Some Linear Diophantine Fuzzy Similarity Measures and Their Application in Decision Making Problem

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ABSTRACT The uncertainty in the data information for decision making is a most challenging and critical fear. In order to reduce the uncertainty in the decision making expert information for decision making problem, the Linear Diophantine fuzzy number is taking more critical part in reducing the uncertainty in information. Therefore the primary aim of this paper is to develop some different types of similarity and distance measures for linear Diophantine fuzzy numbers. With the frequent occurrence of emergency events, emergency decision making (EDM) plays a significant role in the emergency situations. It is essential for decision makers to make reliable and reasonable emergency decisions within a short time period since inappropriate decisions may result in enormous economic losses and chaotic social order. Accordingly, to ensure that EDM problems can be solved effectively and quickly, this paper proposes a new EDM method based on the novel distance and similarity measures under Linear Diophantine fuzzy (LDF) information. The similarity measure is one of the beneficial tools to determine the degree of similarity between objects. It has many crucial applications such as decision making, data mining, medical diagnosis, and pattern recognition. In this study, some novel distances and similarity measures of linear Diophantine fuzzy sets are presented. Then, the Jaccard similarity measure, exponential similarity measure, Cosine and Cotangent function based on similarity measures for LDFSs were proposed. The newly defined similarity measures are applied to medical diagnosis problem for COVID-19 virus and the results are discussed. A comparative study for the new similarity measures is established, and some advantages of the proposed work are discussed.

INDEX TERMS Fuzzy decision, linear diophantine fuzzy set, multi-criteria decision making, similarity measure.

I. INTRODUCTION

By the end of 2019, some cases were reported with the same symptoms in Wuhan city, province Hubei, China. After analyzing these reported cases, they were diagnosed as a novel corona virus (n-COVID-19). The Latin word corona is used for crown (circle of light), that why the virus was called novel corona virus (nCoV), later the world health organization (WHO) called this disease Corona Virus Infectious Disease 2019 (COVID-19). The COVID-19 spread very quickly in China. Later, around 200 countries were affected by COVID-19. COVID-19 spread rapidly in the universe and by the mid of May 2020 about 3.954 million of people were infected with total of 275160 deaths and the recovery of the infected people reached to 1.331 million, see figure 1. COVID-19 is transmitted among people through breathing droplets from sneezes and coughs, and also through the close contact. In the human body, the COVID-19 is a new type of corona-virus strain. The major symptoms of the infected patients are shortness of breath, fever and cough. The minor symptoms are sputum production, muscle pain, abdominal pain, diarrhea and loss of smell. The WHO declared COVID-19 as a world health emergency case due to the rapidly effected people around the world. Due to this situation, the experts have been tried to put plans and strategies to control spreading of COVID-19.
In the real-life decision making, the experts need the data information without vague and uncertainties, the classical set theory doesn’t cover the uncertainty and vagueness in data information. The experts used new theories to describe the uncertainty and vague information. Therefore, in 1965 Zadeh [1] established the concept of fuzzy set (FS) which discussed the membership degree (MD) and later Atanassov’s [2] generalized the notion of FS by adding non-membership degree (NMD). IFS is a very important generalization of fuzzy set and deliberated an appropriate tool to grip uncertain information. IFS has been applied to pattern recognition, decision making, cluster analysis and many other fields.

In the IFS theory, there is a constraint on MD and NMD of an element. The sum of MD and NMD must lie between 0 and 1. Therefore, in some cases, practical application, the decision makers, or decision experts assign the values MD and NMD of an element whose sum of squares of MD and NMD may be greater than one. In such type of cases, the IFS failed to explain accurate information or uncertain information. These types of uncertain information have been solved by Yager by introducing a new concept Pythagorean fuzzy set (PyFS). Yager enhanced the space of MD and NMD using the condition square sum of MD and NMD must lie between 0 and 1. The complicated and the indeterminacy problems in data information have considered by PyFSs [23].

Likewise of IFS, the PyFS was deliberated of the function \( \psi_p \) of MD and \( \psi_p \) of NMD with the condition that square sum of MD and NMD must lie between 0 and 1. This mean that the PyFs is more generalized than the IFS and can be more informative for decision making experts. In case, if expert assign MD = 0.7 and NMD = 0.6 to an object, then this information doesn’t satisfy the condition of IFS but obviously this information can be expressed by PyFS only. Therefore, the IF decision making and SMs are special cases of PyF decision making and SMs. Finally, it can be concluded that PyFS can be more powerfully covenant than IFS. Many researchers and investigators developed various methods and techniques for Decision making [24]–[28]. In [29], Zhang et al. studied the extended version of PyF-TOPSIS method and applied it to MCDM. The new operation of subtraction and division was introduced by Peng and Yang and their fundamental properties were studied [30]. In [31], Reformat and Yager developed the recommender system based on collaborative under PyFS information. Many authors and researchers considered PyFSs and introduced different models for group decision making problems [33]–[36]. The Pythagorean fuzzy SM for two PyFSs was defined by Wei and Wei [37]. The different aggregation operators based on Bonferroni mean (BM) operators, BM with geometric mean and some other aggregation operators can be found in [38]–[43].

In IFS and PyFS, the information for an object is not explained due to the restriction on the MD and NMD. Therefore, in some cases both theories failed to express the information of an object. Therefore, in [44], Yager developed a new notion q-rung orthopair fuzzy set (q-ROFS), Yager expanded the space of MD and NMD by relaxing the condition to sum of qth power of MD \( \psi_p \) and NMD \( \psi_q \) i.e., \( \psi_p^q + \psi_q^q \leq 1 \), the value of \( q \) is equal or less than 1. This means that if \( q = 1 \), then q-ROFS is reduced to IFS and if \( q = 1 \), then q-ROFS is reduced to PyFS. It’s clear that q-ROFS is a more generalized structure to explain the uncertain and vague information than IFS and PyFS. In [45], the algebraic sum and product operations for q-ROFSs were defined by Liu and Wang and they also studied their application in MCDM. The aggregation operators for q-ROFSs based on McLaurin symmetric mean (MSM) has developed in [46]. Some different operators have been developed such as Heronian mean operator [47], Partitioned Bonferroni mean operator [48], and Power Bonferroni mean operator [49]. In [50], Liu and Xu developed a MCDM technique for green supplier selection problems under q-ROFSs.

The SMs for two q-ROFSs have been defined in [51] by Wang et al., and they applied them to Pattern recognition and MCDM problems. In [52], Peng Liu developed distance, information and similarity measure and their relationship with each other. In [53], the Minkowski-type DM, including HM, ED and Chebyshev DM has been developed and debated their applications in MCDM. In [54], two other approaches were discussed by Ali. Yager developed the notion of possibility and confidence as well as credibility and certainty in q-ROFSs. The main motivation of the proposed work is to discuss each extension of fuzzy set (FS); in intuitionistic fuzzy set, the two membership explain the uncertainty of the object but the IFS failed to explain the real life problem, i.e, consider a real life world problem, if the values of MD and NMD are greater than 0.5 i.e 0.6 and 0.7, 0.6 + 0.7 > 1, in this case the Pythagorean fuzzy set describe the real life world problems. Sometime PyFS also failed to describe certain information of real-life problem, then the PyFS does not explain the uncertain information, so we use q-rung orthopair fuzzy set but also q-rung orthopair fuzzy set has failed to explain the uncertainty, therefore the concept of Linear Diophantine fuzzy set (LDFS) has been developed to cover the value of membership function and non-membership function. The concept of linear Diophantine fuzzy set (LDFS) was developed by Riaz and Hashmi in 2020 and they showed that the concept of LDFS is more generalized than IFS, PyFS and q-ROFSs [57]. The LDFS is more informative and effective than the IFS, PyFS and q-ROFSs. The main advantage of the LDFS is having reference parameters (RPs). Due to these RPs, the membership degree (MD) and non-membership degree (NMD) have more space than IFS, PyFS and q-ROFSs. In LDFS, the RPs have some restriction and bounded to some limited space i.e., the sum of RPs must be less than or equal to 1. Many scholars began to study the subject of LDFS theory to better apply it to other fields. Ayub et al., [58] established algebraic properties and relations for LDFS. lampande et al., [59] developed LDFS Einstein’s aggregation operators and applied them to decision-making problems. Further Riaz et al., [60] LDFS prioritized aggregation operators to solve decision-making problems. Kamac [61] developed the complex LDFS
also defined cosine similarity measure and their applications. Almaghrabi et al., defined a new approach to Q-rung linear Diophantine fuzzy sets [62] and Qiyas et al. developed some different distance measure for Q-rung linear Diophantine fuzzy numbers [63]. According to our knowledge and based on the above review, there are no applications with different types of distance measure and similarity measure i.e. Jaccard similarity measure as well as Exponential similarity measure (ESM) for weighted and generalized weighted ESM for two LDFSs.

The main contribution of this work are as follows:

- To extend the theory and applications of LDFS, we work on distance measure and similarity measure of linear Diophantine fuzzy numbers (LDFNs).
- The distance measure for LDFNs has been constructed and their relations with similarity measure have been studied.
- Developed different types of similarity measure; we constructed the Jaccard similarity measure (JMS) of LDFS.
- We also extended the JSM to weighted and generalized weighted JSM for two LDFSs. Also the Exponential similarity measure (ESM) was developed for two LDFSs and also extended for weighted and generalized weighted ESM for two LDFSs.
- All proposed similarity measure has been applied to Pattern recognition to solve the COVID-19 problem as well as compared with the other existence methods of similarity measures.

In this paper, we develop some different types of similarity and distance measure for linear Diophantine fuzzy numbers. With the frequent occurrence of emergency events, emergency decision making (EDM) plays a significant role in the emergency situations. It is essential for decision makers to make reliable and reasonable emergency decisions within a short time period since inappropriate decisions may result in enormous economic losses and chaotic social order. Accordingly, to ensure that EDM problems can be solved effectively and quickly, this paper proposes a new EDM method based on the novel distance and similarity measures under Linear Diophantine fuzzy (LDF) information. The similarity measure is one of the beneficial tools to determine the degree of similarity between objects. It has many crucial applications such as decision making, data mining, medical diagnosis, and pattern recognition. In this study, some novel distances and similarity measures of linear Diophantine fuzzy sets are presented. Then, the Jaccard similarity measure, exponential similarity measure, Cosine and Cotangent function based on similarity measures for LDFSs were proposed. The newly defined similarity measures are applied to medical diagnosis problem for COVID-19 virus and the results are discussed. A comparative study for the new similarity measures is established, and some advantages of the proposed work are discussed.

