Quantum speedups for treewidth

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Abstract

In this paper, we study quantum algorithms for computing the exact value of the treewidth of a graph. Our algorithms are based on the classical algorithm by Fomin and Villanger (Combinatorica 32, 2012) that uses $O(2.616^n)$ time and polynomial space. We show three quantum algorithms with the following complexity, using QRAM in both exponential space algorithms:

- $O(1.618^n)$ time and polynomial space;
- $O(1.554^n)$ time and $O(1.452^n)$ space;
- $O(1.538^n)$ time and space.

In contrast, the fastest known classical algorithm for treewidth uses $O(1.755^n)$ time and space. The first two speed-ups are obtained in a fairly straightforward way. The first version uses additionally only Grover’s search and provides a quadratic speedup. The second speedup is more time-efficient and uses both Grover’s search and the quantum exponential dynamic programming by Ambainis et al. (SODA ’19). The third version uses the specific properties of the classical algorithm and treewidth, with a modified version of the quantum dynamic programming on the hypercube. Lastly, as a small side result, we also give a new classical time-space tradeoff for computing treewidth in $O^*(2^n)$ time and $O^*(\sqrt{2^n})$ space.

1 Introduction

For many NP-complete problems, the exact solution can be found much faster than a brute-force search over the possible solutions; it is not so rare that the best currently known algorithms are exponential [FK10]. Perhaps one of the most famous examples is the travelling salesman problem, where a naive brute-force requires $O^*(n!)$ computational time, but a dynamic programming algorithm solves it exactly only in $O^*(2^n)$ time [Bel62; HK62]. Such algorithms are studied also because they can reveal much about the mathematical structure of the problem and because sometimes in practice they can be more efficient than subexponential algorithms with a large constant factor in their complexity.

With the advent of quantum computing, it is curious how quantum procedures can be used to speed up such algorithms. A clear example is illustrated by the SAT problem: while iterating over all possible assignments to the Boolean formula on $n$ variables gives $O^*(2^n)$ time, Grover’s search [Gro96] can speed this up quadratically, resulting in $O^*(\sqrt{2^n})$ time. Grover’s search can also speed up exponential dynamic programming: recently Ambainis et al. [Amb+19] have shown how to apply Grover’s search recursively together with classical precalculation to speed up the
\(O^*(2^n)\) dynamic programming introduced by Bellman, Held and Karp [Bel62; HK62] to a \(O(1.817^n)\) quantum algorithm. For some problems like the travelling salesman problem and minimum set cover, the authors also gave a more efficient \(O(1.728^n)\) time quantum algorithm by combining Grover’s search with both divide & conquer and dynamic programming techniques. Their approach has been subsequently applied to find a speedup for more NP-complete problems, including graph coloring [SM20], minimum Steiner tree [Miy+20] and optimal OBDD ordering [Tan20].

In this paper, we focus on the NP-complete problem of finding the treewidth of a graph. Informally, the treewidth is a value that describes how close the graph is to a tree; for example, the treewidth is 1 when the graph is a tree, while the treewidth of a complete graph on \(n\) vertices is \(n - 1\). This quantity is prominently used in parameterized algorithms, as many problems are efficiently solvable when treewidth is small, such as vertex cover, independent set, dominating set, Hamiltonian cycle, graph coloring, etc. [AP89]. The applications of treewidth, both theoretical and practical, are numerous, see [Bod05] for a survey. If the treewidth is at most \(k\), it can be computed exactly in \(O(n^{k+2})\) time [ACP87]; 2-approximated in parameterized linear time \(2^O(k) n [Kor21]; O(\sqrt{\log k})\)-approximated in polynomial time [FHL08]; \(k\)-approximated in \(O(k^7 n \log n)\) time [Fom+18].

As for exact exponential time treewidth algorithms, both currently most time and space efficient algorithms were proposed by Fomin and Villanger in [FV12]: the first uses \(O(1.755^n)\) time and space and the second requires \(O(2.616^n)\) time and polynomial space. The crucial ingredient of these algorithms is a combinatorial lemma that upper bounds the number of connected subsets with fixed neighborhood size (Lemma 3), as well as gives an algorithm that lists such sets.

Our main motivation for tackling these algorithms is that although the \(O(1.817^n)\) quantum algorithm from [Amb+19] is applicable to treewidth, it is still less efficient than Fomin’s and Villanger’s. In this paper we show that their techniques are also amenable to quantum search procedures. In particular, we focus on their polynomial space algorithm. This algorithm has two nested procedures: the first procedure uses Lemma 3 to search through specific subsets of vertices to be fixed as a bag of the tree decomposition; the second procedure finds the optimal width of the tree decomposition with \(S\) as its bag.

We find that Grover’s search can be applied to the listing procedure of Lemma 3, thus speeding up the first procedure quadratically. For the second procedure, classically one can use either the \(O^*(2^n)\) time and space dynamic programming algorithm or the \(O^*(4^n)\) time and polynomial space divide & conquer algorithm (Fomin and Villanger use the latter), which both were introduced in [Bod+12]. The divide & conquer algorithm we can also speed up using Grover’s search. Thus, we obtain a quadratic speedup for the polynomial space algorithm:

**Theorem 21.** There is a bounded-error quantum algorithm that finds the exact treewidth of a graph on \(n\) vertices in \(O(1.61713^n)\) time and polynomial space.

Next, using the fact that the \(O^*(2^n)\) dynamic programming algorithm can be sped up to an \(O^*(1.817^n)\) quantum algorithm together with the quadratic speedup of Lemma 3, we obtain our second quantum algorithm:

**Theorem 22.** Assuming the QRAM data structure, there is a bounded-error quantum algorithm that finds the exact treewidth of a graph on \(n\) vertices in \(O(1.55374^n)\) time and \(O(1.45195^n)\) space.

The last theorem suggests a possibility for an even more efficient algorithm by trading some space for time. We achieve this by proving a treewidth property which essentially states that we
can precalculate some values of dynamic programming for the original graph, and reuse these values in the dynamic programming for its subgraphs (Lemma 23). This allows us a global precalculation, which can be used in the second procedure of the treewidth algorithm. To do that, we have to modify the $O(1.817^n)$ algorithm of [Amb+19]. We refer to it as the asymmetric quantum exponential dynamic programming. This gives us the following algorithm:

**Theorem 24.** Assuming the QRAM data structure, there is a bounded-error quantum algorithm that finds the exact treewidth of a graph on $n$ vertices in $O(1.53793^n)$ time and space.

Lastly, we observe that replacing the $O^*(4^n)$ divide & conquer algorithm in the classical $O(2.616^n)$ polynomial space algorithm by the $O^*(2^n)$ dynamic programming only lowers the time complexity to $O^*(2^n)$. However, the interesting consequence is that the space requirement becomes only $O^*(\sqrt{2^n})$. Hence, we obtain a classical time-space tradeoff:

**Theorem 20.** The treewidth of a graph with $n$ vertices can be computed in $O^*(2^n)$ time and $O^*(\sqrt{2^n})$ space.

Time-wise, this is more efficient than the $O(2.616^n)$ time polynomial space algorithm, and space-wise, this is more efficient than the $O(1.755^n)$ time and space algorithm. It also fully subsumes the time-space tradeoffs for permutation problems proposed in [KP10] applied to treewidth.

