Realizability of metamaterials with prescribed electric permittivity and magnetic permeability tensors

Graeme W. Milton
Department of Mathematics, University of Utah, Salt Lake City UT 84112, USA

Abstract

We show that any pair of real symmetric tensors $\varepsilon$ and $\mu$ can be realized as the effective electric permittivity and effective magnetic permeability of a metamaterial at a given fixed frequency. The construction starts with two extremely low loss metamaterials, with arbitrarily small microstructure, whose existence is ensured by the work of Bouchitté and Bourel and Bouchitté and Schweizer, one having at the given frequency a permittivity tensor with exactly one negative eigenvalue, and a positive permeability tensor, and the other having a positive permittivity tensor, and a permeability tensor having exactly one negative eigenvalue. To achieve the desired effective properties these materials are laminated together in a hierarchical multiple rank laminate structure, with widely separated length scales, and varying directions of lamination, but with the largest length scale still much shorter than the wavelengths and attenuation lengths in the macroscopic effective medium.

Keywords: Metamaterials, Realizability, Transformation Optics, Laminates
PACS: 42.70.-a,78.20.-e,78.67.Pt

Introduction

Isotropic materials with negative values of permittivity $\varepsilon$ and/or permeability $\mu$ (with low loss) have some fascinating properties. The colors of some stained glass windows are due to a resonance effect caused by metal spheres, which have a negative permittivity, embedded in the glass [1]. Veselago [2] found that a slab of material with $\varepsilon = \mu = -1$ at a given frequency would have a negative refractive index and act as a lens. Nicorovici, McPhedran and Milton [3] found that a cylindrical shell with permeability $\varepsilon = -1$ would, in the quasistatic limit and for TM waves, magnify the core material and produce a perfect image of a line dipole in the limit as the loss went to zero. Pendry [4] made the remarkable assertion that the Veselago lens would behave as a superlens, not limited by diffraction, providing perfect images of point sources, not just in the quasistatic limit, and now the validity of this assertion is beyond doubt provided one adds an infinitesimal loss to the lens: see, for example, [5, 6, 7, 8, 9]. Polarizable dipoles can be cloaked by such lenses [10, 11, 12, 13, 14], and larger objects can be cloaked by embedding the matching “antiobject” in the lens [15]. However any reasonable losses in the material can almost destroy these effects. From a practical viewpoint losses are always present and even extremely tiny...
losses can prevent the functioning of the Pendry superlens if the wavelength is short compared with the lens thickness $[8, 16]$. However the negative refractive index property is robust: materials with a negative refractive index were manufactured by Shelby, Smith and Schultz $[17]$. 

Other remarkable applications are associated with anisotropic materials having permittivity tensors $\varepsilon$ and permeability tensors $\mu$ with $\varepsilon = \mu$. It has long been known that the time harmonic Maxwell’s equations are invariant under curvilinear coordinate transformations $[18]$ and Dolin $[19]$ realized that the transformed equations can be viewed as a solution of Maxwell’s equations in a new material. Using such transformations empty space with $\varepsilon = \mu = 1$ is transformed in a material with $\varepsilon(x) = \mu(x)$: thus Dolin recognized that certain inhomogeneous inclusions with $\varepsilon(x) = \mu(x)$ are equivalent to empty space and are thus invisible. Pendry, Schurig and Smith $[20]$ went one step further and, similar to what Greenleaf, Lassas and Uhlmann $[21]$ had done in the context of the conductivity equations, found that by using a transformation which maps a point to a sphere one could create a cloak having $\varepsilon(x) = \mu(x)$ which completely shields an object, making it invisible to a fixed frequency of incident radiation. Rigorous mathematical justifications of this cloaking, and its generalizations, have been given $[22, 23, 24]$, and various approximate cloaks have been proposed and some have been physically constructed $[25, 26, 27, 28, 29, 30]$. This and other types of cloaking are surveyed in reviews $[31, 32]$. The growing field of transformation optics uses coordinate transformations to achieve unusual effects, such as rotators $[33]$, concentrators $[34]$, wormholes $[35]$, novel superlenses $[36, 37, 38, 39]$ but requires one to tailor materials with prescribed tensors $\varepsilon$ and $\mu$. In the geometric optics limit one can use transformations of the refractive index to achieve cloaking $[40, 41]$ and novel Eaton lenses $[42]$. 

Extreme values of $\varepsilon$ and $\mu$, near zero or infinity, also have striking applications, leading to new types of circuits $[43, 44, 45]$ and tunneling of energy through narrow channels $[46]$. 

