RENORMALIZATION FOR CUBIC FREQUENCY INVARIANT TORI IN HAMILTONIAN SYSTEMS WITH TWO DEGREES OF FREEDOM

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Abstract. We consider the break-up of invariant tori in Hamiltonian systems with two degrees of freedom with a frequency which belongs to a cubic field. We define and construct renormalization-group transformations in order to determine the threshold of the break-up of these tori. A first transformation is defined from the continued fraction expansion of the frequency, and a second one is defined with a fixed frequency vector in a space of Hamiltonians with three degrees of freedom.

1. Introduction. We consider the dynamics of a particle in a potential \( V \) described by the following Hamiltonian:

\[
H(p, x, t) = \frac{1}{2}p^2 + V(x, t),
\]

where \( V \) is \( 2\pi \)-periodic in \( x \) and \( t \). For \( V = 0 \), the phase space is foliated with invariant tori. For \( V \) sufficiently small and regular, the KAM theorem states that most of these invariant tori persist.

The purpose of this paper is to construct renormalization transformations in order to study the persistence of invariant tori with a given cubic frequency \( \omega \) smaller than one, satisfying

\[
\omega^3 = a\omega^2 + b\omega + 1,
\]

where \((a, b) \in \mathbb{Z}^2\).

For Hamiltonian systems with two and three degrees of freedom, renormalization-group transformations in the framework of Ref. [1] have been successfully applied for the determination of the threshold of the break-up of invariant tori [2, 3, 4]. They have also been used for the analysis of the properties of critical invariant tori (at the threshold of the break-up) [5, 6, 7, 8].

For two degrees of freedom and for reduced quadratic irrational frequencies, renormalization transformations have been defined with a fixed frequency vector, i.e. at each iteration of the transformation, the frequency vector of the torus was unchanged. This feature was based on the fact that the continued fraction expansion of the frequency was periodic. For cubic (and more generally for non-quadratic irrational) frequencies, this is not the case, and thus ergodic renormalization was

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constructed from the entries of the continued fraction of the frequency. This transformation allows the accurate determination of critical couplings by comparison with Greene’s criterion [9] and Laskar’s frequency map analysis [10] (see Ref. [11]).

In order to define a renormalization with a fixed frequency vector, one has to add one degree of freedom to the system as explained below. For degenerate Hamiltonians with three degrees of freedom, a renormalization transformation has been defined for the spiral mean frequency vector. This transformation allows one to determine accurately the thresholds of break-up of tori by comparison with Laskar’s Frequency Map Analysis [3].

In this article, we propose to define and study numerically two renormalization transformations for invariant tori with cubic frequency: one based on the continued fraction of the frequency acting within a space of Hamiltonians with two degrees of freedom, and the other one with a fixed frequency vector acting within a space of degenerate Hamiltonians with three degrees of freedom.

We apply both renormalization methods to the numerical determination of the threshold of invariant tori of a given one-parameter family of Hamiltonians, Escande’s paradigm model [12] which is a forced pendulum model. We consider three cases: the spiral mean frequency, the Tribonacci mean frequency, and the $\tau$-mean frequency.

2. Renormalization from the continued fraction expansion. In order to investigate the torus with frequency $\omega$, we map the time-dependent Hamiltonian (1) into a time-independent Hamiltonian with two degrees of freedom, written in terms of actions $A = (A_1, A_2) \in \mathbb{R}^2$ and angles $\varphi = (\varphi_1, \varphi_2) \in \mathbb{T}^2$:

$$H(A_1, A_2, \varphi_1, \varphi_2) = \frac{1}{2} A_1^2 - A_2 + V(\varphi_1, -\varphi_2).$$

The two-dimensional invariant torus is now located around $A_1 = \omega$. We shift the action $A_1$ in order to locate this torus around $A_1 = 0$. The Hamiltonian becomes:

$$H(A, \varphi) = \omega \cdot A + \frac{1}{2} A_1^2 + V(\varphi_1, -\varphi_2),$$

where the frequency vector of the considered two-dimensional torus is now $\omega = (\omega, -1)$.

