ON TRIVIAL WORDS IN FINITELY PRESENTED GROUPS

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To the memory of Herb Wilf, Pierre Leroux and Philippe Flajolet — all great men of generating functions.

Abstract. We propose a numerical method for studying the cogrowth of finitely presented groups. To validate our numerical results we compare them against the corresponding data from groups whose cogrowth series are known exactly. Further, we add to the set of such groups by finding the cogrowth series for Baumslag-Solitar groups $BS(N,N) = \langle a, b | a^N b = b a^N \rangle$ and prove that their cogrowths are algebraic numbers. We have been unable to find the cogrowth series for other Baumslag-Solitar groups, but we have found recurrences that yield the first few terms of the cogrowth series exponentially faster than is possible by naive methods. Finally we apply our numerical method to several presentations of Thompson’s group $F$ and our results give strong indication that the group is not amenable.

1. Introduction

In this article we consider the function that counts the number of trivial words in a finitely presented group, the so-called cogrowth function. The exponential growth rate of this function is simply called the cogrowth and is intimately related to the amenability of the group [12, 19]. Amenability is an active area of current research, and cogrowth is just one of many characterisations. The amenability of one group in particular – Richard Thompson’s group $F$ – has been the subject of intensive research and conjecture.

In this article we propose a new numerical technique to estimate the cogrowth of finitely presented groups, based on ideas from statistical mechanics, which we show to be quite accurate in predicting the cogrowth for a range of groups for which the cogrowth series and/or amenability is known: these include Baumslag-Solitar groups, a finitely presented relative of the Basilica group, and some free products studied by Kouksov [26]. We apply the method to several different presentations for Thompson’s group $F$, and the evidence obtained points strongly towards the conclusion that $F$ is not amenable.

The present article builds on previous work of a subset of the authors [14], where various techniques, also based in statistical mechanics, were applied to the problem of estimating and computing the cogrowth for Thompson’s group $F$. This in turn built on previous work of Burillo, Cleary and Wiest [8], and Arzhantseva, Guba, Lustig, and Préaux [3], who applied experimental techniques to the problem of deciding the amenability of $F$.

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For the benefit of readers outside of group theory, we start with a precise definition of group presentations and cogrowth.

**Definition 1.1** (Presentations and trivial words). A presentation $\langle a_1, \ldots, a_k | R_1, \ldots, R_m \rangle$ encodes a group as follows.

- The letters $a_i$ are elements of the group and are called *generators*, and the $R_i$ are finite length words over the letters $a_1, \ldots, a_k, a_1^{-1}, \ldots, a_k^{-1}$ and are called *relations* or *relators*.
- A group is called *finitely generated* if it can be encoded by a presentation with the list $a_1, \ldots, a_k$ finite, and *finitely presented* if it can be encoded by a presentation with both lists $a_1, \ldots, a_k$ and $R_1, \ldots, R_m$ finite.
- A word in the letters $a_1, \ldots, a_k, a_1^{-1}, \ldots, a_k^{-1}$ is called *freely reduced* if it contains no subword $a_1^\pm 1 a_j^\pm 1$. The set of all freely reduced words, together with the operation of concatenation followed by free reduction (deleting $a_1^\pm 1 a_j^\pm 1$ pairs) forms a group, called the *free group* on the letters $\{a_1, \ldots, a_k\}$, which we denote by $F(a_1, \ldots, a_k)$.
- Let $N(R_1, \ldots, R_m)$ be the normal subgroup which contains all words of the form $\prod_{j=1}^{m} \rho_j R_j \rho_j^{-1}$ where $\rho_i$ is any word in the free group, and $R_j$ is one of the relators or their inverses. This subgroup is called the *normal closure* of the set of relators, and is the smallest normal subgroup in $F(a_1, \ldots, a_k)$ that contains all words $R_1, \ldots, R_m$.
- The group encoded by the presentation $\langle a_1, \ldots, a_k | R_1, \ldots, R_m \rangle$ is defined to be the quotient group $F(a_1, \ldots, a_k)/N(R_1, \ldots, R_m)$.

We will make extensive use of this last point in the work below. We call a word in $F(a_1, \ldots, a_k)$ that equals the identity element in $G$ a *trivial word*.

The function $c : \mathbb{N} \to \mathbb{N}$ where $c(n)$ is the number of freely reduced words in the generators of a finitely generated group that represent the identity element is called the *cogrowth function* and the corresponding generating function is called the *cogrowth series*. The rate of exponential growth of the cogrowth series is the *cogrowth* of the group (with respect to a chosen finite generating set). Grigorchuk and independently Cohen [12, 19] proved that a finitely generated group is amenable if and only if its cogrowth is twice the number of generators minus 1.

For more background on amenability and cogrowth see [30, 34], and on Thompson’s group $F$ see [9, 10]. The free group on two or more letters, as defined above, is known to be non-amenable. Also, subgroups of amenable groups are also amenable. It follows that if a group contains a subgroup isomorphic to the free group on 2 generators ($F(a_1, a_2)$ above), then it cannot be amenable. Thompson’s group $F$ has no such subgroup, but at the same time, no simple proof of amenability has been forthcoming – hence the intense interest in this example.

The article is organised as follows. In Section 2 we adapt an algorithm designed to sample self-avoiding polygons to the problem of estimating the growth rate of...
trivial words in finitely presented groups. The algorithm we describe actually works when the group is finitely generated but has infinitely many relations – in this case its application is more subtle (in the way one samples relators from an infinite list). To further validate our algorithm we test it on groups whose cogrowth series are known exactly. In Section 3 we add to this pool of results by finding the cogrowth series of the Baumslag-Solitar groups $BS(N,N) = \langle a,b \mid a^N b = b a^N \rangle$. We apply our algorithm and analyse the resulting data in Section 4 and summarise our results in Section 5.

2. Metropolis Sampling of Freely Reduced Trivial Words in Groups

The dynamical implementation of our algorithm is inspired by the BFACF algorithm [1, 2, 6], which was developed to sample lattice self-avoiding walks and polygons from stretched Boltzmann distributions. The self-avoiding walk is a model of polymer entropy, a celebrated unsolved problem in polymer physics and chemistry [13, 17]. Details about the implementation of the BFACF algorithm can also be found in [23, 29].

Our algorithm will be implemented to sample words in a group $G$ along a Markov chain using the Metropolis algorithm [31]. States will be sampled by the algorithm by generating new states from a current state via elementary moves. These elementary moves will be defined in more detail below – they are local changes made in a systematic manner to a freely reduced trivial word $w$ to obtain a new freely reduced trivial word $v$.

The approach is as follows: Let $w_n$ be the current state of the algorithm (so that $w_n$ is a freely reduced trivial word of $G$). Choose an elementary move from a set of available elementary moves and create a trial word $w'_n$ by implementing the elementary move on $w_n$ (where $w'_n$ is also a freely reduced trivial word). Accept $w'_n$ as the next state in the Markov chain with probability $P(w_n \rightarrow w'_n)$, in which case the next state is $w_{n+1} = w'_n$. If $w'_n$ is rejected, then the next state is by default $w_{n+1} = w_n$. This rejection technique is characteristic of the Metropolis algorithm and it ensures that the sampling is aperiodic.

This implementation samples words $\{w_n\}$ for $n = 0, 1, 2, \ldots$ along a Markov chain which is initiated at a state $w_0$. The initial state $w_0$ may be chosen arbitrarily, but must be a freely reduced trivial word of $G$. It is convenient to choose $w_0$ from the set of relators of $G$.

2.1. Elementary moves for sampling trivial words in a group $G$. Let $G = \langle a_1, a_2, \ldots, a_k \mid R_1, R_2, \ldots, R_m, \ldots \rangle$ be a group on $k$ generators with finite length relators $R_i$. The number of relators may be finite, or infinite. Let $w$ be a freely reduced trivial word in $\{a_1, a_1^{-1}, \ldots, a_k, a_k^{-1}\}$. Denote the length of $w$ by $|w|$. And finally, let $S$ be the set of relators $R_i$, their inverses $R_i^{-1}$ and all cyclic permutations of relators and their inverses. Note that $S$ is an infinite set if and only if $G$ has an infinite set of relators.

Suppose that we have sampled along a Markov chain $\{w_n\}$ and that the current state is a freely reduced trivial word $w = w_n$ of length $|w| = |w_n|$. A new trivial word $w'$ is constructed from $w$ by choosing from the following two elementary moves:

- **Conjugation** — Let $x$ to be one of the $2k$ possible generators (and their inverses) chosen uniformly and at random. Put $w' = x w x^{-1}$ and perform free reductions on $w'$ to produce $w''$. 
• **Insertion** — Let $R \in S$ be one of the relators or their inverses or any cyclic permutations of those relators or their inverses\(^1\). Choose an integer $m \in \{0, 1, \ldots, |w|\}$ with uniform probability and partition $w$ into two subwords $u$ and $v$, with $|u| = m$. Form $w' = uRv$, and freely reduce this word to get $w''$. If $m = 0$, then $R$ is prepended to $w$, and if $m = |w|$, then $R$ is appended to $w$.

The elementary moves are implemented by choosing a conjugation with probability $p_c$, and otherwise an insertion.

The two elementary moves produce freely reduced trivial words $w''$ by acting on $w$. A Metropolis style Monte Carlo algorithm can be implemented using these moves provided that they are uniquely reversible.

One may verify that conjugations are uniquely reversible. Unfortunately, insertions are not, and this must be accounted for in the implementation of the algorithm by conditioning the use of insertion moves such that they become uniquely reversible.

We show by example that insertions are not reversible: Let $R \in S$ and consider the insertion of $R^{-1}$ to the right of $R$ in the word $a^\ell Ra^{-\ell}$. This will reduce the word to the empty word, but there is no elementary move which will produce $a^\ell Ra^{-\ell}$ from the empty word by inserting any relator on the empty word (here we assume $a^\ell Ra^{-\ell}$ are not relators). This difficulty can be overcome by rejecting proposed moves as a result of inserting $R$ if it changes the length of the word by more than $|R|$.

