ON THE COUNTABLE, MEASURE PRESERVING RELATION INDUCED ON AN HOMOGENEOUS QUOTIENT, BY THE ACTION OF A DISCRETE GROUP

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ABSTRACT. In this paper we consider a countable discrete group $G$ acting ergodically and a.e. free, by measure preserving transformation on an infinite measure space $(\mathcal{X}, \nu)$, with $\sigma$-finite measure $\nu$. Let $\Gamma \subseteq G$ be an almost normal subgroup, with fundamental domain $F \subseteq \mathcal{X}$ of finite measure. We consider the countable, measurable equivalence relation $R_G$ on $\mathcal{X}$ induced by the orbits of $G$, and let $R_G|F$ be its restriction to $F$ (thus two points in $F$ are equivalent if and only if they are on the same orbit of $G$). The $C^*$-algebra groupoid structure corresponding to such a quotient was studied in ([LLN], [RP]).

In this paper we analyze the generators and relations for the above algebra. We prove that the composition formula is an averaged version of the product formula for double cosets and we obtain an algebraic presentation for the quotient algebra.

In the case $G = \text{PGL}_2(\mathbb{Z}[\frac{1}{p}]), \Gamma = \text{PSL}_2(\mathbb{Z}), p$ a prime number, the relation $R_G|F$ is the equivalence relation associated to a free, measure preserving action, on $F$, of a free group with $(p + 1)/2$ generators. The Hecke operators associated to this action of the free group are the Hecke operators associated to the action of double cosets of $G$ on $F$.

Let $G$ be a countable, discrete group acting ergodically, by measure preserving transformation on an infinite measure space $(\mathcal{X}, \nu)$, with $\sigma$-finite measure $\nu$. In addition we assume that $\Gamma \subseteq G$ is an almost normal subgroup, that has a fundamental domain $F$ of finite measure in $\mathcal{X}$. We consider the countable measurable equivalence relation $R_G$ on $\mathcal{X}$ induced by the orbits of $G$,

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and let \( \mathcal{R}_G|F \) be its restriction to \( F \) (thus two points in \( F \) are equivalent if and only if they are on the same orbit of \( G \)). The \( C^* \)-algebra groupoid structure corresponding to such a quotient was studied in ([LLN],[RP]) .

In this paper we analyze the generators and relations for this algebra. In Theorem 4 we establish a precise generators and relations presentation for the above quotient algebra. The abelian subalgebra is the infinite measure space construction of the space associated to a Schlichting ([Sch], [KLM], [Tz]) completion of \( G \), the measure is an invariant measure deduced from the fundamental domain of the action, and the action of the generators is a Fock space type of action on the infinite measure space.

Moreover, we give an explicit description for the action of the generators of \( \mathcal{R}_G|F \) on \( \mathcal{X} \), which is context free (does not depend on \( \mathcal{X} \)) in the sense of symbolic dynamics.

In the case \( G = \text{PGL}_2(\mathbb{Z}[\frac{1}{p}]) \), \( \Gamma = \text{PSL}_2(\mathbb{Z}) \) we use the precise formula of the composition relations for the generators for the \( * \)-algebra associated to the equivalence relation \( \mathcal{R}_G|F \). This proves that for \( G = \text{PGL}_2(\mathbb{Z}[\frac{1}{p}]) \), \( p \geq 3 \) a prime, \( \Gamma = \text{PSL}_2(\mathbb{Z}) \), if the action is a.e. free, then \( \mathcal{R}_G|F \) is treeable and has cost \( \frac{p+1}{2} \). Hence, by the results of Hjorth, the equivalence relation is implemented by the action of the free group with \( \frac{p+1}{2} \) generators on \( F \). Moreover, the radial algebra of the free group (the algebra generated by convolutors in the words on \( F_{\frac{p+1}{2}} \) of equal length have equal weight) will coincide with the Hecke algebra corresponding to \( G \), \( \Gamma \) and to the action on \( \mathcal{X} \).

In particular, we prove that in analogy with the measured equivalence for groups ([Ga]), that \( \text{PGL}_2(\mathbb{Z}[\frac{1}{p}]) \) is infinitesimally orbit equivalent to \( F_{\frac{p+1}{2}} \), \( p \geq 3 \) (see Corollary 7 for the definition of infinitesimally orbit equivalence).

We start with the construction of a family of generators for the relation \( \mathcal{R}_G|F \).

**Proposition 1.** Let \( G \) be a discrete group acting by measure preserving transformations, almost everywhere free \( \mathcal{X} \). Assume that \( \Gamma \) is an almost normal subgroup, having a fundamental domain \( F \subseteq \mathcal{X} \), of finite measure.

Let as above \( \mathcal{R}_G|F \) be the countable equivalence relation on \( F \), defined by requiring that \( x \sim y \) if \( Gx = Gy \).

For \( g \) in \( G \), define the transformation \( \hat{\Gamma}g \) on \( F \) as follows:

Let \( x \) be in \( F \), since \( F \) is a fundamental domain, there exists a unique \( \gamma \in \Gamma \) and \( x_1 \) in \( F \) such that \( gx = \gamma x_1 \).

We define \( \hat{\Gamma}g x = x_1 = \gamma^{-1} gx \). Clearly, \( \hat{\Gamma}g \) depends only on \( \Gamma g \), the left \( \Gamma \)-coset class of \( g \).
Then $\mathcal{R}_G|_F$ is generated by the transformations $\hat{\Gamma}g$, $g$ running through a system of representatives for left cosets of $\Gamma$ (in the sense that $x \sim y$ iff and only if there exists $g \in G$ such that $\hat{\Gamma}gx = y$).

