Completeness Theorems for First-Order Logic
Analysed in Constructive Type Theory

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LFCS’20, Deerfield Beach, Florida
January 6, 2020

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Is the completeness theorem of FOL algorithmic?

**Gödel 1930, Henkin 1949**
Classical completeness proofs

**Kreisel 1962**
Standard completeness requires Markov’s principle

**Veldman 1976**
Modified semantics yield fully constructive completeness

**Herbelin/Ilik 2016**
Computational analysis in constructive type theory
Constructive type theory as a formal meta-logic

**Reification**
Deductions $\vdash \varphi$ can be extracted from meta-level proofs of $\models \varphi$

**Conservativity**
Meta-logic and formal object-logic “agree” on first-order formulas

**Admissibility**
Particular instances of completeness may be provable

**Computability**
Internal notion of computation allows direct analysis
Main Contributions

Constructive analysis of completeness theorems:

- **Model-theoretic semantics**: uniform analysis of standard, exploding, and minimal semantics of the $\rightarrow, \forall, \bot$-fragment; admissibility results

- **Algebraic semantics**: constructive completeness proofs for algebras with explicit atom interpretations based on Scott ’08

- **Game semantics**: instantiation of a general isomorphism\(^1\) between winning strategies and deductions to intuitionistic first-order logic; streamlined representation of dialogues as state transition systems

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1Sørensen and Urzyczyn ’08

Reusable Coq library for first-order logic hyperlinked with the PDF: https://www.ps.uni-saarland.de/extras/fol-completeness/
Constructive Type Theory
Constructive Type Theory

Features as implemented in Coq’s type theory:

- Inductive types: $\mathbb{B}$, $\mathbb{N}$, lists $\mathcal{L}(X)$, vectors $X^n$, ...
- Standard type formers: $X \to Y$, $X \times Y$, $X + Y$, $\forall x. F x$, $\Sigma x. F x$
- Propositional universe $\mathbb{P}$ with logical connectives: $\to$, $\land$, $\lor$, $\forall$, $\exists$

All definable functions are computable!

So standard notions from computability theory can be synthesised:

Definition

Let $X$ be a type and $p : X \to \mathbb{P}$ be a predicate. We say that $p$ is

- **decidable** if there is $f : X \to \mathbb{B}$ with $\forall x. p x \iff f x = \text{tt}$
- **enumerable** if there is $f : \mathbb{N} \to X$ with $\forall x. p x \iff \exists n. f n = \mathllap{\lceil} x \mathllap{\rceil}$
Markov’s Principle: 2 Versions

"Termination is stable (under double negation)"

For the computation internal to constructive type theory:

\[ MP := \forall f : \mathbb{N} \rightarrow \mathbb{B}. \neg \neg (\exists n. f \ n = \text{tt}) \rightarrow \exists n. f \ n = \text{tt} \]

For the cbv. lambda calculus L as a formal model of computation:\(^2\)

\[ MP_L := \forall s : \mathcal{L}. \neg \neg \mathcal{E} \ s \rightarrow \mathcal{E} \ s \quad (\mathcal{E} \ s := "s terminates") \]

- MP implies MP\(_L\) (since \(\mathcal{E}\) is enumerable)
- MP is independent but admissible in Coq’s type theory\(^3\)
- MP\(_L\) is independent but admissible in Coq’s type theory

\(^2\) Plotkin '75, Forster/Smolka '17
\(^3\) Coquand/Manna '17, Pédrot/Tabareau '18
First-order Logic
Syntax

Represented as inductive type over signature $\Sigma$ providing function symbols $f : \mathcal{F}_\Sigma$ and relation symbols $P : \mathcal{P}_\Sigma$ with arities $|f|$ and $|P|:

$$
t : T ::= x \mid f \overrightarrow{t} \quad (x : \mathbb{N})
$$

$$
\varphi, \psi : \mathcal{F} ::= \bot \mid P \overrightarrow{t} \mid \varphi \rightarrow \psi \mid \varphi \land \psi \mid \varphi \lor \psi \mid \forall \varphi \mid \exists \varphi
$$

We set $\neg \varphi ::= \varphi \rightarrow \bot$ and denote the $\rightarrow, \forall, \bot$-fragment by $\mathcal{F}^\ast$.