## 2 Preliminaries

We denote the set of integers from 1 to $n$ by $[n]$. For a set $S$, denote the set of all its subsets by $2^S$. We call a permutation of a set of vertices $S \subseteq V$ a bijection $\pi: S \rightarrow [|S|]$. We denote the set of permutations of $S$ by $\Pi(S)$. For a permutation $\pi \in \Pi(S)$, let $\pi_{<v} = \{w \mid \pi(w) < \pi(v)\}$ and $\pi_{>v} = \{w \mid \pi(w) > \pi(v)\}$.

We write $O(f(n)) = \text{poly}(n)$ if $f(n) = O(n^{c})$ for some constant $c$. Also let $O(\text{poly}(n)f(m)) = O^*(f(m))$. This is useful since our subprocedures will often have some running time $f(m)$ times some function that depends on the size of the input graph $G$ on $n$ vertices. In this paper, we are primarily concerned with the exponential complexity of the algorithms, hence, we are interested in the $f(m)$ value of an $O^*(f(m))$ complexity.

**Graph notation.** For a graph $G = (V, E)$ and a subset of vertices $S \subseteq V$, denote $G[S]$ as the graph induced in $G$ on $S$. For a subset of vertices $S \subseteq V$, let $N(S) = \{v \in V - S \mid u \in S, \{u, v\} \in E\}$ be its neighborhood. We call a subset $S \subseteq V$ connected if $G[S]$ is connected, and $C \subseteq V$ a clique if $G[C]$ is a complete graph. Later on we also mention the notions of potential maximum cliques and minimal separators, which are specific subsets of $V$, but we don’t rely on them; for their definitions and properties, see e.g. [FV12].

**Treewidth.** A tree decomposition of a graph $G = (V, E)$ is a pair $(X, T)$, where $T = (V_T, E_T)$ and $X = \{X_i \mid i \in V_T\} \subseteq 2^V$ such that:

- $\bigcup_{X_i \in X} X_i = V$;
- for each edge $\{u, v\} \in E$, there exists $X_i \in X$ such that $u, v \in X_i$;
- for any vertex $v \in V$ in $G$, the set of vertices $\{X \mid v \in X\}$ forms a connected subtree of $T$. 


We call the subsets $\chi \in X$ bags and the vertices of $T$ nodes. The width of $T$ is defined as the minimum size of $\chi \in X$ minus 1. The treewidth of $G$ is defined as the minimum width of a tree decomposition of $G$ and we denote it by $\text{tw}(G)$.

We also consider optimal tree decompositions given that some subset $\chi \in V$ is a bag of the tree. We denote the smallest width of a tree decomposition of $G$ among those that contain $\chi$ as a fixed bag by $\text{tw}(G, \chi)$.

**Approximations.** For the binomial coefficients, we use the following well-known approximation:

**Theorem 1 (Entropy approximation).** For any $k \in [0, 1]$, we have

$$\binom{n}{k} \leq 2^{H\left(\frac{k}{n}\right)\cdot n},$$

where $H(\epsilon) = -\epsilon \log_2(\epsilon) + (1 - \epsilon) \log_2(1 - \epsilon)$ is the binary entropy function.

**Quantum subroutines.** Our algorithms use a well-known variation of Grover’s search, quantum minimum finding:

**Theorem 2 (Theorem 1 in [DH96]).** Let $A : N \to [n]$ be an exact quantum algorithm with running time $T$. Then there is a bounded-error quantum algorithm that computes $\min_{i \in [N]} A(i)$ in $O^*(T\sqrt{N})$ time.

Two of our algorithms use the QRAM data structure [GLM08]. This structure stores $N$ memory entries and, given a superposition of memory indices together with an empty data register $\sum_{i \in [N]} |i\rangle|0\rangle$, it produces the state $\sum_{i \in [N]} |i\rangle|\text{data}_i\rangle$ in $O(\log N)$ time. In our algorithms, $N$ will always be exponential in $n$, which means that a QRAM operation is going to be polynomial in $n$. Thus, this factor will not affect the exponential complexity, which we are interested in.

In our algorithms, we will often have a quantum algorithm that takes exact subprocedures (like in Theorem 2), and give it bounded-error subprocedures. Since we always going to take $O(\exp(n))$ number of inputs, this issue can be easily solved by repeating the subprocedures $\text{poly}(n)$ times to boost the probability of correct answer to $1 - O(1/\exp(n))$: it can be then shown that the branch in which all the procedures have correct answers has constant amplitude. The final bounded-error algorithm incurs only a polynomial factor, and does not affect the exponential complexity. We also note that on a deeper perspective, all our quantum subroutines are based on the primitive of Grover’s search [Gro96]; an implementation of Grover’s search with bounded-error inputs that does not incur additional factors in the complexity has been shown in [HMW03].

We also are going to encounter an issue that sometimes we have some real parameter $\alpha \in [0, 1]$ and we are examining $\binom{n}{\alpha n}$. Since $\alpha n$ is not integer, this value is not defined; however, we can take this to be any value between $\binom{n}{\lfloor \alpha n \rfloor}$ or $\binom{n}{\lceil \alpha n \rceil}$, as they differ only by a factor of $n$. Thus, this does not produce an issue for the exponential complexity analysis. Henceforward we abuse the notation and simply write $\binom{n}{\alpha n}$.

### 3 Combinatorial lemma

In this section we describe how the main combinatorial lemma of [FV12] can be sped up quantumly quadratically using Grover’s search.
Lemma 3 (Lemmas 3.1. and 3.2. in [FV12]). Let $G = (V, E)$ be a graph. For every $v \in V$ and $b, f \geq 0$, the number of connected subsets $B \subseteq V$ such that

1. $v \in B$,
2. $|B| = b + 1$, and
3. $|N(B)| = f$

is at most $\binom{b+f}{b}$. There also exists an algorithm that lists all such sets in $O^*(\binom{b+f}{b})$ time and polynomial space.

Informally, this lemma is used in the treewidth algorithm to search for a set, such that, if fixed as a bag of the tree decomposition, the remaining graph breaks down into connected components of bounded size; then, the optimal width of the tree decomposition with this bag fixed can be solved using algorithms from Section 4. The lemma gives an upper bound on the number of sets to consider.

Their proof of this lemma essentially gives a branching algorithm that splits the problem into several problems of the same type, and solves them recursively. The idea for applying Grover’s search to such a branching algorithm is simple. The algorithm that generates all sets can be turned into a procedure that, given a number $i \in \mathbb{Z}$ of the set we need to generate, generates this set in polynomial time. Then, we can run Grover’s search over all integers in $\binom{b+f}{b}$ on this procedure.

This was formalized by Shimizu and Mori:

Lemma 4 (Lemma 4 in [SM20]). Let $P$ be a decision problem with parameters $n_1, \ldots, n_{\ell}$. Suppose that there is a branching rule $b(P)$ that reduces $P$ to $m(b(P))$ problems $P_1, \ldots, P_{m(b(P))}$ of the same class. Here, $P_i$ has parameters $f^{b(P)}_i(n_j)$ for $j \in [\ell]$, where $f^{b(P)}_i \leq n_j$. At least one of the parameters of $P_i$ must be strictly smaller than the corresponding parameter of $P$. The solution for $P$ is equal to the minimum of the solutions for $P_1, \ldots, P_{m(b(P))}$.

