A SUPERCRITICAL SOBOLEV TYPE INEQUALITY DRIVEN BY THE K-HESSIAN EQUATION

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Abstract. Our main purpose in this paper is to establish a supercritical Sobolev type inequality for the \( k \)-Hessian operator acting on \( \Phi^k_0, \text{rad}(B) \), the space of radially symmetric \( k \)-admissible functions on the unit ball \( B \subset \mathbb{R}^N \). Besides, we prove both the existence of optimizers for the associated variational problem and the solvability of a related \( k \)-Hessian equation with supercritical growth.

1. Introduction

It is well known that the Sobolev-type inequalities and the corresponding variational problems play an important role in many branches of mathematics such as analysis, partial differential equations, geometric analysis, and calculus of variations. The classical Sobolev inequality provides an optimal embedding from the Sobolev space \( H^1(\Omega) \) into the Lebesgue spaces \( L^p(\Omega) \) with \( p \leq 2^* = 2N/(N-2) \), where \( \Omega \subset \mathbb{R}^N \), with \( N \geq 3 \) is a bounded smooth domain. By restricting to the Sobolev space of radially symmetric functions about the origin \( H^1_{0, \text{rad}}(B) \), where \( B \) is the unit ball in \( \mathbb{R}^N \), J.M. do Ó, B. Ruf, and P. Ubilla in [14] were able to prove a variant of the Sobolev inequality giving an embedding into non-rearrangement invariant spaces \( L^p(x)(B) \), the variable exponent Lebesgue spaces, which goes beyond the critical exponent \( 2^* \).

Namely, it was proven that
\[
U_{N,\alpha} = \sup \left\{ \int_B |u|^{2^*+|x|^\alpha} \, dx \mid u \in H^1_{0, \text{rad}}(B), \| \nabla u \|_{L^2(B)} = 1 \right\} < \infty
\]
for all \( \alpha > 0 \). In addition, the supremum in (1.1) is attained, when \( 0 < \alpha < \min \{ N/2, N-2 \} \).

As an application the authors are also able to prove that the following supercritical elliptic equation
\[
\begin{aligned}
-\Delta u &= |u|^{2^*+|x|^\alpha-2} u, \quad \text{in } B \\
u &= 0, \quad \text{on } \partial B
\end{aligned}
\]
admits at least one positive solution for all \( 0 < \alpha < \min \{ N/2, N-2 \} \), which is somewhat surprising since the nonlinearities have strictly supercritical growth except in the origin. Based on these results, the authors in [4] were able to show the existence of infinitely many nodal solutions for problem (1.2). The inequality (1.1) and its applications have captured attention recently. In the recent work [20], it was proved by Q.A. Ngô and V.H. Nguyen that the inequality (1.1) and its extremal problem can be extended for higher order Sobolev spaces \( H^m_{0, \text{rad}}(B) \), \( m \geq 1 \), while in [11] suitable extension including \( W^{1,p}_{0, \text{rad}}(B) \), \( p \geq 2 \) has been done, which is motivated by the classical Hardy inequality [16]. For more results related to this class of problems, the reader can see [8, 10, 17].

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One of the main purposes in this work is to provide a version of the inequality (1.1) for the k-Hessian operator ($k \geq 1$). We will also analyze the existence of extremals for the associated extremal problem.

In order to state our main results, we briefly introduce the essential notation. Let $F_k$, $1 \leq k \leq N$ be the k-Hessian operator defined by

$$F_k[u] = \sum_{1 \leq i_1 < \ldots < i_k \leq N} \lambda_{i_1} \ldots \lambda_{i_k},$$

where $\lambda = (\lambda_1, \ldots, \lambda_N)$ are the eigenvalues of the real symmetric Hessian matrix $D^2 u$ of a function $u \in C^2(\Omega)$. Alternatively, $F_k[u]$ is the sum of all $k \times k$ principal minors of the Hessian matrix $D^2 u$, which coincides with the Laplacian $F_1[u] = \Delta u$ if $k = 1$ and the Monge-Ampère operator $F_N[u] = \det(D^2 u)$ if $k = N$. The k-Hessian operator have a divergence structure (see for instance [26]), but the fully nonlinear operators $F_k$, $k = 2, \ldots, N$ are not elliptic in the whole space $C^2(\Omega)$. In order to overcome this difficulty, in the pioneer and seminal work by L. Caffarelli, L. Nirenberg, and J. Spruck [3] they suggested to consider the operator $F_k$ restricted to the class of $k$-admissible functions. Namely, we say a function $u \in C^2(\Omega)$ is $k$-admissible if $F_j[u] \geq 0$, $j = 1, \ldots, k$. We will denote by $\Phi^k(\Omega)$ the set of all $k$-admissible functions in $\Omega$, and by $\Phi^k_0(\Omega)$ the set of all $k$-admissible function vanishing on $\partial \Omega$.

The $k$-admissible function space $\Phi^k_0(\Omega)$ has been used by several authors to study the k-Hessian equation. For existence, multiplicity, uniqueness, and asymptotic behavior of radially symmetric $k$-admissible solutions to the k-Hessian equation we recommend [5,9,21,23,27,28]; while details on the space $\Phi^k_0(\Omega)$ and more general results can be found in [3,6,19,24–26] and references therein.

As observed in [26], the expression

$$(1.3) \quad \|u\|_{\Phi^k_0} = \left( \frac{1}{\Omega} \int (-u) F_k[u] \, dx \right)^{\frac{1}{k^*}}, \quad u \in \Phi^k_0(\Omega)$$

defines a norm on $\Phi^k_0(\Omega)$. In addition, the following Sobolev type inequality holds: there exists a constant $C = C(N, k, p, \Omega)$ such that

$$(1.4) \quad \|u\|_{L^p(\Omega)} \leq C \|u\|_{\Phi^k_0}, \forall u \in \Phi^k_0(\Omega)$$

for any $1 \leq p \leq k^*$, where

$$k^* = \frac{N(k + 1)}{N - 2k}, \quad 1 \leq k < \frac{N}{2}$$

is the optimal exponent of the inequality (1.4).

Our first main result reads as follows.

**Theorem 1.1.** Let $\alpha > 0$ be real number and assume $1 \leq k < N/2$. Then

$$(1.5) \quad \sup \left\{ \int_B |u(x)|^{k^* + |x|^\alpha} \, dx \mid u \in \Phi^k_{0,\text{rad}}(B), \|u\|_{\Phi^k_0} = 1 \right\} < \infty,$$

where $\Phi^k_{0,\text{rad}}(B)$ is the subspace of radially symmetric functions in $\Phi^k_0(B)$.

The above result represents the counterpart of (1.1) to the fully nonlinear case $F_k$, $k \geq 2$ (recall $F_1[u] = \Delta u$).

In the same line of [14], one can see that the Theorem 1.1 ensures the continuous embedding of the $k$-admissible function space $\Phi^k_{0,\text{rad}}(B)$ into the variable exponent Lebesgue space $L_{k^*+|x|^\alpha}(B)$. Precisely,
Corollary 1.2. Let $1 \leq k < N/2$ and $\alpha > 0$. Then the following embedding is continuous
\[
\Phi^k_{0,\text{rad}}(B) \hookrightarrow L^{k^*+|x|^\alpha}(B),
\]
where $L^{k^*+|x|^\alpha}(B)$ denotes the variable exponent Lebesgue space defined by
\[
L^{k^*+|x|^\alpha}(B) = \left\{ u : B \to \mathbb{R} \text{ is measurable} \mid \int_B |u(x)|^{k^*+|x|^\alpha} \, dx < +\infty \right\},
\]
with the norm
\[
\|u\|_{L^{k^*+|x|^\alpha}(B)} = \inf \left\{ \lambda > 0 \mid \int_B \frac{|u(x)|^{k^*+|x|^\alpha}}{\lambda} \, dx \leq 1 \right\}.
\]

On the attainability, we can prove the following:

Theorem 1.3. For $\alpha > 0$ be real number and $1 \leq k < N/2$, we set
\[
(1.6) \quad U_{k,N,\alpha} = \sup \left\{ \int_B |u(x)|^{k^*+|x|^\alpha} \, dx \mid u \in \Phi^k_{0,\text{rad}}(B), \|u\|_{\Phi^k_0} = 1 \right\}.
\]

Then, $U_{k,N,\alpha}$ is attained provided that $0 < \alpha < \min \{N/(k+1), (N-2k)/k\}$.

