NEW EXAMPLES
OF SANDWICH GRAVITATIONAL WAVES
AND THEIR IMPULSIVE LIMIT

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Abstract

Non-standard sandwich gravitational waves are constructed from the homogeneous pp vacuum solution and the motions of free test particles in the space-times are calculated explicitly. They demonstrate the caustic property of sandwich waves. By performing limits to impulsive gravitational wave it is demonstrated that the resulting particle motions are identical regardless of the “initial” sandwich.

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1 Introduction

Plane-fronted gravitational waves with parallel rays (pp waves) are characterized by the existence of a quadruple Debever-Penrose null vector field which is covariantly constant. In suitable coordinates (cf. [1]) the metric of vacuum pp waves can be written as

$$ds^2 = 2d\zeta d\bar{\zeta} - 2dudv - (f + \bar{f})du^2,$$  \hspace{1cm} (1)

where $f(u, \zeta)$ is an arbitrary function of $u$, analytic in $\zeta$. The only non-trivial components of the curvature tensor are proportional to $f_{,\zeta\zeta}$ so that (1) represents flat Minkowski space-time when $f$ is linear in $\zeta$. The simplest case for which the metric describe gravitational waves arise when $f$ is of the form

$$f(u, \zeta) = d(u)\zeta^2,$$  \hspace{1cm} (2)
where \( d(u) \) is an arbitrary function of \( u \); such solutions are called homogeneous \( pp \) waves (or "plane" gravitational waves). Performing the transformation (cf. (2))

\[
\begin{align*}
\zeta &= \frac{1}{\sqrt{2}} (Px + iQy), \\
v &= \frac{1}{2} (t + z + PP'x^2 + QQ'y^2), \\
u &= t - z,
\end{align*}
\]

where real functions \( P(u) \equiv P(t - z), \, Q(u) \equiv Q(t - z) \) are solutions of differential equations

\[
P'' + d(u) P = 0, \quad Q'' - d(u) Q = 0,
\]

(4) (here prime denotes the derivation with respect to \( u \)) the metric can simply be written as

\[
ds^2 = -dt^2 + P^2 dx^2 + Q^2 dy^2 + dz^2.
\]

This form of the homogeneous \( pp \) waves is suitable for physical interpretation. Considering two free test particles standing at fixed \( x, y \) and \( z \), their relative motion in the \( x \)-direction is given by the function \( P(u) \) while it is given by \( Q(u) \) in the \( y \)-direction. The motions are unaffected in the \( z \)-direction which demonstrate transversality of gravitational waves. The coordinate \( u = t - z \) can now be understood as a "retarded time" and the function \( d(u) \) as a "profile" of the wave. Note also that functions \( P, Q \) may have a higher degree of smoothness than the function \( d \) so that relative motions of particles are continuous even in the case of a shock wave (with a step-function profile, \( d(u) \sim \Theta(u) \)), or an impulsive wave (with a distributional profile, \( d(u) \sim \delta(u) \)).

### 2 Standard sandwich wave

A sandwich gravitational wave [3, 4] is constructed from the homogeneous \( pp \) solution (1), (2) if the function \( d(u) \) is non-vanishing only on some finite interval of \( u \), say \( u \in [u_1, u_2] \). In such a case the space-time splits into three regions: a flat region \( u < u_1 \) ("Beforezone"), a curved region \( u_1 < u < u_2 \) ("Wavezone"), and another flat region \( u_2 < u \) ("Afterzone"). In the region \( u < u_1 \) where \( d(u) = 0 \) it is natural to choose solutions of Eqs. (4) such that \( P = 1 \) and \( Q = 1 \) so that the metric (3) is explicitly written in Minkowski form. The form of the metric (3) for \( u > u_1 \) is then given by solutions of Eqs. (4) where the function \( P, Q \) are chosen to be continuous up to the first derivatives at \( u_1 \) and \( u_2 \).
A standard example of a sandwich wave can be found in textbooks (cf. [3]). The “square” profile function \(d(u)\) is given simply by

\[
d(u) = \begin{cases} 
0, & u < 0 \\
 a^{-2}, & 0 \leq u \leq a^2 \\
0, & a^2 < u 
\end{cases}
\]

