A theory of incremental compression

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Abstract

The ability to find short representations, i.e. to compress data, is crucial for many intelligent systems. We present a theory of incremental compression showing that arbitrary data strings, that can be described by a set of features, can be compressed by searching for those features incrementally, which results in a partition of the information content of the string into a complete set of pairwise independent pieces. The description length of this partition turns out to be close to optimal in terms of the Kolmogorov complexity of the string. At the same time, the incremental nature of our method constitutes a major step toward faster compression compared to non-incremental versions of universal search, while still staying general. We further show that our concept of a feature is closely related to Martin-Löf randomness tests, thereby formalizing the meaning of “property” for computable objects.

Keywords: compression, autoencoder, feature, Kolmogorov complexity, universal search, Occam’s razor

1. Introduction

In the machine learning community it has long been known that short representations of data lead to high generalization abilities [3, 19]. The larger the model, for example the neural network, the less it is able to generalize, hence to predict.

Similar ideas have been expressed in cognitive neuroscience, psychology and linguistics. In neuroscience, the efficient coding hypothesis [2] states that the spikes in the sensory system form a neural code minimizing the number of spikes needed to transmit a given signal. Sparse coding [18] is the representation of items by the strong activation of a relatively small set of neurons. In psychology, it is not a recent idea to regard perception from the perspective of Bayesian inference [22], which is known to entail Occam’s razor [17, Chapter 28]. In linguistics, various principles of information maximization and communication cost minimization have been proposed (see [11] for a review). In the philosophy of science, Occam’s razor entails that scientists should strive for explanations / theories that are as simple as possible while capturing and explaining as much data as possible.

Kolmogorov, Solomonoff and Chaitin went further and formalized those ideas, which has led to the development of the algorithmic information theory [13, 23, 24]. The amount of information contained in an object $x$ is defined as the length $K(x)$ of its shortest possible description, which has become known as the (prefix) Kolmogorov complexity. $K(x)$ is the length of the shortest program that prints $x$ when executed on a universal (prefix) Turing machine $U$, i.e. a computer. For example, if $x$ is a string of one million zeros, we can write a short program $q$ that can print it while being much shorter than the data, $l(q) \ll l(x)$.

Finding short descriptions for data is what data compression is all about. After all, the original data can be unpacked again from its short description. Furthermore, compression is closely tied to prediction. For example, a short program implementing a zero printing loop could just as well continue printing more than a million of them, which would constitute a prediction of the continuation of the sequence. Indeed, Solomonoff’s theory of universal induction proves formally that compressing data leads to the best possible predictor [25]. In order to do so, the so-called universal prior of a data string $x$ is defined:

$$M(x) = \sum_{q : U(q) = x} 2^{-l(q)}$$  \hspace{1cm} (1)

where $x, q \in \mathcal{B}^*$ are finite strings defined on a finite alphabet $\mathcal{B}$, $U$ is a universal Turing machine that executes program $q$ and prints $x$ and $l(q)$ is the length of program $q$. Given already seen history $x_{<k} \equiv x_1 \cdots x_{k-1}$ the
predictor’s task is to compute a probability distribution over \( x_k \), which is given by the conditional distribution
\[ M(x_k | x < k) = M(x_{1:k}) / M(x_{<k}). \]

The Solomonoff predictor has been shown to converge quickly \(^{25}\) to the true data generating distribution, allowing it to predict future data with the least possible loss in the limit. Note that eq. (1) weighs each “explanation” \( q \) for the data with \( 2^{-l(q)} \), which directly expresses Occam’s razor (i.e., compression): even though we should consider all explanations, the shorter/simpler ones should receive the highest weight. Remarkably, optimal induction and prediction requires halving the prior probability of an explanation for every additional bit in the explanation length.

In the context of artificial intelligence, Hutter went further and attached the Solomonoff predictor to a reinforcement learning agent \(^{10}\). If general intelligence is defined as the ability to achieve goals in a wide range of environments \(^{14}\), the resulting AIXI agent has been shown to exhibit maximal general intelligence \(^{10, \text{Theorems } 5.23 \text{ and } 5.24}\). This result formally ties the conceptual problem of artificial general intelligence to efficient data compression, making the search for its solution even more urgent.

In this paper, we would like to attack this problem by introducing our theory of incremental compression.\(^1\) It is an approach to compress data incrementally and potentially more efficiently, while still staying general, i.e. without assuming the data to come from a narrow class. Although there are several variants of universal search in the literature \(^{9, 15, 20}\), to the best of our knowledge there are no incremental procedures in the sense proposed here. We proceed as follows. First, we introduce the main idea and examine the properties of a single compression step in Chapters 2.1–2.3. Iterating this step many times leads to the formulation of our compression scheme in Chapters 2.4–2.8. Finally, we set up a bridge to Martin-Löf’s theory of randomness in Chapter 2.9.

## 2. Results

### 2.1. Basic idea

In practice, describing an object often comes down to identifying particular properties or features of the object, such as color, shape, size, location etc. Intuitively, a description appears most accomplished if those features are independent, i.e. do not contain information about each other. For example, knowing the color of an object does not contain information about its shape and vice versa. This independence appears to allow us to find the features one by one, without having to find the full description at once.

In order to formalize this idea, we represent any data string \( x \) by a composition of functions, \( x = (f_1 \circ \cdots \circ f_s)(r_s) \), by looking for stacked autoencoders \(^7\). The idea of an autoencoder is to use a descriptive map \( f' \) to project input data \( x \) on a shorter residual description \( r \), from which a feature map \( f \) can reconstruct it, see Fig. 1.

![Autoencoder diagram](image)

Figure 1: An autoencoder. For example, consider a string of the form \( x = 1^n0y \) (where \( 1^n := 1 \ldots 1 \)). Then, a descriptive map could be a function computing the number \( n \) of initial ones and copying \( y \) to the residual description, \( f'(x) = (n, y) =: r \). A feature could be a function taking the residual \( r = (n, y) \) and map it back to \( x \), \( f(r) = x \). Clearly, the residual description is much shorter for large enough \( n \), \( l(r) \approx l(n) + l(y) \ll l(x) = n + 1 + l(y) \), meaning that some data compression has been achieved (encoding a number \( n \) as a binary string takes only \( l(n) = \lfloor \log(n+1) \rfloor \ll n \) bits).

The key advantage of an autoencoder is that a shorter representation \( r \) of the data \( x \) can be computed from \( x \) instead of employing blind search for \( r \). At the same time, information in \( r \) about \( x \) is preserved by requiring the

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\(^1\)Some theorems on this idea have already been published in a conference paper \(^{11}\), which unfortunately contained several mistakes.
ability to reconstruct it back from \( r \). Instead of searching for a potentially long description \( r \), the search is focused on finding a pair of hopefully simple functions \((f, f')\) that manage to compress \( x \) at least a little bit, which we call the compression condition. A little compression shall be enough, since this process is repeated for the residual description \( r \) in the role of input data for the next autoencoder required to compress \( r \) a little further. The process iterates until step \( s \) where no compression is possible.

2.2. Definitions and basic properties

Consider strings made up of elements of the set \( \mathcal{B} = \{0, 1\} \) with \( \epsilon \) denoting the empty string. \( \mathcal{B}^* \) denotes the set of finite strings over \( \mathcal{B} \). Denote the length of a string \( x \) by \( l(x) \). Since there is a bijective map \( \mathcal{B} \rightarrow \mathcal{N} \) of finite strings onto natural numbers, strings and natural numbers are used interchangeably. We define the prefix-codes \( E_1(x) = \pi = 1^{l(x)}0x \) and \( E_2(x) = l(x)x \).

The universal, prefix Turing machine \( U \) is defined by

\[
U((y, \langle i, q \rangle)) = T_i((y, q)),
\]

(2)

where \( T_i \) is \( i \)-th machine in some enumeration of all prefix Turing machines and \( \langle \cdot, \cdot \rangle \) is an one-to-one mapping \( \mathcal{N} \times \mathcal{N} \rightarrow \mathcal{N} \), defined by \( \langle x, y \rangle \equiv \pi y \). This means that \( f = \langle i, q \rangle \) describes some program on a prefix Turing machine and consists of the number \( i \) of the machine, its input \( q \) and some additional parameter \( y \). We will use the shortcut \( y(z) := U(\langle z, y \rangle) \) defining how strings, interpreted as Turing machines (partial recursive functions), can execute other strings. The conditional (prefix Kolmogorov) complexity is given by

\[
K(x \mid r) := \min_f \{l(f) : U((r, f)) \equiv f(r) = x\}
\]

(3)

This means that \( K(x \mid r) \) is the length of the shortest function \( f \) able to compute \( x \) from \( r \) on a universal prefix Turing machine. The unconditional (prefix Kolmogorov) complexity is defined by \( K(x) \equiv K(x \mid \epsilon) \). We define the information in \( x \) about \( y \) as \( I(x : y) := K(y) - K(y \mid x) \). Up to this point, we have followed the standard definitions as given in [10].

