Abstract

We build on a working program initiated by Pudlák [Pud17] and construct an oracle relative to which each coNP-complete set has P-optimal proof systems and NP ∩ coNP does not have complete problems.

1 Introduction

The main motivation for the present paper is an article by Pudlák [Pud17] who lists several major conjectures in the field of proof complexity and discusses their relations. Among others, Pudlák conjectures the following assertions (note that within the present paper all reductions are polynomial-time-bounded):

- **CON** (resp., **SAT**): coNP (resp., NP) does not contain many-one complete sets that have P-optimal proof systems

- **CON**_N: coNP does not contain many-one complete sets that have optimal proof systems, (note that **CON**_N is the non-uniform version of **CON**)

- **DisjNP** (resp., **DisjCoNP**): The class of all disjoint NP-pairs (resp., coNP-pairs) does not have many-one complete elements,

- **TFNP**: The class of all total polynomial search problems does not have complete elements,

- **NP ∩ coNP** (resp., **UP**): NP ∩ coNP (resp., UP, the class of problems accepted by NP machines with at most one accepting path for each input) does not have many-one complete elements.

Pudlák asks for oracles separating corresponding relativized conjectures. Recently there has been made some progress in this working program [Kha19, DG19, Dos19a, Dos19b] which is documented by the following figure representing the current state of the art.

In the figure Thm 3.2 denotes the result of the present paper. It shows that there is no relativizable proof for the implication **NP ∩ coNP** ⇒ **CON**. So the conjectures **NP ∩ coNP** and **CON** cannot be shown equivalent with relativizable proofs.
Figure 1: Solid arrows mean implications. All implications occurring in the graphic have relativizable proofs. A dashed arrow from one conjecture A to another conjecture B means that there is an oracle X against the implication A \( \Rightarrow \) B, i.e., relative to X, it holds A \( \land \neg \) B. Pudlák [Pud17] also defines the conjecture RFN\(_1\) and lists it between CON \( \lor \) SAT and P \( \neq \) NP, i.e., CON \( \lor \) SAT \( \Rightarrow \) RFN\(_1\) \( \Rightarrow \) P \( \neq \) NP. Khaniki [Kha19] even shows CON \( \lor \) SAT \( \iff \) RFN\(_1\), which is why we omit RFN\(_1\) in the figure. For a definition of RFN\(_1\) we refer to [Pud17].

2 Preliminaries

Throughout this paper let \( \Sigma \) be the alphabet \( \{0, 1\} \). We denote the length of a word \( w \in \Sigma^* \) by \( |w| \). Let \( \Sigma^{*<n} = \{w \in \Sigma^* \mid |w| < n\} \) for \( \leq \in \{\leq, <, =, >, \geq\} \). The empty word is denoted by \( \varepsilon \) and the \( i \)-th letter of a word \( w \) for \( 0 \leq i < |w| \) is denoted by \( w(i) \), i.e., \( w = w(0)w(1)\cdots w(|w| - 1) \). For \( k \leq |w| \) let \( \text{pr}_k(w) = w(0)\cdots w(k-1) \) be the length \( k \) prefix of \( w \). A word \( v \) is a prefix of \( w \) if there exists \( k \leq |w| \) such that \( v = \text{pr}_k(w) \). If \( v \) is a prefix of \( w \), then we write \( v \preceq w \) if \( w \) \( \supseteq v \). If \( v \subseteq w \) and \( |v| < |w| \), then we write \( v \subseteq w \) or \( w \supseteq v \). For each finite set \( Y \subseteq \Sigma^* \), let \( \ell(Y) \equiv \sum_{w \in Y} |w| \).

Given two sets \( A \) and \( B \), \( A - B \) denotes the set difference between \( A \) and \( B \), i.e., \( A - B = \{a \in A \mid a \notin B\} \). The complement of a set \( A \) relative to the universe \( U \) is denoted by \( \overline{A} = U - A \). The universe will always be apparent from the context. Furthermore, the symmetric difference is denoted by \( \triangle \), i.e., \( A \triangle B = (A - B) \cup (B - A) \) for arbitrary sets \( A \) and \( B \).

\( \mathbb{Z} \) denotes the set of integers, \( \mathbb{N} \) denotes the set of natural numbers, and \( \mathbb{N}^+ = \mathbb{N} - \{0\} \). The set of primes is denoted by \( \mathbb{P} = \{2, 3, 5, \ldots\} \) and \( \mathbb{P}^{\geq 3} \) denotes the set \( \mathbb{P} - \{2\} \).

We identify \( \Sigma^* \) with \( \mathbb{N} \) via the polynomial-time computable, polynomial-time invertible bijection \( w \mapsto \sum_{i=1}^{2|w|+1-1} (1 + w(i))^2 \), which is a variant of the dyadic encoding. Hence, notations, relations, and operations for \( \Sigma^* \) are transferred to \( \mathbb{N} \) and vice versa. In particular, \( |n| \) denotes the length of \( n \in \mathbb{N} \). We eliminate the ambiguity of the expressions \( 0^i \) and \( 1^i \) by always interpreting them over \( \Sigma^* \).

Let \( \langle \cdot \rangle : \bigcup_{i \geq 0} \mathbb{N}^i \to \mathbb{N} \) be an injective, polynomial-time computable, polynomial-time invertible pairing function such that \( |\langle u_1, \ldots, u_n \rangle| = 2(|u_1| + \cdots + |u_n| + n) \).
The domain and range of a function \( t \) are denoted by \( \text{dom}(t) \) and \( \text{ran}(t) \), respectively.

FP, P, and NP denote standard complexity classes \([\text{Pap94}]\). Define \( \text{coC} = \{ A \subseteq \Sigma^* \mid \overline{A} \in \mathcal{C} \} \) for a class \( \mathcal{C} \).

We also consider these complexity classes in the presence of an oracle \( D \) and denote the corresponding classes by \( \text{FP}^D \), \( \text{P}^D \), and \( \text{NP}^D \). Moreover, we define \( \text{coC}^D = \{ A \subseteq \Sigma^* \mid \overline{A} \in \mathcal{C}^D \} \) for a class \( \mathcal{C} \).

Let \( M \) be a Turing machine. \( M^D(x) \) denotes the computation of \( M \) on input \( x \) with \( D \) as an oracle. For an arbitrary oracle \( D \) we let \( L(M^D) = \{ x \mid M^D(x) \text{ accepts} \} \), where as usual in case \( M \) is nondeterministic, the computation \( M^D(x) \) accepts if and only if it has at least one accepting path.

For a deterministic polynomial-time Turing transducer (i.e., a Turing machine computing a function), depending on the context, \( F^D(x) \) either denotes the computation of \( F \) on input \( x \) with \( D \) as an oracle or the output of this computation.

**Definition 2.1** A sequence \( (M_i)_{i \in \mathbb{N}^+} \) is called standard enumeration of nondeterministic, polynomial-time oracle Turing machines, if it has the following properties:

1. All \( M_i \) are nondeterministic, polynomial-time oracle Turing machines.
2. For all oracles \( D \) and all inputs \( x \) the computation \( M_i^D(x) \) stops within \( |x|^i + i \) steps.
3. For every nondeterministic, polynomial-time oracle Turing machine \( M \) there exist infinitely many \( i \in \mathbb{N} \) such that for all oracles \( D \) it holds that \( L(M^D) = L(M_i^D) \).
4. There exists a nondeterministic, polynomial-time oracle Turing machine \( M \) such that for all oracles \( D \) and all inputs \( x \) it holds that \( M^D((i, x, 0|x|^i)) \) nondeterministically simulates the computation \( M_i^D(x) \).

Analogously we define standard enumerations of deterministic, polynomial-time oracle Turing transducers.

Throughout this paper, we fix some standard enumerations. Let \( M_1, M_2, \ldots \) be a standard enumeration of nondeterministic polynomial-time oracle Turing machines. Then for every oracle \( D \), the sequence \( (M_i)_{i \in \mathbb{N}^+} \) represents an enumeration of the languages in \( \text{NP}^D \), i.e., \( \text{NP}^D = \{ L(M_i^D) \mid i \in \mathbb{N} \} \). Let \( F_1, F_2, \ldots \) be a standard enumeration of polynomial time oracle Turing transducers.

By the properties of standard enumerations, for each oracle \( D \) the problem

\[
K^D = \{ \langle 0^t, 0^t, x \rangle \mid i, t, x \in \mathbb{N}, i > 0, \text{ and } M_i^D(x) \text{ accepts within } t \text{ steps} \}
\]

is \( \text{NP}^D \)-complete (in particular it is in \( \text{NP}^D \)) and consequently, \( K^D \) is \( \text{coNP}^D \)-complete.

In the present article we only use polynomial-time-bounded many-one reductions. Let \( D \) be an oracle. For problems \( A, B \subseteq \Sigma^* \) we write \( A \leq_m^P B \) (resp., \( A \leq_m^{P,D} B \)) if there exists \( f \in \text{FP} \) (resp., \( f \in \text{FP}^D \)) with \( \forall x \in \Sigma^* (x \in A \iff f(x) \in B) \). In this case we say that \( A \) is polynomially many-one reducible to \( B \).

