ON THE SUPPORT OF RELATIVE $D$-MODULES

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Abstract. In this article we prove that although cyclic relative $D$-modules are not finitely generated as modules over the polynomial ring, their support is open and Zariski dense in the vanishing set of their annihilator. As a consequence we obtain an alternative proof of a conjecture of Budur which was recently proven by Budur, van der Veer, Wu and Zhou.

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1. Introduction

The purpose of this article is to give an alternative proof of the following theorem:

Theorem A (Conjectured in [Bud15], proven in [BvVWZ21]). Denote by $\text{Exp} : \mathbb{C}^p \to (\mathbb{C}^*)^p$ the coordinate-wise exponential map. Let $F = (f_1, \ldots, f_p)$ be a tuple of polynomials on $\mathbb{C}^n$, and denote by $B_F$ the Bernstein-Sato ideal of the tuple, and by $\psi_F(\mathbb{C}^n)$ the specialization complex of $F$. Then

$$\text{Exp}(\mathbb{Z}(B_F)) = \text{supp}_{(\mathbb{C}^*)^p}(\psi_F(\mathbb{C}^n)).$$

We refer to [Bud15], [BvVWZ21] for background on $B_F$ and $\psi_F$. The inclusion

$$\text{Exp}(\mathbb{Z}(B_F)) \supset \text{supp}_{(\mathbb{C}^*)^p}(\psi_F(\mathbb{C}^n))$$

was proven already in [Bud15]. To analyse the reverse inclusion the following criterion can be extracted from the proof of [Bud15] Proposition 1.7].

Proposition 1.1 ([Bud15]). If $\alpha \in \mathbb{Z}(B_F)$ and

$$\frac{D_n[s_1, \ldots, s_p]f_1^s \ldots f_p^s \otimes \mathbb{C}[s_1, \ldots, s_p]}{D_n[s_1, \ldots, s_p]f_1^{s+1} \ldots f_p^{s+1} \otimes \mathbb{C}[s_1, \ldots, s_p]} \not\in \mathfrak{m}_{\alpha},$$

then $\text{Exp}(\alpha) \in \text{supp}_{(\mathbb{C}^*)^p}(\psi_F(\mathbb{C}^n)).$

2020 Mathematics Subject Classification. 14F10 (primary), 16Z10 (secondary).
Key words and phrases. Groebner basis; Weyl algebra; Bernstein-Sato ideal; $b$-function.
Since supp($C^*P(\psi_F(C^n_\mathbb{C}))$) is a closed subset of ($C^*$)$P$, Theorem B follows if we can prove (1) for $\alpha$ in an open Zariski dense subset of $Z(B_F)$. With this in mind the following theorem is our main result.

**Theorem B.** Let $J \subset D_n[s_1, \ldots, s_p]$ be a left ideal, where $D_n$ is the Weyl algebra. Let $p$ be a minimal prime divisor of $J \cap \mathbb{C}[s_1, \ldots, s_p]$. Then there exists a polynomial $h \in \mathbb{C}[s_1, \ldots, s_p]$ such that for all $\alpha \in \mathbb{Z}(p) \setminus \mathbb{Z}(h)$ with maximal ideal $m_\alpha$,

$$\left(\frac{D_n[s_1, \ldots, s_p]}{J}\right) \otimes_{\mathbb{C}[s_1, \ldots, s_p]} \left(\frac{\mathbb{C}[s_1, \ldots, s_p]}{m_\alpha}\right) \neq 0.$$  

Theorem A was recently proven in [BvVWZ21] where it was deduced from the following theorem.

**Theorem C ([BvVWZ21]).** Let $F = (f_1, \ldots, f_p)$ be a tuple of polynomials on $\mathbb{C}^n$. For every codimension 1 irreducible component $H$ of $Z(B_F)$ there is a Zariski open subset $V \subset H$ such that for every $\alpha \in V$ with maximal ideal $m_\alpha$,

$$\frac{D_n[s_1, \ldots, s_p]}{J} f_1^{s_1} \ldots f_p^{s_p} \otimes_{\mathbb{C}[s_1, \ldots, s_p]} \mathbb{C}[s_1, \ldots, s_p] / m_\alpha \neq 0.$$  

Theorem C is a special case of Theorem B since it only deals with codimension 1 components. This restriction meant that in [BvVWZ21] results from [Mai16] were necessary to conclude A from Theorem C. More precisely, the following result from [Mai16] was used to prove Theorem A in [BvVWZ21]:

**Theorem D ([Mai16]).** Every irreducible component of $Z(B_F)$ can be translated along an integer vector into a codimension 1 component of $Z(B_F)$.

