Virtual Compton Scattering—Generalized Polarizabilities of Nucleons and Pions

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Abstract. Virtual Compton scattering off nucleons and pions at low energies is discussed. Predictions for the generalized polarizabilities of the nucleon are presented within the framework of heavy-baryon chiral perturbation theory and the linear sigma model. First results for the generalized polarizabilities of the charged pion in chiral perturbation theory at $O(p^4)$ are shown.

1 Introduction

Real Compton scattering (RCS) off a stable particle has a long history as a means to study the dynamics responsible for the internal structure of a system. In this context, low-energy theorems (LET) play an important role in defining a reference point for the accuracy which has to be achieved in order to obtain information beyond global properties of the target and, thus, to potentially distinguish between different models. The famous low-energy theorem of Low [1] and Gell-Mann and Goldberger [2] for Compton scattering of real photons off a nucleon is based on the requirements of gauge invariance, Lorentz covariance, crossing symmetry, and the discrete symmetries. The low-energy amplitude is specified up to and including terms linear in the photon energy. The Taylor series coefficients are expressed in terms of the charge, mass, and magnetic moment of the target. Terms of second order in the frequency, which are not determined by this LET, can be parametrized in terms of two new structure constants, the electric and magnetic polarizabilities $\alpha$ and $\beta$ (see, e.g., [3, 4]).

Recently, the investigation of low-energy virtual Compton scattering (VCS) as tested in, e.g., the reactions $e^- p \rightarrow e^- p \gamma$ [5] and $\pi^- e^- \rightarrow \pi^- e^- \gamma$ [6], has attracted a lot of interest. The possibilities to investigate the structure of the target increase substantially, if virtual photons are used since (a) photon energy and momentum can be varied independently and (b) longitudinal
components of the transition current are accessible. For the nucleon, the model-independent properties of the low-energy VCS amplitude have been identified in [8, 9] whereas the spin-zero case has been discussed in Ref. [10]. In [8] the model-dependent part beyond the LET was analyzed in terms of a multipole expansion. Keeping only terms linear in the energy of the final photon, the corresponding amplitude was parametrized in terms of ten so-called generalized polarizabilities (GPs) which are functions of the three-momentum transfer of the virtual photon in the VCS process. The number of independent GPs reduces to six, if charge-conjugation invariance is imposed [11, 12]. Predictions for the generalized polarizabilities of the nucleon have been obtained in various frameworks [8, 13, 14, 15, 16, 17, 18] (for an overview, see Ref. [19]).

2 Kinematics and LET

The invariant amplitude consists of a Bethe-Heitler piece, where the real photon is emitted by the initial or final electrons, and the VCS contribution (see Fig. 1),

\[ M = M_{BH} + M_{VCS}. \] (1)

\[ M_{VCS} = -ie^2 \epsilon_\mu \epsilon_\nu^* \epsilon^{\mu \nu} = -ie^2 \epsilon_\mu M^\mu = ie^2 \left( \epsilon_T \cdot M_T + \frac{q^2}{\omega^2} \epsilon_z M_z \right), \] (2)

where \( \epsilon_\mu = e \bar{u} \gamma_\mu u / q^2 \) is the polarization vector of the virtual photon \( (e > 0, e^2 / 4\pi \approx 1/137) \), and where use of current conservation has been made. In the center-of-mass system, using the Coulomb gauge for the final real photon, the transverse (longitudinal) part of \( M_{VCS} \) for the nucleon can be expressed in terms of eight (four) independent structures [8, 10].

\[ \epsilon_T \cdot M_T = \epsilon_T^\prime \cdot \epsilon_T A_1 + \cdots, \quad M_z = \epsilon_T^\prime \cdot \hat{q} A_9 + \cdots, \] (3)
where the functions \( A_i \) depend on three kinematical variables, \(|q|, \omega' = |q'|, z = \hat{q} \cdot \hat{q}'\). For the spin zero case only one longitudinal and two transverse structures are required \([10, 11]\).