Moreover, $\hat{\Gamma}g$ is not injective, but the number of preimages of each point in the image is bounded by $[\Gamma : \Gamma_g]$, where $\Gamma_g$ is the subgroup $\Gamma \cap g\Gamma g^{-1}$. In addition, if $\Gamma g s_i$ are the left $\Gamma$-cosets contained in $\Gamma g \Gamma$, then every point $x$ in $F$ will show up exactly $[\Gamma : \Gamma_g]$-times in the reunion of the images of the maps $\hat{\Gamma} g s_i$. Moreover, the same is true for preimages (with $[\Gamma : \Gamma_g^{-1}]$ instead of $[\Gamma : \Gamma_g]$).

Note that with the above notations one could define a two cocycle $\alpha = (\alpha_1, \alpha_2) : G \times F$ with values in $\Gamma \times F$, by defining $\alpha(g, x) = (\gamma_1, x_1)$. This cocycle could then be used to construct the adelic action of $G$, which would be implemented by the same cocycle.

**Proof.** The only thing to prove here is the statement about counting images and preimages. But this follows from the fact that the domain $F_0 = \bigcup g s_i F$ is fundamental domain for $\Gamma g^{-1}$ and it covers $[\Gamma : \Gamma_g]$ times the set $F$. Here $s_i$ are coset representatives for $\gamma g$ into $\Gamma$. $\square$

The transformations $\hat{\Gamma} g$, $g \in G$ have a natural composition rule. The composition rules are similar to the multiplication rules from the Hecke algebra, just that they have to be taken on pieces.

**Proposition 2.** With the previous hypothesis, let $g_1, g_2$ be arbitrary elements in $G$. Assume that $r_j$, $j = 1, 2, \ldots, [\Gamma : \Gamma_{g_1}^{-1}]$ are the right coset representatives for $\Gamma_{g_1}^{-1}$ in $\Gamma$. Thus $\Gamma g_1 = \bigcup \Gamma_{g_1}r_j$, and hence $\Gamma g_1 \Gamma = \bigcup \Gamma g_1 r_j$.

Let $A^r_{g_1, g_2}$ be the subset of $F$ defined as

$$\{ f \in F \mid r_j g_2 f \in \Gamma_{g_1}^{-1} F \} = (r_j g_2)^{-1} \Gamma_{g_1}^{-1} F \cap F, \quad j = 1, 2, \ldots, [\Gamma : \Gamma_{g_1}^{-1}].$$

Let $\chi_{A^r_{g_1, g_2}}$ be the characteristic functions of these sets.

Then

$$\hat{\Gamma} g_1 \hat{\Gamma} g_2 = \sum_j \hat{\Gamma} g_1 r_j g_2 \chi_{A^r_{g_1, g_2}}.$$

Moreover, the sets $A^r_{g_1, g_2}$, $j = 1, 2, \ldots, [\Gamma : \Gamma_{g_1}^{-1}]$ are a partition of unity of $F$.

**Proof.** Since $G$ acts freely almost everywhere, we may simply work on an orbit of $G$. So we may assume that $X = G$, and that $F = S$ is a system of
coset representatives for $\Gamma \setminus G$. (The only point where the initial data would enter would be in the measure of the sets in the partitions from the previous proposition.)

Given $s \in S$ and two left cosets $\Gamma g_1, \Gamma g_2$ we calculate the composition $\Gamma g_1 \Gamma g_2$.

Thus assume that $g_2s = \gamma_2s_2$ for some $\gamma_2 \in \Gamma$, $s_2 \in S$ and thus $\Gamma g_2s = s_2$.

Then $s_2 = \gamma_2^{-1}g_2s$ and hence

$$g_1(\Gamma g_2s) = g_1s_2 = g_1\gamma_2^{-1}g_2s.$$ 

We need to identify to which coset of $\Gamma g_1^{-1}$ the element $\gamma_2^{-1}$ belongs. Assume thus that $\gamma_2^{-1}$ belongs to $\Gamma g_1^{-1}r_j$ for some fixed $j$, thus assume $\gamma_2^{-1} = \theta r_j$ for some $\theta \in \Gamma g_1^{-1}$. Then $g_1\gamma_2^{-1}g_2s$ is further equal to $(\gamma_1\theta\gamma_1^{-1})g_1r_jg_2s$. But $\theta' = \gamma_1\theta\gamma_1^{-1}$ belongs to $\Gamma g_2 \subseteq \Gamma$ (since $g\Gamma g^{-1} = g(\Gamma g^{-1})g^{-1} = \Gamma g$). Thus

$$g_1(\Gamma g_2s) = g_1\gamma_2^{-1}g_2s = \theta'(g_1r_jg_2)s.$$ 

On the other hand, there exists $\gamma_1 \in \Gamma$ such that $g_1r_jg_2s = \gamma_1s_1$, $s_1 \in S$. Thus $\Gamma g_1r_jg_2s = s_1$. From the above formula we conclude

$$g_1(\Gamma g_2s) = \theta'\gamma_1s_1$$

and hence

$$\tilde{\Gamma}g_1 \Gamma g_2 s = s_1 = \Gamma g_1r_jg_2s.$$

We have to determine the conditions that we have to impose $s$, so that $\gamma_2$ belongs to $\Gamma g_1^{-1}r_j$. But the relation defining $s_2$ was

$$g_2s = \gamma_2s_2.$$ 

Thus for $\gamma_2^{-1}$ to be in $\Gamma g_1^{-1}r_j$, which is equivalent to $\gamma_2 \in r_j^{-1}\Gamma g_1^{-1}$, is necessary and sufficient that $g_2s$ belongs to $r_j^{-1}\Gamma g_1^{-1}S$.

