Unpredictable Solutions of Linear Differential Equations

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Abstract

In this study, the existence and uniqueness of the unpredictable solution for a non-homogeneous linear system of ordinary differential equations is considered. The hyperbolic case is under discussion. New properties of unpredictable functions are discovered. The presence of the solutions confirms the existence of Poincaré chaos. Simulations illustrating the chaos are provided.

Keywords: Unpredictable solutions; Poincaré chaos; Linear non-homogeneous systems

1 Introduction and Preliminaries

The concept of the unpredictable functions was introduced in the paper \cite{2}. The description of such functions relies on the dynamics of unpredictable points, which were presented in the study \cite{1} for the first time in the literature. Considering unpredictable points the authors extended the limits of the theory of classical dynamical systems, which was founded by H. Poincaré and G. Birkhoff. The definition in \cite{2} considers the functions as points of the Bebutov dynamics \cite{6}. The metric of the dynamics is not convenient for applications in the theory of differential equations. Therefore, in paper \cite{4}, it was suggested to utilize the topology of uniform convergence on compact subsets of the real axis. This definition is possibly the most effective one for methods of qualitative analysis. The paper \cite{4} was devoted to the investigation of sufficient conditions for the existence of unpredictable solutions of quasilinear differential equations in the case that matrices of coefficients admit all eigenvalues with negative real parts as well as discrete equations.

The present study has two principal novelties with respect to the previous results in the field. The first one is that we consider the hyperbolic case, when the eigenvalues of the matrix of coefficients can admit positive real parts. The second one is that we propose a simpler and more comprehensible proof for the unpredictability property this time. As it was confirmed in papers by Akhmet and Fen \cite{1}-\cite{4}, the

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existence of unpredictable solutions simultaneously means the presence of Poincaré chaos, i.e., unpredictable solutions are “irregular”. This makes the subject attractive for applications. Finally, we consider new properties of the functions. Sufficient conditions are provided such that a linear transformation of an unpredictable function is unpredictable, and it is proved that the sum of an unpredictable function and a periodic function is an unpredictable function.

In the remaining part of the paper, we will make use of the usual Euclidean norm for vectors and the norm induced by the Euclidean norm for square matrices.

The definition of unpredictable functions is as follows.

**Definition 1.1** [4] A uniformly continuous and bounded function \( \vartheta : \mathbb{R} \to \mathbb{R}^m \) is unpredictable if there exist positive numbers \( \epsilon_0, \delta \) and sequences \( \{t_n\}, \{u_n\} \) both of which diverge to infinity such that 
\[
\|\vartheta(t + t_n) - \vartheta(t)\| \to 0 \quad \text{as} \quad n \to \infty
\]
uniformly on compact subsets of the axis and 
\[
\|\vartheta(t + t_n) - \vartheta(t)\| \geq \epsilon_0 \quad \text{for each} \quad t \in [u_n - \delta, u_n + \delta] \quad \text{and} \quad n \in \mathbb{N}.
\]

For the convenience of the next discussion, we will call the convergence of the function’s shifts on compact subsets as Poisson stability and the existence of the number \( \epsilon_0 \) as unpredictability property of the function. Thus, a function is unpredictable, if it is Poisson stable and admits the unpredictability property.

It is worth noting that in the literature a large number of results are obtained for periodic, quasi-periodic and almost periodic solutions of differential equations due to the established mathematical methods and important applications. On the other hand, recurrent and Poisson stable solutions are also crucial for the theory of differential equations [5, 6]. The proposal by Akhmet and Fen can revive interest of specialists in differential equations theory for two reasons. The first one is related to the verification of the unpredictability which requests more sophisticated technique than for recurrent and Poisson stable solutions. Thus the problem of the existence of unpredictable solutions is a challenging one. Another crucial reason is that the presence of unpredictable solutions is necessarily accompanied by the existence of chaos, which is called Poincaré chaos in papers [1]-[4]. Consequently, the research of differential equations with unpredictable solutions will definitely help to activate study and applications of chaos.

