Congruences modulo 4 for Rogers–Ramanujan–Gordon type overpartitions

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Abstract. In a recent work, Andrews defined the singular overpartitions with the goal of presenting an overpartition analogue to the theorems of Rogers–Ramanujan type for ordinary partitions with restricted successive ranks. As a small part of his work, Andrews noted two congruences modulo 3 for the number of singular overpartitions prescribed by parameters $k = 3$ and $i = 1$. It should be noticed that this number equals the number of the Rogers–Ramanujan–Gordon type overpartitions with $k = i = 3$ which come from the overpartition analogue of Gordon’s Rogers–Ramanujan partition theorem introduced by Chen, Sang and Shi. In this paper, we derive numbers of congruence identities modulo 4 for the number of Rogers–Ramanujan–Gordon type overpartitions.

Keywords: overpartition; congruence; dissection

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1 Introduction

In a recent work, Andrews\textsuperscript{3} introduced the singular overpartitions to present a Rogers–Ramanujan type theorem for overpartitions and denoted $Q_{k,i}(n)$ to be the number of singular overpartitions of $n$ subject to overlining conditions prescribed by $k$ and $i$. As a part of his work, Andrews noted two congruences modulo 3 for $Q_{k,i}(n)$ with $k = 3$ and $i = 1$, which equals the number of overpartitions of $n$ into parts not divisible by 3. This number is also equal to the number $A_{3,3,1}(n)$, where $A_{k,i,1}(n)$ is defined to be the number of Rogers–Ramanujan–Gordon type overpartitions of $n$ with even moduli.

The aim of this paper is to derive some congruences for the number of Rogers–Ramanujan–Gordon type overpartitions modulo 4. We shall prove a number of results by constructing
3-dissection and 4-dissection of the generating function for the Rogers–Ramanujan–Gordon type overpartitions.

Let us give an overview of some definitions. A partition \( \lambda \) of a positive integer \( n \) is a non-increasing sequence of positive integers \( \lambda_1 \geq \cdots \geq \lambda_s > 0 \) such that \( n = \lambda_1 + \cdots + \lambda_s \). The partition of zero is the partition with no parts. An overpartition \( \lambda \) of a positive integer \( n \) is also a non-increasing sequence of positive integers \( \lambda_1 \geq \cdots \geq \lambda_s > 0 \) such that \( n = \lambda_1 + \cdots + \lambda_s \) and the first occurrence of each integer may be overlined. We denote the number of overpartitions of \( n \) by \( \overline{p}(n) \). For example, there are 8 overpartitions of 3:

\[
3; \overline{3}; 2 + 1; 2 + \overline{1}; 2 + \overline{1}; 1 + 1 + 1; 1 + 1 + 1
\]

For a partition or an overpartition \( \lambda \) and for any integer \( l \), let \( f_l(\lambda)(f_l(\lambda)) \) denote the number of occurrences of non-overlined (resp. overlined) \( l \) in \( \lambda \). Let \( V_\lambda(l) \) denote the number of overlined parts in \( \lambda \) that are less than or equal to \( l \). We shall adopt the common notation as used in Andrews [2]. Let

\[
(a)_\infty = (a; q)_\infty = \prod_{i=0}^{\infty} (1 - aq^i),
\]

and

\[
(a)_n = (a; q)_n = \frac{(a)_\infty}{(aq^n)_\infty}.
\]

We also write

\[
(a_1, \ldots, a_k; q)_\infty = (a_1; q)_\infty \cdots (a_k; q)_\infty.
\]

The Rogers–Ramanujan-Gordon theorems for overpartition are motivated by the Rogers–Ramanujan–Gordon type partition. The first combinatorial generalization of the Rogers–Ramanujan identities was derived by Gordon in 1961 [12]. And then, in 1967, Andrews [1] gave the analytic generalization of Rogers–Ramanujan identities, which is also a generating function form of Gordon’s theorem. These two theorems both concern odd moduli. In 1980 Bressoud [4] extended their results to all moduli.

