A CONDITION FOR WEAK MIXING
OF INDUCED IETS

MICHAEL BOSHERNITZAN

Abstract. Let \( f : X \rightarrow X, X = [0, 1), \) be an ergodic IET (interval exchange transformation) relative to the Lebesgue measure on \( X. \) Denote by \( f_t : X_t \rightarrow X_t \) the IET obtained by inducing \( f \) to the subinterval \( X = [0, t), 0 < t < 1. \) We show that

\[ X_{wm} = \{ 0 < t < 1 \mid f_t \text{ is weakly mixing} \} \]

is a residual subset of \( X \) of full Lebesgue measure. The result is proved by establishing a generic Diophantine sufficient condition on \( t \) for \( f_t \) to be weakly mixing.

1. IETs: minimality, ergodicity and mixing

Denote by \( \lambda \) the Lebesgue measure on the real line \( \mathbb{R}. \) Denote by \( \mathbb{Z}, \mathbb{N} = \{ k \in \mathbb{Z} \mid k \geq 1 \} \) the sets of integers and of natural numbers, respectively. We write \( \sharp(S) \) for the cardinality of a set \( S. \)

An IET (interval exchange transformation) is a pair \((X, f)\) where \( X = [0, b) \subset \mathbb{R} \) is a bounded interval, and \( f \) is a right continuous bijection \( f : X \rightarrow X \) of it with a finite set \( D \) of discontinuities and such that \( f'(x) = 1 \) for all \( x \in X \setminus D. \) (The last conditions means that \( f \) is a local translation at every its continuity point). We often refer to the map \( f \) itself as an IET.

Let \( \sharp(D) = r - 1, r \geq 2. \) Then without loss of generality

\[ D = \{ d_k \}_{k=1}^{r-1}; \quad d_0 = 0 < d_1 < \ldots < d_{r-1} < b = d_r. \]

(The conventions \( d_0 = 0, d_r = b \) are used for convenience. Note that \( 0, b \notin D. \))

An IET \((X, f)\) (or \( f \)) with \( \sharp(D) = r - 1 \) is also called (more specifically) an \( r \)-IET referring to the fact that \( f \) exchanges the \( r \) intervals \( X_k = [d_k-1, d_k) \subset X, 1 \leq k \leq r, \) according to some permutation \( \rho \in S_r. \)

An \( r \)-IET is completely determined by this permutation \( \rho \in S_r \) and the lengths \( \lambda_k = d_k - d_{k-1} > 0, 1 \leq k \leq r, \) of exchanged subintervals \( X_k. \) Thus \( r \)-IETs can be identified with pairs \((\bar{\lambda}, \rho)\) where \( \bar{\lambda} \in (\mathbb{R}^+)^r \) and \( \rho \in S_r: (X, f) = (\bar{\lambda}, \rho). \)

A permutation \( \rho \in S_r \) is called irreducible if \( \rho(\{1, 2, \ldots, k\}) \neq (\{1, 2, \ldots, k\}), \) for all \( k < r. \)

An IET \((X, f)\) is called minimal if all \( f \)-orbits are dense in \( X. \) Note that the irreducibility of \( \rho \) is a necessary condition for \( f \) to be minimal because otherwise \( X \) splits into two \( f \)-invariant subintervals.

Keane \[16\] proved that if \( r \geq 2 \) and if \( \rho \in S_r \) is irreducible then the IET \((\bar{\lambda}, \rho)\) is minimal provided that the lengths \( \lambda_k \) of exchanged intervals are linearly independent over the rationals (a generic assumption on \( \bar{\lambda} \)).
Masur [21] and Veech [26] independently proved the following result (conjectured by Keane in [16]): If $r \geq 2$ and $\rho \in S_r$ is irreducible, then for Lebesgue almost all $\vec{\lambda} \in (\mathbb{R}^+)^r$ the IET $(\vec{\lambda}, \rho)$ is uniquely ergodic (all its orbits are uniformly distributed). Alternative approaches to Keane’s conjecture were later given by Rees [22], Kerchoff [20] and Boshernitzan [3]. (Boshernitzan exhibited some Diophantine conditions, including a generic one, Property P, for unique ergodicity of IETs, see [3], [5] and [28], an improvement by Veech. The idea to use a suitable generic Diophantine condition to establish a metric result is also central in the present paper).

Avila and Forni [1] proved that Lebesgue almost all IETs are weakly mixing (assuming that $\rho$ is irreducible and not a rotation). Partial results in this directions were obtained earlier by Katok and Stepin [15] who established weak mixing of generic 3-IETs and Veech [27] who proved generic weak mixing for a large class of permutations (Veech permutations). It was later shown by Boshernitzan and Nogueira [8] that if an IET $f = (\vec{\lambda}, \rho)$ satisfies property P (a generic condition used in [3]), or even a weaker condition [28], and if $\rho$ is a Veech permutation, then $f$ is weakly mixing.

Note that Katok proved that IETs are never (strongly) mixing [13]. On the other hand, Chaika [9] constructed a 4-IET which is topologically mixing. Boshernitzan and Chaika [7] showed that 3-IETs are never topologically mixing.

