A survey on partial Nambu-Poisson structures in the convenient setting

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Abstract

This paper is a survey (may be incomplete) on (partial) Nambu-Poisson structures in infinite dimension, mainly in the convenient setting. These ones can be seen as a generalization of both partial Poisson and Nambu-Poisson structures. We also study the properties of the associated characteristic distribution. Finally, we are interested in the projective and direct limits of such structures.

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1 Introduction

In finite dimension, the notion of Poisson structure corresponds to a smooth manifold $M$ whose algebra of smooth functions $\mathcal{C}^\infty(M)$ is endowed with a bracket $\{\cdot,\cdot\}$, i.e. a bilinear skew-symmetric map $\mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ which both satisfies Jacobi and Leibniz identities. To any smooth function $f$, the bracket gives rise to a derivation $g \mapsto \{f,g\}$ which corresponds to a vector field $X_f$ called the Hamiltonian vector field associated to the Hamiltonian $f$.

An extension of such a structure to the Banach context was firstly defined in [OdRa03] and studied in a series of papers by A. Odzijewicz, T. Ratiu and their collaborators (see for instance [OdRa08] and [Rat11]). In the case
of a manifold modelled on a non-reflexive Banach space, the existence of the Hamiltonian vector field $X_f$ requires an additional condition on the Poisson tensor $P$ associated to the bracket.

In the convenient setting, F. Pelletier and P. Cabau consider

- the algebra $\mathfrak{A}(M)$ of smooth functions $f$ on $M$ whose differential $df$ induces a section of a subbundle $T^b M$ of the kinematic cotangent bundle $T' M$;
- a bundle morphism $P : T^b M \to TM$ such that
  \[
  \{ f, g \}_P = dg(P(df))
  \]
  defines a Poisson bracket on $\mathfrak{A}(M)$.

In the Banach setting, A. B. Tumpach considers a notion of generalized Banach Poisson manifold in [Tum20] which is a triple $(M, F, \pi)$ consisting of

- a smooth Banach manifold $M$;
- a subbundle $F$ of the cotangent bundle $T^* M$ where each fibre $F_x$ separates points in $T_x M$;
- $\pi$ is a section of $\wedge^2 F$ which is a Poisson tensor in the sense of Definition 3.5 in [Tum20].

When the inclusion of $F$ into $T^* M$ is continuous, we recover the previous definition. This situation occurs for instance in the framework of Banach Poisson Lie groups as nicely exposed and illustrated in this paper.

When $M$ is a smooth manifold modelled on a locally convex topological vector space, K.H. Neeb, H. Sahlmann and T. Thiemann ([NST14]) consider a unital subalgebra $\mathcal{A}$ of $C^\infty(M)$ provided with a Poisson bracket satisfying a separation condition and assume the existence of a Hamiltonian vector field for any function of $\mathcal{A}$.

The extension of this structure to functional spaces in order to study evolution equations reveals some difficulties. Since the product of two functionals is not well defined (cf. [Olv93], p. 357), the Leibniz rule has no counterpart in this situation.
On the other hand, in finite dimension, another type of generalization of Poisson structures was introduced by Nambu in 1973 ([Nam73]) where the setting of standard Hamilton equations of motion in even dimensional phase space with a single Hamiltonian is extended to a phase space of three (and more generally, arbitrary) dimensions with two Hamiltonians. The compatibility of such functions gives rise to a bi-Hamiltonian structure on the manifold (cf. [MaMo84]). Such a generalization have a large application on mechanics and in particular in fluid mechanics (see for example [Nam73], [NeBl93], [Takh94], [Gau96], [BlBa15], [Guh04], [Mor20], ...). Note that the reader can find in [Vais99] a global survey in finite dimension of Nambu-Poisson structure.

The principal purpose of this paper is to give a survey (may be incomplete) on Nambu-Poisson structures in finite and, above all, infinite dimension. Essentially, we introduce the notion of partial Nambu-Poisson structure in the convenient setting. This appears as a generalization of both partial Poisson and Nambu-Poisson structures.

More precisely, at first, we recall some classical examples of Nambu-Poisson structures in finite dimension and we reformulate the different definitions and properties required for a Nambu-Poisson bracket. (cf. section 2).

Then, in section 3 we define the notion of partial Nambu-Poisson structure on a convenient manifold and study the properties of the characteristic distribution associated to such a structure. Essentially we show that, as in finite dimension, such a Nambu-Poisson structure is very rigid and we recover the essential results as in finite dimension (cf. Theorem 3.15 and Corollary 3.19):

− a local model around a regular point (as a Darboux Theorem for symplectic forms);

− the characteristic distribution is a finite dimensional involutive distribution, not necessarily integrable in the convenient setting, but always integrable in the Banach framework;

− locally, a partial Nambu-Poisson structure can always be the restriction of a Nambu-Poisson one.

− in the Banach setting, a Leibniz algebroid structure on \( \wedge^{r-1} T^*M \) is associated to a (partial) \( r \)-Nambu-Poisson structure.
As an illustration of the first three previous points, we give examples of (partial) \(r\)-Nambu-Poisson structures on a Banach-Lie group, on the Fréchet manifold of loop spaces and on the convenient manifold of volume preserving form on a compact manifold. (cf. section 4).

To end this work, we study some type of projective (resp. direct) limits of \(r\)-Nambu-Poisson Banach manifolds. If we assume that one of these \(r\)-Nambu-Poisson structures has at least a regular point, any of them also has a regular point and we obtain an \(r\)-Nambu-Poisson structure on the projective (resp. direct) limit. This structure has the same properties as in the Banach setting. In particular, its characteristic foliation is directly obtained as the pull-back (resp. the union) of the characteristic distribution of any component (resp. of all components) of the \(r\)-Nambu-Poisson Banach manifolds (cf. section 5).

In order to be as self-contained as possible, the reader will find in the appendices all needed definitions and properties about projective and direct limits and results about the Schouten bracket on the subbundle \(T^0 M\).

2 Partial Nambu-Poisson brackets

2.1 Nambu-Poisson bracket

Nambu structures introduced by Nambu in [Nam73] are a generalization of Poisson structures. In this paper, he proposed the following system of equations of motion for the flows of a point \((x^1, x^2, x^3)\) in the space \(\mathbb{R}^3\)

\[
\frac{dx^i}{dt} = \{H, K, x^i\}, \quad i \in \{1, 2, 3\}
\]

where the (Nambu-Poisson) bracket on the r.h.s. is defined for any triple of functions \((F_1, F_2, F_3)\) on the phase space in terms of three-dimensional Jacobian

\[
\{F_1, F_2, F_3\} = \frac{\partial (F_1, F_2, F_3)}{\partial (x^1, x^2, x^3)} = \varepsilon^{ijk} \frac{\partial F_1}{\partial x^j} \frac{\partial F_2}{\partial x^k} \frac{\partial F_3}{\partial x^l}.
\]

Therefore, the vector fields \(X_i = \{H, K, x^i\} \ (i \in \{1, 2, 3\})\) generating the lines of flows in the phase space can be expressed with both hamiltonians \(H\) and \(K\):

\[
X_i = \varepsilon^{ijk} \frac{\partial H}{\partial x^j} \frac{\partial K}{\partial x^k}
\]

This leads to a natural extension of the equations of Hamilton-Jacobi (cf. [Yon17]).
An important development was laid by L. Takhtajan who, in 1994, gave a formalism in terms of brackets on the algebra of smooth functions on a finite dimensional manifold. From the geometric point of view, the so-called Nambu-Poisson structures introduced in [ILMM97] constitutes an interesting generalization of Poisson geometry.

Let \( r \) be an integer such that \( r \geq 2 \).

- Let \( V \) be a vector space. An \( r \)-bracket on \( V \) is a \( k \)-linear skew-symmetric map
  \[
  V^r \rightarrow V \\
  (f_1, \ldots, f_r) \mapsto \{f_1, \ldots, f_r\}
  \]
  This bracket satisfies the fundamental identity (or Filippov identity) if
  \[
  \{f_1, \ldots, f_{r-1}, \{g_1, \ldots, g_r\}\} = \sum_{i=1}^{r} \{g_1, \ldots, g_{i-1}, \{f_1, \ldots, f_{r-1}, g_i\}, g_{i+1}, \ldots, g_r\}
  \]
  (FI)
  for every \( f_1, \ldots, f_{r-1}, h, g_1, \ldots, g_r \) in \( V \).
  For \( r = 2 \), (FI) is nothing but the Jacobi identity.

- Let \( A \) be an associative and commutative algebra.
  The \( r \)-bracket satisfies the Leibniz identity if
  \[
  \{f_1, \ldots, f_{r-1}, gh\} = \{f_1, \ldots, f_{r-1}, g\} h + g \{f_1, \ldots, f_{r-1}, h\}
  \]
  (L)
  for every \( f_1, \ldots, f_{r-1}, g \) in \( A \).

**Remark 2.1.** According to Definition [C.8], the Leibniz property for an \( r \)-bracket on an algebra \( A \) implies that, for any fixed \( f_1, \ldots, f_{r-1} \) in \( A \), the map
  \[
  g \mapsto \{f_1, \ldots, f_{r-1}, g\}
  \]
  is an \((r-1)\)-alternating derivative on \( A \). Note that the Leibniz property cannot be defined on a vector space \( A \) without a ring structure. However, the Filippov identity may occur even if \( A \) has not a ring structure (cf. Example [2.10] and Remark [2.11]).

**Definition 2.2.** Let \( r \) be an integer such that \( r \geq 2 \).

1. An \( r \)-bracket on a linear vector space algebra \( V \) is called an \( r \)-Lie bracket if it satisfies the fundamental identity (FI).
2. An $r$-bracket on a associative and commutative algebra $A$ is called an *almost $r$-Nambu-Poisson bracket* if it satisfies the Leibniz property (L).

3. An $r$-bracket on an associative and commutative algebra $A$ is called an *$r$-Nambu-Poisson bracket* if it both satisfies (FI) and (L).

### 2.2 Partial almost Nambu-Poisson bracket

In contrast to the finite dimensional manifolds context, any local smooth function on a convenient manifold $M$, modelled on a convenient space $\mathbb{M}$, may not be extended to a global one, if there does not exist bump functions on $\mathbb{M}$. Therefore, the algebra $\mathcal{F}(M)$ of smooth functions on $M$ restricted to an open set $U$ is exactly the algebra $\mathcal{F}_U$ of smooth functions on $U$. Such a situation also exists in the Banach setting. Since many classical examples of convenient manifolds do not have bump functions, we will consider Nambu-Poisson brackets on sets of smooth functions on open sets in $M$.

**Definition 2.3.** Let $M$ be a convenient manifold and let $r$ be an integer such that $r \geq 2$.

Consider $\mathcal{V}$ (resp. $\mathcal{A}$) a linear subspace (resp. a sub-algebra) of the vector space (resp. associative and commutative algebra) $C^\infty(M)$ of smooth functions on $M$.

1. A *partial $r$-Lie bracket* on $M$ is an $r$-Lie bracket on $\mathcal{V}$.

2. A *partial almost Nambu-Poisson bracket* on $M$ is an almost Nambu-Poisson bracket on $\mathcal{A}$.

3. A *partial Nambu-Poisson bracket* on $M$ is a Nambu-Poisson bracket on $\mathcal{A}$.

**Definition 2.4.** Let $M$ be a convenient manifold and let $r$ be an integer such that $r \geq 2$. For any open set $U$ in $M$, we set $\mathcal{V}_U = C^\infty(U)$ (resp. $\mathcal{A}_U = C^\infty(U)$) the vector space (resp. commutative and associative algebra) of smooth functions on $U$.

1. An *$r$-Lie algebra structure* on $U$ is an $r$-Lie algebra structure on $\mathcal{A}_U$.

2. An *almost Nambu-Poisson structure* on $U$ is an almost Nambu-Poisson bracket on $\mathcal{V}_U$. 
3. A Nambu-Poisson structure of order $r$ or $r$-Nambu-Poisson bracket on $U$ is an $r$-Nambu-Poisson bracket on $A_U$.

**Remark 2.5.** According to Remark 2.1, Leibniz property in $F_U$ implies that, for every $(f_1,\ldots, f_{r-1}) \in F_{r-1}$,

$$D_{f_1,\ldots,f_{r-1}} : g \mapsto \{f_1,\ldots,f_{r-1},g\}$$

is an $(r-1)$ alternating derivation of the algebra $F_U$. Since, for any $x \in U$, we have $T^*_x M = \{d_x f, f \in F_U\}$, when $D_{f_1,\ldots,f_{r-1}}$ is a derivation of 1st-order (cf. Definition C.3), then we can define a smooth $r$-skew-symmetric tensor $\Lambda$ on $U$ by

$$\Lambda_x(d_x f_1,\ldots,d_x f_r) = \{f_1,\ldots,f_r\}$$

and since $\{d_x f, f \in F_U\}$ is separating, then

$$x \mapsto D_{f_1,\ldots,f_{r-1}} = \Lambda(df_1,\ldots,df_{r-1},\cdot)$$

defines a vector field on $U$.

Such a situation is always true in finite dimension. Unfortunately in the Banach setting, $D_{f_1,\ldots,f_{r-1}}$ can be an $(r-1)$-alternating derivation of order greater than 2 and so no vector field can be associated with such a derivation (cf. [BGT18]).

**Example 2.6.** Nambu bracket on the dual of a semi-simple $n$-dimensional Lie algebra.— We present an explicit construction method realized in [BiMo91] (see also [SBH11]).

Note that an analogous triple bracket can be found in [BMR12], 4.2.1 in the framework of metriplectic systems on a quadratic Lie algebra.

Let $g$ be a semi-simple $n$-dimensional Lie algebra endowed with the bracket $[,]$ and let $(e_1,\ldots,e_n)$ be a basis of $g$. The structure constants $c^k_{ij}$ of the Lie algebra $g$ are defined by

$$[e_i,e_j] = \sum_{k=1}^{n} c^k_{ij} e_k.$$
If \((x_i)_{i \in \{1, \ldots, n\}}\) denote coordinates on the dual space \(g^*\), the Lie-Poisson bracket is given by

\[
\forall (F_1, F_2) \in (C^\infty (g^*))^2, \quad \{F_1, F_2\}_{LP} = \sum_{i,j,k=1}^n c_{ij}^k x_k \frac{\partial F_1}{\partial x_i} \frac{\partial F_2}{\partial x_j}.
\]

Every element \(x\) of \(g\) defines the adjoint endomorphism \(\text{ad}_x\) of \(g\) where

\[
\text{ad}_x(y) = [x, y]
\]

The trace of the compositions of two such endomorphisms defines a symmetric bilinear form \(K\) as

\[
K(x, y) = \text{trace} (\text{ad}_x \circ \text{ad}_y)
\]

and called the Killing form.

Its expression in coordinates is given by

\[
K_{ij} = \sum_{k,l=1}^n c_{ik}^j c_{jl}^k
\]

A Nambu-Poisson bracket can then be defined as

\[
\forall (F_1, F_2, F_3) \in (C^\infty (g))^3, \quad \{F_1, F_2, F_3\} = \sum_{i,j,k=1}^n N_{ijk} \frac{\partial F_1}{\partial x_i} \frac{\partial F_2}{\partial x_j} \frac{\partial F_3}{\partial x_k}
\]

where

\[
\forall (i, j, k) \in \{1, \ldots, n\}^3, \quad N_{ijk} = \sum_{l=1}^n c_{ij}^l K_{lk}.
\]

Since the Lie algebra \(g\) is semi-simple, then the Killing form is invertible (Cartan’s criterion for semi-simplicity) and the expression of its inverse in coordinates is denoted \(K^{ij}\).

The function \(C : (x_i, x_j) \mapsto \frac{1}{2} \sum_{i,j=1}^n K^{ij} x_i x_j\) is a Casimir function for the Lie-Poisson bracket, i.e. \(\{C, \cdot\}_{LP} = 0\). We then have the following relation

\[
\{F_1, F_2, C\} = \{F_1, F_2\}_{LP}.
\]

An illustration in Mechanics of such a situation is given by the dual \(\mathfrak{so}(3)^*\) of the Lie algebra \(\mathfrak{so}(3)\) associated to the Lie group \(SO(3)\) which corresponds to the configuration space of the rigid body. The Lie group \(SO(3)\)
is the connected component of the orthogonal Lie group $O(3)$ containing the identity; so we have

$$SO(3) = \left\{ M \in \mathcal{M}_3(\mathbb{R}) : \begin{cases} M \times M^t = \text{Id}_{\mathbb{R}^3} \\ \det M = 1 \end{cases} \right\}.$$  

Differentiating at $\text{Id}_{\mathbb{R}^3}$ any smooth path $t \mapsto M(t)$ in $SO(3)$ passing through this neutral element of the group, it is easy to see that the Lie algebra $\mathfrak{so}(3)$ consists of skewsymmetric $3 \times 3$ matrices.

The vectors of this 3-dimensional vector space are called *angular velocities* (cf. [ArKh98], Ch. 1, § 2).

A basis of this Lie algebra is $(R_x, R_y, R_z)$ where

$$R_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad R_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix}, \quad R_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

are the matrices of rotation about $x$-axis, $y$-axis and $z$-axis respectively.

We have the following expressions of the brackets

$$[R_x, R_y] = R_z, \quad [R_y, R_z] = R_x, \quad [R_z, R_x] = R_y.$$

If $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$ is an orthonormal basis of the Euclidean vector space $\mathbb{R}^3$, there exists an isomorphism

$$j : \mathbb{R}^3 \rightarrow \mathfrak{so}(3)$$

$$a\vec{e}_x + b\vec{e}_y + c\vec{e}_z \mapsto \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$$

equivariant with respect to the natural action of $SO(3)$ on $\mathbb{R}^3$ and its adjoint action to its Lie algebra $\mathfrak{so}(3)$ (cf. [Marl20]).

A straightforward computation gives

$$P = -2 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3}$$

where $(x_1, x_2, x_3)$ is an adapted system of coordinates on $\mathfrak{g}^*$.

### 2.3 Some classical examples of partial Nambu-Poisson brackets

**Example 2.7.** Lie-Poisson bracket on the dual of a Banach-Lie algebra not necessarily reflexive.— Let $G$ be a Banach-Lie group whose Lie algebra $\mathfrak{g}$ is not necessarily reflexive and let $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$. 

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In order to define the Lie-Poisson bracket, we must restrict to the set $A$ of regular functions, i.e. functions $f \in C^\infty (\mathfrak{g}^*)$ whose derivatives $df(m) : \mathfrak{g}^* \to \mathbb{R}$ belongs to the subspace $\mathfrak{g}$ of $\mathfrak{g}^{**}$.

Then the Lie-Poisson bracket is defined as follows:

$$\forall (f,g) \in A^2, \forall m \in \mathfrak{g}^*, \{f,g\}_{LP}(m) = \langle m, [df(m), dg(m)] \rangle$$

where $\langle ., . \rangle$ is the natural pairing $\mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ and $df(m)$ and $dg(m)$ are considered as elements of $\mathfrak{g}$.

This makes sense because $[df(m), dg(m)]$ can also be seen as an element of $\mathfrak{g}$ and also because $\{f,g\}_{LP} \in A$.

This bracket satisfies both Leibniz and Jacobi identities. So, $(\mathfrak{g}^*, A(M), \{., .\}_{LP})$ is a partial 2-Poisson-Nambu manifold.

Note that if $G$ is a finite dimensional Lie group, this 2-Poisson-Nambu structure is nothing but the classical Kirillov-Kostant-Souriau Poisson structure on $\mathfrak{g}^*$.

**Example 2.8.** The weak symplectic manifold $(l^\infty \times l^1, \omega).$— For this example, the reader is referred to [OdRa08]. A weak symplectic form $\omega$ on a Banach manifold $M$ is a closed 2-form ($d\omega = 0$) such that, for every $x \in M$, the map

$$\omega^\flat_x : T_x M \to T_x M^*$$

is injective. A weak symplectic manifold $(M, \omega)$ is a Banach manifold $M$ endowed with a weak symplectic form $\omega$.

Since the map $\omega^\flat_x$ in only injective, the Poisson bracket $\{f, g\}$ cannot be defined for arbitrary $(f, g) \in C^\infty (M)^2$. In order to define the Hamiltonian vector field associated to a smooth function $f$ by $\iota_{X_f} \omega = df$, it is necessary that for every $x$ in $M$, $df(x) \in \omega^\flat_x(T_x M)$.

$$\mathcal{A} = \{ f \in C^\infty (M) : \forall x \in M, \ df(x) \in \omega^\flat_x(T_x M) \}$$

is an algebra.

If $(f, g) \in \mathcal{A}^2$, the Hamiltonian vector fields $X_f$ and $X_g$ exist and one defines the Poisson bracket by

$$\{f, g\}_\omega = \omega (X_f, X_g).$$

$\mathcal{A}$ is then a Poisson algebra, i.e. an algebra relative to multiplication of functions, a Lie algebra relative to the Poisson bracket and the Leibniz
identity holds. As an example of such a situation consider the Banach space \( \ell^\infty \times \ell^1 \) where

\[
\ell^\infty = \left\{ q = (q_n)_{n \in \mathbb{N}} : \|q\|_\infty = \sup_{n \in \mathbb{N}} |x_n| < +\infty \right\}
\]
is the Banach space of bounded real sequences and

\[
\ell^1 = \left\{ p = (p_n)_{n \in \mathbb{N}} : \|p\|_1 = \sum_{n=0}^{+\infty} |x_n| < +\infty \right\}
\]
is the Banach space of absolutely convergent real sequences. The strongly non degenerate duality pairing

\[
\forall (q,p) \in \ell^\infty \times \ell^1, \quad \langle q,p \rangle = \sum_{n=0}^{+\infty} q_n p_n
\]
corresponds to the Banach space isomorphism \((\ell^1)^* = \ell^\infty\).

The weak symplectic form \( \omega \) is the canonical one given by

\[
\forall (q,q',p,p') \in (\ell^\infty)^2 \times (\ell^1)^2, \quad \omega ((q,p),(q',p')) = \langle q,p' \rangle - \langle q',p \rangle
\]
We then have

\[
\mathcal{A} = \left\{ f \in C^\infty (\ell^\infty \times \ell^1) : \left( \frac{\partial f}{\partial q_n} \right)_{n \in \mathbb{N}} \in \ell^1 \right\}
\]
and the Poisson bracket is given by

\[
\{f,g\}_\omega = \sum_{n=0}^{+\infty} \left( \frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} - \frac{\partial g}{\partial q_k} \frac{\partial f}{\partial p_k} \right).
\]
Again in this example the bracket satisfies the Leibniz property and is associated to a 2-skew-symmetric tensor (cf. 2.5).

### 2.4 Some classical examples of \( r \)-Lie brackets

**Example 2.9.** The first attempt to make a connection between Nambu mechanics and the Euler equation of incompressible fluid was proposed by Névir and Blender in [NeBl93].

For the motion of an ideal and incompressible fluid on a compact domain \( \Omega \), in the Lagrangian framework, the phase space is the cotangent bundle of
the Lie group of volume preserving smooth diffeomorphisms. In the Eulerian
approach, the phase space is the dual of the Lie algebra $\mathfrak{sdiff}(\Omega)$.
The dynamics of the flow can be written in terms of the vorticity (cf. [SBH11] and [BiBa15]).
For a two-dimensional, non-divergent flow $(u, v)$ in the $(x, y)$ plane on a
periodic domain $\Omega$ derived from the stream-function $\psi$:

$$ u = -\frac{\partial \psi}{\partial y} \text{ and } v = \frac{\partial \psi}{\partial x} $$

the vorticity $\zeta$ is given by

$$ \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \nabla^2 \psi $$

The first conservation law is the kinetic energy

$$ H = \frac{1}{2} \int_\Omega \left( u^2 + v^2 \right) dA $$

which can be written under some periodic boundary conditions as

$$ H(\zeta) = -\frac{1}{2} \int_\Omega \zeta \psi dA $$

whose functional derivative is

$$ \frac{\delta H}{\delta \zeta} = -\frac{1}{2} \psi $$

The second conservation law is the enstrophy

$$ E(\zeta) = \frac{1}{2} \int_\Omega \zeta^2 dA $$

whose functional derivative is

$$ \frac{\delta E}{\delta \zeta} = \zeta $$

The associated 2D-Euler equation is given by the $3$-Lie bracket\footnote{usually called Nambu-Poisson bracket in fluid mechanics literature} associated to the dynamics

$$ \{F, E, H\}(\zeta) = \int_\Omega \frac{\partial F}{\partial \zeta} J \left( \frac{\partial H}{\partial \zeta}, \frac{\partial E}{\partial \zeta} \right) dA $$

where $J$ is the Jacobi operator given by

$$ J(u, v) = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x}. $$
Example 2.10. Lie-Poisson bracket on the regular dual of the Lie algebra of the Fréchet-Lie group $\text{Diff} (S^1)$.— We mainly follow the papers [Kol07a] and [Kol07b].

Let $\text{Diff} (S^1)$ be the group of smooth, orientation preserving diffeomorphisms of the circle $S^1$. This group is naturally equipped with a Fréchet manifold structure modelled on the Fréchet vector space $C^\infty (S^1)$. The Fréchet structure on $C^\infty (S^1)$ is the projective limit of the sequence of Banach spaces $(C^k(S^1))_{k \in \mathbb{N}}$ (see [Omo97] for structure on inverse limits of Lie groups).

Since the composition and the inverse are smooth maps for this structure, $\text{Diff} (S^1)$ is a Fréchet-Lie group (cf. [Mil84]). Its Lie algebra corresponds to $\text{Vect} (S^1)$; it is isomorphic to the space $C^\infty (S^1)$ of periodic functions endowed with the Lie bracket given by

$$[u, v] = u_x v - u v_x$$

The topological dual of the Fréchet space $\text{Vect} (S^1)$ is isomorphic to the space of distributions on the circle and it is not anymore a Fréchet space. We then define the regular dual of $\text{Vect} (S^1)$, denoted $\text{Vect}^{(*)} (S^1)$, which also has a structure of Fréchet space, as the subspace of linear functionals $F$ of the form

$$F : u \mapsto \int_{S^1} f(x) u(x) dx$$

where $f \in C^\infty (S^1)$ is a smooth density of $F$. The $L^2$-inner product

$$(u, v) = \int_{S^1} u(x) v(x) dx$$

defines an isomorphism between $\text{Vect} (S^1)$ and its regular dual $\text{Vect}^{(*)} (S^1)$.