Theorem D is a corollary of Theorem A since it is known from [BLSW17, Theorem 1.3] that supp($C^*P(\psi_F(C^n_\mathbb{C}))$) is a union of codimension 1 torsion translated subtori of ($C^*$)$P$. In particular, our methods provide a new proof of Theorem D.

Another advantage of Theorem B over the methods in [BvVWZ21] is that Theorem B does not depend on the module $D_n[s_1, \ldots, s_p]/J$ being relatively holonomic, as defined in [BvVWZ21].

The proof strategy for Theorem B is to specialize a suitable Gröbner basis for $J + p$ to a Gröbner basis for $(D_n[s_1, \ldots, s_p]/J) \otimes_{\mathbb{C}[s_1, \ldots, s_p]} (\mathbb{C}[s_1, \ldots, s_p]/m_\alpha)$. Using a result on specializing Gröbner bases from [Ley01] and [Oak97] we can then conclude Theorem B. This method of proof works in every ring where Gröbner basis methods are available.

In Section 2 we start with some technical preliminaries that allow us to conclude in Lemma 2.4 that $(J + p) \cap \mathbb{C}[s_1, \ldots, s_p] = p$. This result will be used in Section 3 to construct the Gröbner basis we need and to control the specialization.

**Acknowledgement.** We would like to thank Nero Budur and Alexander Van Werde for the helpful comments and suggestions.

The author is supported by a PhD Fellowship of the Research Foundation - Flanders.
2. A LEMMA IN COMMUTATIVE ALGEBRA, AND A NON-COMMUTATIVE COROLLARY

We denote \( A = \mathbb{C}[s_1, \ldots, s_p] \). An ideal \( q \subset A \) is called primary if for all \( x, y \in A \) with \( xy \in q \), either \( x \in q \) or \( y \in \sqrt{q} \). If \( q \) is primary, then \( p = \sqrt{q} \) is a prime ideal. When we say that \( q \) is \( p \)-primary we mean that \( q \) is primary and \( \sqrt{q} = p \).

Lemma 2.1. Let \( q \subset A \) be a \( p \)-primary ideal. If \( q \neq p \), then there exists an \( f \in p \setminus q \) such that \( fp \subset q \).

Proof. Let \( N \subset A/q \) be the nilradical, which is non-zero since \( q \neq p \). Let \( g_1, \ldots, g_m \in N \) be a set of generators. Let \( f_0 = 1 \in A/q \). Using induction we define for \( i = 1, \ldots, m \):

\[
f_{i+1} = f_i g_{k_i+1}^{k_i+1-1},
\]

where \( k_{i+1} \in \mathbb{Z} \) is the smallest integer for which

\[
f_i g_{k_i+1} = 0.
\]

Notice that such \( k_{i+1} \) always exists, since each \( g_i \) is nilpotent and that \( k_{i+1} \) is always at least 1 since \( f_i \) is not zero, by induction. By construction we have that for each \( i = 1, \ldots, m \),

\[
g_i f_m = 0.
\]

Let \( f \) be a lift of \( f_m \) to \( A \). Since \( f_m \neq 0 \), \( f \not\in q \), and since \( f_m \) is nilpotent, \( f \in p \). Let \( p \in p \). Then in \( A/q \) we can write \( p + q = \sum_{i=1}^m a_i g_i \) for some \( a_i \in A/q \). Then

\[
fp + q = \sum_{i=1}^m a_i g_i f_m = 0,
\]

so that \( fp \in q \). \( \square \)

Every ideal \( \mathfrak{J} \subset A \) has a primary decomposition. This means that we can write \( \mathfrak{J} = \bigcap_{i=1}^m q_i \) such that

1. every \( q_i \) is primary, and
2. for all \( 1 \leq j \leq m, \bigcap_{i=1}^m q_i \not\subset \bigcap_{i\neq j}^m q_i \), and
3. The prime ideals \( \sqrt{q_1}, \ldots, \sqrt{q_m} \) are pairwise distinct.

The minimal (under the inclusion order) elements of the set \( \{ \sqrt{q_i} \} \) are uniquely determined by \( \mathfrak{J} \), and are called the minimal prime divisors of \( \mathfrak{J} \).

