Constraining Low-Energy Proton Capture on Beryllium-7 through Charge Radius Measurements

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Abstract

In this paper, we point out that a measurement of the charge radius of Boron-8 provides indirect access to the S-factor for radiative proton capture on Beryllium-7 at low energies. We use leading-order halo effective field theory to explore this correlation and we give a relation between the charge radius and the S-factor. Furthermore, we present important technical aspects relevant to the renormalization of pointlike P-wave interactions in the presence of a repulsive Coulomb interaction.
I. INTRODUCTION

A frequent problem encountered in nuclear astrophysics is that observables relevant to stellar and cosmological evolution cannot be measured directly in the laboratory. In many cases, this is due to the Coulomb barrier, which leads to very small cross sections at the low energies at which these processes occur in the stellar environment. Specifically, electroweak reactions relevant to big bang nucleosynthesis or solar astrophysics that involve either neutrinos or soft photons display this problem. A common strategy is then to carry out experiments of the desired observable at higher energies and to use models to extrapolate to the low-energy region. One process that has been analyzed frequently this way is radiative proton capture on $^7$Be into $^8$B. In the solar pp fusion reaction network, this reaction proceeds at $\sim 10$ keV energies, and the subsequent beta decay determines the intensity of the high-energy part of the solar neutrino spectrum. The currently accepted threshold value of the corresponding S-factor

$$S(0) = (20.8 \pm 0.7\text{(expt.)} \pm 1.4\text{(theor.)}) \text{ eV b}$$

has been obtained by extrapolation of experimental data [1].

Effective field theory (EFT) provides an alternative way to obtain information on such observables. The EFT algorithm correlates a finite number of physical observables at each order, based on a power counting scheme that captures the relevant scales of the system. The simplest example might be the correlation between the scattering length $a$ and the binding energy of two particles with mass $m$, $B_2 = 1/(ma^2)$, when $a$ is positive and much larger than the range of the interaction $R$. The application of these ideas to nuclear physics has led to the development of the so-called pionless EFT, which is applicable for typical momenta below the pion mass. (See Refs. [2–4] for reviews and references to earlier work.) If one also introduces field operators for certain tightly bound clusters of nucleons, such as $\alpha$ particles, the pionless EFT is usually referred to as halo EFT [5, 6]. It has been applied to describe the structure and reactions of a number of known and suspected one- and two-neutron halo systems in the Helium [7, 8], Lithium [9–13], Beryllium [9, 10, 14], Carbon [9, 10, 15–17], and Calcium isotope chains [18].

For reactions with two or more charged particles, Coulomb effects are important. Although the Coulomb interaction is perturbative for intermediate and higher energies, it has to be treated nonperturbatively close to threshold. In the two-nucleon sector this was first
discussed by Kong and Ravndal for the proton–proton channel [19, 20] and later extended to next-to-next-to-leading order by Ando et al. [21]. A renormalization-group analysis of proton–proton scattering in a distorted-wave basis was performed in Refs. [22, 23] and αα scattering, which is governed by a shallow S-wave resonance close to threshold, was discussed in Ref. [24]. The extension to describe the structure and charge radii of S-wave proton halo bound states with repulsive Coulomb interaction was recently carried out in Ref. [25].

Here we focus on the ⁷Be-proton system which has been considered previously in halo EFT by Zhang, Nollett and Phillips [26]. They calculated the S-factor of the reaction ⁷Be(p, γ)⁸B at leading order (LO) and obtained \( S(0) = (18.2 \pm 1.2(\text{ANC only})) \) eV b consistent with the recommended value in Eq. (1). Moreover, they performed a detailed error analysis and investigated the sensitivity of their result to the input parameters. The scope of our work is somewhat different. Modern isotope shift measurements facilitate an extremely precise determination of the nuclear charge radius. We will show below that the charge radius of ⁸B and the astrophysical S-factor of the reaction ⁷Be(p, γ)⁸B are free of short-distance counterterms at LO in halo EFT. Thus, these two observables are correlated at this order and a measurement of the charge radius can provide independent information on the S-factor.

Our strategy is twofold: First, we match our low-energy coupling constants to the asymptotic normalization coefficients (ANCs) determined from ab initio calculations and transfer reaction measurements [27–29] and the proton separation energy of ⁸B in order to calculate the S-factor at threshold. Second, we consider the charge radius of ⁸B and demonstrate that it provides complementary information on the astrophysical S-factor. In this connection we employ two different field theories; with and without explicit inclusion of the excited-core state. The addition of the excited-core field corresponds to an inclusion of short-distance physics, and therefore allows to study convergence properties of the EFT approach in LO calculations.

The paper is structured as follows. In Sec. II, we introduce the halo EFT for the ⁸B system. We provide a detailed discussion of the renormalization in a P-wave channel with Coulomb interactions and the relevant low-energy constants at LO. In Sec. III, we calculate the charge form factor of ⁸B. The radiative capture reaction, ⁷Be(p, γ)⁸B, is considered in Sec. IV, and the correlation with the charge radius is discussed in Sec. V. We conclude with a summary in Sec. VI.
II. HALO EFT FOR P-WAVE INTERACTIONS BETWEEN CHARGED PARTICLES

To describe the proton-⁷Be system, we employ halo EFT. The Lagrangian describing the P-wave interaction between the proton and core is given by [26]

\[
\mathcal{L} = p_\sigma^\dagger \left( iD_t + \frac{D^2}{2m} \right) p_\sigma + \bar{c}_\sigma^\dagger \left( iD_t + \frac{D^2}{2M} \right) c_\sigma \\
+ \bar{c}_\sigma^\dagger \left( iD_t + \frac{D^2}{2M} - E_* \right) c_\sigma + d_\alpha \left[ \Delta + \nu \left( iD_t + \frac{D^2}{2M_{\text{tot}}} \right) \right] d_\alpha \\
- g_1 \left[ d_\alpha^\dagger C_\alpha^{\sigma\sigma} c_\alpha \left( \left( 1-f \right) i \nabla_k - fi \nabla_k \right) \right] p_\sigma + \text{h.c.} \\
- g_2 \left[ d_\alpha^\dagger C_\alpha^{\sigma\beta} c_\alpha \left( \left( 1-f \right) i \nabla_k - fi \nabla_k \right) \right] p_\sigma + \text{h.c.} \\
- g_\ast \left[ d_\alpha^\dagger C_\alpha^{\sigma\chi} \bar{c}_\chi \left( \left( 1-f \right) i \nabla_k - fi \nabla_k \right) \right] p_\sigma + \text{h.c.} + \ldots .
\]  

(2)

