A Bochner formula on path space for the Ricci flow

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Received: 6 April 2020 / Accepted: 2 November 2022 / Published online: 12 January 2023
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Abstract

We generalize the classical Bochner formula for the heat flow on evolving manifolds $(M, g_t)_{t \in [0, T]}$ to an infinite-dimensional Bochner formula for martingales on parabolic path space $PM$ of space-time $M = M \times [0, T]$. Our new Bochner formula and the inequalities that follow from it are strong enough to characterize solutions of the Ricci flow. Specifically, we obtain characterizations of the Ricci flow in terms of Bochner inequalities on parabolic path space. We also obtain gradient and Hessian estimates for martingales on parabolic path space, as well as condensed proofs of the prior characterizations of the Ricci flow from Haslhofer–Naber (J Eur Math Soc 20(5):1269–1302, 2018a). Our results are parabolic counterparts of the recent results in the elliptic setting from Haslhofer–Naber (Commun Pure Appl Math 71(6):1074–1108, 2018b).

Mathematics Subject Classification

53C44 · 53E20

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Communicated by P. Topping.

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1 Introduction

The goal of this paper is to prove a Bochner formula on path space for the Ricci flow, and to discuss some applications. This generalizes the Bochner formula on path space for Einstein metrics from Haslhofer and Naber [8].

Throughout this paper, we shall use the convention that an evolving family of manifolds is a smooth and complete family of Riemannian manifolds \((M^n, g_t)_{t \in I}\) such that

\[
\sup_{M \times I} (|\text{Rm}| + |\partial_t g| + |\nabla \partial_t g|) < \infty. \tag{1.1}
\]

1.1 Background on characterizations of Einstein metrics

To begin, let us recall some well-known characterizations of when a Riemannian manifold \((M, g)\) is a supersolution to the Einstein equations. Let \(H_t f\) denote the heat flow of a function \(f : M \to \mathbb{R}\). Then its gradient satisfies the Bochner formula

\[
(\partial_t - \Delta) |\nabla H_t f|^2 = -2|\nabla^2 H_t f|^2 + 2Rc(\nabla H_t f, \nabla H_t f). \tag{1.2}
\]

Using this, an equivalence between supersolutions of the Einstein equations, the classical Bochner inequality and the gradient estimate readily follows, i.e.

\[
Rc \geq 0 \iff (\partial_t - \Delta) |\nabla H_t f|^2 \leq -2|\nabla^2 H_t f|^2 \tag{1.3}
\]

\[
\iff |\nabla H_t f| \leq H_t |\nabla f|, \tag{1.4}
\]

for all test functions \(f : M \to \mathbb{R}\).

Until recently, however, there was no analogous characterization of solutions to the Einstein equations. Such a characterization was discovered by Naber [9] by employing the analytic properties of path space \(PM = C([0, \infty), M)\). This path space is naturally endowed with a family of Wiener measures \(\{P_x\}\) of Brownian motion starting at \(x \in M\). One then introduces a notion of stochastic parallel transport and the corresponding family of parallel gradients \(\nabla_s^\parallel\). Using this foundation, Naber [9] developed an infinite-dimensional generalization of the gradient estimate (1.4) to characterize solutions of the Einstein equations. Namely, he proved that

\[
Rc = 0 \iff |\nabla x \int_{PM} F dP_x| \leq \int_{PM} |\nabla_0^\parallel F| dP_x, \tag{1.5}
\]

for all test functions \(F : PM \to \mathbb{R}\).

Interesting variants of these characterizations and estimates have been obtained in [3–5, 11, 12].

Later, Haslhofer and Naber [8] proved an infinite-dimensional generalization of (1.3). Namely, they showed

\[
Rc = 0 \iff d|\nabla_s F_t|^2 \geq \langle \nabla_t |\nabla_s F_t|^2, dW_t \rangle \tag{1.6}
\]

for all martingales \(F_t : PM \to \mathbb{R}\).

Using the infinite-dimensional Bochner formula (1.6), they gave a simpler proof of the infinite-dimensional gradient estimate (1.5) in a similar vein to how the classical Bochner formula (1.3) readily implies the classical gradient estimate (1.4).
1.2 Background on characterizations of Ricci flow

To motivate the characterization of solutions of the Ricci flow, let us first recall characterizations of supersolutions, namely evolving Riemannian manifolds \((M, g_t)_{t \in I}\) such that

\[
\partial_t g_t \geq -2\text{Rc}_{g_t}.
\]  

(1.7)

To begin, consider the heat flow \(H_{st} f\) on this evolving background, namely the solution of the heat equation \(\partial_t u = \Delta_{g_t} u\) with initial condition \(f\) at time \(t = s\). Then its gradient satisfies the Bochner formula

\[
(\partial_t - \Delta_{g_t})|\nabla H_{st} f|^2 = -2|\nabla^2 H_{st} f|^2 + (\partial_t g_t + 2\text{Rc}_{g_t})(\nabla H_{st} f, \nabla H_{st} f).
\]  

(1.8)

Using this, an equivalence between supersolutions of the Ricci flow, the Bochner inequality and the gradient estimate readily follows, i.e.

\[
\partial_t g_t \geq -2\text{Rc}_{g_t} \iff (\partial_t - \Delta_{g_t})|\nabla H_{st} f|^2 \leq -2|\nabla^2 H_{st} f|^2
\]  

(1.9)

\[
|\nabla H_{st} f| \leq H_{st}|\nabla f|,
\]  

(1.10)

for all test functions \(f : M \to \mathbb{R}\).

To generalize the inequality (1.10) to an infinite dimensional estimate, Haslhofer and Naber [7] considered space-time \(\mathcal{M} = M \times I\) equipped with the space-time connection defined on vector fields by

\[
\nabla_X Y = \nabla^g_{X} Y, \quad \nabla_t Y = \partial_t Y + \frac{1}{2} \partial_t g_t (Y, \cdot)^{g_t}.
\]  

(1.11)

The main difference, compared to the infinite dimensional estimate that characterizes Einstein metrics, is that the parabolic path space \(P_T \mathcal{M}\) only consists of continuous space-time curves \(\{y_T = (T - \tau, x_\tau)\}\) that move backwards along the time-axis with unit speed and start at fixed time \(T \in I\). This path space is naturally endowed with a family of parabolic Wiener measures \(\{\mathbb{P}_{(x, T)}\}\) of Brownian motion starting at \((x, T) \in \mathcal{M}\) and parabolic stochastic parallel gradients \(\{\nabla|\nabla\}_{\sigma \geq 0}\) defined via (1.11). Using this framework, Haslhofer and Naber proved an infinite-dimensional generalization of the gradient estimate (1.10) that characterizes solutions of the Ricci flow. Namely, they proved that

\[
\partial_t g_t = -2\text{Rc}_{g_t} \iff \nabla x \int_{P_T \mathcal{M}} F \ d\mathbb{P}_{(x, T)} \leq \int_{P_T \mathcal{M}} |\nabla|\nabla F| \ d\mathbb{P}_{(x, T)}
\]  

(1.12)

for all test functions \(F : P_T \mathcal{M} \to \mathbb{R}\).

Some nice variants of these characterizations have been obtained by Cheng and Thalmaier [4]. Moreover, Cabezas-Rivas and Haslhofer [2] found an interesting link between estimates in the elliptic and parabolic settings.

However, there is no analogous treatment of the Bochner inequality (1.6) in the time-dependent setting. The primary goal of this paper shall be to prove such an equivalent notion.

1.3 Bochner formula on parabolic path space

Let \((M, g_t)_{t \in I}\) be a family of evolving manifolds and let \(\mathcal{M} = M \times I\) be its space-time equipped with the space-time connection defined on vector fields via (1.11). Next, as in Section 1.2, we consider the parabolic path space \(P_T \mathcal{M}\), given by

\[
P_T \mathcal{M} := \{(x_\tau, T - \tau)_{\tau \in [0,T]} | x \in C([0, T], M)\},
\]  

(1.13)
and endow this space with the parabolic Wiener measure of Brownian motion on space-time, \( \mathbb{P}_{(x,T)} \), based at \((x,T) \in \mathcal{M}\) as well as the associated parabolic parallel gradients \(\nabla_\sigma \| \cdot \| \) defined via stochastic parallel transport on space-time \(\mathcal{M}\). To explain these notions in more detail, first recall that the solution to the heat equation \( \partial_t u = \Delta_g u \) with initial condition \( f \) at time \( t = s \) is given by convolving with the heat kernel i.e.

\[
H_{st} f(x) = \int_{\mathcal{M}} H(x, t| y, s) f(y) \, dV_g(y). \tag{1.14}
\]

The Wiener measure \( \mathbb{P}_{(x,T)} \) is then uniquely characterized in terms of the heat kernel by

\[
\mathbb{P}_{(x,T)} \left[ X_{\tau_1} \in U_1, \ldots, X_{\tau_k} \in U_k \right] = \int_{U_1} \cdots \int_{U_k} H(x, T|x_1, T-\tau_1) \cdots H(x_{k-1}, T-\tau_{k-1}| x_k, T-\tau_k) \, d\text{Vol}_{g_{T-\tau_1}}(x_1) \cdots d\text{Vol}_{g_{T-\tau_k}}(x_k) \tag{1.15}
\]

where \( X_\tau \) is a Brownian motion on \( M \) starting at \( x \). Moreover, the stochastic parallel gradient \( \nabla_\sigma F(\gamma) \in (T_x \mathcal{M}, g_T) \) of a function \( F : P_T \mathcal{M} \to \mathbb{R} \), is expressed in terms of the Fréchet derivative by

\[
D_{V_\sigma} F(\gamma) = \langle \nabla_\sigma \| F(\gamma), v \rangle_{(T_x \mathcal{M}, g_T)}, \tag{1.16}
\]

where \( V_\sigma \) is the vector field along \( \gamma \) defined by \( V_\sigma^T = P_T^{-1} v \|_{\| \cdot \|_{(T_x \mathcal{M}, g_T)}} \) and \( \{ P_t \} \), a family of isometries, referred to as stochastic parallel transport.

With the aim of generalizing (1.9) to an infinite-dimensional estimate, we consider martingales on parabolic path space, i.e. \( \Sigma_\tau \)-adapted integrable processes \( F_\tau : P_{(s,T)} \mathcal{M} \to \mathbb{R} \) that satisfy

\[
F_{\tau_1} = \mathbb{E}[F_{\tau_2} | \Sigma_{\tau_1}], \tag{1.17}
\]

where \( \mathbb{E}[\cdot | \Sigma_\tau] \) denotes the conditional expectation with respect to the \( \sigma \)-algebra \( \Sigma_\tau \) of events observable at time \( \tau \).

For example, if \( F(\gamma) = f(\pi_1 \gamma_{\tau_1}) \), where \( f : \mathcal{M} \to \mathbb{R} \) and \( \pi_1 : \mathcal{M} \times I \to I \), then the induced martingale \( F_\tau = \mathbb{E}[F | \Sigma_\tau] \) for \( \tau < \tau_1 \) is given by

\[
F_\tau(\gamma) = H_{T-\tau_1,T-\tau} f(\pi_1 \gamma_{\tau_1}) \tag{see example 2.18}. \tag{1.18}
\]

Specifically, martingales generalize heat flow. This analogy between martingales and heat kernels will motivate our development of the following generalized Bochner formula on \( P\mathcal{M} \).

**Theorem 1.1** (Generalized Bochner Formula on \( P\mathcal{M} \)) Let \( F_\tau : P_{(s,T)} \mathcal{M} \to \mathbb{R} \) be a martingale on the parabolic path space of space-time. If \( \sigma \geq 0 \) is fixed, then

\[
d(\| \nabla_\sigma F_\tau \|^2) = \langle \| \nabla_\sigma F_\tau \|^2, dW_\tau \rangle + (\dot{g} + 2Rc_\tau)(\| F_\tau \|, \| F_\tau \|) \, d\tau + 2\nabla_\sigma \| F_\tau \|^2 \, d\tau + 2\nabla_\sigma \| F_\tau \|^2 \, d\delta_\sigma(\tau), \tag{1.19}
\]

where \( (\dot{g} + 2Rc_\tau)(v, w) = (\dot{g}_t + 2Rc_{g_t})|_{t=T-\tau} \in (P_T^{-1}v, P_T^{-1}w) \) and \( \dot{g} = \frac{d}{dt} g \).

