Existence, multiplicity and regularity of solutions of elliptic problem involving non-local operator with variable exponents and concave-convex nonlinearity

Reshmi Biswas* and Sweta Tiwari†
Department of Mathematics,
Indian Institute of Technology Guwahati
Guwahati, Assam-781039, India

Abstract
In this paper, first we introduce the variable exponent fractional Sobolev space $W^{s(x,y), p(x,y)}(\Omega)$. Then, using variational methods we study the existence and multiplicity of solution of the following variable order non-local problem involving concave-convex type nonlinearity:

$$(-\Delta)^{s(x,y)}_{p(x,y)} u(x) = \lambda |u(x)|^{\alpha(x)-2} u(x) + f(x,u), \quad x \in \Omega,$$

$$u = 0, \quad x \in C\Omega := \mathbb{R}^n \setminus \Omega,$$

where $\lambda > 0$, $p \in C(\Omega \times \Omega, (1, \infty))$, $s \in C(\Omega \times \Omega, (0,1))$ and $\alpha, q \in C(\Omega, (1, \infty))$ and $f : \Omega \times \mathbb{R} \to [0, \infty)$ is a Carathéodory function with subcritical growth. We also prove the uniform estimate for the solution of the above problem.

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Non-local problem, fractional Laplacian with variable exponent, concave-convex non-linearity, uniform estimate

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up to a constant, for continuous functions $s : \overline{\Omega} \times \overline{\Omega} \to (0, 1)$ and $p : \overline{\Omega} \times \overline{\Omega} \to (1, \infty)$ with $s(x, y)p(x, y) < n$ for all $(x, y) \in \overline{\Omega} \times \overline{\Omega}$. Here we consider the parameter $\lambda > 0$ and sub-critical growth assumption on $f(x, t)$ as described later.

In recent years, problems involving non-local operators have gained a lot of attentions due to their occurrence in real-world applications, such as, the thin obstacle problem, optimization, finance, phase transitions and also in pure mathematical research, such as, minimal surfaces, conservation laws etc. The celebrated work of Nezza, Palatucci and Valdinoci [11] provides the necessary functional set-up to study these non-local problems using variational method. We refer [2] and references there in for more details on problems involving semi-linear fractional-Laplacian operator.

In continuation to this, problems involving quasilinear non-local fractional $p$-Laplacian are extensively studied by many researchers including Squassina, Palatucci, Mosconi, Rădulescu et al. (see [3], [4], [5], [6]). Here the authors have studied various aspects, viz, existence, multiplicity and regularity of solutions of quasilinear non-local fractional $p$-Laplacian problem. Concerning the evolution equation involving nonlocal operator we refer [7], [8], [9] and references therein.

In analogy to the Sobolev spaces with variable exponent (see [10], [11]), recently Kaufmann et al. have introduced the fractional Sobolev spaces with variable exponent in [12]. As the variable growth of the exponent $p$ in the local $p(x)$-Laplacian operator defined as $\text{div}(|\nabla u|^{p(x) - 2} \nabla u)$ makes it more suitable for modeling problems like image restoration, obstacle problems compared to $p$-Laplacian operator, henceforth, it is natural inquisitiveness to substitute the nonlocal fractional $p$-Laplacian with the non-local operator with variable exponents as defined in [12] and expect better modeling. Some results involving this type of operator associated variable exponent Sobolev spaces are studied in [13], [14], [15].

In present work, we study the existence, multiplicity and regularity of the solutions of problem involving non-local operator $(-\Delta)^{s(x,y)}_{p(x,y)}$ with variable exponents $s$ and $p$. Here we would like to emphasis that in our work we have consider the variable growth on the exponent $s$ as well. Also we would like to mention that the the embedding of the variable exponent fractional Sobolev space $W^{s(x,y), q(x,y)}(\Omega)$ (see Section 2 for definition) into $L^{q(x)}(\Omega)$ makes it more suitable for modeling problems like image restoration, obstacle problems compared to $p$-Laplacian operator.

Motivated by the pioneer work of Cerami et al. [16] of problems involving concave-convex nonlinearity in case of local operator and [17] in case of non-local operator, in present work we study the existence and multiplicity of solutions of non-local problem with variable exponents involving concave-convex nonlinearity. Also we establish the $L^\infty$ bound on the weak solution of (1.1) using elegant bootstrap technique. The $L^\infty$ bound on the weak solution of the problem involving fractional $p$-Laplacian has been studied by Franzina and Palatucci in [3].

Now we precisely state the main results to be proved in this paper. For any function $\phi : \overline{\Omega} \to \mathbb{R}$ (or $\phi : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}$) we set

$$\phi^- := \inf_{\overline{\Omega}} \phi(x) \quad \text{or} \quad \phi^- := \inf_{\overline{\Omega} \times \overline{\Omega}} \phi(x, y) \quad \text{and} \quad \phi^+ := \sup_{\overline{\Omega}} \phi(x) \quad \text{or} \quad \phi^+ := \sup_{\overline{\Omega} \times \overline{\Omega}} \phi(x, y).$$

We also define the function space $C_+ (\overline{\Omega}) := \{g \in C(\overline{\Omega}, \mathbb{R}) : 1 < g^- \leq g^+ < \infty\}$. Now, we assume that the continuous variable exponent $s : \overline{\Omega} \times \overline{\Omega} \to (0, 1)$ satisfies the following properties.

(S1) $s$ is symmetric, i.e., $s(x, y) = s(y, x)$ for all $(x, y) \in \overline{\Omega} \times \overline{\Omega}$.

(S2) $0 < s^- \leq s(x, y) \leq s^+ < 1$ for all $(x, y) \in \overline{\Omega} \times \overline{\Omega}$.
In this section we introduce fractional Sobolev spaces with variable exponent and establish the preliminary lemmas and embeddings associated with these spaces. For this we assume that the suitably chosen $\lambda^*$ for suitably chosen $\lambda^* > 0$ while the regularity of the associated eigen function is established by Theorem 1.3.

2 Variable exponent fractional Sobolev spaces and functional setting

In this section we introduce fractional Sobolev spaces with variable exponent and establish the preliminary lemmas and embeddings associated with these spaces. For this we assume that the
variable exponents $s(\cdot, \cdot)$ satisfies $(S1) - (S2)$ and $p(\cdot, \cdot)$ satisfies $(P1) - (P2)$. We also assume that $q \in C_+(\Omega)$. Recalling the definition of the Lebesgue spaces with variable exponent in [10], we introduce the variable exponent fractional Sobolev space as follows:

$$W = W^{s(x,y), q(x), p(x,y)}(\Omega) := \left\{ u \in L^q(x)(\Omega) : \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^p(x,y)}{|x - y|^{n + s(x,y)p(x,y)}} dxdy < \infty, \text{ for some } \lambda > 0 \right\}.$$ 

Set $[u]_{W^{s(x,y), p(x,y)}} := \inf \left\{ \lambda > 0 : \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{n + s(x,y)p(x,y)}} dxdy < 1 \right\}$. Then $(W, \| \cdot \|_W)$ is a Banach space equipped with the norm $\|u\|_W := \|u\|_{L^q(x)(\Omega)} + [u]_{W^{s(x,y), p(x,y)}}$.

