Abstract
We consider a linear Boltzmann equation that arises in a model for quantum friction. It describes a particle that is slowed down by the emission of bosons. We study the stochastic process generated by this Boltzmann equation and we show convergence of its spatial trajectory to a multiple of Brownian motion with exponential scaling. The asymptotic position of the particle is finite in mean, even though its absolute value is typically infinite. This is contrasted to an approximation that neglects the influence of fluctuations, where the mean asymptotic position is infinite.

1 Introduction
1.1 Motivation
One of the interesting themes in non-equilibrium physics is the role of fluctuations. For example, ratchets (also known as molecular motors) function because they can convert fluctuations into a directed motion, something that would be impossible in an equilibrium (detailed balance) process. The model described in this paper offers another very simple illustration of the fundamental role of fluctuations. It describes a quantum-mechanical particle (hereafter called “tracer particle”) interacting with the atoms in an ideal Bose gas at zero temperature exhibiting Bose–Einstein condensation. The effective dynamics of the tracer particle is described by a linear Boltzmann equation, i.e., by a stochastic process. Lately, the analysis of such transport equations has received considerable attention; see [13, 1, 15], and [6] for a general introduction. They can appear as linearizations of nonlinear Boltzmann equations close to equilibrium, or as models of a single entity interacting with a medium in thermal equilibrium, as in our example. Below, we describe the stochastic process corresponding to our linear Boltzmann equation and we discuss some of its properties, postponing a sketch of its derivation from a physical model to Section 1.4. (Actually, the physics origin of the equation studied in this note has no importance for our analysis.)
### 1.2 Jump process

Let \((X_t, K_t) \in \mathbb{R}^3 \times \mathbb{R}^3\) stand for the position/momentum of the tracer particle at time \(t\). The momentum \(K_t\) is a Markov jump process defined by a jump kernel \(q(\cdot, \cdot)\) as follows: For each momentum \(k \in \mathbb{R}^3\), \(q(k, \cdot)\) is a finite measure on \(\mathbb{R}^3\); its total weight, \(\Sigma(k) := q(k, \mathbb{R}^3)\), is the rate at which a jump in momentum space occurs, starting from momentum \(k\), and \(q(k, B)/\Sigma(k)\) is the probability for such a jump to land in the set \(B \subset \mathbb{R}^3\). Formally, the jump kernel \(q\) is given by

\[
q(k, dk') = w \delta(\varepsilon(k') - \varepsilon(k) - \omega(k' - k)) \, dk',
\]

where the dispersion laws

\[
\varepsilon(k) = \frac{1}{2m} |k|^2, \quad \omega(q) = \frac{1}{2M} |q|^2, \quad (k, q \in \mathbb{R}^3),
\]

correspond to the kinetic energies, as functions of the momenta, of the non-relativistic tracer particle and the Bose atoms, respectively, \(m\) and \(M\) are their masses, and \(w\) is the effective coupling strength of the interaction between the tracer particle and an atom in the Bose gas. In what follows, we will choose units such that \(w = 1\). Physically, the delta function in (1.1) expresses conservation of energy in a momentum-preserving process where the tracer particle excites an atom in the Bose–Einstein condensate to a state of momentum \(k - k'\) and energy \(\omega(k - k')\). (This can be viewed as the emission of a sound wave into the Bose gas.)

The position \(X_t\) evolves according to

\[
\frac{dX_t}{dt} = \frac{K_t}{m}.
\]

Note that \(X_t\) is a random variable, because \(K_t\) is random. Obviously, since, at zero temperature, the particle can only emit (but not absorb) sound waves, thereby lowering its energy, we expect that \(K_t \to 0\), i.e., that the particle experiences friction. Our main interest is in the behavior of the position \(X_t\), including its asymptotics as \(t \to \infty\). The following scaling law (cf. (3.4) below) is crucial for our result:

\[
\lambda q(\lambda k, \lambda B) = q(k, B), \quad \text{for any } \lambda > 0.
\]

When combined with rotation invariance of the kernel, it leads to the formal relation

\[
\mathbb{E}(K_{t+dt} - K_t | K_t) = -\eta |K_t| K_t dt,
\]

where \(\mathbb{E}(\cdot | K_t)\) is the expectation of the momentum process conditioned on \(K_t\), and \(\eta > 0\) is some friction coefficient depending on \(m, M\).

### 1.3 Mean field approximation vs. fluctuations

A first guess concerning the behavior of our process is that one may neglect random fluctuations. This amounts to omitting the expectation \(\mathbb{E}\) in (1.5) and pretending that \(K_{t+dt} - K_t\) is nonrandom and hence that \(K_t\) satisfies the differential equation

\[
\frac{dK_t}{dt} = -\eta |K_t| K_t, \quad \text{with } K_{t=0} = K_0.
\]
whose solution is given by

\[ K_t = (\eta t + 1/|K_0|)^{-1} \frac{K_0}{|K_0|}. \quad (1.7) \]

As a consequence, we find that the position \( X_t \) diverges logarithmically

\[ |X_t| \sim \text{const} \log t, \quad \text{as } t \to \infty. \quad (1.8) \]

However, after including fluctuations, the behavior of \( X_t \) turns out to be different: \( \mathbb{E}(X_t) \) remains finite, while typical trajectories, \( (X_t)_{0 \leq t < \infty} \), escape to infinity in random directions, as \( t \to \infty \). In fact, the scaling relation (1.4) implies that the typical time elapsing before the momentum jumps again, starting from a momentum \( K_t \), is of order \( |K_t|^{-1} \), hence the typical distance the tracer particle travels between two successive momentum jumps remains bounded, as \( K_t \to 0 \). Rescaling time, we can alternatively think of the path \( (X_t)_{0 \leq t \leq T} \) as the path of a particle traveling with unit speed and randomly changing its direction of motion at a uniform rate, which is stopped at a time of order \( \log T \). In particular, after appropriate rescaling, \( X_t \) can be expected to converge to a multiple of Brownian motion. This is, in fact, our main result, which is stated precisely in Theorem 2.1 below.

### 1.4 Derivation from first principles

The physical system motivating our study of the particular model described above consists of a non-relativistic, quantum tracer particle of mass \( m \) interacting with non-relativistic atoms of mass \( M \) in an ideal Bose gas at zero temperature exhibiting Bose–Einstein condensation. The Bose–Einstein condensate serves as a reservoir of atoms that, through soft collisions, the tracer particle can lift to traveling wave states of non-vanishing momentum and positive kinetic energy. This corresponds to an emission of Čerenkov radiation of sound waves into the Bose gas — similarly to a well known phenomenon of light emission observed when a charged particle moves through an optically dense medium at a speed larger than the speed of light in the medium; see, e.g., [12]. One may thus expect that the tracer particle experiences friction and slows down until, asymptotically, it comes to rest.

