A Relational Approach to Quantum Mechanics
Part II: Measurement

Jianhao M. Yang
Qualcomm, San Diego, CA 92121, USA
(Dated: July 4, 2018)

The work presented here is a continuation of the relational formulation of quantum mechanics. Relational formulation of quantum mechanics is based on the idea that relational properties among quantum systems, instead of the independent properties of a quantum system, are the most fundamental elements to construct quantum mechanics. In the earlier works, basic framework is formulated to derive quantum probability, Born’s Rule, and Schrödinger Equations. This paper further develops the formulations for ideal quantum measurement and general quantum operation. The formulation is compatible with the traditional quantum mechanics mathematically but gives new insights on several important concepts. First, the formulation shows how mutual information is exchanged during measurement. Second, it asserts that Schrödinger Equations alone cannot explain the measurement process. The “collapse” of wave function is simply an update of relational matrix based on measurement outcome. Third, the relational nature of a quantum state is discussed in depth that leads to a resolution of the EPR paradox. Completeness of quantum mechanics and locality can coexist by recognizing that a quantum state is observer-dependent. Lastly, objectivity of a quantum state can be preserved via synchronization of information among observers.

PACS numbers: 03.65.Ta, 03.65.-w

I. INTRODUCTION

Quantum mechanics was originally developed as a physical theory to explain the experimental observations of a quantum system in a measurement. In the early days of quantum mechanics, Bohr had emphasized that the description of a quantum system depends on the measuring apparatus [1–3]. In more recent development of quantum interpretations, the dependency of a quantum state on a reference system was further recognized. The relational state formulation of quantum mechanics [4–6] asserts that a quantum state of a subsystem is only meaningful relative to a given state of the rest of the system. Similarly, in developing the theory of decoherence induced by environment [7–9], it is concluded that correlation information between two quantum systems is more basic than the properties of the quantum systems themselves. Relational Quantum Mechanics (RQM) further suggests that a quantum system should be described relative to another system, there is no absolute state for a quantum system [10, 11]. Quantum theory does not describe the independent properties of a quantum system. Instead, it describes the relation among quantum systems, and how correlation is established through physical interaction during measurement. The reality of a quantum system is only meaningful in the context of measurement by another system.

The idea that relational properties are more basic than the independent properties of a quantum system is profound. It should be considered a starting point for constructing the formulation of quantum mechanics. However, traditional quantum mechanics always starts with an observer-independent quantum state. It is of interest to see if a quantum theory constructed based on relational properties can address some of the unanswered fundamental questions mentioned earlier. Such reconstruction program was initiated [10] and had some successes, for example, in deriving the Schrödinger Equation.

Recently, a similar reformulation of quantum mechanics was proposed [12]. The reformulation is based on two basic ideas. 1.) Relational properties between the two quantum systems are the most fundamental elements to formulate quantum mechanics. 2.) A physical measurement of a quantum system is a probe-response interaction process. Thus, the framework to calculate the probability of an outcome when measuring a quantum system should model this bidirectional process. This implies the probability can be derived from product of two quantities with each quantity associated with a unidirectional process. Such quantity is defined as relational probability amplitude. Specifically, the probability of a measurement outcome is proportional to the summation of probability amplitude product from all alternative measurement configurations. The properties of quantum systems, such as superposition and entanglement, are manifested through the rules of counting the alternatives. As results, Born’s rule is recovered, wave function is found to be summation of relational probability amplitudes, and Schrödinger Equation is derived when there is no entanglement in the relational probability amplitude matrix. An explicit calculation of the relational probability amplitude is shown using path integral method and gives consistent results with traditional path integral formulation. The formulation in Ref. [12] is mathematically compatible to the traditional quantum mechanics. In essence, quantum mechanics demands a new set of rules to calculate measurement probability from an interaction process.

The most important outcome of Ref. [12] is that quan-
quantum mechanics can be constructed based on the relational properties between the measured system and the apparatus. The entanglement measure between the measured system and the apparatus quantifies the difference between time evolution and measurement. Ref. [12] shows that Schrödinger Equation can be derived when the entanglement measure between the observed quantum system and the observing system is zero and unchanged. When there is change in the entanglement measure, we expect to derive the quantum measurement theory, which is missing in Ref. [12]. This paper is intended to complete the formulation for quantum measurement and quantum operation in the relational context. Furthermore, the conceptual importance of the relational nature of a quantum state is not fully discussed in Ref. [12]. There is no super observer who is capable of knowing a measurement outcome for a quantum system at a remote distance. This understanding is key to resolve the EPR paradox [13] and will be analyzed in detailed.

