Quaternion Generalization of the
Laughlin State and the Three
Dimensional Fractional QHE.

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ABSTRACT: The 3D state of strongly correlated electrons is proposed, which in the external magnetic field $\vec{B}$ exhibits the fractional quantum Hall effect, with the zero temperature conductivity tensor $\sigma_{ij} = (e^2/h)(1/m) \sum_k \epsilon_{ijk} B^k / |\vec{B}|$. The analog of Landau and Laughlin states in 3D are given using quaternion coordinates as generalization of complex coordinates. We discuss the notion of the fractional statistics in 3D introduced recently by Haldane.

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The concept of “anyons” or “fractional statistics” \(^1\) particles has been intensively investigated recently. Soon after experimental discovery of the Fractional QHE (FQHE)\(^2\) Laughlin proposed variational wave function which describes the incompressible electron liquid in 2D in external magnetic field at fractional filling factors, which naturally leads to the “fractional statistics” of the quasiparticles \(^3\). The holomorphic structure of the Laughlin state rely essentially on the fact that the coordinate space of electron liquid is 2D. Mathematically 2D space allows the existence of the abelian representations of the braid group instead of permutation group in higher dimensions \(^4\). Physically Wilczek’s construction of anyons as the bound state of particles with fluxes is very natural namely in 2D space.

However we were led to a conclusion that some aspects of the 2D FQHE state can be generalized on to strongly correlated states of electrons in higher dimensions, namely the 3D. I will present the variational wave function of the isotropic 3D electron liquid subjected in to strong magnetic field \(\vec{B}\). This state can be considered as a generalization of the Laughlin state on 3D electron liquid. The conductivity tensor of this state generalizes the FQHE conductivity:

\[
\sigma_{ij} = \frac{e^2}{h} \frac{1}{m} \sum_{k} \epsilon_{ijk} e^k
\]

with \(m\)-odd integer, and \(\epsilon_{ijk}\) - antisymmetric tensor. Analogous formula for IQHE in 3D has been obtained by Halperin\(^5\) and Kunz\(^6\). Detailed numerical calculations for noninteracting particles were done by Montambaux and Kohmoto \(^7\).

This state supports the fractional statistics of the quasiparticle excitations, which have nontrivial braiding phases. Presence of external magnetic field \(\vec{B}\) provides the external axis in otherwise isotropic 3D space what allows the braid group classification of the particles paths in the plane \(\pi_{\vec{B}}\), perpendicular to the magnetic field \(\vec{B}\).

I will also consider the vortex-like quasiparticle wave function in 3D which is a candidate for fractional statistics in Haldane’s definition \(^8\).
Generalization of the Landau state and IQHE.

Consider first the generalization of the lowest Landau level (LLL) wave function on 3D space. In 2D case the Landau wave function is written in terms of holomorphic coordinates $z_j = x_j + iy_j, j = 1,...N$ is the particle number. The LLL wave function

$$\Psi_{LL}^{2D} = \prod_{i<j} (z_i - z_j) e^{-\frac{1}{4}|z_i|^2}$$

satisfies the following conditions:

(a) it is odd under any permutations;

(b) it is isotropic;

(c) it has a Jastrow form and is homogenous polynomial.

To construct the analog of Landau state in 3D space, now consider coordinates $x_j, y_j, z_j$ in 3D. (There should be no confusion between $z_i$ - the third coordinate in 3D space and complex 2D coordinate $z_j = x_j + iy_j$. To generalize 2D holomorphic coordinates on 3D space, introduce complex quaternions $q \in \mathcal{H}$.

