Nonlinear Maximal Monotone extensions of symmetric operators

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1. Introduction

Let $S : \mathcal{D}(S) \subseteq \mathcal{H} \to \mathcal{H}$, $S \geq 0$, be a linear symmetric positive operator on the Hilbert space $\mathcal{H}$.

By the famed Birman-Kreĭn-Vishik theory we know how to find its positive self-adjoint extensions.

**Question**: is there some nonlinear analogue of such a theory?

If $A \geq 0$ is a linear self-adjoint extension of $S$ then $e^{-tA}$, $t \geq 0$, is a continuous semi-group of contractions in $\mathcal{H}$, i.e.

$$
\| e^{-tA} u \| \leq \| u \| \quad \text{(equivalently } \| e^{-tA} u - e^{-tA} v \| \leq \| u - v \| \text{)}).
$$

Thus in the nonlinear case we are lead to look for nonlinear extensions which are generators of continuous nonlinear semi-groups of contractions, i.e.

$$
S_t, t \geq 0, \text{ such that } \| S_t(u) - S_t(v) \| \leq \| u - v \|.
$$
By the theory of one-parameter continuous nonlinear semi-groups of contractions there follows that $S_t$ has a nonlinear generator $A$ given by a monotone operator which is the principal section $A^0$ of a maximal monotone relation $A \subseteq \mathcal{H} \times \mathcal{H}$.

Since maximal monotonicity can be characterized in terms of nonlinear resolvents and since, in the linear case, the theory of self-adjoint extensions can be formulated in terms of the famed Kreĭn’s resolvent formula, one is led to look for a nonlinear version of such a formula.
A nonlinear operator $A : D(A) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ in the real Hilbert space $\mathcal{H}$ is said to be \textit{monotone of type $\omega$} (\textit{monotone} if $\omega = 0$) whenever

$$\forall u, v \in D(A), \quad \langle A(u) - A(v), u - v \rangle \geq -\omega \|u - v\|^2,$$

and \textit{maximal monotone of type $\omega$} if for some $\lambda > \omega$ (equivalently for any $\lambda > \omega$) one has

$$\text{Range}(A + \lambda) = \mathcal{H}.$$
A nonlinear operator $\tilde{A}: \mathcal{D}(\tilde{A}) \subseteq \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$, $\tilde{\mathcal{H}}$ the complex Hilbert space $\tilde{\mathcal{H}} = \mathcal{H} + i \mathcal{H}$, is said to be monotone of type $\omega$ whenever

$$\forall u, v \in \mathcal{D}(\tilde{A}), \quad \Re\langle \tilde{A}(u) - \tilde{A}(v), u - v \rangle \geq -\omega \|u - v\|^2.$$ 

Defining $A_1$, $A_2$ by

$$\tilde{A}(u_1 + iu_2) = A_1(u_1, u_2) + iA_2(u_1, u_2),$$

one has that $\tilde{A}$ is monotone in $\tilde{\mathcal{H}}$ if and only if $A$ defined by

$$A(u_1 \oplus u_2) := A_1(u_1, u_2) \oplus A_2(u_1, u_2)$$

is monotone in the real Hilbert space $\mathcal{H} \oplus \mathcal{H}$. Similarly $\tilde{A}$ is maximal monotone if and only if $A$ is maximal monotone. Thus the whole theory of maximal monotone operators in real Hilbert spaces extends, with the obvious modifications, to complex Hilbert spaces.
If $A$ is monotone of type $\omega$ then

$$\langle (A + \lambda)(u) - (A + \lambda)(v), u - v \rangle \geq (\lambda - \omega)\|u - v\|^2.$$ 

and so if $A$ is maximal then

$$(A + \lambda) : \mathcal{D}(A) \to \mathcal{H}$$

is bijective for any $\lambda > \omega$ and the nonlinear resolvent

$$(A + \lambda)^{-1} : \mathcal{H} \to \mathcal{H}, \quad \lambda > \omega,$$

is monotone and Lipschitz with Lipschitz constant $(\lambda - \omega)^{-1}$. 
The notion of maximal monotone operator can be generalized to multi-valued maps:

$A \subseteq \mathcal{H} \times \mathcal{H}$ is said to be a *monotone relation of type $\omega$* (monotone relation in case $\omega = 0$) if

$$\forall (u, \tilde{u}), (v, \tilde{v}) \in A, \quad \langle \tilde{u} - \tilde{v}, u - v \rangle \geq -\omega \|u - v\|^2$$

and is said to be a *maximal monotone relation of type $\omega$* if it is not properly contained in any other monotone relation of type $\omega$.