The dynamics of the physical system described above is determined by the many-body Schrödinger Hamiltonian

\[ H^{(N)} = -\frac{\Delta X}{2m} + \sum_{n=1}^{N} \left\{ -\frac{\Delta x_n}{2M} + gW(x_n - X) \right\}, \quad (1.9) \]

where \( \Delta \) denotes the Laplacian (with suitable boundary conditions imposed at the boundary of a cube in \( \mathbb{R}^3 \) to which the system is confined), \( gW(x_n - X) \) is the two-body interaction potential when the tracer particle is at position \( X \) and the \( n^{th} \) atom of the Bose gas at position \( x_n, n = 1, \ldots, N \), and \( g \) is a coupling constant. We assume that \( W \) is bounded and of rapid decrease at \( \infty \), and that the density of the gas is positive and kept constant, as the thermodynamic limit is approached. The operator \( H^{(N)} \) introduced in (1.9) is self-adjoint and bounded below on the usual \( L^2 \)—space of orbital \((N + 1)\)—particle wave functions and generates the time evolution of the system, for all \( N < \infty \).

We are interested in studying the “effective time evolution” of the tracer particle when the Bose gas is initially in a state of thermal equilibrium at some temperature \( \beta^{-1} \geq 0 \). This evolution
is obtained by taking the expectation over the degrees of freedom of the Bose gas corresponding
to an equilibrium state, conditioned on the state of the tracer particle, and taking \( N \to \infty \) to
eliminate finite-size effects. In \([9]\), Erdős has shown that, for positive temperatures, \( \beta^{-1} > 0 \),
and after rescaling \( x \mapsto g^{-2}x \), \( t \mapsto g^{-2}t \) (“weak kinetic scaling”), the limit \( g \to 0 \) of the effective
dynamics of the tracer particle exists. Moreover, in this so-called kinetic limit, the evolution of the
Wigner distribution of the state of the tracer particle is given by a solution of the linear Boltzmann
equation whose collision operator is given by the kernel

\[
q(k, dk') = \lvert \hat{W}(k - k') \rvert^2 \left[ (N(k - k') + 1)\delta(\varepsilon(k) - \varepsilon(k') - \omega(k - k'))dk' + N(k - k')\delta(\varepsilon(k') - \varepsilon(k) - \omega(k' - k'))dk' \right].
\]

(1.10)

Here \( \hat{W} \) is the Fourier transform of the potential \( W \), and \( N(q) \) is given by

\[
N(q) = \frac{e^{-\beta \omega(q)}}{1 - e^{-\beta \omega(q)}} = n(\beta \omega(q)), \quad n(x) = \frac{1}{e^x - 1},
\]

(1.11)

with \( n(x) \) the Bose–Einstein distribution function. The results in \([9]\) only hold for positive temperatures, \( \beta^{-1} > 0 \). However, one may formally pass to zero temperature and finds that the collision
kernel then reduces to

\[
q(k, dk') = \lvert \hat{W}(k - k') \rvert^2 \delta(\varepsilon(k) - \varepsilon(k') - \omega(k - k'))dk'.
\]

(1.12)

Furthermore, since we are interested in phenomena at small momenta, only the behavior of \( \hat{W} \) near
0 matters. As \( W \) has been assumed to have rapid decay at \( \infty \), \( \hat{W}(k) \) is smooth near \( k = 0 \), and, to
simplify matters, it is reasonable to replace the function \( \hat{W} \) by the constant \( w = \lvert \hat{W}(0) \rvert^2 \), which
is henceforth chosen to be equal to unity. This then yields the jump kernel introduced in (1.1).
(The replacement of \( \hat{W} \) by a constant, \( \hat{W}(0) \), is made in order to avoid uninteresting technical complications. We expect, however, that all our results hold for an arbitrary \( \hat{W} \) that is continuous
at 0. )

As our discussion may have made plausible, a rigorous understanding of friction in the realm
of unitary quantum dynamics or classical Hamiltonian mechanics still remains an excellent mathematical challenge. Some results in this direction may be found in \([1, 5, 10]\).

2 Result

To begin with, we carefully introduce the linear Boltzmann equation and the associated stochastic
process that have been described in Section 1.2.

On the space of functions \( f \in C^1(\mathbb{R}_x^3 \times \mathbb{R}_k^3) \) (we will sometimes write \( \mathbb{R}_x^3 \), instead of \( \mathbb{R}^3 \), to emphasize that the variable in \( \mathbb{R}_x^3 = \mathbb{R}^3 \) is denoted by \( x \) ), we define a linear operator \( L \) by setting

\[
L f = \frac{k}{m} \cdot \nabla_x f + M f,
\]

(2.1)

where \( M \) is the linear collision operator

\[
M f(k) = \int_{\mathbb{R}_k^3} (f(k') - f(k))\delta(\varepsilon(k) - \varepsilon(k') - \omega(k - k')) \, dk'.
\]

(2.2)
The δ-function in the integrand enforces conservation of energy. It is the composition of an ordinary δ-function with the function

\[ F(k') = F_k(k') = \varepsilon(k) - \varepsilon(k') - \omega(k - k') \quad (2.3) \]

and is defined, more precisely, by

\[
\int f(k')\delta(F(k')) \, dk' = \int_{F^{-1}(0)} f(k') |\nabla F(k')|^{-1} H(\,dk'),
\]

(2.4)

where \( H \) is the unnormalized surface (Hausdorff) measure on \( F^{-1}(0) \).

It is a standard fact that the operator \( L \) introduced in (2.1) generates a Markovian semigroup. We adopt the usual probabilistic point of view in which the semigroup is thought to act on observables ("Heisenberg picture"), so that the expectation of a test function \( f: \mathbb{R}^d_x \times \mathbb{R}^d_k \to \mathbb{R} \) at time \( t \) is given by \( \mathbb{E}(f_t(X_0, K_0)) \), where \( f_t \) is the unique solution to the forward equation

\[
f_0 = f, \quad \partial_t f_t = Lf_t, \quad (t > 0), \quad (2.5)
\]

and \( \mathbb{E}(\cdot) \) denotes the expectation corresponding to the random variable \((X_0, K_0)\).

In the following, \( S^2 \subset \mathbb{R}^3 \) denotes the unit sphere, and \( U \) is a uniform random variable on \( S^2 \). Let \((X_t, K_t)\) be a realization of the Markov process with generator (2.1), started from a deterministic point \((X_0, K_0) = (x, k)\) (i.e., \( \mathbb{E}(f(X_t, K_t)) = f_t(x, k) \)), and let \((B_s)\) be standard Brownian motion on \( \mathbb{R}^3_x \). The main result in this note is the following theorem.

