CHARACTERISING APPROXIMABLE ALGEBRAS.

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Abstract. In [2], Huayi Chen introduces the notion of an approximable graded algebra, which he uses to prove a Fujita-type theorem in the arithmetic setting, and asked if any such algebra is the graded ring of a big line bundle on a projective variety. This was proved to be false in [8]. Continuing the analysis started in [8], we here show that whilst not every approximable graded algebra is a sub algebra of the graded ring of a big line bundle on a projective variety, it is the case that to any approximable graded algebra $B$ we can associate a projective divisor $X(B)$ and an infinite divisor $D(B) = \sum_{i=1}^{\infty} a_i D_i$ with $a_i \to 0$ such that $B$ is included in

$$R(D(B)) = \oplus_n H^0(X(B), nD(B)).$$

We also establish a partial converse to these results by showing that if the infinite divisor $D = \sum_i a_i D_i$ converges in the space of numerical classes then any full-dimensional sub-graded algebra of $\oplus_m H^0(X, |mD|)$ is approximable.

1. Introduction

The Fujita approximation theorem, [4], is an important result in algebraic geometry. It states that whilst the section ring associated to a big line bundle $L$ on an algebraic variety $X$

$$R(L) \overset{\text{def}}{=} \oplus_m H^0(mL, X)$$

is typically not a finitely generated algebra, it can be approximated arbitrarily well by finitely generated algebras. More precisely, we have that

Theorem 1.1 (Fujita). Let $X$ be an algebraic variety and let $L$ be a big line bundle on $X$. For any $\epsilon > 0$ there exists a birational modification

$$\pi : \hat{X} \to X$$

and a decomposition of $\mathbb{Q}$ divisors, $\pi^*(L) = A + E$ such that

- $A$ is ample and $E$ is effective,
- $\text{vol}(A) \geq (1 - \epsilon) \text{vol}(L)$.

In [7], Lazarsfeld and Mustata used the Newton-Okounkov body associated to $A$ to give a simple proof of Fujita approximation. The
Newton-Okounkov body, constructed in \cite{6} and \cite{7}, building on previous work of Okounkov \cite{9}, is a convex body $\Delta_{Y^*}(L, X)$ in $\mathbb{R}^d$ associated to the data of

- a $d$-dimensional variety $X$
- an admissible flag $Y^*$ on $X$
- a big line bundle $L$ on $X$.

This convex body encodes information on the asymptotic behaviour of the spaces of global sections $H^0(nL)$ for large values of $L$.

Lazarsfeld and Mustata’s simple proof of Fujita approximation is based on the equality of volumes of Newton-Okounkov bodies

\[
\text{vol}(L) = d! \text{vol}(\Delta_{Y^*}(L, X))
\]

where we recall that the volume of a big line bundle on a $d$-dimensional variety is defined by

\[
\text{vol}(L) = \lim_{n \to \infty} \frac{d! h^0(nL)}{n^d}.
\]

One advantage of their approach to the Fujita theorem is that Newton-Okounkov bodies are not only defined for section algebras $R(L)$, but also for any graded sub-algebra of section algebras. Lazarsfeld and Mustata give combinatorical conditions (conditions 2.3-2.5 of \cite{7}) under which equation (1) holds for a graded sub-algebra $B = \bigoplus B_m \subset R(L)$ and show that these conditions hold if the graded subalgebra $B$ contains an ample series.

Di Biagio and Pacenzia in \cite{3} subsequently used Newton-Okounkov bodies associated to restricted algebras to prove a Fujita approximation theorem for restricted linear series, i.e. subalgebras of $\bigoplus H^0(mL|_V, V)$ obtained as the restriction of the complete algebra $\bigoplus H^0(mL, X)$, where $V \subset X$ is a subvariety.

In \cite{2}, Huayi Chen uses Lazarsfeld and Mustata’s work on Fujita approximation to prove a Fujita-type approximation theorem in the arithmetic setting. In the course of this work he defines the notion of approximable graded algebras, which are exactly those algebras for which a Fujita-type approximation theorem hold.

**Definition 1.** An integral graded algebra $B = \bigoplus B_m$ with $B_0 = k$ a field is approximable if and only if the following conditions are satisfied.

1. all the graded pieces $B_m$ are finite dimensional over $k$.
2. for all sufficiently large $m$ the space $B_m$ is non-empty.

\[\text{Ie. if there exists an ample divisor } A \leq L \text{ such that } \bigoplus H^0([mA]) \subset B\]
(3) for any $\epsilon$ there exists a $p_0$ such that for all $p \geq p_0$ we have that
\[
\liminf_{n \to \infty} \frac{\dim(\text{Im}(S^n B_p \to B_{np}))}{\dim(B_{np})} > (1 - \epsilon).
\]

In his paper [2], Chen asks whether any graded approximable algebra is in fact a subalgebra of the algebra of sections of a big line bundle. A counter-example was given to this in [3], where a counter example is constructed in which the graded approximable algebra is equal to the section ring of an infinite divisor\(^2\). This begs the question: is any approximable algebra a subalgebra of the section ring of an infinite divisor?

In the current paper we will prove that the answer is yes by establishing the following theorem.

**Theorem 1.2.** Let $B = \oplus_m B_m$ be a graded approximable algebra whose first graded piece $B_0$ is an algebraically closed field of characteristic zero. There is then a projective variety $X(B)$ and an infinite divisor $D(B) = \sum_{i=1}^{\infty} a_i D_i$ such that $a_i \to 0$ and there is a natural inclusion of graded algebras
\[
B \hookrightarrow \oplus_m H^0(X(B), mD(B)).
\]

Furthermore, in the other direction we prove the following.

**Theorem 1.3.** Let $X$ be a complex algebraic variety and let $D = \sum a_i D_i$ be an infinite Weil divisor on $X$ such that the sum of divisor classes $\sum_i a_i[D_i]$ converges to a finite real big cohomology class. Any graded subalgebra of $\oplus_m H^0(mD)$ such that
\[
\left(\frac{\text{rk}(B_m)}{m^{d(C)}}\right)
\]
does not converge to zero is then an approximable algebra.

2. **Notation and two preliminary results.**

In this section we will fix some notation and recall an essential preliminary lemma from [2].