In the sequel, the elements of $\text{Vect} (S^1)$ will be denoted $u, v, ...$ and the elements of its regular dual $\text{Vect}^{(*)} (S^1)$ by $m, n, ...$, although they can be identified with elements of $C^\infty (S^1)$.

For the notion of jet bundles, the reader is referred to [Take79], [Sau04] and [Mul96]. $u^{[r]}_x$ denotes the $r$-jet of $u \in C^k(S^1)$ at $x \in S^1$, for $r \in \{0, 1, \ldots, k\}$.

A function $F$ on $\text{Vect}^{(*)} (S^1) \simeq C^\infty (S^1)$ is called a local functional if it can be written as

$$F(m) = \int_{S^1} f(x, m, \ldots m^{(r)}_x) dx$$

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for some smooth function $f : C^r(S^1) \to \mathbb{R}$ (called the Lagrangian density of $F$) and some $r \in \mathbb{N}$.

We denote $\mathcal{A}$ the set of local functionals.

The directional derivative of $F$ at $m \in \text{Vect}^r(S^1)$ in the direction $X \in C^\infty(S^1)$ is

$$D_X F(m) = \frac{d}{dt} F(m + tX)_{|t=0} = \int_{S^1} \sum_{k=0}^r \frac{\partial^k f}{\partial m^k}(x, m^r_x) \times X^{(k)}(x) dx$$

Integrating $\sum_{k=0}^r \frac{\partial^k f}{\partial m^k}(x, m^r_x) \times X^{(r)}(x)$ $r$-times by parts, we obtain

$$D_X F(m) = \int_{S^1} E(m)(x) X(x) dx$$

where $E$ is the Euler operator (cf. [Olv93], Definition 4.3) given by

$$E = \sum_{k=0}^{+\infty} (-D_x)^k \frac{\partial}{\partial m^k} = \frac{\partial}{\partial m} - D_x \frac{\partial}{\partial m_x} + D^2_x \frac{\partial}{\partial m_{xx}} - \cdots$$

where $D_x$ is the total derivative with respect to $x$ ([Olv93], Proposition 2.35) defined by

$$D_x = \frac{\partial}{\partial x} + m_x \frac{\partial}{\partial m} + m_{xx} \frac{\partial}{\partial m_x} + \cdots$$

The map

$$m \mapsto \frac{\delta F}{\delta m} = E(m)$$

can be considered as a vector field on $C^\infty(S^1)$; it is called the $L^2$-gradient of $F$. So a local functional on $C^\infty(S^1)$ has a smooth $L^2$-gradient.

As a typical example, for a linear functional $F : m \mapsto \int_{S^1} u m dx$ where $u \in C^\infty(S^1)$, we have $\frac{\delta F}{\delta m}(m) = u$.

We define an equivalence relation $\sim$ on $C^\infty(S^1)$ as follows

$$f \sim g \iff \exists \varphi \in C^\infty(S^1) : f = g + D_x \varphi$$

The space $\mathcal{A}$ of local functionals corresponds to

$$C^\infty(S^1) / \sim = C^\infty(S^1) / D_x (C^\infty(S^1))$$

(cf [Olv93], p. 356).
In order to define a 2-Lie bracket on $\mathcal{A}$, we consider a 1-parameter family $P_m (m \in C^\infty (S^1))$ of linear operators

$$
\{F,G\}(m) = \int_{S^1} \frac{\delta F}{\delta m} P_m \frac{\delta G}{\delta m} dx
$$

also written as

$$
\{F,G\}(m) = \int_{S^1} E(f).P_m E(g)dx
$$

if $f$ (resp. $g$) is the density of $F$ (resp. $G$).

The family $P_m$ defines a 2-Lie algebra structure if

- $\forall (F,G) \in \mathcal{A}^2$, $\{F,G\} \in \mathcal{A}$
- $\{\ldots\}$ is skew-symmetric:
  $$
  \forall (F,G) \in \mathcal{A}^2, \{G,F\} = -\{F,G\}
  $$
- $\{\ldots\}$ satisfies the Jacobi identity:
  $$
  \forall (F,G,H) \in \mathcal{A}^3, \{\{F,G\},H\} + \{\{G,H\},F\} + \{\{H,F\},G\} = 0
  $$

The skew-symmetry is relative to the $L^2$ inner product. Since the expression for the bracket $\{F,G\}$ is a local functional, the class $\mathcal{A}$ is closed under this bilinear operation.

Since the proof of the Jacobi identity is tedious in practice, following Olver (\cite{Ol}, Example 4.7), the technique of \textit{functional bivectors} is better suited to our situation.

The canonical Lie-Poisson structure on the regular dual $\text{Vect}^{(s)}(S^1)$ is given by

$$
\{F,G\}(m) = m \left( \left\langle \frac{\delta F}{\delta m}, \frac{\delta G}{\delta m} \right\rangle \right) = - \int_{S^1} \left( \frac{\delta F}{\delta m} (mD + Dm) \frac{\delta G}{\delta m} \right)
$$

where $D = \frac{d}{dx}$ (cf. \cite{Ko}, Example 4.7).

\textbf{Remark 2.11.} Note that in the previous example, $\mathcal{A}$ has a structure of vector space structure but not a structure of ring; so $\mathcal{A}$ is not an algebra. However, we have a 2-Lie bracket defined on $\mathcal{A}$

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\textsuperscript{4}In the classical physics mathematical literature, such a bracket is simply called a Poisson bracket.
3 Partial Nambu-Poisson structure on a convenient manifold

3.1 Some preliminaries

Let $M$ be a convenient manifold modelled on a convenient space $\mathbb{M}$. We denote by $p_M : TM \rightarrow M$ its kinematic tangent bundle (KrMi97, 28.12) and by $p'_M : T'M \rightarrow M$ its kinematic cotangent bundle (KrMi97, 33.1).

As we have seen at the beginning of §2.2, on a convenient manifold $M$, in general, the algebra $\mathcal{F}(M)$ of smooth functions on $M$ restricted to an open set $U$ gives rise to a strict subset of the algebra $\mathcal{F}_U$ of smooth functions on $U$, even in the Banach setting. So, in order to obtain local results for (partial) Nambu-Poisson structures, a sheaf of algebras of smooth functions on a convenient manifold $M$ will be used.

Let $p^\flat : T^\flat M \rightarrow M$ be a weak subbundle\(^5\) of $p'_M : T'M \rightarrow M$. For any open set $U$ in $M$, we denote by $\mathcal{A}_U$ the set of smooth functions $f \in \mathcal{F}_U$ such that each iterated derivative $d^k f(x) \in L^k(T_x M, \mathbb{R})$ for $k \in \mathbb{N}^*$ satisfies:

$$\forall x \in U, \forall (u_2, \ldots, u_k) \in (T_x M)^{k-1}, d^k_x f(\ldots, u_2, \ldots, u_k) \in T'_x M. \quad (1)$$

Then $\{\mathcal{A}_U \subset \mathcal{F}_U, U \text{ open in } M\}$ is defined on a sub-sheaf $\mathcal{A}_M$ of algebras of $\mathcal{F}_M$ (cf. CaPe). Of course, if $T^\flat M = T'M$, then $\mathcal{A}_M = \mathcal{F}_M$.

**Definition 3.1.** Let $M$ be a convenient manifold and let $r$ be an integer such that $r \geq 2$.

A *sheaf of Nambu-Poisson structures* (resp. *sheaf of partial Nambu-Poisson structures*) of order $r$ (resp. *sheaf of $r$-Lie algebra structures*) on $M$ is an $r$-Nambu-Poisson bracket (resp. almost $r$-Nambu-Poisson bracket) (resp. $r$-Lie algebra structure) on each $\mathcal{A}_U$, for all open sets $U$ in $M$.

On can find in CaPe, the following result:

**Lemma 3.2.** For any $\alpha \in T^\flat_x M$, there exist an open neighbourhood $U$ of $x$ and a function $f \in \mathcal{A}_U$ such that $d_x f = \alpha$.

Let $E$ be a convenient space. We denote

- $E^\sharp$ the algebraic dual of $E$;

\(^5\)i.e. the canonical injection $\iota : T^\flat M \rightarrow T'M$ is a convenient bundle morphism.
• \(E^*\) the topological dual of \(E\);
• \(E'\) the bornological dual of \(E\).

The algebraic tensor product \(u_1 \otimes \cdots \otimes u_k\) of \(k\) elements \(u_1, \ldots, u_k\) of \(E\) is given by

\[
\begin{align*}
    u_1 \otimes \cdots \otimes u_k : & \quad (E^*)^k \to \mathbb{R} \\
    & \quad (\alpha_1, \ldots, \alpha_k) \mapsto \prod_{i=1}^{k} \alpha_i (u_i)
\end{align*}
\]

The bornological \(k\)-tensor product \(E \otimes_\beta \cdots \otimes_\beta E\) is the vector space spanned by all the elements \(u_1 \otimes \cdots \otimes u_k\) endowed with the finest locally convex topology for which the map

\[
    (E^*)^k \to E \otimes_\beta \cdots \otimes_\beta E
\]

is bounded.

The wedge product is generated from the bounded map alternator:

\[
\begin{align*}
    \text{alt} : & \quad \otimes^k E \to \otimes^k E \\
    & \quad x_1 \otimes \cdots \otimes x_k \mapsto \frac{1}{k!} \sum_{\sigma \in \sigma_k} \text{sign}(\sigma) x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}
\end{align*}
\]

where \(\sigma_k\) is the set of permutations of \(\{1, \ldots, k\}\).

A 0-neighbourhood basis of the associated topology is given by those absolutely convex sets which absorb the set

\[
B_1 \wedge \cdots \wedge B_k = \{u_1 \wedge \cdots \wedge u_k, u_1 \in B_1, \ldots, u_k \in B_k\}
\]

for all bounded sets \(B_1, \ldots, B_k\) in \(E\). Note that this topology is bornological. The completion of this bornological space is denoted by \(\bigwedge^k E\) (cf. [KrMi97]). In particular \(\bigwedge^k E\) is a convenient set.

**Remark 3.3.** \(E\) is a closed convenient subspace of the bornological bi-dual \(E''\) of \(E\). It follows that each \(\Omega \in \bigwedge^k E\) induces an element of \(\bigwedge^k E''\).

To the bundle \(TM\) is associated the convenient vector bundle \(\bigwedge^k TM\) whose typical fibre is \(\bigwedge^k M\). For any open set \(U\) in \(M\), if \(\mathfrak{X}(U)\) is the \(\mathcal{F}_U\)-module of vector fields on \(U\), we denote by \(\bigwedge^k \mathfrak{X}(U)\) the set of sections of \(\bigwedge^k TM\) over \(U\).
We denote by \((T^0 M)'\) the dual bundle of \(T^0 M\) and \(\bigwedge^k (T^0 M)'\) the associate skew-symmetric bundle. Given \(k\) vector fields \(X_1, \ldots, X_k\) in \(\Gamma(U)\), then \(X_1 \wedge \cdots \wedge X_k\) is a decomposable element of \(\bigwedge^k \mathfrak{X}(U)\). Therefore \(X_1 \wedge \cdots \wedge X_k\) is a section of \(\bigwedge^k (T^0 M)'\) over \(U\) defined by:

\[
(X_1 \wedge \cdots \wedge X_k) (d_x f_1, \ldots, d_x f_k) = (d_x f_1 \wedge \cdots \wedge d_x f_k) (X_1, \ldots, X_k)
\]

for all \(x \in U\) and all germs of functions \(f_1, \ldots, f_k\) in \(A_V\) for some open neighbourhood \(V\) of \(x\).

Thus, for all \(k \geq 1\), \(\bigwedge^k \mathfrak{X}(U)\) can be considered as a \(\mathcal{F}_U\)-module of sections of \(\bigwedge^k (T^0 M)'\) over \(U\) and so any section \(\Omega\) of \(\bigwedge^k T^0 M\) over \(U\) gives rise to a section of \(\bigwedge^k (T^0 M)'\) over \(U\) (cf Lemma 3.2).

### 3.2 Partial Nambu-Poisson anchors

Let \(p^0 : T^0 M \to M\) be a weak subbundle of \(p'_M : T' M \to M\).

Consider the bilinear crossing \(< , >\) between \(T' M\) and \(TM\) it induces a bilinear crossing between \(T^0 M\) and \(TM\) again denoted \(< , >\).

**Definition 3.4.** Let \(r\) be an integer such that \(r \geq 2\).

1. Let \(M\) be a convenient space and \(M'\) its bornological dual. We consider a convenient vector space \(M^0\) contained in \(M'\) such that the inclusion of \(M^0\) in \(M'\) is bounded. A bounded linear map \(P : (M^0)^{r-1} \to M\) is called \(r\)-skewsymmetric or \(r\)-alternating if it satisfies, for all \(\alpha_0, \alpha_1, \ldots, \alpha_{r-1}\) of \(M^0\) and for all \(j > i\), the relations

\[
\langle \alpha_0, P(\alpha_1, \ldots, \alpha_j, \ldots, \alpha_i, \ldots, \alpha_{r-1}) \rangle = -\langle \alpha_0, P(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_j, \ldots, \alpha_{r-1}) \rangle
\]

\[
\langle \alpha_0, P(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(r-1)}) \rangle = (-1)^{\varepsilon(\sigma)} \langle \alpha_0, P(\alpha_1, \ldots, \alpha_{r-1}) \rangle
\]

where \(\sigma\) is any permutation of \(\{1, \ldots, r-1\}\) whose parity is denoted \(\varepsilon(\sigma)\).

2. A convenient bundle morphism \(P : (T^0 M)^{r-1} \to TM\) is called a partial almost \(r\)-Nambu-Poisson anchor if, for each \(x \in M\),

\[
P_x : \left( T^0_x M \right)^{r-1} \to T_x
\]

is a bounded \(r\)-alternating map.

If \(T^0 M = T' M\), \(P\) is simply called an almost \(r\)-Nambu-Poisson anchor.

The following result summarizes the elementary properties of a partial almost \(r\)-Nambu-Poisson anchor:
Proposition 3.5. A partial almost r-Nambu-Poisson anchor

\[ P : \left( T^o M \right)^{r-1} \rightarrow TM \]

has the following properties:

1. For all \((\alpha_0, \alpha_1, \ldots, \alpha_{r-1})\) in \((T^o M)^r\) and for all \(x \in M\),

\[ \Lambda : (\alpha_0, \alpha_1 \ldots \alpha_{r-1}) \mapsto \langle \alpha_0, P(\alpha_1, \ldots, \alpha_{r-1}) \rangle \]

defines a smooth bounded section of \(\Lambda^r T^o M\). 
\(\Lambda\) is called the associated partial almost r-Nambu-Poisson tensor.

2. \(P\) (resp. \(\Lambda\)) induces a natural morphism of bundles

\[ \Lambda^\sharp : \bigwedge^{r-1} T^o M \rightarrow TM \]

such that

\[ P(\alpha_1, \ldots, \alpha_{r-1}) = \Lambda^\sharp (\alpha_1 \wedge \cdots \wedge \alpha_{r-1}). \] (3)

3. Conversely, consider any global smooth bounded section \(\Lambda\) of \(\bigwedge^k TM\) and \(\Lambda^\sharp : \bigwedge^{r-1} T^o M \rightarrow TM\). Then we have an associated partial almost r-Nambu-Poisson anchor defined by the relation (3). The partial almost r-Nambu-Poisson tensor associated to \(P\) is precisely \(\Lambda\).

4. For any fixed \((\beta_1, \ldots, \beta_{r-k})\) in \((T^o M)^{r-k}\), the map

\[ P_k : \left( T^o M \right)^{k-1} \rightarrow TM \]

\[ (\alpha_1, \ldots, \alpha_{k-1}) \mapsto P(\beta_1, \ldots, \beta_{r-k}, \alpha_1, \ldots, \alpha_{k-1}) \]

defines a partial almost k-Nambu-Poisson anchor for \(1 \leq k \leq r\).

5. For any smooth map \(f\) on \(M\) then \(fP\) is also a partial almost r-Nambu-Poisson anchor defined on \((T^o M)^{r-1}\).

Given a partial almost r-Nambu-Poisson anchor \(P : (T^o M)^{r-1} \rightarrow TM\), for each open set \(U\) in \(M\), we define the following bracket on \(A_U\)

\[ \{f_1, \ldots, f_{r-1}, g\}_P = \langle dg, P(df_1, \ldots, df_{r-1}) \rangle \] (4)

According to Proposition[C.17] and Proposition[C.18] the \(r\)-bracket \(\{\ldots, \ldots\}_P\) takes values in \(A(M)\) and satisfies the Leibniz property.
Remark 3.6. To the bracket defined in (4), is canonically associated a unique admissible section $\Lambda \in \bigwedge^k \Gamma^*(T^\flat M_U)$ (cf. Proposition C.18) and conversely, to any admissible section $\Lambda \in \bigwedge^k \Gamma^*(T^\flat M_U)$ is canonically associated such a bracket. From now on such a bracket will be also denoted $\{\ldots,\ldots\}_\Lambda$.

Definition 3.7. Let $p^\flat : T^\flat M \to M$ be a weak subbundle of $p'_M : T'M \to M$ and $P : (T^\flat M)^{r-1} \to TM$ be a partial almost $r$-Nambu-Poisson anchor. We say that $(M, A_M, \{\ldots,\ldots\}_P)$ is a partial $r$-Nambu-Poisson structure on $M$ or that $(M, A_M, \{\ldots,\ldots\}_P)$ is a partial $r$-Nambu-Poisson manifold if the $r$-bracket $\{\ldots,\ldots\}_P$ satisfies the Filippov identity

$$\{f_1, \ldots, f_{r-1}, \{g_1 , \ldots, g_r\}\} = \sum_{i=1}^r \{g_1 , \ldots, g_{i-1}, \{f_1 , \ldots, f_{r-1}, g_i\}, g_{i+1} , \ldots, g_r\}$$

for every $(f_1, \ldots, f_{r-1}, g_1, \ldots, g_r)$ in $(A_U)^{2r-1}$ and any open set $U$.

If $T^\flat M = T'M$, we simply say that $(M, A_M, \{\ldots,\ldots\}_P)$ is a $r$-Nambu-Poisson structure.

Remark 3.8. In terms of $r$-Nambu-Poisson tensor, the relation (FI) can be written

$$\Lambda (df_1, \ldots, df_{r-1}, d\{\Lambda (dg_1, \ldots, dg_r)\}) = \sum_{i=1}^r \Lambda (dg_1, \ldots, dg_{i-1}, d\{\Lambda (df_1, \ldots, df_{r-1}, dg_i)\}, dg_{i+1} , \ldots, dg_r).$$

Remark 3.9. Consider a partial almost $r$-Nambu-Poisson anchor

$$P : (T^\flat M)^{r-1} \to TM$$

If $r = 2s$, then as in finite dimension, $P$ defines a partial $r$-Nambu Poisson structure if and only if the Shouten bracket $[P, P]'_S = 0$. Such a condition is equivalent to the condition (FI) (cf. Theorem C.24 5. and the proof of such a result in finite dimension).

If $(M, A_M, \{\ldots,\ldots\}_P)$ is a partial $r$-Nambu-Poisson (resp. almost partial $r$-Nambu-Poisson) manifold, then $P$ induces on $A_M$ a sheaf of Nambu-Poisson (resp. almost Nambu-Poisson) brackets.

$$P_{f_1,\ldots,f_k}(dg_1, \ldots, dg_{r-k-1}) = P(df_1, \ldots, df_k, dg_1, \ldots, dg_{r-k-1})$$

(5)
As in finite dimension, for any \((f_1, \ldots, f_{r-1}) \in \mathcal{A}_U^{r-1}\) the vector field
\[
X_{f_1, \ldots, f_{r-1}} = P(df_1, \ldots, df_{r-1})
\]
is called the Hamiltonian vector field associated to the Hamiltonian \((f_1, \ldots, f_{r-1})\).

From (4), we then have
\[
X_{f_1, \ldots, f_{r-1}}(g) = \{f_1, \ldots, f_{r-1}, g\}_P
\]
for all \(g \in \mathcal{A}_U\).

According to the Filippov identity, we then have:
\[
X_{f_1, \ldots, f_{r-1}}(X_{g_1, \ldots, g_{r-1}}(h)) = \sum_{i=1}^{r-1} \{g_1, \ldots, g_{i-1}, \{f_1, \ldots, f_{i-1}, g_i\}_P, g_{i+1}, \ldots, g_{r-1}, h\}_P
\]
\[
+ X_{g_1, \ldots, g_{r-1}}(X_{f_1, \ldots, f_{r-1}}(h))
\]
\[
= \sum_{i=1}^{r-1} \{g_1, \ldots, g_{i-1}, dg_i(X_{f_1, \ldots, f_{r-1}}), g_{i+1}, \ldots, g_{r-1}, h\}_P
\]
\[
+ X_{g_1, \ldots, g_{r-1}}(X_{f_1, \ldots, f_{r-1}}(h))
\]
(6)
for all \(h \in \mathcal{A}_U\). Finally, we obtain
\[
[X_{f_1, \ldots, f_{r-1}}, X_{g_1, \ldots, g_{r-1}}](h) = \sum_{i=1}^{r-1} X_{g_1, \ldots, g_{i-1}, dg_i(X_{f_1, \ldots, f_{r-1}}), g_{i+1}, \ldots, g_{r-1}}(h)
\]
(7)
for all \(h \in \mathcal{A}_U\).

**Remark 3.10.**

1. In fact, the relation (7) is equivalent to the Filippov identity.

2. Let \(\Lambda\) be the \(r\) multi-vector associated to a partial almost \(r\)-Nambu-Poisson \(P : (T^9M)^{r-1} \rightarrow TM\). Since we have
\[
L_{X}\Lambda(dg_1, \ldots, dg_r) = d(\Lambda(dg_1, \ldots, dg_r))(X) - \sum_{i=1}^{r} \Lambda(dg_1, \ldots, d(dg_i(X)), \ldots, dg_r)
\]
thus, as in finite dimension (cf. [Vais99]), \(P\) is a partial \(r\)-Nambu-Poisson anchor if and only if
\[
L_{X_{f_1, \ldots, f_{r-1}}} \Lambda|_{T^9M} = 0
\]
for all \((f_1, \ldots, f_{r-1}) \in (\mathcal{A}_U)^{r-1}\) and any open set \(U\) in \(M\).
Definition 3.11. Let \((M, A_M, \{\ldots, \}^P)\) be a partial \(r\)-Nambu-Poisson manifold. A point \(x_0 \in M\) is called regular if the dimension of the range of \(P\) is strictly positive. Otherwise, \(x_0\) is called singular.

Lemma 3.12. If the set of regular points is not empty, then this set is open and dense in \(M\). For any regular point \(x \in M\), the dimension of the range of \(P\) at \(x\) is at least \(r\).

Proof. For each \(x \in M\), let \(n(x) \in \mathbb{N} \cup \{\infty\}\) be the dimension of the range of \(P\) at \(x\). This map is lower semi-continuous. So the set \(\{x \in M, n(x) > 0\}\) is open and dense in \(M\). Thus, if \(x\) is a regular point of \(P\), there exists a neighbourhood \(U\) of \(x\) on which \(P\) is regular.

For any \((\alpha_1, \ldots, \alpha_r)\) in \((T^*_x M)^r\), according to Proposition 3.5, if the range of \(P\) is a finite dimensional vector space, it is equal to the range of \(\Lambda^2\). Thus if the range of \(P\) is infinite, it follows that \(x\) is a regular point and so the dimension of the range of \(P\) is greater than \(r\). Assume that \(\dim(\text{Im } P) < r\); it follows that \(\dim(\text{Im } \Lambda^2) < r\).

We denote by \(A(\alpha_1, \ldots, \alpha_r)\) the finite vector space of \((T^*_x M)^{r-1}\) generated by \(\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_r\) for \(1 \leq i \leq r\). Assume that \(\dim(A(\alpha_1, \ldots, \alpha_r)) = r\) if and only if \(\alpha_1 \wedge \cdots \wedge \alpha_r \neq 0\) and so the range of \(P\) has a dimension at least \(r\) in \(x\) and so \(x\) is a regular point.

Now, if \(\dim(A(\alpha_1, \ldots, \alpha_r)) < r\), we must have \(\alpha_1 \wedge \cdots \wedge \alpha_r = 0\) and, in this case, from the skew-symmetry of \(\Lambda\), we must have \(\Lambda(\alpha_1, \ldots, \alpha_r) = 0\). It follows that the dimension of the range of \(\Lambda^2\) must be zero and so \(x\) is a singular point, which ends the proof.

Let \(H_U(P)\) be the linear space generated by the set of all Hamiltonian vector fields \(X_{f_1, \ldots, f_{r-1}} = P(df_1, \ldots, df_{r-1})\). Unfortunately, as in finite dimension, any vector field in \(H_U(P)\) is not necessarily a Hamiltonian vector field (cf. [Vais99]). However, if \(P\) is regular on \(U\), from (7), \([X_{f_1, \ldots, f_{r-1}}, X_{g_1, \ldots, g_{r-1}}]\) belongs to \(H_U(P)\). Thus we have:

**Lemma 3.13.** The distribution \(D_U\) generated by \(H_U(P)\) is involutive on each open \(U\) on which \(P\) is regular.

Remark 3.14.

1. If the dimension of the range of \(P\) is \(r\) at \(x\), there exists a neighbourhood \(U\) of \(x\) and functions \(f_1, \ldots, f_r\) in \(A_U\) such that \(d_x f_1 \wedge \cdots \wedge d_x f_r \neq 0\) and so that the range of \(P\) at \(x\) is generated by the set \(\{X_{f_1, \ldots, f_i} \, | \, i \in \{1, \ldots, r\}\}\) of independent Hamiltonian vector fields.
2. If \( x \) is a singular point of \( P \), the distribution \( \mathcal{D} \) may be not involutive at \( x \). For example (cf. [GrMa99]) on \( \mathbb{R}^n \) provided with coordinates \((x_1, \ldots, x_n)\), for \( n \geq 3 \), take \( \Lambda = \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \wedge (x_1 \frac{\partial}{\partial x_3}) \). Then the Lie bracket \( \left[ \frac{\partial}{\partial x_1}, (x_1 \frac{\partial}{\partial x_3}) \right] = \frac{\partial}{\partial x_3} \) which does not belong to \( \mathcal{D} \) at the singular point 0.

