Ductile damage model for metal forming simulations including refined description of void nucleation

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Abstract

We address the prediction of ductile damage and material anisotropy accumulated during plastic deformation of metals. A new model of phenomenological metal plasticity is proposed which is suitable for applications involving large deformations of workpiece material. The model takes combined nonlinear isotropic/kinematic hardening, strain-driven damage and rate-dependence of the stress response into account. Within this model, the work hardening and the damage evolution are fully coupled. The description of the kinematics is based on the double multiplicative decomposition of the deformation gradient proposed by Lion. An additional multiplicative decomposition is introduced in order to account for the damage-induced volume increase of the material. The model is formulated in a thermodynamically admissible manner. Within a simple example of the proposed framework, the material porosity is adopted as a rough measure of damage.

A new simple void nucleation rule is formulated based on the consideration of various nucleation mechanisms. In particular, this rule is suitable for materials which exhibit a higher void nucleation rate under torsion than in case of tension.

The material model is implemented into the FEM code Abaqus and a simulation of a deep drawing process is presented. The robustness of the algorithm and the performance of the formulation is demonstrated.

Keywords: ductile damage, Bauschinger effect, kinematic hardening, finite strain elasto-plasticity, multiplicative plasticity, void nucleation

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Nomenclature

1 identity tensor
\( \mathbf{F} \), \( \mathbf{F}^{\text{ep}} \) deformation gradient and its elasto-plastic part
\( \mathbf{F}_i \), \( \mathbf{F}_{ii} \) dissipative parts of deformation
\( \mathbf{F}_e \), \( \mathbf{F}_{ie} \) conservative parts of deformation
\( \mathbf{C} \), \( \mathbf{C}^{\text{ep}} \) total and elasto-plastic right Cauchy-Green tensors
\( \mathbf{C}_i \), \( \mathbf{C}_{ii} \) tensors of right Cauchy-Green type (dissipative)
\( \mathbf{C}_e \), \( \mathbf{C}_{ie} \) tensors of right Cauchy-Green type (conservative)
\( \mathbf{L} \) velocity gradient tensor
\( \mathbf{L}_i \) inelastic velocity gradient
\( \mathbf{L}_{ii} \) inelastic velocity gradient of substructure
\( \mathbf{D} \) strain rate tensor (stretching tensor)
\( \mathbf{D}_i \) inelastic strain rate
\( \mathbf{D}_{ii} \) inelastic strain rate of substructure
\( s \), \( s_d \), \( s_e \) inelastic arc-length and its parts
\( \mathbf{T} \), \( \mathbf{\tilde{T}} \) Cauchy stress and 2nd P-K stress
\( \mathbf{S} \), \( \mathbf{\dot{S}} \) Kirchhoff stress and its pull-back to \( \mathbf{\hat{K}} \)
\( \mathbf{S}^{\text{ep}} \), \( \mathbf{\dot{S}}^{\text{ep}} \), \( \mathbf{T}^{\text{ep}} \) Kirchhoff-like stress and its pull-backs to \( \mathbf{\hat{K}} \) and \( \mathbf{\hat{K}}^{\text{ep}} \)
\( \mathbf{\hat{X}} \), \( \mathbf{\dot{X}} \), \( \mathbf{\ddot{X}} \) backstresses on \( \mathbf{\hat{K}} \), \( \mathbf{\dot{K}} \), \( \mathbf{\ddot{K}} \)
\( \mathbf{\Sigma} \), \( \mathbf{\Xi} \) effective stress on \( \mathbf{\hat{K}} \) and Mandel-like backstress on \( \mathbf{\dot{K}} \)
\( R \) isotropic hardening (stress)
\( \mathbf{A}^b \) deviatoric part of a second-rank tensor
\( \mathbf{\overline{A}} \) unimodular part of a second-rank tensor
\( \mathbf{A}^T \) transposition of a second-rank tensor
\( \text{sym}(\mathbf{A}) \) symmetric part of a second-rank tensor
\( \| \mathbf{A} \| \) Frobenius norm of a second-rank tensor
\( \mathbf{A} : \mathbf{B} \) scalar product of two second-rank tensors
\( \rho_R \), \( \rho_{\text{por}} \) mass densities in \( \mathbf{\hat{K}} \) and \( \mathbf{\hat{K}}^{\text{ep}} \)
\( \Phi \) damage-induced volume change
\( N \) void number per unit volume in \( \mathbf{\hat{K}} \)

1. Introduction

Dealing with metal forming applications, it is often necessary to assess the mechanical properties of the resulting engineering components, including the remaining bearing capacity, accu-
mulated defects, and residual stresses. Thus, a state-of-the-art model for metal forming simulations should account for various nonlinear phenomena. If the residual stresses and the spring back are of particular interest, such a model should include the combined isotropic/kinematic hardening. For some metals, however, the influence of ductile damage induced by plastic deformation should be taken into account as well. In spite of widespread applications involving large plastic deformations accompanied by kinematic hardening and damage, only few material models cover these effects (cf. Simo and Ju (1989); Menzel et al. (2005); Lin and Brocks (2006); Grammenoudis et al. (2009)).

In the current study we advocate the approach to plasticity/viscoplasticity based on the multiplicative decomposition of the deformation gradient. This approach is gaining popularity due to its numerous advantages like the absence of spurious shear oscillations, the absence of non-physical dissipation in the elastic range and the invariance under the change of the reference configuration (Shutov and Ihlemann, 2014). The main purpose of the current publication is to promote the phenomenological damage modeling within the multiplicative framework. Toward that end, the model of finite strain viscoplasticity proposed by Shutov and Kreißig (2008) is extended to account for ductile damage. The original viscoplastic model takes the isotropic hardening of Voce type and the kinematic hardening of Armstrong-Frederick type into account. The model is based on a double multiplicative split of the deformation gradient, considered by Lion (2000). Both the original model and its extension are thermodynamically consistent. We aim at the simplest possible extension which takes the following effects into account:

- nonlinear kinematic and isotropic hardening;
- inelastic volume change due to damage-induced porosity;
- damage-induced deterioration of elastic and hardening properties.

In this vein, we assume that the elastic properties deteriorate with increasing damage and remain isotropic at any stage of deformation. The assumption of elastic isotropy is needed to exclude the plastic spin from the flow rule, thus reducing the flow rule to six dimensions.

The interaction between damage and strain hardening is explicitly captured by the extended model. It is known that the mechanisms of the isotropic and kinematic hardening are not identical (cf. Barlat et al. (2003)). Therefore, the isotropic and kinematic contributions to the overall hardening deteriorate differently with increasing damage. Concerning the coupling in the

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1 The seminal idea of the double split was already used by several authors to capture the nonlinear kinematic hardening (Helm, 2001; Tsakmakis and Willuweit, 2004; Dettmer and Reese, 2004; Hartmann et al., 2008; Henann and Anand, 2009; Brepols et al., 2013; Zhu et al., 2013; Brepols et al., 2014). It was also implicitly adopted by Menzel et al. (2005); Johansson et al. (2005). A simple extension to distortional hardening was presented by Shutov et al. (2011).
opposite direction, the accumulated plastic anisotropy has a clear impact on the rate of damage accumulation upon the strain path change.

Since the proposed model is a generalization of the existing viscoplasticity model, some well-established numerical procedures can be used. The model is formulated as an open framework, suitable for further extension. Physically motivated relations describing void nucleation, growth, and coalescence can be implemented in a straightforward way, thus enriching the constitutive formulation. The restrictions imposed on these relations by the second law of thermodynamics are obtained in an explicit form.

A new refined void nucleation rule is introduced in this paper for porous ductile metals with second phases. This evolution law contains dependencies on the eigenvalues of the (effective) stress deviator. The rule is based on consideration of various nucleation mechanisms, like debonding of second phase particles under tensile and shear loading or crushing of the particles under high hydrostatic stress. Therefore, the material parameters possess a clear mechanical interpretation, which simplifies the parameter identification based on microstructural observations or molecular dynamics simulations.

The material model is validated using the experimental flow curves presented by Horstemeyer (1998) for a cast A356 aluminium alloy as well as some experimental data on void nucleation published by Horstemeyer et al. (2000).

2. Material model

2.1. Finite strain kinematics

As a starting point for model construction, we consider the rheological model shown in Fig. 1. This model represents a modified Schwedoff-body connected in series with an idealized porosity element. The modified Schwedoff body exhibits nonlinear phenomena which are similar to plasticity with Bauschinger effect (Lion, 2000; Shutov and Kreißig, 2008). The porosity element is introduced to account for the effect of damage-induced volume change.

In this paper we consider a system of constitutive equations which qualitatively reproduces the rheological model. Following the procedure described by Lion (2000), the displacements and forces in rheological elements are formally replaced by strains and stresses. Any connection of rheological bodies in series will be represented by a multiplicative split of certain tensor-valued quantities. Let \( F \) be the deformation gradient from the local reference configuration \( \tilde{\mathbf{F}} \) to the

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2Note that this rheological model has a similar structure to that of thermoplasticity with kinematic hardening (Lion, 2000; Shutov and Ihlemann, 2011). In the current study, the element of thermal expansion is replaced by the porosity element.