A. RELATED WORK

The similarity measure (SM) is a very central part in the multi-criteria decision making (MCDM) theory and pattern recognition. Moreover, SM is a very powerful tool to determine the degree of similarity, dissimilarity between two objects. The theory and applications of SMs have been studied for the last few decades. In [3]–[6], the author introduced some different types of SMs for two IFSs. Liu and Cheng’s [7], applied the new proposed SM’s for pattern recognition and they also verified the proposed method by using numerical application. The Liu and Cheng’s [7], SMs have been modified by Mitchel [8] and also have been applied to MCDM. Li and Park et al. [9], [10] evaluated the generalized form of fuzzy Hamming distance measure (HDM), also developed an IFHDM and also presented different SMs to be applied on the MCDM problems. Torra and Narukawa [11], developed some new SMs based on Hausdorff distance and they hence discussed some applications. In [12], Xia and Xu developed a new type of DM and SM operators by using the geometric aggregation operators and applied them to the MCDM problem. Ye [13] used the cosine function to develop IF cosine SMs and applied them to the MCDM problem. The likelihood based on an SM for IFS was defined by Ku-Chen [14] and has been applied to medical diagnosis (MD). The modified form of CSMs was presented by Shi and Ye in [15]. The cotangent SM between two IFS has been defined by Tian [16] for MD. The modified version of cotangent SMs has been defined by Rajarajawari and Uma [17] as they considered MD, NMD and the indeterminacy degree designated in IFS. The distance measure (DM) of IFSs was introduced by Szmidt and SMs [18] were developed by him. Some generalized distance measure and SMs of IFSs are found in [19]–[22].

The main motivation of the proposed work is to discuss each extension of fuzzy set (FS); in intuitionistic fuzzy set, the two membership explain the uncertainty of the object but the IFs failed to explain the real life problem, i.e. consider a real life world problem, if the values of MD and NMD are greater than 0.5 i.e 0.6 and 0.7, 0.6 + 0.7 > 1, in this case the Pythagorean fuzzy set describe the real life world problems. Sometime PyFS also failed to describe certain information of real-life problem, then the PyFS does not explain the uncertain information, so we use q-rung orthopair fuzzy set but also q-rung orthopair fuzzy set has failed to explain the uncertainty, therefore the concept of Linear Diophantine fuzzy set (LDFS) has been developed to cover the value of membership function and non-membership function. The concept of linear Diophantine fuzzy set (LDFS) was developed by Riaz and Hashmi in 2020 and they showed that the concept of LDFS is more generalized than IFs, PyFS and q-ROFSs [57]. The LDFS is more informative and effective than the IFs, PyFS and q-ROFSs. The main advantage of the LDFS is having reference parameters (RPs). Due to these RPs, the membership degree (MD) and non-membership degree (NMD) have more space than IFs, PyFS and q-ROFSs. In LDFS, the RPs have some restriction and bounded to some limited space.
I. PRELIMINARIES

Definition 1 [2]: Consider a non-empty universal set \( \varphi \). Then, an IFS \( \mathbb{I} \) of \( \varphi \) is defined by the following mathematical equation:

\[
\mathbb{I} = \{ (\rho, (\varphi_L(\rho), \psi_L(\rho)) : \rho \in \varphi \},
\]

where \( \varphi_L \) and \( \psi_L \) are representing degree of membership (DM) and degree of non-membership (DNM) respectively and \( \varphi_L(\rho), \psi_L(\rho) \in [0, 1] \) with subject to \( \varphi_L(\rho) + \psi_L(\rho) \leq 1 \).

Definition 2: Consider a non-empty universal set \( \varphi \). Then, an intuitionistic fuzzy set (IFS) \( \mathbb{I} \) of \( \varphi \) is defined by the following mathematical equation:

\[
\mathbb{I} = \{ (\rho, (\varphi_L(\rho), \psi_L(\rho)) : \rho \in \varphi \},
\]

where \( \varphi_L \) and \( \psi_L \) are representing degree of membership (DM) and degree of non-membership (DNM) respectively and \( \varphi_L(\rho), \psi_L(\rho) \in [0, 1] \) with subject to \( \varphi_L(\rho) + \psi_L(\rho) \leq 1 \).

In some cases, the experts give the DM and degree of DNM to an object whose sum is greater than 1, for this case Yager [23] defined Pythagorean fuzzy sets as the following:

Definition 3: Consider a fixed set \( \varphi \). The PyFS is denoted by \( \mathbb{P} \) and mathematically is defined as

\[
\mathbb{P} = \{ (\rho, (\varphi_L(\rho), \psi_L(\rho)) : \rho \in \varphi \},
\]

where \( \varphi_L \) and \( \psi_L \) are representing degree of membership (DM) and degree of non-membership (DNM) respectively and \( \varphi_L(\rho), \psi_L(\rho) \in [0, 1] \) with subject to \( \varphi_L(\rho)^2 + \psi_L(\rho)^2 \leq 1 \).

Definition 4: Consider a fixed set \( \varphi \). The PyFS is denoted by \( \mathbb{P} \) and mathematically is defined as

\[
\mathbb{P} = \{ (\rho, (\varphi_L(\rho), \psi_L(\rho)) : \rho \in \varphi \},
\]

where \( \varphi_L \) and \( \psi_L \) are representing degree of membership (DM) and degree of non-membership (DNM) respectively and \( \varphi_L(\rho), \psi_L(\rho) \in [0, 1] \) with subject to \( \varphi_L(\rho)^2 + \psi_L(\rho)^2 \leq 1 \).

The IFS and PyFS become limited in the situation when sum of MG and NMG and the sum of its squares is also larger than one, hence Yager [44] launched a generalization of IFS and PyFS called q-ROFS.

Definition 5 [57]: Consider a non-empty universal set \( \varphi \). Then, a LDFS \( \mathcal{L} \) of \( \varphi \) is defined by the following mathematical form:

\[
\mathcal{L} = \{ (\rho, (\varphi_L(\rho), \psi_L(\rho)) : \rho \in \varphi \},
\]

where \( \varphi_L(\rho) \) and \( \psi_L(\rho) \) represent the degree membership and the non-membership of an element \( \rho \) in \( \varphi \).

Definition 6: Consider a non-empty universal set \( \varphi \). Then, an LDFS \( \mathcal{L} \) of \( \varphi \) is defined by the following mathematical form:

\[
\mathcal{L} = \{ (\rho, (\varphi_L(\rho), \psi_L(\rho)) : \rho \in \varphi \},
\]

where \( \varphi_L(\rho) \), \( \psi_L(\rho) \in [0, 1] \) and \( \varphi, \sigma \in [0, 1] \) represent the degree membership (DM) and non-membership (DNM) of element \( \rho \) in \( \varphi \) and the reference parameters (RPs).

In the next definition, we define the distance measure and similarity measure of fuzzy sets.

Definition 7: Consider a family of FSs \( F(X) \). Then a mapping \( d(\mathcal{L}_1, \mathcal{L}_2):F(X) \times F(X) \rightarrow [0, 1] \) is said to be a distance measure (DM) between fuzzy sets, if the following conditions hold. Consider \( \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 \in F(X) \). Then,

1) \( 0 \leq d(\mathcal{L}_1, \mathcal{L}_2) \leq 1 \)
2) \( d(\mathcal{L}_1, \mathcal{L}_2) = d(\mathcal{L}_2, \mathcal{L}_1) \)
3) \( d(\mathcal{L}_1, \mathcal{L}_2) = 0 \) iff \( \mathcal{L}_1 = \mathcal{L}_2 \)

Definition 8: Consider a family of FSs \( F(X) \). Then a mapping \( S : F(X) \times F(X) \rightarrow [0, 1] \) is said to be similarity measure (SM) between two fuzzy sets, if the following conditions hold. Consider \( \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 \in F(X) \). Then,

1) \( 0 \leq S(\mathcal{L}_1, \mathcal{L}_2) \leq 1 \)
2) \( S(\mathcal{L}_1, \mathcal{L}_2) = S(\mathcal{L}_2, \mathcal{L}_1) \)
3) \( S(\mathcal{L}_1, \mathcal{L}_2) = 1 \) iff \( \mathcal{L}_1 = \mathcal{L}_2 \)
4) \( \mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \mathcal{L}_3 \), Then, \( S(\mathcal{L}_1, \mathcal{L}_3) \leq S(\mathcal{L}_2, \mathcal{L}_3) \)

Distance Measure of Linear Diaphantine Fuzzy Sets: This section provide the detail description of the distance measure (DM) between two LDFSs and their fundamental properties. We define the Hausdroff DM.

Definition 9: Consider a family of LDFSs \( LDF(\mathcal{L}) \). Then a mapping \( d(\mathcal{L}_1, \mathcal{L}_2):LDF(\mathcal{L}) \times LDF(\mathcal{L}) \rightarrow [0, 1] \) is said to be distance measure (DM), if the following conditions hold. Consider \( \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 \in LDF(\mathcal{L}) \). Then,

1) \( 0 \leq d(\mathcal{L}_1, \mathcal{L}_2) \leq 1 \)
2) \( d(\mathcal{L}_1, \mathcal{L}_2) = d(\mathcal{L}_2, \mathcal{L}_1) \)
3) \( d(\mathcal{L}_1, \mathcal{L}_2) = 0 \) iff \( \mathcal{L}_1 = \mathcal{L}_2 \)
4) \( \mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \mathcal{L}_3 \), Then, \( d(\mathcal{L}_1, \mathcal{L}_3) \leq d(\mathcal{L}_1, \mathcal{L}_2) + d(\mathcal{L}_2, \mathcal{L}_3) \)

Consider a discrete non-empty set \( \varphi = \{ \rho_1, \rho_2, \rho_3, \ldots, \rho_n \} \) and two sets of LDFSs \( \mathcal{L}_1 = \{ (\varphi_L(\rho_1), \psi_L(\rho_1)), (\varphi, \sigma) \} \) and \( \mathcal{L}_2 = \{ (\varphi_L(\rho_1), \psi_L(\rho_1)), (\varphi, \sigma) \} \). Now we define a normalized Hamming distance and a normalized Euclidean distance measure between two LDFSs \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) in the coming definitions.
**TABLE 1.** Diagnosis-symptoms of decision information matrix-II (for $\varrho = 0.3, \sigma = 0.3$).

|       | (M1,S1) | (M1,S2) | (M1,S3) | (M2,S1) | (M2,S2) | (M2,S3) | (M3,S1) | (M3,S2) | (M3,S3) |
|-------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| P1    | (0.8,0.9) | (0.8,0.9) | (1.1)   | (0.9,1) | (0.9,1) | (0.7,0.7) | (0.8,0.9) | (0.9,1) | (0.7,0.8) |
| P2    | (0.7,0.9) | (0.7,0.3) | (0.4,0.8) | (0.3,0.3) | (0.1,0.3) | (0.4,0.8) | (0.5,0.9) | (0.1,0.3) | (0.2,0.8) |
| P3    | (0.3,0.9) | (0.3,0.2) | (0.9,0.5) | (0.7,0.9) | (0.9,1) | (0.7,0.3) | (0.1,0.6) | (0.5,0.6) | (0.3,0.8) |
| P4    | (0.2,1) | (0.2,0.9) | (0.3,0.7) | (0.4,0.2) | (0.2,0.9) | (0.5,0.6) | (0.2,0.9) | (0.7,0.7) |
| P5    | (0.8,0.9) | (0.8,0.9) | (1.1) | (0.9,0.9) | (0.5,0.3) | (0.8,0.9) | (0.3,0.3) | (0.5,0.3) | (0.4,0.7) |
| P6    | (0.8,0.3) | (0.9,0.4) | (0.3,0.6) | (0.8,0.5) | (0.1,0.5) | (0.5,0.3) | (0.8,0.5) | (0.1,0.5) | (0.3,0.6) |
| P7    | (0.7,0.6) | (0.8,0.3) | (0.5,0.8) | (0.4,0.4) | (0.8,0.9) | (0.8,0.5) | (0.9,0.9) | (0.8,0.9) | (0.2,0.9) |
| P8    | (0.3,0.8) | (0.9,0.2) | (0.4,0.7) | (0.9,0.8) | (1.1) | (0.8,0.6) | (1.0,3) | (1.1) | (0.5,0.6) |
| P9    | (0.6,0.7) | (1.0,3) | (0.6,1) | (0.8,0.7) | (0.9,0.1) | (0.8,0.6) | (0.6,0.9) | (0.9,1) | (0.5,0.4) |

**Definition 10:** Consider a discret non-empty set $\varphi = \{\rho_1, \rho_2, \rho_3, \ldots, \rho_n\}$ and two sets of LDFSs $L_1 = \{\psi_{\varphi_1}(\rho_i), \psi_{\varphi_1}(\rho_i), (\varphi_i, \sigma_i)\}$ and $L_2 = \{\psi_{\varphi_2}(\rho_i), \psi_{\varphi_2}(\rho_i), (\varphi_i, \tau_i)\}$.