**Definition 1.** Let \( f \) and \( x \) be finite strings and \( D_f(x) \) the set of **descriptive maps** of \( x \) given \( f \):

\[
D_f(x) = \{f' : f'(f^*(x)) = x, l(f) + l(f^*(x)) < l(x)\}
\]

(4)

If \( D_f(x) \neq \emptyset \) then \( f \) is called a **feature** of \( x \). The strings \( r \equiv f'(x) \) are called **residual descriptions** of the feature \( f \). \( f^* \) is called **shortest feature** of \( x \) if it is one of the strings fulfilling

\[
l(f^*) = \min \{l(f) : D_f(x) \neq \emptyset\}
\]

(5)

and \( f^{**} \) is called **shortest descriptive map** of \( x \) given \( f^* \) if

\[
l(f^{**}) = \min \{l(g) : g \in D_{f^*}(x)\}
\]

(6)

The so-called **compression condition**

\[
l(f) + l(r) < l(x)
\]

(7)

is required to avoid the case of \( f \) and \( f' \) being the identity functions, leading to the useless transformation \( r = id(x) = x \).

Note that in eq. (7) \( f \) has to be part of the description of \( x \), since otherwise \( r \) can be made arbitrarily small by inserting all the information into \( f \). Our first result shows that features of compressible strings always exist, since the shortest description of \( x \) could be coded into a single function \( f \) using an empty residual \( r = \epsilon \). We shall call this a **total feature** of the string:

**Lemma 1.** If \( x \) is compressible, that is \( K(x) < l(x) \), then there exists a total feature \( f \), and its residual \( r = \epsilon \) such that \( f(r) = x, l(f) + l(r) = K(x) < l(x) \). The length of the shortest feature is \( l(f^*) \leq K(x) \).

On the other extreme, the universal Turing machine \( U \) itself could function as a feature function, since nothing prevents a universal machine to simulate another universal machine. Such features shall be called universal, since they are able to compute any computable \( x \). However, \( x \) has to be compressible enough to accommodate the length of \( U \) in order to fulfill the compression condition.

**Definition 2.** A string \( f_0 \) shall be called **universal feature**, if there is a constant \( C \), such that \( f_0 \) is a feature of any string compressible by more than \( C \) bits.

**Lemma 2.** There exists a constant \( C \) and a universal feature \( f_0 \) with length \( l(f_0) = C \) such that \( f_0 \) is a feature of any string \( x \) compressible by more than \( C \) bits: \( K(x) < l(x) - C \). The residual description \( r \) is a shortest description of \( x \), \( l(r) = K(x) \).
2.3. Properties of a single compression step

In the following, we will show that the information $K(x)$ inside $x$ is partitioned by the shortest features $f_1^*, \ldots, f_s^*$ and the last residual $r_s$ by the process of incremental compression. The main proof strategy is illustrated in Fig. 2.

What is the consequence of picking the shortest possible feature? It turns out that similarly to shortest descriptions per se, the shortest features are themselves incompressible. After all, if the shortest feature was compressible, there would be a shorter program $q$ to compute it. But then, $q$ can just as well take $r$ and go on computing $x$ while becoming a feature itself, shorter than the shortest feature, $l(q) < l(f^*)$, which is a contradiction. Theorem 1 formalizes this reasoning. In order to prove it, we will need the following lemma:

Lemma 3. Let $f^*$ and $f'^*$ be the shortest feature and shortest descriptive map of a finite string $x$, respectively. Further, let $r \equiv f'^*(x)$. Then $l(f^*) = K(x | r)$ and $l(f'^*) = K(r | x)$.

Theorem 1 (Feature incompressibility). The shortest feature $f^*$ of a finite string $x$ is incompressible:

$$l(f^*) - O(1) \leq K(f^*) \leq K(l(f^*)) + O(1) \leq l(f^*) + O(\log l(f^*)).$$

(8)

The next theorem shows that the number of shortest features is actually quite small. Intuitively, this is due to the fact that the compression condition requires the features to do some actual compression work, which is a rare capacity.

Lemma 4. Let $f$ be a feature of a finite string $x$ and $r$ its residual description. Then,

$$K(x | f) \leq l(r) + O(\log l(r)) < l(x) - l(f) + O(\log l(x)).$$

(9)
Lemma 5. Let $f$ be a feature of a finite string $x$ and $r$ its residual description. If $I(f : x) := K(x) - K(x | f)$ \[ K(f | x) \leq O(\log l(x)) \] \[ (10) \]

Theorem 2 (The number of shortest features). The number of shortest features of a binary string $x$ grows at most polynomially with $l(x)$. Moreover, $K(f^* | x) \leq O(\log l(x))$ for any shortest feature $f^*$ of $x$.

Now we move to a central result that a shortest feature does not contain information about the residual and vice versa. This is important, since otherwise, there would be an informational overlap between them, indicating that we would be wasting description length on the overlap. Figuratively speaking, if some “bits” inside $f^*$ describe the same information as some “bits” in $r$, we could choose $f^*$ to be even shorter and avoid this redundancy. Interestingly, the number of wasted bits is simply bounded by the difference in lengths, $l(f) - l(f^*)$, between the chosen feature and the shortest one:

Theorem 3 (Independence of features and residuals, general case). Let $f$ be a feature of a finite string $x$ and $r$ its residual description. Further, let $f^*$ be a shortest feature and $d := l(f) - l(f^*)$. Then,

\[ I(r : f) := K(f) - K(f | r) \leq d + K(l(f)) + O(1) \leq d + O(\log l(f)), \]

\[ K(r) + K(f) - K(f, r) \leq d + K(l(f)) + K(K(r) | r) + O(1) \]

\[ \leq d + O(\log l(f)) + O(\log l(r)) \]

This theorem is valid for features in general. In particular, we obtain for shortest features the following relationships:

Corollary 1 (Independence of features and residuals). Let $f^*$ be the shortest feature a finite string $x$ and $r$ its residual description. Then,

\[ K(f^* | r) = K(f^*) + O(\log l(f^*)), \]

\[ K(r | f^*) = K(r) + O(\log l(f^*)) + O(\log l(r)), \]

\[ K(f^*, r) = K(f^*) + K(r) + O(\log l(f^*)) + O(\log l(r)) \]

We conclude that features and residuals do not share information about each other, therefore the description of the $(f^*, r)$-pair breaks down into the simpler task of describing $f^*$ and $r$ separately. Since Theorem 3 implies the incompressibility of $f^*$ and $U((r, f^*)) = x$, the task of compressing $x$ is reduced to the mere compression of $r$. Hence, we can sloppily write $K(f^*, r) \approx l(f^*) + K(r)$, where the “$\approx$” sign denotes equality up to additive logarithmic terms.

However, there is still a possibility that the $(f^*, r)$-pair contains more information than necessary to compute $x$. As we shall see this is not the case: no description length is wasted. Intuitively, since the shortest feature $f^*$ neither overlaps with $r$, nor contains redundant bits due to its incompressibility, the only way for $f^*$ to contain superfluous information is to contain totally unrelated noise with respect to $r$ and $x$. But that also does not make sense if the shortest feature is picked. The following theorem substantiates this reasoning:

Theorem 4 (No superfluous information). The shortest feature $f^*$ of string $x$ and its residual $r$ do not contain much superfluous information

\[ K(f^*, r | x) = O(\log l(x)) \]

\[ (16) \]

the shortest descriptive map generally does not contain much information

\[ l(f^*) = O(\log l(x)) \]

\[ (17) \]

and the description of $x$ is partitioned into $f^*$ and $r$:

\[ K(x) = K(f^*) + K(r) + O(\log l(x)) \]

\[ (18) \]

We can summarize the results by stating:

1. $f^*$ and $r$ do not contain superfluous information: $K(f^*, r | x) \approx 0$

2. The shortest descriptive map is very short: $l(f^*) \approx 0$
3. The description of $x$ breaks down into the separate description of $f^*$ and $r$: $K(x) \approx K(f^*) + K(r)$

Finally, Theorem 1 leads to the main result of a single compression step: $K(x) \approx l(f^*) + K(r)$. We do not have to bother about the feature, we can greedily continue to compress the residual. It this is done well, $x$ will also be compressed well. The following sections explore this compression scheme.

2.4. Incremental compression scheme

The iterative application of the just described compression step is called incremental compression. Denote $r_0 \equiv x$ and let $f^*_i$ be a shortest feature of $r_{i-1}$, $f^*_i$ a shortest (corresponding) descriptive map with $r_i = f^*_i(r_{i-1})$. The iteration $i = 1, 2, \ldots$ continues until some $r_s$ is not compressible (for example, $r_s = \epsilon$) any more. This leaves us with the composition of functions $x = f^*_1(f^*_2(\cdots f^*_s(r_s)))$.

A consequence of such a compression procedure is the pairwise orthogonality of features obtained this way. After all, if $f^*_i$ doesn’t know much about $r_i$, and $f^*_{i+1}$ is part of $r_i$, then $f^*_i$ doesn’t know much about $f^*_{i+1}$ either. Formally, we define the information in $x$ about $y$ as $I(x: y) \equiv K(y) - K(y | x)$, and obtain:

**Lemma 6.** Let $x, y, z$ be arbitrary finite strings. Then $I(x: z) \leq I(x: y) + K(z | y) + O(\log l(z))$.

This result can be applied to all features during incremental compression:

**Theorem 5 (Orthogonality of features).** Let $x$ be a finite string that is incrementally compressed by a sequence of shortest features $f^*_1, \ldots, f^*_s$ that is $x = f^*_1(f^*_2(\cdots f^*_s(r_s)))$. Then, the features are pairwise orthogonal in terms of the algorithmic information:

$$I(f^*_i : f^*_j) = O(|i - j| \log l(x))$$

(19)

for all $i \neq j$.