Throughout this article, $k$ will be an algebraically closed field of characteristic zero. $B = \oplus_m B_m$ will be a graded approximable algebra such that $B_0 = k$: we will say that $B$ is a graded approximable algebra over $k$. For any natural numbers $k$ and $n$ we will denote by $\text{Sym}^n(B_k)$ the $n$-th symmetric power of the vector space $B_k$ and by $S^n(B_k)$ the image of $\text{Sym}^n(B_k)$ in $B_{nk}$.\(^2\)

\(^2\)Infinite in this context meaning an infinite sum of Weil divisors with real coefficients $\sum_i a_i D_i$. 
For any $k$ we will denote by $\langle B_k \rangle$ the subalgebra $\oplus_n S^n(B_k) \subset B$ and by $B_k$ the subalgebra $\oplus_n B_{nk}$.

We now recall a result from Chen on approximable algebras which will be necessary in what follows.

**Lemma 1.** [Chen, Proposition 2.4] Let $B = \oplus_{m \geq 0} B_m$ be an integral graded algebra which is approximable. There then exists a constant $a \in \mathbb{N}^*$ such that, for any sufficiently large integer $p$, the algebra $\langle B_p \rangle$ has Krull dimension $a$. Furthermore, let us denote by denoted by $d(B)$ the number $a - 1$. The sequence $v_n$ defined by

$$v_n = \left( \frac{\text{rk} B_n}{n^{d(B)/d(B)}} \right)_{n \geq 1}$$

then converges in $\mathbb{R}_+$.

Naturally, the number $d(B)$ represent the dimension of the algebra $B$ and the limit of the sequence in equation (2) is its volume.

**Definition 2.** Let $B = \oplus_{m \geq 0} B_m$ be an integral graded algebra over a field $k$ which is approximable. We then define the **dimension** of $B$ to be the number $d(B)$ whose existence is guaranteed by Lemma 1. Furthermore, we define the **volume** of $B$, denoted $\text{vol}(B)$ to be the limit

$$\text{vol}(B) = \lim_{n \to \infty} \left( \frac{\text{rk} B_n}{n^{d(B)/d(B)}} \right)$$

We will say that a graded, not a priori approximable, algebra $\oplus_m B_m$ is of dimension $d$ and has volume $v$ if the sequence

$$\lim_{n \to \infty} \left( \frac{\text{rk} B_n}{n^{d(B)/d(B)}} \right)$$

converges to the real number $v$.

Note that the condition (3) in the definition of an approximable algebra tells us that

$$\lim_{p \to \infty} (\text{vol}(\langle B_p \rangle)) = \text{vol}(B).$$

The following lemma from Chen (Corollary 2.5) will also be useful.

**Lemma 2.** Let $B$ be a graded approximable algebra. We then have that for any $r \in \mathbb{N}$

$$\lim_{n \to \infty} \frac{\text{rk}(B_{n+r})}{\text{rk}(B_n)} = 1.$$
3.1. **Construction of** $X(B)$. We will define the variety $X$ using the homogeneous field of fractions of $B = \oplus_mB_m$, which we now define.

**Definition 3.** Let $B = \oplus_mB_m$ be a graded algebra such that $B_0 = k$. Then we define its homogeneous fraction field by

$$K_{\text{hom}}(B) = \left\{ \frac{b_1}{b_2} | \exists m \text{ such that } b_1, b_2 \in B_m, b_2 \neq 0 \right\} / \sim$$

where $\sim$ is the equivalence relation

$$\frac{b_1}{b_2} \sim \frac{c_1}{c_2} \iff b_1c_2 = c_1b_2.$$

Note that $k$ is included in $K_{\text{hom}}(B)$ via the map $\lambda \to \lambda f$ for any $f \in B_m$.

Choose $m$ large enough that $B_m$ and $B_{m+1}$ are both non-trivial. Choose $f_1 \in B_n$ and $f_2 \in B_{n+1}$. For any $m$ we can then identify $B_m$ with a subspace of $K_{\text{hom}}(B)$ via the identification

$$b_m \to \frac{b_m f_1^m}{f_2^m}.$$

Throughout what follows, we will consider the space $B_m$ as a subvector space in $K_{\text{hom}}(B)$.

The idea of our definition is that $X(B)$ will be an algebraic variety whose function field will be $K_{\text{hom}}(B)$. Before being able to pose this definition, we will need to show that $K_{\text{hom}}(B)$ is finitely generated as a field extension of $k$.

**Proposition 1.** Let $B$ be an approximable graded algebra over an algebraically closed field $k$ of characteristic zero. The field $K_{\text{hom}}(B)$ is then a finitely generated field over $k$, whose transcendence degree is equal to the dimension $d(B)$.

**Proof of Proposition 1.** Suppose that $B$ is an approximable graded algebra, and let $p_0$ be such that for any $p > p_0$ we have that

$$\liminf_n \frac{\dim(S^nB_p)}{\dim(B_{np})} > \frac{2}{3}$$

We claim that $K_{\text{hom}}(B)$ is then generated as a field by $B_p \subset K_{\text{hom}}(B)$.

Indeed, consider an arbitrary element

$$\frac{b_1}{b_2} \in K_{\text{hom}}(B)$$
with $b_1, b_2 \in B_m$ for some sufficiently large $m$. After multiplication by an element of the form $f/f$ we may assume that $m = kp$ is divisible by $p$. For $n$ large enough we have that
\[
\dim (b_1 \cdot S^n(B_p)) > \left( \frac{2}{3} \dim(B_{np}) \right) > \left( \frac{2}{3} - \epsilon \right) \dim(B_{np+m})
\]
where the last equality follows from Lemma \[2\]. Similarly, we have that
\[
\dim (\text{Sym}^{n+k}(B_p)) > \left( \frac{2}{3} \dim(B_{np+m}) \right)
\]
and hence the space
\[
S^{n+k}(B_p) \cap b_1 \cdot S^n(B_p)
\]
is of strictly positive dimension. Take a non-zero element $b_3$ of this space: we then have that
\[
b_3 = b_1 P_1 = P_2
\]
for some elements $P_1 \in S^n(B_p)$ and $P_2 \in S^{n+k}(B_p)$. Similarly, there are elements $Q_1 \in S^n(B_p)$ and $Q_2 \in S^{n+k}(B_p)$ such that $b_2 Q_1 = Q_2$. But it then follows that
\[
\frac{b_1}{b_2} = \frac{P_2 Q_1}{P_1 Q_2}
\]
in $K^{\text{hom}}(B)$, and since $P_1, P_2, Q_1, Q_2$ are all generated by $B_p$ this completes the proof of the first part of Proposition \[1\].

