Mean Field Limit of Interacting Filaments and Vector Valued Non-linear PDEs

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Abstract Families of $N$ interacting curves are considered, with long range, mean field type, interaction. They generalize models based on classical interacting point particles to models based on curves. In this new set-up, a mean field result is proven, as $N \to \infty$. The limit PDE is vector valued and, in the limit, each curve interacts with a mean field solution of the PDE. This target is reached by a careful formulation of curves and weak solutions of the PDE which makes use of 1-currents and their topologies. The main results are based on the analysis of a nonlinear Lagrangian-type flow equation. Most of the results are deterministic; as a by-product, when the initial conditions are given by families of independent random curves, we prove a propagation of chaos result. The results are local in time for general interaction kernel, global in time under some additional restriction. Our main motivation is the approximation of 3D-inviscid flow dynamics by the interacting dynamics of a large number of vortex filaments, as observed in certain turbulent fluids; in this respect, the present paper is restricted to smoothed interaction kernels, instead of the true Biot–Savart kernel.

Keywords Currents · Mean field theory · Propagation of chaos · (random) vortex filaments · Vector-valued PDEs

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1 Introduction

1.1 From Interacting Particles to Interacting Curves

Classical mean field theory deals with pointwise particles in $\mathbb{R}^d$, described by their position $X_{t}^{i,N}$, that satisfy dynamics of the form

$$\frac{dX_{t}^{i,N}}{dt} = \frac{1}{N} \sum_{j=1}^{N} K \left(X_{t}^{i,N} - X_{t}^{j,N}\right)$$

(1)

governed by an interaction kernel $K : \mathbb{R}^d \to \mathbb{R}^d$ (often a stochastic analog is considered but here we deal with the deterministic case). Denoting by $\delta_x$ the Dirac measure concentrated at $x$, one considers the empirical measure $S_N^t = \frac{1}{N} \sum_{i=1}^{N} \delta_{X_{t}^{i,N}}$ associated to system (1). If $K$ is bounded Lipschitz continuous and $S_N^t$ weakly converges to a probability measure $\mu_0$, Dobrushin [11] proved that $S_N^t$ weakly converges to a measure-valued solution $\mu_t$ of the mean field equation

$$\frac{\partial \mu_t}{\partial t} + \text{div} \left( (K * \mu_t) \mu_t \right) = 0$$

with initial condition $\mu_0$, where $K * \mu_t$ is the vector field in $\mathbb{R}^d$ with $i$-component given by the convolution $K_i * \mu_t$.

Our aim is to prove an analogous result in the case when interacting points are replaced by interacting curves. We consider curves which vary in time, but always parametrized by $\sigma \in [0, 1]$. Hence our curves depend on two parameters, $t \geq 0$ and $\sigma \in [0, 1]$. Given $t \geq 0$, let

$$\left\{ \gamma_{t}^{i,N}(\sigma) ; \sigma \in [0, 1], \quad i = 1, ..., N \right\}$$

be a family of $N$ curves in $\mathbb{R}^d$; for shortness we shall write $\left\{ \gamma_{t}^{i,N} \right\}$. We assume that these curves vary in time and interact through the equations

$$\frac{\partial \gamma_{t}^{i,N}(\sigma)}{\partial t} = \sum_{j=1}^{N} \alpha_j^{N} \int_{0}^{1} K \left( \gamma_{t}^{i,N}(\sigma) - \gamma_{t}^{j,N}(\sigma') \right) \frac{\partial}{\partial \sigma'} \gamma_{t}^{j,N}(\sigma') d\sigma'$$

(2)

where $\alpha_j^{N}$ play the role of the factors $\frac{1}{N}$ in (1) and where now $K : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ is a smooth matrix-valued function (precisely, we need $K$ of class $U\mathcal{C}^3_b(\mathbb{R}^d, \mathbb{R}^{d \times d})$, see Sect. 2 for the definition). The time-evolution of each curve is determined by both the position and the tangent vectors of the other curves. Notice also that each curve is subject to a self-interaction (the term in the sum with the same index of the curve) and a mutual interaction. As remarked below, the structure of equations (2) is inspired by the interaction between vortex curves in 3D fluids.

As explained in the next subsection, we associate to such a family of curves a distributional vector field (called a 1-current). Our main result will be that the distributional vector field associated to the family $\left\{ \gamma_{t}^{i,N} \right\}$ of curves converges, in appropriate topologies, to the unique solution of a vector valued nonlinear PDE, of the form

$$\frac{\partial \xi_t}{\partial t} + \text{div} \left( (K * \xi_t) \xi_t \right) = \xi_t \cdot \nabla (K * \xi_t)$$

(3)
In the case when the interaction kernel $K$ is divergence free, this equation becomes
\[
\frac{\partial \xi_t}{\partial t} + [(K * \xi_t) \cdot \nabla] \xi_t = \xi_t \cdot \nabla (K * \xi_t)
\]
which is the form of the vorticity equation in the theory of incompressible fluids. The result will be local in time, under general assumptions on $K$, global under restricted assumptions.

In order to give a precise statement we need the language of 1-currents, that we introduce in the next subsection.

1.2 Formulation as Lagrangian Dynamics of 1-Currents

A 1-current is a distributional vector field, namely an element of the dual of smooth vector fields (see [20,21] for general definitions); for us, it is sufficient to consider 1-currents which are elements of the dual space $\mathcal{M} := C_b \left( \mathbb{R}_d; \mathbb{R}_d \right)'$.

Using the language of 1-currents it is possible to associate an empirical current, analogue of the empirical measure, to a family of curves $\{ \gamma_i^{t,N} \}$. At any time $t$, we define the 1-current $\xi_t^{N} \in \mathcal{M}$ as
\[
\xi_t^{N} = \sum_{j=1}^{N} \alpha_j^{N} \int_{0}^{1} \delta_{\gamma_i^{t,N}(\sigma)} \frac{\partial}{\partial \sigma} \gamma_i^{t,N}(\sigma) d\sigma.
\]
Its action on a test function $\theta \in C_b \left( \mathbb{R}_d; \mathbb{R}_d \right)$ is $\xi_t^{N}(\theta) = \sum_{j=1}^{N} \alpha_j^{N} \int_{0}^{1} \theta \left( \gamma_i^{t,N}(\sigma) \right) \cdot \frac{\partial}{\partial \sigma} \gamma_i^{t,N}(\sigma) d\sigma$. If $K : \mathbb{R}_d \to \mathbb{R}_d \times d$ is continuous and bounded, we may define the vector field $K * \xi_t^{N} : \mathbb{R}_d \to \mathbb{R}_d$ as $(K * \xi_t^{N}(x))_i = \xi_t^{N}(K_i(x - \cdot))$, $i = 1, \ldots, d$ (See Sect. 1 for more details on convolution). Then Eq. (2) may be rewritten as
\[
\frac{\partial}{\partial t} \gamma_i^{t,N}(\sigma) = \left( K * \xi_t^{N} \right) \left( \gamma_i^{t,N}(\sigma) \right).
\]
This fact motivates the idea to investigate the following ODE
\[
\frac{dx_t}{dt} = (K * \xi_t)(x_t)
\]
where $\xi_t$ is an arbitrary time dependent current (not necessarily $\xi_t^{N}$ defined above). Assume that under suitable hypothesis there is a flow of diffeomorphism $\varphi^{t,K*\xi}$ associated to Eq. (6). Then from (5) we have
\[
\gamma_i^{t,N}(\sigma) = \varphi^{t,K*\xi}(\gamma_i^{t,N}(\sigma)).
\]
With a little additional effort, it is possible to express the current $\xi_t^{N}$ at time $t$ by means of $\varphi^{t,K*\xi}$ and the initial current $\xi_0^{N}$; such a result requires the concept of push-forward of a 1-current by a $C^1$ map, that we shall describe below in the technical part of the paper. The result takes the form
\[
\xi_t^{N} = \varphi_{\varphi^{t,K*\xi}}^{t,K*\xi} \xi_0^{N}
\]
where $\varphi_{\varphi^{t,K*\xi}}^{t,K*\xi}$ denotes the operation of push-forward. In other words, we have the following preliminary result:

**Lemma 1** The dynamics (1) of $N$ interacting curves is equivalent to equation (7) for the corresponding currents.
The precise statement is given in Theorem 19. It turns out that also the vector-valued PDE (3), weakly interpreted for 1-currents, has a Lagrangian reformulation. Consider the following equation which generalizes (7):

$$\xi_t = \varphi_t^{\xi,K*\xi_0}$$ (8)

where $\xi_0$ is a current at time zero, $\varphi_t^{\xi,K*\xi}$ is the flow associated to equation (6) with the drift given by $K*\xi$. $\varphi_t^{\xi,K*\xi_0}$ is the push-forward of $\xi_0$ by $\varphi_t^{\xi,K*\xi}$ and the solution of the equation is a time-dependent current $\xi_t$ that satisfies identity (8). We have (see Theorem 12):

Lemma 2 Given $\xi_0 \in M$, a time dependent current $\xi_t$ is a weak solution of Eq. (3) if and only if it is a solution of Eq. (8).

It follows, in particular, that the empirical current $\xi^N_t$ itself is a solution of the PDE (3) (as it happens for the classical empirical measure, in the case of point particles).

1.3 Well Posedness of the Lagrangian Dynamics and Its Consequences

We call non-linear Lagrangian dynamics, or flow equation, Eq. (8) defined by the flow and its push-forward. The previous two lemmas clarify that it plays a basic role in the understanding of the curve dynamics and the PDE. Thus, all our results are deduced from a careful investigation of the Lagrangian dynamics (8), in suitable classes of time-dependent 1-currents; this is an extension of the new approach developed by Dobrushin [11].

Let us summarize the main results on the flow equation (8), given in details in Theorems 8, 15, 18. On the space $M$ of 1-currents, we introduce a concept of weak convergence, as described below in the technical sections, and define $M_w$ as the space $M$ endowed with such a weak topology. For the global result we stress only the most special condition, on the Fourier transform of $K$ and point out to the theorem in the technical section for additional assumptions.

Theorem 3

(i) (Local solutions) Assume $K$ of class $UC^3_b(\mathbb{R}^d,\mathbb{R}^{d\times d})$. Then, for every $\xi_0 \in M$, there is a unique maximal solution $\xi$ of the flow equation (8) in $C([0,T_{\xi_0});M)]$.
(ii) (Global solutions) If the Fourier transform of $K$ has compact support and the other technical assumptions of Theorem 15 below hold true, then $T_{\xi_0} = +\infty$.
(iii) (Continuous dependence) If $\xi^n_0 \to \xi_0$ in $M_w$ and $[0,T]$ is a common time interval of existence and uniqueness for the flow equation (20), then for the corresponding solutions $\xi^n$ and $\xi$ we have $\xi^n \to \xi$ in $C([0,T];M_w)$.

By the two lemmas of the previous section, one easily deduces the following main results on the curve dynamics and the corresponding PDE:

Corollary 4

(i) (Well posedness) The local and global well posedness results stated in Theorem 3 hold true for the curve dynamics (2) and for the PDE (3).
(ii) (Continuum limit) Let $\xi_t^N$ be the 1-current associated to the curves $\gamma_i^t$ by definition (4) and let $\xi_0 \in M$. If $\xi_0^N \to \xi_0$ in $M$ as $N \to \infty$, then $\xi^N_t$ converges to the unique current-solution of the PDE (3) with initial condition $\xi_0$ (convergence takes place in $C([0,T];M)$ over any time interval $[0,T]$ where all solutions exist).
(iii) (Mean field) For every $i$, $\gamma_i^t$ converges as $N \to \infty$ to the solution of the equation

$$\frac{\partial}{\partial t} \gamma_i^t (\sigma) = (K*\xi_t) (\gamma_i^t (\sigma))$$

namely it is coupled with the mean field $\xi_t$ in the limit as $N \to \infty$. 
Again, some aspects of these claims require additional details to be precise, see in particular Theorem 22. Finally, from these deterministic results we deduce a propagation of chaos result, when the initial conditions are assumed to be random independent curves. The statement is rather technical and we postpone it to Sect. 7.3.

1.4 Relations with Fluid Dynamics and Comments on the Literature

The investigation made here of interacting curves and the associated mean field PDE is motivated by the theory of vortex filaments in turbulent fluids. Starting from the simulations of [27], a new vision of a three dimensional turbulent fluid appeared as a system composed of a large number of lower dimensional structures, in particular thin vortex structures. The idea is well described for instance by A. Chorin in his book [10]. For the purpose of turbulence, the investigation of large families of filaments was related to statistical properties, as we shall recall below. But, in parallel to statistical investigations, one of the natural questions is the relation between these families of filaments and the equations of fluid dynamics, the Euler or Navier-Stokes equations. In dimension 2, it is known that a proper mean field limit of point vortices leads to the 2D Euler equation. In dimension 3 this is an open problem, see for instance [24]. Our mean field result here is a contribution in this direction. We do not solve the true fluid dynamic problem, since we cannot consider Biot-Savart kernel $K$ yet, but at least for relatively smooth kernels we show that the expected result holds true.

Having mentioned the link with fluid dynamics and works on vortex filaments, let us give more details and some references. As we have already said, the importance of thin vortex structures in 3D turbulence has been discussed intensively, especially after the striking simulations of [27]. While the situation in the two-dimensional case is pretty understood, this is not the case in the three-dimensional case. Chorin [10] has emphasized both the similarities and differences between statistical theories for heuristic models for ensembles of three-dimensional vortex filaments and the earlier two-dimensional statistical theories for point vortices.

Some computational methods based on vortex structures are introduced in [1,25] and the references therein, and, in a more computational settings, in [22]. In particular in [1], a new algorithm with very high order of accuracy and stability has been implemented in order to simulate three dimensional incompressible fluid flows. This algorithm couples a vortex blob method with a local segment approximation which incorporates in a Lagrangian fashion the local vortex stretching (this is in contrast with Chorin’s algorithm that was incorporated in an Eulerian fashion, see [10]). Let us mention that the approximations in these references are more in the spirit of numerical methods and that these discrete approximations are more similar to the two dimensional case. Moreover, no mean field results are mentioned.

Some probabilistic models of vortex filaments based on the paths of stochastic processes have been proposed in [12–14,16–19,24,26], and the investigation of their statistics in [15, 23]. The importance of these models for the statistics of turbulence or for the understanding of 3D Euler equations is of high importance. Let us mention that the existence and uniqueness of solutions for the dynamics of vortex filaments has been investigated in [2] and for a random vortex filament in [4,7,8] in the case of fractional Brownian motion. Of course, all the previous references mentioned deal with a smoothened version of the dynamics which is related to a mollified version of the Biot–Savart formula.