Extending the method of Gell-Mann and Goldberger \([3]\) to VCS, model-independent predictions for the functions \( A_i \) were obtained in \([3]\). For example, the result for \( A_1 \) up to second order in \(|q|\) and \( \omega' \) is

\[
A_1 = -\frac{1}{M} + \frac{z}{M^2} |q| - \left( \frac{1}{8M^3} + \frac{7r_E^2}{6M} - \frac{\kappa}{4M^3} - \frac{4\pi \alpha}{e^2} \right) \omega'^2
\[
+ \left( \frac{1}{8M^3} + \frac{7r_E^2}{6M} - \frac{z^2}{M^3} + \frac{(1+\kappa)\kappa}{4M^3} \right) |q|^2.
\]

To this order, all \( A_i \) can be expressed in terms of \( M, \kappa, G_E, G_M, r_E^2, \alpha \), and \( \beta \). For \(|q| = \omega'\), Eq. (4) reduces to the well-known RCS result.

In Ref. \([10]\) the low-energy behavior of the VCS amplitude of \( \pi^- (p_i) + \gamma^+ (\epsilon, q) \rightarrow \pi^- (p_f) + \gamma (\epsilon', q') \) was found to be of the form

\[
\mathcal{M}_{\text{VCS}} = -2ie^2 F(q^2) \left[ \frac{p_f \cdot \epsilon^* (2p_i + q) \cdot \epsilon}{s - m_\pi^2} + \frac{p_i \cdot \epsilon^* (2p_f - q) \cdot \epsilon}{u - m_\pi^2} - \epsilon \cdot \epsilon^* \right] + \mathcal{M}_R,
\]

where \( F(q^2) \) is the electromagnetic form factor of the pion, \( s = (p_i + q)^2 \), and \( u = (p_i - q')^2 \). The residual term \( \mathcal{M}_R \) is separately gauge invariant and at least of second order, i.e. \( \mathcal{O}(qq, qq', q'q') \).

### 3 Generalized Polarizabilities

For the purpose of analyzing the model-dependent terms specific to VCS, the invariant amplitude \( \mathcal{M}_{\text{VCS}} \) is split into a (generalized) pole piece \( \mathcal{M}_A \) and a residual part \( \mathcal{M}_B \). For the nucleon, the s- and u-channel pole diagrams are calculated using electromagnetic vertices of the form

\[
\Gamma^\mu(p', p) = \gamma^\mu F_1(q^2) + i\frac{q^\mu q_\nu}{2M} F_2(q^2), \quad q = p' - p,
\]

where \( F_1 \) and \( F_2 \) are the Dirac and Pauli form factors, respectively. The corresponding amplitude \( \mathcal{M}_A^{\gamma^\mu \gamma} \) contains all irregular terms as \( q \rightarrow 0 \) or \( q' \rightarrow 0 \) and is separately gauge invariant \([10, 11]\). For the pion, the situation is somewhat more complicated due to the fact that even for real photons the s- and u-channel pole diagrams are not separately gauge-invariant. A natural starting point is given by Eq. (3) with \( \mathcal{M}_B \equiv \mathcal{M}_R \).

The generalized polarizabilities in VCS \([3]\) result from an analysis of \( \mathcal{M}_B^{\gamma^\mu \gamma} \) in terms of electromagnetic multipoles \( H^{(\rho)L, (\rho)L'} (\omega', |q|) \), where \( \rho (\rho') \) denotes the type of the initial (final) photon (\( \rho = 0 \): charge, C; \( \rho = 1 \): magnetic, M; \( \rho = 2 \): electric, E). The initial (final) orbital angular momentum is denoted by \( L \) (\( L' \)), and \( S \) distinguishes between non-spin-flip (\( S = 0 \)) and spin-flip (\( S = 1 \)) transitions. For the pion, only the case \( S = 0 \) applies. According to
the LET for VCS, $\mathcal{M}_B^{\gamma^*\gamma}$ is at least linear in the energy of the real photon. A restriction to the lowest-order, i.e. linear terms in $\omega'$ leads to only electric and magnetic dipole radiation in the final state. Parity and angular-momentum selection rules (see Table 3.1) then allow for 3 scalar multipoles ($S = 0$) and 7 vector multipoles ($S = 1$). The corresponding ten GPs, $P^{(01,01)}_0, ..., P^{(11,11)}_1$,

