I. INTRODUCTION

The inspiral and merger of compact binary systems of black holes are important sources of gravitational waves (GWs) for the proposed space-based GW missions such as the Laser Interferometer Space Antenna (LISA). Eccentricity adds orbital harmonics to the Fourier transform of the GW signal, and relativistic pericenter precession leads to a three-way splitting of each harmonic peak. We study the parameter estimation accuracy for such waveforms with different initial eccentricity using the Fisher matrix method and a Monte Carlo sampling of the initial orbital orientation. The eccentricity improves the parameter estimation by breaking degeneracies between different parameters. In particular, we find that the source localization precision improves significantly for higher-mass binaries due to eccentricity. The typical sky position errors are \( \sim 1 \) deg for a nonspinning, \( 10^7 M_\odot \) equal-mass binary at redshift \( z = 1 \), if the initial eccentricity 1 yr before merger is \( e_0 \sim 0.6 \). Pericenter precession does not affect the source localization accuracy significantly, but it does further improve the mass and eccentricity estimation accuracy systematically by a factor of 3–10 for masses between \( 10^6 M_\odot \) and \( 10^7 M_\odot \) for \( e_0 \sim 0.3 \).

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Parameter estimation for inspiraling eccentric compact binaries including pericenter precession

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Inspiraling supermassive black hole binary systems with high orbital eccentricity are important sources for space-based gravitational wave (GW) observatories like the Laser Interferometer Space Antenna (LISA). Eccentricity adds orbital harmonics to the Fourier transform of the GW signal, and relativistic pericenter precession leads to a three-way splitting of each harmonic peak. We study the parameter estimation accuracy for such waveforms with different initial eccentricity using the Fisher matrix method and a Monte Carlo sampling of the initial orbital orientation. The eccentricity improves the parameter estimation by breaking degeneracies between different parameters. In particular, we find that the source localization precision improves significantly for higher-mass binaries due to eccentricity. The typical sky position errors are \( \sim 1 \) deg for a nonspinning, \( 10^7 M_\odot \) equal-mass binary at redshift \( z = 1 \), if the initial eccentricity 1 yr before merger is \( e_0 \sim 0.6 \). Pericenter precession does not affect the source localization accuracy significantly, but it does further improve the mass and eccentricity estimation accuracy systematically by a factor of 3–10 for masses between \( 10^6 M_\odot \) and \( 10^7 M_\odot \) for \( e_0 \sim 0.3 \).

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The inspiral and merger of compact binary systems of black holes are important sources of gravitational waves (GWs) for the proposed space-based GW missions such as the Laser Interferometer Space Antenna (LISA). Eccentricity adds orbital harmonics to the Fourier transform of the GW signal, and relativistic pericenter precession leads to a three-way splitting of each harmonic peak. We study the parameter estimation accuracy for such waveforms with different initial eccentricity using the Fisher matrix method and a Monte Carlo sampling of the initial orbital orientation. The eccentricity improves the parameter estimation by breaking degeneracies between different parameters. In particular, we find that the source localization precision improves significantly for higher-mass binaries due to eccentricity. The typical sky position errors are \( \sim 1 \) deg for a nonspinning, \( 10^7 M_\odot \) equal-mass binary at redshift \( z = 1 \), if the initial eccentricity 1 yr before merger is \( e_0 \sim 0.6 \). Pericenter precession does not affect the source localization accuracy significantly, but it does further improve the mass and eccentricity estimation accuracy systematically by a factor of 3–10 for masses between \( 10^6 M_\odot \) and \( 10^7 M_\odot \) for \( e_0 \sim 0.3 \).

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that eccentricity can significantly bias the recovered parameters of the source for LISA if circular templates are used even if the eccentricity is as small as $e \sim 10^{-4}$. More recently, Key and Cornish [52] extended that study by using an effective $1.5\text{PN}$ waveform for inspiraling eccentric SMBHs \cite{key-cornish1} taking into account eccentricity and spin effects in the template model. They found that the eccentricity measurement errors are of order $\Delta e \sim 10^{-3}$ for a range of mass ratios and a particular choice of angular parameters.

Since the parameter space is large, 17-dimensional for an eccentric spinning binary, state-of-the-art MCMC calculations are numerically too expensive to explore the full range of source parameters. However, for a large signal-to-noise ratio (SNR), the PDF may be well approximated by an ellipsoid, and the parameter measurement errors can be estimated very efficiently using the Fisher matrix method \cite{yunes-kidder2009}. Using this method, it has been shown that different source inclinations and sky locations lead to a wide range of parameter measurement errors sub-tending many orders of magnitude $\sim 10^4$. In this study, we carry out a Fisher matrix analysis to investigate the possible range of parameter estimation errors for eccentric binaries.

Only a few studies have investigated the LISA parameter estimation accuracy for eccentric inspiraling sources using the Fisher matrix method \cite{yunes-kidder2009}. Barack and Cutler \cite{barack-cutler2007} investigated the LISA errors for highly eccentric stellar mass compact objects inspiraling into a SMBH. They found that the influence of eccentricities on $\Delta M/M \sim 10^{-4}$ (error of the chirp mass), $\Delta e_0 \sim 10^{-4}$ (error of initial eccentricity) and $\Delta \Omega_s \sim 10^{-4}$ (angular resolution error) is not substantial; the error estimates do not differ much from those obtained for circular orbits \cite{yunes-kidder2009}. However, they assumed only an arbitrarily chosen, single set of orientations, which may not be representative of the typical errors. Yunes et al. \cite{yunes-et-al2011} provided ready-to-use analytic expressions for the Fourier waveforms of moderately eccentric sources. They have shown that eccentricity increases the detectable mass range of GW detectors toward higher masses by enhancing the orbital harmonics \cite{yunes-et-al2011}. Yagi and Tanaka \cite{yagi-tanaka2011} investigated the LISA errors for various alternative theories of gravity for spinning, small-eccentricity inspiraling SMBH binaries ($e_0 \sim 0.01$ at 1 yr before merger), using restricted 2 PN waveforms, neglecting higher orbital harmonics and apsidal precession in the waveform. They have found that the eccentricity and the spin-orbit interaction reduce the parameter errors by an order of magnitude for spinning SMBHs in massive graviton theories, but not in Brans-Dicke-type theories.

Neither of the previous systematic Fisher matrix studies of parameter errors included the effects of relativistic pericenter precession for eccentric sources. However, precession effects introduce an additional feature in the waveform, and have the potential to break the degeneracy between parameter errors \cite{yagi-tanaka2011}. In particular, spin-orbit precession has been shown to improve the source localization precision substantially during the last day of the inspiral \cite{yagi-tanaka2011}. Similarly, GR pericenter precession may also be expected to improve the LISA parameter measurement accuracy. In fact, since pericenter precession enters at a lower PN order, this improvement could take place well before the binary reaches merger. Localizing the source before merger could be used to provide triggers for electromagnetic (EM) facilities to search for the EM counterpart \cite{yagi-tanaka2011}. A coincident GW and EM observation of the same source could have far-reaching astrophysical implications \cite{yagi-tanaka2011}.