(6)

where \(a\) is a constant. It is easy to show that the corresponding functions \(P,Q\) are given by

\[
P(u) = \begin{cases} 
1, & u \leq 0 \\
 \cos(u/a), & 0 \leq u \leq a^2 \\
 -u \sin a/a + \cos a + a \sin a, & a^2 \leq u 
\end{cases}
\]

(7)

\[
Q(u) = \begin{cases} 
1, & u \leq 0 \\
 \cosh(u/a), & 0 \leq u \leq a^2 \\
u \sinh a/a + \cosh a - a \sinh a, & a^2 \leq u 
\end{cases}
\]

(8)

Therefore, particles which were in rest initially accelerate within the wave in such a way that they approach in \(x\)-direction and move apart in \(y\)-direction. Behind the wave they move uniformly (see Fig. 1a).

3 Non-standard sandwich waves

Now we construct some other (non-trivial) sandwich waves. Our work is motivated primarily by the possibility of obtaining impulsive gravitational waves by performing appropriate limits starting from different sandwich waves (see next Section). This also enables us to study particle motions in such radiative space-times. Moreover, the standard sandwich wave given by (6) is very special and “peculiar” since it represents a radiative space-time containing stationary regions. Indeed, for \(d(u)\) being a positive constant, the Killing vector \(\partial_a\) is timelike where \(|Re \zeta| > |Im \zeta|\). This “strange” property remained unnoticed in literature so far. It may be useful to introduce more general sandwich waves which are not stationary.

a) Sandwich wave with “∧” profile

Let us consider a solution (1), (2) for which the function \(d(u)\) takes the form

\[
d(u) = \begin{cases} 
0, & u \leq -a \\
b(a+u)/a, & -a \leq u \leq 0 \\
b(a-u)/a, & 0 \leq u \leq a \\
0, & a \leq u 
\end{cases}
\]

(9)
where \(a, b\) are arbitrary real (positive) constants. The wave has a “wedge” profile illustrated in Fig. 1b. Straightforward but somewhat lengthy calculations give the following form of the functions \(P(u), Q(u)\) (continuous up to the second derivatives everywhere including the points \(u = -a, u = 0\) and \(u = a\)):

\[
P(u) = \begin{cases} 
1, & u \leq -a \\
\frac{c}{\sqrt{u_1}} J_{-\frac{1}{3}} \left(\frac{2}{3} u_1^{3/2}\right), & -a \leq u \leq 0 \\
\sqrt{u_2} \left[A J_{\frac{1}{3}} \left(\frac{2}{3} u_2^{3/2}\right) + B J_{-\frac{1}{3}} \left(\frac{2}{3} u_2^{3/2}\right)\right], & 0 \leq u \leq a \\
C u + D, & a \leq u
\end{cases}
\]

\[
Q(u) = \begin{cases} 
1, & u \leq -a \\
\frac{c}{\sqrt{u_1}} I_{-\frac{1}{3}} \left(\frac{2}{3} u_1^{3/2}\right), & -a \leq u \leq 0 \\
\sqrt{u_2} \left[E I_{\frac{1}{3}} \left(\frac{2}{3} u_2^{3/2}\right) + F I_{-\frac{1}{3}} \left(\frac{2}{3} u_2^{3/2}\right)\right], & 0 \leq u \leq a \\
G u + H, & a \leq u
\end{cases}
\]

where \(c = 3^{-1/3} \Gamma(\frac{2}{3})\) (\(\Gamma\) being the gamma function), \(J_n\) is the Bessel function, \(I_n\) is the modified Bessel function (cf. [6]),

\[
u_1 = \sqrt[3]{\frac{b}{a} (a + u)} , \quad \nu_2 = \sqrt[3]{\frac{b}{a} (a - u)} ,
\]

and \(A, B, C, D, E, F, G, H\) are real constants given by the relations

\[
A = -2cZ\beta\delta , \quad B = cZ(\beta\gamma + \alpha\delta) , \\
C = -A \sqrt[3]{\frac{9b}{a} / \Gamma(1/3)} , \quad D = B/c - Ca , \\
E = -2cZ\nu\sigma , \quad F = cZ(\nu\rho + \mu\sigma) , \\
G = -E \sqrt[3]{\frac{9b}{a} / \Gamma(1/3)} , \quad H = F/c - Ga ,
\]

