Coulomb Force as an Entropic Force

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Motivated by Verlinde’s theory of entropic gravity, we give a tentative explanation to Coulomb’s law with an entropic force. When trying to do this, we find the equipartition rule should be extended to charges and the concept of temperature should be reinterpreted. If one accepts the holographic principle as well as our generalizations and reinterpretations, then Coulomb’s law, the Poisson equation and the Maxwell equations can be derived smoothly. Our attempt can be regarded as a new way to unify the electromagnetic force with gravity, from the entropic origin. Possibly some of our postulates are related to the D-brane picture of black hole thermodynamics.

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I. MOTIVATION

Starting from the holographic principle and an equipartition rule, E. Verlinde proposed an intriguing explanation for Newton’s law of gravity as an entropic force [1]. This idea turns out to be very powerful. From it one can derive both Newton’s laws and Einstein’s equations. In this paper, we try to understand its implication to the Reissner-Nordström (RN) spacetime. With a few generalizations and reinterpretations, we find the same idea can produce Coulomb’s law and Maxwell’s equations.

To make our narration clear, for the most part we will concentrate on the 4-dimensional spacetime, in which the RN solution is

\[
\begin{align*}
\text{ds}^2 &= -f(r)\text{dt}^2 + \frac{1}{f(r)}\text{dr}^2 + r^2d\Omega^2 \\
f(r) &= 1 - \frac{2GM}{c^2r} + \frac{G^2Q^2}{c^4r^2}, \quad M \geq |Q|.
\end{align*}
\]

For conciseness, we work in the “geometrized” unit of charge so that the Coulomb force between point charges \(Q, q\) at large separation \(r\) in flat space takes the form

\[
F_{\text{em}} = \frac{G_N Q q}{r^2}
\]

If one feels uneasy with such a unit, the more familiar form can be recovered by replacing \(Q \rightarrow Q/\sqrt{4\pi\varepsilon_0}G_N\), \(q \rightarrow q/\sqrt{4\pi\varepsilon_0}G_N\) for electric charges, where \(\varepsilon_0\) is the electric constant and \(G_N\) is the Newton’s constant.

Now let us show a puzzle regarding the equipartition rule. On the one hand, in [1] it was proposed that the temperature \(T\) is determined by

\[
Mc^2 = \frac{1}{2}Nk_B T
\]

as the average energy per “bit” (in certain unit of information and in terms of the Shannon entropy). Here

\[
N = \frac{Ac^3}{G_N \hbar}
\]

is the number of bits on the boundary with area \(A\), as suggested by the holographic principle. One would be puzzled when checking the validity of (4) on the horizon of the RN black hole, where

\[
A_H = \frac{4\pi G_N^2}{c^2} \left( M + \sqrt{M^2 - Q^2} \right)^2, \\
T_H = \frac{2G_N \hbar}{k_B c A_H} \sqrt{M^2 - Q^2}.
\]

Obviously the relation (4) is violated as long as \(Q \neq 0\). It would be violated even if we go away from the event horizon of an RN black hole.

Actually this puzzle has driven us to pursue the entropic origin of Coulomb force. The puzzle will be solved in subsequent sections. To do this, we will reinterpret the temperature and generalize the equipartition relation in section II. In the same section, we will derive Coulomb’s law in two different but related ways. As we will see, there are promising relations between our postulates and the D-brane picture of black hole thermodynamics. In section III taking Coulomb force as an entropic force, we will derive the Poisson equation and the Maxwell equations. The holographic principle and the generalized partition rules are vital in our derivation. Finally, in section IV we will comment on our results and open questions.
II. GENERALIZED EQUIPARTITION RULE AND COULOMB’S LAW

Our analysis is very similar to Verlinde’s [1]. We will assume the readers are familiar with the ideas and skills in reference [1]. In this way, we can pay attention to our new ingredients and results, while the readers can better reflect on our ideas.