Statistical ensembles of vortex filaments arise many questions. One of them, approached with success by Onsager and subsequent authors in dimension 2, is the mean field limit of a dense collection of many interacting vortices. In dimension 3 this question has been investigated successfully by P. L. Lions and A. Majda. In [24], they develop the first mathematically
rigorous equilibrium statistical theory for three-dimensional vortex filaments in the context of a model involving simplified asymptotic equations for nearly parallel vortex filaments. Their equilibrium Gibbs ensemble is written down exactly through function space integrals; then a suitably scaled mean field statistical theory is developed in the limit of infinitely many interacting filaments. The mean field equations involved a novel Hartree-like problem. A similar approach has been used for stochastic vortex filaments in [5,6] where the Gibbs measure was based on a previous rigorous definition introduced in [13]. The mean field was proved to be solution of a variational formulation but given in an implicit form.

The present work is also a contribution to the investigation of models based on geometric objects of Hausdorff dimension 1, compared to the classical case of models based on objects of Hausdorff dimension 0. Families of deterministic and random curves, and their time-evolution, appear in different fields of Physics and Biology and may deserve more intense research. See [9] for a review of definitions, examples and problems.

1.5 Content of the Paper

Section 2 is devoted to the introduction of the space of currents (1-forms) endowed with its strong and weak topologies. The push forward of 1-currents is defined and some properties given. More details are given in the Appendix. In Sect. 3, Lagrangian current dynamics are introduced. A flow equation for the current is defined by taking the push forward of an initial current under the flow of diffeomorphisms generated by a general differential equation. The existence and uniqueness of maximal solutions for the flow are proved under some assumptions by means of a fixed point argument. Section 4 is devoted to the Eulerian current dynamics. In particular, we prove that the two formulations are equivalent. The well posedness of the Lagrangian formulation translates into the well posedness of the Eulerian formulation and viceversa. In Sect. 5 follows a discussion on global solutions for the flow equation and as a consequence for the PDE, since they are in one to one correspondence. This result is obtained under some more restrictive assumptions on the kernel $K$. In Sect. 6, continuous dependence on initial conditions is proved, that will be later used for proving a mean field result. A sequence of interacting curves (filaments) are defined in Sect. 7. These curves are solutions of a system of differential equations (with a scaling $\alpha N$), that describe our flow of diffeomorphism. Here we are using a smooth kernel which could be a mollified version of the Biot-Savart formula. To this family of curves, we associate a current defined in the vein of empirical measures. We prove a mean field result when the number of filaments $N \to \infty$. A similar result is also proved when the filaments are random. Moreover, under the assumptions that the family of filaments is symmetric, we deduce a result of propagation of chaos in Sect. 7.3. An example of such a family is given in Sect. 7.4. Finally in the Appendix we give some technical details on the completeness of the spaces of currents under the relevant norms used in the paper; we define the convolution of a current with a matrix-valued function; we give a description of currents centered on curves; and finally we analyze the push-forward of a vector field interpreted as a current.

2 Preliminaries on 1-Currents

Given $k, d, m \in \mathbb{N}$, we denote by $C^k_b(\mathbb{R}^d, \mathbb{R}^m)$ the space of all functions $f : \mathbb{R}^d \to \mathbb{R}^m$ that are of class $C^k$, bounded together with their derivatives of order up to $k$. By $UC^3_b(\mathbb{R}^d, \mathbb{R}^m)$, we denote the subset of $C^3_b(\mathbb{R}^d, \mathbb{R}^m)$ of those functions $f$ such that $f$, $Df$ and $D^2f$ are also uniformly continuous.
2.1 Generalities

Currents of dimension 1 (called 1-currents here) are linear continuous mappings on the space $C_0^\infty (\mathbb{R}^d, \mathbb{R}^d)$ of smooth compact support vector fields of $\mathbb{R}^d$, see for instance [20, 21]. Thereafter, we shall only consider 1-currents which are continuous in the $C_b (\mathbb{R}^d, \mathbb{R}^d)$ topology. On the space $C_b (\mathbb{R}^d, \mathbb{R}^d)$ of continuous and bounded vector fields on $\mathbb{R}^d$, denote the uniform topology by $\| \cdot \|_\infty$. Throughout the paper we shall always deal with the following Banach space of 1-currents:

$$\mathcal{M} := C_b (\mathbb{R}^d, \mathbb{R}^d).$$

The topology induced by the duality will be denoted by $| \cdot |_\mathcal{M}$:

$$| \xi |_{\mathcal{M}} := \sup_{\| \theta \|_\infty \leq 1} | \xi (\theta) |.$$

We are interested in the weak topology too, essential to deal with approximation by “filaments”. We define

$$\| \xi \| = \sup \{ | \xi (\theta) | \mid \| \theta \|_\infty + \text{Lip} (\theta) \leq 1 \}$$

where Lip(\theta) is the Lipschitz constant of \(\theta\). We set

$$d (\xi, \xi') = \| \xi - \xi' \|$$

for all $\xi, \xi' \in \mathcal{M}$. The number $\| \xi \|$ is well defined and

$$\| \xi \| \leq | \xi |_{\mathcal{M}}$$

and $d (\xi, \xi')$ satisfies the conditions of a distance. Convergence in the metric space $(\mathcal{M}, d)$ corresponds to weak convergence in $\mathcal{M}$ as dual to $C_b (\mathbb{R}^d, \mathbb{R}^d)$.

We shall denote by $\mathcal{M}_w$ the space $\mathcal{M}$ endowed by the metric $d$.

The unit ball in $(\mathcal{M}, | \cdot |_{\mathcal{M}})$ is complete with respect to $d$, see Appendix.

2.2 Push-Forward

In this section we recall the definition of push-forward of a current with respect to a differentiable function and we show how the push-forward is defined in the case of specific currents, i.e. vector field and curves, since we will deal with these objects later in the paper.

Let $\theta \in C_b (\mathbb{R}^d, \mathbb{R}^d)$ be a vector field (test function) and $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ be a map. When defined, the pull-back of $\theta$ is

$$\left( \varphi_\# \theta \right) (x) = D \varphi (x)^T \theta (\varphi (x)) .$$

If $\varphi$ is of class $C^1 (\mathbb{R}^d, \mathbb{R}^d)$, then $\varphi_\#$ is a well defined bounded linear map from $C_b (\mathbb{R}^d, \mathbb{R}^d)$ to itself.

Given a 1-current $\xi \in \mathcal{M}$ and a smooth map $\varphi : \mathbb{R}^d \to \mathbb{R}^d$, recall that the push-forward $\varphi_\# \xi$ is defined as the current

$$\left( \varphi_\# \xi \right) (\theta) := \xi (\varphi_\# \theta), \quad \theta \in C_b (\mathbb{R}^d, \mathbb{R}^d).$$

In the (see “1-Currents Associated with Curves” and “1-currents Associated with Vector Fields” sections of Appendix) we deal with the push-forward of currents associated with curves or vector fields respectively.
3 Lagrangian Current Dynamics

In order to prove that the nonlinear vector-valued PDE (3) with initial condition \(\xi_0 \in \mathcal{M}\), has unique local solutions in the space of currents, we adopt a Lagrangian point of view: we examine the ordinary differential equation

\[
\frac{d\xi_t}{dt} = (K \ast \xi_t) (x_t),
\]

consider the flow of diffeomorphisms \(\varphi^t_{K \ast \xi}\) generated by it and take the push forward of \(\xi_0\) under this flow:

\[
\xi_t = \varphi^t_{K \ast \xi} \ast \xi_0, \quad t \in [0, T].
\]

The pair of equations (9)–(10) defines a closed system for \((\xi_t)_{t \in [0, T]}\) which, for small \(T\), has a unique solution. We shall prove then that current-valued solutions of the PDE (3) are in one-to-one correspondence with current-valued solutions of the flow system (9)–(10) and thus we get local existence and uniqueness for (3).

Since the specific linear form \(K \ast \xi_t\) for the drift of equation (9) is irrelevant, we replace it with a more general, possibly non-linear, map. Thus we investigate a “flow equation” of the form

\[
\xi_t = \varphi^t_{B(\xi)} \ast \xi_0, \quad t \in [0, T]
\]

where \(B(\xi_t)\) is a time-dependent vector field in \(\mathbb{R}^d\), associated to the time-dependent current \(\xi_t\), and \(\varphi^t_{B(\xi)}\) is the flow associated to \(B(\xi)\) by the equation

\[
\frac{dx_t}{dt} = B(\xi_t)(x_t).
\]

3.1 Assumptions on the Drift

Let us discuss the general assumptions that we impose on the drift \(B\) of equation (11). We assume

\[
B : \mathcal{M}_w \to C^2_b \left(\mathbb{R}^d, \mathbb{R}^d\right)
\]

to be a continuous map such that for every \(\xi, \xi' \in \mathcal{M}\)

\[
\|B(\xi)\|_{C^2_b} \leq C_B \left(\|\xi\| + 1\right)
\]

(12)

\[
\|B(\xi) - B(\xi')\|_{\infty} \leq C_B \|\xi - \xi'\|
\]

(13)

\[
\|DB(\xi) - DB(\xi')\|_{\infty} \leq C_B \|\xi - \xi'\|
\]

(14)

We denote by \(DB\) and \(D^2B\) the derivatives of \(B\) in the \(x \in \mathbb{R}^3\) variable.

Our main example of \(B\) is the linear function

\[
B(\xi) = K \ast \xi
\]

where \(K : \mathbb{R}^d \to \mathbb{R}^{d \times d}\) [see (45)]. The necessary regularity of \(K\) is specified by the next lemma.

**Lemma 5** Let \(K \in \mathcal{UC}^3_b(\mathbb{R}^d, \mathbb{R}^{d \times d})\). Then \(B(\xi) = K \ast \xi\) maps continuously \(\mathcal{M}\) in to \(C^2_b \left(\mathbb{R}^d, \mathbb{R}^d\right)\) and satisfies assumptions (12)–(14).
Proof Since $K \in C_b(\mathbb{R}^d, \mathbb{R}^{d \times d})$, $K \ast \xi : \mathbb{R}^d \to \mathbb{R}^d$ is a well defined function, for every $\xi \in \mathcal{M}$. From (45) and the uniform continuity of $K$ it follows that $K \ast \xi$ is a continuous function: if $x_n \to x$, then $K_i \ast (x_n - \cdot) \to K_i \ast (x - \cdot)$ uniformly, for every $i = 1, \ldots, d$. It is bounded, since

$$\left| (K \ast \xi)_i (x) \right| \leq \|\xi\| (\|K\|_\infty + \|DK\|_\infty)$$

Moreover, the linear map $B : \mathcal{M}_w \to C_b(\mathbb{R}^d, \mathbb{R}^d)$, just defined is continuous in the weak topology of $\mathcal{M}$: from the previous inequality it follows

$$\|(K \ast \xi)_i - (K \ast \xi')_i\|_\infty \leq \|\xi - \xi'\| (\|K\|_\infty + \|DK\|_\infty).$$

Let us show that all the same facts extend to the first derivatives of $K \ast \xi$. Since $DK$ is uniformly continuous and bounded, from

$$\left| K_{ij}(x + \epsilon h) - K_{ij}(x) \frac{\epsilon}{\epsilon} - DK_{ij}(x) \cdot h \right| = \left| \int_0^1 \epsilon DK_{ij}((1 - \alpha)x + \alpha(x + \epsilon h)) \cdot \epsilon h \, d\alpha - \int_0^1 DK_{ij}(x) \cdot h \, d\alpha \right|$$

$$\leq \int_0^1 \left| DK_{ij}((1 - \alpha)x + \alpha(x + \epsilon h)) - DK_{ij}(x) \right| \, d\alpha$$

it follows that the incremental ratio of $K_{ij}$ in a direction $h$ converges uniformly to $DK_{ij} \cdot h$. From

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \xi(K_i \ast (x + \epsilon h - \cdot)) - \xi(K_i \ast (x - \cdot)) = \lim_{\epsilon \to 0} \xi \left( \frac{K_i \ast (x + \epsilon h - \cdot) - K_i \ast (x - \cdot)}{\epsilon} \right)$$

it follows that $K \ast \xi$ is differentiable at every point and

$$D (K \ast \xi)_i (x) \cdot h = \xi (DK_{ij}(x) \cdot h).$$

The arguments now are similar to those already exposed above and iterate: this first derivatives are continuous bounded functions and $B : \mathcal{M}_w \to C^1_b(\mathbb{R}^d, \mathbb{R}^d)$ is continuous.

Iterating again, based on the uniform continuity of $D^2 K$ and the property $K \in C^3_b$, we get that $B : \mathcal{M}_w \to C^2_b(\mathbb{R}^d, \mathbb{R}^d)$ is well defined and continuous. Property (12) comes from (15) and the similar inequalities for first and second derivatives; the last one requires $K$ of class $C^3_b$. Property (13) follows from (15). Finally, property (14) is proved similarly, using the analogous bound on the second derivative.

\[ \square \]

3.2 Properties of the Flow

For any $b \in C \left( [0, T]; C^2_b(\mathbb{R}^d, \mathbb{R}^d) \right)$, consider the ODE in $\mathbb{R}^d$

$$X'_t = b(t, X_t)$$

and denote by $\varphi^{t,b}(x)$ the associated flow. It is differentiable and

$$\frac{d}{dt} \varphi^{t,b} (x) = Db \left( t, \varphi^{t,b}(x) \right) D\varphi^{t,b} (x), \quad D\varphi^{0,b} (x) = Id.$$
The computations in the proof of the following lemma are classical; however, it is important for Theorem 10 below that we carefully make the estimates (16) and (17) depend only on one of the two current-valued processes, say \( \xi \); this asymmetric dependence is less obvious, although common to other problems like the theorems of weak-strong uniqueness.

**Lemma 6** If \( \xi \in C ([0, T]; \mathcal{M}_w) \), then the flow \( \varphi^{t,B(\xi)} : \mathbb{R}^d \to \mathbb{R}^d \) is twice differentiable and satisfies, for all \( t \in [0, T] \),

\[
\left\| D\varphi^{t,B(\xi)} \right\|_{\infty} \leq e^{C_B T(\|\xi\|_{T}+1)} \tag{16}
\]

\[
\left\| \varphi^{t,B(\xi)} - \varphi^{t,B(\xi')} \right\|_{\infty} \leq C_B T e^{C_B T(\|\xi\|_{T}+1)} \| \xi - \xi' \|_T \tag{17}
\]

and for every \( x \in \mathbb{R}^d \)

\[
\left\| D\varphi^{t,B(\xi)} - D\varphi^{t,B(\xi')} \right\|_{\infty} \leq C_B T e^{C_B (T+1)(\|\xi\|_{T}+\|\xi'\|_{T})+2} \left( 1 + C_B T(\|\xi\|_{T}+1)e^{C_B T(\|\xi\|_{T}+1)} \right) \| \xi - \xi' \|_T \tag{18}
\]

Moreover, for every \( x, y \in \mathbb{R}^d \),

\[
|D\varphi^{t,B(\xi)}(x) - D\varphi^{t,B(\xi)}(y)| \leq C_B T (\|\xi\|_{T}+1) e^{C_B (2T+1)(\|\xi\|_{T}+1)} |x - y|. \tag{19}
\]

**Proof** We have, from \( \frac{d}{dt} D\varphi^{t,B(\xi)}(x) = D B (\xi_t) \left( \varphi^{t,B(\xi)}(x) \right) D\varphi^{t,B(\xi)}(x), \)

\[
\left\| D\varphi^{t,B(\xi)}(x) \right\| \leq e^{\int_t^0 \| D B (\xi_s) \|_{\infty} ds} \]

\[
\left\| D\varphi^{t,B(\xi)} \right\|_{\infty} \leq e^{\int_t^0 \| D B (\xi_s) \|_{\infty} ds}. \]

Now, using the assumption (12) on \( B \) we get (16).