The graph

$$\text{Graph}(A) := \{(u, \tilde{u}) \in \mathcal{H} \times \mathcal{H} : u \in D(A), \ \tilde{u} = A(u)\}$$

of a maximal monotone operator of type $\omega$ is a maximal monotone relation of type $\omega$. 
Any $A \subset H \times H$ defines a set-valued operator by

$$
u \mapsto A(u) := \{ \tilde{u} \in H : (u, \tilde{u}) \in A \}$$

with domain

$$\mathcal{D}(A) := \{ u \in H : A(u) \neq \emptyset \}$$

If $A$ is maximal monotone then $A(u)$ is closed and convex and so

$$\exists! u_{\min} \in H \text{ such that } \|u_{\min}\| = \inf\{ \|v\| : v \in A(u) \}.$$ 

Therefore the single-valued nonlinear operator

$$A^0 : \mathcal{D}(A) \subseteq H \to H, \quad A^0(u) := u_{\min}$$

is well defined; it is called the principal section of $A$.

The principal section is unique: $A_1^0 = A_2^0 \implies A_1 = A_2$. 

While the domain of a linear maximal monotone relation is necessarily dense, in the nonlinear case this can be false.

\[ \mathcal{A} \text{ is maximal monotone} \implies \overline{\mathcal{D}(\mathcal{A})} \text{ is a convex set.} \]

Let \( \mathcal{C} \) be a closed convex nonempty subset of \( \mathcal{H} \). The family of nonlinear operators \( S_t : \mathcal{C} \to \mathcal{C}, \ t \geq 0 \), is said to be a one-parameter nonlinear continuous semi-group of type \( \omega \) (of contractions in case \( \omega = 0 \)) on \( \mathcal{C} \) if

\[
S_0 = \text{Id}, \quad S_{t_1} \circ S_{t_2} = S_{t_1 + t_2},
\]

\[
\forall u \in \mathcal{C}, \quad \lim_{t \downarrow 0} \| S_t(u) - u \| = 0,
\]

\[
\forall u, v \in \mathcal{C}, \quad \| S_t(u) - S_t(v) \| \leq e^{\omega t} \| u - v \|.
\]
The generator of the semigroup \( S_t \) is defined by

\[
A : \mathcal{D}(A) \subseteq \mathcal{H} \to \mathcal{H}, \quad -A(u) := \lim_{t \downarrow 0} \frac{1}{t} (S_t(u) - u),
\]

where \( \mathcal{D}(A) \subseteq \mathcal{C} \) is the set of \( u \) such that the above limit exists.

\( \mathcal{D}(A) \) is dense in \( \mathcal{C} \) and invariant.

For all \( u \in \mathcal{D}(A) \), \( u(t) := S_t(u) \) is the unique solution of the Cauchy problem

\[
\begin{cases}
\frac{d}{dt} u(t) = -A(u(t)), & \text{a.e. } t > 0 \\
u(0) = u.
\end{cases}
\]
Theorem. (Komura-Kato)

A maximal monotone of type \( \omega \)

\[\Downarrow\]

A generates a strongly continuous semigroup of type \( \omega \) on \( \overline{\mathcal{D}(A)} \)

\[\Downarrow\]

A is the principal section of a maximal monotone relation \( \mathcal{A} \) of type \( \omega \).
Given $A$ maximal monotone the corresponding one-parameter nonlinear continuous semi-group $S_t$ is constructed in the following way: defining the nonlinear Yosida approximation

$$A_\lambda := \frac{1}{\lambda} \left(1 - (1 + \lambda A)^{-1}\right),$$

maximal monotonicity implies that $A_\lambda$ is a Lipschitz map and that

$$\forall u \in \mathcal{D}(A), \quad \lim_{\lambda \to 0} A_\lambda(u) = A(u).$$

