Universality Classes for Force Networks in Jammed Matter

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Abstract. We study the geometry of forces in some simple models for granular stackings. The information contained in geometry is complementary to that in the distribution of forces in a single inter-particle contact, which is more widely studied. We present a method which focuses on the fractal nature of the force network and find good evidence of scale invariance. The method enables us to distinguish universality classes characterized by critical exponents. Our approach can be applied to force networks in other athermal jammed systems.

1 Jammed Matter and Force Networks

Aggregates of particles can be found in a disordered solid-like state resulting from the phenomenon of jamming [1–5]. Granular materials, colloidal suspensions and molecular liquids are but a few examples of such systems that present a non-zero yield stress while trapped in one of many accessible metastable states. If thermal fluctuations are irrelevant, the forces on each particle must balance. Each stable configuration is thus characterized by a highly irregular network of forces spanning the entire system.

Experimental [6–10] and numerical [11–19] studies have identified two main distinctive features of these force networks. Firstly, strong fluctuations are found in the magnitudes of inter-particle forces. The associated distribution function $P(F)$ displays two characteristic properties: (i) it decays exponentially at large forces and (ii) it exhibits a plateau or small peak at small forces, which has been identified as a signature of jamming. The second experimental observation is that large forces are concentrated along tenuous paths, which have been deemed “force chains”. While $P(F)$ has been commonly used also as a characterization of these force chains, strictly speaking it provides no information about the spatial organization of forces. In fact, so far force chains have been identified mainly visually, and a quantitative characterization seems to be lacking.

By drawing an analogy with percolation, in this Letter we develop a geometrical description which associates a set of critical exponents with an ensemble of force networks. We apply this approach to three different models of static granular media under uniform pressure. We find that they belong to different geometrical universality classes although $P(F)$ displays similar features in all three of them.
2 Force clusters

Consider an ensemble of configurations of a fixed number of jammed particles, obtained numerically or experimentally. Each configuration defines a contact graph $G$, where nodes correspond to particle centers and edges connect particles in contact. Assuming there is no friction, the inter-particle forces are normal to the particle surface, and the underlying force network can be represented by associating with each edge $i$ of $G$ the corresponding force magnitude $F_i$. To investigate the geometry of forces, rather than the underlying geometry of contacts, we choose a threshold $f$ and look at the subgraph $\bar{G}(f)$ of $G$ obtained by selecting only the edges with $F_i > f$. For $f$ small, $\bar{G}(f)$ consists of a single connected component, but as $f$ increases, $\bar{G}(f)$ breaks up into a number of disconnected clusters. An ensemble of force networks thus induces a family of probability distributions of cluster sizes $\rho(s, f)$ for different thresholds $f$, the cluster size $s$ being defined as the number of edges in a cluster.

If the forces $F_i$ were distributed independently for each $i$, e.g. uniformly between 0 and 1, then the force clusters would simply be bond percolation clusters [20]. In that case, in the thermodynamic limit $N \to \infty$, a phase transition occurs at a critical value $f_c$ of $f$: an infinite cluster exists with probability 1 for $f < f_c$, and with probability 0 for $f > f_c$. At $f_c$, the cluster sizes are power-law distributed, $\rho(s, f_c) \propto s^{-\tau}$, and the correlation length diverges as $\xi \propto |f - f_c|^{-\nu}$ near the threshold. The scaling exponents $\tau$ and $\nu$ are universal, they are independent of the underlying geometry, and in fact they do not depend on the local distribution of forces $P(F)$ or even their correlations, as long as these are short-ranged.

In an ensemble of force networks corresponding to a jammed system, force and torque balance on each particle cause dependence and long-range correlations between bonds. Nevertheless, if the average forces are uniform over the extent of the system, we expect to find a critical threshold $f_c$ and an associated set of universal scaling exponents. The analogy with percolation moreover suggests that these exponents are independent of $P(F)$ and thus provide a new, complementary characterization of force networks.

3 Criticality and finite size scaling.

An efficient method is necessary to study the existence of scale invariance around the critical threshold from numerical and experimental data. While, strictly speaking, the system becomes scale-invariant only in the thermodynamic limit, $f_c$ and the associated critical exponents can be extracted from data on systems of finite size using finite size scaling [21]. This describes the scaling of an observable with the system size close to criticality: if a quantity $X$ is expected to diverge as $|f - f_c|^{-\chi}$ near $f_c$ in an infinite system, then in a system of size $N$, it obeys the scaling law

$$X(N, f) = N^\phi \tilde{X}((f - f_c)N^{1/d\nu})$$  (1)
Fig. 1. Examples of force networks (the thickness of the lines is proportional to the force magnitude) and corresponding force clusters close to the critical threshold: packing of 400 grains in Model A (top) and packing of 200 grains in Model B (bottom).

with $d$ the spatial dimension and $\phi = \chi/d\nu$. The scaling function $\tilde{X}$ depends on a single rescaled variable $x = (f - f_c)N^{1/d\nu}$, and for $x \gg 1$ it behaves as $x^{-\chi}$, while for $x \to 0$ it remains finite.

