1. Introduction

Let $\mathbb{C}$ be the field of complex numbers. If $z \in \mathbb{C}$ then we write $\bar{z}$ for its complex-conjugate and denote by $\iota: \mathbb{C} \to \mathbb{C}$ the corresponding element of the group $\text{Aut} (\mathbb{C})$ of automorphisms of $\mathbb{C}$. We write $\mathbb{Q} \subset \mathbb{C}$ for the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. It is well-known that the subfield $\bar{\mathbb{Q}}$ is $\text{Aut} (\mathbb{C})$-stable and the natural homomorphism

$$\text{Aut}(\mathbb{C}) \to \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$$

is surjective. If $W$ is a $\mathbb{Q}$-vector space, $\mathbb{Q}$-algebra or $\mathbb{Q}$-Lie algebra then we write $WC$ for the corresponding $\mathbb{C}$-vector space (respectively, $\mathbb{C}$-algebra or $\mathbb{C}$-Lie algebra) $W \otimes \mathbb{Q} \mathbb{C}$.

Let $f(x) \in \mathbb{C}[x]$ be a polynomial of degree $n \geq 2$ without multiple roots. Suppose that $p$ is a prime that does not divide $n$ and a positive integer $q = p^r$ is a power of $p$. As usual, $\varphi(q) = (p - 1)p^{r-1}$ denotes the Euler function. Let us fix a primitive $q$th root of unity $\zeta_q \in \mathbb{C}$. We write $C_{f,q}$ for the superelliptic curve $y^q = f(x)$ and $J(C_{f,q})$ for its jacobian. Clearly, $J(C_{f,q})$ is an abelian variety and

$$\dim(J(C_{f,q})) = \frac{(n - 1)(q - 1)}{2}.$$

The periodic automorphism $(x, y) \mapsto (x, \zeta_q y)$ of $C_{f,q}$ induces by Albanese functoriality the periodic automorphism of $J(C_{f,q})$ that we denote by $\delta_q$. It is known [13, 25] that $\delta_q$ gives rise to an embedding of the product $\prod_{i=1}^{\varphi(q)} \mathbb{Q}(\zeta_{p^i})$ of cyclotomic fields into the endomorphism algebra $\text{End}^0(J(C_{f,q}))$ of $J(C_{f,q})$. (If $q = p$ then we actually get an embedding $\mathbb{Z}[\delta_p] \to \text{End}(J(C_{f,p}))$ that sends $\zeta_p$ to $\delta_p$.) More precisely, if $q \neq p$ then the map $(x, y) \to (x, y^p)$ defines the map of curves $C_{f,q} \to C_{f,q/p}$, which induces (by Albanese functoriality) the surjective homomorphism $J(C_{f,q}) \to J(C_{f,q/p})$ of complex abelian varieties; we write $J(f,q)$ for the identity component of its kernel. (If $q = p$ then we put $J(f,p) = J(C_{f,p})$.) One may check [25] that $J(C_{f,q})$ is isogenous to the product $\prod_{i=1}^{\varphi(q)} J(f,p^i)$ and $\delta_q$ gives rise to an embedding

$$\mathbb{Z}[\zeta_q] \to \text{End}(J(f,q)).$$

In a series of papers [21, 23, 24, 25, 27], one of the authors (Y.Z.) was able to prove that

$$\text{End}(J(f,q)) = \mathbb{Z}[\zeta_q], \quad \text{End}^0(J(f,q)) = \mathbb{Q}(\zeta_q)$$

assuming that $n \geq 5$ and there exists a subfield $K \subset \mathbb{C}$ such that all the coefficients of $f(x)$ lie in $K$ and the Galois group of $f(x)$ over $K$ is either the full symmetric group $S_n$ or the alternating group $A_n$. In particular, $\text{End}^0(J(C_{f,q})) \cong \prod_{i=1}^{\varphi(q)} \mathbb{Q}(\zeta_{p^i})$. (The same assertion holds true if $n = 4$, the prime $p$ is odd, $\zeta_q \in K$ and the Galois group is $S_4$.)
Our goal is to study the (reductive $\mathbb{Q}$-algebraic connected) Hodge group $\text{Hdg}(J(f,q))$ of $J(f,q)$. Notice that when $q = 2$ (i.e., in the hyperelliptic case) this group was completely determined in [22] (when $f(x)$ has “large” Galois group); in particular, in this case the Hodge group is simple and the center of its Lie algebra is $\{0\}$. So, further we assume that $q > 2$ and therefore $\mathbb{Q}(\zeta_q)$ is a CM-field. So, if $\text{End}^0(J(f,q)) = \mathbb{Q}(\zeta_q)$ then (see Remark 3.6 below) the center $\delta^0$ of the $\mathbb{Q}$-Lie algebra $\text{hdg}$ of $\text{Hdg}(J(f,q))$ lies in

$$\mathbb{Q}(\zeta_q)^{-} := \{ e \in \mathbb{Q}(\zeta_q) \mid \bar{e} = -e \} \subset \mathbb{Q}(\zeta_q).$$

If $q = 2$ then $\mathbb{Q}(\zeta_q) = \mathbb{Q}$ and $\mathbb{Q}(\zeta_q)^{-} = \{0\}$. In particular, its dimension does not exceed $\varphi(q)/2$; the equality holds if and only if $q > 2$ and $\delta^0$ coincides with $\mathbb{Q}(\zeta_q)^{-}$.

Let

$$\mathbb{Q}(\zeta_q)^{+} := \{ e \in \mathbb{Q}(\zeta_q) \mid \bar{e} = e \} \subset \mathbb{Q}(\zeta_q)$$

be the maximal totally real subfield of $\mathbb{Q}(\zeta_q)$. If $q > 2$ then $[\mathbb{Q}(\zeta_q)^{+} : \mathbb{Q}] = \varphi(q)/2$. We write $R_{\mathbb{Q}(\zeta_q)}G_m$ and $R_{\mathbb{Q}(\zeta_q)^{+}}G_m$ for the algebraic $\mathbb{Q}$-tori obtained by the Weil restriction of scalars of the multiplicative group $G_m$ to $\mathbb{Q}$ from $\mathbb{Q}(\zeta_q)$ and $\mathbb{Q}(\zeta_q)^{+}$ respectively. The norm map $\mathbb{Q}(\zeta_q) \to \mathbb{Q}(\zeta_q)^{+}$ induces the natural homomorphism of algebraic $\mathbb{Q}$-tori and we denote by $U_q = T_{\mathbb{Q}(\zeta_q)}$ its kernel, i.e., the corresponding norm torus [15]. It is well known that $U_q$ is an algebraic $\mathbb{Q}$-torus (in particular, it is connected) and

$$U_q(\mathbb{Q}) = \{ e \in \mathbb{Q}(\zeta_q) \mid \bar{e}e = 1 \} \subset \mathbb{Q}(\zeta_q).$$

The embedding $\mathbb{Q}(\zeta_q) \to \text{End}^0(J(f,q))$ allows us to identify $\mathbb{Q}(\zeta_q)$ with a certain $\mathbb{Q}$-subalgebra of $\text{End}_\mathbb{Q}(H^1(J(f,q)))$ and consider $R_{\mathbb{Q}(\zeta_q)}G_m$ and therefore $U_q$ as certain $\mathbb{Q}$-algebraic subgroups of the general linear group $\text{GL}(H^1(J(f,q)), \mathbb{Q})$ over $\mathbb{Q}$. Then the $\mathbb{Q}$-Lie algebras of $R_{\mathbb{Q}(\zeta_q)}G_m$ and $U_q$, viewed as $\mathbb{Q}$-Lie subalgebras of $\text{End}_\mathbb{Q}(H^1(J(f,q)))$, coincide with $\mathbb{Q}(\zeta_q)$ and $\mathbb{Q}(\zeta_q)^{-}$ respectively.

Recall that $J(f,q)$ is an abelian subvariety of the jacobian $J(C_{f,q})$ and consider the $(\delta_q)$-invariant) polarization $\lambda_r$ on $J(f,q)$ induced by the canonical principal polarization on $J(C_{f,q})$. The polarization $\lambda_r$ gives rise to a certain $\delta_q$-invariant non-degenerate alternating $\mathbb{Q}$-bilinear form

$$\psi_r : H_1(J(f,q), \mathbb{Q}) \times H_1(J(f,q), \mathbb{Q}) \to \mathbb{Q}$$

(This form is the imaginary part of the Riemann form of $\lambda_r$ [11, 14].) The $\delta_q$-invariance implies that $\psi_r(e.x, y) = \psi_r(x, \bar{e}y) \forall e \in \mathbb{Q}(\zeta_q); \ x, y \in H_1(J(f,q), \mathbb{Q})$. If $q > 2$ then we choose a nonzero element $\beta_r \in \mathbb{Q}(\zeta_q)$ and a standard construction (see, for instance, [13, p. 531]) gives us a nondegenerate Hermitian $\mathbb{Q}(\zeta_q)$-sesquilinear form

$$\phi_r : H_1(J(f,q), \mathbb{Q}) \times H_1(J(f,q), \mathbb{Q}) \to \mathbb{Q}(\zeta_q)$$

such that $\text{Tr}_{\mathbb{Q}(\zeta_q)}/\mathbb{Q}(\zeta_q)Q(\delta_r, \phi_r) = \psi_r$. We write $U(\mathbb{H}_1(J(f,q), \mathbb{Q}), \phi_r)$ for the unitary group of $\phi_r$, viewed as an algebraic (reductive) $\mathbb{Q}$-subgroup of $\text{GL}(H_1(J(f,q), \mathbb{Q}))$ (via Weil’s restriction of scalars from $\mathbb{Q}(\zeta_q)^{+}$ to $\mathbb{Q}$ (ibid). Then the center of $U(\mathbb{H}_1(J(f,q), \mathbb{Q}), \phi_r)$ coincides with $U_q$.

Since the Hodge group of $J(f,q)$ respects the polarization and commutes with endomorphisms of $J(f,q)$,

$$\text{Hdg}(J(f,q)) \subset U(\mathbb{H}_1(J(f,q), \mathbb{Q}), \phi_r).$$
Recall that the centralizer of $\text{Hdg}(J^{(f,q)})$ in $\text{End}_K(H_1(J^{(f,q)}, K))$ coincides with $\text{End}_K^0(J^{(f,q)})$. This implies that if $\text{End}_K^0(J^{(f,q)})$ coincides with $K(\zeta_q)$ then the center of $\text{Hdg}(J^{(f,q)})$ lies in $U_q$.

**Remark 1.1.** Let $\text{Hdg}^s = [\text{Hdg}, \text{Hdg}]$ be the derived subgroup of $\text{Hdg}$. Let $\mathfrak{z}$ be the center of $\text{Hdg}$ and $\mathfrak{z}^0$ the identity component of $\mathfrak{z}$. Since the Hodge group is connected reductive, $\text{Hdg}^s$ is a semisimple connected algebraic $\mathbb{Q}$-group, $\mathfrak{z}^0$ an algebraic $\mathbb{Q}$-torus and the natural morphism of linear algebraic $\mathbb{Q}$-groups $\text{Hdg}^s \times \mathfrak{z}^0 \to \text{Hdg}$ is an isogeny. It follows that the $\mathbb{Q}$-Lie algebra $\text{Lie}(\mathfrak{z})$ of $\mathfrak{z}$ coincides with the $\mathbb{Q}$-Lie algebra $\text{Lie}(\mathfrak{z}^0)$ of $\mathfrak{z}^0$ and equals $\mathfrak{z}^0$.

**Theorem 1.2.** Assume that $n \geq 3$ and $p$ does not divide $n$. Let $f(x) \in \mathbb{C}[x]$ be a degree $n$ polynomial without multiple roots. If $q > 2$ then the center $\mathfrak{z}^0$ of the $\mathbb{Q}$-Lie algebra $\text{Lie}(\text{Hdg}(J^{(f,q)}))$ has $\mathbb{Q}$-dimension greater or equal than $\varphi(q)/2$. In other words, the center of $\text{Hdg}(J^{(f,q)})$ has dimension greater or equal than $\varphi(q)/2$.

As an application, we obtain the following statement.

**Theorem 1.3.** Assume that $n \geq 4$ and $p$ does not divide $n$. Let $K$ be a subfield of $\mathbb{C}$ that contains all the coefficients of $f(x)$. Suppose that $f(x)$ is irreducible over $K$ and the Galois group $\text{Gal}(f)$ of $f(x)$ over $K$ is either $S_n$ or $A_n$. Assume additionally that either $n \geq 5$ or $n = 4$ and $\text{Gal}(f) = S_4$.

If $q > 2$ then the center $\mathfrak{z}^0$ of the $\mathbb{Q}$-Lie algebra $\text{Lie}(\text{Hdg}(J^{(f,q)}))$ has $\mathbb{Q}$-dimension $\varphi(q)/2$ and coincides with $\mathbb{Q}(\zeta_q)_-$. In addition, the center of $\text{Hdg}(J^{(f,q)})$ coincides with $U_q$.

**Example 1.4.** Suppose that $n, p, f(x)$ enjoy the conditions of Theorem 1.3. Assume additionally that $p$ is odd. Since $J(C_{f,p}) = J^{(f,p)}$, we conclude that the center of $\text{Hdg}(J(C_{f,p}))$ coincides with $U_p$.

**Remark 1.5.** In Theorem 1.3 we prove that the center of the Hodge group of $J^{(f,q)}$ is “as large as possible”, taking into account that the endomorphism algebra of $J^{(f,q)}$ coincides with $\mathbb{Q}(\zeta_q)$. In fact, our goal was (and still is) to prove that (under the assumptions of Theorem 1.3) the whole Hodge group is “as large as possible”, i.e., coincides with $U(H_1(J^{(f,q)}, \mathbb{Q}), \phi_r)$, which would imply that all Hodge classes on each self-product of $J^{(f,q)}$ can be presented as linear combinations of products of divisor classes and, in particular, the validity of the Hodge conjecture for all the self-products [14] p. 528 and 531. Since the Hodge group is connected reductive, the problem splits naturally in two parts: to prove that the center of $\text{Hdg}(J^{(f,q)})$ is “as large as possible” (i.e., coincides with $U_q$) and that the derived subgroup (semisimple part) of $\text{Hdg}(J^{(f,q)})$ is “as large as possible” (i.e., coincides with the corresponding special unitary group). Theorem 1.3 settles the first one. (The second problem is solved in [19] under certain additional conditions on $n$ and $q$.)