2. The results.

In what follows let $(X, f)$ be a fixed aperiodic $r$-IET, $X = [0, b)$, $r \geq 2$. (Aperiodicity of $f$ means absence of $f$-periodic points).

Denote by $X_t$, $0 < t < b$, the subinterval $[0, t) \subset X$ and by $f_t$ the IET obtained by inducing $f$ to $X_t$. It is well known that each $(X_t, f_t)$ is an $s$-IET with $s = s(t) \leq n + 1$ (see [10] or [16]). (In fact, $s(t) \geq 2$ due to the aperiodicity assumption).

The central result of the paper is the following. Recall that $\lambda$ stands for the Lebesgue measure on $X$.

**Theorem 1.** Let $(X, f)$ be a $\lambda$-ergodic IET, $X = [0, b)$. Then the set

$$(2.1a) \quad X_{wm} = X_{wm}(\lambda) = \{0 < t < b \mid f_t \text{ is weakly mixing (relative $\lambda$)}\}$$

is a residual set of full measure: $\lambda(X_{wm}) = 1$.

Because of the above result, we refer to the complement set

$$(2.1b) \quad X_{nwm} = (0, b) \backslash X_{wm}$$

as “the exceptional set for (weakly mixing induction of) $(X, f)$”.

**Remark 1.** Two additional versions of Theorem [11] (for minimal but not uniquely ergodic IETs) are given in Section [7].

We need some notation. Recall that $D = \{d_k\}_{k=1}^{r-1}$ stands for the set of discontinuities of $f$. Denote by

$D_0 = D \cup \{0, b\} = \{d_k\}_{k=0}^{r}$
the set of \((r+1)\) points in (1.1), and denote by
\[ D' = \bigcup_{k=-\infty}^{\infty} f^k(D) \]
the set of points whose orbits hit \(D\).

The aperiodicity of \(f\) implies that \(D'\) is a dense countable subset of \(X = [0,b)\) containing 0 (see e.g. [5, Section 2]).

For \(x \in X\) and \(n \geq 1\), we set
\[ \rho(x) = \text{dist}(x, D_0) = \min_{0 \leq k \leq r} |x - d_k| \]
\[ \rho_n(x) = \min_{-n \leq k \leq n-1} \rho(f^k(x)) \]
\[ \Delta_n(x) = \min_{|p|, |q| \leq n} |f^p(x) - f^q(x)| \]
and
\[ \rho'_n(x) = \min(\rho_n(x), \frac{1}{2}\Delta_n(x)). \]

Note that for all \(x \in D'\) both \(\rho_n(x)\) and \(\rho'_n(x)\) vanish for large \(n\), while for \(x \in X \setminus D'\) we have \(\rho_n(x) \geq \rho'_n(x) > 0\) due to the aperiodicity of \(f\).

Important interpretations of the values \(\rho_n(x)\) and \(\rho'_n(x)\) are given by Propositions 1 and 2 in the next section. The following two functions
\[ \phi: X \to \mathbb{R}; \quad \phi(x) = \phi_f(x) = \limsup_{n \to \infty} n\rho_n(x); \]
and
\[ \psi: X \to \mathbb{R}; \quad \psi(x) = \psi_f(x) = \limsup_{n \to \infty} n\rho'_n(x); \]
will play central role in the paper. (We usually suppress the dependence on \(f\) by writing \(\phi, \psi\) rather than \(\phi_f, \psi_f\) due to the standing assumption that \((X,f)\) is a fixed aperiodic IET).

**Theorem 2 (A sufficient condition for weak mixing of \(f_t\)).** Let \(f: X \to X\) be a \(\lambda\)-ergodic IET, \(X = [0,b)\). If \(\psi(t) > 0\) for some \(t \in (0,b)\), then the induced IET \(f_t: X_t \to X_t, X_t = [0,t)\), is weakly mixing.

For an aperiodic IET \(f: X \to X, X = [0,b)\), the set
\[ X_{\psi=0} = \{ t \in (0,b) | \psi(t) = 0 \} \]
will be referred as “the critical set for \((X,f)\)”. We also adopt similar notation
\[ X_{\psi>0} = (0,b) \setminus X_{\psi=0} = \{ t \in (0,b) | \psi(t) > 0 \} \]
for the complement of this set.

Theorem 2 claims that for an \(\lambda\)-ergodic IET \((X,f)\) the inclusion \(X_{\text{nwm}} \subset X_{\psi=0}\) takes place. (In other words, every exceptional point must be critical, see (2.1b) and (2.5b)).

Equivalently, \(X_{\psi>0} \subset X_{\text{nwm}}\) (see (2.1a) and (2.5a)).

The importance of Theorem 2 is twofold. First, it provides a generic condition (Theorem 3) sufficient to establish Theorem 1 claiming that the exceptional set \(X_{\text{nwm}}\) is small in both measure and topology categories.
Secondly, it allows (under certain Diophantine conditions on $t$ and $f$) to establish more
delicate information on the “smallness” of the exceptional set $X_{nwm}$. In particular, there are
examples of ergodic IETs $(X, f)$ for which one can show that the critical set $X_{\psi=0}$ coincide
with $D'$, and hence $X_{nwm} \subset D'$ is at most countable (see Section 8).