The renormalization we define relies upon the continued fraction expansion of $\omega$:

$$\omega = \frac{1}{a_0 + \frac{1}{a_1 + \cdots}} \equiv [a_0, a_1, \ldots].$$

This transformation will act within a space of Hamiltonians $H$ of the form

$$H(A, \varphi) = \omega \cdot A + V(\Omega \cdot A, \varphi),$$

where $\Omega = (1, \alpha)$ is a vector not parallel to the frequency vector $\omega$. We assume that Hamiltonian (2) satisfies a non-degeneracy condition: If we expand $V$ into

$$V(\Omega \cdot A, \varphi) = \sum_{\nu \in \mathbb{Z}^2} V_{(2)}^{(k)}(\Omega \cdot A)^k e^{i\omega \cdot \varphi},$$

we assume that $V_{(2)}^{(2)}$ is non-zero.

The transformation contains essentially two parts: a rescaling and an elimination procedure. We follow the renormalization scheme defined in Ref. [6]. Similar
Renormalization transformations have been defined in Refs. [13, 14] for vector fields.

(1) Rescaling: The first part of the transformation is composed by a shift of the resonances, a rescaling of time and a rescaling of the actions. It acts on a Hamiltonian $H$ as $H' = H \circ T$ (see Ref. [6] for details):

$$H'(A, \varphi) = \lambda \omega^{-1} H \left( -\frac{1}{\lambda} N_a A, -N_a^{-1} \varphi \right),$$

with

$$\lambda = 2\omega^{-1}(a + \alpha)^2 \nu_0^{(2)},$$

and

$$N_a = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix},$$

and $a$ is the integer part of $\omega^{-1}$ (the first entry in the continued fraction expansion). This change of coordinates is a generalized (far from identity) canonical transformation and the rescaling $\lambda$ is chosen in order to ensure a normalization condition (the quadratic term in the actions has a mean value equal to 1/2). For $H$ given by Eq. (2), this expression becomes

$$H'(A, \varphi) = \omega' \cdot A + \sum_{\nu,k} V'(k) (\Omega' \cdot A)^k e^{i\nu \cdot \varphi},$$

where

$$\omega' = (\omega', -1) \text{ with } \omega' = \omega^{-1} - a,$$

$$\Omega' = (1, \alpha) \text{ with } \alpha' = 1/(a + \alpha),$$

$$V'(k) = r_k V^{(2)} \text{ with } r_k = (-1)^k 2^{1-k} \omega^{k-2}(a + \alpha)^{2-k} \left( \nu_0^{(2)} \right)^{1-k}.$$

We notice that the frequency of the torus is changed according to the Gauss map:

$$\omega \mapsto \omega' = \omega^{-1} - \left[ \omega^{-1} \right],$$

where $[\omega^{-1}]$ denotes the integer part of $\omega^{-1}$. Expressed in terms of the continued fraction expansion, it corresponds to a shift to the left of the entries

$$\omega = [a_0, a_1, a_2, \ldots] \mapsto \omega' = [a_1, a_2, a_3, \ldots].$$

Remark: After renormalization (one or more iterations) the frequency vector $\omega = (\omega, -1)$ is changed into $\omega' = \mu M \omega$ with $M \in GL_2(\mathbb{Z})$. In order to define a renormalization with a fixed frequency vector $\omega' = \omega$, the frequency $\omega$ has to be a quadratic irrational. For a cubic frequency (e.g., satisfying $\omega^3 = a \omega^2 + b \omega + 1$), such renormalization cannot be defined with a fixed frequency vector. The idea is to add one more dimension and consider frequency vectors of the form $(\omega, -1, \omega_1)$. If $\omega_1$ is carefully chosen, there exists $\mu$ and $M \in GL_3(\mathbb{Z})$ such that $\omega = \mu M \omega$, and then one can define a similar renormalization with the exception that now there is an additional degree of freedom corresponding to the additional frequency $\omega_1$.

