Fault-Tolerant Additive Weighted Geometric Spanners

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Abstract. Let \(S\) be a set of \(n\) points and let \(w\) be a function that assigns non-negative weights to points in \(S\). The additive weighted distance \(d_w(p,q)\) between two points \(p,q \in S\) is defined as \(w(p) + d(p,q) + w(q)\) if \(p \neq q\) and it is zero if \(p = q\). Here, \(d(p,q)\) denotes the (geodesic) Euclidean distance between \(p\) and \(q\). A graph \(G(S,E)\) is called a \(t\)-spanner for the additive weighted set \(S\) of points if for any two points \(p\) and \(q\) in \(S\) the distance between \(p\) and \(q\) in graph \(G\) is at most \(td_w(p,q)\) for a real number \(t > 1\). Here, \(d_w(p,q)\) is the additive weighted distance between \(p\) and \(q\). For some integer \(k \geq 1\), a \(t\)-spanner \(G\) for the set \(S\) is a \((k,t)\)-vertex fault-tolerant additive weighted spanner, denoted with \((k,t)\)-VFTAWS, if for any set \(S' \subseteq S\) with cardinality at most \(k\), the graph \(G \setminus S'\) is a \(t\)-spanner for the points in \(S' \setminus S'\). For any given real number \(\epsilon > 0\), we obtain the following results:

- When the points in \(S\) belong to Euclidean space \(\mathbb{R}^d\), an algorithm to compute a \((k,(2 + \epsilon))\)-VFTAWS with \(O(kn)\) edges for the metric space \((S,d_w)\). Here, for any two points \(p, q \in S\), \(d(p,q)\) is the Euclidean distance between \(p\) and \(q\) in \(\mathbb{R}^d\).

- When the points in \(S\) belong to a simple polygon \(P\), for the metric space \((S,d_w)\), one algorithm to compute a geodesic \((k,(2 + \epsilon))\)-VFTAWS with \(O(kn \log n)\) edges and another algorithm to compute a geodesic \((k,(\sqrt{10} + \epsilon))\)-VFTAWS with \(O(kn(k \log n)^2)\) edges. Here, for any two points \(p, q \in S\), \(d(p,q)\) is the geodesic Euclidean distance along the shortest path between \(p\) and \(q\) in \(P\).

- When the points in \(S\) lie on a terrain \(T\), an algorithm to compute a geodesic \((k,(2 + \epsilon))\)-VFTAWS with \(O(kn \log n)\) edges.

Keywords: Computational Geometry, Geometric Spanners, Approximation Algorithms

1 Introduction

When designing geometric networks on a given set of points located in a metric space, it is desirable for the network to have short paths between any pair of nodes while being sparse with respect to the number of edges. Let \(G(S,E)\) be an edge-weighted geometric graph on a set \(S\) of \(n\) points in \(\mathbb{R}^d\). The weight of any edge \((p,q) \in E\) is the Euclidean distance \(|pq|\) between \(p\) and \(q\). The distance in \(G\) between any two nodes \(p\) and \(q\), denoted by \(d_G(p,q)\), is defined as the length of a shortest (that is, minimum-weighted) path between \(p\) and \(q\) in \(G\). The graph \(G\) is called a \(t\)-spanner for some \(t > 1\) if for any two points \(p, q \in S\) we have \(d_G(p,q) \leq t|pq|\). The smallest \(t\) for which \(G\) is a \(t\)-spanner is called the stretch factor of \(G\) and the number of edges of \(G\) is called its size. Althöfer et al. \(^{11}\) first attempted to study sparse spanners on edge-weighted graphs that have the triangle-inequality property. Narasimhan and Smid \(^{11}\) gives a detailed account of geometric spanners, including a \((1 + \epsilon)\)-spanner for the set \(S\) of \(n\) points in \(\mathbb{R}^d\) with only \(O(n^{\frac{1+\epsilon}{d}})\) edges for any \(\epsilon > 0\).

As mentioned in Abam et al. \(^{3}\), the cost of traversing a path in a network is not only determined by the lengths of the edges on the path but also by the delays occurring at the nodes on the path. These delays are modeled in \(^{3}\) with the additive weighted metric. Let \(S\) be a set of \(n\) points in \(\mathbb{R}^d\). For every \(p \in S\), let \(w(p)\) be the non-negative weight associated to \(p\). The following additive weighted distance function \(d_w\) on \(S\) defining the metric space \((S,d_w)\) is considered in \(^{3}\): for any \(p, q \in S\),

\[
d_w(p,q) = \begin{cases} 
0 & \text{if } p = q, \\
|w(p)| + |pq| + w(q) & \text{if } p \neq q.
\end{cases}
\]

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For any fixed constant \( \epsilon > 0 \), [3] devises an algorithm to compute a \((5 + \epsilon)\)-spanner with a linear number of edges for \((S, d_w)\). Recently, Abam et al. [1] showed that there exists a \((2 + \epsilon)\)-spanner with a linear number of edges for the metric space \((S, d_w)\) that has bounded doubling dimension \(O(\pi)\). And, [3] gives a lower bound on the stretch factor, showing that \((2 + \epsilon)\) stretch is nearly optimal. Bose et al. [3] studied the problem of computing spanners for a weighted set of points. They considered the points lying on the plane to have positive weights associated with them; and defined the distance \(d_w\) between any two distinct points \(p, q \in S\) as \(d(p, q) - w(p) - w(q)\). Under the assumption that the distance between any pair of points is non-negative, they showed the existence of a \((1 + \epsilon)\)-spanner with \(O(\frac{2}{\epsilon})\) edges.

The free space \(F(D)\) of the given polygonal domain \(D\) is defined as the region interior to \(P_T\) but excludes the interior of all the holes of \(D\). Given a set \(S\) of \(n\) points in the free space \(F(D)\) defined by the polygonal domain \(D\), computing geodesic spanners is considered in Abam et al. [2]. In computing paths in polygonal domains, interior of holes and the simple polygon that contains these holes are considered as obstacles. Moreover, for any two distinct points \(p, q \in S\), the \(d_g(p, q)\) is defined as the geodesic Euclidean distance along a shortest path in \(F(D)\) between \(p\) and \(q\). Abam et al. [2] showed that for the metric space \((S, \pi)\), for any constant \(\epsilon > 0\), there exists a \((5 + \epsilon)\)-spanner of size \(O(\sqrt{hn}(\log n)^2)\). Further, for any constant \(\epsilon > 0\), [2] gave a \((\sqrt{10} + \epsilon)\)-spanner with \(O(n(\log n)^2)\) edges when \(h = 0\) i.e., the polygonal domain is a simple polygon with no holes. Given a set \(S\) of \(n\) points on a polyhedral terrain \(T\), the geodesic Euclidean distance between any two points \(p, q \in S\) is the length of a shortest path on \(T\) between \(p\) and \(q\). Abam et al. [1] showed that for a set of unweighted points on a polyhedral terrain, for any constant \(\epsilon > 0\), there exists a \((2 + \epsilon)\)-geodesic spanner with \(O(n(\log n))\) edges.

A network may need to be vertex fault-tolerant i.e., when a subset of nodes are removed, the induced network on the remaining nodes may require to be connected. Formally, a graph \(G(S, E)\) is a \(k\)-vertex fault-tolerant \(t\)-spanner, denoted by \((k, t)\)-VFTS, for a set \(S\) of \(n\) points in \(\mathbb{R}^d\) if for any subset \(S'\) of \(S\) with size at most \(k\), the graph \(G' \setminus S'\) is a \(t\)-spanner for the points of \(S \setminus S'\). [9,10,6,0] devised algorithms to compute a \((k, t)\)-VFTS for the set \(S\) of points located in \(\mathbb{R}^d\). These algorithms are presented in [11] as well. [9] devised an algorithm to compute a \((k, t)\)-VFTS of size \(O\left(\frac{n}{(t-1)^{\frac{d}{d-1} + 1}}\right)\) in \(O\left(\frac{n \log n}{(t-1)^{d-1}}\right)\) time and another algorithm to compute a \((k, t)\)-VFTS with \(O(k^2 n)\) edges in \(O\left(\frac{kn \log n}{(t-1)^{d-1}}\right)\) time. [10] gives an algorithm to compute a \((k, t)\)-VFTS of size \(O\left(\frac{k n}{(t-1)^{d-1} \log n}\right)\) in \(O\left(\frac{1}{(t-1)^{d-1}} (n \log d - 1 + n \log k + kn \log \log n)\right)\) time. The algorithm given in [3] computes a \((k, t)\)-VFTS having \(O\left(\frac{k n}{(t-1)^{d-1}}\right)\) edges in \(O\left(\frac{1}{(t-1)^{d-1}} (kn \log^d n + nk^2 \log k))\right)\) time with total weight of edges upper bounded by \(O\left(\frac{k^2 \log n}{(t-1)^{d-1}}\right)\) multiplicative factor of the weight of MST of the given set of points.

**Our results.** The spanners computed in this paper are first of their kind as we combine fault-tolerance with the additive weighted set of points. We devise the following algorithms for computing the vertex fault-tolerant additive weighted geometric spanners (VFTAWS) for any \(\epsilon > 0\) and \(k \geq 1:\)

* Given a set \(S\) of \(n\) additive weighted points lying in \(\mathbb{R}^d\), our first algorithm presented herewith computes a \((k, 2 + \epsilon)\)-VFTAWS having \(O(kn)\) edges. We incorporate fault-tolerance to the recent results of Abam et al. [4] while retaining the same stretch factor and increasing the number of edges in the spanner by a multiplicative factor of \(O(k)\).

* Given a set \(S\) of \(n\) additive weighted points in a simple polygon, we propose two algorithms: one computes a \((k, 2 + \epsilon)\)-VFTAWS with \(O\left(\frac{kn}{\log n}\right)\) edges and the other computes a \((k, \sqrt{10} + \epsilon)\)-VFTAWS with \(O(kn(\log n)^2)\) edges. For the first algorithm, we combine the clustering based algorithm from [2] and [3], and with the careful addition of more edges, we show that \(k\) fault-tolerance is achieved. The second algorithm uses the semi-separated pair decomposition (SSPD) based algorithm from [2]. As part of this, in computing clusters we use the result from [12].

