AFFINE LINKING NUMBERS AND CAUSALITY RELATIONS
FOR WAVE FRONTS

VLADIMIR V. CHERNOV (TCHERNOV) AND YULI B. RUDYAK

Abstract. Two wave fronts $W_1$ and $W_2$ that originated at some points of the manifold $M^n$ are said to be causally related if one of them passed through the origin of the other before the other appeared. We define the causality relation invariant $\text{CR}(W_1, W_2)$ to be the algebraic number of times the earlier born front passed through the origin of the other front before the other front appeared. Clearly, if $\text{CR}(W_1, W_2) \neq 0$, then $W_1$ and $W_2$ are causally related. If $\text{CR}(W_1, W_2) = 0$, then we generally can not make any conclusion about fronts being causally related. However we show that for front propagation given by a complete Riemannian metric of non- positive sectional curvature, $\text{CR}(W_1, W_2) \neq 0$ if and only if the two fronts are causally related. The models where the law of propagation is given by a metric of constant sectional curvature are the famous Friedmann Cosmology models.

The classical linking number $\text{lk}$ is a $\mathbb{Z}$-valued invariant of two zero homologous submanifolds. We construct the affine linking number generalization $\text{AL}$ of the $\text{lk}$ invariant to the case of linked $(n-1)$-spheres in the total space of the unit sphere tangent bundle $(STM)^{2n-1} \rightarrow M^n$. For all $M$, except of odd-dimensional rational homology spheres, $\text{AL}$ allows one to calculate the value of $\text{CR}(W_1, W_2)$ from the picture of the two wave fronts at a certain moment. This calculation is done without the knowledge of the front propagation law and of their points and times of birth. Moreover, in fact we even do not need to know the topology of $M$ outside of a part $\overline{M}$ of $M$ such that $W_1$ and $W_2$ are null-homotopic in $\overline{M}$.

INTRODUCTION

In this paper the word “smooth” means $C^\infty$. Throughout this paper $M$ is a smooth connected oriented Riemannian $n$-dimensional manifold (not necessarily compact). In our paper the wave fronts on $M$ are assumed to be parametrized by smooth mappings of manifolds. They are not assumed to be immersed and are allowed to have various singularities including degenerations of the differential. The popular case of wave fronts being projections to $M$ of Legendrian mappings of $N^{n-1}$ (parameterizing the fronts) to the unit cotangent bundle $(ST^*M)^{2n-1} \subset T^*M = TM$ of $M$ is easily obtained from our paper as a particular, though a very important case. (The front stays smooth,
though not immersed, at the cusp points and other singularities of the projections to $M$ of Legendrian mappings $N^{m-1} \to (ST^*M)^{2n-1}$.)

Let $W_1$ and $W_2$ be two wave fronts which are propagating in $M$. (Generally, we assume that the fronts have different propagation laws.) We define a *dangerous intersection* between the fronts $W_1(t)$ and $W_2(t)$ at some moment of time $t$ to be a point $x$ where the fronts intersect and have the same direction of propagation (see Section 1 for the precise definition).

A passage of the front $W_1(t)$ through the birth point of the front $W_2$ at the moment of time $t$ before the front $W_2$ originated is called the *baby-intersection*.

It turns out that we can associate to each dangerous intersection as well as to each baby-intersection a sign (i.e. a number $\pm 1$). The sum of the signs up to a moment $t$ is called a *causality relation invariant* and is denoted by $\text{CR}(W_1(t), W_2(t))$.

In particular, assume that there are no dangerous intersections of $W_1(t)$ and $W_2(t)$ for all $t$. (Many such examples are constructed in Section 1 and, for example, light and sound wave fronts have this property.) Then the causality relation invariant tells us the algebraic number of times the earlier-born wave front passed through the birth point of the other front before the other front originated.

Two wave fronts $W_1$ and $W_2$ that originated at some points of the manifold $M^n$ are said to be *causally related* if one of them passed through the origin of the other before the other appeared. Clearly, if $\text{CR}(W_1, W_2) \neq 0$ and no dangerous intersections occurred during the propagation, then $W_1$ and $W_2$ are causally related.

If $\text{CR}(W_1, W_2) = 0$, then we generally can not make any conclusion about fronts being causally related. However we show that in case of front propagation given by a complete Riemannian metric of non-positive sectional curvature, $\text{CR}(W_1, W_2) \neq 0$ if and only if the two fronts are causally related, see 2.5. The models where the law of propagation is given by a metric of constant sectional curvature are the famous Friedmann Cosmology models.

We are interested in reconstructing the value $\text{CR}(W_1(t), W_2(t))$ from the current shape of the wave fronts only, without the knowledge of the propagation laws, of the birth-points of the fronts, of the topology of $M$ etc.

It turns out that, having the current picture only, we can evaluate $\text{CR}(W_1(t), W_2(t)) \in \mathbb{Z}$ modulo a certain $m \in \mathbb{Z}$ that depends on $M$. This $m$ is zero if $M$ is not an odd-dimensional rational homology sphere, see Proposition 3.8, and when $m = 0$ we can completely reconstruct the value of $\text{CR}(W_1(t), W_2(t)) \in \mathbb{Z}$. Furthermore, for $M$
odd-dimensional this $m$ is divisible by the order of $\pi_1(M)$, see Proposition 3.8. In particular, if $\pi_1(M)$ is infinite then $m = 0$, i.e. we can completely evaluate CR from the current picture. The really bad case $m = 1$ (when we can not say anything about CR) appears only when $M$ is an odd-dimensional homotopy sphere. In particular, for Friedmann cosmology models based on a Riemannian metric of non-positive sectional curvature our methods allow one to detect precisely whether two wave fronts are causally related or not from the current picture of the wave fronts only, see 4.10.

To evaluate CR modulo $m$, we introduce an invariant $AL \in \mathbb{Z}/m\mathbb{Z}$, the so-called affine linking invariant, which depends on the current picture only. Then we notice that CR and AL are congruent modulo $m$. Here the biggest technical difficulty appears, since in order to define AL we must define the “linking number” for two spheres that are non-homologous to zero. (This affine linking number can be shown to be a particular case of a very general affine linking invariant discussed in our later work [10]. However, results of this paper are independent from the results of [10] and are not corollaries of our results obtained later.)

This theory has the following physical interpretation. Let $\overline{M}$ be the part of the manifold (the universe) $M$ such that $\overline{M}$ contains the current picture of wave fronts $W_1, W_2$, and $W_1, W_2$ are contractible in $\overline{M}$.

We transform the wave fronts via certain allowable moves to trivial fronts, i.e. small spherical fronts with the canonical orientation and coorientation, located far away from each other. The allowable moves should be thought of as generalized Reidemeister moves: they are the passages through generic singularities (in both directions) of wave fronts and dangerous intersection moves. We count the change of the invariant CR that occurs in the process of this formal deformation, and it turns out that this change is congruent modulo $m$ with the (unknown!) value $\text{CR}(W_1, W_2)$ of the current picture. In particular, as we have already mentioned, if $M$ is not an odd-dimensional rational homology sphere or if $\pi_1(M)$ is infinite, then we can completely compute CR from the current picture, without any knowledge of the propagations, moments and points of birth of the fronts, and topology of $M$ outside of $\overline{M}$.

The following observation seems to be interesting. Suppose that we have two pictures of two pairs of fronts $(W_1, W_2)$ and $(W_1', W_2')$ made at two unknown moments of time $t_0$ and $t_1$. (We assume that both pairs are free of dangerous intersection points.) Assume that we know that the propagation laws for the two fronts are such that the dangerous intersection points cannot appear during the propagation and
that \( \text{CR}(W_1(t_0), W_2(t_0)) \) and \( \text{CR}(W'_1(t_1), W'_2(t_1)) \) are not comparable modulo \( m \). Then we can conclude that the pairs \( (W_1(t_0), W_2(t_0)) \) and \( (W'_1(t_1), W'_2(t_1)) \) of wave fronts are not the pictures of the same pair of fronts taken at different moments of time.

Note that in these calculations we disregard the dangerous self-intersections of wave fronts. (In a sense this is similar to the theory of link homotopy where different components of links are not allowed to intersect through possible deformations, but self-intersections are allowed.) The study of self-intersections of fronts on surfaces was initiated by the groundbreaking work of Arnold [4], see also [1, 2, 11, 15, 16, 18, 19, 25, 26, 6, 7, 9, 30]. The methods developed in this paper allow us to calculate the algebraic number of dangerous self-intersection points that arise under the propagation of fronts on manifolds of arbitrary dimensions, we do it in a next paper.

The following physical speculations related to the CR invariant seem to be possible. Assume that the space-time is topologically a product \( M^n \times \mathbb{R} \), and that the observable universe \( \overline{M} \) is so big that we are not able to see the current picture of wave fronts (due to the finiteness of the speed of light). The propagating fronts define the mapping of the cones \( C_1, C_2 \) (over the sphere \( S^{n-1} \) parameterizing the fronts at every moment of time) into \( M \times \mathbb{R} \). Let \( \text{sec} : \overline{M} \rightarrow M \times \mathbb{R} \) be a section of the projection \( p_M : \overline{M} \times \mathbb{R} \rightarrow \overline{M} \), and let \( \overline{W}_i = C_i \cap \text{sec}(\overline{M}), i = 1, 2 \). Assume that the law of propagation is such that dangerous intersections do not occur. Then, similarly to the above, we can restore the number of baby-intersections from the picture of images of \( \overline{W}_i \) under the projection \( p_M^{-1} \). The section \( \text{sec} \) can be thought of as the picture of the universe that we see as the light from the points of \( \overline{M} \) reaches the observer, and thus \( \overline{W}_i \) can be regarded as the picture of fronts that we actually see.

