Intricacies of cosmological bounce in polynomial metric $f(R)$ gravity for flat FLRW spacetime

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Abstract. In this paper we present the techniques for computing cosmological bounces in polynomial $f(R)$ theories, whose order is more than two, for spatially flat FLRW spacetime. In these cases the conformally connected Einstein frame shows up multiple scalar potentials predicting various possibilities of cosmological evolution in the Jordan frame where the $f(R)$ theory lives. We present a reasonable way in which one can associate the various possible potentials in the Einstein frame, for cubic $f(R)$ gravity, to the cosmological development in the Jordan frame. The issue concerning the energy conditions in $f(R)$ theories is presented. We also point out the very important relationships between the conformal transformations connecting the Jordan frame and the Einstein frame and the various instabilities of $f(R)$ theory. All the calculations are done for cubic $f(R)$ gravity but we hope the results are sufficiently general for higher order polynomial gravity.

Keywords: physics of the early universe, alternatives to inflation, cosmic singularity

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1 Introduction

Due to advancement in technology we can presently probe most of the information stored in the cosmic microwave background radiation (CMBR) field. The average temperature of CMBR and the fluctuations on it gives us a hint of the pre-CMBR phase of the universe. The data stored in CMBR hints at the possibility that the very early phase of the universe may have gone through an inflationary phase. The theory of cosmic inflation [1, 2] tries to address various problems in cosmology, as the horizon problem, the entropy problem, the flatness problem and others. In spite of the fact that inflationary theories can solve most of the problems of the very early universe they have an inherent problem, termed as the trans-Planckian problem [3–7]. To address the difficulties of the trans-Planckian problem and the major problem of the big-bang singularity there are proposals for an alternative way to deal with the very early universe. The alternative paradigm about the cosmological dynamics of the early universe deals with the idea of a cosmological bounce. In the bouncing universe scenario there exists a contracting phase of the universe, prior to the time of the big-bang singularity. The contracting universe never actually reaches the singular point but bounces to an expanding phase. Technically the scale factor of the FLRW spacetime never vanishes but it attains its minimum value during the bounce. Some authors have proposed cosmological bounces in the general relativistic framework [8, 9] and others have used theories using modified gravity/particle physics to model such cosmological bounces [10–18].

Cosmological bounces can be thought of as an interesting alternative to inflation as it resolves the horizon problem by making most part of the observable universe to be causally connected during the contracting phase of the universe. The perturbation length scales corresponding to these observations became superhorizon during the contracting phase. These perturbations later entered the causally connected universe in the expanding phase. One
may assume that far away from the bouncing regime the dynamics of the cosmos is guided by general relativity (GR) and only when the Ricci scalar attains a considerable value (near bounce) the theory of gravity attains quantum corrections [19] and an effective $f(R)$ theory of gravity emerges.\(^1\) One can use $f(R)$ theory of gravity near the bouncing regime to understand the dynamics near the bounce. It must be noted that in $f(R)$ gravity theories one can have non-zero, and presumably, high values of the Ricci scalar, $R$, in a primarily radiation dominated (early) phase of the universe unlike GR, where the Ricci scalar is zero in radiation domination. The reason being that the dynamic equation for evolution of the metric changes in $f(R)$ theories (the standard Einstein equation of GR is modified/altered). Due to this fact one can indeed think of high Ricci scalar values near a radiation dominated bouncing regime.

Some preliminary work on bouncing cosmologies using $f(R)$ theories can be found in refs. [23–25]. In a previous paper ref. [26] the present authors proposed an interesting cosmological bounce in quadratic $f(R)$ theory in presence of hydrodynamic matter. In the previous work it was shown that for spatially flat FLRW spacetimes one cannot get cosmological bounces in both the Jordan frame, where the $f(R)$ theory lives, and the conformally connected Einstein frame. In spite of this fact one can use the Einstein frame to solve the bouncing problem. In the present work we extend our earlier work by touching upon the point related to energy conditions. We show that one can always use the Einstein frame to calculate the bouncing dynamics in general polynomial gravity without breaking the weak energy condition and the null energy conditions. It is also pointed out that some energy conditions as the strong energy condition and the dominant energy condition may be violated near the bouncing point. Actually the cosmological bounce only happens in the Jordan frame. In this paper when we refer to the bouncing point in the Einstein frame we mean the cosmological coordinates in the Einstein frame which corresponds to the cosmological variables in the Jordan frame at the time of bounce. The energy conditions are perfectly defined in the Einstein frame where the mathematical description of the cosmological process follows the known route of GR. In the Jordan frame the energy conditions are violated but this violation may not be taken very seriously as the energy conditions are not uniquely formulated for $f(R)$ theories as the energy momentum tensor gets contribution from curvature terms. In this paper we explicitly deal with spatially flat FLRW spacetimes where the dynamics of the scale factor is guided by cubic $f(R)$ theory in the Jordan frame. The interested reader can see ref. [27] where the peculiarities of the two conformal frames are discussed.

In this paper we address the important question relating to the conformal correspondence of the Jordan frame and Einstein frame where the form of $f(R)$ is a polynomial function, whose order is more than two. This question is particularly important in $f(R)$ theories as because the correspondence between the two conformal frames for such higher order $f(R)$ theories becomes many-to-one. While there is only one $f(R)$ theory in the Jordan frame, for a polynomial gravity theory, there can be various different possibilities of cosmological evolution in the Einstein frame. It appears that there are multiple Einstein frame descriptions of a single $f(R)$ theory. This fact emerges from the conformal transformations connecting the two frames. For quadratic gravity this correspondence was unique and there was no confusion in solving the bouncing problem in the Einstein frame and then converting the results into Jordan frame language. But as soon as one starts with cubic gravity the problems of the conformal frames becomes apparent. In this work we address the problem of the emergence of multiple scalar potentials in the Einstein frame for a single polynomial gravity theory and

\(^1\)Most of the common techniques of $f(R)$ theories which we will use in this article are appropriately summed up in the review articles in, ref. [20], ref. [21] and ref. [22].
specify how one can identify the origin of the multiple potentials in the Einstein frame. The discussion regarding multiple Einstein frame description, of a unique \( f(R) \) theory, is general in nature and has no particular connection to bouncing cosmologies. We hope our results will give a new way of interpreting polynomial \( f(R) \) theories.

In this paper we point out that there is a close relationship between the instabilities of \( f(R) \) theories and the conformal transformations connecting the Jordan frame and the Einstein frame. It is well known that if \( f'(R) \leq 0 \) the gravitational theory becomes difficult to handle as the effective gravitational constant in \( f(R) \) theory diverges or becomes negative. On the other hand if \( f'(R) \leq 0 \) the conformal transformations connecting the two conformal frames becomes ill defined. Going one step further, it was shown by Dolgov and Kawasaki in ref. [27] that for low curvature \( f(R) \) gravity the condition \( f''(R) < 0 \) sets up a new form of instability where the effective gravitational interactions increases without an upper bound. Dolgov and Kawasaki’s result deals with low curvature cases and consequently it was not clear whether their results hold for \( f(R) \) theories applied in the very early universe where the Ricci scalar may not have a small value. In general in ref. [27] the authors also proclaim that \( f''(R) > 0 \) is a stability condition for their cosmological model. In this article we show that in the case of the very early universe, where the gravitational dynamics is dictated by a cubic theory of gravity, the epoch when \( R \) attains the value \( R_c \), where \( R_c \) is the solution of \( f''(R) = 0 \), is not connected with other cosmological epochs. In other words a bouncing universe will take an infinite time to attain \( R = R_c \) after the bounce. Consequently the very early universe can exist in two complimentary branches, in one branch \( f''(R) > 0 \) and in the other branch \( f''(R) < 0 \). In the branch where \( f''(R) < 0 \) one can have perfectly well behaved cosmological bounces but this branch also shows the embryo of gravitational instability, as in this branch the Ricci scalar \( R \) can become arbitrarily small and consequently Dolgov-Kawasaki like instabilities can affect the theory in the low curvature limit.\(^2\) On the other hand as we are applying \( f(R) \) theory to study cosmological bounce, which can only happen in the very early universe and at high values of the Ricci scalar, the necessity of such a modified theory of gravity may decrease in the low curvature limit. Consequently near the neighborhood of \( R = 0 \) the theory of gravity may cross-over to conventional GR. If one uses cubic gravity in the other branch\(^3\) then this theory does not have any apparent instabilities but the lowest value of Ricci scalar attainable in this branch is not zero and consequently it may be difficult to transform this effective \( f(R) \) theory into a conventional theory of gravity as GR. The two complimentary branches of cosmological existence meets at \( R = R_c \). It is interesting to note that the Einstein frame description of \( f(R) \) gravity also breaks down precisely at the point \( R = R_c \). In this article we provide some general analysis on the relationship of the conformal transformations and the instabilities which plague any \( f(R) \) theory of gravity. As the instabilities are inherently related with the signs and zeros of the first and second derivative of \( f(R) \), with respect to \( R \), we will analyze the properties of these derivatives and show how these properties affect the conformal transformations.

The material in the present paper are organized as follows. In the next section we briefly recapitulate the conformal relationship between the Jordan frame and the Einstein frame for general \( f(R) \) theories. In this section we present the basics of the bouncing cosmological problem in \( f(R) \) theory and opine on the very important issues related to the signs and zeros of the first and second order derivatives (with respect to \( R \)) of \( f(R) \) theories and their

\(^2\)This branch will be governed by the scalar potential \( V_2(\phi) \) in the Einstein frame as shown later in this article and this branch includes the point \( R = 0 \) where \( R \) is the Jordan frame Ricci scalar.

