Rigid body motion in viscous flows using the Finite Element Method

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Abstract

A new model for the numerical simulation of a rigid body moving in a viscous fluid flow using FEM is presented. One of the most interesting features of this approach is the small computational effort required to solve the motion of the rigid body compared to a pure fluid solver. The model is based on the idea of extending the fluid velocity inside the rigid body and solving the flow equations with a penalization term to enforce rigid motion inside the solid. In order to get the velocity field in the fluid domain the Navier-Stokes equations for an incompressible viscous flow are solved using a fractional-step procedure combined with the two-step Taylor-Galerkin for the fractional linear momentum. Once the velocity field in the fluid domain is computed, calculation of the rigid motion is obtained by averaging translation and angular velocities over the solid. One of the main challenges when dealing with the fluid-solid interaction is the proper modelling of the interface which separates the solid moving mass from the viscous fluid. In this work the combination of the level set technique and the two-step Taylor-Galerkin algorithm for tracking the fluid-solid interface is proposed. The good properties exhibited by the two-step Taylor-Galerkin, minimizing oscillations and numerical diffusion, make this method suitable to accurately advect the solid domain avoiding distortions at its boundaries, and thus preserving the initial size and shape of the rigid body. The proposed model has been validated against empirical solutions, experimental data and numerical simulations found in the literature. In all tested cases, the numerical results have shown to be accurate, proving the potential of the proposed model as a valuable tool for the numerical analysis of the fluid-solid interaction.

Keywords

Finite Element Method; projection method; fractional-step; two-step Taylor-Galerkin; level set algorithm; newtonian fluid; fluid-solid interaction.

1. Introduction

Movement description of immersed bodies is present in several manufacturing processes (i.e. sedimentation procedures, metallurgical industry, etc.), but it is also of great interest in different scientific areas, such as geological sedimentation processes, asteroid impacts or fluvial transport and deposition, to name a few. In most of these cases, it becomes necessary the study of the interaction among a set of rigid bodies immersed in a fluid flow. To this end, and as a starting point, one single rigid body dynamics is usually analysed. Despite of the large number of works addressing this issue in the last years, development of new models capable to accurately deal with this problem is still a challenge nowadays.

Modelling the motion of an immersed body in a fluid flow requires the solution of the coupled fluid and solid dynamics. In addition to that, a procedure must be provided for the tracking of
the solid position in time, preserving its volume and shape. Many efforts have been devoted in the last years to solve these issues.

In order to track the fluid-solid interface, some formulations have been proposed in the past within the frame of classical mesh-based methods. The deformable spatial domain [1] and the arbitrary Lagrangian-Eulerian formulations [2,3,4] intended to describe the body position by just moving the mesh. However, these methods presented some problems when a high relative motion among the bodies occurred. To circumvent this issue, an improved mesh updated method was introduced by Behr and Tezduyar [5]. Nevertheless, many alternative formulations were proposed in order to avoid the numerical invonveniences of remeshing. In this context the immersed boundary method [6] and the fictitious domain method [7, 8, 9] are specially remarkable. In the fictitious domain method, the fluid motion equations are extended inside the solid particles domains. Lagrangian multipliers are defined in the rigid bodies domains to match over these regions the fluid flow and solid motion velocities [7-13]. Some other formulations, however, combine techniques to describe the surface of the particles along with the Lattice Boltzmann method to solve the dynamics of the problem in a fixed mesh [14-16]. In addition, a wide variety of meshfree methods have been proposed over the last years: diffuse element method [17], element free Galerkin method [18,19], reproducing kernel particle method [20,21], h-p cloud method [22], partition of unity method [23], meshless local Petrov–Galerkin method [24], finite point method [25], radial basis function [26,27], Smoothed Particle Hydrodynamics (SPH) [28-34] or Taylor-SPH [35-39], among others. The main advantage of meshfree methods is that numerical solutions can be achieved without using a computational grid, thus dealing in a straightforward manner with particle tracking and therefore avoiding some of the difficulties encountered in classical mesh-based methods. However, most of these meshless methods present other numerical issues, such as smearing in their solutions, lack of consistency or problems when dealing with boundary conditions.

In this paper a numerical model for the direct computational simulation of freely moving rigid bodies in fluids using the Finite Element Method (FEM) is presented. The proposed computational algorithm is based on the idea of extending the fluid velocity inside the rigid body and solving the flow equations with a penalization term to enforce rigid motion inside the solid [12,13]. One of the most interesting features of this approach is the low computational effort required to solve the motion of the rigid body compared to a pure fluid solver [12]. In order to get the velocity field in the fluid domain the Navier-Stokes equations for an incompressible viscous flow are be solved using the fractional-step procedure proposed by Chorin [40,41] combined with the two-step Taylor-Galerkin [42-51] to solve the fractional linear momentum equation. Once the velocity field in the fluid domain is computed, calculation of the rigid motion is obtained by averaging translation and angular velocities over the solid. One of the main problems found in the fluid-solid interaction is the proper modelling of the interface which separates the solid moving mass from the viscous fluid. In this work the combination of the level set technique [13,52-55] and the two-step Taylor-Galerkin algorithm for tracking the fluid-solid interface is adopted [41]. The good properties exhibited by the two-step Taylor-Galerkin [42-51], minimizing oscillations and numerical diffusion, make this method suitable to accurately advect the solid domain avoiding distortions at its boundaries, and thus preserving the initial size and shape of the rigid body.

