ON THE RATE OF CONVERGENCE TO EQUILIBRIUM
FOR COUNTABLE ERGODIC MARKOV CHAINS

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Abstract. Using elementary methods, we prove that for a countable Markov
chain \( P \) of ergodic degree \( d > 0 \) the rate of convergence towards the stationary
distribution is subgeometric of order \( n^{-d} \), provided the initial distribution sat-
sifies certain conditions of asymptotic decay. An example, modelling a renewal
process and providing a markovian approximation scheme in dynamical system
theory, is worked out in detail, illustrating the relationships between conver-
gence behaviour, analytic properties of the generating functions associated to
transition probabilities and spectral properties of the Markov operator \( P \) on the
Banach space \( \ell_1 \). Explicit conditions allowing to obtain the actual asymptotics
for the rate of convergence are also discussed.

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properties; Markov approximations; renewal theory; recurrence; intermittency

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0. INTRODUCTION.

Let $S$ be a countable set and $P : S \times S \to [0, 1]$ be a transition probability matrix. With no loss we may set $S = \mathbb{N}$. We shall assume that $P$ governs an irreducible, recurrent and aperiodic Markov chain $X = (x_n)_{0}^{\infty}$ with state space $S$. To be more precise, we set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and let $\Omega$ denote the subset of $S^{\mathbb{N}_0}$ given by all sequences $\omega = (\omega_i)_{i \in \mathbb{N}_0}$ which satisfy for any integer $i$:

$$p_{\omega_i, \omega_{i+1}} \equiv P(\omega_i, \omega_{i+1}) > 0.$$  

For any $n \in \mathbb{N}_0$ we let $x_n$ be the projection on the $n$th coordinate, i.e. $x_n(\omega) = \omega_n$. Let moreover $P_\nu$ be the probability measure with initial distribution $\nu$ (that of $x_0$) on $\Omega$, i.e.

$$P_\nu \{x_n(\omega) = j\} = \sum \nu ip_{ij}^n = (\nu P^n)_j$$  \hspace{1cm} (0.1)

where $p_{ij}^n \equiv P^{(n)}(i, j)$. Our sample space will be $\Omega$ equipped with the restriction of the product $\sigma$-field and with probability measure $P_\nu$ for some initial distribution $\nu$. We shall denote by $E_\nu$ the expectation w.r.t. $P_\nu$. In particular, if $\nu = \delta_i$, where $i$ is some reference state chosen from the outset, we have

$$P_i \{x_n(\omega) = j\} = P \{x_n(\omega) = j|x_0(\omega) = i\} = p_{ij}^n.$$  \hspace{1cm} (0.2)

Let $m_i$ be the $P_i$-expectation of $\min\{n \in \mathbb{N}_0, x_n(\omega) = i\}$, the time of the first visit at $i$. It is well known (see e.g. [Chu]) that if $P$ is irreducible and aperiodic then

$$\lim_{n \to \infty} p_{ij}^n = \frac{1}{m_i},$$  \hspace{1cm} (0.3)

where the r.h.s. is taken to be zero in the transient and null recurrent cases when $m_i = \infty$. If instead $m_i$ is finite for some (and hence for all) $i \in S$ then $P$ is called ergodic, or positive recurrent, and there is a (unique) probability distribution $\pi$ on $S$ given by $\pi = (\pi_i)_{1}^{\infty} = (1/m_i)_{1}^{\infty}$ which is a solution to $\pi = \pi P$ and thus defines a stationary distribution. This paper is devoted to the study of the rate of convergence in (0.3) for ergodic chains and more generally to the rate convergence of a given initial distribution $\nu$ to the stationary distribution $\pi$. It is divided into two main parts. In the first part (Sections 1 and 2) general
convergence results are stated and proved, which relate the rate of convergence to a parameter $d$ called the ergodic degree. Roughly speaking, the ergodic degree controls in a continuous fashion the number of finite moments possessed by the time of the first visit at a given state $i \in S$ (see Definition 1). The fact that the speed of convergence for countable state Markov chains is connected to the number of moments of first passage times has been put forward by several works starting with Feller [Fe1]. In particular, using the technique of coupling, Pitman [Pi] proved that if first passage times have finite $r$-th moment, with $r$ a given positive integer with $r \geq 2$, then the rate of convergence in (0.3) is $o(n^{-(r-1)})$. For other results of the same nature we refer to [Pop] and [TT]. In Theorem 1 stated below an improvement of the above results is achieved in that for any real positive value of the ergodic degree $d$, which is assumed to be finite, it is possible to prove subgeometric convergence to equilibrium of order $n^{-d}$. This amounts to obtaining subgeometric lower bounds as well, which are here proved using elementary generating functions techniques. The relevance of obtaining sharp bounds is further discussed in Section 3, where an example modelling a renewal process is worked out in detail using a different (although similar in spirit) method which makes use of matrix-valued analytic functions and allows to further sharpen the general results of Section 1 under suitable conditions. The main motivation is that of illustrating the relationships between convergence behaviour, analytic properties of the generating functions associated to transition probabilities and spectral properties of the Markov operator $P$ on the Banach space $\ell_1$. A second motivation is discussed in the Appendix and comes from the fact that this example provides a markovian approximation scheme in dynamical system theory, where the question of obtaining sharp subgeometric bounds for the decay of correlations appears to be particularly relevant (see [Is1] and [Sa]).
1. ERGODIC DEGREE AND GENERAL CONVERGENCE RESULTS.

In the sequel we identify sequences \( \nu = (\nu_i)_{i=1}^{\infty} \in \ell_1(S) \), the corresponding row vectors \( \nu = (\nu_0, \nu_2, \ldots) \), and finite signed measures on \( S \), and define

\[
\|\nu\| = \sum_{i=1}^{\infty} |\nu_i|.
\]

A signed measure \( \nu \) satisfying \( \sum \nu_l = 1 \) will be called a signed distribution. Similarly, we shall identify sequences \( u = (u_i)_{i=1}^{\infty} \in \ell_\infty(S) \), the corresponding column vectors \( u = (u_0, u_2, \ldots)^t \), and bounded functions on \( S \).

We introduce the classical taboo quantities:

\[
f^n_{ij} = \mathbb{P}\{x_l(\omega) \neq j, 0 < l < n, x_n(\omega) = j \mid x_0(\omega) = i\},
\]

\[
k^p^n_{ij} = \mathbb{P}\{x_l(\omega) \neq k, 0 < l < n, x_n(\omega) = j \mid x_0(\omega) = i\},
\]

\[
f^*_n = \sum_{n=1}^{\infty} f^n_{ij}, \quad k^p^*_n = \sum_{n=1}^{\infty} k^p^n_{ij},
\]

Clearly \( j^p^n_{ij} = f^n_{ij} \). Since we have a unique recurrent class, \( f^*_n = 1 \) for all \( i, j \in S \).

Moreover, for an ergodic chain we have (Chu, Chap. I.9, Thm. 5)

\[
\lim_{n \to \infty} \frac{\sum_{k=0}^{n} p^n_{ij}}{\sum_{k=0}^{n} p^n_{ii}} = \frac{\pi_j}{\pi_i} = \pi^*_i, \quad (1.1)
\]

The last quantity can also be viewed as the \( \pi_i \)-mean number of visits to the state \( j \) before return to \( i \). The relation between the \( f_{ij} \)'s and the transition probabilities \( p_{ij} \) is given by (Chu, Chap. I.5, Thm. 2)

\[
p^0_{ij} = \delta_{ij}, \quad f^0_{ij} = 0
\]

and

\[
p^n_{ij} = \sum_{k=1}^{n} f^n_{ij} p^{n-k}_{jj}. \quad (1.2)
\]

We also have

\[
\pi_i = \frac{1}{\sum_{n=1}^{\infty} n f^n_{ii}}. \quad (1.3)
\]
For \( k \in \mathbb{N}, i \in S \), let \( t_k^{(i)} \) be the time of the \( k \)-th entrance into state \( i \), and let
\[
  r_k^{(i)}(\omega) = t_{k+1}^{(i)} - t_k^{(i)}, \quad k \geq 0
\]
be the sequence of times between returns (set \( t_0^{(i)} = 0 \)). Clearly we have \( r_0^{(i)} \geq 0 \) and \( r_k^{(i)} > 0 \) for \( k \geq 1 \). Moreover, the state \( i \) being recurrent, \( r_1^{(i)}, r_2^{(i)}, \ldots \) are i.i.d. random variables under the probability \( P_i \). Their common distribution is given by
\[
  P_i \{ r_k^{(i)}(\omega) = n \} = f_{ii}^n, \quad k \geq 1. \tag{1.5}
\]

On the other hand, having fixed an initial distribution \( \nu \) and a reference state \( i \), the random variable \( r_0^{(i)} \) (the delay in the embedded renewal process) is distributed according to \( P_\nu \). More specifically,
\[
  P_\nu \{ r_0^{(i)} = n \} = \nu_i \delta_{n0} + \sum_{l \neq i} \nu_l f_{li}^n.
\]

For \( \gamma \geq 0, i, j \in S \) (and \( k \geq 1 \)), we set
\[
  M_{ij}^{(\gamma)} := E_i(\{ r_k^{(j)} | \gamma \}) = \sum_{n=1}^{\infty} n^\gamma f_{ij}^n. \tag{1.6}
\]

Notice that \( m_i \equiv M_{ii}^{(1)} \). Given a signed distribution \( \nu \) on \( S \), we also set,
\[
  M_{ii}^{(\gamma)} := E_\nu(\{ r_0^{(i)} | \gamma \}) = \sum_{l \neq i} \nu_l \sum_{n=1}^{\infty} n^\gamma f_{li}^n = \sum_{l \neq i} \nu_l M_{ii}^{(\gamma)}. \tag{1.7}
\]

The next result extends ([SKS], Thm. 9.65) to arbitrary (i.e. not necessarily integer) \( \gamma \)-values.

