Optimal Control under Controlled-Loss Constraints via Reachability Approach and Compactification

Géraldine Bouveret
Smith School and Wadham College
University of Oxford
geraldine.bouveret@smithschool.ox.ac.uk

Athena Picarelli
Department of Mathematics
Imperial College London
a.picarelli@imperial.ac.uk

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Abstract

We study a family of optimal control problems under a set of controlled-loss constraints holding at different deterministic dates. It is well known that the characterization of the related value function by a Hamilton-Jacobi-Bellman equation usually calls for additional strong assumptions involving an interplay between the dynamics of the processes involved and the set of constraints. To treat this problem in absence of those assumptions we first translate it into a state-constrained stochastic target problem and then apply a level-set approach to describe the reachable set. Using this approach the state constraints can be easily handled through an exact penalization technique. However, this stochastic target problem involves a new set of state and control variables. In particular, those controls are unbounded. A “compactification” of the problem is then performed.

Keywords: Hamilton-Jacobi-Bellman Equations, Viscosity Solutions, Optimal Control, Controlled-Loss Constraints, State Constraints, Singular Controls.

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1 Introduction

On a filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) and for \((t, z) \in [0, T] \times \mathbb{R}^{d+1}\), with \(0 \leq T < \infty\) fixed, we consider the process \(Z_s^{t, z, \nu}, t \leq s \leq T\), valued in \(\mathbb{R}^{d+1}\) and with initial conditions \((t, z)\). This process is controlled by some \(\nu \in \mathcal{U}\), with \(\mathcal{U}\) the set of \(\{\mathcal{F}_t\}\)-predictable processes valued in a given compact set \(U\), and is a strong solution to the following stochastic differential equation

\[
Z_s^{t, z, \nu} = z + \int_t^s \mu_Z(r, Z_r^{t, z, \nu}, \nu_r) \, dr + \int_t^s \sigma_Z(r, Z_r^{t, z, \nu}, \nu_r) \, dW_r, \quad s \geq t,
\]

for some \(d\)-dimensional Brownian motion \(W\). We then consider the time grid \(t_0 = 0 \leq \ldots \leq t_i \leq \ldots \leq t_n = T\), with \(n \in \mathbb{N}\) fixed. For any \(t \in [t_i, t_{i+1})\), \(0 \leq i \leq n-1\), and \(z \in \mathbb{R}^{d+1}\), the objective of the paper is to solve the stochastic optimal control problem associated with the cost

\[
\mathbb{E} \left[ f(Z_T^{t, z, \nu}) + \int_t^T \ell(s, Z_s^{t, z, \nu}, \nu_s) \, ds \right], \tag{1.1}
\]

for some functions \((f, \ell)\) satisfying some continuity and growth conditions, under the following constraints in expectation

\[
\mathbb{E} \left[ \Psi(Z_{i+k}^{t, z, \nu}) \right] \leq p_k, \quad i + 1 \leq k \leq n. \tag{1.2}
\]

It is well known that in the unconstrained case, and under some general assumptions, the value function associated with this type of optimal control problems can be characterized as the unique continuous viscosity solution of a second-order Hamilton-Jacobi-Bellman (HJB) equation (see e.g. \([30, 53]\) and the references therein). However in many practical applications, the question of optimization under state constraints arises. We suggest to study here the case of constraints in expectation holding on a set of deterministic dates. The function \(\Psi\) appearing in (1.2) typically represents a loss function, e.g. \(\Psi(z) = (z)^-\), so that the restriction given by (1.2) is often referred in the literature as a controlled-loss constraint.

This type of problems has attracted a great attention in the finance literature (see e.g. \([27, 19]\)). In particular, some authors have worked in a non-Markovian setting on utility maximization problems under an expected shortfall constraint. For instance, a similar problem has been considered in \([31]\) where the constraint includes all coherent risk measures, but with a utility function satisfying some regularity assumptions. The latter results have then been extended in \([24]\) to the case where the utility function is applied to positive gains only while the risk measure is applied to negative shortfall. They provide a full solution under a complete market setting. This paper actually
extends the previous results to several constraints, a Markovian and incomplete market setting, and to functions satisfying only some mild regularity conditions.

There exists a huge literature on state-constrained optimal control problems and their associated HJB equations. We refer the reader to [40, 38, 5, 35, 29, 14] for stochastic control problems and to [20, 48, 49, 33] for deterministic ones. In this case, the characterization of the value function as a viscosity solution of a HJB equation is intricate and usually requires a delicate interplay between the dynamics of the processes involved and the set of constraints. First, the value function can take infinite values in the regions where (1.2) cannot be satisfied and some “viability” assumptions are therefore required to ensure its finiteness. Second, specific properties on the dynamics of the processes at the boundaries of the viability domain and on the admissible controls must hold to ensure the continuity of the value function and its partial differential equation (PDE) characterization. This often makes the problem not treatable.

From a mathematical perspective, the state-constrained problem described in (1.1)-(1.2) is non-standard as the constraints are expressed in expectation and are not imposed on \([t, T]\). In the discrete-time setting, optimal control problems with an expectation constraint imposed at time \(T\) only, have been studied in [44]. In continuous time, [17] has recently provided a treatment of similar problems but with the constraint expressed in probability. In particular, the analysis in [17] builds on [14] and proves that the value function is a solution of a constrained HJB equation. However, this approach still involves strong assumptions on the controls and processes involved. In particular, the results require to have \(U = \mathbb{R}^d\).

This paper therefore aims at providing an alternative way for fully characterizing the associated value function under a general framework. This objective is achieved at the price of augmenting the state and control space by \(n + 1\) additional components and considering unbounded controls. More precisely, following the ideas developed in [11] for the case of a constraint holding almost surely on the entire time interval \([t, T]\), our approach relies on two strategies. First, we reformulate the original problem as a stochastic target problem involving almost-sure constraints and unbounded controls. Then, we characterize this auxiliary target problem by means of a level-set approach where the state constraints are managed through the use of an exact penalization technique.

The first reformulation builds on the arguments developed in [13, 15]. It involves one unbounded control that comes from the martingale representation of the quantity in (1.1) and \(n\) unbounded controls that emanate from the martingale representation of the quantity in (1.2) for every \(i + 1 \leq k \leq n\). In particular, compared with [11] where a single
\( \mathbb{R}^d \)-valued unbounded control is sufficient to provide the stochastic target formulation, the number of unbounded controls involved here depends on the number of constraints and therefore changes with the time interval considered.

Stochastic target problems have been extensively studied via a Hamilton-Jacobi-Bellman (HJB) characterization [50, 52, 15, 18] or a dual approach [12, 16]. The HJB characterization is favored here as a dual approach relies on specific assumptions on the coefficients of the diffusion and suffers from a lack of tractability of computations when several dates are involved in the constraint. Still, a direct treatment of the derived stochastic target problem remains challenging because of the nature and number of constraints involved. In particular, with the equivalence result derived in [13], we expect to retrieve the HJB equations that would be obtained with a direct treatment of the original optimal control problem. However, we know from the work done by [14, 17] and commented above, that such characterization would involve strong regularity assumptions that would considerably restrict the scope of our study. Accordingly, at this stage, a direct resolution of the stochastic target problem associated with the original optimal control problem seems unsatisfactory.

Therefore, for solving the auxiliary stochastic target problem mentioned above, we use a different approach, the so-called level-set approach. Initially introduced by Osher and Sethian in [43] to model some deterministic front propagation phenomena, the level-set approach has been used in many applications related to (non)linear controlled systems (see e.g. [28, 39, 41, 9, 10] and the references therein). The connection between stochastic target problems and level-set characterization has already been pointed out in [51]. In our case, the level-set approach links the stochastic target problem to an auxiliary optimal control problem, referred as the level-set problem. The latter is defined on an augmented state and control space, but without state constraints. This level-set problem has the great advantage to allow a complete treatment of the original problem. More precisely, under very mild assumptions, the associated value function can be proved to be continuous on each time interval \([t_i, t_{i+1})\), \(0 \leq i \leq n-1\), and to admit a complete PDE characterization. In particular, this function is defined differently on each interval \([t_i, t_{i+1})\), \(0 \leq i \leq n-1\), and shows discontinuity at \(t_{i+1}\). Its HJB characterization must then be obtained by induction arguments over \(i = n, \ldots, 0\).

As intimated in [11], this approach relies on one necessary condition which is the existence of an optimizer for the level-set problem. The conditions under which this assumption holds are not fully investigated here and are left for further research. However, we highlight that the nature of those conditions often relates to convexity properties of the dynamics, the cost functions, and the set of controls. Nonetheless, they are com-
pletely independent of the aforementioned viability and regularity assumptions needed to deal with the original problem by the direct application of dynamic programming arguments.

When it comes to the characterization of the value function associated with the level-set problem, the unbounded controls, coming from the stochastic target formulation of the original optimal control problem, typically lead to discontinuous Hamiltonians. Following the arguments developed in [8], a compactification of the differential operator is applied in [11]. We follow here a different route. More precisely, we apply a suitable time change in the second step of our reformulation to obtain a level-set problem with bounded controls. Thanks to this transformation, the characterization of the associated value function involves a continuous HJB operator, and the assumption on the existence of an optimizer mentioned above can be relaxed (compare (H4) in [11] with Assumption 3.2.1 below). The performed compactification by time change is another important contribution of the paper being of general interest for several problems presenting a similar structure (see, for instance, [1, 4]). Analogous techniques have been introduced in e.g. [46] for deterministic control systems, and in [26, 42] for stochastic control problems in which the unbounded control acts on the drift term of the dynamics involved. In our case, the unbounded controls appear linearly in the diffusion terms as they result from the application of the martingale representation theorem. Interestingly, the full PDE characterization of a time-changed problem also seems to be a contribution in itself.

The rest of the paper is organized as follows: in Section 2 we formally state the problem and give the precise set of assumptions under consideration. In Section 3 we formulate the optimal control problem as a constrained stochastic target problem and apply the level-set approach together with the compactification of the set of control values. A complete characterization of the obtained level-set function as the unique viscosity solution of a suitable HJB equation is obtained in Section 4. In Section 5 the extension to the case of next-period controlled-loss constraints and probability constraints is discussed. An appendix contains proofs of some technical results.

**Notations.** We let $d \geq 1$ be an integer. Any vector $x$ of $\mathbb{R}^d$ is viewed as a column vector unless otherwise stated. We denote by $|x|$ the Euclidean norm of $x$, and by $x^\top$ its transpose. Moreover, $M^d$ is the set of $d$-dimensional square matrices whose each element belongs to $\mathbb{R}$, and $S^d$ is the subset of elements of $M^d$ that are symmetric. We set $M^\top$ the transpose of $M \in M^d$, while $\text{Tr}[M]$ is its trace. We also introduce $I_d \in S^d$ the identity matrix. Additionally, we introduce for a given function $g$ defined on a subset $B$ of $\mathbb{R}^d$ the set $\text{Im}(g) := \{g(r), r \in B \text{ s.t. } g(r) \in \mathbb{R}\}$. We also denote by $B_{r}(t,x)$ the open ball of radius $r > 0$ centered at $(t,x) \in [0,T] \times \mathbb{R}^d$. Moreover, we define $S_d$ the sphere of
\( \mathbb{R}^d \) of radius 1, and \( D_d \) the set of vectors \( \beta \in \mathcal{S}_d \) such that their first component \( \beta_1 = 0 \). Finally, given a countable set \( J \), we write \( \text{Card}(J) \) for its cardinality. In this paper, the constant \( C > 0 \) is, unless otherwise stated, a generic denomination of a constant term (possibly depending on \( d \) and \( n \)), and the abbreviation “s.t.” (resp. “w.r.t.”) stands for “such that” (resp. “with respect to”). All over the paper, inequalities between random variables have to be understood in the \( \mathbb{P} \)-a.s. sense.

2 Setting and main assumptions

In this manuscript we consider \( \Omega \), the space of \( \mathbb{R}^d \)-valued continuous functions \( (\omega_t)_{t \leq T} \) on \( [0, T] \), \( d \geq 1 \), endowed with the Wiener measure \( \mathbb{P} \). We introduce \( W \) the coordinate mapping, i.e. \( (W(\omega_t))_{t \leq T} \) for \( \omega \in \Omega \) so that \( W \) is a \( d \)-dimensional Brownian motion on the canonical filtered probability space \( (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}) \). In particular, \( \mathcal{F} \) is the Borel tribe of \( \Omega \) and \( \mathbb{F} := \{ \mathcal{F}_t, 0 \leq t \leq T \} \) is the \( \mathbb{P} \)-augmentation of the filtration generated by \( W \). We define \( \mathcal{U} \) as the collection of \( \{ \mathcal{F}_t \} \)-predictable processes valued in \( U \), a compact subset of \( \mathbb{R}^d \). For \( t \in [0, T], z := (x, y) \in \mathbb{R}^{d+1} \) and for \( \nu \in \mathcal{U} \), we define the process \( Z^{t,z,\nu} := (X^{t,x,\nu}, Y^{t,z,\nu}) \) as the unique (strong) solution to the following stochastic differential equations (SDE)

\[
X^{t,x,\nu}_s = x + \int_t^s \mu_X(r, X^{t,x,\nu}_r, \nu_r) \, dr + \int_t^s \sigma_X(r, X^{t,x,\nu}_r, \nu_r) \, dW_r \quad \text{on } \mathbb{R}^d,
\]

\[
Y^{t,z,\nu}_s = y + \int_t^s \mu_Y(r, Z^{t,z,\nu}_r, \nu_r) \, dr + \int_t^s \sigma_Y(r, Z^{t,z,\nu}_r, \nu_r) \, dW_r \quad \text{on } \mathbb{R},
\]

where \( (\mu_X, \sigma_X) : (t, x, u) \in [0, T] \times \mathbb{R}^d \times U \mapsto \mathbb{R}^d \times \mathcal{S}^d \) (resp. \( (\mu_Y, \sigma_Y) : (t, z, u) \in [0, T] \times \mathbb{R}^{d+1} \times U \mapsto \mathbb{R} \times \mathcal{S}^d \) are continuous functions being, in particular, Lipschitz continuous in \( x \) (resp. \( z \)). In what follows we consider \( \mu_Z : [0, T] \times \mathbb{R}^{d+1} \times U \mapsto \mathbb{R}^{d+1} \) and \( \sigma_Z : [0, T] \times \mathbb{R}^{d+1} \times U \mapsto \mathcal{M}^{d+1,d} \) (where \( \mathcal{M}^{d+1,d} \) is the matrix of size \((d+1) \times d\)) defined as

\[
\mu_Z(t, z, u) := \begin{pmatrix} \mu_X(t, x, u) \\ \mu_Y(t, z, u) \end{pmatrix}, \quad \sigma_Z(t, z, u) := \begin{pmatrix} \sigma_X(t, x, u) \\ \sigma_Y(t, z, u) \end{pmatrix}.
\]

The particular form of the dynamics is adapted to financial applications where \( X \) models the evolution over time of the price of some underlying asset and \( Y \) represents a portfolio process. However, our main results hold for any \( Z \) solution of a SDE. We now introduce three maps: \( f : z \in \mathbb{R}^{d+1} \mapsto f(z) \in \text{Im}(f) \), a Lipschitz continuous function; \( \ell : (t, z, u) \in [0, T] \times \mathbb{R}^{d+1} \times U \mapsto \ell(t, z, u) \in \text{Im}(\ell) \), a continuous function being, in particular, Lipschitz continuous in \( z \); and \( \Psi : z \in \mathbb{R}^{d+1} \mapsto \Psi(z) \in \text{Im}(\Psi) \), a Lipschitz continuous
function. We assume that $\text{Im}(\ell)$ (resp. $\text{Im}(f)$) := $\mathbb{R}^+$ and $\text{Im}(\Psi)$ := $[\gamma, \infty)$ for some $\gamma \in \mathbb{R}$. We point out that the non-negativity requirement on $f$ and $\ell$ is not restrictive here and can be easily substituted with a lower-boundedness condition.

We fix $n \in \mathbb{N}$ and consider the time grid $t_0 = 0 \leq \cdots \leq t_i \leq \cdots \leq t_n = T$.

To alleviate notations in this manuscript we denote for $0 \leq i \leq n - 1$, $\mathcal{C}_i := [t_i, t_{i+1}) \times \mathbb{R}^{d+(n-i+1)}$ as well as $\text{cl}(\mathcal{C}_i) := [t_i, t_{i+1}] \times \mathbb{R}^{d+(n-i+1)}$. From now on (and in what precedes), we omit the dependency in $n$ and $i$ to ease notations, unless strictly necessary.

For $s \in [0, T]$ and $t_k \in [s, T]$, $1 \leq k \leq n$, we introduce $\mathcal{A}_k^s$, the collection of $\{\mathcal{F}_t\}$-predictable processes $L$ valued in $\mathbb{R}^d$ and s.t. $|L| := \mathbb{E}[|\int_t^s |L|dv|^2] < \infty$.

The objective of the paper is to solve on $\mathcal{C}_i$, $0 \leq i \leq n - 1$, the following stochastic optimal control problem

$$V(t, z, p_{i+1}, \ldots, p_n) := \inf_{\nu \in \mathcal{U}_t, z, p_{i+1}, \ldots, p_n} \mathbb{E}\left[ f(Z^t_{T, \nu}) + \int_t^T \ell(s, Z^s_{\nu}, \nu_s) ds \right],$$

(2.1)

where

$$\mathcal{U}_t, z, p_{i+1}, \ldots, p_n := \left\{ \nu \in \mathcal{U} : \mathbb{E}\left[ \Psi(Z^t_{i_k, \nu}) \right] \leq p_k, i + 1 \leq k \leq n \right\}.$$ 

On $\{T\} \times \mathbb{R}^{d+1}$, we set $V(T, z) = f(z)$. We use the convention $V(t, z, p_{i+1}, \ldots, p_n) = \infty$ whenever $\mathcal{U}_t, z, p_{i+1}, \ldots, p_n = \emptyset$. Observe that $\mathcal{U}_t, z, p_{i+1}, \ldots, p_n = \emptyset$ whenever there exists $i + 1 \leq k \leq n$ s.t. $p_k < \gamma$. We underline that the problem can be treated similarly if we consider different loss functions at each date.

### 3 Problem reduction via reachability, compactification, and level-set approach

In the spirit of [11], our approach articulates in two steps. First, we reformulate (2.1) as a constrained stochastic target problem (see Proposition 3.1). Then, this stochastic target problem is described by a level-set approach where the constraints are handled using an exact penalization technique (see Proposition 3.3). This links the backward reachable set associated with the stochastic target problem to the zero level-set of a value function associated with an auxiliary unconstrained optimal stopping/control problem. In [11] the auxiliary problem is an unconstrained optimal control problem defined on a space of unbounded controls. Here, we manage to define the auxiliary value function on a set of controls taking values in relatively compact subsets of $\mathbb{R}^d$ (see Proposition 3.2). This is done via a compactification argument “à la” Dufour and Miller in [26] and [25] (see also [42]).
3.1 Associated stochastic target problem

We investigate here the link between (2.1) and a suitable stochastic target problem under almost sure constraints. We recall that this has been previously investigated in [13], in the unconstrained case, and in [11], in the case of an almost sure constraint imposed pointwise on \([t, T]\). More precisely, we prove in Proposition 3.1 that (2.1) can be formulated as a stochastic target problem involving a set of (hedging) constraints holding almost surely.

