All Conformal Effective String Theories are Isospectral to Nambu-Goto Theory

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It is shown that all Polchinski-Strominger effective string theories are isospectral to Nambu-Goto theory. The relevance of these results to QCD-Strings is discussed.

I. INTRODUCTION

String-like defects or solitons occur in a wide variety of physical systems. Some well-known examples are vortices in superfluids, the Nielsen-Olesen vortices of quantum field theories, vortices in Bose-Einstein condensates and QCD-strings (for reviews see [1]). Under suitable conditions these objects can behave quantum-mechanically. The challenge then is to find consistent quantum mechanical descriptions in arbitrary dimensions. It should be recalled that fundamental string theories are consistent only in the so called critical dimensions. Polyakov [2] gave formulations of string theories in sub-critical dimensions; his ideas play a central role in what follows. A pragmatic approach to such a quantum description would be to treat these objects in some effective manner much as interactions of pions are so successfully described in terms of chiral effective field theories without any pretenses about them being fundamental at all scales.

Two such approaches to effective string theories exist in the literature. One due to Lüscher and collaborators [3], is entirely in terms of the $D-2$ transverse physical degrees of freedom. It is a case where the gauge is fixed completely without any residual invariance left. The other approach is the one pioneered by Polchinski and Strominger [4] where the theories are invariant under conformal transformations and the physical states are obtained by requiring that the generators of conformal transformations annihilate them. These too are gauge-fixed theories but with leftover residual invariances characterized by conformal transformations. The physical basis of both approaches is that the physical degrees of freedom are transverse. In the light of the results obtained in this paper, it has become important to reexamine this physical basis.

Lüscher was the first to show [3] that the leading correction to ground state energy of a (closed) string of length $2\pi R$ takes the form $V(R) = \sigma R - \frac{(D-2)^2 \pi}{24}$. He drew attention to the fact that the leading correction, subsequently named the Lüscher term, is universal depending only on space-time dimensionality. Polchinski and Strominger [4] showed, through explicit construction of an approximately conformally invariant action for effective string theories that not only can string-like defects be quantized in arbitrary dimensions but also that the leading correction is the Lüscher term. Very recently Lüscher and Weisz, using their path-breaking multilevel algorithm, showed clear numerical evidence for this term [10] in Lattice QCD. Subsequent large scale numerical simulations of the string-like behaviour of Yang-Mills flux tubes by Hari Dass and Majumdar [5] pointed to the strong possibility that even the subleading $R^{-3}$ terms in $D = 3$ and $D = 4$ were universal and what is more, coincided with similar terms of Nambu-Goto theory [2].

This rather unexpected result was analytically explained by Drummond and, by Hari Dass and Matlock [6, 7]. Focus then shifted to finding ways of understanding even higher order corrections, and possibly an analysis to all orders. The first result we obtained in this direction was the proof that the action that Polchinski and Strominger used in [4] extended to be exactly conformally invariant to all orders in $11$, has the same spectrum as the Nambu-Goto theory [13] to all orders. We had called this extended action by the name Polyakov-Liouville action. Drummond [8] had shown that the next level at which candidate actions could be found was only at $R^{-6}$. However, he did not identify the conformal transformations leaving invariant his actions, four in number, which we have called Drummond Actions. With the help of our covariant formalism we had shown that only two of these are linearly independent and we had identified their invariance transformation laws [11]. In the next all order result we had shown that effective string actions with these Drummond terms also do not change the spectrum! [14]. In this paper we extend our proof to all classes of Polchinski-Strominger effective string theories to all orders.

II. ACTIONS

The total action for effective string theories has the form

$$S = S_0 + S_\beta + \sum_j S^j_{\text{cov}} \quad (1)$$

where

$$4\pi a^2 S_0 = \int d\tau^+ d\tau^- L = \int d^2\tau \partial_+ X^\nu \partial_- X_\nu \quad (2)$$

is the action for the free bosonic string theory and is consistent quantum mechanically only in $D = 26$ space-
time dimensions. $S_\beta$ is the Polyakov-Liouville action

$$S_\beta = \frac{\beta}{4\pi} \int d^2\tau \left\{ \frac{\partial_+ (\partial_+ X \cdot \partial_+ X)}{(\partial_+ X \cdot \partial_+ X)^2} + \delta^0 (\tau^\pm) \partial_\pm X^\mu \right\}. \quad (3)$$

