What is the Gravitational Field of a Mass in a Spatially Nonlocal Quantum Superposition?

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The study of the gravitational field produced by a spatially non-local, superposed quantum state of a massive particle is a thrilling area of modern physics. One question to be answered is whether the gravitational field behaves as the classical superposition of two particles separated by a spatial distance with half the mass located at each position or as a quantum superposition with a far more interesting and subtle behaviour for the gravitational field. Quantum field theory is ideally suited to probe exactly this kind of question. We study the scattering of a massless scalar on such a spatially nonlocal, quantum superposition of a massive particle. We compute the differential scattering cross section corresponding to the interaction coming from the exchange of one graviton. We find that the scattering cross section is not at all represented by the Schrödinger-Newton picture of potential scattering from two localized sources with half the mass at each source. We discuss how our result would be lethal to the Schrödinger-Newton description of gravitation interacting with quantum matter and would be conducive to considering the gravitational field to be quantized. We comment on the experimental feasibility of observing such effects.

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Introduction

At the core of modern physics lie the two most successful theory of physics: General Relativity (GR) on one side and Quantum Field Theory (QFT) on the other. Consistently combining these theories is one of the main goals of modern theoretical physics. The most common approach is to consider the quantum theory to be more fundamental and try to quantize gravity [1]. This choice is also justified by some indirect evidence [2]. The opposite direction has also been advocated, of “gravitizing” quantum mechanics, with some interesting arguments, see Penrose [3, 4].

However, we shall be interested in the standard ideas of a quantum theory of gravity. Indeed, quantum gravity as a low energy effective field theory makes perfect sense [5–8], although the directly observable predictions of such an effective quantum field theory are still not remotely experimentally observable. An intriguing possibility for the detection of noise due to quantum fluctuations in the gravitational field at gravitational wave detectors such as LIGO [9] has been suggested recently [10]. Within the effective field theory point of view, gravitation must be treated as a fully quantum mechanical field with all the complexities ensuing from the possibilities of arbitrary linear superpositions of quantum states. It is in this context that we examine the gravitational scattering of a massless particle from the effective quantum gravitational field of a massive particle in a spatially non-local quantum superposition.

Scattering on a spatially nonlocal wave packet

The usual formalism for scattering in quantum field theory corresponds to computing the amplitude for a given scattering process and converting the amplitude into an experimentally observable differential scattering cross section. The formalism is almost as old as the advent of quantum field theory, however, generally, certain assumptions are made so that the dependence on the initial and final wave packets disappears; see for example [11]. However, here we wish to consider a process where the cross section strongly depends on the wave packet of one of the initial states, a situation which is not usually considered.

We consider as the scatterer a particle of mass M in a linearly superposed state described by a spatially non-local wave function, which of course correspond to a particular momentum space wavefunction $\phi_1(k_1)$, causing the scattering of a massless particle supposed to be in a state $\phi_2(k_2)$. While $\phi_2(k_2)$ is assumed to be sharply peaked at momentum $p_2$, the wavefunction $\phi_1(k_1)$ is not peaked at any $p_1$, but nevertheless assumed for simplicity to be symmetrically distributed around $p_1 = 0$, i.e. $\phi_1(-k_1) = \phi_1(k_1)$. This means that the calculation is done in the convenient reference frame where $p_1 = 0$, i.e., the ”centre of mass” frame of the wavefunction.
The formula for the differential of the cross section is given by:

$$
\frac{d\sigma}{dE} = \left( \prod_i \frac{d^3p_i}{(2\pi)^3} \frac{1}{2E_i} \right) \left( \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} \frac{|\mathcal{M}(k_1, k_2 \to \{p_f\})|^2}{2E_1 2E_2 |v_1 - v_2|} \right)
\times |\phi_1(k_1)|^2 |\phi_2(k_2)|^2 \frac{1}{(2\pi)^4 \delta^4(k_1 + k_2 - \sum p_f)}.
$$

For our needs, we will use the formalism of Kotkin et al [12] as elaborated in Karlovets [13] that employs the Wigner function formalism [14]. We will take the incident massless particles in on-shell momentum space wave packets $\phi_2(k_2)$ centred and highly peaked on a momentum $p_2$. These massless particles will be incident on particles of mass $M$ which are in on-shell, spatially nonlocal wave packets. When expressed in terms of momentum space wave packets $\phi_1(k_1)$, these wave packets are not highly peaked on any specific momentum as noted above. We will be interested in the inclusive scattering cross section for the massless particle, $p_2 \to p_4$, while the non-trivial wave packet will in principle scatter to all possible allowed final states, which will be integrated over.

