What is the spirit of the cylindric paradigm, as opposed to that of the polyadic one?

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Abstract

The question posed in the title is more of a philosophical nature. Even more, departing from the materialistic point of view that spirits do not exist, it is meta-physical, and so from the positivist point of view it is meaningless.

However, we approach the problem in a completely rigorous mathematical way; and we give an answer that to our mind is quite satisfactory using deep concepts in algebraic logic wrapped in the language of arrows, better known as category theory.

For a start, the following two questions are investigated for cylindric-like algebras:

1. Given ordinals \( \alpha < \beta \) and an algebra of dimension \( \alpha \), does it (neatly) embed into the \( \alpha \) reduct of a \( \beta \) dimensional algebra? And if it does, does it neatly embed into the \( \alpha \) reduct of a \( \beta + k \) dimensional algebra for some \( k \geq 1 \).

2. Suppose that \( \mathfrak{A} \) has the neat embedding property, so that \( \mathfrak{A} \) actually embeds into the neat \( \alpha \) reduct of an algebra \( \mathfrak{A} \) in \( \omega \) extra dimensions, is this last algebra, called a dilation, uniquely determined by \( \mathfrak{A} \) in some sense?

For the first question we show that the answer is no for many cylindric-like algebras of relations (like quasi-polyadic algebras), for both finite and infinite dimensions. We give a categorial answer to the question in the title, encompassing an answer to the second question. We show that the uniqueness of the minimal dilation obtained when the small algebra generates the dilation, depends on the adjointness of the neat embedding operator, viewed as a functor. For polyadic algebras the neat reduct functor (that has to do with compressing dimensions) is strongly invertible, while for cylindric algebras, and its likes, it does not even have a right adjoint (a functor that stretches dimensions.)

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1 Introduction

In our treatment of (notation and concepts on) neat reducts we follow [47] whose notation is consistent with [24]. We have only strayed from this principle only when we felt there was a compelling reason, and this happens only once. We denote the restriction of a function \( f \) to a set \( X \) by \( f \upharpoonright X \) and not the other way round as done in [24]. Let \( K \) be a cylindric-like class (like for instance quasi-polyadic algebras), so that for all \( \alpha \), \( K_\alpha \) is a variety consisting of \( \alpha \) dimensional algebras, and for \( \alpha < \beta \), \( Nr_\alpha \mathcal{B} \) and \( Rd_\alpha \mathcal{B} \) are defined, so that the former, the \( \alpha \) neat reduct of \( \mathcal{B} \), is a subalgebra of the latter, the \( \alpha \) reduct of \( \mathcal{B} \) and the latter is in \( K_\alpha \).

We address the following two questions on neat embeddings:

1. Given ordinals \( \alpha < \beta \) and \( A \in K_\alpha \) is there a \( B \in K_\beta \) such that
   (i) \( A \) embeds into the \( \alpha \) reduct of \( B \)?
   (ii) \( A \) embeds into the \( \alpha \) neat reduct of \( B \)?

2. Assume that it does neatly embed into \( B \), and assume further that \( A \) (as a set) generates \( B \), is then \( B \) unique up to isomorphisms that fix \( A \) pointwise?

If \( \mathcal{B} \) and \( A \) are like in the second item, then \( \mathcal{B} \) is called a minimal dilation of \( A \).

Our second question is motivated by the following quote of Henkin Monk and Tarski [24]:

Unless specified to the contrary ordinals considered are always infinite. It will be shown in Part II that for each \( \alpha, \beta \) such that \( \beta \geq \alpha \geq \omega \) there is a CA\( \alpha \) \( A \) and a CA\( \beta \) \( \mathcal{B} \) such that \( A \) is a generating subreduct of \( \mathcal{B} \) different from \( Nr_\alpha \mathcal{B} \); in fact, both \( \mathcal{A} \) and \( \mathcal{B} \) can be taken to be representable. Thus DC\( \alpha \) cannot be replaced by CA\( \alpha \) in Theorem 2.6.67 (ii); it is known that this replacement also cannot be made in certain consequences of 2.6.67, namely 2.6.71 and 2.6.72.

This result was not proved in [25] as promised, but it was proved by the present author with a precursor; a joint publication with Istvan Németi [48]. The solution is announced in [39] and presented briefly in [47].

The main result in [39] is that minimal dilations for representable algebras that are not dimension complemented are not unique up to isomorphisms that fix the base algebra pointwise. We will see that this question has an elegant categorial formulation and answer.

Much of the beauty of mathematics, and in particular, category theory, is that it affords abstraction. Not only does it allow one to see the forest rather than the trees, but it also offers the possibility for study of the structure of the
entire forest in preparation for the next stage of abstraction comparing forests
and then, perhaps even, comparing forests of forests.

Category theory provides an entirely new language, a language that pro-
vides economy of thought and expression as well as allowing easier commun-
ication among investigators in different areas, it is a language that brings to
the forefront the common basic ideas underlying ostensibly unrelated theorems
and hence a language that gives a new context in which to view old problems.
In a wider perspective, this new context allows vieweing the old problem in a
wider framework, with new insight.

As category theory was not mature enough, at the time part one of the
monograph cylindric algebras was published, at the present time the question
raised by Henkin et all can be formulated more succintly using categorial jarg-
on. The question, in retrospect, is essentially equivalent to the more elegant
and concise question as to whether the neat reduct operator viewed (in a nat-
ural way) as functor, that compresses dimensions, has - using categorial jargon
- a right adjoint, or using polyadic algebra jargon, if you like, a dilation. The
answer is no.

Here we investigate the analogous question for many cylindric -like alge-
bras. We will discover that the answer depends essentially on the invertibility
of the neat reduct operator, viewed as a functor. For cylindric-like algebras
this functor is not even weakly invertible but for polyadic-like ones it is, and
strongly so.

The paper intends to replace trips into algebraic territory by the use of
category theory. Even so, the trade between algebraic logic, logic, and category
theory remains interesting, even when it is not a matter of applying concrete
theorems, but exporting more universal ideas.

Category theory has also the supreme advantage of putting many existing
results scattered in the literature, in their proper perspectives highlighting in-
terconnections, illuminating differences and similarities, despite the increasing
tendencies toward fragmentation and specialization, in mathematical logic in
general, and in even more specialized fields like algebraic logic.

Throughout this article, the high level of abstraction embodied in category
theory and in dealing with highly abstract notions, like systems of varieties
definable by a schema, is motivated and exemplified by well known concrete
examples, so that this level of abstraction can be kept from becoming a high
level of obfuscation.

The categorial approach adopted here is not merely a formal wrapping, on
the contrary, it is an emphasis that general mathematical sophistication, or
essayistic common sense, is the more appropriate road towards insight than
elaborate logical formal systems. Insights found in category theory really live
at some higher generic abstraction level that can often be brought out better
in an approach originating from algebraic logic, and indeed from most concrete
techniques available in connection to the notion of representability.

The notion of neat reducts \[47\] are intimately related to representability (the most important notion in algebraic logic). Henkin’s completeness proof, has radically influenced model theory and for that matter algebraic logic, and more. This technique which exists almost everywhere, whenever we encounter completeness or interpolation, or an omitting types setting, for various predicate logics, is now called simply a Henkin construction. It has since unfolded into a sophisticated and versatile proof technique, in many branches in (algebraic) logic, and beyond. Its re-incarnation in algebraic logic has come under the name of the Neat Embedding Theorem, which is an algebraic version of a completeness theorem for certain fairly standard extensions for first order logic that are in some sense more basic.

On the border line, there are cylindric-like algebras for which the neat reduct functor is strongly invertible, too. Some of these are classical in the sense that, like cylindric algebras, their cylindrifiers commute, others are obtained by removing the ‘Rosser condition’ of commutativity of cylindrifiers, a theme that can be traced back to the Andréka-Resek-Thompson result, inspired by Leon Henkin (in analogy to considering two sorted first order logic to provide semantics for second order logic) and severely boosted by Ferenczi, in his recent inspiring work on neat embeddings of non-commutative algebras that are representable only by relativized set algebras, cf. \[6\], \[7\], \[8\], \[9\], \[11\]. Such results can be also viewed as a fruitful contact between neat embedding theorems and relativized representations. The algebraisation process is a powerful strategy, and it works modulo modest requirements on the base logic. But as with general models, the conspicuous possibilities lie in between. This typically involves varying the ’semantic parameter’; this was started by Leon Henkin and his student Resek, and has culminated in incredibly sophisticated representation theorems \[11\].

Let us start from the very beginning. The first natural question that can cross one’s mind is: Is it true that every algebra neatly embeds into another algebra having only one extra dimension?, having \(k\) extra dimension, \(k > 1\) (\(k\) could be infinite) ? And could it possibly happen that an \(\alpha\) dimensional algebra neatly embeds into \(\alpha + k\) dimensions but does not neatly embed into \(\alpha + k + 1\) extra dimension? These are all fair questions, and indeed difficult to answer. Such questions have provoked extensive research that have engaged algebraic logicians for years, and they were all (with the exception of the infinite dimensional case solved here for cylindric algebras using existing finite dimensional constructions) settled by the turn of the millennium after thorough dedicated trials, and dozens of publications providing partial answers. We will show that this is indeed the case for finite dimensions \(\geq 3\), this is a known result for cylindric algebras due to Hirsch, Hodkinson and Maddux, as well as for infinite dimensions, which will follow from the finite dimensional
case using an ingenious lifting argument of Monk’s. The infinite dimensional case was also proved by Robin Hirsch and Sayed Ahmed for other algebras (Like Pinter’s substitution algebras) for all dimensions, using the same lifting argument here to pass from the finite to the transfinite.

To make our argument as general as much as possible, we introduce the new notion of a system of varieties of Boolean algebras with operators definable by a schema. It is like the definition of a Monk’s schema, except that we integrate finite dimensions, in such a way that the \( \omega \) dimensional case, uniquely determining higher dimensions, is a natural limit of all \( n \) dimensional varieties for finite \( n \). This is crucial for our later investigations. The definition is general enough to handle our algebras, and narrow enough to prove what we need. At the final section we will define a system definable by a schema that covers also polyadic algebras.

But for the time being as we are only dealing with the cylindric paradigm, we slightly generalized Monk’s schemas allowing finite dimensions, but not necessarily all, so that our systems are indexed by all ordinals \( \geq m \) and \( m \) could be finite. (We can allow also proper infinite subsets of \( \omega \), but we do not need that much.) The main advantage in this approach is that it shows that a lot of results proved for infinite dimensions (like non-finite schema axiomatizability of the representable algebras) really depend on the analogous result proved for every finite dimension starting at a certain finite \( n \) which is usually 3. The hard work is done for the finite dimensional case. The rest is a purely syntactical ingenious lifting process invented by Monk.

**Definition 1.**
(i) Let \( 2 \leq m \in \omega \). A finite \( m \) type schema is a quadruple \( t = (T, \delta, \rho, c) \) such that \( T \) is a set, \( \delta \) and \( \rho \) maps \( T \) into \( \omega \), \( c \in T \), and \( \delta c = \rho c = 1 \) and \( \delta f \leq m \) for all \( f \in T \).

(ii) A type schema as in (i) defines a similarity type \( t_n \) for each \( n \geq m \) as follows. The domain \( T_n \) of \( t_n \) is

\[
T_n = \{(f, k_0, \ldots, k_{\delta f - 1}) : f \in T, k \in \delta f n\}.
\]

For each \( (f, k_0, \ldots, k_{\delta f - 1}) \in T_n \) we set \( t_n(f, k_0 \ldots, k_{\delta f - 1}) = \rho f \).

(iii) A system \( (K_n : n \geq m) \) of classes of algebras is of type schema \( t \) if for each \( n \geq m \) \( K_n \) is a class of algebras of type \( t_n \).

**Definition 2.** Let \( t \) be a finite \( m \) type schema.

(i) With each \( m \leq n \leq \beta \) we associate a language \( L'_n \) of type \( t_n \); for each \( f \in T \) and \( k \in \delta f n \), we have a function symbol \( f_{k_0 \ldots k_{\delta f - 1}} \) of rank \( \rho f \)

(ii) Let \( m \leq \beta \leq n \), and let \( \eta \in ^\beta n \) be an injection. We associate with each term \( \tau \) of \( L'_n \) a term \( \eta^+ \tau \) of \( L'_n \). For each \( \kappa, \omega, \eta^+ v_k = v_k \). if
$f \in T, k \in \delta f \alpha$, and $\sigma_1 \ldots \sigma_{\rho f - 1}$ are terms of $L^t_\beta$, then

$$\eta^+ f_{k(0)} \ldots k(\delta f - 1) \sigma_0 \ldots \sigma_{\rho f - 1} = f_{\eta(k(0))} \ldots \eta(\delta f - 1) \eta^+ \sigma_0 \ldots \eta^+ \sigma_{\rho f - 1}. $$

Then we associate with each equation $\epsilon = \sigma = \tau$ of $L^t_\beta$ the equation $\eta^+ \sigma = \eta^+ \tau$ of $L^t_\alpha$, which we denote by $\eta^+(\epsilon)$.

(iii) A system $\mathbf{K} = (\mathbf{K}_n : n \geq m)$ of finite $m$ type schema $t$ is a complete system of varieties definable by a schema, if there is a system $(\Sigma_n : n \geq m)$ of equations such that $\text{Mod}(\Sigma_n) = \mathbf{K}_n$, and for $n \leq m < \omega$ if $e \in \Sigma_n$ and $\rho : n \rightarrow m$ is an injection, then $\rho^+ e \in \Sigma_m$; $(\mathbf{K}_\alpha : \alpha \geq \omega)$ is a system of varieties definable by schemes and $\Sigma_\omega = \bigcup_{n \geq m} \Sigma_n$.

Definition 3. (1) Let $\alpha, \beta$ be ordinals, $\mathfrak{A} \in \mathbf{K}_\beta$ and $\rho : \alpha \rightarrow \beta$ be an injection. We assume for simplicity of notation that in addition to cylindrifiers, we have only one unary function symbol $f$ such that $\rho(f) = \delta(f) = 1$. (The arity is one, and $f$ has only one index.) Then $\mathfrak{M}_\alpha^\rho \mathfrak{A}$ is the $\alpha$ dimensional algebra obtained for $\mathfrak{A}$ by defining for $i \in \alpha$, $f_i$ by $f_i^{(n)}$. $\mathfrak{M}_\alpha^\rho \mathfrak{A}$ is $\mathfrak{M}_\alpha^{\rho_i} \mathfrak{A}$ when $\rho$ is the inclusion.

(2) As in the first part we assume only the existence of one unary operator with one index. Let $\mathfrak{A} \in \mathbf{K}_\beta$, and $x \in A$. The dimension set of $x$, denoted by $\Delta x$, is the set $\Delta x = \{ i \in \alpha : c_i x \neq x \}$. We assume that if $\Delta x \subseteq \alpha$, then $\Delta f(x) \leq \alpha$. Then $\mathfrak{N}_\alpha \mathfrak{B}$ is the subuniverse of $\mathfrak{M}_\alpha \mathfrak{B}$ consisting only of $\alpha$ dimensional elements.

(3) For $K \subseteq \mathbf{K}_\beta$ and an injection $\rho : \alpha \rightarrow \beta$, then $\mathfrak{M}_\alpha^\rho K = \{ \mathfrak{M}_\alpha^\rho \mathfrak{A} : \mathfrak{A} \in K \}$ and $\mathfrak{N}_\alpha K = \{ \mathfrak{N}_\alpha \mathfrak{A} : \mathfrak{A} \in K \}$

The class $\mathfrak{N}_\alpha K_{\alpha + \omega}$ has special significance since it coincides in the most known cases to the class of representable algebras. In the next theorem, we show how properties that hold for all finite reducts of an infinite dimensional algebra forces it to have the neat embedding property. The proof does not use any properties not formalizable in systems of varieties definable by a schema; it consists of non-trivial manipulation of reducts and neat reducts via ultraproduct constructions, used to ‘stretch’ dimensions.

In the following theorem, we use a very similar argument of lifting to solve problem 2.12 in [21] for infinite dimensions. So let us warm up by the first lifting argument; for the second, though in essence very similar, will be more involved technically.

Theorem 4. Let $\mathfrak{A} \in \mathbf{K}_\alpha$ such that for every finite injective map $\rho$ into $\alpha$, and for every $x, y \in A, x \neq y$, there is a function $h$ and $k < \alpha$ such that $h$ is an endomorphism of $\mathfrak{M}_\alpha^\rho \mathfrak{A}$, $k \in \alpha \sim \text{Rng}(\rho)$, $c_k \circ h = h$ and $h(x) \neq h(y)$. then $\mathfrak{A} \in \text{Up} \mathfrak{N}_\alpha K_{\alpha + \omega}$.
We now show that there exists $B$ for every injection $\rho : k \to \alpha$, and every $x, y \in A$, $x \neq y$, there exists $\sigma, h$ such that $\sigma : k + l \to \alpha$ is an injection, $\rho \subseteq \sigma$, $h$ is an endomorphism of $\mathcal{R}_d^{\sigma}A$, $c_{\sigma} \circ h = h$, whenever $k \leq u \leq k + l$, and $h(x) \neq h(y)$.

We proceed by induction on $l$. This holds trivially for $l = 0$, and it is easy to see that it is true for $l = 1$. Suppose now that it holds for given $l \geq 1$. Consider $k, \rho$, and $x, y$ satisfying the premisses. By the induction hypothesis there are $\sigma, h$, $\sigma : \alpha + l \to \alpha$ an injection, $\rho \subseteq \sigma$, $h$ is an endomorphism of $\mathcal{R}_d^{\sigma}A$, $c_{\rho} \circ h = h$ whenever $k \leq u \leq k + l$, and $h(x) \neq h(y)$. But then there exist $x, y$ such that $k$ is an endomorphism of $\mathcal{R}_d^{\sigma + l}A$, $v \in \alpha \sim Rg\sigma$, $c_v \circ k = k$ and $k \circ h(x) \neq k \circ h(y)$. Let $\sigma'$ be defined by $\sigma' \upharpoonright \alpha + (l + k) = \sigma$ and $\sigma'(k + l) = v$, and let $h' = = \sigma h$. It is easy to check that $\sigma'$ and $h'$ complete the induction step.

We have $h \in \text{Hom}(\mathcal{R}_d^{\alpha}A, \mathcal{R}_d^{\sigma}B)$ where $B = \mathcal{R}_d^{\rho + l}A$. Then $\mathcal{R}_d^{\alpha}A \subseteq \mathcal{S}_{\mathcal{R}_d^{\alpha}}A$. For brevity let $\mathcal{D} = \mathcal{R}_d^{\alpha}A$. For each $\beta < \omega$, let $\mathcal{G}_\beta \in K_{\beta + l}$ such that $\mathcal{D} \subseteq \mathcal{R}_d^{\alpha}B$. For all such $\beta$, let $C_\beta$ be an algebra have the same similarity type as of $K_\omega$ be such $B_\beta = \mathcal{R}_d^{\alpha + l}C_\beta$. Let $F$ be a non-principal ultrafilter on $\omega$, and let $\mathcal{G} = \Pi_{\eta<\omega}C_\beta/F$. Let

$$G_n = \{ \Gamma \cap (\omega \sim n) : \Gamma \in F \}.$$ 

Then for all $\mu < \omega$, we have

$$\mathcal{R}_d^{\alpha + \mu}B = \Pi_{\eta<\omega}\mathcal{R}_d^{\alpha + \mu}C_\eta/F \cong \Pi_{\mu \leq \eta<\omega}\mathcal{R}_d^{\alpha + \mu}C_\eta/G_\mu = \Pi_{\mu \leq \eta<\omega}\mathcal{R}_d^{\beta + \mu}B_\eta/G_\mu.$$ 

We have shown that $\mathcal{G} \in K_\omega$. Define $h$ from $\mathcal{D}$ to $\mathcal{G}$, via

$$x \to (x : \eta < \omega)/F.$$ 

Then $h$ is an injective homomorphism from $\mathcal{D}$ into $\mathcal{R}_d^{\alpha}B$. We have $\mathcal{G} \in K_\omega$.

We now show that there exists $\mathcal{B}$ in $K_{\alpha + \omega}$ such that $\mathcal{D} \subseteq \mathcal{R}_d^{\alpha}B$. (This is a typical instance where reducts are used to 'stretch dimensions', not to compress them). One proceeds inductively, at successor ordinals (like $\omega + 1$) as follows. Let $\rho : \omega + 1 \to \omega$ be an injection such that $\rho(i) = i$, for each $i \in \alpha$. Then $\mathcal{R}_d^{\rho}B \subseteq K_{\omega + 1}$ and $\mathcal{G} = \mathcal{R}_d^{\rho}B$. At limits one uses ultraproducts like above.

Thus $\mathcal{R}_d^{\alpha}A \subseteq \mathcal{R}_d^{\alpha}B$ for some $\mathcal{B} \in K_{\alpha + \omega}$. Let $\sigma$ be a permutation of $\alpha + \omega$ such that $\sigma \upharpoonright k = \rho$ and $\sigma(j) = j$ for all $j \geq \omega$. Then

$$\mathcal{R}_d^{\alpha}A \subseteq \mathcal{R}_d^{\alpha}B = \mathcal{R}_d^{\sigma}A \mathcal{R}_d^{\sigma^{-1}}B.$$ 

Then for any $u$ such that $\sigma[k] \subseteq u \subseteq \alpha + \omega$, we have

$$\mathcal{R}_d^{\sigma}A \mathcal{R}_d^{\sigma^{-1}}B \subseteq \mathcal{R}_d^{\sigma_{\omega}}\mathcal{R}_d^{\sigma^{-1}}B.$$
Thus $\mathcal{R}_k^\alpha \mathfrak{A} \in S\mathcal{R}_k^\alpha \mathfrak{M}_\alpha K_{\alpha+\omega}$, and this holds for any injective finite sequence $\rho$.

Let $I$ be the set of all finite one to one sequences with range in $\alpha$. For $\rho \in I$, let $M_{\rho} = \{ \sigma \in I : \rho \subseteq \sigma \}$. Let $U$ be an ultrafilter of $I$ such that $M_{\rho} \in U$ for every $\rho \in I$. Exists, since $M_{\rho} \cap M_{\sigma} = M_{\rho \cup \sigma}$. Then for $\rho \in I$, there is $\mathfrak{B}_\rho \in K_{\alpha+\omega}$ such that $\mathcal{R}_k^\alpha \mathfrak{A} \subseteq \mathcal{R}_k^\alpha \mathfrak{B}_\rho$. Let $C = \prod \mathfrak{B}_\rho / U$; it is in $U_p K_{\alpha+\omega}$. Define $f : \mathfrak{A} \rightarrow \prod \mathfrak{B}_\rho$ by $f(a) = a$, and finally define $g : \mathfrak{A} \rightarrow C$ by $g(a) = f(a) / U$. Then $g$ is an embedding, and we are done.

2 The first question

Our first question addresses reducts and neat reducts. Handling reducts are usually easier. Problems concerning neat reducts tend to be messy, in the positive sense.

So lets get over with the easy part of reducts. The infinite dimensional case follows from the definition of a system of varieties definable by schema, namely, for any such system we have $K_{\alpha} = HSP \mathcal{R}_k^\alpha K_{\beta}$ for any pair of infinite ordinals $\alpha < \beta$ and any injection $\rho : \alpha \rightarrow \beta$. But for all algebras considered $S\mathcal{R}_k^\alpha K_{\beta}$ is a variety, hence the desired conclusion; which is that every algebra is a subreduct of an algebra in any preassigned higher dimension. We can strengthen this to:

**Theorem 5.** For any pair of infinite ordinals $\alpha < \beta$, we have $K_{\alpha} = El \mathcal{R}_k^\alpha K_{\beta}$

*Proof.* For simplicity we assume that we have one unary operation $f$, with $\rho(f) = 1$. The general case is the same. Let $\mathfrak{A} \in K_{\alpha}$. Let $I = \{ \Gamma : \Gamma \subseteq \beta, |\Gamma| < \omega \}$. Let $I_{\Gamma} = \{ \Delta \subseteq I, \Gamma \subseteq \Delta \}$, and let $F$ be an ultrafilter such that $I_{\Gamma} \in F$ for all $\Gamma \in I$. Notice that $I_{\Gamma_1} \cap I_{\Gamma_2} = I_{\Gamma_1 \cup \Gamma_2}$ so this ultrafilter exists. For each $\Gamma \in I$, let $\rho(\Gamma)$ be an injection from $\Gamma$ into $\alpha$ such that $Id \upharpoonright \Gamma \cap \alpha \subseteq \rho(\Gamma)$, and let $\mathfrak{B}_{\Gamma}$ be an algebra having same similarity type as $K_{\beta}$ such that for $k \in \Gamma$ $f^{\mathfrak{B}_{\Gamma}}_k = f^{\mathfrak{A}}_{\rho(\Gamma)[k]}$. Then $D = \prod \mathfrak{B}_{\Gamma} / F \in K_{\beta}$ and $f : \mathfrak{A} \rightarrow \mathcal{R}_k^\alpha D$ defined via $a \mapsto (a : \Gamma \in I) / F$ is an elementary embedding.

Things are different for finite dimensions. Here we give an example for quasi-polyadic equality algebras, modelled on a construction of Henkin for cylindric algebras reported in [24]. The construction essentially depends on the presence of diagonal elements. We do not know whether an analogous result hold for quasi-polyadic algebras.

**Example 6.** Let $n \geq 2$ and $\mathfrak{A} \in \mathcal{QEA}_n$. Let $e$ be the equation defined in lemma 2.6.10, in [25]. Lemma 2.6.13, provides a cylindric algebra $\mathfrak{C}$ in $\mathcal{R}_k^\alpha CA_{\beta}$ for any finite $\beta$, such that $\mathfrak{C}$ is generated by a set with cardinality $\beta$, and $e$ fails in this algebra.
Now this algebra, is based on the algebra constructed in lemma 2.6.12; so we need to define substitutions on this last algebra, which is a product of the Boolean part two cylindric algebras. One simply sets \( p_{ij}(x, y) = (p_{ij}z, p_{ij}y) \); but then the algebra \( C \) is in \( \mathfrak{M}_\alpha\mathfrak{PEA}_\beta \).

The proofs of theorems 2.6.14 and 2.6.16, work verbatim by replacing cylindric algebras with polyadic algebras. And so we have the proper inclusions, for \( 2 \leq \alpha < \beta \leq \omega \):

\[
HSP\mathfrak{M}_\alpha\mathfrak{QEA}_{\beta+1} \subset HSP\mathfrak{M}_\alpha\mathfrak{QEA}_\beta \subset \mathfrak{QEA}_\alpha
\]

(The inclusion follows from the fact that a reduct of a reduct is a reduct).

The neat reduct part, as we shall see, is profoundly more involved. The following is our main result in this section. It is not the case that every algebra in \( \mathfrak{CA}_m \) is the neat reduct of an algebra in \( \mathfrak{CA}_n \), nor need it even be a subalgebra of a neat reduct of an algebra in \( \mathfrak{CA}_n \). Furthermore, \( S\mathfrak{Mr}_m\mathfrak{CA}_{m+k+1} \neq S\mathfrak{Mr}_m\mathfrak{CA}_m \), whenever \( 3 \leq m < \omega \) and \( k < \omega \).

The hypothesis in the following theorem presupposes the existence of certain finite dimensional algebras, not chosen haphazardly at all, but are rather an abstraction of cylindric algebras existing in the literature witnessing the last proper inclusions. The main idea, that leads to the conclusion of the theorem, is to use such finite dimensional algebras to obtain an an analogous result for the infinite dimensional case. Accordingly, we streamline Monk’s argument who did exactly that for cylindric algebras, but we do it in the wider context of systems of varieties definable by a schema. (Strictly speaking Monk’s lifting argument is weaker, the infinite dimensional constructed algebras are merely non-representable, in our case they are not only non-representable, but are also subneat reducts of algebras in a given pre assigned dimension; this is a technical difference, that needs some non-trivial fine tuning in the proof). The inclusion of finite dimensions in our formulation, was therefore not a luxury, nor was it motivated by aesthetic reasons, and nor was it merely an artefact of Monk’s definition. It is motivated by the academic worthiness of the result (for infinite dimensions).

**Theorem 7.** Let \( (K_\alpha : \alpha \geq 2) \) be a complete system of varieties definable by a schema. Assume that for \( 3 \leq m < n < \omega \), there is an \( m \) dimensional algebra \( \mathfrak{C}(m, n, r) \) such that

1. \( \mathfrak{C}(m, n, r) \in S\mathfrak{Mr}_mK_n \)
2. \( \mathfrak{C}(m, n, r) \notin S\mathfrak{Mr}_mK_{n+1} \)
3. \( \prod_{r \in \omega} \mathfrak{C}(m, n, r) \in S\mathfrak{Mr}_mK_n \)
4. For \( m < n \) and \( k \geq 1 \), there exists \( x_n \in \mathfrak{C}(n, n + k, r) \) such that \( \mathfrak{C}(m, m + k, r) \cong \mathfrak{M}_\alpha \mathfrak{C}(n, n + k, r) \).
Then for any ordinal $\alpha \geq \omega$, $S\mathfrak{M}_\alpha \mathbf{K}_{\alpha+k+1}$ is not axiomatizable by a finite schema over $S\mathfrak{M}_\alpha \mathbf{K}_{\alpha+k}$.

**Proof.** The proof is a lifting argument essentially due to Monk, by 'stretching' dimensions using only properties of reducts and ultraproducts, formalizable in the context of a system of varieties definable by a schema.

It is divided into 3 parts:

1. Let $\alpha$ be an infinite ordinal, let $X$ be a finite subset of $\alpha$, let $I = \{\Gamma : X \subseteq \Gamma \subseteq \alpha, |\Gamma| < \omega\}$. For each $\Gamma \in I$ let $M_\Gamma = \{\Delta \in I : \Delta \supseteq \Gamma\}$ and let $F$ be any ultrafilter over $I$ such that for all $\Gamma \in I$ we have $M_\Gamma \in F$ (such an ultrafilter exists because $M_\Gamma \cap M_{\Gamma'} = M_{\Gamma \cap \Gamma'}$). For each $\Gamma \in I$ let $\rho_\Gamma$ be a bijection from $|\Gamma|$ onto $\Gamma$. For each $\Gamma \in I$ let $A_\Gamma, B_\Gamma$ be $K_\alpha$-type algebras. If for each $\Gamma \in I$ we have $\mathfrak{A}_\Gamma = \mathfrak{B}_\Gamma$ then $\Pi_{F,A_\Gamma} = \Pi_{F,B_\Gamma}$. Standard proof, by Łos' theorem. Note that the base of $\Pi_{F,A_\Gamma}$ is identical with the base of $\Pi_{F,B_\Gamma}$, which is identical with the base of $\Pi_{F,B_\Gamma}$, by the assumption in the lemma. Each operator $o$ of $K_\alpha$ is the same for both ultraproducts because $\{\Gamma \in I : \dim(o) \subseteq \operatorname{rng}(\rho_\Gamma)\} \in F$.

Furthermore, if $\mathfrak{A}_\Gamma \in K_{|\Gamma|}$, for each $\Gamma \in I$ then $\Pi_{F,A_\Gamma} \in K_\alpha$. For this, it suffices to prove that each of the defining axioms for $K_\alpha$ holds for $\Pi_{F,A_\Gamma}$. Let $\sigma = \tau$ be one of the defining equations for $K_\alpha$, the number of dimension variables is finite, say $n$. Take any $i_0, i_1, \ldots, i_{n-1} \in \alpha$, we must prove that $\Pi_{F,A_\Gamma} \models \sigma(i_0, \ldots, i_{n-1}) = \tau(i_0, \ldots, i_{n-1})$. If they are all in $\operatorname{rng}(\rho_\Gamma)$, say $i_0 = \rho_\Gamma(j_0), i_1 = \rho_\Gamma(j_1), \ldots, i_{n-1} = \rho_\Gamma(j_{n-1})$, then $\mathfrak{A}_\Gamma \models \sigma(j_0, \ldots, j_{n-1}) = \tau(j_0, \ldots, j_{n-1})$, since $\mathfrak{A}_\Gamma \in K_{|\Gamma|}$, so $\mathfrak{A}_\Gamma \models \sigma(i_0, \ldots, i_{n-1}) = \tau(i_0, \ldots, i_{n-1})$. Hence $\{\Gamma \in I : \mathfrak{A}_\Gamma \models \sigma(i_0, \ldots, i_{n-1}) = \tau(i_0, \ldots, i_{n-1})\} \supseteq \{\Gamma \in I : \rho_\Gamma \in \operatorname{rng}(\rho_\Gamma) \in F\}$, hence $\Pi_{F,A_\Gamma} \models \sigma(i_0, \ldots, i_{n-1}) = \tau(i_0, \ldots, i_{n-1})$. Thus $\Pi_{F,A_\Gamma} \in K_\alpha$.

2. Let $k \in \omega$. Let $\alpha$ be an infinite ordinal. Then $S\mathfrak{M}_\alpha \mathbf{K}_{\alpha+k+1} \subset S\mathfrak{M}_\alpha \mathbf{K}_{\alpha+k}$. Let $r \in \omega$. Let $I = \{\Gamma : \Gamma \subseteq \alpha, |\Gamma| < \omega\}$. For each $\Gamma \in I$, let $M_\Gamma = \{\Delta \in I : \Gamma \subseteq \Delta\}$, and let $F$ be an ultrafilter on $I$ such that $\forall \Gamma \in I, M_\Gamma \in F$. For each $\Gamma \in I$, let $\rho_\Gamma$ be a one to one function from $|\Gamma|$ onto $\Gamma$. Let $C_\Gamma$ be an algebra similar to $K_\alpha$ such that $\mathfrak{A}_\Gamma C_\Gamma = C(|\Gamma|, |\Gamma| + k, r)$.

Let

$$\mathfrak{B} = \prod_{\Gamma/F \in I} C_\Gamma^r.$$ 

We will prove that

1. $\mathfrak{B} \in S\mathfrak{M}_\alpha \mathbf{K}_{\alpha+k}$ and
2. $\mathcal{B}^r \notin S\mathcal{M}_r K_{a+k+1}$.

The theorem will follow, since $\mathcal{R}_r^a \mathcal{B}^r \in S\mathcal{M}_r K_{a+k} \setminus S\mathcal{M}_r K_{a+k+1}$.

For the first part, for each $\Gamma \in I$ we know that $C(|\Gamma| + k, |\Gamma| + k + r) \in K_{|\Gamma|+k}$ and $\mathcal{R}_r |\Gamma| C(|\Gamma| + k, |\Gamma| + k, r) \cong C(|\Gamma|, |\Gamma| + k, r)$. Let $\sigma_\Gamma$ be a one to one function $\langle |\Gamma| + k \rangle \to \langle \alpha + k \rangle$ such that $\rho_\Gamma \subseteq \sigma_\Gamma$ and $\sigma_\Gamma(|\Gamma| + i) = \alpha + i$ for every $i < k$. Let $A_\Gamma$ be an algebra similar to a $K_{a+k}$ such that $\mathcal{R}_r^{\sigma_\Gamma} A_\Gamma = C(|\Gamma| + k, |\Gamma| + k, r)$. By the second part with $\alpha + k$ in place of $\alpha$, $m \cup \{\alpha + i : i < k\}$ in place of $X$, $\{\Gamma \subseteq \alpha + k : |\Gamma| < \omega, X \subseteq \Gamma\}$ in place of $I$, and with $\sigma_\Gamma$ in place of $\rho_\Gamma$, we know that $\Pi_{\Gamma/F} A_\Gamma \in K_{a+k}$.

We prove that $\mathcal{B}^r \subseteq \mathcal{M}_r \Pi_{\Gamma/F} A_\Gamma$. Recall that $\mathcal{B}^r = \Pi_{\Gamma/F} \mathcal{C}_r^\Gamma$ and note that $\mathcal{C}_r^\Gamma \subseteq A_\Gamma$ (the base of $\mathcal{C}_r^\Gamma$ is $C(|\Gamma|, |\Gamma| + k, r)$, the base of $A_\Gamma$ is $C(|\Gamma| + k, |\Gamma| + k + r)$). So, for each $\Gamma \in I$,

$$
\mathcal{R}_r^{\sigma_\Gamma} \mathcal{C}_r^\Gamma = C(|\Gamma|, |\Gamma| + k, r)
\cong \mathcal{M}_r |\Gamma| C(|\Gamma| + k, |\Gamma| + k, r)
= \mathcal{M}_r |\Gamma| \mathcal{R}_r^{\sigma_\Gamma} A_\Gamma
= \mathcal{R}_r^{\sigma_\Gamma} \mathcal{M}_r A_\Gamma
= \mathcal{R}_r^{\sigma_\Gamma} \mathcal{M}_r \Pi_{\Gamma/F} A_\Gamma
$$

By the first part of the first part we deduce that $\Pi_{\Gamma/F} \mathcal{C}_r^\Gamma \cong \Pi_{\Gamma/F} \mathcal{M}_r A_\Gamma \subseteq \mathcal{M}_r \Pi_{\Gamma/F} A_\Gamma$, proving (1).

Now we prove (2). For this assume, seeking a contradiction, that $\mathcal{B}^r \in S\mathcal{M}_r K_{a+k+1}$, $\mathcal{B}^r \subseteq \mathcal{M}_r C$, where $C \in K_{a+k+1}$. Let $3 \leq m < \omega$ and $\lambda : m + k + 1 \to \alpha + k + 1$ be the function defined by $\lambda(i) = i$ for $i < m$ and $\lambda(m + i) = \alpha + i$ for $i < k + 1$. Then $\mathcal{R}_m^\lambda (C) \in K_{m+k+1}$ and $\mathcal{R}_m \mathcal{B}^r \subseteq \mathcal{M}_m \mathcal{R}_m^\lambda (C)$. For each $\Gamma \in I$, let $I_{|\Gamma|}$ be an isomorphism $C(m, m + k + r) \cong \mathcal{M}_{x_{|\Gamma|}} \mathcal{R}_m C(|\Gamma|, |\Gamma| + k, r)$.