In the present paper, we carry out a systematic parameter estimation study for inspiraling SMBH binaries, taking into account both orbital eccentricity and the relativistic pericenter precession effect. We account for the evolution of the semimajor axis and eccentricity in our waveforms to leading order due to GW emission \cite{yunes-et-al2011, yagi-tanaka2011, yunes-et-al2011, yagi-tanaka2011}, but we neglect higher-order PN contributions and spin effects. We compute the waveform in the frequency domain using the stationary phase approximation (SPA, see Refs. \cite{yunes-et-al2011, yagi-tanaka2011, yunes-et-al2011, yagi-tanaka2011}) and derive the signal-to-noise ratio (SNR) and the Fisher information matrix using a Fourier-Bessel analysis for the parameter estimation of eccentric sources. To explore the possible range of parameter errors, we generate a Monte Carlo sample of binaries with random orientations and vary the masses and initial eccentricities systematically over a wide range relevant for LISA. We calculate the parameter errors for the standard three-arm LISA/NGO configuration, as well as for a descoped detector configuration, where one of the two independent interferometers is removed.

In Sec. II, we summarize the basic formulas describing eccentric waveforms in the leading quadrupole approximation, using a Fourier-Bessel decomposition. In Sec. III, we derive the frequency domain waveforms and the LISA detector response. After a brief introduction of parameter estimation using the Fisher matrix method in Sec. IV, we present results for specific systems in Sec. V. We summarize our conclusions in Sec VI. Some details of the calculations are described in Appendices A and B.

We use geometrical units $G = c = 1$.

II. TIME-DEPENDENT ECCENTRIC WAVEFORMS

To leading order, the waveform emitted by a binary moving on a Keplerian orbit can be computed by the quadrupole approximation. In this approach the observer (i.e. the interferometric detector) is assumed to be far from the source and higher-order contributions; e.g., the effects of the spins and higher multipole moments are neglected, but the orbit is corrected for the effect of pericenter precession. For such precessing Keplerian orbits, the eccentric waveforms are given in Ref. \cite{yunes-et-al2011}. We have rewritten the leading-order quadrupole tensor and transformed to the transverse-traceless gauge, which gives

$$c = 1.$$
Here \( \phi \) is the true anomaly, which describes the azimuthal angle from the pericenter along the orbit as shown in Fig. 1. The value \( \gamma \) is the azimuthal angle of the pericenter relative to the coordinate system \( x \) axis in the orbital plane, \( \epsilon \) is the orbital eccentricity, \( a \) is the semimajor axis, \( D_L \) is the luminosity distance, \( \Theta \) is the inclination (the angle between the orbital plane and the line of sight to the observer), and \( m = m_1 + m_2 \), \( \mu = m_1 m_2 / m \) are the total and reduced masses (Fig. 1). Using the well-known Fourier-Bessel decomposition, the polarization states can be expressed as a sum of harmonics of the orbital frequency:

\[
\tilde{h}_x(t) = -h \cos \Theta \sum_n \left[ B_{n}^+ \sin \Phi_{n+}^t + B_{n}^- \sin \Phi_{n-}^t \right],
\]

\[
\tilde{h}_+(t) = -\frac{h}{2} \sum_n \left[ \sin^2 \Theta A_n \cos \Phi_n^t + (1 + \cos^2 \Theta) \left( B_{n}^+ \cos \Phi_{n+}^t - B_{n}^- \cos \Phi_{n-}^t \right) \right].
\]

Here \( h = 4 \mu m(a D_L)^{-1} \) is the amplitude, and \( B_{n}^+ = (S_n \pm C_n) / 2 \) and \( A_n \) are linear combinations of the Bessel functions of the first kind \( [J_n(ne)] \) and their derivatives,

\[
S_n = -\frac{2}{e} \left( 1 - e^2 \right)^{1/2} n^{-1} J_n'(ne) + \frac{2}{e} \left( 1 - e^2 \right)^{3/2} n J_n(ne),
\]

\[
C_n = -\frac{2}{e^2} J_n(ne) + \frac{2}{e} \left( 1 - e^2 \right) J_n'(ne),
\]

\[
A_n = J_n(ne),
\]

where \( \gamma \) is the pericenter precession. The phase functions in Eqs. (3-4) are

\[
\Phi_n^t = nl,
\]

\[
\Phi_{n+}^t = nl \pm 2\gamma,
\]

where \( l \) is the mean anomaly which is defined by the Kepler equation

\[
l = \xi - e \sin \xi = 2\pi \nu (t - t_0).
\]

In the Kepler equation \( \xi \) is the eccentric anomaly, \( \nu = T^{-1} \) is the Keplerian orbital frequency (here \( T = 2\pi m^{-1/2}a^{3/2} \) is the Newtonian radial orbital period), and \( t_0 \) is the time of pericenter passage (Hereafter, we set \( t_0 = 0 \)). Equations (6,7) show that the phase splits into a triplet due to the pericenter position \( \gamma \). If the pericenter precesses, a triplet of frequencies appears in Fourier space for each harmonic [61,62]. Note that Eq. (8) is approximately valid during an orbit as long as \( \nu / c \ll 1 \) and \( \nu \) is constant, but this equation requires modifications on large time scales where the binary inspirals (see Eqs. [12,13] below), or at small separations, where the 1 PN treatment breaks down.

Pericenter precession leads to a time-dependent angle of pericenter, which may be written as \( \gamma(t) = \gamma_0 + \gamma(t) \) where \( \gamma_0 \) is the initial angle of the pericenter (Fig. 1). Henceforth, we adopt pericenter precession from the classical relativistic motion and assume the adiabatic evolution of the orbital parameters. These effects are averaged over one radial oscillation period, i.e. \( \langle \dot{\gamma} \rangle = \Delta \gamma / T \), where \( \Delta \gamma = 6\pi m(a(1 - e^2))^{-1} \) is the angle of precession for an eccentric orbit governed by the geodesic equation of the Schwarzschild geometry (see e.g. Ref. [64]). In the following we shall drop \( \langle \cdot \rangle \) for the average quantities, so we write

\[
\dot{\gamma} = \frac{3m^{3/2}}{a^{5/2} (1 - e^2)} = \frac{3m^{2/3} (2\pi \nu)^{5/3}}{(1 - e^2)}. \tag{9}
\]

The 2.5 PN leading-order adiabatic evolution of the orbital parameters due to gravitational radiation averaged over one radial period are [21]

\[
\dot{\nu} = \frac{48\mathcal{M}^{5/3}(2\pi \nu)^{11/3}}{5\pi (1 - e^2)^{7/2}} \left( 1 + \frac{73}{24}e^2 + \frac{37}{96}e^4 \right), \tag{10}
\]

\[
\dot{e} = \frac{304\mathcal{M}^{5/3}(2\pi \nu)^{8/3}}{15(1 - e^2)^{3/2}} \left( 1 + \frac{121}{304}e^2 \right), \tag{11}
\]

where \( \mathcal{M} = \mu^{3/5}m^{2/5} \) is the chirp mass. (We used Kepler’s third law, i.e. \( \nu = (2\pi)^{-1} m^{1/2}a^{-3/2} \).)

