Path integral formulation of the tunneling dynamics of a superfluid Fermi gas in an optical potential

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Abstract

To describe the tunneling dynamics of a stack of two-dimensional fermionic superfluids in an optical potential, we derive an effective action functional from a path integral treatment. This effective action leads, in the saddle point approximation, to equations of motion for the density and the phase of the superfluid Fermi gas in each layer. In the strong coupling limit (where bosonic molecules are formed) these equations reduce to a discrete nonlinear Schrödinger equation, where the molecular tunneling amplitude is reduced for large binding energies. In the weak coupling (BCS) regime, we study the evolution of the stacked superfluids and derive an approximate analytical expression for the Josephson oscillation frequency in an external harmonic potential. Both in the weak and intermediate coupling regimes the detection of the Josephson oscillations described by our path integral treatment constitutes experimental evidence for the fermionic superfluid regime.

I. INTRODUCTION

Recent experiments have demonstrated that both degenerate Fermi gases and Bose-Einstein condensates (BECs) can be loaded in one-dimensional optical lattices created by standing laser waves [1–5]. The atoms are trapped in the valleys of the periodic potential, and form a stack of ‘pancake’ shaped clouds weakly coupled to each other. When the laser power is large enough, the gases in the separate valleys become quasi two-dimensional
(quasi-2D). An additional parabolic potential, provided by external magnetic fields, and with corresponding oscillator length much larger than the period of the periodic potential, can be applied. In the ground state of this system, the fermionic and/or bosonic atoms are distributed in the lattice sites near the bottom of the additional parabolic trap.

The possibility to load a BEC in a periodic potential has led to the observation of the Mott-Insulator phase transition [6] and the detection of Bloch oscillations and Josephson currents through the potential barriers separating the layers of superfluid [2–4]. To observe the Josephson effect, Cataliotti et al. suddenly displaced the additional parabolic potential, placing the stack of quasi-2D BECs out of equilibrium [2]. Josephson currents allow the superfluid to tunnel between different layers and perform pendulum-like oscillations around the equilibrium position, driven by the external harmonic potential.

In Ref. [2], a critical Josephson current was found when the BEC was moved too far out of equilibrium, indicating the breakdown of superfluidity across the layers [7]. The existence of the Josephson currents is a direct manifestation of phase coherence across layers. For a one-dimensional array of dilute Fermi gases the superfluid regime is predicted to be accessible [8], and also in this case the Josephson effect will be a signature of superfluidity.

In this paper, we derive an effective action, starting from the path integral representation of the partition function of the coupled quasi-2D layers filled with two different species of fermions at temperature zero. From this effective action, we derive equations of motion that allow us to study the dynamics of the phase and the density of the fermionic superfluid. Based on our results for the equations of motions of a fermionic superfluid in a one-dimensional potential, we describe what would happen if a fermionic superfluid instead of a BEC would be subject to the experiment of Ref. [2] as illustrated in Fig. 1. We show that when the interatomic interactions are weak there are loosely bound BCS-pairs that can tunnel coherently through the barriers that separate the potential valleys. When the interatomic interaction becomes stronger, confinement induced quasi-2D bosonic molecules with high binding energy are formed and the superfluid becomes a Bose-Einstein condensate of these molecules and we calculate their tunneling energy.

II. THE EFFECTIVE ACTION

In the derivation of the effective action, we follow rather closely the approach suggested by S. De Palo et al. in Ref. [9] and start from the path integral representation of the partition function for a system consisting of layers of 2D fermions

\[ Z = \int D\psi^\dagger_{j,\sigma}(x)D\psi_{j,\sigma}(x) \exp \left\{ -S \left[ \psi^\dagger_{j,\sigma}(x), \psi_{j,\sigma}(x) \right] \right\}, \]