It is most convenient to consider the wave function of the scattered massive particle, that was initially in the spatially nonlocal wave function, to be scattered to wave functions that are eigenstates of momentum $p_3$ and to integrate over this momentum, as these do correspond to a complete set of final states. In practice, the integration is of course not required as energy-momentum conservation for given $k_1, p_2, p_4$ fixes the value of $p_3$. Thus the scattering will give rise to final momenta $k_1 \to k_3$ and $p_2 \to p_4$ with integration over $k_1$ smeared with wave function $\phi_1(k_1)$ understood. Then straightforwardly simplifying the formula in [13], to the case of a particle in a momentum eigenstate scattering on a particle in a non-trivial wave packet, we find

$$
\frac{d\sigma}{dE} = \int \frac{d^3k_1}{(2\pi)^3} T_{PW}(\{k_1, p_2\} \to \{p_3, p_4\}) \frac{|\phi_1(k_1)|^2 (2\pi)^4 \delta^4(k_1 + p_2 - p_3 - p_4) d^3p_3 d^3p_4}{(2\pi)^3 (2\pi)^3 \delta^4(k_1 + k_2 - \sum p_f)}
$$

where the plane wave amplitude $T_{PW}(\{k_1, p_2\} \to \{p_3, p_4\})$ is given by

$$
T_{PW}(\{k_1, p_2\} \to \{p_3, p_4\}) = \frac{\mathcal{M}(\{k_1, p_2\} \to \{p_3, p_4\})}{\sqrt{2\epsilon_1 2\epsilon_2 2\epsilon_3 2\epsilon_4}}
$$

with $\mathcal{M}(\{k_1, p_2\} \to \{p_3, p_4\})$ the invariant matrix element for the momentum space scattering process [11], and $\epsilon_i$ the energy of particle $i$. We note the usual factor of $|v_1 - v_2|$ in the particle flux simplifies to unity, the velocity of the massless particle, because of the assumed symmetry of the wave function $\phi_1(k_1)$.

The normalized wave function for the particle of type 1, which is taken to be spatially nonlocal, in momentum space has the form

$$
|\phi_1(k_1)|^2 = 4 \left( \pi \sigma^2 \right)^{3/2} e^{-\sigma^2 |k_1|^2} \frac{2 + e^{2i\tau \cdot k_1} + e^{-2i\tau \cdot k_1}}{1 + e^{-|\tau|^2/(\sigma^2)}}
$$

where $\sigma$ is the width of a Gaussian wave packet that is superposed at spatial position $\tau$ and $-\tau$.

The 1-graviton exchange scattering amplitude

The amplitude is easily computed using the linearized gravitational theory and subsequent graviton propagator and matter vertices, following Donoghue [6, 7] (see also [8, 15–19]), as prescribed by the Feynman diagram Fig. (1).
Applying momentum conservation, we get for the amplitude

\[
\mathcal{M} = \frac{(-i)^2 \kappa^2 (k_1^a p_2^a + k_1^b p_2^b - (p_2 \cdot p_4) \eta^{\mu \nu}) (\eta_{\mu \alpha} \eta_{\nu \beta} + \eta_{\mu \beta} \eta_{\nu \alpha} - \eta_{\mu \nu} \eta_{\alpha \beta}) (p_2^a p_4^a + p_2^b p_4^b - (p_2 \cdot p_4) \eta^{\alpha \beta})}{2q^2} \\
= \kappa^2 \left[ 2(k_1 \cdot p_2)(k_1 \cdot p_4) + (k_1 \cdot p_2)(p_2 \cdot p_4) - (k_1 \cdot p_4)(p_2 \cdot p_4) - (p_2 \cdot p_4)^2 \right] \frac{1}{p_2 \cdot p_4}
\]