### 3.3 Characteristic distribution of a partial \( r \)-Nambu-Poisson structure

For \( r > 2 \), the properties of an \( r \)-Nambu-Poisson tensor in finite dimension (cf. for instance [Gau96] among many other references) are again valid in the convenient setting. In fact, the proof given by Ph. Gautheron in [Gau96] does not use local coordinates and so can be adapted to our context. More precisely we have:

**Theorem 3.15.** Let \((M, \mathcal{A}_M, \{.,...,\}_P)\) be a partial \( r \)-Nambu-Poisson manifold. We have the followings

1. Around each regular point \( x \in M \), there exists a neighbourhood \( U \) of \( x \), \( r \) functions \( f_1, \ldots, f_r \) in \( \mathcal{A}_U \) such that if, for \( i \in \{1, \ldots, r\} \), \( X_i \) is the Hamiltonian field associated to \((f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_r)\), then for the Hamiltonian fields \( X_1, \ldots, X_r \), we have
   \[
   \Lambda = X_1 \wedge \cdots \wedge X_r.
   \]

2. If \( \Sigma \) is the open dense set of regular points of \( P \), then the distribution \( \mathcal{D} \) defined by all the local Hamiltonian sets \( \mathcal{H}_U(P) \) (for all open sets \( U \) in \( M \)) is involutive and \( r \)-dimensional.
   
   If, around each \( x \in \Sigma \), there exists a local basis \( X_1, \ldots, X_r \) of \( \mathcal{D} \) where each \( X_i \) has a local flow for \( 1 \leq i \leq r \), then \( \mathcal{D} \) defines a \( r \)-dimensional foliation \( \mathcal{F} \) of \( \Sigma \). This is in particular true if \( M \) is a Banach manifold.

**Definition 3.16.** The distribution \( \mathcal{D} \) (resp. the foliation \( \mathcal{F} \) if \( \mathcal{D} \) is integrable) is called the characteristic distribution (resp. foliation) of the partial Nambu-Poisson structure.

**Remark 3.17.** As in finite dimension, for \( r = 2 \), Theorem 3.15 1. is not true, if we do not impose that \( \Lambda \) is decomposable. Although, according to results in [Pel18], the characteristic distribution can be infinite dimensional (cf. [BRT07]).

\[ \text{see Remark 3.17 for a justification of such an assumption} \]
The proof of Theorem 3.15 uses the same arguments as in finite dimension (cf. proof of Theorem 2 in [Gau96] and [MVV98], 4). However, we give a sketch of this proof. Essentially, we point out the adaptations needed to our context and, for the comparison, we use the same notations as in these references. For the context of this proof, we refer to §3.1.

At first, we recall an algebraic result of [MVV98] whose proof is given in finite dimensional vector spaces but which is also true without this assumption.

**Proposition 3.18.** Let $E$ be a convenient vector space and $\Lambda \in \wedge^r E$ for $r \geq 3$. Considering $\Lambda$ as an element of $\wedge^r E'$ and given $(a_1, \ldots, a_k) \in E^k$, we denote by $\Lambda_{a_1, \ldots, a_k}$ the induced element of $\wedge^{r-k} E$ defined by

$$(b_{k+1}, \ldots, b_r) \mapsto \Lambda(a_1, \ldots, a_k, b_{k+1}, \ldots, b_r).$$

Assume that we have, for all elements $a, b, c_1, \ldots, c_{r-1}$ of $E'$:

$$\Lambda_{c_1, \ldots, c_{r-2}} \wedge a + \Lambda_{c_1, \ldots, c_{r-2}, b} \wedge \Lambda = 0.$$

Then $\Lambda$ is decomposable.

At first, we will apply this result on $E = T_x M$.

**Proof of Theorem 3.15**

In the one hand, let us consider $X \in T_x M$ and $\Lambda \in \wedge^r T_x M$. Then the definition of the wedge product $X \wedge \Lambda$ implies the following relation

$$(X \wedge \Lambda)(\alpha_0, \alpha_1, \ldots, \alpha_r) = X(\alpha_0)\Lambda(\alpha_1, \ldots, \alpha_r) - \sum_{i=1}^{r} X(\alpha_i) \Lambda(\alpha_1, \ldots, \alpha_{i-1}, \alpha_0, \alpha_{i+1} \ldots \alpha_r)$$

where $\Lambda_{\alpha_0}(\alpha_1, \ldots, \alpha_{r-1}) = \Lambda(\alpha_0, \alpha_1, \ldots, \alpha_{r-1})$.

This implies that if $X \neq 0$ and $X \wedge \Lambda = 0$, then there exists a bounded $(r-1)$ multi-vector $\Lambda' \in \wedge^{r-1} T_x M$ such that $\Lambda = X \wedge \Lambda'$.

On the other hand, consider the relation (1) in $\mathcal{A}_U$ in which we replace $f_{r-1}$ by the product $h.k$ of functions $h$ and $k$ defined on $U$; we then obtain
(cf. [Gau96], p 108):

\[
0 = \Lambda (d_x f_1, \ldots, d_x f_{r-2}, d_x h, d_x k) \cdot \Lambda (d_x g_1, \ldots, d_x g_r) \\
- \sum_{i=1}^{r} \Lambda (d_x f_1, \ldots, d_x f_{r-2}, d_x h, d_x g_i) \cdot \Lambda (d_x g_1, \ldots, d_x g_{i-1}, d_x k, d_x g_{i+1}, \ldots, d_x g_r) \\
+ \Lambda (d_x f_1, \ldots, d_x f_{r-2}, d_x k, d_x h) \cdot \Lambda (d_x g_1, \ldots, d_x g_r) \\
- \sum_{i=1}^{r} \Lambda (d_x f_1, \ldots, d_x f_{r-2}, d_x k, d_x g_i) \cdot \Lambda (d_x g_1, \ldots, d_x g_{i-1}, d_x h, d_x g_{i+1}, \ldots, d_x g_r).
\]

But according to the definition of the Nambu-Poisson tensor \( \Lambda \), for \( \varphi_1, \ldots, \varphi_{r-1} \) in \( \mathcal{A}_U \), recall that \( P(d\varphi_1, \ldots, d\varphi_{r-1}) \) is a vector field \( X_\varphi \). Thus, according to the relation (8), the second member of (9) can be written in terms of wedge product and so we obtain

\[
(X_{f_1, \ldots, f_{r-2}, h} \wedge \Lambda) (dk, d_x g_1, \ldots, d_x g_r) + (X_{f_1, \ldots, f_{r-2}, k} \wedge \Lambda) (dh, d_x g_1, \ldots, d_x g_r) = 0.
\]

for all \((g_1, \ldots, g_r)\) in \((\mathcal{A}_U)^r\).

On the other hand, according to the notations of Proposition 3.18 for any \( x \in U \), we have \( X_{f_1, \ldots, f_{r-2}, h}(x) = \Lambda_{d_x f_1, \ldots, d_x f_{r-2}, d_x h} \) and so the term \((X_{f_1, \ldots, f_{r-2}, h}(x) \wedge \Lambda) (d_x k, d_x g_1, \ldots, d_x g_r)\) can be written

\[
(\Lambda_{d_x f_1, \ldots, d_x f_{r-2}, d_x h} \wedge \Lambda_{d_x k}) (d_x g_1, \ldots, d_x g_r).
\]

In the same way, \((X_{f_1, \ldots, f_{r-2}, k}(x) \wedge \Lambda) (d_x h, d_x g_1, \ldots, d_x g_r)\) can be written

\[
(\Lambda_{d_x f_1, \ldots, d_x f_{r-2}, d_x k} \wedge \Lambda_{d_x h}) (d_x g_1, \ldots, d_x g_r).
\]

So, if \( x \) is a regular point of \( P \), then \( \Lambda \) is decomposable at \( x \). Now, from Lemma 3.12 if \( x \) is a regular point there exists an open neighbourhood \( U \) of \( x \) such that all points in \( U \) are regular and so the dimension of the range of \( P \) is \( r \) at any point on \( U \).

From Remark 3.14 there exists a neighbourhood \( U \) of \( x \) and \( r \) functions \( f_1, \ldots, f_r \) in \( \mathcal{A}_U \) such that \( d_x f_1 \wedge \cdots \wedge d_x f_r \neq 0 \) and so that the range of \( P \) at \( x \) is generated by the set \( \{X_i = X_{f_1, \ldots, f_{i-1}, f_i}, \ i \in \{1, \ldots, r\}\} \) of independent Hamiltonian vector fields. Thus after restricting \( U \), if necessary, we may assume that \( df_1 \wedge \cdots \wedge df_r \neq 0 \) on \( U \). In the one hand, we have

\[
\Lambda(df_1, \ldots, df_r) = \{f_1, \ldots, f_r\}_P = \sum_{i=1}^{r} (-1)^{r-i} < df_i, X_i >
\]
On the other hand, for \( j \neq i \)

\[
< df_j, X_i > = \Lambda(df_j, df_1, \ldots, \widehat{df_i}, \ldots, df_r) = 0
\]  

(12)
since \( \Lambda \) is skew-symmetric. Thus finally we have

\[
\Lambda(df_1, \ldots, df_r) \det(df_j(x_i)) = (X_1 \wedge \cdots \wedge X_r)(df_1, \ldots, df_r)
\]

According to Lemma 3.2, we have \( \Lambda = X_1 \wedge \cdots \wedge X_r \) on \( U \).

Now, from relation (6), the Lie bracket of two Hamiltonian vector fields is also a Hamiltonian vector field. It follows that, on the set \( \Sigma \) of regular points of \( P \), the distribution generated by \( \mathcal{H}_U(P) \) for all open sets \( U \) in \( \Sigma \) is a regular involutive distribution of rank \( r \). If, around each \( x \), there exists a local basis \( X_1, \ldots, X_r \) of \( \mathcal{D} \) of vector fields, each one having a local flow, then \( \mathcal{D} \) is integrable from Teichmann Frobenius theorem in the convenient setting (cf. [Teich01]). In particular, if \( M \) is a Banach manifold, this assumption is always satisfied.

Now, as in finite dimension, in the Banach setting, we have:

**Corollary 3.19.** Assume that \((M, \mathcal{A}_M, \{\ldots, \})_P\) is a partial \( r \)-Nambu-Poisson manifold where \( M \) is a Banach manifold modelled on \( \mathbb{M} \). Let \( x \) be a regular point. Then, \( \mathbb{M} \) is isomorphic to \( \mathbb{R}^r \times \mathcal{M}_2 \) and there exists a Frobenius chart \((U, \phi = (\phi_1, \phi_2))\) such that \( \phi(U) = V_1 \times V_2 \) where \( V_1 \) (resp. \( V_2 \)) is an open set of \( \mathbb{R}^r \) (resp. \( \mathcal{M}_2 \)) and such that \( \phi_*(\Lambda) = \frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_r} \)

where \((t_1, \ldots, t_r)\) is some coordinates system on \( \mathbb{R}^r \).

**Proof.** According the Theorem 3.15, there exists functions \( f_1, \ldots, f_r \) in \( \mathcal{A}_U \) for some open set \( U \) around \( x \) such that \( df_1, \ldots, df_r \) are independent sections of \( T^oM \) over \( U \) and such that, if \( X_i \) is the Hamiltonian field \( X_{f_1,\ldots,f_r} \) then \( \Lambda = X_1 \wedge \cdots \wedge X_r \). So \( \mathcal{D} \) is generated by \( X_1, \ldots, X_r \) on \( U \). Now, from [AMR88], there exists a chart \((U, \phi)\) around \( x \) such that \( \phi(U) = I \times V \subset \mathbb{R} \times \mathcal{M}_1 \equiv \mathcal{M} \) where \( \phi(x) = (0,0) \) and \( \phi_*(X_r) = \frac{\partial}{\partial t_1} \). Thus \( \bar{f}_r = \phi \circ t_1 \) satisfies the relation \( df_r(X_r) = 1 \) on \( U \).

Now, since \( \mathcal{D} \) is an involutive distribution, according to the classical Frobenius theorem on Banach spaces, after restricting \( U \) if necessary, we have a chart \((U, \phi = (\phi_1, \phi_2))\) from \( U \) to \( \mathcal{M} \equiv \mathbb{R}^r \times \mathcal{M}_2 \) such that \( \phi(x) = (0,0) \) and \( \phi_1(U) = V_1 \) (resp. \( \phi_2(U) = V_2 \)) is an open neighbourhood of \( 0 \) in \( \mathbb{R}^r \) (resp. in \( \mathcal{M}_2 \)) and the foliation induced on \( U \) is the set \( \{\phi_1^{-1}(V_1) \times \phi_2^{-1}(y)\}_{y \in V_2} \). In particular, \( T_x \phi_1 \) is an isomorphism from \( \mathcal{D}_x \) to \( T_0 \mathbb{R}^r \). After restricting
But from Theorem 3.15 there exists functions \( f_1, \ldots, f_r \) in \( \mathcal{A}_U \) such that \( \{ df_1, \ldots, df_r \} \) is a basis of \( H^* \), it follows that \( H^* \) is contained in \( T^*M \). Moreover, over \( U \), if \( X_i \) is the Hamiltonian vector field of \( (f_1, \ldots, \hat{f}_i, \ldots, f_r) \), then \( \Lambda(\alpha_1, \ldots, \alpha_r) = \det (\alpha_j(X_i)) \). Now, each \( \alpha \in T^*M_U \) has a unique decomposition \( \alpha = \alpha_H + \alpha_G \) where \( \alpha_H \) and \( \alpha_G \) are the respective components of \( \alpha \) on \( H^0 \) and \( G^0 \) and so \( \alpha_G \) belongs to \( T^*M \). By the way, over \( U \), we can extend \( \Lambda \) to \((T^*M)^r \) by

\[
\tilde{\Lambda}(\alpha_1, \ldots, \alpha_r) := \Lambda((\alpha_1)_G, \ldots, (\alpha_r)_G)
\]  

(14)

Note that if \( (\alpha_1, \ldots, \alpha_r) \) belongs to \( T^*M_U \) then

\[
\Lambda(\alpha_1, \ldots, \alpha_r) = \det (\alpha_j(x_i)) = \det ((\alpha_j)_G(X_i))
\]

and so the restriction of \( \tilde{\Lambda} \) to \((T^*M)^r \) is exactly \( \Lambda \).

This implies that \( \tilde{\Lambda} \) is an extension to \( T^*M_U \) of \( \Lambda \) and so is an \( r \)-Nambu-Poisson tensor. So, without loss of generality, from now on we may assume that \( \Lambda \) is \( r \)-Nambu-Poisson on \( U \). In particular, we may assume that \( \mathcal{A}_U = \mathcal{F}_U \).

Fix a sequence of functions \( f_1, \ldots, f_r \) which has the property of Theorem 3.15. As we have seen previously, we can choose \( f_r \) such that \( df_r(X_r) = 1 \) and so \( \Lambda(df_1, \ldots, df_r, df_r) = 1 \). Therefore, without loss of generality, we can replace \( f_i \) by \( \hat{f}_i = (-1)^{r-i} f_i \) for \( i \in \{1, \ldots, r-1\} \) and set \( \hat{f}_r = f_r \). Again, we have \( df_1 \wedge \cdots \wedge df_r \neq 0 \). From relations (11) and (12), we get \( df_i(X_j) = \delta_{ij} \), which means that these vector fields are independent.

Now, from relation (6), we have

\[
[X_i, X_j](h) = \sum_{k=1}^{r-1} \left\{ g_1, \ldots, g_{k-1}, \{ f_1, \ldots, \hat{f}_i, \ldots, f_r, g_k \} p, g_{k+1}, \ldots, g_{r-1}, h \right\} p
\]

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with \( g_1 = f_1, \ldots, g_{j-1} = f_{j-1}, g_j = f_{j+1}, \ldots, g_{r-1} = f_r \). We will show that \([X_i, X_j] = 0\). Without loss of generality, we may assume that \(i < j\). Under all these assumptions, we have

\[
\{f_1, \ldots, \hat{f}_i, \ldots, f_r, g_k\}_P = 0 \quad \text{for } k \neq i
\]
\[
\{f_1, \ldots, f_i, \ldots, f_{r-1}, g_k\}_P = \pm 1 \quad \text{for } k = i
\]

This implies that

\[
\{g_1, \ldots, g_{j-1}, f_1, \ldots, \hat{f}_i, \ldots, f_r, g_k, g_{k+1}, \ldots, g_{r-1}, h\}_P = 0
\]

for all \(h \in \mathcal{F}_U\).

Finally, we have \([X_i, X_j] = 0\) for all \(1 \leq i, j \leq r\).

But from the choice of our Frobenius chart, recall that \(T_z \phi_1\) is an isomorphism from \(\mathcal{D}_z\) to \(T_{\phi_1(z)} \mathbb{R}^r\), for all \(z \in U\). It follows that \([\tilde{X}_1 = (\phi_1)_*(X_1), \ldots, \tilde{X}_r = (\phi_1)_*(X_r)]\) is a family of independent commuting vector fields on \(V_1\). After restricting \(U\) again if necessary, there exists a diffeomorphism \(\phi'_1\) of \(V_1\) which fixed 0 and such that \((\phi'_1)_*(\tilde{X}_i) = \frac{\partial}{\partial t_i}\) for \(i \in \{1, \ldots, r\}\). It follows that \((U, \phi' = (\phi'_1 \circ \phi_1, \phi_2))\) is a chart such that \(\phi'_*(X_i) = \frac{\partial}{\partial t_i}\). Since by construction \(\Lambda = X_1 \wedge \cdots \wedge X_r\), this ends the proof.

\[\square\]

**Remark 3.20.** From Corollary 3.19 under the notations in the previous proof, since \(\phi_1\) is a submersion, \(V_1 = \phi_1(U)\) is an open neighbourhood of \(0 \in \mathbb{R}\) and we have \((\phi_1)_*(X_i) = \frac{\partial}{\partial t_i}\) and, for all \(1 \leq i, j \leq r\), \(df_i(X_j) = \delta_{ij}\), thus \(d(\phi_1 \circ f_i)(\frac{\partial}{\partial t_i}) = \delta_{ij}\). This implies that \(dt_i = d(\phi_1 \circ f_i)\) and since \(t_i(0) = 0\), we must have \(t_i = \phi_1 \circ \tilde{f}_i\) if \(\tilde{f}_i(z) = f_i(z) - f(x)\) for all \(z \in U\). Note that \(X_i\) is also the Hamiltonian field of \(\tilde{f}_1, \ldots, \tilde{f}_i, \ldots, \tilde{f}_r\).

**Remark 3.21.** From Corollary 3.19 for any partial \(r\)-Nambu-Poisson structure on a Banach space, around each regular point, there exists a chart such that, in associated local coordinates, the \(r\)-Nambu-Poisson tensor is the canonical non degenerate \(r\)-Nambu-Poisson tensor on \(\mathbb{R}^r\). This can be seen as a generalization of the Darboux Theorem to the Banach setting as in finite dimension.
Note that around a singular point, even in finite dimension, there is no such model without additional assumption. For a classification of linear Nambu-Poisson structures and a linearization of such structures in finite dimension, the reader is referred to [Duf2000] and [DuZu99]).

**Remark 3.22.** As we have seen in the proof of Corollary 3.19, a partial \( r \)-Nambu-Poisson tensor \( \Lambda \) can be extended to an \( r \)-Nambu-Poisson tensor \( \overline{\nabla} \) around any regular point. Thus, if \( M \) is a smooth paracompact Banach manifold, any partial regular \( r \)-Nambu-Poisson structure can be extended to a regular \( r \)-Nambu-Poisson structure. So, in this case, the notion of partial \( r \)-Nambu-Poisson structure associated to an anchor \( \rho : (T^pM)^{r-1} \to TM \) is nothing but else that the restriction to \( (T^pM)^{r-1} \) of a regular \( r \)-Nambu-Poisson structure defined by an anchor \( \rho : (T^pM)^{r-1} \to TM \). However, we think that, in the Banach setting, such a result is true only locally but not globally, without such an assumption, but, unfortunately, we have no example of such a situation.

### 3.4 Leibniz algebroids and \( r \)-Nambu-Poisson manifolds

In this section, we will adapt the definition of a Leibniz algebroid to the convenient setting and show that to an \( r \)-Nambu-Poisson Banach manifold is associated a Leibniz algebroid as in finite dimension.

**Definition 3.23.** A **Leibniz algebra** is a module \( \mathcal{L} \) over a ring \( \mathcal{R} \) provided with a bilinear bracket \( [\cdot,\cdot] \) which satisfies the identity:

\[
[[a,b],c] = [[a,c],b] + [a,[b,c]] \quad \text{(LeibAlg)}
\]

for all \((a,b,c) \in \mathcal{M}^3\).

When the bracket is skew-symmetric, the relation (LeibAlg) gives the Jacobi identity and so \( \mathcal{L} \) is then a Lie algebra.

**Definition 3.24.** Let \( \pi : E \to M \) be a convenient vector bundle and \( \rho : E \to TM \) an anchor. A **Leibniz algebroid structure** on \( E \) is the datum of a sheaf \( \{\Gamma(E_U),[\cdot,\cdot]_U, U \text{ open in } M\} \) of Leibniz algebras where \( \Gamma(E_U) \) is the module of sections of \( E \) over \( U \) such that, for all \( s_1, s_2 \) in \( \Gamma(E_U) \) and any smooth function \( f \) on \( U \):

- \( \text{(LA 1)} \) \( \rho([s_1,s_2]_U) = [\rho(s_1),\rho(s_2)]_U \);
- \( \text{(LA 2)} \) \([s_1,f s_2]_U = f[s_1,s_2]_U + df(\rho(s_1))s_2 \).

\(^7\)Right multiplication by \( c \) can be seen as a derivation
The family \{[\cdot, \cdot]_U, \ U \text{ open set in } M\} is called a sheaf of Leibniz brackets and is denoted \([\cdot, \cdot]_E\).

The triple \((E, [\cdot, \cdot]_E, \rho)\) is called a Leibniz algebroid.

Note that if each \([\cdot, \cdot]_U\) is skew-symmetric, then \((E, [\cdot, \cdot]_E, \rho)\) is a Lie algebroid.

Recall that if \(P\) is a Poisson structure on a finite dimensional manifold then the cotangent bundle \(T^*M\) has a structure of Lie algebroid. Unfortunately, if \(P\) is a partial Poisson structure, \(T^\flat M\) does not have a Lie algebroid structure in general even if \(T^*M = T^*M\). However, it is true if \(P\) is injective, in particular, in some cases of weak symplectic manifolds. In general, we can associate to the Poisson anchor \(P\) a sheaf of almost Lie brackets on \(T^\flat M\) but the Jacobi identity is not satisfied on the sheaf of local sections of \(T^\flat M\).

Now, in finite dimension, to an \(r\)-Nambu-Poisson structure is canonically associated a Leibniz algebroid structure on \(\bigwedge^{r-1} T^*M\) (cf. [ILMP99] for instance). For a partial \(r\)-Nambu-Poisson structure on a Banach manifold, we will also define a sheaf of brackets which induces a structure of Leibniz algebra on the module \(\bigwedge^{r-1} (\Gamma(T^*M_U))\) of local sections of \(\bigwedge^{r-1} T^\flat M\) defined on an open set \(U\) of \(M\).

From now to the end of this paragraph, \(M\) is a Banach manifold and \((M, A_M, \{\cdot, \cdot, \cdot\}_P)\) is an \(r\)-Nambu-Poisson structure associated to the Nambu-Poisson anchor \(P : (T^*M)^{r-1} \to TM\).

We again denote by \(\Lambda\) the section of \(\bigwedge^{r-1} TM\) associated to \(P\) and \(\Lambda^\sharp : \bigwedge^{r-1} T^\flat M \to TM\) the associated bundle morphism. At first, as in finite dimension (cf. [ILMP01]), we have:

**Proposition 3.25.** Under the previous assumptions, fix some open set \(U\) contained in the regular set of \(P\). Then, for all \(\alpha, \beta\) in \(\bigwedge^{r-1} \Gamma(T^*M_U)\) we have

\[ [\Lambda^\sharp(\alpha), \Lambda^\sharp(\beta)] = \Lambda^\sharp \left( L_{\Lambda^\sharp(\alpha)} \beta + (-1)^r (\text{id}_\alpha \Lambda) \beta \right) \]  

where \(L_{\Lambda^\sharp(\alpha)}\) is the Lie derivative according to the vector field \(\Lambda^\sharp(\alpha)\) and \(\text{id}_\alpha\) is the inner product by \(d\alpha\).

**Remark 3.26.** If \(x\) is a singular point, then \(\Lambda^\sharp(x) = 0\), so the r.h.s. of (15) is zero. Now, since for any \(y \in M\), we have \((\Lambda^\sharp(\alpha))(y) = \Lambda(y)(\alpha(y))\), it follows that for any local sections \(\alpha\) and \(\beta\) defined around \(x\), then \(\Lambda^\sharp(\alpha)(x) = \Lambda^\sharp(\beta)(x) = 0\) and so the l.h.s. of the relation (15) is also zero at \(x\). Since the Lie bracket at \(x\) of vector fields which are zero in \(x\) is also zero, the l.h.s.
of the relation (15) is also zero. Thus, the relation (15) is also true at a singular point.

**Proof.** We fix some regular point $x$. Since such a result is local, we can choose a chart $(U, \phi = (\phi_1, \phi_2))$ as in the Corollary 3.19 and so, without loss of generality, we may assume that $U = V_1 \times V_2 \subset \mathbb{R}^r \times M_2$. By the way, $\Lambda^2$ is an isomorphism from $\bigwedge^{r-1} T^* V_1$ to its range in $TU$ which is exactly $TV_1$ and so $\bigwedge^{r-1} T^* V_1$ is a supplementary of $\ker \Lambda^2$. According to this decomposition, if $\alpha_1$ is the projection of $\alpha \in \bigwedge^{r-1} \Gamma(T^* U)$ onto $\bigwedge^{r-1} \Gamma(T^* V_1)$, it follows that $\Lambda^2(\alpha) = \Lambda^2(\alpha_1)$. Now, in the r.h.s. of (15), $L_{\Lambda^2(\alpha_1)} \beta = L_{\Lambda^2(\alpha_1)} \beta_1 + L_{\Lambda^2(\alpha_1)} \beta_0$ where $\beta_0$ belongs to $\Gamma(\ker \Lambda^2_U)$. On the other hand, note that $\beta$ belongs to $\Gamma(\ker \Lambda^2_U)$, if and only if

$$\beta \left( \frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_i}, \ldots, \frac{\partial}{\partial t_r} \right) = 0$$

for all $1 \leq i \leq r$.