A. Scheme One

There are two schemes to solve the puzzle we brought out. The first choice is to naively replace the equipartition rule (4) with

\[ e^2 \sqrt{M^2 - Q^2} = \frac{1}{2} N k_B T. \]  

(7)

This relation is well-satisfied on the horizon of the RN black hole. In the past, we are familiar with the equipartition rule for energy. When writing down (7), we generalized the rule to \( \sqrt{M^2 - Q^2} \). On the event horizon, the temperature \( T \) is identical to the Bekenstein-Hawking temperature. Outside the horizon, \( T \) can be considered as a generalized Bekenstein-Hawking temperature on the holographic screen. But such a generalization makes sense only if \( M \geq |Q| \). This is a shortcoming of rule (7).

Now suppose there is a test particle with mass \( m \) and charge \( q \) near a holographic screen which encloses a volume with total mass \( M \) and total charge \( Q \). In reference [1], without a charge, it was shown that the Newton’s law emerges as an entropic force

\[ F = T \partial_x S, \]  

(8)

where \( x \) is the emergent coordinate, i.e., the direction perpendicular to the holographic screen. Generalizing the assumption of [1], we write down

\[ \partial_x S = \frac{2 \pi k_B c}{\hbar} \frac{Mm - Qq}{\sqrt{M^2 - Q^2}}. \]  

(9)

When \( Q = 0 \), it reduces to the assumption made in [1], also in the third equation of (12) in the next subsection. In [1] there is an elegant argument for (12), but for (9) we have not found such a nice argument. This is another shortcoming of scheme one.

Nevertheless, by substituting (5), (7) and (9) into (8), we can unify gravity and Coulomb force as one entropic force

\[ F = \frac{G N}{r^2} (Mm - Qq). \]  

(10)

It would be interesting to notice that in the limit \( m \ll M \) and \( q \ll Q \), equation (9) can be reformed as

\[ \partial_x S \simeq \frac{2 \pi k_B c}{\hbar} \sqrt{M^2 - Q^2} = \frac{\pi k_B^2}{\hbar} \delta(NT) \]  

(11)

if we identify \( \delta M = m \), \( \delta Q = q \). This might give us a hint to incorporate the angular momentum, but we will not go that far in this paper.

In the present scheme, masses and charges appear in a mixed form, or namely, in a better-unified form. In the coming subsection, we will propose a scheme where the mass and the charge are “disassociated”.

B. Scheme Two

The second choice is by postulating equipartition rules and entropy changes in the emergent direction as following:

\[ M c^2 = \frac{1}{2} N k_B T_g, \quad \partial_x S_g = \frac{2 \pi k_B mc}{\hbar}, \]

\[ Q c^2 = \frac{1}{2} N k_B T_{em}, \quad \partial_x S_{em} = -\frac{2 \pi k_B q c}{\hbar}. \]  

(12)

Remember that, as mentioned in the very beginning of section [1], we are working in the geometrized unit of charge, in which the Coulomb’s law takes almost the same form as the Newton’s law except for the difference in signature. In our assumptions, the temperature and entropy should be reinterpreted. In this scheme, the gravitational and the electromagnetic parts have their own “temperatures” and “entropies”, as indicated by the subscripts.

The first line of (12) corresponds to the gravitational part, which has been investigated by Verlinde in [1]. The new ingredients are these given in the second line, which are essentially independent of the gravitational part. This line corresponds to the electromagnetic part. Even the holographic screen and the emergent direction for electromagnetic force can be different from those for gravity. Here again we generalized the equipartition rule, to the electric or magnetic charge \( Q \). The electric or magnetic “temperature” \( T_{em} \) is dictated by the average charge per bit. It is important to notice that \( T_{em} \) can be positive or negative, depending on the signature of \( Q \).

This sounds bizarre. However, as will be explained in the next subsection, we can never observe \( T_{em} \) directly. Amusingly, the definition of entropy change in the second line implies a new interpretation for the electric or magnetic charge, parallel to Verlinde’s interpretation for mass [1].

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2 We would like to regard (5) as the definition of entropic force rather than the first law of thermodynamics. Some people take it as the first law of thermodynamics, then the second law of Newton should be implicit, because \( F = \partial_x E = \partial_x (mv^2/2) = mva/v = ma \). If we took their point of view, then Verlinde’s derivation for the second law of Newton [1] would look like circular reasoning.
With these assumptions at hand, it is straightforward to derive Coulomb’s law from the entropic force $F_{em} = T_{em} \partial_r S_{em}$. When the emergent directions of gravity and electromagnetic force coincide, we can reobtain equation (10) as

$$F_g + F_{em} = T_g \partial_x S_g + T_{em} \partial_x S_{em} = \frac{G}{r^2} (Mm - Qq). \quad (13)$$

Now return to our puzzle. In view of this scheme, the temperature $T$ in (13) should be interpreted as gravitational temperature $T_g$, and it does not equal to the Bekenstein-Hawking temperature $T_h$ on the horizon. Then the puzzle is solved.

