Some properties of the Sturm-Liouville operator in $L_p(R)$

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ABSTRACT. We consider the boundary problem  

$$-y''(x) + q(x)y(x) = f(x), \quad x \in R$$  \hspace{1cm} (1)  

$$\lim_{|x| \to \infty} y^{(i)}(x) = 0, \quad i = 0, 1$$  \hspace{1cm} (2)  

where $f(x) \in L_p(R)$, $p \in [1, \infty]$, $1 \leq q(x) \in L_1^{loc}(R)$. For this boundary problem we obtain: 1) necessary and sufficient conditions for unique solvability and a priori properties of the solution; 2) a criterion for the resolvent to be compact in $L_p(R)$, $p \in [1, \infty]$ and some a priori properties of the spectrum.

§1. Introduction.  

In this paper we consider the boundary problem (1.1) - (1.2):

$$-y''(x) + q(x)y(x) = f(x), \quad x \in R$$  \hspace{1cm} (1.1)  

$$\lim_{|x| \to \infty} y^{(i)}(x) = 0, \quad i = 0, 1$$  \hspace{1cm} (1.2)  

where $f(x) \in L_p(R)$, $p \in [1, \infty]$ ($\| \cdot \|_\infty$ is defined as essup), and

$$1 \leq q(x) \in L_1^{loc}(R).$$  \hspace{1cm} (1.3)  

Our aim is to give a detailed description of the Sturm-Liouville operator $\mathcal{L}_p$ in $L_p(R)$, $p \in [1, \infty]$ (see below). Usually, the operator $\mathcal{L}_p$ is defined as the closure in $L_p(R)$ of a differential expression $\ell$:

$$\ell y = -y''(x) + q(x)y(x)$$  \hspace{1cm} (1.4)
with \( y(x) \) belonging to \( C_0^\infty(R) \) [10, Ch.VII, §5], [1, Ch.II, §1]. Such an approach allows one to develop a sufficiently complete theory of the operator \( L_p \), but some problems are left in the background or not considered at all. This "neglect" is quite natural since the above approach is built upon general functional methods rather than on concrete properties of the differential operator \( \ell \). Below we add some new facts to already known ones using a more detailed study of the operator \( \ell \) and its inversion.

**Theorem 1.1.**

A) For \( p \in [1, \infty) \) for any \( f(x) \in L_p(R) \), the boundary problem (1.1)-(1.2) has a unique solution \( y(x) \in D_p \), where

\[
D_p = \{ z(x) : z(x) \in C^{(1)}(R) \cap L_p(R), \ z'(x) \in AC^{loc}(R), (\ell z)(x) \in L_p(R), \\
\lim_{|x| \to \infty} z^{(i)}(x) = 0, \ i = 0, 1 \}
\]

B) Let \( p = \infty \). For any \( f(x) \in L_\infty(R) \), the boundary problem (1.1)-(1.2) has a solution (moreover, a unique solution) if and only if (1.5) holds:

\[
\lim_{|x| \to \infty} \int_{x-a}^{x+a} q(t) dt = \infty \quad \text{for any } a \in (0, \infty).
\]

In the latter case, the solution \( y(x) \) to (1.1)-(1.2) belongs to \( D_\infty^0 \), where

\[
D_\infty^0 = \{ z(x) : z(x) \in C^{(1)}(R), \ z'(x) \in AC^{loc}(R), (\ell z)(x) \in L_\infty(R), \\
\lim_{|x| \to \infty} z^{(i)}(x) = 0, \ i = 0, 1 \}.
\]

C) Let \( p = \infty \), and suppose that (1.5) fails. For any \( f(x) \in L_\infty(R) \), equation (1.1) has a unique solution \( y(x) \in D_\infty \) where

\[
D_\infty = \{ z(x) : z(x) \in C^{(1)}(R), \ z'(x) \in AC^{loc}(R), (\ell z)(x) \in L_\infty(R) \}.
\]

In the sequel, by the Sturm-Liouville operator \( \mathcal{L}_p \), we mean the differential expression (1.4) defined on \( D_p \). Thus \( \mathcal{L}_p : D_p \to L_p(R) \), \( p \in [1, \infty] \), and there exists \( \mathcal{L}_p^{-1} : D(\mathcal{L}_p^{-1}) = L_p(R), p \in [1, \infty] \) (see Theorem 1.1). The following fact is well known for \( p = 2 \) (see [8]).
Theorem 1.2. The operator $L_p^{-1} : L_p(R) \to L_p(R)$ is bounded for $p \in [1, \infty]$. Moreover, for any $p \in [1, \infty]$ it is compact if and only if (1.5) holds. In the latter case, the spectrum $\sigma(L_p)$ of the operator $L_p$ is purely discrete for all $p \in [1, \infty]$ and does not depend on $p$ for $p \in [1, \infty)$. In particular, $\sigma(L_p) = \sigma(L_2) \subset [1, \infty)$ for $p \in [1, \infty)$.

Concluding remarks and acknowledgements. Some of our results have been proved in [13] under the additional assumption

$$1 \leq q(x) \in L^{1_{oc}}(R), \quad p \in [1, \infty].$$

This paper was stimulated by crucial remarks of the referee concerning our paper [4]. In [4] we used results of [13], and the referee noticed that they can be made more precise. In particular, he suggested the definition of the classes $D_p, D_\infty^0$ as "a natural $L_p$-framed for (1.1)" (we cite the report). Theorem 1.2 (under condition (1.6)) was a subject of the authors' discussion with Prof. V.G. Maz'ya in 1989. It was V.G. Maz'ya who noticed that Theorem 1.2 must hold under the weakest requirement (1.5) replacing (1.6). The authors thank the referee of [4] and V.G. Maz'ya for their attention, benevolent criticism, and formulation of the problems.