Let $x = (x_{|\Gamma|} : \Gamma)/F$ and let $\iota(b) = (I_{|\Gamma|} b : \Gamma)/F$ for $b \in C(m, m + k, r)$. Then $\iota$ is an isomorphism from $C(m, m + k, r)$ into $\mathcal{R}_m \mathcal{B}^r$. Then $\mathcal{R}_x \mathcal{R}_m \mathcal{B}^r \in S\mathcal{M}_m K_{m+k+1}$. It follows that $C(m, m + k, r) \in S\mathcal{M}_m K_{m+k+1}$ which is a contradiction and we are done.

$\Box$

2.0.1 Monk’s algebras

Monk’s seminal result proved in 1969, showing that the class of representable cylindric algebras is not finitely axiomatizable had a shattering effect on algebraic logic, in many respects. The conclusions drawn from this result, were
that either the extra non-Boolean basic operations of cylindrifiers and diagonal elements were not properly chosen, or that the notion of representability was inappropriate; for sure it was concrete enough, but perhaps this is precisely the reason, it is far too concrete.

Research following both paths, either by changing the signature or/and altering the notion of concrete representability have been pursued ever since, with amazing success. Indeed there are two conflicting but complementary facets of such already extensive research referred to in the literature, as 'attacking the representation problem'. One is to delve deeply in investigating the complexity of potential axiomatizations for existing varieties of representable algebras, the other is to try to sidestep such wild unruly complex axiomatizations, often referred to as taming methods.

Those taming methods can either involve passing to (better behaved) expansions of the algebras considered, or even completely change the signature bearing in mind that the essential operations like cylindrifiers are term definable or else change the very notion of representatibility involved, as long as it remains concrete enough.

The borderlines are difficult to draw, we might not know what is not concrete enough, but we can judge that a given representability notion is satisfactory, once we have one.

One can find well motivated appropriate notions of semantics by first locating them while giving up classical semantical prejudices. It is hard to give a precise mathematical underpinning to such intuitions. What really counts at the end of the day is a completeness theorem stating a natural fit between chosen intuitive concrete-enough, but not too concrete, semantics and well behaved axiomatizations. The move of altering semantics has radical philosophical repercussions, taking us away from the conventional Tarskian semantics captured by Fregean-Godel-like axiomatization; the latter completeness proof is effective but highly undecidable; and this property is inherited by finite varibale fragments of first order logic as long as we insist on Tarskian semantics.

Monk defined the required algebras, witnessing the non finite axiomtizability of $\mathbf{RCA}_n$, $n \geq 3$, via their atom structure. An $n$ dimensional atom structure is a triple $\mathfrak{B} = (G, T_i, E_{ij})_{i,j \in n}$ such that $T_i \subseteq G \times G$ and $E_{ij} \subseteq G$, for all $i,j \in n$. An atom structure so defined, is a cylindric atom structure if its complex algebra $\mathfrak{C}a\mathfrak{B} \in \mathbf{CA}_n$. $\mathfrak{C}a\mathfrak{C}$ is the algebra

$$(\wp(G), \cap, \sim, T_i^*, E_{ij}^*)_{i,j \in n},$$

where

$$T_i^*(X) = \{ a \in G : \exists b \in X : (a,b) \in T_i \}$$

and

$$E_{i,j}^* = E_{i,j}.$$
Cylindric algebras are axiomatized by so-called Sahlqvist equations, and therefore it is easy to spell out first order correspondants to such equations characterizing atom structures of cylindric algebras.

**Definition 8.** For $3 \leq m \leq n < \omega$, $G_{m,n}$ denotes the cylindric atom structure such that $G_{m,n} = (G_{m,n}, T_i, E_{i,j})_{i,j < m}$ of dimension $m$ which is defined as follows: $G_{m,n}$ consists of all pairs $(R, f)$ satisfying the following conditions:

1. $R$ is equivalence relation on $m$,
2. $f$ maps $\{(\kappa, \lambda) : \kappa, \lambda < n, \kappa \not R \lambda\}$ into $n$,
3. for all $\kappa, \lambda < m$, if $\kappa \not R \lambda$ then $f_{\kappa \lambda} = f_{\lambda \kappa}$,
4. for all $\kappa, \lambda, \mu < m$, if $\kappa \not R \lambda R \mu$ then $f_{\kappa \lambda} = f_{\kappa \mu}$,
5. for all $\kappa, \lambda, \mu < n$, if $\kappa \not R \lambda \not R \mu$ then $|f_{\kappa \lambda}, f_{\kappa \mu}, f_{\lambda \mu}| \neq 1$.

For $\kappa < m$ and $(R, f), (S, g) \in G(m, n)$ we define

$$(R, f)T_\kappa(S, g) \text{ iff } R \cap 2(n \setminus \{\kappa\}) = S \cap 2(m \setminus \{\kappa\})$$

and for all $\lambda, \mu \in m \setminus \{\kappa\}$, if $\lambda \not R \mu$ then $f_{\lambda \mu} = g_{\lambda \mu}$.

For any $\kappa, \lambda < m$, set

$$E_{\kappa \lambda} = \{(R, f) \in G(m, n) : \kappa R \lambda\}.$$

Monk proves that this indeed defines a cylindric atom structure, he defines the $m$ dimensional cylindric algebra $C(m, n) = C_a(G(m, n))$, then he proves:

**Theorem 9.** (1) For $3 \leq m \leq n < \omega$ and $n - 1 \leq \mu < \omega$, $\mathcal{M}_m C(n, \mu) \cong C(m, \mu)$. In particular, $C(m, m + k) \cong \mathcal{M}_m C(n, n + k)$.

2. Let $x_n = \{(R, f) \in G_{n, n+k}; R = (R \cap 2(n \sim m))$ for all $u, v, uRv, f(u, v) \in n + k$, and for all $\mu \in n \sim m, v < \mu$, $f(\mu, v) = \mu + k\}$.

Then $C(n, n + k) \cong \mathcal{M}_x \mathcal{M}_n C(m, m + k)$.

**Proof.** \cite{25}, theorems 3.2.77 and 3.2.86.

**Theorem 10.** The class $\text{RCA}_\alpha$ is not axiomatized by a finite schema.

**Proof.** By $\text{RCA}_\alpha = S \mathcal{M}_\alpha \text{CA}_{\alpha+\omega}$. Let $r \in \omega$. Then $\mathcal{B}^r$, call it $\mathcal{B}_k$ constructed above, from the finite dimensional algebras increasing in dimension, is in $S \mathcal{M}_\alpha \text{CA}_{\alpha+k}$ but it is not in $S \mathcal{M}_\alpha \text{CA}_{\alpha+k+1}$ least representable. Then the
The ultraproduct of the $\mathcal{B}_k$’s over a non-principal ultrafilter will be in $\mathcal{SN}_{\alpha} \mathcal{CA}_{\alpha+\omega}$, hence will be representable.

Johnsson defined a polyadic atom structure based on the $\mathcal{G}_{m,n}$. First a helpful piece of notation: For relations $R$ and $G$, $R \circ G$ is the relation

$$\{ (a, b) : \exists c (a, c) \in R, (c, b) \in S \}.$$

Now Johnson extended the atom structure $\mathcal{G}(m,n)$ by

$$(R, f) \equiv_{ij} (S, g)$$

iff $f(i, j) = g(j, i)$ and if $(i, j) \in R$, then $R = S$, if not, then $R = S \circ [i,j]$, as composition of relations.

Strictly speaking, Johnsson did not define substitutions quite in this way; because he has all finite transformations, not only transpositions. Then, quasipolyadic algebras was not formulated in schematizable form, a task accomplished by Sain and Thompson [29] much later.

**Theorem 11.** (Sain-Thompson) $RQA_{\alpha}$ and $RQEA_{\alpha}$ is not finite schema axiomatizable

**Proof.** One proof uses the fact that $RQA_{\alpha} = \mathcal{SN}_{\alpha} \mathcal{QA}_{\alpha+\omega}$, and that the diagonal free reduct Monk’s algebras (hence their infinite dilations) are not representable. Another proof uses a result of Robin Hirsch and Tarek Sayed Ahmed that there exists finite dimensional quasipolyadic algebras satisfying the hypothesis of theorem [7]. A completely analogous result holds for Pinter’s algebras, using also finite dimensional Pinter’s algebras satisfying the hypothesis of theorem [7].

## 3 The methods of splitting applied to quasipolyadic equality algebras

More severe negative results on potential universal axiomatizations of cylindric and quasi polyadic equality were obtained by Andréka and Sayed Ahmed, we give one in what follows. Such results use a different technique called splitting, although there are similarities with Monk’s ideas.

The idea, traced back to Jonsson for relation algebras, consists of constructing for every finite $k \in \omega$ a non-representable algebra, all of whose $k$-generated subalgebras are representable.

Andréka ingeniously transferred such an idea to cylindric algebras, and to fully implement it, she invented the nut cracker method of splitting. The subtle splitting technique invented by Andréka can be summarized as follows. In the presence of only finitely many substitutions, we take a fairly simple representable algebra generated by an atom, and we break up or split the atom into enough (finitely many) $k$ atoms, forming a larger algebra, that is in
fact non-representable; in fact, its cylindric reduct will not be representable, due to the incompatibility between the number of atoms, and the number of elements in the domain of a representation. However, the ”small” subalgebras namely, those generated by $k$ elements of such an algebra will be representable.

This does have affinity to Monk’s construction witnessing non finite axiomatizability for the class of representable cylindric algebras. The key idea of the construction of a Monk’s algebra is not so hard. Such algebras are finite, hence atomic, more precisely their Boolean reducts are atomic. The atoms are given colours, and cylindrifications and diagonals are defined by stating that monochromatic triangles are inconsistent. If a Monk’s algebra has many more atoms than colours, it follows from Ramsey’s Theorem that any representation of the algebra must contain a monochromatic triangle, so the algebra is not representable.

For $CA\kappa$’s, for each $k$ only one splitting into $k$ atoms are required, as done by Andrêka, For $RQEA\kappa$, things are more complicated, one has to perform infinitely many finite splittings (that is into a pre assigned finite $k$), one for every reduct containing only finitely many substitutions. (not just one which is done in $[29]$; relative though to infinitely many atoms which is much more than needed), increasing in number but always finite, constructing infinitely many algebras, whose similarity types contain only finitely many substitutions. Such constructed non-representable algebras, form a chain, and our desired algebra will be their directed union. The easy thing to do is to show that “small” subalgebras of every non-representable algebra in the chain is representable; the hard thing to do is to show that “small” subalgebras of the non-representable limit remain representable. (The error in Sain’s Thompson paper is claiming that the small subalgebras of the non-representable algebra, obtained by performing only one splitting into infinitely many atoms, are representable; this is not necessarily true).

The cylindric reduct of the algebras forming the chain is of $CA\omega$ type; in particular, it contains infinitely many cylindrifications and diagonal elements. The combinatorial argument of counting depends essentially on the presence of infinitely many diagonal elements. Indeed, it can be shown that the splitting technique adopted to prove complexity results concerning axiomatizations of $RQEA\omega$, simply does not work in the absence of diagonals. This can be easily distilled from our proof since our constructed non-representable quasipolyadic equality algebras, in fact have a representable quasipolyadic reduct. An open problem here, that can be traced back to to Sain’s and Thompson’s paper $[29]$, is whether $RQA\omega$ can be axiomatized by a necessarily infinite) set of formulas using only finitely many variables. This seems to be a hard problem, and the author tends to believe that there are axiomatizations that contain only finitely many variables, but further research is needed in this area.

On the other hand, the algebra constructed by this method of splitting is
‘almost representable’, in the sense that if we enlarge the potential domain of a representation, then various reducts of the algebra, obtained by discarding some of the operations (for example diagonal elements or infinitely many cylindrifications), turn out representable; and this gives relative non-finitizability results. Here we are encountered by a situation where we cannot have our cake and eat. If we want a quasipolyadic equality algebras that is only barely representable, then we cannot obtain non-representability of some of its strict reducts like its quasipolyadic reduct. Throughout, we will be tacitly assuming that quasipolyadic (equality) algebras are not only term-definitionally with finitary polyadic (equality) algebras as proved in [29] p.546, but that they are actually the same. This means that in certain places we consider only substitutions corresponding to transpositions rather than all substitutions corresponding to finite transformations which is perfectly legitimate. Also we understand representability of reducts of quasipolyadic equality algebras, when we discard some of the substitution operations, in the obvious sense.

If $A$ has a cylindric reduct, then $\mathcal{Rd}_{ca}A \in \mathbf{RCA}_\omega$ denotes this reduct. Our next theorem corrects the error mentioned above in Sain’s Thompson’s seminal paper [29], generalizes Theorem 6 in [1] p. 193 to infinitely many dimensions, and answers a question by Andreka in op cit also on p. 193.

**Theorem 12.** The variety $\mathbf{RQEA}_\omega$ cannot be axiomatized with a set $\Sigma$ of quantifier free formulas containing finitely many variables. In fact, for any $k < \omega$, and any set of quantifier free formulas $\Sigma$ axiomatizing $\mathbf{RQEA}_\omega$, $\Sigma$ contains a formula with more than $k$ variables in which some diagonal element occurs.

**Proof.** The proof consists of two parts. In the first part we construct algebras $A_{k,n}$ with certain properties, for each $n,k \in \omega \sim \{0\}$. In the second part we form a limit of such algebras as $n$ tends to infinity, obtaining an algebra $A_k$ that is not representable, though its $k$-generated subalgebras are representable. This algebra will finish the proof.

**Part I**

Let $k, n \in \omega \sim \{0\}$. Let $G_n$ be the symmetric group on $n$. $G_n$ is generated by the set of all transpositions $[[i,j]: i,j \in n]$ and for $n \leq m$, we can consider $G_n \subseteq G_m$. We shall construct an algebra $A_{k,n} = (A_{k,n}, +, -, c_i, s_\tau, d_{ij})_{i,j \in \omega, \tau \in G_n}$ with the following properties.

(i) $\mathcal{Rd}_{ca}A_{k,n} \notin \mathbf{RCA}_\omega$.

(ii) Every $k$-generated subalgebra of $A_{k,n}$ is representable.

(iii) There is a one to one mapping $h : A_{k,n} \to (\mathcal{B}(\omega^2W), c_i, s_\tau, d_{ij})_{i,j \in \omega, \tau \in G_n}$ such that $h$ is a homomorphism with respect to all operations of $A_{k,n}$ except for the diagonal elements.
Here $k$-generated means generated by $k$ elements. The proof for finite reducts uses arguments very similar to the proof of Andréka of Theorem 6 in [1], and has affinity with the proof of theorem 3.1 in [43]. However, there are two major differences. Our cylindric reducts are infinite dimensional, and our proof is more direct and, in fact, far easier to grasp. The proof of the above cited theorem of Andréka’s goes through the route of certain finite expansions by so-called permutation invariant unary operations that are also modalities (distributive over the boolean join), and these are more general than substitutions. Substitutions are more concrete, and therefore our proof is less abstract.

(1) Let $m \geq 2^{k.n+1}$, $m < \omega$ and let $\langle U_i : i < \omega \rangle$ be a system of disjoint sets such that $|U_i| = m$ for $i \geq 0$ and $U_0 = \{0, \ldots, m-1\}$. Let

$$U = \bigcup \{U_i : i \in \omega\},$$

let

$$R = \prod_{i < \omega} U_i = \{s \in \omega U : s_i \in U_i\},$$

and let $\mathfrak{A}'$ be the subalgebra of $\langle \mathfrak{B}(\omega U), c_i, d_{ij}, s_\tau \rangle_{\tau \in G_n}$ generated by $R$. Then $s_\tau R$ is an atom of $\mathfrak{A}'$ for any $\tau \in G_n$. Indeed for any two sequences $s, z \in R$ there is a permutation $\sigma : U \to U$ of $U$ taking $s$ to $z$ and fixing $R$, i.e $\sigma \circ s = z$ and $R = \{\sigma \circ p : p \in R\}$. $\sigma$ fixes all the elements generated by $R$ because the operations are permutation invariant. Thus if $a \in \mathfrak{A}'$ and $s \in a \cap R$ then $R \subseteq a$ showing that $R$ is an atom of $\mathfrak{A}'$. Since $\tau$ is a bijection, it follows that $s_\tau R$ is also an atom of $\mathfrak{A}$ and, it is easy to see that all these atoms are pairwise disjoint. That is if $\tau_1 \neq \tau_2$, then $s_{\tau_1} R \cap s_{\tau_2} R = \emptyset$. We now split each $s_\tau R$ into abstract atoms $s_\tau R_j$, $j \leq m$ and $\tau \in G_n$. Let $(R_j : j \leq m)$ be a set of $m+1$ distinct elements, and let $\mathfrak{A}_{k,n}$ be an algebra such that

1. $\mathfrak{A}' \subseteq \mathfrak{A}_{k,n}$, the Boolean part of $\mathfrak{A}_{k,n}$ is a Boolean algebra,
2. $R = \sum (R_j : j \leq m)$,
3. $s_\tau R_j$ are pairwise distinct atoms of $\mathfrak{A}_{k,n}$ for each $\tau \in G_n$ and $j \leq m$ and $c_i s_\tau R_j = c_i s_\tau R$ for all $i < \omega$ and all $\tau \in G_n$,
4. each element of $\mathfrak{A}_{k,n}$ is a join of element of $\mathfrak{A}'$ and of some $s_\tau R_j$’s,
5. $c_i$ distributes over joins,
6. The $s_\tau$’s are Boolean endomorphisms such that $s_\tau s_\sigma a = s_{\tau \circ \sigma} a$.

The existence of such algebra is easy to show; furthermore they are unique up to isomorphim, see [1], the comment right after the definition on p.168. Now we show that $\mathfrak{R}_\omega \mathfrak{A}_{k,n}$ cannot be representable. This part of the proof is identical to Andréka’s proof but we include it for the sake
of completeness. The idea is that we split \( R \) into \( m + 1 \) distinct atoms but \( U_0 \) has only \( m \) elements, and those two conditions are incompatible in case there is a representation. The substitutions have to do with permuting the atoms and they do not contribute to this part of the proof. For \( i, j < \omega, i \neq j \), recall that \( s_j^i x = c_i(d_{ij}, x) \). Let

\[
\tau(x) = \prod_{i \leq m} s_i^0 c_1 \ldots c_m x \cdot \prod_{i < j \leq m} -d_{ij}
\]

Then \( \mathfrak{A}' \models \tau(R) = 0 \). Indeed we have

\[
c_1 \ldots c_m R = mU \times U_{m+1} \times \ldots
\]

\[
s_i^0 c_1 \ldots c_m R = U \times \ldots U_0 \times U \times U_{m+1} \ldots
\]

\[
\bigcap s_i^0 c_1 \ldots c_m R = m+1U_0 \times U_{m+1} \times .
\]

Then by \( |U_0| \leq m \) there is no repetition free sequence in \( m+1U_0 \). Thus as claimed \( \mathfrak{A}' \models \tau(R) = 0 \). Then \( \mathfrak{A}_{k,n} \models \tau(R) = 0 \). Assume that \( \mathfrak{A}_{k,n} \) is represented somehow. Then there is a homomorphism \( h : \mathfrak{A}_{k,n} \rightarrow (\mathfrak{B}^{(W)}, c_i, d_{ij})_{i<\omega} \) for some set \( W \) such that \( h(R) \neq \emptyset \).

By \( h(R) \neq \emptyset \) there is some \( s \in h(R) \). By \( R \leq c_0 R_i \) we have \( h(R) \subseteq h(R_i) \), so there is a \( w_i \) such that \( s(0|w_i) \in h(R_i) \) for all \( i \leq m \). These \( w_i \)'s are distinct since the \( R_i \)'s are pairwise disjoint (they are distinct atoms) and so are the \( h(R_i) \)'s. Consider the sequence

\[
z = \langle w_0, w_1, \ldots w_m, s_{m+1}, \ldots \rangle.
\]

We show that \( z \in \tau(h(R)) \). Indeed let \( i, j \leq m, i \neq j \), then \( z \in -d_{ij} \) by \( w_i \neq w_j \). Next we show that \( z \in s_i^0 c_1 \ldots c_m h(R) \). By definition, \( \langle w_i, s_1 \ldots \rangle \in h(R_i) \subseteq h(R) \) so \( \langle w_i, w_1, \ldots w_m, s_{m+1}, \ldots \rangle \in c_i \ldots c_m h(R) \) and thus \( z \in c_0(d_{0i} \cap c_1 \ldots c_m h(R)) = s_i^0 c_1 \ldots c_m h(R) \). This contradicts that \( \mathfrak{A}_{k,n} \models \tau(R) = 0 \).

Next we show that the \( k \) generated subalgebras of \( \mathfrak{A}_{k,n} \) are representable. Let \( G \) be given such that \( |G| \leq k \). The idea is to use \( G \) and define a “small” subalgebra of \( \mathfrak{A}_{k,n} \) that contains \( G \) and is representable. Define \( R_i \equiv R_j \) iff

\[
(\forall g \in G)(\forall \tau \in G_n)((s_\tau R_i \leq g \iff s_\tau R_j \leq g)).
\]

This is similar to the equivalence relation defined by Andréka [1] p. 157; the difference is that substitutions have to come to the picture [1] p.189. Then \( \equiv \) is an equivalence relation on \( \{ R_j : j \leq m \} \) which has \( \leq 2^{k.n!} \) blocks by \( |G| \leq k \) and \( G_n = n! \). Let \( p \) denote the number of blocks of
because transformations considered are bijections we have $p = |\{R_j / \equiv : j \leq m\}| \leq 2^{k,n!} \leq m$. Now that $R$ is split into $p < m + 1$ atoms, the incompatibility condition above no longer holds. Indeed, let

$$B = \{a \in A_{k,n} : (\forall i, j \leq m)(\forall \tau \in G_n)(R_i \equiv R_j \text{ and } s_\tau R_i \leq a \implies s_\tau R_j \leq a)\}.$$ 

We first show that $B$ is closed under the operations of $\mathfrak{A}_{k,n}$, then we show that, unlike $\mathfrak{A}_{k,n}$, $B$ is the universe of a representable algebra. Let $i < l < \omega$ clearly $B$ is closed under the Boolean operations. The diagonal element $d_{it} \in \mathfrak{B}$ since $s_\tau R_j \not\equiv d_{it}$ for all $j \leq m$ and $\tau \in G_n$. Also $A' \subseteq B$ since $s_\tau R$ is an atom of $\mathfrak{A}'$ and $c_i a \in A'$ for all $a \in A_{k,n}$. Thus $c_i b \in B$ for all $b \in B$. Assume that $a \in B$ and let $\tau \in G_n$. Suppose that $R_i \equiv R_j$ and $s_\tau R_i \leq s_\tau a$. Then $s_\tau s_\sigma R_i \leq a$, so $s_\tau s_\sigma R_i \leq a$. Since $a \in B$ we get that $s_\tau s_\sigma R_i = s_\tau s_\sigma R_i \leq a$, and so $s_\tau R_i \leq s_\tau a$. Thus $B$ is also closed under substitutions. Let $\mathfrak{B} \subseteq \mathfrak{A}_{k,n}$ be the subalgebra of $\mathfrak{A}_{k,n}$ with universe $B$. Since $G \subseteq B$ it suffices to show that $\mathfrak{B}$ is representable. Let $\{y_j; j < p\} = \{(\sum(R_j / \equiv) : j \leq m\}$. Then $\{y_j : j < p\}$ is a partition of $R$ in $\mathfrak{B}$, $c_i y_j = c_i R$ for all $j < p$ and $i < \omega$ and every element of $\mathfrak{B}$ is a join of some element of $\mathfrak{A}'$ and of finitely many of $s_\tau y_j$'s. Recall that $p \leq m$. We now split $R$ into $m$ ‘real’ atoms, cf. [1] p.167, lemma 2. We define an equivalence relation on $R$. For any $s, z \in R$

$$s \sim z \iff |\{i \in \omega : s_i \neq z_i\}| < \omega.$$ 

Let $S \subseteq R$ be a set of representatives of $\sim$. Consider the group $Z_m$ of integers modulo $m$. (Any finite abelian group with $m$ elements will do.) For any $s \in S$ and $i \in \omega$ let $f_i^s : U_i \to Z_m$ be an onto map such that $f_i^s(s_i) = 0$. For $j < m$ define

$$R_j^s = \{z \in R : \sum\{f_i^s : i \in \omega\} = j\}$$

and

$$R_j'' = \bigcup\{R_j^s : s \in S\}.$$ 

Then $\{R_0'' \ldots R_m''\}$ is a partition of $R$ such that $c_i R_j'' = c_i R$ for all $i < \omega$ and $j < m$. Let $\mathfrak{A}''$ be the subalgebra of $\langle \mathfrak{B}(\sim U), c_i, d_{ij}, s_\tau \rangle_{i,j<\omega, \tau \in G_n}$ generated by $R_0'', \ldots, R_m''$. Let

$$\mathcal{R} = \{s_\sigma R_j'' : \sigma \in G_n, j < m\}.$$ 

Let

$$H = \{a + \sum X : a \in A', X \subseteq \mathcal{R}\}.$$ 

Clearly $H \subseteq A''$ and $H$ is closed under the boolean operations. Also because transformations considered are bijections we have

$$c_i s_\sigma R_j = c_i s_\sigma R$$ for all $j < m$ and $\sigma \in G_n$. 

19
Thus $H$ is closed under $c_i$. Also $H$ is closed under substitutions. Finally $d_{ij} \in A' \subseteq H$. We have proved that $H = A''$. This implies that every element of $R$ is an atom of $A''$. We now show that $B$ is embeddable in $A''$, and hence will be representable. Define for all $j < p - 1$,

$$R'_j = R''_j,$$

and

$$R'_{p-1} = \bigcup \{ R''_j : p - 1 \leq j < m \}$$

Then define for $b \in B$:

$$h(b) = (b - \sum_{\tau \in G_n} s_\tau R) \cup \bigcup \{ s_\tau R'_j : \tau \in G_n, j < p, s_\tau y_j \leq b \}.$$ 

It is clear that $h$ is one one, preserves the Boolean operations and the diagonal elements and is the identity on $A'$. Now we check cylindrifications and substitutions.

$$c_i h(b) = c_i[(b - \sum_{\tau \in G_n} s_\tau R) \cup \bigcup \{ s_\tau R'_j : \tau \in G_n, j < p, s_\tau y_j \leq b \}]
= c_i(b - \sum_{\tau \in G_n} s_\tau R) \cup \bigcup \{ c_i s_\tau R'_j : \tau \in G_n, j < p, s_\tau y_j \leq b \}
= c_i(b - \sum_{\tau \in G_n} s_\tau R) \cup \bigcup \{ c_i s_\tau y_j : \tau \in G_n, j < p, s_\tau y_j \leq b \}
= c_i[(b - \sum_{\tau \in G_n} s_\tau R) \cup \bigcup \{ s_\tau y_j, \tau \in G_n, j < p, s_\tau y_j \leq b \}]
= c_i b.$$ 

On the other hand

$$h c_i(b) = (c_i b - \sum_{\tau \in G_n} s_\tau R) \cup \bigcup \{ s_\tau R'_j : s_\tau y_j \leq c_i b \} = c_i b.$$ 

Preservation of substitutions follows from the fact that the substitutions are Boolean endomorphisms. In more detail, let $\sigma \in G_n$, then:

$$s_\sigma h(b) = s_\sigma[(b - \sum_{\tau \in G_n} s_\tau R) \cup \bigcup \{ s_\tau R'_j : \tau \in G_n, j < p, s_\tau y_j \leq b \}]
= (s_\sigma b - \sum_{\tau \in G_n} s_\sigma s_\tau R) \cup \bigcup \{ s_\sigma s_\tau R'_j : \tau \in G_n, j < p, s_\tau y_j \leq b \}
= (s_\sigma b - \sum_{\tau \in G_n} s_\sigma s_\tau R) \cup \bigcup \{ s_\sigma s_\tau R'_j : \tau \in G_n, j < p, s_\sigma y_j \leq b \}
= (s_\sigma b - \sum_{\tau \in G_n} s_\tau R) \cup \bigcup \{ s_\tau R'_j : \tau \in G_n, j < p, s_\tau y_j \leq b \}.$$ 

20
But for fixed $\sigma$, we have $\{\sigma \circ \tau : \tau \in G_n\} = G_n$ and so
$$s_\sigma h(b) = h(s_\sigma(b)).$$

For every $k,n < \omega$ we have constructed an algebra $\mathfrak{A}_{k,n}$ such that $\mathfrak{A}_{k,n} \notin \text{RCA}_\omega$ and the $k$-generated subalgebras of $\mathfrak{A}_{k,n}$ are representable. We should point out that the “finite dimensional version” of the $\mathfrak{A}_{k,n}$’s were constructed in [43], and their construction can be recovered from the proof of Theorem 6 in [1] which addresses the finite dimensional case but in a more general setting allowing arbitrary unary additive permutation invariant operations expanding those of $\text{RCA}_n$.

We note that the latter result does not survive the infinite dimensional case. There are easy examples, cf. [1] p.192 and [41].

(2) We show that $\mathfrak{A}_{k,n}$ has a representation which preserves all operations except for the diagonal elements. That is, its quasipolyadic reduct is representable. The proof is analogous to that of Andréka’s on of Claim 16 on p.194 of [1]. Let $\mathfrak{A}_{k,n}$ be the algebra obtained by splitting the atom $R$ in $\mathfrak{A}'$ as in the above proof. Then $\mathfrak{A}_{k,n}$ is not representable, but its $k$ generated subalgebras are representable. We show that there is a representation of $\mathfrak{A}_{k,n}$ in which all operations are preserved except for the diagonal elements. Let $\mathfrak{A}_{k,n}$ be the algebra obtained by splitting the atom $R$ in $\mathfrak{A}'$ as in the above proof. Then $\mathfrak{A}_{k,n}$ is not representable, but its $k$ generated subalgebras are representable. We show that there is a representation of $\mathfrak{A}_{k,n}$ in which all operations are preserved except for the diagonal elements. Let $U_i$, $i < \omega$ be a sequence of pairwise disjoint sets such that $|U_0| = m \geq 2^{kn!+1}$ and $|U_i| \geq m + 1$. Let $R$ be as above except that it is defined via the new $U_i$’s. Let $(R_j : j \leq m)$ be the splitting of $R$ in $\mathfrak{A}_{k,n}$. Let $W \supset U$ (properly). Let $W_0 = U_0 \cup (W \sim U)$, and $W_i = U_i$ for $0 < i < \omega$. First we define a function $h : \varphi(\omega U) \to \varphi(\omega W)$ with the desired properties and $h(R) = \prod_{i<\omega} W_i$. Let $t : W \to U$ be a surjective function which is the identity on $U$ and which maps $W_0$ to $U_0$. Define $g : \omega W \to \omega U$ by $g(s) = t \circ s$ for all $s \in \omega W$ and for all $x \subseteq \omega U$, define
$$h(x) = \{s \in \omega U : g(s) \in x\}.$$

Since $|W_i| \geq m + 1$ for all $i < \omega$, the incompatibility condition between the number of atoms splitting $R$ and the number of elements in $|W_0|$ used in the representation vanishes, so there is a real partition $(S_j : j \leq m)$ of $S = \prod_{i<\omega} W_i$ such that $c_i S_j = c_i S$ for all $i < \omega$ and $j \leq m$. Then $(s_\sigma S_j : j \leq m)$ is an analogous partition of $s_\sigma S$ for $\sigma \in G_n$. Let $X_{\sigma,j} = s_\sigma^{kn} R_j$ for $j \leq m$. Define $\bar{h} : \mathfrak{A}_{k,n} \to \varphi(\omega W)$ by
$$\bar{h}(a) = h(a), \quad a \in A'$$
$$\bar{h}(X_{\sigma,j}) = s_\sigma S_j, \sigma \in G_n, j \leq m$$
and
$$\bar{h}(x + y) = \bar{h}(x) + \bar{h}(y), x, y \in A_{k,n}.$$
It is easy to check using lemma (iv) in [1] that $\bar{h}$ is as desired. In fact $\bar{h}$ preserves all the quasipolyadic operations including substitutions corresponding to replacements, which are now no longer definable, because we have discarded diagonal elements. The reasoning is as follows [1] p.194. For $i, j \in n$, the quantifier free formula $x \leq -d_{ij} \rightarrow s^i_j x$ is valid in representable algebras hence it is valid in $A_{k,n}$ since its $k$ generated subalgebra are representable. Let $\sigma \in G_n, l \leq m$. Then $s^i_j(X_{\sigma,l}) = 0$ in $A_{k,n}$. Now

$$\bar{h}(s^i_j(X_{\sigma,l})) = \bar{h}(0) = 0 = s^i_j s \sigma S_j = s^i_j h(X_{\sigma,l}).$$

Assume that $a \in A_{k,n}$. Then

$$\bar{h}(s^i_j a) = h(s^i_j a) = s^i_j h(a) = s^i_j \bar{h}(a).$$

Since both $\bar{h}$ and $s^i_j$ are additive we get the required.

**Part II**

1. Here is where we really start the non-trivial modification of Andréka’s splitting. For $n \in \omega$ and $m = 2^{k,n+1}$, we denote $A_{k,n}$ by $\text{split}(\mathcal{A}', R, m, n)$. This is perfectly legitimate since the algebra $A_{k,n}$ is determined uniquely by $R, \mathcal{A}', m$ and $n$. Recall that $m$ is the number of atoms splitting $R$, while $n$ is the finite number of substitutions available. For $n_1 < n_2$, we denote by $\mathcal{A}_{n_1}$ $\text{split}(\mathcal{A}', R, m, n_2)$ the reduct of $\text{split}(\mathcal{A}', R, m, n_2)$ obtained by restricting substitutions to $G_{n_1}$. Let $m_1 < m_2$ and $n_1 < n_2$. Then we claim that $\text{split}(\mathcal{A}', R, m_1, n_1)$ embeds into $\mathcal{A}_{n_1}$ $\text{split}(\mathcal{A}', R, m_2, n_2)$.

This part of the proof is analogous to Andréka’s proofs in [1], lemma 3, on splitting elements in cylindric algebras. Indeed, let

$$\chi : m_1 \rightarrow m_2$$

be such that the set $\chi(j), j < m_1$ are non empty and pairwise disjoint, and

$$\bigcup \{\chi(j) : j < m_1\} = m_2.$$

For $x \in \text{split}(\mathcal{A}, R, m_1, n_1)$, let

$$J_\tau(x) = \{j < m_1 : s \tau R_j \leq x\}.$$

Let $(R_i : i \leq m_2)$ be the splitting of $R$ in $\text{split}(\mathcal{A}', R, m_2, n_2)$. Define

$$h(x) = (x - \sum s \tau R) + \sum \{s \tau R_i : \tau \in G_{n_1}, i \in \bigcup \{\chi(j) : j \in J_\tau(x)\}\}. $$

22
Here we are considering \( G_{n_1} \) as a subset of \( G_{n_2} \). It is easy to check that \( h(x) \) is a Boolean homomorphism and that \( h(x) \neq 0 \) whenever \( 0 \neq x \leq s_{\tau}R \), for \( \tau \in G_{n_1} \). Thus \( h \) is one to one. Let \( i \in \omega \) and \( x \in \text{split}(\mathfrak{A}, R, m_1, n_1) \). If \( x \cdot s_{\tau}R = 0 \) for all \( \tau \), then \( x \in A' \), hence \( h(c_i x) = c_i h(x) \). So assume that there is a \( \tau \in G_{n_1} \) such that \( x \cdot s_{\tau}R \neq 0 \). Then \( c_i(x \cdot s_{\tau}R) = c_is_{\tau}R \) and \( c_i h(x \cdot s_{\tau}R)) = c_is_{\tau}R \) by \( 0 \neq h(x \cdot s_{\tau}R) \leq s_{\tau}R \).

Now

\[
h(c_i x) = h(c_i (x - s_{\tau}R) + c_i(x \cdot s_{\tau}R)) \\
= c_i(x - s_{\tau}R) + c_i R.
\]

\[
c_i h(x) = c_i(h(x - s_{\tau}R) + x \cdot s_{\tau}R)) \\
= c_i(h(x - s_{\tau}R)) + h(x \cdot s_{\tau}R) \\
= c_i(x - s_{\tau}R) + c_i R.
\]

We have proved that

\[
c_i h(x) = h(c_i x).
\]

Now we turn to substitutions. Let \( \sigma \in G_{n_1} \). Then we have

\[
s_{\sigma} h(x) = \sigma[(x - \sum_{\tau \in G_{n_1}} s_{\tau}R) + \sum \{s_{\tau}R_i : \tau \in G_{n_1}, i \in \bigcup \{\chi(j) : j \in J_{\tau}x\}\}] \\
= (\sigma x - \sum_{\tau \in G_{n_1}} s_{\sigma \tau}R) + \sum \{s_{\sigma \tau}R_i : \tau \in G_{n_1}, i \in \bigcup \{\chi(j) : j \in J_{\tau}x\}\}.
\]

Since \( \{\sigma \circ \tau : \tau \in G_{n_1}\} = G_{n_1} \), then we have:

\[
s_{\sigma} h(b) = h(s_{\sigma}(b))
\]

(2) We have a sequence of algebras \( (\mathfrak{A}_{k,i} : i \in \omega \sim 0) \) such that for \( n < m \), we can assume by the embeddings proved to exist in the previous item that \( \mathfrak{A}_{k,n} \) is a subreduct (subalgebra of a reduct) of \( \mathfrak{A}_{k,m} \). Form the natural direct limit of such algebras which is the (reduct directed) union call it \( \mathfrak{A}_{k} \). That is \( A_{k} = \bigcup_{n \in \omega} A_{k,n} \), and the operations are defined the obvious way. For example if \( i < \omega \), and \( a \in A_{k} \), then \( i \in n \) and \( a \in A_{n,k} \) for some \( n \); set \( c_{i}^{A_{k}}a = c_{i}^{A_{n,k}}a \). These are well defined. The other operations are defined analogously, where we only define the \( s_{i,j} \)'s for \( i, j \in \omega \). Clearly, \( \mathfrak{A}_{\omega} \) is not representable, for else \( \mathfrak{A}_{\omega} \mathfrak{A}_{k} \) would be representable for all \( n \in \omega \).