**Theorem 1.1** Given positive integer \( k \), \( j = 0 \) or \( 1 \) and integral \( i \) such that \( 0 < i < (2k + j)/2 \). Let \( A_{k,i,j}(n) \) denote the number of partitions of \( n \) that no part is congruent to 0 or \( \pm i \) modulo \( 2k + j \). Let \( B_{k,i,j}(n) \) denote the number of partitions of \( n \) of the form \( \lambda_1 + \lambda_2 + \cdots + \lambda_s \) such that:

(i) \( f_1(\lambda) \leq i - 1 \);

(ii) \( f_i(\lambda) + f_{i+1}(\lambda) \leq k - 1 \);

(iii) if \( f_i(\lambda) + f_{i+1}(\lambda) = k - 1 \), then \( i f_1(\lambda) + (l + 1) f_{l+1}(\lambda) \equiv i - 1 \pmod{2 - j} \).

Then we have

\[
A_{k,i,j}(n) = B_{k,i,j}(n).
\]

Bressoud [5] also gave the analytic form of this theorem.
Theorem 1.2  Given positive integer $k$, $j = 0$ or $1$ and integral $i$ such that $0 < i < (2k + j)/2$, 

$$
\sum_{N_1 \geq N_2 \geq \cdots \geq N_{k-1} \geq 0} \frac{q^{N_1^2 + N_2^2 + \cdots + N_{k-1}^2 + N_{k+1} + \cdots + N_{k-1}}}{(qN_1 - N_2 \cdots (qN_{k-2} - N_{k-1})(q^{2-j}; q^{2-j})_{N_{k-1}}}} = \frac{(q^i, q^{2k+j-i}, q^{2k+j}; q^{2k+j})_{\infty}}{(q)_{\infty}}. 
$$

(1.2)

One can see that, when $j = 1$ Theorem 1.1 becomes the Gordon’s generalization in [12] and identity (1.2) becomes the analytic form of Andrews obtained in [1].

In 2013, Chen, Sang and Shi [8] found an overpartition analogue of the Rogers–Ramanujan–Gordon theorem, more precisely, an overpartition analogue of Theorem 1.1 in the case $j = 1$ and also the generating function form. In 2015, Chen, Sang and Shi [9] obtained the overpartition analogue of Theorem 1.1 in the case $j = 0$. The generating function form was also derived by Sang and Shi [13]. Here we state these two theorems in a unified form together with its generating function form.

Theorem 1.3  Let $j = 0$ or $1$, and $k$ and $i$ be integers, such that $0 < i < (2k + j)/2$. Let $B_{k,i,j}(n)$ denote the number of overpartitions of $n$ of the form $\lambda_1 + \lambda_2 + \cdots + \lambda_s$, such that

(i) $f_1(\lambda) \leq i - 1$;

(ii) $f_1(\lambda) + f_T(\lambda) + f_{i+1}(\lambda) \leq k - 1$;

(iii) if the equality in Condition (ii) is attained at $l$, i.e., $f_1(\lambda) + f_T(\lambda) + f_{i+1}(\lambda) = k - 1$, then $lf_1(\lambda) + lf_T(\lambda) + (l + 1)f_{i+1}(\lambda) \equiv V_3(l) + i - 1 \pmod{2 - j}$.

For $i \neq k$ nor $j \neq 1$, let $A_{k,i,j}(n)$ denote the number of overpartitions of $n$ whose non-overlined parts are not congruent to 0, ±$i$ modulo $2k - 1 + j$. Let $A_{k,k,1}(n)$ denote the number of overpartitions of $n$ with parts not divisible by $k$. Then for all $n \geq 0$, we have

$$
A_{k,i,j}(n) = B_{k,i,j}(n). 
$$

(1.3)

The generating function form of Theorem 1.3 which is an Andrews–Gordon type identity for overpartitions with all moduli can be stated as follows.