Let $U(n_1, \ldots, n_{\ell})$ be an upper bound on the number of leaves in the computational tree. Assume that the running time of computing $b(P)$, $P_i$, $f^{b(P)}_i$ and $U(n_1, \ldots, n_{\ell})$ is polynomial w.r.t. $n_1, \ldots, n_{\ell}$. Suppose that $U(n_1, \ldots, n_{\ell}) \geq \sum_{i=1}^{m(b(P))} U(f^{b(P)}_i(n_1), \ldots, f^{b(P)}_i(n_\ell))$. Also suppose that $T$ is the running time for the computation at each of the leaves in the computational tree. Then there is a bounded-error quantum algorithm that computes $P$ and has running time $\text{poly}(n_1, \ldots, n_{\ell}) \sqrt{U(n_1, \ldots, n_{\ell})} T$.

We apply this lemma to Lemma 3:

Lemma 5. Let $G = (V, E)$ be a graph, and $A : 2^V \rightarrow [n]$ be an exact quantum algorithm with running time $T$. For every $v \in V$ and $b, f \geq 0$, let $B_{v,b,f}$ be the set of connected subsets $B \subseteq V$ satisfying the conditions of Lemma 3. Then there is a bounded-error quantum algorithm that computes $\min_{B \in B_{v,b,f}} A(B)$ in time

$$O^*\left(\sqrt{\binom{b+f}{b}}\right).$$

Proof. According to the proof of Lemma 3 in [FV12], we have that

- $\ell = 2$, $n_1 = b$, $n_2 = f$. 


• $b(P)$ splits the problem into $m_{b(P)} = f + b$ problems.

• $P_i$ has parameters $f_1^{b(P),i}(b) = b - 1$ and $f_2^{b(P),i}(f) = f - i + 1$.

• $U(b, f) = \binom{b+f}{f}$.

• $\sum_{i=1}^{m_b(P)} U\left(f_1^{b(P),i}(b), f_2^{b(P),i}(f)\right) = \sum_{i=1}^{f+b} \left(f+b-i\right) = \sum_{i=0}^{f+b-1} \left(f+b-1-i\right) = \binom{b+f}{f} = U(b, f)$.

• Computing $b(P), f_1^{b(P),i}, f_2^{b(P),i}$ and $U(b, f)$ takes time polynomial in $b$ and $f$; computing $P_i$ involves contracting two vertices in the graph and can be done in poly($n$) time.

\[ \square \]

4 Fixed bag treewidth algorithms

In this section we describe algorithms that calculate the optimal treewidth of a graph with the condition that a subset of its vertices is fixed as a bag of the tree decomposition. We then show ways to speed them up quantumly. Both approaches were given by Bodlaender et al. [Bod+12].

4.1 Treewidth as a linear ordering

Both of these algorithms use the fact that treewidth can be seen as a graph linear ordering problem. For a detailed description, see Section 2.2 of [Bod+12], from where we also borrow a lot of notation. We will also use the properties of this formulation in our improved quantum algorithm.

A linear ordering of a graph $G = (V, E)$ is a permutation $\pi \in \Pi(V)$. The task of a linear ordering problem is finding $\min_{\pi \in \Pi(V)} f(\pi)$, for some known function $f$.

For two vertices $v, w \in V$, define a predicate $P_\pi(v, w)$ to be true iff there is a path from $v$ to $w$ in $G$ such that all internal vertices in that path are before $v$ and $w$ in $\pi$. Then define $R_\pi(v)$ to be the number of vertices $w$ such that $\pi(w) > \pi(v)$ and $P_\pi(v, w)$ holds. The following proposition gives a description of treewidth as a linear ordering problem:

**Proposition 6** (Proposition 3 in [Bod+12]). Let $G = (V, E)$ be a graph, and $k$ a non-negative integer. The treewidth of $G$ is at most $k$ if there is a linear ordering $\pi$ of $G$ such that for each $v \in V$, we have $R_\pi(v) \leq k$.

For a set of vertices $S \subseteq V$ and a vertex $v \notin S$, define

$Q_G(S, v) = |\{w \in V - S - \{v\} | v \text{ and } w \text{ are connected by a path in } G[S \cup \{v, w\}]\}|$.

Note that $R_\pi(v) = |Q_G(\pi_{<v}, v)|$, and $|Q_G(S, v)|$ can be computed in poly($n$) time using, for example, depth-first search.

Then define

$TWR_G(L, S) = \min_{\pi \in \Pi(V)} \max_{v \in S} |Q_G(L \cup \pi_{<v}, v)|$.

Also define

$TW_G(S) = \min_{\pi \in \Pi(V)} \max_{v \in S} |Q_G(\pi_{<v}, v)|$.

These notations are connected by the relation

$TW_G(S) = TWR_G(\emptyset, S)$. 

\[ 6 \]
Note that $\text{tw}(G)$ is equal to $\min_{\pi \in \Pi(V)} \max_{v \in V} R_\pi(v) = \text{TW}_G(V)$.

The following lemma gives a way to find optimal fixed bag tree decompositions using the algorithms for finding the optimal linear arrangements:

**Lemma 7** (Lemma 11 in [Bod+12]). Let $C \subseteq V$ induce a clique in a graph $G = (V, E)$. The treewidth of $G$ equals $\max(\text{TW}_G(V - C), |C| - 1)$. Essentially, this lemma tells us that $C$ can be placed in the end of the optimal arrangement.

**Lemma 8.** Let $G = (V, E)$ be a graph, and $\chi \subseteq V$ a subset of its vertices. Then

$$\text{tw}(G, \chi) = \max(\text{TW}_G(V - \chi), |\chi| - 1).$$

**Proof.** Completing a bag of a tree decomposition into a clique does not change the width of the tree decomposition. The claim then follows from Lemma 7.

In the final treewidth algorithms, we will also use the following fact:

**Lemma 9.** Let $G = (V, E)$ be a graph and $\chi \subseteq V$ a subset of its vertices. Let $C$ be the set of connected components of $G[V - \chi]$. Then

$$\text{tw}(G, \chi) = \max_{C \in C} \text{tw}(G[C \cup \chi], \chi).$$

**Proof.** Let $(X, T)$ be a tree decomposition with the smallest width $w$ that contains $\chi$ as a bag. For a connected component $C \in C$, examine the tree decomposition $(X_C, T_C)$ obtained from $(X, T)$ by removing all vertices not in $\chi$ or $C$ from all bags. Clearly, this is a tree decomposition of $G[C \cup \chi]$ with $\chi$ as a bag: as we only have possibly removed some vertices, its width is at most $w$. Now, examine the tree decomposition obtained by taking all $(X_C, T_C)$ and making $\chi$ its common bag. This is a valid tree decomposition, since no two vertices in distinct connected components of $C$ are connected by an edge. Its width is the maximal width of $(X_C, T_C)$, therefore at most $w$. \qed

### 4.2 Divide & Conquer

The first algorithm is based on the following property:

**Lemma 10** (Lemma 7 in [Bod+12]). Let $G = (V, E)$ be a graph, $S \subseteq V$, $|S| \geq 2$, $L \subseteq V$, $L \cap S = \emptyset$, $1 \leq k < |S|$. Then

$$\text{TWR}_G(L, S) = \min_{S' \subseteq S} \max_{|S'| = k} \left( \text{TWR}_G(L, S'), \text{TWR}_G(L \cup S', S - S') \right).$$

Note that $\text{TWR}_G(L, \{v\}) = |Q_G(L, v)|$ can be calculated in polynomial time. The value we wish to calculate is $\text{TWR}_G(\emptyset, V - \chi)$. Picking $k = |S|/2$ in Lemma 10 and applying Lemma 8, we obtain a $\text{poly}(|V|4^{|V| - |\chi|})$ deterministic algorithm with polynomial space:

**Theorem 11** (Theorem 8 in [Bod+12]). Let $G = (V, E)$ be a graph on $n$ vertices and $\chi \subseteq V$ a subset of its vertices. There is an algorithm that calculates $\text{tw}(G, \chi)$ in $O^*(4^{n-|\chi|})$ time and polynomial space.