An IET $(X, f)$ is called persistently weakly mixing if $X_{nwm} = \emptyset$. It is easy to see that there
are no persistently weakly mixing $r$-IETs with $r < 4$ (because then $f_t$ become rotations for
a countable set of $t$).

We believe that there exist persistently weakly mixing IETs.

**Question.** What is the “size” (in the sense of measure, category and cardinality) of $X_{nwm}$
for “most” 4-IETs with permutation $\rho = (4321)$?

The answers for the same questions for $r$-IETs with $r = 2$ or 3 are known (see Section 8
for the answers without proofs).

**Theorem 3.** Let $f : X \to X$ be an aperiodic IET, $X = [0, b)$. Then the critical set $X_{\psi=0}$ is
meager and has Lebesgue measure 0.

Theorem 1 follows immediately from Theorems 2 and 3, the proofs of which are presented
in Sections 5 and 4, respectively.

3. Some notation, terminology and lemmas.

The discussion in this section continues under the assumption that $(X, f)$ is a fixed ape-
periodic $r$-IET, $X = [0, b)$, $r \geq 2$. Note that $f^{-1}$ (the compositional inverse of $f$) is also an
aperiodic $r$-IET on $X$.

**Definition 1.** An open subinterval $Y \subset X$ is called $f$-basic if the following equivalent
conditions are met:

(b1) $f|_Y$ is a translation;
(b2) $f|_Y$ is continuous;
(b3) $Y \cap D = \emptyset$.

Given an $f$-basic interval $Y$, we write $Y^+(f)$ for the translation constant $f|_Y(y) - y$.

Observe that if an interval $Y \subset X$ is $f$-basic then $f(Y)$ is an $f^{-1}$-basic interval.

**Definition 2.** A sequence $\vec{Y} = (Y_k)_{k=1}^n$ of subsets of $X$ is called an $f$-stack if the following conditions are met:

(s1) Each of the sets $Y_k$, $1 \leq k \leq n - 1$, is an $f$-basic interval;
(s2) $f(Y_k) = Y_{k+1}$, for $1 \leq k \leq n - 1$.

An $f$-stack $\vec{Y} = (Y_k)_{k=1}^n$ is called distinct if the sets $Y_k$ are pairwise disjoint. Given an
$f$-stack $\vec{Y} = (Y_k)_{k=1}^n$, we use the following terminology:

- The width of $\vec{Y}$: $\omega(\vec{Y}) = \lambda(Y_1)$ (in fact, all $\lambda(Y_k)$ are equal);
- The support of $\vec{Y}$: $\text{supp}(\vec{Y}) = \bigcup_{k=1}^n Y_k \subset X$;
- The length of $\vec{Y}$: $h(\vec{Y}) = n$;
- The measure of $\vec{Y}$: $\lambda(\vec{Y}) = \lambda(\text{supp}(\vec{Y}))$.

Note that if $\vec{Y} = (Y_k)_{k=1}^n$ is a distinct $f$-stack, then $\lambda(\vec{Y}) = \omega(\vec{Y})h(\vec{Y})$. 
Observe that if \((Y_k)_{k=1}^n\) is an \(f\)-stack then the inverted sequence \((Y_{n+1-k})_{k=1}^n\) forms an \(f^{-1}\)-stack; in particular, the last set \(Y_n\) must also be an open subinterval of \(X\) (but not necessarily an \(f\)-basic one).

For \(x \in \mathbb{R}\) and \(\varepsilon > 0\), denote by \(B_\varepsilon(x) = (x - \varepsilon, x + \varepsilon)\) the \(\varepsilon\)-neighborhood of \(x \in \mathbb{R}\).

**Proposition 1.** Let \((X,T)\) be an aperiodic IET. For \(\varepsilon > 0\), \(x \in X \setminus D'\) and \(n \geq 1\), the following three conditions are equivalent:

1. The sequence of intervals \(\left(B_\varepsilon(T^k(x))\right)_{k=-n}^n\) forms an \(f\)-stack (of length \((2n+1)\)).
2. \(\varepsilon \leq \rho(x)\).
3. There exists an \(f\)-stack \(Z_k\) with \(Z_0 = B_\varepsilon(x)\).

**Proof.** Follows from the definition of \(\rho_n(x)\) (see (2.3a)).

**Proposition 2.** Under the assumptions and notations as in Proposition 1, assume that the equivalent conditions (1a), (1b) and (2c) hold. Then the following three conditions are equivalent:

1. The \(f\)-stack \(\left(B_\varepsilon(T^k(x))\right)_{k=-n}^n\) is distinct;
2. \(\varepsilon \leq \rho_n(x)\).
3. There exists a distinct \(f\)-stack \((Z_k)_{k=-n}^n\) with \(Z_0 = B_\varepsilon(x)\).