For the estimate (17), notice that

\[
\frac{d}{dt} \left( \varphi^{t,B(\xi)}(x) - \varphi^{t,B(\xi')}(x) \right) = B (\xi_t) \left( \varphi^{t,B(\xi)}(x) \right) - B (\xi'_t) \left( \varphi^{t,B(\xi)}(x) \right)
\]

\[
= B (\xi_t) \left( \varphi^{t,B(\xi)}(x) \right) - B (\xi_t) \left( \varphi^{t,B(\xi')}(x) \right)
\]

\[
+ B (\xi'_t) \left( \varphi^{t,B(\xi')}(x) \right) - B (\xi'_t) \left( \varphi^{t,B(\xi')}(x) \right)
\]

hence

\[
\left| \varphi^{t,B(\xi)}(x) - \varphi^{t,B(\xi')}(x) \right| \leq \int_0^t \| D B (\xi_s) \|_{\infty} \left| \varphi^{s,B(\xi)}(x) - \varphi^{s,B(\xi')}(x) \right| ds
\]

\[
+ \int_0^t \| B (\xi_s) - B (\xi'_s) \|_{\infty} ds.
\]

Hence, using Gronwall’s Lemma we get that

\[
\left| \varphi^{t,B(\xi)}(x) - \varphi^{t,B(\xi')}(x) \right| \leq \int_0^t \| B (\xi_s) - B (\xi'_s) \|_{\infty} e^{\int_s^t \| DB (\xi_r) \|_{\infty} dr} ds.
\]

Now, using again assumptions (12) and (13), we deduce (17).
Now, let us prove (18). Let us notice that
\[
\frac{d}{dt} \left( D\varphi^{t,B}(\xi) (x) - D\varphi^{t,B}(\xi') (x) \right) \\
= DB (\xi) \left( \varphi^{t,B}(\xi) (x) \right) D\varphi^{t,B}(\xi) (x) - DB (\xi') \left( \varphi^{t,B}(\xi') (x) \right) D\varphi^{t,B}(\xi') (x) \\
= DB (\xi) \left( \varphi^{t,B}(\xi) (x) \right) D\varphi^{t,B}(\xi) (x) - DB (\xi) \left( \varphi^{t,B}(\xi) (x) \right) D\varphi^{t,B}(\xi) (x) \\
+ DB (\xi) \left( \varphi^{t,B}(\xi) (x) \right) D\varphi^{t,B}(\xi) (x) - DB (\xi') \left( \varphi^{t,B}(\xi) (x) \right) D\varphi^{t,B}(\xi') (x) .
\]
For the first term,
\[
\left| DB (\xi) \left( \varphi^{t,B}(\xi) (x) \right) D\varphi^{t,B}(\xi) (x) - DB (\xi') \left( \varphi^{t,B}(\xi) (x) \right) D\varphi^{t,B}(\xi') (x) \right| \\
\leq \| DB (\xi') \|_{\infty} \left| D\varphi^{t,B}(\xi) (x) - D\varphi^{t,B}(\xi') (x) \right| .
\]
For the second term,
\[
\left| DB (\xi) \left( \varphi^{t,B}(\xi) (x) \right) D\varphi^{t,B}(\xi) (x) - DB (\xi) \left( \varphi^{t,B}(\xi) (x) \right) D\varphi^{t,B}(\xi') (x) \right| \\
\leq \left| DB (\xi) \left( \varphi^{t,B}(\xi) (x) \right) - DB (\xi') \left( \varphi^{t,B}(\xi) (x) \right) \right| \left| D\varphi^{t,B}(\xi') (x) \right| \\
\leq \left| DB (\xi) \left( \varphi^{t,B}(\xi) (x) \right) - DB (\xi) \left( \varphi^{t,B}(\xi) (x) \right) \right| \left| D\varphi^{t,B}(\xi') (x) \right| \\
+ \left| DB (\xi) \left( \varphi^{t,B}(\xi) (x) \right) - DB (\xi) \left( \varphi^{t,B}(\xi) (x) \right) \right| \left| D\varphi^{t,B}(\xi') (x) \right| \\
\leq \|| DB (\xi) - DB (\xi') \|_{\infty} \left| D\varphi^{t,B}(\xi) (x) \right| \left| D\varphi^{t,B}(\xi') (x) \right| .
\]
Hence, using assumption (12), (14) and the estimates (16) and (17), we get that
\[
\left| D\varphi^{t,B}(\xi) (x) - D\varphi^{t,B}(\xi') (x) \right| \\
\leq CB (\| \xi \|_{T} + 1) \int_{0}^{t} \left| D\varphi^{s,B}(\xi) (x) - D\varphi^{s,B}(\xi') (x) \right| ds \\
+ CB T e^{C_B T (\| \xi \|_{T} + 1)} \left( 1 + C_B T (\| \xi \|_{T} + 1) e^{C_B T (\| \xi \|_{T} + 1)} \right) \| \xi - \xi' \|_{T}
\]
which implies, by Gronwall’s Lemma,
\[
\left| D\varphi^{t,B}(\xi) (x) - D\varphi^{t,B}(\xi') (x) \right| \\
\leq e^{TC_B (\| \xi \|_{T} + 1)} CB T e^{C_B T (\| \xi \|_{T} + 1)} \left( 1 + C_B T (\| \xi \|_{T} + 1) e^{C_B T (\| \xi \|_{T} + 1)} \right) \| \xi - \xi' \|_{T} .
\]
It is left to prove (19).
\[
\left| D\varphi^{t,B}(\xi) (x) - D\varphi^{t,B}(\xi) (y) \right| \\
\leq \int_{0}^{t} \left| DB (\xi) (\varphi^{s,B}(\xi) (x)) D\varphi^{t,B}(\xi) (x) - DB (\xi) (\varphi^{s,B}(\xi) (y)) D\varphi^{t,B}(\xi) (y) \right| ds \\
\leq \int_{0}^{t} \left| DB (\xi) (\varphi^{s,B}(\xi) (x)) D\varphi^{s,B}(\xi) (x) - DB (\xi) (\varphi^{s,B}(\xi) (y)) D\varphi^{s,B}(\xi) (y) \right| ds \\
+ \left| DB (\xi) (\varphi^{s,B}(\xi) (x)) D\varphi^{s,B}(\xi) (y) - DB (\xi) (\varphi^{s,B}(\xi) (y)) D\varphi^{s,B}(\xi) (y) \right| ds .
\]
\[ \leq \sup_{s \in [0, t]} \| DB(\xi_s) \|_\infty \int_0^t \left| D\phi^{t, B(\xi)}(x) - D\phi^{t, B(\xi)}(y) \right| ds \\
+ t \sup_{s \in [0, t]} \left( \left\| D\phi^{t, B(\xi)}(\xi) \right\|_\infty \right)^2 \| DB(\xi_s) \|_\infty \| x - y \|. \]

We now apply Gronwall’s Lemma and we get
\[ |D\phi^{t, B(\xi)}(x) - D\phi^{t, B(\xi)}(y)| \leq T \sup_{s \in [0, T]} \left( \left\| D\phi^{t, B(\xi)}(\xi) \right\|_\infty \right)^2 \| DB(\xi_s) \|_\infty \| x - y \|. \]

Now, using (6) and (12) we get (19).

\[ \Box \]

3.3 Well Posedness of the Flow Equation

We are now ready to consider the closed loop \( \xi \mapsto \phi^{t, B(\xi)} \mapsto \xi_t := \phi^{t, B(\xi)}(\xi_0), \) namely the equation:
\[ \xi_t = \phi^{t, B(\xi)}(\xi_0), \quad t \in [0, T]. \]  

(20)

Let us prove it has a unique solution in the space \( C([0, T]; \mathcal{M}) \) by using a fixed point argument. Indeed, let \( \xi_0 \in \mathcal{M} \) be the initial current, at time \( t = 0 \). Given \( \xi = (\xi_t)_{t \in [0, T]} \in C([0, T]; \mathcal{M}_w) \), let \( \Gamma(\xi) = \eta = (\eta_t)_{t \in [0, T]} \) be the time-dependent current defined as
\[ \eta_t = \phi^{t, B(\xi)}(\xi_0), \quad t \in [0, T]. \]

Lemma 7 Given \( \xi_0 \in \mathcal{M} \), set \( R = 2 |\xi_0|_{\mathcal{M}} \). Then there exists \( T_R^0 > 0 \), depending only on \( R \), such that \( \Gamma(B_R) \subset B_R \), where \( B_R \) is the set of all \( \xi = (\xi_t)_{t \in [0, T]} \in C([0, T]; \mathcal{M}_w) \) such that \( \sup_{t \in [0, T]} |\xi_t|_{\mathcal{M}} \leq R \). Similarly if \( B_R \) is defined by the norm \( C([0, T]; \mathcal{M}_w) \), \( \mathcal{M}_w \) is endowed with the norm \( ||\cdot|| \).

Proof First we prove the first statement. To do it, we must estimate the strong norm of \( \eta = \Gamma(\xi) \):
\[ \left| \phi^{t, B(\xi)}(\xi_0)(\theta) \right| = \left| \xi_0 \left( \phi^{t, B(\xi)}(\xi_0)(\theta) \right) \right| \leq |\xi_0|_{\mathcal{M}} \left\| \phi^{t, B(\xi)}(\xi_0)(\theta) \right\|_\infty \\
= |\xi_0|_{\mathcal{M}} \left\| D\phi^{t, B(\xi)}(\cdot) \right\|_\infty \left\| \theta \right\|_\infty \]
\[ \leq |\xi_0|_{\mathcal{M}} \left\| D\phi^{t, B(\xi)}(\cdot) \right\|_\infty \left\| \theta \right\|_\infty \]

which implies that
\[ \sup_{t \in [0, T]} |\eta_t|_{\mathcal{M}} \leq |\xi_0|_{\mathcal{M}} \sup_{t \in [0, T]} \left\| D\phi^{t, B(\xi)}(\cdot) \right\|_\infty. \]

(21)

Using (16) and \( ||\xi_0|| \leq |\xi_0|_{\mathcal{M}} = \frac{R}{2} \), we get that
\[ \sup_{t \in [0, T]} |\eta_t|_{\mathcal{M}} \leq \frac{R}{2} e^{C_B(R+1)T}. \]

If \( T \) satisfies \( e^{C_B(R+1)T} \leq 2 \), we get \( \Gamma(B_R) \subset B_R \) and the proof is complete.

To prove the second statement we first see, from the definition of the norm \( ||\cdot|| \), that it holds
\[ \sup_{t \in [0, T]} \left| \eta_t \right| \leq \sup_{t \in [0, T]} \left\| \phi^{t, B(\xi)}(\xi_0) \right\| = \sup_{t \in [0, T]} \sup_{\eta \in [0, T]} \left\| \xi_0(\phi^{t, B(\xi)}(\xi_0)(\theta)) \right\| \left\| \theta \right\|_\infty + Lip(\theta) \leq 1 \].

\[ \Box \]
Now, proceeding as in the previous part, we estimate $|\xi_0(\varphi_{\#}^t B(\xi) \theta)|$ and the proof follows similarly.

\textbf{Theorem 8} For every $\xi_0 \in \mathcal{M}$, there is $T_R > 0$, depending only on $|\xi_0|\mathcal{M}$, such that there exists a unique solution $\xi$ of the flow equation (20) in $C([0,T_R];\mathcal{M})$.

\textbf{Proof} \textbf{Step 1.} Let $R = 2 |\xi_0|\mathcal{M}$ and $T_R^0$ be given by the previous lemma; let $T \in [0,T_R^0]$ to be chosen below. Let $\xi,\xi' \in C([0,T];\mathcal{M}_w)$, $\eta = \Gamma (\xi)$, $\eta' = \Gamma (\xi')$. We have, for every Lipschitz function $\theta$,

$$
|\eta_t - \eta'_t(\theta)| \leq |\xi_0|\mathcal{M}\|\varphi_{\#}^{t,B(\xi)} \theta - \varphi_{\#}^{t,B(\xi')} \theta\|_\infty
$$

(22)

Now

$$
\|\varphi_{\#}^{t,B(\xi)} \theta - \varphi_{\#}^{t,B(\xi')} \theta\|_\infty
$$

$$
= \|D\varphi_{\#}^{t,B(\xi)} (\cdot)^T \theta (\varphi_{\#}^{t,B(\xi)} (\cdot)) - D\varphi_{\#}^{t,B(\xi')} (\cdot)^T \theta (\varphi_{\#}^{t,B(\xi')} (\cdot))\|_\infty
$$

$$
\leq \|D\varphi_{\#}^{t,B(\xi)} (\cdot)^T \theta (\varphi_{\#}^{t,B(\xi)} (\cdot)) - D\varphi_{\#}^{t,B(\xi')} (\cdot)^T \theta (\varphi_{\#}^{t,B(\xi')} (\cdot))\|_\infty
$$

$$
+ \|D\varphi_{\#}^{t,B(\xi)} (\cdot)^T \theta (\varphi_{\#}^{t,B(\xi')} (\cdot)) - D\varphi_{\#}^{t,B(\xi')} (\cdot)^T \theta (\varphi_{\#}^{t,B(\xi')} (\cdot))\|_\infty
$$

$$
\leq \|D\varphi_{\#}^{t,B(\xi)}\|_\infty \|D\theta\|_\infty \|\varphi_{\#}^{t,B(\xi)} - \varphi_{\#}^{t,B(\xi')}\|_\infty
$$

$$
+ \|\theta\|_\infty \|D\varphi_{\#}^{t,B(\xi)} - D\varphi_{\#}^{t,B(\xi')}\|_\infty.
$$

By definition, $\|\eta_t - \eta'_t\|$ is less than or equal to the supremum of (22), taken over the Lipschitz functions $\theta$ such that $\|\theta\|_\infty + \text{Lip}(\theta) \leq 1$. Hence,

$$
\sup_{t \in [0,T]} \|\eta_t - \eta'_t\|
$$

$$
\leq |\xi_0|\mathcal{M}\sup_{t \in [0,T]} \|D\varphi_{\#}^{t,B(\xi)}\|_\infty \|\varphi_{\#}^{t,B(\xi)} - \varphi_{\#}^{t,B(\xi')}\|_\infty
$$

$$
+ |\xi_0|\mathcal{M}\sup_{t \in [0,T]} \|D\varphi_{\#}^{t,B(\xi)} - D\varphi_{\#}^{t,B(\xi')}\|_\infty
$$

Hence, using (16)

$$
\sup_{t \in [0,T]} \|\eta_t - \eta'_t\|
$$

$$
\leq e^{CBT(R+1)} |\xi_0|\mathcal{M}\sup_{t \in [0,T]} \|\varphi_{\#}^{t,B(\xi)} - \varphi_{\#}^{t,B(\xi')}\|_\infty
$$

$$
+ |\xi_0|\mathcal{M}\sup_{t \in [0,T]} \|D\varphi_{\#}^{t,B(\xi)} - D\varphi_{\#}^{t,B(\xi')}\|_\infty.
$$

\textbf{Step 2.} Using the estimates given in Lemma 6 and summarizing

$$
\sup_{t \in [0,T]} \|\eta_t - \eta'_t\|
$$

$$
\leq CBTE^{2CBT(R+1)} |\xi_0|\mathcal{M}\sup_{t \in [0,T]} \|\xi_t - \xi'_t\|
$$
Therefore, for $T$ small enough, $\Gamma$ is a contraction in $C([0, T]; \mathcal{M}_w)$.