where the action is given by

\[ S \left[ \psi^\dagger_{j,\sigma}(x), \psi_{j,\sigma}(x) \right] = \sum_j \int_0^\beta d\tau \int d^2x \left[ \sum_\sigma \psi^\dagger_{j,\sigma}(x) \left( \partial_\tau - \frac{\nabla^2}{2m} + V_{\text{ext}}(j) - \mu \right) \psi_{j,\sigma}(x) \\
- U \psi^\dagger_{j,\uparrow}(x) \psi^\dagger_{j,\downarrow}(x) \psi_{j,\downarrow}(x) \psi_{j,\uparrow}(x) \\
+ t_1 \sum_\sigma \left( \psi^\dagger_{j,\sigma}(x) \psi_{j+1,\sigma}(x) + \psi^\dagger_{j+1,\sigma}(x) \psi_{j,\sigma}(x) \right) \right]. \]
Here, $\beta = 1/(k_B T)$ where $T$ denotes the temperature and $k_B$ the Boltzmann constant. The three-vector notation $x = (x, \tau)$ is used. The field $\psi_{j,\sigma} (x)$ belongs to a fermion of mass $m$ in layer $j$ and spin $\sigma$ ($\uparrow$ or $\downarrow$). The potential $V_{\text{ext}} (j)$ is an additional external potential that the fermions are subjected to. The attraction strength between the fermions is determined by $U$. The interlayer tunneling energy for a fermion is denoted by the real number $t_1$ that can be calculated with the approximate formula from Ref. [10] for the tunneling energy in a optical potential $V (z) = V_0 \sin^2 (2\pi z/\lambda)$ with wave length $\lambda$ and depth $V_0$:

$$t_1 = \frac{m\omega^2 L^2}{8\pi^2} \left[ \frac{\pi^2}{4} - 1 \right] e^{-\lambda/(4\ell_0)^2},$$

where $\omega_L = \sqrt{8\pi^2 V_0 / (m\lambda^2)}$ and $\ell_L = \sqrt{1/m\omega_L}$ are respectively the trapping frequency and the oscillator length that an atom feels in the $z$-direction.

In appendix A, the reduction of (2) to an effective action is given and here we only give a summary. In order to grasp the most important part of the path integral in the superfluid state, the interaction between the fermions is decoupled by the Hubbard-Stratonovich (HS) transformation with the complex HS-field $\Delta^{HS} (x)$. After integration over the fermion fields, one is left with an effective action in terms of the HS-fields.

However, no information about the physical density of the system can be read off from such an effective action. In order have access to this quantity in the effective action, we introduce it by multiplying the partition function with the constant

$$C = \int \mathcal{D} \zeta^{HS} (x) \mathcal{D} \rho_j (x) \exp \left\{ - \sum_j \int_0^\beta d\tau \int d^2 x \, i \zeta_j^{HS} (x) \right.$$} \begin{equation}
\times \left[ \rho_j (x) - \psi_{j,\uparrow}^\dagger (x) \psi_{j,\uparrow} (x) - \psi_{j,\downarrow}^\dagger (x) \psi_{j,\downarrow} (x) \right]\right\}.
\end{equation}

Carrying out the functional integral over $\zeta_j^{HS} (x)$ alone gives $\delta [\rho_j (x) - \psi_{j,\uparrow}^\dagger (x) \psi_{j,\uparrow} (x) - \psi_{j,\downarrow}^\dagger (x) \psi_{j,\downarrow} (x)]$ and thus $\rho_j (x)$ corresponds to the physical density of the system along any path. Next, the complex field $\Delta_j^{HS} (x)$ is separated in a modulus and a phase. This phase is important for the low energy dynamics and therefore it is advantageous to introduce it explicitly

$$\Delta_j^{HS} (x) = \left| \Delta_j^{HS} (x) \right| e^{i\theta_j (x)}.$$  

We then arrive at the following expression for the partition function

$$Z \propto \int \mathcal{D} \left| \Delta_j^{HS} (x) \right| \mathcal{D} \zeta_j^{HS} (x) \mathcal{D} \theta_j (x) \mathcal{D} \rho_j (x) \exp \left[ -S_{\text{eff}} \right],$$

where $S_{\text{eff}}$ is given in appendix A, expression (A10).

To describe the low-energy dynamics of the density and the phase of the superfluid, the paths along which $\theta_j (x)$ and $\rho_j (x)$ vary slowly in comparison to the fermionic frequencies (Fermi energy and binding energy) will be of importance. Along these paths, we make a saddle point approximation for the remaining fields $\left| \Delta_j^{HS} (x) \right|$ and $\zeta_j^{HS} (x)$ in appendix B. The fluctuations $\delta \left| \Delta_j^{HS} (x) \right|$ and $\delta \zeta_j^{HS} (x)$ around the saddle point values $\left| \Delta_j^{(0)} (x) \right|$ and $\zeta_j^{(0)} (x)$
can be treated perturbatively. The saddle point value for the effective action is calculated in the appendix, and given by expression (B9). The saddle point equations are