Then, the normalized Hamming distance is denoted by $HD_{LDFS}(L_1, L_2)$ and is given in equation (1):

$$HD_{LDFS}(L_1, L_2) = \frac{1}{4n} \sum_{i=1}^{n} \left( |\psi_{\varphi_1}(\rho_i) - \psi_{\varphi_2}(\rho_i)| + |\varphi_i - \sigma_i| + |\varphi_i - \tau_i| \right)$$  \hspace{1cm} (1)

**Definition 11:** Consider a discret non-empty set $\varphi = \{\rho_1, \rho_2, \rho_3, \ldots, \rho_n\}$ and two sets of LDFSs $L_1 = \{\psi_{\varphi_1}(\rho_i), \psi_{\varphi_1}(\rho_i), (\varphi_i, \sigma_i)\}$ and $L_2 = \{\psi_{\varphi_2}(\rho_i), \psi_{\varphi_2}(\rho_i), (\varphi_i, \tau_i)\}$.

Then, the normalized Hamming distance is denoted by $ED_{LDFS}(L_1, L_2)$ and is given in equation (2), as shown at the bottom of the page.

We further generalize the HDM and EDM for two LDFSs $L_1$ and $L_2$, using a parameter $Y \geq 1$ to equation (1) and (2), respectively. Then, generalized distance measure (GDM) of two LDFSs $L_1$ and $L_2$ is given in equation (3)

$$GDM_{LDFS}(L_1, L_2) = \left( \frac{1}{4n} \sum_{i=1}^{n} \left( |\psi_{\varphi_1}(\rho_i) - \psi_{\varphi_2}(\rho_i)|^Y + |\varphi_i - \sigma_i|^Y + |\varphi_i - \tau_i|^Y \right) \right)^{\frac{1}{Y}}$$  \hspace{1cm} (3)

**Remark 1:**
1) If we put $Y = 1$, then $GDM_{LDFS}(L_1, L_2)$ is reduced to $HD_{LDFS}(L_1, L_2)$.
2) If we put $Y = 2$, then $GDM_{LDFS}(L_1, L_2)$ is reduced to $ED_{LDFS}(L_1, L_2)$.

Next, we use the weight of each $\rho_i$ with the weight vector $w = (w_1, w_2, w_3, \ldots, w_n)$ such that $\sum_{i=1}^{n} w_i = 1$. Then, we propose a new DM, so called generalized weighted distance measure (GWM) of $L_1$ and $L_2$ which is denoted by $GWM_{LDFS}(L_1, L_2)$ and defined in equation (4)

$$GWM_{LDFS}(L_1, L_2) = \left( \frac{1}{4n} \sum_{i=1}^{n} \left( |\psi_{\varphi_1}(\rho_i) - \psi_{\varphi_2}(\rho_i)|^Y + |\varphi_i - \sigma_i|^Y + |\varphi_i - \tau_i|^Y \right) \right)^{\frac{1}{Y}}$$  \hspace{1cm} (4)

**III. SIMILARITY MEASURE OF LINEAR DIOPHANTINE FUZZY SETS**

In this section, we define different types of similarity measure (SM) of LDFS and also we discuss the basic properties of the proposed SMs. We will use the notion of the cosine, exponential, square root cosine and Jaccard functions to develop the similarity measures between LDFSs.

First we define the similarity measures between two LDFSs.

**Definition 12:** Consider a family of LDFSs L DFS. Then a mapping $M:LDFS \times LDFS \rightarrow [0, 1]$ is said to be

$$ND_{LDFS}(L_1, L_2) = \sqrt{\frac{1}{4n} \sum_{i=1}^{n} \left( |\psi_{\varphi_1}(\rho_i) - \psi_{\varphi_2}(\rho_i)|^2 + |\varphi_i - \sigma_i|^2 + |\varphi_i - \tau_i|^2 \right)}$$  \hspace{1cm} (2)
similarity measure, if the following conditions hold. Consider $L_1, L_2, L_3 \in LDF (\mathcal{L})$.

1) $0 \leq \mathcal{M}(L_1, L_2) \leq 1$
2) $\mathcal{M}(L_1, L_2) = 1$ if and only if $L_1 = L_2$
3) $\mathcal{M}(L_1, L_2) = \mathcal{M}(L_2, L_1)$
4) If $L_1 \subseteq L_2$ and $L_2 \subseteq L_3$, then $\mathcal{M}(L_1, L_3) = \min\{\mathcal{M}(L_1, L_2), \mathcal{M}(L_2, L_3)\}$

A. Jaccard Similarity Measure of LDFS

This subsection considers a novel SM between LDFS which is called Jaccard similarity measure (JSM) between two LDFSs. The JSM tells us some information about the two LDFSs whether they are similar or not. The JSM provides information of similarity from 0 to 1 for the two LDFSs. If the information of similarity of two LDFSs is near to 1, then both are similar to each other and if the information of similarity is near to 0, then both are dissimilar from each other. The JSM is very informative for similarity and dissimilarity for LDFSs, so the JSM is useful in the decision making problem and pattern recognition. In the next definition, we define the JSM for two LDFSs.

Definition 13: Consider a discret non-empty set $\varphi = \{\rho_1, \rho_2, \rho_3, \ldots, \rho_n\}$ and two sets of LDFSs $L_1 = \{(\psi_{L_1}(\rho_1), \psi_{L_1}(\rho_2)), (\varphi_1, \varphi_1)\}$ and $L_2 = \{(\psi_{L_2}(\rho_1), \psi_{L_2}(\rho_2)), (\varphi_2, \varphi_1)\}$. Then Jaccard similarity measure is defined as the equation (5), as shown at the bottom of the page. In the coming Theorem, equation (1) satisfies the condition of similarity measure of LDFSs.

Theorem 1: Consider $L_1 = \{(\psi_{L_1}(\rho_1), \psi_{L_1}(\rho_2)), (\varphi_1, \varphi_1)\}$ and $L_2 = \{(\psi_{L_2}(\rho_1), \psi_{L_2}(\rho_2)), (\varphi_2, \varphi_1)\}$ be two LDFSs. Then, $\mathcal{J}(L_1, L_2)$ defined in equation (1) hold the following conditions:

1) $0 \leq \mathcal{J}(L_1, L_2) \leq 1$
2) $\mathcal{J}(L_1, L_2) = \mathcal{J}(L_2, L_1)$
3) $\mathcal{J}(L_1, L_2) = 1$ if and only if $L_1 = L_2$

Proof: Let $L_1$ and $L_2$ be two LDFSs. Since we know that $a^2 + b^2 \geq 2ab$, so

$\varphi_{L_1}(\rho_1) \psi_{L_2}(\rho_2) + \psi_{L_1}(\rho_2) \varphi_{L_2}(\rho_1) + \varphi_{L_1} \varphi_{L_2} + \varphi_{L_1} \varphi_{L_2} \geq 0$

and

$[\varphi_{L_1}(\rho_1) + (\psi_{L_1}(\rho_1) + \varphi_{L_1}) + \varphi_{L_1} \varphi_{L_2} + \varphi_{L_1} \varphi_{L_2} \geq 0]$

Thus, $\mathcal{J}(L_1, L_2) \geq 0$ and also from Eq. (1) summation up to $n$ and normalized, we have $\mathcal{J}(L_2, L_1) \leq 1$. Hence, $0 \leq \mathcal{J}(L_1, L_2) \leq 1$.

2) There is no need proof for $\mathcal{J}(L_1, L_2) = \mathcal{J}(L_2, L_1)$, which is trivial.
3) Suppose that $L_1 = L_2$. Then $\varphi_{L_1} = \varphi_{L_2}$. Then, $\mathcal{J}(L_1, L_2)$ becomes as the following JSM ($L_2, L_1$), as shown at the bottom of the next page, the proof of the converse is easy.

This completes the required proof.

We can consider the refusal degree of LDFS and use this refusal degree to define JSM. In the next definition we define a new JSM using refusal degree of LDFS.

Definition 14: Consider $L_1 = \{(\psi_{L_1}(\rho_1), \psi_{L_1}(\rho_2)), (\varphi_1, \varphi_1)\}$ and $L_2 = \{(\psi_{L_2}(\rho_1), \psi_{L_2}(\rho_2)), (\varphi_2, \varphi_1)\}$ be two LDFSs and $R_1(x_i) = 1 - \varphi_{L_1}(\rho_i) \varphi_1 + \psi_{L_1}(\rho_1) \varphi_1$ be the refusal degree of $L_1$ and $R_2(x_i) = 1 - \varphi_{L_2}(\rho_i) \varphi_1 + \psi_{L_2}(\rho_2) \varphi_1$. Then, we propose another JSM using refusal degree of LDFSs as the following (6), as shown at the bottom of the next page. The proposed JSM in Eq. 2 also satisfies the conditions of Theorem 1, therefore we don’t need the proof as all the conditions are in Theorem 1 of Eq. 2. In the coming SMs that are satisfying the properties of Theorem 1, we don’t proof the properties for the coming section SM because the space of constraint and similarity satisfies the properties of the all conditions of SM.

Next we use the weight of each $\rho_i$ with the weight vector $w = (w_1, w_2, w_3, \ldots, w_n)$ subject to $w_i \in [0, 1]$ such that $\sum_{i=1}^{n} w_i = 1$. Then, we propose a new SM, so-called weighted Jaccard similarity measure of $L_1$ and $L_2$ which is denoted by $\mathcal{JSMw}(L_1, L_2)$ and defined in equation (7), as shown at the bottom of the next page.

The proposed $\mathcal{JSMw}(L_1, L_2)$ also satisfies the conditions of Theorem 1. If we use the refusal degree of LDFSs in the weighted JSM $\mathcal{JSMw}(L_1, L_2)$, then weighted JSM of $L_1$ and $L_2$ with refusal degrees of LDFS $L_1$ and $L_2$ is denoted by $\mathcal{JSMw}(L_1, L_2)$ and defined in an equation (8), as shown at the bottom of the next page.

The properties of Theorem 1 are also satisfied by Eq. 4.