§2. Preliminaries.

Throughout the sequel we denote by $c$ absolute positive constants whose values are not essential for exposition and may differ even within a single chain of calculations.

Lemma 2.1. Consider an equation

$$z''(x) = q(x)z(x), \quad x \in R.$$ (2.1)

There exists a fundamental system of solutions (FSS) $\{u(x), v(x)\}$ of (2.1) such
that
\[ u(x) > 0, \quad v(x) > 0, \quad u'(x) < 0, \quad v'(x) > 0, \quad x \in \mathbb{R} \]
\[ v'(x)u(x) - u'(x)v(x) = 1, \quad u(x) = v(x)\int_x^\infty \frac{dt}{v^2(t)}, \quad x \in \mathbb{R} \]
\[ \lim_{x \to \infty} u(x) = \lim_{x \to \infty} u'(x) = \lim_{x \to -\infty} v(x) = \lim_{x \to -\infty} v'(x) = 0 \quad (2.2) \]
\[ \lim_{x \to \infty} v(x) = \lim_{x \to \infty} v'(x) = \lim_{x \to -\infty} u(x) = \lim_{x \to -\infty} |u'(x)| = \infty. \]

Under condition (1.3), equation (2.1) does not oscillate at \( \pm \infty \), and therefore \( u(x), v(x) \) are determined up to constant factors as principal solutions to (2.1) on \((0, \infty)\) and \((-\infty, 0)\), respectively [7,p.355]. We call a FSS of (2.1) with properties (2.2) a principal FSS of (2.1) (PFSS, [3]). The proof of Lemma 2.1 can be also found in [3].

**Lemma 2.2.** [5] For \( x \in \mathbb{R} \), a PFSS of (2.1) admits the following representation:
\[ v(x) = \sqrt{\rho(x)} \exp \left( \frac{1}{2} \int_{x_0}^x \frac{dt}{\rho(t)} \right), \quad u(x) = \sqrt{\rho(x)} \exp \left( -\frac{1}{2} \int_{x_0}^x \frac{dt}{\rho(t)} \right), \quad x \in \mathbb{R} \quad (2.3) \]
where \( \rho(x) \overset{\text{def}}{=} u(x)v(x) \), \( x \in \mathbb{R} \) and \( x_0 \) is the unique root of the equation \( u(x) = v(x) \).

Moreover
\[ |\rho'(x)| < 1, \quad x \in \mathbb{R}. \quad (2.4) \]

Formulae of type (2.3) were also mentioned in [14, pp.419-420]. Lemma 2.2 can be found in the above presentation in [3].

For a fixed \( x \in \mathbb{R} \), consider the following equations in \( d \geq 0 \):
\[ 1 = \int_0^\sqrt{2d} \int_{x-t}^x q(\xi)d\xi dt, \quad 1 = \int_0^\sqrt{2d} \int_x^{x+t} q(\xi)d\xi dt, \quad 2 = d \int_{x-d}^{x+d} q(t)dt. \quad (2.5) \]
For any \( x \in \mathbb{R} \), each of the equations (2.5) has a unique finite positive solution [3]. Denote the solutions by \( d_1(x), \quad d_2(x), \quad d(x) \) respectively.

**Lemma 2.3.** The function \( d(x) \) is continuous for \( x \in \mathbb{R} \) and
\[ 0 < d(x) \leq 1, \quad x \in \mathbb{R}. \quad (2.6) \]
Moreover, for \( \varepsilon \in [0, 1] \), \( x \in \mathbb{R} \) the following inequalities hold:

\[
(1 - \varepsilon)d(x) \leq d(t) \leq (1 + \varepsilon)d(x), \quad t \in [x - \varepsilon d(x), x + \varepsilon d(x)].
\]

(2.7)

Lemma 2.3 and the definition of \( d(x) \) are due to M.O. Otelbaev (see [9]). The functions \( d_1(x), d_2(x) \) were introduced in [2]. Lemma 2.3 was proved in the above presentation in [4].

**Theorem 2.1** [3]. For \( x \in \mathbb{R} \) one has the following estimates (see (2.3)):

\[
\frac{1}{\sqrt{2}} \leq \frac{v'(x)}{v(x)} d_1(x) \leq \sqrt{2}, \quad \frac{1}{\sqrt{2}} \leq \frac{|u'(x)|}{u(x)} d_2(x) \leq \sqrt{2},
\]

(2.8)

\[
\frac{1}{\sqrt{2}} \frac{d_1(x) d_2(x)}{d_1(x) + d_2(x)} \leq \rho(x) \leq \sqrt{2} \frac{d_1(x) d_2(x)}{d_1(x) + d_2(x)},
\]

(2.9)

\[
4^{-1}d(x) \leq \rho(x) \leq 3 \cdot 2^{-1}d(x).
\]

(2.10)

**Lemma 2.4.** [13] Condition (1.5) holds if and only if \( \lim_{|x| \to \infty} d(x) = 0 \).

**Theorem 2.2.** [6] Let \( p \in [1, \infty) \). A set \( K \subset L_p(\mathbb{R}) \) is precompact if and only if the following conditions hold:

1) \( \sup_{f \in K} \|f\|_p < \infty \);

2) \( \lim_{\delta \to 0} \sup_{f \in K} \sup_{|t| \leq \delta} \|f(\cdot + t) - f(\cdot)\|_p = 0 \)

3) \( \lim_{N \to \infty} \sup_{f \in K} \int_{|x| \geq N} |f(x)|^p dx = 0 \).

**Theorem 2.3.** [11] Let \( X, Y \) be normed spaces, \( T : X \to Y \) a linear bounded operator, \( T^* \) the adjoint operator. Then \( T \) is compact if and only if \( T^* \) is compact.