Recall now Lemma 27. The space of currents of class $C([0, T]; \mathcal{M}_w)$ with $sup_{t \in [0, T]} |\xi_t|_\mathcal{M} \leq R$ is a complete metric space, and $\Gamma$ is a contraction in this space, for $T$ small enough. It follows that there exists a unique fixed point of $\Gamma$ in this space. Finally, the fixed point is also in $C([0, T]; \mathcal{M})$ since the output of the push forward is in this space. \hfill \Box

### 3.4 Maximal Solutions

Given $\xi_0 \in \mathcal{M}$, let $\Upsilon_{\xi_0}$ be the set of all $T > 0$ such that there exists a unique current-valued solution on $[0, T]$ for the flow equation (20) with initial condition $\xi_0$. Let $T_{\xi_0} = sup_{\xi_0} \Upsilon_{\xi_0}$; on $[0, T_{\xi_0})$ a unique solution $\xi$ is well defined; it is called the maximal solution. We have $\xi \in C([0, T_{\xi_0}); \mathcal{M})$. An easy fact is:

**Lemma 9** If $T_{\xi_0} < \infty$, then

$$\lim_{t \rightarrow T_{\xi_0}^-} |\xi_t|_\mathcal{M} = +\infty.$$

**Proof** We prove the claim by contradiction. Assume there is a sequence $t_n \rightarrow T_{\xi_0}^{-}$ and a constant $C > 0$ such that $|\xi_{t_n}|_\mathcal{M} \leq C$ for every $n \in \mathbb{N}$. We may apply the existence and uniqueness theorem on the time interval $[t_n, t_n + T]$ where $T$ depends only on $C$. Hence a unique solution exists up to time $t_n + T$. For large $n$ this contradicts the definition of $T_{\xi_0}$, if it is finite. \hfill \Box

Taking into account that we only have $\|\xi\| \leq |\xi|_\mathcal{M}$, we have the following interesting criterion.

**Theorem 10** If $T_{\xi_0} < \infty$, then

$$\lim_{t \rightarrow T_{\xi_0}^-} \|\xi_t\| = +\infty.$$

**Proof** For $t \in [0, T_{\xi_0})$ we have (the proof is the same as estimate (21) in Lemma 7)

$$|\xi_t|_\mathcal{M} \leq |\xi_0|_\mathcal{M} \left\| D\varphi^{t, B(\xi)} \right\|_\infty \leq |\xi_0|_\mathcal{M} e^{C_B t(\|\xi\|_L + 1)}$$

having used (16) in the last step. Hence ($\xi_0 \neq 0$, otherwise $T_{\xi_0} = +\infty$), for $t \in (0, T_{\xi_0})$

$$\|\xi\| \geq \frac{1}{C_B t} \log \frac{|\xi_t|_\mathcal{M}}{|\xi_0|_\mathcal{M}} - 1.$$  

From $\lim_{t \rightarrow T_{\xi_0}^-} |\xi_t|_\mathcal{M} = +\infty$ it follows $\lim_{t \rightarrow T_{\xi_0}^-} \|\xi_t\| = +\infty$. \hfill \Box

### 4 Eulerian Current Dynamics

Given an operator $B$ with the assumptions exposed at the beginning of Section 3.2, and taking values in the set of divergence free vector fields, consider the non-linear PDE

$$\begin{cases} \frac{\partial \xi}{\partial t} + \text{div} (B(\xi) \times \xi) = (\xi \cdot \nabla)B(\xi) \\ \xi(0) = \xi_0. \end{cases}$$ (24)
**Definition 11** We say that $\xi \in C([0, T]; \mathcal{M})$ is a current-valued solution for the PDE (24) if for every $\theta \in C^1_b(\mathbb{R}^d; \mathbb{R}^d)$ and every $t \in [0, T]$, it satisfies

$$
\xi_t(\theta) - \int_0^t \xi_s (D \theta \cdot B(\xi_s)) \, ds = \xi_0(\theta) + \int_0^t \xi_s (DB(\xi_s)^T \cdot \theta) \, ds.
$$

The definition on an open interval $[0, T)$ (possibly infinite) is similar. The aim of this section is to prove the following result.

**Theorem 12** Given $\xi_0 \in \mathcal{M}$, there exists a unique current-valued solution for the PDE (24), defined on a maximal interval $[0, T_{\xi_0})$. It is given by the unique maximal current-valued solution of the flow equation (20).

The proof consists in proving that $\xi \in C([0, T_{\xi_0}); \mathcal{M})$ is a current-valued solution for the PDE (24) if and only if it is a solution of the flow equation (20); when this is done, all statements of the theorem are proved, because we already know that Eq. (20) has a unique local solution in the space of currents.

In order to prove the previous claim of equivalence between (24) and (20) consider, for given $T > 0$ and $b \in C\left([0, T]; C^2_b(\mathbb{R}^d, \mathbb{R}^d)\right)$, the auxiliary linear PDE

$$
\begin{cases}
\frac{d\xi}{dt} + \text{div}(b \xi) = (\xi \cdot \nabla)b \\
\xi(0) = \xi_0
\end{cases}
$$

(26)

The definition of solution is analogous to the nonlinear case, just replacing $B(\xi)$ by $b$. For this equation we shall prove:

**Lemma 13** A function $\xi \in C([0, T]; \mathcal{M})$ is a current-valued solution for the PDE (26) if and only if it is given by

$$
\xi_t = \varphi^{t,b}_\xi \xi_0.
$$

The proof of this lemma occupies the next two subsections. When this result is reached, we can prove Theorem 12 with the following simple argument: if $\xi \in C([0, T]; \mathcal{M})$ is a current-valued solution for the PDE (24), then it is a current-valued solution for the PDE (26) with $b := B(\xi)$, hence $\xi_t = \varphi^{t,b}_\xi \xi_0 = \varphi^{t,B(\xi)}_\xi \xi_0$, namely $\xi$ solves the flow equation (20). Conversely, if $\xi$ solves the flow equation (20), namely $\xi_t = \varphi^{t,B(\xi)}_\xi \xi_0$, setting $b := B(\xi)$ we have that $\xi_t = \varphi^{t,b}_\xi \xi_0$, hence by the lemma it solves the PDE (26) with $b = B(\xi)$, hence it solves (24).

**4.1 From the Flow to the PDE**

In this subsection we prove one half of Lemma 13, precisely that $\xi_t$ defined by $\xi_t = \varphi^{t,b}_\xi \xi_0$ is a current-valued solution of the PDE (26). Let $\theta$ be a test function, so that

$$
\xi_t(\theta) = \varphi^{t,b}_\xi \xi_0(\theta) = \xi_0 \left( \varphi^{t,b}_\xi \theta \right) = \xi_0 \left( DB^{t,b}(\cdot)^T \theta \left( \varphi^{t,b}(\cdot) \right) \right).
$$
Hence (the time derivative commutes with $\xi_0$ since $\xi_0$ is linear continuous)

$$\frac{d}{dt}\xi_t(\theta) = \frac{d}{dt}\xi_0 \left( D\varphi^{t,b}_t(\cdot)^T \theta \left( \varphi^{t,b}_t(\cdot) \right) \right)$$

$$= \xi_0 \left( \frac{d}{dt} \left( D\varphi^{t,b}_t(\cdot)^T \theta \left( \varphi^{t,b}_t(\cdot) \right) \right) \right) + \xi_0 \left( D\varphi^{t,b}_t(\cdot)^T \frac{d}{dt} \left( \theta \left( \varphi^{t,b}_t(\cdot) \right) \right) \right)$$

$$= \xi_0 \left( D\varphi^{t,b}_t(\cdot)^T D\varphi_t(\varphi^{t,b}_t(\cdot))^T \theta \left( \varphi^{t,b}_t(\cdot) \right) \right)$$

$$+ \xi_0 \left( D\varphi^{t,b}_t(\cdot)^T D\theta(\varphi^{t,b}_t(\cdot)) \frac{d}{dt} \left( \varphi^{t,b}_t(\cdot) \right) \right)$$

$$= \xi_0 \left( D\varphi^{t,b}_t(\cdot)^T D\varphi_t(\varphi^{t,b}_t(\cdot))^T \theta \left( \varphi^{t,b}_t(\cdot) \right) \right) + \xi_0 \left( D\varphi^{t,b}_t(\cdot)^T D\theta(\varphi^{t,b}_t(\cdot)) b_t(\varphi^{t,b}_t(\cdot)) \right)$$

$$= I_1 + I_2$$

we have

$$I_1 = \xi_0 \left( \varphi^{t,b}_t(\cdot)^T \left( D\varphi_t(\theta) \right) \right) = \varphi^{t,b}_t(\cdot)^T \xi_0(D\varphi_t(\theta)) = \xi_t(D\varphi_t(\theta)).$$

And

$$I_2 = \xi_0 \left( \varphi^{t,b}_t(\cdot)^T D\varphi_t(b) \right) = \varphi^{t,b}_t(\cdot)^T \xi_0(D\varphi_t(b)) = \xi_t(D\varphi_t(b)).$$

Hence

$$\frac{d}{dt}\xi_t(\theta) = \xi_t(D\varphi_t(\theta)) + \xi_t(D\varphi_t(b)).$$

This is Eq. (25) (with $b$ in place of $B(\xi)$), which completes the proof.

### 4.2 From the PDE to the Flow

In this subsection we prove the other half of Lemma 13: if $\xi$ is a current-valued solution of the PDE (26), then $\xi_t = \varphi^{t,b}_t(\cdot)^T \xi_0$. To prove this we assume that $\text{div}(K) = 0$, this implies that also $b$ is divergence free and that the Jacobian of the flow, $D\varphi^{t,b}_t$, is a unitary matrix.

Since the computation, by means of regularizations and commutator lemma, may obscure the underlying argument, let us first provide the proof in the case smooth fields. In such a case, from Proposition 29 we have (we denote $\varphi^{t,b}_t$ by $\varphi_t$ for simplicity)

$$\xi_t(x) = D\varphi_t(\varphi^{-1}_t(x))\xi_0(\varphi^{-1}_t(x))$$

which is equivalent to

$$\frac{\partial}{\partial t} \left[ (D\varphi_t)^{-1}(x)\xi_t(\varphi_t(x)) \right] = 0. \tag{27}$$

To compute this derivative we will make use of the following rule

$$d(D\varphi_t)^{-1} = -(D\varphi_t)^{-1} D\varphi_t.$$ 

Let us compute the derivative (27),

$$\frac{\partial}{\partial t} \left[ (D\varphi_t)^{-1}(x)\xi_t(\varphi_t(x)) \right] = (D\varphi_t)^{-1} \left[ -D\varphi_t(\varphi_t(\cdot))\xi_t(\varphi_t) + \partial_t \xi_t(\varphi_t) + \xi_t(\varphi_t) b_t(\varphi_t) \right]$$

$$= (D\varphi_t)^{-1} \left[ -(\xi_t \cdot \nabla) b_t + \partial_t \xi_t + (b_t \cdot \nabla)\xi_t \right](\varphi_t) \tag{28}$$
and the term in the brackets is equal to zero when $\xi_i$ solves Eq. (26). In the previous computations and also below it is convenient to keep in mind that, given a vector field $\theta : \mathbb{R}^d \to \mathbb{R}^d$, its Jacobian matrix is given by

$$(D\theta)_i := \nabla \theta_i^T.$$ 

Let us now go back to currents. Given a current-valued solution $\xi$ of the PDE (26), we regularize it as

$$v_\epsilon^i(t, x) := (\xi_i \ast \theta^\epsilon e_i)(x) = \xi_i(\theta^\epsilon(x - \cdot)e_i), \quad \text{for } 1 \leq i \leq d$$

Here $\theta^\epsilon(x) = e^{-d \theta(\epsilon^{-1} x)}$ is a mollifier, and $\{e_i\}_{1 \leq i \leq d}$ is the canonical base of $\mathbb{R}^d$.

Using equation (25) (with $b$ in place of $B(\xi)$), we see that $v_\epsilon$ solves

$$v_\epsilon^i(t, x) = v_\epsilon^i(0, x) + \int_0^t \left\{ \xi_s(\nabla(\theta^\epsilon(x - \cdot) \cdot b_s e_i) + [(\xi_s \cdot \nabla)b_s](\theta^\epsilon(x - \cdot)e_i) \right\} ds$$

Define now

$$\left( R^\epsilon_i[b, \xi_i](x) \right)_i := \xi_i(\nabla(\theta^\epsilon(x - \cdot) \cdot b_i e_i) + (b_i(x) \cdot \nabla)\xi_i(\theta^\epsilon(x - \cdot)e_i) \quad 1 \leq i \leq d \quad (29)$$

so that

$$v_\epsilon^i(t, x) = v_\epsilon(0, x)$$

$$\quad + \int_0^t \left\{ R^\epsilon_i[b, \xi_s](x) + R^\epsilon_i[b, b_s](x) - (b_s \cdot \nabla)v_\epsilon^i(s, x) + (v_\epsilon^i(s, x) \cdot \nabla)b_s \right\} ds$$

which means (provided continuity in $t$ of the integrand),

$$\partial_t v_\epsilon^i(t, x) + (b_t \cdot \nabla)v_\epsilon^i(t, x) - (v_\epsilon^i(t, x) \cdot \nabla)b_t = R^\epsilon_i[b, \xi_t](x) + R^\epsilon_i[b, b_t](x)$$

Plugging this last equation into Eq. (28), we obtain

$$\left[(D\varphi_t)^{-1}(x)v_\epsilon^i(t, \varphi_t(x))\right] = \int_0^t (D\varphi_x)^{-1}(x) \left( R^\epsilon_i[b, \xi_s] + R^\epsilon_i[b, b_s] \right) (\varphi_s(x))ds \quad (30)$$

If we want (27) to hold, we must verify that the left-hand side goes to the left hand side of (27) and the right-hand side goes to 0, as $\epsilon \to \infty$. It suffices to obtain this convergence weakly, thus we test (30) against a test function $\rho \in C^1_b(\mathbb{R}^d, \mathbb{R}^d)$ [we need $\rho$ to be differentiable because of Lemma 14 and (31)].

$$\int \left[(D\varphi_t)^{-1}(x)v_\epsilon^i(t, \varphi_t(x))\right] \cdot \rho(x)dx$$

$$\quad = \int \int_0^t (D\varphi_x)^{-1}(x) \left( R^\epsilon_i[b, \xi_s] + R^\epsilon_i[b, b_s] \right) (\varphi_s(x))ds \cdot \rho(x)dx$$

If we have a closer look at the right-hand side, we see that we need that the commutator goes to zero when tested against the function

$$\overline{\varphi}_s(x) = (D\varphi_x)^{-1}(\varphi_s^{-1}(x))\rho(\varphi_s^{-1}(x)) \quad (31)$$

If this test function satisfies the assumptions on Lemma 14, we can conclude. In particular, we ask that it is bounded together with its first derivative,

$$\|\overline{\varphi}_s\|_\infty \leq \|(D\varphi_x)^{-1}\|_{\infty} \|\rho_s\|_{\infty}$$
\[ \| D\bar{\rho}_s \|_\infty \leq \| D^2 (\varphi_s^{-1}) \|_\infty \| \rho \|_\infty + \| (D\varphi_s)^{-1} \|_\infty \| D\rho \|_\infty \]

\textbf{Lemma 14} Let \( \rho, b \in C^1_b(\mathbb{R}^d, \mathbb{R}^d) \), there exists a constant \( C \) independent of \( \epsilon \) such that,
\[
\left| \int \mathcal{R}_1^\epsilon [b, \xi](x) \cdot \rho(x) dx \right| \leq \epsilon C \| \xi \|_{\mathcal{M}} \| D\rho \|_\infty \| Db \|_\infty
\]

Let \( \rho \in C_b(\mathbb{R}^d, \mathbb{R}^d) \), \( b \in C^2_b(\mathbb{R}^d, \mathbb{R}^d) \). It holds
\[
\left| \int \mathcal{R}_2^\epsilon [\xi, b](x) \cdot \rho(x) dx \right| \leq \epsilon C \| \xi \|_{\mathcal{M}} \| \rho \|_\infty \| D^2 b \|_\infty
\]

where the constant \( C \) is the same as before.