Before stating the first results, we define for \(p(t, z, m) \in \mathbb{R}^{d+2}, 0 \leq i \leq n - 1,\) and \((\nu, \eta) \in \mathcal{U} \times \mathcal{A}_i\), we define

\[
M^{t, z, m, \nu, \eta} := m - \int_t^T \ell(s, Z^{t, z, \nu}_{s}, \nu_s) \, ds + \int_t^T \eta_s \, dW_s \text{ on } [t, T].
\]

Finally, we introduce for \((t, z, p_i+1, ..., p_n) \in \mathcal{C}_i, 0 \leq i \leq n - 1,\) the new set of controls

\[
\mathcal{U}_{t, z, p_i+1, ..., p_n} := \left\{ (\nu, \alpha) \in \mathcal{U} \times \hat{\mathcal{A}}^i \text{ s.t. } P^{t, p_k, \alpha_k} \geq \Psi(Z^{t, z, \nu}_{t_k}, i + 1 \leq k \leq n) \right\},
\]

with \(\alpha := (\alpha_i+1, ..., \alpha_n)^\top\) and \(\hat{\mathcal{A}}^i := \mathcal{A}_i^1 \times ... \times \mathcal{A}_i^n.\)

**Remark 3.1.** We observe that on \(\mathcal{C}_i, 0 \leq i \leq n - 1, \mathcal{U}_{t, z, p_i+1, ..., p_n} = \emptyset\) if there exists \(i + 1 \leq k \leq n \) s.t. \(p_k < \gamma\) since, by the martingale property of \(P^{t, p_k, \alpha_k}, P^{t, p_k, \alpha_k} < \gamma\) on \([t, t_k]\) whenever \(p_k < \gamma.\)

We can now state the following result.

**Proposition 3.1.** Fix \(0 \leq i \leq n - 1,\) and \((t, z, p_{i+1}, ..., p_n) \in \mathcal{C}_i.\) Then

\[
V(t, z, p_{i+1}, ..., p_n) = \inf \left\{ m \geq 0 : \exists (\nu, \alpha) \in \mathcal{U}_{t, z, p_{i+1}, ..., p_n} \times \mathcal{A}_i \text{ s.t. } M^{t, z, m, \nu, \eta} \geq f(Z^{t, z, \nu}_{t}) \right\}.
\]

**Proof.** We fix \(0 \leq i \leq n - 1,\) and \((t, z, p_{i+1}, ..., p_n) \in \mathcal{C}_i.\) Arguing as in [13], one can easily prove that

\[
V(t, z, p_{i+1}, ..., p_n) = \inf \left\{ m \geq 0 : \exists \nu \in \mathcal{U}_{t, z, p_{i+1}, ..., p_n} \text{ s.t. } m \geq \mathbb{E} \left[ f(Z^{t, z, \nu}_{t}) + \int_t^T \ell(s, Z^{t, z, \nu}_{s}, \nu_s) \, ds \right] \right\}.
\]
We then prove the equivalence between the two following statements
\[
(i) \exists \nu \in \mathcal{U}_{i, t, p_{i+1}, \ldots, p_n} \text{ s.t. } m \geq \mathbb{E} \left[ f(Z_T^{i,z,\nu}) + \int_t^T \ell(s, Z_s^{i,z,\nu}, \nu_s) \, ds \right],
\]
\[
(ii) \exists (\nu, \alpha, \eta) \in \mathcal{U} \times \mathcal{A}^1 \times \mathcal{A}^1_n, \text{ s.t. } \begin{cases} 
M_{T}^{i,m,\nu,\eta} = m - \int_t^T \ell(s, Z_s^{i,z,\nu}, \nu_s) \, ds + \int_t^T \eta_s \, dW_s \geq f(Z_T^{i,z,\nu}), \\
\text{and} \\
P_{t,k}^{i,p_k,\alpha_k} = p_k + \int_t^{t_k} \alpha_k \, dW_s \geq \Psi(Z_{t_k}^{i,z,\nu}), i + 1 \leq k \leq n
\end{cases}
\]
for any \( m \geq 0. \)

To this aim we appeal to similar techniques as those exploited in [15, 12]. The implication (ii) \( \Rightarrow \) (i) follows by taking the expectation in (ii) and using the martingale property of the stochastic integrals. On the other hand, the implication (i) \( \Rightarrow \) (ii) follows from the martingale representation theorem (see e.g. [37, Theorem 4.15, Chapter 3]). More precisely, from the assumptions on the coefficients of \( X \) and \( Y \), as well as the growth conditions on \( f, \ell \) and \( \Psi \), there exists, for any \( \nu \in \mathcal{U} \), \( (\hat{\alpha}_{i+1}, \ldots, \hat{\alpha}_n, \hat{\eta}) \in \mathcal{A}^1 \times \mathcal{A}^1_n \) s.t.
\[
\begin{aligned}
M_{T}^{i,m,\nu,\eta} := m - \int_t^T \ell(s, Z_s^{i,z,\nu}, \nu_s) \, ds + \int_t^T \eta_s \, dW_s \geq f(Z_T^{i,z,\nu}) \\
\text{and} \\
P_{t,k}^{i,p_k,\alpha_k} := p_k + \int_t^{t_k} \alpha_k \, dW_s \geq \Psi(Z_{t_k}^{i,z,\nu}), i + 1 \leq k \leq n
\end{aligned}
\]
leading to the result.

\[\square\]

**Remark 3.2.** Fix \( 0 \leq i \leq n - 1 \), and let \((t, z) \in \[t_i, t_{i+1}\] \times \mathbb{R}^{d+1}. \) We observe from the proof of Proposition 3.1 above that we could consider the control \( \eta \in \mathcal{A}_n \) and for \( i + 1 \leq k \leq n \), \( \alpha_k \in \mathcal{A}_k \) s.t.
\[
\int_t^T \eta_s \, dW_s = f(Z_T^{i,z,\nu}) + \int_t^T \ell(s, Z_s^{i,z,\nu}, \nu_s) \, ds - \mathbb{E} \left[ f(Z_T^{i,z,\nu}) + \int_t^T \ell(s, Z_s^{i,z,\nu}, \nu_s) \, ds \right],
\]
and
\[
\int_t^T \alpha_k \, dW_s = \Psi(Z_{t_k}^{i,z,\nu}) - \mathbb{E} \left[ \Psi(Z_{t_k}^{i,z,\nu}) \right].
\]

Therefore appealing to the assumptions on the coefficients of \( X \) and \( Y \), as well as the growth conditions on \( f, \ell \) and \( \Psi \), the admissible set of controls \( \eta \) (resp. \( \alpha_k \), \( i + 1 \leq k \leq n \)) can be reduced to the subset of controls for which there exists \( C > 0 \) s.t.
\[
\mathbb{E} \left[ \int_t^T |\eta_s|^2 \, ds \right] \leq C(1 + |z|^2) \quad \text{resp. } \mathbb{E} \left[ \int_t^{t_k} |\alpha_k|^2 \, ds \right] \leq C(1 + |z|^2), \forall i + 1 \leq k \leq n.
\]

(3.2)
From now on we assume that $A^t_k$ (resp. $A^t_k$, $i + 1 \leq k \leq n$) integrates this constraint. This will be of great importance in the derivation of the time-boundary conditions for the auxiliary value function (see Proposition 3.4).

### 3.2 Compactification and level-set approach

We apply here the so-called level-set approach. In particular, we intend to link $V$ to the zero level-set of a suitable function (see Proposition 3.3). The idea underlying this approach is to represent a reachable set (here, the argument of the infimum in (3.1)) as the level set of a continuous function (here, the function $\hat{w}$ defined in (3.8) below) solution of a suitable PDE. This technique has been introduced by Osher and Sethian in [43] to describe the propagation of fronts and has then been exploited for instance in [28, 39, 41, 51, 9, 10] to solve different types of target problems.

For $0 \leq i \leq n - 1$, and $(t, z, p_{i+1}, \ldots, p_n, m) \in \mathcal{C}_i \times \mathbb{R}$, we define the following optimal control problem

$$w(t, z, p_{i+1}, \ldots, p_n, m) := \inf_{(\nu, \alpha, \eta) \in \mathcal{U}^t \times \mathcal{A}^i \times \mathcal{A}^n} J^{\nu, \alpha, \eta}(t, z, p_{i+1}, \ldots, p_n, m),$$

with

$$J^{\nu, \alpha, \eta}(t, z, p_{i+1}, \ldots, p_n, m) := \mathbb{E} \left[ \left( f(Z^t_{i+1, z, \nu}) - M^t_{i+1, z, m, \nu, \eta} \right)^+ + \sum_{k=1}^{i} \left( \Psi(Z^t_{i+1, z, \nu}) - F^t_{i+1, z, \nu, \eta} \right)^+ \right].$$

On $\{T\} \times \mathbb{R}^{d+1} \times \mathbb{R}$, we set $w(T, z, m) = (f(z) - m)^+$.

Arguing as in [11, Theorem 3.1], one can easily show that, if (3.3) admits an optimizer, (3.11) in Proposition 3.3 below holds for $\hat{w}$ replaced by $w$, i.e. $w$ can be used as the level-set function. However, (3.3) is a singular and weakly coercive optimal control problem being characterized by a discontinuous Hamiltonian. As a result, the HJB characterization must be obtained passing through a reformulation of the differential operator for a comparison result to hold (see e.g. [8, 11, 18]). Therefore, before detailing the level-set approach, we work towards defining a value function $\hat{w}$ that takes the same values as $w$ but with the property that the controls are valued in relatively compact sets. More precisely, we “compactify” the optimal control problem by means of a suitable time change. This compactification leads to the definition of an optimal stopping/control problem whose associated value function is given by $\hat{w}$ in (3.8) and which is proved to be equivalent to the problem defined in (3.3) (see Proposition 3.2). The HJB operator involved in the PDE characterization of $\hat{w}$ is therefore continuous, facilitating the arguments. Moreover, the proposed compactification allows us to obtain the reformulation of $V$ in Proposition 3.3 under the relaxed assumption of existence of
an optimal control for either $\tilde{w}$ or $w$ (see Assumption 3.2.1). We underline that, in this paper, the question of existence of optimal controls is not fully investigated and is left for further research. However, we refer the reader to Remark 3.5 for a discussion on this topic.

The compactification presented in this section is an interesting contribution of the paper as it extends some of the results in [42, 26, 25] to the case where the unbounded controls act on the diffusion term. This is a very common structure of the dynamics, see for instance [15, 14, 1, 4]. However, in [42, 26, 25], a weak formulation of the optimal control problem is considered (i.e. the probability space is not fixed, see [53, Definition 4.2, Chapter 2]) since the final interest of the paper is to provide an existence result. In the present work, it seems more natural to work under the strong formulation of the problem and we have therefore adapted the arguments in this regard.

We start with the following definition.

**Definition 3.1.** We fix $t \in [t_i, t_{i+1})$, $0 \leq i \leq n - 1$. We let $J$ be a given subset of \{i + 1, \ldots, n\} and $j := \text{Card}(J)$. An auxiliary control is a triple $(\Gamma, \beta, \nu)$ s.t. (i)-(viii) below hold.

(i) The process $(\Gamma_s)_{s \geq 0}$ is a continuous and (strictly) increasing time change s.t. $\lim_{s \to \infty} \Gamma_s = \infty$.

(ii) The vector $\beta := (\beta_1, \beta_5, \beta_7)^\top$ is valued in the set $S_{j+1,d+1} \setminus D_{(j+1)d+1}$.

(iii) The process $0 < \beta_1 \leq 1$ is of the form $\vartheta_{\Gamma}$, for some $\{\mathcal{F}_s\}$-predictable process $\vartheta$, and is therefore an $\{\mathcal{F}_{\Gamma_s}\}$-predictable process.

(iv) The vector $\beta^5$ is a $j\text{-dimensional $\{\mathcal{F}_{\Gamma_{s}}\}$-predictable process s.t. if $J \neq \emptyset$, $\beta^5 := (\beta^5_k)_{k \in J}$.

(v) The vector $\beta_7$ is a $d$-dimensional $\{\mathcal{F}_{\Gamma_{s}}\}$-predictable process.

(vi) If $J \neq \emptyset$, there exists $C > 0$ s.t.

$$E \left[ \int_0^{\Gamma_k} |\beta^5_k|^2 \, ds \right] \leq C, \forall k \in J,$$

with $\Gamma$ the dual of $\Gamma$, i.e. $\Gamma_s := \inf\{u \geq 0 : \Gamma_u > s\}$, $s \geq t$.

(vii) There exists $C > 0$ s.t.

$$E \left[ \int_0^{\Gamma_k} |\beta^7_k|^2 \, ds \right] \leq C,$$

with $\Gamma$ defined in (vi) above.
(viii) The control $\nu$ belongs to $\tilde{U}$, the set of $\{\mathcal{F}_\tau\}$-predictable processes valued in $U$.

Finally we define by $\tilde{A}_1^t$ the set of controls $\beta_1$ satisfying (iii) above, $\tilde{A}_2^tJ$ the set of controls $(\beta_1, \beta^\prime)$ valued in $S_{j_2+1} \setminus D_{j_2+1}$ and satisfying (iii)-(iv) and (vi) above, and $\tilde{A}_1^tJ$ the set of controls $\beta$ satisfying (ii)-(vii) above. In particular, we denote $\tilde{A}_1^tJ$ the set of controls satisfying (iii)-(vii) above and s.t. $\beta_1$ is allowed to be zero. We define similarly $\tilde{A}_1^t$ and $\tilde{A}_2^tJ$. To lighten the notation, we will omit the dependency on $J$ of the previous sets when $J = \{i + 1, ..., n\}$.

In the following remark we provide some properties that are directly derived from the definition of $\Gamma$ in the previous definition.

**Remark 3.3.** Fix $t \in [t_i, t_{i+1})$, $0 \leq i \leq n - 1$.

(i) The time change $(\Gamma_s)_{s \geq 0}$ is an $\{\mathcal{F}_s\}$-stopping time (see [6, Definition 1.1]).

(ii) The filtration $\{\mathcal{F}_\tau\}$ is increasing, right-continuous and complete on $(\Omega, \mathcal{F}, \mathbb{P})$ (see [47, Proposition 1.1, Chapter V]).

(iii) The dual $\hat{\Gamma}$ is a continuous and (strictly) increasing time change s.t. $\lim_{s \to \infty} \hat{\Gamma}_s = \infty$. Besides, $(\hat{\Gamma}_s)_{s \geq t}$ is an $\{\mathcal{F}_\tau\}$-stopping time (see [47, Proposition 1.1, Chapter V]).

(iv) For $t \leq s < \infty$, $\hat{\Gamma}_s = s$ while for $0 \leq s < \infty$, $\hat{\Gamma}_s = s$.

We then state the following proposition showing the equivalence between the optimal control problem (3.3) and the optimal stopping/control problem (3.8).

**Proposition 3.2.** Fix $(t, z, p_{i+1}, ..., p_n, m) \in \mathcal{C}_i \times \mathbb{R}$, $0 \leq i \leq n - 1$.

(i) For each $(\Gamma, \beta, \nu)$ satisfying Definition 3.1 s.t.

$$\Gamma_s = \Gamma_s^{0,t,\beta_1} := t + \int_0^s (\beta_1_r)^2 \, dr \geq t, \quad 0 \leq s < \infty,$$

there exists $(\bar{\nu}, \alpha, \eta) \in \bar{U} \times \hat{A}^t \times \hat{A}_n^t$ s.t.

$$J^\nu,\alpha,\eta(t, z, p_{i+1}, ..., p_n, m) = \bar{\gamma}^\nu,\beta(t, z, p_{i+1}, ..., p_n, m),$$

where

$$\bar{\gamma}^\nu,\beta(t, z, p_{i+1}, ..., p_n, m)$$

:= $\mathbb{E} \left[ \left( f(\tilde{\gamma}^0 z, \nu) - \tilde{M}^0 z, \nu, \beta^{\prime} \right)^+ + \sum_{k=1}^n \left( \Psi(\tilde{\gamma}^k z, \nu) - \tilde{M}^k z, \nu, \beta^{\prime} \right)^+ \right].$
Therefore where on the dependency on \((t, \beta_1)\) of \(\hat{Z}, \hat{M}, \text{and } \hat{\Gamma}\) with \(\hat{M}^{0,z,m,\nu,\beta}\) and \(\hat{Z}^{0,z,\nu} := (\hat{X}^{0,x,\nu}, \hat{Y}^{0,z,\nu})\) (resp. \(\hat{P}^{0,p_k,\beta_k^*}\), \(i + 1 \leq k \leq n\)) an \(\mathbb{R}\)-valued and \(\mathbb{R}^{d+1}\)-valued (resp. \(\mathbb{R}\)-valued) \(\{\mathcal{F}_\tau\}\)-adapted process being the unique solution on \([0, \hat{\Gamma}_T]\) (resp. \([0, \hat{\Gamma}_k]\)) to

\[
\hat{X}^{0,x,\nu} := x + \int_0^t (\beta_1)^2 \mu_x(\mathcal{H}_r, \hat{X}^{0,x,\nu}, \nu_r) \, dr + \int_0^t \beta_1 \sigma_x(\mathcal{H}_r, \hat{X}^{0,x,\nu}, \nu_r) \, d\hat{W}_r, \\
\hat{Y}^{0,z,\nu} := y + \int_0^t (\beta_1)^2 \mu_y(\mathcal{H}_r, \hat{Z}^{0,z,\nu}, \nu_r) \, dr + \int_0^t \beta_1 \sigma_y(\mathcal{H}_r, \hat{Z}^{0,z,\nu}, \nu_r) \, d\hat{W}_r, \\
\hat{M}^{0,z,m,\nu,\beta} := m - \int_0^t (\beta_1)^2 \ell(\mathcal{H}_r, \hat{Z}^{0,z,\nu}, \nu_r) \, dr + \int_0^t (\beta_1)^2 \, d\hat{W}_r, \tag{3.6}
\]

(resp. \(\hat{P}^{0,p_k,\beta_k^*} := p_k + \int_0^t (\beta_k)^2 \, d\hat{W}_r\), \(\tag{3.7}\)

) with \(\hat{W}_s := \int_t^s \frac{1}{\beta_{F_{s_u}}} \, dW_u\), a \(d\)-dimensional \(\{\mathcal{F}_s\}\)-Brownian motion.

(ii) For each \((\nu, \alpha, \eta) \in \mathcal{U} \times \hat{\mathcal{A}}^l \times \mathcal{A}_n^l\), there exists \((\gamma, \beta, \hat{z})\) satisfying Definition 3.1 and s.t. (3.4) and (3.5) hold.

Therefore

\[
w(t, z, p_{i+1}, \ldots, p_n, m) = \hat{w}(t, z, p_{i+1}, \ldots, p_n, m),
\]

where on \(\mathcal{C}_i \times \mathbb{R}\),

\[
\hat{w}(t, z, p_{i+1}, \ldots, p_n, m) := \inf_{(\nu, \beta) \in \mathcal{U} \times \hat{\mathcal{A}}^l} \hat{J}^{\nu, \beta}(t, z, p_{i+1}, \ldots, p_n, m), \tag{3.8}
\]

and on \(\{T\} \times \mathbb{R}^{d+1} \times \mathbb{R}\), we set \(\hat{w}(T, z, m) = (f(z) - m)^+\).

**Proof.** We only prove (i) and (ii) as the last assertion is their direct consequence. We fix \((t, z, p_{i+1}, \ldots, p_n, m) \in \mathcal{C}_i \times \mathbb{R}, 0 \leq i \leq n - 1, \) and start by proving (i). Consider \((\Gamma, \beta, \nu)\) satisfying Definition 3.1 and s.t. (3.4) holds. We first perform a time change in (3.4) and appeal to [47, Proposition 1.4, Chapter V] to obtain that for \(t \leq s < \infty\),

\[
\hat{\Gamma}_s := \int_t^s \frac{1}{(\beta_1)^2} \, dr. \tag{3.9}
\]
We then introduce the process \( N_s := \int_0^s \frac{1}{\beta_{\Gamma_r}} \, dW_s \), with \( \{W_s\} \) the \( d \)-dimensional \( \{\mathcal{F}_s\} \)-Brownian motion. The latter is an \( \{\mathcal{F}_s\} \)-continuous local martingale s.t.

\[
\forall s \geq t, \ \forall (i, j) \in \mathbb{N}^2, \ <N^i, N^j>_s = 0 \text{ when } i \neq j
\]

and \( <N^i, N^j>_s = \Gamma_s \text{ when } i = j \).