Theorem 2.2. Let \( \mathfrak{J} \subset A \) be an ideal with primary decomposition \( \mathfrak{J} = \bigcap_{i=1}^m q_i, m > 1 \). Let \( \sqrt{q_j} \) be a minimal prime divisor. Then there exists an \( h \in A \setminus q_j \) such that \( h \sqrt{q_j} \subset \mathfrak{J} \).

Proof. We assume without loss of generality that \( j = 1 \). We claim that \( \bigcap_{i=2}^m q_i \not\subset \sqrt{q_1} \). If we would have \( \bigcap_{i=2}^m q_i \subset \sqrt{q_1} \) then there is some \( q_i \) contained in \( \sqrt{q_1} \), since the latter is prime. Hence also \( \sqrt{q_i} \subset \sqrt{q_1} \). Since \( \sqrt{q_1} \) is a minimal prime, this must be an equality. However, by definition of the primary decomposition, \( \sqrt{q_i} \neq \sqrt{q_1} \), and this contradiction proves the claim. Let \( g \in \bigcap_{i=2}^m q_i \setminus \sqrt{q_1} \).
If \( \sqrt{q_1} = q_1 \), then \( h = g \) satisfies the condition of the theorem. Namely, let \( q \in q_1 \). Then \( gj \subseteq (\bigcap_{i=2}^m q_i) q_1 \subseteq \bigcap_{i=1}^m q_i = \mathfrak{J} \).

If \( \sqrt{q_1} \neq q_1 \), we get from Lemma 2.1 an \( f \in \sqrt{q_1} \setminus q_1 \) such that \( f \sqrt{q_1} \subseteq q_1 \). Set \( h = fg \). We claim that this \( h \) satisfies the conditions of the theorem. If \( h \in q_1 \), then since \( q_1 \) is primary, \( f \in q_1 \) or \( g \in \sqrt{q_1} \), neither of which is possible by choice of \( f \) and \( g \). This means that indeed \( h \in A \setminus q_1 \). Let \( p \in \sqrt{q_1} \). By choice of \( f, pf \in q_1 \), and thus \( ph = gpf \in (\bigcap_{i=2}^m q_i) q_1 \subseteq \bigcap_{i=1}^m q_i = \mathfrak{J} \), which concludes the proof. \( \square \)

**Corollary 2.3.** Let \( M \) be an \( A \)-module, and let \( p \) be a minimal prime divisor of the ideal \( \text{Ann}_A(M) \). Then \( \text{Ann}_A(M \otimes_A (A/p)) = p \).

**Proof.** Let \( \text{Ann}_A(M) = \bigcap_{i=1}^m q_i \) be a primary decomposition of \( \text{Ann}_A(M) \) with \( \sqrt{q_1} = p \). From Theorem 2.2 we get an \( h \in A \setminus q_1 \) such that \( hp \subseteq \text{Ann}_A(M) \).

Let \( g \in \text{Ann}_A(M \otimes_A (A/p)) \). This means that for every \( m \in M \), \( gm \subseteq pM \). In other words, for every \( m \in M \), there exists an \( n \in M \) and \( p \in p \) such that \( gm = pn \).

We multiply this equation by \( h \) on both sides to find \( ghm = hpn \). By construction, \( hp \subseteq \text{Ann}_A(M) \), so that \( ghm = 0 \) for all \( m \), and thus \( gh \subseteq \text{Ann}_A(M) \). In particular, \( gh \in q_1 \), so that either \( h \in q_1 \) or \( g \in \sqrt{q_1} \). Since \( h \not\in q_1 \) by construction, we conclude that \( g \in p \), which shows that \( \text{Ann}_A(M \otimes_A (A/p)) \subseteq p \). The other inclusion is obvious, and this concludes the proof. \( \square \)

**Lemma 2.4.** Let \( \mathfrak{J} \subseteq D_n[s_1, \ldots, s_p] \) be a left ideal. Let \( p \) be a minimal prime divisor of \( \mathfrak{J} \cap A \). Then \( (\mathfrak{J} + Rp) \cap A = p \).

