow 0$ is analyzed.

A new interesting feature of ground states in TMT we have revealed in the present paper concerns the so-called global Einstein symmetry (GES), equations (35)–(38), which turns out generically to be explicitly broken in all models with non-trivial dynamics. The surprising result we have discovered here on the basis of a number of models is that the GES is restored in the course of transitions to the ground state in all models considered. Hence its subgroup of the sign inversions of $g_{\mu\nu}$ and $\Phi$, equation (39), is also restored. Therefore the oscillations of $g_{\mu\nu}$ and $\Phi$ around zero in the course of transition to a $\Lambda = 0$ ground state provoke a wish to compare this dynamical effect with the attempts to solve the old CC problem developed in [42–44]. The main idea of these approaches is that the field theory or at least the ground state [43] should be invariant under transformations of a discrete symmetry. According to [42–44] it might be either an invariance under the metric reversal symmetry or under the spacetime coordinate transformations with the imaginary unit $i$: $x^A \rightarrow ix^A$. In contrast with these approaches, in TMT there is no need to postulate such exotic enough symmetries. Nevertheless, we have seen that sign inversions of $g_{\mu\nu}$ emerge as a dynamical effect in the course of the cosmological evolution and this effect has indeed a relation to the resolution of the old CC problem.

10.1.2. The case of a tiny CC. In the scalar field models of dark energy, an interesting feature of TMT consists of the possibility of providing a small value of the CC. If in the model of section 8.2, the parameter $V_2 < 0$ and $|V_2| \gg b_\xi V_1$ then the CC can be very small without the need for $V_1$ and $V_2$ to be very small. For example, if $V_1$ is determined by the energy scale of electroweak symmetry breaking $V_1 \sim (10^3 \text{ GeV})^4$ and $V_2$ is determined by the Planck scale $V_2 \sim (10^{18} \text{ GeV})^4$ then $\Lambda_1 \sim (10^{-3} \text{ eV})^4$. Along with such a seesaw mechanism [25, 45], there exists another way to explain the smallness of the CC applicable in all types of scenarios discussed in sections 8.2–8.4 (see also appendix A). As one can see from equations (72), (85) and (89), the value of $\Lambda_1$ appears to be inverse proportional to the dimensionless parameter $b_\xi$ which characterizes the relative strength of the ‘manifold’ and ‘metrical’ parts of the gravitational action. If for example $V_1 \sim (10^3 \text{ GeV})^4$ then for getting $\Lambda_1 \sim (10^{-3} \text{ eV})^4$ one should assume that $b_\xi \sim 10^{60}$. Such a large value of $b_\xi$ (see equation (9)) permits one to formulate a correspondence principle [33] between TMT and regular (i.e. one-measure) field theories: when $\zeta/b_\xi \ll 1$ then one can neglect the gravitational term in $L_1$ with respect to that in $L_2$ (see equation (9) or equation (17) or equation (45)). More detailed analysis shows that in such a case the manifold volume measure $\Phi = \zeta \sqrt{-g}$ has no a dynamical effect and TMT is reduced to GR. This happens, e.g. in the model of section 8.3 where the late time evolution proceeds in a quintessence-like manner: the energy density decreases to the cosmological constant, equation (86), and $\zeta \rightarrow 0$, equation (84). Another example is the model of section 8.4 where $\Phi = 0$ in the ground state, equation (91), while $\sqrt{-g}$ is finite. However generically $\zeta/b_\xi$ is not small, as it happens for example in the quintessence-like scenario of the late time universe in the model of section 8.2 (see equation (74)).

10 In the pure gravity model, section 3, $\Lambda$ is proportional to $b_\xi$. 

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10.2. Possibilities for predictions of new physical effects

10.2.1. Short review of the TMT model with spontaneously broken dilatation symmetry in the presence of matter. It would be interesting to find out other possible physical manifestations of the sign indefiniteness of the manifold volume measure. In fact, the model with spontaneously broken dilatation symmetry studied in sections 7–9 and in [33] allows extensions which include fermion and gauge fields [29–31] or, alternatively, dust as a phenomenological matter model [34]. In the former case, for example, the constraint (52) is modified to the following:

\[
\frac{1}{(\zeta + b)^2}((b - \zeta)(M^4 e^{-2\alpha\phi/M_P} + V_1) - 2V_2) = \frac{\mu(\xi - \zeta)(\xi - \zeta^2)}{2(\xi + k)^2(\xi + b)^{3/2}} \bar{\Psi}\Psi,
\]

