On the class of pseudo-Riemannian geometries which can not be locally described using curvature scalars solely: a necessity analysis

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Abstract
A classic, double problem with intriguing implications at the level of both applied differential geometry and theoretical physics is dealt with in this short work: Is there any criterion in order to decide whether a pseudo-Riemannian space can be locally described using curvature scalars solely? Also: In the case where such a description is impossible, does the Cartan-Karlhede algorithm constitute the only refuge? Surprisingly enough, the first question is susceptible of a very simple and elegant answer, while a naive scheme carries the ambition of providing (modulo specific restrictions) a negative answer to the second question. In order to avoid unnecessary complexity, the analysis is restricted to local rather than global considerations –without any loss of not only the generality but also the insights to the initial problem.

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1 Introduction
During the last four decades a constantly increasing interest in geometry motivated physical theories has been observed; indeed, even a bird’s eye view on the theoretical physics literature could verify it. Of course, there is a subsequent interaction with the various relevant fields of mathematics; algebraic, differential geometry and topology would be the most prominent examples. A consequence of such an interaction is the emergence of various problems to be addressed, endowed with a versatility in both nature and applications. Moreover, in many cases, old problems, solutions or approaches —irrelevant, at first sight, to a particular modern physical theory— have the chance to be reconsidered from a different point of view.

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One of the most important paradigms reflecting this situation is the problem of equivalence, adapted to pseudo-Riemannian geometry—seen, of course, as an exterior differential system (EDS). The core essence of this problem is to distinguish—at a local level—between two given, albeit arbitrary, metric tensor fields (or, according to the relevant terminology, to invariantly describe a pseudo-Riemannian geometry).

The well established Cartan-Karlhede (CK) algorithm can, in principle (i.e., barring practical calculational difficulties), give—amongst various other pieces of information, like the isometry and isotropy groups, etc.—the fundamental set of Cartan scalars (or properly named invariants) which can serve as the ultimate criterion on whether two metric tensor fields are equivalent or not—even when discrete symmetries are taken into account. This is possible simply because the Cartan scalars (and the relations amongst them) contain only the non spurious (cf. non absorbable) parts of a metric tensor field i.e., essential constants and fundamental functions along with their (ordinary/partial) derivatives.

Nevertheless, as mentioned before, the CK algorithm can be quite entangled (or even practically impossible) in actual examples, since three independent factors are involved in the entire procedure: the dimension of the underlying manifold, the signature of the metric tensor field (determining the gauge group in the (co)tangent bundle), and the functional form of the metric components. Exactly due to this complexity an alternative, in some loose sense, approach (practically—at least) used to be the mere implementation of curvature scalars, i.e., scalars constructed from the metric tensor field elements, through the use of Riemann, Ricci, and Weyl tensors and their (covariant) derivatives—along with standard tensorial calculus. Of course, by definition, that approach does not provide the wealth of information given by Cartan’s method—hence the ‘loose sense’ referred to above.

Also, the discovery (originally, withing the context of general relativity) of some exceptional families of pseudo-Riemannian geometries, like the pp-waves, most decisively showed the general insufficiency of that approach. Or, to be more precise, the area of applicability of that approach was restricted only to the case of Riemannian geometry (i.e., for Euclidean signature)—something which is indirectly implied, especially, in Weyl’s work. The reason behind this insufficiency is that some fundamental functions (along with their derivatives) contained in the metric tensor fields—are missing from the (infinite) totality of the curvature scalars. For instance, in the case of e.g., vacuum pp-waves, there is only one fundamental function appearing in the metric tensor field, yet all the curvature scalars are zero. Thus, based on curvature scalars solely, no one is able to distinguish between a vacuum pp-wave and Minkowski space-time.

So, it seems that under some certain circumstances, which have to do with a quite complex mingling of three independent factors (i.e., the dimension of the underlying manifold, the signature of the metric tensor field, and the functional form of the metric components), an interesting phenomenon can occur where some fundamental functions contained in the metric tensor field do not appear in any curvature scalar—acting as ‘phantoms’ (of course, every fundamental piece of information will always be appearing in the Cartan scalars). Prompted by the case of pp-waves, the explanation for this behaviour is the non compactness of the Lorentz group—as opposed to the compactness of the rotation group in a Euclidean space. Indeed, from Weyl’s work two related points are obvious. First, out of the three aforementioned factors, that of the metric signature is the
most important. Second, for any non Euclidean signature the non compactness of the corresponding (in the (co)tangent bundle) gauge group could, in principle, permit for the occurrence of this phenomenon. The non compactness of the gauge group gives an answer as to why; it does not give an answer as to when exactly this phenomenon takes place though. In a series of papers, the corresponding authors —through a long string of notions and definitions, and by developing an interesting mathematical machinery— tried to systematically deal with this intriguing problem. Ideally, the goal would be to have a statement in the form of a necessary and sufficient condition. But, until now, that is not the case. Their most important results, thus far, could be very coarsely —due to the bulk of the research material— summarised as follows.

1. When the signature of the metric is Lorentzian, then only the degenerate Kundt family of geometries exhibits such a behaviour, as long as the dimension of the underlying manifold is equal or greater than three.

