SUBDIFFERENTIABLE FUNCTIONS AND PARTIAL DATA COMMUNICATION IN A DISTRIBUTED DETERMINISTIC ASYNCHRONOUS DYKSTRA’S ALGORITHM

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Abstract. We described a decentralized distributed deterministic asynchronous Dykstra’s algorithm that allows for time-varying graphs in an earlier paper. In this paper, we show how to incorporate subdifferentiable functions into the framework using a step similar to the bundle method. We point out that our algorithm also allows for partial data communications. We discuss a standard step for treating the composition of a convex and linear function.

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1. Introduction

Consider a connected graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) where a closed convex function \( f_i : \mathbb{R}^m \to \mathbb{R} \cup \{\infty\} \) is defined on each vertex \( i \in \mathcal{V} \). A problem of interest that occurs in problems with data too large to be stored in a single location is to minimize the sum

\[
\min_{x \in \mathbb{R}^m} \sum_{i \in \mathcal{V}} f_i(x)
\]

in a distributed manner so that the communications of data occur only along the edges of the graph. In our earlier paper [Pan18a], we consider the regularized problem

\[
\min_{x \in \mathbb{R}^m} \sum_{i \in \mathcal{V}} [f_i(x) + \frac{1}{2}||x - [\bar{x}]_i||^2]
\]

instead, where \( \bar{x} \in [\mathbb{R}^m]^{[\mathcal{V}]} \).

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1.1. Distributed optimization algorithms. Since this paper builds on [Pan18a], we shall give a brief introduction. Our algorithm is for the case when the edges are undirected. But we remark that notable papers on the directed case. A notable paper based on the directed case using the subgradient algorithm is [SLWY15], and surveys are [Ned15] and [Ned17]. The papers [NOL15] and [NOS17] further touch on the case of time-varying graphs. The algorithm in [BCN+17] uses a Newton-Raphson method to design a distributed algorithm for directed graphs. Naturally, the communication requirements for directed graphs need to be more stringent that the requirements for undirected graphs.

From here on, we discuss only algorithms for undirected graphs. A product space formulation on the ADMM leads to a distributed algorithm [BPC+10, Chapter 7]. Such an algorithm is decentralized and distributed, but is not asynchronous and so can get slowed down by slow vertices. An approach based on [CE18] allows for asynchronous operation, but is not decentralized.

Moving beyond deterministic algorithms, distributed decentralized asynchronous algorithms were proposed, but many of them involve some sort of randomization. For example, the work [BCH13, BHI14] and the generalization [PXYY16] are based on monotone operator theory (see for example the textbook [BCH11]), and require the computations in the nodes to follow specific probability distributions.

We now look at asynchronous distributed algorithm with deterministic convergence (rather than probabilistic convergence). Other than subgradient methods, we mention that the paper [AFJ16] is an algorithm for strongly convex problems that is primal in nature, so can’t handle constraint sets as is. The method in [AH16] may arguably be considered to have these properties.

1.2. Dykstra’s algorithm and the corresponding distributed algorithm. Again, we shall be brief with the introduction, and defer to [Pan18a] for a more detailed introduction. Dykstra’s algorithm was first studied in [Dyk83] for projecting a point onto the intersection of a number of closed convex sets. The convergence proof without the existence of dual solutions was established in [BD85] and rewritten in terms of duality in [GM89], and is sometimes called the Boyle-Dykstra theorem. Dykstra’s algorithm was independently noted in [Han88] to be block coordinate minimization on the dual problem, but their proof depends on the existence of a dual solution. (For an example of a problem without dual solutions, look at [Han88, page 9] where two circles in \( \mathbb{R}^2 \) intersect at only one point.) We pointed out in [Pan17] that the Boyle-Dykstra theorem can be extended to the case of minimization problems of the form \( \min_x \frac{1}{2} \| x - \bar{x} \|^2 + \sum_{i=1}^k f_i(x) \). For more on the background on Dykstra’s algorithm, we refer to [BC11, BB96, Deu01, ER11].

Dykstra’s algorithm was extended to a distributed algorithm in [PB17], and they highlight the works [AH16, LN13, RNV10, ONP10] on distributed optimization. The work in [PB17] is vastly different from how Dykstra’s algorithm is studied in [BD85] and [GM89].

It turns out that [NN17] also makes use of the same Dykstra’s algorithm setting, but they solve with a randomized dual proximal gradient method. The differences between their setup and ours is detailed in [Pan18a].

In [Pan18a], we rewrote (1.2) in a form similar to (1.1) (see Remark 3.3 for an explanation of the differences) and applied an extended Dykstra’s algorithm. We list down the features of the distributed Dyksyra’s algorithm:
(1) distributed (with communications occurring only between adjacent agents $i$ and $j$ connected by an edge),
(2) decentralized (i.e., there is no central node coordinating calculations),
(3) asynchronous (contrast this to synchronous algorithms, where the faster agents would wait for slower agents before they can perform their next calculations),
(4) able to allow for time-varying graphs in the sense of [NO15, NOS17] (to be robust against failures of communication between two agents),
(5) deterministic (i.e., not using any probabilistic methods, like stochastic gradient methods),
(6) able to allow for constrained optimization, where the feasible region is the intersection of several sets (this largely rules out primal-only methods),
(7) able to incorporate proximable functions naturally.

Since Dykstra’s algorithm is also dual block coordinate ascent, the following property is obtained:

(8) choosing large number of dual variables to be maximized over gives a greedier increase of the dual objective value.

Also, the distributed Dykstra’s algorithm does not require the existence of a dual minimizer provided that the functions $f_i(\cdot)$ are proximable. Moreover, if some of the $f_i(\cdot)$ were defined to be the indicator functions of closed convex sets, then a greedy step for dual ascent [Pan16] is possible. For the rest of this paper, we shall just refer to the algorithm in [Pan18a] as the distributed Dykstra’s algorithm.

1.3. Main contribution of this paper. This paper builds on [Pan18a]. We now describe the main contribution of this paper without assuming any prior knowledge of [Pan18a].

For each node $i \in \mathcal{V}$, recall the function $f_i : \mathbb{R}^m \rightarrow \mathbb{R}$ in (1.2). Let $f_i : [\mathbb{R}^m]^\mathcal{V} \rightarrow \hat{\mathbb{R}}$ be defined by $f_i(x) = f_i(x_i)$ (i.e., $f_i$ depends only on $i$-th variable, where $i \in \mathcal{V}$). Recall the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Let the set $\hat{\mathcal{E}}$ be defined to be

$$\hat{\mathcal{E}} := \mathcal{E} \times \{1, \ldots, m\}.$$  

For $x \in [\mathbb{R}^m]^\mathcal{V}$, the component $x_i \in \mathbb{R}^m$ is straightforward. We let $[x_i]_k \in \mathbb{R}$ be the $k$-th component of $x_i \in \mathbb{R}^m$. For each $((i, j), k) \in \hat{\mathcal{E}}$, the hyperplane $H_{((i, j), k)} \subset [\mathbb{R}^m]^\mathcal{V}$ is defined to be

$$H_{((i, j), k)} := \{x \in [\mathbb{R}^m]^\mathcal{V} : [x_i]_k = [x_j]_k\}. \quad (1.3)$$

We can see that the regularized problem (1.2) is equivalent to

$$\min_{x \in [\mathbb{R}^m]^\mathcal{V}} \frac{1}{2}\|x - \bar{x}\|^2 + \sum_{((i, j), k) \in \hat{\mathcal{E}}} \delta_{H_{((i, j), k)}}(x) + \sum_{i \in \mathcal{V}} f_i(x). \quad (1.4)$$

We let the functions $f_\alpha : [\mathbb{R}^m]^\mathcal{V} \rightarrow \hat{\mathbb{R}}$ be as defined in (1.4) for all $\alpha \in \hat{\mathcal{E}} \cup \mathcal{V}$. The (Fenchel) dual of (1.4) is

$$\max_{z_\alpha \in [\mathbb{R}^m]^\mathcal{V}, \alpha \in \hat{\mathcal{E}} \cup \mathcal{V}} F(\{z_\alpha\}_\alpha \in \hat{\mathcal{E}} \cup \mathcal{V}), \quad (1.5)$$
where
\[
F(\{\mathbf{z}_\alpha\}_{\alpha \in \mathcal{E} \cup \mathcal{V}}) := -\frac{1}{2} \left\| \mathbf{x} - \sum_{\alpha \in \mathcal{E} \cup \mathcal{V}} \mathbf{z}_\alpha \right\|^2 + \frac{1}{2} \left\| \mathbf{x} \right\|^2 - \sum_{\alpha \in \mathcal{E} \cup \mathcal{V}} f^*_\alpha(\mathbf{z}_\alpha).
\] (1.6)

To give further insight on the problems (1.4)-(1.6), we note that if \( f_i \equiv 0 \), then the problems (1.4) reduces to the averaged consensus algorithm in [BGPS06, DKMT10], where the primal variable \( \mathbf{x} \in [\mathbb{R}^m]^{|\mathcal{V}|} \) converges to the vector \( \mathbf{x}^* \in [\mathbb{R}^m]^{|\mathcal{V}|} \) (where \( \mathbf{x}^* \) is defined so that each \( \mathbf{x}^*_i \in \mathbb{R}^m \) is the average \( \frac{1}{|\mathcal{V}|} \sum_{i \in \mathcal{V}} \mathbf{x}_i \)) at a linear rate dependent on the properties of the graph \((\mathcal{V}, \mathcal{E}))\.

In [Pan18a], we applied the techniques of [GM89, HD97] to prove that a block coordinate optimization applied to (1.6) leads to the increase of the objective value \( F(\cdot) \) in (1.6) to the maximal value, which is also the objective value of (1.4) since not all of properties (1)-(8) are satisfied by the subgradient algorithm.

As it stands, the algorithm in [Pan18a] does not handle the case when \( f_i(\cdot) \) are smooth for all \( i \in \mathcal{V} \). Given an affine minorant of \( f_i(\cdot) \), say \( \tilde{f}_i(\cdot) \), the conjugate \( \tilde{f}^*_i(\cdot) \) satisfies \( \tilde{f}^*_i(\cdot) \geq f^*_i(\cdot) \). The main contribution of this paper is to show that for the dual function (1.6), if the \( f^*_i(\cdot) \) are replaced by \( \tilde{f}^*_i(\cdot) \) whenever \( f_i(\cdot) \) is a subdifferentiable function and \( \tilde{f}_i(\cdot) \) is defined as an affine minorant of \( f_i(\cdot) \), then the minorized dual functions would have the values ascending and converging to the optimal objective value of the dual problem (1.5). This extends the algorithm in [Pan18a] to give an algorithm with properties (1)-(8) and also incorporating subdifferentiable \( f_i(\cdot) \) naturally. (A more traditional method of majorizing \( f^*_i(\cdot) \) through \( f^*_i(\mathbf{z}_i, \cdot) + \langle \mathbf{x}_i, \mathbf{z}_i - \mathbf{z}_i \cdot \rangle \) would be problematic because a strongly convex modulus \( \sigma \) of \( f^*_i(\cdot) \) may not even exist, which is the case when \( f_i(\cdot) \) is affine.) As far as we are aware, distributed algorithms for subdifferentiable functions include methods based on the subgradient algorithm as mentioned earlier as well as [WB13]. (Since the problems we treat in this paper are strongly convex, it would be unfair to bring out the fact that subgradient methods are slow for problems that are not strongly convex due to the need of using diminishing stepsizes. But still, our dual approach has other advantages compared to the subgradient algorithm since not all of properties (1)-(8) are satisfied by the subgradient algorithm.)

In Section 2, we first show that this procedure is sound for the sum of one subdifferentiable function and a regularizing quadratic with convergence rates compatible with standard first order methods. In Section 3, we integrate this algorithm into our distributed Dykstra’s algorithm.
1.4. Other contributions of this paper. In [Pan18a], we had used the hyper-planes $H_{(i,j)} := \{ x \in \mathbb{R} : \| x \|_i = \| x \|_j \}$ for all $(i,j) \in \mathcal{E}$ instead of (1.3). We point out that using $H_{(i,j,k)}$ instead of $H_{(i,j)}$ allows for part of the data to be communicated at one time step to achieve convergence to the optimal solution, which in turn means that computation will not be held back by communications between nodes. See Subsection 3.1 and Example 3.9 for more details.

Finally, in Subsection 5.5 we point out that a standard step allows us to reduce matrix operations whenever the function $f_i(\cdot)$ of the form $\tilde{f}_i \circ A_i$ for some closed convex function $\tilde{f}_i(\cdot)$ and linear map $A_i$, although such a step now introduces additional regularizing functions.

1.5. Notation. For much of the paper, we will be looking at functions with domain either $\mathbb{R}^m$ or $[\mathbb{R}^m]^{|V|}$. We reserve bold letters for functions with domain $[\mathbb{R}^m]^{|V|}$ (for example, (1.4)), and we usually use non-bold letters for functions with domain $\mathbb{R}^m$ (for example, (2.1)). For a vector $z \in [\mathbb{R}^m]^{|V|}$, $z_i \in \mathbb{R}^m$ and $[z]_i \in \mathbb{R}^m$ are both understood to be the $i$-th component of $z$, where $i \in |V|$. Furthermore, $[z]_i |_{k}$ and $[z]_i |_{k} \in \mathbb{R}$ are both understood to be the $k$-th component of $[z]_i$.

We say that $f(\cdot)$ is proximable if the problem $\arg \min_x f(x) + \frac{1}{2} \| x - \bar{x} \|_2^2$ is easy to solve for any $\bar{x}$. For a closed convex set $C$, the indicator function is denoted by $\delta_C(\cdot)$. All other notation are standard.