De Bruijn encoding of quantifiers to avoid naming conflicts:

$$
"P \ x \ y \rightarrow \ \forall x. \ \exists y. \ P \ x \ y" \quad \sim\sim \quad P \ \overrightarrow{7} \ \overrightarrow{4} \ \rightarrow \ \forall \ \exists \ \overrightarrow{P} \ \overrightarrow{1} \ \overrightarrow{0}
$$

The variables 7 and 4 in this example are free and variables that do not occur freely are fresh. A formula with no free variables is closed.
Deduction Systems

Represented as inductive predicates of the form $\mathcal{L}(F) \rightarrow F \rightarrow P$:

$\Gamma \vdash \varphi \quad$ Al $\quad \frac{\Gamma \vdash \forall \varphi}{\Gamma \vdash \forall \varphi}$

$\Gamma \vdash \forall \varphi \quad$ AE $\quad \frac{\Gamma \vdash \varphi[t]}{\Gamma \vdash \forall \varphi}$

$\Gamma \vdash \varphi[t] \quad$ EI $\quad \frac{\Gamma \vdash \exists \varphi \quad \uparrow, \varphi \vdash \psi}{\Gamma \vdash \psi}$

$\Gamma \vdash \exists \varphi \quad$ EE $\quad \frac{\Gamma \vdash \psi}{\Gamma \vdash \psi}$

\ldots

Given $\Gamma$, $\varphi$, and $\psi$ one can compute a fresh variable $x$ such that

1. $\uparrow \Gamma \vdash \varphi$ iff $\Gamma \vdash \varphi[x]$ and
2. $\uparrow \Gamma, \varphi \vdash \psi$ iff $\Gamma, \varphi[x] \vdash \psi$.

Classical variant $\Gamma \vdash_c \varphi$ obtained by adding $\Gamma \vdash_c ((\varphi \rightarrow \psi) \rightarrow \varphi) \rightarrow \varphi$.
Natural generalisation to $\mathcal{T} \vdash \varphi$ for theories $\mathcal{T} : F \rightarrow P$. 
Computational Properties

If $\Sigma$ is a data type (enumerable and discrete), then:

- $T$ and $F$ are data types
- $\Gamma \vdash_c \varphi$ is enumerable
- $MP_L$ is equivalent to the stability of $\Gamma \vdash_c \varphi$
- Stability of $\Gamma \vdash_c \varphi$ is independent but admissible
- $T \vdash_c \varphi$ for enumerable $T$ behaves similarly wrt. MP

Strategic consequences:

1. Analysing completeness up to the stability of deduction suffices
2. Completeness can be analysed on two levels ($MP$ and $MP_L$)
3. Some formulations of completeness may be admissible
Model-Theoretic Semantics
(Considering enumerable $\Sigma$ and the $\mathbb{F}^*$-fragment)
A (Tarski) model $\mathcal{M}$ over a domain $D$ is a family of functions

$$f^\mathcal{M} : D|f| \to D \quad \text{and} \quad P^\mathcal{M} : D|P| \to \mathcal{P}.$$ 

Assignments $\rho : \mathbb{N} \to D$ are extended to term evaluations $\hat{\rho} : T \to D$. The relation $\mathcal{M} \models_\rho \varphi$ for formulas $\varphi : \mathbb{F}^*$ is defined recursively by:

$$\mathcal{M} \models_\rho \bot := \bot \quad \mathcal{M} \models_\rho \varphi \to \psi := \mathcal{M} \models_\rho \varphi \to \mathcal{M} \models_\rho \psi$$

$$\mathcal{M} \models_\rho P \vec{t} := P^\mathcal{M}(\hat{\rho} \vec{t}) \quad \mathcal{M} \models_\rho \forall \varphi := \forall a : D. \mathcal{M} \models_{a;\rho} \varphi$$

