Stable Elliptical Vortices in a Cylindrical Geometry

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We show that, in a two-dimensional (2d) ideal fluid (also applies to a column of quasi-2d non-neutral plasma in an axial magnetic field), large elliptical vortices in a finite disk are stable. The stability is established by comparison between energy of elliptical and symmetrical states to satisfy a sufficient condition, without dynamical eigen-analysis. Analytical small ellipticity expansion of energy and exact numerical values for finite ellipticity are both obtained. The expansion indicates stable linear $l = 2$ diocotron modes for large vortices (or plasma columns). Numerical simulations of the 2d Euler equation are also performed. They not only confirm the sufficient condition, but also show that the stability persists to smaller vortex sizes. The reason why decaying $l = 2$ modes were obtained by Briggs, Daugherty, and Levy [Phys. Fluids 13, 421 (1970)] using eigen-analysis is also discussed.

The two-dimensional (2d) incompressible Euler equation

$$\frac{\partial \omega}{\partial t} + (\mathbf{u} \cdot \nabla)\omega = 0, \quad (1)$$

not only describes an incompressible 2d ideal fluid, but also governs the behavior of a long non-neutral plasma column confined by a uniform axial magnetic field $\mathbf{B}$. Here $\mathbf{u}(x, y)$ is the 2d velocity field and $\omega(r)$ is the vorticity field, $\omega \equiv (\nabla \times \mathbf{u}) \cdot \hat{z}$. The incompressibility condition, $\nabla \cdot \mathbf{u} = 0$, can be automatically satisfied by defining the stream function $\phi$ as $\mathbf{u} \equiv (\partial \phi/\partial y, -\partial \phi/\partial x)$. The stream function and vorticity are related by the Poisson equation $\nabla^2 \phi = -\omega$. In a pure electron plasma, $\omega$ corresponds to the electron density and $\phi$ to the electrical potential.

Stability problems of coherent vortex states in this system are long being interesting and important questions. In a free space, there exist exact nonlinear elliptical (Kirchoff) patch solutions $\mathbf{u}$. In a cylindrical geometry Briggs, Daugherty, and Levy showed that, using dynamical eigen-analysis, resonance between fluid elements and wave modes will lead to damping of $l \geq 2$ diocotron modes. Here $l$ denotes the mode number as the perturbation to a symmetric stream function is written as $\phi(r) \exp[i(\Omega t - \theta)]$. By solving the initial value problem of linearized equations and properly treating analytical continuation in complex $\Omega$ plane, they obtained formulation for complex eigenvalue $\omega$. In particular, for a vorticity distribution very close to a step function but negative radial derivative at all places, $\omega$ with a positive second derivative of entropy against all possible perturbations. This of course is using a similar principle as the method mentioned above.

A stability argument based on global constraints has also been applied to the 2d vortex system $\omega$. The logic of this analysis is to show that a functional $W[\omega]$ which is conserved by the 2d Euler equation is a maximum at a particular $\omega(r)$ against all other states that are accessible under incompressible flows. At this maximum, no further changes in $\omega(r)$ are possible and the state is then stable. For example, Davidson and Lund showed that a state in a cylindrical geometry following a relation $\omega(r) = \omega(\phi(r))$ and $\partial \omega(\phi)/\partial \phi \geq 0$ is nonlinearly stable. In another example, O’Neil and Smith demonstrated that an off-center coherent vortex (linearly an $l = 1$ perturbation) in a disk is also stable. However, no results on the stability of an $l = 2$ mode using this method have been given in the literatures.

Thermal equilibrium has been studied in 2d ideal fluids $\omega$. Since the coarse-grained entropy will not decrease due to the dynamical vorticity mixing, it is proposed that the system will reach a maximum coarse-grained entropy state at long time. Mean field equations governing these states have been derived, and solutions in some situations were obtained. Once a mean-field equilibrium state is obtained, its stability can be assured by showing a positive second derivative of entropy against all possible perturbations. This of course is using a similar principle as the method mentioned above.