For an inspiraling system, the phase functions are \( \Phi_n^t = 2\pi n \int_{t_0}^{t} \nu(t') dt' \) and \( \Phi_{n+}^t = \Phi_n \pm 2\gamma_0 \pm 2 \int_{t_0}^{t} \dot{\gamma}(t') dt' \). Equations (6, 7), are generalized as (here the \( t \) index is suppressed in \( \Phi_n^t, \Phi_{n+}^t \))

\[
\Phi_n = 2\pi n \int_{-\infty}^{\nu(t)} \frac{\nu}{\nu} d\nu, \tag{12}
\]

\[
\Phi_{n+} = \Phi_n \pm 2\gamma_0 \pm 2 \int_{-\infty}^{\nu(t)} \dot{\gamma} d\nu. \tag{13}
\]
FIG. 1. The geometry of an eccentric orbit. The coordinate system \((x, y, z)\) is defined by the initial orbit, where the \(x\) axis points in the direction of the pericenter and the \(z\) axis is parallel to the orbital angular momentum vector. In the reduced Kepler problem the body with mass \(m = m_1 + m_2\); the separation vector is \(r = a_0(1 - e^2)/(1 + e \cos \phi)\), where \(e\) is the orbital eccentricity; \(a_0 = m^{3/2}(2\pi)^{2/3}\) (here \(\nu_0\) is the orbital frequency) is the semimajor axis; \(\phi\) is the true anomaly (the angle between the pericenter and the \(z\) axis), \(\gamma\) is the eccentric anomaly \([\tan \gamma/2 = \sqrt{(1 - e^2)/(1 + e^2)} \tan \phi/2]\). The Kepler equation determines the evolution of the time parameter: \(\xi = e_0 \sin \xi = 2\pi\nu_0(t - t_0)\), where \(\xi\) is the eccentric anomaly \([\tan \xi/2 = \sqrt{(1 - e^0)/(1 + e^0)} \tan \phi/2]\). The adiabatic evolution of the eccentric orbit is driven by the pericenter precession \((1\,PN\) effect\) and the inspiral \((2.5\,PN\) effect\) of the compact binary due to gravitational radiation.

where \(\Phi_{n \pm}\) are phase functions which arise due to pericenter precession. Note that here one must incorporate the evolution in the eccentricity by solving Eqs. \([10,11]\), i.e. \(\dot{\nu} = \dot{\nu}(\nu) = \dot{\nu}[\nu, e(\nu)]\), and similarly for \(\dot{\gamma}\) [see Eq. \([36]\) below].

III. FOURIER TRANSFORMATION OF THE ECCENTRIC INSPIRAL WAVEFORM

The sensitivity of a GW detector is usually given in Fourier space. Thus, to estimate the detection signal-to-noise ratio and measurement accuracy, we construct the Fourier transform of the waveform as

\[
h(f) = \int_{-\infty}^{\infty} \tilde{h}(t) e^{2\pi i tf} dt ,
\]

where \(f\) is the Fourier frequency. These integrals cannot be evaluated analytically without further assumptions. However, since the orbital parameters \((a, e)\) evolve very slowly relative to the GW phase, the stationary phase approximation (SPA) can be utilized \([62]\) (Appendix B). We account for the adiabatic time evolution during the inspiral in the Fourier-transformed waveform \(h(f)\) using Eqs. \([12,13]\) and the SPA. In this approximation the Fourier transformation of the waveform becomes a discrete sum over the harmonics of orbital frequency, \(f_n = n\nu\). When the pericenter precession is taken into account, each harmonic \(f_n\) is split into a triplet \(f \equiv (f_n, f_{n\pm})\) and therefore the waveform consists of the sum over these triplets of Fourier frequencies:

\[
h_{\times}(f) = -\frac{h_0}{2} \sum_n \cos \Theta \left[ B_n^+ \Lambda^+ e^{i(\Psi_n^+ + \pi/4)} + B_n^- \Lambda^- e^{i(\Psi_n^- + \pi/4)} \right], \tag{15}
\]

\[
h_{+}(f) = -\frac{h_0}{4} \sum_n \left[ \sin^2 \Theta A_n \Lambda e^{i(\Psi_n - \pi/4)} + (1 + \cos^2 \Theta) \left( B_n^+ \Lambda^+ e^{i(\Psi_n^- - \pi/4)} - B_n^- \Lambda^- e^{i(\Psi_n^+ - \pi/4)} \right) \right], \tag{16}
\]

where \(f_n = n\nu, f_{n\pm} = n\nu \pm \frac{\pi}{\gamma}\), \(h_0 = 4\mathcal{M}^{5/3} (2\pi\nu)^{2/3}/D_L\) is the amplitude corresponding to the orbital frequency; and \(\Psi_n = 2\pi ft_n - \Phi_n, \Psi_{n\pm} = 2\pi ft_{n\pm} - \Phi_{n\pm}\) are phase functions (where \(t_n, t_{n\pm}\) are the time parameters of the SPA; see Appendix B). We have introduced the notation \(\Lambda = (n\nu \pm \dot{\gamma}/\pi)^{-1/2}\) and \(\Lambda = (n\nu)^{-1/2}\). The phases \(\Psi_n\) and \(\Psi_{n\pm}\) depend on the corresponding Fourier frequencies \(f_n, f_{n\pm}\), respectively.

We recall that for circular orbits (i.e. \(e \to 0\)) the waveforms in Eqs. \([15,16]\) simplify as

\[
h_{\times}(f) = -2\sqrt{\frac{5}{96}} \frac{\mathcal{M}^{5/6} f^{-7/6}}{\pi^{5/3} D_L} \cos \Theta \dot{\epsilon} e^{i\Psi^+_n}, \tag{17}
\]

\[
h_{+}(f) = -\sqrt{\frac{5}{96}} \frac{\mathcal{M}^{5/6} f^{-7/6}}{\pi^{5/3} D_L} \left( 1 + \cos^2 \Theta \right) e^{i\Psi^-_n}, \tag{18}
\]

where \(f = 2\nu\) is the (circular) Fourier frequency, and \(\Psi^n = 2\pi ft_c - \Phi_c \pm \pi/4 + (3/4)(8\pi\mathcal{M} f)^{-5/3}\) is the well-known phase function.