Theorem 1.4  Let $j = 0$ or $1$ and $k$ and $i$ be integers, such that $0 < i < (2k + j)/2$, we have

$$
\sum_{N_1 \geq N_2 \geq \cdots \geq N_{k-1} \geq 0} \frac{q^{(N_1+1)N_1 + N_2^2 + \cdots + N_{k-1}^2 + N_{k+1} + \cdots + N_k}(-q)^{N_1-1}(1 + q^{N_1})}{(q)_{N_1 - N_2} \cdots (q)_{N_{k-2} - N_{k-1}}(q^{2-j}; q^{2-j})_{N_{k-1}}(q^{2k-j}; q^{2k-j})_{N_k}} = \frac{(-q)_{\infty}(q^i, q^{2k-1-i+j}, q^{2k-1+j}; q^{2k-1+j})_{\infty}}{(q)_{\infty}}. 
$$

(1.4)

In this paper, we shall derive a number of congruence properties of $A_{k,i,j}(n)$, which can be called the number of overpartitions of Rogers–Ramanujan–Gordon type. Indeed, two congruence properties for a number equal to $A_{3,3,1}(n)$ have been obtained by Andrews.
In 2015, Andrews \cite{3} defined a new type of overpartitions, called singular overpartition to give the overpartition analogous to Rogers–Ramanujan type theorems for ordinary partitions with restricted successive ranks and denoted $\overline{Q}_{k,i}(n)$ to be the number of singular overpartitions of $n$ subject to overlining conditions prescribed by $k$ and $i$. The generating function of $\overline{Q}_{k,i}(n)$ was derived as follows:

$$\sum_{n=0}^{\infty} \overline{Q}_{k,i}(n)q^n = \frac{(q^{k}, -q^i, -q^{k-i}; q^k)_{\infty}}{(q; q)_{\infty}}.$$ (1.5)

As a small part of his work, Andrews gave the following result:

**Theorem 1.5**

$$\overline{Q}_{3,1}(9n + 3) \equiv \overline{Q}_{3,1}(9n + 3) \equiv 0 \pmod{3}.$$ (1.6)

It should be noticed that

$$\overline{Q}_{3,1}(n) = \overline{B}_{3,3,1}(n) = \overline{A}_{3,3,1}(n),$$

that is to say,

$$\overline{A}_{3,3,1}(9n + 3) \equiv \overline{A}_{3,3,1}(9n + 3) \equiv 0 \pmod{3}.$$ (1.7)

We shall consider more congruence properties of $\overline{A}_{k,i,j}(n)$. To make the computation easier, we define $S_{k,i}(n)$ as follows.

$$S_{2k-1+j,i}(n) = \overline{A}_{k,i,j}(n),$$

then for $1 \leq i \leq k/2$,

$$\sum_{n \geq 0} S_{k,i}(n)q^n = \frac{(-q)_{\infty}(q^i, q^{k-i}, q^k; q^k)_{\infty}}{(q)_{\infty}}.$$ (1.7)

The proofs of the congruences of $S_{k,i}(n)$ involve the following definitions and results. Let us employ the Ramanujan’s general theta function $f(a,b)$ defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^n b^{n(n+1)/2}, \quad |ab| < 1.$$ (1.8)

One special case is, in Ramanujan’s notation,

$$f(-q) = f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}.$$ (1.9)

By using Jacobi’s triple product identity that

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = (-zq; q^2)_{\infty}(-q/z; q^2)_{\infty}(q^2; q^2)_{\infty},$$ (1.10)

it follows

$$f(-q) = (q; q)_{\infty}.$$ (1.11)

Throughout this paper, we use $f_n$ to denote $f(-q^n)$, that is,

$$f_n := f(-q^n) = (q^n; q^n)_{\infty}.$$ (1.11)
As noted by Corteel and Lovejoy [10], the generating function of $\overline{p}(n)$ is given by
\[
\sum_{n=0}^{\infty} \overline{p}(n)q^n = \frac{(-q; q)_{\infty}}{(q)_{\infty}} = \frac{f_2}{f_1^2}.
\] (1.12)

It can be seen that the generating function $S_{k,i}(n)$ can be written in terms of Ramanujan’s theta function as follows.
\[
\sum_{n \geq 0} S_{k,i}(n)q^n = \frac{(-q)_{\infty}(q^i, q^{k-i}, q^k; q^k)_{\infty}}{(q)_{\infty}} = f(-q^i, -q^{k-i}) \frac{f_2}{f_1^2}.
\] (1.13)

In [13], Hirschhorn and Sellers obtained 2-, 3- and 4-dissections of the generating function of $\overline{p}(n)$ and derived a number of congruences for $\overline{p}(n)$ modulo 4, 8 and 64 including $\overline{p}(8n+7) \equiv 0 \mod 64$, for $n \geq 0$.