**Theorem 2.1.** There are explicit positive constants \( \theta < 1 \) and \( \sigma \) such that, for an arbitrary choice of initial conditions \((X_0, K_0) = (x, k) \in \mathbb{R}^3_x \times \mathbb{R}^3_k \),

\[
\left( \frac{1}{\sqrt{n}} X_{\theta^{-n}s} \right)_s \to (\sigma B)_s, \quad \text{as } n \to \infty,
\]

(2.6)

in distribution, in the topology of uniform convergence on bounded intervals of the variable \( s \).

This result shows in particular that, no matter what the initial conditions are, the position of the tracer particle is not confined to any bounded region in \( \mathbb{R}^3 \), but that the distance of its position from the initial position only grows very slowly. Its motion is diffusive on an exponential time scale. The mean position of the particle remains, however, finite, as time tends to \( \infty \), because the momentum of the particle rapidly loses memory of its initial values. More precisely, from the proof of Theorem 2.1 we will infer the following result.

**Theorem 2.2.**

\[
|K_t| = |K_0|t^{-1+o(1)} \quad \text{almost surely},
\]

(2.7)

and

\[
\lim_{t \to \infty} \mathbb{E}|X_t| = \infty \quad \text{but} \quad \sup_{t \geq 0} \mathbb{E}|X_t| < \infty.
\]

(2.8)

The proofs of these theorems are probabilistic and rely on a decomposition of the continuous-time Markov process \((X_t, K_t)\) into its discrete-time skeleton process and on a process of jump times. For the jump times, we prove a law of large numbers on an exponential scale, and for the
skeleton process we derive a functional central limit theorem. The theorems stated above follow from these two results by standard arguments.

The constants $\theta \in (0, 1)$ and $\sigma > 0$ in Theorem 2.1 are explicit functions of $m$ and $M$ and are given by

$$\log \theta = \frac{1}{2} \left( \frac{(1 - 2a)^2 \log |1 - 2a|}{2(a - 1)a} - 1 \right), \quad \sigma^2 = \frac{2}{3(1 - b)} \frac{(m + M)^4}{16\pi^2 m^4 M^4},$$

(2.9)

where

$$a := \frac{m}{m + M}, \quad b = \frac{1}{2} \frac{|1 - 2a| (2a^2 + a - 1) - 3a + 1}{3(a - 1)a^2}. \quad (2.10)$$

In our proofs, the constants $b$ and $\theta$ arise from the equations

$$by = \frac{1}{4\pi} \int_{S^2} ay + (1 - a)z \ H(dz), \quad (2.11)$$

$$\log \theta = \frac{1}{4\pi} \int_{S^2} \log(|ay + (1 - a)z|) \ H(dz), \quad (2.12)$$

for any $y \in S^2$, as can be verified by using polar coordinates on $S^2$.

3 Proof of Theorem 2.1

3.1 Stochastic decomposition

Recall that $a = m/(m + M)$. A straightforward calculation shows that $F_k(k') = 0$, with $F$ given by (2.3), is equivalent to

$$k' \in ak + (1 - a)|k|S^2 \subset \mathbb{R}^3; \quad (3.1)$$

see Figure 1. This observation and the equation

$$\nabla_{k'} F_k(k') = -\nabla \varepsilon(k') + \nabla \omega(k - k') = \frac{1}{M} \left( k - \frac{k'}{a} \right)$$

(3.2)

Figure 1: The surfaces $k' \in ak + (1 - a)|k|S^2$ for $a < \frac{1}{2}$ (left) and $a > \frac{1}{2}$ (right).
imply that, for $F_k(k') = 0$,
\[
|\nabla_{k'} F_k(k')| = \frac{1 - a}{aM} |k| = \frac{|k|}{m}.
\] (3.3)
Let \(\{u(k, dk')\}\) be the family of unnormalized surface (Hausdorff) measures on the surfaces given by \(ak + (1 - a)|k|S^2\), for \(k \in \mathbb{R}^3\). They are given by affine transformations of the uniform measure on the unit sphere. Expressed in terms of these measures, the kernel of the collision operator (2.2) is given by
\[
q(k, dk') = \frac{m}{|k|} u(k, dk').
\] (3.4)
For a given \(k\), the scattering rate \(\Sigma(k)\) is the total weight of \(q(k, \cdot)\):
\[
\Sigma(k) = q(k, \mathbb{R}^3) = \frac{m}{|k|} u(k, \mathbb{R}^3) = m(1 - a)^2 |S^2||k| = 4\pi \frac{M^2 m}{(m + M)^2} |k|.
\] (3.5)
The normalized transition kernel,
\[
p(k, dk') = \Sigma(k)^{-1} q(k, dk'),
\] (3.6)
is simply the uniform probability measure on the sphere \(ak + (1 - a)|k|S^2\).

Let \((\Omega, \Sigma, \mathbb{P})\) be a probability space \(\Omega\) with sigma-algebra \(\Sigma\) and probability measure \(\mathbb{P}\) on which the following random variables are defined: \((K_n)\) is a Markov chain on \(\mathbb{R}^3\) with \(K_0 = k\) and transition probabilities given in (3.6), and \((\lambda_n)\) is a sequence of independent random variables with exponential probability distribution with mean 1. We also let \(X_0\) be the deterministic random variable \(X_0 = x\). Let \(\Sigma_n\) be the sigma-algebra generated by \((\lambda_j, K_j)_{j \leq n}\). Let
\[
T_n = \sum_{j=0}^{n-1} \frac{\lambda_j}{\Sigma(K_j)},
\] (3.7)
and let \(N_t\) be its right-continuous inverse. Then \(T_n\) is the time at which the \(n\)th jump occurs, and \(N_t\) is the number of jumps up to time \(< t\). We set
\[
K_t = K_{N_t}, \quad X_t = X_0 + \int_0^t \frac{K_t}{m} dt,
\] (3.8)
where, with some abuse of notation, \(K\) is an abbreviation for both the original continuous-time process \(K_t\) and the discrete time process \(K_n\); (when using a subscript \(t \in \mathbb{R}_+\) we mean the former process, while, for a subscript \(n \in \mathbb{N}\), the latter process is meant).

**Proposition 3.1.** \((X_t, K_t)_{t \geq 0}\) is a strong Markov process. Let \((\Gamma_t)\) be its semigroup defined by \((\Gamma_t f)(x, k) = \mathbb{E}^{x,k}(f(X_t, K_t))\), where \(\mathbb{E}^{x,k}\) is the expectation with \((X_0, K_0) = (x, k)\). Then the generator of \((\Gamma_t)_{t \geq 0}\) is the operator \(L\) given by (2.1).