Thus $s$ should belong to $A_{g_1,g_2}^j = g_2^{-1}r_j^{-1}\Gamma g_1^{-1}S \cap S$. Thus the relation * holds on $A_{g_1,g_2}^j$.

Since the cosets $r_j^{-1}\Gamma g_1^{-1}$ are disjoint and $S$ is a set representatives, it follows that $\gamma S \cap S = \phi$ for all $\gamma \neq e$ and hence $\gamma_1S \cap \gamma_2S = \phi$ if $\gamma_1 \neq \gamma_2$. It follows that that the intersection $r_j^{-1}\Gamma g_1^{-1}S \cap r_k^{-1}\Gamma g_1^{-1}S$ is void, if $j \neq k$.

From here it follows that the sets $A_{g_1,g_2}^j$, $j = 1, 2, \ldots, [\Gamma : \Gamma g_1^{-1}]$ are forming a partition of $S$ (since $\bigcup A_{g_1,g_2}^j = g\Gamma S \cap S = gGS \cap S = G \cap S = S$).
Note that obviously the decomposition
\[ \tilde{\Gamma}g_1 \cdot \tilde{\Gamma}g_2 = \sum_j \tilde{\Gamma}g_1 r_j g_2 x_{g_2^{-1} \Gamma g_1^{-1} S \cap S} \]
depends only on the class \( g_1 \) of \( g_1 \) (as \( g_1^{-1} = g_1^{-1} \Gamma g_1 \cap \Gamma \)).

The formula does not depend either of the choice the representative \( g_2 \) in \( \Gamma g_2 \), since changing \( g_2 \) into \( \gamma' g_2 \) would have the effect of permuting the sum, since
\[ \Gamma g_1^{-1} r_j \gamma' = \Gamma g_1^{-1} r_{\pi_{\gamma'}(j)} \]
for some partition \( \pi_{\gamma'} \) of \( \{1, 2, \ldots, [\Gamma : \Gamma g_1^{-1}]\} \). By using the methods from the previous proof, we prove the following.

**Lemma 3.** Let \( S \) be as in the proof of the previous lemma. Let \( g \in G \) and let \( \alpha_i \) be a system of right representatives for cosets of \( \Gamma g \) in \( \Gamma \) (that is \( \Gamma = \bigcup \Gamma \alpha_i \), or \( \Gamma g \Gamma = \bigcup \Gamma g \alpha_i \)). Then for every \( \alpha_i \) the image through \( \tilde{\Gamma}g \) of the set \( g^{-1} \Gamma g \alpha_i S \cap S = \{s \in S \mid gs \in \Gamma g \alpha_i S\} \) is \( \alpha_i^{-1} \Gamma g S \cap S = B_{\alpha_i, g} \).

Note that as before the sets \( A_{\alpha_i, \Gamma g} \) are partition of \( S \), while the sets \( B_{\alpha_i, g} \) are not a partition in general; they may have overlaps in \( S \).

Moreover, \( \tilde{\Gamma}g|_{A_{\alpha_i, \Gamma g}} \) is bijective and the inverse is \( \tilde{\Gamma}g^{-1} \alpha_i \), acting on \( \alpha_i^{-1} \Gamma g S \cap S \).

**Proof.** The fact that the image through \( \tilde{\Gamma}g \) of the set \( A_{\alpha_i, \Gamma g} \) is \( \alpha_i^{-1} \Gamma g S \cap S \), is proved as follows.

Let \( s \) be an element in
\[ A_{\alpha_i, \Gamma g} = g^{-1} \Gamma g \alpha_i^{-1} S \cap S = \{s \in S \mid gs \in \Gamma g \alpha_i S\} \]
Thus \( gs = \theta s_1 \) for some \( s_1 \in s, \theta \in \Gamma g \), but then \( s_1 = \alpha_i = \theta g \) which belongs to \( \alpha_i^{-1} \Gamma g S \cap S \).

To verify the inverse formula we have to calculate \( \tilde{\Gamma}g^{-1} \alpha_i \tilde{\Gamma}g \). By the previous proposition we have to chose \( r_j \), a system of right representatives for \( \Gamma (g^{-1} \alpha_i)^{-1} \) in \( \Gamma \), that is \( \Gamma = \bigcup_i \Gamma (g^{-1} \alpha_i)^{-1} r_j \). But \( \Gamma (g^{-1} \alpha_i)^{-1} = \Gamma \alpha_i^{-1} g = \alpha_i^{-1} \Gamma g \alpha_i \), thus \( \Gamma = \bigcup (\alpha_i^{-1} \Gamma g \alpha_i) r_j \).

