CONFORMAL ANOMALY FOR FREE SCALAR PROPAGATION ON CURVED BOUNDED MANIFOLDS

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ABSTRACT. The trace anomaly for free propagation in the context of a conformally invariant scalar field theory defined on a curved manifold of positive constant curvature with boundary is evaluated through use of an asymptotic heat kernel expansion. In addition to their direct physical significance the results are also of relevance to the holographic principle and to Quantum Cosmology.

I. Introduction

The fundamental physical significance of bounded manifolds has been amply demonstrated in the framework of Euclidean Quantum Gravity and, more recently, in the context of the holographic principle and the AdS/CFT correspondence. An issue of immediate importance on such manifolds is the evaluation of the effective action and, by extension, of the conformal anomaly relevant to the dynamical behaviour of quantised fields. The contribution to the conformal anomaly which emerges from free propagation on a curved manifold is the exclusive result of the gravitational backreaction on the manifold’s geometry and has a distinct character from that which emerges from matter-to-matter interactions. In the case of bounded manifolds the conformal reaction also receives a simultaneous contribution from the non-trivial boundary. In what follows a brief outline of the techniques relevant to the evaluation of the free-propagation-related conformal anomaly on a general bounded manifold will be presented as the incipient point of an analysis which advances from the general to the concrete case of the bounded manifold of positive constant curvature stated herein.

For reasons of technical convenience the analysis will be performed on $C_4$, a segment of the Euclidean sphere bounded by a hypersurface of positive extrinsic curvature, with homogeneous Dirichlet-type boundary conditions. Such a choice allows for a direct use of the results hitherto attained on such a manifold [1], [2], [3], [4]. The results obtained herein have general significance for bounded manifolds of the same topology both in terms of the general structure of the effective action and in terms of the interaction between boundary and surface terms. Notwithstanding that, such results as are obtained herein on $C_4$ deserve attention in their own merit due to their additional significance for the Hartle-Hawking approach to the quantisation of closed cosmological models.

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II. Trace Anomaly and Free Scalar Propagation on $C_4$

The scalar component of the bare action defining a theory for a free, conformal, massless field $\Phi$ specified on $C_n$ - a manifold of positive constant curvature embedded in a $(n+1)$-dimensional Euclidean space with embedding radius $a$ and bounded by a $(n-1)$-sphere of positive constant extrinsic curvature $K$ (diverging normals) - at $n = 4$ is 

$$S[\Phi_0] = \int_C d^4 \eta \left[ \frac{1}{2} \Phi_0 \left( \frac{L^2}{2a^2} - \frac{1}{2} n(n-2) \right) \right] + \oint_{\partial C} d^3 \eta K \Phi_0^2$$

with the subscript $C$ signifying integration in the interior of $C_4$ and with the subscript $\partial C$ signifying integration exclusively on its boundary. In either case the subscript 4 has been omitted as the integration itself renders the associated dimensionality manifest. In (1) $\eta$ is the position vector in the embedding $(n+1)$-dimensional Euclidean space signifying the coordinates $\eta_\mu$ and

$$L_{\mu\nu} = \eta_\mu \frac{\partial}{\partial \eta_\nu} - \eta_\nu \frac{\partial}{\partial \eta_\mu}$$

is the generator of rotations. In addition, the Ricci scalar $R$ relates to the constant embedding radius $a$ through

$$R = \frac{n(n-1)}{a^2}$$

As stated, the bare action (1) is associated by choice with the homogeneous Dirichlet condition $\Phi|_{\partial C_4} = 0$ for the scalar field.

The gravitational component of the bare action on $C_n$ at $n \rightarrow 4$ is

$$S_g = \frac{1}{16\pi G_0} \int_C d^4 \eta \left[ 2 \Lambda_0 - R \right] - \frac{1}{8\pi G_0} \oint_{\partial C} d^3 \eta K$$

The boundary term in (3) is essentially the Gibbons-Hawking term in the gravitational action functional [4]. At one-loop order it signifies a quantum correction to the Einstein-Hilbert action which emerges as the result of the influence which the background curvature and boundary have on free propagation.

In the context of the theory pursued herein the one-loop vacuum effective action and associated trace anomaly shall be obtained, in what follows, through use of heat kernel asymptotic expansions on a general bounded manifold $M$ [5] by specifying the geometry to be that of $C_4$ with homogeneous Dirichlet conditions on $\partial C_4$ and the coupling between matter and gravity to be a conformal coupling between a scalar field and the stated geometry.

