A Remark on Formal KMS States in Deformation Quantization

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Abstract

In the framework of deformation quantization we define formal KMS states on the deformed algebra of power series of functions with compact support in phase space as $\mathbb{C}[[\lambda]]$-linear functionals obeying a formal variant of the usual KMS condition known in the theory of $C^*$-algebras. We show that for each temperature KMS states always exist and are up to a normalization equal to the trace of the argument multiplied by a formal analogue of the usual Boltzmann factor, a certain formal star exponential.
1 Introduction

The concept of deformation quantization has been set up in [4] and the existence of formal associative deformations of the pointwise multiplication in the space of all complex-valued smooth functions on a symplectic manifold, the so-called star products, has been proved in [12]. The deformed algebra can be seen as a module over the formal power series ring $\mathbb{C}[[\lambda]]$ where the deformation parameter $\lambda$ corresponds to Planck’s constant $\hbar$ and the constructions can be made such that the pointwise complex conjugation becomes an antilinear involutive antiautomorphism of the deformed algebra. Using the natural ring ordering in the subring $\mathbb{R}[[\lambda]]$ of real power series it is possible to define formal positive $\mathbb{C}[[\lambda]]$-linear functionals on the deformed algebra and to imitate the GNS construction known in the theory of complex $C^{\ast}$-algebras (see [7, 8]) which gives a notion of formal states in the theory of deformation quantization yielding physically reasonable representations such as the Schrödinger picture or the WKB expansion for cotangent bundles (see [6, 8]).

Having a notion of formal states it is natural to consider problems of quantum statistical physics in this light. In the modern approach based on the quantum observable algebra (taken to be a complex $C^{\ast}$-algebra) the analogue of a Gibbs state of inverse temperature $\beta$ is a positive linear functional $\mu$ on the algebra obeying the so-called KMS condition (see for example the books by Bratteli-Robinson [9], Haag [17], or Connes [10] or Section 8 of this paper for a precise definition). Originally, the KMS condition appeared as a boundary condition for complex times for thermal Green functions in the papers of Kubo and Martin & Schwinger (see [19] and [20, p.1357, p.1359]) and was cast into the $C^{\ast}$-algebra language by Haag, Hugenholtz, and Winnink [18]. This condition proved to be rather useful in the development of the statistical theory based on $C^{\ast}$-algebras, and it is believed that the nonuniqueness of KMS states for a certain temperature is related to the existence of several different thermodynamic phases (see [18], or [17, p.213], or [10, p.41] for a discussion).

Beside the usual approaches in quantum field theory investigations in this directions in the setting of classical mechanics of infinitely many degrees of freedom has been made in e. g. [1, 15, 16] where the situation of infinitely many particles moving in flat $\mathbb{R}^n$ is considered by using sequences of coordinates and momenta for the particles and measure theoretical techniques to describe the KMS states.

More than ten years ago [2, 3] have already given a treatment of the KMS condition in the framework of deformation quantization: the inverse temperature $\beta$ is incorporated in the deformed algebra by an equivalence transformation (having zeroth order term not equal to the identity) which is a left multiplication by an invertible, $\beta$-dependent function (such as an analogue of the Boltzmann factor). Rigidity and equivalence of such $\beta$-dependent star products have been further discussed in [3]. The connection to the KMS condition is made by assuming the existence of some complex topological subalgebra $\mathcal{A}$ of the deformed algebra and the existence of a complex continuous linear functional $\mu$ on $\mathcal{A}$ such that the KMS condition for a Hamiltonian $H$ makes sense, and by deriving a condition on the $\beta$-dependent star product (see [2], Section 3, p. 490, in particular eqn (3.3), and eqn (3.10) on p. 492).

More recently the classical KMS condition in the context of general Poisson manifolds has been discussed in [23]: the starting point is a positive density $\mu$ on the manifold whose Lie derivative with respect to a Hamiltonian vector field gives rise to a unique vector field $\phi_{\mu}$ the so-called modular vector field which can be regarded as an infinitesimal version of the modular automorphisms in the Tomita-Takesaki theory of von Neumann algebras.