Finally, we can turn to the interesting question on the optimality of our compression scheme. One of our main theoretical results shows that incremental compression finds a description whose length coincides with the Kolmogorov complexity up to logarithmic terms, i.e. achieves near optimal compression:

**Theorem 6 (Optimality of incremental compression).** Let $x$ be a finite string. Define $r_0 \equiv x$ and let $f^*_i$ be a shortest feature of $r_{i-1}$ and $r_i$ its corresponding residual description. The compression scheme described above leads to the description $x = f^*_1(f^*_2(\cdots f^*_s(r_s)))$, encoded as $D_s := \langle s, r_s, f^*_1, \cdots, f^*_s \rangle$, for which the following expression holds:

$$K(x) = \sum_{i=1}^{s} l(f^*_i) + K(r_s) + O(s \cdot \log l(x))$$

(20)

2.5. On the number of compression steps

The number of compression steps in eq. (20) is in general difficult to estimate. In the worst case, we could be compressing merely 1 bit at every step. Then, $s = O(l(x))$ and $l(D_s) = K(x) + O(l(x) \cdot \log l(x))$ which is not satisfactory. In order to bound the residual term $O(s \cdot \log l(x))$ we impose an additional condition on the number of iterations $s$: we demand that apart from the compression condition, the residual shall be at least $b$ times smaller than $l(x)$, where $b \geq 1$.

**Definition 3.** Let $f$ and $x$ be finite strings. Denote $D_{f, b}(x)$ as the set of $b$-descriptive maps of $x$ given $f$, the $b$-feature $f$, $b$-residuals $r$ and $b$-descriptive map $f'$ similarly to Definition 1 adding the condition $l(f'(x)) \leq l(x)/b$ to the compression condition:

$$D_{f, b}(x) = \left\{ f' : f'(f'(x)) = x, l(f) + l(f'(x)) < l(x), l(f'(x)) \leq l(x)/b \right\}$$

(21)

As can be verified by going through the proofs, the most results about shortest features remain valid for the shortest $b$–features:

**Lemma 7.** Lemma 3, Lemma 9, Theorem 4, Lemma 9, Lemma 11, Theorem 6, Theorem 8, Corollary 4, Theorem 10, Theorem 12 hold for the shortest $b$-feature and shortest $b$-descriptive map of a finite string $x$, respectively.

Applying the compressibility by factor $b$ to each residual, $l(r_i) \leq l(r_{i-1})/b$, leads to $l(r_s) \leq l(x)/b^{s-1}$. Since $r_{s-1}$ is compressible it cannot be the empty string $\epsilon$, hence $l(r_{s-1}) \geq 1$ and we obtain the bound $s \leq \log_b(l(x)/l(r_{s-1})) + 1 \leq \log_b(l(x)) + 1$. In this way, the estimation error $O(s \cdot \log l(x))$ in Theorem 6 becomes $O((\log l(x))^2)$, which is quite small.
2.6. A special case: well-compressible strings

If we are aiming for a practical compression algorithm, it is reasonable to assume that the strings actually are compressible. In this section, we would like to take a look at this special case. Let $x$ be compressible by a factor $b$: $K(x) \leq l(x)/b$, $b > 1$. Clearly, by Lemma 2, a universal feature exists if $x$ is long enough. Then the shortest feature is bounded by the length of the universal feature: a constant. Conversely, if $x$ is not long enough, say $l(x) < a$, then the length of the shortest features is also bounded by a constant, since $l(f) < l(x) < a$ by the compression condition. This is substantiated by

**Theorem 7.** Let $x$ be a $b$-compressible string for some fixed factor $b > 1$. Then the length of any shortest feature $f^*$ of $x$ is bounded, $l(f^*) = O(1)$. More precisely, $l(f^*) \leq \max\{C, \frac{C}{b-1}\} =: C_0$. The same statement holds for the shortest $b$-feature. Moreover, if additionally $l(x) > \frac{C_0}{b-1}$ then the universal feature $f_0$ is both a feature and a $b$-feature of $x$, and $l(f^*) \leq C = l(f_0)$.

Note that if $b \geq 2$ then $\frac{C}{b-1} \leq C$, so the length of shortest feature of a $b$-compressible string is always limited by $C = l(f_0)$. This theorem implies that if the $b$-compressibility assumption holds, we do not require $x$ to be sufficiently long. The existence of a short feature, i.e. $l(f) \leq C_0$, is guaranteed. Therefore, many of the above theorems simplify considerably. We obtain not just $K(f^* | x) \approx 0$ (Theorem 4), but a much stronger proposition $K(f^*) = O(1)$ and $l(f^*) = O(1)$ in case of $b$-compressible $x$. This circumstance demonstrates that features can be very short. Incompressibility of $f^*$ (Theorem 1) and independence between $f^*$ and $r$ (Corollary 1) follow trivially. The number of shortest features (Theorem 2) is also bounded by a constant $2C_0$. It turns out that the shortest descriptive map is also short:

**Theorem 8 (Short descriptive maps, no superfluous information in short features).** Let $f$ be a short feature of a compressible $x$, hence $f(r) = x$, $l(f) + l(r) < l(x)$, $l(f) = O(1)$. Then there exists a residual $q$ such that $f(q) = x$, $l(f) + l(q) < l(x)$ and $K(q | x) = O(1)$. If $f^*$ is a shortest descriptive map given $f$, then $K(f^*(x) | x) = l(f^*) = O(1)$.

2.7. Incremental compression scheme for $b$-compressible strings

What are the implications for the whole compression scheme? Suppose $x$ is a $b$-compressible string and $l(x) > \frac{C_0}{b-1}$. Denote $r_0 \equiv x$ and start an iterative process of compression: let $f_{i+1}^*$ be a shortest $b$-feature of $r_i$, $f_{i+1}^*$ a shortest corresponding descriptive map and $r_{i+1} = f_{i+1}^*(r_i)$. We continue this process until either $l(r_i) \leq \frac{C_0}{b^i}$ or $r_i$ is not $b$-compressible for some $i$. Denote this $i$ by $s$. Just like in Sect. 2.4 we obtain $x = f_1^*(f_2^*(\cdots f_s^*(r_s)))$ and $D_s := (s, r_s, f_s^* \cdots f_1^*)$ is a description of $x$.

**Theorem 9.** Given the compression scheme above the following relationships hold:

$$K(x) - O(1) \leq l(D_s) \leq K(x) + b + O(\log l(r_s)) + O(s) \leq K(x) + b + O(\log l(x))$$

$$l(f_i^*) \leq C \quad \text{for} \quad i = 1, \ldots, s$$

$$s = O(\log l(x))$$

Note that the constant in the $O$-notation here depends on $b$ but not on $x$. If $b$ is close to 1 then $l(D_s)$ will be close to $K(x)$, making $D_s$ a quite short description of $x$.

2.8. Comparison of compression schemes

We present three incremental compression schemes, summarized in Table 1. All three are based on a greedy selection of the shortest feature or $b$-feature, which leads to a decomposition of information into an incompressible feature and a residual information that is to be compressed further. The first scheme does not possess any free parameters and the absolutely shortest feature is searched for. The applicability condition of this scheme is the existence of a feature per se, i.e. the compressibility of initial data $r_0 \equiv x$ or the current residual $r_i$. As long as the current residual $r_i$ fulfills the inequality $l(r_i) - K(r_i) > C$, all shortest features $f_i^*$ will be short: $l(f_i^*) \leq C$ and the overhead description at this step will be small: $l(f_i^*) + K(r_{i+1}) - K(r_i) = O(1)$. In the end, however, when the condition $l(r_i) - K(r_i) > C$ is violated, the boundedness of feature lengths is not guaranteed any more, and the overhead costs become logarithmic $l(f_i^*) + K(r_{i+1}) - K(r_i) = O(\log l(r_i)) \leq O(\log l(x))$. The main problem of this scheme is the lack of an estimate of the number of compression steps $s$. In the worse case, each subsequent $r_i$ is compressed by merely $O(1)$ bit, which leads to $s = \Omega(l(x))$ compression steps making the estimate in eq. (20) unsatisfactory.
Using O\textsuperscript{K} incompressibility:

\[ l(f_{i+1}^r) + l(r_{i+1}) < l(r_i) \]

is due to this bound that the discrepancy between our final description and the overhead bounds become logarithmic. The main difference to the first scheme is that now the number \( r \) of steps is bounded by \( l(r_i) \) and \( l(r_{i+1}) \leq l(r_i) / b \)

### Definition 4 (Randomness test).

Recall the definition of a uniform test for randomness [16, Definition 2.4.1] (\( b \)-feature) of a set:

\[ D_b := \langle s, r_s, f_s^r \ldots f_1^r \rangle \]

**Plain incremental compression**

\[ l(f_{i+1}^r) + l(r_{i+1}) < l(r_i) \]

**Using \( b \)-features**

\[ l(f_{i+1}^r) + l(r_{i+1}) < l(r_i) \text{ and } l(r_{i+1}) \leq l(r_i) / b \]

**Using \( b \)-features, early termination**

\[ l(f_{i+1}^r) + l(r_{i+1}) < l(r_i) \text{ and } l(r_{i+1}) \leq l(r_i) / b \]

| Compression condition | Plain incremental compression | Using \( b \)-features | Using \( b \)-features, early termination |
|-----------------------|-------------------------------|------------------------|----------------------------------------|
| Halting condition     | \( K(r_i) \geq l(r_i) \)     | \( K(r_i) \geq l(r_i) \) | \( l(r_i) \leq \frac{C_B}{b} \text{ or } K(r_i) > l(r_i) / b \) |
| Number of steps (worst case) | \( O(l(x)) \) | \( O(\log l(x)) \) | \( O(\log l(x)) \) |
| Overhead on each step | \( O(\log l(r_i)) \) | \( O(\log l(r_i)) \) | \( O(1) \) |
| Length of description | \( K(x) + O(l(x) \log l(x)) \) | \( K(x) + O((\log l(x))^2) \) | \( \leq K(x) b + O(\log l(x)) \) |

