Bi-differential calculus and the KdV equation

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Abstract

A gauged bi-differential calculus over an associative (and not necessarily commutative) algebra $\mathcal{A}$ is an $\mathbb{N}_0$-graded left $\mathcal{A}$-module with two covariant derivatives acting on it which, as a consequence of certain (e.g., nonlinear differential) equations, are flat and anticommute. As a consequence, there is an iterative construction of generalized conserved currents. We associate a gauged bi-differential calculus with the Korteweg-de-Vries equation and use it to compute conserved densities of this equation.

1 Introduction

A distinguishing feature of soliton equations and other completely integrable models is the existence of an infinite set of conservation laws. For the special case of two-dimensional (principal) chiral or $\sigma$-models, a simple iterative construction of conserved currents and charges had been presented in [1]. In [2, 3, 4] some generalizations of this work in the framework of noncommutative geometry have been achieved. In a recent work [5], the existence of an infinite set of conserved currents in several completely integrable classical models, including chiral and Toda models, as well as the KP and self-dual Yang-Mills equations, has been traced back to a simple construction of an infinite chain of closed (respectively, covariantly constant) 1-forms in a (gauged) bi-differential calculus. A bi-differential calculus consists of a graded algebra on which two anticommuting differential maps act. In a gauged bi-differential calculus these maps are extended to covariant derivatives which, as a consequence of, e.g., nonlinear differential equations, are flat and anticommuting.
Section 2 introduces a mathematical scheme which may be regarded as the crucial structure behind the appearance of an infinite chain of conserved currents in the abovementioned completely integrable models (see also [3]). Section 3 shows how to realize such a scheme in terms of bi-differential calculi and covariant derivatives. Section 4 treats the case of the Korteweg-deVries equation in some detail. Section 5 contains some conclusions.

2 The central mathematical construction

Let \( \mathcal{A} \) be an associative algebra over \( \mathbb{R} \) or \( \mathbb{C} \) with a unit \( \mathbf{1} \). In the following, a linear map is meant to be linear over \( \mathbb{R} \), respectively \( \mathbb{C} \). We consider an \( \mathbb{N}_0 \)-graded left \( \mathcal{A} \)-module \( \mathcal{M} = \sum_{r \geq 0} \mathcal{M}^r \), on which two linear maps \( D, \mathcal{D} : \mathcal{M}^r \to \mathcal{M}^{r+1} \) act such that

\[
D^2 = 0, \quad \mathcal{D}^2 = 0, \quad \mathcal{D}D = g \mathcal{D} \mathcal{D}
\]  

(2.1)

with some \( g \in \mathcal{A} \). Furthermore, we assume that, for some \( s > 0 \), there is a (nonvanishing) \( \chi^{(0)} \in \mathcal{M}^{s-1} \) with

\[
\mathcal{D}\chi^{(0)} = 0 .
\]  

(2.2)

Then

\[
J^{(1)} = \mathcal{D}\chi^{(0)}
\]  

(2.3)

is \( \mathcal{D} \)-closed, i.e.,

\[
\mathcal{D}J^{(1)} = g \mathcal{D} \mathcal{D}\chi^{(0)} = 0 .
\]  

(2.4)

If every \( \mathcal{D} \)-closed element of \( \mathcal{M}^s \) is \( \mathcal{D} \)-exact, then

\[
J^{(1)} = \mathcal{D}\chi^{(1)}
\]  

(2.5)

with some \( \chi^{(1)} \in \mathcal{M}^{s-1} \). Now let \( J^{(m)} \in \mathcal{M}^s \) satisfy

\[
\mathcal{D}J^{(m)} = 0, \quad J^{(m)} = \mathcal{D}\chi^{(m-1)} .
\]  

(2.6)

Then

\[
J^{(m)} = \mathcal{D}\chi^{(m)}
\]  

(2.7)

with some \( \chi^{(m)} \in \mathcal{M}^{s-1} \) (which is determined only up to addition of some \( \beta \in \mathcal{M}^{s-1} \) with \( \mathcal{D}\beta = 0 \)), and

\[
J^{(m+1)} = \mathcal{D}\chi^{(m)}
\]  