$\mathcal{M}$ is called classical if $\mathcal{M} \models ((\varphi \to \psi) \to \varphi) \to \varphi$ for all $\varphi, \psi : \mathbb{F}^*$. We write $\mathcal{T} \models \varphi$ if $\mathcal{M} \models_\rho \varphi$ for every classical $\mathcal{M}$ and $\rho$ with $\mathcal{M} \models_\rho \mathcal{T}$. 
Tarski Semantics: Standard Completeness

The central model existence theorem has a constructive proof:⁴

**Theorem**

*Every consistent (and closed) theory is satisfied in a classical model.*

Model existence yields **quasi-completeness**: \( \mathcal{T} \models \varphi \) implies \( \neg \neg (\mathcal{T} \vdash_c \varphi) \)

\( \Rightarrow \) Single applications of MP and MP\(_L\) yield completeness statements

\( \Rightarrow \) These completeness statements are admissible

Leads to the following characterisations:

**Theorem**

- *Completeness of* \( \Gamma \vdash_c \varphi \) *is equivalent to* MP\(_L\).
- *Completeness of* \( \mathcal{T} \vdash_c \varphi \) *for enumerable* \( \mathcal{T} \) *is equivalent to* MP.
- *Completeness of* \( \mathcal{T} \vdash_c \varphi \) *for arbitrary* \( \mathcal{T} \) *is equivalent to* EM.

⁴Herbelin/Ilik ’16
Tarski Semantics: Constructive Completeness

Generalise semantics to admit exploding models:⁵

- Extend models with falsity interpretation $\bot^M : \mathbb{P}$
- $\mathcal{M}$ is exploding if $\mathcal{M} \models \bot \rightarrow \varphi$ for all $\varphi : \mathbb{F}^*$
- $\mathcal{T} \models_e \varphi$ if $\mathcal{M} \models \rho \varphi$ for all exploding classical $\mathcal{M}$ and $\rho$ with $\mathcal{M} \models \rho \mathcal{T}$

Generalised model existence yields constructive completeness:

**Theorem**

For every (closed) theory $\mathcal{T}$ there is an exploding classical model $\mathcal{M}$ and an assignment $\rho$ such that (1) $\mathcal{M} \models \rho \mathcal{T}$ and (2) $\mathcal{M} \models \rho \bot$ implies $\mathcal{T} \vdash_c \bot$.

**Corollary**

$\mathcal{T} \models_e \varphi$ implies $\mathcal{T} \vdash_c \varphi$ (for closed $\mathcal{T}$ and $\varphi$).

⁵Veldman '76
Kripke Semantics (cf. Herbelin/Lee ’09)

Analogous properties hold for Kripke entailment $\Gamma \models \varphi$.

Universal Kripke models yield completeness as follows:

- $\Gamma \models_e \varphi$ implies $\Gamma \vdash \varphi$
- $\Gamma \models \varphi$ implies $\neg\neg(\Gamma \vdash \varphi)$

These in fact hold for a cut-free sequent calculus $\Gamma \Rightarrow \varphi$, therefore establishing a normalisation-by-evaluation procedure.

Again, the stability assumptions are necessary, so we conclude:

**Theorem**

- Completeness of $\Gamma \vdash \varphi$ is equivalent to $\text{MP}_L$.
- Completeness of $\mathcal{T} \vdash \varphi$ for enumerable $\mathcal{T}$ implies MP.
- Completeness of $\mathcal{T} \vdash \varphi$ for arbitrary $\mathcal{T}$ implies EM.
Algebraic Semantics
Given a complete Heyting algebra \((\mathcal{H}, \leq, 0, \bigwedge, \bigvee, \Rightarrow)\) we extend atom interpretations \([P \vec{t}] : \mathcal{H}\) to all formulas in \(\mathbb{F}\):

\[
\begin{align*}
[\bot] & := 0 \\
[\varphi \land \psi] & := [\varphi] \cap [\psi] \\
[\varphi \Rightarrow \psi] & := [\varphi] \Rightarrow [\psi] \\
[\varphi \lor \psi] & := [\varphi] \cup [\psi] \\
[\exists \varphi] & := \bigvee_{t} [\varphi[t]] \\
[\forall \varphi] & := \bigwedge_{t} [\varphi[t]]
\end{align*}
\]