A second difficulty may arise with insertions, and we show again by example that an insertion may not be uniquely reversible, even if it it changes the length of a word by at most $|R|$. Consider the group $\mathbb{Z}^2 = \langle a, b \mid bab^{-1}a^{-1} \rangle$ and insert the relator $R = bab^{-1}a^{-1}$ into the word $uba^{-1}b^{-1}aba^{-1}v$ at the position marked by $\ast$ below:

\[
uba^{-1}b^{-1} \ast aba^{-1}v \mapsto uba^{-1}b^{-1} \cdot bab^{-1}a^{-1} \cdot aba^{-1}v \mapsto uba^{-1}v \tag{2.1}
\]

This move can be reversed in 2 ways. First insert $ba^{-1}b^{-1}a$ (which is a cyclic permutation of the inverse of a relation) at the $\ast$:

\[
u * ba^{-1}v \mapsto u \cdot ba^{-1}b^{-1}a \cdot ba^{-1}v \tag{2.2}
\]

and then we could also insert another relation of $\mathbb{Z}^2$, $b^{-1}aba^{-1}$, at the $\ast$:

\[
uba^{-1} * v \mapsto uba^{-1} \cdot b^{-1}aba^{-1} \cdot v \tag{2.3}
\]

This will disturb the detailed balance condition required for Metropolis style algorithms with the result that the algorithm will sample from an incorrect stationary distribution.

We show how to account for the above by modifying the insertion move as follows: Reject all attempted insertions of $R \in S$ in a word if either there are cancellations to the right, or if it changes the length of the word by more than $|R|$. Attempted insertions which neither cancel to the right, nor change the length of the word by more than $|R|$ will be called valid, and we call an insertion a left-insertion if cancellations of generators only occurs to the left and if the insertion is valid.

\(^1\)For example, in $BS(2, 3)$ defined in Section 3, the relator $a^2ba^{-3}b^{-1}$ yields $2 \times 7 = 14$ possible choices.
• **Left-Insertion** — Let $R \in S$ be one of the relators or their inverses or any cyclic permutation of those relators and their inverses. Choose an integer $m \in \{0, 1, 2, \ldots, |w|\}$ uniformly and partition $w$ into two subwords $u$ and $v$, with $|u| = m$. If $m = 0$ then prepend $w' = Rw$ and note that this is valid only if there are no cancellations of generators. If this is valid, then put $w'' = w'$, otherwise put $w'' = w$. If $m = |w|$, then append $w' = uR$ and this is valid even if there are cancellations to the left. Freely reduce $w'$ to obtain $w''$. Otherwise, form $w' = uRv$. If $R$ cancels to the right with $v$ then reject the proposed move and keep $w$. Otherwise, freely reduce $w'$ to obtain $w''$. If $|w''| < |w| - |R|$ then reject the move (and keep $w$).

Left-insertions are uniquely reversible, and are suitable as an elementary move in a Metropolis style Monte Carlo algorithm for sampling freely reduced trivial words in $G$.

**Lemma 2.1.** Left-insertions are uniquely reversible.

**Proof.** Let $w = uv$ be a freely reduced trivial word in the group $G$ and let $R \in S$. Form $w' = uRv$ via a left-insertion, where $u$ or $v$ may be the empty word.

- Suppose there are no possible cancellations to the left or right — then $w'' = uRv$, and the move can be uniquely reversed by inserting $R^{-1}$ (which must also be a relator in the group) to the right of $R$. This gives $uRR^{-1}v \mapsto uv$. Further cancellations cannot occur because $w = uv$ was freely reduced. Note that this is unique because any other insertion must cancel $R$, and to do so would require cancellations to the right and so would not be a left-insertion.
- Suppose there are some cancellations to the left when $R$ is inserted in $w$. In particular, in this case one has $w = u'sv$ and $R = st$ for some freely reduced words $u'$, $s$ and $t$ (where $t$ may be the empty word). Inserting $R$ to the right of $s$ and freely reducing the word gives $w'' = u'tv$ (and $t$ may be empty). This move is uniquely reversible by inserting $R^{-1} = ts$ to the right of $t$. This gives $u'tstv \mapsto utsv = w$. No further cancellations are possible because the original word was freely reduced. Again, by a similar argument, this move is unique — all other possibilities require cancellation to the right.

The conjugation and left-insertion elementary moves can be implemented in a Metropolis algorithm to sample freely reduced trivial words in $G$.

**2.2. Metropolis style implementation of the elementary moves.** Conjugations and left-insertions may be used to sample along a Markov chain in the state space of freely reduced trivial words of a group $G$ on $k$ generators.

The algorithm is implemented as follows: Let $p_c \in [0, 1]$, $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^+$ be parameters of the algorithm and assume that $\beta$ is small. As above, let $S$ be the set of all cyclic permutations of the relators and their inverses and recall that $S$ may be finite or infinite.

Define $P$, a probability distribution over $S$, so that $P(R)$ is the probability of choosing $R \in S$ with $\sum_{R \in S} P(R) = 1$. Further, assume that $P(R) > 0$ for all $R \in S$ and also that $P(R) = P(R^{-1})$ (we shall eventually require these two conditions).
In the case that $S$ is finite we choose $P$ to be the uniform distribution, although we are free to choose other distributions.

Suppose that $w_n$ is the current state, a freely reduced trivial word produced by the algorithm, and inductively construct the next state $w_{n+1}$ as follows:

- With probability $p_c$, choose a conjugation move and otherwise choose a left-insertion.
- If the move is a conjugation, then choose one of the $2k$ possible conjugations randomly with uniform probability: Say that the pair $(c, c^{-1})$ is chosen where $c$ is a generator or its inverse. Put $u = cw_n c^{-1}$ and freely reduce $u$ to obtain $w'$. Construct $w_{n+1}$ from $w'$ and $w_n$ as follows:

$$w_{n+1} = \begin{cases} w', & \text{with probability } p = \min \left\{ 1, \frac{(|w'|+1)^{1+\alpha}}{|w|+1} \beta |w'|-|w| \right\}; \\ w_n, & \text{otherwise}. \end{cases}$$

(2.4)

- If the move is a left-insertion, then choose an element $R \in S$ with probability $P(R)$. Choose a location $m \in \{0, 1, 2, \ldots, |w_n|\}$ in the word $w_n$ uniformly. This is the location where the left-insertion will be attempted. Attempt a left insert of $R$ at the location $m$. Construct $w_{n+1}$ as follows:

$$w_{n+1} = \begin{cases} w_n, & \text{if the left-insertion of } R \text{ is not valid}; \\ w', & \text{if } R \text{ is valid and with probability } p = \min \left\{ 1, \frac{(|w'|+1)^{1+\alpha}}{|w|+1} \beta |w'|-|w| \right\}; \\ w_n, & \text{otherwise}. \end{cases}$$

(2.5)

Let $w$ and $v$ be two words and suppose that $v$ was obtained from $w$ by a conjugation as implemented above. Then the transition probability $P_r(w \rightarrow v)$ is given by

$$P_r(w \rightarrow v) = \frac{1}{2k} \left( \frac{(|v| + 1)^{1+\alpha}}{|w| + 1} \beta |w'|-|w| \right).$$

(2.6)

since a conjugation is chosen uniformly from $2k$ possibilities, and provided that $p < 1$ in equation (2.4). Otherwise, the transition probability of the reverse transition is $P_r(v \rightarrow w) = 1/2k$. This, in particular, shows the condition of detailed balance for conjugation moves:

$$P_r(w \rightarrow v) = \left( \frac{(|v| + 1)^{1+\alpha}}{|w| + 1} \beta |w'|-|w| \right) P_r(v \rightarrow w)$$

(2.7)

which simplifies to the symmetric presentation

$$|v| + 1)^{1+\alpha} \beta |w| P_r(w \rightarrow v) = (|v| + 1)^{1+\alpha} \beta |v| P_r(v \rightarrow w).$$

(2.8)

In the alternative case that $w$ and $v$ are two words and $v$ was obtained from $w$ by a left-insertion of $R \in S$ as implemented above, the transition probability is given by

$$P_r(w \rightarrow v) = \frac{P(R)}{|w| + 1} \left( \frac{(|v| + 1)^{\alpha}}{|w| + 1} \beta |v'|-|w| \right)$$

(2.9)

where the element $R \in S$ is selected with probability $P(R)$, the location for the left-insertion of $R$ is chosen with probability $1/(|w| + 1)$, and we have assumed (without loss of generality) that $p < 1$ in equation (2.5).
Similarly, the transition probability of $v \rightarrow w$ via a left-insertion of $R^{-1} \in S$ is
\begin{equation}
Pr(v \rightarrow w) = \frac{P(R^{-1})}{|v| + 1}.
\end{equation}
This gives a second condition on the probability distribution $P$ over $S$, namely that $P(R) = P(R^{-1})$ for all elements $R \in S$. In this event a comparison of the last two equations, and simplification, gives
\begin{equation}
(|w| + 1)^{1+\alpha \beta |w|} Pr(w \rightarrow v) = (|v| + 1)^{1+\alpha \beta |v|} Pr(v \rightarrow w)
\end{equation}
as a condition of detailed balance for left-insertions. This is the identical condition obtained for conjugation in equation (2.8). The above is a proof of the following lemma.

**Lemma 2.2.** Let $\{w_n\}$ be a Markov chain in the state space of freely reduced words in $G$, and suppose the transition of state $w_n$ to $w_{n+1}$ is due to a transition by a conjugation move with probability $p_c$, and due to a left-insertion with probability $q_c = 1 - p_c$. Then the Markov chain samples from the stationary distribution
\[ Pr(w) = \frac{(|w| + 1)^{1+\alpha \beta |w|}}{N} \]
over its state space, where $N$ is a normalising factor.

**Proof.** This lemma is a corollary of the Perron-Frobenius theorem (see [5] for example), and follows by summing the conditions of detailed balance in equations (2.8) and (2.11) over $v$. \qed

**2.3. Irreducibility of the elementary moves.** In this subsection we examine the state space of the Markov chain in Lemma 2.2 by determining the irreducibility class of trivial freely reduced words in $G$ with respect to the elementary moves of the algorithm.