\[
\frac{1}{U} = \int \frac{d^2k}{(2\pi)^2} \frac{1 - 2n_F [E_j(k)]}{2E_j(k)}, \tag{7}
\]
\[
\rho_j(x) = \int \frac{d^2k}{(2\pi)^2} \left( \frac{k^2}{2m} - i\zeta_j^{(0)}(x) \right) \left\{ \frac{2n_F [E_j(k)] - 1}{E_j(k)} + 1 \right\}, \tag{8}
\]

with \(n_F(E) = 1/(e^{\beta E} + 1)\) the Fermi-Dirac distribution function and \(E_j(k)\) the local BCS energy defined by

\[
E_j(k) = \sqrt{\left( \frac{k^2}{2m} - i\zeta_j^{(0)}(x) \right)^2 + \left| \Delta_j^{(0)}(x) \right|^2}. \tag{9}
\]

Equations (8) and (9) show that the saddle point value \(\zeta_j^{(0)}(x)\) can be interpreted as a chemical potential \(\zeta_j(x) = i\zeta_j^{(0)}(x)\). The first saddle point equation (7) corresponds to the BCS gap equation, whereas the second saddle point equation leads to the BCS equation fixing the chemical potential \(\zeta_j(x)\) in layer \(j\) as a function of the density \(\rho_j(x)\) in layer \(j\).

As we have introduced a momentum independent contact-interaction, equation (7) has to be regularized as described in [11], after which it becomes

\[
-\frac{1}{T_{00}(E)} = \int \frac{d^2k}{(2\pi)^2} \left[ \frac{2n_F [E_j(k)] - 1}{2E_j(k)} - \frac{1}{k^2/m - E + i\varepsilon} \right], \tag{10}
\]

where \(T_{00}(E)\) is the low-momentum limit of the \(T\)-matrix. This equation has no ultraviolet divergences. In two dimensions, at low energy, \(T_{00}(E)\) is given by [12,13]

\[
\frac{1}{T_{00}(E)} = \frac{m}{4} \left[ \frac{-1}{\pi} \ln \frac{E}{E_b} + i \right], \tag{11}
\]

where \(E_b\) is the energy of the 2D bound state that always exists in two dimensions. For the optical potential \(V_0 \sin^2 (2\pi z/\lambda)\) described earlier, the binding energy of the quasi-2D bound state is given by [16]

\[
E_b = \frac{C\hbar\omega L}{\pi} \exp \left( \frac{\sqrt{2\pi}\ell_L}{a} \right), \tag{12}
\]

with \(a\) the scattering length of the fermionic atoms and \(C \approx 0.915\).

III. JOSEPHSON CURRENT AT \(T = 0\)

A. Equations of motion for density and phase

We now proceed with an analysis at \(T = 0\) and with the assumption that the energy \(t_1\) is small compared to the other energies, such that a perturbational expansion with \(t_1\) as a
small parameter is possible. In this case, equations (8) and (10) can be solved analytically for $|\Delta_j^{(0)}(x)|$ and $z_j(x)$ (see also [12]):

$$\left|\Delta_j^{(0)}(x)\right| = \sqrt{\frac{2\pi \rho_j(x)}{m}} E_b,$$

$$z_j(x) = \frac{\pi \rho_j(x)}{m} - \frac{E_b}{2}. \tag{14}$$

As we want to study the current perpendicular to the layers in which the atoms are confined, we have to calculate the terms in the effective action that couple the different layers. In our perturbational expansion of the effective action (B9), the lowest order contribution comes from the term in the self energy (B7) proportional to $t_1$. We find that the contribution equals

$$-2t_1^2 \sum_j \frac{1}{\beta} \sum_{\omega_n=(2n+1)\pi/\beta} \int \frac{d^2k}{(2\pi)^2} \frac{1}{\omega_n^2 + E_j^2(k)} \left[\omega_n^2 + E_{j+1}^2(k)\right]$$

$$\times \left\{-\omega_n^2 + \xi_{j+1}(k) \xi_j(k) + \left|\Delta_j^{(0)}(x)\right| \left|\Delta_j^{(0)}(x)\right| \cos \left[\theta_{j+1}(x) - \theta_j(x)\right]\right\}, \tag{15}$$

with $\xi_j(k) = k^2/(2m) - z_j(x)$ the free fermion dispersion and $E_j$ the BCS dispersion relation (9). In this expression, the part proportional to $\cos [\theta_{j+1}(x) - \theta_j(x)]$ is responsible for the Josephson tunneling between the layers and we write it symbolically as

$$S_{\text{tunnel}} = -\sum_j \int_0^\beta d\tau \int dx \ T_{j+1,j} \cos [\theta_{j+1}(x) - \theta_j(x)]. \tag{16}$$