A. LISA detector response

With its three arms, LISA represents a pair of two orthogonal arm detectors, \(I\) and \(II\), producing two linearly independent signals. The frequency domain waveforms are

\[
h^{I,II}(f) = \frac{\sqrt{2}}{2} \left[ F_{\times}^{I,II} h_{\times}(f) + F_{+}^{I,II} h_{+}(f) \right], \tag{19}
\]

with the antenna beam pattern functions

\[
F_{\times}^I = \frac{1 + \mu^2}{2} \cos 2\phi_S \sin 2\psi_S + \mu \sin 2\phi_S \cos 2\psi_S, \tag{20}
\]

\[
F_{+}^I = \frac{1 + \mu^2}{2} \cos 2\phi_S \cos 2\psi_S - \mu \sin 2\phi_S \sin 2\psi_S, \tag{21}
\]
where \( \mu_{S,L} = \cos \theta_{S,L} \) with \((\theta_S, \phi_S)\) being spherical angles of the source in the detector-based coordinate system. The angle \( \psi_S \) is the polarization angle that can be expressed by the position of the detector and the orbital plane. The other antenna beam pattern functions are \( F^I_{\pm, \times} = F^I_{\pm, \times}(\phi_S - \pi/4) \). The quantities \( \theta_S, \phi_S, \) and \( \psi_S \) are time dependent, because the LISA constellation moves around the Sun, and these explicit time evolutions are \[ \phi_S = \alpha_1(t) + \frac{\pi}{12} + \arctan \frac{\sqrt{3} \bar{s}_L + \bar{\lambda}_S \cos \bar{\phi}_L}{2 \bar{s}_S \sin \bar{\phi}_S}, \]

\[ \psi_S = \arctan \frac{\bar{\mu}_L - \sqrt{3} \bar{\lambda}_L \cos \bar{\phi}_L - \cos \Theta (\bar{\mu}_S - \sqrt{3} \bar{\lambda}_S \cos \bar{\phi}_S)}{2 \bar{\kappa}} \]

where \( \bar{\lambda}_{S,L} = \sin \bar{\theta}_{S,L}, \bar{\mu}_{S,L} = \cos \theta_{S,L} \) and \( \bar{\phi}_{S,L} = \phi(t) - \bar{\phi}_{S,L} \), with \( \bar{\theta}, \bar{\phi}, \bar{\psi} \) being the spherical angles of the source’s position. The angles \( \bar{\theta}_L, \bar{\phi}_L \) correspond to the direction of orbital angular momentum in the barycenter frame. In Eqs. (22-24), \( \Theta = \arccos \left( \frac{\bar{\mu}_L \bar{\mu}_S + \sqrt{3} \bar{\lambda}_S \cos \bar{\phi}_L - \bar{\lambda}_L \cos \bar{\phi}_S}{\bar{s}_S \bar{s}_L} \right) \) is the inclination \([8]\) and the explicit time dependences are \( \alpha_1(t) = 2\pi t/T - \pi/12 + \alpha_0, \phi(t) = \bar{\phi}_0 + 2\pi t/T, \) and

\[ K = \frac{\lambda}{2} \left[ \left( \sin(\bar{\phi}_L - \bar{\phi}_S) \right) + \frac{\sqrt{3}}{2} \cos \bar{\phi}(t) (\bar{\mu}_L \bar{\lambda}_S \sin - \bar{\mu}_S \bar{\lambda}_L \sin \bar{\phi}_L) \right] \]

\[ - \frac{\sqrt{3}}{2} \left[ \sin \bar{\phi}(t) (\bar{\mu}_S \bar{\lambda}_L \cos \bar{\phi}_L - \bar{\mu}_L \bar{\lambda}_S \cos \bar{\phi}_S) \right]. \]

We note that \( \bar{\theta}_L, \bar{\phi}_L \) are generally not constants for spinning binaries due to spin-orbit effects, but we neglect these effects here.

We carry out the analysis for the single-detector case (I only) and the full two-detector configuration \((I + II)\).

In practice, the measured signal in Eq. \([19]\) is truncated at minimum and maximum frequencies corresponding to the start of the observation and the last stable orbit for each harmonic, respectively \(\text{see Sec. V below).}\)

### IV. PARAMETER ESTIMATION

In this section we review the basics of Bayesian parameter estimation. The measured signal \( \bar{s}(t) \) is made up of the GW \( \bar{h}(t) \) and the noise \( \bar{n}(t) \)

\[ \bar{s}(t) = \bar{h}(t) + \bar{n}(t). \]

We assume that the noise is stationary, Gaussian, and statistically independent at different frequencies. Then each Fourier component has a Gaussian probability distribution and the different Fourier components of the noise are "uncorrelated," i.e.

\[ p(n = n_0) \propto e^{-(n_0 - n_0)^2}, \]

\[ \langle n(f) n^*(f') \rangle = \frac{1}{2} \delta(f - f') S(f). \]

In Eqs. \([27, 28]\) \( p(n) \) is the probability for the noise, the inner product is defined by

\[ (g | k) = 4R \int_{0}^{\infty} \frac{g(f) k^*(f)}{S(f)} \, df, \]

\( k^* \) is complex conjugation and \( S(f) \) is the one-sided spectral noise density. The definition of the signal-to-noise ratio (SNR) of \( h \) is

\[ \rho^2 = (h | h) = 4R \int_{0}^{\infty} \frac{h(f) h^*(f)}{S(f)} \, df. \]

In the circular case \( h(f) \) depends on the parameters \( \lambda^a \) which characterize the source. For a large SNR, the errors \( \Delta \lambda^a \) have the Gaussian probability distribution

\[ p(\Delta \lambda^a) = p_0 e^{-\Gamma_{ab} \Delta \lambda^a \Delta \lambda^b / 2}, \]

where \( p_0 \) is the normalization factor and \( \Gamma_{ab} \) is the Fisher information matrix defined by

\[ \Gamma_{ab} = \langle \partial_a h | \partial_b h \rangle = 4R \int_{0}^{\infty} \frac{\partial_a h(f) \partial_b h^*(f)}{S(f)} \, df, \]

with \( \partial_a = \partial / \partial \lambda^a \). The inverse of the Fisher matrix is approximately the \( \Sigma_{ab} \) variance-covariance matrix for \( \rho \gg 1 \), which gives the accuracy of each parameter and is defined by \( \Sigma_{ab} = (\Gamma_{ab})^{-1} = \langle \Delta \lambda^a \Delta \lambda^b \rangle \).

The root-mean-square errors of the parameters \( \lambda^a \) are \( \Delta \lambda^a = \sqrt{\Sigma_{aa}} \).

For example, the error of the sky position solid angle is

\[ \Delta \Omega_S = 2\pi \sqrt{\langle \Delta \phi S \Delta \phi S \rangle - \langle \Delta \theta S \Delta \theta S \rangle - \langle \Delta \phi S \Delta \theta S \rangle. \]

The source localization sky area is an ellipse with semiminor and major axes \((a_S, b_S)\) given by Eq. \(4.12\) in Ref. \([11]\). The SNR and Fisher matrix for the LISA configuration are

\[ \rho^2 = \rho_0^2 + \rho_1^2, \]

\[ \Gamma_{ab} = \Gamma_{ab}^I + \Gamma_{ab}^{II}, \]

where the \( I, II \) subscripts distinguish the \( h^I, h^{II} \) waveforms in Eq. \([19]\).