The main object of the present paper is the following system of linear differential equations,

\[
x'(t) = Ax(t) + g(t),
\]

where \( x \in \mathbb{R}^n \), \( n \) is a fixed natural number and \( \mathbb{R} \) is the set of all real numbers. Moreover, we assume that all eigenvalues of the constant matrix \( A \in \mathbb{R}^{n \times n} \) have nonzero real parts, and the function \( g : \mathbb{R} \to \mathbb{R}^n \) is uniformly continuous and bounded.

Assume that \( \Re \lambda_i < 0, i = 1, 2, \ldots, q \), and \( \Re \lambda_i > 0, i = q + 1, \ldots, p, 1 \leq q < p \), where \( \lambda_i, i = \ldots \)
In what follows we will denote $\parallel$ the property is proved.

The proof of the lemma immediately follows the inequalities $\parallel \phi(t) \parallel \leq \parallel B^{-1} \parallel \parallel g(t) \parallel\parallel f(t) \parallel$ and $\parallel f(t) - g(t) \parallel$.

For further applications of the main result of the paper, the following lemma can be useful.

**Lemma 1.1** The function $f(t) = B^{-1}g(t)$ is unpredictable.

The proof of the lemma immediately follows the inequalities $\parallel f(t) - f(t) \parallel \leq \parallel B^{-1} \parallel \parallel g(t) - g(t) \parallel$ and $\parallel g(t) - g(t) \parallel$.

**Lemma 1.2** Assume that $g(t)$ is an unpredictable function, and a function $f(t)$ is continuous and periodic. Then the sum $g + f$ is an unpredictable function.

**Proof.** Consider the sum $h(t) \equiv g(t) + f(t)$. Let $t_n$ be the sequence such that $g(t) + f(t)$ converges to $g(t)$ uniformly on all compact subsets of the axis. Since of periodicity of $f(t)$, one can find a subsequence of $t_n$ (we will assume, without loss of generality that it is the sequence $t_n$ itself) that satisfy $f(t) - f(t) \rightarrow 0$ uniformly on the axis. Consequently, $h(t) + f(t) - f(t) = (g(t) - g(t)) + (f(t) - f(t)) \rightarrow 0$ uniformly on compact subsets of the axis as $n \rightarrow \infty$. The Poisson stability is proved.

Consider, now, the sequence $u_n \rightarrow \infty$ and positive numbers $\eta, \sigma$, such that $\parallel g(t) - g(t) \parallel \geq \eta$ for $t \in [u_n - \sigma, u_n + \sigma]$. Again, due to the periodicity, one can find a subsequence of $u_n$, let say the sequence itself, such that $\parallel f(t) - f(t) \parallel < \eta/2$ if $t \in [u_n - \sigma, u_n + \sigma]$. Then, $\parallel h(t) + f(t) - f(t) \parallel \geq \parallel g(t) - g(t) \parallel - \parallel f(t) - f(t) \parallel > \eta - \eta/2 = \eta/2$ for $t \in [u_n - \sigma, u_n + \sigma]$. The unpredictability property is proved. \[ \square \]

## 2 Main result

In what follows we will denote $g = (g_-, g_+)$, where the vector-functions $g_-$ and $g_+$ are of dimensions $q$ and $p - q$, respectively.

As it is known from the theory of differential equations, system (1.1) admits a unique solution $\varphi(t) = (\varphi_-(t), \varphi_+(t))$ which is bounded on $\mathbb{R}$, where

$$\varphi_-(t) = \int_{-\infty}^{t} e^{A_- (t-s)} g_-(s) ds, \quad \varphi_+(t) = -\int_{t}^{\infty} e^{A_+ (t-s)} g_+ (s) ds. \quad (2.3)$$
One can confirm that the bounded solution is periodic, quasi-periodic, or almost periodic if the perturbation function $g$ is respectively of the same type.