By the Lipschitz property the Cauchy problem

$$\begin{cases}
\frac{d}{dt} u_\lambda(t) = A_\lambda(u_\lambda(t)) \\
u_\lambda(0) = u \in \mathcal{H}
\end{cases}$$

has a unique solution $t \mapsto u_\lambda(t)$ which defines the semi-group $S_\lambda^t(u) := u_\lambda(t)$. Finally

$$\forall T \geq 0, \quad \forall u \in \mathcal{D}(A), \quad \lim_{\lambda \to 0} \sup_{0 \leq t \leq T} \|S_\lambda^t(u) - S_t(u)\| = 0.$$
Let $\varphi : \mathcal{H} \to (-\infty, +\infty]$ be a proper (i.e. not identically $+\infty$) convex function. Its sub-differential $\partial \varphi \subset \mathcal{H} \times \mathcal{H}$ is defined by

$$\partial \varphi := \{(u, \tilde{u}) \in \mathcal{H} \times \mathcal{H} : \forall v \in \mathcal{H}, \varphi(u) \leq \varphi(v) + \langle \tilde{u}, u - v \rangle \}$$

Notice that $(u, 0) \in \partial \varphi$ if and only if $u$ is a minimum point of $\varphi$.

If $\varphi$ is Gâteaux-differentiable at $u$ then $\partial \varphi(u) = \nabla \varphi(u)$.

Sub-differentials are the main source of maximal monotone operators:

$\varphi$ convex, lower semi-continuous $\implies \partial \varphi$ is maximal monotone.
Let $S^\varphi_t$ be the nonlinear semigroup generated by $A = \partial \varphi$. Then one has the following regularity results:

\[ \forall u \in \mathcal{D}(A), \ \forall t > 0, \quad S^\varphi_t(u) \in \mathcal{D}(A), \]

\[ \forall u \in \mathcal{D}(A), \ \forall v \in \mathcal{D}(A), \ \forall t > 0, \quad \left\| \frac{d}{dt} S^\varphi_t(u) \right\| \leq \|A v\| + \frac{1}{t} \|u - v\|, \]

\[ \forall u \in \mathcal{D}(A), \ \forall T > 0, \quad \int_0^T \left\| \frac{d}{dt} S^\varphi_t(u) \right\|^2 dt < +\infty, \]

\[ \forall u : \varphi(u) < +\infty, \ \forall T > 0, \quad \int_0^T \left\| \frac{d}{dt} S^\varphi_t(u) \right\|^2 dt < +\infty, \]

\[ \forall u \in \mathcal{D}(A), \ \forall T > 0, \quad \int_0^T |\varphi(S^\varphi_t(u))| dt < +\infty, \]

\[ \forall u : \varphi(u) < +\infty, \ \forall T > 0, \quad \int_0^T \left| \frac{d}{dt} \varphi(S^\varphi_t(u)) \right| dt < +\infty. \]
3. Nonlinear maximal monotone extensions

Let $S \geq -\omega$ be a densely defined, symmetric lower bounded operator. It is linear monotone of type $\omega$ but is not maximal monotone since its Friedrich’s extensions $A_0 \geq -\omega$ is a proper monotone extension. We want to define nonlinear maximal monotone operators $A$ such that

$$S \subset A \subset S^*.$$ 

Without loss of generality we can suppose that $S = A|\mathcal{N}$, where $\mathcal{N}$ is the (dense in $\mathcal{H}$) kernel of a continuos (w.r.t. the graph norm of $A_0$) surjective linear map

$$\tau : \mathcal{D}(A_0) \rightarrow \mathfrak{h},$$

$\mathfrak{h}$ being an auxiliary Hilbert space.
For any $\lambda > \omega$ we pose $R_\lambda^0 := (A_0 + \lambda)^{-1}$ and define the bounded linear operator

$$G_\lambda : \mathfrak{h} \to \mathcal{H} \ , \quad G_\lambda := (\tau R_\lambda^0)^* .$$