C. Relations between Schemes and to D-brane

Scheme two has more general applications than scheme one, because scheme one can be applied only if $M \geq |Q|$ and the distribution of Newtonian potential coincides with the distribution of Coulomb potential. It is quite revealing to derive scheme one from scheme two when these conditions are satisfied. This can be done quickly in virtue of the relations below:

$$T^2 = T^2_g - T^2_{em}, \quad (14)$$
$$T \partial_x S = T_g \partial_x S_g - T_{em} \partial_x S_{em}. \quad (15)$$

In subsection II A we mentioned that the temperature in (4) is a natural generalization of the Bekenstein-Hawking entropy outside the event horizon of the RN black hole. From this point of view, only the temperature $T$ on the left hand side of (14) is possibly observable, while $T_{em}$ on the right hand side is never. $T_g$ is not observable if $T_{em} \neq 0$. But the value of observable $T$ is determined by $T_g$ and $T_{em}$ together.

Up to some numerical coefficients, expression (6) holds well in a higher dimensional spacetime. As a result, in appropriate units, expressions (14) and (15) will hold in higher dimensional spacetimes.

Relation (14) reminds us of the temperatures defined in reference [3] in the D-brane picture, where the thermodynamic property of a 5-dimensional near-extremal RN black hole has been studied. Here we copy some relations from [3] in their convention of notations and normalization. In that reference, they defined the left-moving temperature and the right-moving temperature as

$$T_L = \frac{2}{\pi R} \sqrt{\frac{N_L}{Q_1 Q_5}},$$
$$T_R = \frac{2}{\pi R} \sqrt{\frac{N_R}{Q_1 Q_5}}, \quad (16)$$

where $N_L$ is the quantized number of left-moving momentum and $N_R$ is the right-moving one. In the case $N_L \gg N_R \gg 1$, they found the Bekenstein-Hawking temperature is identical to the right-moving temperature $T_R$.

Stimulated by the D-brane picture, we conjecture that the temperatures in (14) can be identified as $T_g = T_L$ and $T = T_R$. Then making use of some relations in [6], we find

$$T^2_{em} = T^2_L - T^2_R = \left(\frac{2}{\pi R}\right)^2 \frac{N_L - N_R}{Q_1 Q_5} = \left(\frac{2}{\pi r_e}\right)^2, \quad (17)$$
in which $r_e$ is horizon radius of the near-extremal black hole. The same result can be obtained by generalizing [12] to five dimensions.

III. FROM ENTROPIC FORCE TO MAXWELL’S EQUATIONS

A. To Poisson’s Equation

In scheme two of the previous section, we treat the electromagnetic force and gravity independently. This scheme can be extended to general matter and charge distributions. As argued in [1], the holographic screens correspond equipotential surfaces, so it seems natural to define the gravitational temperature generally by the gradient of Newtonian potential

$$k_B T_g = \frac{\hbar}{2 \pi c} \nabla \Phi_g. \quad (18)$$

If one generalizes relation (5) to the form

$$dN = \frac{c^3}{G_N \hbar} dA, \quad (19)$$

the Poisson equation for gravity can be perfectly derived [1].

Likewise, we define the electromagnetic temperature generally by the gradient of Coulomb potential

$$k_B T_{em} = -\frac{\hbar}{2 \pi c} \nabla \Phi_{em}. \quad (20)$$

Once again we choose the geometrized unit. To recover the familiar unit, one has to replace $\Phi_{em} \rightarrow \sqrt{4 \pi \varepsilon_0 G_N} \Phi_{em}$. Then making use of the integral expression of the equipartition rule for charge

$$Q c^2 = \frac{1}{2} k_B \int_{\partial V} T_{em} dN \quad (21)$$

and the Gauss’ theorem, we straightforwardly work out

$$Q = -\frac{1}{4 \pi G_N} \int_V \nabla^2 \Phi_{em} dV \quad (22)$$

and hence the Poisson equation

$$\nabla^2 \Phi_{em} = -4 \pi G_N \rho_{em}. \quad (23)$$

Transformed from the geometrized unit to the ordinary unit, it takes the familiar form

$$\nabla^2 \Phi_{em} = -\frac{1}{\varepsilon_0} \rho_{em}. \quad (24)$$
B. To Maxwell’s Equations

