QCD Factorization and PDFs from Lattice QCD Calculation

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In this talk, we review a QCD factorization based approach to extract parton distribution and correlation functions from lattice QCD calculation of single hadron matrix elements of quark-gluon operators. We argue that although the lattice QCD calculations are done in the Euclidean space, the nonperturbative collinear behavior of the matrix elements are the same as that in the Minkowski space, and could be systematically factorized into parton distribution functions with infrared safe matching coefficients. The matching coefficients can be calculated perturbatively by applying the factorization formalism on to asymptotic partonic states.

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1. Introduction

Parton distribution and correlation functions describe the fascinating relation between a hadron and the quarks and gluons (or partons) within it. They carry rich information on hadron’s mysterious partonic structure that cannot be calculated by QCD perturbation theory. Parton distribution functions (PDFs) are the simplest of all correlation functions, \( f_{i/q}(x, \mu^2) \), defined as the probability distributions to find a quark, an antiquark, or a gluon \( (i = q, \bar{q}, g) \) in a fast moving hadron to carry the hadron’s momentum fraction between \( x \) and \( x + dx \), probed at the factorization

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scale $\mu$. They carry an invaluable source of information on the partonic structure and the confinement-sensitive partonic dynamics of colliding hadron(s), and they also play an essential role to connect the cross sections of colliding hadron(s) to the short-distance scattering between quarks and gluons. Without them, we would not be able to understand the hard probes, cross sections with large momentum transfers, in high energy hadronic collisions, as well as the discovery of Higgs particles in proton-proton collisions at the LHC. PDFs are nonperturbative, but, universal, and have been traditionally extracted from QCD global analysis of all existing high energy scattering data in the framework of QCD factorization 1–4.

Unlike cross sections, PDFs are not direct physical observables. Their extraction from data on hard probes in high energy scattering always rely on the validity of QCD factorization 5 or the approximation to neglect all contributions suppressed by inverse powers of the large momentum transfer. On the other hand, PDFs are hadronic matrix elements of well-defined operators, which are made of quark or gluon fields in QCD, along with a proper ultraviolet renormalization specified for these composite operator 6. It might be possible to derive these PDFs directly from \textit{ab initio} calculations of QCD, such as lattice QCD, and to compare the calculated PDFs with the extracted ones. However, it is extremely difficult to calculate PDFs in lattice QCD since the PDFs are defined by hadronic matrix elements of non-local operators in the Minkowski space, while all lattice QCD calculations are done with an Euclidean time. The moments of PDFs, $\int_0^1 dx x^{n-1} f_i/h(x, \mu^2)$, given by the hadronic matrix elements of local operators, have been traditionally investigated by the lattice QCD community. Although progress has been made, the comparison between the lattice calculations and the moments of extracted PDFs has not reached to a satisfactory accuracy 7, 8.

Recently, Ji 9 introduced a set of quasi-PDFs, defined in terms of hadronic matrix elements of equal time correlators, calculable in lattice QCD 10, and suggested that the quasi-PDFs become the normal PDFs when the hadron momentum $P_z$ is boosted to the infinity. However, since the hadron momentum in lattice QCD calculation is effectively bounded by the lattice spacing, the $P_z \to \infty$ limit is hard to achieve in lattice calculations, Ji introduced the large-momentum effective field theory of QCD 11 and suggested that it provides a frame work to evaluate the difference between PDFs and quasi-PDFs due to a finite $P_z$. The connection between the PDFs and quasi-PDFs is further complicated by the fact that the operator defining the quasi-PDFs are power ultra-violet (UV) divergent, while the operators defining the normal PDFs have only logarithmic UV divergence.

In this talk, we review a QCD factorization approach, proposed recently by us 12, for extracting PDFs from lattice QCD calculations of single hadron matrix elements of quark-gluon correlators. For all QCD factorization treatments of hadronic cross sections with large momentum transfer(s), PDFs were introduced to absorb all leading power partonic collinear (CO) divergences associated with the colliding hadron(s). For example, all leading power partonic CO divergences of the lepton-hadron deep inelastic scattering (DIS) cross section are universal and can
be completely absorbed into the PDFs, if we neglect the corrections suppressed by
the inverse powers of the large momentum transfer $Q = \sqrt{-q^2}$, the virtuality of
the exchange photon of momentum $q$. All leading power partonic CO divergences
of the DIS cross sections, which is proportional to a single hadron matrix element
of two conserved electromagnetic currents, $\langle h(P)|j_\mu(\xi)j_\nu(0)|h(P)\rangle$, come from the
region of phase space where all active partons’ transverse momenta, $k_{i\perp}^2 \to 0$, where
\( i = 1, 2, \ldots \), with respect to the colliding hadron’s momentum $P^\mu = (P_0, 0_\perp, P_z)$.