### Table 1: Comparison of compression schemes

In order to limit the number of steps we have introduced a second scheme in which it is necessary to fix a number \( b > 1 \) that determines the minimal compression of the residual on each step. The applicability condition of this scheme is the existence of a \( b \)-feature of the current residual \( r_i \), i.e. the just mentioned compression condition \( l(r_{i+1}) \leq l(r_i) / b \) on the current residual. As long as the conditions \( l(r_i) > \frac{C_B}{b} \) and \( K(r_i) \leq l(r_i) / b \) are fulfilled all shortest \( b \)-features \( f_{i+1}^r \) will be short: \( l(f_{i+1}^r) \leq C_0 \) (Theorem [1]) and the overhead at this step will be small: \( l(f_{i+1}^r) + K(r_{i+1}) - K(r_i) = 0(1) \). Ultimately, however, this bound on the length can not be guaranteed any more, and the overhead becomes logarithmic. The main difference to the first scheme is that now the number of steps is bounded by \( \log l(x) + 1 \), allowing us to regulate the maximum number of steps by changing \( b \). It is due to this bound that the discrepancy between our final description and \( K(x) \) does not differ by more than \( O((\log l(x))^2) \).

The third scheme is analogous to the second, but we stop at an earlier step as soon as there is no short \( b \)-feature any more. Since up to this point the overhead is constant, the overall estimation error is merely \( O(\log l(x)) \). However, the last residual \( r_s \) may remain compressible, albeit not by more than factor \( b \).

This concludes our presentation of incremental compression schemes and we turn to an attempt to deepen our understanding of what a feature is. Since features describe general non-random aspects of a string, it turns out, there is a close relationship to the celebrated Martin-Löf theory of randomness.

### 2.9. Relationship to Martin-Löf randomness

Recalling the definition of a feature, it comes to mind that it could serve as a general algorithmic definition of an object’s properties. The expression \( U((r, f)) = x \) means that the feature is part of the description of object \( x \).

Properties could be viewed as partial descriptions. The compression condition demands that the property is not trivial in some sense. For example, the partial description “begins with 011010100” would violate the compression condition, but it is not particularly interesting and begs the question whether this partial description should be called a property at all. After all, a property should be something that demarcates a particular class of objects from all other objects in a non-trivial way. In other words, an object possessing a property should be rare in some sense and therefore compressible.

This idea is closely tied to the idea of Martin-Löf randomness. A string is called Martin-Löf random, if it passes all randomness tests. Recall the definition of a uniform test for randomness [16, Definition 2.4.1] (\( d(\cdot) \) measures the cardinality of a set):

**Definition 4 (Randomness test).** A total function \( \delta : \mathcal{N} \to \mathcal{N} \) is a uniform Martin-Löf test for randomness if \( \delta \) is lower semicomputable and \( d(\{ x : l(x) = n, \delta(x) \geq m \}) \leq 2^{n-m} \), for all \( n \).

If \( \delta \) measures some non-random aspect of \( x \), for example the number of initial zeros, then the fraction of random strings with high values of \( \delta(x) \) should be low. Otherwise, the string is unlikely to be random. In the present paper, the task of a feature is to map out some non-random aspect of \( x \). Therefore, there should be some relationship between features and randomness tests, substantiated by the following theorems:
Theorem 10 (From features to randomness tests). For each feature $f$ of some finite string, the function

$$\delta(x) := \begin{cases} \max \{l(x) - l(r) - 1 : f(r) = x, l(f) + l(r) < l(x)\} & \text{if such an } r \text{ exists} \\ 0 & \text{otherwise} \end{cases}$$

(25)

is a uniform Martin-Löf test for randomness.

Example 1. Let $x = 0^n w$. Clearly, there is a feature $f$ that takes residual $r = \tilde{a}w$ and computes $x$, if $a$ is big enough to fulfill the compression condition. Since $l(r) = 2l(a) + 1 + l(w)$ and $l(x) = a + l(w)$, we obtain $\delta(x) \geq a - 2l(a) - 2$. For example, we can simply set $\delta(x) = a$, which is clearly a randomness test that counts the number of leading zeros of $x$ [16, Example 2.4.1].

Example 2. Consider a string that has $a$ 1’s at odd positions, $x = 1x_21x_41x_61x_8...z$ and continues with some arbitrary string $z$ eventually. There is a feature $f$ that takes residual $r = \tilde{a}x_2x_4...x_{2a}z$ and computes $x$, if $a$ is big enough to fulfill the compression condition. Since $l(r) = 2l(a) + 1 + a + l(z)$ and $l(x) = 2a + l(z)$ we obtain $\delta(x) \geq a - 2l(a) - 2$ again. We can choose

$$\delta(x) := a = \max \{i : x_1 = x_3 = \cdots = x_{2i-1} = 1\}$$

(26)

which is a randomness test [16] Example 2.4.3.

Conversely, there is also a map in the reverse direction, from randomness tests to features:

Theorem 11 (From randomness tests to features). For each uniform and unbounded Martin-Löf test for randomness $\delta$, there is a feature $f$ such that it is a feature of all $x$ fulfilling $\delta(x) > l(f)$.

Intuitively, all strings $x$ of fixed length that fulfill $\delta(x) \geq m$ for some test $\delta$ possess some non-random aspect, and there are few of them by definition of $\delta$. Therefore, a feature can be constructed to enumerate them.

Example 3. Consider again Example 2, now departing from the particular test in eq. (26) encoded by number $y$ in the standard enumeration of tests. There are $2^{l(z) - a}$ strings of length $l(x)$ with 1’s at the first $a$ odd positions. Clearly, if we encode $y$ and $a$ into a function $f$, it can print $x$ given the index $j$ of $x$ in that set.

3. Discussion

We have presented a theory of incremental compression of arbitrary finite data. It applies to any compressible data $x$ and suggests to decompose the compression endeavor into small independent pieces: the features. The main result, illustrated in Fig. 2(b), shows that ultimately the length of the obtained description of $x$ by features $f_1, \ldots, f_s$ and last residual $r_s$ reaches the optimal Kolmogorov complexity $K(x)$ to logarithmic precision, if the shortest possible features $f_i$ are selected at each incremental compression step.

3.1. Efficiency of incremental compression

The main motivation is to be able to compress data faster, since even by exhaustive search a set of short pieces can be expected to be easier to find than the full description at once. Indeed, if we neglect execution time, an exhaustive search for a program of length $K(x)$ takes about $T_{\text{non-incremental}} := 2^{K(x)}$ time steps. Inserting our result from eq. (26) leads to

$$T_{\text{non-incremental}} \approx 2^{K(r_s)} \prod_{i=1}^s 2^{l(f'_i)}$$

(27)

In contrast to that, searching the shortest features incrementally can be done in a greedy fashion: at every compression step we look for the shortest feature and can be certain to approximate the Kolmogorov complexity of the string well, without the necessity of backtracking. Since the shortest descriptive maps are very short, $l(f'_s) \approx 0$ (Theorem 4), the incremental search is expected to take merely

$$T_{\text{incremental}} \approx 2^{K(r_s)} + \sum_{i=1}^s 2^{l(f'_i)} \ll T_{\text{non-incremental}}$$

(28)

time steps.
3.2. Incomputability

Nevertheless, we can not really neglect the execution time which is the cornerstone of the halting problem. It is well-known that the Kolmogorov complexity is not only incomputable, but it can not even be approximated to any precision [follows from [10] Theorem 2.3.2]. The reason why we have found an approximation to \( K(x) \) is that our own algorithm is incomputable, due to the incomputability of the shortest features and the impossibility to effectively check for the incompressibility of the residual.

Unfortunately, a computable theory of incremental compression is not available yet, although some practical success can be claimed by WILLIAM – our Python-based implementation of incremental compression [5, 6]. Nevertheless, the presented line of reasoning provides reasons to conjecture that a computable incremental compression scheme would be much more efficient than other versions of universal search (e.g. Levin Search), while still staying general.

3.3. Generality

Speaking of generality, there is probably a price to be paid for the gained efficiency. The theory is only applicable to data possessing any properties, i.e. features at all. As the relationship to Martin-Löf randomness has demonstrated, properties – i.e. non-random aspects – guarantee the existence of features. Even though it is hard to imagine compressible data without any features at all, it might exist in abundance. Of course, compressible strings always possess at least one feature – the universal feature if the string is long enough (Lemma 1) or the total feature engulfing all of the information about \( x \) (Lemma 2). These degenerate features constitute extreme cases and contain either nonsignificant or all of the information about \( x \). However, the interesting question is how many nondegenerate cases exist, in which the information in \( x \) is divided into several chunks of intermediate size, which is expected to lead to the highest boost in efficiency, according to eq. \( \ref{eq:2} \). Unfortunately, in order to estimate this number, we would need to know how feature lengths are distributed. If strings with these nondegenerate, medium-sized features constitute a genuine subset of all data, it would render our theory non-universal. From a practical perspective however, the universe we inhabit seems to be teeming with features, wherever we look. In this sense, our theory may not be a theory of universal compression, but a compression theory for our universe.