A formula \(\varphi\) is valid whenever \(x \leq [\varphi]\) for all \(x\) in every \(\mathcal{H}\).

The Lindenbaum algebra \(\mathcal{L} = (\mathbb{F}, \varphi \vdash \psi, \bot, \land, \lor, \Rightarrow)\) can be completed via a standard construction which then witnesses \([\varphi] \equiv \lambda \psi. \psi \vdash \varphi\), so:

**Theorem**

*If \(\varphi\) is valid in every complete Heyting algebra, then \(\vdash \varphi\).*

This generalises to \(\Gamma \vdash_c \varphi\) and complete Boolean algebras.
Game Semantics
We represent intuitionistic E-dialogues parametrised over \((\mathcal{F}', \mathcal{F}^0, A, \triangleright, \mathcal{D})\).

The attacks and defenses for the concrete instance of FOL are as follows:

- \(a_\perp \triangleright \perp\) \hspace{1cm} \(D_\perp = \{\emptyset\}\)
- \(a_t \triangleright \forall \varphi\) \hspace{1cm} \(D_t = \{\varphi[t]\}\)
- \(a_\exists \triangleright \exists \varphi\) \hspace{1cm} \(D_\exists = \{\varphi[t] \mid t : T\}\)
- \(a_\rightarrow \triangleright \varphi \rightarrow \psi\) \hspace{1cm} \(D_\rightarrow = \{\psi\}\)
- \(a_\leftarrow \triangleright \varphi \leftarrow \psi\) \hspace{1cm} \(D_\leftarrow = \{\varphi, \psi\}\)
- \(a_L \triangleright \varphi \land \psi\) \hspace{1cm} \(D_L = \{\varphi\}\)

- The player and opponent take turns in manipulating the game state \((A, a)\) containing the opponent’s current admissions \(A\) and attack \(a\).
- The player may \textbf{defend} against \(a\) or \textbf{attack} any formula from \(A\).
- The opponent reacts by \textbf{attacking} the player’s defense or by \textbf{defending/countering} against the player’s attack.
- A state is \textbf{winning} if for all allowed player moves every possible opponent reaction leads to a winning state.
- A formula is \textbf{E-valid} if \(([\varphi], a)\) is winning for all initial attacks \(a\) on \(\varphi\).
Game Semantics (cf. Sørensen and Urzyczyn ’08)

A sequent calculus LJD of type $L(F') \rightarrow (F' \rightarrow P) \rightarrow P$ is defined by:

- $\varphi \in S$ justified $\Gamma \varphi \quad \forall a \mid \psi \triangleright \varphi. \quad \Gamma,\psi \Rightarrow_D D_a$
  \[ \Gamma \Rightarrow_D S \quad \text{R} \]

- $\varphi \in \Gamma$ justified $\Gamma \psi \quad a \mid \psi \triangleright \varphi \quad \forall \theta \in D_a. \quad \Gamma,\theta \Rightarrow_D S \quad \forall a' \mid \tau \triangleright \psi. \quad \Gamma,\tau \Rightarrow_D D_{a'}$
  \[ \Gamma \Rightarrow_D S \quad \text{L} \]

Winning strategies and LJD derivations are easily shown isomorphic, so:

**Theorem**

Any formula $\varphi$ is E-valid if and only if one can derive $\{\varphi\} \Rightarrow_D \varnothing$.