2. The algorithm for one function

In this section, we consider the problem

$$\min_{x \in \mathbb{R}^m} f(x) + \frac{1}{2} \| x - \bar{x} \|_2^2,$$  

(2.1)

where $f : \mathbb{R}^m \to \mathbb{R}$ is a subdifferentiable convex function such that $\text{dom}(f) = \mathbb{R}^m$. We define our first dual ascent algorithm to solve (2.1) before we show how to integrate it into the distributed Dykstra’s algorithm for solving (1.4) through the increasing the dual objective value in (1.5)-(1.6). Consider Algorithm 2.1 on the following page which is somewhat like the bundle method.

We shall prove that each function of the form (2.2) is a lower approximation of $f(\cdot)$ in Lemma 2.2. With a sequence of such lower approximations like in the bundle method, we can then solve (2.1). We prove some lemmas before proving the convergence of Algorithm 2.1.

Lemma 2.2. In Algorithm 2.1, the functions $h_w(\cdot)$ are such that $h_w(\cdot) \leq f(\cdot)$.

Proof. We prove our result by induction. Note that $h_0(\cdot)$ was defined so that $h_0(\cdot) \leq f(\cdot)$. It is also clear from the definition of $\bar{h}_w(\cdot)$ in (2.3) that $\bar{h}_w(\cdot) \leq f(\cdot)$ for all $w \geq 0$. Suppose that $h_w(\cdot) \leq f(\cdot)$. We now show that $h_{w+1}(\cdot) \leq f(\cdot)$. We have

$$\max\{\bar{h}_w, h_w\}(\cdot) \leq f(\cdot).$$  

(2.5)

The functions $\bar{h}_w(\cdot)$ and $h_w(\cdot)$ are convex, so $\max\{\bar{h}_w, h_w\}(\cdot)$ is convex. Since the minimum of $\max\{\bar{h}_w, h_w\}(\cdot) + \frac{1}{2} \| \cdot - \bar{x} \|_2^2$ is attained at $\bar{x}_{w+1}$, it follows that $0 \in \partial \max\{\bar{h}_w, h_w\}(\bar{x}_{w+1}) + \bar{x}_{w+1} - \bar{x}$, or that $\bar{x} - \bar{x}_{w+1} \in \partial \max\{\bar{h}_w, h_w\}(\bar{x}_{w+1})$. The construction of $h_{w+1}(\cdot)$ implies that $h_{w+1}(\bar{x}_{w+1}) = \max\{\bar{h}_w, h_w\}(\bar{x}_{w+1})$ and $h_{w+1}(\cdot) \leq \max\{\bar{h}_w, h_w\}(\cdot)$. Together with (2.5), this implies $h_{w+1}(\cdot) \leq f(\cdot)$, which completes the proof. \qed
Algorithm 2.1. In this algorithm, we want to solve (2.1)
Let \( h_0 : \mathbb{R}^m \to \mathbb{R} \) be an affine function such that \( h_0(\cdot) \leq f(\cdot) \) defined by the parameters \((\bar{x}_0, \bar{f}_0, \bar{y}_0) \in \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^m\) where for all \( w \geq 0 \), \( h_w : \mathbb{R}^m \to \mathbb{R} \) is defined through \((\bar{x}_w, \bar{f}_w, \bar{y}_w) \in \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^m\) by

\[
h_w(x) = \bar{y}_w^T(x - \bar{x}_w) + \bar{f}_w. \tag{2.2}
\]

Without loss of generality, let \( \bar{x}_0 \) be the minimizer to \( \min_x h_0(x) + \frac{1}{2} \|x - \bar{x}\|^2 \).

01 For \( w = 0, \ldots \)
02 Recall \( \bar{x}_w \) is the minimizer to \( \min_x h_w(x) + \frac{1}{2} \|x - \bar{x}\|^2 \).
03 Evaluate \( f(\bar{x}_w) \) and find a subgradient \( \bar{s}_w \in \partial f(\bar{x}_w) \).
04 Construct the affine function \( \bar{h}_w : \mathbb{R}^m \to \mathbb{R} \) to be

\[
\bar{h}_w(x) = \bar{s}_w^T(x - \bar{x}_w) + f(\bar{x}_w). \tag{2.3}
\]

05 Consider

\[
\min \{ \max \{ \bar{h}_w, h_w \}(x) + \frac{1}{2} \| x - \bar{x} \|^2 \}. \tag{2.4}
\]

06 Let \( \bar{x}_{w+1} \) be the minimizer of (2.4).
07 Let \( \bar{f}_{w+1} = \max \{ \bar{h}_w, h_w \}(\bar{x}_{w+1}) \).
08 Let \( \bar{y}_{w+1} \) be \( x = \bar{x}_{w+1} \).
09 Define \( h_{w+1}(\cdot) \) through \((\bar{x}_{w+1}, \bar{f}_{w+1}, \bar{y}_{w+1}) \) and (2.2).
10 End for

Let

\[
\bar{\alpha}_w := f(\bar{x}_w) - h_w(\bar{x}_w). \tag{2.6}
\]

Let the minimizer of (2.4) be \( x^* \). We have

\[
h_w(\bar{x}_w) + \frac{1}{2} \| \bar{x}_w - \bar{x} \|^2 \leq f(x^*) + \frac{1}{2} \| x^* - \bar{x} \|^2 \tag{2.7}
\]

\[\text{where for all } x, \quad \alpha_w \text{ solves (2.6)} \]

Let the real number \( \alpha_w \) be

\[
\alpha_w := \left[ f(x^*) + \frac{1}{2} \| x^* - \bar{x} \|^2 \right] - \left[ h_w(\bar{x}_w) + \frac{1}{2} \| \bar{x}_w - \bar{x} \|^2 \right]. \tag{2.8}
\]

It is clear to see that (2.7) translates to \( 0 \leq \alpha_w \leq \bar{\alpha}_w \).

Lemma 2.3. Recall the definitions of \( \alpha_w \) and \( \bar{\alpha}_w \) in (2.6) and (2.8).

1. We have \( \alpha_{w+1} \leq \alpha_w - \frac{1}{2} t^2 \), where \( \frac{1}{2} t^2 + \ell \| \bar{s}_w + \bar{x}_w - \bar{x} \| = \bar{\alpha}_w \).
2. Next, \( \frac{1}{2} \| \bar{s}_w + \bar{x}_w - \bar{x} \|^2 \alpha_{w+1} + \alpha_{w+1} \leq \alpha_w \).

Proof. Since the function \( x \mapsto \bar{h}_w(x) + \frac{1}{2} \| x - \bar{x} \|^2 \) is convex, it is bounded from below by its linearization at \( \bar{x}_w \) using \( \bar{s}_w \in \partial f(\bar{x}_w) \) via (2.8). In other words, for all \( x \in \mathbb{R}^m \), we have

\[
\bar{h}_w(x) + \frac{1}{2} \| x - \bar{x} \|^2 \geq \left( \bar{s}_w + \bar{x}_w - \bar{x} \right)^T (x - \bar{x}_w) + f(\bar{x}_w) + \frac{1}{2} \| \bar{x}_w - \bar{x} \|^2,
\]

\[l(\bar{s}_w, \bar{x}_w)(x) \tag{2.10}
\]
where $l(\tilde{x}_w, \tilde{s}_w)(\cdot)$ as defined above is the affine function derived from taking a subgradient of $\hat{h}_w(\cdot) + \frac{1}{2}\|x - \bar{x}\|^2$ at $\tilde{x}_w$. Consider the problem
\[
\min_x \max_{h_w(x)} \left\{ h_w(x) + \frac{1}{2}\|x - \bar{x}\|^2, l(\tilde{x}_w, \tilde{s}_w)(x) \right\},
\] (2.11)
where $h_w^+(\cdot)$ is as defined above.

We first look at the case when $\tilde{s}_w + \tilde{x}_w - \bar{x} = 0$. We have $h_w(\tilde{x}_w) \leq f(\tilde{x}_w)$ by Lemma 2.2. So
\[
f(\tilde{x}_w) + \frac{1}{2}\|\tilde{x}_w - \bar{x}\|^2 \leq h_w^+(\tilde{x}_w) \leq \min_x l(\tilde{x}_w, \tilde{s}_w)(x) \leq \min_x h_w^+(x)
\] (2.11)
We then have $h_w(\tilde{x}_w) = f(\tilde{x}_w)$, or $\alpha_w = 0$. Also,
\[
0 = \tilde{s}_w + \tilde{x}_w - \bar{x} \in \partial f(\tilde{x}_w) + \partial (\frac{1}{2}\|x - \bar{x}\|^2)(\tilde{x}_w),
\]
so $\tilde{x}_w = x^*$. The remaining techniques in this proof shows that $\alpha_w' = 0$ for all $w' \geq w$, which implies the claims in this lemma. Thus, we assume $\tilde{s}_w + \tilde{x}_w - \bar{x} \neq 0$.

Since $l(\tilde{s}_w, \tilde{x}_w)(\cdot)$ is affine and $h_w(\cdot) + \frac{1}{2}\|\cdot - \bar{x}\|^2$ is a quadratic with minimizer $\tilde{x}_w$ and Hessian $I$, some elementary calculations will show that the minimizer of (2.11) is of the form $\tilde{x}_w - t\frac{\tilde{s}_w + \tilde{x}_w - \bar{x}}{\|\tilde{s}_w + \tilde{x}_w - \bar{x}\|}$ for some $t \geq 0$. Let $\tilde{d}_w := \frac{\tilde{s}_w + \tilde{x}_w - \bar{x}}{\|\tilde{s}_w + \tilde{x}_w - \bar{x}\|}$, and let this minimizer be $\tilde{x}_w - t\tilde{d}_w$. We can see that $t > 0$, because if $t = 0$, then $\tilde{x}_w - t\tilde{d}_w = \tilde{x}_w$, and $\tilde{x}_w$ would once again be the minimizer of $f(\cdot) + \frac{1}{2}\|\cdot - \bar{x}\|^2$. With $t > 0$, the function values in (2.11) are equal, which gives
\[
h_w(\tilde{x}_w - t\tilde{d}_w) + \frac{1}{2}\|\tilde{x}_w - t\tilde{d}_w - \bar{x}\|^2 = h_w(\tilde{x}_w) + \frac{1}{2}\|\tilde{x}_w - \bar{x}\|^2 + \frac{1}{2}t^2,
\] (2.12)

and
\[
l(\tilde{x}_w, \tilde{s}_w)(\tilde{x}_w - t\tilde{d}_w) = h_w(\tilde{x}_w) + \frac{1}{2}\|\tilde{x}_w - \bar{x}\|^2 + \alpha_w - t\|\tilde{s}_w + \tilde{x}_w - \bar{x}\|.
\]
Equating the last two formulas gives
\[
\frac{1}{2}t^2 + t\|\tilde{s}_w + \tilde{x}_w - \bar{x}\| = \alpha_w.
\] (2.13)

Next, we have
\[
h_w(\tilde{x}_w + t\tilde{d}_w) + \frac{1}{2}\|\tilde{x}_w + t\tilde{d}_w - \bar{x}\|^2 = \max\{h_w, \hat{h}_w\}(\tilde{x}_w + t\tilde{d}_w) + \frac{1}{2}\|\tilde{x}_w + t\tilde{d}_w - \bar{x}\|^2
\] (2.14)
\[
\geq h_w^+(\tilde{x}_w + t\tilde{d}_w)
\] (2.15)
\[
\geq h_w^+(\tilde{x}_w - t\tilde{d}_w)
\] (2.16)
\[
h_w(\tilde{x}_w - t\tilde{d}_w) + \frac{1}{2}\|\tilde{x}_w - t\tilde{d}_w - \bar{x}\|^2
\] (2.17)
\[
h_w(\tilde{x}_w - t\tilde{d}_w) + \frac{1}{2}\|\tilde{x}_w - \bar{x}\|^2 + \frac{1}{2}t^2.
\] (2.18)

The formulas (2.13) and (2.14) imply the first part of our lemma. Next, let $t_2$ be the positive root of
\[
\frac{1}{2}t_2^2 + t_2\|\tilde{s}_w + \tilde{x}_w - \bar{x}\| = \alpha_w.
\] (2.19)
Since \( \alpha_w \leq \hat{\alpha}_w \), we have \( t_2 \leq t \). Recalling the definition of \( \alpha_w \) in (2.8), we have

\[
\alpha_{w+1} \leq \alpha_w - \frac{1}{2} t_2^2 \leq \alpha_w - \frac{1}{2} t_2^2 \leq \frac{\|s_w + \bar{x}_w - \bar{x}\|}{t_2},
\]

or \( \frac{\alpha_{w+1}}{\|s_w + \bar{x}_w - \bar{x}\|} \leq t_2 \). Substituting this into (2.15) gives (2.16) as needed. \( \square \)

**Remark 2.4.** It is clear that \( h_0(\cdot) \) in Algorithm 2.1 can be defined as an affine function based on the evaluation of \( f(x) \) and a subgradient in \( \partial f(x) \) for some point \( x \). In line 9, instead of \( h_{w+1}(\cdot) \) defined there, one can use the maximum of a number of affine functions like in the bundle method. We shall only limit to the easy case of using one affine function to model \( h_{w+1}(\cdot) \) for pedagogical reasons.

We need the following result proved in [BT13] and [Bec15].

