GELFAND-KIRILLOV DIMENSIONS OF HIGHEST WEIGHT
HARISH-CHANDRA MODULES FOR $SU(p, q)$

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Abstract. Let $(G, K)$ be an irreducible Hermitian symmetric pair of non-
compact type with $G = SU(p, q)$, and let $\lambda$ be an integral weight such that
the simple highest weight module $L(\lambda)$ is a Harish-Chandra $(g, K)$-module.
We give a combinatorial algorithm for the Gelfand-Kirillov dimension of $L(\lambda)$.
This enables us to prove that the Gelfand-Kirillov dimension of $L(\lambda)$ decreases
as the integer $\langle \lambda + \rho, \beta^\vee \rangle$ increases, where $\rho$ is the half sum of positive roots and
$\beta$ is the maximal noncompact root. Finally by the combinatorial algorithm,
we obtain a description of the associated variety of $L(\lambda)$.

1. Introduction

In the representation theory of Lie groups and Lie algebras, the Gelfand-Kirillov
dimension is an important invariant to measure the size of an infinite-dimensiona
module. This kind of invariant was first introduced by Gelfand and Kirillov [6].
In this paper, we are concerned with the Gelfand-Kirillov dimensions of highest
weight $(g, K)$-modules, where $g$ is the Lie algebra of the Hermitian type Lie group
$G = SU(p, q)$ and $K$ is the maximal compact subgroup $K = S(U(p) \times U(q))$ with
$n = p + q$.

To state our problem, we need some notations from [4]. Let $(G, K)$ be an irre-
ducible Hermitian symmetric pair of non-compact type. We denote their Lie alge-
bras by $(g_0, k_0)$, and denote their complexification by $g = g_0 \otimes \mathbb{C}$ and $k = k_0 \otimes \mathbb{C}$. Then $\mathfrak{t} = \mathbb{C} H \oplus \mathfrak{t}$ with $\text{ad}(H)$ having eigenvalues $0, 1, -1$ on $g$. If we denote $p^\pm = \{ X \in g \mid [H, X] = \pm X \}$. Then $g = p^- \oplus \mathfrak{t} \oplus p^+$. Let $h_0 \subseteq k_0$ be a Cartan subalgebra. Let $\Phi$ denote the set of roots with respect to $(g, h)$, let $W$ denote
the Weyl group of $g$, and let $\Phi_c$ (resp. $\Phi_n$) denote the set of compact roots (resp. non-
compact roots). Choose a Borel subalgebra $b$ containing $h$. Let $\beta$ denote the unique
maximal non-compact root of $\Phi^+$. Now choose $\zeta \in h^*$ so that $\zeta$ is orthogonal to
$\Phi_c$ and $\langle \zeta, \beta^\vee \rangle = 1$. For $\lambda \in h^*$, the simple highest weight module $L(\lambda)$ is a $(g, K)$-
module if and only if $\lambda$ is $\Phi_c^+$-dominant integral (i.e $\lambda$ is dominant and integral as a
$\mathfrak{t}$-weight) and $\langle \lambda + \rho, \beta^\vee \rangle \in \mathbb{R}$. In this paper, we call such a module $L(\lambda)$ a highest
weight $(g, K)$-module for simplicity. Note that if $L(\lambda)$ is a $(g, K)$-module then we
can write $\lambda = \tilde{\lambda} + z\zeta$, for some $z \in \mathbb{R}$ and for an integral and $\Phi_c^+$-dominant weight
$\tilde{\lambda}$ such that $\langle \tilde{\lambda} + \rho, \beta^\vee \rangle = 0$; in fact $z = \langle \lambda + \rho, \beta^\vee \rangle$ and $\tilde{\lambda} = \lambda - z\zeta$.

The motivation of this paper origins from the study on unitary highest weight
$(g, K)$-modules; see [2]. In [2], the Gelfand-Kirillov dimension of a unitary highest

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weight module $L(\lambda)$ is expressed explicitly in terms of $z := \langle \lambda + \rho, \beta^\vee \rangle$. From this result, one can see that the Gelfand-Kirillov dimension decreases as $z$ takes unitary reduction points and increases. Then a natural question is to ask whether a similar result holds for all highest weight $(\mathfrak{g}, K)$-modules. We will obtain a positive answer in this paper for the Lie group $SU(p, q)$.

In 1978, Joseph [11] found that the GK-dimension of a highest weight $\mathfrak{sl}(n)$-module $L_w := L(-w\rho - \rho)$ with $w \in W$ can be computed by Robinson-Schensted correspondence. This result suggests a combinatorial algorithm for highest weight $(\mathfrak{g}, K)$-modules in the case $G = SU(p, q)$. It is known that $\text{GKdim} L(\lambda)$ for any weight $\lambda$ can be reduced to $L_w$ for some $w \in W$ by using Jantzen’s translation functors [9], but we fail to find a concrete result in published literatures. In the first part of this paper, we will formulate an explicit algorithm for $\text{GKdim} L(\lambda)$ for $\mathfrak{g} = \mathfrak{sl}(n)$ and any weight $\lambda \in \mathfrak{h}^*$. A main tool that we use is Lusztig’s $a$-function on the Weyl group.

The $a$-function is defined in a combinatorial way using the Kazhdan-Lusztig basis of the Hecke algebra; see [13] or §3.2. There is a formula connecting the $a$-function and the Gelfand-Kirillov dimension

$$\text{GKdim} L_w = |\Phi^+| - a(w) \text{ for } w \in W,$$

which is formulated in [14]. Thus computations of Gelfand-Kirillov dimensions can be ultimately reduced to that of Lusztig’s $a$-functions. In the case $\mathfrak{g} = \mathfrak{sl}(n)$, the Weyl group is isomorphic to the symmetric group $\mathfrak{S}_n$ in $n$ letters, and the $a$-functions on $\mathfrak{S}_n$ can be easily determined by using some known properties about two-side cells. Our results can be stated as follows.

**Proposition 1.1** (See Prop. 3.8). Let $\mathfrak{g}$ be a simple complex Lie algebra. For $\lambda \in \mathfrak{h}^*$, let $W_{[\lambda]}$ be the integral Weyl group relative to $\lambda$, i.e the subgroup of $W$ generated by the set of reflections $\{s_a | \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}\}$ and let $a_{[\lambda]} : W_{[\lambda]} \to \mathbb{N}$ be the Lusztig’s $a$-function associated to $W_{[\lambda]}$. Then we have

$$\text{GKdim} L(\lambda) = |\Phi^+| - a_{[\lambda]}(w)$$

where $w \in W_{[\lambda]}$ is the element of minimal length such that $w^{-1}\lambda$ is antidominant (under the dot action).

Now we assume that $\mathfrak{g} = \mathfrak{sl}(n)$. By the proposition above, we have an explicit algorithm of $\text{GKdim} L(\lambda)$. For a weight $\lambda \in \mathfrak{h}^*$ we write $\lambda + \rho = (\lambda_1, \lambda_2, \ldots, \lambda_n)$; see §4.2. We associated to $\lambda$ a set $P(\lambda)$ of some Young tableaux as follows. Let $\lambda X : \lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_r}$ be a maximal subsequence of $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $\lambda_{i_k}, 1 \leq k \leq r$ are congruent to each other modulo $\mathbb{Z}$. Then the Young tableau associated to the subsequence $\lambda X$ using Schensted insertion algorithm (see §4.1) is a Young tableau in $P(\lambda)$. If $\lambda$ is an integral weight then $P(\lambda)$ consists of only one Young tableau and we identify $P(\lambda)$ with this Young tableau in this case.

If $Y$ is a Young tableau, we define $A(Y) := \sum_{i \geq 1} c_i c_{i+1}$ where $c_i$ is the number of entries in the $i$-th column of $Y$. And we define $A(P(\lambda)) := \sum_{Y \in P(\lambda)} A(Y)$.

**Theorem 1.2** (See Thm. 4.6). Let $\mathfrak{g} = \mathfrak{sl}(n)$. For any $\lambda \in \mathfrak{h}^*$, we have

$$\text{GKdim} L(\lambda) = \frac{n(n-1)}{2} - A(P(\lambda)).$$
Theorem 1.5

Let $\lambda$ be a $(p,q)$-dominant weight, i.e., $\lambda_i - \lambda_j \in \mathbb{Z}_{>0}$ for all $1 \leq i < j \leq p$ and $p+1 \leq i < j \leq p+q$ where $\lambda + \rho = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. Now we can concentrate on the Gelfand-Kirillov dimension $L(\lambda)$ for a $(p,q)$-dominant weight.

Theorem 1.4 (See Thm. 5.2). Assume that $\lambda$ is $(p,q)$-dominant with $\lambda + \rho = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. Then

(i) If $\lambda_1 - \lambda_{p+1} \in \mathbb{Z}$, i.e. $\lambda$ is an integral weight, then $P(\lambda)$ is a Young tableau with at most two columns. In this case we have $\text{GKdim} L(\lambda) = m(n-m)$ where $m$ is the number of entries in the second column of $P(\lambda)$.

(ii) If $\lambda_1 - \lambda_{p+1} \notin \mathbb{Z}$, then $P(\lambda)$ consists of two Young tableaux with single column. In this case we have $\text{GKdim} L(\lambda) = pq$.