We further generalize the weighted Jaccard similarity measure of two LDFSs $L_1$ and $L_2$, $\mathcal{JSMw}(L_1, L_2)$ by using a parameter $Y \in [0, 1]$ to equation 3. Then Equation (3) will be converted to a generalized weighted Jaccard similarity measure of LDFSs $L_1$ and $L_2$, $\mathcal{JGSMw}(L_1, L_2)$. The $\mathcal{JGSMw}(L_1, L_2)$ is defined as the following equation (9), as shown at the bottom of the next page, where $Y \in [0, 1]$, the condition of the Theorem 1 is satisfied by equation of $\mathcal{JGSMw}(L_1, L_2)$. $\mathcal{JGSMw}(L_1, L_2)$ is the most general similarity measure of two LDFS $L_1$ and $L_2$. Now, we will discuss some cases of $\mathcal{JGSMw}(L_1, L_2)$.
Remark 1: The following are some special cases of a generalized weighted Jaccard similarity measure by changing the values of parameter $Y$:

i) If we put $Y = 0.5$ in equation (9), then $GJSM_w (\mathcal{L}_1, \mathcal{L}_2)$ will be converted to $JSM_w (\mathcal{L}_1, \mathcal{L}_2)$.

ii) If we put $Y = 0$, in equation (9), then $GJSM_w (\mathcal{L}_1, \mathcal{L}_2)$ will be converted to the following weighted Jaccard similarity measure. i.e; (10), as shown at the bottom of the next page.

iii) If we put $Y = 1$, in equation (9), then $GJSM_w (\mathcal{L}_1, \mathcal{L}_2)$ will be converted to the following weighted Jaccard similarity measure. i.e; (11), as shown at the bottom of the next page.

Now consider generalized weighted Jaccard similarity measure with refusal degrees of LDFSs. Then $GJSM_w (\mathcal{L}_1, \mathcal{L}_2)$ can be evaluated with refusal degrees of LDFSs $\mathcal{L}_1$ and $\mathcal{L}_2$ by equation (12), as shown at the bottom of the next page, where $Y \in [0, 1]$. The condition of the Theorem 1 is satisfied by
GJSM\textsuperscript{R}(\mathcal{L}_1, \mathcal{L}_2) is the most generalized similarity measure with referral degrees of two LDFSs \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \). Now we discuss some cases of GJSM\textsuperscript{R}(\mathcal{L}_1, \mathcal{L}_2) in Remark 2.

Remark 2: The following are some special cases of a generalized weighted Jaccard similarity measure by changing the values of parameter \( Y \):

i) If we put \( Y = 0.5 \) in equation (12), then GJSM\textsuperscript{R}(\mathcal{L}_1, \mathcal{L}_2) will be converted to JSM\textsuperscript{R}(\mathcal{L}_1, \mathcal{L}_2).

ii) If we put \( Y = 0 \), in equation (12), then GJSM\textsuperscript{R}(\mathcal{L}_1, \mathcal{L}_2) will be converted to the following weighted Jaccard similarity measure. i.e; (13), as shown at the bottom of the next page.

iii) If we put \( Y = 1 \), in equation (12), then GJSM\textsuperscript{R}(\mathcal{L}_1, \mathcal{L}_2) will be converted to the following weighted Jaccard similarity measure. i.e; (14), as shown at the bottom of the next page.

### B. Exponential Similarity Measure

This section provides new similarity measure between two LDFSs \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) by using exponential function with Euclidean and Hamming distance between two LDFSs. In 1987, Shepard anticipated a universal law about the distance measure and similarity measure. Both are related to an exponential function. The similarity measure based on exponent function using distance measure is defined in equation (15).

\[
EM_d (\mathcal{L}_1, \mathcal{L}_2) = e^{-d(\mathcal{L}_1, \mathcal{L}_2)} \tag{15}
\]

Now, we discuss the properties of distance measure which is called axioms of distance measure, every distance measure satisfies the following axioms;

1) Equality: \( d(\mathcal{L}_1, \mathcal{L}_1) = d(\mathcal{L}_2, \mathcal{L}_2) \) for every \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \). Thus, \( EM_d (\mathcal{L}_1, \mathcal{L}_1) = EM_d (\mathcal{L}_2, \mathcal{L}_2) \) for every \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \).

2) Minimality: \( d(\mathcal{L}_1, \mathcal{L}_2) \geq d(\mathcal{L}_1, \mathcal{L}_1) \) for every \( \mathcal{L}_1 \neq \mathcal{L}_2 \). Thus, \( EM_d (\mathcal{L}_1, \mathcal{L}_2) < EM_d (\mathcal{L}_1, \mathcal{L}_1) \) for every \( \mathcal{L}_1 \neq \mathcal{L}_2 \).

3) Symmetry; \( d(\mathcal{L}_1, \mathcal{L}_2) = d(\mathcal{L}_2, \mathcal{L}_1) \) for every \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \). Thus, \( EM_d (\mathcal{L}_1, \mathcal{L}_2) = EM_d (\mathcal{L}_2, \mathcal{L}_1) \) for every \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \).

4) Triangle Inequality: \( d(\mathcal{L}_1, \mathcal{L}_2) + d(\mathcal{L}_2, \mathcal{L}_3) \geq d(\mathcal{L}_1, \mathcal{L}_3) \) for all \( \mathcal{L}_1, \mathcal{L}_2, \) and \( \mathcal{L}_3 \). Thus, we can say that the dissimilarities of the any three LDFS satisfy the above properties. We can also say that if \( \mathcal{L}_1 \) is similar to \( \mathcal{L}_2 \) and \( \mathcal{L}_2 \) is similar to \( \mathcal{L}_3 \), then \( \mathcal{L}_1 \) should be similar to \( \mathcal{L}_3 \).

Next, we use the normalized Hamming distance (NHD) of two LDFSs in Eq. (15) and we will get the new type of SM based on NHD. The exponential similarity measure (ESM) based on NHD is given in Equation (18):

\[
EM_{HD} (\mathcal{L}_1, \mathcal{L}_2) = e^{-\frac{1}{\pi} \sum_{i=1}^{n} \left( |(\psi_{\mathcal{L}_1}(\rho_i)-\psi_{\mathcal{L}_2}(\rho_i))+(\psi_{\mathcal{L}_1}(\tau_i)-\psi_{\mathcal{L}_2}(\tau_i))| \right)} \tag{16}
\]
Theorem 2: Consider $\mathcal{L}_1 = \{(\varphi_{\mathcal{L}_1}(\rho_1), \varphi_{\mathcal{L}_2}(\rho_1)), (\sigma_1, \tau_1)\}$ and $\mathcal{L}_2 = \{(\varphi_{\mathcal{L}_2}(\rho_2), \varphi_{\mathcal{L}_2}(\rho_2)), (\sigma_2, \tau_2)\}$ be two LDFSSs. Then, $EM_{HD}(\mathcal{L}_1, \mathcal{L}_2)$ defined in equation (18) satisfies the following conditions:

1) $0 \leq EM_{HD}(\mathcal{L}_1, \mathcal{L}_2) \leq 1$, for every $\mathcal{L}_1$ and $\mathcal{L}_2$,

2) $EM_{HD}(\mathcal{L}_1, \mathcal{L}_2) = EM_{HD}(\mathcal{L}_2, \mathcal{L}_1)$, for every $\mathcal{L}_1$ and $\mathcal{L}_2$,

3) $EM_{HD}(\mathcal{L}_1, \mathcal{L}_2) = 1$ if $\mathcal{L}_1 = \mathcal{L}_2$, for every $\mathcal{L}_1$ and $\mathcal{L}_2$.

Proof: Since we know from the definition of LDFSS, $0 \leq \varphi_{\mathcal{L}_1}(\rho_1), \varphi_{\mathcal{L}_2}(\rho_1), \sigma_1, \sigma_2 \leq 1$ and $0 \leq \varphi_{\mathcal{L}_2}(\rho_2), \varphi_{\mathcal{L}_2}(\rho_2), \sigma_1, \sigma_2 \leq 1$. Thus, we can write the following:

$\frac{1}{4n} \sum_{i=1}^{n} \left[ (|\varphi_{\mathcal{L}_1}(\rho_1) - \varphi_{\mathcal{L}_2}(\rho_1)| + |\varphi_{\mathcal{L}_1}(\rho_1) - \varphi_{\mathcal{L}_2}(\rho_1)| + |\sigma_1 - \sigma_2| + |\sigma_1 - \sigma_2|) \right]$

$\geq 0$

$- \left[ \frac{1}{4n} \sum_{i=1}^{n} \left[ (|\varphi_{\mathcal{L}_1}(\rho_1) - \varphi_{\mathcal{L}_2}(\rho_1)| + |\varphi_{\mathcal{L}_1}(\rho_1) - \varphi_{\mathcal{L}_2}(\rho_1)| + |\sigma_1 - \sigma_2| + |\sigma_1 - \sigma_2|) \right] \right]$

$\leq 0$.

We also know from the exponential properties that the zero power of $e$ is equal to 1 and the value of the negative power of $e$ is less than 1. In this case, we have the equation as shown at the bottom of the next page. Therefore, $EM_{HD}(\mathcal{L}_1, \mathcal{L}_2)$ for every $\mathcal{L}_1$ and $\mathcal{L}_2$.

2) This property is proved by using the symmetric property of HD. We don’t need to prove this property.

3) Suppose that $\mathcal{L}_1 = \mathcal{L}_2$. Then, $\varphi_{\mathcal{L}_1}(\rho_1) = \varphi_{\mathcal{L}_2}(\rho_1), \varphi_{\mathcal{L}_2}(\rho_1) = \varphi_{\mathcal{L}_2}(\rho_1), \sigma_1 = \sigma_2, \sigma_2 = \tau_1$. Using equation (18), we have

$EM_{HD}(\mathcal{L}_1, \mathcal{L}_2) = e^{-\left[ \frac{1}{4n} \sum_{i=1}^{n} (|\varphi_{\mathcal{L}_1}(\rho_1) - \varphi_{\mathcal{L}_2}(\rho_1)| + |\varphi_{\mathcal{L}_1}(\rho_1) - \varphi_{\mathcal{L}_2}(\rho_1)| + |\sigma_1 - \sigma_2| + |\sigma_1 - \sigma_2|) \right]}$

$= e^{0} = 1$

Hence, $EM_{HD}(\mathcal{L}_1, \mathcal{L}_2) = 1$.

Next, we use the normalized Euclidean distance (ED) of two LDFSSs in Eq. (16) and we will get the new type of SM based on ED. The Exponential Similarity Measure (ESM) based on ED is given in Eq. (17):

$EM_{ED}(\mathcal{L}_1, \mathcal{L}_2) = e^{-\left[ \frac{1}{4n} \sum_{i=1}^{n} (|\varphi_{\mathcal{L}_1}(\rho_1) - \varphi_{\mathcal{L}_2}(\rho_1)| + |\varphi_{\mathcal{L}_1}(\rho_1) - \varphi_{\mathcal{L}_2}(\rho_1)| + |\sigma_1 - \sigma_2| + |\sigma_1 - \sigma_2|) \right]}$

Theorem 2: Consider $\mathcal{L}_1 = \{(\varphi_{\mathcal{L}_1}(\rho_1), \varphi_{\mathcal{L}_1}(\rho_1)), (\sigma_1, \tau_1)\}$ and $\mathcal{L}_2 = \{(\varphi_{\mathcal{L}_2}(\rho_2), \varphi_{\mathcal{L}_2}(\rho_2)), (\sigma_2, \tau_2)\}$ be two LDFSSs. Then, $EM_{ED}(\mathcal{L}_1, \mathcal{L}_2)$ defined in equation (19) satisfies the following conditions:

1) $0 \leq EM_{khED}(\mathcal{L}_1, \mathcal{L}_2) \leq 1$, for every $\mathcal{L}_1$ and $\mathcal{L}_2$,

2) $EM_{ED}(\mathcal{L}_1, \mathcal{L}_2) = EM_{ED}(\mathcal{L}_2, \mathcal{L}_1)$, for every $\mathcal{L}_1$ and $\mathcal{L}_2$,

3) $EM_{ED}(\mathcal{L}_1, \mathcal{L}_2) = 1$ if $\mathcal{L}_1 = \mathcal{L}_2$, for every $\mathcal{L}_1$ and $\mathcal{L}_2$.