The following theorem is concerned with the unpredictable solution of system (1.1).

**Theorem 2.1** If all eigenvalues of the matrix $A$ admit non-zero real parts, and the function $g(t)$ is unpredictable, then the system (1.1) possesses a unique unpredictable solution. If additionally, all the eigenvalues are with negative real parts, then the unpredictable solution is uniformly asymptotically stable.

**Proof.** From the conditions of the theorem it implies that the system admits a bounded solution and it is uniformly asymptotically stable if the matrix of coefficients admits all eigenvalues with negative real parts. This is why, it is sufficient to prove that the solution (2.3) is an unpredictable function.

We will show, first, that the solution is Poisson stable. Since the eigenvalues of the matrix $A$ in system (1.1) have non-zero real parts, there exist numbers $K \geq 1$ and $\alpha > 0$ such that $\|e^{A-t}\| \leq Ke^{-\alpha t}$ for $t \geq 0$ and $\|e^{A+t}\| \leq Ke^{\alpha t}$ for $t \leq 0$.

Since the function $g$ is Poisson stable there exists a sequence $t_n \to \infty$ such that $\|g(t+t_n) - g(t)\| \to 0$ uniformly on compact subsets of the axis. One can easily find that

$$\|\varphi_-(t + t_n) - \varphi_-(t)\| = \|\int_{-\infty}^{t} e^{A-(t-s)}g_-(s + t_n)ds - \int_{-\infty}^{t} e^{A-(t-s)}g_-(s)ds\|$$

$$= \|\int_{-\infty}^{t} e^{A-(t-s)}[g_-(s + t_n) - g_-(s)]ds\| \leq \int_{-\infty}^{t} \|e^{A-(t-s)}\||g_-(s + t_n) - g_-(s)||ds$$

$$\leq \int_{-\infty}^{t} Ke^{-\alpha(t-s)}||g_-(s + t_n) - g_-(s)||ds,$$

and

$$\|\varphi_+(t + t_n) - \varphi_+(t)\| = \|\int_{t}^{\infty} e^{A+(t-s)}g_+(s + t_n)ds - \int_{t}^{\infty} e^{A+(t-s)}g_+(s)ds\|$$

$$= \|\int_{t}^{\infty} e^{A+(t-s)}[g_+(s + t_n) - g_+(s)]ds\| \leq \int_{t}^{\infty} \|e^{A+(t-s)}\||g_+(s + t_n) - g_+(s)||ds$$

$$\leq \int_{t}^{\infty} Ke^{\alpha(t-s)}||g_+(s + t_n) - g_+(s)||ds.$$

Fix an arbitrary positive number $\epsilon$ and a section $[a, b], -\infty < a < b < \infty$, of the real axis. We will show that for sufficiently large $n$ it is true that $\|\varphi(t + t_n) - \varphi(t)\| < \epsilon$ on $[a, b]$. Denote $M = \sup_{\mathbb{R}} \|g\|$, and choose numbers $c < a, b < d, \xi > 0$, such that $\frac{2MK}{\alpha}e^{-\alpha(a-c)} < \frac{\epsilon}{4}, \frac{2MK}{\alpha}e^{-\alpha(d-b)} < \frac{\epsilon}{4}$ and $\frac{K\xi}{\alpha} < \frac{\epsilon}{4}$.

Consider $n$ sufficiently large such that $\|g(t + t_n) - g(t)\| < \xi$ on $[c, b]$. Then for all $t \in [a, b]$

$$\|\varphi_-(t + t_n) - \varphi_-(t)\| \leq \int_{-\infty}^{c} Ke^{-\alpha(t-s)}||g_-(s + t_n) - g_-(s)||ds +$$

$$\int_{c}^{t} Ke^{-\alpha(t-s)}||g_-(s + t_n) - g_-(s)||ds \leq \int_{-\infty}^{c} Ke^{-\alpha(t-s)}2Mds +$$