2. Assuming that the signature is non Euclidean, and the dimension of the underlying manifold is greater than three, then if a space can not be characterised by its curvature scalars (weakly or strongly), then there exists no analytical continuation of it to a Riemannian space.

3. Assuming that the signature is non Euclidean, and the dimension of the underlying manifold is greater than three, then there are at least two different families of geometries exhibiting such a behaviour: the degenerate Kundt and some particular Walker metrics.

Again, no answer as to when exactly is given in this series of papers. On the other hand, the bulk of the information to be found there lines up along the direction of sufficiency rather than necessity. For example, using the boost weight analysis, the authors do provide an extremely useful tool towards making the decision as to whether a given metric tensor field contains ‘phantom’ elements or not.

The objective of the present work is threefold.

$O_1$ To strictly cover this logical direction: what are the necessary conditions for a metric tensor field to contain ‘phantom’ elements?

$O_2$ To answer the question: is it possible to give the generic metric tensor field, endowed with this property?

$O_3$ To try to provide a ‘patch’ to the approach of curvature scalars, in order to render it valid (always, modulo its inherent weakness over Cartan’s method) in all circumstances.

In Section 2 a concise account on both the local differential geometry in the language of differential forms, and curvature scalars is given. In Section 3, by using the machinery exhibited in the previous section, a necessity analysis is exhibited having as a crescendo the desired results. By exploiting an observation regarding some fine points in the analysis, a naive procedure is proposed as a practical alternative to the CK algorithm—which can cover even the cases where the phenomenon of ‘phantom’ elements manifests itself. Finally, a Discussion summarising the basic results concludes the paper.
2 Elements of (local) differential geometry à la Cartan, and a concise account on curvature scalars

For the sake of both completeness and logical continuity, a brief exposition of the elements (i.e., basic notions, definitions, formulae, and results) from the Cartan’s formulation of differential geometry —in terms of moving frames— will be given prior to the general discussion on curvature scalars. It should be meant, of course, that the purpose of this (short) presentation is not to give a detailed account on the theory; it rather serves as a reference collection. Standard references on the subject are the first volume of the classical work by Kobayashi and Nomizu and, of course, the original reference by Cartan himself.

Conventions. Lower case Latin letters destined for index variables are used for any coordinate space, while capital Latin letters destined for index variables are used for any (co)tangent space—always, in $n$ dimensions. Both classes of indices have as their domain of definition the set \{1, 2, \ldots, n\}. The abstract index notation is implemented throughout. More precisely, any tensorial quantity, when being thought of in its abstract entirety, is denoted by a boldface italic letter (like e.g., $X$ for a vector field), whilst its set of components is written with italic letters—both for the kernel symbol and its indices—(like e.g., $X^a$ in the previous example). Alternatively, the indexed quantity may stand for the entire tensorial quantity. Within this context, the symbol $\nabla$ (nabla) refers to the covariant derivative operator. Thus, e.g., if $X$ denotes a vector field, then the quantity $\nabla X$ corresponds to $\nabla_a X^a$. The Lie derivative with respect to a vector field $X$ is denoted by $\mathcal{L}_X$. Finally, the Einstein’s summation convention is also in use.

Let a pseudo-Riemannian space be described by the pair $(\mathcal{M}, g)$, where $\mathcal{M}$ is an $n$ dimensional, simply connected Hausdorff, and $C^\infty$ manifold and $g$ is a $C^\infty$ metric tensor field on it that is a non degenerate, covariant tensor field of order 2, with the property that at each point of $\mathcal{M}$ one can choose a frame of $n$ real vectors \{$e_1, \ldots, e_n$\}, such that $g(e_A, e_B) = g_{AB}$ where $g$ (called frame metric) is a symmetric matrix with prescribed signature. The totality of the sets \{$e_A$\} (i.e., the sets for every point on the manifold) determines the $GL(n, \mathbb{R})$ frame bundle over $\mathcal{M}$ and defines the tangent bundle $T(\mathcal{M})$ of $\mathcal{M}$. Thus, the matrix $g$ simply reflects the inner products of the vectors in the tangent bundle.

Another fundamental notion is that of the cotangent bundle $T^*(\mathcal{M})$ of $\mathcal{M}$ which, as a linear vector space, is the dual to $T(\mathcal{M})$. Indeed, if \{$\theta^A$\} denotes the basis of the cotangent space at a point on the manifold, then in a similar manner, the totality of the sets \{$\theta^A$\} (i.e., the sets for every point on the manifold) determines the $GL(n, \mathbb{R})$ coframe bundle over $\mathcal{M}$ and defines the cotangent bundle $T^*(\mathcal{M})$ of $\mathcal{M}$. The duality relation is realised through a linear operation called contraction ($\lrcorner$)

$$e_A \lrcorner \theta^B = \delta_A^B$$

(2.1)

where $\delta_A^B$ is the Kronecker delta.
Cartan has given a very elegant and powerful formulation of local differential geometry in terms of some basic $p$-forms (i.e., totally antisymmetric $(p)$ tensors; by definition, the coframe vectors are 1-forms) and the four basic operations acting upon them: the wedge product ($\wedge$), the exterior differentiation ($d$), the contraction ($\mathcal{L}$) with a frame vector (field) $e$, and the Lie derivative ($\mathcal{L}_e$) with respect to a frame vector (field) $e$.