**Lemma 1.** If \( \gamma \geq 0 \), then \( M_{ii}^{(\gamma+1)} < \infty \) if and only if \( M_{ii}^{(\gamma)} < \infty \).

**Proof.** Using the last identity in (1.1) and the decomposition \( i p_{ii}^{n+m} = \sum_{l \neq i} i p_{ii}^m i p_{li}^n \) we get
\[
  M_{ii}^{(\gamma)} = \sum_{n=1}^{\infty} n^\gamma \sum_{l \neq i} \pi_l f_{li}^n = \pi_i \sum_{n=1}^{\infty} n^\gamma \sum_{l \neq i} i p_{li}^n f_{li}^n
\]
\[
  = \pi_i \sum_{n=1}^{\infty} n^\gamma \sum_{m=1}^{n} i p_{ii}^m i p_{li}^n = \pi_i \sum_{n=1}^{\infty} n^\gamma \sum_{m=1}^{\infty} i p_{ii}^{n+m}
\]
\[
  = \pi_i \sum_{n=1}^{\infty} n^\gamma \sum_{m>n} f_{ii}^m = \pi_i \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n-1} k^\gamma \right) f_{ii}^n
\]

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and we finish the proof by noting that $\sum_{k=1}^{n-1} k^\gamma \sim n^{\gamma+1}/(\gamma + 1)$ as $n \to \infty$.  

Remark. It is well known that, for a recurrent chain, if $M_{ii}^{(\gamma+1)} < \infty$ for some state $i$ then $M_{ij}^{(\gamma+1)} < \infty$, for all pairs (distinct or not) $i, j \in S$ (see, e.g., [Chu], Chap. I.11, Cor. 1). Notice however that even though $M_{ii}^{(\gamma)} < \infty$ for all $i \in S$, the series $\sum_{i \in S} \pi_i M_{ii}^{(\gamma)}$ is divergent. To see this, consider for example $\gamma = 1$. Assuming $M_{ii}^{(2)} < \infty$ let us suppose that

$$\sum_{i \in S} \pi_i M_{ii}^{(1)} = \sum_i \pi_i \sum_{l \neq i} \pi_l M_{li}^{(1)} < \infty$$

Then, since the double series has positive terms we would have

$$\sum_l \pi_l \sum_{i \neq l} \pi_i M_{li}^{(1)} < \infty,$$

as well. But this is impossible because $M_{ii}^{(1)} + M_{li}^{(1)} = (1 + \pi_i^*) \pi_i^{-1}$ ([Chu], p.65) and $\lim_{i \to \infty} (M_{ii}^{(1)}/M_{li}^{(1)}) = 0$ for all $l \in S$ ([Chu], Chap. I.11, Thm. 6; see also [H1]).

We now state the following definition.

**Definition 1.** Given a recurrent Markov chain $P$ with state space $S$, the ergodic degree of $P$ is the number

$$d = \inf \{ \gamma : M_{ii}^{(\gamma+1)} = \infty \text{ for some (and then for all) } i \in S \}$$

Notice that $M_{ii}^{(0)} = 1$ so that the degree satisfies $d > -1$. In the following we shall refer to an ergodic chain as a chain for which $d$ is strictly positive. If $M_{ii}^{(\gamma)} < \infty$ for every $\gamma$, one says that $P$ has infinite ergodic degree. This happens for instance if the coefficients $f_{ii}^n$ decay geometrically with $n$. In this case the corresponding chain is accordingly called geometrically ergodic. We refer to [FMM] for related convergence results in the geometrically ergodic case.

The preceding observations and Lemma 1 motivate the next definition.

**Definition 2.** Given an ergodic chain $P$ with state space $S$ and a signed distribution $\nu$ on $S$, the $P$-order of $\nu$ is the number

$$\sup \{ \gamma > 0 : M_{\nu i}^{(\gamma)} < \infty \text{ for some (and then for all) } i \in S \}$$
**Remark.** Lemma 1 implies that the ergodic degree of an ergodic chain $P$ coincides with the $P$-order of its stationary distribution $\pi$.

**Notations:** Here and in the sequel, for two sequences $a_n$ and $b_n$ we shall write $a_n \sim b_n$ if the quotient $a_n/b_n$ tends to unity as $n \to \infty$. Moreover, the notation $a_n = O_\epsilon (n^{-d})$ means that $a_n = o(n^{-(d-\epsilon)})$, $\forall \epsilon > 0$, or, which is the same, that $a_n \cdot n^d$ grows slower than any power of $n$ as $n \to \infty$. This condition is satisfied if, for example, $a_n$ decays as $C n^{-d} L(n)$ where $L(n)$ is some function slowly varying at infinity, i.e. $L(cn) \sim L(n)$ for every positive $c$.

We now state the main result of this Section.

**Theorem 1.** Suppose $P$ has ergodic degree $d > 0$. Then, for any initial signed distribution $\nu$ of $P$-order at least $d$, we have

$$||\nu P^n - \pi|| = O_\epsilon (n^{-d}).$$

In addition, if $M^{(d+1)}_{ii} = \infty$ for some (and then for all) $i$ and the $P$-order of $\nu$ is strictly larger than $d$, then the above bound is sharp, i.e. $n^d \cdot ||\nu P^n - \pi||$ varies slower than any power of $n$.

We let $\tau$ be the shift transformation on $\Omega$, that is $x_k \circ \tau(\omega) = \omega_{k+1}$. With $P$ and $\pi$ one can define a $\tau$-invariant Markov random field $\mu = \mu(P, \pi)$ supported by $\Omega$ as follows:

$$\mu(\{x_k(\omega) = \xi_0, \ldots, x_{k+n}(\omega) = \xi_n\}) = \pi_{\xi_0} \prod_{j=1}^{n} p_{\xi_{j-1} \xi_j} \quad (1.8)$$

We shall say that $\mu$ has ergodic degree $d$ whenever $P$ (and $\pi$) has the same property. We then have the following,

**Corollary 1.** Suppose $\mu$ has ergodic degree $d > 0$. Then, for any pair of bounded vectors $u, v : S \to \mathbb{R}$,

$$|\mu(u(x_n)v(x_0)) - \mu(u(x_0)) \mu(v(x_0))| = O_\epsilon (n^{-d})$$
2. PROOFS

We shall prove Theorem 1 and its Corollary through several Lemmas. We start with few technical results which will be used several times in the sequel.

**Lemma A.** (see, e.g., [Chu], Chap. I.5) Let \( \{a_n\}_{n \geq 0} \) be a sequence of nonnegative numbers not all vanishing and such that \( a_n / \left( \sum_{m=0}^{n} a_m \right) \to 0, \ n \to \infty. \) Then, whenever the sequence \( \{b_n\}_{n \geq 0} \) of real numbers has a limit, we have

\[
\lim_{n \to \infty} \frac{\sum_{m=0}^{n} a_m b_{n-m}}{\sum_{m=0}^{n} a_m} = \lim_{n \to \infty} b_n.
\]

**Lemma B.** Let \( D(z) = \sum_{n=0}^{\infty} d_n z^n \) be absolutely convergent and \( D(z) \neq 0 \) for \( |z| \leq 1. \) Let moreover \( d_n = O_\epsilon(n^{-\gamma}) \) for some \( \gamma \geq 1. \) Then

\[
C(z) = \frac{1}{D(z)} = \sum_{n=0}^{\infty} c_n z^n
\]

is also absolutely convergent for \( |z| \leq 1 \) and \( c_n = O_\epsilon(n^{-\gamma}). \) The assertion remains valid if \( O_\epsilon(n^{-\gamma}) \) is replaced by \( o(n^{-\gamma}). \)

If, in addition, \( d_0 = 1, \) \( d_n > 0 \) and \( d_n/d_{n-1} \) is increasing, then \( c_0 = 1 \) and \( \sum_{k=0}^{n} c_k > 0 \) decreases monotonically to \( D(1)^{-1} \leq 1. \)

**Proof.** The first statement is a consequence of a theorem of Wiener and its proof can be found in [Ro], Lemma 3.II. For the last statement see, e.g., [H2], Thm. 22. \( \Box \)

**Lemma C.** Let \( D(z) \) and \( C(z) \) be as in the first part of Lemma B with \( d_n = O_\epsilon(n^{-\gamma}) \) for some \( \gamma > 1. \) Assume furthermore that \( d_n \geq 0 \) and \( \sum_{k=0}^{n} d_k = \infty. \) Given a sequence \( e_n, \ n \geq 0, \) let \( h_n = \sum_{k=0}^{n} c_k e_{n-k}, \) or else

\[
\sum_{n=0}^{\infty} h_n z^n = \frac{\sum_{n=0}^{\infty} e_n z^n}{\sum_{n=0}^{\infty} d_n z^n}.
\]

(a) If \( e_n = O_\epsilon(\sum_{\ell > n} d_\ell), \) then \( h_n = O_\epsilon(\sum_{\ell > n} d_\ell), \) and the assertion remains valid if \( O_\epsilon(\sum_{\ell > n} d_\ell) \) is replaced by \( o(\sum_{\ell > n} d_\ell). \)

(b) If, in addition, \( e_n = \sum_{\ell > n} d_\ell \) then \( h_n \sim D(1)^{-1} e_n. \)
Proof. First, since \( d_n = O_\epsilon(n^{-\gamma}) \) we have from Lemma B that \( c_n = O_\epsilon(n^{-\gamma}) \) as well. To show (a) we then notice that

\[
|h_n| \leq \max_{0 \leq k \leq n/2} |e_{n-k}| \sum_{k=0}^{n/2} |c_k| + \max_{n/2 \leq k \leq n} |c_k| \sum_{k=0}^{n/2} |e_k|.
\]

Therefore, if \( \sum |e_k| < \infty \) then \( h_n = O(\max\{|c_n|, |e_n|\}) \), otherwise \( h_n = O(e_n) \).