In order to describe our results for the whole $J(C_{f,q})$ when $q > p$, let us put

$$E_{p,i} := \mathbb{Q}(\zeta_{p^i}), \quad E_{p,0} := \mathbb{Q}(\zeta_{p^0})_-,$$

$$E_{p,r} := \{ (e_i)_{i=1}^r \in \oplus_{i=1}^r E_{p,i} \mid \text{Tr}_{E_{p,i+1}/E_{p,i}}(e_{i+1}) = e_i \forall i < r \} \subset \oplus_{i=1}^r E_{p,i}.$$

**Theorem 1.6.** Assume that $n \geq 4$ and $p$ does not divide $n$. Let $K$ be a subfield of $\mathbb{C}$ that contains all the coefficients of $f(x)$. Suppose that $f(x)$ is irreducible over $K$ and the Galois group $\text{Gal}(f)$ of $f(x)$ over $K$ is either $S_n$ or $A_n$. Assume
Remark 1.7. Let us fix an isogeny \( E \) and its first rational homology group \( H_1(Z, \mathbb{Q}) = \bigoplus_{r=1}^\infty \mathbb{H}_1(J^{(f,p^r)}_Z) \). If \( p^r > 2 \) then the center \( \mathfrak{o}_2 \) of the \( \mathbb{Q} \)-Lie algebra \( \text{hdg}_Z \) of the Hodge group \( \text{hdg}(Z) \) of \( Z \) has \( \mathbb{Q} \)-dimension \( \varphi(p^r)/2 \) and coincides with
\[
\mathcal{E}^{p,r} \subset \bigoplus_{i=1}^r \mathcal{E}^{p,i}_G \subset \bigoplus_{i=1}^r \mathbb{Q}(\mathfrak{o}_p) \subset \bigoplus_{i=1}^r \mathbb{Q}H_1(J^{(f,p^r)}_Z) \subset \text{End}_{\mathbb{Q}}(H_1(J^{(f,p^r)}_Z, \mathbb{Q})) \subset \text{End}_{\mathbb{Q}}(H_1(Z, \mathbb{Q})).
\]

Remark 1.8. Let us fix an isogeny \( \alpha : J(C_{f,p^r}) \to \prod_{i=1}^r J^{(f,p^r)}_Z = Z \). Then \( \alpha \) induces an isomorphism of \( \mathbb{Q} \)-vector spaces \( \alpha : \text{hdg}(J(C_{f,p^r})) \cong \text{hdg}(Z) \). Clearly, the Hodge group of \( J(C_{f,p^r}) \) coincides with \( \alpha^{-1}\text{hdg}(Z)\alpha \). This implies that if \( q > 2 \) then the center of the \( \mathbb{Q} \)-Lie algebra of \( \text{hdg}(J(C_{f,p^r})) \) has \( \mathbb{Q} \)-dimension \( \varphi(p^r)/2 \) and coincides with \( \alpha^{-1}\mathcal{E}^{p,r}_G \alpha \).

Another application of Theorem 1.2 is the following statement.

Theorem 1.9. Assume that \( n \geq 3 \) and \( p \) does not divide \( n \). Let \( f(x) \in \mathbb{C}[x] \) be a degree \( n \) polynomial without multiple roots. Assume also that \( q > 2 \).

(i) If \( p \) is odd then \( J^{(f,q)}_Z \) contains a simple complex abelian subvariety \( T \) with
\[
\dim(T) \geq \varphi((p-1)p^{r-1}) \geq \varphi(p-1).
\]
In particular, \( \dim(T) \geq \varphi(p-1) \cdot (p-1)p^{r-2} \) when \( r \geq 2 \).

(ii) If \( p = 2 \) and \( r \geq 3 \) then \( J^{(f,q)}_Z \) contains a simple complex abelian subvariety \( T \) with \( \dim(T) \geq 2^{r-3} \).

Remark 1.10. Actually, our proof gives a little bit more, namely, that the center \( \mathfrak{e}_T \) of \( \text{End}^0(T) \) is a CM-field such that \( \mathfrak{e}_T : \mathbb{Q} \) is greater or equal than the lower bound given in Theorem 1.9. (Notice that \( \mathfrak{e}_T \) is a direct summand of the center of \( \text{End}(J^{(f,q)}) \)).

Corollary 1.11 (Corollary to Theorem 1.9). Suppose that \( n \geq 3 \) and \( d \) is a positive integer such that \( (d,n) = 1 \). Let \( f(x) \in \mathbb{C}[x] \) be a degree \( n \) polynomial without
multiple roots. Assume that \( d \geq 5 \) and \( d \) is neither 6 nor 8 nor 12 nor 24. Let us consider the superelliptic curve \( C_{f,d} : y^d = f(x) \) and let \( J(C_{f,d}) \) be its jacobian.

Then \( J(C_{f,d}) \) is not isogenous to a product of elliptic curves.

**Proof.** Clearly, \( d \) has a divisor \( q \) such that either \( q \) is a prime \( \geq 5 \) or \( q = 9 \) or \( q = 16 \). The existence of the covering of algebraic curves \( C_{f,d} \rightarrow C_{f,q}, (x,y) \mapsto (x,y^{d/q}) \) implies that \( J(C_{f,d}) \) has a quotient isomorphic to \( J(C_{f,q}) \). Now the result follows from Theorem 1.9 if we take into account that \( J(f,q) \) is an abelian subvariety of \( J(C_{f,q}) \). \( \square \)

**Remark 1.12.** Corollary 1.11 implies that if \( p \geq 5 \) then none of jacobians of \( C_{f,p} \) is totally split in a sense of \([5]\). The same is true for the jacobians of \( C_{f,16} \) and \( C_{f,9} \).

**Remark 1.13.** Recently D. Ulmer \([17]\), using a construction of L. Berger \([1]\), found out that the rank of the Mordell-Weil group of the jacobian of the curve \( f(x) - tf(y) = 0 \) over the function field \( \mathbb{C}(t^1/q) \) is closely related to the endomorphism algebras of \( J^{[f,p]} \) (for \( i \leq r \)). One may hope that our results could be useful for the study of the rank of abelian varieties in infinite towers of function fields.

The paper is organized as follows. In Section 2 we discuss auxiliary results related to CM-fields. Section 3 treats complex abelian varieties with multiplication by CM-fields. Section 4 contains the proof of main results modulo some arithmetic properties of certain (non-vanishing) Fourier coefficients with respect to the finite commutative group \( (\mathbb{Z}/q\mathbb{Z})^* \); those properties are proved in Sections 5 and 6. Last section contains an auxiliary result from semilinear algebra.

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### 2. Field embeddings

**2.1.** If \( V \) is a finite-dimensional \( \mathbb{Q} \)-vector space (resp. \( \mathbb{Q} \)-algebra) then we write \( V_C \) for the corresponding finite-dimensional \( \mathbb{C} \)-vector space (resp. \( \mathbb{C} \)-algebra) \( V \otimes_{\mathbb{Q}} \mathbb{C} \); clearly

\[
\dim_{\mathbb{Q}}(V) = \dim_{\mathbb{C}}(V_C).
\]

The group \( \text{Aut}(\mathbb{C}) \) acts tautologically on \( \mathbb{C} \) and the subfield of \( \text{Aut}(\mathbb{C}) \)-invariants coincides with \( \mathbb{Q} \). This allows us to define the *tautological* semilinear action on \( V_C \) as follows.

\[
s(v \otimes z) = v \otimes s(z) \quad \forall \ s \in \text{Aut}(\mathbb{C}), v \in V, z \in \mathbb{C}.
\]

The semilinearity means that

\[
s(zv) = s(z)s(v) \quad \forall \ s \in \text{Aut}(\mathbb{C}), v \in V_C, z \in \mathbb{C}.
\]

Clearly, the \( \mathbb{Q} \)-subspace of all \( \text{Aut}(\mathbb{C}) \)-invariant elements in \( V_C \) coincides with \( V \otimes 1 = V \). It is also clear that if \( W \subset V \) is a \( \mathbb{Q} \)-vector subspace then \( W_C \) is a \( \text{Aut}(\mathbb{C}) \)-stable complex vector subspace in \( V_C \). Conversely, if \( W \) is a \( \text{Aut}(\mathbb{C}) \)-stable complex vector subspace in \( V_C \) then there exists exactly one \( \mathbb{Q} \)-vector subspace \( W \subset V \) such that \( W = W_C \); in addition, \( W \) is the \( \mathbb{Q} \)-vector subspace of all \( \text{Aut}(\mathbb{C}) \)-invariant elements in \( W \). (See Sect. 7)
2.2. Let $E$ be a number field. Let $\Sigma_E$ be the set of all field embeddings $\sigma : E \hookrightarrow \mathbb{C}$. Clearly, $\sigma(E) \subset \bar{\mathbb{Q}}$ for all $\sigma$. It is well-known that $\Sigma_E$ consists of $[E : \mathbb{Q}]$ elements. The group $\text{Aut}(\mathbb{C})$ acts naturally on $\Sigma_E$. Namely, if $\sigma : E \hookrightarrow \mathbb{C}$ is a field embedding and $s$ is an automorphism of $\mathbb{C}$ then we define $s(\sigma) : E \hookrightarrow \mathbb{C}$ as the composition $s \sigma : E \hookrightarrow \mathbb{C} \rightarrow \mathbb{C}.

If $\sigma \in \Sigma_E$ then we write $\bar{\sigma}$ for the complex-conjugate of $\sigma$, i.e., for the composition $\iota \sigma : E \hookrightarrow \mathbb{C}$. Clearly, the action of $\text{Aut}(\mathbb{C})$ on $\Sigma_E$ factors through the natural surjection $\text{Aut}(\mathbb{C}) \twoheadrightarrow \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

Let us consider the $[E : \mathbb{Q}]$-dimensional $\mathbb{C}$-algebra $\mathbb{C}^{\Sigma_E}$ of all functions $\phi : \Sigma_E \rightarrow \mathbb{C}$. The action of $\text{Aut}(\mathbb{C})$ induces the semilinear action of $\text{Aut}(\mathbb{C})$ on $\mathbb{C}^{\Sigma_E}$ as follows.

$$\phi \mapsto \{ \sigma \mapsto s(\phi(s^{-1}\sigma)) \} \quad \forall \ s \in \text{Aut}(\mathbb{C}).$$

The semilinearity means that $s(z \phi) = s(z) s(\phi)$ $\forall \ z \in \mathbb{C}$.

Let us consider a $\mathbb{C}$-linear map of $[E : \mathbb{Q}]$-dimensional $\mathbb{C}$-algebras

$$\kappa_E : E \otimes_{\mathbb{Q}} \mathbb{C} \rightarrow \mathbb{C}^{\Sigma_E}, \ e \otimes z \mapsto \{ \sigma \mapsto z \sigma(e) \} \quad \forall \ e \in E, z \in \mathbb{C}$$

(see [4, (2.2.2)]). Clearly, $\kappa_E$ is $\text{Aut}(\mathbb{C})$-equivariant. (Here $\text{Aut}(\mathbb{C})$ acts on $E \otimes_{\mathbb{Q}} \mathbb{C}$ through its second factor in the obvious way.) It follows from Artin’s theorem on linear independence of multiplicative characters [9, Ch. VI, Sect. 4, Th. 4.1] that $\kappa_E$ is injective; now the coincidence of dimensions implies that $\kappa_E$ is an isomorphism of $\mathbb{C}$-vector spaces that commutes with $\text{Aut}(\mathbb{C})$-actions. On the other hand, for each $\sigma \in \Sigma_E$ the natural surjection

$$E \otimes_{\mathbb{Q}} \mathbb{C} \twoheadrightarrow E \otimes_{E,\sigma} \mathbb{C} =: C_\sigma = \mathbb{C}$$

obviously coincides with the composition of $\kappa_E$ and

$$\mathbb{C}^{\Sigma_E} \rightarrow \mathbb{C}, \phi \mapsto \phi(\sigma).$$

This allows us to identify $\mathbb{C}^{\Sigma_E}$ and

$$\bigoplus_{\sigma \in \Sigma_E} C_\sigma = \bigoplus_{\sigma \in \Sigma_E} C$$

and we may view $\kappa_E$ as an isomorphism

$$E \otimes_{\mathbb{Q}} \mathbb{C} \cong \bigoplus_{\sigma \in \Sigma_E} C_\sigma = \bigoplus_{\sigma \in \Sigma_E} E \otimes_{E,\sigma} \mathbb{C}.$$ \hfill (1)

Further we will identify $E \otimes_{\mathbb{Q}} \mathbb{C}$ with

$$\mathbb{C}^{\Sigma_E} = \bigoplus_{\sigma \in \Sigma_E} C$$

via $\kappa_E$.

2.3. Let $G$ be a (finite) automorphism group of the field $E$ and let $F = E^G$ be the subfield of $G$-invariants. One may view $G$ as a certain group of automorphisms of the $\mathbb{C}$-algebra $E \otimes_{\mathbb{Q}} \mathbb{C}$ where $G$ acts through the first factor. Clearly,

$$(E \otimes_{\mathbb{Q}} \mathbb{C})^G = E^G \otimes_{\mathbb{Q}} \mathbb{C} = F \otimes_{\mathbb{Q}} \mathbb{C};$$

in addition, if $z \in E = E \otimes 1 \subset E \otimes_{\mathbb{Q}} \mathbb{C}$

then

$$w = \text{Tr}_{E/F}(z) \in F = F \otimes 1 \subset F \otimes_{\mathbb{Q}} \mathbb{C}.$$
where

\[ \text{Tr}_{E/F} : E \to F \]

is the trace map that corresponds to the finite field extension \( E/F \).

It is also clear that the corresponding action of \( G \) on \( C^{\Sigma_E} \) induced by \( \kappa_E \) could be described as follows. Every \( s \in G \) sends a function \( \phi : \Sigma_E \to C \) to the function \( \sigma \mapsto \phi(\sigma s) \).

If \( z \in E \otimes_Q C \) then \( w = \sum_{s \in G} s z \in (E \otimes_Q C)^G = F \otimes_Q C \). If \( \kappa_E(z) \) is a function \( \phi \) on \( \Sigma_E \) and \( \kappa_F(w) \) is a function \( \psi \) on \( \Sigma_F \) then one may easily check that for each field embedding \( \sigma_F : F \to C \)

\[ \psi(\sigma_F) = \sum \phi(\sigma) \]

where the sum is taken over all field embeddings \( \sigma : E \to C \), whose restriction to \( F \) coincides with \( \sigma_F \). In other words, if \( \sigma \) is one of those embeddings then

\[ \psi(\sigma_F) = \sum_{s \in G} \phi(\sigma s). \]

2.4. Assume that \( E \) is a CM-field and let \( c_0 \in \text{Aut}(E/Q) \) be the “complex conjugation”, i.e., the involution, whose subfield of invariants consists of all totally real elements of \( E \). Since \( E \) is CM, we have

\[ \sigma c_0 = i \sigma = \bar{\sigma} \forall \sigma \in \Sigma_E. \]

Let us consider the \( [E : Q]/2 \) -dimensional \( Q \)-vector subspace

\[ E_- := \{ e \in E \mid c_0(e) = -e \} \subset E \]

of \( c_0 \)-antiinvariants. The involution \( c_0 \) gives rise to the involutions of \( C \)-algebras

\[ E \otimes_Q C \to E \otimes_Q C, \ e \otimes z \mapsto c_0(e) \otimes z; \]

\[ C^{\Sigma_E} \to C^{\Sigma_E}, \ \phi(\sigma) \mapsto \phi(\sigma c_0) = \phi(\bar{\sigma}), \]

which we still denote by \( c_0 \). Clearly, \( \kappa_E \) is \( c_0 \)-equivariant. It is also clear that the \( C \)-subspace of \( c_0 \)-antiinvariants in \( E \otimes_Q C \) coincides with \( E_- \otimes_Q C \) and the \( C \)-subspace of \( c_0 \)-antiinvariants in \( C^{\Sigma_E} \) coincides with the subspace \( X_{E,C} \) of all functions \( \phi \) that satisfy

\[ \phi(\bar{\sigma}) = -\phi(\sigma) \forall \sigma \in \Sigma_E. \]

Let \( X_E \subset X_{E,C} \) be the \( Q \)-vector subspace that consists of all functions \( \phi : \Sigma_E \to Q \subset C \) with

\[ \phi(\bar{\sigma}) + \phi(\sigma) = 0 \forall \sigma \in \Sigma_E. \]

Clearly, \( X_E \) is a \( \text{Aut}(C) \)-invariant \( Q \)-vector subspace of \( C^{\Sigma_E} \) and we get the natural homomorphism

\[ \text{Aut}(C) \to \text{Gal}(\bar{Q}/Q) \to \text{Aut}_Q(X_E). \]

Clearly, \( \iota \) acts on \( X_E \) as multiplication by \(-1\).