Low [20, 21, 22] attacked a similar problem for \( M = \mathbb{R}^3 \), where he considered a linking invariant for linked cones \( C_1, C_2 \) as above. In this case the linking numbers can be constructed directly via the approach of Tabachnikov [30], because \( \mathbb{R}^3 \) has the topological end. (Some very interesting results relating causality for fronts on \( \mathbb{R}^2 \) and linking were obtained recently by Natario and Todd [23].)

The paper is organized as follows. In Section 1 we discuss some preliminary information, in Section 2 we define the invariant \( \text{CR} \), in Section 3 we prove homotopy theoretical results which we use in order to define the invariant \( \text{AL} \), in Section 4 we define the invariant \( \text{AL} \) and state the relation between \( \text{CR} \) and \( \text{AL} \), in Section 5 we treat the case of propagation with respect to a certain Riemannian metric, in Section 6 we give some examples and applications.
1. Preliminaries: propagation laws, propagations and dangerous intersections

We denote by $p_T : TM \to M$ the tangent bundle over $M$. Let $s : M \to TM$ be the zero section of the tangent bundle. We set $\overline{TM} = TM \setminus s(M)$. The multiplicative group $\mathbb{R}^+$ of positive real numbers acts fiberwise on $TM \setminus s(M)$ by multiplication, and we set

$$STM = (TM \setminus s(M))/\mathbb{R}^+.$$  

Let $\overline{p} : \overline{TM} \to STM$ be the quotient map. Clearly, the projection $p_T : TM \to M$ yields the commutative diagram

$$\begin{array}{ccc}
TM & \xrightarrow\gamma & \overline{TM} \\
\downarrow p_T & & \downarrow \overline{p} \\
M & \xrightarrow{pr} & M
\end{array}$$

It is easy to see that $pr : STM \to M$ is a locally trivial bundle with the fiber $S^{n-1}$, we call this bundle the spherical tangent bundle.

Given $x \in M^n$, we denote by $S^{n-1}_x$ (or just by $S_x$) the fiber $pr^{-1}(x)$ over $x$ of the spherical tangent bundle and by $T_xM$ the tangent space to $M$ at $x$.

Since $M$ is orientable, the bundle $pr : STM \to M$ is also orientable, and in order to orient $STM$ it suffices to orient the fiber $S^{n-1}_x$. We do it as follows. Choose an orientation preserving chart for $M$ centered at $x$ and let $S$ be a small $(n-1)$-sphere centered at $x$. We equip $S$ with the unique orientation $o$ by requiring that the pair $(o, \text{outer normal vector to } S)$ gives us the orientation of $M$.

Given $s \in S$, the radius-vector from $x$ to $s$ can be regarded as a nonzero tangent vector to $M$ at $x$, i.e., as a point of $S^{n-1}_x$. In this way we get a diffeomorphism $\psi : S \to S_x$ which gives us an orientation of $S^{n-1}_x$. It is easy to see that this orientation of $S_x$ does not depend of choice of the chart. Now, the pair (the orientation of $M$, the orientation of $S_x$) gives us an orientation of $STM$ which we fix forever.

1.1. Definition. We define a propagation law on $M$ to be a smooth map

$$L : \overline{TM} \times \mathbb{R} \times \mathbb{R} \to \overline{TM}$$

(a time-dependent flow on $\overline{TM}$). Here $L(u,s,t) \in \overline{TM}$ should be thought of as the point that corresponds to the position and the velocity vector at moment $s+t$ of a perturbation whose position and the velocity vector at moment $s$ was $u$. (We assume that a velocity of movement of a perturbation is either zero all the time or nonzero all the time.)
Furthermore we assume that \( L(u,s,t) \) satisfies the following natural conditions:

- **a:** \( L(u,s,0) = u \) for all \( u \in TM \);
- **b:** \( \forall s,t \in \mathbb{R} \) the map \( L_{s,t} : TM \to TM \) defined as \( L_{s,t}(u) = L(u,s,t) \) is a diffeomorphism;
- **c:** \( L(u,s,t_1 + t_2) = L(L(u,s,t_1),s+t_1,t_2), \forall s,t_1,t_2 \in \mathbb{R} \);
- **d:** \( \forall u \in TM \) and \( \forall s_0,t_0 \in \mathbb{R} \)

\[
\frac{d}{dt} p_T(L(u,s_0,t)) \bigg|_{t=t_0} = L(u,s_0,t_0).
\]

1.2. **Definition.** A propagation is a quadruple \( P = (L,x,T,V) \) where \( L \) is a propagation law, \( x \in M, T \in \mathbb{R} \) and \( V : S_n^{-1} \to (T_x M \setminus s(x)) \) is a smooth section of the \( \mathbb{R}^+ \)-bundle \( (T_x M \setminus s(x)) \to S_n^{-1} \). We fix an orientation preserving diffeomorphism \( S_n^{-1} \to S_n^{-1} \) and further in the text regard \( V \) as a mapping \( V : S_n^{-1} \to (T_x M \setminus s(x)) \).

A propagation \( P = (L,x,T,V) \) produces a wave front \( W(t) : S_n^{-1} \to M, t \geq T \), as follows. Informally speaking, we assume that at a moment of time \( T \) something happens at a point \( x \in M \) and the perturbation caused by this event starts to radiate from the point \( x \) in all the directions according to a propagation law \( L \) with the initial velocities of propagation in \( T_x M \) described by \( V \). Formally, for \( t \geq T \) we define the front \( W(t) \) to be the mapping

\[
W(t) := p_T(L(V,T,t-T)) : S_n^{-1} \to M.
\]

We put \( \overline{W}(t) = L(V,T,t-T) \) and \( \widetilde{W}(t) = \varphi \circ \overline{W}(t) \). In this case we also say that the wave front has originated from the event \( (x,T) \). Initially a front of an event is a smooth embedded sphere (because of 1.1(d)), but generically it soon acquires double points, folds, cusps, swallow tails, and other complicated singularities. Generally, singular values of the front form a codimension two subset of \( M \).

We denote by \( \varepsilon_x : S_n^{-1} \to STM \) any map of the form

\[
S_n^{-1} \overset{h}{\longrightarrow} S_x^{-1} \subset STM
\]

where \( h \) is a map of degree 1. Clearly, the homotopy class of \( \varepsilon_x \) is well-defined and does not depend on \( x \).

Let \( \mathcal{S} \) be the space of smooth maps \( S_n^{-1} \to STM \) that are homotopic to a map \( \varepsilon_x \) as in (1.1). Then \( \mathcal{S} \times \mathcal{S} \) is the space of ordered pairs \((f_1,f_2)\) with \( f_i \in \mathcal{S} \).

Put \( \Sigma \) to be the discriminant in \( \mathcal{S} \times \mathcal{S} \), i.e. the subspace that consists of pairs \((f_1,f_2)\) such that there exist \( y_1,y_2 \in S_n^{-1} \) with \( f_1(y_1) = f_2(y_2) \).
We do not include into $\Sigma$ the maps that are singular in the common sense but do not have double points between the two different spheres.

1.3. **Definition.** We define $\Sigma_0$ to be a subset (stratum) of $\Sigma$ consisting of all the pairs $(f_1, f_2)$ such that there exists precisely one pair of points $y_1, y_2 \in S^{n-1}$ such that:

- **a:** $f_1(y_1) = f_2(y_2)$. And moreover this pair of points is such that:
- **b:** $y_i$ is a regular point of $f_i$, $i = 1, 2$;
- **c:** $(df_1)(T_{y_1}) \cap (df_2)(T_{y_2}) = 0$. Here $df_i$ is the differential of $f_i$ and $T_{y_i}$ is the tangent space to $S^{n-1}$ at $y_i$.

1.4. **Construction.** Let $\rho : (a, b) \to S \times S$ be a path which intersects $\Sigma_0$ in a point $\rho(t_0)$. We also assume that $\rho(t_0 - \delta, t_0 + \delta) \cap \Sigma_0 = \rho(t_0)$ for $\delta$ small enough. We construct a vector $v = v(\rho, t_0, \delta)$ as follows. We regard $\rho(t_0)$ as a pair $(f_1, f_2) \in S \times S$ and consider the points $y_1, y_2$ as in 1.3. Set $z = f_1(y_1) = f_2(y_2)$. Choose a small $\delta > 0$ and regard $\rho(t_0 + \delta)$ as a pair $(g_1, g_2) \in S \times S$. Set $z_i = g_i(y_i), i = 1, 2$. Take a chart for $STM$ that contains $z$ and $z_i, i = 1, 2$ and set

$$v(\rho, t_0, \delta) := \overrightarrow{zz_1} - \overrightarrow{zz_2} \in T_z STM.$$

1.5. **Definition.** Let $\rho : (a, b) \to S \times S$ be a path as in 1.4. We say that $\rho$ intersects $\Sigma_0$ transversally for $t = t_0$ if there exists $\delta_0 > 0$ such that

$$v(\rho, t_0, \delta) \notin (df_1)(T_{y_1} S^{n-1}) \oplus (df_2)(T_{y_2} S^{n-1}) \subset T_z STM$$

for all $\delta \in (0, \delta_0)$.