We have the following Sobolev embedding theorem for $W$.

**Theorem 2.1.** Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. Let $s(\cdot, \cdot)$ and $p(\cdot, \cdot)$ satisfy $(S1) - (S2)$ and $(P1) - (P2)$, respectively and $q \in C_+(\Omega)$ such that $q(x) \geq p(x, x)$ for all $x \in \Omega$. Assume that $\beta \in C_+(\Omega)$ such that $\beta^* \leq \beta(x) < p^*_x(x)$ for $x \in \Omega$. Then there exits a constant $K = K(n, s, p, q, \beta, \Omega) > 0$ such that for every $u \in W$, $\|u\|_{L^p(\Omega)} \leq K\|u\|_W$. Moreover, this embedding is compact.

**Proof.** Here we follow the approach as in [12]. As we have $p$, $q$, $\beta$, $s$ are continuous on $\Omega$, it follows that $\inf_{x \in \Omega} \frac{np(x, x)}{n - s(x, x)p(x, x)} - \beta(x) = k_1 > 0$. From this expression and continuity of the exponents $p, q, \beta$ and $s$, it follows that there exists a finite family of disjoint open balls $\{B_i\}_{i=1}^k$ with radius $\epsilon = \epsilon(p, q, \beta, s, k_1)$ satisfying $\Omega \subseteq \bigcup_{i=1}^k B_i$ such that

$$\frac{np(z, y)}{n - s(z, y)p(z, y)} - \beta(x) = \frac{k_1}{2} > 0$$

(2.6)

and

$$q(x) \geq p(z, y)$$

(2.7)

for all $(z, y) \in B_i \times B_i$ and $x \in B_i$, $i = 1, 2, \ldots, k$, where $B_i = \Omega \cap B_i$ for each $i = 1, 2, \ldots, k$.

We set

$$p_i := \inf_{(z, y) \in B_i \times B_i} (p(z, y) - \delta) \text{ and } s_i := \inf_{(z, y) \in B_i \times B_i} s(z, y).$$

(2.8)

Again, using continuity of $p, q, \beta$ and $s$ we can choose $\delta = \delta(k_1)$, with $p^* - 1 > \delta > 0$, $t_i \in (0, s_i)$ and $\epsilon > 0$ such that (2.6), (2.7) and (2.9) give us

$$p_i^* := \frac{np_i}{n - t_ip_i} \geq \frac{k_1}{3} + \beta(x)$$

(2.9)

and

$$q(x) \geq p(x, x) > p_i$$

(2.10)

for all $x \in B_i$, $i = 1, 2, \ldots, k$. Indeed, from (2.8) as $p_i = \inf_{(z, y) \in B_i \times B_i} p(z, y) - \delta(k_1) < p(x, x) \leq q(x)$ for each $x \in B_i$, we have (2.10). Using Sobolev embedding results (Theorem 6.7, Theorem 6.9 in [11]) for constant exponent, we get a constant $C = C(n, p_i, t_i, \epsilon, B_i) > 0$ such that

$$\| u \|_{L^{p_i^*}(B_i)} \leq C \left\{ \|u\|_{L^{p_i}(B_i)} + \left( \int_{B_i \times B_i} \frac{|u(x) - u(y)|^{p_i}}{|x - y|^{n + t_ip_i}} dxdy \right)^{\frac{1}{p_i}} \right\}$$

(2.11)
As \(|u(x)| = \sum_{i=1}^{k} |u(x)|_{\mathcal{X}_{B_i}}\), we note that

\[
\|u\|_{L^{\beta(x)}(\Omega)} \leq \sum_{i=1}^{k} \|u\|_{L^{\beta_i}(B_i)}.
\] (2.12)

From (2.9), we get \(\beta(x) < p_i^{\ast}\) for all \(x \in B_i, i = 1, \ldots, k\). Hence we can take \(a_i \in C_+(\Omega)\) such that \(\frac{1}{\beta(x)} = \frac{1}{p_i^{\ast}} + \frac{1}{\pi_i(x)}\). Now applying Hölder’s inequality, we obtain

\[
\|u\|_{L^{\beta(x)}(B_i)} \leq k_2 \|u\|_{L^{p_i^{\ast}}(B_i)} \|1\|_{L^{\pi_i(x)}(B_i)} \leq k_3 \|u\|_{L^{p_i^{\ast}}(B_i)}
\] (2.13)

for some constants \(k_2, k_3 > 0\). From (2.12) and (2.13), we deduce

\[
\|u\|_{L^{\beta(x)}(\Omega)} \leq k_4 \sum_{i=1}^{k} \|u\|_{L^{p_i^{\ast}}(B_i)},
\] (2.14)

where \(k_4 > 0\) is a constant. Again, from (2.10), \(p_i < q(x)\) for all \(x \in B_i, i = 1, \ldots, k\), we argue in a similar way as above to obtain

\[
\sum_{i=1}^{k} \|u\|_{L^{p_i}(B_i)} \leq k_5 \|u\|_{L^{q(x)}(\Omega)}
\] (2.15)

for some constant \(k_5 > 0\). Next, for each \(i = 1, \ldots, k\), taking \(b_i \in C_+(B_i \times B_i)\) such that \(\frac{1}{p_i} = \frac{1}{p_i} + \frac{1}{\pi(x)}\) and considering the measure in \(B_i \times B_i\), given by \(d\mu(x, y) = \frac{1}{|x - y|^{n+(\mu_i - s(x))} p_i}\), we observe using Hölder’s inequality that there exist some constants \(k_6, k_7 > 0\) such that

\[
\left\{ \int_{B_i \times B_i} \frac{|u(x) - u(y)|^{p_i}}{|x - y|^{n+\mu_i - p_i}} \, dxdy \right\}^{\frac{1}{p_i}} = \left\{ \int_{B_i \times B_i} \left( \frac{|u(x) - u(y)|}{|x - y|^{s(x)} p_i} \right)^{p_i} \, dxdy \right\}^{\frac{1}{p_i}}
\]

\[
= \left[ \int_{B_i \times B_i} (U(x, y))^{p_i} \, d\mu(x, y) \right]^{\frac{1}{p_i}}
\]

\[
\leq k_6 \|U\|_{L^{(p_i, s_i)(\mu, B_i \times B_i)}} \|1\|_{L^{(p_i, s_i)(\mu, B_i \times B_i)}}
\]

\[
\leq k_7 \|U\|_{L^{(p_i, s_i)(\mu, B_i \times B_i)}}.
\] (2.16)

where the function \(U\) is defined in \(B_i \times B_i\) as \(U(x, y) = \frac{|u(x) - u(y)|}{|x - y|^{s(x)} p}, x \neq y\). Now, let \(\lambda > 0\) be such that

\[
\int_{B_i \times B_i} \frac{|u(x) - u(y)|^{p(x, y)}}{\lambda^{p(x, y)} |x - y|^{n+s(x, y)} p(x, y)} \, dxdy < 1.
\] (2.17)