**Proof.** We denote \( R = D_n[s_1, \ldots, s_p] \). We regard \( R/\mathfrak{J} \) as an \( A \)-module. As such we claim that \( \text{Ann}_A(R/\mathfrak{J}) = \mathfrak{J} \cap A \). To see this, let \( f \in \text{Ann}_A(R/\mathfrak{J}) \), so that \( f \cdot (1+\mathfrak{J}) = f + \mathfrak{J} \) is zero in \( R/\mathfrak{J} \), which means that \( f \in \mathfrak{J} \cap A \). For the other inclusion let \( f \in \mathfrak{J} \cap A \). Then

\[
(f \cdot (P + \mathfrak{J}) = fP + \mathfrak{J} = Pf + \mathfrak{J},
\]

where in the second equality we use that \( A \) is contained in the center of \( R \). Since \( f \in \mathfrak{J} \), which is a left ideal, we conclude that \( Pf \in \mathfrak{J} \), so that \( f \cdot (P + \mathfrak{J}) = 0 \) and hence indeed \( f \in \text{Ann}_A(R/\mathfrak{J}) \).

We apply Corollary 2.3 to find that \( \text{Ann}_A((R/\mathfrak{J}) \otimes_A (A/p)) = p \). The lemma follows when we prove that \( \text{Ann}_A((R/\mathfrak{J}) \otimes_A (A/p)) = (\mathfrak{J} + Rp) \cap A \), which in turn follows, as above, when we prove that we have an isomorphism

\[
(R/\mathfrak{J}) \otimes_A (A/p) \cong R/(\mathfrak{J} + Rp).
\]

There is a well-defined map from the left hand side to the right hand side given by

\[
(P + \mathfrak{J}) \otimes (f + p) \mapsto Pf + (\mathfrak{J} + Rp).
\]

Note that this is well-defined because \( A \) is contained in the center of \( R \). In the other direction we have the map

\[
P + (\mathfrak{J} + Rp) \mapsto (P + \mathfrak{J}) \otimes (1 + p).
\]
These maps are inverse to each other, which proves the claim.  

3. Gröbner bases for \(D_n\)-modules

In this section we recall some facts about Gröbner bases for ideals in \(R = D_n[s_1, \ldots, s_p]\) and \(D_n\) and then prove Theorem 13. We denote as in the previous section \(A = \mathbb{C}[s_1, \ldots, s_p]\). We will always implicitly regard \(A\) and \(D_n\) as subsets of \(R\).

In \(D_n\), we consider the set of standard monomials of the form \(x^\alpha \partial^{\beta}, \alpha, \beta \in \mathbb{Z}_{\geq 0}^n\). On these monomials, we consider the lexicographical order with

\[
\partial_1 > \cdots > \partial_n > x_n > \cdots > x_1.
\]

Any operator in \(D_n\) is a finite \(\mathbb{C}\)-linear combination of standard monomial in a unique way. Writing \(P \in D_n\) as a linear combination \(P = \sum \alpha, \beta c_{\alpha, \beta} x^\alpha \partial^{\beta}\) with non-zero \(c_{\alpha, \beta}\) we denote by \(\text{lm}_{D_n}(P)\) the largest monomial \(x^\alpha \partial^{\beta}\) with respect to the lexicographical order. For an ideal \(\mathcal{I} \subset D_n\), we denote \(\text{lm}_{D_n}(\mathcal{I}) = \{\text{lm}_{D_n}(P) \mid P \in \mathcal{I}\}\). A finite generating set \(G \subset \mathcal{I}\) is called a Gröbner basis for \(\mathcal{I}\) if the following is true: for every \(m \in \text{lm}_{D_n}(\mathcal{I})\) there exists a \(P \in G\) such that \(\sigma(\text{lm}_{D_n}(P)) \mid \sigma(m)\) where \(\sigma\) denotes the operation of taking the principal symbol. More explicitly, this says that when \(x^\alpha \partial^{\beta} \in \text{lm}_{D_n}(\mathcal{I})\) there exists a \(P \in G\) with \(\text{lm}_{D_n}(P) = x^\alpha \partial^{\beta}\) and \((\alpha, \beta')\) is entry-wise less than or equal to \((\alpha, \beta)\). To simply the notation we will write this divisibility notion simply as \(x^\alpha \partial^{\beta} \mid x^\alpha \partial^{\beta}\), but we emphasise that this does not mean that there exists some \(P \in D_n\) for which \(Px^\alpha \partial^{\beta'} = x^\alpha \partial^{\beta}\). It follows immediately from the definition that \(\mathcal{I} = D_n\) if and only if any Gröbner basis for \(\mathcal{I}\) contains a unit, i.e. an element of \(\mathbb{C}\).

