On uniform distribution modulo one

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Abstract
We introduce an elementary argument to the theory of distribution of sequences modulo one.

2000 Mathematics Subject Classification: 11J71, 11K06

1 Introduction
Throughout the paper $x_1, x_2, \ldots$ denotes a sequence of real numbers with their fractional parts $\{x_1\}, \{x_2\}, \ldots$. For $0 \leq \alpha < \beta \leq 1$ we use $F(N, x_n; \alpha, \beta)$ to denote the number of terms of this sequence with the condition

$$\alpha \leq \{x_n\} < \beta, \quad n \leq N.$$ 

The sequence $x_n$ is called uniformly distributed modulo one if

$$\lim_{N \to \infty} \sup_{0 \leq \alpha < \beta \leq 1} \left| \frac{F(N, x_n; \alpha, \beta)}{N} - (\beta - \alpha) \right| = 0.$$ 

The central place in the theory of uniform distribution modulo one belongs to the Weyl criterion. Its most nontrivial part reads as follows: if for any integer $h \neq 0$ we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} = 0,$$ 

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then $x_n$ is uniformly distributed modulo one. It is easy to see that the opposite statement is also true.

The traditional method to obtain quantified versions of the Weyl criterion is Vinogradov’s lemma on “little glasses”, see Vinogradov [3, Lemma 2, Chapter II] or Karatsuba [1, Lemma A, Chapter I]. The well known Erdős-Turán inequality claims that for any $H \geq 1$,

$$
\sup_{0 \leq \alpha < \beta \leq 1} \left| \frac{F(N, x_n; \alpha, \beta)}{N} - (\beta - \alpha) \right| \ll \frac{1}{H} + \frac{1}{N} \sum_{h=1}^{H} \frac{1}{h} \left| \sum_{n=1}^{N} e^{2\pi ihx_n} \right|
$$

see Montgomery [2, Corollary 1.1, Chapter I]. In [2, Theorem 1, Chapter I] the following estimate has been proved:

$$
\left| \frac{F(N, x_n; \alpha, \beta)}{N} - (\beta - \alpha) \right| \ll \frac{1}{H} + \frac{1}{N} \sum_{h=1}^{H} \min \left( \beta - \alpha, \frac{1}{h} \right) \left| \sum_{n=1}^{N} e^{2\pi ihx_n} \right| . \quad (1)
$$

The advantage of (1) over the Erdős-Turán inequality is that it gives more precise information on distribution of $\{x_n\}$ in small intervals.

The aim of the present paper is to introduce an elementary self-contained argument to investigate the problem of uniform distribution of sequences modulo one.

Throughout the paper we use the following simple identity:

$$
\frac{1}{m} \sum_{h=0}^{m-1} e^{2\pi i h \frac{u}{m}} = \begin{cases} 
0, & \text{if } u \not\equiv 0 \pmod{m}, \\
1, & \text{if } u \equiv 0 \pmod{m}.
\end{cases}
$$

In particular, if $\mathcal{X} \in \{0, 1, \ldots, m-1\}$ is a set with $|\mathcal{X}|$ elements, then

$$
\frac{1}{m} \sum_{h=0}^{m-1} \left| \sum_{x \in \mathcal{X}} e^{2\pi i h \frac{x}{m}} \right|^2 = \frac{1}{m} \sum_{h=0}^{m-1} \sum_{x_1 \in \mathcal{X}} \sum_{x_2 \in \mathcal{X}} e^{2\pi i h \frac{x_1-x_2}{m}} = m|\mathcal{X}|.
$$

We also note that for any $h, 1 \leq h \leq m/2$, and any integers $L$ and $M \geq 1$ one has

$$
\left| \sum_{u=L+1}^{L+M} e^{2\pi i h \frac{u}{m}} \right| = \frac{\left| \sin(\pi M/m) \right|}{\left| \sin(\pi h/m) \right|} \leq \frac{1}{\left| \sin(\pi h/m) \right|} \leq \frac{m}{2h}.
$$
2  A quantified version of the Weyl criterion

Denote
\[ D(N, x_n) = \sup_{0 \leq \alpha < \beta \leq 1} \left| \frac{F(N, x_n; \alpha, \beta)}{N} - (\beta - \alpha) \right|. \]

We describe our method in proving the following statement.