In this paper we shall use the following 3-dissection and 4-dissection of the generating function of $\overline{p}(n)$ given by Hirschhorn and Seller [13] to compute the congruences of $S_{k,i}(n)$.

**Theorem 1.6** The 3-dissection of the generating function of $\overline{p}(n)$ is
\[
\sum_{n=0}^{\infty} \overline{p}(n)q^n = \frac{f_{19}}{f_{14} f_{16}} = \frac{f_4 f_6 f_8}{f_3 f_9 f_{10}}, + 2q \frac{f_3 f_6 f_9}{f_3} + 4q^2 \frac{f_2 f_6 f_{10}}{f_3}, \] (1.14)

and the 4-dissection is
\[
\sum_{n=0}^{\infty} \overline{p}(n)q^n = \frac{f_8}{f_{14} f_{16}} = \frac{f_7}{f_4} \frac{f_8}{f_{16}} + 2q \frac{f_7}{f_4} \frac{f_8}{f_{16}} + 4q^2 \frac{f_7}{f_4} \frac{f_8}{f_{16}} + 8q^3 \frac{f_8}{f_{16}}. \] (1.15)

Our main goal in this paper is to prove numerous arithmetic relations satisfied by $S_{k,i}(n)$. The techniques we employ are elementary, by involving dissections of $q$-series. In Section 2, we shall compute the 3-dissection and 4-dissection of $f(-q^i, -q^{k-i})$. In Section 3 and 4, we will discuss the congruences of $S_{k,i}(n)$ by considering 3-dissection and 4-dissection of the generating function of $S_{k,i}(n)$.

## 2 The dissections of $f(-q^i, -q^{k-i})$

In this section, by using Jacobi’s triple product identity (1.10) we give the 3-dissection and 4-dissection of $f(-q^i, -q^{k-i})$.

**Lemma 2.1** The 3-dissection of $f(-q^i, -q^{k-i})$ is as follows
\[
f(-q^i, -q^{k-i}) = f(-q^{3i+3k}, -q^{6k-3i}) - q^j (-q^{3i+6k}, -q^{3k-3i}) + q^{2i+k} f(-q^{-3i}, -q^{3i+9k}). \] (2.16)

**Proof.** By definition (1.9),
\[
f(-q^i, -q^{k-i}) = \sum_{n=-\infty}^{\infty} (-1)^n q^{kn(n-1)/2} = \sum_{n=-\infty}^{\infty} (-1)^n q^{kn^2+(2i-k)n}/2,
\]
so that the 3-dissection is 
\[
f(-q^i, -q^{k-i}) = \sum_{n=0}^{\infty} (-1)^n q^{[kn^2+3(2i-k)n]/2} + \sum_{n=0}^{\infty} (-1)^{n+1} q^{[kn^2+(6i+3k)n+2i]/2} 
\]
\[
= \sum_{n=0}^{\infty} (-1)^{n+2} q^{[kn^2+(6i+9k)n+(4i+2k)]/2}. 
\]

By employing Jacobi’s triple identity, we have 
\[
f(-q^i, -q^{k-i}) = \left( q^{3i+3k}, q^{6k-3i}; q^9 \right)_\infty - q^i \left( q^{3i+6k}, q^{3k-3i}; q^9 \right)_\infty 
\]
\[
+ q^{2i+k} \left( q^{-3i}, q^{3i+9k}; q^9 \right)_\infty 
\]
\[
= f(-q^{3i+3k}, -q^{6k-3i}) - q^i f(-q^{3i+6k}, -q^{3k-3i}) + q^{2i+k} f(-q^{-3i}, -q^{3i+9k}). 
\]

Similarly as the 3-dissection of \( f(-q^i, -q^{k-i}) \), the following 4-dissection of \( f(-q^i, -q^{k-i}) \) follows.

