STOCHASTIC PDES IN $S'$ FOR SDES DRIVEN BY LÉVY NOISE

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ABSTRACT. In this article we show that a finite dimensional stochastic differential equation driven by a Lévy process can be formulated as a stochastic partial differential equation. We prove the existence and uniqueness of strong solutions of such stochastic PDEs. The solutions that we construct have the ‘translation invariance’ property. The special case of this correspondence for diffusion processes was proved in [Rajeev, Translation invariant diffusion in the space of tempered distributions, Indian J. Pure Appl. Math. 44 (2013), no. 2, 231–258].

1. Introduction

In this article we show that a finite dimensional stochastic differential equation (abbrev. SDE) driven by a Lévy process can be formulated as a stochastic partial differential equation (abbrev. SPDE). The goal of this article is to prove the existence and uniqueness of strong solutions of such SPDEs.

Given a Brownian motion $\{B_t\}$ and an independent Poisson random measure $N$ driven by a Lévy measure $\nu$, we consider an SDE in $\mathbb{R}^d$, of the form

$$U_t = \kappa + \int_0^t \bar{b}(U_s; \xi) \, ds + \int_0^t \bar{\sigma}(U_s; \xi) \cdot dB_s + \int_0^t \int_{|x|<1} \bar{F}(U_s, x; \xi) \tilde{N}(dsdx)$$

and the corresponding SPDE in the space of tempered distributions $S'$, (more specifically, in a Hermite-Sobolev space $S_{-p}$) viz.

$$Y_t = \xi + \int_0^t A(Y_s-) \cdot dB_s + \int_0^t \tilde{L}(Y_s-) \, ds$$

$$+ \int_0^t \int_{|x|<1} (\tau_F(Y_{s-}, x) - Id) Y_{s-} \tilde{N}(dsdx) + \int_0^t \int_{|x|\geq 1} (\tau_G(Y_{s-}, x) - Id) Y_{s-} N(dsdx),$$

where $\xi$ is an $S_{-p}$ valued $\mathcal{F}_0$-measurable random variable and $\kappa$ is an $\mathbb{R}^d$ valued $\mathcal{F}_0$-measurable random variable (see Sections 2, 3 and 4 for notations).

The study of such correspondence was initiated in [20, 21] for diffusion processes with a deterministic initial condition. In [4], this was extended to random initial conditions, which require some technical conditions on the coefficients of the diffusion processes. For diffusion processes this correspondence together with the pathwise uniqueness of (1.2) actually leads to strong solutions of
when the diffusion and drift coefficients are ‘rough’ (see Proposition 4.1). The question of when the solutions of an SPDE can be realised on finite dimensional submanifolds has also been studied recently in the context of the HJM model in finance (see [8], [9], [12]). A key feature of our correspondence and of independent interest is that the diffusion, drift and jump coefficients viz. $\bar{\sigma}$, $\bar{b}$, $\bar{F}$, $\bar{G}$ for the finite dimensional SDE can be written as a convolution involving the initial condition $\xi$ of the SPDE (see [21], Remark 3.7). The correspondence between SPDE’s and SDE’s also extends to the flows generated by the SDE’s (see [3], [24]).

In this paper we show the local existence and uniqueness of strong solutions to the SPDE (Theorems 4.3 and 4.7). The existence is shown by an explicit construction of the solution $\{Y_t\}$ of the SPDE as a translate of the initial condition $\xi$ by the solutions $\{U_t\}$ of the finite dimensional SDE i.e. $Y_t = \tau_{U_t}(\xi)$. Here $\tau_x : \mathbb{R}^d \to \mathbb{R}^d, x \in \mathbb{R}^d$ denote the translation operators. This requires a generalisation of the Itô formula in [20] for continuous semi-martingales with a non-random initial condition to semi-martingales with jumps and in particular to Lévy processes. This was done in [3], where an existence theorem for the SPDE (1.2) was also proved for a sub class of SDE’s than those considered here (3 Theorem 4.7, [2]). See also [25] for a related Itô formula.

The uniqueness result uses the technique of the ‘Monotonicity inequality’ (see [6], [14], [19]). The main difference between the present case and the cases treated earlier in the references above is the addition of the jump terms in the SPDE. The large jumps are easily handled by a boundedness assumption. Estimate for the small jump terms (see Term 2 in (1.9)) require the 'Monotonicity inequality' and involves a second order Taylor expansion of the functions of the form $v \in [0, 1] \to <\tau_z \psi, \phi>$ where $\psi$ is a tempered distribution, $\phi$ is a suitable test function and $z \in \mathbb{R}^d$, thereby reducing it to the case of second order constant coefficient differential operators as in [6]. If the Lévy measure $\nu$ is bounded, then the required estimate follows provided the coefficient appearing in the small jump terms is bounded (see Remark 4.5).

The proofs for the existence and uniqueness of local strong solutions of the SDEs use standard techniques (e.g. those used in [1], [15]), but requires growth and continuity assumptions of the coefficients which involve an additional parameter. The assumptions are stated in (σb), (loc-Lip), (F1), (F2), (F3) and (G1) and the proofs using these assumptions are given in [7].

Our proof of local existence and uniqueness for the SPDE involves conditions (F1), (F2), (F3) and (G1), (G2) on the coefficients involving the small and large jumps (Theorem 4.3). In Theorem 4.3, using the well known ‘interlacing technique’ for Lévy processes, we eliminate the condition (G2) involving the large jumps.

In this article we restrict ourselves to the study of the relationship between (1.1) and (1.2). Our results are proved in the framework of the Hermite-Sobolev spaces. For formulations of SPDEs in these spaces see [17], [18]. We also refer to [27] for an SPDE associated to branching measure valued processes and canonically linked to Brownian motion, and to [19], [11], [11] for the connection between measure valued processes and finite dimensional diffusions. In [26], an analogous class of processes arise in the study of interacting particle systems wherein the finite dimensional SDE represents the microscopic motion of a ‘tagged’ particle and the SPDE describes the macroscopic behaviour of a system of particles.

2. Preliminaries

2.1. Topology. Let $S$ be the space of rapidly decreasing smooth functions on $\mathbb{R}^d$ with dual $S'$, the space of tempered distributions (see [17]). Let $\mathbb{Z}^d_+ := \{n = (n_1, \ldots, n_d) : n_i \text{ non-negative integers}\}$. If $n \in \mathbb{Z}^d_+$, we define $|n| := n_1 + \cdots + n_d$. 
For $p \in \mathbb{R}$, consider the increasing norms $\| \cdot \|_p$, defined by the inner products

$$
\langle f, g \rangle_p := \sum_{n \in \mathbb{Z}^d_+} (2n + d)^{2p} \langle f, h_n \rangle \langle g, h_n \rangle, \quad f, g \in \mathcal{S}.
$$

In the above equation, $\{h_n : n \in \mathbb{Z}^d_+\}$ is an orthonormal basis for $\mathcal{L}^2(\mathbb{R}^d, dx)$ given by the Hermite functions and $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathcal{L}^2(\mathbb{R}^d, dx)$. For $d = 1$, $h_n(t) := (2\pi n!)^{-1/2} \exp(-t^2/2)H_n(t)$, where $H_n, t \in \mathbb{R}$ are the Hermite polynomials (see [17]). For $d > 1$, $h_n(x_1, \cdots, x_d) := h_n(x_1) \cdots h_n(x_d)$ for all $(x_1, \cdots, x_d) \in \mathbb{R}^d, n \in \mathbb{Z}^d_+$, where the Hermite functions on the right-hand side are one-dimensional. We define the Hermite-Sobolev spaces $\mathcal{S}_p, p \in \mathbb{R}$ as the completion of $\mathcal{S}$ in $\| \cdot \|_p$. Note that the dual space $\mathcal{S}_p'$ is isometrically isomorphic with $\mathcal{S}_{-p}$ for $p \geq 0$ and $\langle \cdot, \cdot \rangle$ extends the $\mathcal{L}^2$ inner product to the duality between $\mathcal{S}$ and $\mathcal{S}'$. We also have $\mathcal{S} = \bigcap_p (\mathcal{S}_p, \| \cdot \|_p), \mathcal{S}' = \bigcup_{p > 0} (\mathcal{S}_{-p}, \| \cdot \|_{-p})$ and $\mathcal{S}_0 = \mathcal{L}^2(\mathbb{R}^d)$.

Consider the derivative maps denoted by $\partial_i : \mathcal{S} \to \mathcal{S}$ for $i = 1, \cdots, d$. We can extend these maps by duality to $\partial_i : \mathcal{S}' \to \mathcal{S}'$ as follows: for $\psi \in \mathcal{S}'$,

$$
\langle \partial_i \psi, \phi \rangle := -\langle \psi, \partial_i \phi \rangle, \quad \forall \phi \in \mathcal{S}.
$$

Let $\{e_i : i = 1, \cdots, d\}$ be the standard basis vectors in $\mathbb{R}^d$. Then for any $n = (n_1, \cdots, n_d) \in \mathbb{Z}^d_+$ we have (see [16] Appendix A.5)]

$$
\partial_i h_n = \sqrt{\frac{n_i}{2} h_{n-e_i} - \sqrt{\frac{n_i + 1}{2}} h_{n+e_i},}
$$

with the convention that for a multi-index $n = (n_1, \cdots, n_d)$, if $n_k < 0$ for some $i$, then $h_n \equiv 0$. The above recurrence relation implies that $\partial_i : \mathcal{S}_p \to \mathcal{S}_{p-\frac{d}{2}}$ is a bounded linear operator. For $x \in \mathbb{R}^d$, let $\tau_x$ denote the translation operators on $\mathcal{S}$ defined by $(\tau_x \phi)(y) := \phi(y - x), \forall y \in \mathbb{R}^d$. These operators can be extended to $\mathcal{S}' : \mathcal{S}' \to \mathcal{S}'$ by

$$
\langle \tau_x \phi, \psi \rangle := (\phi, \tau_{-x} \psi), \forall \psi \in \mathcal{S}.
$$

For $x \in \mathbb{R}^d, |x|$ will denote its Euclidean norm.