Here \( p_\sigma \) denotes the spin-1/2 proton field with mass \( m \), \( c_\alpha \) (\( \bar{c}_\alpha \)) the spin-3/2 core (spin-1/2 excited core) field with mass \( M \) and \( E_* \) denotes the energy of the excited state. The parameter \( f \) denotes the mass ratio \( m/M_{\text{tot}} \), where \( M_{\text{tot}} = M + m \). The covariant derivatives are defined as \( D_\mu = \partial_\mu + ie\hat{Q}A_\mu \). The halo field \( d_\alpha \) has mass \( M_{\text{tot}} \) and the vertices where the halo field breaks up into a proton and a core have strengths \( g_1 \) and \( g_2 \) for the \( S = 1, 2 \) channels, respectively. Furthermore, the vertex where the halo field breaks up into a proton and an excited core in the \( S = 1 \) channel has strength \( g_\ast \). We define the coupling \( g^2 = g_1^2 + g_2^2 \), since the \( S = 1, 2 \) channels will always appear together in this combination for the observables we consider. The Clebsch-Gordan coefficients in Eq. (2) are defined as \( C_\alpha^{\beta\chi} \left( \begin{array}{c} 2 \beta 1 k \\ \alpha \end{array} \right) \) = \( 2\beta 1 k \left( \begin{array}{c} 2 \alpha \end{array} \right) \), \( C_\beta^{\sigma\chi} \left( \begin{array}{c} 1 \alpha 1 \end{array} \right) \) = \( 1 \alpha 1 \left( \begin{array}{c} 11 \end{array} \right) \), \( C_{\alpha\sigma}^{\beta\chi} \left( \begin{array}{c} \frac{3}{2} \alpha 1 \frac{1}{2} \chi \end{array} \right) \) = \( \frac{3}{2} \alpha 1 \frac{1}{2} \chi \left( \begin{array}{c} \frac{3}{2} \beta \chi \end{array} \right) \) and \( C_{\sigma\chi}^{\chi} \left( \begin{array}{c} \frac{1}{2} \sigma 1 \chi \end{array} \right) \) = \( \frac{1}{2} \sigma 1 \chi \left( \begin{array}{c} \frac{1}{2} \chi \end{array} \right) \). The combinations \( C_{\alpha\beta}^{\alpha\beta} \), \( C_{\alpha\beta}^{\alpha\beta} \) and \( C_{\alpha\beta}^{\alpha\beta} \) give the two spin channels \( S = 1, 2 \) and the \( S = 1 \) channel for the excited core state and the \( \nabla_k \) in the P-wave interaction vertex is understood to be written as a spherical tensor, that is \( \nabla_{-1} = (\nabla_x - i\nabla_y)/\sqrt{2} \), \( \nabla_0 = \nabla_z \) and \( \nabla_1 = (\nabla_x + i\nabla_y)/\sqrt{2} \). The possible values of the spin indices are: \( \alpha, \beta = -2, -1, 0, 1, 2, a = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, i, j, k = -1, 0, 1 \) and \( \sigma, \chi = -\frac{1}{2}, \frac{1}{2} \).

We include the Coulomb interaction to all orders between the proton and the core by using the Coulomb Green’s function \( G_C \). In terms of Coulomb wavefunctions we can write \( G_C \) in coordinate space as

\[
\langle \mathbf{r}_1 | G_C(E) | \mathbf{r}_2 \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{\psi_p^\dagger(\mathbf{r}_1) \psi_p(\mathbf{r}_2)}{E - p^2/(2m_R) + i\varepsilon},
\]  

(3)
where $m_R = mM/M_{\text{tot}}$ is the reduced mass. Note that we are using round brackets to indicate coordinate-space states. The full details of the Coulomb wavefunctions are given in Appendix A.

A. The irreducible self-energy

To describe the halo bound state we need the irreducible self-energy, which is given in terms of Feynman diagrams in Fig. 1. It is required for the full form of the dressed halo propagator and the resulting LSZ-factor for the halo bound state. The irreducible self-energy, for the ground-state core field, is proportional to two powers of the P-wave vertex $i(g_1 C_{i}^{\sigma_a} C_j^{k_i} + g_2 C_{\beta_i}^{\sigma_a} C_{\alpha j}^{k_i})$ and it is diagonal in the spin index since the Coulomb interaction does not change the spin of the particles. For convenience, we define $\Sigma = \delta^{\alpha\beta} \Sigma^{\alpha\beta} / 5$, and we note that the P-wave vertex squared and traced over spin indices evaluates to

$$
\frac{1}{5} \left( g_1 C_{i}^{\sigma_a} C_j^{k_i} + g_2 C_{\beta_i}^{\sigma_a} C_{\alpha j}^{k_i} \right) \left( g_1 C_{i'}^{\sigma_a} C_j^{k_i'} + g_2 C_{\beta_i'}^{\sigma_a} C_{\alpha j'}^{k_i'} \right) = \frac{1}{3} (g_1^2 + g_2^2) k \cdot k'.
$$

(4)

Remember that for all observables we consider in this paper, the couplings $g_1$ and $g_2$ will always appear in the combination $g^2 = g_1^2 + g_2^2$.

Evaluating the irreducible self-energy for the ground state core field using Eq. (B11) from the Appendix, we find

$$
i \Sigma(E) = \frac{i g^2}{3} \int \frac{d^3 k_1 d^3 k_2}{(2\pi)^6} k_2 \cdot k_1 \langle k_2 | G_C(E) | k_1 \rangle
$$

$$
= \frac{i g^2}{3} \int \frac{d^3 p \cdot X(E_p) \cdot X^*(E_p)}{(2\pi)^3 E - p^2/(2m_R)}.
$$

(5)

The properties of the vector function $X(E_p) = \int d^3 k \, k \psi_p(k)/(2\pi)^3$, where $\psi_p$ is the Coulomb wavefunction, are discussed in Appendix B (cf. Eq. (B1)). Writing $k^2 = 2m_R E$ and using

Figure 1: Irreducible self-energy loop-diagram. The solid line represents the proton, the dashed line the core and the dotted line the excited core field. The shaded blob indicates the Coulomb Green’s function.
Eqs. (B11) and (A11) from the Appendix, we have the integral,

\[ i\Sigma(E) = -ig^2 m_R \int dp \frac{C(1, \eta')^2 p^4}{p^2 - k^2} \]
\[ = -ig^2 m_R \int dp C_{\eta'}^2 \left[ p^2 + k^2 + k^2 C + \frac{k^2 k^2 + k^4}{p^2 - k^2} \right] \]
\[ = -ig^2 \left[ L_3 + (k^2 + k^2) L_1 + (k^2 k^2 + k^4) J^{\text{fin}}(k) \right], \quad (6) \]

where we have defined

\[ L_n = \frac{m_R}{3\pi^2} \int dp C_{\eta'}^2 p^{n-1} \]
\[ J^{\text{fin}}(k) = \frac{m_R}{3\pi^2} \int dp \frac{C_{\eta'}^2}{p^2 - k^2} \]

The \( L_n \) are formally infinite integrals that we will absorb in the couplings \( \Delta \) and \( g \). We have introduced the Sommerfeld factor \( C_{\eta'}^2 = 2\pi \eta'/(\exp(2\pi \eta') - 1) \), with \( \eta' = k_C/p \), and the Coulomb momentum \( k_C = Z_c \alpha m_R \), which is the inverse Bohr radius of the system.

The irreducible self-energy \( \Sigma^* \) computed from the loop diagram involving the excited core field is given by

\[ i\Sigma^*(E) = -ig^2 \left[ L_3 + (k^2 + k^2) L_1 + (k^2 k^2 + k^4) J^{\text{fin}}(k^*) \right], \quad (9) \]

where \( k^* = \sqrt{2m_R (E - E^*)} \).

We can evaluate the finite integral \( J^{\text{fin}} \) using the integral representation of the polygamma function \( \psi [30] \). Writing \( \eta = k_C/k \), we have

\[ \psi(i\eta) + \frac{1}{2i\eta} - \log(i\eta) = -2 \int_0^\infty d\eta' \frac{\eta'}{\eta'^2 - \eta^2} \exp(2\pi \eta') - 1 \]
\[ = \frac{k^2}{\pi k_C} \int_0^\infty dp \frac{1}{p^2 - k^2} \exp(2\pi k_C/p) - 1 \]
\[ = \frac{3\pi k^2}{k_C m_R} J^{\text{fin}}(k), \quad (11) \]

and thus

\[ J^{\text{fin}}(k) = \frac{k_C}{6\pi E} \left( \psi(i\eta) + \frac{1}{2i\eta} - \log i\eta \right). \quad (12) \]

This completes the calculation of the self energies \( \Sigma \) and \( \Sigma^* \).
Figure 2: Integral equation for the full halo propagator. The thick (thin) double line denotes the full (bare) halo propagator.

