A Quaternionic Structure as a Landmark for Symplectic Maps

Hugo Jiménez-Pérez *

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Abstract

We use a quaternionic structure on the product of two symplectic manifolds for relating Liouvillian forms with linear symplectic maps obtained by the symplectic Cayley’s transformation.

1 Introduction

One of the main difficulties for constructing symplectic maps by the method of generating functions is the resolution of the Hamilton-Jacobi equation. Instead of solving such an equation, in this paper we consider a local quaternionic structure on the symplectic product manifold and the three different symplectic forms induced by this structure. Symplectic maps are constructed using the primitive Liouvillian forms related to these symplectic forms.

2 Local symplectic maps from Liouvillian forms

Let $(M, \omega)$ be a $2n$-dimensional symplectic manifold with symplectic form $\omega$. A symplectomorphism is a diffeomorphism $\phi : (M, \omega) \rightarrow (M, \omega)$ preserving the symplectic structure $\phi^* \omega = \omega$, where the star stands for the pull-back of differential forms. When the symplectic structure has a global primitive linear form $\theta$, then $(M, d\theta)$ is called an exact symplectic manifold. Main representatives are cotangent bundles $(T^*Q, \alpha)$ which possesses a canonical or tautological form $\alpha$ called the Liouville form. We define a Liouvillian form in an exact symplectic manifold, as any representative $\theta \in [\alpha]$ in $\Omega^1(M)$, and a Liouville vector field $Z$ by the implicit equation $\theta = (i_Z \omega)$.

We are interested in symplectic maps for constructing symplectic integrators, then we can consider maps defined on convex balls $B \subset (M, \omega)$ containing at the same time, the points $z_0, z_h$ and the full path of symplectic diffeomorphisms.

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\( \phi^t = \phi(t) \) connecting them \( z_h = \phi^h(z_0) \). Then \( \phi([0,h]) \mapsto B \) is an embedded segment of curve. In a convex ball, there always exist primitive 1-forms \( \theta \) by Poincaré’s lemma and consequently we can apply this procedure locally on any symplectic manifold.

2.1 The product manifold

Define the product \( P = M_1 \times M_2 \) of two copies of an exact symplectic manifold \( (M, \omega = d\theta) \), which we denote by \( (M_1, \omega_1) \) and \( (M_2, \omega_2) \), respectively. Each copy corresponds to the flow of a (Hamiltonian) system at two different times \( t = 0 \) and \( t = h \) for small \( h \). The canonical projections \( \pi_i : P \to M_i \) for \( i = 1, 2 \) let us define the forms \( \theta_\circ \) and \( \omega_\circ \) on \( P \) by, \( \theta_\circ = \pi_1^* \theta_1 - \pi_2^* \theta_2 \), and \( \omega_\circ = \pi_1^* \omega_1 - \pi_2^* \omega_2 \). It is well known that \( (P, \omega_\circ) \) is a symplectic manifold of dimension \( 4n \) [8]. The graph of any symplectic map \( \phi : (M_1, \omega_1) \to (M_2, \omega_2) \), defined by

\[
\Gamma_\phi = \{ (z, \phi(z)) \in P \mid z \in M_1, \phi(z) \in M_2 \},
\]

is a Lagrangian submanifold in \( (P, \omega_\circ) \).

Consider \( \Gamma_\phi \) as an embedding \( j : \Lambda \hookrightarrow P \) with \( j(\Lambda) = \Gamma_\phi \subset P \), and by abuse of notation we identify \( \Lambda \) and its image \( j(\Lambda) \). In terms of this embedding we have \( j^* \omega_\circ \equiv 0 \). In addition to the symplectic form \( \omega_\circ \), for every \( x \in P \) there exists an induced endomorphism on \( T_x P \) which becomes the associated complex structure to \( \omega_\circ \) given by \( J_\circ = J_1 \oplus J_2^x \), where \( J_i \), are the associated complex structures to \( \omega_i \), \( i = 1, 2 \).