Evaluating this term with the supposition that the gap $\left|\Delta_j^{(0)}(x)\right|$ and chemical potential $z_j(x)$ vary slowly with $j$ leads to

$$T_{j+1,j} = \frac{t_1^2 m}{4\pi} \left(1 + \frac{z_j(x)}{\sqrt{\left|\Delta_j^{(0)}(x)\right|^2 + z_j^2(x)}}\right) = \frac{t_1^2 \rho_j(x)}{2\pi \rho_j(x)/m + E_b}. \tag{17}$$

A similar contribution to the energy can be obtained in a BCS-approach, following Ref. [14].

We now take as an approximation for the effective action

$$S_1[\theta_j(x), \rho_j(x)] = S_{\text{eff}} + S_{\text{tunnel}}, \tag{18}$$

with $S_{\text{eff}}$ given by (B9) from which $|\Delta_j^{0}(x)|$ and $\zeta_j^{0}(x)$ have been eliminated using equations (13),(14). Having obtained an expression for the action that only depends on $\theta_j(x)$ and $\rho_j(x)$ we can finally proceed with deriving equations of motion by extremizing $S_1$ with respect to these fields. Because we want to give a dynamical interpretation to these equations, we write them in real time $(i\partial_\tau \to \partial_t)$. Extremizing with respect to the phase field $\theta_j(x)$ results in

$$\partial_t \frac{\rho_j(x)}{2} = -\frac{\nabla \theta_j(x) \cdot \nabla \rho_j(x)}{4m}$$

$$+ T_{j,j-1} \sin \left[\theta_j(x) - \theta_{j-1}(x)\right]$$

$$- T_{j+1,j} \sin \left[\theta_{j+1}(x) - \theta_j(x)\right], \tag{19}$$

$$\partial_t \theta_j(x) = \frac{\rho_j(x)}{m} - \frac{E_b}{2}. \tag{20}$$
and the derivative with respect to $\rho_j(x)$ yields

$$-\partial_j \frac{\theta_j(x)}{2} = \frac{[\nabla \theta_j(x)]^2}{8m} + V_{\text{ext}}(j) + z_j - \mu$$

$$-\frac{\partial T_{j+1,j}}{\partial \rho_j(x)} \cos[\theta_{j+1}(x) - \theta_{j}(x)]$$

$$-\frac{\partial T_{j,j-1}}{\partial \rho_j(x)} \cos[\theta_j(x) - \theta_{j-1}(x)].$$

(21)

We have calculated the derivative $\partial T_{j+1,j}/\partial \rho_j$ if the density varies smoothly with the layer index ($\rho_{j+1} \approx \rho_j \approx \rho_{j-1}$) and is constant in the plane:

$$\frac{\partial T_{j+1,j}}{\partial \rho_j(x)} = \frac{t^2_1}{2} \frac{E_b}{[2\pi \rho_j(x)/m + E_b]^2}. \quad (22)$$

B. Oscillations of the superfluid in an optical lattice

We now introduce the wave function

$$\psi_j(t) = \sqrt{\frac{\rho_j(t)}{2}} e^{i\theta_j(t)}, \quad (23)$$

with $\rho_j$ and $\theta_j$ only depending on time and layer index $j$. That is, we assume that within a layer, the density and the phase are homogeneous, but that they can still vary over time and over layers. The wave function (23) obeys the Schrödinger equation

$$i \frac{d}{dt} \psi_j = (2V_{\text{ext}}(j) + 2z_j - 2\mu) \psi_j - \frac{t^2_1}{2\pi \rho_j(x) + E_b} \psi_j \left(e^{i(\theta_{j-1} - \theta_j)} + e^{i(\theta_{j+1} - \theta_j)}\right)$$

$$-\psi_j t^2_1 \frac{4\pi \rho_j/m}{(E_b + 2\pi \rho_j/m)^2} \left[\cos(\theta_{j+1} - \theta_j) + \cos(\theta_j - \theta_{j-1})\right], \quad (24)$$