This generalized Bochner formula proves to be a fundamental tool in characterizing the Ricci flow. Note that, if \( (M, g_t)_{t \in \mathcal{I}} \) evolves by Ricci flow, this formula reduces to

\[
d(\| \nabla_\sigma F_\tau \|^2) = \langle \| \nabla_\sigma F_\tau \|^2, dW_\tau \rangle + 2\nabla_\sigma \| F_\tau \|^2 \, d\tau + 2\nabla_\sigma \| F_\tau \|^2 \, d\delta_\sigma(\tau). \tag{1.20}
\]
and this trivially implies the following infinite-dimensional generalization of Bochner inequality (1.9) in the time-dependent setting
\[
d(|\nabla_{\sigma} F_{\tau}|^2) \geq \left( \nabla_{\tau} |\nabla_{\sigma} F_{\tau}|^2, dW_{\tau} \right) + 2|\nabla_{\tau} |\nabla_{\sigma} F_{\tau}|^2| \, d\tau + 2|\nabla_{\sigma} F_{\tau}|^2 \, d\delta_{\sigma}(\tau). \tag{1.21}
\]
In contrast to the heat flow Bochner inequality, this generalized martingale Bochner inequality (1.21) as well as the estimates that follow from it are strong enough to help exhibit solutions and not just supersolutions of the Ricci flow.

Specifically, Theorem 1.1 has four main applications:

- a characterization of the Ricci flow via Bochner inequalities for martingales on parabolic path space;
- gradient estimates for martingales on parabolic path space;
- Hessian estimates for martingales on parabolic path space;
- a new and much simpler proof of the characterization of solutions of the Ricci flow by Haslhofer and Naber in 2018 [7, Theorem 1.22], which will be discussed in Sect. 1.4.

To explain the meaning of Theorem 1.1 in the simplest example, this generalized Bochner formula on \( P_M \) directly reduces to the standard Bochner formula in the case of 1-point functions, i.e. when \( F_{\tau}(\gamma) \) satisfies Eq. (1.18). That is, the evolution of \( |\nabla H_{T-\tau_1, T-\tau} f|^2 \) for \( \tau \leq \tau_1 \) is calculated as
\[
\left( -\partial_\tau - \Delta_{g_{T-\tau}} \right) |\nabla H_{T-\tau_1, T-\tau} f|^2 \leq -2|\nabla^2 H_{T-\tau_1, T-\tau} f|^2 \tag{1.22}
\]
in Corollary 3.5. Setting \( s = T - \tau_1 \) and \( t = T - \tau \), this explicitly recovers (1.9) from Sect. 1.2.

1.4 Applications

We will conclude with some main applications of our Bochner inequality (1.21). First, we shall develop a new characterization of the Ricci flow.

**Theorem 1.2** (New characterizations of the Ricci flow) For an evolving family of manifolds \( (M^n, g_t)_{t \in I} \), the following are equivalent to solving the Ricci flow \( \partial_t g_t = -2\text{Rc}_{g_t} \):

(C1) Martingales on parabolic path space satisfy the full Bochner inequality
\[
d(|\nabla_{\sigma} F_{\tau}|^2) \geq \left( \nabla_{\tau} |\nabla_{\sigma} F_{\tau}|^2, dW_{\tau} \right) + 2|\nabla_{\tau} |\nabla_{\sigma} F_{\tau}|^2| \, d\tau + 2|\nabla_{\sigma} F_{\tau}|^2 \, d\delta_{\sigma}(\tau) \tag{1.23}
\]

(C2) Martingales on parabolic path space satisfy the dimensional Bochner inequality
\[
d(|\nabla_{\sigma} F_{\tau}|^2) \geq \left( \nabla_{\tau} |\nabla_{\sigma} F_{\tau}|^2, dW_{\tau} \right) + \frac{2}{n} |\nabla_{\sigma, t} F_{\tau}|^2 \, d\tau + 2|\nabla_{\sigma} F_{\tau}|^2 \, d\delta_{\sigma}(\tau) \tag{1.24}
\]

(C3) Martingales on parabolic path space satisfy the weak Bochner inequality
\[
d(|\nabla_{\sigma} F_{\tau}|^2) \geq \left( \nabla_{\tau} |\nabla_{\sigma} F_{\tau}|^2, dW_{\tau} \right) + 2|\nabla_{\sigma} F_{\tau}|^2 \, d\delta_{\sigma}(\tau) \tag{1.25}
\]

(C4) Martingales on parabolic path space satisfy the linear Bochner inequality
\[
d(|\nabla_{\sigma} F_{\tau}|) \geq \left( \nabla_{\tau} |\nabla_{\sigma} F_{\tau}|, dW_{\tau} \right) + |\nabla_{\sigma} F_{\tau}| \, d\delta_{\sigma}(\tau) \tag{1.26}
\]

(C5) If \( F_{\tau} \) is a martingale, then \( \tau \to |\nabla_{\sigma} F_{\tau}| \) is a submartingale for every \( \sigma \geq 0 \).

Second, we shall obtain gradient estimates for martingales on parabolic path space.
We have the Poincaré Hessian estimate

\[ |\nabla_\sigma^\| F_{\tau_1}^\| \leq E_{(x, T)} \left[ |\nabla_\tau^\| F_{\tau_2}^\| \Sigma_{\tau_1} \right]. \quad (1.27) \]

(G2) For any \( F \in L^2(P.M) \), \( \sigma \) fixed and \( \tau_1 \leq \tau_2 \), the induced martingale satisfies the gradient estimate

\[ |\nabla_\sigma^\| F_{\tau_1}^\|^2 \leq E_{(x, T)} \left[ |\nabla_\tau^\| F_{\tau_2}^\|^2 \Sigma_{\tau_1} \right] . \quad (1.28) \]

Note that in the case of \( \sigma = \tau_1 = 0 \), (G1) reduces to the infinite-dimensional gradient estimate (1.12).

Next, we shall obtain Hessian estimates for martingales on parabolic path space.

**Theorem 1.4** (Hessian estimates for martingales on parabolic path space) For an evolving family of manifolds \( (M^n, g_t)_{t \in I} \) that solve the Ricci flow \( \partial_t g_t = -2Rc_{g_t} \) and a function \( F \in L^2(P.M) \), it holds that:

(H1) For each \( \sigma \geq 0 \), we have the estimate

\[ E_{(x, T)} \left[ |\nabla_\sigma^\| F_{\sigma}^\|^2 \right] + 2E_{(x, T)} \int_0^T \left[ |\nabla_\tau^\| \nabla_\sigma^\| F_{\tau}^\|^2 \right] d\tau \leq E_{(x, T)} \left[ |\nabla_\sigma^\| F_{\tau}^\|^2 \right]. \quad (1.29) \]

(H2) We have the Poincaré Hessian estimate

\[ E_{(x, T)} \left[ (F - E_{(x, T)}[F])^2 \right] + 2 \int_0^T \int_0^T E_{(x, T)} \left[ |\nabla_\tau^\| \nabla_\sigma^\| F_{\tau}^\|^2 \right] d\sigma d\tau \leq \int_0^T E_{(x, T)} \left[ |\nabla_\sigma^\| F_{\tau}^\|^2 \right] d\sigma . \quad (1.30) \]

(H3) We have the log-Sobolev Hessian estimate

\[ E_{(x, T)} \left[ F^2 \ln(F^2) \right] - E_{(x, T)}[F^2] \ln \left( E_{(x, T)}[F^2] \right) + 2 \int_0^T \int_0^T E_{(x, T)} \left[ (F^2)_{\tau} |\nabla_\tau^\| \nabla_\sigma^\| (F^2)_{\tau}^2 \right] d\sigma d\tau \leq 4 \int_0^T E_{(x, T)} \left[ |\nabla_\sigma^\| F_{\tau}^\|^2 \right] d\sigma . \quad (1.31) \]

Finally, our generalized Bochner formula on parabolic path space leads to a simpler proof of the characterization of solutions of the Ricci flow found by Haslhofer and Naber [7].

**Theorem 1.5** (Characterization of solutions of the Ricci flow) [7, Theorem 1.22] For an evolving family of manifolds \( (M^n, g_t)_{t \in I} \), the following are equivalent:

(R1) \( (M^n, g_t)_{t \in I} \) solves the Ricci flow \( \partial_t g_t = -2Rc_{g_t} \).

(R2) For every \( F \in L^2(P.M) \), we have the gradient estimate

\[ |\nabla_{x} E_{(x, T)}[F]| \leq E_{(x, T)}[|\nabla_0^\| F|]. \quad (1.32) \]

(R3) For every \( F \in L^2(P.M) \), the induced martingale \( \{F_{\tau}\}_{\tau \in [0, T]} \) satisfies the quadratic variation estimate

\[ E_{(x, T)} \left[ d[F, F]_{\tau} \right] \leq 2E_{(x, T)} \left[ |\nabla_\tau^\| F_{\tau}^\|^2 \right] . \quad (1.33) \]
The Ornstein-Uhlenbeck operator \( \mathcal{L}_{(\tau_1, \tau_2)} \) on parabolic path space \( L^2(P, \mathcal{M}) \) satisfies the log-Sobolev inequality
\[
\mathbb{E}_{(x, T)} \left[ (F^2)_{\tau_2} \log((F^2)_{\tau_2}) - (F^2)_{\tau_1} \log((F^2)_{\tau_1}) \right] \leq 2 \mathbb{E}_{(x, T)} \left[ \langle F, \mathcal{L}_{(\tau_1, \tau_2)} F \rangle_{\tau_1} \right].
\]  
(1.34)

The article is organized as follows:

- In Sect. 2, we shall discuss the geometric and probabilistic preliminaries needed for the proofs of our main theorems.
- In Sect. 3, we shall prove Theorem 1.1, the Bochner formula for martingales on parabolic path space.
- In Sect. 4, we shall discuss the four aforementioned applications of our analysis on path space, i.e. Theorems 1.2, 1.3, 1.4 and 1.5.

\section*{2 Preliminaries}

\subsection*{2.1 Geometric preliminaries}

To begin this section, we shall recall the concepts relevant to the construction of the frame bundle on evolving manifolds. An expression of the canonical horizontal \( (H_a \text{ and } D_t) \) and vertical \( (V_{ab}) \) vector fields and their commutators will complete this preliminary section.

In time-independent geometry, given a complete Riemannian manifold \( M \), one considers the orthonormal frame bundle \( \pi : F \to M \), where the fibres are orthonormal maps \( F_\xi := \{u : \mathbb{R}^n \to T_x M \text{ orthonormal}\} \). To each curve \( x_t \in M \), one can associate a horizontal lift \( u_t \in F \). In particular, to each vector \( X \in T_x M \), given \( u \in \pi^{-1}(x) \), one can associate its horizontal lift \( X^u \in T_u F \).

We shall now explain, following [6, 7, 10], how these notions can be adapted to the time-dependent setting.

To make the appropriate adjustment, we begin by defining space-time \( \mathcal{M} \) and the equipped connection \( \nabla \) as follows:

\textbf{Definition 2.1 (Space-time)} Let \((M, g_t)_{t \in I}\) be an evolving family of Riemannian manifolds. The space-time is then defined as \( \mathcal{M} = M \times I \) equipped with the space-time connection defined on vector fields by \( \nabla_X Y = \nabla_X^{g_t} Y \) and \( \nabla_t Y = \delta_t Y + \frac{1}{2} \delta_t g_t(Y, \cdot) \).
Also observe that this choice of connection is compatible with the metric, namely
\[
\frac{d}{dt} (X, Y)_{g_t} = \langle \nabla_t X, Y \rangle_{g_t} + \langle X, \nabla_t Y \rangle_{g_t}.
\] (2.1)

Generalizing the earlier time-independent construction, we consider the \(O_n\)-bundle \(\pi : \mathcal{F} \rightarrow \mathcal{M}\), where the fibres are given by \(\mathcal{F}_{(x,t)} := \{u : \mathbb{R}^n \rightarrow (T_x M, g_t)\}\) orthonormal.

