Short Message Noisy Network Coding with Rate Splitting

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Abstract—For noisy network with multiple relays, short message noisy network coding with rate splitting (SNNC-RS) scheme is presented. It has recently been shown by Hou and Kramer that mixed cooperative strategies in which relays in favorable positions decode-and-forward (DF) and the rest quantize via short message noisy network coding (SNNC) can outperform existing cooperative strategies (e.g., noisy network coding (NNC) and decode-and-forward). A drawback of such schemes is that forwarding of quantized signals at SNNC relays introduces interference to the rest of the relays. Our proposed relaying scheme has a capability to reduce such interference and thereby improve the rate performance. In the proposed scheme, superposition coding is incorporated into SNNC encoding to enable partial interference cancellation at DF relays. The achievable rate with proposed scheme is derived for the discrete two-relay network and evaluated in the Gaussian case where gains over the rate performance achievable with rate splitting are demonstrated.

I. INTRODUCTION

General capacity achieving encoding strategies for wireless networks are unknown. Two coding strategies, noisy network coding (NNC) [1], [2] and short message NNC (SNNC) [3] have recently been shown to outperform other cooperative strategies for multicast networks (e.g., decode-and-forward (DF)) [2], [3]. Both NNC and SNNC are based on compression at relays, first used in compress-and-forward (CF) [4]. However, in networks with many relays receiving signals of different strengths, constraining all relays to perform the same cooperative scheme may not be optimal. Relays in favorable positions can perform DF thereby removing the noise otherwise partially propagated via compression based schemes. On the other hand, decoding requirement at a DF relay can severely limit the transmission rate if the links over which the relay is receiving are weak. It has recently been shown by Hou and Kramer that mixed cooperative strategies that allow some relays to perform DF while the rest deploy SNNC, can outperform strategies in which all relays use the same relaying scheme (e.g., DF, CF, NNC or SNNC) [3]. NNC and SNNC can achieve the same transmission rate [3]; the main difference between them is that in SNNC, the source sends independent "short" messages in each block, whereas in NNC, the same message of higher encoding rate is repeatedly sent over many blocks. The lower encoding rate of short messages allows relays which have strong received signals to perform DF, while other relays still deploy SNNC [5], thus enabling the use of mixed strategies.

In this paper, we present an encoding scheme that improves performance achievable with the mixed cooperative strategies analyzed in [3]. To motivate our approach, consider a multihop relay network in which all relays use DF, shown in Fig. 1. A relay decodes a message based on signals received from all "upstream" nodes. At the same time, although all relays simultaneously transmit, "downstream" relays do not cause interference to the upstream relays because the latter ones know the messages sent by the downstream relays and can cancel created interference [6]. The limitation of this scheme is that, if the received signal is weak at even one DF relay, the end-to-end rate will be decreased; relays with weak received signals should compress their signals. However, when some of the relays compress, interference cancellation at DF relays is no longer possible because DF nodes do not know the compressed signals. This can decrease the rate at DF relays and thus the overall performance. To overcome this drawback of mixed cooperative strategies, we present the SNNC with rate splitting scheme that allows DF relays to partially decode interference created by SNNC relays. The idea is to incorporate superposition coding [7] into SNNC encoding. In particular, in the proposed scheme, a relay performing SNNC will, once it determines the compression index it wants to use, split the compression index into two indexes each of a lower rate and use superposition coding to encode them. The lower encoding rate enables DF relays to decode one part of the quantization index and hence cancel part of interference. This in turn increases the end-to-end rate.

The main contributions of this paper are the following:

1) A novel relaying strategy is proposed and analyzed.
2) It is demonstrated that, in a single-relay channel [4], rate splitting does not reduce SNNC rate. Note that in this channel, rate splitting cannot bring gains because there are no multiple relays deploying different schemes.
3) For the general discrete memoryless two-relay network (shown in Fig. 2), an achievable rate of the proposed cooperative strategy is derived.
4) The obtained rate is evaluated for Gaussian two-relay
networks; rate gains of the proposed SNNC-RS coding scheme compared to other coding schemes are demonstrated.