Table 3.1. Multipolarities of initial and final states

| $J^P$ | final real photon | initial virtual photon |
|------|------------------|-----------------------|
| $\frac{1}{2}^-$ | E1 | C1,E1 |
| $\frac{3}{2}^-$ | E1 | C1,E1,M2 |
| $\frac{1}{2}^+$ | M1 | C0,M1 |
| $\frac{3}{2}^+$ | M1 | C2,E2,M1 |

are functions of $|q|^2$, where mixed-type polarizabilities, $\hat{P}^{(\rho',L,S)}(|q|^2)$, have been introduced which are neither purely electric nor purely Coulomb type (see [8] for details). Only six of the above ten GPs are independent, if charge-conjugation symmetry is imposed [11, 12]. For example, for a charged pion, the constraint for the Compton tensor reads [10, 11]

$$M_{\mu\nu}^{\pi} = M_{\mu\nu}^{\pi} - M_{\mu\nu}^{\pi}(-p_i, q'; -p_f, q),$$

(7)

generating one relation between originally three GPs [11, 15]. Relations between the GPs at $|q| = 0$ and the four spin-dependent RCS polarizabilities $\gamma_i$ of Ref. [20] were discussed in [12]:

$$\gamma_3 = -\frac{e^2}{4\pi} \frac{3}{\sqrt{2}} P^{(01,12)}_1(0), \quad \gamma_2 + \gamma_4 = -\frac{e^2}{4\pi} \frac{3\sqrt{3}}{2\sqrt{2}} P^{(11,02)}_1(0),$$

(8)

i.e., only two of the four $\gamma_i$ can be related to GPs at $|q| = 0$.

4 Generalized Polarizabilities of the Nucleon

In the following we will discuss the predictions for the generalized polarizabilities of the nucleon obtained within the heavy-baryon formulation of chiral perturbation theory (HBChPT) and the linear sigma model.

In Ref. [16] the VCS amplitude was calculated using HBChPT to third order in the external momenta. At $O(p^3)$, contributions to the GPs are generated by nine one-loop diagrams and the $\pi^0$-exchange $t$-channel pole graph (see [16]). For the loop diagrams only the leading-order Lagrangians are required,

$$\tilde{\mathcal{L}}^{(1)}_{\pi N} = \tilde{N}_v (iv \cdot D + g_A S \cdot u) N_v, \quad \mathcal{L}^{(2)}_{\pi \pi} = \frac{F_\pi^2}{4} \text{Tr} [\nabla_\mu U (\nabla^\mu U)^\dagger],$$

(9)

where $N_v$ represents a non-relativistic nucleon field, and $U = \exp(i\tau \cdot \pi / F_\pi)$ contains the pion field. The covariant derivatives $\nabla_\mu U$ and $D_\mu N_v$ include the
coupling to the electromagnetic field, and $u_\mu$ contains in addition the derivative coupling of a pion. In the heavy-baryon formulation the spin matrix is given by $S_\mu = i \gamma^5 \sigma_{\mu\nu} v_\nu$, where $v_\mu$ is a four-vector satisfying $v^2 = 1, v_0 \geq 1$. Finally, for the $\pi^0$-exchange diagram one requires in addition to Eq. (9) the $\pi^0\gamma\gamma^*$ vertex provided by the Wess-Zumino-Witten Lagrangian,

$$L_{\gamma\gamma\pi^0}^{(WZW)} = - \frac{e^2}{32\pi^2 F_\pi} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \pi^0,$$

where $e_{0123} = 1$ and $F_{\mu\nu}$ is the electromagnetic field strength tensor. At $O(p^3)$, the LET of VCS is reproduced by the tree-level diagrams obtained from Eq. (9) and the relevant part of the second- and third-order Lagrangian,

$$\hat{L}_{\pi N}^{(2)} = - \frac{1}{2M} \bar{N}_\nu \left[ D\cdot D + \frac{e}{2} \left( \mu_S + \tau_3 \mu_V \right) \varepsilon_{\mu
u\rho\sigma} F^{\mu\nu} v^\rho S^\sigma \right] N_\nu,$$

$$\hat{L}_{\pi N}^{(3)} = \frac{i e \varepsilon_{\mu
u\rho\sigma} F^{\mu\nu}}{8M^2} \bar{N}_\nu \left[ \mu_S - \frac{1}{2} + \tau_3 (\mu_V - \frac{1}{2}) \right] S^\rho D^\sigma N_\nu + \text{h.c.},$$