B. Elastic scattering

The low-energy coupling constants can be related to the effective-range parameters by considering elastic scattering. To achieve this, we first need to write down the full halo propagator with the self-energy bubble summed to all orders.

The bare propagator $D_{\text{bare}}$ in its center of mass frame is given by

$$iD_{\text{bare}}(E) = \frac{i}{\Delta + \nu E + i\varepsilon},$$

and the full propagator is given by the geometric series

$$iD(E) = iD_{\text{bare}}(E) + iD_{\text{bare}}(E) i \left[ \Sigma(E) + \Sigma^*(E) \right] iD(E),$$

which is written in closed form as

$$iD(E) = \frac{iD_{\text{bare}}(E)}{1 + [\Sigma(E) + \Sigma^*(E)] D_{\text{bare}}(E)} = \frac{i}{\Delta + \nu E + \Sigma(E) + \Sigma^*(E) + i\varepsilon}.$$  

Note that the full halo field propagator is given in terms of irreducible self-energy bubbles from both the ground-state core field and the excited-state core field.

Using the P-wave integrals from Eq. (B11) for incoming (–) and outgoing (+) Coulomb wavefunctions, the P-wave elastic scattering t-matrix $T_1$ can now be written as,

$$iT_1(E) = ig^2 D(E) \left( X^+(E)^* \cdot X^-(E) \right)$$

$$= ig^2 D(E) \exp \left( 2i\sigma_1 \right) p^2 \mathcal{C}(1,\eta)^2,$$

where the P-wave pure Coulomb phaseshift is given by $\sigma_1$. The t-matrix is given in terms of observables according to

$$T_1(E) = \frac{6\pi}{m_R} \frac{k^2 \exp \left( 2i\sigma_1 \right)}{k^3 (\cot \delta_1 - i)}.$$
where $\delta_1$ denotes the P-wave phaseshift due to the strong interaction. We can then match Eqs. (16) and (17) to arrive at

$$k^3C(1, \eta)^2(\cot \delta_1 - i) = \frac{6\pi}{m_R} \left[ g^2 D(E) \right]^{-1}.$$  

(18)

The Coulomb-modified effective range expansion (ERE) for the P-wave proton-$^7$Be system, with both the ground state and excited state core fields included, is\(^1\) [31]

$$k^3C(1, \eta)^2(\cot \delta_1 - i) + 2kCh_1(\eta) + \frac{g^2}{g^2}2kCh_1(\eta_*) = -\frac{1}{a_1} + \frac{1}{2}r_1 k^2 + \ldots ,$$  

(19)

where $\eta_* = kC/k_*$ and the function $h_l$ is defined as

$$h_l(\eta) = k^2\frac{C(l, \eta)^2}{C'(0, \eta)^2}\left(\psi(i\eta) + \frac{1}{2i\eta} - \log(i\eta)\right),$$  

(20)

with $\psi$ being the polygamma function. For $l = 1$, this reduces to

$$h_1(\eta) = k^2(1 + \eta^2)\left(\psi(i\eta) + \frac{1}{2i\eta} - \log(i\eta)\right).$$  

(21)

Thus also note that, using Eq. (12),

$$\frac{6\pi}{m_R}k^4(1 + \eta^2)J_{\text{fin}}(E) = 2kCh_1(\eta).$$  

(22)

Combining Eqs. (18) and (19) and using that the $J_{\text{fin}}$ and $h_1$ terms cancel out, according to Eq. (22), we match order by order in $k^2$

$$a_1 = -\frac{m_R}{6\pi} \left[ \frac{\Delta}{g^2} - \left(1 + \frac{g_*^2}{g^2}\right) L_3 - \left(k_C^2 + k_C^2 \frac{g_*^2}{g^2} - 2m_R E^* \frac{g_*^2}{g^2}\right) L_1 \right]^{-1},$$  

(23)

$$r_1 = \frac{12\pi}{m_R} \left[ \frac{\nu}{2m_R g^2} - \left(1 + \frac{g_*^2}{g^2}\right) L_1 \right].$$  

(24)

\(^1\) Note that $g^2$ and $g_*^2$ are both inversely proportional to the wavefunction renormalization $Z$ and their ratio is therefore finite.
Table I: Asymptotic Normalization Coefficients (ANCs) calculated by Nollett and Wiringa [27] and Zhang et al. [26], by Navrátil et al. [28], and extracted from a proton-transfer reaction by Tabacaru et al. [29]. The ANCs are given in fm$^{-1/2}$ for the two spin-channels $S = 1, 2$ ($A_1, A_2$) and for the $S = 1$ channel with an excited core ($A_\ast$).

| Ref.            | $A_1$            | $A_2$            | $A_\ast$           |
|-----------------|------------------|------------------|--------------------|
| “Nollett” [27], [26] | $-0.315(19)$     | $-0.662(19)$     | $0.3485(51)$       |
| “Navrátil” [28]  | $-0.294$         | $-0.650$         | $-$                |
| “Tabacaru” [29]  | $0.294(45)$      | $0.615(45)$      | $-$                |

These are the P-wave scattering parameters for the $J = 2$ channel. The terms above in Eqs. (23) and (24) define the two renormalization conditions needed in the case of a P-wave interaction in the presence of the Coulomb interaction.

We can now calculate the LSZ-residue factor required for wavefunction renormalization.

The residue is given by

$$Z = \left[ \frac{d(D^{-1})}{dE} \right]^{-1} \bigg|_{E = -B}$$

$$= \left[ \nu + \Sigma' + \Sigma'_\ast \right]^{-1} \bigg|_{E = -B}$$

$$= \frac{6\pi}{g^2m_R^2} \left[ r_1 - \frac{2k_C}{m_R} \frac{d}{dE} \left( h_1(\eta) + \frac{g^2}{g_C^2}h_1(\eta_\ast) \right) \right]^{-1} \bigg|_{E = -B},$$

where we have used Eqs. (15), (6), (9) and (24).

C. Matching to the Asymptotic Normalization Coefficients

Above, we derived matching conditions that relate the low-energy coupling constants $\Delta$ and $g$ to the parameters of the Coulomb-modified ERE. In practice, we require the one-proton separation energy and the effective range as input for our calculations. We can obtain the effective range from the ANCs, which are defined as the coefficient of the bound-state wavefunction outside the range $R$ of the interaction. The solution to the radial wavefunction $w(r)$ is proportional to a Whittaker $W$-function in the Coulomb case. We can therefore write

$$w_l(r) = A \cdot W_{-i\eta_{l+1/2}}(2\kappa r), \quad \text{for } r > R.$$  (27)
The P-wave effective range for the $J = 2$ channel is then obtained by using the ANCs either from \textit{ab initio} microscopic calculations, or from a transfer reaction experiment. In this work, we compare three different sets of input ANCs. Firstly, we employ ANCs from variational Monte Carlo calculations by Nollett and Wiringa [27] and by Zhang et al. [26] for the excited-core channel. Secondly, we employ ANCs from an NCSM/RGM calculation by Navrátil et al. [28]. In this case, no error estimate was provided for the published result. Finally, we use ANCs extracted from a proton-transfer experiment by Tabacaru et al. [29]. These three sets of ANCs are denoted “Nollett”, “Navrátil”, and “Tabacaru”, respectively.