The elementary moves above may be represented as a multigraph $M$ on the freely reduced words of $G$: Two freely reduced words $w, v$ form an arc $wv$ for each elementary move (a conjugation or a left-insertion) which takes $w$ to $v$. Since each elementary move is uniquely reversible, $M$ may be considered undirected. The irreducibility class of a freely reduced trivial word $w$ in $G$ is the collection of freely reduced trivial words in the largest connected component $M_w$ of $M$ which contains $w$. The algorithm will be said to be irreducible on freely reduced trivial words in $G$ if the words in $M_w$ form exactly the family of freely reduced trivial words in $G$.

**Lemma 2.3.** Consider the group $G = \langle a_1, \ldots, a_k | R_1 \ldots R_m \ldots \rangle$ with $k$ generators. If $0 < p_c < 1$ and $P(R) > 0$ for all $R \in S$, then the elementary moves defined above are irreducible on the set of all freely reduced trivial words in that group.

**Proof.** Consider a relator of $G$, say $R_1 \in S$. Observe that left-insertions can be used to change $R_1$ into any other relator $R_m$ of $G$. Hence, all the relators $R_m$ of $G$ are in the irreducibility class $M$ of $R_1$. It follows that all cyclic permutations of the $R_m$, and inverses and their cyclic permutations are also in $M$. Hence, $S \subseteq M$.

Next, let $C = \langle w_n \rangle$ be a realisation of a Markov chain with initial state $w_0 = R_1$. All words $w_n$ sampled by $C$ are obtained by conjugation or by left-insertions by elements of $S$, and so they are all trivial and freely reduced. Thus $C \subseteq M$ if $C$ is initiated by $R_1$. 

It remains to show that any trivial and freely reduced word can occur in a realisation of a Markov chain $C$ with initial state $R_1 \in S$.

A word $w \in \{a_1^{\pm 1}, \ldots, a_k^{\pm 1}\}^*$ represents the identity element in the group if and only if it is the product of conjugates of the relators $R_i^{\pm 1}$. So $w$ is the word

$$\prod_{j=1}^s \rho_j r_j \rho_j^{-1}$$

after free reduction, where $\rho_j \in \{a_1^{\pm 1}, \ldots, a_k^{\pm 1}\}^*$ and $r_j = R_i^{\pm 1}$.

We can obtain $w$ using conjugation and left-insertion as follows:

1. set $w = r_1$;
2. conjugate by $\rho_2^{-1} \rho_1$ one letter at a time to obtain $w = \rho_2^{-1} \rho_1 r_1 \rho_1^{-1} \rho_2$ after free reduction;
3. insert (append) $r_2$ on the right;
4. repeat the previous two steps (conjugating by $\rho_{j+1}^{-1} \rho_j$ then inserting $r_j$ on the left) until $r_s$ is inserted;
5. conjugate by $\rho_s$.

Since we only ever append $r_j$ to the end of the word, there are no right cancellations, and at most $|r_j|$ left cancellations.

This completes the proof.

**Corollary 2.4.** The Monte Carlo algorithm is aperiodic, provided that $P(R) = P(R^{-1}) > 0$ and $0 < p_c < 1$.

**Proof.** Let $P_r(w \to v)$ be the one step transition probability from state $w$ to state $v$ in the Monte Carlo algorithm. The probability of achieving a transition $w \to v$ in $N$ steps is denoted by $P_r^N(w \to v)$, and by Lemma 2.3 there exists an $N_0$ such that $P_r^N(w \to v) > 0$, if both $w, v$ are freely reduced trivial words.

The rejection technique used in the definition of both the conjugation and left-insertion elementary moves immediately implies that if $P_r^N(w \to v) > 0$ then $P_r^M(w \to v) > 0$ for all $M \geq N_0$. Hence the algorithm is aperiodic.

A Monte Carlo algorithm which is aperiodic, and irreducible on its state space, is said to ergodic. Hence, the algorithm above is ergodic on the state space of freely reduced trivial words. In these conditions, the fundamental theorem of Monte Carlo Methods implies the algorithm samples along a Markov chain $C = \langle w_n \rangle$ asymptotically from the stationary distribution given in Lemma 2.2.

**2.4. Analysis of Variance.** The algorithm was implemented and tested for accuracy. The stationary distribution of the algorithm (see Lemma 2.2) shows that the expectation value of the mean length of words sampled for given parameters $(\alpha, \beta)$ is

$$\mathbb{E}(|w|) = \frac{\sum_w |w| (|w| + 1)^{1+\alpha} \beta^{|w|}}{\sum_w (|w| + 1)^{1+\alpha} \beta^{|w|}}$$

where the summations are over all freely reduced trivial words in $G$.

We observe two points: The first is that increasing $\beta$ will increase $\mathbb{E}(|w|)$. In fact, there is a critical point $\beta_c$ such that $\mathbb{E}(|w|) < \infty$ if $\beta < \beta_c$, and $\mathbb{E}(|w|)$ is

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2 As check on our coding, the algorithm was coded independently by two of the authors (AR and EJJvR), and the results were compared. Further, we ran the simulations making lists of observed trivial words for short lengths and then compared these against exhaustive enumerations.
divergent if $\beta > \beta_c$. Observe that $\beta_c$ is independent of $\alpha$. The second point is that increasing $\alpha$ will generally increase the value of $E(|w|)$. This is convenient when one seeks to estimate the location of $\beta_c$.

Equation (2.12) is a log-derivative of the cogrowth series and will be finite for $\beta$ below the reciprocal of the cogrowth (being the critical point of the associated generating function) and divergent above it. Because of this, we identify $\beta_c$ with the reciprocal of the cogrowth. Hence the convergence of this statistic gives us a sensitive test of the cogrowth and so the amenability of the group. For example, if the mean length of words sampled from a group with 2 generators at $\beta = \epsilon + 1/3$ is finite, then the group is not amenable.

The realisation of a Markov chain $C = \{w_0, w_1, \ldots, w_n, \ldots\}$ by the algorithm produces a correlated sequence of an observable (for example the length of words). We denote the sequence of observables by $\{O(w_1), O(w_2), O(w_3), \ldots, O(w_n)\}$. The sample average of the observable over the realised chain is given by

$$\langle \langle O(w) \rangle \rangle_n = \frac{1}{n} \sum_{i=1}^{n} O(w_i). \quad (2.13)$$

This average is asymptotically an unbiased estimator distributed normally about the expected value $E(O(w))$, given by

$$E(O(w)) = \frac{\sum_{w} O(w) (|w| + 1)^{1+\alpha} \beta^{|w|}}{\sum_{w} (|w| + 1)^{1+\alpha} \beta^{|w|}}. \quad (2.14)$$

Hence, $\langle \langle O(w) \rangle \rangle_n$ may be computed to estimate the expected value $E(O(w))$.

It is harder to determine the variance in the distribution of $\langle \langle O(w) \rangle \rangle_n$ about $E(O(w))$. Although the Markov chain produces a time series of identically distributed states, they are not independent, and autocorrelations must be computed along the time series to determine confidence intervals about averages.

The dependence of an observable along a time series is statistically measured by an autocorrelation function. The autocorrelation function usually decays at an exponential rate measured by the autocorrelation time $\tau_O$ along the time series. In particular, the measured connected autocorrelation function of the algorithm is defined by

$$S_O(k) = \langle \langle O(w_i) O(w_{i+k}) \rangle \rangle_n - \langle \langle O(w) \rangle \rangle_n^2, \quad (2.15)$$

and is dependent on $n$, the length of the chain. If $n$ becomes very large, then $S_O(k)$ measures the correlations between states a distance of $k$ steps apart. The Markov chain is asymptotically homogeneous (independent of its starting point); this implies that $\langle \langle O(w_i) \rangle \rangle_n \simeq \langle \langle O(w) \rangle \rangle_n$ if both $n$ and $i$ are large, and if $i \ll n$. Thus, for large values of $n$ and $i$, the autocorrelation time $S_O(k)$ is only dependent on the separation $k$ between the observables $O(w_i)$ and $O(w_{i+k})$.

Normally, the autocorrelation function of a homogeneous chain is expected to decay (to leading order) at an exponential rate given by

$$S_O(k) \simeq C_O e^{-k/\tau_O} \quad (2.16)$$

where $\tau_O$ is the exponential autocorrelation time of the observable $O$. The autocorrelation time $\tau_O$ sets a time scale for the decay of correlations in the time series $\{O(w_i)\}$: If $k >> \tau_O$, then the states $O(w_i)$ and $O(w_{i+k})$ are for all practical statistical purposes independent. These observations allow us to compute statistical confidence intervals on the average $\langle \langle O(w) \rangle \rangle_n$ in a systematic way.
Suppose that a time series of length \( N \) of observables \( \{O(w_i)\} \) were realised by the Markov chain Monte Carlo algorithm. Cut the times series in blocks of size \( M \ll N \), but with \( M \gg \tau_O \). Then one may determine \( \lfloor N/M \rfloor \) averages estimating \( \langle\langle O(w) \rangle\rangle_n \) over the blocked data, given by

\[
|O(w)|_i = \frac{1}{M} \sum_{j=1}^{M} O(w_{iM+j})
\]

for \( i = 0, 1, \ldots, \lfloor N/M \rfloor - 1 \).

The sequence of estimates \( \{[O(w)]_0, [O(w)]_1, \ldots, [O(w)]_{\lfloor N/M \rfloor-1}\} \) is itself a time series, and if these are independent estimates, then for large \( M \ll N \) its variance is estimated by determining

\[
s^2_{M,O} = \langle|O|^2 \rangle - \langle|O|\rangle^2,
\]

where canonical averages \( \langle \cdot \rangle \) are taken. So if the \( [O(w)]_i \) are treated as independent measurements of \( E(O(w)) \), then the 67% statistical confidence interval \( \sigma_{M,O} \) is given by

\[
\sigma^2_{M,O} = \frac{s^2_{M,O}}{\lfloor N/M \rfloor - 1}.
\]

In practical applications the above is implemented by increasing \( M \ll N \) until \( \sigma_{M,O} \) is insensitive to further increases. In this event one has \( M \gg \tau_O \), and \( \sigma_{M,O} \) is the estimated 67% statistical confidence interval on the average computed in equation (2.13).