By the denseness hypothesis on $\mathcal{N}$ one has

$$\text{Range}(G_\lambda) \cap \mathcal{D}(A_0) = \{0\}$$

and, by first resolvent identity,

$$(\lambda - \mu) R_\mu^0 G_\lambda = G_\mu - G_\lambda .$$
We try to define a nonlinear extension $A$ by producing its nonlinear resolvent. Given a nonlinear resolvent $R_\lambda = (A + \lambda)^{-1}$, one has

$$R_\lambda^{-1} - \lambda = A = R_\mu^{-1} - \mu,$$

which is equivalent to the nonlinear resolvent identity

$$R_\lambda = R_\mu \circ (1 - (\lambda - \mu)R_\lambda).$$

Thus if $R_\lambda : \mathcal{H} \to \mathcal{H}$, $\lambda > \omega$, is a family of monotone and injective nonlinear maps which satisfies the nonlinear resolvent identity, then

$$A := (R_\lambda^{-1} - \lambda) : \mathcal{D}(A) \subseteq \mathcal{H} \to \mathcal{H}, \quad \mathcal{D}(A) := \text{Range}(R_\lambda),$$

is a $\lambda$-independent, maximal monotone nonlinear operator of type $\omega$. 
Therefore we need to produce a family $R_\lambda$, $\lambda > \omega$, of monotone and injective nonlinear maps which satisfies the nonlinear resolvent identity.

Kreĭn's linear resolvent formula suggests us to write the presumed resolvent as

$$R_\lambda = R^0_\lambda + G_\lambda V_\lambda \circ G^*_\lambda,$$

where the nonlinear map $V_\lambda : \mathfrak{h} \to \mathfrak{h}$ has to be determined. Since $R^0_\lambda$ is monotone,

$$\langle R_\lambda(u) - R_\lambda(v), u - v \rangle \geq \langle V_\lambda(G^*_\lambda u) - V_\lambda(G^*_\lambda v), G^*_\lambda u - G^*_\lambda v \rangle,$$

so that $R_\lambda$ is monotone whenever

$$\forall \xi, \zeta \in \mathfrak{h}, \quad \langle V_\lambda(\xi) - V_\lambda(\zeta), \xi - \zeta \rangle \geq 0,$$

namely whenever $V_\lambda$ is monotone.
Lemma. Let $V_\lambda : \mathfrak{h} \to \mathfrak{h}$ be monotone. Then

$$R_\lambda = R_\lambda^0 + G_\lambda V_\lambda \circ G_\lambda^*$$

satisfies the nonlinear resolvent identity if and only if there exists a family of maximal monotone relations $\Gamma_\lambda \subset \mathfrak{h} \times \mathfrak{h}$ such that $\Gamma_\lambda^{-1} = V_\lambda$ and

$$\Gamma_\lambda - \Gamma_\mu = (\lambda - \mu) G_\mu^* G_\lambda .$$  \hspace{1cm} (1)

Lemma. Let $\Theta \subset \mathfrak{h} \times \mathfrak{h}$ be a maximal monotone relation and let $\lambda_0 > \omega$. Then

$$\Gamma_{\lambda_0}^\Theta := \Theta + (\lambda - \lambda_0) G^* G_\lambda , \quad \lambda > \omega , \quad G := G_{\lambda_0} ,$$

is a maximal monotone relation for any $\lambda \geq \lambda_0$. It fulfills (1) and it has a single-valued monotone inverse for any $\lambda > \lambda_0$. 
By collecting the above results one gets the following nonlinear version of Kreĭn’s resolvent formula:

**Theorem.**

Let \( \lambda_0 > \omega \) and let \( \Theta \subset \mathfrak{h} \times \mathfrak{h} \) be a maximal monotone relation. Then

\[
R^\Theta_\lambda := R^0_\lambda + G_\lambda(\Theta + (\lambda - \lambda_0)G^*G_\lambda)^{-1} \circ G^*_\lambda, \quad \lambda > \lambda_0
\]

is the resolvent of a nonlinear maximal monotone operator \( A_\Theta \) of type \( \lambda_0 \); \( A_\Theta \) is monotone of type \( \omega \) whenever \( \Theta^{-1} \) is single-valued. Such an operator is defined by

\[
\mathcal{D}(A_\Theta) := \{u \in \mathcal{H} : u = u_0 + G\xi_u, \ u_0 \in \mathcal{D}(A_0), \ (\xi_u, \tau u_0) \in \Theta\},
\]

\[
A_\Theta(u) := A_0u_0 - \lambda_0 G\xi_u.
\]
Remarks.

