We give algorithms to factorize large integers in the duality computer. We provide three duality algorithms for factorization based on a naive factorization method, the Shor algorithm in quantum computing, and the Fermat’s method in classical computing. All these algorithms are polynomial in the input size.

PACS numbers: 03.67.Hk,03.65.Ud,03.67.Dd,03.67.-a

I. INTRODUCTION

Factorizing large integers has been a great problem in mathematics and has important applications in information security. Many efforts have been made to find polynomial algorithms. The most naive method is that we divide the input state $N$ by each integer ranging from 2 to $\sqrt{N}$, so it needs $\sqrt{N}$ operations which needs $2^{\frac{1}{2}\log N}$ steps, an exponential function of the input size $\log N$. Until now, there has been no known efficient polynomial algorithm for factoring a large integer in classical computing. The most efficient factoring algorithms in classical computing are the Fermat’s factoring algorithm[1].

With the advent of quantum computer[2, 3], Shor gave a remarkable polynomial-time quantum algorithm for the factoring problem, and its computational complexity has been proved to be $O((\log N)^2(\log \log N)(\log \log \log N))$[4]. Recently, a new computing model, the duality computer(DC), was proposed based on the general principle of quantum interference[5]. The DC exploits the particle wave duality properties of the quantum world, and provides polynomial algorithms for NP-complete problems. Finding a marked item from an unsorted database requires only a single step in the duality computer, which achieves the holy grail in the unsorted database problem. Recently, the mathematical theory of the duality computer has been worked out[6, 7]. In the DC, there are quantum wave divider and combiner operations in addition to the usual unitary quantum gate operations. The divider and combiner operators make the connection between duality computing and classical computing very close, and it is possible that many classical algorithms can be adopted to the duality computing with exponential speedup. In this paper, we explore this connection with the problem of prime factorization as the subject.

This paper is organized as follows. In Sec.II, we briefly review the DC and the implementation of the naive factorization algorithm. In Sec.III, we briefly review Shor’s algorithm and its implementation in the DC. Compared to the quantum computer, the algorithm in the DC acquires some improvement though both algorithms are polynomial ones. In Sec.IV, we design two algorithms using DC based on both the Fermat’s methods and analyze their complexities. The results are discussed and summarized in Sec.V.

II. THE NAIVE FACTORIZATION DC ALGORITHM

The general principle of quantum interference for the quantum system was proposed and based on it, the DC was proposed[2]. The DC makes use of not only the particle nature but also the wave nature of microscopic objects. The quantum wave of the DC are split into several sub-waves by the quantum wave divider(QWD) and they pass through different routes where different computing gate operations are performed. These sub-waves are then re-combined to interfere to give the computational results at the quantum wave combiner(QWC).

The naive factorization algorithm can be easily implemented using the Long algorithm for unsorted database search[5]. We need a two register DC. The DC algorithm is given as follows

**Step 0:** foundlist=[null]

**Step 1:** Prepare the state of the first DC register in the equally distributed state and the second register in the 0 state,

$$|\varphi_0\rangle = \frac{1}{\sqrt{\sqrt{N}}}(|2\rangle + |3\rangle + \cdots + |\sqrt{N} + 1\rangle)|0\rangle,$$  \hspace{1cm} (1)
where \([k]\) means the integer nearest to \(k\).

Step 2: Let the DC go through QWD, so that it divides the wave into two sub-waves,

\[
|\varphi_u\rangle = \frac{1}{2} \frac{1}{\sqrt{N}} (|2\rangle + |3\rangle + \cdots + |p_i\rangle + \cdots + |\sqrt{N} + 1\rangle)|0\rangle,
\]

\[
|\varphi_d\rangle = \frac{1}{2} \frac{1}{\sqrt{N}} (|2\rangle + |3\rangle + \cdots + |p_i\rangle + \cdots + |\sqrt{N} + 1\rangle)|0\rangle.
\]

Step 3: Perform the following function to the lower-path sub-wave,

\[
f(N, i) = \begin{cases} 
1 & \text{if } N \mod i = 0 \\
0 & \text{otherwise}
\end{cases}
\]

Then the state of the lower-path becomes

\[
|\varphi_d'\rangle = \frac{1}{2} \frac{1}{\sqrt{N}} (|2\rangle|0\rangle + |3\rangle|0\rangle + \cdots + |p_i\rangle|1\rangle + \cdots + |\sqrt{N} + 1\rangle)|0\rangle.
\]

Step 4: Apply a query to the lower-path wave: if the second register of the item is 1 retain the sign of the item, reverse the sign of other items and the items in the foundlist.