**Theorem 2.4.** [12] Let \( (X, \Sigma, \mu) \) be a space with a \( \Sigma \)-infinite measure, \( L_0 \) the space of piecewise constant \( \mu \)-integrable functions on \( X \), \( M \) the space of all measure functions on \( X \), \( T : L_0 \to M \) a linear operator having continuous extensions \( T_p : L_p(\mu) \to L_p(\mu), T_1 : L_1(\mu) \to L_1(\mu) \) where \( 1 \leq p \leq q \leq \infty \). Let the spectra \( \sigma(T_p) \) and \( \sigma(T_q) \) be zero-dimensional (say, countable). Then for all \( r \in [p, q] \) one has \( \sigma(T_r) = \sigma(T_q) \). (The operator \( T \) has a continuous extension \( T_r : L_r(\mu) \to L_r(\mu), r \in [p, q] \) by the Riesz-Torin Theorem (see [6, Ch.VI, §10].)
3. Unconditional solvability of boundary problem in $L_p(R), \ p \in [1, \infty)$.

In this section, we prove part A) of Theorem 1.1 We need Lemmas 3.1 – 3.4.

**Lemma 3.1.** For a PFSS of (2.1) one has the following inequalities:

\[
v'(x) \geq v(x), \quad |u'(x)| \geq u(x), \quad x \in R. \tag{3.1}
\]

\[
v(x) \geq \exp(x - t)v(t) \quad \text{for} \quad x \geq t; \quad u(x) \geq \exp(t - x)u(t) \quad \text{for} \quad x \leq t. \tag{3.2}
\]

\[
\rho(x) = u(x)v(x) \leq 1, \quad x \in R. \tag{3.3}
\]

**Proof.** Let \( \hat{\gamma}(t) = \exp(t) \). Then for \( t \in R \) we obtain

\[
[v'(t)\hat{\gamma}(t) - \hat{\gamma}'(t)v(t)]' = v''(t)\hat{\gamma}(t) - \hat{\gamma}''(t)v(t) = (q(t) - 1)v(t)\hat{\gamma}(t). \tag{3.4}
\]

Taking into account (1.3) and (2.2), we deduce inequality (3.1) from (3.4) for \( v(x) \):

\[
\hat{\gamma}(x)[v'(x) - v(x)] = v'(x)\hat{\gamma}(x) - v'(x)v(x) = \int_{-\infty}^{x} (q(t) - 1)v(t)\hat{\gamma}(t)dt \geq 0, \quad x \in R.
\]

Together with (2.2), this implies (3.2) for \( v(\cdot) \):

\[
\ln \frac{v(x)}{v(t)} = \int_{t}^{x} \frac{v'(\xi)}{v(\xi)} d\xi \geq x - t, \quad x \geq t.
\]

The following chain of calculations is based on (2.2) and (3.1) and leads to (3.3):

\[
\rho(x) = v^2(x) \int_{x}^{\infty} \frac{d\xi}{v^2(\xi)} = -v^2(x) \int_{x}^{\infty} \frac{1}{v'(\xi)} d\left(\frac{1}{v(\xi)}\right) \\
= -v^2(x) \left[ \frac{1}{v'(\xi)v(\xi)} \right]_{x}^{\infty} + \int_{x}^{\infty} \frac{1}{v(\xi)} v''(\xi) d\xi \\
= \frac{v(x)}{v'(x)} - v^2(x) \int_{x}^{\infty} \frac{q(\xi)d\xi}{v'(\xi)^2} \leq \frac{v(x)}{v'(x)} \leq 1.
\]

The proof of (3.1) – (3.2) for \( u(\cdot) \) is similar to the above proof for \( v(\cdot) \). \( \square \)

Let us introduce the Green function \( G(x, t) \) and the Green operator \((Gf)(x)\) :

\[
G(x, t) = \begin{cases}
  u(x)v(t), & x \geq t \\
  u(t)v(x), & x \leq t
\end{cases} \tag{3.5}
\]

\[
(Gf)(x) = \int_{-\infty}^{\infty} G(x, t)f(t)dt, \quad x \in R, \quad f(\cdot) \in L_p(R)
\]
Lemma 3.2. For \( x, t \in \mathbb{R} \) one has the inequalities

\[
0 < G(x, t) \leq \exp(-|t - x|), \quad 0 < G(x, t) \leq \frac{3}{4} d(x) \exp(-|t - x|). \tag{3.6}
\]

\[
\left| \frac{\partial}{\partial x} G(x, t) \right| \leq \exp(-|t - x|). \tag{3.7}
\]

Proof. Let \( x \geq t \). Then (3.5), (3.3), and (3.2) imply (3.6):

\[
G(x, t) = u(x)v(t) = \rho(x) \frac{v(t)}{v(x)} \leq \frac{v(t)}{v(x)} \leq \exp(-|t - x|).
\]

The case \( x \leq t \) can be considered in a similar way. If in both cases (\( x \geq t \) and \( x \leq t \)) we use (2.10) instead of (3.3), we obtain the second inequality of (3.6). Let \( x > t \). Then (3.5), (2.2), and (3.2) imply (3.7):

\[
\left| \frac{\partial}{\partial x} G(x, t) \right| = |u'(x)|v(t) = |u'(x)|v(x) \frac{v(t)}{v(x)} \leq \frac{v(t)}{v(x)} \leq \exp(-|t - x|).
\]

The case \( x < t \) can be treated in a similar way; the case \( x = t \) follows from (2.4).