**Lemma 2.2** We have
\[
f(-q^i, -q^{k-i}) = f(-q^{6k+4i}, -q^{10k-4i}) - q^i f(-q^{10k+4i}, -q^{6k-4i}) 
\]
\[
+ q^{2i+k} f(-q^{14k+4i}, -q^{2k-4i}) - q^{3i+3k} f(-q^{18k+4i}, -q^{-2k-4i}). \tag{2.17} 
\]

**Proof.**
\[
f(-q^i, -q^{k-i}) = \sum_{n=-\infty}^{\infty} (-1)^n q^{[kn^2+(2i-k)n]/2} 
\]
\[
= \sum_{n=-\infty}^{\infty} q^{8kn^2+2(2i-k)n} - q^i \sum_{n=-\infty}^{\infty} q^{8kn^2+(4i+2k)n} 
\]
\[
+ q^{2i+k} \sum_{n=-\infty}^{\infty} q^{8kn^2+(6k+4i)n} - q^{3i+3k} \sum_{n=-\infty}^{\infty} q^{8kn^2+(10k+4i)n} 
\]
\[
= f(-q^{6k+4i}, -q^{10k-4i}) - q^i f(-q^{10k+4i}, -q^{6k-4i}) 
\]
\[
+ q^{2i+k} f(-q^{14k+4i}, -q^{2k-4i}) - q^{3i+3k} f(-q^{18k+4i}, -q^{-2k-4i}). 
\]

In next section, we shall consider this 3-dissection of the generating function of \( S_{k,i}(n) \) corresponding to the parameters \( k \) and \( i \), and get arithmetic properties of \( S_{k,i}(n) \) by employing the 3-dissection of \( \mathfrak{g}(n) \).

### 3 Arithmetic properties as the consequence of the 3-dissection of the generating function of \( S_{k,i}(n) \)

Combining \((2.16)\) and \((1.14)\), we get the following 3-dissection of the generating function of \( S_{k,i}(n) \):
\[
\sum_{n \geq 0} S_{k,i}(n)q^n = f(-q^i, -q^{i-1}) \frac{f_2}{f_1} \\
= \frac{f(-q^{3i+3k}, -q^{6k-3i}) - q^i f(-q^{3i+6k}, -q^{3k-3i}) + q^{2i+k} f(-q^{-3i}, -q^{3i+9k})}{f_8 f_9}
\]
\[
\times \left( \frac{f_4 f_6}{f_3} + 2q \frac{f_2 f_3}{f_3} + 4q^2 \frac{f_2 f_3^3}{f_3} \right)
\]
\[
= \frac{f(-q^{3i+3k}, -q^{6k-3i}) f_4 f_6}{f_3^3 f_18} - q^i f(-q^{3i+6k}, -q^{3k-3i}) f_4 f_6 f_9 + q^{2i+k} f(-q^{-3i}, -q^{3i+9k}) f_6 f_9 f_3 f_18
\]
\[
+ 2q f(-q^{3i+3k}, -q^{6k-3i}) f_6 f_9 f_3^3 f_18 - 2q^{i+1} f(-q^{3i+6k}, -q^{3k-3i}) f_6 f_9 f_3^3
\]
\[
+ 2q^{2i+k+1} f(-q^{-3i}, -q^{3i+9k}) f_6 f_9 f_3^3 f_18
\]
\[
+ 4q^2 f(-q^{3i+6k}, -q^{3k-3i}) f_6 f_9 f_3^3 f_18 + 4q^{2i+k+2} f(-q^{-3i}, -q^{3i+9k}) f_6 f_9 f_3^3 f_18. \quad (3.18)
\]

3.1 The congruences of \( S_{k,i}(n) \) in the case of \( k \equiv 2 \text{ (mod 3)} \), \( i \equiv 1 \text{ (mod 3)} \)