The proof is a standard argument, presented for completeness in Appendix [3]. We study the skeleton process \((K_n)\) in terms of its polar decomposition
\[
Y_n = K_n/|K_n|, \quad R_n = |K_n|.
\] (3.9)
The main observation is that \((Y_n)\) is a Markov chain on \(S^2\), and
\[
X_t = X_0 + \frac{1}{m} \sum_{j=0}^{N_t-1} (T_{j+1} - T_j) K_j + \frac{1}{m} (t - T_{N_t}) K_{N_t},
\]
\[
= X_0 + (m + M)^2 \frac{2}{4\pi m^2 M^2} \sum_{j=0}^{N_t-1} \lambda_j Y_j + \frac{1}{m} (t - T_{N_t}) K_{N_t}.
\]
(3.10)

Theorem 2.1 is a consequence of the following two propositions for the sum over \(j\) of \(\lambda_j Y_j\) and for the number of jumps \(N_t\). In the statements of the propositions, \(\theta, b \in (0, 1)\) are the constants defined in (2.9)–(2.10), and \(D([0, s_0], \mathbb{R}^3)\) is the Skorohod space of right-continuous functions with left-limits (càdlàg functions) \([0, s_0] \to \mathbb{R}^3\), endowed with the Skorohod topology; see, e.g., [3].

Proposition 3.2. For \(s \in [0, s_0]\), where \(s_0 > 0\) is arbitrary,
\[
\left( \frac{1}{\sqrt{n}} \sum_{j=0}^{[ns]} \lambda_j Y_j \right)_s \overset{D}{\to} \left( \sqrt{\frac{2}{3(1-b)}} B_s \right)_s, \quad \text{as } n \to \infty,
\]
(3.11)
in distribution in the Skorohod space \(D([0, s_0], \mathbb{R}^3)\).

Proposition 3.3. Uniformly for \(s \in [0, s_0]\), where \(s_0 > 0\) is arbitrary,
\[
N_{\theta^{\alpha_{ns}}} / n \to s, \quad \text{as } n \to \infty, \quad \text{almost surely.}
\]
(3.12)

The proofs of the propositions are deferred to Sections 3.2–3.3. The left-hand side of (3.11) is an additive functional of an exponentially mixing Markov process, and there are many approaches to proving such a functional central limit theorem; we use the martingale method. Given these two propositions, Theorem 2.1 is proved as follows.

Proof of Theorem 2.1. Fix \(s_0 > 0\). Since the topology of uniform convergence on \(C([0, s_0], \mathbb{R}^3)\) coindices with the Skorohod topology restricted to continuous functions, and since \(s \mapsto X_{\theta^{\alpha_{ns}}}\) is continuous almost surely, which is evident from (3.8), it suffices to show that \(\frac{1}{n} X_{\theta^{\alpha_{ns}}} \to \sigma B_s\) as processes in \(D([0, s_0], \mathbb{R}^3)\). Let
\[
Z_n(s) = \frac{1}{\sqrt{n}} \left( \frac{m + M)^2}{4\pi m^2 M^2} \sum_{j=0}^{[ns]} \lambda_j Y_j \right)_s, \quad \Phi_n(s) = \frac{N_{\theta^{\alpha_{ns}}} - 1}{n}.
\]
(3.13)
We claim that \(Z_n \circ \Phi_n \to \sigma B\) in distribution in \(D([0, s_0], \mathbb{R}^3)\). This claim can be shown following [3] Section 17: Let \(D_0\) be the subspace of \(D([0, s_0], [0, \infty))\) of nondecreasing functions (endowed with the relative topology). Let \(\Phi(s) = s\). Then \(\Phi_n, \Phi \in D_0\), and Proposition 3.3 with the fact that the Skorohod topology is weaker than the uniform topology implies \(\Phi_n \to \Phi\) almost surely as elements of \(D_0\). By [3] Theorem 4.4, with \(\Phi_n \to \Phi\) a.s. in \(D_0\) and Proposition 3.2, it follows that, in distribution in \(D \times D_0\),
\[
(Z_n, \Phi_n) \to (\sigma B, \Phi).
\]
(3.14)
Since \(B\) and \(\Phi\) are continuous, this implies that \(Z_n \circ \Phi_n \to \sigma B\); see [3] Section 17.
It remains to argue that the last term in (3.10) is negligible. By [3, Theorem 4.1], it suffices to show that
\[
\frac{1}{\sqrt{n}} \sup_{s \in [0, \theta]} \left| (s - T_{N_\theta - n}) K_{N_\theta - n} \right| \leq \frac{1}{\sqrt{n}} \max_{j \leq N_\theta - n} \lambda_j \to 0 \quad (\text{in probability}) \tag{3.15}
\]
Since \( E(\max\{\lambda_1, \ldots, \lambda_k\}) = \sum_{i=1}^{k} \frac{1}{i} = O(\log k) \) and by Proposition 3.3
\[
P\{ \max_{j \leq N_\theta - n} \lambda_j \geq \epsilon \sqrt{n} \} \leq P\{ \max \lambda_j \geq \epsilon \sqrt{n} \} + P\{ N_\theta - n \geq n^2 \} = o(1), \tag{3.16}
\]
showing (3.15), as claimed.

It remains to prove Propositions 3.2–3.3 and to prove Theorem 2.2, a task addressed in the rest of the paper.

### 3.2 Proof of Proposition 3.3

Before giving the proof of Proposition 3.3, we briefly sketch its main idea. It will be shown that \( \frac{1}{n} \log \Sigma(R_n) \) concentrates at \( \log \theta \), as \( n \to \infty \), so that \( \Sigma(R_n) \approx \theta^n \). Since \( \theta^{-n} \) grows exponentially, as \( n \to \infty \), the last term in
\[
\sum_{j=0}^{n} \lambda_j \Sigma(R_j)^{-1} \tag{3.17}
\]
is dominant. In particular, a lower bound on this sum can be obtained by dropping all terms except for the last one. On the other hand, using that \( \Sigma(R_j) \) is decreasing, an upper bound is obtained by replacing \( \Sigma(R_j)^{-1} \) by \( \Sigma(R_n)^{-1} \) for \( j = 0, \ldots, n-1 \). Thus, on the exponential scale, the above sum can be approximated well by \( \theta^{-n} \). (Very crude estimates suffice here, due to the exponential scaling.) The claim can be deduced from such considerations, recalling that, by definition,
\[
N_{\theta^{-s}} = \inf \left\{ n \geq 0 : \sum_{j=0}^{n} \lambda_j \Sigma(R_j)^{-1} \geq \theta^{-s} \right\}. \tag{3.18}
\]