In this Letter we shall discuss the simplest case of finitely many degrees of freedom: using the formal subalgebra $C^\infty_0(M)[[\lambda]]$ of series of smooth complex-valued functions having compact support
in a connected symplectic manifold $M$ we show first that for any Hamiltonian function $H$ the KMS-condition can be formulated in terms of $\mathbb{C}[[\lambda]]$-linear functionals $\mu$ on $C_0^\infty(M)[[\lambda]]$. Secondly, we prove —without making any a priori assumptions on the continuity of the functionals with respect to the standard locally convex topology— that there is always a unique (up to normalizations in $\mathbb{C}[[\lambda]]$) formal KMS state $\mu$ on $C_0^\infty(M)[[\lambda]]$ given by the following analogue of the Boltzmann factor

$$\mu(f) = c \text{tr} \left( \text{Exp}(-\beta H) \ast f \right),$$

where $\text{tr}$ is a nonzero trace on $C_0^\infty(M)[[\lambda]]$ (a $\mathbb{C}[[\lambda]]$-linear functional on $C_0^\infty(M)[[\lambda]]$ vanishing on commutators) and $\text{Exp}(-\beta H)$ is the star-exponential of $-\beta H$ where no formal convergence problem arises since there is no $\frac{1}{\lambda}$ in front of $H$ in the exponent. In case the complex conjugation is an antilinear antiautomorphism of the star product the prefactor can be chosen such that the KMS states become formally positive. Thirdly, we show that for $\beta \neq 0$ there is no nonzero KMS state in case the quantum time development is induced by a symplectic, but non-Hamiltonian vector field.

Assuming for a moment that phase transitions are related to the nonuniqueness of KMS states for a given inverse temperature $\beta$ we see that our result is physically reasonable insofar that phase transitions become mathematically visible only when some kind of thermodynamic limit is performed where particle number and configuration space volume are both sent to infinity while the average particle density is kept fixed: hence for finite-dimensional symplectic manifolds one would not expect phase transitions on physical grounds. For future investigations one would have to incorporate the technically more involved formulation of thermodynamic limits (possibly based on the work of [1, 15, 16]) in the analysis and again look for formal KMS states in a more general infinite-dimensional setting.

The paper is organized as follows: Firstly we remember some basic facts concerning the notion of time development in deformation quantization as well as traces, i. e. $\mathbb{C}[[\lambda]]$-linear functionals which vanish on star product commutators. Here we refer to the existence and uniqueness of traces established in [21] and give an alternative simple proof for the uniqueness of the traces which we shall need afterwards. In the following section we define formal KMS states after a short discussion of the KMS condition used in the context of $C^*$-algebras. Finally section [4] contains the two main theorems: we prove the existence and uniqueness of KMS states for any star product on a connected symplectic manifold for any inverse temperatur $\beta$ with respect to the time development induced by an arbitrary Hamiltonian vector field. Moreover we show that for symplectic but non-Hamiltonian vector fields no KMS states exist for $\beta \neq 0$ exist.

2 Some basic concepts: time evolution and traces

In this section we shall briefly remember some basic facts concerning time evolution and traces as well as star exponentials in deformation quantization. Firstly we shall fix some notation: we consider a symplectic manifold $(M, \omega)$ and a symplectic vector field $X$. Then $i_X \omega = \alpha$ is a closed one-form and any closed one-form determines a symplectic vector field via this equation. By $\phi_t$ we denote the flow of $X$ where we assume for simplicity that $X$ has a complete flow. Then the classical time evolution of the observables, i. e. the complex-valued functions $C^\infty(M)$, with respect to $X$ is given by the pull-back $\phi^*_t : C^\infty(M) \to C^\infty(M)$ and for any initial condition $f \in C^\infty(M)$ the time evolution $f(t)$ through $f(0) = f$ is uniquely determined by