* Given a set \(S\) of \(n\) additive weighted points lying on a terrain, we devise an algorithm to compute a \((k, 2 + \epsilon)\)-VFTAWS having \(O\left(\frac{kn}{\log n}\right)\) edges. For achieving the fault-tolerance, our algorithm adds a minimal set of edges to the spanner constructed in [4].
Unless specified otherwise, the points are assumed to be in Euclidean space \( \mathbb{R}^d \). The Euclidean distance between two points \( p \) and \( q \) is denoted by \( |pq| \). The distance between two points \( p, q \) in the metric space \( X \) is denoted by \( d_X(p, q) \). The length of the shortest path between \( p \) and \( q \) in a graph \( G \) is denoted by \( d_G(p, q) \).

Section 2 details the algorithm and its analysis to compute a \((k, 2 + \epsilon)\)-VFTAWS when the input additive weighted points are lying in \( \mathbb{R}^d \). When the input additive weighted points are in a simple polygon, Subsections 3.1 and 3.2 of Section 3 respectively devise algorithms to compute a \((k, 2 + \epsilon)\)-VFTAWS and \((k, \sqrt{10} + \epsilon)\)-VFTAWS. Section 4 devises an algorithm to compute a \((k, 2 + \epsilon)\)-VFTAWS when the input points are on a terrain. Conclusions are in Section 5.

2 Vertex fault-tolerant Spanner for additive weighted set of points

We devise an algorithm to compute a \((k, t)\)-VFTAWS for the set \( S \) of \( n \) additive weighted points, where \( t > 1 \) and \( k \geq 1 \). Following the algorithm given in [1], we first partition all the points belonging to \( S \) into at least \( k + 1 \) clusters. For creating these clusters, the points in \( S \) are sorted in non-decreasing order of their weights. Then the first \( k + 1 \) points in this sorted list are taken as the centres of \( k + 1 \) distinct clusters. As the algorithm progress, more points are added to these clusters and more clusters (with cluster centers) may also be created. In any iteration of the algorithm, for any point \( p \) in the remaining sorted list, among the current set of cluster centres, we determine the cluster centre \( c_j \) nearest to \( p \). Let \( C_j \) be the cluster to which \( c_j \) is the center. It adds \( p \) to the cluster \( C_j \) if \( |pc_j| \leq \epsilon \cdot w(p) \); otherwise, a new cluster \( C_p \) with \( p \) as its centre is initiated. Let \( C = \{c_1, \ldots, c_z\} \) be the final set of cluster centers obtained through this procedure. For every \( i \in [1, z] \), the cluster to which \( c_i \) is the center is denoted by \( C_i \). Using the algorithm from [12], we compute a \((k, (1 + \epsilon))\)-VFTS \( B \) for the set \( C \) of cluster centers. We compute a new graph \( G \), which is initialized to \( B \) unioned with points in \( S \setminus C \) as additional vertices. Our algorithm to compute a \((k, 2 + \epsilon)\)-VFTAWS differs from [1] with respect to both the algorithm used in computing \( B \) and the set of edges added to \( B \). The latter part is described now. For any node \( v \) of \( B \), let \( B_v \) be the set comprising \( k \) nearest neighbors of \( v \) in \( B \). For every \( i \in [1, z] \), let \( C'_i \) be the set comprising of \( \min\{k + 1, |C_i|\} \) least weighted points of cluster \( C_i \). For each point \( p \in S \setminus C \), if \( p \) belongs to cluster \( C_l \), for each \( v \in B_l \cup C'_l \), our algorithm introduces an edge between \( p \) and \( v \) with weight \( |pv| \); in the following lemma, we prove that the graph \( G \) is indeed a \((k, 2 + \epsilon)\)-VFTAWS with \( O(kn) \) edges.

**Theorem 1.** Let \( S \) be a set of \( n \) additive weighted points in \( \mathbb{R}^d \) with non-negative weights associated to points with weight function \( w \). For any fixed constant \( \epsilon > 0 \), the graph \( G \) is a \((k, (2 + \epsilon))\)-VFTAWS with \( O(kn) \) edges for the metric space \((S, d_w)\).

**Proof:** From [12], the number of edges in \( B \) is \( O(k |C|) \), which is essentially \( O(kn) \). From each point in \( S \setminus C \), we are adding at most \( 2k + 1 \) edges. Hence, the number of edges in \( G \) is \( O(kn) \).

For showing that \( G \) is a \((k, (2 + \epsilon))\)-VFTAWS for the metric space \((S, d_w)\), we show that for any set \( S' \subset S \) with \( |S'| \leq k \) and for any two points \( p, q \in S \setminus S' \) there exists a \((2 + \epsilon)\)-spanner path between \( p \) and \( q \) in \( G \setminus S' \). Following are the possible cases based on the role \( p \) and \( q \) play with respect to clusters formed and their centers:

**Case 1:** Both \( p \) and \( q \) are cluster centres of two distinct clusters i.e., \( p, q \in C \).

Since \( B \) is a \((k, (1 + \epsilon))\)-VFTS for the set \( C \),
\[
\begin{align*}
\text{Case 1: } & \quad d_G(S', p, q) = d_{G, S'}(p, q) \\
& \quad \leq t_B \cdot d_w(p, q)
\end{align*}
\]

**Case 2:** Both \( p \) and \( q \) in the same cluster \( C_i \) and one of them, say \( p \), is the centre of \( C_i \). Since \( p \) is the least weighted point in \( C_i \), there exists an edge joining \( p \) and \( q \) in \( G \). Hence,
\[
\begin{align*}
\text{Case 2: } & \quad d_G(S', p, q) = d_w(p, q)
\end{align*}
\]
Case 3: Both $p$ and $q$ are in the same cluster, say $C_i$; $p \neq c_i$, $q \neq c_i$; and, $c_i \notin S'$.

\[ d_{G \setminus S'}(p, q) = d_w(p, c_i) + d_w(c_i, q) \]
\[ = w(p) + |pc_i| + w(c_i) + |c_iq| + w(q) \]
\[ \leq w(p) + \epsilon \cdot w(p) + w(c_i) + \epsilon \cdot w(q) + w(q) \]
\[ [\text{since a point } x \text{ is added to cluster } C_i \text{ only if } |xc_i| \leq \epsilon \cdot w(x)] \]
\[ \leq w(p) + \epsilon \cdot w(p) + w(c_i) + w(q) + \epsilon \cdot w(q) + w(q) \]
\[ \text{[since the points are sorted in the non-decreasing order of their weights and the first point added to any cluster is taken as its center]} \]
\[ = (2 + \epsilon) \cdot [w(p) + w(q)] \]
\[ < (2 + \epsilon) \cdot [w(p) + |pq| + w(q)] \]
\[ = (2 + \epsilon) \cdot d_w(p, q) \]

Case 4: Both $p$ and $q$ are in the same cluster, say $C_i$; $p \neq c_i$, $q \neq c_i$; and, $c_i \in S'$.

In the case of $|C_i| \leq k$, there exists an edge between $p$ and $q$ in $G$. Hence, suppose that $|C_i| > k$. Let $S''$ be the set of $k + 1$ least weighted points from $C_i$. If $p, q \in S''$ then there exists an edge between $p$ and $q$ in $G$.

If $p \in S''$ and $q \notin S''$ then as well there exists an edge between $p$ and $q$. (Argument for the other case in which $q \in S''$ and $p \notin S''$ is analogous.) Now consider the case in which both $p, q \notin S''$. Since $p$ and $q$ are connected to every point in $S''$ and $|S''| = k + 1$, there exists an $r \in S''$ such that $r \notin S'$ and the edges $(p, r)$ and $(r, q)$ belong to $G \setminus S'$.

\[ d_{G \setminus S'}(p, q) = d_w(p, r) + d_w(r, q) \]
\[ = w(p) + |pr| + w(r) + w(r) + |rq| + w(q) \]
\[ \leq w(p) + |pc_i| + |c_ir| + w(r) + w(r) + |rc_i| + |c_iq| + w(q) \]
\[ \text{[by triangle inequality]} \]
\[ \leq w(p) + \epsilon \cdot w(p) + \epsilon \cdot w(r) + w(r) + \epsilon \cdot w(r) + \epsilon \cdot w(q) + w(q) \]
\[ \text{[since a point } x \text{ is added to cluster } C_i \text{ only if } |xc_i| \leq \epsilon \cdot w(x)] \]
\[ \leq w(p) + \epsilon \cdot w(p) + w(p) + w(q) + \epsilon \cdot w(q) + w(q) \]
\[ \text{[since for any point the edges are added to the } k + 1 \text{ least weighted points of the cluster to which it belongs]} \]
\[ = (2 + 2\epsilon) \cdot [w(p) + w(q)] \]
\[ < (2 + 2\epsilon) \cdot [w(p) + |pq| + w(q)] \]
\[ = (2 + 2\epsilon) \cdot d_w(p, q) \]

Case 5: Points $p$ and $q$ belong to two distinct clusters, say $p \in C_i$ and $q \in C_j$. In addition, $p \neq c_i$ and $q \neq c_j$, and neither of the cluster centres belong to $S'$.

Then,

\[ d_{G \setminus S'}(p, q) = d_w(p, c_i) + d_B(c_i, c_j) + d_w(c_j, q) \]
\[ = w(p) + |pc_i| + w(c_i) + d_B(c_i, c_j) + w(c_j) + |c_jq| + w(q) \]
\[ \leq w(p) + \epsilon \cdot w(p) + w(c_i) + d_B(c_i, c_j) + w(c_j) + \epsilon \cdot w(q) + w(q) \]
\[ \text{[since a point } x \text{ is added to cluster } C_i \text{ only if } |xc_i| \leq \epsilon \cdot w(x)] \]
\[ \leq (1 + \epsilon) \cdot [w(p) + w(q)] + w(c_i) + w(c_j) + t_B \cdot d_w(c_i, c_j) \]
\[ \text{[since } B \text{ is a } (k, t_B)\text{-vertex fault-tolerant spanner for the set } C_i\text{]} \]
Case 6: Both the points $p$ and $q$ are in two distinct clusters, say $p \in C_i$ and $q \in C_j$, one of them, say $p$ is the centre of $C_i$ (i.e., $p = c_i$), and $c_j \notin S'$.