We now enrich the above sketch to a proof. Since \( p(k, \cdot) \) is the uniform probability measure on \( ak + (1 - a)k|S^2 \), it follows that \( K_{j+1} = aK_j + (1 - a)|K_j|U_j \), where \( U_j \) is a uniform random variable on \( S^2 \). Thus \( R_{j+1} = R_j aY_j + (1 - a)U_j \). In distribution, \( aY_j + (1 - a)U_j \) is equal to \( |ay + (1 - a)U_j| \), with any \( y \in S^2 \), and therefore \( D_j = \log R_j - \log R_{j-1} \) are i.i.d. random variables with distribution \( \gamma_s u \) where, for an arbitrarily fixed \( y \in S^2 \),
\[
\gamma(z) = \log |ay + (1 - a)z|, \quad (z \in S^2), \tag{3.19}
\]
u is the uniform probability measure on \( S^2 \), and \( \gamma_s u \) is its push-forward by \( \gamma \). Let \( \Lambda \) be the logarithmic moment generating function of \( D_j \), i.e.,
\[
\Lambda(\xi) = \log E(e^{\xi D_j}) = \int_{S^2} |ay + (1 - a)z|^\xi u(dz). \tag{3.20}
\]
As is evident from Figure 11 \( |ay + (1 - a)z| \) is bounded above and below for \( a \neq \frac{1}{2} \), and thus then \( \Lambda(\xi) < \infty \), for all \( \xi \in \mathbb{R} \). For \( a = \frac{1}{2} \), using polar coordinates, it can be seen that \( \Lambda(\xi) < \infty \), for
$\xi > -1$. In both cases $\Lambda$ is convex on $\mathbb{R}$, strictly convex in a neighborhood of 0, and differentiable in a neighborhood of 0. By general properties of generating functions, we deduce that the Legendre transform of $\Lambda$,

$$I(x) = \sup_{\xi \in \mathbb{R}} (x\xi - \Lambda(\xi)), \quad (3.21)$$

satisfies $I(\log \theta) = \min_{x \in \mathbb{R}} I(x) = 0$ with $\log \theta = E(D_j) = \Lambda'(0) < \infty$ and $I$ is finite, continuous and strictly convex in a neighborhood of $\log \theta$. In particular, $I(x)$ is positive (or $+\infty$) if $x \neq \log \theta$.

By Cramér’s theorem \[7\], the sum of i.i.d. random variables

$$R_1. \text{ In both cases } \Lambda \text{ is convex on } \mathbb{R}, \text{ strictly convex in a neighborhood of } 0, \text{ and differentiable in a neighborhood of } 0. \text{ By general properties of generating functions, we deduce that the Legendre transform of } \Lambda,$$

$$I(x) = \sup_{\xi \in \mathbb{R}} (x\xi - \Lambda(\xi)), \quad (3.21)$$

satisfies $I(\log \theta) = \min_{x \in \mathbb{R}} I(x) = 0$ with $\log \theta = E(D_j) = \Lambda'(0) < \infty$ and $I$ is finite, continuous and strictly convex in a neighborhood of $\log \theta$. In particular, $I(x)$ is positive (or $+\infty$) if $x \neq \log \theta$.

By Cramér’s theorem \[7\], the sum of i.i.d. random variables

\begin{equation}
\frac{1}{n} \sum_{j=1}^{n} D_j = \frac{1}{n} (\log R_n - \log R_0) \quad (3.22)
\end{equation}

satisfies the large deviation estimates

\begin{align}
\limsup_{n \to \infty} \frac{1}{n} \log P\{\log(R_n) - \log(R_0) \geq xn\} &\leq -I(x) \quad (x \geq \log \theta), \quad (3.23) \\
\limsup_{n \to \infty} \frac{1}{n} \log P\{\log(R_n) - \log(R_0) \leq xn\} &\leq -I(x) \quad (x \leq \log \theta). \quad (3.24)
\end{align}

**Proof of Proposition 3.3.** It suffices to show that, for $x \neq 1$, there exists some $J(x) > 0$ such that, as $s \to \infty$,

\begin{align}
P\{N_{\theta^{-s}} \geq xs\} &\leq e^{-sJ(x)+o(s)} \quad (x > 1), \quad (3.25) \\
P\{N_{\theta^{-s}} \leq xs\} &\leq e^{-sJ(x)+o(s)} \quad (x < 1). \quad (3.26)
\end{align}

As we show further below, this implies the claim. We first observe that $\log \Sigma(R_n) = \log R_n - \log R_0 + \log \Sigma(R_0)$, by (3.3). Since $\log \Sigma(R_0) = \Sigma(r)$ is a constant, (3.23)–(3.24) and continuity of $I(x)$ near $\log \theta$ imply that, as $n \to \infty$,

\begin{align}
P\{\log \Sigma(R_n) \geq xn\} &\leq e^{-nI(x)+o(n)} \quad (x > \log \theta), \quad (3.27) \\
P\{\log \Sigma(R_n) \leq xn\} &\leq e^{-nI(x)+o(n)} \quad (x < \log \theta). \quad (3.28)
\end{align}

Also, by (3.18),

\begin{align}
\{N_{\theta^{-s}} \geq r\} &\subseteq \left\{ \sum_{j=0}^{[r]-1} \lambda_j \Sigma(R_j)^{-1} \leq \theta^{-s} \right\}, \quad (3.29) \\
\{N_{\theta^{-s}} \leq r\} &\subseteq \left\{ \sum_{j=0}^{[r]+1} \lambda_j \Sigma(R_j)^{-1} \geq \theta^{-s} \right\}. \quad (3.30)
\end{align}

To show (3.23), let $x > 1$ and choose $x'$ such that $x > x' > 1$. By (3.29), and using $\Sigma(R_j) \geq 0$, $\lambda_j \geq 0$, and the union bound,

\begin{align}
P\{N_{\theta^{-s}} \geq xs\} \leq &\ P\left\{ \sum_{j=0}^{[xs]-1} \frac{\lambda_j}{\Sigma(R_j)} \leq \theta^{-s} \right\} \leq P\left\{ \frac{\lambda_{[xs]-1}}{\Sigma(R_{[xs]-1})} \leq \theta^{-s} \right\} \\
&\leq P\left\{ \lambda_{[xs]-1} \leq \theta^{(x' - 1)s} \right\} + P\left\{ \Sigma(R_{[xs]-1}) \geq \theta^{x's} \right\}. \quad (3.31)
\end{align}
Since \( \lambda_{[x]} \) is an exponential random variable and \( \theta x' < 1 \), the first term goes to 0 exponentially as \( s \to \infty \). The second term is estimated by

\[
P\{ \Sigma(R_{[x]}+1) \geq \theta^{-s} \} \leq P\{ \log(\Sigma(R_{[x]}+1)) \geq x' \log \theta \} \leq e^{-x'sI(x') \log \theta} + o(s) \quad (3.32)
\]
as \( s \to \infty \). From these inequalities, (3.23) follows, with some \( J(x) > 0 \).