Then the previous formula gives that
\[ \tilde{\Gamma}g^{-1} \alpha_i \tilde{\Gamma}g = \sum_j \tilde{\Gamma}g^{-1} \alpha_i r_j g x_{(g^{-1} r_j^{-1} \Gamma \alpha_i^{-1} g S \cap S)} \].
In the above formula, we get the identity exactly when \( \alpha_ir_j \) belongs to \( \Gamma_g \). Then the identity term will occur on the set
\[
g^{-1}r_j^{-1}\Gamma^{-1}_{\alpha_i^{-1}}gS \cap S = g^{-1}r_j^{-1}\alpha_i^{-1}\Gamma_g\alpha_iS \cap S,
\]
when \( \alpha_ir_j \) belongs to \( \Gamma_g \). But in this case the set is
\[
g(\alpha_ir_j)^{-1}\Gamma_g\alpha_iS \cap S = g^{-1}\Gamma_g\alpha_iS \cap S.
\]
Thus the inverse of \( \Gamma g \) on \( g^{-1}\Gamma_g\alpha_iS \cap S \) is \( \Gamma g^{-1}\alpha_i \).

It is easy to see that this formula is consistent, that is if we apply formula this to \( \Gamma g^{-1}\alpha_i \) on \( \alpha_i\Gamma_gS \cap S \) we get the same result.

**Observation 4.** In general if we want to compute the inverses of all \( \Gamma g^{-1} \), where \( r_j \) are a system of left coset representatives for \( \Gamma^{-1}_g \) in \( \Gamma \) (thus \( \Gamma = \bigcup \Gamma_{g^{-1}r_j} \)) and then \( \Gamma g\Gamma = \bigcup \Gamma gr_j \). Then by the above result (and since \( \Gamma gr_j = \Gamma_g \)) we obtain that the inverse of \( \Gamma g^{-1} \) on the set \( (gr_j)^{-1}\Gamma_g^{-1}\alpha_i^{-1}S \cap S = \Gamma g^{-1}\alpha_i^{-1}S \cap S \) is \( \Gamma g^{-1}\alpha_i = \Gamma g^{-1}\alpha_i \) acting on the set \( \alpha_i^{-1}\Gamma_g^{-1}r_jS \cap S = \alpha_i^{-1}g\Gamma_{g^{-1}r_j}S \cap S \). (Here \( \Gamma = \bigcup \Gamma_g\alpha_i = \bigcup \Gamma_{g^{-1}r_j} \).) Note that the sets \( g^{-1}r_j^{-1}\Gamma_g\alpha_iS \cap S \) are disjoint after \( i \), while the sets \( \alpha_i^{-1}g\Gamma_{g^{-1}r_j}S \cap S \) are disjoint after \( j \).

In the following we want to describe the algebra \( B \) of subsets of of \( F \) that are invariated by the transformations \( \Gamma g \) taken on their domains of bijectivity. (Here again, for simplicity, as in the proof of Proposition 2 we work directly on an orbit of \( G \) and thus an instead of the set \( F \) we simply work on a subset \( S \) in \( G \) of \( \Gamma \)-cosets representatives).

Clearly, \( B \) contains first all sets of the form \( \alpha_i^{-1}\Gamma_gS \cap S \), and \( g^{-1}\Gamma_g\alpha_iS \cap S \), for all \( g \in G \), \( \Gamma = \bigcup \Gamma_g\alpha_i \).

The sets \( g^{-1}\Gamma_g\alpha_iS \cap S \) are easily written in the form
\[
\{ s \in \Gamma_g\alpha_iS \cap S \}
\]
while the sets \( \alpha_i^{-1}\Gamma_gS \cap S = \alpha_i^{-1}g\Gamma_{g^{-1}}S \cap S \) are
\[
\{ s \in S \mid \alpha_i^{-1}gs \in \Gamma_{g^{-1}}S \}.
\]
It is clear that by decomposing \( \Gamma g \) with respect to smaller normal subgroup \( \Gamma_0 \subseteq \Gamma_g \) as \( \Gamma_g = \bigcup \gamma_a\Gamma_0 \), the sets in formula (1) are also of the form
\[
\{ s \mid gs \in \Gamma_\alpha\Gamma_0 S \cap S \}.
\]
In the following theorem we describe explicitly, in terms of the generators introduced in Proposition 2, the structure of the equivalence relation. We will also give an abstract description of the Borel algebra $B$.

**Theorem 5.** Let $K$ be the profinite completion of $\Gamma$ with respect to the lattice $L$ generated by the subgroups of the form $\sigma \Gamma \sigma^{-1} \cap \Gamma$, $\sigma \in G$ and let $S$ be the disjoint union of the cosets $KgK$, where $\Gamma g \Gamma$ runs over a system of representatives for cosets of $\Gamma$ in $G$. Then $S$ is locally compact, totally disconnected group with a canonical Haar measure.

Let $Y$ be the infinite product of the measure spaces $S$. Let $F$ be the fundamental domain of $\Gamma$ in $X$ considered above. We define a measure $\nu_F$ on subsets of $Y$ which are products of sets with a finite number of factors, as follows: let $C_i$ be closures, in $S$, of the cosets $C_i = g_i \Gamma_i$, $i = 1, 2, \ldots, n$, where $g_i$ are elements of $G$ and $\Gamma_i$ are subgroups in $L$. We define

$$\nu_F(C_1 \times C_2 \times \ldots \times C_n) = \nu(F \cap C_1 F \cap C_2 F \cap \ldots \cap C_n F).$$

Then, the algebra of the quotient relation of the action of the group $G$ on $X$ will be generated by the elements $\hat{\Gamma}g$, $g \in G$ and the algebra $L^\infty(Y, \nu_F)$.