**Theorem 1.** For any fixed real numbers \( a \) and \( b \) with \( a \geq 2b, 0 \leq b < 2 \), the estimate
\[ D(N, x_n) \ll \left( \sum_{h=1}^{\infty} h^{- \frac{2 + a - 2b}{2 - b}} \left( \frac{1}{N} \left| \sum_{n=1}^{N} e^{2\pi i h x_n} \right| \right)^2 \right)^{\frac{2-b}{2+a-b}} \]
holds, where the implied constant may depend only on \( a \) and \( b \).

In particular, taking \( b = 1 \) one has for any fixed \( a \geq 2 \)
\[ D(N, x_n) \ll \left( \sum_{h=1}^{\infty} h^{- a} \left( \frac{1}{N} \left| \sum_{n=1}^{N} e^{2\pi i h x_n} \right| \right)^2 \right)^{\frac{1}{a+1}}. \]

If we take in the latter estimate \( a = 2 \), we obtain (apart from the constant factor) LeVeque’s inequality [2, p.9].

Taking \( a = 2b = 4(1 - \frac{1}{c}) \), one obtains for any fixed \( c > 1 \)
\[ D(N, x_n) \ll \left( \sum_{h=1}^{\infty} h^{- c} \left( \frac{1}{N} \left| \sum_{n=1}^{N} e^{2\pi i h x_n} \right| \right)^{c} \right)^{\frac{1}{2c-1}}. \]

Taking \( a = 2, b = 2(1 - \frac{1}{c}) \), one obtains for any fixed \( c > 1 \)
\[ D(N, x_n) \ll \left( \sum_{h=1}^{\infty} h^{-2} \left( \frac{1}{N} \left| \sum_{n=1}^{N} e^{2\pi i h x_n} \right| \right)^{c} \right)^{\frac{1}{c+1}}. \]

**Proof.** It is easy to see that if we prove
\[ \frac{F(N, x_n; \alpha, \beta)}{N} - (\beta - \alpha) \ll \left( \sum_{h=1}^{\infty} h^{- \frac{2 + a - 2b}{2 - b}} \left( \frac{1}{N} \left| \sum_{n=1}^{N} e^{2\pi i h x_n} \right| \right)^2 \right)^{\frac{2-b}{2+a-b}} \] (2)
in the case $1/4 \leq \beta - \alpha \leq 1/2$, then we are done. Indeed, if $1/2 \leq \beta - \alpha \leq 1$, then $1/4 \leq (\beta - \alpha)/2 \leq 1/2$. Therefore, (2) can be applied to the intervals $[\alpha, \alpha + (\beta - \alpha)/2]$ and $[\alpha + (\beta - \alpha)/2, \beta]$.

This yields the required estimate for any $\alpha, \beta$ with $1/2 \leq \beta - \alpha \leq 1$.

If $0 < \beta - \alpha < 1/4$, then consider the sequence $\{x_n\} - \alpha$ and apply (2) with this sequence instead of $x_n$ to the interval $[\beta, 1)$. Then it remains to note that

$$F(N, x_n, \alpha, \beta) = N - F(N, \{x_n\} - \alpha; \beta - \alpha, 1)$$

which follows from the fact that for any given $n, 1 \leq n \leq N$, either $\alpha \leq \{x_n\} < \beta$ or $\beta - \alpha \leq \{x_n\} - \alpha < 1$.

We now proceed to prove (2) for $\alpha, \beta$ with $1/4 \leq \beta - \alpha \leq 1/2$. We may suppose that $0 \leq x_n < 1$.