Let

\[ E^+ = \{ e \in E \mid c_0(e) = e \} \]

be the maximal totally real subfield of \( E \). Clearly, \( E \) is a quadratic extension of \( E^+ \) with the Galois group \( \{ 1, c_0 \} \). The corresponding norm map \( E \to E^+ \) coincides with the map

\[ e \mapsto e \cdot c_0(e). \]

Let us extend \( c_0 \) by \( C \)-linearity to the \( C \)-linear algebra automorphism

\[ E_C \to E_C, \ e \otimes z \mapsto c_0(e) \otimes z, \]
which we continue to denote by \( c_0 \). The corresponding automorphism of \( \mathbb{C}^{\Sigma_E} \) (via \( \kappa_E \) sends a function \( h : \Sigma_E \rightarrow \mathbb{C} \) to the function \( \sigma \mapsto h(\sigma c_0) \).

Let \( R_{E/\mathbb{Q}G_m} \) and \( R_{E^+/\mathbb{Q}G_m} \) be the algebraic \( \mathbb{Q} \)-tori obtained by the Weil restriction of scalars from the multiplicative group \( G_m \) to \( \mathbb{Q} \) from \( E \) and \( E^+ \) respectively. For every commutative \( \mathbb{Q} \)-algebra \( A \)

\[
R_{E/\mathbb{Q}G_m}(A) = (A \otimes \mathbb{Q}E)^*, \quad R_{E^+/\mathbb{Q}G_m}(A) = (A \otimes \mathbb{Q}E^+)^*.
\]

Clearly, \( R_{E^+/\mathbb{Q}G_m} \) is an algebraic \( \mathbb{Q} \)-subgroup of \( R_{E/\mathbb{Q}G_m} \). Again let us define the \( A \)-linear algebra automorphism

\[
A \otimes \mathbb{Q}E \rightarrow A \otimes \mathbb{Q}E, \quad a \otimes e \mapsto a \otimes c_0(e),
\]

which we continue denote by \( c_0 \). Clearly, the subalgebra of \( c_0 \)-invariants coincides with \( A \otimes \mathbb{Q}E^+ \). The homomorphisms

\[
(A \otimes \mathbb{Q}E)^* \rightarrow (A \otimes \mathbb{Q}E^+)^*, \quad b \mapsto b \cdot c_0(b)
\]
gives rise to the \( \mathbb{Q} \)-homomorphism of algebraic \( \mathbb{Q} \)-tori

\[
R_{E/\mathbb{Q}G_m} \rightarrow R_{E^+/\mathbb{Q}G_m},
\]

whose kernel \( T_E \) is called the norm torus. By definition,

\[
T_E(A) = \{ b \in (A \otimes \mathbb{Q}E)^* \mid b \cdot c_0(b) = 1 \}.
\]

In particular,

\[
T_E(\mathbb{C}) = \{ u \in E_{\mathbb{C}} \mid u \cdot c_0(u) = 1 \} = \kappa_E^{-1}\{ h : \Sigma_E \rightarrow \mathbb{C} \mid h(\sigma)h(\sigma c_0) = 1 \quad \forall \ \sigma \}.
\]

It is well known [18] that the norm torus is an algebraic \( \mathbb{Q} \)-torus; in particular, it is a connected algebraic \( \mathbb{Q} \)-group.

Let \( \mathbb{Q}[\epsilon] = \mathbb{Q} \oplus \mathbb{Q} : \epsilon \) be the \( \mathbb{Q} \)-algebra of dual numbers: \( \epsilon^2 = 0 \). One may naturally identify the \( \mathbb{Q} \)-Lie algebra \( \text{Lie}(R_{E/\mathbb{Q}G_m}) \) with \( E \): namely, each \( e \in E \) corresponds to \( 1 + \epsilon \otimes e \in (\mathbb{Q}[\epsilon] \otimes \mathbb{Q}E)^* \). The corresponding \( \mathbb{Q} \)-Lie subalgebras of \( R_{E^+/\mathbb{Q}G_m} \) coincide with \( E^+ \) and \( E^- \) respectively.

Suppose that \( E \) is a CM field that is normal over \( \mathbb{Q} \) and fix a field embedding \( E \hookrightarrow \mathbb{Q} \subset \mathbb{C} \). Further, we view \( E \) as a subfield of \( \mathbb{C} \). Then \( \sigma(E) = E \) for all \( \sigma \), the involution \( c_0 \) coincides with the restriction of the complex conjugation \( \iota \) to \( E \).

In addition, \( c_0 \) is a central element of the Galois group \( \text{Gal}(E/\mathbb{Q}) \). The set \( \Sigma_E \) “coincides” with \( \text{Gal}(E/\mathbb{Q}) \). In addition, the action of \( \text{Aut}(\mathbb{C}) \) on \( \Sigma_E = \text{Gal}(E/\mathbb{Q}) \) factors through \( \text{Gal}(E/\mathbb{Q}) \) and corresponds to the left translations. The action of \( \text{Aut}(\mathbb{C}) \) on \( X_E \) factors through \( \text{Gal}(E/\mathbb{Q}) \) and this action admits the following description.

\[
\tau(f)(\sigma) = f(\tau^{-1}\sigma) \quad \forall \tau \in \text{Gal}(E/\mathbb{Q}), \ \sigma \in \Sigma_E = \text{Gal}(E/\mathbb{Q}), \ f \in X_E.
\]

If we consider the \( \mathbb{Q} \)-vector (sub)space

\[
E_- = \{ e \in E \mid c_0(e) = -e \} \subset E
\]

then

\[
\kappa_E(E_- \otimes \mathbb{Q} \mathbb{C}) = X_{E, \mathbb{C}}.
\]

Clearly,

\[
\dim_{\mathbb{Q}}(E_-) = \frac{1}{2} |E : \mathbb{Q}| = \dim_{\mathbb{Q}}(X_E).
\]
Caution: Although $\kappa_E(E_- \otimes \mathbf{Q} \mathbf{C}) = X_{E, \mathbf{C}}$, it is not true that $\kappa_E(E_-) = X_E$ unless both are 0. Indeed, for any nonzero $e \in E_-$, the function $\kappa_E(e)$ takes value $\sigma(e)$ at $\sigma \in \Sigma_E$, which is never real, while $X_E$ consists of $\mathbf{Q}$-valued functions.

Remark 2.5. The $\text{Gal}(E/\mathbf{Q})$-module $X_E$ is faithful. Indeed, let us consider the function $f$ on $\Sigma_E = \text{Gal}(E/\mathbf{Q})$ that takes on value 1 on the identity element of $\text{Gal}(E/\mathbf{Q})$, value $-1$ on $c_0$ and zero elsewhere. Then $f \in X_E$ but $\tau(f) \neq f$ if $\tau$ is not the identity element of $\text{Gal}(E/\mathbf{Q})$.

Definition 2.6. We write $\max(E)$ for the largest $\mathbf{Q}$-dimension of simple $\text{Gal}(E/\mathbf{Q})$-submodules of $X_E$. Clearly, $\max(E) \leq \dim_{\mathbf{Q}}(X_E)$; the equality holds if and only if $X_E$ is simple.

Lemma 2.7. Let $G = \text{Gal}(E/\mathbf{Q})$ and $W$ be a simple $\mathbf{Q}[G]$-module such that the involution $c_0$ acts on $W$ as multiplication by $-1$. Then there exists an injective homomorphism of $\mathbf{Q}[G]$-modules $W \hookrightarrow X_E$. In particular, $X_E$ contains a $\mathbf{Q}[G]$-submodule that is isomorphic to $W$.

Proof. Fix a nonzero linear function $\lambda \in \text{Hom}_{\mathbf{Q}}(W, \mathbf{Q})$ and consider the $\mathbf{Q}$-linear map

$$\pi_{\lambda} : W \rightarrow \mathbf{Q}^{\Sigma_E}, \quad x \mapsto \{\sigma \mapsto \lambda(\sigma^{-1}x)\}.$$ 

Clearly $\pi_{\lambda}$ is nonzero. For all $x \in W$, $\sigma \in \Sigma_E$ and $\tau \in G$, we have

$$\pi_{\lambda}(\tau x)(\sigma) = \lambda(\sigma^{-1}\tau x) = \lambda((\tau^{-1}\sigma)^{-1}x) = \pi_{\lambda}(x)(\tau^{-1}\sigma) = \tau(\pi_{\lambda}(x))(\sigma).$$

Hence $\pi_{\lambda}$ is a map of $\mathbf{Q}[G]$-modules. In particular, if we choose $\tau$ to be the involution $c_0$, then

$$c_0(\pi_{\lambda}(x)) = \pi_{\lambda}(c_0 x) = \pi_{\lambda}(-x) = -\pi_{\lambda}(x).$$

It follows that $\pi_{\lambda}(W) \subseteq X_E$ and we have a map of $\mathbf{Q}[G]$-modules $W \rightarrow X_E \subseteq \mathbf{Q}^{\Sigma_E}$ that is still denoted by $\pi_{\lambda}$. Now the lemma follows since $W$ is simple and $\pi_{\lambda}$ is nonzero.

Examples 2.8. (i) Suppose that $\text{Gal}(E/\mathbf{Q}) = \langle c_0 \rangle \times H$ where $H$ is a cyclic subgroup of order $M$ in $G$. Let us consider the $G$-module $\mathbf{Q}(\zeta_M)$ where the group $H \cong \mu_M$ acts via multiplication by $M$th roots of unity and $c_0$ acts as multiplication by $-1$. Clearly, $\mathbf{Q}(\zeta_M)$ is simple and $\dim_{\mathbf{Q}}(\mathbf{Q}(\zeta_M)) = \varphi(M)$. It follows from Lemma 2.7 that the $G$-module $X_E$ contains a submodule that is isomorphic to $\mathbf{Q}(\zeta_M)$. In particular, $\max(E) \geq \varphi(M)$. (In fact, one may prove that $\max(E) = \varphi(M)$.)

(ii) Suppose that $G$ is a cyclic group of order $2M$. Then $c_0$ is its only element of order 2. Let us consider the $G$-module $\mathbf{Q}(\zeta_{2M})$ where the group $G \cong \mu_{2M}$ acts via multiplication by $2M$th roots of unity. Clearly, $c$ acts on $\mathbf{Q}(\zeta_{2M})$ as multiplication by $-1$. It is also clear that the $G$-module $\mathbf{Q}(\zeta_{2M})$ is simple. It follows again that the $G$-module $X_E$ contains a submodule that is isomorphic to $\mathbf{Q}(\zeta_{2M})$. In particular, $\max(E) \geq \varphi(2M)$. (In fact, one may prove that $\max(E) = \varphi(2M)$.)

Lemma 2.9. Let $E$ be a CM-field that is normal over $\mathbf{Q}$ and let us fix an embedding $E \hookrightarrow \mathbf{C}$. (Further we view $E$ as a subfield of $\mathbf{C}$.) Let $h : \Sigma_E \rightarrow \mathbf{Q} \subseteq \mathbf{C}$ be a $\mathbf{Q}$-valued function on $\Sigma_E$ that lies in $X_E$. Let $W$ be the $\mathbf{Q}$-vector subspace of $X_E$ generated by all $\tau(h) : \sigma \mapsto h(\tau^{-1}\sigma)$ where $\tau$ runs through $\text{Gal}(E/\mathbf{Q})$. Let $q$ be the smallest $\mathbf{Q}$-vector (sub)space of $E_-$ such that $\kappa_E(q_{\mathbf{C}})$ contains $h$. Then

$$\dim_{\mathbf{Q}}(q) = \dim_{\mathbf{Q}}(W).$$

In particular, $q = E_-$ if and only if $W = X_E$. 
Proof. By definition, $W \subset X_E$ is the $\mathbb{Q}$-vector subspace generated by functions $h(s^{-1}\sigma)$, $s \in \text{Gal}(E/\mathbb{Q})$. Clearly, $\dim_{\mathbb{Q}}(W)$ coincides with the rank of the matrix $(a_{s,\sigma}) = (h(s^{-1}\sigma))$ over the rationals with $s \in \Sigma_E$. Let $\tilde{W} \subset X_{E/C}$ be the $\mathbb{C}$-vector (sub)space generated by functions $h(s^{-1}\sigma)$, $s \in \text{Gal}(E/\mathbb{Q})$. Clearly, $\dim_{\mathbb{C}}(\tilde{W})$ coincides with the rank of the matrix $(a_{s,\sigma}) = (h(s^{-1}\sigma))$ over the complex numbers with $s \in \text{Gal}(E/\mathbb{Q})$, $\sigma \in \Sigma_E$. In particular,
\[
\dim_{\mathbb{Q}}(W) = \dim_{\mathbb{C}}(\tilde{W}).
\]

It is also clear that $\tilde{W}$ is the smallest $\text{Aut}(\mathbb{C})$-invariant complex vector subspace of $C^{\Sigma_E}$ that contains $h(\sigma)$. It follows that there exists a $\mathbb{Q}$-vector subspace $q' \subset E$ such that $q'_C = q' \otimes_{\mathbb{Q}} \mathbb{C}$ coincides with $\tilde{W}$. In particular
\[
\dim_{\mathbb{Q}}(q') = \dim_{\mathbb{C}}(\tilde{W}).
\]

The minimality property of $q$ implies that $q \subset q'$ and therefore
\[
\tilde{W} = qC \subset q'_C = \tilde{W}.
\]

The minimality property of $\tilde{W}$ implies that $qC = \tilde{W}$ and therefore $qC = q'_C$. Since $q \subset q'$, we conclude that $q = q'$. In order to finish the proof, one has only to recall that
\[
\dim_{\mathbb{Q}}(q) = \dim_{\mathbb{C}}(\tilde{W}) = \dim_{\mathbb{Q}}(W).
\]

\[\square\]

2.10. Let $t$ be a positive integer and suppose that for each positive $j \leq t$ we are given a number field $E_j$. For the sake of simplicity, let us assume that every $E_j$ is normal over $\mathbb{Q}$ and write $\text{Gal}(E_j/\mathbb{Q})$ for the corresponding Galois group. Further, we fix an embedding of $E_j$ into $\mathbb{C}$; this allows us to identify $\Sigma_{E_j}$ and $\text{Gal}(E_j/\mathbb{Q})$.

