NEGATIVE DIMENSIONAL INTEGRATION FOR MASSIVE FOUR POINT FUNCTIONS–I: THE STANDARD SOLUTIONS

Alfredo T. Suzuki and Alexandre G. M. Schmidt

Instituto de Física Teórica, Universidade Estadual Paulista, R.Pamplona, 145,
São Paulo SP, CEP 01405-900, Brazil

Abstract

Feynman diagrams are the best tool we have to study perturbative quantum field theory. For this very reason the development of any new technique which allows us to compute Feynman integrals is welcome. By the middle of the 80’s, Halliday and Ricotta suggested the possibility of using negative dimensional integrals to tackle the problem. The aim of this work is to revisit the technique as such and check up on its possibilities. For this purpose, we take a box diagram integral contributing to the photon-photon scattering amplitude in quantum electrodynamics using the negative dimensional integration method. The reason for this choice of ours is twofold: Firstly, it is a well-studied integral with well-known results, and secondly because it bears in its integrand the complexities associated with four massive propagators of the intermediate states.

Key words: Feynman Integrals, Negative Dimensional Integration Method, Feynman Box Diagrams.
PACS: 02.90+p, 03.70+k, 12.20.Ds

1 Introduction.

Scattering amplitudes, radiative corrections, $\beta$ functions of renormalization group, etc., all require the computation of Feynman integrals[1,2], which are the more complex to evaluate the more loops in a given diagram one has. Since this approach is still the best technique we have to study quantum field theory (QFT) perturbatively, solving Feynman integrals becomes basic to do any serious study on physical processes of interest involving those quantities. Such computations become harder to do not only with increasing number of
loops, but also with increasing number of massive particles in the intermediate states. This subject becomes even more compelling as we realize that increasingly higher energies are disposable at particle accelerators and tests for the standard electroweak theory are pushed to new limits. It is then clear that in order to study radiative corrections there will unavoidably lead us to the necessity of dealing with Feynman integrals containing massive intermediate vector bosons, and our ability to perform them will play a key role.

The standard way of solving such integrals starts with the introduction of Feynman parameters, Wick rotation and then finally, integration. This method is somewhat tedious and sometimes it is not possible to solve exactly the parametric integrals. For this reason physicists developed several other techniques to calculate Feynman integrals[3]. A technique known as negative dimensional integration method (NDIM) [4–6] has also been considered to tackle the problem.

Our aim in this work is to further check up on NDIM as a useful tool for the referred task. For this purpose, we use NDIM to evaluate the box diagram contributing to the photon-photon scattering (see fig.1) in quantum electrodynamics (QED). The outline for our paper is as follows: In section 2 we give a brief review of the methodology to be employed, while in section 3 we compute the integral proper writing down the two well-known hypergeometric series representations for it. In section 4 we consider two particular cases of the given integral, namely, integrals with three and two massive propagators respectively. Finally, in the last section, we make a few comments about the six new results we have for this diagram, which will be the subject to be addressed in our shortly forthcoming paper. Also we mention more complicated integrals, like the ones arising in one-loop correction to Bhabha scattering in QED, and off-shell two-loop three point graphs[7].

2 Integration in Negative Dimensions.

NDIM was introduced by Halliday and Ricotta[4] some years ago. In this section we present a brief review of this technique. Basically what one does is to perform an analytic continuation

\[
\int \frac{d^Dq}{(A)(B)(C)\ldots} \xrightarrow{AC} \int d^Dq(A)(B)(C)\ldots
\]  

(1)
so that one gets a polynomial integral in $D < 0$ from a rather complicated one in $D > 0$. We then solve it in $D < 0$ and go back [5] to $D > 0$, through another analytic continuation. One of the advantages of NDIM is that simultaneously we get several hypergeometric series representations for the integral in $D > 0$, i.e., we obtain expressions for all the possible regions, physical and non-physical alike, of the external momenta.

We start from the relation[4–6] between a gaussian integral and its counterpart in negative dimensions

$$
\int d^Dq \exp (-\lambda q^2) = \left( \frac{\pi}{\lambda} \right)^{D/2} = \sum_{j=0}^{\infty} \frac{(-1)^j \lambda^j}{j!} \int d^Dq (q^2)^j
$$

where in the last step we have expanded the exponential function in Taylor series. Just like in dimensional regularization[9] we take this expression as the definition of the negative $D$—dimensional integral [10]. The middle term is an analytic function of $D$ so the integral on the right hand side is also an analytic

1 The negative dimensional linear operator object here can be seen from the viewpoint of positive dimensional fermionic integration[8].
function of $D$ [11–13].