(2.8)
is also \( \mathcal{D} \)-closed:

\[
\mathcal{D} J^{(m+1)} = g D \chi^{(m)} = g D J^{(m)} = g D^2 \chi^{(m-1)} = 0 .
\]

(2.9)

In this way one obtains an infinite tower of \( \mathcal{D} \)-closed elements \( J^{(m)} \in \mathcal{M}^s \) and elements \( \chi^{(m)} \in \mathcal{M}^{s-1} \) which satisfy

\[
\mathcal{D} \chi^{(m+1)} = D \chi^{(m)} .
\]

(2.10)

In certain cases this construction may break down at some level \( m > 0 \) or become trivial in some sense (see also [5]). In terms of

\[
\chi = \sum_{m=0}^{\infty} \lambda^m \chi^{(m)}
\]

(2.11)

with a parameter \( \lambda \), the set of equations (2.10) leads to

\[
\mathcal{D} \chi = \lambda D \chi .
\]

(2.12)

Conversely, if the last equation holds for all \( \lambda \), we recover (2.10).

3 Bi-differential calculi and covariant derivatives

In this section we consider realizations of the structure introduced in the last section in terms of covariant exterior derivatives.

Definition 1. A \textit{graded algebra} over \( \mathcal{A} \) is an \( \mathbb{N}_0 \)-graded associative algebra \( \Omega(\mathcal{A}) = \bigoplus_{r \geq 0} \Omega^r(\mathcal{A}) \) such that \( \Omega^0(\mathcal{A}) = \mathcal{A} \) and the unit \( \mathbf{1} \) of \( \mathcal{A} \) extends to a unit of \( \Omega(\mathcal{A}) \), i.e., \( \mathbf{1} w = w \mathbf{1} = w \) for all \( w \in \Omega(\mathcal{A}) \).

Definition 2. A \textit{differential calculus} \( (\Omega(\mathcal{A}), d) \) over \( \mathcal{A} \) consists of a graded algebra \( \Omega(\mathcal{A}) \) over \( \mathcal{A} \) and a linear map \( d : \Omega^r(\mathcal{A}) \to \Omega^{r+1}(\mathcal{A}) \) with the properties

\[
d^2 = 0 ,
\]

(3.13)

\[
d(w w') = (d w) w' + (-1)^r w d w'
\]

(3.14)

where \( w \in \Omega^r(\mathcal{A}) \) and \( w' \in \Omega(\mathcal{A}) \). We also require that \( d \) generates \( \Omega(\mathcal{A}) \) in the sense that \( \Omega^{r+1}(\mathcal{A}) = \mathcal{A} (d \Omega^r(\mathcal{A})) \).

Definition 3. A triple \( (\Omega(\mathcal{A}), d, \delta) \) consisting of a graded algebra \( \Omega(\mathcal{A}) \) over \( \mathcal{A} \) and two linear maps \( d, \delta : \Omega^r(\mathcal{A}) \to \Omega^{r+1}(\mathcal{A}) \) with the properties (3.13), (3.14) and

\[
\delta d + d \delta = 0
\]

(3.15)

\(^1\text{The identity } \mathbf{1} \mathbf{1} = \mathbf{1} \text{ then implies } d \mathbf{1} = 0.\)
is called a **bi-differential calculus**.

Let \((\Omega(\mathcal{A}), d, \delta)\) be a bi-differential calculus, and \(A, B\) two \(N \times N\)-matrices of 1-forms (i.e., the entries are elements of \(\Omega^1(\mathcal{A})\)). We introduce

\[
D = d + A \quad \mathcal{D} = \delta + B
\]

which act from the left on \(N \times M\)-matrices with entries in \(\Omega(\mathcal{A})\). The latter form an \(\mathbb{N}_0\)-graded left \(\mathcal{A}\)-module \(\mathcal{M} = \bigoplus_{r \geq 0} \mathcal{M}^r\). Then the conditions \((2.1)\) with \(g = -1\) can be expressed in terms of \(A\) and \(B\) as follows,

\[
D^2 = 0 \iff F = dA + AA = 0 ,
\]

\[
\mathcal{D}^2 = 0 \iff F = \delta B + BB = 0 ,
\]

\[
D \mathcal{D} + \mathcal{D} D = 0 \iff dB + \delta A + BA + AB = 0 .
\]

If these conditions are satisfied, we speak of a **gauged bi-differential calculus**.