In \(R\), we consider the set of standard monomials of the form \(x^\alpha \partial^{\beta} s^\gamma, \alpha, \beta \in \mathbb{Z}_{\geq 0}^n, \gamma \in \mathbb{Z}_{\geq 0}^p\). On these monomials, we consider the lexicographical order with

\[
\partial_1 > \cdots > \partial_n > x_n > \cdots > x_1 > s_p > \cdots > s_1.
\]

Any operator in \(R\) is a finite \(\mathbb{C}\)-linear combination of standard monomial in a unique way. We denote by \(\text{lm}_R(P)\) the leading monomial of an operator \(P \in R\) with respect to the lexicographical order. For an ideal \(\Omega \subset R\), we denote \(\text{lm}_R(\Omega) = \{\text{lm}_R(P) \mid P \in \Omega\}\). A finite generating set \(G \subset \Omega\) is called a Gröbner basis for \(\Omega\) if the following Gröbner property is true: for every \(m \in \text{lm}_R(\Omega)\) there exists a \(P \in G\) such that \(\sigma(\text{lm}_R(P)) \mid \sigma(m)\) where \(\sigma\) denotes the operation of taking the principal symbol. More explicitly, this says that when \(x^\alpha \partial^{\beta} s^\gamma \in \text{lm}_R(\Omega)\) there exists a \(P \in G\) with \(\text{lm}_R(P) = x^\alpha \partial^{\beta} s^\gamma\) and \((\alpha', \beta', \gamma')\) is entry-wise less than or equal to \((\alpha, \beta, \gamma)\). Again we denote this divisibility relation simply by \(x^\alpha \partial^{\beta} s^\gamma \mid x^\alpha \partial^{\beta} s^\gamma\). Regarding the \(s_1, \ldots, s_p\) as parameters we will also need the following. Any \(P \in R\) can be written uniquely as

\[
P = \sum_{\alpha, \beta} h_{\alpha, \beta}(s_1, \ldots, s_p)x^\alpha \partial^{\beta},
\]
with $h_{α,β} ≠ 0$. We denote the *parametric leading monomial* of $P$ by $\text{plm}(P) = x^α∂^β$ and the *parametric leading coefficient* $\text{plc}(P) = h_{α,β}$, where $x^α∂^β$ is the largest monomial occurring in (4) with respect to the monomial order (2).

In the commutative ring $A$ consider the lexicographical monomial order with

\[(5)\]

$s_p > \cdots > s_1$.

We denote by $\text{lm}_A(f)$ the leading monomial of $f ∈ A$.

In all three rings $D_n, A$ and $R$ a Gröbner basis for an ideal can be obtained by applying Buchberger’s algorithm to an arbitrary generating set. We note the following relations between the leading monomials. For $f ∈ A$:

$\text{lm}_R(f) = \text{lm}_A(f)$.

For $P ∈ D_n$, $\text{lm}_D_n(P) = \text{lm}_R(P)$.

For $P ∈ R$, $\text{lm}_A(\text{plc}(P)) \cdot \text{plm}(P) = \text{lm}_R(P)$. Because of these equalities we will from now on suppress the notation of the ring and simply write $\text{lm}$ for leading monomials.

**Lemma 3.1.** Let $\mathfrak{Q} ⊂ R$ be an ideal and let $G$ be a Gröbner basis for $\mathfrak{Q}$ with respect to the order (3). Then $G ∩ A$ is a Gröbner basis for $\mathfrak{Q} ∩ A$ with respect to the order (5).

**Proof.** Let $f ∈ \mathfrak{Q} ∩ A$. There exists a $P ∈ G$ such that $\text{lm}(P) | \text{lm}(f)$. This means that $\text{lm}(P) ∈ A$, and thus by definition of the order (3), $P ∈ A$. We conclude that for every $f ∈ \mathfrak{Q} ∩ A$ there exists a $g ∈ G ∩ A$ such that $\text{lm}(g) | \text{lm}(f)$. It only remains to show that $G ∩ A$ generates $\mathfrak{Q} ∩ A$. This follows from the preceding statement, since we can reduce $f$ to zero modulo $G ∩ A$ by iteratively canceling leading monomials.