(3) Let \( |G| \leq k \). Then \( G \subseteq \mathfrak{A}_{k,n} \) for some \( n \). If \( \mathfrak{G}^{A_{k}}G \) is not representable then there exists \( l \geq n \) such that \( G \subseteq \mathfrak{A}_{k,l} \) and \( \mathfrak{G}^{A_{k,l}}G \) is not representable, contradiction. To see this we can show directly that \( \mathfrak{G}^{A_{k}}G \) has to be representable. We show that every equation \( \tau = \sigma \) valid in the
variety \( \text{RQEA}_\omega \) is valid in \( \mathcal{G}^3_k \mathcal{G} \). Let \( v_1, \ldots, v_k \) be the variables occurring in this equation, and let \( b_1, \ldots, b_k \) be arbitrary elements of \( \mathcal{G}^3_k \mathcal{G} \). We show that \( \tau(b_1, \ldots, b_k) = \sigma(b_1, \ldots, b_k) \). Now there are terms \( \eta_1 \ldots \eta_k \) written up from elements of \( \mathcal{G} \) such that \( b_1 = \eta_1 \ldots b_k = \eta_k \), then we need to show that \( \tau(\eta_1, \ldots, \eta_k) = \sigma(\eta_1, \ldots, \eta_k) \). This is another equation written up from elements of \( \mathcal{G} \), which is also valid in \( \text{RQEA}_\omega \). Let \( n \) be an upper bound for the indices occurring in this equation and let \( l > n \) be such that \( \mathcal{G} \subseteq \mathfrak{A}_{k,l} \). Then the above equation is valid in \( \mathcal{G}^3_{\omega,n} \mathcal{G} \) since the latter is representable. Hence the equation \( \tau = \sigma \) holds in \( \mathcal{G}^3_{\omega} \mathcal{G} \) at the evaluation \( b_1, \ldots, b_k \) of variables.

(4) Let \( \Sigma_n \) be the set of universal formulas using only \( n \) substitutions and \( k \) variables valid in \( \text{RQEA}_\omega \), and let \( \Sigma_n^d \) be the set of universal formulas using only \( n \) substitutions and no diagonal elements valid in \( \text{RQEA}_\omega \).

By \( n \) substitutions we understand the set \( \{ s_{i,j} \colon i, j \in \mathbb{N} \} \). Then \( \mathfrak{A}_{k,n} \models \Sigma_n \cup \Sigma_n^d \), \( \mathfrak{A}_{k,n} \models \Sigma_n^d \) because the \( k \) generated subalgebras of \( \mathfrak{A}_{k,n} \) are representable, while \( \mathfrak{A}_{k,n} \models \Sigma_n \) because \( \mathfrak{A}_{k,n} \) has a representation that preserves all operations except for diagonal elements. Indeed, let \( \phi \in \Sigma_n^d \), then there is a representation of \( \mathfrak{A}_{k,n} \) in which all operations are the natural ones except for the diagonal elements. This means that (after discarding the diagonal elements) there is a one to one homomorphism \( h : \mathfrak{A}^d \to \mathfrak{P}^d \) where \( \mathfrak{A}^d = (A_{k,n}, +, \cdot, c_k, s_{[i,j]}, s_{[i,j]}^j)_{k \in \omega,i,j \in \mathbb{N}} \) and \( \mathfrak{P}^d = (\mathfrak{B}(\omega^W), c_{k}^W, s_{[i,j]}^W, s_{[i,j]}^W)_{k \in \omega,i,j \in \mathbb{N}} \), for some infinite set \( W \). Now let \( \mathfrak{P} = (\mathfrak{B}(\omega^W), c_{k}^W, s_{[i,j]}^W, s_{[i,j]}^W, d_{kl})_{k \in \omega,i,j \in \mathbb{N}} \). Then we have that \( \mathfrak{P} \models \phi \) because \( \phi \) is valid and so \( \mathfrak{P}^d \models \phi \) due to the fact that no diagonal elements occur in \( \phi \). Then \( \mathfrak{A}^d \models \phi \) because \( \mathfrak{A}^d \) is isomorphic to a subalgebra of \( \mathfrak{P}^d \) and \( \phi \) is quantifier free. Therefore \( \mathfrak{A}_{k,n} \models \phi \).

Let

\[
\Sigma^u = \bigcup_{n \in \omega} \Sigma_n^u \quad \text{and} \quad \Sigma^d = \bigcup_{n \in \omega} \Sigma_n^d
\]

Hence \( \mathfrak{A}_k \models \Sigma^u \cup \Sigma^d \). For if not then there exists a quantifier free formula \( \phi(x_1, \ldots, x_m) \in \Sigma^u \cup \Sigma^d \), and \( b_1, \ldots, b_m \) such that \( \phi[b_1, \ldots, b_n] \) does not hold in \( \mathfrak{A}_k \). We have \( b_1, \ldots, b_m \in \mathfrak{A}_{k,i} \) for some \( i \in \omega \). Take \( n \) large enough \( \geq i \) so that \( \phi \in \Sigma_n^u \cup \Sigma_n^d \). Then \( \mathfrak{A}_{k,n} \) does not model \( \phi \), a contradiction. Now let \( \Sigma \) be a set of quantifier free formulas axiomatizing \( \text{RQEA}_\omega \), then \( \mathfrak{A}_k \) does not model \( \Sigma \) since \( \mathfrak{A}_k \) is not representable, so there exists a formula \( \phi \in \Sigma \) such that \( \phi \notin \Sigma^u \cup \Sigma^d \). Then \( \phi \) contains more than \( k \) variables and a diagonal constant occurs in \( \phi \).

\[\Box\]

We immediately get the following answer to Andréka’s question formulated on p. 193 of [1].

24
Corollary 13. The variety $\text{RQEA}_\omega$ is not axiomatizable over $\text{RQA}_\omega$ with a set of universal formulas containing infinitely many variables.

One can show, using the modified method of splitting here, that all theorems in [1] on complexity of axiomatizations generalize to $\text{RQEA}_\omega$ with the sole exception of Theorem 5, which is false for infinite dimensions [11]. As a sample we give the following theorem which can be proved by some modifications of the cited theorems in the proof; this modifications are not hard, most of them can be found in [13], proof of theorem 3.1. The basic idea in the proof is to show that certain cylindric homomorphisms (between cylindric algebras) remain to be quasipolyadic equality algebra homomorphisms (between their corresponding natural quasipolyadic equality expansions), that is when we add substitutions corresponding to transpositions.

Theorem 14. Let $\Sigma$ be a set of equations axiomatizing $\text{RQEA}_\omega$. Let $l, k, k' < \omega$. Then $\Sigma$ contains infinitely equations in which $-$ occurs, one of $+$ or $\cdot$ occurs a diagonal or a permutation with index $l$ occurs, more than $k'$ cylindrifications and more than $k$ variables occur.

Sketch of Proof. Let $n, k \in \omega \sim \{0\}$. Let $A_{k,n}$ be the non-representable algebra constructed above obtained by splitting $A'$ into $m \geq 2^k n! + 1$ atoms and we require that $m \geq k'$ as well. One then shows that the complementation free reduct $A^-_{k,n}$ of $A_{k,n}$ is a homomorphic image of a subalgebra $C$ of the complementation free reduct $P'$ of a a representable $P$, cf. Theorem 7, p.163. The algebra $A_{k,n}$ can be represented such that every operation except for $\cup$ and $\cap$ are the natural ones, cf. [1] p.200 and [43] for the necessary modifications. For any $I \subseteq \omega$, $|I| = m$ there is an infinite set $W$ an an embedding from $A_{k,n} \to (B^{(\omega W)}, c_i, d_{ij}, s_{\tau})_{i,j \in \omega, \tau \in G_n}$ which is a homomorphism with respect to all operations of $A_{k,n}$ except for $c_i, i \notin I$, cf. [11] p.172, Theorem 3. One just has to show that the map $h : A' \to \varphi(\omega W)$ defined on p. 174 preserves substitutions, which is straightforward from the definition of the map $g$ defined on p.173. There is an infinite set $W$, such that there is an embedding $h : A \to (B^{(\omega W)}, c_i, d_{ij}, s_{\tau})_{i,j < n, \tau \in G_n}$ such that $h$ is a homomorphism preserving all operations except for $d_l$ and $s_{[i,l]}$ if $i, l \in n$, cf. p.176 Claim 6. This will prove the theorem because of the following reasoning. By $n$ substitutions we understand the set $\{s_{[i,j]} : i, j \in n\}$. Let $\Sigma_n^-$ denote the set of equations without complementation in which only $n$ substitutions occur, $\Sigma_n^c$ be the set of equations which contains at most $k$ variables in which only $n$ substitutions occur, $\Sigma_n^d$ be the set of equations in which only $k'$ cylindrifications and $n$ substitutions occur, $\Sigma_n^{ds}$ be the set of equations in which at most $n$ substitutions occur and no diagonal nor substitutions with index $l$ occurs, and $\Sigma_n^{\text{Bool}}$ the set of equations that does not contain $\cdot$ nor $+$ and $n$ substitutions occur, all valid in $\text{RQEA}_\omega$. Then $A_{k,n} \models \Sigma_n^- \cup \Sigma_n^c \cup \Sigma_n^d \cup \Sigma_n^{\text{ds}} \cup \Sigma_n^{\text{Bool}}$. Indeed, the algebra $A_{k,n} \models \Sigma_n^-$ because of the following reasoning. Let $C$ and $P$ be as above. Then
$P \models \Sigma_n$ because $P$ is representable. So $P^\neg \models \Sigma_n^\neg$ because $-$ does not occur in $\Sigma_n^\neg$. Now $C \models \Sigma_n^\neg$ by $C \subseteq P^\neg$, and so $U_{k,n} \models \Sigma_n^\neg$ since $U_{k,n}$ is a homomorphic image of $C$ and $\Sigma_n^\neg$ consists of equations. Then $C_{k,n} \models \Sigma_n^\neg$. This together with previous reasoning proves that $\Sigma_n \equiv \Sigma_{k,n} \models \Sigma_n^\neg$. Then we can infer that $C \models \bigcup_{n \in \omega} \Sigma_n = \Gamma$. For if not, then we can choose $n$ large enough such that $C_{k,n}$ does not model $\Sigma_n$. But $C_{k,n}$ is not representable hence any equational axiomatization of the representable algebras contain a formula that is outside $\Gamma$. Thus the required follows.

For axiomatization with universal formulas, using the same ideas as above, except for those involving complementation, we obtain the slightly weaker:

**Theorem 15.** Let $\Sigma$ be a set of quantifier free formulas axiomatizing $RQEA_\omega$. Let $l, k, k' < \omega$. Then $\Sigma$ contains infinitely equations in which one of $+$ or $\cdot$ occurs a diagonal or a permutation with index $l$ occurs, more than $k'$ cylindrications and more than $k$ variables occur.

Our corollary 2 is evidence that $RQA_\omega$ can be axiomatized by an infinite set of universal formulas containing only finitely many variables.

Next we generalize Theorem 2 in [1] to the class $S_{\text{n}}\mathcal{QEA}_{\omega+p}$ for $p \geq 2$. Let $n \in \omega$ and $m = 2^{k,n+1}$. Let

$$e_n = \prod_{i \leq m} c_0(x \cdot x_i \cdot \prod_{i \neq j \leq m} -x_j) \leq c_0 \cdot \ldots \cdot c_m \prod_{i,j \leq m, i \neq j} s_i^0 c_1 \ldots c_m x_i \cdot d_{ij}. $$

Note that $e_n$ is equivalent to the above set of equations. Then $U_{k,n}$ does not model $e_n$, and so $U_k$ does not model $e_n$, for all $n \in \omega$, but $S_{\text{n}}\mathcal{QEA}_{\omega+p} \models e_n$ for all $n \in \omega$. This is done like the CA case since every equation that holds in $S_{\text{n}}\mathcal{CA}_{n+2}$ holds in $S_{\text{n}}\mathcal{QEA}_{n+2}$. Using the reasoning on p.163, one obtains:

**Theorem 16.** Let $p \geq 2$. Then $S_{\text{n}}\mathcal{QEA}_{\omega+p}$ is not axiomatizable with any set of quantifier free formulas containing only finitely many variables.

### 3.0.2 Monks algebras modified, by Hirsch and Hodkinson

Now we prove the conclusion of theorem[7] for cylindric algebras and quasipolyadic equality, solving the infinite dimensional version of the famous 2.12 problem in algebraic logic. The finite dimensional algebras we use are constructed by Hirsch and Hodkinson; and they based on a relation algebra construction. Such combinatorial algebras have affinity with Monk’s algebras. Related algebras were constructed by Robin Hirsch and the present author (together with the
above lifting argument) to prove theorem the analogue of theorem \(7\) holds for
various equality free algebraisations of first order logic.

We recall the construction of Hirsch and Hodkinson. They prove their
result for cylindric algebras. Here, by noting that their atom structures are
also symmetric; it permits expansion by substitutions, we slightly extend the
result to polyadic equality algebras. Define relation algebras \(A(n, r)\) having two
parameters \(n\) and \(r\) with \(3 \leq n < \omega\) and \(r < \omega\). Let \(\Psi\) satisfy \(n, r \leq \Psi < \omega\).
We specify the atom structure of \(A(n, r)\).

- The atoms of \(A(n, r)\) are \(id\) and \(a^k(i, j)\) for each \(i < n - 1, j < r\) and \(k < \psi\).
- All atoms are self converse.
- We can list the forbidden triples \((a, b, c)\) of atoms of \(A(n, r)\)- those such
  that \(a.(b, c) = 0\). Those triples that are not forbidden are the consistent
  ones. This defines composition: for \(x, y \in A(n, r)\) we have

\[
x; y = \{ a \in At(A(n, r)) ; \exists b, c \in AtA : b \leq x, c \leq y, (a, b, c) \text{ is consistent} \}
\]

Now all permutations of the triple \((Id, s, t)\) will be inconsistent unless
\(t = s\). Also, all permutations of the following triples are inconsistent:

\[
(a^k(i, j), a^{k'}(i, j), a^{k''}(i, j')),
\]

if \(j \leq j' < r\) and \(i < n - 1\) and \(k, k', k'' < \Psi\). All other triples are
consistent.

Hirsch and Hodkinson invented means to pass from relation algebras to
\(n\) dimensional cylindric algebras, when the relation algebras in question have
what they call a hyperbasis.

Unless otherwise specified, \(A = (A, +, \cdot, -, 0, 1, \sim; Id)\) will denote an arbi-
trary relation algebra with \(\sim\) standing for converse, and \(\cdot\) standing for composition,
and \(Id\) standing for the identity relation.

**Definition 17.** Let \(3 \leq m \leq n \leq k < \omega\), and let \(\Lambda\) be a non-empty set. An
\(n\) wide \(m\) dimensional \(\Lambda\) hypernetwork over \(A\) is a map \(N : \leq^n m \to \Lambda \cup AtA\)
such that \(N(\bar{x}) \in AtA\) if \(|\bar{x}| = 2\) and \(N(\bar{x}) \in \Lambda\) if \(|\bar{x}| \neq 2\), with the following
properties:

- \(N(x, x) \leq Id\) ( that is \(N(\bar{x}) \leq Id\) where \(\bar{x} = (x, x) \in \leq^n 2\).
- \(N(x, y) \leq N(x, z); N(z, y)\) for all \(x, y, z < m\)
- If \(\bar{x}, \bar{y} \in \leq^n m, |\bar{x}| = |\bar{y}|\) and \(N(x_i, y_i) \leq Id\) for all \(i < |\bar{x}|\), then \(N(\bar{x}) = N(\bar{y})\)
• when \( n = m \), then \( N \) is called an \( n \) dimensional \( \Lambda \) hypernetwork.

**Definition 18.** Let \( M, N \) be \( n \) wide \( m \) dimensional \( \Lambda \) hypernetworks.

1. For \( x < m \) we write \( M \equiv_x N \) if \( M(\vec{y}) = N(\vec{y}) \) for all \( \vec{y} \in \preceq^n(\vec{m} \sim \{x\}) \)

2. More generally, if \( x_0, \ldots, x_{k-1} < m \) we write \( M \equiv_{x_0, \ldots, x_{k-1}} N \) if \( M(\vec{y}) = N(\vec{y}) \) for all \( \vec{y} \in \preceq^n(\vec{m} \sim \{x_0, \ldots, x_{k-1}\}) \).

3. If \( N \) is an \( n \) wide \( m \) dimensional \( \Lambda \)-hypernetwork over \( A \), and \( \tau : m \rightarrow m \) is any map, then \( N \circ \tau \) denotes the \( n \) wide \( m \) dimensional \( \Lambda \) hypernetwork over \( A \) with labellings defined by

\[
N \circ \tau(\vec{x}) = N(\tau(\vec{x})) \text{ for all } \vec{x} \in \preceq^n \vec{m}.
\]

That is

\[
N \circ \tau(\vec{x}) = N(\tau(x_0), \ldots, \tau(x_{l-1}))
\]

**Lemma 19.** Let \( N \) be an \( n \) dimensional \( \Lambda \) hypernetwork over \( \mathfrak{A} \) and \( \tau : n \rightarrow n \) be a map. Then \( N \circ \tau \) is also a network.

**Proof.** [26] lemma 12.7

**Definition 20.** The set of all \( n \) wise \( m \) dimensional hypernetworks will be denoted by \( H^m_n(\mathfrak{A}, \Lambda) \). An \( n \) wide \( m \) dimensional \( \Lambda \) hyperbasis for \( \mathfrak{A} \) is a set \( H \subseteq H^m_n(\mathfrak{A}, \Lambda) \) with the following properties:

1. For all \( a \in At\mathfrak{A} \), there is an \( N \in R \) such that \( N(0,1) = a \)

2. For all \( N \in R \) all \( x, y, z < n \) with \( z \neq x, y \) and for all \( a, b \in At\mathfrak{A} \) such that \( N(x, y) \leq a; b \) there is \( M \in R \) with \( M \equiv_x N, M(x, z) = a \) and \( M(z, y) = b \)

3. For all \( M, N \in H \) and \( x, y < n \), with \( M \equiv_{xy} N \), there is \( L \in H \) such that \( M \equiv_x L \equiv_y N \)

4. For a \( k \) wide \( n \) dimensional hypernetwork \( N \), we let \( N|^{k}_{m} \) the restriction of the map \( N \) to \( \preceq^k \vec{m} \). For \( H \subseteq H^m_n(\mathfrak{A}, \Lambda) \) we let \( H|^{m}_{k} = \{N|^{k}_{m} : N \in H\} \).

5. When \( n = m \), \( H_n(\mathfrak{A}, \Lambda) \) is called an \( n \) dimensional hyperbases.

We say that \( H \) is symmetric, if whenever \( N \in H \) and \( \sigma : m \rightarrow m \), then \( N \circ \sigma \in H \).

We note that \( n \) dimensional hyperbasis are extensions of Maddux’s notion of cylindric basis.
Theorem 21. If $H$ is a $m$ wide $n$ dimensional $\Lambda$ symmetric hyperbases for $\mathfrak{A}$, then $\mathfrak{C}aH \in \mathbf{PEA}_n$.

Proof. Let $H$ be the set of $m$ wide $n$ dimensional $\Lambda$ symmetric hypernetworks for $\mathfrak{A}$. The domain of $\mathfrak{C}a(H)$ is $\varphi(H)$. The Boolean operations are defined as expected (as complement and union of sets). For $i, j < n$ the diagonal is defined by

$$d_{ij} = \{ N \in H : N(i, j) \leq \text{Id} \}$$

and for $i < n$ we define the cylindrifier $c_i$ by

$$c_i S = \{ N \in H : \exists M \in S (N \equiv_i M) \}.$$

Now the polyadic operations are defined by

$$p_{ij} X = \{ N \in H : \exists M \in S (N = M \circ [i, j]) \}$$

Then $\mathfrak{C}a(H) \in \mathbf{PEA}_n$. Furthermore, $\mathfrak{A}$ embeds into $\mathfrak{R}a(\mathfrak{C}a(H))$ via $a \mapsto \{ N \in H : N(0, 1) \leq a \}$.

\[\blacksquare\]

Theorem 22. Let $3 \leq m \leq n \leq k < \omega$ be given. Then $\mathfrak{C}a(H|_{m}) \cong \mathfrak{N}r_m(\mathfrak{C}a(H))$

Proof. \[\blacksquare\] 12.22

The set $C = H^{n+1}_n(\mathfrak{A}(n, r), \Lambda)$ aff all $(n + 1)$ wide $n$ dimensional $\Lambda$ hypernetworks over $\mathfrak{A}(n, r)$ is an $n + 1$ wide $n$ dimensional symmetric $\Lambda$ hyperbasis. $H$ is symmetric, if whenever $N \in H$ and $\sigma : m \to m$, then $N \circ \sigma \in H$. Hence $\mathfrak{A}(n, r)$ embeds into the $\mathfrak{R}a$ reduct of $C$.

Theorem 23. Assume that $3 \leq m \leq n$, and let

$$\mathfrak{C}(m, n, r) = \mathfrak{C}a(H^{n+1}_m(\mathfrak{A}(n, r), \omega)).$$

Then the following hold:

1. For any $r$ and $3 \leq m \leq n < \omega$, we have $\mathfrak{C}(m, n, r) \in \mathfrak{N}r_m \mathbf{PEA}_n$.

2. For $m < n$ and $k \geq 1$, there exists $x_n \in \mathfrak{C}(n, n + k, r)$ such that $\mathfrak{C}(m, m + k, r) \cong \mathfrak{R}l_x \mathfrak{C}(n, n + k, r)$.

3. $\mathfrak{S}\mathfrak{N}r \mathfrak{C}A_{\alpha+k+1}$ is not axiomatizable by a finite schema over $\mathfrak{S}\mathfrak{N}r \mathfrak{C}A_{\alpha+k}$

Proof.

1. $H^{n+1}_m(\mathfrak{A}(n, r), \omega)$ is a wide $n$ dimensional $\omega$ symmetric hyperbases, so $\mathfrak{C}aH \in \mathbf{PEA}_n$. But $H^{n+1}_m(\mathfrak{A}(n, r), \omega) = H^{n+1}_m$. Thus

$$\mathfrak{C}_r = \mathfrak{C}a(H^{n+1}_m(\mathfrak{A}(n, r), \omega)) = \mathfrak{C}a(H^{n+1}_m) \cong \mathfrak{N}r_m \mathfrak{C}aH.$$
(2) For $m < n$, let

$$x_n = \{ f \in F(n, n + k, r) : m \leq j < n \rightarrow \exists i < m f(i, j) = Id \}.$$ 

Then $x_n \in C(n, n + k, r)$ and $c_i x_n \cdot c_j x_n = x_n$ for distinct $i, j < m$.

Furthermore

$$I_n : C(m, m + k, r) \cong \mathfrak{R}_{x_n} \mathfrak{R}_m C(n, n + k, r).$$

via

$$I_n(S) = \{ f \in F(n, n + k, r) : f \upharpoonright m \times m \in S, \forall j(m \leq j < n \rightarrow \exists i < m f(i, j) = Id) \}.$$ 

(3) Follows from theorem [7]

\[\square\]

### 3.1 The class of neat reducts proper

Let $K$ be any of cylindric algebra, polyadic algebra, with and without equality, or Pinter’s substitution algebra. We give a unified model theoretic construction, to show the following:

1. For $n \geq 3$ and $m \geq 3$, $\mathfrak{Nr}_n K_m$ is not elementary, and $S_c \mathfrak{Nr}_n K_\omega \not\subseteq \mathfrak{Nr}_n K_m$.

2. For any $k \geq 5$, $\mathfrak{RaCA}_k$ is not elementary and $S_c \mathfrak{RaCA}_\omega \not\subseteq \mathfrak{RaCA}_k$.

$S_c$ stands for the operation of forming complete subalgebras. The relation algebra part formulated in the abstract reproves a result of Hirsch in [33], and answers a question of his posed in op.cit. For CA and its relatives the idea is very much like that in [35], the details implemented, in each separate case, though are significantly distinct, because we look for terms not in the clone of operations of the algebras considered; and as much as possible, we want these to use very little spare dimensions.

The relation algebra part is more delicate. We shall construct a relation algebra $\mathfrak{A} \in \mathfrak{RaCA}_\omega$ with a complete subalgebra $\mathfrak{B}$, such that $\mathfrak{B} \notin \mathfrak{RaCA}_k$, and $\mathfrak{B}$ is elementary equivalent to $\mathfrak{A}$. (In fact, $\mathfrak{B}$ will be an elementary subalgebra of $\mathfrak{A}$.)

Roughly the idea is to use an uncountable cylindric algebra in $\mathfrak{Nr}_3 CA_\omega$, hence representable, and a finite atom structure of another cylindric algebra. We construct a finite product of the the uncountable cylindric algebra; the product will be indexed by the atoms of the atom structure; the $\mathfrak{Ra}$ reduct of the former will be as desired; it will be a full $\mathfrak{Ra}$ reduct of an $\omega$ dimensional algebra and it has a complete elementary equivalent subalgebra not in $\mathfrak{RaCA}_k$. 

30
This is the same idea for \( \text{CA} \), but in this case, and the other cases of its relatives, one spare dimension suffices.

This subalgebra is obtained by replacing one of the components of the product with an elementary countable algebra. First order logic will not see this cardinality twist, but a suitably chosen term \( \tau_k \) not term definable in the language of relation algebras will, witnessing that the twisted algebra is not in \( \text{RaCA}_k \). For \( \text{CA}' \)’s and its relatives, as mentioned in the previous paragraph, we are lucky enough to have \( k \) just \( n + 1 \), proving the most powerful result.

We concentrate on relation algebras. Let \( \tau_k \) be an \( m \)-ary term of \( \text{CA}_k \), with \( k \) large enough, and let \( m \geq 2 \) be its rank. We assume that \( \tau_k \) is not definable of relation algebras (so that \( k \) has to be \( \geq 5 \)); such terms exist. Let \( \tau \) be a term expressible in the language of relation algebras, such that \( \text{CA}_k \models \tau_k(x_1, \ldots x_m) \leq \tau(x_1, \ldots x_m) \). (This is an implication between two first order formulas using \( k \)-variables.) Assume further that whenever \( \mathfrak{A} \in \text{Cs}_k \) (a set algebra of dimension \( k \)) is uncountable, and \( R_1, \ldots R_m \in A \) are such that at least one of them is uncountable, then \( \tau^3_k(R_1 \ldots R_m) \) is uncountable as well. (For \( \text{CA}' \)’s and its relatives, \( k = n + 1 \), for \( \text{CA}' \)’s and \( \text{Scs} \), it is a unary term, for polyadic algebras it also uses one extra dimension, it is a generalized composition hence it is a binary term.)

**Lemma 24.** Let \( V = (\text{At}, \equiv, d_{ij})_{i,j<3} \) be a finite cylindric atom structure, such that \( |\text{At}| \geq 3 \). Let \( L \) be a signature consisting of the unary relation symbols \( P_0, P_1, P_2 \) and uncountably many tenary predicate symbols. For \( u \in V \), let \( \chi_u \) be the formula \( \bigwedge_{u \in V} P_{u_i}(x_i) \). Then there exists an \( L \)-structure \( M \) with the following properties:

1. \( M \) has quantifier elimination, i.e. every \( L \)-formula is equivalent in \( M \) to a boolean combination of atomic formulas.
2. The sets \( P^M_i \) for \( i < n \) partition \( M \),
3. For any permutation \( \tau \) on \( 3 \), \( \forall x_0 x_1 x_2[R(x_0, x_1, x_2) \leftrightarrow R(x_\tau(0), x_\tau(1), x_\tau(2))], \)
4. \( M \models \forall x_0 x_1(R(x_0, x_1, x_2) \rightarrow \bigvee_{u \in V} \chi_u) \), for all \( R \in L \),
5. \( M \models \exists x_0 x_1 x_2(\chi_u \land R(x_0, x_1, x_2) \land \lnot S(x_0, x_1, x_2)) \) for all distinct tenary \( R, S \in L \), and \( u \in V \).
6. For \( u \in V, i < 3, M \models \forall x_0 x_1 x_2(\exists x_i \chi_u \leftrightarrow \bigvee_{v \in V, v \equiv i} \chi_v) \),
7. For \( u \in V \) and any \( L \)-formula \( \phi(x_0, x_1, x_2) \), if \( M \models \exists x_0 x_1 x_2(\chi_u \land \phi) \) then \( M \models \forall x_0 x_1 x_2(\exists x_i \chi_u \leftrightarrow \exists x_i(\chi_u \land \phi)) \) for all \( i < 3 \)

**Proof.** We cannot apply Frasie’s theorem to our signature, because it is uncountably infinite. What we do instead is that we introduce a new \( 4 \)-ary relation symbol, that will be used to code the uncountably many tenary
relation symbols. Let $\mathcal{L}$ be the relational signature containing unary relation symbols $P_0, \ldots, P_3$ and a 4-ary relation symbol $X$. Let $\mathbf{K}$ be the class of all finite $\mathcal{L}$-structures $\mathfrak{D}$ satisfying

1. The $P_i$’s are disjoint: $\forall x \bigwedge_{i<j<4}(P_i(x) \land \bigwedge_{j \neq i} \neg P_j(x))$.
2. $\forall x_0x_1x_2x_3(X(x_0, x_1, x_2, x_3) \rightarrow P_3(x_3) \land \bigvee_{u \in V} \chi_u)$. 

Then $\mathbf{K}$ contains countably many isomorphism types. Also it is easy to check that $\mathbf{K}$ is closed under substructures and that $\mathbf{K}$ has the amalgamation Property. From the latter it follows that it has the Joint Embedding Property. Then there is a countably infinite homogeneous $\mathcal{L}$-structure $\mathcal{N}$ with age $\mathbf{K}$. $\mathcal{N}$ has quantifier elimination, and obviously, so does any elementary extension of $\mathcal{N}$. $\mathbf{K}$ contains structures with arbitrarily large $P_3$-part, so $P_3^\mathcal{N}$ is infinite.

Let $\mathcal{N}^*$ be an elementary extension of $\mathcal{N}$ such that $|P_3^{\mathcal{N}^*}| = |\mathcal{L}|$, and fix a bijection $*$ from the set of binary relation symbols of $\mathcal{L}$ to $P_3^{\mathcal{N}^*}$. Define an $\mathcal{L}$-structure $\mathfrak{M}$ with domain $P_0^{\mathcal{N}^*} \cup P_1^{\mathcal{N}^*} \cup P_2^{\mathcal{N}^*} \cup \ldots P_3^{\mathcal{N}^*}$, by $P_i^{\mathfrak{M}} = P_i^{\mathcal{N}^*}$ for $i < 3$ and for binary $R \in \mathcal{L}$, and $\tau \in S_3$

$$\mathfrak{M} \models R(\tau \circ \bar{a}) \iff \mathcal{N}^* \models X(\bar{a}, R^\ast).$$

If $\phi(\bar{x})$ is any $\mathcal{L}$-formula, let $\phi^*(\bar{x}, \bar{R})$ be the $\mathcal{L}$-formula with parameters $\bar{R}$ from $\mathcal{N}^*$ obtained from $\phi$ by replacing each atomic subformula $R(\bar{x})$ by $X(\bar{x}, R^\ast)$ and relativizing quantifiers to $\neg P_n$, that is replacing $(\exists x)\phi(x)$ and $(\forall x)\phi(x)$ by $(\exists x)(\neg P_3(x) \rightarrow \phi(x))$ and $(\forall x)(\neg P_3(x) \rightarrow \phi(x))$, respectively. A straightforward induction on complexity of formulas gives that for $\bar{a} \in \mathfrak{M}$

$$\mathfrak{M} \models \phi(\bar{a}) \iff \mathcal{N}^* \models \phi^*(\bar{a}, \bar{R}).$$

We show that $\mathfrak{M}$ is as required. For quantifier elimination, if $\phi(\bar{x})$ is an $\mathcal{L}$-formula, then $\phi^*(\bar{x}, \bar{R}^\ast)$ is equivalent in $\mathcal{N}^*$ to a quantifier free $\mathcal{L}$-formula $\psi(\bar{x}, \bar{R}^\ast)$. Then replacing $\psi$’s atomic subformulas $X(x, y, z, R^\ast)$ by $R(x, y, z)$, replacing all $X(t_0, \ldots, t_3)$ not of this form by $\bot$, replacing subformulas $P_3(x)$ by $\bot$, and $P_i(R^\ast)$ by $\bot$ if $i < 3$ and $\top$ if $i = 3$, gives a quantifier free $\mathcal{L}$-formula $\psi$ equivalent in $\mathfrak{M}$ to $\phi$. (2) follows from the definition of satisfiability.

Let

$$\sigma = \forall x(\neg P_3(x) \rightarrow \bigvee_{i<3} (P_i(x) \land \bigwedge_{j \neq i} \neg P_j(x))).$$

Then $K \models \sigma$, so $\mathfrak{M} \models \sigma$ and $\mathcal{N}^* \models \sigma$. It follows from the definition that $\mathfrak{M}$ satisfies (3); (4) is similar.

For (5), let $u \in V$ and let $r, s \in P^\mathfrak{M}_3$ be distinct. Take a finite $\mathcal{L}$-structure $D$ with points $a_i \in P_u^D(i < 3)$ and distinct $r', s' \in P^D_3$ with

$$D \models X(a_0, a_1, a_2, r') \land \neg X(a_0, a_1, a_2, s').$$
Then $D \in K$, so $D$ embeds into $\mathfrak{M}$. By homogeneity, we can assume that the embedding takes $r'$ to $r$ and $s'$ to $s$. Therefore

$$\mathfrak{M} \models \exists \bar{x}(\chi_u \land X(\bar{x}, r) \land \neg X(\bar{x}, s)),$$

where $\bar{x} = \langle x_0, x_1, x_2 \rangle$. Since $r, s$ were arbitrary and $\mathcal{N}^*$ is an elementary extension of $\mathfrak{M}$, we get that

$$\mathcal{N}^* \models \forall yz(\mathcal{P}_3(y) \land \mathcal{P}_3(z) \land y \neq z \rightarrow \exists \bar{x}(\chi_u \land X(\bar{x}, y) \land \neg X(\bar{x}, z))).$$

The result for $\mathfrak{M}$ now follows.

Note that it follows from (4,5) that $P_i \not\equiv \emptyset$ for each $i < 3$. So it is clear that

$$\mathfrak{M} \models \forall x_0 x_1 x_2 (\exists x_i \chi_u \leftrightarrow \bigvee_{v \in V_i \equiv i} \chi_v);$$

giving (6).

Finally consider (7). Clearly, it is enough to show that for any $\mathcal{L}$-formula $\phi(\bar{x})$ with parameters $\bar{r} \in P_3^\mathfrak{M}, u \in S_3, i < 3$, we have

$$\mathfrak{M} \models \exists \bar{x}(\chi_u \land \phi) \rightarrow \forall \bar{x}(\exists x_i(\chi_u \rightarrow \exists x_i(\chi_u \land \phi))).$$

For simplicity of notation assume $i = 2$. Let $\bar{a}, \bar{b} \in \mathfrak{M}$ with

$$\mathfrak{M} \models (\chi_u \land \phi)(\bar{a}) \text{ and } \mathfrak{M} \models \exists x_2(\chi_u(\bar{b})).$$

We require

$$\mathfrak{M} \models \exists x_2(\chi_u \land \phi)(\bar{b}).$$

It follows from the assumptions that

$$\mathfrak{M} \models P_{u_0}(a_0) \land P_{u_1}(a_1) \land a_0 \neq a_1, \text{ and } \mathfrak{M} \models P_{u_0}(b_0) \land P_{u_1}(b_1) \land b_0 \neq b_1.$$

These are the only relations on $a_0 a_r \bar{r}$ and on $b_0 b_1 \bar{r}$ (cf. property (4) of Lemma), so

$$\theta^- = \{(a_0, b_0)(a_1, b_1)(r_l, r_l) : l < |\bar{r}|\}$$

is a partial isomorphism of $\mathfrak{M}$. By homogeneity, it is induced by an automorphism $\theta$ of $\mathfrak{M}$. Let $c = \theta(\bar{a}) = (b_0, b_1, \theta(a_2))$. Then $\mathfrak{M} \models (\chi_u \land \phi)(\bar{c})$. Since $\bar{c} \equiv_2 \bar{b}$, we have $\mathfrak{M} \models \exists x_2(\chi_u \land \phi)(\bar{b})$ as required.  

