A note on the rate of convergence for sequences of random polarizations

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January 23, 2024

Abstract

It is shown in [3] that the expected $L^1$ distance between $f^*$ and $n$ random polarizations of an essentially bounded function $f$ with support in a ball of radius $L$ is bounded by $2dm(B_{2L})\|f\|_{\infty}n^{-1}$. This note complements [3]. The expected $L^1$ distance is bounded by $c_n n^{-1}$ with $\limsup_{n \to \infty} c_n \leq 2^{d+1} |\nabla f|_1$ for every $f \in W_{1,1}(B_L) \cap L^\infty(B_L)$. Furthermore, the expected $L^1$ distance is $O(n^{-1/q})$ for $f \in L^p(B_L)$ with $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. The rate $n^{-1}$ is best possible: $n$ times the measure of the symmetric difference between the random polarizations of a ball and its corresponding Schwarz symmetrization converges in distribution to a random variable with moments that are derived. It is also shown that the expected symmetric difference between the random polarizations of a measurable set in $B_L$ and its corresponding Schwarz symmetrization is slower than $n^{-r}$ for any $r > 2d$ and if the rate is $n^{-1}$ then $n$ times the measure of the symmetric difference between the random polarizations of the set and its corresponding Schwarz symmetrization converges in distribution. A new sequence of random polarizations is introduced such that the transition probability depends on the state of the underlying Markov chain. For compact sets with finite perimeter, the rate of convergence is $O(n^{-3/2})$ when $d = 1$ and $O(n^{-(1-\frac{d+1}{2(d+1)})})$ for $d > 1$. Finally it is shown that for every compact set $A$ in $\mathbb{R}$ with finite perimeter there exists a sequence of polarizations $A_n$ of $A$ converging exponentially to its Schwarz symmetrization.

1 Introduction

It is well known that there exists sequences of polarizations (a rearrangement defined below) which can be applied iteratively to any initial function $f \in L^p$ ($1 \leq p < \infty$) to generate a sequence of functions (polarizations of $f$) which converge

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in $L^p$ to $f^*$ — the symmetric decreasing rearrangement of $f$. The convergence is uniform when applied to continuous functions with compact support and, when applied to compact sets, the convergence also holds with respect to the Hausdorff distance (see [3] for a detailed overview). The first result on rates of convergence of polarizations to the symmetric decreasing rearrangement appears in [3, p.19]: if $f \in L^1(\mathbb{R}^d)$ and bounded with support in $B_L$, then

$$
E[\|f^{\sigma_1 \cdots \sigma_n} - f^*\|_1] \leq 2dm(B_{2L})\|f\|_{\infty} n^{-1}.
$$

(1)

The purpose of this note is to expand on (1).

2 Notation, rearrangements, results

2.1 Notation

In what follows, $m$ is the Lebesgue measure with sigma algebra $\mathcal{M}$; $B_{x,r}$ is the ball of radius $r$ centered at $x$; $B_r$ is the ball of radius $r$ centered at the origin; $\kappa_d$ is the volume of $B_1$ and $\omega_d$ is the surface area of $B_1$. The space of reflections that do not map the origin to the origin will be denoted by $\Omega$ and $\sigma_{x,y}$ will denote the unique reflection that maps $x$ to $y$.

2.2 Rearrangements

A rearrangement $T$ is a map $T : \mathcal{M} \to \mathcal{M}$ that is both monotone ($A \subset B$ implies $T(A) \subset T(B)$) and measure preserving ($m(T(A)) = m(A)$ for all $A$). If

$$
\mu_f(t) = m(\{f > t\}) < \infty
$$

(2)

for all $t > 0$, then $f$ is said to vanish at infinity and we can define its rearrangement $Tf$ by using the “layer cake principle”:

$$
Tf(x) = \int_{0}^{\infty} \mathbb{1}_{T(\{f > t\})}(x)dt = \sup\{t : x \in T(\{f > t\})\}.
$$

(3)

2.2.1 Polarization

The polarization of $f$ with respect to $\sigma \in \Omega$ is defined as

$$
\sigma f(x) = \begin{cases} f(x) \lor f(\sigma(x)) & \text{if } x \in X^\sigma_+ \\ f(x) \land f(\sigma(x)) & \text{if } x \in X^\sigma_- \\ f(x) & \text{if } x \in X^\sigma_0 \end{cases}
$$

(4)

with $X^\sigma_0$ the hyperplane invariant under $\sigma$ which splits $\mathbb{R}^d$ into two disjoint half-spaces: $X^\sigma_+$, the half-space containing 0, and $X^\sigma_-$, the half-space not containing
0. If $A$ is an arbitrary set, then its polarization with respect to $\sigma$ is simply the polarization of $\mathbb{1}_A$ and is denoted by $A^\sigma$:

$$A^\sigma = (\sigma(A \cap X^\sigma_0) \cap A^c) \cup (\sigma(A \cap X^\sigma_0) \cap A) \cup (A \cap X^\sigma) \cup (A \cap X^\sigma_0). \quad (5)$$

In other words, $A^\sigma$ is the same as $A$ except that the part of $A$ contained in $X^\sigma$ whose reflection does not lie in $A$ is replaced by its reflection in $X^\sigma$. As a result, polarization is measure preserving. It is clear from (4) that $f^\sigma \leq g^\sigma$ for all $\sigma \in \Omega$ whenever $f \leq g$ and thus polarization is monotone, i.e., polarization is a rearrangement. One can check directly that $\{f^\sigma > t\} = \{f > t\}^\sigma$ for all $\sigma \in \Omega$ and, by (3), $f^\sigma$ is the rearrangement of $f$ with respect to the polarization rearrangement.

2.2.2 Schwarz Symmetrization

For any $A \in \mathcal{M}$ there exists a unique open ball centered at the origin $A^*$ with the same measure as $A$ called the Schwarz symmetrization of $A$. If $f(x)$ vanishes at infinity then its Schwarz rearrangement is denoted by $f^*(x)$. It is clear that $f^*(x)$ is radially decreasing: $f^*(x) \leq f^*(y)$ for $|x| \geq |y|$ and $f(x) = f(y)$ for $|x| = |y|$. In the literature, $f^*$ is also called the symmetric decreasing rearrangement of $f$. If $f$ vanishes at infinity, we let

$$r_f(t) = \left(\frac{\mu_f(t)}{\kappa_d}\right)^{1/d} \quad (6)$$

denote the radius of the open ball $\{f > t\}^*$. The distribution function $\mu_f(t)$ is always right continuous and thus so is $r_f(t)$. In particular, we have

$$\{f^* > t\} = \{f > t\}^* \quad (7)$$

for all $t \geq 0$ and thus $f^*$ is right continuous.