(6)

where \( \kappa \) is the gravitational coupling constant, \( \omega_1 \) and \( \omega_4 \) are short hand notation for \( p_2^4 \) and \( p_4^4 \) and the momentum transfer is \( q = p_3 - k_1 = p_2 - p_4 \) with \( q^2 = -2p_2 \cdot p_4 = -2\omega_2\omega_4 (1 - \hat{p}_2 \cdot \hat{p}_4) \) as \( p_2 \) and \( p_4 \) are massless and on shell.

The scattering cross section

The amplitude must now be folded in with the wave function, the energy denominators and factors of \( 2\pi \) as in Eqn. (3) and then integrated over \( d^3 p_3 \) which removes the spatial delta functions and integrated over \( d\omega_4 \) which removes the temporal delta function yielding the differential scattering cross section

\[
\frac{d\sigma}{d\Omega_4} = \frac{d^3 k_1}{2\pi^3} |\phi_1(k_1)|^2 \frac{|M|^2}{\epsilon_1 \omega_2 \epsilon_3 \omega_4} \frac{\omega_4^2}{|f_4(\omega_4)|}.
\]

(8)

The energy conserving delta function is given by \( \delta(f_\delta(\omega_4)) = \delta(\epsilon_1 + \omega_2 - \epsilon_3 - \omega_4) \) where the complications arise because \( \epsilon_3 = \sqrt{M^2 + (k_1 + (p_2 - \omega_4 \hat{p}_4)^2)} \). \( \epsilon_3 = \sqrt{M^2 + k_1^2} \) is the energy of the Fourier component corresponding to momentum \( k_1 \) of the spatially nonlocal particle wave function and \( \omega_2 \) is the energy of the incoming massless particle. The full expression for the cross section is a complicated, unenlightening jumble, however its multipole expansion does shed some light on the gravitational interaction.

Multipole expansion of the scattering cross section

We can write the cross section Eqn. (8) as

\[
\frac{d\sigma}{d\Omega_4} = \int \frac{d^3 k_1}{2\pi^3} |\phi_1(k_1)|^2 g(k_1) |M|^2
\]

(9)

and then \( g(k_1) |M|^2 \) admits an expansion in powers of \( k_1 \) as

\[
g(k_1) |M|^2 = g(k_1) |M|^2 \bigg|_{k_1 = 0} + \frac{1}{2} \beta_{k_1} \partial_{k_1} \left( g(k_1) |M|^2 \right) \bigg|_{k_1 = 0} k_1^i k_{1j} + \cdots
\]

(10)

where the terms odd in \( k_1 \) are absent due to symmetry. Then the scattering cross section admits the multipole expansion

\[
\frac{d\sigma}{d\Omega_4} = \alpha \int |\phi_1(k_1)|^2 \frac{d^3 k_1}{(2\pi)^3} + \beta^{ij} \int k_{1i} k_{1j} |\phi_1(k_1)|^2 \frac{d^3 k_1}{(2\pi)^3}
\]

\[
= \alpha + \frac{\beta^{ij} \delta_{ij}}{3} \int |k_1|^2 |\phi_1(k_1)|^2 \frac{d^3 k_1}{(2\pi)^3} + \beta^{ij} \left( k_{1i} k_{1j} - \frac{|k_1|^2}{3} \delta_{ij} \right) |\phi_1(k_1)|^2 \frac{d^3 k_1}{(2\pi)^3} + \cdots
\]

(11)

\[
= \alpha + \frac{\beta^{ij} \delta_{ij}}{3} \left( \frac{3}{2\sigma^2} - \frac{|r|^2}{\sigma^2 (1 + e^{r^2/\sigma^2})} \right) \beta^{ij} \left( \frac{3}{2\sigma^2} \delta_{ij} - r_i r_j \right) \frac{1}{\sigma^4 (1 + e^{r^2/\sigma^2})}
\]

\[
\int k_{1i} k_{1j} |\phi_1(k_1)|^2 \frac{d^3 k_1}{(2\pi)^3} = \frac{\delta_{ij}}{2\sigma^2} - \frac{r_i r_j}{\sigma^2 (1 + e^{r^2/\sigma^2})}
\]