\(^3\)This branch is governed by the potential \( V_1(\phi) \) in the Einstein frame.
relationship with the conformal transformations. The second section ends with a detailed discussion on the energy conditions in \( f(R) \) theories keeping the discussion focussed around cosmological bounces. The next section 3 addresses the problem of multiple scalar potentials which arise in the Einstein frame when \( f(R) \) is a polynomial of order more than two. In this section we choose to work with cubic gravity where the problem of multiple scalar potentials first shows its signature. In this same section the analysis of the cosmological dynamics of \( f(R) \) theories around \( R = R_c \) is presented. In section 4 the ways to understand the multiple potentials are discussed and explicit numerical solutions giving cosmological bounces in cubic gravity are presented. This section ends with a detailed discussion of the bouncing phenomena in cubic gravity. The last section concludes the paper.

2 Metric \( f(R) \) gravity and its Einstein frame description

In this section we briefly describe the relationship of the two conformally related frames important for us. One is traditionally referred as the Jordan frame where the metric \( f(R) \) theory is defined and the other is the Einstein frame where one can express the gravitational evolution of the cosmological system defined in the Jordan frame. In this article we will be particularly interested in flat FLRW spacetimes and consequently the discussions will be based on equations of cosmological evolution where the curvature constant \( k \) has been set to zero. The elaborate relation between these two frames were described in an earlier paper ref. [26]. Here we briefly express the most important results which connect these two frames.

Using the FLRW spacetime in terms of the scale-factor \( a(t) \),

\[
ds^2 = -dt^2 + a^2(t) \left[ dr^2 + r^2(d\theta^2 + \sin^2 \theta \, d\phi^2) \right],
\]

one can write the field equations in metric \( f(R) \) gravity as [21]:

\[
3H^2 = \frac{\kappa}{F(R)} \rho_{\text{eff}},
\]

\[
3H^2 + 2\dot{H} = -\frac{\kappa}{F(R)} P_{\text{eff}},
\]

where \( H \) is the conventional Hubble parameter defined as \( H \equiv \dot{a}/a \) and \( k \) stands for the constant specifying the curvature of the 3-dimensional spatial hypersurface. The constant \( \kappa = 8\pi G \) where \( G \) is the universal gravitational constant. In the above equations

\[
F(R) \equiv \frac{df(R)}{dR}.
\]

The dot specifies a derivative with respect to cosmological time \( t \). The effective energy density, \( \rho_{\text{eff}} \), and pressure, \( P_{\text{eff}} \), are defined as:

\[
\rho_{\text{eff}} \equiv \rho + \rho_{\text{curv}}, \quad P_{\text{eff}} \equiv P + P_{\text{curv}},
\]

where \( \rho_{\text{curv}} \) and \( P_{\text{curv}} \) are given by

\[
\rho_{\text{curv}} \equiv \frac{RF - f}{2\kappa} - \frac{3H\dot{R}F'(R)}{2\kappa},
\]

\[
P_{\text{curv}} \equiv \frac{\dot{R}F'' + 2H\dot{R}F' + \ddot{R}F'}{\kappa} - \frac{RF - f}{2\kappa},
\]

\[-4\]
which are curvature induced energy-density and pressure. In the above equations the primes designate a derivative with respect to the Ricci scalar $R$. The curvature induced thermodynamic variables exists in absence of any hydrodynamic matter, in contrast to the conventional $\rho$ and $P$ in

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}, \quad (2.8)$$

which has the information of hydrodynamic matter. In this article we assume the fluid to be barotropic so that its equation of state is

$$P = \omega \rho, \quad (2.9)$$

where $\omega$ is a constant and its value is zero for dust and one-third for radiation. It must be noted that $u_\mu$ in eq. (2.8) is the four-velocity of a fluid element and $u_\mu u^\mu = -1$. Till now the description of the gravitational field equation and energy momentum tensor was specified in the Jordan frame.

To understand the dynamics of $f(R)$ gravity one can recast the problem in the Einstein frame by applying a conformal transformation on the Jordan frame metric $g_{\mu\nu}$ as

$$\tilde{g}_{\mu\nu} = F(R) g_{\mu\nu}, \quad (2.10)$$

and simultaneously defining a new scalar field $\phi$ as

$$\phi = \sqrt{\frac{3}{2\kappa}} \ln F(R). \quad (2.11)$$

This scalar field plays an important role in the Einstein frame. The conformally transformed line element in the Einstein frame is

$$d\tilde{s}^2 = -(\tilde{t}^2 + a^2(\tilde{t}) \left[ dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right]), \quad (2.12)$$

where the time coordinate, $\tilde{t}$, and the scale factor, $\tilde{a}$, in the Einstein frame are related to their corresponding Jordan frame terms via the relations $d\tilde{t} = \sqrt{F(R)} dt$ and $\tilde{a}(t) = \sqrt{F(R)} a(t)$. Using these transformations one can formulate the gravitational dynamics of $f(R)$ gravity in the Einstein frame in presence of matter and the scalar field $\phi$ acting as sources. The energy-momentum tensor in the Einstein frame, which is related to $T^{\mu\nu}$ in the Jordan frame, turns out to be

$$\tilde{T}_{\mu\nu} = (\tilde{\rho} + \tilde{P})\tilde{u}_\mu \tilde{u}_\nu + \tilde{P} \tilde{g}_{\mu\nu}, \quad (2.13)$$

where $\tilde{\rho} = \rho/F^2(R)$, $\tilde{P} = P/F^2(R)$ and $\tilde{u}_\mu = \sqrt{F(R)} u_\mu$. In the Einstein frame $\tilde{g}^{\mu\nu} \tilde{u}_\mu \tilde{u}_\nu = -1$. Except $\tilde{T}^{\mu\nu}$, the energy-momentum tensor for the scalar field also acts as source of curvature in the Einstein frame and it is given as

$$S^\alpha _{\nu} = \partial_\alpha \phi \partial_\nu \phi \tilde{g}^{\alpha\mu} - \delta^\alpha _{\nu} \mathcal{L}(\phi), \quad (2.14)$$

where the scalar field Lagrangian is

$$\mathcal{L}(\phi) = \frac{1}{2} \partial_\alpha \phi \partial_\beta \phi \tilde{g}^{\alpha\beta} + V(\phi). \quad (2.15)$$
The scalar field potential in the Einstein frame turns out to be
\[ V(\phi) = \frac{RF - f}{2\kappa F^2}, \] 
where one has to express \( R = R(\phi) \), from eq. (2.11) by inverting it, and then express \( V(\phi) \) as an explicit function of \( \phi \). From the form of \( S^\mu_\nu \) and the signature of the metric used, \( S^0_0 = -\rho \) and \( S^i_i = P \) yielding:
\[
\rho_\phi = \frac{1}{2} \left( \frac{d\phi}{dt} \right)^2 + V(\phi), \quad P_\phi = \frac{1}{2} \left( \frac{d\phi}{dt} \right)^2 - V(\phi),
\] 
where the scalar field \( \phi \) is assumed to be a function of time only. The total energy-momentum tensor responsible for gravitational effects in the Einstein frame is \( \tilde{T}^\mu_\nu + S^\mu_\nu \) which is a mixed tensor with only diagonal components.

The time coordinate, \( t \), and the Hubble parameter, \( H \), in the the Jordan frame are related to the time coordinate, \( \tilde{t} \), and Hubble parameter \( \tilde{H}(\equiv \frac{1}{\tilde{a}} \frac{d\tilde{a}}{d\tilde{t}}) \), in the Einstein frame via the relations:
\[
\tilde{t} = \int_{t_0}^{t} \sqrt{F(R)} dt', \quad H = \sqrt{F} \left( \tilde{H} - \frac{\kappa}{6} \frac{d\phi}{dt} \right).
\] 
As we will be interested mainly in bouncing cosmologies \( t_0 \) will be set to zero. The instant \( t_0 = 0 \) is the bouncing time in the Jordan frame. The Einstein frame description of cosmology can be tackled like FLRW spacetime in presence of a fluid and a scalar field. The presence of the Scalar field potential \( V(\phi) \) gives one a pictorial understanding of the physical system which is lacking in the Jordan frame. Seeing the nature of the potential and the initial conditions of the problem one gets a hint about the possible time development of the system. The time evolution of the scalar field in the Einstein frame is dictated by the equation
\[
\frac{d^2 \phi}{dt^2} + 3\tilde{H} \frac{d\phi}{dt} + \frac{dV}{d\phi} = \frac{\sqrt{\kappa}}{6} (1 - 3\omega)\tilde{\rho},
\] 
where the equation of state of the fluid in the Jordan frame is \( P = \omega \rho \). It is interesting to note that the equation of state of the fluid remains the same in the Einstein frame. The evolution of the energy density in the Einstein frame is given by
\[
\frac{d\tilde{\rho}}{dt} + \frac{\sqrt{\kappa}}{6} (1 - 3\omega)\rho \frac{d\phi}{dt} + 3\tilde{H} \tilde{\rho}(1 + \omega) = 0.
\] 
The above two equations dictate the time evolution of \( \tilde{\rho} \) and \( \phi \) in the Einstein frame. To generate proper bouncing solution from the above two equations one requires the values of only two quantities at the bouncing time, they are \( \phi(t_0), \left( \frac{d\phi}{dt} \right)_{t_0} \), the other parameters are determined from these two at the bouncing time [26]. In general for a flat FLRW spacetime these two values of the respective quantities are enough to solve the whole system in the Einstein frame. The expression of the Hubble parameter and its rate of change in the Einstein frame are given as
\[
\tilde{H}^2 = \frac{\kappa}{3} (\rho_\phi + \tilde{\rho}),
\] 
\[
\frac{d\tilde{H}}{dt} = -\frac{\kappa}{2} \left[ \left( \frac{d\phi}{dt} \right)^2 + \frac{4}{3} \tilde{\rho} \right].
\]
In the next section we will use the above results to formulate a version of energy conditions which can be applied in metric $f(R)$ gravity.