Some very important properties are (for any $p$-form $\alpha$, $q$-form $\beta$, and frame vector(s) $e$)

\[ e \mathcal{L} (\alpha \wedge \beta) = (e \mathcal{L} \alpha) \wedge \beta + (-1)^p \alpha \wedge (e \mathcal{L} \beta) \quad (2.2a) \]
\[ d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta \quad (2.2b) \]
\[ \mathcal{L}_e \alpha = e \mathcal{L} \alpha + d(e \mathcal{L} \alpha) \quad (2.2c) \]
\[ \mathcal{L}_e (\alpha \wedge \beta) = (\mathcal{L}_e \alpha) \wedge \beta + \alpha \wedge (\mathcal{L}_e \beta) \quad (2.2e) \]
\[ \mathcal{L}_1 (e_2 \mathcal{L} \alpha) = (\mathcal{L}_1 e_2) \mathcal{L} \alpha + e_2 \mathcal{L} (\mathcal{L}_1 \alpha) \quad (2.2f) \]

Returning to the description of local differential geometry, the necessary equations are provided by *Cartan’s structure equations* (CSEs)

(A) The first set reads
\[ d\theta^A + \Gamma^A_B \wedge \theta^B = T^A \quad (2.3a) \]
where $\Gamma^A_B$ defines the (linear) matrix valued connexion 1-form through the *Ricci rotation coefficients* $\gamma^A_{BM}$

\[ \Gamma^A_B = \gamma^A_{BM} \theta^M \quad (2.3b) \]

and $T^A$ defines the vector valued torsion 2-form as

\[ T^A = \frac{1}{2} \tau^A_{MN} \theta^M \wedge \theta^N \quad (2.3c) \]

At this point, it should be noted that
\[ \mathcal{L}_A e_B \equiv [e_A, e_B] = -C^M_{AB} e_M \quad \Rightarrow \quad d\theta^A = \frac{1}{2} C^A_{MN} \theta^M \wedge \theta^N \quad (2.4a) \]

where the quantities $C^A_{BM}$ are called *structure functions*.

(B) The second set reads
\[ d\Gamma^A_B + \Gamma^A_M \wedge \Gamma^M_B = \Omega^A_B \quad (2.5a) \]
where the $\Omega^A_B$ defines the matrix valued curvature 2-form

\[ \Omega^A_B = \frac{1}{2} R^A_{BMN} \theta^M \wedge \theta^N \quad (2.5b) \]

and the quantities $R^A_{BMN}$ constitute the Riemann tensor.

(C) Taking into account the EDC, the action of a $d$ upon the first set —by virtue of the first two sets themselves— results in the third set
\[ dT^A + \Gamma^A_B \wedge T^B = \Omega^A_B \wedge \theta^B \quad (2.6a) \]

which are nothing but a generalisation of the *Jacobi identities*.\[19\]
(D) Taking into account the EDC, the action of a $d$ upon the second set —by virtue of the second set itself— results in the fourth set

$$d\Omega^A_B + \Gamma^A_{MA} \wedge \Omega^M_B - \Omega^A_M \wedge \Gamma^M_B = 0 \quad (2.7a)$$

which are the Bianchi identities.

(E) The frame metric $g$, which is used to raise and lower (co)frame indices, usually (but not necessarily always) is subject to the metricity condition

$$\nabla g = 0 \iff dg_{AB} - \Gamma^A_{AB} - \Gamma^A_{BA} = 0 \quad (2.8)$$

It is also used to define the (first) fundamental form $\Phi$

$$\Phi = g_{AB} \theta^A \otimes \theta^B \quad (2.9)$$

According as the torsion vanishes or not, or the metricity condition holds or not, various branches of local differential geometry emerge.

| Branch             | A linear connexion is defined | A metric tensor is defined | $\nabla g = 0$ | $\Omega = 0$ | $T = 0$ |
|--------------------|------------------------------|---------------------------|----------------|-------------|---------|
| Affine             | ✓                            | ×                         | ×              | ×           | ×       |
| Weyl               | ✓                            | ✓                         | ×              | ×           | ×       |
| Riemann-Cartan     | ✓                            | ✓                         | ✓              | ×           | ×       |
| Weitzenbőck        | ✓                            | ✓                         | ✓              | ✓           | ×       |
| pseudo-Riemannian  | ✓                            | ✓                         | ✓              | ×           | ✓       |

The previous formulae, along with the exterior calculus can provide all the needed quantities. For instance, all the covariant derivatives of any tensor can be calculated by a procedure entailing a successive and alternating implementation of both contraction and exterior differentiation operations, as follows (using the Riemann tensor as example).