The concrete instance of LJD for FOL is equivalent to the standard sequent calculus LJ defined by judgements $\Gamma \Rightarrow_J \varphi$:

**Theorem**

Any formula $\varphi$ is E-valid if and only if one can derive $\{\varphi\} \Rightarrow_J \varnothing$. 
Discussion
What else is in the paper?

- Constructive completeness proofs for minimal logic (\(\to, \forall\)-fragment)
  - by a generalised model existence theorem for Tarski semantics
  - by the same exploding universal model for Kripke semantics

- Completeness of \(\mathcal{T} \vdash_c \varphi\) for formulas with free variables

- Independence of MP\(_L\) as special case of Pédrot/Tabareau '18:
  MP\(_L\) with independence of premise rule IP yields a decider for \(\mathcal{E}\)

- Reductions establishing the equivalence of MP\(_L\) and stability of ND:
  - \(\mathcal{E}\) reduces to \(\vdash_c\) (chaining reductions from previous work\(^6\))
  - \(\vdash\) reduces to \(\mathcal{E}\) (by showing that \(\vdash\) is L-enumerable\(^7\))
  - \(\vdash_c\) reduces to \(\vdash\) (double negation translation)

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\(^6\)Wuttke '18, Forster et al. ’18, and Forster et al. ’19.

\(^7\)Using the extraction framework from Forster/Kunze ’19.
Coq Formalisation

- About 7500 lines of code
- All main results parametric in the signature $\Sigma$
- De Bruijn encoding of binders supported by Autosubst
- Usage of type classes, parametric deduction systems, Equations package, special ND tactics, etc. to ease mechanisation
- Part of a growing Coq library of undecidability proofs

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8https://www.ps.uni-saarland.de/~kstark/autosubst2/
9https://github.com/uds-psl/coq-library-undecidability

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Future Work

Extend the completeness library:
- Model-theoretic semantics including $\lor$ and $\exists$
- Strong completeness ($\mathcal{T} \models \varphi \rightarrow \mathcal{T} \vdash \varphi$) for more semantics
- Hybrid semantics such as mix of model-theoretic and algebraic parts
- Second-order and higher-order logic

Other related ideas:
- Analyse the (upward) Löwenheim-Skolem theorem
- Study first-order axiom systems (undecidability, incompleteness, ...)
- Implement a proof extraction procedure
- Separate MP from MP$_L$
- Find an effective model with classical reasoning in $\mathbb{P}$
Related Work I

- Walter Felscher.
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  *Annals of pure and applied logic, 28(3):217–254, 1985.*

- Georg Kreisel.
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  *The Journal of Symbolic Logic, 27(2):139–158, 1962.*

- Wim Veldman.
  An intuitionistic completeness theorem for intuitionistic predicate logic 1.
  *The Journal of Symbolic Logic, 41(1):159–166, 1976.*

- Danko Ilik.
  *Constructive completeness proofs and delimited control.*
  PhD thesis, Ecole Polytechnique X, 2010.
Related Work II

Hugo Herbelin and Danko Ilik.
An analysis of the constructive content of Henkin’s proof of Gödel’s completeness theorem.
Draft, 2016.

Hugo Herbelin and Gyesik Lee.
Forcing-based cut-elimination for Gentzen-style intuitionistic sequent calculus.
In *International Workshop on Logic, Language, Information, and Computation*, pages 209–217. Springer, 2009.

Victor N. Krivtsov.
Semantical completeness of first-order predicate logic and the weak fan theorem.
*Studia Logica*, 103(3):623–638, 2015.

Yannick Forster and Gert Smolka.
Weak call-by-value lambda calculus as a model of computation in Coq.
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Yannick Forster and Fabian Kunze.
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Related Work IV

Morten Heine Sørensen and Pawel Urzyczyn.
Sequent calculus, dialogues, and cut elimination.
*Reflections on Type Theory, λ-Calculus, and the Mind*, pages 253–261, 2007.