Choose

\[
d_i := \sup \left\{ 1, \sup_{(x, y) \in B_i \times B_i} |x - y|^{s(x, y) - \mu_i} \right\} \quad \text{and} \quad \lambda_i = \lambda d_i.
\] (2.18)

Combining (2.17) and (2.18), we deduce

\[
\int_{B_i \times B_i} \left( \frac{U(x, y)}{\lambda_i} \right)^{p(x, y)} \, d\mu(x, y) = \int_{B_i \times B_i} \left( \frac{|u(x) - u(y)|}{\lambda_i |x - y| s(x, y)} \right)^{p(x, y)} \, dxdy
\]

\[
\leq \int_{B_i \times B_i} \frac{|u(x) - u(y)|^{p(x, y)}}{d_i^{p(x, y)} \lambda_i^{p(x, y)} |x - y|^{n+s(x, y)} p(x, y)} \, dxdy
\]

\[
\leq \int_{B_i \times B_i} \frac{|u(x) - u(y)|^{p(x, y)}}{\lambda_i^{p(x, y)} |x - y|^{n+s(x, y)} p(x, y)} \, dxdy < 1.
\]
Therefore from the above, we obtain $\|U\|_{L^p(x;\mu, B_i \times B_i)} \leq \lambda d_i$, which implies
\[
\|U\|_{L^p(x;\mu, B_i \times B_i)} \leq k_8 \|u\|_{L^p(B_i)}^{s(x,y), \mu} \leq k_8 \|u\|_{L^p(\Omega)}^{s(x,y), \mu},
\]
(2.19)
where $k_8 = \max_{i=1,2,...,k} \{d_i\} > 1$. Taking into account (2.16) and (2.19), we get
\[
\frac{1}{m} \sum_{i=1}^m \left\{ \int_{B_i \times B_i} \frac{|u(x) - u(y)|^p}{|x - y|^{n+s(x,y)}} \, dx \, dy \right\}^{\frac{1}{p}} \leq k_9 \|u\|_{L^p(\Omega)}^{s(x,y), \mu},
\]
(2.20)
for some constant $k_9 > 0$. Thus using (2.14), (2.16) and (2.20), we deduce
\[
\|u\|_{L^{\beta(x)}(\Omega)} \leq k_4 \sum_{i=1}^k \|u\|_{L^{\beta(x)}(B_i)}
\]
\[
\leq k_{10} \left\{ \|u\|_{L^p(B_i)} + \left( \int_{B_i \times B_i} \frac{|u(x) - u(y)|^p}{|x - y|^{n+s(x,y)}} \, dx \, dy \right)^{\frac{1}{p}} \right\}
\]
\[
\leq k_{11} \left\{ \|u\|_{L^q(\Omega)} + \|u\|_{L^p(\Omega)}^{s(x,y), \mu} \right\} = K \|u\|_{W},
\]
where the constants $k_{10}, k_{11}$ and $K > 0$. This proves that the space $W$ is continuously embedded in $L^{\beta(x)}(\Omega)$. The compactness of this embedding in the bounded domain $\Omega$ can be established by suitably extracting a convergent subsequence in $L^{\beta(x)}(B_i)$ for each $i = 1, ..., k$ of a bounded sequence $\{u_m\}$ in $W$ and arguing as above.

Next, for studying nonlocal problems involving the operator $(-\Delta)^{s(x,y)}$ with Dirichlet boundary data via variational methods, we introduce another new fractional type Sobolev spaces with variable exponents. One can refer [2] and references there in for this type of spaces in fractional $p$-Laplacian framework. Since the variable exponents $p, s$ are continuous in $\overline{\Omega} \times \overline{\Omega}$ and $q$ is continuous in $\overline{\Omega}$, using Tietze extension theorem, we can extend $p, s$ to $\mathbb{R}^n \times \mathbb{R}^n$ and $q$ to $\mathbb{R}^n$ continuously such that (S1) – (S2), (P1) – (P2) hold for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$ and $q \in C_+(\mathbb{R}^n)$.

Set $Q := (\mathbb{R}^n \times \mathbb{R}^n) \setminus (C \Omega \times C \Omega)$ and define
\[
X = X^{s(x,y), q(x), p(x,y)}(\Omega)
\]
\[
= \left\{ u : \mathbb{R}^n \to \mathbb{R} : u|_\Omega \in L^q(x)(\Omega) \text{ and } \int_Q \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{n+s(x,y)p(x,y)}} \, dx \, dy < \infty \right\}.
\]

The space $X$ is a normed linear space equipped with the following norm:
\[
\|u\|_X := \|u\|_{L^{s(x)}(\Omega)} + \inf \left\{ \lambda > 0 : \int_Q \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)} |x - y|^{n+s(x,y)p(x,y)}} \, dx \, dy < 1 \right\}.
\]

Next we define a subspace $X_0$ of $X$ as
\[
X_0 = X_0^{s(x,y), q(x), p(x,y)}(\Omega) := \{ u \in X : u = 0 \text{ a.e. in } C \Omega \}.
\]
Lemma 2.2. Let \( u \in X_0 \) and \( \rho_{X_0} \) be defined as in (2.21). Then, we have the following:

(i) \( \|u\|_{X_0} < 1 (= 1; > 1) \iff \rho_{X_0}(u) < 1 (= 1; > 1) \).

(ii) If \( \|u\|_{X_0} > 1 \), then \( \|u\|_{X_0}^p \leq \rho_{X_0}(u) \leq \|u\|_{X_0}^p \).

(iii) If \( \|u\|_{X_0} < 1 \), then \( \|u\|_{X_0}^p \leq \rho_{X_0}(u) \leq \|u\|_{X_0}^p \).

Lemma 2.3. Let \( u, u_m \in X_0 \), \( m \in \mathbb{N} \). Then the following statements are equivalent:

(i) \( \lim_{m \to \infty} \|u_m - u\|_{X_0} = 0 \),

(ii) \( \lim_{m \to \infty} \rho_{X_0}(u_m - u) = 0 \).

The proofs of Lemma 2.2 and Lemma 2.3 follow in the same line as the proofs of Theorem 3.1 and Theorem 3.2, respectively, in [10]. Now, we study the following Sobolev embedding theorem for the space \( X_0 \).

Theorem 2.4. Let \( \Omega \subset \mathbb{R}^n \) be a smooth bounded domain. Let \( s(\cdot, \cdot) \) and \( p(\cdot, \cdot) \) satisfy (S1) – (S2) and (P1) – (P2) respectively such that \( s(x, y) p(x, y) < n \) for all \( (x, y) \in \overline{\Omega} \times \Omega \) and \( q \in C(\Omega) \) such that \( p(x, x) \leq q(x) < p_s^*(x) \) for all \( x \in \Omega \). Then for any \( \beta \in C(\Omega) \) such that \( 1 < r(x) < p_s^*(x) \) for all \( x \in \Omega \), there exits a constant \( C = C(n, s, p, q, \beta, \Omega) > 0 \) such that for every \( u \in X_0 \),

\[
\|u\|_{L^q(\Omega)} = \|u\|_{L^q(\Omega)} \leq C\|u\|_{X_0}.
\]

Moreover, this embedding is compact.

