THE SET OF ALL ORTHOGONAL COMPLEX
STRUCTURES ON THE FLAT 6-TORI

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Abstract. In [2], Borisov, Salamon and Viaclovsky constructed non-standard orthogonal
complex structures on flat tori $T^n_R$ for any $n \geq 3$. We will call these examples BSV-tori. In
this note, we show that on a flat 6-torus, all the orthogonal complex structures are either the
complex tori or the BSV-tori. This solves the classification problem for compact Hermitian
manifolds with flat Riemannian connection in the case of complex dimension three.

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1. INTRODUCTION

Given a Hermitian manifold $(M^n, g)$, there are several canonical metric connections on it
that are well-studied. The Riemannian (or Levi-Civita) connection $\nabla$ which is torsion free,
and the Chern (aka Hermitian) connection $\nabla^c$ which is compatible with the complex structure,
and the Bismut connection $\nabla^b$, which is compatible with the almost complex structure and has
skew-symmetric $(3,0)$ torsion. When $g$ is Kähler, all three connections coincide, but when $g$
is not Kähler, the three are mutually distinct. Let us denote by $R, R^c$, and $R^b$ the corresponding
curvature tensors, respectively.

From the differential geometric point of view, it is very natural to study the curvature of each
of these connections, and ask what kind of manifolds are “space forms” with respect to a given
connection. In particular, one could ask what kind of compact complex manifolds will admit a
Hermitian metric with flat Riemannian or Chern or Bismut connection?

For the Chern connection $\nabla^c$, Boothby [1] proved in 1958 that compact Hermitian manifolds
with $R^c = 0$ identically are exactly the compact quotients of complex Lie groups equipped with
left invariant metrics. Such manifolds can be non-Kähler when $n \geq 3$. H.-C. Wang’s complex
parallisable manifolds [16] form an important subset in this class.

For the Bismut connection $\nabla^b$, in a recent work [17], we were able to show that compact
Hermitian manifolds $(M^n, g)$ with flat Bismut connections are exactly those covered by Samel-
son spaces, namely, $G \times \mathbb{R}^k$ equipped with a bi-invariant metric and a left invariant complex
structure. Here $G$ is a simply-connected compact semisimple Lie group, and $0 \leq k \leq 2n$. In par-
ticular, compact non-Kähler Bismut flat surfaces are exactly those isosceles Hopf surfaces, and
in dimension three their universal cover is either a central Calabi-Eckmann threefold $S^3 \times S^3$,
or $(\mathbb{C}^2 \setminus \{0\}) \times \mathbb{C}$. We refer the readers to [17] for more details.
So now we are left with the question of answering what kind of compact Hermitian manifolds \((M^n, g)\) will have identically zero Riemannian curvature tensor? By Bieberbach Theorem, we know that such manifolds admit finite unbranched cover that is a flat torus \(T^n_{R}\). So the question boils down to what kind of orthogonal complex structures are there on a flat \(T^n_{R}\)?

Given a flat 2n-torus \(M = T^n_{R}\), first of all, there are always compatible complex structures \(J\) on \(M\) that makes \(M\) a complex \(n\)-torus. All such complex structures (compatible with the orientation) are parameterized by the Hermitian symmetric space \(Z_n = SO(2n)/U(n)\). Clearly, for a complex structure \(J\) on \(M\) compatible with the flat metric \(g\), if \(J\) makes \(g\) a Kähler metric, then \((M, J)\) is a complex torus. In this case we will call this \(J\) a standard complex structure. When \(J\) makes the metric \(g\) non-Kähler, we will call such a complex structure non-standard.

When \(n = 2\), the classification theory for compact complex surfaces implies that any complex structure on \(T^n_{2}\) must be a complex 2-torus, thus there are no non-standard complex structures. For \(n \geq 3\), however, there are non-standard complex structures on some flat 2n-torus for each \(n \geq 3\). In [2], Borisov, Salamon, and Viaclovsky constructed non-standard orthogonal complex structures on some flat \(T^n_{2}\) for any \(n \geq 3\). We will call these examples warped tori of Borisov-Salamon-Viaclovsky, or BSV-tori for short. In Section 3, we will give some explicit discussion of BSV-tori in dimension 3 and their generalizations. In particular, BSV-tori in dimension 3 are defined as follows:

**Definition (BSV 3-tori).** For \(i = 1\) and \(2\), let \((M_i, g_i)\) be the flat torus of real dimension 2 and 4, respectively, and let \((M, g)\) be their product. Let \(J_1\) be the complex structure determined by \(g_1\), which makes \(M_1\) an elliptic curve. Let \(f\) be a non-constant holomorphic map \(f : M_1 \to \mathbb{P}^1\). Since \(\mathbb{P}^1 = SO(4)/U(2)\) is the set of all complex structures on the flat 4-torus \((M_2, g_2)\) compatible with the metric and the orientation, one may consider almost complex structures \(J\) on \(M\) defined by

\[
J = J_1 + J_{f(y_1)}
\]

at the point \((y_1, y_2)\) in \(M = M_1 \times M_2\). It is shown in [2] that \(J\) is integrable since \(f\) is holomorphic, so \((M, g, J)\) becomes a Hermitian manifold with everywhere zero Riemannian curvature. The metric \(g\) is not Kähler with respect to these complex structures (since \(f\) is non-constant), so they are all non-standard.

Note that any BSV-3-torus is always a product of a flat 2-torus with a flat 4-torus as a Riemannian manifold, while a generic flat 6-torus does not split. Also, as a complex manifold, a BSV 3-torus \(M^3\) is a holomorphic submersion over an elliptic curve, whose fibers are complex 2-tori, but the fibers are not all biholomorphic to each other.

The main purpose of this article is to show that, in complex dimension three, BSV-tori actually give all the possible orthogonal complex structures on the flat torus \(T^n_{R}\), besides the standard complex tori. In other words, we have the following:

**Theorem 1.** Let \((M^3, g)\) be a compact Hermitian manifold whose Riemannian curvature tensor is identically zero. Then a finite unbranched cover of \(M\) is holomorphically isometric to either a flat complex torus or a BSV-torus.

As the proof shall indicate, in higher dimensions, Riemannian flat compact Hermitian manifolds are still rather special and should form a highly restrictive class which contains all BSV tori. But perhaps a generalization of BSV tori should be formulated and organized before a classification statement can be made and proved. For \(n \geq 4\), the algebraic behavior of the Chern torsion tensor is much more complicated than the \(n = 3\) case, and we intend to pursue these higher dimensional cases as the next project.

One property worth noticing is that, these BSV 3-tori are actually non-Kählerian, namely, they do not admit any Kähler metric:

**Proposition 2.** Let \(M^3 = M_1 \times M_2\) be a BSV 3-tori, where \(M_1\) is a flat 2-torus and \(M_2\) a flat 4-torus. Then \(M\) admits no pluri-closed Hermitian metrics, in particular, it is non-Kählerian. Its Kodaira dimension is \(-\infty\), and its total torsion, namely, the \(L^2\)-norm of the Chern torsion...
of \( M \) with respect to the standard flat metric \( g \), is equal to \( 32\pi v_2 d \), where \( v_2 \) is the volume of \( M_2 \) and \( d \) the degree of the map \( f : M_1 \to \mathbb{P}^1 \).

In Section 3 we will prove a slightly more general version of Proposition 2, where \( M_1 \) is replaced by any compact Riemann surface with positive genus. All statements are valid except the one on Kodaira dimension. We should point it out it is already proved in [2] (Proposition 5.3 on P.144 [2]) that the flat metric \( (M^3, J, g) \) is not Kähler if the holomorphic map \( f \) in the definition is non-constant. Here we emphasize that \( (M^3, J) \) is non-Kählerian in the sense that it does not admit any Kähler metric.