$S_1$ The first covariant derivative of the Riemann tensor can be read off, with the operation of contraction, from the 1-form

$$\begin{align*}
(\nabla_M R^A_{BCD})\theta^M &= dR^A_{BCD} + R^Z_{BCD} \Gamma^A_Z - R^A_{ZCD} \Gamma^Z_B \\
&\quad - R^A_{BZD} \Gamma^Z_C - R^A_{BCZ} \Gamma^Z_D
\end{align*}$$

$S_2$ The second covariant derivative of the Riemann tensor can be read off, again with the operation of contraction, from the 1-form

$$\begin{align*}
(\nabla_N \nabla_M R^A_{BCD})\theta^N &= d(\nabla_M R^A_{BCD}) + (\nabla_M R^Z_{BCD}) \Gamma^A_Z \\
&\quad - (\nabla_M R^A_{ZCD}) \Gamma^Z_B - (\nabla_M R^A_{BZD}) \Gamma^Z_C \\
&\quad - (\nabla_M R^A_{BCZ}) \Gamma^Z_D - (\nabla_M R^A_{ZCD}) \Gamma^Z_M
\end{align*}$$

etc. . .

These alternating steps can go on resulting in higher covariant derivatives. Since the example above involves a tensor of a mixed type (i.e., $(\frac{1}{3})$), the needed proper
sing alterations for fully covariant or contravariant tensors are obvious –cf. e.g., (2.8). It is also worth mentioning that the above procedure is valid in any affine geometry (i.e., in principle, no metric tensor field needs to exist).

Before shifting attention to curvature scalars, the following definition will be useful in the next section.

**Definition 2.1.** If $X = X^A e_A$ is a vector field, the associated 1-form is defined to be $\theta^A = g_{AB} X^A ;$ and vice versa.

The following two definitions are crucial in the discussion on curvature scalars.

**Definition 2.2.** The (infinite) collection $C_T$ of curvature tensors is defined as

$$C_T = \{ \text{The tensors, of any order, constructed out of } g, \varepsilon, R \}$$

where the symbol $\varepsilon$ stands for the kernel of the Levi-Civita tensor\(^{20}\) It is meant that the process of construction involves only the four fundamental tensorial operations; i.e., addition, outer product, contraction of indices, and covariant differentiation.

**Definition 2.3.** The (infinite) collection $C_S$ of curvature scalars is defined as

$$C_S = \{ \text{The scalars constructed out of members of } C_T \}$$

where the construction, here, implies a full and meaningful, according to tensor algebra, contraction of all the indices.

A few comments are pertinent here.

**C\,1** One can easily see that all the well known tensors (like Ricci, Weyl, Hodge dual tensors, their covariant derivatives of any order, etc.) are included in the first definition.

**C\,2** The first collection ($C_T$) constitutes a class and not a set\(^{21}\) Two reasons are responsible for this: due to index symmetries, no one can avoid having equivalent or connected formations. In general, use of group theoretical methods (like the Young tableaux\(^{22}\) could help in reducing —to some extent— the redundancy but full elimination is impossible. The second reason, mainly of topological nature, is the existence of some constraints, amongst —virtually all — the members of the $C_T$ class called dimensional dependent identities\(^{23}\).

**C\,3** By construction, given its relation to $C_T$, the collection $C_S$ also constitutes a class, and not a set (leaving aside the issue of functional dependence) —exactly for analogous reasons.

**C\,4** The $C_S$ class contains scalars involving the Levi-Civita tensor —something which, usually, is not found in the relevant literature.

**C\,5** It is precisely this nature of the collection $C_S$ (i.e., that it is a class) which constitutes the major difficulty in finding a ‘base’ —even for the algebraic (i.e., constructed only from Riemann, Ricci, and Weyl tensors) scalars— so difficult. Indeed there is no consensus on neither such a ‘base’ nor what algebraic completeness should mean in this case\(^{24}\).
The scalars, members, of the $CS$ class are just a portion of the collection of Cartan scalars (always, barring functional dependence). Indeed, there is yet another—(conceptually) complementary to $CS$—collection of curvature scalars, denoted $CR$, each member of which is defined as the ‘ratio’ between two tensors (of the same valence), members, of $CT$. The collection $CR$ also constitutes a class. Unfortunately, there is no systematic way in finding this complementary class—although Brans gives some very indirect hints.

In general, curvature scalars are useful for a variety reasons: e.g., in the characterisation and (local) classification of spaces, in the coordinate invariant characterisation of certain geometrical properties, in the study of singularities, or in the study of inverse problems like that of (locally) determining a space for a given Riemann tensor in writing a Lagrangian for a physical theory, etc. Nevertheless, all these efforts fail when dealing with metric tensor fields with ‘phantom’ elements, and the analysis is based on the corresponding $CS$ class solely.

3 On those geometries which can not be locally described using the $CS$ class solely: a theorem and a naive procedure alternative to the CK algorithm

In this section a necessity analysis, towards the determination of those (pseudo-Riemannian) geometries which can not be locally described using the $CS$ class solely, is exhibited. An alternative statement of the problem is the following:

Under which circumstances two (or more) essentially different (i.e., according to the equivalence principle) geometries share the same $CS$ class?