3.4. Machine learning and compression

Machine learning models generalize better when the number of degrees of freedom of the employed models is small and the size of the data set large. In concordance with the reviewed Solomonoff theory of universal induction the generalization ability of a model is firmly tied to the idea of data compression. In fact, it is not an exaggeration to say that data compression is an essential property of machine learning in general, sometimes disguised as the minimum description length principle, bias-variance trade-off, various regularization techniques and model selection criteria [1, 3, 21]. For example, deep belief networks (DBNs) [8] consist of stacked autoencoders. In fact our theory of incremental compression can be viewed as an algorithmic generalization of deep belief networks, maybe even of deep learning in general, in so far as to show that compressing data in small incremental steps (such as neuronal layers) is a reasonable thing to do. Any transformation \( f \) from a description \( r \) to the data \( x \) can be viewed as a feature as long as some compression is achieved. In the context of machine learning, often \( f \) is fixed after learning and is required to represent as set of data sets \( x_1, \ldots, x_n \) with the respective descriptions \( r_1, \ldots, r_n \): \( f(r_i) = x_i \). In that respect, as long as \( n \) is large and \( l(r_i) < l(x_i) \), compression is achieved. Our theory predicts that compression (and thereby the generalization properties), will be best if the model \( f \) and the description \( r \) of data in that model (including noise) do not carry mutual information. This can be achieved by picking the simplest possible model \( f^* \) achieving compression. In the context of DBNs, \( f \) is the one-layer neural network generating visual neuron patterns \( x \) from the hidden neuron patterns \( r \). This observation raises doubts about whether a one-layer network can cover a broad enough set of features for arbitrary data and is not too biased toward a narrow class of transformations.

3.5. Searching for features in practice

Therefore, it is important to ask how features can be found in practice. If exhaustive search is to be used, the search might turn out to be slow. Interestingly, if we use some parametrized family of functions employing e.g. gradient descent on \( E(w) := (f_w(r) - x)^2 \), then an even shorter feature \( f \) can be found since the parameter \( w \) can be moved into the residual: \( f(w, r) = x \) since \( l(f) < l(f_w) \) and in practice probably even \( l(f) \ll l(f_w) \). This observation undermines the attempt to look for shortest features using a parametrized family of functions and emphasizes that exhaustive search might not be a bad idea, since the shortest features appear to be very short and it allows the discovery of very different functions. It does appear to work in practice [5, 6].

Nevertheless, the incomputability of shortest features raises the question about the consequences of picking not the shortest feature. From Theorem 8 we know that if \( f \) is not the shortest, we risk wasting \( d := l(f) - l(f^*) \) bits.
of description length on the data: \( K(f) + K(r) \leq K(x) + d \). If the model class \( f_w \) from which we pick the models is not appropriate for the data, then the shortest model from this class might still be much longer than the shortest model from the Turing complete class, \( l(f'_w) \gg l(f) \), and much description length would be wasted. Or, such as in the case of a single-layered neural network, the reconstruction \( f(r) \) might be far from the actual data \( x \) risking substantial information loss.

Apart from that, \( f \) not being shortest might spoil its incompressibility (Theorem 1) which might require compressing \( f \) further in order to achieve the smallest total description.

Note also that we should strive for lossless compression since the residual \( r \) contains both model parameters and noise. The reconstruction \( f(r) = x \) consists of computing the model prediction and adding the noise to retrieve the data \( x \) exactly. The “noise” is part of \( r \) since it may, in general, contain information not captured by the first model \( f_1 \), but which might in turn be captured by the next model \( f_2 \). This is reminiscent of principal component analysis, where at each step the principal component of the a residual description \( r_i \) is searched for.

Another consequence of not picking the shortest feature (i.e. the simplest model) could be that the residual description will be somehow messed up. In other words, it could contain noise unrelated to \( x \). Algorithmically expressed, it is a question about the value of \( K(r \mid x) \). We have encountered this term in Theorem 4 about superfluous information. Interestingly, this can not happen to a severe extent, even if the feature is not the shortest. This follows from the proof of the above-mentioned theorem where so-called first features of a fixed length are search for. If we fix the length of the feature \( l(f) = n \), then \( r \) can be computed from \( x \). Therefore, \( K(r \mid x) \leq K(r \mid x, n) + K(n) + O(1) \leq K(n) + O(1) = O(\log(l(f))) \), which is not much. For example, in the context of DBNs, if the first layer consists of weights \( w_1, \ldots, w_N \), then \( l(f) \approx \sum_{i=1}^N l(D(w_i)) \) where \( D \) is some way to describe floats to some precision. If the number of data-description pairs \((x_i, r_i)\) is large compared to that, the superfluous information in \( r \) will be small. Thus, while we could spoil some description length by overlap \( d \), the residual will not be spoiled much, meaning that by compressing the residual further we will be mostly describing the original data \( x \) and not irrelevant information. It might be confounded by information in \( f \), but since the superfluous information in the latter is also bounded by \( O(\log(l(f))) \), it does not matter much.

Another concern might be that an unfortunate autoencoder \((f_1, f'_1)\) might transform \( x \) into a description \( r_1 \) whose shortest feature \( f'_2 \) might be long and therefore difficult to find. After all, there is no guarantee, that there always exists a decomposition of a large set of short features, which would be optimal from the point of view of search efficiency. However, as we have seen in Theorem 7 a string will always have a feature whose length is bounded by a constant, if it is well-compressible. If it is not well-compressible, not much can be done anyway, since we are dealing with an almost random string.

It all sounds like good news, if we merely care about the amount of information but not computation time. For example, if an unfortunate, highly non-linear transformation \( f' \) distorts \( x \) into \( r \), the amount of “distortion” in \( r \) is not high only because we assume that we can find the reverse, “fixing” operation \( f \) by merely providing its length \( l(f) \) to an algorithm that searches through all \( 2^{l(f)} \) such reverse transformations until it finds the one that generates \( x \) from the distorted \( r \). In practice, \( l(f) \) is not available and executing \( 2^{l(f)} \) functions in parallel might simply be too computationally expensive. Therefore, even if the shortest program computing \( r \), with length \( K(r) \), is not much longer than \( K(x) - l(f') \approx K(r) - K(f'_r \mid x) \), we might end up spending much time “unwinding” the superfluous information \( K(f'_r \mid x) \) that has been inserted into \( r \). It is hence advisable to use short descriptive maps \( f' \) that are not able to inject much superfluous information into \( r \), due to the relationship \( l(f'_r) = K(r \mid x) \) in Lemma 3. In practice, for example, in the case of linear regression \( f' \) might be the least squares algorithm that finds the parameters of a linear function such as to minimize the reconstruction error. Apart from the parameters, the residual \( r \) will mostly consist of the reconstruction error which is not spoiled much by the transformation and can be compressed further if it contains information not captured by the linear function.

3.6. Conclusions

As we have seen in the discussion, the presented theory appears to lay a foundation for a more efficient and fairly general compression algorithm, aiming to be applied in practice, although several open questions remain. Most importantly, since many machine learning algorithms rely on good data compression the generality of the proposed compression scheme could help out those algorithms to overcome their narrowness and improve their performance. In the context of the theory of universal intelligence (10) this theory could be a fruitful way to derive more efficient formulations of the generally intelligent AIXI agent.

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## Appendix

**Proof of Lemma 7**

Let \( K(x) = K(x \mid e) < l(x) \), then there exists a string \( f \) such that \( U((\epsilon, f)) = f(\epsilon) = x, l(f) = K(x) \). Clearly, \( f \) is a feature, since with \( r = \epsilon \) the compression condition \( l(f) + l(r) = K(x) + 0 < l(x) \) is fulfilled. Thus for the shortest feature \( f^* \) we obtain \( l(f^*) \leq l(f) = K(x) \).

**Proof of Lemma 8**

Consider some finite string \( x \) and its shortest description \( \langle i, t \rangle \). Then \( U(\langle \langle i, t \rangle, \epsilon \rangle) = T_i(\langle \epsilon, t \rangle) = x \) and \( l(\langle i, t \rangle) = K(x) \). Define \( h(\langle \langle i, t \rangle, \epsilon \rangle) = T_i(\langle \epsilon, t \rangle) \), which is a kind of universal Turing machine. Let its number be \( j \) in the enumeration of Turing machines, \( h = T_j \). We obtain

\[
U(\langle \langle i, t \rangle, \langle \epsilon, t \rangle \rangle) = T_j(\langle \langle i, t \rangle, \epsilon \rangle) = T_i(\langle \epsilon, t \rangle) = x.
\]

(29)

Denote \( f_0 = \langle \epsilon, t \rangle, r = \langle i, t \rangle, C = l(f_0) \). Then, \( f_0(r) = x, l(r) = K(x), \) and \( l(f_0) + l(r) = C + K(x) < l(x) \), since \( x \) is compressible by more than \( C \) bits by assumption. Therefore, \( f_0 \) is a feature of \( x \).