Step 5: Recombine the two sub-waves \(|\varphi_u\rangle\) and \(|\varphi_d'\rangle\) at the QWC. Then \(|\varphi_f\rangle = |p_1\rangle + |p_2\rangle + \cdots + |p_m\rangle\) apart from a normalization constant, namely \(|\varphi_f\rangle\) contains all the factors except those in the foundlist.

Step 6: Make a read-out measurement, and one of the unfound prime factors \(p_i\) is found out with probability 1. If nothing is found, then go to step 8.

Step 7: Add \(p_i\) to foundlist.

Step 8: Stop.

If \(N = pq\), then the above algorithm runs from step 0 to step 7 to give the smaller factor, and then it goes to step 0 and flows to step 6, and then goes to step 8 and stops.

III. THE DC FACTORIZATION ALGORITHM BASED ON THE SHOR ALGORITHM

The Shor algorithm finds the period of the function \(f_{a,N} = a^x \mod N\). In the quantum computer, the registers of the quantum computer is divided into two, and the Shor algorithm contains the following steps:

Step 1: Prepare the first register in the evenly distributed state and the second register in state 0

\[
|\psi_1\rangle = \sqrt{\frac{q}{N}} \sum_{i=0}^{q-1} |i\rangle|0\rangle,
\]

where \(N^2 < q \leq 2N^2\).

Step 2: Apply the function \(f_{a,N}\) to the state and store the results in the second register,

\[
|\psi_2\rangle = \sqrt{\frac{q}{N}} \sum_{i=0}^{q-1} |i\rangle|a^x \mod N\rangle.
\]

Step 3: Apply a Fourier transform to the first register.

Step 4: Makes a measurement on the first register and an integer will be found.

Repeat steps 1 to 4 a sufficient number of times and from the measured results a period is inferred, from which the period of the function \(f_{a,N}(x)\) is determined.

In the DC there is no need to apply the Fourier transform on the first register. Here we present a modified Shor algorithm in the DC with \(N = 21\) and \(a = 2\) as an example. Taking \(q = 512\), after applying the function \(f_{a,N}\) on the DC, we let wave go through the QWD, so that its wave function is split into two parts, and in the lower path we
We denote the integer bigger than or equal to \( N \) as \( \varphi \). The upper and lower path wave functions are now

\[
|\varphi_u\rangle = \frac{1}{2\sqrt{512}}(|0\rangle|1\rangle + |1\rangle|2\rangle + |2\rangle|4\rangle + |3\rangle|8\rangle + |4\rangle|16\rangle + |5\rangle|11\rangle + \\
|6\rangle|1\rangle + |7\rangle|2\rangle + |8\rangle|4\rangle + |9\rangle|8\rangle + |10\rangle|16\rangle + |11\rangle|11\rangle + \\
|12\rangle|1\rangle + \cdots ),
\]

(6)

\[
|\varphi_d\rangle = \frac{1}{2\sqrt{512}}(-|0\rangle|1\rangle - |1\rangle|2\rangle - |2\rangle|4\rangle - |3\rangle|8\rangle - |4\rangle|16\rangle - |5\rangle|11\rangle + \\
|6\rangle|1\rangle - |7\rangle|2\rangle - |8\rangle|4\rangle - |9\rangle|8\rangle - |10\rangle|16\rangle - |11\rangle|11\rangle + \\
|12\rangle|1\rangle - \cdots ),
\]

(7)

respectively.

Next step, we combine the sub-waves at the QWC, and apart from a normalization constant the wave becomes

\[
|\varphi\rangle = (|6\rangle|1\rangle + |12\rangle|1\rangle + \cdots + |504\rangle|1\rangle + |510\rangle|1\rangle)
\]

= (|6\rangle + |12\rangle + \cdots + |504\rangle + |510\rangle)|1\rangle.
\]

(8)

Make a read-out measurement of the first register, one will obtain one of the following numbers: 6, 12, 18, 24 ... 504, 510 with equal probability. Repeat the procedure a number of times, the smallest increment will be found and hence the required period.