where we can take in the approximation of slowly varying density that we used before, namely $\psi_{j-1} \approx \psi_j e^{i(\theta_{j-1} - \theta_j)}$ and $\psi_{j+1} \approx \psi_j e^{i(\theta_{j+1} - \theta_j)}$, so that we have in the limit $\pi \rho_j/m \ll E_b$ that

$$i \frac{d}{dt} \psi_j = (2V_{\text{ext}}(j) + 2z_j - 2\mu) \psi_j - \frac{t^2_1}{2\pi \rho_j(x) + E_b} (\psi_{j-1} + \psi_{j+1}), \quad (25)$$

which is the nonlinear discrete Schrödinger equation for an array of Bose-Einstein condensates [15], where the boson tunneling matrix element is given by $t^2_1 / (2\pi \rho_j/m + E_b)$. This is a decreasing function of $E_b$, which has the physical consequence that for increasing molecular binding energy the coupling between the layers becomes weaker and that in the Bose-Einstein limit ($E_b \to \infty$), no tunneling of molecules occurs. This may be related to the fact that the binding together of atoms in molecules is strongly influenced by the the confinement potential and the quasi-2D nature of the gas (as can be seen from eq. 12). In the intermediate states in the calculation of the amplitude of the tunneling process, the molecular state is
unlikely to survive as a bound state. Hence, the molecular binding energy is to be added to
the energy barrier for tunneling.

We know from [2], that in the bosonic limit equation (25) allows for oscillations where
the phase difference $\theta_{j+1} - \theta_j$ is locked to a constant value $\Delta \theta$ and the differential
equation governing the dynamics of the center of mass coordinate and this phase difference $\Delta \theta$
is of the pendulum type. Our numerical simulations of (24) suggest that in the BCS-regime there
is a similar current through the array.

From equations (20) and (21), we can derive a simplified analytical formula to estimate
the oscillation frequency in an external harmonic potential $V_{ext}(j) = \Omega j^2$. Assuming that
the phase difference between neighboring layers is constant, this becomes

$$\partial_t R = 2 \langle T_{j,j-1} \rangle \sin(\theta_j - \theta_{j-1}),$$ \hspace{1cm} (26)

where

$$\langle T_{j,j-1} \rangle = \frac{1}{\sum_j \rho_j(x)} \sum_j T_{j,j-1}$$

is the average of $T_{j,j-1}$ over the lattice sites. The initial density profile $\rho_j(t=0)$ is calculated
from the chemical potentials $z_j$ through (14) and these chemical potentials are derived from
(21), $z_j = \mu - V_{ext}(j)$, for a particular choice of $\mu$. The last two terms from equation (24)
seem to have only a minor influence on the frequency so that we omit them in the analytic
calculation. Equation (21) then becomes

$$\partial_t (\theta_j - \theta_{j-1}) = -4\Omega R.$$ \hspace{1cm} (27)

For small oscillations, (26) and (27) lead to the frequency

$$\omega = \sqrt{8\Omega \langle T_{j,j-1} \rangle}.$$ \hspace{1cm} (28)

Taking $^{40}$K atoms with a central density of $n = 10^9$ cm$^{-2}$, an optical wavelength $\lambda = 754$
nm, and an axial frequency $\omega_a = 2\pi \times 24$ Hz, we plot in Fig. 2 the analytical frequency
(curves) obtained from equation (28) and compare it with a numerical calculation (symbols)
based on equation (24). The oscillation frequency is plotted as a function of the inverse scattering length which appears in expression (12) of the binding energy. Fig. 2 shows that the estimation (28) agrees almost perfectly with the numerical results in the bosonic limit,
and that also in the BCS regime there is reasonable quantitative agreement. The inset of Fig. 2 shows that far in the bosonic regime ($a > 0$), the oscillation frequency decreases rapidly
as an exponential function of $1/a$. Nevertheless in the cross-over regime, the oscillation
frequencies are high enough for the observation of Josephson currents to be useful as a tool
to investigate the superfluidity of an ultracold Fermi gas, analogous to the experiments of
Ref. [2] in the bosonic case.