**Lemma 25.** (1) For $\mathfrak{A} \in \mathcal{C}A_3$ or $\mathfrak{A} \in \mathcal{S}C_3$, there exist a unary term $\tau_{3}(x)$ in the language of $\mathcal{S}C_4$ and a unary term $\tau(x)$ in the language of $\mathcal{C}A_3$ such that $\mathcal{C}A_4 \models \tau_3(x) \leq \tau(x)$, and for $\mathfrak{A}$ as above, and $u \in \mathfrak{A}t = \mathcal{S}_3$, $\tau^\mathfrak{A}(\chi_u) = \chi_{\tau^\mathfrak{A}(u)}(u)$.  

33
(2) For $A \in \text{PEA}_3$ or $A \in \text{PA}_3$, there exist a binary term $\tau_4(x,y)$ in the language of $\text{SC}_4$ and another binary term $\tau(x,y)$ in the language of $\text{SC}_3$ such that $\text{PEA}_4 \models \tau_4(x,y) \leq \tau(x,y)$, and for $A$ as above, and $u, v \in \text{At} = 3^3$, $\tau^A(\chi_u, \chi_v) = \chi_{\tau^\text{at}(u,v)}$.

(3) Let $k \geq 5$. Then there exist a term $\tau_k(x_1, \ldots, x_m)$ in the language of $\text{CA}_k$ and a term $\tau(x_1, \ldots, x_m)$ in the language of $\text{RA}$, expressible in $\text{CA}_3$, such that $\text{CA}_k \models \tau_k(x_1, \ldots, x_m) \leq \tau(x_1, \ldots, x_m)$, and for $A$ as above, and $u_1, \ldots, u_m \in \text{At}$, $\tau^A(\chi_{u_1}, \ldots, \chi_{u_m}) = \chi_{\tau^\text{at}(u_1, \ldots, u_m)}$.

Proof. (1) For all reducts of polyadic algebras, these terms are given in [34], and [35]. For cylindric algebras $\tau_4(x) = 3s(0,1)x$ and $\tau(x) = s_1c_1.x.s_0c_0x$. For polyadic algebras, it is a little bit more complicated because the former term above is definable. In this case we have $\tau(x,y) = c_1(c_0x.s_1c_1y).c_1x.c_0y$, and $\tau_4(x,y) = c_3(s_3c_3x.s_3c_3y)$.

(2) For relation algebras, we take the term corresponding to the following generalization of Johnson’s $Q$’s. Given $1 \leq n < \omega$ and $n^2$ tenary relations we define

$$Q(R_{ij} : i, j < n)(x_0, x_2, x_3) \iff \exists z_0 \ldots z_{n-1}(z_0 = x \land z_1 = y \land z_2 = z \land \bigwedge_{i,j,l < n} R_{ij}(z_i, z_j, z_l)).$$

The term $\tau$ is not difficult to find.

Theorem 26. (1) There exists $A \in \text{Nr}_3\text{QEA}_\omega$ with an elementary equivalent cylindric algebra, whose $\text{SC}$ reduct is not in $\text{Nr}_3\text{SC}_4$. Furthermore, the latter is a complete subalgebra of the former.

(2) There exists a relation algebra $A \in \text{RaCA}_\omega$, with an elementary equivalent relation algebra not in $\text{RaCA}_k$. Furthermore, the latter is a complete subalgebra of the former.

Proof. Let $L$ and $M$ as above. Let $\mathfrak{A}_\omega = \{\phi^M : \phi \in L\}$. Clearly $\mathfrak{A}_\omega$ is a locally finite $\omega$-dimensional cylindric set algebra. For the first part, we prove the theorem for $\text{CA}_3$ and its relatives.

Then $\mathfrak{A} \cong \mathfrak{N}_3\mathfrak{A}_\omega$, the isomorphism is given by

$$\phi^{\text{at}} \mapsto \phi^{\text{at}}.$$

Quantifier elimination in $M$ guarantees that this map is onto, so that $\mathfrak{A}$ is the full $\text{Ra}\mathfrak{A}$ reduct.

For $u \in V$, let $\mathfrak{A}_u$ denote the relativisation of $\mathfrak{A}$ to $\chi^\text{at}_u$ i.e.

$$\mathfrak{A}_u = \{x \in A : x \leq \chi^\text{at}_u\}. $$
$\mathfrak{A}_u$ is a boolean algebra. Also $\mathfrak{A}_u$ is uncountable for every $u \in V$ because by property (iv) of the above lemma, the sets $(\chi_u \land R(x_0, x_1, x_2)^{3^n})$, for $R \in L$ are distinct elements of $\mathfrak{A}_u$.

Define a map $f : \mathfrak{B} \mathfrak{I} \mathfrak{A} \rightarrow \prod_{u \in V} \mathfrak{A}_u$, by

$$f(a) = (a \cdot \chi_u)_{u \in V}.$$  

Here, and elsewhere, for a relation algebra $\mathfrak{E}$, $\mathfrak{B} \mathfrak{I} \mathfrak{E}$ denotes its boolean reduct. We will expand the language of the boolean algebra $\prod_{u \in V} \mathfrak{A}_u$ by constants in such a way that the relation algebra reduct of $\mathfrak{A}$ becomes interpretable in the expanded structure. For this we need.

Let $\mathfrak{P}$ denote the following structure for the signature of boolean algebras expanded by constant symbols $1_u$ for $u \in V$ and $d_{ij}$ for $i, j \in 3$: We now show that the relation algebra reduct of $\mathfrak{A}$ is interpretable in $\mathfrak{P}$. For this it is enough to show that $f$ is one to one and that $\text{Rng}(f)$ (Range of $f$) and the $f$-images of the graphs of the cylindric algebra functions in $\mathfrak{A}$ are definable in $\mathfrak{P}$. Since the $\chi_u^{3^n}$ partition the unit of $\mathfrak{A}$, each $a \in A$ has a unique expression in the form $\sum_{u \in V} (a \cdot \chi_u^{3^n})$, and it follows that $f$ is boolean isomorphism: $\text{bool}(\mathfrak{A}) \rightarrow \prod_{u \in V} \mathfrak{A}_u$. So the $f$-images of the graphs of the boolean functions on $\mathfrak{A}$ are trivially definable. $f$ is bijective so $\text{Rng}(f)$ is definable, by $x = x$. For the diagonals, $f(d_{ij}^{3^n})$ is definable by $x = d_{ij}$.

Finally we consider cylindrifications for $i < 3$. Let $S \subseteq V$ and $i < 3$, let $t_S$ be the closed term

$$\sum \{1_v : v \in V, v \equiv_i u \text{ for some } u \in S\}.$$  

Let

$$\eta_i(x, y) = \bigwedge_{S \subseteq V} \left( \bigwedge_{u \in S} x.1_u \neq 0 \land \bigwedge_{u \in V \setminus S} x.1_u = 0 \rightarrow y = t_S \right).$$

We claim that for all $a \in A$, $b \in P$, we have

$$\mathfrak{P} \models \eta_i(f(a), b) \iff b = f(c_i^3 a).$$

To see this, let $f(a) = (a_u)_{u \in V}$, say. So in $\mathfrak{A}$ we have $a = \sum u a_u$. Let $u$ be given; $a_u$ has the form $(\chi_i \land \phi)^{3^n}$ for some $\phi \in L^3$, so $c_i^3(a_u) = (\exists x_i(\chi_u \land \phi))^{3^n}$. By property (vi), if $a_u \neq 0$, this is $(\exists x_i(\chi_u)^M$; by property 5, this is $(\bigvee_{v \in V, v \equiv_i u} \chi_v)^{3^n}$. Let $S = \{u \in V : a_u \neq 0\}$. By normality and additivity of cylindrifications we have,

$$c_i^3(a) = \sum_{u \in V} c_i^3 a_u = \sum_{u \in S} c_i^3 a_u = \sum_{u \in S} \sum_{v \in V, v \equiv_i u} \chi_v^{3^n}$$

$$= \sum \{\chi_v^{3^n} : v \in V, v \equiv_i u \text{ for some } u \in S\}.$$
So $\mathcal{P} \models f(c_i^a) = t_\delta$. Hence $\mathcal{P} \models \eta_i(f(a), f(c_i^a))$. Conversely, if $\mathcal{P} \models \eta_i(f(a), b)$, we require $b = f(c_i a)$. Now $S$ is the unique subset of $V$ such that

$$\mathcal{P} \models \bigwedge_{u \in S} f(a) \cdot 1_u \neq 0 \land \bigwedge_{u \in V \setminus S} f(a) \cdot 1_u = 0.$$  

So we obtain

$$b = t_\delta = f(c_i^a).$$

The rest is the same as in [35] while for other relatives, the idea implemented is also the same; one just uses the corresponding terms as in lemma [25].

For relational algebras we proceed as follows: Now the $\mathcal{RA}$ reduct of $\mathfrak{A}$ is a generalized reduct of $\mathfrak{A}$, hence $\mathcal{P}$ is first order interpretable in $\mathcal{RA}_\mathfrak{A}$, as well. It follows that there are closed terms $1_{u,v}, d_{i,j}$ and a formula $\eta$ built out of these closed terms such that

$$\mathcal{P} \models \eta(f(a), b, c)$$

iff $b = f(a \circ c)$, where the composition is taken in $\mathcal{RA}_\mathfrak{A}$.

We have proved that $\mathcal{RA}_\mathfrak{A}$ is interpretable in $\mathcal{P}$. Furthermore it is easy to see that the interpretation is two dimensional and quantifier free.

For each $u \in V$, choose any countable boolean elementary complete subalgebra of $\mathfrak{A}_u$, $\mathfrak{B}_u$ say. Let $u_i : i < m$ be elements in $V$ and let

$$Q = \left( \prod_{u : i < m} \mathfrak{A}_u \right) \times \left( \prod_{u \in V \setminus \{u_1, \ldots, u_m\}} \mathfrak{A}_u \right) \times \left( \prod_{u, v, d_{i,j} \in V, i,j < 3} \mathfrak{A}_u \right) 1_{u,v,d_{i,j}} \equiv$$

$$\prod_{u \in V} \mathfrak{A}_u, 1_{u,v}, d_{i,j} u, v, i, j < 3 = P.$$

Let $\mathfrak{B}$ be the result of applying the interpretation given above to $Q$. Then $\mathfrak{B} \equiv \mathcal{RA}_\mathfrak{A}$ as relation algebras, furthermore $\mathfrak{B} \mathfrak{I} \mathfrak{B}$ is a complete subalgebra of $\mathfrak{B} \mathfrak{A}$. Assume for contradiction that $\mathfrak{B} = \mathcal{RA}_D$ with $D \in C\mathfrak{A}_k$. Let $u_1, \ldots, u_m \in V$ be such that $\tau_k^D(\chi_{u_1}, \ldots, \chi_{u_m})$, is uncountable in $D$.

Because $\mathfrak{B}$ is a full $\mathcal{RA}$ reduct, this set is contained in $\mathfrak{B}$.

For simplicity assume that $\tau^C_{u, v}(u_1 \ldots u_m) = Id$. On the other hand for $x_i \in B$, with $x_i \leq \chi_{u_i}$, we have

$$\tau_k^D(x_1, \ldots, x_m) \leq \tau(x_1 \ldots x_m) \in \tau(\chi_{u_1}, \ldots, \chi_{u_m}) = \chi_{\tau(u_1 \ldots u_m)} = \chi_{Id}.$$

But this is a contradiction, since $B_{Id} = \{ x \in B : x \leq \chi_{Id} \}$ is countable. \square
4 Second question

In the previous section we showed that it can be the case that an algebra does not neatly embed in another algebra in one extra dimension (for finite as well as for infinite dimensional algebras). Not only that, but if an algebra neatly embeds into \( k \) extra dimensions, there is no guarantee whatsoever, that it neatly embeds into \( k + 1 \) extra dimensions, and in fact we know that there are cases that it cannot. There are known examples for finite dimensions, and these can be used to prove the analogous result for infinite dimensions. Such results are related to completeness results for modifications of first order logic, or rather incompleteness ones.

Here we address a variation, or rather a natural extension of this question, namely, suppose that an algebra does neatly embed into extra dimensions, is the big algebra, or the dilation, uniquely defined over the small algebra, if the latter as a set generates the former using all the extra dimensions? The more spare dimensions we have, the more likely that the original algebra has more control on the dilation, because it codes more hidden dimensions, and so in a sense, it 'defines' more, it has a larger expressive power.

The strongest case would be the neat embedding in \( \omega \) extra dimensions (this often implies representability of the algebra in question) and is equivalent, implementing a standard ultraproduct construction, to embedding into an algebra having any transfinite larger ordinal as its dimension. (In cylindric algebra, for any two infinite ordinals \( \alpha < \beta \) and \( n \in \omega \), \( \forall \tau_n CA_\alpha = \forall \tau_n CA_\beta \), so that getting to \( \omega \) is getting over the hurdle).

We will see that such uniqueness is actually equivalent to that the neat reduct operator formulated in an appropriate way as a functor has a right adjoint, and that this in turn is strongly related to various amalgamation properties of the class in question. So while in the first case we deal with completeness theorems, or rather the lack thereof, in the second we deal with the interpolation property for the corresponding algebraisable logic.

This algebraic approach via neat embeddings suggests that the two notions are not unrelated. The mere fact that a Henkin construction can prove both completeness and interpolation, not only in the context of first order logic, also emphasizes this strong tie.

Another algebraic manifestation of this link, is the recurrent phenomena in algebraic logic, that for several algebrasations of variants of first order logic, completeness and interpolation come hand in hand. This happens for example in Keislers logic \[5\], \[38\] and their countable reducts studied by Sain \[30\].

To prove a completeness theorem using the methodology of algebraic logic, one needs to show that a certain abstract class of algebras defined syntactically (via a simple set of first order axioms preferably equations), consists solely of representable algebras providing a complete semantics. This often appeals to
the neat embedding theorem of Henkin. First one proves that the algebra in question neatly embeds into an ω dilation, and in such an abundance of spare dimensions one can construct so-called Henkin ultrafilters that eliminate cylindrifiers, and then the representations becomes really definable by reducing it basically to the propositional part.

In such a context interpolation is provided by showing that this neat embedding is faithful, the small algebra essentially determines the structure of the bigger one. Expressed otherwise, for the interpolation property to hold a necessary (and in some cases also sufficient) condition is that algebraic terms definable using the added spare dimensions, to eliminate cylindrifiers, are already term definable. The typical situation (even for cylindric algebras with no restriction whatsoever, like local finitenes) an interpolant can always be found, but the problem is that it might, and indeed there are situations where it must, resort to extra dimensions (variables). This happens in the case for instance of the so called finitary logics of infinitary relations \[45\]. This interpolant then becomes term definable in higher dimensions, but when these terms are actually coded by ones using only the original amount of dimensions, we get the desired interpolant. One way of getting around the unwarranted spare dimensions in the interpolant, is to introduce new connectives that code extra dimensions, in which case in the original language any implication can be interpolated by a formula using the same number of variables (and common symbols) but possibly uses the new connectives. In \[45\] this is done to prove an interpolation theorem for the severely incomplete typless logics studied in \[25\].

Categorially, and indeed intuitively, this means that the dilation functor is invertible, the interpolant is found in a dilation, but the inverse, the neat reduct functor gets us back to our base, to our original algebra. To formulate our results, we need some preparations.

**Definition 27.** (1) $K$ has the Amalgamation Property if for all $A_1, A_2 \in K$ and monomorphisms $i_1 : A_0 \to A_1$, $i_2 : A_0 \to A_2$ there exist $D \in K$ and monomorphisms $m_1 : A_1 \to D$ and $m_2 : A_2 \to D$ such that $m_1 \circ i_1 = m_2 \circ i_2$.

(2) If in addition, $(\forall x \in A_j)(\forall y \in A_k)(m_j(x) \leq m_k(y) \implies (\exists z \in A_0)(x \leq i_j(z) \wedge i_k(z) \leq y))$ where $\{j, k\} = \{1, 2\}$, then we say that $K$ has the superamalgamation property (SUPAP).

**Definition 28.** An algebra $\mathfrak{A}$ has the strong interpolation theorem, SIP for short, if for all $X_1, X_2 \subseteq A$, $a \in \mathcal{S}_g^A X_1$, $c \in \mathcal{S}_g^A X_2$ with $a \leq c$, there exist $b \in \mathcal{S}_g^A (X_1 \cap X_2)$ such that $a \leq b \leq c$.

For an algebra $\mathfrak{A}$, $\text{Co}\mathfrak{A}$ denotes the set of congruences on $\mathfrak{A}$.
Definition 29. An algebra $\mathfrak{A}$ has the congruence extension property, or CP for short, if for any $X_1, X_2 \subset A$ if $R \in Co\mathcal{S}g^\mathfrak{A} X_1$ and $S \in Co\mathcal{S}g^\mathfrak{A} X_2$ and
\[
R \cap \mathcal{S}g^\mathfrak{A} (X_1 \cap X_2) = S \cap \mathcal{S}g^\mathfrak{A} (X_1 \cap X_2),
\]
then there exists a congruence $T$ on $\mathfrak{A}$ such that
\[
T \cap \mathcal{S}g^\mathfrak{A} X_1 = R \text{ and } T \cap \mathcal{S}g^\mathfrak{A} (X_2) = S.
\]

Maksimova and Madárasz [16], [17], proved that if interpolation holds in free algebras of a variety, then the variety has the superamalgamation property. Using a similar argument, we prove this implication in a slightly more general setting. But first an easy lemma:

Lemma 30. Let $K$ be a class of BAO’s. Let $\mathfrak{A}, \mathfrak{B} \in K$ with $\mathfrak{B} \subseteq \mathfrak{A}$. Let $M$ be an ideal of $\mathfrak{B}$. We then have:

1. $Ig^\mathfrak{A} M = \{ x \in A : x \leq b \text{ for some } b \in M \}$
2. $M = Ig^\mathfrak{A} M \cap \mathfrak{B}$
3. if $C \subseteq \mathfrak{A}$ and $N$ is an ideal of $C$, then $Ig^\mathfrak{A} (M \cup N) = \{ x \in A : x \leq b + c \text{ for some } b \in M \text{ and } c \in N \}$
4. For every ideal $N$ of $\mathfrak{A}$ such that $N \cap \mathfrak{B} \subseteq M$, there is an ideal $N'$ in $\mathfrak{A}$ such that $N \subseteq N'$ and $N' \cap \mathfrak{B} = M$. Furthermore, if $M$ is a maximal ideal of $\mathfrak{B}$, then $N'$ can be taken to be a maximal ideal of $\mathfrak{A}$.

Proof. Only (iv) deserves attention. The special case when $n = \{0\}$ is straightforward. The general case follows from this one, by considering $\mathfrak{A}/N$, $\mathfrak{B}/(N \cap \mathfrak{B})$ and $M/(N \cap \mathfrak{B})$, in place of $\mathfrak{A}, \mathfrak{B}$ and $M$ respectively.

The previous lemma will be frequently used without being explicitly mentioned.

Theorem 31. Let $K$ be a class of BAO’s such that $HK = SK = K$. Assume that for all $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in K$, inclusions $m : \mathfrak{C} \to \mathfrak{A}$, $n : \mathfrak{C} \to \mathfrak{B}$, there exist $\mathfrak{D}$ with SIP and $h : \mathfrak{D} \to \mathfrak{C}$, $h_1 : \mathfrak{D} \to \mathfrak{A}$, $h_2 : \mathfrak{D} \to \mathfrak{B}$ such that for $x \in h^{-1}(\mathfrak{C})$,
\[
h_1(x) = m \circ h(x) = n \circ h(x) = h_2(x).
\]
Then $K$ has SUPAP. In particular, if $K$ is a variety and the free algebras have SIP then $V$ has SUPAP.
Proof. Let $\mathcal{D}_1 = h_1^{-1}(\mathfrak{A})$ and $\mathcal{D}_2 = h_2^{-1}(\mathfrak{B})$. Then $h_1 : \mathcal{D}_1 \to \mathfrak{A}$, and $h_2 : \mathcal{D}_2 \to \mathfrak{B}$.

Let $M = \ker h_1$ and $N = \ker h_2$, and let $\tilde{h}_1 : \mathcal{D}_1/M \to \mathfrak{A}, \tilde{h}_2 : \mathcal{D}_2/N \to \mathfrak{B}$ be the induced isomorphisms.

Let $l_1 : h^{-1}(\mathfrak{C})/h^{-1}(\mathfrak{E}) \cap M \to \mathfrak{C}$ be defined via $\tilde{x} \mapsto h(x)$, and $l_2 : h^{-1}(\mathfrak{C})/h^{-1}(\mathfrak{E}) \cap N$ to $\mathfrak{C}$ be defined via $\tilde{x} \mapsto h(x)$. Then those are well defined, and hence $k^{-1}(\mathfrak{E}) \cap M = h^{-1}(\mathfrak{E}) \cap N$. Then we show that $P = \mathfrak{I}(M \cup N)$ is a proper ideal and $\mathfrak{D}/P$ is the desired algebra. Now let $x \in \mathfrak{I}(M \cup N) \cap \mathcal{D}_1$. Then there exist $b \in M$ and $c \in N$ such that $x \leq b + c$. Thus $x - b \leq c$. But $x - b \in \mathcal{D}_1$ and $c \in \mathcal{D}_2$, it follows that there exists an interpolant $d \in \mathcal{D}_1 \cap \mathcal{D}_2$ such that $x - b \leq d \leq c$. We have $d \in N$ therefore $d \in M$, and since $x \leq d + b$, therefore $x \in M$. It follows that $\mathfrak{I}(M \cup N) \cap \mathcal{D}_1 = M$ and similarly $\mathfrak{I}(M \cup N) \cap \mathcal{D}_2 = N$. In particular $P = \mathfrak{I}(M \cup N)$ is a proper ideal.

Let $k : \mathcal{D}_1/M \to \mathcal{D}/P$ be defined by $k(a/M) = a/P$ and $h : \mathcal{D}_2/N \to \mathcal{D}/P$ by $h(a/N) = a/P$. Then $k \circ m$ and $h \circ n$ are one to one and $k \circ m \circ f = h \circ n \circ g$. We now prove that $\mathcal{D}/P$ is actually a superamalgam. i.e we prove that $K$ has the superamalgamation property. Assume that $k \circ m(a) \leq h \circ n(b)$. There exists $x \in \mathcal{D}_1$ such that $x/P = k(m(a))$ and $m(a) = x/M$. Also there exists $z \in \mathcal{D}_2$ such that $z/P = h(n(b))$ and $n(b) = z/N$. Now $x/P \leq z/P$ hence $x - z \in P$. Therefore there is an $r \in M$ and an $s \in N$ such that $x - r \leq z + s$. Now $x - r \in \mathcal{D}_1$ and $z + s \in \mathcal{D}_2$, it follows that there is an interpolant $u \in \mathcal{D}_1 \cap \mathcal{D}_2$ such that $x - r \leq u \leq z + s$. Let $t \in \mathfrak{C}$ such that $m \circ f(t) = u/M$ and $n \circ g(t) = u/N$. We have $x/P \leq u/P \leq z/P$. Now $m(f(t)) = u/M \geq x/M = m(a)$. Thus $f(t) \geq a$. Similarly $n(g(t)) = u/N \leq z/N = n(b)$, hence $g(t) \leq b$. By total symmetry, we are done. \qed

For a cardinal $\beta > 0$, $L \subseteq K_\alpha$ and $\rho : \beta \to \varphi(\alpha)$, $\mathfrak{f}_\rho L$ stands for the dimension restricted $L$ free algebra on $\beta$ generators. The sequence $\langle \eta/\mathfrak{C}^\rho L : \eta < \beta \rangle$ $L$-freely generates $\mathfrak{f}_\rho L$, cf. [24] Theorem 2.5.35. $\mathfrak{f}_\rho L \mathfrak{A}_\alpha$ is treated in [28] under the name of free algebras over $L$ subject to certain defining relations, cf. [28] Definition 1.1.5. The super amalgamation property, due to Maksimova, is rarely applied to algebraisations of first order logic; yet in this direction we have:

**Theorem 32.** Let $\kappa$ be any ordinal $> 1$. Let $M = \{ \mathfrak{A} \in K_{\kappa + \omega} : \mathfrak{A} = \mathfrak{G}^3 \mathfrak{N}_\kappa \mathfrak{A} \}$. Then $M$ has SUPAP.

Proof. First one proves that dimension restricted free algebras have the strong interpolation property, and then shows that they satisfy the conditions in the previous theorem. The first part is proved in [15]. For the second part we proceed as follows. Let $\kappa$ be an arbitrary ordinal $> 0$. Let $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$ be in $K$ and $f : \mathfrak{C} \to \mathfrak{A}$ and $g : \mathfrak{C} \to \mathfrak{B}$ be monomorphisms. We want to find an amalgam. Let $\langle a_i : i \in I \rangle$ be an enumeration of $\mathfrak{A}$ and $\langle b_i : i \in J \rangle$ be
Assume that.

Proof. Let $k = I \cup J$. Let $\xi$ be a bijection from $k$ onto a cardinal $\mu$. Let $\rho \in \wp(\kappa + \omega)$ be defined by $\rho i = \Delta a_i$ for $i \in I$ and $\rho j = \Delta b_j$ for $j \in J$. Then $\rho$ is well defined. Let $\beta = \kappa + \omega$ and let $\mathfrak{F} = \mathfrak{F}_{\mu}^{\omega}K_{\beta}$. Let $\mathfrak{F}'$ be the subalgebra of $\mathfrak{F}$ generated by $\{\xi i/\mathfrak{C}r_\mu^\omega K_{\beta} : i \in I\}$ and let $\mathfrak{F}''$ be the subalgebra generated by $\{\xi i/\mathfrak{C}r_\mu^\omega K_{\beta} : i \in J\}$. To avoid cumbersome notation we write $\xi i$ instead of $\xi i/\mathfrak{C}r_\mu^\omega K_{\beta}$ and similarly for $\xi j$. No confusion is likely to ensue. Then there exists a homomorphism from $\mathfrak{F}'$ onto $\mathfrak{A}$ such that $\xi i \mapsto a_i$ $(i \in I)$ and similarly a homomorphism from $\mathfrak{F}''$ into $\mathfrak{B}$ such that $\xi j \mapsto b_j$ $(j \in I)$.

The cylindric algebra analogue of the following theorem was proved for cylindric algebras using a very strong result, namely that the class of representable algebras does not have AP. This result is proved for quasi polyadic algebras by the present author [12] so the same thread of argument reported in [47] together with the latter result gives the required.

However, here we follow a different route that is admittedly slightly long and winding. But it has many advantages. First it avoids such a strong result. Second, the proof shows explicitly the connections between the unique neat embedding property, existence of universal maps, interpolation properties and congruence extension properties in free algebras and ultimately the amalgamation property.

QEA denotes quasi-polyadic algebras and RQEA denotes the representable QEAs.

**Theorem 33.** For $\alpha \geq \omega$, the following hold:

(i) There exists $\mathfrak{A} \in \text{RQEA}_\alpha$, $\mathfrak{B} \in \text{QEA}_{\alpha+\omega}$ such that $\mathfrak{A} \subseteq \mathfrak{N}_\alpha \mathfrak{B}$ $\mathfrak{A}$ generates $\mathfrak{B}$ but $\mathfrak{A} \neq \mathfrak{N}_\alpha \mathfrak{B}$.

(ii) There exist $\mathfrak{A} \in \text{RQEA}_\alpha$, $\mathfrak{B} \in \text{QEA}_{\alpha+\omega}$ and an ideal $J \subseteq \mathfrak{B}$, such that $\mathfrak{A} \subseteq \mathfrak{N}_\alpha \mathfrak{B}$, $\mathfrak{A}$ generates $\mathfrak{B}$, but $\mathfrak{I}g_\mathfrak{B}(J \cap A) \neq \mathfrak{B}$.

(iii) There exist $\mathfrak{A, A'} \in \text{RQEA}_\alpha$, $\mathfrak{B, B'} \in \text{QEA}_{\alpha+\omega}$ with embeddings $e_A : \mathfrak{A} \rightarrow \mathfrak{N}_\alpha \mathfrak{B}$ and $e_{A'} : \mathfrak{A'} \rightarrow \mathfrak{N}_\alpha \mathfrak{B'}$ such that $\mathfrak{S}g_\mathfrak{B} e_A(A) = \mathfrak{B}$ and $\mathfrak{S}g_{\mathfrak{B}'} e_{A'}(A) = \mathfrak{B'}$, and an isomorphism $i : \mathfrak{A} \rightarrow \mathfrak{A'}$ for which there exists no isomorphism $\tilde{i} : \mathfrak{B} \rightarrow \mathfrak{B'}$ such that $\tilde{i} \circ e_A = e_{A'} \circ i$.

**Proof.** Assume that (i) is false. Then we can prove (*) (the negation of (ii)).

(*) For all $\mathfrak{A} \in \text{RQEA}_\alpha$, $\mathfrak{B} \in \text{QEA}_{\alpha+\omega}$ and ideal $J \subseteq \mathfrak{B}$, if $\mathfrak{A} \subseteq \mathfrak{N}_\alpha \mathfrak{B}$, and $\mathfrak{A}$ generates $\mathfrak{B}$, then $\mathfrak{I}g_\mathfrak{B}(J \cap A) = J$.

From (*) we prove that (iii) is false, from which we reach a contradiction. This will prove (i) and (ii) and (iii).
Since $\mathfrak{A}$ generates $\mathfrak{B}$ we have $\mathfrak{A} = \mathfrak{Nr}_\alpha \mathfrak{B}$. Clearly $\mathfrak{g}^2_\alpha(J \cap A) \subseteq J$. Conversely, let $x \in J$. Then $\Delta x \sim \alpha$ is finite, call it $\Gamma$. So $c_{(\Gamma)} x \in \mathfrak{Nr}_\alpha \mathfrak{B}$, so, by assumption, it is in $\mathfrak{A}$. Hence $c_{(\Gamma)} x \in \mathfrak{A} \cap J$. But $x \leq c_{(\Gamma)} x$ we get the first required. Now we can assume (**). Let $f$ be a bijection from $\mathfrak{A}$ into $\mathfrak{B}$ and let $i : \mathfrak{A} \rightarrow \mathfrak{A}'$ be an isomorphism. Let $\mu = |\mathfrak{A}|$. Let $x$ be a bijection from $\mu$ onto $\mathfrak{A}$. Let $y$ be a bijection from $\mu$ onto $\mathfrak{A}'$, such that $i(x_j) = y_j$ for all $j < \mu$. Let $\rho = \langle \Delta^{(\alpha)} x_j : j < \mu \rangle$, $\mathfrak{D} = \mathfrak{g}^{\mathfrak{R}_{\mu}} \mathfrak{QEA}_\beta$, $g_\xi = \xi/C_{\mathfrak{R}_{\mu}}^{(\alpha)} \mathfrak{QEA}_\beta$ for all $\xi < \mu$ and $\mathfrak{C} = \mathfrak{g}^{\mathfrak{R}_{\mu}} \mathfrak{D}\{g_\xi : \xi < \mu\}$. Then $\mathfrak{C} \subseteq \mathfrak{Nr}_\alpha \mathfrak{D}$, $\mathfrak{C}$ generates $\mathfrak{D}$ and $\mathfrak{C} \in \mathfrak{RQE}_{\alpha}$. There exist $f \in Hom(\mathfrak{D}, \mathfrak{B})$ and $f' \in Hom(\mathfrak{D}, \mathfrak{B}')$ such that $f(g_\xi) = e_A(x_\xi)$ and $f'(g_\xi) = e_{A'}(y_\xi)$ for all $\xi < \mu$. Note that $f$ and $f'$ are both onto. We now have $e_A \circ i^{-1} \circ e_A^{-1} \circ (f' \upharpoonright \mathfrak{C}) = f \upharpoonright \mathfrak{C}$. Therefore $Ker f' \cap \mathfrak{C} = Ker f \cap \mathfrak{C}$. Hence $\mathfrak{g}(Ker f' \cap \mathfrak{C}) = \mathfrak{g}(Ker f \cap \mathfrak{C})$. So, by assumption, $Ker f' = Ker f$. Let $y \in B$, then there exists $x \in D$ such that $y = f(x)$. Define $\tilde{i}(y) = f'(x)$. The map is well defined and is as required. That is, $\tilde{i} \circ e_A = e_{A'} \circ i$. From now on, we assume that (ii) is false. Hence we assume the following (**):

For all $\mathfrak{A}, \mathfrak{A}' \in \mathfrak{RQE}_{\alpha}$, $\mathfrak{B}, \mathfrak{B}' \in \mathfrak{QEA}_\beta$ $e_A, e_{A'}$ are embeddings from $\mathfrak{A}, \mathfrak{A}'$ into $\mathfrak{Nr}_\alpha \mathfrak{B}, \mathfrak{Nr}_\alpha \mathfrak{B}'$, respectively, such that $\mathfrak{g}^2(\mathfrak{A}(A)) = \mathfrak{B}$ and $\mathfrak{g}^2(\mathfrak{A}'(A')) = \mathfrak{B}'$, and isomorphism $i : \mathfrak{A} \rightarrow \mathfrak{A}'$, there exist $\tilde{i} : \mathfrak{B} \rightarrow \mathfrak{B}'$ such that $\tilde{i} \circ e_A = e_{A'} \circ i$. We will arrive at a contradiction.

We start with the following claim that free algebras satisfy. Recall that $S \in Co\mathfrak{A}$ means that $S$ is a congruence relation on $\mathfrak{A}$.

**Claim 1.** Let $\mu$ be a cardinal $> 0$. Let $\mathfrak{A} = \mathfrak{g}^{\mathfrak{R}_{\mu}} \mathfrak{QEA}_\alpha$. For any $X_1, X_2 \subseteq \mu$ if $R \in Co\mathfrak{A}(X_1)$ and $S \in Co\mathfrak{A}(X_2)$ and

$$R \cap A^{(X_1 \cap X_2)} = S \cap A^{(X_1 \cap X_2)},$$

then there exists a congruence $T$ on $\mathfrak{A}$ such that

$$T \cap A^{(X_1)} = R \text{ and } T \cap A^{(X_2)} = S.$$  

**Proof.** For $R \in Co\mathfrak{A}$ and $X \subseteq A$, by $(\mathfrak{A}/R)^{(X)}$ we understand the subalgebra of $\mathfrak{A}/R$ generated by $\{x/R : x \in X\}$. Let $\mathfrak{A}$, $X_1$, $X_2$, $R$ and $S$ be as specified in the claim. Define

$$\theta : \mathfrak{A}^{(X_1 \cap X_2)} \rightarrow \mathfrak{A}^{(X_1)}/R$$

by

$$a \mapsto a/R.$$ 

Then $ker \theta = R \cap A^{(X_1 \cap X_2)}$ and $Im \theta = (\mathfrak{A}^{(X_1)}/R)^{(X_1 \cap X_2)}$. It follows that

$$\tilde{\theta} : \mathfrak{A}^{(X_1 \cap X_2)}/R \cap A^{(X_1 \cap X_2)} \rightarrow (\mathfrak{A}^{(X_1)}/R)^{(X_1 \cap X_2)}$$

with $\tilde{\theta}(a) = a/R$. Therefore $\tilde{\theta}(R \cap A^{(X_1 \cap X_2)}) = S \cap A^{(X_1 \cap X_2)}$. Hence $R \cap A^{(X_1 \cap X_2)} = S \cap A^{(X_1 \cap X_2)}$, and $T \cap A^{(X_1)} = R$ and $T \cap A^{(X_2)} = S$.
defined by
\[ a/R \cap ^2A^{X_1 \cap X_2} \mapsto a/R \]
is a well defined isomorphism. Similarly
\[ \overline{\psi} : \mathfrak{A}(X_1 \cap X_2)/S \cap ^2A^{(X_1 \cap X_2)} \to (\mathfrak{A}(X_2)/S)(X_1 \cap X_2) \]
defined by
\[ a/S \cap ^2A^{X_1 \cap X_2} \mapsto a/S \]
is also a well defined isomorphism. But
\[ R \cap ^2A^{(X_1 \cap X_2)} = S \cap ^2A^{(X_1 \cap X_2)} \]
Hence
\[ \phi : (\mathfrak{A}(X_1)/R)(X_1 \cap X_2) \to (\mathfrak{A}(X_2)/S)(X_1 \cap X_2) \]
defined by
\[ a/R \mapsto a/S \]
is a well defined isomorphism. Now \((\mathfrak{A}(X_1)/R)(X_1 \cap X_2)\) embeds into \(\mathfrak{A}(X_1)/R\) via the inclusion map; it also embeds in \(\mathfrak{A}(X_2)/S\) via \(i \circ \phi\) where \(i\) is also the inclusion map. For brevity let \(\mathfrak{A}_0 = (\mathfrak{A}(X_1)/R)(X_1 \cap X_2)\), \(\mathfrak{A}_1 = \mathfrak{A}(X_1)/R\) and \(\mathfrak{A}_2 = \mathfrak{A}(X_2)/S\) and \(j = i \circ \phi\). Then \(\mathfrak{A}_0\) embeds in \(\mathfrak{A}_1\) and \(\mathfrak{A}_2\) via \(i\) and \(j\) respectively. We now use (**)) to show that there exists \(\mathfrak{B} \in \text{RQEA}_\alpha\) and monomorphisms \(f\) and \(g\) from \(\mathfrak{A}_1\) and \(\mathfrak{A}_2\) respectively to \(\mathfrak{B}\) such that \(f \circ i = g \circ j\). By the Neat embedding Theorem, there exist \(\mathfrak{A}_0^+, \mathfrak{A}_1^+, \mathfrak{A}_2^+ \in \text{CA}_{\alpha+\omega}\), \(e_1 : \mathfrak{A}_1 \to \mathfrak{N}_\alpha \mathfrak{A}_1^+\), \(e_2 : \mathfrak{A}_2 \to \mathfrak{N}_\alpha \mathfrak{A}_2^+\) and \(e_0 : \mathfrak{A}_0 \to \mathfrak{N}_\alpha \mathfrak{A}_0^+\). We can assume that \(\mathfrak{G}g^{\mathfrak{A}_1^+}e_1(A_1) = \mathfrak{A}_1^+\) and similarly for \(\mathfrak{A}_2^+\) and \(\mathfrak{A}_0^+\). Let \(i(A_0)^+ = \mathfrak{G}g^{\mathfrak{A}_1^+}e_1(i(A_0))\) and \(j(A_0)^+ = \mathfrak{G}g^{\mathfrak{A}_2^+}e_2(j(A_0))\), then by (**)) there exist \(\overline{i} : \mathfrak{A}_0^+ \to i(A_0)^+\) and \(\overline{j} : \mathfrak{A}_0^+ \to j(A_0)^+\) such that \((e_1 \mid i(A_0)) \circ \overline{i} = i \circ e_0\) and \((e_2 \mid j(A_0)) \circ \overline{j} = j \circ e_0\). Now \(K\) as in theorem 5 has SUPAP, hence there is a \(\mathfrak{D}^+\) in \(\mathfrak{K}\) and \(k : \mathfrak{A}^+ \to \mathfrak{D}^+\) and \(h : \mathfrak{B}^+ \to \mathfrak{D}^+\) such that \(k \circ \overline{i} = h \circ \overline{j}\). Let \(\mathfrak{D} = \mathfrak{N}_\alpha \mathfrak{D}^+\). Then \(f = k \circ e_1 : \mathfrak{A}_1 \to \mathfrak{N}_\alpha \mathfrak{D}\) and \(g = h \circ e_2 : \mathfrak{A}_2 \to \mathfrak{N}_\alpha \mathfrak{D}\) are one to one and \(k \circ e_1 \circ i = h \circ e_2 \circ j\). ■
Figure 1

\[ D^+ \xrightarrow{k \circ e_1} Nr_{\alpha}D^+ \]

\[ A_1^+ \xrightarrow{e_1} A_1 \quad A_1 \xrightarrow{i} A_0 \]

\[ Sg^{A_1^+}(e_1 i(A_0)) \xrightarrow{e_1 \upharpoonright i(A_0)} i(A_0) \]

\[ A^+_0 \xrightarrow{i} A_0 \]

\[ A_0 \xrightarrow{j} Sg^{A_2^+}(e_2 j(A_0)) \]

\[ A_2 \xrightarrow{e_2} A_2^+ \]

\[ h \circ e_2 \]

\[ Id \]

\[ h \]
Let \( \bar{f} : \mathfrak{A}(X_1) \rightarrow \mathfrak{B} \)
be defined by
\[
a \mapsto f(a/R)
\]
and
\[
\bar{g} : \mathfrak{A}(X_2) \rightarrow \mathfrak{B}
\]
be defined by
\[
a \mapsto g(a/R).
\]
Let \( \mathfrak{B}' \) be the algebra generated by \( \text{Im} f \cup \text{Img} \).
Then \( \bar{f} \cup \bar{g} |_{X_1 \cup X_2} \) is a function since \( \bar{f} \) and \( \bar{g} \) coincide on \( X_1 \cap X_2 \). By freeness of \( \mathfrak{A} \), there exists \( h : \mathfrak{A} \rightarrow \mathfrak{B}' \) such that \( h |_{X_1 \cup X_2} = \bar{f} \cup \bar{g} \). Let \( T = \ker h \). Then it is not hard to check that
\[
T \cap \mathfrak{A} := R \text{ and } T \cap \mathfrak{A} = S.
\]
Note that \( T \) is the congruence generated by \( R \cup S \).