**Proposition 2.1.** The translation operators $\tau_x, x \in \mathbb{R}^d$ have the following properties:

(a) For $x \in \mathbb{R}^d$ and any $p \in \mathbb{R}$, $\tau_x : \mathcal{S}_p \to \mathcal{S}_p$ is a bounded linear map. In particular, there exists a real polynomial $P_k$ of degree $k = 2(\|p\| + 1)$ such that

$$
\|\tau_x \phi\|_p \leq P_k(\|x\|)\|\phi\|_p, \forall \phi \in \mathcal{S}_p.
$$

(b) For any $x \in \mathbb{R}^d$ and any $i = 1, \cdots, d$ we have $\tau_x \partial_i = \partial_i \tau_x$.

(c) Fix $\phi \in \mathcal{S}_p$, for some $p \in \mathbb{R}$. The map $x \in \mathbb{R}^d \mapsto \tau_x \phi \in \mathcal{S}_p$ is continuous.

**Proof.** See [23] Theorem 2.1 for the proof of part (a) and the proof of [24] Proposition 3.1 for the proof of part (c). Part (b) is well-known. \(\square\)

**Proposition 2.2** (H Proposition 3.8)). Let $p > d + \frac{1}{2}$. Then for any $\psi \in \mathcal{S}_{p+\frac{1}{2}}$ and any positive integer $n$, there exists a constant $D(n) > 0$ such that for all $x_1, x_2 \in \mathbb{R}^d$ with $|x_1|, |x_2| \leq n$, we have

$$
\|\tau_{x_1} \psi - \tau_{x_2} \psi\|_p \leq D(n)\|\psi\|_{p+\frac{1}{2}}|x_1 - x_2|.
$$

In particular, for any bounded set $\mathcal{K}$ in $\mathcal{S}_{p+\frac{1}{2}}$ and any positive integer $n$, there exists a constant $D(\mathcal{K}, n) > 0$ such that for all $x_1, x_2 \in \mathbb{R}^d$ with $|x_1|, |x_2| \leq n$, we have

$$
\|\tau_{x_1} \psi - \tau_{x_2} \psi\|_p \leq D(\mathcal{K}, n)|x_1 - x_2|, \forall \psi \in \mathcal{K}.
$$
Proof. The proof is contained in the proof of [1] Proposition 3.8, specifically, in the arguments after [1] equation (3.16)].

Let \( \sigma = (\sigma_{ij}) \) be a constant \( d \times r \) matrix with \( (a_{ij}) = (\sigma \sigma^t)_{ij} \) and \( b = (b_1, ..., b_d) \in \mathbb{R}^d \). For \( \phi \in S \), we define

\[
L \phi := \frac{1}{2} \sum_{i,j=1}^d a_{ij} \partial_{ij}^2 \phi - \sum_{i=1}^d b_i \partial_i \phi,
\]

\[
A_i \phi := -\sum_{j=1}^d \sigma_{ji} (\partial_j \phi), \quad i = 1, \ldots, r
\]

\[
A \phi = (A_1 \phi, \ldots, A_r \phi)
\]

**Theorem 2.3** ([11] Theorem 2.1 and Remark 3.1). For every \( p \in \mathbb{R} \), there exists a positive constant \( C = C(p, d, (\sigma_{ij}), (b_j)) \), such that

\[
2 \langle \phi, L \phi \rangle_p + \|A \phi\|_{H^p(S)}^2 \leq C \|\phi\|_p^2
\]

for all \( \phi \in S \), where \( \|A \phi\|_{H^p(S)}^2 := \sum_{i=1}^r \|A_i \phi\|_p^2 \). Furthermore, by density arguments the above inequality can be extended to all \( \phi \in S_{p+1} \). The constant \( C \) depends on \( \sigma_{ij}, b_i \) through the maximum of \( |\sigma_{ij}|, |b_i| \) and hence the inequality can be extended to the case where \( \sigma, b \) are bounded processes parametrized by some set.

**2.2. An Itô formula.** Let \((\Omega, F, \{F_t\}_{t \geq 0}, P)\) be a filtered complete probability space satisfying the usual conditions viz. \( F_0 \) contains all \( A \in F \), s.t. \( P(A) = 0 \) and \( F_t = \bigcap_{s \geq t} F_s, t \geq 0 \). Given two real valued semimartingales \( \{X^1_t\} \) and \( \{X^2_t\} \), let \( \{(X^1_t, X^2_t)\} \) denote the continuous part of the covariation process \( \{(X^1_t, X^2_t)\} \).

**Theorem 2.4.** Let \( p > 0 \). Let \( \xi \) be an \( S_{-p} \) valued random variable. Let \( \{X_t\} \) be an \( \mathbb{R}^d \) valued \( F_0 \) semimartingale with \( X_t = (X^1_t, \cdots, X^d_t) \). Then \( \{\tau(X_t, \xi)\} \) is an \( S_{-p} \) valued semimartingale and

\[
\sum_{s \leq t} \left[ \tau(X_s, \xi) - \tau(X_{s-}, \xi) + \sum_{i=1}^d (\Delta X_s^i \partial_i \tau(X_{s-}, \xi)) \right]
\]

is an \( S_{-p-1} \) valued process of finite variation and we have the following equality in \( S_{-p-1}, \text{a.s.} \)

\[
\tau(X_t, \xi) = \tau(X_0, \xi) - \sum_{i=1}^d \int_0^t \partial_i \tau(X_{s-}, \xi) \, dX_s^i + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \partial_{ij}^2 \tau(X_{s-}, \xi) \, d[X^i, X^j]_s + \sum_{s \leq t} \left[ \tau(X_s, \xi) - \tau(X_{s-}, \xi) + \sum_{i=1}^d (\Delta X_s^i \partial_i \tau(X_{s-}, \xi)) \right], \quad t \geq 0.
\]

**Proof.** The case when \( \xi \) is deterministic was proved in [3] Theorem 4.5. We indicate the proof for a random \( \xi \) via two observations.

(i) Recall that \( \partial_i : S_q \to S_{q-\frac{1}{2}}, 1 \leq i \leq d \) are bounded linear operators for every \( q \in \mathbb{R} \). By Proposition 2.1, the processes \( \{\tau(X_t, \xi)\}, \{\partial_i \tau(X_t, \xi)\}, \{\partial_{ij} \tau(X_t, \xi)\}, 1 \leq i, j \leq d \) are locally norm bounded predictable processes with values in \( S_{-p}, S_{-p-\frac{1}{2}}, 1 \leq i \leq d \) respectively. Hence the integrals \( \int_0^t \partial_i \tau(X_{s-}, \xi) \, dX_s^i \), \( \int_0^t \partial_{ij} \tau(X_{s-}, \xi) \, d[X^i, X^j]_s \), \( 1 \leq i, j \leq d \) exist.

(ii) Given any \( F_0 \) measurable set \( F \), an \( S_{-p} \) valued predictable step process \( \{G_t\} \) and an \( \mathbb{R}^d \) valued rcll semimartingale \( \{X_t\} \), we have a.s.

\[
\mathbbm{1}_F \int_0^t G_s dX_s = \int_0^t 1_F G_s dX_s, \quad t \geq 0.
\]
This equality can be extended to the case involving locally norm-bounded $S_{-p}$ valued predictable process $\{G_t\}$. Again, given any $F_0$ measurable set $F$, $\phi \in S_{-p}$, $\psi \in S$ and $x \in \mathbb{R}^d$ we have

$$\langle 1_F \tau_x \phi, \psi \rangle = 1_F \langle \tau_x \phi, \psi \rangle = 1_F \langle \phi, \tau_x \psi \rangle = \langle 1_F \phi, \tau_x \psi \rangle$$

(2.7)

and hence $1_F \tau_x \phi = \tau_x (1_F \phi)$. Similarly $1_F \tau_x \phi = \tau_x (1_F \phi)$.

Since $\Omega = \bigcup_{M=1}^{\infty} \{\omega : \|\xi(\omega)\|_{-p} \leq M\}$, it is enough to establish (2.5) for almost every $\omega$ in $\{\omega : \|\xi(\omega)\|_{-p} \leq M\}$ for every fixed positive integer $M$. Multiplying (2.5) by $1_{\{\|\xi\|_{-p} \leq M\}}$, it is enough to establish the result when $\xi$ is norm bounded.

