Analysis of Impulsive $\varphi$–Hilfer Fractional Differential Equations

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Abstract

This paper is concerned with the existence and uniqueness, and Ulam–Hyers stabilities of solutions of nonlinear impulsive $\varphi$–Hilfer fractional differential equations. Further, we investigate the dependence of the solution on the initial conditions, order of derivative and the functions involved in the equations. The outcomes are acquired in the space of weighted piecewise continuous functions by means of fixed point theorems and the generalized version of Gronwall inequality.

Key words: $\varphi$–Hilfer fractional derivative; Fixed point theorem; Ulam–Hyers Stability; Dependence of solutions.

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1 Introduction

The qualitative theory of impulsive fractional differential equations (FDEs) is of extraordinary significance in view of its applications in exhibiting numerous characteristic of physical phenomena which are appearing in the field of medicine, biology, mechanics and electrical engineering. Consequently numerous scientists [11–12] engaged with working on impulsive FDEs and examining the existence and uniqueness, dependence and various kinds of stabilities of solution.

On the other hand, the representation formula for the solution of impulsive FDEs of the form:

$$
\begin{align*}
&cD^\alpha u(t) = f(t, u(t)), \quad t \in [0, T], t \neq t_k, \\
&\Delta u|_{t=t_k} = I_k(y(t^-_k)) \\
&u(0) = u_0 \in \mathbb{R},
\end{align*}
$$

have different approaches [1, 2]. Consequently analysis of impulsive FDEs have been done with different representation formula of the solution. Wang et al. [3, 5, 13, 14] have analyzed nonlinear impulsive differential equations with Caputo fractional derivative for existence, uniqueness and data dependence of solutions via generalized singular Gronwall inequalities. Benchohra et al. [6, 15, 16, 17] investigated the sufficient conditions for the existence of solutions for different kinds of impulsive FDEs. Mophou [7] investigated the existence and uniqueness of a mild solution to impulsive semilinear FDEs. Zhang and Wang [8] examined the existence of solutions for an anti-periodic boundary value problem of nonlinear impulsive FDEs via fixed point theorems. For other interesting work on analysis of different class of impulsive fractional differential equations we refer the reader to [18–21].
On the other hand, Harrat et al. [22], utilizing the tools of fixed point technique, semigroup theory and multivalued analysis, investigated the solvability and optimal controls of an impulsive nonlinear delay evolution inclusion in Banach spaces with Hilfer fractional derivative. Harikrishnan et al. [23], established existence and stability of solutions for impulsive Hilfer FDEs using fixed point theorem of Banach and Schaefer. Ahmed et al. [24] examined existence and approximate controllability for Sobolev-type impulsive FDEs involving the Hilfer derivative.

However, as talked about in [25], numerous definitions of fractional derivatives and integrals have been proposed in the literature. One can see that existence, uniqueness and many other essential qualitative properties of solution for FDEs have been demonstrated for similar type of nonlinear FDEs with different fractional order derivative operators. Hence it is have to research nonlinear FDEs with more general fractional operators which incorporates all the specific fractional derivative operators including Caputo and Riemann-Liouville derivatives and all other well known fractional derivative operators such as Hadamard derivative, Katugampola derivative, Hilfer derivative, Chen derivative, Prabhakar derivative, Erdelyi-Kober derivative, Riesz derivative, Feller derivative, Weyl derivative, Cassar derivative etc. Hence it imperative to analyze the impulsive FDEs with broad class of fractional derivative operator that incorporate various definitions of well known fractional derivatives.

The definition of the Riemann-Liouville fractional integral have been extended to a fractional integral of a function with respect to the another function viz. $\varphi$-Riemann-Liouville fractional integral [26, Chapter 2]. Using concept of generalized fractional integral, the Riemann-Liouville and Caputo version of fractional derivative have been introduced namely $\varphi$-Riemann-Liouville fractional derivative [26, Chapter 2] and the $\varphi$-Caputo fractional derivative [27]. Properties of these generalized fractional derivatives one can find in [26, 27]. Jarad and coauthors [28-30] analyzed distinctive class of FDEs for existence, uniqueness and the Ulam-Hyers stabilities of solution. Following the technique of [26, 27], Sousa and Olivera [31, 32] presented a Hilfer version of fractional derivative viz. $\varphi$-Hilfer fractional derivative which incorporate the Hilfer fractional derivative [33] as well as includes a wide class of well known fractional derivatives including most widely used Caputo and Riemann-Liouville derivative.

The list of all possible fractional derivatives which are the particular cases of $\varphi$-Hilfer fractional derivatives have been provided in [31].

In [34], Sousa et al. established generalized Gronwall inequality involving $\varphi$-Riemann-Liouville fractional integral and utilized it to investigate the qualitative properties of solutions such as uniqueness, continuous dependence of solution on different data for the Cauchy-type problem with $\varphi$-Hilfer fractional derivative. Kucche et al. [35] investigated the existence and uniqueness, dependence of solution for $\varphi$-Hilfer FDEs in the weighted space of functions through Weissinger fixed point theorem. Further derived representation formula for the solution of linear Cauchy problem for $\varphi$-Hilfer FDEs obtained by using Picards successive approximation method.

In [36], Kucche et al. investigated the formula for the solution of nonlinear $\varphi$-Hilfer impulsive FDEs and established the existence and uniqueness results. It is proved that the obtained formula for the solution of nonlinear $\varphi$-Hilfer impulsive FDEs includes the formula for the solution of impulsive FDEs involving Riemann-Liouville and Caputo deriva-
tives. Sousa et al. [37] derived sufficient conditions to ensure existence and uniqueness of solutions and \(\delta\)-Ulam–Hyers–Rassias stability of an impulsive FDEs involving \(\varphi\)-Hilfer fractional derivative. Liu et al. [38] presented existence, uniqueness, and Ulam–Hyers–Mittag–Leffler stability of solutions to a class of \(\varphi\)-Hilfer fractional-order delay differential equations through Picard operator theory and a generalized Gronwall inequality.

In the present paper, we consider the nonlinear impulsive \(\varphi\)-Hilfer fractional differential equation (\(\varphi\)-HFDE) with initial condition of the form:

\[
\begin{aligned}
H^{\alpha, \nu; \varphi}_a u(t) &= f(t, u(t)), \quad t \in J = [a, T] - \{t_1, t_2, \cdots, t_m\} \\
\Delta I^{1-\sigma; \varphi}_a u(t_k) &= J_k(u(t_k^-)), \quad k = 1, 2, \cdots, m, \\
I^{1-\sigma; \varphi}_a u(a) &= u_a \in \mathbb{R}, \quad \sigma = \rho + \nu - \rho \nu,
\end{aligned}
\]  

(1.2)

where \(\varphi \in C^1(I, \mathbb{R})\) be an increasing function with \(\varphi'(x) \neq 0\), for all \(x \in I\), \(H^{\alpha, \nu; \varphi}_a (\cdot)\) is the \(\varphi\)-Hilfer fractional derivative [31] of order \(\rho (0 < \rho < 1)\) and type \(\nu (0 \leq \nu \leq 1)\) defined by

\[
H^{\alpha, \nu; \varphi}_a f(t) = I^{\nu(1-\rho)}_a; \varphi \left( \frac{1}{\varphi'(t)} \right) \frac{d}{dt} I^{(1-\nu)(1-\rho)}_a; \varphi f(t),
\]

and \(I^{1-\sigma; \varphi}_a\) is left sided \(\varphi\)-Riemann Liouville fractional integration operator [31], which defined for any \(\rho > 0\)

\[
I^{\rho, \nu; \varphi}_a f(t) := \frac{1}{\Gamma(\rho)} \int_a^t \varphi'(\sigma) (\varphi(t) - \varphi(\sigma))^{\rho-1} f(\sigma) d\sigma.
\]

Let \(a = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T\), \(\Delta I^{1-\sigma; \varphi}_a u(t_k) = I^{1-\sigma; \varphi}_a u(t_k^+) - I^{1-\sigma; \varphi}_a u(t_k^-)\), \(I^{1-\sigma; \varphi}_a u(t_k^+) = \lim_{\epsilon \to 0^+} I^{1-\sigma; \varphi}_a u(t_k + \epsilon)\) and \(I^{1-\sigma; \varphi}_a u(t_k^-) = \lim_{\epsilon \to 0^-} I^{1-\sigma; \varphi}_a u(t_k + \epsilon)\). The functions \(f : [a, T] \times \mathbb{R} \to \mathbb{R}\) and \(J_k : \mathbb{R} \to \mathbb{R}\) are appropriate functions specified latter.

The motivation for the work presented in the present paper is originated from [3][14][36]. Main objective of the present paper is to establish the existence results for the impulsive \(\varphi\)-Hilfer fractional differential equation (impulsive \(\varphi\)-HFDE) with initial condition via Schaefer fixed point theorem. Additionally, by utilizing the generalized version of Gronwall inequality, we examine the uniqueness of solution, dependence of the solution on the initial conditions, order of the \(\varphi\)-Hilfer derivative and the functions involved in the equations. Further, we investigate the Ulam–Hyers and Ulam–Hyers–Rassias stabilities via generalized version of Gronwall inequality.

The remainder of this paper is organized as follows. In Section 2, existence results are exhibited. In Section 3, the dependence of the solution associated with initial conditions, the functions involved in the equations and the order of \(\varphi\)-Hilfer derivative have been examined. Section 4, deals with the Ulam–Hyers and Ulam–Hyers–Rassias stabilities for (1.2). Finally in section 5, we give an example to illustrate the obtained results.

### 2 Preliminaries

Consider the weighted space [31] defined by

\[
C_{1-\sigma; \varphi} (I) = \{ u : (a, b] \to \mathbb{R} : (\varphi(t) - \varphi(a))^{1-\sigma} u(t) \in C(I) \}, \quad 0 < \sigma \leq 1.
\]
Define the weighted space of piecewise continuous functions as

\[ \mathcal{PC}_{1-\sigma; \varphi}(I, \mathbb{R}) = \{ u : (a, b] \to \mathbb{R} : u \in C_{1-\sigma; \varphi}((t_k, t_{k+1}], \mathbb{R}), k = 0, 1, 2, \ldots, m, \]  
\[ \mathbf{1}_{a^+}^{1-\sigma; \varphi}u(t_k^+) \text{ exists and } \mathbf{1}_{a^+}^{1-\sigma; \varphi}u(t_k^-) = \mathbf{1}_{a^+}^{1-\sigma; \varphi}u(t_k) \text{ for } k = 1, 2, \ldots, m \} \]

Clearly, \( \mathcal{PC}_{1-\sigma; \varphi}(I, \mathbb{R}) \) is a Banach space with the norm

\[ \| u \|_{\mathcal{PC}_{1-\sigma; \varphi}(I, \mathbb{R})} = \sup_{t \in I} \left| (\varphi(t) - \varphi(a))^{1-\sigma}u(t) \right| . \]

Note that for \( \sigma = 1 \), we get \( \mathcal{PC}_{0; \varphi}(I, \mathbb{R}) = PC(I, \mathbb{R}) \) a particular case of the space \( \mathcal{PC}_{1-\sigma; \varphi}(I, \mathbb{R}) \), whose details are given in [15] [4] [39].