In this paper we establish the stability of a large elliptical vortex (comparing to the system size) against relaxation to a symmetrical state using neither of the above two methods with eigen-analysis and global maximum. We will first deduce a stable sufficient condition and then show that it is satisfied by elliptical vortices larger than a critical radius. The method is to compare energy of proper states, not by evaluating second derivatives, actually not even finding any equilibrium states. We further perform numerical simulations of the 2d Euler equation to test our predictions. Simulations not only confirm the sufficient condition, but also show that elliptical vortices are stable to lower radii.

Basic argument of the sufficient condition goes as: Consider initially a uniform-vorticity elliptical vortex sit-
tang at the center of a unit disk, with unit vorticity level without losing generosity. Now consider its possible dynamics toward an axis-symmetrical vortex. This will be a state with a linear \( l = 2 \) diocotron mode if infinitesimal ellipticity.

The Euler equation conserves the total vorticity \( Q \), angular momentum \( M \), and energy \( E \) of the initial ellipse, which are given by

\[
Q = \int \omega(\mathbf{r}) d\mathbf{r}, \quad M = \int r^2 \omega(\mathbf{r}) d\mathbf{r}, \quad E_c = \frac{1}{2} \int \phi(\mathbf{r}) \omega(\mathbf{r}) d\mathbf{r}
\]

Furthermore, dynamical vorticity mixing ensures that the vorticity level of the resulting symmetrical vortex will never exceed one (the original uniform value). Under this restriction and given \( Q \) and \( M \) from the initial ellipse, there must be a maximum energy state with its energy denoted as \( E_\text{e} \) among all possible symmetrical distributions. With conservation of energy, this condition then immediately follows:

\[
E_c < E_\text{e} \quad \text{is necessary for the ellipse to ever evolve to a symmetrical vortex;}
\]

\[
E_c > E_\text{e} \quad \text{is the sufficient condition for the ellipse not evolving to a symmetrical state.}
\]

Applied to infinitesimal ellipticity, the \( l = 2 \) diocotron mode will not decay when \( E_c > E_\text{e} \).

It should be noted here that this condition only try to exclude symmetrical states from possible evolutions, a limitation purely physically motivated. For example, it seems unlikely that an ellipse at the disk center will break the symmetry and relax to an off-center vortex, although we believe that the energy of off-center vortices could be larger than \( E_c \). The conjecture (not decaying to off-center states) is confirmed by numerical simulations which will be discussed later.

To test the above condition, our first task is to calculate the energy \( E_c \) of a uniform elliptical vortex, which we define as a vorticity distribution \( \omega_c(\mathbf{r}) \) in the polar coordinate \((r, \theta)\)

\[
\omega_c(r, \theta; r_0, \epsilon) = 1 - s(r - r_0(1 + \epsilon \cos 2\theta)), \quad (2)
\]

with \( s(x) \) the usual step function. The parameter \( r_0 \) defines a base vortex size and \( \epsilon \) its ellipticity. To compute the energy of this vortex in a unit disk, first we consider the Green function in a disk for the Poisson equation, \( \nabla^2 G(\mathbf{r}; \mathbf{r'}) = -\delta(\mathbf{r} - \mathbf{r'}) \), with zero boundary condition at \( r = 1 \). Using an opposite-charged image charge sitting at \( r' \equiv (1/r', \theta') \), the Green function can be written as \( G(\mathbf{r}; \mathbf{r'}) = -\frac{1}{2\pi} \left( \ln|\mathbf{r} - \mathbf{r'}| - \ln|\mathbf{r} - \mathbf{r'}| - \ln r' \right) \). The energy of the uniform elliptical vortex is then

\[
E_c(r_0, \epsilon) = \frac{1}{2} \int \phi(\mathbf{r}) \omega(\mathbf{r}) d\mathbf{r}
\]