The linear form \( \alpha = j^* \theta_\circ \) on \( \Lambda \) is closed since its differential satisfies

\[
\alpha = j^* \omega_\circ = 0.
\]

Applying Poincare’s lemma, \( \alpha \) is locally exact on \( \Lambda \) and there (locally) exists a function \( S : \Lambda \to \mathbb{R} \) defined on \( \Lambda \) such that its differential coincides with the pullback of \( \theta_\circ \) to \( \Lambda \), i.e. \( dS = \alpha = j^* \theta_\circ \). The function \( S : \Lambda \to \mathbb{R} \) is called a generating function for the symplectic map \( \phi : (M, \omega) \to (M, \omega) \). In fact the generating function is a function \( S : P \to \mathbb{R} \) defined on \( P \) and the function \( S \) is the composition \( S = S \circ j : \Lambda \to \mathbb{R} \).

An embedding of a 2n-dimensional manifold \( j : \Lambda \hookrightarrow (P, \omega_\circ) \) is called Lagrangian if \( j^* \omega_\circ \equiv 0 \). Given a Lagrangian embedding \( j : \Lambda \hookrightarrow (P, \omega_\circ) \), there exists an open neighborhood \( \Lambda \subset U \subset P \) around \( \Lambda \) and a projection \( \pi : U \to \Lambda \), such that the composition \( \pi_j : U \to \Lambda \) satisfies \( \pi \circ j = id_\Lambda \). This fact is just Weinstein’s theorem saying that \( U \) is locally symplectomorphic to an open neighborhood of the zero section in \( T^* \Lambda \). A Liouvillean form \( \theta \) on \( (P, \omega_\circ = d\theta) \) is related to the generating function \( S : \Lambda \to \mathbb{R} \) by the identity \( dS = j^* \theta \), and it satisfies \( ker \theta \subset ker \pi^*(dS) \), equivalently \( ker \theta \subset j_*(TA) \). The last relation is all we need to know to construct symplectic maps from Liouvillean forms.

2.2 A quaternionic structure on the product manifold

The method of generating functions uses two different symplectic structures on \( P \), usually denoted by \( \omega_\circ \) and \( \omega_\circ \), for working with Lagrangian submanifolds [7].
3. It implicitly uses a *twist* diffeomorphism known as the canonical isomorphism for cotangent bundles, relating $T^* Q_1 \times T^* Q_2 \cong T^* (Q_1 \times Q_2)$.

For the construction of symplectic maps, a different twist diffeomorphism is applied solving an alternative Hamiltonian system [3]. This diffeomorphism relates the product manifold with the double cotangent bundle $T^* Q \times T^* Q \cong T^* (T^* Q)$, and defines a projection by composition

$$T^* Q_1 \times T^* Q_2 \overset{\Phi}{\to} T^* (T^* Q) \overset{\pi \circ \Phi}{\to} T^* Q.$$  

The way we select the twist $\Phi$ will define a different projection which, by the way, determines a particular type of generating function.

In this paper, we avoid the twist diffeomorphisms and the uncomfortable situation of working with different symplectomorphic manifolds. Instead of the twist diffeomorphisms, we consider only the product manifold $P$, and we define a quaternionic or almost hypercomplex structure on $P$ given by \{$(I_4^n, I, J, K)$\} $\subset End(TP)$ [2]. In local coordinates, we have the matricial representation

$$I = \begin{pmatrix} 0_{2n} & -I_{2n} \\ I_{2n} & 0_{2n} \end{pmatrix}, \quad J = \begin{pmatrix} J_{2n} & 0_{2n} \\ 0_{2n} & J_{2n} \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 0_{2n} & J_{2n} \\ J_{2n} & 0_{2n} \end{pmatrix}, \quad (1)$$

satisfying

$$I^2 = J^2 = K^2 = IJ = -K, \quad IJ = K, \quad JK = I, \quad KI = J. \quad (2)$$

We obtain an equivalent framework to the usual one, and it is easy to prove that it just corresponds to a relabeling of coordinates.

Let $g$ be the Riemannian structure on $P$ which pointwise corresponds to the Euclidean structure $\langle \cdot, \cdot \rangle$ on $T_x P$, $x \in P$ and define three symplectic forms by

$$\omega_I(\cdot, \cdot) = g(\cdot, I\cdot), \quad \omega_J(\cdot, \cdot) = g(\cdot, J\cdot) \quad \text{and} \quad \omega_K(\cdot, \cdot) = g(\cdot, K\cdot),$$

(in particular $\omega_J \equiv \omega \circ \Phi$, $\omega \equiv J^n_{1n}$).