Despite these tantalizing results, many of which are summarized in the book of Cai and Shalaev $[47]$, to my knowledge no one has either experimentally, numerically or theoretically shown that any given pair of prescribed symmetric real tensors $\varepsilon$ and $\mu$, and in particular given tensor pairs with $\varepsilon = \mu$, including isotropic materials with $\varepsilon = \mu = -1$, can be almost realized at a fixed frequency. (Isotropic materials with negative real permittivity and permeability have been realized $[48]$, but it is unclear if they can be realized with $\varepsilon = \mu = -1$.) This paper accomplishes that goal from a theoretical perspective. It shows there are no hidden constraints restricting the possible real pairs $(\varepsilon, \mu)$ that can exist at one given frequency. The construction uses hierarchical microstructures with structure on many widely separated length scales, and component metamaterials with arbitrarily low loss. It seems likely that the same effective properties could be achieved with more realistic designs, but it is doubtful that their properties could be calculated analytically.
The starting materials and Tartar’s formulae for the effective tensors of laminates

In this paper we assume an idealized world where the continuum, and Maxwell’s equations, extend down to arbitrarily small length scales and we assume the existence of two metamaterials. Material A has arbitrarily small microstructure and an effective permittivity tensor with a bounded real part \((\varepsilon^A)’\) which is diagonal, with exactly one negative and two positive diagonal elements and an effective permeability tensor with a real part \((\mu^A)’\) which is not necessarily diagonal but bounded and strictly positive definite, i.e. there exist constants \(\beta_A > \alpha_A > 0\) such that

\[ \beta_A I > (\mu^A)’ > \alpha_A I. \] \(1\)

The imaginary parts \((\varepsilon^A)''\) and \((\mu^A)''\) of these two tensors are assumed to be vanishingly small, but non-zero, i.e. there exist constants \(\eta > \nu > 0\) such that

\[ \eta I > (\varepsilon^A)'' > \nu I, \quad \eta I > (\mu^A)'' > \nu I, \] \(2\)

where the loss parameters \(\eta\) and \(\nu\) are vanishingly small. [Strictly speaking we should talk about a sequence of materials A, parameterized by \(\eta\) and consider what happens in the limit as \(\eta \to 0\), when \((\varepsilon^A)’\) and \((\mu^A)’\) do not depend on \(\eta\), while \(\nu\) and the scale and material constants of the microstructure within the metamaterial, do depend on \(\eta\).]

Material B has arbitrarily small microstructure and an an effective permittivity tensor with a real part \((\varepsilon^B)’\) which is strictly positive definite, i.e. satisfying the bounds

\[ \beta_B I > (\varepsilon^B)’ > \alpha_B I, \] \(3\)

for some choice of \(\beta_B > \alpha_B > 0\), and with an effective permeability with a bounded real part \((\mu^B)’\) which is diagonal, with exactly one negative and two positive diagonal elements. The imaginary parts \((\varepsilon^B)''\) and \((\mu^B)''\) of these two tensors are assumed to be vanishing small but non-zero, i.e. to satisfy the bounds

\[ \eta I > (\varepsilon^B)'' > \nu I, \quad \eta I > (\mu^B)'' > \nu I, \] \(4\)

where, without loss of generality \(\eta\) and \(\nu\) are the same parameters as appear in \(2\). [Again strictly speaking we should consider a sequence of materials B, parameterized by \(\eta\) and consider what happens in the limit as \(\eta \to 0\).]

Material A could be a metamaterial comprised of a cubic lattice of well separated cubes, where each cube has a microstructure of highly conducting rods aligned in the \(x_1\) direction. A rigorous mathematical proof that such a material has the desired effective properties, with a more precise description of the needed microgeometry, has been given by Bouchitté and Bourel [49]. Based on the results of Pendry et.al. [50] and Bouchitté and Felbacq [51] one might think that an periodic array of highly conducting thin rods aligned in the \(x_1\) direction would suffice for material A. However, Bouchitté and Felbacq [52] show that in a finite sample of such a material the associated effective equations can have a non-local behavior.

Material B could be a metamaterial comprised of a periodic lattice of highly conducting split ring resonators with axes aligned in the \(x_1\) direction with one rod per unit
cell. The split rings behave like polarizable magnetic dipoles and if one is just above resonance these can have a negative permeability in the $x_1$ direction. This means of creating artificial magnetism was discussed by Schelkunoff and Friis [53], and although their formula showed a negative permeability in the $x_1$ direction, they did not draw attention to this fact. The microstructure was rediscovered by Pendry et al. [54], who realized the significance of the negative permeability. A rigorous mathematical proof that such a material has the desired effective properties, has been given by Bouchitté and Schweizer [55], following earlier work of [56, 57, 58, 59, 60] (see also the introduction of [61]). Bouchitté and Schweizer also prove that one can obtain isotropic effective magnetic permeability tensors with a negative permeability.