To verify (3.26), let \( x < 1 \) and choose \( x' > 0 \) such that \( x < x' < 1 \). Then, by (3.30), since \( R_j \) is decreasing, and using the union bound,

\[
P\{ N_{\theta^{-s}} \leq xs \} \leq P\left\{ \sum_{j=0}^{[x]+1} \frac{\lambda_j}{\sum(R_j)} \geq \theta^{-s} \right\} \leq P\left\{ \sum_{j=0}^{[x]+1} \lambda_j \geq \theta^{-s} \sum(R_{[x]+1}) \right\}
\]

\[
\leq P\left\{ \sum_{j=0}^{[x]+1} \lambda_j \geq \theta^{-(1-x')s} \right\} + P\{ \sum(R_{[x]+1}) \leq \theta^{x's} \}. \quad (3.33)
\]

Since \( E(\sum_{j=0}^{[x]+1} \lambda_j) = O(xs) \), by Markov’s inequality, the first term on the right-hand side of (3.33) tends to 0 exponentially. As in (3.32), the second term on the right-hand side of (3.33) converges to 0 exponentially since, by (3.28),

\[
P\{ \log(\Sigma(R_{[x]+1})) \leq x's \log \theta \} \leq e^{-x'sI(x') \log \theta} + o(s) \quad (3.34)
\]
as \( s \to \infty \). Thus we conclude (3.26) for some \( J(x) > 0 \).

It remains to show that (3.23) – (3.26) indeed imply compact convergence of \( \Phi_n(s) = N_{\theta^{-n}s}/n \to s \) almost surely. For this, we proceed as in the proof of [1, Proposition 8.2]. Fix \( s_0 > 0 \), \( \varepsilon > 0 \). Setting \( m > 2s_0/\varepsilon \), \( s_k = ks_0/m \), since \( \Phi_n(s) \) is nondecreasing in \( s \),

\[
P\left\{ \sup_{s \in [0,s_0]} (\Phi_n(s) - s) \geq \varepsilon \right\} \leq \sum_{k=1}^m P\left\{ \sup_{s \in [s_{k-1},s_k]} (\Phi_n(s) - s) \geq \varepsilon \right\}
\]

\[
\leq \sum_{k=1}^m P\{ \Phi_n(s_k) - s_k \geq \varepsilon - \frac{s_0}{m} \geq \frac{1}{2} \varepsilon \}
\]

\[
\leq \sum_{k=1}^m P\{ N_{\theta^{-s_k}} \geq n k (1 + \frac{1}{2} \varepsilon m/s_0) \} \quad (3.35)
\]

which is bounded by \( c^{-1}e^{-cn} \to 0 \) as \( n \to \infty \), for some \( c = c_{\varepsilon,s_0} > 0 \). Analogously,

\[
P\left\{ \sup_{s \in [0,s_0]} (s - \Phi_n(s)) \geq \varepsilon \right\} \leq c^{-1}e^{-cn}. \quad (3.36)
\]

Since the right-hand sides are summable, the Borel–Cantelli Lemma immediately implies that \( \sup_{x \in [0,t_0]} |\Phi_n(s) - s| \to 1 \) with probability 1, as claimed.

\[\square\]

### 3.3 Proof of Proposition 3.2

The proof of Proposition 3.2 follows the well-known route to prove convergence to Brownian motion by martingale approximation; see, e.g., the introduction of [14]. We recall that \((Y_n)\) is a Markov
chain on $S^2$ and that $(\lambda_n)$ is a sequence of i.i.d. unit exponential random variables, independent of $(Y_n)$. We find it convenient to consider the pair $(Y_n, \lambda_n)$ as a Markov chain on $S^2 \times (0, \infty)$, and denote its transition operator by $P$ and its kernel by $p(y, l; dy', dl')$, $y, y' \in S^2, l, l' > 0$. When the exponential random variables are irrelevant, we also write transition kernel of $(Y_n)$ as $p(y; dy')$.

Then, with $b$ as in (2.11),

$$\int l' y' \ p(y, l; dy', dl') = \int y' \ p(y; dy') = by. \quad (3.37)$$

Define $h, V : S^2 \times \mathbb{R}_+ \to \mathbb{R}^3$ by

$$h(y, l) = ly + \frac{b}{1 - b} y, \quad V(y, l) = ly, \quad (y \in S^2, l > 0) \quad (3.38)$$

so that $V = (1 - P)h$, by (2.11). Define

$$Y'_n = h(Y_n, \lambda_n) - Ph(Y_{n-1}, \lambda_{n-1}). \quad (3.39)$$

Then $\mathbb{E}(Y'_{n+1} | \Sigma_n) = 0$, i.e., $(Y'_n, \Sigma_n)$ is a martingale difference sequence.

**Lemma 3.4.**

$$\frac{1}{n} \sum_{j=1}^{n} \mathbb{E}(Y'_j \alpha Y'_j \beta | \Sigma_{j-1}) \to \frac{2\delta_{\alpha \beta}}{3(1 - b)} \quad (in \ probability). \quad (3.40)$$

**Proof.** Let $q_{\alpha \beta}(y) = \mathbb{E}(Y'_j \alpha Y'_j \beta | Y_{j-1} = y) = \mathbb{E}(Y'_j \alpha Y'_j \beta | Y_{j-1} = y, \lambda_{j-1} = l)$ with arbitrary $l$. In a short calculation, we first show that

$$C_{\alpha \beta} := \mathbb{E}(q_{\alpha \beta}(U)) = \frac{2\delta_{\alpha \beta}}{3(1 - b)}, \quad (3.41)$$

where $U$ is a uniform random variable on $S^2$. Indeed, by (3.37), (3.39),

$$Y'_j = (\lambda_j + \frac{b}{1 - b})Y_j - \frac{b}{1 - b} Y_{j-1}. \quad (3.42)$$

Since $\lambda_j$ is a unit exponential random variable,

$$\mathbb{E}(\lambda_j + \frac{b}{1 - b}) = 1 + \frac{b}{1 - b} = \frac{1}{1 - b}, \quad \mathbb{E}((\lambda_j + \frac{b}{1 - b})^2) = \frac{1 + (1 - b)^2}{(1 - b)^2}. \quad (3.43)$$