Since the degree of the map \( f \) can be any positive integer greater than 1, we know that on \( T^3_g \), there are infinitely many complex structures with mutually distinct first Chern class, and there is no uniform bound on the total torsion, even though all complex structures are balanced in this case ([10], [2]).

In 1958 Calabi [5] discovered that \( M_1 \times T^3_g \) where \( M_1 \) is a hyperelliptic Riemann surface with odd genus \( g \geq 3 \) and \( T^3_g \) a real 4-torus, can be given a complex structure \( J \) such that the resulting threefold \( (M^3, J) \) admits no Kähler metric and has vanishing first Chern class. The complex structure Calabi used is related to vector cross product in the space of purely Cayley numbers. In Section 4, we show that Calabi’s construction is a special case of the BSV type warped complex structures on \( M_1 \times T^3_g \). The induced Hermitian metrics from Calabi’s construction is also a special case of balanced metrics which are product Riemannian metrics.

It seems natural to ask whether Theorem 1 is also true when \( M_1 \) is a Riemann surface with genus \( g \geq 2 \) with its standard hyperbolic metric. In the end of paper we formulate the problem and leave it to the future studies.

2. The kernel spaces of the torsion

Let us start with a Hermitian manifold \( (M^n, g) \). Following the notations of [18], we will denote by \( \nabla, \nabla^c \) the Riemannian (aka Levi-Civita) or the Chern (aka Hermitian) connection, respectively. Denote by \( R, R^c \) the curvature tensors of these two connections, and by \( T^c \) the torsion tensor of \( \nabla^c \). Under a local unitary frame \( e \) of type \( (1, 0) \) tangent vectors, \( T^c \) has components

\[
T^c(e_i, e_j) = \sum_{k=1}^{n} 2 T^c_{ij} e_k, \quad T^c(e_i, \overline{e_j}) = 0.
\]

By Lemma 7 of [18], we have the following

\[
\begin{align*}
(1) \quad 2T^c_{ij, l} &= R^c_{jikl} - R^c_{jikl}, \\
(2) \quad R_{ijkl} &= T^c_{ij, k} + T^c_{ik, j} - T^c_{ij, k} - T^c_{ij, l}, \\
(3) \quad R_{ijkl} &= T^c_{ij, kl} - T^c_{ij, kl}, \\
(4) \quad R_{ijkl} &= T^c_{ij, kl} - T^c_{ij, kl}.
\end{align*}
\]

for any indices \( i, j, k, l \). Here and below, \( r \) is summed from 1 to \( n \), and the index after the comma stands for covariant derivative with respect to \( \nabla^c \).

Now let us denote by \( T^c_{ij, l}, T^c_{ij, l} \) the covariant derivatives with respect to \( \nabla \). Following the notations of [18], we have

\[
\begin{align*}
\nabla_{e_i} e_i &= \nabla^c_{e_i} e_i + \gamma_{ir}(e_i) e_r = \nabla^c_{e_i} e_i + T^c_{ij} e_r, \\
\nabla_{e_i} e_j &= \nabla^c_{e_i} e_j + \gamma_{ir}(e_i) e_r + (\theta_2)_{ir}(e_i) e_r = \nabla^c_{e_i} e_j + T^c_{ij} e_r + T^c_{ir} e_r.
\end{align*}
\]

By a straightforward computation, we obtain the following identities:

\[
\begin{align*}
T^c_{ij, l} &= T^c_{ij, l} - T^c_{ir, l} + T^c_{ir, l}, \\
T^c_{ij, l} &= T^c_{ij, l} - T^c_{ijkl} + T^c_{ijkl} - T^c_{ijkl}.
\end{align*}
\]
From the last equality, we get
\begin{equation}
T^k_{ij;\ell} - T^l_{ij;k} = T^k_{ij;\ell} - T^l_{ij;k} = -2T^l_{ij;k\ell} + T^l_{rj;r}\hat{T}^{k}_{ij} - T^l_{rj;r}\hat{T}^{k}_{ij} - T^l_{rj;r}\hat{T}^{k}_{ij} + T^l_{rj;r}\hat{T}^{k}_{ij}.
\end{equation}

Combining (2) and (7), or comparing (3) with (9), we get
\begin{equation}
T^k_{ij;l} = T^r_{ij;r} - R_{ij;k},
\end{equation}
\begin{equation}
T^k_{ij;\ell} - T^l_{ij;k} = -R_{ij;\ell}.
\end{equation}

So for Hermitian manifold \((M^n, g)\) with \(R = 0\) everywhere, we have the following

**Lemma 1.** On a Hermitian manifold \((M^n, g)\) with identically zero Riemannian curvature, let \(T^k_{ij}\) be the components of (half of) the torsion of the Chern connection, under a local unitary frame \(e\). Their covariant derivatives with respect to the Riemannian connection \(\nabla\) satisfy
\begin{equation}
\sum_{r} T^r_{ij} T^k_{rl},
\end{equation}
\begin{equation}
T^k_{ij;\ell} - T^l_{ij;k} = T^l_{ij;\ell} - T^l_{ij;k} = -R_{ij;\ell},
\end{equation}
for any \(i, j, k, l\) between 1 and \(n\).

Now if \(M\) is also compact, then since \(R = 0\), by the equality case of the main theorem of [10], or by Theorem 3 of [18], we know that \(M\) is balanced. That is, \(\sum_i T^i_{ij} = 0\) for any \(i\). So by (12) we have \(\sum_i T^i_{ij;\ell} = 0\) for any \(i, j\).

Let us fix a point \(p \in M^n\). Denote by \(W \cong \mathbb{R}^{2n}\) the real tangent space of \(M\) at \(p\), and by \(V \cong \mathbb{C}^n\) the space of type \((1,0)\) complex tangent vectors at \(p\), and \(J\) the almost complex structure of \(M\). Since \(T^c(e_i, e_j) = 0\) and \(T^c(e_i, e_j) = 2\sum_{k=1}^n T^k_{ij} e_k\) under any unitary frame \(e\), we have
\begin{equation}
T^c(Jx, y) = T^c(x, Jy), \quad T^c(Jx, y) = J T^c(x, y)
\end{equation}
for any \(x, y\) in \(W\). Consider linear subspaces \(K_1, K_2\) in \(W\) defined by
\begin{align*}
K_1 &= \{ x \in W \mid T^c(x, u) = 0, \forall u \in W \}, \\
K_2 &= \{ x \in W \mid \langle T^c(u, v), x \rangle = 0, \forall u, v \in W \}.
\end{align*}
Clearly \(K_1, K_2\) are both \(J\)-invariant. Let \(K_0 = K_1 \cap K_2\), and for \(i = 1, 2\), write \(K'_i = K_0 \oplus K_i\). Then we have orthogonal decomposition \(K_i = K_0 \oplus K'_i\) for \(i = 1, 2\). We claim that

**Lemma 2.** If the components of the torsion tensor under a unitary frame \(e\) at \(p\) satisfy the condition
\begin{equation}
\sum_{r=1}^n T^r_{ij} T^k_{rl} = 0
\end{equation}
for any \(i, j, k, l\), then at the point \(p\) we have the orthogonal decomposition
\[ W = K_0 \oplus K'_1 \oplus K'_2. \]

**Proof.** Note that all the subspaces \(K_0, K_i, K'_i\) are \(J\)-invariant, so we may consider their corresponding complex subspaces \(N_0, N_i, N'_i\) in \(V\) instead, where \(i = 1, 2\). Clearly, \(N_1\) consists of all \(X \in V\) such that \(T^c_X = 0\), and \(N_2\) consists of all \(X \in V\) such that \(T^c_{XX} = 0\).