In the case of \( k \equiv 2 \text{ (mod 3)} \) and \( i \equiv 1 \text{ (mod 3)} \) we rearrange (3.18) to get the following 3-dissection:

\[
\sum_{n \geq 0} S_{k,i}(n)q^n = f(-q^i, -q^{k-i}) \frac{f_2}{f_1}
\]
\[
= \left[ f(-q^{3i+3k}, -q^{6k-3i}) \frac{f_4 f_6}{f_3^3 f_18} - 4q^{i+2} f(-q^{3i+6k}, -q^{3k-3i}) \frac{f_6 f_3}{f_3} + 4q^{2i+k+2} f(-q^{-3i}, -q^{3i+9k}) \frac{f_6 f_3}{f_3} \right]
\]
\[
+ q \left[ -q^{i-1} f(-q^{3i+6k}, -q^{3k-3i}) \frac{f_4 f_6}{f_3^3 f_18} + q^{2i+k-1} f(-q^{-3i}, -q^{3i+9k}) \frac{f_6 f_3}{f_3} + 2f(-q^{3i+3k}, -q^{6k-3i}) \frac{f_6 f_9}{f_3^3} \right]
\]
\[
+ q^2 \left[ -2q^{i-1} f(-q^{3i+6k}, -q^{3k-3i}) \frac{f_6 f_3}{f_3} + 2q^{2i+k-1} f(-q^{-3i}, -q^{3i+9k}) \frac{f_6 f_3}{f_3} + 4f(-q^{3i+3k}, -q^{6k-3i}) \frac{f_6 f_9}{f_3^3} \right].
\]

On the right-hand side of the above identity in each square brackets, the powers of \( q \) are all multiples of 3.

Then, we can get the generating function of \( S_{k,i}(3n) \) as follows

\[
\sum_{n \geq 0} S_{k,i}(3n)q^n = f(-q^{i+k}, -q^{2k-i}) \frac{f_4 f_6}{f_1 f_6} - 4q^{i+k} f(-q^{i+2k}, -q^{k-i}) \frac{f_4 f_6}{f_1 f_6} + 4q^{2i+k+2} f(-q^{-i}, -q^{i+3k}) \frac{f_4 f_6}{f_1 f_6},
\]

which implies that

\[
\sum_{n \geq 0} S_{k,i}(3n)q^n \equiv f(-q^{i+k}, -q^{2k-i}) \frac{f_4 f_6}{f_1 f_6} \text{ (mod 4)}.
\]

Applying (1.14) to \( f_4^4 / f_1^8 \), we have

\[
\frac{f_4^4}{f_1^8} = \left( \frac{f_2}{f_1} \right)^4 \left( \frac{f_4 f_6}{f_3^3 f_18} + 2q \frac{f_2 f_3}{f_3} + 4q^2 \frac{f_2 f_3^3}{f_3} \right)^4 \equiv \left( \frac{f_6 f_9}{f_3^3 f_18} \right)^4 \text{ (mod 4)}. \quad (3.19)
\]
So we have

\[ \sum_{n \geq 0} S_{k,i}(3n)q^n \equiv f(-q^{i+k}, -q^{2k-i}) \left( \frac{f_6 f_9}{f_3 f_2} \right)^4 \frac{f_3}{f_6} \quad (\text{mod } 4). \quad (3.20) \]

It can be checked that all exponents of \( q \) in the right-hand side of (3.20) are multiples of 3. Then one can verify that

\[ S_{k,i}(9n + 3) \equiv S_{k,i}(9n + 6) \equiv 0 (\text{mod } 4), \]

with \( k \equiv 2 \) (mod 3), \( i \equiv 1 \) (mod 3).