It is to be noted that the prescription for transforming from the Jordan frame to the Einstein frame is limited and fails in a particular case. In this paper we will show an example of this failure. Moreover the two conformal frames may not be equivalent for all cosmological scenarios, especially for bouncing cosmologies in flat FLRW cases in the Jordan frame there corresponds no bounce in the Einstein frame [26].

2.1 Cosmological bounce in Jordan frame for polynomial $f(R)$

As in this paper we will be mainly talking about cosmological bounces so it is pertinent to give the conditions for a bounce. The conditions for a cosmological bounce are:

$$H(t_0) = 0, \quad \dot{H}(t_0) > 0,$$

(2.23)

where $H$ is the Hubble parameter in the Jordan frame where the gravitational effects are dictated by some $f(R)$ theory of gravity. We will assume $t = t_0$ to be the bouncing time. In this paper most of the time we will set the origin of cosmic time at the bouncing point and consequently $t_0 = 0$. From eq. (2.18) it can be seen that $\tilde{t} = 0$ in the Einstein frame at the time of bounce, $t = 0$, in the Jordan frame.

The condition that the Hubble parameter vanishes at the bouncing time transforms to the condition:

$$\rho_0 + \frac{R_0 f_0' - f_0}{2\kappa} = 0,$$

(2.24)

for a flat FLRW spacetime. In the above equation we have used subscripts zero to denote the values of the quantities at the bouncing time when $t = 0$. If one assumes the hydrodynamic matter in the cosmological background satisfies the conditions:

$$4\rho \geq 0, \quad \rho + P \geq 0,$$

(2.25)

then the other bouncing condition becomes

$$\ddot{R}_0 f_0'' + \dddot{R}_0 f_0' < 0,$$

(2.26)

for the flat FLRW solution. In ref. [26] it was shown that if one takes $f(R) = R + \alpha R^2$ then the bouncing conditions invariably constrains $\alpha < 0$ and a successful cosmological bounce only happens in the presence of hydrodynamic matter.

In this paper we will focus on cubic gravity where

$$f(R) = R + \beta R^2 + \gamma R^3,$$

(2.27)

where $\beta$ and $\gamma$ are real numbers. Using this form of $f(R)$ one can easily show that there can be two kinds of bounces, one without any presence of matter and the other a bounce in

\footnote{It is to be noted that the conditions on the energy density and pressure assumed here does not stem from any energy condition and one cannot strictly call it the weak energy condition as the authors did in [26]. Although the conditions can be derived by assuming $T_{\mu\nu}u^\mu u^\nu \geq 0$ in the Jordan frame where $u^\mu$ is any time-like 4-vector, but then also we cannot strictly call this conditions the weak energy condition as $T_{\mu\nu}$ is not the only source of curvature, as implicitly assumed in Einstein gravity, as there are effective Ricci scalar dependent energy density and pressure. The only way to justify the conditions is by assuming no exotic hydrodynamic matter during the bounce phase. Discussions on the energy conditions in $f(R)$ gravity will be presented later in this paper.}
presence of hydrodynamic matter. In particular for the flat FLRW spacetime in absence of any matter the bouncing condition becomes

$$(R_0 f'_0 - f_0) = 0,$$  
(2.28)

predicting that

$$R_0 = -\frac{\beta}{2\gamma}.$$  
(2.29)

As $R_0$ is positive definite at the bouncing point for flat FLRW spacetimes, one concludes that $\gamma$ and $\beta$ must have different signs for cosmological bounces taking place in the absence of any hydrodynamic matter. In the present case if one demands that $f'(R) > 0$, the coefficients $\beta$ and $\gamma$ must satisfy the following inequality

$$0 < \beta^2 \leq 3\gamma.$$  
(2.30)

From eq. (2.29) it was seen that $\beta$ and $\gamma$ should have opposite sign whereas from the above equation it can be uniquely said that for a bouncing solution $\gamma > 0$ and $\beta < 0$.

2.2 A discussion on the signs and zeros of the first and second order derivatives of $f(R)$, with respect to $R$, and their relationship with the conformal transformations

In this article we will be analyzing bouncing phenomena with polynomial $f(R)$. Polynomial $f(R)$ theory themselves have some interesting features. To discuss those features we first state the basic issues about stability of a $f(R)$ gravity theory. From eqs. (2.2) and (2.3) it is seen that $\kappa/F(R)$ acts like an effective gravitational constant in $f(R)$ theory and consequently for $F(R) > 0$ the theory is well defined. On the other hand if $F(R) \leq 0$ the theory of gravitation becomes either ill defined or produces a negative effective gravitational constant which makes the theory unstable. On the other hand if $F(R) > 0$ but $F'(R) < 0$ there can be other instabilities in the theory in the low curvature limit. This second kind of instability is generally called a Dolgov-Kawasaki instability [28]. The authors who first analyzed this kind of instability used a weak gravitational field and applied $f(R)$ theory where as in our case we will never use a weak gravitational field approximation. Consequently, we do not expect Dolgov-Kawasaki like instability to affect the stability of the cosmological bounce phenomena. Unlike the Dolgov-Kawasaki case for low curvature gravity where one obtains unstable solutions for $F'(R) < 0$, in the early universe one can have both $F'(R) > 0$ as well as $F'(R) < 0$, both the branches giving rise to perfectly well behaved bounces. It will be shown later that in higher derivative cosmology guided by $f(R)$ theory the universe avoids to attain the value of $R$ which is a solution of $F'(R) = 0$. As a consequence the two branches are complimentary to each other. If the universe is in the $F'(R) > 0$ branch it will remain so for infinite time and if it is on the other branch it will remain there for infinite time.

In $f(R)$ theory the signs and zeros, of the first and second derivatives of $f(R)$, plays an important part in the cosmological dynamics in the Jordan frame and the conformal transformations connecting these two frames. The effect of the sign of the first derivative of $f(R)$ has been discussed in the last paragraph, henceforth we discuss the effects of the sign of the second derivative of $f(R)$ on the conformal transformation. More over it will be seen that the roots of $F'(R) = 0$ plays an important role as far as the conformal transformations are concerned. Here we present three general results about polynomial $f(R)$ theory and the
conformal transformations relating the Jordan frame and the Einstein frame. Henceforth in this subsection we will assume \( f(R) \) to be a polynomial of order \( n \) so that \( F(R) \) is a polynomial of order \( n - 1 \) and \( F'(R) \) is a polynomial of order \( n - 2 \). The first point about an effective Einstein frame description of an \( f(R) \) theory is as follows. Any Einstein frame description of gravity arising from an odd order polynomial \( f(R) \) where the order of the polynomial \( n > 1 \) must have multiple Einstein frame potentials. To prove this statement we first notice that if one wants to write an unique potential \( V(\phi) \), as given by eq. (2.16), for the scalar field in the Einstein frame one has to invert the relation which gives \( \phi \) as a function of \( R \) in eq. (2.11) and express \( R = R(\phi) \). On the other hand it is seen from eq. (2.11) that \( \phi(R) = \sqrt{\frac{3}{2\pi} \ln F(R)} \) is not uniquely invertible if \( F(R) \) is a polynomial of degree greater than or equal to two. It will be shown later that if \( F(R) \) is an odd order polynomial (hence \( f(R) \) is an even order polynomial) one can circumvent this problem related to invertibility. Only when \( F(R) \) is an even order polynomial (where \( f(R) \) is an odd order polynomial) the problem related to invertibility becomes really an acute one. If \( F(R) \) is a polynomial of degree greater than two then the inverse function \( R = R(\phi) \) can be made single valued by partitioning it into various branches where in each branch the relation \( R = R(\phi) \) remains a single valued monotonic increasing/decreasing function of \( R \). For all these separate branches there will be separate forms of \( V(\phi) \). If \( f(R) \) is an odd order polynomial, then \( F(R) \) is an even order polynomial in \( R \) and thus \( F(R) \) has at least one extremum for some finite value of \( R \). Consequently, the Einstein frame description for such an \( f(R) \) theory will have multiple scalar potentials \( V(\phi) \). If \( F(R) \) has \( n \) extrema, then it has \( n + 1 \) different Einstein frame pictures separated by the points of extrema. If \( f(R) \) is an even order polynomial then \( F'(R) \) is also an even order polynomial which may not have any real zeros if the parameters of the theory are chosen properly.\(^5\) In such a case the question of multiple potentials may be avoided. In figure 1 we present the graph of the scalar field \( \phi \) versus the Jordan frame Ricci scalar \( R \) for cubic \( f(R) \) gravity as given in eq. (2.27). The graph clearly shows the multi valued nature of the inverse function. The minimum of \( \phi \) appears on the right hand side, near \( R_c = 5.0 \times 10^{-13} \) where \( \phi = \phi_c \). Here \( R_c \) is the solution of the equation \( F'(R) = 0 \) for cubic gravity. The inverse function is invertible in the right hand branch or the left hand branch of the minima where the branches bifurcate at the minima. In plotting the above graph we have used Planck units where \( \beta = -10^{26} \text{ GeV}^{-2} = -10^{12} / M_P^2 \), where \( M_P = 10^{19} \text{ GeV} \) is the approximate Planck energy. To represent the numerical values in a brief and compact way all the dimensional quantities are scaled by \( M_P \) in this paper. Henceforth when we give the values of dimensional quantities it will be assumed that we are using the Planck units.