The one-loop effective action $W_0$ associated with the free scalar propagation on a general manifold $M$ is, generally, given by the expression

$$W_0 = \frac{1}{2} Tr \ln D$$
where $D$ is the operator associated with the scalar propagator on $M$, acting on an abstract Hilbert space of states $|n>$ subject to orthonormality conditions with eigenvalues $\lambda_n$ \cite{6}. Introducing the generalised $\zeta$-function as \cite{5}

\begin{equation}
\zeta(s) \equiv Tr[D^{-s}] = \sum_n \lambda_n^{-s}
\end{equation}

it follows that in the case of the scalar field it is

\begin{equation}
W_0 = -\frac{1}{2} \frac{\partial}{\partial s} \zeta(s) |_{s=0} - \frac{1}{2} \zeta(0) \ln(\mu^2)
\end{equation}

On the grounds of general theoretical considerations the mean value of the stress energy-momentum tensor in some vacuum state is

\begin{equation}
<T_{\mu\nu}> = \frac{2}{\sqrt{-g}} \frac{\delta W_0}{\delta g_{\mu\nu}}
\end{equation}

Such counterterms contained perturbatively in the bare gravitational action as are necessary to cancel the divergences which appear in $W_0$ on a general manifold $M$ are local in the metric field and conformally invariant in four dimensions. In effect, the trace of the renormalised stress tensor in (7) receives a contribution from $W_0$ which can be seen from (6) to relate to $\zeta(s)$ through \cite{5}

\begin{equation}
\int_M d^4x \sqrt{-g} < T^c_\ell > = \zeta(0)
\end{equation}

On a general manifold $M$, for that matter, the conformal anomaly emerging from free propagation at one-loop level for conformally invariant theories is specified by $\zeta(0)$. This result remains valid in the presence of a non-trivial boundary on the understanding that integration over $M$ also includes $\partial M$.

In order to evaluate the trace anomaly associated with free propagation on $M$ it is necessary to consider the asymptotic expansion \cite{5}

\begin{equation}
G(t) \sim \sum_{k=0}^{\infty} A_k t^{\frac{k+1}{2}}, \; t \to 0^+
\end{equation}

of the supertrace

\begin{equation}
G(t) = \int_C tr K_D(x, x, t)d^4x = \sum_n e^{-\lambda_n t} = Tr e^{-tD}
\end{equation}

of the heat kernel

\begin{equation}
K_D(x, x', t) = \sum_n <n|x'|<x|n> e^{-\lambda_n t}
\end{equation}
associated with the bounded elliptic operator $D$ in (4) through the heat equation

\begin{equation}
(\partial_t + D)K_D(x, x', t) = 0
\end{equation}

The supertrace relates to $\zeta(s)$ through an inverse Mellin transform as

\begin{equation}
\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}G(t)dt
\end{equation}

The asymptotic expansion in (9) yields, in the context of (12), the result

\begin{equation}
\zeta(0) = A_4
\end{equation}

which, as (8) reveals, reduces the issue of the conformal anomaly due to free propagation of matter on $M$ to the issue of evaluating the constant coefficient $A_4$ in (9).

The general asymptotic expansion in (9) is characterised by the exclusive presence of even-order coefficients $A_{2k}$ on any manifold $M$ for which $\partial M = 0$. The presence of a non-trivial $\partial M$ has the effect of generating an additional boundary-related component for each even-order coefficient as well as non-vanishing boundary-related values for all $A_{2k+1}$ in (9). In general, the coefficients for the supertrace of the heat kernel associated with the relevant elliptic operator on a bounded four-dimensional manifold $M$ admit, in the context of (9), the form

\begin{equation}
A_{2k} = \int_M a_{2k}^{(0)} \sqrt{g}d^4x + \int_{\partial M} a_{2k}^{(1)} \sqrt{h}d^3x
\end{equation}

\begin{equation}
A_{2k+1} = \int_{\partial M} a_{2k+1}^{(1)} \sqrt{h}d^3x
\end{equation}

with $h$ being the induced metric on the boundary.

The local interior coefficients $a_{2k}^{(0)}$ are specified by the same local invariants as in the unbounded manifold of the same local geometry and do not, for that matter, depend on the boundary conditions. The far more complicated boundary coefficients $a_{2k}^{(1)}$ necessitate knowledge of the geometry of $\partial M$ and of the boundary conditions on it in addition to knowledge of the geometry of $M$.