2.3 Random Polarizations

2.3.1 Construction of probability measures

Probability measures for $\Omega$ are easily constructed by mapping $\Omega$ to $\mathbb{R}^d$ via the invertible map $\varphi(\sigma) = \sigma(0)$: if $(\mathbb{R}^d, \mathcal{F}, \mu)$ is a probability space, then $\mathbb{P}(\varphi^{-1}(A)) = \mu(A)$ yields a corresponding probability space for $\Omega$. A particular good choice is $\mathcal{F} = \mathcal{M}$ and $d\mu = |x|^{-(d-1)} \mathbb{1}_{B_{2L}} dm$:

$$\mathbb{P}(\varphi^{-1}(A)) = (2L \omega_d)^{-1} \int_{A \cap B_{2L}} |x|^{-(d-1)} dx \quad (8)$$

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2.3.2 Generating random polarizations

We will be working with an i.i.d sequence of random polarizations $\sigma_n$:

$$P(\bigcap_{i=1}^{n} \{ \sigma_i \in \phi^{-1}(B_i) \}) = \prod_{i=1}^{n} P(\phi^{-1}(B_i)).$$

(10)

Given $f$ in $L^1$, we can iteratively apply the $n$ polarizations $\sigma_1, \ldots, \sigma_n$ to $f$ resulting in the random polarization $f^{\sigma_1 \cdots \sigma_n}$. Since the $\sigma_n$ are independent, it is clear that $f^{\sigma_1 \cdots \sigma_n}$ is a Markov chain.

2.4 Results

2.4.1 Rate of convergence estimates

Theorem 1. Define $\rho_f(t) := \text{per}(\{f > t\})$. The following holds:

(i) If $\|f\|_\infty < \infty$, then $E[\|f^{\sigma_1 \cdots \sigma_n} - f^*\|_1]$ is bounded by

$$2dm(B_L)\|f\|_\infty (1 + L^{-1}(\|f\|_\infty \omega_d) \frac{d-1}{d-1} \|\rho_f\|_1) - 1 n^{-1}. \quad (11)$$

(ii) Suppose $f \in L^p(B_L)$ with $p \geq 1$ and let $x_n$ denote the unique fixed point of the function $dLn^{-1}(1 + x^2)^{d-1}$. If $p > 1$ and $n$ is large enough that

$$x_n^{1/d} < r_f(0), \text{ then } E[\|f^{\sigma_1 \cdots \sigma_n} - f^*\|_1] \text{ is bounded by}$$

$$C_d,p,L\|f\|_p n^{-1} \left( \int_{L^{-1} x_n^{1/d}}^1 \frac{(1 + r)^{(d-2)/2}}{r^{(d+1)/2}} dr \right)^{1/q} + 2\|f^*\|_{|x| \leq x_n^{1/d}} \quad (12)$$

with $\frac{1}{p} + \frac{1}{q} = 1$ and $C_d,p,L = 2(d-1)\omega_d^{1/q} L^{d/q+1/p}$. If $p = 1$ and $n$ is large enough that

$$x_n^{1/d} < r_f(0), \text{ then } E[\|f^{\sigma_1 \cdots \sigma_n} - f^*\|_1] \text{ is bounded by}$$

$$2(d-1) L^{d-1}\|f\|_1 n^{-1} (1 + L^{-1} x_n^{1/d} - 2 x_n^{(d-2)/d} + 2\|f^*\|_{|x| \leq x_n^{1/d}} \quad (13)$$

(iii) If $m(\partial\{f > t\}) = 0$ for almost every $t$ and $f \in L^\infty(B_L)$, then

$$\limsup_{n \to \infty} n E[\|f^{\sigma_1 \cdots \sigma_n} - f^*\|_1] \leq 2d^1 L \|\rho_f\|_1. \quad (14)$$
(iv) For any measurable set $A \subset B_L$, we must have

$$\sum_{n=1}^{\infty} nE[X_n]^{1/d} = \infty.$$ 

In particular, $n' E[X_n] \to \infty$ for $r > 2d$.

(v) If $\lim \inf_{n \to \infty} n E[X_n] > 0$ then $nX_n$ converges in distribution.

Corollary 1. If $f \in W^{1,1}(B_L) \cap L^{\infty}(B_L)$, then

$$\limsup_{n \to \infty} n E[\|f^{\sigma_1 \cdots \sigma_n} - f^*\|_1] \leq 2^{d+1} L \|\nabla f\|_1.$$ 

Corollary 2. If $f \in L^p(B_L)$ with $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $E[\|f^{\sigma_1 \cdots \sigma_n} - f^*\|_1] = O(n^{-1/q})$.

Proof. If $f \in L^p(B_L)$ with $p > 1$, then Hölder's inequality and $x_n \sim dLn^{-1}$ gives

$$E[\|f^{\sigma_1 \cdots \sigma_n} - f^*\|_1] = O(n^{-1}(\int_{x_n}^{L} r^{-(d-1)q/p} dr)^{1/q}) + O(\|f\|_{L^q(x_n^{1/d})})$$

$$= O(n^{-1(1+1/d)}) + O(n^{-1/q})$$

$$= O(n^{-1/q}).$$

Corollary 3. If $d = 1$ then $nX_n$ converges in distribution.

Proof. It is shown in [2] that $X_n$ is always greater than or equal to $X'_n$, where $X'_n$ is the measure of the symmetric difference between the random polarizations of a non-centered ball, with the initial condition $X'_0 = X_0$, and its Schwarz symmetrization. Theorem (2) gives that $nX'_n$ converges in distribution, so $\liminf_{n \to \infty} nX_n > 0$. 

\[\square\]
2.4.2 Random polarizations of balls

We study the rate of convergence of the random polarization of balls (see [1] for a different approach). Let $A$ denote a ball of radius $r$ contained in $B_L$, $A_n = A^{\sigma_1, \ldots, \sigma_n}$ and $X_n$ the distance from the origin of the centre of the ball $A_n$.

**Theorem 2.** Let $u$ denote any unit vector. The following holds:

(i) The moments of $X_n$ can be computed exactly:

$$
E[X_n^j] = \sum_{k=0}^{\lfloor j/2 \rfloor} \binom{n}{k} (-1)^k X_0^{k-j} \prod_{i=0}^{k} c_i$$

where $c_0 = (2L\omega d)^{-1} \int_{|y|<1} (1-|y|^\alpha)|u-y|^{-(d-1)} dy$ and $c_0 = 1$.

(ii) The moment $E[X_n^k]$ is increasing in $X_0$ for all $k$ and all $n$.

(iii) If $d = 1$, then $E[X_n] = 2L \int_0^{X_0/2L} (1-t)^{\alpha} dt$ and $nX_n$ converges in distribution to an exponential distribution with scale parameter $2L$.

(iv) If $\alpha \geq 1$ then $E[X_n^\alpha] \leq (X_0^{-1} + c_0\alpha^{-1}n)\alpha$.

(v) If $d > 1$ then $E[X_n] \geq (X_0^{-1} + \ell_d n)^{-1}$ where

$$
\ell_d = (2L\omega d)^{-1} \int_{|y|<1} (1-|y|)|y|^{-1}|u-y|^{-(d-1)} dy.
$$

(vi) $nX_n$ converges in distribution to a random variable $Y$ with moments

$$
E[Y^k] = \frac{\alpha_d (2L)^k (k-1)!}{\prod_{i=1}^{k-1} \tau_i},
$$

where

$$
\alpha_d = \lim_{n \to \infty} n \sum_{k=0}^{\lfloor j/2 \rfloor} \binom{n}{k} (-1)^k 2^{-(k+1)} \prod_{i=0}^{k} \tau_i
$$

and

$$
\tau_k = \omega d^{-1} \int_{|y|<1} (1-|y|^k)|u-y|^{-(d-1)} dy
$$

and $\tau_0$ is set to 1.