Here and from now on we adopted the convention that \(T^X_{ij} = \sum_k X_k T^k_{ij}\) for \(X = \sum_k X_k e_k\) in \(V\). This is because \(T^X_{ij}\) is conjugate linear in the upper position.

As in the proof of Theorem 2 of [18], for \(X = \sum_i X_i e_i \in V\), we will denote by \(A_X\) the linear transformation from \(V\) to \(V\) defined by
\[ A_X(e_i) = \sum_{j=1}^n T^X_{ij} e_j = \sum_{k,j=1}^n X_k T^j_{ki} e_j. \]

With this notation, (14) is simply saying that \(A_X A_Y = 0\) for any \(X, Y\) in \(V\). In particular, \((A_X)^2 = 0\). So \(N_2\) is the orthogonal complement of \(\sum_{X \in V} \text{Im}(A_X)\), where \(\text{Im}(A_X)\) stands for the image space of \(A_X\). In the mean time, it is clear that \(N_1 = \bigcap_{X \in V} \ker(A_X) = \{ X \in V \mid A_X = 0 \}. \)
Since $A_X A_Y = 0$ for any $X, Y$ in $V$, we have $\sum_{X \in V} \text{Im}(A_X) \subseteq \bigcap_{X \in V} \ker(A_X)$. So $N_2^* \subseteq N_1$. Therefore, $V = N_0 \oplus N_1^1 \oplus N_2^2$, where $N_1 = N_0 \oplus N_1^1$, $N_2 = N_0 \oplus N_2^2$, and all the direct sums are orthogonal. This completes the proof of the lemma.

**Remark:** (1). This lemma says that, when the equation (14) holds, or equivalently $T_{ij;i}^k = 0$ by (12), the torsion tensor obeys a nice decomposition which resembles those on a warped torus of the BSV type [2].

(2). Notice that for any $0 \neq X \in N_1^1$, there exists some $Y, Z$ in $V$ (necessarily in $N_2^2$) such that $T_X^Y Z \neq 0$, as otherwise $X$ would be in $N_2$ by definition. Similarly, for any $0 \neq X \in N_2^2$, there must be $Y$ and $Z$ (where $Y \in N_1^1$ and $Z \in N_2^2$ necessarily) such that $T_X^Y Z \neq 0$.

Next, let us examine the behavior of the almost complex structure under the above decomposition. We have the following

**Lemma 3.** Let $(M^n, g)$ be a Hermitian manifold with $R = 0$ identically, and assume that (14) holds everywhere. In an open subset of $M$ where $K_0, K_1^1$ and $K_2^2$ form distributions, we can write $J = J_0 + J_1 + J_2$ for the decomposition of the almost complex structure under the decomposition $W = K_0 \oplus K_1^1 \oplus K_2^2$. Then we have $\nabla_x J_0 = \nabla_x J_1 = 0$ for any $x \in W$, $\nabla_y J_2 = 0$ for any $y \in K_2$, and $\nabla_y J_2 \neq 0$ for any $0 \neq y \in K_1^1$.

**Proof.** Under any local unitary frame $e$ in $M$, by using formula (5) and (6), we get through a straight forward computation the following:

\[
(\nabla_{e_i} J)(e_j) = 0
\]

\[
(\nabla_{\tau e_i} J)(e_j) = 2\sqrt{-1} \sum_{k=1}^n T_{jk}^i \bar{e}_k
\]

Then the lemma is a direct consequence of (15), (16) and the remarks above, so we will omit the details here.

Now let us focus on the 3-dimensional case. In this case we will show that equation (14) always holds:

**Lemma 4.** Let $(M^3, g)$ be a compact Hermitian manifold with $R = 0$ identically. Then the equality (14) holds everywhere.

**Proof.** Since $M$ is compact and $R = 0$, by the equality case of Gauduchon’s inequality in [10], we know that $g$ is balanced. So $\sum_k T_{jk}^i k = 0$. By letting $k = i$ and sum up in (12), we get $\sum_{i,k} T_{jk}^i T_{lk}^i = 0$ for any $j, l$. In other words, we have

\[
\text{tr}(A_X A_Y) = 0, \quad \forall \ X, Y \in V
\]

We will show that, when $n = 3$, the above equality (17) actually implies $A_X A_Y = 0$ for any $X, Y$ in $V$, which is (14).

Let $e$ be a unitary frame. Write $a_i = T_{jk}^i$, $b_i = T_{ij}^i$ where $(ijk)$ is a cyclic permutation of (123). These 6 terms are all the components of $T^c$ since $g$ is balanced. We have:

\[
A_{e_1} = \begin{bmatrix} 0 & 0 & 0 \\ b_2 & b_1 & a_3 \\ -b_3 & -a_2 & -b_1 \end{bmatrix}, \quad A_{e_2} = \begin{bmatrix} -b_2 & -b_1 & -a_3 \\ 0 & 0 & 0 \\ a_1 & b_3 & b_2 \end{bmatrix}, \quad A_{e_3} = \begin{bmatrix} b_3 & a_2 & b_1 \\ -a_1 & -b_3 & -b_2 \\ 0 & 0 & 0 \end{bmatrix}
\]

Therefore,

\[
\text{tr}(A_{e_i}^2) = 2(b_i^2 - a_j b_k) = 0, \quad \text{tr}(A_{e_i} A_{e_j}) = 2(a_k b_i - b_i b_j) = 0
\]

where $(ijk)$ is any cyclic permutation (123). Now let us fix a point $p$ and also fix $e_1$, and rotate $\{e_2, e_3\}$ if necessary, we may assume that $T_{i1}^2 = 0$. That is, we may assume that $b_1 = 0$. The above equalities implies that $a_2 a_3 = b_2 b_3 = a_2 b_3 = a_3 b_3 = 0$.

If $a_3 \neq 0$, then we have $b_3 = a_2 = 0$. So the only possibly non-zero terms are $a_1, a_3,$ and $b_2$. Also, $b_2^2 = a_1 a_3$. From this, it is easy to check that $A_{e_i} A_{e_m} = 0$ for any $1 \leq i, m \leq 3$. So (14)
holds. When both $a_2 = a_3 = 0$, then the only possibly non-zero term would be $a_1$. In this case clearly (14) holds.

So for a compact Hermitian threefold $(M^3, g)$ with $R = 0$, we have the orthogonal decomposition $T_M = N_0 \oplus N_0^\perp \oplus N_2^\perp$ at any $p \in M$, where $T_M = V$ is the holomorphic tangent space of $M$ at $p$, and $N_i$, $N_i^\perp$ are the complex subspaces of $V$ corresponding to the real kernel spaces $K_i$, $K_i^\perp$.

Now let us assume that $g$ is not Kähler, and let $U \subseteq M^3$ be the open subset where $T^c \neq 0$. For any $p \in U$, since $N_1^\perp$ needs to be at least one dimensional, and $N_2^\perp$ needs to be at least two dimensional, so we must have $N_0 = 0$ and $T_M = N_1 \oplus N_2$. Let us choose a local unitary frame $e$ such that $e_3 \in N_1$. Then $T^c_{12} = 0$ is the only non-zero components of $T^c$. By (12)-(14), we have

$$T^c_{ij} = 0, \quad T^c_{12,1} = T^c_{12,2} = 0.$$

In the open subset $U \subseteq M$, let $V = N_1 \oplus N_2$ be the decomposition of the holomorphic tangent bundle $T_M$, and $W = K_1 \oplus K_2$ be the corresponding $J$-invariant orthogonal decomposition of the real tangent space of $M$. We make the following claims:

Claim 1: In $U$, $K_2$ is a totally geodesic foliation with complete leaves.