From [3, Theorem 4.5] and (2.6), (2.7) we can establish the required result when $\xi$ is an $S_{-p}$ valued simple $F_0$ measurable random variable. A limiting argument then proves the result when $\xi$ is norm bounded. This completes the proof. \qed

3. Finite dimensional SDEs

3.1. Setup and Notations. We use the following notations throughout the paper.

- The set of positive integers will be denoted by $\mathbb{N}$. Recall that for $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm. The transpose of any element $x \in \mathbb{R}^{n \times m}$ will be denoted by $x^t$.
- For any $r > 0$, define $O(0,r) := \{x \in \mathbb{R}^d : |x| < r\}$. Then $O(0,r) = \{x \in \mathbb{R}^d : |x| \leq r\}$.
- Let $(\Omega, F, \{F_t\}_{t \geq 0}, P)$ be a filtered complete probability space satisfying the usual conditions viz. $F_t$ contains all $A \in F$, s.t. $P(A) = 0$ and $F_t = \bigcap_{s \geq t} F_s$, $t \geq 0$.
- Let $p > 0$. Let $\sigma = (\sigma_{ij})_{d \times d}$, $b = (b_1, \cdots, b_d)^t$ be such that $\sigma_{ij}, b_i : \Omega \to S_p$ are $F_0$ measurable and

$$\beta := \sup\{\|\sigma_{ij}(\omega)\|_p, \|b_i(\omega)\|_p : \omega \in \Omega, 1 \leq i,j \leq d\} < \infty.$$

- Define $\bar{\sigma} : \Omega \times \mathbb{R}^d \times S_{-p} \to \mathbb{R}^{d \times d}$ and $\bar{b} : \Omega \times \mathbb{R}^d \times S_{-p} \to \mathbb{R}^d$ by $\bar{\sigma}(\omega, z; y) := (\sigma(\omega), \tau_z y)$ and $\bar{b}(\omega, z; y) := (b(\omega), \tau_z y)$, where $((\sigma(\omega), \tau_z y))_{ij} := (\sigma_{ij}(\omega), \tau_z y)$ and $((b(\omega), \tau_z y))_i := (b_i(\omega), \tau_z y)$.
- Let $F : \Omega \times S_{-p} \times O(0,1) \to \mathbb{R}^d$ and $G : \Omega \times S_{-p} \times O(0,1)^c \to \mathbb{R}^d$ be $F_0 \otimes B(S_p) \otimes B(O(0,1))$, measurable respectively. Here $B(K)$ denotes the Borel $\sigma$-field of set $K$.
- Define $\bar{F} : \Omega \times \mathbb{R}^d \times O(0,1) \times S_{-p} \to \mathbb{R}^d$, $\bar{G} : \Omega \times \mathbb{R}^d \times O(0,1)^c \times S_{-p} \to \mathbb{R}^d$ by $\bar{F}(\omega, z; x; y) := F(\omega, \tau_z y, x)$, $\bar{G}(\omega, z; x; y) := G(\omega, \tau_z y, x)$.
- Let $\{B_t\}$ denote a standard Brownian motion and let $N$ denote a Poisson random measure driven by a Lévy measure $\nu$. $\bar{N}$ will denote the corresponding compensated random measure. We also assume that $B$ and $N$ are independent.
- In our arguments, at times we use time intervals of the form $[0,T]$. In such cases, $T$ will always assumed to be finite i.e. $[0, T]$ will be a finite time interval.
- Given a process $\{X_t\}$ and a stopping time $\eta$, the stopped process $\{X_t^\eta\}$ is defined as $X_t^\eta := X_{t \wedge \eta}$.
Consider the following SDE in $\mathbb{R}^d$,

$$
dU_t = b(U_{t-}; \xi)dt + \sigma(U_{t-}; \xi) \cdot dB_t + \int_{(0<|x|<1)} \bar{F}(U_{t-}, x; \xi) \tilde{N}(dtdx)
$$

(3.1)

$$
+ \int_{(|x|\geq 1)} G(U_{t-}, x; \xi) N(dtdx), \quad t \geq 0
$$

$$
U_0 = \kappa,
$$

where $\xi$ is an $\mathcal{S}_{-p}$ valued $\mathcal{F}_0$-measurable random variable and $\kappa$ is an $\mathbb{R}^d$ valued $\mathcal{F}_0$-measurable random variable. Unless stated otherwise, $\xi$ and $\kappa$ will be taken to be independent of the noise $B$ and $N$. Note that the $i$-th component of $\int_0^t \sigma(U_{s-}; \xi) \cdot dB_s$ is $\sum_{j=1}^d \int_0^t \sigma_{ij}(U_{s-}; \xi) dB^j_s$. We list some hypotheses.

(F1) For all $\omega \in \Omega$ and $x \in \mathcal{O}(0, 1)$ there exists a constant $C_x \geq 0$ s.t.

$$
|F(\omega, y_1, x) - F(\omega, y_2, x)| \leq C_x \|y_1 - y_2\|_{-p-\frac{1}{2}}, \forall y_1, y_2 \in \mathcal{S}_{-p}.
$$

(3.2)

We assume $C_x$ to depend only on $x$ and independent of $\omega$. Since $\|y\|_{-p-\frac{1}{2}} \leq \|y\|_{-p}, \forall y \in \mathcal{S}_{-p}$, we have

$$
|F(\omega, y_1, x) - F(\omega, y_2, x)| \leq C_x \|y_1 - y_2\|_{-p}, \forall y_1, y_2 \in \mathcal{S}_{-p}.
$$

(F2) The constant $C_x$ mentioned above has the following properties, viz.

$$
\sup_{|x|<1} C_x < \infty, \quad \int_{(0<|x|<1)} C_x^2 \nu(dx) < \infty.
$$

(F3) $\sup_{\omega \in \Omega, |x|<1} |F(\omega, 0, x)| < \infty$ and $\sup_{\omega \in \Omega} \int_{(0<|x|<1)} |F(\omega, 0, x)|^2 \nu(dx) < \infty$.

(G1) The mapping $y \mapsto G(\omega, y, x)$ is continuous for all $x \in \mathcal{O}(0, 1)^c$ and $\omega \in \Omega$.

(G2) For every bounded set $\mathcal{K}$ in $\mathcal{S}_{-p},$

$$
\sup_{\omega \in \Omega, y \in \mathcal{K}, x \in \mathcal{O}(0, 1)^c} |G(\omega, y, x)| < \infty.
$$

Example 3.1. Examples of coefficient $F$ satisfying (F1) (F2) and (F3) can be constructed as follows. Choose a function $h : \Omega \to \mathbb{R}$ which is bounded and $\mathcal{F}_0$ measurable. Next choose Borel measurable $f_1 : \mathcal{O}(0, 1) \to \mathbb{R}$ with $f_1 \in L^2(\mathcal{O}(0, 1), \nu) \cap L^\infty$. Fix $\gamma_1, \cdots, \gamma_d \in \mathcal{S}_{-\frac{1}{2}}$ and consider the function $f_2 : \mathcal{S}_{-p} \to \mathbb{R}^d$ defined by $f_2(y) := (\langle \gamma_1, y \rangle, \cdots, \langle \gamma_d, y \rangle)^t$. Note that $\langle \gamma_1, y \rangle$ etc. are duality actions, since $\mathcal{S}_{-p} \subset \mathcal{S}_{-\frac{1}{2}} \cong (\mathcal{S}_{\frac{1}{2}})^\prime$ and hence $f_2$ is Lipschitz in the $\|\cdot\|_{-p-\frac{1}{2}}$ norm. Then the function $F(\omega, y, x) := h(\omega)f_1(x)f_2(y)$ satisfies the required assumptions. Examples of coefficient $G$ satisfying (G1) and (G2) can be constructed as follows. Take any bounded Borel measurable function $g_1 : \mathcal{O}(0, 1)^c \to \mathbb{R}$ and let $h$ be as above. Fix $\gamma_1, \cdots, \gamma_d \in \mathcal{S}_p$ and consider the function $g_2 : \mathcal{S}_{-p} \to \mathbb{R}^d$ defined by $g_2(y) := (\langle \gamma_1, y \rangle, \cdots, \langle \gamma_d, y \rangle)^t$. Then the function $G(\omega, y, x) := h(\omega)g_1(x)g_2(y)$ satisfies the required assumptions. Finite linear combinations of such functions are also examples of $F$ and $G$.

We also require certain Lipschitz regularity of the coefficients of (3.1). For the sake of convenience, we state the hypothesis here.
Lemma 3.2. Let \([\mathbf{F1}], \mathbf{F2}\) and \(\mathbf{F3}\) hold. Then, for any bounded set \(K\) in \(S_{-p}\) the following are true.

(i) \(\sup_{\omega \in \Omega, y \in K, |x| < 1} |F(\omega, y, x)| < \infty\).

(ii) \(\sup_{\omega \in \Omega, y \in K} \int_{0 < |x| < 1} |F(\omega, y, x)|^2 \nu(dx) =: \alpha(K) < \infty\).

(iii) \(\sup_{\omega \in \Omega, y \in K} \int_{0 < |x| < 1} |F(\omega, y, x)|^4 \nu(dx) ds < \infty\) for all 0 \(\leq t < \infty\).

Proof. For \(y \in S_{-p}\),

\(\tag{3.3} |F(\omega, y, x)| \leq |F(\omega, y, x) - F(\omega, 0, x)| + |F(\omega, 0, x)| \leq C_x \|y\|_{-p} + |F(\omega, 0, x)|.