### V. MEASURING ECCENTRIC INSPIRALING SMBH BINARIES

We focus on comparable-mass SMBH binaries in the range \((10^4 - 10^7) M_{\odot}\), which corresponds to the measured frequency range \(10^{-4} \text{ to } 10^{-1} \text{Hz}\). For initial configurations \(1 \text{ yr} \) before merger, we assume that the binary has orbital eccentricity \( e_0 \) and pericenter position \( \gamma_0 \). The ten-dimensional parameter space is

\[ \lambda^a = \{ \ln D_L, \ln M, t_c, \Phi_c, \bar{\phi}_S, \bar{\mu}_S, \bar{\phi}_L, \bar{\mu}_L, e_0, \gamma_0 \} \]

In the circular case \( e_0 \) and \( \gamma_0 \) do not appear. Note that only one mass parameter, the chirp mass \( M \), enters the leading-order waveform. Our assumptions are as follows:
To examine the effects of eccentricity and pericenter precession, we neglect higher-order post-Newtonian (beyond 1 PN) orders and spins; we use the heuristic pericenter precession in phase described above.

In all cases, we take \( t_c = \Phi_c = \gamma_0 = 0 \). (We use the \( \alpha_0, \phi_0 = 0 \) choice, as in Ref. [8].)

We assume that the observation time is 1 yr before the merger—more precisely, before the Newtonian last stable orbit (LSO), which is defined by [54]

\[
\nu_{LSO}^N = \frac{1}{2\pi m} \left( \frac{1 - c_{LSO}^2}{6 + 2c_{LSO}^2} \right)^{3/2},
\]

where \( c_{LSO} \) is the final eccentricity at the last stable orbit (\( \nu_{LSO}^N = \nu_{LSO} \)).

For the \( n \)th orbital harmonic, the limits of integration are taken to be \( \nu_{\text{max}} = \nu_{LSO} \) and \( \nu_{\text{min}} = \max\{\nu_0, f_c/n\} \), where \( \nu_0 \) is the frequency 1 yr before the LSO and \( f_c = 0.03 \text{ mHz} \) is the cutoff frequency of the LISA detector.

We assume that the luminosity distance to the source is \( D_L = 6.4 \text{ Gpc} \), corresponding to a cosmological redshift \( z = 1 \), and we use the comoving masses as free parameters, \( m_1^2 = (1 + z)m_1 \). [4] We do not take into account the Doppler phase due to the varying light travel during the LISA constellation’s orbit around the Sun.

We parametrize the evolution of the orbital frequency with the instantaneous eccentricity following Ref. [38] (Appendix A):

\[
\nu(e) = \nu_0 \frac{\sigma(e)}{\sigma(e_0)},
\]

where \( \nu_0 \) and \( e_0 \) are the initial orbital frequency and eccentricity, and \( \sigma(e) \) follows from Ref. [21].

We truncate the harmonics at \( n_{\text{max}} \), where 99% of the signal power corresponds to [38]

\[
n_{\text{max}} = \left[ \frac{5(1 + e_0)^{1/2}}{(1 - e_0)^{3/2}} \right].
\]

where the bracket \( [\ ] \) denotes the floor function (integer part of nonnegative argument). Here \( n_{\text{max}} = \{9, 24\} \) for \( e_0 = \{0.3, 0.6\} \), respectively.

We analyze \( 10^4 \) SMBH binaries where the angular variables were chosen randomly, i.e. for \( \phi_S, \phi_L \) in the range \( (0, 2\pi) \) and for \( \theta_S, \theta_L \) in the range \( (-\pi/2, \pi/2) \).

The computation of SNR and the Fisher matrix with the above general definition [Eq. 14] is numerically expensive for a large set of binaries. We resort to the SPA waveform. The SNR and the Fisher information matrix consist of three terms for each orbital harmonic which correspond to \( (f_n, f_{n\pm}) \), respectively:

\[
\hat{\rho}_n = \sum_n \left( \hat{\rho}_n^2 + \hat{\rho}_{n+}^2 + \hat{\rho}_{n-}^2 \right)
\]

\[
\hat{\Gamma}_{ab} = \sum_n \left( \hat{\Gamma}_{ab}^n + \hat{\Gamma}_{ab}^{n+} + \hat{\Gamma}_{ab}^{n-} \right)
\]

where we have introduced the notations \( \hat{\rho}_{n,n+,n-}^2 = (h_{n,n+,n-} - h_{n,n+,n-})^2 \), \( \hat{\Gamma}_{ab}^{n,n+,n-} = (\partial_a h_{n,n+,n-} - \partial_b h_{n,n+,n-}) \), and \( h_{n,n+,n-} = h(f_{n,n+,n-}) \). Here we neglect the cross terms between different harmonics \( n, n+, \) and \( n- \), in \( \hat{\rho} \) and \( \hat{\Gamma}_{ab} \). We use the LISA sensitivity curve generator [65]. In the SPA, we can change the integration variables from \( f_n, f_{n\pm} \) to \( e \):

\[
\hat{\rho}_n^2 = 4R \int_{\nu_{\text{min}}}^{\nu_{\text{max}}} \frac{h_n(e)h_n^*(e) \, nd\nu}{S[\nu(e)]} \, de \, de,
\]

\[
\hat{\Gamma}_{ab} = 4R \int_{\nu_{\text{min}}}^{\nu_{\text{max}}} \frac{\partial_a h_n(e)\partial_b h_n^*(e) \, nd\nu}{S[\nu(e)]} \, de \, de,
\]

where \( \nu(e) \) and \( \nu(d\nu) \) are given by Eqs. [11, 3] and \( \nu_{\text{max}} = \nu_{LSO} \), \( \nu_{\text{min}} = \min\{\nu_c(n), e_0\} \). (Here \( \nu_c(n) \) corresponds to \( f_c/n \), where \( f_c = 0.03 \text{ mHz} \) is the cutoff frequency for the LISA detector.)

![SNR vs E eccentricity](image)

**FIG. 2.** (color online) Smooth probability density function of SNR for various initial eccentricities \( e_0 = 0.15, 0.3, 0.45, 0.6 \) and masses \( \{10^6 - 10^9\} M_\odot \). The eccentricity dependence of SNR is almost negligible.