II. MAIN RESULT

We denote random variables with upper case letters and their realizations with the corresponding lower case letters. We drop subscripts of probability distributions if the arguments of the distributions are lower case versions of the random variables. Subscripts of probability distributions if the arguments of the realizations with the corresponding lower case letters. We drop

Consider a discrete memoryless two-relay channel shown in Fig. 2. This is the smallest network that captures gains from our proposed scheme. The source wishes to send messages from the message set $W$ to the destination node $30$. The channel is described by the conditional probability $P(x_1, y_2 | x_1, x_2, x_3)$ where $x_i \in \mathcal{X}_i$, $i = 1, 2, 3$ and $y_j \in \mathcal{Y}_j$, $j = 2, 3, 4$ and $\mathcal{X}_i$ and $\mathcal{Y}_j$ are respective input and output alphabets at nodes $i$ and $j$.

A $(R, n)$ code for the two-relay network consists of the message set $W = \{1, \ldots, 2^nR\}$, encoding functions at the source $X_i^n = f(W)$, and at the relays $X_{k,i} = f_{k,i}(Y_{k-1}^{k-1})$, $k = 2, 3$ and the decoding function $\hat{W} = g(Y_4^n)$. The average error probability of the code is given by $P_e = P[\hat{W} \neq W]$. A rate $R$ is achievable if, for any $\epsilon > 0$, there exists, for a sufficiently large $n$, a code $(R, n)$ such that $P_e < \epsilon$. The capacity is the supremum of all achievable $R$.

We analyze an encoding strategy in which relay 1 deploys DF, and relay 2 deploys SNNC-RS. The code construction, encoding and decoding are specified in the Appendix. Quantization at relay 2 is performed as in SNNC; the difference between SNNC-RS and SNNC schemes is in the addition of superposition coding in SNNC-RS which is used to encode the quantization index at relay 2. Specifically, instead of using a single codebook to encode the quantization index (say at rate $R_3$), relay 2 splits the quantization index into two indexes and encodes them via superposition coding using two codebooks at rates $R_{30}$ and $R_{31}$, such that $R_3 = R_{30} + R_{31}$.

Theorem 1: Rate splitting at the relay does not decrease the SNNC rate in the single-relay channel.

The proof can be found in [8].

We have the following result for the two-relay channel.

Theorem 2: (Joint decoding at relay 1) The achievable rate with coding scheme in which relay 1 performs DF and SNNC-RS and joint decoding at the DF relay in the two-relay channel satisfies:

\[
R < I(X_1; Y_2 | X_2 X_{30})
\]

\[
R < I(\hat{X}; Y_3 Y_4 | X_{30} X_{31})
\]

\[
R < I(\hat{X} X_{30}; X_3 | Y_4) - I(\hat{Y}_4; Y_3 | X_{30} X_{31} Y_4)
\]

\[
R < I(X_{31}; Y_4 | \hat{X} X_{30}) - I(\hat{Y}_3; Y_3 | \hat{X} X_{30} X_{31} Y_4)
\]

\[
+ I(X_1 X_{30}; Y_2 | X_2)
\]

\[
2R < I(\hat{X} X_{31}; Y_4 | X_{30}) - I(\hat{Y}_3; Y_3 | \hat{X} X_{30} X_{31} Y_4)
\]

\[
+ I(X_1 X_{30}; Y_2 | X_2)
\]

for any joint distributions that factors as

\[
P(x_1 x_2) P(x_{30} x_{31}) P(y_3 y_4 | x_{30} x_{31} y_3)\]

and where we introduced the notation $X = (X_1, X_2)$.

Proof: The proof outline is given in the Appendix.