(2) Elimination: The second step is a (connected to identity) canonical transformation $U$ that eliminates the non-resonant modes of the perturbation in $H'$. Following Ref. [6], we consider the set $I^- \subset \mathbb{Z}^2$ of non-resonant modes as the set of integer vectors $\nu = (\nu_1, \nu_2)$ such that $|\nu_2| > |\nu_1|$. Other choices of resonant modes can be done without affecting the convergence and speed of the algorithm.
A mode which is not an element of $I^-$, will be called resonant. The canonical transformation $U$ is such that $H'' = H' \circ U$ does not have any non-resonant mode, i.e. it is defined by the following equation:

$$\Gamma^-(H' \circ U) = 0,$$

where $\Gamma^-$ is the projection operator on the non-resonant modes; it acts on a Hamiltonian $H$ as:

$$\Gamma^- H = \sum_{\nu \in I^-, k \geq 0} V^{(k)}(\Omega \cdot A)^k e^{i\nu \cdot \varphi}.$$

We solve Eq. (11) by a Newton’s method following Refs. [2, 6]. We notice that in Refs. [2, 6], a renormalization procedure acting within a space of quadratic Hamiltonians in the actions has been constructed. This renormalization is the one used in this paper for numerical purposes. This transformation gave the same accuracy than the renormalization defined for Hamiltonians in power series in the actions for the numerical computation of the parameters that characterize the break-up of invariant tori.

These rescaling and elimination procedures define the renormalization transformation acting on a space of Hamiltonians $H$ with two degrees of freedom as $H'' = \mathcal{R}(H) = H \circ \mathcal{T} \circ U$. The main point is that the frequency $\omega$ is changed according to the Gauss map (10) each time we iterate the transformation.

### 3. Renormalization with a fixed frequency vector

In order to study a torus with frequency $\omega$, we consider another frequency $\omega_1$ such that $(1, \omega, \omega_1)$ is a complete integral basis of the cubic field to which $\omega$ belongs to. We consider the invariant torus with frequency vector $\omega = (\omega, -1, \omega_1)$. The Hamiltonian (10) is mapped into the time independent Hamiltonian with three degrees of freedom written in terms of actions $A = (A_1, A_2, A_3) \in \mathbb{R}^3$ and angles $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in \mathbb{T}^3$:

$$H(A_1, A_2, A_3, \varphi_1, \varphi_2, \varphi_3) = \frac{1}{2} A_1^2 - A_2 + V(\varphi_1, -\varphi_2).$$

We shift the action $A_1$ by $\omega$ and we add $\omega_1 A_3$ since $A_3(t)$ is constant ($H$ does not depend on $\varphi_3$). The Hamiltonian becomes:

$$H(A, \varphi) = \omega \cdot A + \frac{1}{2} A_1^2 + V(\varphi_1, -\varphi_2).$$

We assume that $\omega$ is an eigenvector of a matrix $N$ with integer coefficients, determinant $\pm 1$, and a real eigenvalue $\mu$ of modulus smaller than one. For instance, if $\omega_1 = \omega^2$,

$$N = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ b & -1 & a \end{pmatrix},$$

and $\mu = \omega$.

This transformation will act within a space of Hamiltonians $H$ of the form

$$H(A, \varphi) = \omega \cdot A + V(\Omega \cdot A, \varphi),$$

where $\Omega \in \mathbb{R}^3$ is a vector of Euclidean norm one ($\|\Omega\| = 1$), not parallel to the frequency vector $\omega$. We assume that Hamiltonian (12) satisfies a non-degeneracy condition ($V^{(2)}_0$ is non-zero).