Let $\Lambda$ be a $2n$-dimensional manifold and $j : \Lambda \hookrightarrow P$ an embedding in the product manifold $P$. Consider a tubular neighborhood $\Lambda \subset U \subset P$ around $\Lambda$ being diffeomorphic to an open neighborhood around the zero section in $T^* \Lambda$ such that the projection $\pi : U \to \Lambda$ is well-defined and $\pi \circ j = id_\Lambda$.

The following result characterizes the submanifolds $\Lambda$ which are adapted for constructing non-degenerated local symplectic maps.

**Theorem 2.1** If the image $\Lambda \subset U \subseteq P$ is a Lagrangian submanifold with respect to both $\omega_I$ and $\omega_J$ then:

1. it is a symplectic submanifold \footnote{We use $I = J^n_{1n}$ in accordance to complex geometry. See the discussion in \cite{7} Rmk. 3.1.6.} with respect to $\omega_K$,
2. the kernel of the projection $\pi : U \to \Lambda$ defines a local symplectic map by the equation

$$\pi_*(J(v)) = \pi_*(I(v)) = 0, \quad x \in \Lambda, \quad v \in T_x \Lambda. \quad (3)$$

\footnote{Note the similarity of the conditions on $\Lambda \subset P$ with those for *Special Lagrangian submanifolds* in Kähler or Calabi-Yau manifolds. See in particular \cite{5} Sec 8.1.1].}
We obtain the Liouvillian forms

\[ \theta \]

vector field \( Z \) with the expanding vector field corresponds to the midpoint rule.

In \([4]\) it is proved that the symplectic integrator constructed \( Z \) for primitive forms \( \omega \) for all the three symplectic forms

\[ \omega \]

Remark 1 The expanding vector field

\[ \theta \]

for \( \omega \) and \( \omega \) and

\[ \omega \]

to the identity map. In \([4]\) it is proved that the symplectic integrator constructed \( Z \) using Liouvillian forms, consider an element

\[ \{ \theta \} \]

Lemma 2.2 Let \( \{ x_i \}_{i=1}^{4n} \) be local coordinates on \( P \). The “expanding” or “Euler” vector field \( Z_0 \in \Gamma ( TP ) \), given in these coordinates by

\[ Z_0 = \frac{1}{2} \sum x_i \frac{\partial}{\partial x_i} \]

is Liouville for all the three symplectic forms \( \omega \), \( \omega \) and \( \omega \).

Proof. A direct verification shows that

\[ d \circ i_\chi ( \omega ) = \omega , \quad C \in \{ I, J, K \} \]

and \( \theta \) is a Liouville form for \( ( P, \omega ) \).

Remark 1 The expanding vector field \( Z_0 \) is a degenerated case which corresponds to the identity map. In \([4]\) it is proved that the symplectic integrator constructed with the expanding vector field corresponds to the midpoint rule.
We proceed by looking for Liouville vector fields $Z \in \ker \theta_T \cap \ker \theta_J$ close to the expanding vector field. This is achieved by the addition of a component to the vector fields which is Hamiltonian with respect to $\omega_T$ and $\omega_J$. We solve this problem in the linear case.

A linear vector field $Y = \sum_i A_{ij} x_j \frac{\partial}{\partial x_i}$ on $\mathcal{P}$ is Hamiltonian for $\omega_C$, $C \in \{ \mathcal{I}, \mathcal{J}, \mathcal{K} \}$, if $A = (A_{ij})$ is a Hamiltonian matrix for the corresponding complex structure, i.e. if $A^T C + CA = 0$ holds.

**Lemma 2.3** Let $S, R \in \mathbb{M}_{2n \times 2n}(\mathbb{R})$ be a symmetric and a Hamiltonian matrix respectively, for the $2n$-dimensional symplectic manifold $(M, \omega)$. Then the matrix $A \in \mathbb{M}_{4n \times 4n}(\mathbb{R})$ given by

$$A = \begin{pmatrix} R & S \\ -JSJ & -R^T \end{pmatrix}.$$  \hspace{1cm} (5)

is Hamiltonian for $(\mathcal{P}, \omega_T)$ and $(\mathcal{P}, \omega_J)$.

**Proof.** The matrix $A$ is Hamiltonian for both $\omega_T$ and $\omega_J$ if it satisfies simultaneously: i) $A^T \mathcal{I} + \mathcal{I} A = 0$ and ii) $A^T \mathcal{J} + \mathcal{J} A = 0$.