We separate \( E_c \) into three terms in the last equation. Here \( \phi_0(r; r_0) \) is the stream function of a uniform circular vortex with radius \( r_0 \), \( \frac{1}{r} \frac{d}{dr}(r \frac{d\omega}{dr}) = -\omega_0, \quad \omega_0(r; r_0) = 1 - s(r - r_0) \) and \( E_0 \) its corresponding energy,

\[
E_0 = \frac{1}{2} \int \phi_0(r; r_0) \omega_0(r; r_0) 2\pi r dr = \pi r_0^4 \left( -\frac{1}{4} \ln r_0 + \frac{1}{16} \right).
\]

We know of no way to integrate Eq. (3) analytically. Nevertheless, we can study the linear stability of an \( l = 2 \) diocotron mode from small \( \epsilon \) behavior of \( E_c(r_0, \epsilon) \). Since the vortex is defined by \( r_0(1 + \epsilon \cos 2\theta) \), the lowest order dependence on \( \epsilon \) must be \( \epsilon^2 \). Correct to the order of \( \epsilon^2 \), the second term in Eq. (3) is quickly found to be,

\[
\int_0^{2\pi} \frac{1}{2} \left[ r_0 \phi_0'(r_0; r_0) + \phi_0(r_0; r_0) \right] r_0^2 \epsilon^2 \cos 2\theta d\theta = -\frac{1}{4} \pi r_0^4 \left( 1 + \ln r_0 \right) \epsilon^2.
\]

Here prime denotes the derivative respected to \( r \). Evaluation of the third term in Eq. (3) is more difficult. Again correct to the order of \( \epsilon^2 \), the integration becomes

\[
\frac{1}{2} r_0^4 \epsilon^2 \int_0^{2\pi} d\theta \int_0^{2\pi} d\theta' \cos 2\theta \cos 2\theta' G(r_0, \theta; r_0, \theta').
\]

Using the Green function and changing to new variables \( u \equiv \theta + \theta', v \equiv \theta - \theta' \), after some algebra, we reach

\[
\frac{1}{8} r_0^4 \epsilon^2 \left[ \pi + \int_0^{2\pi} \ln(a - \cos v) \cos 2vdv \right],
\]

with \( a \equiv \frac{1}{2} (r_0^2 + 1/r_0^2) \geq 1 \). The integration \( I \equiv \int_0^{2\pi} \ln(1 - \cos v) \cos 2vdv = -\pi \) has also been used in reaching Eq. (4).

The integration in Eq. (3) is computed by first integrating its derivative respective to \( a \), and then using \( I \) to determine the constant arising from integration of \( a \). Eventually Eq. (4) is found to be \( \frac{1}{8} \pi r_0^4 (1 - r_0^2) \epsilon^2 \), and the energy of the elliptical vortex becomes

\[
E_c(r_0, \epsilon) = E_0 + \frac{1}{4} \pi r_0^4 (\frac{4}{3} r_0^2 - \frac{1}{2} - \ln r_0) \epsilon^2 + O(\epsilon^4).
\]

(5)
The energy \( E_e(r_0, \epsilon) \) is now to be compared with the energy \( E_s \) of the maximum-energy symmetrical state with the same values of total vorticity \( Q \) and angular momentum \( M \). Its vorticity must also be equal or less than one. To see what this state is, first it is favorable to have all the vorticity stay together, i.e., a uniform unit-valued circular vortex with radius \( r_0 = (Q/\pi)^{1/2} \), to gain as much as energy. However this circular vortex has a fixed angular momentum \( \frac{1}{2}\pi r_0^4 \), and the uniform ellipse always has a larger value. To satisfy the requirement of both \( Q \) and \( M \), as well as achieving a maximum energy, the state will have a vorticity distribution \( \omega_s(r) \) as,

\[
\omega_s(r) = \begin{cases} 
1 & \text{for } 0 < r < \alpha \text{ & } \beta < r < 1 \\
0 & \text{for } \alpha < r < \beta.
\end{cases}
\]  