Hence by [37, Theorem 4.13, Chapter 3], \( \{\tilde{W}_s := N_{\Gamma_s}\} \) is a \( d \)-dimensional \( \{\mathcal{F}_s\} \)-Brownian motion.

Now, appealing to [47, Proposition 1.4, Chapter V] and (3.9), we obtain, after performing a time change, that for all \( s \in [0, \hat{\Gamma}_T] \),

\[
\int_0^s (\beta_{\Gamma_r})^2 \mu_X(\Gamma_r, \tilde{X}^{0,x,\nu}_{\Gamma_r}, \nu_{\Gamma_r}) \, d\Gamma_r = \int_t^{\Gamma_s} (\beta_{\Gamma_r})^2 \mu_X(r, \tilde{X}^{0,x,\nu}_{\Gamma_r}, \nu_{\Gamma_r}) \, d\Gamma_r = \int_t^{\Gamma_s} \mu_X(r, \tilde{X}^{0,x,\nu}_{\Gamma_r}, \nu_{\Gamma_r}) \, d\Gamma_r.
\]

On the other hand, observing that \( dN_s = \frac{1}{\beta_{\Gamma_s}} \, dW_s \) and appealing to [37, Proposition 4.8, Chapter 3], we obtain that for all \( s \in [0, \hat{\Gamma}_T] \),

\[
\int_0^s \beta_{\Gamma_r} \sigma_X(\Gamma_r, \tilde{X}^{0,x,\nu}_{\Gamma_r}, \nu_{\Gamma_r}) \, d\tilde{W}_r = \int_t^{\Gamma_s} \beta_{\Gamma_r} \sigma_X(r, \tilde{X}^{0,x,\nu}_{\Gamma_r}, \nu_{\Gamma_r}) \, dN_r = \int_t^{\Gamma_s} \sigma_X(r, \tilde{X}^{0,x,\nu}_{\Gamma_r}, \nu_{\Gamma_r}) \, dW_r.
\]

In particular, \( \tilde{\nu} := \nu_{\Gamma_k} \) is an \( \{\mathcal{F}_s\} \)-adapted predictable process (see [32, Theorem 3.52] and [37, Proposition 2.18, Chapter 1]) valued in \( U \). Therefore,

\[
\tilde{X}^{0,x,\nu}_{\Gamma_s} = \tilde{X}^{0,x,\nu}_{t,s} = x + \int_t^{\Gamma_s} \mu_X(r, \tilde{X}^{0,x,\nu}_{\Gamma_r}, \nu_{\Gamma_r}) \, d\Gamma_r + \int_t^{\Gamma_s} \sigma_X(r, \tilde{X}^{0,x,\nu}_{\Gamma_r}, \nu_{\Gamma_r}) \, dW_r.
\]

As a result, \( \tilde{X}^{0,x,\nu}_{\Gamma_s} = X^{0,x,\nu}_{\Gamma_s} \) on \( [0, \hat{\Gamma}_T] \) by the uniqueness of the solution to the preceding SDE and is therefore \( \{\mathcal{F}_s\} \)-adapted (see [37, Proposition 2.18, Chapter 1]) and valued in \( \mathbb{R}^d \). We then repeat the arguments for the process \( (\tilde{X}^{0,z,\nu}_{\Gamma_s}, (\tilde{\beta}_{k,\Gamma_s})_{i+1 \leq k \leq n}, \tilde{M}^{0,z,\nu}_{\Gamma_s}, \beta_{\Gamma_s}) \) and consider

- for \( s \in [t, k] \) and \( i + 1 \leq k \leq n \), \( \alpha_{k,s} := \frac{\beta_{k,\Gamma_s}^i}{\beta_{\Gamma_s}^i} \),

- for \( s \in [t, T] \), \( \eta_s := \frac{\beta_{T,\Gamma_s}^i}{\beta_{\Gamma_s}^i} \).

These processes are \( \{\mathcal{F}_s\} \)-predictable controls (see [32, Theorem 3.52] and [37, Proposition 2.18, Chapter 1]) each valued in \( \mathbb{R}^d \). We then prove that (3.2) holds for the obtained controls (recall Remark 3.2). We have, after appealing to [47, Proposition 1.4, Chapter V] and (3.4),

\[
\mathbb{E}\left[ \int_t^T |\eta_r|^2 \, d\Gamma_r \right] = \mathbb{E}\left[ \int_t^{\Gamma_s} |\eta_{\Gamma_r}|^2 \, d\Gamma_r \right] = \mathbb{E}\left[ \int_0^{\Gamma_s} |\beta_{\Gamma_r}^i|^2 \, d\Gamma_r \right] \leq C, \ C > 0,
\]
by Definition 3.1 (vii). We proceed similarly for \((\alpha_k)_{i+1 \leq k \leq n}\). Finally (3.5) is a direct consequence of what precedes.

We now prove (ii) and consider

\[
\hat{\Gamma}_s := \int_t^s (1 + |\zeta_r|^2) \, dr, \quad \forall t \leq s < \infty ,
\]

where \(\zeta_r := (\alpha_{i+1 \land t_i+1}, \ldots, \alpha_{n \land T}, \eta_{r \land T})\). We recall that, by definition

\[
\Gamma_s := \inf\{u \geq t : \hat{\Gamma}_u > s\}, \quad \forall 0 \leq s < \infty .
\]

Hence one can prove, after performing a time change in (3.10) and appealing to [47, Proposition 1.4, Chapter IV], that (3.4) holds for \(0 \leq s < \infty\) with \(\beta_1 := \frac{1}{\sqrt{1 + |\zeta_r|^2}}\). The proof then follows the same lines as those for (i) and considers for \(0 \leq s < \infty\),

\[
\hat{\nu}_s := \nu_{T_s}, \quad \beta^s := \left(\frac{\alpha_{i+1 \land t_i+1 \land r}}{\sqrt{1 + |\zeta_r|^2}}, \ldots, \frac{\alpha_{n \land r}}{\sqrt{1 + |\zeta_r|^2}}\right) \quad \text{and} \quad \beta^2 := \frac{\eta_{r \land T}}{\sqrt{1 + |\zeta_r|^2}}.
\]

The proof is then completed observing that for \(i + 1 \leq k \leq n\),

\[
E\left[\int_0^{t_{k-1}} |\beta^2_k|^2 \, dr\right] \leq E\left[\hat{\Gamma}_{t_{k-1}}\right] \leq (t_{k-1} - t) + E\left[\int_{t_{k-1}}^{t_k} |\zeta_r|^2 \, dr\right] \leq C,
\]

as well as

\[
E\left[\int_0^{T} |\beta^2_T|^2 \, dr\right] \leq E\left[\hat{\Gamma}_T\right] \leq (T - t) + E\left[\int_t^{T} |\zeta_r|^2 \, dr\right] \leq C,
\]

with \(C > 0\) depending on \(z\), by Remark 3.2.

\[\square\]

**Remark 3.4.** The proof of the previous proposition actually shows that we could work with controls \((\Gamma, \beta, \nu)\) satisfying Definition 3.1 (i), (iii)-(viii) and (3.4) and s.t. \(|\beta|^2 \leq 1\). This remark also holds for the functions \(\hat{\nu}_x_{T_s}, \hat{\nu}_{t_{k-1}}\) and \(\hat{\nu}_{t_{k-1}}\), defined in Section 4.2 and \(\hat{\nu}_1\) defined in Section 4.2, we could work with controls \((\Gamma, \beta_1, \nu)\) satisfying Definition 3.1 (i), (iii), (viii) and (3.4) and s.t. \(\beta_1 = 1\).

We introduce an assumption that is key for proving Proposition 3.3 below.

**Assumption 3.2.1.** On \(\mathcal{G}_i \times \mathbb{R}, \quad 0 \leq i \leq n - 1\), the problem defined by (3.3) (or resp. (3.8)) admits an optimal control in \(\mathcal{U} \times \tilde{\mathcal{A}}^i \times \mathcal{A}^i_{t_0}\) (or resp. in \(\tilde{\mathcal{U}} \times \tilde{\mathcal{A}}^i\)).

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Remark 3.5.

(i) Assumption 3.2.1 holds, for example, for (3.3) when the set $U$ is convex, the functions $f$ and $\Psi$ are convex in $z$, the function $\ell$ is convex in $(z,u)$, and when either the coefficients of the diffusion are of the form

$$
\mu_Z(t,z,u) \equiv A(t)z + B(t)u \text{ and } \sigma_Z(t,z,u) \equiv C(t)z + D(t)u ,
$$

with $A, B, C$ and $D$ matrices of suitable size, or $X$ is independent of $\nu$ and the coefficients of $Y$ are of the form

$$
\mu_Y(t,z,u) \equiv A(t,x)y + B(t,x)u \text{ and } \sigma_Y(t,z,u) \equiv C(t,x)y + D(t,x)u ,
$$

with $A, B, C$ and $D$ matrices of suitable size. This can be proved following the arguments in [53, Theorem 5.2, Chapter 2] (see also [11, Appendix A]).

(ii) In [42], a compactification of the same nature (but with processes involving unbounded controls in the drift term rather than the diffusion term) is the first step towards the proof of existence results. However, those results (as many others in the literature) are derived under a weak formulation of the optimal control problem. In the present work, if we were not fixing the filtered probability space, the reformulation in terms of a stochastic target problem would involve the martingales $M$ and $(P^k)_{i+1 \leq k \leq n}$. Their respective representation would be unknown as the filtration of interest might differ from the Brownian one (see [13]). It would be certainly interesting to push the research in this direction and investigate whether the existence results in [42] can be extended to our framework.

We can now state the main result of this section.

Proposition 3.3. On $\mathcal{C}_i$, 0 $\leq$ $i$ $\leq$ $n$ $-$ 1, and under Assumption 3.2.1, one has

$$
V(t,z,p_{i+1},...,p_n) = \inf \{m \geq 0 : \hat{w}(t,z,p_{i+1},...,p_n,m) = 0 \} . \quad (3.11)
$$

Remark 3.6.

(i) Observe that on $\mathcal{C}_i$, 0 $\leq$ $i$ $\leq$ $n$ $-$ 1, $\{m \geq 0 : \hat{w}(t,z,p_{i+1},...,p_n,m) = 0 \} = \emptyset$ if there exists $i + 1 \leq k \leq n$ s.t. $p_k < \gamma$ as, by the martingale property of $\hat{F}^{0,p_k} \land \hat{F}_k$, $\hat{P}^{0,p_k} \land \hat{F}_k < \gamma$ on $[0, \hat{\Gamma}_k]$ whenever $p_k < \gamma$.

(ii) The result in (3.11) extends to $\{T\} \times \mathbb{R}^{d+1}$ as $\inf \{m \geq 0 : \hat{w}(T,z,m) = 0 \} = f(z) = V(T,z)$. 

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Proof of Proposition 3.3. We argue as in the proof of [11, Theorem 3.1]. We fix $0 \leq i \leq n - 1$, and denote by $\bar{V}$ the right-hand side of (3.11).

We define for $t \in [t_i, t_{i+1})$, the backward reachable set of the augmented differential system

$$
\mathcal{R}_t := \left\{ (z, p_{i+1}, ..., p_n, m) \in \mathbb{R}^{d + (n-i+2)} : \exists (\nu, \alpha_{i+1}, ..., \alpha_n, \eta) \in \mathcal{U}_{t,z,p_{i+1},...,p_n} \times \mathcal{A}_n \text{ s.t. } M^z_i : z, m, \nu, \eta \geq f(Z_T^i) \right\},
$$

and notice from Proposition 3.1 that for $(t, z, p_{i+1}, ..., p_n) \in \mathcal{C}_i$,

$$V(t, z, p_{i+1}, ..., p_n) := \inf \{ m \geq 0 : (z, p_{i+1}, ..., p_n, m) \in \mathcal{R}_t \}. \quad (3.12)$$

**Step 1. Proof of $V \geq \bar{V}$.** For any $(z, p_{i+1}, ..., p_n, m) \in \mathcal{R}_t$, there exists $(\nu, \hat{\alpha}_{i+1}, ..., \hat{\alpha}_n, \hat{\eta}) \in \mathcal{U} \times \hat{\mathcal{A}} \times \mathcal{A}_n$ s.t.

$$
\left( f(Z_T^i, \nu) - M_T^i : z, m, \nu, \hat{\eta} \right)_+ = 0, \quad (3.13)
$$

$$
\Psi(Z_T^i, \nu) - P_{t_k}^i : p_k, \hat{\alpha}_k = 0, \quad \text{for all } i + 1 \leq k \leq n, \quad (3.14)
$$

leading to $w(t, z, p_{i+1}, ..., p_n, m) = 0$ by the non-negativity of each of these terms and therefore to $\tilde{w}(t, z, p_{i+1}, ..., p_n, m) = 0$ by Proposition 3.2. Therefore $V \geq \bar{V}$.

**Step 2. Proof of $V \leq \bar{V}$.** We assume that for $(t, z, p_{i+1}, ..., p_n, m) \in \mathcal{C}_i \times \mathbb{R}$, we have $\tilde{w}(t, z, p_{i+1}, ..., p_n, m) = 0$. We appeal to Assumption 3.2.1 and consider the case where the problem defined in (3.8) admits an optimal control in $\tilde{U} \times \tilde{\mathcal{A}}$ (the case where (3.3) admits an optimal control being treated similarly up to obvious modifications). We know from Proposition 3.2 that there exists an optimal control $(\tilde{\nu}, \hat{\alpha}_{i+1}, ..., \hat{\alpha}_n, \hat{\eta}) \in \tilde{U} \times \hat{\mathcal{A}} \times \mathcal{A}_n$ s.t. $w(t, z, p_{i+1}, ..., p_n, m) = 0$ and therefore s.t. (3.13)-(3.14) hold by the non-negativity of each term. Hence $(z, p_{i+1}, ..., p_n, m) \in \mathcal{R}_t$. We therefore conclude from (3.12) that $\bar{V} \geq V$.

Proposition 3.3 is critical here as it allows the reformulation of $V$ in terms of the optimal control problem described by $\tilde{w}$. In particular, problem (3.8) satisfies the following key properties: it is an unconstrained optimal stopping/control problem, all controls take values in relatively compact sets, the value function satisfies important regularity properties (see Proposition 3.4 below).

As a consequence, it will be possible in Section 4 to obtain a complete PDE characterization for $\tilde{w}$. Observe that the cost functional associated with $\tilde{w}$ changes on each time interval $[t_i, t_{i+1})$, $0 \leq i \leq n - 1$, and since it has to be adapted to the decreasing number of constraints involved. As a result, a discontinuity at each point $t_i$, $1 \leq i \leq n$, arises
(see Proposition 3.4 below) and, unlike [11], the HJB characterization of \( \tilde{w} \) requires the use of an induction argument over \( i \) for \( i = n, \ldots, 0 \).

We end this section with a result on the growth and regularity properties of the auxiliary value function \( \tilde{w} \) which is important for the comparison result and characterization respectively derived in Section 4.3 and Section 4.4.

**Proposition 3.4.** On \([t_i, t_{i+1}) \times \mathbb{R}^{d+1} \times [\gamma, \infty)^{n-i} \times \mathbb{R}^+\), \( 0 \leq i \leq n-1 \), there exists \( C > 0 \) s.t.

\[
|\tilde{w}(t, z, p_{i+1}, \ldots, p_n, m)| \leq C(1 + |z|). \tag{3.15}
\]

Moreover, on \( \mathcal{G}_i \times \mathbb{R} \), \( 0 \leq i \leq n-1 \), \( \tilde{w} \) is continuous w.r.t. the variable \( t \) (locally uniformly in \( z \)) and Lipschitz continuous w.r.t. \( (z, p_{i+1}, \ldots, p_n, m) \) and satisfies

\[
\lim_{t \to t_{i+1}} \tilde{w}(t, z, p_{i+1}, \ldots, p_n, m) = \tilde{w}(t_{i+1}, z, p_{i+2}, \ldots, p_n, m) + (\Psi(z) - p_{i+1})^+. \tag{3.16}
\]

**Proof.** We prove the proposition for \( w \) and then recall Proposition 3.2 to conclude that the results hold for \( \tilde{w} \). We start proving the first assertion. Fix \((t, z, p_{i+1}, \ldots, p_n, m) \in [t_i, t_{i+1}) \times \mathbb{R}^{d+1} \times [\gamma, \infty)^{n-i} \times \mathbb{R}^+, 0 \leq i \leq n-1 \). Using the definition of \( w \) we write

\[
w(t, z, p_{i+1}, \ldots, p_n, m) \leq \inf_{(\alpha_{i+1}, \ldots, \alpha_n, \eta) \in \{0, \ldots, 0\}} \mathbb{E} \left[ \left( f(Z^t, z, \nu_s) \right) ds - m^+ + \sum_{k=i+1}^n \left( \Psi(Z^t) - p_k^+ \right) \right],
\]

and therefore

\[
w(t, z, p_{i+1}, \ldots, p_n, m) \leq \inf_{\nu \in \mathcal{P}} \mathbb{E} \left[ \int_t^{t_{i+1}} |\ell(s, Z^t, \nu_s)| ds + |f(Z^t, \nu)| + \sum_{k=i+1}^n \left( |\Psi(Z^t)| + |\gamma| \right) \right],
\]

where we used the non-increasing property of \( w \) in \( m \in \mathbb{R}^+ \) and \( p_k \in [\gamma, +\infty) \), \( i + 1 \leq k \leq n \). Hence it follows from the growth conditions on \( f, \ell \) and \( \Psi \) and the assumptions on the coefficients of \( X \) and \( Y \), that (3.15) holds for \( w \). On \( \mathcal{G}_i \times \mathbb{R} \), \( 0 \leq i \leq n-1 \), the Lipschitz continuity of \( w \) w.r.t. \((z, p_{i+1}, \ldots, p_n, m)\) is straightforward. We now prove (3.16) for \( w \). We first notice that for \( h > 0 \),

\[
\inf_{\alpha \in \mathcal{A}_{i+1}^{t_i}} \mathcal{J}^{i, \alpha, \eta}(t_i+1, z, p_{i+2}, \ldots, p_n, m) = \inf_{\alpha \in \mathcal{A}_{i+1}^{t_i-h}} \mathcal{J}^{i, \alpha, \eta}(t_i+1, z, p_{i+2}, \ldots, p_n, m),
\]

to obtain

\[
|w(t_{i+1} - h, z, p_{i+1}, \ldots, p_n, m) - w(t_{i+1}, z, p_{i+2}, \ldots, p_n, m) - (\Psi(z) - p_{i+1})^+| 
\leq \sup_{\alpha \in \mathcal{A}_{i+1}^{t_i-h}} \mathbb{E} \left[ \left| f(Z^t_{i+1-h}, z, \nu_s) - M^t_{i+1-h, z, m, \nu_s} \right|^+ \right]
\]