Since $\left[ \frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j} \right] = 0$ for all $1 \leq i, j \leq r$, it follows that if $\beta \in \Gamma(\ker \Lambda^2_U)$ then $d\beta \in \Gamma(\ker \Lambda^2_U)$ and also, for any vector field $X$ on $U$ tangent to $\mathcal{D}$, then $i_X \beta \in \Gamma \left( \ker \Lambda^2_U \right)$. From the Cartan formulae of the Lie derivative, it follows that $L_{\Lambda^2(\alpha_1)} \beta_0$ belongs to $\Gamma \left( \ker \Lambda^2_U \right)$. Finally, since

$$\Lambda^2((-1)^r (i_{da\Lambda}) \beta) = (-1)^r (i_{da\Lambda}) \Lambda^2(\beta_1)$$

it follows that (15) is equivalent to

$$[\Lambda^2(\alpha_1), \Lambda^2(\beta_1)] = \Lambda^2 \left( L_{\Lambda^2(\alpha_1)} \beta_1 + (-1)^r (i_{da\Lambda}) \beta_1 \right) \quad (16)$$

for all $\alpha$ and $\beta$ in $\bigwedge^{r-1} \Gamma(T^* U)$ where $\alpha_1$ (resp. $\beta_1$) is the projection of $\alpha$ (resp. $\beta$) on $\bigwedge^{r-1} \Gamma(T^* V_1)$.

Therefore, the result is obtained by application of Proposition 3.3 in [ILMP99] to $\phi_*(\Lambda)$ on $V_1$. 

**Definition 3.27.** Let $(M, A_M, \{\ldots, \})_P$ be a Banach $r$-Nambu-Poisson structure and $\Lambda$ the associated Nambu-Poisson tensor. For any open set $U$ in $M$, the **sheaf of $P$-brackets** on $\bigwedge^{r-1} T^* M$ is the sheaf of the bilinear operations

$$[\ldots]_P : \bigwedge^{r-1} \Gamma(T^* M_U) \times \bigwedge^{r-1} \Gamma(T^* M_U) \to \bigwedge^{r-1} \Gamma(T^* M_U)$$

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defined by
\[
[\alpha, \beta]_P = L_{\Lambda^*(\alpha)} \beta + (-1)^r (\text{id}_\alpha \Lambda) \beta
\]  
(17)
for any \(\alpha, \beta\) in \(\bigwedge^{r-1} \Gamma(T^*M_U)\).

Finally, as in [ILMP99], we have:

**Theorem 3.28.** Let \((M, \mathcal{A}_M, \{., ., .\}_P)\) be a Banach \(r\)-Nambu-Poisson structure and \(\Lambda\) the associated Nambu-Poisson tensor.

(1) The sheaf of \(P\)-brackets has the following properties, for any open set \(U\) in \(M\):

(a) for all \(f_1, \ldots, f_{r-1}, g_1, \ldots g_{r-1} \in \mathcal{F}_U\), we have:
\[
[df_1 \wedge \cdots \wedge df_{r-1}, dg_1 \wedge \cdots \wedge dg_{r-1}]_P = \sum_{i=1}^{r-1} dg_1 \wedge \cdots \wedge d\{f_1, \ldots, f_{i-1}, g_i\}_P \wedge \cdots \wedge dg_{r-1}
\]  
(18)

(b) for any \(f \in \mathcal{F}_U\) and \(\alpha, \beta \in \bigwedge^{k-1} \Gamma(T^*M_U)\), we have:
\[
[\alpha, f \beta]_P = f[\alpha, \beta]_P + df \left( \Lambda^*(\alpha) \right) \beta
\]  
(19)
\[
[f \alpha, \beta]_P = f[\alpha, \beta]_P - i_{\Lambda^*(\alpha)} (df \wedge \beta)
\]  
(20)

(2) The bracket \([.,.]_P\) provides \(\bigwedge^{r-1} \Gamma(T^*M_U)\) with a Leibniz algebra structure and so the triple \(\left( \bigwedge^{r-1} T^*M, \Lambda^*, [.,.]_P \right)\) has a Leibniz algebroid structure.

**Sketch of proof.** On the open dense set \(\Sigma\) on which \(P\) is non singular, according to the equivalence between (15) and (16) and using arguments of projection as in the end of the proof of Proposition 3.25, it follows that the Theorem is then an application of Theorem 3.5 and Theorem 3.6 in [ILMP99] over \(\Sigma\). Now, using arguments exposed in Remark 3.26, continuity and density arguments, it follows that the results are also true at singular points.

**Remark 3.29.** Given a finite dimension manifold \(M\) provided with an \(r\)-Nambu-Poisson tensor \(\Lambda\), in [Hag02], relation (14), the author also provides the anchored bundle \(\bigwedge^{r-1} T^*M\) with the following bracket
\[
[[\alpha, \beta]] = L_{\Lambda^*(\alpha)} \beta - i_\beta (d\alpha)
\]  
(21)
so that \( (\Lambda^{r-1} T^* M, \Lambda^2, [[,]] ) \) has a Leibniz algebroid structure. This Leibniz algebroid structure does not coincide with the one defined in [HMP99], Theorem 3.6. Using analogue arguments of projection as in the proof of Proposition 15 and of Theorem 3.28, we can also provide the anchored bundle \( (\Lambda^{r-1} T^* M, \Lambda^2) \) with a Leibniz algebroid structure characterized by a sheaf of Leibniz algebras on \( \Lambda^{r-1} \Gamma(T^* M_U) \) defined with a Leibniz bracket on it given by the formula (21).

4 Examples of (partial) \( r \)-Nambu-Poisson for \( r > 2 \) manifold in the convenient setting

4.1 Left-invariant partial \( r \)-Nambu Poisson structures on a Banach-Lie Group

In this section, we adapt some results of [Nak2000] to the Banach setting. We consider a Banach-Lie group \( G \) whose Lie algebra is denoted \( \mathfrak{g} \) and \( \mathfrak{g}^* \) its dual. Let \( \mathfrak{g}^b \) be a Banach subspace of \( \mathfrak{g}^* \) and \( p : (\mathfrak{g}^b)^{r-1} \to \mathfrak{g} \) be a continuous linear \((r-1)\) skew symmetric map.

Proposition 4.1. We denote by \( L_g \) the left translation by \( g \) on \( G \). For \( r \geq 3 \) we have:

1. The subset of \( T^* G \) defined by

\[
\forall g \in G, \ T^*_g G := T_e^* L_g^{-1}(\mathfrak{g}^b)
\]

is a closed subbundle of \( T^* G \).

2. For all \( g \in G \) and \( \alpha_1, \ldots, \alpha_{r-1} \) in \( T_g G \), we set

\[
P_g(\alpha_1, \ldots, \alpha_{r-1}) = T_e L_g \circ p(L^*_g(\alpha_1), \ldots, L^*_g(\alpha_{r-1}))
\]

Then \( P : (T^*_b G)^{r-1} \to TG \) is a partial almost \( r \)-Nambu-Poisson anchor.

3. Let \( \mathfrak{h} = p((\mathfrak{g}^b)^{r-1}) \). Then \( \mathfrak{h} \) is either reduced to \( \{0\} \) or is \( r \) dimensional. If \( \mathfrak{h} \) is \( r \) dimensional, then \( P \) is a partial \( r \) Nambu-Poisson anchor if and only if \( \mathfrak{h} \) is a Banach Lie subalgebra of \( \mathfrak{g} \). In this case, there exists a basis \( \mathfrak{a}_1, \ldots, \mathfrak{a}_r \) of \( \mathfrak{h} \) such that the \( r \)-Nambu-Poisson tensor \( \Lambda \) associated to \( P \) is equal to \( X_1 \wedge \cdots \wedge X_r \) if \( X_j \) is the left invariant vector field on \( G \) defined by \( \mathfrak{a}_j \) for all \( j \in \{1, \ldots, r\} \).
Corollary 4.2. (cf. [Nak2000]). There is a one to one correspondence, up to a multiplicative constant, between the set of left invariant Nambu-Poisson tensors of order $r$ on $G$ and the set of $r$-dimensional Lie subalgebras of $\mathfrak{g}$.

Definition 4.3. Let $G$ be a Banach-Lie group and $(G, A_G, \{\ldots, \})_P$ be a Banach $r$-Nambu-Poisson structure on $G$. We say that this structure is a left invariant $r$-Nambu-Poisson Lie group if the associated Nambu tensor is left invariant.

Remark 4.4. In a symmetric way, the notion of right invariant $r$-Nambu-Poisson tensor on a Banach Lie group can be defined.

Proof of Proposition 4.1

1. It is well known that the map $\Phi: G \times \mathfrak{g} \rightarrow TG$ 
\[ (g, X) \mapsto T_e L_g(X) \]

is a diffeomorphism. Since $(L_g)^{-1} = L_{g^{-1}}$, then $\Psi: G \times \mathfrak{g}^* \rightarrow T^*G$ 
\[ (g, \alpha) \mapsto L_{g^{-1}}^*(\alpha) \]

is also a diffeomorphism. But, by definition, we have $T^\flat G = \Psi(G \times \mathfrak{g}^*)$, which ends the proof of 1.

2. By definition, we have
\[ P(g, \alpha_1, \ldots, \alpha_{r-1}) = \Phi(g, p(L_g^*(\alpha_1), \ldots, L_g^*(\alpha_{r-1}))) \]

and so $P$ must be a vector bundle morphism. But since $p$ is a bounded $r$-alternating map, the same is true for $P_g$, which implies Point 2.

3. Assume that $P$ is a partial $r$-Nambu-Poisson anchor on $G$. From Lemma 3.12 and Theorem 3.15 at point $e \in G$, since $\mathfrak{h}$ is the range of $P_e = p$, thus $\mathfrak{h}$ is either zero or $r$-dimensional. If $\mathfrak{h}$ is $r$-dimensional, the range of $P_g$ is $\Phi(\{g\} \times \mathfrak{h}) = T_e L_g(\mathfrak{h})$ from the definition of $P$ and is also $r$-dimensional. Now, from Theorem 3.15, the subbundle $H = \Phi(G \times \mathfrak{h})$ of $TG$ is integrable; this implies that $\mathfrak{h}$ is a Banach Lie subalgebra of $\mathfrak{g}$. Now, from the definition of $P$, this implies that the associated $r$-Nambu-Poisson tensor $\Lambda$ is given by
\[ \Lambda_g(\alpha_0, \alpha_1, \ldots, \alpha_{r-1}) = \Lambda_e(L_g^*(\alpha_0), L_g^*(\alpha_1), \ldots, L_g^*(\alpha_{r-1})) \]  \hspace{1cm} (22)

and so is left invariant considered as a section of $\bigwedge^r (T^*G)^*$. Now, at $e$, $\Lambda$ is regular. Thus from Theorem 3.15 there exist $r$ vector fields $Y_1, \ldots, Y_r$ around $e$ such that $\Lambda = Y_1 \wedge \cdots \wedge Y_r$. But $a_1 = (Y_1)_e, \ldots, a_r = (Y_r)_e$ must be a basis of $\mathfrak{h}$. We denote by $X_j$ the left invariant vector field defined by $a_j$ for $j \in \{1, \ldots, r\}$. Then $\overline{\Lambda} = X_1 \wedge \cdots \wedge X_r$ satisfies

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\[ \Lambda_g(\alpha_0, \alpha_1, \ldots, \alpha_{r-1}) = \det \langle \alpha_i, (L_g)_* a_j \rangle \]
\[ = \det \langle L_g^* \alpha_i, a_j \rangle \]
\[ = \Lambda_e(L_g^* \alpha_0, L_g^* \alpha_1, \ldots, L_g^* \alpha_{r-1}) \]
\[ = \Lambda_g(\alpha_0, \alpha_1, \ldots, \alpha_{r-1}) \]

Thus \( \Lambda = X_1 \wedge \cdots \wedge X_r \).

Conversely, assume that \( \mathfrak{h} \) is a Banach Lie subalgebra of \( \mathfrak{g} \). Since \( P \) is of rank \( r \), the restriction to \( \mathfrak{g} \) of the \( r \)-Nambu-Poisson tensor \( \Lambda \) associated to \( P \) is equal to its restriction to \( \mathfrak{h} \) and so is decomposable (cf Lemma 1 in [MVV98]). Therefore there exists a basis \( a_1, \ldots, a_r \) of \( \mathfrak{h} \) such that \( \Lambda_e = a_1 \wedge \cdots \wedge a_r \). But since \( P \) is left invariant, by the same previous arguments, if \( X_j \) is the left invariant vector field on \( G \) defined by \( a_j \) for \( j \in \{1, \ldots, r\} \), we have \( \Lambda = X_1 \wedge \cdots \wedge X_r \). But again since \( P \) is left invariant then its range is a \( r \) dimensional vector bundle \( H \) and as \( \mathfrak{h} \) is integrable so is \( H \).

According to Remark 3.10 it remains to show that \( L_{X_1, \ldots, X_{r-1}} \Lambda|_{T^* M} = 0 \) for all \((f_1, \ldots, f_{r-1}) \in (A_U)^{r-1} \) and all open sets \( U \) in \( M \).

Consider an open neighbourhood \( U \) of \( g \in G \) and smooth functions for \( j \in \{1, \ldots, r\} \). If \( Z_i = P(df_1, \ldots, \hat{df}_i, \ldots, df_r) \), we have

\[ L_{Z_i} \Lambda(dg_1, \ldots, dg_r) = d(\Lambda(dg_1, \ldots, dg_r))(Z_i) - \sum_{j=1}^{r} \Lambda(dg_1, \ldots, dg_j(Z_i), \ldots, dg_r). \]

But

\[ \Lambda(dg_1, \ldots, dg_r) = \sum_{\sigma \in S_r} (-1)^\sigma <dg_1, X_{\sigma(1)}> \cdots <dg_r, X_{\sigma(r)}>. \]

We then obtain:

\[ d(\Lambda(dg_1, \ldots, dg_r))(Z_i) \]
\[ = \sum_{\sigma \in S_r} (-1)^\sigma \left( \sum_{j=1}^{r} <dg_1, X_{\sigma(1)}> \cdots <dg_j, X_{\sigma(i)}> \right) (Z_i) \cdots <dg_r, X_{\sigma(r)}> \]
\[ = \sum_{j=1}^{r} \Lambda(dg_1, \ldots, dg_j(Z_i), \ldots, dg_r). \]
Remark 4.5. This example shows that there always exists a partial $r$ Nambu-Poisson structure manifold on any Banach Lie group and for any $r \in \mathbb{N}$. Note that on a Banach Lie group, a partial Nambu-Poisson structure is also the restriction of some Nambu-Poisson structure without the smooth paracompacity assumption.

In [Vais00], the author defines a notion of multiplicative Nambu-Lie group in finite dimension. Of course, this notion can be also adapted to the Banach setting for (partial) Nambu-Poisson structures on a Banach Lie group. Such a generalization would need to introduce the notion of multiplicative tensor. It would be one more type of example, but as in finite dimension, such concrete examples are very specific and seem irrelevant for $r > 2$.

### 4.2 Nambu-Poisson structure on a loop space

According to [Ham82], the set $\mathcal{C}^\infty(S^1, M)$ of smooth loops in $M$ has a structure of Fréchet manifold. Recall that it is a smooth paracompact manifold and so, according to Remark 3.22, any partial Nambu-Poisson structure on this manifold is obtained as the restriction of some Nambu-Poisson structure.

Assume that $M$ is provided with an $r$-Nambu-Poisson tensor $\Lambda$ and let $\{\ldots,\} \Lambda$ be the associated Nambu-Poisson bracket. We will provide $\mathcal{C}^\infty(S^1, M)$ with an $r$-Nambu-Poisson structure as follows.

Recall that, for $\gamma \in \mathcal{C}^\infty(S^1, M)$, an element $\alpha \in T^*_\gamma \mathcal{C}^\infty(S^1, M)$ (resp. $X \in T\gamma \mathcal{C}^\infty(S^1, M)$) is a section of the pull-back $\gamma^* (T^* \mathcal{C}^\infty(S^1, M))$ (resp. $\gamma^* (TC^\infty(S^1, M))$) over $\gamma : S^1 \rightarrow M$. We also have a dual pairing between these spaces given by

$$\langle \alpha, X \rangle = \int_{S^1} \langle \alpha(t), X(t) \rangle \, dt$$

We set:

$$\Lambda^L_\gamma (\alpha_1, \ldots, \alpha_r) = \int_{S^1} \langle \alpha_r(t), P_{\gamma^*_t}(\alpha_1(t), \ldots, \alpha_{r-1}(t)) \rangle \, dt >$$

$$= \int_{S^1} \Lambda_{\gamma^*_t}(\alpha_r(t), \alpha_1(t), \ldots, \alpha_{r-1}(t)) \, dt$$

Using analog arguments as for a symplectic manifold $(M, \omega)$ in [Pel18], we can show that this relation defines a smooth $r$-tensor $\Lambda^L$ on $T^* \mathcal{C}^\infty(S^1, M)$.  

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By the way, the almost Lie Nambu-Poisson bracket associated to $\Lambda^L$ is given by:

$$\{f_1, \ldots, f_r\}_\Lambda^L(\gamma) = \int_{S^1} \Lambda_{\gamma(t)}(df_1 \circ \gamma(t), \ldots, df_r \circ \gamma(t)) \, dt$$  \hspace{1cm} (23)$$

The fundamental relation only depends on the differential of functions on $M$ at a point of $M$ and the 1-jet of $\Lambda$ at this point. Thus, by integration along $\mathbb{S}^1$, we obtain that (FI) is also satisfied at any point $\gamma \in C^\infty(S^1, M)$. Thus $(C^\infty(S^1, M), \{\ldots, \}_\Lambda^L)$ is an r-Nambu-Poisson structure on $C^\infty(S^1, M)$.

**Proposition 4.6.** A point $\gamma \in C^\infty(S^1, M)$ is singular if and only if $\gamma : \mathbb{S}^1 \to M$ is a constant loop whose range is a singular point in $M$. In particular, $(C^\infty(S^1, M), \{\ldots, \}_\Lambda^L)$ is a regular r-Nambu-Poisson manifold if and only if $(M, \{\ldots, \}_\Lambda^L)$ is so.

**Proof.** From (23), it is clear that if $\gamma : \mathbb{S}^1 \to M$ is a constant loop whose range is a singular point in $M$, then $\gamma$ is a singular point in $C^\infty(S^1, M)$. Consider now a loop $\gamma : \mathbb{S}^1 \to M$ and assume that its range contains a regular point $x_0 = \gamma(t_0)$ in $M$. This means that there exist non zero 1-forms $\alpha_1, \ldots, \alpha_r$ in $T_{x_0}M$ such that $\Lambda_{x_0}(\alpha_1, \ldots, \alpha_r) \neq 0$. After changing $\alpha_1$ into $-\alpha_1$ if necessary, we can assume that $\Lambda_{x_0}(\alpha_1, \ldots, \alpha_r) > 0$. For $i \in \{1, \ldots, r\}$, we can extend each $\alpha_i$ to a 1-form (again denoted $\alpha_i$) around $x_0$ such that the support of $\Lambda(\alpha_1, \ldots, \alpha_r)$ in restriction to $\mathbb{S}^1$ is contained in some interval $I = [t_1, t_2] \subseteq S^1$ and $\Lambda(\alpha_1, \ldots, \alpha_r) \geq 0$ on $I$. From (23), it follows that $\gamma$ is a regular point of $\Lambda^L$. \hfill $\Box$

**Remark 4.7.** Of course, the characteristic distribution on the open dense set of regular points in $C^\infty(S^1, M)$ is involutive from Theorem 3.15. But, in general, such a distribution is not integrable, even if the r-Nambu-Poisson structure on $C^\infty(S^1, M)$ is regular as justified below. According to Pel18., the set $L^k_p(S^1, M)$ of Sobolev loops of class $L^k_p$ has a structure of Banach manifold. Using the same arguments as previously, each r-Nambu-Poisson structure on $M$ provides $L^k_p(S^1, M)$ with an r-Nambu-Poisson structure and again the Proposition 4.6 is true for this Sobolev manifold. So, by application of Theorem 3.15 3., the characteristic distribution is integrable. Note that for $p = 2$ the $L^2_2(S^1, M)$ is an Hilbert manifold and the Fréchet structure on $C^\infty(S^1, M)$ is the projective limit of $L^k_2(S^1, M)$ (cf. Wur95). Therefore $C^\infty(S^1, M)$ has an ILH structure in Omori’s sense (cf. Omo97). Assume that, on each level $L^k_2(S^1, M)$, the r-Nambu-Poisson structure is the restriction of the r-Nambu structure on $L^k_2(S^1, M)$ for each $k < k'$. It is easy to see that the open set $\Sigma$ of regular points in $C^\infty(S^1, M)$ is the ILH limit of the open dense sets $\Sigma_k$ in $L^k_2(S^1, M)$. But if $\gamma = \lim_{k \to \infty} \gamma_k$ in $\Sigma$
the sequence $L_k$ of characteristic leaves through $\gamma_k$ is a projective sequence of $r$ finite dimensional manifold. Unfortunately, the projective limit of finite dimensional manifold does not have a finite dimensional manifold structure (cf. [AbMa99]).

4.3 Nambu-Poisson structure on $P^\infty(M)$

In this section, we refer essentially to [Lot18]. Consider a closed finite dimensional Riemannian $(M,g)$. We denote by $d\text{vol}_M$ the volume form on $M$ associated to $g$. We consider the set (cf. [Lot18] and reference inside)

$$P^\infty(M) = \{ \rho \text{ dvol} : \rho \in C^\infty(M), \rho > 0, \int_M \rho \text{ dvol}_M = 1 \}.$$  

Then $P^\infty(M)$ has a convenient manifold structure ([Lot18]) which can be specified as follows.

The set $C^\infty(M)$ has a structure of convenient space (cf. [KrMi97], Proposition 6.1). Consider the smooth function $I : \phi \mapsto \int_M \phi \text{ dvol}_M$. The differential $d\rho I$ is $\phi \mapsto -\int_M \text{div}(\rho \nabla \phi) d\text{vol}_M$ which is non singular. Since the set $C^\infty_+(M)$ of smooth strictly positive functions is an open set of $C^\infty(M)$, it follows that $I^{-1}(1) \cap C^\infty_+(M)$ has a structure of convenient manifold and its tangent space at $\rho$ is the convenient space of equation

$$\int_M \text{div}(\rho \nabla \phi) d\text{vol}_M = 0.$$  

Now, for any $\phi \in C^\infty(M)$, we consider the function $F_\phi \in C^\infty(P^\infty(M))$ given by:

$$F_\phi(\rho \text{ dvol}_M) = \int_M \phi \rho \text{ dvol}_M.$$  

For the sake of simplicity, we will denote $\rho \text{ dvol}_M$ by $\rho_M$.

On the other hand, the gradient of $\phi \in C^\infty(M)$ is denoted $\nabla \phi$ and is associated to a vector field $V_\phi$ on $P^\infty(M)$ defined by:

$$V_\phi(F)(\rho_M) = \left. \frac{d}{d\tau} \right|_{\tau=0} F(\rho_M - \tau \text{div}(\rho \nabla \phi) d\text{vol}_M)$$  

for any $F \in C^\infty(P^\infty(M))$.

Consider the equivalence relation $\phi \sim \phi' \iff \phi = \phi' + \text{cte}$ on $C^\infty(M)$ and denote by $C^\infty(M)/\mathbb{R}$ the associated quotient space. Then the map $\phi \mapsto V_\phi$
is an isomorphism from $C^\infty(M)/\mathbb{R}$ to $T_{\rho_M}P^\infty(M)$ (cf. \cite{Ot01} and \cite{Lot18}). In particular, $P^\infty(M)$ is modelled on $C^\infty(M)/\mathbb{R}$.

**Remark 4.8.**

1. From \cite{KrMi97}, § 6.4, $C^\infty(M)$ is reflexive and so is any closed subspace of it. Thus the convenient manifold $P^\infty(M)$ is modelled on a reflexive convenient space.

2. As asserted in \cite{Lot18}, the map $\phi \mapsto F_\phi$ gives an injection of $C^\infty(M)$ into the dual $C^\infty(M)'$ and so the set of functions $\{F_\phi, \phi \in C^\infty(M)\}$ separates points in $P^\infty(M)$.

By the way, the Riemannian metric $\bar{g}$ on $P^\infty(M)$ defined in \cite{Ot01} is given by

$$\bar{g}_{\rho_M}(V_\phi, V_\psi) = \int_M g(\nabla \phi, \nabla \psi) \rho_M.$$ 

Then, for any $\phi \in C^\infty(M)$, the gradient of $\bar{\nabla}F_\phi$ relative to $\bar{g}$ is precisely $V_\phi$ (cf. Lemma 2.7 in \cite{Lot18}). Of course, $\bar{g}$ is a weak Riemannian metric. Therefore, if $T'P^\infty(M)$ is the kinematic cotangent bundle of $P^\infty(M)$, the map $X \mapsto \bar{g}(X, \cdot)$ is an injective morphism $\bar{g} : TP^\infty(M) \to T'P^\infty(M)$. It follows that the range $T^\rho P^\infty(M)$ of $\bar{g}$ is a subbundle of $T'P^\infty(M)$. In particular, for any $\phi \in C^\infty(M)$ the differential $dF_\phi$ is the restriction to $T_{\rho \text{dvol}_M}P^\infty(M)$ of the linear map $\mu \mapsto \int_M \mu \text{dvol}_M$ and so $dF_\phi$ is a differential 1-form on $P^\infty(M)$. Moreover, $dF_\phi$ is a section of $T^\rho P^\infty(M)$ for all $\phi \in C^\infty(M)$.

Note that, according to Remark 4.8, using the Hahn-Banach Theorem for locally convex topological spaces, it follows that $T^\rho P^\infty(M)$ is dense in the cotangent bundle $T'P^\infty(M)$.

We denote by $\mathcal{A}(M)$ the vector space of functions $\{F_\phi, \phi \in C^\infty(M)\}$ and provide it with an algebra structure by the product given by $F_\phi F_\psi = F_{\phi \psi}$. Thus the map $C^\infty(M) \to \mathcal{A}(M)$ is an algebra morphism. It is shown in \cite{Lot18} that if we have a Poisson structure on $M$ there exists a canonical Poisson structure on $P^\infty(M)$ which induces on $\mathcal{A}(M)$ a Poisson bracket such that the previous morphism is a Poisson Lie algebra morphism. We will show that if we have an $r$-Nambu-Poisson structure on $M$, this implies an analog result:

**Proposition 4.9.** Let $(M, C^\infty(M), \{\cdot, \ldots, \cdot\}_P)$ be an $r$-Nambu-Poisson on $M$. 

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1. If $\Lambda$ is the $r$-Nambu-Poisson tensor on $M$ associated to $P$, then the relation

$$\bar{\Lambda}_{\rho M}(dF_{\phi_1}, \ldots, dF_{\phi_r}) = \int_M \Lambda(d\phi_1, \ldots, d\phi_r) \rho_M$$

defines an $r$-tensor $\bar{\Lambda}$ which belongs to $C^\infty(\wedge^r TP^\infty(M))$. We set

$$\bar{P}(\alpha_1, \ldots, \alpha_{r-1}) = \bar{\Lambda}(., \alpha_1, \alpha_{r-1}).$$

(26)

2. The bracket

$$\{F_{\phi_1}, \ldots, F_{\phi_r}\} = \bar{\Lambda}(dF_{\phi_1}, \ldots, dF_{\phi_r})$$

(27)

provides $\mathcal{A}(M)$ with a Nambu-Poisson bracket.