(6)

Here \( \alpha \) and \( \beta \) depend on \( Q \) and \( M \), which are determined by \( r_0 \) and \( \epsilon \). In this distribution, a certain amount of vorticity is put as far away from center as possible, i.e., at the disk boundary, to account for the excess angular momentum and maximum amount of vorticity is left to concentrate at the center to acquire a maximum energy \[13\]. Here we see how the system size comes into play in a delicate manner. At small \( \epsilon \), \( \alpha = r_0 \left( 1 + \frac{1}{4}\frac{3\pi^2}{1 - r_0^2} \epsilon^2 \right) \)

and \( \beta = 1 - \frac{r_0^2}{2(1 - r_0^2)} \epsilon^2 \), and the energy \( E_s \) is expanded as

(It involves only straightforward algebra to solve \( \phi_s \) and then integrate \( E_s \).

\[
E_s = E_0 - \frac{1}{4}\pi r_0^4 \frac{1 - 3\epsilon^2}{1 - r_0^2} \ln r_0 \epsilon^2 + \mathcal{O}(\epsilon^4).
\]

Now we obtain the energy difference between \( E_e \) and \( E_s \) as

\[
E_e - E_s = \frac{\pi}{4} r_0^4 \left( \frac{1 - 3\epsilon^2}{1 - r_0^2} \ln r_0 \right) \epsilon^2 + \mathcal{O}(\epsilon^4).
\]

Evaluation of \( \epsilon^2 \) term reveals that there is a critical value of \( r_0, r_c \approx 0.586 \), such that \( E_e < E_s \) for \( r_0 < r_c \) and \( E_e > E_s \) for \( r_0 > r_c \).

So applying the energy condition, this indicates: the \( l = 2 \) mode perturbation of a circular vortex in a finite disk will not decay if the vortex is large enough (larger than 0.586 times the disk radius). This result seems contradict that of Briggs, Daugherty, and Levy \[3\] where decaying modes were calculated from eigen-analysis for all \( l \geq 2 \) modes of a circular vortex with a smooth profile very close to \( \omega_0(r; r_0) \) (a step at \( r_0 \)) but negative \( \omega'(r) \) at all \( r \). The resolution lies at that in the calculation of Briggs, Daugherty, and Levy, the symmetrical vortex is assumed as a monotonic decreasing function of \( r \). This seems a reasonable and harmless condition. However, as Eq. \[3\] shows, this condition is very restrictive and always violated by the uniform ellipse and hence their results no longer apply.

To further determine the stability of an ellipse with finite ellipticity, we need go beyond the expansion and calculate the energy for arbitrary \( \epsilon \). Here we resort to numerical calculations of integration \( E = \frac{1}{2} \int \phi \omega dr \). Since the Green function using image charges has logarithmic functions and is not easy to handle numerically, we rewrite the Green function as a summation of Fourier components in the azimuthal direction,

\[
G(r; r') = \sum_{m=0}^{\infty} g_m(r; r') \cos(m(\theta - \theta'))
\]

with \( g_m \) power \( \pm m \) of both \( r \) and \( r' \). The energy now becomes a summation on \( m \) of four-dimensional \((r, \theta, r', \theta')\) integrals. The integration on \( r \) and \( r' \) can be carried out analytically and the energy simplifies to a summation of double integrals on \( \theta \) and \( \theta' \). The integrals are then calculated numerically, and results are checked to conform to Eq. \[3\] at small \( \epsilon \).