Immediately we can prove a quadratic quantum speedup using Grover’s search:
Theorem 12. Let $G = (V, E)$ be a graph on $n$ vertices and $\chi \subseteq V$ a subset of its vertices. There is a bounded-error quantum algorithm that calculates $\text{tw}(G, \chi)$ in $O^*(2^{n-|\chi|})$ time and polynomial space.

Proof. We can apply quantum minimum finding to check sets $S'$ in Lemma 10, in order to obtain a quadratic speedup over Theorem 11. To avoid the accumulation of error in the recursion, we can use the implementation of Grover’s search with bounded-error inputs [HMW03].

4.3 Dynamic programming

The second algorithm is based on the following recurrence:

Lemma 13 (Lemma 5 in [Bod+12]). Let $G = (V, E)$ be a graph and $S \subseteq V$, $S \neq \emptyset$. Then

$$\text{TW}_G(S) = \min_{v \in S} \max(\text{TW}_G(S - \{v\}), |Q_G(S - \{v\}, v)|).$$

Note that in fact Lemma 13 is a special case of Lemma 10 with $L = \emptyset$ and $k = |S| - 1$. This lemma together with Lemma 8 and the dynamic programming technique by Bellman, Held and Karp [Bel62; HK62] gives the following algorithm:

Theorem 14 (Theorem 6 in [Bod+12]). Let $G = (V, E)$ be a graph on $n$ vertices and $\chi \subseteq V$ a subset of its vertices. There is an algorithm that calculates $\text{tw}(G, \chi)$ in $O^*(2^{n-|\chi|})$ time and space.

This algorithm calculates the values of $\text{TW}_G(S)$ for all sets $S$ in order of increasing size of the sets, and also stores them all in memory. Such dynamic programming can be sped up quantumly: Ambainis et al. [Amb+19] have shown an $O(1.817^n)$ time and space quantum algorithm with QRAM for such problems. Therefore, this gives the following quantum algorithm:

Theorem 15. Let $G = (V, E)$ be a graph on $n$ vertices and $\chi \subseteq V$ a subset of its vertices. Assuming the QRAM data structure, there is a bounded-error quantum algorithm that calculates $\text{tw}(G, \chi)$ in $O^*(1.81691^{n-|\chi|})$ time and space.

Note that this algorithm can be used to calculate $\text{TWR}_G(L, S)$. Firstly, $\text{TWR}_G(L, \emptyset) = 0$ and

$$\text{TWR}_G(L, S) = \min_{v \in S} \max(\text{TWR}_G(L, S - \{v\}), \text{TWR}_G(L \cup (S - \{v\}), \{v\}))$$

by Lemma 10. As already mentioned earlier, the value $|Q_G(L \cup (S - \{v\}), v)|$ can be calculated in polynomial time. Hence this recurrence is of the same form as Lemma 13.

Theorem 16. Let $G = (V, E)$ be a graph on $n$ vertices and $L, S \subseteq V$ be disjoint subsets of vertices. Assuming the QRAM data structure, there is a bounded-error quantum algorithm that calculates $\text{TWR}_G(L, S)$ in $O^*(1.81691^{|S|})$ time and space.

5 Fomin’s and Villanger’s algorithm

In this section, we first describe the polynomial space treewidth algorithm of [FV12]. Afterwards, we analyze the time complexity for the classical algorithm and then for the same algorithm sped up by the quantum tools presented above.

The algorithm relies on the following, shown implicitly in the proof of Theorem 7.3. of [FV12].
Lemma 17. Let $G = (V, E)$ be a graph, and $\beta \in [0, \frac{1}{3}]$. There exists an optimal tree decomposition $(X, T)$ of $G$ so that at least one of the following holds:

(a) there exists a bag $\Omega \in X$ such that $\Omega$ is a potential maximum clique and there exists a connected component $C$ of $G[V - \Omega]$ such that $|C| \leq \beta n$;

(b) there exists a bag $S \in X$ such that $S$ is a minimal separator and there exist two disjoint connected components $C_1, C_2$ of $G[V - S]$ such that $N(C_1) = N(C_2) = S$ and $|C_2| \geq |C_1| \geq \beta n$.

The idea of the algorithm then is to try out all possible potential maximum cliques and minimal separators that conform to the conditions of this lemma, and for each of these sets, to find an optimal tree decomposition of $G$ given that the examined set is a bag of the decomposition using the algorithm from Theorem 11. The treewidth of $G$ then is the minimum width of all examined decompositions.

The potential maximum clique generation is based on the following lemma.

Lemma 18 (Lemma 7.1. in [FV12]). Let $G = (V, E)$ be a graph. The number of maximum potential cliques $\Omega$ of size $p$ such that there exists a connected component $C$ of $G[V - \Omega]$ of size $c$ is at most $n^{\binom{n-c}{p-1}}$. The set of all these cliques can also be generated in time $O^*(\binom{n-c}{p-1})$.

For the minimal separators, suppose that the size of $S$ is fixed, denote it by $s$. Note that since $C_1$ in Lemma 17 is a connected component such that $N(C_1) = S$, then instead of generating minimal separators, we can generate the sets of vertices $C$ with neighborhood size equal to $s$. The set of sets generated in this way contains all of the minimal separators of size $s$ that we are interested in, and for those sets that are not, the fixed-bag treewidth algorithm will still find some tree decomposition of the graph, albeit not an optimal one. The generation is done using Lemma 3: for a fixed size $c$ of $C$, the number of such $C$ with exactly $s$ neighbors is at most $n^{\binom{c+s}{c}}$ (the factor of $n$ comes from trying each of $n$ vertices as the fixed vertex $v \in B$). The algorithm generating all such $C$ requires time $O^*(\binom{c+s}{c})$. For a set $C$, we then find an optimal tree decomposition of $G$ containing $N(C)$ as a fixed bag using the algorithm from Theorem 11. In this way we work through all $c$ from $\beta n$ to $n - s - |C_2| \leq (1 - \beta)n - s$.

Algorithm 1 The polynomial space algorithm for treewidth.

1. For $c$ from 0 to $\beta n$ and $p$ from 1 to $n - c$ generate the set of potential maximal cliques $\Omega$ of size $p$ with a connected component of size $c$ using Lemma 18. For each of these $\Omega$, find $\text{tw}(G, \Omega)$ using Theorem 11.