**Proof.** Follows from the definition of \(\rho'_n(x)\) (see (2.3d)).

The following lemma will be used in the proof of Theorem 3. (Various versions of it are well known).

**Lemma 1.** Let \((X,f)\) be a minimal \(r\)-IET, \(X = [0,b)\). Then for every \(N\), there exists a distinct \(f\)-stack \(\tilde{Y}\) of length at least \(N\) and of measure at least \(\frac{b}{r^N}\).

**Remark.** The above lemma holds under the weaker assumption that \((X,f)\) is aperiodic (rather than minimal). We do not use the stronger version, and its proof is not included.

**Proof of Lemma 1.** Pick a small subinterval \(Y \subset X\), \(0 < \lambda(Y) < \frac{b}{r^N}\), so that the induced map \(g\) on \(Y\) is an \(s\)-IET, with some \(2 \leq s \leq r\). Let \(Y_k \subset Y\), \(1 \leq k \leq s\), be the subintervals of \(Y\) exchanged by \(g\): \(g(Y_k) = f^{n_k}(Y_k)\). By minimality, the images \(f^n(Y_k)\) cover \(X = [0,b)\) before returning to \(Y\).

More precisely, the family of subintervals \(\{f^n(Y_k) \mid 1 \leq k \leq s, 0 \leq n \leq n_k\}\) partitions the interval \(X = [0,b)\). Thus \(\lambda\left(\bigcup_{n=0}^{n_k} f^n(Y_k)\right) \geq \frac{b}{s} \geq \frac{b}{r^N}\), for some \(k \in [1,s]\). To satisfy the conditions of the Lemma 1, one takes \(\tilde{Y} = \left(f^n(Y_k)\right)_{n=0}^{n_k}\).

We would need the following notation. For an open finite interval \(Y\), denote by \(\Theta(Y)\) the middle third subinterval of \(Y\) defined as the interval with the same center but 3 times shorter:

\[
(3.1) \quad \Theta(B_\varepsilon(x)) = B_{\varepsilon/3}(x); \quad \Theta((a,b)) = \left(\frac{2a+b}{3}, \frac{a+2b}{3}\right); \quad \lambda(\Theta(Y)) = \frac{1}{3} \lambda(Y).
\]
4. PROOF OF THEOREM 3

One easily validates $f$-invariance of $\psi$: $\psi(x) = \psi(f(x))$, so the Borel $f$-invariant set $X_{\psi>0}$ must have Lebesgue measure either 0 or 1, in view of the ergodicity of $f$.

Since $(X, f)$ is an ergodic IETs, it is minimal, so Lemma 1 applies. It follows that there exists a sequence $(\vec{Y}_n)_{n \geq 1}$ of distinct $f$-stacks with lengths $h(n) = h(\vec{Y}_n)$ approaching infinity and measures $\lambda(\vec{Y}_n) \geq \frac{b}{f}$ (see Definition 2). Let

$$\vec{Y}_n = (Y_{n,k})_{k=1}^{h(n)} = (Y_{n,1}, Y_{n,2}, \ldots, Y_{n,h(n)}), \quad n \in \mathbb{N}.$$  

We may assume that all $h(n) \geq 6$. Let $p(n) = \left\lceil \frac{h(n)}{3} \right\rceil$ and $q(n) = h(n) - p(n) + 1$.

Consider the following sequence of distinct $f$-stacks $(\vec{Z}_n)_{n \geq 1}$:

$$\vec{Z}_n = \left( \Theta(Y_{n,k}) \right)_{k=p(n)+1}^{q(n)-1} = \left( \Theta(Y_{n,p(n)+1}), \Theta(Y_{n,p(n)+2}), \ldots, \Theta(Y_{n,q(n)-1}) \right), \quad n \in \mathbb{N},$$

and set

$$Z_n = \text{supp}(\vec{Z}_n) \subset X, \quad n \in \mathbb{N},$$

(see Definition 2 for notation) and

$$Z = \bigcap_{n=1}^{\infty} \left( \bigcup_{k=n}^{\infty} Z_k \right) = \{ z \in X \mid z \in Z_n, \text{ for infinitely many } n \geq 1 \}.$$  

The set $Z$ is clearly a residual subset of $X$. (In fact, it is a dense $G_\delta$ subset of $X$).

Observe the following inequalities for the length and the width of $\vec{Z}_n$:

$$h(\vec{Z}_n) = q(n) - p(n) - 1 \geq \frac{h(n)}{3}, \quad \omega(\vec{Z}_n) = \frac{1}{3} \omega(\vec{Y}_n) \quad (n \in \mathbb{N}).$$

It follows that

$$\lambda(Z_n) = \lambda(\vec{Z}_n) \geq \frac{1}{9} \lambda(\vec{Y}_n) \geq \frac{b}{9} \quad (n \in \mathbb{N}),$$

and therefore

$$\lambda(Z) \geq \limsup_{n \to \infty} \lambda(\vec{Z}_n) \geq \frac{b}{9}.$$  

**Lemma 2.** $\psi(z) > 0$ for all $z \in Z$.

Since the function $\psi$ is easily seen to be Borel measurable and $f$-invariant ($\psi(x) = \psi(f(x))$, for all $x \in X$), the ergodicity of $f$ implies that $\lambda(X_{\psi>0}) \in \{0, 1\}$. The proof of Lemma 2 would imply that $\lambda(X_{\psi>0}) = 1$, completing the proof of Theorem 3.