**Lemma 2.5.** (Sequence convergence rate) Let \( \alpha > 0 \). Suppose the sequence of nonnegative numbers \( \{a_k\}_{k=0}^\infty \) is such that

\[
a_k \geq a_{k+1} + \alpha a_{k+1}^2 \text{ for all } k \in \{1, 2, \ldots\}.
\]

(1) [BT13] Lemma 6.2] If furthermore, \( a_1 \leq \frac{1.5}{\alpha} \) and \( a_2 \leq \frac{1.5}{\alpha} \) , then

\[
a_k \leq \frac{1.5}{\alpha k} \text{ for all } k \in \{1, 2, \ldots\}.
\]

(2) [Bec15] Lemma 3.8] For any \( k \geq 2 \),

\[
a_k \leq \max \left\{ \left( \frac{1}{2} \right)^{(k-1)/2} a_0, \frac{4}{\alpha^2(k-1)} \right\}.
\]

In addition, for any \( \epsilon > 0 \), if

\[
k \geq \max \left\{ \frac{2}{\ln(\epsilon)} \left[ \ln(a_0) + \ln(1/\epsilon) \right], \frac{4}{\alpha^2} \right\} + 1,
\]

then \( a_k \leq \epsilon \).

Theorem 2.6 shows that Algorithm 2.1 has convergence rates consistent with standard first order methods.

**Theorem 2.6.** (Convergence rate) Suppose Algorithm 2.1 is used to solve (2.1), and \( \text{dom}(f) = \mathbb{R}^m \).

1. There is a \( O(1/w) \) convergence rate.
2. If in addition, \( \nabla f(\cdot) \) is Lipschitz with constant \( L_1 \), then there is a linear rate of convergence.

**Proof.** Recall that \( h_w(\cdot) \leq f(\cdot) \) from Lemma 2.2, so

\[
f(x^*) + \frac{1}{2} \|x^* - \bar{x}\|^2 \geq h_w(x^*) + \frac{1}{2} \|x^* - \bar{x}\|^2.
\]

Alg 2.1 line 2

\[
\geq h_w(\bar{x}_w) + \frac{1}{2} \|\bar{x}_w - \bar{x}\|^2 + \frac{1}{2} \|x^* - \bar{x}_w\|^2.
\]

Therefore, \( \|\bar{x}_w - x^*\| \leq \sqrt{2\alpha_w} \). We have \( 0 \in \partial [f(\cdot) + \frac{1}{2} \|\cdot - \bar{x}\|^2](x^*) \) and \( s_w + \bar{x}_w - \bar{x} \in \partial [f(\cdot) + \frac{1}{2} \|\cdot - \bar{x}\|^2](\bar{x}_w) \).

**Case 1: General case**

Since \( \text{dom}(f) = \mathbb{R}^m \) and \( \bar{x}_w \) lies in a compact set, there is some constant \( L > 0 \) such that \( \|\bar{s}_w + \bar{x}_w - \bar{x}\| \leq L \). Hence

\[
\frac{1}{2L} \alpha_{w+1}^2 + \alpha_{w+1} \leq \frac{1}{2L} \frac{1}{\|\bar{s}_w + \bar{x}_w - \bar{x}\|^2} \alpha_{w+1}^2 + \alpha_{w+1} \leq \alpha_w.
\]
This recurrence together with Lemma 2.3 gives us the $O(1/w)$ convergence rate we need.

**Case 2: Smooth case.** Let $L_1$ be the Lipschitz constant on the gradient of $f(x) + \frac{1}{2} \|x - x^*\|^2$. We then have $\|s_w + \bar{x}_w - \bar{x}\| \leq L_1 \|\bar{x}_w - x^*\| \leq L_1 \sqrt{2\alpha_w}$. Using the formula (2.9) gives us $\frac{1}{4L_1 \alpha_w} \alpha_{w+1}^2 + \alpha_{w+1} \leq \alpha_w$, or

$$\frac{1}{4L_1 \alpha_w} \left( \frac{\alpha_{w+1}}{\alpha_w} \right)^2 + \frac{\alpha_{w+1}}{\alpha_w} \leq 1.$$  

This gives us the linear convergence as needed.

**Remark 2.7.** (Minimizing (2.4)) The quadratic program can be solved easily by noting that the minimizer must be a minimizer of one of the problems

$$\min_x h_w(x) + \frac{1}{2} \|x - x^*\|^2, \quad \min_x h_w(x) + \frac{1}{2} \|x - \bar{x}\|^2,$$

or

$$\min_x \{h_w(x) + \frac{1}{2} \|x - x^*\|^2 : h_w(x) = h_w(x)\},$$

all of which are rather easy to solve.

We now state a proposition that will be useful for the proof of convergence.

**Proposition 2.8.** Consider the problem

$$\min_{x \in \mathbb{R}^m} \frac{1}{2} \|\bar{x} - x\|^2 + f(x),$$

and its corresponding (Fenchel) dual

$$\max_{y \in \mathbb{R}^m} -\frac{1}{2} \|\bar{y} - y\|^2 + \frac{1}{2} \|\bar{y}\|^2 - f^*(y).$$

The optimal solutions $x^*$ and $y^*$ are related by $x^* + y^* = \bar{x}$, and strong duality holds.

**Proof.** This result can be seen to be Moreau’s decomposition theorem. 

In view of Proposition 2.8, we now explain that Algorithm 2.1 can be interpreted as a dual ascent algorithm. We can see that Algorithm 2.1 finds $h_w(\cdot)$ for $w = 0, 1, \ldots$ such that $h_w(\cdot) \geq f^*(\cdot)$ and dual iterates $\{\tilde{y}_w\}_{w=0}^{\infty}$ so that $\{-h_w^*(\tilde{y}_w) + \frac{1}{2} \|\tilde{x}\|^2 - \frac{1}{2} \|\tilde{y}_w - \tilde{x}\|^2\}_{w=0}^{\infty}$ is a monotonically nondecreasing sequence that converges to $-f^*(y^*) + \frac{1}{2} \|\tilde{x}\|^2 - \frac{1}{2} \|y^* - \tilde{x}\|^2$, where $y^*$ is the optimal dual variable for (2.1). This interpretation shall be exploited in our subdifferentiable distributed Dykstra’s algorithm.

3. Deterministic Distributed Asynchronous Dykstra Algorithm

We now proceed to integrate the dual ascent algorithm in Section 2 into the distributed Dykstra’s algorithm for the problem (1.2). We partition the vertex set $\mathcal{V}$ as the disjoint union $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3 \cup \mathcal{V}_4$ so that

- $f_i(\cdot)$ are proximable functions for all $i \in \mathcal{V}_1$.
- $f_i(\cdot)$ are indicator functions of closed convex sets for all $i \in \mathcal{V}_2$.
- $f_i(\cdot)$ are proximable functions such that $\text{dom}(f_i) = \mathbb{R}^m$ for all $i \in \mathcal{V}_3$.
- $f_i(\cdot)$ are subdifferentiable functions (i.e., a subgradient is easy to obtain) such that $\text{dom}(f_i) = \mathbb{R}^m$ for all $i \in \mathcal{V}_4$. 

With the definition of the hyperplanes $H$ paper is that Algorithm 3.5 supports the partial communication of data. We lay down the foundations of the parts of Algorithm 3.5 relevant for this insight.

3.1. Partial communication of data. One insight that we point out in this paper is that Algorithm [3.5] supports the partial communication of data. We lay down the foundations of the parts of Algorithm 3.5 relevant for this insight.

Let $D \subset [R^m]\cdot|\mathcal{V}|$ be the diagonal set defined by

$$D := \{ x \in [R^m]\cdot|\mathcal{V}| : x_1 = x_2 = \cdots = x_{|\mathcal{V}|} \}. \quad (3.2)$$

With the definition of the hyperplanes $H_{((i,j),\bar{k})}$ in (1.3) and $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ being a connected graph, we have

$$\bigcap_{((i,j),\bar{k}) \in \mathcal{E}} H_{((i,j),\bar{k})} = D \text{ and } \sum_{((i,j),\bar{k}) \in \mathcal{E}} H_{((i,j),\bar{k})}^\perp = D^\perp = \left\{ z \in [R^m]\cdot|\mathcal{V}| : \sum_{i \in \mathcal{V}} z_i = 0 \right\}. \quad (3.3)$$

**Proposition 3.1.** Suppose $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a connected graph. Let $H_{((i,j),\bar{k})}$ be the hyperplane (1.3). Let $\mathcal{E}'$ be a subset of $\mathcal{E}$. The following conditions are equivalent:

1. $\bigcap_{((i,j),\bar{k}) \in \mathcal{E}'} H_{((i,j),\bar{k})} = D$
2. $\sum_{((i,j),\bar{k}) \in \mathcal{E}'} H_{((i,j),\bar{k})}^\perp = D^\perp$.
3. For each $\bar{k} \in \{1, \ldots, m\}$, the graph $\mathcal{G}' = (\mathcal{V}, \mathcal{E}'_{\bar{k}})$ is connected, where $\mathcal{E}'_{\bar{k}} := \{ (i,j) \in \mathcal{E} : ((i,j),\bar{k}) \in \mathcal{E}' \}$.

**Proof.** The equivalence between (1) and (3) is easy, and the equivalence between (1) and (2) is simple linear algebra. \qed

**Definition 3.2.** We say that $\mathcal{E}' \subset \mathcal{E}$ connects $\mathcal{V}$ if any of the equivalent properties in Proposition 3.1 is satisfied.

**Remark 3.3.** (Change from [Pan18a]) The change in this paper from [Pan18a] is that each hyperplane $H_{((i,j),\bar{k})}$ is now of codimension 1. In [Pan18a], we defined the hyperplanes $H_{(i,j)} := \{ x \in [R^m]\cdot|\mathcal{V}| : x_i = x_j \}$ of codimension $m$ which are indexed by $(i,j) \in \mathcal{E}$ instead. The advantage of introducing the additional variables is that we can have a partial transfer of the data between two vertices rather than a full transfer. This will be elaborated in Example 3.9.
Lemma 3.4. (Expressing $v$ as a sum) Recall the definitions of $D$ and $H_{((i,j),k)}$ in (3.2) and (3.3). There is a $C_1 > 0$ such that for all $v \in D^1$ and $E' \subset E$ such that $E'$ connects $V$, we can find $z_{((i,j),k)} \in H_{((i,j),k)}^1$ for all $((i,j),k) \in E'$ such that $\sum_{((i,j),k) \in E'} z_{((i,j),k)} = v$ and $\|z_{((i,j),k)}\| \leq C_1 \|v\|$ for all $((i,j),k) \in E'$.

Proof. This is elementary linear algebra. We refer to [Pan18a] for a proof of a similar result.

3.2. Algorithm description and preliminaries. In this subsection, we present Algorithm 3.5 below and recall some of the results that were presented in [Pan18a] that are necessary for further discussions.

Recall that in the one node case in Section 2, the subdifferentiable function $f_i(\cdot)$ is handled using lower approximates. In addition to (1.6), we need to consider the function

$$F_{n,w}(\{z_\alpha\}_{\alpha \in \hat{E} \cup V}) := -\frac{1}{2} \left\| x - \sum_{\alpha \in \hat{E} \cup V} z_\alpha \right\|^2 + \frac{1}{2} \|x\|^2 - \sum_{\alpha \in \hat{E} \cup V} f^*_{\alpha,n,w}(z_\alpha),$$

(3.4)

where for all $n \geq 1$ and $w \in \{1, \ldots, \bar{w}\}$, $f_{\alpha,n,w} : [\mathbb{R}^m]^{\mathcal{V}} \rightarrow \mathbb{R}$ satisfies

$$f_{\alpha,n,w}(\cdot) = f_{\alpha}(\cdot) \quad \text{for all } \alpha \in [\hat{E} \cup V \setminus \mathcal{V}_4]$$

and

$$f_{\alpha,n,w}(\cdot) \leq f_{\alpha}(\cdot) \quad \text{for all } \alpha \in \mathcal{V}_4.$$

(3.5a)

(3.5b)

So $F_{n,w}(\cdot) \leq F(\cdot)$. We present Algorithm 3.5 on the next page.

Even though Algorithm 3.5 is described so that each node $i \in \mathcal{V}$ and $(i,j), k \in \hat{E}$ is associated with a dual variable $z_\alpha \in [\mathbb{R}^m]^{\mathcal{V}}$, we point out that the size of the dual variable $z_\alpha$ that needs to be stored in each node and edge is small due to sparsity.

Proposition 3.6. (Sparsity of $z_\alpha$) We have $[z_{i,j}^{n,w}]_j = 0$ for all $j \in \mathcal{V} \setminus \{i\}$, $n \geq 1$ and $w \in \{0, 1, \ldots, \bar{w}\}$. Similarly, for all $n \geq 1$, $w \in \{0, 1, \ldots, \bar{w}\}$ and $(e,k) \in \hat{E}$, the vector $z_{(e,k)}^{n,w} \in [\mathbb{R}^m]^{\mathcal{V}}$ satisfies $[z_{(e,k)}^{n,w}]_j = 0$ unless $k = k'$ and $i$ is an endpoint of $e$.

Proof. The proof of this result is similar to the corresponding result in [Pan18a]. The claim for $z_{i,j}^{n,w}$ relies on the fact that $f_{i,n,w}(\cdot)$ depends only on the $i$-th component, and the claim for $z_{(e,k)}^{n,w}$ relies on the fact that $f_{(e,k)}(\cdot) = \partial H_{(e,k)}^\perp (\cdot)$, with $H_{(e,k)}^\perp$ containing vectors that are zero in all but 2 coordinates.

Dykstra’s algorithm is traditionally written in terms of solving for the primal variable $x$. For completeness, we show the equivalence between (3.7) and the primal minimization problem. The proof is easily extended from [Pan17].