The above claim is an algebraic version of Robinson’s joint consistency
property. The next one, is, on the other hand, an algebraic version of the
Craig interpolation property.

**Claim 2.** Let \( \mu \) be a cardinal \( > 0 \). Let \( \mathfrak{A} = \mathfrak{R}^\mu \mathfrak{Q} \mathfrak{E} \mathfrak{A}_\alpha \). Then for any \( X_1, X_2 \subseteq \mu \), if \( x \in \mathfrak{A}(X_1) \) and \( z \in \mathfrak{A}(X_2) \) and \( x \leq z \) then there is a \( y \in \mathfrak{A}(X_1 \cap X_2) \), a
finite \( \Gamma \subseteq \alpha \) such that
\[
x \leq y \leq c(\Gamma) z.
\]

**Proof.** We call \( y \) in the claim an interpolant of \( x \) and \( z \) and we say that \( x \leq z \)
can be interpolated inside \( \mathfrak{A}(X_1 \cap X_2) \). Now let \( x \in \mathfrak{A}(X_1) \), \( z \in \mathfrak{A}(X_2) \) and assume that \( x \leq z \). Then
\[
x \in (\mathfrak{I} g^{\mathfrak{A}}(\{ z \}) \cap \mathfrak{A}(X_1)).
\]
Let
\[
M = \mathfrak{I} g^{\mathfrak{A}}(X_1) \{ z \} \text{ and } N = \mathfrak{I} g^{\mathfrak{A}}(X_2)(M \cap A(\{ X_1 \cap X_2 \})).
\]
Then
\[
M \cap A(\{ X_1 \cap X_2 \}) = N \cap A(\{ X_1 \cap X_2 \}).
\]
By identifying ideals with congruences, and using the congruence extension
property, there is a an ideal \( P \) of \( \mathfrak{A} \) such that
\[
P \cap A(\{ X_1 \}) = N \text{ and } P \cap A(\{ X_2 \}) = M.
\]
It follows that
\[
\mathfrak{I} g^{\mathfrak{A}}(N \cup M) \subseteq P \cap A(\{ X_1 \}) = N.
\]
Hence
\[(\mathcal{I}g^{(A)}\{z\}) \cap A^{(X_1)} \subseteq N.\]
and we have
\[x \in \mathcal{I}g^{(A_1)}[\mathcal{I}g^{(A_2)}\{z\} \cap A^{(X_1 \cap X_2)}].\]
This implies that there is an element \(y\) such that
\[x \leq y \in A^{(X_1 \cap X_2)}\]
and \(y \in \mathcal{I}g^{(X_2)}\{z\}\). But \(\mathcal{I}g^{(X_2)}\{z\} = \{x \in A : x \leq c(\Gamma)z : \text{for some finite } \Gamma \subseteq \alpha\}\). Indeed, let \(H\) denote the set of elements on the right hand side. It is easy to check \(H \subseteq \mathcal{I}g^{(X_2)}\{z\}\). Conversely, assume that \(y \in H, \Gamma \subseteq \omega \alpha\). It is clear that \(c(\Gamma)y \in H\). Now let \(x, y \in H\). Assume that \(x \leq c(\Gamma)z\) and \(y \leq c(\Delta)z\), then
\[x + y \leq c(\Gamma \cup \Delta)z.\]
Therefore there exists \(\Gamma \subseteq \omega \alpha\) such that
\[x \leq y \leq c(\Gamma)z.\]

\[\text{Claim 3}.
\]
(1) Let \(\mathfrak{A} = \mathfrak{H}_4\text{RCA}_\alpha\). Let \(r, s\) and \(t\) be defined as follows:

\[r = c_0(x \cdot c_1y) \cdot c_0(x \cdot -c_1y),\]
\[s = c_0c_1(c_1z \cdot s^0_1c_1z \cdot -d_01) + c_0(x \cdot -c_1z),\]
\[t = c_0c_1(c_1w \cdot s^0_1c_1w \cdot -d_01) + c_0(x \cdot -c_1w),\]
where \(x, y, z,\) and \(w\) are the first four free generators of \(\mathfrak{A}\). Then \(r \leq s \cdot t\)

(2) Let \(\mathfrak{A} = \mathfrak{H}_5\text{RQA}_\alpha\) on 5 generators. Let \(r, s\) and \(t\) be defined as follows:

\[r = c_0(x \cdot c_1y) \cdot c_0(x \cdot -c_1y),\]
\[s = c_0c_1(c_1z \cdot s^0_1c_1z \cdot -m) + c_0(x \cdot -c_1z),\]
\[t = c_0c_1(c_1w \cdot s^0_1c_1w \cdot -m) + c_0(x \cdot -c_1w),\]
where \(x, y, z, w\) and \(m\) are the five generators of \(\mathfrak{A}\). Then \(r \leq s \cdot t\).
Proof. Put

\[ a = x \cdot c_1 y - c_0 (x \cdot -c_1 z), \]
\[ b = x \cdot -c_1 y - c_0 (x \cdot -c_1 z). \]

Then we have

\[ c_1 a \cdot c_1 b \leq c_1 (x \cdot c_1 y) \cdot c_1 (x \cdot -c_1 y) \] by [24] 1.2.7
\[ = c_1 x \cdot c_1 y \cdot c_1 x \cdot -c_1 y \] by [24] 1.2.11

and so

\[ c_1 a \cdot c_1 b = 0. \] (2)

From the inclusion \( x \cdot -c_1 z \leq c_0 (x \cdot -c_1 z) \) we get

\[ x \cdot -c_0 (x \cdot -c_1 z) \leq c_1 z. \]

Thus \( a, b \leq c_1 z \) and hence, by [24] 1.2.9,

\[ c_1 a, c_1 b \leq c_1 z. \] (3)

We now compute:

\[ c_0 a \cdot c_0 b \leq c_0 c_1 a \cdot c_0 c_1 b \] by [24] 1.2.7
\[ = c_0 c_1 a \cdot c_1 s_0^1 c_1 b \] by [24] 1.5.8 (i), [24] 1.5.9 (i)
\[ = c_1 (c_0 c_1 a \cdot s_0^1 c_1 b) \]
\[ = c_0 c_1 (c_1 a \cdot s_0^1 c_1 b) \]
\[ = c_0 c_1 [c_1 a \cdot s_0^1 c_1 b \cdot (-d_01 + d_01)] \]
\[ = c_0 c_1 [(c_1 a \cdot s_0^1 c_1 b \cdot -d_01) + (c_1 a \cdot c_1 b \cdot d_01)] \]
\[ = c_0 c_1 [(c_1 a \cdot s_0^1 c_1 b \cdot -d_01) \cdot (c_1 a \cdot c_1 b \cdot d_01)] \] by [24] 1.5.5
\[ = c_0 c_1 (c_1 a \cdot s_0^1 c_1 b \cdot -d_01) \] by (2)
\[ \leq c_0 c_1 (c_1 z \cdot s_0^1 c_1 z \cdot -d_01) \] by (3), [24] 1.2.7

We have proved that

\[ c_0 [x \cdot c_1 y \cdot -c_0 (x \cdot -c_1 z)] \cdot c_0 [x \cdot -c_1 y \cdot -c_0 (x \cdot -c_1 z)] \leq c_0 c_1 (c_1 z \cdot s_0^1 c_1 z \cdot -d_01). \]

In view of [24] 1.2.11 this gives

\[ c_0 (x \cdot c_1 y) \cdot c_0 (x \cdot -c_1 y) \cdot -c_0 (x \cdot -c_1 z) \leq c_0 c_1 (c_1 z \cdot s_0^1 c_1 z \cdot -d_01). \]

The conclusion of the claim now follows.
By proving the next claim, we reach a contradiction with Claim 2, and thus our theorem will be proved.

**Claim 4.** Let everything be as in the previous claim. Then the inequality \( r \leq s \cdot t \) cannot be interpolated by an element of \( A(\{x\}) \).

**Proof.** This part is taken from [42]. We include it for the sake of completeness. Let

\[
\mathcal{B} = (\wp(\alpha), \cup, \cap, \sim, \emptyset, \alpha, C_{\kappa}, D_{\kappa\lambda}, \mathcal{P}_{ij})_{\kappa, \lambda < \alpha},
\]

that is \( \mathcal{B} \) is the full set algebra in the space \( \alpha \alpha \). Let \( E \) be the set of all equivalence relations on \( \alpha \), and for each \( R \in E \) set

\[
X_R = \{ \varphi : \varphi \in \alpha \alpha \text{ and, for all } \xi, \eta < \alpha, \varphi_\xi = \varphi_\eta \text{ iff } \xi R \eta \}.
\]

Let

\[
C = \{ \bigcup_{R \in L} X_R : L \subseteq E \}.
\]

\( C \) is clearly closed under the formation of arbitrary unions, and since

\[
\sim \bigcup_{R \in L} X_L = \bigcup_{R \in E \sim L} X_R
\]

for every \( L \subseteq E \), we see that \( C \) is closed under the formation of complements with respect to \( \alpha \alpha \). Thus \( C \) is a Boolean subuniverse (indeed, a complete Boolean subuniverse) of \( \mathcal{B} \); moreover, it is obvious that

\[
X_R \text{ is an atom of } (C, \cup, \cap, \sim, \emptyset, \alpha) \text{ for each } R \in E.
\]

For all \( \kappa, \lambda < \alpha \) we have \( D_{\kappa\lambda} = \bigcup\{ X_R : (\kappa, \lambda) \in R \in E \} \) and hence \( D_{\kappa\lambda} \in B \). Also,

\[
C_\kappa X_R = \bigcup\{ X_S : S \in E, 2(\alpha \sim \{\kappa\}) \cap S = 2(\alpha \sim \{\kappa\}) \cap R \}
\]

for any \( \kappa < \alpha \) and \( R \in E \). Thus, because \( C_\kappa \) is completely additive (cf.[24] 1.2.6(i)) and the remark preceding it), we see that \( C \) is closed under the operation \( C_\kappa \) for every \( \kappa < \alpha \). Also it is straightforward to see that \( C \) is closed under substitutions. For any \( \tau = [i, j] \in \alpha \alpha \),

\[
S_\tau X_R = \bigcup\{ X_S : S \in E, \forall i, j < \omega (i R j \leftrightarrow \tau(i) S \tau(j)) \}.
\]

Therefore, we have shown that

\[
C \text{ is a subuniverse of } \mathcal{B}.
\]

48
To prove that \( r \leq s \cdot t \) can’t be interpolated by an element of \( A(\{x\}) \), it suffices to show that there is a subset \( Y \) of \( \alpha \) such that
\[
X_{Id} \cap f(r) \neq 0 \quad \text{for every} \quad f \in Hom(A, B)
\]
such that \( f(x) = X_{Id} \) and \( f(y) = Y \).

and also that for every finite \( \Gamma \subseteq \alpha \), there are subsets \( Z, W \) of \( \alpha \) such that
\[
X_{Id} \sim C(\Gamma)g(s \cdot t) \neq 0 \quad \text{for every} \quad g \in Hom(A, B)
\]
such that \( g(x) = X_{Id}, g(z) = Z \) and \( g(w) = W \).

For suppose, on the contrary, that these conditions are not sufficient. Then there exists a finite \( \Gamma \subseteq \alpha \), and an interpolant \( u \in A(\{x\}) \) and there also exist \( Y, Z, W \subseteq \alpha \) such that (6) and (7) hold. Take any \( h \in Hom(A, B) \) such that \( h(x) = X_{Id}, h(y) = Y, h(z) = Z \), and \( h(w) = W \). This is possible by the freeness of \( A \). Then using the fact that \( X_{Id} \cap h(r) \) is non-empty by (6) we get
\[
X_{Id} \cap h(u) = h(x \cdot u) \supseteq h(x \cdot r) \neq 0.
\]
And using the fact that \( X_{Id} \sim C(\Gamma)h(s \cdot t) \) is non-empty by (7) we get
\[
X_{Id} \sim h(u) = h(x \cdot -u) \supseteq h(x \cdot -c(\Gamma)(s \cdot t)) \neq 0.
\]
However, in view of (8), it is impossible for \( X_{Id} \) to intersect both \( h(u) \) and its complement since \( h(u) \in C \) and \( X_{Id} \) is an atom; to see that \( h(u) \) is indeed contained in \( C \) recall that \( u \in A(\{x\}) \), and then observe that because of (5) and the fact that \( X_{Id} \in C \) we must have
\[
h[A(\{x\})] \subseteq C \tag{8}
\]
Therefore, (6) and (7) are sufficient conditions for \( r \leq s \cdot t \) not to be interpolated by an element of \( A(\{x\}) \). The next part of the proof is taken verbatim from [28] p. 340-341. Let \( \sigma \in \alpha \) be such that \( \sigma_0 = 0 \), and \( \sigma_\kappa = \kappa + 1 \) for every non-zero \( \kappa < \alpha \) and otherwise \( \sigma(k) = k \). Let \( \tau = \sigma \upharpoonright (\alpha \sim \{0\}) \cup \{(0, 1)\} \). Then \( \sigma, \tau \in X_{Id} \). Take
\[
Y = \{\sigma\}.
\]
Then
\[
\sigma \in X_{Id} \cap C_1 Y \quad \text{and} \quad \tau \in X_{Id} \sim C_1 Y
\]
and hence
\[
\sigma \in C_0(X_{Id} \cap C_1 Y) \cap C_0(X_{Id} \sim C_1 Y). \tag{9}
\]
Therefore, we have \( \sigma \in f(r) \) for every \( f \in Hom(A, B) \) such that \( f(x) = X_{Id} \) and \( f(y) = Y \), and that (6) holds. We now want to show that for any given
finite $\Gamma \subseteq \alpha$, there exist sets $Z, W \subseteq ^{\alpha} \alpha$ such that (7) holds; it is clear that no generality is lost if we assume that $0, 1 \in \Gamma$, so we make this assumption. Take

$$Z = \{ \varphi : \varphi \in X_{Id}, \varphi_0 < \varphi_1 \} \cap C(\Gamma)\{Id\}$$

and

$$W = \{ \varphi : \varphi \in X_{Id}, \varphi_0 > \varphi_1 \} \cap C(\Gamma)\{Id\}.$$ 

We show that

$$Id \in X_{Id} \sim C(\Gamma)g(s \cdot t)$$

(10)

for any $g \in Hom(\mathfrak{A}, \mathfrak{B})$ such that $g(x) = X_{Id}, g(z) = Z$, and $g(w) = W$; to do this we simply compute the value of $C(\Gamma)g(s \cdot t)$. For the purpose of this computation we make use of the following property of ordinals: if $\Delta$ is any non-empty set of ordinals, then $\bigcap \Delta$ is the smallest ordinal in $\Delta$, and if, in addition, $\Delta$ is finite, then $\bigcup \Delta$ is the largest element ordinal in $\Delta$. Also, in this computation we shall assume that $\varphi$ always represents an arbitrary sequence in $^{\alpha}\alpha$. Then, setting

$$\Delta \varphi = \Gamma \sim \varphi[\Gamma \sim \{0, 1\}]$$

for every $\varphi$, we successively compute:

$$C_1Z = \{ \varphi : |\Delta \varphi| = 2, \varphi_0 = \bigcap \Delta \varphi \} \cap C(\Gamma)\{Id\},$$

$$(X_{Id} \sim C_1Z) \cap C(\Gamma)\{Id\} =$$

$$\{ \varphi : |\Delta \varphi| = 2, \varphi_0 = \bigcup \Delta \varphi, \varphi_1 = \bigcap \Delta \varphi \} \cap C(\Gamma)\{Id\},$$

and, finally,

$$C_0(X_{Id} \sim C_1Z) \cap C(\Gamma)\{Id\} =$$

$$\{ \varphi : |\Delta \varphi| = 2, \varphi_1 = \bigcap \Delta \varphi \} \cap C(\Gamma)\{Id\}. \quad (11)$$

Similarly, we obtain

$$C_0(X_{Id} \sim C_1W) \cap C(\Gamma)\{Id\} =$$

$$\{ \varphi : |\Delta \varphi| = 2, \varphi_1 = \bigcup \Delta \varphi \} \cap C(\Gamma)\{Id\}.$$ 

The last two formulas together give

$$C_0(X_{Id} \sim C_1Z) \cap C_0(X_{Id} \sim C_1W) \cap C(\Gamma)\{Id\} = 0. \quad (12)$$

Continuing the computation we successively obtain:

$$C_1Z \cap D_{01} = \{ \varphi : |\Delta \varphi| = 2, \varphi_0 = \varphi_1 = \bigcap \Delta \varphi \} \cap C(\Gamma)\{Id\},$$

50
\[ S_1^0C_1 Z = \{ \varphi : |\Delta \varphi| = 2, \varphi_1 = \bigcap \Delta \varphi \} \cap C_{(\Gamma)} \{ Id \}, \]

\[ C_1 Z \cap S_1^0C_1 Z = \{ \varphi : |\Delta \varphi| = 2, \varphi_0 = \varphi_1 = \bigcap \Delta \varphi \} \cap C_{(\Gamma)} \{ Id \}, \]

hence we finally get

\[ C_0C_1(C_1 Z \cap S_1^0C_1 Z \cap \sim D_{01}) = C_0C_10 = 0, \quad (13) \]

and similarly we get

\[ C_0C_1(C_1 W \cap S_1^0C_1 W \cap \sim D_{01}) = 0 \quad (14) \]

Now take \( g \) to be any homomorphism from \( \mathfrak{A} \) into \( \mathfrak{B} \) such that \( g(x) = X_{Id} \), \( g(z) = Z \) and \( g(w) = W \). Let \( a = g(s \cdot t) \). Then \( a = C_0(X_{Id} \sim C_1 Z) \cap C_0(X_{Id} \sim C_1 W) \). Then from \((12)\), we have

\[ a \cap C_{(\Gamma)} \{ Id \} = \emptyset. \]

Then applying \( C_{(\Gamma)} \) to both sides of this equation we get

\[ C_{(\Gamma)}a \cap C_{(\Gamma)} \{ Id \} = \emptyset. \]

Thus \((10)\) holds and we are done. \( \blacksquare \)

Using the techniques above, one can prove the following new theorem

**Theorem 34.** Let \( \alpha \) be an infinite ordinal. Then for any \( n \in \omega \) the variety \( V = S\mathfrak{fr}_n \text{QEA}_{\alpha+n} \) does not have AP.

**Proof.** In what follows by \( \mathfrak{A}^{(x)} \) we denote the subalgebra of \( \mathfrak{A} \) generated by \( X \), and we write \( \mathfrak{A}^{(x)} \) for \( \mathfrak{A}^{(\{x\})} \). Seeking a contradiction, assume that \( V \) has AP with respect Let \( \mathfrak{A} = \mathfrak{fr}_4V \), the free \( V \) algebra on 4 generators. Let \( r, s \) and \( t \) be defined as follows:

\[ r = c_0(x \cdot c_1y) \cdot c_0(x \cdot -c_1y), \]

\[ s = c_0c_1(c_1z \cdot s_1^0c_1z \cdot -d_{01}m) + c_0(x \cdot -c_1z), \]

\[ t = c_0c_1(c_1w \cdot s_1^0c_1w \cdot -d_{01}) + c_0(x \cdot -c_1w), \]

where \( x, y, z, w \) and \( m \) are the five generators of \( \mathfrak{A} \). Then \( r \leq s \cdot t \). Let \( X_1 = \{ x, y \} \) and \( X_2 = \{ x, z, w \} \). Then

\[ \mathfrak{A}^{(X_1 \cap X_2)} = \mathfrak{G}g^{\mathfrak{A}} \{ x \}. \quad (15) \]

We have

\[ r \in A^{(X_1)} \text{ and } s, t \in A^{(X_2)}. \quad (16) \]

51
Let \( \{x', y', z', w'\} \) be the first four generators of \( \mathcal{D} = \mathfrak{R}_4 \mathbf{RQEA}_\alpha \). Let \( h \) be the homomorphism from \( \mathfrak{A} \) to \( \mathcal{D} \) such that \( h(i) = i' \) for \( i \in \{x, y, w, z\} \). Let \( J \) be the kernel of \( h \). Then

\[
\mathfrak{A}/J \cong \mathcal{D} \tag{17}
\]

We work inside the algebra \( \mathfrak{A} \). Since \( r \leq s \cdot t \) we have

\[
r \in \mathfrak{g}^{\mathfrak{A}} \{s \cdot t\} \cap A^{(x_1)}. \tag{18}
\]

Let

\[
M = \mathfrak{g}^{\mathfrak{A}}\{s \cdot t\} \cup (J \cap A^{(x_2)}); \tag{19}
\]

\[
N = \mathfrak{g}^{\mathfrak{A}}\{s \cdot t\} \cup (J \cap A^{(x_1)}). \tag{20}
\]

Then we have

\[
J \cap A^{(x_2)} \subseteq M \text{ and } J \cap A^{(x_1)} \subseteq N. \tag{21}
\]

From the first of these inclusions we get

\[
M \cap A^{(x_1 \cap x_2)} \supseteq (J \cap A^{(x_2)}) \cap A^{(x_1 \cap x_2)} = (J \cap A^{(x_1)}) \cap A^{(x_1 \cap x_2)}.
\]

Then

\[
N \cap A^{(x_1 \cap x_2)} = M \cap A^{(x_1 \cap x_2)}. \tag{22}
\]

For \( R \) an ideal of \( \mathfrak{A} \) and \( X \subseteq A \), by \( (\mathfrak{A}/R)^{\langle X \rangle} \) we understand the subalgebra of \( \mathfrak{A}/R \) generated by \( \{x/R : x \in X\} \). Replacing \( S \) by \( N \) and \( R \) by \( M \) in the first part of the proof and using that \( V \) has \( AP \), let \( P \) be the ideal corresponding to \( kerh \) as defined above. Then, as before:

\[
P \cap A^{(x_1)} = N, \tag{22}
\]

and

\[
P \cap A^{(x_2)} = M. \tag{23}
\]

Now \( s \cdot t \in P \) and so \( r \in P \). Consequently we get \( r \in N \), and so there exist elements

\[
u \in M \cap A^{(x_1 \cap x_2)} \tag{24}
\]

and \( b \in J \) such that

\[
r \leq u + b. \tag{25}
\]
Since $u \in M$ by (7) there is a $\Gamma \subseteq \omega \alpha$ and $c \in J$ such that

\[ u \leq c_{(\Gamma)}(s \cdot t) + c. \]

Recall that $h$ is the homomorphism from $\mathcal{A}$ to $\mathcal{D}$ such that $h(i) = i'$ for $i \in \{x, y, w, z\}$, and that $\ker h = J$. Then $h(b) = h(c) = 0$. It follows that

\[ h(r) \leq h(u) \leq c_{(\Gamma)}(h(s) \cdot h(t)). \]

And this is impossible. \qed

5 Adjointness of the neat reduct functor

Now it is high time to find the spirit!

For this purpose we put category theory in use. The main advantage of category theory is that it allows one not to miss the forest for the trees.

In our investigations, we have a certain dichotomy; we have two trees, the polyadic one and the cylindric one; or perhaps even two forests (indeed each paradigm is huge enough). One way of formulating our investigations in this paper, in a nut shell, is where is, or rather what is the forest, or, perhaps, the forest encompassing the two forests?

Between the lines, one can see that we are basically proving that the superamalgamation property for a subclass of representable algebras is equivalent to invertibility of the dilation functor, while only the existence of a right adjoint is equivalent to amalgamation. This is the general picture. But the details are intricate.

Such results will be also proved in a much more general setting in the final section, when we apply category theory to one way (most probably not the only one) of locating the universal forest (the common spirit of the two spirits), a generalized systems of varieties covering also the polyadic paradigm.

In category theory what really counts are the formulation of the definitions. The proofs come later in priority of importance. The most important versatile concept in category theory is that of adjoint situations which abound in all branches of pure mathematics.

A special case of adjoint situations is equivalence of two categories; this is most interesting and intriguing when this equivalence can be implemented by the contravariant Hom functor using a co-separator in the target category. Examples include Boolean algebras and Stone spaces, cylindric algebras and Sheaves, locally compact abelian groups and abelian groups and C Star algebras and compact Hausdorff space.

It is definitely most inspiring and exciting to discover that two seemingly unrelated areas are nothing more than two sides of the same coin. Here, our adjoint situation proved in the polyadic case, shows that the category of
algebras in $\omega$ dimensions is actually equivalent to that in $\omega + \omega$ dimensions. This also holds for the countable reducts studied by Sain as a solution to the so-called finitizability problem.

The aim of this problem that casts its shadow over the entire field, is to capture infinitely many dimensions in a finitary way, which seems paradoxical at first sight. But this can be done, with some ingenuity in usual set theory for infinite dimensions, and by changing the ontology to non-well-founded set theories for finite dimensions. (Here the extra dimensions are generated 'downwards').

Categorically this is expressed by the fact that the hitherto established equivalence says that the infinite gap can be finitized, but alas, even more, it actually says that it does not exist at all. The apparent gap happens to be there, because it is either a historical accident or an unintended repercussion of the original formulation of such algebras.

In our categorial notation we follow [27]

Definition 35. Let $L$ and $K$ be two categories. Let $G : K \to L$ be a functor and let $\mathfrak{B} \in Ob(L)$. A pair $(u_B, \mathfrak{A}_B)$ with $\mathfrak{A}_B \in Ob(K)$ and $u_B : \mathfrak{B} \to G(\mathfrak{A}_B)$ is called a universal map with respect to $G$ (or a $G$ universal map) provided that for each $\mathfrak{A}' \in Ob(K)$ and each $f : \mathfrak{B} \to G(\mathfrak{A}')$ there exists a unique $K$ morphism $\bar{f} : \mathfrak{A}_B \to \mathfrak{A}'$ such that

\[ G(\bar{f}) \circ u_B = f. \]

\[ \begin{array}{ccc}
\mathfrak{B} & \xrightarrow{u_B} & G(\mathfrak{A}_B) \\
\downarrow f & & \downarrow G(f) \\
G(\mathfrak{A}') & \xrightarrow{f} & \mathfrak{A}'
\end{array} \]

The above definition is strongly related to the existence of adjoints of functors. For undefined notions in the coming definition, the reader is referred to [27] Theorem 27.3 p. 196.

Theorem 36. Let $G : K \to L$.

1. If each $\mathfrak{B} \in Ob(K)$ has a $G$ universal map $(\mu_B, \mathfrak{A}_B)$, then there exists a unique adjoint situation $(\mu, \epsilon) : F \to G$ such that $\mu = (\mu_B)$ and for each $\mathfrak{B} \in Ob(L)$, $F(\mathfrak{B}) = \mathfrak{A}_B$.

2. Conversely, if we have an adjoint situation $(\mu, \epsilon) : F \to G$ then for each $\mathfrak{B} \in Ob(K)$ $(\mu_B, F(\mathfrak{B}))$ have a $G$ universal map.

Now we apply this definition to the 'neat reduct functor' from a certain subcategory of $\text{CA}_{\alpha+\omega}$ to $\text{RCA}_\alpha$. More precisely, let

\[ L = \{ \mathfrak{A} \in \text{CA}_{\alpha+\omega} : \mathfrak{A} = Sg^\alpha \mathfrak{A} \}. \]
Note that $L \subseteq RCA_{\alpha+\omega}$. The reason is that any $\mathfrak{A} \in L$ is generated by $\alpha$-dimensional elements, so is dimension complemented (that is $\Delta x \neq \alpha$ for all $x$), and such algebras are representable. Consider $\mathfrak{R}_\alpha$ as a functor from $L$ to $CA_\alpha$, but we restrict morphisms to one to one homomorphisms; that is we take only embeddings. By the neat embedding theorem $\mathfrak{R}_\alpha$ is a functor from $L$ to $RCA_\alpha$. (For when $\mathfrak{A} \in CA_{\alpha+\omega}$, then $\mathfrak{R}_\alpha \mathfrak{A} \in RCA_\alpha$). The question we address is: Can this functor be “inverted”. This functor is not dense since there are representable algebras not in $\mathfrak{R}_\alpha CA_{\alpha+\omega}$, as the following example, which is a straightforward adaptation of a result in [48] shows:

**Example 37.** (1) Let $\mathfrak{F}$ be a field of characteristic 0. Let

$$V = \{ s \in \mathfrak{F}^\alpha : |\{ i \in \alpha : s_i \neq 0 \}| < \omega \},$$

Let

$$\mathcal{C} = (\cup(V), \cup, \cap, \exists, \emptyset, V, c_i, d_{ij})_{i,j \in \alpha},$$

with cylindrifiers and diagonal elements restricted to $V$. Let $y$ denote the following $\alpha$-ary relation:

$$y = \{ s \in V : s_0 + 1 = \sum_{i>0} s_i \}.$$

Note that the sum on the right hand side is a finite one, since only finitely many of the $s_i$'s involved are non-zero. For each $s \in y$, we let $y_s$ be the singleton containing $s$, i.e. $y_s = \{ s \}$. Define $\mathfrak{A} \in CA_\alpha$ as follows:

$$\mathfrak{A} = \mathfrak{S} g^\mathcal{C} \{ y, y_s : s \in y \}.$$

Then it is proved in [48] that

$$\mathfrak{A} \notin \mathfrak{R}_\alpha CA_{\alpha+1}.$$

That is for no $\mathfrak{B} \in CA_{\alpha+1}$, it is the case that $\mathfrak{S} g^\mathcal{C} \{ y, y_s : s \in y \}$ exhausts the set of all $\alpha$ dimensional elements of $\mathfrak{B}$.

(2) Let $\mathfrak{A}$ be as in above. Then since $\mathfrak{A}$ is a weak set algebra, it is representable. Hence $\mathfrak{A} \in S\mathfrak{R}_\alpha CA_{\alpha+\omega}$. Let $\mathfrak{B} \in CA_{\alpha+\omega}$ be an algebra such that $\mathfrak{A} \subseteq \mathfrak{R}_\alpha \mathfrak{B}$. Let $\mathfrak{B}'$ be the subalgebra of $\mathfrak{B}$ generated by $\mathfrak{A}$. Then $\mathfrak{A}$ generates $\mathfrak{B}$ but $\mathfrak{A}$ is not isomorphic to $\mathfrak{R}_\alpha \mathfrak{B}$.

Item (2) in the above example says that there are two non isomorphic algebras, namely $\mathfrak{A}$ and $\mathfrak{R}_\alpha \mathfrak{B}'$ that generate the same algebra $\mathfrak{B}'$ using extra dimensions [39]. If $\mathfrak{A} \subseteq \mathfrak{R}_\alpha \mathfrak{B}$ then $\mathfrak{B}$ is called a dilation of $\mathfrak{A}$. $\mathfrak{B}$ is a minimal dilation if $\mathfrak{A}$ generates $\mathfrak{B}$, in which case $\mathfrak{A}$ is called a generating subreduct of $\mathfrak{B}$. In the previous example $\mathfrak{A}$ is a generating subreduct of $\mathfrak{B}$. One would expect that the “inverse” of the Functor $\mathfrak{R}$ would be the functor that takes $\mathfrak{A}$ to a minimal dilation, and lifting morphisms. But this functor is not even a right adjoint.
Corollary 38. Let \( L = \{ A \in RCA_{\alpha+\omega} : A = \mathcal{S}g^3\text{Nr}_\alpha A \} \). Then the neat reduct functor \( \text{Nr}_\alpha \) from \( L \) to \( RCA_\alpha \) with morphisms restricted to injective homomorphisms does not have a right adjoint.

**Proof.** UNEP is equivalent to existence of universal maps; the former does not hold for the representable algebras. \( \square \)

Corollary 39. If \( A \) has a universal map with respect to the above functor, then \( A \) belongs to the amalgamation base of \( \text{RK}_\alpha \).

For \( A \in \text{PA}_\alpha \), a polyadic algebra and \( \beta > \alpha \), a \( \beta \) dilation of \( A \) is an algebra \( B \in \text{PA}_\beta \) such that \( A \subseteq \text{Nr}_\alpha B \). \( B \) is a minimal dilation of \( A \) if \( A \) generates \( B \). Let \( L = \{ A \in \text{PA}_\beta : \mathcal{S}g\text{Nr}_\alpha A = A \} \). Then \( \text{Nr}_\alpha : L \to \text{PA}_\alpha \) is an equivalence. To prove this we first note that polyadic algebras do not satisfy (i) of ???. But before that we need a lemma. For \( X \subseteq A \), \( \mathcal{J}g^3X \) denotes the ideal generated by \( A \):

**Lemma 40.** Let \( \alpha < \beta \) be infinite ordinals. Let \( B \in \text{PA}_\beta \) and \( A \subseteq \text{Nr}_\alpha B \).

1. if \( A \) generates \( B \) then \( A = \text{Nr}_\alpha B \)
2. If \( A \) generates \( B \), and \( I \) is an ideal of \( B \), then \( \mathcal{J}g^3(I \cap A) = I \)

**Proof.**

1. Let \( A \subseteq \text{Nr}_\alpha B \) and \( A \) generates \( B \) then \( B \) consists of all elements \( s^B_\sigma x \) such that \( x \in A \) and \( \sigma \) is a transformation on \( \beta \) such that \( \sigma \upharpoonright \alpha \) is one to one. Now suppose \( x \in \text{Nr}_\alpha \mathcal{S}g^3X \) and \( \Delta x \subseteq \alpha \). There exists \( y \in \mathcal{S}g^3X \) and a transformation \( \sigma \) of \( \beta \) such that \( \sigma \upharpoonright \alpha \) is one to one and \( x = s^B_\sigma y \). Let \( \tau \) be a transformation of \( \beta \) such that \( \tau \upharpoonright \alpha = Id \) and \( (\tau \circ \sigma) \alpha \subseteq \alpha \). Then \( x = s^B_\tau s^B_\sigma y = s^B_{\tau \circ \sigma} y = s^3_{\tau \circ \sigma \upharpoonright \alpha} \).