Proof: Consider $\mathcal{L}_1 = \{(\varphi_{\mathcal{L}_1}(\rho_1), \varphi_{\mathcal{L}_1}(\rho_1)), (\sigma_1, \tau_1)\}$ and $\mathcal{L}_2 = \{(\varphi_{\mathcal{L}_2}(\rho_2), \varphi_{\mathcal{L}_2}(\rho_2)), (\sigma_2, \tau_2)\}$ to be two LDFSSs and since $0 \leq \varphi_{\mathcal{L}_1}(\rho_1), \varphi_{\mathcal{L}_1}(\rho_1), \sigma_1, \sigma_2 \leq 1$ and $0 \leq \varphi_{\mathcal{L}_2}(\rho_2), \varphi_{\mathcal{L}_2}(\rho_2), \sigma_1, \sigma_2 \leq 1$ we get

$E^{-\left[ \frac{1}{4n} \sum_{i=1}^{n} (|\varphi_{\mathcal{L}_1}(\rho_1) - \varphi_{\mathcal{L}_2}(\rho_1)| + |\varphi_{\mathcal{L}_1}(\rho_1) - \varphi_{\mathcal{L}_2}(\rho_1)| + |\sigma_1 - \sigma_2| + |\sigma_1 - \sigma_2|) \right]}$

$= e^{0} = 1$.
We also know from the exponential properties that the zero power of e is equal to 1 and the value of the negative power of e is less than 1. In this case, we have

\[ 0 \leq E_{M_{ED}}(L_1, L_2) \]

\[ = e^{-\frac{1}{\psi} \sum_{i=1}^{n} \left( \left| \psi_{L_1}(\rho_i) - \psi_{L_2}(\rho_i) \right| + |\psi_{L_1}(\rho_i) - \psi_{L_2}(\rho_i)| + |\psi_{L_1}(\rho_i) - \psi_{L_2}(\rho_i)| + |\psi_{L_1}(\rho_i) - \psi_{L_2}(\rho_i)| \right) \}} \]

\[ \leq 1 \]

Hence, \( 0 \leq E_{M_{ED}}(L_1, L_2) \leq 1 \).

2) The proof of this property is followed by using the symmetric property of HD, so we don’t need to proof this property.

3) Suppose that \( L_1 = L_2 \). Then, \( \psi_{L_1}(\rho_1) = \psi_{L_2}(\rho_1), \psi_{L_1}(\rho_i) = \psi_{L_2}(\rho_i), \theta_i = \zeta_i, \zeta_i = \tau_i \). Using equation (18), we have

\[ E_{M_{ED}}(L_1, L_2) \]

\[ = e^{-\frac{1}{\psi} \sum_{i=1}^{n} \left( \left| \psi_{L_1}(\rho_i) - \psi_{L_2}(\rho_i) \right| + |\psi_{L_1}(\rho_i) - \psi_{L_2}(\rho_i)| + |\psi_{L_1}(\rho_i) - \psi_{L_2}(\rho_i)| + |\psi_{L_1}(\rho_i) - \psi_{L_2}(\rho_i)| \right) \}} \]

\[ = e^{0} \]

\[ = 1 \]

Hence, \( E_{M_{ED}}(L_1, L_2) = 1 \).

If we consider the refusal degree of LDFS and use this refusal degree in Eq. (16) and (17), then we will get some new \( E_{M_{D}}(L_1, L_2) \) with refusal degree based on HD and ED in equation (18) and (19), respectively. In the next definitions, we define a new \( E_{M_{R}}(L_1, L_2) \) and \( E_{M_{R}}^R(\rho_1, \rho_2) \) of \( L_1 \) and \( L_2 \) with refusal degrees of LDFS.

Definition 17: Consider a discrete non-empty set \( \mathcal{L} = \{ \rho_1, \rho_2, \rho_3, \ldots, \rho_n \} \) and two sets of LDFSs \( L_1 = \{ \psi_{L_1}(\rho_1), \psi_{L_1}(\rho_i), \zeta_i, \sigma_i \} \) and \( L_2 = \{ \psi_{L_2}(\rho_1), \psi_{L_2}(\rho_i), \zeta_i, \tau_i \} \). Then, ESM based on NHD is generated by using Eq. (16) in Eq. (15), So the ESM based on NHD is given in Eq. (18):

\[ E_{M_{R}}^R(\rho_1, \rho_2) \]

\[ = \left[ \exp\left( -\frac{1}{\psi} \sum_{i=1}^{n} \left( \left| \psi_{L_1}(\rho_i) - \psi_{L_2}(\rho_i) \right| + |\psi_{L_1}(\rho_i) - \psi_{L_2}(\rho_i)| + |\psi_{L_1}(\rho_i) - \psi_{L_2}(\rho_i)| + |\psi_{L_1}(\rho_i) - \psi_{L_2}(\rho_i)| \right) \} \right) \]
The weighted ESM based on ED is denoted by \( WEM_{ED} (L_1, L_2) \) and given in Eq. (21):

\[
WEM_{ED} (L_1, L_2) = e^{- \frac{1}{n} \sum_{i=1}^{n} \left( \frac{1}{2} \left( \psi(L_1, \rho_i) - \psi(L_2, \rho_i) \right)^2 + \left( \psi(L_1, \sigma_i) - \psi(L_2, \sigma_i) \right)^2 \right)}
\]

(21)

If we use the refusl degree of LDFSs \( L_1 \) and \( L_2 \) in \( WEM_{HD} (L_1, L_2) \) and \( WEM_{ED} (L_1, L_2) \), then we get another type of weighted exponention similarity measure based on HD and ED with refusl degree of \( L_1 \) and \( L_2 \), respectively. The new WESM is denoted by \( WEM_{ED}^R (L_1, L_2) \) and \( WEM_{ED} (L_1, L_2) \) given in Eq. (22) and (23), respectively.

\[
WEM_{HD} (L_1, L_2) = e^{- \frac{1}{n} \sum_{i=1}^{n} \left( \left( \psi(L_1, \rho_i) - \psi(L_2, \rho_i) \right)^2 + \left( \psi(L_1, \sigma_i) - \psi(L_2, \sigma_i) \right)^2 \right)}
\]

(22)

\[
WEM_{ED} (L_1, L_2) = e^{- \frac{1}{n} \sum_{i=1}^{n} \left( \left( \psi(L_1, \rho_i) - \psi(L_2, \rho_i) \right)^2 + \left( \psi(L_1, \sigma_i) - \psi(L_2, \sigma_i) \right)^2 \right)}
\]

(23)

C. SIMILARITY MEASURE BASED ON COSINE AND COTANGENT FUNCTIONS

This section is devoted for the SM based on Cosine and Cotangent functions for LDFSs. The proposed SM is also know as cosine similarity measure (CSM) and Cotangent similarity measure (CtSM) between two LDFS \( L_1 \) and \( L_2 \). The CSM is generated by using the cosine function and CtSM is generated by using cotangent function. This section has two subsection, one is for CSM and the other is for CtSM.

1) COSINE SIMILARITY MEASURE FOR LDFSs

In this section, the authors use the cosine function to develop a CSM for two LDFSs \( L_1 \) and \( L_2 \). We also develop the weighted CSM for two LDFS \( L_1 \) and \( L_2 \) and also prove the properties of the CSM and WCSM with distance measure.

Definition 19: Consider a discrete non-empty set \( \varphi = \{ \rho_1, \rho_2, \rho_3, \ldots, \rho_n \} \) and two sets of LDFSs \( L_1 = \{ (\psi(L_1, \rho_1), \psi(L_1, \sigma_1)) \} \) and \( L_2 = \{ (\psi(L_2, \rho_1), \psi(L_2, \sigma_1)) \} \). Then Cosine similarity measure between two LDFSs \( L_1 \) and \( L_2 \) is defined as the following (24), as shown at the bottom of the next page.

Proof: The first two properties are easy, we omit the proof of first two properties.

3) Assume that \( L_1 = L_2 \), then \( \psi(L_1, \rho_i) = \psi(L_2, \rho_i) \), \( \psi(L_1, \sigma_i) = \psi(L_2, \sigma_i) \), \( \varphi_1 = \varphi_2 \), \( \tau_1 = \tau_2 \). Then Eq. (24) implies CSM \( L_1, L_2 \) is 1.

Thus, CSM \( L_1, L_2 \) is 1.

Now to prove the last property, consider \( L_1 \subseteq L_2 \subseteq L_3 \), so the angle of \( L_1, L_3 \) is larger than the angle of \( L_1, L_2 \) and \( L_2, L_3 \). Thus from the fact, we can write CSM \( L_1, L_3 \) ≤ CSM \( L_1, L_2 \) and CSM \( L_1, L_2 \) ≤ CSM \( L_2, L_3 \).

Next, we present a relation between distance measure (DM) and CSM. We explore the DM between two LDFSs using CSM measure. Consider a discrete non-empty set \( \varphi = \{ \rho_1, \rho_2, \rho_3, \ldots, \rho_n \} \) and two sets of LDFSs \( L_1 = \{ (\psi(L_1, \rho_1), \psi(L_1, \rho_1), \varphi_1, \tau_1) \} \) and \( L_2 = \{ (\psi(L_2, \rho_1), \psi(L_2, \rho_1), \varphi_2, \tau_2) \} \). Then the DM between two LDFSs is given by the following equation;

\[
d(L_1, L_2) = \cos^{-1} (CSM (L_1, L_2))
\]

(25)

Now, we will prove that Eq. (25) satisfies the conditions of the DM;

1) If \( 0 \leq CSM (L_1, L_2) \leq 1 \), then \( d(L_1, L_2) \geq 0 \),
2) If \( CSM (L_1, L_2) = CSM (L_2, L_1) \), then \( d(L_1, L_2) = d(L_2, L_1) \),
3) If \( CSM (L_1, L_2) = 1 \) if \( L_1 = L_2 \), then \( d(L_1, L_2) = 0 \),
4) If \( L_1 \subseteq L_2 \subseteq L_3 \), then \( d(L_1, L_3) \leq d(L_1, L_2) + d(L_2, L_3) \).