Proposition 3.7. (On solving (3.7)) If a minimizer $\{z_{\alpha}^{n,w}\}_{\alpha \in S_{n,w}}$ for (3.7) exists, then the $x^{n,w}$ in (3.14) satisfies

$$x^{n,w} = \arg \min_{x \in [\mathbb{R}^m]^{\mathcal{V}}} \sum_{\alpha \in S_{n,w}} f_{\alpha,n,w}(x) + \frac{1}{2} \left\| x - \sum_{\alpha \in S_{n,w}} z_\alpha^{n,w} \right\|^2 .$$

(3.8)

Conversely, if $x^{n,w}$ solves (3.8) with the dual variables $\{z_\alpha^{n,w}\}_{\alpha \in S_{n,w}}$ satisfying

$$z^{n,w}_\alpha \in \partial f_{\alpha,n,w}(x^{n,w})$$

and

$$x^{n,w} - x + \sum_{\alpha \in S_{n,w}} z^{n,w}_\alpha + \sum_{\alpha \in S_{n,w}} \breve{z}^{n,w}_\alpha = 0,$$

(3.9)

then $\{\breve{z}^{n,w}_\alpha\}_{\alpha \in S_{n,w}}$ solves (3.7).
Algorithm 3.5. (Distributed Dykstra’s algorithm) Consider the problem (1.4) along with the associated dual problem (1.5).

Let \( w \) be a positive integer. Let \( C_1 > 0 \) satisfy Lemma 3.3. For each \( \alpha \in [\hat{\mathcal{E}} \cup \mathcal{V}] \setminus \mathcal{V}_n \), let \( f_{\alpha,n,w} : [\mathbb{R}^m]^{[\mathcal{V}]} \to \mathbb{R} \) be as defined in (3.5). Our distributed Dykstra’s algorithm is as follows:

01 Let

- \( z_i^{1,0} \in [\mathbb{R}^m]^{[\mathcal{V}]} \) be a starting dual vector for \( f_i(\cdot) \) for each \( i \in \mathcal{V} \) so that \( [z_i^{1,0}]_j = 0 \in \mathbb{R}^m \) for all \( j \in \mathcal{V} \setminus \{i\} \).
- \( v_H^{1,0} \in D^\perp \) be a starting dual vector for (1.5).
  - Note: \( \{z_{i(j),k}^{n,0}\}_{(i,j) \in \mathcal{E}} \) is defined through \( v_H^{n,0} \) in (3.6).

02 For each \( i \in \mathcal{V}_n \), let \( f_{i,1,0} : [\mathbb{R}^m]^{[\mathcal{V}]} \to \mathbb{R} \) be a function such that \( f_{i,1,0}(\cdot) \leq f_i(\cdot) \)

03 For \( n = 1, 2, \ldots \)

04 Let \( \hat{\mathcal{E}}_n \subset \hat{\mathcal{E}} \) be such that \( \hat{\mathcal{E}}_n \) connects \( \mathcal{V} \) in the sense of Definition 3.2.

05 Define \( \{z_{i(j),k}^{n,0}\}_{(i,j) \in \mathcal{E}} \) so that:

\[
\begin{align*}
z_{i(j),k}^{n,0} &= 0 \text{ for all } ((i,j), k) \notin \hat{\mathcal{E}}_n \quad (3.6a) \\
z_{i(j),k}^{n,0} &\in H^\perp_{((i,j), k)} \text{ for all } ((i,j), k) \in \hat{\mathcal{E}}_n \quad (3.6b) \\
\|z_{i(j),k}^{n,0}\| &\leq C_1 v_H^{n,0} \text{ for all } ((i,j), k) \in \hat{\mathcal{E}}_n \quad (3.6c) \\
\sum_{(i,j),k} z_{i(j),k}^{n,0} &\equiv v_H^{n,0} \quad (3.6d)
\end{align*}
\]

(This is possible by Lemma 3.3.)

06 For \( w = 1, 2, \ldots , \hat{w} \)

07 Choose a set \( S_{n,w} \subset \hat{\mathcal{E}}_n \cup \mathcal{V} \) such that \( S_{n,w} \neq \emptyset \).

08 If \( S_{n,w} \subset \mathcal{V}_n \), then

09 Apply Algorithm 3.4

10 else

11 Set \( f_{i,n,w}(\cdot) := f_{i,n,w-1}(\cdot) \) for all \( i \in \mathcal{V}_n \).

12 Define \( \{z_{\alpha}^{n,w}\}_{\alpha \in S_{n,w}} \) by

\[
\{z_{\alpha}^{n,w}\}_{\alpha \in S_{n,w}} = \arg \min_{z_{\alpha} \in S_{n,w}} \frac{1}{2} \left\| x - \sum_{\alpha \in S_{n,w}} z_{\alpha}^{n,w-1} - \sum_{\alpha \in S_{n,w}} z_{\alpha} \right\|^2 + \sum_{\alpha \in S_{n,w}} f_{\alpha,n,w}(z_{\alpha}) 
\]

(3.7)

13 end if

14 Set \( z_{\alpha}^{n,w} := z_{\alpha}^{n,w-1} \) for all \( \alpha \notin S_{n,w} \).

15 End For

16 Let \( z_i^{n+1,w} := z_i^{n,w} \) for all \( i \in \mathcal{V} \) and \( v_H^{n+1,0} = v_H^{n,w} \).

17 Let \( f_{i,n+1,0}(\cdot) := f_{i,n,w}(\cdot) \) for all \( i \in \mathcal{V}_n \).

18 End For

Remark 3.8. (Irrelevance of \( z_{i(j),k}^{n,w} \)) In [Pan18a], we explained that each node \( i \in \mathcal{V} \) needs to keep track of just \( \|x - v_H^{n,w}\| \in \mathbb{R}^m \) and \( [z_i^{n,w}]_i \in \mathbb{R}^m \), and does
not have to keep track of any part of the vectors $z^{n,w}_{(i,j),k} \in \mathbb{R}^m$ for $(i,j),k) \in \mathcal{E}$. The same is true for Algorithms 3.9 and 3.10 here. The reason for introducing $z^{n,w}_{(i,j),k} \in [\mathbb{R}^m]^{|V|}$ is that the proof of the convergence result in Theorem 3.10 needs (A.9), which in turn needs the variables $z^{n,w}_{(i,j),k}$.

**Example 3.9.** (Partial communication of data) Fix some $(i,j) \in \mathcal{E}$ and some set $K \subset \{1,\ldots,m\}$. Suppose the set $S_{n,w}$ is chosen to be $\{(i,j),k) : k \in K\}$. Then $x^{n,w}$ is obtained from (3.8), which tells that $x^{n,w}$ is the projection of $[\bar{x} - \sum_{\alpha \notin S_{n,w}} z^{n,w}_{\alpha}] \cap _{i,j} H_{(i,j),k}$. Since $H_{(i,j),k}$ are all affine spaces with normals $z^{n,w-1}_{(i,j),k}$, $x^{n,w}$ is also the projection of

$$[x^{n,w-1}]_{\bar{v}} = \begin{cases} 1/2\left([x^{n,w-1}]_k + [x^{n,w-1}]_{\bar{k}}\right) & \text{if } i' \in \{i,j\} \text{ and } \bar{k} \in K \text{.} \\ \left([x^{n,w-1}]_{\bar{v}}\right) & \text{otherwise.} \end{cases}$$

As mentioned in Remark 3.8, there is no need to keep track of the dual variables $z^{n,w}_{(i,j),k}$ to run Algorithm 3.5. So the larger $K$ is, the more variables are updated. Thus in Algorithm 3.10 computations can be performed continuously even when not all the data is communicated. In other words, communications will not be a bottleneck for Algorithm 3.5.

3.3. **Subroutine for subdifferentiable functions.** If $\mathcal{V}_4 = \emptyset$, then Algorithm 3.5 corresponds mostly to the algorithm in [Pan18a] because there are no subdifferentiable functions. In this subsection, we present and derive Algorithm 3.10, which is a subroutine within Algorithm 3.5 to handle subdifferentiable functions.

We state some notation necessary for further discussions. For any $\alpha \in \mathcal{E} \cup \mathcal{V}$ and $n \in \{1,2,\ldots\}$, let $p(n,\alpha)$ be

$$p(n,\alpha) := \max \{ w' : w' \leq \bar{w}, \alpha \in S_{n,w'} \}.$$ 

In other words, $p(n,\alpha)$ is the index $w'$ such that $\alpha \in S_{n,w'}$ but $\alpha \notin S_{n,k}$ for all $k \in \{w' + 1,\ldots, \bar{w}\}$. It follows from line 14 in Algorithm 3.5 that

$$z^{n,p(n,\alpha)}_{\alpha} = z^{n,p(n,\alpha)+1}_{\alpha} = \cdots = z^{n,w}_{\alpha} \text{ for all } \alpha \in \mathcal{E} \cup \mathcal{V}. \quad (3.10)$$

Moreover, $(i,j),\bar{k} \notin \mathcal{E}_n$ implies $(i,j),\bar{k} \notin S_{n,w}$ for all $w \in \{1,\ldots, \bar{w}\}$, so

$$0 \leq z^{n,0}_{(i,j),\bar{k}} = z^{n,1}_{(i,j),\bar{k}} = \cdots = z^{n,w}_{(i,j),\bar{k}} \text{ for all } ((i,j),\bar{k}) \in \mathcal{E} \setminus \mathcal{E}_n. \quad (3.11)$$

We present Algorithm 3.10 on the following page.

We make three assumptions that will be needed for the proof of convergence of Theorem 3.11.

**Assumption 3.11.** *(Start of Algorithm 3.10)* Recall that at the start of Algorithm 3.11, $S_{n,w} \subset \mathcal{V}_4$. We make three assumptions.

1. Suppose $(n,w)$ is such that $w > 1$ and $S_{n,w} \subset \mathcal{V}_4$ so that Algorithm 3.11 is invoked. Then for all $i \in S_{n,w}$, $z^{n,w-1}_{i} \in \mathbb{R}^m$ is the optimizer to the problem

$$\min_{z \in \mathbb{R}^m} \frac{1}{2} \|x - v^{n,w-1}_{H}i - z\|^2 + f^{*}_{i,n,w-1}(z). \quad (3.16)$$
Algorithm 3.10. (Subalgorithm for subdifferentiable functions) This algorithm is run when line 9 of Algorithm [3.9] is reached. Suppose $S_{n,w} \subset V_4$ and Assumption [3.11] holds.

01 For each $i \in S_{n,w}$
02 For $f_{i,n,w-1} : \mathbb{R}^m \to \mathbb{R}$ defined in (3.21), consider
03 Let the primal and dual solutions of (3.12) be $x_i^+$ and $z_i^+$
04 Define $f_{i,n,w} : \mathbb{R}^m \to \mathbb{R}$ to be the affine function
05 In other words, $f_{i,n,w}(\cdot)$ is chosen such that the primal and dual optimizers to (3.12) coincide with that of
06 Define the function $f_{i,n,w} : [\mathbb{R}^m]^{|V|} \to \mathbb{R}$ and the dual vector $z_{n,w}^i \in [\mathbb{R}^m]^{|V|}$ to be
07 End for
08 For all $i \in V_4 \setminus S_{n,w}$, $f_{i,n,w}(\cdot) = f_{i,n,w-1}(\cdot)$.

In other words, suppose $w_i = 1$ is the largest $w$ such that $i \in S_{n,n'}$ and $i \not\in S_{n,w}$ for all $\tilde{w} \in \{w' + 1, w' + 2, \ldots, w - 1\}$. Then for all $\tilde{w} \in \{w_i + 1, \ldots, w - 1\}$, $(e, \tilde{k}) \not\in S_{n,\tilde{w}}$ if $i$ is an endpoint of $e$.

(2) Suppose that for all $i \in V_4$ and $\tilde{w} \in \{n(i) + 1, \ldots, \tilde{w}\}$, $(e, \tilde{k}) \not\in S_{n,\tilde{w}}$ if $i$ is an endpoint of $e$. (This implies $x_i^{n_i,p_i(n_i)} = x_i^{n,w}$.)

(3) Suppose that $S_{n,1} = V_4$ for all $n > 1$.

Remark 3.12. We need Assumption [3.11](1) for Proposition [3.12] which is in turn needed for the proof of Theorem [3.15]. We need Assumption [3.11](2) so that the analogue of Lemma [3.10](1) holds, which in turn is used in the proof of Theorem [3.19](iv). Also, Assumption [3.11](1) is seen to be satisfied if $S_{n,w} \subset S_{n,w-1}$ if $S_{n,w} \subset V_4$.