If $M$ is specified to be a Riemannian manifold of positive constant curvature then $M$ reduces to $S_4$ in the absence of a boundary and to $C_4$ if $\partial C_4$ is, itself, specified to be a Euclidean three-sphere of constant extrinsic curvature. In either case, the elliptic operator $D$ in (4) associated with free propagation of a massless scalar field conformally coupled to the background metric is the operator which appears as the d’Alembertian in (1)

\begin{equation}
D = \frac{L^2 - \frac{1}{2}n(n-2)}{2a^2}
\end{equation}
This elliptic operator is unbounded on the Euclidean de Sitter space $S_4$ and its zeta-
function evaluation results in the one-loop effective action $[^7, ^8]$

\[ W_0 = \frac{1}{90} \frac{1}{\epsilon} + O(\epsilon^0); \quad \epsilon = 4 - n \]

for a conformal scalar field with an associated anomalous trace contribution

\[ < T^c >_r = -\frac{1}{90} \frac{1}{a^4 \Omega_5} \]

The elliptic operator in (17) is bounded on $C_4$. The evaluation of the trace anomaly
due to free scalar propagation on that manifold necessitates the asymptotic expansion
(9) of the supertrace

\[ G(t) = \int_C trK_D(\eta, \eta, t)d^4\eta \]

of the heat kernel $K_D(\eta, \eta', t)$ associated with the bounded elliptic operator $D$ in (16).

It is worth emphasising that, despite appearances stemming from the homogeneous
Dirichlet condition $\Phi|_{\partial C_4} = 0$, the boundary term

\[ \int_{\partial C} d^3\eta K_\Phi^2 \]

in (1) does not vanish. Such a non-vanishing effect arises as a result of the boundary
condition

\[ K_D(\eta, \eta', t = 0) = \delta^{(4)}(\eta - \eta') \]

- imposed on the solution to the heat equation on $C_4$

\[ \left( \frac{\partial}{\partial t} + D \right) K_D(\eta, \eta', t) = 0 \]

- which offsets the effect of the homogeneous Dirichlet condition on $\partial C_4$ $[^9]$.

The expressions (14) and (15) for the expansion coefficients $A_k$ in (9) reduce, respec-
tively, to

\[ A_{2k}(D, C_4) = \int_C c_{2k}^{(2)} d^4\eta + \int_{\partial C} c_{2k}^{(1)} d^3\eta_B \]

and

\[ A_{2k+1}(D, C_4) = \int_{\partial C} c_{2k+1}^{(1)} d^3\eta_B \]

on $C_4$, with $D$ specified in (16) and with the embedding coordinate vector $\eta_B$ specifying
the spherical boundary hypersurface of maximum colatitude $\theta_0$. As stated in the context
of (8) and (13), the trace anomaly on $C_4$ is associated with the $A_4(D, C_4)$ coefficient in (22).

If the curvature of a Riemannian manifold $M$ satisfies the vacuum Einstein equations with a cosmological constant

\[ R_{\mu\nu} = \Lambda g_{\mu\nu} \]

then the coefficients $a_4^{(0)}$ and $a_4^{(1)}$ specifying $A_4$ in (14) will admit the expressions

\[ a_4^{(0)} = \alpha_0 \Lambda^2 + \alpha_2 R_{abcd} R^{abcd} \]

and

\[ a_4^{(1)} = \beta_1 \Lambda k + \beta_2 k^3 + \beta_3 k_{ab} k_{ab} + \beta_4 k_{b} k_{c} k_{d} + \beta_5 C_{abcd} k_{ac} n^b n^d \]

respectively, where in (26) $k_{ab}$ is the extrinsic curvature of $\partial M$ and $n$ is the vector normal to $\partial M$. The expressions in (25) and (26) essentially disentangle the geometry-related contributions to the $A_4$ coefficient from those contributions which depend on the elliptic operator and boundary conditions. Specifically, the coefficients $\alpha_0$ and $\alpha_2$ multiplying respectively the geometry-dependent expressions in (25) depend exclusively on the operator in whose heat-kernel asymptotic expansion $A_4$ is the constant coefficient. Likewise, the five coefficients $\beta_i$ which respectively multiply the geometry-related expressions in (26) depend only on the same operator and the conditions specified on the boundary.

If, in the case of $\Lambda > 0$, a boundary condition imposed on (24) is that of a compact four-geometry then the solution to (24) can be either the spherical cap $C_4$ or the Euclidean four-sphere $S_4$. The former case emerges if the remaining boundary condition corresponds to the specification of the induced three-geometry as a three-sphere. The latter case emerges if the remaining boundary condition corresponds to the absence of a boundary. In addition, the former case reduces to a disk $D$ at the limit of boundary three-spheres small enough to allow for their embedding in flat Euclidean four-space. These three solutions are aspects of the Hartle-Hawking no-boundary proposal for the quantisation of closed universes. For the stated boundary conditions these solutions to (24) also coincide with the corresponding solutions to the Euclidean Einstein field equations in the presence of a massless scalar field conformally coupled to gravity on the additional Dirichlet condition of a constant field on $\partial C_4$, in the present case of $C_4$ as well as in that of $D$. Such a coincidence is a consequence of a vanishing stress tensor for the conformal scalar field.