Proof: The first three properties of DM are easy to prove, we only prove the last property of Eq (25), consider \( L_1, L_2 \) and \( L_3 \) such that \( L_1 \subseteq L_2 \subseteq L_3 \). Then, DM of \( L_1, L_2 \) and \( L_3 \) are given as;

\[
d(L_1, L_2) = \cos^{-1} (CSM (L_1, L_2))
\]

\[
d(L_2, L_3) = \cos^{-1} (CSM (L_2, L_3))
\]

\[
d(L_1, L_3) = \cos^{-1} (CSM (L_1, L_3))
\]

Therefore, we can write the above equations as following CSM \( L_1, L_2 \), CSM \( L_2, L_3 \), and CSM \( L_1, L_3 \), as shown at the bottom of the next page.
TABLE 3. Diagnosis-symptoms of decision information matrix-III (for $\rho = 0.3$, $\sigma = 0.3$).

|     | (M11, S1) | (M11, S2) | (M11, S3) | (M12, S1) | (M12, S2) | (M12, S3) | (M13, S1) | (M13, S2) | (M13, S3) |
|-----|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| P1  | (0.7, 0.8) | (0.2, 0.9) | (0.8, 0.9) | (0.9, 0.8) | (0.8, 0.9) | (1, 1)    | (0.8, 0.9) | (0.8, 0.9) | (0.8, 0.9) |
| P2  | (0.2, 0.8) | (0.3, 0.8) | (0.2, 0.5) | (0.8, 0.8) | (0.7, 0.9) | (0.4, 0.8) | (0.3, 0.8) | (0.2, 0.5) | (0.7, 0.9) |
| P3  | (0.3, 0.8) | (0.7, 0.1) | (0.9, 0.3) | (0.6, 0.8) | (0.3, 0.9) | (0.3, 0.5) | (0.7, 1)  | (0.9, 0.3) | (0.3, 0.9) |
| P4  | (0.7, 0.8) | (0.4, 0.9) | (0.1, 0.5) | (0.9, 0.9) | (0.2, 1)   | (0.3, 0.7) | (0.4, 0.9) | (0.1, 0.5) | (0.2, 1)   |
| P5  | (0.4, 0.7) | (0.5, 0.6) | (0.2, 0.6) | (0.6, 0.6) | (0.7, 0.2) | (0.2, 1)   | (0.5, 0.6) | (0.2, 0.6) | (0.7, 0.2) |
| P6  | (0.3, 0.6) | (0.6, 0.4) | (0.3, 0.6) | (0.4, 0.9) | (0.8, 0.3) | (0.3, 0.6) | (0.6, 0.4) | (0.3, 0.6) | (0.8, 0.3) |
| P7  | (0.2, 0.9) | (0.7, 0.8) | (0.8, 0.9) | (0.9, 0.4) | (0.7, 0.6) | (0.4, 0.8) | (0.7, 0.8) | (0.8, 0.9) | (0.7, 0.6) |
| P8  | (0.5, 0.6) | (0.3, 0.8) | (0.9, 0.7) | (0.7, 0.5) | (0.3, 0.8) | (0.4, 0.7) | (0.3, 0.8) | (0.9, 0.7) | (0.3, 0.8) |
| P9  | (0.5, 0.4) | (0.2, 0.9) | (0.8, 0.9) | (0.9, 0.4) | (0.6, 0.7) | (0.6, 1)   | (0.2, 0.9) | (0.8, 0.9) | (0.6, 0.7) |

Let $L_1 = \{(\varphi_{L_1}(\rho), \psi_{L_1}(\rho_i), (\varphi_i, \sigma_i))\}$, $L_2 = \{(\varphi_{L_2}(\rho), \psi_{L_2}(\rho_i), (\varphi_i, \sigma_i))\}$ and $L_3 = \{(\varphi_{L_3}(\rho), \psi_{L_3}(\rho_i), (\varphi_i, \xi_i))\}$ be three vectors such that $L_1 \subseteq L_2 \subseteq L_3$. Then by using triangular inequality, we have $d(L_1, L_3) \leq d(L_1, L_2) + d(L_2, L_3)$.

Now, consider weight $w_i$ for each $\rho_i$, the weight vector for all values of $\rho_i$, for all $I$. Then we proposed weighted CSM for two LDFS.

**Definition 20:** The weighted CSM (WCSM) is defined for two LDFS $L_1 = \{(\varphi_{L_1}(\rho), \psi_{L_1}(\rho_i), (\varphi_i, \sigma_i))\}$ and $L_2 = \{(\varphi_{L_2}(\rho), \psi_{L_2}(\rho_i), (\varphi_i, \xi_i))\}$ as following (26), as shown at the bottom of the next page, where $w = (w_1, w_2, w_3, \ldots, w_n)$ is a weight vector of each $\rho_i$ such that $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. If we take $w_i = \frac{1}{n}$, then Eq. (26) becomes Eq 20. The WCSM is generalized form of CSM. The WCSM is also satisfied for two LDFS.

1) $0 \leq \text{WCSM} (L_1, L_2) \leq 1$,

2) $\text{WCSM} (L_1, L_2) = \text{WCSM} (L_2, L_1)$,

3) $\text{WCSM} (L_1, L_2) = 1$ iff $L_1 = L_2$.

**Proof:** The proof follows from the proof of Eq (24).

Now we will use the cosine function to propose new SMs and WSMs based on cosine function. The similarity measure based on cosine function is given in Eq. (26).

**Definition 21:** The SMs based on cosine function for two LDFS $L_1$ and $L_2$ is denoted by $\text{SM}_{\cos} (L_1, L_2)$ and defined by the following equation:

$$
\text{SM}_{\cos}^1 (L_1, L_2) = \frac{1}{n} \sum_{i=1}^{n} \cos \left( \left( \varphi_{L_1}(\rho) - \varphi_{L_2}(\rho_i) \right) \vee \left| \psi_{L_1}(\rho) - \psi_{L_2}(\rho_i) \right| \right)
$$

$$
= \frac{\pi}{2} \left( \left( \varphi_{L_1}(\rho) - \varphi_{L_2}(\rho_i) \right) \vee \left| \psi_{L_1}(\rho) - \psi_{L_2}(\rho_i) \right| \right)
$$

(27)

$$
\text{CSM} (L_1, L_2) = \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{\varphi_{L_1}(\rho) \varphi_{L_2}(\rho_i) + \psi_{L_1}(\rho) \psi_{L_2}(\rho_i) + \varphi_i \sigma_i + \sigma_i \tau_i}{\sqrt{\varphi_{L_1}^2(\rho) + \psi_{L_1}^2(\rho_i) + \varphi_i^2 + \sigma_i^2}} \right]
$$

(24)

$$
\text{CSM} (L_2, L_3) = \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{\varphi_{L_2}(\rho) \varphi_{L_3}(\rho_i) + \psi_{L_2}(\rho) \psi_{L_3}(\rho_i) + \varphi_i \xi_i + \xi_i \tau_i}{\sqrt{\varphi_{L_2}^2(\rho) + \psi_{L_2}^2(\rho_i) + \varphi_i^2 + \xi_i^2 + \tau_i^2}} \right]
$$

$$
\text{CSM} (L_1, L_3) = \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{\varphi_{L_1}(\rho) \varphi_{L_3}(\rho_i) + \psi_{L_1}(\rho) \psi_{L_3}(\rho_i) + \varphi_i \kappa_i + \kappa_i \xi_i}{\sqrt{\varphi_{L_1}^2(\rho) + \psi_{L_1}^2(\rho_i) + \varphi_i^2 + \kappa_i^2 + \xi_i^2}} \right]
$$

$$
\text{CSM} (L_1, L_2) = \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{\varphi_{L_1}(\rho) \varphi_{L_2}(\rho_i) + \psi_{L_1}(\rho) \psi_{L_2}(\rho_i) + \varphi_i \sigma_i + \sigma_i \tau_i}{\sqrt{\varphi_{L_1}^2(\rho) + \psi_{L_1}^2(\rho_i) + \varphi_i^2 + \sigma_i^2}} \right]
$$

$$
\text{CSM} (L_2, L_3) = \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{\varphi_{L_2}(\rho) \varphi_{L_3}(\rho_i) + \psi_{L_2}(\rho) \psi_{L_3}(\rho_i) + \varphi_i \xi_i + \xi_i \tau_i}{\sqrt{\varphi_{L_2}^2(\rho) + \psi_{L_2}^2(\rho_i) + \varphi_i^2 + \xi_i^2 + \tau_i^2}} \right]
$$

$$
\text{CSM} (L_1, L_3) = \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{\varphi_{L_1}(\rho) \varphi_{L_3}(\rho_i) + \psi_{L_1}(\rho) \psi_{L_3}(\rho_i) + \varphi_i \kappa_i + \kappa_i \xi_i}{\sqrt{\varphi_{L_1}^2(\rho) + \psi_{L_1}^2(\rho_i) + \varphi_i^2 + \kappa_i^2 + \xi_i^2}} \right]
$$
Also, we prove the following properties for Eq. (27) and (28):

1) The SM $\sigma_{L_{i}}$ must lie between 0 and 1.

2) Consider three LDFSs such that $\sigma_{L_{i}} \subseteq \sigma_{L_{i+1}}$, where $i = 1, 2, ..., n$. Thus, $\sigma_{L_{i}} \subseteq \sigma_{L_{i+1}}$.

3) In order to prove the third property, we consider two LDFSs $\sigma_{L_{i}} \subseteq \sigma_{L_{i+1}}$. Then, $\psi_{L_{i}}(\rho_i) = \psi_{L_{i+1}}(\rho_i)$, $\psi_{L_{i+1}}(\rho_i) = \psi_{L_{i+1}}(\rho_{i+1})$, $\sigma_{i} = \sigma_{i+1}$, and $\sigma_{i} = \tau_{i}$. Thus, $\sigma_{L_{i}} \subseteq \sigma_{L_{i+1}}$.

4) Consider three LDFSs such that $\sigma_{L_{i}} \subseteq \sigma_{L_{i+1}}$, then

$$\psi_{L_{i}}(\rho_i) \leq \psi_{L_{i+1}}(\rho_i) \leq \psi_{L_{i+1}}(\rho_{i+1}) \leq \psi_{L_{i+1}}(\rho_{i+2})$$

$$\sigma_{i} \leq \sigma_{i+1} \leq \sigma_{i+2}$$

Then,

$$\left| \psi_{L_{i}}(\rho_i) - \psi_{L_{i+1}}(\rho_i) \right| \leq \left| \psi_{L_{i}}(\rho_i) - \psi_{L_{i+1}}(\rho_i) \right|$$

$$\left| \psi_{L_{i+1}}(\rho_i) - \psi_{L_{i+1}}(\rho_{i+1}) \right| \leq \left| \psi_{L_{i+1}}(\rho_i) - \psi_{L_{i+1}}(\rho_{i+1}) \right|$$

$$\left| \psi_{L_{i+1}}(\rho_{i+1}) - \psi_{L_{i+1}}(\rho_{i+2}) \right| \leq \left| \psi_{L_{i+1}}(\rho_{i+1}) - \psi_{L_{i+1}}(\rho_{i+2}) \right|$$

Since, the cosine function is decreasing function in $[0, \pi]$. Thus, $\sigma_{L_{i}} \leq \sigma_{L_{i+1}}$ and $\sigma_{L_{i+1}} \leq \sigma_{L_{i+2}}$.