We explore what effective permittivity and permeability tensors can be generated from composites of these two metamaterials $A$ and $B$ and all rotations of them within classical homogenization in the limit as $\eta$ tends to zero. The first step will be to recover the result of Bouchitté and Bourel [49] that a material with any real symmetric permittivity tensor $\varepsilon_*$ can be approximately achieved. Then the same argument applies to show that a material with any real symmetric permeability tensor $\mu_*$ can be approximately achieved. Finally by combining these two results we show that any pair of real symmetric tensors $(\varepsilon_*, \mu_*)$ can be approximately achieved.

The composites we consider are multiple rank laminates. This class of composite was first introduced by Maxwell [62] and has been extensively studied in the homogenization literature: see, for example, [63] and references therein. They are obtained by laminating the starting (rank 0) materials together on an extremely small length scale to obtain rank 1 laminates and then laminating these rank 1 laminates with other rank 1 or rank 0 laminates on a much larger, but still extremely small length scale, using a possibly different direction of lamination, to obtain rank 2 laminates and so forth. The advantage of considering this class of composites is that their effective tensors can be explicitly calculated, in the limit in which the ratio of successive length scales approaches infinity. The bounds (2) and (4) ensure that one can use reiterated homogenization theory: to calculate the effective tensor in the limit of widely separated scales of a multiple rank laminate which is a simple laminate of two (possibly multiple) rank laminates $C$ and $D$ one can replace $C$ and $D$ by homogeneous materials with material tensors the same as the effective tensors of $C$ and $D$.

The use of classical homogenization, except inside the metamaterials $A$ and $B$, should be valid provided, when we replace materials $A$ and $B$ by homogeneous materials with material tensors the same as the effective tensors of $A$ and $B$, the wavelengths and attenuation lengths within each substructure should be much larger than the microstructure at that level. For any fixed $\eta > 0$ this should be ensured by (2) and (4), in the limit in which the overall microstructure tends to zero, and the ratio between scales tends to infinity.

If a set of materials are laminated in direction $x_1$ then the formulae obtained by Tartar [64] (using an idea that goes back to Backus [65]) for the effective permittivity and permeability are

$$\tilde{\varepsilon}^* = \langle \tilde{\varepsilon} \rangle, \quad \tilde{\mu}^* = \langle \tilde{\mu} \rangle,$$

where $\varepsilon(x_1)$ and $\mu(x_1)$ are the local permittivity and permeability tensors (that could themselves be effective tensors) which only depend on $x_1$ and the angular brackets denote volume averages and for any symmetric matrix $\mathbf{C}$ with elements $c_{ij}$ the matrix $\tilde{\mathbf{C}}$ is
symmetric and has elements

\[
\tilde{c}_{11} = -1/c_{11}, \quad \tilde{c}_{1k} = c_{1k}/c_{11}, \quad \tilde{c}_{k\ell} = c_{k\ell} - c_{k1}c_{1\ell}/c_{11},
\]

(6)

for all \( k \neq 1, \ell \neq 1 \). Conversely if a symmetric matrix \( \tilde{C} \) is given, then \( C \) is symmetric with elements

\[
c_{11} = -1/\tilde{c}_{11}, \quad c_{1k} = -\tilde{c}_{1k}/\tilde{c}_{11}, \quad \tilde{c}_{k\ell} = \tilde{c}_{k\ell} - \tilde{c}_{k1}\tilde{c}_{1\ell}/\tilde{c}_{11},
\]

(7)

for all \( k \neq 1, \ell \neq 1 \).

In the case where \( \varepsilon \) is diagonal, (5) and (6) reduce to the familiar harmonic and arithmetic averages

\[
1/\varepsilon_{11}^* = \langle 1/\varepsilon_{11} \rangle, \quad \varepsilon_{kk}^* = \langle \varepsilon_{kk} \rangle \quad k \neq 1,
\]

(8)

with similar formulae applying when \( \mu \) is diagonal.

In the limit as the imaginary part of \( \varepsilon \) approaches zero then (5) implies that the imaginary part of \( \varepsilon^* \) will also tend to zero unless one is at resonance where the real part of \( \langle 1/\varepsilon_{11} \rangle \) or the real part of \( \langle 1/\mu_{11} \rangle \) vanishes. We will always avoid this happening. Then the limiting tensors \( \varepsilon^* \) and \( \mu^* \) can be obtained by replacing in (5) \( \varepsilon \) and \( \mu \) with their real parts, i.e. the vanishingly small imaginary parts do not effect the effective tensors except at resonance. From now on we will drop the primes and treat \( \varepsilon \) and \( \mu \) as real, while remembering that they do have vanishingly small imaginary parts to ensure that we can apply the rules of classical reiterated homogenization.