If \(B = 0\), the conditions \((3.17)-(3.19)\) become \(F = 0\) and \(\delta A = 0\). There are two obvious ways to further reduce the latter equations:

(i) We can solve \(F = 0\) by setting \(A = g^{-1} dg\) with an invertible \(N \times N\)-matrix \(g\) with entries in \(\mathcal{A}\). The remaining equation reads \(\delta (g^{-1} dg) = 0\) which resembles the field equation of principal chiral models (cf. [5]).

(ii) We can solve \(\delta A = 0\) via \(A = \delta \phi\) with a matrix \(\phi\). Then we are left with the equation \(d(\delta \phi) + (\delta \phi)^2 = 0\) which generalizes the so-called ‘pseudodual chiral models’ (cf. [3] and references cited there).

### 4 Example: conserved densities of the Korteweg-de-Vries equation

Let \(\mathcal{A}_0 = C^\infty(\mathbb{R} \times \mathcal{I})\) be the algebra of smooth functions of coordinates \(t, x\), where \(\mathcal{I}\) is an interval, and \(\mathcal{A}\) the noncommutative algebra generated by the elements of \(\mathcal{A}_0\) and the partial derivative \(\partial_x = \partial/\partial x\) such that \(\partial_x f = f_x + f \partial_x\) for \(f \in \mathcal{A}\). Here, \(f_x\) denotes the partial derivative of \(f\) with respect to \(x\). Furthermore, let \(\Omega^1(\mathcal{A})\) be the \(\mathcal{A}\)-bimodule generated by two elements \(\tau\) and \(\xi\) which commute with all elements of \(\mathcal{A}\). With

\[
\tau \xi = -\xi \tau, \quad \tau \tau = 0 = \xi \xi
\]

we obtain a graded algebra \(\Omega(\mathcal{A}) = \bigoplus_{r=0}^2 \Omega^r(\mathcal{A})\) over \(\mathcal{A}\). Now

\[
df = [\partial_t + 4f_3 x, f] \tau - 6 [\partial_x^3, f] \xi
\]

\[
hf = (f_t + 4f_{xxx} + 12 f_{xx} \partial_x + 12 f_x \partial_x^2) \tau - 6 (f_{xx} + 2f_x \partial_x) \xi ,
\]

\[
\delta f = -\frac{1}{2} [\partial_{x}^2, f] \tau + [\partial_{x}, f]\xi = -\frac{1}{2} (f_{xx} + 2f_x \partial_x) \tau + f_x \xi
\]
and
\[ d(f \tau + h \xi) = (df) \tau + (dh) \xi, \quad \delta(f \tau + h \xi) = (\delta f) \tau + (\delta h) \xi \] (4.4)
define two linear maps \(d, \delta : \Omega^r(A) \to \Omega^{r+1}(A)\), and \((\Omega(A), d, \delta)\) becomes a bi-differential calculus over \(A\).

**Remark.** The above calculus is *noncommutative* in the sense that differentials do not, in general, commute with elements of \(A\), even with those of the commutative subalgebra \(A_0\). In particular, we have \(x \delta x = (\delta x) x + \tau\). A (noncommutative) differential calculus is a basic structure in ‘noncommutative geometry’.