VI. RESULTS AND DISCUSSION

We have found that the LISA parameter estimation accuracy depends sensitively on the initial eccentricity and
TABLE I. The initial and final frequencies ($\nu_0$ and $\nu_1 = \nu_{LSO}$) for various initial eccentricities ($e_0$) and comoving masses ($m_1$, $m_2$ with redshift $z = 1$) for a 1 yr inspiral before LSO. We use the shorthand notation $e_1 = e_{LSO}$ for the final eccentricity. We have completed with a dimensionless semimajor axis $\bar{r} = a/m$ at the initial ($\bar{r}_0$) and final points ($\bar{r}_1$).

| SMBH [$M_\odot$] | $e_0 = 0$ | $e_0 = 0.3$ | $e_0 = 0.6$ |
|------------------|-----------|-------------|-------------|
| $10^7 - 10^7$ | $\nu_0 = 3.47 \mu$Hz, $\bar{r}_0 = 37.84$ | $\nu_0 = 3.05 \mu$Hz, $\bar{r}_0 = 41.21$ | $\nu_0 = 1.92 \mu$Hz, $\bar{r}_0 = 56.17$ |
|                 | $\nu_1 = 54.96 \mu$Hz, $\bar{r}_1 = 6.00$ | $\nu_1 = 54.47 \mu$Hz, $\bar{r}_1 = 6.04$ | $\nu_1 = 53.78 \mu$Hz, $\bar{r}_1 = 6.09$ |
| $10^6 - 10^6$ | $\nu_0 = 14.64 \mu$Hz, $\bar{r}_0 = 67.23$ | $\nu_0 = 12.88 \mu$Hz, $\bar{r}_0 = 73.28$ | $\nu_0 = 8.09 \mu$Hz, $\bar{r}_0 = 99.87$ |
|                 | $\nu_1 = 549.59 \mu$Hz, $\bar{r}_1 = 6.00$ | $\nu_1 = 547.75 \mu$Hz, $\bar{r}_1 = 6.01$ | $\nu_1 = 554.22 \mu$Hz, $\bar{r}_1 = 6.03$ |
| $10^5 - 10^5$ | $\nu_0 = 61.73 \mu$Hz, $\bar{r}_0 = 119.64$ | $\nu_0 = 54.31 \mu$Hz, $\bar{r}_0 = 130.30$ | $\nu_0 = 34.13 \mu$Hz, $\bar{r}_0 = 177.59$ |
|                 | $\nu_1 = 5495.90 \mu$Hz, $\bar{r}_1 = 6.00$ | $\nu_1 = 5488.93 \mu$Hz, $\bar{r}_1 = 6.01$ | $\nu_1 = 5479.18 \mu$Hz, $\bar{r}_1 = 6.01$ |
| $10^4 - 10^4$ | $\nu_0 = 260.30 \mu$Hz, $\bar{r}_0 = 212.75$ | $\nu_0 = 229.02 \mu$Hz, $\bar{r}_0 = 231.72$ | $\nu_0 = 143.94 \mu$Hz, $\bar{r}_0 = 315.80$ |
|                 | $\nu_1 = 54959 \mu$Hz, $\bar{r}_1 = 6.00$ | $\nu_1 = 54934 \mu$Hz, $\bar{r}_1 = 6.00$ | $\nu_1 = 54896 \mu$Hz, $\bar{r}_1 = 6.01$ |

FIG. 3. (color online). Smooth probability density function of SNR for various equal-mass binaries (for initial eccentricity $e_0 = 0.3$). The SNR is $\mathcal{O}(10^2)$ for low-mass binaries ($10^4 - 10^5$) $M_\odot$. In the other cases, the SNR is $\mathcal{O}(10^3)$.

pericenter precession, and we have also examined the distribution of parameter errors for a wide range of initial binary parameters and masses. The four angular parameters ($\phi_S$, $\mu_S$, $\phi_L$, $\mu_L$) are chosen randomly in a Monte Carlo sampling, and the cosmological redshift and luminosity distance are fixed at $z = 1$ and $D_L = 6.4$ Gpc. Figures [1][10] show the histograms of the expected measurement errors of the binary parameters for the chirp mass $\Delta M/M$, initial eccentricity $\Delta e_0$, and angular resolution $\Delta \Omega_S$ for equal-mass binaries with $10^6 M_\odot$ or $10^7 M_\odot$ each. Our parametrization of the orbit is singular at $e_0 = 0$. To get around this, we use $e_0 = 10^{-6}$ for circular orbits. We have presented three representative cases for the initial eccentricity: a nearly circular orbit with $e_0 = 10^{-6}$ (see Table [1] and Fig. [3]), and orbits with medium $e_0 = 0.3$ and high $e_0 = 0.6$ eccentricities. Our computations correspond to a 1 yr inspiral before LSO. The initial and final orbital frequencies ($\nu_0$ and $\nu_{LSO}$) vary for the three kinds of initial eccentricities and different equal-mass SMBH binaries as shown in Table [1]. If the initial eccentricity $e_0$ increases, the initial frequency $\nu_0$ decreases 1 yr before LSO, while the final frequency $\nu_{LSO}$ does not change significantly, due to the fact that $e_{LSO}$ is close to zero.

Representative values are shown in Table [1] for equal-mass SMBHs for a fixed set of angular configurations ($\phi_S = 4.642$, $\mu_S = -0.3185$, $\phi_L = 4.724$ and $\mu_L = -0.3455$). The table shows that accounting for the eccentricity in the waveform improves some of the parameter errors such as the errors of the angular resolution $\Delta \Omega_S$, the initial eccentricity $\Delta e_0$, and the chirp mass $\Delta M/M$ for higher-mass SMBH binaries ($10^6 - 10^5$) $M_\odot$. For lower masses, i.e. ($10^4 - 10^5$) $M_\odot$, the eccentricity and precession have no essential effects on parameter estimation. For masses of $10^5 M_\odot$, high eccentricity has no significant effect on the parameters $\Delta M/M$ or $\Delta \Omega_S$. However, the initial eccentricity errors ($\Delta e_0$) are improved for smaller masses typically by factors of 3–10 and they are greatly improved for larger initial eccentricities by orders of magnitude. Similarly, the source localization angular resolution $\Delta \Omega_S$ decreases with increasing eccentricity and mass. However, pericenter precession does improve the parameter errors for higher-mass SMBHs. It can be seen that the eccentricity, compared to the circular orbit case, does improve the error of luminosity distance $\Delta D_L/D_L$, but there is no essential change between the high and medium eccentricities with the inclusion of pericenter precession. The error of $t_c$ is not affected by the eccentricity or by pericenter precession. It is interesting to note that there are degeneracies ($\Delta t_c, \Delta \gamma_0 > 1$) for errors of $\Phi_c$ and $\gamma_0$ in the nearly circular case, which
can be explained by the fact that our parametrization of the orbit is singular at \( e_0 = 0 \). For eccentric orbits (medium and high initial eccentricities) this degeneracy disappears (the errors of \( t_c, \Phi_c \) and \( \gamma_0 \) are not presented in Table 1).

Figures 2 and 3 show the distribution of the SNR for different binary orientations, for various eccentricities and masses. The SNR is similar for equal-mass binaries with \( 10^5 M_\odot \leq M \leq 10^7 M_\odot \), but significantly smaller for a SMBH of \( 10^4 M_\odot \) or less. Remarkably, the SNR does not change significantly with the initial eccentricity, which is consistent with previous studies for small eccentricities [32]. This shows that the systematic improvement of the parameter estimation accuracy for eccentric sources is due to the breaking of correlations between different parameter errors instead of an overall change in the SNR.

Figure 4 shows the distribution of the major and/or minor axes of the sky position error ellipse for the nearly circular, medium and high initial eccentricity orbits. The shape of the error ellipse is important in coordinating GW observations with telescopes [11, 19]. It can be seen that the error of the major and/or minor axes is improved for highly eccentric binaries.