(101)

where \(\Psi\) is the fermion field in the Einstein frame and for simplicity we have chosen \(\delta = 0\) (that is \(b_\phi = b_\chi = b\)); \(\zeta_{1,2}\) are defined by

\[
\zeta_{1,2} = \frac{1}{2} \left[ k - 3h \pm \sqrt{(k - 3h)^2 + 8b(k - h) - 4kh} \right]
\]

(102)

and the dimensionless parameters \(k\) and \(h\) appear in the underlying action in the total volume measures of the fermion kinetic term

\[
\int d^4 x e^{2\alpha\phi/M_P} (\Phi + k \sqrt{-g}) \frac{i}{2} \bar{\Psi} (\gamma^a e^\mu_a \gamma^\mu - \bar{\gamma}^\mu e^\mu_a) \Psi
\]

(103)

and the fermion mass term

\[
- \int d^4 x e^{2\alpha\phi/M_P} (\Phi + h \sqrt{-g}) \mu \bar{\Psi}\Psi,
\]

(104)

respectively. Note that the fermion equation in the Einstein frame has a canonical form but the mass of the fermion turns out \(\xi\) dependent

\[
m(\xi) = \frac{\mu(\xi + h)}{(\xi + k)(\xi + b)^{3/2}}.
\]

(105)

The constraint (101) describes the local balance between the fermion energy density and the scalar field \(\phi\) contribution to the dark energy density in the spacetime region where the wavefunction of the primordial fermion is not equal to zero. By means of this balance the constraint determines the scalar \(\xi(x)\).

In the case of dust as a phenomenological matter model, the rhs of the constraint (101) looks like

\[
\frac{\xi - b_m + 2b}{2\sqrt{\zeta + b}} m \tilde{n},
\]

(106)

where the dimensionless parameter \(b_m\) appears in the total volume measure of the dust contribution to the underlying action

\[
S_m = \int (\Phi + b_m \sqrt{-g}) L_m d^4 x
\]

(107)

\[
L_m = -m \sum_i \int e^{2\alpha\phi/M_P} \sqrt{g_{\mu\nu}} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \delta^4(x - x_i(\lambda)) \sqrt{-g} d\lambda
\]

(108)

and \(m\) is the mass parameter.

The wonderful feature of these models in the Einstein frame consists of the exact coincidence of the following three quantities: (a) the noncanonical (in comparison with GR) terms in the energy–momentum tensor; (b) the effective coupling ‘constant’ of the dilaton
\( \phi \) to the matter (up to the factor \( \alpha/M_p \)); (c) the expressions in the rhs of the above-mentioned constraints (101) and (106) for fermionic matter and dust, respectively. For matter in normal conditions, the local matter energy density (i.e. in the spacetime region occupied by the matter) is many orders of magnitude larger than the vacuum energy density. Detailed analysis [29, 30, 31, 34] shows that when the matter is in the normal conditions, the balance dictated by the constraint becomes possible if \( \zeta \) with very high accuracy takes the constant values: \( \zeta \approx \zeta_1 \) or \( \zeta \approx \zeta_2 \) for fermions (and therefore the fermion masses become constant)) and \( \zeta \approx b_m - 2b \) for the dust. Then the mentioned three quantities simultaneously become extremely small. Besides for the matter in normal conditions the gravitational equations are reduced to the canonical GR equations. The practical disappearance of the dilaton-to-matter coupling ‘constant’ for the matter in normal conditions which occurs without fine tuning of the parameters allows us to assert that in such type of models the fifth force problem is resolved [31, 34].

It does not mean however that matter does not interact with the dilaton at all. When the matter is in states different from normal, the effect of dilaton-to-matter coupling may yield new very interesting phenomena. One of such effects appears when the neutrino energy density decreases to the order of magnitude close to the vacuum energy density. The latter can happen due to spreading of the neutrino wave packet. Then the cold gas of uniformly distributed nonrelativistic neutrinos causes a reconstruction of the vacuum to a state with \( \zeta \rightarrow |k| \) and as a result the neutrino gas rapidly transmute into an exotic state called neutrino dark energy (see, e.g. [47]). This effect was studied in detail in [31] where we have shown that transmutation from the pure scalar field dark energy to the neutrino dark energy regime is favorable from the energetic point of view.