2. For $s$ from 1 to $(1 - 2\beta)n$ and for $c$ from $\beta n$ to $(1 - \beta)n - s$ generate the set of subsets $C$ such that $|C| = c$ and $N(C) = s$ using Lemma 3. Let $S = N(C)$; then $\text{tw}(G, S)$ is equal to the maximum of $\text{tw}(G[S \cup C], S)$ and $\text{tw}(G[V - C], S)$ by Lemma 9. Use the algorithm from Theorem 11 to compute these values.

3. Output the minimum width of all examined tree decompositions.

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The original lemma gives an upper bound if the size $\Omega$ is not fixed, but our statement follows from their proof. We need to fix $|\Omega|$ because in the quantum algorithms, Grover's search will be called for fixed $|\Omega|$ and $|C|$.
5.1 Classical complexity

**Theorem 19** (Theorem 7.3. in [FV12]). Algorithm 1 computes the treewidth of a graph with \( n \) vertices in \( O^*(2.61508^n) \) time and polynomial space.

**Proof.** The algorithms from Lemma 3 and Theorem 11 both require polynomial space, hence it holds also for Algorithm 1.

Now we analyze the time complexity; Stage 1 of the algorithm requires time

\[
O^* \left( \sum_{c=0}^{\beta n} \sum_{p=1}^{n-c} \binom{n-c}{p-1} 4^{n-p} \right) = O^* \left( \sum_{c=0}^{\beta n} 2^{n-c} 4^c \right) = O^* \left( \frac{\beta n}{\max_{c=0} 2^n+c} \right) = O^* \left( 2^{(1+\beta)n} \right). \tag{1}
\]

Stage 2 of the algorithm requires time

\[
O^* \left( \sum_{s=1}^{(1-2\beta)n} \sum_{c=\beta n}^{(1-\beta)n-s} \binom{c+s}{c} 4^{n-c-s} \right)
\]

Note that we can assume that \( C = C_1 \) and \( V - C - S \) contains \( C_2 \) (we can check this in polynomial time by finding the connected components of \( G[V - S] \)); since \( |C_2| \geq |C_1| \), we can assume that \( n-c-s \geq c \). Hence the complexity becomes

\[
O^* \left( \sum_{s=1}^{(1-2\beta)n} \sum_{c=\beta n}^{(1-\beta)n-s} \binom{c+s}{c} 4^{n-c-s} \right) = O^* \left( \frac{(1-2\beta)n}{\max_{s=1} 4^n} \sum_{c=\beta n}^{(1-\beta)n-s} \binom{c+s}{c} 4^{n-c-s} \right).
\]

Now denote \( d = n - c - s \), then \( c + s = n - d \) and we can rewrite the complexity as

\[
O^* \left( \frac{(1-\beta)n}{\max_{d=\beta n} 4^n} \max_{c=\beta n} \binom{n-d}{c} 4^d \right).
\]

For any \( d \), the maximum of \( \binom{n-d}{c} \) over \( c \geq \beta n \) can be one of two cases: if \( \beta n \leq \frac{n-d}{2} \), it is equal to \( \Theta^*(2^{n-d}) \); otherwise it is equal to \( \binom{n-d}{\beta n} \). In the first case, for the interval \( c \in [\beta n, \frac{n-d}{2}] \), the function being maximized becomes \( 2^{n-d} 4^d = 2^{n+d} \). Since this function is increasing in \( d \), its maximum is covered by the second case with the smallest \( c \) such that \( c = \frac{n-d}{2} \) (in case \( \beta n \leq \frac{n-d}{2} \)). Therefore, the complexity of Stage 2 of the algorithm becomes

\[
O^* \left( \frac{(1-\beta)n}{\max_{d=\beta n} 2^n} \binom{n-d}{\beta n} 4^d \right). \tag{2}
\]

Now we are searching for the optimal \( \beta \in [0, \frac{1}{2}] \) that balances the complexities (1) and (2). We solve it numerically and obtain \( \beta \approx 0.38685 \), giving complexity \( O^*(2.61508^n) \).

\( \square \)

5.1.1 A time-space tradeoff

One might ask whether replacing the \( O^*(4^n) \) divide & conquer algorithm from Theorem 11 with the \( O^*(2^n) \) dynamic programming algorithm from Theorem 14 in Algorithm 1 can give any interesting complexity. Indeed, we can show the following previously unexamined classical time-space tradeoff.
Theorem 20. The treewidth of a graph with \( n \) vertices can be computed in \( O^*(2^n) \) time and \( O^*(\sqrt{2^n}) \) space.

Proof. First, we look at the time complexity. The time complexity of Stage 1 now is equal to

\[
O^*\left(\sum_{c=0}^{\beta n} 2^{n-c}2^c\right) = O^*(2^n).
\]

The time complexity of Stage 2 is equal to

\[
O^*\left((1-\beta)^n \max_{d=\beta n} \left(\frac{n-d}{\beta n}\right)2^d\right) = O^*(2^{n-d}2^d) = O^*(2^n).
\]

The space complexity of Stage 1 is equal to

\[
O^*\left(\max_{c=0}^{\beta n} 2^c\right) = O^*(2^{\beta n}).
\]

The space complexity of Stage 2 is equal to

\[
O^*\left((1-\beta)^n \max_{d=\beta n} 2^d\right) = O^*(2^{(1-\beta)n}).
\]

Therefore, the time complexity of this algorithm is \( O^*(2^n) \) and, taking \( \beta = \frac{1}{2} \), the space complexity is equal to \( O^*(\sqrt{2^n}) \).

We can compare this to the existing treewidth algorithms. The most time-efficient treewidth algorithm runs in time and space \( O^*(1.755^n) \) [FV12], which is more than \( O^*(\sqrt{2^n}) \). The polynomial space \( O^*(2.616^n) \) algorithm, of course, is slower than \( O^*(2^n) \). The time-space tradeoffs for permutation problems from [KP10] give

\[
TS \gtrsim 3.93,
\]

where \( T \geq 2 \) and \( \sqrt{2} \leq S \leq 2 \) are the time and space complexities (bases of the exponent \( n \)) of the algorithm. In this case, \( TS = 2^{\frac{3}{2}} \approx 2.83 \), \( T = 2 \) and \( S = \sqrt{2} \). Therefore, Theorem 20 fully subsumes their tradeoff for treewidth. We also note that we cannot “tune” our tradeoff directly for less time and more space, since the first stage with \( c = 0 \) already requires \( \Theta^*(2^n) \) time for any \( \beta \).

5.2 Quantum complexity

Now we are ready to examine the quantum versions of the algorithm. First, we consider the analogue of Algorithm 1 with its procedures sped up quadratically using Grover’s search.

Theorem 21. There is a bounded-error quantum algorithm that finds the treewidth of a graph on \( n \) vertices in \( O(1.61713^n) \) time and polynomial space.

Proof. In Algorithm 1, we replace the algorithms from Lemmas 3 and 18 with the quantum algorithm from Lemma 5; the algorithm from Theorem 11 is replaced with the algorithm from Theorem 12. Since all exponential subprocedures now are sped up quadratically, the time complexity becomes

\[
O\left(\sqrt{2.61508^n}\right) = O(1.61713^n).
\]

The space complexity is still polynomial, since Grover’s search additionally uses only polynomial space.
Similarly, we can replace the algorithm from Theorem 14 with the quantum dynamic programming algorithm from Theorem 15:

**Theorem 22.** Assuming the QRAM data structure, there is a bounded-error quantum algorithm that finds the treewidth of a graph on \( n \) vertices in \( O(1.55374^n) \) time and \( O(1.45195^n) \) space.

