Lévy systems and moment formulas for mixed Poisson integrals \(^*\dagger\)

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Abstract

We propose Mecke-Palm formula for multiple integrals with respect to the Poisson random measure and its intensity measure performed, or mixed, in an arbitrary order. We apply the formulas to mixed Lévy systems of Lévy processes and obtain moment formulas for mixed Poisson integrals.

1 Introduction

The Mecke-Palm formula is an important identity in stochastic analysis of Poisson random measures. In this work we propose its generalization named the (multiple) mixed-type Mecke-Palm formula. We show that the generalization is useful and has a considerable scope of applications.

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Part of our motivation comes from recent results on moments of stochastic integrals [21], [7]. These were obtained for 1-processes in [21] by using combinatorics of the binomial convolution to undo the usual compensation in stochastic integration against Poisson random measures [12], [11]; and they were extended in [7, Theorem 3.1] to ensembles of integrals of 1-processes.

By compensation in the previous paragraph we mean integration against the difference of the random Poisson measure and its intensity, or control, measure. It is well-known that such integration fits well into the framework of $L^2$ Hilbert spaces [19]. In opposition, the results of this paper mainly concern iterated integrations against the (uncompensated) Poisson random measure interlaced, or mixed, with integrations against the control measure. Such integrations preserve nonnegativity and are performed under nonnegativity or absolute integrability conditions, rather then the square-integrability conditions (for which see Lemma 4.8 below or [19]). In both settings, however, the main feature of the iterated stochastic integration is the impact of the diagonals in the corresponding Cartesian products of the state space, which cannot be ignored because the random measure has atoms. The impact is accounted for by using partitions of the set of coordinates. We shall see below that in the setting of the uncompensated stochastic integration the description is simpler than in the compensated, or $L^2$, setting, for which we refer the reader to [19, Chapter 5]. In fact, the integrals against the compensated Poisson measure can be considered as (limits of) linear combinations of mixed integrals with respect to the Poisson random measure and its control measure, which explains the added complexity. Moreover, we may consider the results obtained in both settings as consequences of the mixed Mecke-Palm formula and the structure of the family of partitions. Our presentation is essentially self-contained in that it relies on the mixed Mecke-Palm formula, which we explain from the first principles. We should also remark that the integrands we consider are random, and in this respect they are more general than those in [19]. A complete survey of results on integration with respect to random measures is beyond the scope of this paper, but for more information we like to refer the reader to [13, 14, 17].

Below we first prove the mixed Mecke-Palm formula and use its along with the so-called linearization to obtain moments of stochastic integrals in more generality than known before: we consider moments of $k$-processes with arbitrary integer $k \geq 1$, and we allow Poisson stochastic integrations to be interlaced, or mixed, up to arbitrary multiplicity and order, with integrations against the intensity measure of the Poisson random measure. Our proofs are more direct as compared to [21] and [7], because they easily follow from the mixed-type Mecke-Palm formula.

When the random measure is given by the jumps of a Lévy process, the mixed
Mecke-Palm formula translates into multiple Lévy systems of mixed type, which is our second main application. By the multiple Lévy systems of mixed type we mean identities for expectations of functionals defined by accumulated summations indexed by the jumps of the Lévy process and integrations against the product of the linear Lebesgue measure on the time scale and the Lévy measure of the process in space. They generalize the classical (single) Lévy system \([3, 8, 4]\), which is an important tool in the study of jump-type Markov processes. The multiple variants have interesting applications and we indicate some of them.

The structure of the paper is as follows. In Theorem 2.4 of Section 2 we give the mixed Mecke-Palm formula for \(k\)-processes. In Theorem 3.1 of Section 3 we derive general moment formulas for ensembles of \(k\)-processes. These are illustrated by the moments formulas for 1-processes and 2-processes in Section 3.2 and Section 3.3, respectively. In Theorem 4.3 of Section 4 we present the multiple mixed Lévy systems for Lévy processes in \(\mathbb{R}^d\). In Section 4.2 we present several applications of the Lévy systems including applications that merge the topics and techniques from Section 3 and Section 4. Some of the results are known, but even then the presentation may be of interest. In Section 5 we give a proof of the simple Mecke-Palm formula, to make the paper more self-contained.

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2 Mixed Mecke-Palm formulas

A direct approach to calculus of Poisson random measures is based on the configuration space: Given a locally compact separable metric space \(X\), any locally finite subset of \(X\) is called a configuration on \(X\). The configuration space is defined as \(\Omega = \Omega_X = \{\omega \subset X : \omega\) is a configuration on \(X\}\) [20]. The elements of \(\Omega\) can be identified with the class of locally finite, nonnegative-integer valued
measures: if \( \omega = \{y_1, y_2, \ldots\} \), where \( y_i \in X \) are all different, then we also write

\[
\omega = \sum_i \delta_{y_i},
\]

where \( \delta_y \) is the probability measure concentrated at \( y \in X \). According to this identification, \( \omega \) will have two meanings depending on the context: a configuration on \( X \) or a measure on \( X \). We equip \( \Omega \) with a \( \sigma \)-algebra \( \mathcal{F} \), which is the smallest sigma-algebra of subsets of \( \Omega \) making the maps \( \omega \mapsto \omega(A) \) measurable for each Borel set \( A \subset X \) cf. [12, Chapter 10]. A jointly measurable map

\[
f : (X)^k \times \Omega \ni (x_1, \ldots, x_k; \omega) \mapsto f(x_1, \ldots, x_k; \omega) \in \mathbb{R}
\]

is called a process or, more specifically, a \( k \)-process. Here \( \mathbb{R} = \mathbb{R} \cup \{-\infty, \infty\} \), \( k \in \mathbb{N}_0 = \{0, 1, \ldots\} \), and when \( k = 0 \), i.e., \( f : \Omega \ni \omega \mapsto f(\omega) \in \mathbb{R} \), we call \( f \) a random variable. We also note that for every Borel function \( \phi \geq 0 \) on \( X \), the map

\[
\omega \mapsto \int \phi(x) \omega(dx)
\]

is well-defined and measurable, hence a random variable. We say that a \( k \)-process \( f \) depends only on \( X \subseteq X \), if \( f(x_1, \ldots, x_k; \omega) = f(x_1, \ldots, x_k; \omega \cap X) \) for all \( \omega \in \Omega \) and \( x_1, \ldots, x_k \in X \).

We let \( X^n_{\text{diag}} = \{x = (x_1, \ldots, x_n) \in X^n : x_i = x_j \text{ for some } i \neq j\} \) and \( X^n_\neq = X^n \setminus X^n_{\text{diag}} \), where \( X^1_\neq = X \). Given a \( k \)-process \( f \) and \( n \in \mathbb{N} \) we define the \( n \)-th coefficient \( f(n) \) of \( f \) as a function \( f(n) : X^k \times X^n_\neq \mapsto \mathbb{R} \) such that

\[
f(n)(x_1, \ldots, x_k; y_1, \ldots, y_n) = f(x_1, \ldots, x_k; \omega), \quad \text{where } \omega = \{y_1, \ldots, y_n\},
\]

We also let \( f(0)(x_1, \ldots, x_k) = f(x_1, \ldots, x_k; \emptyset) \). Thus, coefficients \( f(n) \) are Borel functions on \( X^k \times X^n_\neq \) invariant upon permutations of the last \( n \) coordinates. In particular, for random variables (0-processes) \( f \) we simply have \( f(n)(y_1, \ldots, y_n) = f(\{y_1, \ldots, y_n\}) \), where \( y_1, \ldots, y_n \) are all different, and \( f(0) = f(\emptyset) \). Of course, if \( f \) is a \( k \)-process, then \( \omega \mapsto f(x_1, \ldots, x_k; \omega) \) is a random variable for every choice of \( x_1, \ldots, x_k \in X \), and the \( n \)-th coefficient of this random variable is \( f(n)(x_1, \ldots, x_k; y_1, \ldots, y_n) \), provided \( (y_1, \ldots, y_n) \in X^n_\neq \).

Now we define a Poisson probability measure \( \mathbb{P} \) on \( (\Omega, \mathcal{F}) \) and the corresponding expectation \( \mathbb{E} \). Notice that \( N(A, \omega) := \omega(A) \) is a random measure on \( X \) under any probability measure on \( \Omega \), but we will consider the probability \( \mathbb{P} \) which makes \( N \) a Poisson random measure with intensity measure \( \sigma(A) = \mathbb{E}N(A) \). Here is a
construction of $\mathbb{P}$. The main analytic datum is a non-atomic measure $\sigma$ finite on compact subsets of $\mathbb{X}$. If $\mathcal{X}$ is a Borel subset of $\mathbb{X}$ and $\sigma(\mathcal{X}) < \infty$, then the corresponding probability, say $\mathbb{P}_{\mathcal{X}}$, is concentrated on finite configurations $\Omega_{\mathcal{X}}$ on $\mathcal{X}$ and defined by

$$
\mathbb{E}_{\mathcal{X}} f = \int_{\Omega_{\mathcal{X}}} f(\omega) \mathbb{P}_{\mathcal{X}}(d\omega) = e^{-\sigma(\mathcal{X})} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathcal{X}^n} f(n(y_1, \ldots, y_n)) \sigma(dy_n) \cdots \sigma(dy_1), \tag{2.1}
$$

cf. [20, p. 196]. Here the first term on the rightmost of (2.1) is $e^{-\sigma(\mathcal{X})} f(0)$, according to a general convention.