(vii) $n \cdot m(A_n \triangle A^*)$ converges in distribution to $4\kappa_d-1 n^d Y$.

2.5 Proof of theorems

We first consider a measurable set $A \subset B_L$ with $m(A) < m(B_L)$. Define the following random sequences:
• $A_n = A^{\sigma_1 \cdots \sigma_n}$.
• $X_n = m(A_n/A^*)$.
• $Y_n = (2L\omega_d)^{-1}m(A_n/A^*)^{-2} \int_{A_n/A^*} \int_{A^*/A_n} |x-y|^{(d-1)} dy dx$.
• $r_n = (X_n/\kappa_d)^{1/d}$.

**Lemma 1.** The following holds:

(i) $E[X_n - X_{n+1} | X_n] = Y_n X_n^2$.
(ii) $E[X_n - X_{n+1} | X_n] \geq Y_n X_n^k$ for all $k \geq 1$.
(iii) $E[X_n^k] \leq (X_0^{-1} + k^{-1} \sum_{i=1}^n E[Y_i^{-1}]^{-1/k})^{-k}$.
(iv) $E[X_n - X_{n+1} | X_n] \leq r_n L^{-1} X_n$.

**Proof.**

(i) Fubini’s Theorem and (9) gives [3, p.19]:

$$E[X_n - X_{n+1} | X_n] = E[m(A_n/A^* \cap \sigma^{-1}(A^*/A_n))]$$
$$= \int_{A_n/A^*} E[\mathbb{1}_{\sigma^{-1}(A^*/A_n)}(x)]dx$$
$$= Y_n X_n^2.$$  

(ii) We proceed by induction. The case $k = 1$ is covered by (i). Suppose $k \geq 1$. We have

$$E[X_n^{k+1} - X_{n+1}^{k+1} | X_n] = E[(X_n^k - X_{n+1}^k)X_n - (X_{n+1} - X_n)X_n^k | X_n]$$
$$\geq E[(X_n^k - X_{n+1}^k)X_n | X_n]$$
$$\geq Y_n X_n^{k+1}.$$  

(iii) By the mean value theorem and Jensen’s inequality:

$$E[X_n^k]^{-1/k} - X_0^{-1} \geq k^{-1} \sum_{i=1}^n \frac{E[X_i^{k+1} | Y_{i-1}]}{E[X_i^{k+1} | Y_{i-1}]^{1/k}} \geq k^{-1} \sum_{i=1}^n \frac{E[Y_i^{-k}]}{E[Y_i^{-k}]^{1/k}}$$

which is equivalent to

$$E[X_n^k] \leq (X_0^{-1} + k^{-1} \sum_{i=1}^n E[Y_i^{-k}]^{-1/k})^{-k}$$

7
Apply the Riesz rearrangement inequality:

\[
\mathbb{E}[X_n - X_{n+1} | X_n] \leq (2L\omega_d)^{-1} \int_{B_{r_n}} \int_{B_{r_n}} |x - y|^{-(d-1)}\,dy\,dx
\]

\[
= r_n^{d+1} L^{-1} (2\omega_d)^{-1} \int_{B_1} \int_{B_1} |x - y|^{-(d-1)}\,dy\,dx
\]

\[
\leq X_n r_n L^{-1}.
\]

\[\square\]

We define the following random sequences associated with \( f \):

- \( A_{n,t} = \{ f > t \}^{\sigma_1 \ldots \sigma_n} \).
- \( X_{n,t} = m(A_{n,t}/A_{n,t}^*) \).
- \( Y_{n,t} = (2L\omega_d)^{-1} m(A_{n,t}/A_{n,t}^*)^{-2} \int_{A_{n,t}/A^*} \int_{A^*/A_{n,t}} |x - y|^{-(d-1)}\,dy\,dx \).

To relate the rate of convergence for random polarizations of functions to that of measurable sets, we will make use of the following convenient formula:

\[
\|f^{\sigma_1 \ldots \sigma_n} - f^*\|_1 = \int_0^\infty m(A_{n,t} \triangle A_{n,t}^*)dt. \tag{17}
\]

### 2.5.1 Proof of Theorem 1

Proof. (17) and the previous lemma (1) give:

\[
\mathbb{E}[\|f^{\sigma_1 \ldots \sigma_n} - f^*\|_1] \leq 2 \int_0^b \left( \sum_{i=1}^n \mathbb{E}[Y_{i-1,t}]^{-1} \right)^{-1}dt + 2 \int_b^\infty \mu_f(t)dt \tag{18}
\]

\[
\leq 2\omega_d L^{-1} \int_0^b (L + r_f(t))^{d-1}dt + 2 \int_b^\infty \mu_f(t)dt \tag{19}
\]

for every \( b \geq 0 \).

Set \( b = \|f\|_\infty < \infty \). The binomial theorem and Jensen’s inequality gives

\[
\int_0^b (L + r_f(t))^{d-1}dt \leq bL^{d-1} \sum_{k=0}^{d-1} \binom{d-1}{k} L^{-k} b^{\frac{d-k}{d}} \left( \int_0^b r_f^{d-1}(t)dt \right)^{\frac{k}{d-1}}
\]

\[
= bL^{d-1} \left( 1 + L^{-1}(b\omega_d)^{\frac{1}{d-1}}\|\rho_f\|_1^{\frac{1}{d-1}} \right)^{d-1}.
\]

Now suppose that \( f \in L^p(B_L) \) with \( p \geq 1 \) and \( f^*(r) \) is absolutely continuous. Using the change of variable \( t = f^*(r) \) and setting \( b = f^*(x) \), the right
hand side of (19) equals

\[-2L\omega_d n^{-1} \int_x^{r_f(0)} (L + r)^{d-1} |f^*(r)| dr - 2\kappa_d \int_0^x r^d |f^*(r)| dr. \]  \hspace{1cm} (20)

Integration by parts shows that (20) equals

\[\gamma(x; n) + 2L\omega_d n^{-1} (d-1) \int_x^{r_f(0)} (L + r)^{d-2} f^*(r) dr + 2\omega_d \int_0^x f^*(r) r^{d-1} dr \]  \hspace{1cm} (21)

with

\[\gamma(x; n) = 2\kappa_d f^*(x)(dLn^{-1}(L + x)^{d-1} - x^d).\]

Recalling the sequence \(x_n\) from statement (ii), we have \(\gamma(x_n^{1/d}; n) = 0\). Assume that \(p > 1\). By Hölder’s inequality \((\frac{1}{p} + \frac{1}{q} = 1)\), the first integral in (21) is bounded by

\[\|f\|_p \omega_d^{-1/p} \left( \int_{x_n^{1/d}}^L (L + r)^{(d-2)q} r^{-(d-1)q/p} dr \right)^{1/q} \]  \hspace{1cm} (22)

for \(x = x_n^{1/d} < r_f(0)\). For \(p = 1\), we have the upper bound

\[\int_{x_n^{1/d}}^{r_f(0)} (L + r)^{d-2} f^*(r) dr \leq \omega_d^{-1} \|f\|_1 (L + x_n^{1/d})^{d-2} x_n^{-(d-1)/d}. \]  \hspace{1cm} (23)