**Proof.** The time complexity of the first stage is now equal to

\[
O^* \left( \sum_{c=0}^{\beta n} \sqrt{2^n - c} \cdot 1.81691^c \right) = O^* \left( \sqrt{2^n} \cdot 1.28475\beta n \right).
\]

For the second stage, the time is given by

\[
O^* \left( \frac{(1-\beta)n}{d=\beta n} \sqrt{\frac{n-d}{\beta n}} \cdot 1.81691^d \right).
\]

We can numerically find that \( \beta \approx 0.3755 \) balances these complexities, which then are both \( O(1.55374^n) \). The space complexity is

\[
O^* \left( 1.81691^{\max(\beta n, (1-\beta)n)} \right) = O^* \left( 1.81691^{(1-\beta)n} \right) = O(1.45195^n). \]

6 Improved quantum algorithm

We can see that in Theorem 22 we still have some room for improvement by trading space for time. This can be done using an additional technique. The main idea is to make a global precalculation for \( \text{TW}_{G[S]} \) for all subsets \( S \subseteq V \) of size at most \( \alpha n \), for some constant parameter \( \alpha \). Then, as we will see later, these values can be used in all calls of the quantum dynamic programming because of the properties of treewidth. For many such calls, this reduces the \( O^*(1.817^d) \) running time to something smaller, which in turn reduces the overall time complexity.

6.1 Asymmetric quantum dynamic programming on the hypercube

We describe our modification to the quantum dynamic programming algorithm by Ambainis et al. [Amb+19]. First, we prove the following lemma that allows us to reutilize the precalculated DP values on the original graph \( G \) in the DP calculation in the subgraphs examined by our algorithms.

**Lemma 23.** Let \( G = (V, E) \) be a graph, and \( \chi \subseteq V \) a subset of its vertices. Suppose that \( C \) is a union of some connected components of \( G[V-\chi] \). Then for any \( S \subseteq C \), we have \( \text{TW}_{G[C\cup\chi]}(S) = \text{TW}_G(S) \).

**Proof.** Examine the permutations \( \pi \) achieving \( \text{TW}_{G[C\cup\chi]}(S) = \min_{\pi \in \Pi(C\cup\chi)} \max_{v \in S} Q_G(\pi_{<v}, v) \).

As a direct consequence of Lemma 13, there exists such a permutation \( \pi \) with the property that \( S \) is its prefix. Now let \( \pi' \in \Pi(V) \) be a permutation obtained by adding the vertices of \( V-C-\chi \) at the end of \( \pi \) in any order. Examine any vertex \( u \in V-C-\chi \) and any \( v \in S \). Since \( u \) and \( v \) are located in different connected components of \( G[C-\chi] \), any path from \( u \) to \( v \) in \( G \) passes through some vertex of \( \chi \). However, \( \pi_{<v} \cap \chi = \emptyset \), as \( \pi_{<v} \subseteq S \). Then we can conclude that \( Q_G[C\cup\chi](\pi'_{<v}, v) = Q_G(\pi'_{<v}, v) \), as \( u \) cannot contribute to \( Q \). Therefore, \( \text{TW}_G(S) \leq \text{TW}_{G[C\cup\chi]}(S) \).

On the other hand, \( \text{TW}_G(S) \geq \text{TW}_{G[C\cup\chi]}(S) \), as additional vertices cannot decrease \( \text{TW} \).
We call these collections layers \( TWR \) weight, see Figure 1. For all sets \( S \) with \( \lambda \) values of \( TW \) where \( \alpha \) values of \( TW \) path from \( v \) to \( s \) the vertex \( s \) search independently for the best path from the hypercube. First, we use Grover’s search over the vertex \( v \) to \( s \) the optimal path between the vertices \( v \) is some fixed parameter. Suppose that we have fixed a subset \( \chi \subseteq V \), and our task is to calculate \( tw(G \cup \chi, \chi) \) for a union \( C \) of some connected components of \( G[V - \chi] \). By Lemma 8, it is equal to \( \max(TW_{G[U \cup \chi]}(C), |\chi| - 1) \). Since \( |\chi| \) is known, our goal is to compute \( TW_{G[U \cup \chi]}(C) \).

Let \( n = |V| \) and \( n' = |C| \). If \( n' \leq \alpha n \), then \( TW_{G[U \cup \chi]}(C) = TW_G(C) \) by Lemma 23 and is known from the precalculated values. Hence, assume that \( n' > \alpha n \). Pick some natural \( k \), we will call this the number of layers. Let \( \lambda_1 = \frac{\alpha n}{n'} \), and pick constants \( \lambda_2 < \ldots < \lambda_k \) such that \( \lambda_k < \mu < \rho_k < \ldots < \rho_1 < 1 \), with \( \lambda_1 < \lambda_2 \). Then define collections

\[
L_i = \{ S \subseteq C \mid |S| = \lambda_i n' \}, \\
M = \{ S \subseteq C \mid |S| = \mu n' \}, \\
R_i = \{ S \subseteq C \mid |S| = \rho_i n' \}.
\]

We call these collections layers: we can represent subsets \( S \subseteq C \) as vertices on the hypercube of dimension \( n' \); then these layers are defined as the subsets of vertices with some fixed Hamming weight, see Figure 1. For all sets \( S \) corresponding to the vertices in the crosshatched area (such that \( |S| \leq \alpha n \), the value of \( TW_{G[U \cup \chi]}(S) = TW_G(S) \) is known from the assumed precalculation.

![Figure 1: A schematic representation of layers in the Boolean hypercube with \( k = 3 \).](image)

Now we are ready to describe our quantum dynamic programming procedure. Suppose that all values of \( TW_G(S) \) for sets with \( |S| \leq \alpha n \) have been precalculated beforehand and stored in QRAM, where \( \alpha \in \left[0, \frac{1}{2}\right] \) is some fixed parameter. Suppose that we have fixed a subset \( \chi \subseteq V \), and our task is to calculate \( tw(G \cup \chi, \chi) \) for a union \( C \) of some connected components of \( G[V - \chi] \). By Lemma 8, it is equal to \( \max(TW_G(C), |\chi| - 1) \). Since \( |\chi| \) is known, our goal is to compute \( TW_{G[U \cup \chi]}(C) \).

We call these collections layers: we can represent subsets \( S \subseteq C \) as vertices on the hypercube of dimension \( n' \); then these layers are defined as the subsets of vertices with some fixed Hamming weight, see Figure 1. For all sets \( S \) corresponding to the vertices in the crosshatched area (such that \( |S| \leq \alpha n \), the value of \( TW_{G[U \cup \chi]}(S) = TW_G(S) \) is known from the assumed precalculation.

Now we will describe the quantum procedure. Denote \( C' = G[U \cup \chi] \). Also denote \( TW_{G'}(S) = TWR_{G'}(S, C - S) \) and note that \( TW_{G'}(S) = TWR_{G'}(\emptyset, S) \). Informally, calculating \( TW_{G'}(S) \) means finding the best ordering for the vertices \( S \) as a prefix of the permutation, and \( TW_{G'}(S) \) means finding the best ordering for the vertices \( C - S \), where \( C - S \) is in the middle of permutation, followed by some ordering of \( \chi \).