**Lemma 3.2.** Let $\mathfrak{Q} ⊂ R$ be an ideal. Then there exists a Gröbner basis $G$ for $\mathfrak{Q}$ with respect to the order (3) such that for all $P ∈ G \setminus A$, $\text{plc}(P) ∉ \mathfrak{Q} ∩ A$.

**Proof.** Let $P ∈ G \setminus A$ and write

$P = \text{plc}(P) \cdot \text{plm}(P) + Q$.

By Lemma 3.1 $G ∩ A = \{f_1, \ldots, f_m\}$ is a Gröbner basis for $\mathfrak{Q} ∩ A$. Using the division algorithm [CLO15, Proposition 1.6.1] we find $q_1, \ldots, q_m, r$ such that

$\text{plc}(P) = \sum_{i=1}^m q_i f_i + r$,

where $\text{lm}(r)$ is not divisible by any $\text{lm}(f_i)$. This means that

$P = \left( \sum_{i=1}^m q_i f_i + r \right) \cdot \text{plm}(P) + Q ≡ r \text{plm}(P) + Q =: P'$,

where $≡$ denotes equivalence modulo $\mathfrak{Q}$. If $P' = 0$ we define $G' = G \setminus \{P\}$ and if $P' ≠ 0$ then we define $G' = G \setminus \{P\} ∪ \{P'\}$.

We claim that $G'$ is still a Gröbner basis for $\mathfrak{Q}$. This is clear if $P = P'$ so assume that $P ≠ P'$. We first remark that $G'$ clearly still generates $\mathfrak{Q}$, and hence we need only verify the Gröbner property. If $\text{lm}(P') = \text{lm}(P)$ then the Gröbner property is clearly still satisfied, so assume that $\text{lm}(P') ≠ \text{lm}(P)$. This assumption means that
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For any $\alpha \in \mathbb{C}^p$ there is a specialization map $q_\alpha : R \rightarrow D_\alpha$ defined by

$$q_\alpha(P(s_1, \ldots, s_p)) = P(\alpha_1, \ldots, \alpha_p).$$

Notice that for any ideal $\Omega \subset R$ we have

$$(R/\Omega) \otimes_A (A/m_\alpha) \cong D_\alpha/q_\alpha(\Omega),$$

where $m_\alpha$ denotes the maximal ideal corresponding to $\alpha$.

**Theorem 3.3** ([Ley01], Lemma 2.5). Let $\Omega \subset R$ be an ideal, let $p \subset \Omega$ be a second ideal, and let $G \subset \Omega$ be a Gröbner basis for $\Omega$ with respect to the order $(3)$. Let $h = \prod_{P \in G \setminus p} \text{plc}(P)$. Let $\alpha \in Z(p \cap A) \setminus Z(h)$. Then $q_\alpha(G \setminus p)$ is a Gröbner basis for $q_\alpha(\Omega)$ with respect to the order $(2)$.

**Proof.** Apply [Ley01] Lemma 2.5 without any $y$-parameters.

We remark that a statement similar to Theorem 3.3 also appeared in [Oak97], and that the comprehensive Gröbner bases from [KW91] can also be used to give a simple proof of this result.

**Proof of Theorem 3**. Let $G$ be a Gröbner basis for $\Omega = J + Rp$ with respect to the order $(3)$. By Lemma 3.2 we can assume that for all $P \in G \setminus A$,

$$\text{plc}(P) \notin \Omega \cap A = p,$$

where this equality follows from Corollary 2.3.

Since $\Omega \cap A = p$, clearly we have $G \cap A \subset p$, and hence $G \cap p = G \cap A$, and hence also

$$G \setminus A = G \setminus p.$$

Putting the preceding two statements together we conclude that for all $P \in G \setminus p$, $\text{plc}(P) \notin p$. Since $p$ is prime, this means that

$$h = \prod_{P \in G \setminus p} \text{plc}(P) \notin p.$$

Since $G \cap p = G \cap A$, we have that $(G \setminus p) \cap A = \emptyset$, and hence for every $P \in G \setminus p$, $\text{plm}(P) \notin \mathbb{C}$. By choice of $h$, for all $\alpha \in Z(p) \setminus Z(h)$ and $P \in G \setminus p$, $\text{plm}(P) = \text{lm}(q_\alpha(P))$. 

lm(plc(P)) was divisible by lm($f_i$) for some $f_i$. To see this, note that if this were not the case then by the definition of the division algorithm, lm(r) = lm(plc(P)), and hence lm(P) = lm($P'$), contradicting our assumption. This means that lm(P) = lm(plc(P))plm(P) is divisible by some lm($f_i$). Hence any element $Q \in \Omega$ for which lm(P) | lm(Q) also satisfies lm($f_i$) | lm(Q). It follows that $G'$ is still a Gröbner basis, since $P$ was redundant for the Gröbner basis property.