(12)

where evidently \( \alpha = \frac{1}{2\pi^2} g(k_1) |M|^2 \bigg|_{k_1 = 0} \) and \( \beta^{ij} = \frac{1}{2\pi^2} \partial_{k_1} \partial_{k_1} \left( g(k_1) |M|^2 \right) \bigg|_{k_1 = 0} \) and where we have used the integral
The leading terms in the expansion about $k_1 = 0$ are found after a somewhat tedious calculation. For the second derivative we will use:

$$
\frac{\partial k_i \partial k_j}{k_i} \left(g(k_1) |\mathcal{M}|^2\right) |_{k_1=0} = \\
\mathcal{M}^2 \partial^2 g |_{k_1=0} + 2M(\partial^2 g \partial k_i |\mathcal{M}| + \partial^2 g \partial k_i |\mathcal{M}| + \partial g \partial k_i |\mathcal{M}|) |_{k_1=0} 
$$

We will need the expression for $\omega_4$ obtained from the energy conservation delta function $\omega_4 = \epsilon_1 + \omega_2 - \epsilon_3$, and its derivatives. We find that the leading terms in the expansion in powers of $1/M$ of $\omega_4$ and its derivatives are given by

$$
\omega_4 = \omega_2 - \frac{1}{M} (\omega_2^2 + k_1 \cdot p_2 - \omega_2 (k_1 + p_2) \cdot \hat{p}_4) + \\
\frac{1}{M^2} (\omega_2^3 - 2\omega_2^2 (k_1 + p_2) \cdot \hat{p}_4 + \omega_2 (k_1 \cdot \hat{p}_4 + p_2 \cdot \hat{p}_4)^2 + (k_1 \cdot p_2) (\omega_2 - k_1 \cdot \hat{p}_4 - p_2 \cdot \hat{p}_4)) 
$$

Then we find, using the above, relatively easily, to leading order in powers of $M$

$$
\mathcal{M}_{k_1=0} = \kappa^2 \frac{2M^2}{1 - \hat{p}_2 \cdot \hat{p}_4} 
$$

$$
\partial_i \mathcal{M}_{k_1=0} = \kappa^2 \frac{2}{1 - \hat{p}_2 \cdot \hat{p}_4} (-2M(\hat{p}_{2i} + \hat{p}_{4i})) 
$$

$$
\partial_i \partial_j \mathcal{M}_{k_1=0} = \frac{\kappa^2}{1 - \hat{p}_2 \cdot \hat{p}_4} (4\delta_{ij} + 2(\hat{p}_{2i} \hat{p}_{4j} + \hat{p}_{2j} \hat{p}_{4i})). 
$$

A lengthier and more tedious calculation, which includes the calculation of $1/|\mathcal{M}|$, gives $g$ and its derivatives, again expanded in powers of $1/M$,

$$
g(\omega_4, k_1) = \frac{\omega_4}{M^2 \omega_2} \left(1 - \frac{\omega_2 (1 - \hat{p}_2 \cdot \hat{p}_4) - k_1 \cdot \hat{p}_4}{M} + \frac{\omega_2^2 - |k_1|^2 - 2\omega_2 (k_1 + p_2) \cdot \hat{p}_4 + (k_1 \cdot \hat{p}_4 + p_2 \cdot \hat{p}_4)^2}{M^2}\right) 
$$

$$
g_{k_1=0} = \frac{1}{M^2} 
$$

$$
\partial_i g_{k_1=0} = \frac{1}{M^2} (2\hat{p}_{4i} - \hat{p}_{2i}) 
$$

$$
\partial_i \partial_j g_{k_1=0} = \frac{2}{M^4} (3\hat{p}_{4i} \hat{p}_{4j} - \hat{p}_{2i} \hat{p}_{4j} - \hat{p}_{2j} \hat{p}_{4i} - \delta_{ij}) 
$$