Let us consider the product
\[
E = \prod_{j=1}^t E_j = \bigoplus_{j=1}^t E_j.
\]

Clearly, $E$ is a finite-dimensional semisimple commutative $\mathbb{Q}$-algebra and the set $\Sigma_E$ of algebra homomorphisms $E \to \mathbb{C}$ that send 1 to 1 could be naturally identified with the disjoint union $\bigcup_{j=1}^t \Sigma_{E_j}$ of $\Sigma_{E_j}$'s. Taking the product of $\kappa_{E_j}$'s, we get the natural isomorphism of $\mathbb{C}$-algebras
\[
\kappa_E : E_C \cong C^{\Sigma_E},
\]

which sends $\{e_j\}_{j=1}^t \otimes z$ to the function
\[
\Sigma_E = \prod_{j=1}^t \Sigma_{E_j} \to \mathbb{C}
\]

that coincides with $\sigma \mapsto \sigma(e_j)z$ on $\Sigma_{E_j}$. As above, we identify $E_C$ with the space of functions $C^{\Sigma_E}$ via $\kappa_E$. Again, there is the natural semilinear action of $\text{Aut}(\mathbb{C})$ on $E_C$, whose subalgebra of invariants coincides with $E \otimes 1 = E$. An automorphism $\tau \in \text{Aut}(\mathbb{C})$ sends function $\Sigma_E \to \mathbb{C}$ to the function $\tau(h) := \{\sigma \mapsto \tau(h(\sigma^{-1}\sigma))\}$. We have a $\text{Aut}(\mathbb{C})$-invariant splitting
\[
C^{\Sigma_E} = \bigoplus_{j=1}^t C^{\Sigma_{E_j}}.
\]

Clearly, every function $h : \Sigma_E \to \mathbb{C}$ may be viewed as a collection $\{h_j\}_{j=1}^t$ of functions $h_j : \Sigma_{E_j} \to \mathbb{C}$. The $\mathbb{Q}$-vector (sub)space $Q^{\Sigma_E}$ of $\mathbb{Q}$-valued functions
is $\text{Aut}(C)$-invariant; in addition, the action of $\text{Aut}(C)$ on $Q^E$ factors through $\text{Gal}(Q/Q)$ and for all $Q$-valued functions $h$

$$\tau(h)(\sigma) = h(\tau^{-1}\sigma) \forall \sigma \in \Sigma_E, \tau \in \text{Gal}(Q/Q).$$

We have a $\text{Gal}(\bar{Q}/Q)$-invariant splitting

$$Q^E = \oplus_{j=1}^t Q^{E_j}.$$ 

**Lemma 2.11.** Let $h = \{h_j\}_{j=1}^t$ be a function on $\Sigma_E$ that takes on only rational values, i.e., $h_j(\Sigma_{E_j}) \subset Q \forall j$. Let $W$ (resp. $W_j$) be the $Q$-vector subspace generated by all $\tau(h)$ (resp. $\tau(h_j)$) where $\tau$ runs through $\text{Gal}(Q/Q)$. We have

$$W \subset Q^{\Sigma_E}, \ W_j \subset Q^{\Sigma_{E_j}}, \ W \subset \bigoplus_{j=1}^t W_j.$$

On the other hand, let $q$ (resp. $q_j$) be the smallest $Q$-vector subspace of $E$ (resp. of $E_j$) such that $k_E(q_C)$ contains $h$ (resp. $k_{E_j}(q_j \otimes Q C)$ contains $h_j$). Then

$$\dim_Q(W) = \dim_Q(q); \ \dim_Q(W_j) = \dim_Q(q_j) \forall j.$$

**Proof.** The proof could be carried out by the same arguments as the proof of Lemma 2.9 and is left to the reader.

If $F$ is a subfield of $E_t$ then we write $Tr_{E_t/F}: E_t \to F$ for the corresponding $Q$-linear trace map. Extending $Tr_{E_t/F}$ by $C$-linearity, we get a $C$-linear map

$$E_t \otimes Q C \to F \otimes Q C,$$

which we still denote by $Tr_{E_t/F}$.

**Lemma 2.12.** Assume that for all $j$ the field $E_j$ contains $E_t$. Let

$$x = \{x_j\}_{j=1}^t \in \prod_{j=1}^t (E_j)C = E_C.$$

Suppose that for all $j$

$$x_j = Tr_{E_t/E_j}(x_t).$$

Let $q$ (resp. $q_t$) be the smallest $Q$-vector subspace of $E$ such that $q_C$ contains $x$ (resp. the smallest $Q$-vector subspace of $E_t$ such that $(q_t)_C$ contains $x_t$). Then

$$q = \{(e_j)_{j=1}^t \in \prod_{j=1}^t E_j = E \mid e_t \in q_t, e_j = Tr_{E_t/E_j}(e_t) \ \forall j\}.$$ 

In particular, $\dim_Q(q) = \dim_Q(q_t)$.

**Proof.** Let us put

$$q' = \{(e_j)_{j=1}^t \in E \mid e_t \in q_t, e_j = Tr_{E_t/E_j}(e_t) \ \forall j\}.$$ 

Clearly, $\dim_Q(q') = \dim_Q(q_t)$ and

$$q'_C = \{(z_j)_{j=1}^t \in \prod_{j=1}^t (E_j)C = E_C \mid z_t \in (q_t)_C, z_j = Tr_{E_t/E_j}(z_t) \ \forall j\}.$$ 

It is also clear that $q'_C$ contains $x$. The minimality property of $q$ implies that $q \subset q'$ and therefore

$$\dim_Q(q) \leq \dim_Q(q') = \dim_Q(q_t);$$
the equality holds if and only if \( q = q' \). On the other hand, let us consider the projection map \( q \subset E = \prod_{j=1}^t E_j \rightarrow E_t \). The minimality properties for \( q \) and \( q_t \) imply that the image of \( q \) coincides with \( q_t \); in particular,
\[
\dim Q(q) \geq \dim Q(q_t).
\]
This proves that
\[
\dim Q(q) = \dim Q(q_t) = \dim Q(q'),
\]
and therefore \( q = q' \).

**Theorem 2.13.** Keep the notation and assumptions of Lemma 2.11. Assume additionally that for all \( j \) the field \( E_t \) contains \( E_j \) and for each \( \sigma_j \in \Sigma_{E_j} \) we have
\[
h_j(\sigma_j) = \sum_\sigma h_\sigma(\sigma) \quad \text{where the sum is taken across all } \sigma : E_t \hookrightarrow C, \quad \text{whose restriction to } E_j \text{ coincides with } \sigma_j.
\]
Then
\[
q = \{(e_j)_{j=1}^t \in E \mid e_t \in q_t, e_j = \text{Tr}_{E_t/E_j}(e_t) \quad \forall j\},
\]
In particular, \( \dim Q(q) = \dim Q(q_t) \).

**Proof.** Since \( E_j \) is normal over \( Q \), field extension \( E_t / E_j \) is also normal and we may view the Galois group \( G_j := \text{Gal}(E_t/E_j) \) as a normal subgroup of \( G := \text{Gal}(E_t/Q) \).

Recall that group \( G \) acts naturally by \( C \)-linear automorphism on \( (E_t)_C \) by
\[
s(e \otimes z) = s(e) \otimes z \quad \forall s \in G, \ e \in E_t, \ z \in C
\]
and on \( C^{\Sigma_{E_t}} \) via
\[
(su)(\sigma) = u(\sigma s) \quad \forall s \in G, \sigma \in \Sigma_{E_t}, \ u \in C^{\Sigma_{E_t}}.
\]
Clearly, the isomorphism \( \kappa_{E_t} \) is \( G \)-equivariant.

It is also clear that if \( \sigma_j : E_j \hookrightarrow C \) is a field embedding and \( \sigma_t : E_t \hookrightarrow C \) is a field embedding that extends \( \sigma_j \) then the coset \( \sigma_t G_j \) coincides with the set of all field embeddings \( \sigma : E_t \hookrightarrow C \), whose restriction to \( E_j \) coincides with \( \sigma_j \). It follows that
\[
h_j(\sigma_j) = \sum_{s \in G_j} h_\sigma(\sigma s).
\]
This implies that if we put \( x = \kappa_{E_t}^{-1}(h) \) then
\[
x = \{(x_j)_{j=1}^t \in \prod_{j=1}^t (E_j)_C = \mathcal{E}_C \}
\]
satisfies
\[
x_j = \text{Tr}_{E_t/E_j}(x_t) \quad \forall j.
\]
Now the result follows from Lemma 2.12. \( \square \)

3. **Complex abelian varieties**

3.1. Let \( Z \) be a complex abelian variety of positive dimension. We write \( \mathcal{C}_Z \) for the center of the semisimple finite-dimensional \( Q \)-algebra \( \text{End}^0(Z) \). Let us choose a polarization on \( Z \) and let
\[
\text{End}^0(Z) \rightarrow \text{End}^0(Z), \ u \mapsto u'
\]
be the corresponding Rosati involution. It is well-known that \( \mathcal{C}_Z \) is stable under the Rosati involution and its restriction
\[
\mathcal{C}_Z \rightarrow \mathcal{C}_Z, \ u \mapsto u'
\]
does not depend on the choice of polarization. In addition, if $\mathcal{C}_Z$ is a CM-field $E$ then the Rosati involution on $E$ coincides with the complex conjugation $c_0$.

3.2. Let $H_1(Z, \mathbb{Q})$ be the first rational homology group of $Z$: it is a $2\dim(Z)$-dimensional $\mathbb{Q}$-vector space. The $\mathbb{Q}$-algebra $\text{End}^0(Z)$ acts by functoriality on $H_1(Z, \mathbb{Q})$ and this action gives rise to the embedding of $\mathbb{Q}$-algebras

$$\text{End}^0(Z) \hookrightarrow \text{End}_\mathbb{Q}(H_1(Z, \mathbb{Q}))$$

that sends the identity automorphism $1_Z$ of $Z$ to the identity automorphism $\text{Id}$ of $H_1(Z, \mathbb{Q})$. It follows easily [16] Ch. II that if $E \subset \text{End}^0(Z)$ is a subfield that contains $1_Z$ then $E$ is a number field and the embedding

$$E \subset \text{End}^0(Z) \hookrightarrow \text{End}_\mathbb{Q}(H_1(Z, \mathbb{Q}))$$

provides $H_1(Z, \mathbb{Q})$ with the natural structure of an $E$-vector space of dimension

$$d = d(Z, E) := \frac{2\dim(Z)}{|E : \mathbb{Q}|}.$$

We write $\text{End}_E(H_1(Z, \mathbb{Q})) \subset \text{End}_\mathbb{Q}(H_1(Z, \mathbb{Q}))$ for the $E$-algebra of $E$-linear operators in $H_1(Z, \mathbb{Q})$ and

$$\text{Tr}_E : \text{End}_E(H_1(Z, \mathbb{Q})) \to E$$

for the corresponding trace map. Clearly, $\text{Tr}_E$ is a $\mathbb{Q}$-Lie algebra homomorphism (even an $E$-Lie algebra homomorphism). Here $E$ is viewed as a commutative Lie algebra.

Let us consider the first complex homology group of $Z$

$$H_1(Z, \mathbb{C}) = H_1(Z, \mathbb{Q}) \otimes \mathbb{C},$$

which is a $2\dim(Z)$-dimensional complex vector space. If $E$ is as above then $H_1(Z, \mathbb{C})$ carries the natural structure of a free $E_C := E \otimes \mathbb{C}$-module of rank $d(Z, E)$. We write $\text{End}_{E_C}(H_1(Z, \mathbb{C})) \subset \text{End}_\mathbb{C}(H_1(Z, \mathbb{C}))$ for $E_C$-algebra of endomorphisms of the free $E_C$-module $H_1(Z, \mathbb{C})$ and

$$\text{Tr}_{E_C} : \text{End}_{E_C}(H_1(Z, \mathbb{C})) \to E_C$$

for the corresponding trace map. For example,

$$\text{Tr}_{E_C} (\text{Id}_E) = d(Z, E) = d.$$

Here $\text{Id}_E$ stands for the identity automorphism of $H_1(Z, \mathbb{C})$.

The group $\text{Aut}(\mathbb{C})$ acts tautologically on $H_1(Z, \mathbb{C}) = H_1(Z, \mathbb{Q}) \otimes \mathbb{C}$ by semilinear automorphisms through the second factor. The natural homomorphism of $\mathbb{C}$-algebras

$$\text{End}_{\mathbb{Q}}(H_1(Z, \mathbb{Q})) \otimes \mathbb{C} \to \text{End}_{\mathbb{C}}(H_1(Z, \mathbb{Q}) \otimes \mathbb{C}) = \text{End}_{\mathbb{C}}(H_1(Z, \mathbb{C}))$$

is an isomorphism that will allow us to identify $\mathbb{C}$-algebras $\text{End}_{\mathbb{Q}}(H_1(Z, \mathbb{Q}) \otimes \mathbb{C}$ and $\text{End}_{\mathbb{C}}(H_1(Z, \mathbb{C}))$. The group $\text{Aut}(\mathbb{C})$ acts tautologically on $\text{End}_{\mathbb{C}}(H_1(Z, \mathbb{C})) = \text{End}_{\mathbb{Q}}(H_1(Z, \mathbb{Q})) \otimes \mathbb{C}$ by semilinear automorphisms.

3.3. There is a canonical Hodge decomposition ([11 chapter 1], [8 pp. 52–53])

$$H_1(Z, \mathbb{C}) = H^{-1,0} \oplus H^{0,-1}$$

where $H^{-1,0} = H^{-1,0}(Z)$ and $H^{0,-1} = H^{0,-1}(Z)$ are mutually “complex conjugate” $\dim(Z)$-dimensional complex vector spaces. This splitting is $\text{End}^0(Z)$-invariant.
(and the $\text{End}^0(Z)$-module $H^{-1,0}$ is canonically isomorphic to the commutative Lie algebra $\text{Lie}(Z)$ of $Z$). Let

$$f_H = f_{H,Z} : H_1(Z, \mathbb{C}) \to H_1(Z, \mathbb{C})$$

be the $\mathbb{C}$-linear operator in $H_1(Z, \mathbb{C})$ defined as follows.

$$f_H(x) = -x \quad \forall x \in H^{-1,0}; \quad f_H(x) = 0 \quad \forall x \in H^{0,-1}.$$ Clearly, $f_H$ commutes with $\text{End}^0(Z)$.