$$\int_{c}^{t} Ke^{-\alpha(t-s)}\xi ds \leq \frac{2MK}{\alpha}e^{-\alpha(a-c)} + \frac{K\xi}{\alpha} \leq \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}. $$
and similarly one can show that
\[
\|\varphi(t + t_n) - \varphi(t)\| \leq \int_t^{t_n} Ke^{\alpha(t-s)}\|g_+(s + t_n) - g_+(s)\|ds + \int_t^t Ke^{\alpha(t-s)}\|g_+(s + t_n) - g_+(s)\|ds \\
\int_t^t Ke^{\alpha(t-s)}\|g_+(s + t_n) - g_+(s)\|ds \leq \frac{K\xi}{\alpha} + \frac{2MK}{\alpha}e^{-\alpha(b-t)} \leq \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}.
\]

We have obtained that for sufficiently large \(n\) it is true that
\[
\|\varphi(t + t_n) - \varphi(t)\| \leq \|\varphi_+(t + t_n) - \varphi_+(t)\| + \|\varphi_-(t + t_n) - \varphi_-(t)\| < \epsilon
\]
for \(t \in [a, b]\).

The Poisson stability of the solution \(\varphi\) is proved.

Let us check that the solution possesses the unpredictability property. Since the function \(g\) is unpredictable, there exist a sequence \(u_n \to \infty\) and positive numbers \(\eta, \kappa\) such that \(\|g(u_n + t) - g(u_n)\| \geq \eta\) for \(t \in [u_n - \kappa, u_n + \kappa]\). One can easily check that either \(\|g_-(u_n + t) - g_-(u_n)\| \geq \eta/2\) or \(\|g_+(u_n + t) - g_+(u_n)\| \geq \eta/2\) for the same \(t\). Assume that the first inequality is valid, since another case can be considered very similarly. Then, there is a number, without loss of generality \(\kappa\) itself, such that \(\|g_-(t + t_n) - g_-(t)\| \geq \eta/4\) for all \(t \in [u_n - \kappa, u_n + \kappa]\). Without loss of generality we assume that \(\|g_-(t + t_n) - g_-(t)\| \geq \eta\) for all \(t \in [u_n - \kappa, u_n + \kappa]\).

At first we assume that \(\inf_n \|\varphi(u_n + t_n) - \varphi(u_n)\| = 0\). This contradicts the equality
\[
\varphi(t + t_n) - \varphi(t) = \varphi(u_n + t_n) - \varphi(u_n) + \int_{u_n}^{t_n} A(\varphi(s + t_n) - \varphi(s))ds + \int_{u_n}^{t_n} (g(s + t_n) - g(s))ds,
\]

since the last integral is a positive number larger than \(t\eta\) and other terms tend to zero as \(n \to \infty\) for each positive fixed moment \(t\) from the interval \([u_n - \kappa, u_n + \kappa]\).

This is why, it is true that \(\inf_n \|\varphi(u_n + t_n) - \varphi(u_n)\| = 2\epsilon_0\), where \(\epsilon_0\) is a positive number. Fix a positive number \(\sigma\) such that \(2\eta\sigma e^{\|A\|\kappa} < \epsilon_0\).

Now, we have that
\[
\|\varphi(t + t_n) - \varphi(t)\| = \|\varphi(u_n + t_n) - \varphi(u_n) + \int_{u_n}^{t_n} A(t-s)\|g(s + t_n) - g(s)\|ds \geq 2\epsilon_0 - \eta\sigma e^{\|A\|\kappa} > \epsilon_0,
\]
for \(t \in [u_n - \sigma, u_n + \sigma]\). The unpredictability is proved.

The proof of the theorem is completed. \(\Box\)
3 Examples

One of the possible ways to confirm the presence of chaos is through simulations. The concept of unpredictable solutions maintain the series of oscillators, but from the other side the chaos accompanies unpredictability. Consequently, we can look for a confirmation of the results for unpredictability observing irregularity in simulations. The approach is effective for asymptotically stable unpredictable solutions, and it is just illustrative for hyperbolic systems with unstable solutions. In the latter case we rely on the fact that any solution becomes unpredictable ultimately.