Abusing notation we write \( A \) for \( \mathcal{S}g^3X \) and \( B \) for \( \mathcal{S}g^3X \). Then \( B \) is a minimal dilation of \( A \). Each element of \( B \) has the form \( s^B_\sigma a \) for some \( a \in A \), and \( \sigma \) a transformation on \( \beta \) such that \( \sigma \upharpoonright \alpha \) is one to one. We claim that \( \text{Nr}_\alpha B \subseteq A \). Indeed let \( x \in \text{Nr}_\alpha B \). Then by the above we have \( x = s^B_\sigma y \), for some \( y \in A \) and \( \sigma \in \beta \). Let \( \tau \in \beta \beta \) such that

\[
\tau \upharpoonright \alpha \subseteq Id \text{ and } (\tau \circ \sigma) \alpha \subseteq \alpha.
\]

Such a \( \tau \) clearly exists. Since \( x \in \text{Nr}_\alpha B \), it follows by definition that \( c_{(\beta \sim \alpha)} x = x \). From

\[
\tau \upharpoonright \beta \sim (\beta \sim \alpha) = \tau \upharpoonright \alpha = Id \upharpoonright \alpha = Id \upharpoonright \beta \sim (\beta \sim \alpha),
\]

we get from the polyadic axioms that

\[
s^B_\tau x = s^B_\tau c_{(\beta \sim \alpha)} x = s^B_{Id} c_{(\beta \sim \alpha)} x = s^B_{Id} x = x.
\]
Therefore
\[ x = s_{\tau}^{\alpha} x = s_{\sigma}^{\alpha} s_{\tau}^{\alpha} x = s_{\rho \sigma}^{\alpha} x. \]  (27)

Let
\[ \mu = \tau \circ \sigma \upharpoonright \alpha \text{ and } \bar{\mu} = \mu \cup Id \upharpoonright (\beta \sim \alpha). \]

Since
\[ \bar{\mu} \upharpoonright \beta \sim (\beta \sim \alpha) = \bar{\mu} \upharpoonright \alpha = \mu = \tau \circ \sigma \upharpoonright \beta \sim (\beta \sim \alpha), \]
we have
\[ s_{\bar{\mu}}^{\alpha} c_{(\beta \sim \alpha)} y = s_{\rho \sigma}^{\alpha} c_{(\beta \sim \alpha)} y. \]

Since \( \mathfrak{A} \subseteq \mathfrak{M}_\alpha \mathfrak{B} \) and \( y \in A \), we have
\[ s_{\mu}^{\alpha} y = s_{\bar{\mu}}^{\alpha} y \text{ and } c_{(\beta \sim \alpha)} y = y. \]

Therefore
\[ s_{\mu}^{\alpha} y = s_{\mu}^{\alpha} y = s_{\bar{\mu}}^{\alpha} c_{(\beta \sim \alpha)} y = s_{\rho \sigma}^{\alpha} c_{(\beta \sim \alpha)} y = s_{\rho \sigma}^{\alpha} y. \]  (28)

From (27) and (28) we get \( x = s_{\mu}^{\alpha} y \in \mathfrak{A} \). By this the proof is complete since \( x \) was arbitrary.

(2) Let \( x \in \mathcal{Ig}^{\alpha}(I \cap A) \). Then \( c_{(\Delta x \sim \alpha)} x \in \mathfrak{M}_\alpha \mathfrak{B} = \mathfrak{A} \), hence in \( I \cap A \). But
\[ x \leq c_{(\Delta x \sim \alpha)} x, \]
and we are done.

The previous lemma fails for cylindric algebras in general \[39\], but it does hold for \( \mathbf{Dc}_a \)'s, see theorem 2.6.67, and 2.6.71 in \[24\].

**Theorem 41.** Let \( \alpha < \beta \) be infinite ordinals. Assume that \( \mathfrak{A}, \mathfrak{A}' \in \mathbf{PA}_\alpha \) and \( \mathfrak{B}, \mathfrak{B}' \in \mathbf{PA}_\beta \). If \( \mathfrak{A} \subseteq \mathfrak{M}_\alpha \mathfrak{B} \) and \( \mathfrak{A} \subseteq \mathfrak{M}_\alpha \mathfrak{B}' \) and \( A \) generates both then \( \mathfrak{B} \) and \( \mathfrak{B}' \) are isomorphic, then \( \mathfrak{B} \) and \( \mathfrak{B}' \) are isomorphic with an isomorphism that fixes \( \mathfrak{A} \) pointwise.

**Proof.** \[25\] theorem 2.6.72. We prove something stronger, we assume that \( \mathfrak{A} \) embeds into \( \mathfrak{M}_\alpha \mathfrak{B} \) and similarly for \( \mathfrak{A}' \). So let \( \mathfrak{A}, \mathfrak{A}' \in \mathbf{PA}_\alpha \) and \( \beta > \alpha \).

Let \( \mathfrak{B}, \mathfrak{B}' \in \mathbf{PA}_\beta \) and assume that \( e_A, e_{A'} \) are embeddings from \( \mathfrak{A}, \mathfrak{A}' \) into \( \mathfrak{M}_\alpha \mathfrak{B}, \mathfrak{M}_\alpha \mathfrak{B}' \) respectively, such that \( \mathcal{S}g^{\alpha}(e_A(A)) = \mathfrak{B} \) and \( \mathcal{S}g^{\alpha}(e_{A'}(A')) = \mathfrak{B}' \), and let \( i : \mathfrak{A} \rightarrow \mathfrak{A}' \) be an isomorphism. We need to “lift” \( i \) to \( \beta \) dimensions. Let \( \mu = |A| \). Let \( x \) be a bijection from \( \mu \) onto \( A \). Let \( y \) be a bijection from \( \mu \) onto \( A' \), such that \( i(x_j) = y_j \) for all \( j < \mu \). Let \( \mathcal{D} = \mathfrak{M}_\beta \mathbf{PA}_\beta \) with generators \( (\xi_i : i < \mu) \). Let \( C = \mathcal{S}g^{\beta \mathfrak{M}_\beta \mathcal{D}} \{ \xi_i : i < \mu \} \). Then \( C \subseteq \mathfrak{M}_\alpha \mathcal{D} \), \( C \) generates \( \mathcal{D} \) and so by the previous lemma \( \mathcal{C} = \mathfrak{M}_\alpha \mathcal{D} \). There exist \( f \in \text{Hom}(\mathcal{D}, \mathfrak{B}) \) and \( f' \in \text{Hom}(\mathcal{D}, \mathfrak{B}') \) such that \( f(g_\xi) = e_A(x_\xi) \) and \( f'(g_\xi) = e_{A'}(y_\xi) \) for all \( \xi < \mu \). Note that \( f \) and \( f' \) are both onto. We now have
\[ e_A \circ i^{-1} \circ e_{A'}^{-1} \circ (f' \upharpoonright C) = f \upharpoonright C. \]
Therefore \( Ker f' \cap C = Ker f \cap C \). Hence by \( \mathcal{Ig}(Ker f' \cap C) = \mathcal{Ig}(Ker f \cap C) \). So, again by the the previous lemma,
Let $y \in B$, then there exists $x \in D$ such that $y = f(x)$. Define $\hat{i}(y) = f'(x)$. The map is well defined and is as required.

\[ D \xrightarrow{f} B \xrightarrow{e_A} A \]

\[ f' \downarrow \hat{i} \downarrow \hat{i} \]

\[ B' \xleftarrow{e_{A'}} A' \]

**Corollary 42.** Let $\mathfrak{A}$, $\mathfrak{A}'$, $i, e_A, e_{A'}$, $\mathfrak{B}$ and $\mathfrak{B}'$ be as in the previous proof. Then if $i$ is a monomorphism form $\mathfrak{A}$ to $\mathfrak{A}'$, then it lifts to a monomorphism $\tilde{i}$ from $\mathfrak{B}$ to $\mathfrak{B}'$.

\[ B \xrightarrow{e_A} A \]

\[ \tilde{i} \downarrow \tilde{i} \]

\[ B' \xleftarrow{e_{A'}} A' \]

**Proof.** Consider $i : \mathfrak{A} \rightarrow i(\mathfrak{A})$. Take $C = \mathfrak{S}g^{\mathfrak{B}'}(e_A' \hat{i}(A))$. Then $\tilde{i}$ lifts to an isomorphism $\tilde{i} \rightarrow C \subseteq \mathfrak{B}$.

**Theorem 43.** Let $\beta > \alpha$. Let $L = \{ \mathfrak{A} \in \mathfrak{PA}_\beta : \mathfrak{A} = \mathfrak{S}g^{\mathfrak{A}} \mathfrak{Mr}_\alpha \mathfrak{A} \}$. Let $\mathfrak{Mr} : L \rightarrow \mathfrak{PA}_\alpha$ be the neat reduct functor. Then $\mathfrak{Mr}$ is invertible. That is, there is a functor $G : \mathfrak{PA}_\alpha \rightarrow L$ and natural isomorphisms $\mu : 1_L \rightarrow G \circ \mathfrak{Mr}$ and $\epsilon : \mathfrak{Mr} \circ G \rightarrow 1_{\mathfrak{PA}_\alpha}$.

**Proof.** The idea is that a full, faithful, dense functor is invertible, [27] theorem 1.4.11. Let $L$ be a system of representatives for isomorphism on $\mathfrak{Ob}(L)$. For each $\mathfrak{B} \in \mathfrak{Ob}(\mathfrak{PA}_\alpha)$ there is a unique $\mathfrak{S}(B)$ in $L$ such that $\mathfrak{Mr}(G(\mathfrak{B})) \cong \mathfrak{B}$. $G(\mathfrak{B})$ is a minimal dilation of $\mathfrak{B}$. Then $G : \mathfrak{Ob}(\mathfrak{PA}_\alpha) \rightarrow \mathfrak{Ob}(L)$ is well defined. Choose one isomorphism $\epsilon_B : \mathfrak{Mr}(G(B)) \rightarrow \mathfrak{B}$. If $g : \mathfrak{B} \rightarrow \mathfrak{B}'$ is a $\mathfrak{PA}_\alpha$ morphism, then the square

\[ \begin{array}{ccc}
\mathfrak{Mr}(G(B)) & \xrightarrow{\epsilon_B} & B \\
\epsilon_B \downarrow \mathfrak{Mr}(g) \circ \epsilon_B & & \downarrow g \\
\mathfrak{Mr}(G(B')) & \xrightarrow{\epsilon_B} & B'
\end{array} \]

commutes. By corollary 45 there is a unique morphism $f : G(\mathfrak{B}) \rightarrow G(\mathfrak{B}')$ such that $\mathfrak{Mr}(f) = e_{\mathfrak{B}}^{-1} \circ g \circ \epsilon$. We let $G(g) = f$. Then it is easy to see that $G$ defines a functor. Also, by definition $\epsilon = (\epsilon_{\mathfrak{B}})$ is a natural isomorphism from $\mathfrak{Mr} \circ G$ to $1_{\mathfrak{PA}_\alpha}$. To find a natural isomorphism from $1_L$ to $G \circ \mathfrak{Mr}$, observe
that \( e_{FA} : \mathcal{N} \circ G \circ \mathcal{N}(A) \to \mathcal{N}(A) \) is an isomorphism. Then there is a unique \( \mu_A : A \to G \circ \mathcal{N}(A) \) such that \( \mathcal{N}(\mu_A) = e_{FA}^{-1} \). Since \( e^{-1} \) is natural for any \( f : A \to A' \) the square

\[
\begin{array}{ccl}
\mathcal{N}(A) & \xrightarrow{\epsilon_{\mathcal{N}(A)}^{-1} \circ \mathcal{N}(\mu_A)} & \mathcal{N} \circ G \circ \mathcal{N}(A) \\
m(f) & \downarrow & \mathcal{N} \circ G \circ \mathcal{N}(f) \\
\mathcal{N}(A') & \xrightarrow{\epsilon_{FA}^{-1} = \mathcal{N}(\mu_{A'})} & \mathcal{N} \circ G \circ \mathcal{N}(A')
\end{array}
\]

commutes, hence the square

\[
\begin{array}{ccc}
A & \xrightarrow{\mu_A} & G \circ \mathcal{N}(A) \\
f & \downarrow & \mathcal{N} \circ G \circ \mathcal{N}(f) \\
A' & \xrightarrow{\mu_{A'}} & G \circ \mathcal{N}(A')
\end{array}
\]

commutes, too. Therefore \( \mu = (\mu_A) \) is as required. \( \blacksquare \)

To summarize we have theorems 2.6.67 (ii), 2.6.71-72 of [24] formulated for \( \mathcal{D}c \)'s do not hold for \( K \in \{ \text{SC}, \text{CA}, \text{QA}, \text{QEA} \} \); in fact they do not hold for \( \text{RK}_\alpha \) but they hold for \( \text{PA}_\alpha \)'s. Here \( \alpha \) is an infinite ordinal. This establishes yet another dichotomy between the \( \text{CA} \) paradigm and the \( \text{PA} \) paradigm.

Let \( C \) be the reflective subcategory of \( \text{RCA}_\alpha \) that has universal maps. Then \( \mathcal{D}c_\alpha \subseteq \mathcal{L} \). And indeed we have:

**Theorem 44.** Let \( \alpha \geq \omega \). Let \( \mathfrak{A}_0 \in \mathcal{D}c_\alpha, \mathfrak{A}_1, \mathfrak{A}_2 \in \text{RCA}_\alpha \) and \( f : \mathfrak{A}_0 \to \mathfrak{A}_1 \) and \( g : \mathfrak{A}_0 \to \mathfrak{A}_2 \) be monomorphisms. Then there exists \( \mathfrak{D} \in \mathcal{N}_\alpha \text{CA}_{\alpha + \omega} \) and \( m : \mathfrak{A}_1 \to \mathfrak{D} \) and \( n : \mathfrak{A}_2 \to \mathfrak{D} \) such that \( m \circ f = n \circ g \). Furthermore \( \mathfrak{D} \) is a super amalgam.

**Proof.** Looking at figure 1, assuming that the base algebra \( \mathfrak{A}_0 \) is in \( \mathcal{D}c_\alpha \), we obtain \( \mathfrak{D} \in \mathcal{N}_\alpha \text{CA}_{\alpha + \omega} \) \( m : \mathfrak{A}_1 \to \mathfrak{D} \), and \( n : \mathfrak{A}_2 \to \mathfrak{D} \) such that \( m \circ i = n \circ j \).

Here \( m = k \circ e_1 \) and \( n = h \circ e_2 \). Denote \( k \) by \( m^+ \) and \( h \) by \( n^+ \). Now we further want to show that if \( m(a) \leq n(b) \), for \( a \in A_1 \) and \( b \in A_2 \), then there exists \( t \in A_0 \) such that \( a \leq i(t) \) and \( j(t) \leq b \). So let \( a \) and \( b \) be as indicated. We have \( m^+ \circ e_1(a) \leq n^+ \circ e_2(b) \), so \( m^+(e_1(a)) \leq n^+(e_2(b)) \). Since \( L \) has \( SUPA_{\mathfrak{P}} \), there exist \( z \in A_0^+ \) such that \( e_1(a) \leq \tilde{i}(z) \) and \( \tilde{j}(z) \leq e_2(b) \). Let \( \Gamma = \Delta z \setminus \alpha \) and \( z' = c_{(\Gamma)}z \). (Note that \( \Gamma \) is finite.) So, we obtain that \( e_1(1_{(\Gamma)}a) \leq \tilde{i}(c_{(\Gamma)}z) \) and \( \tilde{j}(c_{(\Gamma)}z) \leq e_2(c_{(\Gamma)}b) \). It follows that \( e_A(a) \leq \tilde{i}(z') \) and \( \tilde{j}(z') \leq e_B(b) \). Now \( z' \in \mathcal{N}_\alpha \mathfrak{A}_0^+ = \mathfrak{A}_0 \mathcal{N}_\alpha \mathfrak{A}_0^+(e_{\mathfrak{A}_0}(A_0)) = A_0 \). Here we use [24] 2.6.67. So, there exists \( t \in C \) with \( z' = c_C(t) \). Then we get
$e_1(a) \leq \bar{i}(e_0(t))$ and $j(e_1(t)) \leq e_2(b)$. It follows that $e_1(a) \leq e_A \circ i(t)$ and $e_2 \circ j(t) \leq e_2(b)$. Hence, $a \leq i(t)$ and $j(t) \leq b$. We are done.

Now we have seen that polyadic algebras have a nice representation theorem. But is the standard modelling of polyadic algebras appropriate for certain phenomena of reasoning? The received conclusions about complexity of the axiomatizations of such algebras, are not warranted, they are an artefact of these modellings, which are not mandatory in anyway. The complexity of axiomatizations of polyadic algebras is highly complex from the recursion theory point of view [32]. This is one negative aspect of polyadic algebras, that is highly undesirable.

6 Cylindric-polyadic algebras

Now what?

Let us reflect on our earlier investigations. We have formulated an adjoint situation that cylindric algebras do not satisfy while polyadic algebras do.

Cylindric algebras are nice in many respects. They are definable by a finite simple schema, and the there exists recursive axiomatizations of the representable algebras, which is the least that can be said about polyadic algebras. On the other hand polyadic algebra has a strong Stone like representability result, which cylindric algebras lack, in a very resilient way.

Can we amalgamate the positive properties of both paradigms? Can we tame non-finite axiomatizability results of cylindric algebras, and at the same time obtain positive results of polyadic algebras like interpolation?

The question is certainly fair, and worthwhile pondering about, even though it does not really have a mathematical exact formulation, and hence can lend itself to different interpretations. Indeed, it is more of a philosophical question, but in history, it often happened that what was a philosophical question at one point of time became a mathematical one, with a rigorous mathematical answer at a later time. (For example Greeks talked about atoms). Furthermore, vagueness could be an asset not a liability. On the other hand, insisting on rigour can occasionally be counterproductive.

We can even go further than philosophy, and use the title of this article. Metaphysically, what is the "spirit" of cylindric algebras? Why, is there this feeling in the air, that quasi-polyadic algebras belong to the cylindric paradigm, while polyadic algebras do not. Can we get this feeling down to earth, can we pin it down. This will be the aim of our later investigations.

One natural way to approach our general problem is to experiment with signatures and see what happens. This is not a novel approach, it was already implemented by Sain studying countable reducts of polyadic algebras. Only finite cylindrifiers are available, but the algebras intersect the polyadic paradigm
because of the presence of infinitary substitutions (that is substitutions moving infinitely many points) though a finite number of them only, and it can be proved that two is enough.

Now we study a very natural amalgam of cylindric and polyadic algebras. We allow all substitutions, and restrict cylindrifies only to finite ones. We do not alter the notion of representability; representable algebras are those that can be represented on disjoint unions of cartesian squares, so we keep the Tarskian (semantical) spirit. We prove both a completeness and an interpolation result. And we also show that the neat reduct functor is strongly invertible, which is utterly unsurprising, because we have so many substitutions (these can be used to code extra dimensions).

We prove only the interpolation property (completeness is discernible below the surface of the proof) but in the presence of full fledged commutativity of cylindrifiers. The proof is basically a Henkin construction; algebraically a typical neat embedding theorem.

**Lemma 45.** Let \( \mathfrak{A} \) be a polyadic algebra of dimension \( \alpha \). Then for every \( \beta > \alpha \) there exists a polyadic algebra of dimension \( \beta \) such that \( \mathfrak{A} \subseteq \mathfrak{N}_\alpha \mathfrak{B} \), and furthermore, for all \( X \subseteq \mathfrak{A} \) we have

\[
\mathfrak{S} g^\alpha X = \mathfrak{S} g^{\mathfrak{N}_\alpha^\alpha} X = \mathfrak{N}_\alpha \mathfrak{S} g^\beta X.
\]

In particular, \( \mathfrak{A} = \mathfrak{N}_\alpha \mathfrak{B} \). \( \mathfrak{S} g^\beta \mathfrak{A} \) is called the minimal dilation of \( \mathfrak{A} \).

**Proof.** The proof depends essentially on the abundance of substitutions; we have all of them, which makes stretching dimensions possible. We provide a proof for cylindric polyadic algebras; the rest of the cases are like the corresponding prof in [5] for Boolean polyadic algebras.

We extensively use the techniques in [5], but we have to watch out, for we only have finite cylindrifications. Let \((\mathfrak{A}, \alpha, S)\) be a transformation system. That is to say, \( \mathfrak{A} \) is a Heyting algebra and \( S : \L o \alpha \rightarrow \text{End}(\mathfrak{A}) \) is a homomorphism. For any set \( X \), let \( F(\alpha \alpha, \mathfrak{A}) \) be the set of all functions from \( \alpha \alpha \) to \( \mathfrak{A} \) endowed with Heyting operations defined pointwise and for \( \tau \in \L o \alpha \) and \( f \in F(\alpha \alpha, \mathfrak{A}) \), \( s_\tau f(x) = f(x \circ \tau) \). This turns \( F(\alpha \alpha, \mathfrak{A}) \) to a transformation system as well. The map \( H : \mathfrak{A} \rightarrow F(\L o \alpha, \mathfrak{A}) \) defined by \( H(p)(x) = s_\alpha p \) is easily checked to be an isomorphism. Assume that \( \beta \supseteq \alpha \). Then \( K : F(\L o \alpha, \mathfrak{A}) \rightarrow F(\L o \beta, \mathfrak{A}) \) defined by \( K(f)(x) = f(x \uparrow \alpha) \) is an isomorphism. These facts are straightforward to establish, cf. theorem 3.1, 3.2 in [5]. \( F(\L o \beta, \mathfrak{A}) \) is called a minimal dilation of \( F(\L o \alpha, \mathfrak{A}) \). Elements of the big algebra, or the cylindrifier free dilation, are of form \( s_\sigma p, p \in F(\L o \beta, \mathfrak{A}) \) where \( \sigma \) is one to one on \( \alpha \), cf. [5] theorem 4.3-4.4.

We say that \( J \subseteq I \) supports an element \( p \in A \), if whenever \( \sigma_1 \) and \( \sigma_2 \) are transformations that agree on \( J \), then \( s_{\sigma_1} p = s_{\sigma_2} p \). \( \mathfrak{N}_J \mathfrak{A} \), consisting of the elements that \( J \) supports, is just the neat \( J \) reduct of \( \mathfrak{A} \); with the operations
defined the obvious way as indicated above. If $\mathfrak{A}$ is an $\mathfrak{B}$ valued $I$ transformation system with domain $X$, then the $J$ compression of $\mathfrak{A}$ is isomorphic to a $\mathfrak{B}$ valued $J$ transformation system via $H : \mathfrak{N}_J \mathfrak{A} \to F^J X, \mathfrak{A}$ by setting for $f \in \mathfrak{N}_J \mathfrak{A}$ and $x \in JX$, $H(f)x = f(y)$ where $y \in X^I$ and $y \upharpoonright J = x$, cf. \cite{5} theorem 3.10.

Now let $\alpha \subseteq \beta$. If $|\alpha| = |\beta|$ then the required algebra is defined as follows. Let $\mu$ be a bijection from $\beta$ onto $\alpha$. For $\tau \in ^\beta \beta$, let $s_\tau = s_{\tau \mu \mu^{-1}}$ and for each $i \in \beta$, let $c_i = c_{\mu(i)}$. Then this defined $\mathfrak{B} \in GPHA_\beta$ in which $\mathfrak{A}$ neatly embeds via $s_\mu|\alpha$, cf. \cite{5} p.168. Now assume that $|\alpha| < |\beta|$. Let $\mathfrak{A}$ be a given polyadic algebra of dimension $\alpha$; discard its cylindrifications and then take its minimal dilation $\mathfrak{B}$, which exists by the above. We need to define cylindrifications on the big algebra, so that they agree with their values in $\mathfrak{A}$ and to have $\mathfrak{A} \cong \mathfrak{N}_\alpha \mathfrak{B}$. We let (\*):

$$c_\kappa s_\sigma^\mathfrak{A} = s_\rho^{-1} c_{\rho |\kappa \cap \sigma \alpha} s_{\mu \sigma |\alpha}^\mathfrak{A} P.$$

Here $\rho$ is a any permutation such that $\rho \circ \sigma(\alpha) \subseteq \sigma(\alpha)$. Then we claim that the definition is sound, that is, it is independent of $\rho, \sigma, p$. Towards this end, let $q = s_\sigma^\mathfrak{A} P = s_{\sigma_1}^\mathfrak{A} P$ and $(\rho_1 \circ \sigma_1)(\alpha) \subseteq \alpha$.

We need to show that (**)

$$s_\rho^{-1} c_{|\rho |\kappa \cap \sigma \alpha}^\mathfrak{A} s_{\rho \sigma |\alpha}^\mathfrak{A} P = s_\rho^{-1} c_{|\rho_1 |\kappa \cap \sigma_1 \alpha}^\mathfrak{A} s_{\rho_1 \sigma_1 |\alpha}^\mathfrak{A} P.$$

Let $\mu$ be a permutation of $\beta$ such that $\mu(\sigma(\alpha) \cup \sigma_1(\alpha)) \subseteq \alpha$. Now applying $s_\mu$ to the left hand side of (**), we get that

$$s_\mu s_\rho^{-1} c_{|\rho |\kappa \cap \sigma \alpha}^\mathfrak{A} s_{\rho \sigma |\alpha}^\mathfrak{A} P = s_\mu s_\rho^{-1} c_{|\rho |\kappa \cap \sigma \alpha}^\mathfrak{A} s_{\rho \sigma |\alpha}^\mathfrak{A} P.$$

The latter is equal to $c_{|\mu |\kappa \cap \sigma \alpha}^\mathfrak{A} s_\mu^\mathfrak{A} q$. Now since $\mu(\sigma(\alpha) \cap \sigma_1(\alpha)) \subseteq \alpha$, we have $s_\mu^\mathfrak{A} P = s_{\mu \sigma_1 |\alpha}^\mathfrak{A} P = s_{\mu \sigma_1 |\alpha}^\mathfrak{A} P$. It thus follows that

$$s_\rho^{-1} c_{|\rho |\kappa \cap \sigma \alpha}^\mathfrak{A} s_{\rho \sigma |\alpha}^\mathfrak{A} P = c_{|\mu |\kappa \cap \sigma \alpha}^\mathfrak{A} s_{\mu \sigma_1 |\alpha}^\mathfrak{A} P.$$

By exactly the same method, it can be shown that

$$s_{\rho_1^{-1}} c_{|\rho_1 |\kappa \cap \sigma \alpha}^\mathfrak{A} s_{\rho_1 \sigma_1 |\alpha}^\mathfrak{A} P = c_{|\mu |\kappa \cap \sigma \alpha}^\mathfrak{A} s_{\mu_1 |\alpha}^\mathfrak{A} P.$$

By this we have proved (**).

Furthermore, it defines the required algebra $\mathfrak{B}$. Let us check this. Since our definition is slightly different than that in \cite{5}, by restricting cylindrifications to be only finite, we need to check the polyadic axioms which is tedious but routine. The idea is that every axiom can be pulled back to its corresponding axiom holding in the small algebra $\mathfrak{A}$. We check only the axiom

$$c_\kappa(q_1 \land c_\kappa q_2) = c_\kappa q_1 \land c_\kappa q_2.$$
We follow closely [5] p. 166. Assume that \( q_1 = s_{\rho}^{|A|} p_1 \) and \( q_2 = s_{\rho}^{|A|} p_2 \). Let \( \rho \) be a permutation of \( I \) such that \( \rho(\sigma_1 I \cup \sigma_2 I) \subseteq I \) and let

\[
p = s_{\rho}^{|A|} [q_1 \land c_k q_2].
\]

Then

\[
p = s_{\rho}^{|A|} q_1 \land s_{\rho}^{|A|} c_k q_2 = s_{\rho}^{|A|} s_{\sigma_1} p_1 \land s_{\rho}^{|A|} c_k s_{\sigma_2} p_2.
\]

Now we calculate \( c_k s_{\sigma_2} p_2 \). We have by (*)

\[
c_k s_{\sigma_2} p_2 = s_{\sigma_2}^{-1} c_{\rho(\{k\}) \cap \sigma_2 I} s_{(\rho \sigma_2 \tau(I))}^{|A|} p_2.
\]

Hence

\[
p = s_{\rho}^{|A|} s_{\sigma_1} p_1 \land s_{\rho}^{|A|} s_{\sigma_2}^{-1} c_{\rho(\{k\}) \cap \sigma_2 I} s_{(\rho \sigma_2 \tau(I))}^{|A|} p_2.
\]

\[
= s_{\rho \sigma_1}^{|A|} p_1 \land s_{\rho}^{|A|} s_{\sigma_2}^{-1} c_{\rho(\{k\}) \cap \sigma_2 I} s_{(\rho \sigma_2 \tau(I))}^{|A|} p_2.
\]

\[
= s_{\rho \sigma_1}^{|A|} p_1 \land s_{\rho}^{|A|} c_{\rho(\{k\}) \cap \sigma_2 I} s_{(\rho \sigma_2 \tau(I))}^{|A|} p_2.
\]

Now

\[
c_k s_{\rho^{-1}} p = c_k s_{\rho^{-1}} s_{\rho}^{|A|} (q_1 \land c_k q_2) = c_k (q_1 \land c_k q_2)
\]

We next calculate \( c_k s_{\rho^{-1}} p \). Let \( \mu \) be a permutation of \( I \) such that \( \mu \rho^{-1} I \subseteq I \). Let \( j = \mu(\{k\} \cap \rho^{-1} I) \). Then applying (*), we have:

\[
c_k s_{\rho^{-1}} p = s_{\mu^{-1}} c_{j} s_{(\mu \rho^{-1} I)}^{|A|} p.
\]

\[
= s_{\mu^{-1}} c_{j} s_{(\mu \rho^{-1} I)}^{|A|} s_{\rho \sigma_1}^{|A|} p_1 \land c_{\rho(\{k\}) \cap \sigma_2 I} s_{(\rho \sigma_2 \tau(I))}^{|A|} p_2.
\]

\[
= s_{\mu^{-1}} c_{j} [s_{\mu \sigma_1}^{|A|} p_1 \land r].
\]

where

\[
r = s_{\mu \rho^{-1}} c_{j} s_{\rho \sigma_2 \tau(I)}^{|A|} p_2.
\]

Now \( c_k r = r \). Hence, applying the axiom in the small algebra, we get:

\[
s_{\mu^{-1}} c_{j} [s_{\mu \sigma_1}^{|A|} p_1] \land c_k q_2 = s_{\mu^{-1}} c_{j} [s_{\mu \sigma_1}^{|A|} p_1 \land r].
\]

But

\[
c_{\mu(\{k\}) \cap \rho^{-1} I} s_{(\mu \sigma_1 \tau(I))}^{|A|} p_1 = c_{\mu(\{k\}) \cap \sigma_1 I} s_{(\mu \sigma_1 \tau(I))}^{|A|} p_1.
\]

So

\[
s_{\mu^{-1}} c_{k} [s_{\mu \sigma_1}^{|A|} p_1] = c_k q_1,
\]

and we are done. The second part, is exactly like theorem [40].

**Theorem 46.** Let \( \beta \) be a cardinal, and \( \mathfrak{A} = \mathfrak{F} \mathfrak{r}_{\beta} \mathfrak{F} \mathfrak{P} \mathfrak{A}_\alpha \) be the free algebra on \( \beta \) generators. Let \( X_1, X_2 \subseteq \beta \), \( a \in \mathfrak{F} g^{|A|} X_1 \) and \( c \in \mathfrak{F} g^{|A|} X_2 \) be such that \( a \leq c \). Then there exists \( b \in \mathfrak{F} g^{|A|} (X_1 \cap X_2) \) such that \( a \leq b \leq c \).
Proof. Let \( a \in \mathbb{S}g^\alpha X_1 \) and \( c \in \mathbb{S}g^\alpha X_2 \) be such that \( a \leq c \). We want to find an interpolant in \( \mathbb{S}g^\alpha(X_1 \cap X_2) \). Assume that \( \kappa \) is a regular cardinal \( > \max(|\alpha|,|A|) \). Let \( \mathfrak{B} \in \mathsf{FPA}_\kappa \) such that \( \mathfrak{A} = \mathfrak{N}_\alpha \mathfrak{B} \), and \( A \) generates \( \mathfrak{B} \). Let \( H_\kappa = \{ \rho \in {}^\kappa \kappa : |\rho(\alpha) \cap (\kappa \sim \alpha)| < \omega \} \). Let \( S \) be the semigroup generated by \( H_\kappa \). Let \( \mathfrak{B}' \in \mathsf{FPA}_\kappa \) be an ordinary dilation of \( \mathfrak{A} \) where all transformations in \( {}^\kappa \kappa \) are used. (This can be easily defined like in the case of ordinary polyadic algebras). Then \( \mathfrak{A} = \mathfrak{N}_\alpha \mathfrak{B}' \). We take a suitable reduct of \( \mathfrak{B}' \). Let \( \mathfrak{B} \) be the subalgebra of \( \mathfrak{B}' \) generated from \( A \) be all operations except for substitutions indexed by transformations not in \( S \). Then, of course \( A \subseteq \mathfrak{B} \); in fact, \( \mathfrak{A} = \mathfrak{N}_\alpha \mathfrak{B} \), since for each \( \tau \in {}^\alpha \alpha \), \( \tau \cup \text{Id} \in S \). It can be checked inductively that for \( b \in B \), if \( |\Delta b \sim \alpha| < \omega \), and \( \rho \in S \), then \( |\rho(\Delta b) \sim \alpha| < \omega \). Then there exists a finite \( \Gamma \subseteq \kappa \sim \alpha \) such that \( a \leq c_\Gamma b \leq c \) and

\[
c_\Gamma b \in \mathfrak{N}_\alpha \mathbb{S}g^\alpha(X_1 \cap X_2) = \mathbb{S}g^\alpha \mathfrak{B}(X_1 \cap X_2) = \mathbb{S}g^\alpha(X_1 \cap X_2).
\]

The rest of the proof is similar to that in [36], except that the latter reference deals with countable algebras, and here our algebras could be uncountable, hence the condition of regularity on the cardinal \( \kappa \). Arrange \( \kappa \times \mathbb{S}g^\xi(X_1) \) and \( \kappa \times \mathbb{S}g^\xi(X_2) \) into \( \kappa \)-termed sequences:

\[
\langle (k_i, x_i) : i \in \kappa \rangle \text{ and } \langle (l_i, y_i) : i \in \kappa \rangle \text{ respectively.}
\]

Since \( \kappa \) is regular, we can define by recursion \( \kappa \)-termed sequences:

\[
\langle u_i : i \in \kappa \rangle \text{ and } \langle v_i : i \in \kappa \rangle
\]

such that for all \( i \in \kappa \) we have:

\[
u_i \in \kappa \setminus (\Delta a \cup \Delta c) \cup \cup_{j \leq i}(\Delta x_j \cup \Delta y_j) \cup \{ u_j : j < i \} \cup \{ v_j : j < i \}
\]

and

\[
v_i \in \kappa \setminus (\Delta a \cup \Delta c) \cup \cup_{j \leq i}(\Delta x_j \cup \Delta y_j) \cup \{ u_j : j \leq i \} \cup \{ v_j : j < i \}.
\]

For a boolean algebra \( \mathfrak{C} \) and \( Y \subseteq \mathfrak{C} \), we write \( f\ell \mathfrak{C} Y \) to denote the boolean filter generated by \( Y \) in \( \mathfrak{C} \). Now let

\[
Y_1 = \{ a \} \cup \{ -c_i x_i + s_{a_i}^i x_i : i \in \omega \},
\]

\[
Y_2 = \{ -c \} \cup \{ -c_i y_i + s_{a_i}^i y_i : i \in \omega \},
\]

\[
H_1 = f\ell \mathfrak{B}\mathbb{S}g^\alpha(X_1) Y_1, \quad H_2 = f\ell \mathfrak{B}\mathbb{S}g^\alpha(X_2) Y_2,
\]

and

\[
H = f\ell \mathfrak{B}\mathbb{S}g^\alpha(X_1 \cap X_2)[(H_1 \cap \mathbb{S}g^\alpha(X_1 \cap X_2) \cup (H_2 \cap \mathbb{S}g^\alpha(X_1 \cap X_2))].
\]
Then $H$ is a proper filter of $\mathcal{S}g^B(X_1 \cap X_2)$. This is proved by induction with the base of the induction being no interpolant exists in $\mathcal{S}g^B(X_1 \cap X_2)$, cf. [36] Claim 2.18 p.339. Let $H^*$ be a (proper boolean) ultrafilter of $\mathcal{S}g^B(X_1 \cap X_2)$ containing $H$. We obtain ultrafilters $F_1$ and $F_2$ of $\mathcal{S}g^B X_1$ and $\mathcal{S}g^B X_2$, respectively, such that

$$H^* \subseteq F_1, \quad H^* \subseteq F_2$$

and (**)

$$F_1 \cap \mathcal{S}g^B(X_1 \cap X_2) = H^* = F_2 \cap \mathcal{S}g^B(X_1 \cap X_2).$$

Now for all $x \in \mathcal{S}g^B(X_1 \cap X_2)$ we have

$$x \in F_1 \text{ if and only if } x \in F_2.$$

Also from how we defined our ultrafilters, $F_i$ for $i \in \{1, 2\}$ satisfy the following condition: (*) For all $k < \mu$, for all $x \in \mathcal{S}g^B X_i$ if $c_k x \in F_i$ then $s_k^l x$ is in $F_i$ for some $l \notin \Delta x$.