Lemma 3.3. Let \( r(x) = \frac{1}{d(x)} \), \( x \in \mathbb{R} \). The functions \((Gf)(x)\), \( \frac{d}{dx}(Gf)(x) \), where \( f(x) \in L_p(\mathbb{R}), \quad p \in [1, \infty] \) are absolutely continuous and satisfy the inequalities:

\[
\|r(x)(Gf)(x)\|_p \leq c\|f\|_p, \quad \|r(x)(Gf)(x)\|_{C(\mathbb{R})} \leq c\|f\|_p. \tag{3.8}
\]

\[
\left\| \frac{d}{dx}(Gf)(x) \right\|_p \leq c\|f\|_p, \quad \left\| \frac{d}{dx}(Gf)(x) \right\|_{C(\mathbb{R})} \leq c\|f\|_p. \tag{3.9}
\]

Proof. For \( f(x) \in L_p(\mathbb{R}), \quad p \in [1, \infty] \) consider the integrals

\[
\mathcal{T}_1(x) = \int_{-\infty}^{x} v(t)f(t)dt, \quad \mathcal{T}_2(x) = \int_{x}^{\infty} u(t)f(t)dt, \quad x \in \mathbb{R}. \tag{3.10}
\]

These integrals converge absolutely since (3.2) and Hölder’s inequality imply:

\[
|\mathcal{T}_1(x)| \leq \int_{-\infty}^{x} v(t)|f(t)|dt \leq v(x) \int_{-\infty}^{x} \exp(-|t - x|)|f(t)|dt \leq cv(x)\|f\|_p. \tag{3.11}
\]

\[
|\mathcal{T}_2(x)| \leq \int_{x}^{\infty} u(t)|f(t)|dt \leq u(x) \int_{x}^{\infty} \exp(-|t - x|)|f(t)|dt \leq cu(x)\|f\|_p. \tag{3.12}
\]
Since \((Gf)(x) = u(x)T_1(x) + v(x)T_2(x)\), we conclude that both \((Gf)(x)\) and \((Gf)'(x) = u'(x)T_1(x) + v'(x)T_2(x)\) are absolutely continuous. The following estimates can be derived from (3.6) and Hölder’s inequality:

\[
\begin{align*}
|r(x)|(Gf)(x)| & \leq r(x) \int_{-\infty}^{\infty} G(x, t)|f(t)|dt \leq c \int_{-\infty}^{\infty} \exp(-|t - x|)|f(t)|dt \\
& \leq c \left( \int_{-\infty}^{\infty} \exp(-|t - x|)dt \right)^{1/p'} \left( \int_{-\infty}^{\infty} \exp(-|t - x|)|f(t)|^pdt \right)^{1/p} \\
& \leq c \left( \int_{-\infty}^{\infty} \exp(-|t - x|)|f(t)|^pdt \right)^{1/p} \leq c\|f\|_p.
\end{align*}
\]

In particular, (3.13) and (2.6) imply the following inequalities:

\[
\|Gf(x)\|_{C(R)} \leq \|r(x)(Gf)(x)\|_{C(R)} \leq c\|f\|_p, \quad p \in [1, \infty].
\]

From (3.13), (2.6) and Fubini’s theorem we get

\[
\begin{align*}
\|Gf(x)\|_p^p & \leq \|r(x)(Gf)(x)\|_p^p \leq c \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} |f(t)|^p \exp(-|t - x|)dt \right] dx \\
& = c \int_{-\infty}^{\infty} |f(t)|^p \left[ \int_{-\infty}^{\infty} \exp(-|t - x|)dx \right] dt \leq c\|f\|_p^p.
\end{align*}
\]

We thus obtained estimates (3.8). Inequalities (3.9) can be found in a similar way by applying (3.7) instead of (3.6).

**Corollary 3.3.1.** Let \(f(x) \in L_p(R), p \in [1, \infty]\). The function \(y(x) = (Gf)(x), x \in R\) is the unique solution to (1.1) in the class \(C^{(1)}(R)\). In particular, part c) of Theorem 1.1 holds.

**Proof.** It is an immediate consequence of Lemma 3.3 and (2.2)

**Lemma 3.4.** Let \(f(x) \in L_p(R), p \in [1, \infty]\). Then

\[
\lim_{|x| \to \infty} (Gf)(x) = 0, \quad \lim_{|x| \to \infty} \frac{d}{dx}(Gf)(x) = 0.
\]

**Proof.** Let \(A \in (0, \infty)\). From (3.13) and (2.6) \(\Rightarrow\) (3.17), \(\Rightarrow\) (3.18):

\[
\begin{align*}
|(Gf)(x)| & \leq c \int_{-\infty}^{\infty} |f(t)|^p \exp(-|t - x|)dt \leq c \int_{|t-x| \leq A} |f(t)|^p \exp(-|t - x|)dt \\
& + c \int_{|t-x| \geq A} |f(t)|^p \exp(-|t - x|)dt \leq c \int_{|t-x| \leq A} |f(t)|^p dt + c\exp(-A)\|f\|_p^p.
\end{align*}
\]
\[ 0 \leq \lim_{|x| \to \infty} |(Gf)(x)|^p \leq \lim_{|x| \to \infty} |(Gf)(x)|^p \leq c \exp(-A)\|f\|_p^p. \quad (3.18) \]

In (3.18) we pass to limit as \( A \to \infty \) and obtain the first equality of (3.16). The second equality of (3.16) is checked similarly using (3.7).

\[
\square
\]

Theorem 1.1 A) follows from Lemma 3.1, Corollary 3.3.1 and (3.16).

\[
\square
\]

§4. A criterion for solvability of a singular boundary problem in \( L_\infty(R) \).

In this section we prove Part B) of Theorem 1.1.