To each curve \(\gamma_t \in \mathcal{M}\), we can now associate a horizontal lift \(u_t \in \mathcal{F}\). Namely, given \(u_0 \in \pi^{-1}(\gamma_0)\), the curve \(u_t\) is the unique solution of \(\pi(u_t) = \gamma_t\) and \(\nabla_{\dot{\gamma}_t}(u_t e_a) = 0\) for \(a \in \{1, 2, \ldots, n\}\), where \(\nabla\) is the space-time connection from Definition 2.1. More explicitly, we provide the following formal definition:

**Definition 2.2 (Horizontal lift)** Given a vector \(\alpha X + \beta \partial_t \in T_{(x,t)} \mathcal{M}\) and a frame \(u \in \mathcal{F}_{(x,t)}\), there is a unique horizontal lift \(\alpha X^* + \beta D_t\) satisfying \(\pi_u(\alpha X^* + \beta D_t) = \alpha X + \beta \partial_t\). In particular, \(X^*\) is the horizontal lift of \(X \in T_x M\) with respect to the fixed metric \(g_t\).

Note that there are \(n + 1\) canonical horizontal vector fields on \(\mathcal{F}\), namely the time-like horizontal vector field \(D_t\) defined as the horizontal lift of \(\partial_t\) and the space-like horizontal vector fields \(\{H_a\}_{a=1}^n\) defined by \(H_a(u) = (u e_a)^*\). Also note the notion of vertical vector fields given by \(V_{ab}(u) = \frac{d}{dt} \big|_{t=0}(u \exp(\varepsilon A_{ab}))\) where \((A_{ab})_{cd} = (\delta_{ac}\delta_{bd} - \delta_{bc}\delta_{ad}) \in M_n(\mathbb{R})\).

We now want to express these horizontal and vertical vector fields in local coordinates as follows:

**Definition 2.3 (Local coordinates)** We view \(\mathcal{F}\) as a sub-bundle of the \(GL_n\)-bundle \(\pi : G \rightarrow \mathcal{M}\) where \(G_{(x,t)} := \{u : \mathbb{R}^n \rightarrow (T_x M, g_t)\}\) invertible, linear. Then, when given local coordinates \((x^1, \ldots, x^n, t)\) on \(\mathcal{M}\), we get local coordinates \((x^i, t, e^j_a)\) on \(G\) where \(e^j_a\) is defined by \(ue_a = e^j_a \frac{\partial}{\partial x^j}\).

Also note that on \(\mathcal{F}\) we have \(\delta_{ab} = g(ue_a, ue_b) = g_{ij} e^i_a e^j_b\) and thus we can express the inverse metric as
\[
\begin{align*}
g^{ij} &= e^i_a e^j_b. 
\end{align*}
\] (2.2)

It now remains in this section to both write out the canonical vector fields explicitly in local coordinates and derive some commutator relations between them.

**Lemma 2.4** (cf. [6]) In local coordinates, the canonical horizontal vector fields \(H_a\) and \(D_t\) and canonical vertical vector fields \(V_{ab}\) can be expressed as
\[
\begin{align*}
H_a &= e^j_a \frac{\partial}{\partial x^j} - e^j_b e^k_b \Gamma^j_{jk} \frac{\partial}{\partial x^j}, \\
V_{ab} &= e^j_b \frac{\partial}{\partial e^j_a}, \\
D_t &= \partial_t - \frac{1}{2} \tilde{\partial}_t g_{ab} e^j_b \frac{\partial}{\partial e^j_a},
\end{align*}
\] (2.3)

where \((\tilde{\partial}_t g)_{ab}(u) := (\partial_t g)_{\pi(u)}(ue_a, ue_b)\).

**Proof** The canonical horizontal vector fields, \(H_a\), are exactly the same as in [6].

Next, considering the curve \(u(\varepsilon) = u(\varepsilon A_{ab})\), recall that \(\varepsilon^j_a\) and \(A_{ab} e_c\) are defined via the relations \(ue_c = \varepsilon^j_a \frac{\partial}{\partial x^j}\) and \(A_{ab} e_c = \delta_{ca} e_b - \delta_{cb} e_a\). Then derive
\[
\dot{u}(0) e_c = \varepsilon^j_a (0) \frac{\partial}{\partial x^j} = u A_{ab} e_c = \delta_{ac} u e_b - \delta_{bc} u e_a = \left(\delta_{ac} e^j_b - \delta_{bc} e^j_a\right) \frac{\partial}{\partial x^j},
\] (2.4)
whence
\[ V_{ab} = \dot{u}(0) = \dot{e}^i_a(0) \frac{\partial}{\partial e^i_a} = \left( \delta_{ac} e^j_b - \delta_{bc} e^j_a \right) \frac{\partial}{\partial e^i_a} = e^j_b \frac{\partial}{\partial e^i_a} - e^j_a \frac{\partial}{\partial e^i_b}. \] (2.5)

Finally we recall that \( D_t \) is defined as the horizontal lift of \( \partial_t \). More explicitly, given \( u_0 \in F \), suppose \( \pi(u_0) = (x_0, t_0) \) and \( \gamma_t := (x_0, t_0 + t) \) and let \( u_t \) be the horizontal lift of \( \gamma_t \). Then, we have that \( D_t(u_0) = \left. \frac{d}{dt} \right|_{t=0} u_t \). Recalling Definition 2.1, and using the tensorial transformation rule \( \tilde{\gamma}^{(g)}_{ab} = \partial_t g_{jkl} e^j_a e^k_b \) (see Eq. (2.9) below), we compute
\[
\nabla_t \left( e^j_a \frac{\partial}{\partial x^j} \right) = \frac{d(e^j_a)}{dt} \frac{\partial}{\partial x^j} + e^j_a \nabla_t \left( \frac{\partial}{\partial x^j} \right) = \frac{d(e^j_a)}{dt} \frac{\partial}{\partial x^j} + e^j_a \nabla_t \left( \partial_t - \frac{1}{2} \tilde{\gamma}^{(g)}_{ab} \frac{\partial}{\partial x^a} \right) = 0.
\] (2.7)

By exhibiting \( D_t(u_0) \) in local coordinates
\[
D_t(u_0) = 0 \cdot \frac{\partial}{\partial x^j} + 1 \cdot \frac{\partial}{\partial t} + \left. \frac{d}{dt} \right|_{t=0} (e^j_a(t)) \frac{\partial}{\partial e^j_a} = 0 - \frac{1}{2} \tilde{\gamma}^{(g)}_{ab} \frac{\partial}{\partial e^j_a} = 0.
\] (2.8)
we conclude the proof.

We now recall that the time-dependent tensor fields \( T \) correspond to equivariant functions \( \tilde{T} \) on \( F \). For example, a function \( f : M \rightarrow \mathbb{R} \) corresponds to the invariant function \( \tilde{f} = f \circ \pi : F \rightarrow \mathbb{R} \) and a time-dependent two-tensor \( T = T(x, t) dx^i \otimes dx^j \) corresponds to an equivariant function \( \tilde{T} = (\widetilde{T}_{ab}) : F \rightarrow \mathbb{R}^{n \times n} \) via \( \tilde{T}_{ab}(u) = T_{\pi(u)}(ue_a, ue_b) \). Note that identities \( ue_a = e^j_a \frac{\partial}{\partial x^j} \) and \( ue_b = e^k_b \frac{\partial}{\partial x^k} \) yield the transformation rule
\[
\tilde{T}_{ab} = T_{ij} e^j_a e^i_b.
\] (2.9)

Also observe that using Eqs. (2.3) and (2.9), one obtains the formula
\[
V_{ab} \tilde{T}_{cd} = \tilde{T}_{bd} \delta^a_c - \tilde{T}_{ad} \delta^b_c + \tilde{T}_{cb} \delta^a_d - \tilde{T}_{ca} \delta^b_d.
\] (2.10)

**Proposition 2.5 (Derivatives) [7]** From the correspondence with equivariant functions, the first and second order derivatives of tensor fields can be computed as follows
\[
\begin{align*}
\nabla_X \tilde{T} &= X^* \tilde{T} \\
\nabla_t \tilde{T} &= D_t \tilde{T} \\
\Delta \tilde{T} &= \sum_{a=1}^n H_a H_a \tilde{T} =: \Delta_H \tilde{T} \\
(\nabla^2 f)(ue_a, ue_b) &= H_a H_b \tilde{f}
\end{align*}
\] (2.11)
Proof Except for the fourth identity regarding the Hessian, these are either classical results from differential geometry or have already been proven in Lemmas 3.1 and 3.3 of [7]. For this last identity, write the canonical horizontal vector fields in local coordinates and compute

\[ H_a H_b \tilde{f} = \left( e^j_a \frac{\partial}{\partial x^j} - e^j_a e^k_b \Gamma^\ell_{jk} \frac{\partial}{\partial e^\ell_b} \right) e^p_b \frac{\partial}{\partial x^p} \tilde{f} \]

\[ = e^j_a e^k_b \left( \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^k} \tilde{f} - \Gamma^p_{jk} \frac{\partial}{\partial x^p} \tilde{f} \right) \]

\[ = e^j_a e^k_b \nabla_j \nabla_k f \]

\[ = \nabla^2 f (ue_a, ue_b), \quad (2.12) \]

thereby proving the proposition. \(\square\)

Next we proceed to prove a few commutator relations between the newly defined vector field, \(D_t\), and the canonical horizontal and vertical vector fields.

Lemma 2.6 (cf. [6]) The fundamental vectors fields on the frame bundle satisfy the following commutator relations

\[
\begin{align*}
[H_a, H_b] &= \frac{1}{2} R_{abcd} V_{cd} \\
[V_{ab}, H_c] &= \delta_{ac} H_b - \delta_{bc} H_a \\
[V_{ab}, V_{cd}] &= \delta_{bd} V_{ac} - \delta_{ad} V_{bc} + \delta_{ac} V_{bd} - \delta_{bc} V_{ad} \\
[D_t, H_a] &= -\frac{1}{2} \tilde{\partial}_t g_{ad} V_{bd} + \frac{1}{2} H_b \tilde{\partial}_t g_{ac} V_{cb} \\
[D_t, V_{ab}] &= 0.
\end{align*}
\]

(2.13)

Proof We admit without proof the commutator relations not involving \(D_t\) as they can be derived from basic differential geometry. It remains to check the final two commutator relations. To prove the first of these, between \(D_t\) and the horizontal vector field \(H_a\), we compute

\[
[D_t, e^j_a \frac{\partial}{\partial x^j}] = -e^j_a e^k_b \frac{\partial}{\partial x^j} \nabla_k \frac{\partial}{\partial e^\ell_b} \]

\[
= -e^j_a e^k_b \left( \Gamma^\ell_{jk} \frac{\partial}{\partial e^\ell_b} - \frac{\partial}{\partial x^k} \nabla_j \frac{\partial}{\partial e^\ell_b} \right)
\]

\[= -\frac{1}{2} \Gamma_{jk}^{\ell} e^\ell_a e^k_b \left( \nabla_j \nabla_k \right) \frac{\partial}{\partial e^\ell_b} \]

\[= -\frac{1}{2} \nabla^2 f (ue_a, ue_b), \quad (2.14) \]
and

\[ [D_t - \partial_t, H_a] = -\frac{1}{2} (\partial_t \widetilde{g}_{cd}) e^e_d \frac{\partial}{\partial e^e_c}, H_a \]

\[ = \frac{1}{2} \partial_t \widetilde{g}_{cd} \left[ e^e_d \frac{\partial}{\partial e^e_c}, H_a \right] + \frac{1}{2} H_a (\partial_t \widetilde{g}_{cd}) e^e_d \frac{\partial}{\partial e^e_c} \]

\[ = \frac{1}{2} \partial_t \widetilde{g}_{cd} \delta^e_a H_d + \frac{1}{2} H_a (\partial_t \widetilde{g}_{bc}) e^e_c \frac{\partial}{\partial e^e_b} \]

\[ = -\frac{1}{2} \partial_t \widetilde{g}_{ad} H_d + \frac{1}{2} H_a (\partial_t \widetilde{g}_{bc}) e^e_c \frac{\partial}{\partial e^e_b} . \]  