Remark 3.13. (On the problem (3.12)) Consider the case where $S_{n,w} = \{i\}$ first, where $i \in V_4$. If $i$ were in $V \setminus V_4$ instead, $z_{i,n,w}^i$ is the minimizer of

$$\min_{z_i \in [\mathbb{R}^m]^{|V|}} \frac{1}{2} \left\| x - v_H^{n,w-1} - \sum_{j \in V \setminus \{i\}} z_j^{n,w-1} - z_i \right\|^2 + f_i^*(z_i).$$

(3.17)

When $i \in V_4$, we use $f_{i,n,w-1}(\cdot)$, where $f_{i,n,w-1}(\cdot) \leq f_i(\cdot)$, instead of $f_i(\cdot)$. This gives $f_{i,n,w}^*(\cdot) \geq f_i^*(\cdot)$. Instead of (3.17), we now have

$$\min_{z_i \in [\mathbb{R}^m]^{|V|}} \frac{1}{2} \left\| x - v_H^{n,w-1} - \sum_{j \in V \setminus \{i\}} z_j^{n,w-1} - z_i \right\|^2 + f_{i,n,w-1}^*(z_i).$$

(3.18)
The dual of (3.18) is (up to a constant independent of $\mathbf{x}$)

$$\min_{\mathbf{x} \in [\mathbb{R}^m]|\mathcal{V}|^2} \frac{1}{2} \left\| \mathbf{x} - \mathbf{v}_H^{n,w-1} - \sum_{j \in \mathcal{V}\setminus\{i\}} \mathbf{z}_j^{n,w-1} \right\|^2 + f_{i,n,w-1}(\mathbf{x}). \quad (3.19)$$

Since $\mathbf{z}_i^{n,w-1} \in [\mathbb{R}^m]|\mathcal{V}|$ and $\mathbf{z}_i^{n,w} \in [\mathbb{R}^m]|\mathcal{V}|$ are such that the components in $\mathcal{V}\setminus\{i\}$ are all zero by Proposition 3.6, the problem (3.19) reduces to

$$\min_{\mathbf{x} \in [\mathbb{R}^m]|\mathcal{V}|^2} \frac{1}{2} \left\| \mathbf{x} - \mathbf{v}_H^{n,w-1} \right\|^2 + f_{i,n,w-1}(\mathbf{x}). \quad (3.20)$$

Suppose that the minimizer of (3.18) is $\mathbf{z}_i^{n,w-1}$, which is the case when Assumption 3.11 holds. Then the minimizer of (3.20) is $[\mathbf{x}^{n,w-1}]_i$, which is also $[\mathbf{x} - \mathbf{v}_H^{n,w-1} - \mathbf{z}_i^{n,w-1}]_i$ by (3.1). Construct $\tilde{f}_{i,n,w-1} : [\mathbb{R}^m] \rightarrow \mathbb{R}$ by

$$\tilde{f}_{i,n,w-1}(x) := f_i([\mathbf{x} - \mathbf{v}_H^{n,w-1} - \mathbf{z}_i^{n,w-1}]_i) + \langle s, x - [\mathbf{x} - \mathbf{v}_H^{n,w-1} - \mathbf{z}_i^{n,w-1}]_i \rangle, \quad (3.21)$$

where $s \in \partial f_i([\mathbf{x} - \mathbf{v}_H^{n,w-1} - \mathbf{z}_i^{n,w-1}]_i)$. The primal problem that we now consider is (3.12).

**Remark 3.14.** (On the condition $S_{n,1} = V_4$) Throughout this paper, we assumed $S_{n,1} = V_4$ in Assumption 3.11 with this condition would not be truly asynchronous, but it is relatively easy to enforce this condition. One way to enforce this condition is to use a global clock. Another way to enforce this condition is to use the sparsity of $\mathbf{z}_\alpha$ in Proposition 3.6. Suppose that $\{S_{n,w}\}_{w=1}^\wbar$ is such that for all $i \in V_4$, $S_{n,w_i} = \{i\}$ for some $w_i \in \{1, \ldots, \wbar\}$. Suppose also that for all $i, j \in V_4$ such that $w_i < w_j$:

$$\ast \text{ There are no } (e, k) \in \hat{E} \text{ such that } i \text{ and } j \text{ are the two endpoints of } e \text{ and } (e, k) \in S_{n,w'} \text{ for some } w' < w_j. $$

If condition $\ast$ holds for some $i, j \in V_4$, then the sparsity of $\mathbf{z}_\alpha^{n,w}$ implies that if we changed from $S_{n,w_i} = \{i\}$ and $S_{n,w_j} = \{j\}$ to $S_{n,w_i} = \{i, j\}$ and $S_{n,w_j} = \emptyset$, then the iterates $\{\mathbf{x}^{n,w}\}_{w=1}^\wbar$ obtained will remain equivalent. It is possible to ensure $\ast$ for all $i, j \in V_4$ using a signal from a fixed node in $V$ propagated as computations in the algorithm are carried out.

As mentioned in Remark 2.7, the problem (3.12) is still easy to solve if $f_{i,n,w-1}(\cdot)$ and $f_{i,n,w-1}(\cdot)$ are affine functions with the known parameters $[\mathbf{x} - \mathbf{v}_H^{n,w-1}]_i$ and $\mathbf{z}_i^{n,w-1}$.

Next, for the primal optimizer $x^+_i$ defined in line 3 of Algorithm 3.10, we can construct the affine function $f_{i,n,w} : [\mathbb{R}^m] \rightarrow \mathbb{R}$ to be such that

$$\min_{\mathbf{x} \in [\mathbb{R}^m]} \frac{1}{2} \left\| \mathbf{x} - \mathbf{v}_H^{n,w-1} \right\|^2 \left[1 + f_{i,n,w}(x) \right]$$

has the same minimizer and objective value as (3.12). The function $f_{i,n,w}(\cdot)$ can be checked to be (3.12). It is clear to see that $f_{i,n,w}(\cdot) \leq \max\{f_{i,n,w-1}(\cdot), f_{i,n,w-1}(\cdot)\}$. Since both $f_{i,n,w-1}(\cdot)$ and $f_{i,n,w-1}(\cdot)$ are both by definition lower approximates of $f_i(\cdot)$, $f_{i,n,w}(\cdot)$ will also be a lower approximate of $f_i(\cdot)$. The function $f_{i,n,w} : [\mathbb{R}^m]|\mathcal{V}| \rightarrow \mathbb{R}$ is constructed to be

$$f_{i,n,w}(\mathbf{x}) = f_{i,n,w}([\mathbf{x}]_i).$$
Similarly, if Assumption 3.11(2) and (3) hold, then Proposition 2.8 implies that the problem (3.16) corresponds to (3.21), the function (2.3) to (3.21), the problem (2.4) to (3.12), and the function $h_{w+1}(\cdot)$ in line 9 of Algorithm 3.1 to (3.13).

One way to understand Proposition 2.8 is to see that any change in the primal objective value gives the same change in the dual objective value. We have the following result.

**Proposition 3.16.** Suppose $(n,w)$ is such that $w > 1$ and $S_{n,w} \subset V_i$ so that Algorithm 3.10 is run, and Assumption 3.11(1) holds. Then we have

$$\frac{1}{2}\|x^{n,w} - x^{n,w-1}\|^2 \leq F_{n,w}((z^{n,w}_{\alpha})_{\alpha \in \mathcal{E} \cup \mathcal{V}}) - F_{n,w-1}((z^{n,w-1}_{\alpha})_{\alpha \in \mathcal{E} \cup \mathcal{V}}).$$

**Corollary 3.15.** (Similarities to the one node case) Note that the problem (2.22) corresponds to (3.10), the function (2.23) to (3.21), the problem (2.4) to (3.12), and the function $h_{w+1}(\cdot)$ in line 9 of Algorithm 2.1 to (3.13).

Proof. Recall Proposition 3.6 on the sparsity of the $z^{w}_{i,n} \in [\mathbb{R}^{n}]^{\mathcal{V}}$. Recall that in line 3 of Algorithm 3.10 the primal and dual optimal solutions of (3.12) are $x_1^+$ and $z_1^+$. We can see that $x_1^+ = [x^{n,w}]_1$ and $z_1^+ = [z^{n,w}]_1$. Let the dual and primal optimal solutions of (3.21) be $z_1^0$ and $z_1^-$, which are $z_1^0 = [z^{n,w-1}]_1$ and $z_1^- = [x^{n,w-1}]_1$ respectively. By Proposition 2.8 and the forms of the problems (3.12) and (3.20), we have $x_1^+ + z_1^+ = x_1^- + z_1^-$. Thus $z_1^+ - z_1^- = -(x_1^- - x_1^+)$. In other words,

$$[x^{n,w} - x^{n,w-1}]_1 = -[z^{n,w} - z^{n,w-1}]_1.$$  

Note that since $S_{n,w} \cap \mathcal{E} = \emptyset$, $v^{n,w}_H = v^{n,w-1}_H$. We have the following inequality chain, which we explain in a moment.

$$f_{i,n,w}([x^{n,w}]_i) + \frac{1}{2}\|[\bar{x} - v^{n,w}_H]_i - [x^{n,w}]_i\|^2 \leq f_{i,n,w-1}([x^{n,w}]_i) + \frac{1}{2}\|[\bar{x} - v^{n,w}_H]_i - [x^{n,w}]_i\|^2 \leq f_{i,n,w-1}([x^{n,w-1}]_i) + \frac{1}{2}\|[\bar{x} - v^{n,w-1}_H]_i - [x^{n,w-1}]_i\|^2 + \frac{1}{2}\|[x^{n,w-1}]_i - [x^{n,w}]_i\|^2.$$

The equation in (3.25) comes from the fact that $[x^{n,w}]_1$ be the minimizing of (3.12) is such that $f_{i,n,w-1}([x^{n,w}]_i) = f_{i,n,w-1}([x^{n,w}]_i)$, and $f_{i,n,w}([x^{n,w}]_i)$ is designed through (3.14) so that $f_{i,n,w}([x^{n,w}]_i) = f_{i,n,w-1}([x^{n,w}]_i)$. The inequality in (3.25) follows from the design of $f_{i,n,w-1}([x^{n,w}]_i)$ through (3.14), which implies that $[x^{n,w-1}]_i$ is the minimizer of $f_{i,n,w-1}([x^{n,w}]_i) + \frac{1}{2}\|[\bar{x} - v^{n,w}_H]_i - [x^{n,w}]_i\|^2$.

Since $S_{n,w} \cap \mathcal{E} = \emptyset$, we have $v^{n,w-1}_H = v^{n,w}_H$. Let $\beta_i$ be defined by

$$\beta_i := (f_{i,n,w}([z^{n,w}]_i) + \frac{1}{2}\|[\bar{x} - v^{n,w}_H]_i - [z^{n,w}]_i\|^2) - (f_{i,n,w-1}([z^{n,w-1}]_i) + \frac{1}{2}\|[\bar{x} - v^{n,w-1}_H]_i - [z^{n,w-1}]_i\|^2).$$

**Proposition 2.8** implies that

$$f_{i,n,w}([z^{n,w}]_i) + \frac{1}{2}\|[\bar{x} - v^{n,w}_H]_i - [z^{n,w}]_i\|^2 = -f_{i,n,w}([x^{n,w}]_i) + \frac{1}{2}\|[\bar{x} - v^{n,w}_H]_i - [x^{n,w}]_i\|^2 - \frac{1}{2}\|[\bar{x} - v^{n,w}_H]_i - [x^{n,w}]_i\|^2.$$
An equation similar to (3.27) involving $f_v$ which leads to our result. The proof of the second statement is exactly the same.

3.17. the convergence proof.

3.18. quadratics (3.12) analytically, but doing so without Assumption 3.11 would affect that we can apply the observation in Remark 2.7 to minimize the maximum of two shares many similarities to the original proof in [Pan18a], we describe the new steps of the proof in this subsection that were not already covered and defer the rest of the proof to the appendix.

We remark on the design of Algorithm 3.5.

**Proof.** There are two cases. The first case is when (3.7) is invoked. By taking the optimality conditions in (3.7) with respect to $z^\alpha$ for $\alpha \in S_{n,w}$ and making use of (3.1) to get $x^{n,w} = \bar{x} - \sum_{\alpha \in \bar{E} \cup V} z^{\alpha,n,w}$, we deduce (a). The second case is when Algorithm 3.10 is invoked, and is similar. The equivalence of (a), (b) and (c) is standard.

For all valid $(n, w)$, since $f_{\alpha,n,w}(\cdot) \leq f_{\alpha}(\cdot)$ for all $\alpha \in \mathcal{V}_4$, we have $f_{\alpha,n,w}(\cdot) \geq f_{\alpha}^*(\cdot)$. Let $D_{\alpha,n}$ and $E_{\alpha,n}$ be defined to be

$$D_{\alpha,n} := f_{\alpha,n,p(n,\alpha)}^{\ast}(z^{n,p(n,\alpha)}_{\alpha}) - f_{\alpha}^*(z^{n,p(n,\alpha)}_{\alpha}) \geq 0 \quad (3.28a)$$

and

$$E_{\alpha,n} := f_{\alpha}(x - v^A_{p(n,\alpha)}) - f_{\alpha,n,p(n,\alpha)}(x - v^A_{p(n,\alpha)}) \geq 0. \quad (3.28b)$$

When $\alpha \in (\bar{E} \cup V) \setminus \mathcal{V}_4$, then $E_{\alpha,n} = D_{\alpha,n} = 0$ for all $n$. Next, we have

$$f_{\alpha}^*(z^{n,p(n,\alpha)}_{\alpha}) + f_{\alpha}(x - v^A_{p(n,\alpha)}) = f_{\alpha,n,p(n,\alpha)}(z^{n,p(n,\alpha)}_{\alpha}) + f_{\alpha,n,p(n,\alpha)}(x - v^A_{p(n,\alpha)}) + E_{\alpha,n} - D_{\alpha,n} \quad (3.29)$$

We now state the main convergence theorem of this paper.

**Theorem 3.19.** (Convergence to primal minimizer) Consider Algorithm 3.5. Assume that the problem (1.4) is feasible, and for all $n \geq 1$, $\bar{E}_n = \bigcup_{w=1}^n S_{n,w} \cap \bar{E}$, and $[\bigcup_{w=1}^n S_{n,w}] \supseteq \mathcal{V}$. Suppose that Assumption 3.11 holds.