Stated this way, it is apparent that there are three aspects of this problem.

$A_1$ Two essentially different metrics can share the same $CS$ class due to the existence of ‘phantom’ elements in any of these two—cf. e.g., pp-waves and Minkowski spaces.

$A_2$ Two essentially different metrics can share the same $CS$ class due to proper signature change—cf. e.g., Euclidean and Minkowski spaces.

$A_3$ A combination of $A_1$ and $A_2$.

The third aspect has no independent meaning, while—a somehow elaborate—treatment of the second aspect can be found elsewhere perhaps a more neat study could be the objective of a future work. So, the attention is focus on the first aspect.

Before proceeding two comments are necessary. In the literature there are various approaches regarding the choice of frame. For the purposes of the present work the most suitable choice would be that of the coordinate frame; then all the previous formulae reduce, essentially, to tensor calculus.

All the following considerations are based and valid on the (aggregation of the least) assumptions, regarding the pseudo-Riemannian space, as the pair $(M, g)$,
to be found at the beginning of Section 2. Thus, for example, the fact that the metric tensor is a $C^\infty$ field implies not only that all the members of the $CS$ class can be defined, but also that they are $C^\infty$ functions as well—something which can be seen by both the construction process and, \textit{a posteriori}, by the Cartan’s method.

Now, one can prove the following:

**Theorem 3.1.** The necessary condition for a metric tensor field $g$, part of an $n$-dimensional pseudo-Riemannian space $(M, g)$, not to be characterised by the corresponding $CS$ class of curvature scalars solely is the existence of a non-diverging, normal, and null vector field $N$:

$$\nabla_a N^a = 0 \quad N_{[a} \nabla_b N_{c]} = 0 \quad N^a N_a = 0$$

or, in terms of the associated 1-form $n$:

$$*d(*n) = 0 \quad n \wedge dn = 0 \quad N \hook n = 0$$

where the $*$ denotes the Hodge dual. As a consequence, there exists a local coordinate patch, spanned by the set $\{x^a\} = \{u, v, x^1, \ldots, x^{n-2}\}$ (the coordinates $\{u, v\}$ being null) where the corresponding fundamental form can be written as

$$\Phi = 2du \otimes (Adu + dv + B_k dx^k) + \gamma_{ij} dx^i \otimes dx^j, \quad k, i, j \in \{1, \ldots, n-2\}$$

$$\partial_v \det \gamma = 0$$

Modulo the constraint on the determinant of the matrix $\gamma$, all the functions entailed (i.e., $\{A, B_k, \gamma_{ij}\}$) depend, in general, on all the coordinates.

**Proof.** Before beginning the proof, a comment is in order. Using all the CSE, it is a quite simple exercise to show that in two dimensions, the Riemann tensor is completely characterised by the Ricci scalar (a member of the $CS$ class) —a well known topological fact. Therefore, the initial problem acquires meaning in three or more dimensions.

Let a local coordinate patch spanned by, say, the set $\{y^a\} = \{y^1, \ldots, y^n\}$. It is assumed that the metric tensor field contains some ‘phantom’ elements (i.e., functions not appearing in any of the members of the corresponding $CS$ class) in specific positions of the tensor components. Of course, there is no need for these elements to be entire components; they equally might or might not be. For reasons to become apparent in a few lines, it would be much more convenient to consider the corresponding positions in the contravariant form of the metric tensor field. Thus, let these ‘phantom’ elements occupying (perhaps parts of) the set of positions $\mathcal{P} = \{(i_1, j_1), (i_2, j_2), \ldots\}$ in $g^{ab}$.

Next, let consider any arbitrary element, say $\varphi$, of the corresponding (infinite) $CS$ class of curvature scalars. By the initial hypothesis, $\varphi$ does not contain some particular functions appearing in specific —perhaps— parts of the metric tensor field components (in contravariant form). An integrability (necessity) condition easily follows. Indeed, the quantity (known as the \textit{first Beltrami operator})

$$\Delta_1(\varphi, \varphi) \equiv g^{ab}(\nabla_a \varphi)(\nabla_b \varphi) = g^{11}(\partial_1 \varphi)(\partial_1 \varphi) + \ldots + 2g^{12}(\partial_1 \varphi)(\partial_2 \varphi) + \ldots \quad (3.1)$$

should also not contain the previously mentioned elements. Nevertheless, the only way for those elements to enter this quantity is \underline{only} through the metric
components (in contravariant form). This forces one to conclude that some (and according to a minimal ‘factoring’ in the elements of the set \( \mathcal{P} \)) of the partial derivatives of \( \varphi \) must vanish:

\[
\partial_i \varphi \equiv \partial_{y^i} \varphi = 0, \quad i \in \{1, \ldots, S\}
\]  

(3.2)