10.2.2. Prediction of the strong gravity effect in high energy physics experiments. For the solutions \( \zeta \approx \zeta_1 \) or \( \zeta \approx \zeta_2 \) of the constraint (101), the lhs of the constraint has the order of magnitude close to the vacuum energy density. There exists, however, another solution if one allows the possibility that in the core of the support of the fermion wavefunction the local dark energy density may be much bigger than the vacuum energy density. Such a solution turns out to be possible as fermion density is very big and \( \zeta \) becomes negative and close enough to the value \( \zeta \approx -b \). Then the solution of the constraint (101) looks like [29]

\[
\frac{1}{\sqrt{\zeta + b}} \approx \left[ \frac{\mu(b - h)}{4M^4b(b - k)} \sqrt{\Psi} e^{2\alpha \phi / M_p} \right]^{1/3}.
\]

(109)

In such a case, instead of constant masses, as it was for \( \zeta \approx \zeta_{1,2} \), equation (105) results in the following fermion self-interaction term in the effective fermion Lagrangian:

\[
L_{\text{ferm selfint}} = 3 \left[ \frac{1}{b} \left( \frac{\mu(b - h)}{4M(b - k)} \sqrt{\Psi} \right)^4 e^{2\alpha \phi / M_p} \right]^{1/3}.
\]

(110)

It is very interesting that the described effect is the direct consequence of the strong gravity. In fact, in the regime where \( \zeta + b \ll 1 \) the effective Newton constant in the gravitational term of underlying action (45)

\[
S_{\text{grav}} = - \int d^4x \sqrt{-g} \frac{\zeta + b}{kb} R(\Gamma, g) e^{2\alpha \phi / M_p}
\]

(111)

becomes anomalously large. Recall that for simplicity we have chosen here \( b_\phi = b_\xi = b \). But if one do not imply this fine tuning then one can immediately see from equations (49)–(51) that in the Einstein frame the regime of the strong gravity dictated by the dense fermion matter is manifested for the dilaton too.
The coupling constant in equation (110) is dimensionless and depends exponentially on the dilaton $\phi$ if one can regard $\phi$ as a background field $\bar{\phi}$. But in a more general case equation (110) may be treated as describing an anomalous dilaton-to-fermion interaction very much different from the above discussed case of the interaction of the dilaton to the fermion matter in normal conditions where the coupling constant practically vanishes. Such an anomalous dilaton-to-fermion interaction should result in the creation of quanta of the dilaton field in processes with very heavy fermions. The probability of these processes is of course proportional to the Newton constant $M_p^{-2}$. But the new effect consists of the fact that the effective coupling constant of the anomalous dilaton-to-fermion interaction is proportional to $e^{2\alpha \bar{\phi}/3M_p}$. If the dilaton is the scalar field responsible for the quintessential inflation type of the cosmological scenario [48] then one should expect an exponential amplification of the effective coupling of this interaction in the present day universe in comparison with the early universe. One can hope that the described effect of the strong gravity might be revealed in the LHC experiments in the form of missing energy due to the multiple production of quanta of the dilaton field (recall that coupling of the dilaton to fermions in normal conditions practically vanishes and therefore the dilaton will not be observed after being emitted).

10.2.3. Some other possible effects.

(1) Dark matter as an effect of gravitational enhancement. In the case of dust as a phenomenological matter model, the constraint (101) with the rhs (106) is the fifth degree algebraic equation with respect to $\sqrt{\xi + b}$. There are some indications that in a certain region of the parameters a solution of the constraint exists which could provide a very interesting effect of an amplification of the gravitational field of visible diluted galactic and intergalactic dust or/and neutrinos. Such an effect might imply that the dark matter is not a new sort of matter but it is just a result of a so far unknown enhancement of the gravitational field of low density states of the usual matter.

