Structured Discrete Shape Approximation

Andreas M. Tillmann  
Visual Computing Institute & Chair of Operations Research, RWTH Aachen University, Germany  
e-mail: andreas.tillmann@cs.rwth-aachen.de

Leif Kobbelt  
Visual Computing Institute, RWTH Aachen University, Germany  
e-mail: kobbelt@cs.rwth-aachen.de

Abstract—We consider approximating a 2D shape contour using discrete graph-assembly systems, which consist of limited sets of node and edge types along with edge length and orientation restrictions. This task is related to, e.g., minimum-link path, art gallery and watchman route, robot motion planning, VLSI routing, or polygonal approximation problems. We show that already deciding feasibility of such approximation problems is NP-hard and then devise an algorithmic framework that combines shape sampling with exact $\ell_0$/cardinality-minimization to obtain good reconstructions using few graph components within reasonable time, in spite of the problem’s intractability. As a particular (2D) application, we approximate shape contours using the classical Zometool construction kit.

I. INTRODUCTION AND PRELIMINARIES

We define a discrete graph-assembly system (DGS) as $G := (V, E, B, D)$, where $V$ and $E$ are sets of node and edge types, respectively, $B := \{b_e \in \mathbb{Z}_+ \cup \{\infty\} : e \in E\}$ contains edge availability budgets, and $D := \{(d_i, V_i, E_i) : V_i \subseteq V, E_i \subseteq E, i = 1, \ldots, k\}$ is a collection of admissible (edge) orientations $d_i$, (given, e.g., by angles or direction vectors in $\mathbb{R}^2$) along with compatible node and edge types (i.e., to any node $v \in V_i$, edges from $E_i$ can be attached with orientation $d_i$). In general, not every orientation is allowed at every node, and not every edge type is allowed for every orientation. Budget-observing DGS constructions corresponding to planar graphs are called valid.

Approximating a given shape contour $F \subset \mathbb{R}^2$ (or curve segment) by means of a DGS is related to a variety of problems (see Abstract) which seek paths within a given (often polygonal) shape between two specified points; optimization is usually done w.r.t. edge number or total path length, and possible constraints include, in particular, a discrete set of admissible edge orientations. An overview of many such problems and algorithms for them is given in the recent survey [1], see also the extensive list of references therein and, e.g., [2], [3], [4]. Notably, the key feature of finitely many edge types/lengths seems to be missing in all these and similar previously considered problems. The only appearance we are aware of is in the context of simulated annealing for (3D) shape approximation using the Zome(tool) system (cf. [5], [6]), which indeed is a special DGS with a single node type, three edge types (red, blue, yellow ”struts”) with three length variations each, and orientation restrictions associated with the colors, see Fig. 1 and [7], [8] for details. In general, DGSs like the Zome system allow for a huge (exponential) number of component combinations, and indeed, the intractability results put forth in the following provide theoretical evidence that they are hard to handle algorithmically.

II. PROBLEM FORMULATION AND COMPLEXITY

Shape contour and curve segment approximation problems can be formalized in various ways; here, we state only one natural version that also forms the basis for our algorithm. Note that the intractability result pertains to feasibility alone, so that associated optimization problems are hard regardless of the objective.

DGS CONTOUR APPROXIMATION (DCA): Given a shape contour $F \subset \mathbb{R}^2$, sampling points $\{p_1, \ldots, p_k\}$ on $F$, a $\delta \geq 0$, and a DGS $G$, does there exist an $F$-resembling $G$-cycle, i.e., a valid simple cycle built from $G$ with at least one node in each $\mathcal{N}_i^G := \{p \in \mathbb{R}^2 : \|p - p_i\|_\infty \leq \delta\}, i \in \{1, \ldots, k\}$?

Theorem II.1. The DCA problem is NP-hard in the strong sense, even if restricting to piecewise linear, rectangular shape contours with integral-coordinate turning points and sampling points, and DGSs with only one node type, horizontal and vertical orientations, and integral-length edge types.

