1. Introduction

Since a terrain-following coordinate ($\sigma$-coordinate) (Phillips 1957) can transform the complex surface of Earth into a regular coordinate surface, the $\sigma$-coordinate becomes a common choice for atmospheric and oceanic models. However, the computational form of pressure gradient force (PGF) in a $\sigma$-coordinate is two-term (the so-called classic method). These two terms are opposite in sign and of the same order near steep terrain, which induces significant numerical errors; namely, the PGF errors of the classic method are large. Finally, the effects of three factors on inducing the PGF errors of the classic method are validated by a series of idealized experiments using various terrain types and pressure fields. The experimental results also demonstrate that the PGF errors of the covariant method are affected little by the three factors.

Many methods have been proposed to reduce the PGF errors, which can be categorized into three types. The first type is to reduce the PGF errors based on the two-term PGF, and includes high-order schemes (Blumberg and Mellor 1987; Corby, Gilchrist, and Newson 1972; Maher 1984; Qian and Zhou 1994), standard stratification deduction (Zeng 1979), and so on. The second type is to overcome the PGF errors through various coordinate transformations based on the non-orthogonal $\sigma$-coordinate (Li, Wang, and Wang 2012; Qian and Zhong 1986; Smagorinsky et al. 1967; Yan and Qian 1981; Yoshio 1968). And the third type is to design an orthogonal terrain-following coordinate to bypass the PGF errors (Li et al. 2014). Both methods proposed by Li, Wang, and Wang (2012) and Li et al. (2014) create a one-term computational form of PGF, while Li, Wang, and Wang (2012) proposed to use the covariant scalar equations of the $\sigma$-coordinate (the covariant method). Note that studies on the factors inducing the PGF errors are less common than efforts made to reduce the PGF errors.
For the classic method, there are two factors inducing the PGF errors; namely, the terrain slope and the nonlinear vertical pressure gradient (Yan and Qian 1981; Zeng and Ren 1995). Recently, Klemp (2011) proposed that the PGF errors could be minimized when the vertical pressure gradient is nearly linear. The idealized experiments implemented by Li, Li, and Wang (2015) demonstrated that the PGF errors increased according to the increasing terrain slope. However, for the covariant method, few analyses of the factors inducing the PGF errors have been carried out.

In this study, we further explore the factors inducing the PGF errors of the classic and covariant method. In Section 2 we use geometric analysis to identify all the possible factors inducing the PGF errors of the classic method. In Section 3 we implement a series of idealized experiments of various terrain types and pressure fields to investigate the effects of all the factors identified in Section 2 that induce the PGF errors of the classic and covariant method.

2. Geometric analysis of the PGF errors of the classic method

The two-term expression of PGF of the classic method is given by

$$\frac{\partial p}{\partial x} = \left( \frac{\partial p}{\partial x} \right)_\sigma + \left( -\frac{\partial p}{\partial z} \left( \frac{\partial z}{\partial x} \right)_\sigma \right).$$

We abbreviate the first and second term on the RHS of Equation (1) as PGF1 and PGF2, respectively. According to Equation (1), the PGF of the classic method is relevant to three factors:

1. The pressure gradient along each vertical layer $\left( \frac{\partial p}{\partial x} \right)_\sigma$;
2. The vertical pressure gradient $\frac{\partial p}{\partial z}$;
3. The slope of each vertical layer $\left( \frac{\partial z}{\partial x} \right)_\sigma$.

Since the terrain slope equals the slope of the bottom vertical layer in a model, the effect of terrain slope can be included in the effect of each vertical layer on inducing the PGF errors. The effect of the vertical pressure gradient on inducing the PGF errors has been analyzed by many researchers (e.g. Zeng and Ren 1995). Therefore, we only investigate the effects of the pressure gradient along each vertical layer and slope of each vertical layer.

Geometric schematics representing the PGF and its components of different coordinates are illustrated in Figures 1(b)–(d). The AC represents $\nabla p$, AD is the $\left( \frac{\partial p}{\partial x} \right)_\sigma$, and CD is the vertical pressure gradient:

$$CD = \frac{\partial p}{\partial z}. \tag{2}$$

The pressure gradient along each vertical layer is AB:

$$AB = \left( \frac{\partial p}{\partial x} \right)_\sigma. \tag{3}$$

Figure 1. Schematic diagrams of PGF vectors and their components of different coordinates. Notes: Green curves represent a certain vertical layer; blue lines are its tangent and normal directions. $S_1$, $S_2$, and $S_3$ in (a) indicate the areas of different directions of PGF. Panels (b–d) are schematic diagrams of PGF in $S_1$, $S_2$, and $S_3$, respectively. Red lines with solid arrows (AC) represents $\nabla p$. Black- and red-arrowed lines are the components of the PGF of the $\sigma$- and $z$-coordinate, respectively.
the direction of PGF (α) is \( \angle CAB \), and the slope of each vertical layer (φ) is given by

\[
\phi = \arctan \left( \frac{dz}{dx} \right),
\]