Our next observation regarding polynomial \( f(R) \) theory is as follows. One cannot simultaneously have both \( F(R) > 0 \) and \( F'(R) > 0 \) for all possible values of \( R \) in a polynomial \( f(R) \) theory of gravity. To prove the above statement one must first note that if \( f(R) \) is an even order polynomial, then \( F(R) \) must be an odd order polynomial and consequently it must have at least one real zero. As a result \( F'(R) > 0 \) does not hold for all \( R \). On the other hand suppose \( f(R) \) is such an odd order polynomial so that the even order polynomial \( F(R) \) has no real roots and the condition \( F'(R) > 0 \) holds for all \( R \). But then \( F'(R) \) is an odd order polynomial and has at least one real root and consequently one cannot have \( F'(R) > 0 \) for all \( R \).

\(^5\)In such cases there will be even number of roots of the equation \( F'(R) = 0 \) and one may in principle chose half of the roots to be complex numbers by tuning the parameters appearing in the algebraic equation. The other half of them will be complex conjugates of the earlier roots.
Figure 1. Einstein frame scalar field as function of Jordan frame Ricci scalar. We have taken \( \beta = -10^{12}, \gamma = \frac{2}{3} \beta^2 \) in Planck units.

The third observation regarding the Einstein frame description of \( f(R) \) gravity follows. The Einstein frame description of \( f(R) \) gravity dynamics breaks down at the extrema of \( F(R) \). This observation about \( f(R) \) gravity is more general in nature and also holds for non-polynomial nature of \( f(R) \). This observation was known previously in various forms and in ref. [30] the authors gave a proof of the above statement. Here we present a different version of the proof which is particularly suited for \( f(R) \) theories governing cosmological dynamics. To prove it one must first note that the time evolution of the scalar field \( \phi \) in the Einstein frame is given by eq. (2.19) which contains a term \( \frac{dV}{d\phi} \) on the left hand side. One can calculate \( \frac{dV}{d\phi} \) as

\[
\frac{dV(\phi)}{d\phi} = \frac{dV}{dR} \frac{dR}{d\phi},
\]

where eq. (2.16) can be used to calculate \( dV/dR \) and one can invert eq. (2.11) to calculate \( dR/d\phi \). If it happens that \( F(R) \) has an extremum for some value of \( R \) then at that extremum point \( d\phi/dR = 0 \) and a result \( dR/d\phi \) is divergent. In general at these values of \( R \) where \( \phi \) has extremas, \( dV/dR \) are well behaved. Consequently \( dV/d\phi \) becomes singular at the extremum point and the scalar field dynamics becomes ill defined near the extremum point making the Einstein frame description of \( f(R) \) gravity invalid.

2.3 A brief discussion on energy conditions in \( f(R) \) gravity and their conformal frame dependence

The energy conditions on the energy-momentum tensor, for a perfect fluid, are most well understood in Einsteins gravity where the energy-density, \( \rho \), and pressure, \( P \), of the hydrodynamic fluid acts as source of curvature in space time.\(^6\) The curvature of space time back

\(^6\)In this section the variables \( \rho \) and \( P \) are used as generic symbols of energy-density and pressure in the Einstein frame. The same symbols were used earlier to specify energy-density and pressure in the Jordan frame. We are deliberately using these symbols as we do not want to increase the number of mathematical variables used in the paper and we think the reader can understand there role in the present context.
reacts on the source, via Einstein equation, and modifies them. As a consequence, in cosmology guided by GR, both the metric and the fluid properties evolve in time. This evolution by itself can produce various forms of $\rho$ and $P$ which can produce peculiar situations where the energy density and pressure does not behave as they should for standard matter for which the energy conditions hold in GR. In such circumstances one may invoke exotic matter to justify the properties of $\rho$ and $P$. The standard energy conditions in GR can be briefly described as [31]:

1. The null energy condition (NEC) which is valid when $\rho + P \geq 0$.
2. The weak energy condition (WEC) which is valid when $\rho \geq 0$ and $\rho + P \geq 0$.
3. The strong energy condition (SEC) which is valid when $\rho + P \geq 0$ and $\rho + 3P \geq 0$.
4. The dominant energy condition (DEC) which is valid when $\rho > 0$ and $\rho \geq |p|$.

In the context of $f(R)$ gravity there are various complexities in interpreting and applying energy conditions as the source of curvature of spacetime may itself contain the Ricci scalar $R$ as seen in eq. (2.2) and eq. (2.3). In ref. [32] the authors tried to implement the WEC and DEC in the Jordan frame using $\rho_{\text{eff}}$ and $P_{\text{eff}}$ as defined in eq. (2.5), while the NEC and SEC were derived using the properties of Raychaudhuri’s equation. In refs. [33, 34] the authors give a much vivid discussion on the properties of energy conditions in $f(R)$ gravity. In these references the authors note that the energy conditions may not be simultaneously satisfied in both the Jordan and Einstein frames.

On the other if one started with minimally coupled scalar field in the Einstein frame and then tried to conformally transform the theory in the Jordan frame then matter becomes non-minimally coupled to curvature in Jordan frame. In ref. [35] the authors tried to reformulate the NEC in the Jordan frame in such a way that it reduces to the normal NEC in the Einstein frame under appropriate conditions.

In this subsection we will specifically give an example where most of the energy conditions are violated in the Jordan frame, at the time of bounce, and none of the energy conditions are violated in the Einstein frame, at the corresponding time.

The structures of eq. (2.2) and eq. (2.3) and the bouncing conditions in eq. (2.23) immediately shows that for a successful bounce in the Jordan frame:

- one must have at the time of bounce $(\rho_{\text{eff}})_0 = (\rho_{\text{curv}})_0 + \rho_0 = 0$ where we have assumed $\rho > 0$ in eq. (2.25). The subscript zero specifies the value of the quantities at the time of bounce.
- The other condition at the bouncing point is $(P_{\text{eff}})_0 = P_0 + (P_{\text{curv}})_0 < 0$. This implies $(P_{\text{curv}})_0 < 0$ as we have chosen $P_0 > 0$.

These conditions show that the curvature energy is at its lowest value during bounce, as the energy density of normal matter is maximum during the bounce. In general after bounce $\rho$ gradually fades away but $\rho_{\text{curv}}$ increases and drives the expansion of the universe. Consequently one can say that a cosmological bounce in presence of matter violates the energy conditions in the Jordan frame, if one tries to understand the energy conditions there in terms of the effective energy density and effective pressure. On the other hand for a bounce in the absence of hydrodynamic matter $\rho_{\text{curv}} = 0$ and $P_{\text{curv}} < 0$ at the bouncing point. Consequently in this case also all the energy conditions are violated in the Jordan frame if
one tries to understand the energy conditions there in terms of \( \rho_{\text{eff}} \) and \( P_{\text{eff}} \). On the other hand in the Einstein frame the analysis of the energy conditions are different. We give the Einstein frame analysis of the energy conditions below.

The correspondence between the \( f(R) \) theory variables in the Jordan frame in presence of a perfect fluid \( T^{\mu\nu} \) and the conformally related Einstein frame variables was presented before. As we have assumed normal hydrodynamic matter in the Jordan frame, whose properties were given in eq. (2.25), it can easily be shown that in the Einstein frame one must also have

\[
\tilde{\rho} > 0, \quad \tilde{\rho} + \tilde{P} \geq 0. \tag{2.31}
\]

The above condition does not guarantee that the various energy conditions will be maintained in the Einstein frame. The reason being that in the Einstein frame one has to apply the energy conditions on \( \tilde{T}^{\mu\nu} + S^{\mu\nu} \) and consequently the quantities which go inside the energy conditions are

\[
\rho_E = \tilde{\rho} + \rho_\phi \quad \text{and} \quad P_E = \tilde{P} + P_\phi
\]

where \( \rho_\phi \) and \( P_\phi \) are the energy density and pressure of the time dependent scalar field given as:

\[
\rho_\phi = \frac{1}{2} \left( \frac{d\phi}{dt} \right)^2 + V(\phi), \quad P_\phi = \frac{1}{2} \left( \frac{d\phi}{dt} \right)^2 - V(\phi). \tag{2.32}
\]

From the above equations one can notice that \( \rho_\phi + P_\phi \geq 0 \) always in the Einstein frame but one cannot guarantee that \( \rho_\phi > 0 \) in the Einstein frame and consequently the total energy density \( \rho_E \) may turn out to be negative in the Einstein frame. This can happen because \( \rho_\phi \) consists of the potential \( V(\phi) \) as given in eq. (2.16) which may attain arbitrary small negative values. If one imposes the condition that a meaningful description of Jordan frame cosmology can only be described by its Einstein frame description if \( V(\phi) > 0 \) then one can be sure that the WEC and NEC will be satisfied in the Einstein frame for most of the cosmological processes. For bouncing cosmologies, where the bounce happens in the Jordan frame, one cannot impose this condition in the Einstein frame, for polynomial \( f(R) \) gravity, as in these cases the Einstein frame potential generally is always negative for some values of \( \phi \). In spite of this fact one can work out the cosmological bounce problem in \( f(R) \) gravity using the Einstein frame, without breaking any energy conditions at the bouncing point. One can always work in the Einstein frame in such a way that the cosmological system respects all the energy conditions at the “bouncing point” (which corresponds to the actual bouncing point in the Jordan frame), but slightly away from the bouncing point some of the energy conditions, as the SEC or the DEC may be violated.