To obtain the effective range from the ANCs we use the relation derived by König et al. in Ref. [32]. For the ground state ANCs we obtain

$$A_1^2 + A_2^2 = 2\gamma^2 \Gamma(2 + kC/\gamma)^2 \left[ -r_1 + \frac{2kC}{mR} \frac{d}{dE} \left( h_1(\eta) + \frac{g^2}{g^2} h_1(\eta*) \right) \right]_{E=-B}^{-1},$$

and for the excited state

$$A_2^2 = 2\gamma^2 \Gamma(2 + kC/\gamma*)^2 \left[ -\frac{g^2}{g^2} r_1 + \frac{2kC}{mR} \frac{d}{dE} \left( \frac{g^2}{g^2} h_1(\eta) + h_1(\eta*) \right) \right]_{E=-B}^{-1}.$$  

Using the calculated and measured ANCs for $^8\text{B}$ in the $S = 1, 2$ channels in Table I, we find the effective range

$$r_1 = \begin{cases} 
- (60 \pm 4) \text{ MeV} & \text{(Nollett ANCs)} \\
-63 \text{ MeV} & \text{(Navrátil ANCs)} \\
- (69 \pm 13) \text{ MeV} & \text{(Tabacaru ANCs)} 
\end{cases},$$  

where we have used the excited state ANC calculated by Zhang et al. [26] for all three results presented in Eq. (30). The change in the effective range from including the excited-core state corresponds to an 7-8% increase.

\section{III. THE CHARGE FORM FACTOR}

We are now in the position to obtain results for the charge radius. The charge radius is obtained from the charge form factor $F_C$, that is related to the matrix element

$$\langle \mathbf{p}_2 | J^0 | \mathbf{p}_1 \rangle = (Z_c + 1)eF_C(Q),$$

\footnote{Here we corrected a typographical error in Eq. (85) of Ref. [32].}
Figure 4: Feynman diagram for $\Gamma_{\text{tree}}$ diagram. The curly (double) line indicates the photon (halo) field.

evaluated in the Breit frame. The external photon momentum (or momentum exchange) is given by $Q = p_2 - p_1$. The irreducible diagrams that we need to evaluate are a loop-diagram $\Gamma_{\text{loop}}$, where the $A_0$ photon couples to either the core or the proton, and the tree-diagram $\Gamma_{\text{tree}}$ with the $A_0$ photon coupling to the halo field.

We have calculated the charge radius at LO, using both the EFT that contains only the ground state of the core and one that also includes the excited state of the core. This leads to small differences in the result that will be discussed below. Furthermore, we are limited to a LO calculation since at NLO the short-range operator $d_\alpha^\dagger \nabla^2 A_0 d_\alpha$ enters, which requires a form factor datum for renormalization.

A. Evaluation of $\Gamma_{\text{tree}}$

The photon can couple directly to the halo field at tree level. This diagram, which is shown in Fig. 4, is given by

$$i\Gamma_{\text{tree}} = i\nu e (Z_c + 1).$$  \hspace{1cm} (32)

B. Evaluation of $\Gamma_{\text{loop}}$

In total we have four loop diagrams contributing to the LO charge form factor. These are shown in Fig. 5. They are loop diagrams in which an external $A_0$ photon couples either to the proton, the core or to the excited-core field.

Let us start by writing out the integral for the ground state proton-core loop. The momentum space integral, for momentum transfer $Q$, is given by

$$i\Gamma_{\text{loop}, \text{g.s.}}(Q) = -\frac{g^2 e Z_c}{3} \int \frac{d^3k_1 d^3k_2 d^3k_3}{(2\pi)^9} \mathbf{k}_3 \cdot \mathbf{k}_1 \langle \mathbf{k}_3 | G_C(-B) | \mathbf{k}_2 - fQ/2 \rangle$$

$$\times \langle \mathbf{k}_2 + fQ/2 | G_C(-B) | \mathbf{k}_1 \rangle$$

$$+ \left[ (f \rightarrow 1 - f), \ (Z_c \rightarrow 1) \right]. \hspace{1cm} (33)$$
where \( k_1 \) and \( k_3 \) are the loop-momenta in the outermost loops in the diagrams in Fig. 5, and \( k_2 \) is the loop-momentum in the middle loop. The \([(f \to 1 - f), (Z_c \to 1)]\) term is present since the photon can couple to both the ground state core field and to the proton. Continuing by doing Fourier transforms on the \( k_2 \) momentum, and using Eq. (C7), we write this as

\[
i\Gamma_{\text{loop}, \text{g.s.}}(Q) = -i\frac{g^2 e Z_c}{3} \int \frac{d^3 k_1 d^3 k_2 d^3 k_3}{(2\pi)^9} d^3 r_1 d^3 r_2 k_3 \cdot k_1 \langle k_3 | G_C(-B) | r_1 \rangle \\
x \exp (i k_2 \cdot (r_2 - r_1)) \exp (i f Q \cdot (r_1 + r_2)/2)(r_2 | G_C(-B) | k_1) \\
+ [(f \to 1 - f), (Z_c \to 1)] \\
= -i 3g^2 e Z_c \int d^3 r \exp (i f Q \cdot r) \left| \lim_{r' \to 0} \left( \frac{G_C^{(1)}(-B; r', r)}{r'} \right) \right|^2 \\
+ [(f \to 1 - f), (Z_c \to 1)]. \tag{34}\]

Replacing the limit of the partial-wave projected Coulomb Green’s function by Eq. (A15) we arrive at

\[
\Gamma_{\text{loop}, \text{g.s.}}(Q) = -\frac{e(Z_c + 1)g^2 m_R^2 \Gamma(2 + k_C/\gamma)^2 \gamma^2}{3\pi} \\
x \int dr \left[ 1 - \left( (1 - f)^2 + Z_c f^2 \right) \frac{r^2 Q^2}{6(Z_c + 1)} + \ldots \right] W_{-k_C/\gamma, 3/2}(2\gamma r)^2, \tag{35}\]

where \( W \) is the Whittaker W-function. The order \( Q^2 \) integral in Eq. (35) can be solved numerically.

Using the same steps as above we can also derive the contribution from the excited-core diagrams

\[
\Gamma_{\text{loop}, \text{c.s.}}(Q) = -\frac{e(Z_c + 1)g^2 m_R^2 \Gamma(2 + k_C/\gamma^*_s)^2 \gamma^*_s^2}{3\pi} \\
x \int dr \left[ 1 - \left( (1 - f)^2 + Z_c f^2 \right) \frac{r^2 Q^2}{6(Z_c + 1)} + \ldots \right] W_{-k_C/\gamma^*_s, 3/2}(2\gamma^*_s r)^2. \tag{36}\]

### C. The Charge Radius

The charge radius \( r_C \) is defined through the expansion of the charge form factors in even powers of \( Q \)

\[
F_C(Q) = 1 - \frac{r_C^2}{6} Q^2 + \ldots \tag{37}\]
Figure 5: Feynman diagrams for $\Gamma_{\text{loop}}$ where the photon couples either to the proton (solid line), the core (dashed line) or excited core (dotted line) in the proton-core loop.