(2.15)

Next, we sum Eqs. (2.14) and (2.15) to compute the desired commutator relation

\[ [D_t, H_a] = [\partial_t, H_a] + [D_t - \partial_t, H_a] = -\frac{1}{2} \partial_t \widetilde{g}_{ad} H_d + \frac{1}{2} H_b (\partial_t \widetilde{g}_{ac}) V_{cb}. \]  

(2.16)

Finally, using Eqs. (2.3) and (2.10), the commutator of \( D_t \) and \( V_{ab} \) is

\[ [D_t, V_{ab}] = \left[ \partial_t - \frac{1}{2} (\partial_t \widetilde{g}_{cd}) e^e_d \frac{\partial}{\partial e^e_c}, V_{ab} \right] \]

\[ = -\frac{1}{2} \partial_t \widetilde{g}_{cd} \left[ e^e_d \frac{\partial}{\partial e^e_c}, V_{ab} \right] + \frac{1}{2} V_{ab} (\partial_t \widetilde{g}_{cd}) e^e_d \frac{\partial}{\partial e^e_c} \]

\[ = \frac{1}{2} \partial_t \widetilde{g}_{cd} \left( e^e_d \frac{\partial}{\partial e^e_c} \delta^a_b + e^e_b \frac{\partial}{\partial e^e_b} \delta^a_d - e^e_d \frac{\partial}{\partial e^e_a} \delta^b_b - e^e_b \frac{\partial}{\partial e^e_a} \delta^d_d \right) \]

\[ + \frac{1}{2} \left( \partial_t \widetilde{g}_{bd} \delta^e_a + \partial_t \widetilde{g}_{cb} \delta^a_d - \partial_t \widetilde{g}_{ad} \delta^b_b - \partial_t \widetilde{g}_{cd} \delta^d_d \right) e^e_d \frac{\partial}{\partial e^e_c} \]

\[ = 0 , \]  

(2.17)

thereby proving this lemma on commuting canonical vector fields.

\[ \square \]

**Corollary 2.7** If \( \tilde{f} : \mathcal{F} \to \mathbb{R} \) is an orthonormally invariant function, then

\[ \begin{align*}
H_a H_b \tilde{f} - H_b H_a \tilde{f} &= 0 \\
\Delta_H H_a \tilde{f} - H_a \Delta_H \tilde{f} &= \tilde{R}_{ab} H_b \tilde{f},
\end{align*} \]

(2.18)

where \( \tilde{R}_{ab}(u) = R_{\pi(u)}(ue_a, ue_b) \).

**Proof** This is a direct application of the commutator relations from Lemma 2.6.

\[ \square \]

**Proposition 2.8** Let \( \tilde{f} : \mathcal{F} \to \mathbb{R} \) be an orthonormally invariant function. Then

\[ [D_t - \Delta_H, H_a] \tilde{f} = -\frac{1}{2} (\partial_t \tilde{g} + 2\tilde{R}_{ab}) H_b \tilde{f} . \]  

(2.19)

**Proof** It readily follows from Lemma 2.6 and Corollary 2.7 that

\[ [D_t - \Delta_H, H_a] \tilde{f} = [D_t, H_a] \tilde{f} - [\Delta_H, H_a] \tilde{f} \]

\[ = -\frac{1}{2} \partial_t \widetilde{g}_{ad} H_d \tilde{f} + \frac{1}{2} H_b (\partial_t \widetilde{g}_{ac}) V_{cb} \tilde{f} - \tilde{R}_{ab} H_b \tilde{f} \]

\[ = -\frac{1}{2} (\partial_t \tilde{g} + 2\tilde{R}_{ab}) H_b \tilde{f} . \]  

(2.20)

thereby proving the proposition.

\[ \square \]
2.2 Probabilistic preliminaries

The principal goal of this section is to recall the notions of Brownian motion and stochastic parallel transport in the setting of evolving manifolds as developed in [1, 7].

We first remark that it shall hereafter be assumed that in addition to the Riemannian manifolds \( \{M_t\} \) being complete as in the previous section, they will also satisfy

\[
\sup_{M_t} (|\text{Rm}| + |\partial_t g| + |\nabla \partial_t g|) < \infty. \tag{2.21}
\]

Horizontal curves \( \{u_\tau\}_{\tau \in [0, T]} \in \mathcal{F} \), where \( \pi(u_\tau) = (x_\tau, T - \tau) \), correspond to curves \( \{w_\tau\}_{\tau \in [0, T]} \in \mathbb{R}^n \) (also known as the anti-development of \( u_\tau \)) via the following initial value problem

\[
\begin{dcases}
\frac{du_\tau}{d\tau} = D_\tau + H_\tau(u_\tau) \frac{dw_\tau^a}{d\tau} \\
w_0 = 0.
\end{dcases} \tag{2.22}
\]

This definition of the anti-development in the time-dependent geometry setting appropriately motivates the following stochastic differential equation in the case of evolving manifolds

\[
\begin{dcases}
dU_\tau = D_\tau d\tau + H_\tau(U_\tau) \circ dW^a_\tau \\
U_0 = u.
\end{dcases} \tag{2.23}
\]

We make a short note on notation that \( W_\tau \sim \sqrt{2}B_\tau \) refers to the Brownian motion in \( \mathbb{R}^n \) with rescaling by a factor of \( \sqrt{2} \) such that it has quadratic variation \( d[W, W]_\tau = 2d[B, B]_\tau = 2d\tau \) and \( \circ \) refers to the Stratonovich integral in differential notation.

Next, by demonstrating that this equation satisfies existence and uniqueness criterion as well as Itô’s lemma, the notions of Brownian motion, via projection onto \( M \), and stochastic parallel transport can be formalized.

**Proposition 2.9** (Existence, uniqueness and Itô’s lemma) The stochastic differential equation (2.23) has a unique solution \( \{U_\tau\}_{\tau \in [0, T]} \) that satisfies \( \pi_2(U_\tau) = T - \tau \). Moreover, given \( \tilde{f}: \mathcal{F} \rightarrow \mathbb{R} \) is of class \( C^2 \), then the solution \( U_\tau \) satisfies

\[
d\tilde{f}(U_\tau) = D_\tau \tilde{f}(U_\tau) d\tau + \langle (H \tilde{f})(U_\tau), dW_\tau \rangle + \Delta_H(\tilde{f})(U_\tau) d\tau. \tag{2.24}
\]

**Proof** This result has already been proven in Proposition 3.7 of [8]. \( \square \)

We shall now continue with defining the notions of Brownian motion and stochastic parallel transport, from [7], in the setting of time-evolving families of Riemannian manifolds.

**Definition 2.10** (Brownian motion on space time) We call \( \pi(U_\tau) = (X_\tau, T - \tau) \) the Brownian motion on space time \( \mathcal{M} = M \times I \) with base point \( \pi(u) = (x, T) \).

**Definition 2.11** (Stochastic parallel transport) The family of isometries

\[
\{ P_\tau = U_0U_\tau^{-1} : (T_x, M, g_{T - \tau}) \rightarrow (T_x M, g_T) \} \tag{2.25}
\]

is called the stochastic parallel transport along the Brownian curve \( X_\tau \).

This Brownian motion now inherits a path based space, diffusion measure and filtration. First, we denote by \( P_{0,0} \mathbb{R}^n \) the based path space on \( \mathbb{R}^n \), namely the space of continuous curves \( \{w_\tau | w_0 = 0\}_{\tau \in [0, T]} \subset \mathbb{R}^n \).
Definition 2.12 (Based path spaces) Let $P_u\mathcal{F}$ and $P_{(x,T)}\mathcal{M}$ be the spaces of continuous curves, $\{u_t|u_0=u, \pi_2(u_t)=T-t\}_{t\in[0,T]} \subset \mathcal{F}$ and $\{\gamma_t=(x_t, T-t)|\gamma_0=(x,T)\}_{t\in[0,T]}$ respectively.

To construct the Wiener measure, we first observe that solving the stochastic differential equation (2.23) yields a map $U: P_0\mathbb{R}^n \to P_u\mathcal{F}$. Moreover, the projection map $\pi: \mathcal{F} \to \mathcal{M}$ induces a map $\Pi: P_u\mathcal{F} \to P_{(x,T)}\mathcal{M}$.

Definition 2.13 (Wiener measure) Let $\mathbb{P}_0$ be the Wiener measure on path space $P_0\mathbb{R}^n$. We then say that $\mathbb{P}_u := U_\ast(\mathbb{P}_0)$ and $\mathbb{P}_{(x,T)} := \Pi_\ast \mathbb{P}_u$ are the Wiener measures of horizontal Brownian motion on $\mathcal{F}$ and Brownian motion on space-time $\mathcal{M}$ respectively.

Moreover, we can uniquely characterize the Wiener measure in terms of the heat kernel.

Proposition 2.14 [7] Let $\{\tau_j\}_{j=1}^k$ be a partition of $[0, T]$, $U_j \subseteq M$ and $\gamma_0 = (x, T)$. Then

$$\mathbb{P}_{(x,T)}[X_{\tau_j} \in U_j, \forall j \in 1, \ldots, k] = \int_{\bigotimes_j U_j} \prod_j H(x_{\tau_{j-1}}, T-\tau_{j-1}|x_j, T-\tau_j) \otimes d\text{Vol}_{\mathbb{R}^n}(\gamma_j)$$

uniquely characterizes the Wiener measure on $P_{(x,T)}\mathcal{M}$.

Proof The proof follows as in Proposition 3.31 of [7]. ☐

Next, we recall that the path space $P_0\mathbb{R}^n$ comes equipped with an intrinsic filtration $\Sigma^u_\tau$ generated by evaluation maps $e_\sigma: P_0\mathbb{R}^n \to \mathbb{R}^n|e_\sigma(u) = u_\sigma, \sigma \leq \tau$.

Definition 2.15 (Filtrations on $P_u\mathcal{F}$ and $P_{(x,T)}\mathcal{M}$) The filtrations on $P_u\mathcal{F}$ and $P_{(x,T)}\mathcal{M}$ are simply the respective push-forwards $\Sigma^\mathcal{M} := (\Pi \circ U)_\ast \Sigma^u_\tau$ and $\Sigma^\mathcal{F} := U_\ast \Sigma^u_\tau$.

A short reiteration of induced martingales as well as parallel and Malliavin gradients are constructed in the time-dependent setting will now complete this section.

Definition 2.16 (Induced martingale) Let $F: P_{(x,T)}\mathcal{M} \to \mathbb{R}$ be integrable. Then, we define the induced martingale as $F_\tau(\gamma) := \mathbb{E}[F|\Sigma_\tau](\gamma)$.

Using this definition, the conditional expectation can now be characterized by a representation formula.

Proposition 2.17 (Conditional expectation) [7] Suppose the conditional expectation is as defined. Then, for almost every Brownian curve $\{\gamma_t\}_{t\in[0,T]}$,

$$F_\tau(\gamma) := \mathbb{E}[F|\Sigma_\tau](\gamma) = \int_{P_{\gamma_\tau}\mathcal{M}} F(\gamma|_{[0,T]} \ast \gamma') d\mathbb{P}_{\gamma_\tau}(\gamma'),$$

where we integrate over all Brownian curves $\gamma'$ in the based path space $P_{\gamma_\tau}\mathcal{M}$ with respect to Wiener measure $\mathbb{P}_{\gamma_\tau}$ and $\ast$ denotes concatenation of the two curves $\gamma|_{[0,T]}$ and $\gamma'$.