A naive (i.e., non rigorous, but susceptible of a scrupulous analysis) sketch of a practical example will help to clarify this point. Suppose e.g., that —say, in 5 dimensions— \( \mathcal{P} = \{(1,1), (1,4), (2,4), (3,3), (5,5)\} \) in \( g^{ab} \). Then the expression \( (3.1) \) —as an entity— will not contain ‘phantom’ elements provided that the quantity

\[
(g^{11}(\partial_1 \varphi) + 2g^{14}(\partial_4 \varphi)) + 2g^{24}(\partial_2 \varphi)(\partial_4 \varphi) + g^{33}(\partial_3 \varphi)(\partial_3 \varphi) + g^{55}(\partial_5 \varphi)(\partial_5 \varphi)
\]

vanishes. Because each position reflects different ‘phantom’ function(s) (or, at least, this should be the generic case), each term in the previous expression must vanish separately. Regarding the first term two possibilities exist. Either

\[
g^{11}(\partial_1 \varphi) + 2g^{14}(\partial_4 \varphi) = 0
\]

or

\[
\partial_1 \varphi = 0
\]

The first possibility leads to a quite involved, in general, constraint entailing mostly the ‘non phantom’ (i.e., all the rest) metric tensor field elements, since those will do enter this expression through the arbitrary curvature scalar \( \varphi \). On the other hand, these constraints not only will be infinitely many (each for a different curvature scalar) but also they will generate further (secondary) constraints—through their integrability conditions. Those, in turn, they will generate more (tertiary) constraints—through their integrability conditions—etc. This unending procedure exhibits the degenerate (i.e., overdetermined) charactered of the doubly infinite system of constraints—which is nothing but a system of (non linear) partial differential equations. From this point of view, it would be reasonable to exclude this first possibility. Therefore, one is left with the second possibility, i.e.,

\[
\partial_1 \varphi = 0
\]

In a similar way of thinking, one ends up with the conditions

\[
\partial_1 \varphi = 0 \quad \partial_2 \varphi = 0 \quad \partial_3 \varphi = 0 \quad \partial_5 \varphi = 0
\]

or

\[
\partial_1 \varphi = 0 \quad \partial_3 \varphi = 0 \quad \partial_4 \varphi = 0 \quad \partial_5 \varphi = 0
\]

Of course, both cases ought to be studied separately—although qualitatively are the same, since both are characterised by the existence of four annihilating conditions.
Finally, it should be mentioned that, more complex combinations—reflecting a higher dimension, $n$, of the underlying manifold or a richer set $\mathcal{P}$—could also be considered.

Now, returning to the set of fundamental conditions (3.2) a few points should be noticed.

$P_1$ Although they were deduced in a particular, albeit arbitrary, coordinate patch, they constitute tensorial statements and as such are valid in any other patch within the atlas.

$P_2$ The upper limit $S$ must, obviously, be less or equal to the number of dimensions of the underlying manifold; i.e., $n$. Yet, this bound is not sharp— but more on this in the Discussion.

$P_3$ The most generic case would be to consider only one such condition, simply because it is the minimum requirement— even when they are many of them.

Thus, based on points $P_1$ and $P_3$, one concludes— by disengaging (at the same time) from the particular, albeit arbitrary, coordinate patch considered at the beginning—that the desired necessary condition is the existence of a vector field $\mathbf{N}$ which annihilates all the curvature scalars:

$$\mathcal{L}_N \phi = 0, \quad \forall \phi \in \mathcal{CS}$$  \hspace{1cm} (3.3)

The first implication of this important constraint is

$$\mathcal{L}_N \phi = 0, \quad \forall \phi \in \mathcal{CS} \Rightarrow \mathcal{L}_N \int_U \phi dV = 0, \quad \forall \phi \in \mathcal{CS}$$  \hspace{1cm} (3.4)

where $U$ is, an arbitrary, simply connected region of $\mathcal{M}$ and $dV$ the volume element. The previous relation can be further worked out

$$\mathcal{L}_N \int_U \phi dV = 0 \Rightarrow \int_U \phi \mathcal{L}_N dV = 0 \Rightarrow \int_U \phi \nabla_a N^a dV = 0$$  \hspace{1cm} (3.5)

by the properties of the Lie derivative for densities (here, the volume density). But, since both $\phi$ and (most importantly) $U$ are arbitrary, it is deduced (e.g., by implementing Riemann’s definition for definite integrals and remembering the initial assumptions) that

$$\nabla_a N^a = 0$$  \hspace{1cm} (3.6)

thus the vector field is non diverging.

The constraint (3.3) must be valid for all the underlying geometries. Indeed, the very statement of the problem concerns, essentially different (à la Cartan) geometries sharing the same $\mathcal{CS}$ class. Of course, all these are to be embodied in the metric tensor field considered at the beginning of the proof. Thus all the properties, referring to the vector field $\mathbf{N}$, must remain invariant as one sweeps the (infinite) totality of those essentially different geometries. For instance, the second implication of the constraint (3.3) is that the vector field $\mathbf{N}$ can freely be rescaled. The most interesting point here is that such a rescaling does not compensate neither any coordinate transformation nor any change in the
underlying geometry—since none of these two took place. Therefore, it should always be possible for such a rescaling to be canceled out by absorbing it. This, in turn, necessarily implies that the vector field $N$ must be normal:

$$N_{[a} \nabla_b N_{c]} = 0$$  \hspace{1cm} (3.7)