\[
+ \sup_{\alpha \in \mathcal{A}_{i+1}^{t_i-h}} \mathbb{E} \left[ \left| \Psi(Z^t_{i+1-h}) - P^t_{i+1-h} p_{i+2, \alpha} \right|^+ \right].
\]
Thanks to the Lipschitz continuity of \( f \) and \( \Psi \), we have
\[
|w(t_{i+1} - h, z, p_{i+1}, \ldots, p_n, m) - w(t_{i+1}, z, p_{i+2}, \ldots, p_n, m) - (\Psi(z) - p_{i+1})^+| \\
\leq \sup_{\nu \in \mathcal{A}^n_{i+1}, h} 2\mathbb{E} \left[ \sum_{k=i+1}^{n} \left( |Z^{k}_{t_{k}^{i+1}-h,z,\nu} - Z^{k}_{t_{k}^{i+1},z,\nu}| + |P^{i+1}_{t_{k}^{i+1}-h,p_{i+1},\alpha_k} - P^{i+1}_{t_{k}^{i+1},p_{i+1},\alpha_k}| \right) \right].
\]
(3.17)
Moreover, standard arguments involving the assumptions on the coefficients of \( Z \) give
\[
\mathbb{E} \left[ \sup_{\nu \in \mathcal{A}^n_{i+1}, h} |Z^{i+1}_{s} - Z^{i+1}_{s',\nu}|^2 \right] \leq C h(1 + |z|^2), \quad C > 0. \quad (3.18)
\]
Similarly, using the definition of \( M \) and \( P \) as well as the Lipschitz property of \( \ell \), one has with (3.18), Itô's isometry and Fubini's theorem
\[
\sup_{\nu \in \mathcal{A}^n_{i+1}, h} \mathbb{E} \left[ |M^{i+1}_{t_{k}^{i+1}-h,z,\mu,\nu} - M^{i+1}_{t_{k}^{i+1},z,\mu,\nu}|^2 \right] + \sum_{k=i+1}^{n} \mathbb{E} \left[ |P^{i+1}_{t_{k}^{i+1}-h,p_{i+1},\alpha_k} - P^{i+1}_{t_{k}^{i+1},p_{i+1},\alpha_k}|^2 \right] \\
\leq C h(1 + |z|^2) + \sup_{\nu \in \mathcal{A}^n_{i+1}, h} \mathbb{E} \left[ \int_{t_{i+1}^{i+1}}^{t_{i+1}} \eta^\top_s dW_s^2 \right] + \sum_{k=i+1}^{n} \sup_{\nu \in \mathcal{A}^n_{i+1}, h} \mathbb{E} \left[ \int_{t_{i+1}^{i+1}}^{t_{i+1}} \eta^\top_s dW_s^2 \right] \\
\leq C h(1 + |z|^2) + \sup_{\nu \in \mathcal{A}^n_{i+1}, h} \int_{t_{i+1}^{i+1}}^{t_{i+1}} \mathbb{E} [|\eta_s|^2] ds + \sum_{k=i+1}^{n} \sup_{\nu \in \mathcal{A}^n_{i+1}, h} \int_{t_{i+1}^{i+1}}^{t_{i+1}} \mathbb{E} [|\alpha_s|^2] ds.
\]
(3.19)
In virtue of the uniform bound in the \( L^2 \)-norm of the controls \( \eta \) and \( \alpha_k \), \( i+1 \leq k \leq n \) (see Remark 3.2), the right-hand side of (3.19) converges to zero as \( h \downarrow 0 \). The right-hand side of (3.18) also converges to zero as \( h \downarrow 0 \). Therefore, appealing to (3.17) we obtain that \( \lim_{h \downarrow 0} |w(t_{i+1} - h, z, p_{i+1}, \ldots, p_n, m) - w(t_{i+1}, z, p_{i+2}, \ldots, p_n, m) - (\Psi(z) - p_{i+1})^+| \to 0. \) Finally, on \( \mathcal{G}_t \times \mathbb{R}, 0 \leq i \leq n - 1 \), the continuity of \( w \) w.r.t. \( t \) is obtained through a straightforward adaptation of the preceding argument.

The rest of this paper is dedicated to characterizing \( \tilde{w} \) as the solution of a suitable HJB equation (Section 4) and discussing some possible extensions (Section 5).

4 A complete PDE characterization for \( \tilde{w} \)

To alleviate notations in this manuscript we introduce for \( 0 \leq i \leq n - 1 \) the sets:
\[
\mathcal{B}_i := [t_i, t_{i+1}) \times \mathbb{R}^{d-1} \times \mathcal{G}^{n-i}, \quad \mathcal{D}_i := \mathcal{B}_i \times \mathcal{G}^+ \quad \text{We also define } \text{int}(\mathcal{B}_i) := [t_i, t_{i+1}) \times \mathbb{R}^{d-1} \times (\mathcal{G} \times (0, \infty)).
\]
4.1 On the interior of the domain

The main ingredient towards the PDE characterization of \( \hat{w} \) is the following result (see Appendix A for the proof).

**Theorem 4.1** (Dynamic Programming Principle). Fix \( (t, z, p_{i+1}, \ldots, p_n, m) \in \mathcal{C}_i \times \mathbb{R}, 0 \leq i \leq n - 1 \), and let \( 0 \leq \theta \leq \hat{\Gamma}_{t_{i+1}} \) be a stopping time. Then

\[
\hat{w}(t, z, p_{i+1}, \ldots, p_n, m) = \inf_{(\nu, \beta) \in \mathcal{U} \times \tilde{A}} \mathbb{E} \left[ \Phi \left( \Gamma_{\theta, \tilde{Z}^{0,z,\nu}_{\theta}, \tilde{P}^{0,p_{i+1},\nu}_{\theta}, \ldots, \tilde{P}^{0,p_{n},\nu}_{\theta}, \tilde{M}^{0,z,m,\nu,\beta}_{\theta} \right) \right],
\]

(4.1)

where for \( (t, z, p_{i+1}, \ldots, p_n, m) \in \text{cl}(\mathcal{C}_i) \times \mathbb{R}, 0 \leq i \leq n - 1 \),

\[
\Phi(t, z, p_{i+1}, \ldots, p_n, m) := \begin{cases} 
\hat{w}(t, z, p_{i+1}, \ldots, p_n, m), & t < t_{i+1} \\
\hat{w}(t_{i+1}, z, p_{i+2}, \ldots, p_n, m) + (\Psi(z) - p_{i+1})^+, & t = t_{i+1} \text{ and } i < n - 1 \\
(f(z) - m)^+ + (\Psi(z) - p_n)^+, & t = t_{i+1} \text{ and } i = n - 1 
\end{cases}
\]

For \( \Theta := (t, z, p_{i+1}, \ldots, p_n, m, q, A) \in \mathcal{D}_i \times \mathbb{R}^{d+(n-i-2)} \times \mathbb{S}^{d+(n-i-2)}, 0 \leq i \leq n - 1 \), with

\[
q := (q^z, q^{p_{i+1}}, \ldots, q^{p_{n}}, q^m)^\top \text{ and } A := \begin{pmatrix} A^{zz} & A^{zp} & A^{zm} \\ A^{zp\top} & A^{pp} & A^{pm} \\ A^{zm\top} & A^{pm\top} & A^{mm} \end{pmatrix},
\]

(4.2)

and for \( c \in \mathbb{R}, u \in U \) and \( b \in S_{(n-i+1)d+1}, \) we define the following operator

\[
H^{u,b}_{(\Theta, c)} := \begin{cases} 
(b_1)^2 \left( -c + L^{u,\tilde{b},\tilde{b}}_{\theta} (\Theta) \right), & b \in S_{(n-i+1)d+1} \setminus \mathcal{D}_{(n-i+1)d+1} \\
-\frac{1}{2} \sum_{k=i+1}^n |b_k|^2 A p_k - \frac{1}{2} |b|^2 A_{mm} \\
- \sum_{k=i+1}^n b_k^T \tilde{b}_k A p_k, & b \in \mathcal{D}_{(n-i+1)d+1}
\end{cases},
\]

where \( \tilde{b} := \frac{\tilde{b}}{b_1} \in \mathbb{R}^{(n-i)d} \) and similarly \( \tilde{b}^+ := \frac{\tilde{b}^+}{b_1^+} \in \mathbb{R}^d, \) and

\[
L^{u,\tilde{b},\tilde{b}}_{\theta} (\Theta) := \begin{cases} 
-\mu z (t, z, u) q^z + \ell(t, z, u) q^m \\
-\frac{1}{2} \text{Tr} [\sigma_{z} \sigma_{z\top} z(t, z, u) A_{zz}] - \frac{1}{2} \sum_{k=i+1}^n |\tilde{b}_k|^2 A p_k - \frac{1}{2} |\tilde{b}|^2 A_{mm} \\
- \sum_{k=i+1}^n \tilde{b}_k^T \sigma^T z(t, z, u) A_{zz} - \tilde{b}^T \sigma^T z(t, z, u) A_{zm} - \sum_{k=i+1}^n \tilde{b}_k^T \tilde{b}_k A p_k
\end{cases}.
\]

**Remark 4.1.** The operator \( b \mapsto H^{u,b}_{(\Theta, c)} \) is continuous on \( S_{(n-i+1)d+1}, \) in particular,

\[
\sup_{(u,b) \in U \times S_{(n-i+1)d+1}} H^{u,b}_{(\Theta, c)} = \sup_{(u,b) \in U \times S_{(n-i+1)d+1} \setminus \mathcal{D}_{(n-i+1)d+1}} H^{u,b}_{(\Theta, c)}.
\]

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For the reader’s convenience, we will write for any test function $\varphi$ defined on $\mathcal{D}$, $0 \leq i \leq n-1$,

$$H^{u,b}\varphi(t,z,p_{i+1},...,p_n,m) \text{ for } H^{u,b}(\cdot,D\varphi(\cdot),D^2\varphi(\cdot),\partial_t\varphi(\cdot))(t,z,p_{i+1},...,p_n,m).$$

A similar writing holds for the operator $L^{p,\tilde{\mu},\tilde{\beta}}$ and the operators $H^{u,b}_{\kappa,\lambda}$ and $H^{u,b}_{\kappa,\lambda}$ respectively defined in (4.6) and Section 5.1.2.

**Theorem 4.2.** On $\text{int}(\mathcal{D})$, $0 \leq i \leq n-1$, $\tilde{w}$ is a viscosity solution of

$$\sup_{(u,b)\in U \times \mathcal{S}(n-i+1)d+1} H^{u,b}\varphi = 0.$$

**Proof.** The result is a consequence of Theorem 4.1. Fix $0 \leq i \leq n-1$.

**Step 1. Proof of the sub-solution property.** The proof is an adaptation of the arguments in [45, Proposition 4.3]. We consider $\varphi$ a smooth function s.t.

$$\max(\tilde{w} - \varphi) = (\text{strict}) \max_{\text{int}(\mathcal{D})} (\tilde{w} - \varphi) = (\tilde{w} - \varphi)(\tilde{t},\tilde{z},\tilde{p}_{i+1},...,\tilde{p}_n,\tilde{m}) = 0. \quad (4.3)$$

We now choose $(\tilde{u},\tilde{b}) \in U \times \mathcal{S}(n-i+1)d+1 \setminus \mathcal{D}(n-i+1)d+1$ and $(\hat{\nu},\hat{\beta}) \in \tilde{U} \times \tilde{A}$ the control identically equal to $(\tilde{u},\tilde{b})$. We let

$$(\Gamma^{0,\tilde{t},\tilde{b}_1}_s,\tilde{Z}^{0,\tilde{z},\tilde{\nu}}_s,\tilde{P}^{0,\tilde{p}_{i+1},\tilde{\beta}_1}_s,\ldots,\tilde{P}^{0,\tilde{p}_n,\tilde{\beta}_n}_s,\tilde{M}^{0,\tilde{z},\tilde{\mu},\tilde{\nu},\tilde{\beta}}_s),$$

be the associated controlled process. We define the set $\mathcal{O} := B_{\varepsilon}(\tilde{t},\tilde{z},\tilde{p}_{i+1},...,\tilde{p}_n,\tilde{m}) \subset \text{int}(\mathcal{D})$, for $\varepsilon > 0$, and the stopping time

$$\theta := \inf \left\{ s \geq 0 : (\Gamma^{0,\tilde{t},\tilde{b}_1}_s,\tilde{Z}^{0,\tilde{z},\tilde{\nu}}_s,\tilde{P}^{0,\tilde{p}_{i+1},\tilde{\beta}_1}_s,\ldots,\tilde{P}^{0,\tilde{p}_n,\tilde{\beta}_n}_s,\tilde{M}^{0,\tilde{z},\tilde{\mu},\tilde{\nu},\tilde{\beta}}_s) \notin \mathcal{O} \right\},$$

(observe that $0 < \theta < \infty$ since $\Gamma_s$ is (strictly) increasing). Define $\tau := \theta \land r$, $r > 0$.

Theorem 4.1 and the inequality $\varphi \geq \tilde{w}$ (recall (4.3)) give for $\varepsilon$ small enough

$$\varphi(\tilde{t},\tilde{z},\tilde{p}_{i+1},...,\tilde{p}_n,\tilde{m}) \leq \mathbb{E}\left[ \varphi \left( \Gamma^{0,\tilde{t},\tilde{b}_1}_\tau,\tilde{Z}^{0,\tilde{z},\tilde{\nu}}_\tau,\tilde{P}^{0,\tilde{p}_{i+1},\tilde{\beta}_1}_\tau,\ldots,\tilde{P}^{0,\tilde{p}_n,\tilde{\beta}_n}_\tau,\tilde{M}^{0,\tilde{z},\tilde{\mu},\tilde{\nu},\tilde{\beta}}_\tau \right) \right],$$

leading with Itô’s lemma to

$$\frac{1}{r} \mathbb{E}\left[ \int_0^\tau (\hat{\beta}_1)_s^2 \left( \left(-\partial_t\varphi(\cdot) + L^{p,\tilde{\mu},\tilde{\beta}}\varphi(\cdot) \right) \right) ds \right] \leq 0.$$

As a result, the proof is completed sending $r$ to zero and appealing to the dominated convergence theorem and the mean-value theorem as well as to the arbitrariness of
\((\hat{u}, \hat{b}) \in U \times \mathcal{S}_{(n-i+1)d+1} \setminus \mathcal{D}_{(n-i+1)d+1}\) and Remark 4.1.

**Step 2. Proof of the super-solution property.** The proof adapts the arguments in [14, Section 6.2]. Let \(\varphi\) be a smooth function s.t.

\[
\min_{\mathcal{G}_i}(\hat{w} - \varphi) = (\text{strict}) \min_{\text{int}(\mathcal{G}_i)} (\hat{w} - \varphi) = (\hat{w} - \varphi)(\hat{t}, \hat{z}, \hat{p}_i, \ldots, \hat{p}_n, \hat{m}) = 0. \tag{4.4}
\]

We assume to the contrary that

\[
-2\rho := \sup_{(u, b) \in U \times \mathcal{S}_{(n-i+1)d+1}} H^{u, b}_{\varphi}(\hat{t}, \hat{z}, \hat{p}_i, \ldots, \hat{p}_n, \hat{m}) < 0,
\]

and work towards a contradiction. By continuity of \(H^{u, b}\), we can find \(\varepsilon > 0\) s.t. for all \((u, b) \in U \times \mathcal{S}_{(n-i+1)d+1} \setminus \mathcal{D}_{(n-i+1)d+1}\),

\[
(b_1)^2 \left( -\partial_t \varphi(t, z, p_{i+1}, \ldots, p_n, m) + L^{u, b, \beta'} \varphi(t, z, p_{i+1}, \ldots, p_n, m) \right) \leq -\rho \quad \text{on} \quad \mathcal{O} \subset \text{int}(\mathcal{G}_i),
\]

with \(\mathcal{O} := B(\hat{t}, \hat{z}, \hat{p}_i, \ldots, \hat{p}_n, \hat{m})\). We arbitrarily consider \((\nu, \beta) \in \hat{U} \times \hat{\mathcal{A}}^1\), and denote

\[
\theta := \inf \left\{ s > 0 : \left( r_{\nu}^{0, \hat{t}, \beta_1}, z_{\nu}^{0, \hat{z}, \nu}, p_{\nu}^{0, \hat{p}_{i+1}, \beta_{i+1}} \ldots, p_{\nu}^{0, \hat{p}_n, \beta_n}, n_{\nu}^{0, \hat{m}, \nu, \beta} \right) \notin \mathcal{O} \right\},
\]

(again notice that \(0 < \theta < \infty\)). Using Itô’s Lemma and (4.5) we get

\[
\varphi(\hat{t}, \hat{z}, \hat{p}_{i+1}, \ldots, \hat{p}_n, \hat{m}) \leq \mathbb{E} \left[ \varphi \left( r_{\theta}^{0, \hat{t}, \beta_1}, z_{\theta}^{0, \hat{z}, \nu}, p_{\theta}^{0, \hat{p}_{i+1}, \beta_{i+1}} \ldots, p_{\theta}^{0, \hat{p}_n, \beta_n}, n_{\theta}^{0, \hat{m}, \nu, \beta} \right) \right],
\]

\[
\leq \mathbb{E} \left[ \hat{w} \left( r_{\theta}^{0, \hat{t}, \beta_1}, z_{\theta}^{0, \hat{z}, \nu}, p_{\theta}^{0, \hat{p}_{i+1}, \beta_{i+1}} \ldots, p_{\theta}^{0, \hat{p}_n, \beta_n}, n_{\theta}^{0, \hat{m}, \nu, \beta} \right) \right] - \xi,
\]

where

\[
\xi := \min_{\partial_{\nu} \mathcal{O}} (\hat{w} - \varphi) > 0,
\]

with \(\partial_{\nu} \mathcal{O}\) the parabolic boundary of \(\mathcal{O}\). Recalling (4.4), this contradicts Theorem 4.1. \(\square\)

We now consider two functions \(\kappa : l \in \mathbb{R} \mapsto \mathbb{R}^+ \setminus \{0\}\) as well as \(\lambda : l \in \mathbb{R} \mapsto \mathbb{R}^+ \setminus \{0\}\) and define for \(\Theta := (t, z, p_{i+1}, \ldots, p_n, m, q, A) \in \mathcal{G}_i \times \mathbb{R}^{d+(n-i+2)} \times \mathbb{S}^{d+(n-i+2)}, 0 \leq i \leq n-1\) (see (4.2)), \(c \in \mathbb{R}, u \in U\) and \(b \in \mathcal{S}_{(n-i+1)d+1}\),

\[
H^{u, b}_{\kappa, \lambda}(\Theta, c) := \begin{cases} (b_1)^2 \left( -c + L_{n, \lambda}^{u, b, \beta'} (\Theta) \right), & b \in \mathcal{S}_{(n-i+1)d+1} \setminus \mathcal{D}_{(n-i+1)d+1} \\
-\frac{1}{2} \sum_{k=1}^{n} \kappa(p_k)^2 |b_k|^2 A_{p_k} p_{k} - \frac{1}{2} \lambda(m)^2 |\beta'|^2 A^{mm} \\
- \sum_{k=1}^{n} \kappa(p_k) \lambda(m) b^T b^T A_{p_k} p_{k}, & b \in \mathcal{D}_{(n-i+1)d+1} 
\end{cases},
\]

(4.6)
where we recall that $(\tilde{\theta}, \tilde{\nu}) := (\tilde{\theta}_1, \tilde{\nu}_1) \in \mathbb{R}^{(n-i+1)d}$, and

\[
L_{\kappa, \lambda}^{u, \tilde{\theta}, \tilde{\nu}} (\Theta) := \begin{cases} 
-\frac{1}{2} \sum_{k=1}^{n} \kappa(p_k) \frac{1}{2} A^{p_k} - \frac{1}{2} \sum_{k=1}^{n} \kappa(p_k) b_k^T A^{p_k} \frac{1}{2} \sigma^2_{\tilde{Z}}(t, z, u) A^{p_k} \\
-\lambda(m) \sum_{k=1}^{n} \kappa(p_k) \frac{1}{2} A^{p_k} \frac{1}{2} \sigma^2_{\tilde{Z}}(t, z, u) A^{p_k} \\
-\frac{1}{2} \sum_{k=1}^{n} \kappa(p_k) \frac{1}{2} A^{p_k} - \frac{1}{2} \lambda(m) \frac{1}{2} A^{p_k} \frac{1}{2} \sigma^2_{\tilde{Z}}(t, z, u) A^{p_k} 
\end{cases}.
\]

We can state the following corollary which is in the spirit of [11, Remark 4.4] and of important use in Lemma 4.1. The proof is provided in Appendix B.