Suppose that $M_T = M_{TZ} \subset \text{GL}_Q(H_1(Z, \mathbb{Q}))$ is the Mumford-Tate group of (the rational Hodge structure $H_1(Z, \mathbb{Q})$ and of) $Z$ (20, 14, 20). It is a connected reductive algebraic $\mathbb{Q}$-group that contains scalars and could be described as follows (20, section 6.3). Let $m_t \subset \text{End}_Q(H_1(Z, \mathbb{Q}))$ be the $\mathbb{Q}$-Lie algebra of $M_T$; it is a reductive algebraic linear $\mathbb{Q}$-Lie algebra which contains scalars and its natural faithful representation in $H_1(Z, \mathbb{Q})$ is completely reducible. In addition, $m_t$ is the smallest $\mathbb{Q}$-Lie subalgebra in $\text{End}_Q(H_1(Z, \mathbb{Q}))$ that enjoys the following property: its complexification

$$m_t \subset \text{End}_Q(H_1(Z, \mathbb{Q}))$$

contains scalars and $f_H$. It is well-known that the centralizer of $M_T$ (and therefore of $m_t$) in $\text{End}_Q(H_1(Z, \mathbb{Q}))$ coincides with $\text{End}^0(Z)$. This implies that the center $c$ of $m_t$ lies in $C_Z$. Since $m_t$ is reductive, it splits into a direct sum

$$m_t = m_t^{ss} \oplus c$$

of $c$ and a semisimple $\mathbb{Q}$-Lie algebra $m_t^{ss}$.

Since $m_t^{ss}$ is semisimple, and $E$ is commutative,

$$\text{Tr}_E(m_t) = \text{Tr}_E(c) \subset E.$$ This implies easily that

$$\text{Tr}_{E_{\mathbb{Q}}}(m_t) = \text{Tr}_E(c) \subset E \otimes \mathbb{C}.$$ In particular, since $f_H \in m_t$, we have $\text{Tr}_{E_{\mathbb{Q}}}(f_H) \in \text{Tr}_E(c) \subset E \otimes \mathbb{C}$.

3.4. We refer to [14, 20] Sect. 6.6.1 and 6.6.2 for the definition and basic properties of the Hodge group $Hdg = Hdg_Z$ of the rational Hodge structure $H_1(Z, \mathbb{Q})$ and of $Z$. Recall that $Hdg$ is a normal connected algebraic subgroup of $M_T$; in addition, $Hdg$ lies in the special general linear group $\text{SL}_Q(H_1(Z, \mathbb{Q}))$ of $H_1(Z, \mathbb{Q})$ and the natural homomorphism-product

$$Hdg \times G_m \to MT$$

is an isogeny of connected algebraic $\mathbb{Q}$-groups. Here $G_m = G_m \cdot \text{Id} \subset \text{GL}_Q(H_1(Z, \mathbb{Q}))$ is the group of homotheties. It follows easily that $Hdg$ is reductive and if

$$hdg = hdg_Z \subset \text{End}_Q(H_1(Z, \mathbb{Q}))$$

is the $\mathbb{Q}$-Lie algebra of $Hdg$ then it is reductive, its semisimple part coincides with $m_t^{ss}$ and

$$m_t = Q \cdot \text{Id} \oplus hdg, \quad hdg = m_t \cap sl(H_1(Z, \mathbb{Q})).$$ (Here $sl(H_1(Z, \mathbb{Q}))$ is the (simple) $\mathbb{Q}$-Lie algebra of $\mathbb{Q}$-linear operators in $H_1(Z, \mathbb{Q})$ with zero trace.) In particular, if $c^0 = c^0_Z$ is the center of (reductive) $hdg$ then

$$c = c^0 \oplus Q \cdot \text{Id}, \quad hdg = m_t^{ss} \oplus c^0, \quad m_t = m_t^{ss} \oplus c^0 \oplus Q \cdot \text{Id}.$$
Clearly,
\[ \text{mt}_C = \text{hdg}_C \oplus C \cdot \text{Id}_C, \quad f_H = \left( f_H + \frac{1}{2} \text{Id}_C \right) - \frac{1}{2} \text{Id}_C = f_H^0 - \frac{1}{2} \text{Id}_C \]
where
\[ f_H^0 := f_H + \frac{1}{2} \text{Id}_C \in \mathfrak{s}(H_1(Z, C)). \]

It follows easily that hdg is the smallest \( \mathbb{Q} \)-Lie subalgebra in \( \text{End}_{\mathbb{Q}}(H_1(Z, \mathbb{Q})) \) that enjoys the following property: its complexification
\[ \text{hdg}_C = \text{hdg} \otimes_{\mathbb{Q}} C \subset \text{End}_C(H_1(Z, C)) \]
contains \( f_H^0 \). Clearly,
\[ (1) \quad \text{Tr}_E(\text{hdg}) = \text{Tr}_E(\text{mt}^{ss} \oplus \mathcal{O}) = \text{Tr}_E(\mathcal{O}). \]
The choice of the polarization on \( Z \) gives rise to an alternating non-degenerate \( \mathbb{Q} \)-bilinear form
\[ \psi : H_1(Z, \mathbb{Q}) \times H_1(Z, \mathbb{Q}) \to \mathbb{Q} \]
that is Hdg-invariant; in addition
\[ \psi_{\mathbb{Q}}(ux, y) = \psi_{\mathbb{Q}}(x, u'y) \quad \forall u \in \text{End}^0(Z), \ x, y \in H_1(Z, \mathbb{Q}). \]
The Hdg-invariance of \( \psi_{\mathbb{Q}} \) means that
\[ \psi_{\mathbb{Q}}(ux, y) + \psi_{\mathbb{Q}}(x, uy) = 0 \quad \forall u \in \text{hdg}, \ x, y \in H_1(Z, \mathbb{Q}). \]
If \( u \in \mathcal{O} \subset \text{hdg} \) then \( u \in \mathcal{C}_Z \) and we have
\[ \psi_{\mathbb{Q}}(ux, y) = \psi_{\mathbb{Q}}(x, u'y), \ \psi_{\mathbb{Q}}(ux, y) + \psi_{\mathbb{Q}}(x, uy) = 0. \]
Since \( (u')' = u \), we have \( \psi_{\mathbb{Q}}(u'x, y) = \psi_{\mathbb{Q}}(x, uy) \) and therefore
\[ 0 = \psi_{\mathbb{Q}}(ux, y) + \psi_{\mathbb{Q}}(x, uy) = \psi_{\mathbb{Q}}(ux, y) + \psi_{\mathbb{Q}}(u'x, y) = \psi_{\mathbb{Q}}((u + u')x, y). \]
The non-degeneracy of \( \psi_{\mathbb{Q}} \) implies that \( u + u' = 0 \), i.e., \( u' = -u \). This means that
\[ \mathcal{O} \subset \{ u \in \mathcal{C}_Z \mid u' = -u \} \subset \mathcal{C}_Z. \]

**Remark 3.5.** It is well known \cite{11} that if the center \( \mathcal{C}_Z \) is a field then it is either a totally real number field or a CM-field. If \( \mathcal{C}_Z \) is a totally real number field then the Rosati involution acts on \( \mathcal{C}_Z \) as identity map, \( \{ u \in \mathcal{C}_Z \mid u' = -u \} = \{ 0 \} \) and therefore
\[ \mathcal{O} = \{ 0 \}. \]

Suppose that \( \mathcal{C}_Z \) is a CM field, i.e., a totally imaginary quadratic extension of a totally real number field \( F_2 \). Then the Rosati involution acts on on \( \mathcal{C}_Z \) as the “complex conjugation” \cite{11}; in particular, it is \( F_2 \)-linear and \( \{ u \in \mathcal{C}_Z \mid u' = -u \} \) is a one-dimensional \( F_2 \)-vector subspace of \( \mathcal{C}_Z \) and therefore its \( \mathbb{Q} \)-dimension equals \( [\mathcal{C}_Z : \mathbb{Q}] / 2 \). This implies that
\[ \mathcal{O} \subset \{ u \in \mathcal{C}_Z \mid u' = -u \}, \quad \dim_{\mathbb{Q}}(\mathcal{O}) \leq \frac{1}{2}[\mathcal{C}_Z : \mathbb{Q}]. \]

If \( Z = \prod_{j=1}^t Z_j \) is a product of abelian varieties \( Z_j \)'s then there is an inclusion \( \oplus_{j=1}^t \text{End}^0(Z_j) \subset \text{End}^0(Z) \) and therefore \( \mathcal{C}_Z \subset \oplus_{j=1}^t \mathcal{C}_{Z_j} \).
3.6. Suppose that a CM field $E$ is the center of $\text{End}^0(Z)$. As in Subsect. [27], we write $c_0$ for the “complex conjugation” on $E$ and $T_E$ for the corresponding norm torus. Clearly, the center of the $C$-algebra
\[
\text{End}^0(Z)_C = \text{End}^0(Z) \otimes_Q C \subset \text{End}_Q(H_1(Z, Q)) \otimes_Q C = \text{End}_C(H_1(Z, C))
\]
coincides with $E_C$.

Let $3$ be the center of $\text{Hdg}$.

The inclusion $E \subset \text{End}^0(Z) \subset \text{End}_Q(H_1(Z, Q))$ gives rise to the embedding of $\mathbf{Q}$-algebraic groups $R_{E/Q}G_m \subset GL(H_1(Z, Q))$. Since $T_E \subset R_{E/Q}G_m$, we have
\[
T_E \subset R_{E/Q}G_m \subset GL(H_1(Z, Q)),
\]
\[
R_{E/Q}G_m(Q) = E^* \subset Aut_Q(H_1(Z, Q)), \ T_E(Q) = \{e \in E \mid e c_0(e) = 1\}.
\]

Clearly, the $\mathbf{Q}$-Lie algebras of $T_E$ and $R_{E/Q}G_m$, viewed as $\mathbf{Q}$-Lie subalgebras of $\text{End}_Q(H_1(Z, Q))$, coincide with $E_e$ and $E$ respectively. Since $H_1(Z, C) = H_1(Z, Q) \otimes_Q C$, we have
\[
R_{E/Q}G_m(C) = E^*_C \subset Aut_C(H_1(Z, C)),
\]
\[
T_E(C) = \{u \in E^*_C \mid u \cdot c_0(u) = 1\} \subset E^*_C \subset Aut_C(H_1(Z, C)).
\]

Since the centralizer of $\text{hdg}$ in $\text{End}_Q(H_1(Z, Q))$ coincides with $\text{End}^0(Z)$, it follows that the centralizer of the $C$-Lie algebra $\text{hdg}_C$ in $\text{End}_C(H_1(Z, C))$ coincides with $\text{End}^0(Z)_C$. Since the $C$-Lie subalgebra
\[
\text{hdg}_C \subset \text{End}_C(H_1(Z, C))
\]
coincides with the $C$-Lie algebra of the connected complex algebraic subgroup $\text{Hdg}(C) \subset Aut_C(H_1(Z, C))$, it follows that the centralizer of $\text{Hdg}(C)$ in $\text{End}_C(H_1(Z, C))$ also coincides with $\text{End}^0(Z)_C$. This implies that the center $3(C)$ of $\text{Hdg}(C)$ lies in the center of $\text{End}^0(Z)_C$. It follows that
\[
3(C) \subset E^*_C = R_{E/Q}G_m(C).
\]

This implies that
\[
3 \subset R_{E/Q}G_m.
\]

We want to prove that $3 \subset T_E$. In order to do that, let us extend $\psi_Q$ by $C$-linearity to $H_1(Z, Q) \otimes_Q C = H_1(Z, C)$. We get a non-degenerate alternating $C$-bilinear form
\[
\psi_C : H_1(Z, C) \times H_1(Z, C) \rightarrow C,
\]
which is $\text{Hdg}(C)$-invariant. Clearly,
\[
\psi_C(ux, y) = \psi_C(x, c_0(u)y) \quad \forall u \in E_C, \ x, y \in H_1(Z, C).
\]

This implies that
\[
\psi_C(ux, uy) = \psi_C(x, c_0(u)uy) = \psi_C(x, uc_0(u)y), \quad \forall u \in E_C.
\]

This implies that if $u \in E_C$ then $\psi_C$ is $u$-invariant if and only if $uc_0(u) = 1$, i.e., $u \in T_E(C)$. It follows that $3(C) \subset T_E(C)$, i.e.,
\[
3 \subset T_E.
\]
3.7. The \( \dim(Z) \)-dimensional complex vector space \( \Omega^1(Z) \) of the differentials of the first kind on \( Z \) carries the natural structure of \( E \otimes Q \text{C} \)-module [24, Sect. 2].

Clearly,

\[
\Omega^1(Z) = \bigoplus_{\sigma \in \Sigma_E} C_{\sigma} \Omega^1(Z) = \bigoplus_{\sigma \in \Sigma_E} \Omega^1(Z)_{\sigma}
\]

where \( \Omega^1(Z)_{\sigma} := C_{\sigma} \Omega^1(Z) = \{ x \in \Omega^1(Z) \mid ex = \sigma(e)x \quad \forall e \in E \} \). Let us put

\[
n_{\sigma} = n_{\sigma}(Z,E) = \dim_{\text{C}_{\sigma}} \Omega^1(Z)_{\sigma} = \dim_{\text{C}} \Omega^1(Z)_{\sigma}.
\]

It follows (compare with [24, p. 260]) that \( \text{Tr}_{E\text{C}}(f_H) = (-n_{\sigma})_{\sigma \in \Sigma_E} \). This implies that

\[
(n_{\sigma})_{\sigma \in \Sigma_E} = -\text{Tr}_{E\text{C}}(f_H) \in \text{Tr}_E(\text{c}) \otimes Q \text{C} = \text{Tr}_E(\text{mt}) \otimes Q \text{C}.
\]

**Remarks 3.8.**

(i) It is well-known [24, Sect. 2] that

\[
n_{\sigma} + \bar{n}_{\sigma} = d = 2\dim(Z)/[E : Q] \forall \sigma.
\]

This means that the function

\[
\Sigma_E \to Q, \quad \sigma \mapsto d - n_{\sigma}
\]

lies in \( X_E \).