Then, for any bounded set \(K\) in \(S_{-p}\),

\(\tag{3.4} \sup_{\omega \in \Omega, y \in K, |x| < 1} |F(\omega, y, x)| \leq (\sup_{x \in K} C_x) \cdot (\sup_{y \in K} \|y\|_{-p}) + \sup_{\omega \in \Omega, |x| < 1} |F(\omega, 0, x)| < \infty.

Now,

\(\tag{3.5} |F(\omega, y, x)|^2 \leq 2C_x^2 \|y\|_{-p}^2 + 2|F(\omega, 0, x)|^2.

Then for \(y \in K\),

\(\tag{3.6} \int_{0 < |x| < 1} |F(\omega, y, x)|^2 \nu(dx) \leq 2 \sup_{y \in K} \|y\|_{-p}^2 \int_{0 < |x| < 1} C_x^2 \nu(dx) + 2 \int_{0 < |x| < 1} |F(\omega, 0, x)|^2 \nu(dx).

From (3.6), (ii) follows. Combining part (i) and (ii), (iii) follows. This completes the proof. \(\square\)

Using the continuity result in Proposition 2.1 the next result follows.

Lemma 3.3. Suppose \(\mathbf{G1}\) holds. Then the map \(z \in \mathbb{R}^d \to \tilde{G}(\omega, z, \tau; \xi(\omega)) = G(\omega, \tau; \xi(\omega), x) \in \mathbb{R}^d\) is continuous for all \(x \in \mathcal{O}(0, 1)^c\) and \(\omega \in \Omega\).

3.2. Global Lipschitz coefficients. We first consider the existence and uniqueness of solutions for the reduced equation, viz.

\(\tag{3.7} dU_t = \tilde{b}(U_{t-}; \xi) dt + \tilde{\sigma}(U_{t-}; \xi) \cdot dB_t + \int_{0 < |x| < 1} \tilde{F}(U_{t-}, x; \xi) \cdot \tilde{N}(dt dx), \quad t \geq 0\)

\(U_0 = \kappa;\)

with \(\xi\) and \(\kappa\) as in (3.1).

Theorem 3.4 \((\mathbb{F})\). Let \(\mathbf{(\sigma b)}, \mathbf{F1}, \mathbf{F2}\) and \(\mathbf{F3}\) hold. Suppose the following conditions are satisfied.

(i) \(\kappa, \xi\) are \(\mathcal{F}_0\) measurable, as stated in \((\mathbb{G})\).
(ii) (Global Lipschitz in $z$, locally in $y$) For every bounded set $K$ in $S_{-p}$, there exists a constant $C(K) > 0$ such that for all $z_1, z_2 \in \mathbb{R}^d$, $y \in K$ and $\omega \in \Omega$
\[|\bar{b}(\omega, z_1; y) - \bar{b}(\omega, z_2; y)|^2 + |\bar{\sigma}(\omega, z_1; y) - \bar{\sigma}(\omega, z_2; y)|^2\]
\[+ \int_{(0<|x|<1)} |\bar{F}(\omega, z_1; y) - \bar{F}(\omega, z_2; x; y)|^2 \nu(dx) \leq C(K) |z_1 - z_2|^2.\]
(3.8)

Then (3.7) has an $(\mathcal{F}_t)$ adapted strong solution $\{X_t\}$ with roll paths. Pathwise uniqueness of solutions also holds, i.e. if $\{X'_t\}$ is another such solution, then $P(X_t = X'_t, t \geq 0) = 1$.

We now consider the SDE (3.1). The next result follows by the interlacing technique (see [1, Example 1.3.13, pp. 50-51]). The arguments run similar to [1, Theorem 6.2.9].

**Theorem 3.5 ([1]).** Suppose all the assumptions of Theorem 3.4 hold. In addition, assume that $(G1)$ holds. Then there exists a unique roll adapted solution to (3.1).

### 3.3. Local Lipschitz coefficients.

Let $\widehat{\mathbb{R}}^d := \mathbb{R}^d \cup \{\infty\}$ be the one point compactification of $\mathbb{R}^d$. The next result is an extension of Theorem 3.6 for ‘global Lipschitz’ coefficients to ‘local Lipschitz’ coefficients.

**Theorem 3.6 ([1]).** Let $(\hat{\sigma}, \hat{b}, (F1), (F2), (F3), (loc-Lip))$ and $(G1)$ hold. Then there exists an $(\mathcal{F}_t)$ stopping time $\eta$ and an $(\mathcal{F}_t)$ adapted $\mathbb{R}^d$ valued process $\{X_t\}$ with roll paths such that $\{X_t\}$ solves (3.1) upto time $\eta$ and $X_\eta = \infty$ for $t \geq \eta$. Further $\eta$ can be identified as follows: $\eta = \lim_m \theta_m$ where $\{\theta_m\}$ are $(\mathcal{F}_t)$ stopping times defined by $\theta_m := \inf\{t \geq 0 : |X_t| \geq m\}$.

This is also pathwise unique in this sense: if $\{\{X'_t\}, \eta'\}$ is another such solution, then $P(X_t = X'_t, 0 \leq t < \eta \wedge \eta') = 1$.

Using Proposition 2.2 we now give explicit regularity assumptions on $\sigma, b, F$ which imply the ‘local Lipschitz’ regularity $(loc-Lip)$ of $\hat{\sigma}, \hat{b}, \hat{F}$. The argument here is a variant of [1, Proposition 3.8]. For the sake of convenience, we state the result with deterministic $\sigma, b, F$.

**Proposition 3.7.** Let $p > d + \frac{1}{4}$. Fix deterministic $b_i, \sigma_{ij} \in S_{-\frac{p}{2}}, 1 \leq i, j \leq d$ and $y \in S_{-p}$. Assume $(F1)$ and $(F2)$. Then for any positive integer $n$, there exists a constant $D_n > 0$ such that for all $z_1, z_2 \in \mathcal{O}(0, n)$ and $0 < |x| < 1$
\[|\bar{b}(z_1; y) - \bar{b}(z_2; y)| \leq \|y\|_{-p} D_n \sup_i |b_i|_{p + \frac{1}{2}} |z_1 - z_2|,
\]
\[|\bar{\sigma}(z_1; y) - \bar{\sigma}(z_2; y)| \leq \|y\|_{-p} D_n \sup_{i,j} |\sigma_{ij}|_{p + \frac{1}{2}} |z_1 - z_2|,
\]
\[|\hat{F}(z_1; x; y) - \hat{F}(z_2; x; y)| \leq C_2 D_n \|y\|_{-p} |z_1 - z_2|.
\]
(3.9)

In particular $(loc-Lip)$ follows, i.e. for any bounded set $K$ in $S_{-p}$ and any positive integer $n$, there exists a constant $D(K, n) > 0$ such that for all $z_1, z_2 \in \mathcal{O}(0, n)$ and $y \in K$,
\[|\bar{b}(z_1; y) - \bar{b}(z_2; y)| \leq D(K, n) |z_1 - z_2|,
\]
\[|\bar{\sigma}(z_1; y) - \bar{\sigma}(z_2; y)| \leq D(K, n) |z_1 - z_2|,
\]
\[|\hat{F}(z_1; x; y) - \hat{F}(z_2; x; y)| \leq D(K, n) |z_1 - z_2|^2.
\]
(3.10)

**Proof.** We prove the result for $\hat{F}$. The results for $\hat{\sigma}, \hat{b}$ follow similarly.

If $z \in \mathbb{R}^d$ takes values in a bounded set, then using Proposition 2.1, we conclude that corresponding $\tau_z y$ also takes values in some bounded set. Then we have for all $z_1, z_2 \in \mathcal{O}(0, n)$,
\[|\hat{F}(z_1; x; y) - \hat{F}(z_2; x; y)| = |\hat{F}(\tau_z y, x) - \hat{F}(\tau_z y, x)|
\]
This proves the inequality for $\tilde{F}$ in (4.1). The other inequality for $\tilde{F}$ follows from (F2). \hfill \Box

4. INFINITE DIMENSIONAL SPDE

We continue with the same notations and hypotheses as in Section 3. In this section, we study the existence and uniqueness of strong solutions to the following SPDE, viz.

$$Y_t = \xi + \int_0^t A(Y_{s-}) \cdot dB_s + \int_0^t \tilde{L}(Y_{s-}) \, ds$$

(4.1)

$$+ \int_0^t \int_{0<|x|<1} (\tau_{F(Y_{s-},x)} - Id) \, Y_{s-} \, \tilde{N}(dx\,ds)$$

$$+ \int_0^t \int_{|x|\geq 1} (\tau_{G(Y_{s-},x)} - Id) \, Y_{s-} \, N(dx\,ds),$$

where $\xi$ is an $S_{-p}$ valued $\mathcal{F}_0$-measurable random variable, $A = (A_1, \cdots, A_d)$ with $A_j : S_{-p} \rightarrow S_{-p-\frac{1}{2}} \subset S_{-p-1}, j=1,2,\cdots, d$ and $\tilde{L} : S_{-p} \rightarrow S_{-p-1}$ are defined as follows, for $\rho \in S_{-p}$

$$A_j \rho := -\sum_{i=1}^d (\sigma, \rho)_{ij} \partial_i \rho,$$

(4.2)

$$\tilde{L}(\rho) := L\rho + \int_{0<|x|<1} \left( \tau_{F(\rho,x)} - Id + \sum_{i=1}^d F^i(\rho, x) \partial_i \right) \rho \, \nu(dx),$$

$$L\rho := \frac{1}{2} \sum_{i,j=1}^d (\langle \sigma, \rho \rangle_{\sigma} \, \delta_{ij} \rho - \sum_{i=1}^d \langle b, \rho \rangle_{ij} \partial_i \rho.$$