$$\frac{d}{dt} f(t) = \mathcal{L}_X f(t), \quad f(0) = f$$

where $\mathcal{L}_X$ denotes the Lie derivative with respect to $X$. In the case where $\alpha = dH$ is exact the symplectic vector field $X$ is a Hamiltonian vector field with Hamiltonian function $H$ and [3] can
be rewritten as
\[ \frac{d}{dt} f(t) = \{ f(t), H \}, \quad f(0) = f \] (3)
where \( \{ \cdot, \cdot \} \) denotes the Poisson bracket induced by \( \omega \). Now in deformation quantization (see e. g. [3]) the classical Poisson algebra \( C^\infty(M) \) of observables is deformed into an associative star product algebra \( (C^\infty(M)[[\lambda]], *) \) where the star product \( * \) is given by the formal power series in the formal parameter \( \lambda \)
\[ f * g = \sum_{r=0}^{\infty} \lambda^r M_r(f, g) \] (4)
with \( M_0(f, g) = fg \) and \( M_1(f, g) - M_1(g, f) = i\{f, g\} \) and all \( M_r \) are bidifferential operators on \( M \) vanishing on the constants for \( r \geq 1 \). Here the deformation parameter \( \lambda \) corresponds directly to Planck’s constant \( \hbar \) whence it is considered to be real, i.e., \( \lambda = \lambda \). In the case of convergence \( \lambda \) may be substituted by \( h \in \mathbb{R} \). In the case of a Hamiltonian vector field \( X \) the quantum analogue to (3) is given by Heisenberg’s equation of motion
\[ \frac{d}{dt} f(t) = \frac{i}{\lambda} \text{ad}(H) f(t) \quad f(0) = f \] (5)
where \( \text{ad}(H)g := H * g - g * H \) as usual and computing the first order in \( \lambda \) of (5) for any initial condition (see e. g. [14, Sec. 5.4], [6, App. B], [7, Sec. 5]):

**Proposition 2.1** Let \((M, \omega)\) be a symplectic manifold and \(X\) a symplectic vector field with complete flow \(\phi_t\). Moreover let \(*\) be a star product for \(M\) then the Heisenberg equation of motion
\[ \frac{d}{dt} f(t) = \frac{i}{\lambda} \delta_X f(t), \quad f(0) = f \in C^\infty(M)[[\lambda]] \] (6)
has a unique solution denoted by \( f(t) = A_t f \) and the map \( A_t : C^\infty(M)[[\lambda]] \rightarrow C^\infty(M)[[\lambda]] \) is a \( \mathbb{C}[[\lambda]]\)-linear automorphism of \(*\) and has the following properties:

i.) \( A_t = \phi_t^* \circ T_t \) where \( T_t = \text{id} + \sum_{r=1}^{\infty} \lambda^r T_t^{(r)} \) and \( T_t^{(r)} \) is a differential operator vanishing on constants.

ii.) \( A_t \circ \delta_X = \delta_X \circ A_t \) and \( A_t \) is a one-parameter group of automorphisms of the star product \(*\).

iii.) If the complex conjugation is an antilinear anti-automorphism of \(*\), i.e. \( \overline{f * g} = \overline{g} * \overline{f} \) where \( \overline{\lambda} := \lambda \) then \( A_t \) is a real automorphism, i.e. \( \overline{A_t f} = A_t \overline{f} \).

In the following we shall often make use of a particular form of the star exponential [3] of a function \( H \in C^\infty(M) \). In our case the star exponential can be defined as a solution of a differential equation which is shown to exist. In fact all relevant properties can be proved easily this way. We consider the differential equation
\[ \frac{d}{d\beta} f(\beta) = H * f(\beta), \quad \beta \in \mathbb{R}. \] (7)
Lemma 2.2 Let $(M, \omega)$ be a symplectic manifold and $\ast$ a star product for $M$ and let $H \in C^\infty(M)$. Then there exists a unique solution $f(\beta)$ of (4) in $C^\infty(M)[[\lambda]]$ with initial condition $f(0) = 1$. This solution is denoted by Exp$(\beta H)$ and one has the following properties for all $\beta, \beta' \in \mathbb{R}$:

i.) $\text{Exp}(\beta H) = e^{\beta H} \left(1 + \sum_{r=1}^{\infty} \lambda^r g^{(r)}_\beta \right)$ where $g^{(r)}_\beta \in C^\infty(M)$.

ii.) $\text{Exp}(\beta H) \ast H = H \ast \text{Exp}(\beta H)$ and $\text{Exp}(\beta H) \ast \text{Exp}(\beta' H) = \text{Exp}((\beta + \beta')H)$.