Further, let Borel sets $\mathcal{X}_1, \mathcal{X}_2, \ldots \subset \mathbb{X}$ be such that $\bigcup_m \mathcal{X}_m = \mathbb{X}$, $\mathcal{X}_m \cap \mathcal{X}_n = \emptyset$ for $m \neq n$, and $\sigma(\mathcal{X}_m) < \infty$ for every $m$. We identify $\Omega_\mathbb{X}$ with $\otimes \mathcal{X}_m$ by identifying $\omega$ with $(\omega \cap \mathcal{X}_m)_m$. Then $\mathbb{P}$ is unambiguously defined as the product measure,

$$
\mathbb{P} = \otimes_m \mathbb{P}_{\mathcal{X}_m}.
$$

For $\mathcal{X} \subset \mathbb{X}$, $\mathbb{P}_{\mathcal{X}}$ may be considered as a marginal distribution of $\mathbb{P}$, and for random variables $f_1, f_2$ depending only on disjoint $\mathcal{X}_1, \mathcal{X}_2 \subset \mathbb{X}$, respectively, we have

$$
\mathbb{E}[f_1(\omega) f_2(\omega)] = \mathbb{E}_{\mathcal{X}_1}[f_1(\omega)] \mathbb{E}_{\mathcal{X}_2}[f_2(\omega)]. \tag{2.2}
$$

Here the notions of independence of a function from a set of arguments, and the probabilistic independence happily meet. In what follows $\mathbb{E}$ and $\mathbb{P}$ are always the expectation and distribution making $\omega$ a Poisson random measure with control measure $\sigma$ (in Section 4 we make additional structure assumptions on $\mathbb{X}$ and $\sigma$).

In what follows we denote $\omega_1 = \omega$, $\omega_0 = \sigma$, for $\omega \in \Omega$. For a 1-process $f \geq 0$ and $\epsilon \in \{0, 1\}$, we have

$$
\mathbb{E} \int_{\mathbb{X}} f(x; \omega) \omega_\epsilon(dx) = \int_{\mathbb{X}} \mathbb{E} f(x; \omega + \epsilon \delta_x) \sigma(dx). \tag{2.3}
$$

Indeed, for $\epsilon = 0$ the identity follows from Fubini-Tonelli, and if $\epsilon = 1$, then it is the celebrated Mecke-Palm formula, see also [15, (2.10)]. (For the reader’s convenience a direct proof of the Mecke-Palm formula is given in Section 5.)

We say that a $k$-process $f$ vanishes on the diagonals if for all $\omega \in \Omega = \Omega_\mathbb{X}$ we have $f(x_1, \ldots, x_k; \omega) = 0$ whenever $(x_1, \ldots, x_k) \in \mathbb{X}_{\text{diag}}^k$, i.e. whenever $x_i = x_j$ for some $1 \leq i < j \leq k$. This condition is restrictive only if $k \geq 2$. We propose the following mixed Mecke-Palm formula.
Lemma 2.1. If $f \geq 0$ vanishes on the diagonals and $\epsilon_1, \ldots, \epsilon_k \in \{0, 1\}$, then
\[
\mathbb{E} \int_{\mathbb{X}^k} f(x_1, \ldots, x_k; \omega) \omega_{\epsilon_1}(dx_1) \cdots \omega_{\epsilon_k}(dx_k)
\]
\[
= \int_{\mathbb{X}^k} \mathbb{E} f(x_1, \ldots, x_k; \omega + \sum_{i=1}^{k} \epsilon_i \delta_{x_i}) \sigma(dx_1) \cdots \sigma(dx_k).
\]

Proof. Case $k = 0$ is trivial: $\mathbb{E} f(\omega) = \mathbb{E} f(\omega)$. Case $k = 1$ is precisely (2.3). For $k > 1$ we define
\[
g(x; \omega; \epsilon_1, \ldots, \epsilon_{k-1}) = \int_{\mathbb{X}^{k-1}} f(x_1, \ldots, x_{k-1}, x; \omega) \omega_{\epsilon_1}(dx_1) \cdots \omega_{\epsilon_{k-1}}(dx_{k-1}).
\]
Since $f(x_1, \ldots, x_{k-1}, x; \omega)$ vanishes on $\mathbb{X}_\text{diag}^k$, we get
\[
g(x; \omega + \delta_x; \epsilon_1, \ldots, \epsilon_{k-1})
= \int_{\mathbb{X}^{k-1}} f(x_1, \ldots, x_{k-1}, x; \omega + \delta_x) (\omega_{\epsilon_1} + \epsilon_1 \delta_x)(dx_1) \cdots (\omega_{\epsilon_{k-1}} + \epsilon_{k-1} \delta_x)(dx_{k-1})
= \int_{\mathbb{X}^{k-1}} f(x_1, \ldots, x_{k-1}, x; \omega + \delta_x) \omega_{\epsilon_1}(dx_1) \cdots \omega_{\epsilon_{k-1}}(dx_{k-1}).
\]
By (2.3), (2.5) and induction we obtain
\[
\mathbb{E} \int_{\mathbb{X}^k} f(x_1, \ldots, x_k; \omega) \omega_{\epsilon_1}(dx_1) \cdots \omega_{\epsilon_k}(dx_k)
= \mathbb{E} \int_{\mathbb{X}} g(x_k; \omega; \epsilon_1, \ldots, \epsilon_{k-1}) \omega_{\epsilon_k}(dx_k) = \int_{\mathbb{X}} \mathbb{E} g(x_k; \omega + \epsilon_k \delta_{x_k}; \epsilon_1, \ldots, \epsilon_{k-1}) \sigma(dx_k)
= \int_{\mathbb{X}} \mathbb{E} \int_{\mathbb{X}^{k-1}} f(x_1, \ldots, x_{k-1}, x_k; \omega + \epsilon_k \delta_{x_k}) \omega_{\epsilon_1}(dx_1) \cdots \omega_{\epsilon_{k-1}}(dx_{k-1}) \sigma(dx_k)
= \int_{\mathbb{X}} \mathbb{E} f(x_1, \ldots, x_k; \omega + \sum_{i=1}^{k} \epsilon_i \delta_{x_i}) \sigma(dx_1) \cdots \sigma(dx_k),
\]
which proves (2.4). \qed

Remark 2.2. Lemma 2.1 extends to signed processes $f$ satisfying
\[
\int_{\mathbb{X}^k} \mathbb{E} \left| f(x_1, \ldots, x_k; \omega + \sum_{i=1}^{k} \epsilon_i \delta_{x_i}) \right| \sigma(dx_1) \cdots \sigma(dx_k) < \infty,
\]
because of the decomposition \( f = f_+ - f_- \), where \( f_+ = \max(f, 0) \) and \( f_- = \max(-f, 0) \). In what follows we leave such extensions to the reader.

**Remark 2.3.** The assumption in Lemma 2.1 that \( f \) should vanish on the diagonals is essential. Indeed, take \( k = 2 \) and (deterministic) \( f(x_1, x_2; \omega) = 1_{\{x_1 = x_2\}} \) for \( (x_1, x_2) \in \mathbb{X}^2 \). Considering the atoms of \( \omega \) we have

\[
\int_{\mathbb{X}^2} f(x_1, x_2; \omega) \omega(dx_1)\omega(dx_2) = \sum_{x_1 \in \omega} \sum_{x_2 \in \omega} 1_{x_1 = x_2} = \omega(\mathbb{X}),
\]

hence

\[
\mathbb{E} \int_{\mathbb{X}^2} f(x_1, x_2; \omega) \omega(dx_1)\omega(dx_2) = \sigma(\mathbb{X}).
\]

On the other hand \( \sigma \) is non-atomic, therefore

\[
\int_{\mathbb{X}^2} \mathbb{E} f(x_1, x_2; \omega) \sigma(dx_1)\sigma(dx_2) = \int_{\mathbb{X}^2} f(x_1, x_2; \omega)\sigma(dx_1)\sigma(dx_2) = 0.
\]

Motivated by the above example we shall give a version of the multiple Mecke-Palm formula for processes which do not necessarily vanish on the diagonals. This calls for a notation that can handle *partitions*: For integers \( k, n \geq 1 \) we consider a family of pairwise disjoint nonempty sets (blocks) of integers \( P = \{P_1, \ldots, P_k\} \), such that \( \bigcup_{i=1}^k P_i = \{1, \ldots, n\} \). Thus, \( P \) is a partition of \( \{1, \ldots, n\} \). We denote by \( \mathcal{P}_n \) the set of all such partitions. We will use partitions to describe effects of interlaced Poisson integrations on the diagonals of \( \mathbb{X}^n \), in a manner which resembles the approach to multiple Itô integrals and compensated Poisson integrals in [19]. For \( P \in \mathcal{P}_n \) we let

\[
\mathbb{X}_P = \left\{ (x_1, \ldots, x_n) \in \mathbb{X}^n : x_i = x_j \text{ iff } i, j \in P_s \text{ for some } s \in \{1, \ldots, k\} \right\}.
\]