Necessity. Let \( y(x) = (Gf)(x), \ f(x) \in L_\infty(R) \). From Corollary 3.3.1 it follows that the general solution of (1.1) has the form \( z(x) = \alpha u(x) + \beta v(x) + y(x), \ x \in R, \ \alpha, \beta - \text{const.} \) From (2.2) and (3.8) we conclude that if \( z(x) \) is a solution to (1.1) – (1.2), then \( \alpha = \beta = 0 \) and \( z(x) \equiv y(x), \ x \in R \). Let now \( f(x) \equiv 1, \ x \in R \). In this case, the solution to (1.1) – (1.2) is of the form

\[
y(x) = u(x) \int_{-\infty}^{x} v(t)dt + v(x) \int_{x}^{\infty} u(t)dt \overset{\text{def}}{=} u(x)I_1(x) + v(x)I_2(x).
\]

From (2.2) we easily derive lower estimates for

\[
I_1(x) = \int_{-\infty}^{x} v(t)dt = \int_{-\infty}^{x} v(t)[v'(t)u(t) - u'(t)v(t)]dt \geq \int_{-\infty}^{x} v(t)v'(t)u(t)dt
\]

\[
\geq u(x) \int_{-\infty}^{x} v(x)v'(t)dt = \frac{u(x)v^2(x)}{2}, \ x \in R,
\]

\[
I_2(x) = \int_{x}^{\infty} u(t)dt = \int_{x}^{\infty} u(t)[v'(t)u(t) - u'(t)v(t)]dt \geq -\int_{x}^{\infty} v(t)u'(t)u(t)dt
\]

\[
\geq -v(x) \int_{x}^{\infty} u'(x)u(t)dt = \frac{v(x)u^2(x)}{2}, \ x \in R,
\]

Hence \( y(x) \geq (u(x)v(x))^2 = \rho^2(x) \). Then (1.2), (2.10) and Lemma 2.4 imply (1.5).

Sufficiency. By Corollary 3.3.1, the function \( y(x) = (Gf)(x), \ f(x) \in L_\infty(R) \) satisfies (1.1) almost everywhere, \( y(x) \in C^{(1)}(R), \ y'(x) \in AC^{loc}(R) \); and by (3.6)

\[
|y(x)| \leq \int_{-\infty}^{\infty} G(x, t)|f(t)|dt \leq cd(x) \int_{-\infty}^{\infty} \exp(-|t-x|)dt\|f\|_\infty \leq cd(x)\|f\|_\infty, \ x \in R.
\]

Hence by Lemma 2.4, \( y(x) \to 0 \) as \( |x| \to \infty \).
Lemma 4.1. Condition (1.5) is equivalent to either of the equalities (4.1)

\[ \lim_{|x| \to \infty} d_1(x) = 0, \quad \lim_{|x| \to \infty} d_2(x) = 0. \]  

Proof. Let us verify that (1.5) is equivalent to the first equality of (4.1). (For the second equality of (4.1) this equivalence can be established similarly.) Suppose that (1.5) holds. Note that from (1.3) it follows that \( d_1(x) \leq 1 \) for \( x \in \mathbb{R} \):

\[ 1 = \int_0^{\sqrt{2}d_1(x)} \int_{x-t}^x q(\xi) d\xi dt \geq \int_0^{\sqrt{2}d_1(x)} \int_{x-t}^x d\xi dt = d_1^2(x) \Rightarrow 0 < d_1(x) \leq 1. \]  

(4.2)

Let \( a = \frac{2\sqrt{2}}{3} \), \( b = \frac{\sqrt{2}}{3} \), \( \varpi = x - bd_1(x) \). Then (see [3]):

\[ 2 = 2 \int_0^{\sqrt{2}d_1(x)} \int_{x-t}^x q(\xi) d\xi dt \geq 2 \int_{ad_1(x)}^{\sqrt{2}d_1(x)} \int_{x-t}^x q(\xi) d\xi dt \]
\[ \geq 2bd_1(x) \int_{x-2bd_1(x)}^x q(\xi) d\xi \geq (bd_1(x)) \int_{\varpi+(bd_1(x))}^{\varpi-(bd_1(x))} q(\xi) d\xi. \]

Hence \( d(\varpi) \geq bd_1(x), \ x \in \mathbb{R} \), i.e.

\[ 3.2^{-1/2}d(x - \sqrt{2} \cdot 3^{-1}d_1(x)) \geq d_1(x) > 0, \quad x \in \mathbb{R}. \]  

(4.3)

Then Lemma 2.4, (4.2) and (4.3) imply (4.1). Conversely, suppose that (4.1) holds.

By the definition of \( d_1(x) \),

\[ 2 = 2 \int_0^{\sqrt{2}d_1(x)} \int_{x-t}^x q(\xi) d\xi dt \leq 2\sqrt{2}d_1(x) \int_{x-\sqrt{2}d_1(x)}^x q(\xi) d\xi \leq (2\sqrt{2}d_1(x)) \int_{x-(2\sqrt{2}d_1(x))}^{x+(2\sqrt{2}d_1(x))} q(\xi) d\xi. \]

Hence \( 2\sqrt{2}d_1(x) \geq d(x) > 0 \) for \( x \in \mathbb{R} \). Therefore \( d(x) \to 0 \) as \( |x| \to \infty \), and it remains to use Lemma 4.2.