3This technique can be successfully adopted even for two-dimensional rheological bodies (Shutov et al., 2011). For a general framework dealing with one-dimensional rheological elements, we refer the reader to Bröcker and Matzenmiller (2014).
current configuration $\mathcal{K}$ (see Fig. 2). First, we decompose it into an elasto-plastic part $F_{\text{ep}}^{\text{por}}$ and a porosity-induced expansion $F_{\text{por}}^{\text{por}}$ in the following way

$$F = F_{\text{ep}}^{\text{por}} F_{\text{por}}^{\text{por}}, \quad F_{\text{por}}^{\text{por}} = \Phi^{1/3} I, \quad \Phi \geq 1. \quad (1)$$

Following Bammann and Aifantis (1989), the expansion $F_{\text{por}}^{\text{por}}$ is assumed to be isotropic, where the damage-induced volume change is given by $\Phi$

$$\det F_{\text{por}}^{\text{por}} = \Phi. \quad (2)$$

The decomposition $F_{\text{por}}^{\text{por}}$ yields a configuration of porous material $\mathcal{K}^{\text{por}} := F_{\text{por}}^{\text{por}} \mathcal{K}$. Let $\rho_{R}$ be the mass density in the reference configuration. The mass density of unstressed porous material is given by $\rho_{\text{por}} := \rho_{R} / \Phi$. Next, we consider the well-known decomposition of the elasto-plastic part into an inelastic part $F_{\text{i}}^{\text{por}}$ and an elastic part $\hat{F}_{e}^{\text{por}}$

$$F_{\text{ep}}^{\text{por}} = \hat{F}_{e}^{\text{por}} F_{\text{i}}^{\text{por}}. \quad (3)$$

Following the approach of Lion (2000) to the nonlinear kinematic hardening, we decompose the inelastic part into a dissipative part $F_{\text{ii}}^{\text{por}}$ and a conservative part $\hat{F}_{\text{ie}}^{\text{por}}$

$$F_{\text{i}}^{\text{por}} = \hat{F}_{\text{ie}}^{\text{por}} F_{\text{ii}}^{\text{por}}. \quad (4)$$

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4 A general introduction to the kinematics within the geometrically exact setting can be found in Haupt (2002); Hashiguchi and Yamakawa (2012).
We adopt this decomposition as a phenomenological assumption, without explicit discussion of its microstructural origins (Clayton et al., 2014). Intuitively, one can relate the dissipative parts $F_i$ and $F_{ii}$ to the deformations of the St.-Venant body and the modified Newton body, respectively (cf. Fig. 1). The conservative parts $\hat{F}_e$ and $\hat{F}_{le}$ are associated with the deformation of the elastic springs (cf. Fig. 1). The commutative diagram shown in Fig. 2 is useful to summarize the introduced multiplicative decompositions. The fictitious local configurations $\hat{\mathcal{K}}$ and $\check{\mathcal{K}}$ are referred to as stress-free intermediate configuration and configuration of kinematic hardening, respectively.

**Remark 1.** Combining (1) and (3) we arrive at

$$F = \hat{F}_e F_i F_{\text{por}}.$$  

(5)

Due to the isotropy of $F_{\text{por}}$, this decomposition is equivalent to the decomposition considered by Bammann and Aifantis (1989):

$$F = \hat{F}_e F_{\text{por}} F_i.$$  

(6)

In this paper we prefer to operate with (5) rather than (6) in order to retain the structure of the viscoplasticity model proposed by Shutov and Kreißig (2008).

**Remark 2.** There exists an abundance of literature considering ductile damage modeling within multiplicative elasto-plasticity based on the split (3) (see, for example, Bammann and Aifantis (1989); Steinmann et al. (1994); Menzel et al. (2005); Soyarslan and Tekkaya (2010); Bammann and Solanki (2010)). Unfortunately, only few damage models incorporate the multiplicative split (1) which is used to capture the nonlinear kinematic hardening. For instance, Grammenoudis et al. (2003); Bröcker and Matzenmiller (2014) have coupled ductile damage with a model of finite strain plasticity by adopting the concept of effective stresses.

$\hat{\mathcal{K}}$ $\check{\mathcal{K}}$ $F_{\text{por}}$ $\hat{F}_e$ $\hat{F}_{le}$ $\check{\mathcal{K}}$ $F$ $F_i$ $F_{\text{por}}$ $F_{ii}$

Figure 2: Commutative diagram: triple multiplicative decomposition of the deformation gradient.

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As will be seen from the following, the concept of effective stresses is too restrictive in some cases.
Let us consider the right Cauchy-Green tensor (RCGT) $\mathbf{C} := \mathbf{F}^T \mathbf{F}$. Based on the decompositions (1) and (3), we introduce the following kinematic quantities of the right Cauchy-Green type

$$
\mathbf{C}_{ep} := \mathbf{F}_{ep}^T \mathbf{F}_{ep}, \quad \mathbf{C}_{i} := \mathbf{F}_{i}^T \mathbf{F}_{i}, \quad \hat{\mathbf{C}}_{e} := \hat{\mathbf{F}}_{e}^T \hat{\mathbf{F}}_{e}.
$$

Here, the elasto-plastic RCGT $\mathbf{C}_{ep}$ and the inelastic RCGT $\mathbf{C}_{i}$ operate on the porous configuration $\mathbf{K}^p$; the elastic RCGT $\hat{\mathbf{C}}_{e}$ operates on the stress-free configuration $\hat{\mathbf{K}}$. Next, based on (4), we define the inelastic RCGT of substructure $\mathbf{C}_{ii}$ and the elastic RCGT of substructure $\hat{\mathbf{C}}_{ie}$

$$
\mathbf{C}_{ii} := \mathbf{F}_{ii}^T \mathbf{F}_{ii}, \quad \hat{\mathbf{C}}_{ie} := \hat{\mathbf{F}}_{ie}^T \hat{\mathbf{F}}_{ie}.
$$

The tensors $\mathbf{C}_{ii}$ and $\hat{\mathbf{C}}_{ie}$ operate on $\mathbf{K}^p$ and $\hat{\mathbf{K}}$ respectively. Combining (1) and (7), we have

$$
\mathbf{C}_{ep} = \Phi^{-2/3} \mathbf{C}.
$$

In order to describe the inelastic flow we consider the inelastic velocity gradient $\mathbf{\hat{L}}_i$ and the inelastic strain rate $\mathbf{\hat{D}}_i$, both operating on $\hat{\mathbf{K}}$

$$
\mathbf{\hat{L}}_i := \left( \frac{d\mathbf{F}_{i}}{dt} \right)^{-1}, \quad \mathbf{\hat{D}}_i := \text{sym}(\mathbf{\hat{L}}_i).
$$

Analogously, the inelastic flow of the substructure is captured using the following tensors operating on $\hat{\mathbf{K}}$

$$
\mathbf{\hat{L}}_{ii} := \left( \frac{d\mathbf{F}_{ii}}{dt} \right)^{-1}, \quad \mathbf{\hat{D}}_{ii} := \text{sym}(\mathbf{\hat{L}}_{ii}).
$$

In terms of the rheological analogy shown in Fig. 1, the strain rates $\mathbf{\hat{D}}_i$ and $\mathbf{\hat{D}}_{ii}$ can be associated with the deformation rates of the St.-Venant body and the modified Newton body, respectively.

The accumulated inelastic arc-length (Odqvist parameter) is a strain-like internal variable defined through

$$
s(t) := \int_0^t \dot{s}(\tau) d\tau, \quad \dot{s} := \sqrt{\frac{2}{3} \| \mathbf{\hat{D}}_i \|}.
$$

Along with $s$ we introduce its dissipative part $s_d$ such that $s_e := s - s_d$ controls the isotropic hardening.}

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For simplicity, the rheological model in Fig. 1 does not include the isotropic hardening. The reader interested in idealized rheological elements of isotropic hardening is referred to Bröcker and Matzenmiller (2013).
2.2. Stresses

Let $T$ and $S := \det(F)T$ be respectively the Cauchy and Kirchhoff stresses. We introduce a Kirchhoff-like stress tensor operating on the current configuration $\mathcal{K}$

$$S_{\text{ep}} := \det(F_{\text{ep}})T = \Phi^{-1}S. \quad (13)$$

The elastic pull-back of this tensor yields its counterpart operating on the stress-free configuration $\hat{\mathcal{K}}$

$$\hat{S}_{\text{ep}} := \hat{F}_{e}^{-1}S_{\text{ep}}\hat{F}_{e}^{-T}. \quad (14)$$

The 2nd Piola-Kirchhoff tensor is obtained applying a pull-back to $S$

$$\hat{T} := F^{-1}SF^{-T}. \quad (15)$$

Analogously, the elasto-plastic pull-back of $S_{\text{ep}}$ yields a Kirchhoff-like tensor operating on the porous configuration $\text{por}$. At the same time, this tensor is identical to the inelastic pull-back of $\hat{S}_{\text{ep}}$:

$$\text{por}T_{\text{ep}} := \text{por}F_{\text{ep}}^{-1}S_{\text{ep}}\text{por}F_{\text{ep}}^{-T} = \text{por}F_{i}^{-1}\hat{S}_{\text{ep}}\text{por}F_{i}^{-T}. \quad (16)$$