Indeed, the above proof establishes not only that $K^{\text{hom}}(B)$ is finitely generated, but moreover that it is equal to $K^{\text{hom}}(\langle B_p \rangle)$ for any sufficiently large $p$.

It remains to show that the transcendence degree of $K^{\text{hom}}(B)$ is equal to the dimension of $B$ as an approximable algebra. Since the algebra $\langle B_p \rangle$ is a finitely generated algebra over $k$, taking $p$ large enough if necessary we have by \[3\], Theorem A (p. 223), that
\[
\text{trdeg}_k K(\langle B_p \rangle) = a
\]
Note that the above field is the total field of fractions of $\langle B_p \rangle$, not just the homogeneous part. We have that
\[
K(\langle B_p \rangle) = K^{\text{hom}}(\langle B_p \rangle)(f)
\]
for any function $f \in K^{\text{hom}}(\langle B_p \rangle)$ of the form
\[
f = \frac{f_2}{f_1}
\]
with $f_1 \in B_n$ and $f_2 \in B_{n+1}$. Moreover, $f$ is transcendent over $K^{\text{hom}}(\langle B_p \rangle)$ by degree considerations, so it follows that
\[
\text{trdeg}_k K^{\text{hom}}(\langle B_p \rangle) = a - 1 = d(B).
\]
This completes the proof of Proposition 1.

Now, we are ready to define the variety \( X(B) \).

**Definition 4.** The variety \( X(B) \) is a smooth projective \( k \)-variety such that the function field \( K(X(B)) = K^{\text{hom}}(B) \).

**Remark 1.** The variety \( X(B) \) is here defined only up to birational equivalence. It can of course be chosen smooth by Hironaka’s resolution of singularities.

**Remark 2.** Since the dimension of any algebraic variety \( X \) is equal to the transcendence degree of its function field, we have that \( d(B) = \dim(X) \).

**Remark 3.** It follows from the proof of Proposition 1 that \( K(X(B)) = K(\langle B_p \rangle) \), and in particular, the map defined on \( X(B) \) by the linear series \( B_p \) is birational onto its image.

**Definition 5.** For any \( b_m \in B_m \) we denote by \( (b_m)_X \) the principal divisor on \( X(B) \) cut out by the rational function \( b_m \). It negative part will be denoted by \( (b_m)_X^- \) and its positive part will be denoted by \( (b_m)_X^+ \), so that
\[
(b_m)_X = (b_m)^+_X - (b_m)^-_X.
\]

### 3.2. Construction of \( D_m \)

The divisor \( D(B) \) will be constructed as the limit of the sequence of divisors \( D_m/m \), where the divisors \( D_m \) are constructed as poles of the rational functions \( b_m \in B_m \).

**Definition 6.** For any \( m \) such that \( B_m \) is non-empty we define the effective divisor \( D_m \) on \( X(B) \) by
\[
D_m = \sup_{b_m \in B_m} \left( (b_m)_X^- \right).
\]

where the supremum is taken with respect to the natural partial order on \( \text{Weil}(X(B)) \).

We note that for any \( b_m, b'_m \in B_m \) and for generic \( \lambda \in k \) we have
\[
(b_m + \lambda b'_m)_X^- = \sup \left( (b_m)_X^-, (b'_m)_X^- \right)
\]
so this supremum is actually a maximum. It follows that \( D_m \) is indeed a finite divisor.

Another possible characterisation of \( D_m \) is the following

**Definition 7.** \( D_m \) is the smallest divisor on \( X(B) \) with the property that \( D_m + (b_m) \geq 0 \) for all \( b_m \in B_m \).

The construction of \( D \) as the limit of the normalised divisors \( D_m/m \) will depend on the following lemma.
**Lemma 3.** Let $m_1$ and $m_2$ be two natural numbers, and let the divisors $D_m$ be as constructed above. If $m_1 | m_2$ then $\frac{D_{m_1}}{m_1} \leq \frac{D_{m_2}}{m_2}$

**Proof of Lemma 3**

We have that $D_{m_1} = (b_{m_1})_X$ for some $b_{m_1} \in B_{m_1}$. Set $m_2 = rm_1$. We then have that

$$m_2D_{m_1} = rm_1(b_{m_1})_X = m_1(b_{m_1}')_X \leq m_1D_{m_2}.$$  

This completes the proof of Lemma 3.

In the next section we will recall some technical tools - multivaluations and Newton-Okounkov bodies - that will be useful in the construction of $D(\mathcal{B})$.

4. Multivaluations and Newton-Okounkov bodies.

The proof of the various properties of the divisor $D(\mathcal{B})$ will depend on the use of multivaluations on the function field of $X(\mathcal{B})$ and Newton-Okounkov constructions in order to estimate volumes of algebras using convex bodies in $\mathbb{R}^d$ associated to admissible flags.

4.1. Multivaluations. We consider a $k$-variety $X$ of dimension $d$ equipped with an admissible flag. By an admissible flag on $X$ we mean data of a flag of varieties

$$\hat{X} = Y_0 \supset Y_1 \supset Y_2 \supset Y_d$$

such that

1. $\hat{X} \to X$ is a birational modification.
2. Each of the varieties $Y_i$ is reduced, irreducible and of dimension $d - i$,
3. Each of the varieties $Y_i$ is smooth in a neighbourhood of the point $Y_d$.