Proof: This lemma is proved by first factorizing $f(\beta) = e^{\beta H} g(\beta)$ and then rewriting the induced differential equation for $g(\beta)$ as integral equation which can be uniquely solved by recursion since the integral operator raises the degree in $\lambda$. Then the other properties easily follow using the uniqueness and (4). \qed

Now we consider again a symplectic vector field $X$ and the corresponding derivation $\delta_X$. Since clearly $\delta_X = -i\lambda L_X + \cdots$ the map $\delta_X$ raises the $\lambda$-degree at least by one which implies that the series

$$e^{\beta \delta_X} := \sum_{r=0}^{\infty} \frac{1}{r!} (\beta \delta_X)^r$$

is a well-defined formal power series of maps for $\beta \in \mathbb{R}$ and one easily shows that $e^{\beta \delta_X}$ is a one-parameter group of automorphisms of the star product. Moreover one has the following lemma:

Lemma 2.3 Let $(M, \omega)$ be a symplectic manifold and $\ast$ a star product for $M$ and let $X$ be a symplectic vector field. Then the one-parameter group $e^{\beta \delta_X}$ of automorphisms of $\ast$ where $\beta \in \mathbb{R}$ is inner iff $i_X \omega = dH$ is exact and in this case for all $f \in C^\infty(M)[[\lambda]]$

$$e^{\beta \delta_X}(f) = \text{Exp}(\beta H) \ast f \ast \text{Exp}(-\beta H).$$

Proof: Let us first assume that $e^{\beta \delta_X}$ is an inner automorphism for some $\beta \neq 0$, i.e. we assume that there exist elements $b = \sum_{r=0}^{\infty} \lambda^r b_r$ and $c = \sum_{r=0}^{\infty} \lambda^r c_r$ where $b_r, c_r \in C^\infty(M)$ (depending on $\beta$) such that

$$e^{\beta \delta_X}(f) = b \ast f \ast c \quad \text{and} \quad b \ast c = c \ast b.$$ 

By straightforward computation of the first order in $\lambda$ of the relation $e^{\beta \delta_X}(f) - e^{-\beta \delta_X}(f) = b \ast f \ast c - c \ast f \ast b$ one obtains $\beta L_X f = c_0 \{f, b_0\}$. Since $b \ast c = 1$ one has $b_0 c_0 = 1$ and thus $b_0$ is a non-vanishing function on $M$. Now define $H := \frac{1}{2\beta} \log(b_0 b)$ which is clearly a smooth function on $M$. We obtain $\{f, H\} = L_X f$ which shows that $X$ is in fact Hamiltonian and thus $e^{\beta \delta_X}$ has only a chance to be inner if $i_X \omega = dH$ is exact. If on the other hand $i_X \omega = dH$ then (4) is a simple computation using lemma 2.2 and (4). \qed

A last important structure needed in the following is the notion of a trace. Though in deformation quantization traces are usually considered in the setting of formal Laurent series (e.g. in [12, 21, 14]) which allows a more suitable normalization motivated by either physical reasons (‘to get dimensions right’) or by analogy to traces of pseudo-differential operators we shall stay for simplicity in the category of formal power series.

Definition 2.4 Let $(M, \omega)$ be a symplectic manifold and $\ast$ a star product for $M$. A $\mathbb{C}[[\lambda]]$-linear functional $\text{tr} : C^\infty_0(M)[[\lambda]] \to \mathbb{C}[[\lambda]]$ is called a trace iff $\text{tr}(f \ast g - g \ast f) = 0$ for all $f, g \in C^\infty_0(M)[[\lambda]]$.

Proposition 2.5 (Existence and uniqueness of traces [21, 14]) Let $(M, \omega)$ be a connected and symplectic manifold and $\ast$ a star product for $M$. Then the set of traces forms a $\mathbb{C}[[\lambda]]$-module which is one-dimensional over $\mathbb{C}[[\lambda]]$. 

For a proof of the existence we refer to e. g. [21] where also the uniqueness up to normalization by elements in $\mathbb{C}[[\lambda]]$ is shown. For the existence of strongly closed star products as defined in [1] see [23]. For later use we shall give here an elementary proof of the uniqueness since we need in particular the lowest (non-trivial) order of the traces. Though the following lemma should be well-known we shall give for completeness a short proof since the result is crucial for the following:

**Lemma 2.6**

i.) Let $\varphi \in C^\infty_0(\mathbb{R}^n)$ ($n \geq 1$) be a smooth complex valued function with support contained in an open ball $B_R(0)$ around 0 with radius $R > 0$ such that $\int_{\mathbb{R}^n} \varphi(x) d^n x = 0$ then there exist functions $h_i \in C^\infty_0(\mathbb{R}^n)$ with $\text{supp} h_i \subset B_R(0)$ for $i = 1, \ldots, n$ such that $\varphi = \sum_{i=1}^n \frac{\partial h_i}{\partial x_i}$.

ii.) Let $B_R(0) \subseteq \mathbb{R}^n$ be an open ball around 0 with radius $R > 0$ (where $R = \infty$ is also allowed) and let $\mu : C^\infty_0(B_R(0)) \rightarrow \mathbb{C}$ be a $\mathbb{C}$-linear (not necessarily continuous) functional such that $\mu \left( \frac{\partial f}{\partial x_i} \right) = 0$ for all $i = 1, \ldots, n$ and $f \in C^\infty_0(B_R(0))$ then $\mu$ is a distribution and given by

$$\mu(f) = c \int_{B_R(0)} f(x) d^n x$$

with some constant $c \in \mathbb{C}$.