### 3.2 \( k \equiv 2 \) (mod 3), \( i \equiv 2 \) (mod 3)

In the case \( k \equiv 2 \) (mod 3) and \( i \equiv 2 \) (mod 3) we rearrange (3.19) to get the following 3-dissection:

\[
\sum_{n \geq 0} S_{k,i}(n)q^n = f(-q^i, -q^{k-i}) \frac{f_2}{f_1} \\
= \left[ f(-q^{3i+3k}, -q^{6k-3i}) \frac{f_3^2 f_3}{f_3^2 f_3} + q^2 f(-q^{3i+3k}, -q^{6k-3i}) \frac{f_3^2 f_3}{f_3^2 f_3} - 2q^i f(-q^{3i+6k}, -q^{3k-3i}) \frac{f_3^2 f_3}{f_3^2 f_3} \right] \\
+ q \left[ 2f(-q^{3i+3k}, -q^{6k-3i}) \frac{f_3^2 f_3}{f_3^2 f_3} + 2q^{2i+k} f(-q^{3i+3k}, -q^{3i-9k}) \frac{f_3^2 f_3}{f_3^2 f_3} - 4q^{i+1} f(-q^{3i+6k}, -q^{3k-3i}) \frac{f_3^2 f_3}{f_3^2 f_3} \right] \\
+ q^2 \left[ -q^{i-2} f(-q^{3i+6k}, -q^{3k-3i}) \frac{f_3^2 f_3}{f_3^2 f_3} + 4f(-q^{3i+3k}, -q^{6k-3i}) \frac{f_3^2 f_3}{f_3^2 f_3} + 4q^{2i+k} f(-q^{3i+3k}, -q^{3i+9k}) \frac{f_3^2 f_3}{f_3^2 f_3} \right].
\]

By this 3-dissection for \( k \equiv 2 \) (mod 3), \( i \equiv 2 \) (mod 3), we can get the generating function of \( S_{k,i}(3n + 2) \).

\[
\sum_{n \geq 0} S_{k,i}(3n + 2)q^n = 4f(-q^{i+k}, -q^{2k-i}) \frac{f_2^2 f_3}{f_1} + 4q^{(2i+k)/3} f(-q^{-1}, -q^{i+3k}) \frac{f_2^2 f_3}{f_1} - q^{(i-2)/3} f(-q^{i+2k}, -q^{k-i}) \frac{f_2^2 f_3}{f_1}.
\]

and the following arithmetic property, that,

\[
\sum_{n \geq 0} S_{k,i}(3n + 2)q^n \equiv -q^{(i-2)/3} f(-q^{i+2k}, -q^{k-i}) \frac{f_2^2 f_3}{f_1 f_2} \quad (\text{mod } 4) \quad (3.22)
\]

As a consequence of (3.19), we have

\[
\sum_{n \geq 0} S_{k,i}(3n + 2)q^n \equiv -q^{(i-2)/3} f(-q^{i+2k}, -q^{k-i}) \left( \frac{f_3^4 f_6^2}{f_3^4 f_2} \right)^4 \frac{f_3^4}{f_3^6} \quad (\text{mod } 4). \quad (3.23)
\]

Then, for \( k \equiv 2 \) (mod 3), we have
1. if $i \equiv 2 \pmod{9}$, then
\[ S_{k,i}(9n + 5) \equiv S_{k,i}(9n + 8) \equiv 0 \pmod{4}; \quad (3.24) \]

2. if $i \equiv 5 \pmod{9}$, then
\[ S_{k,i}(9n + 2) \equiv S_{k,i}(9n + 8) \equiv 0 \pmod{4}; \quad (3.25) \]

3. if $i \equiv 8 \pmod{9}$, then
\[ S_{k,i}(9n + 2) \equiv S_{k,i}(9n + 5) \equiv 0 \pmod{4}. \quad (3.26) \]

3.3 $k \equiv 0 \pmod{3}$, $i \equiv 0 \pmod{3}$

Similar with the above cases we can get the following identity
\[ \sum_{n=0}^{\infty} S_{k,i}(3n+2)q^n = 4\frac{f_2^2 f_6^3}{f_1^6} \bigl[f(-q^{i+k}, -q^{2k-i}) - q^{i/3} f(-q^{i+2k}, -q^{k-i}) + q^{(2i+k)/3} f(-q^{-i}, -q^{i+3k})\bigr]. \]

It is easy to see that, for $k \equiv i \equiv 0 \pmod{3}$, we have
\[ S_{k,i}(3n + 2) \equiv 0 \pmod{4}. \quad (3.27) \]