Claim 2: For any $p \in U$, the leaves of $K_2$ near $p$ are parallel to each other.

Fix any $p \in U$. In a small neighborhood of $U$, let $e$ be a unitary frame such that $e_3$ lies in $N_1$. This is the unique type $(1, 0)$ tangent direction $X$ (up to scalar multiple) such that $T^c_{iX} = 0$ for any $i, j$. Denote by $\varphi$ the coframe dual to $e$. As in [18], write $\nabla e = \theta_1 e + \overline{\theta_2} \overline{e}$ for the connection form, then the condition $R = 0$ is the same as

$$\begin{align*}
\Theta_1 &= d\theta_1 - \theta_1 \theta_1 - \overline{\theta_2} \theta_1 = 0 \\
\Theta_2 &= d\theta_2 - \theta_2 \theta_1 - \overline{\theta_1} \theta_2 = 0
\end{align*}$$

Since $0 \neq \lambda = T^c_{12}$ is the only non-zero component of $T^c$ under $e$, by Lemma 2 of [18], we have

$$\theta_2 = \begin{bmatrix} \beta E & 0 & 0 \end{bmatrix}, \quad \text{where} \quad E = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad \beta = \overline{\lambda} \varphi_3.$$

Let us write

$$\theta_1 = \begin{bmatrix} \chi & \xi \\ -\xi^* & \alpha \end{bmatrix}.$$

Since $\theta_1$ is skew-Hermitian, and $E \chi + \lambda E = \text{tr}(\chi) E$, we get from (18) and (19) that

$$\begin{align*}
d\chi &= \chi \xi - \xi^* \alpha, \\
d\xi &= \chi \xi + \xi \alpha, \\
d\alpha &= -\xi^* \xi;
\end{align*}$$

(20)

$$d\beta = \beta \wedge \text{tr}(\chi), \quad \beta \wedge E \xi = 0.$$  

(21)

From the second equation in (21), we know that the entries of $\xi$ are multiples of $\varphi_3$:

$$\xi = v \varphi_3 = \begin{bmatrix} a \\ b \end{bmatrix} \varphi_3.$$

By the structure equation $d\varphi = -\theta_1 \varphi - \theta_2 \overline{\varphi}$, we obtain

$$d\varphi_3 = -(a \varphi_1 + b \varphi_2 + \alpha) \wedge \varphi_3.$$  

(22)

Since $K_2$ is the distribution annihilated by $\{\varphi_3, \overline{\varphi_3}\}$, the above identity and its conjugation show that $K_2$ is a foliation.

To see that $K_2$ is a totally geodesic foliation, we need to show that $\langle \nabla X Y, e_3 \rangle = 0$ for any $X$, $Y$ in $K_2$, or equivalently,

$$(\theta_1)_{i3}(e_j) = (\theta_1)_{i3}(e_j) = (\theta_2)_{i3}(e_j) = (\theta_2)_{i3}(e_j) = 0$$

for any $i, j$ in $\{1, 2\}$. As $(\theta_2)_{i3} = 0$, and $(\theta_1)_{i3}$ is given by $\xi$ which is proportional to $\varphi_3$, we know that $K_2$ is a totally geodesic foliation in $U$. 

Since $T^3_{12;k} = 0$ for any $k$ and $T^3_{12;\mathfrak{T}} = T^3_{12;\mathfrak{T}} = 0$, we know that along any leaf of $K_2$, $\lambda$ is a constant function thus remains non-zero, so the leaves of $K_2$ are complete in $U$. This concludes the proof of Claim 1.

Next let us prove Claim 2. It is equivalent to $K_1$ being a foliation, and equivalent to the condition that within $U$, the decomposition $W = K_1 \oplus K_2$ gives a local metric product splitting. It suffices to show that $\xi = 0$ at $p$.

Let $\sigma : \mathbb{R} \to U$ be the constant-speed geodesic contained in the leaf of $K_2$ through $p$, so that $\sigma'(0) = e_1 + \mathfrak{T}$. Write $\sigma'(t) = X$. By Lemma 3, $J_2 = J|_{K_2}$ is constant along the leaves of $K_2$, so we may choose our unitary frame $e$ in a neighborhood of $\sigma$ such that $e_1, e_2$ are parallel along $\sigma$. This implies that $\chi(X) = 0$. We also have $\alpha(X) = 0$ since $\mathfrak{T} = -\alpha$, and $\varphi_1(X) = 1$, $\varphi_2(X) = \varphi_3(X) = 0$. The second equation in (20) now gives
\[ dv\varphi_3 - v(a\varphi_1 + b\varphi_2 + \alpha) = \chi\xi + \xi\alpha, \]
when applied to the vectors $(X, e_3)$, we get
\[ X(a) - a^2 = 0. \]
So $a(t)$ satisfies the Riccati equation along the geodesic $\sigma$. Since solutions to the equation blows up in finite time unless the initial condition is trivial, we know that $a$ must be zero at $p$. Similarly, $b = 0$ at $p$, and this completes the proof of Claim 2.

**Claim 3:** The universal covering space $\pi : \tilde{M} \to M$ admits a product structure $\tilde{M} = Y_1 \times Y_2$, where $Y_1 \cong \mathbb{R}^2$ and $Y_2 \cong \mathbb{R}^4$, such that within the open subset $\pi^{-1}(U)$, the $Y_2$ factor are given by the leaves of $K_2$.

Since $M$ is a complex manifold, and it is well known that a flat metric $g$ is real analytic, any local splitting spreads to a global splitting on the universal cover. Here, however, we want to make sure that the extended splitting again respect the condition that $T^3_{12}$ is the only possibly non-zero component of $T^e$ when $e_3$ is in the $Y_1$ direction. To see this, let $\{U_a\}_{a \in A}$ be the connected components of $\pi^{-1}(U)$. Each $U_a$ is isometric to the product $\Sigma_a \times L_a$ where $L_a \cong \mathbb{R}^4$, $\Sigma_a$ is an open subset of the flat $\mathbb{R}^2$ and the $L_a$ factor are given by the leaves of $K_2$.

Given any $a, b \in A$, we claim that the affine subspaces $L_a$ and $L_b$ in $\tilde{M} = \mathbb{R}^6$ are parallel to each other. To this end, let $\sigma$ be a line segment in $\mathbb{R}^6$ which is the shortest path connecting $L_a$ and $L_b$. Then $\sigma$ is perpendicular to both $L_a$ and $L_b$. Consider the tangent vector field $X = \sigma'(t)$ along $\sigma$. Within $U_a$, as $X$ lives in $K_1$, $JX$ is parallel along $\sigma \cap U_a$ by Lemma 3. Since $g$ is real analytic, $JX$ is parallel along the entire $\sigma$. Now as both $L_a$ and $L_b$ are perpendicular to $X$ and $JX$, they must be parallel to each other.

Note that by Claim 3 and Lemma 3, we know that the complex structure on $\tilde{M}$ is actually a warped complex structure in the sense of [2], namely, if we write $J = J_1 + J_2$ for the decomposition of the almost complex structure, then $J_1$ is constant, and makes $Y_1$ the flat $\mathbb{C}$, and at any $(y_1, y_2) \in \tilde{M}$, $J_2$ is given by $J_{f(y_1)} \in Z_2$ where $Z_2 = SO(4)/U(2) \cong \mathbb{P}^1$ is the space of all complex structures on $\mathbb{R}^4$ compatible with the metric and the orientation, and $f : Y_1 \cong \mathbb{C} \to Z_2$ is a smooth map. As proved in [2], the integrability of $J$ corresponds to the holomorphy of $f$. (See also the next section for an explicit calculation of this). Clearly, when the flat metric $g$ is not Kähler with respect to $J$, $f$ can not be a constant.