Remark 1: In the special case of no rate-splitting, i.e., for $X_{30} = \emptyset, X_{31} = X_3$, the rate of Thm. 2 is the DF-SNNC rate without rate splitting given by [3, Eq. (56)]. For this case, [4-5] are loose and [1-3] coincide with the achievable rate [8 Eq. (56)] given by

\[
R < I(X_1; Y_2 | X_2)
\]

\[
R < I(X_1 X_2; \hat{Y}_3 Y_4 | X_3)
\]

\[
R < I(X_1 X_2 X_3; Y_4) - I(\hat{Y}_3; Y_3 | X_1 X_2 X_3 Y_4)
\]

for any joint distribution factors that factors as [6].

Remark 2: When condition

\[
I(X_{30}; Y_4 | \hat{X}) < I(X_{30}; Y_2 | \hat{X})
\]

is satisfied, relay 1 can decode the full quantization index sent by relay 2.

Remark 3: When

\[
I(X_3; Y_4) < I(X_3; Y_2 | X_2)
\]

bound [5] is loose compared to the sum of the rates given by [1] and [3].

Remark 4: When

\[
I(\hat{X} X_{30}; Y_4) < I(X_1 X_{30}; Y_2 | X_2)
\]

bound [3] is loose compared to [6].

In the case that the relay 1 uses successive decoding whereby it first decodes the part of the quantization index and then the source message, the corresponding achievable region with mixed DF and SNNC-RS is given by the next theorem.

Theorem 3: (Successive decoding at relay 1) The achievable rate with mixed strategy of DF and SNNC-RS and
sequential decoding at the DF relay in the two-relay channel satisfies:

\[
R < I(X_1; Y_2 | X_2 X_{30}) \\
R < I(X_1; Y_3 | X_30 X_{31}) \\
R < I(X_1 X_{30} X_{31}; Y_4) - I(Y_3; Y_3 | X_30 X_{31} Y_4) \\
R < I(X_1 X_{31}; Y_3 | X_{30} Y_4) - I(Y_3; Y_3 | X_30 X_{31} Y_4) + I(X_{30}; X_2 | Y_2) \\
I(X_{31}; Y_3 | X_{30} X_{31} Y_4) + I(X_{30}; X_2 | Y_2) \\
- I(Y_3; Y_3 | X_{30} X_{31} Y_4) > 0
\]  

(11)

for any joint distribution given by (6).
The proof follows the same steps as the proof for Theorem 2.

We next evaluate the rate given by Theorem 2 for Gaussian channels. We then compare it to the rate achieved with the mixed encoding scheme that does not use rate splitting, as well as to the schemes in which both relays decode-and-forward or perform noisy network coding.

III. GAUSSIAN CHANNEL

We evaluate the obtained rate (1)-(5) in Gaussian channels

\[
Y_2 = h_{12} X_1 + h_{32} X_3 + Z_2 \\
Y_3 = h_{13} X_1 + h_{23} X_2 + Z_3 \\
Y_4 = h_{14} X_1 + h_{24} X_2 + h_{34} X_3 + Z_4
\]  

(12)

where \(Z_i \sim \mathcal{N}(0, 1)\) is additive white Gaussian noise. Channel gain from node \(i\) to node \(j\) is denoted as \(h_{ij}\). We assume average power constraint \(P_i\) at each node \(i = 1, 2, 3\) given by \(E[X_i^2] \leq P_i\). We choose Gaussian inputs \(X_i \sim \mathcal{N}(0, P_i)\). We denote \(C(x) = \log_2(1 + x)\). Rate bounds (1)-(5) evaluate to

\[
R < C \left( \frac{\beta h_{12}^2 P_1}{1 + \hat{\alpha} h_{32}^2 P_3} \right) \\
R < C \left( P_1 (\frac{h_{13}^2}{1 + N_3} + h_{14}^2) + P_2 (\frac{h_{23}^2}{1 + N_3} + h_{24}^2) + 2\sqrt{\beta P_1 P_2 (\frac{h_{13} h_{23}}{1 + N_3} + h_{14} h_{24})} + \frac{\beta P_1 P_2}{1 + N_3} (h_{13} h_{24} - h_{23} h_{14})^2 \right)
\]