The paper is structured as follows. First, the mathematical model and the proposed computational algorithm are presented in Section 2. Next, in Section 3, the Navier-Stokes equations are discretized, velocity in the solid domain is obtained and the rigid body properties are advected by means of the level set technique. To demonstrate the performance of the proposed method, some numerical examples are presented in Section 4, which is followed by the final conclusions in Section 5.
2. Mathematical model

Let us consider a rigid solid \( S(t) \) submerged in an incompressible flow contained in a domain \( \Omega \). Thus \( F(t) = \Omega - S(t) \) is the fluid domain. The fluid-solid interaction problem can be modeled by the Navier-Stokes equations in \( F(t) \) and the rigid motion of the solid \( S(t) \):

\[
\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \, \text{div} \left( \mathbf{u} \times \mathbf{u} \right) = \text{div} \mathbf{\sigma} + \rho \, \mathbf{g} \quad \text{for } \mathbf{x} \in F(t) \text{ and } t > 0
\]

\[
\mathbf{u} = \mathbf{V} + \mathbf{W} \times (\mathbf{x} - \mathbf{x}_G) \quad \text{for } \mathbf{x} \in S(t) \text{ and } t > 0
\]

\[
M \ddot{\mathbf{V}} = - \int_{\Sigma} \mathbf{\sigma}(\mathbf{u}, p) \, n \, d\Omega + M \mathbf{g}
\]

\[
I \ddot{\mathbf{W}} = - \int_{\Sigma} \mathbf{\sigma}(\mathbf{u}, p) \times (\mathbf{x} - \mathbf{x}_G) \, d\Omega
\]

where \( \Sigma(t) \) is the fluid-solid interface, \( n \) is the normal pointing towards the solid, \( \mathbf{g} \) is gravity; \( \rho(\mathbf{x}, t), \mathbf{u}(\mathbf{x}, t) \) and \( p(\mathbf{x}, t) \) are the density, velocity and pressure fields; and \( M, \mathbf{x}_G \) and \( I \) are the mass, center of gravity and inertia matrix of the solid, respectively.

Equations (3) and (4) translate the solid acceleration as a result of gravity and fluid forces at the interface, where the linear and angular velocities are noted as \( \mathbf{V} \) and \( \mathbf{W} \). The stress tensor \( \mathbf{\sigma} \) is defined as

\[
\sigma_{ij}(\mathbf{u}, p) = \frac{\mu}{\rho} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - p \delta_{ij}
\]

being \( \mu \) the dynamic viscosity of the fluid.

The above system has to be complemented by the initial and boundary conditions on \( \Gamma \).

2.2. Computational algorithm

The proposed computational algorithm is based on Patankar’s projection method [12,13]. The idea is to extend the fluid velocity inside the solid body and to solve the flow equations with a penalization term to enforce rigid motion inside the solid. Considering a penalization parameter \( \lambda >> 1 \), the momentum equation applicable in the entire domain \( \Omega \) can be written as:

\[
\rho(\mathbf{x}, t) \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{x}, t) \text{div} (\mathbf{u} \times \mathbf{u}) = \text{div} \mathbf{\sigma} + \rho(\mathbf{x}, t) \mathbf{g} + \lambda \rho(\mathbf{x}, t) H(\phi)(\bar{\mathbf{u}} - \mathbf{u})
\]

coupled with the incompressibility condition:

\[
\text{div} \mathbf{u} = 0
\]

In the proposed approach, the density field, \( \rho(\mathbf{x}, t) \), can be expressed in terms of an indicator function, \( \phi(\mathbf{x}, t) \), which identifies the portion of the domain occupied by either the solid or the fluid. Thus, the density field can be calculated as

\[
\rho(\mathbf{x}, t) = \rho_f + (\rho_s - \rho_f) H(\phi(\mathbf{x}, t))
\]

being

\[
H(\phi) = \begin{cases} 0 & \text{if } \phi \leq 0 \quad (\text{fluid}) \\ 1 & \text{if } \phi > 0 \quad (\text{solid}) \end{cases}
\]
The calculation of the rigid motion \( \vec{u} \), is obtained by averaging translation and angular velocities over the solid:

\[
\vec{u} = \frac{1}{M} \int_{\Omega} \rho(x, t) H(\phi) \, d\Omega + \left( I^{-1} \int_{\Omega} (x - x_G) \times \rho(x, t) H(\phi) \, d\Omega \right) \times (x - x_G) \tag{10}
\]

The rigid body moves with an advection velocity that can be chosen to be either the flow velocity \( u \) or the rigid motion \( \vec{u} \) (both values are equivalent just in the limit \( \lambda \to \infty \)). However, the choice to follow the solid phase with the velocity \( \vec{u} \) instead of \( u \) is recommended here, since a strictly rigid motion of the solid is guaranteed, independently on numerical errors, a feature which is desirable in practice.

In order to track the solid motion, advection of the solid properties is required. Since density is a material property moving with the flow, its material derivative is zero, and therefore:

\[
\frac{D(\rho)}{D_t} = \frac{\partial(\rho)}{\partial t} + \vec{u} \, \nabla(\rho) = 0 \quad \text{for } x \in \Omega \text{ and } t > 0 \tag{11}
\]

In order to solve above system of equations (6)-(11), the following fractional-step algorithm is proposed:

Given a time step \( \Delta t \), and being \( t^n = n\Delta t \), it is possible to go from \( u^n \approx u(\cdot, t^n) \) to \( u^{n+1} \) by:

Step 1: Solving Navier-Stokes equation (6)-(7) with \( \lambda = 0 \) for a time step \( \Delta t \); \( \vec{u}^n \) will be obtained.

Step 2: Computing \( \vec{u}^n \) from (10) after replacing \( u \) by the result \( \vec{u}^n \) obtained in the previous step.

Step 3: Solving

\[
\rho(x, t) \frac{\partial u}{\partial t} = \lambda \rho(x, t) H(\phi)(\vec{u} - u) \tag{12}
\]

for a time step \( \Delta t \). Choosing \( \lambda = 1/\Delta t \), this step returns

\[
u^{n+1} = \begin{cases} 
\vec{u}^n & \text{in } F(t) \\
u^n & \text{in } S(t)
\end{cases} \tag{13}
\]

It is important to note here that one of the advantages of considering the penalization formulation is that the penalization equation (12) can be discretized in time in an implicit manner enabling larger penalization coefficients and therefore better accuracy in satisfying the interface boundary conditions. On the contrary, explicit time discretization of (12) would require smaller values for \( \lambda \) than \( 1/\Delta t \).