The paper is organized as following. We first briefly review the relational formulation of quantum mechanics in Section II. In Section III we present the measurement theory based on the relational formulation. These are the main results of the paper. Section IV describes a formulation for general quantum operation. It turns out that Schrödinger Equation, formulations for selective and non-selective measurement, can all be derived from the general quantum operation. Section V shows that the Open Quantum System theory is equivalent to the theory presented in the paper. The notion of observer dependent quantum state during measurement is best illustrated in resolving the EPR paradox, which is analyzed in detailed in Section VI. Lastly, we discuss some of the conceptual subtleties and summarize the conclusions in Section VII.

II. RELATIONAL FORMULATION OF QUANTUM MECHANICS

A. Terminologies

A Quantum System, denoted by symbol \( S \), is an object under study and follows the laws of quantum mechanics. An Apparatus, denoted as \( A \), can refer to the measuring devices, the environment that \( S \) is interacting with, or the system from which \( S \) is created. All systems are quantum systems, including any apparatus. Depending on the selection of observer, the boundary between a system and an apparatus can change. For example, in a measurement setup, the measuring system is an apparatus \( A \), the measured system is \( S \). However, the composite system \( S + A \) as a whole can be considered a single system, relative to another apparatus \( A' \). In an ideal measurement to measure an observable of \( S \), the apparatus is designed in such a way that at the end of the measurement, the pointer state of \( A \) has a distinguishable, one to one correlation with the eigenvalue of the observable of \( S \).

The definition of Observer is associated with an apparatus. An observer, denoted as \( O \), is a person who can operate and read the pointer variable of the apparatus. Whether or not this observer (a person) is a quantum system is irrelevant in our formulation. However, there is a restriction that is imposed by the principle of locality. An observer is defined to be physically local to the apparatus he associates with. This prevents the situation that \( O \) can instantaneously read the pointer variable of the apparatus that is space-like separated from \( O \). Receiving the information from \( A \) at a speed faster than the speed of light is prohibited. This locality requirement is crucial in resolving the EPR argument [11, 13]. An observer cannot be associated with two or more apparatuses in the same time if these apparatuses are space-like separated.

In the traditional quantum measurement theory proposed by von Neumann [14], both the quantum system and the measuring apparatus follow the same quantum mechanics laws. Von Neumann further distinguished two separated measurement stages, Process 1 and Process 2. Mathematically, an ideal measurement process is expressed as

\[
\ket{\Psi}_{SA} = \ket{\psi_S}_{A_0} = \sum_i c_i \ket{s_i}_{A_0} \rightarrow \sum_i c_i \ket{s_i}_{A_i} \rightarrow \ket{s_n}_{A_n}
\]

Initially, both \( S \) and \( A \) are in a product state described by \( \ket{\Psi}_{SA} \). In Process 2, referring to the first arrow in Eq. (1), the quantum system \( S \) and the apparatus \( A \) interact. However, as a combined system they are isolated from interaction with any other system. Therefore, the dynamics of the total system is determined by the Schrödinger Equation. Process 2 establishes a one to one correlation between the eigenstate of observable of \( S \) and the pointer state of \( A \). After Process 2, there are many possible outcomes to choose from. In the next step which is called Process 1, referring to the second arrow in Eq. (1), one of these possible outcomes (labeled with eigenvalue \( n \)) emerges out from the rest. An observer knows the outcome of the measurement via the pointer variable of the apparatus. Both systems encode information each other, allowing an observer to infer measurement results.

1 Traditional quantum mechanics does not provide a theoretical description of Process 1. In the Copenhagen Interpretation, this is considered as the “collapse” of the wave function into an eigenstate of the measured observable. The nature of this wave function collapse has been debated over many decades. Recent interpretations of quantum theory advocate that the wave function simply encodes the information that an observer can describe on the quantum system. Therefore, it is an epistemic, rather than ontological, variable. In this view, the collapse of wave function is just an update of the observer’s description on the condition of knowing the measurement outcome. For example, Quantum Bayesian theory [13] formulates how Bayesian theorem can be utilized to describe such process. The relational argument of the wave function “collapse” is presented in Section III.
of $S$ by reading pointer variable of $A$. Quantum measurement is a question-and-answer bidirectional process. The measuring system interacts (or, disturbs) the measured system. The interaction in turn alters the state of the measuring system. As a result, a correlation is established, allowing the measurement result for $S$ to be inferred from the pointer variable of $A$.