The stress tensors $\hat{T}$ and $\text{por}T_{\text{ep}}$ are related by

$$\hat{T} = \Phi^{1/3}\text{por}T_{\text{ep}}. \quad (17)$$

Next, let us consider the backstresses which are typically used to capture the Bauschinger effect.\footnote{A backstresses-free approach to the Bauschinger effect was presented in Barlat et al. (2011).} Intuitively, the backstresses correspond to stresses in the modified Maxwell body shown in Fig. 1. Let $\hat{X}$ be the backstress operating on $\hat{\mathcal{K}}$, which is a stress measure power conjugate to the strain rate $\hat{D}_{ii}$. A backstress measure operating on the stress-free configuration $\hat{\mathcal{K}}$ is obtained using a push-forward operation as follows

$$\hat{X} := \hat{F}_{ie}\hat{X}\hat{F}_{ie}^{T}. \quad (18)$$

Its counterpart operating on the porous configuration $\text{por}\mathcal{K}$ is given by

$$\text{por}X := \text{por}F_{il}^{-1}\text{por}X_{il}\text{por}F_{il}^{-T} = \text{por}F_{i}^{-1}\text{por}X_{i}\text{por}F_{i}^{-T}. \quad (19)$$

A Mandel-like backstress tensor on $\hat{\mathcal{K}}$ is defined through

$$\hat{\Xi} := \hat{C}_{ie}\hat{X}. \quad (20)$$
For what follows, it is instructive to introduce the so-called effective stress operating on $\hat{\Sigma} := \hat{C}_e \hat{S}_{ep} - \hat{X}$.  

(21)

In other words, the effective stress represents the difference between the Mandel-like stress $\hat{C}_e \hat{S}_{ep}$ and the backstress $\hat{X}$. For this definition, a Kirchhoff-like stress $S_{ep}$ is adopted instead of the Kirchhoff stress $S$ in order to keep the structure of the damage model close to the viscoplasticity model presented by Shutov and Kreißig (2008). The effective stress represents the local force driving the inelastic deformation. In terms of the rheological model, this stress corresponds to the load acting on the St.-Venant body.

2.3. Free energy

Only small pore volume fractions will be considered in this paper, such that the applicability domain of the model is restricted to $\Phi - 1 \ll 1$. The material porosity will be seen as a rough measure of damage (cf. Bammann and Aifantis (1989)). Aiming at a thermodynamically consistent formulation, we postulate the free energy per unit mass in the form

$$\psi = \psi(\hat{C}_e, C_{ie}, s_e, \Phi) = \psi_{el}(\hat{C}_e, \Phi) + \psi_{kin}(\hat{C}_{ie}, \Phi) + \psi_{iso}(s_e, \Phi),$$

(22)

where $\psi_{el}$ corresponds to the energy stored due to macroscopic elastic deformations, components $\psi_{kin}$ and $\psi_{iso}$ correspond to the energy storage associated with kinematic and isotropic hardening. The dependence of $\psi_{el}$, $\psi_{kin}$, and $\psi_{iso}$ on $\Phi$ is introduced in order to capture the damage-induced deterioration of elastic and hardening properties, as is common for metals. In this study, the system of constitutive equations will be formulated in case of general isotropic energy-storage functions $\psi_{el}(\hat{C}_e, \Phi)$ and $\psi_{kin}(\hat{C}_{ie}, \Phi)$. However, some concrete assumptions will be needed for the numerical computations. To be definite, we postulate

$$\rho_R \psi_{el}(\hat{C}_e, \Phi) = \frac{k(\Phi)}{2} \left( \ln \sqrt{\det \hat{C}_e} \right)^2 + \frac{\mu(\Phi)}{2} \left( \text{tr} \hat{C}_e - 3 \right),$$

(23)

$$\rho_R \psi_{kin}(\hat{C}_{ie}, \Phi) = \frac{c(\Phi)}{4} \left( \text{tr} \hat{C}_{ie} - 3 \right), \quad \rho_R \psi_{iso}(s_e) = \frac{\gamma(\Phi)}{2} (s_e)^2,$$

(24)

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8 A distinction should be made between the effective stress concept which is typically used in continuum damage mechanics (Rabotnov, 1969) and the notion of effective stress adopted within plasticity with kinematic hardening. In fact, the second notion is implemented in the current study.

9 More precisely, $\psi_{kin} + \psi_{iso}$ is a part of the free energy stored in the defects of the crystal structure. Remaining parts of the “defect energy” which are detached from any hardening mechanism (Shutov and Ihlemann, 2011) are neglected here. An accurate description of the energy storage becomes especially important within thermoplasticity with nonlinear kinematic hardening (Canadija and Brnic, 2004; Canadija and Mosler, 2011; Shutov and Ihlemann, 2011).
where the overline \( (\cdot) \) denotes the unimodular part of a second-rank tensor
\[
\overline{\mathbf{A}} := (\det \mathbf{A})^{-1/3} \mathbf{A}.
\] (25)

The reduction (deterioration) of the bulk modulus \( k(\Phi) \) and the shear modulus \( \mu(\Phi) \) is postulated in the form
\[
k(\Phi) = k_0 \exp \left( -BRR \cdot (\Phi - 1) \right), \quad \mu(\Phi) = \mu_0 \exp \left( -SRR \cdot (\Phi - 1) \right), \tag{26}
\]
where constant parameters \( BRR \geq 0 \) and \( SRR \geq 0 \) stand for “bulk reduction rate” and “shear reduction rate” respectively\(^{10}\). Analogously, for the degradation of the hardening mechanisms we postulate
\[
c(\Phi) = c_0 \exp \left( -KRR \cdot (\Phi - 1) \right), \quad \gamma(\Phi) = \gamma_0 \exp \left( -IRR \cdot (\Phi - 1) \right), \tag{27}
\]
with \( KRR \geq 0 \) and \( IRR \geq 0 \). As follows from (26) and (27), the material parameters \( k_0 > 0 \), \( \mu_0 > 0 \), \( c_0 > 0 \), and \( \gamma_0 > 0 \) correspond to the reference state with \( \Phi = 1 \).

**Remark 3.** Even for \( k = \text{const} \) and \( \mu = \text{const} \), a weak damage-elasticity coupling is predicted by (23) due to the mass density reduction\(^{11}\).

Now we introduce the following relations of hyperelastic type which express the stresses \( \hat{\mathbf{S}}_{\text{ep}} \), the backstresses \( \hat{\mathbf{X}} \), and the isotropic hardening \( R \) as a function of introduced kinematic variables
\[
\hat{\mathbf{S}}_{\text{ep}} = 2\rho_{\text{pos}} \frac{\partial \psi_{\text{el}}(\hat{\mathbf{C}}_e, \Phi)}{\partial \hat{\mathbf{C}}_e}, \quad \hat{\mathbf{X}} = 2\rho_{\text{pos}} \frac{\partial \psi_{\text{kin}}(\hat{\mathbf{C}}_{\text{ie}}, \Phi)}{\partial \hat{\mathbf{C}}_{\text{ie}}}, \quad R = \rho_{\text{pos}} \frac{\partial \psi_{\text{iso}}(s_e, \Phi)}{\partial s_e}. \tag{28}
\]

The relations \((28)_1\) and \((28)_2\) can be motivated by the rheological model shown in Fig. (1), if a hyperelastic response is assumed for both elastic springs. On the other hand, these relations will play a crucial role in the proof of the thermodynamic consistency (cf. Section 2.4). Another implication of (28) is that \((\hat{\mathbf{S}}_{\text{ep}}, \hat{\mathbf{C}}_e), (\hat{\mathbf{X}}, \hat{\mathbf{C}}_{\text{ie}}), \) and \((R, s_e)\) form conjugate pairs.

### 2.4. Clausius-Duhem inequality

Here we consider the Clausius-Duhem inequality, which states that the internal dissipation \( \delta_i \) is always non-negative. For isothermal processes studied in this paper, this inequality takes the reduced form (cf., for instance, Haupt (2002))
\[
\delta_i := \frac{1}{\rho_R} \mathbf{T} : \dot{\mathbf{E}} - \dot{\psi} \geq 0. \tag{29}
\]

\(^{10}\)For some applications, a deterioration of elastic properties with intact bulk modulus can be sufficient (cf. Steinmann et al. (1994)), which would imply \( BRR=0 \) in our case.