Then,
\[
d_{\mathcal{P} \setminus S'}(p, q) = d_B(c_i, c_j) + d_w(c_j, q)
\]
\[
\leq t_B \cdot d_w(c_i, c_j) + d_w(c_j, q)
\]
[since $B$ is a $(k, t_B)$-vertex fault-tolerant spanner for the set $C$]
\[
= t_B \cdot [w(c_i) + |c_i c_j| + w(c_j)] + w(q) + |c_j q| + w(q)
\]
\[
\leq t_B \cdot [w(p) + |c_i c_j| + w(c_j)] + w(q) + |c_j q| + w(q)
\]
[since the points are sorted in the non-decreasing order of their weights and the first point added to any cluster is taken as its center]
\[
\leq t_B \cdot [w(p) + |c_i c_j| + w(q)] + w(q) + \epsilon \cdot w(q) + w(q)
\]
[since a point $x$ is added to cluster $C_l$ only if $|x c_l| \leq \epsilon \cdot w(x)$]
\[
\leq t_B \cdot [w(p) + |c_i c_j| + w(q)] + w(q) + \epsilon \cdot w(q) + w(q)
\]
[by triangle inequality]
\[
\leq t_B \cdot [w(p) + \epsilon \cdot w(p) + \epsilon \cdot w(q) + w(q)] + w(q) + \epsilon \cdot w(q) + w(q)
\]
[since a point $x$ is added to cluster $C_l$ only if $|x c_l| \leq \epsilon \cdot w(x)$]
\[
= t_B \cdot [(1 + \epsilon) \cdot [w(p) + w(q)] + |pq|] + (2 + \epsilon) \cdot w(q)
\]
\[
\leq t_B \cdot [(1 + \epsilon) \cdot [w(p) + w(q)] + |pq|] + (2 + \epsilon) \cdot [w(p) + w(q)]
\]
[since each point has non-negative weight associated with it]
\[
\leq t_B \cdot (2 + \epsilon) \cdot |w(p) + w(q)| + t_B \cdot |pq|
\]
\[
\leq t_B \cdot (2 + \epsilon) \cdot d_w(p, q)
\]

Case 7: Both the points $p$ and $q$ are in two distinct clusters, say $p \in C_i$ and $q \in C_j$; $p \neq c_i$, $q \neq c_j$; and, one of these centers, say $c_j$, belongs to $S'$ and the other center $c_i \notin S'$. 

\[
\leq (1 + \epsilon) \cdot [w(p) + w(q)] + w(p) + w(q) + t_B \cdot d_w(c_i, c_j)
\]
[since the points are sorted in the non-decreasing order of their weights and the first point added to any cluster is taken as center of that cluster]
\[
= (2 + \epsilon) \cdot [w(p) + w(q)] + t_B \cdot [w(c_i) + |c_i c_j| + w(c_j)]
\]
\[
\leq (2 + \epsilon) \cdot [w(p) + w(q)] + t_B \cdot [w(p) + |c_i c_j| + w(q)]
\]
[since the points are sorted in the non-decreasing order of their weights and the first point added to any cluster is taken as its center]
\[
\leq (2 + \epsilon) \cdot [w(p) + w(q)] + t_B \cdot [w(p) + w(q)] + \epsilon \cdot w(p) + |pq| + \epsilon \cdot w(q)
\]
[by triangle inequality]
\[
= (2 + \epsilon) \cdot [w(p) + w(q)] + t_B \cdot [(1 + \epsilon) \cdot [w(p) + w(q)] + |pq|]
\]
\[
< (2 + \epsilon) \cdot [w(p) + w(q)] + |pq| + t_B \cdot (1 + \epsilon) \cdot [w(p) + w(q)] + |pq|
\]
\[
< t_B (2 + \epsilon) \cdot [w(p) + w(q)] + |pq|
\]
[since each point has non-negative weight associated with it]
Since \( q \) is connected to \( k \) nearest neighbor of \( c_j \) in \( B \), there exists a \( c_r \in C \) such that \( c_r \notin S' \) and the edge 
\((q, c_r)\) belongs to \( \mathcal{G} \setminus S' \). Therefore,
\[
d_{\mathcal{G} \setminus S'}(p, q) = d_w(p, c_i) + dg(c_i, c_r) + d_w(c_r, q) \\
= w(p) + |pc_i| + w(c_i) + dg(c_i, c_r) + w(c_r) + |c_rq| + w(q) \\
\leq w(p) + \epsilon \cdot w(p) + w(c_i) + dg(c_i, c_r) + w(c_r) + |c_rq| + w(q) \\
\text{[since a point } x \text{ is added to cluster } C \text{ only if } |xc_l| \leq \epsilon \cdot w(x)] \\
\leq w(p) + \epsilon \cdot w(p) + w(c_i) + dg(c_i, c_r) + w(c_r) + |c_rq| + w(q) \\
\text{[by triangle inequality]}
\]
\[
\leq w(p) + \epsilon \cdot w(p) + w(c_i) + dg(c_i, c_r) + w(c_r) + |c_rq| + w(q) \\
\text{[since a point } x \text{ is added to cluster } C_i \text{ only if } |xc_l| \leq \epsilon \cdot w(x)]
\]
\[
\leq (1 + \epsilon) \cdot (w(p) + w(q)) + w(c_i) + dg(c_i, c_r) + w(c_r) + |c_rc_r|
\]
\[
(1)
\]

Since \( B \) is a \((k, t_B)\)-VFTS, there are at least \( k + 1 \) vertex disjoint \( t_B \)-spanner paths between \( c_j \) and \( c_i \) in \( B \). As \( c_r \) is among the \( k \) nearest neighbors of \( c_j \) in \( B \), one of these \( k + 1 \) paths from \( c_j \) to \( c_i \) must use \( c_r \).
\[
d_{B}(c_i, c_r) + d_w(c_r, c_j) < t_B \cdot d_w(c_i, c_j) \\
\text{[since LHS is } d_{B}(c_i, c_j)]
\]
\[
\Rightarrow d_{B}(c_i, c_r) + w(c_i) + |c_rc_j| + w(c_j) < t_B \cdot d_w(c_i, c_j) \\
\Rightarrow d_{B}(c_i, c_r) + |c_rc_j| + w(c_r) < t_B \cdot d_w(c_i, c_j) - w(c_j)
\]
\[
(2)
\]

Substituting (2) in (1),
\[
d_{\mathcal{G} \setminus S'}(p, q) < (1 + \epsilon) \cdot (w(p) + w(q)) + w(c_i) + t_B \cdot d_w(c_i, c_j) - w(c_j) \\
\leq (1 + \epsilon) \cdot (w(p) + w(q)) + w(c_i) + t_B \cdot d_w(c_i, c_j)
\]
\[
\text{[since weight associated with any point is non-negative]}
\]
\[
= (1 + \epsilon) \cdot (w(p) + w(q)) + w(c_i) + t_B \cdot (w(c_i) + |c_ic_j| + w(c_j))
\]
\[
\leq (1 + \epsilon) \cdot (w(p) + w(q)) + w(p) + t_B \cdot (w(p) + |c_ic_j| + w(c_j))
\]
\[
\text{[since the points are sorted in the non-decreasing order of their weights and the first}
\]
\[
\text{point added to any cluster is taken as its center]}
\]
\[
\leq (1 + \epsilon) \cdot (w(p) + w(q)) + w(p) + t_B \cdot (w(p) + |c_ip| + |pq| + |qc_j| + w(q))
\]
\[
\text{[by triangle inequality]}
\]
\[
\leq (1 + \epsilon) \cdot (w(p) + w(q)) + w(p) + t_B \cdot (w(p) + \epsilon \cdot w(p) + |pq| + \epsilon \cdot w(q) + w(q))
\]
\[
\text{[since a point } x \text{ is added to cluster } C_i \text{ only if } |xc_l| \leq \epsilon \cdot w(x)]
\]
\[
\leq (2 + \epsilon) \cdot (w(p) + w(q)) + t_B \cdot (1 + \epsilon) \cdot (w(p) + w(q)) + |pq|
\]
\[
\leq t_B \cdot [(2 + \epsilon) \cdot w(p) + w(q)] + |pq|
\]
\[
\leq t_B \cdot (2 + \epsilon) \cdot (w(p) + w(q)) + t_B \cdot w(p) + |pq| + w(q)
\]
\[
\text{[since weight associated with any point is non-negative]}
\]
\[
\leq t_B \cdot (2 + \epsilon) \cdot d_w(p, q)
\]

**Case 8:** Points \( p \) and \( q \) are in two distinct clusters, say \( p \in C_i \) and \( q \in C_j \); \( p \neq c_i, q \neq c_j \); and both \( c_i, c_j \in S' \). Since \( p \) (resp. \( q \)) is connected to \( k \) nearest neighbor of \( c_i \) (resp. \( c_j \)), there exists \( c_r, c_l \in C \) such that \( c_r, c_l \notin S' \) and the edges \((p, c_r)\) and \((c_l, q)\) belong to \( \mathcal{G} \setminus S' \).
Considering the analysis in all these cases proves that \( G \) is a \((k, t_B)\)-VFTS, there are at least \(k + 1\) vertex disjoint \(t_B\)-spanner paths between \(c_j\) and \(c_i\) in \( B \). As \( c_r \) (resp. \( c_j \)) is among the \(k \) nearest neighbors of \( c_i \) (resp. \( c_j \)) in \( B \), one of these \(k + 1\) paths from \(c_j\) to \(c_i\) must use \(c_r\) (resp. \(c_j\)).

\[
d_{G \setminus S}(p, q) = d_w(p, c_r) + d_B(c_r, c_i) + d_w(c_i, q) = w(p) + |pc_r| + w(c_r) + d_B(c_r, c_i) + w(c_i) + |c_iq| + w(q) \leq w(p) + |pc_r| + |c_i c_r| + w(c_r) + d_B(c_r, c_i) + w(c_i) + |c_iq| + w(q) + w(c_r) + |c_j c_i| + |c_j q| + w(q) \quad \text{(3)}
\]

[by triangle inequality]

Since \( B \) is a \((k, t_B)\)-VFTS, there are at least \(k + 1\) vertex disjoint \(t_B\)-spanner paths between \(c_j\) and \(c_i\) in \( B \).

\[
d_w(c_i, c_r) + d_B(c_r, c_i) + d_w(c_i, c_j) < t_B \cdot d_w(c_i, c_j) \quad \text{[since LHS is } d_B(c_i, c_j)]
\]

\[
\Rightarrow w(c_r) + |c_i c_r| + w(c_i) + d_B(c_r, c_i) + w(c_j) + |c_j c_i| + w(c_i) < t_B \cdot d_w(c_i, c_j)
\]

\[
\Rightarrow w(c_r) + |c_i c_r| + d_B(c_r, c_i) + w(c_i) + |c_j c_i| < t_B \cdot d_w(c_i, c_j) - w(c_i) - w(c_j) \quad \text{(4)}
\]