In the same spirit, one could imagine various successive rescalings and then taking the norm of the differently rescaled—each time—vector field with respect to various essentially different geometries—members of the unifying underlying geometry. Since the field is a given entity with specific features, one concludes that the only reasonable fact regarding its norm, is that the latter must be zero:

$$N^a N_a = 0$$  \hspace{1cm} (3.8)

Now, using the properties of normality (3.7), and nullity (3.8) it is a mere exercise to prove, another established result i.e., that the vector field under consideration must also be geodetic:

$$N^b \nabla_b N^a = 0$$  \hspace{1cm} (3.9)

since the choice of an affine parameter, say $v$, is always feasible. By virtue of (3.7), it is inferred that—after a proper rescaling—there always exists a scalar quantity $u$ such that

$$n = du$$  \hspace{1cm} (3.10)

for the associated, to $N$, 1-form $n$. A new coordinate patch naturally emerges: $\{x^a\} = \{u, v, x^1, \ldots, x^{n-2}\}$, by completing the system. In this patch, it is

$$N^a \equiv \frac{dx^a}{dv} = \{0, 1, 0, \ldots, 0\}$$  \hspace{1cm} (3.11a)

$$N_a = \{1, 0, 0, \ldots, 0\}$$  \hspace{1cm} (3.11b)

The last expressions, along with the non divergence property (3.6) result in for the metric tensor field

$$g_{ab} = \begin{pmatrix} 2A & 1 & B_j \\ 1 & 0 & 0 \\ B_i & 0 & \gamma_{ij} \end{pmatrix}, \quad i, j \in \{1, \ldots, n-2\}$$  \hspace{1cm} (3.12a)

$$\partial_v \det \gamma = 0$$  \hspace{1cm} (3.12b)

Barring the constraint on the determinant of $\gamma_{ij}$, all the functions appearing in the metric tensor field depend on all the coordinate variables.

One final point. There is no need to consider other integrability conditions (say involving higher derivatives) than (3.11). The reason lies on the fact that the scalar $\varphi$, being completely arbitrary, covers the need for higher derivatives.

All these conclude the proof. Q.E.D.

It is, perhaps, needless to say that the family of geometries, described in the theorem, includes all the results found—thus far—in the relevant literature. On the other hand, one might object on the existence of some redundancy in the fundamental form, provided by the theorem. For example, that form contains (amongst other things) not only the degenerate Kundt family and the some
Walker metrics—all endowed with the desired property—but also ‘regular’ geometries (like the simple Kundt spaces, etc.). The answer to such an objection would be that this redundancy is the price to be paid for the encapsulation, in one family, of various and very versatile geometries characterised by the property under consideration.

The circle has now closed: the theorem has narrowed down the possibilities to a given family. A geometry not characterised by its curvature scalars must necessarily conform with the demands of the theorem and its covariant criterion. Thus, if one has a given geometry at hand, then one has to follow a simple, straightforward procedure with two steps.

1. First, to implement the covariant statement of the theorem 13.1 and search for the existence of a non diverging, normal, and null vector field. If the outcome is negative then the geometry is susceptible of a description based on curvature scalars. If the outcome is positive then one can go to the next step.

2. Given the positive outcome, implementation of the boost weight analysis can help one to find which (fundamental) functions appearing in the metric tensor field act as ‘phantoms’.

Still, one might ask whether the Cartan-Karlhede algorithm is the only refuge in the case where a given geometry can not be characterised by its corresponding $CS$ class. The answer depends on the quality of the requested information. As stated in the Introduction, the equivalence method is the ultimate framework towards the invariant description of spaces, and this can not be changed—at least not easily. Besides, it can provide pieces of information not accessible from the path of the curvature scalars—not even for ‘regular’ geometries. On the other hand it is much simpler to calculate some curvature scalars, rather than carry the full equivalence method out. So, in those cases where one is pleased with the approach based on curvature scalars, the question remains valid.

A scrutinised analysis of the proof of the theorem 13.1 will exhibit that two important points could render its truth vacuous. Indeed, both are valid only in one case; when the involved connexion is symmetric (i.e., a Riemannian connexion). Even by a simple inspection, it is easy for one to see that in the Riemann-Cartan branch both points would also entail the antisymmetric part of the connexion, namely the torsion, since using (2.3a), (2.3b), (2.3c), and (2.8) in a cyclic order—with proper overall sings— one can easily get

$$\gamma_{AMN} = \frac{1}{2} \left\{ \left( g_{AM|N} + g_{AN|M} - g_{MN|A} \right) \right\} + \left( C_{AMN} + C_{MNA} - C_{NAM} \right) \right\}$$

(3.13)
for a general frame (where the bar symbol (|) denotes partial differentiation adapted to the frame i.e., directional derivative) and this reduces to

$$\gamma^{amn} = \frac{1}{2} \left\{ (g^{am,n} + g^{an,m} - g^{mn,a}) - (\tau^{amn} + \tau^{nma} - \tau^{nam}) \right\}$$

for any coordinate frame (where a comma, as usual, denotes partial coordinate differentiation). It should be noted that, in general, torsion does contribute to the symmetric part of the connexion.