The form factor can also be written as the sum of the three contributions discussed above

$$F_C(Q) = \frac{Z}{e(Z_c + 1)} \left( \Gamma_{\text{tree}}^0 + \Gamma_{\text{loop, g.s.}}^0(Q) + \Gamma_{\text{loop, e.s.}}^0(Q) \right),$$

where $Z$ is the wavefunction renormalization in Eq. (26). We evaluate this expression for $Q = 0$ to show that the form factor is normalized correctly. At $Q = 0$, $\Gamma_{\text{loop, g.s.}}$ in Eq. (33) simplifies to

$$i \Gamma_{\text{loop, g.s.}}^0(0) = -ie(Z_c + 1) \int \frac{d^3p}{(2\pi)^3} \frac{X(E_p) \cdot X^*(E_p)}{-B - p^2/(2m_R)^2}.$$

where we have used the $k$-space version of Eq. (3), the orthonormality of the Coulomb wavefunctions and Eq. (B1). Now, comparison to Eq. (5) gives

$$\Gamma_{\text{loop, g.s.}}^0(0) = e(Z_c + 1)\Sigma'(-B).$$

(39)

For the excited core contribution we similarly have

$$\Gamma_{\text{loop, e.s.}}^0(0) = e(Z_c + 1)\Sigma'_e(-B).$$

(40)

The Eqs. (39) and (40) together with Eqs. (25), (32) and (38) lead to the correct normalization

$$F_C(0) = 1.$$ 

(41)

To obtain the charge radius we first define

$$\Gamma_{\text{loop, g.s.}}(Q) + \Gamma_{\text{loop, e.s.}}(Q) = \Gamma_{\text{loop}}^0 + \Gamma_{\text{loop}}^{(2)}Q^2 + \ldots$$

(42)

and use this to expand Eq. (38) in powers of $Q^2$

$$F_C(Q) = \frac{Z}{e(Z_c + 1)} \left( \Gamma_{\text{tree}} + \Gamma_{\text{loop}}^0 + \Gamma_{\text{loop}}^{(2)}Q^2 \right) + \ldots$$

$$= 1 + \frac{Z}{e(Z_c + 1)} \Gamma_{\text{loop}}^{(2)}Q^2 + \ldots,$$

(43)
Table II: The relevant scales and parameters in the $^8B$ system.

| $k_C$  | $\gamma$  | $\gamma_*$ | $r_1$  | $k_\alpha$ | $1/R_{^7Be}$ | $f$ | $Z_c$ |
|--------|------------|-------------|--------|-------------|--------------|-----|------|
| 23.79 MeV | 14.97 MeV  | 30.39 MeV  | \sim 60-70 MeV | 50.86 MeV  | 74.55 MeV    | 1/8 | 4    |

where we used Eq. (41) in the last step. The charge radius is therefore given by

$$r_C^2 = -\frac{Z}{e(Z_c + 1)} \Gamma_{\text{loop}}^{(2)}.$$  \hspace{1cm} (44)

Evaluating the order $Q^2$ integrals in Eqs. (35) and (36) we arrive at the charge radius

$$r_C^2 = \begin{cases} (2.56 \pm 0.08 \text{ fm})^2 & (\text{Nollett ANCs}) \\ (2.50 \text{ fm})^2 & (\text{Navrátil ANCs}) \\ (2.41 \pm 0.18 \text{ fm})^2 & (\text{Tabacaru ANCs}) \end{cases},$$  \hspace{1cm} (45)

using the parameter values for the $^8B$ system in Table II and the ANCs of Nollett and Wiringa [27], Navrátil et al. [28] and Tabacaru et al. [29], and the excited core ANC given in Zhang et al. [26].

The errors given in Eq. (45) are due to the errors on the calculated or experimentally extracted ANCs. We obtain an estimate for the LO halo EFT error by assuming that the breakdown scale of the theory is given by the $\alpha$-threshold of $^7$Be. It is at $E_{\text{thresh}} = 1.5866$ MeV [33] which defines a break-down momentum scale $k_{hi} \sim k_\alpha = 50.86$ MeV. The low-momentum scale is defined by the binding momentum $k_{lo} \sim \gamma = 14.97$ MeV. The expected error for the charge radius squared is therefore $k_{lo}/k_{hi} \sim 30\%$ which is larger than the numerical/experimental errors quoted above.

At the next order, the short-range operator $d_\alpha^\dagger \nabla^2 A_0 d_\alpha$ enters the electromagnetic current. The coefficient of this operator can be fit to the charge radius or the charge form factor. We can analyze, however, the convergence properties of the EFT without a full higher order calculation, by only including the minimal set of explicit degrees of freedom at first and then add more physics by adding additional information into the EFT.

If we remove the excited-core field from the field theory we would arrive at the LO result $r_C^2 = (2.32 \text{ fm})^2$, using the central value of the ANCs by Nollett and Wiringa [27] only. For such a field theory, an estimate of the break-down scale would be the momentum scale corresponding to the core excited-state energy $\sqrt{2m_R E_*} = 26.4$ MeV, and thus the expected EFT error for the charge radius squared would be 57%. Comparing the “Nollett ANCs”
charge radius result in Eq. (45), \( r_C = (2.56 \pm 0.34(\text{EFT})) \) fm, with the result using only the
ground state core field \( r_C = (2.32 \pm 0.53(\text{EFT})) \) fm, we see a convergence due to the inclusion
of the excited state of the \(^7\text{Be}\) core. Therefore, even though we formally only consider LO,
we can in this way see convergence properties of the EFT. This is since the inclusion of
the excited core field corresponds to the inclusion of additional short-range physics and is
therefore comparable to an actual higher-order calculation.

The charge radius of \(^8\text{B}\) has been calculated by Pastore \textit{et al.} [34] with quantum Monte
Carlo techniques. The authors find the point-proton charge radius to be \( r_{\text{pt-p}} = 2.48 \) fm.
Folding the point-proton charge radius with the proton and neutron charge radii, and adding
relativistic plus spin-orbit corrections as in Ref. [35], gives a charge radius of \( r_C = 2.60 \) fm.
This value is compatible with our LO result within the estimated error of 30\% for the radius
squared.

IV. RADIATIVE CAPTURE

We will now consider low-energy radiative proton capture on \(^7\text{Be}\). We will not consider
initial-wave scattering since it is exponentially suppressed at threshold \( E_{\text{c.m.}} = 0 \) due to
the Coulomb repulsion. Note, however, that we do consider pure Coulomb scattering to all
orders for the incoming proton-core pair.

Without initial-wave scattering we are left with the diagrams shown in Figures 6 and
7. Below we discuss these diagrams in detail. Since the bound state is a \( J = 2^+ \) due to a
P-wave interaction, the E1 capture is through an incoming proton-core pair in a relative S-
or D-wave, with total spin either \( S = 1 \) or \( S = 2 \). These are the relevant channels since
the E1 operator changes the angular momentum by one, and the spin of the constituent
\(^7\text{Be}\) core is \( 3/2^- \) while the proton has spin \( 1/2^+ \). The two channels are defined by the two
different vertices in the Lagrangian Eq. (2), \( g_1 C_{\alpha}^k C_{\alpha a}^k \) for the spin-1 channel \( (k = -1, 0, 1) \) and
\( g_2 C_{\alpha}^\beta C_{\alpha a}^\beta \) for the spin-2 channel \( (\beta = -2, -1, 0, 1, 2) \), where the Clebsch-Gordan coefficients
guarantee that the spins of the constituents couple to either \( S = 1 \) or \( S = 2 \) and that the
total spin couples to the E1 operator to give a \( J = 2 \) final state. For the threshold capture
cross section the S-wave dominates and we therefore drop the D-wave contribution.