$S_\beta$ is exactly, in the sense to all orders in $R^{-1}$, invariant under the conformal transformations

$$\delta^0 X^\mu = \epsilon^\pm (\tau^\pm) \partial_\pm X^\mu. \quad (4)$$

under which the leading action $S_0$ is also exactly invariant. The action and transformations laws originally used in [4] are related by a field redefinition [8]. The PS-action, equivalently $S_\beta$, is what yields quantum consistency in all dimensions. The so called manifestly covariant actions $S_{\text{cov}}$ are such that their integrands transform as scalar densities. They can be constructed systematically with the help of the Covariant Calculus developed in [11], and are therefore exactly invariant under eqn. (4). An important difference from the algorithm given in [4] and elaborated in [8, 9] is that terms proportional to EOM can no longer be dropped as that would entail field redefinitions which could change eqn. (4). In the next section we describe some essentials.

III. COVARIANT CALCULUS

A. General Considerations

We start with the general form of manifestly covariant action terms, more specifically, terms that transform as scalar densities. A systematic procedure for construction of such terms to any desired order in $1/R$ is given in [11].

$$I_{\text{cov}} = \sqrt{\Omega} D_{\alpha_1 \beta_1} \ldots X^{\mu_1} D_{\alpha_2 \beta_2} \ldots X^{\mu_2} \cdot A^{\alpha_1 \beta_1 \ldots \alpha_2 \beta_2 \ldots \mu_1 \mu_2 \ldots} \quad (5)$$

where $A^{\alpha_1 \beta_1 \ldots \alpha_2 \beta_2 \ldots}$ is composed of suitable factors of Levi-Civita and metric tensors on the two-dimensional world sheet and $B_{\mu_1 \mu_2}$ made up of $\eta_{\mu \nu}$ and Levi-Civita tensors in target space. In the spirit of the PS-construction, the covariant calculus is constructed based on the induced metric on the world-sheet given by $g_{\alpha \beta} = \partial_\alpha X \cdot \partial_\beta X$. In the conformal gauge, $g_{++} = g_{--} = 0$, this construction amounts to stringing together a number of covariant derivatives so that there are equal net numbers of $(+, -)$ indices, and use sufficient inverse powers of $g_{++} = L = \partial_+ X \cdot \partial_+ X$ to make the expression transform as (1, 1). The residual transformations maintaining the conformal gauge result in the invariance of these actions under

$$\delta_\pm X^\mu = -\epsilon^\pm (\tau^\pm) \partial_\pm X^\mu \quad (6)$$

$L$ transforms as a $(1, 1)$-tensor and $L^{-1}$ as a $(-1, -1)$ tensor. The non-vanishing components of the Christoffel connection are:

$$\Gamma^{(1)}_{++} = \partial_+ \ln L \quad \Gamma^{(1)}_{--} = \partial_- \ln L \quad D_\pm L = 0 \quad (7)$$

The last of eqn. (7) is just the covariant constancy of the metric tensor. We give explicit expressions for some covariant derivatives of interest to this paper:

$$D_\pm X^\mu = \partial_\pm X^\mu$$
$$D_{++} X^\mu = \partial_{++} X^\mu - \partial_+ \ln L \partial_+ X^\mu$$
$$D_{--} X^\mu = \partial_{--} X^\mu - \partial_- \ln L \partial_- X^\mu$$
$$D_{+-} X^\mu = D_{-+} X^\mu = \partial_\pm X^\mu \quad (8)$$

If $T_{\mu_1 \ldots \mu_n}$ is a tensor with $m_\pm$ indices of type $\pm$,

$$D_\pm T_{\mu_1 \ldots \mu_n} = \partial_\pm T_{\mu_1 \ldots \mu_n} - m_\pm \partial_\pm \ln L T_{\mu_1 \ldots \mu_n} \quad (9)$$

Another important property is that $D_\pm X \cdot D_\pm X$ are linear combinations of the gauge fixing conditions $g_{\pm} = \partial_\pm X \cdot \partial_\pm X$ and their derivatives.