It is easy to see that the concept of transversal intersection does not depend on the choice of the chart.

1.6. **Definition.** A path $\rho : (a, b) \to S \times S, -\infty \leq a < b \leq \infty$ is said to be *generic* if

- **a:** $\rho(a, b) \cap \Sigma = \rho(a, b) \cap \Sigma_0$;
- **b:** the set $J = \{t | \rho(t) \cap \Sigma_0 \neq \emptyset\} \subset (a, b)$ is an isolated subset of $\mathbb{R}$;
- **c:** the path $\rho$ intersects $\Sigma_0$ transversally for all $t \in J$.

As one can expect, every path can be turned into a generic one by a small deformation. We leave a proof to the reader.
Let \( P_1 = (L_1, x_1, T_1, V_1) \) and \( P_2 = (L_2, x_2, T_2, V_2) \) be two propagations. They define mappings \( r_i : \mathbb{R} \to S, \ i = 1, 2 \) as follows.

\[
  r_i(t) = \begin{cases} 
    \overline{p} \circ V_i & \text{for } t \leq T_i, \\
    \overline{W_i(t)} & \text{for } t > T_i. 
  \end{cases}
\]

1.7. Definition. A pair of propagations \( \{P_1, P_2\} \) is said to be \textit{generic} if the path \( r = (r_1, r_2) : \mathbb{R} \to S \times S \) is generic and \( r(T_i) \notin \Sigma, i = 1, 2 \).

1.8. Definition. Let \( \{P_1, P_2\} \) be a generic pair of propagations and let \( r : \mathbb{R} \to S \times S \) be as above. Then a moment \( t \in \mathbb{R} \) such that \( r(t) \in \Sigma \) corresponds either to the baby-intersection or to the case where \( t > \max(T_1, T_2) \) and there exists \( y_1, y_2 \in S^{n-1} \) such that \( W_1(t)(y_1) = W_2(t)(y_2) = z \) and \( \overline{W_1(t)}(y_1) = \overline{W_2(t)}(y_2) \in S^{n-1}_z \), i.e. to the case where there is a double point of the two fronts \( W_1(t) \) and \( W_2(t) \) at which the directions of the propagations of the two fronts coincide. Such a double point of two fronts is called \textit{a point of dangerous intersection}. Notice that we do not exclude situations in which the two fronts are tangent, the so-called \textit{Arnold’s dangerous tangencies}, cf. [4].

For many pairs of propagations the dangerous intersection points do not occur. Such pairs of propagations are called \textit{dangerous intersections free}. Now we describe a source of examples of such pairs.

1.9. Source of Examples of propagations without dangerous intersections. Let \( L : \overline{TM} \times \mathbb{R} \to \overline{TM} \) be a propagation law. Suppose that there exists a section

\[
  \tilde{s} : STM \times \mathbb{R} \times \mathbb{R} \to \overline{TM} \times \mathbb{R} \times \mathbb{R}
\]

of the map \( \overline{p} \times 1 \times 1 \) such that \( \text{Im}(\tilde{s}) \) consists of the trajectories of \( L \), i.e. if \( (u, s_0, 0) \in \text{Im}(\tilde{s}) \) for some \( u \in \overline{TM} \) and \( s_0 \in \mathbb{R} \), then \( L(u, s_0, t) \in \text{Im}(\tilde{s}) \), for every \( t \in \mathbb{R} \).

Let \( P_1 = (L, x_1, T_1, V_1) \) and \( P_2 = (L, x_2, T_2, V_2) \) be propagations such that \( (\text{Im}(V_1), T_1, 0) \subset \text{Im}(\tilde{s}|_{S_{x_1}, T_1, 0}) \), \( (\text{Im}(V_2), T_2, 0) \subset \text{Im}(\tilde{s}|_{S_{x_2}, T_2, 0}) \) and \( r(T_i) \notin \Sigma, i = 1, 2 \).

It is easy to see that \textit{the pair} \( \{P_1, P_2\} \) \textit{is dangerous intersections free}.

1.10. Example. \textit{Propagations that are defined by a Riemannian metric}. An interesting class of examples comes from the propagation defined by the geodesics of a complete Riemannian metric \( g \) on \( M \). In this case \( L(u, s, t) \) is just a point on \( \overline{TM} \) that corresponds to a velocity vector at moment \( s + t \) of a geodesic curve that had a velocity vector \( u \) at the moment \( s \). Thus the wave front \( W(t) \) corresponding to a propagation \( (L, x, T, V) \) can be described as \( W(t) : S^{m-1} \to M \) with
\( W(t)(y) = \exp_x((t - T)V(y)) \), \( y \in S^{m-1} \), where \( \exp_x : T_x M \to M \) is the exponent map corresponding to the Riemannian metric \( g \).

It is easy to see that in these examples if \( V(S^{m-1}) \) is a sphere of some radius \( r \), then at every moment of time the velocity vectors of the points on the wave front are perpendicular to the image of the front. Thus, in this case the dangerous intersections are precisely the dangerous tangencies.

Furthermore, if both \( \text{Im} V_1 \) and \( \text{Im} V_2 \) are spheres of the same radius \( r \), then the dangerous intersections (= dangerous tangencies) do not occur, since spheres of radius \( r \) in all the tangent spaces produce the section \( \tilde{s} \) described above.

1.11. Example. Propagation in a non-homogeneous and non-isotropic medium whose structure does not depend on time.

Assume that \( M \) is a Riemannian manifold and \( \mu : STM \to TM \) is a smooth section of the corresponding \( \mathbb{R}^+ \)-bundle such that \( \text{Im}(\mu|_S) \) bounds a strictly convex domain in \( T_x M \) for all \( x \in M \). The radius vector from \( s(x) \) to \( \text{Im}(\mu|_S) \) in the given direction is the velocity vector of the distortion traveling in the direction. This information allows us to calculate for every smooth curve \( \gamma : [t_1, t_2] \to M \) the total time \( \tau(\gamma) \) needed for the distortion to travel along this curve.

Assume \( \text{Im}(V_1) \subset \text{Im}\mu \) and \( \text{Im}(V_2) \subset \text{Im}\mu \) and that propagation occurs according to the Huygens principle, i.e. distortion travels along the extremal curves of the functional \( \tau \) on the space of smooth curves on \( M \). It is clear that here we have a special case of the situation described in 1.9, and so the dangerous intersection points do not occur for such a pair of propagation. On the other hand, if the propagation happens according to the Huygens principle then at every point \( W(t)(x) = z \) the normal vector to the wave front is conjugate with respect to \( \mu|_S \) to the direction of the extremal curve along which the information traveled to this point, Arnold [5]. In particular, in this case the dangerous tangencies do not occur under the wave fronts propagation, since they are the dangerous intersections.

2. The causality relation invariant

Recall that the standard sphere \( S^{n-1} \) is assumed to be oriented. We say that a tangent frame \( \mathbf{r} \) to \( S^{n-1} \) is positive if it gives us the standard orientation of \( S^{n-1} \).
2.1. Definition. Let \( \rho \) be a path in \( S \times S \) that intersects \( \Sigma \) transversally in one point \( \rho(t_0) \in \Sigma_0 \). We associate a sign \( \tilde{\sigma}(\rho, t_0) \) to such a crossing as follows.

We regard \( \rho(t_0) \) as a pair \( (f_1, f_2) \in S \times S \) and consider the points \( y_1, y_2 \in S^{n-1} \) such that \( f_1(y_1) = f_2(y_2) \). Set \( z = f_1(y_1) = f_2(y_2) \). Let \( \tau_1 \) and \( \tau_2 \) be frames which are tangent to \( S^{n-1} \) at \( y_1 \) and \( y_2 \), respectively, and both are assumed to be positive. Consider the frame
\[
\{ df_1(\tau_1), v, df_2(\tau_2) \}
\]
at \( z \in STM \) where \( v \) is a vector described in 1.4. We put \( \tilde{\sigma}(\rho, t_0) = 1 \) if this frame gives us the orientation of \( STM \), otherwise we put \( \tilde{\sigma}(\rho, t_0) = -1 \).

Because of the transversality and condition (c) from 1.3, the family \( \{ df_1(\tau_1), v, df_2(\tau_2) \} \) is really a frame.

Notice also that the vector \( v \) is not well-defined, but the above defined sign \( \tilde{\sigma} \) is.

Clearly if we traverse the path \( \rho \) in the opposite direction then the sign of the crossing changes.

2.2. Definition. Suppose that a front \( W \) passes through a point \( x \in M \) at the moment of time \( t_0 \) in such a way that the velocity vector \( v_x \) of the front \( W(t_0) \) at \( x \) is transverse to \( W(t_0) \), \( W(t_0) \) restricted to a small neighborhood \( U \) of \( W^{-1}(t_0)(x) \) is an embedding, and \( x \) has only one preimage under \( W(t_0) \).