$$q = \hat{i}x + \hat{j}y + \hat{k}z, \bar{q} = -q$$

$$(3a)$$

$$\hat{i}\hat{j} = -\hat{j}\hat{i} = \hat{k}, \hat{j}\hat{k} = \hat{i} = -\hat{k}\hat{j}, \hat{k}\hat{i} = \hat{j} = -\hat{i}\hat{k}$$

$$(3b)$$

$$\hat{i}^2 = \hat{j}^2 = \hat{k}^2 = -1$$

Complex quaternions realize the mapping of configuration of the particles in 3D space onto the $\mathcal{H}$ space of quaternions. For each position of spinless particles $x_i, y_i, z_i, i = 1,...N$, consider corresponding weight with $q_i = \hat{i}x_i + \hat{j}y_i + \hat{k}z_i$

$$\Psi_{LLL}^{3D}(r_i) =: \prod_{i<j} (q_i - q_j)$$

$$(4)$$
Where because of noncommutativity of multiplication in $\mathcal{H}$, the ordering of multiplication is introduced. For example, one may choose to multiply $q_1$ with all $q_i, i > 1$ first, then $q_2$ with $q_i, i > 2$, etc. in the ordered way in (4). The “wave function” $\Psi_{LLL}^{3D}$ is a 3D generalization of the Landau state (its holomorphic part more precisely). It obviously satisfies the conditions a) - c) mentioned above. However it has no physical meaning itself because $\Psi_{LLL}^{3D}$ is $SU(2)$ valued. We will project out extra degrees of freedom in it using the fact that external magnetic field provides natural coordinate axis. This operation corresponds to taking $U(1)$ subgroup of $SU(2)$.

It is convenient to use the isomorphism of quaternion algebra $\mathcal{H}$ - to the algebra of Pauli matrices:

\[
\hat{i} = -i\sigma^1; \hat{z} = -i\sigma^2; \hat{k} = -i\sigma^3
\] (5)

Then $\Psi_{LLL}^{3D}$ can be rewritten as

\[
\Psi_{LLL}^{3D}(r_i) =: \prod_{i<j} | \vec{r}_i - \vec{r}_j | e^{-i\vec{\sigma} \cdot \vec{h}_{ij} \pi / 2} : \exp(-\frac{1}{4} | q_i |^2)
\] (6)

\[
=: \prod_{i<j} (\sigma^1 x_{ij} + \sigma^2 y_{ij} + \sigma^3 z_{ij}) : \exp(-\frac{1}{4} | q_i |^2)
\]

with $\vec{h}_{ij} = \frac{\vec{r}_i - \vec{r}_j}{| \vec{r}_i - \vec{r}_j |}$, $x_{ij} = x_i - x_j$, $y_{ij} = y_i - y_j$ and $z_{ij} = z_i - z_j$.

To make the plasma analogy, so useful in 2D, applicable, the $\Psi_{LLL}^{3D}(r_i)$ is multiplied by a scalar factor $\prod_i \exp(-\frac{1}{4} | q_i |^2)$. This will ensure the incompressibility of this state after projection (see below) and enables to construct vortex like quasiparticles. Eq.(6) can be generalized to $\Psi_{LLL}^{3D}(r_i) =: \prod_{i<j} f(| \vec{r}_i - \vec{r}_j |) e^{-i\vec{\sigma} \cdot \vec{h}_{ij} \pi / 2} : \exp(-\frac{1}{4} | q_i |^2)$ with $f(x)$ being some general scalar function. For $f(x) = e^{-\frac{1}{4}x}$ the plasma analogy can be applied to 3D state. Hamiltonian $H_{pl} = -\frac{1}{\beta} \ln(\langle \Psi_{LLL}^{3D}(r_i) | \Psi_{LLL}^{3D}(r_i) \rangle)$ becomes a one-component 3D plasma Hamiltonian.
Next crucial step is to project the “wave function” $\Psi_{LLL}^{3D}(r_i)$ down on to space of complex numbers, to construct wave function of 3D system in external magnetic field. So far $\Psi_{LLL}^{3D}(r_i)$ is matrix valued (SU(2) essentially), as follows from Eq. (6). Introduce the projectors:

$$P_z = \sigma^1 + i\sigma^2, \quad Tr(P_z q_{ij}) = x_{ij} + iy_{ij}$$

$$P_x = \sigma^2 + i\sigma^3, \quad Tr(P_x q_{ij}) = y_{ij} + z_{ij}$$

(7)

$$P_y = \sigma^3 + i\sigma^1, \quad Tr(P_y q_{ij}) = z_{ij} + ix_{ij}$$

and define

$$P_z \Psi_{LLL}^{3D}(r_i) \equiv \prod_{i<j} (Tr P_z q_{ij}) : e^{-\frac{1}{4}|Tr P_z q_{ij}|^2}$$