with \(Z = \frac{2\pi}{3\sqrt[3]{3}} (a\sqrt{b})^{1/3}\),

\[
\alpha = J_{\frac{1}{3}}(\kappa) , \quad \beta = J_{-\frac{1}{3}}(\kappa) , \\
\gamma = (a\sqrt{b})^{2/3} J_{-\frac{1}{3}}(\kappa) , \quad \delta = -(a\sqrt{b})^{2/3} J_{\frac{1}{3}}(\kappa) , \\
\mu = I_{\frac{1}{3}}(\kappa) , \quad \nu = I_{-\frac{1}{3}}(\kappa) , \\
\rho = (a\sqrt{b})^{2/3} I_{-\frac{1}{3}}(\kappa) , \quad \sigma = (a\sqrt{b})^{2/3} I_{\frac{1}{3}}(\kappa) ,
\]

\(\kappa = \frac{2}{3}a\sqrt{b}\) (note that \(\beta\gamma - \alpha\delta = 1/Z = \nu\rho - \mu\sigma\)). Typical behaviour of the particles in the above sandwich space-time is shown in Fig. 1b.
b) Sandwich wave with “/” profile

Another sandwich wave can be obtained using the function \( d(u) \) such that

\[
d(u) = \begin{cases} 
0, & u \leq -a \\
b(a + u)/a, & -a \leq u < 0 \\
0, & 0 < u
\end{cases}
\]  

(15)

where \( a, b \) are again constants. In fact, it has a “saw” profile (see Fig. 1c) which is one “half” of the sandwich discussed above. It is non-symmetric and contains two discontinuities of different types. The functions \( d(u) \) defined by Eq. (9) and (15) coincides for \( u \leq 0 \) so that the functions \( P, Q \) are identical in both cases. It is only necessary to join the solution at \( u = 0 \) differently:

\[
P(u) = \begin{cases} 
1, & u \leq -a \\
c \sqrt{u_1} J_{-\frac{1}{3}}(\frac{2}{3} u_1^{3/2}), & -a \leq u \leq 0 \\
K u + L, & 0 \leq u
\end{cases}
\]  

(16)

\[
Q(u) = \begin{cases} 
1, & u \leq -a \\
c \sqrt{u_1} I_{-\frac{1}{3}}(\frac{2}{3} u_1^{3/2}), & -a \leq u \leq 0 \\
M u + N, & 0 \leq u
\end{cases}
\]  

(17)

where

\[
K = c\delta \sqrt[3]{b/a}, \quad L = c\beta \sqrt[3]{a^2 b}, \quad M = c\sigma \sqrt[3]{b/a}, \quad N = c\nu \sqrt[3]{a^2 b}.
\]  

(18)

Relative motions of test particles are illustrated in Fig. 1c. Qualitatively, they resemble motions in both previous cases (cf. Fig. 1a and Fig. 1b), only the relative velocities of particles in the region behind the wave (given in \( x \)-direction by \( -\sin a/a, C \), and \( K \), respectively; in \( y \)-direction by \( \sinh a/a, G \), and \( M \)) depend differently on particular choice of the parameters \( a \) and \( b \).

c) Asymptotic sandwich wave

Let us also consider the function \( d(u) \) of the form

\[
d(u) = \frac{n}{2} \exp(-n|u|),
\]

(19)

(shown in Fig. 1d) where \( n \) is an arbitrary real positive constant. Now there are no flat regions in front of the wave and behind it. The space-time is curved everywhere (it is of
Petrov type N and therefore radiative), it becomes flat only asymptotically as \( u \to \pm \infty \). For this reason we choose the functions \( P, Q \) such that \( P(u \to -\infty) \to 1, P'(u \to -\infty) \to 0 \) and similarly \( Q(u \to -\infty) \to 1, Q'(u \to -\infty) \to 0 \). Then it can be shown that the functions are given by

\[
P(u) = \begin{cases} J_0 \left( \sqrt{\frac{2}{n}} \exp \left( \frac{n}{2} u \right) \right), & u \leq 0 \\ A_1 J_0 \left( \sqrt{\frac{2}{n}} \exp \left( -\frac{n}{2} u \right) \right) + A_2 Y_0 \left( \sqrt{\frac{2}{n}} \exp \left( -\frac{n}{2} u \right) \right), & 0 \leq u \end{cases}
\]