To any such flag we can associate a multivaluation on $K(X)$, i.e., a map

$$\nu_{\mathcal{Y}} : K(X) \setminus 0 \to \mathbb{Z}^d$$

such that

1. $\nu_{\mathcal{Y}}(k) = 0$.
2. For any $g_1, g_2 \in K(X)$ we have that $\nu_{\mathcal{Y}}(g_1g_2) = \nu_{\mathcal{Y}}(g_1) + \nu_{\mathcal{Y}}(g_2)$,
3. For any $g_1, g_2 \in K(X)$ we have that

$$\nu_{\mathcal{Y}}(g_1 + g_2) \geq \min(\nu_{\mathcal{Y}}(g_1), \nu_{\mathcal{Y}}(g_2)),$$

where the order used on $\mathbb{Z}^d$ is the lexicographic order. Moreover, this inequality is an identity whenever $\nu_{\mathcal{Y}}(g_1) \neq \nu_{\mathcal{Y}}(g_2)$

We recall the definition of this multivaluation.
Definition 8. Choose functions $f_1, \ldots, f_d \in K(X)$, regular in a neighbourhood of the point $Y_d$, such that in a neighbourhood of $Y_d$ we have for all $i$

$$Y_i|_{Y_{i-1}} = \text{zero}(f_i|_{Y_{i-1}})$$

scheme-theoretically. Now, for any $g \in K(X)$ we define inductively functions $g_i \in K(Y_i)$ and integers $\nu_i$ by $g_0 = g$ and for all $i \geq 1$

1. $\nu_i = \text{degree of vanishing of } g_{i-1} \text{ along } Y_i$
2. $g_i = \left( \frac{g_{i-1}}{(g_{i-1})^{\nu_i}} \right) |_{Y_i}$

The multivaluation $\nu_Y$ on $K(X)$ is then defined by

$$\nu_Y(g) = (\nu_1, \nu_2, \ldots, \nu_d).$$

It is immediate that this function satisfies conditions (1)- (3). Following the argument in [7], we can see that it also satisfies the condition

Lemma 4. For any finite dimensional sub $k$-vector space $V \subset K(X)$ of finite dimension we have that

$$\#\{\nu_Y(V \setminus 0)\} = \dim(V).$$

Proof of Lemma 4.

We proceed by induction on the dimension of $V$. The case $\dim(V) = 1$ is immediate. Set $\mu = \max(\nu_Y(V))$ and consider the subspace $V_\mu = \{f \in V | \nu_Y(f) = \mu\}$.

Choose $V'$, a complement of $V_\mu$ in $V$. By the induction hypothesis, $\nu_Y(V') = \dim(V')$ and hence by (3) $\nu_Y(V) = \nu_Y(V') + \mu$. It is therefore enough to show that $\dim(V_\mu) = 1$. If not, then $V_\mu$ contains two independent elements $f_1$ and $f_2$ and there exists a $\lambda$ such that

$$\nu_Y(f_1 + \lambda f_2) > \mu,$$

which is impossible by definition of $\mu$. This completes the proof of Lemma 4.

4.2. Construction of the Newton-Okounkov body. The set of points associated by $\nu_Y$ to a subspace $B$ will be an important invariant in what follows.

Definition 9. For any $k$-variety $X$, any subspace $B \subset K(X)$ and any choice of admissible flag $Y_\bullet$ on $X$ we denote by $\Gamma_{Y_\bullet}(B)$ the set

$$\nu_Y(B \setminus 0).$$

We note that if $B_m$ ($m \in \mathbb{N}$) is a family of subspaces of $K(X)$ with the property that $B_0 = k$ and $\oplus_m B_m$ is a graded algebra, then the following set

$$\{(m, a_1, \ldots, a_d) \in \mathbb{Z}^{d+1} | \exists f \in B_m \text{ such that } (a_1, \ldots, a_d) = \nu_Y(f)\}$$

which can of course be negative if $g_{i-1}$ has a pole along $Y_i$
is a semigroup, which we denote by $\Gamma_{Y^\bullet}(\oplus_m B_m)$. We may now define the Newton-Okounkov body of the graded algebra $\oplus_m B_m$ with respect to the flag $Y^\bullet$.

**Definition 10.** Let $X$ be a $k$-variety and $(B_m) \subset K(X)$ be a family of $k$-vector spaces such that $B_0 = k$ and $\oplus_m B_m$ is a graded algebra. Let $Y^\bullet$ be an admissible flag on $X$. The Newton-Okounkov body of the graded algebra $\oplus_m B_m$ is given by

$$\Delta_{Y^\bullet}(\oplus B_m) = \text{Cone}(\Gamma_{Y^\bullet}(\oplus B_m)) \cap \{(1, v) | v \in \mathbb{R}^d\}$$

where the closure is taken with respect to the Euclidean topology on $\mathbb{R}^d$.

We will need the following lemma, first proved by [6]. (The given here is the version given in [7], where proof relies heavily on previous work by Kaveh and Khovanskii).

**Lemma 5.** Suppose that the valuation $\nu_{Y^\bullet}$ is such that the semigroup $\Gamma_{Y^\bullet}(\oplus B_m) \subset \mathbb{N}^{d+1}$ and moreover the following conditions are satisfied.

1. $\Gamma_{Y^\bullet}(B_0) = \{0\}$
2. There exists a finite set of vectors $(v_i, 1)$ spanning a semigroup containing $\Gamma_{Y^\bullet}(\oplus B_m)$.
3. $\Gamma_{Y^\bullet}(\oplus B_m)$ generates $\mathbb{Z}^{d+1}$ as a group.

Then $\oplus_m B_m$ has a volume and moreover

$$\text{vol}(\oplus_m B_m) = d! \text{vol}(\Delta_{Y^\bullet}(\oplus B_m))$$

where the volume appearing in the right hand side is simply the standard Euclidean volume on $\mathbb{R}^d$.

5. **Definition of the infinite divisor $D(B)$.**

We define the infinite divisor $D(B)$ to be the limsup of the normalisations of the divisors $D_m$ constructed above.

**Definition 11.** Let $B$ be an approximable algebra over $k$, let $X(B)$ be the associated $k$-variety and for every $m$ such that $B_m$ is non-trivial let $D_m$ be the divisor defined above. We then define

$$D = \limsup_m \left( \frac{D_m}{m} \right).$$

We will first check that this definition makes sense, ie., that there is no prime divisor $C \subset X$ such that $\limsup_m \text{coeff}(C, \left( \frac{D_m}{m} \right)) = +\infty$. Our proof will depend on the following proposition.

**Proposition 2.** Let $B$ be an approximable algebra over the field $k$, $X(B)$ the associated algebraic variety and $Y^\bullet$ an admissible flag on $X$. The Newton-Okounkov body $\Delta_{Y^\bullet}(B)$ is then a bounded subset of $\mathbb{R}^d$.