**Proof of Lemma 9**

**Proof of \( l(f^*) = K(x \mid r) \).** Suppose the opposite \( K(x \mid r) = \min \{ l(z) : z(r) = x \} \neq l(f^*) \). This means that there exists a shorter program \( g \) with \( l(g) < l(f^*) \) such that \( g(r) = f^*(r) = x \). We have to check whether \( D_g(x) \neq \emptyset \) holds. First, we take a descriptive map \( g' := f^* \) and prove that \( g' \in D_g(x) \): We have \( g(g'(x)) = g(f^*(x)) = g(r) = x \) and \( l(x) > l(f^*) + l(f^*(x)) > l(g) + l(g'(x)) \). Therefore, \( g' \in D_g(x) \) and \( D_g(x) \neq \emptyset \) implying that \( g \) is a feature of \( x \). However, since \( f^* \) is defined as the shortest feature, the assumption \( l(g) < l(f^*) \) constitutes a contradiction.

**Proof of \( l(f^*) = K(r \mid x) \).** Suppose the opposite \( K(r \mid x) = \min \{ l(z) : z(x) = r \} \neq l(f^*) \). This means that there exists a shorter program \( g' \) with \( l(g') < l(f^*) \) such that \( g'(x) = f^*(x) = r \). Then \( g' \in D_f(x) \) since \( f^*(g'(x)) = f^*(r) = x \) and \( l(f^*) + l(g'(x)) = l(f^*) + l(f^*(x)) < l(x) \), which contradicts the assumption that \( f^* \) is the shortest program in \( D_f(x) \).

**Proof of Theorem 7**

Since \( x \) can be obtained from \( f^* \) and \( r \), we get

\[
l(f^*) = K(x \mid r) \leq K(f^* \mid r) + O(1) \leq K(f^*) + O(1)
\]

(30)

Moreover, the general property of prefix Kolmogorov complexity, \( K(f^*) \leq l(f^*) + K(l(f^*)) + O(1) \leq l(f^*) + O(\log l(f^*)) \), holds.

**Proof of Lemma 12**

If \( f \) is a feature then by definition there exists a residual \( r \) such that \( f(r) = x \) and \( l(f) + l(r) \leq l(x) \). Consider \( K(x \mid f) \) which is somewhat analogous to \( K(x \mid r) \) in Lemma 8. We can exchange \( f \) and \( r \) by running \( U(\langle f, (i, E_2(r)) \rangle) = T_i(\langle f, E_2(r) \rangle) = U(\langle r, f \rangle) = x \), where \( T_i \) swaps \( f \) and \( r \). In order to be able to do so, \( r \) has to be encoded in a self-delimiting way with \( E_2(r) \) for example. Using the invariance theorem, we find that \( K(x \mid f) \leq l(E_2(r)) + O(1) = l(r) + 2l(l(r)) + O(1) \) leading to

\[
K(x \mid f) \leq l(r) + O(\log l(r)) < l(x) - l(f) + O(\log l(x))
\]

(31)
Proof of Lemma 5

Using the symmetry of algorithmic information [10, Lemma 3.9.2]

\[ |I(f : x) - I(x : f)| \leq \log K(f) + \log K(x) + 2 \log \log K(f) + 2 \log \log K(x) + O(1) \] (32)

and inequalities \( K(x) \leq O(l(x)) \), and \( K(f) \leq O(l(f)) \) we obtain

\[ |I(f : x) - I(x : f)| \leq O(\log l(x)). \] (33)

It follows that \( K(f) = K(f \mid x) = I(x : f) \geq I(x : f) - O(\log l(x)) \geq I(f) - O(\log l(x)) \). Therefore, \( K(f \mid x) \leq K(f) - l(f) + O(\log l(x)) \leq O(\log l(f)) + O(\log l(x)) \leq O(\log l(x)) \), since \( K(f) \leq l(f) + O(\log l(f)) \) and \( l(f) < l(x) \).

\( \square \)

Proof of Theorem 2

If \( f \) is a feature then by Lemma 4 we have \( K(x \mid f) \leq l(x) - l(f) + O(\log l(x)) \). Let us consider two cases: if \( l(x) - K(x) > C \) for some constant \( C = l(f_0) \) then a universal feature \( f_0 \) of string \( x \) appears (see Definition 2 and Lemma 2), which limits number of shortest features by some constant (at most \( 2^{C^2+1} - 1 \)) since their length is not greater than \( l(f_0) = C \). In this case, \( K(f^* \mid x) \leq K(f^*) + O(1) \leq l(f^*) + 2 \log l(f^*) + O(1) \leq C + 2 \log C + O(1) = O(1) \).

Otherwise, if \( l(x) - K(x) \leq C \) for some constant \( C \), then \( K(x \mid f) \leq l(x) - l(f) + O(\log l(x)) \leq K(x) + C - l(f) + O(\log l(x)) = K(x) - l(f) + O(\log l(x)) \) since the constant \( C \) can be subsumed into \( O(\log l(x)) \). Thus, \( I(f : x) = K(x) - K(x \mid f) \geq l(f) - O(\log l(x)) \) and by Lemma 5 \( K(f \mid x) \leq O(\log l(x)) \). We know that the number of strings \( z \) fulfilling \( K(z \mid x) \leq a \) can not exceed the number of descriptions of maximal length \( a \), which is \( 2^{a+1} - 1 \). Therefore, for number of all features is not greater than \( 2^{O(\log l(x)) + 1} = 2^{O(\log l(x))} \), which implies at most polynomial growth of the number of all features \( O(l(x)^a) \) for some constant \( a \).

\( \square \)

Proof of Theorem 3

Proof of eq. (11). We know that \( f(r) = x \). Consider the shortest string \( \hat{f} \) with \( \hat{f}(r) = x \) and \( l(\hat{f}) = K(x \mid r) \). Then, \( l(\hat{f}) \leq l(f) \) and the compression condition \( l(\hat{f}) + l(r) \leq l(f) + l(r) \leq l(x) \) holds, since \( f \) is a feature.

Therefore, \( \hat{f} \) is also a feature. But since \( f^* \) is a shortest feature we know \( l(f^*) \leq l(\hat{f}) \leq l(f) + l(f) - l(\hat{f}) \leq l(f) - l(f^*) =: d \). Further, observe that

\[ K(f) \leq l(f) + K(l(f)) + O(1) \leq d + l(\hat{f}) + K(l(f)) + O(1) = d + K(x \mid r) + K(l(f)) + O(1) \] (34)

\[ d + K(x \mid r) + K(l(f)) + O(1) \leq d + K(f \mid r) + K(l(f)) + O(1) \] (35)

Here, the first and last inequalities are direct consequences of general properties of the prefix Kolmogorov complexity and of the equation \( f(r) = x \). Thus, \( I(r : f) := K(f) - K(f \mid r) \leq d + K(l(f)) + O(1) \leq d + O(\log l(f)) \).

Proof of eq. (13). Using the general expansion \( K(f \mid r) \leq K(f \mid r, K(r)) + K(K(r) \mid r) + O(1) \) and (11) we have

\[ K(f, r) = K(r) + K(f \mid r, K(r)) + O(1) \geq K(r) + K(f \mid r) - K(K(r) \mid r) - O(1) \geq K(r) + K(f) - d - K(l(f)) - K(r \mid r) - O(1) \geq K(r) + K(f) - d - O(\log l(f)) - O(\log l(r)) \] (36)

\[ K(r + K(f) - d - K(l(f)) - K(r \mid r) - O(1) \geq K(r) + K(f) - d - O(\log l(f)) - O(\log l(r)) \] (37)

\( \square \)

Proof of Corollary 4

Proof of eq. (15). We have \( K(f^*) \leq K(f^* \mid r) + O(\log l(f^*)) \leq K(f^*) + O(\log l(f^*)) \). The first inequality follows from Theorem 3 in the case \( f = f^* \), thus \( d = 0 \). The second inequality is a general property of Kolmogorov complexity.

Proof of eq. (14). Consider the Kolmogorov-Levin theorem to logarithmic precision (see, e.g. [22, Theorem 21] and note that the prefix and plain complexities coincide at logarithmic precision):

\[ K(f^*, r) = K(f^*) + K(r \mid f^*) + O(\log l(f^*)) = K(r) + K(f^* \mid r) + O(\log l(r)) \] (38)

Using the just proven equality we obtain: \( K(r \mid f^*) = K(r) + K(f^* \mid r) - K(f^*) + O(\log l(r)) + O(\log l(f^*)) = K(r) + O(\log l(r)) + O(\log l(f^*)) \)
Proof of eq. 13. We insert the just obtained result into eq. 38: $K(r, f^*) = K(f^*) + K(r | f^*) + O(\log l(f^*)) = K(f^*) + K(r) + O(\log l(r)) + O(\log l(f^*))$ □

Proof of Theorem 4

If $x$ and $l(f^*)$ is given, then consider all combinations of strings $g$ with fixed length $l(g) = l(f^*)$ and strings $q$ with $l(q) < l(x) − l(f^*)$ of which there are finitely many. Execute every $(g, q)$-pair as $g(q)$ in parallel, until one of them prints $x$. We know that some pair will halt and print $x$ at some point, since by assumption we know that there exists a $r$ such that $f^*(r) = x$ halts and that pair is part of the executed set. Let $(\tilde{f}, \tilde{r})$ be the first halting pair. Then the described algorithm implies

$$K(\tilde{f}, \tilde{r} | x, l(f^*)) = O(1)$$

while the compression condition $l(\tilde{f}) + l(\tilde{r}) < l(x)$ holds by construction. Therefore, $K(\tilde{f}, \tilde{r} | x) \leq K(l(f^*)) + O(1) = O(\log l(f^*)) = O(\log l(x))$. The boundedness by $O(\log l(x))$ is guaranteed, since $l(f^*) < l(x)$ by the compression condition.