Lemma 4.2. Suppose that (1.5) holds. Then

\[ \lim_{|x| \to \infty} \int_{-\infty}^{\infty} \left| \frac{\partial}{\partial x} G(x, t) \right| dt = 0. \]  

(4.4)
Proof. From (3.5), (2.2), (2.8), and (2.9) we obtain for \( x \in \mathbb{R} \):

\[
\int_{-\infty}^{\infty} \left| \frac{\partial}{\partial x} G(x, t) \right| dt = |u'(x)| \int_{-\infty}^{x} v(t) dt + v'(x) \int_{x}^{\infty} u(t) dt
\]

\[
= \frac{|u'(x)|}{u(x)} \rho(x) \int_{-\infty}^{x} v(t) dt + \frac{v'(x)}{v(x)} \rho(x) \int_{x}^{\infty} u(t) dt
\]

\[
\leq 2 \left\{ \frac{1}{v(x)} \int_{-\infty}^{x} v(t) dt + \frac{1}{u(x)} \int_{x}^{\infty} u(t) dt \right\}. \quad (4.5)
\]

The equalities

\[
\lim_{|x| \to \infty} \frac{1}{v(x)} \int_{-\infty}^{x} v(t) dt = 0, \quad \lim_{|x| \to \infty} \frac{1}{u(x)} \int_{x}^{\infty} u(t) dt = 0. \quad (4.6)
\]

are verified in the same way, and therefore we only check the first one. From (2.2) and (3.1) we obtain:

\[
v(x) \to \infty \text{ as } x \to \infty \Rightarrow \int_{-\infty}^{x} v(t) dt \to \infty \text{ as } x \to \infty. \quad (4.7)
\]

\[
v'(x) \geq v(x) \Rightarrow v(x) \geq \int_{-\infty}^{x} v(t) dt, \ x \in \mathbb{R} \Rightarrow \int_{-\infty}^{x} v(t) dt \to 0 \text{ as } x \to \infty. \quad (4.8)
\]

From Lemmas 2.4 and 4.1 it follows that (4.1) holds, and therefore by (2.8),

\[
\lim_{|x| \to \infty} \frac{v(x)}{v'(x)} = 0. \text{ By L’hopital’s rule, taking into account (4.7) and (4.8), we get}
\]

\[
\lim_{|x| \to \infty} \frac{\int_{-\infty}^{x} v(t) dt}{v(x)} = \lim_{|x| \to \infty} \frac{v(x)}{v'(x)} = 0. \quad (4.9)
\]

Equality (4.4) now follows from (4.9) and (4.5). \( \square \)

To end the proof of the theorem, it remains to show that \( y'(x) \to 0 \) as \( |x| \to \infty \) and to prove that (1.1) – (1.2) has a unique solution. From (4.4) it follows that

\[
|y'(x)| \leq \int_{-\infty}^{\infty} \left| \frac{\partial}{\partial x} G(x, t) \right| |f(t)| dt \leq \int_{-\infty}^{\infty} \left| \frac{\partial}{\partial x} G(x, t) \right| dt \cdot \|f\|_{\infty} \to 0 \text{ as } |x| \to \infty
\]

and therefore \( y(x) \in D_{0}^{0}. \) The uniqueness in the class \( D_{0}^{0} \) follows from (2.2). \( \square \)
§5. Properties of the Green operator in $L_p(R)$. 

This section is devoted to Theorem 1.2. From Theorem 1.1 and the definition of the operator $L_p$ (see §1) it follows that $L_p^{-1} = G$ (see (3.5)) By (2.6) and (3.8), $\|G\|_{p \to p} \leq c < \infty$, $p \in [1, \infty]$. To verify that the condition of the theorem on compactness of $G$ is necessary, we need the following Lemmas 5.1 and 5.2.

**Lemma 5.1.** For $x \in R$, $t \in \left[ x - \frac{d(x)}{2}, x + \frac{d(x)}{2} \right]$ one has the inequalities

$$c^{-1}v(x) \leq v(t) \leq cv(t), \quad c^{-1}u(x) \leq u(t) \leq cu(x). \quad (5.1)$$

**Proof.** From (2.3), (2.7) (for $\varepsilon = \frac{1}{2}$), and (2.10), we get

$$\frac{v(t)}{v(x)} \leq \sqrt{\frac{\rho(t)}{\rho(x)}} \exp \left( \frac{1}{2} \left| \int_x^t \frac{d\xi}{\rho(\xi)} \right| \right) = \sqrt{\frac{\rho(t)}{\rho(x)}} \cdot \frac{d(t)}{d(x)} \cdot \frac{d(x)}{\rho(x)} \exp \left( \frac{1}{2} \left| \int_x^t \frac{d(\xi)}{\rho(\xi)} \cdot \frac{d(x)}{\rho(\xi)} \cdot \frac{d\xi}{\rho(\xi)} \right| \right) \leq c.$$

$$\frac{v(t)}{v(x)} \geq \sqrt{\frac{\rho(t)}{\rho(x)}} \exp \left( - \frac{1}{2} \left| \int_x^t \frac{d\xi}{\rho(\xi)} \right| \right) = \sqrt{\frac{\rho(t)}{\rho(x)}} \cdot \frac{d(t)}{d(x)} \cdot \frac{d(x)}{\rho(x)} \exp \left( - \frac{1}{2} \left| \int_x^t \frac{d(\xi)}{\rho(\xi)} \cdot \frac{d(x)}{\rho(\xi)} \cdot \frac{d\xi}{\rho(\xi)} \right| \right) \geq c^{-1}.$$

Inequalities (5.1) for $u(\cdot)$ are checked similarly. \(\Box\)