Example 1.3. If $\lambda + \rho = (3, 3.5, 2, 1.5, -1, 5.5, -1, 0, 1.1)$, then $P(\lambda)$ has three Young tableaux:

\[
\begin{array}{ccc}
-1 & -1 & 0 \\
2 & 1.5 & 5.5 \\
3 & 3.5 & 1.1
\end{array}
\]

By the theorem, $\text{GKdim} L(\lambda) = \binom{5}{2} - A(P(\lambda)) = \binom{9}{2} - \binom{5}{2} = 36 - 3 - 1 = 32$.

Example 1.6. Let $\lambda$ be a $(5, 5)$-dominant weight such that

$\lambda + \rho = (5, 4, 3, 2, 1, 9, 8, 7, 6, 2)$.

Then the line of balls associated to $\lambda$ is as follows:

\[
\begin{array}{cccccccccccc}
\odot & \odot & \odot & \odot & \# & \# & \# & \# & \odot & \odot & \odot & \# & \#
\end{array}
\]

The number of adjacent white-black pairs that we can remove from this line of balls is 5. Hence $m = 5$, and $\text{GKdim} L(\lambda) = 5 \times (10 - 5) = 25$.

From this combinatorial model, we can reprove a result of [2] on Gelfand-Kirillov dimensions of unitary highest weight modules (in the case $G = SU(p,q)$); see Prop. 6.2. Furthermore, we can prove the following result.
Theorem 1.7 (See Thm. 6.3). Assume that $G = SU(p, q)$ and $\tilde{\lambda}$ is a $(p, q)$-dominant weight such that $\langle \lambda + \rho, \beta^\vee \rangle = 0$. Then $\text{GKdim } L(\lambda + z\zeta)$ decreases to 0 as $z$ increases in $\mathbb{Z}$. And $\text{GKdim } L(\lambda + z\zeta) = pq$ if $z \in \mathbb{R} \setminus \mathbb{Z}$.

From Vogan [19], we know that the associated variety of any highest weight $(\mathfrak{g}, K)$-module is the closure of some $K_C$ orbit in $p^{+} \cong (\mathfrak{k} + p^{+})^{\ast}$. The closures of $K_C$ orbits in $p^{+}$ form a linear chain of varieties:

\begin{equation}
\{0\} = \bar{O}_{0} \subseteq \bar{O}_{1} \subseteq ... \subseteq \bar{O}_{r-1} \subseteq \bar{O}_{r} = p^{+},
\end{equation}  

where $r$ is the rank of the Hermitian symmetric space $G/K$. In the case of $G = SU(p, q)$, $r = \min\{p, q\}$.

Theorem 1.8 (See Thm. 6.4). If $G = SU(p, q)$ and $\lambda$ is an integral weight such that $L(\lambda)$ is a highest weight $(\mathfrak{g}, K)$-module, then the associated variety of $L(\lambda)$ is the $m$-th closure $\bar{O}_{m}$ in (1.1), where $m$ is the number of entries in the second column of the Young tableau $P(\lambda)$.

As pointed out by one of our referees, Garfinkle [5] also obtained Gelfand-Kirillov dimensions and associated varieties of irreducible Harish-Chandra modules with trivial infinitesimal character for $SU(p, q)$ by a different method. One of advantages of our method in this paper is that we can uniformly and directly obtain these invariants from the highest weight of any simple highest weight Harish-Chandra module. We thank the referee for pointing out Garfinkle’s work.

2. Preliminaries on Gelfand-Kirillov Dimension

In this section we recall the definition and some properties of the Gelfand-Kirillov dimension. The details can be found in [3, 10, 12, 16, 18, 19].

Definition 2.1. Let $A$ be an algebra generated by a finite-dimensional subspace $V$. Let $V^n$ denote the linear span of all products of length at most $n$ in elements of $V$. The Gelfand-Kirillov dimension of $A$ is defined by:

\[
\text{GKdim}(A) = \limsup_{n \to \infty} \frac{\log \dim(V^n)}{\log n}.
\]

It is well-known that the above definition is independent of the choice of the finite dimensional generating subspace $V$ (see [3, 12]). Clearly $\text{GKdim}(A) = 0$ if and only if $\dim(A) < \infty$.

The notion of Gelfand-Kirillov dimension can be extended for left $A$-modules. In fact, we have the following definition.

Definition 2.2. Let $A$ be an algebra generated by a finite-dimensional subspace $V$. Let $M$ be a left $A$-module generated by a finite-dimensional subspace $M_0$. The Gelfand-Kirillov dimension $\text{GKdim}(M)$ of $M$ is defined by:

\[
\text{GKdim}(M) = \limsup_{n \to \infty} \frac{\log \dim(V^n M_0)}{\log n}.
\]

Lemma 2.3. Let $\mathfrak{g}$ be a finite dimensional Lie algebra, $M$ be a finitely generated $U(\mathfrak{g})$-module and $F$ be a finite dimensional $U(\mathfrak{g})$-module. Then we have $\text{GKdim}(M \otimes F) = \text{GKdim}(M)$. 
3. Gelfand-Kirillov dimensions and Lustig’s a-functions

3.1. Translation functors. We first recall the category $\mathcal{O}$ of a semisimple Lie algebra; see for example [7]. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ and $\mathfrak{h}$ be a Cartan subalgebra. Let $\Phi \subseteq \mathfrak{h}^*$ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Choose a set $\Phi^+$ of positive roots. Let $\Delta$ be the set of simple roots and let $\Phi^\pm = -\Phi^\pm$. We have a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ where $\mathfrak{n}^\pm = \bigoplus_{\alpha \in \Phi^\pm} \mathfrak{g}_\alpha$ and $\mathfrak{n}^- = \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha$. Let $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$ (resp. $\mathfrak{b}^- = \mathfrak{h} \oplus \mathfrak{n}^-$) be the positive (resp. negative) Borel subalgebra with respect to $\Delta$.

Translation functors. Let $\lambda, \alpha \in \Phi$ be the half sum of all positive roots. The shifted action $\rho \in \mathfrak{h}^*$ is called the $W$-action on $\mathfrak{h}^*$ by

$$\sigma_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha \quad \text{for all } \lambda \in \mathfrak{h}^*.$$

Then $W$ is a Coxeter group with $S = \{s_\alpha \mid \alpha \in \Delta\}$ being the set of generators. Let $\rho$ be the half sum of all positive roots. The shifted action

$$w.\lambda := w(\lambda + \rho) - \rho, \quad w \in W, \lambda \in \mathfrak{h}^*$$

is called the dot action of $W$ on $\mathfrak{h}^*$. If we denote $w_0$ by the longest element of $W$, then $w_0.0 = -2\rho$, and for any $w \in W$ we have $w.(-2\rho) = -w\rho - \rho$.

The Verma module of $\mathfrak{g}$ with highest weight $\lambda \in \mathfrak{h}^*$ is defined to be

$$M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{h}^+)} \mathbb{C}_\lambda$$

where $\mathbb{C}_\lambda$ is a $U(\mathfrak{h}^+)$-module such that

$$h.m = \lambda(h)m, \quad n^+.m = 0, \quad \text{for } h \in \mathfrak{h}^*, m \in \mathbb{C}_\lambda.$$

The Verma module $M(\lambda)$ has a unique simple quotient, which is denoted by $L(\lambda)$. These modules $L(\lambda)$ with $\lambda \in \mathfrak{h}^*$ are called the simple highest weight $\mathfrak{g}$-modules.

Let $\Lambda = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Phi^+\}$, whose elements are called integral weights. A weight $\lambda \in \mathfrak{h}^*$ is called antidominant if $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{\geq 0}$ for all $\alpha \in \Phi^+$. Let

$$\Phi_\lambda := \{\alpha \in \Phi \mid s_\alpha \lambda - \lambda \in \mathbb{Z}\alpha\} = \{\alpha \in \Phi \mid \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}\},$$

and let $W_{[\lambda]}$ be the subgroup of $W$ generated by $\{s_\alpha \mid \alpha \in \Phi_\lambda\}$. Note that $W_{[\lambda]}$ is still a Coxeter group, but in general it is not generated by a subset of $S$. There is one and only one antidominant weight in $W_{[\lambda]}$.\lambda.

The category $\mathcal{O}$ consists of $\mathfrak{g}$-modules which are

- semisimple as $\mathfrak{h}$-modules;
- finitely generated as $U(\mathfrak{g})$-modules;
- locally $\mathfrak{n}$-finite.

It is known that $\mathcal{O}$ is an abelian category which is both notherian and artinian, and has enough projective objects and injective objects. The simple objects of $\mathcal{O}$ are precisely those $L(\lambda)$ with $\lambda \in \mathfrak{h}^*$. A block of $\mathcal{O}$ is an indecomposable summand of $\mathcal{O}$ as an abelian subcategory. Let $\mathcal{O}_\lambda$ be the block containing the simple module $L(\lambda)$. Then we have $\mathcal{O}_\lambda = \mathcal{O}_{\mu}$ if and only if $\mu \in W_{[\lambda]} \lambda$; equivalently, the simple modules in $\mathcal{O}_\lambda$ are precisely $L(\mu)$, $\mu \in W_{[\lambda]} \lambda$. Since there is a unique antidominant weight in $W_{[\lambda]} \lambda$, we can assume that $\lambda$ is antidominant when considering a block.
$\mathcal{O}_\lambda$. By definition we have

$$\mathcal{O} = \bigoplus_{\lambda \text{ antidominant}} \mathcal{O}_\lambda.$$  

When $\lambda = 0$, $\mathcal{O}_\lambda = \mathcal{O}_0$ is called the principal block. The simple modules in $\mathcal{O}_0$ are denoted by

$$L_w := L(w(-2\rho)), w \in W.$$  

In what follows, we recall Jantzen’s translation functors which are used to compare two blocks. Here we only consider the case of integral weights, which is enough for the purpose of this paper.