Proof. First we note that, as \( p, s \) are continuous in \( \overline{\Omega} \times \Omega \) and \( q, \beta \) are continuous in \( \Omega \), using Tietze extension theorem, we can extend \( p, s \) to \( \mathbb{R}^n \times \mathbb{R}^n \) and \( q, \beta \) in \( \mathbb{R}^n \) such that \( p(x, x) \leq q(x) \) and \( r(x) \leq p_s^*(x) \) for all \( x \in \mathbb{R}^n \). Next, we claim that there exists a constant \( C' > 0 \) such that

\[
\|u\|_{L^q(\Omega)} \leq \frac{1}{C'} \|u\|_{X_0} \text{ for all } u \in X_0.
\] (2.22)

This is equivalent to proving that \( \inf_{u \in A} \|u\|_{X_0} \), where \( A := \{u \in X_0 : \|u\|_{L^q(\Omega)} = 1\} \) is achieved. Let \( \{u_m\} \subset A \) be a sequence such that \( \|u_m\|_{X_0} \downarrow \inf_{u \in A} \|u\|_{X_0} := C' \) as \( m \to \infty \). This implies that \( \{u_m\} \) is bounded in \( X_0 \) and \( L^{q(x)}(\Omega) \) and hence in \( W \). Therefore, up to a subsequence \( u_m \rightharpoonup u_0 \) in \( W \) as \( m \to \infty \). Now, from Theorem 2.4 it follows that \( u_m \to u_0 \) strongly in \( L^{q(x)}(\Omega) \) as \( m \to \infty \).
We extend \( u_0 \) to \( \mathbb{R}^n \) by setting \( u_0(x) = 0 \) on \( x \in \mathbb{C} \). This implies \( u_m(x) \to u_0(x) \) a.e. \( x \in \mathbb{R}^n \) as \( m \to \infty \). By using Fatou’s Lemma, we have

\[
\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u_0(x) - u_0(y)|^{p(x,y)}}{|x-y|^{n+\beta(x,y)p(x,y)}} \, dx \, dy \leq \liminf_{n \to \infty} \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u_m(x) - u_m(y)|^{p(x,y)}}{|x-y|^{n+\beta(x,y)p(x,y)}} \, dx \, dy,
\]

which implies that \( \|u_0\|_{X_0} \leq \liminf_{n \to \infty} \|u_m\|_{X_0} = C' \) and hence \( u_0 \in X_0 \). Also, as \( \|u_0\|_{L^{q(x)}(\Omega)} = 1 \), we have \( u_0 \in A \). Therefore \( \|u_0\|_{X_0} = C' \). This proves our claim and hence (2.22). From (2.22), it follows that

\[
\|u\|_W = \|u\|_{L^{q(x)}(\Omega)} + \|u\|_{L^{q(x)}(\Omega)} + \|u\|_{X_0} \leq (1 + \frac{1}{C})\|u\|_{X_0},
\]

which implies that \( X_0 \) is continuously embedded in \( W \). Also as from Theorem 2.1, \( W \) is continuously embedded in \( L^{\beta(x)}(\Omega) \), there exists a constant \( C(n, s, p, r, \beta, \Omega) > 0 \), such that

\[
\|u\|_{L^{\beta(x)}(\mathbb{R}^n)} = \|u\|_{L^{\beta(x)}(\Omega)} \leq C(n, s, p, r, \beta, \Omega)\|u\|_{X_0}.
\]

To prove that the embedding given in (2.23) is compact, let \( \{v_n\} \) be a bounded sequence in \( X_0 \). This implies that \( \{v_n\} \) is bounded in \( W \). Hence by using Theorem 2.1 we infer that there exists \( v_0 \in L^{\beta(x)}(\Omega) \) such that up to a subsequence \( v_n \to v_0 \) strongly in \( L^{\beta(x)}(\Omega) \) as \( m \to \infty \). This completes the theorem.

Using the above Sobolev embedding result together with Theorem 2.4.14 and Theorem 3.4.9 in [11] we have the following proposition.

**Proposition 2.5.** \((X_0, \|\cdot\|_{X_0})\) is a uniformly convex and reflexive Banach space.

**Remark 2.** From now onwards we take \( q(x) = p(x, x) \) and consider the function space \( X_0^{s(x,y), p(x,x), p(x,y)}(\Omega) \). For brevity we still denote \( X_0^{s(x,y), p(x,x), p(x,y)}(\Omega) \) by \( X_0 \).

## 3 Proof of main results

In this section we give the proof of Theorem 1.2 and Theorem 1.3. First, we study the existence of multiple solutions of (1.1) with concave-convex nonlinearity. By the standard critical point theory, the weak solutions of (1.1) are characterized by the critical points of the associated energy functional \( J_\lambda : X_0 \to \mathbb{R} \) given as

\[
J_\lambda(u) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^{p(x,y)}}{p(x, y)} \, dx \, dy - \int_{\Omega} \frac{|u|^{\alpha(x)}}{\alpha(x)} \, dx - \int_{\Omega} F(x, u(x)) \, dx.
\]

(3.24)

Note that \( J_\lambda \) is well-defined and Gâteaux differentiable on \( X_0 \). Also \( J_\lambda \) admits the mountain-pass geometry. Precisely, we have the following lemma.

**Lemma 3.1.** Let \( J_\lambda \) be defined as in (3.24). Then we have the followings.

(i) There exists \( \lambda^* > 0 \) such that for any \( \lambda \in (0, \lambda^*) \) we can choose \( R > 0 \) and \( 0 < \delta < 1 \) such that \( J_\lambda(u) \geq R > 0 \) for all \( u \in X_0 \) with \( \|u\|_{X_0} = \delta \).

(ii) There exists \( \phi \in X_0, \phi > 0 \) such that \( J(t\phi) \to -\infty \) as \( t \to +\infty \).