Algorithm 2 is exactly the algorithm of [Amb+19], with the exception that the precalculation is performed only for suffixes (and the precalculation for prefixes comes “for free”). The idea is to find the optimal path between the vertices \( s \) and \( t \) with the smallest and highest Hamming weight in the hypercube. First, we use Grover’s search over the vertex \( v_{k+1} \) in the middle layer \( M \). Then we search independently for the best path from \( s \) to \( v_{k+1} \) and from \( v_{k+1} \) to \( t \); the optimal path from \( s \) to \( t \) is their concatenation. To find the best path from \( s \) to \( v_{k+1} \), we use Grover’s search over the vertex \( v_k \) on the layer \( L_k \) such that there exists a path from \( v_k \) to \( v_{k+1} \). Then we find the best path from \( v_k \) to \( v_{k+1} \) by recursively using the \( O^*(1.817^{n'}) \) algorithm (where \( n' \) is the dimension of
the hypercube with $v_k$ and $v_{k+1}$ being the smallest and largest weight vertices, respectively. We combine it with the best path from $s$ to $v_k$, which we find in the same way (fixing $v_{k-1}, \ldots, v_1$). The value of the optimal path from $s$ to $v_1$ is known from the global precalculation we assumed took place before the algorithm. The optimal path from $v_{k+1}$ to $t$ is found analogously; only to know the value of the best path from vertices in $R_1$ to $t$, we have to precalculate these values “from the back” using Bellman & Held-Karp dynamic programming in the beginning of the algorithm. The formal description of this algorithm for treewidth is given in Algorithm 2.

Algorithm 2 Asymmetric quantum dynamic programming algorithm.

1. For all $S \in R_1$, calculate and store in QRAM the values $TW'_{G'}(S)$ using the recurrence

$$TW'_{G'}(S) = \min_{v \in C - S} \max(TW'_{G'}(S \cup \{v\}), |Q_{G'}(S, v)|).$$

This follows from Lemma 10 with $k = 1$.

2. Use quantum minimum finding over sets $S \in \mathcal{M}$ to find the answer,

$$TW_{G'}(C) = \min_{S \in \mathcal{M}} \max(TW_{G'}(S), TW'_{G'}(S)).$$

This also follows from Lemma 10 with $k = \mu n'$.

- To find $TW_{G'}(S)$, we use the recursive procedure $\text{BestPrefix}_i(G', S)$. Its value is equal to $TW_{G'}(S)$, and it requires $S \in \mathcal{L}_i$ (if $i = k + 1$, then $S \in \mathcal{M}$). The needed value is then given by $\text{BestPrefix}_{k+1}(G', S)$. The description of $\text{BestPrefix}_i(G', S)$:
  - If $i = 1$, return $TW_{G'}(S) = TW_G(S)$ that is stored in QRAM.
  - If $1 < i \leq k + 1$, then use quantum minimum finding over the sets $T \in \mathcal{L}_{i-1}$ to find

$$TW_{G'}(S) = \min_{T \in \mathcal{L}_{i-1}} \max(\text{BestPrefix}_{i-1}(G', T), \text{TWR}_{G'}(T, S - T)).$$

Again, this recurrence follows from Lemma 10 with $k = \lambda_i - 1 n'$. The value of $\text{TWR}_{G'}(T, S - T)$ is calculated by the quantum dynamic programming from Theorem 16 and requires $O^*(1.817^{|S| - |T|})$ time and QRAM space.

- To find $TW'_{G'}(S)$, we similarly use the recursive procedure $\text{BestSuffix}_i(G', S)$. Its value is equal to $TW'_{G'}(S)$, and it requires $S \in \mathcal{R}_i$ (if $i = k + 1$, then $S \in \mathcal{M}$). The needed value is then given by $\text{BestSuffix}_{k+1}(G', S)$. The description of $\text{BestSuffix}_i(G', S)$:
  - If $i = 1$, return $TW'_{G'}(S)$ stored in QRAM from the precalculation in Step 1.
  - If $1 < i \leq k + 1$, then use quantum minimum finding over the sets $T \in \mathcal{R}_{i-1}$ to find

$$TW'_{G'}(S) = \min_{T \in \mathcal{R}_{i-1}} \max(\text{TWR}_{G'}(S, T - S), \text{BestSuffix}_{i-1}(G', T)).$$

Again, this recurrence follows from Lemma 10 with $k = \rho_i - 1 n' - \rho_i n'$. The value of $\text{TWR}_{G'}(S, T - S)$ is calculated by the quantum dynamic programming from Theorem 16 and requires $O^*(1.817^{|T| - |S|})$ time and QRAM space.
Finally, note that with $\lambda_1 \approx 0.28448$, the time complexity of Algorithm 2 becomes $O^*(1.817^{n'})$, as this is the same parameter for the precalculation layer as in [Amb+19]. Thus if it happens that $\alpha n < 0.28448n'$, the asymmetric version of the algorithm will have time complexity larger than $O^*(1.817^{n'})$, so in that case it is better to call the algorithm from Theorem 15. Therefore, our procedure for calculating $\text{TW}_{G[C]}(C)$ is as follows:

**Algorithm 3**  
Quantum algorithm calculating $\text{tw}(G[C \cup \chi], \chi)$ assuming global precalculation.  

Assume that $\text{TW}_G(S)$ are stored in QRAM for all $|S| \leq \alpha n$.

- If $n' \leq \alpha n$, fetch $\text{TW}_{G[C \cup \chi]}(C) = \text{TW}_G(C)$ from the global precalculation.
- Else if $\alpha n \leq 0.28448n'$, find $\text{TW}_{G[C \cup \chi]}(C)$ using the $O^*(1.817^{n'})$ algorithm of Theorem 15.
- Else calculate $\text{TW}_{G[C \cup \chi]}(C)$ using Algorithm 2.

Return $\text{tw}(G[C \cup \chi], \chi) = \max(\text{TW}_{G[C \cup \chi]}(C), |\chi| - 1)$.

### 6.2 Complexity of the quantum dynamic programming

We will estimate the time complexity of Algorithm 2. The space complexity will not be necessary, because for the final treewidth algorithm it will be dominated by the global precalculation, as we will see later.