(4)

In addition, according to the geometric relationship in Figures 1(b)–(d), we obtain

\[
BD = CD \cdot \tan \phi;
\]

(5)

and

\[
AB = CD \cdot \left( \tan \phi + \frac{1}{\tan \alpha} \right).
\]

(6)

Equation (6) manifests that the pressure gradient along each vertical layer (AB) can be represented by the vertical pressure gradient (CD), the direction of PGF (α) and the slope of each vertical layer (φ). Therefore, the possible factors inducing the PGF errors of the classic method are as follows: (1) the direction of PGF (α); (2) the slope of each vertical layer (φ); (3) the vertical pressure gradient (CD).

According to the different directions, \( \angle CAB \), of PGF (α) in Figure 1(a), we define three situations: (1) \( S_1: \alpha \in \left[ 0, \frac{\pi}{2} \right] \) the yellow area; (2) \( S_2: \alpha \in \left[ \frac{\pi}{2}, \frac{\pi}{2} + \phi \right] \) the purple area; (3) \( S_3: \alpha \in \left[ \frac{\pi}{2} + \phi, \pi \right] \) the grey area. The corresponding schematics of PGF and its components of different coordinates in \( S_1, S_2, \) and \( S_3 \) are shown in Figures 1(b)–(d), respectively.

The direction of PGF (α) in \( S_1 \) is given by

\[
\alpha = \angle CAB = \arctan \left( \frac{\partial p}{\partial z} \right);
\]

(7)

while both of them in \( S_2 \) and \( S_3 \) are given as follows:

\[
\alpha = \angle CAB = \pi - \arctan \left( \frac{\partial p}{\partial z} \right).
\]

(8)

Substituting Equations (2), (3), (5), and (7)/(8) into Equation (1), we obtain the expression of PGF in \( S_1, S_2, \) and \( S_3 \), respectively:

\[
\left( \frac{\partial p}{\partial x} \right)_{S_1} = AB - BD;
\]

(9)

\[
\left( \frac{\partial p}{\partial x} \right)_{S_2} = BD - AB;
\]

(10)

\[
\left( \frac{\partial p}{\partial x} \right)_{S_3} = AB + BD.
\]

(11)

According to Equation (11), the expression of PGF in \( S_3 \) is a summation of \( AB \) and \( BD \); namely, the PGF errors are consistently small regardless of the terrain slope. Therefore, we only investigate the PGF errors in \( S_1 \) and \( S_2 \) in the following calculation.

First, using Equations (5) and (6), we calculate the proportion between the PGF2 and PGF1 in \( S_1 \) (BD and AB in Equation (9)) as follows:

\[
\frac{BD}{AB} = \frac{\tan \phi \cdot \tan \alpha}{1 + \tan \phi \cdot \tan \alpha}.
\]

(12)

Equation (12) manifests that \( 0 \leq \frac{BD}{AB} \leq 1 \). Furthermore, if \( 0.1 \leq \frac{BD}{AB} \leq 1 \), the BD and AB are of the same order: namely, the PGF errors are large; otherwise, they are small. Moreover, we abbreviate \( \tan \phi \cdot \tan \alpha \) in Equation (12) as TT, and define a critical value of TT (TTc) for inducing the PGF errors as follows: if \( TT > TTc \), \( 0 < TT < TTc \), the PGF1 and PGF2 in Equation (1) are (are not) of the same order.

Substituting Equation (12) into 0.1 \( \leq \frac{BD}{AB} \leq 1 \), we obtain

\[
TT \geq \frac{1}{9}.
\]

(13)

The 1/9 on the RHS of Equation (13) is the TTc in \( S_1 \).

Second, using Equations (5) and (6), we calculate the proportion between PGF2 and PGF1 in \( S_2 \) (AB and BD in Equation (10)) as follows:

\[
\frac{AB}{BD} = \frac{1 + \tan \phi \cdot \tan \alpha}{\tan \phi \cdot \tan \alpha}.
\]

(14)

Substituting Equation (14) into 0.1 \( \leq \frac{AB}{BD} \leq 1 \), we obtain

\[
TT \leq -\frac{10}{9}.
\]

(15)

The \( 10/9 \) on the RHS of Equation (15) is the TTc in \( S_2 \). The expressions of PGF in \( S_1, S_2, \) and \( S_3 \), and TTc in both \( S_1 \) and \( S_2 \) are all summarized in Table 1.