IV. DC FACTORIZATION ALGORITHM BASED ON FERMAT’S METHOD

On classical computers, Fermat’s method for factoring an odd integer \( N = pq \) where \( q \) and \( p \) are prime integers consists of finding positive integers \( X \) and \( Y \) that satisfies \( N = X^2 - Y^2 \). The key points lie in finding those \( X \) and \( Y \) satisfying \( N = X^2 - Y^2 \) so that we can get the result \( p = (X + Y)/2, q = (X - Y)/2 \). Fermat’s method is quite efficient if \( p/q \) is near 1, but less efficient if \( p/q \) is not near 1. The computational complexity for Fermat’s method is \( O(q - \sqrt{N}) \), and in worst conditions is \( O(N/2 - \sqrt{N}) \).

We design an algorithm under worst conditions for a DC based on Fermat’s method and analyze its complexity. We denote the integer bigger than or equal to \( (N/2 - \sqrt{N}) \) as \( M \)

1. Make the initial state

\[
|\varphi_0\rangle = \frac{1}{\sqrt{M}} \sum_{i=0}^{M} |x_i\rangle = \frac{1}{\sqrt{M}}(|x_0\rangle + |x_1\rangle + \cdots + |x_M\rangle) = \frac{1}{\sqrt{M}}(|\sqrt{N}\rangle + |\sqrt{N} + 1\rangle + \cdots + |N/2\rangle)
\]

(9)

2. Let \( |\varphi_0\rangle \) go through the QWD and it is divided into:

\[
|\varphi_u\rangle = \frac{1}{2\sqrt{M}} \sum_{i=0}^{N/2} |x_i\rangle = \frac{1}{2\sqrt{M}}(|x_0\rangle + |x_1\rangle + \cdots + |x_M\rangle) = \frac{1}{2\sqrt{M}}(|\sqrt{N}\rangle + |\sqrt{N} + 1\rangle + \cdots + |N/2\rangle)
\]

(10)

\[
|\varphi_d\rangle = \frac{1}{2\sqrt{M}} \sum_{i=0}^{N/2} |x_i\rangle = \frac{1}{2\sqrt{M}}(|x_0\rangle + |x_1\rangle + \cdots + |x_M\rangle) = \frac{1}{2\sqrt{M}}(|\sqrt{N}\rangle + |\sqrt{N} + 1\rangle + \cdots + |N/2\rangle)
\]

(11)

3. Apply on the upper-sub wave the following query function \( f_{x,N} \)

\[
f_{x,N} = \begin{cases} 
+1 & \text{if } \sqrt{x^2 - N} \text{ is an integer} \\
-1 & \text{otherwise}
\end{cases}
\]

(12)

4. Let the two sub-waves go through the QWC and it is combined into \( |X\rangle \), then we can make a measurement to read out this \( X \), and the \( Y \) is then \( \sqrt{X^2 - N} \). Then \( p \) and \( q \) are obtained.

The complexity for our algorithm is \( O(\log (N/2)) \).
In this paper, we propose three duality algorithms to factorize large integer number. It has been shown that Shor’s algorithm in a duality computer can be simplified a little. The naive factorization algorithm can be implemented in a duality computer polynomially. We also show that the Fermat’s method in classical computing can be adopted in duality computing with a computational complexity of $O(\log N)$.

Compared to quantum computing, the computational complexity of prime-factorization algorithm is the same which is not remarkable. The important point shown in this work is that exponential algorithms in classical computing can be implemented in duality computer with a polynomial computational complexity. This provides a convenient avenue in constructing duality computing algorithms.

**Acknowledgments**

This work is supported by the National Fundamental Research Program Grant No. 001CB309308, China National Natural Science Foundation Grant No. 10325521, 60433050, and the SRFDP program of Education Ministry of China.

---

[1] Donald E. Knuth, *The Art of Computer Programming Volume II: Seminumerical Algorithms*, Addison Wesley (2002).
[2] R. P. Feynman, *Simulating physics with computers*, Int. J. Theo. Phys., 21, 467-488, (1982).
[3] P. Benioff, *Quantum mechanical models of Turing machines that dissipate no energy*, Phys. Rev. Lett. 48, 1581 (1982).
[4] P. Shor, *Algorithms for quantum computation: discrete logarithms and factoring*. In Proceedings of the 35th Annual Symposium on Foundations of Computer Science, pages 124–134, 1994.
[5] G. L. Long, *The general quantum interference principle and the duality computer*, Commun. Theor. Phys. 34 (2006) 825-844, also available as quant-ph/0512120.
[6] S. Gudder, *Duality quantum computers*, preprint M06/04, http://www.math.du.edu/.
[7] G. L. Long, *Mathematical theory of duality computer in the density matrix formalism*, quant-ph/0605087.