Finally, we study the existence of $k$-admissible solutions for $k$-Hessian equation involving supercritical growth.

Theorem 1.4. Suppose $1 \leq k < N/2$ and $0 < \alpha < \min \{N/(k+1), (N-2k)/k\}$, then equation
\[
(1.7) \quad \begin{cases}
F_k[u] = (-u)^{k^*+|x|^\alpha-1} & \text{in } B \\
u < 0 & \\
u(x) = 0 & \text{on } \partial B
\end{cases}
\]

admits at least one radially symmetric $k$-admissible solution.

To prove the existence of a radially symmetric solution of problem (1.7) is equivalent to find a solution of the following boundary value problem
\[
(1.8) \quad \begin{cases}
\mathcal{C}^k_N (\tau^{N-k}(w')^k)' = N\tau^{N-1}(-w)^{k^*+|x|^\alpha-1} & \text{in } (0,1) \\
w < 0 & \\
w(0) = 0, \ w'(0) = 0
\end{cases}
\]

where $w(x) = u(|x|)$, and $\mathcal{C}_m^m = n!/(n-m)!m!$ is the combinatorial constant. See [5,9,28] for more details. It was recently shown in [10] that the equation (1.8) admits at least one solution $w \in C^2(0,1)$. The new in the Theorem 1.4 is to guarantee the $k$-admissibility of the function $u(x) = w(|x|)$, for which the Lemma A.1 (below) plays a crucial role.

The rest of this paper is arranged as follows. In Section 2, we show the Theorem 1.1 and its consequence the Corollary 1.2. The Section 3 is devoted to the study of an auxiliary extremal problem. The proof of Theorem 1.3 is given in Section 4. In Section 5, we ensure the existence radially symmetric $k$-admissible solution for the nonlinear equation (1.7). In Appendix A, we establish a technical result that gives the behavior of a suitable class of functions at the origin.
2. The inequality: Proof of Theorem 1.1

This section is devoted to show the Theorem 1.1. The first step is to prove the following radial type Lemma in $\Phi^k_{0,\text{rad}}(B)$ (cf. [22]). We will use throughout the paper the notation $|x| = r$.

**Lemma 2.1.** Assume $1 \leq k < N/2$. Then, for any $0 < r \leq 1$

\[
|u(r)| \leq \frac{C (1 - r)^{\frac{k}{N - 2k}}}{r^{\frac{N - 2k}{k + 1}}} \|u\|_{\Phi^k_0}, \quad \forall \ u \in \Phi^k_{0,\text{rad}}(B),
\]

where $C$ is a positive constant depending only on $N$ and $k$.

**Proof.** Let $u \in \Phi^k_{0,\text{rad}}(B)$ be arbitrary. It follows that (see for instance [26])

\[
\|u\|_{\Phi^k_0} = \left(\omega_{N,k} \int_0^1 r^{N-k}|u'|^{k+1}dr\right)^{\frac{1}{k+1}},
\]

with $\omega_{N,k}$ defined by

\[
\omega_{N,k} = \frac{\omega_{N-1} C_N^k}{N},
\]

where $\omega_{N-1}$ is the area of the unit sphere in $\mathbb{R}^N$, and $C_N^k = N!/(N-k)!k!$. Using the Hölder inequality one has

\[
|u(r)| \leq \int_r^1 |u'(s)|ds = \int_r^1 (\omega_{N,k}^{\frac{1}{k+1}} s^{\frac{N-k}{k+1}} |u'(s)|) (\omega_{N,k}^{\frac{1}{k+1}} s^{-\frac{N-k}{k+1}}) ds
\]

\[
\leq \left(\omega_{N,k} \int_r^1 s^{N-k}|u'|^{k+1}ds\right)^{\frac{1}{k+1}} \left(\omega_{N,k} \int_r^1 s^{-\frac{N-k}{k}} ds\right)^{\frac{k}{k+1}}
\]

\[
\leq \|u\|_{\Phi^k_0} \left(-\frac{k\omega_{N,k}^{\frac{1}{k+1}}}{N - 2k} \left(1 - r^{-\frac{N-2k}{k}}\right)\right)
\]

\[
= \left(\frac{k\omega_{N,k}^{\frac{1}{k+1}}}{N - 2k} \left(1 - r^{-\frac{N-2k}{k}}\right)\right)^{\frac{k}{k+1}} \|u\|_{\Phi^k_0} r^{\frac{N-2k}{k+1}}.
\]

Since, for all $r \in [0, 1]$ we have

\[
1 - r^{-\frac{N-2k}{k}} \leq \max\left\{1, \frac{N - 2k}{k}\right\} (1 - r).
\]

Then (2.1) holds with $C = \left(\frac{k\omega_{N,k}^{\frac{1}{k+1}}}{N - 2k} \max\left\{1, \frac{N-2k}{k}\right\}\right)^{\frac{1}{k+1}}$. \hfill $\square$

2.1. **Proof Theorem 1.1.** Initially, let us denote by

\[
\Sigma_{k,N} = \sup \left\{ \int_B |u(x)|^k dx \mid u \in \Phi^k_{0,\text{rad}}(B), \|u\|_{\Phi^k_0} = 1 \right\}
\]
the best constant of Sobolev type inequality (1.4), with Ω = B and p = k∗ (cf. [26]). Let u ∈ Φ^k_{0,rad}(B), with ∥u∥_{Φ^k_0} = 1. We can write

\begin{equation}
\frac{1}{\omega_{N-1}} \int_B |u(x)|^{k^*+|x|^\alpha} dx = \int_0^\rho r^{N-1}|u|^{k^*+r^\alpha} dr + \int_0^1 r^{N-1}|u|^{k^*+r^\alpha} dr,
\end{equation}

where 0 < ρ < 1 will be chosen later. Firstly,

\begin{equation}
\int_0^\rho r^{N-1}|u|^{k^*+r^\alpha} dr = \int_0^\rho r^{N-1}|u|^{k^*}(|u|^{r^\alpha} - 1) dr + \int_0^\rho r^{N-1}|u|^{k^*} dr
\end{equation}

\begin{equation}
\leq \int_0^\rho r^{N-1}|u|^{k^*}(|u|^{r^\alpha} - 1) dr + \frac{1}{\omega_{N-1}} \Sigma_{k,N}.
\end{equation}

From Lemma 2.1, we can choose a constant C = C(k, N) > 0 such that

\begin{equation}
|u(r)| \leq \frac{C}{r^{\frac{k^*}{k^*+\alpha}}}, \quad 0 < r \leq 1,
\end{equation}

for each u ∈ Φ^k_{0,rad}(B), with ∥u∥_{Φ^k_0} = 1. By using the definition of k* and (2.7), it follows that

\begin{equation}
\int_0^\rho r^{N-1}|u|^{k^*}(|u|^{r^\alpha} - 1) dr \leq c_0 \int_0^\rho \frac{1}{r} [\exp (c_1 r^\alpha + c_2 r^\alpha |\log r|) - 1] dr,
\end{equation}

for positive constants c_0, c_1 and c_1 independent on u. Noticing that

\begin{equation}
\lim_{r \to 0^+} (c_1 r^\alpha + c_2 r^\alpha |\log r|) = 0 \quad \text{and} \quad \lim_{s \to 0} \frac{e^s - 1}{s} = 1
\end{equation}

there exists sufficiently small ρ > 0 such that

\begin{equation}
\exp (c_1 r^\alpha + c_2 r^\alpha |\log r|) - 1 \leq 2 [c_1 r^\alpha + c_2 r^\alpha |\log r|], \quad \forall \ r \in (0, \rho).
\end{equation}

It follows that

\begin{equation}
\int_0^\rho \frac{1}{r} [\exp (c_1 r^\alpha + c_2 r^\alpha |\log r|) - 1] dr \leq C_1 \int_0^\rho r^{\alpha-1} dr + C_2 \int_0^\rho r^{\alpha-1} |\log r| dr
\end{equation}

for some positive constants C_1 and C_2. From (2.8) and (2.6), there exists C > 0 (independent on u) such that

\begin{equation}
\int_0^\rho r^{N-1}|u|^{k^*+r^\alpha} dr < C.
\end{equation}

It remains to estimate the third integral in (2.5). Directly from (2.7), we can choose C > 1 such that

\begin{equation}
|u(r)| \leq \frac{C}{r^{\frac{k^*}{k^*+\alpha}}}, \quad 0 < r \leq 1.
\end{equation}