Let $E := \mathbb{Z}\Phi \otimes_{\mathbb{Z}} \mathbb{R}$. It is a Euclidean space, on which the bilinear form is just the restriction to $E$ of the bilinear form $(\ , \ )$ on $\mathfrak{h}^*$ defined above. One can see that $\Lambda \subseteq E$. The Weyl group $W$ also acts on $E$ by dot action. Consider the hyperplanes $H_\alpha = \{\lambda \in E \mid (\lambda + \rho, \alpha^\vee) = 0\}$, $\alpha \in \Phi^+$. These hyperplanes give rise to some facets. Precisely, a facet $F$ is a maximal subset of $E$ such that for any $\lambda, \mu \in F$, $\langle \lambda + \rho, \alpha^\vee \rangle = 0$ (resp. $< 0$, or $> 0$) if and only if $\langle \mu + \rho, \alpha^\vee \rangle = 0$ (resp. $< 0$, or $> 0$) for all $\alpha \in \Phi^+$. The facets of maximal dimension are the connected components of $E \setminus \bigcup_{\alpha \in \Phi^+} H_\alpha$, which are called chambers. Denote by $C^+$ (resp. $C^-$) the chamber consisting of $\lambda$ such that $\langle \lambda + \rho, \alpha^\vee \rangle > 0$ (resp. $< 0$) for all $\alpha \in \Phi^+$.

For a facet $F$ with $\lambda \in F$, let $\Phi_{>0}(F) = \{\alpha \in \Phi^+ \mid \langle \lambda + \rho, \alpha^\vee \rangle > 0\}$, $\Phi_{=0}(F) = \{\alpha \in \Phi^+ \mid \langle \lambda + \rho, \alpha^\vee \rangle = 0\}$, $\Phi_{<0}(F) = \{\alpha \in \Phi^+ \mid \langle \lambda + \rho, \alpha^\vee \rangle < 0\}$. The upper closure of $F$ is defined to be the set of $\mu$ such that

$$\langle \mu + \rho, \alpha^\vee \rangle > 0 \text{ for } \alpha \in \Phi_{>0}(F),$$

$$\langle \mu + \rho, \alpha^\vee \rangle = 0 \text{ for } \alpha \in \Phi_{=0}(F),$$

and $$\langle \mu + \rho, \alpha^\vee \rangle \leq 0 \text{ for } \alpha \in \Phi_{<0}(F).$$

For example, the upper closure of $C^-$ is the closure $C^-$, and the upper closure of $C^+$ is $C^+$ itself.

**Lemma 3.1.** The space $E$ is the disjoint union of upper closures of chambers. For any integral weight $\lambda \in \Lambda$, there is a unique $w \in W$ such that $\lambda$ is in the upper closure of the chamber $w.C^- \cap \{\lambda\}$ containing $w(-2\rho)$. Moreover, the element $w$ here is characterized as the unique element of $W$ of minimal length such that $w^{-1}.\lambda$ is antidominant. Generally, for any weight $\lambda \in \mathfrak{h}^*$, there is a unique element $w \in W_{\lambda}[\lambda]$ of minimal length such that $w^{-1}.\lambda$ is antidominant.

For $\lambda, \mu \in \Lambda$, we set $\gamma = \mu - \lambda$. We can find a weight $\bar{\gamma} \in W\gamma$ such that $\langle \bar{\gamma}, \alpha^\vee \rangle \geq 0$ for all $\alpha \in \Phi^+$. Then $L(\bar{\gamma})$ is finite dimensional. The Jantzen’s translation functor

$$T^\mu_\lambda : \mathcal{O}_\lambda \rightarrow \mathcal{O}_\mu$$

is an exact functor given by $T^\mu_\lambda(M) := \text{Pr}_\mu(L(\bar{\gamma}) \otimes M)$, where $M \in \mathcal{O}_\lambda$ and $\text{Pr}_\mu$ is the natural projection $\mathcal{O} \rightarrow \mathcal{O}_\mu$. From [9, §2.10-2.11], we have the following lemma.

**Lemma 3.2.** Let $\lambda, \mu \in \Lambda$.

(i) $T^\mu_\lambda$ is biadjoint to $T^\lambda_\mu$.

(ii) If $\lambda \in \Lambda$ is in a facet $F$ and $\mu \in \Lambda$ is in the closure $\overline{F}$, then

$$T^\lambda_\mu(L(\lambda)) \cong \begin{cases} L(\mu), & \text{if } \mu \text{ is in the upper closure of } F; \\ 0, & \text{otherwise.} \end{cases}$$
Corollary 3.3. Let \( \lambda \in \Lambda \). We have
\[
T_{\mu}^\lambda(-2\rho)(L_w) = L(\lambda)
\]
where \( w \in W \) is the unique element of minimal length such that \( w^{-1}\lambda \) is antidominant.

Proof. By Lemma 3.1, \( \lambda \) is in the upper closure of the facet that contains \( w(-2\rho) \). Then the corollary follows from Lemma 3.2 (ii). \( \square \)

Lemma 3.4. Let \( \lambda \in \Lambda \) be an element of a facet \( F \) and \( \mu \in \Lambda \) is in the upper closure of \( F \), then
\[
\text{GKdim } L(\lambda) = \text{GKdim } L(\mu).
\]

Proof. By definition, we have \( \text{GKdim } T_{\lambda}^\mu L(\lambda) \leq \text{GKdim } L(\lambda) \) and \( \text{GKdim } T_{\mu}^\lambda L(\mu) \leq \text{GKdim } L(\mu) \).

By Lemma 3.2 (ii), \( \text{GKdim } L(\mu) = \text{GKdim } T_{\lambda}^\mu L(\lambda) \leq \text{GKdim } L(\lambda) \). By Lemma 3.2 (i), we have \( \text{Hom}(L(\lambda), T_{\lambda}^\mu L(\mu)) = \text{Hom}(T_{\lambda}^\mu L(\lambda), L(\mu)) = \text{Hom}(L(\mu), L(\mu)) \neq 0 \), which implies that \( \text{GKdim } L(\lambda) \leq \text{GKdim } T_{\mu}^\lambda L(\mu) \leq \text{GKdim } L(\mu) \). Then the lemma follows. \( \square \)

3.2. Lusztig’s a-function. Recall that the Weyl group \( W \) of \( g \) is a Coxeter group generated by \( S = \{ s_\alpha \mid \alpha \in \Delta \} \). Then we have a Hecke algebra \( \mathcal{H} \) over \( \mathcal{A} := \mathbb{Z}[v, v^{-1}] \), which is generated by \( T_w, w \in W \) with relations
\[
T_w T_{w'} = T_{w w'}, \text{ if } l(w w') = l(w) + l(w'),
\]
and \((T_s + v^{-1})(T_s - v) = 0\) for any \( s \in S \).

If \( \lambda \in \mathfrak{h}^* \), associated to the Weyl group \( W_{[\lambda]} \) of \( g_\lambda \), we have a Hecke algebra \( \mathcal{H}_{[\lambda]} \). One need to note that in general \( \mathcal{H}_{[\lambda]} \) is not a subalgebra of \( \mathcal{H} \), since \( W_{[\lambda]} \) is not necessarily generated by a subset of \( S \).

The Kazhdan-Lusztig basis \( C_w, w \in W \) of \( \mathcal{H} \) are characterized as the unique elements \( C_w \) such that
\[
\overline{C_w} = C_w, \quad C_w \equiv T_w \mod \mathcal{H}_{<0}
\]
where \( - : \mathcal{H} \to \mathcal{H} \) is the bar involution such that \( \overline{q} = q^{-1}, \overline{T_w} = T_{w^{-1}} \), and \( \mathcal{H}_{<0} = \bigoplus_{w \in W} \mathcal{A}_{<0} T_w, \mathcal{A}_{<0} = v^{-1}[v^{-1}] \).

If \( C_y \) occurs in the expansion of \( hC_w \) with respect to the KL-basis for some \( h \in \mathcal{H} \), then we write \( y \leftarrow_L w \). And we extend \( \leftarrow_L \) to a preorder \( \prec_L \) on \( W \). Define the equivalence relation \( \sim_L \) by that \( x \sim_L y \) if and only if \( x \prec_L y \) and \( y \prec_L x \). An equivalence class of \( \sim_L \) on \( W \) is called a left cell of \( W \). We define \( x \prec_R y \) by \( x^{-1} \prec_L y^{-1} \) and define \( x \sim_R y \) by \( x^{-1} \sim_L y^{-1} \). The equivalence class of \( \sim_R \) is called a right cell. Define \( \prec_{LR} \) to be the preorder generated by \( \prec_L \) and \( \prec_R \). And we have an equivalence relation \( \sim_{LR} \) on \( W \) corresponding to \( \prec_{LR} \); a equivalence class of \( \sim_{LR} \) is called a two-sided cell of \( W \).