**Lemma 2.1 (\( \mathcal{PC}_{1-\sigma; \varphi } \) type Arzela-Ascoli Theorem, [36])** Let \( X \) be a Banach space and \( W_{1-\sigma; \varphi} \subset \mathcal{PC}_{1-\sigma; \varphi}(J, X) \). If the following conditions are satisfied:

(a) \( W_{1-\sigma; \varphi} \) is uniformly bounded subset of \( \mathcal{PC}_{1-\sigma; \varphi}(J, X) \);

(b) \( W_{1-\sigma; \varphi} \) is equicontinuous in \( (t_k, t_{k+1}), k = 0, 1, 2, \ldots, m \), where \( t_0 = a, t_{m+1} = T \);

(c) \( W_{1-\sigma; \varphi}(t) = \{ u(t) : u \in W_{1-\sigma; \varphi}, t \in J - t_1, \ldots, t_m \}, W_{1-\sigma; \varphi}(t_k^+ = \{ u(t_k^+) : u \in W_{1-\sigma; \varphi} \} \) and \( W_{1-\sigma; \varphi}(t_k^-) = \{ u(t_k^-) : u \in W_{1-\sigma; \varphi} \} \) are relatively compact subsets of \( X \),

then \( W_{1-\sigma; \varphi} \) is a relatively compact subset of \( \mathcal{PC}_{1-\sigma; \varphi}(J, X) \).

**Theorem 2.2 (Schaefer, [40])** Let \( F : C(J, \mathbb{R}) \to C(J, \mathbb{R}) \) be a completely continuous operator. If the set

\[ G(F) = \{ u \in C(J, \mathbb{R}) : u = \lambda F(u), \text{for some } \lambda \in (0, 1) \} \]

is bounded, then \( F \) has at least one fixed point.

**Lemma 2.3 ([41])** Let \( U \in PC_{1-\sigma; \varphi}(J, \mathbb{R}) \) satisfying the following inequality

\[ U(t) \leq V(t) + g(t) \int_a^t \varphi(s)(\varphi(t) - \varphi(s))^{\sigma-1}U(s)ds + \sum_{a^+ < t_b < t} \beta_b U(t_b^-), t > a, \]

where \( g \) is a continuous function, \( V \in PC_{1-\sigma; \varphi}(J, \mathbb{R}) \) is non-negative, \( \beta_k > 0 \) for \( k = 1, 2, \ldots, m \), then we have

\[ U(t) \leq V(t) \left[ \prod_{i=1}^k \{1 + \beta_i E_{\varphi}(g(t)\Gamma(\varphi)(\varphi(t_i) - \varphi(a))^{\sigma})\} \right] \times E_{\varphi}(g(t)\Gamma(\varphi)(\varphi(t) - \varphi(a))^{\sigma}), t \in (t_k, t_{k+1}] \]
To analyze the impulsive \( \varphi \)-HFDE (1.2), we utilize its equivalent fractional integral given in the following Lemma.

**Lemma 2.4 ([36])** Let \( h : J \rightarrow \mathbb{R} \) be a continuous function. Then a function \( u \in PC_{1-\sigma; \varphi}(J, \mathbb{R}) \) is a solution of impulsive \( \varphi \)-HFDE (1.2) if and only if \( u \) is a solution of the following fractional integral equation

\[
    u(t) = \begin{cases} 
        \dfrac{(\varphi(t)-\varphi(a))^{\sigma-1}}{\Gamma(\sigma)} u_a + \int_a^t \varphi h(s) \, ds, & t \in [a, t_1], \\
        \dfrac{(\varphi(t)-\varphi(a))^{\sigma-1}}{\Gamma(\sigma)} \left( u_a + \sum_{i=1}^k \mathcal{J}_i(u(t^-_i)) \right) + \int_a^t \varphi h(s) \, ds, & t \in (t_k, t_{k+1}], \quad k = 1, 2, \ldots, m. 
    \end{cases}
\]

(2.1)

3 Existence results

In this section, we derive the existence of solution to the problem (1.2) by means of Schaefer’s fixed point theorem.

**Theorem 3.1** Assume that:

\( (H_1) \) The function \( f : (a, T] \times \mathbb{R} \rightarrow \mathbb{R} \) is continuous and satisfy the following conditions:

(i) \( (\cdot, u(\cdot)) \in PC_{1-\sigma; \varphi}(J, \mathbb{R}) \) for any \( u \in PC_{1-\sigma; \varphi}(J, \mathbb{R}) \),

(ii) \( |f(t, u(t)) - f(t, v(t))| \leq (\varphi(t)-\varphi(a))^{1-\sigma} |u(t)-v(t)| \), \( t \in J \), \( u, v \in PC_{1-\sigma; \varphi}(J, \mathbb{R}) \),

\( (H_2) \) The functions \( \mathcal{J}_k : \mathbb{R} \rightarrow \mathbb{R} \), \( (k = 1, 2, \ldots, m.) \) satisfy the condition

(i) \( |\mathcal{J}_k(u(t^-_k)) - \mathcal{J}_k(v(t^-_k))| \leq (\varphi(t^-_k)-\varphi(a))^{1-\sigma} |u(t^-_k)-v(t^-_k)| \), \( u, v \in PC_{1-\sigma; \varphi}(J, \mathbb{R}) \),

(ii) there exist \( \zeta_k > 0 \) such that \( |\mathcal{J}_k(u(t^-_k))| \leq \zeta_k \), \( u \in PC_{1-\sigma; \varphi}(J, \mathbb{R}) \).

Then, the problem (1.2) has at least one solution in the space \( PC_{1-\sigma; \varphi}(J, \mathbb{R}) \).

**Proof:** Consider the operator \( F \) defined on \( PC_{1-\sigma; \varphi}(J, \mathbb{R}) \) by

\[
    (F u)(t) = \dfrac{(\varphi(t)-\varphi(a))^{\sigma-1}}{\Gamma(\sigma)} \left( u_a + \sum_{a < t_k < t} \mathcal{J}_k(u(t^-_k)) \right) + \int_a^t \varphi f(s, u(s)) \, ds, \quad t \in J. \tag{3.1}
\]

Then as proved in [36], \( F \) is mapping from \( PC_{1-\sigma; \varphi}(J, \mathbb{R}) \) to itself. Further, our problem of finding the solution to (1.2) is reduced to find a fixed point to the operator \( F \). Utilizing Schaefer fixed point theorem, we prove that the operator \( F \) has fixed point, and the proof of same given in following four steps.
Step 1: \( F \) is continuous.
Let \( \{u_n\} \) be a sequence such that \( u_n \to u \) in \( \mathcal{PC}_{1-\sigma;\varphi}(J, \mathbb{R}) \). Then for each \( t \in J \),
\[
(\varphi(t) - \varphi(a))^{1-\sigma} |(F u_n)(t) - (F u)(t)|
\]
\[
= \left| \frac{1}{\Gamma(\sigma)} \left( u_a + \sum_{a < t_k < t} J_k(u_n(t_k^-)) \right) + (\varphi(t) - \varphi(a))^{1-\sigma} \mathbf{1}_{a+}^\varphi f(t, u_n(t)) \right|
\]
\[
- \frac{1}{\Gamma(\sigma)} \left( u_a + \sum_{a < t_k < t} J_k(u(t_k^-)) \right) - (\varphi(t) - \varphi(a))^{1-\sigma} \mathbf{1}_{a+}^\varphi f(t, u(t)) \right|
\]
\[
\leq \frac{1}{\Gamma(\sigma)} \sum_{a < t_k < t} \left| J_k(u_n(t_k^-)) - J_k(u(t_k^-)) \right|
\]
\[
+ (\varphi(t) - \varphi(a))^{1-\sigma} \left| \mathbf{1}_{a+}^\varphi (f(t, u_n(t)) - f(t, u(t))) \right|
\]
Since, \( f \) and \( J_k, (k = 1, \cdots, n) \) are continuous functions, we have
\[
\left| J_k(u_n(t_k^-)) - J_k(u(t_k^-)) \right| \to 0
\]
and
\[
\left| (f(t, u_n(t)) - f(t, u(t))) \right| \to 0
\]
as \( n \to \infty \). Therefore,
\[
(\varphi(t) - \varphi(a))^{1-\sigma} |(F u_n)(t) - (F u)(t)| \to 0 \text{ as } n \to \infty,
\]
which gives
\[
\|F u_n - F u\|_{\mathcal{PC}_{1-\sigma;\varphi}(J, \mathbb{R})} \to 0 \text{ as } n \to \infty.
\]
This proves the operator \( F : \mathcal{PC}_{1-\sigma;\varphi}(J, \mathbb{R}) \to \mathcal{PC}_{1-\sigma;\varphi}(J, \mathbb{R}) \) is continuous.

Step 2: \( F \) maps bounded sets into bounded sets.
To prove this we shall show that for \( \delta > 0 \) there exists \( \eta > 0 \) such that for any \( u \in \mathcal{D}_\delta = \left\{ u \in \mathcal{PC}_{1-\sigma;\varphi}(J, \mathbb{R}) : \|u\|_{\mathcal{PC}_{1-\sigma;\varphi}(J, \mathbb{R})} \leq \delta \right\} \) one has \( \|F u\|_{\mathcal{PC}_{1-\sigma;\varphi}(J, \mathbb{R})} \leq \eta \). Let \( \mathcal{M}^* = \sup_{s \in J} |f(s, 0)| \) and \( \zeta = \sum_{k=1}^m \zeta_k \). Then, using the hypotheses \((H_1)(ii)\) and \((H_2)(ii)\), for each \( t \in J \), we have
\[
(\varphi(t) - \varphi(a))^{1-\sigma} |(F u)(t)|
\]
\[
\leq \frac{\|u_a\|}{\Gamma(\sigma)} + \frac{1}{\Gamma(\sigma)} \sum_{a < t_k < t} \left| J_k(u(t_k^-)) \right| + (\varphi(t) - \varphi(a))^{1-\sigma} \left| \mathbf{1}_{a+}^\varphi f(t, u(t)) \right|
\]
\[
\leq \frac{\|u_a\|}{\Gamma(\sigma)} + \frac{1}{\Gamma(\sigma)} \sum_{a < t_k < t} \zeta_k + \frac{(\varphi(t) - \varphi(a))^{1-\sigma}}{\Gamma(\varphi)} \int_a^t \frac{\varphi'(s)(\varphi(t) - \varphi(s))^{\varphi-1}}{\Gamma(\varphi)} |f(s, u(s)) - f(s, 0)| \, ds
\]
\[
+ \frac{(\varphi(t) - \varphi(a))^{1-\sigma}}{\Gamma(\varphi)} \int_a^t \frac{\varphi'(s)(\varphi(t) - \varphi(s))^{\varphi-1}}{\Gamma(\varphi)} |f(s, 0)| \, ds
\]
\[
\leq \frac{\|u_a\|}{\Gamma(\sigma)} + \frac{\zeta}{\Gamma(\sigma)} + \frac{(\varphi(t) - \varphi(a))^{1-\sigma}}{\Gamma(\varphi)} \int_a^t \frac{\varphi'(s)(\varphi(t) - \varphi(s))^{\varphi-1}}{\Gamma(\varphi)} (\varphi(s) - \varphi(a))^{1-\sigma} |u(s)| \, ds
\]
\[
+ \mathcal{M}^* (\varphi(t) - \varphi(a))^{1-\sigma} \int_a^t \frac{\varphi'(s)(\varphi(t) - \varphi(s))^{\varphi-1}}{\Gamma(\varphi)} \, ds
\]
Let bounded set is completed. This gives \( \|u\|_{PC_{1-\sigma},\varphi(J,\mathbb{R})} \leq \eta \). Thus the proof of the operator \( \mathbb{F} \) maps bounded sets into bounded set is completed.