Hence

\begin{equation}
\int_0^1 r^{N-1}|u|^{k^*+r^\alpha} dr \leq c_* \int_0^1 r^{N-1} \frac{1}{r^{N+\frac{k}{k^*+\alpha}}} dr = c_* \int_0^1 \frac{dr}{r^{1+\frac{k}{k^*+\alpha}} < \infty.
\end{equation}

Thus, (2.5) yields our result.
2.2. Proof Corollary 1.2. Let $u \in \Phi_{0, \text{rad}}^k(B)$ be arbitrary nonzero function. Then the Theorem 1.1 gives

$$\int_B \left| \frac{u(x)}{\|u\|_{\Phi_0^k}} \right|^{k^* + |x|^\alpha} \, dx \leq C,$$

for some $C > 0$ independent on $u$. Thus, for any $\lambda > 1$

$$\int_B \left| \frac{u(x)}{\lambda \|u\|_{\Phi_0^k}} \right|^{k^* + |x|^\alpha} \, dx = \frac{1}{\lambda} \int_B \left| \frac{u(x)}{\|u\|_{\Phi_0^k}} \right|^{k^* + |x|^\alpha} \, dx \leq \frac{1}{\lambda} \leq \frac{C}{\lambda^{k^*}}.$$

Consequently, by taking large enough $\lambda_*$ such that $C/\lambda_*^{k^*} \leq 1$ one has

$$\|u\|_{L^{k^* + |x|^\alpha}(B)} \leq \lambda_* \|u\|_{\Phi_0^k}$$

and the proof is completed.

3. An auxiliar extremal problem

Since $\Phi_0^k(B)$ is just a convex cone, it is no easy to use directly variational methods. Then some strategies will be implemented. In fact, we are going to use an auxiliary function space defined as follows: Let $AC_{\text{loc}}^1(0, R)$ be the set of all locally absolutely continuous functions on interval $(0, R)$, and set $X_R = X_{1,k+1}^R$ the set of all functions $v \in AC_{\text{loc}}^1(0, R)$ satisfying

$$\lim_{r \to R} v(r) = 0, \quad \int_0^R r^{N-k} |v'|^{k+1} \, dr < \infty \quad \text{and} \quad \int_0^R r^{N-1} |v|^{k+1} \, dr < \infty.$$

If $0 < R < \infty$, then $X_R$ is a Banach space endowed with the gradient norm

$$\|v\|_{X_R} = \left( \int_0^R r^{N-k} |v'|^{k+1} \, dr \right)^{\frac{1}{k+1}},$$

where $\omega_{N,k}$ is given by (2.3). For more details on the weighted Sobolev space above as well as its applications, we refer the reader to [10–13] and to the recent works [11, 13] where an inherent discussion has been done.

As a byproduct of a Hardy type inequality, which is essentially due to A. Kufner and B. Opic [18] (see also [7]), the following continuous embedding holds:

$$X_R \hookrightarrow L^q_{N-1}, \quad \text{if} \quad q \in (1, k^*] \quad \text{(compact if} \quad q < k^*),$$

where $L^q_{N-j} = L^q_{N-j}(0, R), q \geq 1, j \in \{1, 2, \cdots, N\}$ is the weighted Lebesgue space composed by all measurable functions $v$ on $(0, R)$ such that

$$\|v\|_{L^q_{N-j}} = \left( \int_0^R r^{N-j} |v|^q \, dr \right)^{\frac{1}{q}} < +\infty.$$
Let \( u \in \Phi^k_{0,\text{rad}}(B) \) be arbitrary and define \( v(r) = u(x), r = |x| \). We clearly have \( v \in X_1 \) satisfying

\[
\int_B |u|^{k^*+|x|\alpha} \, dx = \omega_{N-1} \int_0^1 r^{N-1}|v|^{k^*+r\alpha} \, dr,
\]

and

\[
\|u\|_{\Phi^k_0} = \|v\|_{X_1}.
\]

The aim of this section is to prove the existence of maximizers for the following extremal problem:

\[
\mathcal{V}_{k,N,\alpha} = \sup_{\|v\|_{X_1} = 1} \int_0^1 r^{N-1}|v|^{k^*+r\alpha} \, dr.
\]

We observe that \( \mathcal{U}_{k,N,\alpha} \leq \omega_{N-1} \mathcal{V}_{k,N,\alpha} \), and thus, the upper bound to \( \mathcal{V}_{k,N,\alpha} \) cannot be directly obtained by Theorem 1.1. However, in [10, Proposition 2.4] has been proven there exists a positive constant \( c \) such that \( \mathcal{V}_{k,N,\alpha} < c \).

Now, let us denote by \( \mathcal{V}_{k,N} \) the best constant to the embedding (3.2), for \( q = k^* \). Namely,

\[
\mathcal{V}_{k,N} = \sup_{\|v\|_{X_1} = 1} \int_0^1 r^{N-1}|v|^{k^*} \, dr.
\]

It is well-known the supremum in (3.6) is not attained (cf. [7]), nevertheless we will be able to show the following result:

**Proposition 3.1.** Suppose \( 1 \leq k < N/2 \) and \( 0 < \alpha < \min \{N/(k+1), (N-2k)/k\} \). Then \( \mathcal{V}_{k,N,\alpha} \) is attained for some \( v \in X_1 \), with \( \|v\|_{X_1} = 1 \).

By normalized concentrating sequence at the origin in \( X_1 \) we mean \( (v_n) \subset X_1, n \in \mathbb{N} \) satisfying

\[
\|v_n\|_{X_1} = 1, \quad v_n \to 0 \quad \text{weakly in } X_1 \quad \text{and} \quad \int_{r_0}^1 r^{N-k}|v_n'|^{k+1} \, dr \to 0, \quad \forall r_0 > 0.
\]

To prove Proposition 3.1, it is sufficient to show the following three steps:

**Step 1:** The strict inequality \( \mathcal{V}_{k,N} < \mathcal{V}_{k,N,\alpha} \) holds;

**Step 2:** If \( (v_n) \subset X_1 \) is any normalized concentrating sequence at origin, then

\[
\limsup_n \int_0^1 r^{N-1}|v_n|^{k^*+r\alpha} \, dr \leq \mathcal{V}_{k,N};
\]

**Step 3:** Let \( (v_n) \subset X_1 \) be any maximizing sequence for \( \mathcal{V}_{k,N,\alpha} \) in \( X_1 \). Then, either \( (v_n) \) is normalized concentrating at origin or \( \mathcal{V}_{k,N,\alpha} \) is attained.

The rest of this section is devoted to prove that these three steps hold.
3.1. Proof of Step 1. Firstly, for each $0 < R \leq \infty$, we define

$$S_0(k^*, R) = \inf \left\{ \int_0^R r^{N-k}|v'|^{k+1}dr \mid v \in X_R, \int_0^R r^{N-1}|v|^kdr = 1 \right\}.$$  

It is known that $S_0(k^*, R)$ is independent of $R$, and that it is achieved when $R = +\infty$ (see [7], for more details). In addition, the functions

$$v_\varepsilon^*(r) = \frac{\hat{c}\varepsilon^s}{(\varepsilon^2 + r^2)^{1/m}}, \quad \varepsilon > 0$$

with

$$s = \frac{N-2k}{(k+1)k}, \quad m = \frac{2k}{N-2k}, \quad \hat{c} = \left[ N \left( \frac{N-2k}{k} \right)^{\frac{N-2k}{2k}} \right]$$

satisfy

$$S_{N2}^k = \int_0^\infty r^{N-k}|(v_\varepsilon^*)'|^{k+1}dr = \int_0^\infty r^{N-1}|v_\varepsilon^*|^kdr,$$

where $S$ denotes the value of $S_0(k^*, R)$ for $R = +\infty$, and then for any $R > 0$. We also observe the relation

$$\omega_{N,k} S_{N2}^k = V_{k,N} = 1.$$  

Let us consider $\eta \in C_0^\infty(0, 1)$ be a fixed cut-off function satisfying

$$0 \leq \eta \leq 1, \quad \eta(r) \equiv 1, \quad \forall \ r \in (0, r_0] \quad \text{and} \quad \eta(r) \equiv 0, \quad \forall \ r \in [2r_0, 1],$$

for some $0 < r_0 < 2r_0 < 1$. In the same line of H. Brezis and L. Nirenberg [2], the following result was recently shown in [10]:

**Lemma 3.2.** The family $(v_\varepsilon^*)_{\varepsilon > 0}$ given by (3.9) satisfies:

(a) $\int_0^1 r^{N-k}|(\eta v_\varepsilon^*)'|^{k+1}dr = S_{N2}^k + O(\varepsilon^{\frac{N-2k}{2k}}), \quad \text{as} \ \varepsilon \to 0,$

(b) $\int_0^1 r^{N-1}|\eta v_\varepsilon^*|^kdr = S_{N2}^k + O(\varepsilon^{\frac{N}{2k}}), \quad \text{as} \ \varepsilon \to 0,$

where $\eta$ is the cut-off function given by (3.13).