In addition, when the slope of the vertical layer (φ) or direction of PGF (α) is close to the vertical direction \( \left( \frac{\pi}{2} \right) \), the absolute value of TT increases. According to Equations (12) and (14), the ratio of \( \frac{BD}{AB} \) tends to be a constant one with increasing (decreasing) TT; namely BD is close to AB. Consequently, the difference between the order of AB (BD) and that of AB – BD in \( S_1 \) or BD – AB in \( S_2 \) is large, and then the PGF errors are large too. Note that, in reality, typhoons, squall lines, and topographic blocking near steep terrain, in which the direction of PGF may be close to the vertical direction, may create large TT, therefore also inducing large PGF errors.

In conclusion, there are three factors inducing the PGF errors of the classic method: (1) the direction of PGF (α); (2) the slope of each vertical layer (φ); (3) the
vertical pressure gradient. The effects of $\alpha$ and $\varphi$ can be quantified by $\tan \varphi \cdot \tan \alpha$ (Table 1). Specifically, when $\tan \varphi \cdot \tan \alpha \geq \frac{1}{9}$ or $\tan \varphi \cdot \tan \alpha \leq -\frac{10}{9}$, the PGF errors of the classic method are large; and the closer $\alpha$ or $\varphi$ is to the vertical direction, the larger the PGF errors of the classic method are.

### 3. Idealized experiments

In order to investigate the effects of the three factors obtained in Section 2 on inducing the PGF errors of the classic and covariant method, idealized experiments using various terrain types and pressure fields are performed. For consistency, we use the same parameters as Li, Wang, and Wang (2012), except for the terrain and pressure fields. The basic parameters of all the experiments are introduced in Section 3.1. The experiments investigating the effects of the slope of each vertical layer and the direction of PGF are illustrated in Section 3.2. The experiments analyzing the effect of the vertical pressure gradient are demonstrated in Section 3.3.

#### 3.1. Basic parameters

We use the central spatial discretization in the horizontal direction, and the forward scheme in the vertical direction, for the PGF of both methods. The expressions are given as follows:

$$\frac{\partial p_{i,k}}{\partial x} = \frac{p_{i+1,k} - p_{i-1,k}}{2\Delta x} - \frac{p_{i,k+1} - p_{i,k}}{\Delta z} \cdot (\frac{\partial z}{\partial x})_a \quad (\text{classic method});$$

$$\left(\frac{\partial p_{i,k}}{\partial x}\right)_a = \frac{p_{i+1,k} - p_{i-1,k}}{2\Delta x} \quad (\text{covariant method}).$$

A 2D bell-shaped terrain type,

$$h(x) = H \cdot \frac{\alpha^2}{(x - h_0)^2 + \alpha^2},$$

is used (Figure 2), where $H$ is the maximum height, $\alpha = 5$ km is the half width, and $h_0 = 50$ km is the middle point of the terrain. We use two types of idealized pressure fields, defined as follows (Figure 2): (i) a pressure field with a linear vertical pressure gradient (LP),

$$p^{LP} = \frac{(p_0 - p_1)(z - H)}{H - p_1} + p_1;$$

and (ii) a pressure field with an exponential vertical pressure gradient (EP),

$$p^{EP} = p_0 \cdot e^{-\left(\frac{z - H}{H - h(x)}\right)}.$$
Figure 2. Pressure fields given by Equations (19) and (20).
Notes: The pressure scale (color bar at the bottom) is in hPa. The black curve in each panel indicates the terrain.

Figure 3. Average of PGF1 and PGF2 at different TT: (a) The variation of PGF1 and PGF2 according to different TT; (b–g) The PGF1 and PGF2 at TT = −2.728, −0.902, and −0.757, respectively.
Note: Black dots in (a) represent the TT of the experiments shown in (b–g).
by the slope of each vertical layer), but also the direction of PGF, needs to be considered in terms of inducing the PGF errors of the classic method, and both effects can be quantified by TT.

Second, we use the experiments of $H = 14$ km ($TT_c = -1.121$, closest to $-10/9$) as an example to further illustrate the effect of TT. Figure 3(a) shows the variation around its analytical value, $TT_c = -\frac{10}{9}$, obtained in Section 2, and the average value is $-1.116$. Moreover, whatever the terrain slope, the minimum and maximum relative errors (REs) of the PGF of the classic method are approximately 0.187 and 0.437, respectively. These results validate that not only the terrain slope (included by the slope of each vertical layer), but also the direction of PGF, needs to be considered in terms of inducing the PGF errors of the classic method, and both effects can be quantified by TT.