**Proof:** For the first part the case $n = 1$ is readily checked by noting that there is a primitive $h_1$ of $\varphi$ having compact support. Assume that $n \geq 2$. Choose three pairwise distinct concentric closed balls $B_1 \subset B_2 \subset B_3$ in $B_R(0)$ such that $\text{supp} \varphi \subset B_1$. Embed $B_R(0)$ as an open subset in the sphere $S^n$ and extend $\varphi$ to a smooth complex-valued function on $S^n$ vanishing outside the embedded $B_R(0)$. We can assume that the embedding is volume preserving. Hence the integral of the extended $\varphi$ over $S^n$ (with some suitable volume $\mu$) is zero. Since the $n$th de Rham cohomology group of $S^n$ is well-known to be one-dimensional it follows by the de Rham Theorem that $\varphi \mu$ is exact, hence equal to $d\alpha$ where $\alpha$ is some $n - 1$-form on $S^n$. Now $\varphi$ vanishes on the complement of the embedded ball $B_1$ in $S^n$ which is diffeomorphic to $\mathbb{R}^n$ hence $\alpha$ is closed on that set. By the Poincaré Lemma there is an $n - 2$-form $\beta$ on that subset such that $\alpha = d\beta$ on that subset. Choose a nonnegative smooth function $\chi$ on the sphere being zero on the embedded $B_2$ and 1 on the complement of the interior of the embedded $B_3$ it follows that $\alpha' := \alpha - d(\chi \beta)$ is a globally defined $n - 2$-form on the sphere with support in the embedded $B_3$ such that $\varphi \mu = d\alpha'$. Pulling this back to the ball $B_R(0)$ we get the desired functions $h_i$ by the components of the pulled-back $\alpha'$. For the second part notice that part one shows that the linear space of smooth functions of compact support in the ball generated by derivatives of such functions is of codimension one hence all the linear functionals having this subspace in their kernel must be multiples of the integral. \[\square\]

**Corollary 2.7** Let $(M, \omega)$ be a connected symplectic manifold and let $\mu : C^\infty_0(M) \rightarrow \mathbb{C}$ be a linear functional vanishing on Poisson brackets, i. e. $\mu(\{f, g\}) = 0$ for all $f, g \in C^\infty_0(M)$ then there exists a complex number $c \in \mathbb{C}$ such that

$$\mu(f) = c \int_M f \Omega$$

where $\Omega = \omega \wedge \cdots \wedge \omega$ is the symplectic volume form.

Now we consider a non-trivial trace $\text{tr}$ for a connected symplectic manifold $(M, \omega)$ with star product $\ast$. Firstly we remember that any $\mathbb{C}[[\lambda]]$-linear functional of $C^\infty_0(M)[[\lambda]]$ can be written as

$$\text{tr} = \sum_{r=0}^\infty \lambda^r \mu_r$$
with $\mu_r : C^0_0(M) \to \mathbb{C}$ due to [13, Prop. 2.1] and we can assume without restriction that $\mu_0 \neq 0$. Then the trace property of $\text{tr}$ obviously implies that $\mu_0$ vanishes on Poisson brackets and hence there exists a complex number $c_0 \neq 0$ such that

$$\mu_0(f) = c_0 \int_M f \, \Omega. \quad (10)$$

Now if $\tilde{\mu}$ is another trace for $\ast$ then $\tilde{\mu}_0(f) = \tilde{c}_0 \int_M f \Omega$ and thus $\tilde{\mu} - \frac{\tilde{c}_0}{c_0} \text{tr}$ is again a trace starting at least with order $\lambda^1$. Thus one can construct recursively a formal power series $c = \frac{\tilde{c}_0}{c_0} + \cdots$ such that $\tilde{\mu} = c \text{tr}$ which proves the uniqueness of traces in the connected case up to normalization.