From now on we denote by

$$w_\varepsilon(r) = B\eta v_\varepsilon^* = A\eta(r)\frac{\varepsilon^s}{(\varepsilon^2 + r^2)^{1/m}},$$

where

$$B = \left( \omega_{N,k} S_{N2}^k \right)^{-\frac{1}{k+1}} \quad \text{and} \quad A = B\hat{c}.$$  

As an easy consequence of Lemma 3.2, it follows that

$$\|w_\varepsilon\|_{X_1} = 1 + O(\varepsilon^{\frac{N-2k}{k}})$$

and (see (3.12))

$$\int_0^1 r^{N-1}|w_\varepsilon|^kdr = (\omega_{N,k} S_{N2}^k)^{-\frac{k}{k+1}} + O(\varepsilon^{\frac{N}{2k}}) = V_{k,N} + O(\varepsilon^{\frac{N}{2k}}).$$
Lemma 3.3. For \((w_\varepsilon)_\varepsilon\) defined as in (3.14), we have

\[
\int_0^1 r^{N-1} |w_\varepsilon|^{k^*+r^\alpha} \, dr \geq \int_0^1 r^{N-1} |w_\varepsilon|^{k^*} \, dr + C|\log \varepsilon|\varepsilon^\alpha + O\left(\varepsilon^{\frac{N}{k+1}}\right), \quad \text{as } \varepsilon \to 0,
\]

for some constant \(C > 0\).

Proof. It follows from the definition of \(v_\varepsilon^*\) in (3.9) that \(Bv_\varepsilon^*(r) \leq 1\) if and only if

\[
r \geq \sqrt{A m \varepsilon^{sm} - \varepsilon^2} = \varepsilon^{\frac{1}{k+1}} \sqrt{A m - \varepsilon^{2/k}} := a_\varepsilon.
\]

In particular, if \(\varepsilon > 0\) is small enough, we have

\[
0 < \varepsilon < a_\varepsilon < 1.
\]

We now write

\[
\int_0^1 r^{N-1} |w_\varepsilon|^{k^*+r^\alpha} \, dr = \int_0^{a_\varepsilon} r^{N-1} |w_\varepsilon|^{k^*+r^\alpha} \, dr + \int_{a_\varepsilon}^1 r^{N-1} |w_\varepsilon|^{k^*+r^\alpha} \, dr.
\]

Noticing that \(B\eta v_\varepsilon^* \leq 1\) in \((a_\varepsilon, 1)\), it follows that

\[
\int_{a_\varepsilon}^1 r^{N-1} |w_\varepsilon|^{k^*+r^\alpha} \, dr = \int_{a_\varepsilon}^1 r^{N-1} |B\eta v_\varepsilon^*|^{k^*+r^\alpha} \, dr
\]

\[
\leq \int_{a_\varepsilon}^1 r^{N-1} |Bv_\varepsilon^*|^{k^*} \, dr
\]

\[
= \int_{a_\varepsilon}^1 r^{N-1} \left| \frac{A \varepsilon^s}{(\varepsilon^2 + r^2)^{1/m}} \right|^{k^*} \, dr
\]

\[
\leq A^{k^*} \varepsilon^{\frac{N}{k}} \int_{a_\varepsilon}^1 r^{N-1-\frac{2k^*}{m}} \, dr
\]

\[
= A^{k^*} \varepsilon^{\frac{N}{k}} \left[ \frac{k}{N} \left( a_\varepsilon^{-\frac{N}{k}} - 1 \right) \right]
\]

\[
= O\left(\varepsilon^{\frac{N}{k+1}}\right).
\]

Hence,

\[
\int_{a_\varepsilon}^1 r^{N-1} |w_\varepsilon|^{k^*+r^\alpha} \, dr = O\left(\varepsilon^{\frac{N}{k+1}}\right).
\]

Analogously, we can see that

\[
\int_{a_\varepsilon}^1 r^{N-1} |w_\varepsilon|^{k^*} \, dr = O\left(\varepsilon^{\frac{N}{k+1}}\right).
\]
Since \(|Bv_\varepsilon|^{k + r\alpha} - |Bv_\varepsilon'|^{k*}\) ≥ 0 on \([\varepsilon, a_\varepsilon]\), the estimates (3.21) and (3.22) yield

\[
\int_0^1 r^{N-1}|w_\varepsilon|^{k + r\alpha} \, dr
= \int_0^1 r^{N-1}|w_\varepsilon|^{k*} \, dr + \int_0^1 r^{N-1}(|w_\varepsilon|^{k + r\alpha} - |w_\varepsilon|^{k*}) \, dr
= \int_0^1 r^{N-1}|w_\varepsilon|^{k*} \, dr + \int_0^{\varepsilon} r^{N-1}(|w_\varepsilon|^{k + r\alpha} - |w_\varepsilon|^{k*}) \, dr + O\left(\frac{\varepsilon^N}{k+1}\right)
\geq \int_0^1 r^{N-1}|w_\varepsilon|^{k*} \, dr + \int_0^{\varepsilon} r^{N-1}(|Bv_\varepsilon|^{k + r\alpha} - |Bv_\varepsilon'|^{k*}) \, dr + O\left(\frac{\varepsilon^N}{k+1}\right),
\]
for sufficiently small \(\varepsilon > 0\). Let us denote

\[
I_{1,\varepsilon} = \int_0^\varepsilon r^{N-1}(|Bv_\varepsilon'|^{k*} - |Bv_\varepsilon|^{k + r\alpha}) \, dr.
\]

For \(r \in [0, \varepsilon]\), it follows that

\[
|Bv_\varepsilon'|(r) = \left|A\frac{\varepsilon^s}{(\varepsilon^2 + r^2)^{1/m}}\right| \geq \left(\frac{A}{2^{1/m}}\right)\varepsilon^{(s-\frac{2}{m})} = d\varepsilon^{-\frac{N-2k}{k+1}},
\]
where \(d = (A/2^{1/m})\). Thus,

\[
I_{1,\varepsilon} = \int_0^\varepsilon r^{N-1}|Bv_\varepsilon'|^{k*}(|Bv_\varepsilon|^{r\alpha} - 1) \, dr
\geq d^{k*}\varepsilon^{-N} \int_0^\varepsilon r^{N-1}(|Bv_\varepsilon|^{r\alpha} - 1) \, dr
\geq d^{k*}\varepsilon^{-N} \int_0^\varepsilon r^{N-1} \left[\left(\varepsilon^{-\frac{N-2k}{k+1}}\right)^{r\alpha} - 1\right] \, dr
= d^{k*}\varepsilon^{-N} \int_0^\varepsilon r^{N-1} \left[\varepsilon^{(\log d + \frac{N-2k}{k+1}\log |\varepsilon|)r\alpha} - 1\right] \, dr
\geq d^{k*}\varepsilon^{-N} \left(\log d + \frac{N-2k}{k+1}\right) \int_0^\varepsilon r^{N+\alpha-1} \, dr
= \frac{d^{k*}}{N + \alpha} \left[\log d + \frac{N-2k}{k+1}\right] |\log \varepsilon|^{\varepsilon^{\alpha}}
\geq C|\log \varepsilon|^{\varepsilon^{\alpha}},
\]

for some \(C > 0\), if \(\varepsilon\) is small enough. By combining (3.23) and (3.24) the estimate (3.17) is proved. \(\square\)

In the next result we provide the expansion of \(\|w_\varepsilon\|_{X_1}^{-(k* + r\alpha)}\) in terms of \(\varepsilon\).