Remark 5: As pointed out in the previous section, when condition (13) is met, full decoding of quantization index at relay 1 is possible. This is easily observed in the considered Gaussian channel (12) because (8) evaluates to

\[
h_{32} > h_{34}.
\]  

(14)

Fig. 5 shows the comparison of the proposed scheme with the corresponding DF-SNNC scheme without rate splitting (7), for a network in Fig. 4. Also shown are the rate performance of strategies in which both relays perform DF [6, Sec. IV.C] or NNC, as well as the cut-set bound. Note that, for this network, condition (14) (and hence (8)) is not always satisfied. The quantization noise variance \(N_3\) is optimized numerically. We observe that the proposed scheme reduces the gap to the cut-set bound and outperforms the other strategies for the considered topology. The increasing gap between the proposed scheme and DF-SNNC without rate splitting illustrates the gain of the
former: without rate-splitting, the mixed strategy is limited by the rate to the DF relay that suffers interference from the NNC relay.

IV. DISCUSSION AND FUTURE WORK

We presented a new relaying strategy that combines SNNC with superposition coding. In the proposed encoding scheme, superposition coding is used to encode the quantization index sent by a SNNC relay, in order to facilitate partial decoding of quantization index at relays performing DF. We demonstrated that this relaying strategy can bring rate gains compared to NNC, DF, as well as mixed strategies in which some of the nodes use DF and others SNNC. Analysis of the proposed SNNC-RS encoding scheme in larger networks where multiple nodes can benefit from this scheme and therefore further increase the achievable rate, is the topic of our future work.

V. ACKNOWLEDGMENT

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VI. APPENDIX: PROOF OF THEOREM 2

Proof:

Fix a distribution \(p(x_1, x_2)p(x_3)p(y_3|x_3, y_3)\). The source message \(w\) containing \(nBR\) bits is split into messages \(w_1, w_2, \ldots, w_B\) of \(nR\) bits each. Transmission is performed over \(B + 2\) blocks. The last block is used to transmit index \(u_B\) from node 3 to the destination. Table I shows the encoding at the nodes for \(B = 3\) blocks.

**Codebook generation:** For each block \(b = 1, \ldots, B + 1\) generate \(2^{nR}\) codewords \(x_{2b}(w_b)\) \(w_b = 1, \ldots, 2^{nR}\) according to \(\prod_{i=1}^{n} P_{X_i}(x_{2b})\). For each \(x_{2b}(w_b)\), generate \(2^{nR}\) codewords \(x_{1b}(w_b, w'_b)\), \(w'_b = 1, \ldots, 2^{nR}\) according to \(P_{X_1}X_2\). Generate \(2^{nR_{30}}\) codewords \(x_{3b}(w_{0b})\), \(u_{0b} = 1, \ldots, 2^{nR_{30}}\) according to \(\prod_{i=1}^{n} P_{X_{30}}(x_{3b})\). For each \(x_{3b}(w_{0b})\), generate \(2^{nR_{31}}\) codewords \(x_{3b}(u_{1b}, w_{0b})\), \(u_{1b} = 1, \ldots, 2^{nR_{31}}\) according to \(\prod_{i=1}^{n} P_{X_{31}}(x_{3b}|x_{3b})\). We denote \(u_b = (u_{0b}, u_{1b})\) and \(R_3 = R_{30} + R_{31}\).

**Encoding:** The source: In each block \(b = 1, \ldots, B + 1\), the source transmits \(x_{1b}(w_b, w_{0b})\). By convention, we set \(w_0 = 0, w_{B+1} = 1\).