**Definition 2.2** ([CR79]) A function \( f \in \text{FP} \) is called proof system for the set \( \text{ran}(f) \). For \( f, g \in \text{FP} \) we say that \( f \) is simulated by \( g \) (resp., \( f \) is P-simulated by \( g \)) denoted by \( f \leq g \) (resp., \( f \leq_P g \)), if there exists a function \( \pi \) (resp., a function \( \pi \in \text{FP} \)) and a polynomial \( p \) such that \( |\pi(x)| \leq p(|x|) \) and \( g(\pi(x)) = f(x) \) for all \( x \). A function \( g \in \text{FP} \) is optimal (resp., P-optimal), if \( f \leq g \) (resp., \( f \leq_P g \)) for all \( f \in \text{FP} \) with \( \text{ran}(f) = \text{ran}(g) \). Corresponding relativized notions are obtained by using \( \text{FP}^D \), \( \text{FP}^D \), and \( \leq_{P,D} \) in the definitions above.
The following proposition states the relativized version of a result by Köbler, Messner, and Torán [KMT03], which they show with a relativizable proof.

**Proposition 2.3 ([KMT03])** For every oracle \( D \), if \( A \) has a \( \text{P}^D \)-optimal (resp., optimal) proof system and \( B \leq_{\text{m}}^D A \), then \( B \) has a \( \text{P}^D \)-optimal (resp., optimal) proof system.

**Corollary 2.4** For every oracle \( D \), if there exists a \( \leq_{\text{m}}^D \)-complete \( A \in \text{coNP}^D \) that has a \( \text{P}^D \)-optimal (resp., optimal) proof system, then all sets in \( \text{coNP}^D \) have \( \text{P}^D \)-optimal (resp., optimal) proof systems.

Let us introduce some (partially quite specific) notations that are designed for the construction of oracles [DG19]. The support \( \text{supp}(t) \) of a real-valued function \( t \) is the subset of the domain that consists of all values that \( t \) does not map to 0. We say that a partial function \( t \) is injective on its support if \( t(i, j) = t(i', j') \) for \( (i, j), (i', j') \in \text{supp}(t) \) implies \( (i, j) = (i', j') \). If a partial function \( t \) is not defined at point \( x \), then \( t \cup \{ x \mapsto y \} \) denotes the extension of \( t \) at \( x \) has value \( y \).

If \( A \) is a set, then \( A(x) \) denotes the characteristic function at point \( x \), i.e., \( A(x) = 1 \) if \( x \in A \), and 0 otherwise. An oracle \( D \subseteq \mathbb{N} \) is identified with its characteristic sequence \( D(0)D(1) \cdots \), which is an \( \omega \)-word. In this way, \( D(i) \) denotes both, the characteristic function at point \( i \) and the \( i \)-th letter of the characteristic sequence, which are the same. A finite word \( w \) describes an oracle that is partially defined, i.e., only defined for natural numbers \( x < |w| \). We can use \( w \) instead of the set \( \{ i \mid w(i) = 1 \} \) and write for example \( A = w \cup B \), where \( A \) and \( B \) are sets.

For nondeterministic oracle Turing machines \( M \) we use the following phrases: a computation \( M^w(x) \) **definitely accepts**, if it contains a path that accepts and all queries on this path are \( < |w| \). A computation \( M^w(x) \) **definitely rejects**, if all paths reject and all queries are \( < |w| \).

For a nondeterministic Turing machine \( M \) we say that the computation \( M^w(x) \) is **defined**, if it definitely accepts or definitely rejects. For a polynomial-time oracle transducer \( F \), the computation \( F^w(x) \) is **defined** if all queries are \( < |w| \).

## 3 Oracle Construction

The following lemma is a slightly adapted variant of a result from [DG19].

**Lemma 3.1** For all \( y \leq |w| \) and all \( v \supseteq w \) it holds \( (y \in K^v \Leftrightarrow y \in K^w) \).

**Proof** We may assume \( y = \langle 0^i, 0^t, x \rangle \) for suitable \( i \in \mathbb{N}^+ \) and \( t, x \in \mathbb{N} \), since otherwise, \( y \notin K^w \) and \( y \notin K^u \). For each \( q \) that is queried within the first \( t \) steps of \( M^w_i(x) \) or \( M^u_i(x) \) it holds that \( |q| \leq t < |y| \) and thus, \( q < y \). Hence, these queries are answered the same way relative to \( w \) and \( v \), showing that \( M^w_i(x) \) accepts within \( t \) steps if and only if \( M^u_i(x) \) accepts within \( t \) steps. \( \square \)

**Theorem 3.2** There exists an oracle \( O \) such that the following statements hold:

- \( \text{NP}^O \cap \text{coNP}^O \) does not have \( \leq_{\text{m}}^O \)-complete problems.
- \( \overline{\text{K}}^O \) has \( \text{P}^O \)-optimal proof systems.

**Proof of Theorem 3.2** Let \( D \) be a (possibly partial) oracle and \( p \in \text{P}^{\geq 3} \). We define

\[
A_p^{D} := \{ 0^k \mid k \in \mathbb{N}^+, \exists x \in \Sigma^k \, x \in D \text{ and } x \text{ odd} \} \cup \{ 0^k \mid k \in \mathbb{N}^+ \}
\]

\[
B_p^{D} := \{ 0^k \mid k \in \mathbb{N}^+, \exists x \in \Sigma^k \, x \in D \text{ and } x \text{ even} \}
\]

4
Note that $A^D_p, B^D_p \in \text{NP}^D$ and $A^D_p = \overline{B^D_p}$ if $|\Sigma^k \cap D| = 1$ for each $k \in \mathbb{N}^+$. In that case $A^D_p \in \text{NP}^D \cap \text{coNP}^D$.

For the sake of simplicity, let us call a pair $(M_i, M_j)$ a $\text{NP}^D \cap \text{coNP}^D$-machine if $L(M^D_i) = L(M^D_j)$. Note that throughout this proof we sometimes omit the oracles in the superscript, e.g., we write $\text{NP}$ or $A^p$, instead of $\text{NP}^D$ or $A^D_p$. However, we do not do that in the “actual” proof but only when explaining ideas in a loose way in order to give the reader the intuition behind the occasionally very technical arguments.

**Preview of construction.** We sketch some very basic ideas of our construction.

1. For all $i > 0$ we try to ensure that $F_i$ is not a proof system for $\overline{K}$ relative to the final oracle. If this is possible, we do not have to consider $F_i$ anymore. If it is not possible, then $F_i$ inherently is a proof system for $\overline{K}$. In that case we start to encode the values of $F_i$ into the oracle. This way we easily obtain a $\text{P}$-optimal proof system for $\overline{K}$ in the end. Note that it is crucial that we allow to also encode values of functions $F_j$ into the oracle before we try —as described above— to make sure that these functions are not proof systems for $\overline{K}$. Hence, the final oracle also contains encodings of values of functions that are not proof systems for $\overline{K}$.

2. Similarly, for each pair $(i, j)$ with $i \neq j$ we first try to make sure that $(M_i, M_j)$ is not a $\text{NP} \cap \text{coNP}$-machine. If this is not possible, then $(M_i, M_j)$ inherently is an $\text{NP} \cap \text{coNP}$-machine. In this case we choose a prime $p$ and ensure in the further construction that $A_p = \overline{B_p}$, i.e., $A_p \in \text{NP} \cap \text{coNP}$. Moreover, we diagonalize against all $\text{FP}$-functions $F_r$ in order to make sure that $F_r$ does not reduce $A_p$ to $L(M_i)$.

For $i \in \mathbb{N}^+$ and $x, y \in \mathbb{N}$ we write $c(i, x, y) := (0^i, 0|x|^{i+1}, 0|x|^{|+i}, x, y, y)$. Note that $|c(i, x, y)|$ is even and by the properties of the pairing function $\langle \cdot \rangle$,

$$\forall i \in \mathbb{N}^+, x, y \in \mathbb{N} \; |c(i, x, y)| > 4 \cdot \max(|x|^i + i, |y|).$$

(1)

**Claim 3.3** Let $w \in \Sigma^*$ be an oracle, $i \in \mathbb{N}^+$, and $x, y \in \mathbb{N}$ such that $c(i, x, y) \leq |w|$. Then the following holds.

1. $F^w_i(x)$ is defined and $F^w_i(x) < |w|$.
2. $(F^w_i(x) \in K^w \Leftrightarrow F^w_i(x) \in K^v)$ for all $v \supseteq w$.

**Proof** As the running time of $F^w_i(x)$ is bounded by $|x|^i + i < |c(i, x, y)| < c(i, x, y) \leq |w|$, the computation $F^w_i(x)$ is defined and its output is less than $|w|$. Hence, 1 holds. Consider 2. It suffices to show that $K^v(q) = K^w(q)$ for all $q < |w|$ and all $v \supseteq w$. This holds by Lemma 3.1. \(\square\)

During the construction we maintain a growing collection of requirements that is represented by a partial function belonging to the set

$$\mathcal{T} = \left\{ t : (\mathbb{N}^+)^2 \rightarrow \mathbb{Z} \mid \text{dom}(t) \text{ is finite, } t \text{ is injective on its support, and} \right\}$$

- $t(\{(i, i) \mid i \in \mathbb{N}^+\}) \subseteq \{0\} \cup \mathbb{N}^+$
- $t(\{(i, j) \mid i, j \in \mathbb{N}^+, i \neq j\}) \subseteq \{0\} \cup \{-p \mid p \in \mathbb{P}^{\geq 3}\}$.