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4 By definition of an approximable algebra, this holds for all large enough $m$. 
Proof of Proposition 2

We start with the following lemma.

Lemma 6. Let $B$, $X(B)$ and $Y_*$ be as in the statement of Proposition 2. There is then a natural number $k$ such that the set $\Gamma_{Y_*}(B_k)$ contains a set of $d+1$ points, $v_1, \ldots, v_{d+1} \in \mathbb{R}^d$, whose convex hull is a $d$-dimensional simplex of non-empty interior.

Proof of Lemma 6

Choose an integer $k'$ such that $\langle B_{k'} \rangle$ is an algebra of positive volume, which is possible by the definition of an approximable algebra. This algebra is a sub-algebra of $\bigoplus_n H^0(nD_k)$, so by [7] Lemma 1.10, its Newton-Okounkov body is a compact body. Moreover, by Lemma (4), we have that $\Delta_{Y_*}(\langle B_{k'} \rangle)$ has non-empty interior. The lemma follows on setting $k = rk'$ for sufficiently large $r$. This completes the proof of Lemma 6.

We return now to the proof of Proposition 2. For any $k$ we have that $\Delta_{Y_*}(B_k) = k\Delta_{Y_*}(B)$ so after passing to the algebra $B_k$ we may assume that $k = 1$. We denote by $v_1, \ldots, v_{d+1}$ the vectors whose existence is guaranteed by Lemma 6.

Note that for any $m$ the map

$$\left\{(a_1, \ldots, a_{d+1}) | a_i \geq 0, \sum_{i=1}^{d+1} a_i = m\right\} \rightarrow \sum_{i=1}^{d+1} a_i v_i$$

is injective. We set

$$M = \sup_i \{||v_i||\}$$

Suppose that $\Delta_{Y_*}(B)$ is not bounded. In other words, for any $N \in \mathbb{R}^+$ there exists a natural number $m(N)$ and an element $b \in B_{m(N)}$ such that $||\nu(b)|| > Nm(N)$. For simplicity we will consider only values of $N$ which are exact multiples of $M$.

Let us now consider for any $p$ the set $\Gamma_{Y_*}(B_{pm(N)})$: our aim is to show that this set is too large. Note that $\Gamma_{Y_*}(B_{pm(N)})$ contains the set

$$Z(p, m(N)) = \left\{a_0 \nu(b) + \sum_{i=1}^{d+1} a_i v_i | a_i \geq 0, m(N) a_0 + \sum_{i=1}^{d+1} a_i = pm(N)\right\}.$$

Once again, for the sake of simplicity we assume that $p$ is a multiple of $N$. The map

$$(a_0, \ldots, a_{d+1}) \rightarrow a_0 \nu(b) + \sum_{i=1}^{d+1} a_i v_i$$
is not a priori injective, but it becomes injective if we make the additional assumption that that \(a_0\) be a multiple of \(2pM/N\). It follows that

\[
\dim(B_{pm(N)}) = \# \Gamma_*(B_{pm(N)})
\]

\[
\geq \# \left\{ (a_0, \ldots, a_{d+1}) | a_i \geq 0, a_0 \text{ is a multiple of } 2pM/N \right. + \left. \sum_{i=1}^{d+1} a_i = pm(N) \right\}.
\]

To find a lower bound on \(\dim(B_{pm(N)})\) it will be enough to find a lower bound on the size of the set

\[
\left\{ (a_0, \ldots, a_{d+1}) | a_i \geq 0, a_0 \text{ is a multiple of } 2pM/N \right. + \left. \sum_{i=1}^{d+1} a_i = pm(N) \right\}.
\]

On setting \(c = \frac{a_0N}{2pm}\) we see that the size of this set is equal to

\[
\sum_{c=0}^{N/2M} \left( pm - 2pmMc/N + d \right)/d \geq \sum_{c=0}^{N/2M} \frac{1}{d!}(p^d m^d(1 - 2Mc/N)^d)
\]

\[
= \frac{1}{d!}(p^d m^d) \sum_{c=0}^{N/2M} (1 - 2Mc/N)^d \geq \frac{1}{d!}(p^d m^d) \sum_{c=0}^{N/4M} (1 - 2Mc/N)^d
\]

\[
\geq \frac{Np^d m^d}{2d+2d!M}
\]

from which it follows that

\[
\text{vol}(B) \geq \frac{N}{M2^{d+2}},
\]

for any \(N\). This is impossible since \(B\) has finite volume. This completes the proof of Proposition 2.

Given this proposition, it is fairly easy to deduce the following.

**Proposition 3.** Let \(B\) and \(X(B)\) be as above. The divisor \(D(B)\) constructed above is then well-defined, i.e., for any prime divisor \(C \subset X\) the set

\[
\{ \text{coeff} (C; D_m/m) | m \in \mathbb{N} \}
\]

is bounded.

**Proof of Proposition 3.**

We argue by contradiction: suppose that \(C\) is a divisor on \(X\) such that \(\text{coeff} (C; D_m/m)\) can be arbitrarily large. After blow-up, we may assume that \(C\) is a smooth divisor in \(X\) and choose an admissible flag \(Y_*\) on \(X(B)\) whose first element is \(C\) - the Newton-Okounkov body of \(B\) with respect to \(Y_*\) is then unbounded, contradicting Proposition 2. This completes the proof of Proposition 3.
In order to have a reasonable level of control of the algebra \( R(D(B)) \) it will be useful to have the following result.

**Proposition 4.** Let \( B, X(B) \) and \( D(B) \) be as above. For any \( m \in \mathbb{N} \) the divisor \( \lfloor mD \rfloor \) is in fact a finite divisor.

In other words, if \( D(B) = \sum a_i D_i \) then the sequence \( a_i \) converges to 0.

**Proof of Proposition 4.**

We know that for \( k \) sufficiently large the map associated to the linear system \( B_k \) is birational. Note that if \( \lfloor mD \rfloor \) is not finite then \( \lfloor mkD \rfloor \) is not finite either, so on passing to the approximable algebra \( B_k \) we may assume that \( k = 1 \).

Let \( U \) be the open set on which this birational map associated to the linear system \( B_1 \) is an isomorphism. Proposition 4 will follow from the following result.