Indeed, the condition \( \sum |e_k| = \infty \) entails \( 1 < \gamma \leq 2 \). Since \( d_n = O_\epsilon(n^{-\gamma}) \) and \( e_n = O_\epsilon(\sum_{\ell>n} d_\ell) = O_\epsilon(n^{-\gamma+1}) \), Lemma B implies that the last term in the r.h.s. of the above expression is \( O_\epsilon(n^{-2(\gamma-1)}) = o(\sum_{\ell>n} d_\ell) \).

Let us now prove assertion (b). Under the assumption stated there, Lemma A yields \( h_n - e_n \sum_{k=0}^{n} c_k = o(\sum_{k=0}^{n} c_k) = o(1) \). But we can say more. The conditions \( d_n \geq 0 \) and \( \sum n^{\gamma-1} d_n = \infty \) imply that \( n^{\gamma-1} \cdot e_n \) decays slower than any inverse power of \( n \). Moreover, let us note that since \( e_n = \sum_{\ell>n} d_\ell \) we have

\[
D(1) - \sum_{n=0}^{\infty} d_n z^n = (1 - z) \sum_{n=0}^{\infty} e_n z^n.
\]

We then write

\[
\sum_{n=0}^{\infty} h_n z^n = D(1)^{-1} \sum_{n=0}^{\infty} e_n z^n + \frac{(1 - z) \left( \sum_{n=0}^{\infty} e_n z^n \right)^2}{D(1) \sum_{n=0}^{\infty} d_n z^n}.
\]

The proof of (b) then reduces to show that the coefficients of the last power series are \( o(e_n) \). To this end we use the following easily checked fact:

\[
\sum_{k=0}^{n} k^{1-\gamma} (n-k)^{1-\gamma} = \begin{cases} 
O(n^{3-2\gamma}), & \text{for } 1 < \gamma < 2, \\
O(\log n/n), & \text{for } \gamma = 2, \\
O(n^{1-\gamma}), & \text{for } \gamma > 2.
\end{cases}
\]

By the above, the coefficients of the power series \( (1 - z) \left( \sum_{n=0}^{\infty} e_n z^n \right)^2 \) are \( O_\epsilon(n^{-2(\gamma-1)}) \) for \( 1 < \gamma \leq 2 \) and \( O_\epsilon(n^{-\gamma}) \) for \( 2 < \gamma \), therefore \( o(e_n) \) in both cases. The claim now follows by applying the same reasoning as in the proof of (a) to the coefficients of \( (1 - z) \left( \sum_{n=0}^{\infty} e_n z^n \right)^2 \sum_{n=0}^{\infty} c_n z^n \). \( \diamond \)

In the following Lemma we shall establish an asymptotic equivalence which determines the speed of convergence of the diagonal transition probabilities \( p_{ii}^n \) to the stationary distribution \( \pi_i \) in terms of the \( \mathbf{P}_i \)-distribution of the first return time \( r_1^{(i)} \). This will be prove useful to obtain sharp bounds under appropriate conditions.
Lemma 2. For a (finitely) ergodic chain $P$ with state space $S$ and stationary distribution $\pi$, we have, for any $i \in S$,

\[ p_{ii}^n - \pi_i \sim \frac{1}{m_i^2} \sum_{\ell > n} P_i \{ r_1^{(i)}(\omega) > \ell \} . \]

Proof. We introduce the generating functions

\[ P_{ij}(z) = \sum_{n=0}^{\infty} p_{ij}^n z^n, \quad F_{ij}(z) = \sum_{n=0}^{\infty} f_{ij}^n z^n \]  \hspace{1cm} (2.1)

and from (1.2) we get the relations (we set $f_{ij}^n = 0$ for $n = 0$)

\[ P_{ii}(z) = \frac{1}{1 - F_{ii}(z)}, \quad P_{ij}(z) = F_{ij}(z) P_{jj}(z), \quad i \neq j. \]  \hspace{1cm} (2.2)

We first show that the function $P_{ii}(z)$ is analytic in $|z| < 1$ and converges at every point of the unit circle besides $z = 1$. Indeed, recurrence of the state $i$ implies $F_{ii}(1) = 1$, so that $|F_{ii}(z)| < 1$ for $|z| < 1$ because $f_{ii}^n \geq 0$. Moreover, $|F_{ii}(z)| < 1$ also for $|z| = 1, z \neq 1$. This follows from the fact that, since the chain is aperiodic, \( \text{g.c.d.}\{n, f_{ii}^n \neq 0\} = 1 \). Now set

\[ D_{ii}(z) = \sum_{n=0}^{\infty} d_{ii}^{(n)} z^n, \quad d_{ii}^{(n)} := \sum_{k>n} f_{ii}^k = P_i \{ r_1^{(i)}(\omega) > n \} \]  \hspace{1cm} (2.3)

and notice that $D_{ii}(z)$ converges absolutely in $|z| \leq 1$ and has no zeros on $|z| = 1$. In addition $\sum_{n=0}^{\infty} d_{ii}^{(n)} = m_i$. It then follows from Lemma B that the function

\[ \frac{1}{D_{ii}(z)} = (1 - z) P_{ii}(z) \]  \hspace{1cm} (2.4)

has a power series expansion which converges absolutely in the closed unit disk and, moreover, its value at $z = 1$ is $m_i^{-1} = \pi_i$. Set

\[ \frac{1}{D_{ii}(z)} = \sum_{n=0}^{\infty} c_{ii}^{(n)} z^n, \quad \sum_{n=0}^{\infty} |c_{ii}^{(n)}| < \infty. \]  \hspace{1cm} (2.5)

We now observe that the ergodicity assumption implies that $d_{ii}^{(n)} = P_i \{ r_1^{(i)}(\omega) > n \} = o(n^{-1})$. We may then use again Lemma B to obtain $c_{ii}^{(n)} = o(n^{-1})$ as well. By an Abelian theorem (see, e.g., [Chu], p.55) we then have

\[ p_{ii}^n = \sum_{k=0}^{n} c_{ii}^{(k)} \rightarrow \pi_i, \quad n \rightarrow \infty. \]  \hspace{1cm} (2.6)
To obtain more information, we first observe that \( m_i p_i^n - 1 \) is the coefficient of \( z^n \) in
\[
H_{ii}(z) := m_i P_{ii}(z) - \frac{1}{1 - z} = \frac{E_{ii}(z)}{D_{ii}(z)}
\]
(2.7)
where
\[
E_{ii}(z) = \sum_{n=0}^{\infty} e_{ii}(n) z^n, \quad e_{ii}(n) := \sum_{\ell > n} d_{ii}^{(\ell)} = \sum_{\ell > n} P_i \{ r_1^{(i)}(\omega) > \ell \}.
\]
(2.8)

Now, if the ergodic degree \( d \) is finite the conditions of Lemma C-(b) are satisfied for the sequences \( d_{ii}^{(n)} \), \( c_{ii}^{(n)} \) and \( e_{ii}^{(n)} \). Whence we conclude that
\[
m_i^2 (p_{ii}^n - \pi_i) \sim \sum_{\ell > n} P_i \{ r_1^{(i)}(\omega) > \ell \}.
\]
(2.9)
This finishes the proof. ♦

**Lemma 3.** Suppose \( M_{ii}^{(\gamma)} < \infty \) for some (and hence for all) \( i \in S \) and for some \( \gamma \geq 1 \). Then,
\[
||\delta_i P^n - \pi|| = o(n^{-(\gamma-1)}).
\]

**Proof.** We start noticing that the assumption \( M_{ii}^{(\gamma)} < \infty \) implies \( \pi_i \{ r_1^{(i)}(\omega) > n \} = o(n^{-\gamma}) \) and therefore, by Lemma 2, we have
\[
|p_{ii}^n - \pi_i| = o(n^{-(\gamma-1)}).
\]
(2.10)

More generally, it follows from (1.2), \( \sum_n f_{ij}^n = 1 \) and and Lemma A that \( p_{ij}^n \to \pi_j \) as \( n \to \infty \). Furthermore, as already remarked, the condition \( M_{ii}^{(\gamma)} < \infty \) implies that \( \sum_{n=1}^{\infty} n^\gamma f_{ij}^n < \infty \), for all pairs (distinct or not) \( i, j \in S \). This and Lemma A, along with the inequality
\[
|p_{ij}^n - \pi_j| \leq \sum_{k=1}^{n} f_{ij}^k |p_{jj}^{n-k} - \pi_j| + \pi_j \sum_{k > n} f_{ij}^k,
\]
(2.11)

imply that the rate of convergence to zero of \( |p_{ij}^n - \pi_j| \) is the same as in (2.10). These properties entail that \( P^n \) tends to the matrix whose rows are \( (\pi_1, \pi_2, \ldots) \).