From this equation we get,

$$\int d^Dq(q^2)^j = (-1)^j \pi^{D/2} \delta_{D/2,-j} \Gamma(1 + j)$$  \hspace{1cm} (3)

In a similar way, we can solve, e.g.,

$$J(i, j, k, l; m) = \int d^Dq \left( (q^2 - m^2)^i \left[ (q - p)^2 - m^2 \right]^j \left[ (q - k_1)^2 - m^2 \right]^k \right. \right. \times \left. \left. \frac{1}{(q - k_2)^2 - m^2} \right|^l \right)$$  \hspace{1cm} (4)

whose counterpart in $D > 0$ is the integral

$$K(i, j, k, l; m) = \int \frac{d^Dq}{(q^2 - m^2)^i \left[ (q - p)^2 - m^2 \right]^j \left[ (q - k_1)^2 - m^2 \right]^k} \times \frac{1}{(q - k_2)^2 - m^2} \right|^l \right)$$  \hspace{1cm} (5)

This is one of the integrals that contributes to the photon-photon scattering amplitude in QED and it is the one we want to evaluate in our “lab test” for NDIM. Of course, since the external photons are real particles, they are on-shell, i.e., we consider here that $k_1^2 = k_2^2 = (p - k_1)^2 = (p - k_2)^2 = 0$ (see fig.1).

So, to begin with, let our “gaussian integral” be

$$I = \int d^Dq \exp \left( -\alpha (q^2 - m^2) - \beta \left[ (q - p)^2 - m^2 \right] - \gamma \left[ (q - k_1)^2 - m^2 \right] - \omega \left[ (q - k_2)^2 - m^2 \right] \right)$$ \hspace{1cm} (6)

Completing the square, integrating over $q$ and expanding the exponential, we get

$$I = \pi^{D/2} \sum_{n_i=0}^{\infty} \frac{(-s)^{n_1}(-t)^{n_2}(-m^2)^{n_3} \alpha^{n_2 + n_4} \beta^{n_2 + n_5} \gamma^{n_1 + n_6} \omega^{n_1 + n_7}}{n_1! n_2! n_3! n_4! n_5! n_6! n_7!} \times \Gamma(1 + n_3 - n_1 - n_2 - \frac{D}{2})$$ \hspace{1cm} (7)
where \( s \) and \( t \) are the Mandelstam variables (see fig.1) and \( m \) here is the mass of the virtual matter fields. Since we use a multinomial expansion the sum index above must satisfy the constraint

\[
n_4 + n_5 + n_6 + n_7 = n_3 - n_1 - n_2 - \frac{D}{2}
\]

On the other hand, expanding the exponential of (6), we have

\[
I = \sum_{i,j,k,l=0}^{\infty} \frac{(-1)^{i+j+k+l} \alpha^i \beta^j \gamma^k \omega^l}{\Gamma(1+i) \Gamma(1+j) \Gamma(1+k) \Gamma(1+l)} J(i, j, k, l; m)
\]  \hspace{1cm} (8)

and comparing the expressions (7) and (8) we obtain a general relation for the integral \( J(i, j, k, l; m) \) and a system of linear algebraic equations linking the sum indices \( n_i \), with five equations and seven variables, i.e., we have twenty-one distinct solutions for this system. So, NDIM transfers the problem of calculating Feynman integrals to solving systems of linear algebraic equations, which is an easier task to perform, of course.

And for the particular system we are dealing with above, there are six trivial solutions, which are of no interest at all and therefore discarded, while five are of the hypergeometric type, which will give the known results. The other ten solutions — the new results — will be the subject of our shortly forthcoming paper.