Given an $S_{-p}$ valued adapted process $\{Y_t\}$ with rcll paths, the integrals (with respect to $B, \tilde{N}$ and $\nu$) appearing in (4.1) exist. For example, to show the existence of the integral with respect to $\tilde{N}$, we need to establish $\mathbb{E} \sup_{t \geq 0} \int_0^t \int_{0<|x|<1} \| (\tau_{F(Y_{s-},x)} - Id) \, Y_{s-} \|_{-p-1} \nu(dx) \, ds < \infty$, for some increasing sequence of stopping times $\{\pi_n\}$ with $\pi_n \uparrow \infty$ a.s. Now,

$$\| (\tau_{F(Y_{s-},x)} - Id) \, Y_{s-} \|_{-p-1}$$

$$= \sum_{m \in \mathbb{Z}^d_+} (2|m| + d)^{-2(p+1)} \left| f(1; F(Y_{s-},x), Y_{s-}, m) - f(0; F(Y_{s-},x), Y_{s-}, m) \right|^2$$
and Lemma 3.2, we can show

\[ f(v; z, \psi, m) := \langle \tau_{vz} \psi, h_m \rangle, \quad v \in [0, 1], \]

for all fixed \( z \in \mathbb{R}^d, \psi \in S_{-p}, m \in \mathbb{Z}^d_+ \). Using the fact that

\[ f'(v; z, \psi, m) = -\sum_{i=1}^d \langle z_i \partial_i \tau_{vz} \psi, h_m \rangle \]

and Lemma 3.2 we can show

\[
\mathbb{E} \sup_{t \geq 0} \int_0^{t \wedge \pi_n} \int_{|x| < 1} |(\tau_{F(Y_{s-}, x)} - Id) Y_{2-} - \|F(\tau_{F(Y_{s-}, x)} - Id) Y_{2-} - \|_p \nu(dx) ds
\]

\[
\leq C(n) \mathbb{E} \sup_{t \geq 0} \int_0^{t \wedge \pi_n} \int_{|x| < 1} |F(Y_{s-}, x)|^2 \nu(dx) ds < \infty,
\]

where, \( \pi_n := \inf \{ t : \|Y_t\|_{-p} \geq n \} \) \( n \) and \( \{C(n)\} \) denotes a sequence of positive real numbers. We omit the details here and provide the details (see (4.12) and (4.14) below) for a similar estimate involving a second order Taylor expansion.

Let \( \delta \) be an arbitrary state, viewed as an isolated point of \( \hat{S}_{-p} := S_{-p} \cup \{\delta\} \). We make two definitions extending [21] Definition 3.1 and Definition 3.3.

**Definition 4.1.** Let \( \xi \) be an \( S_{-p} \) valued \( \mathcal{F}_0 \)-measurable random variable. By an \( \hat{S}_{-p} \) valued local strong solution of SPDE (4.1), we mean a pair \( \{\hat{Y}_t\}, \eta \) where \( \eta \) is an \( \langle \mathcal{F}_t \rangle \) stopping time and \( \{\hat{Y}_t\} \) an \( \hat{S}_{-p} \) valued \( \langle \mathcal{F}_t \rangle \) adapted rcll process such that

\( 1 \) for all \( \omega \in \Omega \), the map \( Y(\omega) : [0, \eta(\omega)) \rightarrow S_{-p} \) is well-defined and \( Y_t(\omega) = \delta, \quad t \geq \eta(\omega) \).

\( 2 \) a.s. the equality (4.1) holds in \( S_{-p} \) for \( 0 \leq t < \eta \).

We say local strong solutions of SPDE (4.1) are unique or pathwise unique, if given any two \( \hat{S}_{p}(\mathbb{R}^d) \) valued strong solutions \( \{\hat{Y}_t^1\}, \eta^1 \) and \( \{\hat{Y}_t^2\}, \eta^2 \), we have \( P(Y_t^1 = Y_t^2, 0 \leq t < \eta^1 \wedge \eta^2) = 1 \).

**4.1. Existence of solutions.** The next few results, viz. Theorem 4.2, Theorem 4.3 and Lemma 4.4 establish the existence and uniqueness of equation (4.1) and also exhibit the translation invariance of the solutions.

**Theorem 4.2.** Let \( (s, b), \langle F1 \rangle, \langle F2 \rangle, \langle F3 \rangle, \langle \text{loc-Lip} \rangle \) and \( \langle G1 \rangle \) hold. Consider the SDE (5.1) with \( \kappa = 0 \) and let \( \{U_t\}, \eta \) denote the unique local strong solution obtained by Theorem 7.2. Then the \( S_{-p} \) valued process \( \{U_t\} \) defined by \( \hat{Y}_t := \tau_{U_t} \xi, t < \eta \) solves the SPDE (4.1). We set \( \hat{Y}_t := \delta, \quad t \geq \eta \) so that \( \{\hat{Y}_t\}, \eta \) is a local strong solution of (4.1).

**Proof.** Note that \( \hat{Y}_t = \tau_{U_t} \xi, t < \eta \). By the Itô formula in Theorem 2.4, a.s.

\[
\tau_{U_t} \xi = \xi - \sum_{i=1}^d \int_0^t \partial_i \tau_{U_s} \xi dU_s^i + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \partial_{ij}^2 \tau_{U_s} \xi d[U^i, U^j]_s
\]

\[
+ \sum_{s \leq t} \left[ \tau_{U_s} \xi - \tau_{U_s} \xi + \sum_{i=1}^d (\Delta U_s^i \partial_i \tau_{U_s} \xi) \right], \quad t < \eta.
\]

\[ (4.3) \]
Observe that

\[
\frac{\Delta}{t} U^i_t = F^i(U_{t-}, \Delta N^i_t; \xi) \mathbb{I}_{\{0 < |\Delta N^i_t| < 1\}} + G^i(U_{t-}, \Delta N^i_t; \xi) \mathbb{I}_{\{|\Delta N^i_t| \geq 1\}}
\]

and hence

\[
\tau_{U^i} - \tau_{U^i - \mathbb{E}_se_{1} + \mathbb{E}_{s+1}e_{1}} + \sum_{i=1}^{d}(\Delta X^i \partial_{t} \tau_{U^i - \mathbb{E}_se_{1} + \mathbb{E}_{s+1}e_{1}})
\]

\[
= (\tau_{U^i} - Id) \tau_{U^i - \mathbb{E}_se_{1} + \mathbb{E}_{s+1}e_{1}} + \sum_{i=1}^{d}(\Delta X^i \partial_{t} \tau_{U^i - \mathbb{E}_se_{1} + \mathbb{E}_{s+1}e_{1}})
\]

\[
= \mathbb{I}_{\{0 < |\Delta N^i_t| < 1\}} \left( \tau_{F^i(U_{t-}, \Delta N^i_t; \xi)} - Id + \sum_{i=1}^{d} F^i(U_{t-}, \Delta N^i_t; \xi) \partial_{i} \right) \tau_{U^i - \mathbb{E}_se_{1} + \mathbb{E}_{s+1}e_{1}}
\]

\[
+ \mathbb{I}_{\{|\Delta N^i_t| \geq 1\}} \left( \tau_{G^i(U_{t-}, \Delta N^i_t; \xi)} - Id + \sum_{i=1}^{d} G^i(U_{t-}, \Delta N^i_t; \xi) \partial_{i} \right) \tau_{U^i - \mathbb{E}_se_{1} + \mathbb{E}_{s+1}e_{1}}.
\]

This observation yields a simplification of the fourth term of the right-hand side of (4.3), viz.

\[
\sum_{s \leq t} \left[ \tau_{U^i} - \tau_{U^i - \mathbb{E}_se_{1} + \mathbb{E}_{s+1}e_{1}} + \sum_{i=1}^{d}(\Delta X^i \partial_{t} \tau_{U^i - \mathbb{E}_se_{1} + \mathbb{E}_{s+1}e_{1}}) \right]
\]

\[
= \int_{0}^{t} \int_{0 < |x| < 1} \left( \tau_{F^i(U_{t-}, x; \xi)} - Id + \sum_{i=1}^{d} F^i(U_{t-}, x; \xi) \partial_{i} \right) \tau_{U^i - \mathbb{E}_se_{1} + \mathbb{E}_{s+1}e_{1}} N(dx ds dx)
\]

\[
+ \int_{0}^{t} \int_{|x| > 1} \left( \tau_{F^i(U_{t-}, x; \xi)} - Id + \sum_{i=1}^{d} F^i(U_{t-}, x; \xi) \partial_{i} \right) \tau_{U^i - \mathbb{E}_se_{1} + \mathbb{E}_{s+1}e_{1}} \bar{N}(dx ds dx)
\]

\[
= \int_{0}^{t} \int_{0 < |x| < 1} \left( \tau_{F^i(U_{t-}, x; \xi)} - Id + \sum_{i=1}^{d} F^i(U_{t-}, x; \xi) \partial_{i} \right) \tau_{U^i - \mathbb{E}_se_{1} + \mathbb{E}_{s+1}e_{1}} \nu(dx ds)
\]

\[
+ \int_{0}^{t} \int_{0 < |x| < 1} \left( \tau_{F^i(U_{t-}, x; \xi)} - Id + \sum_{i=1}^{d} F^i(U_{t-}, x; \xi) \partial_{i} \right) \tau_{U^i - \mathbb{E}_se_{1} + \mathbb{E}_{s+1}e_{1}} N(dx ds dx)
\]

\[
+ \int_{0}^{t} \int_{|x| > 1} \left( \tau_{F^i(U_{t-}, x; \xi)} - Id + \sum_{i=1}^{d} F^i(U_{t-}, x; \xi) \partial_{i} \right) \tau_{U^i - \mathbb{E}_se_{1} + \mathbb{E}_{s+1}e_{1}} \bar{N}(dx ds dx)
\]

Substituting (4.5) into (4.3) and simplifying the equality, we obtain (4.4). Here we have used the fact that the coefficients satisfy relations like \( \bar{F}(U_{t-}, x; \xi) = F(U_{t-}, x; \xi) = F(Y_{t-}, x) \) etc..  