**Lemma 5.2.** Let $f_x(\xi)$ be the characteristic function of the segment $\Delta(x) = \left[ x - \frac{d(x)}{2}, x + \frac{d(x)}{2} \right]$, $x \in R$. Then for $t \in \Delta(x)$ one has the inequalities

$$(Gf_x)(t) \geq c^{-1}d^2(x), \quad x \in R. \quad (5.2)$$

**Proof.** For $t \in \Delta(x)$, by (5.1) and (2.10), we get

$$(Gf_x)(t) = u(t) \int_{x - \frac{d(x)}{2}}^t v(\xi)d\xi + v(t) \int_t^{x + \frac{d(x)}{2}} u(\xi)d\xi$$

$$= \left[ \frac{u(t)}{u(x)} \int_{x - \frac{d(x)}{2}}^t \frac{v(\xi)}{v(x)}d\xi \right] \rho(x) + \left[ \frac{v(t)}{v(x)} \int_t^{x + \frac{d(x)}{2}} \frac{u(\xi)}{u(x)}d\xi \right] \rho(x) \geq c^{-1} \rho(x)d(x) \geq c^{-1}d^2(x). \quad \Box$$
Let $p \in [1, \infty)$ and $f_x(\xi), x \in \mathbb{R}$ be the functions from Lemma 5.2. By (2.6), one has $\|f_x\|_p \leq 1, x \in \mathbb{R}$. If $G : L_p(\mathbb{R}) \to L_p(\mathbb{R})$ is compact, then by Theorem 2.2 we get

$$0 = \lim_{N \to \infty} \sup_{\|f\|_p \leq 1} \left( \int_{-N}^{-\infty} |(Gf)(t)|^p dt + \int_{N}^{\infty} |(Gf)(t)|^p dt \right)$$

$$\geq \lim_{N \to \infty} \sup_{|x| \geq N} \left( \int_{-N}^{-\infty} |(Gf_x(t)|^p dt + \int_{N}^{\infty} |(Gf_x)(t)|^p dt \right)$$

$$\geq \lim_{N \to \infty} \sup_{x \leq -N} \left[ \int_{x-d(x)}^{x} |(Gf_x(t)|^p dt \right] + \lim_{N \to \infty} \sup_{x \geq N} \left[ \int_{x}^{\infty} \frac{d(x)}{2} |Gf_x(t)|^p dt \right]$$

$$\geq \lim_{N \to \infty} \sup_{|x| \geq N} c^{-1}d(x)^{2p+1}.$$ 

Hence \( \lim_{|x| \to \infty} d(x) = 0 \), and (1.5) holds by Lemma 2.4. To prove that the condition of the theorem on compactness of $G$ is sufficient, we need Theorem 5.1.

**Theorem 5.1.** Let $p \in [1, \infty)$. The operator $G : L_p(\mathbb{R}) \to L_p(\mathbb{R})$ is compact if

$$\lim_{|x| \to \infty} \int_{-\infty}^{\infty} G(x, t)dt = 0. \quad (5.3)$$

We divide the proof of the theorem into several separate assertions.

**Lemma 5.3.** Suppose that (5.3) holds. Then

$$\lim_{N \to \infty} \sup_{x \in \mathbb{R}} \left( \int_{-\infty}^{-N} G(x, t)dt \right) = 0, \quad \lim_{N \to \infty} \sup_{x \in \mathbb{R}} \left( \int_{N}^{\infty} G(x, t)dt \right) = 0. \quad (5.4)$$

**Proof.** Equalities (5.4) are checked in the same way. Let us prove, say, the second one. For given $x$ and $N$, in the cases $x \leq N$ and $x \geq N$, respectively, we establish inequalities proving (5.4):

$$H(x, N) \overset{\text{def}}{=} \int_{N}^{\infty} G(x, t)dt = v(x) \int_{N}^{\infty} u(t)dt \leq v(N) \int_{N}^{\infty} u(t)dt \leq \int_{-\infty}^{\infty} G(N, t)dt$$

$$\leq \sup_{|x| \geq N} \int_{-\infty}^{\infty} G(x, t)dt$$

$$H(x, N) = u(x) \int_{N}^{\infty} v(t)dt + v(x) \int_{x}^{\infty} u(t)dt \leq \int_{-\infty}^{\infty} G(x, t)dt \leq \sup_{|x| \geq N} \int_{-\infty}^{\infty} G(x, t)dt. \quad \square$$
Lemma 5.4. If (5.3) holds, then for $p \in [1, \infty)$ one has

$$\lim_{N \to \infty} \sup_{\|f\|_p \leq 1} \int_{|x| \geq N} |(Gf)(x)|^p dx = 0. \quad (5.5)$$

Proof. By Hölder’s inequality, Fubini’s theorem and (3.6), we get

$$\sup_{\|f\|_p \leq 1} \int_{|x| \geq N} |(Gf)(x)|^p dx \leq \sup_{\|f\|_p \leq 1} \left\{ \int_{|x| \geq N} \left[ \int_{-\infty}^{\infty} G(x,t) dt \right]^{p/p'} \left[ \int_{-\infty}^{\infty} G(x,t)|f(t)|^p dt \right] dx \right\}$$

$$\leq c \sup_{\|f\|_p \leq 1} \int_{|x| \geq N} \left[ \int_{-\infty}^{\infty} G(x,t)|f(t)|^p dt \right] dx$$

$$\leq c \sup_{\|f\|_p \leq 1} \int_{-\infty}^{\infty} |f(t)|^p \left[ \int_{|x| \geq N} G(x,t) dx \right] dt \leq c \sup_{t \in \mathbb{R}} \int_{|x| \geq N} G(x,t) dx.$$

The lemma follows from (5.3), (5.4) taking into account that $G(x,t)$ is symmetric.