Step 3: \( \mathbb{F} \) is equicontinuous.
Let \( u \in C_{1-\sigma}, \varphi(J, \mathbb{R}) \) and \( t_1, t_2 \in J \) with \( a < t_1 < t_2 < T \), we get

\[
|\mathbb{F}(u)(t_2) - \mathbb{F}(u)(t_1)| = \left| \varphi(t_2) - \varphi(t_1) \right| \frac{1}{\Gamma(\sigma)} \left( u_a + \sum_{a < t_k < t_2} \mathcal{J}_k(u(t_k^-)) \right) + \frac{\varphi(t_2) - \varphi(t_1)}{\Gamma(\sigma)} \left( u_a + \sum_{a < t_k < t_1} \mathcal{J}_k(u(t_k^-)) \right) - \frac{\varphi(t_2) - \varphi(t_1)}{\Gamma(\sigma)} \sum_{a < t_k < t_1} \mathcal{J}_k(u(t_k^-)) \right|
\]

\[
+ \left| \varphi(t_2) - \varphi(t_1) \right| \frac{1}{\Gamma(\sigma)} \left( u_a + \sum_{a < t_k < t_2} \mathcal{J}_k(u(t_k^-)) \right) - \frac{\varphi(t_2) - \varphi(t_1)}{\Gamma(\sigma)} \sum_{a < t_k < t_1} \mathcal{J}_k(u(t_k^-)) \right|
\]

\[
+ \left| \varphi(t_2) - \varphi(t_1) \right| \frac{1}{\Gamma(\sigma)} \left( u_a + \sum_{a < t_k < t_2} \mathcal{J}_k(u(t_k^-)) \right) - \frac{\varphi(t_2) - \varphi(t_1)}{\Gamma(\sigma)} \sum_{a < t_k < t_1} \mathcal{J}_k(u(t_k^-)) \right|
\]

\[
+ \left| \varphi(t_2) - \varphi(t_1) \right| \frac{1}{\Gamma(\sigma)} \left( u_a + \sum_{a < t_k < t_2} \mathcal{J}_k(u(t_k^-)) \right) - \frac{\varphi(t_2) - \varphi(t_1)}{\Gamma(\sigma)} \sum_{a < t_k < t_1} \mathcal{J}_k(u(t_k^-)) \right|
\]

\[
+ \left| \varphi(t_2) - \varphi(t_1) \right| \frac{1}{\Gamma(\sigma)} \left( u_a + \sum_{a < t_k < t_2} \mathcal{J}_k(u(t_k^-)) \right) - \frac{\varphi(t_2) - \varphi(t_1)}{\Gamma(\sigma)} \sum_{a < t_k < t_1} \mathcal{J}_k(u(t_k^-)) \right|
\]

\[
+ \left| \varphi(t_2) - \varphi(t_1) \right| \frac{1}{\Gamma(\sigma)} \left( u_a + \sum_{a < t_k < t_2} \mathcal{J}_k(u(t_k^-)) \right) - \frac{\varphi(t_2) - \varphi(t_1)}{\Gamma(\sigma)} \sum_{a < t_k < t_1} \mathcal{J}_k(u(t_k^-)) \right|
\]
This shows that $F$ is equi-continuous on $J$. By utilizing $PC_{1-\sigma;\varphi}$ type of Arzela-Ascoli Theorem 2.1, the operator $F$ is completely continuous.

Step 4: Finally, we show that the set

$$\mathcal{D} = \{ u \in PC_{1-\sigma;\varphi}(J, \mathbb{R}) : u = \lambda Fu, \text{ for some } \lambda \in (0, 1) \}$$

is bounded. Let any $u \in \mathcal{D}$. Then

$$|u(t)| = |\lambda Fu(t)| < |Fu(t)|.$$

Let $N^* = \max_{a < t_k < t} |J_k(0)|$. Then, using hypotheses $(H_1)(i)$ and $(H_2)(i)$, for each $t \in J$, we have

$$(\varphi(t) - \varphi(a))^{1-\sigma}|u(t)|$$

$$\leq \frac{|u_a|}{\Gamma(\sigma)} + \frac{1}{\Gamma(\sigma)} \sum_{a < t_k < t} |J_k(u(t_k^-))| + (\varphi(t) - \varphi(a))^{1-\sigma} |f(t, u(t))|$$

$$\leq \frac{|u_a|}{\Gamma(\sigma)} + \frac{1}{\Gamma(\sigma)} \sum_{a < t_k < t} |J_k(u(t_k^-)) - J_k(o(t_k^-))| + \frac{1}{\Gamma(\sigma)} \sum_{a < t_k < t} |J_k(o(t_k^-))|$$

$$+ (\varphi(t) - \varphi(a))^{1-\sigma} |f(t, u(t)) - f(t, 0)|$$

$$+ (\varphi(t) - \varphi(a))^{1-\sigma} |f(t, 0)|$$

$$\leq \frac{|u_a|}{\Gamma(\sigma)} + \frac{1}{\Gamma(\sigma)} \sum_{a < t_k < t} (\varphi(t_k) - \varphi(a))^{1-\sigma}|u(t_k^-)| + \frac{1}{\Gamma(\sigma)} \sum_{a < t_k < t} |J_k(0)|$$

$$+ \frac{(\varphi(t) - \varphi(a))^{1-\sigma}}{\Gamma(\sigma)} \int_a^t \varphi'(s)(\varphi(t) - \varphi(s))^{\sigma-1} (\varphi(s) - \varphi(a))^{1-\sigma} |u(s)| \, ds$$

$$+ \frac{(\varphi(t) - \varphi(a))^{1-\sigma}}{\Gamma(\sigma)} \int_a^t \varphi'(s)(\varphi(t) - \varphi(s))^{\sigma-1} |f(s, 0)| \, ds$$

$$\leq \left( \frac{|u_a|}{\Gamma(\sigma)} + \frac{mN^*}{\Gamma(\sigma)} + \frac{\mathcal{M}^*(\varphi(T) - \varphi(a))^{1-\sigma+\theta}}{\Gamma(\sigma + 1)} \right)$$

$$+ \frac{(\varphi(t) - \varphi(a))^{1-\sigma}}{\Gamma(\sigma)} \int_a^t \varphi'(s)(\varphi(t) - \varphi(s))^{\sigma-1} (\varphi(s) - \varphi(a))^{1-\sigma} |u(s)| \, ds$$

$$+ \frac{1}{\Gamma(\sigma)} \sum_{a < t_k < t} (\varphi(t_k) - \varphi(a))^{1-\sigma}|u(t_k^-)|.$$
\[ \beta_k = \frac{1}{\Gamma(\sigma)}. \]

we obtain
\[
(\varphi(t) - \varphi(a))^{1-\sigma} |u(t)| \\
\leq \left( \frac{|u_0|}{\Gamma(\sigma)} + \frac{m N^*}{\Gamma(\sigma)} + \frac{M^*(\varphi(T) - \varphi(a))^{1-\sigma+\varepsilon}}{\Gamma(\theta + 1)} \right) \\
\times \left[ \prod_{i=1}^{k} \left( 1 + \frac{1}{\Gamma(\sigma)} E_\theta (\frac{(\varphi(t) - \varphi(a))^{1-\sigma}}{\Gamma(\theta)} (\varphi(t_i) - \varphi(a))) \right) \right] \\
\times E_\theta \left( (\varphi(T) - \varphi(a))^{1-\sigma+\varepsilon} \right).
\]

Since \( \varphi \) is increasing, \( (\varphi(t_i) - \varphi(a))^{1-\sigma} \leq (\varphi(t) - \varphi(a))^{1-\sigma}, \ k = 1, 2, \cdots, m \). Therefore, we have
\[
(\varphi(t) - \varphi(a))^{1-\sigma} |u(t)| \leq \left( \frac{|u_0|}{\Gamma(\sigma)} + \frac{m N^*}{\Gamma(\sigma)} + \frac{M^*(\varphi(T) - \varphi(a))^{1-\sigma+\varepsilon}}{\Gamma(\theta + 1)} \right) \times \left( \prod_{i=1}^{k} \left( 1 + \frac{1}{\Gamma(\sigma)} \right) E_\theta (\frac{(\varphi(T) - \varphi(a))^{1-\sigma}}{\Gamma(\theta)} (\varphi(t_i) - \varphi(a))) \right) \\
\times E_\theta \left( (\varphi(T) - \varphi(a))^{1-\sigma+\varepsilon} \right).
\]

Therefore for each \( t \in J \), we have
\[
(\varphi(t) - \varphi(a))^{1-\sigma} |u(t)| \leq \left( \frac{|u_0|}{\Gamma(\sigma)} + \frac{m N^*}{\Gamma(\sigma)} + \frac{M^*(\varphi(T) - \varphi(a))^{1-\sigma+\varepsilon}}{\Gamma(\theta + 1)} \right) A_{m,\varepsilon},
\]

where
\[
A_{m,\varepsilon} = \left( 1 + \frac{1}{\Gamma(\sigma)} E_\theta (\frac{(\varphi(T) - \varphi(a))^{1-\sigma+\varepsilon}}{\Gamma(\theta)} (\varphi(T) - \varphi(a))) \right)^m E_\theta \left( (\varphi(T) - \varphi(a))^{1-\sigma+\varepsilon} \right).
\]

Thus
\[
\|u\|_{PC_{1-\sigma}; \varphi(J, \mathbb{R})} \leq A_{m,\varepsilon} \left( \frac{|u_0|}{\Gamma(\sigma)} + \frac{m N^*}{\Gamma(\sigma)} + \frac{M^*(\varphi(T) - \varphi(a))^{1-\sigma+\varepsilon}}{\Gamma(\theta + 1)} \right).
\]

This shows that the set \( D \) is bounded subset of \( PC_{1-\sigma}; \varphi(J, \mathbb{R}) \). Hence by Schaefer’s fixed point theorem \( F \) has at least one fixed point in \( PC_{1-\sigma}; \varphi(J, \mathbb{R}) \) which is the solution of problem (1.2).