Using measurements of $X$ in systems of finite sizes, the parameters $\phi$, $\nu$ and $f_c$ can be obtained from (1) in two steps. Assuming that $X(N, f)$ as function of $f$ displays a maximum $X_m(N)$, from (1) the maxima for different $N$ all correspond to the same maximum of $\tilde{X}$, hence $X_m(N) \propto N^\phi$. Plotting the amplitudes of the maxima versus $N$, we get the exponent $\phi$. The values of $f_c$ and $\nu$ can then be obtained by determining the best data collapse in the region around the maximum.

4 Models studied

Combining the finite-size scaling method with Monte-Carlo simulations, we studied force-cluster criticality in three two-dimensional models of static granular matter under uniform pressure [22,23]. As all three models – which we will call A, B, and C for further reference – have been introduced earlier in other contexts, here we only define them briefly, without motivating in detail their relevance to granular matter. In our view, they are the simplest implementations of two fundamental ingredients of force networks, namely force balance on each grain and force randomness.
4.1 Snooker model

To start with, we consider the “snooker-triangle packing” studied in [26,27]. It consists of a hexagonal packing of frictionless spherical grains confined within a triangular domain, with the same confining pressure applied on all sides of the triangle. A force network on this packing consists of repulsive forces in vectorial balance on each grain and consistent with the applied pressure. These constraints however do not define a single configuration of forces, but a whole set. Following Edwards’ prescription [28], all such force networks are taken to be equally likely, similarly to a micro-canonical ensemble. We sample this ensemble with a Metropolis algorithm, using the parametrization of force networks developed in Ref. [29]. In Fig. 1 we show an example of a force network in this model and the corresponding force clusters for a threshold $f = 0.94$.

4.2 Independent $q$-model

We next consider consider the scalar $q$-model [24], one of the first models introduced to account for the fluctuations of forces and appearance of force chains in a granular packing. Here we consider the massless $q$-model on a periodic tilted square lattice, which can be interpreted as a packing of rectangular bricks [25]. A uniform pressure is applied on the top of the packing and on each site a brick supports a weight $W_{ij}$. Each brick transfers vertical forces $F_{ij}^{(l)}$ and $F_{ij}^{(r)}$ to its bottom left and right neighbors respectively. Vertical force balance is automatically satisfied by considering $F_{ij}^{(l)}$ and $F_{ij}^{(r)}$ respectively as fractions $q_{ij}$ and $1 - q_{ij}$ of $W_{ij}$, and randomness in force transfer is implemented by taking the $q_{ij}$ uniformly distributed between 0 and 1, independently for each site. Fig. 1 shows a force network in this model and the corresponding force clusters for a threshold $f = 0.7$ (for unit external pressure).

4.3 Microcanonic $q$-model

Our third model is a variation on the $q$-model. We consider the same packing as in Sec. 4.2, but now, following Edwards’ prescription, all allowed force networks – consisting of sets of vertical forces $\{F_{ij}^{(l)}, F_{ij}^{(r)}\}$ – are equally likely. As shown in Ref. [30], this is equivalent to having the $q_{ij}$ distributed with the joint probability distribution $\prod_i W_{i,j}$. The aim is to examine the influence of the form of the probability distribution by comparing independent and microcanonic $q$-models, and the difference between scalar and vectorial conservation laws by comparison with the snooker model.

5 Results

A convenient observable to study is the second moment of the distribution of cluster sizes, $\langle s^2(N,f) \rangle = \int s^2 \rho(s,f_c)$, where the contribution from the largest cluster in each configuration is omitted, and the system size $N$ is defined as the
total number of edges. In all three models defined above, we find that \( \langle s^2(N, f) \rangle \) displays a maximum as function of \( f \) for fixed \( N \). The amplitudes of the maxima as functions of \( N \) follow sharp power-laws shown in Fig. 2 (a), thus confirming the existence of a critical threshold in each model. The corresponding critical exponent \( \phi \) is related via the hyper-scaling relation [20] to \( \tau \), the exponent of the cluster-size distribution at criticality, and \( D \), the fractal dimension of the incipient cluster as \( \phi = \frac{3 - \tau}{\tau - 1} = D - 1 \). Higher moments \( \langle s^n(N, f) \rangle \) display a similar scaling with exponents \( \phi_n = \frac{n + 1 - \tau}{\tau - 1} \), implying that the full distribution \( \rho(s, f_c) \) approaches a scaling form around the critical threshold.

The value of the critical threshold \( f_c \) depends on the scale set by the external pressure. Under unit pressure, we found a different \( f_c \) for each model. Fig. 3 displays the scaling functions obtained by collapse of the data. The estimated values of the critical thresholds and exponents are summarized in Table 1, where the two-dimensional percolation exponents are also included for reference.