We keep repeating this procedure until no $P \in G \setminus A$ is changed by applying this reduction. This is clearly a finite process. Once we are done we thus find that for all $P \in G \setminus A$, lm(plc(P)) is not divisible by any lm($f_i$). Since the $\{f_1, \ldots, f_m\}$ form a Gröbner basis for $\Omega \cap A$, this means that for all $P \in G \setminus A$, plc(P) $\notin \Omega \cap A$. □

For any $\alpha \in \mathbb{C}^p$ there is a specialization map $q_\alpha : R \rightarrow D_\alpha$ defined by

$$q_\alpha(P(s_1, \ldots, s_p)) = P(\alpha_1, \ldots, \alpha_p).$$

Notice that for any ideal $\Omega \subset R$ we have

$$(R/\Omega) \otimes_A (A/m_\alpha) \cong D_\alpha/q_\alpha(\Omega),$$

where $m_\alpha$ denotes the maximal ideal corresponding to $\alpha$.

**Theorem 3.3** ([Ley01], Lemma 2.5). Let $\Omega \subset R$ be an ideal, let $p \subset \Omega$ be a second ideal, and let $G \subset \Omega$ be a Gröbner basis for $\Omega$ with respect to the order $(3)$. Let $h = \prod_{P \in G \setminus p} \text{plc}(P)$. Let $\alpha \in Z(p \cap A) \setminus Z(h)$. Then $q_\alpha(G \setminus p)$ is a Gröbner basis for $q_\alpha(\Omega)$ with respect to the order $(2)$.

**Proof.** Apply [Ley01] Lemma 2.5 without any $y$-parameters. □

We remark that a statement similar to Theorem 3.3 also appeared in [Oak97], and that the comprehensive Gröbner bases from [KW91] can also be used to give a simple proof of this result.

**Proof of Theorem 3**. Let $G$ be a Gröbner basis for $\Omega = J + Rp$ with respect to the order $(3)$. By Lemma 3.2 we can assume that for all $P \in G \setminus A$,

$$\text{plc}(P) \notin \Omega \cap A = p,$$

where this equality follows from Corollary 2.3.

Since $\Omega \cap A = p$, clearly we have $G \cap A \subset p$, and hence $G \cap p = G \cap A$, and hence also

$$G \setminus A = G \setminus p.$$

Putting the preceding two statements together we conclude that for all $P \in G \setminus p$, $\text{plc}(P) \notin p$. Since $p$ is prime, this means that

$$h = \prod_{P \in G \setminus p} \text{plc}(P) \notin p.$$

Since $G \cap p = G \cap A$, we have that $(G \setminus p) \cap A = \emptyset$, and hence for every $P \in G \setminus p$, $\text{plm}(P) \notin \mathbb{C}$. By choice of $h$, for all $\alpha \in Z(p) \setminus Z(h)$ and $P \in G \setminus p$, $\text{plm}(P) = \text{lm}(q_\alpha(P))$, 

lm(plc(P)) was divisible by lm($f_i$) for some $f_i$. To see this, note that if this were not the case then by the definition of the division algorithm, lm(r) = lm(plc(P)), and hence lm(P) = lm($P'$), contradicting our assumption. This means that lm(P) = lm(plc(P))plm(P) is divisible by some lm($f_i$). Hence any element $Q \in \Omega$ for which lm(P) | lm(Q) also satisfies lm($f_i$) | lm(Q). It follows that $G'$ is still a Gröbner basis, since $P$ was redundant for the Gröbner basis property.