The manifestly covariant Drummond actions we had analysed in [14] are linear combinations of $S_1^D, S_2^D$ where

$$S_1^D = \int d\tau^+ d\tau^- \mathcal{M}_1^D \quad (10)$$

where

$$\mathcal{M}_1^D = \frac{(D_{++} X \cdot D_{--} X)^2}{L^2} \quad (11)$$
$$\mathcal{M}_2^D = \frac{(D_{++} X \cdot D_{++} X)(D_{--} X \cdot D_{--} X)}{L^3} \quad (12)$$

It was shown there that these actions are scalar densities under eqn. (11) and they have vanishing on-shell $T_{--}$. The main result of this paper is in extending these results to all $S_{\beta}$. The Polyakov action ($R(\xi)$ is the Ricci scalar)

$$S_{\text{Polya}} = \int d^2\xi \sqrt{|g(\xi)|} R(\xi)(\frac{1}{\sqrt{2}} R(\xi)) \quad (13)$$

This is the so-called WZNW effective action for conformal anomaly in two dimensions. In [11] we had given a general proof that for WZNW effective actions of eqn. (14).
the integrand can never be manifestly covariant. The basic reason is that the solution \( \phi(\xi) \) to
\[
\nabla^2 \phi(\xi) = R(\xi)
\]
is not a scalar even though \( R \) is and \( \nabla^2 \) is the scalar Laplacian. In fact \( \phi(\xi) = \ln L \). A related two-dimensional peculiarity is that the equation \( \partial_\alpha f(\xi) = V_\alpha \) where \( V_\alpha \) transforms as a vector, admits non-scalar solutions for \( f(\xi) \).

As happens in the theory of anomalous effective actions, after isolating \( S_\beta \) in eqn.(15), all other terms will be manifestly covariant. Because of this special nature of \( S_\beta \), the analysis of its spectral content has to use techniques other than what is presented in [13]. This has been done in [13].

IV. GENERAL ANALYSIS

Our analysis begins with a derivation of the full EOM and \( T_{-} \) corresponding to the total action \( S_{\text{tot}} \) of eqn.(13). These are given by
\[
E^\mu = \frac{\delta S}{\delta X^\mu} = 0 \quad \delta S = \frac{1}{2\pi} \int d^2 \partial_+ (\tau^-, \tau^+) T_{-}
\]
the second part of which is obtained by the usual Nöether procedure after treating \( \tau^- \) as depending on \( \tau^+ \). The following relation between the off-shell \( T_{-} \) and \( E^\mu \) is an important one:
\[
\partial_+ T_{-} = -2\pi E \cdot \partial_- X
\]
We now introduce the decomposition
\[
X^\mu = X^\mu_{\text{ih}} + F^\mu(\tau^+) + G^\mu(\tau^-) + H^\mu(\tau^+, \tau^-)
\]
where \( X^\mu_{\text{ih}} = e^\mu_+ R\tau^+ + e^\mu_- R\tau^- (e^2_+ = e^2_- = 0 \) and \( e^\mu_- = -i/2 \) \) is a classical solution of \( S_0 \); \( F, G \) are anti-holomorphic and holomorphic functions respectively. \( H \) is purely non-holomorphic and by construction it does not have any purely holomorphic and anti-holomorphic parts. The form of the full EOM given in eqn.(16) can be iteratively solved to give \( H \) in terms of \( F_+, G_- \) and their higher derivatives. The advantage of introducing the decomposition of eqn.(18) is that off-shell \( T_{-} \) of eqn.(16) can be uniquely split as
\[
T_{-} = T_{-}^h + T_{-}^{nh}
\]
where \( T_{-}^{nh} \) is purely non-holomorphic in the sense that it has no holomorphic parts. The on-shell \( T_{-} \) can be obtained by simply setting \( T_{-}^{nh} = 0 \) which follows as a consequence of eqn.(17). It is only \( T_{-}^h \) that is relevant for the spectrum of the theory.