**Corollary 4.1.** On \(\text{int}(\mathcal{P}_i), 0 \leq i \leq n - 1\), \(\tilde{w}\) is a viscosity solution of

\[
\sup_{(u, b) \in U \times S_{(n-i+1)d+1}} H_{\kappa, \lambda}^{u, b} \varphi = 0.
\]

We conclude proving a convexity property of \(\tilde{w}\) on \(\text{int}(\mathcal{P}_i)\).

**Proposition 4.1.** On \(\text{int}(\mathcal{P}_i), 0 \leq i \leq n - 1\), the function \(\tilde{w}\) is convex in \((p_{i+1}, ..., p_n, m)\).

**Proof.** Let \(\varphi\) be a smooth function s.t.

\[
\max_{\mathcal{P}_i} (\tilde{w} - \varphi) = (\text{strict}) \max_{\text{int}(\mathcal{P}_i)} (\tilde{w} - \varphi) = (\tilde{w} - \varphi)(\tilde{t}, \tilde{z}, \tilde{p}_{i+1}, ..., \tilde{p}_n, \tilde{m}) = 0.
\]

We first observe that the sub-solution property implies that the Hessian matrix of \(\varphi\) in \((p_{i+1}, ..., p_n, m)\) is the matrix of a quadratic form that is positive semi-definite for all \((p_{i+1}, ..., p_n, m) \in (\gamma, \infty)^{n-i} \times (0, \infty)\). The convexity of \(\tilde{w}\) then follows from the same arguments as in [23, Proposition 5.2]. \[\square\]

### 4.2 On the space boundaries

We study here the boundary conditions in \(m\) and \(p_k, 1 \leq k \leq n\). We first divide \(\mathcal{C}_i, 0 \leq i \leq n - 1\), into different regions corresponding to the different boundaries associated with the level of controlled loss. More precisely, given \(0 \leq i \leq n - 1\), we define \(\mathcal{P}_i := \{ I : I \subseteq \{i + 1, ..., n\}, I \neq \emptyset \}\), and \(C_{i, I} := \{(p_{i+1}, ..., p_n) \in \mathbb{R}^{n-i} : p_k \leq \gamma \text{ for } k \in I \text{ and } p_k > \gamma \text{ for } k \notin I \}\), as well as \(\mathcal{C}_{i, I} := \{t, t_{i+1}) \times \mathbb{R}^{d+1} \times C_{i, I}\). In particular, \(\mathcal{C}_i = \cup_{I \in \mathcal{P}_i} \mathcal{C}_{i, I} \cup [t, t_{i+1}) \times \mathbb{R}^{d+1} \times (\gamma, \infty)^{n-i}\). We define \(J_I := \{I \neq I\}\) and recall Definition 3.1.

The results appeal to the following functions. We define

- On \(\mathcal{C}_i, 0 \leq i \leq n - 1\),

\[
\tilde{w}_2(t, z, p_{i+1}, ..., p_n) := \inf_{(\nu, \beta) \in \mathcal{T} \times \mathbb{R}^{d+1}} \mathbb{E} \left[ f(\hat{Z}^{0, z, \nu}_{\Gamma_T}) - \hat{M}^{0, z, \nu, 0}_{\Gamma_T} + \sum_{k=i+1}^n \left( \Psi(\hat{Z}^{0, z, \nu}_{\Gamma_k}) - \hat{P}^{0, p_k, \beta_k}_{\Gamma_k, \nu} \right)^+ \right],
\]

where on \(\{T\} \times \mathbb{R}^{d+1}\), we set \(\tilde{w}_2(T, z) = f(z)\).
• On $[t_i, t_{i+1}) \times \mathbb{R}^{d+1} \times (\gamma, \infty)^{\text{Card}(J_t)} \times \mathbb{R}, \ 0 \leq i \leq n - 1$,
\[
\tilde{w}_{J_t}(t, z, (p_t)_{t \in J_t}, m) := \inf_{\mu \in \hat{\mathcal{U}}^{\gamma}_{J_t}} \mathbb{E} \left[ \left( f\left( \tilde{Z}_{t}^{0, z, \nu} - \tilde{M}_{J_t}^{0, z, m, \nu, \beta^T} \right)^+ + \sum_{t \in J_t} \left( \Psi(\tilde{Z}_{t}^{0, z, \nu}) - \tilde{F}_{t}^{0, m, \beta^T} \right)^+ + \sum_{\ell \notin J_t} \Psi\left( \tilde{Z}_{J_t}^{0, z, \nu} \right) \right] \right],
\]
where on $\{T\} \times \mathbb{R}^{d+1} \times \mathbb{R}$, we set $\tilde{w}_{J_t}(T, z, m) = (f(z) - m)^+.$

• On $[t_i, t_{i+1}) \times \mathbb{R}^{d+1} \times (\gamma, \infty)^{\text{Card}(J_t)}, \ 0 \leq i \leq n - 1$,
\[
\tilde{w}_{2,J_t}(t, z, (p_t)_{t \in J_t}) := \inf_{\nu \in \hat{\mathcal{U}}^{\gamma}_{J_t}} \mathbb{E} \left[ f\left( \tilde{Z}_{t}^{0, z, \nu} - \tilde{M}_{J_t}^{0, z, 0, \nu, 0} \right)^+ + \sum_{t \in J_t} \left( \Psi(\tilde{Z}_{t}^{0, z, \nu}) - \tilde{F}_{t}^{0, p_t, \beta^T} \right)^+ + \sum_{\ell \notin J_t} \Psi\left( \tilde{Z}_{J_t}^{0, z, \nu} \right) \right],
\]
where on $\{T\} \times \mathbb{R}^{d+1} \times \mathbb{R}$, we set $\tilde{w}_{2,J_t}(T, z) = f(z)$.

• On $[t_i, t_{i+1}) \times \mathbb{R}^{d+1}, \ 0 \leq i \leq n - 1$,
\[
\tilde{w}_1(t, z) := \inf_{(\nu, \beta) \in \hat{\mathcal{U}} \times \hat{A}} \mathbb{E} \left[ f\left( \tilde{Z}_{t}^{0, z, \nu} - \tilde{M}_{t}^{0, z, 0, \nu, 0} + \sum_{k = i+1}^{n} \Psi(\tilde{Z}_{t_k}^{0, z, \nu}) \right) \right],
\]
where on $\{T\} \times \mathbb{R}^{d+1}$, we set $\tilde{w}_1(T, z) = f(z)$.

**Remark 4.2.** Using the same techniques as the ones developed in this paper to study the function $\tilde{w}$ (see above and hereinafter) one can easily characterize the previous functions on their respective domain and prove their continuity.

### 4.2.1 Boundary in $m$

**Proposition 4.2.** On $\mathcal{C}_i \times (-\infty, 0]$, $0 \leq i \leq n - 1$,
\[
\tilde{w}(t, z, p_{i+1}, ..., p_n, m) = \tilde{w}_2(t, z, p_{i+1}, ..., p_n) - m.
\]

**Proof.** Fix $0 \leq i \leq n - 1$. On one hand, on $\mathcal{C}_i \times \mathbb{R}$, one has
\[
\tilde{w}(t, z, p_{i+1}, ..., p_n, m) \geq \tilde{w}_2(t, z, p_{i+1}, ..., p_n) - m,
\]
from the martingale property of $\int_0^{\Gamma_T} \beta_s \tilde{d}W_s$ as $\beta \in \hat{A}$ and with Remark 3.4. On the other hand, on $\mathcal{C}_i \times (-\infty, 0]$
\[
\tilde{w}(t, z, p_{i+1}, ..., p_n, m)
\leq \inf_{(\nu, \beta) \in \hat{\mathcal{U}} \times \hat{A}} \mathbb{E} \left[ \left( f\left( \tilde{Z}_{t}^{0, z, \nu} - \tilde{M}_{t}^{0, z, m, \nu, 0} \right)^+ + \sum_{k = i+1}^{n} \left( \Psi(\tilde{Z}_{t_k}^{0, z, \nu}) - \tilde{F}_{t_k}^{0, p_k, \beta^T} \right)^+ \right] \right],
\]
leading to $\tilde{w}(t, z, p_{i+1}, ..., p_n, m) \leq \tilde{w}_2(t, z, p_{i+1}, ..., p_n) - m$ as $m \leq 0$. \qed

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4.2.2 Boundary in \( p_k, 1 \leq k \leq n \)

The following proposition is stated without proof as its arguments follow those in the proof of Proposition 4.2 above.

**Proposition 4.3.** On \( \mathcal{C}_{i,i} \times \mathbb{R}, 0 \leq i \leq n - 1, \)

\[
\tilde{w}(t, z, p_{i+1}, ..., p_n, m) = \tilde{w}_{J_i}(t, z, (p_i)_{i \in J_i}, m) - \sum_{i \in J_i} p_i.
\]

In particular, on \( \mathcal{C}_{i,i} \times (-\infty, 0], 0 \leq i \leq n - 1, \)

\[
\tilde{w}_{J_i}(t, z, (p_i)_{i \in J_i}, m) = \tilde{w}_{2,J_i}(t, z, (p_i)_{i \in J_i}) - m.
\]

Moreover, if \( I = \{i + 1, ..., n\}, \) then \( \tilde{w}_{2,J_i}(t, z, (p_i)_{i \in J_i}) = \tilde{w}_I(t, z). \)

4.3 A Comparison principle for (4.7)

In this section, we work with the operator involved in (4.7) considering \( \lambda : m \in \mathbb{R} \mapsto \lambda(m) := 1 \vee m > 0 \) as well as \( \kappa : p \in \mathbb{R} \mapsto \kappa(p) := (1 + |\gamma|) \vee (p + |\gamma|) > 0. \) Such an operator is non-standard as it involves a non-linearity in the time-derivative. However, appealing to a strict super-solution approach (see for instance, Ishii and Lions in [34] and Cheridito, Soner and Touzi in [21]), we can prove a comparison theorem for this non-linear PDE. We therefore need to introduce beforehand the following lemma.

**Lemma 4.1** (Strict super-solution property). Fix \( 0 \leq i \leq n - 1. \) Let us define on \( [t_i, t_{i+1}) \times \mathbb{R}^{n-i+1} \) the smooth positive function

\[
\phi(t, p_{i+1}, ..., p_n, m) := e^{(t_{i+1}-t)} \left( 1 + \sum_{k=i+1}^n \ln(1 + p_k + 2|\gamma|) + \ln(1 + m) \right).
\]

Let \( v \) be a lower semi-continuous super-solution of (4.7). Then the function \( v + \xi \phi, \xi > 0, \) is a viscosity solution of

\[
\sup_{(\eta, \nu) \in U \times S(n-i+1)d+1} H_{\kappa, \lambda}^{u,b} \varphi \geq \frac{1}{\xi} > 0 \text{ on } \text{int}(\mathcal{D}_i).
\]

**Proof.** Fix \( 0 \leq i \leq n - 1, \) and let \( \varphi \) be a smooth function s.t. \( \min_{\mathcal{D}_i} ((v + \xi \phi) - \varphi) = ((v + \xi \phi) - \varphi)(\bar{t}, \bar{z}, \bar{p}_{i+1}, ..., \bar{p}_n, \bar{m}) = 0 \) with \( (\bar{t}, \bar{z}, \bar{p}_{i+1}, ..., \bar{p}_n, \bar{m}) \in \text{int}(\mathcal{D}_i), \xi > 0. \)

Since \( \phi \) is a smooth function, the function \( \psi := \varphi - \xi \phi \) is a test function for \( v \) at \( (\bar{t}, \bar{z}, \bar{p}_{i+1}, ..., \bar{p}_n, \bar{m}). \)

We consider \( b \in S_{(n-i+1)d+1} | \mathcal{D}_{(n-i+1)d+1} \) and \( u \in U. \) Using the definition of \( H_{n,\lambda}^{u,b}, \)
we obtain

\[
H_{\kappa, \lambda}^{u,b}(\bar{t}, \bar{z}, \bar{p}_{i+1}, ..., \bar{p}_n, \bar{m}) \geq H_{n,\lambda}^{u,b}(\bar{t}, \bar{z}, \bar{p}_{i+1}, ..., \bar{p}_n, \bar{m}) + \mathfrak{A},
\]

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where

$$\mathcal{A} = \xi(b_1)^2 \left( -\tilde{c}_t \phi(\cdot) + \ell(\overline{t}, \overline{\tilde{z}}, \overline{u})D_m \phi(\cdot) - \frac{1}{2} \left( -\frac{1}{2} \sum_{k=1}^{n} \kappa(\overline{p}_k)^2 |\overline{b}|^2 D_{p_k \overline{p}_k} \phi(\cdot) - \frac{1}{2} \lambda(\overline{m})^2 |\overline{b}|^2 D_{mm} \phi(\cdot) \right) \right) (\overline{t}, \overline{\tilde{z}}, \overline{p}_{i+1}, ..., \overline{p}_n, \overline{m}).$$

We now provide a lower bound for $\mathcal{A}$ and compute

$$\mathcal{A} = \xi(b_1)^2 e^{(t_{i+1} - t)} \left( 1 + \sum_{k=i+1}^{n} \ln(1 + \overline{p}_k + 2|\overline{\gamma}|) + \ln(1 + \overline{m}) \right)$$

$$+ \xi(b_1)^2 e^{(t_{i+1} - t)} \left( \frac{\ell(\overline{t}, \overline{\tilde{z}}, \overline{u})}{1 + \overline{m}} + \sum_{k=i+1}^{n} \kappa(\overline{p}_k)^2 |\overline{b}|^2 \right) + \frac{\lambda(\overline{m})^2 |\overline{b}|^2}{2(1 + \overline{m})^2}$$

$$\geq \xi(b_1)^2 \left( 1 + \frac{1}{8} \left( |\overline{\gamma}|^2 + |\overline{b}|^2 \right) \right),$$

where we used the non-negativity of the function $\ell$, and the fact that $\overline{m} \geq 0$, $\overline{p}_k \geq \gamma$, $i + 1 \leq k \leq n$, and that for any $l \in \mathbb{R}$ and $a \in \mathbb{R}$,

$$\frac{(1 + |a|)^2}{(1 + l + 2|a|)^2} \geq \frac{1}{4}.$$

Noticing that $(b_1)^2(1 + |\overline{\gamma}|^2 + |\overline{b}|^2) = |\overline{b}|^2 = 1$, we finally obtain

$$\mathcal{A} \geq \xi \frac{1}{8}. \quad (4.12)$$

Thanks to the arbitrariness of $u$ and $b$ and (4.11) and (4.12), we deduce, after appealing to the super-solution property of $v$ (see Corollary 4.1), that

$$\sup_{(u,b) \in U \times \mathcal{S}(n-i+1)_{d+1} \setminus \mathcal{D}(n-i+1)_{d+1}} H^{u,b}_{k,\lambda,\gamma}(\overline{t}, \overline{\tilde{z}}, \overline{p}_{i+1}, ..., \overline{p}_n, \overline{m}) \geq \xi \frac{1}{8}.$$

We finally conclude the proof recalling Remark 4.1. \hfill \Box

We can now state a comparison result holding for viscosity solutions of (4.7) whose proof is postponed to Appendix C and is based on Lemma 4.1. To this aim, and similarly to Section 4.2, we define for $0 \leq i \leq n - 1$, $B_{i,I} := \{(p_{i+1}, ..., p_n) \in [\gamma, \infty)^{n-i} : p_k = \gamma$ for $k \in I$ and $p_k > \gamma$ for $k \notin I\}$, as well as $\mathcal{B}_{i,I} := [\overline{t}_i, \overline{t}_{i+1}] \times \mathbb{R}^{d+1} \times B_{i,I}$. In particular, $\mathcal{B}_i = \cup_{I \in \mathcal{B}_i} \mathcal{B}_{i,I} \cup \text{int}(\mathcal{B}_i)$.

**Theorem 4.3.** (Comparison Principle) Fix $0 \leq i \leq n - 1$. Let $V$ (resp. $U$) be a lower semi-continuous (resp. upper semi-continuous) map satisfying the following growth condition

$$|V(t, z, p_{i+1}, ..., p_n, m)| + |U(t, z, p_{i+1}, ..., p_n, m)| \leq C (1 + |z|) \text{ on } \mathcal{B}_i.$$

Moreover, assume that
• on \( \text{int}(\mathcal{D}_i) \), \( V \) (resp. \( U \)) is a viscosity super-solution (resp. sub-solution) of (4.7),
• on \( \text{int}(\mathcal{D}_i) \times \{0\} \) and \( \mathcal{B}_{i,I} \times \mathbb{R}^+ \), \( V(\cdot) \geq U(\cdot) \),
• on \( \mathbb{R}^{d+1} \times [\gamma, \infty)^{n-i} \times \mathbb{R}^+ \), \( V(t_{i+1}, \cdot) \geq U(t_{i+1}, \cdot) \),
then \( V \geq U \) on \( \text{int}(\mathcal{D}_i) \).

4.4 Complete characterization of \( \hat{w} \)

Thanks to the results in the previous sections we can now obtain a full characterization of \( \hat{w} \) by the HJB equation.

**Theorem 4.4.** (Complete characterization of \( \hat{w} \)) The function \( \hat{w} \) is the unique viscosity solution of (4.7) on \( \text{int}(\mathcal{D}_i) \), \( 0 \leq i \leq n-1 \), in the class of functions being continuous on \( \mathcal{D}_i \) and satisfying the growth condition (3.15), as well as the boundary conditions

\[
\lim_{t \downarrow t_{i+1}} \hat{w}(t,z,p_{i+1},...,p_n,m) = \hat{w}(t_{i+1},z,p_{i+2},...,p_n,m) + (\Psi(z) - p_{i+1})^+,
\]
on \( \mathbb{R}^{d+1} \times [\gamma, \infty)^{n-i} \times \mathbb{R}^+ \),

\[
\hat{w} = \hat{w}_2 \text{ on } \text{int}(\mathcal{D}_i) \times \{0\} \quad \text{and} \quad \hat{w} = \hat{w}_{J_I} - \gamma \times \text{Card}(I) \text{ on } \mathcal{B}_{i,I} \times \mathbb{R}^+,
\]
and

\[
\hat{w}(T,z,m) = (f(z) - m)^+ \text{ on } \mathbb{R}^{d+1} \times \mathbb{R}^+.
\]

**Proof.** The result is proved by induction and is a direct consequence of Corollary 4.1, Propositions 3.4, 4.2 and 4.3 and Theorem 4.3.

As a direct consequence of Theorem 4.4, we can state the following corollary which is the counterpart, in our setting, of [42, Theorem 4.2, (eq. 30)] proved here by a PDE argument rather than a probabilistic one. For any \( 0 \leq i \leq n-1 \), let us define on \( \mathcal{C}_i \times \mathbb{R} \),

\[
\hat{w}(t,z,p_{i+1},...,p_n,m) := \inf_{(\nu,\beta) \in \mathcal{U} \times \mathcal{A}^+} \hat{J}^{\nu,\beta}(t,z,p_{i+1},...,p_n,m),
\]
where on \( \{T\} \times \mathbb{R}^{d+1} \times \mathbb{R} \), \( \hat{w}(T,z,m) = (f(z) - m)^+ \).