\(^{11}\)This type of weak coupling is also exhibited by the well-known Rousselier model.
Let us specialize this inequality for the free energy given by (22). First, we decompose the stress power into a power of the hydrostatic stress component on the damage-induced volume expansion and the remaining part

\[ \frac{1}{\rho_R} \overline{T} : \dot{E} = \frac{1}{\rho_{por}} \left( \Phi_{por} \frac{\Phi_{por}}{3\Phi} T_{ep} : C_{ep} + T_{ep} : \frac{1}{2} \dot{C}_{ep} \right). \]  

(30)

Recalling (9) and (17) we obtain the following relation for the hydrostatic stress component

\[ T_{ep} : C_{ep} = \Phi^{-1} \overline{T} : C. \]  

(31)

Let us take a closer look at the stress power \( T_{ep} : \frac{1}{2} \dot{C}_{ep} \). As is usually done in multiplicative inelasticity (cf. equation (2.28) in Lion (2000) or equation (13.258) in Haupt (2002)), this quantity is split into two components. More precisely, it follows from (3) that

\[ T_{ep} : \frac{1}{2} \dot{C}_{ep} = \dot{S}_{ep} : \frac{1}{2} \dot{C}_e + (\dot{C}_e \dot{S}_{ep}) : \dot{L}_i. \]  

(32)

Since \( \psi_{el} \) is isotropic, relation (28) implies that the tensors \( \dot{C}_e \) and \( \dot{S}_{ep} \) commute. In that case, the Mandel tensor \( \dot{C}_e \dot{S}_{ep} \) is symmetric and (32) yields

\[ T_{ep} : \frac{1}{2} \dot{C}_{ep} = \dot{S}_{ep} : \frac{1}{2} \dot{C}_e + (\dot{C}_e \dot{S}_{ep}) : \dot{D}_i. \]  

(33)

Next, it follows from (28) that \( \dot{C}_{ic} \) and \( \dot{X} \) commute as a result of the isotropy assumption made for \( \psi_{kin} \). Therefore, the Mandel-like tensor \( \ddot{X} = \dot{C}_{ic} \dot{X} \) is symmetric. Taking the multiplicative split (4) into account, we obtain a relation for the backstress power, which has a similar structure to (33)

\[ \dot{X} : \dot{D}_i = \dot{X} : \frac{1}{2} \dot{C}_{ie} + (\dot{C}_{ie} \dot{X}) : \dot{D}_i. \]  

(34)

Rearranging the terms, we arrive at

\[ 0 = -\dot{X} : \dot{D}_i + \dot{X} : \frac{1}{2} \dot{C}_{ie} + (\dot{C}_{ie} \dot{X}) : \dot{D}_i. \]  

(35)

Summing both sides of (33) and (35) we obtain the stress power as

\[ T_{ep} : \frac{1}{2} \dot{C}_{ep} = \dot{S}_{ep} : \frac{1}{2} \dot{C}_e + (\dot{C}_e \dot{S}_{ep}) : \dot{D}_i - \dot{X} : \dot{D}_i + \dot{X} : \frac{1}{2} \dot{C}_{ie} + (\dot{C}_{ie} \dot{X}) : \dot{D}_i. \]  

(36)

\[ ^{12} \text{A similar decomposition is obtained for some models of thermoplasticity due to the similar description of kinematics [Lion 2000; Shutov and Ihlemann 2011].} \]
On the other hand, for the rate of the free energy we have
\[ \dot{\psi} = \frac{\partial \psi_{el}(\hat{C}_e, \Phi)}{\partial \hat{C}_e} : \dot{\hat{C}}_e + \frac{\partial \psi_{kin}(\hat{C}_{ie}, \Phi)}{\partial \hat{C}_{ie}} : \dot{\hat{C}}_{ie} + \frac{\partial \psi_{iso}(s_e, \Phi)}{\partial s_e} \dot{s}_e + \frac{\partial \psi(\hat{C}_e, \hat{C}_{ie}, s_e, \Phi)}{\partial \Phi} \dot{\Phi}. \]  
(37)

Substituting (30) and (37) into (29) and taking (36) into account we rewrite the internal dissipation in the form
\[ \delta_i = \left( \frac{1}{\rho_{por}} \frac{1}{3\Phi} T_{ep} : C_{ep} - \frac{\partial \psi(\hat{C}_{ie}, \hat{C}_{ie}, s_e, \Phi)}{\partial \Phi} \right) \dot{\Phi} + \left( \frac{1}{2\rho_{por}} \hat{S}_{ep} - \frac{\partial \psi_{el}(\hat{C}_e, \Phi)}{\partial \hat{C}_e} \right) : \dot{\hat{C}}_e \\
+ \left( \frac{1}{2\rho_{por}} \hat{X} - \frac{\partial \psi_{kin}(\hat{C}_{ie}, \Phi)}{\partial \hat{C}_{ie}} \right) : \dot{\hat{C}}_{ie} + \frac{1}{\rho_{por}} (\hat{C}_e \hat{S}_{ep} - \hat{X}) : \dot{\hat{D}}_i + \frac{1}{\rho_{por}} (\hat{C}_{ie} \hat{X}) : \dot{\hat{D}}_{ii} \\
- \frac{\partial \psi_{iso}(s_e, \Phi)}{\partial s_e} \dot{s}_e. \]  
(38)

Taking the potential relations (28) into account and recalling the abbreviations (20), (21), the internal dissipation takes the following simple form
\[ \delta_i = \left( \frac{1}{\rho_{por}} \frac{1}{3\Phi} T_{ep} : C_{ep} - \frac{\partial \psi(\hat{C}_{ie}, \hat{C}_{ie}, s_e, \Phi)}{\partial \Phi} \right) \dot{\Phi} + \frac{1}{\rho_{por}} \left( \hat{\Sigma} : \dot{\hat{D}}_i + \hat{\Xi} : \dot{\hat{D}}_{ii} - R \dot{s}_e \right). \]  
(39)

Damage-induced energy dissipation is given by the first term on the right-hand side of (39). The remaining part corresponds to the energy dissipation induced by plasticity. In other words, this relation encompasses the concept that both plasticity and damage are dissipative processes. As a sufficient condition for the Clausius-Duhem inequality, we assume that both contributions are non-negative
\[ \delta_i^{\text{damage}} := \left( \frac{1}{\rho_{por}} \frac{1}{3\Phi} T_{ep} : C_{ep} - \frac{\partial \psi(\hat{C}_{ie}, \hat{C}_{ie}, s_e, \Phi)}{\partial \Phi} \right) \dot{\Phi} \geq 0, \]  
(40)

\[ \delta_i^{\text{plasticity}} := \frac{1}{\rho_{por}} \left( \left( \hat{\Sigma} : \dot{\hat{D}}_i - R \dot{s} \right) + \hat{\Xi} : \dot{\hat{D}}_{ii} + R \dot{s}_d \right) \geq 0. \]  
(41)

Note that the terms \( \hat{\Sigma} : \dot{\hat{D}}_i \) and \( \hat{\Xi} : \dot{\hat{D}}_{ii} \) are related to the energy dissipation in the friction element and the rate-independent dashpot, shown in Fig. \( \square \)

2.5. Yield condition and evolution equations

Ductile damage influences both the plastic flow and the size of the elastic domain in the stress space. To account for these effects, we consider the yield function as follows
\[ f := \| \hat{\Sigma}^D \| - \sqrt{\frac{2}{3}} [K(\Phi) + R], \]  
(42)
where \( K(\Phi) \geq 0 \) stands for the damage-dependent yield stress. As a sufficient condition for the inequality (41), we postulate the following evolution equations for the inelastic flow and the inelastic flow of the substructure

\[
\dot{\mathbf{D}}_i = \lambda_i \frac{\hat{\Sigma}^D}{\| \hat{\Sigma}^D \|}, \quad \dot{\mathbf{D}}_{ii} = \lambda_i \zeta(\Phi) \hat{\Xi}^D,
\]

\[
\dot{s} = \sqrt{\frac{2}{3}} \lambda_i, \quad \dot{s}_d = \frac{\beta(\Phi)}{\gamma(\Phi)} \dot{s}_R,
\]

(43)

where \( \zeta(\Phi) \geq 0 \) and \( \beta(\Phi) \geq 0 \) are damage-dependent hardening parameters and \( \lambda_i \geq 0 \) is the inelastic multiplier. It follows from (43) that \( \lambda_i = \| \dot{\mathbf{D}}_i \| \). Thus, this quantity corresponds to the intensity of the inelastic flow. In this paper we describe it based on Peryzna’s overstress theory

\[
\lambda_i = \frac{1}{\eta} \langle f \rangle^m, \quad \langle x \rangle := \max(x, 0).
\]

(45)

Here, \( f_0 > 0 \) is used to obtain a non-dimensional term in the angle bracket. For simplicity, we assume that the viscosity parameters \( \eta \geq 0 \) and \( m \geq 1 \) are not affected by damage.

The evolution equation (43) complies with the normality flow rule. Both inelastic flows governed by (43) are incompressible. Indeed, after some computations we arrive at

\[
\frac{d}{dt} (\det \mathbf{F}_i) = \frac{d}{dt} (\det \mathbf{F}_{ii}) = 0, \quad \frac{d}{dt} (\det \mathbf{C}_i) = \frac{d}{dt} (\det \mathbf{C}_{ii}) = 0.
\]

(46)

Therefore, under appropriate initial conditions we have

\[
\det \mathbf{F}_i = \det \mathbf{F}_{ii} = \det \mathbf{C}_i = \det \mathbf{C}_{ii} = 1.
\]

(47)

In particular, the volume changes are accommodated by the elastic bulk strain and the damage-induced expansion of the material, but not by the inelastic deformation. Thus, \( \det \mathbf{F} = \det \mathbf{F}_{el} \Phi \).