3 The formal KMS condition in deformation quantization

After these preliminaries we can now discuss the meaning of the KMS condition in deformation quantization which was first discussed in this context in [4]. In our approach we try to stay completely in the formal category and avoid any assumptions about convergence of the formal power series. Moreover we restrict ourselves to finite-dimensional phase spaces.

Firstly we shall shortly remember the well-known definition of KMS states used e. g. in algebraic quantum field theory in the context of $C^\ast$-algebras (see e. g. [4, 13]). Here the observable algebra $\mathcal{A}$ is a net of local $C^\ast$-algebras with inductive limit topology and the time development operator $\alpha_t : \mathcal{A} \to \mathcal{A}$ is a one-parameter group of $^\ast$-automorphisms of $\mathcal{A}$ and a state $\mu$ (i. e. a positive linear functional) of $\mathcal{A}$ is called a KMS state for the inverse temperature $\beta = \frac{1}{kT}$ (where $k$ is Boltzmann’s constant and $T$ the absolute temperature) if for any two observables $a, b \in \mathcal{A}$ there exists a continuous function $F_{ab} : S_\beta \to \mathbb{C}$ which is holomorphic inside the strip $S_\beta := \{ z \in \mathbb{C} \mid 0 \leq \text{Im} \, z \leq \hbar \beta \}$ such that for any real $t$

$$F_{ab}(t) = \mu(\alpha_t(a)b) \quad \text{and} \quad F_{ab}(t + \hbar \beta) = \mu(\alpha_t(a)b). \quad (11)$$

This formulation replaces in a mathematically reasonable way the more intuitive requirement that for any two observables $a, b \in \mathcal{A}$ the state $\mu$ should satisfy

$$\mu(\alpha_t(a)b) = \mu(\beta_{t+\hbar \beta}(a))$$

which is obviously not well-defined in general since there is a priori no sense of the complexification of the time development operator $\alpha_t$ to define $\alpha_{t+\hbar \beta}$. Nevertheless we shall see that in deformation quantization there is indeed a reasonable notion of such a ‘complexification’ which avoids the usage of the holomorphic functions $F_{ab}$ which is not suitable in the formal setting since we want to treat $\hbar$ as formal! The key ingredient is the following simple lemma which follows directly from proposition [2,13] and the definition of $e^{\beta \delta \chi}$:

**Lemma 3.1** Let $(M, \omega)$ be a symplectic manifold with star product $\ast$ and let $X$ be a symplectic vector field on $M$ with complete flow and let $A_t$ be the corresponding time development operator. Then the map

$$A_{t+\hbar \lambda \beta} := A_t \circ e^{\beta \delta \chi} \quad (12)$$

where $t, \beta \in \mathbb{R}$ is an automorphism of the star product $\ast$ and $A_{t+\hbar \lambda \beta} \circ A_{t' + \hbar \lambda \beta'} = A_{t+t' + \hbar \lambda (\beta + \beta')}$ for all $t, t', \beta, \beta' \in \mathbb{R}$.

This seems to be a reasonable definition for the ‘complexification’ of $A_t$ in this particular situation. Note that this would make no longer sense in general if we tried to define $A_{t+\hbar \beta}$ for $\beta \in \mathbb{R}$.
Now we can define formal KMS states in deformation quantization in the following way: firstly we remember that even in the formal setting there is a both mathematically and physically reasonable notion of positive linear functionals in the case where the star product satisfies $f \ast g = g \ast f$ using the natural ring ordering of $\mathbb{R}[[\lambda]]$. Such positive linear functionals give raise to a formal GNS construction as defined in details in [7]. Hence it appears natural to consider only such star products and search for formal KMS states within these positive linear functionals. But it will turn out that the formal KMS condition will in fact essentially imply (in the connected case after suitable normalization) the positivity and hence we shall not proceed this way but state the following definition:

**Definition 3.2** Let $(M, \omega)$ be a symplectic manifold with star product $\ast$ and let $X$ be a symplectic vector field on $M$ and let $\mu : C^\infty_0(M)[[\lambda]] \to \mathbb{C}[[\lambda]]$ be a $\mathbb{C}[[\lambda]]$-linear functional.