A. Analysis of \( S_{\text{cov}}^i \)

It is easy to see that the only way to get a \( T_{-}^h \) is for the Lagrangean to be made up of two factors one of which, called \( L^h \) has holomorphic terms in it and another, called \( L^{hv} \) whose Nöether variation has holomorphic parts in it. We first enlist possible candidates for \( L^h \). To this end we note that all the covariant derivatives \( (D_-)^n X^\mu \) are such that they have holomorphic parts. Though it has a +-derivative, \( D_+ X = Re_+ \ldots \) actually has a holomorphic part. As can be seen from using eqn.(18) in eqn.(8), none of the higher covariant derivatives \( (D_+)^n X \) contains holomorphic pieces. It then follows that no tensor with two or more +-indices contains holomorphic pieces. Likewise \( D_- X \) and all its higher covariant derivatives also do not contain any holomorphic pieces. In these conclusions we have used the fact that \( L \) and \( \partial_- L \) contain holomorphic pieces but not \( \partial_+ L \).

To construct \( L^h \) one forms scalar (in target space) dot products among all the tensors containing holomorphic pieces; at first sight \( D_+ X \cdot (D_-)^n X \) appears to be one such. Consider
\[
(D_-)^{n-1} L = (D_-)^n X \cdot D_+ X + \ldots
\]
where the additional terms contain \( D_- X \) and its higher derivatives. Since the lhs vanishes due to \( D_- L = 0 \), and all additional terms are not holomorphic, one concludes that actually \( D_+ X \cdot (D_-)^n X \) does not contain holomorphic pieces. Thus \( L^h \) can contain \( (D_-)^n X \cdot (D_-)^m X \) and arbitrary products of such terms. Of these \( D_- X \cdot D_- X \) and \( D_- X \cdot D_- X \) should not be considered as they are proportional to \( g_- \) and its derivatives, which should vanish in the conformal gauge. This means that every term in \( L^h \) has four or more negative indices.

Constructing \( L^{hv} \) involves a little work. To that end the following Nöether variations are useful:
\[
\delta D_+ X^\mu = \partial_+ (\delta X^\mu) = \partial_+ \epsilon^\mu - \partial_- X^\mu + \epsilon^\mu \partial_+ X^\mu
\]
\[
\delta \ln L = \partial_+ \epsilon^- - \epsilon^\mu \partial_\mu L + \frac{\partial_+ \epsilon^-}{L} \partial_- X \cdot \partial_- X
\]
From this it follows that among all the higher rank covariant derivatives only \( D_+ X, D_+ X \), and their \( D_- \) derivatives, can have holomorphic Nöether variations. It is also clear that the integrand of \( S_{\text{cov}}^i \) can not contain more than one higher (than rank one in +) covariant derivative as a factor. The way to construct such integrands is to string together target space scalar products with equal number of \((+, -)\) indices and divide by sufficient numbers of \( L = D_+ X \cdot D_- X \) (it was for this reason we did not include \( L \) among \( L^h \)). But this would require an element of \( L^{hv} \) to have at least four \( \text{net} \) +-indices. Every scalar product must necessarily include either a \( D_+ X \) or \( (D_-)^n X \) factor. It is clearly optimal to choose a \( D_+ X \) factor. Thus the \( \text{single} \) higher rank covariant derivative factor must involve \( D_+ X \) or higher +-derivatives. But none of them can contain a holomorphic Nöether variation.

Thus we come to the startling conclusion that for each of the \( S_{\text{cov}}^i \) there is no holomorphic part to \( T_{-} \) to all orders! In other words they can not change the spectrum from what the \( S_0, S_\beta \) actions give. But we have already
shown in [13] that $S_\beta$ does not correct the spectrum of $S_0$ to all orders. Hence the conclusion of this paper, namely, all conformally invariant effective string theories are isospectral to the Nambu-Goto theory, $S_0$.