3 Hypergeometric Series Representations.

Solving the system we find five hypergeometric series which can be divided into two sets according to its variables, \( \{I_1\} \), and \( \{I_2, I_3, I_4, I_5\} \). The solution in the first category is

\[
I_1 = \left( -\frac{\pi}{2} \right)^{D/2} \frac{2\sqrt{\pi}(-2m^2)^\sigma \Gamma(-\sigma)}{\Gamma(\frac{1}{2} - \frac{\sigma}{2} + \frac{D}{4}) \Gamma(-\frac{\sigma}{2} + \frac{D}{4})} \sum_{n_1,n_2=0}^{\infty} \left( \frac{s}{4m^2} \right)^{n_1} \times \left( \frac{t}{4m^2} \right)^{n_2} \frac{(-i|n_1)(-j|n_1)(-k|n_2)(-l|n_2)(-\sigma|n_1 + n_2)}{n_1!n_2!(\frac{\sigma}{2} + \frac{D}{4}|n_1 + n_2)(\frac{1}{2} - \frac{\sigma}{2} + \frac{D}{4}|n_1 + n_2)}
\]  \hspace{1cm} (9)

where we define \( \sigma = i + j + k + l + \frac{D}{2} \) and use the Pochhammer symbol

\[
(a|k) \equiv (a)_k = \frac{\Gamma(a + k)}{\Gamma(a)}.
\]
Substituting \( i = j = k = l = -1 \), we get the integral (5) with exponents corresponding to the one we want to calculate for box diagrams. Then, the first hypergeometric series representation yields (in four dimensions)

\[
J_1(-1, -1, -1, -1; m) = \frac{\pi^2}{6m^4} F_3\left(1, 1, 1, 1; \frac{5}{2}, \frac{s}{4m^2}, \frac{t}{4m^2}\right)
\]

(10)

This result is exactly what was already obtained by Davydychev[14] using the Mellin-Barnes’ representation for massive propagators[3]. Note that since we are in Euclidean space there is an overall factor \( i \) difference when compared to Davydychev’s result, obtained in Minkowski space (his result has the extra factor \( i \)). This expression is symmetric in \( s \) and \( t \) and is nonvanishing for \( s = t = 0 \). It will allow us, in section 4, to read off the particular cases where the integral has three and two propagators respectively, and also the analytic continuation to other regions of external momenta. This expression is valid in the region of convergence of the series which defines the \( F_3 \) hypergeometric function[15,16],

\[
F_3(\alpha, \alpha', \beta, \beta'; |x, y|) = \sum_{j,k=0}^{\infty} \frac{x^j y^k (\alpha | j)(\alpha' | k)(\beta | j)(\beta' | k)}{(\gamma | j + k)}
\]

(11)

where \(|x| < 1\) and \(|y| < 1\). In other words, it is valid below the threshold of pair production. It is also suitable for studying the non-relativistic limit since the Mandelstam variable \( s \) must be less than \( 4m^2 \). We note that \( s = 4m^2 \) defines the point where the process changes its nature, that is, there exists the possibility of pair creation and this fact manifests itself in the amplitude as a branch point in the Feynman integral[2,13].

The hypergeometric function in (10) can be expressed in terms of its double integral representation[15], and from there one can arrive at the standard result expressed in terms of a rather cumbersome sum of logarithms and dilogarithms[14,17].

The next set of solutions, also obtained by Davydychev[14], has variables \( 4m^2/s \) and \( 4m^2/t \), the inverse of the ones in the first solution. In the following we write down these four solutions:

\[
I_2 = \frac{2\pi^{D/2}(-t)^i(-s)^j(-m^2)^{D/2+i+k}(-i | j)(-k | l)}{(1 + i - j + k - l | \frac{D}{2} + j + l)} \sum_{n_1,n_2=0}^{\infty} \frac{1}{n_1!n_2!} \left( \frac{4m^2}{s} \right)^{n_1} \times \left( \frac{4m^2}{t} \right)^{n_2} \frac{(-j | n_1)(-l | n_2)}{(1 + i - j | n_1)(1 + k - l | n_2)(1 + i + k + \frac{D}{2} | n_1 + n_2)} \times \left( 1 + \frac{D}{2}(i - j + k - l) | n_1 + n_2 \right).
\]

(12)
Note that while the Feynman integral as just the linear combination \( I_3 \) takes the values \( \{a, b, c, d\} \) and \( \sum_{n_1, n_2=0}^{\infty} \frac{1}{n_1!n_2!} \left( \frac{4m^2}{s} \right)^{n_1} \times \left( \frac{4m^2}{t} \right)^{n_2} \) for the special case when \( J \) is valid in the non-relativistic case and \( J_2 \) in the relativistic one, they do not cover all the possible regions of external momenta.