Dana Scott.
The algebraic interpretation of quantifiers: Intuitionistic and classical.
In V.W. Marek A. Ehrenfeucht and M. Srebrny, editors, *Andrzej Mostowski and Foundational Studies*. IOS Press, 2008.

Dominik Wehr.
A Constructive Analysis of First-Order Completeness Theorems in Coq, 2019.
Bachelor’s thesis, Saarland University.

Maximilian Wuttke.
Verified Programming Of Turing Machines In Coq, 2018.
Bachelor’s thesis, Saarland University.
Details on the Coq Formalisation

| Section                          | Specification | Proofs |
|---------------------------------|---------------|--------|
| Preliminaries Autosubst         | 169           | 53     |
| Preliminaries for $\mathbb{F}^*$| 680           | 599    |
| Tarski Semantics                | 655           | 682    |
| Kripke Semantics                | 342           | 255    |
| On Markov’s Principle           | 593           | 978    |
| Preliminaries for $\mathbb{F}$  | 523           | 430    |
| Heyting Semantics               | 297           | 456    |
| Dialogue Semantics              | 312           | 488    |
| **Total**                       | **3571**      | **3941** |
Completeness implies stability

Fact

\[(\forall \Gamma \varphi. \Gamma \models \varphi \rightarrow \Gamma \vdash c \varphi) \rightarrow \forall \Gamma \varphi. (\neg \neg \Gamma \vdash c \varphi) \rightarrow \Gamma \vdash c \varphi\]

Proof.

Assume completeness and let \(\neg \neg (\Gamma \vdash c \varphi)\). To conclude \(\Gamma \vdash c \varphi\) it suffices to show \(\Gamma, \neg \varphi \vdash c \bot\) and by completeness \(\Gamma, \neg \varphi \models \bot\). So for \(M \models \rho \Gamma, \neg \varphi\) we have to prove \(\bot\), meaning that we may use \(\neg \neg (\Gamma \vdash c \varphi)\) positively as \(\Gamma \vdash c \varphi\). But then \(\Gamma \models \varphi\) by soundness, contradicting \(M \models \rho \Gamma, \neg \varphi\). \(\square\)
Lemma

For every closed theory $\mathcal{T}$ there is an extension $\mathcal{T}' \supseteq \mathcal{T}$ s.t.:

- $\mathcal{T}'$ maintains consistency, i.e. $\mathcal{T} \vdash_c \bot$ whenever $\mathcal{T}' \vdash_c \bot$.
- $\mathcal{T}'$ is deductively closed, i.e. $\varphi \in \mathcal{T}'$ whenever $\mathcal{T}' \vdash_c \varphi$.
- $\mathcal{T}'$ respects implication, i.e. $\varphi \rightarrow \psi \in \mathcal{T}'$ iff $\varphi \in \mathcal{T}' \rightarrow \psi \in \mathcal{T}'$.
- $\mathcal{T}'$ respects universal quantification, i.e. $\forall \varphi \in \mathcal{T}'$ iff $\forall t. \varphi[t] \in \mathcal{T}'$.

Extension in two subsequent steps:

1. $\mathcal{H} := \mathcal{T} \cup \{ \varphi_n[n] \rightarrow \forall \varphi_n \mid n : \mathbb{N} \}$ is Henkin and maintains consistency

2. $\Omega \supseteq \mathcal{H}$ which is maximal, i.e. $\varphi \in \Omega$ if $\mathcal{O}, \varphi \vdash_c \bot$ implies $\mathcal{O} \vdash_c \bot$:

$$\Omega_0 := \mathcal{H} \quad \Omega_{n+1} := \Omega_n \cup \{ \varphi_n \mid \Omega_n, \varphi_n \vdash_c \varphi_\bot \rightarrow \Omega_n \vdash_c \varphi_\bot \} \quad \Omega := \bigcup_{n : \mathbb{N}} \Omega_n$$

$\Rightarrow$ If $\mathcal{T}$ is consistent, then its extension $\mathcal{T}'$ yields a syntactic model of $\mathcal{T}$. 
Lemma