Substituting (4) in (3), we get

\[
d_{G \setminus S}(p, q) < w(p) + |pc_i| + t_B \cdot d_w(c_i, c_j) - w(c_i) - w(c_j) + |c_j q| + w(q) \leq w(p) + |pc_i| + t_B \cdot d_w(c_i, c_j) + |c_j q| + w(q) \quad \text{[since the weight associated with each point is non-negative]}
\]

\[
= w(p) + |pc_i| + t_B \cdot [w(c_i) + |c_i c_r| + w(c_r)] + |c_j q| + w(q) \leq w(p) + |pc_i| + t_B \cdot [w(c_i) + |c_i p| + |pq| + |qc_j| + w(c_j)] + |c_j q| + w(q) \quad \text{[by triangle inequality]}
\]

\[
\leq w(p) + \epsilon \cdot w(p) + t_B \cdot [w(c_i) + \epsilon \cdot w(p) + |pq| + \epsilon \cdot w(q) + w(c_j)] + \epsilon \cdot w(q) + w(q) \quad \text{[since a point } x \text{ is added to cluster } C_i \text{ only if } |xc_i| \leq \epsilon \cdot w(x)]
\]

\[
\leq w(p) + \epsilon \cdot w(p) + t_B \cdot [w(p) + \epsilon \cdot w(p) + |pq| + \epsilon \cdot w(q) + w(q)] + \epsilon \cdot w(q) + w(q) \quad \text{[since the points are sorted in the non-decreasing order of their weights and the first point added to any cluster is taken as its center]}
\]

\[
\leq (1 + \epsilon) \cdot [w(p) + w(q)] + t_B \cdot [(1 + \epsilon) \cdot [w(p) + w(q)] + |pq|] \leq t_B \cdot (2 + 2\epsilon) \cdot [w(p) + w(q)] + t_B \cdot |pq| \leq t_B \cdot (2 + 2\epsilon) \cdot [w(p) + w(q)] + |pq| = t_B \cdot (2 + 2\epsilon) \cdot d_w(p, q)
\]

Considering the analysis in all these cases proves that \( G \) is a \(k\) VFTAWS with stretch \( t \) upper bounded by \( t_B \cdot (2 + 2\epsilon) \). Since \( t_B \) is \((1 + \epsilon), t = (1 + \epsilon) \cdot (2 + 2\epsilon) \leq (2 + 6\epsilon)\).

Hence, \( G \) is a \((k, 2 + 6\epsilon)\)-VFTAWS for the metric space \((S, d_w)\). \(\Box\)

3 Vertex fault-tolerant additive weighted spanner for points in simple polygon

Given a set \( S \) of \( n \) points in a simple polygon \( P \), for any two points \( p, q \in S \), the path of shortest length between \( p \) and \( q \) in \( P \) is denoted by \( \pi(p, q) \), and the length of that path is denoted by \( d_w(p, q) \). A geodesic spanner of \( S \) is a graph \( G \) that has \( S \) as its vertex set and for any edge \( e \) with endpoints \( p \) and \( q \), the weight of \( e \) equals to \( d_w(p, q) \). Here, we devise an algorithm to compute a geodesic vertex fault-tolerant additive weighted spanner for the set \( S \) of \( n \) additive weighted points located inside a simple polygon \( P \).
The distance function \(d_{\pi,w}(p,q)\) on \(S\) as defined below is considered in \([3]\), and it was shown that \((S,d_{\pi,w})\) is a metric space. For any \(p,q \in S\),

\[
d_{\pi,w}(p,q) = \begin{cases} 
0 & \text{if } p = q, \\
w(p) + d_\pi(p,q) + w(q) & \text{if } p \neq q.
\end{cases}
\]

Here, for every \(x \in S\), \(w(x)\) is the non-negative weight associated to \(x \in S\).

We devise a divide-and-conquer based algorithm to compute a \((k,2+\epsilon)\)-VFTAWS for the metric space \((S,d_{\pi,w})\). Following \([2]\), we define few terms. Let \(S'\) be a set of points contained in a simple polygon \(P'\). A vertical line segment that splits \(P'\) into two simple sub-polygons of \(P'\) such that each sub-polygon contains at most two-thirds of the points of \(S'\) is termed a splitting segment with respect to \(S'\) and \(P'\). (The \(S'\) and \(P'\) are not mentioned if they are clear from the context.) The geodesic projection \(p_l\) of a point \(p\) onto a splitting segment \(l\) is a point on \(l\) that has the minimum geodesic Euclidean distance from \(p\) among all the points of \(l\).

By extending \([2]\), we devise an algorithm to compute a \((k,2+\epsilon)\)-VFTAWS \(G\) for the metric space \((S,d_{\pi,w})\). Our algorithm partitions \(P\) containing points in \(S\) into two simple sub-polygons \(P'\) and \(P''\) with a splitting segment \(l\). For every point \(p \in S\), we compute its geodesic projection \(p_l\) onto \(l\) and assign \(w(p) + d_\pi(p,p_l)\) as the weight of \(p_l\). Let \(S_l\) be the set comprising of all the geodesic projections of \(S\) onto \(l\). Also, let \(d_{l,w}\) be the additive weighted metric associated with points in \(S_l\). We use the algorithm from Section 2 to compute a \((k,2+\epsilon)\)-VFTAWS \(G_l\) for the metric space \((S_l,d_{l,w})\). For every edge \((r,s)\) in \(G_l\), we add an edge between \(p\) and \(q\) to \(G\) with weight \(d_\pi(p,q)\), wherein \(r\) (resp. \(s\)) is the geodesic projection of \(p\) (resp. \(q\)) onto \(l\). Let \(S''\) be the set of points contained in the sub-polygon \(P''\) (resp. \(P''\)) of \(P\). We recursively process \(P'\) (resp. \(P''\)) with points in \(S'\) (resp. \(S''\)) unless \(|S'\|\) (resp. \(|S''\|\)) is less than or equal to one.

### 3.1 Computing a \((k,2+\epsilon)\)-VFTAWS

We prove that the graph \(G\) is a \((k, (6+18\epsilon))\)-VFTAWS for the metric space \((S,d_{\pi,w})\). We show that by removing any subset \(S'\) with \(|S'| \leq k\) from \(G\), for any two points \(p\) and \(q\) in \(S \setminus S'\), there exists a path between \(p\) and \(q\) in \(G \setminus S'\) such that the \(d_G(p,q)\) is at most \((6+18\epsilon)d_{\pi,w}(p,q)\). First we note that there exists a splitting segment \(l\) at some iteration of the algorithm so that \(p\) and \(q\) are on different sides of \(l\). Let \(r\) be a point belonging to \(l \cap \pi(p,q)\). Let \(S'\) be the set comprising of geodesic projections of points in \(S'\) on \(l\). Since \(G_l\) is a \((k, (2+6\epsilon))\)-VFTAWS for the metric space \((S_l,d_{l,w})\), there exists a path \(Q\) between \(p_l\) and \(q_l\) in \(G_l \setminus S'\) whose length is upper bounded by \((2+6\epsilon) \cdot d_{l,w}(p_l,q_l)\). Let \(Q'\) be a path between \(p\) and \(q\) in \(G \setminus S'\) which is obtained by replacing each vertex \(v_l\) of \(Q\) by \(v\) in \(S\) such that the point \(v_l\) is the geodesic projection of \(v\) on \(l\). In the following, we show that the length of \(Q'\), which is \(d_{G \setminus S'}(p,q)\), is upper bounded by \((6+18\epsilon) \cdot d_{\pi,w}(p,q)\).

For every \(x, y \in S\),

\[
d_{\pi,w}(x,y) = w(x) + d_\pi(x,y) + w(y)
\leq w(x) + d_\pi(x,x_l) + d_\pi(x_l,y_l) + d_\pi(y_l,y) + w(y)
\text{[by triangle inequality]}
= w(x_l) + d_\pi(x_l,y_l) + w(y_l)
\text{[since the weight associated with projection } z_l \text{ of every point } z \text{ is } w(z) + d_\pi(z,z_l)]
= d_{l,w}(x_l,y_l)
\tag{5}
\]
This implies
\[ d_{\pi\setminus l}(p, q) \leq \sum_{x_i, y_i \in Q} d_{\pi, w}(x, y) \]
\[ \leq \sum_{x_i, y_i \in Q} d_{l, w}(x_i, y_i) \]
[from (5)]
\[ \leq (2 + 6\epsilon) \cdot d_{l, w}(p_i, q_i) \]  \hspace{1cm} (6)
[since \( G_l \) is a \((k, (2 + 6\epsilon))\)-vertex fault-tolerant geodesic spanner]
\[ d_{\pi\setminus l}(p, q) \leq \sum_{x_i, y_i \in Q} d_{\pi, w}(x, y) \]
\[ \leq (2 + 6\epsilon) \cdot d_{l, w}(p_i, q_i) \]  \hspace{1cm} (from (6))
[since \( G_l \) is a \((k, (2 + 6\epsilon))\)-vertex fault-tolerant geodesic spanner]
\[ = (2 + 6\epsilon) \cdot [w(p_i) + d_{\pi}(p_i, q_i) + w(q_i)] \]
\[ = (2 + 6\epsilon) \cdot [w(p_i) + d_{\pi}(p_i, q_i) + w(q_i)] \]
[since \( P \) contains \( l \), shortest path between \( p_i \) and \( q_i \)
is same as the geodesic shortest path between \( p_i \) and \( q_i \)]
\[ = (2 + 6\epsilon) \cdot [w(p) + d_{\pi}(p, p_i) + d_{\pi}(p_i, q_i) + d_{\pi}(q_i, q) + w(q)] \]  \hspace{1cm} (7)
[since the weight associated with projection \( z \) of every point \( z \) is \( w(z) + d_{\pi}(z, z_l) \)]

Since \( r \) is a point belonging to both \( l \) as well as to \( \pi(p, q) \),
\[ d_{\pi}(p, p_i) \leq d_{\pi}(p, r) \text{ and } d_{\pi}(q, q_i) \leq d_{\pi}(q, r) \]  \hspace{1cm} (8)