To elaborate the above mentioned points we specifically show the various kinds of possible bounces in the Jordan frame and their Einstein frame analogue. First let us concentrate bounce in the presence of matter. From eq. (2.21) one can see that at the time of bounce, one must have

\[
V(\phi) = -\tilde{\rho}. \tag{2.33}
\]

As \( \rho > 0 \) we have \( \tilde{\rho} > 0 \) in the Einstein frame and consequently at the bouncing time, \( V(\phi) < 0 \) in the Einstein frame. If near the bounce point one can ensure \( \tilde{\rho} + \frac{1}{2} \left( \frac{d\phi}{dt} \right)^2 > |V(\phi)| \) then \( \rho_E > 0 \), and as a result the energy conditions will be maintained in the Einstein frame and a successful bounce can happen in the Jordan frame where the energy conditions, interpreted in terms of the effective thermodynamic variables, are violated.
For bounce in the absence of matter, where \( \rho = \dot{\rho} = 0 \), which is a pure curvature driven bounce we have from eq. (2.33), the potential at the time of bounce as \( V(\phi) = 0 \).\footnote{Although a cosmological bounce, in the absence of any hydrodynamic matter, in the Jordan frame requires \( V(\phi) = 0 \) but the vanishing of the Einstein frame potential does not necessarily imply a cosmological bounce, in the absence of hydrodynamic matter, in the Jordan frame in general.} In this case at the time of bounce, in general \( \phi(0) \neq 0 \) in the Einstein frame. As bounce in absence of matter requires \( V(\phi) = 0 \) at the time of bounce, one can note that the equation of state for such a scalar field in the Einstein frame will be

\[
\omega_\phi = \frac{1}{2} \phi'^2(0) - V(\phi) \frac{1}{2} \phi'^2(0) + V(\phi)
\]

which implies

\[
\omega_\phi(0) \rightarrow 1,
\]

which implies a kinetic energy driven free scalar field in the Einstein frame. This result is in general true for all matter less bounces, which can be tackled in the Einstein frame, for spatially flat FLRW spacetimes. More over as in this case \( V(\phi) = 0 \) at the time of bounce \( \tilde{t} = 0 \),

\[
\rho_\phi = \frac{1}{2} \phi'^2(0).
\]

If one puts the condition that \( \phi'(0) = 0 \) then it should have been a symmetric bounce in the Jordan frame but the peculiarity of cosmological bounces, in absence of hydrodynamic matter, makes these symmetrical bounces in the Jordan frame very rare. The reason being that if \( \phi'(0) = 0 \) then \( \rho_\phi = \frac{1}{2} \phi'^2(0) + V(\phi) < 0 \) just near the bounce point as there \( \phi'(0) \rightarrow 0 \) and \( V(\phi) < 0 \) breaking the energy conditions in the Einstein frame, which makes the dynamic development of the system impossible. Consequently any pure curvature driven bounce, where energy conditions are maintained in the Einstein frame, mostly are asymmetric in nature in the the Jordan frame.

From the expressions of \( \rho_\phi \) and \( P_\phi \) it is clear that \( P_\phi > \rho_\phi \) if \( V(\phi) < 0 \) and consequently DEC may be broken near the bounce point, in the Einstein frame for pure curvature driven bounce. On the other hand \( P_\phi \) may be negative if \( V(\phi) > 0 \) and consequently in such regions SEC may be violated. It must be noted that the NEC and WEC are always maintained in the Einstein frame and none of the violations, of the other energy conditions, happen at the bouncing point, but may be violated near to it.

The above examples show that one can have perfect cosmological bounces in the Jordan frame, for the spatially flat FLRW spacetimes, for which the analogous Einstein frame descriptions may not break any of the energy conditions at the bouncing point. The energy conditions in the Einstein frame may dictate the kind of bounce in the Jordan frame.

3 The issue about multiple scalar potentials in the Einstein frame

In this section we present discussions on the most difficult issue regarding the Einstein frame description of polynomial \( f(R) \) theories. The issue is related with the emergence of multiple scalar potentials \( V(\phi) \) in the Einstein frame for a single \( f(R) \) theory. To track the dynamics of \( f(R) \) gravity in the Einstein frame one has to solve the field equation for \( \phi \) which requires the knowledge of \( V(\phi) \). From eq. (2.19) it is seen that if one can construct a potential \( V(\phi) \) in the Einstein frame using eq. (2.16) then one can uniquely determine the behavior of \( \phi \) in
the Einstein frame and once this is done one can transform the results back to the Jordan frame. This program becomes difficult if one is unable to find out an unique scalar potential in the Einstein frame corresponding to a unique $f(R)$ theory in the Jordan frame. In this section we will first discuss how multiple potentials arise, in the Einstein frame, in the case of cubic $f(R)$ theory of gravity. After pointing out the difficulty posed by this multiple Einstein frame description of polynomial $f(R)$ theories we will try to address how these difficulties can be addressed.

To illustrate the difficulties inherently related to polynomial $f(R)$ theories we study the particular case of cubic gravity. In this case we have

$$f(R) = R + \beta R^2 + \gamma R^3,$$

as given in eq. (2.27). In this case one can easily check, from the relation

$$\phi = \sqrt{\frac{3}{2\kappa}} \ln F(R),$$

that

$$R^2 + \frac{2\beta}{3\gamma} R + \frac{1}{3\gamma} \left(1 - e^{\sqrt{2}\kappa/3} \phi\right) = 0,$$

gives $R$ in terms of $\phi$. This relation is not linear and yields two roots as

$$R_{1,2} = -\left(\frac{\beta}{3\gamma}\right) \pm \sqrt{\left(\frac{\beta}{3\gamma}\right)^2 - \frac{1}{3\gamma} \left(1 - e^{\sqrt{2}\kappa/3} \phi\right)}, \quad (3.1)$$

which shows where $R_1 > R_2$ where the plus sign goes with the first branch. For a cosmological bounce in the Jordan frame it was discussed before that $\gamma > 0$ and $\beta < 0$. The above results show that $R = R(\phi)$ is not a single valued function. One can tackle this problem by using two branches for $R$ where we have for branch one $R_1 = R_1(\phi)$ and for the second branch $R_2 = R_2(\phi)$. Each of these branches are single valued functions of $\phi$. It is interesting to note that on the first branch $f''(R_1) = F'(R_1) \geq 0$ and for the second branch $f''(R_2) = F'(R_2) \leq 0$. These two branches meet at a particular $\phi$ called $\phi_c$ whose value will be deduced shortly. These two branches gives two different possible values of $V(\phi)$ in the Einstein frame if one uses eq. (2.16) to express the form of the potential in the Einstein frame.

In the case of cubic gravity

$$RF - f = \beta R^2 + 2\gamma R^3.$$

There is an interesting approximation using which one may avoid the problems arising out of the multiple possible scalar potentials in the Einstein frame. If one just neglects the cubic term in $R$ while inverting the relation between $\phi$ and $R$ then this gives rise to the approximate but unique result

$$R \sim \frac{1}{2\beta} \left(e^{\sqrt{2}\kappa/3} \phi - 1\right).$$

Now one can write an unique scalar potential for cubic gravity in the Einstein frame as

$$V(\phi) \sim \frac{1}{2\kappa} \left[\frac{1}{4\beta} e^{-\sqrt{2}\kappa/3} 2\phi \left(e^{\sqrt{2}\kappa/3} \phi - 1\right)^2 + \frac{\gamma}{4\beta^3} e^{-\sqrt{2}\kappa/3} 2\phi \left(e^{\sqrt{2}\kappa/3} \phi - 1\right)^3\right], \quad (3.2)$$
as obtained in ref. [29]. From the form of the above potential one can verify that $V(\phi) \leq 0$ for all $\phi$ as $\beta < 0$ and $\gamma > 0$. The above potential has only one zero for finite values of $\phi$ and that zero is at $\phi = 0$. From now onwards we will call this Einstein frame potential as the approximate unique Einstein frame potential. This approximate unique Einstein frame potential is defined for all values of $\phi$ although it truly reflects reality only in the neighborhood of $\phi = 0$. The nature of the approximate unique Einstein frame potential for cubic gravity is shown in figure 2. In this potential only the region near $R = 0$ is relevant and later we will show that this region also admits a cosmological bouncing solution in the corresponding Jordan frame.