Since $\mathbb{E}(Y'_j \alpha | \Sigma_{j-1}) Y'_j \beta - bY'_j \alpha Y'_j \beta$ by (3.37), it follows that

$$\mathbb{E}(Y'_j \alpha Y'_j \beta | \Sigma_{j-1}) = \frac{1}{(1 - b)^2}((1 + (1 - b)^2)\mathbb{E}(Y'_j \alpha Y'_j \beta | \Sigma_{j-1}) - b^2 Y'_j \alpha Y'_j \beta), \quad (3.44)$$

and therefore, since $\mathbb{E}(U^\alpha U^\beta) = \frac{1}{3}\delta_{\alpha \beta}$,

$$C_{\alpha \beta} = \frac{1}{(1 - b)^2}((1 + (1 - b)^2 - b^2)\mathbb{E}(U^\alpha U^\beta) = \frac{2\delta_{\alpha \beta}}{3(1 - b)}. \quad (3.45)$$
as claimed. Let \( p^n(y, dz) \) be the \( n \)-step transition probability of the Markov chain \((Y_n)\), defined from the one-step transition probability \( p(y, dz) \) by the Chapman-Kolmogorov equations. From the definition of \( p(y, dz) \) above, it can be seen (cf. Figure 1) that there exists \( n > 0 \) such that the density of the absolutely continuous component, with respect to the uniform measure on \( S^2 \), of \( p^n(y, \cdot) \) is bounded below uniformly by a strictly positive constant. Thus the Markov chain \( Y_j \) satisfies Doeblin’s condition [8, p.197, condition (D')]. It follows that, if \( Y_j \) is an independent instance of this Markov chain with initial stationary distribution \( u \), the total variation distance between the distributions of \( Y_1 \) and \( Y_j \) tends to 0, exponentially fast. Since the sequence \( Y_j \) is stationary and ergodic, \( \frac{1}{n} \sum_{j=0}^{n-1} q(Y_j) \to q(U) = C \) almost surely, by the ergodic theorem, and the proof follows. \( \square \)

**Proof of Proposition 3.2.** Given Lemma 3.4, the proof is a straightforward consequence of a standard functional central limit theorem for martingales [11, Theorem 3.3]. For the reader’s convenience, we restate a special case of it as Theorem A.1. Let

\[ Z'_n := \sum_{j=0}^{n} Y'_j = \sum_{j=1}^{n} V(Y_j, \lambda_j) - Ph(Y_0, \lambda_0) + Ph(Y_n, \lambda_n). \]  

(3.46)

Since

\[ |Y'_j| = \left| (\lambda_j + \frac{b}{1-b}) Y_j - \frac{b}{1-b} Y_{j-1} \right| \leq \lambda_j + \frac{2b}{1-b}, \]  

(3.47)

it follows that

\[ \mathbb{E}(Y'^{\alpha_0}Y'^{\beta_1} | Y'_j | \geq \sqrt{n} \varepsilon) \leq \mathbb{P} \{ \lambda_j \geq \sqrt{n} \varepsilon - O(1) \} \to 0. \]  

(3.48)

Together with the fact that \((Y'_j, \Sigma_j)\) is a martingale difference sequence, all conditions of Theorem A.1 are satisfied for \( M_j = Y'_j \). Setting \( \tilde{\sigma} = \frac{2}{3}(1-b)^{-1} \), it follows that \((\frac{1}{\sqrt{n}} Z'_n) \to (\tilde{\sigma} B_s)\) in the Skorohod space \( D([0, s_0], \mathbb{R}^3) \), for any \( s_0 > 0 \). Since

\[ \sum_{j=0}^{n} \lambda_j Y_j = \sum_{j=1}^{n} V(Y_j, \lambda_j), \]  

(3.49)

the proof is concluded by the observation that the two “boundary terms” in (3.46) are negligible in the limit after multiplication by \( \frac{1}{\sqrt{n}} \). This is trivial since

\[ |Ph(Y_0, \lambda_0) + Ph(Y_n, \lambda_n)| = \left| \frac{b}{1-b} Y_0 - \frac{b}{1-b} Y_n \right| \leq \frac{2b}{1-b}, \]  

(3.50)

so the boundary terms are in fact bounded. \( \square \)

### 3.4 Proof of Theorem 2.2

**Proof of (2.7).** By Proposition 3.3, \( N_{t-s}/s \to 1 \) as \( s \to \infty \) a.s., i.e., \( N_t = (\log t/(-\log \theta))(1+o(1)) \). Moreover, by (3.23)-(3.24), \( \log |K_n| - \log |K_0| = n \log \theta(1+o(1)) \) almost surely. It follows that

\[ \log |K_t| - \log |K_0| = \frac{-\log t}{N_t \log \theta} (\log |K_{N_t}| - \log |K_0|)(1+o(1)) = -\log t(1+o(1)) \]  

(3.51)

almost surely, as claimed. \( \square \)
Proof of (2.8). We start from (3.10). Since \((t - T_N) | K_N | \leq \lambda_N \) is finite almost surely, and using  
again the bounds on the random time change (Proposition 3.3), it suffices to prove  

\[
\sup_n \left| \mathbb{E} \left( \sum_{j=1}^{n} \lambda_j Y_j \right) \right| < \infty, \quad \text{and} \quad \mathbb{E} \left| \sum_{j=1}^{n} \lambda_j Y_j \right| \to \infty. \tag{3.52}
\]

The first claim follows from the fact that the distribution of \( \lambda_j Y_j \) converges to that of \( \lambda U \) 
exponentially fast, in total variation distance (see proof of Lemma 3.4), and  
\( \mathbb{E}(\lambda U) = \mathbb{E}(U) = 0 \). To verify the second claim, we apply Markov’s inequality,  

\[
\mathbb{E} \left| \sum_{j=1}^{n} \lambda_j Y_j \right| \geq \sqrt{n} \mathbb{P} \left\{ \frac{1}{\sqrt{n}} \left| \sum_{j=1}^{n} \lambda_j Y_j \right| \geq 1 \right\} \tag{3.53}
\]

and can then invoke Proposition 3.2 to argue that \( \mathbb{P}\{\cdot\} \) on the right hand side is bounded away from 0. \qed

Acknowledgement

Most of the research that has led to this paper was carried out while the three authors were at the Institut für Theoretische Physik, ETH Zürich. The work of R.B. has also been supported by the National Science Foundation under grant No. DMS-1128155. This paper was completed during a stay of R.B. and J.F. at the Institute for Advanced Study in Princeton. The stay of J.F. at IAS has been supported by ‘The Fund for Math’ and ‘The Robert and Luisa Fernholz Visiting Professorship Fund’.

A Martingale functional central limit theorem

The following theorem is a special case of [11, Theorem 3.3].