Now, we exploit Theorem 3 that entails that the number of shortest features $d(F)$ is bounded by $2^{O(\log l(x))}$. In particular, any shortest feature $f^*$ can be encoded by an index $i_{f^*}$ bounded by $i_{f^*} = O(\log d(F))$. Therefore, $K(f^*, r | x, l(f^*), i_{f^*}) + O(1) = O(\log l(x))$ (40) by a similar line of reasoning. Let $f^*$ be shortest descriptive map corresponding to $f^*$. Using Lemma 3 we observe

$$l(f^*) = K(r | x) \leq K(f^*, r | x) + O(1) = O(\log l(x))$$

yielding the second result (eq. 17).

Finally, since $f^*(r) = x$, on the one hand, $K(x) \leq K(f^*, r) + O(1) \leq K(f^*) + K(r) + O(1)$. On the other hand, using Corollary 1 we obtain

$$K(f^*) + K(r) - O(\log l(f^*)) - O(\log l(r)) = K(f^*, r) + O(1) \leq K(f^*, r | x) + O(1) = K(x) + O(\log l(x))$$

(42)

Since $O(\log l(f^*)) + O(\log l(r)) = O(\log l(x))$ due to the compression condition, the third result (eq. 18) follows. □

Proof of Lemma 6

Using general properties of the prefix complexity, we expand up to additive constants

$$K(z) + K(y | z, K(z)) = K(z, y) = K(z | y, K(y)) + K(y) \leq K(y) + K(z | y)$$

(43)

and also look at the conditional version [10 eq. (3.22)]:

$$K(z | x) + K(y | z, K(z | x), x) = K(z, y | x) = K(z | y, K(y | x), x) + K(y | x) \geq K(y | x)$$

(44)

We subtract the inequalities and obtain:

$$I(x : z) \leq I(x : y) + K(y | z, K(z | x), x) - K(y | z, K(z)) + K(z | y)$$

(45)

Observe that $K(y | z, K(z | x), x) \leq K(y | z) \leq K(y | z, K(z)) + K(K(z) | z)$ (up to additive constants) leading to:

$$I(x : z) \leq I(x : y) + K(z | y) + K(K(z) | z) \leq I(x : y) + K(z | y) + O(\log l(z)).$$

(46) □
Proof of Theorem 5

First, we prove orthogonality for the case \( j > i \). We denote \( r_i = (f_{i+1}^* \circ f_{i+2}^* \circ \ldots \circ f_j^*)(r_j) \), where \( x = (f_{i+1}^* \circ f_{i+2}^* \circ \ldots \circ f_j^*)(r_i) \). Using the core idea of Theorem 4, we can prove that not much information is required to encode the shortest features \( f_{i+1}^*, f_{i+2}^*, \ldots, f_j^* \) given \( r_i \). \( f_j^* \) can be found by iteratively exploiting the algorithm described in above-mentioned theorem, substituting \( r_m \) for \( x \) and \( f_{m+1}^* \) for the \( n \)-th shortest feature \( f^* \), where \( m \) takes values from \( i \) to \( j - 1 \). Formally:

\[
K(f_j^* | r_i) \leq K\{f_j^* | r_i, l(f_{i+1}^*), \ldots, l(f_j^*), n_{i+1}, \ldots, n_j\} + \sum_{m=i+1}^{j} K(l(f_m^*)) + O((j - i) \log l(x)) \leq O((j - i) \log l(x))
\]

(47)

From Corollary 4, we know that \( I(f_j^* : r_i) = O(\log l(f_j^*)) + O(\log l(r_i)) = O(\log l(x)) \) due to the compression condition. We insert this into Lemma 6 (replacing \( x \rightarrow f_i^* \), \( y \rightarrow r_i \) and \( z \rightarrow f_j^* \)) to get an upper bound on \( I(f_i^* : f_j^*) \):

\[
I(f_i^* : f_j^*) \leq I(f_i^* : r_i) + K(f_j^* | r_i) + O(\log l(f_j^*)) \leq O((j - i) \log l(x))
\]

(48)

For the case \( j < i \) we first rename \( i \leftrightarrow j \): \( I(f_j^* : f_i^*) = O((i - j) \log l(x)) \). We exploit the symmetry of information [16, Lemma 3.9.2] and obtain:

\[
I(f_i^* : f_j^*) \leq \log K(f_j^*) + \log K(f_i^*) + 2 \log \log K(f_j^*) + 2 \log \log K(f_i^*) + O((i - j) \log l(x)) = O((i - j) \log l(x))
\]

(49)

(50)

Proof of Theorem 6

Iterating the relationship \( K(x) = l(f^*) + K(r) + O(\log l(x)) \) from Theorem 4 gives us the required result, since \( l(r_i) < l(x) \) for all \( i = 1, \ldots, s \) due to the iterated compression condition.

We show that \( D_s := \langle s, r_s, f_s^* \cdots f_1^* \rangle \) is a description of \( x \). A program of constant length can take the number of features \( s \), the last residual \( r_s \) and execute \( U(\bar{r}_s f_s^* \cdots f_1^*) \). By construction, \( U(\bar{r}_s f_s^*) \equiv U(\langle r_s, f_s^* \rangle) \) will halt with the input head at the start of the remainder \( f_{s-1}^* \cdots f_1^* \). Thus, no self-delimiting code of the features is necessary. The remainder can be concatenated to \( r_{s-1} \) and executed on \( U \) again: \( U(\bar{r}_{s-1} f_{s-1}^* \cdots f_1^*) \). This procedure is iterated \( s \) times until \( x \) is printed.

Proof of Lemma 7

In the following, we will not repeat all the proofs for \( b \)-features, but only highlight the parts in which special care has to be taken.

Lemma 1. The total feature \( f \) is always a \( b \)-feature, since its residual \( r = \epsilon \), and thus \( 0 = l(r) \leq \frac{l(x)}{b} \) holds trivially for any \( b \).

Lemma 2. To prove the equality \( l(f^*) = K(x | r) \) for the shortest \( b \)-feature we need to check whether \( D_{g,b}(x) \neq \emptyset \) holds. But \( g' = f^* \in D_{g,b}(x) \) since \( g' \in D_g(x) \) by the proof of Lemma 3 and \( l(g'(x)) = l(f^*(x)) = l(r) \leq l(x)/b \) since \( f^* \) is a \( b \)-feature by assumption of modified Lemma 3. Using similar arguments in the proof of \( l(f^*) = K(r | x) \), we show \( g' \in D_{f^*,b}(x) \).

Theorem 4. Theorem 1 is direct consequence of Lemma 3 and general properties of Kolmogorov complexity.

Lemma 3 and 4. Lemmas 4 and 5 hold for any features, and hence hold for \( b \)-features as well.
Theorem 2. Let us consider the analogue of Theorem 2 with shortest $b$-features instead of shortest features. If $l(x) - K(x) \leq C$ for some constant $C$ we know from Theorem 2 that the number of all features is not greater than $O(l(x)^n)$ for some constant $\alpha$ and $K(f | x) \leq O(\log l(x))$ for any feature, so this holds for $b$-features as well. If $l(x) - K(x) > C = l(f_0)$ and $K(x) \leq l(x)/b$ then the universal feature $f_0$ is also a $b$-feature (since for its residual $l(r) = K(x) \leq l(x)/b$). Thus, similarly to Theorem 2 the number of shortest features is limited by a constant $2^{C+1} - 1$ and for them $K(f^* | x) = O(1)$.

Consider the remaining case $K(x) > l(x)/b$ and denote $d = K(x) - \lfloor l(x)/b \rfloor$, where $\lfloor z \rfloor$ is the maximal integer not greater than the real number $z$. Divide shortest description $g$ of $x$ in two parts $g = g'g''$ where $l(g') = \lfloor l(x)/b \rfloor$ and $l(g'') = d$. Define some function $h((\epsilon, q)) := U((\epsilon, yq))$. Let $j$ be the number of $h$ in the standard enumeration of Turing machines, $h = T_j$. Thus,

$$U((g', (j, g''))) = T_j((g', g'')) = h((g', g'')) = U((\epsilon, g'g'')) = U((\epsilon, g)) = x,$$

(51)
since $g$ is a description of $x$. Therefore, $\hat{f} := (j, g'')$ is a $b$-feature with its residual $\hat{r} = g'$ if the compression condition holds: $l(\hat{f}) + l(\hat{r}) < l(x)$, since the other condition $l(\hat{r}) \leq l(x)/b$ holds by construction of $g'$. Let us calculate $l(\hat{f}) = 2l(j) + 1 + l(g'')$ and define a constant $\hat{C} = 2l(j) + 1$, so that $l(\hat{f}) = \hat{C} + d$. Therefore, $l(\hat{f}) + l(\hat{r}) = \hat{C} + d + l(g') = \hat{C} + K(x) - \lfloor l(x)/b \rfloor + \lfloor l(x)/b \rfloor = K(x) + \hat{C}$. The case $l(x) - K(x) \leq \hat{C}$ is similar to the already reviewed case $l(x) - K(x) \leq C$. Thus, consider the case $l(x) - K(x) > \hat{C}$. Then $l(\hat{f}) + l(\hat{r}) = K(x) + \hat{C} < l(x) - \hat{C} + \hat{C} = l(x)$ and $\hat{f}$ is indeed a $b$-feature of length $l(\hat{f}) = \hat{C} + d$.