**Claim 4:** The leaves of $K_2$ are compact in $M$.

Let us denote by $\Gamma$ the deck transformation group of $M$. Replacing $M$ by a finite unbranched cover of it if necessary, we may assume that $\Gamma \cong \mathbb{Z}^6$ acting as translations in $\mathbb{R}^6$. For $i = 1, 2$, let $p_i : \Gamma \to \Gamma_i$ be the projection into the isometry group of the factors $Y_i$, with $\Gamma_i$ being the image.

For any $\gamma(y_1, y_2) = (y_1 + a, y_2 + b)$ in $\Gamma$, since the complex structure on $\tilde{M}$ is preserved by $\gamma$, we have $J_{f(y_1)} = J_{f(y_1 + a)}$, where $f : Y_1 \cong \mathbb{C} \to Z_2 = \mathbb{P}^1$ is the holomorphic map characterizing
$J$ as a warped complex structure. That says that any $\Gamma_1$-orbit is contained in a level set of $f$, which is necessarily discrete in $Y_1 = \mathbb{C}$. So $\Gamma_1$ is discrete, which will imply that the leaves of $Y_2$ close up in $M$.

Indeed, let us take a leaf $F$ of the foliation of $Y_2$ in $M$, if $F$ is not compact, then there will be a sequence $x_i$ in $F$ that converges to a point $x_0 \in M$, such that $x_0 \not\in F$. Take a sufficiently small neighborhood $U$ of $x_0$, inside $U$ the foliation can be parameterized by $F_t$, where $t$ belongs to a small open subset $V \subset Y_1$. We may assume that $F_0$ is the one through $x_0$. By assumption $F_0$ is not in $F$, but there exists $t_i \to 0$ such that $F_{t_i}$ is a part of $F$.

Now let us look at the picture on the universal cover. Take a point $0$ over $x_0$ and a small neighborhood $\tilde{U}$ over $U$. The pre-image $\tilde{\pi}^{-1}(F)$ is equal to the union of $\Gamma_1 \times Y_1$. So if $\Gamma_1$ is discrete, then $\tilde{\pi}^{-1}(F)$ would be closed in the universal cover; however in $\tilde{U}$, we have the same picture of $F_{t_i}$ and $F$ as in $U$. This leads to a contradiction.

To summarize, we have proved that, if $(M^3, g)$ is a compact, non-Kähler, Hermitian manifold with flat Riemannian connection, then a finite unbranched cover $M'$ of $M$ is isometric to $M_1 \times M_2$, where $(M_1, g_1)$ is a flat 2-torus and $(M_2, g_2)$ is a flat 4-torus, and the complex structure $J$ on $M'$ is given by

$$J = J_1 + J_f(x_1)$$

at the point $(x_1, x_2) \in M'$, where $J_1$ is a constant complex structure on $M_1$ compatible with $g_1$, and makes $M_1$ an elliptic curve, and $f : M_1 \to Z_2 \cong \mathbb{P}^1$ is a holomorphic map from the elliptic curve into the space of oriented orthogonal complex structures on $\mathbb{R}^4$. In other words, $M'$ is a BSV 3-torus. This completes the proof of Theorem 1.

3. The BSV-tori in dimension three

In this section, let us give a more detailed discussion on the BSV-tori in dimension three, and show that they are indeed non-Kählerian, namely, as a complex manifold they do not admit any Kähler metric. The readers are referred to [2] for a much broader discussion on the subject, and here we will try to be explicit and also focus on the differential-geometric aspect.

Following [2], let $\mathbb{P}^3$ be the set of all constant complex structures on $\mathbb{R}^3$ compatible with a fixed flat metric and orientation. Its elements are skew-symmetric orthogonal $4 \times 4$ real matrices, and with a choice of orientation, they can be expressed as

$$J_{(a,b,c)} = \begin{pmatrix} aE & bE + cI \\ bE - cI & -aE \end{pmatrix}, \text{ where } E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$I$ is the identity matrix, and $a^2 + b^2 + c^2 = 1$. Under the identification $\mathbb{P}^2 \cong \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, we have

$$a = \frac{2x}{r^2 + 1}, \quad b = \frac{2y}{r^2 + 1}, \quad c = \frac{r^2 - 1}{r^2 + 1}, \quad \text{where } r = |z|, \quad z = x + iy \in \mathbb{C} \cup \{\infty\}.$$  \hspace{1cm} (23)

We will write the above $J_{(a,b,c)}$ simply as $J_z$.

Now suppose that $(M_1, J_1, g_1)$ is a compact Hermitian manifold, and $f : M \to \mathbb{P}^1$ a smooth map. Let $(M_2, g_2)$ be a flat 4-torus, and consider the manifold $M = M_1 \times M_2$, equipped with the Riemannian product metric $g = g_1 \times g_2$, and the warped almost complex structure $J$ on $M$ giving by

$$J = J_1 + J_{f(y_1)}$$

at $(y_1, y_2) \in M$. Clearly, $J$ is orthogonal with respect to $g$, and as proved in [2] and also in [4], the integrability of $J$ is equivalent to the holomorphicity of the map $f$. Let us verify the equivalence in this explicit special case, namely, let us prove the following

**Lemma 5.** The almost complex structure $J$ defined on $M = M_1 \times M_2$ as above is integrable if and only if the map $f : M_1 \to \mathbb{P}^1$ is holomorphic.
is a straight forward computation to see that the above system is equivalent to the following
\[ N_J(X, Y) := [X, Y] - [JX, JY] + J[JX, Y] + J[X, JY] \]
vanishes identically, for any vector fields \( X, Y \) in \( M \). Since \([X, Y] = \nabla_X Y - \nabla_Y X \), \( J_1 \) is integrable, and \( J \) is constant along the \( M_2 = T^2_R \) direction, we get \( N_J(X, Y) = 0 \) if \( X, Y \) are both in the \( M_1 \) direction or both in the \( M_2 \) direction. So it suffices to verify \( N_J(X, Y) = 0 \) for \( X \) in \( M_1 \) and \( Y \) in \( M_2 \). Since \( \nabla_Y J_1 = 0 \) and \( \nabla_J Y J_1 = 0 \), we get
\[ N_J(X, Y) = \nabla_X Y + J\nabla_J X Y - J\nabla_J X Y + J\nabla_J Y J_1. \]
Now let \( \{\epsilon_1, \ldots, \epsilon_4\} \) be the standard parallel frame on \( M_2 \). By taking \( Y \) to be any \( \epsilon_i \) in the above equality, we know that \( N_J \) vanishes on \( M \) if and only if
\[ \nabla_{J_1 X} J_1 \epsilon_i = J\nabla_X J_1 \epsilon_i \]
for any \( 1 \leq i \leq 4 \) and any tangent vector \( X \) in \( M_1 \). Let us write \( J = J_z \) the \( 4 \times 4 \) matrix in (23), where \( z = f(y_1) \), and denote by \( J' \), \( J' \) its derivative in the direction \( J_2 X \), \( X \), respectively. Then the identity (27) is simply
\[ J' = J\dot{J}. \]
Using the expression of \( J \) in (23), and the fact \( a\dot{a} + b\dot{b} + c\dot{c} = 0 \), we get
\[ \begin{cases} 
    a' = cb - bc \smallskip 
    b' = ac - ca \smallskip 
    c' = ba - ab
\end{cases} \]
Now if we use the coordinate \( z = x + iy \) and the stereographic projection formula (24), then it is a straight forward computation to see that the above system is equivalent to the following
\[ \begin{cases} 
    x' = y \\
    y' = -\dot{x}
\end{cases} \]
which is just the Cauchy-Riemann equation. So \( J \) is integrable if and only if the map \( z = f(y_1) \) is holomorphic. \( \square \)

Of course the torus \( T^2_R \) in above lemma can be replaced by \( T^{2k}_R \) for any \( k \geq 2 \), and the lemma is still valid. This is Proposition 5.2 in [2] or Proposition 5.1 in [4].