Step 4: Advecting the indicator function, \( \phi \), with velocity \( \vec{u} \) will allow the determination of the new density field (8), and thus the new position for the solid at \( t^{n+1} \)

\[
\frac{\partial \phi}{\partial t} + \vec{u} \, \nabla \phi = 0 \tag{14}
\]

2.3. Modelling the fluid-solid interface via level set technique

One of the main problems found in the fluid-solid interaction is the proper modelling of the interface which separates the solid moving mass from the viscous fluid. In this work a level set technique has been adopted [41,52-55], which will be described next.

The unsteady flow of the interacting fluid and solid is modelled using equations (6)-(11).
Figure 1: Example of description of each material subdomain using one indicator function.

For the example illustrated in Figure 1, the material properties (i.e. densities) in both subdomains may be calculated from \( \phi \) as

\[
P(\mathbf{x}, t) = P_w + (P_s - P_w)H(\phi(\mathbf{x}, t))
\]  

(15)

where \( P \) is a material property, and the subscripts \( w, s \) stand for water and solid respectively.

Since \( P \) is a material property moving with the flow, its material derivative is zero

\[
\frac{D(P)}{Dt} = \frac{\partial(P)}{\partial t} + \mathbf{u} \cdot \nabla(P) = 0
\]  

(16)

and considering the dependence of the material properties on the indicator function, (15), this condition may be written as

\[
\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + \mathbf{u} \cdot \nabla \phi = 0
\]  

(17)

which states that the indicator is purely advected by the flow and requires the function \( H(\cdot) \) to be smooth.

In the case the indicator function, \( \phi \) is a linear function of its position, \( \mathbf{x} \), its second order spatial derivative is zero and equation (17) is exact, and also its numerical approximations. Thus, considering \( \phi \) as linear, also benefits the numerical solution of (17) as the front smearing caused by low order numerical schemes or the oscillations induced in the high order case will not appear. The simplest linear function is that with slope unity, i.e. the distance function: \( |\nabla \phi| = 1 \).

As regards the definition of \( H(\cdot) \), this equation works well only for small density ratios. For high density ratios it results in unwanted instabilities in the pressure field that may destroy the calculated problem solution. This problem is related to the numerical solution of a badly conditioned Poisson equation for the pressure [56]. To avoid abrupt changes in the material properties when crossing the interface, it is interpolated through a constant thickness tube of total width \( 2\delta \) surrounding the interface [55], where \( \delta \) is taken of the order of the mesh size. There exist different alternatives for the definition of function \( H(\cdot) \), such as the simple linear interpolation
\[ \begin{align*}
H(\phi) &= \begin{cases} 
0 & \text{for } \phi \leq -\delta \\
(\phi + \delta) & \text{for } -\delta < \phi < \delta \\
1 & \text{for } \phi \geq \delta
\end{cases} \quad (18)
\end{align*} \]

However, extra smoothing can be gained considering other functions. This paper considers the sine function given below [52]

\[ \begin{align*}
H(\phi) &= \begin{cases} 
0 & \text{for } \phi \leq -\delta \\
\sin \left( \frac{\pi(\phi + \delta)}{4\delta} \right) & \text{for } -\delta < \phi < \delta \\
1 & \text{for } \phi \geq \delta
\end{cases} \quad (19)
\end{align*} \]

This definition of the interpolation function, based on the distance to the interface, requires keeping the indicator function as a distance function. In effect, equation (17) states that the indicator is advected by the fluid velocity: as the fluid velocity is not uniform in the domain, the initial distance function will be distorted as time progresses and, after some time, it will not be a distance function any longer.

This issue can be addressed by using the fluid flow velocity as proposed in equation (17) to advect the indicator. Once the indicator is advected, it must be corrected in order to comply with the \(|\text{grad} \phi| = 1\) condition. This can be achieved by solving at any time \(t\), the following problem to steady state [55]:

\[ \frac{\partial \phi(\tau)}{\partial \tau} + S(\phi(t)) |\text{grad} \ \phi(\tau)| = S(\phi(t)) \quad (20) \]

with initial conditions

\[ \phi(x, \tau = 0) = \phi(x, t) \quad (21) \]

being \(S(.)\) the sign function and \(\tau\) a fictitious time.

Clearly, the steady-state solution of this problem is compliant with the condition \(|\text{grad} \phi| = 1\) and the zero level set of \(\phi(\tau \to \infty)\) matches that of \(\phi(t)\).

Equation (20) may be written also as

\[ \frac{\partial \phi(\tau)}{\partial \tau} + S(\phi^n) \frac{\text{grad} \ \phi(\tau)}{|\text{grad} \ \phi(\tau)|} |\text{grad} \ \phi(\tau)| = S(\phi^n) \quad (22) \]

Which is an advection problem with velocity

\[ v = S(\phi^n) \frac{\text{grad} \ \phi(\tau)}{|\text{grad} \ \phi(\tau)|} \quad (23) \]

This equation for the velocity indicates that the problem characteristics initiate at the interface position and travel with velocity \(\pm 1\). Therefore, reconstruction of the indicator function as a distance function initiates at the interface position and progresses along its outward normal direction. Therefore, the critical zone, surrounding the interface position, is reconstructed in the first fictitious-time, \(\tau\), iterations of the solution of problem (22).

This idea may be easily extended to \(M\) solids in a multiphase flow of \(N\) immiscible fluids by incorporating \(N+M-1\) indicator functions, as it is shown in Table 1. However, for a system of \(M\) rigid bodies interacting in an incompressible multi-fluid flow, a special method to handle collisions should be designed, which does not fall within the scope of this paper. Thus, in the following, only one solid immersed in a viscous fluid will be considered.
3. Numerical model

The numerical method used to solve the Navier–Stokes equations (6)-(7) and the advection (17) and correction (22) of the indicator function is the two-step Taylor-Galerkin method. This method is described in [44], where the interested reader can find the details of the derivation.

3.1. Step 1: Solving Navier-Stokes equations with $\lambda = 0$

In order to solve Navier-Stokes equations (6)-(7) with $\lambda = 0$ for a time step $\Delta t$, the fractional-step procedure proposed by Chorin [40] is followed.