**Corollary 4.2.** Assume that the coefficients \( (\mu_Z,\sigma_Z) \) and the function \( \ell \) are \( \iota \)-Hölder continuous in the time variable for some \( \iota > 0 \). Then, on \( \mathcal{D}_i \), \( 0 \leq i \leq n-1 \), one has \( \hat{w} = \tilde{w} = \hat{w} \).
Proof. We first observe that by definition \( w(T, \cdot) = \tilde{w}(T, \cdot) = \hat{w}(T, \cdot) \). Fix \( 0 \leq i \leq n-1 \). Applying the dynamic programming argument we can show, exactly as done for \( \hat{w} \), that \( \hat{w} \) is a viscosity solution of (4.7) on \( \text{int}(D_i) \). Therefore, if (1) on \( D_i \), \( \hat{w} \) is continuous and satisfies the growth condition (3.15); (2) on \( \mathbb{R}^{d+1} \times [\gamma, \infty)^{n-i} \times \mathbb{R}^+ \),

\[
\lim_{t\uparrow t_{i+1}} \hat{w}(t, z, p_{i+1}, ..., p_n, m) = \hat{w}(t_{i+1}, z, p_{i+2}, ..., p_n, m) + (\Psi(z) - p_{i+1})^\tau; \quad (4.13)
\]

(3) on \( \text{int}(D_i) \times \{0\} \) and \( \mathcal{B}_{i,I} \times \mathbb{R}^+ \), \( \hat{w} \) respectively satisfies

\[
\hat{w} = \hat{w}_2 \text{ and } \hat{w} = \hat{w}_{J_i} - \gamma \times \text{Card}(I), \quad (4.14)
\]

with \( \hat{w}_2 \) and \( \hat{w}_{J_i} \) defined in (4.8) and (4.9);

the equality follows from Theorem 4.4. The rest of this proof (where only the dependency on \( \beta \) will be omitted) is therefore divided into three steps.

Step 1. Proof of (1). Arguing as in [42, Lemma 3.1], one can prove that there exists \( C > 0 \) s.t. for \( u \geq t_{i+1}, h > 0, z, z' \in \mathbb{R}^{d+1} \) and \( \nu \in \mathcal{U} \),

\[
\mathbb{E} \left[ \sup_{s \in [0,\theta]} \left| \tilde{Z}_s^{0,u-h,z,\nu} - \tilde{Z}_s^{0,u-h,z',\nu} \right|^2 \right] \leq C(1 + |z|^2), \quad (4.15)
\]

\[
\mathbb{E} \left[ \sup_{s \in [0,\theta]} \left| \tilde{Z}_s^{0,u-h,z,\nu} - \tilde{Z}_s^{0,u-h,z',\nu} \right|^2 \right] \leq C|z - z'|^2, \quad (4.16)
\]

\[
\mathbb{E} \left[ \sup_{s \in [0,\theta]} \left| \tilde{Z}_s^{0,u-h,z,\nu} - \tilde{Z}_s^{0,u,z,\nu} \right|^2 \right] \leq C\Gamma^{1\wedge 1}, \quad (4.17)
\]

where \( \theta := \tilde{T}_\tau \). The growth (resp. continuity) condition on \( D_i \) is therefore a direct consequence of (4.15) (resp. (4.15)-(4.17)) and of the assumptions on \( f, \Psi \) and \( \ell \).

Step 2. Proof of (2). Following the lines of Proposition 3.4, we must prove that, on \( \mathbb{R}^{d+1} \times [\gamma, \infty)^{n-i} \times \mathbb{R}^+ \), the following quantity

\[
\sup_{(\nu, \beta) \in \mathcal{U} \times \mathcal{A}^{i+1-h^+}} \mathbb{E} \left[ \sum_{k=i+1}^{n} \left( \left| \tilde{Z}_{t_{i+1}+1-k}^{0,t_{i+1}-h,z,\nu} - \tilde{Z}_{t_{i+1}+1-k}^{0,t_{i+1},z,\nu} \right| + \left| \tilde{P}_{t_{i+1}+1-k}^{0,t_{i+1}-h,p_{k}\beta_k} - \tilde{P}_{t_{i+1}+1-k}^{0,t_{i+1},p_{k}\beta_k} \right| ight) \right],
\]

\[
(4.18)
\]
tends to zero as $h \downarrow 0$. After appealing to Young’s inequality, we first obtain that, for any $\nu \in \tilde{U}$ and $\beta \in \tilde{A}^{t_{i+1}-h,+}$, there exists $C > 0$ s.t.

$$
\begin{align*}
&\mathbb{E}\left[|M^{0,t_{i+1}-h,z,m,\nu,\beta}_{t_i} - \tilde{M}^{0,t_{i+1}-h,z,m,\nu,\beta}_{t_i}|^2\right] \\
&\leq C\mathbb{E}\left[\int_0^{\tilde{t}_{i+1}-h} (\beta_{\nu})^2 |\ell(\Gamma^{0,t_{i+1}-h}_r, \tilde{Z}^{0,t_{i+1}-h,z,\nu}_r, \nu_r) - \ell(\Gamma^{0,t_{i+1}-h}_r, \tilde{Z}^{0,t_{i+1}-h,z,\nu}_r, \nu_r)| dr\right] \\
&\quad + \mathbb{E}\left[\int_{\tilde{t}_{i+1}-h}^{\tilde{t}_{i+1}-h} (\beta_{\nu})^2 |\ell(\Gamma^{0,t_{i+1}-h}_r, \tilde{Z}^{0,t_{i+1}-h,z,\nu}_r)| dr\right] + \mathbb{E}\left[\int_{\tilde{t}_{i+1}-h}^{\tilde{t}_{i+1}-h} (\beta_{\nu}^2) \tilde{dW}_r\right].
\end{align*}
$$

Now, using Burkholder-Davis-Gundy’s inequality and performing a time change (recall [47, Proposition 1.4, Chapter V] and (3.9)), the last expectation becomes

$$
\mathbb{E}\left[\int_{\tilde{t}_{i+1}-h}^{\tilde{t}_{i+1}-h} (\beta_{\nu}^2) \tilde{dW}_r\right]^2 \leq C \int_{T-h}^T \mathbb{E}[|\eta_r|^2] dr \leq Ch, \quad C > 0, \quad (4.19)
$$

with $\eta_r := \frac{\beta_{\nu}^2_{t_{i+1}-h}}{\beta_{t_{i+1}-h}}$ and where the last inequality follows from

$$
\begin{align*}
\mathbb{E}\left[\int_{t_{i+1}-h}^{T} |\eta_r|^2 dr\right] &= \mathbb{E}\left[\int_0^{\tilde{t}_{i+1}-h} \frac{|\beta_{\nu}^2|^2}{(\beta_{\nu})^2} d\Gamma^{0,t_{i+1}-h}_r\right] = \mathbb{E}\left[\int_0^{\tilde{t}_{i+1}-h} |\beta_{\nu}^2|^2 dr\right] \leq C,
\end{align*}
$$

where a time-change argument (recall [47, Proposition 1.4, Chapter V] and (3.4)) and the definition of $\tilde{A}^{t_{i+1}-h,+}$ have been used.

Moreover, performing a time change (recall [47, Proposition 1.4, Chapter V] and (3.9)) and appealing to the $r$-Hölder and Lipschitz property of $(\mu_Z, \sigma_Z, \ell)$ and to Cauchy-Schwarz’s inequality, we can find $C > 0$ s.t.

$$
\begin{align*}
&\mathbb{E}\left[\int_0^{\tilde{t}_{i+1}-h} (\beta_{\nu})^2 |\ell(\Gamma^{0,t_{i+1}-h}_r, \tilde{Z}^{0,t_{i+1}-h,z,\nu}_r, \nu_r) - \ell(\Gamma^{0,t_{i+1}-h}_r, \tilde{Z}^{0,t_{i+1}-h,z,\nu}_r, \nu_r)| dr\right] \\
&\leq C \left(h^{2n} + \int_{t_{i+1}-h}^{T-h} \mathbb{E}[|\tilde{Z}^{0,t_{i+1}-h,z,\nu}_\Gamma - \tilde{Z}^{0,t_{i+1}-h,z,\nu}_\Gamma|^2] dr\right) \\
&\leq Ch^{t_{i+1}}, \quad (4.20)
\end{align*}
$$

where the last inequality follows from (4.17). Finally, proceeding as for the first expectation and using the growth property of $\ell$ and (4.15) instead of its Hölder- and
Lipschitz-continuity property and (4.17), we have
\[
\mathbb{E}\left[ \int_{\Gamma_t^{0,1-h,0}} (\beta_1)^2 \rho_t^{0,1-h,z,\nu} \, dr \right] \leq C h (1 + |z|^2), \quad C > 0. \quad (4.21)
\]
Combining (4.19)-(4.21), we finally obtain
\[
\mathbb{E}\left[ \left| \frac{\partial z}{\partial \beta} \right|^2 \rho_t^{0,1-h,0} \right] \leq C h^{1/2} (1 + |z|^2), \quad C > 0.
\]
Proceeding similarly for the first and second term in (4.18) we conclude that the quantity in (4.18) tends to zero as \( h \to 0 \) and therefore that (4.13) holds.

**Step 3. Proof of (3).** Proceeding as in the proof of Proposition 4.2 (resp. 4.3), we can prove that on \( \text{int} (\mathcal{R}) \times \{0\} \) (resp. \( \mathcal{R} \times \mathbb{R}^+ \)), \( \hat{w} = \hat{w}_2 \) (resp. \( \hat{w} = \hat{w}_{J_1} - \gamma \times \text{Card}(I) \)), where \( \hat{w}_2 \) (resp. \( \hat{w}_{J_1} \)) is defined by replacing \( \hat{A}^I \) with \( \hat{A}^{I_+} \) in (4.8) (resp. (4.9)). We conclude that (4.14) holds after noticing that an adaptation of the arguments presented in this proof would give \( \hat{w}_2 = \hat{w}_2 \) (resp. \( \hat{w}_{J_1} = \hat{w}_{J_1} \)).

## 5 Extensions

In this section, we discuss possible extensions of our approach. We first discuss the case where the controlled-loss constraints hold from one period to another. These are the so-called *next-period* controlled-loss constraints, with the terminology of [36]. We then show how the level-set approach can be applied to probability constraints.

### 5.1 Next-period controlled-loss constraints

Next-period probability constraints have been studied for stochastic target problems under a complete market setting in [16]. We intend to explore the next-period controlled-loss counterpart for optimal control problems under a general framework.

Under the setting of Section 2, we consider on \( \mathcal{C}_i, 0 \leq i \leq n - 1 \), the following stochastic optimal control problem
\[
\hat{V}(t, z, p_{i+1}, ..., p_n) := \inf_{\nu \in \hat{U}, z, p_{i+1}, ..., p_n} \mathbb{E} \left[ f(Z_t^{i+1, \nu}) + \int_t^T \ell(s, Z_s^{i, \nu}, \nu_s) \, ds \right],
\]
where
\[
\hat{U}, z, p_{i+1}, ..., p_n := \left\{ \nu \in \hat{U} : \begin{array}{ll}
\mathbb{E} \left[ \Psi(Z_t^{i+1, \nu}) | F_t \right] \leq p_{i+1}, \\
\mathbb{E} \left[ \Psi(Z_t^{k, \nu}) | F_{t_k-1} \right] \leq p_k, \ i + 2 \leq k \leq n
\end{array} \right\}.
\]
On \( \{T\} \times \mathbb{R}^{d+1} \), we set \( \hat{V}(T, z) = f(z) \).
5.1.1 Associated stochastic target problem and level-set approach

Proceeding as in Section 3.1, one can prove the following proposition.

**Proposition 5.1.** Fix $0 \leq i \leq n - 1$, and $(t, z, p_{i+1}, \ldots, p_n) \in \mathcal{C}_i$. Then

$$
\dot{V}(t, z, p_{i+1}, \ldots, p_n) = \inf \left\{ m \geq 0 : \exists (\nu, \alpha, \eta) \in \dot{\mathcal{U}}_{t, z, p_{i+1}, \ldots, p_n} \times \mathcal{A}_n \right\},
$$

with

$$
\dot{\mathcal{U}}_{t, z, p_{i+1}, \ldots, p_n} := \left\{ (\nu, \alpha) \in \mathcal{U} \times \dot{\mathcal{A}}_i \text{ s.t. } \begin{align*}
\Psi(Z_{t+1}^{i, \nu}) &\leq P_{t+1}^{t_{i+1}, \alpha_{i+1}}, \\
\Psi(Z_{t+1}^{i, \nu}) &\leq P_{k}^{t, p_{k}} \text{, } i + 2 \leq k \leq n,
\end{align*} \right\},
$$

where $\alpha := (\alpha_i, \ldots, \alpha_n)$ and $\dot{\mathcal{A}}_i := \mathcal{A}_{i+1} \times \mathcal{A}_{i+2} \times \ldots \times \mathcal{A}_n$.

We now introduce for $0 \leq i \leq n - 1$, and $(t, z, p_{i+1}, \ldots, p_n, m) \in \mathcal{C}_i \times \mathbb{R}$, the following optimal control problem

$$
\hat{w}(t, z, p_{i+1}, \ldots, p_n, m) := \inf_{(\nu, \alpha, \eta) \in \mathcal{U} \times \dot{\mathcal{A}}_i} \hat{j}^{\nu, \alpha, \eta}(t, z, p_{i+1}, \ldots, p_n, m), \quad (5.1)
$$

with

$$
\hat{j}^{\nu, \alpha, \eta}(t, z, p_{i+1}, \ldots, p_n, m) := \mathbb{E} \left[ \left( f(Z_{t+1}^{i, \nu}) - M_T^{t, z, m, \nu, \eta} \right)^+ + \left( \Psi(Z_{t+1}^{i, \nu}) - P_{t+1}^{t, p_{i+1}, \alpha_{i+1}} \right)^+ \right],
$$

On $\{T\} \times \mathbb{R}^{d+1}$, we set $\hat{w}(T, z, m) = (f(z) - m)^+$. We define on $\mathcal{C}_i \times \mathbb{R}$, $0 \leq i \leq n - 1$, the auxiliary optimal control problem

$$
\hat{w}(t, z, p_{i+1}, \ldots, p_n, m) := \inf_{(\nu, \beta) \in \mathcal{U} \times \mathcal{A}_i} \hat{j}^{\nu, \beta}(t, z, p_{i+1}, \ldots, p_n, m), \quad (5.2)
$$

where

$$
\hat{j}^{\nu, \beta}(t, z, p_{i+1}, \ldots, p_n, m) := \mathbb{E} \left[ \left( f(\hat{Z}_{T}^{0, \nu, \beta}) - \hat{M}_T^{0, z, m, \nu, \beta} \right)^+ + \left( \hat{\Psi}(\hat{Z}_{T}^{0, \nu, \beta}) - \hat{P}_{T}^{0, p_{i+1}, \beta_{i+1}} \right)^+ \right] + \sum_{k=i+2}^{n} \left( \hat{\Psi}(\hat{Z}_{T}^{0, \nu, \beta}) - \hat{P}_{T}^{k-1, p_{k}, \beta_{k}} \right)^+,
$$

with $(\hat{Z}_{T}^{0, z, \nu}, \hat{M}_T^{0, z, m, \nu, \beta}, \hat{P}_{T}^{0, p_{i+1}, \beta_{i+1}}, \hat{P}_{T}^{k-1, p_{k}, \beta_{k}})$ defined according to (3.6)-(3.7). On $\{T\} \times \mathbb{R}^{d+1} \times \mathbb{R}$, we set $\hat{w}(T, z, m) = (f(z) - m)^+$. Proceeding as in Section 3.2, we get the following level-set characterization of $\hat{V}$.

**Proposition 5.2.** On $\mathcal{C}_i$, $0 \leq i \leq n - 1$, if the problem defined by (5.1) (or resp. (5.2)) admits an optimal control in $\mathcal{U} \times \dot{\mathcal{A}}_i$, (or resp. in $\dot{\mathcal{U}} \times \dot{\mathcal{A}}_i$), then one has

$$
\hat{V}(t, z, p_{i+1}, \ldots, p_n) = \inf \left\{ m \geq 0 : \hat{w}(t, z, p_{i+1}, \ldots, p_n, m) = 0 \right\}. \quad (5.2)
$$

In particular, the latter equality extends to $\{T\} \times \mathbb{R}^{d+1}$.
5.1.2 Complete characterization of $\hat{\tilde{w}}$

Similarly to what has been done in Section 4.2, we define the following functions (we remind that $J_r := \{ l \notin I \}$).

- On $\mathcal{C}_i$, $0 \leq i \leq n - 1$,

$$\hat{\tilde{w}}_2(t, z, p_{i+1}, \ldots, p_n) := \inf_{\nu \in \mathcal{D}^+} \mathbb{E} \left[ f \left( \tilde{Z}^0_{\Gamma^{-1}} - \tilde{M}^0_{\Gamma^{-1}} + \sum_{k=0}^{n} \Psi \left( \tilde{Z}^0_{\Gamma_k^{-1}} - \tilde{P}_{\Gamma_k^{-1}}^{p_k, \beta^i} \right) \right) \right],$$

where on $\{ T \} \times \mathbb{R}^{d+1}$, we set $\hat{\tilde{w}}_2(T, z) = f(z)$.

- On $[t_i, t_{i+1}) \times \mathbb{R}^{d+1} \times \{ \gamma, \infty \}^{\operatorname{Card}(J)} \times \mathbb{R}$, $0 \leq i \leq n - 1$,

$$\hat{\tilde{w}}_1(t, z, (p_l)_{l \in J_i}, m) := \inf_{\nu \in \mathcal{D}^+} \mathbb{E} \left[ f \left( \tilde{Z}^0_{\Gamma^{-1}} - \tilde{M}^0_{\Gamma^{-1}} + \sum_{l \in J_i, i \geq i+1} \Psi \left( \tilde{Z}^0_{\Gamma_l^{-1}} - \tilde{P}_{\Gamma_l^{-1}}^{p_l, \beta^i} \right) \right) \right],$$

where on $\{ T \} \times \mathbb{R}^{d+1} \times \mathbb{R}$, we set $\hat{\tilde{w}}_1(T, z, m) = (f(z) - m)^+$. Additionally, we consider two functions $\kappa : l \in \mathbb{R} \mapsto \mathbb{R}^+ \setminus \{0\}$ as well as $\lambda : l \in \mathbb{R} \mapsto \mathbb{R}^+ \setminus \{0\}$ and introduce the operator $\hat{\mathbb{H}}_{\kappa, \lambda}^{u,b}$, where $\Theta := (t, z, p_{i+1}, \ldots, p_n, m, q, A) \in \mathcal{D}_i \times \mathbb{R}^{d+3} \times \mathbb{R}^{d+3}$, $0 \leq i \leq n - 1$, $c \in \mathbb{R}$, $u \in U$ and $b \in \mathcal{S}_{2d+1}$,

$$\hat{\mathbb{H}}_{\kappa, \lambda}^{u,b}(\Theta, c) := \left\{ \begin{array}{ll}
(b_1)^2 \left( -c + \mathbb{H}_{\kappa, \lambda}^{u,b} \left( \Theta, c \right) \right), & b \in \mathcal{D}_{2d+1} \setminus \mathcal{D}_{2d+1}^0 \vspace{1em} \\
-\frac{1}{2} \kappa(p_{i+1})^2 [b_r^2] A_p^{p_{i+1}} + \frac{1}{2} \lambda(m)^2 [b_r^2] A_{mm} - \lambda(m) \kappa(p_{i+1}) b_r^T b_r A_{mm}, & b \in \mathcal{D}_{2d+1}
\end{array} \right.,$$

where

$$\mathbb{H}_{\kappa, \lambda}^{u,b} \left( \Theta, c \right) := \begin{cases} 
-\frac{1}{2} \kappa(p_{i+1})^2 [b_r^2] A_p^{p_{i+1}} + \frac{1}{2} \lambda(m)^2 [b_r^2] A_{mm} - \lambda(m) \kappa(p_{i+1}) b_r^T b_r A_{mm}, \\
-\lambda(m) b_r^T \sigma_{Z}^2 (t, z, u) A_m - \lambda(m) \kappa(p_{i+1}) b_r^T b_r A_{mm} \end{cases}. $$

Proceeding as in Section 4 one can prove the following complete characterization for $\hat{\tilde{w}}$.