To finish the proof we proceed as follows. Having fixed a state \( k \in S \) we use
(1.1) along with standard decomposition formulae (see [Chu], Chap. I.9) to write 
\[ p_{ij}^n - \pi_j = \sum_{m=1}^{n-1} kp_{kj}^n (p_{ik}^m - \pi_k) + k p_{ij}^n - \pi_k \sum_{m=n}^{\infty} kp_{kj}^m = I + II + III. \]

Recalling that \( \sum_j kp_{ij}^n = \sum_{m \geq n} f_{ik}^m \) and summing over \( j \in S \) we immediately obtain \( \sum_{j \in S} |II| = o(n^{-\gamma}) \) and \( \sum_{j \in S} |III| = o(n^{-(\gamma-1)}) \). For the first term we have 
\[ \sum_{j \in S} |I| \leq \sum_{m=1}^{n-1} |p_{ik}^m - \pi_k| \sum_{r \geq n-m} f_{kk}^r. \]

Let us multiply both sides of the above inequality by \( n^{-(\gamma-1)} \). Using the fact that 
\[ n \leq m(n+1-m) \text{ if } 1 \leq m \leq n \] 
we get 
\[ n^{\gamma-1} \sum_{m=1}^{n-1} |p_{ik}^m - \pi_k| \sum_{r \geq n-m} f_{kk}^r \leq \sum_{m=1}^{n-1} |p_{ik}^m - \pi_k| m^{\gamma-1} \sum_{r \geq n-m} f_{kk}^m (n+1-m)^{\gamma-1}. \]

Since \( \lim_{p \to \infty} (p+1)^{\gamma-1} \sum_{r \geq p} f_{kk}^r = 0 \) and \( \lim_{m \to \infty} |p_{ik}^m - \pi_k| m^{\gamma-1} = 0 \), from Lemma A it follows that the r.h.s. tends to zero as \( n \to \infty \) and therefore 
\[ \sum_{j \in S} |I| = o(n^{-(\gamma-1)}) \] 
We have thus found that 
\[ \sum_{j \in S} |p_{ij}^n - \pi_j| = o(n^{-(\gamma-1)}) \]
and the proof of Lemma 3 is complete. \( \diamond \)

**Lemma 4.** For any initial signed distribution \( \nu \) such that \( M_{\nu_i}^{(\gamma-1)} < \infty \) for some (and hence for all) \( i \in S \) (and \( \gamma \geq 1 \)) and under the hypotheses of Lemma 3, we have 
\[ ||\nu P^n - \pi|| = o(n^{-(\gamma-1)}) \] 

**Proof.** Putting \( \nu = \sum \nu_l \delta_l \) and using the fact that \( \nu \) is normalized, i.e. \( \sum \nu_l = 1 \), we write 
\[ \nu P^n - \pi = \sum_l \nu_l (\delta_l P^n - \pi) + \sum_{l \neq i} \nu_l (\delta_l P^n - \delta_i P^n) \]
\[ = (\delta_i P^n - \pi) + \sum_{l \neq i} \nu_l (\delta_l P^n - \delta_i P^n). \]
The $\ell_1$-norm of the first term in the r.h.s. is then estimated by Lemma 3. For the second term we have $\|\delta_l P^n - \delta_i P^n\| = \sum_j |p^n_{ij} - p^n_{ij}|$. Using the decompositions $p^n_{ij} = i p^n_{ij} + \sum_{k=1}^{n-1} f^n_{li} p^n_{ij} \approx P^n - \delta_i P^n$ ([Chu], Chap. I.9, Thm. 1) and $p^n_{ij} = \sum_{k=n}^\infty f^n_{li} p^n_{ij}$, and noting that $\sum_j i p^n_{ij} = \sum_{k\geq n} f^n_{li}$ and $\sum_j p^n_{ij} = 1$, we obtain

$$\|\delta_l P^n - \delta_i P^n\| \leq 2 \sum_{k=n}^\infty f^n_{li} + \sum_{k=1}^{n-1} f^n_{li} \sum_j |p^n_{ij} - p^n_{ij}|.$$ 

Thus, by (1.6), the norm of the last term in the r.h.s. of (2.15) is bounded by

$$2 \sum_{k=n}^\infty f^n_{li} + \sum_{k=1}^{n-1} f^n_{li} \sum_j |p^n_{ij} - p^n_{ij}|.$$ 

The assumption that $M_{\nu}^{(\gamma-1)} < \infty$ immediately implies that the first term in the above expression is $o(n^{-(\gamma-1)})$. As far as the second term is concerned, we may use the inequality

$$\sum_j |p^n_{ij} - p^n_{ij}| \leq \sum_j |p^n_{ij} - \pi_j| + \sum_j |p^n_{ij} - \pi_j|,$$

and it will suffice to estimate the expression

$$\sum_{k=1}^{n-1} f^n_{li} \sum_j |p^n_{ij} - \pi_j|.$$ 

Now, the assumption $M_{\nu}^{(\gamma-1)} < \infty$ implies that $\lim_{k\rightarrow\infty} k^{\gamma-1} \sum_j |p^n_{ij} - \pi_j| = 0$ and, under the assumptions of Lemma 3, $\lim_{m\rightarrow\infty} m^{\gamma-1} \sum_j |p_m^{ij} - \pi_j| = 0$. We may then repeat the argument given at the end of the proof of Lemma 3 to see that the above expression is $o(n^{-(\gamma-1)})$.

\[\Box\]

**Proof of Theorem 1.** The conditions on the ergodic degree of $P$ and on the $P$-order of $\nu$ imply that the assumptions of Lemmas 3 and 4 are satisfied for $\gamma = d + 1 - \epsilon$, $\forall \epsilon > 0$. This gives a rate of convergence $o(n^{-(d-\epsilon)})$, $\forall \epsilon > 0$, that is $O_\epsilon(n^{-d})$. But we can say more. Indeed, the condition $\sum n^{d+1} f^n_{ii} = \infty$ and Lemma 2 entail that $|p^n_{ii} - \pi_i| \cdot n^d$, and thus $||\delta_l P^n - \pi|| \cdot n^d$, decays slower
than any inverse power of \( n \). On the other hand, from the proof of Lemma 4 we see that the condition that \( \nu \) has \( P \)-order strictly larger than \( d \) implies that the norm of \( \sum_{l \neq i} \nu_l (\delta_l P^n - \delta_i P^n) \) is \( O(\epsilon^{-d'}) \), for some \( d' > d \). This prevents from possible cancellations among the two terms in the r.h.s. of (2.15).

\[ \diamond \]

**Remark.** The proof given above brings out the meaning of the condition on the \( P \)-order of the initial distribution \( \nu \). This is related to the fact that the behaviour of \( |p^n_{ij} - \pi_j| \) and hence of \( \|\delta_i P^n - \pi\| \) is necessarily not uniform in the departing state index \( i \). Indeed, according to the above discussion, such uniformity would imply the existence of two positive constants \( C_1, C_2 \) and an integer \( n_0 \), which do not depend on \( i \) and \( l \), such that, for all \( n \geq n_0 \)

\[ C_1 \leq \frac{\sum_{k \geq n} f^k_{li}}{\sum_{k \geq n} f^k_{il}} \leq C_2. \]

This, in turn, would imply that the ratio \( M^{(1)}_{ii}/M^{(1)}_{il} \) satisfies a similar bound.

On the other hand, as already observed, \( \lim_{l \to \infty} (M^{(1)}_{ii}/M^{(1)}_{il}) = 0 \), for all \( i \in S \).

**Proof of Corollary 1.** For any pair \( u \in \ell_\infty(S) \), \( \rho \in \ell_1(S) \) we define \( \overline{\rho u} = (\rho(1)u(1), \rho(2)u(2), \ldots) \) and \( \rho \cdot u = \sum_{i \in S} \rho(i) u(i) \). Thus \( \overline{\rho u} \cdot 1 = \rho \cdot u \), and the unit column vector \( 1 = (1,1,\ldots)^t \) satisfies \( P1 = 1 \). For definiteness and without loss, suppose that \( \mu(u) \mu(v) \neq 0 \). Then we have

\[ |\mu(u(x_n)v(x_0)) - \mu(u(x_0))\mu(v(x_0))| = |\overline{\pi v} P^n \cdot u - (\pi \cdot v)(\pi \cdot u)| \]

\[ = |(\overline{\pi v} P^n - \pi (\overline{\pi v} \cdot 1)) \cdot u| \]

\[ \leq \|u\|_\infty \|v\|_\infty \|\nu P^n - \pi\| \]

where \( \nu \) denotes the normalized \( \ell_1 \) row vector \( \overline{\pi v}/(\pi \cdot v) \). The result now follows putting together Lemma 1 and Theorem 1.

\[ \diamond \]

**3. CONVERGENCE VS ANALYTIC AND SPECTRAL PROPERTIES. AN EXAMPLE.**

As we have seen, the dependence on the departing state \( i \) of the behaviour of \( \|\delta_i P^n - \pi\| \), although not explicitly indicated in Lemma 3, is what makes our assumptions on the \( P \)-order of the initial distribution \( \nu \) necessary.
Moreover, from our discussion it follows that the rate of convergence to zero of \( \|\delta_i P^n - \pi\| \) is connected with the analytic properties of the generating functions \( P_{ij}(z) \) in the vicinity of the singular point \( z = 1 \).