Proof The proof follows as in Proposition 3.19 of [7]. ☐

To define the two notions of gradients, we first recall that cylinder functions are of the form $u \circ e_\sigma$, where $e_\sigma: P_{(x,T)}\mathcal{M} \to M^k$ are $k$-point evaluation maps, namely $e_\sigma: \gamma \mapsto (\pi_1\gamma_{\sigma_1}, \ldots, \pi_1\gamma_{\sigma_k})$, and $u: M^k \to \mathbb{R}$ is compactly supported.
Example 2.18 Let $F(\gamma) := f \circ e_{\tau_1}(\gamma) = f(\pi_1\gamma_{\tau_1})$. Then the induced martingale of $F$ is given for $\tau > \tau_1$ by

$$F_\tau(\gamma) = \int_{P_{\gamma},\mathcal{M}} F(\gamma|_{[0,\tau]} \ast \gamma') \, dP^\gamma_{\tau}(\gamma')$$

$$= \int_{P_{\gamma},\mathcal{M}} f(\pi_1\gamma_{\tau_1}) \, dP^\gamma_{\tau}(\gamma')$$

$$= f(X_{\tau_1}),$$

(2.28)

and for $\tau < \tau_1$ by

$$F_\tau(\gamma) = \int_{P_{\gamma},\mathcal{M}} F(\gamma|_{[0,\tau]} \ast \gamma') \, dP^\gamma_{\tau}(\gamma')$$

$$= \int_{P_{\gamma},\mathcal{M}} f(\gamma_{\tau_1-\tau}) \, dP^\gamma_{\tau}(\gamma')$$

$$= \int_{M} f(y)H(X_{\tau}, T - \tau |y, T - \tau_1) \, dV_{g_{T-\tau_1}}(y)$$

$$= H_{T-\tau_1,T-\tau} f(\pi_1\gamma_{\tau}).$$

(2.29)

Definition 2.19 (Parallel gradient) Let $\sigma \in [0, T]$ and let $F: P(x, T)M \rightarrow \mathbb{R}$ be a cylinder function. Then the $\sigma$-parallel gradient is the almost everywhere uniquely defined function $\nabla||_{\sigma} F: P(x, T)M \rightarrow (T_x M, g_T)$ such that

$$DV^\sigma F(\gamma) = \langle \nabla||_{\sigma} F(\gamma), v \rangle (T_x M, g_T),$$

(2.30)

for almost every Brownian curve $\gamma$ and $v \in (T_x M, g_T)$, where $V^\sigma_{\tau} = P^{-1}_{\tau} \tau_1 \| \sigma, T \rangle(\tau)$. Here, $DV$ denotes the Fréchet derivative.

Example 2.20 Let $F = u \circ e_{\tau}$ be a $k$-point function with partition $\{\tau_j\}_{j=1}^k$. Then the parallel gradient of $F$ is given by

$$\nabla||_{\sigma} F = e^*_{\tau} \left( \sum_{\tau_j \geq \sigma} P_{\tau_j} \text{grad}^{(j)}_{g_{T-\tau_j}} u \right).$$

(2.31)

Finally, we let $\mathcal{H}$ be the Hilbert space of $W^{1,2}$ curves in $(T_x M, g_T)$ with $v_0 = 0$ equipped with the natural Sobolev inner product given by

$$\langle u, v \rangle_{\mathcal{H}} := \int_0^T \langle \dot{u}_\tau, \dot{v}_\tau \rangle (T_x M, g_T) \, d\tau.$$  

(2.32)

Definition 2.21 (Malliavin gradient) Let $F: P(x, T)M \rightarrow \mathbb{R}$ be a cylinder function. The Malliavin gradient is the almost everywhere uniquely defined function $\nabla^\mathcal{H} F: P(x, T)M \rightarrow \mathcal{H}$ such that

$$DV F(\gamma) = \langle \nabla^\mathcal{H} F(\gamma), v \rangle_{\mathcal{H}},$$

(2.33)

for almost every Brownian curve $\gamma$ and $v \in \mathcal{H}$, where $V_{\tau} = P^{-1}_{\tau} v_{\tau}$.

Definition 2.22 (Skorokhod integral) The adjoint of the Malliavin gradient, also known as the Skorokhod integral, is the uniquely defined operator $(\nabla^\mathcal{H})^*: L^2(P, \mathcal{M}) \rightarrow L^2(P, \mathcal{M})$ such that

$$\mathbb{E}[F(\nabla^\mathcal{H})^* g] = \mathbb{E} \left[ \langle \nabla^\mathcal{H} F, g \rangle_{\mathcal{H}} \right].$$

(2.34)
for all $F, g \in L^2(P, M)$.

**Definition 2.23** *(Ornstein-Uhlenbeck operator)* The Ornstein-Uhlenbeck operator is defined as

$$\mathcal{L}_{(\tau_1, \tau_2)} := (\nabla^{\mathcal{H}})^* \nabla^{\mathcal{H}}$$  \hspace{1cm} (2.35)

where $\nabla^{\mathcal{H}}$ and $(\nabla^{\mathcal{H}})^*$ are the Malliavin gradient and its adjoint from (2.33) and (2.34) respectively.

### 3 Bochner formula on parabolic path space

For convenience of the reader, we shall first recall and prove the statement of Bochner’s formula in the time-dependent setting.

**Lemma 3.1** *(Bochner)* First let $(M, g_t)_{t \in I}$ be a family of Riemannian manifolds and $\nabla^{\mathcal{H}_{g_t}} = g_{ij}^{t} \partial_j$ where $\Delta_{g_t}$ be the Laplace-Beltrami operator. Then the evolution of $|\nabla u|_{g_t}^2$ is given by

$$\frac{1}{2}(-\partial_t + \Delta_{g_t})(|\nabla u|_{g_t}^2) = \langle \nabla u, \nabla (-\partial_t + \Delta_{g_t})u \rangle + |\nabla^2 u|^2 + \frac{1}{2}(\partial_t g_{ij} + 2Rc_{g_t})(\text{grad} u, \text{grad} u).$$  \hspace{1cm} (3.1)

**Proof** We evaluate both

$$\frac{1}{2} \Delta_{g_t}(|\nabla u|_{g_t}^2) = \frac{1}{2} \nabla_i \nabla_i (\nabla_j u \nabla_j u) = (\nabla_i \nabla_j u)(\nabla_i \nabla_j u) + (\nabla_j u)(\nabla_i \nabla_i \nabla_j u) = |\nabla^2 u|^2 + (\nabla_j u)(\nabla_j \Delta_{g_t} u) + Rc_{g_t}(\text{grad} u, \text{grad} u)$$ \hspace{1cm} (3.2)

and

$$\frac{1}{2} \partial_t(|\nabla u|_{g_t}^2) = \frac{1}{2} \partial_t g_{ij}^{t} \nabla_i u \nabla_j u + g_{ij}^{t} \nabla_i \partial_t(\nabla_j u) = -\frac{1}{2} \partial_t g_{ki}^{t} g_{\ell j}^{t} \nabla_i \nabla_j u + g_{ij}^{t} \nabla_i \partial_t (\nabla_j u) = \langle \nabla u, \nabla (\partial_t u) \rangle - \frac{1}{2} \partial_t g_{ij}(\text{grad} u, \text{grad} u).$$ \hspace{1cm} (3.3)

We then deduce the Bochner formula as the difference of the two results. \hfill \square

**Theorem 3.2** *(Martingale representation theorem)* If $F_{\tau} : P_{(x, T)}\mathcal{M} \to \mathbb{R}$ is a martingale on parabolic path space and $F_{\tau} \in \mathcal{D}(\mathcal{V}^{\mathcal{H}})$, then $F_{\tau}$ solves stochastic differential equation

$$\begin{cases}
    dF_{\tau} = \langle \mathcal{V}^{\mathcal{H}} \rangle F_{\tau}, dW_{\tau} \\
    F|_{\tau=0} = F_0.
\end{cases}$$  \hspace{1cm} (3.4)

**Proof** By approximation (cf. [7, Sec 2.4]), it suffices to prove the theorem in the case where $F_{\tau}$ is a martingale induced by a $k$-point cylinder function. Namely, let $F(\gamma) = f(\pi_1 \gamma_1, \ldots, \pi_1 \gamma_k)$, where $f : \mathcal{M}^k \to \mathbb{R}$ and we recall that $\gamma_{\tau} = (X_{\tau}, T - \tau)$. Also
let $F_{\tau} = \mathbb{E}(\tau, T)\{F|\Sigma_{\tau}\}$ be the induced martingale. Then, for $\tau \in (\tau_{\ell}, \tau_{\ell+1})$ by Propositions 2.17 and then 2.14, we calculate

$$ F_{\tau}(y) = \int_{\mathcal{P}_{\tau}^{\ell}} F(y|_{[0, \tau]} * y') \, d\mathbb{P}_y(y') $$

$$ = \int_{\mathcal{P}_{\tau}^{\ell}} f(\pi_1y_{\tau_1}, \ldots, \pi_1y_{\tau_{\ell}}, \pi_1y_{\tau_{\ell+1}-\tau}, \ldots, \pi_1y_{\tau_{\tau_{k}-\tau}}) \, d\mathbb{P}_y(y') $$

$$ = \int_{\mathcal{M}^{k-\ell}} f(x_{\tau_1}, \ldots, x_{\tau_{\ell}}, y_{\tau_{\ell+1}}, \ldots, y_{\gamma})H(x_{\tau}, T - \tau)y_{\tau+1}, T - \tau_{\ell+1}) $$

$$ H(y_{\tau+1}, T - \tau_{\ell+1}|y_{\tau+2}, T - \tau_{\ell+2}) \ldots H(y_{\gamma-1}, T - \tau_{\gamma-1}|y_{\gamma}, T - \tau_{\gamma}) $$

$$ dV_{\gamma-\gamma}(y_{\gamma}) =: f_{\tau}(x_{\tau_1}, \ldots, x_{\tau_{\ell}}, x_{\tau}). \quad (3.5) $$

Note that, for $(x_1, \ldots, x_{\ell})$ fixed, $(x, \tau) \rightarrow f_{\tau}(x_1, \ldots, x_{\ell}, x)$ is uniformly Lipschitz in $\tau$ and solves $(\partial_{\tau} + \Delta^{(\ell+1)}_{\gamma})f_{\tau} = 0$, where $\Delta^{(\ell+1)}$ acts on the last entry.

Consider the lift $\tilde{f}_{\tau} := f_{\tau} \circ \mathcal{P}_{\tau}^{\ell+1} \pi_1 \circ \mathcal{P}_{\tau}^{\ell+1} \pi$. Also let $\tilde{F}_{\tau} := F_{\tau} \circ \mathcal{P}$, where $\mathcal{P} : P\mathcal{F} \rightarrow P\mathcal{M}$. Then we have that $\tilde{F}_{\tau}(U) = \tilde{f}_{\tau}(U_{\tau_1}, \ldots, U_{\tau_{\ell}}, U_{\tau})$, which satisfies $(D_{\tau} + \Delta^{(\ell+1)}_H)\tilde{f}_{\tau} = 0$ by applying Proposition 2.9. Also note that herein we shall denote the vector $(H_{\tau} \tilde{f}, \ldots, H_{\tau} \tilde{f})$ by $H \tilde{f}$.