\textbf{Proof} We show the bound on the first commutator. By (29), we have
\[
\int \mathcal{R}_1^\epsilon [b, \xi](x) \cdot \rho(x) dx = \sum_{i=1}^d \int \left[ \xi_i (\nabla (\theta^\epsilon (x - .)) \cdot b \, e_i) + (b(x) \cdot \nabla) \xi_i (\theta^\epsilon (x - .) e_i) \right] \cdot \rho(x) dx
\]

If we consider \( \xi \) to be a \( d \)-dimensional measure \( (d\xi_1, \cdots, d\xi_d) \), we obtain
\[
\int \mathcal{R}_1^\epsilon [b, \xi](x) \cdot \rho(x) dx = \sum_{i=1}^d \int \int \nabla \theta^\epsilon (x - y) \cdot (b(x) - b(y)) \rho_i(x) d\xi_i(y) dx
\]
\[
= -\sum_{i=1}^d \int \int \theta^\epsilon (b(x) - b(y)) \nabla \rho_i(x) d\xi_i(y) dx
\]

If we assume that the current can be swapped with (1) the integration, (2) the derivative in \( x \), and (3) the scalar product by \( b(x) \) we can repeat the same reasoning to obtain
\[
\int \mathcal{R}_1^\epsilon [b, \xi](x) \cdot \rho(x) dx = -\sum_{i=1}^d \xi_i \left( \int \theta^\epsilon (x - y) (b(x) - b(y)) \nabla \rho_i(x) dx \right)
\]

Taking the absolute value on both sides we get
\[
\left| \int \mathcal{R}_1^\epsilon [b, \xi](x) \cdot \rho(x) dx \right| \leq d |\xi|_{\mathcal{M}} \| D\rho \|_\infty \| \nabla b \|_\infty \sup_{y \in \mathbb{R}^d} \left( \int \theta^\epsilon (x - y)|x - y|dx \right)
\]

Now, a change of variables in the integral does the trick and we obtain the desired estimation with the constant equal to
\[
C := \int \theta(x)|x| dx
\]

To show the second inequality one notice that
\[
\int \mathcal{R}_2^\epsilon [\xi, b](x) \cdot \rho(x) dx = \sum_{i=1}^d \xi_i \left( \int \theta^\epsilon (x - .) (\partial_i b(\cdot) - \partial_i b(x)) \cdot \rho(x) dx \right)
\]

and the proof follows in the same way. \( \square \)
5 Global Solutions

In this section we show how to extend all the results in the previous sections to an arbitrary time interval \([0, T]\) provided that \(\xi_0\) has compact support. In Sect. 3.4 we stated a criterion to obtain global solutions to the flow equation (10). Now, we will show that the solution to equation (24) satisfies this criterion, under suitable assumptions on \(K\). We can work indistinguishably with the solution of either equation. Indeed in Sect. 4, we showed that the solutions to the flow equation (10) are in one to one correspondence with the solutions to Eq. (24). This correspondence is independent on the time interval on which the solutions are defined, meaning that, as soon as one of the two equations has a unique solution \(\xi\) on an interval \([0, T]\), this current \(\xi\) is the unique solution to the other equation on the same interval.

Theorem 10 states that a maximal solution is global, provided that one can get a bound, independent from time, for the weak norm \(\|\xi_t\|\). We will show that the weak norm of the solution is controlled by its energy and that this energy is conserved. The details are showed below in the proof of the following theorem and the related lemmas. The results in this section are inspired by the global existence of solutions proved in [3].

**Theorem 15** Let \(\xi_0\) be any current such that \(|\xi_0|_\mathcal{M} \leq R\), for some \(R > 0\). Assume the following on the kernel \(K\):

- \(\text{div}(K) = 0\), here the divergence is to be intended row-wise.
- \(\hat{K}\) has compact support, where \(\hat{K}\) is the Fourier transform of \(K\).
- \(\xi_0\) has compact support.

Moreover, assume that \(K * \xi_0 \in L^2(\mathbb{R}^d, \mathbb{R}^d)\). Under the previous assumptions the maximal time of definition \(T_{\xi_0}\) of a solution \(\xi_t\) of Eq. (10) is infinite.

**Proof** Let \(\xi_t\) be any maximal solution to the flow equation starting from \(\xi_0\) and defined on the interval \([0, T_{\xi_0}]\). By Eq. (10) and the definition of push-forward we have, for any \(\theta\) Lipschitz vector field,

\[
|\xi_t(\theta)| = |\xi_0(\varphi_z^{t,K*\xi}\theta)| = \left| \xi_0 \left( (D\varphi_z^{t,K*\xi})^T \theta(\varphi_z^{t,K*\xi}) \right) \right|
\]

Recalling that \(\|\xi_t\| = \sup_{\|\theta\|_{\infty} + \text{Lip}(\theta) \leq 1} |\xi_t(\theta)|\), we have that

\[
\|\xi_t\| \leq |\xi_0| \mathcal{M} \|\theta\|_{\infty} \|D\varphi_z^{t,K*\xi}\|_{\infty} \leq R \cdot 1 \cdot \|D\varphi_z^{t,K*\xi}\|_{\infty}
\]

Hence, it is enough to control the supremum of the derivative of the flow to control the weak norm. The derivative of the flow satisfies, as already noted at the beginning of the proof of Lemma 6, the following equation

\[
\frac{d}{dt} D\varphi_z^{t,K*\xi}(x) = (DK * \xi_t)(\varphi_z^{t,K*\xi}(x)) D\varphi_z^{t,K*\xi}(x)
\]

The solution of this equation is controlled by \(\|DK * \xi_t\|_{\infty}\). Notice that, by Remark 30, the push-forward \(\xi_t\) has compact support. Hence, we can apply Lemmas 17 and 16 to obtain

\[
\|DK * \xi_t\|_{\infty} \lesssim H(\xi_t)^{\frac{1}{2}} = H(\xi_0)^{\frac{1}{2}}
\]

This concludes the proof. Notice that the assumptions on \(K\) are needed in order to apply the two lemmas. In particular \(\text{div}(K) = 0\) is used in Lemma 16 and \(\text{supp}(\hat{K})\) compact is used in Lemma 17. 

\(\square\)
Lemma 16 Let $\xi_t$ be a solution to the Eq. (24) and assume that $\text{div}(K) = 0$ and that $K \ast \xi_0 \in L^2(\mathbb{R}^d, \mathbb{R}^d)$. Then the total energy of $\xi_t$, defined as

$$H(\xi_t) := \|K \ast \xi_t\|_{L^2}^2$$

is constant in time. Namely, for any $t$,

$$H(\xi_t) = H(\xi_0)$$

Proof We compute the derivative in time of $H(\xi_t)$ and we verify that it is null. Using (24),

$$\frac{d}{dt} H(\xi_t) = 2 \sum_{i=1}^{d} \langle \xi(K_i \cdot), \frac{d}{dt} \xi(K_i) \rangle_{L^2}$$

$$= 2 \sum_{i=1}^{d} \langle \xi(K_i), \xi(\nabla K_i \cdot (K \ast \xi)) \rangle_{L^2} + 2 \sum_{i=1}^{d} \langle \xi(K_i), [(\xi \cdot \nabla)(K \ast \xi)] K_i \rangle_{L^2}$$

$$= I + II$$

We compute $I$ and $II$ separately and we check that they are equal with opposite sign. Since $D(K \ast \xi)_i(x) \cdot h = \xi(DK_i(x) \cdot h)$, we get that

$$I = 2 \sum_{i=1}^{d} \langle \xi(K_i), \xi(\nabla K_i(x - \cdot) \cdot (K \ast \xi)) \rangle_{L^2} = 2 \sum_{i=1}^{d} \langle \xi(K_i), D(K \ast \xi)_i(x) \cdot (K \ast \xi) \rangle_{L^2}.$$

On the other side

$$[(\xi \cdot \nabla)(K \ast \xi)] K_i(x - \cdot) = (K \ast [(\xi \cdot \nabla)(K \ast \xi)])_i(x)$$

$$= (K \ast \xi)_i(x) \cdot \nabla(K \ast \xi)$$

$$= \text{div} \left([K \ast \xi)_i(x)(K \ast \xi)\right)$$

if we assume that $\text{div} K = 0$. Now, integrating by parts on $II$ and using again $\text{div} K = 0$, we get that

$$II = 2 \sum_{i=1}^{d} \langle \xi(K_i), \text{div} \left([K \ast \xi)_i(x)(K \ast \xi)\right) \rangle_{L^2}$$

$$= -2 \sum_{i=1}^{d} \langle \xi(K_i), D(K \ast \xi)_i(x) \cdot (K \ast \xi) \rangle_{L^2}$$

$$= -I$$

This concludes the proof.

Lemma 17 Assume that $\xi$ and the Fourier transform of $K$ have compact support. Then there exists a constant $C > 0$, depending only on the radius of the support of $K$, such that

$$|(K \ast \xi)(x)| \leq CH(\xi)^{1/2}$$

for every $\xi \in \mathcal{M}$.

Proof Step 1. It is sufficient to prove the inequality of the lemma at $x = 0$:

$$|(K \ast \xi)(0)| \leq C \|K \ast \xi\|_{L^2}.$$
Let \( \xi_n \) be a sequence of \( L^2(\mathbb{R}^d, \mathbb{R}^d) \) vector fields such that \( \xi_n \to \xi \) in \( \mathcal{M} \), this can be done because \( \xi \) has compact support. Then \((K \ast \xi_n)(0) \to (K \ast \xi)(0)\) and \(K \ast \xi_n \to K \ast \xi\) in \( L^2(\mathbb{R}^d, \mathbb{R}^d)\). If we prove

\[
|(K \ast \xi_n)(0)| \leq C \|K \ast \xi_n\|_{L^2}
\]

then we deduce the result. Thus it is sufficient to prove our claim for currents \( \xi \) of class \( L^2(\mathbb{R}^d, \mathbb{R}^d)\).

**Step 2.** The inequality to be proved is equivalent to the following one: for every smooth complex-valued function \( p(k) \) with compact support, we have

\[
\left| \int \hat{K}(k) p(k) \, dk \right|^2 \leq C \int |\hat{K}(k)|^2 |p(k)|^2 \, dk.
\]  

(32)

Indeed, We have already reduced our claim to the proof that

\[
|(K \ast \xi)(0)| \leq C \|K \ast \xi\|_{L^2}
\]

is true for every \( \xi \in L^2(\mathbb{R}^d, \mathbb{R}^d) \). We have, with the notation \( K^-(x) = K(-x) \),

\[
(K \ast \xi)(0) = \langle K^-, \xi \rangle_{L^2} = \langle \hat{K}^-, \xi \rangle_{L^2}
\]

\[
\|K \ast \xi\|_{L^2}^2 = \|\hat{K} \ast \xi\|_{L^2}^2 = \|\hat{K} \cdot \hat{\xi}\|_{L^2}^2
\]

hence, noticing that \( \hat{K}^-(k) = \hat{K}(k) \) we have to prove that

\[
\left| \int \hat{K}(k) \hat{\xi}(k) \, dk \right|^2 \leq C \int |\hat{K}(k)|^2 |\hat{\xi}(k)|^2 \, dk.
\]

Call \( p(k) = \hat{\xi}(k) \) and notice that this inequality is equivalent to (32). Until now \( p = \hat{\xi} \in L^2(\mathbb{R}^d, \mathbb{R}^d) \), but now we may approximate \( p \) by smooth functions with compact support and restrict the proof of (32) to them.

**Step 3.** Let us prove (32). Let \( B(z, R) \) be a ball that contains the support of \( \hat{K}(k) \). We have

\[
\left| \int \hat{K}(k) p(k) \, dk \right|^2 = \left| \int 1_{B(z,R)} \hat{K}(k) p(k) \, dk \right|^2 \leq |B(z,R)| \int |\hat{K}(k)|^2 |p(k)|^2 \, dk.
\]

The proof is complete. \( \square \)

### 6 Continuous Dependence on Initial Conditions

Recall that a local time of existence and uniqueness for the flow equation (20) exists for every \( \xi_0 \in \mathcal{M} \) and its size, in the proof based on contraction principle, depends only on \( |\xi_0|_{\mathcal{M}} \).

Therefore, if we have a sequence \( \xi_0^n \to \xi_0 \), since weakly convergent sequences are bounded, there exists a common time interval \([0, T]\) of existence and uniqueness, with \( T > 0 \).

**Theorem 18** Given \( \xi_0, \xi_0^n \in \mathcal{M} \), \( n \in \mathbb{N} \), with \( \lim_{n \to \infty} \|\xi_0^n - \xi_0\| = 0 \), let \([0, T]\) be a common time interval of existence and uniqueness for the flow equation (20) and denote by \( \xi, \xi^n \in C([0, T]; \mathcal{M}) \) the corresponding solutions. Then \( \xi^n \to \xi \) in \( C([0, T]; \mathcal{M}_w) \) and
more precisely there exists a constant $C > 0$ (depending on the $\| \cdot \|$-norms of $\xi_0, \xi_0^n$ and on $T$) such that

$$\sup_{t \in [0, T]} \| \xi_t^n - \xi_t \| \leq C \| \xi_0^n - \xi_0 \|.$$  

Moreover, if we assume that $K$ and $\xi_0$ satisfy the assumptions of Theorem 15, then (33) holds true up to any time $T > 0$.