For \( P = \{P_1, \ldots, P_k\} \in \mathcal{P}_n \) and \( y \in \mathbb{X}^k_{\neq} \), we define \( y^{[P]} = (y_1^{[P]}, \ldots, y_k^{[P]}) \) by letting \( y_i^{[P]} = y_j \) if \( i \in P_j \). We have, as in Remark 2.3,

\[
\int_{\mathbb{X}^n} f(x_1, \ldots, x_n; \omega)\omega(dx_1)\cdots\omega(dx_n) = \sum_{P = \{P_1, \ldots, P_k\} \in \mathcal{P}_n} \int_{\mathbb{X}_P} f(x; \omega)\omega(dx_1)\cdots\omega(dx_n) = \sum_{P = \{P_1, \ldots, P_k\} \in \mathcal{P}_n} \int_{\mathbb{X}_P^k} f(y^{[P]}; \omega)\omega(dy_1)\cdots\omega(dy_k). \tag{2.6}
\]
As in Remark 2.3 we also note that for $n > 1$ and all $\omega$,

$$
\int_{\mathbb{X}^n} 1_{x_1 = x_2 = \ldots = x_n} \sigma(dx_1) \omega(dx_2) \cdots \omega(dx_n) = 0,
$$

(2.7)
because the first marginal of the product measure is non-atomic. Therefore in view of generalizing (2.6) to interlaced integrations against $\omega_1$ and $\omega_0$, we propose the following notation. For $\epsilon_1, \ldots, \epsilon_n \in \{0, 1\}$ we let $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ and consider the family $\mathcal{P}_n^\epsilon$ of all partitions $P = \{P_1, \ldots, P_k\}$ of $\{1, \ldots, n\}$ such that for every block $P_i \in P$ with $|P_i| > 1$ we have $\epsilon_j = 1$ for all $j \in P_i$. For $P \in \mathcal{P}_n^\epsilon$ we let $\epsilon^{[P]} = (\epsilon_1^{[P]}, \ldots, \epsilon_k^{[P]})$, where $\epsilon_1^{[P]} = \epsilon_{i_1}, \ldots, \epsilon_k^{[P]} = \epsilon_{i_k}$ and $i_1 \in P_1, \ldots, i_k \in P_k$. For $y = (y_1, \ldots, y_k) \in \mathbb{X}^k$ we then let $y_{\epsilon^{[P]}} = \{y_i : \epsilon_i^{[P]} = 1\}$.

In the following extension of Lemma 2.1 we write $x$ for $(x_1, \ldots, x_n) \in \mathbb{X}^n$ and $\sigma^k(dy) = \sigma(dy_1) \cdots \sigma(dy_k)$. The identity (2.8) below gives an algorithm to calculate expectations of Poisson integrals mixed with integrations against the control measure.

**Theorem 2.4.** Let $\mathbb{E}$ be the expectation making configurations $\omega$ on $\mathbb{X}$ a Poisson random measure with control measure $\sigma$. For every $n$-process $f \geq 0$ and $\epsilon_1, \ldots, \epsilon_n \in \{0, 1\}$ we have

$$
\mathbb{E} \int_{\mathbb{X}^n} f(x; \omega) \omega_{\epsilon_1}(dx_1) \cdots \omega_{\epsilon_n}(dx_n)
$$

(2.8)

$$
= \sum_{P = \{P_1, \ldots, P_k\} \in \mathcal{P}_n^\epsilon} \int_{\mathbb{X}^k} \mathbb{E} f(y^{[P]}; \omega \cup y_{\epsilon^{[P]}}) \sigma^k(dy).
$$
Proof. By similar reasons as in (2.6), and by Lemma 2.1,

$$\mathbb{E} \int_{\mathbb{X}^n} f(x; \omega) \omega_{\epsilon_1} (dx_1) \cdots \omega_{\epsilon_n} (dx_n)$$

$$= \sum_{P = \{P_1, \ldots, P_k\} \in \mathcal{P}_n} \mathbb{E} \int_{\mathbb{X}^n_{P_i}} f(x; \omega) \omega_{\epsilon_1} (dx_1) \cdots \omega_{\epsilon_n} (dx_n)$$

$$= \sum_{P = \{P_1, \ldots, P_k\} \in \mathcal{P}_n} \mathbb{E} \int_{\mathbb{X}^n_{P_i}} f(x; \omega) \omega_{\epsilon_1} (dx_1) \cdots \omega_{\epsilon_n} (dx_n)$$

(2.9)

$$= \sum_{P = \{P_1, \ldots, P_k\} \in \mathcal{P}_n} \mathbb{E} \int_{\mathbb{X}^n_{P_i}} f(y[P_i]; \omega) \omega_{\epsilon_1} (dy)$$

$$= \sum_{P = \{P_1, \ldots, P_k\} \in \mathcal{P}_n} \mathbb{E} f(y[P_i]; \omega) \sigma^k(dy).$$

In (2.9) we use (2.7) to eliminate $P \notin \mathcal{P}_n$.

\[\square\]

3  Moments

In this section we give applications of the mixed Mecke-Palm formula to expectations of products of stochastic integrals with respect to the Poisson random measure $\omega$ with control measure $\sigma$ on $\mathbb{X}$ and probability $\mathbb{P}$ and expectation $\mathbb{E}$.

3.1 General moment formulas

Theorem 3.1 below generalizes moment formulas of [21], [7]. As we see in the proof, the result is equivalent to the mixed Mecke-Palm formula (2.8) and is obtained after a simple linearization procedure. Let $S$ be a finite set and $\mathbb{X}^S = \{x : S \to \mathbb{X}\}$. For $x \in \mathbb{X}^S$ and $s \in S$ we write $x_s = x(s)$. We consider $\mathcal{P}(S)$, the class of all the partitions $P = \{P_1, \ldots, P_k\}$ of $S$. Here (blocks) $P_1, \ldots, P_k$ are disjoint, and $\bigcup_{\alpha=1}^{k} P_\alpha = S$. Let $P \in \mathcal{P}(S)$ and consider the $P$-diagonal:

$$\mathbb{X}_P^S = \{x \in \mathbb{X}^S : x_s = x_t \text{ iff there is } i \in \{1, \ldots, k\} \text{ such that } s, t \in P_\alpha\}.$$
For \( \epsilon : S \to \{0, 1\} \) we denote \( \omega_\epsilon(dx) = \otimes_{s \in S} \omega_{\epsilon_s}(dx_s) \). We note that \( \omega_\epsilon \) vanishes on \( \mathcal{X}_\mathcal{P}^\epsilon \) if there is block \( P_\alpha \in \mathcal{P} \) with cardinality \( |P_\alpha| > 1 \) and such that \( \epsilon = 0 \) at some point of \( P_\alpha \). This is so because the product measure has a non-atomic marginal. The set of the remaining partitions will be denoted \( \mathcal{P}_\mathcal{P}^\epsilon \). In particular, if \( \mathcal{P} \in \mathcal{P}_\mathcal{P}^\epsilon \) then \( \epsilon \) is constant on every block of \( \mathcal{P} \), and we may define \( \epsilon^P_\alpha := \epsilon_s \) if \( s \in P_\alpha, \alpha = 1, \ldots, k \). We denote \( \epsilon^P = (\epsilon^P_1, \ldots, \epsilon^P_k) \). For \( y = (y_1, \ldots, y_k) \in \mathcal{X}^k \) we let \( y^P_s = y_s \) if \( s \in P_\alpha \). Thus, \( \epsilon^P \in \{0, 1\}^k \) and \( y^P \in \mathcal{X}^S \). For measurable \( f : \mathcal{X}^S \to \mathbb{R}_+ \) we have

\[
\int_{\mathcal{X}^S} f(x) \omega_\epsilon(dx) = \int_{\mathcal{X}^k} f(y^P) \omega_{\epsilon_1}(dy_1) \cdots \omega_{\epsilon_k}(dy_k),
\]

which follows because \( \omega \) is a sum of Dirac measures supported at different points.

Let \( l \in \mathbb{N}_+ \) and \( r_1, n_1, \ldots, r_l, n_l \geq 1 \). We define

\[
S = \{(\alpha, \beta, \gamma) : 1 \leq \alpha \leq l, 1 \leq \beta \leq r_\alpha, 1 \leq \gamma \leq n_\alpha\}.
\]

If \( 1 \leq \alpha \leq l \) and \( 1 \leq \gamma \leq n_\alpha \), then we let

\[
S_{\alpha, \gamma} = \{(\alpha, \beta, \gamma) \in S : 1 \leq \beta \leq r_\alpha\}.
\]

For \( z \in \mathcal{X}^S \) we write, as usual, \( z_{S_{\alpha, \gamma}} \) for the restriction of \( z \) to \( S_{\alpha, \gamma} \). If \( \mathcal{P} = \{P_1, \ldots, P_k\} \in \mathcal{P}(S) \) and \( y \in \mathcal{X}^k \), then \( y^P_{S_{\alpha, \gamma}} \) denotes \( (y^P_s)_{S_{\alpha, \gamma}} \). In particular, \( y^P_{S_{\alpha, \gamma}} \in \mathcal{X}^{S_{\alpha, \gamma}} \).