(21) and (22) shows that the upper bound (12) is valid for all functions \(f \in L^p(B_L)\) whose symmetric decreasing rearrangement \(f^*\) is absolutely continuous. The set of all such functions is dense in \(L^p(B_L)\): the Pólya-Szegő inequality (24.1) implies that such a set must contain \(W^{1,1}(B_L)\). Suppose that \(f \in L^p(B_L)\), \(f_k\) approaches \(f \in L^p(B_L)\) with \(f_k^*\) absolutely continuous, and \(x_n^{1/d} < r_f(0)\). Convergence in \(L^p\) implies that

\[\lim_{k \to \infty} r_{f_k}(t) = r_f(t)\]

when \(r_f\) is continuous at \(t\). In particular, \(x_n^{1/d} < r_{f_k}(0)\) for sufficiently large \(k\) and, for such \(k\), the first inequality of (i) applies to \(f_k\). Since both polarization and the symmetric decreasing rearrangement are contractive on \(L^p\),

\[\| (f_{\sigma^1 \cdots \sigma_n} - f_k^*) - (f_{\sigma^1 \cdots \sigma_n} - f^*) \|_1 \leq \| f^* - f_k^* \|_1 + \| f - f_k \|_1 \]  \hspace{1cm} (24)

\[\leq 2\| f - f_k \|_1 \]  \hspace{1cm} (25)

for every random sequence \(\sigma_n\). By the bounded convergence theorem:

\[\lim_{k \to \infty} \mathbb{E}[\| f_{\sigma^1 \cdots \sigma_n} - f_k^* \|_1] = \mathbb{E}[\| f_{\sigma^1 \cdots \sigma_n} - f^* \|_1].\]
Since \( \|f_k^*\|_{L^1(B_n)} \) tends to \( \|f^*\|_{L^1(B_n)} \) then the first inequality in (iii) holds for \( f \). The same approximation procedure can be used to extend the second inequality in (iii) to all of \( L^1(B) \).

(iii) We suppose that \( \|f\|_\infty < \infty \) and \( m(\partial \{ f > t \}) = 0 \) for almost every \( t \). Recalling (18):

\[
\mathbb{E}[||f^{\sigma_1,\ldots,\sigma_n}_t - f^*||_{L^1}] \leq 2 \int_0^\infty \sum_{i=1}^n \mathbb{E}[|Y_{i-1,t}|] dt. \tag{26}
\]

We have

\[
Y_{n,t} \geq (2\omega dL)^{-1}[2r_f(t) + \sup_{x \in A_{n,t}} d_H(x, A_n^*)]^{-(d-1)}.
\]

If \( m(\partial \{ f > t \}) = 0 \) then

\[
\sup_{x \in A_{n,t}} d_H(x, A_n^*) \to 0
\]

almost surely. Hence, by the dominated convergence theorem,

\[
\limsup_{n \to \infty} \mathbb{E}[||f^{\sigma_1,\ldots,\sigma_n}_t - f^*||_1] \leq 2^{d+1}L \|\rho_f\|_1.
\]

(iv) We have

\[
\mathbb{P}(X_{n+1} \leq X_n(1-1/n)) = \mathbb{E}[\mathbb{P}(X_{n+1} \leq X_n(1-1/n)|X_n)]
\leq nL^{-1}\kappa^{-1/d}_d \mathbb{E}[X_n^{-1}X_n^{1+1/d}]
\leq nL^{-1}\kappa^{-1/d}_d \mathbb{E}[X_n]^{1/d}.
\]

By the Borel-Cantelli Lemma, if

\[
\sum_{n=1}^\infty n\mathbb{E}[X_n]^{1/d} < \infty
\]

then

\[
X_{n+1} \geq (1-1/n)X_n
\]

for \( n \geq N \) (where the \( N \) depends on the sequence), almost surely. But this would imply that

\[
\liminf_{n \to \infty} nX_n > 0
\]

almost surely, and, by Fatou’s Lemma,

\[
\liminf_{n \to \infty} n\mathbb{E}[X_n] > 0
\]

, which is a contradiction.

(v) We first show that \( n^k\mathbb{E}[X_n^k] \) converges to a non-zero limit for all \( k \). We
have

\[ 0 < \liminf_{n \to \infty} n^k \mathbb{E}[X_n^k] \leq \liminf_{n \to \infty} n^k \mathbb{E}[X_n^k] \leq \limsup_{n \to \infty} n^k \mathbb{E}[X_n^k] < \infty. \]

Hence it suffices to prove that

\[
\lim_{n \to \infty} \frac{\mathbb{E}[X_n^k - X_{n+1}^k]}{\mathbb{E}[X_n^k]} = 0.
\]

We have

\[
\mathbb{E}[\Delta X_n^k] \leq k \mathbb{E}[\Delta X_n X_n^{k-1}]
\]

\[
\leq \mathbb{E}[(\Delta X_n)^2]^{1/2} \mathbb{E}[X_n^{2(k-1)}]^{1/2}
\]

\[
\leq \mathbb{E}[(\Delta X_n)^2]^{1/2} n^{-(k-1)}
\]

and

\[
\mathbb{E}[(\Delta X_n)^2] = \mathbb{E}[X_n^2] - 2 \mathbb{E}[X_n X_{n+1}] + \mathbb{E}[X_{n+1}^2]
\]

\[
\leq 2 \mathbb{E}[X_n^2] - 2 \mathbb{E}[X_{n+1}^2] + 2 \mathbb{E}[X_n \Delta X_n]
\]

\[
\leq 2L^{-1} \kappa_d^{-1/d} \mathbb{E}[X_{n+1}^{2+1/d}]^{2+1/d}
\]

\[
\leq n^{-(2+1/d)}.
\]

Consequently,

\[
\mathbb{E}[X_n^k - X_{n+1}^k] \leq \frac{n^{-(2+1/d)/2-(k-1)}}{n^{-k}} = n^{-1/(2d)} \to 0.
\]

Let \( \mu_k \) equal the limit of \( n^k \mathbb{E}[X_n^k] \). To get convergence in distribution of \( nX_n \), it suffices to prove that

\[
\limsup_{k \to \infty} \frac{\mu_k^{1/2k}}{2k} < \infty
\]

[4 p. 109]. (iii) gives \( \mu_k \leq dm(B_{2L})k^k \) and therefore

\[
\limsup_{k \to \infty} \frac{\mu_k^{1/2k}}{2k} \leq \limsup_{k \to \infty} \frac{(dm(B_{2L})(2k)^{(2k)})^{1/2k}}{2k} = 1.
\]

\( \square \)
2.5.2 Proof of theorem 2

Proof. (i) The function

\[ G(x, \alpha) = (2L\omega_d)^{-1} \int_{|y|<|x|} |y|^\alpha |x-y|^{(d-1)} \, dy \]

has the scaling property \( G(\lambda x, \alpha) = \lambda^{\alpha+1} G(x, \alpha) \) for \( \alpha \geq 0 \). The scaling property implies

\[ \mathbb{E}[X_n^\alpha] - \mathbb{E}[X_{n-1}^\alpha] = -c_{\alpha} \mathbb{E}[X_{n-1}^{\alpha+1}] \tag{27} \]

where

\[ c_{\alpha} = (2L\omega_d)^{-1} \int_{|y|<1} (1-|y|^\alpha)|u-y|^{-(d-1)} \, dy \]

and \( u \) any unit vector. (16) follows directly from the following recurrence relations (which follow from (27)):

\[ \sum_{k=0}^{n} \binom{n}{k} (-1)^k \mathbb{E}[X_{n-k}^j] = \Delta^n(X_0^j) = X_0^{j+n}(-1)^n \prod_{k=0}^{n} c_{k+j-1}. \tag{28} \]