On the other hand if no approximations are made then one has two scalar potentials $V_1(\phi)$ and $V_2(\phi)$ in the Einstein frame corresponding to the cubic gravity Lagrangian in Jordan frame. The potentials can be written as

$$V_{1,2}(\phi) = \frac{e^{-\sqrt{2\kappa/3}2\phi}}{2\kappa} \left[ \left( \frac{\beta}{3\gamma} \right)^{\mp \sqrt{\left( \frac{\beta}{3\gamma} \right)^2 - \frac{1}{3\gamma} \left( 1 - e^{\sqrt{2\kappa/3}\phi} \right)}} \right]^2 \times \left[ \frac{\beta}{3} \pm 2\gamma \sqrt{\left( \frac{\beta}{3\gamma} \right)^2 - \frac{1}{3\gamma} \left( 1 - e^{\sqrt{2\kappa/3}\phi} \right)} \right] ,$$

where subscript 1 corresponds to the upper minus sign and 2 corresponds to the lower plus sign on the right hand side. These potentials correspond to the two branches of $R$ in the $R - \phi$ plane. The above forms of the scalar field potentials in the Einstein frame immediately shows that as soon as

$$e^{\sqrt{2\kappa/3}\phi} < \left( 1 - \frac{\beta^2}{3\gamma} \right) ,$$

the Einstein frame description of cubic gravity ceases to exist as the potentials turn out to be complex. From the expressions of $V_{1,2}(\phi)$ it is clear that $V_2(\phi) \leq 0$ for all values of $\phi$, for the bouncing solution, where $\beta < 0$ and $\gamma > 0$. More over $V_2(\phi)$ has a zero at $\phi = 0$. 

On the other hand $V_1(\phi)$ can have both positive and negative values for the above choice of parameters and it has a zero for $\phi \neq 0$. It can also be noted that both $V_1(\phi)$ and $V_2(\phi)$ are zero as $\phi \to \infty$.

As an exponential function cannot ever be negative so the term on the right hand side of the inequality, in eq. (3.4), cannot be negative if the inequality holds. If one chooses the parameters $\beta$ and $\gamma$ in such a way that the $\left(1 - (\beta^2/3\gamma)\right)$ is negative then the inequality never holds and the Einstein frame description of cubic gravity holds for all values of $\phi$. If one wants to make the Einstein frame description of cubic gravity valid for all values of $\phi$ then another issue related to the conformal transformations become problematic. The Einstein frame description of cubic gravity holds as long as $F(R) > 0$, otherwise one cannot define the scalar field itself. More over if $F(R) \leq 0$ then the original $f(R)$ theory of gravity itself becomes unstable. If one requires $F(R) > 0$ then the inequality in eq. (2.30) has to be satisfied and in that case the right hand side of the inequality in eq. (3.4) become positive and consequently there arises a finite value of $\phi$ as

$$\phi_c = \sqrt{\frac{3}{2\kappa}} \ln \left(1 - \frac{\beta^2}{3\gamma}\right),$$

(3.5)

which sets the lower limit of the scalar field strength in the Einstein frame. The conformal transformations will never be able to generate any Einstein frame dynamics of cubic gravity if $\phi < \phi_c$. At this value of $\phi$, which corresponds to

$$R_c = -\left(\frac{\beta}{3\gamma}\right),$$

(3.6)

one can verify that $F'(R) = 0$ and as a consequence the Einstein frame description of cubic gravity becomes ambiguous as discussed in subsection 2.2. The two potentials $V_1(\phi)$ and $V_2(\phi)$ are in general different from each other and $V_1(\phi) = V_2(\phi)$ only when $\phi = \phi_c$ and $\phi \to \infty$ i.e., at the two extremities of the range of $\phi$. The approximate scalar potential $V(\phi)$, as given in eq. (3.2), approximately equals $V_1(\phi)$ very close to $\phi = 0$. For this choice one can see that the Ricci scalar in the Jordan frame approaches zero as $\phi \to 0$ in the Einstein frame. Consequently the approximate potential, as proposed in ref. [29] only holds near the $R \to 0$ limit in the Jordan frame and for all other finite values of the Ricci scalar one has to rely on the dynamics produced by one of the actual potentials $V_{1,2}(\phi)$ in the Einstein frame. The nature of the two potentials $V_1(\phi)$ and $V_2(\phi)$ for cubic $f(R)$ gravity are shown in figure 3. The plots show that the two potentials meet at $\phi_c$ which is negative for the particular choice of parameters $\beta$ and $\gamma$ in cubic gravity. The potentials also merge at $\phi \to \infty$ which can be inferred from the plots as the two potentials converge as $\phi$ increases in the positive side. The figure shows that $V_2(\phi)$ is always negative except at $\phi = 0$ and $V_1(\phi)$ has a zero for a non-zero $\phi$.

The above arguments show that the Einstein frame description of cubic gravity (defined in Jordan frame) is limited. On the other hand one may partition the Einstein frame description of cubic gravity in two regions dictated by $V_1(\phi)$ and $V_2(\phi)$ and apply the conformal transformations safely but as soon as one reaches $\phi_c$, where these two potentials merge with each other, the Einstein frame description fails. The next section shows that the Jordan frame analysis of the cosmological system near $R = R_c$, where the Einstein frame description fails, reveals an interesting feature of cubic $f(R)$ theory. Using the approximate potential, as given in eq. (3.2), one never feels this constrained nature of cosmological dynamics in the Einstein frame, as in this approximation the potential is valid for all values of $\phi$, although it truly describes reality only near $\phi = 0$. 

\[ \text{JCAP02(2016)030} \]
Figure 3. The two branches of the exact Einstein frame scalar field potential for $\beta = -10^{12}$, $\gamma = \frac{2}{3} \beta^2$ in Planck units. The solid curve is $V_2(\phi)$ and the dashed curve is $V_1(\phi)$. Here $dV_1,2/d\phi \to \infty$ at $\phi = \phi_c$. The potentials do not exist for $\phi < \phi_c$.

3.1 The cosmological behavior of general polynomial $f(R)$ theories very near to $R = R_c$ in the Jordan frame

We have seen in the above discussions that the point $R = R_c$, where $F'(R_c) = 0$, has some special properties as far as conformal transformations are concerned. In this subsection we will elaborate a bit more about the specific nature of cosmological dynamics, in the Jordan frame, when the system is near $R = R_c$. This analysis is necessary as near $R = R_c$ the conformal transformations fail and one cannot generate any Einstein frame analog of the Jordan frame dynamics and consequently one has to rely only on the Jordan frame dynamics of the system. For this analysis we will employ linear perturbation theory near $R = R_c$ where the Hubble parameter is $H_c$. As eq. (2.2) and eq. (2.3) are written purely in terms of $R$, $H$ and $\rho$, for the spatially flat FLRW case, it is enough to specify these parameters near $(R_c, H_c, \rho_c)$ point, in the $R - H - \rho$ space, as

\begin{align*}
R &= R_c + \delta R, \\
H &= H_c + \delta H, \\
\rho &= \rho_c + \delta \rho
\end{align*}

where $\delta R$, $\delta H$ and $\delta \rho$ are small fluctuations of $R$, $H$ and $\rho$ over $R_c$, $H_c$ and $\rho_c$ for any arbitrary $f(R)$ theory. To linear order in fluctuations one can write

\begin{equation}
f(R) = f(R_c) + F(R_c) \delta R,
\end{equation}

near the $(R_c, H_c)$ point where $F(R_c) = f'(R)|_{R=R_c}$. As we have assumed $F'(R_c) = 0$ we have,

\begin{align*}
F(R) &= F(R_c), \\
F'(R) &= F''(R_c) \delta R, \\
F''(R) &= F''(R_c) + F'''(R_c) \delta R,
\end{align*}

near the $(R_c, H_c)$ point. In this analysis we will assume that $F(R) > 0$ as this ensures stability of the $f(R)$ theory. Using the above results in eq. (2.2) and eq. (2.3) and calculating
order by order one obtains

\[ 3H_c^2 = \frac{\kappa}{F(R_c)} \left[ \rho_c + \frac{1}{2\kappa} \{ R_c F(R_c) - f(R_c) \} \right], \quad (3.7) \]

\[ H_c \delta H = \frac{\kappa}{6F(R_c)} \delta \rho, \quad (3.8) \]

\[ \dot{H} = \delta \dot{H} = -\frac{\kappa(1 + \omega)}{2F(R_c)} \delta \rho, \quad (3.9) \]

and

\[ \rho_c (1 + \omega) = 0. \quad (3.10) \]

The above set of equations reveals more information if one rewrites eq. (3.7) in the following form:

\[ 3H_c^2 = \frac{\kappa}{F(R_c)} \left[ \rho_c + F^2(R_c) V(\phi_c) \right], \quad (3.11) \]

where in the above equation we have used the expression of \( V(\phi) \), as given in eq. (2.16), an Einstein frame object deliberately used for a specific reason. For cosmological bouncing scenarios in the absence of any hydrodynamic matter, in cubic gravity, one has \( V(\phi_c) < 0 \). As a consequence the above equation predicts that the universe will never be able to reach the \( R = R_c \) point after bounce as at that particular cosmological epoch \( H_c^2 < 0 \). The discussion below will show that not only for bouncing solutions in vacua, or any other solutions with hydrodynamic matter, where eq. (3.11) gives \( H_c^2 > 0 \), the evolving universe will never be able to reach the phase when the Ricci scalar attains the value \( R_c \).