The threshold radiative capture cross section without initial wave scattering is purely
proportional to the ground state ANCs in the combination \( A_1^2 + A_2^2 \), since the incoming
particles are in their respective ground states. Thus, even though the excited-core field does not contribute to the S-factor result presented here, the excited state is included in the field theory and we may therefore estimate the error from the energy of the $^7$Be $\alpha$-threshold. The results we present are at LO, but since the pole-position is fixed by the ANCs the threshold S-factor should be highly accurate. This is because the higher-order operators in the field theory will enter with additional powers of the energy. Therefore the error for the threshold S-factor is mainly due to the error of the ANCs and at finite energies the error is from the neglected initial-wave interactions, the D-wave component and the higher-order operators.

The amplitudes will have various indices, one from the vector photon $A^i$ and three from the incoming and outgoing particles ($\alpha, \sigma, a$). The first index will thus be dotted by the polarization vectors and the remaining (written in parenthesis) will be traced in the summation of $|A|^2$.

The diagrams are evaluated in the zero-momentum frame of the proton and the core. The incoming proton (core) has thus momentum $p$ ($-p$). The outgoing photon (halo) has momentum $Q$ ($-Q$).

\textit{a. Capture diagram with intermediate halo propagator}  This is the simplest diagram, and is shown in Fig. 6(a). Since the amplitude is proportional to the momentum flowing into the photon-halo field vertex, in Coulomb gauge, then the diagram is identically zero in the zero-momentum frame

$$iA^{(\alpha)}_{1(\sigma a)} = 0.$$  \hfill (46)
Figure 7: Loop-diagram for radiative capture. The photon couples either to the proton or the core.

b. Capture diagram with the photon coupling to the P-wave vertex  This diagram is given in Fig. 6(b) and is simply

\[ iA_{2(\sigma a)}^{i(\alpha)} = -ie\left(1 - f(Z_c + 1)\right) (g_1 C^\alpha_{\beta j} C^\beta_{\sigma a} + g_2 C^\alpha_{\beta j} C^\beta_{\sigma a}) \int \frac{d^3k}{(2\pi)^3} \psi_p(k) \]
\[ = -ie\left(1 - f(Z_c + 1)\right) (g_1 C^\alpha_{\beta j} C^\beta_{\sigma a} + g_2 C^\alpha_{\beta j} C^\beta_{\sigma a}) \psi_p(0). \]  

(47)

The \( \psi_p(0) \) factor implies that this amplitude is purely from an incoming S-wave.

c. Capture loop diagrams  The loop diagrams contributing to the capture process are shown in Fig. 7 and are written in momentum space as

\[ iA_{3(\sigma a)}^{i(\alpha)} = (g_1 C^\alpha_{k j} C^k_{\sigma a} + g_2 C^\alpha_{\beta j} C^\beta_{\sigma a}) \int \frac{d^3k_1 d^3k_2}{(2\pi)^6} ik_2^j \langle k_2 | G_C(-B)| k_1 + fQ \rangle i\frac{feZ_c(-k_1^i)}{m_R} \psi_p(k_1) \]
\[ - [(f \rightarrow 1 - f), (Z_c \rightarrow 1)]. \]  

(48)

It involves an incoming Coulomb wavefunction \( \psi_p(k_1) \), the vector photon-core vertex \( ifeZ_c(-k_1^i)/m_R \), a Coulomb Green’s function \( G_C(-B) \), which defines the propagation down to the bound state, and the P-wave interaction vertex \( ik_2^j \left( g_1 C^\alpha_{k j} C^k_{\sigma a} + g_2 C^\alpha_{\beta j} C^\beta_{\sigma a} \right) \). The photon-proton vertex is given by \( i(1 - f)ek_1^i/m_R \). Fourier transforming to coordinate space and multiplying with the photon polarization vectors \( \epsilon_i^{(n)} \), where \( n = 1, 2 \) and we have chosen \( \epsilon^{(1)} = \hat{x}, \epsilon^{(2)} = \hat{y} \) and \( Q = \omega \hat{z} \), we arrive at

\[ \sum_{i=-1}^1 \sum_{n=1}^2 i\epsilon_i^{(n)} A_{3(\sigma a)}^{i(\alpha)} = -\left( g_1 C^\alpha_{k j} C^k_{\sigma a} + g_2 C^\alpha_{\beta j} C^\beta_{\sigma a} \right) \frac{2feZ_c\gamma}{3p} (\hat{x} + \hat{y})^j \exp(i\sigma_0) \Gamma(2 + k_C/\gamma) \]
\[ \times \int dr rW_{-k_C/\gamma; 3/2}(2\gamma r) j_0(f\omega r) \partial_r \left( \frac{F_0(k_C/p, pr)}{r} \right) \]
\[ - [(f \rightarrow 1 - f), (Z_c \rightarrow 1)], \]  

(49)

where we have dropped the incoming D-wave and a negligible \( j_2 \) spherical Bessel function.
**d. Resulting S-factor**  The LO cross section for radiative capture is now given by the sum of contributions discussed above

\[
\frac{d\sigma}{d\Omega} = \frac{m_R \omega}{8\pi^2 p} \sum_{\alpha,\sigma a} \sum_{n=1}^{1} \sum_{i=-1}^{2} |\epsilon_i^{(n)} \left( A_{1(\sigma a)}^{i(\alpha)} + A_{2(\sigma a)}^{i(\alpha)} + A_{3(\sigma a)}^{i(\alpha)} \right) |^2 ,
\]

(50)

using the expressions for the amplitudes in Eqs. (46), (47) and (49). We present the total cross section in terms of the astrophysical S-factor

\[
S(E) = E \exp \left(2\pi\eta \sigma_{\text{tot}}\right) ,
\]

(51)

which is basically defined by the removal of the low-energy exponential suppression \(C_2^2\).

The resulting S-factor is presented in Fig. 8 together with low-energy direct-capture data from Refs. [36–43]. The threshold value is

\[
S(0) = \begin{cases} 
20.0 \pm 1.4) \text{ eV b (Nollett ANCs)} \\
18.9 \text{ eV b (Navrálí ANCs)} \\
(17.3 \pm 3.0) \text{ eV b (Tabacaru ANCs)}
\end{cases}
\]

(52)

The errors given in Eq. (52) are due to the errors in the ANCs. It is important to note that the EFT errors at threshold are negligible since the higher-order operators enter with additional powers of the photon energy. Such higher-order corrections will, however, influence the shape of the S-factor in the region where comparison with data is possible, resulting in an additional uncertainty when attempting extrapolation to threshold energies. Note that this extrapolation uncertainty is much more serious in potential-model descriptions for which the error is not quantified. Therefore, we present an alternative approach to constrain the threshold S-factor from experimental data: namely to identify its correlation with another observable, the charge radius of \(^8\text{B}\).