We keep repeating this procedure until no $P \in G \setminus A$ is changed by applying this reduction. This is clearly a finite process. Once we are done we thus find that for all $P \in G \setminus A$, lm(plc(P)) is not divisible by any lm($f_i$). Since the $\{f_1, \ldots, f_m\}$ form a Gröbner basis for $\Omega \cap A$, this means that for all $P \in G \setminus A$, plc(P) $\notin \Omega \cap A$. □

For any $\alpha \in \mathbb{C}^p$ there is a specialization map $q_\alpha : R \rightarrow D_\alpha$ defined by

$$q_\alpha(P(s_1, \ldots, s_p)) = P(\alpha_1, \ldots, \alpha_p).$$

Notice that for any ideal $\Omega \subset R$ we have

$$(R/\Omega) \otimes_A (A/m_\alpha) \cong D_\alpha/q_\alpha(\Omega),$$

where $m_\alpha$ denotes the maximal ideal corresponding to $\alpha$.

**Theorem 3.3** ([Ley01], Lemma 2.5). Let $\Omega \subset R$ be an ideal, let $p \subset \Omega$ be a second ideal, and let $G \subset \Omega$ be a Gröbner basis for $\Omega$ with respect to the order $(3)$. Let $h = \prod_{P \in G \setminus p} \text{plc}(P)$. Let $\alpha \in Z(p \cap A) \setminus Z(h)$. Then $q_\alpha(G \setminus p)$ is a Gröbner basis for $q_\alpha(\Omega)$ with respect to the order $(2)$.

**Proof.** Apply [Ley01] Lemma 2.5 without any $y$-parameters. □

We remark that a statement similar to Theorem 3.3 also appeared in [Oak97], and that the comprehensive Gröbner bases from [KW91] can also be used to give a simple proof of this result.

**Proof of Theorem 3**. Let $G$ be a Gröbner basis for $\Omega = J + Rp$ with respect to the order $(3)$. By Lemma 3.2 we can assume that for all $P \in G \setminus A$,

$$\text{plc}(P) \notin \Omega \cap A = p,$$

where this equality follows from Corollary 2.3.

Since $\Omega \cap A = p$, clearly we have $G \cap A \subset p$, and hence $G \cap p = G \cap A$, and hence also

$$G \setminus A = G \setminus p.$$

Putting the preceding two statements together we conclude that for all $P \in G \setminus p$, $\text{plc}(P) \notin p$. Since $p$ is prime, this means that

$$h = \prod_{P \in G \setminus p} \text{plc}(P) \notin p.$$
so that \( q_\alpha(G \setminus p) \) does not contain any units. Since this set is a Gröbner basis for \( q_\alpha(\Omega) \) by Theorem 3.3 we conclude that \( D_n/q_\alpha(\Omega) \neq 0 \).

Now notice that

\[
\frac{D_n}{q_\alpha(\Omega)} \cong \frac{R}{\langle \alpha \rangle} \otimes_A \frac{A}{m_\alpha} \cong \left( \frac{R}{\langle \alpha \rangle} \otimes_A \frac{A}{m_\alpha} \right) \otimes_A \frac{A}{m_\alpha} \cong \frac{R}{\langle \alpha \rangle} \otimes_A \frac{A}{m_\alpha},
\]

which proves the claim. □

**Proof of Theorem A.** We denote

\[
M = \frac{D_n[s_1, \ldots, s_p] f_1^{s_1} \ldots f_p^{s_p}}{D_n[s_1, \ldots, s_p] f_1^{s_1+1} \ldots f_p^{s_p+1}}.
\]

Then \( B_F = \text{Ann}_A(M) \), and \( M \) is a cyclic \( D_n[s_1, \ldots, s_p] \)-module. Let \( C \subset Z(B_F) \) be an irreducible component. By Theorem [1] there is an \( h \in A \) which does not vanish identically on \( C \) such that for all \( \alpha \in C \setminus Z(h) \), \( M \otimes_A (A/m_\alpha) \neq 0 \).

By Proposition [1.1] we conclude that

\[
\text{Exp}(C \setminus Z(h)) \subset \text{supp}_{(\mathbb{C}^*)^p}(\psi_F(\mathbb{C}_p)).
\]

The map \( \text{Exp} : \mathbb{C}^p \to (\mathbb{C}^*)^p \) is continuous for the analytic topology. In the analytic topology we have \( C \setminus Z(h) = C \), and hence

\[
\text{Exp}(C) \subset \text{Exp}(C \setminus Z(h)) \subset \text{supp}_{(\mathbb{C}^*)^p}(\psi_F(\mathbb{C}_p)) = \text{supp}_{(\mathbb{C}^*)^p}(\psi_F(\mathbb{C}_p)),
\]

which is what we wanted to prove. □

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