(iii) There exists \( \psi \in X_0, \psi > 0 \) such that \( J(t\psi) < 0 \) for all \( t \to 0^+ \).
Lemma 3.2. Let $J$ be such that $\lambda > 0$ with $\lambda > \lambda^*$, for $\|u\|_{X_0} < 1$, we have

$$J_\lambda(u) = \int_{\mathbb{R}^n \times [0,1]} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{n+\delta(x,y)p(x,y)}} \, dx \, dy - \lambda \int_{\Omega} u^\alpha(x) \, dx - \int_{\Omega} F(x, u) \, dx$$

$$\geq \frac{1}{p^+} \|u\|_{X_0}^{p^+} + \frac{\lambda}{\alpha} \max \left\{ \|u\|_{L^{r_1}(\Omega)}^{r_1}, \|u\|_{L^{r_2}(\Omega)}^{r_2} \right\} - \frac{M}{r} \left\{ \|u\|_{L^{r_1}(\Omega)}^{r_1}, \|u\|_{L^{r_2}(\Omega)}^{r_2} \right\}$$

$$\geq \frac{1}{p^+} \|u\|_{X_0}^{p^+} - \frac{\lambda c_1}{\alpha} \|u\|_{X_0}^\alpha - \frac{c_2}{r} \|u\|_{X_0}^{-\alpha}$$

$$\geq \left\{ \frac{1}{2p^+} - \frac{\lambda c_1}{\alpha} \|u\|_{X_0}^{-\alpha} - \frac{c_2}{r} \|u\|_{X_0}^{-\alpha} \right\} \|u\|_{X_0}^{p^+},$$

where $c_1, c_2 > 0$ are constants. Now for each $\lambda > 0$, we define the function, $T_\lambda : (0, +\infty) \to \mathbb{R}$ as

$$T_\lambda(t) = c_1 \frac{\lambda}{\alpha} t^{\alpha} - c_2 \frac{1}{r} t^{-\alpha}.$$
Lemma 3.3. The functional $J_\lambda$ satisfies $(PS)_c$ condition for any $c \in \mathbb{R}, c > 0$.

Proof. Let $\{u_m\} \subset X_0$ be a $(PS)_c$ sequence of the functional $J_\lambda$, i.e., $J_\lambda(u_m) \to c$ and $\|J'_\lambda(u_m)\|_{X_0^*} \to 0$ as $m \to \infty$. Note that, $\{u_m\}$ is bounded in $X_0$. Indeed, if $\{u_m\}$ is unbounded in $X_0$, we may assume that $\|u_m\|_{X_0} \to +\infty$ as $m \to +\infty$. Now, for $m$ large enough, using Lemma 2.2 and Theorem 2.4 together with (F3), we have

$$c = J_\lambda(u_m) - \frac{1}{b} \langle J'_\lambda(u_m), u_m \rangle$$

$$= \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u_m(x) - u_m(y)| p(x,y)}{|x - y|^{n+\alpha(x,y)p(x,y)}} dx dy - \lambda \int_{\Omega} \frac{|u_m(x)|^{\alpha(x)}}{\alpha(x)} dx - \int_{\Omega} F(x, u_m) dx$$

$$- \frac{1}{b} \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u_m(x) - u_m(y)| p(x,y)}{|x - y|^{n+\alpha(x,y)p(x,y)}} dx dy + \frac{\lambda}{b} \int_{\Omega} |u_m(x)|^{\alpha(x)} dx + \frac{1}{b} \int_{\Omega} f(x, u_m) u_m dx$$

$$> \left( \frac{1}{b} - \frac{1}{b} \right) \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u_m(x) - u_m(y)| p(x,y)}{|x - y|^{n+\alpha(x,y)p(x,y)}} dx dy - \lambda \left( 1 - \frac{1}{b} \right) \int_{\Omega} |u_m(x)|^{\alpha(x)} dx$$

$$+ \int_{\Omega} \left[ \frac{1}{b} f(x, u_m) u_m - F(x, u_m) \right] dx$$

$$> \left( \frac{1}{b} - \frac{1}{b} \right) \|u_m\|_{X_0^*} - \lambda \left( 1 - \frac{1}{b} \right) \max \left\{ \|u_m\|_{L^{\alpha(x)}(\Omega)}, \|u_m\|_{L^{\alpha(x)}(\Omega)} \right\}$$

$$> \left( \frac{1}{b} - \frac{1}{b} \right) \|u_m\|_{X_0^*} - \lambda c_4 \|u_m\|_{X_0^*}$$

for some constant $c_4 > 0$. As from (F3) and (F4) we get $p^+ < b$ and $\alpha^+ < p^-$, respectively, the above expression gives a contradiction. Therefore the sequence $\{u_m\}$ is bounded in $X_0$. Since $X_0$ is a reflexive Banach space (Proposition 2.5), it follows that there exists $u_1 \in X_0$ such that up to a subsequence still denoted by $\{u_m\}$, $u_m \rightharpoonup u_1$ weakly in $X_0$ as $m \to \infty$. Also by using Theorem 2.4 we get $u_m \to u_1$ in $L^{\alpha(x)}(\Omega)$ and $L^{p(x)}(\Omega)$ strongly and hence $u_m(x) \to u_1(x)$ point-wise a.e. $x \in \mathbb{R}^n$ as $m \to \infty$. We claim that $u_m \to u_1$ strongly in $X_0$ as $m \to \infty$. For $u, \phi \in X_0$, we set the notation

$$\langle I(u), \phi \rangle := \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)| p(x,y)^{-2} (u(x) - u(y)) (\phi(x) - \phi(y))}{|x - y|^{n+\alpha(x,y)p(x,y)}} dx dy. \quad (3.26)$$

Then, clearly $\langle I(u_1), (u_m - u_1) \rangle \to 0$, when $m \to \infty$. Since $\{u_m\}$ is bounded in $X_0$ and $\|J'_\lambda(u_m)\|_{X_0^*} \to 0$ as $m \to \infty$, for $\phi = u_m - u_1$, taking into account (3.24) and (3.26), we deduce

$$o_m(1) = \langle J'_\lambda(u_m), (u_m - u_1) \rangle = \langle I(u_m), (u_m - u_1) \rangle -$$

$$\lambda \int_{\Omega} |u_m(x)|^{\alpha(x)-2} u_m(x)(u_m - u_1)(x) dx - \int_{\Omega} f(x, u_m(x))(u_m - u_1)(x) dx. \quad (3.27)$$

The second term in the right hand side of (3.27) equals to $o_m(1)$. Indeed, using the fact that $u_m \to u_1$ strongly in $L^{\alpha(x)}(\Omega)$ as $m \to \infty$ together with Hölder’s inequality and Lemma 3.2 we get

$$\left| \int_{\Omega} |u_m(x)|^{\alpha(x)-2} u_m(x)(u_m - u_1)(x) dx \right| \leq \int_{\Omega} |u_m(x)|^{\alpha(x)-1} |u_m - u_1(x)| dx$$

$$\leq \|u_m^{\alpha(x)-1}\|_{L^{\alpha(x)-1}(\Omega)} \|u_m - u_1\|_{L^{\alpha(x)}(\Omega)}$$

$$\leq \left\{ \|u_m\|_{L^{\alpha(x)}(\Omega)}^{\alpha(x)-1} + \|u_m\|_{L^{\alpha(x)}(\Omega)}^{\alpha(x)-1} \right\} \|u_m - u_1\|_{L^{\alpha(x)}(\Omega)} = o_m(1). \quad (3.28)$$
Since using (F2), the third term in the right hand side of (3.27) gives us \(\int f(x, u_m(x))(u_m - u_1)(x)dx \leq \int |u_m(x)|^{r(x)-1} |u_m(x) - u_1(x)| dx\), hence arguing same as above, we obtain

\[
\int f(x, u_m(x))(u_m - u_1)(x)dx = o_m(1).
\]