(8)

$$= \Psi_{LLL}^{2D}(r_i)$$

Where in the l.h.s. $P_z$ is an operator, and its action on coordinates is given by Eq.(7). Below we will drop the sign of $Tr$ since it is obvious. The role of these projector operators $P_i, i = x, y, z$ is to discriminate between three equivalent coordinate axis in the presence of external magnetic field $\vec{B}$. Physically the projected state coincides with the Landau state $P_z \Psi_{LLL}^{3D}(r_i) = \Psi_{LLL}^{2D}(r_i)$, and describes the IQHE of the 3D isotropic sample of correlated matter subjected to strong external magnetic field $\vec{B} \parallel \hat{Z}$. The effect of external field on the orbital motion of the particles is assumed stronger than any internal correlations, incorporated in $\Psi_{LLL}^{3D}(r_i)$. For example the correlations along $z$ direction are completely ignored in this projected state.

Because of incompressibility of $\Psi_{LLL}^{2D}$ projected state $P_z \Psi_{LLL}^{3D}(r_i)$ is also incompressible as it should be. It means that chemical potential of the particles lies in the gap. The Hall
conductance of projected state is nothing more than the conductance of the LLL:

\[ \sigma_{xy} = \frac{e^2}{h} \]  (9)

For any unit vector \[ \vec{e} = e_1 \hat{x} + e_2 \hat{y} + e_3 \hat{z} \] define the projector \( P_{\vec{e}} \):

\[ P_{\vec{e}} = e_1 P_x + e_2 P_y + e_3 P_z \]  (10)

Then the corresponding projected state \( P_{\vec{e}} \Psi_{LLL}^{3D}(r_i) \) is:

\[ P_{\vec{e}} \left( \Psi_{LLL}^{3D}(r_i) \right) = \sum_i e_i P_i \left( \Psi_{LLL}^{3D}(r_i) \right) = \]  (11)

\[ = \sum_i e_i \left( \Psi_{LLL}^{2D}(r_i) \text{ for } \vec{B} \parallel \vec{e} \right) \]

From Eqs. (9) - (11) follows that in \( \vec{B} = \vec{e} | \vec{B} | \), conductivity tensor of the state described by Eq. (11) is:

\[ \sigma_{ij} = \frac{e^2}{h} \sum_k \epsilon_{ijk} e_k \]  (12)

Eq. (12) is a generalization of the IQHE conductivity for 3D electron liquid in magnetic field.

Closely related formula has been discussed previously \(^5,6,7\) in the context of the conductivity of 3D electron gas in external periodical potential. Because of continuum limit taken from the beginning, instead of reciprocal lattice vectors in the case of external potential, in Eq. (12) enters the unit vector of external field. It is important to realize that the gap in state \( \Psi_{LLL}^{3D}(r_i) \) and its fractional generalizations (see below) is produced by internal electron correlations, in contrast to gap due to external potential in \(^5,6,7\).

**Generalizations of the Laughlin state and 3D FQHE.**

Generalization of the Laughlin state

\[ \Psi_{3D}^{2D}(r_i) = \prod_{i<j} (z_i - z_j)^m \exp\left( -\frac{1}{4} | z_i |^2 \right) \]  (13)
is not unique and can be done in two distinct ways.