A partial oracle $w \in \Sigma^*$ is called $t$-valid for $t \in \mathcal{T}$ if it satisfies the following properties.
V1 For all \( i \in \mathbb{N}^+ \) and all \( x, y \in \mathbb{N} \), if \( c(i, x, y) \in w \), then \( F^w_i(x) = y \) and \( y \in \overline{K^w} \).

(meaning: if the oracle contains the codeword \( c(i, x, y) \), then \( F^w_i(x) \) outputs \( y \) and \( y \in \overline{K^w} \); hence, \( c(i, x, y) \in w \) is a proof for \( y \in \overline{K^w} \))

V2 For all distinct \( i, j \in \mathbb{N}^+ \), if \( t(i, j) = 0 \), then there exists \( x \) such that (i) \( M^w_i(x) \) and \( M^w_j(x) \) definitely accept or (ii) \( M^w_i(x) \) and \( M^w_j(x) \) definitely reject.

(meaning: for every extension of the oracle, \( (M_i, M_j) \) is not a \( \text{NP} \cap \text{coNP} \)-machine.)

V3 For all distinct \( i, j \in \mathbb{N}^+ \) with \( t(i, j) = -p \) for some \( p \in \mathbb{P}^+ \) and each \( k \in \mathbb{N}^+ \), it holds (i) \( |\Sigma^k \cap w| \leq 1 \) and (ii) if \( w \) is defined for all words of length \( p^k \), then \( |\Sigma^k \cap w| = 1 \).

(meaning: if \( t(i, j) = -p \), then ensure that \( A_p = \overline{B_p} \) (i.e., \( A_p \in \text{NP} \cap \text{coNP} \) relative to the final oracle.)

V4 For all \( i \in \mathbb{N}^+ \) with \( t(i, i) = 0 \), there exists \( x \) such that \( F^w_i(x) \) is defined and \( F^w_i(x) \in \overline{K^w} \) for all \( v \supseteq w \).

(meaning: for every extension of the oracle, \( F_i \) is not a proof system for \( \overline{K} \))

V5 For all \( i \in \mathbb{N}^+ \) and \( x \in \mathbb{N} \) with \( 0 < t(i, i) \leq c(i, x, F^w_i(x)) < |w| \), it holds \( c(i, x, F^w_i(x)) \in w \).

(meaning: if \( t(i) > 0 \), then from \( t(i) \) on, we encode \( F_i \) into the oracle.

Note that V5 is not in contradiction with V3 as \( |c(\cdot, \cdot, \cdot)| \) is even.)

The subsequent claim follows directly from the definition of \( t \)-valid.

Claim 3.4 Let \( t, t' \in \mathcal{T} \) such that \( t' \) is an extension of \( t \). For all oracles \( w \in \Sigma^* \), if \( w \) is \( t' \)-valid, then \( w \) is \( t \)-valid.

Claim 3.5 Let \( t \in \mathcal{T} \) and \( u, v, w \in \Sigma^* \) be oracles such that \( u \subseteq v \subseteq w \) and both \( u \) and \( w \) are \( t \)-valid. Then \( v \) is \( t \)-valid.

Proof \( v \) satisfies V2 and V4 since \( u \) satisfies these conditions. Moreover, \( v \) satisfies V3 as \( w \) satisfies these conditions.

Let \( i \in \mathbb{N}^+ \) and \( x, y \in \mathbb{N} \) such that \( c(i, x, y) \in v \). Then \( c(i, x, y) \in w \) and as \( w \) is \( t \)-valid, we obtain by V1 that \( F^w_i(x) = y \) and \( y \in \overline{K^w} \). Claim 3.3 yields that \( F^w_i(x) \) is defined and \( F^i_v(x) \in \overline{K^w} \Rightarrow F^i_v(x) \in \overline{K^w} \). This yields that \( F^i_v(x) = F^i_v(x) = y \) and \( \overline{K^w}(y) = \overline{K^w}(y) = 0 \).

Thus, \( v \) satisfies V1.

Now let \( i \in \mathbb{N}^+ \) and \( x \in \mathbb{N} \) such that \( 0 < t(i, i) \leq c(i, x, F^w_i(x)) < |v| \). Again, by Claim 3.3, \( F^w_i(x) \) is defined and thus, \( F^i_v(x) = F^i_v(x) \). As \( |v| \leq |w| \) and \( w \) is \( t \)-valid, we obtain by V5 that \( c(i, x, F^w_i(x)) = c(i, x, F^w_i(x)) \in w \). Since \( v \subseteq w \) and \( |v| > c(i, x, F^w_i(x)) \), we obtain \( c(i, x, F^w_i(x)) \in v \), which shows that \( v \) satisfies V5.

Oracle construction. Let \( T \) be an enumeration of \( \mathbb{N}^+ \) with \( \{i, j, r \mid i \neq j, i, j, r \in \mathbb{N}^+ \} \) having the property that \( (i, j) \) appears earlier than \( (i, j, r) \) for all \( i, j, r \in \mathbb{N}^+ \) with \( i \neq j \) (more formally, \( T \) could be defined as a function \( \mathbb{N} \to (\mathbb{N}^+)^2 \cup \{i, j, r \mid i \neq j, i, j, r \in \mathbb{N}^+ \} \) with each \( t \) stands for a task. We treat the tasks in the order specified by \( T \) and after treating a task we remove it and possibly other tasks from \( T \). We start with the nowhere defined function \( t_0 \) and the \( t_0 \)-valid oracle \( w_0 = \varepsilon \). Then we define functions \( t_1, t_2, \ldots \) in \( T \) such that \( t_{i+1} \) is an extension of \( t_i \) and partial oracles \( w_0 \subseteq w_1 \subseteq w_2 \subseteq \cdots \) such that each \( w_i \) is \( t_i \)-valid. Finally, we choose \( O = \bigcup_{i=0}^\infty w_i \) (note that \( O \) is totally defined since in each step we strictly extend the oracle). We describe step \( s > 0 \), which starts with some \( t_{s-1} \in T \) and a \( t_{s-1} \)-valid oracle \( w_{s-1} \) and chooses an extension \( t_s \in T \) of \( t_{s-1} \) and a \( t_s \)-valid \( w_{s} \supseteq w_{s-1} \) (it will be argued later that all these steps are indeed possible). Let us recall that each task is immediately deleted from \( T \) after it is treated.
• task \((i, i)\): Let \(t' = t_{s-1} \cup \{(i, i) \mapsto 0\}\). If there exists a \(t'\)-valid \(v \sqsupseteq w_{s-1}\), then let \(t_s = t'\) and \(w_s\) be the least \(t'\)-valid, partial oracle \(\sqsupseteq w_{s-1}\). Otherwise, let \(t_s = t_{s-1} \cup \{(i, i) \mapsto |w_{s-1}|\}\) and choose \(w_s = w_{s-1}b\) with \(b \in \{0, 1\}\) such that \(w_s\) is \(t_s\)-valid.

(meaning: try to ensure that \(F_i\) is not a proof system for \(K\). If this is impossible, require that from now on the values of \(F_i\) are encoded into the oracle.)

• task \((i, j)\) with \(i \neq j\): Let \(t' = t_{s-1} \cup \{(i, j) \mapsto 0\}\). If there exists a \(t'\)-valid \(v \sqsupseteq w_{s-1}\), then let \(t_s = t'\), define \(w_s\) to be the least \(t'\)-valid, partial oracle \(\sqsupseteq w_{s-1}\), and delete all tasks \((i, j, r)\) from \(T\). Otherwise, let \(z = |w_{s-1}|\), choose some \(p \in \mathbb{P}^{\geq 3}\) greater than \(|z|\) with \(-p \notin \text{ran}(t_{s-1})\), let \(t_s = t_{s-1} \cup \{(i, j) \mapsto -p\}\), and choose \(w_s = w_{s-1}b\) with \(b \in \{0, 1\}\) such that \(w_s\) is \(t_s\)-valid.

(meaning: try to ensure that \((M_i, M_j)\) is not an \(\text{NP} \cap \text{coNP}\)-machine. If this is impossible, then \(L(M_i)\) inherently is in \(\text{NP} \cap \text{coNP}\) and we choose a sufficiently large prime \(p\). It will be made sure in the further construction that \(A_p = \overline{B_p}\) and \(A_p\) cannot be reduced to \(L(M_i)\).)