From these we construct the second hypergeometric series representation for the Feynman integral as just the linear combination \( I_2 + I_3 + I_4 + I_5 \), i.e.,

\[
I_3 = \frac{2\pi^{D/2}(-t)^k(-s)^j(-m^2)^{D+i+l}(-i|j)(-l|k)}{(1+i+j-k-l|D/2+j+k)} \sum_{n_1, n_2=0}^{\infty} \frac{1}{n_1!n_2!} \left( \frac{4m^2}{s} \right)^{n_1} \times \left( \frac{4m^2}{t} \right)^{n_2} \]

\[
I_4 = \frac{2\pi^{D/2}(-t)^k(-s)^j(-m^2)^{D+i+l}(-j|i)(-l|k)}{(1-i+j-k+l|D/2+i+k)} \sum_{n_1, n_2=0}^{\infty} \frac{1}{n_1!n_2!} \left( \frac{4m^2}{s} \right)^{n_1} \times \left( \frac{4m^2}{t} \right)^{n_2} \]

\[
I_5 = \frac{2\pi^{D/2}(-t)^k(-s)^j(-m^2)^{D+i+k}(-j|i)(-k|l)}{(1-i+j+k-l|D/2+i+l)} \sum_{n_1, n_2=0}^{\infty} \frac{1}{n_1!n_2!} \left( \frac{4m^2}{s} \right)^{n_1} \times \left( \frac{4m^2}{t} \right)^{n_2} \]

where the set \( \{a, b, c, d\} \) takes the values \( \{-i, -k, -j, -l\} \), \( \{-i, -l, -j, -k\} \), \( \{-j, -k, -i, -l\} \), \( \{-j, -l, -i, -k\} \) and \( F_2 \) is another hypergeometric function[15,16].

Note that \( J_2 \) is the analytic continuation of \( J_1 \), see for example Erdélyi et al [15]. While \( J_1 \) is valid in the non-relativistic case and \( J_2 \) in the relativistic one, they do not cover all the possible regions of external momenta.

An important point here that one must be aware of is that even though singularities might appear in isolated terms of the RHS, for the special case when
i = j = k = l = −1, the above equation (16) cannot be singular since the corresponding analytic continuation formula (10) is not.

To overcome this difficulty let us introduce small corrections in the parameters $\beta$ and $\beta'$ of the hypergeometric function $F_3(\alpha, \alpha', \beta, \beta'; \gamma|x, y)$[18]. This recourse corresponds to correcting the exponents of propagators[19]. In our case we take

$$
\beta \to 1 + \delta,
$$

$$
\beta' \to 1 + \delta'.
$$

Next we expand all the (gamma) factors which contain $\delta$ and $\delta'$ and the $F_2$ functions in Taylor series around $\delta = 0$ and $\delta' = 0$. In the end we take the limit of vanishing $\delta$ and $\delta'$. This result is valid, like in the first case, within the region of convergence of the series which defines the $F_2$ function[15,16],

$$
F_2(\alpha, \beta, \beta'; \gamma, \gamma'|z_1, z_2) = \sum_{j,k=0}^{\infty} \frac{z_1^j z_2^k}{j!k!} \frac{\left(\alpha|j+k)(\beta|j)(\beta'|k)\right)}{(\gamma|j)(\gamma'|k)}, \quad |z_1| + |z_2| < 1
$$

Following these steps we arrive at Davydychev’s second expression for the Feynman integral[14]. Note that now the Mandelstam’s variables $s$ and $t$ can never be zero. Moreover, like the case of $J_1$, he has shown that the resulting expression for $J_2$ can be converted into the standard result in terms of sum of logarithms and dilogarithms through the use of a double integral representation for $F_2[15]$.

We see that with NDIM we immediately get two hypergeometric functions which are related by analytic continuation. But as we have mentioned before, there are still ten other solutions for the system, so that we expect that these new results will generate solutions for the integral that are also related by analytic continuation. The theory of generalized hypergeometric functions[15,20] tells us that there are three analytic continuations with no restriction in the parameters, namely:

$$
F_3(...|x, y) \longrightarrow F_2(...|1/x, 1/y).
$$

This is the very relation we saw above, connecting the two hypergeometric functions. The other two,

$$
F_3(...|x, y) \longrightarrow H_2(...|1/x, -y)
$$
and

\[ H_2(...) \mid x, y \longrightarrow F_2(...) \mid 1/x, -y \]

must appear in our new results whether in the form of direct or indirect analytic continuation. So these new results for the Feynman integral (5) will cover all the possible physical regions of external momenta.