For the sequence $\{z^{n,w}_{\alpha}\}_{1 \leq \alpha \leq \infty} \subseteq [\mathbb{R}^m]^{\bar{E}}$ for each $\alpha \in \bar{E} \cup \mathcal{V}$ generated by Algorithm 3.3 and the sequences $\{v^H_{n,w}\}_{0 \leq w \leq \infty} \subseteq [\mathbb{R}^m]^{\bar{E}}$ and $\{v^A_{n,w}\}_{0 \leq w \leq \infty} \subseteq [\mathbb{R}^m]^{\mathcal{V}}$ thus derived, we have:
(i) The sum $\sum_{i=1}^{\infty} \sum_{w=1}^{\hat{w}} \|v_A^{n,w} - v_A^{n,w-1}\|^2$ is finite and $\{F_n,w(\{z_{\alpha}^{n,w}\}_{\alpha \in \mathcal{E} \cup \mathcal{V}})\}_{n=1}^{\infty}$ is nondecreasing.

(ii) There is a constant $C$ such that $\|v_A^{n,w}\|^2 \leq C$ for all $n \in \mathbb{N}$ and $w \in \{1, \ldots, \hat{w}\}$.

(iii) For all $i \in \mathcal{V}_3 \cup \mathcal{V}_4$, $n \geq 1$ and $w \in \{1, \ldots, \hat{w}\}$, the vectors $z_i^{n,w}$ are bounded. Suppose also that

(1) There are constants $A$ and $B$ such that

$$\sum_{\alpha \in \mathcal{E} \cup \mathcal{V}} \|z_{\alpha}^{n,w}\| \leq A\sqrt{n} + B \text{ for all } n \geq 0. \quad (3.30)$$

Then

(iv) For all $\alpha \in (\hat{E} \cup \hat{V}) \setminus \mathcal{V}_4$, we have $E_{\alpha,n} = 0$. Also, for all $i \in \mathcal{V}_4$, we have $\lim_{n \to \infty} E_{n,i} = 0$.

(v) There exists a subsequence $\{v_A^{n_k,\hat{w}}\}_{k=1}^{\infty}$ of $\{v_A^{n,w}\}_{n=1}^{\infty}$ which converges to some $v_A^* \in [\mathbb{R}^m]^{\mathcal{V}}$ and that

$$\lim_{k \to \infty} \langle v_A^{n_k,\hat{w}} - v_A^{n_k,\hat{p}(n_k,\alpha)}, z_{\alpha}^{n_k,\hat{w}} \rangle = 0 \text{ for all } \alpha \in (\hat{E} \cup \hat{V}) \setminus \mathcal{V}_4.

(vi) Let $f(\cdot) = \sum_{\alpha \in (\hat{E} \cup \hat{V}) \setminus \mathcal{V}_4} f_{\alpha}(\cdot)$. For the $v_i^*$ in (v), $x_{0} - v_A^*$ is the minimizer of the primal problem (1.3) and

$$\lim_{k \to \infty} F_n,w((\{z_{\alpha}^{n_k,\hat{w}}\}_{\alpha \in \mathcal{E} \cup \mathcal{V}})_{n=1}^{\infty}) = \lim_{k \to \infty} F((\{z_{\alpha}^{n_k,\hat{w}}\}_{\alpha \in \mathcal{E} \cup \mathcal{V}})_{n=1}^{\infty}) = \frac{1}{2}\|v_A^*\|^2 + f(\hat{x} - v_A^*). \quad (3.31)$$

The properties (i) to (vi) in turn imply that $\lim_{n \to \infty} x^{n,\hat{w}}$ exists and equals $\hat{x} - v_A^*$, which is the primal minimizer of (1.3).

The proofs of parts (i), (ii), (v) and (vi) are similar to the proof in [Pan18a], and (iii) and (iv) are new. We shall prove (iii) and (iv) here and defer the rest of the proof to the appendix.

Proof of Theorem 3.19(iii). In view of line 14 in Algorithm 3.5, it suffices to prove that $z_i^{n,w}$ is bounded if $i \in S_{n,w}$. By the sparsity pattern in Proposition 3.6 for each $i \in \mathcal{V}_3 \cup \mathcal{V}_4$, $z_i^{n,w}$ is bounded if and only if $\{z_i^{n,w}\}_{i \in S_{n,w}}$ is bounded. Since $\{[\hat{x} - v_A^*]_{i \in S_{n,w}}\}_{1 \leq n \leq \infty}$ is bounded by (ii), it is clear that $\{z_i^{n,w}\}_{i \in S_{n,w}}$ is bounded if and only if $\{[\hat{x} - v_H^*]_{i \in S_{n,w}}, \hat{w} \leq w \leq \hat{w}, 1 \leq n \leq \infty\}$ is bounded. Seeking a contradiction, suppose $\{[\hat{x} - v_H^*]_{i \in S_{n,w}}\}_{1 \leq n \leq \infty}$ is unbounded. We look at the problem

$$\min_{x \in \mathbb{R}^m} \frac{1}{2}\|\hat{x} - v_H^*\|_2^2 - f_i(x) \quad (3.32)$$

and consider two possibilities. Let $\hat{x}_i^{n,w}$ be the primal solution to (3.32). Note that if $i \in \mathcal{V}_3$, then $\hat{x}_i^{n,w}$ is $[\hat{x}^{n,w}]_i$. If the $\{\hat{x}_i^{n,w}\}_{n,w}$ are bounded, then the dual solution of (3.32) is $[\hat{x} - v_H^*]_i - \hat{x}_i^{n,w}$, which will be unbounded. A standard compactness argument shows that there is a point $\hat{x} \in \mathbb{R}^m$ for which the set $\partial f_i(\hat{x})$ is unbounded, which contradicts $\text{dom}(f_i) = \mathbb{R}^m$.

If the corresponding primal solutions $\hat{x}_i^{n,w}$ are unbounded, consider $\{\hat{z}_i^{n,w}\}_{\alpha \in (\hat{E} \cup \hat{V}) \setminus \{i\}}$, where

$$\hat{z}_i^{n,w} = \begin{cases} [\hat{x} - v_H^*]_i - \hat{x}_i^{n,w} & \text{if } j = i \\ 0 & \text{otherwise.} \end{cases}$$
Let $\tilde{F}_{n,w}(\cdot)$ be defined to be

$$
\tilde{F}_{n,w}(\{z_{\alpha}\}_{\alpha \in E \cup V}) := -\frac{1}{2} \left\| x - \sum_{\alpha \in E \cup V} z_{\alpha} \right\|^2 + \frac{1}{2} \left\| x \right\|^2 + \sum_{\alpha \in (E \cup V) \setminus \{i\}} f_{n,n,w}(z_{\alpha}) - f_{i}(z_{i})
$$

(3.33)

Then $F_{n,w}(\cdot) \leq \tilde{F}_{n,w}(\cdot) \leq F(\cdot)$. Also, Proposition 2.8 shows that $[\check{z}^{n,w}_{i}]$ is a solution to (3.32). So

$$
F_{n,w}(\{z_{\alpha}^{n,w}\}_{\alpha \in E \cup V}) \leq \tilde{F}_{n,w}(\{z_{\alpha}^{n,w}\}_{\alpha \in E \cup V}) \leq \check{F}_{n,w}(\{\check{z}_{\alpha}^{n,w}\}_{\alpha \in E \cup V}) \leq F(\{\check{z}_{\alpha}^{n,w}\}_{\alpha \in E \cup V}).
$$

(3.34)

Next, suppose $x^{*}$ is a solution of (1.4). Then

$$
\frac{1}{2} \left\| x - x^{*} \right\|^2 + \sum_{\alpha \in E \cup V} f_{n}(x^{*}) - F_{n,w}(\{z_{\alpha}^{n,w}\}_{\alpha \in E \cup V})
\geq \frac{1}{2} \left\| x - x^{*} \right\|^2 + \sum_{\alpha \in E \cup V} f_{n}(x^{*}) - F(\{\check{z}_{\alpha}^{n,w}\}_{\alpha \in E \cup V})
$$

(3.35)

$$\geq \frac{1}{2} \left\| x^{*} - \sum_{\alpha \in E \cup V} \check{z}_{\alpha}^{n,w} \right\|^2
\geq \frac{1}{2} \left\| x^{*} - \check{z}_{i}^{n,w} \right\|^2.
$$

The above inequality and the unboundedness of $\check{z}_{i}^{n,w}$ implies that the duality gap would go to infinity, which contradicts part (i). Thus we are done. 

**Proof of Theorem 3.11 (iv).** The first sentence of this claim is immediate from (3.35). We now prove the second sentence. Seeking a contradiction, suppose that $\limsup_{n \to \infty} E_{i,n} > 0$. In Algorithm 3.10 in view of Assumption 3.11(2), $[\check{z}_{i}^{n,w}]_{i}$ is the minimizer of the problem

$$
\min_{x \in \mathbb{R}^{m}} \frac{1}{2} \left\| x - v_{H}^{n,w} \right\|_{i}^2 - z^{*} + f_{i,n,w}(z).
$$

(3.36)

The associated primal problem is, up to a constant independent of $x$,

$$
\min_{x \in \mathbb{R}^{m}} \frac{1}{2} \left\| x - v_{H}^{n,w} \right\|_{i}^2 - x + f_{i,n,w}(x).
$$

(3.37)

The primal solution is $[\bar{x} - v_{H}^{n,w}]_{i} - [z_{i}^{n,w}]_{i} = [\bar{x} - v_{H}^{n,w}]_{i}$. The dual solution is $[\check{z}_{i}^{n,w}]_{i}$. So

$$
[x_{i}^{n,w}]_{i} \in \partial f_{i,n,w}([\bar{x} - v_{H}^{n,w}]_{i}).
$$

(3.38)

Recall Assumption 3.11(3) and $v_{H}^{n,w} = v_{H}^{n+1,0}$ by line 16 in Algorithm 3.10. We now analyze the increase in the dual objective value of each separate problem:

$$
\Delta_{i,n} := [f_{i,n+1,0}^{*}(x_{i}^{n+1,0})]_{i} + \frac{1}{2} \left\| [\bar{x} - v_{H}^{n+1,0}]_{i} - [z_{i}^{n+1,0}]_{i} \right\|^2
\geq -[f_{i,n+1,0}^{*}(x_{i}^{n+1,0})]_{i} + \frac{1}{2} \left\| [\bar{x} - v_{H}^{n+1,0}]_{i} - [z_{i}^{n+1,0}]_{i} \right\|^2.
$$

Recall that $E_{i,n}$ is also $f_{i}(x_{i}^{n,w}) - f_{i,n,w}(x_{i}^{n,w}) = f_{i}(x^{n+1,0}) - f_{i,n+1,0}(x^{n+1,0})$. Proposition 2.8 and Assumption 3.11(2) tell us that

$$
f_{i,n+1,0}^{*}(x_{i}^{n+1,0})_{i} + \frac{1}{2} \left\| [\bar{x} - v_{H}^{n+1,0}]_{i} - [z_{i}^{n+1,0}]_{i} \right\|^2
\geq -f_{i,n+1,0}(x_{i}^{n+1,0})_{i} + \frac{1}{2} \left\| [\bar{x} - v_{H}^{n+1,0}]_{i} - [x_{i}^{n+1,0}]_{i} \right\|^2.
$$
A similar result holds for the problem involving \( f_{i,n+1,1}(\cdot) \). Since \( S_{n+1,1} \cap \bar{E} = \emptyset \) and \( v_H^{n+1,0} = v_H^{n+1,1} \), we have
\[
\Delta_{i,n} = \left[ f_{i,n+1,1}([x^{n+1,1}]_i) + \frac{1}{2} \|[x - v_H^{n+1,1}]_i - [x^{n+1,1}]_i\|^2 \right] \\
- \left[ f_{i,n+1,0}([x^{n+1,0}]_i) + \frac{1}{2} \|[\bar{x} - v_H^{n+1,0}]_i - [x^{n+1,0}]_i\|^2 \right].
\]
The analogue of Lemma 2.3(1) tells us that \( \Delta_{i,n} = \frac{1}{2} t_{i,n}^2 \), where \( t_{i,n} \) is the positive root satisfying
\[
\frac{1}{2} t_{i,n}^2 + s_{i,n,\bar{w}} + [x^{n+1,0}]_i - [x - v_H^{n+1,0}]_i \| t_{i,n} = E_{i,n},
\]
where \( s_{i,n,\bar{w}} \in \partial f_i([x^{n,\bar{w}}]_i) \) is the subgradient used to form the linearization of \( f(\cdot) \) at \([x^{n+1,0}]_i\). Note that \( [x^{n,\bar{w}}]_i - [x - v_H^{n+1,0}]_i = -[z^{n,\bar{w}}_i]_i \), so the term \( \| s_{i,n,\bar{w}} + [x^{n,\bar{w}}]_i - [x - v_H^{n+1,0}]_i \| \) becomes \( \| s_{i,n,\bar{w}} - [z^{n,\bar{w}}_i]_i \| \). Since both \( s_{i,n,\bar{w}} \) and \( [z^{n,\bar{w}}_i]_i \) are bounded, \( \lim_{n \to \infty} t_{i,n} > 0 \), and so \( \lim_{n \to \infty} \Delta_{i,n} > 0 \). We can check from the definitions that
\[
F_{n+1,1}(\{z^{n+1,1}_\alpha\}_{\alpha \in \mathcal{E} \cup \mathcal{V}}) - F_{n+1,0}(\{z^{n+1,0}_\alpha\}_{\alpha \in \mathcal{E} \cup \mathcal{V}}) = \sum_{i \in S_{n,w}} \Delta_{i,n}.
\]
This means that the dual objective value can increase indefinitely, which then implies that the problem (1.2) is infeasible, which is a contradiction. \( \square \)

Proposition 3.20 below shows some reasonable conditions to guarantee (3.30). The ideas of its proof were already present in [Pan17, Pan18a], so we defer its proof to the appendix.