• task \((i, j, r)\) with \(i \neq j\): It holds \(t_{s-1}(i, j) = -p\) for a prime \(p \in \mathbb{P}^{\geq 3}\), since otherwise, this task would have been deleted in the treatment of task \((i, j)\). Define \(t_s = t_{s-1}\) and choose a \(t_s\)-valid \(w_s \sqsupseteq w_{s-1}\) such that for some \(n \in \mathbb{N}^+\) one of the following two statements holds:

- \(0^n \in A_p^w\) for all \(v \sqsupseteq w_s\) and \(M_i^{w_s}(F_r^{w_s}(0^n))\) definitely rejects.
- \(0^n \in B_p^w\) for all \(v \sqsupseteq w_s\) and \(M_j^{w_s}(F_r^{w_s}(0^n))\) definitely rejects.

(meaning: due to V3 it will hold \(A_p = \overline{B_p}\) relative to the final oracle. By construction, relative to the final oracle it will hold \(L(M_i) = \overline{L(M_j)}\). Hence, the treatment of the task \((i, j, r)\) makes sure that it does not hold \(A_p \leq^*_{\text{m}} L(M_i)\) via \(F_r\) relative to the final oracle.)

Observe that \(t_s\) is always chosen in a way such that it is in \(T\). We now show that the construction is possible. For that purpose, we first describe how a valid oracle can be extended by one bit such that it remains valid.

**Claim 3.6** Let \(s \in \mathbb{N}\) and \(w \in \Sigma^*\) be a \(t_s\)-valid oracle with \(w \sqsupseteq w_s\). It holds for \(z = |w|\):

1. If \(|z|\) is odd and for all \(p \in \mathbb{P}^{\geq 3}\) and \(k \in \mathbb{N}^+\) with \(-p \in \text{ran}(t_s)\) it holds \(|z| \neq p^k\), then \(w0\) and \(w1\) are \(t_s\)-valid.

2. If there exist \(p \in \mathbb{P}^{\geq 3}\) and \(k \in \mathbb{N}^+\) with \(-p \in \text{ran}(t_s)\) such that \(|z| = p^k\), \(z \neq 1^{p^k}\), and \(w \cap \Sigma^{p^k} = \emptyset\), then \(w0\) and \(w1\) are \(t_s\)-valid.

3. If there exist \(p \in \mathbb{P}^{\geq 3}\) and \(k \in \mathbb{N}^+\) with \(-p \in \text{ran}(t_s)\) such that \(z = 1^{p^k}\) and \(w \cap \Sigma^{p^k} = \emptyset\), then \(w1\) is \(t_s\)-valid.

4. If \(z = c(i, x, F_i^w(x))\) for \(i \in \mathbb{N}^+\) and \(x \in \mathbb{N}\) and \(0 < t_s(i, i) \leq z\), then \(w1\) is \(t_s\)-valid.

5. If \(z = c(i, x, F_i^w(x))\) for \(i \in \mathbb{N}^+\) and \(x \in \mathbb{N}\), at least one of the three conditions (i) \(t_s(i, i)\) undefined, (ii) \(t_s(i, i) = 0\), and (iii) \(t_s(i, i) > z\) holds, and \(F_i^w(x) \in \overline{K^w}\), then \(w0\) and \(w1\) are \(t_s\)-valid.

6. In all other cases (i.e., none of the assumptions in 1–5 holds) \(w0\) is \(t_s\)-valid.

**Proof** First note that V2 and V4 are not affected by extending the oracle. So we only need to consider V1, V3, and V5 in the following.
Let us show the following assertions.

\( w_0 \) satisfies V1.

If (i) \( z = c(i, x, F^w_i(x)) \) for \( i \in \mathbb{N}^+ \) and \( x \in \mathbb{N} \) with \( F^w_i(x) \in \overline{K^w} \) or (ii) \( z \) has odd length, then \( w_1 \) satisfies V1.

\( w_0 \) satisfies V5 unless there exist \( i \in \mathbb{N}^+ \) and \( x, y \in \mathbb{N} \) such that (i) \( z = c(i, x, y) \), (ii) \( 0 < t_s(i, i) \leq z \), and (iii) \( F^w_i(x) = y \).

\( w_1 \) satisfies V5.

(2) and (3): Let \( i' \in \mathbb{N}^+ \) and \( x', y' \in \mathbb{N} \) such that \( c(i', x', y') \in w \). Then, as \( w \) is \( t_s \)-valid, by V1, \( F^w_{i'} (x') = y' \in \overline{K^w} \) and by Claim 3.3, \( F^w_{i'} (x') \) is defined and \( y' \in K^w \) for all \( v \supseteq w \).

Hence, in particular, \( F^w_{i'} (x') = y' \in K^w \) for all \( b \in \{0, 1\} \). This shows (2). For the proof of (3) it remains to consider \( z \). In case (ii) \( w_1 \) satisfies V1 as \( |z| \) is odd and each \( c(i, x, y) \) has even length. Consider case (i), i.e., \( z = c(i, x, y) \) for \( i \in \mathbb{N}^+ \) and \( x \in \mathbb{N} \) with \( F^w_i(x) \in \overline{K^w} \). Then by Claim 3.3, \( F^w_{i'} (x) = y \in \overline{K^w} \), which shows that \( w_1 \) satisfies V1. This proves (3).

(4) and (5): Let \( i' \in \mathbb{N}^+ \) and \( x', y' \in \mathbb{N} \) such that \( 0 < t_s(i', i') \leq c(i', x', y') \) < \( |w| \). Then by Claim 3.3, \( F^w_{i'} (x') \) is defined and thus, \( F^w_{i'} (x') = F^w_i (x') \) for all \( b \in \{0, 1\} \). As \( w \) is \( t_s \)-valid, it holds \( c(i', x', y') \in w \) and hence, \( c(i', x', y') \in w \subseteq wb \) for all \( b \in \{0, 1\} \). This shows (5).

In order to finish the proof of (4), it remains to consider \( z \). Assume \( z = c(i, x, y) \) for some \( i, x, y \in \mathbb{N} \) with \( i > 0 \) and \( 0 < t_s(i, i) \leq z \) (otherwise, \( w_0 \) clearly satisfies V5). If (iii) is wrong, then \( F^w_i(x) \neq y \). By Claim 3.3, this computation is defined and hence, \( F^w_{i'} (x) \neq y \), which is why \( w_0 \) satisfies V5. This shows (4).

We now prove the statements 1–6.

1. Clearly \( w_0 \) and \( w_1 \) satisfy V3. Moreover, by (2) and (4), the oracle \( w_0 \) satisfies V1 and V5 (recall that the length of each \( c(\cdot, \cdot, \cdot) \) is even). By (3) and (5), the oracle \( w_1 \) satisfies V1 and V5.

2. By (2), (3), (4), and (5), the oracles \( w_0 \) and \( w_1 \) satisfy V1 and V5. As \( z \neq 1^k \) and \( w \) satisfies V3, the oracle \( w_0 \) satisfies V3. As \( w \cap \Sigma^{\#} = \emptyset \), the oracle \( w_1 \) satisfies V3.

3. By (3) and (5), the oracle \( w_1 \) satisfies V1 and V5. As \( w \cap \Sigma^{\#} = \emptyset \), the oracle \( w_1 \) satisfies V3.

4. As \( |z| \) is even, \( w_1 \) satisfies V3. By (5), \( w_1 \) satisfies V5. It remains to argue that \( w_1 \) satisfies V1. In order to apply (3), which will immediately show that \( w_1 \) satisfies V1, it is sufficient to prove \( y := F^w_{i'} (x) \in \overline{K^w} \). For a contradiction assume \( y \in K^w \). Let \( s' \) be the step that treats the task \( (i, i) \). Note \( s' < s \) since \( t_s(i, i) \) is defined. By Claim 3.4, \( w \) is \( t_{s'-1} \)-valid. As by Claim 3.3 the computation \( F^w_{i'} (x) \) is defined and \( y \in K^w \) for all \( v \supseteq w \), the oracle \( w \) is even \( t \)-valid for \( t = t_{s'-1} \cup \{(i, i) \mapsto 0\} \). But then the construction would have chosen \( t_{s'} = t \), in contradiction to \( t_s(i, i) > 0 \).

5. As \( |z| \) is even, \( w_0 \) and \( w_1 \) satisfy V3. By (2), (3), and (5), \( w_0 \) satisfies V1 and \( w_1 \) satisfies both V1 and V5. Moreover, (4) can be applied since each of the conditions (i)–(iii) of statement 5 implies that condition (ii) of (4) does not hold. Thus, \( w_0 \) satisfies V5.

6. By (2), \( w_0 \) satisfies V1. If \( w_0 \) does not satisfy V3, then there exist \( p \in \mathbb{P} \geq 3 \) with \( -p \in \text{ran}(t_s) \) and \( k > 0 \) such that \( w \cap \Sigma^{\#} = \emptyset \) and \( z = 1^k \), but this case is covered by statement 3 of the current claim. If \( w_0 \) does not satisfy V5, then by (4), there exist \( i \in \mathbb{N}^+ \) and \( x, y \in \mathbb{N} \) such that (i) \( z = c(i, x, y) \), (ii) \( 0 < t_s(i, i) \leq z \), and (iii) \( F^w_{i'} (x) = y \). This case, however, is covered by statement 4 of the current claim.
This finishes the proof of Claim 3.6.

In order to show that the above construction is possible, assume that it is not possible and let \( s > 0 \) be the least number, where it fails.