Firstly, consider 3D Laughlin state in strong magnetic field $\vec{B} \parallel \vec{e}$ as a projected state defined by:

$$P_\vec{e}\Psi_{3D}^L(r_i) = \sum_{i=1}^3 e_i P_i \Psi_{3D}^L(r_i) =$$

$$= \sum_i e_i : \prod_{k<\ell}(P_i(q_k - q_\ell))^m : exp\left(-\frac{1}{4} | P_i q_k |^2 \right)$$

(14)

For example, $P_x\Psi_{3D}^L(r_i) = \prod_{k<\ell}(y_{k\ell} + iz_{k\ell})^m exp\left(-\frac{1}{4} (y_{k}^2 + z_{k}^2) \right)$. Projection on each axis leads to the 2D Laughlin state for electrons in magnetic field, parallel to this axis. Because of incompressibility of the projected state $P_\vec{e}\Psi_{3D}^L(r_i)$ for any $\vec{e}$, the chemical potential always lies in the gap induced by electron correlations in this 3D state. The conductivity tensor is given by Eq. (1).

Quasihole wave function is given by the sum of Laughlin’s quasihole wave functions written in proper coordinates for each projection:

$$P_\vec{e}\Psi_{q_o}^3D(r_i) = \sum_i e_i : \left( \prod_k P_i(q_o - q_k) \right) \prod_{k<\ell}(P_i(q_e - q_k))^m : exp\left(-\frac{1}{4} | P_i q_k |^2 \right)$$

(15)

$q_o$ is a position of the quasihole in 3D space. The vortex character of quasihole and quasiparticle as well as their nontrivial braiding properties naturally follows from the form of Eq. (15). For example, from $P_x\Psi_{q_o,q_{o'}}^3D(r_i) = \prod_k(y'_{o\ell} + iz'_{o\ell})(y_{o\ell} + iz_{o\ell}).\prod_{k<\ell}(y_{k\ell} + iz_{k\ell})^m exp\left(-\frac{1}{4} | y_{k}^2 + z_{k}^2 | \right)$, it follows that adiabatic exchange of two quasiholes, located at $q_o$ and $q_{o'}$ in 3D brings the phase $\pi \frac{m}{n}$ per quasihole if the exchange was done along the path having nontrivial projection on the plane $(y, z)$. Indeed, despite in 3D space, it is acceptable to classify the paths of the particles with respect to their winding numbers. The reason for this classification to be well defined, and then leading to the “fractional statistics” is the existence of the magnetic field axis in 3D space what makes the relevant
geometry to be 2D after projection. Clearly the same consideration can be done for any
direction of the magnetic field.

Although the projection of $\Psi_3^D(r_i)$ is defined as Eq. (15), I can not guess the form of
$\Psi_3^D(r_i)$ which leads to Eq. (15). The straightforward generalization of the Laughlin wave
function gives the second possibility to define 3D Laughlin state with different physical
consequences. Define $SU(2)$ (mod scalar function) valued “weight” as:

$$\tilde{\Psi}_3^D(r_i) =: \prod_{i<j} (q_i - q_j)^m : \exp(-\frac{1}{4} | q_i |^2)$$ (16)

Because of $(q_i - q_j)^m = | \vec{r}_i - \vec{r}_j |^{m-1} (q_i - q_j)$ this state is equivalent to

$$\tilde{\Psi}_3^D(r_i) =: \prod_{i<j} | \vec{r}_i - \vec{r}_j |^{m-1} (q_i - q_j) : \exp(-\frac{1}{4} | q_i |^2).$$ (17)

Projected state

$$P e \tilde{\Psi}_3^D(r_i) = \sum_{i=1}^3 e_i : \prod_{k<\ell} | P_i \vec{r}_{k\ell} |^{m-1} (P_i(q_k - q_\ell)) : \exp(-\frac{1}{4} | P_i q_k |^2)$$ (18)

has a form of IQHE state with modified orbital factor, however it does not have a LLL
wave function structure. Probably this state can be obtained using appropriately cho-
sen Haldane’s pseudopotentials \(^\text{11}\) in 2D problem. Thought the holomorphic part of the
wave function has a simple LLL structure, the density of part
icles in this state $\rho(r) = < P e \tilde{\Psi}_3^D(r_i) | \sum_i \delta(\vec{r}_i - \vec{r}) | P e \tilde{\Psi}_3^D(r_i) >$ is $\rho_{LLL}/m$. The quasihole state in $\tilde{\Psi}_3^D(r_i)$ is given by

$$\tilde{\Psi}_{q_o}^3(r_i) =: \prod_{k<\ell} (q_o - q_k)(q_k - q_\ell)^m : \exp(-\frac{1}{4} | q_i |^2).$$ (19)