**Proof of Lemma 2.** Let $z \in Z$. Then there exists an increasing sequence of positive integers $(n_i)_{i=1}^\infty$ such that $z \in Z_{n_i}$ where

$$Z_{n_i} = \text{supp}(\vec{Z}_{n_i}) = \bigcup_{k=p(n_i)+1}^{q(n_i)-1} \Theta(Y_{n_i,k}) \subset \text{supp}(\vec{Y}_{n_i}) = \bigcup_{k=1}^{h(n_i)} Y_{n_i,k}, \quad \text{for all } i \geq 1.$$  

Set $\varepsilon_i := \frac{1}{2} \omega(\vec{Z}_{n_i}) = \frac{1}{6} \omega(\vec{Y}_{n_i})$. Note that for every $i \geq 1$ the sequence

$$\vec{B}_i := \left( f^k(B_{\varepsilon}(z)) \right)_{k=-p(n_i)}^{p(n_i)}$$
forms a distinct $f$-stack due to the fact that $\bar{Y}_{n_i}$ does. By Proposition 2, $\varepsilon_i \leq \rho'_{p(n_i)}(z)$.

Since $\lim_{i \to \infty} n_i = \infty$, we get both $\lim_{i \to \infty} h(n_i) = \infty$ and $\lim_{i \to \infty} p(n_i) = \infty$. One concludes that for all large $i$:

$$\rho'_{p(n_i)}(z) \cdot p(n_i) \geq \varepsilon_i \cdot h(n_i) \cdot \frac{p(n_i)}{h(n_i)} = \frac{1}{6} \cdot \lambda(\bar{Y}_{n_i}) \cdot \frac{p(n_i)}{h(n_i)} > \frac{1}{6} \cdot \frac{1}{\varepsilon_i} = \frac{h}{24\varepsilon}.$$  

The proof of Lemma 2 (and hence of Theorem 3) is completed by direct estimation (see (2.4b)):

$$\psi(z) = \limsup_{n \to \infty} n \rho'(n) \geq \frac{h}{24\varepsilon} > 0. \quad \Box$$

5. Proof of Theorem 2

Denote by $S^1 = \{z \in \mathbb{C} \mid |z| = 1\} = \{e^{it} \mid t \in [0, 2\pi)\}$ the unit circle in the complex plane.

Theorem 2 is derived from the following proposition.

**Proposition 3.** Let $(X, f)$ be an aperiodic IET, $X = [0, b]$. Let $1 \neq \theta \in S^1$. Assume that $\psi(t) > 0$, for some $t \in (0, b)$. Then the equation

$$F(f(x)) = \begin{cases} \theta \cdot F(x) & \text{if } x < t \\ F(x) & \text{if } x \geq t \end{cases}$$

has no (Lebesgue) measurable solutions $F : X \to S^1$.

**Proof of Theorem 2.** Since $(X, f)$ is ergodic, all induced maps $(X_t, f_t)$ also are. If some IET $(X_t, f_t)$ fails to be weakly mixing, it has a nontrivial eigenvalue $\theta \in S^1$, $\theta \neq 1$. Select an eigenfunction $G : X_t \to \mathbb{C}$ corresponding to $\theta$ so that $G(f(x)) = \theta \cdot G(x)$, for $x \in X_t$. The ergodicity of $f_t$ implies that $|G|$ must be a constant which (without loss of generality) is assumed to be 1. Thus $G(X_t) \subset S^1$.

Let $g = f^{-1}$ be the compositional inverse of $f$. Define $F(x) = G(g^{k(x)}(x))$ with $k(x) = \min \left( N_t(x) \right)$ where the set $N_t(x) = \{k \geq 0 \mid g^k(x) \in X_t\}$ is not empty because both $f, f^{-1}$ are ergodic and hence minimal. The constructed function $F : X \to S^1$ is measurable (because $G$ is) and is easily seen to satisfy (5.1). This contradicts the conclusion of Proposition 3 completing the proof of Theorem 2. \Box

6. Proof of Proposition 3

The proof goes by contradiction. Assume to the contrary that there are $\theta$ and $F$ satisfying the conditions of Proposition 3 and that, in particular, (5.1) holds for some $t \in (0, b)$ such that

$$\psi(t) = \limsup_{n \to \infty} n \rho'_n(t) > 0.$$  

Set $\varepsilon_k = \frac{\psi(t)}{2k}$ for $k \geq 1$. Then there exists an infinite subset $\mathbb{M} \subset \mathbb{N}$ of natural numbers such that $\rho'_n(t) > \varepsilon_n > 0$ for all $n \in \mathbb{M}$.