Our proposal is based on the observation that the leading power CO divergences in
the limit $k_{i\perp}^2 \to 0$ are the same regardless if the time defining the hadronic matrix
element is in the Minkowski or the Euclidean space. Our method involves four steps:
1) identify hadronic matrix elements that are both calculable in lattice QCD and
factorizable into PDFs – referred as lattice “cross sections”, 2) g enerate the “data”
of the lattice “cross sections”, 3) evaluate the factorized coefficient functions be-
tween the lattice “cross sections” and the PDFs, and 4) perform the global analysis
of the lattice “data” using the factorization formalisms to extract the PDFs.

Our approach requires a large virtuality – the probe’s momentum transfer,
$Q \gg \Lambda_{QCD}$, to ensure the validity of the QCD factorization of the lattice “cross
sections” into PDFs. The QCD factorization in our approach is an approximation
to neglect corrections suppressed by the inverse powers of $Q$, which is not the same
as the expansion in $1/P_z$ in Ji’s approach. In our proposed approach, the hadron
momentum $P_z$ is a large, but finite, “observed” momentum scale for the lattice
“cross sections”, similar to the collision energy $\sqrt{S}$ for hadronic cross sections, and
is of the order of $Q$ or larger. Our approach is effectively the same as the lead-
ing power QCD factorization approach for extracting PDFs from data of hadronic
cross sections, except the cross sections are replaced by the lattice “cross sections”
evaluated in the Euclidean space.

With the limitation of current lattice size and computing power, it is certainly
too expensive, or even impossible to calculate PDFs at small $x$. However, lattice cal-
culations could certainly provide very valuable information on PDFs in the valence
region, in particular, in the regime when $x \to 1$, where the accuracy of experimen-
tal data is limited, while theoretical techniques to resum large $x$ perturbative
contribution have been developed and improved. Furthermore, with lattice QCD
calculations of the hadronic matrix elements of the same operators on meson states,
or states of baryons other than the proton, this QCD factorization approach could
in principle extract PDFs of mesons or exotic baryons and their partonic structure,
without performing high energy scattering on mesons and exotic baryons, which
could be very difficult if not impossible.

2. Lattice “cross sections” and factorization

We define a lattice “cross section”, $\sigma^{lat}_{h,h}(\hat{x}, 1/a, P_z)$, as the Fourier transform of a
single hadron matrix element, $\langle h(P)|\mathcal{O}(\psi, A)|h(P)\rangle$, with the colliding hadron momen-
tum $\hat{P}$ along $z$-direction and large, $P^0 \approx |P_z| \gg \Lambda_{QCD}$, and an operator $\mathcal{O}(\psi, A)$.
of quark $\psi$ and/or gluon $A$ field, where the transverse lattice spacing $a$ defines the hard scale $\sim 1/a$, and the dimensionless parameter $\tilde{x}$, defined below, and $P_z$ mimic the “rapidity” and “collision energy” of the “cross section”, respectively. A good lattice “cross section”, $\tilde{\sigma}^{\text{lat}}_{h,E}(\tilde{x}, 1/a, P_z)$, should have the following properties:

- It must be calculable in lattice QCD with an Euclidean time, indicated by the superscript “Lat” and the subscript “E”,
- It is infrared (IR) safe if it is calculated in lattice perturbation theory,
- All CO divergences of its continuum limit ($a \to 0$) can be factorized into the PDFs with perturbatively calculable hard coefficient functions.