**Relay 1:** At the end of block \(b\), relay 1 decodes \((w_{b}, u_{b})\) using sliding window decoding, as follows. At the end of block \(b = 2, \ldots, B + 1\), relay 1 determines \((\hat{w}_b, \hat{u}_0(b-1))\) s.t.

\[
(x_{1b}(w_b-1, \hat{u}_b), x_{2b}(w_b-1), x_{3b}(u_0(b-1)), y_{2b}) \in T^r_e(P_{X_1X_2X_3})
\]

where \(u_{00} = 1\) by convention and assuming that, in the previous block, relay 1 had correctly decoded \(u_{b-1}\), i.e., \(\hat{w}_{b-1} = w_{b-1}\). Note that relay 1 knows \(w_0 = 0\) which allows relay 1 to start the sliding window decoding.

**Destination:** Destination uses backward decoding \([6]\) as follows.

At the end of block \(B + 2\), node 4 has reliably decoded \((u_{0(B+1)}, u_{1(B+1)})\) that was transmitted in the block \(B + 2\) from node 3. At the end of block \(B + 1\), node 4 tries to find \((\hat{w}_B, \hat{u}_0(B), \hat{u}_1(B))\) such that

\[
(x_{1(B+1)}(\hat{w}_B, 1), x_{2(B+1)}(\hat{w}_B),
\]

\[
\hat{y}_{3(B+1)}(u_{1(B+1)}, u_{0(B+1)}; \hat{u}_1(B), \hat{u}_0(B)), x_{3(B+1)}(\hat{u}_0(B), \hat{u}_1(B)), y_{4(B+1)}) \in T^r_e(P_{X_1X_2X_{30}X_{31}Y_3})
\]

For blocks \(b = B, B - 1, \ldots, 2\), node 4 tries to find \((\hat{w}_{b-1}, \hat{u}_0(b-1), \hat{u}_1(b-1))\) such that

\[
(x_{1b}(\hat{w}_{b-1}, w_b), x_{2b}(\hat{w}_{b-1}), y_{3b}(u_{1b}, u_{0b}; \hat{u}_1(b-1), \hat{u}_0(b-1)),
\]

\[
(x_{3b}(\hat{u}_0(b-1)), x_{3b}(\hat{u}_0(b-1), \hat{u}_1(b-1)), y_{4b}) \in T^r_e(P_{X_1X_2X_{30}X_{31}Y_3})
\]

where \((w_b, u_{0b}, u_{1b})\) have already been reliably decoded in the previous block \(b + 1\).

**Error Probability:**

Assume without loss of generality that \(w_b = 1\) and \(u_{0b} = 1, u_{1b} = 1\). We denote the event defined by \((I)\) with \(E_{ib}(w_{b-1}, w_b, u_{0b})\).

The decoder at relay 1 makes an error if one of the following error events occur:

\[
E_{r1} = E_{ib}(1, 1, 1)
\]

\[
E_{r2} = \bigcup_{u_{0b} \neq 1} E_{ib}(1, w_b, 1)
\]

\[
E_{r3} = \bigcup_{u_{0b}, u_{1b} \neq (1, 1)} E_{ib}(1, w_b, 1)
\]

Note that the event

\[
E = \bigcup_{u_{0b}, u_{1b} \neq (1, 1)} E_{ib}(1, 1, u_{0b})
\]

is not an error event because relay 1 does not need to reliably decode \(u_b\).

The error event at relay 1 is given by \(E_r = \bigcup_{i=1}^{3} E_{ri}\). By the union bound, the probability of error at relay 1 is given by

\[
E_r \leq \sum_{i=1}^{3} P(E_{ri})
\]
At the destination node, the decoder decodes three indexes as specified by (17), resulting in eight possible error events that can be defined similarly to the case of node 2.