Let \( C_x C_y = \sum_{z \in W} h_{x,y,z} C_z \) with \( h_{x,y,z} \in \mathcal{A} \). Then Lusztig’s a-function \( a : W \to \mathbb{N} \) is defined by
\[
a(z) = \max \{ \deg h_{x,y,z} \mid x, y \in W \} \text{ for } z \in W.
\]

We state a lemma which will be used later.

Lemma 3.5. (i) The equivalence relation \( \sim_{LR} \) is generated by \( \sim_L \) and \( \sim_R \).
(ii) \( a(w) = a(w^{-1}) \) for all \( w \in W \).
(iii) \( a : W \to \mathbb{N} \) is constant on each two-sided cell of \( W \).
If $w_1$ is the longest element of the parabolic subgroup of $W$ generated by a subset $I \subseteq S$, the $a(w_1)$ is equal to the length $l(w_1)$ of $w_1$.

(v) If $W$ is a direct product of Coxeter subgroups $W_1$ and $W_2$, then

$$a(w) = a(w_1) + a(w_2)$$

for $w = (w_1, w_2) \in W_1 \times W_2 = W$.

**Proof.** We use the properties P1-P15 in §14.2 of [15]. The statement (i) follows from P9-P11, (iii) follows from P4, (iv) follows from P12, and (ii),(v) are deduced from the definition.

For $\lambda \in h^*$ the $a$-function of $W_{[\lambda]}$ is denoted by

$$a_{[\lambda]} : W_{[\lambda]} \to \mathbb{N},$$

which is defined using the Hecke algebra $H_{[\lambda]}$. One need to note that $a_{[\lambda]}$ is not the restriction of $a : W \to \mathbb{N}$.

3.3. Gelfand-Kirillov dimensions of simple highest weight modules. By [14, §1], there is a formula connecting the Gelfand-Kirillov dimension and the Lusztig’s $a$-function. Recall that $L_w$ is the simple $\mathfrak{g}$-module in $O_0$ with highest weight $w.\lambda$. (3.3)

**Lemma 3.6** (Lusztig). The Gelfand-Kirillov dimension of $L_w$ is given by

$$\text{GKdim } L_w = \nu_0 - a(w)$$

for $w \in W$, where $\nu_0 = |\Phi^+|$ is the number of positive roots of $\mathfrak{g}$.

**Definition 3.7.** For $\lambda \in h^*$, we define

$$a(\lambda) := a_{[\lambda]}(w)$$

where $w$ is the element of $W_{[\lambda]}$ of minimal length such that $w^{-1}\lambda$ is antidominant.

**Proposition 3.8.** For any $\lambda \in h^*$, we have

$$\text{GKdim } L(\lambda) = \nu_0 - a(\lambda).$$

**Proof.** First, we assume that $\lambda \in \Lambda$. In this case, $W_{[\lambda]} = W$. By Corollary 3.3, Lemma 3.4 and Lemma 3.6, we have

$$\text{GKdim } L(\lambda) \overset{3.3}{=} \text{GKdim } L(w.\lambda) \overset{3.6}{=} |\Phi^+| - a(w),$$

where $w$ is the element of $W$ of minimal length such that $w^{-1}\lambda$ is antidominant. Then we see that the proposition holds for integral weights.

Now we return to the general assumption that $\lambda \in h^*$. In the following, we will reduce the proof to the integral weight case.

Let $\mathfrak{g}_\lambda$ be a reductive Lie algebra such that its Cartan subalgebra coincides with the Cartan subalgebra $h$ of $\mathfrak{g}$ and its root system is given by $\Phi_\lambda$ (see (3.1)). Then $W_{[\lambda]}$ is the Weyl group of $\mathfrak{g}_\lambda$. Note that in general we can not realize $\mathfrak{g}_\lambda$ as a Lie subalgebra of $\mathfrak{g}$ unless $\mathfrak{g}$ is of $ADE$-type.

We denote by $\Theta^0$ the category $\Theta$ for $\mathfrak{g}_\lambda$ with respect to $\Phi_\lambda^+ := \Phi_\lambda \cap \Phi^+$, and denote by $L^0(\mu)$ (resp. $M^0(\mu)$) the simple module (resp. Verma modules) in $\Theta^0$ with highest weight $\mu \in h^*$. Let $\Theta^0_\lambda$ be the block of $\Theta^0$ that containing $L^0(\lambda)$. All the simple modules (resp. Verma module) in $\Theta^0_\lambda$ are given by $L^0(w.\lambda)$ (resp. $M^0(w.\lambda)$) with $w \in W_{[\lambda]}$. By [17, Thm.11], we have an equivalence of categories (3.3)

$$\Theta^0_\lambda \simeq \Theta_\lambda$$
under which $L^0(w,\lambda)$ (resp. $M^0(w,\lambda)$) corresponds to $L(w,\lambda)$ (resp. $M(w,\lambda)$).

Recall that for a semisimple $\mathfrak{h}$-module $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$ with $M_\lambda = \{ m \in M \mid h.m = \lambda(h)m \}$ and $\dim M_\lambda < \infty$, the character of $M$ is defined to be a formal sum $\text{ch}M = \sum_{\mu \in \mathfrak{h}^*} (\dim M_\mu)e_\mu$. For more details about the characters of modules in $\mathcal{O}$, see [7, §1.15]. For $w \in W_\lambda$, we have

$$\text{ch}M(w,\lambda) = \frac{e^{w,\lambda}}{\prod_{\alpha \in \Phi^+}(1 - e^{-\alpha})}, \quad \text{ch}M^0(w,\lambda) = \frac{e^{w,\lambda}}{\prod_{\alpha \in \Phi^+_\lambda}(1 - e^{-\alpha})}.$$ 

Hence, $\text{ch}M(w,\lambda) = \frac{1}{\prod_{\alpha \in \Phi^+ \setminus \Phi^+_\lambda}(1 - e^{-\alpha})} \text{ch}M^0(w,\lambda)$. From the equivalence (3.3), we see that $\text{ch}L(w,\lambda) = \sum_{y \in W_\lambda} a_{y,w}\text{ch}M(y,\lambda)$ for some integers $a_{y,w}$ if and only if $\text{ch}L^0(w,\lambda) = \sum_{y \in W_\lambda} a_{y,w}\text{ch}M^0(\lambda,\lambda)$. Thus,

$$\text{ch}L(w,\lambda) = \frac{1}{\prod_{\alpha \in \Phi^+ \setminus \Phi^+_\lambda}(1 - e^{-\alpha})} \text{ch}L^0(w,\lambda).$$

This implies that

(3.4) \[ \text{GKdim } L(w,\lambda) = |\Phi^+| - |\Phi^+\setminus\Phi^+_\lambda| + \text{GKdim } L^0(w,\lambda). \]

By the definition of $g_\lambda$, $\lambda$ is integral with respect to $g_\lambda$. Then by the first part of the proof, we have

$$\text{GKdim } L^0(\lambda) = |\Phi^+_\lambda| - a(\lambda).$$

Combined with (3.4), we obtain that

$$\text{GKdim } L(\lambda) = |\Phi^+_\lambda| - a(\lambda).$$

Then the proposition follows. \hfill $\Box$

4. Gelfand-Kirillov dimensions for $\mathfrak{sl}(n)$

4.1. Schensted insertion algorithm. A Young diagram is a collection of boxes arranged in left-justified rows, with the row lengths weakly increasing. Let $\Gamma$ be a totally ordered set; in the situation that we will encounter in this paper, $\Gamma$ is taken to be $c + \mathbb{N}$ for some $c \in \mathbb{C}$. A Young tableau $Y$ is a Young diagram with each box filled with an element of $\Gamma$. A Young tableau is called \textit{standard} if the entries in each row and in each column are strictly increasing. And a Young tableau is called \textit{semistandard} if the entries in each row (resp. column) are weakly (resp. strictly) increasing.

\textit{This implication is not obvious. It can be proved in the following way. The enveloping algebra $U(\mathfrak{g})$ is filtered by the subspaces $U_n(\mathfrak{g})$ which is spanned by the monomials of degrees less than or equal to $n$. It is known that $grU(\mathfrak{g}) = \bigoplus_{n \geq 1} U_n(\mathfrak{g})/U_{n-1}(\mathfrak{g})$ is isomorphic to the symmetric algebra $S(\mathfrak{g})$. Let $\mu \in \mathfrak{h}^*$, and $v_\mu \in L(\mu)$ be a highest weight vector. Then $grL(\mu) := \bigoplus_{n \geq 1} U_n(\mathfrak{g})v_\mu/U_{n-1}(\mathfrak{g})v_\mu$ is an $S(\mathfrak{g})$-module generated by $v_\mu$. Since $S(\mathfrak{g})$ is commutative, we have $\mathfrak{n}^+: grL(\mu) = S(\mathfrak{g})\mathfrak{n}^+: v_\mu = 0$.