The statement of the following lemma follows from Proposition 2.
Lemma 3. The sequence \((B_n(f^k(t)))_{k=-n}^n\) forms a distinct f-stack for every integer \(n \in \mathbb{M}\).

Some notation. For \(n \in \mathbb{M}\) and \(|k| \leq n\), set the following open subintervals of \(X\):

\[
\begin{align*}
A^k_n &= (f^k(t) - \varepsilon_n, f^k(t) + \varepsilon_n) = B_{\varepsilon_n}(f^k(t)); \\
B^k_n &= (f^k(t), f^k(t) + \varepsilon_n); \\
C^k_n &= (f^k(t) - \varepsilon_n, f^k(t)).
\end{align*}
\]

Set the constants (all lying in \(D_1 = \{z \in \mathbb{C} \mid |z| \leq 1\}\)):

\[
\alpha^k_n = \mathcal{A}(A^k_n), \quad \beta^k_n = \mathcal{A}(B^k_n), \quad \gamma^k_n = \mathcal{A}(C^k_n),
\]

where

\[
\mathcal{A}(Y) = \frac{1}{\lambda(Y)} \int_Y F(x) \, dx
\]

stands for the average of the function \(F\) over a subinterval \(Y \subset X\).

Since \(F\) is \(S^1\)-valued, the constants in (6.2) lie in the unit disc \(D_1 = \{z \in \mathbb{C} \mid |z| \leq 1\}\).

By a passing to an infinite subset of \(\mathbb{M}\), we may assume that the following six limits

\[
\alpha^i = \lim_{n \to \infty} \alpha^i_n, \quad \beta^i = \lim_{n \to \infty} \beta^i_n, \quad \gamma^i = \lim_{n \to \infty} \gamma^i_n, \quad i \in \{0, 1\}
\]

exist and lie in \(D_1\). We show that in fact all six constants \(\alpha^1, \alpha^0, \beta^1, \beta^0, \gamma^1\) and \(\gamma^0\) must lie in the unit circle \(S^1 = \partial D_1\) (see Lemma 3 below). This will follow from Lemma 4 below.

Recall that, given an f-stack \(\vec{Y}\), we write \(h(\vec{Y})\) and \(\lambda(\vec{Y})\) for the length and the measure of \(\vec{Y}\) (see Definition 2).

Lemma 4. Let \((X, f)\) be an aperiodic IET, \(X = [0, b]\). Let \(\theta \in S^1\) and \(t \in (a, b)\). Assume that a measurable function \(F : X \to S^1\) satisfies the equation (5.1). Let \((\vec{Y}_n)_{n=1}^\infty\) be a sequence of distinct f-stacks

\[
\vec{Y}_n = (Y_{n,k})_{k=1}^h_n = (Y_{n,1}, Y_{n,2}, \ldots, Y_{n,h_n}),
\]

satisfying the following two conditions:

(1A) \(\lim_{n \to \infty} h_n = \infty\) (the lengths of stacks \(h_n = h(\vec{Y}_n)\) approach infinity);

(1B) \(\liminf_{n \to \infty} \lambda(\vec{Y}_n) > 0\) (the measures of stacks \(\lambda(\vec{Y}_n) = \sum_{k=1}^{h_n} \lambda(Y_{n,k})\) stay away from 0);

(1C) \(t \notin Y_{n,k}, \text{ for all } n \text{ and } k \in [1, h_n]\).

Then \(\lim_{n \to \infty} |\mathcal{A}(Y_{n,1})| = 1\) (for notation see (6.3)).

Remark. In the above lemma, under the condition (1C) alone (i.e., without assuming (1A) and (1B)) we obviously have

\[
|\mathcal{A}(Y_{n,1})| = |\mathcal{A}(Y_{n,2})| = \ldots = |\mathcal{A}(Y_{n,h_n})|, \quad \text{for all } n \geq 1,
\]

in view of the equation (5.1) the function \(F\) satisfies. In particular, the conclusion \(\lim_{n \to \infty} |\mathcal{A}(Y_{n,1})| = 1\) in Lemma 4 is equivalent to the relation \(\lim_{n \to \infty} |\mathcal{A}(Y_{n,h_n})| = 1\).
Proof of Lemma 4. For subintervals $Y \subset X$, set $B(Y) = \int_Y |F(x) - A(Y)| \, dx$ where $A(Y)$ stands for the average of $F$ over $Y$ (see (6.3)). Then $\lim_{n \to \infty} \left( \sum_{k=1}^{h_n} B(Y_{n,k}) \right) = 0$ and $\lambda(Y_{n,1}) = \lambda(Y_{n,2}) = \ldots = \lambda(Y_{n,h_n}) \leq \frac{1}{h_n} \to 0$ (as $n \to \infty$), because of the assumption (4A). Since $F$ satisfies (5.1), we have $B(Y_{n,1}) = B(Y_{n,2}) = \ldots = B(Y_{n,h_n})$, and hence $\lim_{n \to \infty} B(Y_{n,1}) \cdot h_n = 0$. (Here we use the assumption (4C)). It follows that $\lim_{n \to \infty} \left( \frac{1}{\lambda(Y_{n,1})} \cdot B(Y_{n,1}) \right) = \lim_{n \to \infty} \left( \frac{1}{\lambda(Y_{n,1})} \cdot B(Y_{n,1}) \cdot \lambda(Y_n) \right) = 0$.