If step \( s \) treats a task \( t \in (\mathbb{N}^+)^2 \), then \( t_{s-1}(t) \) is not defined, since the value of \( t \) is defined in the unique treatment of the task \( t \). Hence, \( t' \) is well-defined. If \( t_s(t) \) is chosen to be 0, then the construction clearly is possible. Otherwise, due to the choice of \( t_s(t) \), the \( t_{s-1} \)-valid oracle \( w_{s-1} \) is even \( t_s \)-valid and Claim 3.6 ensures that there exists a \( t_s \)-valid \( w_{s-1}b \) for some \( b \in \{0,1\} \).

Hence, the construction does not fail in step \( s \), a contradiction.

For the remainder of the proof that the construction above is possible we assume that step \( s \) treats a task \( (i,j,r) \in \{(i,j,r) \mid i \neq j, i,j,r \in \mathbb{N}^+ \} \).

Then \( t_s = t_{s-1} \) and \( t_s(i,j) = -p \) for some \( p \in \mathbb{P}^{\geq 3} \). Let \( \gamma \) be the polynomial given by \( x \mapsto (x^r + r)^{i+j} + i + j \) and choose \( k \in \mathbb{N}^+ \) such that for \( n = p^k \)

\[
2^{n-1} > 2 : \gamma(n)
\]  
(6)

and \( w_{s-1} \) is not defined for any words of length \( n \). Note that \( \gamma(n) \) is greater than the running time of each of the computations \( M_D^p(F^p_D(0^n)) \) and \( M_D^p(F^p_D(0^n)) \) for each oracle \( D \).

We define \( u \supseteq w_{s-1} \) to be the minimal \( t_s \)-valid oracle that is defined for all words of length \( < n \). Such an oracle exists by Claim 3.6.

Moreover, for \( z \in \Sigma^n \), let \( u_z \supseteq u \) be the minimal \( t_s \)-valid oracle with \( u_z \cap \Sigma^n = \{z\} \) that is defined for all words of length \( \leq \gamma(n) \). Such an oracle exists by Claim 3.6: first, starting from \( u \) we extend the current oracle bitwise such that (i) it remains \( t_s \)-valid, (ii) it is defined for precisely the words of length \( \leq n \), and (iii) its intersection with \( \Sigma^n \) equals \( \{z\} \). This is possible by 2, 3, and 6 of Claim 3.6. Then by Claim 3.6, the current oracle can be extended bitwise without losing its \( t_s \)-validity until it is defined for all words of length \( \leq \gamma(n) \).

We define a further oracle \( v \) that will be crucial in the following. Let \( s' \) be the step that treats the task \( (i,j) \). As \( t_s(i,j) \) is defined, it holds \( s' < s \). By Claim 3.4, the oracle \( u \) is \( t_{s' - 1} \)-valid. In order to define \( v \), we need the following two properties (7) and (8) that we also need in different contexts and therefore, define in a general way. Let \( w \supseteq u \) be a \( t_{s' - 1} \)-valid oracle.

We say that \( w \) satisfies property (7) if

\[
\text{for all } i', x \in \mathbb{N} \text{ with } i' > 0, t_s(i', i') > 0 \text{ and } |u| \leq c(i', x, F^{w'}_{i'}(x)) < |w|, \text{ if } F^{w'}_{i'}(x) \in \overline{Kw},
\]

then \( c(i', x, F^{w'}_{i'}(x)) \in w \).

Moreover, \( w \) satisfies property (8) if

\[
\text{for all } p' \in \mathbb{P}^{\geq 3} \text{ with } -p \in \text{ran}(t_s),
\]

\[
\begin{align*}
(i) & \ w \cap \Sigma^{p'^{\kappa}} \subseteq \{1^{p'^{\kappa}}\} \text{ for all } \kappa > 0 \text{ with } n < p'^{\kappa} \text{ and } \\
(ii) & \ w \cap \Sigma^{p'^{\kappa}} = \{1^{p'^{\kappa}}\} \text{ for all } \kappa > 0 \text{ with } n < p'^{\kappa} \text{ and } 1^{p'^{\kappa}} < |w|.
\end{align*}
\]

(8)

Now we define \( v \supseteq u \) to be the minimal \( t_{s' - 1} \)-valid oracle that is defined for all words of length \( \leq \gamma(n) \) and satisfies properties (7) and (8). Let us argue that such an oracle exists. Clearly \( u \) satisfies properties (7) and (8). The second statement of the following claim shows that \( v \) is well-defined.

Claim 3.7

1. For all \( t_{s' - 1} \)-valid oracles \( w \) and \( w' \) with \( u \subseteq w \subseteq w' \), if \( w \) satisfies property (7) and \( w' \) does not satisfy property (7), then there exists \( |w| \leq \alpha < |w'| \) such that \( \alpha = c(i', x, F^{w'}_{i'}(x)) \) for \( i', x \in \mathbb{N} \) with \( i' > 0 \) and \( t_s(i', i') > 0 \), \( F^{w'}_{i'}(x) \in \overline{Kw}' \), and \( \alpha \notin w' \).

2. For each \( t_{s' - 1} \)-valid oracle \( w \supseteq u \) that satisfies properties (7) and (8) there exists \( b \in \{0,1\} \) such that \( wb \) is \( t_{s' - 1} \)-valid and satisfies properties (7) and (8).
3. Let \( w \uplus u \) be \( t_{s'-1} \)-valid. If \( w \) satisfies properties (7) and (8), then each \( w' \) with \( u \subseteq w' \subseteq w \) satisfies properties (7) and (8).

**Proof**

1. Since \( w' \) does not satisfy property (7), there exists \( |u| \leq \alpha = c(i',x,F_{i'}^{w'}(x)) < |w'| \) for \( i',x \in \mathbb{N} \) with \( i' > 0 \) and \( t_s(i',i') > 0 \) such that \( F_{i'}^{w'}(x) \in K^{w'} \) and \( \alpha \notin w' \). For a contradiction we assume \( \alpha < |w| \). Then Claim 3.3 yields \( F_{i'}^{w}(x) = F_{i'}^{w'}(x) \in K^{w} \). From \( \alpha < |w|, w \subseteq w' \), and \( \alpha \notin w' \) it follows \( \alpha \notin w \), which contradicts the assumption that \( w \) satisfies property (7).

2. We study several cases depending on \( \alpha = |w| \) (i.e., \( \alpha \) is the least word that \( w \) is not defined for).

   - If \( \alpha \) is of the form \( c(i',x,F_{i'}^{w}(x)) \) for \( i',x \in \mathbb{N} \) with \( i' > 0 \) and \( t_s(i',i') > 0 \) such that \( F_{i'}^{w}(x) \in K^{w} \), then we choose \( b = 1 \). The statements 4 and 5 of Claim 3.6 state that the oracle \( wb \) is \( t_{s'-1} \)-valid (recall that by construction \( t_s(i',i') \leq |u| \leq |w| = \alpha \) and note that we apply Claim 3.6 for the parameter \( s' - 1 \)).

   - If \( \alpha \) has length \( p^{\kappa} \) for some \( p' \in \mathbb{P}_{\geq 0} \) with \( p' \in \text{ran}(t_s) \) and \( \kappa > 0 \), then we choose \( b = 1 \) if \( \alpha = 1^p \) and \( b = 0 \) otherwise. Since \( w \) satisfies property (8), it holds \( w \cap \Sigma^{p^{\kappa}} = \emptyset \). Hence, the statements 1, 2, and 3 of Claim 3.6 state that the oracle \( wb \) is \( t_{s'-1} \)-valid (again, note that we apply Claim 3.6 for the parameter \( s' - 1 \)).

   - In all other cases Claim 3.6 guarantees that we can choose \( b \in \{0,1\} \) such that \( wb \) is \( t_{s'-1} \)-valid.

By the choice of \( b \), the oracle \( wb \) satisfies property (8). If \( wb \) does not satisfy property (7), then by statement 1 of the current claim, \( \alpha = c(i',x,F_{i'}^{wb}(x)) \) for \( i',x \in \mathbb{N} \) with \( i' > 0 \) and \( t_s(i',i') > 0 \), \( F_{i'}^{wb}(x) \in K^{wb} \), and \( \alpha \notin wb \). Claim 3.3, however, yields that then even \( F_{i'}^{w}(x) = F_{i'}^{wb}(x) \in K^{w} \). But then we would have chosen \( b = 1 \) above, in contradiction to \( \alpha \notin wb \).

3. As \( w' \subseteq w \) and \( w \) satisfies property (8), \( w' \) satisfies property (8). We argue that \( w' \) satisfies property (7). Let \( i',x \in \mathbb{N} \) with \( i' > 0 \) and \( t_s(i',i') > 0 \) such that \( |u| \leq c(i',x,F_{i'}^{w'}(x)) < |w'| \) and \( F_{i'}^{w'}(x) \in K^{w'} \). As \( c(i',x,F_{i'}^{w'}(x)) < |w'| \), Claim 3.3 yields \( F_{i'}^{w}(x) = F_{i'}^{w'}(x) \in K^{w} \). As \( w \) satisfies property (7), it holds \( c(i',x,F_{i'}^{w'}(x)) \in w' \). Since \( w' \subseteq w \) and \( c(i',x,F_{i'}^{w'}(x)) < |w'| \), we obtain \( c(i',x,F_{i'}^{w'}(x)) \in w' \). Hence, \( w' \) satisfies property (7).