FIG. 4. (color online). Distribution of the major (top) and minor (bottom) axes (\( a_s, b_s \)) of the sky position error ellipse (\( \Delta \theta_s = \pi a_s b_s \)) for various eccentric binaries with equal mass. (Here the pericenter precession is neglected.) The two panels correspond to 1 yr observation of \( (10^7 - 10^9) M_\odot \) black hole binaries at \( z = 1 \) (\( D_L = 6.4 \) Gpc) with LISA (2 detector). The angular resolution is improved for high-mass binaries.

FIG. 5. (color online). Estimated distribution of the chirp mass errors in the precessing and nonprecessing cases for the total (I + II, top) and single (I, bottom) detectors. The results are shown for medium (\( e_0 = 0.3 \)) and high (\( e_0 = 0.6 \)) initial eccentricities and higher-mass SMBH binaries \( (10^6 - 10^8) M_\odot \). For precessing sources the \( e_0 = 0.6 \) case is omitted in both figures due to the high degree of overlap with the \( e_0 = 0.3 \) case.

Figures 4 and 5 show that the chirp mass errors are greatly improved for a larger initial eccentricity for \( 10^6 M_\odot \) and \( 10^7 M_\odot \) equal-mass SMBH binaries (see also Ref. [13]). Furthermore, the chirp mass measurement errors are improved by an additional factor of 2–5 due to pericenter precession for relatively massive \( 10^7 M_\odot \) bi-
naries, but not for $10^6 M_\odot$ binaries. The typical chirp mass error is about $10^{-5}$ for $10^7 M_\odot$ and $10^{-4}$ for $10^6 M_\odot$ binaries.

Figures 6 and 7 show that the initial eccentricity errors are also improved for a high eccentricity, as the initial eccentricity parameter can be measured with high accuracy; $\Delta e_0$ is about $10^{-5}$ to $10^{-4}$ for $10^7 M_\odot$ binaries and about $10^{-4}$ to $10^{-3}$ for $10^6 M_\odot$ binaries. Pericenter precession improves the eccentricity errors by a factor of 10 for $10^7 M_\odot$ and by a factor of 2–3 for $10^6 M_\odot$.

Figures 8 and 9 show that the typical source sky localization accuracy for equal-mass binaries at $z = 1$ ranges between $10^{-4}$ and $10^{-3}$ steradians. Consistent with previous studies $[12, 13]$, we find that the errors improve for higher initial eccentricities ($e_0 = 0.6$), compared to the cases of moderate to small initial eccentricities ($e_0 = 0.3$) for equal-mass $10^7 M_\odot$ binaries. The error $\Delta \Omega_S$ in the total two-detector case is about 1 order of magnitude better than for a single detector $[8]$. For high initial eccentricities, the angular resolution of the total detector case is improved more compared to the single detector case for $10^7 M_\odot$ binaries (see Fig. 10). In contrast to the chirp mass and the eccentricity errors, the angular localization capabilities are not improved for eccentric equal-mass $10^6 M_\odot$ binaries but they are improved for $10^7 M_\odot$ binaries. Figures 9 and 10 clearly show that pericenter precession does not affect the sky position error for either mass choice.

A possible explanation for the qualitatively different improvement of the sky position and mass-eccentricity errors is that the sky position is a slow parameter, as opposed to fast parameters like the chirp mass and eccentricity $[12]$. The slow parameters are determined by the slow orbital modulation of the signal by the detector’s motion around the Sun, while the fast parameters also depend on the orbital phase. The correlations between the
slow parameters become large during the last week before merger when the signal-to-noise ratio increases, which prohibits the rapid improvement of the slow parameters’ marginalized errors. Pericenter precession does not vary the binary inclination and cannot effectively break the correlation between slow parameters. However, pericenter precession splits the GW frequency into a triplet for each harmonic which can break degeneracies for the fast parameters and efficiently improve their measurement errors.

VII. CONCLUSIONS

We carried out an extensive study of parameter estimation for eccentric binaries with arbitrary orbital eccentricity. We computed the waveforms in the frequency domain by a new method optimized for taking into account eccentricity, by changing the integration variable for the waveforms from the orbital frequency $\nu(e)$ to the eccentricity variable $e$ [35]. This results in an improvement of numerical precision as compared to standard approaches in the frequency domain, where a Taylor series expansion of the orbital frequency $\nu(e)$ (among others) in the eccentricity $e$ is needed [53]. Our method is well suited for computing the Fisher matrix and the signal-to-noise ratio. Our parameter space is ten dimensional, consisting of four angles, the chirp mass, the luminosity distance, coalescence time and phase, initial eccentricity and pericenter position (compare Fig. 1). The first eight parameters are standard for circular orbits too.

We have examined the LISA parameter estimation errors for GWs emitted by eccentric inspiraling SMBH binaries including the effects of pericenter precession.
It is important to note that the angular resolution is significantly affected by the number of detectors (see Figs. 9 and 10). However, nearly the same parameter estimation accuracy can be obtained for the single and total detector configurations for $(10^6 - 10^7) M_\odot$ binaries for fast parameters like the chirp mass and eccentricity (Figs. 5 and 7). The second detector systematically reduces the errors of these parameters for higher masses $(10^7 - 10^7) M_\odot$.

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Appendix A: Orbital evolution and waveform

According to Eqs. (10,11), the equation

$$\frac{d\nu}{de} = \frac{18\nu}{19} \left(1 + \frac{\dot{e}^2}{\dot{e}^2} \right) \left(1 + \frac{121 e^4}{304} \right)^{-1305/2299},$$

(A1)

can be solved as

$$\nu(e) = C_0 e^{-18/19} \left(1 - e^2\right)^{3/2} \left(1 + \frac{121}{304} e^2 \right)^{-1305/2299},$$

(A2)

where $C_0 = \nu_0 e^{18/19} \left(1 + \frac{121}{304} e^2 \right)^{1305/2299} (1 - e_0^2)^{-3/2}$. $e_0$ is the integration constant that has been chosen to set the initial condition $\nu(e_0) = \nu_0$ for the initial values $e_0$ and $\nu_0$. Then Eq. (A2) is

$$\nu(e) = \nu_0 \frac{\sigma(e)}{\sigma(e_0)},$$

(A3)

where $\sigma(e) = e^{-18/19} \left(1 - e^2\right)^{3/2} \left(1 + \frac{121}{304} e^2 \right)^{-1305/2299}$.