**Proof Step 1.** There exists a constant $C_0 > 0$ such that

$$\| \xi \|_T \leq C_0, \quad \| \xi^n \|_T \leq C_0$$

uniformly in $n \in \mathbb{N}$. Indeed, the time $T$ can be reached in a finite number of small steps $T_k$ related to the contraction principle, namely the application of Theorem 8. On each small interval the uniform-in-time $\| \cdot \|$-norm is controlled by the $\| \cdot \|$-norm of the initial condition of that time interval. The inequalities (33) readily follow. Finally, from (23) it follows also

$$\sup_{t \in [0, T]} |\xi_t|_{\mathcal{M}} \leq C'_0, \quad \sup_{t \in [0, T]} |\xi^n_t|_{\mathcal{M}} \leq C'_0$$

for some $C'_0 > 0$.

**Step 2.**

We have

$$|\xi_t(\theta) - \xi^n_t(\theta)| = |\varphi_t^{t,B(\xi)}(\xi_0(\theta)) - \varphi_t^{t,B(\xi^n)}(\xi^n_0(\theta))|$$

$$\leq |\varphi_t^{t,B(\xi)}(\xi_0(\theta)) - \varphi_t^{t,B(\xi^n)}(\xi^n_0(\theta))| + |\varphi_t^{t,B(\xi^n)}(\xi^n_0(\theta)) - \varphi_t^{t,B(\xi^n)}(\xi^n_0(\theta))|$$

$$= \left| (\xi_0 - \xi^n_0) \left( \varphi_t^{t,B(\xi)}(\theta) \right) \right| + \left| \xi^n_0 \left( \varphi_t^{t,B(\xi^n)}(\theta) \right) \right|$$

$$\leq \| \xi_0 - \xi^n_0 \| \left( \| \varphi_t^{t,B(\xi)}(\theta) \|_\infty + \text{Lip}(\varphi_t^{t,B(\xi^n)}(\theta)) \right)$$

$$+ \| \xi^n_0 \|_{\mathcal{M}} \| \varphi_t^{t,B(\xi^n)}(\theta) - \varphi_t^{t,B(\xi^n)}(\theta) \|_\infty.$$ 

Now, from (33), (16) and (19) there exist $C_{11}, C_{12} > 0$ such that

$$\| \varphi_t^{t,B(\xi)}(\theta) \|_\infty = \| D\varphi^{t,B(\xi)}(\cdot) (\cdot)^T \theta (\varphi^{t,B(\xi)}(\cdot)) \|_\infty \leq \| D\varphi^{t,B(\xi)}(\cdot) \|_\infty \| \theta \|_\infty \leq C_{11}$$

$$\text{Lip}(\varphi_t^{t,B(\xi^n)}(\theta)) \leq \| D^2\varphi^{t,B(\xi^n)}(\cdot) \|_\infty \| \theta \|_\infty + \| D\varphi^{t,B(\xi^n)}(\cdot) \|_\infty \text{Lip}(\theta) \leq C_{12}.$$ 

Moreover,

$$\| \varphi_t^{t,B(\xi)}(\theta) - \varphi_t^{t,B(\xi^n)}(\theta) \|_\infty$$

$$= \| D\varphi^{t,B(\xi)}(\cdot)^T \theta (\varphi^{t,B(\xi)}(\cdot)) - D\varphi^{t,B(\xi^n)}(\cdot)^T \theta (\varphi^{t,B(\xi^n)}(\cdot)) \|_\infty$$

$$\leq \| D\varphi^{t,B(\xi)}(\cdot)^T \theta (\varphi^{t,B(\xi^n)}(\cdot)) - D\varphi^{t,B(\xi^n)}(\cdot)^T \theta (\varphi^{t,B(\xi^n)}(\cdot)) \|_\infty$$

$$+ \| D\varphi^{t,B(\xi^n)}(\cdot)^T \theta (\varphi^{t,B(\xi^n)}(\cdot)) - D\varphi^{t,B(\xi^n)}(\cdot)^T \theta (\varphi^{t,B(\xi^n)}(\cdot)) \|_\infty$$

$$\leq \| D\varphi^{t,B(\xi^n)}(\cdot)^T \|_\infty \text{Lip}(\theta) \| \varphi^{t,B(\xi^n)}(\cdot) - \varphi^{t,B(\xi^n)}(\cdot) \|_\infty + \| D\varphi^{t,B(\xi^n)}(\cdot) - D\varphi^{t,B(\xi^n)}(\cdot) \|_\infty \| \theta \|_\infty$$

$$\leq \| D\varphi^{t,B(\xi^n)}(\cdot)^T \|_\infty \| \varphi^{t,B(\xi^n)}(\cdot) - \varphi^{t,B(\xi^n)}(\cdot) \|_\infty + \| D\varphi^{t,B(\xi^n)}(\cdot) - D\varphi^{t,B(\xi^n)}(\cdot) \|_\infty$$

$$\leq \| t e^{C_B T (\xi_T^{T+1})} \xi \|_T + \xi^n_0 \|_T$$

$$+ \| t C_B e^{C_T (T+1)(\xi_T^{T+1} + \xi^n_0^{T+2})} (1 + C_B T (\xi_T^{T+1} + C_B T (\xi_T^{T+1}) + 1)) \|_T \xi - \xi^n_0 \|_T.$$
Thus there exists $C_{13} > 0$ such that
\[ \| \varphi^{t, B(x)} - \varphi^{t, B(y)} \|_\infty \leq tC_{13} \| x - y \|_t. \]
Collecting these bounds, for every $T_0 \in [0, T]$ we get
\[ \| \xi - \xi^n \|_{T_0} \leq \| \xi_0 - \xi^n \|_{T_0} (C_{11} + C_{12}) + T_0 C_{13} \| \xi^n \|_{M_t} \| \xi - \xi^n \|_{T_0}. \]
This proves the theorem if $T_0$ is small enough, say $T_0 \leq \frac{1}{2C_{13} C_0}$; but the constant $2C_{13} C_0$ does not vary when we repeat the argument on the interval $[T_0, 2T_0]$ and so on (until we cover $[0, T]$) and thus in a finite number of steps we get the result on $[0, T]$. We have thus proved the first statement.

To prove the second statement, one simply notices that the solutions are global in time as a consequence of Theorem (15). Hence, the common time interval $[0, T]$ of existence and uniqueness can be chosen arbitrarily large.

\[ \square \]

7 Interacting Filaments and Their Mean Field Limit

7.1 Interacting Filaments as Current Dynamics

Let $K \in UC^3_b(\mathbb{R}^d, \mathbb{R}^{d \times d})$ be given. An example of $K$ which heuristically motivates the investigation of filaments done here can be, in $d = 3$, a smooth approximation of Biot-Savart kernel. Consider a set of $N$ curves in $\mathbb{R}^d$, $\gamma^{i,N}_t(\sigma)$, parametrized by $\sigma \in [0, 1]$, time dependent, which satisfy the equations
\[ \frac{\partial}{\partial t} \gamma^{i,N}_t(\sigma) = \sum_{j=1}^{N} \alpha_j \int_0^1 K \left( \gamma^{i,N}_t(\sigma) - \gamma^{j,N}_t(\sigma') \right) \frac{\partial}{\partial \sigma'} \gamma^{j,N}_t(\sigma') d\sigma' \] (35)
\[ \gamma^{i,N}_0(\sigma) \text{ given, } i = 1, \ldots, N \]
for some sequence of weights $\{\alpha_j\}$. The dynamics of curves can be reformulated as a dynamic of currents of the form
\[ \xi_t = \varphi^{t, K*\xi}_t \xi_0 \] (36)
for a suitable $\xi_0$. Let us explain this reformulation.

To the family of curves we associate the current $\xi^N_t : C_b(\mathbb{R}^d; \mathbb{R}^d) \to \mathbb{R}$ defined as
\[ \xi^N_t(\theta) := \sum_{j=1}^{N} \alpha_j \int_0^1 \theta \left( \gamma^{i,N}_t(\sigma) \right) \frac{\partial}{\partial \sigma} \gamma^{j,N}_t(\sigma) d\sigma \] (37)
or more formally, in the vein of empirical measures,
\[ \xi^N_t = \sum_{j=1}^{N} \alpha_j \int_0^1 \delta_{\gamma^{i,N}_t(\sigma)} \frac{\partial}{\partial \sigma} \gamma^{j,N}_t(\sigma) d\sigma. \]

**Theorem 19** If $\{\gamma^{i,N}_t(\sigma), i = 1, \ldots, N, \ t \in [0, T], \ \sigma \in [0, 1]\}$ is a family of $C^1([0, T] \times [0, 1] : \mathbb{R}^d)$ functions which satisfies the identities (35), then the current $\xi^N_t : C_b(\mathbb{R}^d; \mathbb{R}^d) \to \mathbb{R}, \ t \in [0, T]$, defined by (37) satisfies Eq. (36). Conversely, if $\xi^N \in C([0, T] ; \mathcal{M}_w)$ is the unique current-valued solution of Eq. (36) with $\xi^N_0$ defined by (37) (for $t = 0$) with respect to a
given family of $C^1$ initial curves $\{\gamma^{i,N}_0(\sigma), i = 1, \ldots, N, \sigma \in [0, 1]\}$, then the representation (37) holds where $\gamma^{i,N}_t(\sigma)$ is defined as

$$
\gamma^{i,N}_t(\sigma) = \varphi^{t,K}\xi^N_0 \left( \gamma^{i,N}_0(\sigma) \right)
$$

and the curves $\gamma^{i,N}_t(\sigma)$ satisfy the identities (35).

**Proof** Let us prove the first direction: from the general definition of push-forward, we get

$$
\left( \varphi^{t,K}\xi^N_0 \xi^N_0 \right)(\theta) = \xi^N_0 \left( \left( D\varphi^{t,K}\xi^N_0 \right)^T \theta \circ \varphi^{t,K}\xi^N_0 \right)
$$

where

$$
\tilde{\theta} = \left( D\varphi^{t,K}\xi^N_0 \right)^T \theta \circ \varphi^{t,K}\xi^N_0
$$

and we want to prove that this is equal to

$$
\xi^N_t(\theta) = \sum_{j=1}^N \alpha_j \int_0^1 \theta \left( \gamma^{j,N}_t(\sigma) \right) \frac{\partial}{\partial \sigma} \gamma^{j,N}_t(\sigma) \, d\sigma.
$$

Thus it is sufficient to prove

$$
\int_0^1 \theta \left( \gamma^{i,N}_t(\sigma) \right) \frac{\partial}{\partial \sigma} \gamma^{i,N}_t(\sigma) \, d\sigma = \int_0^1 \tilde{\theta} \left( \gamma^{j,N}_0(\sigma) \right) \frac{\partial}{\partial \sigma} \gamma^{j,N}_0(\sigma) \, d\sigma.
$$

To prove this, notice that

$$
\left( K * \xi^N_t \right)(x) = \sum_{j=1}^N \alpha_j \int_0^1 K \left( x - \gamma^{j,N}_t(\sigma) \right) \frac{\partial}{\partial \sigma} \gamma^{j,N}_t(\sigma) \, d\sigma.
$$

Therefore the equations (35) for the interaction of curves can be rewritten as

$$
\frac{\partial}{\partial t} \gamma^{i,N}_t(\sigma) = \left( K * \xi^N_t \right) \left( \gamma^{i,N}_t(\sigma) \right).
$$

This means that

$$
\gamma^{i,N}_t(\sigma) = \varphi^{t,K}\xi^N_0 \left( \gamma^{i,N}_0(\sigma) \right).
$$

Now, from this fact, we can deduce the identity above. Indeed,

$$
\int_0^1 \theta \left( \gamma^{i,N}_t(\sigma) \right) \frac{\partial}{\partial \sigma} \gamma^{i,N}_t(\sigma) \, d\sigma = \int_0^1 \theta \left( \varphi^{t,K}\xi^N_0 \left( \gamma^{i,N}_0(\sigma) \right) \right) \frac{\partial}{\partial \sigma} \varphi^{t,K}\xi^N_0 \left( \gamma^{i,N}_0(\sigma) \right) \, d\sigma
$$

$$
= \int_0^1 \left( D\varphi^{t,K}\xi^N_0 \right)^T \left( \gamma^{i,N}_0(\sigma) \right) \left( \theta \circ \varphi^{t,K}\xi^N_0 \right) \times \left( \gamma^{i,N}_0(\sigma) \right) \frac{\partial}{\partial \sigma} \gamma^{i,N}_0(\sigma) \, d\sigma.
$$
Let us now prove the opposite direction. Let us assume that \( \bar{\xi}_t^N \) satisfies Eq. (36) with \( \xi_0^N \) defined by (37) for \( t = 0 \), with respect to a given family of \( C^1 \) initial curves \( \{ \gamma_0^{i,N} (\sigma) , i = 1, \ldots, N, \sigma \in [0, 1] \} \). Then we have that

\[
\bar{\xi}_t^N (\theta) = \left( \varphi_{t,K}^N \xi_0^N \right) (\theta) = \xi_0^N \left( D\varphi_{t,K}^N \xi_0 \right)^T \left( \gamma_0^{i,N} (\sigma) \right) \frac{\partial}{\partial \sigma} \gamma_0^{i,N} (\sigma) d\sigma
\]

\[
= \sum_{j=1}^N \alpha_j^N \int_0^1 \left( D\varphi_{t,K}^N \xi_0 \right)^T \left( \gamma_0^{j,N} (\sigma) \right) \frac{\partial}{\partial \sigma} \gamma_0^{j,N} (\sigma) d\sigma
\]

\[
= \sum_{j=1}^N \alpha_j^N \int_0^1 \left( \varphi_{t,K}^N \xi_0 \right) \left( \gamma_0^{j,N} (\sigma) \right) \frac{\partial}{\partial \sigma} \gamma_0^{j,N} (\sigma) d\sigma
\]

Let us define \( \gamma_i^{j,N} (\sigma) \) by (38). Then the representation (37) holds. Moreover, from (38) we have, for each \( \sigma \),

\[
\frac{\partial}{\partial t} \gamma_i^{j,N} (\sigma) = \left( K * \xi_i^N \right) \left( \gamma_i^{j,N} (\sigma) \right)
\]

which is precisely (35) due to the already established form of \( \xi_i^N \).

The reformulation above provides first of all an existence and uniqueness result:

**Corollary 20** Assume \( K \in \mathcal{UC}^3_b \left( \mathbb{R}^d, \mathbb{R}^{d \times d} \right) \). For every \( N \in \mathbb{N} \) and every family of \( C^1 \) curves \( \{ \gamma_0^{i,N} (\sigma) , i = 1, \ldots, N, \sigma \in [0, 1] \} \), there exists a unique maximal solution in time \( \{ \gamma_i^{j,N} (\sigma) , i = 1, \ldots, N, t \in [0, T_{\xi_0}) , \sigma \in [0, 1] \} \) of equations (35) in the class \( C^1 \left( [0, T_{\xi_0}] \times [0, 1] ; \mathbb{R}^d \right)^N \) functions. Here the maximal time \( T_{\xi_0} \) is determined by the norm \( |\xi_0|_M \) of the empirical measure \( \xi_0 \) associated to the curves at time 0.