**Lemma 3.4.** For all \(r \in (0, 1)\), we have

\[
\|w_\varepsilon\|_{X_1}^{-(k* + r\alpha)} = 1 + O\left(\varepsilon^{\frac{N-2k}{k}}\right),
\]
as \(\varepsilon\) tends to zero.

**Proof.** From (3.15), there exists \(\delta > 0\) such that

\[
0 < 1 - \delta\varepsilon^{\frac{N-2k}{k}} \leq \|w_\varepsilon\|_{X_1} \leq 1 + \delta\varepsilon^{\frac{N-2k}{k}}
\]
for small enough $\varepsilon > 0$. Hence, for $r \in (0, 1)$ there holds
\[
\|w_\varepsilon\|_{X_1}^{k^* + r\alpha} \leq (1 + \delta \varepsilon^{-\frac{k-2k}{k}}) k^* + r\alpha \leq (1 + \delta \varepsilon^{-\frac{k-2k}{k}}) k^* + 1 \leq \delta_1 \varepsilon^{-\frac{k-2k}{k}},
\]
for some constant $\delta_1 > 0$. Analogously,
\[
\|w_\varepsilon\|_{X_1}^{k^* + r\alpha} \geq (1 - \delta \varepsilon^{-\frac{k-2k}{k}}) k^* + r\alpha \geq (1 - \delta \varepsilon^{-\frac{k-2k}{k}}) k^* + 1 \geq 1 - \delta_2 \varepsilon^{-\frac{k-2k}{k}},
\]
for some constant $\delta_2 > 0$. Consequently,
\[
\|w_\varepsilon\|_{X_1}^{k^* + r\alpha} = 1 + O\left(\varepsilon^{-\frac{k-2k}{k}}\right)
\]
and
\[
\|w_\varepsilon\|_{X_1}^{-(k^* + r\alpha)} = 1 + O\left(\varepsilon^{-\frac{k-2k}{k}}\right).
\]

\[\square\]

We are now in a position to complete the proof of the Step 1. Indeed, combining (3.16), (3.17) and (3.25), we obtain
\[
\mathcal{V}_{k,N,\alpha} = \sup_{\|v\|_{X_1} = 1} \int_0^1 r^{N-1}|v|^{k^* + r\alpha} \, dr
\geq \int_0^1 r^{N-1} \frac{|w_\varepsilon|^{k^* + r\alpha}}{\|w_\varepsilon\|_{X_1}} \, dr
= \int_0^1 r^{N-1}|w_\varepsilon|^{k^* + r\alpha} \, dr + O\left(\varepsilon^{-\frac{k-2k}{k}}\right)
\geq \int_0^1 r^{N-1}|w_\varepsilon|^{k^*} \, dr + C|\log \varepsilon|\varepsilon^\alpha + O\left(\varepsilon^\frac{N}{k+1}\right) + O\left(\varepsilon^{-\frac{k-2k}{k}}\right)
= \mathcal{V}_{k,N} + \varepsilon^\alpha|\log \varepsilon| \left[ C + O\left(\frac{\varepsilon^\frac{N}{k}}{\varepsilon^\alpha|\log \varepsilon|}\right) + O\left(\frac{\varepsilon^{\frac{N}{k+1}}}{\varepsilon^{\alpha|\log \varepsilon|}}\right) + O\left(\frac{\varepsilon^{-\frac{k-2k}{k}}}{\varepsilon^{\alpha|\log \varepsilon|}}\right) \right]
\geq \mathcal{V}_{k,N},
\]
for small enough $\varepsilon > 0$, provided that $0 < \alpha < \min\{N/(k+1), (N-2k)/k\}$.

3.2. **Proof of Step 2.** Let $(v_n) \subset X_1$ be a normalized concentrating sequence at the origin. It is sufficient to show that, for each $\varepsilon > 0$, there are $n > 0$ and $n_0 \in \mathbb{N}$ satisfying

(i) $\int_0^\eta r^{N-1}|v_n|^{k^* + r\alpha} \, dr \leq \mathcal{V}_{k,N} + \frac{\varepsilon}{2}$, $\forall \ n \geq n_0$

(ii) $\int_\eta^1 r^{N-1}|v_n|^{k^* + r\alpha} \, dr \leq \frac{\varepsilon}{2}$, $\forall \ n \geq n_0$.

Arguing as in the proof of Lemma 2.1 (see also, (2.7)), we obtain $C > 1$ such that
\[
|v_n(r)| \leq C r^{\frac{2k-N}{k+1}}, \ \forall \ 0 \leq r \leq 1.
\]

In addition, we clearly have
\[
\lim_{r \to 0^+} r^\alpha \log \left(C r^{\frac{2k-N}{k+1}}\right) \leq 0 \quad \text{and} \quad \lim_{s \to 0} \frac{e^s - 1}{s} = 1.
\]
Hence, we conclude that
\[
\int_0^\eta r^{N-1} |v_n|^{k^*} \left( |v_n|^r - 1 \right) \, dr \leq \int_0^\eta r^{N-1} |v_n|^{k^*} \left[ \exp \left( r^\alpha \log \left( Cr^{\frac{2k-N}{k+1}} \right) \right) - 1 \right] \, dr
\]
\[
\leq C \int_0^\eta r^{N-1} |v_n|^{k^* r^\alpha} \log \left( Cr^{\frac{2k-N}{k+1}} \right) \, dr
\]
\[
\leq C_1 \eta^\alpha \left| \log \left( C\eta^{\frac{2k-N}{k+1}} \right) \right| \int_0^\eta r^{N-1} |v_n|^{k^*} \, dr
\]
\[
\leq C_1 \eta^\alpha \left| \log \left( C\eta^{\frac{2k-N}{k+1}} \right) \right| \mathcal{V}_{k,N} ,
\]
by choosing small enough \( \eta > 0 \). Hence, taking some \( \eta = \eta(\alpha, \varepsilon, k, N) > 0 \) small enough such that
\[
C_1 \eta^\alpha \left| \log \left( C\eta^{\frac{2k-N}{k+1}} \right) \right| \mathcal{V}_{k,N} \leq \frac{\varepsilon}{2} ,
\]
we obtain
\[
\int_0^\eta r^{N-1} |v_n|^{k^* + r^\alpha} \, dr = \int_0^\eta r^{N-1} |v_n|^{k^*} \, dr + \int_0^\eta r^{N-1} |v_n|^{k^*} \left( |v_n|^r - 1 \right) \, dr
\]
\[
\leq \mathcal{V}_{k,N} + \frac{\varepsilon}{2} ,
\]
which proves (i).

As in the proof of Lemma 2.1, for all \( r \in (\eta, 1) \), we obtain
\[
|v_n(r)| \leq \int_r^1 |v_n'(s)| \, ds = \int_r^1 s^{\frac{N-k}{k+1}} |v_n'(s)| s^{-\frac{N-k}{k+1}} \, ds
\]
\[
\leq \left( \int_\eta^1 s^{N-k} |v_n'|^{k+1} \, ds \right)^{\frac{1}{k+1}} \left( \int_r^1 s^{-\frac{N-k}{k}} \, ds \right)^{\frac{k}{k+1}}
\]
\[
\leq \delta_n \frac{1}{r^{\frac{N-2k}{k+1}}} ,
\]
where
\[
\delta_n = C \left( \int_\eta^1 s^{N-k} |v_n'|^{k+1} \, ds \right)^{\frac{1}{k+1}} ,
\]
for some \( C = C(k, N) \). Since \((v_n)\) is a concentrating sequence at the origin, we have
\[
\lim_{n \to \infty} \delta_n = 0.
\]
It follows that
\[
\int_\eta^1 r^{N-1} |v_n|^{k^* + r^\alpha} \, dr \leq \int_\eta^1 r^{N-1} \left( \frac{\delta_n}{r^{\frac{N-2k}{k+1}}} \right)^{k^* + r^\alpha} \, dr
\]
\[
\leq \delta_n^{k^*} \int_\eta^1 r^{N-1} \left( \frac{1}{r^{\frac{N-2k}{k+1}}} \right)^{k^* + r^\alpha} \, dr
\]
\[
= \delta_n^{k^*} C(\eta) \leq \frac{\varepsilon}{2} ,
\]
for sufficiently large \( n \). This proves (ii), and consequently the Step 2 holds.
3.3. **Proof of Step 3.** Suppose that the supremum $\mathcal{V}_{k,N,\alpha}$ is not attained. Then we are going to show that every sequence $(v_n) \subset X_1$ satisfying

\[(3.28) \quad \|v_n\|_{X_1} = 1 \quad \text{and} \quad \lim_{n \to \infty} \int_0^1 r^{N-1} |v_n|^{k+r_\alpha} \, dr = \mathcal{V}_{k,N,\alpha}\]

is necessarily concentrated at the origin in $X_1$. Firstly, up to a subsequence, we can assume that

$v_n \to v$ in $X_1$.