### Realizing a material with a desired tensor \( \varepsilon^* \) using lamination

Now let us explore what effective permittivity tensors \( \varepsilon^* \) can be obtained by hierarchically laminating material \( A \) with itself (and its rotations by 90° about the axes) in directions parallel to the axes, calculating the effective tensors at each stage using the harmonic and arithmetic averages (8). Let us assume the axes have been chosen so \( \varepsilon^A = \text{Diag}[-a, b, c] \), with \( a, b \) and \( c \) all positive. The following remark is helpful:

**Remark 1** If \( \varepsilon = \text{Diag}[-a, b, c] \), with \( a \) and \( b \) both positive is realizable then so is \( \varepsilon^* = \text{Diag}[-a/\delta, b\delta, c] \) for any finite \( \delta \neq 0 \).

To prove this remark, let us laminate \( \text{Diag}[-a, b, c] \) with the rotated material \( \text{Diag}[b, -a, c] \) in direction \( x_1 \) in volume fractions \( f_1 \) and \( f_2 = 1 - f_1 \). The resulting material has effective tensor \( \varepsilon^* = \text{Diag}[-a/\delta, b\delta, c] \) with \( \delta = f_1 - f_2(a/b) \) taking all values inside the interval between 1 and \( -a/b \) as \( f_1 \) ranges between 0 and 1, excepting \( \delta = 0 \) where one of the eigenvalues of \( \varepsilon^* \) becomes infinite and one is at resonance. Alternatively, let us laminate these two materials in direction \( x_2 \). The resulting material then has effective tensor \( \varepsilon^* = \text{Diag}[-a/\delta, b\delta, c] \) with \( \delta = [f_1 - f_2(b/a)]^{-1} \) taking all values outside the interval between 1 and \( -a/b \) as \( f_1 \) ranges between 0 and 1. Using one of the two constructions any finite value of \( \delta \neq 0 \) is possible.

Starting from \( \varepsilon^A = \text{Diag}[-a, b, c] \), with \( a, b \) and \( c \) all positive and applying Remark 1, we obtain a material with effective tensor \( \text{Diag}[a/\delta_0, -b\delta_0, c] \), with \( \delta_0 > 0 \). Applying
remark 1 again, we obtain a material with effective tensor \( \text{Diag}[a/\delta_0, -b\delta_0/\delta_1, c\delta_1] \), and by rotation (and replacing \( \delta_1 \) with \( 1/\delta_2 \)) a material with effective tensor \( \text{Diag}[a/\delta_0, c/\delta_2, -b\delta_2\delta_0] \). Laminating these last two materials together in equal proportions in direction \( x_1 \) gives a material with effective tensor \( \text{Diag}[a/\delta_0, -b\delta_0\delta_1/\delta_2, c\delta_1] \), and by rotation (and replacing \( \delta_1 \) with \( 1/\delta_2 \)) a material with effective tensor \( \text{Diag}[a/\delta_0, c/\delta_2, -b\delta_2\delta_0] \).

Given prescribed values of \( e > 0 \) and \( g \neq 0 \) we may choose non-zero \( \delta_0 \)

\[
e = \frac{(c/\delta_2 - b\delta_0/\delta_1)}{2}, \quad g = \frac{(c\delta_1 - b\delta_2\delta_0)}{2}.
\]

so that \( (9) \) is satisfied provided we choose \( \delta_0 \) with

\[
0 < \delta_0 < \frac{|ge|}{(bc)},
\]

to ensure that the roots of \( (11) \) are real. Since \( (11) \) remains valid if we change the sign of \( e \) we can also realize a material with effective tensor \( \text{Diag}[a/\delta_0, -e, g] \), and hence by remark 1, a material with effective tensor \( \text{Diag}[-a/\delta_0, e, g] \). Laminating this material in direction \( x_2 \) with Diag\([a/\delta_0, e, g]\) we can obtain a material with a prescribed effective tensor Diag\([h, e, g]\) with \( e > 0 \) provided \( \delta_0 \) is chosen sufficiently small to satisfy \( (11) \) and the constraint that \( \delta_0 < |a/h| \) which ensures \( h \) lies between \( a/\delta_0 \) and \( -a/\delta_0 \). So any diagonal tensor with non-zero diagonal elements, at least one of which is positive, is realizable.

The only case left to treat is when \( h, e \) and \( g \) are all negative. Let \( t \) be bigger that both \(-e\) and \(-g\). Then the tensors Diag\([h, e + t, g - t]\) and Diag\([h, e - t, g + t]\) are both realizable and by laminating them together in direction \( x_1 \) we see that the tensor Diag\([h, e, g]\) is also realizable as an effective permittivity tensor for any non-zero finite choice of \( h, e \) and \( g \).