Let us first reduce the problem to the case when $x_n$ are rational numbers. Since

$$W(N, x_n) := \left(\sum_{h=1}^{\infty} h^{-2+\alpha-2b} N\left|\sum_{n=1}^{N} e^{2\pi ihx_n}\right|^{2}\right)^{2/2-\alpha} > 0$$

and since for any $L > 10$

$$\sum_{h=L+1}^{\infty} h^{-2+\alpha-2b} \left(\frac{1}{N}\sum_{n=1}^{N} e^{2\pi ihx_n}\right)^{2-\alpha} \leq L^{-\frac{\alpha-b}{2-\alpha}},$$

then there exists a number $\delta > 0$ such that for any sequence $x'_n$ with the condition $|x'_j - x_j| \leq \delta, j = 1, \ldots, N$, we have

$$|W(N, x'_n) - W(N, x_n)| \leq W(N, x_n)/2.$$

Thus

$$W(N, x'_n) < 2W(N, x_n).$$

(3)

Next, if for some $n \leq N, x_n \in [\alpha, \beta)$, then clearly we can choose $x'_n$ to be a rational number such that

$$x_n \leq x'_n \leq x_n + \delta, \quad x'_n \in [\alpha, \beta).$$
Besides, if \( x_n \not\in [\alpha, \beta] \) then we can choose \( x'_n \) to be a rational number such that
\[
x_n \leq x'_n < \min\{1, x_n + \delta\}, \quad x'_n \not\in [\alpha, \beta].
\]
Hence, since any interval of positive length contains a rational number, then we derive that there exists a sequence of rational numbers \( x'_n \) satisfying (4) and such that
\[
F(N, x_n; \alpha, \beta) = F(N, x'_n; \alpha, \beta).
\]
Thus, denoting \( x'_n = s_n/m \), where \( s_n \) and \( m > 10 \) are integers, we conclude that it is indeed sufficient to prove the bound
\[
\frac{F(N, s_n/m; \alpha, \beta)}{N} (\beta - \alpha) \ll \left( \sum_{h=1}^{\infty} h^{-2+2a-2b} \left( \frac{1}{N} \left| \sum_{n=1}^{N} e^{2\pi i h s_n/m} \right| \right)^\frac{2}{2+b-a} \right)^\frac{2-b}{2+a-b}.
\]
We can choose \( m \) to be as large as we wish, just by substituting \( s_n/m \) by \( ks_n/(km) \). In particular, we may assume that
\[
m^{1/2} W(N, s_n/m) > 10, \quad m > (a + 1)^2.
\]
Now observe that \( F(N, s_n/m; \alpha, \beta) \) is equal to the number of solutions of the congruence
\[
s_n \equiv y \pmod{m}, \quad n \leq N, \quad \alpha m \leq y < \beta m.
\]
Set
\[
R = \sum_{h=1}^{\infty} h^{-2+2a-2b} \left( \frac{1}{N} \left| \sum_{n=1}^{N} e^{2\pi i h s_n/m} \right| \right)^\frac{2}{2+b-a}.
\]
If \( R^{(2-b)/(2+a-b)} \geq 1/10 \), then the required estimate becomes trivial. For this reason we suppose that \( R^{(2-b)/(2+a-b)} < 1/10 \). Take \( k = \lceil a \rceil + 1 \) and define \( T = \lceil m R^{(2-b)/(2+a-b)} / k \rceil \). Then
\[
kT < m/10 < (\beta - \alpha)m/2, \quad (\beta - \alpha)m + kT < m, \quad T \geq \lceil 10m^{1/2} / k \rceil \geq 10.
\]
Let \( J_1 \) be the number of solutions of the congruence
\[
s_n \equiv y - y_1 - \ldots - y_k \pmod{m},
\]
where the variables are subject to the restriction
\[ n \leq N, \quad \alpha m \leq y < \beta m + kT, \quad 1 \leq y_1, \ldots, y_k \leq T. \]

Here the length of the interval for \( y \) is less than \((\beta - \alpha)m + kT < m\).

Next, let \( J_2 \) be the number of solutions to the congruence
\[ s_n \equiv y + y_1 + \ldots + y_k \pmod{m}, \]
where the variables are subject to the restriction
\[ n \leq N, \quad \alpha m \leq y < \beta m - kT, \quad 1 \leq y_1, \ldots, y_k \leq T. \]

Here, according to the choice of parameters we have \( \alpha m < \beta m - kT \).