\[\square\]

4 Continuous dependence of solution

This section deals with the continuous dependence of solutions of the problem (1.2) on initial conditions and the functions involved in (1.2) on the right hand sides.
4.1 Continuous Dependence on initial conditions

Theorem 4.1 Suppose that the functions $f : (a, T] \to \mathbb{R}$ and $J_k : \mathbb{R} \to \mathbb{R}$ satisfy the hypotheses (H1)(ii) and (H2)(i) respectively. Let $u, v \in \mathcal{PC}_{1-\sigma; \varphi}(J, \mathbb{R})$ are the solutions of the problem

$$
\begin{aligned}
H_{\alpha^+}^{\varphi; \psi} u(t) &= f(t, u(t)), \quad t \in J - \{t_1, t_2, \cdots, t_m\}, \\
\Delta I_{\alpha^+}^{1-\sigma; \varphi} u(t_k) &= J_k(u(t_k^-)), \quad k = 1, 2, \cdots, m,
\end{aligned}
$$

(4.1)
corresponding to $I_{\alpha^+}^{1-\sigma; \varphi} u(a) = u_a$ and $I_{\alpha^+}^{1-\sigma; \varphi} u(a) = v_a$ respectively. Then,

$$
\|u - v\|_{\mathcal{PC}_{1-\sigma; \varphi}(J, \mathbb{R})} \leq \frac{|u_a - v_a|}{\Gamma(\sigma)} \left(1 + \frac{1}{\Gamma(\sigma)} E_\varphi(\varphi(T) - \varphi(a))^{1-\sigma+\varphi}\right)^m \times E_\varphi (\varphi(T) - \varphi(a))^{1-\sigma+\varphi}.
$$

(4.2)

Proof: Let $u, v \in \mathcal{PC}_{1-\sigma; \varphi}(J, \mathbb{R})$ are the solutions of the problem (4.1) corresponding to $I_{\alpha^+}^{1-\sigma; \varphi} v(a) = u_a$ and $I_{\alpha^+}^{1-\sigma; \varphi} u(a) = v_a$ respectively. Then, in the view of lemma 2.4, we have

$$
u(t) = \frac{(\varphi(t) - \varphi(a))^{\sigma-1}}{\Gamma(\sigma)} \left(\varphi(t) + \sum_{a < t_k < t} J_k(u(t_k^-))\right) + I_{\alpha^+}^{\varphi; f}(t, u(t))
$$

and

$$
v(t) = \frac{(\varphi(t) - \varphi(a))^{\sigma-1}}{\Gamma(\sigma)} \left(\varphi(t) + \sum_{a < t_k < t} J_k(v(t_k^-))\right) + I_{\alpha^+}^{\varphi; f}(t, v(t)).
$$

Using the hypotheses (H1)(ii), (H2)(i) for any $t \in J$, we have

$$
(\varphi(t) - \varphi(a))^{1-\sigma}|u(t) - v(t)|
\leq \frac{|u_a - v_a|}{\Gamma(\sigma)} + \frac{1}{\Gamma(\sigma)} \sum_{a < t_k < t} (\varphi(t_k) - \varphi(a))^{1-\sigma} |u(t_k^-) - v(t_k^-)|
\leq \frac{(\varphi(t) - \varphi(a))^{1-\sigma}}{\Gamma(\sigma)} \int_a^t (\varphi(s) - \varphi(a))^{\sigma-1} (\varphi(s) - \varphi(a))^{1-\sigma}|u(s) - v(s)| ds.
$$

By application of Lemma (2.3) to the above inequality with $U(t)$, $g(t)$ and $\beta_k$ as given in the proof of Theorem 5.2 and $\mathcal{V}(t) = \frac{|u_a - v_a|}{\Gamma(\sigma)}$, we obtain

$$
(\varphi(t) - \varphi(a))^{1-\sigma}|u(t) - v(t)|
\leq \frac{|u_a - v_a|}{\Gamma(\sigma)} \left(1 + \frac{1}{\Gamma(\sigma)} E_\varphi(\varphi(T) - \varphi(a))^{1-\sigma+\varphi}\right)^m E_\varphi (\varphi(T) - \varphi(a))^{1-\sigma+\varphi}, \quad t \in J.
$$

This gives the inequality (4.2). \qed

Remark 4.2 The inequality (4.2) gives dependence of solution of impulsive $\varphi$-HFDE (1.2) on initial conditions. Further, for $u_a = v_a$ the inequality (4.2) gives uniqueness of the solution also.
4.2 Continuous dependence on the functions

Now consider the following impulsive \( \Phi \)-HFDE

\[
\begin{aligned}
H \Phi_{a+}^\sigma v(t) &= \tilde{f}(t, v(t)), \quad t \in J - \{t_1, t_2, \ldots, t_m\}, \\
\Delta \PhiI_{a+}^\sigma v(t_k) &= \tilde{J}_k(v(t_k^-)), \quad k = 1, 2, \ldots, m, \\
\PhiI_{a+}^\sigma v(a) &= v_a \in \mathbb{R},
\end{aligned}
\]

(4.3)

where \( \tilde{f} : (a, T) \to \mathbb{R} \) and \( \tilde{J}_k : \mathbb{R} \to \mathbb{R} \) are the functions other than \( f \) and \( J_k \) specified in the problem (4.2).

**Theorem 4.3** Suppose that the functions \( f : (a, T) \to \mathbb{R} \) and \( J_k : \mathbb{R} \to \mathbb{R} \) satisfy the hypotheses \((H_1)(ii)\) and \((H_2)(i)\) respectively. Further, suppose that there is a constant \( \delta_a > 0 \), \( \varepsilon_f > 0 \), \( \varepsilon_J > 0 \) such that

\[
\begin{align*}
|u_a - v_a| &\leq \delta_a, \\
|f(t, u(t)) - \tilde{f}(t, u(t))| &\leq \varepsilon_f, \\
|J_k(u(t_k^-)) - \tilde{J}_k(u(t_k^-))| &\leq \varepsilon_J.
\end{align*}
\]

Then, the solution \( u \in \mathcal{PC}_{1-\sigma; \phi} (J, \mathbb{R}) \) of (4.2) and the solution \( v \in \mathcal{PC}_{1-\sigma; \phi} (J, \mathbb{R}) \) of (4.3) satisfy the inequality

\[
\|u - v\|_{\mathcal{PC}_{1-\sigma; \phi} (J, \mathbb{R})} \leq \left( \frac{1}{\Gamma(\sigma)} \delta_a + \frac{m}{\Gamma(\sigma)} \varepsilon_J + \frac{\varepsilon_f (\varepsilon_f + \varepsilon_J)}{\Gamma(\sigma) + 1} \right) \left( 1 + \frac{1}{\Gamma(\sigma)} \right) E_\Phi (\phi(T) - \phi(a))^{1-\sigma + \varepsilon_f} m E_\Phi ((\phi(T) - \phi(a))^{1-\sigma + \varepsilon_J}.
\]

(4.4)

(4.5)

**Proof:** Let \( u \in \mathcal{PC}_{1-\sigma; \phi} (J, \mathbb{R}) \) be the solution of (4.2) and \( v \in \mathcal{PC}_{1-\sigma; \phi} (J, \mathbb{R}) \) be the solution of (4.3). Then, their corresponding equivalent integral equations are given by respectively

\[
u(t) = \frac{(\phi(t) - \phi(a))^{\sigma - 1}}{\Gamma(\sigma)} \left( u_a + \sum_{a < t_k < t} J_k(u(t_k^-)) \right) + \PhiI_{a+}^{\phi, \sigma} f(t, u(t))
\]

and

\[
v(t) = \frac{(\phi(t) - \phi(a))^{\sigma - 1}}{\Gamma(\sigma)} \left( v_a + \sum_{a < t_k < t} \tilde{J}_k(v(t_k^-)) \right) + \PhiI_{a+}^{\phi, \sigma} \tilde{f}(t, v(t)).
\]

Using the hypotheses \((H_1)(ii)\) and \((H_2)(i)\), for any \( t \in J \), we have

\[
|u(t) - v(t)| \leq \frac{(\phi(t) - \phi(a))^{\sigma - 1}}{\Gamma(\sigma)} \delta_a + \frac{(\phi(t) - \phi(a))^{\sigma - 1}}{\Gamma(\sigma)} \sum_{a < t_k < t} |J_k(u(t_k^-)) - \tilde{J}_k(v(t_k^-))|
\]
Thus
\[
+ \frac{(\varphi(t) - \varphi(a))^{\sigma-1}}{\Gamma(\sigma)} \sum_{a < t_k < t} \left| \mathcal{J}_k(v(t_k^-)) - \widetilde{\mathcal{J}}_k(v(t_k^-)) \right| \\
+ \left| \mathcal{I}_{a+}^{\varphi}(f(t, u(t)) - f(t, v(t))) \right| + \left| \mathcal{I}_{a+}^{\varphi} \left( f(t, v(t)) - \tilde{f}(t, v(t)) \right) \right| \\
\leq \left( \frac{(\varphi(t) - \varphi(a))^{\sigma-1}}{\Gamma(\sigma)} \delta_a + \frac{(\varphi(t) - \varphi(a))^{\sigma-1}}{\Gamma(\sigma)} m \varepsilon_J + \frac{(\varphi(t) - \varphi(a))^{\sigma-1}}{\Gamma(\sigma)} \frac{m \varepsilon_f}{\Gamma(\theta + 1)} \varepsilon_f \right) \\
+ \frac{(\varphi(t) - \varphi(a))^{\sigma-1}}{\Gamma(\sigma)} \sum_{a < t_k < t} (\varphi(t_k^-) - \varphi(a))^{1-\sigma} \left| u(t_k^-) - v(t_k^-) \right| \\
+ \frac{1}{\Gamma(\theta)} \int_a^t \varphi'(s)(\varphi(t) - \varphi(s))^{\sigma-1} (\varphi(s) - \varphi(a))^{1-\sigma} |u(s) - v(s)| ds.
\]

Thus
\[
(\varphi(t) - \varphi(a))^{1-\sigma} |u(s) - v(s)| \\
\leq \left( \frac{1}{\Gamma(\sigma)} \delta_a + \frac{m}{\Gamma(\sigma)} \varepsilon_J + \frac{(\varphi(t) - \varphi(a))^{1-\sigma+\varepsilon}}{\Gamma(\theta + 1)} \varepsilon_f \right) \\
+ \frac{1}{\Gamma(\sigma)} \sum_{a < t_k < t} (\varphi(t_k^-) - \varphi(a))^{1-\sigma} \left| u(t_k^-) - v(t_k^-) \right| \\
+ \frac{(\varphi(t) - \varphi(a))^{1-\sigma}}{\Gamma(\theta)} \int_a^t \varphi'(s)(\varphi(t) - \varphi(s))^{\sigma-1} (\varphi(s) - \varphi(a))^{1-\sigma} |u(s) - v(s)| ds, \ t \in J
\]

Applying lemma 2.3 to the above inequality with \( U(t), g(t) \) and \( \beta_k \) as given in the proof of Theorem 5.2 and
\[
\mathcal{V}(t) = \left( \frac{1}{\Gamma(\sigma)} \delta_a + \frac{m}{\Gamma(\sigma)} \varepsilon_J + \frac{(\varphi(t) - \varphi(a))^{1-\sigma+\varepsilon}}{\Gamma(\theta + 1)} \varepsilon_f \right),
\]
we obtain
\[
(\varphi(t) - \varphi(a))^{1-\sigma} |u(s) - v(s)| \\
\leq \left( \frac{\delta_a}{\Gamma(\sigma)} + \frac{1}{\Gamma(\sigma)} m \varepsilon_J + \frac{(\varphi(T) - \varphi(a))^{1-\sigma+\varepsilon}}{\Gamma(\theta + 1)} \varepsilon_f \right) \\
\times \left( 1 + \frac{1}{\Gamma(\sigma)} E_\theta (\varphi(T) - \varphi(a))^{1-\sigma+\varepsilon} \right)^m E_\theta ((\varphi(T) - \varphi(a))^{1-\sigma+\varepsilon}), \ t \in J.
\]

This gives the inequality (4.4). \( \square \)

**Remark 4.4**

1. Inequality (4.4) shows that the solution \( u \) of (1.2) depends continuously on the functions involved in right hand side of (1.2).