With \(B = 0\) and \(A \in \Omega^1(A)\), (3.19) becomes \(\delta A = 0\) which is solved by
\[ A = \delta v = -\frac{1}{2} (v_{xx} + 2v_x \partial x) \tau + v_x \xi \] (4.5)
with \(v \in A_0\). Then \(F = 0\) takes the form
\[ v_{tx} + v_{xxxx} - v_x v_{xx} = 0 \] (4.6)
With the substitution
\[ u = -v_x \] (4.7)
this becomes the *Korteweg-de-Vries* equation
\[ u_t + u_{xxx} + u u_x = 0 \] (4.8)

Let \(\mathcal{M} = \Omega(A)\). The general solution of \(\delta \chi^{(0)} = 0\) for \(\chi^{(0)} \in A\) is
\[ \chi^{(0)} = \sum_{n=0}^{\infty} c_n(t) \partial^n_x \] (4.9)
with functions \(c_n\) depending on \(t\) only. A particular solution is given by \(\chi^{(0)} = 1\). The equation (2.12) with \(D = \delta\) is equivalent to the two equations
\[ \chi_x = -\lambda (6 \chi_{xx} + u \chi + 12 \chi_x \partial x), \] (4.10)
\[ -\frac{1}{2} \chi_{xx} = \lambda \left( \chi_t + 4 \chi_{xxx} + \frac{1}{2} u_x \chi + u \chi_x + 6 \chi_{xx} \partial x \right). \] (4.11)

With
\[ \chi = \sum_{n=0}^{\infty} \chi_n \partial^n_x \] (4.12)
the first equation is turned into the following set of equations,

\[
\begin{align*}
\chi_{0,x} + \lambda (6 \chi_{0,xx} + u \chi_0) &= 0 \quad (4.13) \\
\chi_{n,x} + \lambda (6 \chi_{n,xx} + 12 \chi_{n-1,x} + u \chi_n) &= 0 \quad (n > 0) . \quad (4.14)
\end{align*}
\]

Inserting\(^2\)

\[
\chi_0 = e^{-\lambda \varphi}, \quad \varphi = \sum_{m=0}^{\infty} (6\lambda)^m \varphi^{(m)}
\]

(4.15)

(which sets \(\chi^{(0)} = 1\)) in (4.13), we get

\[
\varphi_x = u - 6 \lambda \varphi_{xx} + 6 \lambda^2 (\varphi_x)^2
\]

(4.16)

which in turn leads to

\[
\varphi_x^{(0)} = u, \quad \varphi_x^{(1)} = -u_x
\]

(4.17)

and

\[
\varphi_x^{(m)} = -\varphi_x^{(m-1)} + \frac{1}{6} \sum_{k=0}^{m-2} \varphi_x^{(k)} \varphi_x^{(m-2-k)}
\]

(4.18)

for \(m > 1\). Hence

\[
\begin{align*}
\varphi_x^{(2)} &= u_{xx} + \frac{1}{6} u^2 , \\
\varphi_x^{(3)} &= -(u_{xx} + \frac{1}{3} u^2)_x , \\
\varphi_x^{(4)} &= \frac{1}{6} \left[ \frac{1}{3} u^3 - (u_x)^2 + [u_{xxx} + \frac{1}{2} (u^2)_x]_x \right] , \\
\varphi_x^{(5)} &= -\frac{4}{27} u^3 + \frac{5}{6} (u_x)^2 + \frac{4}{3} u u_{xx} + u_{xxxx} \right]_x , \\
\varphi_x^{(6)} &= \frac{5}{216} \left[ u^4 - 12 u (u_x)^2 + \frac{36}{5} (u_{xx})^2 ight] \\
&\quad + [u_{xxxx} + \frac{5}{3} u u_{xx} + \frac{5}{6} u^2 u_x + 3 u_x u_{xxx}]_x , \\
\varphi_x^{(7)} &= -\frac{2}{27} u^4 + \frac{4}{3} u^2 u_{xx} + \frac{5}{3} u (u_x)^2 + \frac{14}{3} u_x u_{xxx} + 2 u u_{xxxx} \\
&\quad + \frac{10}{3} (u_{xx})^2 + u_{xxxxx}]_x ,
\end{align*}
\]