Recall that the manifold \( M \) is oriented. Let \( o_x \) be the local orientation of \( W(t_0) \) at \( x \) (i.e. the orientation of the tangent plane \( T_x \) to \( W(t_0) \)). We say that the local orientation \( o_x \) is positive, and write \( \sigma(W(t_0), x) = 1 \) if the pair \( (o_x, v_x) \) gives us the orientation of \( M \); otherwise we say that the local orientation of \( W(t) \) at \( x \) is negative and write \( \sigma(W(t_0), x) = -1 \).

Notice that the same wave front \( W(t) \) can contain two points \( x \) and \( y \) such \( o_x \) is positive orientation while \( o_y \) is the negative one, see Figure 1.

Consider a generic pair \( (P_1, P_2) \) of propagations \( P_1 = (L_1, x_1, V_1, T_1) \) and \( P_2 = (L_2, x_2, V_2, T_2) \). In the rest of Section 2 we assume that \( T_1 \leq T_2 \). The case where \( T_1 > T_2 \) is treated in a similar way.

Let \( t > T_2 \) be a generic moment of time, i.e. the one at which dangerous intersections do not occur.

Let \( c_i, i \in I \subset \mathbb{N} \) where \( T_2 < c_i < t \) be moments of time when dangerous intersections did occur.

2.3. Definition. We define \( \sigma(W_1(c_i), W_2(c_i)) \) as the sign \( \tilde{\sigma} \) of the corresponding passage of \( \Sigma_0 \).
Orientations and coorientations at the cusp point of a one-dimensional front

Figure 1

Notice that $\sigma(W_1(c_i), W_2(c_i))$ is symmetric if $n$ is even and skew-symmetric if $n$ is odd.

Let $p_j, j \in J \subset \mathbb{N}$ be the moments of time when the front $W_1$ passed through the point $x_2$ before the front $W_2$ originated. (Notice that $p_j < T_2$ and that $\sigma(W_1(p_j), x_2)$ is well-defined since the pair of propagations is generic.) A straightforward verification shows that

$$\sigma(W_1(p_j), x_2) = \tilde{\sigma}(\rho, t_0).$$

where $\rho(t) = (\tilde{W}_1(t), \varepsilon_{x_2}), t \in (p_j - \delta, p_j + \delta)$.

2.4. Definition. We set

$$\text{CR}(W_1(t), W_2(t)) = \sum_{i \in I} \sigma(W_1(c_i), W_2(c_i)) + \sum_{j \in J} \sigma(W_1(p_j), x_2) \in \mathbb{Z}$$

and call it the causality relation invariant for the fronts $W_1(t)$ and $W_2(t)$ at a given moment of time $t$. (If in fact $T_1 > T_2$, then the second sum should be $\sum_{k \in K} (-1)^{\dim M} \sigma(W_2(q_k), x_1)$, where $q_k, k \in K \subset \mathbb{N}$ are the moments of time when the front $W_2$ passed through the point $x_1$ before the front $W_1$ originated. One can easily verify that $(-1)^{\dim M} \sigma(W_2(q_k), x_1)$ coincides with the sign of the corresponding crossing of $\Sigma_0$ by the path $r = (r_1, r_2)$.)

So, if $\text{CR}(W_1(t), W_2(t)) = k \neq 0$ then the sum of the number of baby-intersections and of the number of dangerous intersections is at least $|k|$. This probably could be interpreted as the quantity that measures either how much faster the first front is than the second so that they could become dangerously intersected; or how many times the first front did pass through the source of the second front before the second front originated.
Now we consider an important special case. If the above pair \((P_1, P_2)\) of propagations is dangerous intersections free, then
\[
\text{CR}(W_1(t), W_2(t)) = \sum_{j \in J} \sigma(W_1(p_j), x_2).
\]

We saw examples of such propagations in 1.9, 1.10, 1.11. It is easy to see that in this case \(\text{CR}(W_1(t), W_2(t))\) does not depend on \(t\) provided \(t > T_2\), and thus it is invariant under the propagation. In particular, if \(\text{CR}(W_1(t), W_2(t))\) is non-zero, then we know for a fact that the perturbation caused by the first signal has reached the source point of the second signal before the second signal originated. Moreover, if \(\text{CR}(W_1(t), W_2(t)) = k \neq 0\), then we can say for sure that the first wave front has passed through the source point of the second front at least \(|k|\) times before the second signal originated. (Of course it could be that it did pass more times, because it could have passed \(k + l\) times with a positive sign and \(l\) times with a negative sign.) Thus, \(W_1\) and \(W_2\) can be thought of as being causally related.

In case \(\text{CR}(W_1, W_2) = 0\) we can not make any conclusions on causality relation between \(W_1\) and \(W_2\), since it could mean that

1: neither of the fronts passed through the origin of the other before the other was born, and thus the fronts are not causally related; or

2: the earlier-born front did pass through the origin of the other front \(2k \neq 0\) times before the other front was born with \(k\) of these passages being positive and \(k\) of the passages being negative. In this case the fronts are causally related.

However, for some Riemannian metrics \(g\) the second situation could not occur, as the following example shows.

2.5. Example (when \(\text{CR}(W_1(t), W(t)) = 0\) if and only if the two fronts are causally unrelated). Consider the case of the propagation law given by a complete Riemannian metric, see 1.10, and assume that the propagations \(P_1 = (L, x_1, T_1, V_1)\) and \(P_2 = (L, x_2, T_2, V_2)\) be such that \(V_1(S^{m-1})\) and \(V_2(S^{m-1})\) are spheres of the same radius \(r\) in \(T_{x_1}M\) and \(T_{x_2}M\), respectively. As it was discussed in 1.10, in this case dangerous intersection points between the two fronts are the dangerous tangency points and they do not occur during the propagation of \(W_1\) and of \(W_2\).

Suppose now that \(M\) is a manifold of non-positive sectional curvature. Then, by the Hadamard Theorem, see for example [13], the exponential map \(\exp_x : T_xM \to M\) is the universal covering map for every \(x \in M\). Thus, the front \(W_i(t) : \exp_{x_i}((t - T_i)V) : S^{m-1} \to M, i = 1, 2,\)
is always an immersion, and the local orientation of a wave front passing through every point \( x \) is positive. In particular, all the passages of \( W_1 \) through \( x_2 \) are positive. Thus these passages can not cancel each other in the definition of CR\((W_1, W_2)\). And we get that for such cases CR\((W_1, W_2)\) \( \neq 0 \) if and only if the fronts are causally related.

Propagation laws given by a Riemannian metric of constant sectional curvature with \( V_1 \) and \( V_2 \) being spheres of unit radius are extremely important in cosmology and they are known as Friedmann Cosmology models, see for example [14], [24], [12], [28]. Thus we conclude that for Friedmann cosmology models given by metrics of constant non-positive sectional curvature CR\((W_1, W_2)\) \( \neq 0 \) if and only if \( W_1 \) and \( W_2 \) are causally unrelated.

3. Homotopy properties of maps to STM

3.1. Definition. Given a map \( \alpha : S^1 \times S^{n-1} \to STM \), we say that \( \alpha \) is special if \( \alpha \big|_{\ast \times S^{n-1}} \) has the form \( \varepsilon_x \) for some \( x \in M \), see (1.1). Here \( \ast \in S^1 \) is the base point.

3.2. Definition. Given an \( n \)-dimensional manifold \( N \) and a map \( \beta : N \to STM \), we define \( d(\beta) \) to be the degree of the map \( pr \circ \beta : N \to M \).

3.3. Lemma. Let \( \alpha : S^1 \times S^{n-1} \to STM \) be a special map. Then there exists a map \( \beta : S^n \to STM \) such that \( d(\beta) = d(\alpha) \).

Proof. We regard \( S^{n-1} \) as a pointed space. Consider a map \( \tilde{\alpha} : S^1 \times S^{n-1} \to STM \) such that:

1: \( \tilde{\alpha} \big|_{\ast \times S^{n-1}} = \alpha \big|_{\ast \times S^{n-1}} \),
2: \( \tilde{\alpha} \big|_{S^1 \times \ast} = \alpha \big|_{S^1 \times \ast} \),
3: \( \tilde{\alpha} \big|_{t \times S^{n-1}} = \varepsilon \big|_{\tilde{\alpha}(t \times \ast)} \).

We regard \( S^1 \times S^{n-1} \) as the CW-complex with four cells \( e^0, e^1, e^{n-1}, e^n \), \( \dim e^k = k \). It is easy to see that the maps \( \tilde{\alpha} \) and \( \alpha \) coincide on the \((n-1)\)-skeleton. Thus, the maps \( \alpha \) and \( \tilde{\alpha} \) (restricted to the \( n \)-cell) together yield a map \( \beta : S^n \to STM \). Clearly \( d(\tilde{\alpha}) = 0 \), and therefore \( d(\beta) = d(\alpha) \).

3.4. Lemma. Suppose that there exists a map \( \beta : S^n \to STM \) with \( d(\beta) \neq 0 \). Then the Euler class \( \chi \in H^n(M) \) of the tangent bundle \( TM \to M \) is zero.