(ii) Recall that the Hodge splitting commutes with \( \text{End}^0(Z) \) and therefore with \( E \). Hence \( f_H \) may be viewed as an endomorphism of the free \( E\text{C} \)-module \( H_1(Z,\text{C}) \) and its trace in \( E\text{C} \) is the tuple

\[
(-n_{\sigma})_{\sigma \in \Sigma_E} \in \prod_{\sigma \in \Sigma_E} C_{\sigma} = E_{\text{C}}
\]

[24 Sect. 2]. It follows that

\[
\text{Tr}_{E\text{C}}(f_H^0) = \text{Tr}_{E\text{C}}(f_H) + \text{Tr}_{E\text{C}}\left( \frac{1}{2} \text{Id}_{E\text{C}} \right) = \text{Tr}_{E\text{C}}(f_H) + \frac{1}{2}d = \left\{ \frac{d}{2} - n_{\sigma} \right\}_{\sigma \in \Sigma_E} \in E_{\text{C}}.
\]

**Lemma 3.9.** \( \text{Tr}_E(\text{hdg}) \) coincides with the smallest \( Q \)-vector subspace \( q \subset E \) such that the \( C \)-vector subspace

\[
q_{\text{C}} = q \otimes Q \text{C} \subset E_{\text{C}} = E \otimes Q \text{C} = \bigoplus_{\sigma \in \Sigma_E} C_{\sigma} = C_{\Sigma_E}
\]

contains \( \left\{ \frac{d}{2} - n_{\sigma} \right\}_{\sigma \in \Sigma_E} \).

**Proof.** Clearly, \( \text{Tr}_E(\text{hdg}) \) contains \( q \), because \( \text{hdg}_E \) contains \( f_H^0 \). On the other hand, if \( \text{Tr}_E(\text{hdg}) \neq q \) then

\[
\text{hdg}' := \{ u \in \text{hdg} \mid \text{Tr}_E(u) \in q \}
\]

is a proper \( Q \)-Lie subalgebra of \( \text{hdg} \), whose complexification contains \( f_H^0 \). This contradicts the minimality property of \( \text{hdg} \) and therefore proves the Lemma. \( \square \)

**Remark 3.10.** It follows from Remarks 3.8 that

\[
\text{Tr}_{E\text{C}}(f_H^0) = \left\{ \frac{d}{2} - n_{\sigma} \right\}_{\sigma \in \Sigma_E}
\]

lies in \( E_{-} \otimes Q \text{C} \). Applying Lemma 3.9, we conclude that \( \text{Tr}_E(\text{hdg}) \subset E_{-} \).
Theorem 3.11. Suppose that $E$ is a CM field that is normal over $\mathbb{Q}$ and fix a field embedding $E \hookrightarrow \mathbb{Q} \subset \mathbb{C}$. Let $W$ be the $\mathbb{Q}[\text{Gal}(E/\mathbb{Q})]$-submodule of $X_E$ generated by the function $h(\sigma) := \left\{ \frac{d}{2} - n_\sigma \right\}_{\sigma \in \Sigma_E}$. Then

$$\dim_{\mathbb{Q}}(\text{Tr}_E(e^0)) = \dim_{\mathbb{Q}}(\text{Tr}_E(hdg)) = \dim_{\mathbb{Q}}(W),$$

and therefore, $\dim_{\mathbb{Q}}(c^0) \geq \dim_{\mathbb{Q}}(W)$. If $W = X_E$ then

$$\text{Tr}_E(e^0) = \text{Tr}_E(hdg) = E_.$$

Proof. Let $q \subset E_-$ be the smallest $\mathbb{Q}$-vector subspace such that $q \subset \mathbb{C}$ contains the function $h$. By Lemma 2.9

$$\dim_{\mathbb{Q}}(q) = \dim_{\mathbb{Q}}(W).$$

The minimality properties of $\text{hdg}$ imply that

$$q = \text{Tr}_E(hdg) = \text{Tr}_E(e^0).$$

This implies that

$$\dim_{\mathbb{Q}}(e^0) = \dim_{\mathbb{Q}}(\text{Tr}_E(hdg)) = \dim_{\mathbb{Q}}(q) = \dim_{\mathbb{Q}}(W).$$

Theorem 3.12. Suppose that $E$ is a CM field that is normal over $\mathbb{Q}$.

(i) If the center $\mathcal{E}_Z$ of $\text{End}^0(Z)$ is a field then $Z$ contains a simple abelian subvariety of dimension $\geq \dim_{\mathbb{Q}} \text{Tr}_E(\text{hdg}_Z)$.

(ii) If $\text{Tr}_E(hdg) = E_-$ then $Z$ contains a simple abelian subvariety of dimension $\geq \max(E)$.

Proof. We may assume that $\text{Tr}_E(hdg) \neq \{0\}$.

(i) Suppose that the center $\mathcal{E}_Z$ of $\text{End}^0(Z)$ is a field. Since the center of $\text{End}^0(Z)$ is a field, there exists a simple complex abelian (sub)variety $T$ (of $Z$) such that $Z$ is isogenous to a self-product $T^m$ of $T$; in particular, $\text{End}^0(Z)$ is the matrix algebra of size $m$ over $\text{End}^0(T)$, which implies that $\mathcal{E}_Z = \mathcal{E}_T$.

If $c^0$ is the center of $\text{hdg}_Z$ then $c^0 \subset \mathcal{E}_T = \mathcal{E}_T$ and $\dim_{\mathbb{Q}}(c^0) \geq \dim_{\mathbb{Q}} \text{Tr}_E(\text{hdg}_Z)$. If $\mathcal{E}_Z = \mathcal{E}_T$ is totally real then the center $c^0$ of $\text{hdg}_Z$ is zero, which is not the case. So, $\mathcal{E}_Z$ is a CM-field and

$$\dim_{\mathbb{Q}}(e^0) \leq \frac{1}{2}[\mathcal{E}_Z : \mathbb{Q}] = \frac{1}{2}[\mathcal{E}_T : \mathbb{Q}].$$

This implies that

$$\frac{1}{2}[\mathcal{E}_T : \mathbb{Q}] \geq \dim_{\mathbb{Q}}(c^0) \geq \dim_{\mathbb{Q}} \text{Tr}_E(\text{hdg}_Z).$$

Since $[\mathcal{E}_T : \mathbb{Q}] \leq 2\dim(T)$, we conclude that $\dim(T) \geq \dim_{\mathbb{Q}} \text{Tr}_E(\text{hdg}_Z)$. This proves (i).

(ii) Let us assume that $\text{Tr}_E(hdg) = E_-$. Clearly, the center $c^0$ of $\text{hdg}_Z$ satisfies

$$\dim_{\mathbb{Q}}(c^0) \geq \dim_{\mathbb{Q}}(E_-) = \frac{1}{2}[E : \mathbb{Q}].$$

The Poincaré reducibility theorem implies that there exist abelian subvarieties $Z_1, \ldots, Z_r$ of $Z$ such that the natural morphism

$$\pi : \prod Z_j \rightarrow Z, \{z_j\} \mapsto \sum z_j$$
is an isogeny, Hom(Z_i, Z_j) = \{0\} for all i ≠ j and each End^0(Z_j) is a simple (but not necessarily central) Q-algebra and its center is a field. In particular,

End^0(Z) = \oplus_j End^0(Z_j).

The morphism π implies the End^0(Z)-equivariant isomorphism of rational Q-structures

H^1(Z, Q) = \oplus_j H^1(Z_j, Q).

In particular, this splitting is E-invariant and we have the embeddings E \hookrightarrow End(Z_j), whose “direct sum” is the E \hookrightarrow End^0(Z). We have

\[ d(Z, E) = \sum_j d(Z_j, E); n_\sigma(Z, E) = \sum_j n_\sigma(Z_j, E) \forall \sigma \in \Sigma_E. \]

It follows that the function

h_{Z, E} : \Sigma_E \to Q, \sigma \mapsto \frac{1}{2} d(Z, E) - n_\sigma(Z, E)

coincides with the sum \( \sum_j h_{Z_j, E} \) where

h_{Z_j, E} : \Sigma_E \to Q, \sigma \mapsto \frac{1}{2} d(Z_j, E) - n_\sigma(Z_j, E)

is the corresponding function attached to Z_j. Clearly, h and all h_j belong to X_E.

Let W_j be the Q[Gal(E/Q)]-submodule of X_E generated by h_j. Since h = \( \sum_j h_j \), the Q[Gal(E/Q)]-submodule \( \sum_j W_j \) contains h. Let W be the Q[Gal(E/Q)]-submodule of X_E generated by h. By Theorem 3.11 \( \dim_Q(W) = \dim_Q(z_\infty) \). Since \( \dim_Q(E) = \dim_Q(X_E) \), we conclude that \( \dim_Q(W) = \dim_Q(X_E) \) and therefore \( X_E = W \). In other words, X_E coincides with its Q[Gal(E/Q)]-submodule generated by h. It follows that X_E = \( \sum_j W_j \). This implies that if W' is a simple Q[Gal(E/Q)]-submodule of X_E then it is isomorphic to a certain Q[Gal(E/Q)]-submodule of W_j for some j; in particular, 1 ≤ \( \dim_Q(W') \leq \dim_Q(W_j) \). On the other hand, by Theorem 3.11 \( \dim_Q(W_j) = \dim_Q(\text{Tr}_E(\text{hdg}_{Z_j})) \). By the already proven case (i), Z_j contains a simple abelian subvariety T_j with \( \dim(T_j) \geq \dim_Q(\text{Tr}_E(\text{hdg}_{Z_j})) \). It follows that

\[ \dim(T_j) \geq \dim_Q(\text{Tr}_E(\text{hdg}_{Z_j})) \geq \dim_Q(W') \]

Clearly, T_j is an abelian subvariety of Z, since Z_j is an abelian subvariety of Z. Now, if we choose W' with \( \dim_Q(W') = \max(E) \) then we get \( \dim(T_j) \geq \max(E) \).

3.13. Let t be a positive integer and suppose that for each positive j ≤ t we are given the following data.

- A number field E_j that is normal over Q: we fix an embedding E_j \hookrightarrow C and consider E_j as the subfield of C.
- A complex abelian variety Z_j of positive dimension.
- An embedding E_j \hookrightarrow End^0(Z_j) that sends 1 to the identity automorphism of Z_j.

Let us consider the corresponding numbers

\[ d_j := d(Z_j, E_j) = \frac{2\dim(Z_j)}{|E_j : Q|} \]

and functions

\[ \Sigma_{E_j} \to Z_+, \sigma \mapsto n_\sigma^{(j)} := n_\sigma(Z_j, E_j) = \dim_C\Omega^1(Z_j)_\sigma. \]
Let us consider the products $Z = \prod_{j=1}^{t} Z_j$ and $E = \prod_{j=1}^{t} E_j = \oplus_{j=1}^{t} E_j$. Clearly, $Z$ is a complex abelian variety and $E$ is is a finite-dimensional semisimple commutative $\mathbb{Q}$-algebra that admits a natural embedding

$$\mathcal{E} \hookrightarrow \text{End}^0(Z)$$

that sends $1 \in \mathcal{E}$ to $1_Z$. The natural (Künneth) isomorphism

$$H_1(Z, \mathbb{Q}) = \oplus_{j=1}^{t} H_1(Z_j, \mathbb{Q})$$

is an isomorphism of rational Hodge structures; in particular, each $H_1(Z_j, \mathbb{Q})$ is a $\text{MT}_Z$-invariant $\mathbb{Q}$-vector space of $H_1(Z, \mathbb{Q})$. Clearly, $H_1(Z, \mathbb{Q})$ carries the natural structure of $\mathcal{E}$-module and

$$\mathcal{E} \subset \text{End}^0(Z) \subset \text{End}_{\mathbb{Q}}(H_1(Z, \mathbb{Q})).$$

It is also clear that

$$\text{End}_\mathcal{E}(H_1(Z, \mathbb{Q})) = \oplus_{j=1}^{t} \text{End}_{E_j}(H_1(Z_j, \mathbb{Q}))$$

and

$$\text{hdg}_Z \subset \oplus_{j=1}^{t} \text{End}_{E_j}(H_1(Z_j, \mathbb{Q})).$$

In particular, the elements of $\text{End}_\mathcal{E}(H_1(Z, \mathbb{Q}))$ are the $t$-tuples $\{u_j\}_{j=1}^{t}$ with $u_j \in \text{End}_{E_j}(H_1(Z_j, \mathbb{Q}))$. Clearly, the map

$$\{u_j\}_{j=1}^{t} \mapsto u_j \mapsto \text{Tr}_{E_j}(u_j) \in E_j$$

is the homomorphism of $\mathbb{Q}$-Lie algebras

$$\text{End}_\mathcal{E}(H_1(Z, \mathbb{Q})) \rightarrow E_j,$$

which we continue to denote by $\text{Tr}_{E_j}$. Since $E_j$ is the commutative Lie algebra, $\text{Tr}_{E_j}$ kills the semisimple part of $\text{hdg}_Z$. On the other hand, the restriction of $\text{Tr}_{E_j}$ to $\mathcal{E}$ coincides with the composition of the projection map

$$\mathcal{E} = \oplus_{j=1}^{t} E_j \rightarrow E_j$$

and multiplication by $d_j$. Let $d$ be the least common multiple of all $d_j$'s. Then the $\mathbb{Q}$-linear map

$$\text{Tr}_\mathcal{E} : \oplus_{j=1}^{t} \text{End}_{E_j}(H_1(Z_j, \mathbb{Q})) \rightarrow \oplus_{j=1}^{t} E_j, \quad \{u_j\}_{j=1}^{t} \mapsto \left\{ \frac{d}{d_j} \text{Tr}_{E_j}(u_j) \right\}_{j=1}^{t}$$

kills the semisimple part of $\text{hdg}_Z$ and acts on $\mathcal{E}$ as multiplication by $d$. It follows that

$$\text{Tr}_\mathcal{E}(\text{hdg}_Z) = \text{Tr}_\mathcal{E}(\mathfrak{c}_Z^0) \subset \mathcal{E};$$

in addition, if $\mathcal{E}_Z \subset \mathcal{E}$ then $\mathfrak{c}_Z^0 \subset \mathcal{E}$ and $\text{Tr}_\mathcal{E}(\mathfrak{c}_Z^0) = \mathfrak{c}_Z^0$, which implies that

$$\text{Tr}_\mathcal{E}(\text{hdg}_Z) = \mathfrak{c}_Z^0 \subset \mathcal{E}.$$
on $\Sigma_{E_j}$.

**Theorem 3.14.** We keep the notation and assumptions of the previous subsection. Suppose that $E_t$ contains all $E_j$’s and for all $j$ and each $\sigma_j \in \Sigma_j$ we have $h_j(\sigma_j) = \sum_\sigma h_\tau(\sigma)$ where the sum is taken across all $\sigma E_t \rightarrow \mathbb{C}$, whose restriction to $E_j$ coincides with $\sigma_j$. Then

$$\text{Tr}_E(hdg_Z) = \{(e_j)_1 \in E | e_t \in \text{Tr}_{E_t}(hdg_{Z_t}), e_j = \text{Tr}_{E_t/E_j}(e_t) \forall j\}.$$ 

In particular, $\dim_Q(q) = \dim_Q(q_t)$.

**Proof.** One has only to notice that $\text{Tr}_E = (d/dt)\text{Tr}_{E_t}$ on $hdg_{Z_t} \subset \text{End}_{E_t}(H_1(Z_t, \mathbb{Q}))$ and apply Theorem 2.13. $\square$

### 4. Proof of Main Results

We keep all notation and assumptions of Section 3.