**Proposition 3.20.** (Growth of \( \sum_{\alpha \in \mathcal{E} \cup \mathcal{V}} \| z^{n,w}_\alpha \|^2 \)) In Algorithm 3.3 suppose:

1. There are only finitely many \( S_{n,w} \) for which \( S_{n,w} \cap (V_1 \cup V_2) \) contains more than one element.
2. There are constants \( M_1 > 0 \) and \( M_2 > 0 \) such that the size of the set\[\{(n',w) : n' \leq n, w \in \{1, \ldots, \bar{w}\}, |S_{n',w} \cap V| > 1\}\]
is bounded by \( M_1 \sqrt{n} + M_2 \) for all \( n \).

Then condition (1) in Theorem 3.19 holds.

### 3.5. Composition with a linear operator.

Suppose some \( f_i(\cdot) \) were defined as \( f_{i,1} \circ A_i(\cdot) \), where \( f : Y \to \mathbb{R} \) is a closed convex function, \( Y \) is another finite dimensional Hilbert space and \( A_i : \mathbb{R}^m \to Y \) is a linear map. One may either still try to take the proximal mapping of \( f_i(\cdot) \), but it may involve some expensive operations on \( A_i \). Alternatively, we can write or we can write \( f_{i,1} \circ A_i(x_i) \) as
\[
f_i(y_i) + \delta_{(x,y) : A_i(x) = y}(x_i, y_i),
\]
which splits into the sum of two functions. Note however that since we require the problem to be strongly convex, creating the new variable \( y \) adds new regularizing terms to the objective function.

### 4. Conclusion

The main contribution in this paper is to show that the distributed Dykstra’s algorithm can be extended to incorporate subdifferentiable functions in a natural manner so that the algorithm converges to the primal minimizer, even if there is no dual minimizer. A next question is to find convergence rates of the algorithm. The derivation of such rates uses rather different techniques from that of this paper, and
requires additional conditions to ensure the existence of a dual minimizer. We defer this to [Pan18b], where we also perform numerical experiments that show that the distributed Dykstra’s algorithm is sound.

**Appendix A. Further proofs**

In this appendix, we completing the parts of the proofs of Theorem 3.19 and 8.20 that we consider to be too similar to the ones in [Pan18a].

The following inequality describes the duality gap between (1.4) and (1.5).

\[
\frac{1}{2} \| \bar{x} - x \|^2 + \sum_{\alpha \in E \cup V} f_\alpha(x) - F(\{z_\alpha\}_{\alpha \in E \cup V}) \quad \text{(A.1)}
\]

**Proof of (i):** We first consider the case when \(S_{n,w} \not\subset V\). From the fact that \(\{z^{n,w}_\alpha\}_{\alpha \in S_{n,w}}\) minimizes (3.7) (which includes the quadratic regularizer) we have

\[
F_{n,w}(\{z^{n,w}_\alpha\}_{\alpha \in E \cup V}) \geq F_{n,w-1}(\{z^{n,w-1}_\alpha\}_{\alpha \in E \cup V}) + \frac{1}{2} \| v^{n,w}_A - v^{n,w-1}_A \|^2.
\quad \text{(A.3)}
\]

(The last term in (A.3) arises from the quadratic term in (3.7).) By line 16 of Algorithm 3.5, \(z^{n+1,0}_i = z^{n,w}_i\) for all \(i \in V\) and \(v^{n+1,0}_H = v^{n,w}_H\) (even though the decompositions (3.3d) of \(v^{n+1,0}_H\) and \(v^{n,w}_H\) may be different).
In the second case when \( S_{n,w} \subset V \), Proposition \([3.16]\) and \([3.1]\) show that the inequality \((A.3)\) holds.

Combining \((A.3)\) over all \( n' \in \{1, \ldots, n\} \) and \( w \in \{1, \ldots, \bar{w}\} \), we have

\[
F_{1,0}(\{z_{\alpha}^{1,0}\}_{\alpha \in E \cup V}) + \sum_{w=1}^{\bar{w}} \sum_{n'=1}^{n} \|v_{n',w}^{*} - v_{A}^{n',w-1}\|^2 \leq F_{n,\bar{w}}(\{z_{\alpha}^{n,\bar{w}}\}_{\alpha \in E \cup V}).
\]

Next, \( F_{n,\bar{w}}(\{z_{\alpha}^{n,\bar{w}}\}_{\alpha \in E \cup V}) \) is bounded from above by weak duality. The proof of the claim is complete.

**Proof of (ii):** From part (i) and the fact that \( F_{n,w}(\cdot) \leq F(\cdot) \), we have

\[
-F_{1,0}(\{z_{\alpha}^{1,0}\}_{\alpha \in E \cup V}) \geq -F_{n,w}(\{z_{\alpha}^{n,w}\}_{\alpha \in E \cup V}) \geq -F(\{z_{\alpha}^{n,w}\}_{\alpha \in E \cup V}). \tag{A.4}
\]

Substituting \( x \) in \((A.1)\) to be the primal minimizer \( x^* \) and \( \{z_{\alpha}\}_{\alpha \in E \cup V} \) to be \( \{z_{\alpha}^{n,w}\}_{\alpha \in E \cup V} \), we have

\[
\frac{1}{2}\|x - x^*\|^2 + \sum_{\alpha \in E \cup V} f_{\alpha}(x^*) - F_{1,0}(\{z_{\alpha}^{1,0}\}_{\alpha \in E \cup V}) \geq \frac{1}{2}\|x - x^*\|^2 + \sum_{\alpha \in E \cup V} f_{\alpha}(x^*) - F(\{z_{\alpha}^{n,w}\}_{\alpha \in E \cup V}) \geq \frac{1}{2}\|x - x^*\|^2 + \|z_{\alpha}^{n,w}\|^2. \tag{A.3}
\]

The conclusion is immediate.

**Proof of (v):** We first make use of the technique in \([BCTI\text{ Lemma 29.1}]\) (which is in turn largely attributed to \([BDS5]\)) to show that

\[
\lim inf_{n \to \infty} \left[ \sum_{w=1}^{\bar{w}} \left( \|v_{n,w}^{*} - v_{A}^{n,w-1}\| \right) \right] = 0. \tag{A.5}
\]

Seeking a contradiction, suppose instead that there is an \( \epsilon > 0 \) and \( \bar{n} > 0 \) such that if \( n > \bar{n} \), then \( \sum_{w=1}^{\bar{w}} \|v_{n,w}^{*} - v_{A}^{n,w-1}\| \geq \epsilon n \). By the Cauchy Schwarz inequality, we have

\[
\bar{n} \leq \left( \sum_{w=1}^{\bar{w}} \|v_{n,w}^{*} - v_{A}^{n,w-1}\| \right)^2 \geq \bar{n} \sum_{w=1}^{\bar{w}} \|v_{n,w}^{*} - v_{A}^{n,w-1}\|^2. \tag{A.6}
\]

This contradicts the earlier claim in (i) that \( \sum_{n=1}^{\infty} \sum_{w=1}^{\bar{w}} \|v_{n,w}^{*} - v_{A}^{n,w-1}\|^2 \) is finite.

Through \((A.5)\), we find a sequence \( \{n_k\}_{k=1}^{\infty} \) such that

\[
\lim_{k \to \infty} \left( \sum_{w=1}^{\bar{w}} \|v_{n_k,w}^{*} - v_{A}^{n_k,w-1}\| \right) = 0. \tag{A.7}
\]

Recalling the assumption \((3.30)\), we get

\[
\lim_{k \to \infty} \left[ \left( \sum_{w=1}^{\bar{w}} \|v_{n_k,w}^{*} - v_{A}^{n_k,w-1}\| \right) \right] = 0 \text{ for all } \alpha \in \bar{E} \cup V. \tag{A.8}
\]

Moreover,

\[
\|v_{n_k,w}^{*} - v_{A}^{n_k,w-1}\| \leq \|v_{n_k,w}^{*} - v_{A}^{n_k,w-1}\| \leq \left( \sum_{w=1}^{\bar{w}} \|v_{n_k,w}^{*} - v_{A}^{n_k,w-1}\| \right) \|z_{\alpha}^{n_k,w}\|. \tag{A.9}
\]

By (ii), there exists a further subsequence of \( \{v_{n_k,w}^{*}\}_{k=1}^{\infty} \) which converges to some \( v_{A}^{*} \in R^m \). Combining \((A.7)\) and \((A.9)\) gives (v).
Proof of (vi): From earlier results, we obtain

\[\sum_{\alpha \in \mathcal{E} \cup \mathcal{V}} f_\alpha (\bar{z} - v_A^*)\]  \hspace{1cm} \text{(A.9)}

\[\leq \frac{1}{2} \|\bar{x} - (\bar{x} - v_A^*)\|^2 - F(\{z_{nk,\bar{w}}^\alpha\}_{\alpha \in \mathcal{E} \cup \mathcal{V}})\]

\[= \frac{1}{2} \|v_A^*\|^2 + \sum_{\alpha \in \mathcal{E}_k \cup \mathcal{V}} f_\alpha^* (z_{nk,\bar{w}}^\alpha)\]

\[+ \sum_{(i,j,k) \notin \mathcal{E}_k} f_\alpha(\bar{z} - v_A^{n_k,p(n_k,\alpha)} - (\bar{x}, v_A^{n_k,\bar{w}}) + \frac{1}{2} \|v_A^{n_k,\bar{w}}\|^2\]

\[\leq \frac{1}{2} \|v_A^*\|^2 + \sum_{\alpha \in \mathcal{E}_k \cup \mathcal{V}} (\bar{x} - v_A^{n_k,p(n_k,\alpha)}, z_{nk,\bar{w}}^\alpha) + \sum_{i \in \mathcal{V}_4} E_{i,n_k} - \sum_{i \in \mathcal{V}_4} D_{i,n_k}\]

\[= \frac{1}{2} \|v_A^*\|^2 - \sum_{\alpha \in \mathcal{E}_k \cup \mathcal{V}} \langle v_A^{n_k,p(n_k,\alpha)} - v_A^{n_k,\bar{w}}, z_{nk,\bar{w}}^\alpha \rangle\]

\[= \|v_A^*\|^2 - \frac{1}{2} \|v_A^{n_k,\bar{w}}\|^2 - \sum_{\alpha \in \mathcal{E}_k \cup \mathcal{V}} \langle v_A^{n_k,p(n_k,\alpha)} - v_A^{n_k,\bar{w}}, z_{nk,\bar{w}}^\alpha \rangle\]

\[- \sum_{\alpha \in \mathcal{E}_k \cup \mathcal{V}} f_\alpha(\bar{x} - v_A^{n_k,p(n_k,\alpha)}) - \langle \bar{x}, v_A^{n_k,\bar{w}} \rangle + \sum_{i \in \mathcal{V}_4} E_{i,n_k} - \sum_{i \in \mathcal{V}_4} D_{i,n_k} + \frac{1}{2} \|v_A^{n_k,\bar{w}}\|^2\]

Since \(\lim_{k \to \infty} v_A^{n_k,\bar{w}} = v_A^*\), we have \(\lim_{k \to \infty} \frac{1}{2} \|v_A^*\|^2 - \frac{1}{2} \|v_A^{n_k,\bar{w}}\|^2 = 0\). The third term in the last group of formulas (i.e., the sum involving the inner products) converges to 0 by (v). The term \(\lim_{k \to \infty} \sum_{i \in \mathcal{V}_4} E_{i,n_k}\) equals to 0 by (iii).

Next, recall that if \(((i,j), \bar{k}) \in \mathcal{E}_k\), by (3.13), we have \(\bar{x} - v_A^{n_k,p(n_k,((i,j),\bar{k}))} \in H((i,j),\bar{k})\). Note that from Claim 3.18 b), we have \(\bar{x} - v_A^{n_k,p(n,((i,j),\bar{k}))} \in H((i,j),\bar{k})\) for all \(((i,j), \bar{k}) \in \mathcal{E}_k\). There is a constant \(\kappa_{\mathcal{E}_k} > 0\) such that

\[d(\bar{x} - v_A^{n_k,\bar{w}}, \cap((i,j), \bar{k}) \in \mathcal{E} H((i,j),\bar{k})) = \kappa_{\mathcal{E}_k} (\cap((i,j), \bar{k}) \in \mathcal{E} H((i,j),\bar{k}))\]

\[\leq d(\bar{x} - v_A^{n_k,\bar{w}}, H((i,j),\bar{k}))\]

\[\leq \kappa_{\mathcal{E}_k} \max_{((i,j), \bar{k}) \in \mathcal{E} \cap H((i,j),\bar{k})} (\|v_A^{n_k,\bar{w}} - v_A^{n_k,p(n_k,((i,j),\bar{k}))}\|).