Proof. We set \( f = pr \circ \beta : S^n \to M \). Clearly, \( M \) is closed and \( H^n(M) = \mathbb{Z} \) because \( d(\beta) \neq 0 \). So, again since \( d(\beta) \neq 0 \), we conclude \( f^* \chi \neq 0 \).
whenever $\chi \neq 0$. Now the result follows because $f^{\ast}\chi$ is the obstruction to the lifting of $f$ to $STM$, while $\beta$ is such a lifting of $f$. \hfill \square

3.5. Lemma. Let $M^n$ be an oriented manifold and $\beta : S^n \to STM$ be a map with $d(\beta) \neq 0$, then $M$ is a closed manifold which is a rational homology sphere.

Proof. We set $f = \text{pr} \circ \beta : S^n \to M$ and $d = d(\beta)$. Clearly $M$ is closed because $d(\beta) \neq 0$. Let $f_i : H_*(M) \to H_*(S^n)$ be the transfer map, see e.g. [27, V.2.12]. Since $f_i(f^*y \cap x) = y \cap f_*x$ for all $x \in H_*(S^n)$ and $y \in H^*(M)$, we conclude that $f_*f_i(z) = dz$, for all $z \in H_*(M)$. In particular, since $H_i(S^n) = 0$, for $0 < i < n$, then $dH_i(M) = 0$, for $0 < i < n$. Thus $H_i(M; \mathbb{Q}) = 0$ for $0 < i < n$ and $M$ is a rational homology sphere. \hfill \square

3.6. Corollary. If $M^{2k}$ is an even-dimensional oriented manifold, then $d(\beta) = 0$ for every $\beta : S^n \to STM$.

Proof. By Lemma 3.5 we get that if $d(\beta) \neq 0$ for some $\beta : S^{2k} \to STM$, then $H_i(M; \mathbb{Q}) = 0$ for $0 < i < 2k$ and $H_0(M, \mathbb{Q}) = H_{2k}(M, \mathbb{Q}) = \mathbb{Q}$. Thus the Euler characteristic of $M$ is 2 and the Euler class of the tangent bundle $TM \to M$ is non-zero (in fact, $\pm 2$). This contradicts to the statement of Lemma 3.3. \hfill \square

Let $\text{deg} : \pi_n(M^n) \to \mathbb{Z}$ be the degree homomorphism, i.e., the homomorphism which assigns the degree $\deg f$ to the homotopy class of a map $f : S^n \to M$. (In fact, it coincides with the Hurewicz homomorphism $h : \pi_n(M) \to H_n(M)$ for $M$ closed and is zero for $M$ non-closed.)

3.7. Definition. Given a connected oriented manifold $M^n$, we define an Abelian group $A(M)$ and a homomorphism $q = q_M : \mathbb{Z} \to A(M)$ as follows. If $n$ is even, then $A(M) = \mathbb{Z}$ and $q = 1_\mathbb{Z}$. If $n$ is odd, then $A(M)$ is the cokernel of the degree homomorphism $\text{deg} : \pi_n(M) \to \mathbb{Z}$ and $q : \mathbb{Z} \to A(M)$ is the canonical epimorphism.

3.8. Proposition. Let $M^n$ be an odd-dimensional manifold as in Definition 3.7. Then the following holds:

(i) If the universal covering space of $M$ is a non-compact manifold, then $A(M) = \mathbb{Z}$.

(ii) If $M$ admits a complete Riemannian metric of non-positive sectional curvature, then $A(M) = \mathbb{Z}$.

(iii) If $M$ is not a rational homology sphere, then $A(M) = \mathbb{Z}$.

(iv) If $\pi_1(M)$ is infinite, then $A(M) = \mathbb{Z}$. 


(v) If \( \pi_1(M) \) is a finite group of order \( k \), then \( A(M) = \mathbb{Z}/m\mathbb{Z} \) where \( k|m \) (the case \( m = 0 \). i.e. \( A = \mathbb{Z} \) is also possible).

(vi) If \( M \) is a closed manifold with \( A(M) = 0 \), then \( M \) is a homotopy sphere.

Proof. (i) This follows because every map \( S^n \to M \) passes through the universal covering \( \tilde{M} \to M \) while \( H_n(\tilde{M}) = 0 \). Therefore the Hurewicz homomorphism \( h: \pi_n(M) \to H_n(M) \) is trivial.

(ii) The Hadamard Theorem says that the universal covering of such \( M^n \) is diffeomorphic to \( \mathbb{R}^n \), see for example [13], and the statement follows from (i).

(iii) This follows immediately from Lemma 3.5.

(iv) This follows from (i) since the universal covering space of \( M \) is non-compact.

(v) This follows because every map \( S^n \to M \) passes through the universal covering map \( p: \tilde{M} \to M \) which is of degree \( k \).

(vi) If \( A(M) = 0 \) then there exists a map \( S^n \to M \) of degree 1. Since every map of degree 1 induces epimorphism of fundamental groups and homology groups, we conclude that \( M \) is a homotopy sphere. \( \square \)

4. THE AFFINE LINKING INVARIANT \( AL \) AS A REDUCTION OF \( CR \)

4.1. Definition. We define \( \Sigma_1 \) to be the subset (stratum) of \( \Sigma \) consisting of all the pairs \((f_1, f_2)\) such that there exists precisely two pairs of points \( y_1, y_2 \in S^{n-1} \) as in 1.3. Here we assume that the two double points of the image are distinct.

Notice that \( \Sigma_i \) is a stratum of codimension \( i \) in \( \Sigma \). In particular, a generic path in \( S \times S \) intersects \( \Sigma_0 \) in a finite number of points, and a generic disk in \( S \times S \) intersects \( \Sigma_1 \) in a finite number of points and does not contain singular mappings that are not in \( \Sigma_0 \cup \Sigma_1 \).

A generic path \( \gamma: [0, 1] \to S \times S \) that connects two points in \( S \times S \setminus \Sigma \) intersects \( \Sigma_0 \) in finitely many points \( \gamma(t_j), j \in J \subset N \) and all the intersection points are of the types described in 2.1. Put

\[
\Delta_{AL}(\gamma) = \sum_{j \in J} \sigma(\gamma, t_j) \in \mathbb{Z}.
\]

We let \( A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\} \), \( B_1 = \{(x, y) \in A \mid xy = 0\} \), \( B_2 = \{(x, y) \in A \mid x = 0\} \), \( B_3 = \{(0, 0)\} \), \( B_4 = \emptyset \).

We define a regular disk in \( S \times S \) as an embedded disk \( D \) such that the triple \( (D, D \cap \Sigma_0, D \cap \Sigma_1) \) is homeomorphic to a triple \( (A, B, C) \), \( A \supset B \supset C \) where \( B \) is one of \( B_i \)'s and \( C \) is a (possibly empty) subset of \( B_3 \).
4.2. Lemma. Let $\beta$ be a generic loop that bounds a regular disk in $S \times S$. Then $\Delta_{AL}(\beta) = 0$.

Proof. Straightforward. \square

4.3. Lemma. Let $\beta$ be a generic loop that bounds a disk in $S \times S$. Then $\Delta_{AL}(\beta) = 0$.

Proof. Without loss of generality we can (using a small deformation of the disk) assume that the disk is the union of regular ones, cf. Arnold [4], [3]. Now the proof follows from Lemma 4.2. \square

Notice that $S \times S$ is path connected.

4.4. Corollary. The invariant $\Delta_{AL}$ induces a well-defined homomorphism $\Delta_{AL}: \pi_1(S \times S, \ast) \to \mathbb{Z}$.

Proof. Since every element of $\pi_1(S \times S, \ast)$ can be represented by a generic loop, the proof follows from Lemma 4.3. \square

Let $x_1, x_2$ be two distinct points of $M$. Let $\alpha : S^1 \times S^{n-1} \to STM$ be a special map (see Definition 3.1) such that the composition

$$S^{n-1} \subset S^1 \times S^{n-1} \xrightarrow{\alpha} STM$$

has the form $\varepsilon_{x_1}$, and let $e : S^1 \times S^{n-1} \to STM$ be the map of the form

$$S^1 \times S^{n-1} \xrightarrow{\text{proj}} S^{n-1} \xrightarrow{\varepsilon_{x_2}} STM$$

Then $(\alpha, e)$ is a loop in $(S \times S, \ast)$.

4.5. Lemma. $\Delta_{AL}[(\alpha, e)] = d(\alpha)$.

Proof. Notice that $\Delta_{AL}[(\alpha, e)]$ is the intersection index of the cycles $\alpha(S^1 \times S^{n-1})$ and $S^{n-1}_{x_2}$. This index coincides with the degree of the map $\text{pr} \circ \alpha$ because the last one is equal to the algebraic number of the preimages of $x_2$. \square

4.6. Definition. Take two different points $x_1, x_2 \in M$ and consider the point $\ast = (\varepsilon_{x_1}, \varepsilon_{x_2}) \in S \times S \setminus \Sigma$, take an arbitrary point $f = (f^1_1, f^1_2) \in S \times S \setminus \Sigma$ and choose a generic path $\gamma$ going from $\ast$ to $f$. We set

$$AL(f) = q(\Delta_{AL}(\gamma)) \in A(M)$$

and call $AL$ the affine linking invariant. Here $q$ is the epimorphism from Definition 3.7.