In this paper we consider the average length of words – that is, \( O(w) = |w| \) for each freely reduced and trivial word \( w \) sampled by the algorithm. We use our algorithm to determine the canonical expected length of freely reduced trivial words with respect to the Boltzmann distribution. This is defined by putting \( \alpha = -1 \) in equation (2.12):

\[
E_C(|w|) = \frac{\sum_w |w| \beta^{|w|}}{\sum_w \beta^{|w|}}
\]

where the summation is over all freely reduced trivial words in \( G \), except the empty word.

An estimator of \( E_C(|w|) \) is obtained by putting \( O(w) = |w|/(|w|+1)^{1+\alpha} \) and \( O(w) = 1/(|w|+1)^{1+\alpha} \) in equation (2.14). This gives

\[
E_C(|w|) = \frac{\mathbb{E}\left(\frac{|w|}{(|w|+1)^{1+\alpha}}\right)}{\mathbb{E}\left(\frac{1}{(|w|+1)^{1+\alpha}}\right)}.
\]

In other words, for arbitrary choice of \( \alpha \), the ratio estimator

\[
\langle|w|\rangle_n = \frac{\langle|w|/(|w|+1)^{1+\alpha}\rangle_n}{\langle1/(|w|+1)^{1+\alpha}\rangle_n}
\]

may be used to estimate the canonical expected length \( E_C(|w|) \) over the Boltzmann distribution on the state space of freely reduced trivial words in \( G \). This is particularly convenient, as one may choose the parameter \( \alpha \) to bias the sampling in order to obtain better numerical results. For example, it is frequently the case that (long) trivial words in the tail of the Boltzmann distribution are sampled with low frequency, and by increasing \( \alpha \) the frequency may be increased. This gives larger
ON TRIVIAL WORDS IN FINITELY PRESENTED GROUPS

2.5. Implementation. The algorithm was implemented using a Multiple Markov chain Monte Carlo algorithm [18, 33] — an approach that is also known as parallel tempering. This greatly reduces autocorrelations in the realised Markov chains and was achieved as follows: Define a sequence of values of $\beta$ such that $0 < \beta_1 < \beta_2 < \ldots < \beta_m < \beta_c$. Separate chains are initiated at each of the $\beta_i$ and run in parallel. States at adjacent values of the $\beta_i$ are compared and swapped. This coupling of adjacent chains creates a composite Markov chain, which is itself ergodic (since each individual chain is) with stationary distribution the product distribution over all the separate chains. This implementation greatly increases the mobility of the Markov chains, and reduces autocorrelations. The analysis of variance follows the outline above. For more detail on a Multiple Markov chain implementation of Metropolis-style Monte Carlo algorithms, see [23] for example.

In practice we typically initiated 100 chains clustered towards larger values of $\beta$ where the mobility of the algorithm is low. Each chain was run for about 1000 blocks, each block a total of $2.5 \times 10^7$ iterations. The total number of iterations over all the chains were $2 \times 10^9$ iterations, which typically took about 1 week of CPU time on a fast desktop Linux station for each group we considered. We also ran each group at five different $\alpha$ values $-1, 0, 1, 2, 3$. The larger values of $\alpha$ will ensure that we sample into the tail of the distribution over trivial words — in practice the different $\alpha$ values gave consistent results. Data were collected and analysed to estimate the cogrowth of each group.

In the next sections, we compare our numerical results with exact analysis of the Baumslag-Solitar groups. This will demonstrate the validity of our numerical approaches above.

3. Exact cogrowth series for Baumslag-Solitar groups

3.1. Equations. Consider the Baumslag-Solitar group

$$BS(N, M) = \langle a, b | a^N b = b a^M \rangle = \langle a, b | a^N b a^{-M} b^{-1} \rangle.$$ 

Our aim is to compute its cogrowth function, or the corresponding generating function. Rather than obtain this directly, we instead consider the set of words (they are not required to be freely reduced) which generate elements in the horocyclic subgroup $\langle a \rangle$ — let $H$ be the set of such words. In what follows we will abuse notation and when a word $w$ generates an element in a subgroup $\langle a^k \rangle$, we shall write $w \in \langle a^k \rangle$.

Consider a normal form $P a^k$, where $k$ is the $a$-exponent, and $P$ is a word in the “alphabet” $\{ b, a b, \ldots a^{N-1} b, b^{-1}, a b^{-1}, \ldots a^{M-1} b^{-1} \}$ that we call the prefix (see [28] p. 181).

Consider a normal form $P a^k$.

- Multiplying this on the right by $a^{\pm 1}$ results in $P a^{k \pm 1}$. Sample sizes on long words, improving the accuracy of the numerical estimates of the canonical expected length of words. For more details, see for example Section 14 in [23].
• If $k = N\ell$ then multiplying on the right by $b$ results in $Pba^{M\ell}$ — if $P$ ends in a $b^{-1}$ then it will shorten and the $a$-exponent will be updated accordingly.

• If $k = M\ell$ then multiplying on the right by $b^{-1}$ results in $Pb^{-1}a^{N\ell}$ — if $P$ ends in a $b$ then it will shorten and the $a$-exponent will be updated accordingly.

• Otherwise multiplying by $b^{\pm 1}$ will change the $a$-exponent and lengthen the prefix.

Now define $g_{n,k}$ to be the number of words in $H$ of length $n$ that generate the element with normal form $a^k$. Clearly we have $g_{n,k} = g_{n,-k}$. Define the generating function

$$G(z; q) = \sum_{n,k} g_{n,k} z^n q^k.$$  

(3.1)

It is very convenient to define the following subsets of $H$ and their corresponding generating functions.

• Let $L$ be the set of words in $H$ that cannot be written as $uv$ where $u$ generates an element with normal from $b^{-1}a^j$ for any $j$.

• Let $K$ be the set of words in $H$ that cannot be written as $uv$ where $u$ generates an element with normal from $ba^j$ for any $j$.

Let the generating functions of these words be $L(z; q)$ and $K(z; q)$ respectively. We note that $L(z; 1) = K(z; 1)$, since the inverse of any word in $L$ gives a word in $K$ and vice versa. We then need to define the operator $\Phi_{d,e}$ which acts on the above generating functions to annihilate all powers of $q$ except those that have $a$-exponent equal to 0 mod $d$ and which maps them to powers of 0 mod $e$.

$$\Phi_{d,e} \circ \sum_n z^n \sum_k c_{n,k} q^k = \sum_n z^n \sum_j c_{n,dj} q^{ej}$$

(3.2)

With these definitions we can write down a set of equations satisfied by the generating functions $G(z; q), K(z; q)$ and $L(z; q)$.

**Proposition 3.1.** The generating functions $G, K, L$ satisfy the following system of equations.

$$L = 1 + z(q + \bar{q})L + z^2L \cdot [\Phi_{N,M} \circ L + \Phi_{M,N} \circ K] - z^2 [\Phi_{M,N} \circ K] \cdot [\Phi_{N,N} \circ L],$$

$$K = 1 + z(q + \bar{q})K + z^2K \cdot [\Phi_{M,N} \circ K + \Phi_{N,M} \circ L] - z^2 [\Phi_{N,M} \circ L] \cdot [\Phi_{M,M} \circ K],$$

and

$$G = 1 + z(q + \bar{q})G + z^2G \cdot [\Phi_{N,M} \circ L + \Phi_{M,N} \circ K]$$

where we have written $G \equiv G(z; q)$ etc.

We remark that these equations can be transformed into equations for the cogrowth series by substituting $z \mapsto \frac{t}{1+3t^2}$ and replacing each generating function $f(z) \mapsto h(t) \left[ \frac{1+3t^2}{1+3t^2} \right]$. We found it easier to work with the equations as stated.

**Proof.** First, we note that the set $H$ is closed by prepending and appending the generator $a$ and $a^{-1}$. We factor $H$ recursively by considering the first letter in any word $w \in H$ (see Figure 1). This gives four cases:

• $w$ is the empty word.
The first letter is $a$ or $a^{-1}$. Then $w = av$ or $w = a^{-1}v$ for some $v \in H$, increasing the length by 1 and altering the $a$-exponent by $\pm 1$. At the level of generating functions this gives $z(q + q^{-1})G(z; q)$.

The first letter is $b$. Factor $w = uv$ where $u$ is the shortest word so that $u \in \langle a \rangle$. Thus, $u = bu'b^{-1}$ for some $u' \in \langle a^N \rangle$. The minimality of $u$ ensures $u' \in L$. Combined, this gives $u \in \langle a^M \rangle$. At the level of generating functions, the maps words counted by $z^nq^{kN}$ to $z^{n+2}q^{kM}$ and resulting in $z^2 \cdot \Phi_{N,M} \circ L(z; q)$.

The first letter is $b^{-1}$. Factor $w = uv$ where $u$ is the shortest word so that $u \in \langle a \rangle$. As per the previous case, $u = b^{-1}u'b$ for some $u' \in \langle a^M \rangle$ with $u' \in K$. Combined, this gives $u \in \langle a^N \rangle$. Similar reasoning gives $z^2 \cdot \Phi_{M,N} \circ K(z; q)$.

Now consider an element $w \in L$, and we note that $L$ (and $K$) is closed under appending the generators $a$ and $a^{-1}$, but not prepending. See Figure 2. In a similar manner to the above, we factor words in $L$ recursively by considering the last letter of $w$.

- $w$ is the empty word.
- The last letter is $a$ or $a^{-1}$. Then $w = va$ or $w = va^{-1}$ for some $v \in L$, increasing the length by 1 and altering the $a$-exponent by $\pm 1$. This yields the term $z(q + q^{-1})L(z; q)$.
- The last letter is $b^{-1}$. Factor $w = uv$ where $u$ is the longest subword such that $u \in \langle a \rangle$ and $v$ is non-empty. This forces $v = bv'b^{-1}$ with the restriction that $v' \in L$. Since both $v, v' \in L$ we must have $v' \in \langle a^N \rangle$ and $v \in \langle a^M \rangle$, and this yields $z^2L(z; q) \cdot \Phi_{N,M} \circ L(z; q)$.
- The last letter is $b$. Factor $w = uv$ where $u$ is the longest subword such that $u \in \langle a \rangle$ and $v$ is non-empty. This forces $v = b^{-1}u'v$ with the restriction that $v' \in K$. Further, $w \in L$ implies the subword $u \not\in \langle a^N \rangle$. Otherwise, $w \not\in L$ as the subword $ub^{-1}$ generates an element with normal form $b^{-1}a^j$ for some $j$.