Let \(\kappa := \max\{\kappa_{\mathcal{E}_k} : \mathcal{E} \text{ connects } \mathcal{V}\}\). We have \(\kappa_{\mathcal{E}_k} \leq \kappa\). Taking limits of (A.10), the RHS converges to zero by (i), so \(d(\bar{x} - v_A^*, \cap((i,j), \bar{k}) \in \mathcal{E} H((i,j),\bar{k})) = 0\), or \(\bar{x} - v_A^* \in \cap((i,j), \bar{k}) \in \mathcal{E} H((i,j),\bar{k})\). So \(\sum_{((i,j), \bar{k}) \in \mathcal{E}} f_\alpha((i,j), \bar{k})(\bar{x} - v_A^*) = 0\). Together with the fact
Therefore for all \( v \) constraints. This also means that for a be any index in \( V \nabla \), we have
\[
\sum_{(i,j),k} f((i,j),k)(x - v_{A_{i,n}}(p_{n,i})) = 0 = \sum_{(i,j),k} f((i,j),k)(x - v^*_A). \tag{A.11}
\]

Lastly, by the lower semicontinuity of \( f(\cdot) \), we have
\[
- \lim_{k \to \infty} \sum_{i \in V} f_i(x - v_{A_{i,n}}(p_{n,i})) \leq - \sum_{i \in V} f_i(x - v^*_A). \tag{A.12}
\]

As mentioned after [A.3], taking the limits as \( k \to \infty \) would result in the first three terms and the 5th term of the last formula in [A.9] to be zero. Hence
\[
- \sum_{\alpha \in E \cup V} f_\alpha(x - v^*_A) \leq \lim_{k \to \infty} \sum_{\alpha \in E \cup V} f_\alpha(x - v_{A_{i,n}}(p_{n,i})) - \lim_{k \to \infty} \sum_{i \in V} D_{i,n_k} \tag{A.11, A.12, 2.25}
\]

So [A.9] becomes an equation in the limit, and \( \lim_{n_k \to \infty} D_{i,n_k} = 0 \) for all \( i \in V \).

The first two lines of [A.9] then gives
\[
\lim_{k \to \infty} F(\{z_{\alpha,n}^{(i,j,w)}\}_{\alpha \in E \cup V}) = \frac{1}{2}\|v^*_A\|^2 + \sum_{i \in V} f_i(x - v^*_A),
\]

which shows that \( x - v^*_A \) is the primal minimizer. Recall that the definitions of \( F_n,w,() \), \( F(\cdot) \) and \( D_{i,n_k} \) in [1.0], [1.3] and [2.25a]. We recall from line 11 of Algorithm [6.5] we have \( f_{a,n,w}(\cdot) = f_{a,n,p(n,a)}(\cdot) \). This gives \( F_n,w(\{z_{\alpha,n}^{(i,j,w)}\}_{\alpha \in E \cup V}) + \sum_{\alpha \in E \cup V} D_{i,n_k} = F(\{z_{\alpha,n}^{(i,j,w)}\}_{\alpha \in E \cup V}) \), from which we deduce the equation on the left of [3.11] as well.

**Proof of Proposition [3.20]** Since this result is used only in the proof of Theorem [3.19] and (ii) in its proof. To address condition (1), we can assume that \( S_{n,w} \cap [V_1 \cup V_2] \) always contains at most one element. Define the sets \( S_{n,1} \) and \( S_{n,2} \) as
\[
S_{n,1} := \{(n',w) : n' \leq n, w \in \{1,\ldots,\bar{w}\}, |S_{n',w} \cap V_1| \leq 1\}
\]
\[
S_{n,2} := \{(n',w) : n' \leq n, w \in \{1,\ldots,\bar{w}\}, |S_{n',w} \cap V_2| > 1\}.
\]

Either \( S_{n',w} \cap [V_1 \cup V_2] = \emptyset \) or \( |S_{n',w} \cap [V_1 \cup V_2]| = 1 \). In the second case, let \( i^* \) be the index such that \( i^* \in S_{n',w} \cap [V_1 \cup V_2] \). Otherwise, in the first case, we let \( i^* \) be any index in \([V_1 \cup V_2] \). We prove claims based on whether \((n',w)\) lies in \( S_{n,1} \) or \( S_{n,2} \).

Without loss of generality, we can assume that \( S_{n',w} \cap \overline{E} \) are linearly independent constraints. This also means that for a \( v_{n',w}^H - v_{n',w-1}^H \), each \( z_{n',w}^H_{(i,j)} - z_{n',w-1}^H_{(i,j)} \) can be determined uniquely with a linear map from the relation
\[
\sum_{\alpha \in \overline{E}} [z_{\alpha,n'}^w - z_{\alpha,n'-1}^w] = \sum_{\alpha \in \overline{E}} [v_{\alpha,n'}^w - v_{\alpha,n'-1}^w]. \tag{A.13}
\]

Therefore for all \( \alpha \in S_{n',w} \cap \overline{E} \), there is a constant \( \kappa_{\alpha,S_{n',w} \cap \overline{E}} > 0 \) such that
\[
\|z_{\alpha,n'}^w - z_{\alpha,n'-1}^w\| \leq \kappa_{\alpha,S_{n',w} \cap \overline{E}} \|v_{\alpha,n'}^w - v_{\alpha,n'-1}^w\|. \tag{A.13}
\]
Thus there is a constant $\kappa > 0$ such that
\[
\sum_{\alpha \in \mathcal{E}} \|z_{\alpha}^{n',w} - z_{\alpha}^{n',w-1}\| \overset{\text{Alg 3.5 line 14}}{=} \sum_{\alpha \in S_{n',w} \cap \mathcal{E}} \|z_{\alpha}^{n',w} - z_{\alpha}^{n',w-1}\| \leq \kappa \|v_{H}^{n',w} - v_{H}^{n',w-1}\|. \tag{A.14}
\]

**Claim 1:** If $(n', w) \in \bar{S}_{n-1}$, then there is a constant $C_2 > 1$ such that
\[
\|v_{H}^{n',w} - v_{H}^{n',w-1}\| + \sum_{\alpha \in \mathcal{E}} \|z_{\alpha}^{n',w} - z_{\alpha}^{n',w-1}\| + \sum_{i \in V} \|z_{i}^{n',w} - z_{i}^{n',w-1}\| \leq C_2 \|v_{A}^{n',w} - v_{A}^{n',w-1}\|. \tag{A.15}
\]

We have
\[
\sum_{i \in V} [v_{A}^{n',w} - v_{A}^{n',w-1}]_{i} \overset{\text{3.4}}{=} \sum_{i \in V} \sum_{\alpha \in S_{n',w}} [z_{\alpha}^{n',w} - z_{\alpha}^{n',w-1}]_{i} \overset{z_{(i,j),k} \in O^{+}}{=} \sum_{i \in V} [z_{i}^{n',w} - z_{i}^{n',w-1}]_{i} \overset{\text{Prop 3.6}}{=} \sum_{i \in V} [z_{i}^{n',w} - z_{i}^{n',w-1}]_{i}. \tag{A.16}
\]

Recall that the norm $\| \cdot \|$ always refers to the 2-norm unless stated otherwise. By the equivalence of norms in finite dimensions, we can find a constant $c_1$ such that
\[
\|v_{A}^{n',w} - v_{A}^{n',w-1}\| \geq c_1 \sum_{i \in V} \|[v_{A}^{n',w} - v_{A}^{n',w-1}]_{i}\| \overset{\text{A.10}}{=} \sum_{i \in V} \|[z_{i}^{n,w} - z_{i}^{n,w-1}]_{i}\| \overset{\text{Alg 3.5 line 14}}{=} c_1 \sum_{i \in V} \|[z_{i}^{n,w} - z_{i}^{n,w-1}]_{i}\|. \tag{A.17}
\]

Next, $v_{H}^{n',w} - v_{H}^{n',w-1} \overset{\text{A.18}}{=} v_{A}^{n',w} - v_{A}^{n',w-1} - (z_{i}^{n',w} - z_{i}^{n',w-1})$, so
\[
\|v_{H}^{n',w} - v_{H}^{n',w-1}\| \overset{\text{A.18}}{=} \|v_{A}^{n',w} - v_{A}^{n',w-1}\| + \|[z_{i}^{n',w} - z_{i}^{n',w-1}]_{i}\| \overset{\text{A.18}}{=} \left(1 + \frac{1}{c_1}\right) \|v_{A}^{n',w} - v_{A}^{n',w-1}\|. \tag{A.18}
\]

We can choose $\{z_{\alpha}^{n',w}\}_{\alpha \in \mathcal{E}}$ such that
\[
\sum_{\alpha \in S_{n',w} \cap \mathcal{E}} [z_{\alpha}^{n',w} - z_{\alpha}^{n',w-1}] \overset{\text{Alg 3.5 line 14}}{=} \sum_{\alpha \in \mathcal{E}} [z_{\alpha}^{n',w} - z_{\alpha}^{n',w-1}] \overset{\text{A.19}}{=} v_{H}^{n',w} - v_{H}^{n',w-1}. \tag{A.19}
\]

Combining (A.17), (A.18) and (A.14) together shows that there is a constant $C_2 > 1$ such that (A.16) holds.

**Claim 2:** If $(n', w) \in \bar{S}_{n,2}$, then there is a constant $C_5 > 0$ such that
\[
\|v_{H}^{n',w} - v_{H}^{n',w-1}\| + \sum_{\alpha \in \mathcal{E}} \|z_{\alpha}^{n',w} - z_{\alpha}^{n',w-1}\| + \sum_{i \in V} \|z_{i}^{n',w} - z_{i}^{n',w-1}\| \leq C_5. \tag{A.20}
\]

We have
By Theorem 3.19(iii), there is a constant $C$ from Theorem 3.19(i), there is a constant $C$

Combining (A.14), (A.23) and (A.24), we can show that Claim 2 holds.

So $\|v\|_{\alpha} \leq H_n \|v\|_{\alpha} \leq H_n \|

From Theorem 3.19(i), there is a constant $C_3 > 0$ such that $\|v_{A}^{n',w} - v_{A}^{n',w-1}\| \leq C_3$. By Theorem 3.19(iii), there is a constant $C_4 > 0$ such that

$$\|z_{i}^{n',w} - z_{i}^{n',w-1}\| \leq C_4 \text{ for all } i \in \mathcal{V} \cup \mathcal{V}_4.$$  \quad (A.22)

So $\|z_{i}^{n',w} - z_{i}^{n',w-1}\| = ||z_{i}^{n',w} - z_{i}^{n',w-1}\|_{\alpha} \leq (|\mathcal{V}_3| + |\mathcal{V}_4|)C_4 + \frac{1}{c_1}C_3$, and

$$\sum_{i \in \mathcal{V}} \|z_{i}^{n',w} - z_{i}^{n',w-1}\| \leq 2(|\mathcal{V}_3| + |\mathcal{V}_4|)C_4 + \frac{1}{c_1}C_3.$$  \quad (A.23)

Next, from (A.14), we have

$$\|v_{A}^{n',w} - v_{A}^{n',w-1}\| = v_{H}^{n',w} - v_{H}^{n',w-1} + \sum_{i \in \mathcal{V}_3 \cup \mathcal{V}_4} |z_{i}^{n',w} - z_{i}^{n',w-1}| \leq C_3 + 2(|\mathcal{V}_3| + |\mathcal{V}_4|)C_4 + \frac{1}{c_1}C_3.$$  \quad (A.24)

Combining (A.14), (A.23) and (A.24), we can show that Claim 2 holds. \triangle

Since $\{z_{\alpha}^{n,0}\}_{\alpha \in \mathcal{E}}$ was chosen to satisfy (3.3), there is some $M > 1$ such that

$$\sum_{\alpha \in \mathcal{E}} \|z_{\alpha}^{n,0}\\| \leq M\|v_{H}^{n,0}\| \leq M \left(\|v_{H}^{1,0}\| + \sum_{n'=1}^{n-1} \sum_{w=1}^{w} \|v_{H}^{n',w} - v_{H}^{n',w-1}\|\right)$$  \quad (A.25)
Now for any \( n \geq 1 \), we have
\[
\sum_{\alpha \in \mathcal{E} \cup \mathcal{V}} \|z^{n,\bar{w}}_\alpha\| \leq \sum_{\alpha \in \mathcal{E}} \|z^{n,0}_\alpha\| + \sum_{\alpha \in \mathcal{V}} \|z^{n,w}_\alpha - z^{n,w-1}_\alpha\| \tag{A.26}
\]
\[
+ \sum_{\alpha \in \mathcal{E}} \sum_{w=1}^n \|z^{n,w}_\alpha - z^{n,w-1}_\alpha\| + \sum_{\alpha \in \mathcal{V}} \|z^{1,0}_\alpha\|
\]
\[
M \|v^{1,0}_H\| + \sum_{\alpha \in \mathcal{V}} \|z^{1,0}_\alpha\| + \sum_{w=1}^n \left( \sum_{\alpha \in \mathcal{E}} \|z^{n,w}_\alpha - z^{n,w-1}_\alpha\| \right)
\]
\[
+ \sum_{\alpha \in \mathcal{V}} \sum_{w=1}^{n-1} \left( M \|v^{n,w}_H - v^{n,w-1}_H\| + \sum_{\alpha \in \mathcal{E}} \|z^{n,w}_\alpha - z^{n,w-1}_\alpha\| \right)
\]
\[
\leq M \|v^{1,0}_H\| + \sum_{\alpha \in \mathcal{V}} \|z^{1,0}_\alpha\| + MC_2 \sum_{n=1}^n w \|v^{n,w}_A - v^{n,w-1}_A\|
\]
\[
+ MC_5 \left( M_1 \sqrt{n} + M_2 \right).
\]
By the Cauchy Schwarz inequality, we have
\[
\sum_{n'=1}^n \sum_{w=1}^w \|v^{n',w}_A - v^{n',w-1}_A\| \leq \sqrt{n} \sqrt{\sum_{n'=1}^n \sum_{w=1}^w \|v^{n',w}_A - v^{n',w-1}_A\|^2}. \tag{A.27}
\]
Since the second square root of the right hand side of (A.27) is bounded by Theorem 3.19(i), we make use of (A.26) to obtain the conclusion 3.30 as needed. \( \square \)

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