(ii) We use induction on \( n \) for fixed \( k \). Suppose \( Z_n \) is like \( X_n \) except that we start at \( Z_0 \) greater than \( X_0 \). The function \( x \mapsto x^k - c_k x^{k+1} \) is increasing in \( x \) for \( x \leq L \) because \( c_k \leq (2L)^{-1} \). Hence the statement is true for \( n = 1 \). We have

\[ \mathbb{E}[Z_{n+1}^k - X_{n+1}^k] = \mathbb{E}[Z_n^k - X_n^k] - c_k \mathbb{E}[Z_n^{k+1} - X_n^{k+1}] \]

\[ \geq \mathbb{E}[Z_n^k - X_n^k] - c_k \mathbb{(X_n + Z_n)(Z_n^k - X_n^k)} \]

\[ \geq \mathbb{E}[Z_n^k - X_n^k] [1 - c_k (X_0 + Z_0)] \]

\[ \geq 0 \]

since \( X_0 + Z_0 \) is less than or equal to \( 2L \).

(iii) For \( d = 1 \), we have

\[ n^j \mathbb{E}[X_n^j] = n^j \sum_{k=0}^{n} \binom{n}{k} (-1)^k X_0^{j+k} (2L)^{-k}(j+k)^{-1} \]

\[ = (2L)^j n^j \int_{0}^{X_n/2L} t^{j-1}(1-t)^n \, dt \]

\[ \rightarrow (2L)^j j! \]

as \( n \) tends to infinity. This shows that \( nX_n \) converges in distribution to an exponential with mean \( 2L \).

(iv) Suppose \( \alpha \geq 1 \). (27), the mean value theorem and Jensen’s inequality
yields
\[
E[X_n^{-\alpha}] - X_0^{-\alpha} \geq \alpha^{-\alpha_\alpha} \sum_{i=1}^{n} E[X_{i-1}^{\alpha+1}]E[X_i^{\alpha}]^{-(1+\alpha)} \geq \alpha^{-\alpha_\alpha} n
\] (29)
which is equivalent to \(E[X_n^{\alpha}] \leq (X_0^{-1} + \alpha c^{-1}n)^{-\alpha}\).

(v) For the lower bound, suppose \(d > 1\) and \(R_n = X_n/X_{n-1}\) then \(E[X_n^{-1}] < \infty\) and
\[
E[X_n^{-1}] - X_0^{-1} = \sum_{i=1}^{n} E[X_i^{-1}]E\left[\frac{1-R_i}{R_i}X_{i-1}\right] = \sum_{i=1}^{n} E[X_i^{-1}X_{i-1}\ell_d] = n\ell_d.
\]
Hence we obtain the desired lower bound \((X_0^{-1} + n\ell_d)^{-1} < \infty\). We first show that \(n^k E[X_n^k]\) converges for all \(k \geq 1\). We have
\[
\frac{(n+1)^k E[X_n^k]}{n^k E[X_n^k]} \sim \frac{E[X_n^{k+1}]}{E[X_n^k]} = 1 - c_k \frac{E[X_n^{k+1}]}{E[X_n^k]}
\],
so
\[
E[X_n^k] \geq E[X_n]^k \geq (n\ell_d)^{-k}
\]
and
\[
E[X_n^{k+1}] = O(n^{-(k+1)}).
\]
Hence
\[
\lim_{n \to \infty} \frac{(n+1)^k E[X_n^k]}{n^k E[X_n^k]} = 1
\]
and \(n^k E[X_n^k]\) converges. Set \(\mu_k\) equal to the limit of \(n^k E[X_n^k]\). We have
\[
n^k E[X_n^k] = n^k c_k \sum_{i=n}^{\infty} E[X_i^{k+1}] \sim n^k c_k \mu_{k+1} \sum_{i=n}^{\infty} i^{-(k+1)} \sim c_k \mu_{k+1} k^{-1}.
\]
Hence we have the recurrence relation
\[
\mu_k = c_k k^{-1} \mu_{k+1}
\]
which implies that
\[
\mu_k = \mu_1 (k-1)! \prod_{i=1}^{k-1} i^{-1}.
\]
To get convergence in distribution of \(nX_n\), it suffices to prove that
\[
\lim_{k \to \infty} \frac{\mu_{2k}}{2k} < \infty
\]
Since $c_i$ is increasing:

$$\limsup_{k \to \infty} \frac{k^{1/2k}}{2k} \leq \lim_{k \to \infty} \frac{(2k-1)c_k^{-1}}{2k} = 2L\omega_d \left( \int_{|y|<1} |u-y|^{-(d-1)} dy \right)^{-1}.$$ 

Hence $nX_n$ converges in distribution to a random variable $Y$ with moments $\mu_k$. We introduce the function $\varphi(x, L)$ which corresponds to $E[Y]$ for starting parameters $X_0 = x$ and $L$. It follows directly from (i) that the function $\varphi$ has the following scaling property:

$$\varphi(x, \lambda L) = \lambda \varphi(\lambda^{-1} x, L).$$

Now if the limiting distribution doesn’t depend on the starting value $X_0 = x$ then $\varphi(\lambda^{-1} x, L) = \varphi(x, L)$ and we have the scaling property

$$\varphi(x, \lambda L) = \lambda \varphi(x, L).$$

Finally, to find $E[Y]$ for a fixed $L$, we use

$$E[Y] = \varphi(x, L) = \varphi(L, L) = L \varphi(1, 1).$$

But we know that

$$\varphi(1, 1) = \lim_{n \to \infty} n \sum_{k=0}^{n} \binom{n}{k} (-2)^k \prod_{i=0}^{k} \tau_i,$$

where

$$\tau_k = \omega_d^{-1} \int_{|y|<1} (1-|y|^k)|u-y|^{-(d-1)} dy$$

for $k > 0$ and $\tau_0$ is set to 1. To complete the proof, we need to prove that the limiting distribution doesn’t depend on $X_0$. Define $X_n$ and $Z_n$ as in the proof of (iii). It follows from the monotonicity property (iii) that

$$E[Z_n] \leq P(Z_N \leq X_0) E[X_{n-N}]$$

for all $n > N$; consequently, we also have

$$\lim_{n \to \infty} nE[Z_n] \leq P(Z_N \leq X_0) \lim_{n \to \infty} \frac{n}{n-N} (n-N)E[X_{n-N}] = P(Z_N \leq X_0) \lim_{n \to \infty} nE[X_n]$$

for all $N$. But since $P(Z_N \leq X_0)$ goes to 1 as $N$ goes to $\infty$,

$$\lim_{n \to \infty} nE[Z_n] \leq \lim_{n \to \infty} nE[X_n] \leq \lim_{n \to \infty} nE[Z_n]$$

as desired.