This finishes the proof of Claim 3.7.

Note that by the choice of \( v \), it holds \( v \cap \Sigma^n = \emptyset \) (cf. Claim 3.6.1 and recall that \( t_{s'-1} \) is not defined for the pair \( (i,j) \)).

**Claim 3.8**

Let \( w \in \{v\} \cup \{u_z \mid z \in \Sigma^n\} \).

1. For each \( \alpha \in w \cap \Sigma^{>n} \) one of the following statements holds.

   - \( \alpha = c(i',x,F_{i'}^{w}(x)) \) for some \( i' \in \mathbb{N}^+ \) and \( x \in \mathbb{N} \) with \( 0 < t_s(i',i') \leq c(i',x,F_{i'}^{w}(x)) \) and \( F_{i'}^{w}(x) \in K^{w} \).
   - \( \alpha = 1^{p^{\kappa}} \) for some \( p' \in \mathbb{P}_{\geq 0} \) with \( p' \in \text{ran}(t_s) \) and some \( \kappa > 0 \).

2. For all \( p' \in \mathbb{P}_{\geq 0} \) with \( p' \in \text{ran}(t_s) \) and all \( \kappa > 0 \), if \( n < p^{\kappa} \leq \gamma(n) \), then \( w \cap \Sigma^{p^{\kappa}} = \{1^{p^{\kappa}}\} \).

3. For all \( z \in \Sigma^n \) and all \( \alpha \in u_z - v \) it holds \( \alpha = c(i',x,F_{i'}^{w_z}(x)) \) for some \( i' \in \mathbb{N}^+ \) and \( x \in \mathbb{N} \) with \( 0 < t_s(i',i') \leq c(i',x,F_{i'}^{w_z}(x)) \) and \( F_{i'}^{w_z}(x) \in K^{w_z} \).
4. For all $z \in \Sigma^n$ and all $\alpha \in v - u_z$ it holds $\alpha = c(i', x, F^u_{i'}(x))$ for some $i' \in \mathbb{N}^+$ and $x \in \mathbb{N}$ with $0 < t_s(i', i') \leq c(i', x, F^u_{i'}(x))$ and $F^u_{i'}(x) \in \overline{Kv}$.

Proof
1. We first argue for the case $w = u_z$ for some $z \in \Sigma^n$. Let $\alpha \in u_z \cap \Sigma^{>n}$. Moreover, let $u'$ be the prefix of $u_z$ that has length $\alpha$, i.e., $\alpha$ is the least word that $u'$ is not defined for. In particular, it holds $u' \cap \Sigma^{\leq n} = u_z \cap \Sigma^{\leq n}$ and thus, $u' \cap \Sigma^n = \{z\}$. As $u \subset u' \subset u_z$ and both $u$ and $u_z$ are $t_s$-valid, Claim 3.5 yields that $u'$ is also $t_s$-valid.

Let us apply Claim 3.6 to the oracle $u'$. If one of the cases 1, 2, 5, and 6 can be applied, then $u''$ is $t_s$-valid and can be extended to a $t_s$-valid oracle $u''$ with $|u''| = |u_z|$ by Claim 3.6. As $u''$ and $u_z$ agree on all words $< \alpha$ and $\alpha \in u_z - u''$, we obtain $u'' < u_z$ and due to $u' \subset u''$ we know that $u'' \cap \Sigma^n = \{z\}$. This is a contradiction to the choice of $u_z$ (recall that $u_z$ is the minimal $t_s$-valid oracle that is defined for all words of length $\leq \gamma(n)$ and that satisfies $u_z \cap \Sigma^n = \{z\}$).

Hence, none of the cases 1, 2, 5, and 6 of Claim 3.6 can be applied, i.e., either (i) Claim 3.6.3 or (ii) Claim 3.6.4 can be applied. Hence, either (i) $\alpha = 1^{|u|''}$ for some $i' \in \mathbb{P}^{2^3}$ and $\kappa > 0$ with $-p' \in \text{ran}(t_s)$ or (ii) $\alpha = c(i', x, F^u_{i'}(x))$ for $i' \in \mathbb{N}^+$ and $x \in \mathbb{N}$ with $0 < t_s(i', i') \leq \alpha$. In the latter case, as $\alpha \in u_z$ and $u_z$ is $t_s$-valid, we obtain from V1 that $F^u_{i'}(x) \in \overline{Ku_z}$.

The arguments for the case $w = v$ are similar: Let $\alpha \in v \cap \Sigma^{>n}$. Moreover, let $v'$ be the prefix of $v$ that has length $\alpha$, i.e., $\alpha$ is the least word that $v'$ is not defined for. As $u \subset v' \subset v$ and both $u$ and $v$ are $t_{s'-1}$-valid, Claim 3.5 yields that $v'$ is also $t_{s'-1}$-valid. Moreover, by Claim 3.7.3, $v'$ satisfies properties (7) and (8).

Let us apply Claim 3.6 to the oracle $v'$ (with the parameter $s' - 1$). If one of the cases 1, 2, 5, and 6 can be applied, then $v''$ is $t_{s'-1}$-valid.

First, assume that it does not hold that $v''$ satisfies properties (7) and (8). If $w_b$ does not satisfy property (8), then $\alpha = 1^{|u|''}$ for some $p' \in \mathbb{P}^{2^3}$ with $-p' \in \text{ran}(t_s)$ and $\kappa > 0$. If $w_b$ does not satisfy property (7), then it holds by Claim 3.7.1 that $\alpha = c(i', x, F^v_{i'}(x))$ for some $i', x \in \mathbb{N}$ with $i' > 0$ and $t_s(i', i') > 0$ such that $F^v_{i'}(x) \notin \overline{Kv}$ by Claim 3.3 and by $|\alpha| \leq |v'|$, we obtain $F^v_{i'}(x) = F^v_{i'}(x) \in \overline{Kv}$. Moreover, by construction, $\alpha > |u| \geq t_{s'}(i', i')$. Hence, under the assumption that $v''$ does not satisfy property (7) or $v''$ does not satisfy property (8), we obtain that $\alpha$ is of the form described by the current claim.

Now we consider the case that $v''$ satisfies properties (7) and (8) and show that this assumption leads to a contradiction. By iteratively applying Claim 3.7.2 we extend $v''$ to a $t_{s'-1}$-valid oracle $v'$ that satisfies $|v''| = |v|$ and properties (7) and (8). As $v''$ and $v'$ agree on all words $< \alpha$ and $\alpha \in v - v''$, it holds $v'' < v$, in contradiction to the choice of $v$ (recall that $v$ is the minimal $t_{s'-1}$-valid oracle $\overline{u}$ that is defined for all words of length $\leq \gamma(n)$ and satisfies properties (7) and (8)).

In order to finish the proof of statement 1, it remains to consider the cases that Claim 3.6.3 or Claim 3.6.4 can be applied to $v'$. This means that either (i) there exist $p' \in \mathbb{P}^{2^3}$ and $\kappa \in \mathbb{N}^+$ with $-p' \in \text{ran}(t_{s'-1}) \subseteq \text{ran}(t_s)$ such that $z = 1^{|u|''}$, or (ii) $\alpha = c(i', x, y)$ for $i', x, y \in \mathbb{N}$ with $0 < t_{s'-1}(i', i') = t_s(i', i') \leq \alpha$. In the latter case, as $\alpha \in v$ and $v$ is $t_{s'-1}$-valid, we obtain from V1 that $F^v_{i'}(x) = y \in \overline{Kv}$.

2. The statement is true in case $w = v$ as $v$ satisfies property (8). Let us argue for the case $w = u_z$ for some $z \in \Sigma^n$. As $-p' \in \text{ran}(t_s)$, $u_z$ is $t_s$-valid, and $u_z$ is defined for all words of length $p'\kappa$, V3 yields that there exists $\beta \in \Sigma^{p'\kappa} \cap u_z$. Let $\beta$ be the minimal element of $\Sigma^{p'\kappa} \cap u_z$. It suffices to show $\beta = 1^{p'\kappa}$. For a contradiction, we assume $\beta < 1^{p'\kappa}$. Let $u'$ be the prefix of $u_z$ that is defined for exactly the words $< \beta$. Then $u \subset u' \subset u_z$ and both $u$ and $u_z$ are $t_s$-valid. Hence, by Claim 3.5, the oracle $u'$ is $t_s$-valid as well.

By Claim 3.6, $u'$ can be extended to a $t_s$-valid oracle $u''$ that satisfies $|u''| = |u_z|$ and
\[ u'' \cap \Sigma^n = \{1^{v''}\} \]. Then \( \beta \in u_z - u'' \). As the oracles \( u'' \) and \( u_z \) agree on all words \( < \beta \), we have \( u'' < u_z \) and \( u'' \cap \Sigma^n = \{z\} \), in contradiction to the choice of \( u_z \) (again, recall that \( u_z \) is the minimal \( t_s \)-valid oracle that is defined for all words of length \( \leq \gamma(n) \) and that satisfies \( u_z \cap \Sigma^n = \{z\} \).