Obviously
\[ \frac{1}{T^k} J_2 \leq F(N, s_n/m; \alpha, \beta) \leq \frac{1}{T^k} J_1. \quad (4) \]

Application of trigonometric sums yields
\[ \frac{J_1}{T^k} = \frac{1}{m T^k} \sum_{h=0}^{m-1} \sum_{n=1}^{N} \sum_{\alpha m \leq y < \beta m + kT} \sum_{y_1=1}^{T} \ldots \sum_{y_k=1}^{T} e^{2\pi i h \frac{s_n+y+y_1+\ldots+y_k}{m}}. \]

Picking up the term corresponding to \( h = 0 \) and observing that for \( y \) there are \((\beta - \alpha)m + kT + \theta\) possible values, where \(|\theta| \leq 1\), we obtain
\[ \left| \frac{J_1}{T^k} - (\beta - \alpha)N \right| \leq \frac{2kTN}{m} + \frac{2}{m T^k} \sum_{1 \leq h \leq m/2} |S_1(h)||S_2(h)||S_3(h)|^k, \quad (5) \]

where
\[ S_1(h) = \sum_{n=1}^{N} e^{2\pi i h \frac{s_n}{m}}, \quad S_2(h) = \sum_{\alpha m \leq y < \beta m + kT} e^{2\pi i h \frac{y}{m}}, \]
\[ S_3(h) = \sum_{y_1=1}^{T} e^{2\pi i h \frac{y_1}{m}}. \]

Now we use the bound
\[ S_2(h) \ll m/h \]
and also
\[ |S_3(h)|^k \leq T^{k-a/2}|S_3(h)|^{a/2} \leq T^{k-a/2} \left( \frac{m}{h} \right)^{a/2-b} |S_3(h)|^b. \]
Here we have used that \( a \geq 2b \). Incorporating this into (5), we obtain
\[
\left| \frac{J_1}{T^k} - (\beta - \alpha)N \right| \ll \frac{TN}{m} + \frac{m^{a/2-b}}{T^{a/2}} \sum_{1 \leq h \leq m/2} h^{-1-a/2+b} |S_1(h)| |S_3(h)|^b.
\]

Next, by Holder’s inequality,
\[
\sum_{1 \leq h \leq m/2} h^{-1-a/2+b} |S_1(h)| |S_3(h)|^b \leq \left( \sum_{h=1}^{\infty} h^{2(a-b)} |S_1(h)|^{2-b} \right)^{(2-b)/2} \left( \sum_{h=0}^{m-1} |S_3(h)|^2 \right)^{b/2} \leq NR^{(2-b)/2} (mT)^{b/2}.
\]

Therefore,
\[
\left| \frac{J_1}{T^k} - (\beta - \alpha)N \right| \ll \frac{TN}{m} + N \left( \frac{m}{T} \right)^{(a-b)/2} R^{(2-b)/2}.
\]

Recalling the choice of \( T \), we obtain
\[
\left| \frac{J_1}{T^k} - (\beta - \alpha)N \right| \ll NR^{(2-b)/(2-b+a)}.
\]

Analogously
\[
\left| \frac{J_2}{T^k} - (\beta - \alpha)N \right| \leq NR^{(2-b)/(2-b+a)}.
\]

Therefore, from (4) we conclude that
\[
|F(N, s_n/m; \alpha, \beta) - (\beta - \alpha)N| \ll NR^{(2-b)/(2-b+a)}.
\]

Theorem (1) is proved.

3 Remarks

Using the same argument one can deduce that if \( 0 < \varepsilon \leq 1 \), \( \beta - \alpha \geq \frac{2A}{\varepsilon} \) and if the estimate
\[
\left| \sum_{n=1}^{N} e^{2\pi i h x_n} \right| \leq \Delta N
\]
holds for any integer \( h \) with \( 1 \leq h \leq \Delta^{-1-\varepsilon} \), then

\[
F(N, x_n; \alpha, \beta) = (\beta - \alpha)N + O(\Delta N \log \frac{\beta - \alpha}{\Delta}),
\]

where the implied constant in the \( O \)--symbol depends only on \( \varepsilon \). This result does not follow from the Erdős-Turán inequality, but it can be derived from (1).

If one would like to have under hands only the proof of Weyl's criterion, without its quantified version, then the argument given in the previous section can be simplified even more. That is, suppose that \( 0 < \varepsilon < 10^{-3} \). We require the following condition:

(i) the inequality

\[
\left| \sum_{n=1}^{N} e^{2\pi i h x_n} \right| \leq \varepsilon^3 N
\]

holds for any integer \( h, 1 \leq h \leq \varepsilon^{-3} \).