4.7. Theorem. The function $AL : \pi_0(S \times S \setminus \Sigma) \to A(M)$ is well-defined and increases by 1 $\in A(M)$ under the positive transverse passage through the stratum $\Sigma_0$. 
4.8. Remark. It is easy to see that AL does not depend on the choice of the pair \((x_1, x_2)\) used to define it. This follows, since all the pairs \((\varepsilon_{x_1}, \varepsilon_{x_2}), x_1, x_2 \in M\) are isotopic.

However, if one chooses a basepoint \(\ast\) in the construction of AL that is not two fibers over two distinct points, then the AL-type invariant obtained this way would be different from AL by an additive constant \(AL(\tilde{\ast}) - AL(\ast)\). This ambiguity is similar to the ambiguity in the choice of the zero vector in an affine vector space, and it is the reason for the adjective “affine” in the name of the invariants. The general theory of affine linking invariants is discussed in our work [10].

Proof of Theorem 4.7. To show that AL is well-defined we must verify that the definition is independent on the choice of the path \(\gamma\) that goes from \(\ast\) to \(f\). This is the same as to show that \(q(\Delta_{AL}(\varphi)) = 0\) for every closed generic loop \(\varphi\) at \(\ast\).

Since \(\Delta_{AL} : \pi_1(S \times S, \ast) \to \mathbb{Z}\) is a homomorphism, it suffices to prove that \(q(\Delta_{AL}(\varphi)) = 0\) for all generators \(\varphi\) of \(\pi_1(S \times S, \ast)\).

The classes \([\langle \alpha, e \rangle]\) and similar classes \([\langle e, \alpha \rangle]\) generate the group \(\pi_1(S \times S, \ast)\). By Lemma 4.5 we have

\[
\Delta_{AL}([\langle \alpha, e \rangle]) = d(\alpha).
\]

Clearly, \(d(\alpha) = 0\) if \(M\) is an non-closed manifold. So, we assume \(M\) to be closed. Now, for \(n\) even \(d(\alpha) = 0\) by Corollary 3.6, while for \(n\) odd \(q(d(\alpha)) = 0\) by Lemma 3.3. □

Let \((P_1, P_2)\) be a generic pair of propagations, and let \(t\) be a moment of time when dangerous intersection do not occur. Let \(q : \mathbb{Z} \to A(M)\) be the epimorphism described in Definition 3.7.

4.9. Theorem. The invariants CR and AL are related as follows:

\[
q(CR(W_1(t), W_2(t))) = AL(\tilde{W}_1(t), \tilde{W}_2(t)).
\]

Proof. The Theorem follows because the signs defined for dangerous intersections (as well as for baby-intersections) are exactly the sign of the corresponding crossings of \(\Sigma_0\). □

The invariant AL works especially nice for even dimensional manifolds, since in these cases \(A(M) = \mathbb{Z}\).

If the pair of propagations is dangerous intersections free then the invariant \(AL(\tilde{W}_1(t), \tilde{W}_2(t)) \in A\) gives us the number of times the earlier-born front had passed through the birth point of the other front before the other front was born. It is also easy to see that if the other wave front did not appear yet, then \(AL(\tilde{W}_1(t), \tilde{W}_2(t))\) is the number of times
the earlier-born front passed through the (future) birth point of the other front by the time moment \( t \).

We illustrate our general results with the following example.

4.10. **Theorem.** Assume that the law of propagation \( L \) is given by a complete Riemannian metric as in 1.10, of non-positive sectional curvature on \( M \) and the wave fronts \( W_i(t), i = 1, 2 \), correspond respectively to propagations \( P_i = (L, x_i, T_i, V_i), i = 1, 2 \), with \( V_i : S^{n-1} \to T_{x_i}M, i = 1, 2 \), being spheres of the same radius. Then \( W_1 \) and \( W_2 \) are causally related if and only if \( AL(\tilde{W}_1(t), \tilde{W}_2(t)) \neq 0 \).

**Proof.** As it was explained in 2.5 for such propagations \( CR(W_1, W_2) \neq 0 \) if and only if \( W_1 \) and \( W_2 \) are causally related. Notice that \( \mathbb{A}(M) = \mathbb{Z} \) if \( M \) is even-dimensional by definition of \( \mathbb{A}(M) \), while \( \mathbb{A}(M) = \mathbb{Z} \) for \( M \) odd-dimensional by Proposition 3.8. Now the result follows from Theorem 4.9. \( \square \)

This Theorem, in particular, says that \( AL(W_1(t), W_2(t)) \) allows one to detect always whether \( W_1 \) and \( W_2 \) are causally related or not in Friedmann models based on metrics of constant non-positive sectional curvature.

5. **Causality relation invariant in the case of the propagation according to Riemannian metrics.**

As it was noticed in 1.10, if a propagation happens according to a complete Riemannian metric and \( \text{Im} \, V \) is a sphere, then the velocities of the points of the front are always orthogonal to the front. So, if each of the two propagations happens according to a complete Riemannian metric, then dangerous tangency points and the dangerous intersection points are the same thing. In this section we deal only with this case. Namely, we provide an explicit way of calculation of the invariant \( CR \). We need some preliminaries.

Let \( W \) be a wave front, and let \( x \in \text{Im} \, W(t) \) be a non-singular point of \( W(t) \). For sake of simplicity we denote \( T_x(\text{Im} \, W(t)) \) just by \( T \). Let \( O \) be a small neighborhood of \( x \) in \( M \), and let \( U = O \cap \text{Im} \, W(t) \). Without loss of generality we can and shall assume that the injectivity radius is big enough (\( \geq 3 \)) for all points of \( O \).

The Riemannian metric \( g \) on \( M \) produces a unique symmetric connection on \( M \). So, for every \( a \in O \), the parallel transport along the geodesic segment (connecting \( x \) and \( a \)) gives us an isomorphism

\[
\tau_a : T_aM \to T_xM
\]
Furthermore, we can regard every sphere $S_a \in STM, a \in M$ as the unit sphere in $T_a M$, and so $STM$ can be regarded as the total space of the unit sphere subbundle of $TM$. Since the connection respects the Riemannian metric, we conclude that $\tau_a(S_a) = S_x$.

5.1. **Definition.** (a) We define

$$\pi : \text{pr}^{-1}(O) \to S_x$$

as follows. A point $z \in \text{pr}^{-1}(O)$ is a pair $(a, \xi)$ with $a = \text{pr}(z)$ and $\xi \in S_a$, and we set $\pi(z) = \tau_a(\xi)$ with $\tau_a$ as in (5.1).

(b) Let $n_u$ be the unit normal vector to $U$ at $u$ that points to the direction of propagation of the front. We define the Gauss map $G = G_W : U \to S_x$ by setting $G(u) = \tau_u(n_u)$.

(c) Given $u \in U$, let $\ell(u) = n_u \in \text{Im} \widetilde{W}$. In this way we get a map $\ell : U \to \text{Im} \widetilde{W} \subset STM$. We set $z = \ell(x)$. Given $e \in T$, we set

$$e^W := d\ell(e), \quad e^W \in T_z \text{Im} \widetilde{W} \subset T_z STM.$$ 

It is clear that $d\ell : T \to T_z \text{Im} \widetilde{W}$ is an isomorphism.

(d) Let $z \in STM$ be the point described in (c). We define the horizontal section $H : O \to STM$ of $\text{pr}$ by setting

$$H(a) = \tau_a^{-1}(z) \in S_a \subset STM.$$ 

Furthermore, given $w \in T_a M, a \in O$, we set

$$w^H = dH(w) \in T_{H(a)} STM.$$ 

Clearly, $w^H$ can be characterized by the properties

$$(d\text{pr})(w^H) = w, \quad d\pi(w^H) = 0.$$ 

(e) Let $z \in STM$ be the point described in (c). Given $w \in T_z M$, we define $w^S \in T_z T_x M$ as follows. We regard $z$ as the vector $z \in T_x M$. Furthermore, we regard $T_x M$ as the affine space $T^\text{aff}$ over the vector space $T_x M$ and consider the parallel shift

$$P_z : T^\text{aff} \to T^\text{aff}, \quad a \mapsto a + z.$$ 

Let $o \in T^\text{aff}$ correspond to the origin of the vector space $T_x M$. Using the obvious identification $T_x M = T_o T^\text{aff}$, we regard $w$ as the tangent vector $w_o \in T_o T^\text{aff}$, and we set

$$w^S = dP_z(w_o) \in T_z T^\text{aff} = T_z T_x M.$$ 

Notice that if $e \in T$ then $e^S \in T_z S_x$. (This is where the notation comes from: $e^S$ is the spherical lifting of $e$.)

5.2. **Lemma.** For every $e \in T$ we have $e^W - e^H = dG(e)$. 