Substituting (8) into (7),
\[ d_{\pi\setminus l}(p, q) \leq (2 + 6\epsilon) \cdot [w(p) + d_{\pi}(p, r) + d_{\pi}(p_i, q_i) + d_{\pi}(r, q) + w(q)] \]
\[ \leq (2 + 6\epsilon) \cdot [w(p) + d_{\pi}(p, r) + w(r) + d_{\pi}(p_i, q_i) + w(r) + d_{\pi}(r, q) + w(q)] \]
[since the weight associated with every point is non-negative]
\[ = (2 + 6\epsilon) \cdot [d_{\pi, w}(p, r) + d_{\pi}(p_i, q_i) + d_{\pi, w}(r, q)] \]
\[ = (2 + 6\epsilon) \cdot [d_{\pi, w}(p, q) + d_{\pi}(p_i, q_i)] \]  \hspace{1cm} (9)
[since \( \pi(p, q) \) intersects \( l \) at \( r \), by optimal substructure property of shortest paths, \( \pi(p, q) = \pi(p, r) + \pi(r, q) \)]

Consider
\[ d_{\pi}(p_i, q_i) \leq d_{\pi}(p_i, p) + d_{\pi}(p, q) + d_{\pi}(q, q_i) \]
[since \( \pi \) follows triangle inequality]
\[ \leq d_{\pi}(r, p) + d_{\pi}(p, q) + d_{\pi}(q, r) \]
[using (8)]
\[ \leq w(r) + d_{\pi}(r, p) + w(p) + d_{\pi}(p, q) + w(q) + d_{\pi}(r, q) + w(r) \]
[since weight associated with every point is non-negative]
\[ = d_{\pi, w}(p, r) + d_{\pi, w}(p, q) + d_{\pi, w}(r, q) \]
\[ = d_{\pi, w}(p, q) + d_{\pi, w}(p, q) \]
[since \( \pi(p, q) \) intersects \( l \) at \( r \), by optimal substructure property of shortest paths, \( \pi(p, q) = \pi(p, r) + \pi(r, q) \)]
\[ = 2d_{\pi, w}(p, q) \]  \hspace{1cm} (10)
Substituting (10) into (9),
\[
d_{G\setminus S}(p, q) \leq 3(2 + 6\epsilon) \cdot d_{\pi,w}(p, q)
\]

Hence, the graph \( G \) computed as described above is a \((k, 6+\epsilon)\)-VFTAWS for the metric space \((S, d_{\pi,w})\). We further improve the stretch factor of \( G \) by applying the refinement given in [3] to above described algorithm. In doing this, for each point \( p \in S \), we compute the geodesic projection \( p_\gamma \) of \( p \) on the splitting line \( \gamma \) and construct a set \( S(p, \gamma) \) as defined hereafter. Let \( \gamma(p) \subseteq \gamma \) be \( \{x \in \gamma : d_{\pi,w}(p, x) \leq (1 + 2\epsilon) \cdot d_{\pi}(p, p_\gamma)\} \). We divide \( \gamma(p) \) into \( O(1/\epsilon^2) \) pieces; each piece is denoted by \( \gamma_j(p) \) where \( 1 \leq j \leq O(1/\epsilon^2) \), and the piece length is at most \( \epsilon \cdot d_{\pi}(p, p_\gamma) \). For every piece \( j \), we compute the point \( p^{(j)}_\gamma \) nearest to \( p \) in \( \gamma_j(p) \). The set \( S(p, \gamma) \) is defined as \( \{p^{(j)}_\gamma : p^{(j)}_\gamma \in \gamma_j(p) \text{ and } 1 \leq j \leq O(1/\epsilon^2)\} \). For every \( r \in S(p, \gamma) \), the non-negative weight \( w(r) \) of \( r \) is set to \( w(p) + d_{\pi}(p, r) \). Let \( S_\gamma \) be \( \bigcup_{p \in S} S(p, \gamma) \).

We replace the set \( S_\gamma \) in computing \( G \) with the set \( S_\gamma \) and compute a \((k, (2 + \epsilon))\)-VFTAWS \( G_l \) using the algorithm from Section 2 for the set \( S_\gamma \) instead. Further, for every edge \((r, s)\) in \( G_l \), we add the edge \((p, q)\) to \( G \) with weight \( d_{\pi}(p, q) \) whenever \( r \in S(p, \gamma) \) and \( s \in S(q, \gamma) \). The rest of the algorithm remains same.

For our analysis, the following lemma from [3] is needed.

**Lemma 1.** (from [3]) Let \( P \) be a simple polygon. Consider two points \( x, y \in P \). Let \( r \) be the point at which shortest path \( \pi(x, y) \) between \( x \) and \( y \) intersects a splitting segment \( \gamma \). If \( r \notin \gamma(x) \), \( x_\gamma \) (resp. \( y_\gamma \)) is set as \( x_\gamma \) (resp. \( y_\gamma \)). Otherwise \( x_\gamma \) (resp. \( y_\gamma \)) is set as the point from \( S(x, \gamma) \) (resp. \( S(y, \gamma) \)) which is nearest to \( x \) (resp. \( y \)). Then \( d_{\pi}(x, x_\gamma) + d_{\pi}(x_\gamma, r) \) (resp. \( d_{\pi}(y, y_\gamma) + d_{\pi}(y_\gamma, y) \)) is less than or equal to \((1 + \epsilon) \cdot d_{\pi}(x, r)\) (resp. \((1 + \epsilon) \cdot d_{\pi}(y, y)\)).

**Theorem 2.** Let \( S \) be a set of \( n \) additive weighted points in simple polygon \( P \) with non-negative weights associated to points with weight function \( w \). For any fixed constant \( \epsilon > 0 \), there exists a \((k, (2 + \epsilon))\)-vertex fault-tolerant additive weighted geodesic spanner with \( O(\frac{k\log n}{\epsilon}) \) edges for the metric space \((S, d_{\pi,w})\).

**Proof:** In constructing a \((k, (2 + 6\epsilon))\)-VFTAWS \( G_l \) for the set \( S_\gamma \) of \( \frac{1}{\epsilon} \) points, we add \( O(\frac{k\log n}{\epsilon}) \) edges to \( G \) in one iteration. Let \( S(n) \) be the size of \( G \). Then \( S(n) = S(n_1) + S(n_2) + \frac{k\log n}{\epsilon} \) where \( n_1, n_2 \) are the number of points in each of the partitions formed by the splitting segment. Since \( n_1, n_2 \geq n/3 \), \( S(n) = O(\frac{k\log n}{\epsilon}) \).

For proving that \( G \) is a \((k, (2 + \epsilon))\)-VFTAWS for the metric space \((S, d_{\pi,w})\), we show that for any set \( S' \subset S \) with \(|S'| \leq k \) and for any two points \( p, q \in S \setminus S' \) there exists a \((2 + \epsilon)\)-spanner path between \( p \) and \( q \) in \( G \setminus S' \). First we note that there there exists a splitting segment \( l \) at some iteration of the algorithm so that \( p \) and \( q \) are on different sides of \( l \). Let \( r \) be a point belonging to \( l \cap \pi(p, q) \). Let \( S'_l \) be the set comprising of geodesic projections of points in \( S' \) on \( l \). Since \( G_l \) is a \((k, (2 + 6\epsilon))\)-VFTAWS for the metric space \((S_l, d_{\pi,w})\), there exists a path \( Q \) between \( p_l \) and \( q_l \) in \( G_l \setminus S'_l \) whose length is upper bounded by \( (2 + 6\epsilon) \cdot d_{\pi,w}(p_l, q_l) \). Let \( Q' \) be a path between \( p \) and \( q \) in \( G \setminus S' \) which is obtained by replacing each vertex \( v_3 \) of \( Q \) by \( v \) in \( S \) such that the point \( v_3 \) is the geodesic projection of \( v \) on \( l \). In the following, we show that the length of \( Q' \), which is \( d_{\pi,w}(p, q) \), is upper bounded by \( (2 + 14\epsilon) \cdot d_{\pi,w}(p, q) \).

Following the Lemma [1] if \( r \notin l(p) \), point \( p'_l \) (resp. \( q'_l \)) is set as \( p_l \) (resp. \( q_l \)). Otherwise \( p'_l \) (resp. \( q'_l \)) is set as the point from \( S(p, l) \) (resp. \( S(q, l) \)) which is nearest to \( p \) (resp. \( q \)).

\[
d_{\pi,w}(p'_l, q'_l) = w(p'_l) + d_{\pi}(p'_l, q'_l) + w(q'_l)
\]
\[
\leq w(p'_l) + d_{\pi}(p'_l, r) + d_{\pi}(r, q'_l) + w(q'_l)
\]
[by triangle inequality]
\[
\leq w(p'_l) + d_{\pi}(p'_l, r) + w(r) + d_{\pi}(r, q'_l) + w(q'_l)
\]
[since the weight associated with each point is non-negative]
\[
= w(p) + d_{\pi}(p, p'_l) + d_{\pi}(p'_l, r) + w(r)
\]
\[
+ w(r) + d_{\pi}(r, q'_l) + d_{\pi}(q'_l, q) + w(q)
\]
[due to weight assigned to geodesic projections]

(11)
Applying Lemma [1] with \( p'_l \) and \( q'_l \),
\[
    d_\pi(p, p'_l) + d_l(x'_l, r) \leq (1 + \epsilon) \cdot d_\pi(p, r) \quad \text{and} \\
    d_l(r, q'_l) + d_\pi(q'_l, q) \leq (1 + \epsilon) \cdot d_\pi(r, y) \tag{12}
\]
Substituting (12) and (13) in (11),
\[
    d_{l,w}(p'_l, q'_l) \leq w(p) + (1 + \epsilon) \cdot \pi(p, r) + w(r) + w(r) + (1 + \epsilon) \cdot \pi(r, q) + w(q) \\
    \leq (1 + \epsilon) \cdot [d_\pi(p, r) + d_\pi(r, q)] \\
    = (1 + \epsilon) \cdot d_\pi(p, q) \\
    \text{[since } r \in l \cap \pi(p, q), \text{ by the optimal substructure property of shortest} \\
    \text{paths, } \pi(p, q) = \pi(p, r) + \pi(r, q)] \tag{14}
\]
Replacing \( p_t \) (resp. \( q_t \)) by \( p'_t \) (resp. \( q'_t \)) in inequality (6),
\[
    d_{\pi,S}(p, q) \leq (2 + 6\epsilon) \cdot d_{l,w}(p'_t, q'_t) \\
    \leq (2 + 6\epsilon)(1 + \epsilon) \cdot d_\pi(p, q) \\
    \text{[from (14)]} \\
    \leq (2 + 14\epsilon) \cdot d_\pi(p, q)
\]
Thus, \( G \) is a \((k, (2 + 14\epsilon))\)-vertex fault-tolerant additive weighted geodesic spanner for the set \( S \) of points in simple polygon \( P \).

\[\square\]

### 3.2 Computing a \((k, \sqrt{10} + \epsilon)\)-VFTAWS

In this subsection, we follow the algorithm from [2] in devising SSPD based algorithm for computing a geodesic \((k, \sqrt{10} + \epsilon)\)-VFTAWS. The spanner being computed has size \( O(kn(\lg n)^2) \) and diameter 2.