In addition, if the assumptions of Theorem 15 are satisfied, the existence and uniqueness of solutions to the system holds up to any time \( T > 0 \).

### 7.2 Mean Field Result

The next theorem proves two important results: first, if a family of initial curves approximates a current at time \( t = 0 \), then the solutions of the filament equations converge to the solution of the vector valued PDE. The second related result is that each curve of the family becomes, in the limit \( N \to \infty \), closer and closer to the solutions \( \bar{\gamma}_t^i \) of equation

\[
\frac{\partial}{\partial t} \bar{\gamma}_t^i \left( \sigma \right) = \left( K * \xi_t \right) \left( \bar{\gamma}_t^i \left( \sigma \right) \right)
\]

\[
\bar{\gamma}_t^i \left( \sigma \right) = \gamma_0^i \left( \sigma \right)
\]

This equation describes the interaction of a filament with the mean field \( \xi_t \). This is the core of the concept of mean field theory.

**Remark 21** It is important to distinguish between two possible scenarios for the choice of \( \gamma_0^i \). A typical example of \( \gamma_0^i \) is that of a realization of an infinite sequence \( \{ \gamma_0^i \}_{i \in \mathbb{N}} \) of random independent initial conditions; see Sect. 7.3 below. In this first case, \( \gamma_0^i \equiv \gamma_0^i \) does not depend on \( N \) when \( i \) is fixed and hence neither does \( \bar{\gamma}_t^i \equiv \bar{\gamma}_t^i \). The second and more...
general case is achieved when the initial condition depends on both \(i\) and \(N\) and this is what we will handle in the next theorem.

When \(\gamma_{0,i}^{i,N}\) depends on \(N\), it is not natural to assume that they converge as \(N \to \infty\); hence we cannot prove that \(\bar{\gamma}_t^{i,N}\) have a limit. For this reason the second part of the next theorem only claims that \(\gamma_{t,i}^{i,N}\) and \(\bar{\gamma}_t^{i,N}\) are close to each other.

In the particular case, however, when \(\gamma_{0,i}^{i,N} \equiv \gamma_0^i\) does not depend on \(N\), and so \(\bar{\gamma}_t^{i,N} \equiv \bar{\gamma}_t^i\), the second part of the next theorem can be reformulated by saying that \(\gamma_{t,i}^{i,N}\) converges to \(\bar{\gamma}_t^i\).

As a technical remark, notice that if \(\xi_t\) exists on a time interval \([0, T]\), then \(K * \xi_t\) satisfies the regularity conditions of Lemma 5 and therefore there exists a unique time-dependent \(C^1\)-curve \(\bar{\gamma}_t^{i,N}\) (for each \(i\)), solution of Eq. (39).

**Theorem 22** Let, for every \(N \in \mathbb{N}, \gamma_{0,i}^{i,N} (\sigma), i = 1, ..., N, \sigma \in [0, 1]\) be a family of \(C^1\) curves. Assume that the associated currents at time zero

\[
\xi_0^N = \sum_{i=1}^{N} \alpha_i^N \int_0^1 \delta_{\gamma_0^{i,N} (\sigma)} \frac{\partial}{\partial \sigma} \gamma_0^{i,N} (\sigma) \, d\sigma
\]

(40)

converge weakly to a current \(\xi_0 \in \mathcal{M}\). Let \(T > 0\) be any time such that on \([0, T]\) there are unique current-valued solutions \(\xi_t^N\) and \(\xi_t\) to Eq. (36) with respect to the initial conditions \(\xi_0^N\) or \(\xi_0\); notice that such a time exists because the initial currents \(\xi_0^N\) and \(\xi_0\) are equibounded (since they converge weakly); moreover, notice that \(\xi_t^N\) has the form

\[
\xi_t^N = \sum_{i=1}^{N} \alpha_i^N \int_0^1 \delta_{\gamma_t^{i,N} (\sigma)} \frac{\partial}{\partial \sigma} \gamma_t^{i,N} (\sigma) \, d\sigma
\]

corresponding to curve-solutions to Eq. (35) and that \(\xi_t\) is the unique solution to the vector-valued PDE (26). Let \(\bar{\gamma}_t^{i,N}\) be the solution to the mean field Eq. (39).

Then:

(i) the currents \(\xi^N\) converge in \(C ([0, T] ; \mathcal{M}_w)\) to the current \(\xi\).

(ii) \(\lim_{N \to \infty} \sup_{(t, \sigma) \in [0, T] \times [0, 1]} |\gamma_t^{i,N} (\sigma) - \bar{\gamma}_t^{i,N} (\sigma)| = 0\).

Moreover, given \(\xi_0\) and \(K\) as in Theorem 15, the results in i) and ii) holds true on any time interval \([0, T]\).

**Proof** Part (i) is a straightforward consequence of Theorem 18 on the continuous dependence on initial conditions.

As to part (ii), denoting as above by \(\xi_t^N\) the current associated to the family \(\gamma_t^{i,N}\) (see (37)), we have

\[
|\gamma_t^{i,N} (\sigma) - \bar{\gamma}_t^{i,N} (\sigma)| \leq \int_0^T \left| (K * \xi_s^N) (\gamma_s^{i,N} (\sigma)) - (K * \xi_s^N) (\bar{\gamma}_s^{i,N} (\sigma)) \right| \, ds
\]

\[
\leq \int_0^T \left| (K * \xi_s^N) (\gamma_s^{i,N} (\sigma)) - (K * \xi_s^N) (\bar{\gamma}_s^{i,N} (\sigma)) \right| \, ds
\]

\[
+ \int_0^T \left| (K * \xi_s^N) (\bar{\gamma}_s^{i,N} (\sigma)) - (K * \xi_s^N) (\bar{\gamma}_s^{i,N} (\sigma)) \right| \, ds.
\]
From two of the properties of “$K \ast \xi$” proved in Lemma 5, we have (taking also the supremum in $\sigma \in [0, 1]$)

$$
\left| \gamma_{i,N}^{j} (\sigma) - \gamma_{i,N}^{j} (\sigma) \right| \leq C \int_{0}^{T} \| DK \ast \xi^{N}_{s} \|_{\infty} \left| \gamma_{i,N}^{j} (\sigma) - \gamma_{i,N}^{j} (\sigma) \right| ds 
+ C \int_{0}^{T} \| \xi^{N}_{s} - \xi_{s} \| ds 
\leq C \int_{0}^{T} \left| \gamma_{i,N}^{j} (\sigma) - \gamma_{i,N}^{j} (\sigma) \right| ds + C \int_{0}^{T} \| \xi^{N}_{s} - \xi_{s} \| ds
$$

where we have denoted a generic constant by $C$ and we have used that $\sup_{t \in [0,T]} \| \xi^{N}_{s} \| < \infty$, as we know from the first part of the proof (e.g. since they converge in $C ([0, T]; \mathcal{M}_{\omega})$). We also know, from the first part, that $\xi^{N}_{s} \to \xi_{s}$ in $C ([0, T]; \mathcal{M}_{\omega})$. Then it is sufficient to apply Gronwall’s Lemma to obtain the claim of part (ii). 

\[ \Box \]

### 7.3 Propagation of Chaos

Sometimes one has a probabilistic framework of the following kind. We have a filtered probability space $(\Omega, (\mathcal{F}_{t})_{t \geq 0}, \mathcal{F}, \mathbb{P})$ and the separable Banach space $\mathcal{C} = C ([0, 1]; \mathbb{R}^{d})$ with the Borel $\sigma$-algebra $\mathcal{B} (\mathcal{C})$; we call random curve in $\mathbb{R}^{d}$ any measurable map from $(\Omega, \mathcal{F}, \mathbb{P})$ to $(\mathcal{C}, \mathcal{B} (\mathcal{C}))$. We use the notation $\gamma$ also for a random curve. The image measure $\mu$ of such map is the law of the random curve. It is a probability measure on $(\mathcal{C}, \mathcal{B} (\mathcal{C}))$.

Then consider, for every $N \in \mathbb{N}$, a family $\left\{ \gamma_{i,N}^{j}, i = 1, ..., N \right\}$ of random curves and consider the associated currents $\xi^{N}_{0}$ defined as in (40), which now are random currents, namely measurable mappings from $(\Omega, F, P)$ to the space $\mathcal{M}$ endowed with its Borel $\sigma$-algebra $\mathcal{B} (\mathcal{M})$. Let $\xi_{0}$ be a random current. For all $\omega \in \Omega$, solve uniquely the flow equation (36) with the initial conditions $\xi^{N}_{0} (\omega)$ and $\xi_{0} (\omega)$ and call $\xi^{N}_{t} (\omega)$ and $\xi_{t} (\omega)$ the corresponding solutions. Assume that the whole family of currents $\xi_{0}^{N} (\omega), \xi_{0} (\omega)$ when $N$ and $\omega$ vary, are equibounded. Then take some $T > 0$ such that unique solutions $\xi^{N}_{t} (\omega)$ and $\xi_{t} (\omega)$ exist. For every $t \in [0, T]$, $\xi^{N}_{t}$ and $\xi_{t}$ are random currents (namely they are measurable), by the continuous dependence on initial conditions, Theorem 18. Assume that $\xi^{N}_{0}$ converges in probability to $\xi_{0}$. Then it is easy to show that, for every $t \in [0, T]$, $\xi^{N}_{t}$ converges in probability to $\xi_{t}$, and also that $\| \xi^{N}_{t} - \xi_{t} \|_{T}$ converges in probability to zero.

In this section we assume that the vorticity is the same for each vortex, namely $\alpha_{j}^{N} = \frac{1}{N}$ for every $j \leq N$, so that the exchangeability is maintained for every $N$ and for every $t > 0$.

To every curve $\gamma \in C^{1} ([0, 1], \mathbb{R}^{d})$, we can associate a current, which will also be called $\gamma$ with a slight abuse of notation, in this way

$$
\gamma (\theta) := \int_{0}^{1} \theta (\gamma (\sigma)) \frac{\partial}{\partial \sigma} \gamma (\sigma) d \sigma
$$

for $\theta \in C_{b} (\mathbb{R}^{d}, \mathbb{R}^{d})$. The definition of the tensor product $\gamma \otimes \gamma'$ is

$$
(\gamma \otimes \gamma') (\theta, \theta') = \gamma (\theta) \gamma' (\theta').
$$

We fix a filtered probability space $(\Omega, (\mathcal{F}_{t})_{t \geq 0}, \mathcal{F}, \mathbb{P})$ and, following the notion given in the previous subsection, we consider random curves. We say that a family $(\gamma_{i})_{1 \leq i \leq N}$ of random curves is symmetric or exchangeable if its law is independent of permutations of the indexes. We start with the following general result, for random currents independent of time.
Theorem 23 Let \( \xi \) be a current and, for every \( N \in \mathbb{N} \), let \( \gamma^N := (\gamma^{i,N})_{1 \leq i \leq N} \) be a symmetric family of random-\( C^1([0,1], \mathbb{R}^d) \) curves. We call \( \xi^N \) the empirical measure associated with the family \( \gamma^N \). Suppose that, for every \( \theta \in C_b(\mathbb{R}^3, \mathbb{R}^3) \),

\[
|\xi|_\mathcal{M} < \infty, \quad |\gamma^{i,N}(\theta)| \leq C \quad \mathbb{P} - \text{a.s.}
\]  
uniformly in \( i, N \) and that

\[
\lim_{N \to \infty} \mathbb{E} \left[ |\xi^N(\theta) - \xi(\theta)| \right] = 0.
\]

Then, for every fixed \( r \in \mathbb{N} \) and for every family of test functions \( (\theta_1, \ldots, \theta_r) \in C_b(\mathbb{R}^d, \mathbb{R}^d)^r \), it holds

\[
\lim_{N \to \infty} \mathbb{E} \left[ \left( \gamma^{1,N} \otimes \cdots \otimes \gamma^{r,N} \right)(\theta_1, \ldots, \theta_r) \right] = \prod_{i=1}^r \xi(\theta_i).
\]

Proof Without loss of generality, we prove the theorem in the case \( k = 2 \). For every \( \theta_1, \theta_2 \) bounded Lipschitz continuous functions in \( \mathbb{R}^d \) we have

\[
|\mathbb{E} \left[ \gamma^{1,N}(\theta_1)\gamma^{2,N}(\theta_2) \right] - \xi(\theta_1)\xi(\theta_2)| \leq |\mathbb{E} \left[ \gamma^{1,N}(\theta_1)\gamma^{2,N}(\theta_2) \right] - \mathbb{E} \left[ \xi^N(\theta_1)\xi^N(\theta_2) \right]| + |\mathbb{E} \left[ \xi^N(\theta_1)\xi^N(\theta_2) \right] - \xi(\theta_1)\xi(\theta_2)|
\]

The second term, \((44)\), goes to zero because of \((41)-(42)\):

\[
|\mathbb{E} \left[ \xi^N(\theta_1)\xi^N(\theta_2) \right] - \xi(\theta_1)\xi(\theta_2)| \leq |\mathbb{E} \left[ \left( \xi^N(\theta_1) - \xi(\theta_1) \right)\xi^N(\theta_2) \right] + |\mathbb{E} \left[ \left( \xi^N(\theta_2) - \xi(\theta_2) \right)\xi(\theta_1) \right]|
\]

\[
\leq C\mathbb{E} \left[ |\xi^N(\theta_1) - \xi(\theta_1)| \right] + |\xi|_\mathcal{M} \mathbb{E} \left[ |\xi^N(\theta_2) - \xi(\theta_2)| \right]
\]

To study \((43)\) we use the symmetry of \( \gamma \),

\[
|\mathbb{E} \left[ \gamma_1(\theta_1)\gamma_2(\theta_2) \right] - \mathbb{E} \left[ \xi^N(\theta_1)\xi^N(\theta_2) \right]| = |\mathbb{E} \left[ \gamma_1(\theta_1)\gamma_2(\theta_2) \right] - \mathbb{E} \left[ \xi^N(\theta_1)\xi^N(\theta_2) \right]| + \frac{1}{N} \mathbb{E} \left[ |\gamma_1(\theta_1)|^2 \right]
\]

The expectations are bounded because of \((41)\), hence the last term goes to zero. \( \square \)

Now we want to apply the previous theorem to our filaments. We verify in the following lemma that the dynamic of filaments satisfies Theorem 23, under suitable assumptions on the initial condition.