**4.1.** Suppose that $n \geq 2$ is an integer, $p$ is a prime that does not divide $n$. Let $r$ be a positive integer and $q = p^r$. Suppose that $E = \mathbb{Q}(\zeta_q)$ and

$$d(Z, E) = n - 1.$$ 

It is well known that $\text{Gal}(E/\mathbb{Q}) = (\mathbb{Z}/q\mathbb{Z})^*$ where $a + q\mathbb{Z} \in (\mathbb{Z}/q\mathbb{Z})^*$ corresponds to the field automorphism

$$s_a : \mathbb{Q}(\zeta_q) \rightarrow \mathbb{Q}(\zeta_q), \; \zeta_q \rightarrow \zeta_q^a.$$ 

It is also well-known that the complex conjugation $c_0$ coincides with $s_{-1}$.

Clearly, $\Sigma_E$ coincides with the set of embeddings $\sigma_a : \mathbb{Q}(\zeta_q) \rightarrow \mathbb{Q}(\zeta_q) \subset \mathbb{C}$ that send $\zeta_q$ to $\zeta_q^{-a}$ with $a + q\mathbb{Z} \in (\mathbb{Z}/q\mathbb{Z})^*$. It is also clear that

$$\bar{\sigma_a} = i\sigma_a = \sigma_{-a} = \sigma_{q-a}$$ 

and

$$s_b(\sigma_a) = \sigma_{ab} \forall a, b.$$ 

**Theorem 4.2.** Suppose that $n_{\sigma_a} = [na/q]$ for all $a$ with $1 \leq a < q,$ $(a, p) = 1$. Then $\text{Tr}_E(c_0) = \text{Tr}_E(hdg) = E_\cdot$.

**Proof.** The following statement will be proven in Section 5 (See Theorem 5.2).

Let us consider the $\mathbb{Q}$-vector space of functions $V_\mathbb{Q} := \{g : (\mathbb{Z}/q\mathbb{Z})^* \rightarrow \mathbb{Q} | g(q - a) = -g(a), \forall a + q\mathbb{Z}\}$ provided with the natural structure of a $(\mathbb{Z}/q\mathbb{Z})^*$-module. Let $n$ be a positive integer that is not divisible by $p$. Let us consider the function $h$ on $(\mathbb{Z}/q\mathbb{Z})^*$ defined by $h(a + q\mathbb{Z}) = \frac{a}{q} - \frac{1}{p} \mathbb{Z}$ for $1 \leq a \leq q - 1$ and $p \nmid a$. Then $h \in V_\mathbb{Q}$ and the $(\mathbb{Z}/q\mathbb{Z})^*$-submodule generated by $h$ coincides with $V_\mathbb{Q}$.

Clearly, $V_\mathbb{Q} = X_E$. Now Theorem 4.2 becomes an immediate corollary of Theorem 3.11 combined with the above result. $\square$
Now we are ready to prove our main theorems listed in Section 1.2.

Proof of Theorem 1.2. By [25] p. 355 and Remark 4.13 on p. 356, [26] Remark 5.14 on p. 383 there exists an embedding \( \mathbb{Q}(\zeta_q) \rightarrow \text{End}^0(J^{(f,q)}) \) such that \( d(J^{(f,q)}, \mathbb{Q}(\zeta_q)) = n - 1 \) and \( n_{a_q} = [na/q] \) for all \( 1 \leq a \leq q - 1 \). Now the result follows from Theorem 4.2.

Proof of Theorem 1.9. If \( p \) is odd then \( \text{Gal}(\mathbb{Q}(\zeta_q)/\mathbb{Q}) \) is a cyclic group of order \( (p - 1)p^{r - 1} \). If \( p = 2 \) and \( q \geq 4 \) then \( \text{Gal}(\mathbb{Q}(\zeta_q)/\mathbb{Q}) = \langle c \rangle \times H \) where \( c \) is the complex conjugation and \( H \) is a cyclic group of order \( 2^{r - 1} \). Now the result follows from Theorem 1.2 combined with Theorem 3.12 and Examples 2.8.

Proof of Theorem 1.12. We know that \( \text{End}^0(J^{(f,q)}) = \mathbb{Q}(\zeta_q) = E \). By the last assertion of Subsect. 3.4 \( E^0 \subset E_- \subset E \) and therefore \( \text{Tr}_E(\iota^0) = \iota^0 \). By Theorem 1.2 \( \text{Tr}_E(\iota^0) = E_- \). This implies that

\[
\iota^0 = \text{Tr}_E(\iota^0) = E_- = \mathbb{Q}(\zeta_q).
\]

Since \( E_- \) coincides with the \( \mathbb{Q} \)-Lie algebra of connected \( T_{\mathbb{Q}(\zeta_q)} \), we conclude that the center \( z \) of \( \text{Hdg}(J^{(f,q)}) \) contains \( T_E \). But we proved in Subsect. 3.0 that \( Z \subset T_E \). It follows that \( Z = T_E = T_{\mathbb{Q}(\zeta_q)} = U_q \).

Proof of Theorem 1.6. We know that \( d_j = d(J^{(f,p^j)}, \mathbb{Q}(\zeta_{p^j})) = n - 1 \) for all \( j \leq r \). The least common multiple \( d \) of all \( d_j \) is also \( n - 1 \). It follows that the function \( h_j : \Sigma_{E_j} \rightarrow \mathbb{Q} \) is defined by

\[
h_j(a + p^j \mathbb{Z}) = \frac{n - 1}{2} - \left\lfloor \frac{na}{p^j} \right\rfloor, \quad \forall 1 \leq a \leq p^j - 1, p \nmid a.
\]

The following relations between the functions \( h_j \) will be proved in Section 4 (See Corollary 4.2).

Let us identify (in the usual way) \( G = (\mathbb{Z}/p^j \mathbb{Z})^* \) with the Galois group \( \text{Gal}(\mathbb{Q}(\zeta_{p^j})/\mathbb{Q}) \) and let us consider its subgroup

\[
G_j = \text{Gal}(\mathbb{Q}(\zeta_{p^j})/\mathbb{Q}(\zeta_{p^{j-1}})) \subset \text{Gal}(\mathbb{Q}(\zeta_{p^j})/\mathbb{Q}) = G.
\]

Then for each \( a \in (\mathbb{Z}/p^j \mathbb{Z})^* \),

\[
h_j(a \mod p^j) = \sum_{b \in G_j} h_r(ab).
\]

The Theorem now follows from Theorem 3.14.

5. Fourier coefficients

5.1. Throughout this section, \( p \) is a prime, \( q = p^r \) is a power of \( p \), and \( n \) is a positive integer that is not divisible by \( p \).

As usual,

\[
S^1 := \{ z \in \mathbb{C} \mid z\bar{z} = 1 \} \subset \mathbb{C}^* \subset \mathbb{C}.
\]

Given a finite group \( G \), its group of characters \( \hat{G} \) is the group \( \text{Hom}(G, S^1) \). (If \( G \) is commutative then \( \hat{G} \) is called the dual of \( G \).) Let \( \mathcal{K} \) be a field that is either \( \mathbb{Q} \) or \( \mathbb{C} \). Recall that the regular representation \( R_{\mathcal{K}} \) of \( G \) over \( \mathcal{K} \) is the space of \( \mathcal{K} \)-valued function on \( G \), where an element \( a \in G \) acts on a function \( f \) by \( (af)(b) = f(ba), \forall b \in G \). Clearly \( R_{\mathcal{K}} = R_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathcal{K} \).

Suppose \( G = (\mathbb{Z}/q\mathbb{Z})^\times \). We write \( V_{\mathcal{K}} \) for the subrepresentation of \( R_{\mathcal{K}} \) consisting all “odd” functions on \( (\mathbb{Z}/q\mathbb{Z})^\times \). Namely,

\[
V_{\mathcal{K}} := \{ f : (\mathbb{Z}/q\mathbb{Z})^\times \rightarrow \mathcal{K} \mid f(q - a) = -f(a), \forall a + q\mathbb{Z} \}.
\]
By definition, $V_\mathbb{C} = \{0\}$ if $q = p = 2$, and $\dim_{\mathbb{K}} V_\mathbb{C} = \varphi(q)/2$ otherwise.

Given a real number $x$, we write $[x]$ for the largest integer less or equal to $x$. Now consider the function $h_r$ defined by $h_r(a) = \frac{n - 1}{2} - \frac{na}{q}$, where $1 \leq a \leq q - 1$ and $p \nmid a$. Since $p \nmid n$, we have
\[
\left\lceil \frac{na}{q} \right\rceil + \left\lfloor \frac{n(q - a)}{q} \right\rfloor = n - 1 \quad \forall 0 < a < q \text{ with } p \nmid a.
\]
Hence $h_r \in V_\mathbb{Q} \subset V_\mathbb{C}$. We also note that $h_r = 0$ if and only if either $q = 2$ or $n = 1$.

The following assertion was used in the proof of Theorem 4.2.

**Theorem 5.2.** Let $p$ be a prime, $q = p^r$, $n \geq 2$ and $p \nmid n$. Let $\mathbb{K}$ be either $\mathbb{Q}$ or $\mathbb{C}$. The function $h_r$ generates the $\mathbb{K}[(\mathbb{Z}/q\mathbb{Z})^\times]$-module $V_\mathbb{K}$.

**Proof.** If $q = 2$, the vector space $V_\mathbb{K} = \{0\}$; if $q = 4$, then $\dim_{\mathbb{K}} V_\mathbb{K} = \varphi(4)/2 = 1$ and $h \neq 0$, hence the theorem is trivial in these cases. Thus we further assume that either $p$ is odd or $p = 2$ and $q = 2^r \geq 8$.

Let $W_\mathbb{K}$ be the submodule of $V_\mathbb{K}$ generated by $h_r$. Clearly $W_\mathbb{K} = W_\mathbb{Q} \otimes \mathbb{C}$. Then $h_r$ generates $\mathbb{Q}[(\mathbb{Z}/q\mathbb{Z})^\times]$-module $V_\mathbb{Q}$ if and only if the same holds true if we replace $\mathbb{Q}$ with $\mathbb{C}$. From now on we work exclusively over the field of complex numbers.

Let $G$ be a finite commutative group. The regular representation $R_\mathbb{C}$ of $G$ decomposes into a direct sum of 1-dimensional irreducible subrepresentations generated by the characters (see [6] Corollary 2.18):
\[
R_\mathbb{C} = \oplus_{\chi \in \hat{G}} \mathbb{C} \cdot \chi.
\]
Recall that $V_\mathbb{C}$ is a subrepresentation of the regular representation for $G = (\mathbb{Z}/q\mathbb{Z})^\times$. Hence
\[
V_\mathbb{C} = \oplus \mathbb{C} \cdot \chi,
\]
where we sum over all characters $\chi \in \hat{G} \cap V_\mathbb{C}$.

A character $\chi$ lies in $V_\mathbb{C}$ if and only if $\chi(-1) = -1$. These characters are mutually orthogonal under the inner product
\[
\langle g_1, g_2 \rangle := \frac{1}{\varphi(q)} \sum_{1 \leq a \leq q-1, (a,p) = 1} g_1(a) \overline{g_2(a)}, \quad \forall g_1, g_2 \in V_\mathbb{C}.
\]
Clearly, $\langle \chi, \chi \rangle = 1$. It follows that the set
\[
B := \hat{G} \cap V_\mathbb{C} = \{ \chi \in \hat{G} \mid \chi(-1) = -1 \}
\]
forms an orthonormal basis of $V_\mathbb{C}$. In particular, there are exactly $\varphi(q)/2$ characters that are in $B$ (which could also be seen from the fact that $\sum_{\chi \in \hat{G}} \chi(-1) = 0$ ). We label the characters in $B$ as $\chi_j$ for $1 \leq j \leq \varphi(q)/2$.

Every function $f \in V_\mathbb{C}$ may be uniquely written as a linear combination of characters $\sum c_j \chi_j$, where
\[
c_j = \langle f, \chi_j \rangle = \frac{1}{\varphi(q)} \sum_{1 \leq a \leq q-1, (a,p) = 1} f(a) \overline{\chi_j(a)}.
\]
If $c_i = 0$ for some $i$, then the $(\mathbb{Z}/q\mathbb{Z})^\times$-submodule generated by $f$ is contained in the proper submodule $\oplus_{j \neq i} \mathbb{C} \cdot \chi_j$. Thus if $f$ generates the $(\mathbb{Z}/q\mathbb{Z})^\times$-module $V_\mathbb{C}$, it is necessary that $c_j \neq 0$ for all $j$. We show that this is also a sufficient condition.
It suffices to show that each \( \chi_i \) lies in the \((\Z/q\Z)^\times\)-submodule generated by \( f \). For each \( j \neq i \), we choose an element \( a_{ij} \in (\Z/q\Z)^\times \) such that \( \chi_i(a_{ij}) \neq \chi_j(a_{ij}) \). Let \( T_i \) be the element in the group \( \C \)-algebra \( \C[(\Z/q\Z)^\times] \) defined by

\[
T_i = \frac{1}{c_i} \prod_{j=1, j \neq i}^{\varphi(q)/2} \frac{a_{ij} - \chi_j(a_{ij})}{\chi_i(a_{ij}) - \chi_j(a_{ij})}.
\]

Clearly \( T_i \chi_j = 0 \) for all \( j \neq i \), and \( T_i \chi_i = \chi_i/c_i \). Then \( T_i f = \chi_i \). Since \( i \) is arbitrary, we conclude that all \( \chi_i \in B \) lie in the \((\Z/q\Z)^\times\)-submodule generated by the function \( f \). Therefore, the theorem follows from the following lemma. \( \square \)

**Lemma 5.3.** Let \( p \) be a prime, \( q = p^r \), \( n \geq 2 \) and \( p \nmid n \). Let \( \chi : (\Z/q\Z)^\times \to \mathbf{S}^1 \subset \C \) be a character of \((\Z/q\Z)^\times \) such that \( \chi(-1) = -1 \). Then the sum \( \sum h_r(a)\chi(a) \neq 0 \), where we sum over all integers \( a \) such that \( 1 \leq a \leq q-1 \) and \( p \nmid a \).

We will prove Lemma 5.3 in Section 6.