Clearly, we have $\|v\|_{X_1} \leq 1$. We claim that $v = 0$ in $X_1$. Arguing by contradiction, we suppose that

\[(3.29) \quad \int_0^1 r^{N-k} |v'|^{k+1} \, dr > 0.\]

By Brezis-Lieb type argument (cf. [1]), we can write

\[(3.30) \quad \int_0^1 r^{N-1} |v_n|^{k+r_\alpha} \, dr = \int_0^1 r^{N-1} |v_n - v|^{k+r_\alpha} \, dr + \int_0^1 r^{N-1} |v|^{k+r_\alpha} \, dr + o(1),\]

and

\[(3.31) \quad 1 = \|v_n\|_{X_1}^{k+1} = \|v_n - v\|_{X_1}^{k+1} + \|v\|_{X_1}^{k+1} + o(1),\]

where $o(1) \to 0$, as $n \to \infty$.

If $\|v\|_{X_1} = 1$, from (3.31) we obtain $v_n \to v$ strongly in $X_1$. In this case, we will prove that $v$ is a maximizer of $\mathcal{V}_{k,N,\alpha}$, which contradicts our assumption. Indeed, from (3.28) and (3.30), it is sufficient to show that

\[(3.32) \quad \limsup_n \int_0^1 r^{N-1} |v_n - v|^{k+r_\alpha} \, dr = 0.\]

By choosing $n$ large enough such that $\|v_n - v\|_{X_1} < 1$, (3.5) yields

\[
\frac{1}{\|v_n - v\|_{X_1}^{k}} \int_0^1 r^{N-1} |v_n - v|^{k+r_\alpha} \, dr \leq \int_0^1 r^{N-1} \frac{|v_n - v|}{\|v_n - v\|_{X_1}}^{k+r_\alpha} \, dr \leq \mathcal{V}_{k,N,\alpha},
\]

which gives (3.32).

Hence, we can assume $\|v\|_{X_1} < 1$. Setting $w_n = v_n - v$ and using (3.29) and (3.31), we have $\|w_n\|_{X_1} < 1$. Hence, (3.28), (3.30) and (3.31) imply

\[
\mathcal{V}_{k,N,\alpha} = \int_0^1 r^{N-1} |w_n|^{k+r_\alpha} \, dr + \int_0^1 r^{N-1} |v|^{k+r_\alpha} \, dr + o(1)
\]

\[
= \int_0^1 r^{N-1} \frac{w_n}{\|w_n\|_{X_1}^{k+r_\alpha}} \|w_n\|_{X_1}^{k+r_\alpha} \, dr
\]

\[
+ \int_0^1 r^{N-1} \frac{v}{\|v\|_{X_1}^{k+r_\alpha}} \|v\|_{X_1}^{k+r_\alpha} \, dr + o(1)
\]

\[
\leq \mathcal{V}_{k,N,\alpha} \left( \|w_n\|_{X_1}^{k} + \|v\|_{X_1}^{k} \right) + o(1)
\]

\[
= \mathcal{V}_{k,N,\alpha} \left( 1 - \|v\|_{X_1}^{k+1} + o(1) \right) \frac{\|v\|_{X_1}^{k}}{\|w_n\|_{X_1}^{k+r_\alpha}} + o(1)
\]

\[
< \mathcal{V}_{k,N,\alpha},
\]
where we have still used \((1 - t)^{k^*/(k+1)} + t^{k^*/(k+1)} < 1\), for all \(0 < t < 1\). This contradiction forces \(v \equiv 0\) in \(X_1\). In order to complete the proof of Step 3, is now sufficient to show that \((v_n)\) satisfies the condition

\[
(3.33) \quad \int_{r_0}^{1} r^{N-k}|v_n'|^{k+1} \, dr \to 0, \quad \forall \, r_0 > 0.
\]

Let us firstly denote by \(X_1([r_0, 1])\) the space \(X_1\) on the interval \([r_0, 1]\) instead of \((0, 1)\). We claim that the embedding

\[
(3.34) \quad X_1([r_0, 1]) \hookrightarrow L^q_{N-1}[r_0, 1]
\]

is compact for any \(q \geq k + 1\). To prove (3.34), we consider the operator \(H : L^{k+1}_{N-k}[r_0, 1] \to L^q_{N-1}[r_0, 1]\) defined by

\[
H(f)(r) = \int_{r_0}^{1} f(s) \, ds.
\]

Using [18, Theorem 7.4], for \(q \geq k + 1\), the operator \(H\) is compact if and only if the following assert holds:

\[
(3.35) \quad \sup_{r \in (r_0, 1)} F(r) < \infty \quad \lim_{r \to r_0} F(r) = 0 \quad \lim_{r \to 1} F(r) = 0,
\]

where

\[
F(r) = \left( \int_{r_0}^{r} s^{N-1} \, ds \right)^{\frac{1}{q}} \left( \int_{r}^{1} s^{-\frac{N-k}{k+1}} \, ds \right)^{\frac{k}{k+1}}.
\]

It is easy to see that (3.35) holds. In addition, the embedding (3.34) can be seen as the composition \(H \circ T\), where

\[
T : X_1([r_0, 1]) \to L^{k+1}_{N-k}[r_0, 1], \quad Tv = -v'.
\]

Since \(T\) is a continuous operator, we conclude the embedding (3.34) is compact. Noticing that \(k^* > k + 1\), (3.26) together with (3.34) ensure

\[
(3.36) \quad \int_{r_0}^{1} r^{N-1}|v_n|^{k^*+\alpha} \, dr \leq C \int_{r_0}^{1} r^{N-1}|v_n|^{k^*} \, dr \to 0, \quad \forall \, r_0 > 0.
\]

Since \((v_n)\) is a maximizing sequence (see (3.28)), the Ekeland’s principle [15, Theorem 3.1] yields

\[
(3.37) \quad \lambda_n \left( \omega_{N,k} \int_{0}^{1} r^{N-k}|v_n'|^{k-1}v_n w' \, dr \right) = \int_{0}^{1} r^{N-1}(k^* + \alpha)|v_n|^{k^*-2+\alpha}v_n w \, dr + \langle o(1), w \rangle
\]

for some multiplier \(\lambda_n\). By choosing \(w = v_n\) one has

\[
\lambda_n \geq k^* \int_{0}^{1} r^{N-1}|v_n|^{k^*+\alpha} \, dr + \langle o(1), v_n \rangle.
\]

It follows that

\[
\liminf_n \lambda_n \geq k^*\gamma_{k,N,\alpha}.
\]
By choosing \( \eta \) a smooth cut-off function satisfying
\[
\eta(r) = \begin{cases} 
0, & \text{if } r \leq r_0/2 \\
1, & \text{if } r \geq r_0
\end{cases}
\]
and, then \( w = \eta v_n \) in (3.37), (3.36) provides
\[
\omega_{N,k} \int_{r_0/2}^1 r^{N-k} |v'_n|^{k-1} v_n(\eta v_n)'\,dr
= \frac{1}{\lambda_n} \int_{r_0/2}^1 r^{N-1}(k^* + r^\alpha)|v_n|^{k^*+2+r^\alpha} v_n(\eta v_n)\,dr + o(1), \eta v_n \to 0.
\]
Using the compact embedding (3.2), we conclude
\[
\int_{r_0}^1 r^{N-k} |v_n|^{k+1}\,dr \leq C \int_{r_0}^1 r^{N-1} |v_n|^{k+1}\,dr \to 0.
\]
Consequently, we get
\[
o(1) = \int_{r_0/2}^1 r^{N-k} |v'_n|^{k-1} v_n(\eta v_n)'\,dr
= \int_{r_0/2}^1 r^{N-k} \eta |v_n|^{k+1}\,dr + \int_{r_0/2}^1 r^{N-k} |v'_n|^{k-1} v_n \eta'\,dr
\geq \int_{r_0}^1 r^{N-k} |v_n|^{k+1}\,dr - C\|v'_n\|_{\infty} \|v_n\|_{X_1}^k \left( \int_{r_0}^1 r^{N-k} |v_n|^{k+1}\,dr \right)^{1/k}
= \int_{r_0}^1 r^{N-k} |v_n|^{k+1}\,dr + o(1),
\]
for some constant \( C(N,k) > 0 \). This completes the proof of (3.33).