Lemma 5.5. Let $p \in [1, \infty)$ and suppose that (5.3) holds. Denote $y(x) = (Gf)(x)$, $f(x) \in L_p(R)$. Then

$$\lim_{\eta \to 0} \sup_{\|f\|_p \leq 1} \|y(\cdot + \eta) - y(\cdot)\|_p = 0. \quad (5.6)$$

Proof. By (3.9), we get for $\|f\|_p \leq 1$:

$$|y(x + \eta) - y(x)| \leq \int_x^{x+\eta} |y'(\xi)| d\xi \leq |\eta| \left\| \frac{d}{dx} G \right\|_{p \to C(R)} \|f\|_p \leq c|\eta|.$$

Let $|\eta| \leq 1$. Then

$$\int_{|x| \geq N} |y(x + \eta) - y(x)|^p dx \leq 2^{p+1} \int_{|x| \geq N-1} |y(x)|^p dx.$$

By Lemma 5.4, for a given $\varepsilon > 0$ there is $N(\varepsilon) \gg 1$ such that

$$2^{p+1} \sup_{\|f\|_p \leq 1} \left( \int_{|x| \geq N(\varepsilon) - 1} |y(x)|^p dx \right) \leq \varepsilon.$$
Therefore, taking into account the above proved inequalities, we get
\[
\lim_{\eta \to 0} \sup_{\|f\|_p \leq 1} \int_{-\infty}^{\infty} |y(x + \eta) - y(x)|^p dx = \lim_{\eta \to 0} \left\{ \int^{-N(\varepsilon)}_{-\infty} |y(x + \eta) - y(x)|^p dx + \int_{|x| \geq N(\varepsilon)} |y(x + \eta) - y(x)|^p dx \right\} 
\leq \varepsilon.
\]

Since \(\varepsilon\) is an arbitrary positive number, (5.6) is proved. \(\square\)

Since the operator \(G : L_p(R) \to L_p(R)\) is bounded (see above), Theorem 5.1 follows from (5.6), (5.5) and Theorem 2.2. \(\square\)

To deduce the sufficiency of the condition of Theorem 1.2 on the compactness of \(G : L_p(R) \to L_p(R)\) for \(p \in [1, \infty)\), note that (3.6) implies
\[
0 \leq \int_{-\infty}^{\infty} G(x, t) dt \leq cd(x) \int_{-\infty}^{\infty} \exp(-|t - x|) dt \leq cd(x). \tag{5.7}
\]
If (1.5) holds, then Lemma 2.4 and (5.7) imply (5.3), and by Theorem 5.1, the operator \(G : L_p(R) \to L_p(R)\) is compact for \(p \in [1, \infty)\). Since \(G(x, t) = G(t, x)\), \(x, t \in R\), we conclude that \(\mathcal{L}_1^{-1} = \mathcal{L}_\infty^{-1}\), and the operator \(\mathcal{L}_\infty^{-1} : L_\infty(R) \to L_\infty(R)\) is compact if and only if \(\mathcal{L}_1^{-1} : L_1(R) \to L_1(R)\) is compact (see Theorem 2.3), i.e., if and only if (1.5) holds. Thus the criterion for compactness of \(\mathcal{L}_p^{-1} : L_p(R) \to L_p(R), p \in [1, \infty]\) is proved. Let us check the other assertions of Theorem 1.1. Under condition (1.5), for any \(p \in [1, \infty]\) the spectrum \(\sigma(\mathcal{L}_p^{-1})\) is at most countable, and for \(p \in [1, \infty]\) the operator \(\mathcal{L}_p^{-1} = G\) satisfies the conditions of Theorem 2.4. Hence in that case \(\sigma(\mathcal{L}_p) = \sigma(\mathcal{L}_2), p \in [1, \infty]\). The operator \(\mathcal{L}_2^{-1}\) is symmetric and bounded, thus it is self-adjoint. Hence \(\sigma(\mathcal{L}_2) \subset R\). Moreover, by (1.3) the operator \(\mathcal{L}_2\) is semi-bounded from below because for \(y(x) \in \mathcal{D}(\mathcal{L}_2) = \mathcal{D}_2\) one has
\[
\langle \mathcal{L}_2 y, y \rangle = \int_{-\infty}^{\infty} (-y''(x) + q(x)y(x))y(x) dx
= y'(x)y(x) \big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} [y'(x)^2 + q(x)y^2(x)] dx
= \int_{-\infty}^{\infty} [y'^2(x) + q(x)y^2(x)] dx \geq \int_{-\infty}^{\infty} |y(x)|^2 dx = \|y\|_2^2
\tag{5.8}
\]
From (5.8) it follows that $\sigma(\mathcal{L}_2) \in [1, \infty)$. Theorem 1.2 is proved.

REFERENCES

1. F.A. Berezin and M.A. Shubin, *Schrödinger Equations*, Moscow Univ. Press, 1983.
2. N. Chernyavskaya and L. Shuster, *On the WKB-method*, Differentsial’ye Uravneniya 25 (1989), no. 10, 1826-1829. (Russian)
3. , *Estimates for Green’s function of the Sturm-Liouville operator*, JDE 111 (1994), no. 2, 410-420.
4. , *Asymptotics on the diagonal of the Green function of a Sturm-Liouville operator and its applications*, J. of the London Math. Soc, submitted.
5. E.B. Davies and E.M. Harrell, *Conformally flat Riemann metrics, Schrödinger operators and semiclassical approximation*, JDE 66 (1987), no. 2, 165-188.
6. N. Dunford and J.T. Schwartz, *Linear Operators, Parts I: General Theory*, New York, 1958.
7. P. Hartman, *Ordinary Differential Equations*, New York, 1964.
8. A.M. Molchanov, *Discrete spectrum conditions for self-adjoint differential equations of the second order*, Trydy Mosk. Mat. Obschestva 2 (1953), 169-199. (Russian)
9. K.T. Mynbaev and M.O. Otelbaev, *Weighted Functional Spaces and the Spectrum of Differential Operators*, Nauka, Moscow, 1988. (Russian)
10. M.A. Naimark, *Linear Differential Operators*, Ungar, New York, 1967.
11. W. Rudin, *Functional Analysis*, New York, 1973.
12. H.H. Schaefer, *Interpolation of spectra*, Int. Eq. and Oper. Th. 3/3 (1980), 463-469.
13. L.A. Shuster, *A priori estimates of solutions of the Sturm-Liouville criterion*, Math. Notes 50 (1991), no. 1, 746-751.
14. E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis*, Cambridge Univ. Press, 1958.