The generating function for $\{ w \in L \mid w \not\in \langle a^N \rangle \}$ is given by $(L - \Phi_{N,N} \circ L)$, and so this last case gives $z^2(L(z; q) - \Phi_{N,N} \circ L(z; q)) \cdot \Phi_{M,N} \circ K(z; q)$.

Putting all of these cases together and rearranging gives the result. The equation for $K$ follows a similar argument. \qed
Figure 2. Any word in $\mathcal{L}$ can be decomposed by considering its last letter. This results in the four possible factorisations we have drawn here. The subwords $L, L' \in \mathcal{L}$, $K \in \mathcal{K}$ and $u$ is a word in $\mathcal{L}$ that generates an element in the subgroup $\langle a \rangle$, but not in the subgroup $\langle a^n \rangle$.

3.2. Solution for $BS(N,N)$. The number of trivial words of length $n$ in $\mathbb{Z}^2 \cong BS(1,1)$ has long been known to be $\left(\frac{n}{2}\right)^2$ (for even $n$). This number grows as $4^{n+1/2}/\pi n$, and the factor of $n^{-1}$ implies that the corresponding generating function is not algebraic (see, for example, section VII.7 of [16]). The generating function does satisfy a linear differential equation with polynomial coefficients and so is D-finite [32] (in fact it can be written in closed form in terms of elliptic integrals). The class of D-finite functions includes rational and algebraic functions and many of the most famous functions in mathematics and physics. Indeed, most of the known group growth and cogrowth series are D-finite (being algebraic or rational). We prove (below) that when $N = M$, the cogrowth series is D-finite and we strongly suspect that when $N \neq M$, the cogrowth series lies outside this class.

Proposition 3.2. When $N = M$ the generating functions $K(z;q) = L(z;q)$ and the generating functions $K = L, G$ satisfy

\[
L = 1 + z(q + \bar{q})L + 2z^2L \cdot [\Phi_{N,N} \circ L] - z^2 [\Phi_{N,N} \circ L]^2
\]

\[
G = 1 + z(q + \bar{q})G + 2z^2G \cdot [\Phi_{N,N} \circ L]
\]

Further, these equations reduce to a set of algebraic equations in $G, L$ and $[\Phi_{N,N} \circ L]$. In particular if we write $L_0(z;q) = [\Phi_{N,N} \circ L]$, and let $\omega = e^{2\pi i/N}$ then we have

\[
NL_0(z;q) = \sum_{j=0}^{n-1} L(z;\omega^j q) = \sum_{j=0}^{n} \frac{1 - z^2L_0(z;q)}{1 - z(\omega q + 1/\omega q) - 2z^2L_0(z;q)}.
\]

For example, for $BS(2,2)$ the generating function $G(z;q)$ satisfies the following cubic equation

\[
1 + 3zQG - (1 - 4z^2 - z^2Q^2)G^2 - zQ(1 - zQ - 2z)(1 - zQ + 2z)G^3 = 0,
\]

where we have written $Q = q + \bar{q}$.

3Perhaps the easiest proof known to the authors is the following. Map any trivial word to a path on the square grid. Now rotate the grid $45^\circ$ and rescale slightly. Each step now changes the $x$-ordinate by $\pm 1$ and similarly each $y$-ordinate by $\pm 1$. In a path of $n$-steps, $n/2$ steps must increase the $x$-ordinate and $n/2$ must decrease it and so giving $\binom{n}{n/2}$ possibilities. The same occurs independently for the $y$-ordinates and so we get $\binom{n}{n/2}^2$ possible trivial words.

4Kouksov proved that the cogrowth series is a rational function if and only if the group is finite [25].
Proof. The proof is a corollary of Proposition 3.1. Setting $N = M$ simplifies the equations considerably and forces $K(z;q) = L(z;q)$. We note that $L_0(z;q) = L_0(z;q + \omega q)$ and the equation for $L_0(z;q)$ follows. Hence both $L(z;q)$ and $G(z;q)$ are also algebraic.

We are not interested in the full generating function $G$, rather we are mainly interested in the coefficient of $q^0$.

Corollary 3.3. For $\text{BS}(N, N)$ the generating function $[q^0]G(z;q) = \sum g_{n,0}z^n$ is D-finite. That is, it satisfies a linear generating function with polynomial coefficients. Furthermore, the cogrowth series (being the generating function of freely reduced words equivalent to the identity) is also D-finite.

It follows that the cogrowth of $\text{BS}(N, N)$ is an algebraic number.

Proof. Every algebraic power series also satisfies a linear differential equation with polynomial coefficients (see [32] for many basic facts about D-finite series). It is known [27] that the constant term of a D-finite series of two variables is a D-finite series of a single variable. Substituting an algebraic series into a D-finite series gives another D-finite series, and so transforming from $[q^0]G(z;q)$ to the cogrowth series (which is done by substituting a rational function) yields another D-finite series.

Finally, if a function satisfies a linear differential equation, then its singularities must correspond to zeros of the coefficient of the highest order derivative. Since the cogrowth series is D-finite, its singularities must be the zeros of the polynomial coefficient of the highest order derivative.

While the results used to prove the above corollary guarantee the existence of such differential equations, they do not give recipes for determining them. There has been a small industry in developing algorithms to do exactly this task (and many other operations on D-finite series) — for example work by Zeilberger, Chyzak and others. Here we have used recent algorithms developed by Chen, Kauers and Singer [11], and we are grateful for Manuel Kauers’ help in the application of these tools.

Applying the algorithms described in [11] to the generating function $G(z;q)$ for $\text{BS}(2, 2)$ which is the solution of equation (3.3) we found a 6th order linear differential equation satisfied by $[q^0]G(z;q)$. Unfortunately the polynomial coefficients of this equation have degrees up to 47. We were also able to guess slightly more appealing equations of higher order with lower degree coefficients, but all are too large to list here.

For $\text{BS}(3, 3)$ and $\text{BS}(4, 4)$ we obtain the following equations for $G(z;q)$ (where $Q = q + \bar{q}$)

\begin{equation}
1 + 4zQG + (6Q^2z^2 - z^2 - 1)G^2 + 2z(Qz + 1)(Q^2z - Q + 2z)G^3
+ z^2(1 - Q)(1 + Q)(Qz + 2z - 1)(Qz - 2z - 1)G^4 = 0
\end{equation}

and

\begin{equation}
1 + 5GQz + (10Q^2z^2 - 2z^2 - 1)G^2 + z(10Q^4z^2 - 6Q^2z^2 - 3Q + 4z)G^3
+ z^2(3Q^4z^2 + 2Q^2z^2 - 3Q^2 + 8Qz - 8z^2 + 2)G^4
- z^3Q(Q^2 - 2)(Qz + 2z - 1)(Qz - 2z - 1)G^5 = 0
\end{equation}

Again applying the same methods, we found an ODE of order 8 with coefficients of degree up to 105 for $\text{BS}(3, 3)$ and for $\text{BS}(4, 4)$ it is order 10 with coefficients of
degree up to 154. Using clever guessing techniques (see [24] for a description) Kauers also found DEs for $N = 5, \ldots, 10$. For $BS(5, 5)$ the DE is order 12 with coefficients of degree up to 301. While that of $BS(10, 10)$ took about 50 days of computer time to guess and is 22nd order with coefficients of degree up to 1153; when written in text file is over 6 Mb! We note that the ODEs found for $N = 2, 3, 4$ have been proved, but it is beyond current techniques\footnote{While there is no theoretical barrier, the time taken by the computations seems to grow quickly with $N$ and exceed the available time.} to prove those found for higher $N$.

Clearly this approach is not a practical means to study the cogrowth for larger $N$—though one can generate series expansions quite quickly using a computer. We are able to determine the radius of convergence of $[q^0]G(z; q)$ for much higher $N$ via the following lemma.

**Lemma 3.4.** For $BS(N, N)$, the generating functions $G(z; 1)$ and $[q^0]G(z; q)$ have the same radius of convergence.

**Proof.** We start with some notation. Write

$$G(z; q) = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} g_{n,k} z^n q^k$$

$G(z; 1) = \sum_{n=0}^{\infty} g_n z^n$

Note that we have $g_{n,-k} = g_{n,k}$ and that $g_{n,k} = 0$ for $|k| > n$. Write $\limsup \frac{1}{n} g_n = \mu$ and $\limsup \frac{1}{n} g_{n,0} = \mu_0$. Since all the $g_{n,k}$ are non-negative, we immediately have $\mu \geq \mu_0$.

To prove the reverse inequality we use a “most popular” argument that is commonly used in statistical mechanics to prove equalities of free-energies (see [22] for example). Fix $n$, then there exists $k^*$ (depending on $n$) so that $g_{n,k^*} \geq g_{n,k}$ — the number $k^*$ is the “most popular” $a$-exponent in all the trivial words of length $n$ contributing to the generating function $G$. We have

$$g_{n,k^*} \leq g_n \leq (2^n + 1) g_{n,k^*}$$

And hence $\limsup \frac{1}{n} g_{n,k^*} = \mu$. Note that numerical experiments show that $k^* = 0$ — the distribution is tightly peaked around 0.

Keeping $n$ fixed, consider a word that contributes to $g_{n,k^*}$ and another that contributes to $g_{n,-k^*}$. Concatenating them together gives a word that contributes to $g_{2n,0}$. So considering all possible concatenation of $M$ such pairs of words gives the following inequality

$$g_{n,k^*} g_{n,-k^*} \leq g_{2Mn,0}$$

Raise both sides to the power $\frac{1}{2nM}$ and let $M \to \infty$ gives

$$g_{n,k^*} \leq \mu_0$$

Letting $n \to \infty$ then shows that $\mu \leq \mu_0$. \hfill $\square$

We have observed that the statement of the lemma appears to hold for Baumslag-Solitar groups $BS(M, N)$ for $M \neq N$ also, however the above proof breaks down in the general case as the number of summands in equation (3.7) grows exponentially with $n$ rather than linearly.