\[ A_\Theta \subset S^* , \]

\[ \mathcal{D}(A_0) \cap \mathcal{D}(A_\Theta) \neq \emptyset \iff 0 \in \mathcal{D}(\Theta) , \]

\[ \mathcal{D}(A_0) \cap \mathcal{D}(A_\Theta) \text{ is convex and closed in } \mathcal{D}(A_0) , \]

\[ \forall u \in \mathcal{D}(A_0) \cap \mathcal{D}(A_\Theta) , \quad A_\Theta(u) = A_0 u , \]

\[ S \subset A_\Theta \iff (0,0) \in \Theta \iff \overline{\mathcal{D}(A_\Theta)} = \mathcal{H} . \]
Theorem. Suppose $\Theta = \partial \varphi$ and $\text{Range}(G) \cap \mathcal{D}((A_0 + \lambda_0)^{\frac{1}{2}}) = \{0\}$. Define the proper convex function $\Phi : \mathcal{H} \to (-\infty, +\infty]$ by

$$
\Phi(u) := \begin{cases} 
\frac{1}{2} \|(A_0 + \lambda_0)^{\frac{1}{2}} u_0\|^2 + \varphi(\xi) & u \in \mathcal{D}(\Phi) \\
+\infty & \text{otherwise,}
\end{cases}
$$

where

$$
\mathcal{D}(\Phi) := \{u \in \mathcal{H} : u = u_0 + G\xi, u_0 \in \mathcal{D}((A_0 + \lambda_0)^{\frac{1}{2}}), \varphi(\xi) < +\infty\}.
$$

Then

$$
A_\Theta + \lambda_0 = \partial \Phi = \partial \bar{\Phi},
$$

where $\bar{\Phi}$ denotes the lower semi-continuous regularization of $\Phi$ i.e. $\Phi$ is the largest lower semi-continuous minorant of $\Phi$:

$$
\bar{\Phi}(v) := \liminf_{u \to v} \Phi(u).
$$
Corollary.
Let $A_0 > 0$ and take $\lambda_0 = 0$. Suppose $\Theta = \partial \varphi$ and that $\xi_0$ is the unique minimum point of $\varphi$. Then $A_\Theta G \xi_0 = 0$ and

$$ \forall u \in \mathcal{D}(A_\Theta), \quad \lim_{t \to +\infty} S_t(u) = G \xi_0. $$

If $\varphi$ is an even function then the above weak limit becomes a strong one.
4. Examples.

Laplacians with nonlinear singular perturbations supported on null sets.

Let $A_0 = -\Delta : H^2(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and let $N \subset \mathbb{R}^n$ be a $d$-set with $2 < n - d < 4$. A Borel set $N \subset \mathbb{R}^n$ is called a $d$-set, if

$$\exists c_1, c_2 > 0 : \forall x \in N, \forall r \in (0, 1), \quad c_1 r^d \leq \mu_d(B_r(x) \cap M) \leq c_2 r^d,$$

where $\mu_d$ is the $d$-dimensional Hausdorff measure and $B_r(x)$ is the closed $n$-dimensional ball of radius $r$ centered at the point $x$. Examples of $d$-sets for $d$ integer are finite unions of $d$-dimensional Lipschitz submanifolds and, in the not integer case, self-similar fractals of Hausdorff dimension $d$. Then we take $\tau = \gamma_N$, where

$$\gamma_N : H^2(\mathbb{R}^n) \to H^s(N), \quad s = 2 - \frac{n - d}{2},$$

is the unique linear continuous and surjective map with coincide on smooth functions with the evaluation at points in $N$. 
Here $H^s(N)$, $0 < s < 1$, is defined as the Hilbert space of functions $f \in L^2(N; \mu_N)$ having finite norm

$$\|f\|_{H^2(N)}^2 := \|f\|_{L^2(N; \mu_N)}^2 + \int_{|x-y|<1} \frac{|f(x) - f(y)|^2}{|x-y|^{d+2s}} \, d\mu_N(x) \, d\mu_N(y),$$

where $\mu_N$ denotes the restriction of the $d$-dimensional Hausdorff measure $\mu_d$ to the set $N$.