Proof. First, notice that \( G = \pi \circ \ell : U \to S_x \). So, \( d\pi(e^W) = dG(e) \). Now,
\[
(d \pi)(e^W - dG(e)) = e - 0 = e
\]
and
\[
d\pi(e^W - dG(e)) = dG(e) - dG(e) = 0.
\]
Thus, \( e^W - dG(e) = e^H \). \( \square \)

5.3. Proposition. Let \( e \in T \), and let \( n \) be the normal vector field to \( U \) in \( M \). Then \( dG(e) = (\nabla_e n)^S \).

Proof. Let \( \gamma : (-\delta, \delta) \to U \) be a curve with \( \dot{\gamma}(0) = e \). We define the curve \( \zeta : (-\delta, \delta) \to S_x \) by setting \( \zeta(t) \) to be the end of the vector \( \tau_{\gamma(t)} n_{\gamma(t)} \). Since
\[
\nabla_e n = \frac{d}{dt} (\tau_{\gamma(t)} n_{\gamma(t)} - n_x) \bigg|_{t=0}
\]
we conclude that \( \dot{\zeta}(0) = (\nabla_e n)^S \). On the other hand, \( G \circ \gamma = \zeta \), and thus
\[
dG(e) = dG(\dot{\gamma}(0)) = (G \circ \gamma)'(0) = \dot{\zeta}(0) = (\nabla_e n)^S.
\]
\( \square \)

5.4. Corollary. \( e^W - e^H = (\nabla_e n)^S \).

Proof. This is the direct consequence of 5.2 and 5.3. \( \square \)

Consider the Weingarten operator
\[
(5.2) \quad A = A_W : T \to T, \quad A(e) = \nabla_e n.
\]
The Corollary 5.4 can now be written as follows:
\[
(5.3) \quad e^W - e^H = (Ae)^S.
\]

Now let \( W_1 \) and \( W_2 \) be two wave fronts, and let \( x \in M \) be a point of dangerous tangency of \( W_1(t) \) and \( W_2(t) \). We assume that the corresponding pair of propagations is generic. Again, we denote by \( T \) the common tangent plane \( T_x W_i(t), i = 1, 2 \). Let \( A_i := A_{W_i} : T \to T, i = 1, 2 \) be the Weingarten operators considered in (5.2). We set \( B = A_1 - A_2 \). It is well known that each \( A_i \) is a self-adjoint operator, \( [29, \text{Ch. 7}] \), and therefore \( B \) is. Let \( k_1, \ldots, k_{n-1} \) be the eigenvalues (with multiplicities) of \( B \).

5.5. Proposition. \( \ker B = 0, \) and so \( k_i \neq 0 \) for all \( i \).
Proof. Let \( e \neq 0 \) be a vector with \( Be = 0 \). Then
\[
e^{W_1} - e^{W_2} = (e^{W_1} - e^H) - (e^{W_2} - e^H) = (A_1e - A_2e)^S = (Be)^S = 0.
\]
i.e. \( T_x \text{Im} \tilde{W}_1 \cap T_x \text{Im} \tilde{W}_2 \neq 0 \). But this is impossible because the pair of propagations is assumed to be generic (see conditions 1.6(b) and 1.3(c)). \( \square \)

In particular, \( \det B = k_1 \cdots k_{n-1} \neq 0 \).

5.6. Definition (Alternative definition of \( \sigma(W_1(t), W_2(t)) \)). We put \( \varepsilon(W_1(t), W_2(t)) = 1 \) if both fronts have the same local orientations at \( x \) (as defined in 2.2) and \( \varepsilon(W_1(t), W_2(t)) = -1 \) if the fronts have opposite local orientations. Now we set
\[
\tilde{\sigma}(W_1(t), W_2(t)) = \varepsilon(W_1(t), W_2(t)) \operatorname{sign}(\det B) \operatorname{sign}(|v_1| - |v_2|)
\]
where \( v_i \) is the velocity vector of \( W_i(t) \) at \( x \).

5.7. Theorem. \( \tilde{\sigma}(W_1(t), W_2(t)) = \sigma(W_1(t), W_2(t)) \).

Proof. Given a vector \( e \in T \), we set \( e' = e^{W_1} \) and \( e'' = e^{W_2} \). Choose a basis \( \{e_1, \ldots, e_{n-1}\} \) of \( T \) containing of the eigenvectors of \( B \), i.e., \( Be_i = k_i e_i \). Because of equality (5.3) we have
\[
e''_i - e'_i = (Be_i)^S = (k_i e_i)^S = k_i e_i^S.
\]
We can and shall assume that the frame \( \{e_1, \ldots, e_{n-1}\} \) gives the positive (local) orientation of \( W_i(t) \), \( i = 1, 2 \) at \( x \). Take the polyvector
\[
p := e'_1 \wedge \cdots \wedge e'_{n-1} \wedge v \wedge e''_1 \wedge \cdots \wedge e''_{n-1}
\]
where \( v \) is the vector defined in 1.4. Then \( p \neq 0 \) since the pair of propagations is assumed to be generic. Notice that \( p \) gives us an orientation of \( STM \), and we say that \( p \) is positive if this orientation coincides with the original one, otherwise we say that \( p \) is negative.

According to Definition 2.3, the sign of \( p \) is equal to
\[
\varepsilon(W_1(t), W_2(t)) \sigma(W_1(t), W_2(t)).
\]
So, we must prove that sign of the polyvector \( p \) is equal to the sign of
\[
(\det B) \operatorname{sign}(|v_1| - |v_2|).
\]
To be definite, we assume that \( |v_1| > |v_2| \) and prove that the sign of \( p \) is equal to the sign of \( \det B \).

Since \( e''_i = e'_i + k_i e_i^S \) and \( \det B = k_1 \cdots k_{n-1} \), we conclude that
\[
e'_1 \wedge \cdots \wedge e'_{n-1} \wedge v \wedge e''_1 \wedge \cdots \wedge e''_{n-1} = (\det B) e'_1 \wedge \cdots \wedge e'_{n-1} \wedge v \wedge e^S_1 \wedge \cdots \wedge e^S_{n-1}.
\]
So, it remains to prove that the polyvector
\[
e'_1 \wedge \cdots \wedge e'_{n-1} \wedge v \wedge e^S_1 \wedge \cdots \wedge e^S_{n-1}
\]
is positive.

Since \(e'_1 \wedge \cdots \wedge e'_{n-1} \wedge v \wedge e^S_1 \wedge \cdots \wedge e^S_{n-1} \neq 0\), we conclude that the family \(\{e^S_1, \ldots, e^S_{n-1}\}\) generate \(T_zS_x\). It is easy to see that the frame \(\{e^S_1, \ldots, e^S_{n-1}\}\) gives the positive local orientation of each of the fronts at \(x\). So, it remains to prove that the polyvector

\[(d\text{pr})(e'_1 \wedge \cdots \wedge e'_{n-1} \wedge v) = e_1 \wedge \cdots \wedge e_{n-1} \wedge (d\text{pr})v\]

is positive, i.e. that it gives the original orientation of \(M\). Recall that the polyvector

\[e_1 \wedge \cdots \wedge e_{n-1} \wedge (v_1 - v_2)\]

is positive since \(|v_1| > |v_2|\). Taking into account that the vector \(v_1 - v_2\) is orthogonal to \(T\) and points into the direction of both propagations at \(x\), it suffices to prove that \(\langle v_1 - v_2, (d\text{pr})(v) \rangle > 0\). But this is clear because \(W_1\) is faster then \(W_2\) at \((x, t)\), and so the point \(\text{pr}(z_1)\) (in notation of 1.4) is further then the point \(\text{pr}(z_2)\) from \(T\). □

The following Proposition 5.8 is useful, when calculating sign \(\det B\).

Let \(\overline{g}\) be another Riemannian metric. Now we define \(\overline{H}, \overline{\pi}, \overline{\ell}, \overline{z}\) and \(\overline{G}\) as in 5.1 with respect to the metric \(\overline{g}\). We also assume that \(\overline{\ell}_1(U_1) \cap \overline{\ell}_2(U_2) = \overline{\pi}\). Note that in this case the image of \(\overline{\ell}\) does not lie in \(\text{Im} \overline{W}\), but we can still define the operator \(B = B(\overline{g})\) as it is done before Proposition 5.5. We say that a metric \(\overline{g}\) is generic if \(d(\overline{\ell}_1)T_x \text{ Im} W_1 \cap d(\overline{\ell}_2)T_x \text{ Im} W_2 = 0\).

5.8. Proposition. If the metric \(\overline{g}\) is generic, then \(\det B(\overline{g}) \neq 0\) and sign \(\det B(\overline{g}) = \text{sign } \det B(\overline{g})\).

5.9. Remark. Probably the most useful case of this proposition is when \(\overline{g}\) is chosen to be locally flat in the neighborhood of the dangerous tangency point and so that one of the fronts is (locally) a totally geodesic submanifold.

Proof of Proposition 5.8. First, notice that the set of all Riemannian metrics is path connected because it is convex. Second, notice that \(\det B(\overline{g}) \neq 0\) for every generic metric \(\overline{g}\). (This is proved in the same way as Proposition 5.5.) Furthermore, the set of non-generic metrics has codimension > 1 in the space of all metrics. So, there exists a continuous family \(g_t, t \in [0, 1]\), of Riemannian metrics such \(g_0 = g, g_1 = \overline{g}\) and each \(g_t\) is generic. Clearly, \(B(g_t)\) depends on \(t\) continuously and \(\det B(g_t) \neq 0\) since \(g_t\) is generic. Thus the sign of \(\det B(g_t)\) is the same for all \(t\). □
5.10. Example (of calculation of $\sigma(\tilde{W_1}(t),\tilde{W_2}(t)))$. Suppose that the fronts propagate as it is shown in Figure 2.