The renormalization is defined by the rescaling and elimination steps.
(1) Rescaling: The first part of the transformation is composed by a shift of the resonances, a rescaling of time and a rescaling of the actions. It acts on a Hamiltonian $H$ as $H' = H \circ T$:

$$H'(A, \varphi) = \lambda \mu^{-1} H \left( \frac{1}{\lambda} \tilde{N} A, N^{-1} \varphi \right),$$

where $\tilde{N}$ denotes the transposed matrix of $N$, and $\lambda$ is the rescaling coefficient:

$$\lambda = 2 \mu^{-1} \|N \Omega\|^2 V_0^{(2)}.$$  

For $H$ given by Eq. (12), this expression becomes

$$H'(A, \varphi) = \omega \cdot A + \sum_{\nu, k} V_{\nu}^{(k)} (\Omega' \cdot A)^k e^{i\nu \cdot \varphi},$$

where

$$\Omega' = \frac{N \Omega}{\|N \Omega\|},$$

$$V_{\nu}^{(k)} = r_k V_{\nu}^{(k)} \tilde{N} \nu$$

with

$$r_k = 2^{1-k} \mu^{-2-k} \|N \Omega\|^{2-k} \left( V_0^{(2)} \right)^{1-k}. $$

We notice that the frequency vector $\omega$ of the torus is kept fixed by the transformation since from Eq. (13), we have

$$\lambda \mu^{-1} \omega \cdot \tilde{N} A = \omega \cdot A.$$  

(2) Elimination: The second step is a canonical transformation $U$ that eliminates the non-resonant modes of the perturbation in $H'$. Following Ref. [3], we consider the set $I^-$ of non-resonant modes to be the set of integer vectors $\nu \in \mathbb{Z}^3$ such that $|\omega \cdot \nu| \geq \frac{1}{\sqrt{2}} \|\omega\| \|\nu\|$. The canonical transformation $U$ is such that $H'' = H' \circ U$ does not have any non-resonant mode, i.e. it is defined by the following equation:

$$\Gamma^-(H' \circ U) = 0.$$  

We solve Eq. (18) by a Newton’s method following Ref. [3], and we apply the procedure acting on a space of Hamiltonians quadratic in the actions.

These two procedures define the renormalization transformation acting on a space of Hamiltonians (12) with three degrees of freedom as $H'' = R H = H \circ T \circ U$. The main point is that in this renormalization the frequency vector $\omega$ is kept fixed at each iteration of the transformation.

4. Numerical results. We consider Escande’s paradigm model [12] which is a forced pendulum model:

$$H_\epsilon(p, x, t) = \frac{1}{2} p^2 - \epsilon (\cos x + \cos(x - t)).$$

We map this Hamiltonian into a time-independent Hamiltonian with two degrees of freedom of the form (12), or into a time-independent Hamiltonian with three degrees of freedom of the form (12). We compute the critical coupling at which the torus is broken by the two renormalization methods $R$. We determine $\epsilon_c$ by iterating the renormalization transformation on $H_\epsilon$. If the coupling $\epsilon$ is smaller than a critical value $\epsilon_c$, the iterations converge to some integrable Hamiltonian $H_0$ (the coefficients $V_{\nu}$ for $\nu \neq 0$ vanish), and if $\epsilon$ is larger than $\epsilon_c$, the iterations diverge:

$$R^n H_\epsilon \rightarrow H_0(A) \quad \text{for} \ |\epsilon| < \epsilon_c,$$

$$R^n H_\epsilon \rightarrow \infty \quad \text{for} \ |\epsilon| > \epsilon_c,$$
as \( n \) tends to infinity. For numerical convenience, each renormalization \( R \) is defined for quadratic Hamiltonians (all the iterations \( R^n H_\varepsilon \) are quadratic in the actions), and at each step, we truncate the Fourier series by neglecting all the modes \( \nu = (\nu_i)_{i=1,2,3} \) such that \( \max_i |\nu_i| > L \). Other way of truncating the Fourier series can be chosen. The one chosen here is the simplest choice for the numerical implementation.