3. This statement follows from the statements 1 and 2.

4. This statement follows from the statements 1 and 2.

This finishes the proof of Claim 3.8.

Let us study the case that both computations \( M^{v}_i(F^{v}_r(0^n)) \) and \( M^{v}_j(F^{v}_r(0^n)) \) reject. Then they even definitely reject as \( v \) is defined for all words of length \( \leq \gamma(n) \). But then \( v \) is not only \( t_{s-1} \)-valid but even \( t \)-valid for \( t = t_{s-1} \cup \{(i, j) \mapsto 0\} \) and then the construction would have chosen \( t_{s'} = t \), in contradiction to \( t_{s}(i, j) = -p < 0 \). Hence one of the computations \( M^{v}_i(F^{v}_r(0^n)) \) and \( M^{v}_j(F^{v}_r(0^n)) \) accepts and thus, even definitely accepts. By symmetry, it suffices to consider the case that \( M^{v}_i(F^{v}_r(0^n)) \) definitely accepts.

Let \( U \) be the set of all those oracle queries of the least accepting path of \( M^{v}_i(F^{v}_r(0^n)) \) that are of length \( \geq n \). Observe \( \ell(U) \leq \gamma(n) \). Moreover, define \( Q_0(U) = U \) and for \( m \in \mathbb{N} \),

\[
Q_{m+1}(U) = \bigcup_{c'(i', x, y) \in Q_m(U)} \left\{ q \in \Sigma^{2^n} \mid q \text{ is queried by } F^{v}_r(x) \right\} \cup \\
\{ q \in \Sigma^{2^n} \mid y = (0^{v''}, 0^{x'}v' + i'', x') \text{ for some } i'' > 0 \text{ and } x' \in M_{v'}(x') \text{ has an accepting path, and } q \text{ is queried by the least such path} \}.
\]

Let \( Q(U) = \bigcup_{m \in \mathbb{N}} Q_m(U) \). Note that all words in \( Q(U) \) have length \( \geq n \).

Claim 3.9 \( \ell(Q(U)) \leq 2\ell(U) \leq 2\gamma(n) \) and the length of each word in \( Q(U) \) is \( \leq \gamma(n) \).

**Proof** We show that for all \( m \in \mathbb{N} \), \( \ell(Q_{m+1}(U)) \leq 1/2 \cdot \ell(Q_m(U)) \). Then \( \sum_{m=0}^{\infty} 1/2^m \leq 2 \) for all \( s \in \mathbb{N} \) implies \( \ell(Q(U)) \leq 2 \cdot \ell(U) \leq 2\gamma(n) \). Moreover, from \( \ell(U) \leq \gamma(n) \) and \( \ell(Q_{m+1}(U)) \leq 1/2 \cdot \ell(Q_m(U)) \) the second part of the claim follows.

Let \( m \in \mathbb{N} \) and consider an arbitrary element \( \alpha \) of \( Q_m(U) \). If \( \alpha \) is not of the form \( c(i', x, y) \) for \( i' \in \mathbb{N}^+ \) and \( x, y \in \mathbb{N} \), then \( \alpha \) generates no elements in \( Q_{m+1}(U) \). Assume \( \alpha = c(i', x, y) \) for \( i' \in \mathbb{N}^+ \) and \( x, y \in \mathbb{N} \) with \( y = (0^{v''}, 0^{x'}v'' + i'', x') \) for \( i'' \in \mathbb{N}^+ \) and \( x' \in \mathbb{N} \). The computation \( F^{v}_r(x) \) runs for at most \( |x|v' + i' \leq |\alpha|/4 \) steps, where “<” holds by (1). Hence, the set of queries \( Q \) of \( F^{v}_r(x) \) satisfies \( \ell(Q) \leq |\alpha|/4 \).

Moreover, the computation \( M^{v}_i(x) \) runs for less than \( |y| < |\alpha|/4 \) steps, where again “<” holds by (1). Hence, for the set \( Q \) of queries of the least accepting path of the computation \( M^{v}_i(x) \) (if such a path exists) we have \( \ell(Q) \leq |\alpha|/4 \).

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which finishes the proof of Claim 3.9. □

For $z$ even we say that $u_z$ and $v$ conflict if there exists $\alpha \in Q(U)$ with $\alpha \in u_z \Delta v$. In that case we say that $u_z$ and $v$ conflict in $\alpha$. As $u_z \supseteq u$ and $v \supseteq u$, either $u_z$ and $v$ conflict in a word of length $\geq n$, or they do not conflict at all.

**Claim 3.10** There exists an even $z \in \Sigma^n$ such that $u_z$ and $v$ do not conflict.

**Proof** Let $z \in \Sigma^n$ be even such that $u_z$ and $v$ conflict in $z$. Let $\alpha \in Q(U)$ be the least word of length $> n$ that $u_z$ and $v$ conflict in. Then $\alpha \in v \Delta u_z$.

We study two cases.

- **Assume $\alpha \in u_z - v$.** By Claim 3.8.3, it holds $\alpha = c(i', x, y)$ for some $i' \in \mathbb{N}^+$ and $x, y \in \mathbb{N}$ with $0 < t_\alpha(i', i') \leq c(i', x, F^{u_z}(x))$ and $F^{u_z}(x) = y \in K^{u_z}$.

  First assume $F^{u_z}_v(x) \neq y$. Then there is one query $q$ of $F^{u_z}_v(x)$ that is in $v \Delta u_z$ (otherwise, $F^{u_z}_v(x)$ and $F^{u_z}_v(x)$ would output the same value). As $v$ and $u_z$ agree on all words of length $< n$, it holds $|q| \geq n$. Hence, by $\alpha \in Q(U)$ and the definition of $Q(U)$, it holds $q \in Q(U)$.

  As $|q| \leq |x|^i + i' < |c(i', x, y)| = |\alpha|$ and $\alpha$ is the least word of length $> n$ in $Q(U)$ that $v$ and $u_z$ conflict in, it holds $|q| = n$. Hence, $v$ and $u_z$ conflict in a word of length $n$.

  Now assume $F^{u_z}_v(x) = y$. As $\alpha \notin v$ and $v$ satisfies property (7), it holds $y \notin K^v$. As $y \in K^v$, $y$ is of the form $(0^i, 0^{x'}^{i''}, x')$ for some $i'' > 0$ and $x' \in \mathbb{N}$. From $y \in K^v$ it follows that the computation $M^{u_z}_v(x')$ has an accepting path and all queries $q$ of length $\geq n$ that are asked on the least such path are in $Q(U)$. However, $y \notin K^{u_z}$ yields that there is some query $q$ on the least accepting path of $M^{u_z}_v(x')$ that is in $v \Delta u_z$ (otherwise, $M^{u_z}_v(x')$ would accept as well). As $v$ and $u_z$ agree on all words of length $< n$, it holds $|q| \geq n$ and by this, $\alpha \in Q(U)$, and the definition of $Q(U)$, it holds $q \in Q(U)$. Since $|q| \leq |x|^i + i'' < |y| < |c(i', x, y)| = |\alpha|$ and $\alpha$ is the least word of length $> n$ in $Q(U)$ that $v$ and $u_z$ conflict in, it holds $|q| = n$. Hence, $v$ and $u_z$ conflict in a word of length $n$.

- **Assume $\alpha \in v - u_z$.** By Claim 3.8.4, it holds $\alpha = c(i', x, F^{u_z}_v(x))$ for some $i' \in \mathbb{N}^+$ and $x \in \mathbb{N}$ with $0 < t_\alpha(i', i') \leq c(i', x, F^{u_z}_v(x))$ and $F^{u_z}_v(x) \in K^{u_z}$ if $F^{u_z}_v(x) = F^{u_z}_v(x)$, then by V5, we have $\alpha \in u_z$, a contradiction. Hence, $F^{u_z}_v(x) \neq F^{u_z}_v(x)$. Then there is one query $q$ of $F^{u_z}_v(x)$ that is in $v \Delta u_z$ (otherwise, $F^{u_z}_v(x)$ and $F^{u_z}_v(x)$ would output the same value). As $v$ and $u_z$ agree on all words of length $< n$, it holds $|q| \geq n$. By this, $\alpha \in Q(U)$, and the definition of $Q(U)$, it holds $q \in Q(U)$. Since $|q| \leq |x|^i + i' < |c(i', x, F^{u_z}_v(x)| = |\alpha|$ and $\alpha$
is the least word of length $> n$ in $Q(U)$ that $v$ and $u_z$ conflict in, it holds $|q| = n$. Hence, $v$ and $u_z$ conflict in a word of length $n$.

In both cases $v$ and $u_z$ conflict in a word of length $n$. As $v \cap \Sigma^n = \emptyset$ and $u_z \cap \Sigma^n = \{z\}$, the oracles $v$ and $u_z$ conflict in $z$ and in particular, $z \in Q(U)$.