If we now consider \( P \) as a bounded linear Markov operator acting on the Banach space \( \ell_1(S) \), its adjoint \( P^* \) is represented by the transposed matrix acting on the dual space \( \ell_1^* = \ell_\infty \). The resolvent \( R_\lambda(P) := (\lambda I - P)^{-1} \) admits, for \( |\lambda| > \|P\| \), the expansion

\[
\lambda R_\lambda(P) = I + \sum_{n=1}^{\infty} \left( \frac{P}{\lambda} \right)^n
\]

which shows that \( 1 - \delta_{ij} + P_{ij}(z) \) is the \((i, j)\)-element of \( \lambda R_\lambda(P) \), with the identification \( z = 1/\lambda \). This, in turn, indicates that the convergence properties of \( \|\delta_i P^n - \pi\| \), the analytic properties of the functions \( P_{ij}(z) \), and the spectral properties of \( P \) in \( \ell_1(S) \) are intimately connected items. In particular, the dependence of the first two from the state index \( i \) plays an important role in determining nature of the latter, as we shall see in the following example\(^1\).

**Example.** Suppose that \( S = \mathbb{N} \) and the transition matrix is

\[
P = \begin{pmatrix}
p_1 & p_2 & p_3 & \cdots \\
1 & 0 & 0 & \cdots \\
0 & 1 & 0 & \cdots \\
0 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

The space \( \Omega \) is then given by all sequences \( \omega \) satisfying the following condition: given \( \omega_i \) then either \( \omega_{i-1} = \omega_i + 1 \) or \( \omega_{i-1} = 1 \). We shall assume that the probability vector \( p = (p_1, p_2, \ldots) \) has the property g.c.d.\( \{n : p_n > 0\} = 1 \). It then follows that the corresponding chain is aperiodic and recurrent. Let the coefficients \( d_n \) be defined by \( d_n := \sum_{i>n} p_i \), \( (n \geq 0) \). The steady-state equation is \( \pi_n = \sum_{i \in S} \pi_i p_{in} \) and is formally solved by \( \pi_n = \pi_1 d_{n-1}, \ (n \geq 1) \). We also have \( f_{11}^n = p_n \). Consequently, the chain is positive-recurrent if and only

\[^1\text{We shall adopt the convention that a matrix } (t_{ij}) \text{ representing an operator } T \text{ acts from the right, that is through the equations } (Tx)_j = \sum_{i \in S} x_i t_{ij}.\]
if $\sum d_n < \infty$, null-recurrent in the opposite case. In the former case, we have

$$\pi_1 = \left(\sum_{n=1}^{\infty} np_n\right)^{-1} = \left(\sum_{n=0}^{\infty} d_n\right)^{-1}.$$  

Notice that the two probability vectors $\pi$ and $p$ coincide if and only if $p_n = 2^{-n}$. On the other hand, if $p_n \sim n^{-(d+2)} L(n)$ with $L(n)$ a suitable function slowly varying at infinity then the chain has ergodic degree $d$.

**Remark 1.** It is not difficult to realize that the $\tau$-invariant Markov random field $\mu = \mu(P, \pi)$ defined in (1.8), with $P$ and $\pi$ as above, can be viewed as an equilibrium state [Ru] for the continuous potential function $V : \Omega \to \mathbb{R}$ defined as

$$V(\omega) = \log p_{\omega_0} - \log p_{\omega_1} + \log P(\omega_0, \omega_1).$$

**Remark 2.** The Markov chain $P$ is a reference model in renewal theory (see [Se]). In particular, the validity of the renewal limit theorem corresponds to the fact that the chain is ergodic. Several estimates on the remainder term in this limit theorem (which corresponds to the speed of convergence to equilibrium) have been obtained. See [Ro] for very accurate results and also [Se], Chap. 24, for a review. These results can be viewed as particular cases (corresponding to $\nu = \delta_i$ and $u_k = \delta_i^k$, for some $i \in \mathbb{N}$) of Theorem 2.III stated below. Moreover, this example has interesting applications in modelling renewal processes arising in dynamical system theory; a situation which has recently become a standard example being that of Markov interval maps modelling temporal intermittency (see, e.g., [Wa]). A brief discussion on the consequences of the results stated below in the context of dynamical systems theory is given in the Appendix at the end of the paper.

**Theorem 2.** Suppose that the chain $P$ defined above has finite ergodic degree $d > 0$. Then,

I. The generating functions $P_{ij}(z)$ defined in (2.1) are analytic in the open unit disk. For $|z| \leq 1$ the functions $1/P_{ij}(z)$ have only one zero at $z = 1$ which is a non-polar singular point for $P_{ij}(z)$. 

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II. The spectrum $\sigma(P)$ of the Markov operator $P$ acting on $\ell_1(\mathbb{N})$ coincides with the closed unit disk and decomposes as follows: $\sigma_p(P) = \{\lambda : |\lambda| < 1\} \cup \{1\}$ and $\sigma_c(P) = \{\lambda : |\lambda| = 1, \lambda \neq 1\}$.

III. For any bounded vector $\mathbf{u}$ and any initial distribution $\nu = (\nu_i)_{i=1}^\infty \in \ell_1(S)$ s.t. $\nu_i = O(\pi_i)$, the quantity $(\nu P^n - \pi) \cdot \mathbf{u}$ decays as $O(n^{-d})$.
Assume furthermore that $p_n \sim n^{-(d+2)} L(n)$ with $L(n)$ slowly varying at infinity and $u_i = o(1)$, $\nu_i = o(\pi_i)$. Then we have

$$(\nu P^n - \pi) \cdot \mathbf{u} \sim C n^{-d} L(n),$$

with $C = (\pi \cdot \mathbf{u})(\nu \cdot 1)/(d(d + 1)m_1)$.

**Remark 1.** Statement I above holds for any aperiodic Markov chain with finite ergodic degree and is well known. On the other hand, it can be considerably improved by specifying further properties of the probability vector $p$. For instance, if the $p_n$ form a monotonically decreasing sequence $p_1 \geq p_2 \geq \cdots$ satisfying the Kaluza property: $p_n^2 > p_{n+1} p_{n-1}$ (with $p_0 = 1$) then using the last part of Lemma B one can show that the generating functions $P_{ij}(z)$ can be continued meromorphically to the entire $z$-plane with a branch cut along the ray $(1, +\infty)$ (see [Is2]).

**Remark 2.** In the null-recurrent case ($d \leq 0$) the statements corresponding to II and III above are modified as follows (see [A]):

II'. The spectrum $\sigma(P)$ of the Markov operator $P$ acting on $\ell_1(\mathbb{N})$ coincides with the closed unit disk and decomposes as: $\sigma_p(P) = \{\lambda : |\lambda| < 1\}$, $\sigma_c(P) = \{\lambda : |\lambda| = 1, \lambda \neq 1\}$ and $\sigma_r(P) = \{1\}$.

III’. Let $\mathbf{v} = (v_i)_{i=1}^\infty \in \ell_\infty(S)$ be the unique (non-normalized) positive invariant vector for $P$ with $v_1 = 1$ (see [De], Thm 1). Here $v_n = d_{n-1}$. For any vector $\mathbf{u} \in \ell_\infty(S)$ such that $\mathbf{u} \cdot \mathbf{v} < \infty$ and any initial distribution $\nu \in \ell_1(S)$ we have

$$\nu P^n \cdot \mathbf{u} \sim (\nu \cdot 1)(\mathbf{u} \cdot \mathbf{v}) p_{11}^n$$

and $p_{11}^n \cdot n^{-d}$ varies slower than any power of $n$. 

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The proof of Theorem 2 will follow from the points I, II and III discussed hereafter.

I. Generating functions.

First, it is easy to check that all entries of the first \( n \) rows of \( P^n \) are positive, the \( i \)-th row of \( P \) being the \((i + n - 1)\)-th of \( P^n \). More specifically, one sees inductively that for \( n > 1 \), \( i > 1 \), \( j \in \mathbb{N} \),

\[
P^n(i, j) = P^{n-1}(i-1, j).
\] (3.1)

For the generating functions of the \( P^n(i, j) \)'s we then obtain the relations

\[
P_{ij}(z) = \delta_{ij} + z^{i-1} P_{1j}(z), \quad j \geq i > 1
\] (3.2)

\[
P_{11}(z) = z^{i-1} P_{11}(z), \quad i \geq 1
\]

\[
P_{ij}(z) = z^{i-j} + z^{i-1} P_{1j}(z), \quad i > j > 1.
\] (3.3)

It then suffice to study the behaviour of the entries of the first row. They satisfy the recurrence relations \( P^n(1, j) = P^{n-1}(1, 1) P(1, j) + P^{n-1}(1, j+1), \; j \geq 1 \) (recall that \( P^0(i, j) = \delta_{ij} \)). This yields

\[
P^n(1, j) = \sum_{k=1}^{n-1} P(1, k) P^{n-k}(1, j) + P(1, j + n - 1).
\] (3.4)

Putting \( j = 1 \) and recalling that \( P(1, k) = p_k = f_{11}^k \) one gets a particular case of equation (1.2). It hence follows that

\[
P_{11}(z) = \frac{1}{1 - \sum_{n=1}^{\infty} p_n z^n} = \frac{1}{(1 - z)D(z)}
\] (3.5)

where \( D(z) = \sum_{n=0}^{\infty} d_n z^n \). More generally, we get for \( j > 1 \)

\[
P_{1j}(z) = z^{1-j} P_j(z) P_{11}(z)
\] (3.6)

where \( P_j(z) = \sum_{n=j}^{\infty} p_n z^n \). Finally, using (1.1)-(1.2) along with (3.2), (3.4) and (3.5) we obtain

\[
F_{ij}(z) = z^{i-j}, \quad i > j,
\]

\[
F_{ij}(z) = \frac{z^{i-j} P_j(z)}{1 - \sum_{0 < n < j} p_n z^n}, \quad j \geq i.
\]
Remark. As an application of the above formulas one can compute the moments $M^{(\gamma)}_{ij}$ of $P$. For instance, if $d > 1$, computing the second derivative at $z = 1$ of $F_{ii}(z)$ yields

$$M^{(2)}_{ii} = \frac{\pi_1}{\pi_i} \left( M^{(2)}_{11} + 2 \sum_{n=1}^{i-1} np_n \right) \sim \frac{2}{\pi_i^2},$$

where $M^{(\gamma)}_{11} = \sum n^\gamma p_n$ and the last asymptotic equivalence holds for $i \to \infty$.