4.2. Uniqueness results via monotonicity arguments. Existence of local strong solutions to SPDE (4.1) follows from Theorem (1.1). Uniqueness of solutions to (4.1) are the focus of this section. The proof of uniqueness is based on the ‘Monotonicity inequality’, (see [13, p. 29], [18, p. 308], [19, Section 3]).

**Theorem 4.3.** Let \( \sigma_b \), \( F_1 \), \( F_2 \), \( F_3 \), \( \text{loc-Lip} \) and \( G_1 \) hold. In addition, assume that \( G_2 \) holds. Then there exists a unique local strong solution to (4.1).
To establish the uniqueness, we first show that any local strong solution to equation (4.1) is of a specific form. Even though this result is only used in the proof of Theorem 4.3, we state it separately in order to keep our arguments transparent.

**Lemma 4.4.** Let the hypotheses of Theorem 4.3 hold. Let \( \{Y_t, \eta\} \) be an \( \hat{S}_p \)-valued local strong solution of (4.1). Define

\[
Z_t := \int_0^t \langle b, Y_s \rangle \, ds + \int_0^t \langle \sigma, Y_s \rangle \cdot dB_s + \int_0^t \int_{(0<|x|<1)} F(Y_s, x) \, \tilde{N}(dsdx) \\
+ \int_0^t \int_{(|x|\geq 1)} G(Y_s, x) \, N(dsdx),
\]

for \( 0 \leq t < \eta \). Then a.s. \( Y_t = \tau_{Z_t} \xi, t < \eta \).

**Proof.** We follow the approach used in [21, Lemma 3.6], but with initial condition \( \xi \) random (see [4, Theorem 3.14]). Define \( V_t := Y_t - \tau_{Z_t} \xi \). As done in Theorem 4.2, we simplify \( \tau_{Z_t} \xi \) using the Itô formula in Theorem 2.4. Then using (4.1), we have a.s. for \( 0 \leq t < \eta \)

\[
V_t = \int_0^t \bar{A}(s)V_{s-} \cdot dB_s + \int_0^t \bar{L}(s)V_{s-} \, ds \\
+ \int_0^t \int_{(0<|x|<1)} \left( \tau_{F(Y_{s-}, x)} - Id + \sum_{i=1}^d F^i(Y_{s-}, x) \partial_i \right) V_{s-} \, \nu(dx) \, ds \\
+ \int_0^t \int_{(|x|\geq 1)} \left( \tau_{G(Y_{s-}, x)} - Id \right) V_{s-} \, \tilde{N}(dsdx) + \int_0^t \int_{(|x|\geq 1)} \left( \tau_{G(Y_{s-}, x)} - Id \right) V_{s-} \, N(dsdx),
\]

where the bounded random linear operators \( \bar{A} = (\bar{A}_1, \cdots, \bar{A}_d) \) with \( \bar{A}_i : \mathcal{S}_p \to \mathcal{S}_{p-d}, i = 1, \cdots, d \) and \( \bar{L} : \mathcal{S}_p \to \mathcal{S}_{p-1} \) are defined as follows, for \( \rho \in \mathcal{S}_p \)

\[
\bar{A}_j(s, \omega)\rho := -\sum_{i=1}^d \langle \sigma(\omega), Y_{s-} \rangle_{ij} \partial_i \rho, \quad j = 1, \cdots, d, \\
\bar{L}(s, \omega)\rho := \frac{1}{2} \sum_{i,j=1}^d \langle \langle \sigma(\omega), Y_{s-} \rangle \sigma(\omega), Y_{s-} \rangle_{ij} \partial_{ij} \rho - \sum_{i=1}^d \langle b(\omega), Y_{s-} \rangle_{i} \partial_i \rho.
\]
The following equation is obtained using Itô formula for the norm \(| \cdot |_{-p}^2\). We have a.s. for \(0 \leq t < \eta\)

\[
(4.8) \quad \|V_t\|_{-p}^2 = 2 \int_0^t \left\langle V_{s-}, \tilde{A}(s)V_{s-} \right\rangle_{-p} \cdot dB_s
\]

\[+ 2 \int_0^t \left\langle V_{s-}, \tilde{L}(s)V_{s-} \right\rangle_{-p} ds + \int_0^t \sum_{j=1}^d \|A_j(s)V_{s-}\|_{-p}^2 ds
\]

\[+ \int_0^t \int_{0<|x|<1} \left[ \|\tau_F(Y_{s-x})V_{s-}\|_{-p}^2 - \|V_{s-}\|_{-p}^2 + 2 \sum_{i=1}^d \left\langle V_{s-}, F^i(Y_{s-x}, x) \partial_t V_{s-} \right\rangle_{-p} \right] \nu(dx) ds
\]

\[+ \int_0^t \int_{0<|x|<1} \left[ \|\tau_G(Y_{s-x})V_{s-}\|_{-p}^2 - \|V_{s-}\|_{-p}^2 \right] \tilde{N}(dsdx)
\]

\[+ \int_0^t \int_{|x|\geq 1} \left[ \|\tau_G(Y_{s-x})V_{s-}\|_{-p}^2 - \|V_{s-}\|_{-p}^2 \right] \tilde{N}(dsdx).
\]

To establish the above equation we execute the following steps.

(i) Recall that \(h_m\) denotes the Hermite functions, where \(m\) denotes multi-indices. Then from \((4.7)\), we obtain the equation satisfied by the real semimartingale \(\{V_t, h_m\}\).

(ii) Applying Itô formula \([1, \text{Theorem 4.4.7}]\) for the function \(x \mapsto x^2\) we get an equation for the process \(\{V_t, h_m\}^2\).

(iii) We multiply the last equation by \((2|m| + d)^{-2(p+1)}\) and sum over \(m\).

Define \(\pi_n := \inf \{t : \max \{\|V_t\|_{-p}, |Z_t| \geq n\} \wedge n \wedge \eta\}\). Note that \(\|V_t\|_{-p} \leq \|Y_t\|_{-p} \leq n\) and \(|Z_t| \leq n\). Hence the process \(\{\|V_t\|_{-p}\}\) is bounded and for \(j = 1, \ldots, d, \tilde{A}_j(t) : \mathcal{S}_{-p} \to \mathcal{S}_{-p-1}, t \leq \pi_n\) are bounded linear operators, bounded uniformly in \(t\). This implies \(\{\int_0^{t \wedge \pi_n} \left\langle V_{s-}, \tilde{A}(s)V_{s-} \right\rangle_{-p} ds, \tilde{B}_s\}\) is an \(\mathcal{L}^2\) martingale.

Again, \(\{\int_0^{t \wedge \pi_n} \int_{0<|x|<1} \left[ \|\tau_F(Y_{s-x})V_{s-}\|_{-p}^2 - \|V_{s-}\|_{-p}^2 \right] \tilde{N}(dsdx)\}\) is an \(\mathcal{L}^2\) martingale, since \(E\sup_{t \geq 0} \int_0^{t \wedge \pi_n} \int_{0<|x|<1} \left[ \|\tau_F(Y_{s-x})V_{s-}\|_{-p}^2 - \|V_{s-}\|_{-p}^2 \right] \nu(dx) ds < \infty\). This integrability condition follows using a first order Taylor expansion of the function \(f\) defined in \((4.13)\). We omit the details here and provide the details (see \((4.12)\) and \((4.14)\) below) for the corresponding second order Taylor expansion used in estimating Term 2 of \((4.9)\). However, \((4.14)\) requires \((4.17)\), while the proof of the integrability condition above involving the first order Taylor expansion uses \((4.17)\) instead. Since a.s. \(V\) has (at most) countably many jumps, taking expectation on both sides of \((4.8)\) we get

\[
(4.9) \quad E\|V_{t \wedge \pi_n}\|_{-p}^2 = E \int_0^{t \wedge \pi_n} \left[ 2 \left\langle V_s, \tilde{L}(s)V_s \right\rangle_{-p} + \|\tilde{A}(s)V_s\|_{HS(-p-1)}^2 \right] ds
\]

\[+ E \int_0^{t \wedge \pi_n} \int_{0<|x|<1} \left[ \|\tau_F(Y_{s-x})V_s\|_{-p}^2 - \|V_s\|_{-p}^2 + 2 \sum_{i=1}^d \left\langle V_s, F^i(Y_{s-x}, x) \partial_t V_s \right\rangle_{-p} \right] \nu(dx) ds
\]

\[+ E \int_0^{t \wedge \pi_n} \int_{|x|\geq 1} \left[ \|\tau_G(Y_{s-x})V_s\|_{-p}^2 - \|V_s\|_{-p}^2 \right] \nu(dx) ds
\]

\[= \text{Term 1} + \text{Term 2} + \text{Term 3}.
\]
We now prove certain estimates of these terms. Some positive constants appearing in these calculations may be written by \( C \) and may change their values from line to line, but will depend on \( n \) and \( d \).