The velocity is decomposed into two parts

$$\Delta \mathbf{u}^n = \Delta \mathbf{u}^{*,n} + \Delta \mathbf{u}^{+,n}$$

such that

$$\rho \frac{\Delta \mathbf{u}^{*,n}}{\Delta t} + \rho \, \text{div} (\mathbf{u}^n \times \mathbf{u}^n) - \text{div} \, \mathbf{r}^n - \rho \mathbf{g}^n = 0$$

$$\rho \frac{\Delta \mathbf{u}^{+,n}}{\Delta t} + \text{grad} \, p^{n+1} = 0$$

along with the continuity equation

$$\text{div} \, \mathbf{u}^{n+1} = 0$$

Concerning the spatial discretization, 2D linear triangles have been chosen because of their numerical efficiency and excellent behaviour in the solution of problems involving strong discontinuities [42-43, 45-47]. However, Babuska-Brezzi condition [57, 58] does not allow the use of the same order of interpolation for both velocity and pressure unless a special stabilization technique is used. As it was shown in [44, 59], the proposed algorithm provides with the required stabilization.

Therefore, above resultant equations from the time discretization, are discretized in space as follows:

**Step I: Fractional velocity discretization.** The fractional linear momentum equation

$$\rho \frac{\Delta \mathbf{u}^{*,n}}{\Delta t} + \rho \, \text{div} (\mathbf{u}^n \times \mathbf{u}^n) - \text{div} \, \mathbf{r}^n - \rho \mathbf{g}^n = 0$$

is discretized in space using the two-step Taylor-Galerkin algorithm:

| Fluid 1 | Fluid 2 | … | Fluid N | Solid 1 | … | Solid M |
|---------|---------|---|---------|---------|---|---------|
| $\phi_1$ | $\phi_1 \leq 0$ | $\phi_1 > 0$ | … | $\phi_1 > 0$ | $\phi_1 > 0$ | … | $\phi_1 > 0$ |
| $\phi_2$ | $\phi_2 \leq 0$ | $\phi_2 \leq 0$ | … | $\phi_2 > 0$ | $\phi_2 > 0$ | … | $\phi_2 > 0$ |
| $\phi_3$ | $\phi_3 \leq 0$ | $\phi_3 \leq 0$ | … | $\phi_3 > 0$ | $\phi_3 > 0$ | … | $\phi_3 > 0$ |
| … | … | … | … | … | … | … | … |
| $\phi_N$ | $\phi_N \leq 0$ | $\phi_N \leq 0$ | … | $\phi_N \leq 0$ | $\phi_N > 0$ | … | $\phi_N > 0$ |
| $\phi_{N+1}$ | $\phi_{N+1} \leq 0$ | $\phi_{N+1} \leq 0$ | … | $\phi_{N+1} \leq 0$ | $\phi_{N+1} \leq 0$ | … | $\phi_{N+1} > 0$ |
| $\phi_{N+2}$ | $\phi_{N+2} \leq 0$ | $\phi_{N+2} \leq 0$ | … | $\phi_{N+2} \leq 0$ | $\phi_{N+2} \leq 0$ | … | $\phi_{N+2} > 0$ |
| … | … | … | … | … | … | … | … |
| $\phi_{N+M-1}$ | $\phi_{N+M-1} \leq 0$ | $\phi_{N+M-1} \leq 0$ | … | $\phi_{N+M-1} \leq 0$ | $\phi_{N+M-1} \leq 0$ | … | $\phi_{N+M-1} > 0$ |

Table 1: Assignment of each material subdomain using N+M-1 indicator functions.
\[(u_i)^{*,n+1/2} = (u_i)^n - \frac{\Delta t}{\tau} \left( \frac{\partial (u_i u_i)}{\partial x_j} - \frac{1}{\rho} \int \nabla \right) \] (29)

\[M \Delta u_i^{*,n} = \Delta t \left\{ \int \nabla \left[ \frac{\partial N^T}{\partial x_j} (u_j u_i)^{*,n+1/2} - \frac{1}{\rho} \int \nabla (\tau_{ij}) \right]^n \right\} d\Omega + \int N^T(g_j)^{*,n+1/2} d\Omega - \int N^T \left( (u_j u_i)^{*,n+1/2} - \frac{1}{\rho} (\tau_{ij}) \right) \cdot n_j d\Gamma \] (30)

**Step II: Continuity equation discretization.** Continuity equation at \(t^{n+1/2}\), is

\[\text{div } u^{n+1} = 0 \] (31)

and using the incremental momentum split,

\[\rho \frac{\partial u^{*,n}}{\partial t} + \nabla p^{n+1} = 0 \] (32)

discretization of the continuity equation can be written as

\[\text{div } u^{*n} - \Delta t \text{ div } \left( \frac{1}{\rho} \nabla p^{n+1} \right) = 0 \] (33)

where \(u^{*n} = u^n + \Delta u^{*,n}\).

Applying the standard Galerkin discretization, spatial discretization of above equation is

\[\int \nabla \left[ \frac{1}{\rho} \nabla \right] d\Omega + \int \frac{1}{\rho} \nabla \right] d\Omega = - \int \nabla \left\{ \text{div } u^{*,n} \right\} d\Omega - \int \frac{1}{\rho} \nabla \right\} d\Omega + \int \left[ \text{grad } p^{n+1} \cdot n \right] N d\Gamma \] (34)

where pressure is prescribed at \(\Gamma_p\).