i.) $\mu$ satisfies the **static formal KMS condition** for the inverse temperature $\beta \in \mathbb{R}$ with respect to $X$ iff for all $f, g \in C^\infty_0(M)[[\lambda]]$

$$\mu(f \ast g) = \mu(g \ast e^{\beta \delta_X}(f)).$$ (13)

ii.) If $X$ has complete flow then $\mu$ satisfies the **dynamic formal KMS condition** for the inverse temperature $\beta \in \mathbb{R}$ with respect to the time development operator $A_t$ iff for all $f, g \in C^\infty_0(M)[[\lambda]]$

$$\mu(A_t(f) \ast g) = \mu(g \ast A_{t+\lambda \beta}(f)).$$ (14)

Clearly $\mu = 0$ satisfies the formal KMS condition trivially and if we consider $\mu \neq 0$ we can assume that the first non-trivial order of $\mu$ is the zeroth order in $\lambda$. Evaluating the first non-trivial order in $\lambda$ of the KMS conditions (13) resp. (14) using (8) and proposition 2.1 one obtains the well-known classical KMS conditions namely the **static classical KMS condition**

$$\mu_0(\{f, g\} - \beta g \mathcal{L}_X f) = 0 \quad \forall f, g \in C^\infty_0(M)$$ (15)

and in the case where $X$ has complete flow the **dynamical classical KMS condition**

$$\mu_0(\{\phi^*_t f, g\} - \beta g \mathcal{L}_X \phi^*_t f) = 0 \quad \forall f, g \in C^\infty_0(M), \forall t \in \mathbb{R}$$ (16)

which where discussed in literature earlier in various ways (See e.g. [1, 2, 13, 16, 23] and references therein).

In the case of a complete flow the dynamical KMS conditions (both quantum and classical) imply clearly the static ones by setting $t = 0$. But since $A_t$ commutes with $\delta_X$ resp. $\phi^*_t$ commutes with $\mathcal{L}_X$ the static KMS conditions imply the dynamical ones by replacing $f$ by $A_t(f)$ resp. $\phi^*_t f$. Hence we shall only consider the static KMS condition in the following and drop the somehow technical assumption of complete flow.

### 4 Existence and uniqueness of formal KMS states

In the case of a Hamiltonian time development, i.e. if $i_X \omega = dH$ with some Hamiltonian function $H \in C^\infty(M)$ the structure of the formal KMS states in the sense of definition 3.2 is completely clarified by the following theorem:

**Theorem 4.1** Let $(M, \omega)$ be a symplectic manifold with star product $\ast$ and let $H \in C^\infty(M)$ be a Hamiltonian function with corresponding Hamiltonian vector field $X$ and let $\beta \in \mathbb{R}$. 


corollary 2.7. □

Part follows immediately by proposition 2.5. The third part is shown the same way by computation and

Proof:

Part one is a simple and straight forward computation using lemma 2.2 and 2.3. Then the second

\[ µ(f) = c \text{tr} (\exp(-\beta H) * f) \]  

(17)

where tr is a non-trivial fixed choice of a trace for * starting with lowest order zero and

c ∈ \mathbb{C}[[\lambda]].

ii.) Let \( \mu_0 : \mathcal{C}_0^\infty(M) \to \mathbb{C} \) be a \( \mathbb{C} \)-linear functional then \( \mu_0 \) satisfies the static classical KMS condition iff the functional \( \tilde{\mu}_0(f) := \mu_0(e^{\beta H} f) \) vanishes on Poisson brackets. Hence if \( M \) is connected \( \mu_0 \) is of the form

\[ \mu_0(f) = c_0 \int_M e^{-\beta H} f \Omega \]  

(18)

with some constant \( c_0 \in \mathbb{C} \).

Proof: Part one is a simple and straight forward computation using lemma 2.2 and 2.3. Then the second part follows immediately by proposition 2.5. The third part is shown the same way by computation and corollary 2.4.

Note that no continuity properties of \( \mu_0 \) had to be assumed for the classical part of the theorem. In fact the algebraic condition (15) implies continuity of \( \mu_0 \) with respect to the standard locally convex topology of \( \mathcal{C}_0^\infty(M) \) since clearly (18) defines a continuous functional.

In the case when the time development is given by a symplectic but not Hamiltonian vector field no non-trivial formal KMS states exist:

Theorem 4.2 Let \( (M, \omega) \) be a connected symplectic manifold with star product * and let \( X \) be a symplectic vector field on \( M \) and let \( 0 \neq \beta \in \mathbb{R} \). If \( \mu \) is a static formal KMS state with respect to \( X \) and inverse temperature \( \beta \) then either \( \mu = 0 \) or \( \alpha := i_X \omega = dH \) is exact.