(3.29)

Thus using the fact \(u_m \to u_1\) as \(m \to \infty\) together with (3.20), (3.27) and (3.29), we deduce that \(\langle I(u_m), (u_m - u_1)\rangle \to 0\) as \(m \to \infty\) and \(\langle I(u_1), (u_m - u_1)\rangle \to 0\) as \(m \to \infty\), which imply

\[
\langle (I(u_m) - I(u_1)), (u_m - u_1)\rangle = o_m(1).
\]

(3.30)

We denote \(v_m = u_m - u_1\). Then for \(1 < p(x, y) < 2\), taking into account Lemma 2.2 H"older's inequality, Lemma 5.2 and Simon's inequality [20], we deduce

\[
\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|v_m(x) - v_m(y)|^{p(x, y)}}{|x-y|^{n+s(x,y)p(x,y)}} dxdy
\]

\[
\leq \frac{1}{(p - 1)\alpha^2} \int_{\mathbb{R}^n \times \mathbb{R}^n} \left\{ \frac{|u_m(x) - u_m(y)|^{p(x, y)}}{|x-y|^{n+s(x,y)p(x,y)}} + \frac{|u_1(x) - u_1(y)|^{p(x, y)}}{|x-y|^{n+s(x,y)p(x,y)}} \right\} dxdy
\]

\[
= c_0 \int_{\mathbb{R}^n \times \mathbb{R}^n} \left\{ g_1(x, y) \frac{p(x, y)}{2} + g_2(x, y) \frac{2-p(x, y)}{2} \right\} dxdy
\]

\[
\leq c_0 \left\{ \left\| \frac{p(x, y)}{2} \right\| \left\| g_1(x, y) \right\| \frac{2-p(x, y)}{2} \right\} + \left\{ g_1(x, y) \frac{p(x, y)}{2} + g_2(x, y) \frac{2-p(x, y)}{2} \right\}
\]

\[
\leq c_0 \left\{ \left\| g_1 \right\|_{L^1(\mathbb{R}^n \times \mathbb{R}^n)} + \left\| g_3 \right\|_{L^1(\mathbb{R}^n \times \mathbb{R}^n)} \right\} \left\| g_2 \right\|_{L^1(\mathbb{R}^n \times \mathbb{R}^n)} + \left\| g_2 \right\|_{L^1(\mathbb{R}^n \times \mathbb{R}^n)}
\]

\[
\leq c_5 \left\{ \left\| g_1 \right\|_{L^1(\mathbb{R}^n \times \mathbb{R}^n)} + \left\| g_3 \right\|_{L^1(\mathbb{R}^n \times \mathbb{R}^n)} \right\} \left\| g_2 \right\|_{L^1(\mathbb{R}^n \times \mathbb{R}^n)} + \left\| g_2 \right\|_{L^1(\mathbb{R}^n \times \mathbb{R}^n)}
\]

(3.31)

where \(c_5 > 0\) is a constant and \(g_i, i = 1, 2, 3\) are defined as follows.

\[
g_1(x, y) = \left[ \frac{|u_m(x) - u_m(y)|^{p(x, y)}}{|x-y|^{n+s(x,y)p(x,y)}} + \frac{|u_1(x) - u_1(y)|^{p(x, y)}}{|x-y|^{n+s(x,y)p(x,y)}} \right],
\]

\[
g_2(x, y) = \left[ \frac{|u_m(x) - u_m(y)|^{p(x, y)}}{|x-y|^{n+s(x,y)p(x,y)}} \right] \text{ and } g_3(x, y) = \left[ \frac{|u_1(x) - u_1(y)|^{p(x, y)}}{|x-y|^{n+s(x,y)p(x,y)}} \right].
Finally using Lemma 2.2 and 3.3, it follows that

\[
\rho_{X_0}(v_m) \leq c_5 \left\{ \| (I(u_m) - I(u_1)) \|_{X_0}^{\rho^+ / 2} + \| (I(u_m) - I(u_1)) \|_{X_0}^{\rho^- / 2} \right\} \left( \| u_m \|_{X_0}^{\rho^+ / 2} + \| u_m \|_{X_0}^{\rho^- / 2} + \| u_1 \|_{X_0}^{\rho^+ / 2} + \| u_1 \|_{X_0}^{\rho^- / 2} \right). \tag{3.32}
\]

Thus from (3.30) and (3.32), we estimate \( \rho_{X_0}(v_m) \to 0 \) as \( m \to \infty \). Hence Lemma 2.3 gives us \( \lim_{m \to \infty} \| u_m - u_1 \|_{X_0} = 0 \). Now for \( p(x,y) > 2 \), taking into account Lemma 2.2, Hölder’s inequality, Lemma 3.2, Simon’s inequality 2.10 and (3.30), we deduce \( \rho_{X_0}(v_m) \leq 2^{\rho^+} (I(u_m) - I(u_1)), (u_m - u_1) = o_m(1) \). Again using the same argument as used for (3.32) we conclude \( u_m \to u_1 \) strongly in \( X_0 \) as \( m \to \infty \). This completes the lemma.

Now, we give the proof of Theorem 1.2.

**Proof of Theorem 1.2.** As from Lemma 3.1 and Lemma 3.3 there exists \( \lambda^* > 0 \) such that for all \( \lambda \in (0, \lambda^*) \), \( J_\lambda \) satisfies the mountain pass geometry and Palais-Smale condition, using mountain pass theorem, we infer that there exists \( u_1 \in X_0 \), a critical point of \( J_\lambda \) with \( J_\lambda(u_1) = \overline{c} > 0 = J_\lambda(0) \).

Hence there exists a non-trivial weak solution of the problem (1.1). Next, we prove the existence of the second weak solution of (1.1). From Lemma 3.1 (iii), we get that \( \inf_{u \in \overline{B}_\delta(0)} J_\lambda(u) = \underline{c} < 0 \), where \( \overline{B}_\delta(0) = \{ u \in X_0 : \| u \|_{X_0} \leq \delta \} \). Now by applying Ekeland’s variational principle, for given any \( \epsilon > 0 \) there exists \( w_\epsilon \in \overline{B}_\delta(\Omega) \) such that \( J_\lambda(w_\epsilon) < \inf_{u \in \overline{B}_\delta(0)} J_\lambda(u) + \epsilon \) and

\[
J_\lambda(w_\epsilon) < J_\lambda(u) + \epsilon \| u - w_\epsilon \|_{X_0}, \text{ for all } u \in B_\delta(0), u \neq w_\epsilon. \tag{3.33}
\]

We choose \( \epsilon > 0 \) such that \( 0 < \epsilon < \inf_{u \in \partial B_\delta(0)} J_\lambda(u) - \inf_{u \in B_\delta(0)} J_\lambda(u) \). This implies \( w_\epsilon \in B_\delta(0) \). Now by taking \( u = w_\epsilon + tv \) for \( t > 0 \) and \( v \in B_\delta(0) \setminus \{0\} \), from (3.33), we deduce that

\[
\lim_{t \to 0} \frac{J_\lambda(w_\epsilon) - J_\lambda(w_\epsilon + tv)}{t} \leq \epsilon \| v \|_{X_0}.
\]