**Theorem 3.1.** Let \( \mathbb{E} \) be the expectation making \( \omega \) a Poisson random measure on \( \mathcal{X} \) with control measure \( \sigma \). Let \( f_0, f_1, \ldots, f_l \geq 0 \) be \( 0, r_1, \ldots, r_l \)-processes, respectively. Let \( \epsilon^{(1)} \in \{0, 1\}^{r_1}, \ldots, \epsilon^{(l)} \in \{0, 1\}^{r_l} \). For \( s = (\alpha, \beta, \gamma) \in S \) we define \( \epsilon_s = \epsilon^{(\alpha)}(\beta) \). Then,

\[
\mathbb{E} \left[ f_0(\omega) \left( \int_{\mathcal{X}^{r_1}} f_1(y; \omega) \omega^{(1)}(dy) \right)^{n_1} \cdots \left( \int_{\mathcal{X}^{r_l}} f_1(y; \omega) \omega^{(l)}(dy) \right)^{n_l} \right] \tag{3.1}
\]

\[
= \sum_{\mathcal{P} = \{P_1, \ldots, P_k\} \in \mathcal{P}(S)} \mathbb{E} \left[ \int_{\mathcal{X}^k} f_0(\omega \cup \bigcup_{\epsilon_s = 1} \{y^P_s\}) \prod_{\alpha = 1}^l \prod_{\gamma = 1}^{n_\alpha} f_\alpha(y^P_{S_{\alpha, \gamma}}; \omega \cup \bigcup_{\epsilon_s = 1} \{y^P_s\}) \sigma^k(dy) \right].
\]

**Proof.** The first transformation in the calculation below we call linearization, and
the last one follows from Theorem 2.4:

\[
\mathbb{E}
\left[
\int_{X^{r}} f_{0}(y; \omega) \omega_{\epsilon_{(1)}}(dy) \right]^{n_{1}} \cdots \left[
\int_{X^{r}} f_{1}(y; \omega) \omega_{\epsilon_{(l)}}(dy) \right]^{n_{l}}
\]

\[
= \mathbb{E}
\left[
\int_{X^{r}} \int_{X^{r}} \cdots \int_{X^{r}} f_{0}(y; \omega) \omega_{\epsilon_{(1)}}(dy_{1}) \cdots \omega_{\epsilon_{(l)}}(dy_{n_{l}})
\right]
\]

\[
= \mathbb{E}
\left[
\int_{X^{s}} \prod_{\alpha=1}^{l} \prod_{\gamma=1}^{n_{\alpha}} f_{\alpha}(y_{\gamma}; \omega) \omega_{\epsilon_{(1)}}(dy_{1}) \cdots \omega_{\epsilon_{(l)}}(dy_{n_{l}})
\right]
\]

\[
= \mathbb{E}
\left[
\sum_{P=\{P_{1}, \ldots, P_{k}\} \in P(S)_{\neq}} \int_{X^{k}} f_{0}(\omega) \prod_{\alpha=1}^{l} \prod_{\gamma=1}^{n_{\alpha}} f_{\alpha}(y_{\gamma}; \omega) \sigma_{P}(dy)
\right]
\]

\[
= \mathbb{E}
\left[
\sum_{P=\{P_{1}, \ldots, P_{k}\} \in P(S)_{\neq}} f_{0}(\omega \cup \bigcup_{\epsilon_{s}=1}^{l} \{y_{P}^{\epsilon_{s}}\}) \prod_{\alpha=1}^{l} \prod_{\gamma=1}^{n_{\alpha}} f_{\alpha}(y_{\gamma}; \omega \cup \bigcup_{\epsilon_{s}=1}^{l} \{y_{P}^{\epsilon_{s}}\}) \sigma_{P}(dy)
\right].
\]

\[\square\]

In concrete computations one may either use Theorem 3.1, along with its somewhat heavy notation, or just follow its proof, i.e. use linearization and the mixed Mecke-Palm formula. For instance in Lemma 3.3 below it is simpler to use the latter approach.

### 3.2 Moment formulas for stochastic integrals of 1-processes

We first specialize to 1-processes. Let \(k, l, n_{1}, \ldots, n_{l} \in \mathbb{N} = \{1, 2, \ldots\}\) and \(n = n_{1} + \ldots + n_{l}\). For \(j = 1, \ldots, l\) and \(P = \{P_{1}, \ldots, P_{k}\} \in P_{n}\) we denote

\[
P_{i,j} = \{d \in P_{i} : \sum_{0<m<i} n_{m} < d \leq \sum_{0<m<j} n_{m}\}.
\]

Let \(|P_{i,j}|\) be the number of elements of \(P_{i,j}\).
Corollary 3.2. For a random variable $f_0 \geq 0$ and 1-processes $f_1, \ldots, f_l \geq 0$,

$$
\mathbb{E}\left[f_0(\omega) \left( \int_X f_1(x; \omega) \omega(dx) \right)^n \cdots \left( \int_X f_l(x; \omega) \omega(dx) \right)^n \right]
$$

(3.2)

$$
= \sum_{P \in \mathcal{P}_n} \mathbb{E} \int_{\mathbb{R}^k} f_0(\omega + \sum_{i=1}^k \delta_{y_i}) f_1^{P_1,1}(y_1; \omega + \sum_{i=1}^k \delta_{y_i}) \cdots f_l^{P_l,1}(y_l; \omega + \sum_{i=1}^k \delta_{y_i}) \times
$$

$$
\times f_1^{P_k,1}(y_k; \omega + \sum_{i=1}^k \delta_{y_i}) \cdots f_l^{P_k,1}(y_k; \omega + \sum_{i=1}^k \delta_{y_i}) \sigma(dy_1) \cdots \sigma(dy_k).
$$

Proof. The result follows from Theorem 3.1. \hfill \Box

For $l = 1$ we recover [21, (1.2)]:

$$
\mathbb{E}\left[v(\omega) \left( \int_X u(x; \omega) \omega(dx) \right)^n \right]
$$

$$
= \sum_{P = \{P_1, \ldots, P_k\} \in \mathcal{P}_n} \mathbb{E} \left[ \int_{\mathbb{R}^k} v(\omega \cup y) u(y_1; \omega \cup y)^{|P_1|} \cdots u(y_k; \omega \cup y)^{|P_k|} \sigma(dy_1) \cdots \sigma(dy_k) \right],
$$

where $y = \{y_1, \ldots, y_k\}$ and $u \geq 0$ is a 1-process. With arbitrary $l$ we obtain an alternative proof of [7, Theorem 3.1] for random Poisson measures. In passing we also refer the reader to recent papers [16] and [6].

3.3 The second moment of stochastic integrals of 2-processes

Moments of arbitrary $k$-processes require formulas of increasing complexity, but they are entirely explicit. Here is a telling example.
Lemma 3.3. If $f \geq 0$ is a $2$-process, then
\[
\mathbb{E}\left( \int_{X^2} f(x_1, x_2; \omega) \omega(dx_1) \omega(dx_2) \right)^2 = \int_X \mathbb{E}f^2(x, \omega \cup \{x\}) \sigma(dx) \tag{3.3}
\]
\[
+ 2 \int_{X^2_{\neq}} \mathbb{E}f(x, x; \omega \cup \{x, y\}) f(x, y; \omega \cup \{x, y\}) \sigma(dx) \sigma(dy)
\]
\[
+ 2 \int_{X^2_{\neq}} \mathbb{E}f(x, x; \omega \cup \{x, y\}) f(y, x; \omega \cup \{x, y\}) \sigma(dx) \sigma(dy)
\]
\[
+ \int_{X^2_{\neq}} \mathbb{E}f(x, x; \omega \cup \{x, y\}) f(y, y; \omega \cup \{x, y\}) \sigma(dx) \sigma(dy)
\]
\[
+ \int_{X^2_{\neq}} \mathbb{E}f^2(x, y; \omega \cup \{x, y\}) \sigma(dx) \sigma(dy)
\]
\[
+ \int_{X^2_{\neq}} \mathbb{E}f(x, y; \omega \cup \{x, y\}) f(y, x; \omega \cup \{x, y\}) \sigma(dx) \sigma(dy)
\]
\[
+ 2 \int_{X^3_{\neq}} \mathbb{E}f(x, x; \omega \cup \{x, y, z\}) f(y, z; \omega \cup \{x, y, z\}) \sigma(dx) \sigma(dy) \sigma(dz)
\]
\[
+ \int_{X^3_{\neq}} \mathbb{E}f(x, x; \omega \cup \{x, y, z\}) f(z, x; \omega \cup \{x, y, z\}) \sigma(dx) \sigma(dy) \sigma(dz)
\]
\[
+ \int_{X^3_{\neq}} \mathbb{E}f(x, y; \omega \cup \{x, y, z\}) f(x, z; \omega \cup \{x, y, z\}) \sigma(dx) \sigma(dy) \sigma(dz)
\]
\[
+ \int_{X^3_{\neq}} \mathbb{E}f(x, y; \omega \cup \{x, y, z\}) f(z, x; \omega \cup \{x, y, z\}) \sigma(dx) \sigma(dy) \sigma(dz)
\]
\[
+ \int_{X^4_{\neq}} \mathbb{E}f(x, y; \omega \cup \{x, y, z, t\}) f(z, t; \omega \cup \{x, y, z, t\}) \sigma(dx) \sigma(dy) \sigma(dz) \sigma(dt).
\]