4 Particular Cases.

If we put in our integral any exponent equal to zero we get an integral with three massive propagators. NDIM must be consistent for any value of the exponents, so, taking for example \( i = 0 \) the result (9) gives

\[
J_1(0, j, k, l; m) = C^{(1)} \times 3F_2 \left( \begin{array}{ccc}
-k, & -l, & -j - k - l - \frac{D}{2} \\
\frac{-j-k-l}{2}, & \frac{1-j-k-l}{2}
\end{array} \mid t \right) \\
4m^2
\]

where \( 3F_2 \) is the generalized hypergeometric function of one variable[15] and

\[
C^{(1)} = \left( \frac{-\pi}{2} \right)^{D/2} \frac{2^{\frac{D}{2}} \Gamma \left( \frac{-2m^2}{2} \right) \Gamma \left( \frac{-j-k-l}{2} \right)}{\Gamma \left( \frac{-j-k-l}{2} \right) \Gamma \left( \frac{-j-k-l}{2} \right) \Gamma \left( \frac{-2m^2}{2} \right)}
\]

The same result arises if one puts \( j = 0 \). For \( k = 0 \) we need simply to replace \( s \leftrightarrow t \) to get

\[
J_1(i, j, 0, l; m) = C^{(2)} \times 3F_2 \left( \begin{array}{ccc}
-i, & -l, & -i - j - l - \frac{D}{2} \\
\frac{-i-j-l}{2}, & \frac{1-i-j-l}{2}
\end{array} \mid s \right) \\
4m^2
\]

where

\[
C^{(2)} = \left( \frac{-\pi}{2} \right)^{D/2} \frac{2^{\frac{D}{2}} \Gamma \left( \frac{-i+j+l+\frac{D}{2}}{2} \right)}{\Gamma \left( \frac{-i+j+l}{2} \right) \Gamma \left( \frac{-i+j+l+\frac{D}{2}}{2} \right)}
\]

Finally, if we take \( l = 0 \) we get the same result as above in virtue of the symmetry \( (k \leftrightarrow l) \) in equation (9).
From these equations above we can also calculate integrals with two massive propagators, i.e., vacuum polarization-like ones. In (17) taking for example $j = 0$, we get

$$J_1(0, 0, k, l; m) = C^{(3)} \ _3F_2 \left( \begin{array}{ccc}
\frac{-k}{2}, & \frac{-l}{2}, & \frac{-k - l - D}{2} \\
\frac{1}{2 - D}, & 1, & 1
\end{array} \bigg| \frac{t}{4m^2} \right)$$

where

$$C^{(3)} = \left( -\pi \right)^{D/2} \frac{2 \sqrt{\pi} \Gamma(-k - l - D/2)}{(-2m^2)^{-k - l - D/2} \Gamma(-k - l) \Gamma(1 - k - l/2)}$$

In an analogous way, if we take $l = 0$ in (18), we have

$$J_1(i, j, 0, 0; m) = C^{(4)} \ _3F_2 \left( \begin{array}{ccc}
\frac{-i}{2}, & \frac{-j}{2}, & \frac{-i - j - D}{2} \\
\frac{1}{2 - D}, & 1, & 1
\end{array} \bigg| \frac{t}{4m^2} \right)$$

where

$$C^{(4)} = \left( -\pi \right)^{D/2} \frac{2 \sqrt{\pi} \Gamma(-i - j - D/2)}{(-2m^2)^{-i - j - D/2} \Gamma(-i - j) \Gamma(1 - i - j/2)}$$

For the particular cases where the exponents are equal to minus one the results above are respectively given by,

$$J_1(-1, -1, 0, -1; m) = \frac{-\pi^2}{2m^2} \ _3F_2 \left( \begin{array}{ccc}
1, & 1, & 1 \\
2, & \frac{3}{2}, & \frac{1}{2}
\end{array} \bigg| \frac{s}{4m^2} \right) \quad (19)$$

which can be written in terms of elementary functions[3]. Here we have already taken the limit $D = 4$, since the result is finite in this limit, and

$$J_1(0, 0, -1, -1; m) = \pi^2(m^2)^2 \ _2F_1 \left( \begin{array}{ccc}
1, & 2 - \frac{D}{2} \\
\frac{3}{2}, & \frac{1}{2}
\end{array} \bigg| \frac{s}{4m^2} \right) \quad (20)$$

where now, of course, there is a well-known single pole in the limit $D = 4$.