In order to compute the boundary integral, equation (32)

\[\frac{1}{\rho} \nabla \right] d\Omega = - \frac{1}{\Delta t} \int \left[ \text{div } u^{*,n} - (u^n + \Delta u^{*,n}) \right] \cdot n d\Gamma \] (35)

is projected along the normal direction, resulting in

\[\int \left[ \text{grad } p^{n+1} \cdot n \right] N d\Gamma = - \frac{1}{\Delta t} \int \left[ \text{grad } p^{n+1} - (u^n + \Delta u^{*,n}) \right] \cdot n N d\Gamma \] (36)

Therefore, the system of equations to be solved for the pressure increment is

\[\int \nabla \left[ \frac{1}{\rho} \nabla \right] d\Omega + \int \frac{1}{\rho} \nabla \right\} d\Omega = - \int \nabla \left\{ \text{div } u^{*,n} \right\} d\Omega - \int \nabla \left\{ \text{grad } p^{n+1} \cdot n \right\} N d\Gamma \] (37)

**Step III: Velocity correction.** Once pressure has been computed in the previous step, the velocity increment, \(\Delta u^*\), must be corrected by discretizing in space equation (32)

\[\int \frac{\rho}{\Delta t} N^T \nabla d\Omega \Delta u^{*,n} + \int N^T \text{grad } p^{n+1} d\Omega = 0 \] (38)

Solving above system of equations \(\Delta u^{*,n}\) is obtained, which is added to \(\Delta u^{*,n}\), resulting in the total velocity increment \(\Delta u^n\):
\[ \Delta \mathbf{u}^n = \Delta \mathbf{u}^*n + \Delta \mathbf{u}^{*,n} \quad (39) \]

and thus,

\[ \mathbf{\bar{u}}^n = \mathbf{u}^n + \Delta \mathbf{u}^n \quad (40) \]

The proposed algorithm can be extended to the 3-dimensional problem by just using linear tetrahedra elements.

The maximum allowed time step in the solution of Navier-Stokes equations is [44]:

\[ C \leq \beta \sqrt{\frac{1}{Pe^2} + \alpha - \frac{1}{Pe}} \quad (41) \]

where:
- \( C \) is the Courant number: \( C = \frac{|\mathbf{u}|}{h/\Delta t} \)
- \( Pe \) is the Peclet number: \( Pe = \frac{|\mathbf{u}| h}{2\mu/\rho} \)
- \( h \) is the element size (minimum height of each triangle has been considered here).
- \( \alpha = 1 \) when using the lumped mass matrix and \( \alpha = 1/3 \) when using the consistent mass matrix in the solution of steps I and III.
- \( \beta \) is a safety coefficient, typically 0.85-0.9

Thus, the global time step limit is calculated as the minimum time step allowed in the mesh. Equation (41) incorporates the effects of viscosity via Peclet number and, in order to account for non-linearities, a safety factor \( \beta \) is considered.

### 3.2. Step 2: Calculating the rigid motion

Recalling equation (10), calculation of the rigid motion \( \mathbf{\bar{u}} \), is obtained by averaging translation and angular velocities over the solid, \( S \):

\[ \mathbf{\bar{u}} = \frac{1}{M} \int_B \rho(x, t) H(\phi) \mathbf{u} \, d\Omega + \left( I^{-1} \int_B (x - x_G) \times \rho(x, t) H(\phi) \mathbf{u} \, d\Omega \right) \times (x - x_G) \quad (42) \]

where the density field, according to equation (15), is given by

\[ \rho(x, t) = \rho_w + (\rho_s - \rho_w) H(\phi(x, t)) \quad (43) \]

and therefore \( \rho(x, t) H(\phi) \) reduces to \( \rho_s \) in the solid domain \( S(t) \).

Applying the Finite Element Method basic theory, the nodal velocity in the solid, \( \mathbf{\bar{u}}^n_e \), is computed as:

\[ \mathbf{\bar{u}}^n_e = \frac{\rho_s}{M} \sum_e \left[ \mathbf{\bar{u}}^n_e \right] \Omega_e + \left[ \frac{\rho_s}{I} \sum_e \left( x_e - x_G \right) \times \mathbf{\bar{u}}^n_e \right] \Omega_e \times (x_i - x_G) \quad (44) \]

where the sub-index \( e \) stands for “element” and \( G \) for gravity center of the element (i.e. the Gauss point position); \( M \) is the mass of the solid, \( \Omega_e \) is the element volume (or surface), and \( I \) is the inertia matrix; \( \mathbf{\bar{u}}^n_e \) is the velocity computed in the previous step calculated at the element level and \( \rho_s \) is the density of the solid.

### 3.3. Step 3: Obtaining the final velocity

As explained in Section 2.2., to get the final velocity at \( t^{n+1} \), it is necessary to solve the following equation:

\[ \rho(x, t) \frac{\partial \mathbf{u}}{\partial t} = \lambda \rho(x, t) H(\phi)(\mathbf{\bar{u}} - \mathbf{u}) \quad (45) \]
where \( \rho = \rho(x, t) \) is given by equation (44).

Choosing \( \lambda = 1/\Delta t \), above equation becomes:

\[
\rho_s \frac{\partial \mathbf{u}}{\partial t} = \rho_s \frac{\mathbf{u}^n - \mathbf{u}^n}{\Delta t}
\]  
(46)

Applying the standard Galerkin discretization:

\[
\int_{\Omega} N^T \rho_s \mathbf{N} d\Omega \Delta \mathbf{u}^n = \int_{\Omega} N^T \rho_s (\mathbf{u}^n - \mathbf{u}^n) \mathbf{N} d\Omega
\]  
(47)

and \( \mathbf{u}^{n+1} \) is obtained in the entire domain, such that \( \mathbf{u}^{n+1} = \mathbf{u}^n + \Delta \mathbf{u}^n \):

\[
\mathbf{u}^{n+1} = \begin{cases} 
\bar{\mathbf{u}}^n & \text{in } F(t) \\
\mathbf{u}^n & \text{in } S(t) 
\end{cases}
\]  
(48)

3.4. Step 4: Advecting the solid properties

The objective of the indicator function is assigning fluid/solid properties at each point within the domain. The algorithm used for tracking the interface is based on (i) advection of the indicator function using the computed velocity field

\[
\frac{D \Phi}{Dt} = \frac{\partial \Phi}{\partial t} + \bar{\mathbf{u}} \text{grad} \Phi = 0
\]

and (ii) correction of the indicator function in order to keep it as distance function. This condition can be achieved by obtaining the stationary solution of:

\[
\begin{aligned}
\left( \frac{\partial \Phi(\hat{\tau})}{\partial \hat{\tau}} + S(\Phi(t)) |\text{grad} \Phi(\hat{\tau})| \right) &= S(\Phi(t)) \\
\Phi(x, \hat{\tau} = 0) &= \Phi(x, t)
\end{aligned}
\]  
(51)

Above equation is indeed an advection equation with velocity.
\[ \mathbf{v} = S(\phi^n) \frac{\text{grad} \phi(\hat{\tau})}{|\text{grad} \phi(\hat{\tau})|} \]

and a source term given by \( S(\phi^n) \).