Finally, quantization at relay 2, (15), demands that

\[ R_{30} + R_{33} > I(\hat{Y}_3; Y_3 | X_3). \]  

(21)

To guarantee that the decoding error at two receivers becomes small as \( n \) gets large, by using the standard procedure to bound the probability of error events [7], it can be shown that the following rate bounds need to be satisfied:

\[ R < I(X_1; Y_2 | X_2 X_3) \]  

(22)

\[ R < I(\bar{X}; \bar{Y}_3 Y_4 | X_3) \]  

(23)

\[ R_{30} + R_{33} < I(\hat{Y}_3; \bar{X} Y_4 | X_3) + I(X_3; Y_4 | \bar{X}) \]  

(24)

\[ R + R_{30} + R_{31} < I(\hat{Y}_3; \bar{X} Y_4 | X_3) + I(\bar{X} X_3; Y_4) \]  

(25)

\[ R_{31} < I(\hat{Y}_3; X Y_4 | X_3) + I(X_{31}; Y_4 | X X_3) \]  

(26)

\[ R + R_{31} < I(\hat{Y}_3; X Y_4 | X_3) + I(\bar{X} X_{31}; Y_4 | X X_3) \]  

(27)

\[ R + R_{30} < I(X_1 X_{30}; Y_2 | X_2) \]  

(28)

\[ R_{30} + R_{33} > I(\hat{Y}_3; Y_3 | X_3) \]  

(29)

for the joint distribution given by (6), where (29) is added due to (21).

Note that we cannot ignore error events in which message is correctly decoded, but one or both quantization indexes are not (which would allow to omit (24) and (26)). This is because correct quantization indexes are needed for backward decoding. On the other hand, we can ignore an error event at relay 1 in which only the quantization index part (but not the message) is decoded in error.

We perform Fourier-Motzkin elimination of \( R_{30} \) to obtain

\[ R < I(X_1; Y_2 | X_2 X_3) \]  

R \[ R < I(\bar{X}; \bar{Y}_3 Y_4 | X_3) \]  

R \[ R < I(\bar{X} X_3; Y_4) - I(\hat{Y}_3; Y_3 | \bar{X} X_3 Y_4) \]  

R \[ R_{31} < I(\hat{Y}_3; X Y_4 | X_3) + I(X_{31}; Y_4 | \bar{X} X_3) \]  

R + R_{31} < I(\hat{Y}_3; X Y_4 | X_3) + I(\bar{X} X_{31}; Y_4 | X X_3) \]  

R - R_{31} < I(X_1 X_{30}; Y_2 | X_2) - I(\hat{Y}_3; Y_3 | X_3) \]  

I(\hat{Y}_3; Y_3 | \bar{X} X_{30} X_{31} Y_4) < I(X_{30} X_{31}; Y_4 | \bar{X}) \]  

(30)

where we omitted loose inequalities.

Finally, by performing Fourier-Motzkin elimination to eliminate \( R_{31} \), we obtain

\[ R < I(X_1; Y_2 | X_2 X_3) \]  

(31)

\[ R < I(\bar{X}; \bar{Y}_3 Y_4 | X_3) \]  

(32)

\[ R < I(\bar{X} X_{30} X_{31}; Y_4) - I(\hat{Y}_3; Y_3 | \bar{X} X_{30} X_{31} Y_4) \]  

(33)

\[ R < I(X_{31}; Y_4 | \bar{X} X_3) - I(\hat{Y}_3; Y_3 | \bar{X} X_{30} X_{31} Y_4) \]  

(34)

\[ 2R < I(\bar{X} X_{30} | Y_3 X_{30}) - I(\hat{Y}_3; Y_3 | \bar{X} X_{30} X_{31} Y_4) \]  

(35)

\[ I(\hat{Y}_3; Y_3 | X_{30} X_{31} Y_4) < I(X_{30} X_{31}; Y_4 | \bar{X}) \]  

(36)

Condition (36) is equivalent to the condition in the single-relay channel given in [5] Eq. 11. When condition (36) is not satisfied, the bound reduces to

\[ R < I(X_1; Y_2 | X_2 X_3) \]  

(37)

\[ R < I(\bar{X}; Y_4) \]  

(38)

which can be achieved by treating the signals from the relay 2 as noise, similarly as in [9]. Therefore, condition (36) can be omitted and we obtain rate bounds given by Thm. 2.

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