With the Poisson equation at hand, the derivation of Maxwell’s equations is quite similar to or even simpler than the derivation of Einstein’s equations in [1], where Jacobson’s method [2] was followed. In this subsection, we will sketch the procedure but not go into details.

In order to derive the Maxwell equations, we need a time-like Killing vector $\xi^a$, just like in [1]. Then $\rho_{em}$ and $\nabla^2 \Phi_{em}$ can be recast into the covariant form

$$\rho_{em} = \xi_a j^a,$$

$$\nabla^2 \Phi_{em} = \frac{1}{\sqrt{-g}} \xi_a \partial_b \left( \sqrt{-g} F^{ab} \right).$$

(25)

Subsequently (24) is generalized to the covariant form

$$\frac{1}{\sqrt{-g}} \xi_a \partial_b \left( \sqrt{-g} F^{ab} \right) = -\frac{1}{\xi_0} \xi_a j^a.$$  

(26)

Following the trick in [1] and applying Jacobson’s reasoning [2] to time-like screens, one can obtain the Maxwell equations with a source current $j^a$.

IV. DISCUSSION

The results in this paper are tentative rather than conclusive. We will make some comments to help the readers better understand our results and ponder on them. After a preliminary version of this paper was posted in the e-Print archive [3], some issues mentioned above were later discussed from different viewpoints in the literature [8–11]. The comments below are made after considering the literature.

We are motivated by the RN black hole where $M \geq |Q|$, but most of the analysis does not rely on the RN solution or the condition $M \geq |Q|$. Actually, in most parts we treat the gravity and electromagnetic force independently. Therefore, as long as the gravitational background is negligible, we can study the pure electrodynamics with the holographic principle and the generalized “equipartition rules”. As clarified in [2], the key point is a straightforward generalization from gravitational “charge” (mass) to electric charge. The “puzzle” proposed in the beginning of this paper was solved in a better way in [10], and it still does not affect the entropic derivation of electromagnetic force here.

In our formulation, the electromagnetic temperature could be negative. This looks bizarre. Especially, this impairs the physical significance of the formally thermodynamical interpretation. At first sight, one may avoid it by focusing on the systems with the same signature of charge. If there is no signature difference in charges, there will be no signature difference in temperatures, and then one can always choose a positive signature. However, in that situation we can only study the repulsive Coulomb force. When physically interpreting our mathematical formulation, the problem of negative temperature is confronted as a serious obstacle. The authors in [11] met the same problem when they try to accommodate inflation in the entropic force scenario. At the moment, we do not have a good solution to this problem. We hope there will be progress in the future.

In the above, we investigated the electrostatics, which is trivial to generalize to incorporate the static magnetic field. Although the Maxwell equations have been derived based on charge integral, the time-dependent case has not been studied and would be more complicated. Of course, the time-dependent gravitational background is even harder to tame.

In subsection [11] we conjectured that our postulates could have a close relation to the D-brane picture of the thermodynamics of RN black holes. We conjectured the correspondence $T_g = T_L, T = T_R$. It will be necessary to do more serious and extensive comparison with the quantities given in [6].

No matter how we formulate the theories and change our logic, the Nature goes in its own way. It is possible that Newton’s law originates from entropic force but Coulomb’s law does not, or neither does. In the past, we have collected a lot of evidence that the gravitational force has a holographic nature. Little for the electromagnetic force. On the other hand, we have a quantum theory of electrodynamics to explain its microscopic degrees of freedom, but this is not the case for gravity. Here the motivation is to reconcile Verlinde’s equipartition rule on the horizon of RN black hole. The other support is given by the apparent similarity between Newton’s law and Coulomb’s law. If they really go in the same way, we hope the formulation in this paper can be regarded as a form of unification of gravity and electromagnetic force.

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