Then, by Proposition 2.9, we calculate

$$ d\tilde{F}_{\tau}(U) = d(\tilde{f}_{\tau}(U_{\tau_1}, \ldots, U_{\tau_{\ell}}, U_{\tau})) = \langle H^{(\ell+1)}(\tilde{f})_\parallel(U_{\tau_1}, \ldots, U_{\tau_{\ell}}, U_{\tau}), dW_{\tau} \rangle $$

$$ + \left(D_{\tau} + \Delta^{(\ell+1)}_H\right)\tilde{f}_{\tau}(U_{\tau_1}, \ldots, U_{\tau_{\ell}}, U_{\tau}) \, d\tau $$

$$ = \langle H^{(\ell+1)}(\tilde{f})_\parallel(U_{\tau_1}, \ldots, U_{\tau_{\ell}}, U_{\tau}), dW_{\tau} \rangle. \quad (3.6) $$

Next, we project down to $\mathcal{M}$ by Proposition 2.5 as follows

$$ H^{(\ell+1)}_{\parallel}(\tilde{f}_{\tau})(U_{\tau_1}, \ldots, U_{\tau_{\ell}}, U_{\tau}) = (U_{\tau}e_a)^*\tilde{f}_{\tau}(U_{\tau_1}, \ldots, U_{\tau_{\ell}}, U_{\tau}) $$

$$ = (U_{\tau}e_a)f_{\tau}(X_{\tau_1}, \ldots, X_{\tau_{\ell}}, X_{\tau}) $$

$$ = \langle U_{\tau}e_a, \text{grad}_{R_{\gamma-\gamma}}^{(\ell+1)}f_{\tau}(X_{\tau_1}, \ldots, X_{\tau_{\ell}}, X_{\tau}) \rangle_{(T_{\gamma}M, g_{\gamma-\gamma})} $$

$$ = \langle P_{\tau}U_{\tau}e_a, P_{\tau}\text{grad}_{R_{\gamma-\gamma}}^{(\ell+1)}f_{\tau}(X_{\tau_1}, \ldots, X_{\tau_{\ell}}, X_{\tau}) \rangle_{(T_{\gamma}M, g_{\gamma-\gamma})} $$

$$ = \langle U_0e_a, \nabla_{\tau}^\parallel F_{\tau}(y) \rangle_{(T_{\gamma}M, g_{\gamma-\gamma})}, \quad (3.7) $$

whence

$$ H^{(\ell+1)}_{\parallel}(\tilde{f}_{\tau}) \, dW_{\tau}^a = \langle \nabla_{\tau}^\parallel F_{\tau}(y), U_0e_a \rangle \, dW_{\tau}^a = \langle \nabla_{\tau}^\parallel F_{\tau}(y) \rangle, \quad (3.8) $$

and we deduce that

$$ dF_{\tau}(y) = d\tilde{F}_{\tau}(U) = \langle \nabla_{\tau}^\parallel F_{\tau}(y), dW_{\tau} \rangle $$

$$ \quad \text{(3.9)} $$

to complete the proof. \hfill \Box

**Theorem 3.3** (Evolution of the parallel gradient) If $F_{\tau} : P(\gamma, T)\mathcal{M} \rightarrow \mathbb{R}$ is a martingale on parabolic path space, and $\sigma \geq 0$ is fixed, then $\nabla_{\tau}^\parallel F_{\tau} : P(\gamma, T)\mathcal{M} \rightarrow (\mathcal{T}(\gamma, T)\mathcal{M}, g_{\gamma})$ satisfies the stochastic differential equation

$$ d(\nabla_{\tau}^\parallel F_{\tau}) = (\nabla_{\tau}^\parallel \nabla_{\tau}^\parallel F_{\tau}, dW_{\tau}) + \frac{1}{2}(\dot{g} + 2Rc)_{\tau}(\nabla_{\tau}^\parallel F_{\tau}) d\tau + \nabla_{\tau}^\parallel F_{\tau}d\delta_{\tau}(\tau), \quad (3.10) $$
where \((\hat{g} + 2\text{Rc})(v, w)_{(T_{\Sigma}, \mathcal{M})} = (\hat{g} + 2\text{Rc}, g)_{|t=T-\tau}(P_{\tau}^{-1}v, P_{\tau}^{-1}w)\) and \(\hat{g} = \frac{d}{dt}g\).

**Proof** As \(F_{\tau}\) is \(\Sigma_t\)-measurable, we have that \(\nabla_{\sigma}^\| F_{\tau} \equiv 0\) for \(\sigma > \tau\). Noting \(d(\nabla_{\sigma}^\| F_{\tau})\) is continuous except for a jump discontinuity at \(\sigma = \tau\), we calculate

\[
d(\nabla_{\sigma}^\| F_{\tau}) = d(\nabla_{\sigma}^\| F_{\tau})_{\text{cont}} + \left(\nabla_{\sigma}^\| F_{\sigma} - \nabla_{\sigma}^\| F_{\sigma}\right) d\delta_{\tau}(\tau).
\]

It remains to show that the identity holds for \(\sigma \leq \tau\). In particular, we’ll show that the continuous parts of the measures agree.

By approximation (cf. [7, Sec 2.4]), it suffices to prove the theorem in the case where \(F_{\tau}\) is a martingale induced by a \(k\)-point cylinder function as in the previous proof. Now, as \(\sigma\) is fixed, it is sufficient for us to consider the evolution equation for \(\tau \in (\tau_{\ell}, \tau_{\ell+1})\), using the parallel gradient from example 2.20,

\[
\nabla_{\sigma}^\| F_{\tau}(y) = \sum_{T_j \geq \sigma} P_{T_j} \nabla^{(j)} f_t(X_{T_1}, \ldots, X_{T_\ell}, X_{\tau}) + P_{\tau} \nabla^{(\ell+1)} f_t(X_{T_1}, \ldots, X_{T_\ell}, X_{\tau}).
\]

which can be lifted to the frame bundle and represented by Proposition 2.5 as

\[
G_{\alpha}(U) := \langle U_0 e_{\alpha}, \nabla_{\sigma}^\| F_{\tau}(\Pi U) \rangle
= \sum_{T_j \geq \sigma} \langle U_{T_j} e_{\alpha}, \nabla^{(j)} f_t(X_{T_1}, \ldots, X_{T_\ell}, X_{\tau}) \rangle
+ \langle U_{T_j} e_{\alpha}, \nabla^{(\ell+1)} f_t(X_{T_1}, \ldots, X_{T_\ell}, X_{\tau}) \rangle
= \sum_{T_j \geq \sigma} H_{\alpha}^{(j)} \tilde{f}_t(U_{T_1}, \ldots, U_{T_\ell}, U_{\tau}) + H_{\alpha}^{(\ell+1)} \tilde{f}_t(U_{T_1}, \ldots, U_{T_\ell}, U_{\tau}).
\]

Applying Propositions 2.8 and 2.9 and the fact that \((D_{\tau} + \Delta_{H}^{(\ell+1)}) \tilde{f}_t = 0\) from the previous proof, we have that

\[
dG_{\alpha}(U) = \sum_{T_k \geq \sigma} \sum_{T_j \geq \sigma} H_{\alpha}^{(\ell+1)} H_{\alpha}^{(k)} \tilde{f}_t(U_{T_1}, \ldots, U_{T_\ell}, U_{\tau}) dW_{\tau}^b
+ \sum_{T_k \geq \sigma} H_{\alpha}^{(\ell+1)} H_{\alpha}^{(k)} \tilde{f}_t(U_{T_1}, \ldots, U_{T_\ell}, U_{\tau}) dW_{\tau}^b
+ \sum_{T_k \geq \sigma} \left(D_{\tau} + \Delta_{H}^{(\ell+1)} \right) H_{\alpha}^{(k)} \tilde{f}_t(U_{T_1}, \ldots, U_{T_\ell}, U_{\tau}) d\tau
+ \left(D_{\tau} + \Delta_{H}^{(\ell+1)} \right) H_{\alpha}^{(\ell+1)} \tilde{f}_t(U_{T_1}, \ldots, U_{T_\ell}, U_{\tau}) d\tau
= \sum_{T_k \geq \sigma} \sum_{T_j \geq \sigma} H_{\alpha}^{(\ell+1)} H_{\alpha}^{(k)} \tilde{f}_t(U_{T_1}, \ldots, U_{T_\ell}, U_{\tau}) dW_{\tau}^b
+ \sum_{T_k \geq \sigma} H_{\alpha}^{(\ell+1)} H_{\alpha}^{(k)} \tilde{f}_t(U_{T_1}, \ldots, U_{T_\ell}, U_{\tau}) dW_{\tau}^b
+ \sum_{T_k \geq \sigma} \left(H_{\alpha}^{(k)} \left(D_{\tau} + \Delta_{H}^{(\ell+1)} \right) + \left[D_{\tau} + \Delta_{H}^{(\ell+1)} \right] H_{\alpha}^{(k)} \right) \tilde{f}_t(U_{T_1}, \ldots, U_{T_\ell}, U_{\tau}) d\tau
+ \left(H_{\alpha}^{(\ell+1)} \left(D_{\tau} + \Delta_{H}^{(\ell+1)} \right) + \left[D_{\tau} + \Delta_{H}^{(\ell+1)} \right] H_{\alpha}^{(\ell+1)} \right) \tilde{f}_t(U_{T_1}, \ldots, U_{T_\ell}, U_{\tau}) d\tau
= \sum_{T_k \geq \sigma} \sum_{T_j \geq \sigma} H_{\alpha}^{(\ell+1)} H_{\alpha}^{(k)} \tilde{f}_t(U_{T_1}, \ldots, U_{T_\ell}, U_{\tau}) dW_{\tau}^b.
\]
Finally, we check that
\[ + H_b^{(\ell+1)} H_a^{(\ell+1)} \tilde{f}_\tau(U_{\tau_1}, \ldots, U_{\tau_\ell}, U_\tau) \, dW_\tau \\
+ \frac{1}{2} (\tilde{g} + 2\tilde{\text{Rc}})_{ab}(U_\tau) H_b^{(\ell+1)} \tilde{f}_\tau(U_{\tau_1}, \ldots, U_{\tau_\ell}, U_\tau) \, d\tau \]

which completes the proof.

\[ \square \]
Theorem 3.4 (Generalized Bochner formula on \( P, M \)) Let \( F_\tau : P_{(x, T)} M \to \mathbb{R} \) be a martingale. If \( \sigma \geq 0 \) is fixed, then \( \nabla^\parallel F_\tau : P_{(x, T)} M \to (T_x M, g_T) \) satisfies
\[
\begin{align*}
d(|\nabla^\parallel F_\tau|^2) &= \langle \nabla^\parallel F_\tau, dW_\tau \rangle + (\dot{g} + 2\text{Rc})_\tau (\nabla^\parallel F_\tau, \nabla^\parallel F_\tau) d\tau \\
&\quad + 2|\nabla^\parallel F_\tau|^2 d\tau + 2|\nabla^\parallel F_\sigma|^2 d\delta_\sigma (\tau),
\end{align*}
\]
where \((\dot{g} + 2\text{Rc})_\tau (v, w) = (\dot{g}_\tau + 2\text{Rc}_g)_t |_{t = \tau} (P_{\tau^{-1}} v, P_{\tau^{-1}} w) \) and \( \dot{g} = \frac{d}{dt} g \).

**Proof** As \( F_\tau \) is \( \Sigma^\tau \)-measurable, we have that \( \nabla^\parallel F_\tau \equiv 0 \) for \( \sigma > \tau \). Noting \( d(|\nabla^\parallel F_\tau|^2) \) is continuous except for the jump discontinuity at \( \sigma = \tau \), we calculate
\[
\begin{align*}
d(|\nabla^\parallel F_\tau|^2) &= 2\langle \nabla^\parallel F_\tau, d(\nabla^\parallel F_\tau) \rangle \\
&\quad + 2|\nabla^\parallel F_\tau|^2 d\delta_\sigma (\tau) \\
&= d(|\nabla^\parallel F_\tau|^2)_{\text{cont}} + 2|\nabla^\parallel F_\sigma|^2 d\delta_\sigma (\tau).
\end{align*}
\]
It remains to show that the identity holds for \( \sigma \leq \tau \). In particular, it remains to show that the continuous parts of the measures agree.