In Fig. 2(b) we show the probability distributions \( P(F) \) of force magnitudes. In the independent \( q \)-model, \( P(F) \) is exactly exponential [31], while in the other two models it is exponential for large forces, and displays a peak at small forces.

![Fig. 2. Results of Monte-Carlo simulations for the three models defined in the text: (a) scaling of the maxima of the second moment \( \langle s^2(N, f) \rangle \) of the distribution of cluster sizes (omitting the largest cluster in every configuration), as function of the total system size \( N \); (b) probability distributions \( P(F) \) of force magnitudes, obtained from 100 samples of systems of \( 10^4 \) particles.](image)

| Model                  | \( f_c \)    | \( \phi \)   | \( \nu \)  |
|------------------------|--------------|--------------|------------|
| Independent \( q \)-model | 0.7 \(\pm 0.01\) | 0.69 \(\pm 0.01\) | 3.1 \(\pm 0.1\) |
| Snooker model          | 0.93 \(\pm 0.01\) | 0.89 \(\pm 0.01\) | 1.65 \(\pm 0.1\) |
| Microcanonic \( q \)-model | 0.585 \(\pm 0.05\) | 0.81 \(\pm 0.01\) | 1.65 \(\pm 0.1\) |
| Percolation             | 0.895(43/48) | 1.33(4/3)    |            |

Table 1. Values of the critical threshold \( f_c \) and the critical exponents \( \phi \) and \( \nu \) obtained from Fig. 2 and the data collapse shown in Fig. 3. For two-dimensional percolation, exact values are shown inside brackets.
6 Discussion

We have introduced a new approach to investigate the geometry of force networks, based on statistics of clusters created by forces larger than a given threshold. The existence of a critical threshold uncovers a scale-invariance of force networks, which we characterized by the critical exponents \( \nu \) and \( \phi \) for the correlation length and the second moment of the cluster size distribution. In particular, in each network we identify a fractal object of dimension \( D = \phi + 1 \), given by the
incipient force cluster at the critical threshold. As shown in Table 1, we found three different sets of critical exponents for the three models we studied, implying that they belong to distinct geometrical universality classes, although their $P(F)$ display similar features. Interestingly, for the snooker model, $\phi$ is very close to the percolation value, but as the values of $\nu$ are further apart and the scaling functions are different, it does not belong to the percolation universality class.

Two distinct universality classes could have been expected a priori for the $q$-models on one hand and the snooker packing on the other. Indeed, the $q$-models are both directed and include only scalar conservation laws, while the snooker model is isotropic with vectorial conservation laws. The reason for the segregation of the independent and microcanonic in two different universality classes is more subtle. They differ only by the form of the probability distribution of forces, but in the independent case the distribution is Markovian from top towards bottom, while in the microcanonic case no such preferred direction of propagation exists.

While in jammed matter the disorder in the underlying contact geometry plays an important role, we considered here only lattice models with fixed contact geometry. The force-cluster method can be applied in a straightforward fashion to ensembles of forces networks resulting from disordered contact networks. By analogy with other critical phenomena, we however do not expect such randomness to modify the universality class. Indeed, the values we have found for $\nu$ in combination with the Harris’ criterion [32] suggest that geometric disorder is irrelevant. A further study indeed shows that introducing quenched disorder in the $q$-models does not modify the universality class, which in turn confirms that the scale-invariance found here is in all aspects similar to equilibrium critical phenomena.

While the results presented here clearly show that our method is able to discriminate between different scaling behaviors, a crucial question is whether any of the models belongs to the same universality class as a realistic two-dimensional system of grains under isotropic pressure. A recent study [33] of packings generated by molecular dynamics simulations showed that packings under isotropic pressure lead to the same scaling behaviour irrespectively of the applied pressure, the polydispersity of the grains, the coefficient of friction and the force law. Remarkably, the corresponding scaling exponents and scaling function appear to be the same as those obtained from the snooker packing.

7 Outlook

The existence of universality classes for force networks raises a number of new questions. First of all, what properties of a jammed system determine the universality class of its force network? Our results suggest that the isotropy of the applied force and the vector nature of the force balance are essential. On the other hand, packings under static shear might lead to another universality
class. Another relevant parameter could be the temperature in thermal systems which exhibit jamming, such as colloids.

Our method based on force cluster criticality is clearly able to discriminate between the many models proposed for force networks [34,35,15,36–38]. In particular, it shows that the Edwards’ hypothesis, which proposes to consider all metastable states of a jammed system equally likely, leads quantitatively to the same scaling properties as found in force networks generated by molecular dynamics simulations.

Finally, the method developed here for force networks in jammed matter is clearly more general. It applies in principle to any ensemble of graphs with continuous variables on the edges, such as flux, transport or metabolic networks [39,40]. The corresponding universality classes could complement the topological characterizations of networks developed in the recent years [41].

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