The generating elements $\hat{\Gamma}g$ are subject to the relations in Proposition 2. More precisely for $g_1, g_2 \in G$, assuming that $r_j$, $j = 1, 2, \ldots, [\Gamma : \Gamma g^{-1}]$ are the right coset representatives for $\Gamma g^{-1}$ in $\Gamma$, we have

$$\hat{\Gamma}g_1 \hat{\Gamma}g_2 = \sum_j \Gamma g_1 r_j g_2 \chi_{(r_j g_2)^{-1} \Gamma g^{-1}}.$$
\((\hat{\Gamma}g)\chi_{g^{-1}\Gamma g\alpha_i}\) maps the characteristic function of the set
\[g^{-1}r_j\alpha_i\Gamma_1 \times g_1^{-1}\Gamma_0 \times g_2^{-1}\Gamma_0 \times \ldots \times g_n^{-1}\Gamma_0\]
on onto the characteristic function of the set
\[\alpha_i^{-1}r_j^{-1}\Gamma_1 g \times \alpha_i^{-1}r_j^{-1}gg_1^{-1}\Gamma_0\].

Moreover, the pairing
\[\hat{\Gamma}g_2^* \cdot \hat{\Gamma}g_1\]
for \(g_1, g_2 \in G\), is a representation of the pairing \(C(S\setminus K) \times C(K/S)\) taking values in the vector space \(C(g_1Kg_2 \mid, g_1, g_2 \in G)\).

**Proof.** It is sufficient to describe the action of the generators \(\hat{\Gamma}g, g \in G\) on a domain of bijectivity. For a fixed \(g \in G\), we let \((\alpha_i)\) be a system of representatives for \(\Gamma g \subseteq \Gamma\), that is \(\Gamma = \bigcup \Gamma_g\alpha_i\) and consider the restriction of \(\hat{\Gamma}g\) to
\[g^{-1}\Gamma_g\alpha_i F \cap F = \{s \in F \mid gs \in \Gamma_g\alpha_i F\}\].

Let \(\Gamma_0\) a subgroup of \(\Gamma\) and let
\[A_{g_1,\ldots,g_n,\Gamma_0} = \{s \in F \mid g_is \in \Gamma_0 F\}\],
be one of the generators of the Borel algebra \(\mathcal{B}\) introduced above.

We will consider the Borel algebra generated by subsets of \(F\) of the form
\[A_{g_1,\ldots,g_n,\Gamma_1,\ldots,\Gamma_n} = \{s \in F \mid g_1s \in \Gamma_1 F, \ldots, g_ns \in \Gamma_n F\}\],
where \(g_1, g_2, \ldots, g_n \in G, \Gamma_1, \ldots, \Gamma_n\) are subgroups of \(\Gamma\) of finite index, in the directed subset \(\mathcal{L}\) of subgroups of \(\Gamma\).

It is clear that by dividing each of the subgroups \(\Gamma_i\) into cosets with respect to a smaller common subgroup \(\Gamma_0\), we arrive at the situation where we only work with the Borel subalgebra generated by subsets of \(F\) of the form
\[A_{g_1,\ldots,g_n,\Gamma_0} = \{s \in F \mid g_is \in \Gamma_0 F\}\].

We may also reduce to the case in which we work only with \(g_i\) in a fixed system \(\mathcal{R}\) of representatives for cosets of \(\Gamma\) in \(G\). Let \(r_j\) be a system of representatives for \(\Gamma_0\) in \(\Gamma\), thus \(\Gamma = \bigcup r_j\Gamma_0, j = 1, 2, \ldots, [\Gamma : \Gamma_0]\).

Then the following sets:
\[A_{g_1,\ldots,g_n,r_j_1,\ldots,r_j_n,\Gamma_0} = \{s \in F \mid g_is \in r_j_i\Gamma_0 F\}, i = 1, 2, \ldots, n\},
where \(g_1, \ldots, g_n\) run over the system of representatives \(\mathcal{R}\), \(j_1, \ldots, j_n\) run over \(\{1, 2, \ldots, [\Gamma : \Gamma_0]\}\) and \(\Gamma_0\) runs over \(\mathcal{L}\), are a system of generators of the
Borel algebra $\mathcal{B}$ of subsets of $F$ generated by the partial transformations and
the sets appearing in the composition formula.

Since $\hat{\Gamma} g$ is bijective on the set
\[(3) \quad \{ s \in F \mid gs \in \Gamma_1 \alpha_i F \},\]
we let $\Gamma_1$ a smaller normal subgroup (we will determine later how small it has
to be taken), and decompose $\hat{\Gamma} g = \bigcup r_j \Gamma_1$.

Hence the set in formula (3) becomes the disjoint union of the sets
\[\{ s \in F \mid gs \in r_j \alpha_i \Gamma_1 F \}.\]

We want to determine the image through $\hat{\Gamma} g$ of the set
\[(4) \quad \{ s \in F \mid gs \in r_j \alpha_i \Gamma_1 F \} \cap \{ s \mid g_i s \in \Gamma_0 F \}.
\]
Note that because of normality of $\Gamma_1$, we have that $r_j \alpha_i \Gamma_1 = \Gamma_1 r_j \alpha_i$.