(vii) The volume of the symmetric difference between two balls of radius $r$
with centers a distance $x$ apart equals
\[
4\kappa_{d-1} r^d \int_{\theta(x)}^{\pi/2} \sin^d(t) \, dt, \quad \theta(x) = \arccos(x/2r) \wedge 1.
\] (30)

We introduce the function $\phi(x) = \int_{\theta(x)}^{\pi/2} \sin^d(t) \, dt$ with $x$ between $-1$ and $1$. On the interval $[-\cos(\theta_0), \cos(\theta_0)]$ with $0 < \theta_0 < \pi/2$, the second derivative is bounded and $\phi'(0) = 1$; consequently, we can write $\phi(x) = x + o(|x|)$ for $|x| \leq \cos(\theta_0)$. Applying (30) to the random sequence $X_n$:
\[
nm(A_n \triangle A^*) = 4\kappa_{d-1} r^d n \phi(X_n/2r \wedge 1)
= 4\kappa_{d-1} r^d n(X_n/2r \wedge \cos(\theta_0)) + no(X_n/2r \wedge \cos(\theta_0))
\Rightarrow 4\kappa_{d-1} r^d Y
\]
where $\Rightarrow$ refers to convergence in distribution.

Remark: For $d = 3$, the constants $\tau_k$ can be computed exactly:
\[
\tau_k = \begin{cases} 
\frac{1}{2}(1 - \frac{2\log(2) + 2}{k+1}), & k \text{ even} \\
\frac{1}{2}(1 - \frac{2\log(2) + 2}{k+1}), & k \text{ odd} 
\end{cases}
\]

3 Random polarizations as a non-homogeneous Markov chain

We introduce another sequence of random polarizations where the transition probability depends on the current state of the Markov chain. Consider a set $A$ of finite measure in $\mathbb{R}^d$. Define the following random sequences:

- $A_n = A^{\sigma_1 \ldots \sigma_n}$
- $X_n = m(A_n/A^*)$
- $\pi_n(x) = X_n - m\left(A_n^{\sigma_0 \ldots \sigma_n}/A^*\right)$
- $g_n(x) \in L^1(\mathbb{R}^d)$.

The probability distribution for $\sigma_n$ is as follows:
\[
P(\sigma_{n+1}(0) \in A | X_n) = \frac{\int_A g_n(x) \pi_n(x) \, dx}{\int_{\mathbb{R}^d} g_n(x) \pi_n(x) \, dx}.
\] (31)
Lemma 2.
\[ \int_{\mathbb{R}^d} g_n(t) \pi_n(t) \, dt = \frac{1}{\sqrt{2}} \int_{A_n/A^*} \int_{A^*/A_n} g_n(u(x,y)) |J_d u(x,y)| \, dx \, dy \]  
(32)

where \( J_d u \) is the \( d \)-dimensional Jacobian of \( u \) with determinant \( |J_d u| \).

Proof. By the coarea formula:
\[ \int_{A_n/A^*} \int_{A^*/A_n} g(u(x,y)) |J_d u(x,y)| \, dx \, dy = \int_{\mathbb{R}^d} g(t) H_d(\{u = t\}) \, dt \]  
(33)

where \( H_d \) is the \( d \)-dimensional Hausdorff measure. The preimage of \( t \) under \( u \) is the graph
\[ \{(x, \sigma_{0,t}(x)) \mid x \in E_{n,t}\} \]
where \( E_{n,t} = A_n/A^* \cap \sigma_{0,t}^{-1}(A^*/A_n) \). The \( d \)-dimensional Hausdorff measure of this graph equals
\[ \int_{E_{n,t}} \sqrt{1 + |\nabla \sigma|^2} \, dx = \sqrt{2} m(E_{n,t}) = \sqrt{2} \pi_n(t) \]
where we have used the fact that \( |\nabla \sigma| = 1 \) since \( \sigma \) is an isometry.

Lemma 3.
\[ |J_d u(x,y)| = \left( \frac{|x|^2 - |y|^2}{|x - y|^2} \right)^{d-1} \sqrt{2} \]  
(34)

Proof. We have
\[ \lim_{\delta \to 0} \delta^{-2d-2} \kappa_d^{-2} \int_{B_{x_0,\delta}} \int_{B_{y_0,\delta}} |J_d u(x,y)| \, dx \, dy = |J_d u(x_0,y_0)| \]  
(35)

Following the proof of the previous lemma, the integral on the left-side also equals
\[ \sqrt{2} \int_{u(B_{x_0,\delta} \times B_{y_0,\delta})} m(E_t) \, dt \]  
(36)

where
\[ E_t = B_{y_0,\delta} \cap \sigma_{0,t}(B_{x_0,\delta}) = B_{y_0,\delta} \cap B_{\sigma_{0,t}(x_0),\delta}. \]

We have
\[ m(E_t) = 4 \delta^d \kappa_{d-1} \phi \left( \frac{|\sigma_{0,t}(x_0) - y_0|}{2 \delta} \right) \]
where
\[ \phi(x) = \int_0^{\arccos(x)} \sin^d(t) \, dt. \]

Making the change of variable \( y = (\sigma_0(x_0) - y_0)/(2\delta) \), the integral in 36 equals

\[ 2^{d+2}\delta^{2d}\kappa_{d-1} \int_{B_1} \left( \frac{|x_0|^2 - |y_0|^2}{|x_0 - y_0 - 2\delta y|^2} \right)^{d-1} \phi(|y|) \, dy \]

and the limit 35 equals

\[ \frac{2^{d+2} \delta^{d-1}}{\kappa^2} \left( \frac{|x_0|^2 - |y_0|^2}{|x_0 - y_0|^2} \right)^{d-1} \int_{B_1} \phi(|y|) \, dy. \] (37)

By Fubini’s theorem:

\[ \int_{B_1} \phi(|y|) \, dy = \kappa_d \int_0^{\pi/2} \sin^d(t) \cos^d(t) \, dt = \frac{\kappa_d \Gamma\left(\frac{d}{2} + \frac{1}{2}\right)^2}{2^d d!}. \] (38)

Using the identity

\[ \frac{\kappa_d}{\kappa_{d-1}} = \frac{\sqrt{\pi} \Gamma\left(\frac{d}{2} + \frac{1}{2} \right)}{\Gamma\left(\frac{d}{2} + 1 \right)} \]

and the duplication identity

\[ \Gamma\left(\frac{d}{2} + 1\right) \Gamma\left(\frac{d}{2} + \frac{1}{2} \right) = 2^{-d} \sqrt{\pi} d! \]

gives the desired result.

\[ \square \]

**Theorem 3.** Suppose \( A \subset B_L \) and set \( g_n(x) = |x|^{-(d-1)} \) for all \( n \). If \( E_n = u(A_n/A^* \times A^*/A_n) \) and

\[ Y_n = \left( \frac{m(E_n)}{k_d} \right)^{1/d} \]

then

\[ \mathbb{E}[X_n - X_{n+1}|X_n] \geq \frac{\sqrt{2}X_n^2}{\text{Per}(B_{2L}) \cdot Y_n}. \] (39)

**Proof.** By Jensen’s inequality:

\[ \mathbb{E}[X_n - X_{n+1}|X_n] = \frac{\int_{E_n} g_n(x) \pi^2_n(x) \, dx}{\int_{E_n} g_n(x) \pi_n(x) \, dx} \geq \frac{\int_{E_n} g_n(x) \pi_n(x) \, dx}{\int_{E_n} g_n(x) \, dx}. \]