From Eqs. (10,11) one can compute the evolution of the time and phase functions $[t - t_c = \int_0^e \frac{d\nu}{\sigma(e)} de, \Phi - \Phi_c = 2\pi \int_0^e \frac{\sigma(e)}{\sigma(e)} de]$ in terms of eccentricity as [see Eqs. (11,13)]

$$t - t_c = -\frac{15}{304 V^{5/3}} \left(\frac{\sigma(e_0)}{2\pi \nu_0}\right)^{8/3} I_t(e),$$

(A4)

$$\Phi - \Phi_c = -\frac{15}{304 V^{5/3}} \left(\frac{\sigma(e_0)}{2\pi \nu_0}\right)^{5/3} I_\phi(e),$$

(A5)

Based on a large set of simulated binary waveforms, we found that there is about 1 order of magnitude improvement compared to circular waveforms in LISA’s angular resolution for highly eccentric sources (e.g., $e_0 = 0.6$) for relatively high SMBH masses $\sim 10^7 M_\odot$. There is however, a much smaller effect for lower-mass binaries in the range $(10^4 - 10^5) M_\odot$. This improves the prospects for identifying the electromagnetic counterparts of relatively high-mass eccentric SMBH mergers with LISA. Similar conclusions have been reached in Refs. 12, 13. However, we found that pericenter precession does not further improve the sky localization accuracy of the source, although it may further improve the measurement errors of mass and eccentricity parameters.
where the $I_t$ and $I_\phi$ integrals are

$$I_t(e) = \int_0^e x^\alpha (1 - \delta x^2)^{-\beta} dx,$$

$$I_\phi(e) = \int_0^e x^\tilde{\alpha} (1 - \delta x^2)^{\tilde{\beta}} dx,$$

with the constants $\alpha = 29/19$, $\beta = -1181/2299$, $\delta = -121/304$, $\tilde{\alpha} = 11/19$ and $\tilde{\beta} = 214/2299$. The integrals in Eqs. (A6) and (A7) can be evaluated with the Appell functions which generalize the hypergeometric functions.

To compute the time ($\Delta T$) and phase ($\Delta \Phi$) differences the binary spends between the initial and final eccentricities $e_0$ and $e_1$ during its evolution, Eqs. (A4), (A5) are used,

$$\Delta T = \frac{15}{304M_\odot^{5/2} \left( \frac{\sigma(e_0)}{2\pi\nu_0} \right)^8} \left[ I_t(e_0) - I_t(e_1) \right]$$

$$\Delta \Phi = \frac{15}{304M_\odot^{5/2} \left( \frac{\sigma(e_0)}{2\pi\nu_0} \right)^{5/3}} \left[ I_\phi(e_0) - I_\phi(e_1) \right]$$

Figures 11 and 12 show the evolution of time and phase for various initial eccentricities, a fixed 1 yr inspiraling time before the LSO and $10^9 M_\odot$ equal-mass binaries. It can be seen that the eccentricity changes significantly near the coalescence, and the accumulated number of orbits is decreasing for high initial eccentricity.

Appendix B: Stationary Phase Approximation

Consider the waveform $h(t) = A(t) \cos \Phi(t)$ with $\hat{A}(t)/A(t) \ll \hat{\Phi}(t)$ and $\Phi(t) \ll \Phi(t)^2$ (see e.g. Ref. [3]).
FIG. 11. The evolution of the eccentricity as a function of time (as "lifetime" for the fixed 1 yr inspiraling time). The eccentricity changes significantly near the coalescence.

FIG. 12. The evolution of the eccentricity in terms of the phase function for the fixed 1 yr inspiraling time.

with its Fourier transform as

$$\mathcal{F}[A(t) \cos \Phi(t)] = \int_{-\infty}^{\infty} A(t) e^{i\Phi(t)} dt .$$

To evaluate the Fourier integral one can use the stationary phase approximation (SPA). For an arbitrary function of the time, $\Phi(t)$, $\int_{-\infty}^{\infty} A(t)e^{\Phi(t)} dt$ satisfies

$$\mathcal{A}(T) = \sqrt{2\pi / \Phi(T)} e^{i[\Phi(T) + \text{sign}[\Phi(T)] \pi / 4]} ,$$

where the saddle point $T$ satisfies $\Phi(T) = 0$. In Eq. (B1), the $e^{\Phi(t)}$ terms have no contributions to the saddle point $T$. Moreover, $\Psi(t) = 2\pi f - \Phi(t)$, and the stationary phase condition $[\dot{\Psi}(T) = 0]$ implies that $f = \dot{\Phi}(T)/(2\pi)$. This provides a relation between the Fourier and orbital frequencies. Carrying out this exercise for an eccentric waveform consisting of many widely separated GW harmonics, the corresponding Fourier frequencies are, respectively, $f_n = n\nu$ and $f_{n\pm} = n\nu \pm \tilde{\gamma}/\pi$ for the terms due to pericenter precession. For circular orbits, the only nonvanishing term has frequency $f = 2\nu$. Therefore, the Fourier transform of harmonic functions with SPA are

$$\mathcal{F}[A(t) \sin \Phi(t)] = \frac{\mathcal{A}[f(T)]}{2} \sqrt{\frac{2\pi}{|\Phi(f(T)|}}} e^{i[\Phi(f(T)) + \tilde{\Phi}]} ,$$

$$\mathcal{F}[A(t) \cos \Phi(t)] = \frac{\mathcal{A}[f(T)]}{2} \sqrt{\frac{2\pi}{|\Phi(f(T)|}}} e^{i[\Phi(f(T)) - \tilde{\Phi}]} ,$$

where $[\Psi(f(T))] = 2\pi f [\nu(T)] - \Phi[\nu(T)]$ is the phase function and $[\nu(T)]$. $\varphi[\nu(T)]$ are derived from radiation reaction by Eqs. (A5). Following Ref. [62], the phase functions for eccentric compact binaries are

$$\Psi_n = 2\pi f t - \Phi_n ,$$

$$\Psi_{n\pm} = 2\pi f t - \Phi_{n\pm} ,$$

where the functions $\Phi_n, \Phi_{n\pm}$ are defined by Eqs. (12,13) and the first time derivatives are expressed as

$$\dot{\Phi}_n = 2\pi f - 2\pi n\nu ,$$

$$\dot{\Phi}_{n\pm} = 2\pi f - 2\pi n\nu \mp 2\dot{\gamma} .$$

There are three saddle points $(t_n, t_{n\pm})$ following from the stationary phase conditions $\dot{\Psi}_n(t_n) = 0$ and $\dot{\Psi}_{n\pm}(t_{n\pm}) = 0$. It follows that there are three Fourier frequencies for each harmonic of the orbital frequency (denoted by $f_n$, $f_{n\pm}$). The second time derivatives of the $\Psi_n$ and $\Psi_{n\pm}$ phase functions are

$$\ddot{\Psi}_n = -2\pi n\dot{\nu} ,$$

$$\ddot{\Psi}_{n\pm} = -2\pi n\dot{\nu} \mp 2\ddot{\gamma} ,$$

where $\ddot{\gamma}$ is the time derivative of $\ddot{\gamma}$ induced by gravitational radiation; see Eqs. (10,11). Then the phase functions of the waveforms, Eqs. (15,16), can be expressed in terms of the time corresponding to the stationary phase and the acceleration of the pericenter precession, formally

$$\Psi_n(f_n) = 2\pi f_n + \Phi_n(f_n) ,$$

$$\Psi_{n\pm}(f_{n\pm}) = 2\pi f_{n\pm} t_{n\pm}(f_{n\pm}) - \Phi_{n\pm}(f_{n\pm}) .$$