Note that in case of constant \( \gamma(\Phi) \) and \( \beta(\Phi) \), the classical Voce rule of isotropic hardening is reproduced by (43). In that case, the parameter \( \beta \) governs the saturation rate. For simplicity, we may postulate \( \beta(\Phi) = \beta_0 = const \). A similar assumption for the kinematic hardening would imply

\[
c(\Phi) \zeta(\Phi) = const = c_0 \zeta_0, \quad \zeta(\Phi) = \zeta_0 \exp \left( K R R (\Phi - 1) \right).
\]

(48)

\[\text{\small\textsuperscript{13}}\] Thus, \( f_0 \) is not a material parameter.

\[\text{\small\textsuperscript{14}}\] We postulate that the porous material obeys the normality rule as soon as the normality rule holds true for the matrix material (Berg, 1969).
Moreover, we note that the yield stress $K(\Phi)$ is connected with the isotropic hardening of the material, since it appears in combination with $R$ (cf. (42)). Therefore, it is natural to assume that $K(\Phi)$ is affected by damage in the same way as the isotropic hardening. More precisely, we assume

$$K(\Phi) = K_0 \exp \left(-IRR \cdot (\Phi - 1)\right).$$

(49)

The damage evolution is postulated as a function of the inelastic strain rate $\dot{D}_i$, the effective stress $\dot{\Sigma}$, the backstress $\dot{X}$, and the damage variable $\Phi$

$$\dot{\Phi} = \dot{\Phi}(\dot{D}_i, \dot{\Sigma}, \dot{X}, \Phi).$$

(50)

We consider ductile damage as a strain-driven process. Therefore, we suppose that $\dot{\Phi}$ is a homogeneous function of $D_i$

$$\dot{\Phi}(\alpha \dot{D}_i, \dot{\Sigma}, \dot{X}, \Phi) = \alpha \dot{\Phi}(\dot{D}_i, \dot{\Sigma}, \dot{X}, \Phi), \quad \text{for all } \alpha \geq 0.$$  

(51)

It follows immediately from this relation that the damage does not evolve if the plastic flow is frozen. Next, for simplicity, we do not consider any material curing in this study. In combination with the inequality (40) we have the following restrictions

$$\dot{\Phi} \geq 0, \quad \dot{\Phi} = 0 \quad \text{for } \frac{1}{\rho_{\text{por}}} \frac{1}{3\Phi} \mathbf{T}_{\text{ep}} : \mathbf{C}_{\text{ep}}^{\text{por}} < \frac{\partial \psi(C_e, C_{ie}, s_e, \Phi)}{\partial \Phi}.$$  

(52)

Taking into account that $\frac{\partial \psi(C_e, C_{ie}, s_e, \Phi)}{\partial \Phi} \leq 0$ (cf. Section 2.3), the restriction (52) states that the damage-induced expansion is impossible whenever the hydrostatic pressure exceeds a certain limit.

Although the presented framework is based on phenomenological assumptions, the damage evolution law (50) provides an entry point for micromechanical approaches to damage modeling (cf. the recent review by Besson (2010)). It may include the modeling of void nucleation, growth, and coalescence (some fundamental approaches are discussed by Marini et al. (1985); Pardoen and Hutchinson (2000); Xue (2007); Brüning et al. (2014)). Since the validation and calibration of such models requires certain microstructural observations, these models may be helpful in extending the applicability range of the purely phenomenological models. Moreover, there is an abundance of literature dealing with specific fracture criteria (see, for example, Khan and Liu (2012) and references therein). Therefore this issue is not addressed in the present publication.

15 This relation is introduced for small pore volume fractions. Some refined modeling is needed at moderate to large volume fractions (Fritzen et al., 2012).

16 It has to be kept in mind that this restriction is a sufficient condition for thermodynamic consistency, but not a necessary one.
2.6. Void nucleation rule

In this subsection we construct a new simple rule of void nucleation for porous ductile metals with second phases. Let \( N \) be the number of voids per unit volume of reference configuration. The nucleation rule is based on consideration of different nucleation mechanisms. We put

\[
\dot{N} = \dot{N}_{\text{tens}} + \dot{N}_{\text{shear}} + \dot{N}_{\text{comp}},
\]

where \( \dot{N}_{\text{tens}} \) and \( \dot{N}_{\text{shear}} \) stand for the nucleation rates due to separation of the matrix material under local tension and shear, respectively; \( \dot{N}_{\text{comp}} \) corresponds to the crushing of inclusions under local hydrostatic compression. It is assumed that void nucleation is driven by the inelastic flow (cf. Gurson (1977); Goods and Brown (1979)). Since the flow is governed by the effective stress tensor \( \hat{\Sigma} \), a similar assumption is made for void nucleation, which is now controlled by the effective stress, and not by the total stress. Let \( \sigma_1, \sigma_2, \) and \( \sigma_3 \) be the eigenvalues of the effective stress \( \hat{\Sigma} \). Additionally, we introduce the norm of its deviatoric part

\[
\hat{\mathfrak{F}} := \| \hat{\Sigma}^D \|.
\]

We postulate the void nucleation rate under local tension as follows

\[
\dot{N}_{\text{tens}} = n_t \lambda_i \left\{ \left( \frac{\sigma_1}{\sqrt{3/2\hat{\mathfrak{F}}}} - K_{\text{tens}} \right) + \left( \frac{\sigma_2}{\sqrt{3/2\hat{\mathfrak{F}}}} - K_{\text{tens}} \right) + \left( \frac{\sigma_3}{\sqrt{3/2\hat{\mathfrak{F}}}} - K_{\text{tens}} \right) \right\}.
\]

Here, \( n_t \geq 0 \) controls the intensity of the void nucleation, \( K_{\text{tens}} < 1 \) represents a certain threshold and \( \langle x \rangle = \max(x, 0) \). The distribution of the void nucleation rate \( \dot{N}_{\text{tens}} \) in the effective stress space is depicted in Fig. 3 for two different values of \( K_{\text{tens}} \). As \( K_{\text{tens}} \to 1 \), only the effective stress states close to uniaxial tension contribute to void nucleation. For smaller values of \( K_{\text{tens}} \), the distribution becomes more uniform. Next, the void nucleation rate under local shear is given by

\[
\dot{N}_{\text{shear}} = n_s \lambda_i \left\{ \left( \frac{|\sigma_1 - \sigma_2|}{\sqrt{2\hat{\mathfrak{F}}}} - K_{\text{shear}} \right) + \left( \frac{|\sigma_2 - \sigma_3|}{\sqrt{2\hat{\mathfrak{F}}}} - K_{\text{shear}} \right) + \left( \frac{|\sigma_1 - \sigma_3|}{\sqrt{2\hat{\mathfrak{F}}}} - K_{\text{shear}} \right) \right\}.
\]

Here, \( n_s \geq 0 \) and \( K_{\text{shear}} < 1 \). Analogously to (55), the threshold \( K_{\text{shear}} \) is used to control the distribution of the nucleation rate in the stress space. As \( K_{\text{shear}} \to 1 \), a distinct single peak near pure shear state is observed (cf. Fig. 4). For \( K_{\text{shear}} < 1 \), the distribution becomes more uniform. Finally, the void nucleation rate under compression is given by

\[
\dot{N}_{\text{comp}} = n_c \lambda_i \langle -\text{tr}(\hat{\Sigma}) - K_{\text{comp}} \rangle,
\]

where \( n_c \geq 0 \) and \( K_{\text{comp}} \) are fixed material parameters. Observe that this type of void nucleation depends solely on the hydrostatic pressure \(-\text{tr}(\hat{\Sigma})/3\) and the inelastic strain rate \( \lambda_i \). This
mechanism contributes to the overall void nucleation only if the pressure exceeds the threshold $K_{\text{comp}}/3$. Note that $\dot{N}_{\text{tens}}, \dot{N}_{\text{shear}}, \dot{N}_{\text{comp}} \geq 0$ for $n_t, n_s, n_c \geq 0$, which forbids any void healing.

**Remark 4.** For some materials, shear induces a higher void nucleation rate than tension (Horstemeyer et al., 2000). Obviously, the nucleation rules (55) and (56) can be adjusted for such materials by appropriate choice of the material parameters. Another void nucleation rule proposed by Horstemeyer and Gokhale (1999) for ductile metals with second phases can describe this effect as well. Interestingly, the rule of Horstemeyer and Gokhale operates with the absolute value of the hydrostatic stress component $|I_1|$. Thus, the sign of $I_1$ is irrelevant. In contrast to the model of Horstemeyer and Gokhale, the evolution equations (55) and (57) are sensitive to the sign of the hydrostatic stress. Another qualitative difference is that the rule of Horstemeyer and Gokhale predicts accelerated void nucleation. In case of monotonic loading, relations (55) and (56) predict nearly linear growth of the void number with increasing accumulated arc-length $s$ (cf. Figure 5).