(20)

\[
Q(u) = \begin{cases} I_0 \left( \sqrt{\frac{2}{n}} \exp \left( \frac{n}{2} u \right) \right), & u \leq 0 \\ B_1 I_0 \left( \sqrt{\frac{2}{n}} \exp \left( -\frac{n}{2} u \right) \right) + B_2 K_0 \left( \sqrt{\frac{2}{n}} \exp \left( -\frac{n}{2} u \right) \right), & 0 \leq u \end{cases}
\]

(21)

where

\[
A_1 = -\frac{\pi}{2} \lambda (\tilde{\alpha} \tilde{\delta} + \tilde{\beta} \tilde{\gamma}) , \quad A_2 = \pi \lambda \tilde{\alpha} \tilde{\gamma} , \\
B_1 = \lambda (\tilde{\mu} \tilde{\sigma} + \tilde{\nu} \tilde{\rho}) , \quad B_2 = 2 \lambda \tilde{\mu} \tilde{\rho} ,
\]

with

\[
\tilde{\alpha} = J_0(\lambda) , \quad \tilde{\beta} = Y_0(\lambda) , \quad \tilde{\gamma} = J_1(\lambda) , \quad \tilde{\delta} = Y_1(\lambda) , \\
\tilde{\mu} = I_0(\lambda) , \quad \tilde{\nu} = -K_0(\lambda) , \quad \tilde{\rho} = I_1(\lambda) , \quad \tilde{\sigma} = K_1(\lambda) ,
\]

\( \lambda = \sqrt{2/n} \) (note also that \( \tilde{\alpha} \tilde{\delta} - \tilde{\beta} \tilde{\gamma} = -\frac{2}{\pi \lambda} \) and \( \tilde{\mu} \tilde{\sigma} - \tilde{\nu} \tilde{\rho} = -\frac{1}{\lambda} \)). Typical behaviour of the functions \( P, Q \) is shown in Fig. 1d. It can be observed that in both asymptotic regions \( u \to \pm \infty \) the particles move uniformly.

### 4 Impulsive limit

Now we can use the above results to construct impulsive gravitational waves. For standard sandwich wave (6)-(8) it is easy to perform the limit \( a \to 0 \). Then the profile function \( d(u) \) approaches the \( \delta \) function distribution and, using \( \sin a/a \to 1, \sinh a/a \to 1 \) we get

\[
P(u) = 1 - u \Theta(u) , \\
Q(u) = 1 + u \Theta(u) ,
\]

(24)

where \( \Theta \) is the Heaviside step function (\( \Theta = 0 \) for \( u < 0 \), \( \Theta = 1 \) for \( u > 0 \)).

For non-standard sandwiches introduced in previous section one has to perform similar limits more carefully. It is well known that the sequence of “\( \land \)” functions given by (9) approach the \( \delta \) function (in a distributional sense) as \( a \to 0 \) if the second parameter is \( b = 1/a \) (so that
the normalization condition \( \int_{-\infty}^{\infty} d(u) du = 1 \) holds for arbitrary \( a \). Considering this limit, \( \kappa = \frac{2}{3} \sqrt{a} \), the parameters (14) are
\[
\alpha \sim \mu \sim a^{1/6} 3^{-1/3} / \Gamma(4/3), \quad \beta \sim \nu \sim a^{-1/6} 3^{1/3} / \Gamma(2/3),
\gamma \sim \rho \sim 3^{2/3} / \Gamma(1/3), \quad \delta \sim -\sigma \sim -a^{2/3} 3^{1/3} / (2 \Gamma(2/3)),
\]
and (13) gives
\[
C \to -1, \quad D \to 1, \quad G \to 1, \quad H \to 1.
\] (25)

Therefore, the functions \( P, Q \) describing relative motions of test particles in the corresponding impulsive wave are again given by (24).