Lattice QCD is a UV finite theory, and the lattice “cross section” calculated in its perturbation theory with a finite $a$ is also UV finite. However, the perturbatively calculated lattice “cross section”, which is needed for extracting the finite coefficient functions when it is factorized into PDFs, might be UV sensitive – perturbatively unstable when the lattice spacing $a$ is sufficiently small, if the operator $O(\psi, A)$ defining the lattice “cross section” does not have a renormalizable continuum limit. That is, a good lattice “cross section” also requires the operator to define its single hadron matrix element to have a renormalizable continuum limit.

Once we identify good lattice “cross sections”, we are able to factorize them in terms of the PDFs,

$$\tilde{\sigma}^{\text{lat}}_{h,E}(\tilde{x}, 1/a, P_z) \approx \sum_i \int_0^1 \frac{dx}{x} f_{i/h}(x, \mu^2) \tilde{C}_i(x, \frac{1}{a}, \mu^2, P_z),$$

where $\tilde{C}_i$'s are perturbative coefficient functions. By applying Eq. (1) to various parton states, $|h(P)\rangle \rightarrow |f(P)\rangle$ with flavor $f = q, \bar{q}, g$, the $\tilde{C}_i$'s can be systematically derived by calculating $\tilde{\sigma}^{\text{lat}}_{f,E}(\tilde{x}, 1/a, P_z)$ on a parton state $f$ in lattice QCD perturbation theory and $f_{i/f}(x, \mu^2)$ of the same parton state in perturbative QCD. As explained in the last section, we can systematically extract the PDFs from “data” of lattice “cross sections” by using the factorization relation in Eq. (1), and perturbatively calculated coefficient functions. The accuracy of the extracted PDFs could be improved perturbatively by more accurate coefficient functions, $\tilde{C}_i$'s.

Our strategy to search for good lattice “cross sections” could be summarized by the following schematic plot,

$$\tilde{\sigma}^{\text{lat}}_{h,E}(\tilde{x}, 1/a, P_z) \leftrightarrow \tilde{\sigma}_{h,E}(\tilde{x}, \tilde{\mu}^2, P_z) \downarrow \tilde{\sigma}_{h,M}(\tilde{x}, \tilde{\mu}^2, P_z) \leftrightarrow f_{i/h}(x, \mu^2),$$

where $\tilde{\sigma}_{h,E}(\tilde{x}, \tilde{\mu}^2, P_z)$ is the Euclidean space continuum limit of $\tilde{\sigma}^{\text{lat}}_{h,E}(\tilde{x}, 1/a, P_z)$ with a proper UV counter term (UVCT) to renormalize its UV divergence, if there is any, at a hard scale $\tilde{\mu}$; and $\tilde{\sigma}_{h,M}(\tilde{x}, \tilde{\mu}^2, P_z)$ is the Minkowski space version of $\tilde{\sigma}_{h,E}(\tilde{x}, \tilde{\mu}^2, P_z)$, as indicated by its subscript “M”. If the operator $O(\psi, A)$ is time-independent, we expect that $\tilde{\sigma}_{h,E}(\tilde{x}, \tilde{\mu}^2, P_z) = \tilde{\sigma}_{h,M}(\tilde{x}, \tilde{\mu}^2, P_z)$. To show the factor-
ization in Eq. (1) is effectively to prove the factorization between \( \sigma_{h,M}(\bar{x}, \mu^2, P_z) \) and \( f_{i/h}(x, \mu^2) \) in the continuous Minkowski space, and to verify the matching between \( \sigma_{h,E}(\bar{x}, 1/a, P_z) \) and \( \sigma_{h,E}(\bar{x}, \mu^2, P_z) \). If the composite operator \( \mathcal{O}(\psi, A) \) is made of conserved currents/tensors, we do not need the UVCT(\( \mu^2 \)) to define the \( \sigma_{h,E}(\bar{x}, \mu^2, P_z) \), and we only need the bottom matching relation with \( \tilde{\mu} = 1/a \). It is therefore very important to show the bottom factorization relation in Eq. (2) for any potential lattice “cross sections” in the Minkowski space,

\[
\sigma_{h,M}(\bar{x}, \mu^2, P_z) \approx \sum_i \int_0^1 \frac{dx}{x} f_{i/h}(x, \mu^2) c_i(\bar{x}, \mu^2, P_z)
\]  

(3)

with corrections suppressed by the inverse powers of factorization scale \( \mu \).