**Remark 5.** For the nucleation mechanisms considered in this section, it is assumed that the voids nucleate at certain inclusions, or, more generally, at second phases which are sites of local stress raisers. Thus, the total number of voids is limited by the number of inclusions of a certain type. But relations (55) – (57) do not take this saturation into account. If needed, the saturation can be introduced by a simple modification, which is not considered here.
In this paper, for simplicity, we neglect the effects of void coalescence and suppose that the
damage evolution stems solely from the void nucleation and growth
\[
\dot{\Phi} = \dot{\Phi}_{\text{nucleation}} + \dot{\Phi}_{\text{growth}}. 
\] (58)
The material expansion due to void nucleation is described by
\[
\dot{\Phi}_{\text{nucleation}} = v_{\text{tens}} \dot{N}_{\text{tens}} + v_{\text{shear}} \dot{N}_{\text{shear}} + v_{\text{comp}} \dot{N}_{\text{comp}}, \tag{59}
\]
where the constant parameters \(v_{\text{tens}}, v_{\text{shear}},\) and \(v_{\text{comp}}\) stand for the characteristic void volumes. Three different parameters are used to account for the effects of different void volumes, depending on the nucleation mechanism. Next, one of the simplest expressions for the material expansion due to the dilatational void growth is given by (cf. Rice and Tracey (1969))
\[
\dot{\Phi}_{\text{growth}} = d_{\text{growth}} (\Phi - \Phi_0) \lambda_i \exp \left( \sqrt{\frac{3}{2}} \frac{\text{tr} \Sigma}{\delta} \right), \quad d_{\text{growth}} = \text{const} \geq 0, \quad \Phi_0 = \text{const} > 0. \tag{60}
\]
Here, the parameter \(\Phi_0\) is adopted to capture the fraction of the pre-existing pores, such that the special case \(\Phi = \Phi_0\) corresponds to the state without voids.

The original model of finite strain viscoplasticity with nonlinear kinematic hardening presented by Shutov and Kreißig (2008) is covered as a special case if \(n_t = n_s = n_c = d_{\text{growth}} = 0\).

3. Numerics

3.1. Transformation of equations to the porous configuration

In order to simplify the numerical treatment of the constitutive equations, they will be transformed to the porous configuration \(\tilde{K}\). First, recalling (3) and (7), we note that
\[
\text{tr} \left( C_i^k e \right) = \text{tr} \left( (C_{\text{ep}} \ C_{\text{i}i}^{-1})^k \right), \quad k = 1, 2, 3. \tag{61}
\]
Analogously, using (4) and (8), we have
\[
\text{tr} \left( \tilde{C}_i^k e \right) = \text{tr} \left( (C_i \ C_{\text{i}i}^{-1})^k \right), \quad k = 1, 2, 3. \tag{62}
\]
Therefore, due to the isotropy, the free energy (22) can be represented in the form
\[
\psi = \psi_{\text{el}} \left( C_{\text{ep}} C_{\text{i}i}^{-1}, \Phi \right) + \psi_{\text{kin}} \left( C_i C_{\text{i}i}^{-1}, \Phi \right) + \psi_{\text{iso}} (s_e, \Phi). \tag{63}
\]
Moreover, note that for any smooth scalar-valued function \(\alpha\) we have
\[
A^T \frac{\partial \alpha (ABA^T)}{\partial (ABA^T) A} = \left. \frac{\partial \alpha (ABA^T)}{\partial B} \right|_{A=\text{const}}. \tag{64}
\]
Applying pull-back transformations to the potential relations (28)$_1$, (28)$_2$, and taking (63), (64) into account, we obtain

\[ \frac{\partial \psi_{el}}{\partial C_i} \bigg|_{C_i=\text{const}} \] and

\[ \frac{\partial \psi_{kin}}{\partial C_i} \bigg|_{C_i=\text{const}}. \]

If the concrete ansatz (23), (24) is adopted, then the stresses are given by

\[ T_{ep} = 2 \rho_{por} \frac{\partial \psi_{el}}{\partial C_i} \bigg|_{C_i=\text{const}}, \quad X = 2 \rho_{por} \frac{\partial \psi_{kin}}{\partial C_i} \bigg|_{C_i=\text{const}}. \]

Substituting (66) into (17), for the 2nd Piola-Kirchhoff stress we obtain

\[ \tilde{T} = k(\Phi) \left( \ln \sqrt{\det(C_{ep})} - \ln \Phi \right) \cdot C^{-1} + \mu(\Phi) \cdot C^{-1} \cdot \left( C_{ep} \cdot C^{-1} \right)^D. \]

Next, we consider the invariants (moments) of the effective stress tensor

\[ \text{tr}(\hat{\Sigma}) = \text{tr}((\hat{C}_e \hat{S}_{ep} - \hat{X})^k) = \text{tr}\left( \frac{\partial \psi_{el}}{\partial C_i} \bigg|_{C_i=\text{const}} \right), \quad k = 1, 2, 3. \]

These relations imply that the eigenvalues of \( \hat{\Sigma} \) coincide with the eigenvalues of \( C_{ep} \cdot T_{ep} - C_i X \).

Moreover, since

\[ \|\Sigma^D\| = \sqrt{\text{tr}[(\Sigma^D)^2]}, \]

we obtain the driving force in the following form

\[ \hat{S} = \|\Sigma^D\| = \sqrt{\text{tr}[(C_{ep} \cdot T_{ep} - C_i X)^D]^2]. \]

For the hydrostatic stress, we have the transformation rule as follows

\[ \text{tr}(\hat{C}_e \hat{S}_{ep}) = \text{tr}(C_{ep} \cdot T_{ep}). \]

Finally, in order to transform the evolution equations, we note that

\[ \frac{d}{dt} C_i = 2 F_i^T \cdot \hat{D}_i \cdot F_i, \quad \frac{d}{dt} C_{ii} = 2 F_{ii}^T \cdot \hat{D}_{ii} \cdot F_{ii}. \]

Substituting (43) into these relations, we have

\[ \frac{d}{dt} C_i = 2 \lambda_i \frac{\partial}{\partial C_{ep}} \left( C_{ep} \cdot T_{ep} - C_i X \right)^D \cdot C_i, \quad \frac{d}{dt} C_{ii} = 2 \lambda_i \cdot \Phi(\cdot) \cdot \left( C_i X \right)^D \cdot C_{ii}. \]

The system of constitutive equations is summarized in Table 1.
Table 1: Summary of the material model

| Equation                                                                 | Description                                                                 |
|--------------------------------------------------------------------------|-----------------------------------------------------------------------------|
| \( \frac{d\mathbf{C}_{i}}{dt} = 2\frac{\lambda_{i}}{\delta_{i}} (\mathbf{C}_{i} \mathbf{T}_{ep} - \mathbf{C}_{i} \mathbf{X}) \mathbf{D} \mathbf{C}_{i} \) | \( \mathbf{C}_{i}|_{t=0} = \mathbf{C}_{i}^{0}, \det \mathbf{C}_{i}^{0} = 1 \), |
| \( \frac{d\mathbf{C}_{ii}}{dt} = 2\lambda_{i}(\Phi)(\mathbf{C}_{i} \mathbf{X}) \mathbf{D} \mathbf{C}_{ii} \)   | \( \mathbf{C}_{ii}|_{t=0} = \mathbf{C}_{ii}^{0}, \det \mathbf{C}_{ii}^{0} = 1 \), |
| \( \dot{s} = \sqrt{\frac{2}{3}} \lambda_{i}, \quad \dot{s}_{d} = \frac{\beta(\Phi)}{\gamma(\Phi)} \dot{s}_{R} \)    | \( s|_{t=0} = s^{0}, \quad s_{d}|_{t=0} = s_{d}^{0} \), |
| \( \mathbf{T}_{ep} = 2\rho_{por} \frac{\partial \psi_{el}(\mathbf{C}_{i}^{-1} \mathbf{X}, \Phi)}{\partial \mathbf{C}_{i}^{-1}} |_{\mathbf{C}_{i} = \text{const}} \) | \( \mathbf{X} = 2\rho_{por} \frac{\psi_{kin}(\mathbf{C}_{i}^{-1} \mathbf{X}, \Phi)}{\partial \mathbf{C}_{i}^{-1}} |_{\mathbf{C}_{i} = \text{const}} \) |
| \( R = \rho_{por} \frac{\partial \psi_{iso}(s - s_{d}, \Phi)}{\partial s} |_{s_{d} = \text{const}} \) | \( \dot{\Phi} = \dot{\Phi}(\mathbf{D}_{i}, \Sigma, \mathbf{X}, \Phi), \quad \mathbf{C}_{ep} = \Phi^{-2/3} \mathbf{C}_{i} \), |
| \( \lambda_{i} = \frac{1}{\eta} \left( \frac{f}{f_{0}} \right)^{m} \), \( f = \Phi - \frac{\sqrt{2}}{3} [K(\Phi) + R] \), \( \Phi = \sqrt{\text{tr}[(\mathbf{C} \mathbf{T} - \mathbf{C} \mathbf{X}) \mathbf{D}]} \) | |

3.2. Mixed explicit/implicit time integration

As a preliminary step, we specify the evolution equation (74) governing the backstress saturation for the case where the energy storage is described by (24). Substituting (67) into (74), we obtain

\[
\frac{d\mathbf{C}_{ii}}{dt} = \Phi^{-1} \lambda_{i}(\Phi) c(\Phi)(\mathbf{C}_{i}^{-1} \mathbf{X}) \mathbf{D} \mathbf{C}_{ii}.
\]  (75)