**Realizing any desired pair of tensors \((\epsilon^*, \mu^*)\)**

By rotating the material just obtained we see that any symmetric matrix \( \epsilon^A \), except possibly those which are singular, is realizable as an effective permittivity tensor. The associated effective permeability tensor \( \mu^A \) (in the limit the loss goes to zero) must satisfy the classical homogenization Weiner bounds \( [66] \)

\[
\langle \mu \rangle \geq \mu^A \geq \langle \mu^{-1} \rangle^{-1},
\]

where \( \mu(x) \) is the local permeability tensor, which is locally a rotation of \( \mu^A \). Since \( \mu^A \) satisfies \( (11) \) we conclude that \( \mu^A \) also satisfies the inequality

\[
\beta_A I > \mu^A > \alpha_A I.
\]

Let us call \( U \) the family of materials thus obtained.

Similarly (because we can interchange the roles of \( \epsilon \) and \( \mu \)) using material \( B \), and its rotations, any symmetric matrix \( \mu^B \), except possibly those which are singular, is
realizable as an effective permeability tensor, and the associated effective permittivity \( \varepsilon^{B*} \) satisfies the bounds

\[
\beta_B I > \varepsilon^{B*} > \alpha_B I.
\]

We let \( V \) denote this family of materials.

Now we are free to take any set of materials in the set \( W = U \cup V \) and laminate them together in direction \( x_1 \) to form a larger set of materials \( LW \) containing \( W \). By an abuse of notation we let \( W \) also denote the set of tensor pairs \((\varepsilon^*, \mu^*)\) deriving from effective tensors \((\varepsilon^*, \mu^*)\) of materials in \( W \), and we let \( LW \) also denote the set of tensor pairs \((\varepsilon^*, \mu^*)\) deriving from effective tensors \((\varepsilon^*, \mu^*)\) of materials in \( LW \). Now (5) implies the tensor pairs in \( LW \) correspond to all convex combinations of the tensor pairs in \( W \), except for possibly those tensor pairs for which either \( \tilde{\varepsilon}_{11} \) or \( \tilde{\mu}_{11} \) are at resonance. So aside from a set of measure zero the set of tensor pairs \( LW \) is a convex set whose convex hull is characterized by the Legendre transform:

\[
f(P, Q) = \inf_{(\varepsilon^*, \mu^*) \in W} \sum_{i=1}^{3} \sum_{j=1}^{3} p_{ij} \tilde{\varepsilon}_{ij}^* + q_{ij} \tilde{\mu}_{ij}^*,
\]

as a function of the symmetric matrices \( P \) and \( Q \). If we can show that \( f(P, Q) \) is minus infinity for every choice of \( P \) and \( Q \), not both zero, then we can conclude that pair of real symmetric matrices \((\varepsilon^*, \mu^*)\) is approximately realizable.

First observe that

\[
f(P, Q) = \min\{f_A(P, Q), f_B(P, Q)\},
\]

where

\[
f_A(P, Q) = \inf_{(\varepsilon^A, \mu^A*) \in U} \sum_{i=1}^{3} \sum_{j=1}^{3} p_{ij} \tilde{\varepsilon}_{ij}^A + q_{ij} \tilde{\mu}_{ij}^A,
\]

\[
f_B(P, Q) = \inf_{(\varepsilon^B, \mu^B*) \in V} \sum_{i=1}^{3} \sum_{j=1}^{3} p_{ij} \tilde{\varepsilon}_{ij}^B + q_{ij} \tilde{\mu}_{ij}^B.
\]

Since almost every matrix \( \varepsilon^A* \) is realizable as a permittivity tensor, it follows that almost every matrix \( \tilde{\varepsilon}^{A*} \) is realizable: given \( \tilde{\varepsilon}^{A*} \) we realize, or almost realize, the tensor \( \varepsilon^{A*} \) whose elements are given according to (7). In particular we can realize a tensor \( \tilde{\varepsilon}^{A*} \approx -\lambda P \), for any value of \( \lambda \) no matter how large. The associated permeability tensor \( \mu^{A*} \) satisfies (13), which implies

\[
\beta_A > \mu_{ii}^{A*} > \alpha_A, \quad \beta_A > |\mu_{ij}^{A*}| \quad \text{for all } i \neq j,
\]

where the latter condition follows from the positivity of the determinant of the associated \( 2 \times 2 \) submatrix of \( \mu^{A*} \). It follows that the elements of \( \tilde{\mu}^{A*} \), given according to (6), satisfy the bounds

\[
|\tilde{\mu}_{11}^{A*}| < 1/\alpha_A, \quad |\tilde{\mu}_{1i}^{A*}| < \beta_A/\alpha_A, \quad |\tilde{\mu}_{ik}^{A*}| < \beta_A + \beta^2/\alpha_A,
\]