Given $f \in H^s(N)$, let $\nu_N(f) \in H^{-2}(\mathbb{R}^n)$ be the signed measure with $\text{supp}(\nu_N(f)) = N$ defined by

$$(\nu_N(f), u)_{-2,2} = \langle f, \gamma_N u \rangle_{H^s(M)},$$

where $(\cdot, \cdot)_{-2,2}$ denotes the $H^{-2}$-$H^2$ duality.

Given $\lambda > 0$, let $g_{\lambda}$ be the kernel of $(-\Delta + \lambda)^{-1}$. Then

$$G_{\lambda} : H^s(N) \to L^2(\mathbb{R}^n), \quad G_{\lambda}f := g_{\lambda} * \nu_N(f).$$
Therefore, given $\lambda_0 > 0$ and posing $g := g_{\lambda_0}$, for any nonlinear maximal monotone relation $\Theta \subset H^s(N) \times H^s(N)$, one gets a nonlinear maximal monotone operator $-\Delta_\Theta$ of type $\lambda_0$ defined by

$$-\Delta_\Theta u = -\Delta u_0 - \lambda_0 g \ast \nu_N(f_u),$$

\[\mathcal{D}(-\Delta_\Theta) := \{ u \in L^2(\mathbb{R}^n) : u = u_0 + g \ast \nu_N(f_u), \ u_0 \in H^2(\mathbb{R}^n), \ (f_u, \gamma_N u_0) \in \Theta \}\]

and with nonlinear resolvent

$$(-\Delta_\Theta + \lambda_0)^{-1} = (-\Delta + \lambda_0)^{-1} + g_\lambda \ast \nu_N((\Theta + \Gamma_{\lambda})^{-1} \circ (\gamma_N(-\Delta + \lambda)^{-1}))),$$

where

$$\Gamma_{\lambda} f = (\lambda - \lambda_0) \gamma_N(g \ast g_\lambda \ast \nu_N(f)).$$
Notice that, since \((-\Delta + \lambda_0)g = \delta_0\), \(-\Delta_\Theta\) can be alternatively defined by

\[
(-\Delta_\Theta + \lambda_0)u := (-\Delta + \lambda_0)u - \nu_N(f_u).
\]

When \(N\) is a Riemannian manifold with volume form \(dv\), since

\[
\nu_N(f) = ((-\Delta_{LB} + \lambda_0)^s f)\delta_N,
\]

where, for any \(f \in H^{-s}(N)\),

\[
(f \delta_N, u)_{-2,2} = \int_N (-\Delta_{LB} + \lambda_0)^{-s/2}f(x)((-\Delta_{LB} + \lambda_0)^{s/2} \gamma_N u)(x)\ dv(x),
\]

one has

\[
(-\Delta_\Theta + \lambda_0)u = (-\Delta + \lambda_0)u - ((-\Delta_{LB} + \lambda_0)^s f_u)\delta_N.
\]
In the case $\Theta^{-1}$ is single-valued one can also write

$$( -\Delta_\Theta + \lambda_0 ) u = ( -\Delta + \lambda_0 ) u - ( ( -\Delta_{LB} + \lambda_0 )^s \Theta^{-1} ( \gamma_N u_0 ) ) \delta_N .$$

If $\Theta = \partial \varphi$, where $\varphi : H^s(N) \to (-\infty, +\infty]$ is a proper lower semicontinuous function, then $-\Delta_\Theta + \lambda_0 = \partial \Phi$, where

$$\Phi(u) := \begin{cases} 
\frac{1}{2} \| (-\Delta + \lambda_0)^{1/2} u_0 \|^2 + \varphi(f) & u \in \mathcal{D}(\Phi) \\
+\infty & \text{otherwise,}
\end{cases}$$

$$\mathcal{D}(\Phi) := \{ u \in L^2(\mathbb{R}^n) : u = u_0 + g*\nu_N(f), u_0 \in H^1(\mathbb{R}^n), \varphi(f) < +\infty \} .$$
The Laplacian with nonlinear boundary conditions on a bounded domain.