If one is interested in other regions of external momenta one has to take the analytic continuation[21] of these results. We do not need to start from the very beginning, eq.(9), to construct them in other regions, because we know from the theory of complex variables that any functional property of one power series is shared by all the others[22].
5 Conclusion.

Using NDIM we have evaluated a massive box diagram integral, namely, a Feynman integral bearing four massive propagators. This integral is the one appearing in the photon-photon scattering in QED and the two well-known results, expressed in terms of hypergeometric functions, have been easily found. So, the computation of such an integral, done as a "lab test" for NDIM, has revealed to us a powerful technique, which transfers the intricacies of performing Feynman integrals in positive dimensions to that of solving a system of linear algebraic equations in negative dimensions, a far simpler task to perform than, e.g., solving parametric integrals. More than that, surprisingly, the technique not only reproduces the standard results, but gives simultaneously, solutions covering other regions of the external momenta. We are studying carefully such solutions and these are going to be the subject of our shortly forthcoming paper. Also more complicated box diagram integrals, such as arising from the one-loop correction to the Bhabha-scattering in QED and an off-shell two-loop triangle diagram integral have already been checked by us.

Acknowledgements

AGMS would like to thank Prof. Andrei Davydychev for helpful hints and very clear discussions of [14]. The work of AGMS is supported by CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico, Brazil).

References

[1] M.E.Peskin, D.V.Schroeder, An Introduction to Quantum Field Theory (Addison-Wesley, 1995).

[2] C.Itzykson, J-B.Zuber, Quantum Field Theory (McGraw-Hill, 1980).

[3] E.E.Boos, A.I.Davydychev, Theor. Math. Phys. 89, 1,(1991) 1052 . For other techniques see references 1-10 therein. A.I.Davydychev, J.Math.Phys. 32,4(1991)1052; J.Math.Phys. 33,1 (1991) 358.

[4] I.G.Halliday, R.M.Ricotta, Phys.Lett.193B,2,(1987)241. R.M.Ricotta, Topics in Field Theory, (Ph.D. Thesis, Imperial College, 1987).

[5] A.T.Suzuki, R.M.Ricotta, XVI Brazilian Meeting on Particles and Fields,(1995)386, C.O. Escobar (Ed.).

[6] A.T.Suzuki, R.M.Ricotta, Topics on Theoretical Physics - Festschrift for P.L.Ferreira, (1995) 219, V.C.Aguilera-Navarro et al (Ed.).
[7] A.T.Suzuki, A.G.M.Schmidt, work in progress(1997).
[8] G.V.Dunne, I.G.Halliday, Phys.Lett.193B (1987)247.
[9] G.'t Hooft, M.Veltman, Nucl. Phys. B 44, (1972) 189; C.G. Bollini, J.J. Giambiagi, Nuovo Cim. B 12, (1972)20.
[10] C.Nash, Relativistic Quantum Fields (Academic Press, 1978).
[11] A.I.Markushevich, Theory of Functions of a Complex Variable (Chelsea Pub.Co, 1977).
[12] E.T.Whittaker, G.N.Watson, A Course of Modern Analysis (Cambridge Univ.Press, 1966).
[13] R.J.Eden, P.V.Landshoff, D.I.Olive and J.C.Polkighorne, The Analytic S-Matrix (Cambridge Univ. Press, 1966).
[14] A.I.Davydychev, Proc. International Conference "Quarks-92" (World Scientific, 1993); hep-ph/9307323.
[15] A.Erdélyi, W.Magnus, F.Oberhettinger and F.Tricomi, Higher Transcendental Functions (McGraw-Hill, 1953).
[16] P.Appel, J. Kampé de Feriet, Fonctions Hypergéométriques et Hypersphériques. Polynomes D'Hermite (Gauthiers-Villars, Paris, 1926).
[17] B. de Tollis, Nuovo Cim.32, (1964)757; 35, (1965)1182.
[18] A.I.Davydychev, private communication(1996).
[19] N.I.Ussyukina, A.I.Davydychev, Phys.Lett.332B,(1994)159.
[20] A.Erdélyi, Proc.Roy.Soc. A 62 (1948)378.
[21] Y.L.Luke,Special Functions and Their Approximations (Vol.I, Academic Press, 1969).
[22] W.Kaplan, Introduction to Analytic Functions (Addison-Wesley, 1966).