First using the algorithm given in [7], we compute a splitting segment \( l \) for \( P \). Let \( P_1 \) be the set comprising of geodesic projections of points in \( P \) onto \( l \). We compute a \( \frac{1}{\pi} \)-SSPD \( R \) for the points in \( P_1 \). For every pair \((A, B)\) in SSPD \( R \) such that \( \text{radius}(A) \leq \text{radius}(B) \), when \(|A| < k + 1\), for every \( p \in A \) and \( q \in B \), we introduce an edge between \( p \) and \( q \) in \( G \). Otherwise, let \( P(A) \) be the set comprising of points in the input set \( S \) of points whose geodesic projections belong to set \( A \). We compute a set \( Q \subseteq P(A) \) with \(|Q| = k + 1\) such that for any \( q \in Q \) the value of \( w(q) + d_\pi(q, q) \) is less than or equal to \( w(q') + d_\pi(q', q') \) for any point \( q' \) not in \( Q \); and, we add an edge \((p, q)\) in \( G \) for every \( r \in A \cup B \) and \( q \in Q \). Further, we recursively add more edges to \( G \) by processing points in \( P \) that are left of \( l \) (including the ones lying on \( l \)) as well as points in \( P \) that are to the right of \( l \).

We restate the following lemma from [2] which is useful in the analysis of our algorithm to compute a \((k, (\sqrt{10} + \epsilon))\)-vertex fault tolerant geodesic spanner for \( S \).

**Lemma 2.** (from [2]) Suppose \( ABC \) is a right triangle with \( \angle CAB = \frac{\pi}{2} \). Let \( H \) be a \( y \)-monotone path between \( B \) and \( D \) such that the region bounded by \( AB, AD, \) and \( H \) is convex where \( D \) is some point on edge \( AC \). We have \( 3d(H) + d(D, C) \leq \sqrt{10}d(B, C) \), where \( d(\cdot) \) denotes the geodesic Euclidean length.

**Theorem 3.** For a set \( S \) on \( n \) additive weighted points lying inside a simple polygon \( P \), there exists a geodesic \((k, (\sqrt{10} + \epsilon))\)-VFTAWS of size \( O(kn(\lg n)^2) \) and diameter 2.

**Proof:** Since every pair of vertices in \( G \) is joined by at most 2 edges, the diameter of \( G \) is 2. Let \( S(n) \) be the size of \( G \). Then \( S(n) = \sum (k(|A| + |B|)) + S(n_1) + S(n_2) \), where \( n_1 + n_2 = n \) and \( n_1, n_2 \geq n/3 \). By SSPD property, \( \sum (|A| + |B|) = O(n \lg n) \), so the size of the spanner is \( O(kn(\lg n)^2) \).

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Consider a set $S' \subseteq S$ such that $|S'| \leq k$ and two arbitrary points $p$ and $q$ from the set $S \setminus S'$. We show that there exists a $(\sqrt{10} + \epsilon)$-spanner path between $p$ and $q$ in $G \setminus S'$. There exists a splitting segment $l$ at some iteration of the algorithm when $p$ and $q$ lie on different sides of $l$. Let $r$ be the point at which a shortest path $\pi(p, q)$ between $p$ and $q$ intersects $l$. At this step, consider a semi separated pair $(A, B)$ such that $p_t \in A$ and $q_t \in B$ or, $q_t \in A$ and $p_t \in B$. W.l.o.g., assume the former holds.

Case 1: Let $|A| \geq k + 1$.

There exists a $c_j \in C$ such that $c_j \notin S'$ since $|C| = k + 1$. 

$\forall p, \exists \pi, \exists w,$

$\quad d_{q \setminus S'}(p, q) = d_{q', w}(p, c_j) + d_{q, w}(c_j, q)$

(15)

Since geodesic shortest paths follow triangle inequality,

$\quad d_{\pi}(p, c_j) \leq d_{\pi}(p, p_t) + |qc_{j_1}| + d_{\pi}(c_{j_1}, c_j)$

(16)

$\quad d_{\pi}(c_j, q) \leq d_{\pi}(c_{j_1}, c_j) + |c_{j_1}q_t| + d_{\pi}(q_t, q)$

(17)

By the triangle inequality,

$\quad |c_{j_1}q_t| \leq |c_{j_1}p_t| + |p_rq| + |rq_t|$

(18)

By the definition of set $Q$,

$\forall p \notin Q, \quad w(c_j) + d_{\pi}(c_j, q) \leq w(p) + d_{\pi}(p, p_t)$

(19)

Then

$\quad d_{q, w}(p, c_j) + d_{q, w}(c_j, q) = w(p) + d_{\pi}(p, c_j) + w(c_j) + w(c_j) + d_{\pi}(c_j, q) + w(q)$

$\leq w(p) + d_{\pi}(p, p_t) + |p_{j_1}c_{j_1}| + d_{\pi}(c_{j_1}, c_j) + w(c_j) + d_{\pi}(c_{j_1}, c_j) + |c_{j_1}q_t| + d_{\pi}(q_t, q) + w(q)$

[Substituting (16) and (17)]

$\leq w(p) + d_{\pi}(p, p_t) + |p_{j_1}c_{j_1}| + w(p) + d_{\pi}(p, p_t) + d_{\pi}(p, p_t) + |c_{j_1}q_t| + d_{\pi}(q_t, q) + w(q)$

[from (19)]

$\leq 3[w(p) + w(q)] + 3d_{\pi}(p, p_t) + |p_{j_1}c_{j_1}| + |c_{j_1}q_t| + d_{\pi}(q_t, q)$

$\leq 3[w(p) + w(q)] + 3d_{\pi}(p, p_t) + |p_{j_1}c_{j_1}| + |c_{j_1}q_t| + |p_{j_1}r| + |q_tq| + d_{\pi}(q_t, q)$

[from (18)]

$= 3[w(p) + w(q)] + 3d_{\pi}(p, p_t) + 2|p_{j_1}c_{j_1}| + |p_{j_1}r| + |q_tq| + d_{\pi}(q_t, q)$

(20)

Let $p'$ (resp. $q'$) be the point at which $\pi(p, q)$ and $\pi(p, p_t)$ (resp. $\pi(q, q_t)$) split. Then

$\quad d_{q, w}(p, c_j) + d_{q, w}(c_j, q) \leq 3[w(p) + w(q)] + 3d_{\pi}(p, p') + 3d_{\pi}(p', p_t) + 2|p_{j_1}c_{j_1}| + |p_{j_1}r| + |q_tq| + d_{\pi}(q_t, q') + d_{\pi}(q', q)$

(21)

[by the triangle inequality]

Now consider the triangles $q'h_{q'}r$ and $p'h_{p'}r$. The path $\pi(p', p_t)$ (resp. $\pi(q', q_t)$) is a $y$-monotone path as well as the region bounded by $\pi(p', p_t), r_{p_t}$ and $p'r$ (resp. $\pi(q', q_t), r_{q_t}$ and $q'r$) is convex. Applying Lemma 2 to the $\Delta(q'h_{q'}r)$,

$\quad d_{\pi}(q', q) + |q_tq| \leq 3d_{\pi}(q', q) + |q_tq| \leq \sqrt{10}d_{\pi}(q', r)$

(22)

And, applying Lemma 2 to the $\Delta(p'h_{p'}r)$,

$\quad 3d_{\pi}(p', p_t) + |p_{j_1}r| \leq \sqrt{10}d_{\pi}(p', r)$

(23)
Substituting (22) and (23) in (21) we get,
\[ d_{\pi,w}(p, c_j) + d_{\pi,w}(c_j, q) \leq 3[w(p) + w(q)] + 3d_{\pi}(p, p') + 3\sqrt{d_{\pi}(p', r)} + 2|p_c c_j| + \sqrt{10}d_{\pi}(q', r) + d_{\pi}(q', q) \]
\[ \leq 3[w(p) + w(q)] + 3d_{\pi}(p, p') + \sqrt{10}d_{\pi,w}(p', r) + 2|p_c c_j| + \sqrt{10}d_{\pi,w}(q', r) + d_{\pi}(q', q) \]
\[ \leq 3[w(p) + w(q)] + 3d_{\pi}(p, p') + \sqrt{10}d_{\pi,w}(p', q') + 2|p_c c_j| + d_{\pi}(q', q) \]
\[ \text{[since weights associated with any point is non-negative]} \]
\[ \leq 3[w(p) + w(q)] + 3d_{\pi}(p, p') + \sqrt{10}d_{\pi,w}(p', q') + 2|p_c c_j| + d_{\pi}(q', q) \]

\[ \text{(24)} \]

Since we compute \( \frac{\pi}{4} \)-SSPD over the set \( P_1 \), for any pair \((X, Y)\) of SSPD the distance between any two points in \( X \) is at most \( \frac{\pi}{4} \) times of the distance between \( X \) and \( Y \). So
\[ |p_c c_j| \leq \frac{\pi}{2} |pq| \]
\[ \text{(25)} \]

\[ |pq| \leq |pr| + |rq| \]
\[ \text{[by triangle inequality]} \]
\[ \leq |pp| + |pr| + |rq| + |qq| \]
\[ \text{[by triangle inequality]} \]
\[ \leq d_{\pi}(p, r) + d_{\pi}(r, q) + d_{\pi}(q, q) \]
\[ \leq d_{\pi}(p, r) + d_{\pi}(r, q) + d_{\pi}(r, q) \]
\[ \text{[by definition of projection of a point on } l \text{]} \]
\[ \leq d_{\pi,w}(p, r) + d_{\pi,w}(p, r) + d_{\pi,w}(r, q) + d_{\pi,w}(r, q) \]
\[ \text{[since weight associated with each point is non-negative.]} \]
\[ = 2d_{\pi,w}(p, q) \]
\[ \text{(26)} \]

\[ \text{[since } r \text{ is the point where } \pi(p, q) \text{ intersects } l \text{]} \]

From (23) and (20), we get
\[ |p_c c_j| \leq \epsilon d_{\pi,w}(p, q) \]
\[ \text{(27)} \]

From (23) and (27),
\[ d_{\pi,w}(p, c_j) + d_{\pi,w}(c_j, q) \leq 3[w(p) + w(q)] + 3d_{\pi}(p, p') + \sqrt{10}d_{\pi,w}(p', q') + \epsilon d_{\pi,w}(p, q) + d_{\pi}(q', q) \]
\[ \leq 3d_{\pi,w}(p, p') + \sqrt{10}d_{\pi,w}(p', q') + \epsilon d_{\pi,w}(p, q) + d_{\pi}(q', q) \]
\[ \text{[since weight associated with each point is non-negative.]} \]
\[ \leq \sqrt{10}d_{\pi,w}(p, q) + \epsilon d_{\pi,w}(p, q) \]
\[ \text{[since } \pi(p, q) = \pi(p, p') + \pi(p', q') + \pi(q', q) \]
\[ \leq (\sqrt{10} + \epsilon)d_{\pi,w}(p, q) \]

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From [15], \( d_{\varphi^1}(p, q) \leq (\sqrt{10} + \epsilon) d_{\pi, w}(p, q) \).