The linear sigma model (LSM) represents a field-theoretical realization of chiral $SU(2)_L \times SU(2)_R$ symmetry. The dynamical degrees of freedom are given by a nucleon doublet $\Psi$, a pion triplet $\pi$, and a singlet $\sigma$:

$$L_S = i \bar{\Psi} \gamma^\mu \partial_\mu \Psi + \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma + \frac{1}{2} \partial_\mu \pi \cdot \partial^\mu \pi$$

$$- g_{\pi N} \left( \bar{\Psi} \gamma^5 (\sigma + i \gamma_5 \tau \cdot \pi) \Psi \right) - \frac{\mu^2}{2} (\sigma^2 + \pi^2) - \frac{\lambda}{4} (\sigma^2 + \pi^2)^2,$$

where $L_{s,b}$ is a perturbation which explicitly breaks chiral symmetry. With an appropriate choice of parameters ($\mu^2 < 0$, $\lambda > 0$) the model reveals spontaneous symmetry breaking, $<0|\sigma|0> = F_\pi = 92.4$ MeV. The spectrum consists of massless pions, a massive sigma and nucleons with masses satisfying the Goldberger-Treiman relation $m_N = g_{\pi N} F_\pi$ with $g_A = 1$. The symmetry breaking of Eq. (14) generates the PCAC relation

$$\partial^\mu A_\mu^a = F_\pi m_\pi^2 \pi^a.$$

The interaction with the electromagnetic field is introduced via minimal substitution in Eq. (13). The generalized polarizabilities have been calculated in the framework of a one-loop calculation [13].

Numerical results for some of the generalized proton polarizabilities are shown in Fig. 2. In both ChPT and the LSM the electric polarizability $\alpha$ decreases considerably faster with $|q|^2$ than in the constituent quark model and the effective Lagrangian approach. Also, the chiral calculations show a distinct behavior for the slope of $\beta$ near the origin. Note that at $O(p^3)$, the results for the GPs are entirely given in terms of the pion mass $m_\pi$, the axial coupling constant $g_A$, and the pion decay constant $F_\pi$. Finally, it has been shown that ChPT and the LSM respect the relations between the GPs derived in Ref. [11, 12].
5 Generalized Polarizabilities of Pions

At the one-loop level, \( \mathcal{O}(p^4) \), of chiral perturbation theory \([21]\), the electromagnetic polarizabilities of the charged pion are entirely determined by an \( \mathcal{O}(p^4) \) counter term \([3]\).

\[
\alpha = -\beta = \frac{e^2}{4\pi} \frac{2}{m_\pi F^2} (2l_5'^- - l_6'^-) = (2.68 \pm 0.42) \times 10^{-4} \text{ fm}^3,
\]

where the linear combination \( 2l_5'^- - l_6'^- \) is predicted through the decay \( \pi^+ \to e^+\nu_e\gamma \). Corrections to this result at \( \mathcal{O}(p^6) \) were shown to be reasonably small, namely 12% and 24% of the \( \mathcal{O}(p^4) \) values for \( \alpha \) and \( \beta \), respectively \([22]\).

Presently, the pion VCS reaction is under investigation as part of the Fermilab E781 SELEX experiment, where a 600 GeV pion beam interacts with atomic electrons in nuclear targets \([7]\). In principle, the different behavior under the substitution \( \pi^- \to \pi^+ \) of \( M_{\text{BH}} \) and \( M_{\text{VCS}} \),

\[
M_{\text{BH}}(\pi^-) = -M_{\text{BH}}(\pi^+), \quad M_{\text{VCS}}(\pi^-) = M_{\text{VCS}}(\pi^+),
\]

may be of use in identifying the contributions due to internal structure by com-
paring the reactions involving a $\pi^-$ and a $\pi^+$ beam for the same kinematics:

$$d\sigma(\pi^+) - d\sigma(\pi^-) \sim 4\text{Re} (\mathcal{M}_{\text{BH}}(\pi^+)\mathcal{M}_{\text{VCS}}(\pi^+)).$$