In the rightly-continuous case by Itô calculus and Theorem 3.3, we calculate the quadratic variation \( d(|\nabla^\parallel F_\tau|, \nabla^\parallel F_\tau) = 2|\nabla^\parallel F_\tau|^2 d\tau \) for \( \sigma \leq \tau \) and then
\[
\begin{align*}
d(|\nabla^\parallel F_\tau|^2) &= 2\langle \nabla^\parallel F_\tau, d(\nabla^\parallel F_\tau) \rangle + d(|\nabla^\parallel F_\tau, \nabla^\parallel F_\tau) \\
&= \langle \nabla^\parallel F_\tau, dW_\tau \rangle + (\dot{g} + 2\text{Rc})_\tau (\nabla^\parallel F_\tau, \nabla^\parallel F_\tau) d\tau \\
&\quad + 2|\nabla^\parallel F_\tau|^2 d\tau,
\end{align*}
\]
which concludes the proof. \( \square \)

Corollary 3.5 (Bochner) The generalized Bochner formula on \( P, M \) (Theorem 3.4) reduces to the standard Bochner formula (Lemma 3.1) in the case of 1-point functions. That is, the evolution of \( |\nabla H_{T-t_1}, T-t f|^2 \) for \( \tau \leq t_1 \) is given by
\[
\begin{align*}
\frac{1}{2} (\partial_\tau + \Delta_{g_{T-t}}) |\nabla H_{T-t_1}, T-t f|^2
&= |\nabla^2 H_{T-t_1}, T-t f|^2 + \frac{1}{2} (\dot{g} + 2\text{Rc})_{|t = \tau} (\nabla H_{T-t_1}, T-t f, \nabla H_{T-t_1}, T-t f).
\end{align*}
\]

**Proof** Fix \( \sigma = 0 \) in the evolution equation from Theorem 3.4. Next, we shall compute the evolution of \( |\nabla^\parallel F_\tau|^2 \), where
\[
F_\tau (\gamma) := \begin{cases} H_{T-t_1}, T-t f (\pi_1 \gamma), & \tau < t_1 \\ f (\pi_1 \gamma), & \tau \geq t_1 \end{cases}
\]
is the martingale induced by \( f (\pi_1 \gamma) \). Then, for \( \tau \in [0, t_1] \), we calculate
\[
|\nabla^\parallel F_\tau (\gamma)| = |\nabla F_\tau (\gamma)| = |\nabla H_{T-t_1}, T-t f (\pi_1 \gamma)|
\]
as well as
\[
|\nabla^\parallel |\nabla F_\tau (\gamma)| = |\nabla^2 H_{T-t_1}, T-t f (\pi_1 \gamma)|.
\]
By Theorem 3.4, we then deduce that
\[
d(|\nabla H_{T-t_1}, T-t f|^2) - \langle \nabla^\parallel |\nabla H_{T-t_1}, T-t f|^2, dW_\tau \rangle \]

\( \square \) Springer
Therefore, by comparing Eqs. (3.26) and (3.27), we conclude that
\[ d(\langle \nabla H_{T-t_1}, T-t f \rangle)^2 = (\partial_t + \Delta_{g_{T-t}}) |\nabla H_{T-t_1}, T-t f|^2 \, d\tau. \] (3.27)

Moreover, for process \( X_t = |\nabla H_{T-t_1}, T-t f|^2 (\pi_1 y_t) \), by applying Itô calculus as in Proposition 2.9, we have that
\[ d(\langle \nabla H_{T-t_1}, T-t f \rangle)^2 - \langle \nabla_{\sigma}^2 \nabla_{\tau}^2 F_{\tau} \rangle_{t = T-t} \langle \nabla H_{T-t_1}, T-t f \rangle^2 = (\partial_t + \Delta_{g_{T-t}}) |\nabla H_{T-t_1}, T-t f|^2 \, d\tau. \] (3.28)

Therefore, by comparing Eqs. (3.26) and (3.27), we conclude that
\[ \frac{1}{2} (\partial_t + \Delta_{g_{T-t}}) |\nabla H_{T-t_1}, T-t f|^2 = |\nabla H_{T-t_1}, T-t f|^2 + \frac{1}{2} (\dot{g} + 2 \text{Re} c) |\nabla H_{T-t_1}, T-t f|^2 \langle \nabla H_{T-t_1}, T-t f, \nabla H_{T-t_1}, T-t f \rangle, \] which completes the proof. □

4 Applications of the Bochner formula on parabolic path space

We shall now proceed by applying the Bochner formula on path space to both characterize the Ricci flow and develop gradient and Hessian estimates for martingales on parabolic path space.

4.1 Proof of the Bochner inequality on parabolic path space

Proof of Theorem 1.2 Using the formalism developed in the last section, we shall prove the equivalencies between the main estimates that characterize the Ricci flow.

(R1) \( \implies \) (C1) \( \implies \) (C2) \( \implies \) (C3): If \((M, g_t)_{t \in I}\) evolves by Ricci flow \(\partial_t g_t = -2 \text{Re} c\) and \(F_t : P_{(x, \tau)} \mathcal{M} \rightarrow \mathbb{R}\) is a martingale on parabolic path space, then Theorem 1.1 gives
\[ d|\nabla_{\sigma} F_{\tau}|^2 = \langle \nabla_{\sigma}^2 |\nabla_{\tau} F_{\tau}|^2, dW_{\tau} \rangle + 2 |\nabla_{\tau}^i \nabla_{\tau}^j F_{\tau}|^2 \, d\tau + 2 |\nabla_{\sigma} F_{\sigma}|^2 d\delta_{\sigma}(\tau), \] thus proving (C1).

Next, to show (C2), calculate
\[ |\Delta_{\sigma, \tau} F_{\tau}|^2 = \left| g^{ij} \left( \nabla_{\sigma}^i \nabla_{\tau}^j F_{\tau} \right) \right|^2 \leq |g^{ij}|^2 |\nabla_{\sigma}^i \nabla_{\tau}^j F_{\tau}|^2 = n |\nabla_{\sigma} F_{\sigma} F_{\tau}|^2, \] (4.2)
and finally show (C3) by simply dropping the non-negative term \(\frac{2}{n} |\Delta_{\sigma, \tau} F_{\tau}|^2\) in (C2).

(C1) \( \implies \) (C4) \( \iff \) (C5): To prove (C4), first apply Itô’s lemma to the left-hand side of the full Bochner inequality (C1) to get
\[ 2 |\nabla_{\sigma} F_{\tau} | (\langle |\nabla_{\sigma} F_{\tau} |, dW_{\tau} \rangle + 2 |\nabla_{\tau}^i \nabla_{\tau}^j F_{\tau}|^2 \, d\tau + 2 |\nabla_{\sigma} F_{\sigma}|^2 d\delta_{\sigma}(\tau) \]
\[ \leq d|\nabla_{\sigma} F_{\tau}|^2 \]
\[ \leq 2 |\nabla_{\sigma} F_{\tau} | d|\nabla_{\sigma} F_{\tau}| + d \left( |\nabla_{\sigma}^i F_{\tau} |, |\nabla_{\sigma}^j F_{\tau} | \right) \]
\[ = 2 |\nabla_{\sigma} F_{\tau} | d|\nabla_{\sigma} F_{\tau}| + 2 |\nabla_{\tau}^i |\nabla_{\tau}^j F_{\tau}|^2 \, d\tau \]
\[\leq 2|\nabla_\sigma^\| F_\tau|^2 d|\nabla_\sigma^\| F_\tau| + 2|\nabla_\tau^\| \nabla_\sigma^\| F_\tau|^2 d\tau. \quad (4.3)\]

Rearranging this inequality and applying (C1), we derive (C4), namely
\[d|\nabla_\sigma^\| F_\tau| \geq \langle \nabla_\tau^\| \nabla_\sigma^\| F_\tau|, dW_\tau \rangle + |\nabla_\sigma^\| F_\sigma| d\delta_\tau(\tau). \quad (4.4)\]

Finally, (C4) is satisfied if and only if \(F_\tau\) is a submartingale (cf. Theorem 3.2) (C5) also holds.

The remaining equivalencies will be proved in tandem with the results in the subsequent few theorems.

\[\square\]

4.2 Proof of gradient estimates for martingales

**Proof of Theorem 1.3** (C5) \(\implies\) (G1) \(\implies\) (G2): The implication of (G1) follows from the definition that if \(\tau \to |\nabla_\sigma^\| F_\tau|\) is a submartingale for every \(\sigma \geq 0\), then for \(\tilde{\tau} \geq \tau\), \(|\nabla_\sigma^\| F_\tilde{\tau}| \leq E\left[|\nabla_\sigma^\| F_\tilde{\tau}| \mid \Sigma_\tau\right]\). Finally, to prove (G2), apply (G1) and Cauchy-Schwarz to get
\[|\nabla_\sigma^\| F_\tilde{\tau}|^2 \leq \left(E\left[|\nabla_\sigma^\| F_\tilde{\tau}| \mid \Sigma_\tau\right]\right)^2 \leq E\left[|\nabla_\sigma^\| F_\tau|^2 \mid \Sigma_\tau\right] \cdot E[1 \mid \Sigma_\tau] = E\left[|\nabla_\sigma^\| F_\tau|^2 \mid \Sigma_\tau\right]. \quad (4.5)\]

The converse implications shall be proven along with later results.

\[\square\]

4.3 Proof of Hessian estimates for martingales

**Proof of Theorem 1.4** (C1) \(\implies\) (H1): To prove (H1), fix \(\sigma \geq 0\) and then integrate (C1) from 0 to \(T\) as well as take expectations
\[
E_{(x,T)}\left[|\nabla_\sigma^\| F|^2\right] - E_{(x,T)}\left[|\nabla_\sigma^\| F_\sigma|^2\right] \\
\overset{(C1)}{=} E_{(x,T)}\left[\int_0^T \langle \nabla_\tau^\| \nabla_\sigma^\| F_\tau|^2, dW_\tau \rangle \right] + 2E_{(x,T)}\left[\int_0^T |\nabla_\tau^\| \nabla_\sigma^\| F_\tau|^2 d\tau \right] \\
= 2E_{(x,T)}\left[\int_0^T |\nabla_\tau^\| \nabla_\tau^\| F_\tau|^2 d\tau \right]. \quad (4.6)\]

(H1) \(\implies\) (H2): To prove (H2), apply Ito isometry and then integrate (H1) from 0 to \(T\) with respect to \(\sigma\) as well as take expectations
\[
E_{(x,T)}\left[(F - E_{(x,T)}(F))^2\right] = E_{(x,T)}\left[\int_0^T |\nabla_\sigma^\| F_\sigma|^2 d\sigma \right] \\
\overset{(H1)}{\leq} E_{(x,T)}\left[\int_0^T |\nabla_\sigma^\| F|^2 d\sigma \right] \\
- 2E_{(x,T)}\left[\int_0^T \int_0^T |\nabla_\tau^\| \nabla_\sigma^\| F_\tau|^2 d\tau d\sigma \right]. \quad (4.7)\]

(R1) \(\implies\) (H3): To prove (H3), let \(G = F^2\) and consider the evolution equation for \(X_\tau := G_\tau^{-1}|\nabla^H G_\tau|^2 - 2G_\tau \log(G_\tau),\) which satisfies
\[
dx_\tau = \langle \nabla_\tau^\| X_\tau, dW_\tau \rangle + 2G_\tau \left(\int_0^T |\nabla_\tau^\| \nabla_\tau^\| \log(G_\tau)|^2 d\sigma \right) d\tau \\
+ G_\tau^{-1} \left(\int_0^T (\dot{g} + 2\Re e)_{\tau}(\nabla_\tau^\| F_\tau, \nabla_\tau^\| F_\tau) d\sigma \right) d\tau \]
\[ (4.8) \]

by Itô calculus and Proposition 3.3 (cf. Proposition 4.23 of [8]). Next, integrate the inequality (4.8) from 0 to \( T \) with respect to \( \tau \) and take expectations to get

\[
\mathbb{E}_{(x, \tau)}[X_T] - \mathbb{E}_{(x, \tau)}[X_0] \geq 2\mathbb{E}_{(x, \tau)} \left[ G_\tau \left( \int_0^T |\nabla_\tau G| \log(G_\tau)^2 \, d\sigma \right) d\tau \right],
\]

and evaluating the two expectations in the difference, namely

\[
\mathbb{E}_{(x, \tau)}[X_0] = \mathbb{E}_{(x, \tau)}[G_0^{-1}|\nabla^h G_0|^2 - 2G_0 \log(G_0)]
= 0 - 2G_0 \log(G_0)
= -2\mathbb{E}_{(x, \tau)}[F^2] \log(\mathbb{E}_{(x, \tau)}[F^2])
\]

and

\[
\mathbb{E}_{(x, \tau)}[X_T] = \mathbb{E}_{(x, \tau)} \left[ G^{-1}|\nabla^h G|^2 - 2G \log(G) \right]
= \mathbb{E}_{(x, \tau)} \left[ F^{-2}|\nabla^h F|^2 \right] - 2\mathbb{E}_{(x, \tau)} \left[ F^2 \log(F^2) \right]
= 4\mathbb{E}_{(x, \tau)} \left[ |\nabla^h F|^2 \right] - 2\mathbb{E}_{(x, \tau)} \left[ F^2 \log(F^2) \right].
\]

Finally we observe that

\[
\mathbb{E}_{(x, \tau)} \left[ |\nabla^h F|^2 \right] = \mathbb{E}_{(x, \tau)} \left[ \int_0^T |\nabla_\sigma F|^2 \, d\sigma \right],
\]

and then combine this and the aforementioned results to prove the claim. \( \square \)

### 4.4 Proof of the characterizations of solutions of the Ricci flow

The following result reproves a theorem by Haslhofer and Naber (cf. Theorem 1.22 of [7]), characterizing solutions of the Ricci flow, using the Bochner formulas on path space that were developed in the previous section.