Case 2:

Since when \(|A| < k + 1\), we have a direct edge between \(p\) and \(q\), \( d_{\varphi^1}(p, q) = d_{\pi, w}(p, q) \).

Hence, it is proved that \(\mathcal{G}\) is a geodesic \((k, (\sqrt{10} + \epsilon))\)-VFTAWS for \(S\). \(\square\)

4 Vertex fault-tolerant additive weighted spanner for points on a terrain

In this section, we devise an algorithm to compute a geodesic \((k, (2 + \epsilon))\)-vertex fault-tolerant spanner with \(\mathcal{O}\left(\frac{k\log n}{\epsilon^2}\right)\) for any given set \(S\) of \(n\) additive weighted points lying on a polyhedral terrain \(T\). We denote the boundary of \(T\) with \(\partial T\). The following distance function \(d_{T, w} : S \times S \to \mathbb{R} \cup \{0\}\) is used to compute the geodesic distance on \(T\) between any two points \(p, q\) from \(S\): \(d_{T, w}(p, q) = w(p) + d_T(p, q) + w(q)\). Here, \(w(p)\) (resp. \(w(q)\)) is the non-negative weight associated with \(p \in S\) (resp. \(q \in S\)). We denote a geodesic shortest path between two points \(x\) and \(y\) on \(T\) with \(\pi(x, y)\). Let \(\pi^+(x, y)\) be the closed region lying to the right of \(\pi(x, y)\) when going from \(x\) to \(y\) including the points lying on the shortest path \(\pi(x, y)\). The projection \(p_\pi\) of a point \(p\) on a shortest path \(\pi\) between two points lying on the polyhedral terrain \(T\) is defined as a point on \(\pi\) that is at minimum geodesic distance from \(p\) among all the points located on \(\pi\). For three points \(u, v, w \in T\), the closed region bounded by shortest paths \(\pi(u, v), \pi(v, w),\) and \(\pi(w, u)\) is termed sp-triangle, denoted with \(\Delta(u, v, w)\). If the points \(u, v, w \in T\) are understood from the context, we simply denote the sp-triangle with \(\Delta\). In the following, we restate a Theorem from [4] which is useful for our analysis.

**Theorem 4 (from [4])**. For any set \(P\) of \(n\) points on a polyhedral terrain \(T\) there is a balanced sp-separator. By balanced sp-separator, it is meant that there is either a shortest path \(\pi(u, v)\) connecting two points \(u, v \in \partial T\) such that \(\frac{2\pi^+(u, v)}{3} \leq |\pi^+(u, v) \cap P| \leq \frac{2\pi^+(u, v)}{3},\) or there is an sp-triangle \(\Delta\) such that \(\frac{2\pi^+(\Delta)}{3} \leq |\Delta \cap P| \leq \frac{2\pi^+(\Delta)}{3}\).

Thus, a sp-separator is either bounded by a shortest path (in the former case) or by three shortest paths (in the latter case). Let \(\gamma\) be a shortest path that belongs to a sp-separator.

Since our algorithm relies on [4], we briefly describe that algorithm to compute spanner \(\mathcal{G}\) first. Herewith, we describe the first recursive step of their algorithm. A balanced sp-separator as given in Theorem [4] is computed. Let \(S_{\text{in}}\) and \(S_{\text{out}}\) be the sets of points defined as follows: if the sp-separator is a shortest path then define \(S_{\text{in}}\) to be \(\gamma^+(u, v) \cap S\); otherwise, \(S_{\text{in}}\) is \(\Delta \cap S\); points in \(S\) that do not belong to \(S_{\text{in}}\) are in \(S_{\text{out}}\). For each \(p \in S\), we compute the projection \(p_\gamma\) of \(p\) on every shortest path \(\gamma\) of sp-separator, and associate a weight \(d_T(p, p_\gamma)\) with \(p_\gamma\). Let \(S_\gamma\) be a set defined as \(\cup_{p \in \text{SP}_{\gamma}}\). Their algorithm computes a \((2 + \epsilon)\)-spanner \(\mathcal{G}_\gamma\) for the additive weighted points in \(S_{\gamma}\). Further, for each edge \((p_\gamma, q_\gamma)\) in \(\mathcal{G}_\gamma\), an edge \((p, q)\) is added to \(\mathcal{G}\), where \(p_\gamma, \) (resp. \(q_\gamma\)) is the projection of \(p\) (resp. \(q\)) on \(\gamma\). The spanners for sets \(S_{\text{in}}\) and \(S_{\text{out}}\) are computed recursively and the edges from these spanners are added to \(\mathcal{G}\). In the base case, if \(\text{card}(S) \leq 3\) then a complete graph on the set \(S\) is constructed.

We closely follow the algorithm from [4] in obtaining a \((k, (6 + \epsilon))\)-vertex fault-tolerant additive weighted spanner for the set \(S\) of points lying on the terrain \(T\). (This construction is later modified to compute a \((k, (2 + \epsilon))\)-VFTAWS.) In specific, with every projected point \(p_\gamma\), instead of associating \(d_T(p, p_\gamma)\) as the weight of \(p_\gamma\), we associate \(w(p) + d_T(p, p_\gamma)\) as the weight of \(p_\gamma\). The rest of the algorithm in constructing \(\mathcal{G}\) remains same.

We prove that the graph \(\mathcal{G}\) is a geodesic \((k, (6 + 18\epsilon))\)-VFTAWS. Consider any set \(S' \subseteq S\) such that \(\text{card}(S') \leq k\) and two arbitrary points \(p\) and \(q\) from the set \(S \setminus S'\). We show that there exists a path between \(p\) and \(q\) in \(\mathcal{G} \setminus S'\) such that the \(d_{\mathcal{G}}(p, q)\) is at most \((6 + 18\epsilon)d_{\pi, w}(p, q)\). The induction hypothesis assumes that for the number of points \(k < n\) in a region of \(T\), there exists a \((6 + 18\epsilon)\)-spanner path between any two points of the given region in \(\mathcal{G} \setminus S'\). As part of inductive step, we extend it to \(n\) points. For the cases when both \(p\) and \(q\) are on the same side of a bounding shortest path \(\gamma\) of the separator, i.e., both are in \(S_{\text{in}}\) or \(S_{\text{out}}\), by induction hypothesis there exists a \((6 + 18\epsilon)\)-spanner path between \(p\) and \(q\) in \(\mathcal{G} \setminus S'\). The
only case remains to be proved is when \( p \) lies on one side of \( \gamma \) and \( q \) lies on the other side of \( \gamma \), i.e., \( p \in S_{in} \) and \( q \in S_{out} \) or, \( q \in S_{in} \) and \( p \in S_{out} \). W.l.o.g., we assume that the former holds. Let \( r \) be a point on \( \gamma \) at which the geodesic shortest path \( \pi(p, q) \) between \( p \) and \( q \) intersects \( \gamma \). Since \( \mathcal{G}_\gamma \) is a \((k, (2 + 6\epsilon))\)-VFTS, there exists a path \( P \) between \( p_\gamma \) and \( q_\gamma \) in \( \mathcal{G}_\gamma \) of length at most \((2 + 6\epsilon)d_{\gamma,w}(p_\gamma, q_\gamma)\). By replacing each vertex \( x_\gamma \) of \( P \) by \( x \in S \) such that \( x_\gamma \) is the projection of \( x \) on \( \gamma \), gives a path \( P' \) between \( p \) and \( q \) in \( \mathcal{G} \setminus S' \). The length \( d_{\mathcal{G} \setminus S'}(p, q) \) of path \( P' \) is less than or equal to the length of path \( P \) in \( \mathcal{G}_\gamma \). In the following, we show that \( d_{\mathcal{G} \setminus S'}(p, q) \leq (6 + 18\epsilon)d_{\mathcal{T,w}(p, q)} \).