Combining Proposition 3.2 and the above lemma we can establish the growth rates of trivial words $\mu$ and the corresponding cogrowths $\lambda$ for the first few values
Table 1. The growth rate $\mu$ of trivial words in $BS(N,N)$ and the corresponding cogrowth $\lambda$. Note that $\mu$ and $\lambda$ are related by $\mu = \lambda + 3/\lambda$, and that the growth rate of trivial words in the free group on 2 generators is $\sqrt{12} = 3.464101615$.

| $N$ | $\mu$         | $\lambda$     |
|-----|---------------|---------------|
| 1   | $3.792765039$ | $2.668565568$ |
| 2   | $3.647639445$ | $2.395062561$ |
| 3   | $3.569497357$ | $2.215245886$ |
| 4   | $3.525816111$ | $2.091305394$ |
| 5   | $3.500607636$ | $2.002421757$ |
| 6   | $3.485775158$ | $1.936941986$ |
| 7   | $3.476962757$ | $1.887871818$ |
| 8   | $3.471710431$ | $1.850717434$ |
| 9   | $3.468586539$ | $1.822458708$ |

of $N$ (see Table 1). Unfortunately we have not been able to find a general form for these numbers. Some simple numerical analysis of these numbers suggests that the growth rate approaches $\sqrt{12}$ exponentially with increasing $N$. This finding agrees with work of Guyot and Stalder [21], discussed below, who examined the limit of the marked groups $BS(M,N)$ as $M, N \to \infty$, and found that the groups tend towards the free group on two letters, which has an asymptotic cogrowth rate of $\sqrt{12}$.

We remark that for $BS(1,1) \cong \mathbb{Z}^2$ the number of trivial words is known exactly and hence so is the dominant asymptotic form

$$g_{n,0} = \left( \frac{n}{n/2} \right)^2 \sim \frac{2}{\pi n} \cdot 4^n$$

for even $n$.

In the case of $N = 2, 3, 4, 5$ we can show from the differential equations found above that

$$g_{n,0} \sim A_N \mu_N^n n^{-2}$$

for even $n$ where $\mu_N$ is given in the previous corollary and we have estimated the amplitudes to be

$$A_2 = 12.47372070225776 \ldots \quad A_3 = 10.81007294255599 \ldots$$
$$A_4 = 12.14125535742978 \ldots \quad A_5 = 14.7314947893552 \ldots$$

Unfortunately we have not been able to identify these constants, but these observations lead to the following conjecture.

**Conjecture 1.** The number of trivial words in $BS(N,N)$ grows as

$$g_{n,0} \sim A_N \mu_N^n n^{-2}$$

for $N \geq 2$.

3.3. **Continued fractions and $BS(1,M)$.** When we set $N = 1$ cancellations occur and the equation for $L$ becomes a $q$-deformation of a Catalan generating
function:
\[ L(z; q) = 1 + z(q + \bar{q})L(z; q) + z^2 L(z; q) L(z; q^M) = \frac{1}{1 - z(q + \bar{q}) - L(z; q^M)}, \]  
\( (3.10) \)

\[ L(z; 1) = \frac{1 - 2z - \sqrt{1 - 4z}}{2z^2}. \]

Setting \( q = 1 \) into the first equation reduces it to algebraic and it is readily solved to give \( L(z; 1) \) which is the generating function of the Catalan numbers. Thus \( L(z; q) \) is a \( q \)-deformation of the Catalan numbers and rearranging the first equation shows that \( L(z; q) \) has a simple continued fraction expansion.

\[ L(z; q) = \frac{1}{1 - z(q + q^{-1}) - \frac{z^2}{1 - z(q^M + q^{-M}) - \frac{z^2}{1 - z(q^{M^2} + q^{-M^2}) - \ldots}}}. \]

\( (3.11) \)

Such continued fraction forms are well known and understood in Catalan objects (see [15] for example). Unfortunately the equation for \( K \) does not simplify:

\[ K = 1 + z(q + \bar{q})K + z^2 L(z; q^M) \cdot [K - \Phi_{M,M} \circ K] + z^2 K \cdot [\Phi_{M,1} \circ K]. \]

\( (3.12) \)

Though as noted above \( K(z; 1) = L(z; 1) \) and so we expect \( K(z; q) \) to be a different \( q \)-deformation of the Catalan numbers. For \( G \) we have made even less progress and we have not found \( G(z; 1) \), let alone \( G(z; q) \), in closed form. Because of the \( q \)-deformed nature of \( L(z; q) \) we conjecture the following

**Conjecture 2.** For Baumslag-Solitar groups \( BS(1, M) \) with \( M > 1 \), the generating functions \( G(z; q) \) and \( [q^{0}]G(z; q) \) are not D-finite.

Since any path that contributes to \( K \) or \( L \) must also contribute to \( G \), it follows that the radius of convergence of \( G(z; 1) \) is at most \( 1/4 \) — and of course cannot be any smaller. Since the groups \( BS(1, N) \) are all amenable, we know that \( g_{n,0} \sim 4^n \). We have been unable to prove any more precise details of the asymptotic form, though it is not unreasonable to expect that

\[ g_{n,0} \sim A4^n n^{-\gamma_M}. \]

\( (3.13) \)

While we have been able to generate the first 50-60 terms of the series for \( M \leq 5 \) by iterating the equations, the series are quite badly behaved and we have been unable to produce reasonable estimates of \( \gamma_M \).

3.4. **When \( N \neq M \).** When \( N \neq M \), we expect that the operators \( \Phi_{N,M} \) and \( \Phi_{M,N} \) in the equations satisfied by \( G, K, L \) give rise to \( q \)-deformations similar to those observed above. In light of this, we extend our previous conjecture:

**Conjecture 2** (Extended from the above). For Baumslag-Solitar groups \( BS(N, M) \) the generating functions \( G(z; q) \) and \( [q^{0}]G(z; q) \) are only D-finite when \( N = M \).

In spite of the absence of D-finite recurrences, we can still use the equations above to determine the first few terms of the cogrowth series. The resulting algorithm to compute the first \( n \) terms of the series requires time and memory that are exponential in \( n \). The coefficient of \( z^n \) is a Laurent polynomial whose degree is exponential in \( n \) and this exponential growth becomes worse as max\{\( N/M, M/N \)\} becomes larger. In spite of this, iteration of these equations to obtain the cogrowth series is
exponentially faster than more naive methods based on say a simple backtracking
exploration of the Cayley graph, or iteration of the corresponding adjacency matrix.

The time and memory requirements can be further improved since we are pri-
marily interested in the constant term; this means that we do not need to keep high
powers of $q$. More precisely if we wish to compute the series to $O(z^n)$, then we
only need to retain those powers of $q$ that will contribute to $[q^0 z^n]G(z; q)$. We must
compute the coefficients of $z^k$ for $k \leq n/2$ exactly, but we can “trim” subsequent
coefficients — the degree of $z^{n/2+k}$ needs only be that of $z^{n/2+k}$.

Simple C++ code using cln\textsuperscript{6} running on a moderate desktop allowed us to generate
about the first 50 terms of $[q^0]G(z; q)$ for $BS(1, 5)$ while over 300 terms for $BS(4, 5)$
were obtained. The series lengths for the other (with $N < M \leq 5$) ranged between
these extremes. We have estimated the growth rate of trivial words using differential
approximants — see Table 2. Again like the $N = 1$ case, we find the series to be
very badly behaved (except when $N = M$) and we have only obtained quite rough
estimates.

3.5. The limit as $N, M \to \infty$. Beautiful work of Luc Guyot and Yves Stalder
[21] demonstrates that in the limit as $N, M \to \infty$ the (marked) group $BS(N, M)$
becomes the free group on 2 generators. We note that we can observe this free
group behaviour in the functional equations we have obtained.

In particular as $N, M \to \infty$, the operators $\Phi_{N,N}, \Phi_{M,M}, \Phi_{N,M}$ and $\Phi_{M,N}$ become
the constant-term operators. So in this limit the equations for $K$ and $L$ from
Proposition 3.1 become

$$L = 1 + z(q + \bar{q})L + z^2 L \cdot [L_0 + K_0] - z^2 K_0 L_0,$$
(3.14)

$$K = 1 + z(q + \bar{q})K + z^2 K \cdot [K_0 + L_0] - z^2 K_0 L_0,$$

where $K_0(z) = [q^0]K(z; q)$ and $L_0(z) = [q^0]L(z; q)$. Clearly $K(z; q) = L(z; q)$ and
so with a little rearranging

$$L(z; q) = \frac{1 - z^2 L_0(z)^2}{1 - z(q + \bar{q}) - 2z^2 L_0(z)} = \frac{1 - z^2 L_0^2}{1 - 2z^2 L_0} \sum_{n \geq 0} \left( \frac{z(q + \bar{q})}{1 - 2z^2 L_0} \right)^n.$$

\textsuperscript{6}An open source C++ library for computations with large integers. At time of writing it is
available from http://www.ginac.de/CLN/
Taking the constant term of both sides then gives

\[(3.16)\]
\[
L_0 = \frac{1 - z^2 L_0^2}{1 - 2z^2 L_0} \sum_{n \geq 0} \binom{2n}{n} \left( \frac{z}{1 - 2z^2 L_0} \right)^{2n} = \frac{1 - z^2 L_0^2}{1 - 2z^2 L_0} \left[ 1 - 4 \left( \frac{z}{1 - 2z^2 L_0} \right)^{2} \right]^{-1/2}
\]

Simplifying this last expression further gives \((3z^2L_0^2 - L_0 + 1)(z^2L_0^2 - L_0 - 1) = 0.\) The only positive term power series solution of this gives \(L_0\) and a similar exercise gives \([q^0]G(z;q)\):

\[(3.17)\]
\[
L_0 = \frac{1 - \sqrt{1 - 12z^2}}{6z^2}, \quad [q^0]G = \frac{3}{1 + 2\sqrt{1 - 12z^2}}
\]

The expression for \([q^0]G\) is the number of trivial words in the free group on 2 generators.