Then we establish the following form of the Weyl criterion: under the condition (i),

\[
\left| \frac{F(N, x_n; \alpha, \beta)}{N} - (\beta - \alpha) \right| \leq \varepsilon.
\]

It is sufficient to show that

\[
\left| \frac{F(N, x_n; \alpha, \beta)}{N} - (\beta - \alpha) \right| \leq \varepsilon/2
\]

in the case \( 1/4 \leq \beta - \alpha \leq 1/2 \). Then by continuity argument the problem is reduced to the case with rational numbers, that is for some integers \( s_n \) and \( m > 100\varepsilon^{-1} \), we have

\[
\left| \frac{F(N, x_n; \alpha, \beta)}{N} - (\beta - \alpha) \right| = \left| \frac{F(N, s_n/m; \alpha, \beta)}{N} - (\beta - \alpha) \right|
\]

and

\[
\left| \sum_{n=1}^{N} e^{2\pi i h x_n} \right| \leq 2\varepsilon^3 N
\]

for any integer \( h, 1 \leq h \leq \varepsilon^{-3} \).
Now $F(N, s_n/m; \alpha, \beta)$ is equal to the number of solutions of the congruence

$$s_n \equiv y \pmod{m}, \quad n \leq N, \quad \alpha m \leq y < \beta m.$$ 

Denote $T = \lfloor \varepsilon m/10 \rfloor$ and set $J_1$ to be the number of solutions of the congruence

$$s_n \equiv y - y_1 \pmod{m}, \quad n \leq N, \quad \alpha m \leq y < \beta m + T, \quad 1 \leq y_1 \leq T.$$

Since $\beta - \alpha \leq 1/2$, then the length of the interval for $y$ is less than $m$.

Next, let $J_2$ be the number of solutions to the congruence

$$s_n \equiv y + y_1 \pmod{m}, \quad n \leq N, \quad \alpha m \leq y < \beta m - T, \quad 1 \leq y_1 \leq T.$$

Since $\beta - \alpha \geq 1/4$, then $\alpha m < \beta m - T$.

Obviously,

$$\frac{J_2}{T} \leq F(N, s_n/m; \alpha, \beta) \leq \frac{J_1}{T}.$$ (6)

For $J_1/T$ we have

$$\frac{J_1}{T} = \frac{1}{mT} \sum_{h=0}^{m-1} \sum_{n=1}^{N} \sum_{\alpha m \leq y < \beta m + T} \sum_{y_1=1}^{T} e^{2\pi i h \frac{s_n - y + y_1}{m}}.$$ 

Picking up the term corresponding to $h = 0$ and observing that for $y$ there are $(\beta - \alpha)m + T + \theta$ possible values, where $|\theta| \leq 1$, we obtain

$$\left| \frac{J_1}{T} - (\beta - \alpha)N \right| \leq \frac{2TN}{m} + \frac{2m}{T} \sum_{1 \leq h \leq m/2} h^{-2} \left| \sum_{n=1}^{N} e^{2\pi i h \frac{s_n}{m}} \right|.$$ 

The sum over $h$ on the left hand side is

$$\leq 2\varepsilon^3 N \sum_{1 \leq h \leq \varepsilon^{-3}} h^{-2} + N \sum_{h \geq \varepsilon^{-3}} h^{-2} \leq 5\varepsilon^3 N.$$ 

Hence, recalling that $T = \lfloor \varepsilon m/10 \rfloor$ and $\varepsilon < 10^{-3}$, we deduce

$$\left| \frac{J_1}{T} - (\beta - \alpha)N \right| \leq \varepsilon N/2.$$

Analogously

$$\left| \frac{J_2}{T} - (\beta - \alpha)N \right| \leq \varepsilon N/2.$$
Therefore, from (6) we conclude that

$$|F(N, s_n/m; \alpha, \beta) - (\beta - \alpha)N| \leq \varepsilon N/2.$$ 

**Acknowledgements.** This work was supported by Project PAPIIT-IN105605 from the UNAM.

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