Since the renormalization for three degrees of freedom involves more Fourier coefficients (proportional to \( L^3 \), compared to \( L^2 \) for the renormalization for two degrees of freedom), we expect the resulting transformation to be much slower than the renormalization for two degrees of freedom. However, it turns out from numerical computations that in order to reach a given accuracy, both methods require approximately the same amount of computational time, i.e. the cut-off parameter \( L \) for three degrees of freedom is chosen smaller than the one for two degrees of freedom. In what follows, the thresholds are computed with approximately the same amount of computational time.

We apply the above procedures for three different frequencies: the spiral mean, the Tribonacci mean, and the \( \tau \)-mean frequencies.

4.1. **spiral mean torus.** We consider an invariant torus with frequency \( \omega = \sigma^{-1} \) where \( \sigma \) is the spiral mean \([12]\), i.e. it is the real solution of \( \sigma^3 = \sigma + 1 \). It is approximately \( \sigma \approx 1.32472 \). The frequency \( \sigma^{-1} \) belongs to the complex cubic field with negative discriminant \(-23\). The break-up of KAM tori with spiral mean frequency of Hamiltonian systems with two degrees of freedom has been previously studied using scaling law analysis of Greene’s residues of nearby elliptic periodic orbits \([16]\).

**Renormalization for** \( \omega = (\sigma^{-1},-1) \). The first entries of the continued fraction expansion of \( \sigma^{-1} \) are \([1,3,12,1,1,3,2,3,2,4,2,141,80,2,5,\ldots] \). For a cut-off parameter \( L = 25 \), the renormalization defined in Sec. 3 gives \( \varepsilon_c \approx 0.0155 \).

**Renormalization for** \( \omega = (\sigma^{-1},-1,\sigma^{-2}) \). The frequency vector \( \omega = (\sigma^{-1},-1,\sigma^{-2}) \) is an eigenvector of the matrix

\[
N = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & -1 \end{pmatrix}.
\]

For a cut-off parameter \( L = 15 \), applying the renormalization defined in Sec. 3 with the above matrix \( N \), we obtain \( \varepsilon_c \approx 0.016 \). If we choose \( \omega_1 = \sigma^{-2} + \sigma^{-1} + 1 \) instead of \( \omega_1 = \sigma^{-2} \), the renormalization defined by the matrix

\[
N = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix},
\]

gives the same result \( (\varepsilon_c \approx 0.016) \) as the renormalization defined for \( \omega_1 = \sigma^{-2} \).

Up to the numerical precision, the renormalizations for two and three degrees of freedom give the same result for the critical threshold \( \varepsilon_c \).

4.2. **Tribonacci torus.** We consider an invariant torus with frequency \( \omega = \varsigma^{-1} \) where \( \varsigma \) is called the Tribonacci number and is the real solution of \( \varsigma^3 = \varsigma^2 + \varsigma + 1 \). Its value is approximately \( \varsigma \approx 1.83928 \). The frequency \( \varsigma^{-1} \) belongs to the complex cubic field with discriminant \(-31\).

\[
\]
Renormalization for $\omega = (\varsigma^{-1}, -1)$. The first entries of the continued fraction expansion of $\varsigma^{-1}$ are $[1, 1, 5, 4, 2, 305, 1, 8, 2, 1, 4, 6, 17, 5, 1, \ldots]$. For a cut-off parameter $L = 25$, we obtain $\varepsilon_c \approx 0.02186$.

Renormalization for $\omega = (\varsigma^{-1}, -1, \varsigma^{-2})$. The frequency vector $\omega = (\varsigma^{-1}, -1, \varsigma^{-2})$ is an eigenvector of the matrix

$$N = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix}.$$  

For a cut-off parameter $L = 10$, the renormalization with the above matrix $N$ gives $\varepsilon_c \approx 0.0219$. We notice that in this case, a very good agreement between both renormalizations is obtained even for small cut-off parameters $L$.

4.3. $\tau$-mean torus. We consider an invariant torus with frequency $\omega = \tau^{-1}$ where $\tau$ is the real solution of $\tau^3 = -\tau^2 + 2\tau + 1$. Its value is $\tau = 2 \cos(2\pi/7) \approx 1.24698$. The frequency $\tau^{-1}$ belongs to the real cubic field with positive discriminant 49.