For every closed formula $\varphi_\bot$ and closed $\mathcal{T}$ there is $\mathcal{T}' \supseteq \mathcal{T}$ s.t.:

- $\mathcal{T}'$ maintains $\varphi_\bot$-consistency, i.e. $\mathcal{T} \vdash_c \varphi_\bot$ whenever $\mathcal{T}' \vdash_c \varphi_\bot$.
- $\mathcal{T}'$ is deductively closed, i.e. $\varphi \in \mathcal{T}'$ whenever $\mathcal{T}' \vdash_c \varphi$.
- $\mathcal{T}'$ respects implication, i.e. $\varphi \rightarrow \psi \in \mathcal{T}'$ iff $\varphi \in \mathcal{T}' \rightarrow \psi \in \mathcal{T}'$.
- $\mathcal{T}'$ respects universal quantification, i.e. $\forall \varphi \in \mathcal{T}'$ iff $\forall t. \varphi[t] \in \mathcal{T}'$.

Extension in three subsequent steps:

0. $\mathcal{E} \supseteq \mathcal{T}$ which is exploding, i.e. $(\varphi_\bot \rightarrow \varphi) \in \mathcal{E}$ for all closed $\varphi$:

$$\mathcal{E} := \mathcal{T} \cup \{ \varphi_\bot \rightarrow \varphi \mid \varphi \text{ closed} \}$$

1. $\mathcal{H} \supseteq \mathcal{E}$ which is Henkin, i.e. $(\varphi_n[n] \rightarrow \forall \varphi_n) \in \mathcal{H}$ for all $n$.

2. $\Omega \supseteq \mathcal{H}$ which is maximal, i.e. $\varphi \in \Omega$ if $\Omega, \varphi \vdash_c \varphi_\bot$ implies $\Omega \vdash_c \varphi_\bot$.

$\Rightarrow$ If $\mathcal{T} \nvdash_c \varphi_\bot$, then its extension $\mathcal{T}'$ resembles a syntactic model of $\mathcal{T}$. 

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A Kripke model $\mathcal{K}$ over a domain $D$ is a preorder $(\mathcal{W}, \preceq)$ with

$$f^K : D^{\lvert f \rvert} \to D \quad P^K : \mathcal{W} \to D^{\lvert P \rvert} \to \mathbb{P} \quad \bot^K : \mathcal{W} \to \mathbb{P}$$

where we require $P^K_v \vec{a} \to P^K_w \vec{a}$ and $\bot^K_v \to \bot^K_w$ whenever $v \preceq w$.

Assignments $\rho$ are extended to formulas $\mathbb{F}^*$:

$$w \models_{\rho} \bot := \bot^K_w \quad w \models_{\rho} \varphi \to \psi := \forall v \geq w. v \models_{\rho} \varphi \to v \models_{\rho} \psi$$

$$w \models_{\rho} P \vec{t} := P^K_w (\hat{\rho} \vec{t}) \quad w \models_{\rho} \forall \varphi := \forall a : D. w \models_{a; \rho} \varphi$$

$\mathcal{K}$ is standard if $\bot^K_w \to \bot$ for all $w$ and exploding if $\mathcal{K} \models \bot \to \varphi$ for all $\varphi$. We write $\mathcal{T} \models \varphi$ if $\mathcal{K} \models_{\rho} \varphi$ for all standard $\mathcal{K}$ and $\rho$ with $\mathcal{K} \models_{\rho} \mathcal{T}$, and $\mathcal{T} \models_{e} \varphi$ when relaxing to exploding models.
Kripke Semantics: Semantic Cut-Elimination