**Proposition 5.** Assume that \( B \) is an approximable algebra such that \( B_1 \) gives rise to a birational map on \( X(B) \). Let \( U \) be the set on which this map is an isomorphism. Choose \( m \) such that

\[
\text{vol}(\langle B_m \rangle) + \frac{\delta d+1}{(1+\delta)^d} \geq \text{vol}(B).
\]

Let \( C \) be a prime divisor that does not appear in \( D_{m'} \) for any \( m' \leq m \) and which is not contained in \( X \setminus U \). Then for all integers \( m'' \) the coefficient of \( C \) in \( D_{m''} \) is less than \( \delta m'' \).

**Proof of Proposition 5.**

We argue by contradiction. Suppose that the conclusion of the proposition is false and that there exists an \( m'' \) such that the coefficient of \( C \) in \( D_{m''} \) is \( > \delta m'' \). Since it then follows that the coefficient of \( C \) in \( D_{rm''} \) is \( > \delta rm'' \) for all \( r \), we may assume that \( m'' \) is a multiple of \( m \).

Choose a point \( x \) contained in \( C \) which satisfies the two following criteria.

1. \( x \) is contained in \( U \)
2. \( x \) is not contained in \( D_{m'} \) for any \( m' \leq m \)

It is possible to choose such an \( x \) because of the assumptions on \( C \). Now, since \( X \) is defined up to birational equivalence, we may blow up the point \( X \) and consider a generic infinitesimal flag centred at the point \( x \), i.e. a flag \( (Y_1, \ldots, Y_d) \) such that \( Y_1 \) is the exceptional divisor of \( X \) over \( x \) and the \( Y_i \)'s for \( i = 2 \ldots d \) are very general linear subspaces of \( Y_1 \).

The various conditions required on the point \( x \) have the following implications.
(1) \( \Gamma_*(B_1) \) contains the vectors \( e_1 = (0, 0, \ldots, 0), e_2 = (1, 0, \ldots, 0), e_3 = (1, 1, \ldots, 0), e_{d+1} = (1, 0, \ldots, 0, 1) \) because the linear system \( B_1 \) defines an isomorphism in a neighbourhood of \( x \).

(2) \( \Gamma_*(\langle B_m \rangle) \) is contained in \( \mathbb{N}^{d+1} \) because \( x \not\in D_m \).

(3) If \( C' \) appears in \( D_{m''} \) with coefficient \( \geq \delta m'' \) then \( \Gamma_*(B_{m''}) \) contains a vector \( v = (v_1, 0, \ldots, 0) \) where \( v_1 \leq -\delta m'' \). (The subsequent coefficients of \( v \) are all zero because of the very general choice of the linear subspaces \( Y_i \)).

Let \( r \) be the integer such that \( m'' = rm \). Now, let us consider for large enough \( p \) the set \( \Gamma_*(B_{pm''}) \), and compare it with the set \( \Gamma_*(S^{\tau p}B_m) \). Recall that these two groups are supposed to be about the same size.

The set \( \Gamma_*(B_{pm''}) \) contains the set of vectors

\[
\left\{ a_0 v + \sum_{i=1}^{d+1} a_i e_i | a_i \geq 0, am'' + \sum_{i=1}^{d+1} a_i = pm'' \right\}
\]

We try to bound below the number of elements of this set that are not contained in \( \Gamma_*(S^{\tau p}B_m) \). We know that \( \Gamma_*(S^{\tau p}B_m) \) does not contain any vectors with negative first coefficient. It follows that the set

\[
\Gamma_*(B_{pm''}) \setminus \Gamma_*(S^{\tau p}B_m)
\]

contains the following set of integral vectors

\[
\left\{ a_0 v + \sum_{i=1}^{d+1} a_i e_i | a_i \geq 0, a_0 \delta m'' \geq \sum_{i=2}^{d+1} a_i, a_0 m'' + \sum_{i=1}^{d+1} a_i = pm'' \right\}
\]

In presence of the equation \( a_0 m'' + \sum_{i=1}^{d+1} a_i = pm'' \) the inequality \( a_0 \delta m'' \geq \sum_{i=2}^{d+1} a_i \) is satisfied whenever

\[
p \geq a_0 \geq \frac{p}{\delta + 1}.
\]

So to obtain a lower bound on \( \text{vol}(B) - \text{vol}(B_{m''}) \) it will be enough to obtain a lower bound on the size of the set

\[
S(m'', p, \delta) = \left\{ a_0 v + \sum_{i=1}^{d+1} a_i e_i | a_i \geq 0, a_0 m'' + \sum_{i=1}^{d+1} a_i = pm'', p \geq a_0 \geq \frac{p}{\delta + 1} \right\}
\]

Let us study first the tuples \( (a_0, \ldots, a_{d+1}) \) and \( (a'_0, \ldots, a'_{d+1}) \) such that

\[
a_0 v + \sum_{i=1}^{d+1} a_i e_i = a'_0 v + \sum_{i=1}^{d+1} a'_i e_i
\]

satisfying the condition

\[
a_0 m'' + \sum_{i=1}^{d+1} a_i = a'_0 m'' + \sum_{i=1}^{d+1} a'_i = pm''
\]
On looking at the $i$ th coordinate of this vector we see that $a_i = a'_i$ for $i \geq 3$. We therefore have that

$$a_0v_1 + a_2 = a'_0v_1 + a'_2$$

so that $a_2 - a'_2$ is a multiple of $|v_1| > \delta m''$. In particular, if we impose the extra condition $a_2, a'_2 \leq \delta m''$ then $a_2 = a'_2$. Note that the condition $a_0m'' + \sum_{i=1}^{d+1} a_i = a'_0m'' + \sum_{i=1}^{d+1} a'_i$ implies that if $a_0 = a'_0$ and $a_i = b'_i$ for $i \geq 2$ then $a_1 = a'_1$.

Pulling the above together we see that on the set

$$Z(m'', p, \delta) = \left\{ (a_0, \ldots, a_{d+1}) | a_i \geq 0, a_0m'' + \sum_{i=1}^{d+1} a_i = pm'', p \geq a_0 \geq \frac{p}{\delta + 1}, 0 \leq a_2 < \delta m'' \right\}$$

the map $(a_0, \ldots, a_{d+1}) \rightarrow a_0v + \sum_{i=1}^{d+1} a_ie_i$ is injective, so the size of $\Gamma(B_{pm''}) \setminus \Gamma((S^pB_m))$ is bounded below by the size of $Z(m'', p, \delta)$.