The proof of the analytic properties of the generating functions $P_{ij}(z)$ now follows a standard path and we therefore omit it.

II. Spectral properties of $P : \ell_1(\mathbb{N}) \to \ell_1(\mathbb{N})$.

From (3.1)-(3.2) we have that the rate of convergence of $P^n(i, j)$ to $\pi_j$ is not uniform in the departing state $i$ (see also the Remark after the proof of Theorem 1). We are now going to see how this fact reflects in the nature of the spectrum of $P$ in $\ell_1$. In particular, the eigenvalue 1 is not isolated, even in the case where the $p_n$'s are exponentially decreasing.

We study the structure of the spectrum of $P$ using the method of generating functions (see, e.g., [VJ]). Setting $x = (x_1, x_2, \ldots)$ and $X(w) = \sum_{n=1}^{\infty} x_n w^n$ the formal solutions to the vector equations

$$(\lambda I - P)x = 0 \quad \text{and} \quad (\lambda I - P^*)x = 0$$

can be written as

$$X(w) = \frac{x_1 w (1 - w) D(w)}{1 - \lambda w} \quad (3.7)$$

and

$$X(w) = \frac{w}{\lambda - w} p \cdot x, \quad (3.8)$$

respectively, where $p \cdot x = \sum_{n \geq 1} x_n p_n$. The equation $1 - \lambda w = 0$ (and its reciprocal $\lambda - w = 0$) entails that the boundary of $\sigma(P)$ (and of $\sigma(P^*)$) is the unit circle. Let us first consider the point $\lambda = 1$. The formal expressions in (3.7) and (3.8) become

$$X(w) = x_1 w D(w) \quad \text{and} \quad X(w) = \frac{w}{1 - w} p \cdot x. \quad (3.9)$$
The latter has the solution \( X(w) = w/(1-w) \) which is the generating function of the unit vector in \( \ell_\infty \). On the other hand, the former is the generating function of an \( \ell_1 \)-vector if and only if \( D(1) < \infty \). Hence, we have that in the positive-recurrent case 1 lies in \( \sigma_p(P) \) (for the null-recurrent chain it lies in \( \sigma_r(P) \)).

More generally, from (3.7) and (3.8) one sees that the open unit disc \( \{ \lambda : |\lambda| < 1 \} \) is always in the point spectrum. Indeed, the function \( (1-w)D(w) = 1 - \sum_{n=1}^{\infty} p_n w^n \) appearing in (3.7) is absolutely convergent for \( |w| \leq 1 \). If \( |\lambda| < 1 \) the same holds true for the function \( w/(1-\lambda w) = \sum_{n=1}^{\infty} \lambda^{n-1} w^n \). Therefore the power series expansion of \( X(w) \), being the product of two absolutely convergent power series, is absolutely convergent at any point of the closed unit disk \( |w| \leq 1 \). More precisely, an easy calculation shows that for \( n \geq 2 \) the coefficient \( x_n \) of \( w^n \) is bounded above by \( |x_1| (|\lambda|^{n-1} + \sum_{k=0}^{n-2} |\lambda|^k p_{n-k-1}) \). This shows that for any \( |\lambda| < 1 \) the function \( X(w) \) is the generating function of a vector \( x \in \ell_1 \). A similar reasoning shows that for any \( |\lambda| < 1 \) the function \( X(w) \) in (3.8) is the generating function of a vector in \( \ell_\infty \), thus proving that \( \{ \lambda : |\lambda| < 1 \} \subseteq \sigma_p(P) \).

We conclude by showing that any \( \lambda \) s.t. \( |\lambda| = 1, \lambda \neq 1 \) lies in \( \sigma_c(P) \). Indeed, take \( \lambda = e^{i\theta} \) with \( 0 < \theta < 2\pi \) and assume that \( (\lambda I - P^*) x = 0 \) for some \( x \in \ell_\infty \). Then the equation in (3.8) gives for the coefficients \( x_n \) the relation \( x_n = e^{-i(n+1)\theta} p \cdot x \). So, if \( x \neq 0 \), then \( p \cdot x \neq 0 \). Multiplying by \( p_n \) and summing over \( n \) we then get \( 1 = \sum_n p_n e^{-i(n+1)\theta} \) which is impossible in our case. If the point \( \lambda \) belongs to the unit circle and is different from \( \lambda = 1 \), then the generating function \( X(w) \) in (3.7) tends to infinity as \( w \) approaches \( \lambda^{-1} \) because \( D(w) \neq 0 \) for any \( |w| = 1 \). But if the solution \( x \) to the equation \( (\lambda I - P) x = 0 \) belongs to \( \ell_1 \), then the generating function \( X(w) \) is absolutely convergent at any point of the unit circle and its absolute value is bounded by \( |x|_1 \). We then see that the point \( \lambda \) does not belong neither to \( \sigma_p(P^*) \) nor to \( \sigma_r(P) \). This means that \( \lambda \in \sigma_c(P) \). In particular, we have found that the eigenvalue 1 is not isolated but is embedded in a continuous spectrum.

III. Convergence properties.

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Next, we discuss the convergence properties of this chain under the hypothesis that it is positive-recurrent. Note that the first part of statement III in Theorem 2 is a consequence of Theorem 1, for 
\[ |(\nu P^n - \pi) \cdot u| \leq \|\nu P^n - \pi\|_1 \cdot \|u\|_\infty. \]
Nevertheless, we shall give an alternative proof which on the one hand yields the actual asymptotic behaviour under the hypotheses stated in the second part of Theorem 2-III and on the other hand allows us to introduce a method which appears to be interesting in its own, for it may be extended to some more general (i.e. non-markovian) mixing Gibbs random fields [Is1].

For \( z \in \mathcal{A} \), consider the matrix \( L_z \) given by

\[
L_z = \begin{pmatrix}
p_1 z & p_2 z & p_3 z & \cdots \\
p_1 z^2 & p_2 z^2 & p_3 z^2 & \cdots \\
p_1 z^3 & p_2 z^3 & p_3 z^3 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

For \( z = 1 \) the matrix \( L_z \) can be viewed as the transition matrix of the process \( r^{(1)}_0, r^{(1)}_1, \ldots \) given by the sequence of times between returns to the state 1 (see (1.4)). The vector equation \( y = L_z x \), takes the generating function form \( Y(w) = p \cdot 1_w X(z) \) where \( Y(w) = \sum_{n=1}^\infty y_n w^n \) and \( 1_w = (w, w^2, w^3, \ldots)^t \). Therefore the power series of \( L_z \) when acting on \( \ell_1(S) \) converges absolutely for any \( z \) in the closed unit disk \( |z| \leq 1 \). In addition, there is a simple algebraic relation between the matrices \( L_z \) and \( P \): let \( Q \) be the transient chain given by the matrix

\[
Q = \begin{pmatrix}
0 & 0 & 0 & \cdots \\
1 & 0 & 0 & \cdots \\
0 & 1 & 0 & \cdots \\
0 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

An easy calculation shows that

\[
(I - zQ)(I - L_z) = (I - zP). \tag{3.10}
\]

This relation entails that if \( u \) is an eigenvector of \( P \) with eigenvalue \( 1/z \), then \( v = u(I - zQ) \) is an eigenvector of \( L_z \) with eigenvalue 1. On the other hand we
already know that $P$, when acting on $\ell_1$, has spectral radius equal to 1 and no eigenvalues on the unit circle besides eventually 1. The choice $z = 1$ gives $u = \pi$ and $v = \pi (I - Q) = \pi_1 p$, as expected.

Let now $u : S \to R$ be a bounded vector and $\nu$ an initial distribution on $S$, which will be assumed to decay not slower than $\pi$ at infinity. The latter condition is equivalent to the assumption made in Theorem 1: if the $P$ has ergodic degree $d > 0$ then $\pi (\nu)$ has $P$-order (at least) $d$.