4. Existence of \( k \)-admissible Extremals: Proof of Theorem 1.3

In order to ensure the existence of an extremal function for the supremum (1.6), we will use the maximizer of the auxiliary problem (3.5), which is ensured by the Proposition 3.1.

Let \( v_0 \in X_1 \) be a maximizer of \( \mathcal{V}_{k,N,\alpha} \). Setting \( u_0(x) = -v_0(|x|) \), (3.3) and (3.4) imply
\[
\int_B |u_0|^{k^*+|x|^\alpha} \,dx = \omega_{N-1} \int_0^1 r^{N-1} |u_0|^{k^*+r^\alpha} \,dr = \omega_{N-1} \mathcal{V}_{k,N,\alpha} \geq \mathcal{U}_{k,N,\alpha},
\]
and
\[
\|u_0\|_{\Phi_{0,\text{rad}}^k} = \|v_0\|_{X_1} = 1.
\]
Thus, in order to ensure the existence of an extremal function of the supremum in (1.6), we only need to show that \( u_0 \) belongs to \( \Phi_{0,\text{rad}}^k(B) \).

To show \( u_0 \in C^2(B) \) or equivalently \( v_0 \in C^2[0,1] \), we shall use the following Lemma which is a consequence of Lemma A.1 in the Appendix A below.

**Lemma 4.1.** Let \( v \in X_1^{1,k+1}, k < N/2 \). Then
\[
\lim_{r \to 0^+} r^\theta |v(r)|^{k^*+r^\alpha-1} = 0, \quad \forall \alpha, \theta > 0.
\]
Since $v_0$ is a maximizer of the supremum (3.5), the Lagrange multipliers theorem yields

$$
\int_0^1 r^{N-k} |v_0'|^{k-1} v_0' h' \, dr = \lambda \int_0^1 r^{N-1} |v_0|^{k+r^{\alpha-1}} \, h \, dr, \quad \forall \ h \in X_1
$$

where

$$
\lambda = \frac{1}{\omega_{N,k} \int_0^1 r^{N-1} |v_0|^{k+r^{\alpha-1}} v_0 \, dr}.
$$

Next, we will explicit expressions for $v_0'$ and $v_0''$. Following the same argument in [7], for each $r \in (0, 1)$ and $\rho > 0$, we consider the function $h_\rho \in \bar{X}_1$ given by

$$
h_\rho(s) = \begin{cases} 1 & \text{if } 0 \leq s \leq r, \\ 1 + \frac{1}{\rho} (r - s) & \text{if } r \leq s \leq r + \rho, \\ 0 & \text{if } s \geq r + \rho. \end{cases}
$$

Applying (4.3) with $h = h_\rho$ and letting $\rho \to 0$, we conclude

$$
r^{N-k} (-|v_0'|^{k-1} v_0') = \lambda \int_0^r s^{N-1} |v_0|^{k+r^{\alpha-1}} \, ds, \quad \text{a.e. on } [0, 1].
$$

It follows from the above equation that $v_0'$ doesn’t change its sign, which is determined by the sign of $\lambda$. Indeed, $\|v_0\|_X = 1$ and the equation (4.3) force $\lambda \neq 0$. If $\lambda > 0$, the above equation gives $v_0' \leq 0$ in $(0, 1)$, and consequently $v_0 \geq 0$ in $(0, 1]$. Similarly, if $\lambda < 0$, one has $v_0$ is a non-decreasing function and such that $v_0 \leq 0$ in $(0, 1]$. Hence, since $\tilde{v}_0 = -v_0$ is also an extremal function of (3.5), without loss of generality we can assume that $v_0$ is a non-increasing function and $v_0 \geq 0$ in $(0, 1]$. Thus, from (4.6) we also can write

$$
-v_0'(r) = \sqrt[N-k]{I(r)}; \quad I(r) = \frac{\lambda}{r^{N-k}} \int_0^r s^{N-1} |v_0|^{k+r^{\alpha-1}} \, ds.
$$

Consequently, we have $v_0 \in C^2(0, 1)$. In addition, from Lemma 4.1 and L’Hospital rule, we get

$$
\lim_{r \to 0^+} I(r) = \frac{\lambda}{N-k} \lim_{r \to 0^+} r^k |v_0|^{k+r^{\alpha-1}} = 0.
$$

Hence, from (4.7)

$$
\lim_{r \to 0^+} v_0'(r) = 0
$$

and thus $v_0 \in C^4[0, 1]$.

In order to get $v_0 \in C^2[0, 1]$, we firstly observe that

$$
\frac{I'(r)}{I(r)} = -\frac{N-k}{r} + \frac{r^{N-1} |v_0|^{k+r^{\alpha-1}}}{\int_0^r s^{N-1} |v_0|^{k+s^{\alpha-1}} ds}, \quad \forall r \in (0, 1].
$$

It follows from (4.7) that

$$
-v_0''(r) = \frac{\sqrt[N-k]{I(r)} I'(r)}{k I(r)} = \frac{-v_0'(r)}{k} \left[ -N-k + \frac{r^{N-1} |v_0|^{k+r^{\alpha-1}}}{\int_0^r s^{N-1} |v_0|^{k+s^{\alpha-1}} ds} \right].
$$
for all \( r \in (0, 1] \). Combining (4.7) with the L'Hospital rule, we obtain

\[
\lim_{r \to 0^+} \frac{v'_0(r)}{r} = \lim_{r \to 0^+} \frac{\sqrt[k]{I(r)}}{r^k} = \frac{\lambda}{N} |v_0(0)|^{k-1} > 0.
\]

In addition, the identity (4.7) yields

\[
\int_0^r s^{N-1} |v_0|^{|k^*|+r^*-1} v'_0 \, ds = \frac{\lambda r^{N-1} |v_0|^{|k^*|+r^*-1} v'_0}{r^{N-k} (-v'_0)^k} = -\lambda \left( \frac{r}{v'_0} \right)^{k-1} |v_0|^{|k^*|+r^*-1},
\]

for small enough \( r > 0 \). From (4.9), (4.10) and (4.11), one gets that there exists \( \lim_{r \to 0^+} v''_0(r) \), and thus \( v_0 \in C^2[0, 1] \) holds.

Now, in order to guarantee \( u_0 \in \Phi^{k, \text{rad}}_0(B) \), it is enough to show that

\[
F_j[u_0] \geq 0 \quad \text{in} \quad B, \quad \forall \ 1 \leq j \leq k.
\]

But, using the \( k \)-Hessian radial expression (cf. [28]) and the definition \( u_0(x) = -v_0(|x|) \), we can reduce the above assert to the following

\[
(r^{N-j} (-v'_0)^j)' \geq 0, \quad \forall \ 1 \leq j \leq k \quad \text{and} \quad r \in (0, 1].
\]

By using the expressions in (4.7) and (4.8), it is easy to see that

\[
(r^{N-j} (-v'_0)^j)' = \left( r^{N-j} [I(r)]^\frac{j}{k} \right)' = (N-j) r^{N-j-1} [I(r)]^\frac{j}{k} + r^{N-j} \frac{j}{k} [I(r)]^\frac{j}{k} \frac{I'(r)}{I(r)} = r^{N-j} \left[ I(r) \right]^\frac{j}{k} \left[ \frac{N-j}{r} + \frac{j}{k} \frac{I'(r)}{I(r)} \right].
\]

To prove (4.12), it is enough to show that

\[
\left[ \frac{N-j}{r} + \frac{j}{k} \frac{I'(r)}{I(r)} \right] \geq 0, \quad \forall \ 1 \leq j \leq k \quad \text{and} \quad r \in (0, 1].
\]

However, from (4.8), we can write

\[
\frac{N-j}{r} + \frac{j}{k} \frac{I'(r)}{I(r)} = \frac{N-j}{r} - \frac{j}{k} \frac{N-k}{r} + \frac{j}{k} \frac{r^{N-1} |v_0|^{|k^*|+r^*-1}}{\int_0^r s^{N-1} |v_0|^{|k^*|+r^*-1} \, ds} = \frac{N-k}{r} \left[ \frac{N-j}{N-k} - \frac{j}{k} \right] + \frac{j}{k} \frac{r^{N-1} |v_0|^{|k^*|+r^*-1}}{\int_0^r s^{N-1} |v_0|^{|k^*|+r^*-1} \, ds}
\]

which is non-negative, for all \( 1 \leq j \leq k \) and \( r \in (0, 1] \).