Introduce the cut-free sequent calculus LJT with focusing $\Gamma ; \varphi \Rightarrow \psi$:

$$
\begin{align*}
\Gamma ; \varphi \Rightarrow \varphi & \quad \text{A} \\
\Gamma ; \varphi \Rightarrow \psi & \quad \varphi \in \Gamma \\
\Gamma & \Rightarrow \psi & \quad \text{C} \\
\Gamma ; \varphi \Rightarrow \psi & \quad \psi \Rightarrow \theta \\
\Gamma & \Rightarrow \varphi & \quad \Gamma ; \varphi \Rightarrow \theta & \quad \text{IL}
\end{align*}
$$

$$
\begin{align*}
\Gamma, \varphi \Rightarrow \psi & \quad \text{IR} \\
\Gamma & \Rightarrow \varphi \Rightarrow \psi \\
\Gamma ; \varphi[t] \Rightarrow \psi & \quad \text{AL} \\
\uparrow \Gamma & \Rightarrow \varphi \\
\Gamma & \Rightarrow \forall \varphi & \quad \text{AR} \\
\Gamma & \Rightarrow \perp & \quad \text{E}
\end{align*}
$$

The following ingredients yield a cut-elimination procedure:

- $\Gamma \vdash \varphi$ implies $\Gamma \models_e \varphi$ (straightforward by induction)
- $\Gamma \models_e \varphi$ implies $\Gamma \Rightarrow \varphi$ (exploiting a universal model)
- Proofs $\Gamma \Rightarrow \varphi$ translate to normal proofs $\Gamma \vdash \varphi$ (by induction)

Herbelin/Lee ’09
Kripke Semantics: Constructive Completeness

The exploding model $\mathcal{U}$ over the domain $\mathbb{T}$ of terms is defined on the type of contexts $\Gamma$ preordered by inclusion $\subseteq$. Further, we set:

$$f^\mathcal{U} \vec{d} := f \vec{d} \quad \quad \quad P^\Gamma \vec{d} := \Gamma \Rightarrow P \vec{d} \quad \quad \quad \bot^\Gamma := \Gamma \Rightarrow \bot$$

We verify $\mathcal{U}$ following the normalisation-by-evaluation structure.\(^{11}\)

**Lemma**

*In the universal Kripke model $\mathcal{U}$ the following hold.*

1. $\Gamma \models_{\sigma} \varphi \rightarrow \Gamma \Rightarrow \varphi[\sigma]$
2. $(\forall \Gamma' \psi. \ \Gamma \subseteq \Gamma' \rightarrow \Gamma' \Rightarrow \psi[\sigma] \Rightarrow \psi \rightarrow \Gamma' \Rightarrow \psi) \rightarrow \Gamma \models_{\sigma} \varphi$

**Corollary**

$\Gamma \models_{e} \varphi$ *implies* $\Gamma \Rightarrow \varphi$.

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\(^{11}\)Berger/Schwichtenberg ‘91
Kripke Semantics: Standard Completeness

The standard model $\mathcal{C}$ over the domain $\mathbb{T}$ of terms is defined on the type of consistent contexts $\Gamma \not\models \bot$ preordered by inclusion $\subseteq$. Further, we set:

$$f^\mathcal{C} \vec{d} := f \vec{d} \quad P^\mathcal{C} \vec{d} := \neg\neg(\Gamma \Rightarrow P \vec{d}) \quad \bot^\mathcal{C} := \bot$$

**Lemma**

*In the universal Kripke model $\mathcal{C}$ the following hold.*

1. $\Gamma \models_\sigma \varphi \rightarrow \neg\neg(\Gamma \Rightarrow \varphi[\sigma])$
2. $(\forall \Gamma' \psi. \Gamma \subseteq \Gamma' \rightarrow \Gamma'; \varphi[\sigma] \Rightarrow \psi \rightarrow \neg\neg(\Gamma' \Rightarrow \psi)) \rightarrow \Gamma \models_\sigma \varphi$

**Corollary**

$\Gamma \models \varphi$ *implies* $\neg\neg(\Gamma \Rightarrow \varphi)$.  

As for $\Gamma \vdash_c \varphi$, completeness of $\Gamma \vdash \varphi$ is admissible and equivalent to MP$_L$. 