irrespective of the choice of \( \lambda \). Consequently, for any fixed choice of \( P \neq 0 \) and \( Q \),

\[
\sum_{i=1}^{3} \sum_{j=1}^{3} p_{ij} \tilde{\varepsilon}_{ij}^{A*} \approx -\lambda \text{Tr}(P^2)
\]
approaches $-\infty$ as $\lambda \to \infty$, while
\[
\sum_{i=1}^{3} \sum_{j=1}^{3} q_{ij} \tilde{\mu}_{ij}^{A*}
\] remains bounded. It follows that $f_A(P, Q)$ is minus infinity unless $P = 0$. Similarly $f_B(P, Q)$ is zero unless $Q = 0$ and thus $f(P, Q)$ is minus infinity unless both $P$ and $Q$ are zero. We conclude that any given pair of real symmetric matrices $(\varepsilon^*, \mu^*)$ is approximately realizable as the (effective permittivity, effective permeability) of a metamaterial.

Acknowledgements

The author is grateful to Guy Bouchitté for stimulating conversations and is thankful for support from the National Science Foundation through grant DMS-070978.

References

[1] J. C. Maxwell Garnett, “Colours in metal glasses and in metallic films”, Philosophical Transactions of the Royal Society of London 203, 385–420 (1904).

[2] V. G. Veselago, “The electrodynamics of substances with simultaneously negative values of $\varepsilon$ and $\mu$”, Uspekhi Fizicheskikh Nauk 92, 517–526 (1967), English translation in Soviet Physics Uspekhi 10:509–514 (1968).

[3] N. A. Nicorovici, R. C. McPhedran, and G. W. Milton, “Optical and dielectric properties of partially resonant composites”, Physical Review B (Solid State) 49, 8479–8482 (1994).

[4] J. B. Pendry, “Negative refraction makes a perfect lens”, Physical Review Letters 85, 3966–3969 (2000).

[5] S. A. Cummer, “Simulated causal subwavelength focusing by a negative refractive index slab”, Applied Physics Letters 82, 1503–1505 (2003).

[6] G. Shvets, “Photonic approach to making a material with a negative index of refraction”, Physical Review B 67, 035109 (2003).

[7] R. Merlin, “Analytical solution of the almost-perfect-lens problem”, Applied Physics Letters 84, 1290–1292 (2004).

[8] V. A. Podolskiy and E. E. Narimanov, “Near-sighted superlens”, Optics Letters 30, 75–77 (2005).

[9] G. W. Milton, N.-A. P. Nicorovici, R. C. McPhedran, and V. A. Podolskiy, “A proof of superlensing in the quasistatic regime, and limitations of superlenses in this regime due to anomalous localized resonance”, Proc. R. Soc. A 461, 3999–4034 (2005).
[10] G. W. Milton and N.-A. P. Nicorovici, “On the cloaking effects associated with anomalous localized resonance”, Proc. R. Soc. A 462, 3027–3059 (2006).

[11] N.-A. P. Nicorovici, G. W. Milton, R. C. McPhedran, and L. C. Botten, “Quasistatic cloaking of two-dimensional polarizable discrete systems by anomalous resonance”, Optics Express 15, 6314–6323 (2007).

[12] G. W. Milton, N.-A. P. Nicorovici, and R. C. McPhedran, “Opaque perfect lenses”, Physica B 394, 171–175 (2007).

[13] O. P. Bruno and S. Lintner, “Superlens-cloaking of small dielectric bodies in the quasistatic regime”, Journal of Applied Physics 102, 124502 (2007).

[14] G. Bouchitté and B. Schweizer, “Cloaking of small objects by anomalous localized resonance”, (2010), Preprint, http://eldorado.tu-dortmund.de:8080/bitstream/2003/26457/1/mathematicalPreprint14-09.pdf.

[15] Y. Lai, H. Chen, Z.-Q. Zhang, and C. T. Chan, “Complementary media invisibility cloak that cloaks objects at a distance outside the cloaking shell”, Physical Review Letters 102, 093901 (2009).

[16] N. A. Kuhta, V. A. Podolskiy, and A. L. Efros, “Far-field imaging by a planar lens: Diffraction versus superresolution”, Physical Review B 76, 205102 (2007).

[17] R. A. Shelby, D. R. Smith, and S. Schultz, “Experimental verification of a negative index of refraction”, Science 292, 77–79 (2001).

[18] E. J. Post, Formal structure of electromagnetics: General covariance and electromagnetics (North-Holland Publishing Co., Amsterdam, 1962), Dover Paperback, 1997.