Now, we have that for large enough $p$

$$#Z(m'', p, \delta) = \sum_{a_0=\frac{p}{\delta + 1}}^{p-\delta} \sum_{a_2=0}^{\delta m''} \frac{1}{(d-1)!}((pm'' - am'' - a_2) + d - 1)$$

$$\geq \sum_{a_0=\frac{p}{\delta + 1}}^{p-\delta} \sum_{a_2=0}^{\delta m''} \frac{1}{(d-1)!}(((pm'' - am'' - \delta m''))^{d-1})$$

$$= \sum_{a_0=\frac{p}{\delta + 1}}^{p-\delta} \frac{1}{(d-1)!}((pm'' - am'' - \delta m''))^{d-1}$$

$$= \sum_{a'=\delta}^{\delta m''} \frac{1}{(d-1)!}((a' - \delta)m''^{d-1}) = \frac{\delta m''^{d}}{(d-1)!} \sum_{a'=\delta}^{\delta m''} \left( \begin{array}{c} p \delta \\ \delta + 1 - \delta \end{array} \right)^{d} \approx \frac{\delta^{d+1}(pm'')^{d}}{d!(1 + \delta)^{d}}.$$ 

In particular,

$$\dim(B_{pm''}) - \dim(S^p(B_m)) \geq \frac{\delta^{d+1}(pm'')^{d}}{d!(1 + \delta)^{d}}$$

$$\limsup_{p \to \infty} \frac{\dim(B_{pm''}) - \dim(S^p(B_m))}{(pm'')^{d}/d!} \geq \frac{\delta^{d+1}}{(1 + \delta)^{d}}$$

$$\text{vol}(B) - \text{vol}((B_m)) \geq \frac{\delta^{d+1}}{d!(1 + \delta)^{d}}.$$
But this contradicts the initial assumptions on $m$. This completes the proof of Proposition 5.

The proof of Proposition 4 follows easily. Choose an integer $l$ and set $\delta = 1/l$. Let $m$ be such that $\text{vol}(\langle B_m \rangle) + \frac{m^{d+1}}{(1+\delta)^d} \geq \text{vol}(\mathcal{B})$. By Proposition 5, any divisor $C$ which appears in $\lfloor lD \rfloor$, is either contained in $X \setminus U$ or in $\cup_{m' \leq m} D_{m'}$. There is only a finite number of such divisors, so this completes the proof of Proposition 4.

Pulling the above together, we have the following theorem.

**Theorem 5.1.** Let $\mathcal{B} = \oplus_m B_m$ be an approximable algebra over a field $k$ of dimension $d(\mathcal{B})$. Then there exists a smooth variety $X(\mathcal{B})$ of dimension $d(\mathcal{B})$ and an infinite Weil divisor on $X(\mathcal{B})$ which we denote $D = \sum_i a_i D_i$ such that

1) for all $m$ the divisor $\lfloor mD \rfloor$ is a finite sum of prime divisors,
2) there is an inclusion of graded algebras

$$\mathcal{B} \hookrightarrow \oplus_m H^0(X, \mathcal{O}_X(\lfloor mD \rfloor))$$

In the final section, we will prove a partial result in the opposite direction.

6. APPROXIMABLE SUBALGEBRAS OF $\oplus_m H^0(X, \mathcal{O}_X(\lfloor mD \rfloor))$.

Given 5.1 it seems reasonable to wonder when a subalgebra of $\oplus_m H^0(X, \lfloor mD \rfloor)$, where $D$ is an infinite sum of Weil divisors on a variety $X$ over an algebraically closed field of characteristic zero, is an approximable algebra. We start with the following result

**Proposition 6.** Let $\mathcal{C} = \oplus_m C_m$ be an approximable algebra of dimension $d(\mathcal{C})$ over an algebraically closed field $k$ of characteristic zero. Let $\mathcal{B} = \oplus_m B_m$ be a sub graded algebra of $\mathcal{C}$ such that

$$\left( \frac{\text{rk}(B_m)}{m^{d(\mathcal{C})}} \right)$$

does not converge to 0 and for all sufficiently large $n$ the space $B_n$ is non-empty. Then the algebra $\mathcal{B}$ is also approximable.

Throughout the proof of this proposition we will denote $d(\mathcal{C})$ by $d$.

**Proof of Proposition 6.**

Conditions 1) and 2) of the definition of an approximable algebra are immediately satisfied, so it remains to check only the condition (3). Pick an admissible flag $Y_\bullet$ on $X = X(\mathcal{C})$ such that $Y_d$ is not contained in $D(\mathcal{C})$, so that $\Gamma_{Y_\bullet}(C_m) \subset \mathbb{N}^d$ for all $m$. We know that the Newton-Okounkov body of $\mathcal{C}$ is compact. It follows that the Newton-Okounkov
body of $B$ is also compact and since $\left( \frac{rk(B_m)}{m^{d+1}\text{vol}(C)} \right)$ does not converge to 0 the Newton-Okounkov body of $B$ has non-empty interior.

There is therefore a $k$ such that $B_k$ contains $(d+1)$ elements $(f_0, \ldots, f_d)$ whose valuation vectors span a $(d+1)$-dimensional simplex in $\mathbb{R}^d$. It follows that the elements $\frac{f_1}{f_0}, \ldots, \frac{f_d}{f_0}$ are algebraically independent in $K^{\text{hom}}(C)$ so the induced rational map $X \to \mathbb{P}^d$ has dense image.