To understand the general nature of cosmological evolution near \( R = R_c \) one has to concentrate on the physical content of eq. (3.10). This equation states that very near to the \((R_c, H_c, \rho_c)\) point either the universe is devoid of hydrodynamic matter or the matter present must be acting like a fluid with an equation of state \( \omega = -1 \). These two cases can be separately analyzed. First we assume that the universe is filled up with matter where \( \rho_c \neq 0 \) but \( \omega = -1 \). In this case one attains a local de Sitter universe around \((R_c, H_c, \rho_c)\) point as now \( \delta \dot{H} = 0 \) from eq. (3.9). Much away from the \((R_c, H_c, \rho_c)\) point it is difficult to predict the nature of the Hubble parameter but one can always predict uniquely the value of \( \omega \) as it is a constant in our analysis. In this case it is observed that if the universe at all comes near the \( R = R_c \) point then that universe must have a local de Sitter phase. If one starts to track the cosmological evolution from some point away from the neighborhood of the \((R_c, H_c, \rho_c)\) point one would see that the time required for such a development is infinite. The reason for this observation can be understood by the equation governing time development which is given by

\[ t = \int_0^t \frac{dH}{\dot{H}}. \]

As the universe nears the \((R_c, H_c, \rho_c)\) point \( \dot{H} \to 0 \) and consequently \( t \to \infty \). This implies that for cubic gravity the universe will take an infinite time to reach the \((R_c, H_c, \rho_c)\) point after bounce.

If one particularly concentrates on a bouncing universe devoid of any matter, we have shown previously that the \((R_c, H_c)\) point is never attained as at that point \( H_c \) turns out to
be complex. In this particular case in between the bouncing point where $H = 0$ and $\dot{H} > 0$ and the ($R_c, H_c$) point the universe has to pass through a phase where $\dot{H} = 0$ as because $H$ was increasing with time after the bounce and must have decreased to zero at some time after bounce so that $H^2$ becomes smaller than zero near the ($R_c, H_c$) point. More over at the point where $\dot{H} = 0$ one must have $H \neq 0$ so that $R \neq 0$. This fact can be understood from the nature of the curve in figure 1, where a bounce in absence of hydrodynamic matter can happen for regions on the right hand side of the minima of $\phi$ (in the $V_1(\phi)$ branch) and their $R \neq 0$. One can now apply the same logic around the point where $\dot{H}$ becomes zero and show that the universe will take an infinite time to reach that point.

On the other hand if the universe had explicit hydrodynamic matter during bounce, whose equation of state $\omega \neq -1$, then one can see from eq. (3.10) that the energy density becomes exactly zero at the ($R_c, H_c, \rho_c$) point. This can only happen when the universe becomes infinite in volume (or the physical volume which corresponds to a unit comoving volume diverges) such that the matter energy density becomes exactly zero. If we assume that at a finite cosmological time after the bounce the physical size of the expanding universe to be finite, then the ($R_c, H_c, \rho_c$) point can only be reached by the universe after an infinite time. The density $\rho$, of hydrodynamic matter, will in general decrease as the universe expands after the bounce and tries to reach the ($R_c, H_c, \rho_c$) point, but it will never be exactly zero at any finite time. Consequently in all of the cases one can observe that the bouncing cosmological system, guided by cubic gravity, will never reach the ($R_c, H_c, \rho_c$) point after a finite cosmological time from the bounce. The ($R_c, H_c, \rho_c$) point in a certain sense will always remain inaccessible to the bouncing solutions in cubic gravity. The Einstein frame description also precisely fails at these points. The discussion of the nature of the cosmological system near the ($R_c, H_c, \rho_c$) point is interesting and we have probed cosmological evolutions near this point for bouncing solutions in cubic gravity. We expect our results to be more general and valid for other forms of polynomial gravity as well.

4 Ways to choose between the three potentials in cubic gravity

The natural question which arises in the case of polynomial $f(R)$ gravity is related to the proper choice of the potential in the Einstein frame. The problem becomes non-trivial in the simplest of the cases in cubic gravity where one can have three different potentials in the Einstein frame which corresponds to a unique $f(R)$ in the Jordan frame. The first amongst the three is the approximate unique Einstein potential as given in eq. (3.2). This potential truly reflects cubic gravity for small values of the Ricci scalar $R$. The other two potentials are $V_1(\phi)$ and $V_2(\phi)$ as given in eq. (3.3). This potentials are in general valid for all values of $R$ when either $f''(R) > 0$ or $f''(R) < 0$. In the following part of this section we show that there can be interesting physics involved in choosing the proper potential in the Einstein frame. We will show that presence of matter during bounce can be an important factor which helps us to choose amongst the three possible potentials in the Einstein frame.

In cubic gravity one can have a cosmological bounce in the Jordan frame in presence of matter. In this case one has to satisfy the condition given in eq. (2.33). Following the discussion after eq. (2.33) in subsection 2.3 it is seen that for a successful cosmological bounce in the Jordan frame one must have $V(\phi) < 0$ in the Einstein frame. As in this case a bounce always happens when $V(\phi) < 0$, we can infer that such a bouncing scenario may be described by both $V_1(\phi)$, $V_2(\phi)$ or $V(\phi)$ as given in eq. (3.2) in the Einstein frame, as all of these potentials can have regions where $V(\phi) < 0$. The behaviors of the scale factor for the flat
Figure 4. Bounce with radiation background in branch $V_2(\phi)$, shown by the dashed curve, and in branch $V_1(\phi)$, shown by the solid curve. In these plots $\beta = -10^{12}$, $\gamma = \frac{2}{3} \beta^2$ in Planck units.

Figure 5. Cosmological bounce in vacuum in branch $V_1(\phi)$ where $\beta = -10^{12}$, $\gamma = \frac{2}{3} \beta^2$ in Planck units.

FLRW spacetime near cosmological bounces in presence of radiation are shown in figure 4. It is seen that both the potentials are capable of producing cosmological bounces in presence of matter.

On the other hand if the universe is devoid of any hydrodynamic matter then from eq. (2.33) one can see that at the time of bounce $V(\phi) = 0$. In this case at the time of bounce, in general $\phi \neq 0$ in the Einstein frame. Consequently, pure curvature driven bounce must happen through the potential $V_1(\phi)$ in the Einstein frame as only $V_1(\phi)$ does have a zero for non-zero $\phi$. In this case if $d\phi/d\tilde{t}$ is non-zero when $V_1(\phi) = 0$, the scalar field climbs to positive values of $V_1(\phi)$ during the bouncing period. The plot showing the evolution of the scale factor $a(t)$, for cubic gravity, near a cosmological bounce in vacuum is shown in figure 5.

From the above description of bouncing phenomenon in the Jordan frame we observe that although the approximate unique Einstein frame potential can describe a cosmological
bounce in presence of matter but it has no information about pure curvature energy dominated bounce. Consequently one can infer that the scope of the approximate unique Einstein frame potential for describing cosmological bouncing phenomenon in the Jordan frame is very limited. It is only a true reflection of cubic gravity for very small values of the Ricci scalar $R$ and more over it can only describe a matter induced bounce in the Jordan frame. The nature of a cosmological bounce in cubic $f(R)$ theory where one employs the approximate unique Einstein frame potential is shown in figure 6. This bounce can only happen in presence of matter background and for our case we have chosen the background hydrodynamic fluid to be comprised of radiation.

4.1 Detailed description of the cosmological bounces in cubic $f(R)$ gravity

Discussions regarding pre-big bang cosmology which discusses about cosmological bounce was noted long ago, in 1992, in the ref. [36]. It is well known that the cosmological behavior can be quite different in the Jordan frame and Einstein frame. In particular, it was mentioned in [26] that a bounce in the Jordan frame is not usually associated with a corresponding bounce in the Einstein frame for flat FLRW cosmologies. In spite of these observations it is surprising that the Jordan frame bounce scenario can be understood from the Einstein frame scalar potential picture.

The evolution of the universe near the bouncing point in the Jordan frame can be understood from the nature of figure 1 where the minimum of $\phi$ occurs at $R = R_c$. On the right of $R_c$ in figure 1 one sees that $\phi$ is an increasing function of $R$. This can only happen for the branch where $\ln F(R)$ is an increasing function of the Ricci scalar, and from the initial discussions in section 3, this happens in the branch $V_1(\phi)$. Consequently the branch $V_1(\phi)$ corresponds to that part of cosmological dynamics in the Jordan frame where $R > R_c$. In this
branch $R$ ranges from $R = R_c$ to $R = \infty$. The other potential branch $V_2(\phi)$ corresponds to the region $R < R_c$ in figure 1 where $\phi$ is a decreasing function of the Ricci scalar. This branch does have the embryo of Dolgov-Kawasaki like instabilities when the Ricci scalar becomes small. On the other hand as because in this branch the Ricci scalar can come close to zero, the original $f(R)$ theory may cross-over to conventional GR in the neighborhood of $R = 0$. From this discussion one can trace the behavior of cosmological bounce in the Jordan frame if one follows the behavior of the scalar field $\phi$ in the Einstein frame.

Let us first discuss about the particular potential branch $V_1(\phi)$ in the Einstein frame, which can support cosmological bounces in presence or absence of hydrodynamic matter. First we discuss about cosmological bounces in the absence of any hydrodynamic matter. Bounce in the absence of matter in the Jordan frame occurs when the scalar potential in the Einstein frame vanishes, as discussed previously. In this branch, at the bouncing time $\tilde{t} = 0$, the value of the scalar field $\phi = \phi_0$ when $V_1(\phi_0) = 0$. In this branch, there are several possible lines of development of the cosmological system. These possibilities depend on the “velocity” $(d\phi/d\tilde{t} \lor \phi')$ of the scalar field at the time of bounce. The contraction phase can start from $\phi < \phi_0$, cross that point (Jordan frame bounce), rise up to a maximum along the potential $V_1(\phi)$ and come down again. If the scalar field has enough initial kinetic energy it can also cross over the maxima of $V_1(\phi)$ and then slowly roll towards $\phi \to \infty$. If it chooses to come down again it crosses $\phi = \phi_0$ again, but this does not correspond to a bounce, which can be understood by a careful look at the Einstein frame initial conditions. Then it goes down towards $\phi_c$. While the universe moves towards $\phi_c$ in the Einstein frame, the cosmological system in the Jordan frame moves so that the Ricci scalar their becomes $R_c$. In doing so the universe will require an infinite time in the Jordan frame. One can also program bounce, in absence of matter, in such a way that the Einstein frame scalar field starts from above $\phi > \phi_0$ during the contracting phase, then the cosmological system crosses the bounce point and goes down towards $\phi_c$. Cosmological bounce in presence of hydrodynamic matter follows a similar course except for the fact that at the time of bounce the scalar field assumes some value $\phi = \phi_0$ where $V_1(\phi_0) < 0$. More over, one must remember that bounce in the presence of matter can also occur in the $V_2(\phi)$ branch of the potential.