By the previous lemmas, the integral \( \int_{E_n} g_n(x) \pi_n(x) \, dx \) equals

\[ \int_{E_n} g_n(x) \, dx = \frac{2^d \delta^{d-1}}{\kappa_d} \int_{B_1} \phi(|y|) \, dy. \]
\[ \sqrt{2} \int_{A^* \setminus A_n} \int_{A_n / A^*} |x - y|^{-(d-1)} \, dx \, dy. \]

The integral above is bounded below by \( \sqrt{2} (2L)^{-(d-1)} X_n^2 \). The Riesz rearrangement inequality gives
\[ \int_{E_n} g_n(x) \, dx \leq \omega_d Y_n \]
where
\[ Y_n = \left( \frac{m(E_n)}{k_d} \right)^{1/d}, \quad E_n = u(A_n / A^* \times A^* / A_n). \]

\[ \square \]

**Lemma 4.** Suppose \( A \) is a compact set and the radius of \( A^* \) is \( r \). If \( d_H(\partial A, \partial A^*) \leq \epsilon \) and \( m(A^*/A) < m(B_{r-\epsilon}) \) then
\[ B_{r-\epsilon} \subset A \subset B_{r+\epsilon}. \]

**Proof.** Suppose \( B_{r-\epsilon} \not\subset A \) then there exists a point \( x \in \overline{A^*/A} \) such that
\[ d_H(x, \partial A^*) = d_H(A^*/A, \partial A^*) > \epsilon. \]
Since \( d_H(\partial A, \partial A^*) \leq \epsilon \), by assumption, then \( x \notin A \). Since \( A \) is a closed set, there exists \( \delta > 0 \) such that \( B_{x,\delta} \subset A^*/A \). But since \( x \) is a point in \( A^*/A \) that is furthest away from the boundary of \( A^* \) then \( x = 0 \). Suppose \( B_{x,\delta_0} \) is the smallest ball intersecting the boundary of \( A \). We must have \( \delta_0 \geq r - \epsilon \) otherwise the distance between the boundary of \( A \) and \( A^* \) would be greater than \( \epsilon \). But this implies that \( m(A^*/A) \geq m(B_{r-\epsilon}) \), a contradiction.

Suppose \( x \) is a point in \( A \) that is furthest away from \( \partial A^* \). If \( x \) is in the interior of \( A \) then there is a ball centered at \( x \) contained in \( A \). But this would imply that there is a point in \( A \) even further away from the boundary of \( A^* \) than \( x \). Hence \( x \) must lie in the boundary of \( A \) and therefore \( A \subset B_{r+\epsilon} \).

For the following theorem, we set
\[ g_n(x) = \begin{cases} |x|^{-(d-1)}, & \text{for } n \text{ odd} \\ |x|^{-(d-1)} \pi_n(x)^{-1}, & \text{for } n \text{ even} \end{cases} \]

**Theorem 4.** Assume \( d = 1 \). If the measure of the boundary of \( A \) is zero then \( \mathbb{E}[X_n] = o(n^{-1}) \). If \( A \) is a compact set with finite Minkowski perimeter then \( \mathbb{E}[X_n] = O(n^{-3/2}) \).

**Proof.** Mimicking the proof of [3] the inequality \( \mathbb{E}[X_n] \) implies that
\[ E[X_n] \leq \frac{1}{\sqrt{2}} \text{Per}(B_{2L})(\sum_{i=1}^{\lfloor n/2 \rfloor} E[Y_{i-1}]^{-1})^{-1}. \quad (40) \]

If the measure of the boundary of \( A \) is assumed to be zero, we can assume that \( A \) is compact. If \( A \) is compact then \( d_H(\partial A_n, \partial A^*) \) converges to zero almost surely. Then, by Lemma 4, we have

\[
m(E_n) = m(A_n/A^* + A^*/A_n) \leq 2d_H(\partial A_n, \partial A^*)
\]

for \( d_H(\partial A_n, \partial A^*) < 2r/3 \) where \( r \) is the radius of \( A^* \). Hence \( Y_n \) goes to zero almost surely and

\[
\lim_{n \to \infty} n(\sum_{i=1}^{n} E[Y_{i-1}]^{-1})^{-1} = \lim_{n \to \infty} E[Y_n] = 0.
\]

Now suppose that \( A \) is a compact set with finite Minkowski perimeter. It is shown in \[3\] that there exists a continuous function \( f \) with support in \( B_{2L} \) such that

\[
d_H(\partial A_n, \partial A^*) = O(\|f^{\sigma_1,\ldots,\sigma_n} - f^*\|_1^{1/3}).
\]

We have

\[
E[m(E_n)] \leq m(B_{2L}) P(d_H(\partial A_n, \partial A^*) \geq 2r/3) + 2E[d_H(\partial A_n, \partial A^*)]
\]

\[
\leq \frac{3}{2r} m(B_{2L}) E[d_H(\partial A_n, \partial A^*)] + 2E[d_H(\partial A_n, \partial A^*)]
\]

\[
= O(E[\|f^{\sigma_1,\ldots,\sigma_n} - f^*\|_1^{1/3}])
\]

\[
= O(E[\|f^{\sigma_1,\ldots,\sigma_n} - f^*\|_1^{1/3}]).
\]

Applying (41) to the level sets of \( f \):

\[
E[Y_n] = O(E[\|f^{\sigma_1,\ldots,\sigma_n} - f^*\|_1^{1/3}]) = O(n^{-1/4})
\]

and

\[
E[X_n] = O((\sum_{i=1}^{n} E[Y_{i-1}]^{-1})^{-1}) = O(n^{-3/4}).
\]

Theorem 4 shows that there exists polarizations \( A_n \) of \( A \) that converge to \( A^* \) at the rate \( O(n^{-3/2}) \), whenever \( A \subset \mathbb{R} \) has finite Minkowski perimeter. The following proposition shows that there exists a class of measurable sets in \( \mathbb{R} \) that admit polarizations converging to their corresponding Schwarz symmetrization at an exponential rate.
Lemma 5. If $d = 1$ and $A = \{f > t\}$ for a non-negative Lipschitz continuous function $f$, with Lipschitz constant $C$ and distribution function that is differentiable at $t$, then there exists a sequence of polarizations $A_n$ of $A$ such that

$$m(A_n \Delta A^*) \leq (1 - C^{-1}(-\mu'_f(t) + C^{-1})^{-1})^n$$

for $n$ sufficiently large.