[19] L. S. Dolin, “To the possibility of comparison of three-dimensional electromagnetic systems with nonuniform anisotropic filling”, Izv. Vyssh. Uchebn. Zaved. Radiofizika 4, 964–967 (1961).

[20] J. B. Pendry, D. Schurig, and D. R. Smith, “Controlling electromagnetic fields”, Science 312, 1780–1782 (2006).

[21] A. Greenleaf, M. Lassas, and G. Uhlmann, “Anisotropic conductivities that cannot be detected by EIT”, Physiological Measurement 24, 413–419 (2003).

[22] A. Greenleaf, Y. Kurylev, M. Lassas, and G. Uhlmann, “Full-wave invisibility of active devices at all frequencies”, Communications in Mathematical Physics 275, 749–789 (2007).

[23] R. Weder, “A rigorous analysis of high-order electromagnetic invisibility cloaks”, Journal of Physics A 41, 065207 (2008).

[24] R. V. Kohn, D. Onofrei, M. S. Vogelius, and M. I. Weinstein, “Cloaking via change of variables for the helmholtz equation”, Communications on Pure and Applied Mathematics (New York) (2010), to appear.
[25] D. Schurig et al., “Metamaterial electromagnetic cloak at microwave frequencies”, Science 314, 977–980 (2006).

[26] W. Cai, U. K. Chettiar, A. V. Kildishev, and V. M. Shalaev, “Optical cloaking with metamaterials”, Nature Photonics 1, 224–227 (2007).

[27] W. Cai, U. K. Chettiar, A. V. Kildishev, V. M. Shalaev, and G. W. Milton, “Non-magnetic cloak with minimized scattering”, Applied Physics Letters 91, 111105 (2007).

[28] J. Li and J. B. Pendry, “Hiding under the carpet: a new strategy for cloaking”, Physical Review Letters 101, 203901 (2008).

[29] R. Liu et al., “Broadband ground-plane cloak”, Science 323, 366–369 (2009).

[30] J. Valentine, J. Li, T. Zentgraf, G. Bartal, and X. Zhang, “An optical cloak made of dielectrics”, Nature Materials (2009), published online doi:10.1038/nmat2461.

[31] A. Alú and N. Engheta, “Plasmonic and metamaterial cloaking: physical mechanisms and potentials”, Journal of Optics A 10, 093002 (2008).

[32] A. Greenleaf, Y. Kurylev, M. Lassas, and G. Uhlmann, “Cloaking devices, electromagnetic wormholes, and transformation optics”, SIAM Review 51, 3–33 (2009).

[33] H. Chen and C. T. Chan, “Transformation media that rotate electromagnetic fields”, Applied Physics Letters 90, 241105 (2007).

[34] M. Rahm et al., “Design of electromagnetic cloaks and concentrators using form-invariant coordinate transformations of maxwell’s equations”, Photonics and Nanostuctures – Fundamentals and Applications 6, 87–95 (2008).

[35] A. Greenleaf, Y. Kurylev, M. Lassas, and G. Uhlmann, “Electromagnetic wormholes and virtual magnetic monopoles from metamaterials”, Physical Review Letters 99, 183901 (2008).

[36] A. V. Kildishev and E. E. Narimanov, “Impedance-matched hyperlens”, Optics Letters 32, 3432–3434 (2007).

[37] G. W. Milton, N.-A. P. Nicorovici, R. C. McPhedran, K. Cherendnichenko, and Z. Jacob, “Solutions in folded geometries, and associated cloaking due to anomalous resonance”, New Journal of Physics 10, 115021 (2008).

[38] M. Yan, W. Yan, and M. Qiu, “Cylindrical superlens by a coordinate transformation”, Physical Review B 78, 125113 (2008).

[39] M. Tsang and D. Psaltis, “Magnifying perfect lens and superlens design by coordinate transformation”, Physical Review B 77, 035122 (2008).

[40] U. Leonhardt, “Optical conformal mapping”, Science 312, 1777–1780 (2006).

[41] U. Leonhardt and T. Tyc, “Broadband invisibility by non-Euclidean cloaking”, Science 323, 110–112 (2009).
[42] T. Tyc and U. Leonhardt, “Transmutation of singularities in optical instruments”, New Journal of Physics 10, 115038 (2008).

[43] N. Engheta, A. Salandrino, and A. Alú, “Circuit elements at optical frequencies: Nanoinductors, nanocapacitors, and nanoresistors”, Physical Review Letters 95, 095504 (2005).

[44] G. W. Milton and P. Seppecher, “Electromagnetic circuits”, Networks and Heterogeneous Media (2010), Submitted, see also arXiv:0805.1079v2 [physics.class-ph] (2008).

[45] G. W. Milton and P. Seppecher, “Hybrid electromagnetic circuits”, Physica B (2010), Available online, doi:10.1016/j.physb.2010.01.007.