Similar consideration holds for the potential branch $V_2(\phi)$ but only for matter induced bounce. In this case $\phi$ assumes some value $\phi = \phi_0$ at the time of bounce, such that $V_2(\phi_0) < 0$. In this branch increasing or decreasing of $\phi$ correspond to decreasing or increasing of the scalar curvature $R$. The particular case of bounce in the $V_2(\phi)$ branch, where $\phi$ starts from some $\phi < \phi_0$ and evolves towards $\phi = 0$, is interesting. In this case the contracting phase may proceed to the bounce point $\phi = \phi_0$ and then crosses this point and goes towards $\phi = 0$. In this branch as $R \to 0$ as $\phi \to 0$, as apparent from figure 1, one can assume that the universe is evolving to a radiation dominated phase where the dynamics is governed by general relativity. The region near the $R \to 0$ can act as a cross-over region from $f(R)$ theory to general relativistic dynamics.

Before we end this discussion we present the various values of the scalar field and its time derivative used in the Einstein frame to numerically solve the equations dictating the bouncing phenomena. The plot in figure 5 shows a cosmological bounce in absence of hydrodynamic matter where $\phi_0 = -0.11$, $\phi'_0 = 10^{-6}$. Note that out of these two conditions only $\phi'$ can be chosen suitably. The value of $\phi$ at bounce is already specified by the relation in eq. (2.29) which gives,

$$\phi_0 = \sqrt{\frac{3}{2\kappa}} \ln \left( 1 - \frac{\alpha^2}{4\beta} \right).$$

(4.1)

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This implies $\phi_0 = -0.05 M_p = -0.05 \times 10^{19}$ GeV; $\phi'_0 = 10^{-6} M_p^2 = 10^{32}$ GeV$^2$. 

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In this case $\phi$ starts from some point where $V_1(\phi) < 0$ (see figure 3), crosses $\phi_0$, rises up to a maximum along the potential and then comes down again.

The plots in figure 4 and figure 6 shows cosmological bounces in presence of radiation in both the branches $V_1(\phi)$, $V_2(\phi)$ and in the approximate unique Einstein frame potential for the same initial conditions $\phi_0 = -0.13$, $\phi'_0 = 10^{-8}$. Unlike the matterless case, here both the initial conditions can be chosen suitably, with the only condition that the scalar field potential at bounce has to be negative. In these cases $\phi$ starts from some point $\phi < -0.13$, crosses the point $\phi = \phi_0$, goes up to a maximum along the potential and then comes down again. The important thing to be noticed in the above mentioned bouncing phenomena, in presence of matter, is that the same conditions at the time of bounce produces three different kinds of time dependence of the scale factor $a(t)$ in the Jordan frame. Consequently these bounces are distinctly different and shows the richness of polynomial $f(R)$ gravity.

5 Conclusion

In this paper we have extended our analysis on cosmological bounces, for the spatially flat FLRW solutions, in higher derivative $f(R)$ theories of gravity. In our earlier work [26] we proposed a scheme by which one can study the phenomena of cosmological bounces, using spatially flat FLRW solutions, in $f(R)$ theories using only the equations of general relativity. The method relied on the validity of the conformal transformations which connected cosmological dynamics in the Jordan frame, where the dynamics is governed by $f(R)$ theory, and the auxiliary Einstein frame which is used to do the calculations related to the problem. Once the problem is solved in the Einstein frame one can map the solutions to the Jordan frame and get the bounce dynamics. This program can be flawlessly applied for the case of quadratic gravity where $f(R) = R + \alpha R^2$ where $\alpha$ is some constant negative real number. In the case of quadratic gravity it turns out that there is only one unique Einstein frame description of the bouncing phenomena. It was immediately noticed that the methods used in the earlier publication were not adequate when one uses higher order polynomials as cubic $f(R)$.

The Einstein frame picture of cubic gravity and other higher order polynomial gravity does not have a one-to-one correspondence with the Jordan frame description of the cosmological evolution and consequently one does not know a priory which picture in the Einstein frame corresponds to the actual picture in the Jordan frame. In the present work we have proposed a consistent method using which one can map the cosmological bouncing problem from the Einstein frame to the Jordan frame. The present work shows that a proper understanding of the conformal transformations yields interesting properties of the cosmological bounces in the Jordan frame. Unlike the case of quadratic gravity, in higher order polynomial gravity there are multiple ways in which cosmological bounces can occur. In the case of quadratic gravity only bounce in presence of hydrodynamic matter was allowed whereas in cubic gravity one can also have cosmological bounces in absence of any form of hydrodynamic matter.

The issue of energy conditions was briefly touched in the present work. We have shown that one can always produce a cosmological bounce in the Jordan frame without violating the weak energy condition and the null energy condition in the Einstein frame. The strong energy condition and the dominant energy conditions may be violated near the bouncing point in the Einstein frame. The interesting thing about the Einstein frame analysis, of the bouncing problem, is that all the energy conditions are valid at $\tilde{t} = 0$, which corresponds to the actual bouncing time in the Jordan frame. For a cosmological bounce the energy conditions in the Jordan frame are violated but one must have to remember that the energy...
conditions are not that clearly and uniquely formulated in the Jordan frame. As a result one may not be very serious about the energy conditions in the Jordan frame.

In this work we have introduced the techniques to understand the origin of multiple scalar potentials in the Einstein frame for cubic or higher order \( f(R) \) gravity. We have explicitly worked out the cosmological bounces arising out of the various potential branches in the Einstein frame. Our discussion also includes the approximate Einstein frame potential which one can use to study cosmological behavior near \( R = 0 \) as given in [29]. As soon one includes the multiple potentials in the Einstein frame the richness of the cosmological scenario becomes apparent. In the penultimate section we have presented the ansatz following which one may map from the Einstein frame picture to the Jordan frame picture without any confusion.

The other important observation which is reported in the present work is related to the cosmological behavior of the universe as it evolves from a bounce to a state where the Ricci scalar attains a value \( R = R_c \) where \( R_c \) is the root of the equation \( f''(R) = 0 \). Here the primes denote derivatives with respect to the Ricci scalar. It was noticed that the conformal transformations between the Jordan frame and the Einstein frame breaks down precisely at the point \( R = R_c \) for cubic gravity. In the present work we show that the state of the universe when \( R = R_c \) can never be connected to the bouncing universe in a finite time. In other words the flat FLRW universe, described by cubic \( f(R) \) theory in the Jordan frame, would take an infinite time to attain \( R = R_c \) after the cosmological bounce. The roots of \( f''(R) = 0 \) bifurcates the region of cosmological existence. We have presented our results for cubic gravity but we hope that our results are general in nature and can hold true for higher order polynomial gravity. This particular issue requires further research. More over the points \( R = R_c \) are interesting, as out of the the two branches emanating from this point in figure 1, the one on the right of \( R = R_c \) has \( f''(R) > 0 \) and the one on the left of \( R = R_c \) has \( f''(R) < 0 \). On the left branch one may expect Dolgov-Kawasaki like instability in the low curvature limit of the theory. But as this branch also includes the point \( R = 0 \) one has the freedom for a cross-over to more conventional GR dynamics as the Ricci scalar becomes small in magnitude. Ideally the Dolgov-Kawasaki instability cannot be applied to the very early universe, in \( f(R) \) theory, as at those times the curvature may be very high and consequently one can always use the left branch to model cosmological bounces happening in the very early universe.

Lastly, we have utilized all the conceptual understanding about cubic gravity to actually solve the cosmological bounce problem. The results are given in figure 4, figure 5 and figure 6. These results are obtained by solving the problem in the Einstein frame and then mapping the solutions back to the Jordan frame. These solutions show various bounce possibilities in the Jordan frame for the case of cubic gravity. We expect the possibilities of various kind of cosmological bounces will increase with the order of polynomial gravity as more and more Einstein frame scalar potentials will emerge in the conformally connected Einstein frame. In this paper we do not present the theory of cosmological perturbations as our main aim was to analyze the complicated and rich background cosmological evolution in polynomial \( f(R) \) theories. We will present the perturbations on this background in a forthcoming publication. Although, the use of the Einstein frame as an auxiliary frame to calculate the background bouncing dynamics was purely a technical choice it turned out that this technical choice yields much valued information about the actual Jordan frame cosmological evolution if one can disentangle the multi valued correspondence between the conformal frames. The technique evolved in the present work can be valuable as one can use pure general relativistic
techniques to solve a problem in a higher derivative $f(R)$ theory of gravity. Although the methods discussed in this article were particularly aimed to tackle the problem of bouncing cosmologies in the Jordan frame, we firmly believe that many of the techniques developed in this article can also be applied to understand the general nature of $f(R)$ theories.

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