Following [2], we will call the above compact Hermitian manifold \((M, g, J)\) a warped torus, and we are particularly interested in the complex dimension three case, namely, when \( M_1 \) is a compact Riemann surface of genus \( g(M_1) \), and \( f \) is a non-constant holomorphic map from \( M_1 \) into \( \mathbb{P}^1 \), or equivalently, a non-constant meromorphic function on the curve \( M_1 \). We will denote this compact Hermitian threefold by \( M^3_f \).

Note that for \( g(M_1) = 1 \) and \( f \) non-constant, these \( M^3_f \) are the BSV 3-tori defined in [2]. When \( M_1 = \mathbb{P}^1 \) and \( f \) is the identity map \( \iota \), then \( M_i \) is the twistor space over the flat 4-torus \( M_2 \). For \( g(M_1) \geq 2 \), such \( M^3_f \) include Calabi’s pioneer construction in [5].

As a complex manifold, it is clear that the projection map \( \pi_1 : M^3_f \to M_1 \) is a holomorphic submersion, and the fibers are flat complex 2-tori, but are not isomorphic to each other in general, so \( \pi_1 \) is not a holomorphic fiber bundle. For any \( y_2 \in M_2 \), the subset \( C_{y_2} = M_1 \times \{y_2\} \) is a totally geodesic complex submanifold of \( M^3_f \) and is holomorphically isometric to \( M_1 \), but \( C_{y_2} \) does not vary holomorphically in \( y_2 \in M_2 \).

It seems that these \( M^3_f \) form a rather interesting class of complex threefolds, and here we will satisfy ourselves by exploring their Hermitian geometry a little bit, and showing that they are always non-Kählerian (for non-constant \( f \)).

First let us choose a convenient local unitary frame \( e \) on \( M^3_f \). Let \( f \) be any non-constant meromorphic function on \( M_1 \), and write \( V_0 = M_1 \setminus \{ f = \infty \} \), \( V_{\infty} = M_1 \setminus \{ f = 0 \} \). Let
$D_1, \ldots, D_m$ be open subsets in $M_1$ such that their union is the entire $M_1$, and on each $D_j$ there exists a $(1,0)$-form $\psi_j$ with unit norm. Then the open subsets

$$U_{j0} = \pi_1^{-1}(D_j \cap V_0), \quad U_{j\infty} = \pi_1^{-1}(D_j \cap V_{\infty}), \quad 1 \leq j \leq m,$$

form an open covering of $M_1^3$. On each $U_{j0}$, we have a unitary coframe $\varphi$ where $\varphi_3 = \pi_1^2 \psi_j$, and

$$\varphi_1 = \frac{1}{\sqrt{2\sqrt{1 + |f|^2}}} \{ f(dx_1 - idx_3) + i(dx_2 - idx_4) \}$$

$$\varphi_2 = \frac{1}{\sqrt{2\sqrt{1 + |f|^2}}} \{ -i(dx_1 + idx_3) + f(dx_2 + idx_4) \}$$

at the point $(y_1, y_2)$ in $U_{j0}$, where $f = f(y_1)$, and $(x_1, \ldots, x_4)$ is the standard Euclidean coordinate on the universal cover $\tilde{M}_2 = \mathbb{R}^4$. Note that away from the poles of $f$, the above expressions are well-defined, and it is easy to check that $\varphi$ is indeed unitary and of type $(1,0)$ as $J$ is defined by (23)-(25). In each $D_{j\infty}$, a coframe can be given in a similar fashion, which we will omit.

In $D_j$, we have the structure equation $d\psi_j = -\xi \psi_j, \, d\xi = \Xi$, where $\Xi$ is the curvature form of $M_1$. Under the unitary coframe $\varphi$ in $U_{j0}$, it is easy to see that the connection forms are given by

$$\theta_1 = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \pi_1^2 \xi \end{bmatrix}, \quad \theta_2 = \begin{bmatrix} 0 & \beta & 0 \\ -\beta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \theta = \begin{bmatrix} \alpha & 0 & -\lambda \varphi_2 \\ 0 & \alpha & \lambda \varphi_1 \\ \pi_1^2 \xi & -\pi_1^2 \xi & 0 \end{bmatrix},$$

where $\lambda = T_{12}^3$, and

$$\alpha = \frac{1}{2(1 + |f|^2)} (fd\overline{f} - \overline{f} df), \quad \beta = \overline{\lambda} \varphi_3 = -\frac{i}{1 + |f|^2} df.$$

We have $da = 2\overline{\alpha} \varphi_3, db = 2\overline{\beta} \varphi_3$. From the structure equation $d\varphi = -\theta_1 \varphi - \theta_2 \overline{\varphi}$, we get

$$d\varphi_1 = -\alpha \varphi_1 + \overline{\beta} \varphi_2, \quad d\varphi_2 = -\alpha \varphi_2 - \beta \varphi_1, \quad d\varphi_3 = -\pi_1^2 \xi \varphi_3.$$

By taking exterior differentiation of $\beta = \overline{\lambda} \varphi_3$, we get

$$(d\overline{\alpha} + 2\overline{\lambda} \alpha - \overline{\lambda} \pi_1^2 \xi) \wedge \varphi_3 = 0,$$

so there will be a local smooth function $\mu$ in $U_{j0}$ such that

$$d\lambda - 2\lambda \alpha + \lambda \pi_1^2 \xi = \mu \varphi_3.$$

We compute the curvature form of the Chern connection $\Theta = d\theta - \theta \wedge \theta$ as follows:

$$\Theta = \begin{bmatrix} |\lambda|^2 (\varphi_2 \varphi_2 + \varphi_3 \overline{\varphi_3}) - |\lambda|^2 (\varphi_4 \overline{\varphi_4}) & |\lambda|^2 (\varphi_3 \overline{\varphi_3} - \mu \varphi_3 \overline{\varphi_3}) & |\lambda|^2 (\varphi_3 \overline{\varphi_3} + \mu \varphi_3 \overline{\varphi_3}) \\ -|\lambda|^2 (\varphi_1 \overline{\varphi_1}) & |\lambda|^2 (\varphi_1 \overline{\varphi_1} + \varphi_3 \overline{\varphi_3}) & |\lambda|^2 (\varphi_3 \varphi_3 + \mu \varphi_3 \overline{\varphi_3}) \\ |\lambda|^2 (\varphi_1 \overline{\varphi_1} - \mu \varphi_3 \overline{\varphi_3}) & |\lambda|^2 (\varphi_2 \varphi_2 + \mu \varphi_3 \overline{\varphi_3}) & \pi_1^2 \Xi - |\lambda|^2 (\varphi_2 \varphi_2 + \varphi_3 \overline{\varphi_3}) \end{bmatrix}.$$

From this, one gets the Chern forms of $M$, and thus the Chern classes. It is easy to see that

$$c_1(M) = 2(1 + \deg(f) - g(M_1)) \pi_1^2 \sigma,$$

where $\sigma \in H^2(M_1, \mathbb{Z}) \cong \mathbb{Z}$ is the positive generator. In particular, $c_1(M) = 0$ if and only if $g(M_1) = 1 + \deg(f)$, and such example with the lowest genus would be $g(M_1) = 3$ and $\deg(f) = 2$, namely, a hyperelliptic curve of genus 3. This includes Calabi’s 3-folds constructed in [5]. In Section 4 we will give a detailed discussion on the connection between Calabi’s 3-folds and BSV type warped complex structures.