Next, we assign the weight of each $\rho_i$ with the weight vector \(w = (w_1, w_2, w_3, ..., w_n)\) subject to $w_i \in [0, 1]$ such that $\sum_{i=1}^{n} w_i = 1$. Then, we propose a new SM based on cosine function, so called weighted SM based on the cosine function of $\sigma_{L_{i}}$ and $\sigma_{L_{i+1}}$ which are denoted by $WSM_{cos}^{1}(\sigma_{L_{i}}, \sigma_{L_{i+1}})$ and $WSM_{cos}^{2}(\sigma_{L_{i}}, \sigma_{L_{i+1}})$, and defined in equation (30) and (31), respectively.

$$WSM_{cos}^{1}(\sigma_{L_{i}}, \sigma_{L_{i+1}}) = \sum_{i=1}^{n} w_i \cos \left( \frac{\pi}{2} \left( \left| \psi_{L_{i}}(\rho_i) - \psi_{L_{i+1}}(\rho_i) \right| \right) \right) \left[ \left| \psi_{L_{i}}(\rho_i) - \psi_{L_{i+1}}(\rho_i) \right| \right]$$

$$WSM_{cos}^{2}(\sigma_{L_{i}}, \sigma_{L_{i+1}}) = \sum_{i=1}^{n} w_i \cos \left( \frac{\pi}{4} \left( \left| \psi_{L_{i}}(\rho_i) - \psi_{L_{i+1}}(\rho_i) \right| \right) \right) \left[ \left| \psi_{L_{i}}(\rho_i) - \psi_{L_{i+1}}(\rho_i) \right| \right]$$

If we put $w_i = \frac{1}{n}$ in Eq. (29) and (30), then they are reduced to Eq. (27) and (28), respectively. So Eq. (29) and (30) are generalized equation of SMs.

Also, the following properties of Eq. (29) and (30) are satisfied.

1) $0 \leq SM_{cos}^{1}(\sigma_{L_{i}}, \sigma_{L_{i+1}}) \leq 1$

2) $SM_{cos}^{1}(\sigma_{L_{i}}, \sigma_{L_{i+1}}) = SM_{cos}^{1}(\sigma_{L_{i}}, \sigma_{L_{i+1}})$
In this section, the authors use the Cotangent function to develop a Cotangent similarity measure (CtSM) for two LDFFs $\mathcal{L}_1$ and $\mathcal{L}_2$. We also develop the weighted CtSM for two LDFF $\mathcal{L}_1$ and $\mathcal{L}_2$ and also prove some properties of the CtSM and WCtSM with distance measure.

**Definition 22**: The SMs based on cotangent function for two LDFF $\mathcal{L}_1$ and $\mathcal{L}_2$ is denoted by $SM_{\cot}(\mathcal{L}_1, \mathcal{L}_2)$ and defined by the following equation:

$$SM^1_{\cot}(\mathcal{L}_1, \mathcal{L}_2) = \frac{1}{n} \sum_{i=1}^{n} cot \left( \frac{\pi}{4} + \frac{\pi}{4} \left( |\psi_{\mathcal{L}_1}(\rho_i) - \psi_{\mathcal{L}_2}(\rho_i)| \vee |\psi_{\mathcal{L}_1}(\rho_i) - \psi_{\mathcal{L}_2}(\rho_i)| \right) \right)$$

$$SM^2_{\cot}(\mathcal{L}_1, \mathcal{L}_2) = \frac{1}{n} \sum_{i=1}^{n} cot \left( \frac{\pi}{4} + \frac{\pi}{4} \left( |\psi_{\mathcal{L}_1}(\rho_i) - \psi_{\mathcal{L}_2}(\rho_i)| + |\psi_{\mathcal{L}_1}(\rho_i) - \psi_{\mathcal{L}_2}(\rho_i)| \right) \right)$$

Also, we prove the following properties for Eqs. (27) and (28):

1. $0 \leq SM^1_{\cot}(\mathcal{L}_1, \mathcal{L}_2) \leq 1$
2. $SM^1_{\cot}(\mathcal{L}_1, \mathcal{L}_2) = SM^1_{\cot}(\mathcal{L}_2, \mathcal{L}_1)$
3. $SM^2_{\cot}(\mathcal{L}_1, \mathcal{L}_2) = 1$ iff $\mathcal{L}_1 = \mathcal{L}_2$
4. $SM^2_{\cot}(\mathcal{L}_1, \mathcal{L}_2) = SM^2_{\cot}(\mathcal{L}_2, \mathcal{L}_1)$ is easy to be proved.

**Proof**: 1) The 1st property is easy because the cosine function values lie between 0 and 1 and also $SM^2_{\cot}(\mathcal{L}_1, \mathcal{L}_2)$ is based on cosine function. Thus, the values of $SM^2_{\cot}(\mathcal{L}_1, \mathcal{L}_2)$ must lie between 0 and 1.

2) For $SM^2_{\cot}(\mathcal{L}_1, \mathcal{L}_2) = SM^2_{\cot}(\mathcal{L}_2, \mathcal{L}_1)$ is easy to be proved.

3) In order to prove the third property, we consider two LDFFs $\mathcal{L}_1 = \mathcal{L}_2$. Then, $\psi_{\mathcal{L}_1}(\rho_i) = \psi_{\mathcal{L}_2}(\rho_i) = \psi_{\mathcal{L}_3}(\rho_i) = \rho_i = \xi_i$ and $\sigma_i = \tau_i$ for all $i = 1, 2, 3, \ldots, n$. Thus, $|\psi_{\mathcal{L}_1}(\rho_i) - \psi_{\mathcal{L}_2}(\rho_i)| = |\psi_{\mathcal{L}_1}(\rho_i) - \psi_{\mathcal{L}_3}(\rho_i)| = |\sigma_i - \tau_i| = 0$. Thus, $SM^2_{\cot}(\mathcal{L}_1, \mathcal{L}_2) = 1$.

4) Consider the three LDFFs such that $\mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \mathcal{L}_3$, then $\psi_{\mathcal{L}_1}(\rho_i) \leq \psi_{\mathcal{L}_2}(\rho_i) \leq \psi_{\mathcal{L}_3}(\rho_i)$, $\psi_{\mathcal{L}_1}(\rho_i) \geq \psi_{\mathcal{L}_2}(\rho_i) \geq \psi_{\mathcal{L}_3}(\rho_i)$, $\sigma_i \leq \xi_i$ and $\sigma_i \geq \tau_i \geq \xi_i$.

Thus,

$$|\psi_{\mathcal{L}_1}(\rho_i) - \psi_{\mathcal{L}_2}(\rho_i)| \leq |\psi_{\mathcal{L}_1}(\rho_i) - \psi_{\mathcal{L}_3}(\rho_i)|$$

$$|\psi_{\mathcal{L}_1}(\rho_i) - \psi_{\mathcal{L}_2}(\rho_i)| \leq |\psi_{\mathcal{L}_1}(\rho_i) - \psi_{\mathcal{L}_3}(\rho_i)|$$

Since, the cotangent function is decreasing function in $[0, \frac{\pi}{4}]$. Thus, $SM^1_{\cot}(\mathcal{L}_1, \mathcal{L}_3) \leq SM^1_{\cot}(\mathcal{L}_1, \mathcal{L}_2)$ and $SM^2_{\cot}(\mathcal{L}_1, \mathcal{L}_3) \leq SM^2_{\cot}(\mathcal{L}_2, \mathcal{L}_3)$.

Next, we assign the weight of each $\rho_i$ with the weight vector $w = (w_1, w_2, w_3, \ldots, w_n)$ subject to $w_i \in [0, 1]$ such that $\sum_{i=1}^{n} w_i = 1$. Then, we propose a new SM based on cosine function, so called weighted SM based on the cosine function of $\mathcal{L}_1$ and $\mathcal{L}_2$ which are denoted by $WSM^1_{\cot}(\mathcal{L}_1, \mathcal{L}_2)$ and $WSM^2_{\cot}(\mathcal{L}_1, \mathcal{L}_2)$, and defined in equation (30) and (31), respectively.

$$WSM^1_{\cot}(\mathcal{L}_1, \mathcal{L}_2) = \sum_{i=1}^{n} w_i \cdot cot \left( \frac{\pi}{4} + \frac{\pi}{4} \left( |\psi_{\mathcal{L}_1}(\rho_i) - \psi_{\mathcal{L}_2}(\rho_i)| \vee |\psi_{\mathcal{L}_1}(\rho_i) - \psi_{\mathcal{L}_2}(\rho_i)| \right) \right)$$

$$WSM^2_{\cot}(\mathcal{L}_1, \mathcal{L}_2) = \sum_{i=1}^{n} w_i \cdot cot \left( \frac{\pi}{4} + \frac{\pi}{4} \left( |\psi_{\mathcal{L}_1}(\rho_i) - \psi_{\mathcal{L}_2}(\rho_i)| + |\psi_{\mathcal{L}_1}(\rho_i) - \psi_{\mathcal{L}_2}(\rho_i)| \right) \right)$$

If we put $w_i = \frac{1}{n}$ in Eq. (29) and (30), then they are reduced to Eq. (27) and (28), respectively. So Eqs. (29) and (30) are generalized equation of SMs.

Also, following properties of Eq. (29) and (30) are satisfied.

1. $0 \leq SM^1_{\cot}(\mathcal{L}_1, \mathcal{L}_2) \leq 1$
2. $SM^1_{\cot}(\mathcal{L}_1, \mathcal{L}_2) = SM^1_{\cot}(\mathcal{L}_2, \mathcal{L}_1)$
3. $SM^1_{\cot}(\mathcal{L}_1, \mathcal{L}_2) = 1$ iff $\mathcal{L}_1 = \mathcal{L}_2$
4. $SM^1_{\cot}(\mathcal{L}_1, \mathcal{L}_2) = SM^1_{\cot}(\mathcal{L}_2, \mathcal{L}_1)$ and $SM^1_{\cot}(\mathcal{L}_1, \mathcal{L}_3) \leq SM^1_{\cot}(\mathcal{L}_2, \mathcal{L}_3)$.