\(^2\)Direct use of the expansion (2.11) for \(\chi_0\) leads to \textit{nonlocal} conserved densities. The transformation from \(\chi_0\) to \(\varphi\) and subsequent expansion of \(\varphi\) leads to \textit{local} expressions, however.
\[
\varphi^{(8)} = \frac{7}{648} \left[ u^5 - 30 u^2 (u_x)^2 + 36 u (u_{xx})^2 - \frac{108}{7} (u_{xxx})^2 \right] \\
+ \frac{7}{3} u u_{xxxx} + \frac{7}{3} u_x u_{xxxx} + \frac{35}{3} u_x u_{xxx} \\
+ \frac{35}{18} u^2 u_{xxx} + \frac{95}{54} (u_x)^3 + \frac{35}{216} (u^4)_x + \frac{7}{2} (u^2 x u_{xx})_x, \\
\tag{4.25}
\]
\[
\varphi^{(9)} = -\frac{16}{405} u^5 + \frac{20}{9} u^2 (u_x)^2 + \frac{32}{27} u^3 u_{xx} + \frac{113}{9} (u_x)^2 u_{xx} + \frac{80}{9} u (u_{xx})^2 \\
+ \frac{112}{9} u u_x u_{xxx} + \frac{8}{3} u^2 u_{xxxx} + \frac{23}{2} (u_{xxx})^2 + \frac{56}{3} u_x u_{xxx} \\
+ 9 u_x u_{xxxx} + \frac{8}{3} u u_{xxxxx} + u_{xxxxxxx} x, \\
\tag{4.26}
\]
\[
\varphi^{(10)} = \frac{7}{1296} \left[ u^6 - 60 u^3 (u_x)^2 + 108 u^2 (u_{xx})^2 - 30 (u_x)^4 - \frac{648}{7} u (u_{xxx})^2 \right] \\
+ \frac{720}{7} (u_{xx})^3 + \frac{216}{7} (u_{xxxx})^2 + \frac{35}{18} u^4 x + \frac{35}{18} u^3 u_{xxx} \\
+ \frac{21}{2} u^2 u_x u_{xx} + \frac{95}{18} u (u_x)^3 + \frac{7}{2} u^2 u_{xxxx} + 20 u u_x u_{xxx} \\
+ \frac{455}{18} (u_x)^2 u_{xxx} + 35 u u_x u_{xxx} + \frac{69}{2} u x (u_{xx})^2 + 3 u u_{xxxxx} \\
+ \frac{35}{3} u_x u_{xxxxx} + 28 u_x u_{xxxx} + \frac{125}{3} u_{xxx} u_{xxxxx}, \\
\tag{4.27}
\]

and so forth. As a consequence of (4.10) and (4.11), we have
\[
\chi_{0,t} + \chi_{0,xxx} + \frac{1}{2} u \chi_{0,x} = 0. \\
\tag{4.28}
\]

In terms of \( \varphi \) this reads
\[
\varphi_t + \varphi_{xxx} - 3 \lambda \varphi_x \varphi_{xx} + \lambda^2 (\varphi_x)^3 + u \varphi_x / 2 = 0, \\
\tag{4.29}
\]

and application of \( \partial_x \) leads to a conservation law for \( \varphi_x \),
\[
\varphi_{xt} = -(\varphi_{xxx} - 3 \lambda \varphi_x \varphi_{xx} + \lambda^2 (\varphi_x)^3 + u \varphi_x / 2)_x. \\
\tag{4.30}
\]

Hence, the \( \varphi^{(m)}_x \) obtained above are conserved densities of the KdV equation. Let
\[
Q^{(m)} = \int_I \varphi^{(m)}_x \, dx \\
\tag{4.31}
\]

where \( dx \) is the ordinary (Lebesgue) measure on \( \mathbb{R} \). Here we assume that either \( u \) is periodic in \( x \) or that \( u \) and its \( x \)-derivatives vanish sufficiently rapidly at the (finite or infinite) boundaries.
of the interval \( I \), so that the above integrals exist (see also [7]). Note that \( \varphi^{(m)} \) will not, in general, be periodic or vanish at the ends of the interval, however. Now we have

\[
\frac{d}{dt} Q^{(m)} = \int_I \varphi_{xt}^{(m)} \, dx = 0 .
\] (4.32)