As is typical for metal viscoplasticity, the system of underlying equations is stiff. Thus, an explicit time integration scheme would be stable only for very small time steps. In case of large \( c(\Phi) \), especially severe restrictions are imposed on the time steps due to the stiff part in equation (75). In this study we benefit from the fact that this equation is similar to the evolution equation governing the Maxwell fluid (Shutov et al., 2013). For this certain type of Maxwell fluid, there exists an explicit update formula within the implicit time integration (Shutov et al., 2013). Equation (75) is treated using the explicit update formula, which leads to (82). The remaining evolution equations are discretized using the explicit Euler forward method with subsequent correction of incompressibility. An obvious advantage of (82) over straightforward explicit discretization is that the solution \( \mathbf{C}_{ii}^{n+1} \) remains bounded even for very large values of \( c(\Phi) \) and \( \Delta t \). \(^{17}\)

Let us consider a typical time step from \( n^{t} \) to \( n^{t+1} \) with \( \Delta t := n^{t+1} - n^{t} > 0 \). Assume that \( \mathbf{C}^{n+1} := \mathbf{C}(n^{t+1}) \) is known. For the viscous case with \( \eta > 0 \), the mixed explicit/implicit

\(^{17}\) Moreover, Silbermann et. al. (2014) have shown by a series of numerical tests that the use of the update formula (82) makes the integration procedure more robust and accurate compared to the fully explicit procedure.
The integration procedure is as follows

$$C_{ep}^{\text{por}} = n\Phi^{-2/3} n^{+1}C,$$  \hspace{1cm} (76)

$$T_{ep}^{\text{por}} = n\Phi^{-1}(k(n\Phi) \ln \det(C_{ep}^{\text{por}} C_{ep}^{-1} + \mu(n\Phi) C_{ep}^{\text{por}} C_{i}^{\text{por}-1} D),$$ \hspace{1cm} (77)

$$X^{\text{por}} = n\Phi^{-1}C(n\Phi) C_{i}^{\text{por}-1} D, \quad R = n\Phi^{-1}(n\Phi)(n s - n s_d),$$ \hspace{1cm} (78)

$$\tilde{f} = \sqrt{\text{tr}[(C_{ep}^{\text{por}} T_{ep}^{\text{por}} - n C_{i}^{\text{por}} X)^{D} D^{-1} C_{i}^{\text{por}}]} f = \tilde{f} - \sqrt{\frac{2}{3}[K(n\Phi) + R], \quad \lambda_i = \frac{1}{\eta} \left( \frac{f}{f_0} \right)^m},$$ \hspace{1cm} (79)

The inelastic flow is frozen for $$\lambda_i = 0$$:

$$n^{+1}C_i^{\text{por}} = n^{+1}C_i, \quad n^{+1}C_{il}^{\text{por}} = n^{+1}C_{il}, \quad n^{+1}s = n s, \quad n^{+1}s_d = n s_d, \quad n^{+1}\Phi = n \Phi$$ for $$\lambda_i = 0.$$ \hspace{1cm} (80)

For $$\lambda_i > 0$$, the internal variables are updated in the following way:

$$n^{+1}C_i^{\text{por}} = n^{+1}C_i + 2\Delta t \frac{\lambda_i}{\tilde{f}} (C_{ep}^{\text{por}} T_{ep}^{\text{por}} - n^{+1}C_i^{\text{por}} X)^{D} n^{+1}C_{il},$$ \hspace{1cm} (81)

$$n^{+1}C_{il}^{\text{por}} = n^{+1}C_{il} + \Delta t n^{+1}\Phi^{-1} \lambda_i = \lambda_i C(n\Phi) C_{i}^{\text{por}-1} D, \quad n^{+1}s = n s, \quad n^{+1}s_d = n s_d, \quad n^{+1}\Phi = n \Phi$$ for $$\lambda_i = 0.$$ \hspace{1cm} (82)

$$\{\sigma_1, \sigma_2, \sigma_3\} = \text{eigenvalues of } (C_{ep}^{\text{por}} T_{ep}^{\text{por}} - n^{+1}C_i^{\text{por}} X), \quad \text{tr} \hat{\Sigma} = \text{tr}(C_{ep}^{\text{por}} T_{ep}^{\text{por}}),$$ \hspace{1cm} (83)

$$\dot{N}_{\text{tens}} = \dot{N}_{\text{tens}}(\lambda_i, \sigma_1, \sigma_2, \sigma_3, \tilde{f}), \quad \dot{N}_{\text{shear}} = \dot{N}_{\text{shear}}(\lambda_i, \sigma_1, \sigma_2, \sigma_3, \tilde{f}).$$ \hspace{1cm} (84)

$$\dot{N}_{\text{comp}} = \dot{N}_{\text{comp}}(\lambda_i, \text{tr} \hat{\Sigma}, \tilde{f}),$$ \hspace{1cm} (85)

$$\dot{\Phi}_{\text{nucleation}} = v_{\text{tens}} \dot{N}_{\text{tens}} + v_{\text{shear}} \dot{N}_{\text{shear}} + v_{\text{comp}} \dot{N}_{\text{comp}},$$ \hspace{1cm} (86)

$$\dot{\Phi}_{\text{growth}} = d_{\text{growth}} (n\Phi - \Phi_0) \lambda_i \exp \left( \frac{3 \text{tr} \hat{\Sigma}}{2 \tilde{f}} \right), \quad \dot{\Phi} = \dot{\Phi}_{\text{nucleation}} + \dot{\Phi}_{\text{growth}},$$ \hspace{1cm} (87)

$$n^{+1}\Phi = n \Phi + \Delta t \dot{\Phi}.$$ \hspace{1cm} (88)

Note that the update formulas (81) and (82) exactly preserve the incompressibility conditions \(\det C_i = 1\) and \(\det C_{il} = 1\) which is advantageous for the prevention of error accumulation (Shutov and Kreißig, 2010).

In case of small time steps, the mixed explicit/implicit integration scheme (76) – (89) is more efficient than the fully implicit scheme, since the solution is given in a closed form and no iteration procedure is required. Due to the reduced computational effort, the presented algorithm is well suited for use within a globally explicit FEM procedure. The algorithm is implemented into the FEM code Abaqus/Explicit adopting the user material subroutine VUMAT.
4. Comparison with experiment

In order to validate the proposed material model, some experimental observations of the Bauschinger effect dependent on accumulated ductile damage are needed. In this study we use a series of quasistatic uniaxial tension-followed-by-compression and compression-followed-by-tension tests presented by Horstemeyer (1998) for a cast A356 aluminium alloy. The flow curves pertaining to six different experiments are shown in Figure 4 (dotted line). As can be seen from this figure, the initial yield stress under tension coincides with the initial yield stress under compression. But, for the plastically deformed samples, the true stresses under tension are essentially smaller than the corresponding stresses under compression. We interpret this tension-compression asymmetry as a consequence of ductile damage, since the damage evolution under tension is typically more intense than under compression. For the considered cast alloy, the damage effect is significant even at low tensile strains. Moreover, a distinct Bauschinger effect is observed in all experiments (cf. Figure 4). It was noted by Jordon et al. (2007) that the Bauschinger effect induced by tensile prestrains is less pronounced than the same effect after compression. Such behavior indicates that the damage evolution is highly dependent on the stress state.

For the validation of the material model, a series of numerical tests is performed. As the reference configuration we choose the initial configuration occupied by the body at \( t = 0 \). Since the initial state is stress free, we put \( \sigma_{i}^{\text{por}}|_{t=0} = 1 \). Furthermore, the initial state of the cast material is assumed to be isotropic. Therefore, we have \( C_{ii}^{\text{por}}|_{t=0} = 1 \). The remaining initial conditions are given by \( s_{i}|_{t=0} = s_{d}|_{t=0} = 0 \), \( \Phi|_{t=0} = 1 \).

We define the set of material parameters as follows. Firstly, void growth is neglected here: \( d_{\text{growth}} = 0 \). Moreover, we neglect void nucleation under compression by putting \( n_{c} = 0 \). Thus, the values of \( K_{\text{comp}} \) and \( v_{\text{comp}} \) become irrelevant. The elasticity parameters \( k_{0} \) and \( \mu_{0} \) are identified using the stress-strain curve in the elastic range. The initial yield stress \( K_{0} \) is estimated at a transition from elasticity to plasticity under quasistatic loading conditions. In general, the viscosity parameters \( m \) and \( \eta \) can be estimated using a series of tests with different loading rates. In this section we neglect the real viscous effects and adopt the parameters \( m \) and \( \eta \) only as numerical regularization parameters. All simulations will be performed with a low loading rate such that the overstress \( f \) will remain below 10 MPa. Next, the nucleation parameters \( K_{\text{tens}} \), \( K_{\text{shear}} \), \( n_{t} \), and \( n_{s} \) should be identified by counting the void nucleation sites as a function of

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18 This alloy plays an important role in the automotive industry. Its microstructural characterization was presented by Gall et al. (1999).
19 The same asymmetry of Bauschinger effect was also observed in porous sintered steels by Deng et al. (2005).
20 Constitutive relations can be adopted to the initial plastic anisotropy by an appropriate choice of the initial conditions (Shutov et al., 2012).
21 Thus, we assume for simplicity that the damage in this case is dominated by void nucleation.
strain under different loading conditions (Horstemeyer et al., 2000). In this section, $v_{\text{tens}}$ and $v_{\text{shear}}$ are roughly estimated by assuming that each void has a volume of approximately $10^{-7}\text{mm}^3$. In general, the parameters $v_{\text{tens}}$ and $v_{\text{shear}}$ should be identified by measuring the volume changes induced by ductile damage. Finally, the remaining parameters can be reliably identified using the experimental flow curves shown in Figure 4. Note that only 7 material parameters ($K_0$, $\gamma_0$, $\beta_0$, $\chi_0$, $c_0$, IRR, KRR) were adjusted to fit the flow curves. The set of material parameters which is used to describe the material response is summarized in Table 2.