**Estimate for Term 1:** By assumption (\( \sigma \), \( \beta \)) the coefficients in \( \bar{L}(s), \bar{A}(s) \) are bounded for \( s \leq \pi_n \). Hence applying the Monotonicity inequality (Theorem 2.3), we get

\[
\mathbb{E} \int_0^{t \wedge \pi_n} \left[ 2 \langle V_s, \bar{L}(s)V_s \rangle_{-p-1} + \| \bar{A}(s)V_s \|_{H^S(-p-1)}^2 \right] ds \leq CE \int_0^{t \wedge \pi_n} \| V_s \|_{-p-1}^2 ds,
\]

where \( C \) is some positive constant.

**Estimate for Term 2:** First we need a special case of the Monotonicity inequality (Theorem 2.3), viz. we need an explicit form of the constant in this special case, to ensure certain integrability conditions. To prove this, we use an alternative proof of the Monotonicity inequality already given in [6].

By [6, Theorem 2.5], for each \( 1 \leq i \leq d \), there exists a bounded operator \( T_i : \mathcal{S}_{-p-1} \to \mathcal{S}_{-p-1} \) such that the adjoint operator \( \partial_i^* \) has the form \( \partial_i^* = -\partial_i + T_i \) on \( \mathcal{S} \). Recall that \( \partial_i : \mathcal{S}_{-p-\frac{1}{2}} \to \mathcal{S}_{-p-\frac{1}{2}} \), \( 1 \leq i \leq d \) are bounded linear operators. Hence it is easy to see that \( \partial_i^* = -\partial_i + T_i \) on \( \mathcal{S}_{-p-\frac{1}{2}} \). Moreover, by [6, Lemma 2.6], the map \( (\partial_i(\cdot), T_j(\cdot)) \) is bounded for \( \mathcal{S} \times \mathcal{S} \to \mathbb{R} \) defined by

\[
(\phi, \psi) \mapsto (\partial_i \phi, T_j \psi)_{-p-1}, \; \forall \phi, \psi \in \mathcal{S}
\]

extends to a bounded bilinear form on \( \mathcal{S}_{-p-1} \times \mathcal{S}_{-p-1} \). Let \( \alpha = (\alpha_1, \ldots, \alpha_d)^t \in \mathbb{R}^d \) and \( \phi \in \mathcal{S}_{-p} \) be chosen arbitrarily. Then there exists a positive constant \( R \), not depending on \( \alpha \) and \( \phi \), such that

\[
\sum_{i,j=1}^d \alpha_i \alpha_j \left[ (\partial_i \phi, \partial_j \phi)_{-p-1} + (\phi, \partial_j^2 \phi)_{-p-1} \right] = \sum_{i,j=1}^d \alpha_i \alpha_j (\partial_i \phi, \partial_j \phi)_{-p-1} \leq R \| \phi \|_{-p-1}^2 \left( \sum_{i=1}^d |\alpha_i|^2 \right)^{\frac{1}{2}} \leq dR \| \phi \|_{-p-1}^2 |\alpha|^2.
\]

To estimate Term 2, we use a second order Taylor expansion described below. Observe that

\[
\| \tau_{F(Y_{s-}, x)} V_s \|_{-p-1}^2 - \| V_s \|_{-p-1}^2 + 2 \sum_{i=1}^d \langle V_s, F^i(Y_{s-}, x) \partial_i V_s \rangle_{-p-1} \]

\[
= \sum_{m \in \mathbb{Z}_+^d} (2|m| + d)^{-2(p+1)} \left[ \langle \tau_{F(Y_{s-}, x)} V_s, h_m \rangle^2 - \langle V_s, h_m \rangle^2 + 2 \sum_{i=1}^d \langle V_s, h_m \rangle \langle F^i(Y_{s-}, x) \partial_i V_s, h_m \rangle \right] \]

\[
= \sum_{m \in \mathbb{Z}_+^d} (2|m| + d)^{-2(p+1)} \left[ f(1; F(Y_{s-}, x), V_s, m) - f(0; F(Y_{s-}, x), V_s, m) \right] \]

\[
= \sum_{m \in \mathbb{Z}_+^d} (2|m| + d)^{-2(p+1)} \int_0^1 \int_0^1 f''(v; F(Y_{s-}, x), V_s, m) dv dr,
\]

where the function \( f \) is given by

\[
f(v; z, \psi, m) := \langle \tau_{uv} \psi, h_m \rangle^2, \; v \in [0, 1],
\]
for all fixed $z \in \mathbb{R}^d, \psi \in \mathcal{S}_{-p}, m \in \mathbb{Z}_+^d$. Note that

$$f'(v; z, \psi, m) = -2 \sum_{i=1}^{d} \langle \tau v \psi, h_m \rangle \langle z_i \partial_i \tau v \psi, h_m \rangle,$$

and

$$f''(v; z, \psi, m) = 2 \sum_{i,j} z_i z_j \left[ \langle \partial_i \tau v \psi, h_m \rangle \langle \partial_j \tau v \psi, h_m \rangle + \langle \tau v \psi, h_m \rangle \langle \partial_{ij} \tau v \psi, h_m \rangle \right].$$

Now, using (4.11) and Proposition 2.1(a), we have

$$\| \tau F(Y_{s-}, x) V_s \|_{p-1}^2 - \| V_s \|_{p-1}^2 + 2 \sum_{i=1}^{d} \langle V_s, F^i(Y_{s-}, x) \partial_i V_s \rangle_{p-1}$$

$$= 2 \int_0^1 \int_0^r \sum_{i,j} F^i(Y_{s-}, x) F^j(Y_{s-}, x)$$

$$\times \left[ \langle \partial_i \tau v F(Y_{s-}, x) V_s, \partial_j \tau v F(Y_{s-}, x) V_s \rangle_{p-1} + \langle \tau v F(Y_{s-}, x) V_s, \partial_{ij} \tau v F(Y_{s-}, x) V_s \rangle_{p-1} \right] dv dr$$

$$\leq C \| F(Y_{s-}, x) \|_{C^2} \int_0^1 \int_0^r \| \tau v F(Y_{s-}, x) V_s \|_{p-1}^2 dv dr$$

$$\leq C \| F(Y_{s-}, x) \|_{C^2} \| V_s \|_{p-1}^2 \int_0^1 \int_0^r (P(|vF(Y_{s-}, x)|))^2 dv dr,$$

where $P$ is some real polynomial of degree $2(|p + 1| + 1)$. Now $\| Y_{t\wedge \pi_n} \|_{-p} \leq n$ and hence, by Lemma 3.2(i), $P(|vF(Y_{s-}, x)|)$ is also bounded for all $s \leq \pi_n$ and $v \in [0, 1]$. Then

$$\mathbb{E} \int_0^{t \wedge \pi_n} \int_{|x| < 1} \left[ \| \tau F(Y_{s-}, x) V_s \|_{p-1}^2 - \| V_s \|_{p-1}^2 + 2 \sum_{i=1}^{d} \langle V_s, F^i(Y_{s-}, x) \partial_i V_s \rangle_{p-1} \right] \nu(dx) ds$$

$$\leq C \mathbb{E} \int_0^{t \wedge \pi_n} \int_{|x| < 1} |F(Y_{s-}, x)| \| V_s \|_{p-1}^2 \int_0^1 \int_0^r (P(|vF(Y_{s-}, x)|))^2 dv dr \nu(dx) ds$$

$$\leq C \mathbb{E} \int_0^{t \wedge \pi_n} \int_{|x| < 1} |F(Y_{s-}, x)| \| V_s \|_{p-1}^2 \nu(dx) ds$$

$$\leq C \mathbb{E} \int_0^{t \wedge \pi_n} \| V_s \|_{p-1}^2 ds.$$

In the last step above, we have used Lemma 3.2(ii).

Estimate for Term 3; Since $\{Y_{t\wedge \pi_n} \}$ is bounded in $\mathcal{S}_{-p}$, using (G2) we get a bound for $G(Y_{s-}, x)$ when $x \in \overline{O(0, 1)}, s \leq \pi_n$. Applying Proposition 2.1, we have

$$\mathbb{E} \int_0^{t \wedge \pi_n} \int_{|x| \geq 1} \left[ \| \tau G(Y_{s-}, x) V_s \|_{p-1}^2 - \| V_s \|_{p-1}^2 \right] \nu(dx) ds \leq C \mathbb{E} \int_0^{t \wedge \pi_n} \| V_s \|_{p-1}^2 ds,$$

where $C$ is some positive constant.