Let $D_i = \mathcal{S}g^A X_i$. For a transformation $\tau \in \alpha \kappa$ let $\bar{\tau} = \tau \cup Id_{\kappa \sim \alpha}$. Define $f_i$ from $D_i$ to the full set algebra $\mathfrak{C}$ with unit $\alpha \kappa$ as follows:

$$f_i(x) = \{ \tau \in \alpha \kappa : s_x x \in F_i \}, \text{ for } x \in D_i$$

Then $f_i$ is a homomorphism by (*), [36] p.343. Without loss of generality, we can assume that $X_1 \cup X_2 = X$. By (**) we have $f_1$ and $f_2$ agree on $X_1 \cap X_2$. So that $f_1 \cup f_2$ defines a function on $X_1 \cup X_2$, by freeness it follows that there is a homomorphism $f$ from $\mathfrak{B}$ to $\mathfrak{C}$ such that $f_1 \cup f_2 \subseteq f$. Then $q \in f(a) \cap f(-c) = f(a - c)$. This is so because $s_{Id} a = a \in F_1$ $s_{Id}(-c) = -c \in F_2$. But this contradicts the premise that $a \leq c$.

6.1 Ferenzci’s algebras

Now we get rid of commutativity of cylindrifes and adopt weaker axioms adding also the so called merry-go-round identities. In set algebras based on cartesian squares cylindrifiers commute, so we have no choice but to alter the notion of representability as well. In modal logic this is termed as relativization. This approach pays, very much so.

So called relativization started as a technique for generalizing representations of cylindric algebras, while also, in some cases, ‘defusing’ undesirable properties, like undecidability or lack of definability (like Beth definability). These ideas have counterparts in logic, and they have been influential in several ways. Relativization in cylindric-like algebras lends itself to a modal perspective where transitions are viewed as objects in their own right, in addition to states, while algebraic terms now correspond to modal formulas defining the essential properties of transitions.
Indeed, why insist on standard models? This is a voluntary commitment to only one mathematical implementation, whose undesirable complexities can pollute the laws of logics needed to describe the core phenomena. Set theoretic cartesian squares modelling as the intended vehicle may not be an orthogonal concern, it can be detrimental, repeating hereditary sins of old paradigms.

Indeed in [24] square units got all the attention and relativization was treated as a side issue. Extending original classes of models for logics to manipulate their properties is common. This is no mere tactical opportunism, general models just do the right thing.

The famous move from standard models to generalized models is Henkin’s turning round second order logic into an axiomatizable two sorted first order logic. Such moves are most attractive when they get an independent motivation.

The idea is that we want to find a semantics that gives just the bare bones of action, while additional effects of square set theoretic modelling are separated out as negotiable decisions of formulation that threatens completeness, decidability, and interpolation.

And indeed by using relativized representations Ferenczi, proved that if we weaken commutativity of cylindrifiers and allow relativized representations, then we get a finitely axiomatizable variety of representable quasi-polyadic equality algebras (analogous to the Resek Thompson CA version); even more this can be done without the merry go round identities. This is in sharp view with our complexity results proved above for quasi poyadic equality algebras.

Now we use two techniques to get positive results. The first is yet again a Henkin construction (carefully implemented because we have changed the semantics, so that Henkin ultrafilters constructed are more involved), the other is inspired by the well-developed duality theory in modal logic between Kripke frames and complex algebras.

This technique was first implemented by Németi in the context of relativized cylindric set algebras which are complex algebras of weak atom structures.

**Theorem 47.** Let $\beta$ be a cardinal, and $\mathfrak{A} = \mathfrak{Fr}_\beta \text{CPA}_\kappa$ be the free algebra on $\beta$ generators. Let $X_1, X_2 \subseteq \beta$, $a \in \mathfrak{S}g^\mathfrak{A} X_1$ and $c \in \mathfrak{S}g^\mathfrak{A} X_2$ be such that $a \leq c$. Then there exists $b \in \mathfrak{S}g^\mathfrak{A} (X_1 \cap X_2)$ such that $a \leq b \leq c$.

**Proof.** Let $a \in \mathfrak{S}g^\mathfrak{A} X_1$ and $c \in \mathfrak{S}g^\mathfrak{A} X_2$ be such that $a \leq c$. We want to find an interpolant in $\mathfrak{S}g^\mathfrak{A} (X_1 \cap X_2)$. Assume that $\kappa$ is a regular cardinal $> \text{max}(|\alpha|, |A|)$. Let $\mathfrak{B} \in \text{CPA}_\kappa$, be as in the previous lemma, such that $\mathfrak{A} = \mathfrak{F}_\alpha \mathfrak{B}$, and $A$ generates $\mathfrak{B}$. Like before we can assume that no interpolant exists in $\mathfrak{B}$. One defines filters in $\mathfrak{S}g^\mathfrak{A} X_1$ and in $\mathfrak{S}g^\mathfrak{A} X_2$ like in Ferenczi [11]. Let

\[ Y_i = \{ s \tau c_j x : \tau \in \text{adm}, j \in \alpha, x \in A \}. \]
\[
H_1 = f t^{B \circ g^{3}}(X_1) Y_1, \quad H_2 = f t^{B \circ g^{3}}(X_2) Y_2,
\]
and
\[
H = f t^{B \circ g^{3}}(X_1 \cap X_2)[(H_1 \cap \mathcal{S} g^{3}(X_1 \cap X_2)) \cup (H_2 \cap \mathcal{S} g^{3}(X_1 \cap X_2))].
\]

Then \(H\) is a proper filter of \(\mathcal{S} g^{3}(X_1 \cap X_2)\). This can be proved by induction with the base of the induction being no interpolant exists in \(\mathcal{S} g^{3}(X_1 \cap X_2)\). Let \(H^*\) be a (proper boolean) ultrafilter of \(\mathcal{S} g^{3}(X_1 \cap X_2)\) containing \(H\). We obtain ultrafilters \(F_1\) and \(F_2\) of \(\mathcal{S} g^{3} X_1\) and \(\mathcal{S} g^{3} X_2\), respectively, such that
\[
H^* \subseteq F_1, \quad H^* \subseteq F_2,
\]
and (**)
\[
F_1 \cap \mathcal{S} g^{3}(X_1 \cap X_2) = H^* = F_2 \cap \mathcal{S} g^{3}(X_1 \cap X_2).
\]
Now for all \(x \in \mathcal{S} g^{3}(X_1 \cap X_2)\) we have
\[
x \in F_1 \text{ if and only if } x \in F_2.
\]
Also from how we defined our ultrafilters, \(F_i\) for \(i \in \{1, 2\}\) are perfect.

Then define the homomorphisms, one on each subalgebra, like in [47] p. 128-129, using the perfect ultrafilters, then freeness will enable use to end these homomorphisms to the set of free generators, and it will satisfy \(h(a - c) \neq 0\) which is a contradiction.

We show using techniques of Marx, that weak polyadic algebras have \(SUPAP\) as well. This works for all varieties of relativized cylindric polyadic algebras studied by Ferenzci and reported in [11].

This follows from the simple observation that such varieties can be axiomatized with positive, hence Sahlqvist equations, and therefore they are canonical; and also we do not have a Rosser condition on cylindrifiers; cylindrifiers do not commute, this allows that the first order correspondants of such equations are clausifiable, see [18] for the definition of this. This proof is inspired by the modal perspective of cylindric-like algebras that suggests a whole landscape below standard predicate logic, with a minimal modal logic at the base ascending to standard semantics via frame constraints. In particular, this landscape contains nice sublogics of the full predicate logic, sharing its desirable meta properties and at the same time avoiding its negative accidents due to its Tarskian 'square frames' modelling. Such mutant logics are currently a very rich area of research.

The technique used here can be traced back to Németi, when he proved that relativized cylindric set algebras have \(SUPAP\); using (classical) duality between atom structures and cylindric algebras. Marx 'modalized' the proof,
and slightly strengthened Németi results, using instead the well-established duality between modal frames and complex algebras.

We consider the noncommutative cylindric polyadic algebras introduced by Ferenzi; but we use the notation $WFPA_\alpha$. A frame is a first order structure $\mathfrak{F} = (V, T_i, S_\tau)_{i \in \alpha, \tau \in \tau^\alpha}$ where $V$ is an arbitrary set and and both $T_i$ and $S_\tau$ are binary relations on $V$ for all $i \in \alpha$; and $\tau \in \tau^\alpha$.

Given a frame $\mathfrak{F}$, its complex algebra will be denoted by $\mathfrak{F}^+$. $\mathfrak{F}^+$ is the algebra $(\mathcal{P}(V), c_i, s_\tau)_{i \in \alpha, \tau \in \tau^\alpha}$ where for $X \subseteq V$, $c_i(X)$ = $\{s \in V : \exists t \in X, (t, s) \in T_i\}$, and similarly for $s_\tau$.

For $K \subseteq WFPA_\alpha$, we let $\mathfrak{S}trK = \{\mathfrak{F} : \mathfrak{F}^+ \in K\}$.

For a variety $V$, it is always the case that $\mathfrak{S}trV \subseteq \text{At}V$ and equality holds if the variety is atom-canonical. If $V$ is canonical, then $\mathfrak{S}trV$ generates $V$ in the strong sense, that is $V = \mathfrak{S}Cm\mathfrak{S}trV$. For Sahlqvist varieties, as is our case, $\mathfrak{S}trV$ is elementary.

**Definition 48.**

Given a family $(\mathfrak{F}_i)_{i \in I}$ of frames, a **zigzag product** of these frames is a substructure of $\prod_{i \in I} \mathfrak{F}_i$ such that the projection maps restricted to $S$ are onto.

**Definition 49.** Let $\mathfrak{F}, \mathfrak{G}, \mathfrak{H}$ be frames, and $f : \mathfrak{G} \to \mathfrak{F}$ and $h : \mathfrak{F} \to \mathfrak{H}$. Then $\text{INSEP} = \{(x, y) \in \mathfrak{G} \times \mathfrak{H} : f(x) = h(y)\}$.

**Lemma 50.** The frame $\text{INSEP} \upharpoonright G \times H$ is a zigzag product of $G$ and $H$, such that $\pi \circ \pi_0 = h \circ \pi_1$, where $\pi_0$ and $\pi_1$ are the projection maps.

**Proof.** [18] 5.2.4

For an algebra $\mathfrak{A}$, $\mathfrak{A}^+$ denotes its ultrafilter atom structure. For $h : \mathfrak{A} \to \mathfrak{B}$, $h^+$ denotes the function from $\mathfrak{B}_+ \to \mathfrak{A}_+$ defined by $h^+(u) = h^{-1}[u]$ where the latter is $\{x \in a : h(x) \in u\}$.

**Theorem 51.** ([18] lemma 5.2.6) Assume that $K$ is a canonical variety and $\mathfrak{S}trK$ is closed under finite zigzag products. Then $K$ has the superamalgamation property.

**Sketch of proof.** Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in K$ and $f : \mathfrak{A} \to \mathfrak{B}$ and $h : \mathfrak{A} \to \mathfrak{C}$ be given monomorphisms. Then $f^+ : \mathfrak{B}_+ \to \mathfrak{A}_+$ and $h^+ : \mathfrak{C}_+ \to \mathfrak{A}_+$. We have $\text{INSEP} = \{(x, y) : f^+(x) = h^+(y)\}$ is a zigzag connection. Let $\mathfrak{F}$ be the zigzag product of $\text{INSEP} \upharpoonright \mathfrak{A}_+ \times \mathfrak{B}_+$. Then $\mathfrak{F}^+$ is a superamalgam.

**Theorem 52.** The variety $WFPA_\alpha$ has SUPAP.

**Proof.** $WFPA_\alpha$ can be easily defined by positive equations then it is canonical. The first order correspondents of the positive equations translated to the class of frames will be Horn formulas, hence clausifiable [18] theorem 5.3.5, and so $\mathfrak{S}trK$ is closed under finite zigzag products. Marx’s theorem finishes the proof. 

68
Let us make philosophy and metaphysics mathematics:

**Definition 53.** A class of algebras is in the cylindric paradigm if the neat reduct functor $\mathfrak{Nr}$ does not have a right adjoint; it is in the polyadic one, if the neat reduct functor is strongly invertible.

More crudely, the polyadic paradigm versus the cylindric one, establishes a dichotomy in algebraic logic; the separating point is the presence of substitutions that move infinitely many points.

Another dichotomy, existing in algebraic logic, that can be traced back to the famous Andréka-Resek-Thompson theorem, is that between square representations and relativized one. Here non-commutativity of cylindrifiers is the separating point. Ferenczi’s recent work on cylindric polyadic sheds a lot of light on this intriguing phenomena; and the connection of neat embeddings to relativized representations is indeed quite a remarkable achievement.

If one views relativized models as the natural semantics for predicate logic rather than some tinkering devise which is the approach adopted in [24], then many well-established taboos of the field must be challenged.

In standard textbooks one learns that predicate logical validity is one unique notion specified once and for all by the usual Tarskian (square) semantics and canonized by Gődel’s completeness theorem. Moreover, it is essentially complex, being undecidable by Church’s theorem.

On the present view, however standard predicate logic has arisen historically by making several ad-hoc semantic decisions that could have gone differently. It’s not all about ‘one completeness theorem’ but rather about several completeness theorems obtained by varying both the semantic and syntactical parameters.

But on the other hand, careful scrutiny of the situation reveals that things are not so clear cut, and the borderlines are hazy. Within the polyadic cylindric dichotomy there is the square relativisation dichotomy, and also vice versa.

An important border line class of algebras are those studied by Sain. Are they in the polyadic paradigm? According to the last definition, they are. But this is not the end of the story. Sain’s algebras introduced in [30] have a unique status. They share the positive properties of both paradigms, the cylindric one, and the polyadic one. First thing they extend first order logic without equality, so that, in particular, cylindrifiers commute (this property is precarious in other contexts, it can kill decidability and amalgamation.)

They are representable as genuine fields of sets, they are finitely axiomatizable over finitely presented semigroups, they have a recursive equational axiomatization, they admit dilations (neat embedding in $\omega$ extra dimensions), they have the super amalgamation property, and they also have **infinitary substitutions, at least two of them**. The main discrepancy between Sain’s algebras and polyadic algebras, say, is that the equational theory of the last has very
high complexity in the recursion theoretic sense, a result of Németi and Sagi [32]. It is hard to place these algebras in either paradigm alone, but we believe that it is fair to say that they belong to the positive part of both.

7 Cylindric and Polyadic algebras rolled into one

Avoiding Platonic complacency, in this section we construct a forest containing the two trees, as opposed to the forest. Other forests are conceivable. We do not claim that we have built a complete philosophical house, but we may have opened a lot of windows.

We give a general definition of a system of varieties definable by a uniform schema, that covers, or rather unifies, Monk’s definition of systems of varieties definable by a schema and the Németi-Sági definition of Halmos’ schemes. In this very general context, we define the operation of forming neat reducts, and we furthermore view the neat reduct operator as a functor that lessens dimensions. Then we prove a general theorem, extending our previous, that relates adjointness of this functor to various forms of the amalgamation property for the system of varieties in question.

The difficulty with the polyadic like algebras, is that there is an algebraic structure on (a part of) the indexing set, the set of all maps from an infinite ordinal to itself, namely, the operation of composition of maps. From a universal algebraic perspective this structure does not manifest itself explicitly; instead it is somewhere up there in the meta language. This is a situation similar to modules over rings. One approach to deal with such structures in first order logic is to allow two sorts, one for the scalars, and the other for the vectors. We adopt, following Németi and Sági, the same philosophy; however, we need three sorts, one for the substitutions, one for sets of ordinals, and one for the first order situation.

Ord is the class of all ordinals. For ordinals \( \alpha < \beta \), \([\alpha, \beta]\) denotes the set of ordinals \( \mu \) such that \( \alpha \leq \mu \leq \beta \). \( I_\alpha \) is the class \( \{ \beta \in \text{Ord} : \beta \geq \alpha \} \) By an interval of ordinals, or simply an interval, we either mean \([\alpha, \beta]\) or \( I_\alpha \).

**Definition 54.**

(i) A type schema is a quantuple \( t = (T, \delta, \rho, c, s) \) such that \( T \) is a set, \( \delta \) maps \( T \) into \( \omega \), \( c, s \in T \), and \( \delta c = \rho c = \delta s = \rho s = 1 \).

(ii) A type schema as in (i) defines a similarity type \( t_\alpha \) for each \( \alpha \) as follows. Sets \( C_\alpha \subseteq \wp(\alpha) \), \( G_\alpha \subseteq ^\alpha \alpha \) are fixed, and the domain \( T_\alpha \) of \( t_\alpha \) is

\[
T_\alpha = \{(f, k_0, \ldots k_{\delta f - 1}) : f \in T \sim \{c, s\}, k \in \delta f \alpha\} \\
\cup \{(c, r) : r \in C_\alpha\} \cup \{(q, r) : r \in C_\alpha\} \cup \{(s, \tau) : \tau \in \alpha \}.
\]
For each \((f, k_0, \ldots, k_{\delta f - 1}) \in T_\alpha\) we set \(t_\alpha(f, k_0, \ldots, k_{\delta f - 1}) = \rho f\) and we set \(\rho(c, r) = \rho(q, r) = \rho(s, \tau) = 1\).

(iii) Let \(\mu\) be an interval of ordinals. A system \((K_\alpha : \alpha \in \mu)\) of classes of algebras is of type schema \(t\) if for each \(\alpha \in \mu\), the class \(K_\alpha\) is a class of algebras of type \(t_\alpha\).

**Definition 55.** Let \(L_T\) be the first order language that consists of countably many unary relational symbols \((Rel)\), countably many function symbols \((Func)\) and countably many constants \((Cons)\), which are \(r_1, r_2 \ldots\) and \(f_1, f_2 \ldots\) and \(n_1, n_2 \ldots\), respectively. We let \(L_T = Rel \cup Cons \cup Func\).

**Definition 56.**

1. A schema is a pair \((s, e)\) where \(s\) is a first order formula of \(L_T\) and \(e\) is an equation in the language \(L_\omega\) of \(K_\omega\). We denote a schema \((s, e)\) by \(s \rightarrow e\). We define \(Ind(L_\omega) = \omega \cup C_\omega \cup G_\omega\). A function \(h : L_T \rightarrow L_\omega\) is admissible if \(h\) is an injection and \(h \upharpoonright Const \subseteq \omega, h \upharpoonright Rel \subseteq C_\omega\) and \(h \upharpoonright Func \subseteq G_\omega\).

2. Let \(g\) be an equation in the language of \(K_\alpha\). Then \(g\) is an \(\alpha\) instance of a schema \(s \rightarrow e\) if there exist an admissible function \(h\), sets, functions and constants

\[
\begin{align*}
  r_1, r_2, f_1, f_2, \ldots &\in G_\alpha, n_1, n_2, \ldots \in \alpha \\
  M &= (\alpha, r_1, r_2, f_1, f_2, n_1, n_2, \ldots) \models s \\
\end{align*}
\]

such

and \(g\) is obtained from \(e\) by replacing \(h(r_i), h(f_i)\) and \(h(n_i)\) by \(r_i, f_i, n_i\), respectively.

**Definition 57.** A system of varieties is a generalized system of varieties definable by a schema, if there exists a strictly finite set of schemes, such that for every \(\alpha\), \(K_\alpha\) is axiomatized by the \(\alpha\) dimensional instances of such schemes.

Given such a system of varieties, we denote algebras in \(K_\alpha\) by

\[
\mathfrak{A} = (\mathfrak{B}, c_{(r)}, q_{(r)}, s_{(r)})_{r \in G_\alpha, \tau \in G_\alpha},
\]

that is, we highlight the operations of cylindrifiers and substitutions, and the operations in \(T \sim \{c, s\}\) (of the Monk’s schema part, so to speak), with indices from \(\alpha\), are encoded in \(\mathfrak{B}\).

Indeed, Monk’s definition is the special case, when we forget the sort of substitutions. That is a system of varieties is definable by Monk’s schemes if \(G_\alpha = \emptyset\) for all \(\alpha\) and each schema the form by \(True \rightarrow e\); see definition below.
Example 58.  (1) Tarski’s cylindric algebras, Pinter’s substitution algebras, Halmos’ quasi-polyadic algebras and Halmos’ quasi-polyadic algebras with equality, all of infinite dimension. Here $G_\alpha = \emptyset$ for all $\alpha \geq \omega$, and $C_\alpha = \{ \{ r \} : r \in \alpha \}$.

(2) Less obvious are Halmos’ polyadic algebras, of infinite dimension, as defined in [25]. Such algebras are axiomatized by Halmos schemes; hence the form a generalized system of varieties definable by a schema of equations. For example, the $\omega$ instance of $(P_{11})$ is:

$$[(\forall y)(r_2(y) \leftarrow \exists z(r_1(z) \land (\forall y, z)(r_2(y) \land r_2(z) \land y \neq z \implies f_1(y) \neq f_1(z), c_{r_1}, s_{f_1}(x) = s_{f_1}c_{r_2}(x))]$$

Example 59. Cylindric-polyadic algebras [11]. These are reducts of polyadic algebras of infinite dimension, where we have all substitutions, but cylindrification is allowed only on finitely many indices. Such algebras have become fashionable lately, with the important recent work of Ferenczi. However, Ferenczi deals with (non-classical) versions of such algebras, where commutativity of cylindrifiers is weakened, substantially, and he proves strong representation theorems on generalized set algebras. Here $C_\alpha$ is again the set of singletons, manifesting the cylindric spirit of the algebras, while $G_\alpha = ^\alpha \alpha$, manifesting, in turn, its polyadic reduct.

Example 60. (a) Sain’s algebras [30]: Such algebras provide a solution to one of the most central problems in algebraic logic, namely, the so referred to in the literature as the fintizability problem. Those are countable reducts of polyadic algebras, and indeed of cylindric-polyadic algebras. Cylindrifies are finite, that is they are defined only on finitely many indices, but at least two infinitary substitutions are there.

Like polyadic algebras, and for that matter cylindric polyadic algebras, such classes algebras, which happen to be varieties, can be easily formulated as a generalized system of varieties definable by a schema on the interval $[\alpha, \alpha + \omega]$, $\alpha$ a countable ordinal.

Here we only have substitutions coming from a countable semigroup $G_\alpha$, and $G_{\alpha+n}$, $n \leq \omega$, is the sub-semigroup of $^{\alpha+n} \alpha + n$ generated by $\tau = \tau \cup Id_{(\alpha+n) - \alpha}$, $\tau \in G_\alpha$. Such algebras, were introduced by Sain, can be modified, in case the semigroups determining their similarity types are finitely presented, providing first order logic without equality a strictly finitely based algebrasisation, see also [36].

(b) Sain’s algebras with diagonal elements [31]. These are investigated by Sain and Gyuris, in the context of fintizizing first order logic with equality. This problem turns out to be harder, and so the results obtained are weaker, because the class of representable algebras $V$ is not elementary; it is not closed under ultraproducts. The authors manage to provide, in this case a generalized
finite schema, for the class of $\mathbf{HV}$; this only implies weak completeness for the corresponding infinitary logics; that is $\models \phi$ implies $\vdash \phi$, relative to a finitary Hilbert style axiomatization, involving only type free valid schemes. However, there are non-empty sets of formulas $\Gamma$, such that $\Gamma \models \phi$, but there is no proof of $\phi$ from $\Gamma$.

It is timely to highlight the novelties in the above definition when compared to Monk’s definition of a system definable by schemes.

1. First, the most striking addition, is that it allows dealing with infinitary substitutions coming from a set $G_\alpha$, which is usually a semigroup. Also infinitary cylindrifiers are permitted. This, as indicated in the above examples, covers polyadic algebras, Heyting polyadic algebras, $MV$ polyadic algebras and Ferenzci’s cylindric-polyadic algebras, together with their important reducts studied by Sain.

2. Second thing, cylindrifiers are not mandatory; this covers many algebraisations of multi dimensional modal logics, like for example modal logics of substitutions. (This will be elaborated upon below, in the new context of complete system of varieties definable by a schema, which integrates finite dimensions).

3. We also have another (universal) quantifier $q$, intended to be the dual of cylindrifiers in the case of presence of negation; representing universal quantification. This is appropriate for logics where we do not have negation in the classical sense, like intuitionistic logic, expressed algebraically by Heyting polyadic algebras.

4. Finally the system could be definable only on an interval of ordinals of the form $[\alpha, \beta]$, while the usual definition of Monk’s schemes defines systems of varieties on $I_\omega$; without this more general condition, we would have not been able to approach Sain’s algebras.

An operator that features prominently in systems of varieties defined earlier is the neat reduct operator, which can be viewed as functor from algebras to algebras of a lesser dimension. The definition of neat reducts for Monk’s schemes is fairly straightforward. By allowing infinitary substitutions possibly moving infinitely many points, the definition becomes more intricate, and it needs caution.

**Definition 61.** (1) Let $(K_\alpha : \alpha \geq \mu)$ be a system of varieties. For $\alpha < \beta$, both in $\mu$, and $\tau \in G_\alpha$ we assume that $\bar{\tau} = \tau \cup Id \in G_\beta$. We also assume that if $r \in C_\alpha$, then $r \in C_\beta$ for every $\beta > \alpha$ in $\mu$ Given $A \in K_\beta$, $A = (B, q(r), c(r), s_\tau)_{r \in C_\alpha, \tau \in G_\beta}$, say, we define $N\alpha A = (N\alpha B, q(r), c(r), s_\tau)_{r \in C_\beta, \tau \in G_\alpha}$.
where $\mathfrak{R}_\alpha \mathcal{B}$ is the reduct obtained from $\mathcal{B}$ by allowing only operations whose indices come from $\alpha$, and discard the rest.

(2) Given $\mathfrak{A} \in K_\beta$ and $x \in A$, $Nr_\alpha \mathfrak{A} = \{x \in A : \forall r \subseteq (\beta \sim \alpha), r \in C_\alpha; c(r)x = x\}$.

(3) We assume that for all $f \in T \sim \{c, s\}$, $\mathfrak{A} \in K_\beta$ and $\alpha < \beta \in \mu$, if $r \subseteq \varphi(\beta \sim \alpha) \cap C_\alpha$ and $c(r)x = x$, then $c(r)f_{i_0, \ldots, i_{n-1}}(\bar{x}) = f_{i_0, \ldots, i_{n-1}}(\bar{x})$; here $n = \delta(f)$ and $|\bar{x}| = \rho(f)$. Same for $q$. Furthermore if $\tau \in G_\beta$ and $\tau \restriction (\beta \sim \alpha) = Id_{\beta \sim \alpha}$, then for all $r \subseteq \varphi(\beta \sim \alpha) \cap C_\alpha$, we have $c(r)s_\tau x = s_\tau x$ (this is a very reasonable condition, because the indices moved by the substitution lie outside the scope of the generalized cylindrifier).

Then $\mathfrak{N}_\alpha \mathcal{B}$ is the subalgebra of $\mathfrak{R}_\alpha \mathcal{B}$ with universe $Nr_\alpha \mathcal{B}$. This is well defined.

(4) For $L \subseteq K_\beta$, and $\alpha < \beta$, $\mathfrak{N}_\alpha L = \{\mathfrak{N}_\alpha \mathfrak{A} : \mathfrak{A} \in K_\beta\}$.

(5) If $\alpha, \alpha + \omega \in \mu$, we set $\mathfrak{K}_\omega = S\mathfrak{N}_\omega \mathfrak{K}_{\omega + \omega}$.

Since in generalized systems, $K_\omega$ specifies higher dimensions uniquely, it is reasonable to formulate our results for only $\omega$ dimensional algebras. This is no real restriction; what can be proved for $\omega$ can be proved for any larger ordinal in the interval defining the system. From now on, we assume that $\omega$ and $\omega + \omega$ are in the interval defining systems addressed.

Call a system of varieties nice if $\mathfrak{K}_\omega$ has the amalgamation property, and call it very nice if $\mathfrak{K}_\omega$ has the superamalgamation property. Our next theorem shows that given that $\mathfrak{M}$ enjoys a strong form of amalgamation (which happens often, like incylindric algebras and quasipolyadic algebras with and without equality and Pinter’s substitution algebras, the amalgamation property is actually equivalent to the adjointness of the neat reduct functor, while the superamalgamation property is equivalent to its strong invertibility.

Using metaphysical jargon, yet again, the next theorem is the heart and soul of this paper, formulated rigorously in the dialect of category theory and algebraic logic:

**Theorem 62.** Let $\mathcal{K} = (K_\alpha : \alpha \in \mu)$ be a system of varieties, such that $\omega$ and $\omega + \omega \in \mu$. Assume that $\mathfrak{M} = \{\mathfrak{A} \in K_{\omega + \omega} : \mathfrak{A} = \mathfrak{N}_\omega \mathfrak{A}\}$ has SUPAP. Assume further that for any injective homomorphism $f : \mathfrak{N}_\alpha \mathcal{B} \to \mathfrak{N}_\alpha \mathcal{B'}$, there exists an injective homomorphism $g : \mathcal{B} \to \mathcal{B'}$ such that $f \subseteq g$. Then the following two conditions are equivalent.

(1) $\mathfrak{K}_\omega$ is (very) nice.

(2) $\mathfrak{N}_\omega$ is (strongly) invertible.
Proof.

(1) Assume that $\mathbf{K}_{\omega}$ has the amalgamation property. We first show that $\mathbf{K}_{\omega}$ has the following unique neat embedding property: If $i_1 : \mathcal{A} \to \mathbf{Nr}_\omega \mathcal{B}_1$, $i_2 : \mathcal{A} \to \mathbf{Nr}_\omega \mathcal{B}_1$ are such that $i_1(A)$ generates $\mathcal{B}_1$ and $i_2(A)$ generates $\mathcal{B}_2$, then there is an isomorphism $f : \mathcal{B}_1 \to \mathcal{B}_2$ such that $f \circ i_1 = i_2$.

By assumption, there is an amalgam, that is there is $\mathcal{D} \in \mathbf{K}_\omega$, $m_1 : \mathbf{Nr}_\omega \mathcal{B}_1 \to \mathcal{D}$, $m_2 : \mathbf{Nr}_\omega \mathcal{B}_2 \to \mathcal{D}$ such that $m_1 \circ i_1 = m_2 \circ i_2$. We can assume that $m_1 : \mathbf{Nr}_\omega \mathcal{B} \to \mathbf{Nr}_\omega \mathcal{D}^+$ for some $\mathcal{D}^+ \in \mathcal{M}$, and similarly for $m_2$. By hypothesis, let $m_1 : \mathcal{B}_1 \to \mathcal{D}^+$ and $m_2 : \mathcal{B}_2 \to \mathcal{D}^+$ be isomorphisms extending $m_1$ and $m_2$. Then since $i_1A$ generates $\mathcal{B}_1$ and $i_2A$ generates $\mathcal{B}_2$, then $m_1 \mathcal{B}_1 = m_2 \mathcal{B}_2$. It follows that $f = m_2^{-1} \circ m_1$ is as desired. From this it easily follows that $\mathbf{Nr}$ has universal maps and we are done.

In fact, the uniqueness property established above, call it $UNEP$, is equivalent to existence of universal maps; this is quite easy to show, hence to prove the converse, we assume $UNEP$, and we set out to prove that $\mathbf{K}_\omega$ has $AP$.

Let $\mathcal{A}, \mathcal{B} \in \mathbf{K}_\omega$. Let $f : \mathcal{C} \to \mathcal{A}$ and $g : \mathcal{C} \to \mathcal{B}$ be injective homomorphisms. Then there exist $\mathcal{A}^+, \mathcal{B}^+, \mathcal{C}^+ \in \mathbf{K}_{\omega + 1}$, $e_A : \mathcal{A} \to \mathbf{Nr}_{\omega + 1} \mathcal{A}^+$, $e_B : \mathcal{B} \to \mathbf{Nr}_{\omega + 1} \mathcal{B}^+$ and $e_C : \mathcal{C} \to \mathbf{Nr}_{\omega + 1} \mathcal{C}^+$. We can assume that $\mathcal{S}g^{\mathcal{A}^+} e_A(A) = \mathcal{A}^+$ and similarly for $\mathcal{B}^+$ and $\mathcal{C}^+$. Let $f(C)^+ = \mathcal{S}g^{\mathcal{A}^+} e_A(f(C))$ and $g(C)^+ = \mathcal{S}g^{\mathcal{B}^+} e_B(g(C))$. Since $\mathcal{C}$ has $UNEP$, there exist $\tilde{f} : \mathcal{C}^+ \to f(C)^+$ and $\tilde{g} : \mathcal{C}^+ \to g(C)^+$ such that $(e_A \upharpoonright f(C)) \circ f = \tilde{f} \circ e_C$ and $(e_B \upharpoonright g(C)) \circ g = \tilde{g} \circ e_C$. Now $\mathcal{M}$ as $SUPAP$, hence there is a $\mathcal{D}^+$ in $\mathcal{M}$ and $k : \mathcal{A}^+ \to \mathcal{D}^+$ and $h : \mathcal{B}^+ \to \mathcal{D}^+$ such that $k \circ \tilde{f} = h \circ \tilde{g}$. Then $k \circ e_A : \mathcal{A} \to \mathbf{Nr}_\omega \mathcal{D}^+$ and $h \circ e_B : \mathcal{B} \to \mathbf{Nr}_\omega \mathcal{D}^+$ are one to one and $k \circ e_A \circ f = h \circ e_B \circ g$.

(2) Now for the second equivalence. Assume that $\mathbf{K}_\omega$ has $SUPAP$. Then, a fortiori, it has $AP$ hence, by the above argument, it has $UNEP$. We first show that if $\mathcal{A} \subseteq \mathbf{Nr}_\omega \mathcal{B}$ and $\mathcal{A}$ generates $\mathcal{B}$ then equality holds, we call this property $NS$, short for neat reducts commuting with forming subalgebras.

If not, then $\mathcal{A} \subseteq \mathbf{Nr}_\omega \mathcal{B}$, $\mathcal{B} \in K$, $A$ generates $\mathcal{B}$ and $\mathcal{A} \neq \mathbf{Nr}_\omega \mathcal{B}$. Then $\mathcal{A}$ embeds into $\mathbf{Nr}_\omega \mathcal{B}$ via the inclusion map $i$. Let $\mathcal{C} = \mathbf{Nr}_\omega \mathcal{B}$. By $SUPAP$, there exists $\mathcal{D} \in \mathbf{K}_\omega$ and $m_1, m_2$ monomorphisms from $\mathcal{C}$ to $\mathcal{D}$ such that $m_1(\mathcal{C}) \cap m_2(\mathcal{C}) = m_1 \circ i(\mathcal{A})$. Let $y \in \mathcal{C} \sim A$. Then $m_1(y) \neq m_2(y)$ for else $d = m_1(y) = m_2(y)$ will be in $m_1(\mathcal{C}) \cap m_2(\mathcal{C})$ but not in $m_1 \circ i(\mathcal{A})$. Assume that $\mathcal{D} \subseteq \mathbf{Nr}_\omega \mathcal{D}^+$ with $\mathcal{D}^+ \in K$. By hypothesis,
there exist injections $\bar{m}_1 : B \to D^+$ and $\bar{m}_2 : B \to D^+$ extending $m_1$ and $m_2$. But $A$ generates $B$ and so $\bar{m}_1 = \bar{m}_2$. Thus $m_1 y = m_2 y$ which is a contradiction.

Now let $\beta = \alpha + \omega$. Let $M = \{ A \in K_\beta : A = S g A, \omega A \}$. Let $\text{Nr} : M \to Kn_\omega$ be the neat reduct functor. We show that $\text{Nr}$ is strongly invertible, namely there is a functor $G : Kn_\omega \to M$ and natural isomorphisms $\mu : 1_M \to G \circ \text{Nr}$ and $\epsilon : \text{Nr} \circ G \to 1_{Kn_\omega}$. Let $L$ be a system of representatives for isomorphism on $Ob(M)$. For each $B \in Ob(Kn_\omega)$ there is a unique $G(B)$ in $M$ such that $\text{Nr}(G(B)) \cong B$. Then $G : Ob(Kn_\omega) \to Ob(M)$ is well defined. Choose one isomorphism $\epsilon_B : \text{Nr}(G(B)) \to B$.

If $g : B \to B'$ is a $Kn_\omega$ morphism, then the square

\[
\begin{array}{ccc}
\text{Nr}(G(B)) & \xrightarrow{\epsilon_B} & B \\
\downarrow{\epsilon_B^{-1} \circ g \circ \epsilon_{B'}} & & \downarrow{g} \\
\text{Nr}(G(B')) & \xrightarrow{\epsilon_{B'}} & B'
\end{array}
\]

commutes. There is a unique morphism $f : G(B) \to G(B')$ such that $\text{Nr}(f) = \epsilon_B^{-1} \circ g \circ \epsilon_{B'}$. We let $G(g) = f$. Then it is easy to see that $G$ defines a functor. Also, by definition $\epsilon = (\epsilon_B)$ is a natural isomorphism from $\text{Nr} \circ G$ to $1_{Kn_\omega}$. To find a natural isomorphism from $1_M$ to $G \circ \text{Nr}$, observe that that for each $A \in Ob(M)$, $\epsilon_A : \text{Nr} \circ G \circ \text{Nr}(A) \to \text{Nr}(A)$ is an isomorphism. Then there is a unique $\mu_A : A \to G \circ \text{Nr}(A)$ such that $\text{Nr}(\mu_A) = \epsilon^{-1}_A$. Since $\epsilon^{-1}$ is natural for any $f : A \to A'$ the square

\[
\begin{array}{ccc}
\text{Nr}(A) & \xrightarrow{\epsilon_A^{-1}} & \text{Nr} \circ G \circ \text{Nr}(A) \\
\downarrow{\text{Nr}(f)} & & \downarrow{\text{Nr} \circ G \circ \text{Nr}(f)} \\
\text{Nr}(A') & \xrightarrow{\epsilon_{A'}^{-1}} & \text{Nr} \circ G \circ \text{Nr}(A')
\end{array}
\]

commutes, hence the square

\[
\begin{array}{ccc}
A & \xrightarrow{\mu_A} & G \circ \text{Nr}(A) \\
\downarrow{f} & & \downarrow{G \circ \text{Nr}(f)} \\
A' & \xrightarrow{\mu_A} & G \circ \text{Nr}(A')
\end{array}
\]

commutes, too. Therefore $\mu = (\mu_A)$ is as required.
Conversely, assume that the functor $\mathcal{N}r$ is invertible. Then we have the $UNEP$ and the $NS$. The $UNEP$ follows from the fact that the functor has a right adjoint, and so it has universal maps. To prove that it has $NS$ assume for contradiction that there exists $\mathfrak{A}$ generating subreduct of $\mathfrak{B}$ and $\mathfrak{A}$ is not isomorphic to $\mathcal{N}r_\alpha \mathfrak{B}$. This means that $Nr$ is not invertible, because had it been invertible, with inverse $Dl$, then $Dl(\mathfrak{A}) = Dl(\mathcal{N}r_\alpha \mathfrak{B})$ and this cannot happen.