Neglecting \( x \)-derivatives (which do not contribute to (4.31)) in the expressions for \( \varphi^{(m)} \), we observe that \( Q^{(m)} = 0 \) for odd \( m \). The nonvanishing conserved charges are

\[
\begin{align*}
Q^{(0)} &= \int_I u \, dx \quad (4.33) \\
Q^{(2)} &= \frac{1}{6} \int_I u^2 \, dx \quad (4.34) \\
Q^{(4)} &= \frac{1}{6} \int_I \left[ \frac{1}{3} u^3 - (u_x)^2 \right] \, dx \quad (4.35) \\
Q^{(6)} &= \frac{5}{216} \int_I \left[ u^4 - 12 u (u_x)^2 + \frac{36}{5} (u_{xx})^2 \right] \, dx \quad (4.36) \\
Q^{(8)} &= \frac{7}{648} \int_I \left[ u^5 - 30 u^2 (u_x)^2 + 36 u (u_{xx})^2 - \frac{108}{7} (u_{xxx})^2 \right] \, dx \quad (4.37) \\
Q^{(10)} &= \frac{7}{1296} \int_I \left[ u^6 - 60 u^3 (u_x)^2 + 108 u^2 (u_{xx})^2 - 30 (u_x)^4 \right. \\
&\quad \left. - \frac{648}{7} u (u_{xxx})^2 + \frac{720}{7} (u_{xx})^3 + \frac{216}{7} (u_{xxxx})^2 \right] \, dx \quad (4.38)
\end{align*}
\]

and so forth. The integrands are in agreement (up to irrelevant constant factors) with \( T_1, \ldots, T_6 \) in [8], equations (5a)-(10a). Using computer algebra, it is easy to compute higher conserved charges. In [8] the uniqueness of the above sequence of conserved polynomial densities of the Korteweg-de-Vries equation has been shown. Therefore, the remaining freedom in the above construction cannot lead to additional polynomial conserved densities.

Remark. Application of the central construction in section 2 to the case under consideration requires that \( \delta \)-closed elements of \( \Omega^1(A) \) are \( \delta \)-exact. \( J \in \Omega^1(A) \) can be written as \( J = a \tau + b \xi \) with \( a, b \in A \). Then \( \delta J = 0 \) means \( a + b_x/2 + b \partial_x = c \), where \( c_x = 0 \). Introducing \( \chi(t, x) = \int^x b(t, x') \, dx' \), we have \( J = (c - \frac{1}{2} \chi_{xx} - \chi_x \partial_x) \tau + \chi_x \xi = \delta \chi + c \tau \). This does not work, however, for periodic boundary conditions on \( I \) (so that \( I \) is actually replaced by the circle \( S^1 \)), since the indefinite integral of a periodic function \( b \) need not be periodic. We still have the problem that the 1-form \( \tau \) is not \( \delta \)-exact in \( \Omega(A) \). But with an extension of \( A \) and \( \Omega(A) \) (see also [8], section 5.3) it becomes exact. This amounts to setting \( \tau = \delta y \) with an additional coordinate \( y \). Then \( \delta \)-closed elements of \( M^1 \) are indeed \( \delta \)-exact. ■
5 Conclusions

The existence of a gauged bi-differential calculus as a (non-trivial) consequence of certain (e.g., differential, difference, or operator) equations may turn out to be a common feature of completely integrable systems. In [5] we have demonstrated that this concept covers many of the known soliton equations and other (in some sense) integrable models. The relation with various notions of complete integrability and approaches towards a classification of integrable models still has to be explored further. Moreover, the notion of a (gauged) bi-differential calculus and its generalization considered in section 2 applies to a large variety of structures (based on noncommutative algebras) most of which are far away from classical completely integrable models. It generalizes a characteristic feature of such models, namely the existence of an infinite set of conserved currents, into a framework of noncommutative geometry where an appropriate notion of complete integrability according to our knowledge is not yet at hand.

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