**Proof of Theorem 1.5** (G1) \( \implies \) (R2): To prove (R2), we evaluate (G1) at \( \sigma = \tau = 0 \),

\[
|\nabla_\bar{x} \mathbb{E}_{(x, \tau)}[F]| = |\nabla_\bar{x} F_0| \overset{(G1)}{\leq} \mathbb{E}_{(x, \tau)} \left[ |\nabla_0^l F| \right] \Sigma_0 = \mathbb{E}_{(x, \tau)} \left[ |\nabla_0^l F| \right].
\]

(G2) \( \implies \) (R3): To prove (R3), we evaluate (G2) at \( \sigma, \tau = 0 \) (and observe that \( d[F, F]_\tau = 2|\nabla_\bar{x}^l F_\tau|^2 \, d\tau \) by Theorem 3.2),

\[
\mathbb{E}_{(x, \tau)} \left[ \frac{d[F, F]_\tau}{d\tau} \right] = 2\mathbb{E}_{(x, \tau)} \left[ |\nabla_\bar{x}^l F_\tau|^2 \right] \overset{(G2)}{\leq} 2\mathbb{E}_{(x, \tau)} \left[ \mathbb{E}_{(x, \tau)} \left[ |\nabla_\bar{x}^l F|^2 \right] \Sigma_\tau \right] \leq 2\mathbb{E}_{(x, \tau)} \left[ |\nabla_\bar{x}^l F|^2 \right].
\]

(G1) \( \implies \) (R4): To prove (R4), we set \( \sigma = \tau \) and take expectations

\[
\mathbb{E}_{(x, \tau)} \left[ |\nabla_\sigma^l F_\sigma| \right] \leq \mathbb{E}_{(x, \tau)} \left[ \mathbb{E}_{(x, \tau)} \left[ |\nabla_\sigma^l F| \Sigma_\sigma \right] \right] = \mathbb{E}_{(x, \tau)} \left[ |\nabla_\sigma^l F| \right].
\]
Then follow the proof of (H3) in Theorem 1.4 and evaluate the expectation $\mathbb{E}_{(x,T)}[X_{T_j}]$ for $j \in \{1, 2\}$, namely

$$\mathbb{E}_{(x,T)}[X_{T_j}] = \mathbb{E}_{(x,T)} \left[ G_{T_j}^{-1} |\nabla^H G_{T_j}|^2 \right] - 2 \mathbb{E}_{(x,T)} \left[ G_{T_j} \log(G_{T_j}) \right]$$

(4.16)

and taking differences, where $\mathbb{E}_{(x,T)}[X_{T_2} | \Sigma_{T_1}] - \mathbb{E}_{(x,T)}[X_{T_1} | \Sigma_{T_1}] \geq 0$, as in the earlier proof. It remains to check that

$$\mathbb{E}_{(x,T)} \left[ G_{T_2}^{-1} |\nabla^H G_{T_2}|^2 | \Sigma_{T_1} \right] - \mathbb{E}_{(x,T)} \left[ G_{T_1}^{-1} |\nabla^H G_{T_1}|^2 | \Sigma_{T_1} \right]$$

$$= 4 \mathbb{E}_{(x,T)} \left[ |\nabla^H F_{T_2}|^2 | \Sigma_{T_1} \right]$$

$$= 4 \mathbb{E}_{(x,T)} \left[ \int_{T_1}^{T_2} |\nabla^H F_{T_2}|^2 \, d\sigma \right]$$

$$\leq 4 \mathbb{E}_{(x,T)} \left[ \int_{T_1}^{T_2} |\nabla^H F|^2 \, d\sigma \right] \quad (\tau \to |\nabla^H F|^2 \text{ is a submartingale})$$

$$= 4 \mathbb{E}_{(x,T)} \left[ (F, \mathcal{L}(\tau_{T_2})F) \epsilon \right].$$

(4.17)

(G2) $\implies$ (R5): To prove (R5), we set $\sigma = \tau$ and take expectations

$$\mathbb{E}_{(x,T)} \left[ |\nabla^H F|^2 \right] \leq \mathbb{E}_{(x,T)} \left[ \mathbb{E}_{(x,T)} \left[ (F, \mathcal{L}(\tau_{T_2})F) \epsilon \right] \right].$$

(4.18)

Then follow the proof of (H2) in Theorem 1.4 and apply Itô isometry

$$\mathbb{E}_{(x,T)} \left[ (F_{T_2} - F_{T_1})^2 | \Sigma_{T_1} \right] = \mathbb{E}_{(x,T)} \left[ \int_0^T |\nabla^H F_{T_2}|^2 \, d\sigma \right]$$

$$\leq \mathbb{E}_{(x,T)} \left[ (F, \mathcal{L}(\tau_{T_2})F) \epsilon \right].$$

(4.19)

The converse implications shall be proven in the next section. 

\[ \square \]

### 4.5 Converse implications

We shall now prove the converse implications below.

**Proof** (C3) $\implies$ (R1): First fix $(x, T) \in \mathcal{M}$ and $v \in (T_x \mathcal{M}, g_T)$ a unit vector and choose a smooth compactly supported $f_1 : M \to \mathbb{R}$ such that

$$f_1(x) = 0, \quad \nabla f_1(x) = v, \quad \nabla^2 f_1(x) = 0$$

(4.20)

using exponential coordinates. Consider the one-point cylinder function given by $F(\gamma) = f_1(\pi_M(\gamma(\epsilon)))$, $F : P_{(x,T)} \mathcal{M} \to \mathbb{R}$ and observe for $\tau \leq \epsilon$ that

$$\nabla^H F_{\tau} = P_{\tau} \nabla H_{T - \tau, T} f_1(\pi_M(\gamma(\epsilon))), \quad |\nabla^H F_{\tau}|^2 = |\nabla^2 H_{T - \tau, T} f_1(\pi_M(\gamma(\epsilon)))|^2.$$  

(4.21)

In particular, $\nabla^H F_{\tau} = v + o(\epsilon)$ and $|\nabla^H F_{\tau}| = o(\epsilon)$. Then, by Theorem 3.4,

$$\tau \to |\nabla^H F_{\tau}|^2 - \int_0^\tau \left( 2 |\nabla^H F_{\rho}|^2 + (\dot{g} + 2 \text{Rc})_\rho (\nabla^H F_{\rho}, \nabla^H F_{\rho}) \right) \, d\rho$$

(4.22)

is a martingale. So, in particular,

$$|\nabla^H F_{T}|^2 = \mathbb{E} \left[ |\nabla^H F_{T}|^2 \right] - \epsilon (\dot{g} + 2 \text{Rc})_\epsilon (v, v) + o(\epsilon).$$

(4.23)
Moreover, since $\tau \to |\nabla^{|F_\tau|}_0|^2$ is a submartingale by (C3), it follows that
\[(\dot{g} + 2\text{Re})_\varepsilon(v, v) \geq \varepsilon^{-1} o(\varepsilon). \tag{4.24}\]

Next choose a smooth compactly supported $f_2 : M \times M \to \mathbb{R}$ such that
\[f_2(x, x) = 0, \quad \nabla^{(1)} f_2(x, x) = 2v, \quad \nabla^{(2)} f_2(x, x) = -v, \quad \nabla^2 f_2(x, x) = 0, \tag{4.25}\]
for example $f_2(y, z) = 2f_1(y) - f_1(z)$. Consider the two-point cylinder function given by $F(\gamma) = f_2(\pi_M(\gamma(0)), \pi_M(\gamma(\tau)))$, $F : P_{(x, \tau)} M \to \mathbb{R}$ and observe for $\tau \leq \varepsilon$ that
\[
\begin{align*}
\nabla^{|F_\tau|}_0 F_\tau &= \nabla^{(1)} f_2(x, \pi_M(\gamma(\tau))) + P_\tau \nabla H^{(2)}_{T-T, \tau} f_2(x, \pi_M(\gamma(\tau))) \\
\nabla^{|F_\tau|}_0 F_\tau &= P_\tau \nabla H^{(2)}_{T-T, \tau} f_2(x, \pi_M(\gamma(\tau))) \\
|\nabla^{|F_\tau|}_0 F_\tau| &\leq |\nabla^2 f_2(x, \pi_M(\gamma(\tau))) + |\nabla^2 H^{(2)}_{T-T, \tau} f_2(x, \pi_M(\gamma(\tau)))|.
\end{align*}
\]
In particular, $\nabla^{|F_\tau|}_0 F_\tau = v + o(\varepsilon), \quad \nabla^{|F_\tau|}_0 F_\tau = -v + o(\varepsilon)$ and $|\nabla^{|F_\tau|}_0 F_\tau| = o(\varepsilon)$. Then, again by Theorem 3.4,
\[
|\nabla^{|F_\tau|}_0 F_\tau|^2 = \mathbb{E} \left[ |\nabla^{|F_\tau|}_0 F_\tau|^2 \right] + \tau (\dot{g} + 2\text{Re})_\varepsilon(v, v) + o(\varepsilon). \tag{4.27}
\]
Moreover, since $\tau \to |\nabla^{|F_\tau|}_0 F_\tau|^2$ is a submartingale by (C3), it follows that
\[(\dot{g} + 2\text{Re})_\varepsilon(v, v) \leq \tau^{-1} o(\varepsilon). \tag{4.28}\]
We can then deduce that (R1) is satisfied by taking $\varepsilon \to 0^+$ in Eqs. (4.24) and (4.28).

To check the remaining converse implications, one can substitute 1-point and 2-point cylinder functions as above. However, there are some alternative tools that can close the loop of equivalences more readily. For example, applying the log-Sobolev equality to $F^2 = 1 + \varepsilon G$ in (R4) gives the Poincaré inequality in (R5). Moreover, dividing by $T - \tau$ and taking $T - \tau \to 0^+$ and using the quadratic variation $d[F, F]_\tau = 2|\nabla^{|F_\tau|}_0 F_\tau|^2 d\tau$ (by Theorem 3.2), (R3) can be derived from (R5). In short, some implications can be done directly without the need to appeal to test functions each time.

\section*{Acknowledgements}
The author has been supported by the Ontario Graduate Scholarship and he acknowledges his supervisor Robert Haslhofer for his invaluable guidance and support in bringing this paper into fruition.

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