The result of these considerations is twofold: on one hand the theorem ceases its validity in the Riemann-Cartan branch (unless the torsion is very special), and on the other hand one can implement the curvature scalar approach, by choosing torsion, in order to discriminate amongst spaces. Thus, for instance the following two choices

$$\tau_{bc}^a = \delta^a_{bc} \psi_{,c} - \delta^a_{c} \psi_{,b}$$
$$\tau_{bc}^a = \frac{1}{(n-3)!} \epsilon^{abc}_{i_1i_2...i_{n-3}} \Psi_{i_1i_2...i_{n-3}}$$

where the ‘test’ scalar function $\psi$ and the ‘test’ $(n-3)$-form $\Psi$ are all supposed to be well defined on $M$, do fulfill their purpose.

The first choice for the torsion is the simplest possible, and most commonly found in the literature. On the other hand, the second choice has the pure geometric advantage of preserving the functional form of the geodesics (more properly named loxodromies in the Riemann-Cartan context) since, now, the torsion —being a completely antisymmetric tensor— does not contribute to the symmetric part of the connexion.

Indeed, the interested reader could use either of these two choices, say for the most ‘famous’ (for its degeneracy) pair example i.e., the vacuum pp-wave and Minkowski geometries. Then, one can find that, for either choice for the torsion, the Minkowski geometry still has vanishing curvature scalars, while the vacuum pp-wave geometries have not. Thus, with the help of the ‘test’ scalar function(s) one can discriminate geometries —with no exception, being confined only by the nature of the approach.

With this final point, the threefold purpose of the paper has been fulfilled.

## 4 Discussion

A triple problem is the central objective of the present paper.

- To find the necessary condition, in the form of a covariant criterion, for a pseudo-Riemannian geometry not to be (locally) characterised based on its curvature scalars class— of course, modulo the inherent weakness of the approach.
- To use this covariant criterion to find the generic form for the metric tensor field endowed with this property.
- To provide a patch to the curvature scalar approach, towards the local classification of geometries, in order for it to be valid in all the cases.
Based on a minimum of notions and definitions and by exploiting simple thoughts, a necessity analysis was carried out and as a result all the three goals have been achieved. It is, perhaps, interesting that not only the first two objectives share a common answer —given by the theorem (3.1)— but also that the answer to the third objective emerged by a simple observation regarding the range of validity of the theorem.

Seen within a broader context, the first two parts of the threefold problem could be considered as the one logical direction of a more restrictive theorem, which could be stated as follows

\textit{a pseudo-Riemannian geometry can be (locally) characterised based on its class of curvature scalars if and only if . . .}

The relevant literature has covered, thus far, an extended part of the other logical direction—but not completely. Nevertheless, the boost weight analysis offers a valuable tool towards this end. On the other hand, the answer given in the present work is plagued by an unavoidable redundancy. As stated before, this is the price to be paid in order to have a (perhaps infinite) manifold of families grouped together in a single, simple form. Moreover, one could realize refinements of the theorem (3.1), by considering the less generic yet more precise cases where more annihilating vector fields exist. In fact, this could be an interesting area for further research (e.g., to prove their normality, nullity and non divergence properties) —although the basic ingredients are already contained here. For example, another tensorial statement is that the vector fields emanating from (3.2) will form an abelian algebra. Also, the cardinality of their set must be equal to the absolute value of the difference between the pluses and minuses in the signature of the underlying geometry—even though no neat proof of this fact is visible.

Regarding the third aspect of the initial problem, it is worth mentioning that the proposed patch is just the classical method—applied within a broader framework. From this point of view, nothing practically impossible is suggested; on the contrary, things become much simpler and practically accessible. Thus, for example, in the case of the pair vacuum pp-waves, and Minkowski geometries, the curvature scalars needed for the discrimination between the two—with the two proposed versions for torsion—are algebraic i.e., formed only by the corresponding Riemann, Ricci, and Weyl tensors. Analogous simplifications can be verified in other cases (like, e.g., the degenerate Kundt family).

Perhaps, the most interesting fact is that three different aspects of an intriguing problem were all attacked by a simple necessity analysis giving, at the same time, some hints for future work on the field.

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Or algebraic Bianchi identities, or cyclic identities.

The Levi-Civita tensor, in $n$ dimensions, is defined by

$\varepsilon_{a_1a_2...a_n} = \sqrt{|\det g|} [a_1a_2...a_n]$

$\varepsilon^{a_1a_2...a_n} = \left(\sqrt{|\det g|}\right)^{-1} \text{sign}(g)[a_1a_2...a_n]$

$[a_1a_2...a_n] = \begin{cases} 
+1 & \text{for even permutation of the indices} \\
-1 & \text{for odd permutation of the indices} \\
0 & \text{in any other case}
\end{cases}$

The totality of its properties follow from this definition; for instance $\varepsilon_{a_1a_2...a_n} \varepsilon^{a_1a_2...a_n} = \text{sign}(g)n!$, etc.

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