Then putting all this together, we find

$$
\alpha = \frac{\kappa^4 M^2}{16\pi^2 (1 - \hat{p}_2 \cdot \hat{p}_4)^2} 
$$

$$
\beta^{ij} = \left(\frac{\kappa^4}{16\pi^2 (1 - \hat{p}_2 \cdot \hat{p}_4)^2}\right) (\delta_{ij} + 3\hat{p}_{2i} \hat{p}_{2j}) 
$$

The term $\alpha$ gives exactly the limiting small momentum transfer gravitational deflection of a massless particles from a massive particle, [19].

**Discussion and Conclusions**

The term proportional to $\alpha$, the lowest order monopole term, correspond to the scattering cross section of a single point like mass $M$, the analog of the Rutherford/Thompson cross section of a massless particle scattering from a
point like Newtonian potential. We see that the higher order terms coming from the $\beta^i_j \delta_{ij} = \left( \frac{k^4}{16\pi^2(1 - \hat{p}_2 \cdot \hat{p}_4)^2} \right)^6$, contribute to the monopole. This means that the scattering cross section is able to probe the non point like nature of the monopole part of the scattering particle’s wave function. This addition to the monopole contribution is given by

$$\left( \frac{k^4}{16\pi^2(1 - \hat{p}_2 \cdot \hat{p}_4)^2} \right)^2 \left( \frac{3}{2\sigma^2} - \frac{|r|^2}{\sigma^4} \right) \frac{1}{1 + e^{r^2/\sigma^2}}.$$  \hspace{1cm} (28)

However the actual quadrupole-type contribution is interestingly nothing like what would be expected if the gravitational field behaved according to the Schrödinger-Newton prescription, [20, 21]. The Schrödinger-Newton prescription would have the corresponding gravitational field as if one half the mass were concentrated at each of the two spatially nonlocal points, [22], a configuration which has a quadrupole moment

$$M \left( \frac{|r|^2}{3} \delta_{ij} - r_i r_j \right).$$  \hspace{1cm} (29)

Such a gravitational field yields a contribution to the (gravitational) scattering cross section

$$\frac{k^4 M^2 \sigma_0^2}{16\pi^2(1 - \hat{p}_2 \cdot \hat{p}_4)^2} (\hat{p}_{4i} - \hat{p}_{2i}) (\hat{p}_{4j} - \hat{p}_{2j}) \left( \frac{|r|^2}{3} \delta_{ij} - r_i r_j \right)$$  \hspace{1cm} (30)

obtained from a presumed Newtonian potential scattering from two point sources of mass $M/2$ located at the two peaks of the spatially nonlocal wave function. This result is not at all what we find from, Eqn.(13)

$$\frac{k^4}{16\pi^2 \sigma^4 (1 + e^{r^2/\sigma^2})} \left( \frac{3 \hat{p}_{2i} \hat{p}_{2j}}{1 - \hat{p}_2 \cdot \hat{p}_4} \right) \left( \frac{|r|^2}{3} \delta_{ij} - r_i r_j \right),$$  \hspace{1cm} (31)

where $\kappa = \sqrt{8\pi G}$ which gives a coefficient $4G^2$. Indeed the quadrupole term from the calculation from one graviton exchange scattering is exponentially small as $|r| \gg \sigma$ compared to the result expected from scattering from a potential where the mass is split between two positions as prescribed by the Newton-Schrödinger formalism. This is lethal to the Schrödinger-Newton formalism, the calculation corresponding to one graviton exchange is clearly more justifiable. Additionally, our result shows that the wave function is only probed by the incoming massless particle’s direction relative to the direction of separation $r$ something that is hopefully experimentally verifiable.