We have calculated the invariant amplitude for VCS at $\mathcal{O}(p^4)$ in the framework of chiral perturbation theory. The result is in complete agreement with the LET of Ref. [10]. Using the procedure developed in Ref. [11] we find for the generalized polarizabilities of the charged pion (see Fig. 3)

$$\alpha(|q|^2) = -\beta(|q|^2),$$

where $E_\pi = \sqrt{m_\pi^2 + |q|^2}$ and

$$J^{(0)\prime}(x) = \frac{1}{x} \left[ 1 - \frac{2}{x \sigma(x)} \ln \left( \frac{\sigma(x) - 1}{\sigma(x) + 1} \right) \right], \quad \sigma(x) = \sqrt{1 - \frac{4}{x}}, \quad x \leq 0.$$

The $|q|^2$ dependence does not contain any $\mathcal{O}(p^4)$ parameter, i.e., it is entirely given in terms of the pion mass and the pion decay constant $F = 92.4$ MeV.

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1This argument works for any particle which is not its own antiparticle such as the $K^+$ or $K^0$. Of course, one could also employ the substitution $e^- \rightarrow e^+$. 

![Figure 3. Generalized polarizability $\alpha$ of the charged pion as function of $|q|^2$.](image)
References

1. F.E. Low: Phys. Rev. 96, 1428 (1954)
2. M. Gell-Mann, M.L. Goldberger: Phys. Rev. 96, 1433 (1954)
3. B.R. Holstein: Comments Part. Nucl. Phys. 19, 221 (1990); Comments Part. Nucl. Phys. 20, 301 (1992)
4. A.I. L’vov: Int. J. Mod. Phys. A8, 5267 (1993)
5. G. Audit et al.: CEBAF Report No. PR 93-050, 1993; J.F.J. van den Brand et al., CEBAF Report No. PR 94-011, 1994; G. Audit et al.: MAMI proposal “Nucleon structure study by Virtual Compton Scattering.” 1995; J. Shaw et al.: MIT–Bates proposal No. 97-03, 1997.
6. N. d’Hose: In these Proceedings
7. M.A. Moinester, A. Ocherashvili: Private Communication
8. P.A.M. Guichon, G.Q. Liu, A.W. Thomas: Nucl. Phys. A591, 606 (1995)
9. S. Scherer, A.Yu. Korchin, J.H. Koch: Phys. Rev. C54, 904 (1996)
10. H.W. Fearing, S. Scherer: Few-Body Syst. 23, 111 (1998)
11. D. Drechsel, G. Knöchelein, A. Metz, S. Scherer: Phys. Rev. C55, 424 (1997)
12. D. Drechsel, G. Knöchelein, A.Yu. Korchin, A. Metz, S. Scherer: Phys. Rev. C57, 941 (1998); nucl-th/9804078, to appear in Phys. Rev. C58 (1998)
13. G.Q. Liu, A.W. Thomas, P.A.M. Guichon: Austral. J. Phys. 49, 905 (1996)
14. M. Vanderhaeghen: Phys. Lett. B368, 13 (1996)
15. A. Metz, D. Drechsel: Z. Phys. A356, 351 (1996); Z. Phys. A359, 165 (1997)
16. T.R. Hemmert, B.R. Holstein, G. Knöchelein, S. Scherer: Phys. Rev. D55, 2630 (1997); Phys. Rev. Lett. 79, 22 (1997)
17. M. Kim, D.-P. Min: Seoul National University report SNUTP-97-046, 1997, hep-ph/9704381.
18. B. Pasquini, G. Salmè: Phys. Rev. C57, 2589 (1998)
19. P.A.M. Guichon, M. Vanderhaeghen: Prog. Part. Nucl. Phys. 41, 125 (1998)
20. S. Ragusa: Phys. Rev. D47, 3757 (1993); Phys. Rev. D49, 3157 (1994)
21. J. Gasser, H. Leutwyler: Ann. Phys. (N.Y.) 158, 142 (1984)
22. U. Bürgi: Phys. Lett. B377, 147 (1996); Nucl. Phys. B479, 392 (1996)
23. C. Unkmeir, S. Scherer, A.I. L’vov, D. Drechsel: In Preparation