This factorization approach for extracting PDFs could be generalized for extracting other parton distribution and correlation functions. For example, for extracting transverse momentum dependent PDFs (TMDs), the lattice “cross sections” are necessarily to have dependence on additional momentum scale(s) different from the hard scale \( \tilde{\mu} \).

3. Case study: the quasi-PDFs

As a case study, we discuss if the quasi-PDFs, introduced by Ji\(^{\text{[9]}}\), could be good lattice “cross sections” for extracting the PDFs.

3.1. Definition

The quasi-quark distribution of a hadron \( h \) of momentum \( P^\mu \) is defined as\(^{\text{[9]}}\)

\[
\tilde{f}_{\bar{g}/h}(\bar{x}, \mu^2, P_z) \equiv \int \frac{d\xi_z}{2\pi} e^{-i\pi P_z \xi_z} \tilde{F}_{\bar{g}/h}(\xi_z, \mu^2, P_z),
\]  

(4)

where \( \tilde{F}_{\bar{g}/h}(\xi_z, \mu^2, P_z) = \langle h(P) | \overline{\psi}(\xi_z) \frac{1}{\mu} \Phi_{n_z}(\{\xi_z, 0\}) \psi(0) | h(P) \rangle + \text{UVCT}(\mu^2) \) and \( \xi_0 = \xi_\perp = 0 \). Similarly, the quasi-gluon distribution is defined as

\[
\tilde{f}_{\bar{g}/h}(\bar{x}, \mu^2, P_z) \equiv \frac{1}{\bar{x} P_z} \int \frac{d\xi_z}{2\pi} e^{-i\pi P_z \xi_z} \tilde{F}_{\bar{g}/h}(\xi_z, \mu^2, P_z),
\]  

(5)

where \( \tilde{F}_{\bar{g}/h}(\xi_z, \mu^2, P_z) = \langle h(P) | F^\nu_z(\xi_z) \Phi_{n_z}(\{\xi_z, 0\}) F_{\nu \mu}(0) | h(P) \rangle + \text{UVCT}(\mu^2) \) with \( \nu \) summing over transverse directions. In Eqs. (4) and (5), \( \tilde{\mu} \) is a renormalization scale, and the gauge links \( \Phi_{n_z}(\{\xi_z, 0\}) = \mathcal{P}\text{exp}[-ig \int_0^{\xi_z} dt_{\eta_z} A^{(\bar{f},a)}_z(\eta_z)] \) where \( \mathcal{P} \) indicates the path ordering, the superscripts, “\( \bar{f} \)” and “\( a \)”, represent the fundamental and adjoint representation of SU(3) color of QCD, respectively, and \( n_z^a = (0, 0_\perp, 1), n_z^2 = -1 \) and \( v \cdot n_z = -v_z \) for any vector \( v^\mu \). Since the operators defining these single hadron matrix elements have no explicit time dependence, the quasi-PDFs could be calculated in lattice QCD\(^{\text{[9]}}\).

To show that these quasi-PDFs could be factorized into the PDFs as in Eq. (3), we need to demonstrate that they are IR safe, UV renormalizable with the UVCTs, and all their perturbative CO divergences can be absorbed into the PDFs. Following
effectively the same arguments used in Ref. [6], it is straightforward to show that these quasi-PDFs of an asymptotic parton state are indeed IR safe.

\[ k_k = C_0 + K_0 + \ldots \]