In particular, $(\mathcal{A}(M), \{., \ldots, .\}_P)$ is an $r$-Lie algebra.

3. The relation:

$$\{F_1, \ldots, F_r\}_P = \bar{\Lambda}(dF_1, \ldots, dF_r)$$

(28)

defines an $r$-Nambu-Poisson bracket on $C^\infty(P^\infty(M))$.

Moreover, $(P^\infty(M), C^\infty(P^\infty(M)), \{., \ldots, .\}_P)$ is a regular Nambu-Poisson manifold.

Remark 4.10. According to Remark 4.8, each tangent space $T_{\rho M}P^\infty(M)$ is reflexive and so $T''_{\rho M}P^\infty(M) = T_{\rho M}P^\infty(M)$. Thus $\bar{P}$ defined by (26) takes values in $TP^\infty(M)$.

Remark 4.11. Note that $\mathcal{A}(M)$ is a vector subspace of $C^\infty(P^\infty(M))$ but for its algebra structure it is not a subalgebra of $C^\infty(P^\infty(M))$. However, $(\mathcal{A}(M), \{., \ldots, .\}_P)$ is an $r$-Lie algebra in the sense of Definition 3.1, 3. Note that, if $M$ is an $r$-Nambu-Poisson manifold, the map $C^\infty(M) \rightarrow \mathcal{A}(M)$ $\phi \mapsto F_{\phi}$ is a morphism of $r$-Lie algebras.

Proof of Proposition 4.9.

1. According to Remark 4.8, using the same argument as in [Lot.18], the map $\phi \mapsto F_{\phi}$ induces an isomorphism from $C^\infty(M)/\mathbb{R}$ to the cotangent space $T''_{\rho M}P^\infty(M)$ for each $\rho_M \in P^\infty(M)$. Since if any function $\phi_i$ is constant, then the RHS of (26) vanishes, this implies the result of Point 1.

2. Since $\{\phi_1, \ldots, \phi_r\}_P = \Lambda(d\phi_1, \ldots, d\phi_r)$, from the RHS of (26) and the definition of the bracket (27), it follows that this bracket takes values in $\mathcal{A}(M)$. In particular, we have

$$\{F_{\phi_1}, \ldots, F_{\phi_r}\}_P(\rho_M) = \int_M \{\phi_1, \ldots, \phi_r\}_P \rho_M$$

(29)
As $F_\phi F_\psi = F_{\phi \psi}$, the Leibniz property of the bracket $\{ \ldots, \}_P$ is a direct consequence of (29). According to (29), the relation (FI) on $A(M)$ is a direct consequence of the relation (FI) on $C^\infty(M)$.

3. The relation (28) is well defined and provides $M$ with a structure of almost $r$-Nambu-Poisson structure. We only have to show that the relation (FI) is satisfied by $\{ \ldots, \} \bar{P}$ for all $F_1, \ldots, F_r$ in $C^\infty(P^\infty(M))$. But since $\phi \mapsto F_\phi$ induces an isomorphism from $C^\infty(M)/R$ to the cotangent space $T^*_\rho M P^\infty(M)$ for each $\rho M \in P^\infty(M)$, for $\rho M$ fixed, and $F_1, \ldots, F_r$ in $C^\infty(P^\infty(M))$, there exist $\phi_1, \ldots, \phi_r$ in $C^\infty(M)$ such that $d_{\rho M} F_i = d_{\rho M} F_\phi_i$ for all $i \in \{1, \ldots, r\}$. Now the relation (FI) is equivalent to the relation (7). Now, each Hamiltonian vector field $\bar{X}_{G_1, \ldots, G_{r-1}} = \bar{P}(dG_1, \ldots, dG_{r-1})$ only depends on the differential of $dG_1, \ldots, dG_{r-1}$ and the relation (7) also only depends on the differential of the functions inside this relation. Therefore, since the relation (FI) is satisfied on $A(M)$, this implies that this relation is also satisfied on $C^\infty(P^\infty(M))$.

It remains to show that the $r$-Nambu Poisson structure $(P^\infty(M), C^\infty(P^\infty(M)), \{ \ldots, \} \bar{P})$ is regular. Fix some regular point $x \in M$. There exist $r$ smooth functions $\phi_1, \ldots, \phi_r$ on some open neighbourhood $U$ of $x$ such that $\{ \phi_1, \ldots, \phi_r \}_P = 1$ on $U$ (cf. [Vais99] for instance). If $\rho M$ is fixed then

$$\int_U \{ \phi_1, \ldots, \phi_r \}_P \cdot \rho M = K \quad (30)$$

Given $\epsilon > 0$, there exists an open set $V_\epsilon$ in $U$ whose closure is contained in $U$ and such that

$$\int_{U \setminus V} \{ \phi_1, \ldots, \phi_r \}_P \cdot \rho M \leq \epsilon \quad (31)$$

Now, there exists a bump function $\theta^\epsilon$ whose support is contained in $U$ and such that $\theta^\epsilon = 1$ on $V_\epsilon$. Each function $\phi^\epsilon_i = \theta^\epsilon \phi_i$ for $i \in \{1, \ldots, r\}$ belongs to $C^\infty(M)$. Since $\theta^\epsilon$ is bounded by 1 on $U$, by Leibniz property and since $\Lambda$ is $r$-linear and bounded on $U$, it follows that $\{ \psi^\epsilon_1, \ldots, \psi^\epsilon_r \}_P$ is bounded on $U$ by some constant $C$ independent of $\epsilon$. Thus from (31), we have

$$\int_{U \setminus V_\epsilon} \{ \phi^\epsilon_1, \ldots, \phi^\epsilon_r \}_P \cdot \rho_M \leq C \epsilon \quad (32)$$

Choose $V_\epsilon$ such that $K - C \epsilon \geq \frac{K}{2}$. According to (30) and the construction of $\phi^\epsilon_1, \ldots, \phi^\epsilon_r$, we obtain:

$$\{ F_{\phi^\epsilon_1}, \ldots, F_{\phi^\epsilon_r} \}_P(\rho M) = \int_U \{ \phi^\epsilon_1, \ldots, \phi^\epsilon_r \}_P \cdot \rho_M \geq \frac{K}{2} \quad (33)$$
It follows that $\rho_M$ is a regular point and so $(P^\infty(M), C^\infty(P^\infty(M)), \{\ldots\}_P)$ is a regular Nambu-Poisson structure.

Since $(P^\infty(M), C^\infty(P^\infty(M)), \{\ldots\}_P)$ is a regular Nambu-Poisson structure, from Theorem 3.15 the associated characteristic distribution is an involutive $r$-dimensional subbundle of $TP^\infty(M)$. In general, it seems that, without more assumptions, this distribution will not be integrable (cf. Remark 4.14). We will now expose a situation in which such a distribution is integrable.

For the sake of simplicity we denote by $\Omega$ the volume form $d\text{vol}_M$ and we consider the group $\mathcal{D}_\Omega(M)$ of diffeomorphisms $\Psi$ of $M$ such that $\Psi^*(\Omega) = \Omega$. Such a diffeomorphism is called a volume preserving diffeomorphism. The group $\mathcal{D}_\Omega(M)$ can be provided with a structure of ILH group in the Omori’s sense (cf. [Omo97] and [Smo07] for a concise presentation).

Following the formalism of symplectic geometry (cf. [DuSa98], § 3.1), a smooth map $[0, 1] \times M \to M$ such that $\Psi_t$ belongs to $\mathcal{D}_\Omega(M)$ for all $t \in [0, 1]$ will be called a volume preserving isotopy. Such an isotopy $\Psi_t$ is the flow of a time-dependent vector field $X_t$ on $M$ which is divergence free, i.e. $L_{X_t}\Omega = 0$. Conversely, any time-dependent divergence free vector field $\{X_t\}_{t \in [0, 1]}$ generates a flow $\Psi_t$ which is a volume preserving diffeomorphism isotopy. We denote by $\mathcal{D}^0_\Omega(M)$ the sub group of $\Psi \in \mathcal{D}_\Omega(M)$ such that there exists a volume preserving diffeomorphism isotopy $\Psi_t$ where $\Psi_0 = \text{Id}_M$ and $\Psi_1 = \Psi$. In fact, $\mathcal{D}^0_\Omega(M)$ is the connected component of the identity in $\mathcal{D}_\Omega(M)$ and so is an open subgroup. Note that since $\Omega$ is preserved by each $\Psi \in \mathcal{D}^0_\Omega(M)$, we have an action of $\mathcal{D}^0_\Omega(M)$ given by $(\Psi, \rho_M) \mapsto \Psi^*(\rho_M)$.

On the other hand, to $\Omega$ is canonically associated an $n$-Nambu-Poisson bracket characterized by (cf. [Gau96], Corollary 1)

$$\Lambda(d\phi_1, \ldots, d\phi_n) = \{\phi_1, \ldots, \phi_n\}_\Omega.$$  \hfill (34)

**Proposition 4.12.** Consider the regular $n$-Nambu-Poisson structure on $P^\infty(M)$ associated to the previous $n$-Nambu-Poisson tensor $\Lambda$. Then its characteristic distribution is integrable and each leaf is an orbit of the action of $\mathcal{D}^0_\Omega(M)$ on $P^\infty(M)$.

**Proof.** At first, note that if $\Psi$ is a diffeomorphism of $M$, from (34), we have

$$\Psi^*(\Lambda) (d\phi_1, \ldots, d\phi_n) = \Lambda (d(\phi_1 \circ \Psi), \ldots, d(\phi_n \circ \Psi)) = \{\phi_1 \circ \Psi, \ldots, \phi \circ \Psi\}_\Omega.$$  

\footnote{In particular, $\mathcal{D}_\Omega(M)$ is a Fréchet manifold.}
On the other hand,
\[ \Psi_*(\Lambda)(d\phi_1, \ldots, d\phi_n) = (\{\phi_1, \ldots, \phi_n\} \circ \Psi). (\Psi^*\Omega) \]

Thus we have
\[ \{\phi_1 \circ \Psi, \ldots, \phi_n \circ \Psi\} = \{\phi_1, \ldots, \phi_n\} \circ \Psi \iff \Psi_*(\Lambda) = \Lambda \iff \Psi^*\Omega = \Omega. \]

This means that
\[ \Psi_*(\Lambda) = \Lambda \iff \Psi \in D_\Omega(M). \]

Therefore, given any Hamiltonian vector field \( X_{\phi_1, \ldots, \phi_{n-1}} \), its flow \( \Psi_t^{X_{\phi_1, \ldots, \phi_{n-1}}} \) for \( t \in [0, 1] \)\(^9\) is a volume preserving isotopy since \( L_{X_{\phi_1, \ldots, \phi_{n-1}}} \Lambda = 0 \).

**Lemma 4.13.** Given \( \phi_1, \ldots, \phi_{n-1} \in C^\infty(M) \), the Hamiltonian flow of \( F_{\phi_1, \ldots, F_{\phi_{n-1}}} \) on \( P^\infty(M) \) is \( \Psi_t^{X_{F_{\phi_1, \ldots, F_{\phi_{n-1}}}}} \) defined by:
\[ \Psi_t^{X_{F_{\phi_1, \ldots, F_{\phi_{n-1}}}}} (\rho, \Omega) = \rho \circ \Psi_t^{X_{\phi_1, \ldots, \phi_{n-1}}} \]
and this flow is defined for any \( t \in \mathbb{R} \).

According to this Lemma and Theorem 3.15, the characteristic distribution of \( \bar{\Lambda} \) is integrable. Let \( L \) be a leaf through \( \rho.\Omega \). Thus it is a finite dimensional manifold. If \( \rho'.\Omega \) belongs to \( L \), there exists a piecewise smooth curve \( c : [0, 1] \rightarrow L \) such that \( c(0) = \rho.\Omega \) and \( c(1) = \rho'.\Omega \) and such that
\[ \text{if } t_0 = 0 \leq t_1 < \cdots < t_{m-1} < t_m = 1 \text{ is a partition of } [0, 1] \text{ such that } c|_{[t_{i-1}, t_i]} \text{ is smooth, then such a curve is tangent to a Hamiltonian vector field on } P^\infty. \]

Thus from Lemma 4.13 \( L \) must be contained in the orbit of the action through \( \rho.\Omega \) of \( D^0_\Omega(M) \) on \( P^\infty(M) \). Conversely, such an orbit is a finite composition of volume preserving isotopies contained in \( D^0_\Omega(M) \). Since such an obit is connected since so is \( D^0_\Omega(M) \), this ends the proof of the Proposition.

**Proof of Lemma 4.13** According to Remark 4.10 for any \( \phi \in C^\infty(M) \) we have:
\[ X_{F_{\phi_1, \ldots, F_{\phi_{n-1}}}}(F_{\phi})(\rho, \Omega) = \int_M \{\phi, \phi_1, \ldots, \phi_{n-1}\} \frac{d}{dt}|_{t=0} (\rho \circ \Psi_t^{X_{\phi_1, \ldots, \phi_{n-1}}} - \rho).\Omega. \]

\( \text{9 since } M \text{ is compact, then the flow is defined for all } t \in \mathbb{R} \)
Therefore the integral curve of $\Psi^X_{t_0} \cdots \phi^n_{t_1}$ through $\rho \Omega$ is $t \mapsto \rho \circ \Psi^X_{t_0} \cdots \phi^n_{t_1}$, for $t$ in a small interval $[0,\epsilon]$. The result follows from the properties of a flow of a vector field.

Remark 4.14. In the upper general context, if the characteristic distribution is integrable, the same arguments work for any flow of Hamiltonian vector field. But such a flow must leave invariant the $r$-Nambu tensor $\Lambda$ and the volume form $\Omega$. The set of typical diffeomorphisms constitute a sub-group of $\mathcal{D}_\Omega(M)$. But such a subgroup does not satisfy the conditions imposed in [Omo97] to have an ILH Lie group structure. To our knowledge, there does not exist other examples of this type of $r$-Nambu-Poisson structure on $P^\infty(M)$ obtained from such a structure on $M$ whose characteristic distribution is integrable. On the other hand, we have no examples of an $r$-Nambu-Poisson structure on $P^\infty(M)$ of the same type, whose characteristic distribution is not integrable.

5 Projective and direct limits of partial Nambu-Poisson structures

5.1 Projective limit of partial Nambu-Poisson structures

The aim of this section is to define a convenient $r$-Nambu-Poisson structure on the projective limit of $r$-Nambu-Poisson Banach structures.

We then use a lot of results of [PeCa19] which can be adapted to our context.

Let $(M_i, A_{M_i}, \{\ldots, \})_{i \in \mathbb{N}}$ be a sequence of $r$-partial Nambu-Poisson Banach manifolds where $p_i^k : T^k M_i \to M_i$ is a Banach subbundle of $p_{M_i} : T^* M_i \to M_i$ and $P : (T^k M)^{k-1} \to TM$ is a skew-symmetric morphism. We denote by $\mathcal{M}_i$ the Banach space on which $M_i$ is modelled, and by $\mathcal{F}_i$ the model of the typical fibre of $p_i^k : T^k M_i \to M_i$ and we assume that $\mathcal{F}_i$ is a Banach subspace of the dual $\mathcal{M}_i^*$ of $\mathcal{M}_i$.

Definition 5.1. The sequence $(M_i, A_{M_i}, \{\ldots, \})_{i \in \mathbb{N}}$ is called a projective sequence of $r$-partial Nambu-Poisson Banach manifolds if there exist mappings $\delta_i^j : M_j \to M_i$ such that $(M_i, \delta_i^j)_{i \in \mathbb{N}}$ is a submersive projective

\textsuperscript{10} i.e. each $\delta_i^j : M_j \to M_i$ is a surjective submersion
sequence of Banach manifolds fulfilling, for all \( i \in \mathbb{N} \), the following properties:

\begin{align*}
    \text{(PSpNPBM 1)} \quad & T^* \delta^i_{i+1}(T^b M_i) \subset T^b M_{i+1} \tag{11} \\
    \text{(PSpNPBM 2)} \quad & P_i = T^* \delta^i_{i+1} \circ P_{i+1} \circ (T^* \delta^i_{i+1})^{r-1} \\
    \text{(PSpNPBM 3)} \quad & \text{Around each } x \in M \text{ there exists a sequence of charts } \left(U_i, \phi_i\right)_{i \in \mathbb{N}} \text{ such that } (U = \lim_{\leftarrow}(U_i), \phi = \lim_{\leftarrow}(\phi_i)) \text{ is a chart of } x \text{ in } M, \text{ so that the charts } (U_{i+1}, \phi_{i+1}) \text{ and } (U_i, \phi_i) \text{ are compatible with Property } \text{(PSpNPBM 1)}. 
\end{align*}

Figure 1: Projective sequence of partial Nambu-Poisson structures

Let \((M_i, A_{M_i}, \{\ldots, \} P_r)_{i \in \mathbb{N}}\) be a projective sequence of \(r\)-partial Nambu-Poisson Banach manifolds.

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\text{As in [PeCa19], this means that for any } y \in M_{i+1} \text{ we have } T^*_y \delta^i_{i+1}(T^b M_i) \subset T^*_y M_{i+1} \]
The projective limit \( M = \lim_{\leftarrow} (M_i) \), is a Fréchet manifold; in particular \( M \) is a convenient manifold. As in [Gal98], the set \( (TM_i, T\delta^j_i) \) is a submersive projective sequence of Banach manifolds and \( TM = \lim_{\leftarrow} (TM_i) \) is the kinematic tangent bundle of the Fréchet manifold \( M \) modelled on \( M \). Now, since \( M \) is a Fréchet manifold, the convenient kinematic cotangent bundle \( p^*_M : T^*M \to M \) is well defined and its typical fibre is the topological dual \( M^* \) of \( M \) (cf. [KrMi97], 33.1).

**Remark 5.2.** Since each \( \delta^j_i \) is a submersion, if each \( \Lambda^i \) is an \( r \)-Nambu-Poisson tensor (that is \( P_i \) is defined on \( T^*_iM \)), then the condition (PSpNPBM 1) is automatically satisfied.

We identify \( M \) with the set
\[
\{ x = (x_i)_{i \in \mathbb{N}} \in \prod_{i \in \mathbb{N}} M_i : x_i = \delta^j_i(x_j), j \geq i \}.
\]

Since for each \( j \geq i \), \( \delta^j_i : M_j \to M_i \) is a submersion, the adjoint map \( T^*\delta^j_i : T^*\delta^j_i(x_j) M_i \to T^*_x M_j \) is a continuous linear injective map whose range is closed, for all \( x_j \in M_j \). We then obtain a submersion \( \delta_i : M \to M_i \) defined by \( \delta_i(x) = x_i \in M_i \) for each \( x \in M \), and so the transpose map \( T^*\delta_i : T^*_\delta_i(x_i) M_i \to T^*_x M \) is a linear continuous injection whose range is closed. Therefore, we have an ascending sequence \( \left( T^*_\delta_i(x_i) M_i \right)_{i \in \mathbb{N}} \) of closed Banach spaces. Since \( T^*_x M \) is the projective limit of the sequence \( \left( T^*_\delta_i(x_i) M_i \right)_{i \in \mathbb{N}} \), each vector space \( \lim_{\leftarrow} (T^*_\delta_i(x_i) M_i) \) is the topological dual of \( T^*_x M \) and is a convenient space (cf. [PeCa19]). In particular, we have \( T^*_x M = \lim_{\leftarrow} (T^*_\delta_i(x_i) M_i) \).

Now from Definition 5.1 (PSpNPBM 1), we have \( T^*\delta^j_i \left( T^*_\delta^j_i(x_j) M_i \right) \subset T^*_x M_j \) for all \( x_j \in M_j \).

On the other hand, with our previous identifications, \( \left( T^*_\delta_i(x_i) M_i \right)_{i \in \mathbb{N}} \) is an ascending sequence of closed Banach spaces contained in \( T^*_x M \). It follows that \( T^*_x M = \lim_{\leftarrow} \left( T^*_\delta_i(x_i) M_i \right) \) is a convenient subspace of \( T^*_x M \) and so is \( (T^*_x M)^{r-1} = \lim_{\leftarrow} \left( T^*_\delta_i(x_i) M_i \right)^{r-1} \). If we set \( T^0 M = \bigcup_{x \in M} T^0_x M \), we

\[ \text{Note that since } \delta^j_i \text{ is a submersion, it follows that } M^* \text{ is usually called the inductive dual if } M^* \]
have \((T^b M)^{r-1} = \bigcup_{x \in M} \left( T^b_x M \right)^{r-1} \). We then consider the associated map

\((\pi_i^{r-1})^b: (T^b M)^{r-1} \to M\) defined by \((\pi_i^{r-1})^b(x_i, (\alpha_1^r, \ldots, \alpha_r^{r-1})) = x_i\).

In [CaPe19], proof of Theorem 3.2.3, it is shown that \(T^b_M M_l = \bigcup_{x \in M} T^b x \delta_l(T^b_{\delta_l(x)} M_l)\) is the total space of a convenient subbundle of \(p^b: T^b M \to M\) with typical fibre \(M^r_l\) and the sequence \((T^b_M M_i, \iota^i_j | (T^b_M M_i))_{i \leq j}\) where \(\iota^i_j\) is the natural injection of \(T^b_M M_i\) into \(T^b_M M_j\), for \(i \leq j\), is a direct sequence of Banach bundles. So \(T^b_M = \lim_{\leftarrow}(T^b_i M_i)\) can be endowed with a structure of convenient bundle with typical fibre \(F = \lim_{\leftarrow}(F_i)\). On the other hand, according to [CaPe19], Theorem 2.3.2, there exists a canonical bundle morphism \(T^b \delta_i: T^b M \to T^b_i M\) associated to the pull-back of the bundle \(p_i: T^b M_i \to M_i\) over \(\delta_i: M \to M_i\):

\[
\begin{array}{c}
T^b_M M_i \\
\downarrow \\
M \xrightarrow{\delta_i} M_i \\
\end{array}
\]

\[
\begin{array}{c}
T^b_i M_i \\
\downarrow \\
p_i \\
\end{array}
\]

such that its restriction to any fibre is an isomorphism \((x, \alpha_i) \mapsto (\delta_i(x), \alpha_i)\). We thus obtain for each \(i \in \mathbb{N}\)

\[
P_i \circ (T^b \delta_i)^{r-1} = T^b \delta_i^j \circ p_j \circ (T^b \delta_j)^{r-1}
\]

such that its restriction to any fibre is an isomorphism \((x, \alpha_i) \mapsto (\delta_i(x), \alpha_i)\). We thus obtain for each \(i \in \mathbb{N}\)

\[
P_i \circ (T^b \delta_i)^{r-1} = T^b \delta_i^j \circ p_j \circ (T^b \delta_j)^{r-1}
\]

Then

\[
P = \lim_{\leftarrow} \left( P_i \circ (T^b \delta_i)^{r-1} \right)
\]

is a bundle morphism from \((T^b M)^{r-1}\) to \(TM\). Moreover, one can prove that this morphism is \(k\)-skewsymmetric.

Now, to each \(P_i\) is associate an \(r\)-Nambu-Poisson tensor \(\Lambda_i\) and from [38] we have \(\Lambda = \lim_{\leftarrow} \Lambda_i\). From the definition of the bracket \(\{ \ldots, \} P_i\), as in [CaPe19], the bracket \(\{ \ldots, \} P\) can be defined on \(A_M\) by

\[
\{ \ldots, \} P = \lim_{\leftarrow} \{ \ldots, \} P_i
\]

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and as each bracket $\{\ldots,\}$ satisfies the Fillipov identity, the same is true for $\{\ldots,\}$.

Finally, we have the following

**Proposition 5.3.** Let $(M_i, A_M, \{\ldots,\})_{i \in \mathbb{N}}$ be a projective sequence of $r$-partial Nambu-Poisson Banach manifolds whose projective limit is the Fréchet manifold $M$.

There exists a convenient weak subbundle $p^\flat : T^b M \to M$ of $p^*_M : T^* M \to M$ and a skew-symmetric morphism $P : (T^b M)^{r-1} \to TM$ such that $(M, A_M, \{\ldots,\})$ is a $r$-partial Nambu-Poisson structure on $M$.

We will now give a version of Theorem 3.28 in this context.

For each $i \in \mathbb{N}$, we denote by $\Lambda_i$ the partial $r$-Nambu-Poisson tensor associated to $P_i$ and $\Lambda_i^\flat : \wedge^{r-1} T^b M_i \to TM_i$. Now since $T^b M = \lim\limits_\leftarrow (T^b M_i)$, from the definition of the bundle $\wedge^{r-1} T^b M_i$ it follows that $\wedge^{r-1} T^b M = \lim\limits_\leftarrow (\wedge^{r-1} T^b M_i)$. Let $\Lambda$ be the $r$-Nambu tensor associated to $P$. Consider a projective limit $U = \lim\limits_\leftarrow (U_i)$ and $j = 1, \ldots, r$, consider $\alpha_j = \lim\limits_\leftarrow (\alpha_i)_j$ where $(\alpha_i)_j$ is a local section of $T^b M_i$ over $U_i$. From the properties of $P_i$ we must have:

$$\Lambda(\alpha_1, \ldots, \alpha_r) = \Lambda(\lim\limits_\leftarrow (\alpha_i)_1, \ldots, \lim\limits_\leftarrow (\alpha_i)_r) = \lim\limits_\leftarrow (\Lambda_i((\alpha_i)_1, \ldots, (\alpha_i)_r)).$$

We have obvious analogue relations for $\Lambda_i^\flat$ relative to $P$ which have analogue properties as $\Lambda_i^\flat$ relatively to $P_i$ and, of course, the same is true for the canonical $r$-skew-symmetric tensor $\Lambda$ associated to $P$.