IV. CONCLUSIONS

We have derived an effective action and the resulting equations of motion to describe
the dynamics of a fermionic superfluid in a layered system and applied this formalism to
study the center of mass motion of an atomic Fermi gas in the potential formed by an optical standing wave. We find that a Fermi gas in the BCS regime can perform superfluid oscillations through the optical lattice, similar to those that have been observed for condensates of bosonic atoms [2–4], when the gas is not in equilibrium in the harmonic trapping potential superimposed on the optical lattice. An analytical approximate expression (28) for the oscillation frequency is derived and the predictions of this expression are tested with numerical simulations of the full equations of motion (24). For a superfluid Fermi gas in the BEC regime, we find that the tunneling is suppressed when the molecular binding energy becomes large. We conclude that superfluidity in Fermi gases can be revealed through Josephson currents in optical lattices if the Fermi gas is either in the BCS regime, or in the weakly-bound molecular BEC regime.

V. ACKNOWLEDGMENTS

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APPENDIX A: DERIVATION OF THE EFFECTIVE ACTION

In order to grasp the most important part of the path integral in the paired-fermion state, the interaction between the fermions is decoupled by the Hubbard-Stratonovich transformation, which transforms (2) into

\[ Z = \int \mathcal{D} \psi_{j,\sigma}^\dagger(x) \mathcal{D} \psi_{j,\sigma}(x) \mathcal{D} \Delta_j^{HS}(x) \mathcal{D} \Delta_j^{HS,\dagger}(x) \exp \left\{ -S^{(1)} \right\}, \quad (A1) \]

with
\[ S^{(1)} = \sum_j \int_0^\beta d\tau \int d^2x \left[ \frac{|\Delta_j^{HS}(x)|^2}{U} \right. \]
\[ + \sum_{\sigma = \pm 1} \psi_{j,\sigma}^\dagger(x) \left( \partial_\tau - \frac{\nabla^2}{2m} + V_{\text{ext}}(j) - \mu \right) \psi_{j,\sigma}(x) \]
\[ - \Delta_j^{HS}(x) \psi_{j,\uparrow}^\dagger(x) \psi_{j,\uparrow}(x) - \Delta_j^{HS,\dagger}(x) \psi_{j,\downarrow}(x) \psi_{j,\downarrow}^\dagger(x) \]
\[ + \bar{t}_1 \sum_{\sigma} \left( \psi_{j,\sigma}^\dagger(x) \psi_{j+1,\sigma}(x) + \psi_{j+1,\sigma}^\dagger(x) \psi_{j,\sigma}(x) \right) \]. \tag{A2} \]

In order to keep track of the total density, we introduce the constant \( C \) (see expression (4)). In order to investigate the BCS gap and the phase we separate the complex field \( \Delta_j^{HS}(x) \) in a modulus and a phase (expression (5)) and also transform the fermion fields as \( \psi_{j,\sigma}(x) \rightarrow \psi_{j,\sigma}(x) e^{i\theta_j(x)/2} \). Additionally, we shift the field \( i\zeta_j^{HS}(x) \) according to
\[ i\zeta_j^{HS}(x) \rightarrow i\zeta_j^{HS}(x) + i\partial_\tau \theta_j(x) + \frac{(\nabla \theta_j(x))^2}{2} + V_{\text{ext}}(j) - \mu \tag{A3} \]
and use the Nambu spinor notation \( \eta_j(x) = (\psi_{j,\uparrow}(x), \psi_{j,\downarrow}^\dagger(x))^T \). After this procedure, the partition function can be written as
\[ Z \propto \int |\Delta_j^{HS}(x)| \mathcal{D}\eta_j^\dagger(x) \mathcal{D}\eta_j(x) \mathcal{D}|\Delta_j^{HS}(x)| \mathcal{D}\theta_j(x) \mathcal{D}\zeta_j^{HS}(x) \mathcal{D}\rho_j(x) \exp \{-S^{(2)}\}, \tag{A4} \]
with the action \( S^{(2)} = S_0 + S^{(3)}[\eta_j^\dagger(x), \eta_j(x)] \) where
\[ S_0 = \sum_j \int_0^\beta d\tau \int d^2x \left\{ \frac{|\Delta_j^{HS}(x)|^2}{U} \right. \]
\[ + \left[ i\zeta_j^{HS}(x) + i\partial_\tau \theta_j(x) + \frac{(\nabla \theta_j(x))^2}{2} + V_{\text{ext}}(j) - \mu \right] \rho_j(x) \right\} \tag{A5} \]
does not contain the fermion fields any more and
\[ S^{(3)} = \sum_j \int_0^\beta d\tau \int d^2x \left\{ \eta_j^\dagger(x) \left[ \left( \partial_\tau - i \frac{\nabla \theta_j(x)}{2} - i \frac{\nabla^2 \theta_j(x)}{4} \right) \sigma_0 \right. \right. \]
\[ + \left( -\frac{\nabla^2}{2} + i \zeta_j^{HS}(x) \right) \sigma_3 - |\Delta_j^{HS}(x)| \sigma_1 \right] \eta_j(x) \]
\[ + \left[ t_1 \eta_j^\dagger(x) e^{i(\theta_{j+1} - \theta_j)\sigma_3/2} \sigma_3 \eta_{j+1}(x) + \text{h.c.} \right] \]. \tag{A6} \]
is the part of the action that still depends on them. The Pauli matrices are denoted by \( \sigma_i \). Since \( S^{(3)} \) is quadratic in the fermion fields, the path integral over these fields can be performed, resulting in
\[ \int \mathcal{D}\eta_j^\dagger(x) \mathcal{D}\eta_j(x) \exp \{-S^{(3)}\} = \det [-G^{-1}], \tag{A7} \]
where the Green’s function \( G \) is a matrix in coordinate space as in layer space and is given by