In view of the assumption (4B), we conclude that

$$\lim_{n \to \infty} \frac{1}{\lambda(Y_{n,1})} \cdot B(Y_{n,1}) = \lim_{n \to \infty} \frac{1}{\lambda(Y_{n,1})} \int_{Y_{n,1}} |F(x) - A(Y_{n,1})| \, dx = 0.$$

Let $\varepsilon > 0$ be given. Then for all sufficiently large $n$, one can select $x_n \in Y_{n,1}$ so that $F(x_n) \in S^1$ and $|F(x_n) - A(Y_{n,1})| < \varepsilon$. This implies that $|A(Y_{n,1}) - 1| < \varepsilon$, and, since $\varepsilon > 0$ is arbitrary, $\lim_{n \to \infty} |A(Y_{n,1})| = 1$, completing the proof of Lemma 4.

Lemma 5. The six constants $\alpha^1, \alpha^0, \beta^1, \beta^0, \gamma^1$ and $\gamma^0$ (see (6.2)) lie in $S^1$.

Proof. Case of $\alpha^1$. Recall that $\alpha^1_n = \lim_{n \in \mathbb{N}} \alpha^1_n$ where $\alpha^1_n = A(A^1_n)$. We claim that the conditions of Lemma 4 are fulfilled with $(\tilde{Y}_n)_{n \geq 1} = \left( (A^k_n)_{k=1}^n \right)_{n \in \mathbb{N}}$. Indeed, in this case $\tilde{Y}$ is a distinct $f$-stack in view of Lemma 3. We also have

$$h(\tilde{Y}_n) = n \quad \text{and} \quad \lambda(\tilde{Y}_n) = 2\varepsilon_n h(\tilde{Y}_n) = 2\varepsilon_n n = \psi(t) = 2\varepsilon_1$$

for $n \in \mathbb{N}$. It follows from Lemma 4 that $|\alpha^1| = 1$.

Case of $\alpha^0$. Similar argument. We set $(\tilde{Y}_n)_{n \geq 1} = \left( (A^0_n)_{k=-n+1}^n \right)_{n \in \mathbb{N}}$ and take in account remark following Lemma 4 to get $|\alpha^0| = 1$.

Case of $\beta^1$. We set $(\tilde{Y}_n)_{n \geq 1} = \left( (B^k_n)_{k=1}^n \right)_{n \in \mathbb{N}}$ and in the same way apply Lemma 3 to get $|\beta^1| = 1$.

Case of $\beta^0$. We set $(\tilde{Y}_n)_{n \geq 1} = \left( (B^k_n)_{k=-n+1}^0 \right)_{n \in \mathbb{N}}$ and in the same way apply Lemma 3 to get $|\beta^0| = 1$.

Cases of $\gamma^1$ and $\gamma^0$. Similar to the preceding two cases.

This completes the proof of Lemma 5.

Since $A(A^i_n) = \frac{A(B^i_n) + A(C^i_n)}{2}$, it follows that, for both $i = 0, 1$, we have $|\beta^i + \gamma^i| = |\alpha^i| = 1$, and hence $\beta^i = \gamma^i = \alpha^i \in S^1$. 

On the other hand, the fact that $F$ satisfies (5.1) implies that
$$\beta^1 = \beta^0; \quad \gamma^1 = \gamma^0.$$ We conclude that $\beta^1 = \gamma^1 = \theta \cdot \gamma^0 = \theta \cdot \beta^0 = \theta \cdot \beta^1$, whence $\theta = 1$, in contradiction with the initial assumption that $\theta \neq 1$.

The proof of Proposition 3 is complete.

7. TWO EXTENSIONS OF THEOREM 1

The following two theorems (Theorem 4 and 5) extends Theorem 1 to arbitrary $f$-invariant ergodic measures $\mu$ (rather than Lebesgue measure $\lambda$). These results are of interest in the case when IETs $(X,F)$ are minimal but not uniquely ergodic. (Such IETs exist, see [17], [18]).

**Theorem 4.** Let $(X,f)$ be a minimal $\mu$-ergodic IET, $X = [0,b)$, where $\mu$ is an $f$-invariant Borel probability measure. Then the set

$$X_{wm}(\mu) = \{0 < t < b \mid f_t \text{ is weakly mixing (relative to } \mu)\}$$

is a residual set of full $\mu$-measure: $\mu(X_{wm}(\mu)) = 1$.

**Proof.** Let $\beta: X \to X$ be an increasing continuous bijection taking measure $\mu$ to $\lambda$. Then the composition $g = \beta^{-1} \circ f \circ \beta$ becomes a $\lambda$-ergodic IET which topologically and combinatorially is $\beta$-isomoirophiic to $f$. (This is the essense of the normalization procedure discussed in [25, Section 1]).