**Theorem A.1.** Let \((M_n, \Sigma_n)\) be an \(\mathbb{R}^d\)-valued martingale difference sequence, i.e., \((\Sigma_n)\) is a filtration of \(\sigma\)-algebras and \(E(M_n+1 | \Sigma_n) = 0\) for 1 \(\leq \alpha \leq d\). Let  

\[
Z_n(s) = \frac{1}{\sqrt{n}} \sum_{j=1}^{[ns]} M_j, \tag{A.1}
\]

Assume there is \(\sigma^2 > 0\) such that for all \(t > 0, 1 \leq \alpha, \beta \leq d, \varepsilon > 0,\)  

\[
\frac{1}{ns} \sum_{j=1}^{[ns]} \mathbb{E}(M_j^\alpha M_j^\beta | \Sigma_{j-1}) \to \sigma^2 \delta_{\alpha\beta}, \tag{A.2}
\]

\[
\frac{1}{ns} \sum_{j=1}^{[ns]} \mathbb{E}(M_j^\alpha M_j^\beta 1_{|M_j| \geq \sqrt{n \varepsilon}} | \Sigma_{j-1}) \to 0, \quad (n \to \infty), \tag{A.3}
\]

in probability Then \((Z_n(s))_s \to (\sigma B(s))_s\) in distribution in \(D([0, s_0], \mathbb{R}^d)\) for all \(s_0 > 0\), where \(D\) is equipped with the Skorohod topology.
B Proof of Proposition 3.1

Proof. The strong Markov property for \((X_t, K_t)\) implies that, for any stopping time \(S\),

\[
\Gamma_t f(x, k) = \mathbb{E}^{x,k}(f(X_t, K_t)) = \mathbb{E}^{x,k}(\Gamma_{t-S} f(X_S, K_S)). \tag{B.1}
\]

We start the semigroup at the time of the first jump, or rather at \(S = \min\{t, T_1\}\):

\[
\Gamma_t f(x, k) = \mathbb{E}^{x,k}(f(X_t, K_t); T_1 > t) + \mathbb{E}^{x,k}((\Gamma_{t-T_1} f)(X_{T_1}, K_{T_1}); T_1 \leq t) \tag{B.2}
\]

In the first term, \(X_t = x + \frac{k}{m} t\) and \(K_t = k\), and hence

\[
\mathbb{E}^{x,k}(f(X_t, K_t); T_1 > t) = f(x + \frac{k}{m} t, k) \mathbb{E}^{x,k}(T_1 > t) = f(x + \frac{k}{m} t, k) e^{-\Sigma(k)t}. \tag{B.3}
\]

In the second term, since \(T_1\) is exponentially distributed with parameter \(\Sigma(k)\),

\[
\mathbb{E}^{x,k}((\Gamma_{t-T_1} f)(X_{T_1}, K_{T_1}); T_1 \leq t) = \int_0^t \left(\int p(k, dk')(\Gamma_{t-\tau} f)(x + \frac{k}{m} \tau, k')\right) e^{-\Sigma(k)\tau} \Sigma(k) d\tau. \tag{B.4}
\]

Using \(q(k, dk') = \Sigma(k)p(k, dk')\), and substituting \(r = t - \tau\), this is

\[
\mathbb{E}^{x,k}((\Gamma_{t-T_1} f)(X_{T_1}, K_{T_1}); T_1 \leq t) = \int_0^t \left(\int q(k, dk')(\Gamma_r f)(x + \frac{k}{m} (t-r), k')\right) e^{-\Sigma(k)(t-r)} dr. \tag{B.5}
\]

Putting both terms together, we obtain

\[
\Gamma_t f(x, k) = f(x + \frac{k}{m} t, k) e^{-\Sigma(k)t} + \int_0^t \left(\int q(k, dk')(\Gamma_r f)(x + \frac{k}{m} (t-r)), k')\right) e^{-\Sigma(k)(t-r)} dr. \tag{B.6}
\]

The last expression is just the Duhamel formula for the semigroup generated by \(L = A - \Sigma + K\), where \(A = \frac{k}{m} \nabla\) is the advection operator, and considering the gain operator \(K\) with kernel \(q\) as a perturbation to the semigroup generated \(A - \Sigma\). Hence, the claim is verified. \(\square\)

References

[1] G. Basile and A. Bovier. Convergence of a kinetic equation to a fractional diffusion equation. Markov Process. Related Fields, 16(1):15–44, 2010.

[2] Roland Bauerschmidt. Quantum friction and Čerenkov radiation. Master’s Thesis, ETH Zürich. Unpublished, 2009.

[3] Patrick Billingsley. Convergence of probability measures. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York, second edition, 1999. A Wiley-Interscience Publication.
[4] Laurent Bruneau and Stephan De Bièvre. A Hamiltonian model for linear friction in a homogeneous medium. *Comm. Math. Phys.*, 229(3):511–542, 2002.

[5] S. Caprino, C. Marchioro, and M. Pulvirenti. Approach to equilibrium in a microscopic model of friction. *Comm. Math. Phys.*, 264(1):167–189, 2006.

[6] Robert Dautray and Jacques-Louis Lions. *Mathematical analysis and numerical methods for science and technology. Vol. 6*. Springer-Verlag, Berlin, 1993. Evolution problems. II, With the collaboration of Claude Bardos, Michel Cessenat, Alain Kavenoky, Patrick Lascaux, Bertrand Mercier, Olivier Pironneau, Bruno Scheurer and Rémi Sentis, Translated from the French by Alan Craig.

[7] Amir Dembo and Ofer Zeitouni. *Large deviations techniques and applications*, volume 38 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, second edition, 1998.

[8] J. L. Doob. *Stochastic processes*. Wiley Classics Library. John Wiley & Sons Inc., New York, 1990. Reprint of the 1953 original, A Wiley-Interscience Publication.

[9] László Erdős. Linear Boltzmann equation as the long time dynamics of an electron weakly coupled to a phonon field. *J. Statist. Phys.*, 107(5-6):1043–1127, 2002.

[10] Jürg Fröhlich, Zhou Gang, and Avy Soffer. Some Hamiltonian models of friction. *J. Math. Phys.*, 52(8):083508, 13, 2011.

[11] Inge S. Helland. Central limit theorems for martingales with discrete or continuous time. *Scand. J. Statist.*, 9(2):79–94, 1982.

[12] John David Jackson. *Classical electrodynamics*. John Wiley & Sons Inc., New York, second edition, 1975.

[13] Milton Jara, Tomasz Komorowski, and Stefano Olla. Limit theorems for additive functionals of a Markov chain. *Ann. Appl. Probab.*, 19(6):2270–2300, 2009.

[14] C. Kipnis and S. R. S. Varadhan. Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions. *Comm. Math. Phys.*, 104(1):1–19, 1986.

[15] Antoine Mellet, Stéphane Mischler, and Clément Mouhot. Fractional diffusion limit for collisional kinetic equations. *Arch. Ration. Mech. Anal.*, 199(2):493–525, 2011.