**V. CORRELATING THE CHARGE RADIUS AND THE THRESHOLD S-FACTOR**

At this stage we are able to demonstrate the relationship between the S-factor at threshold and the charge radius. Instead of fitting to the calculated and measured ground-state ANCs we now let \(A_1^2 + A_2^2\) be a free parameter. We still fit the excited state ANC to the result of Zhang et al. \(A_* = 0.3485(51)\), but use the free parameter \(A_1^2 + A_2^2\) to explore correlations
between the charge radius of $^8B$ and the threshold S-factor of $^7Be(p,\gamma)^8B$. The resulting correlation plot is shown in Fig. 9. The solid line shows the LO one-parameter correlation predicted by halo EFT. The triangle, square and circle show our results for the S-factor with the input ANCs from Nollett, Navrátil and Tabacaru, respectively. They appear in different positions on the correlation line since changing the ANC leads to a different effective range and thus a different charge radius. In turn, the correlation can be used to constrain the ANC from charge radius measurements.

VI. SUMMARY

In this paper, we have calculated the charge radius and the S-factor for radiative proton capture on $^7Be$ at LO in halo EFT. The parameters required to fix the proton-core interaction were obtained from experiment and ab initio calculations. Our prediction for the S-factor at threshold is in agreement with the most recent recommended value for this observable [1] and the EFT calculation by Zhang, Nollett and Phillips [26]. By performing calculations with and without the excited state of $^7Be$ and thereby changing the amount of microscopic physics included in the EFT, we have tested the convergence of our calculation. Furthermore, we have derived a number of universal expressions that can be applied to any proton halo system bound through resonant p-wave interactions.
If the one-proton separation energy of $^8$B is kept fixed, the charge radius is directly correlated to the S-factor at threshold at this order in the calculation. A measurement of the S-factor therefore determines not only the ANCs of the $^7$Be-proton system but also the charge radius. This result provides an excellent example how different observables constrain each other. Note however, that this direct correlation does not exist at the next order in the EFT expansion since two additional counterterms enter. The first counterterm fixes the next contribution in the ERE of $^7$Be-proton scattering. The second counterterm is an operator that couples the photon to the full halo field. Both counterterms could be fitted to the electric form factor either obtained through a measurement or an ab initio calculation. This connection to the electric form factor is particularly useful for the latter case since calculations of static nuclear properties are generally simpler than reactions or scattering observables. A higher order analysis would thus provide additional constraints on the threshold S-factor and does not directly depend on a measurement of the capture cross section and a subsequent extrapolation.

It is worth noting that the additional counterterm appearing in the electromagnetic current is an inherent limitation for any cluster model that tries to describe charge radii as it accounts for microscopic physics not determined by core-proton scattering properties. Previous calculations using cluster models have tried to circumvent this limitation by taking
additional microscopic physics such as core swelling into account, see e.g. [44]. However, the model-dependence of such approaches has not been studied and involves uncontrolled errors. Our EFT approach can therefore provide lower error bounds on any cluster model that employs the same number of degrees of freedom.

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Appendix A: Coulomb wavefunctions and Green’s function

We define the Coulomb wavefunction through its partial wave expansion

$$\psi_k(r) = \sum_{l=0}^{\infty} \phi_l(\eta, \rho)P_l(\hat{k} \cdot \hat{r}),$$  \hspace{1cm} (A1)

with

$$\phi_l(\eta, \rho) = (2l + 1)^{i/2} \exp (i \sigma_l(\eta)) \frac{F_l(\eta, \rho)}{\rho},$$  \hspace{1cm} (A2)

and the Coulomb phase shift \(\sigma_l = \arg \Gamma(l + 1 + i\eta)\). Here we have used the definitions \(\rho = kr\) and \(\eta = k_C/k\), with the Coulomb momentum \(k_C = Z_c\alpha m_R\). The regular Coulomb wavefunctions \(F_l\) can be expressed in terms of the Whittaker M function through

$$F_l(\eta, \rho) = A(l, \eta)M_{\nu_l, l+1/2}(2i\rho),$$  \hspace{1cm} (A3)

with

$$A(l, \eta) = \frac{1}{2} \frac{\Gamma(l + 1 + i\eta) \exp \left(-\pi \eta/2 - i(l + 1)\pi/2\right)}{(2l + 1)!}. \hspace{1cm} (A4)$$
We also give the irregular Coulomb wavefunctions $G_l$ in terms of the Whittaker M and W functions as

$$G_l(\eta, \rho) = iF_l(\eta, \rho) + B_l(\eta)W_{i\eta,l+1/2}(2i\rho) ,$$  \hspace{1cm} (A5)

with

$$B_l(\eta) = \frac{\exp(\pi\eta/2 + il\pi/2)}{\text{arg} \Gamma(l + 1 + i\eta)} .$$  \hspace{1cm} (A6)

Moreover, we have defined

$$|\Gamma(l + 1 + i\eta)|^2 = \Gamma(l + 1 + i\eta)\Gamma(l + 1 - i\eta)$$  \hspace{1cm} (A7)

and

$$\text{arg} \Gamma(l + 1 + i\eta) = \sqrt{\Gamma(l + 1 + i\eta)\Gamma(l + 1 - i\eta)} ,$$  \hspace{1cm} (A8)

such that our expressions make sense for imaginary momentum corresponding to bound states.

The Sommerfeld factor is given by

$$C_\eta^2 = C(0, \eta)^2 = \frac{2\pi\eta}{\exp(2\pi\eta) - 1} = \exp(-\pi\eta)\Gamma(1 + i\eta)\Gamma(1 - i\eta) ,$$  \hspace{1cm} (A9)

and we will also be needing its generalization to higher partial waves

$$C(l, \eta)^2 = \exp(-\pi\eta)\Gamma(l + 1 + i\eta)\Gamma(l + 1 - i\eta) .$$  \hspace{1cm} (A10)

Note in particular that

$$C(1, \eta)^2 = (1 + \eta^2)C_\eta^2 .$$  \hspace{1cm} (A11)

The partial wave projected Coulomb Green’s function for a bound state is given by

$$G_{C}^{(l)}(E; r, r') = -\frac{m_Rp}{2\pi} \frac{F_l(\eta, \rho') \left[ iF_l(\eta, \rho) + G_l(\eta, \rho) \right]}{\rho' \rho} ,$$  \hspace{1cm} (A12)

that is a product of two linear combinations of Coulomb wavefunctions satisfying the bound state boundary conditions both at zero and infinite separation. The normalization of the Coulomb Green’s function Eq. (A12) is set by the Coulomb-Schrödinger equation and the Wronskian of the Coulomb wavefunctions. Using the identity

$$iF_l(\eta, \rho) + G_l(\eta, \rho) = \exp(i\sigma_l + \pi\eta/2 - li\pi/2)W_{-i\eta,l+1/2}(-2i\rho)$$  \hspace{1cm} (A13)

and the limit

$$\lim_{r \to 0} \left( \frac{F_l(\eta, \rho)}{\rho^2} \right) = \frac{1}{3} \exp(-\pi\eta/2) \sqrt{\Gamma(2 + i\eta)\Gamma(2 - i\eta)} ,$$  \hspace{1cm} (A14)
we can then write
\[ \lim_{\rho' \to 0} \left( G^{(1)}_C(E; r', r) \right) = \frac{i m_R \rho}{2 \pi^2} \Gamma(2 + i \eta) \frac{W_{-i \eta, 3/2}(-2i\rho)}{\rho} . \] (A15)

The expression Eq. (A15) is used to evaluate the loop-integrals for the charge form factor and the radiative capture cross section.

**Appendix B: P-wave Integrals**

When we calculate loop diagrams with P-wave interactions, we have to evaluate integrals that involve Coulomb wavefunctions. In this Appendix, we derive a few useful identities for these calculations.