Let $x$ be a point of $X$ at which this rational morphism is well-defined and immersive, and which is not contained in $D_i$ for any $i$. Let $Y\cdot$ be an infinitesimal flag centered at the point $x$. We then know that the set $\Gamma_{Y\cdot}(B_{rk})$ generates $\mathbb{Z}^d$ as group for any $r$ and it follows that for any large enough $n$ the semigroup $\Gamma_{Y\cdot}(B_n)$ generates $\mathbb{Z}^{d+1}$. The conditions of Lemma 5 therefore apply so we can say that vol($B$) exists, as does vol($\langle B_m \rangle$) for any large enough $n$ and moreover

$$\text{vol}(\langle B_n \rangle) = d! \text{vol}(\Delta_{Y\cdot}(\langle B_n \rangle)).$$

Equally, we have that

$$\text{vol}(B) = d! \text{vol}(\Delta_{Y\cdot}(B)).$$

We have only therefore to prove that

$$\lim_n \text{vol}(\Delta_{Y\cdot}(\langle B_n \rangle))/n^d = \text{vol}(\Delta_{Y\cdot}(B)).$$

It is immediate that

$$\limsup_n \text{vol}(\Delta_{Y\cdot}(\langle B_n \rangle))/n^d = \text{vol}(\Delta_{Y\cdot}(B))$$

so it remains only to prove that this lim sup is in fact a limit. But we know that for $k > k_0$

$$\text{vol}(\langle B_{rn+k} \rangle)/(rn+k)^d \geq \left( \frac{rn}{rn+k} \right)^d \text{vol}(\langle B_{rn} \rangle)/(rn)^d$$

$$\geq \left( \frac{rn}{rn+k} \right)^d \text{vol}(\langle B_n \rangle)/(n)^d$$

so for any $n$ we have that

$$\liminf_m \text{vol}(\langle B_m \rangle)/m^d \geq \text{vol}(\langle B_n \rangle)/(n)^d$$

so that $\liminf_m \text{vol}(\langle B_m \rangle)/m^d = \limsup_m \text{vol}(\langle B_m \rangle)/m^d$ and

$$\lim_n \text{vol}(\Delta_{Y\cdot}(\langle B_n \rangle))/n^d = \text{vol}(\Delta_{Y\cdot}(B)).$$

This is completes the proof of Proposition 6.
Proposition 7. Let $X$ be a smooth algebraic variety defined on an algebraically closed field of characteristic zero. If $D = \sum a_i D_i$ is an infinite Weil divisor on $X$ such that the class $[D] = \sum a_i [D_i]$ converges towards a finite numerical big divisor class in $\text{NS}(X)$ then the ring $B = \oplus_m H^0([mD])$ is approximable.

Proof of Proposition 7. Set $D_m = \lfloor mD \rfloor$ and $B_m = H^0(\lfloor mD \rfloor/m)$, so that $B = \oplus_m B_m$. Conditions (1) and (2) of the definition of an approximable algebra are immediately satisfied so our aim is to show that condition (3) is also satisfied.

We have that $K^\text{hom}(B) = K(X)$ since for large enough $m$ the divisor $D_m$ is big. After choosing an element $f \in B_1$ we have an inclusion

$$K^\text{hom}(B) \hookrightarrow K(X).$$

We pick $Y_\bullet$, a very general infinitesimal flag on $X$ centred at $y \in X$, which for this reason is not contained in $D_i$ for any $i$. We can therefore calculate the Newton-Okounkov body of $B$ with respect to $Y_\bullet$. Lemma 1.10 of Lazarsfeld and Mustata shows that $\Delta_{Y_\bullet}(B)$ is compact.

Let us show further that $B$ satisfies the conditions (1)-(3) of Lemma 5. Condition (1) holds by definition and condition (2) is immediate because the Newton Okounkov body is bounded: it remains only to prove condition (3). We note that (3) holds for $B$ if it holds for $\langle B_m \rangle$ for some $m$.

Since the class $[D]$ is big, the class $\lfloor mD \rfloor$ is big for large enough $m$ so for large enough $m$ the map defined by $B_m$ is birational, from which it follows that condition (3) is satisfied for $B$. It follows that $B$ has a well-defined volume, equal to the volume of its Newton-Okounkov body with respect to a general infinitesimal flag. Moreover, for all large enough $m$

$$\text{vol}(\langle B_m \rangle) = d! \text{vol}(\Delta_{Y_\bullet}(\langle B_m \rangle)).$$

It will be enough to show that

$$\lim_m \text{vol}(\Delta_{Y_\bullet}(B_m)/m^d) = \text{vol}(B).$$

It is immediate that

$$\limsup_m \text{vol}(\Delta_{Y_\bullet}(\langle B_m \rangle))/m^d = \text{vol}(B),$$

so it remains only to prove that this lim sup is in fact a limit. But we know that for $k > k_0$

$$\text{vol}(\Delta_{Y_\bullet}(\langle B_{rn+k} \rangle))/(rn+k)^d \geq \left( \frac{rn}{rn+k} \right)^d \text{vol}(\Delta_{Y_\bullet}(\langle B_{rn} \rangle))/(rn)^d$$

$$\text{vol}(\Delta_{Y_\bullet}(\langle B_{rn} \rangle))/(rn)^d$$

will decrease by at least $k^{-d}$.
\[ \geq \left( \frac{r_n}{r_n + k} \right)^d \frac{\operatorname{vol}(\Delta Y_\ast(\langle B_n \rangle))/n^d}{d} \]

so for any \( n \) we have that

\[ \liminf_m \frac{\operatorname{vol}(\Delta Y_\ast(\langle B_m \rangle))/m^d}{d} \geq \frac{\operatorname{vol}(\Delta Y_\ast(\langle B_n \rangle))/n^d}{d} \]

so that \( \liminf_m \frac{\operatorname{vol}(\Delta Y_\ast(\langle B_m \rangle))/m^d}{d} = \limsup_m \frac{\operatorname{vol}(\Delta Y_\ast(\langle B_m \rangle))/m^d}{d} \)

and

\[ \lim_n \frac{\operatorname{vol}(\Delta Y_\ast(\langle B_n \rangle))/n^d}{d} = \operatorname{vol}(\Delta Y_\ast(B)). \]

This completes the proof of the proposition.

Pulling these two results together, we see that we have the following.

**Theorem 6.1.** Let \( X \) be a complex algebraic variety and let \( D = \sum a_iD_i \) be an infinite Weil divisor on \( X \) such that the sum of divisor classes \( \sum_i a_i[D_i] \) converges to a finite real big cohomology class. Any graded subalgebra of \( \bigoplus_m H^0(mD) \) such that

\[ \left( \frac{\operatorname{rk}(B_m)}{m^d(C)} \right) \]

does not converge to zero is then an approximable algebra.

**Remark 4.** The following intriguing question remains open: do there exist approximable algebras \( B \) such that the associated infinite Weil divisor \( D(B) = \sum a_iD_i \) does not converge in the space of numerical classes of divisors?

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