Proof. By linearization,
\[
\left( \int_{X^2} f(x_1, x_2; \omega) \omega(dx_1) \omega(dx_2) \right)^2 = \int_X g(x, y, z, t) \omega(dx) \omega(dy) \omega(dz) \omega(dt),
\]
where $g(x, y, z, t; \omega) = f(x, y; \omega) f(z, t; \omega)$. We will use Theorem 2.4. The partitions involved have $k = 1, 2, 3$ or $4$ blocks, because the number $4$ can be represented as the following sums: $4, 3+1, 2+2, 2+1+1, 1+1+1+1$. In particular,
the partition of \( \{1, 2, 3, 4\} \) with only one block \((k = 1)\), namely \( \{\{1, 2, 3, 4\}\} \), contributes

\[
\mathbb{E} \int_X g(x, x, x; \omega \cup \{x\}) \sigma(dx) = \int_X \mathbb{E} f^2(x, x; \omega \cup \{x\}) \sigma(dx)
\]

to (3.3). Then, partitions with \(k = 2\) blocks are of type \(3 + 1\) and \(2 + 2\). In the first case there are 4 different partitions as there are 4 different choices of the singleton. For instance, \( P = \{\{1, 2\}, \{3\}\} \) contributes

\[
\int_{X^2}^2 \mathbb{E} g(x, x, x, x; \omega \cup \{x\}) \sigma(dx) \sigma(dy)
\]

to (3.3). The contribution to (3.3) from all the partitions of type \(3 + 1\) are the 2nd and the 3rd terms on the right-hand side of (3.3). In the case \(2 + 2\), \( P = \{\{1, 2\}, \{3, 4\}\} \), contributes

\[
\int_{X^2}^2 \mathbb{E} f(x, x; \omega \cup \{x, y\}) f(x, y; \omega \cup \{x, y\}) \sigma(dx) \sigma(dy)
\]

to (3.3), and the contributions from all the partitions of type \(2 + 2\) are precisely the 4th through 6th terms on the right-hand side of (3.3).

For \(k = 3\) we have partitions of type \(2 + 1 + 1\), e.g. \( P = \{\{1, 2\}, \{3\}, \{4\}\} \), which contributes

\[
\int_{X^3}^3 \mathbb{E} f(x, x; \omega \cup \{x, y, z\}) f(y, z; \omega \cup \{x, y, z\}) \sigma(dx) \sigma(dy) \sigma(dz)
\]

to (3.3), and all partitions of type \(2 + 1 + 1\) result in the 7th through 10th terms on the right-hand side of (3.3). Finally, the partition into \(k = 4\) singletons yields

\[
\int_{X^4}^4 \mathbb{E} f(x, y; \omega \cup \{x, y, z, t\}) f(z, t; \omega \cup \{x, y, z, t\}) \sigma(dx) \sigma(dy) \sigma(dz) \sigma(dt)
\]

This finishes the verification of (3.3).

We now investigate the second moment of mixed double stochastic integrals, the ones with respect to the random measures \(\omega \otimes \sigma\) and \(\sigma \otimes \omega\).
Lemma 3.4. If \(f \geq 0\) is a 2-process, then
\[
\mathbb{E} \left( \int_{X^2} f(x_1, x_2; \omega) \sigma(dx_2) \sigma(dx_1) \right)^2
\]
\[
= \mathbb{E} \left( \int_{X^2} f(x_1, x_2; \omega) \sigma(dx_1) \omega(dx_2) \right)^2
\]  
(3.4)
\[
= \mathbb{E} \int_{X^3} f(x, y; \omega \cup \{y\}) f(z, y; \omega \cup \{y\}) \sigma(dx) \sigma(dy) \sigma(dz)
\]  
(3.5)
\[
+ \mathbb{E} \int_{X^4} f(x, y; \omega \cup \{y, t\}) f(z, t; \omega \cup \{y, t\}) \sigma(dx) \sigma(dy) \sigma(dz) \sigma(dt).
\]

Proof. The equation (3.4) follows from Fubini-Tonelli. Then the expectation in (3.4) is written as
\[
\mathbb{E} \int_X f(x, y; \omega) f(z, t; \omega) \sigma(dx) \omega(dy) \sigma(dz) \omega(dt),
\]
and by Theorem 2.4 we get the equality (3.5), as in the proof of Lemma 3.3. \(\square\)

4 Lévy systems

An important motivation for this work is due to the so-called Lévy systems for Lévy processes. These are identities between expectations of sums taken with respect to the jumps of a Lévy process and expectations of integrals taken with respect to the corresponding intensity measure. There exist a considerable variety of (multiple) Lévy systems, as we discuss below.

4.1 General result

We consider (time) \(\mathbb{R}_+ = (0, \infty)\), (space) \(\mathbb{R}^d\) and (space-time) \(\mathbb{R}_+ \times \mathbb{R}^d\).

Let \(\nu\) be a non-zero Lévy measure on \(\mathbb{R}^d\), thus \(\nu(\{0\}) = 0\) and
\[
\int_{\mathbb{R}^d} \min\{1, z^2\} \nu(dz) < \infty.
\]

Let \(X = \{X_t\}_{t \geq 0}\) be a Lévy process in \(\mathbb{R}^d\) with Lévy triplet \((\nu, A, b)\), where \(A\) is a symmetric, nonnegative-definite \(d \times d\) matrix and \(b \in \mathbb{R}^d\) [22]. Let \(\mathbb{P}\) and \(\mathbb{E}\) be the distribution and the expectation of the process and consider
\[
p_t(A) = \mathbb{P}(X_t \in A),
\]
the convolution semigroup of $X$. Let $\Delta X_u = X_u - X_{u-}$ and

$$
\omega = \sum_{u > 0, \Delta X_u \neq 0} \delta_{(u, \Delta X_u)}.
$$

Then $\omega$ is a Poisson random measure with the intensity (control) measure $\sigma(dudz) = du\nu(dz)$ on

$$
\mathbb{X} = \mathbb{R}_+ \times \mathbb{R}^d_0
$$

[10, Section I.9, Section II.3, Example II.4.1] related to $X$ by the Lévy-Itô decomposition [22, Chapter 4], [10, Example II.4.1]. We may and do identify $\omega$, $\mathbb{P}$ and $\mathbb{E}$ with those from Section 2 given by $\sigma(dudz) = du\nu(dz)$. The well-known (simple) Lévy system is the following identity (more comments are given after the proof).

**Lemma 4.1.** If $F : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is nonnegative, then

$$
\mathbb{E} \sum_{0 < u < \infty} F(u, X_{u-}, X_u) = \mathbb{E} \int_0^\infty \int_{\mathbb{R}^d} F(u, x, x + z)\nu(dz)du. \quad (4.1)
$$

**Proof.** First, let $X$ be a compound Poisson process, that is $\nu(\mathbb{R}^d) < \infty$, $X(t) = \sum_{i=1}^{N(t)} Z_i$, where $N(t)$ has Poisson distribution with expectation $t\nu(\mathbb{R}^d)$, and $Z_i$ are i.i.d. random variables with distribution $\nu/\nu(\mathbb{R}^d)$. Therefore

$$
p_t = e^{-|\nu|t}e^{*t\nu} = e^{-|\nu|t} \sum_{n=0}^{\infty} \frac{t^n \nu^n}{n!}.
$$

By Fubini-Tonelli theorem the right-hand side of (4.1) equals

$$
\int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F(u, x, x + z)p_u(dx)\nu(dz)du. \quad (4.2)
$$

Let $S_i = \inf\{t > 0 : N(t) = i\}$, the arrival time of the $i$-th jump of $X$. Recall that $S_i$ has gamma distribution, and clearly $X_{S_i}$ has distribution $\nu^{|i|}$. By Fubini-Tonelli
the left-hand side of (4.1) equals

\[
E \sum_{i=1}^{\infty} F(S_t, X_{S_t-}, X_{S_t}) = \sum_{i=1}^{\infty} \int_{0}^{\infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F(u, x, x+z) \left( e^{-|\nu| u} \sum_{i=1}^{\infty} \frac{u^{i-1} \nu^{i}(i-1)}{(i-1)!} (dx) \right) \nu(dz) du
\]

\[
= \int_{0}^{\infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F(u, x, x+z) \left( e^{-|\nu| u} \sum_{i=1}^{\infty} \frac{u^{i-1} \nu^{i}(i-1)}{(i-1)!} (dx) \right) \nu(dz) du
\]

\[
= \int_{0}^{\infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F(u, x, x+z) p_u (dx) \nu(dz) du.
\]