For every \( x, y \in S \),
\[
d_{\mathcal{T,w}}(x, y) = w(x) + d_{\mathcal{T}}(x, y) + w(y)
\]
\[
\leq w(x) + d_{\mathcal{T}}(x, x_\gamma) + d_{\mathcal{T}}(x_\gamma, y_\gamma) + d_{\mathcal{T}}(y_\gamma, y) + w(y)
\]
[by triangle inequality]
\[
= w(x_\gamma) + d_{\mathcal{T}}(x_\gamma, y_\gamma) + w(y_\gamma)
\]
[since the weight associated with projection \( z_\gamma \) of every point \( z \) is \( w(z) + d_{\mathcal{T}}(z, z_\gamma) \)]
\[
= d_{\gamma,w}(x_\gamma, y_\gamma)
\]

This implies
\[
d_{\mathcal{G} \setminus S'}(p, q) = \sum_{x_\gamma, y_\gamma \in P} d_{\mathcal{T,w}}(x_\gamma, y_\gamma)
\]
\[
\leq \sum_{x_\gamma, y_\gamma \in P} d_{\gamma,w}(x_\gamma, y_\gamma)
\]
[from (28)]
\[
\leq (2 + 6\epsilon)d_{\gamma,w}(p_\gamma, q_\gamma)
\]
[since \( \mathcal{G}_\gamma \) is a \((k, (2 + 6\epsilon))\)-vertex fault tolerant geodesic spanner]
\[
= (2 + 6\epsilon)[w(p_\gamma) + d_{\mathcal{T}}(p_\gamma, q_\gamma) + w(q_\gamma)]
\]
[since \( \gamma \) is a shortest path on \( \mathcal{T} \), shortest path between any two points on \( \gamma \) is a geodesic shortest path on \( \mathcal{T} \)]
\[
= (2 + 6\epsilon)[w(p) + d_{\mathcal{T}}(p, p_\gamma) + d_{\mathcal{T}}(p_\gamma, q_\gamma) + d_{\mathcal{T}}(q_\gamma, q) + w(q)]
\]
[since the weight associated with projection \( z_\gamma \) is \( w(z) + d_{\mathcal{T}}(z, z_\gamma) \)]

By the definition of projection of any point on \( \gamma \), we know that
\[
d_{\mathcal{T}}(p, p_\gamma) \leq d_{\mathcal{T}}(p, r) \quad \text{and} \quad d_{\mathcal{T}}(q, q_\gamma) \leq d_{\mathcal{T}}(q, r)
\]

Substituting (31) into (30),
\[
d_{\mathcal{G} \setminus S'}(p, q) \leq (2 + 6\epsilon)[w(p) + d_{\mathcal{T}}(p, r) + d_{\mathcal{T}}(p_\gamma, q_\gamma) + d_{\mathcal{T}}(r, q) + w(q)]
\]
\[
\leq (2 + 6\epsilon)[w(p) + d_{\mathcal{T}}(p, r) + w(r) + d_{\mathcal{T}}(p_\gamma, q_\gamma) + w(r) + d_{\mathcal{T}}(r, q) + w(q)]
\]
[since weight associated with every point is non-negative]
\[
= (2 + 6\epsilon)[d_{\mathcal{T,w}}(p, r) + d_{\mathcal{T}}(p_\gamma, q_\gamma) + d_{\mathcal{T,w}}(r, q)]
\]
\[
= (2 + 6\epsilon)[d_{\mathcal{T,w}}(p, q) + d_{\mathcal{T}}(p_\gamma, q_\gamma)]
\]
[since \( \pi(p, q) \) intersects \( \gamma \) at \( r \), so by optimal substructure property of shortest paths \( \pi(p, q) = \pi(p, r) + \pi(r, q) \)]
Further,
\[ d_T(p_\gamma, q_\gamma) \leq d_T(p, p) + d_T(p, q) + d_T(q, q_\gamma) \]
by triangle inequality
\[ \leq d_T(r, p) + d_T(p, q) + d_T(q, r) \]
[using (31)]
\[ \leq w(r) + d_T(r, p) + w(p) + w(p) + d_T(p, q) + w(q) + w(q) + d_T(q, r) + w(r) \]

[since weight associated with every point is non-negative]
\[ = d_T,w(p, r) + d_T,w(p, q) + d_T,w(r, q) \]
\[ = d_T,w(p, q) + d_T,w(p, q) \]
\[ \text{since } \pi(p, q) \text{ intersects } \gamma \text{ at } r, \text{ so by optimal substructure property of shortest path } \pi(p, q) = \pi(p, r) + \pi(r, q) \]
\[ = 2d_T,w(p, q) \] (33)

Substituting (33) into (32) yields
\[ d_{\mathcal{G}\backslash S'}(p, q) \leq 3(2 + 6\epsilon)d_T,w(p, q) \]

We improve the stretch factor of \( \mathcal{G} \) by applying the same refinement as the one applied to the first algorithm given in Subsection 3.1. Again, we denote the graph resulted after applying that refinement with \( \mathcal{G} \).

**Theorem 5.** Let \( S \) be a set of \( n \) additive weighted points on a polyhedral terrain \( T \) with non-negative weights associated to points with weight function \( w \). For any fixed constant \( \epsilon > 0 \), there exists a \( (k, (2 + 14\epsilon)) \)-vertex fault-tolerant additive weighted geodesic spanner with \( O(\frac{k n}{\epsilon^2} \log n) \) edges.

**Proof:** Let \( S(n) \) be the size of \( \mathcal{G} \). Since we add \( O(\frac{kn}{\epsilon^2}) \) edges in an iteration, \( S(n) = S(n_1) + S(n_2) + \frac{kn}{2} \), where \( n_1 \) and \( n_2 \) are the number of points in each of the partitions formed. Further, since \( n_1 + n_2 = n \) and \( n_1, n_2 \geq n/3 \), \( S(n) \) is \( O(\frac{k n}{\epsilon^2} \log n) \).

We show that for any set \( S' \subset S \) with \( |S'| \leq k \) and for any two points \( p, q \in S' \), there exists a \((2 + 14\epsilon)\)-spanner path between \( p \) and \( q \) in \( \mathcal{G} \backslash S' \). We induct on the number of points \( k \) from \( S \backslash S' \). The induction hypothesis assumes that for the number of points \( k < n \) in a region of \( T \), there exists a \((6 + 18\epsilon)\)-spanner path between any two points of the given region in \( \mathcal{G} \backslash S' \). For the cases when both \( p \) and \( q \) both are on the same side, a bounding shortest path \( \gamma \) of the separator, i.e., both are in \( S_{in} \) or \( S_{out} \) by induction hypothesis there exists a \((6 + 18\epsilon)\)-spanner path between \( p \) and \( q \) in \( \mathcal{G} \backslash S' \). The case remaining to be proved is when \( p \) and \( q \) lie on distinct sides of \( \gamma \), i.e., \( p \in S_{in} \) and \( q \in S_{out} \) or \( q \in S_{in} \) and \( p \in S_{out} \). W.l.o.g., we assume that the former holds. Let \( r \) be the point at which the geodesic shortest path \( \pi(p, q) \) between \( p \) and \( q \) intersects \( r \). Since \( \mathcal{G}_\gamma \) is a \((k, (2 + 6\epsilon))\)-vertex fault tolerant spanner there exists a path \( P \) between \( p_\gamma \) and \( q_\gamma \) in \( \mathcal{G}_\gamma \) of length at most \((2 + 6\epsilon)d_{\mathcal{G}_\gamma}(p_\gamma, q_\gamma)\). By replacing each vertex \( x_\gamma \) of \( P \) by \( x \in S \) such that \( x_\gamma \) is the projection of \( x \) on \( \gamma \), yields a path \( P' \) between \( p \) and \( q \in \mathcal{G} \backslash S' \). The length \( d_{\mathcal{G}\backslash S'}(p, q) \) of path \( P' \) is less than or equal to the length of path \( P \) in \( \mathcal{G}_\gamma \). Following the Lemma 4 if \( r \notin \gamma(p) \), point \( p'_\gamma \) (resp. \( q'_\gamma \)) is set as \( p_\gamma \) (resp. \( q_\gamma \)). Otherwise, \( p'_\gamma \) (resp. \( q'_\gamma \)) is set as the point from \( S(p, \gamma) \) (resp. \( S(q, \gamma) \)) which is nearest to \( p \) (resp. \( q \)).
\[ d_{\gamma,w}(p'_\gamma, q'_\gamma) = w(p'_\gamma) + d_{\gamma}(p'_\gamma, q'_\gamma) + w(q'_\gamma) \]
\[ \leq w(p'_\gamma) + d_{\gamma}(p'_\gamma, r) + d_{\gamma}(r, q'_\gamma) + w(q'_\gamma) \]
[by triangle inequality]
\[ \leq w(p'_\gamma) + d_{\gamma}(p'_\gamma, r) + w(r) + w(r) + d_{\gamma}(r, q'_\gamma) + w(q'_\gamma) \]
[since the weight associated with each point is non-negative]
\[ = w(p) + d_T(p, p'_\gamma) + d_{\gamma}(p'_\gamma, r) + w(r) + w(r) + d_{\gamma}(r, q'_\gamma) + d_T(q'_\gamma, q) + w(q) \]
(34)

\[ \leq w(p) + d_T(p, r) + d_{\gamma}(r, q'_\gamma) + d_T(q'_\gamma, q) + w(q) \]
[by the assignment of the weight to the projection of any point]
\[ \leq w(p) + d_T(p, r) + d_{\gamma}(r, q'_\gamma) + d_T(q'_\gamma, q) + w(q) \]
(35)

From Lemma 1, we know that \( d_T(p, p'_\gamma) + d_{\gamma}(p'_\gamma, r) \) (resp. \( d_{\gamma}(r, q'_\gamma) + d_T(q'_\gamma, q) \)) is less than or equal to \( (1 + \epsilon).d_T(p, r) \) (resp. \( (1 + \epsilon).d_T(r, q) \)). Hence (34) be written as
\[ d_{T,w}(p'_\gamma, q'_\gamma) \leq w(p) + (1 + \epsilon).d_T(p, r) + w(r) + w(r) + (1 + \epsilon).d_T(r, q) + w(q) \]
\[ \leq (1 + \epsilon).[d_{T,w}(p, r) + d_{T,w}(r, q)] \]
\[ = (1 + \epsilon).d_{T,w}(p, q) \]
[since \( \pi(p, q) \) intersects \( \gamma \) at \( r \), so by optimal substructure property of shortest paths \( \pi(p, q) = \pi(p, r) + \pi(r, q) \)]

Replacing \( p_\gamma \) (resp. \( q_\gamma \)) by \( p'_\gamma \) (resp. \( q'_\gamma \)) in inequality (29),
\[ d_{G \setminus S}(p, q) \leq (2 + 6\epsilon).d_{T,w}(p'_\gamma, q'_\gamma) \]
\[ \leq (2 + 6\epsilon)(1 + \epsilon).d_{T,w}(p, q) \]
[from (35)]
\[ \leq (2 + 14\epsilon).d_{T,w}(p, q) \]

Thus \( G \) is a geodesic \((k, (2 + 14\epsilon))-\)vertex fault tolerant spanner for \( S \). \( \square \)

5 Conclusions

In this paper, we gave algorithms to achieve \( k \) fault-tolerance when the metric is additive weighted. We devised algorithms for the following problems: input points are in \( \mathbb{R}^d \), in a simple polygon, and on a terrain. Apart from the efficient computation, it would be interesting to explore the lower bounds on the number of edges for the fault-tolerant additive weighted spanners. Further, optimizing various spanner parameters, like degree, diameter, and weight of these spanners could be interesting.

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