This implies, for all \( v \in B_\delta(0), \langle -J_\lambda'(w_\epsilon), v \rangle \leq \epsilon \| v \|_{X_0} \). Now replacing \( v \) by \( -v \), it follows that \( \| J_\lambda'(w_\epsilon) \|_{X_0} \leq \epsilon \). This implies the \( \{ w_m \} \subset B_\delta(0) \) is a Palais-Smale sequence at level \( \underline{c} < 0 \). In view of Lemma 3.3, we conclude that there exists \( u_2 \in B_\delta(0) \subset X_0 \) such that \( w_m \to u_2 \) strongly in \( X_0 \) as \( m \to \infty \) and \( u_2 \) is a critical point of \( J_\lambda \) with \( J_\lambda(u_2) = \underline{c} < 0 \). Thus, \( u_2 \) is a nontrivial weak solution of (1.1). Also noticing the fact that \( J_\lambda(|u|) \leq J_\lambda(u) \), we infer that \( u_2 \) is non-negative. Clearly as \( J(u_1) = \overline{c} > 0 \geq \underline{c} = J(u_2), u_1 \neq u_2 \).

Next, we give the proof for our second theorem in this article. We have the following comparison inequality which is useful to handle the variable exponent growth of the non-local operators.

**Lemma 3.4.** Let \( u : \Omega \to \mathbb{R} \) be a function such that \( u(x) > 1 \) a.e. \( x \in \Omega \) and \( \eta(\cdot, \cdot) \) be a symmetric real valued function such that \( 0 \leq \eta(x,y) < \infty \) for all \( (x,y) \in \Omega \times \Omega \). Suppose, \( 0 \leq \eta_0 \leq \eta := \inf_{(x,y) \in \Omega \times \Omega} \eta(x,y) \). Then we have the following inequality \( |u^{\eta_0}(x,y) - u^{\eta}(x,y)| \geq |u^{\eta}(x) - u^{\eta}(y)| \).


Proof. Without loss of generality, we assume $u(x) > u(y) > 1$, $x \neq y$ in $\Omega$. Now,

$$
\begin{align*}
| u^\gamma(x,y)(x) - u^\gamma(x,y)(y) | &= u^\gamma(x,y)(x) - u^\gamma(x,y)(y) \\
&> u^\gamma(x) \left[ 1 - \left\{ \frac{u(y)}{u(x)} \right\}^{\eta_0} \right] \\
&= u^\gamma(x) - u^\gamma(y)
\end{align*}
$$

Now, we recall the following algebraic inequality from [21]:

**Lemma 3.5.** Let $1 < p < +\infty$ and $\kappa \geq 1$. For every $a,b,m \geq 0$ it holds that $| a - b |^{p-2} (a-b)(a_m^m - b_m^m) \geq \frac{k^p_p}{(\kappa + p-1)^p} | a_m^{p-1} - b_m^{p-1} |^p$, where $a_m = \min\{a,m\}$ and $b_m = \min\{b,m\}$.

**Proof of the Theorem 1.3.** Let $u \in X_0$ satisfy (1.4). Without loss of generality we assume that $|u(x)| > 1$ a.e. $x \in \Omega$. First we consider the case $u(x) > 1$ a.e. $x \in \Omega$. For $m > 0$, we define $u_m(x) := \min\{m,u(x)\}$. Therefore, $u_m \in X_0$. By taking $\phi = u_m^\kappa$, $\kappa > 1$ as test function in (1.4) and using Lemma 3.3 and (F2), we deduce

$$
\begin{align*}
\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{k^p_p(x,y)}{(\kappa + p(x,y) - 1)^p | x - y |^{n+s(x,y)p(x,y)}} \left| \frac{u(x) - u(y)}{u_m(x)} - \frac{u(y)}{u_m(y)} \right| d\mu(x,y) \\
&\leq \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{| u(x) |^{p(x,y)}(x) - u_m^{p(x,y)}(y) | x - y |^{n+s(x,y)p(x,y)}}{\kappa + p(x,y) - 1} d\mu(x,y) \\
&= \lambda \int_{\Omega} | u(x) |^{\alpha - 2} u(x) \phi(x) dx + \int_{\Omega} f(x,u(x)) \phi(x) dx \\
&\leq (\lambda + M) \int_{\Omega} | u(x) |^{\gamma(x) - 1 + \kappa} dx.
\end{align*}
$$

(3.34)

Lemma 3.4 and (F2) give $| u_m^\kappa(x) - u_m^\kappa(y) | \leq \frac{\kappa^{p(x,y)-1}}{\kappa + p(x,y) - 1} | x - y |^{n+s(x,y)p(x,y)}$ and the last inequality, it follows that

$$
\begin{align*}
\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\kappa^{\gamma r^+}}{\beta + r^+ - 1} \left| u_m^{r^+}(x) - u_m^{r^+}(y) \right| d\mu(x,y) \\
&\leq (\lambda + M) \int_{\Omega} | u(x) |^{\gamma r^+ - 1 + \kappa} dx \\
&= (\lambda + M) \| u \|^{\gamma r^+}_{L^{\gamma r^+}(\Omega)},
\end{align*}
$$

which implies

$$
\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\kappa^{\gamma r^+}}{\beta + r^+ - 1} \left| u_m^{r^+}(x) - u_m^{r^+}(y) \right| d\mu(x,y) \\
\leq c_6 \gamma^{r^+} \| u \|^{\gamma r^+}_{L^{\gamma r^+}(\Omega)}
$$

(3.35)

for some constant $c_6 = (\lambda + M) \int_{\Omega} (\frac{c}{\lambda})^{r^+} > 0$. Now using the fact $\lim_{m \to \infty} u_m(x) = u(x)$ a.e. $x \in \Omega$, combining with Fatou’s lemma and (3.35), we obtain

$$
\rho X_0(u) \geq \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{| u^\gamma(x) - u^\gamma(y) |^{p(x,y)}}{| x - y |^{n+s(x,y)p(x,y)}} d\mu(x,y) \\
\leq \liminf_{m \to \infty} \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{| u_m^\gamma(x) - u_m^\gamma(y) |^{p(x,y)}}{| x - y |^{n+s(x,y)p(x,y)}} d\mu(x,y) \\
\leq c_7 \gamma^{r^+} \| u \|^{\gamma r^+}_{L^{\gamma r^+}(\Omega)}.
$$

(3.36)
for some \( \theta \in C_{+}(\Omega) \) such that \( \theta(x) < p_{s}^{*}(x) \) for all \( x \in \Omega \) and \( r^{+} < \theta^{-} < p_{s}^{*} \). If \( \| u^{\gamma} \|_{L^{\theta(x)}(\Omega)} < 1 \) in (3.37), then using the assumption \( u(x) > 1 \) a.e. \( x \in \Omega \) together with the comparison results for variable exponent norm and modular function (Theorem 3.2 in [10]), we get

\[
\| u \|_{L^{\theta^{-} \cdot}(\Omega)} \leq \int_{\Omega} (u^{\gamma}(x))^{\frac{\theta}{\gamma}} dx \leq \| u^{\gamma} \|_{L^{\theta(x)}(\Omega)}.
\]