The stress response of the material can be described by the proposed material model with sufficient accuracy (cf. Figure 4). Within the numerical simulation, the real Bauschinger effect is slightly underestimated, which leads to a minor discrepancy between the simulated and the real stress responses shortly after the secondary yielding. Following the ideas of Chaboche (1989), this discrepancy can be reduced by introducing additional backstress.

During the phase of 5% prestrain under tension (within the tension-followed-by-compression test), the theoretical value of $\Phi$ can be up to $\Phi_{0.05} = 1.00428$. The elastic modulus $E_{0.05}$ following

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For a model of multiplicative type it was carried out by Shutov et al. (2010).
Table 2: Material parameters.

| parameter | value | brief explanation |
|-----------|-------|-------------------|
| $k_0$     | 73500 [MPa] | initial bulk modulus |
| $\mu_0$   | 28200 [MPa] | initial shear modulus |
| $c_0$     | 6399.8 [MPa] | kinematic hardening parameter |
| $\alpha_0$| 0.015106 [1/MPa] | kinematic hardening parameter |
| $\gamma_0$| 1442.2 [MPa] | isotropic hardening parameter |
| $\beta_0$ | 1.852 [-] | isotropic hardening parameter |
| $K_0$     | 210 [MPa] | initial yield stress |
| $m$       | 1 [-] | viscosity parameter |
| $\eta$    | 100 [s] | viscosity parameter |
| KRR       | 67.63 [-] | kinematic hardening reduction rate |
| IRR       | 29.97 [-] | isotropic hardening reduction rate |
| BRR       | 45 [-] | bulk modulus reduction rate |
| SRR       | 30 [-] | shear modulus reduction rate |
| $v_{tens}$| $10^{-7}$ [mm$^3$] | void volume nucleated under tension |
| $v_{shear}$| $10^{-7}$ [mm$^3$] | void volume nucleated under shear |
| $n_t$     | 2773000 [mm$^{-3}$MPa$^{-1}$] | void nucleation parameter (tension) |
| $K_{tens}$| 0.79 [-] | void nucleation parameter (tension) |
| $n_s$     | 17188000 [mm$^{-3}$MPa$^{-1}$] | void nucleation parameter (shear) |
| $K_{shear}$| 0.9353 [-] | void nucleation parameter (shear) |
| $n_c$     | 0 [mm$^{-3}$MPa$^{-1}$] | void nucleation parameter (compression) |
| $f_0$     | 1 [MPa] | constant (not a material parameter) |

5% tension and the initial modulus $E_0$ are computed with

$$E_{0.05} = \frac{9k(\Phi_{0.05}) \mu(\Phi_{0.05})}{3k(\Phi_{0.05}) + \mu(\Phi_{0.05})}, \quad E_0 = \frac{9k_0 \mu_0}{3k_0 + \mu_0}. \quad (90)$$

The numerical simulation yields $E_{0.05} = 0.8729E_0$, which corresponds to 12.7% reduction. At the same time, the elastic modulus is unaffected by the 5% compression. As can be seen from Figure 4, the elastic unloading is captured by the model with ample accuracy, both for tensional and compressional prestrains.

Another noteworthy aspect is the following. According to Table 2, $K_{RR} \approx 2 IRR$. This means that the deterioration of the kinematic hardening progresses twice as fast as the deterioration of the isotropic hardening.

Uniaxial experiments are insufficient to validate the void nucleation rule proposed in the current study. The simulated void number $N$ is plotted versus the accumulated inelastic arc-
Figure 5: Simulation results for the void number $N$ as a function of the accumulated inelastic arc-length $s$. Depending on the choice of the material parameters, the void nucleation rate under shear may exceed the nucleation rate under tension.

![Figure 5: Simulation results for the void number $N$ as a function of the accumulated inelastic arc-length $s$. Depending on the choice of the material parameters, the void nucleation rate under shear may exceed the nucleation rate under tension.]

Figure 6: Setup of the axisymmetric deep drawing.

Figure 6: Setup of the axisymmetric deep drawing.

length $s$ in Figure 5 for two different loading cases. In both tests, the initial value of 15000 voids per mm$^3$ is adopted. The simulation results correspond qualitatively to the experimental data previously reported by Horstemeyer et al. (2000) for the cast A356 aluminium alloy.

5. FEM solution of a representative boundary value problem

An axisymmetric deep drawing of a circular sheet metal blank is considered in this section to demonstrate the applicability of the new model and to test the stability of the implemented numerical procedures. The setup is depicted in Fig. 6. The overall forming process consists of three steps:

1. Preliminary step: Forces are gradually applied both to holder and counterholder.
2 Deep drawing: The punch pushes the blank into the die.

3 Spring back: The blank is released from any contact.

The material of the blank is described by the proposed material model with material parameters taken from Table 2. The remaining bodies are assumed to be rigid. Coulomb friction is assumed with a friction coefficient of 0.1. A quasistatic loading rate is chosen such that the kinetic energy is negligible compared to the work of external forces. Mass scaling, which is typically adopted to speed up computation, is not used here. The FE mesh consists of 1950 elements of ABAQUS type CAX4R (4-node bilinear axisymmetric quadrilateral element with reduced integration, hourglass control, linear geometric order). Steps 1, 2 and 3 last 0.01s, 0.10s and 0.12s, respectively. The simulation required 460 563, 4 638 783 and 5 592 654 increments for steps 1, 2 and 3, respectively.

It is known that the deterioration of elastic properties and the Bauschinger effect have a strong impact on the spring back (Wagoner et al., 2013). The deformed state after deep drawing is shown in Fig. 7. Fig. 8 shows the distribution of the von Mises equivalent stress after deep drawing and after the spring back. Fig. 9 provides the distribution of the maximum and minimum principal stresses after the spring back. Note that the residual stresses remain high compared to the initial yield stress ($K_0 = 210$ MPa). By comparison with the resulting inelastic arc length field $s$ (cf. Fig. 10 (left)), it becomes clear that highest residual stresses appear in regions with a considerable gradient of $s$. The distribution of the damage measure $\Phi - 1$ after spring back is depicted in Fig. 10 (right). Note that $\Phi - 1$ remains in the range of a few percent and the damage is localized in the deep-drawn section. The observed damage hot spots are considered as a precursor of failure.

23 An animated version of the FEM solution can be found under http://youtu.be/YhEwI0cpIGQ and http://youtu.be/r6rY2oRUNQ.
after deep drawing after spring back

von Mises stresses

Figure 8: Distribution of von Mises stresses after deep drawing (left) and after spring back (right).

maximum principal stress minimum principal stress

Figure 9: Distribution of maximum and minimum principal stresses after the spring back.

inelastic arc length $s$
damage measure $\Phi - 1$

Figure 10: Distribution of the inelastic arc length $s$ (left) and damage measure $\Phi - 1$ (right) after the spring back.
6. Discussion and conclusion

A thermodynamically consistent approach to ductile damage and kinematic hardening is suggested within the framework of multiplicative plasticity. For simplicity, isotropic damage with no prominent orientation is assumed in this study. As an extension of the current model, some physically-based tensorial damage quantities can be introduced (cf. Murakami and Ohno (1980); Kachanov (1986); Chaboche (1993); Kraicinovic (1996); Br"unig (2001, 2003); Zapara et al. (2010, 2012); Voyiadjis et al. (2012); Tutushkin et al. (2014) among many others). The model is validated using the experimental observations of the Bauschinger effect at different stages of ductile damage.

A relatively large number of material parameters is introduced due to a variety of nonlinear phenomena covered by the model. In particular, refined void nucleation rules (55) – (57) are introduced in this study aiming at accurate description of the complex behavior of porous ductile metals with second phases. In some applications, however, certain parts of the model can be switched off to reduce the number of parameters. For many materials, some elementary or classical void nucleation rules can be adopted (Gurson, 1977). Alternatively, the nucleation rules (55) – (57) can be calibrated using experiments (cf. Su et al. (2010)) or lower-scale models (cf. Gall et al. (2000); Dandekar and Shin (2011)).

Although this formulation has been developed for forming simulations, another promising application includes the prediction of damage and plastic anisotropy in ultra-fine grained materials produced by severe plastic deformations (Wagner et al. (2010); Frint et al. (2011); Neugebauer et al. (2012)). An extension of the framework to models of creep damage mechanics (Altenbach et al. (2002); Altenbach (2003)) is possible as well.

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