We fix $f$ an element in the set (4). Then $gf$ is of the form $\theta_1 r_j \alpha_i f_1$
with $\theta_1$ in $\Gamma_1$ and $f_1 \in F$. Moreover, $g_i f \in \Gamma_0 F$. Then $\hat{\Gamma} g f = f_1$,
and $f = g^{-1} \theta_1 r_j \alpha_i f_1$. The condition that $g_i f \in \Gamma_0 F$ then translates into
$g_i (g^{-1} \theta_1 r_j \alpha_i) f_1 \in \Gamma_0 F$, which is the same as
\[f_1 \in \alpha_i^{-1} r_j^{-1} \theta_1^{-1} g g_i^{-1} \Gamma_0 F.\]
We take $\Gamma_1$ so small that $\Gamma_1 g g_i^{-1} = g g_i^{-1} \Gamma_2$, for all $i$, for a fixed subgroup $\Gamma_2$
of $\Gamma_0$. Hence the condition on $f_1$ is that
\[f_1 \in \alpha_i^{-1} r_j^{-1} g g_i^{-1} \Gamma_0 F.\]
We also have to write down the condition that $f_1$ belongs to the image of
$\{ s \mid gs \in r_j \alpha_i \Gamma_1 F \}$ through $\hat{\Gamma} g$. But for all $j$, we have $f = g^{-1} \theta r_j \alpha_i f_1$ so
\[f_1 = \alpha_i^{-1} r_j^{-1} \theta g f\]
and hence $f_1 \in \alpha_i^{-1} r_j^{-1} \Gamma_2 g F \cap F \subseteq \alpha_i^{-1} \Gamma g \Gamma \cap F$. Thus
the condition on $f_1$ is that it belongs to
\[\{ s \in F \mid r_j \alpha_i s \in \Gamma_1 g F \} \cap \bigcup_{i=1}^{n} \{ g_i g^{-1} r_j \alpha_i s \in \Gamma_0 F \}.\]
Note that the first set in the intersection is also described by the formula
\[\{ s \in F \mid g^{-1} r_j \alpha_i s \in g \Gamma_1 g^{-1} F \},\]
where $g \Gamma_1 g^{-1} \subseteq \Gamma$ since $\Gamma_1 \subseteq \Gamma g^{-1}$.

When translating this into the terms of measure space $L^\infty(\mathcal{Y}, \nu_F)$ we obtain the statement in the theorem.

For the last part of the proof note that if $\pi_{Koop}$ is the Koopman representation of $G$ on $L^2(\mathcal{X}, \nu)$ and $P$ is the projection operator of multiplication
on $L^2(\mathcal{X}, \nu)$ with the characteristic function of $\chi_F$, whose image is $L^2(F, \nu)$, then we have the formula
\[ \hat{\Gamma} g = \sum_{\theta \in \Gamma_g} P_{\pi_{Koop}}(\theta) P, \quad g \in G. \]
The required multiplicativity property was then proved in the paper [Ra1].

In the following proposition we use the properties of the generators of the algebra associated to the equivalence relation, and their multiplication formula from Proposition 2 in the case $G_p = \text{PGL}_2(\mathbb{Z}_p)$, $\Gamma = \text{PSL}_2(\mathbb{Z})$. We obtain very specific formula for the composition of the generators. We also prove the relation of equivalence on the fundamental domain is induced by an a.e. free action of a free group with $(p + 1)/2$ generators, in such a way that the Hecke algebra corresponding to the inclusion $\Gamma \subseteq G$ of Hecke operators acting on $L^2(F, \nu)$ coincides with the image, through the Koopmann representation of the free group, of the radial algebra in the free group.

We are indebted to the anonymous referee of a first version of this paper for pointing out to the author that a related argument was considered in the paper [Ad].

**Theorem 6.** Let $p \geq 3$, be a prime number. We let the group $G$ be $G = G_p = \text{PGL}_2(\mathbb{Z}_p)$ and let $\Gamma = \text{PSL}_2(\mathbb{Z})$. We assume that $G$ acts a. e. freely and ergodically, by measure preserving transformation, on an infinite measure space $(\mathcal{X}, \nu)$, with $\sigma$-finite measure $\nu$. In addition we assume that the action of $\Gamma \subseteq G$ admits a fundamental domain $F$, of finite measure, in $\mathcal{X}$.

We consider the countable measurable equivalence relation $\mathcal{R}_G$ on $\mathcal{X}$ induced by the orbits of $G$, and let $\mathcal{R}_G|F$ be its restriction (see [Ga]) to $F$. Thus two points in $F$ are equivalent if and only if they are on the same orbit of $G$.

Then the set of partial transformations introduced in Proposition 2,
\[ A = \{ \hat{\Gamma}_g |_{\Gamma^p \cap F \cap F} \} \text{ if } \Gamma g \text{ runs through the cosets in } \Gamma \sigma_p \Gamma, \text{ and } \Gamma = \bigsqcup \Gamma_s \Gamma \text{ is closed under taking the inverse (this is also valid in general if } \Gamma \sigma \Gamma = \Gamma \sigma^{-1} \Gamma), \text{ for all } \sigma \in G \}. \text{ Note that the set } A \text{ has cardinality } p + 1. \]

Moreover, any relation $\hat{\Gamma}_g_1 \hat{\Gamma}_g_2 \ldots \hat{\Gamma}_g_nf = f$, for some $f \in F$, is possible if and only if one of the $\hat{\Gamma}_g_i$ is canceled with its consecutive inverse. In particular, the equivalence relation $\mathcal{R}_G|F$ is treeable, of cost $\frac{p+1}{2}$ (its generators and inverses being $\hat{\Gamma}_g$, with $\Gamma g \subseteq \Gamma \sigma_p \Gamma$).
By Hjorth theorem ([Hj]), there exists a free group factor $F_{\frac{p+1}{2}}$ acting freely on $F$, whose orbits are the equivalence relation in $R_G|F$.