- The time of the precalculation Step 1 is dominated by the size of the layer $R_1$. It is equal to $O^*(|R_1|) = O^*(\left(\frac{n'}{\rho_1 n'}\right))$, which by Lemma 1 is
  
  $$O^*(2^{H(\rho_1)n'}).$$

- Let the time of a call of $\text{BestPrefix}_i(G', S)$ be $T_i$, it can be calculated as follows. If $i = 1$,
  
  $$T_1 = O^*(1),$$

  as all we need to do is to fetch the corresponding value $\text{TW}_G(S)$ from QRAM. If $i > 1$, then quantum minimum finding examines all $T \in L_{i-1}$ such that $T \subset S$. The number of such $T$ is $\left(|S|\right) = \left(\frac{\lambda_i n'}{\lambda_{i-1} n'}\right)$ (for generality, denote $\lambda_{k+1} = \mu$). Again, by Lemma 1, this is at most $2^{H(\lambda_{i-1}/\lambda_i)\cdot \lambda_i n'}$. The call to $\text{BestPrefix}_{i-1}(G', T)$ requires time $T_{i-1}$ and calculating $\text{TWR}_{G'}(T, S - T)$ with the algorithm from Theorem 16 requires time $O^*(1.817^{|S| - |T|}) = O^*(1.817^{\lambda_i - \lambda_{i-1})n'})$. Putting these estimates together, we get that for $i > 1$,
  
  $$T_i = O^*\left(\sqrt{2}^{H(\frac{\lambda_{i-1}}{\lambda_i})\cdot \lambda_i n'} \cdot \max(T_{i-1}, 1.817^{\lambda_i - \lambda_{i-1})n'})\right).$$

- The time $T'_i$ for $\text{BestSuffix}_i(G', S)$ is calculated analogously. We can check the precalculated values from Step 1 in
  
  $$T'_1 = O^*(1)$$

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and (taking $\rho_{k+1} = \mu$) for $i > 1$,

$$T'_i = O^*\left(\sqrt{2^H(\frac{1-\rho_{i-1}}{1-\rho_{i}})(1-\rho_{i})^{n'}} \cdot \max\left(T'_{i-1}, 1.817^{(\rho_{i-1}-\rho_{i})^{n'}}\right)\right).$$

- Lastly, the number of sets examined in the first quantum minimum finding in Step 2 is equal to the size of $\mathcal{M}$, which is $\binom{n'}{\mu n'} = 2^{H(\mu n')} \cdot \max(T'_{k+1}, T'_{k+1})$.

For any of the complexities $T$ examined here, let’s look at $\log_2(T)/n'$; since we are interested in the exponential complexity, we need to investigate only the constant $c$ in $O^*(2^{cn'})$. Also note that $\log_2(1.817) \approx 0.862$. This results in the following optimization program

$$\begin{align*}
\text{minimize} & \quad T(\lambda_1) = \max\left(\frac{1}{2}H(\lambda_1), \frac{1}{2}H(\mu) + \max(t_{k+1}, t'_{k+1})\right) \\
\text{subject to} & \quad \lambda_1 < \ldots < \lambda_k = \mu = \rho_{k+1} < \rho_k < \ldots < \rho_1 < 1 \\
& \quad t_i = \frac{1}{2}H\left(\frac{\lambda_{i-1}}{\lambda_i}\right) \cdot \lambda_i + \max(t_{i-1}, 0.862(\lambda_i - \lambda_{i-1})) \quad \text{for } i \in [2, k+1] \\
& \quad t_1 = 0 \\
& \quad t'_{i} = \frac{1}{2}H\left(\frac{1-\rho_{i-1}}{1-\rho_{i}}\right) \cdot (1-\rho_{i}) + \max(t'_{i-1}, 0.862(\rho_{i-1} - \rho_{i})) \quad \text{for } i \in [2, k+1] \\
& \quad t'_{1} = 0
\end{align*}$$

We can solve this program numerically and find the time complexity, depending on the value of $\lambda_1$. Note that for $\lambda_1 \leq 0.28448$ the $O(1.817^{n'})$ symmetric quantum dynamic programming is more efficient, so we don’t have to calculate the complexity in that case. Figure 2 shows the time complexity $T(\lambda_1)$ for $k = 0, 1, 2, 3$. We can see that the advantage of adding additional layers quickly becomes negligible.

Figure 2: Running time of the asymmetric quantum dynamic programming algorithm.
6.3 Final quantum algorithm

Now we can give the improved quantum dynamic programming algorithm for treewidth. It requires two constant parameters: \( \alpha, \beta \in [0, \frac{1}{2}] \). The value \( \alpha n \) gives the limit for the global precalculation, and \( \beta n \) is the cutoff point for the two stages as in Algorithm 1.

\[
\textbf{Algorithm 4 Improved quantum algorithm for treewidth.}
\]

1. Calculate \( \text{TW}_G(S) \) for all subsets \( S \) such that \( |S| \leq \alpha n \) and store them in QRAM.

2. For \( c \) from 0 to \( \beta n \) and \( p \) from 1 to \( n - c \) examine the set of potential maximal cliques \( \Omega \) of size \( p \) with a connected component of size \( c \). Apply Lemma 5 to Lemma 18 to find the minimum of \( \text{tw}(G, \Omega) \) in \( O^*(\sqrt{(n-c)/p}) \) iterations. Calculate the value of \( \text{tw}(G, \Omega) \) using Algorithm 3.

3. For \( s \) from 1 to \( (1-2\beta)n \) and from \( c \) from \( \beta n \) to \( (1-\beta)n - s \) examine the set of subsets \( C \) such that \( |C| = c \) and \( N(C) = s \). Let \( S = N(C) \); then \( \text{tw}(G, S) \) is equal to the maximum of \( \text{tw}(G[S \cup C], S) \) and \( \text{tw}(G[V - C], S) \) by Lemma 9. Find the minimum of \( \text{tw}(G, S) \) using Lemma 5 in \( O^*(\sqrt{(c+s)/s}) \) iterations. Calculate the values of \( \text{tw}(G[S \cup C], S) \) and \( \text{tw}(G[V - C], S) \) using Algorithm 3.

4. Return the minimum width of all examined tree decompositions.

We can now calculate the complexity similarly as in Theorem 22.

\textbf{Theorem 24.} Assuming the QRAM data structure, there is a bounded-error quantum algorithm that finds the exact treewidth of a graph on \( n \) vertices in \( O(1.53793^n) \) time and space.

\textbf{Proof.} First, we choose \( \alpha \) such that it balances the time complexity of the global precalculation (Step 1) and the rest of the algorithm (Steps 2–4). The space complexity of this step asymptotically is equal to its time complexity. Therefore, the space complexity of this algorithm is equal to \( O^*(\binom{n}{\alpha n}) \).

Denote the time complexity of Algorithm 3 with \( |C| = n' \) and some chosen \( \lambda_1 \) by \( O^*(T(\lambda_1)^n) \). For fixed \( \alpha \) and \( n' \), \( \lambda_1 \) is calculated as \( \alpha n/n' \). Then

\[
T(\lambda_1) = \begin{cases} 
1, & \text{if } \lambda_1 \geq 1, \\
1.81691, & \text{if } \lambda_1 \leq 0.28448, \\
T(\lambda_1), & \text{otherwise}.
\end{cases}
\]

Similarly as we have obtained Equations (1, 2) in the proof of Theorem 19, we can also calculate the time complexity here. The time complexity of Step 2 now is equal to

\[
O^* \left( \sum_{c=0}^{\beta n} \sqrt{2^{n-c} \cdot T \left( \frac{\alpha n}{c} \right)^c} \right).
\]

The running time of Step 3 is given by

\[
O^* \left( \max_{d=\beta n} \left( \frac{(1-\beta)n}{\beta n} \right)^{\frac{d}{\beta n}} \cdot T \left( \frac{\alpha n}{d} \right)^d \right).
\]
We can numerically find that $\alpha \approx 0.15447$ and $\beta \approx 0.38640$ balance these complexities, which then are all $O(1.53793^n)$. In our numerical calculation, we have used $k = 3$ for Algorithm 3.

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