\[
-G^{-1}(x, j; x', j') = \delta(x - x') \begin{pmatrix}
\delta_{jj'} & \left( \partial_r - i \frac{\nabla \theta_j(x)}{2m} - i \frac{\nabla^2 \theta_j(x)}{4m} \right) \sigma_0 \\
+ \left( -\frac{\nabla^2}{2m} - i \xi_j^{HS}(x) \right) \sigma_3 - |\Delta_j^{HS}(x)| \sigma_1 \\
+ \delta_{j+1,j'} t_1 e^{i(\theta_{j+1} - \theta_j)\sigma_3/2} \sigma_3 + \delta_{j-1,j'} t_1 e^{-i(\theta_{j+1} - \theta_j)\sigma_3/2} \sigma_3 \end{pmatrix}.
\]  

(A8)

The partition sum can then be written as

\[
Z \propto \int D \frac{|\Delta_j^{HS}(x)| D\theta_j(x) D\xi_j^{HS}(x) DR_j(x)}{\exp \{ -S_{\text{eff}} \}}
\]  

(A9)

where

\[
S_{\text{eff}} = S_0 + \text{Tr} \left[ \ln(-G^{-1}) \right]
\]  

(A10)

with \( S_0 \) given by (A5) and the Green’s function given by (A8).

**APPENDIX B: SADDLE POINT EXPANSION FOR THE EFFECTIVE ACTION**

From the path integrations in (A9), we ultimately want to extract equations of motion for \( \theta_j(x), \rho_j(x) \). To describe the low-energy dynamics, the contributions with \( \theta_j(x), \rho_j(x) \) varying slow in comparison with the fermionic frequencies (Fermi energy and binding energy) will be important. Along the paths of slowly varying \( \theta_j(x), \rho_j(x) \), we can make the saddle point expansion in the fields \( |\Delta_j^{HS}|, \xi_j^{HS} \), setting

\[
|\Delta_j^{HS}(x)| = |\Delta_j^{(0)}(x)| + \delta |\Delta_j^{HS}(x)|,
\]  

(B1)

\[
\xi_j^{HS}(x) = \xi_j^{(0)}(x) + \delta \xi_j^{HS}(x),
\]  

(B2)

where also \( |\Delta_j^{(0)}(x)| \) and \( \xi_j^{(0)}(x) \) will vary slowly in comparison to the fermion frequencies.

Expanding the Green’s function (A8) around the saddle point values \( |\Delta_j^{(0)}| \) and \( \xi_j^{(0)} = -iz_j \) leads to

\[
G^{-1} = G_0^{-1} + \Sigma,
\]  

(B3)

where the saddle point contribution is given by

\[
-G_0^{-1}(x, j; x', j') = \delta(x - x') \delta_{jj'} \left[ \partial_r \sigma_0 + \left( -\frac{\nabla^2}{2m} - z_j \right) \sigma_3 - |\Delta_j^{(0)}| \sigma_1 \right].
\]  

(B4)

This saddle point contribution can be diagonalized by going to Fourier space:

\[
G_0^{-1}(k, \omega; j; k', \omega', j') = \delta_{jj'} \delta_{\omega\omega'} \delta(k - k') \left[ i\omega \sigma_0 - \left( \frac{k^2}{2m} - z_j \right) \sigma_3 + |\Delta_j^{(0)}| \sigma_1 \right].
\]  