In addition, we can arrange that that generators of $F_{\frac{p+1}{2}}$ are built from pieces of the transformations of $\hat{\Gamma} g$, $\Gamma g \subseteq \Gamma \sigma_p \Gamma$, glued together into bijective transformations. Hence the radial elements in group algebra of the group $F_{\frac{p+1}{2}}$ (that is the selfadjoint elements $\chi_n = \text{sum of words in the generators of length } n, n \in \mathbb{N}$) have the property that $\chi_n$ as an operator on $L^2(F)$ coincides with the Hecke operator $T_{\sigma_p n}$, where

$$\sigma_p n = \left( \begin{array}{cc} p^n & 0 \\ 0 & 1 \end{array} \right).$$

Consequently the Hecke operators associated with $\Gamma \subseteq G$ and an action of $G$ with fundamental domain for $\Gamma$ are similar to the Hecke operators considered in [LPS] and associated to the action of a free group on a sphere.

**Proof.** To prove that treability, recall ([Serre]) that the action of $\Gamma \sigma_p \Gamma$ on the cosets in $\Gamma \setminus G_p$ copies exactly the action of the radial algebra on the elements of the free group $F_{\frac{p+1}{2}}$. By this we mean that the Cayley tree of $F_{\frac{p+1}{2}}$ with origin the neutral element $e$ is identified with $\Gamma \setminus G_p$ (with elements of length $n$ corresponding to cosets in $[\Gamma \sigma_p \Gamma] \Psi(w)$). In this way the multivalued action of $\chi_1 = \sum_{i=1}^{\frac{p+1}{2}} s_i s_i^{-1}$, where $s_i$ are the generators of $F_{\frac{p+1}{2}}$, on $F_{\frac{p+1}{2}}$, corresponds bijectively to multiplication by $\Gamma \sigma_p \Gamma$ in the space of cosets. (More precisely, there exists a bijection $\Psi : F_{\frac{p+1}{2}} \rightarrow \Gamma \setminus G_p$ such that $\Psi$ preserves length of coset and the set $\Psi(\{s_i, s_i^{-1}; i = 1, 2, \ldots, \frac{p+1}{2}\}w)$ consists of the set of cosets in $[\Gamma \sigma_p \Gamma] \Psi(w)$, for any word $w$ in $F_{\frac{p+1}{2}}$).

Thus in any sequence $\hat{\Gamma} g_1 \hat{\Gamma} g_2 \ldots \hat{\Gamma} g_n f = f, f \in F$ with $\Gamma g_1, \Gamma g_2, \ldots, \Gamma g_n$ cosets in $[\Gamma \sigma_p \Gamma]$, which will correspond to some cancellation,

$$\gamma g_1 r_j g_2 r_j \ldots g_{n-1} r_j g_n f = f,$$

this will be possible if and only if we have successive cancellations of the form $g_j r_j g_{j+1} \in \Gamma$, which correspond to multiply in the above product, the transformation $\hat{\Gamma} g_j$ with its inverse.

Thus the equivalence relation is treeable, with generators and their inverses being the transformations $\hat{\Gamma} g$, restricted to its domain of bijectivity.

Since this set is closed under inverses and the total area of the domains is $p + 1$ it follows that the cost of the relation is $\frac{p+1}{2}$. (This is easily extended
for $p = 2$.) By Hjorth theorem [Hj], we can find a free group $F_{p+1}$ whose orbits define the relation $R_{G_p}|F$.

Since we have that the set of point images and set of point preimages of the $p + 1$ elements elements in the set $A$ have cardinality exactly $p + 1$ in the set $F$, and $A$ is closed to the inverse operation, it follows that there is no obstruction into measurably breaking the elements in $A$ into partial transformations, and recomposing them into a set $A'$ of $p+1$ bijective transformations, such that the set $A'$ is closed to the operation of taking the inverse. The new elements in the set $A'$ preserve the property of having no relations (except the cancellations due to the inverse operation), and hence the elements in $A'$ generate a free group action. The radial algebra element $\chi_1$ is the sum of the transformation operators induced by the elements in the set $A'$ and this is equal to the analogous sum for the elements in $'$ But this sum is the Hecke operator $[\Gamma \sigma_p \Gamma]$ acting on $F$. □

We introduce the following (definition) corollary of the preceding discussion:

**Corollary 7.** Let $H_1, H_2$ be two discrete groups. We will say that $H_1$ is an infinitesimal orbit reduction of $H_2$, if there exist an infinite ergodic measure preserving free a.e. action of $H_2$, and $F$ a finite measure subset of $Y$, such that if $R_{H_2}$ the countable equivalence relation induce on $Y$ by the orbits of $H_2$, then $R_{H_2}|F$ is orbit equivalent to an action of $H_1$.

Thus, we proved that $F_{p+1}$ is infinitesimal orbit equivalent to $\text{PGL}_2(\mathbb{Z}[[p]])$.