**Theorem 5.4.** Let $(M_i, A(M_i), \{\ldots,\})_{i \in \mathbb{N}}$ be a projective sequence of $r$-partial Nambu-Poisson Banach manifolds whose projective limit is the Fréchet manifold $M$. Assume that $\Lambda_1$ has at least one regular point. We have the following properties

1. A point $x = \lim\limits_\leftarrow (x_i)$ is singular if and only if all $x_i$ are singular.

2. Fix some $x \in \Sigma$. There exists a chart $U = \lim\limits_\leftarrow (U_i), \phi = \lim\limits_\leftarrow \phi_i$ around $x$ in $M$, where $(U_i, \phi)$ is a chart in $M_i$ around $x_i$ which is contained in $\Sigma_i$ for all $i$ with the following properties:
(i) \( \phi(U) \) is a product \( U \times \hat{U} \subset \mathbb{R}^r \times \mathbb{M} \) where \( \hat{U} \) is a Fréchet subspace of \( \mathbb{M} \).

(ii) In this chart, the Nambu-Poisson tensor \( \Lambda \) can be written

\[
\phi_*(\Lambda) = \frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_r}
\]

for some coordinates system \( (t_1, \ldots, t_r) \) on \( \mathbb{R}^r \).

(iii) On \( \Sigma \), the characteristic distribution \( \mathcal{D} \) of \( P \) is integrable and the restriction of \( \delta_i \) to the leaf \( L \) through \( x \in \Sigma \) is a surjective local diffeomorphism onto the leaf \( L_i \) through \( \delta_i(x) \) in \( M_i \) for all \( i \in \mathbb{N} \).

Proof. The notations and results exposed before Theorem 5.3 will be used and the proof will be divided into several steps.

**Step 1.**

For all \( i \in \mathbb{N} \), we denote by \( (\mathcal{D}_i)_{x_i} \) the range of \( P_i \) at \( x_i \in M_i \) and by \( \mathcal{D}_x \) the range of \( P \) at \( x \in M \). Fix some point \( x = \lim(x_i) \) in \( M \). Thus by Theorem 3.28 \( \dim(\mathcal{D}_x) = r \) or 0 and the same is true for \( \dim((\mathcal{D}_i)_{x_i}) \). But we have

\[
P_i \circ (T^\delta_i)^{r-1} = T^\delta_i \circ P_j \circ (T^\delta_j)^{r-1} \quad \text{(cf. (37)).}
\]

Thus, the previous equality implies that \( \dim((\mathcal{D}_i)_{x_i}) \leq \dim((\mathcal{D}_j)_{x_j}) \leq r \) for \( j \geq i \) and in particular \( \dim((\mathcal{D}_i)_{x_i}) \leq \dim(\mathcal{D}_x) \leq r \), for all \( i \in \mathbb{N} \). Therefore, if \( \dim((\mathcal{D}_i)_{x_i}) = r \), this implies that \( \dim((\mathcal{D}_j)_{x_j}) = r \), for all \( j \geq i \). In this case, \( T^\delta_j \) in restriction to \( (\mathcal{D}_j)_{x_j} \) must be an isomorphism for all \( j \geq i \) and in particular \( \dim(\mathcal{D}_x) = r \). Let \( k \) be the smallest integer such that \( x_i \) is regular. Thus, if \( k > 1 \), \( x_i \) is singular for each \( 1 \leq i < k \). For the same reason, if \( \dim(\mathcal{D}_x) = 0 \) this implies that \( \dim((\mathcal{D}_i)_{x_i}) = 0 \) and so each \( x_i \) is singular for all \( i \in \mathbb{N} \). The converse is clear.

Now, it remains to show that if \( M_1 \) contains a regular point, we must have \( k = 1 \).

Indeed, from Theorem 5.3 there exists a neighbourhood \( U_k \) of \( x_k \) in \( M_i \) on which \( P_i \) is regular. Then \( U = \delta_k^{-1}(U_k) \) is an open set in \( M \) and also \( U_i = (\delta_k^{-1})^{-1}(U_k) \) is an open neighbourhood of \( x_i \) for \( i \geq k \), and since \( \delta_k \) is a surjective submersion for \( i \leq k \), then \( U_i = \delta_i^k(U_k) \) is an open neighbourhood of \( x_i \) and so \( U = \lim(U_i) \). After restricting \( U_k \) if necessary, we can assume that there exists a chart \( (U_i, \phi_i) \) around \( x_i \) so that \( (U, \phi = \lim(\phi_i)) \) is a chart for \( M \). Moreover, since we have a submersive projective sequence, we may assume that for each \( i, \phi_i \) and \( \phi_{i+1} \) are so that there exists a decomposition
\[ \mathbb{M} = \ker \delta_i \oplus \tilde{\mathbb{M}}_i \] such that the following diagram is commutative (cf. [CaPe]):

\[
\begin{array}{ccc}
U & \xrightarrow{\phi} & (\ker \delta_i) \oplus \tilde{\mathbb{M}}_i \\
\delta_i & \downarrow & \delta_i \\
U_i & \xrightarrow{\phi_i} & \tilde{\mathbb{M}}_i
\end{array}
\]

where \( \delta_i : \mathbb{M} \to \mathbb{M}_i \) is the projective limit of the family the bounding maps \( \delta_i^j \) associated to the sequence \( (\mathbb{M}_i) \) of Banach spaces. Thus, according to this decomposition, the restriction of \( \delta_i \) to \( \tilde{\mathbb{M}}_i \) is an isomorphism.

Using the same arguments as in Corollary 3.19, applied to \( U_k \), we have a decomposition \( (TM_k)_U = H \oplus G \) with \( T\phi_k(H) = T\mathbb{H} (\mathbb{H} = \mathbb{R}^*) \) and \( T\phi_k(G) = TG \). Now, let \( H^0 \) and \( G^0 \) be the respective annihilator of \( H \) and \( G \) in \( (T^*M_k)_U \). As usually, we can identify \( G^0 \) with \( \mathbb{H}^* \) and, in particular, \( H^* \) is finite dimensional.

By the way, we have

\[ (T^*M_k)_U = H^0 \oplus G^0. \] (40)

Moreover, \( H \) and \( G \) are trivial bundles and \( G^0 \) is a (trivial) sub-bundle of \( (T^*M_k)_U \) compatible with the trivialization of \( (TM_k)_U \) respectively (cf. proof of Corollary 3.19).

Now since \( x_i \) is singular, we must have \( \delta_{k-1}^i(H_{x_i}) = 0 \). But, according to Diagram (39), we must have in one hand \( T\phi_k(H_{x_i}) \subset \ker \delta_{k-1}^i \) and from the triviality of \( H \), on the other hand \( T\phi_k(H) \subset \Phi_k(U_k) \times \ker \delta_{k-1}^i \). But we have seen that \( G^0 \) is contained in \( T^oM_k \) and so from (37), we must have \( \delta_{k-1}^i(H_z) = D_{\delta_{k-1}^i}^i(z) \) for all \( z \in U_{k-1} \). The argument above implies that \( \dim D_z = 0 \) for all \( z \in U_{k-1} \).

Finally, according to our assumption, since \( M_1 \) has a regular point \( y_1 \), this implies that all points in \( \delta_{i-1}^i(y_1) \) are also regular. In particular, \( M_{k-1} \) contains a regular point and then an open dense set of regular points; in particular, \( U_{k-1} \) contains a regular point, which gives a contradiction and so we must have \( k = 1 \).

Finally, the characterization of regular points in \( M \) implies that \( \Sigma = \delta_i^{-1}(\Sigma_i) \), which ends the proof of Point 1.

**Step 2.**

Fix a regular point \( x = \lim x_i \) in \( M \). From Diagram (39) applied for \( i = 1 \), we have a chart \( (U = \lim(U_i), \phi = \lim \phi_i) \) such that \( \phi(U) \) is an open set of \( \mathbb{M} = \ker \delta_1 \oplus \tilde{\mathbb{M}}_1 \). Without loss of generality, we can assume that \( \mathbb{M} = \ker \delta_1 \times \tilde{\mathbb{M}}_1 \) and that \( \phi(U) \) is of type \( U \times \tilde{U} \subset \ker \delta_1 \times \tilde{\mathbb{M}}_1 \). By the way, from this Diagram (39), the restriction of \( \delta_1 \) to \( \tilde{U} \) is diffeomorphism onto
Now, by application of Corollary 4.2, after restriction of $U_1$ if necessary, we have a diffeomorphism $\psi$ from $U_1$ onto some neighbourhood $V_1 \times V'_1$ in $\mathbb{R}^r \times \tilde{M}_1$ such that

$$(\psi \circ \phi_1)_*(\Lambda_1) = \frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_r}$$

for some coordinates system $(t_1, \ldots, t_r)$ on $\mathbb{R}^r$. But from the construction of $P$, we have $(\delta_1)_*(\Lambda) = \Lambda_1$.

Now we can write $\phi$ as a pair $(\bar{\phi}, \hat{\phi})$ where $\bar{\phi} : U \to \ker \delta_1$ and $\hat{\phi} : U \to \tilde{M}_1$. On the other hand, since the restriction of $\delta_1$ to $\hat{M}_1$ is an isomorphism onto $M_1$, we have a decomposition

$$M_1 = \ker \delta_1 \times \mathbb{R}^r \times \tilde{M}_1.$$ 

We set $\tilde{M} = \ker \delta_1 \times \mathbb{R}^r \times \tilde{M}_1$.

By construction, $\phi'$ is a diffeomorphism from $U$ onto an open set of $\mathbb{R}^r \times \tilde{M}$ and $(U, \phi')$ is a chart which satisfies the announced results in Point 2.

Step 3

From the construction of the chart $(U, \phi')$ in Step 2, we can write $\phi'(U) := \tilde{U}_1 \times \tilde{U}_2 \subset \mathbb{R}^r \times \tilde{M}$, and, for all $z \in U$, if $\phi'(z) = (\tau, \zeta) \in \tilde{U}_1 \times \tilde{U}_2$, then $(\phi')^{-1}(\tilde{U}_1 \times \{\zeta\})$ is an integral manifold of $D$ through $z$. Moreover, since $T\delta_1$ in restriction to $D_z$ is an isomorphism onto $(D_1)_{\delta_1(z)}$, it follows that the restriction of $\delta_1$ to a leaf $L$ through $z$ is a local diffeomorphism around $z$ onto the leaf $L_1$ through $\delta_1(z)$.

Now the restriction of $\delta_1$ so such a leaf $L$ is smooth and so continuous (because $L$ is an immersed closed submanifold of $M$, therefore $\delta_1(L)$ is a connected open submanifold of $L_1$).

Consider a point $y_1$ in the closure of $\delta_1(L)$ in $L_1$. There exists a smooth curve $\gamma^1 : [0, 1] \to L_1$ such that $\gamma^1(0) = z_1$ and $\gamma^1(1) = y_1$. We can cover the range of $\gamma_1$ with charts $U^1_1, \ldots, U^m_1$ such that $U^1_1 \cap L_1$ is connected. Since $\delta_1$ is a submersion, we have chart domains $U_1, \ldots, U_m$ in $M$, each one having the property of Diagram (39) for $i = 1$, and such that $U^1_1 \cap U^1_{j+1} \cap L_1 \neq \emptyset$. From the properties of such a diagram, the restriction of $\delta_1$ to $U_1 \cap L$ is an open set contained in $L_1 \cap U^1_1$ and so we have a lift $\bar{\gamma}^1$ from some interval $[0, t_1]$ to $U_1 \cap L$ such that $\bar{\gamma}^1(t_1)$ belongs to $U_2 \cap L$ and $\delta_1 \circ \bar{\gamma}_1 = \gamma^1|[0,t_1]$. By
induction, we can built a lift \( \bar{\gamma} : [0, 1] \to L \) such that \( \bar{\gamma}(0) = z \) and \( \delta \circ \bar{\gamma} = \gamma^1 \).

By the way, \( y = \bar{\gamma}(1) \) belongs to \( L \) and \( \delta_1(y) = y_1 \).

By connectedness argument, it follows that \( \delta_1(L) = L_1 \).

Finally using the same arguments, we can show that such properties are true for any \( i \in \mathbb{N} \).

5.2 Direct limits of partial Nambu-Poisson structures

We consider an ascending sequence \((M_i, \varepsilon_i^{i+1}), i \in \mathbb{N}\) of Banach manifolds (cf. Appendix B.4). We assume that we have a partial \( r \)-Nambu Poisson structure \((M_i, A_{M_i}, \{., ., .\}_i), i \in \mathbb{N}\) on each \( M_i \). We use the same notations as before Definition 5.1 in § 5.1 for each partial Nambu structure.

**Definition 5.5.** The sequence \((M_i, A_{M_i}, \{., ., .\}_i), i \in \mathbb{N}\) is called a direct sequence of partial Nambu-Poisson Banach manifolds if \((M_i), i \in \mathbb{N}\) is an ascending sequence of Banach \(C^\infty\)-manifolds, where \( M_i \) is modelled on the Banach space \( M_i \) such that \( M_i \) is a supplemented Banach subspace of \( M_{i+1} \); moreover, for all \( i \in \mathbb{N} \), we have the following properties:

\[(DSpNPBM 1) \quad T^*\varepsilon_i^{i+1}(T^9 M_{i+1}) \subset T^9 M_i^{13}\]

\[(DSpNPBM 2) \quad P_{i+1} = (T^*\varepsilon_i^{i+1}) \circ P_i \circ (T^*\varepsilon_i^{i+1})^{-1}\]

\[(DSpNPBM 3) \quad T^*_x M_j \subset T^*_x M_{i+1}\]

Let \((M_i, A_{M_i}, \{., ., .\}_i), i \in \mathbb{N}\) be a direct sequence of \( r \)-partial Nambu-Poisson Banach manifolds whose direct limit is the convenient manifold \( M \). Then \( TM \) is a convenient bundle over \( M \) (cf. [CaPe19]). We first recall some arguments and notations used in the proof of Theorem 3.2.3 of [PeCa19] in the context of Direct limits.

Fix \( x \in M \) and let \( h \) be the smallest integer such that \( x \) belongs to \( M_h \).

Since for \( h \leq j \), \( \varepsilon_h^j \) is an injective immersion, then \( T^*_x \varepsilon_h^j \) is a submersion and so we have a projective sequence \((T^*_x M_j, T^*_x \varepsilon_h^j)\) of Banach spaces and we set \( F^*_x M = \lim T^*_x M_j \) which is a Fréchet vector space. Also the sequence \((T^*_x M_j, T^*_x \varepsilon_h^j)\) projective sequence of Banach spaces and we set

\[\text{As in [PeCa19], this means that for any } y \in M_i \text{ we have } T^*_y \varepsilon_i^{i+1}(T^9 M_{i+1}) \subset T^*_y M_i.\]

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Figure 2: Direct sequence of partial Nambu-Poisson structures

\[ F^\flat M = \lim_{\leftarrow} T^\flat M_j \] which is a Fréchet vector sub-space of \( F^* M \). Then for any \( i \in \mathbb{N} \) we set:

\[
T^* M_i M = \bigcup_{x \in M_i} F^* M \quad \text{and} \quad T^\flat M_i M = \bigcup_{x \in M_i} F^\flat M
\]

which are total spaces of Fréchet bundles over \( M_i \) modelled on \( F^* = \lim_{\leftarrow} (M^*_i)_{i \geq h} \) and \( F^\flat = \lim_{\leftarrow} (M^\flat_i)_{i \geq h} \) where \( M^\flat_i \) is the typical fibre of \( T^\flat M_i \) respectively. Now, since \( M_i \) is a closed submanifold of \( M_j \) for all \( i \leq j \), it follows that \( T^* M_i M \) (resp. \( T^\flat M_i M \)) is a closed subbundle of \( T^* M_j M \) (resp. \( T^\flat M_j M \)). Finally we set \( T^* M = \lim_{\leftarrow} T^* M_i M \) (resp. \( T^\flat M = \lim_{\leftarrow} T^\flat M_i M \)) which is the total space of a convenient bundle over \( M \) with typical fibre \( M^* = \lim_{\leftarrow} F^* \) and \( M^\flat = \lim_{\leftarrow} F^\flat \) respectively. Now, from the assumption (DSpNPBM 2), we obtain

\[
P_l = T^{*}_{\varepsilon_{j}^{l}} \circ P_j \circ (T^{*}_{\varepsilon_{j}^{l}})^{-1} : (T^\flat M_j M_i)_{l \geq j} \rightarrow T^* M_i M_l
\]

for all \( l \geq j \). But since \( (T^\flat M_i M_l)^{-1} \subset (T^* \varepsilon^{l}_{j})^{-1} \) for \( i \leq j \), by restriction we
obtain the relation
\[ P_l = T^* \epsilon_l \circ P_j \circ (T^* \epsilon_j)^{r-1} : (T^0_{M_i} M_l)^{r-1} \to T_{M_i} M_l \] (42)

For \( i \leq j \leq l \) we set
\[ P_{ijl} = T^* \epsilon_l \circ P_j \mid M_i : (T^0_{M_i} M_j)^{r-1} \to T_{M_i} M_l \] (43)

In this way \( \{T_{M_i} M_l, T^* \epsilon_i\}_{i \geq l} \) is a direct sequence of Banach bundles and as \( P_{ijl} = T^* \epsilon_l \circ P_j \mid M_i \) we obtain a morphism
\[ \overline{P}_{ij} = \lim_{\to} (P_{ijl})_{l \geq j} : (T^0_{M_i} M_j)^{r-1} \to T_{M_i} M \] (44)

which is \( r \)-skew-symmetric morphism.

As \( \overline{P}_{ij} = \overline{P}_{ii} \circ (T^* \epsilon_j)^{r-1} \) and we have \( \lim_{\to} (T^0_{M_i} M_j)_{j \geq i} = T^0_{M_i} M \) then we can consider the morphism
\[ \overline{P}_i = \lim_{\to} (\overline{P}_{ij})_{j \geq i} : (T^0_{M_i} M)^{r-1} \to T_{M_i} M \] (45)

which is again a \( r \)-skew-symmetric morphism.

Using the fact that \( T^0 M = \bigcup_{j \geq i} (T^0_{M_j} M) \) and \( TM = \bigcup_{j \geq i} (T_{M_j} M) \) and \( \overline{P}_{j \mid M_i} = \overline{P}_i \), finally, we obtain a \( r \)-skew-symmetric morphism
\[ P : (T^0 M)^{r-1} \to TM, \] (46)

given by \( P_x = (\overline{P}_i)_x \) if \( x \) belongs to \( M_i \).

Thus we have the following result:

**Proposition 5.6.** Let \( \left(M_i, A_{M_i}, \{\ldots,\} \right)_{i \in \mathbb{N}} \) be a direct sequence of \( r \)-partial Nambu-Poisson Banach manifolds whose direct limit is the convenient manifold \( M \).

There exists a convenient weak subbundle \( p^b : T^0 M \to M \) of the convenient bundle \( p^*_M : T^* M \to M \) and a skew-symmetric morphism \( P : (T^0 M)^{r-1} \to TM \) such that \( (M, A_M, \{\ldots,\})_P \) is a \( r \)-partial Nambu-Poisson structure on \( M \).

**Remark 5.7.** From (43), we have
\[ P_{iii} = P_i : (T^0_{M_i} M_i)^{r-1} = (T^0 M_i)^{r-1} \to T_{M_i} M_i = TM_i \] (47)
On the other hand, from the relations (43), (44) and (45), we obtain the following commutative diagrams

\[
\begin{array}{ccc}
(T^{\flat}_{M_i}M_i)^{r-1} & \xrightarrow{P_i} & T_{M_i}M_i \\
\downarrow P_i & & \downarrow T_{\varepsilon_i} \\
T_{M_i}M_i & \xrightarrow{(T^{\flat}_{M_i}M_i)^{r-1}} & (T_{M_i}M_i)^{r-1}
\end{array}
\]

According to (47), this implies the following relation:

\[\bar{P}_i = T_{\varepsilon_i} \circ P_i \circ (T^*\varepsilon_i)^{r-1}.\] (49)

As in the previous section, we will now give a version of Theorem 3.28 in the context of direct limit. For each \(i \in \mathbb{N}\), we denote by \(\Lambda_i\) the \(r\)-Nambu tensor associated to \(P_i\) and \(\Lambda^\sharp_i : \bigwedge^{r-1} T^\flat M_i \to TM_i\) the associated morphism. By analogue arguments as the ones used for the definition of \(P\) from \(P_i\), we can build a morphism \(\Lambda^\flat : \bigwedge^{r-1} T^\flat M \to TM\) from \(\Lambda_i\). Clearly we have the same type of properties of \(\Lambda^\flat\) relatively to \(P\) as for \(\Lambda_i\) relatively to \(P_i\). Moreover, if \(\Lambda\) is the \(r\)-skew-symmetric tensor associated to \(P\), then the corresponding morphism from \(\bigwedge^{r-1} T^\flat M\) to \(TM\) is precisely \(\Lambda^\flat\). Thus we have:

**Theorem 5.8.** Let \((M_i, A_{M_i}, \{\ldots, \})_{i \in \mathbb{N}}\) be an ascending sequence of \(r\)-partial Nambu-Poisson Banach manifolds whose direct limit is the convenient manifold \(M\). Assume that \(M_1\) contains a regular point. We have the following properties

1. Let \(x = \lim x_i \in M\) and \(h\) the smallest integer such that \(x_h\) belongs to \(M_h\). Then \(x\) is singular if and only if all \(x_i\) are singular for all \(h \leq i\). The point \(x\) is regular if and only if there exists an integer \(k \geq h\) such that \(x_i\) is regular for all \(i \geq k\) and \(x_i\) is singular for \(h \leq i < k\).

2. For all integer \(k \in \mathbb{N}\), we denote by \(\mathcal{S}_k\) the set

\[\{x = \lim x_i \in M : x_i \text{ regular for all } i \geq k\}\]

\(\mathcal{S}_0\) denotes the set of singular points of \(M\). Then \(\Sigma \neq \emptyset\), for each \(k \in \mathbb{N}\), \(\mathcal{S}_k\) is a convenient non empty open submanifold of \(M\) and \(\Sigma = \bigcup_{k \in \mathbb{N}} \mathcal{S}_k\).
3. Fix some \( x \in \mathcal{S}_k \). There exists a chart \( U = \lim(U_i), \phi = \lim\phi_i \) around \( x \) in \( M \), where \( (U_i, \phi) \) is a chart in \( M_i \) around \( x_i \), which is contained in \( \Sigma_i \) for all \( i \) with the following properties:

(i) \( \phi(U) \) is a product \( \mathbb{U} \times \mathbb{U} \subset \mathbb{R}^r \times \mathbb{M} \) where \( \mathbb{U} \) is a Fréchet subspace of \( \mathbb{M} \).

(ii) In this chart, the Nambu-Poisson tensor \( \Lambda \) can be written

\[
\phi_*(\Lambda) = \frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_r}
\]

for some coordinate system \( (t_1, \ldots, t_r) \) on \( \mathbb{R}^r \).

4. On \( \Sigma \), the characteristic distribution \( \mathcal{D} \) generated by the range of \( P \) is integrable and each leaf \( L \) is contained in one and only one \( \mathcal{S}_k \) characterized by \( L \) contained in \( M_k \) does not intersect \( M_i \) for \( i < k \) if \( k > 1 \).

If \( L \) is such a leaf through \( x = \lim x_i \in \mathcal{S}_k \), then \( L \) is also the leaf through \( \varepsilon_i(x) \) of \( \mathcal{D}_i \) for \( i \geq k \).

In fact, the characteristic foliation of \( P \) on \( M \) is the union of the characteristic foliations of \( P_k \) in \( M_k \) for all \( k \in \mathbb{N} \).

This Theorem is a version of Theorem 5.4 adapted to the context of ascending sequences of \( r \)-Nambu-Poisson Banach manifolds. Therefore its proof is an adaptation to this context of the main arguments used in the proof of Theorem 5.4. For the sake of completeness, we will detail specific arguments for this context; for the others, we refer to the proof of Theorem 5.4.

Proof.

Step 1

Again for all \( i \in \mathbb{N} \), we denote by \( (\mathcal{D}_i)_{x_i} \) the range of \( P_i \) at \( x_i \in M_i \) and by \( \mathcal{D}_x \) the range of \( P \) at \( x \in M \). Fix some point \( x = \lim(x_i) \) in \( M \) and again \( \dim(\mathcal{D}_x) = r \) or 0 and the same is true for \( \dim((\mathcal{D}_i)_{x_i}) \).

Now each \( T\varepsilon_i^l \) is injective, from relation (41), (42) and (43), it follows that

\[
\dim((\mathcal{D}_j)_{x_i}) \leq \dim((\mathcal{D}_l)_{x_i}) \leq r
\]

for all \( j \leq l \) and all \( x_i \in M_i \) with \( i \leq j \). Thus, according to relation (44) and the argument at the beginning, we obtain

\[
\dim((\mathcal{D}_j)_{x_i}) \leq \dim(\mathcal{D}_{x_i}) \leq r
\]
for all \( x_i \in M_i \) and \( i \leq j \). According to relation (46), we must have:

\[
\dim((D_j)_x) \leq \dim(D_x) \leq r \text{ if } x \in M_i \tag{52}
\]

In particular, if \( \dim((D_j)_x) = r \) since \( T\varepsilon_j = \lim T\varepsilon_l \) is injective, its restriction to \( (D_j)_x \) is an isomorphism onto \( (D_l)_x \), for all \( j \leq l \) and so the restriction of \( T\varepsilon_j \) to \( (D_j)_x \) is also an isomorphism onto \( (D)_{x_i} \). Thus, if \( \dim(D_x) = r \), there exists \( i \in \mathbb{N} \) such that \( x \in M_i \) and \( \dim((D_j)_x) = r \) for some \( j \geq i \) and so \( \dim((D_l)_x) = r \) for all \( i \leq j \leq l \) (cf. (50)). Conversely, from (52), if \( \dim((D_j)_x) = r \) for some \( x \in M_i \), and, in this case, the restriction of \( T\varepsilon_j \) to \( (D_j)_x \) is an isomorphism; it follows that \( \dim((D_j)_x) = r \) and, from (51), \( \dim((D_l)_x) = r \), for all \( j \leq l \). Thus \( x = \lim x_i \) is regular if and only if there exists a smallest integer \( k \) such that \( x_k \) belongs to \( M_k \) and \( x_i \) is regular for \( k \leq i \). If \( h \) is the smallest integer of the set \( \{ i \in \mathbb{N}, x \in M_i \} \), it remains to show that \( h = k \).

Assume that \( k > h \).