This yields (4.1) for compound Poisson process \(X\). Now let \(X\) be a general Lévy process. We shall prove that for every \(\epsilon > 0\),

\[
E \sum_{0 < u < \infty} F(u, X_{u-}, X_u) = E \int_{0}^{\infty} \int_{|z| \geq \epsilon} F(u, X_u, X_u + z) \nu(dz) dv. \quad (4.3)
\]

To this end we use the following decomposition,

\[X_t = V_t + Z_t.\]

The terms in the decomposition have the following properties. Process \(V_t\) is a Lévy process with the triplet \((A, \nu|_{|z| < \epsilon}, b)\), on a probability space \((\Omega^V, \mathcal{F}^V, \mathbb{P}^V)\). Here \(\nu|_{|z| < \epsilon}\) is the measure \(\nu\) restricted to \(\{z \in \mathbb{R}^d : |z| < \epsilon\}\). \(Z_t\) is a compound Poisson process on an independent probability space \((\Omega^Z, \mathcal{F}^Z, \mathbb{P}^Z)\), and has the Lévy measure \(\nu|_{|z| \geq \epsilon}\). We denote by \(E^V, E^Z\) and \(\mathbb{P}^V, \mathbb{P}^Z\) the corresponding expectations and probabilities. We may assume that \(\Omega = \Omega^V \times \Omega^Z\) and \(\mathbb{P} = \mathbb{P}^V \otimes \mathbb{P}^Z\), according to the fact that \(V\) and \(Z\) are independent. In what follows we consider

\[
\tilde{F}(v, x, y) = F(v, V_{u-} + x, V_u + y). \quad (4.4)
\]

By Fubini-Tonelli theorem and by (4.1) for the compound Poisson process \(Z\), the
left hand side of (4.3) becomes

\[ E^{VEZ} \sum_{|\Delta(Z_u + V_u)| \geq \epsilon} F(u, V_u + Z_u, V + Z_u) \]

\[ = E^{VEZ} \sum_{|\Delta Z_u| \geq \epsilon} \int \int \int F(u, Z_u, Z_u + z) \nu(dz) du \]

\[ = E \int \int F(u, X_u, X_u + z) \nu(dz) du. \]

We have proved (4.3). Let \( \epsilon \downarrow 0 \). By monotone convergence theorem,

\[ E \sum_{|\Delta Y_u| \geq \epsilon} F(u, X_u, X_u) \rightarrow E \sum_{\Delta Y_u \neq 0} F(u, X_u, X_u), \quad (4.5) \]

and

\[ E \int \int F(u, X_u, X_u + z) \nu(dz) du \rightarrow E \int \int F(u, X_u, X_u + z) \nu(dz) du. \quad (4.6) \]

By (4.6), (4.5) and (4.3) we obtain (4.1).

Lemma 4.1 asserts that the expected sum over the jumps of the Lévy process \( X \) equals to the expectation of the integral with respect to the corresponding intensity measure. As we remarked, the result is well-known, see [3], [8, p. 375], [4, VII.2(d)], but the above direct proof seems original and will be referred to below in extensions which we call multiple mixed Lévy systems. Before presenting them we propose a reformulation of Lemma 4.1.

**Lemma 4.2.** If \( F : \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} \) is nonnegative, then

\[ E \sum_{0 < u \leq \infty \atop \Delta X_u \neq 0} F(u, X_{u-}, \Delta X_u) \rightarrow E \int \int F(u, X_u, z) \nu(dz) du. \]

Here \( \mathbb{R}^+ = (0, \infty) \). The multiple mixed Lévy systems can be described within the framework presented in the previous sections. We consider the “simplex”

\[ \mathbb{X}_n^\n = \{(u_1, z_1; \ldots; u_n, z_n) \in \mathbb{X}^n : 0 < u_1 < \cdots < u_n\}. \]

The following defines the (complete set of the multiple) mixed Lévy systems.
Theorem 4.3. Let $X$ be a Lévy process in $\mathbb{R}^d$ with the Lévy measure $\nu$, the expectation $\mathbb{E}$ and the Poisson random measure of jumps $\omega$. Let $\epsilon_1, \ldots, \epsilon_n \in \{0, 1\}$ and let $F : (\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d)^n \mapsto [0, \infty]$ be measurable. Then,

$$
\mathbb{E} \int_{\mathbb{R}^d_+} F(u_1, X_{u_1-}, z_1; \ldots; u_n, X_{u_n-}, z_n) \omega \epsilon_1(du_1dz_1) \ldots \omega \epsilon_n(du_ndz_n) = \int_{\mathbb{R}^d_+} \mathbb{E} F(u_1, X_{u_1-}, z_1; \ldots; u_n, X_{u_n-} + \sum_{i=1}^{j-1} \epsilon_i z_i, z_j; \ldots; u_n, X_{u_n-} + \sum_{i=1}^{n-1} \epsilon_i z_i, z_n) du_1 \nu(dz_1) \ldots du_n \nu(dz_n).
$$

(4.7)

(4.8)

Proof. We first prove this result for compound Poisson process $X$. By the Lévy-Itô decomposition for $t \geq 0$ we have

$$
X_t = X_t(\omega) = \int_{(0,t]} \int \omega(du) d\nu(\mathbb{R}^d),
$$

and

$$
X_t = X_t(\omega) = \int_{(0,t]} \int \omega(du) d\nu(\mathbb{R}^d).
$$

We note that $X_{t-}$ is a 1-process on $X$, and

$$
1_{\mathbb{R}^d_+} (u_1, z_1; \ldots; u_n, z_n) F(u_1, X_{u_1-}, z_1; \ldots; u_n, X_{u_n-}, z_n)
$$

is an $n$-process, which vanishes on the diagonals. Using the notation from the proof of Lemma 2.1, by Theorem 2.4 we see that the left-hand side of (4.7) equals

$$
\int_{\mathbb{R}^d_+} \mathbb{E} F(u_1, X_{u_1-}(\omega + \sum_{i=1}^{n} \epsilon_i \delta(u_i, z_i)), z_1; \ldots; u_n, X_{u_n-}(\omega + \sum_{i=1}^{n} \epsilon_i \delta(u_i, z_i)), z_n) du_1 \nu(dz_1) \ldots du_n \nu(dz_n).
$$

Since

$$
X_{u_j-}(\omega + \sum_{i=1}^{j} \epsilon_i \delta(u_i, z_i)) = X_{u_j-}(\omega) + \sum_{i=1}^{j-1} \epsilon_i z_i,
$$

we have

$$
\int_{\mathbb{R}^d_+} \mathbb{E} F(u_1, X_{u_1-}(\omega), z_1; \ldots; u_n, X_{u_n-}(\omega), z_n) du_1 \nu(dz_1) \ldots du_n \nu(dz_n).
$$

(4.8)
(4.7) follows. Then we note that the distribution of \( X_{u_i^-} \) is the same as that of \( X_{u_i} \), which is \( p_{u_i} \), and we use Fubini-Tonelli to get (4.8). This resolves the case of compound Poisson processes. The case of general Lévy processes follows as in the proof of Lemma 4.1.

**Corollary 4.4.** If \( X \) is a Lévy process and \( F \) is nonnegative, then

\[
\mathbb{E} \sum_{0 < u_1 < \ldots < u_n < \infty} F(u_1, X_{u_1^-}, X_{u_1}; \ldots; u_n, X_{u_n^-}) \quad (4.9)
\]

\[
\mathbb{E} \int_{0}^{\infty} \ldots \int_{u_{n-1}}^{\infty} \mathbb{E} \int_{0}^{\infty} \ldots \int_{u_n}^{\infty} F(u_1, X_{u_1}, X_{u_1} + z_1; \ldots; u_n, X_{u_n} + z_1 + \ldots + z_n) \nu(dz_1) \ldots \nu(dz_n) du_n \ldots du_1.
\]

**Corollary 4.5.** If \( X \) is a Lévy process and \( F \) is nonnegative, then

\[
\mathbb{E} \sum_{0 < s < \infty} \int_{\Delta Y_s \neq 0} F(s, X_{s^-}, X_{s}; s_1, X_{s_1}, X_{s_1} + z_1) \nu(dz_1) ds_1 \quad (4.10)
\]

\[
= \mathbb{E} \int_{0}^{\infty} \int_{s < \infty} \sum_{\Delta X_{s_1} \neq 0} F(s, X_s, X_s + z; s_1, X_{s_1} + z, X_{s_1} + z) \nu(dz) ds
\]

\[
= \mathbb{E} \int_{0}^{\infty} \int_{s < \infty} \int_{\mathbb{R}^d} F(s, X_s, X_s + z; s_1, X_{s_1} + z, X_{s_1} + z + z_1) \nu(dz_1) \nu(dz) ds_1 ds.
\]

We note in passing that Corollary 4.4 and Corollary 4.5 can also be proved without using Mecke-Palm formula, in a way similar to the first part of the proof of Lemma 4.1, see [24]. The proofs are quite involved and the proof of the general mixed Lévy systems is fraught with problems if similar approach is to be used. On the contrary, Theorem 4.3 offers a clear insight into the structure of multidimensional mixed-type Lévy systems. The structure is explained by accumulating \( z_i \), the \( i \)-th variable of the integrations performed in (4.8), as a jump of the process \( X \) at the moment \( u_i \), but only if \( z_i \) is integrated against the Poisson random measure, rather than its control measure. By accumulation we mean that such jumps are indeed added to the trajectory of the process. We encourage the reader to consider the statement of Corollary 4.5 from this perspective. Notably, the complex machinery of stochastic analysis of general Markov processes, e.g., the notion of
predictability plays little role in the above treatment of Lévy systems for Lévy processes.