In the following examples we will utilize the function $\Theta(t) = \int_{-\infty}^{t} e^{-2(t-s)}\Omega(s)ds$, (3.4)

which was discussed in paper [4]. The function $\Omega(t)$ is defined by $\Omega(t) = \psi_i$ for $t \in [i, i+1)$, $i \in \mathbb{Z}$, where \{\psi_i\}, $i \in \mathbb{Z}$, is an unpredictable solution of the logistic map

$$\lambda_{i+1} = 3.91\lambda_i(1 - \lambda_i)$$

inside the unit interval [0, 1]. The function $\Theta(t)$ is bounded on the whole real axis such that $\sup_{t \in \mathbb{R}} |\Theta(t)| \leq 1/2$, and it is uniformly continuous since its derivative is bounded. It was proven in paper [4] that $\Theta(t)$ is an unpredictable function.

**Example 3.1.**

Consider the system

$$\begin{align*}
x'_1 &= -2x_1 + 2x_2 + 259\Theta(t) - \sin(10t) \\
x'_2 &= x_1 - 3x_2 - 150\Theta(t) + \cos(10t)
\end{align*}$$

(3.6)

where the eigenvalues of the matrix of coefficients are $-2$ and $-0.5$. One can confirm that the perturbation function $(259\Theta(t) - \sin(10t), -150\Theta(t) + \cos(10t))$ is unpredictable in accordance with Lemma 1.2. By the main result of our paper, there is an asymptotically stable unpredictable solution $(\varphi_1(t), \varphi_2(t))$ of system (3.6). Consequently, any solution of the equation behaves irregularly ultimately. This is seen from the simulation of the solution with $x_1(0) = 0.18$, $x_2(0) = 0.01$ in Figures 1 and 2.

The next example is devoted to a system with hyperbolic linear part such that the matrix of coefficients admit both positive and negative eigenvalues.
Figure 1: The time series of the $x_1$ and $x_2$ coordinates of system (3.6) with the initial conditions $x_1(0) = 0, 18$, $x_2(0) = 0, 01$. The figure manifests the presence of Poincaré chaos.

Figure 2: The trajectory of system (3.6) with the initial point $(0, 0, 01)$.

Example 3.2.

Let us take into account the the system

$$
\begin{align*}
u_1' &= -52098u_1 + 7090\varphi_2(t) \\
u_2' &= 9.5u_1 + 0.000000325u_2 + 0.111\varphi_1(t),
\end{align*}
$$

(3.7)

where $(\varphi_1(t), \varphi_2(t))$ is the unpredictable solution of system (3.6).

The eigenvalues of the matrix of coefficients of system (3.7) are $-52098$ and $0.000000325$. According to the result of Theorem 2.1 system (3.7) possesses a unique unpredictable solution.
In order to show the irregular behavior, we consider the system

\[
\begin{align*}
y_1' &= -52098y_1 + 7090x_2(t) \\
y_2' &= 9.5y_1 + 0.0000000325y_2 + 0.111x_1(t),
\end{align*}
\]

(3.8)

where \((x_1(t), x_2(t))\) is the solution of (3.6) depicted in Figures 1 and 2. The simulation results for system (3.8) corresponding to the initial conditions \(y_1(0) = 0\) and \(y_2(0) = 0\) are shown in Figures 3 and 4. Both of the figures confirm the presence of unpredictability in the dynamics of system (3.8).

Figure 3: The time series for the \(y_1\) and \(y_2\) coordinates of system (3.8) with the initial conditions \(y_1(0) = 0\), \(y_2(0) = 0\). The irregular behavior of the solution reveals the presence of an unpredictable solution in the dynamics of (3.8).

Figure 4: The trajectory of the solution of system (3.8) with \(y_1(0) = 0\) and \(y_2(0) = 0\).
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