For the projected quasihole state $P e \tilde{\Psi}_{q_o}^3(r_i)$ the Berry phase argument leads to the statistics $\pi/m$ for quasiparticles. (See Appendix A) This is a somewhat surprising result, since all our insight comes from the Laughlin states, where each electron corresponds to the $m$
fluxes, while quasihole corresponds to unit flux what explains $1/m$ in charge and statistics
of quasiparticles in FQHE. It turns out that the crucial feature of the FQHE to support
“fractional statistics” is *incompressibility* which fixes the particle density deficit at quasiparticle positions to be $\pm \frac{1}{m}$. The original particles does not have to be bound to $m$ fluxes in order to get “fractional statistics” of quasiparticles.

As a possible example of fractional statistics in Haldane’s definition, consider the simplest quasiparticle state in 3D space given by $\tilde{\Psi}^{3D}_{q_o}(r_i)$. Although this object is not a wave function, it is not relevant for the discussion below. Winding properties of quasiparticles at positions $q_o$ and $q'_o$ are trivial - the Berry phase for winding is given by the solid angle $\gamma = \Omega_c$ subtended by a unit vector $\vec{n} = \frac{\vec{r}_o - \vec{r}_{o'}}{|\vec{r}_o - \vec{r}_{o'}|}$ transforming along contour $C$.

For any true permutation $\Omega_c = 2\pi$, what makes quasiparticles to be fermions, as well as underlying particles. We just obtained the well known fact - in 3D the classification of the paths of the particles is given by the permutation group $P_n$, which allows only the fermi and bose statistics. From this point of view fractional statistics is forbidden in 3D. However, recently Haldane proposed the generalization of the notion of the fractional statistics, based on the counting number of states in the Hilbert space for the quasiparticles which can be relevant for this 3D state.

Consider quasiparticle excitation in the form, consistent with 2D FQHE case:

$$\tilde{\Psi}^{3D}_{+q_o}(r_i) =: \prod_{k<\ell} (q_o - q_k)^{-1} \hat{Q}_{q_o}(q_k - q_\ell)^m : \exp(-\frac{1}{4} | q_\ell |^2)$$

with $q_o$ as the position of the quasiparticle and, $\hat{Q}_{q_o}$ as the projection operator reviving states with $(q_k - q_o)^a \exp(-\frac{1}{4} | q_k |^2)$ from the many body state to avoid divergency in Eq.(20). The subscript $+$ in $\tilde{\Psi}^{3D}_{+q_o}$ is introduced to distinguish between quasiparticle state $\tilde{\Psi}^{3D}_{+q_o}$ and quasi-hole state $\tilde{\Psi}^{3D}_{q_o} = \tilde{\Psi}^{3D}_{-q_o}$. Fixing boundary conditions in 3D will lead to the constraint

$$Nm + N_+ - N_- = const$$

analogously to the 2D FQHE case. Where $N$ is the number of particle, $N_\pm$ is the number
of quasiparticles and quasiholes in state $\tilde{\Psi}^{3D}(r_i)$. The constraint Eq.(21) survives after projection operation on any axis, when $P_\epsilon \tilde{\Psi}^{3D}_L(r_i)$ simply becomes the Laughlin state. If the dimension of the Hilbert space for quasiparticles is given by $d = N$, changing the $N^\pm$ by multiples of $m$ will lead to the fractional statistics of quasiparticles in Haldane’s form:

$$\Delta d = \pm \frac{1}{m} \Delta N^\pm \quad (22)$$

To summarize, the isotropic 3D state of correlated electron liquid in external magnetic field is proposed. This state exhibit the fractional Hall conductivity tensor Eq.(1), which is transversal to the external magnetic field at zero temperature. Quite generally the form of the conductivity tensor given by Eq. (1) is fixed by the isotropy of the original 3D state and the presence of the uniaxial field $\vec{B}$. The time reversal symmetry properties of $\sigma_{ij}$ and $\vec{B}$ are the same - they both are odd under $t \rightarrow -t$. $\sigma_{ij}$ and $\vec{B}$ are also odd with respect to parity $P$ transformations. For a 3D electron system with the Fermi level lying in the gap of the energy spectrum the conductivity tensor has to be antisymmetric, as it is given by Eq. (1). Moreover the components of the conductivity tensor, corresponding to the electric field $\vec{E} \parallel \vec{B}$ does not enter since they cannot produce persistent currents and will be screened out.$^{13}$