**Remark 4.6.** We note that Theorem 4.3 may be generalized to allow for \( n \)-processes more complicated than \( F(u_1, X_{u_1 -}, z_1; \ldots; u_n, X_{u_n -}, z_n) \), with similar proofs based on the mixed Mecke-Palm formula. Such extensions may involve modifications by predictable factors, cf. [8, p. 375], [4, VII.2(d)], and integrating processes which are not adapted to the usual filtration associated with the Lévy process.

To illustrate Remark 4.6 we give the following classical extension, cf. [4, VII.2(d)]. An additional discussion is given at the end of Section 4.2.

**Lemma 4.7.** If \( F \geq 0 \) and \( g_t \geq 0 \) is predictable, then

\[
\mathbb{E} \sum_{0 < u < \infty} g_u F(u, X_{u -}, X_u) = \mathbb{E} \int_0^\infty \int_{\mathbb{R}^d} g_u F(u, X_u + z) \nu(dz) du. \tag{4.11}
\]

**Proof.** \( g_u F(u, X_{u -}, X_u) = g_u(\omega) F(u, X_{u -}(\omega), X_u(\omega) + z) \) is a \( 1 \)-process on \( \mathbb{R}_+ \times \mathbb{R}^d \). By predictability, \( g_u(\omega + \delta(u,z)) = g_u(\omega) \) with almost surely. The result follows from the usual Mecke-Palm identity (5.1).

### 4.2 Applications

The purpose of this section is to show usability of our formulas. A typical application of the Lévy system is to the well-known Ikeda-Watanabe formula [9], given as (4.13) below. The formula concerns the situation of the Lévy process \( X \) in \( \mathbb{R}^d \) as it reaches the complement of the open set \( D \subset \mathbb{R}^d \). We shall use the usual Markovian notation: for \( x \in \mathbb{R}^d \) we write \( \mathbb{E}^x \) and \( \mathbb{P}^x \) for the expectation and distribution of \( x + X \), but we use the same symbol \( X \) for the resulting process, cf. [22, Chapter 8]. We write \( p_t(x, A) = p_t(A - x) = \mathbb{P}^x(X_t \in A) \), so that

\[
\mathbb{E}^x \int_0^\infty f(t, X_t) dt = \int_0^\infty \int_{\mathbb{R}^d} f(t, y) p_t(x, dy)
\]

for (Borel) functions \( f \geq 0 \) and \( x \in \mathbb{R}^d \). We consider the time of the first exit of \( X \) from \( D \),

\[
\tau_D = \inf\{t > 0 : X_t \notin D\}.
\]
The Dirichlet kernel $p^D_t(x, dy)$ is defined by
\[
\int_{\mathbb{R}^d} f(y) p^D_t(x, dy) = \mathbb{E}^x[f(X_t); \tau_D > t],
\]
and we have
\[
\mathbb{E}^x \int_0^{\tau_D} f(t, X_t) dt = \int_0^\infty \int_{\mathbb{R}^d} f(t, y) p^D_t(x, dy).
\]
We now consider function $F(u, y, w) = 1_{I(u)} 1_A(y) 1_B(w)$, where $I$ is a bounded interval, and $A \subset D, B \subset (D)^c$ are Borel sets in $\mathbb{R}^d$. We let
\[
M(t) = \sum_{0 < u \leq t \atop |\Delta X_u| \neq 0} F(u, X_{u-}, X_u) - \int_0^t \int_{\mathbb{R}^d} F(u, X_u + z) \nu(dz) dv.
\]
We note that
\[
\mathbb{E} \int_0^t \int_{\mathbb{R}^d} F(u, X_u, X_u + z) \nu(dz) dv \leq |I| \nu(\{|z| > \text{dist}(A, B)|}) < \infty, \quad (4.12)
\]
so by Lemma 4.1, $\mathbb{E} M(t) = 0$. Let $0 \leq s \leq t$. By considering the Lévy process $u \mapsto X_{s+u} - X_s$, independent of $X_r, 0 \leq r \leq s$, we calculate the conditional expectation
\[
\mathbb{E} \left[ \sum_{s < u \leq t \atop |\Delta X_u| \neq 0} F(u, X_{u-}, X_u) - \int_s^t \int_{\mathbb{R}^d} F(u, X_u + z) \nu(dz) dv | X_r, 0 \leq r \leq s \right] = 0,
\]
then we see that $M$ is a uniformly integrable martingale. By stopping at $\tau_D$, we obtain
\[
\mathbb{P}^x[\tau_D \in I, X_{\tau_D-} \in A, X_{\tau_D} \in B] = \int_I \int_{B-y} \int_A p^D_u(x, dy) \nu(dz) du. \quad (4.13)
\]
This defines the joint distribution of $(\tau_D, X_{\tau_D-}, X_{\tau_D})$ restricted to the event $\{X_{\tau_D-} \in D\}$ and calculated under $\mathbb{P}^x$.

As another application we use the double mixed Lévy system to prove the following classical result [10, II (3.9)].
Lemma 4.8. Let $X$ be a Lévy process in $\mathbb{R}^d$ with Lévy measure $\nu$. Let the function $F : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ satisfy
\[
\mathbb{E} \int_{0}^{\infty} \int_{\mathbb{R}^d} F^2(v, X_v, X_v + z) \nu(dz) dv < \infty.
\] (4.14)

For every $t \in [0, \infty)$ the following limit exists in $L^2$
\[
M_t = \lim_{\epsilon \to 0} \left( \sum_{0 < v \leq t, |\Delta X_v| \geq \epsilon} F(v, X_v, X_v) - \int_{0}^{t} \int_{|z| \geq \epsilon} F(v, X_v, X_v + z) \nu(dz) dv \right),
\]
$t \mapsto M_t$ is a martingale with respect to $(\mathcal{F}_t)$, $\mathbb{E}M_t = 0$ and 
\[
\mathbb{E}M_t^2 = \mathbb{E} \int_{0}^{t} \int_{\mathbb{R}^d} F^2(v, X_v, X_v + z) \nu(dz) dv.
\]
Furthermore, the square bracket of $M$ is
\[
[M]_t = \sum_{0 < v \leq t, \Delta X_v \neq 0} F^2(v, X_v, X_v),
\] (4.15)
and the predictable quadratic variation of $M$ is
\[
\langle M \rangle_t = \int_{0}^{t} \int_{|z| \geq \epsilon} F(v, X_v, X_v + z)^2 \nu(dz) dv.
\] (4.16)

Recall that $[M]$ is defined as the unique adapted right-continuous non-decreasing process with jumps $\Delta [M]_t = |\Delta M_t|^2$, and such that $t \mapsto |M|_t^2 - [M]_t$ is a (continuous) martingale starting at 0 ([8, VII.42]). We verify the martingale property of $|M|_t^2 - [M]_t$ by using Corollary 4.4 and Corollary 4.5. Notice that $\mathbb{E}[M]_t = \mathbb{E}\langle M \rangle_t$ by the property of a single Lévy system. More details and applications can be found in [24]. In particular, the square bracket $[M]$ is used in [5] to estimate the $L^p$ norms of Fourier multipliers defined in terms of Lévy processes. We refer the reader to [8, VII-VIII] and [10, II] for further details and reading.