Let us consider the following generating function,

$$S(z) = \sum_{n=0}^{\infty} z^n (\nu P^n - \pi) \cdot u.$$  

Using (3.10) we get for $|z| < 1$,

$$\sum_{n=0}^{\infty} z^n \nu P^n \cdot u = \nu (I - zP)^{-1} \cdot u = \nu (I - L_z)^{-1} (I - zQ)^{-1} \cdot u.$$  

Now observe that $\nu L_z = (\nu \cdot 1_z) p$. Iterating $n$ times we get $\nu L_z^n = (\nu \cdot 1_z) \lambda_z^{n-1} p$, with $\lambda_z = p \cdot 1_z$, and the above expression becomes

$$(\nu \cdot 1_z) p (I - zQ)^{-1} \cdot u + \nu (I - zQ)^{-1} \cdot u = \frac{(\nu \cdot 1_z) (m_1 \pi_z \cdot u)}{1 - \lambda_z} + \nu_z \cdot u$$

where $m_1 = \pi_1^{-1} = D(1)$, $\nu_z = \nu (I - zQ)^{-1}$ and $\pi_z = \pi_1 p (I - zQ)^{-1}$ (in particular $\pi_z|_{z=1} = \pi$). Therefore a short manipulation yields the expression

$$S(z) = (\pi \cdot u) (\nu \cdot 1) H(z) + R(z)$$

where

$$H(z) = \frac{m_1}{1 - \lambda_z} - \frac{1}{1 - z} = \frac{\sum_{n=0}^{\infty} e_n z^n}{\sum_{n=0}^{\infty} d_n z^n}, \quad \text{with} \quad e_n = \sum_{k > n} d_k,$$

and

$$R(z) = \frac{(m_1 \pi \cdot u) (\nu \cdot 1_z - \nu \cdot 1) + (\nu \cdot 1_z) (m_1 \pi_z \cdot u - m_1 \pi \cdot u)}{1 - \lambda_z} + \nu_z \cdot u.$$
Using the above and Lemma C one sees that if $P$ has ergodic degree $d$ then the coefficients of $H(z)$ decay as $\pi_1 e_n = O_\epsilon(n^{-d})$. It remains to examine the behaviour of $R(z)$. We have
\[
\frac{\nu \cdot 1 - \nu \cdot 1_z}{1 - \lambda_z} = \sum_{n=0}^{\infty} \eta_n z^n, \quad \text{with} \quad \eta_n = \sum_{k>n} \nu_k.
\]
Moreover, a straightforward calculation yields
\[
m_1 \pi_z \cdot u = \sum_{n=0}^{\infty} z^n \left( \sum_{k=1}^{\infty} u_k p_{n+k} \right)
\]
and therefore
\[
\frac{m_1 \pi_z \cdot u - m_1 \pi_z \cdot u}{1 - \lambda_z} = \sum_{n=0}^{\infty} \xi_n z^n, \quad \text{with} \quad \xi_n = \sum_{k=1}^{\infty} u_k d_{k+n}.
\]
In addition,
\[
\nu_z \cdot u = \sum_{n=0}^{\infty} \gamma_n z^n, \quad \text{with} \quad \gamma_n = \sum_{k=1}^{\infty} u_k \nu_{n+k}.
\]
On the other hand,
\[
|\xi_n| \leq \|u\|_\infty \sum_{k>n} d_k = \|u\|_\infty e_n, \quad |\gamma_n| \leq \|u\|_\infty \sum_{k>n} \nu_k = \|u\|_\infty \eta_n.
\]
Reasoning as in the proof of Lemma C we have that if $\sum |\xi_n| < \infty$ then the coefficient of $z^n$ of the product $\sum_{n=0}^{\infty} \xi_n z^n \cdot \sum_{n=0}^{\infty} \nu_n z^n$ is $O(\max\{|\xi_n|, |\nu_n|\})$ (recall that $\nu_i = O(\pi_i)$), otherwise it is $O(\xi_n)$. Therefore, by the first estimate above, it is $O(e_n)$ in both cases.

Comparing all the terms above and using again Lemma C we have found that under our assumptions on the distribution $\nu$ and the vector $u$, the quantity $(\nu P^n - \pi) \cdot u$ decays as $O_\epsilon(n^{-d})$.

We conclude by deriving the exact asymptotic behaviour of $(\nu P^n - \pi) \cdot u$ under the additional hypotheses imposed in the last part of Theorem 2.III. First, if $p_n \sim n^{-(d+2)} L(n)$ then we have $d_n \sim (d + 1)^{-1} n^{-(d+1)} L(n)$ and $e_n \sim d^{-1} (d + 1)^{-1} n^{-d} L(n)$. Lemma C then implies that the coefficients of the power series of $H(z)$ are asymptotically equivalent to $d^{-1} (d + 1)^{-1} D(1) n^{-d} L(n)$. Moreover, if $u_i = o(1)$ and $\nu_i = o(\pi_i)$ then $\xi_n = o(e_n)$ and $\eta_n = o(e_n)$. Again by virtue of Lemma C this prevents from possible cancellations among the various coefficients introduced above and yields the claim.
APPENDIX. Renewal chains and Markov approximations of dynamical systems. Let $(X, \rho)$ be a probability space and $f : X \to X$ be a transformation preserving the probability measure $\rho$ which we assume to be ergodic. Given a measurable subset $E \subset X$, the quantity
\[ e_n = \frac{\rho(E \cap f^{-n}E)}{\rho(E)} \] (A.1)
is the probability to observe a return in $E$ after $n$ iterations of $f$ (for the first time or not). The return time function
\[ R_E(x) = \inf \{ n > 0 : f^n(x) \in E \} \] (A.2)
is defined (and finite) for a.e. $x \in E$. $E$ itself becomes a probability space with measure $\rho_E(A) = \rho(A \cap E)/\rho(E)$. One may then define the induced transformation
\[ f_E(x) = f^{R_E(x)}(x) \] (A.3)
for a.e. $x \in E$. Both $R_E$ and $f_E$ are measurable and in fact it is not difficult to check that $f_E$ preserves the measure $\rho_E$ which is of course ergodic. We denote by $E_n = \{ x \in E : R_E(x) = n \}$ the $n$-th levelset of $R_E$. Notice that the above construction yields a countable partition $A = \{ A_n \}$ of $X$ into the sets
\[ A_n = f^{-(n-1)}(E) \setminus (\bigcup_{k=0}^{n-2} f^{-k}(E)) = \bigcup_{k \geq n} f^{k-n+1}(E_k) \] (A.4)
and, $\rho$ being $f$-invariant,
\[ \rho(A_n) = \sum_{k \geq n} \rho(E_k). \] (A.5)
Therefore we have $1 = \rho(X) = \sum \rho(A_n) = \sum n \rho(E_n)$. It hence follows that
\[ \rho_E(R_E) = 1/\rho(E), \] (A.6)
which is a version of Kac’s formula. Now notice that the number $e_n$ may be rewritten as
\[ e_n = \rho_E(f^{-n}E). \] (A.7)
This expression allows us to give another interpretation of $e_n$. For $x \in E$, let $S_n(x) = \sum_{k=0}^{n-1} R_E(f_E^k(x))$ be the total number of iterates of $f$ needed to observe $n$ returns to $E$ and $N_n(x) = \sum_{k=1}^{n} \chi_E(f^k(x))$ the number of returns up to the $n$-th iterate of $f$. A short reflection gives that $\rho_E(S_k \leq n) = \sum_{r=k}^{n} \rho_E(N_n = r)$.

In addition we have $\rho_E(S_k = n) = \rho_E(S_k \leq n) - \rho_E(S_k \leq n - 1)$ for $k < n$ and $\rho_E(S_n = n) = \rho_E(S_n \leq n)$. A straightforward computation using these observations and (A.7) yields (for $n > 0$): 

$$e_n = \sum_{k=1}^{n} \rho_E(S_k = n) = \rho_E(N_n) - \rho_E(N_{n-1}),$$  

(A.8)

where $\rho_E(N_n)$ denotes the mean of the random variable $N_n$ (we set $N_0 = 0$). Thus, $e_n$ may be regarded as the expected number of returns in $E$ per iteration of $f$ (after $n$ iterations). It then turns out that the validity of the renewal theorem for $e_n$, that is [Se]:

$$e_n \rightarrow \frac{1}{\rho_E(R_E)}, \quad n \rightarrow \infty$$  

(A.9)

is equivalent to the (self-)mixing property for the set $E$, that is $e_n \rightarrow \rho(E)$. A further remark is the following. Let us decompose

$$e_n = \sum_{k=1}^{n} \rho_E(f^l(x) \notin E, 0 < l < k, f^k(x) \in E, f^n(x) \in E)$$  

(A.10)

$$= \sum_{k=1}^{n} \rho_E(E_k) \cdot \rho_E(f^n(x) \in E | R_E(x) = k)$$

Now suppose that the process $\{f^n(x)\}$ “renews” itself each time it returns to $E$. In other words, suppose that the random variables $R_E, R_E \circ f_E, R_R \circ f_E^2, \ldots$ defined on the probability space $(E, \rho_E)$ are mutually independent. In this case we would have

$$\rho_E(f^n(x) \in E, |R_E(x) = k) = \rho_E(f^{n-k}(x) \in E) = e_{n-k}$$

so that the $e_n$’s would satisfy the recurrence equation

$$e_n - p_0 e_0 - p_{n-1} e_1 - \cdots - p_1 e_{n-1} = \begin{cases} 1, & \text{for } n = 0, \\ 0, & \text{for } n > 0, \end{cases}$$  

(A.11)
where \( p_n \equiv \rho_E(E_n) \). This would make \( e_0, e_1, e_2, \ldots \) the renewal sequence associated to the sequence \( p_1, p_2, \ldots \). It has been observed \([\text{Fe2}](\text{see also } [\text{Ki}])\) that any renewal sequence, that is any sequence generated as in (A.11) with \( p_1, p_2 \ldots \) satisfying \( p_n \geq 0 \) and \( \sum p_n \leq 1 \), can arise as the diagonal transition probabilities corresponding to a given state in some Markov chain. In our case, a Markov chain which does the job is precisely that discussed in Section 2, with the \( p_n \)'s as above and \( e_n = p_{11}^n \). Indeed, it is not difficult to realize that the Markov chain in question is that with transition probabilities

\[
p_{ij} = \rho(A_i \cap f^{-1}A_j) / \rho(A_i)
\]

and stationary distribution \( \pi_i = \rho(A_i) \), where the sets \( A_i \) are defined in (A.4).