From (3.37) and (3.38), we have \( \| u \|_{L^{\gamma \cdot}(\Omega)} \leq c_{7}^{1/\gamma} < 1 \). Letting \( \gamma \to \infty \), we get \( u \in L^{\infty}(\Omega) \). Similarly for \( \| u^{\gamma} \|_{L^{\gamma(x)}(\Omega)} > 1 \) in (3.37), in view of the assumption \( u(x) > 1 \) a.e. \( x \in \Omega \) together with the comparison results among variable exponent norm and modular function (Theorem 3.2 in [10]), we get

\[
\| u \|_{L^{\gamma^{-} \cdot}(\Omega)} \leq \int_{\Omega} (u^{\gamma}(x))^{\frac{\theta}{\gamma}} dx \leq \| u^{\gamma} \|_{L^{\theta(x)}(\Omega)}.
\]

Combining (3.37) and (3.39), we deduce \( \| u \|_{L^{\gamma \cdot}(\Omega)} \leq \| u \|_{L^{\gamma \cdot}(\Omega)}^{\frac{\gamma^{+}/\theta^{-}}{\gamma^{-}}} \leq (c_{7}^{\frac{\theta^{+}}{\theta^{-}}})^{1/\gamma} \). Again by letting \( \gamma \to \infty \), we obtain \( u \in L^{\infty}(\Omega) \).

**Case II:** For \( \| u^{\gamma} \|_{L^{\infty}(\Omega)} > 1 \), from Lemma 2.2 it follows that \( \| u^{\gamma} \|_{L^{p}X_{0}} \leq \rho X_{0}(u^{\gamma}) \). Thus taking into account this previous inequality together with (3.36) and Theorem 2.4, we estimate

\[
\| u^{\gamma} \|_{L^{p}X_{0}(\Omega)} \leq c_{8}^{\gamma^{+}/p^{-}} \| u \|_{L^{p}X_{0}(\Omega)}^{\gamma^{+}/p^{-}}.
\]

where \( c_{8} > 0 \) is a constant. Thus, for \( \| u^{\gamma} \|_{L^{p}X_{0}(\Omega)} < 1 \) in (3.40), using (3.38), we get

\[
\| u \|_{L^{\gamma^{-} \cdot}(\Omega)} \leq c_{8}^{\gamma^{+}/p^{-}} \| u \|_{L^{\gamma^{-} \cdot}(\Omega)}^{\gamma^{+}/p^{-}}.
\]

Now, we apply bootstrap argument for (3.41). Since \( r^{+} < \theta^{-} \), there exists \( \gamma_{1} > 0 \) such that \( \gamma_{1}r^{+} = \theta^{-} \). Putting \( \gamma = \gamma_{1} \) in (3.41), we obtain

\[
\| u \|_{L^{\gamma_{1} \cdot}(\Omega)} \leq c_{8}^{\gamma^{+}/p^{-}} (\gamma_{1})^{1/\gamma_{1}} r^{+}/p^{-} \| u \|_{L^{\gamma_{1} \cdot}(\Omega)}^{r^{+}/p^{-}}.
\]

Again we can choose \( \gamma = \gamma_{2} \) such that \( \gamma_{2}r^{+} = \gamma_{1} \theta^{-} \). Then from (3.41) and (3.31), we have

\[
\| u \|_{L^{\gamma_{2} \cdot}(\Omega)} \leq c_{8}^{\gamma^{+}/p^{-}} (\gamma_{1}^{1/\gamma_{1}} \gamma_{2}^{1/\gamma_{2}}) r^{+}/p^{-} \| u \|_{L^{\gamma_{2} \cdot}(\Omega)}^{r^{+}/p^{-}}.
\]

Hence by using induction hypothesis, we deduce

\[
\| u \|_{L^{\gamma_{m} \cdot}(\Omega)} \leq c_{8}^{r^{+}/p^{-}} (\gamma_{m}^{1/\gamma_{m}}) r^{+}/p^{-} \| u \|_{L^{\gamma_{m} \cdot}(\Omega)}^{r^{+}/p^{-}}.
\]

where \( \gamma_{m} = (\theta^{-}/r^{+})^{m} \). Now, (3.43) can be re-written as

\[
\| u \|_{L^{\gamma_{m} \cdot}(\Omega)} \leq \sum_{j=1}^{m} \gamma_{j}^{1/\gamma_{j}} \left( \prod_{j=1}^{m} (\gamma_{j}^{1/\gamma_{j}}) \right)^{r^{+}/p^{-}} \| u \|_{L^{\gamma_{j} \cdot}(\Omega)}^{r^{+}/p^{-}}.
\]

Since \( \gamma_{j}^{1/\gamma_{j}} > 1 \) for all \( j \in \mathbb{N} \) and \( \lim_{j \to \infty} \gamma_{j}^{1/\gamma_{j}} = 1 \), there exists a constant \( c_{9} > 1 \), independent of \( m \) such that (3.44) gives us

\[
\| u \|_{L^{\gamma_{m} \cdot}(\Omega)} \leq \sum_{j=1}^{m} \gamma_{j}^{1/\gamma_{j}} \left( c_{9}^{\gamma_{j}^{1/\gamma_{j}}} \right)^{r^{+}/p^{-}} \| u \|_{L^{\gamma_{j} \cdot}(\Omega)}^{r^{+}/p^{-}}.
\]

Note that, \( \gamma_{j} = (\theta^{-}/r^{+})^{j} > 1 \). This yields \( \sum_{j=1}^{m} \gamma_{j}^{1/\gamma_{j}} \) and \( \gamma_{m}^{1/\gamma_{m}} \) are convergent and \( \gamma_{m}^{1/\gamma_{m}} \to \infty \) as \( m \to \infty \). Hence, letting \( m \to \infty \) in (3.44), we get that \( u \in L^{\infty}(\Omega) \).

Next, we consider the case when \( \| u^{\gamma} \|_{L^{\theta(x)}(\Omega)} > 1 \) in (3.30). Then, combining (3.39) and (3.40), we obtain \( \| u \|_{L^{\gamma \cdot}(\Omega)} \leq c_{10}^{\gamma^{+}/p^{-}} \| u \|_{L^{\gamma^{+} \cdot}(\Omega)}^{d} \) for some constant \( c_{10} > 0 \) and \( d = r^{+}/p^{-} \). Again repeating the bootstrap argument as above we can conclude \( u \in L^{\infty}(\Omega) \). For the case \( u(x) < -1 \) for a.e. \( x \in \Omega \), we replace \( u \) by \( -u \) in the above arguments and infer that \( u \in L^{\infty}(\Omega) \). This completes our theorem.
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