Analogously, the limit \( a \to 0 \) of \( / \) sandwiches given by (15) with \( b = 2/a \) gives
\[
\beta \sim \nu \sim (2a)^{-1/6} 3^{1/3} / \Gamma(2/3), \quad \delta \sim -\sigma \sim -(2a)^{2/3} 3^{1/3} / (2 \Gamma(2/3)),
\]
so that the parameters (18) are
\[
K \to -1, \quad L \to 1, \quad M \to 1, \quad N \to 1.
\] (26)

Again, the resulting functions \( P, Q \) can be written in the form (24).

Finally, we can perform a limit \( n \to \infty \) of “asymptotic sandwich waves” given by profile functions (19). For \( u \leq 0 \) it follows immediately from (20), (21) that
\( P \to 1 \) and \( Q \to 1 \). For \( u \geq 0 \) calculations are more complicated: for \( n \to \infty \) we get \( \tilde{\alpha} \sim \tilde{\mu} \sim 1, \quad \tilde{\beta} \sim -\frac{1}{\pi} \ln n, \quad \tilde{\gamma} \sim \tilde{\rho} \sim (2n)^{-1/2}, \quad \tilde{\delta} \sim -\frac{1}{\pi} [2n^{-1/2} \ln n + \sqrt{2n}], \quad \tilde{\nu} \sim -\frac{1}{\pi} \ln n, \quad \tilde{\sigma} \sim \frac{1}{2} [\sqrt{2n} - (2n)^{-1/2} \ln n], \)
so that \( A_1 \sim B_1 \sim 1, \quad A_2 \sim \pi/n \) and \( B_2 \sim 2/n \). Since \( J_0 \sim I_0 \sim 1, \quad Y_0 \sim -\frac{1}{\pi} [nu + \ln n] \) and \( K_0 \sim \frac{1}{2} [nu + \ln n] \) as \( n \to \infty \), in the limit we obtain
\[
P \sim 1 - u, \quad Q \sim 1 + u,
\] (27)
i.e. the relations (24) are revealed once more.

Note finally that using the transformation (3) with the functions \( P \) and \( Q \) given by (24) the metric of the impulsive homogeneous \( pp \) wave
\[
ds^2 = -dt^2 + (1 - u \Theta(u))^2 dx^2 + (1 + u \Theta(u))^2 dy^2 + dz^2,
\] (28)
goers over to
\[
ds^2 = 2 d\zeta d\bar{\zeta} - 2 du dv - \delta(u)(\zeta^2 + \bar{\zeta}^2) du^2,
\] (29)
i.e. the metric (1) with \( f(u, \zeta) = \delta(u) \zeta^2 \). Although this form of the impulsive wave is illustrative with the pulse evidently localized along the hyperplane \( u = 0 \), the metric (28) is more convenient in the sense that the metric system is continuous, \( \delta \) function appearing only in the components of the curvature tensor.

The transformation (3) with (24) also relates to the “scissors-and-paste” approach to the construction of impulsive solutions (2) which recently enabled new impulsive gravitational waves of somewhat different type to be found.
5 Concluding remarks

We constructed three new types of non-standard sandwich $pp$ waves with “wedge” (15), “saw” (15) and “asymptotic” (19) profiles. Contrary to the standard sandwich wave (9) they do not contain stationary regions. Particle motions were calculated explicitly and the corresponding limits to impulsive gravitational wave were performed. It was shown that all these limits give the same result (24). Moreover, for sandwich waves presented above there exist critical values of $u$ for which the function $P(u)$ vanishes so that all the particles initially at rest on the $x$-axis collide. This demonstrates the caustic property of (plane) sandwich waves [4].

Acknowledgments

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Figure Captions

Fig. 1. Typical exact behaviour of functions $P(u)$ and $Q(u)$ determining relative motions of free test particles (initially at rest) in $x$ and $y$-directions, respectively, caused by sandwich gravitational waves of various profiles: a) standard “square”, b) “wedge”, c) “saw”, d) “asymptotic” sandwich wave. In the last case also the limiting procedure $n \to \infty$ leading to an impulsive wave is indicated by corresponding dashed lines.
This figure "Sandwich.gif" is available in "gif" format from:

http://arxiv.org/ps/gr-qc/9801054v1