Since $|T^c|^2 = 8 \sum_{i,j,k} |T_{ij}^k|^2$, in the case of $M_3^3$, it is equal to $16|\lambda|^2$, and we have

$$|\lambda|^2 \varphi_3 \overline{\varphi_3} = \overline{\varphi_3} = \frac{df \overline{df}}{(1 + |f|^2)^2}.$$

On $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, we have

$$\int_{\mathbb{P}^1} \frac{i dz \overline{dz}}{(1 + |z|^2)^2} = 2\pi.$$
So the $L^2$-norm of the Chern torsion of $M^3$ is, or its total Chern torsion, is given by

$$\int_M |T|^2 dv = 32\pi v_2 \deg(f),$$

where $v_2$ is the volume of $M_2$. In other words, the total Chern torsion of $M^3_j$ can be arbitrarily large, when $\deg(f)$ gets bigger and bigger.

Next, let us show that $M^3_j$ is non-Kählerian, namely, it cannot admit any Kähler metric. To see this, let us compute

$$d(\varphi_1 \overline{\varphi}_1) = (\alpha \varphi_1 + \beta \overline{\varphi}_2) \overline{\varphi}_1 - \varphi_1 (\alpha \varphi_1 + \beta \overline{\varphi}_2) = -\beta \overline{\varphi}_1 \varphi_2 + \overline{\beta} \varphi_1 \varphi_2,$$

here we used the fact that $\overline{\alpha} = -\alpha$, thus we get

$$\partial \overline{\partial} (\varphi_1 \overline{\varphi}_1) = d\overline{\partial} (\varphi_1 \overline{\varphi}_1) = d(-\beta \overline{\varphi}_1 \varphi_2) = \beta \overline{\partial} (\varphi_1 \overline{\varphi}_1 + \varphi_2 \overline{\varphi}_2).$$

Similarly,

$$\partial \overline{\partial} (\varphi_2 \overline{\varphi}_2) = \beta \overline{\partial} (\varphi_1 \overline{\varphi}_1 + \varphi_2 \overline{\varphi}_2),$$

Therefore, we get the following

$$\partial \overline{\partial} \omega_g = 2\sqrt{-1} \beta \overline{\partial} (\varphi_1 \overline{\varphi}_1 + \varphi_2 \overline{\varphi}_2) = 2\beta \overline{\partial} \omega_g.$$

Now, if $\omega_h$ is a Hermitian metric on $M^3_j$. Write $\omega_h = \sqrt{-1} \sum h_\sigma \varphi_i \overline{\varphi}_j$. The matrix $(h_\sigma)$ is positive definite. We have

$$\sqrt{-1} \partial \overline{\partial} \omega_g \wedge \omega_h = \frac{1}{3} |\lambda|^2 (h_1 \overline{\varphi}_1 + h_2 \overline{\varphi}_2) \omega_g^3.$$

Clearly, the integral of the right hand side over $M^3_j$ is positive, therefore we conclude that $\omega_h$ cannot satisfy the condition $\partial \overline{\partial} \omega_h = 0$ everywhere. That is, we have

**Lemma 6.** Let $(M_1, J, g_1)$ be any compact Riemann surface and $f$ any non-constant holomorphic map from $M_1$ to $\mathbb{P}^1$. Then the warped complex tori $M^3_j = M_1 \times T^4_\mathbb{R}$ does not admit any Hermitian metric that is pluri-closed. In particular, any such $M^3_j$ is non-Kählerian.

The notion of G-Kähler-like was introduced in [18] and it is equivalent to $\Theta_2 = 0$ (Lemma 5 in [18]). One result in [18] implies any compact G-Kähler-like Hermitian manifold must be balanced. Now it follows from (33) that the product metric $g_1 \times g_2$ on $M^3_j$ has $\Theta_2 = 0$, hence is G-Kähler-like and balanced. We remark that this observation is also implied by a more general result in [2]. (See Proposition 5.3(ii) on P.144 in [2])

Next let us show that $\text{Kod}(M^3_j) = -\infty$ when the base $M_1$ is an elliptic curve, i.e. for any $m \geq 1$, any $s \in H^0(M^3_j, mK_M)$ must be identically 0.

If not, let $D$ be the zero locus of $s$, then $D$ is an effective divisor in $M$. Since $M^3_j, g_1 \times g_2$ is balanced, the integral of $\omega^2$ along $D$ is well-defined, which will be the volume of $D$, thus positive. If $s$ is nowhere zero, then $D$ is the zero divisor and this integral is zero.

On the other hand, we have:

$$\int_D \omega^2 = \int_{M^3_j} -m \frac{\sqrt{-1}}{2\pi} \text{Tr}(\Theta) \wedge \omega^2 = \int_{M^3_j} -\frac{2m}{\pi} |\lambda|^2 \frac{\sqrt{-1}}{2} \varphi_1 \overline{\varphi}_1 \wedge \omega^2 = \frac{-m}{3\pi} \int_{M^3_j} |\lambda|^2 \omega^3,$$

so the integral is always negative, a contradiction.

Note that for compact Riemannian surfaces $M_1$ of genus $g_1$, the last integral in the above equals to $-2m(1 - d - g_1)v_2$, where $d$ is the degree of $f : M_1 \to \mathbb{P}^1$ and $v_2$ the volume of the 4-torus. So the Kodaira dimension of $M^3_j$ will be $-\infty$ if $1 + d - g_1 > 0$. Note that $d$ is always a positive integer, and it can be 1 only when $g_1 = 0$, so for $g_1 \leq 2$ one always has $1 + d - g_1 > 0$.

The above discussion is summarized in the following lemma.

**Lemma 7.** The warped complex tori $M^3_j = M_1 \times T^4_\mathbb{R}$ have Kodaira dimension $-\infty$ when the genus of $M_1$ is 2 or less. In particular, this is the case for all BSV 3-tori.
This completes the proof of Proposition 2.

4. CALABI 3-FOLDS VISITED

In 1958 Calabi [5] discovered that $M_1 \times T^4_2$ where $M_1$ is a hyperelliptic Riemann surface with odd genus $g \geq 3$ and $T^4_2$ a real 4-torus, can be given a complex structure $J$ such that the resulting threefold $(M_1, J)$ admits no Kähler metric and has vanishing first Chern class. In this section, we explore the connection between Calabi’s construction and the BSV type warped complex structures.

4.1. A review of Calabi’s 3-folds. Without specification, all results in this subsection is from Calabi [5]. Let $M_1$ be a hyperelliptic Riemann surface with odd genus $g \geq 3$, then $M_1$ admits a meromorphic function of degree 2, branched over 2g + 2 distinct points on $M_1$. Denote these points by $P_i$, and assume that $z(P_i) \neq \infty$ for each $i$. Then we get a single-valued meromorphic function on $M_1$:

$$w = \sqrt{\prod_{i=1}^{2g+2} (z - z(P_i))}.$$  

It is well-known that $H^{1,0}(M_1) = \text{Span}\{\frac{dz}{w} \mid 0 \leq j \leq g - 1\}$. Let $\phi(z)$ be an arbitrary polynomial in $z$ of degree $\frac{g-1}{2}$. If we view $\phi$ as a meromorphic function on $M_1$, it is of degree $g - 1$.