Now we prove that $Kn_\omega$ has $SUPAP$. We obtain (using the notation in the first part) $D \in \mathcal{N}r_\alpha K_{\alpha + \omega}$ and $m : \mathfrak{A} \to D, n : \mathfrak{B} \to D$ such that $m \circ f = n \circ g$. Here $m = k \circ e_A$ and $n = h \circ e_B$. Denote $k$ by $m^+$ and $h$ by $n^+$. Suppose that $\mathcal{C}$ has $SNEP$. We further want to show that if $m(a) \leq n(b)$, for $a \in A$ and $b \in B$, then there exists $t \in C$ such that $a \leq f(t)$ and $g(t) \leq b$. So let $a$ and $b$ be as indicated. We have $m^+ \circ e_A(a) \leq n^+ \circ e_B(b)$, so $m^+(e_A(a)) \leq n^+(e_B(b))$. Since $\mathcal{M}$ has $SUPAP$, there exist $z \in C^+$ such that $e_A(a) \leq \bar{f}(z)$ and $\bar{g}(z) \leq e_B(b)$. Let $\Gamma = \Delta z \sim \alpha$ and $z' = c_{(\Gamma)}z$. So, we obtain that $e_A(c_{(\Gamma)}a) \leq \bar{f}(c_{(\Gamma)}z)$ and $\bar{g}(c_{(\Gamma)}z) \leq e_B(c_{(\Gamma)}b)$. It follows that $e_A(a) \leq \bar{f}(z')$ and $\bar{g}(z') \leq e_B(b)$. Now by hypothesis

$$z' \in \mathcal{N}r_\alpha \mathcal{C}^+ = \mathcal{S}g^{\mathcal{N}r_\alpha \mathcal{C}^+}(e_C(C)) = e_C(C).$$

So, there exists $t \in C$ with $z' = e_C(t)$. Then we get $e_A(a) \leq \bar{f}(e_C(t))$ and $\bar{g}(e_C(t)) \leq e_B(b)$. It follows that $e_A(a) \leq e_A \circ f(t)$ and $e_B \circ g(t) \leq e_B(b)$. Hence, $a \leq f(t)$ and $g(t) \leq b$.

\section{Another adjoint situation for finite dimensions}

\textbf{Definition 63.} Let $C \in CA_\alpha$ and $I \subseteq \alpha$, and let $\beta$ be the order type of $I$. Then

$$Nr_I C = \{ x \in C : c_i x = x \text{ for all } i \in \alpha \sim I \},$$

$$\mathcal{N}r_I \mathfrak{C} = (Nr_I C, +, \cdot, -, 0, 1, c_{\rho_i}, d_{\rho_i, \rho_j})_{i,j < \beta},$$

where $\beta$ is the unique order preserving one-to-one map from $\beta$ onto $I$, and all the operations are the restrictions of the corresponding operations on $C$. When $I = \{ i_0, \ldots, i_{k-1} \}$ we write $\mathcal{N}r_{i_0 \ldots i_{k-1}} \mathfrak{C}$. If $I$ is an initial segment of $\alpha$, $\beta$ say, we write $\mathcal{N}r_\beta \mathfrak{C}$.

Similar to taking the $n$ neat reduct of a $CA$, $\mathfrak{A}$ in a higher dimension, is taking its $\mathfrak{A} \alpha$ reduct, its relation algebra reduct. This has universe consisting of the 2 dimensional elements of $\mathfrak{A}$, and composition and converse are defined
using one spare dimension. A slight generalization, modulo a reshuffling of the indices:

**Definition 64.** For \( n \geq 3 \), the relation algebra reduct of \( C \in CA_n \) is the algebra

\[
RaC = (Nr_{n-2,n-1}C, +, \cdot, 1, 1')
\]

where \( 1' = d_{n-2,n-1}, \tilde{x} = s_{n-1}^0 s_{n-1}^0 x \) and \( x; y = c_0(s_{n-1}^0 x, s_{n-1}^0 y) \). Here \( s_i^j(x) = c_i(x \cdot d_{ij}) \) when \( i \neq q \) and \( s_i^j(x) = x \).

But what is not obvious at all is that an RA has a CA\(_n\) reduct for \( n \geq 3 \). But Simon showed that certain relations algebras do; namely the QRA\(_s\).

**Definition 65.** A relation algebra \( B \) is a QRA if there are elements \( p, q \) in \( B \) satisfying the following equations:

1. \( \tilde{p}; p \leq 1', q; q \leq 1; \)
2. \( \tilde{p}; q = 1. \)

In this case we say that \( B \) is a QRA with quasi-projections \( p \) and \( q \). To construct cylindric algebras of higher dimensions 'sitting' in a QRA, we need to define certain terms. seemingly rather complicated, their intuitive meaning is not so hard to grasp.

**Definition 66.** Let \( x \in B \in RA \), then dom\(_n\)(\( x \)) = 1'; \( (x; \tilde{x}) \) and ran\(_n\)(\( x \)) = 1'; \( (\tilde{x}; x) \), \( x^0 = 1' \), \( x^{n+1} = x^n \); \( x \). \( x \) is a functional element if \( x; \tilde{x} \leq 1' \).

Given a QRA, which we denote by \( Q \), we have quasi-projections \( p \) and \( q \) as mentioned above. Next we define certain terms in \( Q \), cf. [23]:

\[
\epsilon^n = dom q^{n-1}, \quad \pi^i_n = \epsilon^n; q^i; p, i < n - 1, \pi^{(n)}_{n-1} = q^{n-1},
\]

\[
\xi^{(n)} = \pi^{(n)}_i; \pi^{(n)}_i, \quad \ell^{(n)}_i = \prod_{i \neq j < n} \xi^{(n)}_j, \quad \ell^{(n)} = \prod_{j < n} \xi^{(n)}_j,
\]

\[
c^{(n)}_i x = x; \ell^{(n)}_i, \quad d^{(n)}_{ij} = 1; (\pi^{(n)}_i, \pi^{(n)}_j),
\]

\[
1^{(n)} = 1; \epsilon^{(n)}.
\]

and let

\[
B_n = (B_n, +, \cdot, -0, 1^{(n)}_n, c^{(n)}_i, d^{(n)}_{ij})_{i,j<n},
\]

where \( B_n = \{ x \in B : x = 1; x; \ell^{(n)} \} \). The intuitive meaning of those terms is explained in [23], right after their definition on p. 271.
Theorem 67. Let $n > 1$

1. Then $\mathcal{B}_n$ is closed under the operations.

2. $\mathcal{B}_n$ is a CA$_n$.

Proof. (1) is proved in [23] lemma 3.4 p.273-275 where the terms are definable in a QRA. That it is a CA$_n$ can be proved as [23] theorem 3.9.

Definition 68. Consider the following terms.

$suc(x) = 1; (\bar{p}; x; \bar{q})$

and

$pred(x) = \bar{p}; ran;x; q$.

It is proved in [23] that $\mathcal{B}_n$ neatly embeds into $\mathcal{B}_{n+1}$ via $suc$. The successor function thus codes extra dimensions. The thing to observe here is that we will see that $pred$; its inverse; guarantees a condition of commutativity of two operations: forming neat reducts and forming subalgebras; it does not make a difference which operation we implement first, as long as we implement both one after the other. So the function $suc$ captures the extra dimensions added.

From the point of view of definability it says that terms definable in extra dimensions add nothing, they are already term definable. And this indeed is a definability condition, that will eventually lead to strong interpolation property we want.

Theorem 69. Let $n \geq 3$. Then $suc : \mathcal{B}_n \rightarrow \{a \in \mathcal{B}_{n+1} : c_0a = a\}$ is an isomorphism into a generalized neat reduct of $\mathcal{B}_{n+1}$. Strengthening the condition of surjectivity, for all $X \subseteq \mathcal{B}_n$, $n \geq 3$, we have (*)

$suc(\mathcal{G}\mathcal{G}^\mathcal{B}_nX) \cong \mathcal{N}t_{1,2,\ldots,n}\mathcal{G}\mathcal{G}^\mathcal{B}_{n+1}suc(X)$.

Proof. The operations are respected by [23] theorem 5.1. The last condition follows because of the presence of the functional element $pred$, since we have $suc(predx) = x$ and $pred(sucx) = x$, when $c_0x = x$, [23] lemmas 4.6-4.10.

Theorem 70. Let $n \geq 3$. Let $\mathcal{C}_n$ be the algebra obtained from $\mathcal{B}_n$ by reshuffling the indices as follows; set $c_0^\mathcal{C}_n = c_0^\mathcal{B}_n$ and $c_n^\mathcal{C}_n = c_0^\mathcal{B}_n$. Then $\mathcal{C}_n$ is a cylindric algebra, and $suc : \mathcal{C}_n \rightarrow \mathcal{N}t_{n}\mathcal{C}_{n+1}$ is an isomorphism for all $n$. Furthermore, for all $X \subseteq \mathcal{C}_n$ we have

$suc(\mathcal{G}\mathcal{C}_nX) \cong \mathcal{N}t_{n}\mathcal{G}\mathcal{C}_{n+1}suc(X)$.

Proof. immediate from [69]
**Theorem 71.** Let \( C_n \) be as above. Then \( \text{succ}^m : C_n \to \text{Nr}_n C_m \) is an isomorphism, such that for all \( X \subseteq A \), we have
\[
\text{suc}^m(\text{Sg}^\epsilon_n X) = \text{Nr}_n \text{Sg}^\epsilon_m \text{suc}^{n-1}(X).
\]

*Proof.* By induction on \( n \).

Now we want to neatly embed our QRA in \( \omega \) extra dimensions. At the same we do not want to lose, our control over the stretching; we still need the commutating of taking, now \( \text{Ra} \) reducts with forming subalgebras; we call this property the \( \text{RaS} \) property. To construct the big \( \omega \) dimensional algebra, we use a standard ultraproduct construction. So here we go. For \( n \geq 3 \), let \( C_n^+ \) be an algebra obtained by adding \( c_i \) and \( d_{ij} \)'s for \( \omega > i, j \geq n \) arbitrariness and with \( \text{Ra}_n^+ C_n^+ = B_n \). Let \( C = \prod_{n \geq 3} C_n^+ / G \), where \( G \) is a non-principal ultrafilter on \( \omega \). In our next theorem, we show that the algebra \( A \) can be neatly embedded in a locally finite algebra \( \omega \) dimensional algebra and we retain our \( \text{RaS} \) property.

**Theorem 72.** Let
\[
i : A \to \text{RaC}
\]
be defined by
\[
x \mapsto (x, \text{suc}(x), \ldots \text{suc}^{n-1}(x), \ldots n \geq 3, x \in B_n) / G.
\]
Then \( i \) is an embedding, and for any \( X \subseteq A \), we have
\[
i(\text{Sg}^\epsilon A X) = \text{RaS} \text{g}^\epsilon i(X).
\]

*Proof.* The idea is that if this does not happen, then it will not happen in a finite reduct, and this impossible [47].

**Theorem 73.** Let \( Q \in \text{RA} \). Then for all \( n \geq 4 \), there exists a unique \( A \in S\text{Nr}_3 \text{CA}_n \) such that \( Q = \text{RaA} \), such that for all \( X \subseteq A \), \( \text{Sg}^Q X = \text{RaSg}^\epsilon X \).

*Proof.* This follows from the previous theorem together with \( \text{RaS} \) property.

**Corollary 74.** Assume that \( Q = \text{RaA} \cong \text{RaB} \) then this lifts to an isomorphism from \( A \) to \( B \).

The previous theorem says that \( \text{Ra} \) as a functor establishes an equivalence between QRA and a reflective subcategory of \( \text{Lf}_\omega \). We say that \( A \) is the \( \omega \) dilation of \( Q \). Now we are ready for:

**Theorem 75.** QRA has SUPAP.
Proof. We form the unique dilatons of the given algebras required to be superamalgamated. These are locally finite so we can find a superamalgam \( \mathfrak{D} \). Then \( \mathfrak{RaD} \) will be required superamalgam; it contains quasiprojections because the base algebras does. Let \( \mathfrak{A}, \mathfrak{B} \in \text{QRA} \). Let \( f : \mathfrak{C} \to \mathfrak{A} \) and \( g : \mathfrak{C} \to \mathfrak{B} \) be injective homomorphisms. Then there exist \( \mathfrak{A}^+, \mathfrak{B}^+, \mathfrak{C}^+ \in \mathfrak{CA}_{\alpha+\omega}, e_A : \mathfrak{A} \to \mathfrak{RaA}^+, e_B : \mathfrak{B} \to \mathfrak{RaB}^+ \) and \( e_C : \mathfrak{C} \to \mathfrak{RaC}^+ \). We can assume, without loss, that \( \mathfrak{G}g^{\mathfrak{a}^+}e_A(A) = \mathfrak{A}^+ \) and similarly for \( \mathfrak{B}^+ \) and \( \mathfrak{C}^+ \). Let \( f(C)^+ = \mathfrak{G}g^{\mathfrak{a}^+}e_A(f(C)) \) and \( g(C)^+ = \mathfrak{G}g^{\mathfrak{a}^+}e_B(g(C)) \). Since \( \mathfrak{C} \) has \( \text{UNEP} \), there exist \( \bar{f} : \mathfrak{C}^+ \to f(C)^+ \) and \( \bar{g} : \mathfrak{C}^+ \to g(C)^+ \) such that \( (e_A \upharpoonright f(C)) \circ f = \bar{f} \circ e_C \) and \( (e_B \upharpoonright g(C)) \circ g = \bar{g} \circ e_C \). Both \( \bar{f} \) and \( \bar{g} \) are monomorphisms. Now \( Lf_\omega \) has \( \text{SUPAP} \), hence there is a \( \mathfrak{D}^+ \) in \( K \) and \( k : \mathfrak{A}^+ \to \mathfrak{D}^+ \) and \( h : \mathfrak{B}^+ \to \mathfrak{D}^+ \) such that \( k \circ \bar{f} = h \circ \bar{g} \) and \( k \) and \( h \) are also monomorphisms. Then \( k \circ e_A : \mathfrak{A} \to \mathfrak{RaD}^+ \) and \( h \circ e_B : \mathfrak{B} \to \mathfrak{RaD}^+ \) are one to one and \( k \circ e_A \circ f = h \circ e_B \circ g \). Let \( \mathfrak{D} = \mathfrak{RaD}^+ \). Then we obtained \( \mathfrak{D} \in \text{QRA} \) and \( m : \mathfrak{A} \to \mathfrak{D} n : \mathfrak{B} \to \mathfrak{D} \) such that \( m \circ f = n \circ g \). Here \( m = k \circ e_A \) and \( n = h \circ e_B \). Denote \( k \) by \( m^+ \) and \( h \) by \( n^+ \). Now suppose that \( \mathfrak{C} \) has \( \text{NS} \). We further want to show that if \( m(a) \leq n(b) \), for \( a \in A \) and \( b \in B \), then there exists \( t \in C \) such that \( a \leq f(t) \) and \( g(t) \leq b \). So let \( a \) and \( b \) be as indicated. We have \( (m^+ \circ e_A)(a) \leq (n^+ \circ e_B)(b) \), so \( m^+(e_A(a)) \leq n^+(e_B(b)) \). Since \( K \) has \( \text{SUPAP} \), there exist \( z \in C^+ \) such that \( e_A(a) \leq \tilde{f}(z) \) and \( \tilde{g}(z) \leq e_B(b) \). Let \( \Gamma = \Delta z \sim \alpha \) and \( z' = c_{(\Gamma)}z \). (Note that \( \Gamma \) is finite.) So, we obtain that \( e_A(c_{(\Gamma)}a) \leq \tilde{f}(c_{(\Gamma)}z) \) and \( \tilde{g}(c_{(\Gamma)}z) \leq e_B(c_{(\Gamma)}b) \). It follows that \( e_A(a) \leq \tilde{f}(z') \) and \( \tilde{g}(z') \leq e_B(b) \). Now by hypothesis

\[
z' \in \mathfrak{RaC}^+ = \mathfrak{G}g^{\mathfrak{a}^+}(e_C(C)) = e_C(C).
\]

So, there exists \( t \in C \) with \( z' = e_C(t) \). Then we get \( e_A(a) \leq \tilde{f}(e_C(t)) \) and \( \tilde{g}(e_C(t)) \leq e_B(b) \). It follows that \( e_A(a) \leq (e_A \circ f)(t) \) and \( (e_B \circ g)(t) \leq e_B(b) \). Hence, \( a \leq f(t) \) and \( g(t) \leq b \). We are done. \( \square \)

One can prove the theorem using the dimension restricted free algebra \( B = \mathfrak{F}_1^{\omega} \mathfrak{CA}_\omega \), where \( \rho(0) = 2 \). This corresponds to a countable first order language with a sequence of variables of order type \( \omega \) and one binary relation. The idea is that \( \mathfrak{F}_1 \text{QRA} \cong \mathfrak{RaF}_1^{\omega} \mathfrak{CA}_\omega \). So let \( a, b \in \mathfrak{F}_1 \text{QRA} \) be such that \( a \leq b \). Then there exists \( y \in \mathfrak{G}g^{\mathfrak{a}^+}\{x\} \) were \( x \) is the free generator of both, such that \( a \leq y \leq b \).

But we need to show that pairing functions can be defined in \( \mathfrak{RaF}_1^{\omega} \mathfrak{CA}_\omega \) We have one binary relation \( E \) in our language; for convenience, we write \( x \in y \) instead of \( E(x, y) \), to remind ourselves that we are actually working in the language of set theory. We define certain formulas culminating in formulating the axioms of a finitely undecidable theory, better known as Robinson’s arithmetic in our language. These formulas are taken from Németi [?]. (This is not the only way to define quasi-projections) We need to define, the quasi projections. Quoting Andréka and Németi in [2], we do this by 'brute force'.

81
Now we define the pairing functions:

- \( p_0(x, y) =: \text{pair}(x) \land \{y\} \in x \)
- \( p_1(x, y) =: \text{pair}(x) \land [x = \{y\} \lor \{y\} \notin x \land y \in \bigcup x] \).

\( p_0(x, y) \) and \( p_1(x, y) \) are defined.

8.1 Pairing functions in Németis directed CAs

We recall the definition of what is called weakly higher order cylindric algebras, or directed cylindric algebras invented by Németi and further studied by Sági and Simon. Weakly higher order cylindric algebras are natural expansions of cylindric algebras. They have extra operations that correspond to a certain kind of bounded existential quantification along a binary relation \( R \). The relation \( R \) is best thought of as the ‘element of relation’ in a model of some set theory. It is an abstraction of the membership relation. These cylindric-like algebras are the cylindric counterpart of quasi-projective relation algebras, introduced by Tarski. These algebras were studied by many authors including Andráka, Givant, Németi, Maddux, Sági, Simon, and others. The reference [23] is recommended for other references in the topic. It also has reincarnations in Computer Science literature under the name of Fork algebras. We start by recalling the concrete versions of directed cylindric algebras:

**Definition 76.** (P–structures and extensional structures.)

Let \( U \) be a set and let \( R \) be a binary relation on \( U \). The structure \( \langle U; R \rangle \) is defined to be a P–structure\(^2\) iff for every elements \( a, b \in U \) there exists an element \( c \in U \) such that \( R(d, c) \) is equivalent with \( d = a \) or \( d = b \) (where \( d \in U \) is arbitrary) , that is,

\[
\langle U; R \rangle \models (\forall x, y)(\exists z)(\forall w)(R(w, z) \iff (w = x \lor w = y)).
\]

The structure \( \langle U; R \rangle \) is defined to be a weak P–structure iff

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\(^2\)“P” stands for “pairing” or “pairable”.

82
\[ \langle U; R \rangle \models (\forall x, y)(\exists z)(R(x, z) \text{ and } R(y, z)). \]

The structure \( \langle U; R \rangle \) is defined to be extensional iff every two points \( a, b \in U \) coincide whenever they have the same “\( R \)–children”, that is,
\[ \langle U; R \rangle \models (\forall x, y)(((\forall z)R(z, x) \iff R(z, y)) \Rightarrow x = y). \]

We will see that if \( \langle U; R \rangle \) is a P–structure then one can “code” pairs of elements of \( U \) by a single element of \( U \) and whenever \( \langle U; R \rangle \) is extensional then this coding is “unique”. In fact, in \( \text{RCA}^{\uparrow}_3 \) (see the definition below) one can define terms similar to quasi–projections and, as with the class of \( \text{QRA} \)'s, one can equivalently formalize many theories of first order logic as equational theories of certain \( \text{RCA}^{\uparrow}_3 \)'s. Therefore \( \text{RCA}^{\uparrow}_3 \) is in our main interest. \( \text{RCA}^{\uparrow}_\alpha \) for bigger \( \alpha \)'s behave in the same way, an explanation of this can be found in [32] and can be deduced from our proof, which shows that \( \text{RCA}^{\uparrow}_3 \) has implicitly \( \omega \) extra dimensions.

**Definition 77.** \((\text{Cs}^{\uparrow}_\alpha, \text{RCA}^{\uparrow}_\alpha)\)

Let \( \alpha \) be an ordinal. Let \( U \) be a set and let \( R \) be a binary relation on \( U \) such that \( \langle U; R \rangle \) is a weak P–structure. Then the full w–directed cylindric set algebra of dimension \( \alpha \) with base structure \( \langle U; R \rangle \) is the algebra:
\[ \langle \mathcal{P}(\alpha U); \cap, -, C_i^{\uparrow(R)}, C_i^{\downarrow(R)}, D_{i,j}^U \rangle \] for \( i,j \in \alpha \),

where \( \cap \) and \( - \) are set theoretical intersection and complementation (w.r.t. \( \alpha U \)), respectively, \( D_{i,j}^U = \{ s \in \alpha U : s_i = s_j \} \) and \( C_i^{\uparrow(R)}, C_i^{\downarrow(R)} \) are defined as follows. For every \( X \in \mathcal{P}(\alpha U) \):
\[ C_i^{\uparrow(R)}(X) = \{ s \in \alpha U : (\exists z \in X)(R(z_i, s_i) \text{ and } (\forall j \in \alpha)(j \neq i \Rightarrow s_j = z_j)) \} \]
\[ C_i^{\downarrow(R)}(X) = \{ s \in \alpha U : (\exists z \in X)(R(s_i, z_i) \text{ and } (\forall j \in \alpha)(j \neq i \Rightarrow s_j = z_j)) \} \]

The class of w–directed cylindric set algebras of dimension \( \alpha \) and the class of directed cylindric set algebras of dimension \( \alpha \) are defined as follows.

\( \text{w–Cs}^{\uparrow}_\alpha = \{ \mathcal{A} : \mathcal{A} \text{ is a full w–directed cylindric set algebra of dimension } \alpha \text{ with base structure } \langle U; R \rangle \text{, for some weak P–structure } \langle U; R \rangle \} \)

\( \text{Cs}^{\uparrow}_\alpha = \{ \mathcal{A} : \mathcal{A} \text{ is a full w–directed cylindric set algebra of dimension } \alpha \text{ with base structure } \langle U; R \rangle \text{, for some extensional P–structure } \langle U; R \rangle \} \)

The class \( \text{RCA}^{\uparrow}_\alpha \) of representable directed cylindric algebras of dimension \( \alpha \) is defined to be \( \text{RCA}^{\uparrow}_\alpha = \text{SPCs}^{\uparrow}_\alpha \).
The main result of Sagi in [32] is a direct proof for the following:

**Theorem 78.** \( \text{RCA}_\alpha^\uparrow \) is a finitely axiomatizable variety whenever \( \alpha \geq 3 \) and \( \alpha \) is finite

\( \text{CA}_3^\uparrow \) denotes the variety of directed cylindric algebras of dimension 3 as defined in [32] definition 3.9. In [32], it is proved that \( \text{CA}_3^\uparrow = \text{RCA}_3^\uparrow \). A set of axioms is formulated on p. 868 in [32]. Let \( \mathfrak{A} \in \text{CA}_3^\uparrow \). Then we have quasi-projections \( p, q \) defined on \( \mathfrak{A} \) as defined in [32] p. 878, 879. We recall their definition, which is a little bit complicated because they are defined as formulas in the corresponding second order logic. Let \( \mathcal{L} \) denote the untyped logic corresponding to directed \( \text{CA}_3^\uparrow \)'s as defined p.876-877 in [32]. It has only 3 variables. There is a correspondance between formulas (or formulam schemes) in this language and \( \text{CA}_3^\uparrow \) terms. This is completely analogous to the corresponsonce between \( \text{RCA}_n \) terms and first order formulas containing only \( n \) variables. For example \( v_i = v_j \) corresponds to \( d_{ij} \), \( \exists^i v_i(v_i = v_j) \) correspond to \( \exists i d_{ij} \). In [32] the following formulas (terms) are defined:

**Definition 79.** Let \( i, j, k \in 3 \) distinct elements. We define variable–free \( \text{RCA}_3^\uparrow \) terms as follows:

\[
\begin{align*}
v_i \in_R v_j & \quad \text{is} \quad \exists^i v_j(v_i = v_j), \\
v_i = \{v_j\}_R & \quad \text{is} \quad \forall v_k(v_k \in_R v_j \Leftrightarrow v_k = v_j), \\
\{v_i\}_R \in_R v_j & \quad \text{is} \quad \exists v_k(v_k \in_R v_j \land v_k = \{v_i\}_R), \\
v_i = \{(v_j)_R\}_R & \quad \text{is} \quad \exists v_k(v_k = \{v_j\}_R \land v_i = \{v_k\}_R), \\
v_i \in_R \cup v_j & \quad \text{is} \quad \exists v_k(v_i \in_R v_k \land v_k \in_R v_j).
\end{align*}
\]

Therefore \textit{pair} (a pairing function) can be defined as follows:

\[
\begin{align*}
\exists v_j \forall v_k(\{v_k\}_R \subseteq_R v_i \Leftrightarrow v_j = v_k) \land \\
\forall v_j \exists v_k(v_j \in_R v_i \Rightarrow v_k \in_R v_j) \land \\
\forall v_j \forall v_k(v_j \in_R \cup v_i \land \{v_j\} \notin_R v_i \land v_k \in_R \cup v_i \land \{v_k\} \notin_R v_i \Rightarrow v_j = v_k).
\end{align*}
\]

It is clear that this is a term built up of diagonal elements and directed cylindrifications. The first quasi-projection \( v_i = P(v_j) \) can be chosen as:

\[
\text{pair}_j \land \forall^i v_j \exists^i v_j(v_i = v_j).
\]

and the second quasiprojection \( v_i = Q(v_j) \) can be chosen as:

\[
\text{pair}_j \land (\forall v_i \forall v_k(v_i \in_R v_j \land v_k \in_R v_j \Rightarrow v_i = v_k)) \Rightarrow v_i = P(v_j) \land \\
(\exists v_i \exists v_k(v_i \in_R v_j \land v_k \in_R v_j \land v_i \neq v_k) \Rightarrow (v_i \neq P(v_j) \land \exists^i v_j \exists^i v_j(v_i = v_j)).
\]

84
**Theorem 80.** Let $\mathcal{B}$ be the relation algebra reduct of $\mathcal{A}$; then $\mathcal{B}$ is a relation algebra, and the variable free terms corresponding to the formulas $v_i = P(v_j)$ and $v_j = Q(v_j)$ call them $p$ and $q$, respectively, are quasi-projections.

**Proof.** One proof is very tedious, though routine. One translates the functions as variable free terms in the language of $\mathcal{CA}_3$ and use the definition of composition and converse in the $\mathcal{RA}$ reduct, to verify that they are quasi-projections. Else one can look at their meanings on set algebras, which we recall from Sagi [32]. Given a cylindric set algebra $\mathcal{A}$ with base $U$ and accessibility relation $R$

$$\begin{align*}
(v_i = P(v_j))^A &= \{ s \in ^3U : (\exists a,b \in U)(s_j = (a,b)_R, s_i = a) \} \\
(v_i = Q(v_j))^A &= \{ s \in ^3U : (\exists a,b \in U)(s_j = (a,b)_R, s_i = b) \}.
\end{align*}$$

First $P$ and $Q$ are functions, so they are functional elements. Then it is clear that in this set algebras that $P$ and $Q$ are quasi-projections. Since $\mathcal{RCA}_3^\uparrow$ is the variety generated by set algebras, they have the same meaning in the class $\mathcal{CA}_3^\uparrow$.

Now we can turn the class around. Given a $\mathcal{QRA}$ one can define a directed $\mathcal{CA}_n$, for every finite $n \geq 2$. This definition is given by Németi and Simon in [20]. It is very similar to Simon’s definition above (defining $\mathcal{CA}$ reducts in a $\mathcal{QRA}$, except that directed cylindrifiers along a relation $R$ are implemented.

**Theorem 81.** The concrete category $\mathcal{QRA}$ with morphisms injective homomorphisms, and that of $\mathcal{CA}^\uparrow$ with morphisms also injective homomorphisms are equivalent. in particular $\mathcal{CA}^\uparrow$ of dimension 3 is equivalent to $\mathcal{CA}^\uparrow$ for $n \geq 3$.

**Proof.** Given $\mathcal{A}$ in $\mathcal{QRA}$ we can associate a directed $\mathcal{CA}_3$, homomorphism are restrictions and vice versa; these are inverse Functors. However, when we pass from an $\mathcal{QRA}$ to a $\mathcal{CA}^\uparrow$ and then take the $\mathcal{QRA}$ reduct, we may not get back exactly to the $\mathcal{QRA}$ we started off with, but the new quasi projections are definable from the old ones. Via this equivalence, we readily conclude that $\mathcal{RCA}_3 \to \mathcal{RCA}_n$ are also equivalent.

**References**

[1] Andréka, H., *Complexity of equations valid in algebras of relations*. Annals of Pure and Applied logic, 89 (1997), p.149 - 209.

[2] Reducing first order logic to $Df3$ free algebras In [3]p. 1-15

[3] H. Andreka, M. Ferenczi, I. Nemeti (editors) *Cylindric-like algebras and algebraic logic* Bolyai Society, Mathematical Studies, Springer (2013).
[4] Andréka, H., Németi, I., Sayed Ahmed, T., *Omitting types for finite variable fragments and complete representations of algebras*. Journal of Symbolic Logic **73**(1) (2008) p.65-89

[5] Daigneault, A., and Monk, J.D., *Representation Theory for Polyadic algebras*. Fund. Math. **52**(1963) p.151-176.

[6] Ferenczi, M., *On representation of neatly embeddable cylindric algebras* Journal of Applied Non-classical Logics, **10**(3-4) (2000)

[7] Ferenczi, M., *Finitary polyadic algebras from cylindric algebras*. Studia Logica **87**(1)(2007) p.1-11

[8] Ferenczi, M., *On cylindric algebras satisfying the merry-go-round properties* Logic Journal of IGPL, **15**(2) (2007), p. 183-199

[9] Ferenczi, M., *On the representability of neatly embeddable CA’s by cylindric set algebras*, to appear

[10] Ferenczi, M., *On conservative extensions in logics with infinitary predicates*, to appear

[11] M. Ferenczi *A new representation theory, representing cylindric like algebras by relativized set algebras* In **3** p. 135-162

[12] J. Madarasz, T. Sayed Ahmed *Amalgamation, interpolation and epimorphisms in algebraic logic* in **3**p.91-104

[13] Madárasz J. and Sayed Ahmed T., *Amalgamation, interpolation and epimorphisms*. Algebra Universalis **56**(2) (2007) p. 179-210.

[14] Madárasz J. and Sayed Ahmed T. *Neat reducts and amalgamation in retrospect, a survey of results and some methods. Part 1: Results on neat reducts* Logic Journal of IGPL **17**(4)(2009) p.429-483

[15] Madárasz J. and Sayed Ahmed T., *Neat reducts and amalgamation in retrospect, a survey of results and some methods. Part 2: Results on amalgamation* Logic Journal of IGPL **17**(6) (2009)p. 755-802

[16] Madarász, J. *Interpolation and Amalgamation; Pushing the Limits. Part I* Studia Logica, **61**, (1998) p. 316-345.

[17] Maksimova, L. *Amalgamation and interpolation in normal modal logics*. Studia Logica **50**(1991) p.457-471.

[18] Marx *Algebraic relativization and arrow logic* Ph d Dissertation. University of Amsterdam 1995.
[19] I. Németi *Free algebras and decidability in Algebraic Logic* Hu dissertation with the Hungarian Academy of Sciences (1986)

[20] I. Nemeti, a Simon *Weakly higher order cylindric algebras and finite axiomatization of the representables* Studia Logica 91 (2005) 53-63

[21] I. Sain *On the search of a finitizable algebraisation of first order logic* Logic Journal of *IGPL*, 8 (2000) 495-589

[22] G. Sagi *A completeness theorem for higher order logics* Journal of symbolic Logic(65) (2000) p.857-884

[23] A. Simon *Connections between quasi-projective relation algebras and cylindric algebras* Algebra Universalis (2007) p. 233-301

[24] L. Henkin, D. Monk, A. Tarski *Cylindric algebras, part 1* 1970

[25] L. Henkin, D. Monk, A. Tarski *Cylindric algebras, part 2* 1985

[26] Hirsch and Hodkinson *Relation algebras by Games* Studies in Logic and the Foundations of mathematics, North Holand, 2002.

[27] Herrlich H, Strecker G. *Category theory* Allyn and Bacon, Inc, Boston (1973)

[28] Pigozzi,D. *Amalgamation, congruence extension, and interpolation properties in algebras*. Algebra Universalis. 1(1971) p.269-349.

[29] Sain, I. Thompson, R, *Strictly finite schema axiomatization of quasi-polyadic algebras*. In [?] p.539-571.

[30] Sain, I. *Searching for a finitizable algebraization of first order logic*. 8, Logic Journal of *IGPL*. Oxford University, Press (2000) no 4, p.495–589.

[31] I. Sain Gyuris V *Finite schematizable Algebraic Logic* Logic journal of *IGPL* 5(5) (1997) 600-751

[32] G. Sagi *Polyadic algebras* In [3]p. 376-392

[33] R. Hirsch *Relation algebra reducts of cylindric algebras and complete representations*. Journal of Symbolic Logic (72) (2007) p. 673-703

[34] T. Sayed Ahmed *The class of 2 dimensional neat reducts of polyadic algebras is not elementary*. Fundementa Mathematica (172) (2002) p.61-81

[35] T. Sayed Ahmed *A model theoretic solution to a problem of Tarski* Mathematical Logic Quarterly (48) (2002) p.343-355
[36] Sayed Ahmed, T. On amalgamation of reducts of polyadic algebras. Algebra Universalis, 51 (2004), 301–359

[37] Sayed Ahmed, T., Algebraic Logic, where does it stand today? Bulletin of Symbolic Logic. 11(4)(2005), p.465-516.

[38] Sayed Ahmed, T. The class of polyadic algebras has the superamalgamation property Mathematical Logic Quarterly 56(1)(2010)p.103-112

[39] Sayed Ahmed, T. On neat embeddings of cylindric algebras Mathematical Logic quarterly

[40] T. Sayed Ahmed Classes of algebras without the amalgamation property Logic Journal of IGPL, 192 (2011) p.87-2011.

[41] Sayed Ahmed T. On finite axiomatizability of expansions of cylindric algebras. Journal of Algebra, Number Theory, Advances and Applications, 1(2010), p.19-40.

[42] Sayed Ahmed T Classes of algebras without the amalgamation property. Logic Journal of IGPL, 1(2011), p.87-104.

[43] Sayed Ahmed T. On the complexity of axiomatizations of the class of representable quasi-polyadic equality algebras. Mathematical Logic Quarterly, 4(2011), p. 384-394.

[44] Sayed Ahmed T. Epimorphisms are not surjective, even in simple algebras. Logic Journal of IGPL, 1(2012), p.22-26.

[45] Sayed Ahmed T Three interpolation theorems for typeless logics Logic Journal of IGPL 20(6), 1001-1037 (2012).

[46] Sayed Ahmed T., Completions, complete representations and omiting types In [3] p. 205-222

[47] T. Sayed Ahmed Neat reducts and Neat Embeddings in Cylindric Algebras in [3], p. 105-134

[48] Sayed Ahmed, T. and Németi I, On neat reducts of algebras of logic. Studia Logica, 62 (2) (2001), p.229-262.

[49] Sayed Ahmed, T., and Samir B., A Neat embedding theorem for expansions of cylindric algebras. Logic journal of IGPL 15 (2007) p. 41-51.