In effect, the constant coefficient $A_4(D, C_4)$ in the heat kernel asymptotic expansion for a conformal scalar field on $C_4$, the corresponding $A_4(D, S)$ on $S_4$, as well as the corresponding coefficient $A_4(D, D)$ on $D$ are expected to be inherently related. With $\theta_0$ being the maximum colatitude on $C_4$, which for that matter specifies $\partial C_4$, the stated relation is

\[ A_4(D, C_4) = A_4(D, S)(\frac{1}{2} - \frac{3}{4} \cos^2 \theta_0 + \frac{1}{4} \cos^3 \theta_0) + A_4(D, D) \cos^3 \theta_0 + \frac{9}{8} \beta_1 \cos \theta_0 \sin^2 \theta_0 \]
where, in conformity with (8), (13) and (18), it is

\[(28)\]
\[A_4(D, S) = -\frac{1}{90}\]

Moreover, the corresponding value in the case of \(\Phi_{|\partial D} = 0\) is

\[(29)\]
\[A_4(D, D) = -\frac{1}{180}\]

and the value of the coefficient \(\beta_1\) for the present case of \(C_4\) with \(\Phi_{|\partial C_4} = 0\) is

\[(30)\]
\[\beta_1 = \frac{29}{135}\]

In effect, equation (27) yields \(A_4(D, C_4)\) for free scalar propagation.

In the context of (8) and (13) the result which (27)-(30) signify relates to the conformal anomaly through

\[(31)\]
\[\int_C d^4\eta < T_c^{(C)} + \int_{\partial C} d^3\eta < T_c^{(\partial C)} = A_4(D, C_4)\]

In order to arrive at a local expression for the trace anomaly on \(C_4\) use will be made of the stated fact that on any bounded manifold the local interior coefficients \(a_{2k}^{(0)}\), associated through (14) with the asymptotic expansion of the supertrace of the heat kernel in (9), are specified by the same local invariants as in the unbounded manifold of the same local geometry. This, in turn, reveals in the context of (9) that the local asymptotic expansion of the heat kernel associated with the operator \(D\) in (16) exclusively in the interior of \(C_4\) is in coincidence with the local asymptotic expansion of the heat kernel for the same operator on \(S_4\) so that (22) yields

\[(32)\]
\[c_{2k}^{(0)} = s_{2k}^{(0)}\]

with \(s_{2k}^{(0)}\) being the local coefficients \(a_{2k}^{(0)}\) if \(M\) in (14) is specified as \(S_4\). Setting \(k = 2\) and integrating in the interior of \(C_4\) yields

\[(33)\]
\[\int_C d^4\eta c_4^{(0)} = \int_C d^4\eta s_4^{(0)}\]

and, through (31) and (22)

\[(34)\]
\[< T_c^{(C)} >_{r} = s_4^{(0)}\]

In the context of (14), however, (18) amounts to

\[(35)\]
\[\int_S d^4\eta < T_c^{(r)} >_{r} = -\frac{1}{90} = \int_S d^4\eta s_4^{(0)}\]
so that in view of the constancy of $s^{(0)}_{2k}$ on $S_4$ it is

$$s^{(0)}_4 = -\frac{1}{90} \frac{1}{a^4 \Omega_5}$$

Substituting this result in (34) yields

$$< T^c_r >^{(C)} = -\frac{1}{90} \frac{1}{a^4 \Omega_5}$$

This is the desired local expression for the trace anomaly in the interior of $C_4$. As expected, it coincides with the corresponding expression in (18) for the trace anomaly on $S_4$.

Finally, substituting (37) in (31) yields

$$\int_{\partial C} d^3 \eta < T^c_r >^{(\partial C)} = A_4(D, C_4) + \frac{1}{90} \left( -\frac{1}{3} \sin^2 \theta^0_4 \cos \theta^0_4 - \frac{2}{3} \cos \theta^0_4 + \frac{2}{3} \right)$$

with [1]

$$\int_C d^4 \eta = a^4 2 \pi^2 \left( -\frac{1}{3} \sin^2 \theta^0_4 \cos \theta^0_4 - \frac{2}{3} \cos \theta^0_4 + \frac{2}{3} \right)$$

Again, the boundary-related contribution $< T^c_r >^{(\partial C)}$ is constant on $\partial C_4 \equiv S_3$. Consequently,

$$< T^c_r >^{(\partial C)} = \frac{1}{a^3 4 \pi} A_4(D, C_4) + \frac{1}{90} \frac{1}{a^3 4 \pi} \left( -\frac{1}{3} \sin^2 \theta^0_4 \cos \theta^0_4 - \frac{2}{3} \cos \theta^0_4 + \frac{2}{3} \right)$$

This is the desired local expression for the trace anomaly on $\partial C_4$. As expected, it is contingent upon the specified homogeneous Dirichlet condition through (27)-(30).

In addition to their direct physical significance for the dynamical behaviour of scalar fields on bounded manifolds such results as have been obtained herein are also of relevance to the holographic principle and to Quantum Cosmology.

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