5. \( k \)-admissible solution to the related supercritical equation

To solve the equation (1.8) it was introduced in [10] the following auxiliary equation (see also, [9])

\[
\begin{cases}
-\mathcal{C}_N^{k, \text{rad}} (r^{N-k} |v'|^{k-1} v)' = N r^{N-1} |v|^{|k^*|+r^*-1} \\
v > 0 \\
v(1) = 0, \ v'(0) = 0.
\end{cases}
\]
Following [14] closely, the variant of the well-known mountain pass theorem of Ambrosetti and Rabinowitz without the Palais-Smale condition (see [2, Theorem 2.2]) was applied to get a non-trivial critical point for the associated functional

\[ I(v) = \frac{1}{k+1} \int_0^1 r^{N-k}|v'|^{k+1} dr - \tau \int_0^1 \frac{r^{N-1}}{k^*+r^\alpha} (v^+)^{k^*+r^\alpha} dr : X_1 \to \mathbb{R}, \]

where \( v^+ = \max\{v,0\} \) and \( \tau = N/C_N^k \).

To finish the proof of Theorem 1.4, it is sufficient to prove the following:

**Lemma 5.1.** Let \( v \in X_1 \) be a non-trivial critical point to the functional (5.2). Then

(i) \( v \) is a positive and non-increasing function belonging to \( C^2[0,1] \);

(ii) \( w(x) = -v(|x|), x \in B \) is a radially symmetric \( k \)-admissible function.

**Proof.** We can now proceed analogously to the prove of (4.7). Using the function \( h_\rho \) given in (4.5), the equation \( I'(v) \cdot h_\rho = 0 \) gives

\[ -(|v'|^{k-1}v') = \frac{\tau}{r^{N-k}} \int_0^r s^{N-1}(v^+)^{k^*+s^*-1} ds. \]

Thus, \( v \) is a non-increasing function with \( v(1) = 0 \) and, thus \( v > 0 \) in \( (0,1) \). Also, the above equation ensures \( v \in C^2(0,1) \). In addition, we conclude from the Lemma 4.1 that

\[ \lim_{r \to 0^+} v'(r) = 0. \]

Hence \( v \in C^1[0,1] \). Analogously to (4.9) we can write

\[ v''(r) = \frac{v'(r)}{k} \left[ -\frac{N-k}{r} + \frac{r^{N-1}|v|^{k^*+s^*-1}}{\int_0^r s^{N-1}|v|^{k^*+s^*-1} ds} \right], \quad \forall r \in [0,1). \]

It follows that \( \lim_{r \to 0^+} v''(r) \) exists and finally that \( v \in C^2[0,1] \). This proves (i). Analysis similar to that in the proof of (4.12) shows \( F_j(w) \geq 0 \) in \( B, \forall 1 \leq j \leq k \), which ensures the assert (ii). \( \square \)

**Appendix A. Technical Lemma**

Finally note that the following result can be applied to prove Lemma 4.1:

**Lemma A.1.** Let \( f \in C(0,1) \) such that \( 0 < q_0 \leq f(r) \leq q_1 < +\infty \), for all \( r \in (0,1) \). Then, for any \( v \in X_1^{k+1}, k < N/2 \) we have

\[ \lim_{r \to 0^+} r^\theta |v(r)|^{f(r)} = 0, \]

for each \( \theta > 0 \). In particular, one has

\[ \lim_{r \to 0^+} r^\theta |v(r)|^{k^*+r^\alpha-1} = 0, \quad \forall \alpha, \theta > 0. \]

**Proof.** For each \( 0 < r < s < 1 \), the Hölder inequality implies

\[
\int_r^s |v'(t)|^{\theta/t} dt = \int_r^s |v'(t)|^{\frac{N-k}{k+1} t^{\theta q_1} \frac{N-k}{k+1}} dt \\
\leq \left( \int_0^1 |v'(t)|^{k+1} t^{N-k} dt \right)^{\frac{1}{k+1}} \left( \int_r^s t^{(\frac{\theta}{q_1} - \frac{N-k}{k+1})} dt \right)^{\frac{k}{k+1}} \\
= c_{r,s} \|v\|_{X_1} < \infty,
\]

where \( c_{r,s} \) is a constant depending on \( r, s \) and \( \theta \).
where
\[ c_{r,s} = \left( \frac{1}{\omega_{N,k}} \int_r^s t^{\frac{\alpha}{q_1} - \frac{N-k}{N+1}} dt \right)^{\frac{k}{N+1}}. \]

For each \( r > 0 \) fixed, the integrability of above power function gives \( \lim_{s \searrow r} c_{r,s} = 0 \). In particular, given \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that
\[ (1.1) \quad \int_r^s |v'(t)| t^{\frac{\alpha}{q_1}} dt < \varepsilon, \]
for all \( 0 < r < s \) satisfying \( |s-r| < \delta \). Now, for \( 0 < r < s < 1/2 \), have
\[ |v(r)| = |v(1/2)| + \int_r^s |v'(t)| dt + \int_s^2 |v'(t)| dt. \]

Since \( f \geq q_0 > 0 \) in \( (0,1) \), without loss of generality, we can assume \( \lim \sup_{r \to 0^+} |v(r)| = +\infty \).
Thus, for \( s > 0 \) small enough such that \( 0 < r < s \) implies \( |v(r)| \geq 1 \), the above inequality gives
\[ |v(r)|^{f(r)} \leq |v(r)|^{q_1} \]
\[ \leq K_{q_1} \left( |v(1/2)|^{q_1} + \left( \int_r^s |v'(t)| dt \right)^{q_1} + \left( \int_s^2 |v'(t)| dt \right)^{q_1} \right), \quad \forall 0 < r < s. \]

Hence,
\[ (1.2) \quad r^{\theta} |v(r)|^{f(r)} \leq r^{\theta} |v(r)|^{q_1} \]
\[ \leq K_{q_1} \left[ r^{\theta} |v(1/2)|^{q_1} + \left( \int_r^s |v'(t)| t^{\frac{\alpha}{q_1}} dt \right)^{q_1} + r^{\theta} \left( \frac{1}{s^{N-k}} \int_s^2 |v'(t)| t^{N-k} dt \right)^{q_1} \right]. \]

for all \( 0 < r < s \). By choosing \( 0 < s_0 < \delta \) the inequality (1.1) gives
\[ (1.3) \quad \int_r^{s_0} |v'(t)| t^{\frac{\alpha}{q_1}} dt < \varepsilon, \quad \forall 0 < r < s_0 < \delta. \]

Now, by choosing \( 0 < r_0 < s_0 < \delta \) such that
\[ (1.4) \quad r_0^{\theta} |v(1/2)|^{q_1} < \varepsilon \quad \text{and} \quad r_0^{\theta} \left( \frac{1}{s_0^{N-k}} \int_{s_0}^{2} |v'(t)| t^{N-k} dt \right)^{q_1} < \varepsilon, \]
where in the last choice the continuous embedding \( L^{k+1}_{N-k} \hookrightarrow L^1_{N-k} \) is still used. Hence, (1.2), (1.3) and (1.4) yield
\[ r^{\theta} |v(r)|^{f(r)} \leq K_{q_1} (2\varepsilon + \varepsilon^{q_1}), \quad \forall 0 < r < r_0, \]
which completes the proof.

\[ \square \]

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