The most basic constituent of the integrals that we need to solve is the vector integral
\[ X(E_p) = \int \frac{d^3k}{(2\pi)^3} k \psi_p(k) . \] (B1)

We will evaluate this integral by performing a Fourier transform of \( \psi_p(k) \) to position space followed by a partial integration:
\[ X(E_p) = \int \frac{d^3k}{(2\pi)^3} k \psi_p(k) = \int d^3r \psi_p(r)(-i \nabla)\delta^{(3)}(r) \]
\[ = i \int d^3r \delta^{(3)}(r) \left( \nabla \psi_p(r) \right). \] (B2)

We will now temporarily fix \( p = p \hat{z} \) and do a partial wave expansion of the Coulomb wavefunction. The derivative in Eq. (B2) is then evaluated to (we are suppressing the \( \eta' \) and \( \rho \) dependencies)
\[ \nabla \psi_p(r) = \sum_{l=0}^{\infty} \left[ \hat{r} p \partial_\rho \phi_l P_l(\cos \theta) + \frac{\hat{\theta} P_l}{\rho} \partial_\theta \phi_l P_l(\cos \theta) \right] \] (B3)

Note that for \( l = 0 \) we get
\[ \nabla \psi_p(r) \bigg|_{l=0} = \hat{r} p \partial_\rho \phi_0, \] (B4)
which is an odd function of \( r \). This means that the integration of the \( l = 0 \) term is zero. Furthermore, we have
\[ \lim_{\rho \to 0} \partial_\rho \phi_l = 0, \ l > 1, \] (B5)
and
\[ \lim_{\rho \to 0} \frac{\phi_l}{\rho} = 0, \ l > 1, \] (B6)
which means that only the \( l = 1 \) term will contribute to \( X \):

\[
X(E_p) = i \lim_{\rho \to 0} \left( \cos (\theta) \hat{\rho} \partial_{\rho} \phi_1 - \sin (\theta) \hat{\theta} p \frac{\phi_1}{\rho} \right)
\]

\[
= i p \lim_{\rho \to 0} \left( \frac{\phi_1(\eta', \rho)}{\rho} \right). \quad (B7)
\]

In the last step we used that

\[
\cos (\theta) \hat{r} - \sin (\theta) \hat{\theta} = \hat{z}
\]

and that the limits of \( \partial_{\rho} \phi_1 \) and \( \phi_1 / \rho \) can be written as

\[
\lim_{\rho \to 0} \left( \partial_{\rho} \phi_1(\eta', \rho) \right) = \lim_{\rho \to 0} \left( \frac{\phi_1(\eta', \rho)}{\rho} \right). \quad (B9)
\]

We can also write this limit using the Sommerfeld factor

\[
\lim_{\rho \to 0} \left( \frac{\phi_1(\eta', \rho)}{\rho} \right) = i \exp (i \sigma_1) C(1, \eta'), \quad (B10)
\]

that is

\[
X(E_p) = -p \exp (i \sigma_1) C(1, \eta'). \quad (B11)
\]

**Appendix C: Partial Wave decomposed Coulomb Green’s Function**

We now continue by analyzing the Coulomb Green’s function \((r_1 | G_C | r_2)\). It is useful to express the Green’s function in its partial wave expanded form

\[
(r_1 | G_C(E) | r_2) = \sum_{l=0}^{\infty} (2l + 1) G^{(l)}_C (E; r_1, r_2) P_l(\hat{r}_1 \cdot \hat{r}_2). \quad (C1)
\]

The form of \( G^{(l)}_C \) is derived by expanding the Coulomb wavefunctions in Eq. (3), using spherical harmonics. In the first step we use the orthogonality of the harmonics and in the second step the addition theorem is used:

\[
(r_1 | G_C(E) | r_2) = \sum_{l_{m_1} l_{m_2}} \int \frac{d^3 p}{(2\pi)^3} \frac{(4\pi)^2}{(2l_1 + 1)(2l_2 + 1)} \frac{\phi_{l_1}(\eta, \rho_1) \phi^{*}_{l_2}(\eta, \rho_2)}{E - \frac{p^2}{2m_R}} \times Y^{*}_{l_{m_1}}(\theta_1, \varphi_1) Y^{*}_{l_{m_2}}(\theta_2, \varphi_2)
\]

\[
= \sum_{l m} \int \frac{d^3 p}{(2\pi)^3} \frac{4\pi}{(2l + 1)^2} \frac{\phi_l(\eta, \rho_1) \phi^{*}_l(\eta, \rho_2)}{E - \frac{p^2}{2m_R}} \times Y^{*}_{l m}(\theta_1, \varphi_1) Y^{*}_{l m}(\theta_2, \varphi_2)
\]

\[
= \sum_{l} \frac{P_l(\hat{r}_1 \cdot \hat{r}_2)}{2l + 1} \int \frac{d^3 p}{(2\pi)^3} \frac{\phi_l(\eta, \rho_1) \phi^{*}_l(\eta, \rho_2)}{E - \frac{p^2}{2m_R}}. \quad (C2)
\]
Thus, the Green’s function for a specific partial wave is given by

$$(2l + 1)G_{C}^{(l)}(E; r_1, r_2) = \frac{1}{2l + 1} \int \frac{d^3p}{(2\pi)^3} \frac{\phi_l(\eta, \rho_1)\phi_l^*(\eta, \rho_2)}{E - \frac{p^2}{2m_R}}.$$  \hspace{1cm} (C3)

We are now in a position to simplify the integral

$$I_1(0, r) = \int \frac{d^3k}{(2\pi)^3} k\langle k|G_C(E)|r \rangle .$$  \hspace{1cm} (C4)

Note that it is defined by a mixed matrix element where the Coulomb Green’s function is evaluated between a bra in $k$-space and a ket in $r$-space. Comparing to the S-wave equivalent

$$I_0(0, r) = \int \frac{d^3k}{(2\pi)^3} \langle k|G_C(E)|r \rangle = \langle 0|G_C(E)|r \rangle ,$$  \hspace{1cm} (C5)

we expect that $I_1$ behaves as an $r$-space object. Using Eq. (B7) and doing a partial wave expansion of $\psi_p^*(r)$, we have

$$I_1(0, r) = \int \frac{d^3k d^3p}{(2\pi)^6} k \psi_p(k) \frac{\psi_p^*(r)}{E - \frac{p^2}{2m_R}}$$

$$= \sum_l \int \frac{p^2dp}{(2\pi)^3} \frac{p}{i\hbar} \lim_{\rho' \to 0} \left( \frac{\phi_l(\eta, \rho')}{\rho'} \right) \frac{\phi_l^*(\eta, \rho)}{E - \frac{p^2}{2m_R}} \int d\Omega p P_l(cos \theta)$$

$$= \frac{i}{3} \lim_{r' \to 0} \left[ \frac{1}{r'} \int \frac{d^3p}{(2\pi)^3} \frac{\phi_l(\eta, \rho')\phi_l^*(\eta, \rho)}{E - \frac{p^2}{2m_R}} \right].$$  \hspace{1cm} (C6)

Finally, using Eq. (C3) we find that

$$I_1(0, r) = \int \frac{d^3k}{(2\pi)^3} k\langle k|G_C(E)|r \rangle$$

$$= 3i\hat{r} \lim_{r' \to 0} \left( \frac{G_{C}^{(1)}(E; r', r)}{\rho'} \right).$$  \hspace{1cm} (C7)

This form is useful since the partial wave projected Green’s function can be written in a closed functional form. Eq. (C7) can be further simplified using Eq. (A15).

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