Therefore, equation (51) can be written as

\[ \frac{\partial \phi(\hat{\tau})}{\partial \hat{\tau}} + \mathbf{v} \text{grad} \phi(\hat{\tau}) = S(\phi^n) \]  

(52)

and can be solved using the two-step Taylor-Galerkin algorithm:

\[ \phi^{n+1/2} = \phi^n + \frac{\Delta \hat{\tau}}{2} [S(\phi^n) - \mathbf{v} \text{grad} \phi^n] \]

\[ \mathbf{M} \Delta \phi^n = \Delta \hat{\tau} \left\{ \int_{\Omega} \mathbf{N} S(\phi^{n+1}) d\Omega + \int_{\Omega} (\text{grad} T(\mathbf{vN}))^T \phi^{n+1/2} d\Omega - \int_{\Gamma} \mathbf{N}^T \mathbf{v} \phi^{n+1/2} n d\Gamma \right\} \]  

(53)

Once the indicator function has been advected and corrected, the material properties, \( P \), are interpolated (15) using the sine function given in (19).

Note here that, in order to accelerate convergence of equation (52) to the steady state, the algorithm uses the lumped mass matrix along with an optimum time step for each mesh element [44]:

\[ \Delta \hat{\tau} = \beta \frac{\Delta t}{|\mathbf{v}|} = \beta h \]

being \( \beta \) a safety coefficient lower than 1.

Nevertheless, the correction phase implies a significative computational cost. In order to reduce this computational effort, both phases, advection and correction, are limited to a thin region about the zero level set of the indicator function. Following this procedure, the number of iterations is drastically reduced since condition \( |\text{grad} \phi| = 1 \) must be fulfilled only within this thin region. The size of the mentioned region is about \( 3\delta - 4\delta \) centred at the interface.

4. Examples and applications

In this section, some numerical examples have been selected in order to validate the method since either they have been studied experimentally or they have already been solved using other numerical solutions.

The examples considered next have been chosen in order to illustrate the main advantages of the proposed model:

- The method is capable to reproduce the experimental results of the flow dynamics around a fixed body for a wide range of Reynolds numbers.
- The model is able to track the fluid-solid interface without distortion for a very high number of iterations, maintaining the initial volume and shape of the solid.
- The proposed model is able to predict the dynamics of an immersed body with a high level of accuracy.

4.1 Flow around a circular cylinder

This problem has been widely studied by many researchers in the past [60-71] giving rise to a variety of experimental results, empirical formulae and advanced numerical methods. Thus detailed analysis of the flow around a cylinder is of high interest allowing validation of models.
against experimental and computational results, and becoming the starting point for further applications to the study of more complex geometries such as planes, ships or submarines.

In order to assess the performance of the proposed model when dealing with the flow around a fixed rigid body, the classic problem of the flow around a circular cylinder will be analysed here. This problem consists of a solid circular cylinder of diameter $D = 1$ m and density $\rho_s = 3000$ kg/m$^3$ fixed and immersed in a viscous fluid flow as sketched in Figure 2.

A non-structured mesh of 4712 linear triangles is used for the computation (Figure 3). A horizontal uniform velocity is prescribed in the left boundary of the domain, $v_x = 1$ m/s and $v_y = 0$, while $v_y = 0$ is prescribed in the top and bottom boundaries. Pressure is only prescribed in the right boundary and set equal 0 (see boundary conditions in Figure 3).
The parameters used in the computation are taken as follows: length and height of the fluid domain are $L = 25 \text{ m}$ and $H = 15 \text{ m}$ respectively; the fluid is assumed to be of newtonian type with density $\rho_f = 1000 \text{ kg/m}^3$. The center of the cylinder is placed at a distance of 5 m from the left side of the domain and 7.5 m high. The solid velocity is set to 0 throughout the whole simulation. The time-step used for the calculation is $\Delta t = 5 \times 10^{-2} \text{ s}$.

In order to analyse the dynamics of the flow around the cylinder as a function of the Reynolds number, different values of viscosity $\mu$ will be used in the computations.

For $Re = 10$: A non-oscillatory solution is reached. Behind the cylinder the boundary layer begins to be detached and two eddies start to be formed spinning in opposite directions; behind these two eddies the streamlines get close again parallel and symmetric. These results are the expected ones for Reynolds numbers in the range $1 < Re < 30$ according to the literature [65-66]. Figures 4 and 5 show the velocity vectors and pressure values in the whole domain.

![Figure 4: Velocity vectors in the whole domain for Re = 10](image1)

![Figure 5: Pressure values in the whole domain for Re = 10](image2)
For $\text{Re} = 30$: Behind the cylinder, slight periodic oscillations in the flow begin to be detected. These results are expected for Reynolds numbers within the range of $30 < \text{Re} < 90$ according to [65-66]. Figures 6 and 7 show the velocity field and pressure values in the whole domain for $t = 212$ s. In Figure 8 the detailed velocity field in the rear of the cylinder is depicted.

Figure 6: Velocity vectors in the whole domain for $\text{Re} = 30$ ($t = 212$ s)

Figure 7: Pressure values in the whole domain for $\text{Re} = 30$ ($t = 212$ s)
For $Re = 90$: Two eddies begin to be detached alternatively at both sides of the middle axis in the rear wake, being continuously regenerated with a period of 5.5 s. Figures 9 and 10 show the velocity field and pressure values in the whole domain for $t = 212$ s.
For $Re = 10^3$: The periodic detachment of the eddies at the rear wake continues with a period of 5 s. Figures 11 and 12 depict the velocity field and pressure values in the whole domain for $t = 212$ s. Separation of the rear wake is observed at about $\theta = 80^o$ (Figure 13) as observed experimentally [67,68].
Figure 12: Pressure values in the whole domain for $Re = 10^3$ ($t = 212$ s)

Figure 13: Detailed velocity contours around the cylinder for $Re = 10^3$ ($t = 212$ s)