4. Analysis of random sampling data

4.1. Preliminaries. Using our multiple Markov chain Monte Carlo algorithm we have sampled trivial words from the following groups:

- Thompson’s group \(F\) with the following 3 presentations

\[(4.1)\]
\[
\langle a, b \mid [ab^{-1}, a^{-1}ba], [ab^{-1}, a^{-2}ba^2] \rangle,
\]

\[(4.2)\]
\[
\langle a, b, c, d \mid c = a^{-1}ba, d = a^{-1}ca, [ab^{-1}, c], [ab^{-1}, d] \rangle,
\]

\[(4.3)\]
\[
\langle a, b, c, d, e \mid c = a^{-1}ba, d = a^{-1}ca, e = ab^{-1}, [e, c], [e, d] \rangle.
\]

Note that the generators \(a, b, c, d\) above are often called \(x_0, x_1, x_2, x_3\) respectively in Thompson’s group literature. We have use some simple Tietze transformations (see [28] p. 89) to obtain the second and third presentations from the first (standard) finite presentation of \(F\).

- Baumslag-Solitar groups \(BS(N, M)\) with

\[(4.4)\]
\[
(N, M) = (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3), (3, 5).
\]

- The Basilica group has presentation

\[(4.5)\]
\[
G = \langle a, b \mid [a^n, [a^n, b^n]], [b^n, [b^n, a^{2n}]], n \text{ a power of } 2 \rangle
\]

and embeds in the finitely presented group [20]

\[(4.6)\]
\[
\bar{G} = \langle a, t \mid a^2 = a^2, [[a, t^{-1}], a], a = 1 \rangle.
\]

The groups \(G\) and \(\bar{G}\) are both amenable [4].

We examined two presentations of \(\bar{G}\): The first is obtained from the above by putting \(b = [a, t^{-1}]\), and the second by putting \(b = a^t\). Simplification gives the representations

\[(4.6)\]
\[
\bar{G} = \langle a, b, t \mid [a, t^{-1}] = b, a^2 = aa, [[b, a], a] = 1 \rangle,
\]

\[(4.7)\]
\[
\bar{G} = \langle a, b, t \mid a^t = b, b^t = a^2, b^{-1}aba^{-1}b^{-1}a^{-1}ba = 1 \rangle.
\]

- Other groups for which the cogrowth series is known:

\[(4.8)\]
\[
K_1 = \langle a, b \mid a^2 = b^3 = 1 \rangle,
\]

\[(4.9)\]
\[
K_2 = \langle a, b \mid a^3 = b^3 = 1 \rangle,
\]

\[(4.10)\]
\[
K_3 = \langle a, b, c \mid a^2 = b^2 = c^2 = 1 \rangle.
\]
The exact solutions for $BS(1,1) \cong \mathbb{Z}^2$, $BS(2,2)$ and $BS(3,3)$ are described above, and for the other Baumslag-Solitar groups we have computed series expansions. For the last three groups, the cogrowth series were found by Kouksov [26] and are (respectively)

$$C(t) = \frac{(1 + t) \left( [0, -1, 1, -8, 3, -9] + (2 - t + 6t^2) \sqrt{[1, -2, 1, -6, -8, -18, 9, -54, 81]} \right)}{2(1 - 3t)(1 + 3t^2)(1 + 3t + 3t^2)(1 - t + 3t^2)}$$

$$C(t) = \frac{(1 + t)(-t + \sqrt{1 - 2t - t^2 - 6t^3 + 9t^4})}{(1 - 3t)(1 + 2t + 3t^2)}$$

$$C(t) = \frac{-1 - 5t^2 + 3\sqrt{1 - 22t^2 + 25t^4}}{2(1 - 25t^2)}$$

where $[c_0, c_1, \ldots, c_n] = c_0 + c_1 t + \cdots + c_n t^n$.

In each case we have obtained estimates of the mean length of freely reduced words as a function of $\beta$. More precisely, for each group we estimated

$$E(n^k)(\beta) = \frac{\sum_w |w|^k (|w| + 1)^{1+\alpha} \beta^{|w|}}{\sum_w (|w| + 1)^{1+\alpha} \beta^{|w|}}$$

for $k = \pm 1, \pm 2$ and a range of different $\alpha$ values and where the sum is over all non-empty freely reduced trivial words. These expectations are dependent on $\alpha$, but one can use Equation (2.22) to form $\alpha$-independent estimates of the canonical expectations. Given a knowledge of the cogrowth series we can quickly compute these same means to any desired precision, since we can also write

$$E(n^k)(\beta) = \sum_{n \geq 0} \frac{n^k (n + 1)^{1+\alpha} \beta^n}{(n + 1)^{1+\alpha} \beta^n}$$

where $p_n$ is the number of freely reduced words of length $n$. Note that as $\alpha$ is increased, the samples are biased towards longer words. This expression is convergent for $\beta$ below the reciprocal of the cogrowth (being the critical point of the associated generating function) and divergent above it. The convergence at the critical point depends on the precise details of the asymptotics of $p_n$ and will be effected by $\alpha$. This then points to a simple way to test for amenability:

If the mean length of sampled words from a group on $k$ generators is finite for $\beta$ slightly above $\beta_c = (2k - 1)^{-1}$ then the group is not amenable.

4.2. Amenable groups. We studied the groups $\mathbb{Z}^2 \cong BS(1,1), BS(1,2)$ and $BS(1,3)$. The cogrowth series for $\mathbb{Z}^2$ is known exactly, while we relied on our series expansions to compute statistics for the other two groups — Figure 3 shows the plots of the mean length as a function of $\beta$.

In the case of $BS(1,1) \cong \mathbb{Z}^2$ we see excellent agreement between the numerical estimates generated by our algorithm and the mean length computed from the exact cogrowth series. For $BS(1,2)$ and $BS(1,3)$ we see good agreement for low $\beta$ between our numerical data and mean length computed from the exact cogrowth series. However at larger values of $\beta$ it appears that the cogrowth series systematically underestimates the mean length, compared to the numerical Monte Carlo data. This is, in fact, due to the modest length of the cogrowth series used to compute mean lengths. For $BS(1,2)$ and $BS(1,3)$ we were only able to obtain series of
length 60 and 56 respectively due to memory constraints. Given longer series we expect much better agreement.

One can, for example, compute longer “approximate” cogrowth series by ignoring small terms. When iterating the functional equations given in Proposition 3.1 one can form reasonable approximations by discarding coefficients $g_{n,k}$ which are small compared to nearby coefficients. More precisely we found that if we discard $g_{n,k}$ when $2^{12} \cdot g_{n,k} < \sum_k g_{n,k}$, then we obtain good approximations of the cogrowth series. This means that only the large central coefficients are kept and far less memory used. This made it feasible to approximate the cogrowth series out to

---

7 Rather than iterating the equations for $G(z;q)$ and then transforming the result to get an approximate cogrowth series, we found that our approximation procedure worked best if we iterated the slightly more complicated equations for the cogrowth series directly — see text following Proposition 3.1 for a description of those equations.
around 200 or 300 terms. Of course, the results of these approximation should only be considered a rough guide as we have not bounded the size of any resulting errors. That being said, we see very good agreement between these approximations and our numerical data.

As noted above, we had great difficulty fitting the series data for $BS(1, 2)$ and $BS(1, 3)$. We believe that this is due to the presence of complicated confluent corrections (likely logarithmic terms). Similar corrections also appear to be present in the mean-length data for these groups and we were unable to find convincing or consistent fits to any reasonable functional forms. We did, however, find that the estimated standard error was a good indicator of the location of the singularity: The standard error will diverge as $\beta$ approaches the critical value of $1/3$. We found that linear or quadratic least squares fits of the reciprocal of the error, and finding their $x$-intercept gave consistent, though perhaps slightly low, estimates of the location of the singularity. See Figure 4. The extrapolations give estimates $\beta = 0.330 \pm 0.0002, 0.332 \pm 0.002$ and $\beta = 0.332 \pm 0.002$ for $BS(1, 1), BS(1, 2)$ and $BS(1, 3)$ respectively.

Error bars above were determined by estimating a systematic error in our data. The systematic error was determined by considering the spread of estimates due to our choices of the parameter $\alpha$, the number of data points in the fits, and the chosen functional form for extrapolating the data. We believe that our results give a good indication of quality of the estimates, though we are reluctant to express them as firm confidence intervals. The same general approach to the data for the other groups are followed below.

The HNN-extension of the Basilica group were similarly submitted to Monte Carlo simulation by using the representations (4.6) and (4.7). The canonical expected length of the words, $\langle |w| \rangle$, were computed using the ratio estimator (2.22), and turned out to be remarkably insensitive to the parameter $\beta$ (see Figure 5). This made this group more challenging from a numerical perspective than the Baumslag-Solitar groups discussed above. Putting $\alpha = 5$ finally gave acceptable results: The sample average length show a divergence close to the critical point (since this group is known to be amenable, this is expected to be at $\beta = 0.2$). As in the case of the Baumslag-Solitar groups, the critical value of $\beta$ was determined by extrapolating the reciprocal of the error. Extrapolating the curve corresponding to representation (4.6) gave $\beta_c = 0.217$ and for representation (4.7), $\beta_c = 0.204$. Taking the average and using the absolute difference as a confidence interval gives the estimate $\beta_c = 0.21 \pm 0.01$ to two digits accuracy.

4.3. Non-amenable groups. The groups $BS(N, M)$ with $(N, M) = (2, 2), (2, 3), (3, 3), (3, 5)$ and the groups $K_1, K_2, K_3$ contain a non-abelian free subgroup and so are non-amenable. In the case of the groups $K_1$ and $K_2$ the free subgroups are $F((ab), (ab^{-1}))$, and for $K_3$ the free subgroup is $F((ab), (ac))$.

As noted above, the exact cogrowth series is known exactly for Koukov’s examples and $BS(2, 2), BS(3, 3)$, so we were able to compute the mean length curves exactly — see Figures 6 and 8. As above, we have estimated the location of the dominant singularities for all of these groups — see Figures 7 and 9.