V. DISCUSSION AND CONCLUSIONS

In this paper we have shown that the entire class of conformally invariant Polchinski-Strominger effective string theories has the same spectrum as Nambu-Goto theory to all orders in $R^{-1}$, where $2\pi R$ is the length of the closed string. The proof consisted of two steps the first in demonstrating that $S_\beta$, though contributing non-trivially to the on-shell $T_{-\cir}$ nevertheless does not correct the Nambu-Goto spectrum [13]; next it is shown that no manifestly covariant action even contributes to the on-shell $T_{-\cir}$. An immediate comparison can be made with the results of Aharony and Karzbrun [12] who, following the Lüscher-Weisz approach, showed similar results to order $R^{-3}$ in $D = 3$ but claimed that for $D \geq 4$ there could be corrections. This discrepancy between our results based on the PS-formalism, and their results based on the LW-formalism needs to be understood. If both our calculations are correct, it may point to the possibility that conformal invariance in addition to ensuring only $D - 2$ transverse degrees of freedom, may also be restricting the interactions. It is very important to re-examine whether such features are justifiable from the point of view of the relevant microscopic theory, for example QCD. In other words, the symmetry content of effective string theories needs a fresh look. But such a fresh analysis has to retain the agreement between numerical studies and Nambu-Goto spectrum to order $R^{-3}$.

First principle derivation of the effective actions as done by Akhmedov et al could throw valuable light on these issues [13]. Numerical data [9] clearly shows a deviation from Nambu-Goto theory at intermediate distances. For $D = 4SU(3)$ QCD this was around 0.75fm. This is also the situation with numerical simulation of percolation models [16]. One possibility of reconciling this is if at these intermediate scales the string-like object has not formed at all. If not, one will have to conclude that conformally invariant effective string theories do not provide a good description. Our analysis does not seem to provide any room for extrinsic curvature string effects. Once again it should be stressed that the Polyakov action [17] for such strings does not have the conformal invariance used here. It is also very important to find out what additional physics is coded by the large class of highly non-trivial actions considered here. Studies of scattering amplitudes and Partition functions (as studied in [12]) may be needed for that. It is also desirable to get a deeper physical understanding of our results. It is clear that we are still a long way from understanding QCD-Strings.

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[1] G. S. Bali, Phys. Reports 343 2001, 1; J. Kuti, Lattice QCD and String Theory, PoS(LAT2005) 001.
[2] A.M. Polyakov, Phys. Lett.B103, 1981, 207.
[3] M. Lüscher, K. Symanzik, P. Weisz Nucl. Phys.B173,365(1980); M. Lüscher,Nucl. Phys. B180,317(1981).
[4] J. Polchinski and A. Strominger, Phys.Rev.Lett 67,1681(1991).
[5] J.F. Arvis, Phys. Letts. B127 (1983) 106.
[6] J.M. Drummond,Universal Subleading Spectrum of Effective String Theory,hep-0411017
[7] N.D. Hari Dass and Peter Matlock, Universality of correction to Luscher term in Polchinski-Strominger effective string theories,hep-th/0605265.
[8] N.D. Hari Dass and Peter Matlock, Field Definitions, Spectrum and Universality in Effective String Theories, hep-th/0612291.
[9] N.D. Hari Dass and Pushan Majumdar, String-like behaviour of 4-D Yang-Mills flux tubes, JHEP 0610:020,2006; N. D. Hari Dass and Pushan Majumdar, Phys. Letts. B658: 273-278, 2008;
[10] M. Luscher & P. Weisz, JHEP 0109 (2001) 010; M. Luscher & P. Weisz, JHEP 0207 (2002) 049; M. Luscher & P. Weisz, JHEP 0407 (2004) 014.
[11] N.D. Hari Dass and Peter Matlock, A Covariant Calculus for Effective String Theories, arXiv:0709.1765 [hep-th].
[12] O. Aharony and E. Karzbrun, Effective Action for Confining Strings,JHEP 0906:012, 2009.
[13] N.D. Hari Dass, Peter Matlock and Yashas Bharadwaj, Spectrum to all orders of Polchinski-Strominger Effective String Theory of Polyakov-Liouville Type,hep-th arXiv:0910.5615
[14] N.D. Hari Dass and Yashas Bharadwaj, Spectrum to all orders of Polchinski-Strominger Effective String Theories of Drummond Type,hep-th arXiv:0910.5620
[15] Emil T. Akhmedov, M.N. Chernodub,M.I. Polykarpov and M.A. Zubkov, Phys. Rev. D53:p.2087(1996); Emil T. Akhmedov and M.A. Zubkov, JETP Lett 61:p.351(1995)
[16] P. Giudice, F. Glozzi and S. Lottini, arXiv:0901.0748 [hep-lat].
[17] A.M. Polyakov, Nucl. Phy B268, 406(1986).