On the one hand, using the previous characterization of a regular point, if \( M_1 \) has a regular point, it follows that the set of regular points of \( M_i \) for \( i \geq 1 \) is not empty and is an open dense set \( \Sigma_i \) in \( M_i \), for all \( i \in \mathbb{N} \). Thus, in particular, the set \( \Sigma_{k-1} \) is dense in \( M_{k-1} \).

On the other hand, from the properties of an ascending sequence of manifolds, for each \( i \geq h \), we have (cf. [CaPe19]):

- the map \( \varepsilon_i = \lim_{j \to i} \varepsilon_{i}^{j} : M_i \to M \) is an immersion;

- a decomposition \( M = \bigcup_{j \geq i} M_j = M_i \bigoplus \bigoplus_{i \leq j \leq l} \hat{M}_j^i \) such that \( M_l = M_i \bigoplus \bigoplus_{i \leq j \leq l} \hat{M}_j^i \) for all \( l > i \);

- a chart \( (U = \lim U_i, \phi = \lim \phi_i) \) around \( x \) such that \( (U_i, \phi_i) \) is a chart in \( M_i \) around \( x_i \) such that \( \phi(x) = 0 \) and the following diagrams are
Remark 5.9. Note that if $U_k$ is contained in $\Sigma_k$, necessarily each $U_l \subset \Sigma_l$ for all $l \geq k$ and so from Diagrams (53) (taking $i = k$), and previous arguments, the restriction of $T\varepsilon_k^l$ (resp. $T\varepsilon_l^k$) is a bundle isomorphism form $D_k$ onto $D_l$ (resp. onto $D$). In fact, as sets, we have $U_k \subset U_l \subset U$ and so the restriction of $D$ to each $U_l$ for $l \geq k$ is precisely $D_l$. In particular, $T\phi_l(D_l) \subset \phi(U_l) \times \bar{M}_k$ and $T\phi(D) \subset \phi(U) \times M$.

After restriction of $U$ if necessary, we can apply to $\phi_k(U_k)$, the same argument as in the proof of Corollary 3.19. Thus, there exists a diffeomorphism $\phi' : \phi_k(U_k) \to \bar{M}_k$, with $\phi'(0) = 0$ such that we have a decomposition $T(\phi' \circ \phi_k)(TM_k)_{U_k} = H \oplus G$ with $T(\phi' \circ \phi_k)(H) = T\mathbb{R}$ ($\mathbb{H} = \mathbb{R}^r$) and $T(\phi' \circ \phi_k)(G) = TG$. By the way, $(U_k, \phi'_k = \phi' \circ \phi_k)$ is a chart around $x_k$ such that $T\phi'_k(TM_k)_{U_k} = H \oplus G$.

Now, let $H^0$ and $G^0$ be the respective annihilators of $H$ and $G$ in $(T^*M_k)_{U_k}$. As usually, we can identify $G^0$ with $H^*$ and so, in particular, $H^*$ is finite dimensional. It follows that we have

\[(T^*M_k)_{U_k} = H^0 \oplus G^0.\] (54)

Moreover, $H$ and $G$ are trivial bundles and $G^0$ is a (trivial) sub-bundle of $(T^*M_k)_{U_k}$ compatible with the trivialization of $(TM_k)_{U_k}$ and $(T^*M_k)_{U_k}$ respectively (cf. proof of Corollary 3.19).

According to (41) for $j = k - 1$ and $l = k$, applied at point $x_{k-1}$ and in restriction to $G$, since $x_{k-1}$ is singular, we must have $T_{x_{k-1}}^* \epsilon_{k-1}^k(G^0) = \{0\}$. Via the diffeomorphism $\phi'_k$, this property is equivalent to $G^0$ is contained in the annihilator of $M_{k-1}$ considered as a subset of $M_k = M_{k-1} \oplus \bar{M}_{k-1}^k$. Therefore, $\phi'_k(U) \times \mathbb{G}^0 = \phi'_k(U) \times \mathbb{H}^*$ is contained in the annihilator of
Finally, via the diffeomorphism $\phi'_k$, this means that $G^0$ is contained in the annihilator of $T^k_{U_{k-1}}(TU_{k-1})$ on $U_{k-1}$, and finally this implies that all points of $U_{k-1}$ are singular. But this is a contradiction with the density of $\Sigma_{k-1}$.

**Step 2**

From Point 1 and our assumption on the existence of regular points in $M_1$, it follows that $\Sigma \neq \emptyset$ and $\Sigma = \bigcup k \in \mathbb{N} S_k$.

Fix a regular point $x = \varprojlim_i x_i$ in $S_k$. From Diagram (53) applied for $i = k$ and Remark 5.9, we have a chart $\left( U = \varprojlim(U_i), \phi = \varprojlim\phi_i \right)$ such $U \subset S_k$ and that $\phi(U)$ is an open set of $\mathbb{M} = \bigcup_{l \geq k} M_l = M_k \bigoplus_{l \geq k} \check{M}_l$, which ends the proof of Point 2.

If we set $\check{M}_k = \bigoplus_{l \geq k} \check{M}_l$. Thus, without loss of generality, we can assume that $\mathbb{M} = M_k \times \check{M}_k$ and that $\phi(U)$ is of type $U \times \check{U} \subset M_k \times \check{M}_k$. By the way, from this Diagram (53) and Remark 5.9, the open set $U$ is diffeomorphic to $\phi_k(U_k)$.

To end the proof of Point 3, we apply Corollary 4.2 to $\check{U}$ and use the same arguments as in the same stage in Step 2 of the proof of Theorem 5.4.

**Step 3**

From Point 3 around any $x \in S_k$, we have a chart $\left( U = \varprojlim(U_i), \phi = \varprojlim\phi_i \right)$ such that $U \subset S_k$ and $\phi(U) := U \times \check{U} \subset \mathbb{R}^r \times \check{M} = \check{M}$. Thus, for all $z \in U$, if $\phi(z) = (\tau, \zeta) \in U \times \check{U}$, then $(\phi)^{-1}(U \times \{\zeta\})$ is an integral manifold of $D$ through $z$. Moreover, on $U_l$, for $l \geq k$, according to Remark 5.9, $(D_k)_z = (D_k)_z = D_z$. This implies the result of Point 4.

\[ \square \]

### A Projective Limits

#### A.1 Projective limits in categories

The main reference for this appendix is the book [CaPe].

**Definition A.1.** Let $(I, \succ)$ be a directed set and $\mathcal{C}$ a category. A projective system (or inverse system) is a family $\mathcal{S} = \left\{ (X_i, \delta_i) \right\}_{(i, j) \in I^2, \, j \succ i}$ where $X_i$
is an object of the category $C$ and $\delta^j_i : X_j \to X_i$ is a morphism (bonding map) where:

(PS 1) \[ \forall i \in I, \delta^i_i = \text{Id}_{X_i}; \]

(PS 2) \[ \forall (i, j, k) \in I^3 : k \succeq j \succeq i, \delta^j_i \circ \delta^k_j = \delta^k_i. \]

When $I = \mathbb{N}$ with the usual order relation, countable projective systems are called projective sequences. The notion of projective limit is defined via a universal property.

**Definition A.2.** A cone over $\mathcal{S}$ is a pair $(X, \{\delta_i\}_{i \in I})$ where $X \in \text{ob}(C)$ and $\delta_i : X \to X_i$ is such that $\delta_i = \delta^j_i \circ \delta_j$ whenever $j \succeq i$.

**Definition A.3.** A cone $(X, \{\delta_i\}_{i \in I})$ is a projective limit of $\mathcal{S}$ if, for every cone $(Y, \{\psi_i\}_{i \in I})$ over $\mathcal{S}$, there exists a unique morphism $\vartheta : Y \to X$ such that $\psi_i = \delta_i \circ \vartheta$.

We then write $X = \varprojlim \mathcal{S}$ or $X = \varprojlim X_i$.

Projective limits in the categories of sets, groups and rings admit the following description:

\[ \varprojlim X_i = \left\{ (x_i) \in \prod_{i \in I} E_i : \forall (i, j) \in I^2 : j \succeq i, \ x_i = \delta^j_i(x_j) \right\} \]

is called the projective limit of the system $\{ (X_i, \delta^j_i) \}_{(i, j) \in I^2, j \succeq i}$. In the set category $\text{SET}$, projective limits always exist\(^{14}\) that is $\text{SET}$ is complete.

\(^{14}\)The projective limit of sets can be empty.
Definition A.4. Let \( \{(X_i, \delta^j_i)\}_{(i,j) \in I^2, j \geq i} \) and \( \{(Y_i, \eta^j_i)\}_{(i,j) \in I^2, j \geq i} \) be two projective families with limit \( X \) and \( Y \) respectively. A family \( \{f_i\}_{i \in I} \) of mappings \( f_i : X_i \to Y_i \) satisfying the coherence condition:

\[
\forall (i,j) \in I^2 : j \geq i, f_i \circ \delta^j_i = \eta^j_i \circ f_j.
\]

is called a projective family of mappings.

The limit of the projective family of mappings \( \{f_i\}_{i \in I} \) is the map

\[
f : X = \lim_{\leftarrow} X_i \to Y = \lim_{\leftarrow} Y_i \quad \{x_i\}_{i \in I} \mapsto \{f_i(x_i)\}_{i \in I}
\]

A.2 Projective limits of topological spaces

Definition A.5. A projective sequence of topological spaces is a sequence \( \left( (X_i, \delta^j_i) \right)_{(i,j) \in \mathbb{N}^2, j \geq i} \) where

(PSTS 1) For all \( i \in \mathbb{N} \), \( X_i \) is a topological space;

(PSTS 2) For all \( (i,j) \in \mathbb{N}^2 \) such that \( j \geq i \), \( \delta^j_i : X_j \to X_i \) is a continuous map;

(PSTS 3) For all \( i \in \mathbb{N} \), \( \delta^i_i = Id_{X_i} \);

(PSTS 4) For all \( (i,j,k) \in \mathbb{N}^3 \) such that \( k \geq j \geq i \), \( \delta^j_i \circ \delta^k_j = \delta^k_i \).

Notation A.6. For the sake of simplicity, the projective sequence \( \left( (X_i, \delta^j_i) \right)_{(i,j) \in \mathbb{N}^2, j \geq i} \) will be denoted \( \left( X_i, \delta^j_i \right)_{j \geq i} \).

An element \( (x_i)_{i \in \mathbb{N}} \) of the product \( \prod_{i \in \mathbb{N}} X_i \) is called a thread if, for all \( j \geq i \), \( \delta^j_i (x_j) = x_i \).

Definition A.7. The set \( X = \lim_{\leftarrow} X_i \) of all threads, endowed with the finest topology for which all the projections \( \delta_i : X \to X_i \) are continuous, is called the projective limit of the sequence \( \left( X_i, \delta^j_i \right)_{j \geq i} \).

A basis of the topology of \( X \) is constituted by the subsets \( (\delta_i)^{-1}(U_i) \) where \( U_i \) is an open subset of \( X_i \) (and so \( \delta_i \) is open whenever \( \delta_j \) is surjective).
Let \( \{(X_i, \delta_{i,j})\}_{(i,j)\in \mathbb{N}^2, j\geq i} \) and \( \{(Y_i, \eta_{i,j})\}_{(i,j)\in \mathbb{N}^2, j\geq i} \) be two projective sequence of topological spaces with limit \( X \) and \( Y \) respectively. Consider a family \( \{f_i\}_{i \in I} \) of mappings \( f_i : X_i \to Y_i \) satisfying the coherence condition (CCPM).

The mapping \( f = \lim_{\leftarrow} f_i \) is continuous if all the \( f_i \) are continuous (cf. [?]).

A.3 Projective limits of Banach spaces

Consider a projective sequence \( (E_i, \delta_{i,j})_{j \geq i} \) of Banach spaces.

**Remark A.8.** Since we have a countable sequence of Banach spaces, according to the properties of bonding maps, the sequence \( (\delta_{i,j})_{(i,j)\in \mathbb{N}^2, j\geq i} \) is well defined by the sequence of bonding maps \( (\delta_{i+1})_{i \in \mathbb{N}} \).

Fix some norm \( \| \cdot \|_i \) on the Banach space \( E_i \), for all \( i \in \mathbb{N} \). If \( x = \lim_{\leftarrow} x_i \), then \( p_n(x) = \max_{0 \leq i \leq n} \|x_i\|_i \) is a semi-norm on the projective limit \( E = \lim_{\leftarrow} E_n \) which provides a structure of Fréchet space structure (see [?]).

Following [Dub72] we introduce:

**Definition A.9.** A projective sequence \( (E_i, \delta_{i,j})_{j \geq i} \) of Banach spaces is called reduced if the range of \( \delta_{i+1} \) is dense for all \( i \in \mathbb{N} \).

A.4 Projective limits of Banach manifolds

**Definition A.10.** The projective sequence \( (M_i, \delta_{i,j})_{j \geq i} \) is called projective sequence of Banach manifolds if

1. (PSBM 1) for all \( i \in \mathbb{N} \), \( M_i \) is a manifold modelled on the Banach space \( M_i \);
2. (PSBM 2) \( (M_i, \delta_{i,j})_{j \geq i} \) is a projective sequence of Banach spaces;
3. (PSBM 3) for all \( x = (x_i) \in M = \lim_{\leftarrow} M_i \), there exists a projective sequence of local charts \( (U_i, \varphi_i)_{i \in \mathbb{N}} \) such that \( x_i \in U_i \) where one has the relation
   \[ \varphi_i \circ \delta_{i,j}^i = \delta_{j}^i \circ \varphi_j; \]
4. (PSBM 4) Under the previous assumptions, if \( U = \lim_{\leftarrow} U_i \) and \( \phi = \lim_{\leftarrow} \phi_i \), then \( \phi(U) \) is an open set of \( M = \lim_{\leftarrow} M_i \).
Under the assumptions \((PSBM 1)\) and \((PSBM 2)\) in Definition \(A.10\),
the pair of assumptions \((PSBM 3)\) and \((PSBM 4)\) around \(x \in M\) is called
the \textit{projective limit chart property} around \(x \in M\) and \((U = \lim U_i, \phi = \lim \phi_i)\)
is called a \textit{projective limit chart}.

The projective limit \(M = \lim M_i\) has a structure of Fréchet manifold
modelled on the Fréchet space \(\overrightarrow{M} = \lim M_i\) and is called a \textit{PLB-manifold}.

A.5 Projective limits of Banach vector bundles

Let \(\left(M_i, \delta^j_i\right)_{j \geq i}\) be a projective sequence of Banach manifolds where each
manifold \(M_i\) is modelled on the Banach space \(M_i\).
For any integer \(i\), let \((E_i, \pi_i, M_i)\) be the Banach vector bundle whose type
fibre is the Banach (resp. normed) vector space \(E_i\) where \(\left(E_i, \lambda^j_i\right)_{j \geq i}\) is a
projective sequence of Banach spaces.

\begin{definition}
\(\left((E_i, \pi_i, M_i), \left(f^j_i, \delta^j_i\right)\right)_{j \geq i}\), where \(f^j_i : E_j \rightarrow E_i\) is a mor-
phism of vector bundles, is called a projective sequence of Banach vector
bundles on the projective sequence of manifolds \(\left(M_i, \delta^j_i\right)_{j \geq i}\) if for all \((x_i)\)
there exists a projective sequence of trivializations \((U_i, \tau_i)\) of \((E_i, \pi_i, M_i)\),
where \(\tau_i : (\pi_i)^{-1}(U_i) \rightarrow U_i \times E_i\) are local diffeomorphisms, such that \(x_i \in U_i\)
(open in \(M_i\)) and where \(U = \lim U_i\) is a non empty open set in \(M\) where, for
all \((i, j) \in \mathbb{N}^2\) such that \(j \geq i\), we have the compatibility condition

\[\left(PLBVB\right) \quad \left(\delta^j_i \times \lambda^j_i\right) \circ \tau_j = \tau_i \circ f^j_i.\]

With the previous notations, \((U = \lim U_i, \tau = \lim \tau_i)\) is called a \textit{pro-
jective bundle chart limit}. The triple of projective limit \((E = \lim E_i, \pi = \lim \pi_i, M = \lim M_i)\) is called a \textit{projective limit of Banach bundles} or \textit{PLB-
bundle} for short.

The following proposition generalizes the result of [Gal04] about the
projective limit of tangent bundles to Banach manifolds. Its proof can be
found in [AgSu10].

\begin{proposition}
Let \(\left((E_i, \pi_i, M_i), \left(f^j_i, \delta^j_i\right)\right)_{j \geq i}\) be a projective sequence
of Banach vector bundles.

Then \(\left(\lim E_i, \lim \pi_i, \lim M_i\right)\) is a Fréchet vector bundle (resp. locally convex
vector bundle).
\end{proposition}
B Direct Limits

The main references for this appendix are the paper [CaPe19] and the book [CaPe].

B.1 Direct limits of categories

Definition B.1. Let \((I, \preceq)\) be a directed set and \(C\) a category. A directed system (or inductive system) is a family \(S = \{(Y_i, \varepsilon^j_i)\}_{(i,j) \in I^2, i \preceq j}\) where \(Y_i\) is an object of the category \(C\) and \(\varepsilon^j_i : Y_i \to Y_j\) is a morphism (bonding map) where:

\[
\begin{align*}
&\text{(DS 1)} \quad \forall i \in I, \varepsilon^i_i = \text{Id}_{Y_i}; \\
&\text{(DS 2)} \quad \forall (i, j, k) \in I^3 : i \preceq j \preceq k, \varepsilon^k_j \circ \varepsilon^j_i = \varepsilon^k_i.
\end{align*}
\]

\[
Y_i \xrightarrow{\varepsilon^j_i} Y_j \xrightarrow{\varepsilon^k_j} Y_k
\]

The directed system \(\{(Y_i, \varepsilon^j_i)\}_{(i,j) \in I^2, i \preceq j}\) is a (covariant) functor \(I \to C\) (cf. Appendix B).

When \(I = \mathbb{N}\) with the usual order relation, countable direct systems are called direct sequences.

Definition B.2. An inductive cone (or co-cone) over \(S\) is a pair \((Y, \{\varepsilon_i\}_{i \in I})\) where \(Y \in \text{ob}(C)\) and \(\varepsilon_i : Y_i \to Y\) is such that \(\varepsilon_j \circ \varepsilon^j_i = \varepsilon_i\) whenever \(i \preceq j\).

Definition B.3. An inductive cone \((Y, \{\varepsilon_i\}_{i \in I})\) is a direct limit (or a colimit) of \(S\) if for every inductive cone \((Z, \{\varphi_i\}_{i \in I})\) over \(S\) there exists a unique morphism \(\eta : Y \to Z\) such that \(\eta \circ \varepsilon_i = \varphi_i\). We then write \(Y = \varinjlim S\) or \(Y = \varprojlim Y_i\).
Direct limits in the categories of sets, groups and rings always exist and can be described as follows. Let \( \{ Y_i \}_{i \in I} \) be a family of sets. The disjoint union of this family is the set
\[
\bigsqcup_{i \in I} Y_i = \bigcup_{i \in I} \{ (x, i) : x \in Y_i \}.
\]
Consider the following equivalence relation \( \sim \): for \( x_i \in Y_i \) and \( x_j \in Y_j \),
\[
x_i \sim x_j \iff \exists k \in I : i \precsim k, j \precsim k, \varepsilon_i^k (x_i) = \varepsilon_j^k (x_j).
\]
Then the direct limit is \( \lim_{\rightarrow} Y_i = \bigsqcup_{i \in I} Y_i / \sim \).

Projective and direct limits can be linked via homomorphisms:
\[
\text{Hom} \left( \lim_{\rightarrow} X_i, Y \right) = \lim_{\leftarrow} \text{Hom} (X_i, Y).
\]

Definition B.4. Let \( \left\{ (X_i, \varepsilon_i^j) \right\}_{(i,j) \in I^2, i \precsim j} \) and \( \left\{ (Y_i, \zeta_i^j) \right\}_{(i,j) \in I^2, i \precsim j} \) two directed systems with respective limits \( X \) and \( Y \). A family \( \{ f_i \}_{i \in I} \) of mappings \( f_i : X_i \to Y_i \) satisfying the coherence condition:
\[
\forall (i,j) \in I^2 : i \precsim j, f_j \circ \varepsilon_i^j = \zeta_i^j \circ f_i
\]
is called a direct family of mappings (for \( I = \mathbb{N} \), we have the notion of direct sequence of mappings).

The limit of the direct family of mappings \( \{ f_i \}_{i \in I} \) is the map
\[
f : X = \lim_{\rightarrow} X_i \to Y = \lim_{\rightarrow} Y_i \quad \{ x_i \}_{i \in I} \mapsto \{ f_i (x_i) \}_{i \in I}
\]

B.2 Direct limits of topological spaces

Let \( \left\{ (X_i, \varepsilon_i^j) \right\}_{(i,j) \in I^2, i \precsim j} \) be a direct system of topological spaces and continuous maps. The direct limit \( \{ (X_i, \varepsilon_i) \}_{i \in I} \) of the sets becomes the direct limit in the category \( \text{TOP} \) of topological spaces if \( X \) is endowed with the direct limit topology (DL-topology for short), i.e. the final topology with respect to the inclusion maps \( \varepsilon_i : X_i \to X \) which corresponds to the finest topology which makes the maps \( \varepsilon_i \) continuous. So \( O \subset X \) is open if and only if \( \varepsilon_i^{-1} (O) \) is open in \( X_i \) for each \( i \in I \).
Definition B.5. Let $T = ((X_n, \varepsilon_n^m)_{(m,n)\in\mathbb{N}^2}, n \leq m)$ be a direct sequence of topological spaces such that each $\varepsilon_n^m$ is injective. Without loss of generality, we may assume that we have

$$X_0 \subset X_1 \subset \cdots \subset X_n \subset X_{n+1} \subset \cdots$$

and $\varepsilon_n^{n+1}$ becomes the natural inclusion.

(ASTS) $T$ will be called an ascending sequence of topological spaces.

(SASTS) Moreover, if each $\varepsilon_n^m$ is a topological embedding, then we will say that $T$ is a strict ascending sequence of topological spaces (expanding sequence in the terminology of [Han71]).

B.3 Ascending sequences of Banach spaces

Definition B.6. An ascending sequence $(E_n, i_n^m)_{n \leq m}$ of Banach spaces is a strict ascending sequence of Banach spaces if $E_n$ is a Banach subspace of $E_{n+1}$ for each $n \in \mathbb{N}$.

Definition B.7. A locally convex limit of ascending sequence of Banach spaces is called an (LB)-space.

If the sequence is strict we speak of LB-space or strict (LB)-space.

The following result corresponds to [CaPe19], Proposition 3.7.

Proposition B.8. On an LB-space $E = \lim_{\rightarrow} E_n$, the DL-topology coincides with the $c^\infty$-topology.

B.4 Direct limits of ascending sequences of Banach manifolds

Definition B.9. $\mathcal{M} = (M_n, \varepsilon_n^{n+1})_{n \in \mathbb{N}}$ is called an ascending sequence of Banach manifolds if, for any $n \in \mathbb{N}$, $(M_n, \varepsilon_n^{n+1})$ is a submanifold of $M_{n+1}$.

Proposition B.10. Let $\mathcal{M} = (M_n, \varepsilon_n^{n+1})_{n \in \mathbb{N}}$ be an ascending sequence of Banach manifolds.

Assume that for $x \in M = \lim_{\rightarrow} M_n$, there exists a sequence of charts $((U_n, \phi_n))_{n \in \mathbb{N}}$ of $(M_n)_{n \in \mathbb{N}}$, such that:

(ASC 1) $(U_n)_{n \in \mathbb{N}}$ is an ascending sequence of chart domains;

(ASC 2) $\forall n \in \mathbb{N}, \phi_{n+1} \circ \varepsilon_n^{n+1} = i_{n+1}^n \circ \phi_n$. 

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Then $U = \lim_{n \to \infty} U_n$ is an open set of $M$ endowed with the DL-topology and $\phi = \lim_{n \to \infty} \phi_n$ is a well defined map from $U$ to $\mathbb{M} = \lim_{n \to \infty} M_n$. Moreover, $\phi$ is a continuous homeomorphism from $U$ onto the open set $\phi(U)$ of $\mathbb{M}$.

**Definition B.11.** We say that an ascending sequence $\mathcal{M} = (M_n, \varepsilon_n^{n+1})_{n \in \mathbb{N}}$ of Banach manifolds has the direct limit chart property (DLCP) at $x$ if it satisfies both (ASC 1) and (ASC 2).

We then have the fundamental result (cf. [CaPe19]).

**Theorem B.12.** Let $\mathcal{M} = (M_n)_{n \in \mathbb{N}}$ be an ascending sequence of Banach $C^\infty$-manifolds, modelled on the Banach spaces $\mathbb{M}_n$. Assume that

(ASBM 1) $(M_n)_{n \in \mathbb{N}}$ has the direct limit chart property (DLCP) at each point $x \in M = \lim_{n \to \infty} M_n$;

(ASBM 2) $\mathbb{M} = \lim_{n \to \infty} \mathbb{M}_n$ is an LB-space.

Then there is a unique $n.n.H.$ convenient manifold structure on $M = \lim_{n \to \infty} M_n$ modelled on the convenient space $\mathbb{M}$ such that the topology associated to this structure is the DL-topology on $M$.

In particular, for each $n \in \mathbb{N}$, the canonical injection $\varepsilon_n : M_n \to M$ is an injective conveniently smooth map and $(M_n, \varepsilon_n)$ is a closed submanifold of $M$.

Moreover, if each $M_n$ is locally compact or is open in $M_{n+1}$ or is a paracom pact Banach manifold closed in $M_{n+1}$, then $M = \lim_{n \to \infty} M_n$ is provided with a Hausdorff convenient manifold structure.