Unfortunately we have been unable to solve $BS(2, 3)$ and $BS(3, 5)$, but we used the recurrences of the previous section to compute the first 100 and 120 terms (respectively) of their cogrowth series. And as was the case for $BS(1, 2)$ and $BS(1, 3)$ we also computed an approximation of the cogrowth series using the same method.
Figure 4. Plots of the reciprocal of estimated standard error in the mean length vs beta for $\alpha = 0, 1, 2, 3$ anti-clockwise from the top. We expect that as $\beta$ approaches its critical value, that the standard error will diverge. We see that if we extrapolate the curves then they cross the $x$-axis at $\beta = 0.330 \pm 0.002, 0.332 \pm 0.002$ and $\beta = 0.332 \pm 0.002$ respectively — thus these extrapolations give good estimates of the critical value of $\beta$.

In all cases we see strong agreement between our numerical estimates and the mean length curves computed from series or exact expressions. As was the case with the amenable groups above, fitting the reciprocal of the estimated standard error gives quite acceptable estimates of the location of the dominant singularities and so the cogrowth.

4.4. Thompson's group. Finally we come to Thompson's group for which we examine three different presentations as described above. Repeating the same analysis we used on the previous groups we see no evidence of a singularity in the mean length at the amenable values of $\beta$ — see Figures 12 and 13. Indeed our estimates
Figure 5. Numerical data on the HNN-extension of the Basilica group. Data points indicated by □ corresponds to the representation in equation (4.6) and by × to the representation in equation (4.7). In both simulations \( \alpha = 5 \). On the left is a plot of the canonical expected length \( \langle n \rangle \). These expected lengths are only weakly dependent on \( \beta \). On the right is the reciprocal error bar on our data. This demonstrates that the error diverges as \( \beta \uparrow 0.20 \), consistent with the fact that these this group is amenable.

We have introduced a Markov chain on freely reduced trivial words of any given finitely presented group. The transitions along the chain are defined in terms of conjugations by generators and insertions of relations. These moves are irreducible and satisfy a detailed balance condition; the limiting distribution of the chain is therefore a stretched Boltzmann distribution over trivial words.

In order to validate the algorithm we have implemented it for a range of finitely presented groups for which the cogrowth series is known exactly. We have also added to this set of groups by finding recurrences for the cogrowth series of all Baumslag-Solitar groups. Unfortunately, these recurrences do not have simple closed-form solutions, but can be iterated to obtain far longer series than can be found using brute-force methods. In the case of \( BS(N, N) \), the recurrences simplify significantly and we are able to compute the cogrowth exactly. For \( N = 1, \ldots, 10 \) we have found differential equations satisfied by the cogrowth series which can be used to generate the cogrowth series in polynomial time.

We see excellent agreement between our mean-length estimates and those computed exactly for several groups. As a further check on our simulations, two of the authors independently coded the algorithm and compared the results. We can use our data to estimate the location of the singularity in the generating function of

\[
\beta_c = 0.395 \pm 0.005, 0.172 \pm 0.002 \text{ and } 0.134 \pm 0.004 \text{ respectively.}
\]

These give cogrowths of 2.53 \( \pm \) 0.03, 5.81 \( \pm \) 0.07 and 7.4 \( \pm \) 0.2, all of which are well below the amenable values of 3, 7 and 9. We take this to be very strong evidence that Thompson’s group \( F \) is not amenable.

5. Conclusions

The location of the dominant singularities are

\[
\beta_c = 0.395 \pm 0.005, 0.172 \pm 0.002 \text{ and } 0.134 \pm 0.004 \text{ respectively.}
\]

These give cogrowths of 2.53 \( \pm \) 0.03, 5.81 \( \pm \) 0.07 and 7.4 \( \pm \) 0.2, all of which are well below the amenable values of 3, 7 and 9. We take this to be very strong evidence that Thompson’s group \( F \) is not amenable.
freely reduced trivial words. The location of this singularity is the reciprocal of the cogrowth and so turns out to be an excellent way to predict the amenability of groups. To test this, we used our algorithm on a range of different amenable and non-amenable groups. In each case we found that our numerical estimate of the cogrowth was completely consistent with the known properties of the groups. In particular, where cogrowth is known exactly, our numerics agreed. For each non-amenable group, the numerical “signal” was robust — no evidence of a singularity was seen at the amenable value.

Most importantly, we see absolutely no evidence that the mean length of Thompson’s group is divergent close to the amenable value; i.e. for 2, 4 and 5 generator presentations we see no evidence of a singularity at $\beta = \frac{1}{3}, \frac{1}{7}$ or $\frac{1}{9}$ (respectively). Indeed, in each case, the mean length appears to be very smooth for $\beta$-values some reasonable distance above these points. Varying $\alpha$ or examining other statistics

Figure 6. Mean length of freely reduced trivial words of the indicated groups. The sampled points are indicated with impulses, while the exact values are given by the black line. We have used vertical lines to indicate $\beta = \frac{1}{3}, \frac{1}{3}, \frac{1}{5}$ (respectively) and also the reciprocal of the cogrowth where the statistic will diverge — being $0.3418821478, 0.3664068598$ and $0.2192752634$ respectively. There is excellent agreement between the numerical and exact results, except possibly at the very highest $\beta$ values.
Figure 7. The reciprocal of the estimated standard error vs $\beta$ for the indicated groups with $\alpha = 0, 1, 2, 3$ (clockwise from top in each case). We see that the extrapolations of the curves intersect the x-axis very close, but slightly short, of the indicated critical values of $\beta$ — 0.3418821478, 0.3664068598 and 0.2192752634 respectively. Hence these give good, but slightly low, estimates of the location of the singularities $\beta = 0.340 \pm 0.002$, $0.365 \pm 0.002$ and $0.219 \pm 0.001$.

does not result in any substantial change with the result that values of $\beta$ consistent with amenability are excluded from our estimated error bars. Overall, our numerical data lead us to the conclusion that Thompson’s group $F$ is not amenable with high probability.

To further test this hypothesis we also examined a generalisation of Thompson’s group, namely $F(3)$ (see [7]). We used our algorithm to compute the mean length of freely reduced words in two presentations of $F(3)$, namely

5.1 \langle a, b, c, d, e | d = b^a, e = c^b = d^e = e^d = e^c = e b = e^a \rangle

5.2 \langle a, b, c | b = c^a, (b^a)^c = (b^a) b = (b^a)^a, (c^b)(b^a) = (c^b) c = (c^b) b = (c^b)^a \rangle

where $x^y = y^{-1} x y$. Note that the first presentation can be written as 3 relations of length 4 and 5 of length 6, while the second can be written as 1 relation of length

8As was the case above, the generators $a, b, c, d, e$ are more usually written $x_0, \ldots, x_5$. 
Figure 8. Mean length of freely reduced trivial words in Baumslag-Solitar groups $BS(2, 2)$ and $BS(3, 3)$ at different values of $\beta$. The sampled points are indicated with impulses, while the exact values are given by the black line. We have used vertical lines to indicate $\beta = 1/3$ and also the reciprocal of the cogrowth where the statistic will diverge — being 0.3747331572 and 0.417525628 respectively. We see excellent agreement between our numerical data and the exact results, and our error bars are smaller than the impulses.

Figure 9. The reciprocal of the estimated standard error of the mean length as a function of $\beta$ for $BS(2, 2)$ and $BS(3, 3)$. In both plots we show 4 curves corresponding to simulations at $\alpha = 0, 1, 2, 3$ (anti-clockwise from top) and denote the singular values — 0.3747331572 and 0.417525628 respectively — with vertical lines. Extrapolating the curves give estimates of $\beta_c = 0.372 \pm 0.002$ and $0.416 \pm 0.001$ respectively.

6, 4 of length 10 and 1 of length 14. As was the case for Thompson’s group $F$ we found no evidence of singularities at $\beta = 1/9, 1/5$ respectively.
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Figure 10. The mean length of trivial words in $BS(2,3)$ and $BS(3,5)$ at different values of $\beta$. We see very good for low and moderate values of $\beta$ and by systematic errors for larger $\beta$. Again, this error arises from the modest length of the exact series. The dotted curves in these two cases indicate mean length data generated from longer but approximate series, while the dotted vertical lines indicate the estimated critical value of $\beta$ from series.

Figure 11. The reciprocal of the estimated standard error of the mean length as a function of $\beta$ for $BS(2,3)$ and $BS(3,5)$. In both plots we show 4 curves corresponding to simulations at $\alpha = 0, 1, 2, 3$ (anti-clockwise from top). Extrapolating these curves we estimate $\beta_c = 0.388 \pm 0.02$ and $0.444 \pm 0.002$ for $BS(2,3)$ and $BS(3,5)$ respectively. These are quite close to the estimates from series of $0.393$ and $0.443$ (indicated with vertical lines).

As an additional note, we have applied our methods to a finitely generated, but not finitely presented group — namely the lamplighter group. In this case the algorithm has to be modified slightly. One can no longer choose relations uniformly at random, but instead we choose them from distribution $P(R)$ over the relations. As noted in section 2, this distribution must be positive and and one must have
Figure 12. Mean length of freely reduced trivial words in Thompson’s group $F$ at different values of $\beta$. The solid blue lines indicate the reciprocal of the cogrowth of amenable groups with $k$ generators $\beta_c = \frac{1}{2k-1}$. The dashed blue lines indicate the approximate location of the vertical asymptote. In each case, we see that the mean length of trivial words is finite for $\beta$-values at and slightly above $\beta_c$. This is strong evidence that Thompson’s group is not amenable.

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Figure 13. The reciprocal of the estimated standard error of the mean length as a function of $\beta$ for the three presentations of Thompson’s group. In each plots we show 4 curves corresponding to simulations at $\alpha = 0, 1, 2, 3$ (anti-clockwise from top). Extrapolating these curves leads to estimates of $\beta_* \approx 0.395 \pm 0.005, 0.172 \pm 0.002, 0.134 \pm 0.004$. These are all well above the values of amenable groups and so we take this to be strong evidence that Thompson’s group is not amenable.

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