(B5)
and thus

\[
\text{Tr} \left[ \ln \left( -G_0^{-1} \right) \right] = \sum_j \frac{1}{\beta} \sum_{\omega_n=(2n+1)\pi/\beta} \int \frac{d^2k}{(2\pi)^2} \times \ln \left[ -\omega_n^2 - \left( k^2/(2m) - z_j(x) \right)^2 + \left| \Delta_j^{(0)}(x) \right|^2 \right].
\]

(B6)

In this expression, the \(x\) dependence of \(z_j(x)\) and \(\left| \Delta_j^{(0)}(x) \right|\) has been reintroduced, still assuming that these fields vary slowly in comparison to the relevant fermion frequencies. The self energy \(\Sigma\) equals

\[
-\Sigma(x,j; x', j') = \delta(x - x') \delta_{jj'} \left[ -i\sigma_3 \delta \zeta^{HS}(x) - (\delta \left| \Delta^{HS}(x) \right| ) \sigma_1 \right.
\]

\[
- \left( \frac{i}{2m} \nabla + \frac{i}{4m} \nabla^2 \right) \sigma_0
\]

\[
+ \delta(x - x') \left[ \delta_{j+1,j'} t_1 e^{i(\theta_{j+1}(x) - \theta_j(x)) \sigma_3/2} \sigma_3 \right.
\]

\[
+ \delta_{j-1,j'} t_1 e^{-i(\theta_{j+1}(x) - \theta_j(x)) \sigma_3/2} \sigma_3 \right],
\]

(B7)

so that

\[
\det \left[ -G^{-1} \right] = \exp \left\{ \text{Tr} \left[ \ln \left( -G^{-1} \right) \right] \right\} = \exp \left\{ \text{Tr} \left[ \ln \left( -G_0^{-1} - \Sigma \right) \right] \right\}
\]

\[
= \exp \left\{ \text{Tr} \left[ \ln \left( -G_0^{-1} \right) \right] - \sum_{n=1}^{\infty} \frac{\text{Tr} \left[ \left( -G_0^{-1} \Sigma \right)^n \right]}{n} \right\},
\]

(B8)

where \(\text{Tr}\) denotes the trace over all the variables (coordinates, imaginary time, layer index and Nambu space). Expanding the action, \(S_{\text{eff}}\), expression (A10), around the saddle point values \(\left| \Delta_j^{(0)}(x) \right|\) and \(z_j(x) = i\zeta_j^{(0)}(x)\), and using the result (B8) to lowest order, we find

\[
S_{\text{eff}}^{\text{sp}} = \text{Tr} \left[ \ln \left( -G_0^{-1} \right) \right] + \sum_j \int_0^\beta d\tau \int d^2x \left\{ \frac{\left| \Delta_j^{(0)}(x) \right|^2}{U} \right.
\]

\[
+ \left. z_j(x) + i\partial_\tau \frac{\theta_j(x)}{2} + \frac{\left( \nabla \theta_j(x) \right)^2}{8m} + V_{\text{ext}}(j) - \mu \right] \rho_j(x) \right\}
\]

(B9)

The saddle point values \(\left| \Delta_j^{(0)}(x) \right|\) and \(z_j(x) = i\zeta_j^{(0)}(x)\) are now equal to the extremum of (B9) and they fulfill the equations (7) and (8). The lowest order Green’s function \(G_0\) is given by (B6).
FIGURES

FIG. 1. Illustration of the experiment where the center of the harmonic+optical trap, confining quasi-2D clouds of atoms, is suddenly moved, which causes the oscillation of the superfluid. The grey wave represents the magnetic+optical trapping potential and the ellipsoids are the quasi-2D atomic clouds.

FIG. 2. The center of mass oscillation frequency of a zero temperature fermionic superfluid in a one dimensional optical potential is shown as a function of the inverse scattering length for different values of the optical potential measured in units of the recoil energy $E_R$. The lines are calculated with formula (28) for $V_0/E_R = 6$ (solid line), $V_0/E_R = 8$ (dashed line), $V_0/E_R = 10$ (dashed line). The left of the curve $V_0/E_R = 6$ is shown dotted because there the condition $t_1 \ll |\Delta|$ is not fulfilled anymore. The symbols show the frequency of a sinusoidal fit to the full numerical solution of the equations (24) for $V_0/E_R = 6$ (squares), $V_0/E_R = 8$ (circles), $V_0/E_R = 10$ (triangles).
This figure "ja-fig1.gif" is available in "gif" format from:

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