We point out that under the supposition made above this Markov chain would be isomorphic (mod 0) to the iteration process \( \{f^n(x)\} \). On the other hand, in general the \( R_E \circ f_k^E \) are not mutually independent and we are then led to call the above Markov chain the Markov approximation of the dynamical system \((X, \rho, f)\) w.r.t. the reference set \( E \). Leaving any further detail of this approximation procedure to be discussed elsewhere [Is1], in particular the question of the choice of the reference set \( E \) and that of the “proximity” of \((X, \rho, f)\) and its Markov approximation (see [Che] where this and related questions for a closely related approximation scheme have been dealt with in a far reaching way), we are now going to discuss a simple example (modelling temporal intermittency) where such an approximation is “exact”, in that it is isomorphic to the dynamical system itself.

**Example.** The Markov chain \( P \) studied in Section 2 is isomorphic (mod 0) to the iteration process of the piecewise affine ‘intermittent’ map \( f : [0, 1] \rightarrow [0, 1] \) given by

\[
f(x) = \begin{cases} 
(x - d_1)/\alpha_1, & \text{if } d_1 \leq x \leq d_0 \\
\frac{d_{i-1} + (x - d_i) / \alpha_i}{d_i}, & \text{if } d_i \leq x < d_{i-1}, \ i \geq 2 
\end{cases}
\]

(A.13)

Here the numbers \( d_i = \sum_{l>i} p_l \) are supposed to be all distinct, and \( \alpha_i = p_i/p_{i-1} \), \( i \geq 1 \) (with \( p_0 = 1 \)). In what follows we shall always assume that \( \sum d_i < \infty \).
The partition $\mathcal{A}$ of $[0, 1]$ into the intervals $A_n = [d_n, d_{n-1}]$, $n \geq 1$ is a Markov partition for $f$.

This map is named ‘intermittent’ for, if $\lim \alpha_i = 1$, then $f$ can be viewed as a piecewise affine approximation of a piecewise smooth transformation of $[0, 1]$ which is expanding everywhere but at the fixed point in the origin, where the derivative is equal to one.

Let $\Omega, \pi$ be as in the example of Section 2. One then sees that the map $\phi : \Omega \to [0, 1]$ defined by: $\phi(\omega) = x$ according to $f^j(x) \in A_{\omega_j}$, $j \geq 0$, is a bijection between $\Omega$ and the residual set of points in $(0, 1]$ which are not preimages of 1 w.r.t. the map $f$. Clearly $\phi$ conjugates $f$ with the shift $\tau$ on $\Omega$. Moreover, let $\mu$ be the $\tau$-invariant Markov probability measure on $\Omega$ defined in (1.8) (with $\pi$ and $P$ as above). Then $\rho = \mu \circ \phi^{-1}$ is $f$-invariant and it is easy to see that the $p_{ij}$’s are as in (A.4) – (A.12) with $E = [d_1, d_0]$. Finally, if $f$ is the piecewise affine approximation of a smooth transformation of $[0, 1]$ having a tangency at $x = 0^+$ of order $1+1/\eta$, with $\eta > 0$, then $p_i \sim i^{-(1+\eta)}$ and hence $\alpha_i \sim 1-(1+\eta)/i$. Thus, in order to have $\sum d_i < \infty$ it is necessary that $\eta > 1$, and the corresponding Markov chain $P$ has ergodic degree $d = \eta - 1$.

Let us consider the Perron-Frobenius operator $M : L^1([0, 1], dx) \to L^1([0, 1], dx)$ which satisfies

$$\int_0^1 u \circ f^n(x) v(x) \, dx = \int_0^1 u(x) M^n v(x) \, dx \tag{A.14}$$

for all pairs $u, v \in L^1$. Note that the space $\ell_1(S, p)$ of vectors $u : S \to \mathbb{R}$ such that

$$\|u\|_{1,p} := \sum_{i \in S} |u_i| p_i < \infty$$

is left invariant by the operator $M$, which takes on the matrix representation

$$M(i, j) = \frac{p_i}{p_j} P(i, j), \quad i, j \geq 1. \tag{A.15}$$

The eigenequation $M h = h$ has a solution $h \in \ell_1(S, p)$ given by $h_i = h_1 p_1 d_{i-1}/p_i$, and the vector $p$ satisfies $M^* p = p$. Therefore, recalling that $\pi_i = \pi_1 d_{i-1},$
and putting \( h_1 = \pi_1/p_1 \), we get \( \pi_i = h_i p_i \). One then sees that the vector \( h \in \ell_1(S, \rho) \) corresponds to the (locally constant) density of the absolutely continuous \( f \)-invariant probability measure \( \rho(dx) = h(x) \, dx \), with \( h \in L^1([0, 1], dx) \) and \( h(x) \equiv h_i \) for \( d_i \leq x < d_{i-1} \). Observe that \( \rho(A_i) = \pi_i \). Now, using (A.14) we find

\[
\rho(u \circ f^n v) - \rho(u) \rho(v) = \int_0^1 u(x) [(M^n v h)(x) - \rho(v) h(x)] \, dx \quad (A.16)
\]

Suppose that \( u \) and \( v \) are bounded \( L^1 \)-functions taking constant values \( u_i \) and \( v_i \) on the elements \( A_i \) of the Markov partition \( \mathcal{A} \). We shall denote by \( u = (u_i)_{i=1}^\infty \) and \( v = (v_i)_{i=1}^\infty \) the corresponding vectors in \( \ell_\infty(N) \). Using (A.15), (A.16) and the above observations we get (the notation is as in the proof of Corollary 1),

\[
\rho(u \circ f^n v) - \rho(u) \rho(v) = (\pi v P^n - (\pi \cdot v) \pi) \cdot u. \quad (A.17)
\]

Now set \( u_\infty = \lim u_i \), \( v_\infty = \lim v_i \), and suppose that \( u_\infty \neq 0 \) or \( v_\infty \neq 0 \). Then, setting \( \hat{u} = u - u_\infty 1 \) and \( \hat{v} = v - v_\infty 1 \) have that \( (\pi \cdot \hat{u})(\pi \cdot \hat{v}) \neq 0 \) provided \( \pi \cdot u \neq u_\infty \) and \( \pi \cdot v \neq v_\infty \). Moreover \( \lim \hat{u}_i = 0 \) and \( \lim \hat{v}_i = 0 \). On the other hand we plainly have \( (\pi v P^n - (\pi \cdot \hat{v}) \pi) \cdot \hat{u} = (\pi v P^n - (\pi \cdot v) \pi) \cdot u \). We then see that the conditions \( \rho(u) \equiv \pi \cdot u \neq u_\infty \) and \( \rho(v) \equiv \pi \cdot v \neq v_\infty \) are equivalent to the conditions \( u_i = o(1) \) and \( v_i = o(\pi_i) \) (along with \( (\pi \cdot u)(\nu \cdot 1) \neq 0 \)) assumed in the last statement of Theorem 2, with the identification \( \nu = \pi v / \pi \cdot v \).

The following result is now a direct consequence of Theorem 2 (for related results see [Is2], [LSV], [Mo]; see also [Yo], [Is1] and [Sa] for more general approaches dealing with smooth maps):

**Corollary 2.** Let \( f : [0, 1] \to [0, 1] \) be as in (A.13) and assume that \( \alpha_i \sim 1 - (1 + \eta)/i \) for some \( \eta > 1 \). Then, for any pair of bounded \( L^1 \)-functions \( u, v : [0, 1] \to \mathbb{R} \), locally constant on the Markov partition \( \mathcal{A} \), there is a positive constant \( C = C(u, v) \) such that, for \( n \) large enough,

\[
|\rho(u \circ f^n v) - \rho(u) \rho(v)| \leq C n^{-(\eta-1)}.
\]
Assume furthermore that \( \rho(u) \neq u_\infty \) and \( \rho(v) \neq v_\infty \). Then we have

\[
\rho(u \circ f^n v) - \rho(u) \rho(v) \sim C n^{-(\eta-1)}.
\]

We conclude with a final remark. From the proof of Theorem 2 it follows that if the conditions \( \rho(u) \neq u_\infty \) and \( \rho(v) \neq v_\infty \) are violated, then cancellations may take place to accelerate the convergence rate. A trivial example is obtained by taking \( u, v \) constant on \([0, 1]\). Conversely, one may argue as follows: take \( a > 0 \) and let \( t_a(x) \) be the first entrance time into the set \([a, 1]\). When an orbit falls in a small (compared to \( a \)) neighbourhood of 0 it stays there for a time which can be arbitrarily large before reaching again \([a, 1]\). More precisely, from the above discussion one readily finds that, under the assumptions of Corollary 2,

\[
\rho\{x \in [0, 1] : t_a(x) > n\} \sim C(a) n^{-(\eta-1)}.
\]

Thus, if the condition is satisfied, namely if the average value of the test functions is reached away from the origin, then the term \( \rho(u \circ f^n v) \) cannot approach its asymptotic value \( \rho(u) \rho(v) \) at a rate faster than that given by the statistics of first entrance times given above.

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