From $|Q(U)| \leq \ell(Q(U)) \leq 2\gamma(n)$ (cf. Claim 3.9) we obtain that there are at most $2\gamma(n)$ even words $z \in \Sigma^n$ that $v$ and $u_z$ conflict in. As by (6), it holds $\{|z \in \Sigma^n \mid z \text{ even}|\} = 2^{n-1} > 2\gamma(n)$, the proof of Claim 3.10 is complete.

As guaranteed by Claim 3.10, we can now choose some even $z \in \Sigma^n$ such that $v$ and $u_z$ do not conflict. As all queries of the least accepting path of $M_i^z(F_r^{u_z}(0^n))$ are in $U \subseteq Q(U)$ and $v$ and $u_z$ agree on all these queries, the computation $M_i^z(F_r^{u_z}(0^n))$ accepts. Since $u_z$ is defined for all words of length $\leq \gamma(n)$, the computation even definitely accepts. Note that the computation $M_j^{u_z}(F_r^{u_z}(0^n))$ is defined as well. We study two cases depending on whether this computation accepts or rejects.

- First consider the case that $M_j^{u_z}(F_r^{u_z}(0^n))$ definitely rejects. As $z$ is even, $0^n \in B_p^{u_z}$ and clearly $0^n \in B_p^{u_z}$ for all $w \supseteq u_z$. This, however, contradicts the assumption that step $s$ of the construction treating the task $(i,j,r)$ is not possible.

- Next we consider the case that $M_j^{u_z}(F_r^{u_z}(0^n))$ definitely accepts. Then both $M_i^{u_z}(F_r^{u_z}(0^n))$ and $M_j^{u_z}(F_r^{u_z}(0^n))$ definitely accept. As $u_z$ is $t_{s-1}$-valid by Claim 3.4, we obtain that $u_z$ is even $t$-valid for $t = t_{s-1} \cup \{(i,j) \rightarrow 0\}$. But then the construction would have chosen $t_s = t$, in contradiction to $t_s(i,j) = -p < 0$.

As in both cases we obtain a contradiction, the construction described above is possible. It remains to show that each proof system for $\mathcal{O}$, $\text{NP}^O \cap \text{coNP}^O$ does not have $\leq^{p,O}$-complete problems.

**Claim 3.11** $\overline{\mathcal{O}}$ has $p^O$-optimal proof systems.

**Proof** Let $g \in \text{FP}^O$ be an arbitrary proof system for $\overline{\mathcal{O}}$ and $a$ be an arbitrary element of $\overline{\mathcal{O}}$. Define $f$ to be the following function $\Sigma^* \rightarrow \Sigma^*$:

$$f(z) = \begin{cases} g(z') & \text{if } z = 1z' \\ y & \text{if } z = 0c(i,x,y) \text{ for } i \in \mathbb{N}^+, x,y \in \mathbb{N}, \text{ and } c(i,x,y) \in O \\ a & \text{otherwise} \end{cases}$$

By definition, $f \in \text{FP}^O$ and as $g$ is a proof system for $\overline{\mathcal{O}}$ it holds $f(\Sigma^*) \supseteq \overline{\mathcal{O}}$. We show $f(\Sigma^*) \subseteq \overline{\mathcal{O}}$. Let $z \in \Sigma^*$. Assume $z = 0c(i,x,y)$ for $i \in \mathbb{N}^+$, $x,y \in \mathbb{N}$, and $c(i,x,y) \in O$ (otherwise, clearly $f(z) \in \overline{\mathcal{O}}$). Let $j > 0$ such that $F_j^O$ computes $f$. Let $s$ be large enough such that $w_s$ is defined for $c(i,x,y)$, i.e. $w_s(c(i,x,y)) = 1$. As $w_s$ is $t_s$-valid, we obtain by V1 that $F_i^{w_s}(x) = y \in \overline{\mathcal{O}}$ and by Claim 3.3 that $F_i^{w_s}(x)$ is defined and $y \in \overline{\mathcal{O}}$ for all $\forall v \supseteq w_s$.

Then $F_i^O(x) \in \overline{\mathcal{O}}$. This shows that $f$ is a proof system for $\overline{\mathcal{O}}$.

It remains to show that each proof system for $\overline{\mathcal{O}}$ is $p^{O}$-simulated by $f$. Let $h$ be an arbitrary proof system for $\overline{\mathcal{O}}$. Then there exists $j > 0$ such that $F_j^O$ computes $h$. By construction, $t_s(i,j) > 0$, where $s$ is the number of the step that treats the task $i$. Consider the following function $\pi : \Sigma^* \rightarrow \Sigma^*$:

$$\pi(x) = \begin{cases} 0c(i,x,F_i^O(x)) & \text{if } c(i,x,F_i^O(x)) \geq t_s(i,i) \\ z & \text{if } c(i,x,F_i^O(x)) < t_s(i,i) \text{ and } z \text{ is minimal with } f(z) = F_i^O(x) \end{cases}$$
As \( f \) and \( F_1^O \) are proof systems for \( \overline{K} \), for every \( x \) there exists \( z \) with \( f(z) = F_1^O(x) \). Hence, \( \pi \) is total. Since \( t_s(i, i) \) is a constant, \( \pi \in \text{FP} \subseteq \text{FP}^O \). It remains to show that \( f(\pi(x)) = F_1^O(x) \) for all \( x \in \Sigma^* \). If \( |x| < t_s(i, i) \), it holds \( f(\pi(x)) = F_1^O(x) \). Otherwise, choose \( s' \) large enough such that (i) \( t_s(i, i) \) is defined (i.e., \( t_s(i, i) = t_s(i, i) \)) and (ii) \( w_{s'} \) is defined for \( c(i, x, F_i^{w_{s'}}(x)) \).

Then, as \( w_{s'} \) is \( t_{d'} \)-valid, \( V5 \) yields that \( c(i, x, F_i^{w_{s'}}(x)) \in w_{s'} \). By Claim 3.3, \( F_i^{w_{s'}}(x) \) is defined and hence, \( F_1^O(x) = F_i^{w_{s'}}(x) \) as well as \( c(i, x, F_1^O(x)) \in w_{s'} \subseteq O \). Hence, \( f(\pi(x)) = F_1^O(x) \), which shows \( h = F_1^O \leq_{\text{coNP}} f \). This completes the proof of Claim 3.11.

**Claim 3.12** \( \text{NP}^O \cap \text{coNP}^O \) does not have \( \leq_{\text{m}}^O \)-complete problems.

**Proof** Assume the assertion is wrong, i.e., there exist distinct \( i, j \in \mathbb{N}^+ \) such that \( L(M_i^O), L(M_j^O) \in \text{NP}^O \) with \( L(M_i^O) = L(M_j^O) \) and for every \( A \in \text{NP}^O \cap \text{coNP}^O \) it holds \( A \leq_{\text{m}}^O L(M_i^O) \). From \( L(M_i^O) = L(M_j^O) \) it follows that for all \( s \) there does not exist \( z \) such that both \( M_i^{w_s}(z) \) and \( M_j^{w_s}(z) \) definitely accept or both \( M_i^{w_s}(z) \) and \( M_j^{w_s}(z) \) definitely reject. Hence, for no \( s \) it holds \( t_s(i, j) = 0 \) and thus, by construction \( t_s(i, j) = -p \) for some \( p \in \mathbb{P}^O \) and all sufficiently large \( s \). The latter implies \( |O \cap \Sigma^k| = 1 \) for all \( k > 0 \) (cf. V3), which yields \( A^O_p = B^O_p \), i.e., \( A^O_p \in \text{NP}^O \cap \text{coNP}^O \). Thus, there exists \( r \) such that \( A^O_p \leq_{\text{m}}^O L(M_i^O) \) via \( F_i^O \). Let \( s \) be the step that treats task \( (i, j, r) \). This step makes sure that there exists \( n \in \mathbb{N}^+ \) such that at least one of the following properties holds:

- \( 0^n \in A^O_p \) for all \( v \supseteq w_s \) and \( M_i^{w_s}(F_r^{w_s}(0^n)) \) definitely rejects.
- \( 0^n \in B^O_p \) for all \( v \supseteq w_s \) and \( M_j^{w_s}(F_r^{w_s}(0^n)) \) definitely rejects.

As \( O(q) = w_s(q) \) for all \( q \) that \( w_s \) is defined for, one of the following two statements holds.

- \( 0^n \in A^O_p \) and \( F_r^O(0^n) \notin L(M_i^O) \).
- \( 0^n \in B^O_p = A^O_p \) and \( F_r^O(0^n) \notin L(M_j^O) = L(M_i^O) \).

This is a contradiction to \( A^O_p \leq_{\text{m}}^O L(M_i^O) \) via \( F_i^O \), which completes the proof of Claim 3.12.

This finishes the proof of Theorem 3.2.

**Corollary 3.13** It holds relative to the oracle \( O \) of Theorem 3.2:

- \( \text{NP}^O \cap \text{coNP}^O \) does not have \( \leq_{\text{m}}^O \)-complete problems.
- Each set complete for \( \text{coNP}^O \) has \( \text{P}^O \)-optimal proof systems.

**Proof** This follows from Theorem 3.2 and Corollary 2.4.

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