Theorem 1 applies to $g$ to deduce Theorem 4.

It is known that for any minimal IET $(X,f)$ the set $\mathcal{P}_{erg}(f)$ of ergodic $f$-invariant Borel probability measures on $X$ is finite (see [14] and [25]).

**Theorem 5.** Let $f: X \to X$ be a minimal IET, $X = [0,b)$. Then

$$X_{wm}(\mathcal{P}_{erg}) = \{0 < t < b \mid f_t \text{ is weakly mixing relative to every } \mu \in \mathcal{P}_{erg}(f)\}$$

is a residual subset of $X$.

**Proof.** Follows from Theorem 4 because

$$X_{wm}(\mathcal{P}_{erg}(f)) = \bigcap_{\mu \in \mathcal{P}_{erg}(f)} X_{wm}(\mu)$$

is a finite intersection of residual sets.

8. FINAL COMMENTS.

The discussion in this section is conducted under the assumption that $(X,f)$ is a fixed $\lambda$-ergodic IET, $X = [0,1]$, so that both $X_{nwm}$ and $X_{\psi=0}$ (the the exceptional and the critical sets, respectively) are defined (by (2.1b) and (2.5a)).

By Theorem 2 every exceptional point is critical, i.e., the inclusion

$$X_{nwm}(f) = X_{nwm} \subset X_{\psi=0} = X_{\psi=0}(f)$$
holds, and, by Theorem 3, the critical set \( X_{\psi=0} \) is meager and has Lebesgue measure 0.

In this section we sketch some results on the size of the sets \( X_{\text{nwm}} \) and \( X_{\psi=0} \), concerning their Hausdorff dimensions and cardinalities. The proofs and more detailed description of the results will appear elsewhere.

8.1. Case: Irrational rotation. Let \( a \in \mathbb{R} \) and set \( f(x) = R_a(x) = x + a \pmod{1} \). (Such \( f \) can be viewed as a 2-IET). One verifies that in this case \( f_t \) is a 3-IET provided that \( t \neq D' = \{ na \mid n \in \mathbb{Z} \} \) (see (2.2)). Theorem 3 applies to deduce that \( X_{\psi=0} \) is meager and has Lebesgue measure 0.

It follows from \[8\] that
\[
\{ \mathbb{Q} + \mathbb{Q} a \} \cap (X \setminus D') \cap X_{\psi=0} = \emptyset,
\]
(i.e., no rational linear combination of 1 and \( a \) lying in \( X \setminus D' \) is exceptional).

Both sets \( X_{\text{nwm}}(R_a) \) and \( X_{\psi=0}(R_a) \) are countable if and only if \( a \) is badly approximable (the sequences of partial quotients in its continued fraction expansion is bounded). This fact can be deduced from \[24\]. In fact, if \( a \) is badly approximable then the three sets, \( X_{\text{nwm}}, X_{\psi=0} \) and \( D' \), coincide.

For Lebesgue almost all \( a \), the Hausdorff dimensions of the sets \( X_{\text{nwm}}(R_a) \) and \( X_{\psi=0}(R_a) \) vanish. Nevertheless, for some irrational \( a \), both Hausdorff dimensions can be 1. (This claim is based on an unpublished result by Yitwah Cheung).

8.2. Case: Linearly recurrent IETs. Every minimal \( r \)-IET \( f \) can be naturally coded by a minimal subshift \( \Omega_f \) over the alphabet \( \mathcal{A}_r = \{1, 2, \ldots, r\} \) the following natural way described in \[16\]. (Every point \( x \in X \) corresponds to the infinite word \( (W_x(k))_{k \in \mathbb{Z}} \in (\mathcal{A}_r)^\mathbb{Z} \) determined by the rule: \( W_x(k) = s \) if \( f^k(x) \in X_s = [d_{s-1}, d_s] \), see (1.1)).

A minimal \( r \)-IET \( f \) is said to be linearly recurrent if the corresponding subshift \( \Omega_f \) is linearly recurrent in the sense of \[12\] (see also \[11\]). Linearly recurrent IETs also appear in the literature as “IETs of constant type” (a characterization in terms of Rauzy-Veech induction).

Linearly recurrent IETs include IETs of periodic type (also known as pseudo-Anosov IETs) and, in particular, minimal IETs over quadratic number fields (the IETs with the lengths of exchanged intervals lying in one and the same real quadratic number field). IETs over quadratic number fields always reduce to Pseudo-Anosov IETs (see \[6\]).

One can prove that for linear recurrent IETs \((X, f)\) each of the sets \( X_{\text{nwm}}(f) \) and \( X_{\psi=0}(f) \) is at most countable.

We believe that for \( r \geq 4 \) there are pseudo-Anosov \( r \)-IETs which are persistently weakly mixing (i.e., for which \( X_{\text{nwm}}(f) = \emptyset \)).

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Department of Mathematics, Rice University, Houston, TX 77005, USA

E-mail address: michael@rice.edu