For any shortest $b$-feature $f^*$ we have $K(x | f^*) \leq l(r) + O(\log l(r)) \leq \lfloor l(x)/b \rfloor + O(\log l(x))$ by Lemma 4. Then

$$I(f^* : x) := K(x) - K(x | f^*) \geq K(x) - \lfloor l(x)/b \rfloor - O(\log l(x)) =$$

$$d - O(\log l(x)) = l(\hat{f}) - \hat{C} - O(\log l(x)) \geq l(f^*) - O(\log l(x)),$$

(52)
since $l(f^*) \leq l(\hat{f})$ and the constant $\hat{C}$ can be subsumed into $O(\log l(x))$. By Lemma 5 we conclude $K(f^* | x) \leq O(\log l(x))$ for all shortest $b$-features $f$ of $x$. Similarly to the proof of Theorem 2 the number of shortest $b$-features is not greater than $O(l(x)^n)$ for some constant $\alpha$.

Theorem 3 and Corollary 1. In the modified Theorem 3 $\hat{f}$ is a $b$-feature since $l(r) \leq l(x)/b$ because $f$ is a $b$-feature by assumption of modified Theorem 3. The remainder of the proof is analogous. Therefore, Corollary 1 is also true for shortest $b$-features.

Theorem 4. We need to consider all combinations of strings $g$ with fixed length $l(g) = l(f^*)$ and strings $q$ with $l(q) < l(x) - l(f^*)$ and $l(q) < l(x)/b$ of which there are finitely many. The remainder of the proof is similar to the original.

Theorem 5 and Theorem 6. Since Theorem 5 follows from Theorem 4 Corollary 1 and Lemma 6 it is also true for shortest $b$-features. The same arguments are valid for Theorem 6 which follows from Theorem 4.

Proof of Theorem 7

Let $f_0$ be a universal feature and $C = l(f_0)$. After Lemma 2 its residual $r$ is the shortest description: $l(r) = K(x)$. $f_0$ becomes a feature of $x$ if the compression condition $l(f_0) + l(r) = C + K(x) < l(x)$ is fulfilled, which is the case if $x$ is long enough: $l(x) > \frac{C_0}{b-1}$. After all, since $x$ is $b$-compressible,

$$l(f_0) + l(r) = C + K(x) \leq C + \frac{l(x)}{b} < l(x) \cdot \frac{b-1}{b} + l(x) \cdot \frac{1}{b} = l(x)$$

(53)

Note that $f_0$ is also a $b$-feature, since $l(r) = K(x) \leq l(x)/b$. Let $f^*$ be a shortest feature $x$. Then, $l(f^*) \leq l(f_0) = C$. Since $f_0$ is a $b$-feature, the length of a shortest $b$-feature is also bounded by $C$. Conversely, if $l(x)$ is not long enough, using Lemma 1 $l(f^*) \leq K(x) \leq l(x)/b \leq C$. In both cases, $l(f^*)$ is bounded by a constant for both shortest feature and shortest $b$-feature.
Proof of Theorem 8

Since \( f \) is a feature, there is a residual \( r \) fulfilling \( f(r) = x \) and \( l(r) < l(x) - l(f) \). Consider all strings shorter than \( l(x) - l(f) \) and execute them in parallel on \( f \). Denote the algorithm performing this with \( S(f, x) \). This algorithm will halt at some point, since \( r \) is among the executed strings. Let \( q \) be the first string that prints \( x \). Then \( S(f, x) = q \) and \( K(q \mid f, x) = O(1) \) since \( S \) is an algorithm of constant length. Encode both \( S \) and \( f \) into a descriptive map \( g' \) operating like \( g'((x) := S(f, x) = q \). Since \( l(f) = O(1) \), we conclude \( l(g') = O(1) \) as well and \( K(q \mid x) \leq l(g') = O(1) \). Now define \( f^* \) as a shortest descriptive map, that is \( f^* \in D_f \) and \( l(f^*) \) is minimal. Then \( l(f^*) \leq l(g') = O(1) \) and \( K(f^* \mid x) = l(f^*) = O(1) \). □

Proof of Theorem 10

Since \( D_x := (s, r_s, f_s^1 \cdots f_s^n) \) is a description of \( x \), we get \( l(D_x) \geq K(x) - O(1) \). Since \( l(f_x^s) \leq C \) by Theorem 7 we have

\[
l(D_x) = l((s, r_s, f_s^1 \cdots f_s^n)) = l(r_s) + O(l(r_s)) + O(s)
\]

In the first case, \( l(r_s) \leq \frac{C_b}{b-1} \). Then \( l(D_x) = O(s) \), since \( \frac{C_b}{b-1} \) is a constant independent of \( x \). In the second case, \( r_s \) is not \( b \)-compressible, ergo \( l(r_s) \leq K(r_s) b \leq (K(x) + O(s)) b = K(x) b + O(s) \), since \( r_s \) can be computed from \( x \) by a sequence of descriptive maps, each of which is bounded by a constant (so \( K(r_i+1) \leq K(r_i) + O(1) \) and \( K(r_s) = K(x) + O(s) \)). It follows: \( l(D_x) \leq K(x) b + O(l(r_s)) + O(s) \leq K(x) b + O(l(x)) + O(s) \).

The bound on \( s \) is derived from the application of the factor \( b \) to each residual, \( l(r_i) \leq l(r_{i-1})/b \), leading to \( l(r_{i-1}) \leq l(x)/b^{i-1} \). Since \( r_{i-1} \) is compressible it cannot be the empty string \( \epsilon \), hence \( l(r_{i-1}) \geq 1 \) and we obtain the bound \( s \leq \log_b l(x)/l(r_{i-1}) + 1 \leq \log_b l(x) + 1 = O(l(x)) \). □

Proof of Theorem 11

Fix \( f \). We define \( \phi(t, x) \) as follows: For each \( x \), run feature \( f \) for \( t \) steps on each residual \( r \) of length less than \( l(x) - l(f) \). If for any such input \( r \) the computation halts with output \( x \), then define \( \phi(t, x) := l(x) - l(r) - 1 \) using the shortest such \( r \), otherwise set \( \phi(t, x) := 0 \). Clearly, \( \phi(t, x) \) is recursive, total, and monotonically nondecreasing with \( t \) (for all \( x \), \( \phi(t', x) \geq \phi(t, x) \) if \( t' > t \)). The limit exists, since for each \( x \) either no such \( r \) is found, making \( \phi(t, x) = 0 \) for all \( t \), or a shortest \( r \) is found eventually. Therefore, \( \lim_{t \to \infty} \phi(t, x) = \delta(x) \) and we have shown that \( \delta \) is lower semicomputable.

Consider all \( x \) with length \( n \). The case \( m = 0 \) is trivial, so consider the case \( m \geq 1 \). For each \( x \) that meets condition \( \delta(x) \geq m \) there has to exist some \( r \) with \( f(r) = x \), \( l(f) + l(r) < l(x) \), and \( n - l(r) - 1 \geq m \). Therefore, \( l(r) \leq n - m - 1 \) and the number of such \( r \) is bounded by \( \sum_{i=0}^{n-m-1} 2^i = 2^{n-m} - 1 \). Since different \( x \)’s require different \( r \)’s (they are executed on the same \( f \)), the number of such \( x \) is bounded by the same expression. □

Proof of Theorem 12

Let the set \( V^m_n \) be defined as

\[
V^m_n := \{ x : \delta(x) \geq m, l(x) = n \}
\]

The lower semicomputability of \( \delta \) implies that \( V^m_n \) is recursively enumerable. We have defined \( V^m_n \) such that for any \( x \) fulfilling condition \( \delta(x) \geq m \) we have \( x \in V^m_{l(x)} \) and \( V^m_{l(x)} \leq 2^{l(x) - m} \) with \( m \) to be fixed later. If \( V^m_{l(x)} \) is not empty than \( l(x) - m \geq 0 \). Let \( \delta = \delta_y \) in the standard enumeration \( \delta_1, \delta_2, \ldots \) of tests. Given \( y, m \) and \( l(x) \), we have an algorithm to enumerate all elements of \( V^m_{l(x)} \). Together with the index \( j \) of \( x \) in enumeration order of \( V^m_{l(x)} \), this suffices to find \( x \). We pad the standard binary representation of \( j \) with nonsignificant zeros to a string \( r = 00 \ldots 0j \) of length \( l(x) - m \). This is possible since \( d(V^m_{l(x)}) \leq 2^{l(x) - m} \), thus any index of the element can be encoded by \( l(r) = l(x) - m \) bits. The purpose of changing \( j \) to \( r \) is that now the length \( l(x) \) can be deduced from \( l(r) \) and \( m \). In particular, we can encode \( y \) and \( m \) into a string \( f \) corresponding to a Turing machine that computes \( x \) from input \( r \). This shows the existence of a string \( r \) with \( f(r) = x \) for any \( x \) with \( \delta(x) \geq m \) (for fixed \( m \)).

The compression condition becomes \( l(x) > l(f) + l(r) = l(f) + l(x) - m \), hence we require \( l(f) < m \). Since it takes at most \( 2\log m \) bits to encode \( m \) into \( f \) this inequality can always be fulfilled for large enough \( m \). Moreover, it is possible to fulfill \( l(f) = m - 1 \) by adding some unnecessary bits to \( f \). Let us fix some appropriate \( m \) and \( f \) such that \( l(f) = m - 1 \). Therefore, \( f \) is indeed a feature of all \( x \) with \( \delta(x) \geq m \), which is equivalent to \( \delta(x) > l(f) \). □
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