6. **Explicit formulas**

**Proof of Lemma 5.3.** Using Fourier expansion, one sees that for \( x \not\in \Z \),

\[
x - \lfloor x \rfloor - \frac{1}{2} = \sum_{m \in \Z, m \neq 0} \frac{e(mx)}{2\pi im},
\]

where \( e(x) := e^{2\pi ix} \). If we let \( s(h, \chi) \) to be the sum \( \sum h(a)\chi(a) \), and regard \( \chi \) as Dirichlet character (i.e., we put \( \chi(a) = 0 \) if \( p \mid a \) ), we then get

\[
s(h_r, \chi) = \sum_{a=0}^{q-1} \left( \frac{n-1}{2} - \left\lfloor \frac{na}{q} \right\rfloor \right) \chi(a)
\]

\[
= \sum_{a=0}^{q-1} \left( \frac{n-1}{2} - \frac{na}{q} + \frac{1}{2} - \sum_{m \in \Z, m \neq 0} \frac{e(mna/q)}{2\pi im} \right) \chi(a)
\]

\[
= \frac{n}{2} \sum_{a=0}^{q-1} \chi(a) - \frac{n}{q} \sum_{a=0}^{q-1} a \chi(a) - \sum_{m \in \Z, m \neq 0} \frac{1}{2\pi im} \sum_{a=0}^{q-1} e(mna/q)\chi(a)
\]

\[
= -\frac{n}{q} \sum_{a=0}^{q-1} a \chi(a) - \sum_{m \in \Z, m \neq 0} \frac{\chi(n)}{2\pi im} \sum_{a=0}^{q-1} e(mna/q)\chi(a)
\]

where we used the fact that \( \sum_{a=0}^{q-1} \chi(a) = 0 \). Since \( n \) and \( q \) are coprime, \( na \mod q \) runs through the list of all residue classes modulo \( q \) when \( a \) does so. Then

\[
s(h_r, \chi) = -\frac{n}{q} \sum_{a=0}^{q-1} a \chi(a) - \sum_{m \in \Z, m \neq 0} \frac{\chi(n)}{2\pi im} \sum_{a=0}^{q-1} e(mna/q)\chi(a)
\]

(2)
Let $\tau_q(\chi)$ denote the Gauss sum $\sum e(a/q)\chi(a)$, also set $S_q(\chi) = \sum a \chi(a)$ and $c_\chi(m) = \sum e(ma/q)\chi(a)$, where all sums are taken from $a = 0$ to $q-1$. We then have

$$s(h_r, \chi) = -\frac{n}{q} S_q(\bar{\chi}) - \sum_{m \in \mathbb{Z}, m \neq 0} \frac{\chi(n)c_\chi(m)}{2\pi im}$$

Clearly, (3) works for all $n$ such that $p \nmid n$. In particular, if we set $n = 1$, then $h_r = 0$, hence $s(h_r, \chi) = 0$. That is

$$-\frac{1}{q} S_q(\bar{\chi}) - \sum_{m \in \mathbb{Z}, m \neq 0} \frac{c_\chi(m)}{2\pi im} = 0$$

Combining with (3) we get

$$s(h_r, \chi) = -\frac{n}{q} S_q(\bar{\chi}) + \frac{\chi(n)}{q} S_q(\bar{\chi}) = \frac{1}{q}(\chi(n) - n)S_q(\bar{\chi})$$

When $n \geq 2$, $\chi(n) - n \neq 0$ since $|\chi(n)| = 1$. Thus the lemma follows if we show that $S_q(\bar{\chi}) \neq 0$ for any character $\chi$ with $\chi(-1) = -1$. We prove this by cases.

Case 1. Assume that $\chi$ is a primitive Dirichlet character modulo $q = p^r$. By [10, Theorem 9.7],

$$c_\chi(m) = \chi(m)\tau_q(\bar{\chi}).$$

It follows that

$$S_q(\bar{\chi}) = -q \sum_{m \in \mathbb{Z}, m \neq 0} \frac{c_\chi(m)}{2\pi im} = -\frac{q\tau_q(\bar{\chi})}{2\pi i} \sum_{m \in \mathbb{Z}, m \neq 0} \frac{\chi(m)}{m}$$

$$= -\frac{q\tau_q(\bar{\chi})}{2\pi i} \sum_{m=1}^{\infty} \frac{\chi(m)}{m} = -\frac{q\tau_q(\bar{\chi})}{2\pi i} L(1, \chi)$$

where we used the fact that $\chi(-1) = -1$.

It is well known [10, Theorem 9.7] that the absolute value of the Gauss sum $|\tau_q(\chi)| = \sqrt{q}$ for all primitive characters $\chi$. In particular $\tau_q(\bar{\chi}) \neq 0$. We get

$$L(1, \chi) = -\frac{\pi i S_q(\bar{\chi})}{q\tau_q(\bar{\chi})} = \frac{\pi i}{q^2} \tau_q(\chi)S_q(\bar{\chi})$$

since $\tau_q(\bar{\chi}) = \chi(-1)\tau_q(\bar{\chi}) = -\tau_q(\bar{\chi})$.

It is a classical result that $L(1, \chi) \neq 0$ for all nontrivial Dirichlet characters modulo $q$ (see [2, Theorem 2, Chapter 16]). Hence $S_q(\bar{\chi}) \neq 0$.

The above closed form (6) of $L(1, \chi)$ for $\chi(-1) = -1$ is actually also classical. See [10, Theorem 9.9].

Case 2. Assume that $\chi$ is induced by a primitive character $\chi^*$ modulo $d$ where $d = p^r d'$ with $0 < r_d < r$. Since both $d$ and $q$ are powers of $p$, $\chi(a) = \chi^*(a)$ for all
If we write $a = xd + y$ with $0 \leq x \leq q/d - 1$ and $0 \leq y \leq d - 1$, we then have
\[
S_q(\bar{\chi}) = \sum_{a=0}^{q-1} a \bar{\chi}(a) = \sum_{x=0}^{q/d-1} \sum_{y=0}^{d-1} (xd + y) \bar{\chi}^*(xd + y)
\]
\[
= \sum_{x=0}^{q/d-1} \sum_{y=0}^{d-1} xd \bar{\chi}^*(y) + \sum_{x=0}^{q/d-1} \sum_{y=0}^{d-1} y \bar{\chi}^*(y)
\]
\[
= \frac{q}{d} \sum_{y=0}^{d-1} y \bar{\chi}^*(y) = \frac{q}{d} S_d(\bar{\chi}^*)
\]
By Case 1, $S_d(\bar{\chi}^*) \neq 0$. Thus $S_q(\bar{\chi}) \neq 0$.

**Corollary 6.1.** Let $p$ be a prime, $q$ power of $p$, $n$ an integer coprime to $p$. Suppose that $\chi$ is a Dirichlet character mod $q$ that is induced by a character $\chi^*$ mod $d$ for some $d \mid q$. Then $\sum_{a=0}^{q-1} [na/q] \bar{\chi}(a) = \sum_{b=0}^{d-1} \left[\frac{nb}{d}\right] \bar{\chi}^*(b)$. In particular, if $c_{\chi}(r)$ is the coefficient of $h_r$ with respect to $\chi$ and $c_{\chi}^{(d)}$ is the coefficient of $h_d$ with respect to $\chi^*$ then
\[
\varphi(q)c_{\chi}(r) = \varphi(d)c_{\chi}^{(d)}.
\]
**Proof.** First assume that $\chi^*$ is primitive mod $d$. By previous calculations, if we substitute (7) into (4), then
\[
s(h_r, \chi) = \frac{1}{q} (\chi(n) - n) \frac{q}{d} S_d(\bar{\chi}^*) = \frac{1}{d} (\chi(n) - n) S_d(\bar{\chi}^*).
\]
Applying (4) again,
\[
\frac{1}{d} (\chi(n) - n) S_d(\bar{\chi}^*) = \sum_{b=0}^{d-1} \left( \frac{n-1}{2} - \left[\frac{nb}{d}\right] \right) \bar{\chi}^*(b)
\]
It follows that
\[
\sum_{a=0}^{q-1} \left( \frac{n-1}{2} - \left[\frac{na}{q}\right]\right) \bar{\chi}(a) = s(h_r, \chi) = \sum_{b=0}^{d-1} \left( \frac{n-1}{2} - \left[\frac{nb}{d}\right] \right) \bar{\chi}^*(b).
\]
Equivalently,
\[
\sum_{a=0}^{q-1} \left[\frac{na}{q}\right] \bar{\chi}(a) = \sum_{b=0}^{d-1} \left[\frac{nb}{d}\right] \bar{\chi}^*(b).
\]
If $\chi^*$ is not primitive, then we reduce further to obtain a primitive character $\chi_0^*$ mod $d'$ for some $d' \mid d$ such that $\chi_0^*$ induces both $\chi^*$ and $\chi$. Both sides of the desired equality now equal to $\sum_{c=0}^{d'-1} [nc/d'] \bar{\chi}_0^*(c)$.

The following assertion was used in the proof of Theorem 1.6

**Corollary 6.2.** Let $j \leq r$ be a positive integer. Let us identify (in the usual way) $G = (\mathbb{Z}/p\mathbb{Z})^r$ with the Galois group $\text{Gal}(\mathbb{Q}(\zeta_{p^r})/\mathbb{Q})$ and let us consider its subgroup
\[
G_j = \text{Gal}(\mathbb{Q}(\zeta_{p^r})/\mathbb{Q}(\zeta_{p^j})) \subset \text{Gal}(\mathbb{Q}(\zeta_{p^r})/\mathbb{Q}) = G.
\]
Then for each $a \in (\mathbb{Z}/p^r\mathbb{Z})^*$,
\[
h_j(a \mod p^j) = \sum_{b \in G_j} h_r(ab).
\]
Proof: If $\chi$ is a character that does not kill $G_j$ (i.e., is not induced from $(\mathbb{Z}/p^i\mathbb{Z})^*$ then $\sum_{b \in G_j} \chi(b) = 0$. If $\chi$ is a character that kills $G_j$ (i.e., is induced from $(\mathbb{Z}/p^i\mathbb{Z})^*$) then $\sum_{b \in G_j} \chi(b) = \#(G_j) = \varphi(p^i)/\varphi(p^i)$. Now the result follows from the last assertion of Corollary 6.1.

Examples 6.3. If $q = p \equiv 3 \pmod{4}$, one sees that the Legendre symbol $\chi(a) = (a/p)$ satisfies $(-1/p) = -1$. It is also known ([7] Theorem 1, page 75), [2] formula (19), page 51) that
\begin{align}
\tau_p(\chi) &= \tau_p(\chi) = \sqrt{-p}, \\
S(\chi) &= S(\chi) = -p h_p,
\end{align}
where $h_p$ denotes the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-p})$. Combining (6) and (9), we get
\begin{equation}
L(1, \chi) = \pi h_p/\sqrt{p}.
\end{equation}
Alternatively, one could also deduce (10) by combining the formulas [2] (17), page 50], [2] (3), page 45], and the formula at the bottom of page 52 of [2].

Therefore, in the case $q = p \equiv 3 \pmod{4}$, one has a closed formula
\begin{equation}
\sum_{1 \leq a \leq p-1} h_1(a) \left(\frac{a}{p}\right) = \left(n - \left(\frac{n}{p}\right)\right) h_p.
\end{equation}

\Box

7. Semilinear algebra

Let $Q$ be a field of characteristic zero and $C$ an algebraically closed field that contains $Q$. We write Aut$(C/Q)$ for the group of all automorphisms of $C$ that act identically on $Q$. It is well known that $Q$ coincides with the subfield of Aut$(C/Q)$-invariants in $C$. Let $V$ be a $Q$-vector space of finite positive dimension $n$ and $V_C = V \otimes_Q C$ the corresponding $n$-dimensional $C$-vector space. Further we will identify $V$ with the $Q$-vector subspace $V \otimes 1$ of $V_C$. The group Aut$(C/Q)$ acts on $V_C$ by $C$-semilinear automorphisms:
\[ \sigma(v \otimes c) = v \otimes \sigma(c) \quad \forall \sigma \in \text{Aut}(C/Q), v \in V, c \in C. \]

Clearly, $V$ coincides with the $Q$-vector subspace of Aut$(C/Q)$-invariants in $V_C$.

The following assertion seems to be well-known but we were unable to find a reference. (However, see [2].)

Proposition 7.1. Let $\tilde{W}$ be an Aut$(C/Q)$-invariant $C$-vector subspace of $V_C$. Then there exists a $Q$-vector subspace $W$ of $V$ such that $\tilde{W}$ coincides with $W_C = W \otimes_Q C \subseteq V \otimes_Q C = V_C$. In addition, $W$ coincides with the $Q$-subspace of Aut$(C/Q)$-invariants in $\tilde{W}$.

Proof. Let us pick a basis $\{e_1, \ldots, e_n\}$ of $V$. Let us put $m = \dim_C(\tilde{W})$. Clearly, $m \leq n$ and we may assume that $m \geq 1$. If $m = 1$ then $\tilde{W}$ contains a vector $w' = \sum_{i=1}^n c_i e_i$ such that, at least, one of its coordinates say, $c_j$ is 1. Since $\tilde{W} = C \cdot w'$ is Aut$(C/Q)$-invariant, we conclude that all coordinates $c_i$’s are Aut$(C/Q)$-invariant and therefore lie in $Q$. This means that $w' \in V$ and one may put $W = Q \cdot w'$. On the other hand, if $n = 1$ then $m = 1$ and we are also done.

We use induction by $n$. Assume that $1 < m$ and consider the $C$-subspace $\tilde{W}_0$ that is the intersection of $\tilde{W}$ and the hyperplane $\sum_{i=1}^{n-1} C e_i$. Clearly, $\tilde{W}_0$ is the
By induction assumption, there exists a $Q$-vector subspace $W_0$ of $\{\sum_{i=1}^{n-1} Qe_i\} \otimes_{Q} C$ such that $\tilde{W}_0 = W_0 \otimes_{Q} C$. If $\tilde{W} = \tilde{W}_0$ then we are done. So, assume $\tilde{W} \neq \tilde{W}_0$. Then $\dim_{C}(\tilde{W}_0) = \dim_{C}(\tilde{W}) - 1 = m - 1 > 0$. Since $\dim_{Q}(W_0) = \dim_{C}(\tilde{W}_0)$, we conclude that $\dim_{Q}(W_0) = m - 1 < n - 1$. Let us choose a $(n-m)$-dimensional $Q$-vector subspace $W_1$ of $\{\sum_{i=1}^{n-1} Qe_i\}$ such that $\{\sum_{i=1}^{n-1} Qe_i\} = W_0 \oplus W_1$. We have $V = (W_1 \oplus Qe_n) \oplus W_0$, $W \otimes_{Q} C \subset \tilde{W}$. Notice that $\dim_{Q}(W_1 \oplus Qe_n) = n - (m - 1) < n$. Let us consider the $C$-vector subspace $\tilde{W}_2 = \tilde{W} \cap \{(W_1 \oplus Qe_n) \otimes_{Q} C\}$. Clearly, $\tilde{W} = \tilde{W}_0 \oplus \tilde{W}_2$. By induction assumption applied to the $\Aut(C/Q)$-invariant $C$-subspace $\tilde{W}_2$ of $(W_1 \oplus Qe_n) \otimes_{Q} C$, we conclude that there exists a $Q$-vector subspace $W_2 \subset W_1 \oplus Qe_n$ such that $\tilde{W}_2 = W_2 \otimes_{Q} C$. Now we may put $W = W_0 \oplus W_2$.

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