Since $\mathfrak{g}_\mu$ is a subspace of $\mathfrak{g}$, we have an embedding $S(\mathfrak{g}_\mu) \hookrightarrow S(\mathfrak{g})$. One can see that there is a surjective map $S(\mathfrak{g}) \otimes S(\mathfrak{g}_\mu + \mathfrak{n}^+)^{-1} grL^0(\mu) \twoheadrightarrow grL(\mu)$ of $S(\mathfrak{g})$-algebras, where the $S(\mathfrak{g}_\mu)$-module $grL^0(\mu)$ is extended to an $S(\mathfrak{g}_\mu + \mathfrak{n}^+)$-module by letting $\mathfrak{n}^+: grL^0(\mu) = 0$. This map is actually an isomorphism since $\text{ch}L(\mu) = \frac{1}{\prod_{\alpha \in \Phi^+ \setminus \Phi^+_\lambda}(1 - e^{-\alpha})} \text{ch}L^0(\mu)$. Then by the definition of GK dimension, we have $\text{GKdim } grL(\mu) = |\Phi^+_\lambda| - |\Phi^+_\mu| + \text{GKdim } grL^0(\mu)$. Now (3.4) follows from [12, Prop. 6.6]. (Note that in general we have no map $U(\mathfrak{g}) \otimes U(\mathfrak{g}_\mu + \mathfrak{n}^+) L^0(\mu) \rightarrow L(\mu)$, since $\mathfrak{g}_\mu$ is not necessarily a Lie subalgebra of $\mathfrak{g}$.)}
Let \( \gamma = (\gamma_1, \cdots, \gamma_n) \) be a sequence of elements in \( \Gamma \). The following Schensted insertion algorithm associates \( \gamma \) to a pair \((P(\gamma), \varnothing(\gamma))\) of semistandard Young tableaux.

Let \( P_0, Q_0 \) be two empty Young tableaux. Assume that we have constructed Young tableaux \( P_k, Q_k \) associated to \((\gamma_1, \cdots, \gamma_k)\), \( 0 \leq k < n \). Then \( P_{k+1} \) is obtained by adding \( \gamma_{k+1} \) to \( P_k \) as follows. First add \( \gamma_{k+1} \) to the first row of \( P_k \) by replacing the leftmost entry \( x \) in the first row which is strictly bigger than \( \gamma_{k+1} \). (If there is no such an entry \( x \), we just add a box with entry \( \gamma_{k+1} \) to the right side of the first row, and end this process.) Then add \( x \) to the next row as the same way of adding \( \gamma_{k+1} \) to the first row. The Young \( Q_{k+1} \) is obtained by adding a box with entry \( k+1 \) to \( Q_k \) at the position of \( P_{k+1} \). Then we put \( P(\gamma) = P_n \) and \( Q(\gamma) = Q_n \). Note that \( P(\gamma) \) is a semistandard Young tableau and \( Q(\gamma) \) is a standard Young tableau, and they have the same shape.

For example, if \( \gamma = (3, 5, 2, 2, 1) \), then the Young tableaux produced by this algorithm are:

\[
\begin{array}{c}
3 & \rightarrow & 3 & 5 & \rightarrow & 2 & 2 & 5 & \rightarrow & 2 & 2 & 3 & 5 & \rightarrow & 1 & 2 & 2 & 3 & 5 & = P(\gamma), \\
1 & \rightarrow & 1 & 2 & \rightarrow & 1 & 2 & 3 & 4 & \rightarrow & 1 & 2 & 3 & 4 & 5 & = Q(\gamma).
\end{array}
\]

4.2. Lusztig’s \( a \)-function for \( \mathfrak{g} = \mathfrak{sl}(n) \). In this section, \( \mathfrak{g} \) is assumed to be the special linear Lie algebra \( \mathfrak{sl}(n) \) and the Cartan subalgebra \( \mathfrak{h} \) is taken to be the subalgebra consists of all diagonal matrices in \( \mathfrak{sl}(n) \). Let \( e_i, i \in [1, n] \) be the elements of \( \mathfrak{h}^* \) such that \( e_i(E_{ij}) = \delta_{ij} \) where \( E_{ij} \) is the matrix unit with entry 1 at \((i,j)\)-position. We have a relation \( \sum_{i=1}^n e_i = 0 \) and we have \( \mathfrak{h}^* = (\bigoplus_{i=1}^n \mathbb{C} e_i)/\mathbb{C}(\sum_{i=1}^n e_i) \). The set of positive roots of \( \mathfrak{sl}(n) \) is taken to be \( \Phi^+ = \{e_i - e_j \mid i < j \} \). The set of simple roots is \( \Delta = \{e_i - e_{i+1} \mid i < n \} \). The number of positive roots is \( \nu_0 = \frac{n(n-1)}{2} \).

For a weight \( \lambda \in \mathfrak{h}^* \), we write

\[
\lambda + \rho = (\lambda_1, \lambda_2, \cdots, \lambda_n),
\]

where \( \lambda + \rho = \sum_{i=1}^n \lambda_i e_i, \lambda_i \in \mathbb{C} \). Since \( \sum_{i=1}^n e_i = 0 \) we have \( (\lambda_1 + c, \lambda_2 + c, \cdots, \lambda_n + c) = (\lambda_1, \lambda_2, \cdots, \lambda_n) \) for any \( c \in \mathbb{C} \). For example, \( 2\rho = (n-1, n-3, \cdots, -n+1) \) and \( \rho = (-1, -2, \cdots, -n) \).

Let \( \mathfrak{S}_n \) be the symmetric group in \( n \) letters 1, 2, 3, \cdots, \( n \). Then \( \mathfrak{S}_n \) can be identified with the Weyl group of \( \mathfrak{sl}(n) \) via the action

\[
\sigma(e_i) = e_{\sigma(i)}, \ i \in [1, n], \ \sigma \in \mathfrak{S}_n.
\]

Then simple reflections of \( \mathfrak{S}_n \) with respect to \( \Delta \) are \((i, i+1), 1 \leq i < n \). For an element \( \sigma \in \mathfrak{S}_n \), we use the notation \( \sigma = \left( \begin{array}{cccc} 1 & 2 & \cdots & n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_n \end{array} \right) \) where \( \sigma_i = \sigma(i), \ i \in [1, n] \). We denote by \( P(\sigma) \) and \( Q(\sigma) \) the standard Young tableaux associated to the sequence \( (\sigma_1, \sigma_2, \cdots, \sigma_n) \), which are produced by the Schensted insertion algorithm in §4.1. For example, if \( \sigma = \left( \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{array} \right) \) then \( P(\sigma) = \left( \begin{array}{ccc} 1 & 4 \\ 2 & \end{array} \right) \) and
Lemma 4.1. \( \text{(i)} \) \( P(\sigma) = Q(\sigma^{-1}) \) for any \( \sigma \in S_n \).

\( \text{(ii)} \) \( \sigma, \tau \in S_n \) are in the same right (resp. left) cell if and only if \( P(\sigma) = P(\tau) \) (resp. \( Q(\sigma) = Q(\tau) \)); see § 3.2 for the definition of cells.

\( \text{(iii)} \) \( \sigma, \tau \in S_n \) are in the same two-sided cell if and only if \( P(\sigma), P(\tau) \) have the same shape.

Proof. \( \text{(i)} \) is straightforward from the definition. \( \text{(ii)} \) follows from [1, Thm. A] and \( \text{(i)} \). The “only if” part follows from the “if” part of \( \text{(iii)} \) follows from \( \text{(i)} \) of Lemma 4.2 and the known fact that the number of two-sided cells of \( S_n \) is equal to that of partitions of \( n \).

Lemma 4.2. Let \( Y \) be a Young tableau with \( c_i \) entries in the \( i \)-th column, and \( \sum_i c_i = n \). Let \( \sigma_Y \) be the longest element in the parabolic subgroup of \( S_n \) generated by \( s_k = (k, k + 1) \), \( k \in [1, n] \setminus \{ \sum_{j=1}^i c_j \mid i \geq 1 \} \). Then \( P(\sigma_Y) \) and \( Y \) have the same shape. More precisely, the entries in the \( i \)-th column of \( P(\sigma_Y) \) are the integers \( x \) such that \( \sum_{j=1}^{i-1} c_j < x \leq \sum_{j=1}^i c_j \).

Proof. This is straightforward from the algorithm for \( P(\sigma) \).

Definition 4.3. Let \( Y \) be a Young tableau with \( c_i \) entries in the \( i \)-th column. Define \( A(Y) \) to be the integer \( \sum_{i \geq 1} \frac{c_i(c_i - 1)}{2} \), which is the length of the element \( \sigma_Y \) in the above lemma.

Proposition 4.4. For \( \sigma \in S_n \), we have

\[ a(\sigma) = A(P(\sigma)) \]

Proof. Let \( Y = P(\sigma) \). Since \( P(\sigma_Y) \) and \( P(\sigma) \) are of the same shape (see Lemma 4.2), \( \sigma_Y \) and \( \sigma \) are in the same two-sided cell by Lemma 4.1(iii). Hence \( a(\sigma) = a(\sigma_Y) = l(\sigma_Y) = A(Y) \) by Lemma 3.5 (iii), (iv) and the Definition 4.3.

This proposition gives an efficient algorithm for Lusztig’s function \( a : S_n \to \mathbb{N} \). In next subsection we will give a similar algorithm for \( a(\lambda), \lambda \in \mathfrak{h}^* \) (see Definition 3.7).

4.3. Algorithm for \( a(\lambda) \). Recall that for \( \lambda \in \mathfrak{h}^* \) we write \( \lambda + \rho = (\lambda_1, \lambda_2, \ldots, \lambda_n) \). One can see that \( \lambda \) is an integral weight if and only if \( \lambda_i - \lambda_j \in \mathbb{Z} \) for all \( i, j \), and \( \lambda \) is antidominant if and only if \( \lambda_i \leq \lambda_j \) whenever \( \lambda_i - \lambda_j \in \mathbb{Z}, i < j \). If \( \sigma \in S_n \), then \( \sigma, \lambda + \rho = (\sigma(\lambda + \rho) = \sum_i \lambda_i c(i) = \sum_j \lambda_{\sigma^{-1}(j)} c_j \).