Proof: Since the static formal KMS condition (12) implies the classical one we only have to show that the classical static KMS condition (15) implies either \( \mu_0 = 0 \) or \( \alpha = dH \). Now let \( \mu_0 : \mathcal{C}_0^\infty(M) \to \mathbb{C} \) be a linear functional satisfying (13) then we take an atlas on \( M \) of contractable charts \( \{U_i\}_{i \in I} \) and local functions \( H_i \in \mathcal{C}^\infty(U_i) \) such that \( \alpha|_{U_i} = dH_i \) for all \( i \in I \). Consider \( U_{ij} := U_i \cap U_j \neq \emptyset \) and let \( C_{ij} \in \mathbb{R} \) be the constants such that \( H_i|_{U_{ij}} = H_j|_{U_{ij}} + C_{ij} \). Now define \( \mu^{(i)} : \mathcal{C}_0^\infty(U_i) \to \mathbb{C} \) by \( \mu^{(i)}(f) := \mu(f) \) then \( \tilde{\mu}^{(i)}(f) := \mu(e^{\beta H_i} f) \) is well-defined for \( f \in \mathcal{C}_0^\infty(U_i) \) for any \( i \in I \) and vanishes on Poisson brackets. Hence there exist constants \( C_i \in \mathbb{C} \) such that for any \( f \in \mathcal{C}_0^\infty(U_i) \)

\[ \mu(f) = C_i \int_{U_i} e^{-\beta H_i} f \Omega \]

due to theorem 4.1. Thus for \( U_{ij} \neq \emptyset \) this implies by a standard continuity argument \( C_i e^{-\beta H_i}|_{U_{ij}} = C_j e^{-\beta H_j}|_{U_{ij}} \). Now if \( C_i = 0 \) then for any other \( j \in I \) we obtain \( C_j = 0 \) since \( M \) is connected and hence \( \mu = 0 \). If on the other hand \( C_i \neq 0 \) then \( C_i \neq 0 \) and \( C_i = 0 \). Thus we obtain \( H_i|_{U_{ij}} = H_j|_{U_{ij}} + \frac{1}{\beta} \ln \frac{C_i}{C_j} \), and thus \( C_{ij} = \frac{1}{\beta} \ln \frac{C_i}{C_j} \). Hence the constants \( C_{ij} \) clearly satisfy the cocycle identity which implies that \( \alpha \) is exact.

Finally we shall consider the case where the star products satisfies \( f \star g = \overline{g} \star \overline{f} \) for all \( f, g \in C^\infty(M)[[\lambda]] \) where we set as usual \( \overline{\lambda} := \lambda \). Now let tr be a trace for * then this property of * guarantees that tr := tr + tr (where tr(f) := tr(\overline{f})) is also a trace of * with the additional property that this trace is real in the following sense:

\[ \text{tr}(\overline{f}) = \overline{\text{tr}(f)}. \]  

(19)
In the connected case a real trace $\text{tr}$ is either a positive linear functional, i.e. $\text{tr}(f * f) \geq 0$ in the sense of the ring ordering of $\mathbb{R}[[\lambda]]$ (where $a = \sum_{r=k}^{\infty} \lambda^r a_r \in \mathbb{R}[[\lambda]]$ is called positive iff $a_k > 0$), or $-\text{tr}$ is a positive linear functional:

**Lemma 4.3** Let $(M, \omega)$ be a symplectic connected manifold and let $*$ be a star product for $M$ such that $\overline{f} * g = g * \overline{f}$ and let $\text{tr}$ be a real non-vanishing trace. Then either $\text{tr}$ or $-\text{tr}$ is a positive linear functional and the Gel’fand ideal $J_{\text{tr}} := \{ f \in C_{0}^{\infty}(M)[[\lambda]] \mid \text{tr}(f * f) = 0 \}$ is $\{0\}$.

**Proof:** Since the lowest order of $\text{tr}$ is proportional to the integration over $M$ with volume form $\Omega$ and since $\text{tr}$ is real [7, Lemma 2] implies the lemma. \[\square\]

This lemma implies that in the case where $\overline{f} * g = g * \overline{f}$ a formal KMS state can by rescaled to obtain a positive formal KMS state. Hence the algebraic formal KMS condition (static or dynamic) implies positivity in deformation quantization.

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