To generalize the complex coordinates on to 3D the quaternion algebra is used. Although the 3D weight is matrix valued, the projection on to plane perpendicular to the external field is used to construct true complex wave function which is the weighted sum of the Laughlin states in different planes. The 3D analogs of the quasiparticle and quasihole excitations are found. Again, presence of the magnetic field allows the braid group classification of the particle paths what leads to “fractional statistics”. If to consider the unprojected state as “wave function” for some problem, the winding properties of the quasiparticles in 3D are trivial. However it is argued that these excitations can have “fractional statistics” in Haldane’s definition, using counting of states in the Hilbert space.$^8$
Physically this kind of 3D state can be obtained, if to consider the set of QHE planes, perpendicular, say to \( \hat{z} \), with strong interplanar coupling which makes the problem essentially 3D. Then tilting of the magnetic field may cause the coherent interplanar hopping of electrons in order to build the cyclotron orbits in the external field. The states constructed in this article can not be obtained perturbatively in interplanar hopping matrix element from the Laughlin 2D state, and represent the new phase of 3D electron liquid in external field.

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Appendix A

We describe here the derivation of the statistics of the quasiparticles in projected state $P_{\vec{c}} \tilde{\Psi}^3_{q_0}(r_i)$, Eq. (18). Choose $\vec{c} = \hat{z}$ for simplicity, the result will be independent on orientation of the field, as long as contours involved have nonzero projection onto the plane, perpendicular to the field.

$$P_{\vec{z}} \tilde{\Psi}^3_{q_0}(r_i) = \prod_j (x_{oij} + i y_{oij}) \prod_{j<k} (x_{jki} + i y_{jki}) | P_{\vec{z}} \tilde{r}_{jk} |^{m-1} \times exp(-\frac{1}{4} | x_j^2 + y_j^2 |) \quad (A.1)$$

The Berry phase for adiabatic transport $q_0(t)$ along some contour $C$ in ($xy$) plane is

$$\gamma = i \oint < P_{\vec{z}} \tilde{\Psi}^3_{q_0(t)}(r_i) | \frac{d}{dt} | P_{\vec{z}} \tilde{\Psi}^3_{q_0(t)}(r_i) > dt \quad (A.2)$$

The matrix element of time derivation in (A.2) can be rewritten as

$$< P_{\vec{z}} \tilde{\Psi}^3_{q_0(t)}(r_i) | \frac{d}{dt} \sum_j \ln (x_{oij}(t) + i y_{oij}(t)) | P_{\vec{z}} \tilde{\Psi}^3_{q_0(t)}(r_i) >$$

The Berry phase then is:

$$\gamma = i \oint d\omega \oint d^2 z \rho(z) \frac{1}{z_0 - z} = \pm \frac{2\pi}{m} \rho_{LLL} \text{Area} \quad (A.3)$$

with $z = x + iy$, and using the fact that density of particles $\rho(z) = \frac{1}{m} \rho_{LLL}$ in state $P_{\vec{z}} \tilde{\Psi}^3_{L}(r_i)$:

$$\rho(z) = < P_{\vec{z}} \tilde{\Psi}^3_{L}(r_i) | \sum_i \delta(z - z_i) | P_{\vec{z}} \tilde{\Psi}^3_{L}(r_i) >$$

$$= < \Psi^2_{L}(r_i) | \sum_i \delta(z - z_i) | \Psi^2_{L}(r_i) > = \rho_{1/m} \quad (A.4)$$

And $\rho_{1/m}$ is the density of the Laughlin state Eq.(13).
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