If \( \lambda \in \Lambda \) is an integral weight, we associated to the sequence \( (\lambda_1, \ldots, \lambda_n) \) an semistandard Young tableau \( P(\lambda) \); see § 4.1.

Lemma 4.5. Assume that \( \lambda \) is an integral weight.

\( \text{(i)} \) There is a unique \( \sigma_\lambda \in S_n \) such that for \( i < j \)

\[ \lambda_i \leq \lambda_j \text{ if and only if } \sigma_\lambda(i) < \sigma_\lambda(j) \]

(4.1)

\[ \lambda_i > \lambda_j \text{ if and only if } \sigma_\lambda(i) > \sigma_\lambda(j) \]

(4.2)
(ii) The element $\sigma_\lambda \in S_n$ is of minimal length such that $\sigma_\lambda \cdot \lambda$ is antidominant.

(iii) $P(\lambda)$ and $P(\sigma_\lambda)$ have the same shape. Hence

\[
a(\lambda) = A(P(\lambda));
\]

see Definition 4.3 for the definition of the function $A$.

Proof. (i) is obvious. Let $\sigma = \sigma_\lambda$. Then $\sigma_\lambda + \rho = (\lambda_{\sigma^{-1}(1)}, \lambda_{\sigma^{-1}(2)}, \ldots, \lambda_{\sigma^{-1}(n)})$. If $a = \sigma(p) < \sigma(q) = b$ then by (4.1), (4.2) we always have $\lambda_a \leq \lambda_b$, i.e. $\lambda_{\sigma^{-1}(a)} \leq \lambda_{\sigma^{-1}(b)}$. Thus $\sigma_\lambda$ is antidominant. To prove $\sigma$ is of minimal length we only need to prove that when $s_\alpha \lambda = \lambda$ with $\alpha \in \Phi^+$ we have $\sigma s_\alpha > \sigma$, or equivalently that if $\lambda_i = \lambda_j$, $i < j$ then $\sigma(\epsilon_i - \epsilon_j) \in \Phi^+$, i.e. $\sigma(i) < \sigma(j)$. But this is just (4.1).

(iii). By the construction of $\sigma_\lambda$ and the Schensted insertion algorithm, we see immediately that $P(\lambda)$ and $P(\sigma_\lambda)$ have the same shape. Then

\[
a(\lambda) \overset{\text{Def.}}{=} a(\sigma^{-1}_\lambda) \overset{\text{Lem.} 3.7}{=} a(\sigma_\lambda) \overset{\text{Prop.} 4.4}{=} A(P(\sigma_\lambda)) = A(P(\lambda)).
\]

This completes the proof.

Now we turn to the general case where $\lambda$ is not necessarily integral.

We fix a weight $\lambda \in \mathfrak{h}^*$. Denote by $X_{\lambda}$ or just $X$ the set of subsets $X$ of $[1, n]$ such that $i, j$ are in $X$ if and only if $\lambda_i - \lambda_j \in \mathbb{Z}$. If $X = \{i_1 \leq i_2 \leq \cdots \leq i_k\}$ then denote by $\lambda_X$ the weight such that $\lambda_X + \rho' = (\lambda_{i_1}, \lambda_{i_2}, \cdots, \lambda_{i_k})$, where $\lambda_X$ is viewed as an integral weight of $\mathfrak{sl}(k)$ and $\rho'$ is the half sum of positive roots of $\mathfrak{sl}(k)$.

Then we associate to $\lambda$ the set $P(\lambda)$, which consists of all the Young tableaux $P(\lambda_X)$ with $X \in X$. And we define $A(P(\lambda))$ to be $\sum_{X \in X} A(P(\lambda_X))$.

**Theorem 4.6.** Assume that $g = \mathfrak{sl}(n), \lambda \in \mathfrak{h}^*$. Then

\[
a(\lambda) = A(P(\lambda)).
\]

Hence by Proposition 3.8, we have

\[
\text{GKdim } L(\lambda) = \frac{n(n-1)}{2} - A(P(\lambda)).
\]

This gives rise to a combinatorial algorithm for GKdim $L(\lambda)$.

Proof. First we have

\[
W_{[\lambda]} = \prod_{X \in X} W_X
\]

where $W_X$ is the subgroup of $W = S_n$ generated by $(i, j)$ with $i, j \in X$. Then $w$ is of minimal length in $W_{[\lambda]}$ such that $w^{-1} \cdot \lambda$ is antidominant if and only if $w_X$ is of minimal length in $W_X$ such that $w_X^{-1} \cdot \lambda_X$ is antidominant, where $w = (w_X)_{X \in X}$. Then we have

\[
a(\lambda) = a(\lambda)_w \overset{\text{Lem.} 3.5(v)}{=} \sum_{X \in X} a(\lambda_X)(w_X)
\]

\[
= \sum_{X \in X} a(\lambda_X) \overset{\text{Lem.} 4.5}{=} \sum_{X \in X} A(P(\lambda_X)) = A(P(\lambda)).
\]

This completes the proof of the theorem. \qed
5. GKdim $L(\lambda)$ for $(p, q)$-dominant weights

Let $g = \mathfrak{sl}(n)$, $p + q = n$ with $p, q \in \mathbb{Z}_{\geq 1}$.

**Definition 5.1.** We say $\lambda \in h^*$ is $(p, q)$-dominant if $\lambda_i - \lambda_j \in \mathbb{Z}_{>0}$ for all $i, j$ such that $1 \leq i < j \leq p$ or $p + 1 \leq i < j \leq p + q$, where $\lambda + \rho = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. In particular, $\lambda_1 > \lambda_2 > \cdots > \lambda_p$ and $\lambda_{p+1} > \lambda_{p+2} > \cdots > \lambda_{p+q}$.

The results in this section give rise to a quick algorithm for $\text{GKdim} \ L(\lambda)$ when $\lambda$ is a $(p, q)$-dominant weight.

**Theorem 5.2.** Assume that $\lambda \in h^*$ is $(p, q)$-dominant.

(i) If $\lambda_1 - \lambda_{p+1} \in \mathbb{Z}$, i.e. $\lambda$ is an integral weight, then $P(\lambda)$ is a Young tableau with at most two columns. And in this case $\text{GKdim} \ L(\lambda) = m(n-m)$ where $m$ is the number of entries in the second column of $P(\lambda)$.

(ii) If $\lambda_1 - \lambda_{p+1} \notin \mathbb{Z}$, then $P(\lambda)$ consists of two Young tableaux with single column:

\[
\begin{array}{c}
\lambda_p \\
\vdots \\
\lambda_2 \\
\lambda_1
\end{array}
\quad
\begin{array}{c}
p+1 \\
\vdots \\
p+p \\
p+p+q
\end{array}
\]

And in this case $\text{GKdim} \ L(\lambda) = pq$.

**Proof.** The shape of $P(\lambda)$ follows directly from the definition. And the Gelfand-Kirillov dimension follows from Theorem 4.6.

5.1. **An algorithm for $m$.** By the above theorem, we only need to determine $m$ in the case where $\lambda$ is integral and $(p, q)$-dominant.

**Proposition 5.3.** Assume that $\lambda$ is an integral and $(p, q)$-dominant weight. Then we have the following three cases.

Case (i) $\lambda_{p+1} \geq \lambda_p > \lambda_{p+q}$. Then we can find an integer $1 \leq k \leq q$ such that $\lambda_p \leq \lambda_{p+k}$ and $\lambda_p > \lambda_{p+k+i}$ for all $i > 0$. Let $\lambda'$ be the weight corresponding to $(\lambda_1, \lambda_2, \ldots, \lambda_{p-1}, \lambda_{p+1}, \ldots, \lambda_{p+k-1})$, i.e. deleting the entries $\lambda_p, \lambda_{p+k+j}, j \geq 0$.

Then $P(\lambda)$ is obtained from $P(\lambda')$ in the following way.

$P(\lambda') = \quad \longrightarrow \quad P(\lambda)$
In other words, the second column of $P(\lambda)$ can be obtained from that of $P(\lambda')$ by adding a box containing $\lambda_{p+k}$ on the top.

Case (ii) $\lambda_p \leq \lambda_{p+q}$. Let $\lambda'$ be the weight corresponding to $(\lambda_1, \lambda_2, \cdots, \lambda_{p-1}, \lambda_{p+1}, \cdots, \lambda_{p+q-1})$, i.e. deleting the entries $\lambda_p, \lambda_{p+q}$ from $\lambda$. Then $P(\lambda)$ is obtained from $P(\lambda')$ in the following way.

\[
\begin{array}{c}
\lambda_p \\
\lambda_{p+q}
\end{array}
\]

\[
\begin{array}{c}
P(\lambda') =
\end{array}
\]

\[
= P(\lambda)
\]

In other words, the second column of $P(\lambda)$ is obtained from that of $P(\lambda')$ by adding a box containing $\lambda_{p+q}$ on the top.

Case (iii) $\lambda_p \geq \lambda_{p+1}$. Then $P(\lambda)$ is just a Young tableau with single column.