Now pick the following three linearly independent forms from $H^{1,0}(M_1)$

$$\omega_1 = \frac{\phi^2(z) - 1}{2w} dz, \omega_2 = \frac{\phi(z)}{w} dz, \omega_3 = \frac{\phi^2(z) + 1}{2\sqrt{-1}w} dz.$$  

This is exactly the Weierstrass representation of minimal surfaces, since $\omega_1, \omega_2, \omega_3$ do not vanish simultaneously on $M_1$. This implies that the map $(x_1, x_2, x_3)$ where

$$x_i = \text{Re} \int_{Q_0}^{Q} \omega_i,$$

locally maps $M_1$ to a minimal surface in $\mathbb{R}^3$. Here $Q_0$ is a fixed point on $M_1$. In general, those $x_i$ are not well-defined on $M_1$ globally, and they depend on $\pi_1(M_1, Q_0)$. But after lifting the map to the maximal Abelian covering $\tilde{M}_1$ of $M_1$, one gets a minimal immersion $F_1 : \tilde{M}_1 \rightarrow \mathbb{R}^3$. The image $F_1(\tilde{M}_1)$ might be complicated (e.g., everywhere dense), but the covering transformations in $\tilde{M}_1$ are $\mathbb{Z}^2$ generated by translations in $\mathbb{R}^3$.

Define $F = F_1 \times Id : \tilde{M}_1 \times \mathbb{R}^4 \rightarrow \mathbb{R}^7$, then $F$ defines an immersed hypersurface in $\mathbb{R}^7$. Note that the space of purely Cayley numbers can be identified as $\mathbb{R}^7$, therefore any immersed hypersurface in $\mathbb{R}^7$ can be made an almost complex manifold by defining

$$dF(Ju) = N \times dF(u)$$

for any $u \in T(\tilde{M}_1 \times \mathbb{R}^4)$. Here $\times$ stands for the cross product defined on the space of purely Cayley numbers.

Calabi [5] proved that the almost complex structure on the image of $F_1 \times Id$ in $\mathbb{R}^7$ is integrable when $F_1$ is a minimal immersion. Moreover, such a complex structure is invariant under translations in $\mathbb{R}^7$, thus descends down to the compact quotient $M_1 \times T^4_2$. This is the Calabi’s 3-fold $M^3$. It is proved in [5] that $M^3$ admits no Kähler metric and has $c_1(M^3) = 0$.

Calabi also proved that the induced metric $ds^2 = dF \cdot dF$ from $F(\tilde{M}_1 \times \mathbb{R}^4) \subseteq \mathbb{R}^7$ is compatible with the complex structure $J$ defined above. Therefore the corresponding metric $g$ on $(M^3, J)$ is a Hermitian metric.

Gray [11] studied the curvature properties of Calabi’s 3-fold $(M^3, J, g)$. It is shown in [11] that Calabi’s metric is G-Kähler-like (see [18] for a definition.)
4.2. Calabi 3-folds in terms of BSV-warped complex structures. Let \((M^3, J)\) be the Calabi’s 3-fold in Subsection 4.1, we now explain that Calabi’s construction is exactly a special case of BSV-warped complex structure on \(M_1 \times T_\mathbb{R}^4\).

By a direct calculation, one sees that the image \(F_1(\tilde{M}_1)\) in \(\mathbb{R}^3\) defined above has the unit normal vector

\[
N = (N_1, N_2, N_3) = \left( \frac{2 \Re \phi}{|\phi|^2 + 1}, \frac{|\phi|^2 - 1}{|\phi|^2 + 1}, \frac{2 \Im \phi}{|\phi|^2 + 1} \right).
\]

Therefore, the corresponding image \(F(\tilde{M}_1 \times \mathbb{R}^4)\) in \(\mathbb{R}^7\) has the unit normal vector

\[
N = (N_1, N_2, N_3, 0, 0, 0, 0).
\]

Using the table of the cross product defined on purely Cayley numbers, it is straightforward to write down the action of \(J\) restricted on \(T_p(\tilde{M}_1)\) and on \(T_p(\mathbb{R}^4) = \text{Span}\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} \}\).

For example, the first one is independent of \(T_p(\mathbb{R}^4)\), while the latter takes the following form under the basis \(\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} \}\):

\[
\begin{bmatrix}
0 & N_1 & N_2 & N_3 \\
-N_1 & 0 & -N_3 & N_2 \\
-N_2 & N_3 & 0 & -N_1 \\
-N_3 & -N_2 & N_1 & 0
\end{bmatrix}
\]

Comparing with Formula (23), we see that Calabi’s complex structure can be written as \(J = J_1 + J(\phi(x))\) where \(J_1\) is the complex structure on \(M_1\) and \(J(\phi(x))\) is the complex structure on \(T_\mathbb{R}^4\) defined by the holomorphic map \(\phi: M_1 \to \mathbb{P}^1\). Note that \(\phi\) is of degree \(g - 1\), by (36) we also see that \(c_1(M^3) = 0\).

Another interesting formula from the Weierstrass representation is that the total curvature \(\int_{F_1(\tilde{M}_1)} |K|^2 dA\) equals to \(4\pi\) multiple the degree of the Gauss map determined by \(N\) (i.e. degree of \(\phi\)). This resembles (37) on the total Chern torsion.

4.3. Remarks on the induced Hermitian metric on the Calabi 3-folds. As mentioned in Subsection 4.1, Calabi defined the Hermitian metric \(g\) on \(M_1 \times T^4\) as the one on \(F(\tilde{M}_1 \times \mathbb{R}^4)\) induced from the standard Euclidean metric on \(\mathbb{R}^7\). Note that

\[
F(\tilde{Q}, x_4, x_5, x_6, x_7) = (x_1, x_2, x_3, x_4, x_5, x_6, x_7)
\]

where \(\tilde{Q} \in \tilde{M}_1\). Therefore, the induced metric \(g\) is of the form (here \(u\) is a local holomorphic coordinate on \(\tilde{M}_1\))

\[
g = \sum_{i=1}^3 |\frac{\partial x_i}{\partial u}|^2 + \sum_{i=4}^7 |dx_i|^2.
\]

Apply the formula for the Weierstrass representation, we have

\[
g = \frac{(|\phi|^2 + 1)^2}{8|w|^2} |dz|^2 + \sum_{i=4}^7 |dx_i|^2
\]

This is the metric that Gray [11] proved to be G-Kähler-like. It is a product Riemannian metric, however, in general the factor on the \(M_1\) direction is not the standard hyperbolic metric. Indeed its Gauss curvature has the formula

\[
K = -2\frac{4|\phi'|^2}{(|\phi|^2 + 1)^2}.
\]

It has constant Gauss curvature if and only if there exists a constant \(c > 0\) such that \(c|\phi'|^2 = (|\phi|^2 + 1)^2\). However, there exists no such \(\phi\) which is a polynomial of degree \(\frac{4}{4}\) in terms of \(z\). This can be seen by comparing the growth near \(z = \infty\) determined by the degree.

Motivated by the above discussion, we would like to raise the following question:
Question 3. Let \((M_1, g_1)\) be a compact Riemannian surface of genus \(g(M_1) \geq 2\) equipped with the hyperbolic metric, and let \((T^4_{\mathbb{R}}, g_2)\) be a flat 4-torus. What is the space of all orthogonal complex structures on \(M_1 \times T^4_{\mathbb{R}}\) with respect to \(g_1 \times g_2\)?

In particular, one would like to know the subset with vanishing first Chern class.

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