Figure 4. REs of the PGF of the classic and covariant method: (a) The variation of REs of the two methods according to different TT; (b, c) The patterns of TT and RE of the classic method when the average TT is $-1.247$, respectively; (d, e) The patterns of TT and RE of the classic method when the average TT is $-0.902$, respectively.
of the average values of PGF1 and PGF2 in different TT. Specifically, when TT < TTc, the PGF1 and PGF2 are consistently of the same order and opposite in sign (blue and red dotted lines in Figure 3(a)), and their patterns at TT = −2.728 are shown in Figures 3(b) and (c) as an example of TT < TTc. When TTc < TT < −0.857, the PGF1 and PGF2 are still opposite in sign but no longer of the same order (e.g. the PGF1 and PGF2 at TT = −0.902 shown in Figures 3(d) and (e)). When TT > −0.857 (H > Hp, α is in S4 of Figure 1(a)), the PGF1 and PGF2 become the same in sign (e.g. the PGF1 and PGF2 at TT = −0.757 shown in Figures 3(f) and (g)). These results verify the effects of TTc in S3 and S4 of Table 1.

Finally, we also use the results obtained by the experiments of H = 14 km as an example to analyze the variation of the PGF errors of the classic and covariant method according to the increasing TT (Figure 4). The pattern of TT is consistent with that of the REs of the classic method (Figures 4(b) and (c) with TT = −1.247, and Figures 4(d) and (e) with TT = −0.902). Further, the REs of the classic method increase according to the increasing |TT| (blue line in Figure 4(a)); however, the REs of the covariant method remain almost the same (red line in Figure 4(a)). These results validate that the PGF errors of the classic method increase according to the increasing TT; whereas, the PGF errors of the covariant method are affected little by it.

### 3.3. Effects of vertical pressure gradient

We implement two sets of experiments to calculate the PGF of the classic and covariant method using EP and LP. In addition, the difference between the analytical TTc and the TT obtained in the idealized experiments may be due to using the average TT of all the grids. Further analyses need to be carried out to use the average TT of selected grids, in which PGF1 and PGF2 are of the same order. Besides, the three factors tested by the idealized pressure fields in this study are simultaneously changed in Section 2 are all validated by the idealized experiments; whereas, none of them has the effect of inducing the PGF errors of the covariant method.

### 4. Conclusion and discussion

This study investigates the factors inducing the PGF errors of the classic and covariant method through geometric analysis and idealized experiments. Three factors are considered to induce the PGF errors of the classic method, including the direction of PGF (α), the slope of each vertical layer (φ), and the vertical pressure gradient; however, none of them induces the PGF errors of the covariant method. Moreover, the effects of α and φ can be quantified by tan φ tan α (Table 1).

The geometric analysis first demonstrates that the terrain slope and the vertical pressure gradient can induce the PGF errors of the classic method. Then, the effect of terrain slope is generalized into the effect of the slope of each vertical layer (φ). More importantly, a new factor, the direction of PGF (α), is proposed. When tan φ tan α ≥ \( \frac{\tan \phi \cdot \tan \alpha}{\frac{10}{9}} \) or tan φ tan α ≤ −\( \frac{\tan \phi \cdot \tan \alpha}{\frac{10}{9}} \), the two terms of PGF of the classic method are of the same order and opposite in sign. Subsequently, the PGF errors of the classic method are large (Table 1), and the closer the α or φ is to the vertical direction, the larger the PGF errors of the classic method are.

The effects of all of the three factors on inducing the PGF errors of the classic method are validated by a series of idealized experiments. Results first verify the analytical value of TTc (−\( \frac{\tan \phi \cdot \tan \alpha}{\frac{10}{9}} \)), proposed by the geometric analysis, as well as its effect on inducing the PGF errors of the classic method (Figure 3). It is then found that the PGF errors of the classic method increase according to increasing |TT|, and their patterns are also consistent (Figure 4). Finally, results using LP and EP validate that EP can significantly increase the PGF errors of the classic method (Table 2). Moreover, all the idealized experiments demonstrate that the PGF errors of the covariant method are affected little by the three factors. Note that the comparison of the factors inducing the PGF errors of the classic and covariant methods in this study only considers the PGF term; the true benefit of the covariant method needs to be investigated using the equations of the classic and covariant methods in their entirety.

### Table 2. PGF errors of the classic and covariant methods in LP and EP.

| Method         | LP    | EP   |
|----------------|-------|------|
| Classic method | 1.484 | 3.336|
| Covariant method | 0.117 | 0.113|

* Differences between the PGF errors in EP listed here and those in Li, Wang, and Wang (2012) are due to the revised boundary condition.
due to the analytical expression of pressure, Equation (20). And, only the effects of $TT_c$ in $S_2$ and $S_3$ can be tested by this kind of pressure field. Sensitivity experiments using a discrete pressure field, in which each factor can be independently changed, are needed to further investigate the effects of these three factors. Furthermore, experiments using the real pressure field are needed to verify the effect of $TT_c$.

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