Proof. This directly follows from Schensted insertion algorithm (see §4.1). □

This proposition gives rise to an algorithm for $P(\lambda)$ in the following way. First, we construct a sequence of weights $\lambda(0), \lambda(1), \cdots, \lambda(l)$ for some $l$. Here $\lambda(0) = \lambda$. Assume that we have obtained $\lambda(i)$ in some step. If the entries corresponding to $\lambda(i)$ are not strictly decreasing, then we set $\lambda(i+1) = \lambda(i)'$ as we do in Proposition 5.3(i)(ii). Otherwise, we set $\lambda(i)$ to be the last weight $\lambda(l)$.

Note that $P(\lambda(l))$ is an empty Young tableau or a Young tableau with single column. Then one can obtain $P(\lambda(l-1))$ from $P(\lambda(l))$ by using (i) or (ii) of Proposition 5.3. Repeating this process, we get $P(\lambda(i))$ from $P(\lambda(i+1))$ for all $i$, in particular $P(\lambda) = P(\lambda(0))$. From the process we know that the number $m$ of entries in the second column of $P(\lambda)$ is just $l$ and $m \leq \min(p, q)$. Then GKdim $L(\lambda)$ also follows from this algorithm by using Theorem 5.2(i).

In fact, we do not need to write down the each $P(\lambda(i))$, since we can immediately write down the second column of $P(\lambda)$ from $\lambda(i)$’s; see the following example.

Example 5.4. Let $n = 10, p = 4, q = 6$ and $\lambda + \rho = (6, 5, 3, 2, 9, 8, 7, 4, 2, 1)$.

\[
\begin{array}{c|cc|c|c|c|c}
9 & 8 & 6 & 5 & 3 & 2 & 1 \\
\hline
2 &  &  &  &  &  & \\
\end{array}
\]

Using the above notation, we have

$\lambda(1) \rightarrow (6, 5, 3, 9, 8, 7, 4),$

$\lambda(2) \rightarrow (6, 5, 9, 8, 7),$

$\lambda(3) \rightarrow (6, 9, 8),$. 

\[ \lambda(4) \rightarrow (9). \]

One can see that the second column of \( P(\lambda) \) is \{8, 7, 4, 2\}. Hence we have

\[
P(\lambda) = \begin{bmatrix}
1 & 2 \\
2 & 4 \\
3 & 7 \\
5 & 8 \\
6 & 9
\end{bmatrix}
\]

\[ \text{GKdim } L(\lambda) = 4 \times 7 = 24. \]

5.2. A combinatorial model. Let \( \lambda \) be an integral and \((p, q)\)-dominant weight such that \( \lambda + \rho = (\lambda_1, \lambda_2, \ldots, \lambda_n) \). Let \( a_1 \) be the number of \( \lambda_{p+i}, i \geq 1 \) such that \( \lambda_{p+i} \geq \lambda_1 \) and let \( b_1 \) be the number of \( \lambda_j, j \geq 1 \) such that \( \lambda_j > \lambda_{p+a_1+1} \).

Inductively, let \( a_{k+1} \) be the number such that

\[
a_{k+1} = \begin{cases}
\#(\lambda_{p+i} \mid \lambda \sum_{t=1}^{k} b_t > \lambda_{p+i} \geq \lambda_{(\sum_{t=1}^{k} b_t)+1}) , & \text{if } (\sum_{t=1}^{k} b_t) + 1 \leq p; \\
\#(\lambda_{p+i} \mid \lambda \sum_{t=1}^{k} b_t > \lambda_{p+i} ), & \text{if } \sum_{t=1}^{k} b_t = p; \\
0, & \text{if } \sum_{t=1}^{k} b_t > p.
\end{cases}
\]

and \( b_{k+1} \) the number such that

\[
b_{k+1} = \begin{cases}
\#(\lambda_j \mid \lambda p + \sum_{t=1}^{k+1} a_t \geq \lambda_j > \lambda_{p+(\sum_{t=1}^{k+1} a_t)+1}) , & \text{if } (\sum_{t=1}^{k+1} a_t) + 1 \leq q; \\
\#(\lambda_j \mid \lambda p + \sum_{t=1}^{k+1} a_t \geq \lambda_j ) , & \text{if } \sum_{t=1}^{k+1} a_t = q; \\
0, & \text{if } \sum_{t=1}^{k+1} a_t > q.
\end{cases}
\]

Let \( r \in \mathbb{Z}_{\geq 1} \) be the minimal integer such that \( a_{r+1} = b_{r+1} = 0 \). Then we write

\[ \xi(\lambda) := (a_1, b_1, a_2, b_2, \ldots, a_r, b_r). \]

Note that \( a_i \) and \( b_i \) are positive except that \( a_1, b_1 \) may be 0. For the weight \( \lambda \) in Example 5.4, we have \( \xi(\lambda) = (3, 2, 1, 1, 1, 1, 1, 0) \).

From the definition of \( \xi(\lambda) \) we have the following lemma.

**Lemma 5.5.** If \( \lambda, \mu \) are integral and \((p, q)\)-dominant weights satisfying \( \xi(\lambda) = \xi(\mu) \), then \( P(\lambda) \) and \( P(\mu) \) have the same shape. In particular, \( \text{GKdim } L(\lambda) = \text{GKdim } L(\mu) \).

**Proof.** This follows from Proposition 5.3. \( \square \)

The rest of this section is devoted to a combinatorial model, which enables us to obtain \( m \) directly from \( \xi(\lambda) \).

Assume that we have \( p \) black balls and \( q \) white balls, which are arranged in a line such that from left to right there are \( a_1 \) white balls, \( b_1 \) black balls, \( a_2 \) white balls, \( \ldots, b_r \) black balls. We call two balls an adjacent white-black pair if they are adjacent with the left ball white and the right ball black. We remove the adjacent white-black pairs from this line of balls over and over again until there are no adjacent white-black pairs, i.e the remaining black balls are all in the left side of every white ball. One can check that the number of pairs that are removed in this process are independent of the order, which is denoted by \( G_r \).

**Theorem 5.6.** Let \( \lambda \) be an integral \((p, q)\)-dominant weight such that

\[ \xi(\lambda) = (a_1, b_1, \ldots, a_r, b_r). \]
Let \( m \) be the number of entries in the second column of \( P(\lambda) \). And let \( G_k, 1 \leq k \leq r \) be the number of adjacent white-black pairs in the sequence \((a_1, b_1, \cdots, a_k, b_k)\) of balls, described as above.

(i) We have \( m = G_r \). Hence \( \text{GKdim} \ L(\lambda) = G_r(n - G_r) \) by Theorem 5.2.

(ii) There is a recursive formula

\[
G_{k+1} = G_k + \min\{ \sum_{i=1}^{k} a_i - G_k, b_{k+1} \}, \quad \text{for} \quad k < r.
\]

Proof. (i) follows from Proposition 5.3.

(ii). After deleting all adjacent white-black pairs in the sequence \((a_1, b_1, \cdots, a_k, b_k)\), there are \( \sum_{i=1}^{k} b_i - G_k \) black balls on the left and \( \sum_{i=1}^{k} a_i - G_k \) white balls on the right. If we take \( a_{k+1}, b_{k+1} \) into account, then from left to right we have \( \sum_{i=1}^{k} b_i - G_k \) black balls, \( \sum_{i=1}^{k+1} a_i - G_k \) white balls, and \( b_{k+1} \) black balls. Thus we in addition obtain \( \min\{ \sum_{i=1}^{k} a_i - G_k, b_{k+1} \} \) adjacent white-black pairs. Then (ii) follows. \( \square \)

For the latter use, we formulate the following lemma.

**Lemma 5.7.** Let \( \xi \) be a sequence of white and black balls as above. Then, locally, we can do the following two kinds of operations on \( \xi \)

\[
\begin{array}{c}
\begin{array}{c}
\bigcirc \bigcirc \bigcirc \\
\bigcirc \bigcirc \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\bigcirc \bigcirc \bigcirc \bigcirc \\
\bigcirc \bigcirc \\
\end{array}
\end{array}
\end{array}
\]

such that the number of adjacent white-black pairs in \( \xi \) does not change under these operations.

Proof. It is an easy consequence of the fact that the number of adjacent white-black pairs is independent of the choice of deleting order. \( \square \)

If \( \lambda \) is as in Example 5.4, the sequence \( \xi(\lambda) \) of balls is illustrated as follows:

\[
\begin{array}{c}
\begin{array}{c}
\bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \\
\bigcirc \bigcirc \bigcirc \\
\end{array}
\end{array}
\]

Then we can remove 4 adjacent white-black pairs. And the remaining balls are arranged as follows:

\[
\begin{array}{c}
\begin{array}{c}
\bigcirc \bigcirc \\
\end{array}
\end{array}
\]

So \( m = G_r = 4 \). Using certain operations in Lemma 5.7, \( \xi(\lambda) \) can transform into

\[
\begin{array}{c}
\begin{array}{c}
\bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \\
\end{array}
\end{array}
\]

5.3. Another model. At last, we want to give an algebraic model to compute \( m \).

**Proposition 5.8.** Let \( A \) be a \( \mathbb{Z}[v] \)-algebra generated by \( x, y \) with only relation \( xy = v \) where \( v \) is an indeterminate. It is obvious that \( A \) has a standard \( \mathbb{Z}[v] \)-basis \( \{ y^t x^s \mid s, t \in \mathbb{N} \} \).

Let \( \lambda \) be an integral \((p, q)\)-dominant weight with \( \xi(\lambda) = (a_1, b_1, \cdots, a_r, b_r) \). Then we have

\[
x^{a_1}y^{b_1} \cdots x^{a_r}y^{b_r} = v^m y^{p-m} x^{q-m}
\]

where \( m \) is the number of entries in the second column of \( P(\lambda) \).