From (1.9), (4.10), (4.14) and (4.15), we get

$$\mathbb{E} \| V_{t \wedge \pi_n} \|_{p-1}^2 \leq C \mathbb{E} \int_0^{t \wedge \pi_n} \| V_s \|_{p-1}^2 ds \leq C \mathbb{E} \int_0^t \| V_{\pi_n} \|_{p-1}^2 ds.$$
By Gronwall’s inequality, we have a.s. $V_t^{\Omega} = 0, \forall t$. Hence, a.s. $V_t = 0, t < \eta$ and $Y_t = \tau_{Z_t, \xi}, t < \eta$, where $\{Z_t\}$ is given by (4.6).

**Proof of Theorem 4.3.** Existence of local strong solutions to SPDE (4.1) follows from Theorem 4.2. The uniqueness argument follows as in [21]. Given any local strong solution $\{\{Y_t\}, \eta\}$, by Lemma 4.3 we have $Y_t = \tau_{Z_t, \xi}$ where $\{Z_t\}$ is given by (4.6). Hence, (4.6) becomes

$$Z_t = \int_0^t \bar{b}(Z_s, \xi) \, ds + \int_0^t \bar{\sigma}(Z_s, \xi) \cdot dB_s + \int_0^t \int_{(0<|z|<1)} F(Z_{s-}, x, \xi) \tilde{N}(ds, dx)$$

$$+ \int_0^t \int_{(|z| \geq 1)} \bar{G}(Z_{s-}, x, \xi) N(ds, dx).$$

By Theorem 3.8, $\{\{Z_t\}, \eta\}$ is pathwise unique. Since $Y_t = \tau_{Z_t, \xi}, \{\{Y_t\}, \eta\}$ is also pathwise unique. □

**Remark 4.5.** The estimate of Term 2 in Lemma 4.4 follows easily in simple situations. For example, let $F$ be bounded and the measure $\nu$ be finite. Using the boundedness of the translation operator (Proposition 2.7), the term $\mathbb{E} \int_0^T \int_{(0<|z|<1)} \left[\|\tau_F(\bar{y}, \cdot, x)\bar{V}_s\|_p^2 - \|\bar{V}_s\|_{-p, -1}^2\right] \nu(dx)ds$ can be estimated. To handle the remaining part, observe that $\langle \phi, \partial_t \phi \rangle_{p-1} = -\langle \partial_t \phi, \phi \rangle_{p-1} + \langle \mathcal{T}_t \phi, \phi \rangle_{p-1}$ for all $\phi \in \mathcal{S}_p$ with $\mathcal{T}_t$ as described in the proof of Lemma 4.4. Then

$$2 \langle \phi, \partial_t \phi \rangle_{p-1} = \langle \mathcal{T}_t \phi, \phi \rangle_{p-1} \forall \phi \in \mathcal{S}_p,$$

and the relevant term can be dominated by a constant multiple of $\|\phi\|_{-p, -1}^2$, which in our case gives a bound involving $\|\bar{V}_s\|_{-p, -1}^2$.

**Example 4.6.** Examples of the SPDEs we have considered are as follows.

(1) Consider $F \equiv G \equiv 0$ in (4.1), with deterministic initial condition $\xi = \phi \in \mathcal{S}_p$. Let $\sigma, b \in \mathcal{S}_p$ be deterministic such that $\langle \sigma, \tau_z \phi \rangle$ and $\langle b, \tau_z \phi \rangle$ are locally Lipschitz in $z$. Define $\sigma(\omega, z; y) := \langle \sigma, \tau_z \phi \rangle$ and $b(\omega, z; y) := \langle b, \tau_z \phi \rangle$ for $y \in \mathcal{S}_p, \omega \in \Omega$. Then (loc-Lip) holds. Applying Theorem 3.4, existence and uniqueness of finite dimensional SDEs follow for $\kappa = 0$. By Theorem 4.2 and Theorem 4.3, existence and uniqueness of the corresponding SPDE are established. Our results therefore imply Theorem 3.4 of [21].

(2) Continue with $F \equiv G \equiv 0$ and $\kappa = 0$ as in the previous example, but consider $\xi$ with $\mathbb{E} \|\xi\|_{-p, -1}^2 < \infty$. Then our results imply the results on existence and uniqueness of solutions studied in [4].

(3) We consider $\sigma \equiv b \equiv G \equiv 0$. Let $F : \mathcal{S}_p \times \mathcal{O}(0, 1) \to \mathbb{R}^d$, defined by $F(y, x) := x$. Then we have the existence and uniqueness of the following SPDE

$$Y_t = \xi + \int_0^t \int_{(0<|z|<1)} \left( \tau_x - Id + \sum_{i=1}^d \xi_i \partial_i \right) Y_{s-} \nu(dx)ds + \int_0^t \int_{(0<|z|<1)} (\tau_x - Id) Y_{s-} \tilde{N}(ds, dx).$$

### 4.3. Uniqueness via interlacing

Using results of the previous subsection, the existence and uniqueness of local strong solutions of the reduced equation corresponding to (4.1) follows, which is the case when $G \equiv 0$. In this subsection, we use the result for the reduced equation and use an interlacing argument to attach large jumps to obtain existence and uniqueness of local strong solutions of SPDE (4.1). This approach allows us to drop the assumption (G2) which was used in Theorem 4.3.

**Theorem 4.7.** Let $[\sigma, b]$, (F1), (F2), (F3), (loc-Lip) and (G1) hold. Then we have the existence and uniqueness of the local strong solutions of (4.1).
Proof. Existence of local strong solutions to SPDE (4.1) follows from Theorem 4.2. We use the
interlacing procedure described in Example 1.3.13, pp. 50-51] to establish the uniqueness.

Let \( \{\pi_n\}_{n\in\mathbb{N}} \) be the arrival times for the jumps of the compound Poisson process \( \{P_t\}_{t\geq 0} \), where each \( P_t = \int_{|x|\geq 1} xN(t, dx) \). Let \( \{Y_t, \eta\} \) be a local strong solution of (4.1). Since, a.s. \( \pi_n \uparrow \infty \),
the stochastic interval \([0, \eta]\) can be decomposed as a disjoint union \( \bigcup_{n=0}^{\infty} [\pi_n \wedge \eta, \pi_{n+1} \wedge \eta) \), where \( \pi_0 = 0 \). We now construct an \( \mathbb{R}^d \)-valued adapted rcll process \( \{Z_t\} \) such that the following equalities hold; a.s.

\[
Z_t = \int_0^t \langle b, Y_{s-} \rangle \, ds + \int_0^t \langle \sigma, Y_{s-} \rangle \cdot dB_s + \int_0^t \int_{0<|x|<1} F(Y_{s-}, x) \tilde{N}(dsdx)
\]

(4.18)

\[
+ \int_0^t \int_{|x|\geq 1} G(Y_{s-}, x) N(dsdx), \; t < \eta,
\]

(4.19)

\[
Y_t = \tau_{Z_t} \xi, \; t < \eta,
\]

(4.20)

As pointed out in the proof of Theorem 4.3, (4.20) follows from (4.18) and (4.19). We verify
the claimed equalities on successive time intervals.

Comparing \( \pi_n \)'s and \( \eta \), two cases arise viz. either \( \pi_n \leq \eta < \pi_{n+1} \) for some \( n \geq 0 \) or \( \pi_n < \eta, \forall n \).
We consider the first case. The proof for the second case is similar.

If \( n = 0 \), i.e. \( \pi_0 \leq \eta < \pi_1 \), define \( \{Z_t\} \) by the right hand side of (4.18). Since there is no large
jump, the equality in (4.19) follows as in Lemma 4.4 and (4.20) also follows.

Now assume \( n \geq 1 \), i.e. \( \pi_0 < \pi_1 < \cdots < \pi_n \leq \eta < \pi_{n+1} \).

On \([0, \pi_1]\), define \( \{Z_t\} \) by the right hand side of (4.18). Since there is no large jump, the equality
in (4.19) follows as in Lemma 4.4 and (4.20) also follows.

At \( t = \pi_1 \), define \( Z_t := Z_{\tau_{Z_{\pi_1}}} + \tilde{G}(Z_{\tau_{Z_{\pi_1}}}, \Delta P_{\pi_1}; \xi) \). Then the equality in (4.20) holds true. Note that
\( Y_{\tau_{Z_{\pi_1}}} = \tau_{Z_{\pi_1}} \xi \) on \((0, \pi_1)\]. By Itô formula in Theorem 2.3, the equality in (4.19) follows. Consequently, equality in (4.18) follows.

On \((\pi_1, \pi_2]\), define \( \{Z_t\} \) by the right hand side of (4.18). Observe that there is no contribution
of the large jump at \( \pi_1 \) in the difference \( Y_t - \tau_{Z_{\pi_1}} \xi \). Hence, arguing as in Lemma 4.4, the equality in (4.19)
follows for the time interval \((\pi_1, \pi_2]\). Equality in (4.20) also follows for the same time interval.

At \( t = \pi_2 \), define \( Z_t := Z_{\tau_{Z_{\pi_2}}} + \tilde{G}(Z_{\tau_{Z_{\pi_2}}}, \Delta P_{\pi_2}; \xi) \). We verify the equalities at \( t = \pi_2 \) as in the case
\( t = \pi_1 \).

Continuing this way, we construct \( \{Z_t\} \). Since (4.20) holds, the uniqueness of \( \{Z_t\} \) follows from
Theorem 5.6. Since \( Y_t = \tau_{Z_t} \xi, \{Y_t\} \) is also unique.

This completes the proof.

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