Finally, we describe a computation for the matrix coefficients of the Hecke operators, that is related to the previous considerations. The context is the same as above.

**Theorem 8.** Let $G$ be a countable, discrete group acting ergodicaly, by measure preserving transformation on an infinite measure space $(\mathcal{X}, \nu)$, with $\sigma$-finite measure $\nu$. In addition we assume that $\Gamma \subseteq G$ is an almost normal subgroup, that has a fundamental domain $F$ of finite measure 1, in $\mathcal{X}$. Let $\pi = \pi_{\text{Koop}}$ be the Koopman (see e. g. [Ke]) representation of $G$ into $L^2(\mathcal{X}, \nu)$.

On the $\Gamma$ invariant functions on $\mathcal{X}$ we introduce the (Petersson) scalar product given by integration over $F$. Thus the Hilbert space of $\Gamma$– invariant functions on $\mathcal{X}$, which we are denoting by $L^2(\mathcal{X}, \nu)^{\Gamma}$, is identified with $L^2(F, \nu)$.

We assume that the characteristic function $\chi_F$ is cyclic for $\pi$. We denote the Hecke operator on $L^2(F, \nu)$, associated to a double coset $\Gamma \sigma \Gamma$ by $T^{\Gamma \sigma \Gamma}$,
σ ∈ G. For σ ∈ G consider the Γ-invariant function (depending on the coset \( \Gamma \sigma \)) defined by the formula:

\[
\hat{\chi}_{\Gamma \sigma} = \sum_{\gamma \in \Gamma} \pi(\gamma) \chi_{\sigma F}.
\]

Note that the notation might be misleading since \( \hat{\chi}_{\Gamma \sigma} \) is not a characteristic function, although it is a step function.

Let \( S \) be the linear subspace of \( L^2(\mathcal{X}, \nu)^\Gamma \) generated by the functions \( \hat{\chi}_{\Gamma \sigma} \), where \( \Gamma \sigma \) runs over all cosets of \( \Gamma \) in \( G \). By the hypothesis we assumed on the function \( \chi_F \), the space \( S \) is dense in \( L^2(\mathcal{X}, \nu)^\Gamma \).

Then we have the formulae

\[
T^{\Gamma \sigma \Gamma} \hat{\chi}_{\Gamma \sigma_1} = \sum_{\Gamma \theta \in [\Gamma \sigma \Gamma][\Gamma \sigma_1]} \hat{\chi}_{\Gamma \theta}, \text{ for all } \sigma, \sigma_1 \in G,
\]

where \([\Gamma \theta]\) in the above summation runs over the left cosets in the decomposition of the coset product \([\Gamma \sigma \Gamma][\Gamma \sigma_1]\).

Moreover the Hilbert space scalar product of two functions as above is computed by the formula

\[
\langle T^{\Gamma \sigma \Gamma} \hat{\chi}_{\Gamma \sigma_1}, \hat{\chi}_{\Gamma \sigma_2} \rangle_{L^2(\mathcal{X}, \nu)^\Gamma} = \sum_{\gamma \in \Gamma} \nu(\sigma_1^{-1} \gamma \sigma_2 F \cap F), \sigma_1, \sigma_2 \in G.
\]

Before proving the theorem we note that the above the formulae (6) may be used to define a direct scalar product on the linear space of cosets \( \mathbb{C}(\Gamma \sigma | \sigma \in G) \) having as basis the cosets of \( \Gamma \) in \( G \).

Then the formula (5) proves that the formula of the action the Hecke operators remains the same, while the constant function 1 in \( L^2(F, \nu) \) becomes the identity coset. This copies the action of the Hecke algebra on \( L^2(\Gamma \sigma \Gamma) \), the only difference consisting in the formula of the scalar product given by the positive definite function \( \alpha \).

Obviously in formula (6), if \( \sigma_1 \) is the identity element then the value of the scalar product is 1. Also if, for \( \sigma_1, \sigma_2 \in G \), we decompose the coset \( \sigma_1 \Gamma \sigma_2 \) as a finite disjoint union \( \bigcup_j g_j \Gamma h_j \) of cosets of smaller modular subgroups, where for all \( j, g_j, h_j \in G \), then the computation of the right hand side term in the formula (5) is reduced to the calculation of the distribution:

\[
\nu(\sigma_1 F \cap s_i \Gamma \sigma_2 F), \sigma_1, \sigma_2 \in G,
\]

where \( s_i \) are the right coset representatives for the group \( \Gamma \sigma_2 \) in \( \Gamma \).
Proof. The formula (5) is a consequence of the considerations in Appendix 2 in ([Ra]). To prove formula (5) we note that for $\sigma_1, \sigma_2 \in G$, we have:

$$\langle \chi_{\Gamma \sigma_1}, \chi_{\Gamma \sigma_2} \rangle_{L^2(X, \nu)} = \sum_{\gamma_1, \gamma_2 \in \Gamma} \nu(F \cap \gamma_1 \sigma_1 F \cap \gamma_2 \sigma_2 F).$$

Since $F$ is a fundamental domain, and $\nu$ is a $G$-invariant measure, this sum is further equal to

$$\sum_{\gamma \in \Gamma} \nu(\sigma_1 F \cap \gamma \sigma_2 F).$$

From here we deduce formula (6). If $\sigma_1$ is the identity, the formula computes the measure of $\sigma_2 F$, which is 1 by hypothesis. The rest of the statement is obvious.

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\[\square\]

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