Let \( \Omega \subset \mathbb{R}^n, n > 1, \) be a bounded open set with a regular boundary \( \partial \Omega. \) The continuous and surjective linear operator

\[
\gamma : H^2(\Omega) \rightarrow H^{3/2}(\partial \Omega) \times H^{1/2}(\partial \Omega), \quad \gamma u := (\gamma_0 u, \gamma_1 u),
\]

is defined as the unique bounded linear operator such that, in the case \( u \in C^\infty(\bar{\Omega}), \)

\[
\gamma_0 u(x) = u(x), \quad \gamma_1 u(x) = \frac{\partial u}{\partial n}(x), \quad x \in \partial \Omega,
\]

where \( n \) is the inner normal vector on \( \partial \Omega. \) The map \( \gamma \) can be extended to a bounded linear operator

\[
\hat{\gamma} : \mathcal{D}(\Delta_{max}) \rightarrow H^{-1/2}(\partial \Omega) \times H^{-3/2}(\partial \Omega), \quad \hat{\gamma} \phi = (\hat{\gamma}_0 u \phi, \hat{\gamma}_1 u),
\]

where

\[
\mathcal{D}(\Delta_{max}) := \{ u \in L^2(\Omega) : \Delta u \in L^2(\Omega) \}.
\]
Let $A_0 = -\Delta_D$ be the self-adjoint operator in $L^2(\Omega)$ given by the Dirichlet Laplacian, i.e.

$$\mathcal{D}(\Delta_D) := H^2(\Omega) \cap H_0^1(\Omega), \quad H_0^1(\Omega) := \{ u \in H^1(\Omega) : \gamma_0 u = 0 \}.$$

We take

$$\mathfrak{h} = H^{1/2}(\partial \Omega) \quad \text{and} \quad \tau = \gamma_1 |_{\mathcal{D}(\Delta_D)}.$$

Thus we are looking for nonlinear maximal monotone extensions of the strictly positive symmetric operator $S = -\Delta_{\text{min}}$ given by the minimal Laplacian with domain

$$\mathcal{D}(\Delta_{\text{min}}) := \{ u \in H^2(\Omega) : \gamma_0 u = \gamma_1 u = 0 \}.$$
Let \( \varphi : L^2(\Omega) \to (-\infty, +\infty] \) be a proper lower semicontinuous convex function such that

\[
\text{int}(\{ f \in L^2(\partial\Omega) : \varphi(f) < +\infty \}) \cap H^{1/2}(\partial\Omega) \neq \emptyset.
\]

Defining the maximal monotone relation

\[
\Theta_{\varphi} := (\partial\varphi - P) \circ (-\Delta_{LB} + 1)^{1/2} \subset H^{1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)
\]

where \( P \) is the Dirichlet-to-Neumann operator, one obtains the nonlinear maximal monotone operator \( -\Delta_{\varphi} := -\Delta_{\Theta_{\varphi}} \) defined by

\[
-\Delta_{\varphi} : D(-\Delta_{\varphi}) \subseteq L^2(\Omega) \to L^2(\Omega), \quad -\Delta_{\varphi} u = -\Delta u,
\]

\[
D(-\Delta_{\varphi}) = \{ u \in D(\Delta_{max}) : (\hat{\gamma}_0 u, \hat{\gamma}_1 u) \in \partial \varphi \}.
\]
Moreover $-\Delta \varphi = \partial \Phi$, where

$$
\Phi(u) = \begin{cases}
\frac{1}{2} \| \nabla u \|^2 + \varphi(\gamma_0 u), & u \in \mathcal{D}(\Phi) \\
+\infty, & \text{otherwise,}
\end{cases}
$$

$$
\mathcal{D}(\Phi) = \{ u \in H^1(\Omega) : \varphi(\gamma_0 u) < +\infty \}.
$$

If $\varphi$ has an unique minimum point $f_0 \in L^2(\partial \Omega)$ then, denoting by $S_t^\varphi$ the nonlinear semigroup of contractions generated by $-\Delta \varphi$, one has

$$
\forall u \in \mathcal{D}(-\Delta \varphi), \quad \text{w- lim}_{t \to +\infty} S_t^\varphi(u) = u_0,
$$

where $u_0$ is the unique harmonic function in $\Omega$ such that $\gamma_0 u_0 = f_0$. If $\varphi$ is an even function then the above limit holds in strong sense.