Proof: By reduction from 3-PARTITION (see problem SP15 in [9]). A sketch is shown in Fig. 2; we omit the details.

III. ALGORITHMIC FRAMEWORK

Besides the combinatorial explosion, algorithmic challenges in solving DCA include the global placement of the solution (relative to the input shape), which cannot be fixed a priori, and the fact that error measures w.r.t. the actual input shape can only be evaluated for struts whose spacial placement has already been decided upon (so standard graph-theoretical methods, e.g., shortest paths in a partially extended DGS graph, are not applicable). Also, the input contour may only be given implicitly; here, we work with distance fields from which the ground truth contour can be interpolated. Thus, to circumvent these problems, we propose the following algorithmic scheme for DCA:

1) Divide the input contour into $k$ segments of roughly equal (arc-)length and pick a point with largest curvature from each segment as the sampling points $p_1, \ldots, p_k$.
2) Solve a mixed-integer program (MIP) to find the components of a smallest-cardinality $F$-resembling $G$-cycle.
3) Assemble these components to the final approximation by computing smallest-error strut permutations for each segment (exactly or in a greedy fashion).

The MIP in step 2) has two continuous variables modeling the global positioning, and for each segment between sampling point neighborhoods $\mathcal{N}_i^G$ and $\mathcal{N}_{i+1}^G$ (radius $\delta$ equal to half the longest strut length), we have integer variables counting the number of struts of each type and orientation used to sequentially connect the nodes that are to be placed in each $\mathcal{N}_i^G$. (For the Zome case, connectivity of points amounts to finding an integral linear combination of the elements shown in Fig. 1.) We solve these MIPs with the state-of-the-art solver Gurobi. Figures 3–4 show some results of our algorithm for different input shape contours, scalings and assembly methods in Step 3). The runtimes were of the order of a few seconds to minutes for all these instances.

IV. CONCLUSION

We intend to expand and refine our algorithmic framework. A long-term goal is to transfer the ideas developed here to the 3D case, which will require substantial modifications and extensions.
Fig. 1: The different struts and their lengths and orientations in the most versatile 2-dim. Zome system. Strut thicknesses differ for visualization purposes only, global (i.e., node) orientation is 0, and $\phi := (1 + \sqrt{5})/2 \approx 1.618$ is the golden ratio (recall $\phi^2 = 1 + \phi$). Note that when a Zome node is placed at the origin (0, 0), every node that can be reached via Zome-DGS components has the form $(\alpha_1 + \phi \beta_1, \alpha_2 + \phi \beta_2)$ with $\alpha_i, \beta_i \in \mathbb{Z}$ for all $i \in \{1, 2\}$.

Fig. 2: Schematic illustration of the main construction to prove NP-hardness, for the case of curve segment approximation: The shaded boxes contain all points with $\ell_\infty$-norm distance at most $\delta$ from the resp. sampling points (placed at the "kinks" of the input piecewise linear $x_1$-$x_2$-path $\mathcal{F}$ (solid curve)). All $m$ segments of this path have length $A$. The dashed part of $\mathcal{F}$ indicates a continuation of the discernible pattern in accordance with the actual instance size. A 3-Partition solution exists if and only if the elements, represented as struts, can be fitted three-a-piece between consecutive sampling point neighborhoods.

Fig. 3: Zome-DGS approximations of a cow torso with exact permutation enumeration after MIP, based on 10 roughly equi-distant sampling points (top) or the largest-curvature sample points in each of 10 roughly same-length segments (middle), and with greedy permutation construction based on the latter sampling scheme (bottom). Note that curvature information is pivotal in sample point selection, and that the greedy version yields competitive results w.r.t. the exact one.

Fig. 4: Zome-DGS approximations of the silhouettes of France (left) and Asterix (right), with greedy and exact permutation construction (after MIP), resp., based on the largest-curvature sampling points in each of 10 or 20, resp., roughly same-length segments. Clearly, more sample points or a refined placement is needed here in order to make the pure cardinality minimization work better.

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