For $Re = 4 \times 10^5$: The periodic detachment of the eddies at the rear wake remains unchanged with a period of 5 s. Figures 14 and 15 depict the velocity vectors and pressure values in the whole domain for $t = 212$ s. Separation of the rear wake is observed at about $\theta = 120^\circ$ (Figure 16) as observed experimentally [67,68]. Comparison of Figures 13 and 16 clearly show how the wake width becomes narrower for $Re = 4 \times 10^5$ [67,68].
It is well known that the Strouhal number, $St$, is a dimensionless number characterizing the vortex shedding at the rear wake of the cylinder, which depends on the Reynolds number, $Re$: $St = \frac{D}{T\nu_\infty} = f(Re)$. In this numerical example the $St$ dependency on $Re$ has been analyzed and compared with some empirical models (Figure 17), showing that the proposed numerical approach agrees well with the results found in the literature [60,69,70], especially for Reynolds numbers larger than 160. For $Re < 160$ the proposed model seems to overestimate the empirical models proposed by Roshko [60], Ponta and Aref [69] and V. Strouhal [70], however the agreement is reasonably good comparing with the experimental results given in [71].
Figure 16: Detailed velocity contours around the cylinder for Re = 4 \times 10^5 (t = 212 s)

Figure 17: Strouhal number dependency on the Reynolds number

The theoretical inviscid pressure distribution on a circular cylinder is [68]:

\[ c_p = \frac{p - p_\infty}{\frac{1}{2} \rho v_\infty^2} = 1 - 4 \sin^2 \theta \]

Figure 18 shows the computed pressure distribution around the cylinder, \( c_p = \frac{p - p_\infty}{\frac{1}{2} \rho v_\infty^2} = 1 - 4 \sin^2 \theta \), for different values of the Reynolds number.
4.2 Motion of a rigid square cylinder immersed in a rotational fluid flow

One of the main challenges for the numerical modelling of a rigid body moving inside a viscous fluid is maintaining the volume and shape of the solid throughout its movement. As explained in Sections 2-3, the solution adopted in this work is the combination of the level set technique and the two-step Taylor-Galerkin algorithm for tracking the fluid-solid interface. The good properties exhibited by the two-step Taylor-Galerkin, minimizing oscillations and numerical diffusion, make this method suitable to accurately advect the solid domain avoiding distortions at its boundaries. Illustrating this important feature of the proposed model is the aim of the numerical example presented herein.

This problem consists of a solid square cylinder of $D = 0.2$ m immersed in a rotating viscous fluid whose initial velocity is $v_n = 0$ (normal component) and $v_t = 5\pi r$ m/s (tangential component), being $r$ the distance to the center of the domain (see Figure 20). The boundary conditions are: $v_n = 0$ along the four sides of the domain and $p = 0$ at the middle point of each side (Figure 21).

A non-structured mesh of 11322 linear triangles is used for the computation (Figure 21). The parameters used in the computation are taken as follows: length and height of the fluid domain are $L = 1$ m and $H = 1$ m respectively; the fluid is assumed to be of newtonian type with viscosity $\mu = 1$ Pa s and density $\rho_f = 1000$ kg/m$^3$. The center of the solid square cylinder is placed at the center point of the domain (0.5 m from the left side of the domain and 0.5 m high) and its density is $\rho_s = 1000$ kg/m$^3$. Neither the solid position nor its velocity is restricted, thus both are left free to evolve throughout the simulation. The time-step used for the calculation is $\Delta t = 0.8 \times 10^{-4}$ s.
Figure 19: Geometry

Figure 20: Initial conditions for velocity: (a) Velocity vectors in the whole domain; (b) Velocity vectors in the soil domain.
Figure 21: (a) Computational mesh and boundary conditions; (b) Computational mesh in the solid domain.
Figure 22 shows the position of the fluid-solid interface for 8 different time steps during a total revolution of the solid about its center of mass. It can be observed that the shape and volume of the body are well preserved after a very high number of iterations (5 \times 10^3 iterations). Even though the rigid body is free to move in the domain, the absence of instabilities allow the solid to spin about its center without either translating or distorting.

![Figure 22: Evolution of the fluid-solid interface](image)

According to the initial velocity field, \( v_t = 5\pi \text{ m/s} \), the angular velocity is \( \omega = 5\pi \text{ rad s}^{-1} \). Thus theoretically, in \( t = 0.4 \text{ s} \) the solid must perform one complete rotation about its center of mass, which is in accordance with the obtained numerical result (see Figure 22). Velocity vectors and pressure distribution after one total rotation of the solid are shown in Figures 23 and 24.

![Figure 23: Velocity vectors for t = 0.4 s: (a) Total domain; (b) Solid domain](image)
4.3 Particle settling in a viscous fluid

Finally, to assess the dynamical behaviour of a submerged body inside a viscous flow, the gravity-driven motion of a cylinder in a viscous fluid is considered. The aim of this example is to validate the performance of the model under dynamic conditions, in a case in which experimental and numerical solutions are available.

4.3.1. Settling of a circular cylinder with $\rho_s > \rho_f$

The layout of the problem is sketched in Figure 25. A solid circular cylinder of diameter $D = 0.05$ m and density $\rho_s = 7800$ kg/m$^3$ is immersed in a viscous fluid of newtonian type with viscosity $\mu = 8$ Pa s and density $\rho_f = 1200$ kg/m$^3$ and subject to gravity $g = 9.8$ m/s$^2$. The length and height of the fluid domain are given by $L = 1.4$ m and $H = 2.43$ m respectively. The center of the cylinder is placed at 0.7 m from the left side of the domain and 1.62 m high. No movement restrictions are imposed to the solid. No-slip boundary conditions are assumed along the four boundaries and pressure is prescribed as $p = 0$ in the top corners of the domain. Initial conditions for velocity are $v_x = v_y = 0$ in the whole domain, while hydrostatic initial conditions for pressure are considered.

A non-structured mesh of 9950 linear triangles is used for the computation (Figure 26). The size of mesh elements in the finest area (solid domain) is $h_{\text{min}} = 0.008$ m while $h_{\text{max}} = 0.05$ m in the coarsest part of it. The time-step used for the computation is $\Delta t = 9.66 \times 10^{-4}$ s.