Let $W_1$ be the “right” front. Then, clearly,

$$\sigma(W_1(t), W_2(t)) = -\varepsilon(W_1(t), W_2(t)).$$

The negative sign appears because $W_2$ is faster the $W_1$.

6. Examples

To illustrate the usage of the affine linking invariant consider the following examples.

6.1. Example. Here we show how to apply AL to determining the causality relation. Let $M$ be a smooth oriented $n$-dimensional manifold that is not an odd-dimensional homotopy sphere. Let $W_1, W_2$ be the wave fronts that originated on $M$ long time ago and were propagating according to the dangerous intersections free pair of propagations $\{P_1, P_2\}$.

Assume that the current picture of wave fronts $W_1(t), W_2(t)$ is the one shown in Figure 3 with the velocity vectors normal to the two spheres shown in Figure 3. (Note that after the contracting front contracts to a point it does not appear but rather everts and turns into an expanding spherical front.)

Then a straightforward calculation shows that $AL(\tilde{W_1}(t), \tilde{W_2}(t)) - AL(V_1, V_2) = \pm 1 \neq 0$ (we used the notation as in Theorem 4.9), and thus the first wave front reached the birth point of the second front before the second front originated. (The sign of $\pm 1$ in this example depends on which of the two fronts shown in Figure 3 is $W_1$ in the case where $n$ is odd and is always a plus sign when $n$ is even.)

This seems to demonstrate that AL is a very powerful invariant because in this case we know neither the propagation laws nor when and where the fronts originated. In fact, in this example we can make this
conclusion even without the knowledge of the topology of $M$ outside of the depicted part of it.

6.2. **Example.** Here we show how to apply AL to estimating of the number of times the wave front passed through a given point between the two moments of time.

Assume that we have a wave front $W$ that propagates on $M$ and that $M$ is not an odd-dimensional homotopy sphere.

Assume that at a certain moment of time the picture of the wave front was the one shown in Figure 4.a and later it developed into the shape shown in Figure 4.b. (The Figure 4.b depicts a sphere that can be obtained from the trivially embedded sphere by passing three times through a point and by creation of some singularities far away from $x$.)

Let $\tilde{W}(t_1), \tilde{W}(t_2) : S^{n-1} \to STM$ be the liftings of the fronts shown in Figure 4.a and b respectively. A straightforward calculation shows
that \( \text{AL}(\tilde{W}(t_2), \varepsilon_x) - \text{AL}(\tilde{W}(t_1), \varepsilon_x) = 3 \in A(M) \) for every map \( \varepsilon_x : S^{n-1} \to STM \) as in (1.1).

Thus if the dimension of the ambient manifold is even, or if \( \pi_1(M) \) is infinite, then (see 3.8) \( W \) did pass at least three times through the point \( x \) between the time moments shown in Figure 4.a and 4.b. Once again, this conclusion does not depend on the topology of \( M \) outside of the part of it depicted in Figure 4, on the time passed between the two pictures taken, and on the propagation law.

Acknowledgments: The first author was supported by the free-term research money from the Dartmouth College. The second author was supported by the free-term research money from the University of Florida, Gainesville and by MCyT, projects BFM 2002-00788 and MCyT BFM2003-02068/MATE, Spain.

The authors are very thankful to Robert Caldwell, Jose Natario, and Jacobo Pejsachowicz for useful discussions.

References

[1] F. Aicardi: Topological invariants of Legendrian curves. C. R. Acad. Sci. Paris Ser. I Math. 321 (1995), no. 2, 199–204.
[2] F. Aicardi: Remarks on the symmetries of planar fronts. Rev. Mat. Univ. Complut. Madrid 8 (1995), no. 2, 355–378.
[3] V.I. Arnold: Plane curves, their invariants, perestroikas and classifications, Singularities and Bifurcations (V.I. Arnold, ed.) Adv. Sov. Math., Vol. 21 (1994), pp. 39–91.
[4] V.I. Arnold: Invariants and perestroikas of wave fronts on the plane, Singularities of smooth maps with additional structures, Proc. Steklov Inst. Math., Vol. 209 (1995), pp. 11–64.
[5] V.I. Arnold: Mathematical Methods of Classical Mechanics, Second edition, Graduate Texts in Mathematics, 60, Springer-Verlag, New-York (1989)
[6] V. V. Chernov (Tchernov), Arnold-type invariants of wave fronts on surfaces. Topology 41 (2002), no. 1, 1–45.
[7] V. V. Chernov (Tchernov): Arnold-type invariants of curves on surfaces, J. Knot Theory Ramifications, 8 (1999), No. 1, pp. 71–97
[8] V. V. Chernov (Tchernov): Homotopy groups of the space of curves on a surface, Math. Scand. 86 (2000), no. 1, pp. 36–44.
[9] V. V. Chernov (Tchernov): Shadows of wave fronts and Arnold-Bennequin type invariants of fronts on surfaces and orbifolds, Amer. Math. Soc. Transl. (2) 190 1999, pp. 153–184
[10] V. V. Chernov (Tchernov) and Yu. B. Rudyak: Toward the General Theory of Affine Linking Numbers, preprint math.GT/0302295 at http://www.arxiv.org (2003)
[11] S. Chmutov and V. Goryunov: Polynomial invariants of Legendrian links and plane fronts. Topics in singularity theory, 25–43, Amer. Math. Soc. Transl. Ser. 2, 180, Amer. Math. Soc., Providence, RI, (1997)
[12] N. Cornish and J. Weeks: Measuring the shape of the Universe. Notices AMS 45 (1998) pp. 1463-1471
[13] M. P. do Carmo: Riemannian Geometry, Birkhauser, Boston, 1992
[14] T. Frankel: Gravitational Curvature: An Introduction to Einstein’s Theory. W. H. Freeman and Company (1979)
[15] V. Goryunov: Plane curves, wavefronts and Legendrian knots. Topological methods in the physical sciences (London, 2000). R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci. 359 (2001), no. 1784, 1497–1510
[16] V. Goryunov: Vassiliev type invariants in Arnold’s \( J^+ \)-theory of plane curves without direct self-tangencies. Topology 37 (1998), no. 3, 603–620.
[17] J.W. Hill: Vassiliev-type invariants of planar fronts without dangerous self-tangencies. C. R. Acad. Sci. Paris Ser. I Math. 324 (1997), no. 5, 537–542.
[18] A. V. Inshakov: Invariants of the Types $j^+$, $j^−$ and $st$ for Smooth Curves on Two-dimensional Manifolds Funct. Anal. Appl., Vol. 33 (1999), No. 3, pp. 189–198
[19] A. V. Inshakov: Homotopy groups of spaces of curves on two-dimensional manifolds, Russian Math. Surveys, 53 (1998), No. 2, pp. 390–391
[20] J. R. Low: Twistor linking and causal relations. Classical Quantum Gravity 7 (1990), no. 2, 177–187.
[21] J. R. Low: Celestial spheres, light cones, and cuts. J. Math. Phys. 34 (1993), no. 1, 315–319.
[22] J. R. Low: Twistor linking and causal relations in exterior Schwarzschild space. Classical Quantum Gravity 11 (1994), no. 2, 453–456.
[23] J. Natario and P. Tod: Linking, Legendrian linking and causality, preprint gr-qc/0210036 at http://www.arxiv.org, 30 pages, (2002)
[24] P. J. E. Peebles: Principles of Physical Cosmology. Princeton University Press (1993)
[25] M. Polyak: Invariants of curves and fronts via Gauss diagrams; Topology, Vol. 37 (1998), No. 5, pp. 989–1009
[26] M. Polyak: On the Bennequin invariant of Legendrian curves and its quantization, C. R. Acad. Sci. Paris Ser. I Math. 322 (1996), No. 1, pp. 77–82
[27] Yu. B. Rudyak: On Thom Spectra, Orientability, and Cobordism, Springer, Berlin Heidelberg New York, 1998
[28] Ch. Sormani: Friedmann Cosmology and Closest Isotopy. preprint math.DG/0302244 at http://www.arxiv.org (2003) 45 pages
[29] M. Spivak: A Comprehensive Introduction to Differential Geometry, vol 4, Publish or Perish, Inc, Boston 1975
[30] S. L. Tabachnikov: Calculation of the Bennequin invariant of a Legendre curve from the geometry of its wave front. Funct. Anal. Appl. 22 (1988), no. 3, 246–248 (1989)

V. CHERNOV, DEPARTMENT OF MATHEMATICS, 6188 BRADLEY HALL, DARTMOUTH COLLEGE, HANOVER NH 03755, USA
E-mail address: Vladimir.Chernov@dartmouth.edu

Yu. RUDYAK, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, 358 LITTLE HALL, GAINESVILLE, FL 32611-8105, USA
E-mail address: rudyak@math.ufl.edu