Proof. Suppose $f$ is Lipschitz continuous with Lipschitz constant $C$. If $f(x_0) > t + \lambda$ and $f(y_0) \leq t - \lambda$, then, by the triangle inequality, $f(x) > t$ and $f(y) \leq t$ whenever $|x - x_0| \leq C^{-1}\lambda$ and $|y - y_0| \leq C^{-1}\lambda$. If we add the conditions $|x_0| > r_f(t) + C^{-1}\lambda$ and $|y_0| \leq r_f(t) - C^{-1}\lambda$, then $|x| > r_f(t)$, $|y| \leq r_f(t)$, $x \in A/A^*$, $y \in A^*/A$, and

$$m(A^{\sigma_{x_0,y_0}} \Delta A^*) \leq m(A \Delta A^*) - 4C^{-1}\lambda.$$

Note that

$$A/A^* \cap \{f \leq t + \lambda\} = \{x: t < f(x) \leq t + \lambda, |x| > r_f(t)\}$$

and

$$A^*/A \cap \{f > t - \lambda\} = \{y: t - \lambda < f(y) \leq t, |y| \leq r_f(t)\}.$$

Since the distribution function of $f$ is differentiable at $t$:

$$m(A/A^* \cap \{f \leq t + \lambda\}) \leq \mu_f(t) - \mu_f(t + \lambda) = -\mu'_f(t)\lambda(1 + o(1))$$

and

$$m(A^*/A \cap \{f > t - \lambda\}) \leq \mu_f(t - \lambda) - \mu_f(t) = -\mu'_f(t)\lambda(1 + o(1))$$

as $\lambda \to 0$. Similarly, we have

$$m(A/A^* \cap B_{r_f(t)+C^{-1}\lambda}) \leq m(B_{r_f(t)+C^{-1}\lambda}) - m(B_{r_f(t)}) = 2C^{-1}\lambda$$

and

$$m((A^*/A)/B_{r_f(t)-C^{-1}\lambda}) \leq m(B_{r_f(t)}) - m(B_{r_f(t)-C^{-1}\lambda}) = 2C^{-1}\lambda.$$

Setting

$$\lambda = \frac{1}{4}(-\mu'_f(t) + C^{-1})^{-1}m(A \Delta A^*)$$

we see from the inequalities above that there exists points $x_0$ and $y_0$ such that

$$m(A^{\sigma_{x_0,y_0}} \Delta A^*) \leq m(A \Delta A^*)(1 - C^{-1}(-\mu'_f(t) + C^{-1})^{-1})$$

provided that $m(A \Delta A^*)$ is less than $\delta_f$ where $\delta_f$ depends only on $\mu_f$. The
proof of the proposition is completed by noting that if \( f \) is Lipschitz continuous with Lipschitz constant \( C \) then any polarization of \( f \) is also Lipschitz continuous with Lipschitz constant less than or equal to \( C \).

\[ m(A_n \triangle A^*) \leq \left( \frac{\text{Per}(A)}{1 + \text{Per}(A)} \right)^n \]

for \( n \) sufficiently large.

**Proof.** We can express the interior of \( A \) as the disjoint union of open intervals:

\[ A^\circ = \bigcup_{k=-\infty}^{\infty} (a_k, b_k). \]

Then we define the Lipschitz function \( f \):

\[
f(x) = \frac{L}{2} + \frac{a_k + b_k}{2} - |x - \frac{a_k + b_k}{2}|, \quad x \in [\frac{b_{k-1} + a_k}{2}, \frac{b_k + a_{k+1}}{2}].
\]

The Lipschitz constant is 1 and \( \{f > L/2\} = A^\circ \). Furthermore, we have \(-\mu_f(L/2) = \text{Per}(A)\). Applying the previous lemma completes the proof.

**Theorem 5.** For every compact set \( A \) in \( B_L \) with finite Minkowski perimeter there exists a sequence of polarizations \( A_n \) of \( A \) such that

\[ m(A_n \triangle A^*) \leq \left( \frac{\text{Per}(A)}{1 + \text{Per}(A)} \right)^n \]

for \( n \) sufficiently large.

**Proof.** We can express the interior of \( A \) as the disjoint union of open intervals:

\[ A^\circ = \bigcup_{k=-\infty}^{\infty} (a_k, b_k). \]

Then we define the Lipschitz function \( f \):

\[
f(x) = \frac{L}{2} + \frac{a_k + b_k}{2} - |x - \frac{a_k + b_k}{2}|, \quad x \in [\frac{b_{k-1} + a_k}{2}, \frac{b_k + a_{k+1}}{2}].
\]

The Lipschitz constant is 1 and \( \{f > L/2\} = A^\circ \). Furthermore, we have \(-\mu_f(L/2) = \text{Per}(A)\). Applying the previous lemma completes the proof.

For \( \lambda_n = (L2^{d+1}\omega_d d_H(\partial A_n, \partial A^*)^{(d-1)})^{1/2} \).

**Theorem 6.** Assume \( d > 1 \). If the measure of the boundary of \( A \) is zero then \( E[X_n] = o(n^{-1}) \). If \( A \) is a compact set with finite Minkowski perimeter then \( E[X_n] = O(n^{-(1+\omega(1)\omega_d)}) \).

**Proof.** If \( m(F_n) > m(G_n) \) then
\[ E[X_n - X_{n+1}\mid X_n] \geq \frac{\sqrt{2} \int \int \mathbb{1}_{F_n} |x - y|^{-(d-1)} \, dx \, dy}{(2L) \omega_d} \geq \frac{(2L)^{-(d-1)} \lambda_n^{d-1} m(F_n)}{2^{d+\frac{d}{2}} \omega_d \omega_H (\partial A_n, \partial A^*)^{d-1}} \geq \frac{\lambda_n^{d-1} X_n^2}{2^{d+\frac{d}{2}} \omega_d \omega_H (\partial A_n, \partial A^*)^{d-1}}. \] (43)

And similarly if \( m(G_n) \geq m(F_n) \) then

\[ E[X_n - X_{n+1}\mid X_n] \geq \frac{\sqrt{2} \int \int \mathbb{1}_{G_n} |x - y|^{-(d-1)} \, dx \, dy}{\lambda_n} \geq \frac{X_n^2}{2^{d+\frac{d}{2}} L^{d-1} \lambda_n}. \] (46)

In particular, if we set

\[ \lambda_n = \left( L 2^{d+1} \omega_d \omega_H (\partial A_n, \partial A^*)^{(d-1)} \right)^{\frac{1}{d}} \] (48)

then

\[ E[X_n - X_{n+1}\mid X_n] \geq \frac{X_n^2}{2^{d+\frac{d}{2}} L^{d-1} \lambda_n} = \frac{X_n^2}{C_d \omega_H (\partial A_n, \partial A^*)^{(d-1)/d}} \] (49)

with

\[ C_d = 2^{\frac{1}{d} + \frac{d}{2}} L^{d-1 + \frac{d}{2}} \omega_d^{\frac{1}{d}}. \]

Following previous proofs, we know that this inequality immediately yields the following bound:

\[ E[X_n] \leq C_d \left( \sum_{i=1}^{\lceil n/2 \rceil} E[d_H (\partial A_i, \partial A_i^*)^{(d-1)/d}] \right)^{-1}. \] (50)

If the measure of the boundary of \( A \) is assumed to be zero, we can assume that \( A \) is compact. If \( A \) is compact then \( d_H (\partial A_n, \partial A^*) \) converges to zero almost surely and therefore \( E[X_n] = o(n^{-1}) \).

If \( A \) has finite Minkowski perimeter then it is shown in [3] that there exists a continuous function \( f \) with support in \( B_{2L} \) such that

\[ d_H (\partial A_n, \partial A^*) = O(\|f^{\sigma_1 \ldots \sigma_n} - f^*\|_1^{\frac{1}{n}}). \]

Applying (1) to \( f \), we have
\[ E[H(\partial A_n, \partial A^*)^{(d-1)/d}] = E[O(||f^{\sigma_1 \ldots \sigma_n} - f^*||_1)]^{d-1} = O(\alpha^{(d-1)/d}) \]

and therefore

\[ E[X_n] = O(n^{-\left(1 + \frac{d-1}{d\alpha}\right)}) \]