2. If \( \delta_a \to 0, \varepsilon_J \to 0, \varepsilon_f \to 0 \), then we get the uniqueness of the solution of the problem (1.2).
(3) If \( \varepsilon_f = \varepsilon_J = 0 \), then the inequality (4.4) gives the dependence of solution of the problem (1.2) on the initial condition.

### 4.3 Continuous dependence on the order of \( \varphi \)-Hilfer derivative

Now consider the following impulsive \( \varphi \)-HFDE

\[
\begin{cases}
H \mathcal{D}_{a^+}^{\sigma - \delta, \nu; \varphi} v(t) = f(t, v(t)), \ t \in J - \{t_1, t_2, \ldots, t_m\}, \\
\Delta I_{a^+}^{1-\sigma; \nu} v(t_k) = J_k(v(t^-_k)), \ k = 1, 2, \ldots, m, \\
I_{a^+}^{1-\sigma; \nu} v(a) = v_a \in \mathbb{R},
\end{cases}
\]

where \( f \) and \( J_k \) specified in the problem (1.2) and \( \sigma^* = \sigma + \delta(\nu - 1), \delta > 0 \).

**Theorem 4.5** Suppose that the functions \( f : (a, T] \to \mathbb{R} \) and \( J_k : \mathbb{R} \to \mathbb{R} \) satisfy the hypotheses \( (H_1) \) and \( (H_2) \) respectively. Then, the solution \( u \in \mathcal{P}C_{1-\sigma; \varphi}(J, \mathbb{R}) \) of (1.2) and the solution \( v \in \mathcal{P}C_{1-\sigma^*; \varphi}(J, \mathbb{R}) \) of (4.6) satisfy the inequality

\[
(\varphi(t) - \varphi(a))^{1-\sigma}|u(t) - v(t)| \\
\leq \mathcal{B}(t) \left( 1 + \frac{1}{\Gamma(\sigma)} E_{\varphi}(\varphi(T) - \varphi(a))^{1-\sigma + \nu} \right)^m E_{\varphi} ((\varphi(T) - \varphi(a))^{1-\sigma + \nu}),
\]

where

\[
\mathcal{B}(t) = \frac{u_a}{\Gamma(\sigma)} - \frac{v_a(\varphi(t) - \varphi(a))^{\delta(\beta - 1)}}{\Gamma(\sigma + \delta(\beta - 1))} + \zeta \left| \frac{1}{\Gamma(\sigma)} - \frac{(\varphi(t) - \varphi(a))^{\delta(\beta - 1)}}{\Gamma(\sigma + \delta(\beta - 1))} \right| + \|f\| \mathcal{P}C_{1-\sigma, \varphi} \left\{ \frac{\Gamma(\sigma)(\varphi(t) - \varphi(a))^{\nu}}{\Gamma(\sigma + \nu)} - \frac{\Gamma(\sigma)(\varphi(t) - \varphi(a))^{\nu - \delta}}{\Gamma(\sigma + \nu - \delta)} \right\},
\]

and \( \zeta = \sum_{k=1}^m \zeta_k \).

**Proof:** Let \( u \in \mathcal{P}C_{1-\sigma; \varphi}(J, \mathbb{R}) \) be the solution of (1.2) and \( v \in \mathcal{P}C_{1-\sigma^*; \varphi}(J, \mathbb{R}) \) be the solution of (4.6). Then, their corresponding equivalent integral equations are given by respectively

\[
u(t) = \frac{(\varphi(t) - \varphi(a))^{\sigma - 1}}{\Gamma(\sigma)} \left( u_a + \sum_{a < t_k < t} J_k(u(t^-_k)) \right) + \mathcal{I}_{a^+}^{\sigma; \varphi} f(t, u(t))
\]

and

\[
\nu(t) = \frac{(\varphi(t) - \varphi(a))^{\sigma' - 1}}{\Gamma(\sigma')} \left( v_a + \sum_{a < t_k < t} J_k(v(t^-_k)) \right) + \mathcal{I}_{a^+}^{\sigma' - \delta; \varphi} f(t, v(t)).
\]

Using the hypotheses \( (H_1) \) and \( (H_2) \), for any \( t \in J \), we have

\[
|u(t) - v(t)| \leq \left| \frac{u_a(\varphi(t) - \varphi(a))^{\sigma - 1}}{\Gamma(\sigma)} - \frac{v_a(\varphi(t) - \varphi(a))^{\sigma' - 1}}{\Gamma(\sigma')} \right|.
\]
\[
\begin{align*}
& \leq \left| \frac{u_a(\varphi(t) - \varphi(a))^{\sigma - 1}}{\Gamma(\sigma)} - \frac{v_a(\varphi(t) - \varphi(a))^{(\sigma - 1) + \delta(\beta - 1)}}{\Gamma(\sigma + \delta(\beta - 1))} \right| \\
& \quad + \frac{\varphi(t) - \varphi(a)}{\Gamma(\sigma)} \sum_{a < t_k < t} \left| \mathcal{J}_k(u(t_k^-)) - \mathcal{J}_k(v(t_k^-)) \right| \\
& \quad + \left| \mathcal{J}_a^\varphi f(t, u(t)) - \mathcal{J}_a^{\varphi; \varphi} f(t, v(t)) \right| \\
& \quad + \frac{1}{\Gamma(q)} \int_a^t \varphi'(s)(\varphi(t) - \varphi(s))^{q - 1} \left| f(s, u(s)) - f(s, v(s)) \right| \, ds \\
& \quad + \int_a^t \varphi'(s) \left( \frac{(\varphi(t) - \varphi(s))^{q - 1}}{\Gamma(q)} \frac{a_{\varphi} - (\varphi(t) - \varphi(s))^{q - \delta - 1}}{\Gamma(q - \delta)} \right) \left| f(s, v(s)) \right| \, ds \\
& \quad + \frac{1}{\Gamma(q)} \int_a^t \varphi'(s)(\varphi(t) - \varphi(s))^{q - 1} \left( \varphi(s) - \varphi(a) \right)^{1 - \sigma} \left| u(s) - v(s) \right| \, ds \\
& \quad + \int_a^t \varphi'(s) \left( \frac{(\varphi(t) - \varphi(s))^{q - 1}}{\Gamma(q)} \frac{a_{\varphi} - (\varphi(t) - \varphi(s))^{q - \delta - 1}}{\Gamma(q - \delta)} \right) \left( \varphi(s) - \varphi(a) \right)^{\sigma - 1} \left( \varphi(s) - \varphi(a) \right)^{1 - \sigma} \left| f(s, v(s)) \right| \, ds \\
& \quad + \frac{1}{\Gamma(q)} \int_a^t \varphi'(s)(\varphi(t) - \varphi(s))^{q - 1} \left( \varphi(s) - \varphi(a) \right)^{1 - \sigma} \left| u(s) - v(s) \right| \, ds \\
& \quad + \|f\|_{\mathcal{P}C_{1 - \sigma; \varphi}} \left\{ \frac{\Gamma(\sigma)(\varphi(t) - \varphi(a))^{\sigma + 1 - \sigma}}{\Gamma(\sigma + q)} - \frac{\Gamma(\sigma)(\varphi(t) - \varphi(a))^{q - \delta + \sigma - 1}}{\Gamma(\sigma + q - \delta)} \right\}.
\end{align*}
\]

Therefore, for each \( t \in J \) we have

\[ (\varphi(t) - \varphi(a))^{1 - \sigma} |u(t) - v(t)| \]
where $B(t)$ is as defined in (4.8). Now applying the Lemma 2.3 to (4.9) with $U(t) = (\varphi(t) - \varphi(a))^{1-\sigma}|u(t) - v(t)|$, $V(t) = B(t)$, $g(t) = \frac{(\varphi(t) - \varphi(a))^{1-\sigma}}{\Gamma(\theta)}$, and $\beta_k = \frac{1}{\Gamma(\sigma)}$, we obtain

$$
\left(\varphi(t) - \varphi(a)\right)^{1-\sigma}|u(t) - v(t)| \\
\leq B(t) \left[ \prod_{i=1}^{k} \left\{ 1 + \frac{1}{\Gamma(\sigma)} E^\theta \left( \frac{(\varphi(t) - \varphi(a))^{1-\sigma}}{\Gamma(\theta)} (\varphi(t_i) - \varphi(a))^{\theta} \right) \right\} \right] \\
\times E^\theta \left( \frac{(\varphi(t) - \varphi(a))^{1-\sigma}}{\Gamma(\theta)} (\varphi(t) - \varphi(a))^{\theta} \right) \\
= B(t) \left[ \prod_{i=1}^{k} \left\{ 1 + \frac{1}{\Gamma(\sigma)} E^\theta \left( (\varphi(t_i) - \varphi(a))^{1-\sigma+\theta} \right) \right\} \right] E^\theta \left( (\varphi(t) - \varphi(a))^{1-\sigma+\theta} \right) \\
\leq B(t) \left( 1 + \frac{1}{\Gamma(\sigma)} E^\theta (\varphi(T) - \varphi(a))^{1-\sigma+\theta} \right)^m E^\theta \left( (\varphi(T) - \varphi(a))^{1-\sigma+\theta} \right).
$$

\[\square\]

**Remark 4.6** The inequality (4.7) gives not only the dependence of solution on the order of $\varphi$-Hilfer derivative but also gives the dependency of solution on the initial condition. Indeed,

1. if $u_a \neq v_a$ and taking $\delta \to 0$ then the inequality (4.7) gives dependency of solution on initial condition.

2. if $u_a = v_a$ then the inequality (4.7) gives dependency of solution on the order of $\varphi$-Hilfer derivative.