### B.5 Direct limits of Banach vector bundles

**Definition B.13.** $((E_n, \pi_n, M_n), (\lambda_n^{n+1}, \varepsilon_n^{n+1}))_{n \in \mathbb{N}}$ is called an ascending sequence of Banach vector bundles if the following assumptions are satisfied:

(ASBVB 1) $\mathcal{M} = (M_n)_{n \in \mathbb{N}}$ is an ascending sequence of Banach $C^\infty$-manifolds, where $M_n$ is modelled on the Banach space $\mathbb{M}_n$ such that $\mathbb{M}_n$ is a supplemented Banach subspace of $\mathbb{M}_{n+1}$ and the inclusion $\varepsilon_n^{n+1} : M_n \to M_{n+1}$ is a $C^\infty$ injection such that $(M_n, \varepsilon_n^{n+1})$ is a closed submanifold of $M_{n+1}$;

(ASBVB 2) The sequence $(E_n)_{n \in \mathbb{N}}$ is an ascending sequence such that the sequence of typical fibre $(\mathcal{E}_n)_{n \in \mathbb{N}}$ of $(E_n)_{n \in \mathbb{N}}$ is an ascending sequence of Banach spaces.
(ASBVB 3) For each \( n \in \mathbb{N} \), \( \pi_{n+1} \circ \lambda_{n+1} = \varepsilon_{n+1} \circ \pi_n \) where \( \lambda_{n+1} : E_n \to E_{n+1} \) is the natural inclusion;

(ASBVB 4) Any \( x \in M = \lim\rightarrow M_n \) has the direct limit chart property \((DLCP)\) for \((U = \lim\rightarrow U_n, \phi = \lim\rightarrow \phi_n)\);

(ASBVB 5) For each \( n \in \mathbb{N} \), there exists a trivialization \( \tau_n : (\pi_n)^{-1}(U_n) \to U_n \times \mathbb{E}_n \) such that, for any \( i \leq j \), the following diagram is commutative:

\[
\begin{array}{ccc}
(\pi_i)^{-1}(U_i) & \xrightarrow{\lambda_j} & (\pi_j)^{-1}(U_j) \\
\downarrow \tau_i & & \downarrow \tau_j \\
U_i \times \mathbb{E}_i & \xrightarrow{\varepsilon_i \times \iota_j} & U_j \times \mathbb{E}_j
\end{array}
\]

Moreover, if in assumption \((ASBVB 1)\) each space \( \mathbb{E}_n \) is a supplemented subspace of \( \mathbb{E}_{n+1} \), we say that \( \left( (E_n, \pi_n, M_n), (\lambda_{n+1}^n, \varepsilon_{n+1}^n) \right) \) is a strong ascending sequence of Banach or normed vector bundles.

We then have the following result which is some generalization of Proposition 4.18. in \( [CaPe19] \).

**Proposition B.14.** Let \( (E_n, \pi_n, M_n) \) be an ascending sequence of Banach vector bundles. We have:

(i) \( \lim\rightarrow E_n \) has a structure of \( n.n.H. \) convenient manifold modelled on the \( \LB \)-space \( \lim\rightarrow \mathbb{M}_n \times \lim\rightarrow \mathbb{E}_n \) which has a Hausdorff convenient structure if and only if \( M \) is Hausdorff.

(ii) \( \left( \lim\rightarrow E_n, \lim\rightarrow \pi_n, \lim\rightarrow M_n \right) \) can be endowed with a structure of convenient vector bundle whose typical fibre is \( \lim\rightarrow \mathbb{E}_n \). If this ascending sequence is strong, \( \left( \lim\rightarrow E_n, \lim\rightarrow \pi_n, \lim\rightarrow M_n \right) \) is a convenient bundle whose structural group is the Fréchet topological group \( \mathbb{G}(\mathbb{E}) \).

C Derivations and Schouten bracket on \( T^\flat M \)

In this appendix, we refer to \( [CaPe] \), Ch. 7, § 2.2 where the reader will find the proofs of propositions and theorems.

We fix an almost Poisson anchor \( P : T^\flat M \to TM \). Then we have a structure of strong partial convenient Lie algebroid \( (T^\flat M, p^\flat_M, M, P, \mathfrak{P}_M) \).
on $T^o M$. We will use the definition of the Schouten bracket on a Poisson manifold given in [FeMa14], 1.4, to propose a generalization of the Schouten bracket on $T^o M$.

For any open set $U$ in $M$ we introduce:

**Definition C.1.** Let $\mathfrak{A}(U)$ be the set of smooth functions $f \in C^\infty(U)$ such that each iterated derivative $d^f f(x) \in L^k_{\text{sym}}(T_x M, \mathbb{R})$ ($k \in \mathbb{N}^*$) satisfies:

$$\forall x \in U, \forall (u_2, \ldots, u_k) \in (T_x M)^{k-1}, \ d^f f(\cdot, u_2, \ldots, u_k) \in T^o_x M. \quad (55)$$

**Proposition C.2.** Fix any open set $U$ in $M$,

1. The set $\mathfrak{A}(U)$ is a subalgebra of $C^\infty(U)$.
2. For each $k \in \mathbb{N}$ and local vector fields $X_1, \ldots, X_k$ on $U$, the map $x \mapsto d^f f(X_1, \ldots, X_k)(x)$ belongs to $\mathfrak{A}(U)$.

As a generalization of the notion of derivation on $C^\infty(U)$ we introduce:

**Definition C.3.** Let $U$ be an open set in $M$.

1. For $k \geq 1$, a $k$-alternating derivation on $\mathfrak{A}(U)$ is a $k$-alternating bounded linear map $D : (\mathfrak{A}(U))^k \to \mathfrak{A}(U)$ such that

$$D(f_1, \ldots, f_{i-1}, gh, f_{i+1}, \ldots, f_k) = gD(f_1, \ldots, f_{i-1}, h, f_{i+1}, \ldots, f_k) + D(f_1, \ldots, f_{i-1}, g, f_{i+1}, \ldots, f_k)h$$

for all $i \in \{1, \ldots, k\}$ and all $f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_k$ in $\mathfrak{A}(U)$.
2. A $k$-alternating derivation $D$ on $\mathfrak{A}(U)$ is called of $1$st-order if $D(f_1, \ldots, f_k)$ only depends on the 1-jets of each $f_i$ for $i \in \{1, \ldots, k\}$.

**Notation C.4.** We denote by

- $\text{Der}_k (\mathfrak{A}(U))$ the set of $k$-alternating derivations on $\mathfrak{A}(U)$;
- $\text{Der}_k^1 (\mathfrak{A}(U))$ the subset of $k$-alternating derivations of first order.

**Example C.5.** For $k = 1$, from the definition, a 1-alternating derivation is simply a derivation on $\mathfrak{A}(U)$ in the usual sense. Since $TM$ is a closed subbundle of $T'' M$, each local vector field induces a derivation of $C^\infty(U)$ by

$$X(f) = < df, X > = i_X(df).$$

But from Proposition C.2 if $f$ belongs to $\mathfrak{A}(U)$, then $X(f) \in \mathfrak{A}(U)$. Therefore $X$ defines a derivation of $\mathfrak{A}(U)$ of 1st-order.
Example C.6. Let \((X_1, \ldots, X_k)\) be a finite sequence of vector fields on an open set \(U\) in \(M\). Then we can consider the decomposable non zero multi-vector \(X_1 \wedge \cdots \wedge X_k\). Now, for each \((f_1, \ldots, f_k) \in (\mathfrak{A}(U))^k\), consider the map:

\[
(f_1, \ldots, f_k) \mapsto \det (i_{X_i}(df_j)).
\] (56)

If each \(f_i\) belongs to \(\mathfrak{A}(U)\), then \(i_{X_i}(df_j)\) also belongs to \(\mathfrak{A}(U)\) (cf. Example C.5). Thus the map (56) takes values in \(\mathfrak{A}(U)\). In fact, this maps gives rise to a \(k\)-alternating derivation on \(\mathfrak{A}(U)\) associated to \(X_1 \wedge \cdots \wedge X_k\) and which only depends on the value of this multi-vector and not of such a decomposition. It is a derivation of 1st-order.

Example C.7. More generally, consider a finite sequence \((X_1, \ldots, X_m)\) of decomposable multi-vectors of dimension \(k\) on \(U\). For any function \(f_1, \ldots, f_m \in \mathfrak{A}(U)\), then \(D = \sum_{i=1}^m f_iX_i\) is also a derivation of first order.

All these examples motivate the following definition.

Definition C.8. A derivation \(D \in \text{Der}_k(\mathfrak{A}(U))\) of first order is called a \(k\)-alternating kinematic derivation of \(\mathfrak{A}(U)\) if, for any fixed \(f_2, \ldots, f_k\) in \(\mathfrak{A}(U)\), there exists a vector field \(X\) on \(U\) such that:

\[
D(f, f_2, \ldots, f_k) = df(X)
\] (57)

for any \(f \in \mathfrak{A}(U)\).

Notation C.9. The set of \(k\)-alternating kinematic derivations of \(\mathfrak{A}(U)\) is denoted \(D_k(\mathfrak{A}(U))\).

Remark C.10. If \(D\) is a \(k\)-alternating kinematic derivation, it follows that, for any \(i \in \{1, \ldots, k\}\), the map \(f_i \mapsto D(f_1, \ldots, f_k)\) for \(\{f_1, \ldots, \hat{f}_i, \ldots, f_k\}\) fixed, there exists a vector field \(X_i\) on \(U\) such that \(df_i(X_i) = D(f_1, \ldots, f_k)\). Note that such a vector field may be not unique (cf. Remark C.19).

Of course, Example C.5, Example C.6 and Example C.7 are alternating kinematic derivations. In finite dimension, all derivations of \(\mathcal{C}^\infty(U)\) are kinematic (cf. [FeMa14]). This is not true in the infinite setting, even in the Banach setting as we can see in the following Example.

Example C.11. Consider a non reflexive Banach space \(M\) and set \(T^\circ M = T'M\). Then for any \(c^\infty\) open \(U\) in \(M\), we consider \(\mathfrak{A}(U) = C^\infty(U)\). So
any operational vector field $X$ of order $r$ on $M$ defines a $1$-alternating derivation of order $r$ on $\mathfrak{A}(U)$ which is not kinematic. More generally, if $X_1, \ldots, X_r$ are operational vector fields of order $r$ on $U$, then, in the same way as in Example C.6, $X_1 \wedge \cdots \wedge X_r$ gives rise to a $k$-alternating derivation of order $r$ on $\mathfrak{A}(U)$ which is not kinematic even for $r = 1$.

As in [FeMa14], 1.4, we introduce:

**Definition C.12.** If $D$ belongs to $\text{Der}_k(\mathfrak{A}(U))$ and $D'$ to $\text{Der}_{k'}(\mathfrak{A}(U))$

1. we have

$$D \circ D'(f_1, \ldots, f_{k'}, f_{k'+1}, \ldots, f_{k+k'-1}) = \sum_{\sigma} (-1)^{\text{sign}(\sigma)} D \left( D'(f_{\sigma(1)}, \ldots, f_{\sigma(k')}), f_{\sigma(k'+1)}, \ldots, f_{\sigma(k+k'-1)} \right)$$

for all $f_i \in \mathfrak{A}(U)$, $i \in \{1, \ldots, k + k' - 1\}$ and where the sum is for all $(k', k - 1)$-shuffles $\sigma$.

2. We set

$$[D, D'] = D \circ D' - (-1)^{(k-1)(k'-1)} D' \circ D.$$  \hfill (58)

(3) The wedge product of $D \wedge D'$ of $D$ and $D'$ is defined by:

$$D \wedge D'(f_1, \ldots, f_{k+k'}) = \frac{1}{k! k'!} \sum_{\sigma} (-1)^{\text{sign}\sigma} D(f_{\sigma(1)}, \ldots, f_{\sigma(k)}) D'(f_{\sigma(k+1)}, \ldots, f_{\sigma(k+k')})$$

where the summation is over all $(k, k')$-shuffles $\sigma$ of $\{1, \ldots, k + k'\}$.

Then as in [FeMa14], p. 17, Problem 1.3, Proposition 2.2, we can show:

**Proposition C.13.** Let $U$ be an open set in $M$. Then we have the following properties:

1. $\text{Der}_k(\mathfrak{A}(U))$ has a structure of $\mathfrak{A}(U)$-module and $\text{Der}_1^1(\mathfrak{A}(U))$ is a submodule.

2. If $D$ belongs to $\text{Der}_k(\mathfrak{A}(U))$ and $D'$ to $\text{Der}_{k'}(\mathfrak{A}(U))$ then $[D, D']$ belongs to $\text{Der}_{(k+k'-1)}(\mathfrak{A}(U))$. \footnote{cf [KrMi97], 32}
3. The bracket \([.,.]\) is \(\mathbb{R}\) bilinear on \(\text{Der}(\mathfrak{A}(U))\) and has the following properties:

(i) \([^D,D'] = -(−1)^{(k−1)(k'−1)}[D,D']\].

(ii) (Generalized Jacobi identity)
For all \(D \in \text{Der}_k(\mathfrak{A}(U))\), \(D' \in \text{Der}_{k'}(\mathfrak{A}(U))\) and \(D'' \in \text{Der}_{k''}(\mathfrak{A}(U))\),
\[
(−1)^{(k−1)(k'−1)}[[D,D'],D']+(−1)^{(k'−1)(k''−1)}[[D',D'],D]+(−1)^{(k''−1)(k−1)}[[D'',D],D'] = 0.
\]

Let \(\pi : F \to M\) be a convenient bundle with typical fibre \(\mathbb{F}\). We denote \(L^k_{\text{alt}}(E,F)\) the set of all bounded \(k\)-linear alternating mappings from \(E\) to \(F\) which is a convenient space. Using common atlas for the bundle structure of \(p^\flat_k : \text{Der}^k(T^\flat M; F) = \bigcup_{x \in M} L^k_{\text{alt}}(T^\flat_x M; F) \to M\) is a convenient vector bundle.

The vector space of local sections of \(L^k_{\text{alt}}(T^\flat M; F)\) over \(U\) is denoted by \(\bigwedge^k \Gamma^*(T^\flat M_U; F_U)\) and is called the set of vectorial \(k\)-differential forms on \(T^\flat M\).

The set \(\{\bigwedge^k \Gamma^*(T^\flat M_U; F_U), U\text{ open set in } M\}\) gives rise to a sheaf of modules over the sheaf \(C^\infty_M\) of smooth functions on \(M\). For \(k \geq 1\), the values of a section \(P \in \bigwedge^k \Gamma^*(T^\flat M_U; F_U)\) is characterized by its values on each \(k\)-uple \((df_1, \ldots, df_k)\) for any \((f_1, \ldots, f_k)\) in \((\mathfrak{A}(U))^k\).

When \(F = M \times \mathbb{R}\) we simply denote \(\bigwedge^k \Gamma^*(T^\flat M_U; F_U)\) the set \(\bigwedge^k \Gamma^*(T^\flat M_U; U \times \mathbb{R})\).

Let \(\iota : T^\flat M \to T'M\) be the inclusion morphism. Then \(\iota^*\) is a bundle morphism from the bidual bundle \(T'^\flat M\) to the dual bundle \(T^\flat M'\) of \(T^\flat M\). Note that \(TM\) is a closed subbundle of \(T'^\flat M\) and so \(\iota^*(TM)\) is contained in \(T^\flat M'\) but in general it is not a subbundle.

**Definition C.14.** Let \(U\) be an open set in \(M\).

(i) For \(k = 1\), an element \(\Lambda \in \bigwedge^1 \Gamma^*(T^\flat M_U) = \Gamma^*(T^\flat M_U)\) is admissible if there exists a vector field \(X\) on \(U\) such that \(\Lambda = \iota^* X\).
(ii) For \( k \geq 2 \), a section \( \Lambda \in \bigwedge^k \Gamma^*(T^\oplus M_U) \) will be called admissible if there exists \( P \in \bigwedge^{k-1} \Gamma^*(T^\oplus M_U; TM_U) \) such that we have
\[
\Lambda_x(\alpha_1, \alpha_2, \ldots, \alpha_k) = <\alpha_1, P_x(\alpha_2, \ldots, \alpha_k)>
\]
for all \( \alpha_1, \ldots, \alpha_k \in T^\oplus_x M \).

**Notation C.15.** We denote by \( \Gamma^*_k(\mathfrak{A}(U)) \) the set of admissible elements of \( \bigwedge^k \Gamma^*(T^\oplus M_U) \) and by \( D^*_k(\mathfrak{A}(U)) \) the set of admissible kinematic derivations of \( D_k(\mathfrak{A}(U)) \).

- For \( k \geq 2 \), from now on, the element \( P \) of \( \bigwedge^{k-1} \Gamma^*(T^\oplus M_U; TM_U) \) associated to the admissible section \( \Lambda \in \bigwedge^k \Gamma^*(T^\oplus M_U) \) will be denoted \( \Lambda^\sharp \); we then have
\[
\Lambda_x(\alpha_1, \alpha_2, \ldots, \alpha_k) = \langle \alpha_1, \Lambda^\sharp_x(\alpha_2, \ldots, \alpha_k) \rangle
\]
for all \( \alpha_1, \ldots, \alpha_k \in T^\oplus_x M \) and so \( \iota^* \circ \Lambda^\sharp(\alpha_2, \ldots, \alpha_k) \in T^\oplus M' \).

- For \( k = 1 \), we set \( \Lambda^\sharp = X \) where \( X \) is a fixed vector field such that \( \Lambda = \iota^* X \).

**Example C.16.** To any almost partial Poisson anchor \( P : T^\oplus M \to TM \) is associated a tensor \( \Lambda \in \bigwedge^2 \Gamma^*(T^\oplus M) \) which is admissible.

**Proposition C.17.**

1. \( \Gamma^*_k(\mathfrak{A}(U)) \) is a \( \mathfrak{A}(U) \)-module and the set \( \{ \bigwedge^k \Gamma^*(\mathfrak{A}(U)), U \text{ open in } M \} \) is a \( \mathfrak{A}_M \) sheaf of \( \mathfrak{A}_M \) modules, if \( \mathfrak{A}_M \) denotes the sheaf of rings defined by \( \{ \mathfrak{A}(U), U \text{ open set in } M \} \).

2. For any \( \Lambda \in \Gamma^*_k(\mathfrak{A}(U)) \), \( \Lambda(df_1, \ldots, df_k) \) takes values in \( \mathfrak{A}(U) \) for all \( f_1, \ldots, f_k \in \mathfrak{A}(U) \).

Now, we have the following link between \( \Gamma^*_k(\mathfrak{A}(U)) \) and the set \( D_k(\mathfrak{A}(U)) \) of \( k \)-alternating kinematic derivations on \( \mathfrak{A}(U) \):

**Proposition C.18.** To any \( \Lambda \in \Gamma^*_k(\mathfrak{A}(U)) \) is associated a unique \( k \)-alternating kinematic derivation \( D_{\Omega} \in D_k(\mathfrak{A}(U)) \) defined by
\[
D_{\Lambda}(f_1, \ldots, f_k) = \Lambda(df_1, \ldots, df_k).
\]
for any \( (f_1, \ldots, f_k) \in \mathfrak{A}(U)^k \). Moreover, this correspondence is injective.
Remark C.19. According to Definition C.14 it follows that Examples C.5, C.6 and C.7 give also examples of admissible $k$-forms on $T^oM$ via the morphism $\iota^*$. More generally, let $\bigwedge^k TM$ be the convenient bundle whose typical fibre is $\bigwedge^k M$. Then $\bigwedge^k TM$ is a subbundle of $\bigwedge^k T''M$. Therefore, for any local section $\Xi$ of $\bigwedge^k T^oM$ over $U$, $\iota^*\Xi$ belongs to $\bigwedge^k \Gamma^o(A(U))$. For $k \geq 2$, if $\Xi$ is decomposable we can associate a morphism $\Xi^b : \bigwedge^{k-1} T^oM \to T''M$ defined by

$$\Xi^b(\alpha_2, \ldots, \alpha_k) = \Xi(., \alpha_2, \ldots, \alpha_k)$$

whose range is contained in $T^oM_U$. More generally, if $\Xi$ belongs to the $A(U)$-module generated by decomposable multi-vectors on $U$ then $\iota^*\Xi$ is admissible and we have $\Xi^b = (\iota^*\Xi)^b$. On the other hand, from Proposition C.18 such a section induces a kinematic derivation $D_\Xi$ on $A(U)$ and if $\Xi'$ is another such section then $D_\Xi = D_{\Xi'}$ if and only if $\iota^*(\Xi^b - (\Xi')^b) = 0$.

In particular, if $D$ is a kinematic derivation, two vector fields $X$ and $X'$ induce the same derivation $f \mapsto D(f, f_2, \ldots, f_k)$ on $A(U)$ (for fixed $f_2, \ldots, f_k$ in $A(U)$) if and only if $X - X'$ belongs to $\ker \iota^*_U|T''M_U$.

For a given kinematic derivation $D$ and for $f_2, \ldots, f_k$ fixed, a vector field $X_{f_2, \ldots, f_k}$ such that $df(X_{f_1, \ldots, f_k}) = D(f, f_2, \ldots, f_k)$ is unique if and only if the set

$$\{x \in U : \ker \iota^*_x|T''M_U \cap T_xM_U = \{0\}\}$$

is dense in $U$. In this case the same is true for any previous section $\Xi$.

Remark C.20. The map $\Lambda \mapsto D_\Lambda$ from $\Gamma^o_k(A(U))$ to $D_k(A(U))$ is not surjective in general in infinite dimension for two reasons:

(i) If $M$ is not a regular manifold, in general $A(U)$ could be strictly smaller that $A(V)$ for $V \subset U$ and so the same is true for $D_k(A(U))$. Unfortunately we have no examples of such a situation.

(ii) If $T^oM \neq T'M$, from Remark C.19 if the set

$$\{x \in U : \ker \iota^*_x|T''M_U \cap T_xM_U = \{0\}\}$$

is not dense in $U$, then there exists an open set $V$ in $U$ such that

$$\dim \left( \ker \iota^*_x|T''M_V \cap T_xM_V \right) \geq 1.$$  

After restricting $V$ if necessary, there exists a vector field $Z$ which belongs to $\ker \iota^*_x|T''M_V \cap TM_V$. Thus for instance, any kinematic derivation $D \in D_2(A(V))$, for any $f \in A(V)$, and $g \in A(V)$ fixed, the set of
vector fields $X_g$ on $V$ such that

$$df(X_g) = D(f, g)$$

contains at least the 1-dimensional affine subspace $X_g + \phi Z$ where $\phi \in \mathfrak{A}(V)$. So, if we had $\Lambda \in \Gamma^*_k(\mathfrak{A}(V))$ such that $D\Lambda = D$, then $\Lambda^*_{\Lambda}(dg)$ should belong to

$$X_g(x) + \ker \iota^*_p|T^*_xM \cap T_xM_V$$

and, in particular, could take any value in $X_g(x) + \phi(x)Z(x)$ and so could not be intrinsically defined.

**Lemma C.21.** There exists a unique element $[\Lambda, \Omega]_S \in \Gamma^*_{k+l-1}(\mathfrak{A}(U))$ such that

$$[D\Lambda, D\Omega] = D_{[\Lambda, \Omega]_S}.$$  

Now according to Proposition [C.18] and Lemma [C.21] as in finite dimension (cf. [FeMa14]), we have

**Proposition C.22.** Let $U$ be any open set in $M$. Then, for any $\Lambda \in \Gamma^*_k(\mathfrak{A}(U))$ and $\Omega \in \Gamma^*_l(\mathfrak{A}(U))$, the bracket $[D\Lambda, D\Omega]$ is a kinematic $(k+l-1)$-alternating derivation of $\mathfrak{A}(U)$ and there exists a unique element $[\Lambda, \Omega]_S \in \Gamma^*_{k+l-1}(\mathfrak{A}(U))$ such that

$$D_{[\Lambda, \Omega]_S} = [D\Lambda, D\Omega].$$

According to Proposition [C.22] we can introduce the core notion of Schouten bracket:

**Definition C.23.** For any $\Lambda \in \Gamma^*_k(\mathfrak{A}(U))$ and $\Omega \in \Gamma^*_l(\mathfrak{A}(U))$. Then the Schouten bracket $[\Lambda, \Omega]_S$ is the unique element $[\Lambda, \Omega]_S \in \Gamma^*_{k+l-1}(\mathfrak{A}(U))$ such that

$$D_{[\Lambda, \Omega]_S} = [D\Lambda, D\Omega].$$

Before listing the properties of the Schouten bracket, from Proposition [C.17] we can note that, we have:

$$[\Lambda, \Omega]_S (df_1, df_2, df_3) = [D\Lambda, D\Lambda] (df_1, df_2, df_3)$$

$$= 2 (\{f_1, \{f_2, f_3\}_P\}_P + \{f_2, \{f_3, f_1\}_P\}_P + \{f_3, \{f_1, f_2\}_P\}_P).$$

According to Remark [C.19] we have the following result:

**Theorem C.24.** The Schouten bracket has the following properties:
1. For any vector fields $X$ and $Y$ on $U$, $\iota^*X$ and $\iota^*Y$ belong to $\Gamma^*(\mathfrak{A}(U))$ and we have

$$\iota^*[X,Y] = [\iota^*X, \iota^*Y].$$

2. For any $\Omega \in \Gamma^*(\mathfrak{A}(U))$ and $\Phi \in \Gamma^*_h(\mathfrak{A}(U))$,

$$[\Omega, \Phi]_S = -(-1)^{(k-1)(h-1)}[\Phi, \Omega]_S.$$

3. For any $\Omega \in \Gamma^*_k(\mathfrak{A}(U))$, $\Phi \in \Gamma^*_h(\mathfrak{A}(U))$ and $\Psi \in \Gamma^*_l(\mathfrak{A}(U))$,

$$[\Omega, \Phi \wedge \Psi]_S = [\Omega, \Phi]_S \wedge \Psi + (-1)^{(k-1)h} \Phi \wedge [\Phi, \Psi]_S.$$

4. For any $\Omega \in \Gamma^*_k(\mathfrak{A}(U))$, $\Phi \in \Gamma^*_h(\mathfrak{A}(U))$ and $\Psi \in \Gamma^*_l(\mathfrak{A}(U))$,

$$(-1)^{(k-1)(l-1)}[\Omega, [\Phi, \Psi]_S]_S + (-1)^{(h-1)(k-1)}[\Phi, [\Psi, \Omega]_S]_S + (-1)^{(l-1)(h-1)}[\Psi, [\Omega, \Phi]_S]_S = 0,$$

that is the generalized Jacobi identity.

5. Let $X_1 \wedge \cdots \wedge X_k$ and $Y_1 \wedge \cdots \wedge Y_h$ be decomposable multivectors; then $\iota^*(X_1 \wedge \cdots \wedge X_k)$ and $\iota^*(Y_1 \wedge \cdots \wedge Y_h)$ belong to $\Gamma^*_k(\mathfrak{A}(U))$ and $\Gamma^*_h(\mathfrak{A}(U))$ respectively and we have

$$[\iota^*(X_1 \wedge \cdots \wedge X_k), \iota^*(Y_1 \wedge \cdots \wedge Y_h)]_S$$

$$= \iota^* \left( \sum_{i,j} (-1)^{i+j} [X_i, Y_j] X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_k \wedge Y_1 \wedge \cdots \wedge \hat{Y}_j \wedge \cdots \wedge Y_h \right)$$

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