**Fig. 1.** Ladder expansion of the quasi-quark distribution.

### 3.2. Factorization of CO divergences

Factorization of the partonic CO divergences could be best demonstrated in the light-cone gauge, \( n \cdot A = 0 \), with \( n^\mu = (n^+, n^-, n_\perp) = (0, 1, 0, \perp) \). In this gauge, for example, the leading power flavor non-singlet contribution to the quasi-quark distribution of a parton state of momentum \( p \) can be approximated by a sum of ladder diagrams, as shown in Fig. 1, plus UVCTs [12,15], where \( C_0 \) and \( K_0 \) are two-particle irreducible (2PI) kernels. By definition, \( K_0 \) includes the two quark propagators connecting to the kernel above. We can introduce corresponding renormalized 2PI kernel \( K \), with local UV divergences removed by the counter-terms of renormalized QCD Lagrangian. Since the renormalized 2PI kernels with fixed external momenta are finite [12], all CO divergences of the ladder diagrams in Fig. 1 come from the integration of the loop momentum \( k_i \) between two neighboring 2PI kernels, and corresponding CO divergences are logarithmic. To factorize all leading power CO divergences of the quasi-quark distribution of a parton state into the PDFs of the same parton state, we introduce a projection operator, \( \hat{\mathcal{P}} \) to act on the kernel \( K \) so that \( \hat{\mathcal{P}}K \) picks up the leading logarithmic CO divergence of the kernel \( K \) when \( k_i^2 \perp \rightarrow 0 \), with the corresponding logarithmic UV divergence of the \( K \) when \( k_i^2 \perp \rightarrow \infty \) removed by a local UVCT(\( \mu_i^2 \)) [12]. Use this projection operator, we can sum up all ladder diagrams in Fig. 1 in the following symbolic form [12].

\[
\tilde{f}_{q/p} = \lim_{m \rightarrow \infty} C_0 \sum_{i=0}^m K^i + \text{UVCT} = \left[ C_0 \frac{1}{1 - (1 - \hat{\mathcal{P}})K} + \text{UVCT} \right] + \tilde{f}_{q/p} \hat{\mathcal{P}} K, \quad (6)
\]

where the term in \([\ldots]\) does not have any CO divergence at the leading power of the factorization scale \( \mu_i^2 \). By combining the terms with \( \tilde{f}_{q/p} \), we obtain

\[
\tilde{f}_{q/p} = \left[ C_0 \frac{1}{1 - (1 - \hat{\mathcal{P}})K} + \text{UVCT} \right] \otimes \left[ \frac{1}{1 - \hat{\mathcal{P}}K} \right], \quad (7)
\]
where all leading power CO divergences of the renormalized quasi-quark distribution of a quark of momentum $p$ are now factorized into a “multiplicative” factor $[1/(1-\hat{P}K)]$, which is perturbatively UV finite and equal to the perturbative contribution to the quark distribution. The same factorization arguments for CO divergences can be applied to the flavor singlet quasi-quark and quasi-gluon distribution.  

\[ \mathcal{C}_q(q/p) = \tilde{f}(1)(q/q)(t,\tilde{\mu}_2^2,\mu_2^2,\Lambda_z) - f(1)(q/q)(t,\mu_2^2,\Lambda_z) \]

\[ C_{\mathcal{F}} \frac{\alpha_s}{2\pi} = \left[ \frac{1 + t^2}{1 - t} \ln \frac{\tilde{\mu}_2^2}{\mu_2^2} + 1 - t \right] + \left[ \frac{t\Lambda_1 - t}{1-t} + \frac{\Lambda_t}{1-t} + \text{Sgn}(t)\Lambda_t \right] + \left[ \frac{\text{Sgn}(1-t)}{1-t} \ln \left( 1 + \frac{\Lambda_1}{2|t|} \right) + \text{Sgn}(1-t) \ln \left( 1 + \frac{\Lambda_1}{2|1-t|} \right) \right] \]
where $\Lambda_t = \sqrt{\mu^2/P_z^2 + t^2} - |t|$, Sgn$(t) = 1$ if $t \geq 0$, and $-1$ otherwise. In Eq. (3), the “+”-function is conventional, and the “N”-function is similarly defined as

$$
\int_{-\infty}^{+\infty} dt \left[ g(t) \right]_N h(t) = \int_{-\infty}^{+\infty} dt g(t) \left[ h(t) - h(1) \right],
$$

(9)

where $h(t)$ is any well-behaved function. As expected, the $C_{q/q}^{(1)}$ in Eq. (3), so as the one-loop coefficient functions for all other partonic channels, are free of any UV, IR and CO divergences 14.

4. Summary

In this talk, we reviewed our proposal for a QCD factorization based approach to extract parton distribution and correlation functions from lattice QCD calculations of single hadron matrix elements of quark-gluon correlators – referred as lattice “cross sections”. We presented our strategy to search for good lattice “cross sections”, and their factorization formula to the PDFs. As a case study, we discussed in details the quasi-PDFs as lattice “cross sections”.

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