## 5 Ulam Stabilities of $\varphi$–HFDE

In this section, we investigate the Ulam–Hyers stabilities of the impulsive $\varphi$-HFDE (1.2).

Let $\tilde{u} \in PC_{1-\sigma; \varphi}(J, \mathbb{R})$, $\chi > 0$, $\epsilon > 0$ and $\theta : J \to \mathbb{R}$ be a nondecreasing function. We consider the following inequalities:

$$
\begin{cases}
|^{H}D_{a+}^{\theta; \varphi} \tilde{u}(t) - f(t, \tilde{u}(t))| \leq \epsilon, \ t \in J \\
|\Delta_{a+}^{1-\sigma; \varphi} \tilde{u}(t_k) - J_k(\tilde{u}(t_k^-))| \leq \epsilon, \ k = 1, 2, \cdots, m,
\end{cases}
$$

(5.1)
\[
\begin{aligned}
&\left\{ \begin{array}{l}
|H_{a^+}^{\theta;\varphi} \tilde{u}(t) - f(t, \tilde{u}(t))| \leq \theta(t), \ t \in J \\
|\Delta I_{a^+}^{-\sigma;\varphi} \tilde{u}(t_k) - J_k(\tilde{u}(t_k^-))| \leq \chi, \ k = 1, 2, \ldots, m
\end{array} \right. \\
\text{and} \\
&\left\{ \begin{array}{l}
|H_{a^+}^{\theta;\varphi} \tilde{u}(t) - f(t, \tilde{u}(t))| \leq \epsilon \theta(t), \ t \in J \\
|\Delta I_{a^+}^{-\sigma;\varphi} \tilde{u}(t_k) - J_k(\tilde{u}(t_k^-))| \leq \epsilon \chi, \ k = 1, 2, \ldots, m.
\end{array} \right. \\
\end{aligned}
\tag{5.2}
\]

**Definition 5.1** The problem (5.2) is said to be Ulam–Hyers (UH) stable if for \( \epsilon > 0 \) there exists a constant \( C_{m, \varphi} > 0 \) such that, for every solution \( \tilde{u} \in \mathcal{P}C_{1-\sigma; \varphi}(J, \mathbb{R}) \) of the inequality (5.1), there is a unique solution \( u \in \mathcal{P}C_{1-\sigma; \varphi}(J, \mathbb{R}) \) to the problem (1.2) satisfying

\[
\| \tilde{u} - u \|_{\mathcal{P}C_{1-\sigma; \varphi}(J, \mathbb{R})} \leq C_{m, \varphi} \epsilon, \ t \in J.
\]

**Definition 5.2** The problem (1.2) is said to be generalized Ulam–Hyers (GUH) stable if there exists \( \phi_{m, \varphi} \in C(\mathbb{R}^+, \mathbb{R}^+) \) with \( \phi_{m, \varphi}(0) = 0 \) such that for each solution \( \tilde{u} \in \mathcal{P}C_{1-\sigma; \varphi}(J, \mathbb{R}) \) of the inequality (5.2) there is a unique solution \( u \in \mathcal{P}C_{1-\sigma; \varphi}(J, \mathbb{R}) \) to the problem (1.2) satisfying

\[
\| \tilde{u} - u \|_{\mathcal{P}C_{1-\sigma; \varphi}(J, \mathbb{R})} \leq \phi_{m, \varphi}(\epsilon), \ t \in J.
\]

**Definition 5.3** The problem (1.2) is said to be Ulam–Hyers–Rassias (UHR) stable corresponding to \((\theta, \chi)\) if for every \( \epsilon > 0 \) there exists a real number \( C_{m, \varphi, \theta} > 0 \) such that, for every solution \( \tilde{u} \in \mathcal{P}C_{1-\sigma; \varphi}(J, \mathbb{R}) \) of the inequality (5.3), there is a unique solution \( u \in \mathcal{P}C_{1-\sigma; \varphi}(J, \mathbb{R}) \) to the problem (1.2) satisfying

\[
(\varphi(t) - \varphi(a))^{1-\sigma} |\tilde{u}(t) - u(t)| \leq C_{m, \varphi, \theta} \epsilon (\theta(t) + \chi), \ t \in J.
\]

**Definition 5.4** The problem (1.2) is said to be generalized Ulam–Hyers–Rassias (GUHR) stable corresponding to \((\theta, \chi)\) if there exists a constant \( C_{m, \varphi, \theta} > 0 \) such that, for every solution \( \tilde{u} \in \mathcal{P}C_{1-\sigma; \varphi}(J, \mathbb{R}) \) of the inequality (5.2), there is a unique solution \( u \in \mathcal{P}C_{1-\sigma; \varphi}(J, \mathbb{R}) \) to the problem (1.2) satisfying

\[
(\varphi(t) - \varphi(a))^{1-\sigma} |\tilde{u}(t) - u(t)| \leq C_{m, \varphi, \theta} (\theta(t) + \chi), \ t \in J.
\]

**Remark 5.1** The function \( \tilde{u} \in \mathcal{P}C_{1-\sigma; \varphi}(J, \mathbb{R}) \) is called a solution of the inequality (5.1) if there is a function \( \mathcal{E} \in \mathcal{P}C_{1-\sigma; \varphi}(J, \mathbb{R}) \) together with a sequence \( \{\mathcal{E}_k\}, k = 1, 2, \ldots, m \) depending on \( \tilde{u} \) with
(1) \(|\mathcal{E}(t)| \leq \epsilon, |\mathcal{E}_k| \leq \epsilon, t \in J, k = 1, 2, \cdots, m,\)

(2) \(H \mathbb{D}_{a^+}^{\phi, \nu; \varphi} \bar{u}(t) = f(t, \bar{u}(t)) + \mathcal{E}(t), t \in J,\)

(3) \(\Delta I_{a^+}^{1-\sigma; \varphi} \bar{u}(t_k) = \mathcal{J}_k(u(t_k^-)) + \mathcal{E}_k, k = 1, 2, \cdots, m.\)

Looking towards the Remark 5.1 one can state similar types of remark for the inequalities 5.2 and 5.3.

**Theorem 5.2** If the hypotheses \((H_1)(ii)\) and \((H_2)(i)\) are satisfied then the problem 1.2 is \(UH\) stable.

**Proof:** Let \(\bar{u} \in \mathcal{PC}_{1-\sigma; \varphi}(J, \mathbb{R})\) be the solution of the inequality 5.1. Then in the view of Remark 5.1, we have

\[
\bar{u}(t) = \left(\frac{\varphi(t) - \varphi(a)^{\sigma-1}}{\Gamma(\sigma)} \right) \left( I_{a^+}^{1-\sigma; \varphi} \bar{u}(a) + \sum_{a < t_k < t} \mathcal{J}_k(\bar{u}(t_k^-)) + \sum_{a < t_k < t} \mathcal{E}_k \right) + I_{a^+}^{\sigma; \varphi} f(t, \bar{u}(t)) + I_{a^+}^{\sigma; \varphi} \mathcal{E}(t). \tag{5.4}
\]

Therefore, we have

\[
\left| \bar{u}(t) - \left(\frac{\varphi(t) - \varphi(a)^{\sigma-1}}{\Gamma(\sigma)} \right) \left( I_{a^+}^{1-\sigma; \varphi} \bar{u}(a) + \sum_{a < t_k < t} \mathcal{J}_k(\bar{u}(t_k^-)) + \sum_{a < t_k < t} \mathcal{E}_k \right) - I_{a^+}^{\sigma; \varphi} f(t, \bar{u}(t)) \right|
\]

\[
= \left| \frac{(\varphi(t) - \varphi(a))^{\sigma-1}}{\Gamma(\sigma)} \sum_{a < t_k < t} \mathcal{E}_k + I_{a^+}^{\sigma; \varphi} \mathcal{E}(t) \right|
\]

\[
\leq \frac{\varphi(t) - \varphi(a))^{\sigma-1}}{\Gamma(\sigma)} \sum_{a < t_k < t} \epsilon + \epsilon I_{a^+}^{\sigma; \varphi} (\varphi(t) - \varphi(a))^\theta
\]

\[
\leq m \epsilon (\varphi(t) - \varphi(a))^{\sigma-1} + \epsilon (\varphi(t) - \varphi(a))^\theta \tag{5.5}
\]

Consider the impulsive \(\varphi\)-HFDE

\[
\begin{cases}
H \mathbb{D}_{a^+}^{\phi, \nu; \varphi} u(t) = f(t, u(t)), t \in J - \{t_1, t_2, \cdots, t_m\}, \\
\Delta I_{a^+}^{1-\sigma; \varphi} u(t_k) = \mathcal{J}_k(u(t_k^-)), k = 1, 2, \cdots, m, \\
I_{a^+}^{1-\sigma; \varphi} u(a) = I_{a^+}^{1-\sigma; \varphi} \bar{u}(a).
\end{cases}
\]

Then by existence Theorem 3.1 it has at least one solution and in the view of lemma 2.4, we have

\[
u(t) = \left(\frac{\varphi(t) - \varphi(a)^{\sigma-1}}{\Gamma(\sigma)} \right) \left( I_{a^+}^{1-\sigma; \varphi} \bar{u}(a) + \sum_{a < t_k < t} \mathcal{J}_k(u(t_k^-)) + \sum_{a < t_k < t} \mathcal{E}_k \right) + I_{a^+}^{\sigma; \varphi} f(t, u(t)), t \in J . \tag{5.6}
\]

Utilizing 5.6, 5.7 and the hypotheses \((H_1)(ii)\) and \((H_2)(i)\), we get

\[|\bar{u}(t) - u(t)|\]
we obtain

\[
\begin{aligned}
&= \left| \ddot{u}(t) - \frac{(\varphi(t) - \varphi(a))^{\sigma-1}}{\Gamma(\sigma)} \left( \Gamma_{\alpha^+} \varphi \ddot{u}(a) + \sum_{\alpha < t_k < t} \mathcal{J}_k(u(t_k^-)) \right) - \Gamma_{\alpha^+} \varphi f(t, u(t)) \right| \\
\leq & \left| \ddot{u}(t) - \frac{(\varphi(t) - \varphi(a))^{\sigma-1}}{\Gamma(\sigma)} \left( \Gamma_{\alpha^+} \varphi \ddot{u}(a) + \sum_{\alpha < t_k < t} \mathcal{J}_k(u(t_k^-)) \right) - \Gamma_{\alpha^+} \varphi f(t, \ddot{u}(t)) \right| \\
&+ \frac{(\varphi(t) - \varphi(a))^{\sigma-1}}{\Gamma(\sigma)} \sum_{\alpha < t_k < t} \left| \mathcal{J}_k(u(t_k^-)) - \mathcal{J}_k(u(t_k^-)) \right| \\
&+ \left| \Gamma_{\alpha^+} \varphi \left( f(t, \ddot{u}(t)) - f(t, u(t)) \right) \right| \\
&\leq \frac{m \varepsilon (\varphi(t) - \varphi(a))^{\sigma-1}}{\Gamma(\sigma)} + \frac{\varepsilon (\varphi(t) - \varphi(a))^{\sigma-1}}{\Gamma(\sigma + 1)} \\
&+ \frac{(\varphi(t) - \varphi(a))^{\sigma-1}}{\Gamma(\sigma)} \sum_{\alpha < t_k < t} (\varphi(t_k) - \varphi(a))^{1-\sigma} |\ddot{u}(t_k^-) - u(t_k^-)| \\
&+ \frac{1}{\Gamma(\sigma)} \int_a^t \varphi'(s)(\varphi(t) - \varphi(s))^{\sigma-1} (\varphi(s) - \varphi(a))^{1-\sigma} |\ddot{u}(s) - u(s)| ds.
\end{aligned}
\]