Useful further calculations would be to compute the gravitational contribution to the self energy of a massive particle in a spatially nonlocal wave function. One would look for the amplitude and behaviour of the self energy as a function of the spatial separation $r$ of the nonlocal wave function. A behaviour as $1/r$ of the self energy would correspond to the Newtonian potential and the corresponding $1/r^2$ law of attraction of the two non-local lumps. A calculation of the amplitude would indicate how the two nonlocal lumps behave gravitationally with respect to each other.

It would be very interesting to measure any of these phenomena although probably not technically feasible presently. However, it is not out of the realm of possibility to have a spatially non-local superposition about a $10^9$ atoms. A Bose condensate would be proposed, and it can be imagined that such condensates could be launched in an atom interferometer. Then the gravitational interactions of the two quantum superposed masses might be measurable [23].

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[1] Claus Kiefer. *Quantum Gravity*. Oxford University Press, New York, 2nd ed edition, 2007.
[2] Don N Page and CD Geilker. Indirect evidence for quantum gravity. *Physical Review Letters*, 47(14):979, 1981.
[3] Roger Penrose. On the gravitization of quantum mechanics 1: Quantum state reduction. *Foundations of Physics*, 44(5):557–575, 2014.
[4] Steve Carlip. Is quantum gravity necessary? *Classical and Quantum Gravity*, 25(15):154010, 2008.
[5] John F. Donoghue. Leading quantum correction to the newtonian potential. *Phys. Rev. Lett.*, 72:2996–2999, May 1994.
[6] John F. Donoghue. General relativity as an effective field theory: The leading quantum corrections. *Phys. Rev. D*, 50:3874–3888, Sep 1994.
[7] John F. Donoghue. Introduction to the effective field theory description of gravity. In *Advanced School on Effective Theories*, 6 1995.
[8] C. P. Burgess. Quantum gravity in everyday life: General relativity as an effective field theory. *Living Rev. Rel.*, 7:5–56, 2004.
[9] Gregory M Harry, LIGO Scientific Collaboration, et al. Advanced ligo: the next generation of gravitational wave detectors. *Classical and Quantum Gravity*, 27(8):084006, 2010.
[10] Maulik Parikh, Frank Wilczek, and George Zahariade. Quantum Mechanics of Gravitational Waves. *Phys. Rev. Lett.*, 127(8):081602, 2021.
[11] Michael E. Peskin and Daniel V. Schroeder. *An Introduction to Quantum Field Theory*. Addison-Wesley, Reading, USA, 1995.
[12] GL Kotkin, VG Serbo, and A Schiller. Processes with large impact parameters at colliding beams. *International Journal of Modern Physics A*, 7(20):4707–4745, 1992.
[13] Dmitry V Karlovets. Scattering of wave packets with phases. *Journal of High Energy Physics*, 2017(3):49, 2017.
[14] E. Wigner. On the quantum correction for thermodynamic equilibrium. *Phys. Rev.*, 40:749–759, Jun 1932.
[15] Bryce S DeWitt. Quantum theory of gravity. i. the canonical theory. *Physical Review*, 160(5):1113, 1967.
[16] Bryce S DeWitt. Quantum theory of gravity. ii. the manifestly covariant theory. *Physical Review*, 162(5):1195, 1967.
[17] Bryce S DeWitt. Quantum theory of gravity. iii. applications of the covariant theory. *Physical Review*, 162(5):1239, 1967.
[18] Martin JG Veltman. Quantum theory of gravitation. *Méthodes en théorie des champs*, North-Holland, pages 266–328, 1976.
[19] Michael D Scadron. *Advanced quantum theory*. World Scientific Publishing Company, 2006.
[20] L. Diosi. Gravitation and quantum-mechanical localization of macro-objects. *Physics Letters A*, 105(4):199–202, 1984.
[21] Roger Penrose. On gravity’s role in quantum state reduction. *General Relativity and Gravitation*, 28(5):581–600, 1996.
[22] Lajos Diosi. Nonlinear schrödinger equation in foundations: Summary of 4 catches. In *Journal of Physics: Conference Series*, volume 701, page 012019. IOP Publishing, 2016.
[23] Marc Kasevich. Public communication. *Testing Gravity 2019, Simon Fraser University*, Vancouver, 2019.