(2) Dilaton to photon coupling. Astrophysical observations of the last few years indicate an anomalously large transparency of the universe to gamma rays [49, 50]. It is hard to explain this astrophysical puzzle in the framework of extragalactic background light. Recently a natural mechanism was suggested by De Angelis, Mansutti and Roncadelli [51] in order to resolve this puzzle. The idea is to suppose that there exists a very light spin-zero boson coupled to the photon

$$L_{\phi\gamma} = -\frac{1}{4\mu} F_{\mu\nu} \tilde{F}_{\mu\nu}\phi,$$

(112)

where $\mu$ is a mass parameter. In the context of quintessential scenario such a coupling was studied by Carroll [52]. Then $\gamma \rightarrow \phi \rightarrow \gamma$ oscillations emerge which explain [51] the observed transparency of the Universe to gamma rays in a natural way if mass of the spin-zero boson $m < 10^{-10}$ eV. The crucial feature of this boson is that no other coupling of this scalar to matter exists. In the standard quintessence models this feature seems to be a real problem. But in TMT, as we already mentioned (see also [31, 34]) the dilaton playing the role of quintessence field decouples from matter in normal conditions. At the same time its coupling to the photon in the form (112) is not suppressed.

(3) Creation of a universe in the laboratory. A theoretical attempt by Farhi, Guth and Guven to describe the creation of a universe in the laboratory [53] runs across a need to allow vanishing and changing sign of $\sqrt{-g}$. In [53], this need is naturally regarded as a pathology. If a similar approach to the problem of the creation of a universe in the laboratory could be formulated in the framework of TMT then instead of $\sqrt{-g}$ there should appear a linear combination of $\Phi$ and $\sqrt{-g}$ which, as we already know, is able to
vanish and change sign. In a recent paper [54] by Guendelman and Sakai a model of child universe production without initial singularities was studied. To provide the desirable absence of initial singularity a crucial point is that the energy–momentum tensor of the domain wall should be dominated by a sort of phantom energy. A possible way to realize this idea is to apply the dynamical brane tension [55] obtained when using the modified volume measure similar to the signed measure $\Phi$ of the present paper. So it could be that applying the notions explored in the present paper one can also obtain a framework for formulating non-singular child universe production.

(4) Unparticle physics. $\zeta$ dependence of the fermion mass, equation (105), together with the constraint (101) can be treated as a $\Psi \Psi$ dependence of the fermion mass. This means that in states different from the normal one, the fermion mass spectrum may be continuous, which allows us to think of a possibility to establish a relation with the idea of unparticle physics [56].

Note finally that for the matter in normal conditions the model does not impose essential constraints on the parameters of the model (such as $b_g, b_\phi, b_m, k, h$). But the appropriate constraints should appear when more progress in the study of the listed and another possible new effects will be achieved.

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Appendix A. The ground state with nonzero CC in model I

Let us consider the scalar field model I (equations (17) and (28)) with $\delta = 0$ where we now choose a positive integration constant ($s = +1$) and the parameters $V_2^{(0)} < 0, b_\phi \mu^2 > \mu^2$. Then the ground state is realized for $\phi = 0$ and the vacuum energy is

$$\Lambda = V_{\text{eff}}(0) = \frac{M^8}{4b_g M^4 - V_2^{(0)}}. \quad (A.1)$$

In this ground state, both the measure $\Phi > 0$ and all components of the metric tensor $g_{\mu\nu}$ are regular. Note that the presence of the free dimensionless parameter $b_g$ in the denominator allows us to again reach a small vacuum energy by means of the correspondence principle discussed in item 2 of section 10.

Appendix B. Global Einstein symmetry does not guarantee resolution of the CC problem

In the model (44), the gravitational equations are modified to the following:

$$G_{\mu\nu}(\tilde{g}) = \kappa \left[ \phi,\mu \phi,\nu - \frac{1}{2} \delta_{\mu\nu} X + \frac{b_g [s M^4 + V_1(\phi) + 2\lambda \zeta] - V_2(\phi) - \lambda \zeta^2}{(\zeta + b_g)^2} \right] \quad (B.1)$$

while the form of the scalar field $\phi$ equation remains the same as in equation (25). However the constraint now very much differs from equation (26)

$$4 \lambda \zeta^2 + [s M^4 + V_1(\phi) - 2 b_\phi \lambda] \zeta + 2 V_2(\phi) - b_g [s M^4 + V_1(\phi)] = 0. \quad (B.2)$$
One can see from equation (B.1) that $\zeta$-dependence emerges now in the numerator of the effective potential. Besides, it is evident that in contrast with what was in section 5, the regime with $\zeta \to \infty$ cannot be a solution of the constraint. It is evident that a zero minimum of the effective potential cannot now be reached without fine tuning. Thus although the second term in action (44) is invariant under the GES, by adding this term we lost the ability to resolve the old CC problem.

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