Thus for any \( t \in J \), we have

\[
(\varphi(t) - \varphi(a))^{1-\sigma} |\ddot{u}(t) - u(t)| \\
\leq \varepsilon \left( \frac{m}{\Gamma(\sigma)} + \frac{(\varphi(T) - \varphi(a))^{1-\sigma+\varepsilon}}{\Gamma(\sigma + 1)} \right) + \frac{1}{\Gamma(\sigma)} \sum_{\alpha < t_k < t} (\varphi(t_k^-) - \varphi(a))^{1-\sigma} |\ddot{u}(t_k^-) - u(t_k^-)| \\
+ \frac{(\varphi(t) - \varphi(a))^{1-\sigma}}{\Gamma(\sigma)} \int_a^t \varphi'(s)(\varphi(t) - \varphi(s))^{\sigma-1} (\varphi(s) - \varphi(a))^{1-\sigma} |\ddot{u}(s) - u(s)| ds.
\]

By application of Lemma [2.3] to the above inequality with

\[
U(t) = (\varphi(t) - \varphi(a))^{1-\sigma} |\ddot{u}(t) - u(t)|, \\
V(t) = \varepsilon \left( \frac{m}{\Gamma(\sigma)} + \frac{(\varphi(T) - \varphi(a))^{1-\sigma+\varepsilon}}{\Gamma(\sigma + 1)} \right), \\
g(t) = \frac{(\varphi(t) - \varphi(a))^{1-\sigma}}{\Gamma(\sigma)}, \\
\beta_k = \frac{1}{\Gamma(\sigma)},
\]

we obtain

\[
(\varphi(t) - \varphi(a))^{1-\sigma} |\ddot{u}(t) - u(t)| \leq \varepsilon \left( \frac{m}{\Gamma(\sigma)} + \frac{(\varphi(T) - \varphi(a))^{1-\sigma+\varepsilon}}{\Gamma(\sigma + 1)} \right) \\
\times \left[ \prod_{i=1}^k \left( 1 + \frac{1}{\Gamma(\sigma)} E_{\varphi} \left( \frac{(\varphi(t) - \varphi(a))^{1-\sigma}}{\Gamma(\sigma)} (\varphi(t) - \varphi(a))^{\varepsilon} \right) \right) \right] \\
\times E_{\varphi} \left( \frac{(\varphi(t) - \varphi(a))^{1-\sigma}}{\Gamma(\sigma)} (\varphi(t) - \varphi(a))^{\varepsilon} \right).
\]

Since \( \varphi \) is increasing, \( (\varphi(t_i) - \varphi(a))^{\varepsilon} \leq (\varphi(t) - \varphi(a))^{\varepsilon}, \quad k = 1, 2, \ldots, m \), and hence above inequality reduces to

\[
(\varphi(t) - \varphi(a))^{1-\sigma} |\ddot{u}(t) - u(t)| \leq \varepsilon A_{m,\varphi} \left( \frac{m}{\Gamma(\sigma)} + \frac{(\varphi(T) - \varphi(a))^{1-\sigma+\varepsilon}}{\Gamma(\sigma + 1)} \right), \quad t \in J,
\]
where $A_{m,\varrho}$ is defined in (3.2). Therefore,

$$\|\tilde{u} - u\|_{\mathcal{PC}_{1-\sigma, \varphi}(J, \mathbb{R})} \leq C_{m,\varrho} \epsilon,$$

where

$$C_{m,\varrho} = A_{m,\varrho} \left( \frac{m}{\Gamma(\sigma)} + \frac{(\varphi(T) - \varphi(a))^{1-\sigma+\varrho}}{\Gamma(\varrho+1)} \right)$$

This proves the problem (1.2) is UH stable.

**Corollary 5.3** If the hypotheses (H1)(ii) and (H2)(i) are satisfied then the problem (1.2) is GUH stable.

**Proof:** Proof follows by setting $\phi_{m,\varrho}(\epsilon) = C_{m,\varrho} \epsilon$.

**Theorem 5.4** Suppose that (H1)(ii) and (H2)(i) hold. Moreover, assume that for a non-decreasing function $\theta \in C(J, \mathbb{R})$ there exists $\lambda_0 > 0$ such that $I_{a+}^{\theta} \varphi(t) \leq \lambda_0 \theta(t)$, $t \in J$. Then, the problem (1.2) is UHR stable with respect to $(\theta, \chi)$.

**Proof:** Let $\tilde{u} \in \mathcal{PC}_{1-\sigma, \varphi}(J, \mathbb{R})$ is a solution of the inequality (5.2). Then proceeding as in the proof of Theorem 5.2 we obtain

$$\left| \tilde{u}(t) - (\varphi(t) - \varphi(a))^{\sigma-1} \frac{u_a + \sum_{a < t_k < t} J_k(\tilde{u}(t_k^-))}{\Gamma(\sigma)} - I_{a+}^{\theta} f(t, \tilde{u}(t)) \right| \leq \frac{m \epsilon (\varphi(t) - \varphi(a))^{\sigma-1}}{\Gamma(\sigma)} \left( u_a + \sum_{a < t_k < t} J_k(\tilde{u}(t_k^-)) \right) + \epsilon \lambda_0 \theta(t) \tag{5.7}$$

Utilizing $\tilde{u}(t) - u(t)$

$$\left| \tilde{u}(t) - u(t) \right| = \left| \tilde{u}(t) - (\varphi(t) - \varphi(a))^{\sigma-1} \frac{u_a + \sum_{a < t_k < t} J_k(\tilde{u}(t_k^-))}{\Gamma(\sigma)} - I_{a+}^{\theta} f(t, \tilde{u}(t)) \right|$$

$$+ \frac{(\varphi(t) - \varphi(a))^{\sigma-1}}{\Gamma(\sigma)} \sum_{a < t_k < t} \left| J_k(\tilde{u}(t_k^-)) - J_k(u(t_k^-)) \right|$$

$$+ \left| I_{a+}^{\theta} (f(t, \tilde{u}(t)) - f(t, u(t))) \right|$$

$$\leq \left( \frac{m \epsilon (\varphi(t) - \varphi(a))^{\sigma-1}}{\Gamma(\sigma)} + \epsilon \lambda_0 \theta(t) \right)$$

$$+ \frac{(\varphi(t) - \varphi(a))^{\sigma-1}}{\Gamma(\sigma)} \sum_{a < t_k < t} \left( \varphi(t_k^-) - \varphi(a) \right)^{1-\sigma} \left| \tilde{u}(t_k^-) - u(t_k^-) \right|$$

$$+ \frac{1}{\Gamma(\theta)} \int_a^t \varphi'(s)(\varphi(t) - \varphi(s))^{\sigma-1} (\varphi(s) - \varphi(a))^{1-\sigma} \left| \tilde{u}(s) - u(s) \right| ds.$$
Thus for each \( t \in J \), we obtain
\[
(\varphi(t) - \varphi(a))^{1-\sigma} |\tilde{u}(t) - u(t)| \\
\leq \epsilon \left( \frac{m \chi}{\Gamma(\sigma)} + \lambda_\theta (\varphi(t) - \varphi(a))^{1-\sigma} \theta(t) \right) \\
+ \frac{1}{\Gamma(\sigma)} \sum_{a < t_k < t} (\varphi(t_k) - \varphi(a))^{1-\sigma} |\tilde{u}(t_k^-) - u(t_k^-)| \\
+ \frac{(\varphi(t) - \varphi(a))^{1-\sigma}}{\Gamma(\theta)} \int_a^t (\varphi(s) - \varphi(s))^{\theta - 1} (\varphi(s) - \varphi(a))^{1-\sigma} |\tilde{u}(s) - u(s)| ds.
\]

By application of Lemma (2.3) to the above inequality with \( U(t) \), \( g(t) \) and \( \beta_k \) as given in proof Theorem 5.2 and \( V(t) = \epsilon \left( \frac{m \chi}{\Gamma(\sigma)} + \lambda_\theta (\varphi(t) - \varphi(a))^{1-\sigma} \theta(t) \right) \), we obtain
\[
(\varphi(t) - \varphi(a))^{1-\sigma} |\tilde{u}(t) - u(t)| \leq \epsilon \left( \frac{m \chi}{\Gamma(\sigma)} + \lambda_\theta (\varphi(t) - \varphi(a))^{1-\sigma} \theta(t) \right) A_{m, \epsilon},
\]
where \( A_{m, \epsilon} \) is defined in (3.2). Therefore
\[
(\varphi(t) - \varphi(a))^{1-\sigma} |\tilde{u}(t) - u(t)| \leq C_{m, \epsilon, \theta} \epsilon (\chi + \theta(t)),
\]
where
\[
C_{m, \epsilon, \theta} = \left( \frac{m \chi}{\Gamma(\sigma)} + \lambda_\theta (\varphi(T) - \varphi(a))^{1-\sigma} \right) A_{m, \epsilon}.
\]

Hence the problem (1.2) is UHR stable. \( \square \)

**Corollary 5.5** Let the hypotheses of the Theorem 5.4 hold. Then, the problem (1.2) is GHUR stable with respect to \((\theta, \chi)\).