Lemma 24 Given a family \( \gamma_0 := \{\gamma_0^i\}_{1 \leq i \leq N} \) of random variables on \( C^1(\mathbb{R}^3, \mathbb{R}^3) \), and a current \( \xi_0 \), we assume

1. \( \{\gamma_0^1, \ldots, \gamma_0^N\} \) are exchangeable.
2. \( |\gamma_0^i(\theta)| \leq C \), for a.e. \( \omega \), uniformly in \( i \) and \( N \).
3. \( |\xi_0|_\mathcal{M} < \infty \)
4. \( \lim_{N \to \infty} \mathbb{E} \left[ ||\xi_0^N - \xi_0|| \right] = 0 \)
There exists a time $T > 0$ such that the solutions $\gamma_t := \{\gamma_i^{i,N}\}_{1 \leq i \leq N}$ and $\xi_t$ of Eqs. (35) and (36) starting respectively from $\gamma_0$ and $\xi_0$ satisfy conditions 1–4 at every time $t \in [0, T]$.

Moreover, under the assumptions of Theorem 15, conditions 1–4 are satisfied up to any time $T > 0$.

Proof Exchangeability is clearly preserved by the system of filaments, because there is no other randomness and the dynamics of each filament is perfectly equal to the one of the others.

To prove that $\gamma_i^{i,N}(\theta)$ is bounded, we use (38) and its derivative and we obtain

$$\gamma_i^{i,N}(\theta) = \int_0^1 \theta(\gamma_i^{i,N}) \frac{\partial}{\partial \sigma} \gamma_i^{i,N} d\sigma$$

$$= \int_0^1 \theta(\varphi^K*\xi_t(\gamma_i^t))(\gamma_i^t) \frac{\partial}{\partial \sigma} \gamma_i^t d\sigma = \gamma_i^t(\tilde{\theta})$$

where $\tilde{\theta}(x) := \theta(\varphi^K*\xi_t(x))(\gamma_i^t)(x)$ is bounded continuous because of Lemma 5.

The third property follows immediately from Theorem 8.

The last property, 4), is a direct consequence of Theorem 18.

$\square$

Corollary 25 Under the assumptions of Lemma 24, for every fixed $r \in \mathbb{N}$, every family of test functions $(\theta_1, \ldots, \theta_r) \in C_b(\mathbb{R}^d, \mathbb{R}^d)^r$ and every $t \in [0, T]$, it holds

$$\lim_{N \to \infty} \mathbb{E}\left[\left(\gamma_1^{1,N} \otimes \cdots \otimes \gamma_r^{r,N}\right)(\theta_1, \ldots, \theta_r)\right] = \prod_{i=1}^r \xi_t(\theta_i).$$

7.4 Example

In this section we describe a compact support vector field $\xi$ in $\mathbb{R}^3$ and show that the assumption on the initial current $\xi_0$ that we impose in the mean field results Lemma 24 - the fact that $\xi$ can be approximated by an “empirical” current made of filaments - is satisfied for this vector field. The general idea is to use a finite number of integral lines of the vector field. An approximation theorem for more general vector fields or currents would be interesting but it is out of the scope of this paper.

In $\mathbb{R}^3$, where we write points in the form $x = (x_1, x_2, x_3)$, let us consider a cylinder $C$ of the following form:

$$C = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \in [0, 1], x_2^2 + x_3^2 \leq 1\}$$

$$= [0, 1] \times D$$

where $D \subset \mathbb{R}^2$ is the unitary disk. Let $(x_2, x_3) \mapsto (b(x_2, x_3), c(x_2, x_3))$ be a $C^1$ divergence free ($\partial_2 b + \partial_3 c = 0$) vector field in $D$ which leaves $D$ invariant, in the sense of the dynamics $y_t^x(\sigma)$ defined below (in other words, it must be tangent along the boundary) and let $\xi : C \to \mathbb{R}^3$ be the vector field

$$\xi(x_1, x_2, x_3) = (1, b(x_2, x_3), c(x_2, x_3)).$$

Notice that $\text{div} \xi = 0$. We may consider $\xi$ as a compact support vector field defined over all $\mathbb{R}^3$.

For every $\sigma \in [0, 1]$, call $D_t$ the disk $D$ when seen as a subset of the plane $x_1 = t$:

$$D_t = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = t, x_2^2 + x_3^2 \leq 1\}.$$
For every \( x^0 \in D_0 \), consider the Cauchy problem in \( \mathbb{R}^3 \)
\[
\begin{align*}
  x'(\sigma) &= \xi (x(\sigma)) \quad \text{for } \sigma \in [0, 1] \\
  x(0) &= x^0
\end{align*}
\]
and call \( x^{x^0}(\sigma) \) the unique solution. At the same time, for every \( y^0 \in D \), consider the Cauchy problem in \( \mathbb{R}^2 \)
\[
\begin{align*}
  y'(\sigma) &= (b (y(\sigma)), c (y(\sigma))) \quad \text{for } \sigma \in [0, 1] \\
  y(0) &= y^0.
\end{align*}
\]
Call \( y^{y^0}(\sigma) \) its solution and recall that we have assumed above that \( y^{y^0}(\sigma) \) remains in \( D \).

We have, with \( x^0 = (0, x^0_2, x^0_3) \),
\[
x^{x^0}(\sigma) = \left( \sigma, y^{y^0}(x^0_2, x^0_3)(\sigma) \right).
\]
The cylinder \( C \) is “foliated” by two different families of sets. On one side we have
\[
C = \bigcup_{x^0 \in D_0} \left\{ x^{x^0}(\sigma) ; \sigma \in [0, 1] \right\}
\]
namely \( C \) is foliated by integral lines of the vector field \( \xi \), with the foliation parametrized by points of \( D_0 \). On the other side, we have
\[
C = \bigcup_{t \in [0, 1]} D_t
\]
amely \( C \) is foliated by the surfaces \( D_t \), as \( t \) varies in \([0, 1]\).

Let \( \{ Y^0_i ; i \in \mathbb{N} \} \) be independent random variables with values in \( D \), uniformly distributed. Since \( \partial_2 b + \partial_3 c = 0 \), Lebesgue measure is invariant for the flow associate to the solutions \( y^{y^0}(\sigma) \). Therefore the random variables in \( D \) given by \( y^{y^0}(\sigma) \) are uniformly distributed; and independent. For every \( i \in \mathbb{N} \), consider the random curve
\[
\left\{ y^i(\sigma) ; \sigma \in [0, 1] \right\} := \left\{ (\sigma, y^{y^0}(\sigma)) ; \sigma \in [0, 1] \right\}.
\]
Given \( N \in \mathbb{N} \), consider the random current
\[
\xi^N = \frac{2\pi}{N} \sum_{i=1}^{N} \int_0^1 \delta_{y^i(\sigma)} \frac{\partial}{\partial \sigma} y^i(\sigma) \, d\sigma
\]
Consider also, with a little abuse of notation, the current \( \xi \) associated to the homonymous vector field, namely given by
\[
\xi(\theta) = \int_C \theta(x) \cdot \xi(x) \, dx
\]
for all \( \theta \in C (\mathbb{R}^3, \mathbb{R}^3) \).

**Proposition 26** For every \( \theta \in C (\mathbb{R}^3, \mathbb{R}^3) \), in the sense of almost sure convergence we have
\[
\xi^N(\theta) \to \xi(\theta).
\]
Proof We have
\[
\xi_N^N(\theta) = \frac{2\pi}{N} \sum_{i=1}^{N} \int_{0}^{1} \theta\left(\sigma, y^y_0(\sigma)\right) \cdot \xi\left(\sigma, y^y_0(\sigma)\right) d\sigma
\]
\[
= \int_{0}^{1} \left(\frac{2\pi}{N} \sum_{i=1}^{N} \delta_{y^y_i(\sigma)}(\theta(t, \cdot) \cdot \xi(t, \cdot))\right) d\sigma.
\]

The empirical measure \(\frac{1}{N} \sum_{i=1}^{N} \delta_{y^y_i(\sigma)}\) converges a.s. to the law of \(y^y_0(\sigma)\), which is uniform in \(D\), namely it has density \(\frac{1}{2\pi}\) on \(D\). Hence \(\xi_N^N(\theta)\) converges to
\[
\int_{0}^{1} \left(\int_{D} \theta(\sigma, y) \cdot \xi(\sigma, y) dy\right) d\sigma = \int_{C} \theta(x) \cdot \xi(x) dx.
\]

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Appendix

In this section, we give some properties of 1-currents that have been defined at the beginning of the paper.

Completeness of Strong Balls with Respect to the Weak Norm

Lemma 27 If \(B\) is a closed ball in \((\mathcal{M}, |\cdot|_{\mathcal{M}})\), then \((B, d)\) is a complete metric space.

Proof Let \(\{\xi_n\}_{n \geq 0}\) be a Cauchy sequence in \((B, d)\). This is also a Cauchy sequence in the dual space \(\text{Lip}_b(\mathbb{R}^d, \mathbb{R}^d)'\) with the dual operator norm. Hence it converges to some \(\xi \in \text{Lip}_b(\mathbb{R}^d, \mathbb{R}^d)\). Indeed \(\text{Lip}_b(\mathbb{R}^d, \mathbb{R}^d)\) is a Banach space and \(\|\cdot\|\) is the operator norm on his dual, which is complete.

Now we have an operator \(\xi\) defined on \(\text{Lip}_b(\mathbb{R}^d, \mathbb{R}^d)\), we want to extend it to the bigger space \(C_b(\mathbb{R}^d, \mathbb{R}^d)\) and to show that this extension is a limit to the sequence \(\xi_n\) in the norm \(\|\cdot\|\).

Given \(\theta \in \text{Lip}_b(\mathbb{R}^d, \mathbb{R}^d)\), it holds, for every \(n \in \mathbb{N}\),
\[
|\xi(\theta)| \leq |(\xi - \xi_n)(\theta)| + |\xi_n(\theta)| \leq \|\xi - \xi_n\|(\|\theta\|_{\infty} + \text{Lip}(\theta)) + R\|\theta\|_{\infty}
\]
where \(R\) denotes the radius of \(B\). Hence, as \(n \to \infty\), it holds \(|\xi(\theta)| \leq R\|\theta\|_{\infty}\). We can thus apply Hahn-Banach theorem to obtain a linear functional \(\tilde{\xi}\) defined on \(C_b(\mathbb{R}^d, \mathbb{R}^d)\) such that \(|\tilde{\xi}| \leq R\) and \(\tilde{\xi} \equiv \xi\) on \(\text{Lip}_b(\mathbb{R}^d, \mathbb{R}^d)\).

It only remains to prove that \(\xi_n\) converges to \(\tilde{\xi}\),
\[
\|\tilde{\xi} - \xi_n\| = \sup\{\tilde{\xi}(\theta) - \xi_n(\theta) | \|\theta\|_{\infty} + \text{Lip}(\theta) \leq 1\}
\]
\[
= \sup\{\xi(\theta) - \xi_n(\theta) | \|\theta\|_{\infty} + \text{Lip}(\theta) \leq 1\} = \|\xi - \xi_n\| \to 0, \ \text{as} \ n \to \infty.
\]

\(\square\)
Convolutions of 1-Currents and Matrix Valued Operators

If $\xi \in M$ and $K : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ is a continuous bounded matrix-valued function, then $K * \xi$ is the vector field in $\mathbb{R}^d$ with $i$-component given by

$$(K * \xi)_i(x) = (K_i * \xi)(x) := \xi(K_i(x - \cdot))$$

where $K_i(z)$ is the vector $(K_{ij}(z))_{j=1,...,d}$. We have

$$|(K * \xi)(x)| \leq |\xi|_M \|K\|_{\infty}.$$  

If $K$, in addition, is also of class $C^1_b(\mathbb{R}^d, \mathbb{R}^m)$, then

$$|(K * \xi)(x)| \leq \|\xi\| (\|K\|_{\infty} + \|DK\|_{\infty}).$$

1-Currents Associated with Curves

One can define the current associated to a curve in the following way. Given a curve $\gamma : [0, 1] \rightarrow \mathbb{R}^d$ of class $C^1 (W^{1,1}$ is sufficient), consider the current

$$\xi = \int_0^1 \delta (\cdot - \gamma(\sigma)) \frac{d\gamma}{d\sigma}(\sigma) d\sigma$$

namely the linear functional $\xi : C_b(\mathbb{R}^d, \mathbb{R}^d) \rightarrow \mathbb{R}$ defined as

$$\xi (\theta) = \int_0^1 \theta(\gamma(\sigma)) \cdot \frac{d\gamma}{d\sigma}(\sigma) d\sigma.$$  

Notice that this is exactly how we defined the empirical measure, which is nothing but the current centered on a family of curves. In this case the push forward can be reformulated in a very specific form. For every $\varphi \in C^1(\mathbb{R}^d, \mathbb{R}^d)$, it is

$$\varphi \# \xi = \int_0^1 \delta (\cdot - \varphi(\gamma(\sigma))) D\varphi(\gamma(\sigma)) \frac{d\gamma}{d\sigma}(\sigma) d\sigma.$$  

Remark 28 It is easy to see that the push-forward of $\gamma$ with respect to $\varphi$ is the push-forward of $\varphi \circ \gamma$. This follows from the previous formula and the chain rule, $\frac{d\gamma}{d\sigma}(\sigma) = D\varphi(\gamma(\sigma)) \frac{d\gamma}{d\sigma}(\sigma)$.

1-Currents Associated with Vector Fields

We have seen that $\varphi \# \xi$ has a nice reformulation when $\xi$ is associated to a smooth curve. In Sect. 4 we consider special currents, which are induced by vector fields: we restrict to these particular currents as an intermediate step to prove Lemma 13.

We briefly describe here how to give a reformulation of $\xi$ and its push-forward when the current is associated to a vector field. Thus, with little abuse of notations, let $\xi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be an integrable vector field and denote by $\hat{\xi}$ the associated current defined as

$$\hat{\xi}(\theta) = \int_{\mathbb{R}^d} \theta(x) \cdot \hat{\xi}(x) dx.$$  

Proposition 29 Assume that $\varphi$ is a diffeomorphism of $\mathbb{R}^d$ and $\xi$ is a vector field on $\mathbb{R}^d$ in $\mathbb{R}^d$ of class $L^1$. Then $\varphi \# \xi$ is the following vector field in $\mathbb{R}^d$, of class $L^1$:

$$(\varphi \# \xi)(x) = D\varphi(\varphi^{-1}(x)) \frac{d\gamma}{d\sigma}(\sigma) \det D\varphi^{-1}(x).$$
Proof By definition we have
\[
(\varphi^\# \xi)(\theta) = \xi(\varphi^\# \theta) = \int_{\mathbb{R}^d} D\varphi(x)^T \theta(\varphi(x)) \cdot \xi(x) \, dx
\]
\[
= \int_{\mathbb{R}^d} \theta(\varphi(x)) \cdot D\varphi(x) \xi(x) \, dx
\]
\[
= \int_{\mathbb{R}^d} \theta(y) \cdot D\varphi(\varphi^{-1}(y)) \xi(\varphi^{-1}(y)) \det D\varphi^{-1}(y) \, dy.
\]
In the last inequality we used the change of variable \( y = \varphi(x) \).

\[\square\]

Remark 30 In the case when \( \varphi \) is a diffeomorphism, if \( \xi \) has compact support, then the push-forward \( \varphi^\# \xi \) has compact support.

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