**Proof**: The proof of the above properties follow from the proof of the properties of Eq. (31) and (32).

**IV. APPLICATIONS OF THE PROPOSED SIMILARITY MEASURE**

In the literature the similarity measure of different fuzzy sets has been developed due to the significant role of the SMs in different field such as decision making, clustering analysis, data mining, pattern recognition and medical diagnosing etc. This section provide a detail description of applications of the proposed SM of the present paper. We are going to develop the general multi-attribute group decision support systems for diagnosis of COVID-19 patients. In case of COVID-19 (medical diagnosis), the detail description of the algorithm for COVID-19 patients is given in the following steps;
TABLE 5. Reference indicator of COVID-19 patient (for $\varrho = 0.3, \sigma = 0.3$).

| (M1, S1) | (M1, S2) | (M1, S3) | (M2, S1) | (M2, S2) | (M2, S3) | (M3, S1) | (M3, S2) | (M3, S3) |
|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| P1       | (0.9,0.6) | (0.9,0.5) | (1.0,0.5) | (0.85,7) | (0.84,1) | (0.83,1) | (0.84,1) | (0.79,1) |

TABLE 6. SMs of the each patient from RI by using different SMs (for $\varrho = 0.3, \sigma = 0.3$).

| JSM | $EM_{HD}$ | $EM_{HD}$ | $SM_{cos}$ | $SM_{cos}$ |
|-----|-----------|-----------|------------|------------|
| P1  | 0.678     | 0.797     | 0.846      | 0.926      | 0.701     | 0.733     |
| P2  | 0.433     | 0.503     | 0.643      | 0.762      | 0.588     | 0.565     |
| P3  | 0.485     | 0.489     | 0.669      | 0.757      | 0.605     | 0.524     |
| P4  | 0.446     | 0.511     | 0.657      | 0.727      | 0.563     | 0.574     |
| P5  | 0.478     | 0.495     | 0.675      | 0.744      | 0.611     | 0.523     |
| P6  | 0.434     | 0.510     | 0.689      | 0.68       | 0.619     | 0.569     |
| P7  | 0.683     | 0.805     | 0.871      | 0.933      | 0.712     | 0.781     |
| P8  | 0.622     | 0.782     | 0.838      | 0.914      | 0.681     | 0.682     |
| P9  | 0.616     | 0.767     | 0.826      | 0.902      | 0.658     | 0.689     |

TABLE 7. Diagnosis-symptoms of decision transferred information matrix-IV (for $\varrho = 0.3, \sigma = 0.3$).

| (M1, S1) | (M1, S2) | (M1, S3) | (M2, S1) | (M2, S2) | (M2, S3) | (M3, S1) | (M3, S2) | (M3, S3) |
|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| P1       | (0.7,0.5) | (0.8,0.4) | (0.4,0.8) | (0.9,1)  |
| P2       | (0,6,0.8) | (0.8,0.9) | (0.8,0.9) | (0.7,0.9) |
| P3       | (0.3,0.9) | (0.1,0.3) | (0.9,0.5) | (0.7,0.9) |
| P4       | (0.5,3)   | (0.5,0.6) | (0.3,0.7) | (0.4,0.2) |
| P5       | (0.8,0.7) | (0.2,0.4) | (0.1,1)   | (0.3,0.5) |
| P6       | (0.9,0.5) | (0.7,0.6) | (0.3,0.6) | (0.8,0.5) |
| P7       | (0.3,0.2) | (0.6,0.5) | (0.5,0.8) | (0.4,0.4) |
| P8       | (0.7,0.4) | (0.3,0.2) | (0.4,0.7) | (0.2,0.3) |
| P9       | (0.2,0.3) | (0.4,0.6) | (0.6,1)   | (0.8,0.7) |

Let us consider $m$ decision experts (Doctors) $D = \{D_1, D_2, D_3, \ldots, D_m\}$ and weight vectors of the decision experts is denoted by $V = \{\omega_1, \omega_2, \ldots, \omega_m\}$, where $\sum_{i=1}^{m} \omega_i$ and $0 \leq \omega_i \leq 1$. Consider $n$ patients $P = \{P_1, P_2, \ldots, P_n\}$ having $l$ number symptoms $S = \{s_1, s_2, s_3, \ldots, s_l\}$ and $RI = \{r_1, r_2, r_3, \ldots, r_l\}$ is denoted as reference indicator of the COVID-19. The detail procedure follows for diagnosis of COVID-19 patients;

A. NUMERICAL APPLICATION OF DIAGNOSIS OF COVID-19

Consider a medical diagnosis problem of COVID-19 patients by using SMs. The proposed method is a new technique of SMs between two LDFSs which can be applied to detect whether the patient is suffering from COVID-19 or not.

By the end of 1919, the new virus identified in the capital city of province Hubei, China which is called novel corona virus (n-Covid-19), laterly WHO is called COVID-19. The COVID-19 is very rapidly spreading virus in the universe. The infected peoples on the 11th of May 2020 in the universe reached 3.954 million. The total number of deaths was 275160 and the recovery of the infected peoples reached 1.331 million. The COVID-19 is transmitted among the people through the breathing droplets from sneezes and coughs, and the coss contact.

Now to discuss the symptoms of the infected people from the said virus. The major symptoms of the infected patients are shortness of breath, fever and cough and the minor symptoms are sputum production, muscle pain, abdominal pain, diarrhea and loss of smell. Now consider eight patients $P_1, P_2, \ldots, P_8$ in hospital having the symptoms cough, fever and shortness of breath. Three Physicians visit these patients and the information are given in the diagnosis-symptoms matrixes of each Physicians. The information of the patients in the form LDFVs $L = \langle (\phi_L, \psi_L), (\varrho, \sigma) \rangle$ in the Tables 1, 2, 3; $\phi_L$ is a membership, $\psi_L$ is a non-membership and $\varrho, \sigma$ are set of parameters. We will check the different parameters by changing the values of $\varrho$ and $\sigma$. Now we use the linear Diophantine fuzzy aggregation operators to collect the data information of the three Physicians. The collective or aggregated diagnosis-symptoms data information is given in Table 4.

Now, we consider a reference indicator (RI) of COVID-19. This RI, indicates the symptoms information of the patients who has the most probable symptoms of COVID-19.

We will check the SMs between RI and MI of nine patients by using equations (1), (18), (19), (26), (29) and (33) of SMs.

Algorithm 1

1. To make a committee of the doctors physicians.
2. To collect a data information in the form linguistic terms of the diagnosis-symptoms decision matrix of the patients by each physicians.
3. To find out the collective diagnosis-symptoms decision informations by using the LDFWA operator.
4. Find the similarity measure between patients and RI by using different equations of SMs.
5. The final diagnosis through the maximum SM:
   i) If SM is near to 1, then patient diagnosis COVID-19.
   ii) If SM is near to 0, then patients does not diagnosis COVID-19.
The detail SMs of all patients and the RI from different SMs are given in Table 6.

In the Table 6, the detail description of the SMs of each patient are given. We can see from Table 6, the SMs of these patients are different from the different formulas of SMs. So some of these SMs have optimistic view and some of them pessimistic view. The experts physician can practice these data information of all patients and will easily diagnose the COVID-19 patients. The patients P1, P7, P8, P9 have the maximum similarities among all patients, the SMs of these patients are near to 1, so these patients have the COVID-19. The experts physician will easily diagnose the COVID-19.
patients. The detailed comparisons of each SM with another SM are given in Fig 1-5 and the comparison of the all SMs is given in Fig 6. In Fig 6, the patients P1, P7, P8, P9, the values of similarity are high for every SM formula. This is very clearly that for each formula of the SM, the values of similarity measure of patients P1, P7, P8, P9 are high. Therefore, Patients P1, P7, P8, P9 are diagnosis from COVID19.

**V. COMPARATIVE STUDY**

In this section we provide a comparative study of the proposed method and with the already existing technique. We compared it with the different methods for decision making and pattern recognition. The detailed comparison is given in two subsections;

**A. COMPARATIVE ANALYSIS 1**

In this section, we provide the compairative analysis of the proposed method with the existence methods.

1) VALIDITY TEST

Since different MAGDM approaches give different assessments (ranking), when applied to the same DM problem, which contributes to uncertain results. Thus, Wang and Triantaphyllou [57] presented the following test conditions in order to examine the reliability and validity of the MAGDM approaches:

**Test criteria 1:** The MAGDM approach is effective if the choice of the best alternative stays the same and replace the non-optimal alternative to the other worse alternative without changing the relative value of each decision-attribute.

**Test criteria 2:** An efficient MCGDM approach should obey transitive properties.

**Test criteria 3:** The MCGDM approach is effective when decomposing the MCGDM problem into sub problems and applying the proposed MCGDM approach to these sub-problems for the ranking of alternatives. The cumulative ranking of the alternatives remains the same as the ranking of the actual problem.

The Validity of the proposed solution is checked using the following attributes.

2) VALIDITY CHECK WITH CRITERIA 1

The non-optimal alternative P is replaced by the worst alternative P' in the original decision matrix for each expert to test the validity of the established method with Criteria 1, and the rating values are given in Table 8.

Now, using the LDFWA operator on transferring alternative, we get the computed similarity measure of the alternatives to be P1, P2, P3, P4, P5, P6, P7, P8, P9, respectively. Thus, the final ranking of the alternatives suggests that similarity measure of patients P1, P7, P8, P9 are high and is still the best alternative and that the built method satisfies the test criteria 1.

**TABLE 8. Comparison with some existing methods and their ranking**

| Techniques                      | Ranking |
|---------------------------------|---------|
| Joshi and Kumar [58]            | P1, P4, P5, P6 |
| Babak, et al.[59]               | P1, P3, P7, P9 |
| Liu and Wang [60]               | P1, P2, P4, P6 |

3) VALIDITY CHECK WITH CRITERIA 2 AND 3

In order to test the defined MAGDM method with the Criteria 2 and 3; we decomposed the original decision making problem into sub DM problems, which have included these alternatives (P1,P3,P5,P7,P9), (P1,P2,P4,P6,P8), (P2,P3,P5,P8,P9) and (P3,P4,P6,P8,P9) : When, we apply the suggested MCGDM approach to these sub-problems, we will get the ranking of alternatives. By introducing the ranking of alternatives to these smaller problems, we obtain that the similarity measure of patients P1, P7, P8, P9 are high. This result is the same as the non-decomposed problem and reveals a transitive property. The defined MCGDM method is therefore valid with the Criteria 2 and Criteria 3.

**B. COMPARATIVE ANALYSIS 2**

Within this portion, we compare the output of the defined MAGDM method with some of the current methods of intuitionistic fuzzy set, intuitionistic fuzzy TOPSIS method, q-rung orthopair fuzzy set. On the basis of this environment, we apply the current methods, and their results are given as:

**VI. CONCLUSION**

The new generalized version of all extension of fuzzy sets has developed in [57]. This framework also describes the problem by changing the physical sense of reference. Furthermore, it should be noted that the Diophantine space increases as we give space to the reference parameters and therefore the boundary limits have a greater space for both degrees which can convey a broader variety of the fuzzy data. In this research work, we developed new distance measure and similarity measure for linear Diophantine fuzzy sets. It is essential for decision makers to make reliable and reasonable emergency decisions within a short time of a period since inappropriate decisions may result in enormous economic losses and chaotic social order. Accordingly, to ensure that EDM problems can be solved effectively and quickly. This paper proposes a new EDM method based on the novel distance and similarity measures under Linear Diophantine fuzzy (LDF) information. The similarity measure is one of the beneficial tools to determine the degree of similarity between objects. It has many crucial application areas such as decision making, data mining, medical diagnosis, and pattern recognition. In this study, some novel distances and similarity measures of linear Diophantine fuzzy sets are presented. Then, we propose the Jaccard similarity measure, exponential similarity measure, Cosine and Cotangent functions based on similarity.
measures for LDFSs. The newly defined similarity measures are applied to medical diagnosis problem for COVID-19 virus and the results are discussed. A comparative study of new similarity measures was established, and some advantages of the proposed work are discussed.

We will continue the work in the future in expanding and applying the current operators to other contexts such as spherical fuzzy set, LDF Einstein operators, LDF hesitant operators etc. The proposed MADM problem can also be used to other complicated problems such as risk evaluation, risk aversion, emerging technology, project installation etc.

DATA AVAILABILITY
No data were used to support this study.

CONFLICTS OF INTEREST
Authors declare that they have no conflict of interest.

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