Renormalization for $\omega = (\tau^{-1}, -1)$. The first entries of the continued fraction expansion of $\tau^{-1}$ are $[1, 4, 20, 2, 3, 1, 6, 10, 5, 2, 1, 2, 2, 1, \ldots]$. For a cut-off parameter $L = 25$, we obtain $\varepsilon_c \approx 0.01247$.

Renormalization for $\omega = (\tau^{-1}, -1, \tau^{-2})$. The frequency vector $\omega = (\tau^{-1}, -1, \tau^{-2})$ is an eigenvector of the matrix

$$N = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 1 & -1 & -2 \end{pmatrix},$$

with eigenvalue $\mu = \tau^{-1}$. For any cut-off parameter, the critical value we obtain by the renormalization procedure for this matrix $N (\varepsilon_c \approx 0.48)$ is very far from the one obtained by the previous renormalization. The reason is that the matrix $N$ has not the good properties. In order to define the procedure, the renormalization transformation has to contract all the integer vectors in $I^+$ (the complement of $I^-$ in $\mathbb{Z}^3$) into vectors in $I^- [\mathbf{1}]$. The spectrum of $N$ is composed by $\tau^{-1}, -(1 + \tau^{-1}^{-1}), -(\tau + 1)$. The shift of the resonances is a map acting on integer vectors as $\nu \mapsto N^{-1} \nu$. Since there are two eigenvalues of modulus larger than one for $N^{-1}$ ($\tau$ and $-1 - \tau^{-1}$), there are two unstable directions for the shift of the resonances.

We notice that for the negative discriminant case, this problem cannot occur. If there is one eigenvalue of modulus smaller than one, then the two other complex conjugated eigenvalues are of modulus greater than one since the determinant of $N$ is $\pm 1$.

Renormalization for $\omega = (\tau^{-1}, -1, \tau + 1)$. The frequency vector $\omega = (\tau^{-1}, -1, \tau + 1)$ is an eigenvector of the matrix

$$N = \begin{pmatrix} 2 & -1 & -1 \\ 1 & -1 & -1 \\ 0 & -1 & 0 \end{pmatrix},$$

with eigenvalue $\mu = (1 + \tau)^{-1}$. Here the matrix $N$ has the good property of having only one eigenvalue of modulus smaller than one (the two other eigenvalues are $-\tau$ and $1 + \tau^{-1}$). The renormalization defined in Sec. 3, for a cut-off parameter $L = 7$, gives $\varepsilon_c \approx 0.013$. For this frequency vector, an approximate renormalization has
been set up for degenerate Hamiltonians with three degrees of freedom in Ref. [17] in order to investigate self-similar properties of three-dimensional critical tori. It is defined from the above matrix $N$ and structurally stable dynamics has been found for this approximate renormalization.

**Remark**: For the forced pendulum model [19], the Tribonacci torus is the most robust between the three invariant tori investigated in this paper, and the spiral mean torus is more robust than the $\tau$-mean torus. However, this feature depends on the perturbation in a way that is not understood. The most robust invariant tori are conjectured to be noble tori in general (see Ref. [18] for area-preserving maps and Ref. [13] for the stochastic ionization of Hydrogen driven by microwaves). Therefore, tori with cubic frequency should be less important for the large-scale diffusion of trajectories. However, the results presented in this article would be useful in order to investigate the break-up of invariant tori in Hamiltonian with three degrees of freedom and its comparison with the break-up in two degrees of freedom.

5. **Conclusion.** We have constructed and studied numerically two renormalization transformations in order to compute thresholds of break-up of invariant tori with a cubic frequency in Hamiltonian systems with two degrees of freedom. The main advantage of the procedure acting in a space of degenerate Hamiltonians with three degrees of freedom, is that it is independent of the continued fraction algorithm and it is defined for a fixed frequency vector. Both methods allow an accurate computation of critical couplings.

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