3.4 $k \equiv 2 \pmod{3}$, $i \equiv 0 \pmod{3}$

In this case we also consider the generating function of $S_{k,i}(3n+2)$
\[ \sum_{n=0}^{\infty} S_{k,i}(3n+2)q^n = 4\bigl[f(-q^{i+k}, -q^{2k-i}) - q^{i/3} f(-q^{i+2k}, -q^{k-i})\bigr] \frac{f_2^2 f_6^3}{f_1^6} + q^{(2i+k-2)/3} f(-q^{-i}, -q^{i+3k}) \frac{f_4^6}{f_1^8 f_3^6}. \]

So we have that
\[ \sum_{n=0}^{\infty} S_{k,i}(3n+2)q^n \equiv q^{(2i+k-2)/3} f(-q^{-i}, -q^{i+3k}) \frac{f_4^6}{f_1^8 f_3^6} \pmod{4} \]
\[ \equiv q^{(2i+k-2)/3} f(-q^{-i}, -q^{i+3k}) \left( \frac{f_4^6}{f_3^8 f_1^6} \right)^4 \frac{f_3^6}{f_1^6} \pmod{4}. \]

Then we can verify that

1. for $k - i \equiv 2 \pmod{9}$,
\[ S_{k,i}(9n + 5) \equiv S_{k,i}(9n + 8) \equiv 0 \pmod{4}; \quad (3.27) \]

2. for $k - i \equiv 5 \pmod{9}$,
\[ S_{k,i}(9n + 2) \equiv S_{k,i}(9n + 8) \equiv 0 \pmod{4}; \quad (3.27) \]

2. for $k - i \equiv 8 \pmod{9}$,
\[ S_{k,i}(9n + 2) \equiv S_{k,i}(9n + 5) \equiv 0 \pmod{4}. \quad (3.27) \]
4 Arithmetic properties as the consequence of the 4-dissection of the generating function of $S_{k,i}(n)$

In this section, we shall prove some arithmetic properties according to the 4-dissection of the generating function of $S_{k,i}(n)$. The 4-dissection of the generating function of $S_{k,i}(n)$ is based on the 4-dissection of $f(-q^k, -q^{k-i})$ given in (2.17) and the 4-dissection of $f_2/f_1^2$ given in (1.15).

Recall that the 4-dissection of $f_2/f_1^2$ is

$$\frac{f_2}{f_1^2} = \frac{f_8^{19}}{f_4^{14}f_{16}^6} + 2q\frac{f_8^{13}}{f_4^8f_{16}^2} + 4q^2\frac{f_8^7f_2^2}{f_4^{16}} + 8q^3\frac{f_8f_6^6}{f_2^2}.$$

Combining (2.17) we have that

$$\sum_{n=0}^{\infty} S_{k,i}(n)q^n \equiv [f(-q^{2k+4i}, -q^{2k-4i}) - q^if(-q^{10k+4i}, -q^{6k-4i})
+ q^{2i+k}f(-q^{14k+4i}, -q^{2k-4i}) - q^{3i+3k}f(-q^{18k+4i}, -q^{-2k-4i})]
\times \left(\frac{f_8^{19}}{f_4^{14}f_{16}^6} + 2q\frac{f_8^{13}}{f_4^8f_{16}^2}\right) \pmod{4}. \quad (4.28)$$

We can get the following results

1. $k \equiv 2 \pmod{4}$, $i \equiv 1 \pmod{4}$, $S_{k,i}(4n + 3) \equiv 0 \pmod{4}; \quad (4.29)$

2. $k \equiv 2 \pmod{4}$, $i \equiv 3 \pmod{4}$, $S_{k,i}(4n + 2) \equiv 0 \pmod{4}; \quad (4.30)$

3. $k \equiv 3 \pmod{4}$, $i \equiv 0 \pmod{4}$, $S_{k,i}(4n + 3) \equiv 0 \pmod{4}; \quad (4.31)$

4. $k \equiv 0 \pmod{4}$, $i \equiv 0 \pmod{4}$, $S_{k,i}(4n + 2) = S_{k,i}(4n + 3) \equiv 0 \pmod{4}. \quad (4.32)$

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