Proof. This directly follows from Theorem 5.6 by viewing white (resp. black) balls as \( x \) (resp. \( y \)). \( \square \)
Applying this proposition to Example 5.4, we have $x^3y^2xyz = v^m y^{4-m} x^{6-m}$, which implies that $m = 4$.

6. Application to $(g, K)$-modules

6.1. Application to unitary highest weight $SU(p, q)$-modules. The unitary highest weight $(g, K)$-modules had been classified by [4] and [8]. Let $L(\lambda)$ be a unitary highest weight $(g, K)$-module. We use notations from [4]. Let $\beta$ be the unique maximal root in the positive roots $\Phi^+$ and $\zeta$ be the unique weight orthogonal to $\Phi$ and satisfying $(\zeta, \beta^\vee) = 1$. Then we can write $\lambda = \lambda + z\zeta$, where $\lambda$ is a $t$-dominant weight of $\mathfrak{b}^\ast$ such that $(\lambda + \rho, \beta) = 0$, and $z \in \mathbb{R}$. A formula for the Gelfand-Kirillov dimensions of all unitary highest weight $(g, K)$-modules is found in [2]. Now by our algorithm, we can reprove it in the case $G = SU(p, q)$.

Assume that $\tilde{\lambda}$ is a $(p, q)$-dominant weight such that $\tilde{\lambda}_1 = \tilde{\lambda}_{p+q}$ where $\lambda + \rho = (\tilde{\lambda}_1, \cdots, \tilde{\lambda}_n)$. Without loss of generality, we can assume that all $\tilde{\lambda}_i \in \mathbb{Z}$. We have $\zeta = (1, 1, \cdots, 0, \cdots, 0)$. Let $p'$ be the maximal integer in $[1, p]$ such that $\lambda_1, \lambda_2, \cdots, \lambda_{p+q}$ are consecutive integers. And let $q'$ be the maximal integer in $[1, q]$ such that $\lambda_{p+q-q'+1}, \cdots, \lambda_{p+q}$ are consecutive integers.

From [4, Thm. 7.4], we have the following lemma.

**Lemma 6.1.** Let $z \in \mathbb{R}$. The simple highest weight module $L(\lambda + z\zeta)$ is a unitary $SU(p, q)$-module if and only if $z \in I_{\tilde{\lambda}}$, where

$$I_{\tilde{\lambda}} = \{z \in \mathbb{R} \mid z \leq \max \{p', q'\} \cup \{z \in \mathbb{Z} \mid z \leq p' + q' - 1\} \}.$$

Then we have the following result.

**Proposition 6.2.** Keep the notations as above. When $z \in I_{\tilde{\lambda}}$, the Gelfand-Kirillov dimension of $L(\lambda + z\zeta)$ only depends on $z$. Precisely,

(i) If $z \in I_{\tilde{\lambda}} \setminus \mathbb{Z}$, we have $\text{GKdim } L(\lambda + z\zeta) = pq$.

(ii) If $z \in I_{\tilde{\lambda}} \cap \mathbb{Z}$, we have

$$\text{GKdim } L(\lambda + z\zeta) = \begin{cases} pq, & \text{if } z < \max\{p, q\}; \\ (z + 1)(n - z - 1), & \text{if } \max\{p, q\} \leq z \leq p' + q' - 1. \end{cases}$$

**Proof.** (i) follows from Theorem 5.2(ii).

(ii). Let $\tilde{\mu} + \rho = (p, p - 1, \cdots, 1, p + q - 1, p + q - 2, \cdots, p)$. By using operations in Lemma 5.7 one can check that $P(\lambda + z\zeta)$ and $P(\tilde{\mu} + z\zeta)$ have the same shape when $z \in I_{\tilde{\lambda}} \cap \mathbb{Z}$. Then by Theorem 4.6, we have

$$\text{GKdim } (\lambda + z\zeta) = \text{GKdim } (\tilde{\mu} + z\zeta) \text{ for all } z \in I_{\tilde{\lambda}} \cap \mathbb{Z}.$$ Then (ii) is reduced to the computation of $\text{GKdim } (\tilde{\mu} + z\zeta)$, which explains why $\text{GKdim } L(\lambda + z\zeta)$ only depends on $z$ if $z \in I_{\tilde{\lambda}} \cap \mathbb{Z}$. Now we compute $\text{GKdim } L(\tilde{\mu} + z\zeta)$ for all $z \in \mathbb{Z}$. Using Theorem 5.6, one can check that the number $m$ of entries in the second column of $P(\tilde{\mu} + z\zeta)$ with $z \in \mathbb{Z}$ is

$$m = \begin{cases} \min\{p, q\}, & \text{if } z < \max\{p, q\}; \\ n - 1 - z, & \text{if } \max\{p, q\} \leq z \leq n - 1; \\ 0, & \text{if } z \geq n. \end{cases}$$
Hence by Theorem 5.2, we have
\[
\text{GKdim } L(\tilde{\mu} + z\zeta) = \begin{cases} 
pq, & \text{if } z < \max\{p, q\}; 
(z + 1)(n - 1 - z) & \text{if } \max\{p, q\} \leq z \leq n - 1; 
0 & \text{if } z \geq n.
\end{cases}
\]
which completes the proof. \(\square\)

6.2. Application to highest weight \(SU(p, q)\)-modules. It is proved in [2] that \(\text{GKdim } L(\bar{\lambda} + z\zeta)\) decreases as \(z\) takes unitary points and increases. Now we generalize this result to all highest weight \((\mathfrak{g}, K)\)-modules in the case \(G = SU(p, q)\).

Theorem 6.3. Let \(L(\tilde{\lambda} + z\zeta)\) be a highest weight \(SU(p, q)\)-module. Then
\[
\text{GKdim}(L(\tilde{\lambda} + z\zeta))
\]
will (weakly) decrease as \(z \in \mathbb{Z}\) increases. Moreover, \(\text{GKdim}(L(\tilde{\lambda} + z\zeta)) = 0\) if \(z\) is an integer bigger than \(\tilde{\lambda}_{p+1} - \tilde{\lambda}_p\).

Proof. By the combinatorial model in Theorem 5.6, the value of \(m\) for \(P(\tilde{\lambda} + (z+1)\zeta)\) is less than or equal to the value of \(m\) for \(P(\bar{\lambda} + z\zeta)\). So \(m\) decreases as the integer \(z\) increases. Note that \(m \leq \min(p, q) \leq \left[\frac{p}{2}\right]\). Then Theorem 5.2 implies that \(\text{GKdim } L(\bar{\lambda} + z\zeta)\) decreases as \(m\) decreases. So \(\text{GKdim } L(\bar{\lambda} + z\zeta)\) decreases as the integer \(z\) increases. \(\square\)

Now we turn to consider associated varieties of highest weight \((\mathfrak{g}, K)\)-modules. We have the decomposition \(\mathfrak{g} = \mathfrak{p}^- + \mathfrak{t} + \mathfrak{p}^+\) under the adjoint action of \(K_C\). From Vogan [19], we know that the associated variety of any highest weight \((\mathfrak{g}, K)\)-module \(L(\lambda)\) is the closure of some \(K_C\) orbit in \(\mathfrak{p}^+ \cong (\mathfrak{g}/(\mathfrak{t} + \mathfrak{p}^+))^*\), and closures of these \(K_C\) orbits form a linear chain of varieties:
\[
\{0\} = \breve{O}_0 \subseteq \breve{O}_1 \subseteq \ldots \subseteq \breve{O}_{r-1} \subseteq \breve{O}_r = \mathfrak{p}^+;
\]
where \(r\) is the rank of the Hermitian symmetric space \(G/K\). In [2], it is proved that
\[
\dim \breve{O}_k = k(\rho, \beta^\vee) - k(k - 1)C,
\]
where \(C\) is a constant only depending on \(G\).

In the case \(G = SU(p, q)\), we have \(\langle \rho, \beta^\vee \rangle = n - 1\) and \(C = 1\), and hence \(\dim \breve{O}_k = k(n - k)\). Since \(\text{GKdim } L(\lambda) = m(n - m)\) where \(m\) is the number of entries in the second column of \(P(\lambda)\), then the associated variety of \(L(\lambda)\) must be \(\breve{O}_m\). To summarize, we have the following theorem about Gelfand-Kirillov dimensions and associated varieties of a highest weight \((\mathfrak{g}, K)\)-modules in the case \(G = SU(p, q)\).

Theorem 6.4. Let \(G = SU(p, q)\). Let \(L(\lambda)\) be a highest weight \((\mathfrak{g}, K)\)-module.

(i) If \(\lambda\) is an integral weight, then
- (a) \(P(\lambda)\) is a Young tableau with at most two columns;
- (b) \(\text{GKdim } L(\lambda) = m(n - m)\) where \(m\) is the number of entries in the second column of \(P(\lambda)\);
- (c) The associated variety of \(L(\lambda)\) is \(\breve{O}_m\).

(ii) If \(\lambda\) is not integral, then
- (a) \(P(\lambda)\) consists of two Young tableaux with single column;
- (b) \(\text{GKdim } L(\lambda) = pq\);
- (c) The associated variety of \(L(\lambda)\) is \(\breve{O}_r\), \(r = \min\{p, q\}\).
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