**Proof:** Proof follows by setting \( \epsilon = 1 \). \( \square \)

# 6 Examples

**Example 6.1** Consider the following impulsive \( \varphi \)-HFDE
\[
\begin{align*}
\text{H}_0^{\varphi, \nu; \varphi} u(t) &= (\varphi(t) - \varphi(0))^{1-\sigma} \frac{1}{1 + |u(t)|} + 3 \sin^2(\varphi(t) - \varphi(0)), \ t \in J = [0, 1] - \{ \frac{1}{2} \} \\
\Delta^{1-\sigma} \varphi u(\frac{1}{2}) &= \left( \varphi(\frac{1}{2}) - \varphi(0) \right)^{1-\sigma} \frac{1}{1 + |u(\frac{1}{2})|}, \\
I^{1-\sigma} \varphi u(0) &= \zeta \in \mathbb{R}.
\end{align*}
\]

Define \( f : [0, 1] \times \mathbb{R} \to \mathbb{R} \) by
\[
f(t, u) = \frac{(\varphi(t) - \varphi(0))^{1-\sigma}}{1 + |u|} + 3 \sin^2(\varphi(t) - \varphi(0))
\]
and \( J_1 : \mathbb{R} \to \mathbb{R} \) by
\[
J_1(u) = \frac{(\varphi(\frac{1}{2}) - \varphi(0))^{1-\sigma} |u|}{1 + |u|}.
\]
Note that for any \( u, v \in \mathbb{R} \) and \( t \in [0, 1] \), we have
\[
|f(t, u) - f(t, v)| \leq (\varphi(t) - \varphi(0))^{1-\sigma} \left| \frac{1}{1 + |u|} - \frac{1}{1 + |v|} \right|
\leq (\varphi(t) - \varphi(0))^{1-\sigma} |u - v|
\]
and
\[
|J_1(u) - J_1(v)| = \left( \varphi \left( \frac{1}{2} \right) - \varphi(0) \right)^{1-\gamma} \left( \frac{|u|}{1 + |u|} - \frac{|v|}{1 + |v|} \right) \leq \left( \varphi \left( \frac{1}{2} \right) - \varphi(0) \right)^{1-\sigma} |u - v|,
\]
\[
|J_1(u)| = \frac{(\varphi(\frac{1}{2}) - \varphi(0))^{1-\sigma} |u|}{1 + |u|} \leq \left( \varphi \left( \frac{1}{2} \right) - \varphi(0) \right)^{1-\sigma} = \zeta_1.
\]
Observe that \( f \) and \( J_1 \) satisfy the hypotheses \((H_1)\) and \((H_2)\). By applying the Theorem 3.1, the problem (6.1) has a unique solution on \([0, 1]\). Also by the Theorem 5.2, the problem (6.1) is UH stable. In addition for any solution \( v \in PC_{1-\sigma, \varphi}(J, \mathbb{R}) \) of the inequality
\[
\left\{ \begin{array}{l}
|H^{0+c}_{0+;\varphi} v(t) - f(t, v(t))| \leq \epsilon, \quad t \in J \\
|\Delta^{1-\sigma;\varphi} v(\frac{1}{2}) - J_1(v(\frac{1}{2}^-))| \leq \epsilon,
\end{array} \right.
\]
there exists a unique solution \( u \) of the problem (6.1) such that
\[
\|v - u\|_{PC_{1-\sigma, \varphi}(J, \mathbb{R})} \leq C_{1, \epsilon} \epsilon,
\]
where
\[
C_{1, \epsilon} = \left( \frac{1}{\Gamma(\sigma)} + \frac{(\varphi(1) - \varphi(0))^{1-\sigma+\theta}}{\Gamma(\theta + 1)} \right) \left( 1 + \frac{1}{\Gamma(\sigma)} E_{\phi}(\varphi(1) - \varphi(0))^{1-\sigma+\theta} \right)
\times E_{\phi}(\varphi(1) - \varphi(0))^{1-\sigma+\theta}.
\]

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**References**

[1] M. Feckan, Y. Zhou, J. Wang, On the concept and existence of solution for impulsive fractional differential equations, Commun Nonlinear Sci Numer Simulat, 17(7) (2012) 3050–3060.
[2] G. Wang, B. Ahmad, L. Zhang, J. J. Nieto, Comments on the concept of existence of solution for impulsive fractional differential equations, Commun Nonlinear Sci Numer Simulat 19 (2014) 401-403.

[3] J. Wang, Y. Zhou, Z. Lin, On a new class of impulsive fractional differential equations, Applied Mathematics and Computation 242 (2014), 649–657.

[4] J. Wang, Y. Zhou, M. Fečkan, On recent developments in the theory of boundary value problems for impulsive fractional differential equations, Computers and Mathematics with Applications 64 (2012), 3008-3020.

[5] J. Wang, Y. Zhou, M. Fečkan, On recent developments in the theory of boundary value problems for impulsive fractional differential equations, Computers and Mathematics with Applications 64 (2012), 3008–3020.

[6] M. Benchohra, F. Berhoun, Impulsive fractional differential equations with variable times, Computers and Mathematics with Applications 59 (2010), 1245–1252.

[7] G. M. Mophou, Existence and uniqueness of mild solutions to impulsive fractional differential equations, Nonlinear Analysis 72 (2010), 1604–1615.

[8] L. Zhang, G. Wang, Existence of solutions for nonlinear fractional differential equations with impulses and anti-periodic boundary conditions, Electronic Journal of Qualitative Theory of Differential Equations 7(2011), 1-11.

[9] B. Ahmad, S. Sivasundaram, Existence of solutions for impulsive integral boundary value problems of fractional order. Nonlinear Analysis: Hybrid Systems 4 (2010), 134–141.

[10] Z. Liu, X. Li, Existence and uniqueness of solutions for the nonlinear impulsive fractional differential equations, Commun Nonlinear Sci Numer Simulat 18 (2013), 1362-1373.

[11] M. Fečkan, Y. Zhou, J. Wang, On the concept and existence of solution for impulsive fractional differential equations, Commun Nonlinear Sci Numer Simulat 17 (2012), 3050–3060.

[12] A. Ali, K. Shah, D. Baleanu, Ulam stability results to a class of nonlinear implicit boundary value problems of impulsive fractional differential equations, Advances in Difference Equations (2019) 2019:5.

[13] J. Wang, Y. Zhou, M. Fečkan, Nonlinear impulsive problems for fractional differential equations and Ulam stability, Computers and Mathematics with Applications 64 (2012), 3389-3405.

[14] J. Wang, M. Fečkan, Y. Zhou, Ulams type stability of impulsive ordinary differential equations, Journal of Mathematical Analysis and Applications 395 (2012), 258–264.

[15] M. Benchohra, B. A. Slimani, Existence and Uniqueness of Solutions to Impulsive Fractional Differential Equations, Electronic Journal of Differential Equations, 2009(2009), No. 10, pp. 111.
[16] M. Benchohra, D. Seba, Impulsive fractional differential equations in banach spaces, Electronic Journal of Qualitative Theory of Differential Equations Spec. Ed. I, 8(2009), 1–14.

[17] M. Benchohra, B. Slimani, Existence and uniqueness of solutions to impulsive fractional differential equations, Electronic Journal of Differential Equations 2009(10)(2009), 1–11.

[18] B. Ahmad, S. Sivasundaram, Existence results for nonlinear impulsive hybrid boundary value problems involving fractional differential equations. Nonlinear Anal Hybrid Syst 2009;3:2518.

[19] M. Benchohra, D. Seba, Impulsive fractional differential equations in Banach spaces. Electron J Qual Theory Differ Equ 2009 [Special Edition I, No. 8, 14 pp].

[20] K. Balachandran, S. Kiruthika, Existence of solutions of abstract fractional impulsive semilinear evolution equations. Electron J Qual Theory Differ Equ 2010 [No. 4, 12 pp].

[21] G. Wang, B. Ahmad, L. Zhang, Impulsive anti-periodic boundary value problem for nonlinear differential equations of fractional order. Nonlinear Anal 2011;74:792–804.

[22] A. Harrat, J. J. Nieto, A. Debbouche, Solvability and optimal controls of impulsive Hilfer fractional delay evolution inclusions with Clarke subdifferential. J. Computational Applied Mathematics 344: 725-737 (2018).

[23] S. Harikrishnan, K. Kanagarajan, S. Sivasundaram, Stability analysis and dynamics of impulsive differential equations under Hilfer fractional derivative. Harikrishnan, S., et al. 2018 Nonlinear Studies 25, 403-415.

[24] H. M. Ahmed, M. M. El-Borai, H. M. El-Owaidy, A. S. Ghanem, Impulsive Hilfer fractional differential Equations, Advances in Difference Equations, Ahmed et al. Advances in Difference Equations (2018) 2018:226 https://doi.org/10.1186/s13662-018-1679-7.

[25] A. Fernandez, M. zarslan, D. Baleanu, On fractional calculus with general analytic kernels, Applied Mathematics and Computation, 354 (2019) 248265.

[26] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and applications of fractional differential equations, North–Holland Mathematics Studies, Elsevier, Amsterdam, Vol. 207, 2006.

[27] R. Almeida, A Caputo fractional derivative of a function with respect to another function, Commun. Nonlinear Sci. Numer. Simulat., 44, 460–481 (2017).

[28] R. Ameen, F. Jarad, T. Abdeljawad, Ulam Stability for Delay Fractional Differential Equations with a Generalized Caputo Derivative, Filomat, 32(15) (2018) 5265–5274.

[29] F. Jarad, S. Harikrishnan, K. Shah, K. Kanagarajan, Existence and stability results to a class of fractional random implicit differential equations involving a generalized Hilfer fractional derivative, Discrete and continuous dynamical systems series S,(2018)209–219.
[30] F. Jarad, T. Abdeljawad, Generalized fractional derivatives and Laplace transform, Discrete and continuous dynamical systems series S.,(2019)1775–1786.

[31] J.V.C. Sousa, Oliveira E. Capelas de., On the $\varphi$–Hilfer fractional derivative. Commun.Nonlinear Sci. Numer. Simulat. 60(2018),72-91.

[32] J.V.C. Sousa, Oliveira E. Capelas de., A Gronwall inequality and the Cauchy-type problem by means of $\varphi$- operator, arXiv:1709.03634, (2017).

[33] R. Hilfer, Applications of fractional calculus in Physics, World Scientific, Singapore, 2000.

[34] J.V.C. Sousa, E. Capelas De Oliveira, A Gronwall inequality and the cauchy-type problem by means of $\varphi$-Hilfer operator, arXiv:1709.03634(2017).

[35] K.D. Kucche, A. D. Mali, J.V.C. Sousa, On the nonlinear $\varphi$-Hilfer fractional differential equations Comp. Appl. Math. (2019) 38: 73. https://doi.org/10.1007/s40314-019-0833-5.

[36] K.D. Kucche, J.P. Kharade, J.V.C. Sousa, On the Nonlinear Impulsive $\varphi$-Hilfer Fractional Differential Equations, arXiv:1901.01814, (2019).

[37] J.V.C. Sousa, K. D. Kucche, E. Capelas de Oliveira, Stability of $\varphi$-Hilfer impulsive fractional differential equations, Applied Mathematics Letters 88 (2019) 7380.

[38] K Liu, J. Wang and, D ORegan, Ulam-Hyers-Mittag-Leffler stability for $\varphi$-Hilfer fractional-order delay differential equations, Advances in Difference Equations, (2019) 2019:50 https://doi.org/10.1186/s13662-019-1997-4.

[39] Z. Bai, X. Dong, C. Yin, Existence results for impulsive nonlinear fractional differential equation with mixed boundary conditions,Bai et al. Boundary Value Problems (2016) 2016:63.

[40] Zhou Y.,Basic theory of fractional differential equations. World scientific, 2014.

[41] J.V.C. Sousa, D.S. Oliveira, E. Capelas de Oliveira, A note on the mild solutions of Hilfer impulsive fractional differential equations, arXiv:1811.09256 (2018).