**Theorem 5.1.** (Complete characterization of $\hat{\tilde{w}}$) The function $\hat{\tilde{w}}$ is the unique viscosity solution of

$$\sup_{(u, b) \in U \times \mathcal{S}_{2d+1}} \hat{\mathbb{H}}_{\kappa, \lambda}^{u,b} \varphi = 0 \text{ on } \operatorname{int}(\mathcal{D}_i), 0 \leq i \leq n - 1,$$
in the class of functions being continuous on $D_i$ and satisfying the growth condition (3.15), as well as the boundary conditions

$$\lim_{t \uparrow t_{i+1}} \hat{w}(t, z, p_{i+1}, ..., p_n, m) = \hat{w}(t_{i+1}, z, p_{i+1}, ..., p_n, m) + (\Psi(z) - p_{i+1})^+,\tag{3.15}$$
on $\mathbb{R}^{d+1} \times [\gamma, \infty)^{n-i} \times \mathbb{R}^+$,

$$\hat{w} = \hat{w}_2 \text{ on } \text{int}(D_i) \times \{0\} \text{ and } \hat{w} = \hat{w}_{J_i} - \gamma \times \text{Card}(I) \text{ on } D_i \times \mathbb{R}^+,$$

(with the notations introduced just before Theorem 4.3) and

$$\hat{w}(T, z, m) = (f(z) - m)^+ \text{ on } \mathbb{R}^{d+1} \times \mathbb{R}^+.$$

In particular, if the coefficients $(\mu_Z, \sigma_Z)$ and the function $\ell$ are $\iota$-Hölder continuous in the time variable for some $\iota > 0$, one has $\hat{w} = \hat{w}_2$ on $D_i$, where

$$\hat{w}(t, z, p_{i+1}, ..., p_n, m) := \begin{cases} 
\inf_{(\nu, \beta) \in \tilde{U} \times \tilde{A}^i} \hat{w}^{\nu, \beta}(t, z, p_{i+1}, ..., p_n, m) \text{ on } C_i \times \mathbb{R} \\
\hat{w}(T, z, m) = (f(z) - m)^+ \text{ on } \{T\} \times \mathbb{R}^{d+1} \times \mathbb{R}.
\end{cases}$$

5.2 Probability constraints

We consider the case where constraints in probability hold in the optimal control problem. The techniques developed in this paper may help generalize the results derived in [17, 19]. However, as explained below, the framework of applicability of Proposition 5.4 needs to be further investigated. The arguments that follow focus on lookback-style constraints to widen the framework of applications, but can be easily adapted to next-period or European-style constraints, with the terminology of [36].

Under the setting of Section 2, we consider on $[t_i, t_{i+1}) \times \mathbb{R}^{d+1} \times [0, 1]$, $0 \leq i \leq n-1$, the following stochastic optimal control problem

$$\overline{V}(t, z, p) := \inf_{\nu \in \mathcal{V}_{t,z,p}} \mathbb{E} \left[ f(Z_T^{t, z, \nu}) + \int_t^T \ell(s, Z_s^{t, z, \nu}, \nu_s) ds \right],$$

where

$$\mathcal{V}_{t,z,p} := \left\{ \nu \in \mathcal{U} : \mathbb{P} \left[ \bigcap_{k=i+1}^n \left\{ 1_{X_k^{t, x, \nu}} \geq g(X_k^{t, x, \nu}) \right\} \right] \geq p \right\}, \tag{5.3}$$

with $g : x \in \mathbb{R}^d \rightarrow \mathbb{R}$, a Lipschitz continuous function. On $\{T\} \times \mathbb{R}^{d+1}$, we set $\overline{V}(T, z) = f(z)$. The level given by $g(\cdot)$ can be interpreted as a solvency/minimal performance constraint.

Remark 5.1. A dual algorithm to compute the value function associated with the constraint in (5.3) has been provided in [12] under a complete market framework.
5.2.1 Associated stochastic target problem

Arguing as in Proposition 3.1 and [12, Proposition 2.2], one can prove the following proposition.

**Proposition 5.3.** Fix $0 \leq i \leq n - 1$, and $(t, z, p) \in [t_i, t_{i+1}) \times \mathbb{R}^{d+1} \times [0, 1]$. Then

$$V(t, z, p) = \inf \left\{ m \geq 0 : \exists (\nu, \alpha, \eta) \in \bar{V}_{t, z, p} \times A^i_n \text{ s.t. } M^{i, z, m, \nu, \eta}_T \geq f(Z_T^{i, z, \nu}) \right\} ,$$

where

$$\bar{V}_{t, z, p} := \left\{ (\nu, \alpha) \in U \times A^i_n \text{ s.t. } 1_{\{V_{t, z, p}^{i, x, \nu} \geq g(X_{t_k}^{i, x, \nu})\}} \geq P_{t_k}^{i, p, \alpha} \text{ for } i + 1 \leq k \leq n \right\} ,$$

and

$$A^i_n := \{ \alpha \in A^i_n \text{ s.t. } P^{i, p, \alpha} \in [0, 1] \} .$$

5.2.2 Level-set approach

We intend to define $V$ as a problem involving a zero level-set of a suitable function. Building on the characterization given in Proposition 5.3, a direct application of the results in Section 3.2 would lead to a level-set function involving terms of the form $(p - 1_{\{y \geq g(x)\}})^+$. However, the discontinuity of the indicator function would add an additional difficulty in the PDE characterization of the former function and numerical approximations involving regularization techniques might be needed (see e.g. [2, 3]). We therefore suggest to use an alternative auxiliary optimal control problem.

We define for all $(t, z, p, m) \in [0, T] \times \mathbb{R}^{d+1} \times [0, 1] \times \mathbb{R}$, $0 \leq i \leq n - 1$,

$$v(t, z, p, m) := \inf_{\nu \in U, \alpha \in A^i_n, \eta \in \mathbb{R}} J^{\nu, \alpha, \eta}(t, z, p, m) ,$$

with

$$J^{\nu, \alpha, \eta}(t, z, p, m) := \mathbb{E} \left[ (f(Z_T^{i, z, \nu}) - M^{i, z, m, \nu, \eta}_T) + \sum_{k=i+1}^n P_{t_k}^{i, p, \alpha} (g(X_{t_k}^{i, x, \nu}) - Y_{t_k}^{i, z, \nu}) \right] .$$

**Proposition 5.4.** On $[t_i, t_{i+1}) \times \mathbb{R}^{d+1} \times [0, 1]$, $0 \leq i \leq n - 1$, assuming that the problem defined in (5.4) admits an optimal control in $U \times A^i_n \times A^i_n$, one has

$$V(t, z, p) = \inf \left\{ m \geq 0 : v(t, z, p, m) = 0 \right\} .$$

In particular, the latter equality extends to $\{T\} \times \mathbb{R}^{d+1}$. 

Proof. The proof follows from Proposition 5.3 once observed that for any control \((\nu, \alpha) \in \mathcal{U} \times \mathcal{A}_n\) one has

\[
\left( P_{t_k}^{\nu, \alpha} - 1_{\{X_{t_k}^{t, x, \nu, z, \alpha} \geq g(X_{t_k}^{t, x, \nu, z, \alpha})\}} \right)^+ = 0 \iff P_{t_k}^{\nu, \alpha} \left( g(X_{t_k}^{t, x, \nu, z, \alpha}) - Y_{t_k}^{t, x, \nu, z, \alpha} \right)^+ = 0.
\]

A compactified version of problem (5.4) can be obtained without further difficulty as done in Proposition 3.2. This leads to a slight relaxation of the existence assumption. However, the loss of convexity in the triple \((z, p, m)\) of the cost functional prevents the application of the result in Remark 3.5 (i). At this stage, we are not able to provide a framework ensuring the applicability of Proposition 5.4. Further research in this direction is then necessary, possibly involving the weak formulation of the problem. We observe that the characterization of \(v\) involves here non-trivial boundaries in the \(p\)-variable that can be handled appealing to the techniques in [17, 15].

6 Conclusions

Arguing as in [11, Theorem 3.1] and therefore assuming the existence of an optimizer for the value function \(w\) defined in (3.3), one can prove that the original value function \(V\) in (2.1) writes as the zero level of \(w\). This result has the great advantage of not requiring any viability assumptions on the coefficients of the diffusion. However, \(w\) is associated with an unconstrained optimal control problem involving unbounded controls. Such characteristics raise additional difficulties in the PDE characterization of \(w\) and may make the derivation of the conditions under which the existence assumption holds more convoluted. Adopting a compactification argument "à la" Dufour and Miller [26, 25], we have proved that \(w = \hat{w}\), where \(\hat{w}\) in (3.8) satisfies useful regularity properties and is the value function associated with an unconstrained optimal stopping/control problem involving controls valued in relatively compact sets. Therefore, under Assumption 3.2.1, Proposition 3.3 characterizes \(V\) as the zero level of \(\hat{w}\), extending the results in [11].

Thanks to this transformation, the HJB operator involved in the characterization of \(\hat{w}\) is continuous, and the assumption on the existence of an optimizer can be relaxed to either \(w\) or \(\hat{w}\) (compare \((H4)\) in [11] with Assumption 3.2.1). This transformation also extends the compactification results presented in [42, 26, 25] and can be of interest for the treatment of several problems sharing a similar structure (see, e.g., [1, 4]).

In Theorem 4.4 we have provided a complete characterization of \(\hat{w}\) on each time interval \([t_i, t_{i+1})\), \(0 \leq i \leq n - 1\). Moreover, in Section 5 we have extended the results...
obtained to the case of next-period controlled-loss constraints and initiated a discussion
on the special case of probability constraints.

This paper opens new avenues for further research which include: the investigation of
the conditions under which Assumption 3.2.1 holds, the complete treatment of the case
where the constraint holds in probability, and the study of the numerical approximation
of V appealing to the characterization of \( \hat{w} \).

A Proof of Theorem 4.1

We fix \((t, z, p_{i+1}, \ldots, p_n, m) \in \mathcal{C}_i \times \mathbb{R}, 0 \leq i \leq n-1\). We take \(0 \leq \theta \leq \hat{\Gamma}_{t+i} \) be a stopping
time. When \(\theta = \hat{\Gamma}_{t+i} \) and \(i = n-1\), the result is obvious. We assume that \(\theta = \hat{\Gamma}_{t+i} \)
and \(i < n-1\) as the case \(\theta < \hat{\Gamma}_{t+i} \) can be treated similarly. We denote by \(\hat{w}\) the
right-hand side of (4.1).

**Step 1. Proof of** \(\hat{w} \geq \hat{w} \). Using the Flow property as well as the definition of \(\hat{w}\) in
(3.8) one can prove that

\[
\hat{w}(t, z, p_{i+1}, \ldots, p_n, m) \geq \inf_{(\nu, \beta) \in \hat{U} \times \hat{A}^i} \mathbb{E} \left[ \Phi \left( \Gamma_\theta, \hat{Z}_\theta^{0, z, \nu}, \hat{P}_\theta^{0, p_{i+1}, \beta_i+1}, \ldots, \hat{P}_\theta^{0, p_n, \beta_n}, \hat{M}_\theta^{0, z, m, \nu, \beta^*} \right) \right],
\]

leading to \(\hat{w}(t, z, p_{i+1}, \ldots, p_n, m) \geq \hat{w}(t, z, p_{i+1}, \ldots, p_n, m)\).

**Step 2. Proof of** \(\hat{w} \leq \hat{w} \). We fix \((\hat{\nu}, \hat{\beta}) \in \hat{U} \times \hat{A}^i\), and consider \(\mu\), the measure
induced by \((t_i, \xi, \zeta_{i+1}, \ldots, \zeta_n, \kappa)\) on \([0, T] \times \mathbb{R}^{d-i+2}\) with \((\xi, (\zeta_k)_{i+1 \leq k \leq n, \kappa}) :=
(\hat{Z}_\theta^{0, z, \hat{\nu}}, (\hat{P}_\theta^{0, p_k, \hat{\beta}_k})_{i+1 \leq k \leq n}, \hat{M}_\theta^{0, z, m, \nu, \hat{\beta}^*})\). We appeal to [7, Proposition 7.50, Lemma 7.27]
to prove that, for each \(\epsilon > 0\), we can build \((n-i+2)\) Borel-measurable maps given by
\(\nu_{\mu}^{\epsilon}\) and \(\beta_{\mu}^{\epsilon}\),

\[(\nu_{\mu}^{\epsilon}, \beta_{\mu}^{\epsilon})(t_{i+1}, \xi, \zeta_{i+1}, \ldots, \zeta_n, \kappa) \in \hat{U} \times \hat{A}^{i+1},\]

where \((\beta_{\mu}^{\epsilon}, \beta_{\mu}^{\epsilon})\) has been modified appropriately to satisfy Definition 3.1 (vi)-(vii), and
where

\[
\Phi(t_{i+1}, \xi, \zeta_{i+1}, \ldots, \zeta_n, \kappa) \geq \tilde{\nu}_{\mu}^{\epsilon, \beta_{\mu}^{\epsilon}}(t_{i+1}, \xi, \zeta_{i+2}, \ldots, \zeta_n, \kappa) + (\Psi(\hat{Z}_\theta^{0, z, \hat{\nu}}) - \hat{P}_\theta^{0, p_{i+1}, \beta_{i+1}}) + \epsilon.
\]

(1.1)
We now use [50, Lemma 2.1] to obtain $\nu^\varepsilon$ and $\beta^\varepsilon$ s.t.

$$
\begin{align*}
\nu^\varepsilon 1_{[\theta, \tilde{\Gamma}_T]} &= \nu^\varepsilon_{\mu}(t_{i+1}, \xi, \zeta_{i+1}, ..., \zeta_n, \kappa) 1_{[\theta, \tilde{\Gamma}_T]} dt \times d\mathbb{P}\text{-a.e.}, \\
\beta^\varepsilon_{\Gamma_{t_i}} &= \beta^\varepsilon_{\Gamma_{t_i}, \mu}(t_{i+1}, \xi, \zeta_{i+1}, ..., \zeta_n, \kappa) 1_{[\theta, \tilde{\Gamma}_T]} dt \times d\mathbb{P}\text{-a.e.}, \\
\beta^\varepsilon_{i+2, t_i} &= \beta^\varepsilon_{i+2, t_i, \mu}(t_{i+1}, \xi, \zeta_{i+1}, ..., \zeta_n, \kappa) 1_{[\theta, \tilde{\Gamma}_T]} dt \times d\mathbb{P}\text{-a.e.}, \\
& \vdots \\
\beta^\varepsilon_{n, t_i} &= \beta^\varepsilon_{n, t_i, \mu}(t_{i+1}, \xi, \zeta_{i+1}, ..., \zeta_n, \kappa) 1_{[\theta, \tilde{\Gamma}_T]} dt \times d\mathbb{P}\text{-a.e.},
\end{align*}
$$

implying that $\tilde{\nu} := \tilde{\nu}1_{(0, \theta)} + \nu^\varepsilon 1_{[\theta, \tilde{\Gamma}_T]} \in \tilde{\mathcal{U}}, (\beta^\varepsilon_{1, \Gamma}, \beta^\varepsilon_{2, \Gamma}, ..., \beta^\varepsilon_{n, \Gamma}) \in \tilde{\mathcal{U}}$, where

$$
\begin{align*}
\beta^\varepsilon_{1, \Gamma} &= \beta^\varepsilon_{1}(0, \theta) + \beta^\varepsilon_{1}[\theta, \tilde{\Gamma}_T], \\
\beta^\varepsilon_{i+2, \Gamma} &= \beta^\varepsilon_{i+2}(0, \theta) + \beta^\varepsilon_{i+2}[\theta, \tilde{\Gamma}_T], \\
& \vdots \\
\beta^\varepsilon_{n, \Gamma} &= \beta^\varepsilon_{n}(0, \theta) + \beta^\varepsilon_{n}[\theta, \tilde{\Gamma}_T],
\end{align*}
$$

and (1.1) holds where

$$
\begin{align*}
\tilde{\nu}^\varepsilon, \beta^\varepsilon_{\Gamma_i}(t_{i+1}, \xi, \zeta_{i+1}, ..., \zeta_n, \kappa)
\end{align*}
$$

$$
= E \left\{ \left( f(Z^\varepsilon_{\Gamma_{t_i}}) - M^\varepsilon_{\Gamma_{t_i}} \right)^+ + \sum_{k=i+2}^n \left( \Psi(Z^\varepsilon_{\Gamma_{t_i}}) - P^\varepsilon_{\zeta_{i+1}, \zeta_n, \kappa} \right)^+ \left| (t_{i+1}, \xi, \zeta_{i+1}, ..., \zeta_n, \kappa) \right. \right\}.
$$

We conclude taking the expectation on both sides in (1.1) and using the arbitrariness of $(\tilde{\nu}, \bar{\beta}) \in \tilde{\mathcal{U}} \times \tilde{\mathcal{U}}$ and of $\varepsilon$.