The exact value of \( E_e(r_0, \epsilon) \) now enable us to establish the stability of finite ellipticity. In Figure 1 of the \( r_0-\epsilon \) plane, we plot a solid line indicating the position where \( E_e = E_s \). (Although complicated, again \( E_s \) with arbitrary \( \epsilon \) can be written down analytically from straightforward algebra). To the right of the line, \( E_e > E_s \) and an elliptical vortex will never relax to a symmetrical vortex. The line is almost vertical and only curves a little to the left as \( \epsilon \) is increased. It crosses \( \epsilon = 0 \) at \( r_0 \approx 0.586 \), the value we have obtained from the small \( \epsilon \) expansion. To the left, the present analysis only says that the decay to a symmetric state is allowed, but its occurrence is not implied.

FIG. 1. The vortex size and ellipticity space. The solid line marks the position where \( E_e = E_s \). Squares represent relaxations to elliptical states in simulations, and circles to symmetrical states.

It should be emphasized here that we have proved that the ellipse defined by Eq. \[3\] will not decay to a symmet-
rational state if \(r_0 > r_c\). It is very likely that dynamically it will undergo adjustment and reach an elliptical-like steady state. We cannot say about its exact distribution. Current understanding is that it probably should be a state described by \(\omega(r) = \omega(\phi(r) + \Omega r^2)\), with \(2\Omega\) giving the rigid body rotation frequency around the disk center. With a particular assumption on this functional dependence, an exact distribution can then be computed. One example is the mean field equilibrium [10]. However, whether and when the system will reach the prediction from this maximum-entropy principle (thermal equilibrium) is still not very clear [13].

Although Eq. (2) defines a uniform vortex, we do expect that a smoothly distributed vortex should have similar stability property if not deviated too much from Eq. (2). The exact values of stable radii will of course be different. This is confirmed by the numerical simulations discussed next.

We also perform numerical simulations to test our predictions. Simulations of the Euler equation in the polar coordinate have the difficulty of singularity at the origin due to vanishing grid spacing. To avoid this singularity, we use the functions,

\[
\begin{align*}
    x &= \mu \sqrt{1 - \zeta^2/2} \\
    y &= \zeta \sqrt{1 - \mu^2/2},
\end{align*}
\]

mapping a unit disk in the \(x-y\) plane to a square in \(\mu-\zeta\) plane with \(-1 \leq \mu \leq 1\) and \(-1 \leq \zeta \leq 1\). The simulation is then done in \(\mu-\zeta\) plane with Cartesian coordinate. The resolution is mostly \(256 \times 256\), with a few \(512 \times 512\) runs to test convergence. By avoiding the polar coordinate and hence the singularity at the origin, we need a much smaller numerical viscosity term, \(\nu \nabla^2 \omega\), to stabilize the simulation and hence obtain more reliable long time results. It is also noted that, since it is impossible to use a true step vorticity distribution with finite grid points, the simulation results should not be compared exactly with the predictions based on Eq. (3).

So for an initial ellipse with particular values of \(r_0\) and \(\epsilon\), we run simulations to long time and determine their final states. The results are plotted in Figure 1 as symbols, where squares indicate relaxation to elliptical states and solid circles to symmetrical states. Boundary region between squares and circles represents the conditions where it is difficult to determine final states from simulations. In the figure we see the confirmation of stable elliptical vortices with large vortex sizes. All squares to the right of the solid line shows that simulations are consistent with the predictions. The simulations also show that ellipses are actually stable to a lower radius and the smallest stable size is decreasing with increasing \(\epsilon\). Finally no relaxations to off-center vortices ever happen.

In conclusion we have showed that, from vorticity mixing in time evolution and energy calculations, large elliptical vortices in a finite disk will remain stable. At the infinitesimal ellipticity limit, this indicates stable \(l = 2\) dicrotont modes for large vortices. Numerical simulations not only confirmed these results, but also shows that elliptical states are actually stable to a smaller size. The contradiction to current general idea of decaying \(l = 2\) modes is also indicated due to the incompleteness for considering only monotonic decreasing vorticity by Briggs, Daugherty, and Levy.

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