The proposed model allows the computation of the velocity field and pressure distribution in the domain induced by the rigid body motion. As shown in Figure 27 for $t = 1.26$ s no oscillations or instabilities are present even in the area surrounding the cylinder. Figure 28 shows the vertical velocity distribution along a diametric horizontal line across the domain once the terminal velocity is achieved.
Figure 25: (a) Geometry and (b) initial conditions for pressure.

Figure 26: (a) Computational mesh and boundary conditions; (b) Detailed mesh in the solid domain.
Since no movement restrictions are imposed to the cylinder, the rigid body is free to move in all directions. Evolution of the solid center of mass is plotted in Figure 29. The rigid body follows a vertical path whose deviation from the vertical line is of the order of 2h_{min} (relative error of \( \varepsilon_r = 0.0249 \)). The terminal settling velocity can be derived from Figure 30. For the time interval between \( t = 0.3 \) s and \( t = 1.3 \) s the vertical velocity curve can be approximated by a linear function with a correlation factor \( R^2 = 1 \). The slope of the linear function is the value of the terminal velocity, \( v_{terminal} = 1.079 \) m/s
Some experimental correlations have been proposed by several authors in order to establish the relationship between $C_D$ and $Re$ \cite{68,72}. For $4 < Re < 10$, the following expression can be considered:

$$C_D = 6.8052 \cdot Re^{-0.3755}$$  \hspace{1cm} (54)

For the computed settling velocity, $v_{\text{terminal}} = 1.079$ m/s the Reynolds number is $Re = 8.09$ and therefore, according to \eqref{C_D} the resulting value of the drag coefficient is $C_D = 3.103$.  

Figure 29: Path followed by the solid center of mass

Figure 30: Evolution in time of the y-coordinate of the solid center of mass and linear regression.
This problem has been previously analyzed experimentally and numerically by [73] and the experimental-based values for the terminal settling velocity and the drag coefficient are $v_{\text{terminal}} = 1.067$ m/s and $C_D = 3.12$ respectively, which confirms the good performance of the numerical model proposed herein.

4.3.2. Settling of a circular cylinder with $\rho_s < \rho_f$

In order to test the performance of the model for a rigid body lighter than the fluid, a solid circular cylinder of diameter $D = 0.05$ m and density $\rho_s = 500$ kg/m$^3$ subject to gravity $g = 9.8$ m/s$^2$ is considered. The problem geometry and initial and boundary conditions are the same as in the previous case (Figures 25-26). The fluid properties considered are: viscosity $\mu = 4$ Pa s and density $\rho_f = 1200$ kg/m$^3$. Same non-structured mesh of 9950 linear triangles is used for the computation (Figure 26). The time-step used in the calculation is $\Delta t = 9 \times 10^{-4}$ s.

Velocity field in the domain induced by the rigid body motion is depicted in Figure 31 for $t = 0.55$ s. Again, no oscillations or instabilities are observed. Figure 32 shows the vertical velocity distribution along a diametric horizontal line across the domain once the terminal velocity is achieved.

![Figure 31: Velocity field for $t = 0.55$ s](image)

The rigid body is free to move in all directions, since no movement restrictions are imposed. Evolution of the solid center of mass is plotted in Figure 33. The rigid body follows a vertical path whose deviation from the vertical line is of the order of $2h_{\text{min}}$ (relative error of $\epsilon_r = 0.0236$). The terminal ascending velocity can be derived from Figure 34. For the time interval between $t = 0.53$ s and $t = 1.70$ s the vertical velocity curve can be approximated by a linear function with a correlation factor $R^2 = 0.9999$. The slope of the linear function is used to get the value of the terminal velocity, $v_{\text{terminal}} = 0.3936$ m/s.
Figure 32: Vertical velocity distribution along a diametric horizontal line across the domain for $t = 0.55$ s

Figure 33: Path followed by the solid center of mass

Figure 34: Evolution in time of the $y$-coordinate of the solid center of mass and linear regression.
4.3.1. Settling of a circular cylinder with $\rho_s = \rho_f$

Finally, in order to assess the stability of the model, the same problem is solved for a solid cylinder whose density is $\rho_s = \rho_f = 1200 \text{ kg/m}^3$. The problem geometry and initial and boundary conditions are the same as in the previous cases (Figures 25-26). The fluid viscosity $\mu = 4 \text{ Pa s}$ and gravity $g = 9.8 \text{ m/s}^2$. Same non-structured mesh of Figure 26 is used for the computation. The time-step used for the calculation is $\Delta t = 9 \times 10^{-4} \text{ s}$.

No movement restrictions are imposed to the rigid body, and thus it is free to move in all directions. The positions of the solid at $t = 0$ and $t = 3 \text{ s}$ (3333 iterations afterwards) are compared in Figure 35. It can be observed that the solid does not either move or distort during the computation, which is a consequence of the absence of numerical instabilities in both velocity and pressure. Figures 36 and 37 depict the path followed by the solid center of mass and the evolution of its y-coordinate in time.
5. Conclusions

A new model for the numerical simulation of a rigid body moving in a viscous fluid flow using FEM has been presented. The model is based on Patankar’s projection method [12] combined with the level set algorithm for the tracking of the fluid-solid interface [13,52-55]. The Navier-Stokes equations for a newtonian incompressible fluid are solved using a fractional-step procedure [40,41]. To avoid distortion of the fluid-solid interface the two-step Taylor-Galerkin algorithm is proposed for the solution of the level set advection and correction equations [42-51].

The proposed model has been validated in three different situations:

- Dynamics of a viscous fluid flow moving around a fixed rigid body;
- Dynamics of a freely moving rigid body immersed in a rotational viscous fluid flow;
- Dynamics of a freely moving rigid body immersed in a viscous fluid and subjected to gravity.

In all tested cases, the numerical results have shown to be in good agreement with other empirical solutions, experimental data and numerical simulations found in the literature [60-73], showing the potential of the proposed model as a valuable tool for the numerical analysis of the fluid-solid interaction.

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