### B Proof of Corollary 4.1

The result is a direct consequence of Theorem 4.2 and of the fact that for any continuous functions $\lambda, \kappa : \mathbb{R} \rightarrow \mathbb{R}^+ \setminus \{0\}$, one has for $(t, z, p_{i+1}, ..., p_n, m, c, q, A) \in \mathcal{S}_i \times \mathbb{R} \times \mathbb{R}^{d+(n-i+2)} \times \mathcal{S}^{d+(n-i+2)}, 0 \leq i \leq n - 1, u \in U$ and $b \in \mathcal{S}_{(n-i+1)d+1}$,

$$
\begin{align*}
& b^\top G(t, z, u, c, q, A)b \leq 0 \\
& \Leftrightarrow b^\top Q^\top_{\kappa, \lambda}(p_{i+1}, ..., p_n, m)G(t, z, u, c, q, A)Q_{\kappa, \lambda}(p_{i+1}, ..., p_n, m)b \leq 0,
\end{align*}
$$

(2.1)
with $Q_{c,\lambda}(p_{i+1},...,p_n, m) \in S^{(n-i+1)d+1}$ the diagonal matrix

\[ Q_{c,\lambda}(p_{i+1},...,p_n, m) := \text{diag}(1, \kappa(p_{i+1})I_d, \ldots, \kappa(p_n)I_d, \lambda(m)I_d), \]

and $G(t, z, u, c, q, A) \in S^{(n-i+1)d+1}$ the matrix

\[ G(t, z, u, c, q, A) := \begin{pmatrix} C & -\frac{1}{2} [\sigma_Z^T A^{p_{i+1}}]^T & \ldots & -\frac{1}{2} [\sigma_Z^T A^{p_n}]^T & -\frac{1}{2} [\sigma_Z^T A^{zm}]^T \\ -\frac{1}{2} \sigma_Z^T A^{p_{i+1}z} & -\frac{1}{2} A^{p_{i+1}p_{i+1}}I_d & 0 & \ldots & 0 & -\frac{1}{2} A^{p_{i+1}m}I_d \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & \vdots & \vdots \\ -\frac{1}{2} \sigma_Z^T A^{zm} & 0 & \ldots & 0 & -\frac{1}{2} A^{p_n}I_d & -\frac{1}{2} A^{n}I_d \\ -\frac{1}{2} \sigma_Z^T A^{mz} & -\frac{1}{2} A^{p_{i+1}m}I_d & \ldots & \ldots & -\frac{1}{2} A^{m}I_d & -\frac{1}{2} A^{mm}I_d \end{pmatrix}, \]

where $\sigma_Z \equiv \sigma_Z(t, z, u)$ and $C \equiv C(t, z, u, c, q^*, q^m, A^{zz}) := -c - \mu_Z(t, z, u)q^z + \ell(t, z, u)q^m - \frac{1}{2} \text{Tr}[\sigma_Z^2(t, z, u)A^{zz}]$. To prove equivalence (2.1), we first notice that it reads

\[ \Lambda^+(G) \leq 0 \iff \Lambda^+(Q_{c,\lambda}^T G Q_{c,\lambda}) \leq 0, \]

where for a given symmetric matrix $M$, $\Lambda^+(M)$ denotes the highest eigenvalue of $M$. We also denote by $M^{(k)} \in S^k$ the matrix obtained considering only the first $k \geq 1$ rows and columns of $M$. One can easily check that for any $k \geq 1$,

\[ \det((Q_{c,\lambda}^T G Q_{c,\lambda})^{(k)}) = \lambda(m)^{2\max(k-(n-i)d-1,0)} \prod_{j=1}^{\frac{k-1}{2}+1} \kappa(p_{i+j})^{2\min(d, (1-j)d+k-1)} \det(G^{(k)}). \]

The result then follows.

**C Proof of Theorem 4.3**

We first state the following lemma which is involved in the proof of Theorem 4.3.

**Lemma C.1** (Modulus of continuity). *Fix $0 \leq i \leq n-1$. There exists $\rho > 0$ such that: for any $(t, z, p, q, m, l) \in [t_i, t_{i+1}) \times \mathbb{R}^{2(d+1)} \times \left[\gamma, \infty\right)^2 \times (\mathbb{R}^+)^2$, with $p := (p_{i+1},...,p_n)^T$ and $q := (q_{i+1},...,q_n)^T$, for any $X, Y \in S^{d+(n-i+2)}$ satisfying*

\[ \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \frac{3}{\varepsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 2\varepsilon e^{-\rho t} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \]

(3.1)
for some $(\zeta, \varepsilon) > 0$, with $I = I_{d+(n-i+2)}$ and $\bar{I} := \text{diag}(I_{d+1}, 0, \ldots, 0) \in \mathbb{S}^{d+(n-i+2)}$, and for any $c_1 := -\zeta pe^{-\rho t} (1 + |z|^2) \in \mathbb{R}^-$, $c_2 := \zeta pe^{-\rho t} (1 + |r|^2) \in \mathbb{R}^+$, and

$$\Delta_1 := \begin{pmatrix} \frac{1}{\varepsilon} (z - r) + 2\zeta e^{-\rho t} z \\ \frac{1}{\varepsilon} (p - q) \\ \frac{1}{\varepsilon} (m - l) \end{pmatrix}, \quad \Delta_2 := \begin{pmatrix} \frac{1}{\varepsilon} (z - r) - 2\zeta e^{-\rho t} r \\ \frac{1}{\varepsilon} (p - q) \\ \frac{1}{\varepsilon} (m - l) \end{pmatrix} \in \mathbb{R}^{d+(n-i+2)},$$

one has, with $\Theta_1 := (t, z, p, \Delta_1, X, c_1)$ and $\Theta_2 := (t, r, q, \Delta_2, Y, c_2)$,

$$\sup_{(u,b) \in U \times S(n-i+1)d+1} \mathcal{H}^{u,b}_{\kappa,\lambda}(\Theta_2) - \sup_{(u,b) \in U \times S(n-i+1)d+1} \mathcal{H}^{u,b}_{\kappa,\lambda}(\Theta_1) \leq C \varepsilon (|z - r|^2 + |p - q|^2 + (m - l)^2),$$

for some constant $C > 0$.

**Proof.** We remind that we do not keep track of constants unless otherwise stated. Consider $\Theta_1$ and $\Theta_2$ defined in the theorem. We notice that

$$\sup_{(u,b) \in U \times S(n-i+1)d+1} \mathcal{H}^{u,b}_{\kappa,\lambda}(\Theta_2) - \sup_{(u,b) \in U \times S(n-i+1)d+1} \mathcal{H}^{u,b}_{\kappa,\lambda}(\Theta_1) \leq \sup_{(u,b) \in U \times S(n-i+1)d+1} \left\{ \mathcal{H}^{u,b}_{\kappa,\lambda}(\Theta_2) - \mathcal{H}^{u,b}_{\kappa,\lambda}(\Theta_1) \right\},$$

(recall Remark 4.1 and Corollary 4.1).

For $b \in S(n-i+1)d+1 \setminus D(n-i+1)d+1$ and $u \in U$, appealing to the definition of $\mathcal{H}^{u,b}_{\kappa,\lambda}$, we compute

$$\mathcal{H}^{u,b}_{\kappa,\lambda}(\Theta_2) - \mathcal{H}^{u,b}_{\kappa,\lambda}(\Theta_1) \leq (b_1)^2 (\mathfrak{A} + \mathfrak{B} + \mathfrak{C}),$$

where

$$\mathfrak{A} = \frac{1}{\varepsilon} (\mu_Z(t, z, \nu) - \mu_Z(t, r, \nu))^\top (z - r) + \frac{1}{\varepsilon} |m - l| \ell(t, r, \nu) - \ell(t, z, \nu),$$

$$\mathfrak{B} = 2\zeta e^{-\rho t} \mu_Z(t, z, \nu) z + 2\zeta e^{-\rho t} \mu_Z(t, r, \nu) r - \zeta pe^{-\rho t} (2 + |z|^2 + |r|^2),$$

and

$$\mathfrak{C} = -\frac{1}{2} \text{Tr} \left[ \bar{\sigma} \bar{\sigma}^\top (t, r, q, l, u, b) \right] + \frac{1}{2} \text{Tr} \left[ \bar{\sigma} \bar{\sigma}^\top (t, z, p, m, u, b) \right],$$

where for any $(t, r, q, l) \in [t_i, t_{i+1}] \times \mathbb{R}^{d+1} \times [\gamma, \infty)^{n-i} \times \mathbb{R}^+$ and $(u, b) \in U \times S(n-i+1)d+1 \setminus D(n-i+1)d+1$,

$$\hat{\sigma}(t, r, q, l, u, b) := \begin{pmatrix} \sigma_Z(t, r, u) \\ \kappa(q_{i+1})(\hat{b}_{i+1})^\top \\ \kappa(q_{i})(\hat{b}_{i})^\top \\ \lambda(l)(\hat{b})^\top \end{pmatrix}. $$

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Using the Lipschitz property of $\mu_Z$ and $\ell$ we obtain the existence of $C > 0$ s.t.

$$\mathfrak{A} \leq \frac{C}{\varepsilon} (|z - r|^2 + (m - t)^2) .$$

Moreover, using the growth property of $\mu_Z$, we obtain the existence of $\hat{C} > 0$ s.t.

$$\mathfrak{B} \leq \zeta \hat{C} e^{-\rho t} (1 + |z|^2 + |r|^2) - \zeta \rho e^{-\rho t} (1 + |z|^2 + |r|^2) .$$

For the third term $\mathfrak{C}$, we use (3.1) and the Lipschitz continuity of $\sigma_Z$, $\kappa$ and $\lambda$ to obtain the existence of $\bar{C} > 0$ s.t.

$$\mathfrak{C} \leq C \left( \frac{1}{\varepsilon} \left( 1 + |\bar{b}|^2 + |\bar{b}'|^2 \right) (|z - r|^2 + |p - q|^2 + (m - t)^2) + \zeta e^{-\rho t} (1 + |z|^2 + |r|^2) \right) .$$

Taking $\rho \geq \hat{C} + \bar{C} + 1$ for instance, we obtain for some $C > 0$,

$$\mathfrak{B} + \mathfrak{C} \leq \frac{C}{\varepsilon} \left( 1 + |\bar{b}|^2 + |\bar{b}'|^2 \right) (|z - r|^2 + |p - q|^2 + (m - t)^2) .$$

The proof is concluded by observing that $(b_1)^2(1 + |\bar{b}|^2 + |\bar{b}'|^2) = 1$. \hfill $\square$

We can now prove Theorem 4.3.

**Proof of Theorem 4.3.** Fix $0 \leq i \leq n - 1$. For $\xi > 0$, we introduce on $(t_i, t_{i+1}) \times \mathbb{R}^{d+1} \times [\gamma, \infty)^{n-i} \times \mathbb{R}^+$ the following auxiliary functions

$$V_\xi(t, z, p, m) := (V + \xi \phi)(t, z, p, m) + \xi \left( \frac{1}{t - t_i} \right) ,$$

$$U_\xi(t, z, p, m) := (U - \xi \phi)(t, z, p, m) ,$$

where $p := (p_{i+1}, \ldots, p_n)^T$ and with $\phi$ defined in Lemma 4.1. Appealing to Lemma 4.1, one can easily check that $V_\xi$ is a strict super-solution of (4.7) satisfying (4.10). Analogously $U_\xi$ can be proved to be a sub-solution of (4.7).

We prove that $U - V \leq 0$ on $\mathscr{D}_i$. To this aim we first show arguing by contradiction that for all $\xi > 0$, $(U_\xi - V_\xi) \leq 0$ on $(t_i, t_{i+1}) \times \mathbb{R}^{d+1} \times [\gamma, \infty)^{n-i} \times \mathbb{R}^+$, and the proof is completed sending $\xi$ to zero.

**Step 1.** We assume to the contrary that we can find $\xi > 0$ s.t.

$$\sup_{(t_i, t_{i+1}) \times \mathbb{R}^{d+1} \times [\gamma, \infty)^{n-i} \times \mathbb{R}^+} (U_\xi - V_\xi) > 0 . \quad (3.2)$$

We define on $(t_i, t_{i+1}) \times \mathbb{R}^{d+1} \times [\gamma, \infty)^{n-i} \times \mathbb{R}^+$,

$$\Phi_{\xi \zeta}(t, z, p, m) := (U_\xi - V_\xi)(t, z, p, m) - 2\zeta e^{-\rho t} (1 + |z|^2) ,$$
for $\zeta > 0$ and with $\rho > 0$ defined in Lemma C.1. Using the growth conditions and semi-continuity of $U$ and $V$ as well as (3.2) we obtain that for $\xi > 0$, and $\zeta > 0$ small enough

$$0 < M := \sup_{(t_i, t_{i+1}) \times \mathbb{R}^{d+1} \times [\gamma, \infty)^{n-i} \times \mathbb{R}^+} \Phi_{\xi, \zeta}(t, z, p, m) < \infty.$$  

For $\varepsilon > 0$, we introduce on $(t_i, t_{i+1}) \times \mathbb{R}^{2d+2} \times [\gamma, \infty)^{2(n-i)} \times (\mathbb{R}^+)^2$,

$$\Psi_{\xi, \zeta, \epsilon}(t, z, r, p, q, m, l) := U_\xi(t, z, p, m) - V_\xi(t, r, q, l) - \zeta \epsilon^{\rho \delta}(1 + |z|^2) - \zeta \epsilon^{\rho \delta}(1 + |r|^2) - \frac{1}{2\varepsilon} (|z - r|^2 + |p - q|^2 + (m - l)^2),$$  

with $q := \left(q_{i+1}, ... , q_n\right)^\top$. Again, the growth conditions and semi-continuity of $U$ and $V$ ensure that for $(\xi, \zeta, \epsilon) > 0$ the function $\Psi_{\xi, \zeta, \epsilon}$ admits a maximum $M_\epsilon$ at $(t\xi, z\xi, r\xi, p\xi, q\xi, m\xi, l\xi)$, with $p\xi := \left(p_{i+1}, ... , p_n\right)^\top$ and $q\xi := \left(q_{i+1}, ... , q_n\right)^\top$, on $(t_i, t_{i+1}) \times \mathbb{R}^{2d+2} \times [\gamma, \infty)^{2(n-i)} \times (\mathbb{R}^+)^2$ (we omit the dependency on $(\xi, \zeta, \epsilon)$ for sake of clarity). Using standard arguments (see, for instance, the proof of [45, Theorem 4.4.4] and [22, Lemma 3.1]), one can prove that there exists $(\bar{t}, \bar{z}, \bar{p}, \bar{m}) \in (t_i, t_{i+1}) \times \mathbb{R}^{d+1} \times [\gamma, \infty)^{n-i} \times \mathbb{R}^+$, with $\bar{p} := \left(\bar{p}_{i+1}, ... , \bar{p}_n\right)^\top$, such that

$$\begin{cases} 
\lim_{\epsilon \downarrow 0} t\xi = \bar{t}, \lim_{\epsilon \downarrow 0} z\xi = \bar{z}, \lim_{\epsilon \downarrow 0} p\xi = \bar{p}, \lim_{\epsilon \downarrow 0} m\xi, l\xi = \bar{m}, \\
\lim_{\epsilon \downarrow 0} \frac{1}{\varepsilon} (|z\xi - r\xi|^2 + |p\xi - q\xi|^2 + (m\xi - l\xi)^2) = 0, \quad (3.3) \\
\lim_{\epsilon \downarrow 0} M_\epsilon = M = \Phi_{\xi, \zeta}(\bar{t}, \bar{z}, \bar{p}, \bar{m}).
\end{cases}$$  

Moreover, it follows from the boundaries assumptions on $V$ and $U$ that $(\bar{t}, \bar{z}, \bar{p}, \bar{m}) \in (t_i, t_{i+1}) \times \mathbb{R}^{d+1} \times (\gamma, \infty)^{n-i} \times (0, \infty)$. As a consequence we assume that, up to a subsequence, $(t\xi, z\xi, r\xi, p\xi, q\xi, m\xi, l\xi) \in (t_i, t_{i+1}) \times \mathbb{R}^{2d+2} \times (\gamma, \infty)^{2(n-i)} \times (0, \infty)^2$.

**Step 2.** Using [22, Remark 2.7] and Ishii’s Lemma (see [22, Theorem 8.3]) we obtain the existence of real coefficients $c_{1,\epsilon}, c_{2,\epsilon}$, two vectors $\Delta_{1,\epsilon}, \Delta_{2,\epsilon}$ and two symmetric matrices $X_\epsilon$ and $Y_\epsilon$ being s.t.

$$(c_{1,\epsilon}, \Delta_{1,\epsilon}, X_\epsilon) \in \mathcal{F}_{\bar{O}}^+ U_\xi(t\xi, z\xi, p\xi, m\xi) \quad \text{and} \quad (c_{2,\epsilon}, \Delta_{2,\epsilon}, Y_\epsilon) \in \mathcal{F}_{\bar{O}}^- V_\xi(t\xi, r\xi, q\xi, l\xi),$$  

with $\bar{O} := (t_i, t_{i+1}) \times \mathbb{R}^{d+1} \times (\gamma, \infty)^{n-i} \times (0, \infty)$ and $\mathcal{F}^+$ (resp. $\mathcal{F}^-$) the limiting second-order super-jet (resp. sub-jet) of $U_\xi$ (resp. $V_\xi$) at $(t\xi, z\xi, p\xi, m\xi) \in \bar{O}$ (resp.
\((t_\varepsilon, r_\varepsilon, q_\varepsilon, l_\varepsilon) \in \mathcal{O}\) and where

\[
c_{1,\varepsilon} - c_{2,\varepsilon} := -\zeta \rho e^{-\rho t_\varepsilon}(1 + |z_\varepsilon|^2) - \zeta \rho e^{-\rho t_\varepsilon}(1 + |r_\varepsilon|^2),
\]

\[
\Delta_{1,\varepsilon} := \begin{pmatrix}
\frac{1}{\varepsilon}(z_\varepsilon - r_\varepsilon) + 2\zeta e^{-\rho t_\varepsilon} z_\varepsilon \\
\frac{1}{\varepsilon}(p_\varepsilon - q_\varepsilon) \\
\frac{1}{\varepsilon}(m_\varepsilon - l_\varepsilon)
\end{pmatrix},
\]

\[
\Delta_{2,\varepsilon} := \begin{pmatrix}
\frac{1}{\varepsilon}(z_\varepsilon - r_\varepsilon) - 2\zeta e^{-\rho t_\varepsilon} r_\varepsilon \\
\frac{1}{\varepsilon}(p_\varepsilon - q_\varepsilon) \\
\frac{1}{\varepsilon}(m_\varepsilon - l_\varepsilon)
\end{pmatrix},
\]

\[
\begin{pmatrix}
X_\varepsilon & 0 \\
0 & -Y_\varepsilon
\end{pmatrix} \leq \frac{3}{\varepsilon} \begin{pmatrix}
I & -\bar{I} \\
-\bar{I} & I
\end{pmatrix} + 2\zeta e^{-\rho t_\varepsilon} \begin{pmatrix}
\bar{I} & 0 \\
0 & \bar{I}
\end{pmatrix},
\]

where \(I \equiv I_{d+(n-i+2)}\) and with \(\bar{I}\) defined in Lemma C.1. We know from the definition of \(U_\xi\) and \(V_\xi\) that they are respectively sub-/super-solution of \((4.7)\). As a result, appealing to Lemma 4.1 we obtain

\[
\sup_{(u,b) \in U \times S^{(n-i+1)d+1}} H^{u,b}_{\kappa,\lambda} (t_\varepsilon, r_\varepsilon, q_\varepsilon, l_\varepsilon, \Delta_{2,\varepsilon}, Y_\varepsilon, c_{2,\varepsilon})
\]

\[
- \sup_{(u,b) \in U \times S^{(n-i+1)d+1}} H^{u,b}_{\kappa,\lambda} (t_\varepsilon, z_\varepsilon, m_\varepsilon, \Delta_{1,\varepsilon}, X_\varepsilon, c_{1,\varepsilon}) \geq \frac{1}{8} > 0.
\]

**Step 3.** On the other hand we know from Lemma C.1 and \((3.4)\) that

\[
\sup_{(u,b) \in U \times S^{(n-i+1)d+1}} H^{u,b}_{\kappa,\lambda} (t_\varepsilon, r_\varepsilon, q_\varepsilon, l_\varepsilon, \Delta_{2,\varepsilon}, Y_\varepsilon, c_{2,\varepsilon})
\]

\[
- \sup_{(u,b) \in U \times S^{(n-i+1)d+1}} H^{u,b}_{\kappa,\lambda} (t_\varepsilon, z_\varepsilon, m_\varepsilon, \Delta_{1,\varepsilon}, X_\varepsilon, c_{1,\varepsilon}) \leq \frac{C}{\varepsilon} (|z_\varepsilon - r_\varepsilon|^2 + |p_\varepsilon - q_\varepsilon|^2 + (m_\varepsilon - l_\varepsilon)^2),
\]

for some constant \(C > 0\). We now send \(\varepsilon\) to zero and use \((3.3)\) to obtain that the last inequality is non positive. This contradicts \((3.5)\) and we conclude that \((U_\xi - V_\xi) \leq 0\) for all \(\xi > 0\) on \((t_i, t_{i+1}) \times \mathbb{R}^{d+1} \times [\gamma, \infty)^{n-i} \times \mathbb{R}^+\).  

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