[46] M. Silveirinha and N. Engheta, “Tunneling of electromagnetic energy through subwavelength channels and bends using $\varepsilon$-near-zero materials”, Physical Review Letters 97, 157403 (2006).

[47] W. Cai and V. Shalaev, Optical Metamaterials: Fundamentals and Applications (Springer, Dordrecht, 2010).

[48] A. L. Efros and A. L. Pokrovsky, “Dielectric photonic crystal as medium with negative electric permittivity and magnetic permeability”, Solid State Communications 129, 643–647 (2004).

[49] G. Bouchitté and C. Bourel, “Homogenization of finite metallic fibers and 3D-effective permittivity tensor”, Communications in Computational Physics (2010), To appear.

[50] J. B. Pendry, A. J. Holden, W. J. Stewart, and I. Youngs, “Extremely low frequency plasmons in metallic mesostructures”, Physical Review Letters 76, 4773–4776 (1996).

[51] G. Bouchitté and D. Felbacq, “Homogenization of a set of parallel fibers”, Waves in random media 7, 1–12 (1997).

[52] G. Bouchitté and D. Felbacq, “Homogenization of a wire photonic crystal: the case of small volume fraction”, SIAM Journal on Applied Mathematics 66, 2061–2084 (2006).

[53] S. A. Schelkunoff and H. T. Friis, Antennas: the theory and practice, pages 584–585, John Wiley and Sons, New York / London / Sydney, Australia, 1952.

[54] J. Pendry, A. J. Holden, D. J. Robbins, and W. J. Stewart, “Magnetism from conductors and enhanced nonlinear phenomena”, IEEE Transactions on Microwave Theory and Techniques 47, 2075–2084 (1999).

[55] G. Bouchitté and B. Schweizer, “Homogenization of Maxwell’s equations with split rings”, SIAM Journal on Multiscale Modeling and Simulation (2010), To appear.
[56] V. V. Zhikov, “On an extension and an application of the two-scale convergence method”, Matematicheskii Sbornik 191, 31–72 (2000), English translation in Sbornik: Mathematics, 191 (7), 9731014 (2000).

[57] V. V. Zhikov, “Gaps in the spectrum of some elliptic operators in divergent form with periodic coefficients”, Algebra i Analiz 16, 34–58 (2004), English translation in St. Petersburg Mathematical Journal, 16 (5), 773–790 (2005).

[58] G. Bouchitté and D. Felbacq, “Homogenization near resonances and artificial magnetism from dielectrics”, Comptes Rendus des Séances de l’Académie des Sciences. Série I. Mathématique 339, 377–382 (2004).

[59] D. Felbacq and G. Bouchitté, “Theory of mesoscopic magnetism in photonic crystals”, Physical Review Letters 94, 183902 (2005).

[60] R. V. Kohn and S. P. Shipman, “Magnetism and homogenization of microresonators”, SIAM Journal on Multiscale Modeling and Simulation 7, 62–92 (2008).

[61] V. P. Smyshlyaev, “Propagation and localization of elastic waves in highly anisotropic periodic composites via two-scale homogenization”, Mechanics of Materials 41, 434–447 (2009).

[62] J. C. Maxwell, A Treatise on Electricity and Magnetism, volume 1, pages 371–372, Clarendon Press, Oxford, United Kingdom, 1873, Article 322.

[63] G. W. Milton, The Theory of Composites (volume 6 of Cambridge Monographs on Applied and Computational Mathematics Cambridge University Press, Cambridge, United Kingdom, 2002).

[64] L. Tartar, Estimation de coefficients homogénéisés. (French) [Estimation of homogenization coefficients], in Computing Methods in Applied Sciences and Engineering: Third International Symposium, Versailles, France, December 5–9, 1977,, edited by R. Glowinski and J.-L. Lions, volume 704 of Lecture Notes in Mathematics, pages 364–373, Berlin / Heidelberg / London / etc., 1979, Springer-Verlag, English translation in Topics in the Mathematical Modelling of Composite Materials, pp. 9–20, ed. by A. Cherkaev and R. Kohn. ISBN 0-8176-3662-5.

[65] G. E. Backus, “Long-wave elastic anisotropy produced by horizontal layering”, Journal of Geophysical Research 67, 4427–4440 (1962).

[66] O. Wiener, “Die Theorie des Mischkörpers für das Feld des stationären Strömung. Erste Abhandlung die Mittelwertsätze für Kraft, Polarisation und Energie. (German) [The theory of composites for the field of steady flow. First treatment of mean value estimates for force, polarization and energy]”, Abhandlungen der mathematisch-physischen Klasse der Königlich Sächsischen Gesellschaft der Wissenschaften 32, 509–604 (1912).