Consider the matrix $A = (A_1, A_2)$ and solving equation $A^T \mathcal{I} + \mathcal{I} A = 0$ gives the conditions $A_2 = A_1^T$, $A_3 = A_1^T$ and $A_4 = -A_1^T$. It means i) requires that $A_2$ and $A_3$ be symmetric and it relates $A_1$ with $A_3$. On the other hand the equation $A^T \mathcal{J} + \mathcal{J} A = 0$ gives the conditions $A_1^T J + JA_1 = 0$ and $A_2 = -JA_1^T J$. It means ii) requires that $A_1$ and $A_4$ be Hamiltonian and it relates $A_2$ and $A_3$. If we denote $R = A_1$ and $S = A_2$ then $A_3 = -JSJ$ and $A_4 = -R^T$. Finally, $R$ must be Hamiltonian for $\omega$ on $M$, and $S$ symmetric. This gives $A$ by expression (5) which proves the lemma.

If we consider that $A$ is not Hamiltonian for $\omega_K$ then $A^T \mathcal{K} + \mathcal{K} A \neq 0$. This produces the additional conditions $R \neq -R^T$ or $S \neq ST$. Since $S = ST$ is already a constraint from Lemma 2.3 then $R$ cannot be antisymmetric. In particular, for $R$ a symmetric, Hamiltonian matrix for $(M, \omega)$ this problem has solutions.

For the following result, we need local coordinates for each one of the factors in the product manifold $(\{x_i\}_{i=1}^{2n}, \{X_i\}_{i=1}^{2n}) \in \mathcal{P} = M_1 \times M_2$. In these coordinates, the pointwise element $v = Z(x, X)$ is expressed in matricial form by

$$v = \frac{1}{2} \begin{pmatrix} I+R & S \\ -JSJ & I-R^T \end{pmatrix} \begin{pmatrix} x \\ X \end{pmatrix}. \hspace{1cm} (6)$$

We are in measure of proving second part of Theorem 2.1.

**Proof of 2.** Theorem 2.1 Consider the vector $v = Z(x, X)$ given in matricial form by (5). This vector is tangent to $\Lambda$ by hypothesis and it is in the kernel of $\theta_T$ and $\theta_J$ by construction. Since $\Lambda \subset \mathcal{P}$ is Lagrangian with respect to $\omega_J$ then $\mathcal{J} T(v)$ belongs to the normal bundle $(T_{(x,X)} \Lambda)^{\perp}$. Applying the complex structure $\mathcal{K}$ we have $(\mathcal{K} \circ \mathcal{J} T)(v) = \mathcal{I} (v) \in (T_{(x,X)} \Lambda)^{\perp}$, with expression

$$\mathcal{I} (v) = \frac{1}{2} \begin{pmatrix} -JSJ & I-R^T \\ -I-R & -S \end{pmatrix} \begin{pmatrix} x \\ X \end{pmatrix}. \hspace{1cm} (7)$$
The equation $\pi_*(\mathcal{I}(v)) = 0$ in these local coordinates becomes

$$[-JSJ(x) + (I - RT)(X)] + [-(I + R)(x) - S(X)] = 0.$$ 

Rearranging we obtain the matricial equation

$$[I - (RT + S)] X = [(I + R + JSJ)] x.$$ 

Solving for $X$ is possible if $RT + S$ is a non-exceptional matrix. We consider the case where $R = RT$ and $S = JSJ$, it means both matrices are symmetric and Hamiltonian. Consequently, $H := RT + S = R + JSJ$ is well-defined and it is a non-exceptional, Hamiltonian matrix for $(M, \omega)$. We solve for $X$ and we obtain

$$X = (I - H)^{-1}(I + H)x.$$ 

The Cayley’s transformation assures that the matrix $S = (I - H)^{-1}(I + H)$ is symplectic if, and only if $H$ is Hamiltonian, and consequently the map

$$x \mapsto (I - H)^{-1}(I + H)x$$

is a linear symplectic transformation. \hfill \square

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3A matrix $A \in GL(n)$ is said to be non-exceptional if $\det(I \pm A) \neq 0$, where $I$ is the identity matrix in $GL(n)$. 