As the third application we will calculate moments of the Lévy integral. Let $X_t = (\eta_t, \xi_t)$, where $t \geq 0$, be a Lévy process in $\mathbb{R}^2$. To simplify the discussion we further assume that $\eta$ and $\xi$ are (possibly dependent) subordinators with no
Let \( \nu \) be the Lévy measure of \( X \). Of course, \( \nu \) is concentrated on \( \mathbb{R}^2_+ := (0, \infty) \times (0, \infty) \). Let \( \phi \) be the Laplace exponent of \( \eta \):

\[
    \mathbb{E} \left[ e^{-x\eta} \right] = e^{-t \phi(x)}, \quad x \geq 0.
\]

The following expression is called the Lévy integral,

\[
    Z = \int_0^\infty e^{-\eta t} d\xi_t = \sum_{\Delta X_t \neq 0} e^{-\eta t} \Delta \xi_t.
\]

Lévy integrals represent stationary distributions of generalized Ornstein-Uhlenbeck process (see [18] for details, applications and references). By Lemma 4.1,

\[
    \mathbb{E} [\xi_1] = \mathbb{E} \sum_{0 < t \leq 1, \Delta X_t \neq 0} \Delta \xi_t = \int_{\mathbb{R}^2_+} y \, d\nu(x, y).
\]

We can use the multiple Lévy systems to calculate the moments of \( Z \). The first three moments of \( Z \) take on the following form

\[
    \mathbb{E}[Z] = \frac{\int y \, d\nu(x, y)}{\phi(1)}, \\
    \mathbb{E}[Z^2] = \frac{2 \int e^{-x} y \, d\nu(x, y) \int y \, d\nu(x, y)}{\phi(1)\phi(2)} + \frac{\int y^2 \, d\nu(x, y)}{\phi(2)}, \\
    \mathbb{E}[Z^3] = \frac{6 \int y \, d\nu(x, y) \int e^{-x} y \, d\nu(x, y) \int e^{-2x} y \, d\nu(x, y)}{\phi(1)\phi(2)\phi(3)} + \frac{\int y^3 \, d\nu(x, y)}{\phi(3)} + \frac{3 \int y^2 \, d\nu(x, y) \int e^{-x} y \, d\nu(x, y)}{\phi(2)\phi(3)} + \frac{3 \int y \, d\nu(x, y) \int e^{-2x} y^2 \, d\nu(x, y)}{\phi(1)\phi(3)}.
\]

Indeed, by Lemma 4.1,

\[
    \mathbb{E}[Z] = \mathbb{E} \left[ \sum_{\Delta X_t \neq 0} e^{-\eta_t} - \Delta \xi_t \right] = \mathbb{E} \left[ \int_0^\infty e^{-\eta_t} y \, d\nu(x, y) \, dt \right] = \frac{\int y \, d\nu(x, y)}{\phi(1)}.
\]

For the higher moments we use linearization, as in Section 3, e.g., we obtain

\[
    \mathbb{E}[Z^2] = \mathbb{E} \left[ \left( \sum_{\Delta X_t \neq 0} e^{-\eta_t} - \Delta \xi_t \right)^2 \right] = \mathbb{E} \left[ \left( \sum_{\Delta X_s \neq 0} e^{-\eta_s} - \Delta \xi_s \right) \left( \sum_{\Delta X_t \neq 0} e^{-\eta_t} - \Delta \xi_t \right) \right] = \mathbb{E} \left[ 2 \sum_{s < t} e^{-\eta_s - \eta_t} - \Delta \xi_s \Delta \xi_t \right] + \mathbb{E} \left[ \sum_{\Delta X_t \neq 0} \left( e^{-\eta_t} - \Delta \xi_t \right)^2 \right] = 2I + II,
\]

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where, by Corollary 4.4,

\[
I = \mathbb{E} \left[ \int_0^\infty \int_s^\infty \int \int e^{-\eta s} y_1 e^{-\eta x_1} y_2 \, d\nu(x_1, y_1) \, d\nu(x_2, y_2) \, dt \, ds \right]
\]

\[
= \int \int e^{-x_1} y_1 y_2 \, d\nu(x_1, y_1) \, d\nu(x_2, y_2) \mathbb{E} \left[ \int_0^\infty \int_s^\infty e^{-(\eta - \eta s) - 2\eta s} \, dt \, ds \right]
\]

\[
= \int e^{-x} y \, d\nu(x, y) \int y \, d\nu(x, y) \int_0^\infty \int_s^\infty \mathbb{E} [e^{-\eta s}] \mathbb{E} [e^{-2\eta s}] \, dt \, ds
\]

\[
= \frac{\int e^{-x} y \, d\nu(x, y) \int y \, d\nu(x, y)}{\phi(1) \phi(2)},
\]

and

\[
II = \mathbb{E} \left[ \int_0^\infty \int y^2 e^{-2\eta t} \, d\nu(x, y) \, dt \right] = \int \frac{y^2 \, d\nu(x, y)}{\phi(2)}.
\]

The third and the higher moments are obtained analogously. We note that [2, Theorem 3.1] gives the first and the second moments of \(Z\), but not the higher moments, which are cumbersome to obtain by the methods of [2] (private communication).

By our methods one also compute moments of anticipating integrals such as

\[
Y := \int_0^\infty e^{-\eta \xi t} = \sum_{\Delta X_t \neq 0} e^{-\eta - \Delta \eta t} \Delta \xi_t.
\]

Here, similar calculations as for \(Z\) yield

\[
\mathbb{E}[Y] = \int \frac{e^{-x} y \, d\nu(x, y)}{\phi(1)},
\]

and higher moments of \(Y\) can be obtained analogously. Notice the difference between the formulas for the expectations of \(Z\) and \(Y\).

5 Appendix

The following Mecke-Palm identity holds for 1-processes \(f(x; \omega) \geq 0\) [20],

\[
\mathbb{E} \int \mathcal{X} f(x; \omega) \, \omega(dx) = \int \mathcal{X} \mathbb{E} f(x; \omega \cup \{x\}) \, \sigma(dx).
\]  (5.1)
For the reader's convenience we give a direct proof of (5.1) in the setting of Section 2. We first consider \( \sigma(\mathcal{X}) < \infty \) and nonnegative process \( f(x; \omega) = f(x; \omega \cap \mathcal{X}) \), i.e. depending only on \( \mathcal{X} \). If \( \omega = \{y_1, \ldots, y_n\} \), a set with \( n \) elements, then
\[
\int_{\mathcal{X}} f(x; \omega)(dx) = \sum_{i=1}^{n} f_{(n)}(y_i; y_1, \ldots, y_n).
\]
The above quantity is invariant upon permutations of \( y_1, \ldots, y_n \), in fact it is the \( n \)-th coefficient of the random variable \( \int_{\mathcal{X}} f(x; \omega)(dx) \). By (2.1), the left-hand side of (5.1) equals
\[
e^{-\sigma(\mathcal{X})} \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i=1}^{n} \int_{\mathcal{X}^n} f_{(n)}(y_i; y_1, \ldots, y_n) \sigma(dy_1) \cdots \sigma(dy_n). \tag{5.2}
\]
If \( \omega = \{y_1, \ldots, y_n\} \), a set with \( n \) elements, and \( x \notin \omega \), then
\[
f(x; \omega \cup \{x\}) = f_{n+1}(x; x, y_1, \ldots, y_n) = \ldots = f_{n+1}(x; y_1, \ldots, y_n, x)
= \frac{1}{n+1} \sum_{i=1}^{n+1} f_{n+1}(x; y_1, \ldots, y_i, x, y_{i+1}, \ldots, y_n).
\]
Since \( \sigma \) is non-atomic, we have \( \mathbb{P}(x \in \omega) = 0 \) for every \( x \in \mathcal{X} \), cf. (2.1). Therefore, by (2.1), the right-hand side of (5.1) equals
\[
\mathbb{E} \int_{\mathcal{X}} u(x; \omega \cup \{x\}) \sigma(dx)
= e^{-\sigma(\mathcal{X})} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathcal{X}} \int_{\mathcal{X}^n} \frac{1}{n+1} \sum_{i=1}^{n+1} f_{n+1}(x; y_1, \ldots, y_n, x) \sigma(dy_1) \cdots \sigma(dy_n) \sigma(dx).
\]
This verifies (5.1) when \( \sigma(\mathcal{X}) < \infty \), e.g., if \( \mathcal{X} \subset \mathbb{X} \) is compact; we note in passing that (5.2) is an explicit representation of either side of (5.1).

We next let \( \mathbb{X} = \bigcup_{m} \mathcal{X}_m \) be a countable decomposition of \( \mathbb{X} \) into disjoint Borel sets with \( \sigma(\mathcal{X}_m) < \infty \). For arbitrary process \( f(x; \omega) \geq 0 \) we have
\[
\int_{\mathbb{X}} f(x; \omega)(dx) = \sum_{m} \int_{\mathcal{X}_m} f(x; \omega)(dx). \tag{5.3}
\]
For fixed \( m \), we write \( \omega_* = \omega \cap \mathcal{X}_m \), \( \omega^* = \omega \setminus \mathcal{X}_m \), and denote by \( \mathbb{E}_* \) and \( \mathbb{E}^* \) the expectation \( \mathbb{E} \) when restricted to random variables depending only on \( \mathcal{X}_m \) and
\( X \setminus \mathcal{X}_m \), respectively. By (2.2) and by (5.1) for \( \mathcal{X}_m \),
\[
\mathbb{E} \int_{\mathcal{X}_m} f(x; \omega) \omega(dx) = \mathbb{E}^* \mathbb{E}_* \int_{\mathcal{X}_m} f(x; \omega_* \cup \omega^*) \omega_*(dx)
\]
\[
= \mathbb{E}^* \int_{\mathcal{X}_m} \mathbb{E}_* f(x; \omega_* \cup \{x\} \cup \omega^*) \sigma(dx) = \int_{\mathcal{X}_m} \mathbb{E} f(x; \omega \cup \{x\}) \sigma(dx).
\]
This yields (5.1) in the general case, cf. (5.3).

Needless to say, (5.1) also holds for signed processes \( f \) under the assumption of absolute integrability, because we can decompose both sides of (5.1) according to \( f = f_+ - f_- \), where \( f_+ = \max\{f, 0\} \) and \( f_- = \max\{-f, 0\} \).

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