A Quantum State of $S$ describes the complete information an observer $O$ can know about $S$. From the examination on the measurement process and the interaction history of a quantum system, we consider a quantum state encodes the information relative to the measuring system or the environment that the system previously interacted with. In this sense, the quantum state of $S$ is described relative to $A$. It is equivalent to say that the quantum state is relative to an observer $O$ because there is no space-like separation between $O$ and $A$. $O$ operates $A$, reads the measurement outcomes of $A$, and has the complete control of $A$. The idea that a quantum state encodes information from previous interactions is also proposed in Ref. [11]. The information encoded in the quantum state is complete knowledge an observer can say about $S$, as it determines the possible outcomes of next measurement. When next measurement with another apparatus $A'$ is completed, the description of quantum state is updated to be relative to $A'$.

**B. Basic Formulation**

The relational formulation of quantum mechanics [12] is based on a detailed analysis of the interaction process during measurement of a quantum system. First, from experimental observations, a measurement of a variable on a quantum system yields multiple possible outcomes randomly. Each potential outcome is obtained with a certain probability. We call each measurement with a distinct outcome a quantum event. Denote these alternatives events with a set of kets $\{|s_i\}$ for $S$, where (i = 0, ..., N − 1), and a set of kets $\{|a_j\}$ for $A$, where (j = 0, ..., M − 1). A potential measurement outcome is represented by a pair of kets $\{|s_i\rangle, |a_j\rangle\}$. Second, a physical measurement is a bidirectional process, the measuring system and the measured system interact and modify the state of each other. The probability of finding a potential measurement outcome represented by a pair of kets $\{|s_i\rangle, |a_j\rangle\}$, $p_{ij}$, should be calculated by modeling such bidirectional process. This implies $p_{ij}$ can be expressed as product of two numbers,

$$p_{ij} \propto Q_j^S R_{ij}^S.$$  

(2)

$q_{ji}^S$ and $r_{ij}^S$ are not necessarily real non-negative number since each number alone only models a unidirectional process which is not a complete measurement process. On the other hand, $p_{ij}$ is a real non-negative number since it models an actual measurement process. To satisfy such requirement, we further assume

$$q_{ji}^S = (r_{ij}^S)^*.$$  

(3)

Written in a different format, $Q_j^S = (R_{ij}^S)^\dagger$. This means $Q_j^S = (R_{ij}^S)^\dagger$. Eq. (2) then becomes

$$p_{ij} = |R_{ij}^S|^2 / \Omega$$  

(4)

where $\Omega$ is a real number normalization factor. $Q_j^S$ and $R_{ij}^S$ are called relational probability amplitudes. Given the relation in Eq. (3), we will not distinguish the notation $R$ versus $Q$, and only use $R$. The relational matrix $R^S$ gives the complete description of $S$. It provides a framework to derive the probability of future measurement outcome.

$R_{ij}^S$ can be explicitly calculated using the path Integral formulation. In the context of path integral, $R_{ij}^S$ is defined as the sum of quantity $e^{iS_p/\hbar}$, where $S_p$ is the action of the composite system $S + A$ along a path. Physical interaction between $S$ and $A$ may cause change of $S_p$, which is the phase of the probability amplitude. But $e^{iS_p/\hbar}$ itself is a probabilistic quantity. Although $R_{ij}^S$ is a probability amplitude, not a probability real number, we assume it follows certain rules in the classical probability theory, such as multiplication rule, and sum of alternatives in the intermediate steps.

The set of kets $\{|s_i\}$, representing distinct measurement events for $S$, can be considered as eigenbasis of Hilbert space $\mathcal{H}_S$ with dimension $N$, and $|s_i\rangle$ is an eigenvector. Since each measurement outcome is distinguishable, $\langle s_i|s_j\rangle = \delta_{ij}$. Similarly, the set of kets $\{|a_j\}$ is eigenbasis of Hilbert space $\mathcal{H}_A$ with dimension $N$ for the apparatus system $A$. The bidirectional process $|a_j \equiv |s_i\rangle$ is called a potential measurement configuration in the joint Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_A$.

To derive the properties of $S$ based on the relational $R$, we examine how the probability of measuring $S$ with a particular outcome of variable $q$ is calculated. It turns out such probability is proportional to the sum of weights from all applicable measurement configurations, where the weight is defined as the product of two relational probability amplitudes corresponding to the applicable measurement configuration. Identifying the applicable measurement configuration manifests the properties of a quantum system. For instance, before measurement is actually performed, we do not know that which event will occur to the quantum system since it is completely probabilistic. It is legitimate to generalize the potential measurement configuration as $|a_j \rightarrow |s_m\rangle \rightarrow |a_k\rangle$. In other words, the measurement configuration in the joint Hilbert space starts from $|a_j\rangle$, but can end at $|a_k\rangle$, or any other event, $|a_k\rangle$. Indeed, the most general form of measurement configuration in a bipartite system can be $|a_j\rangle \rightarrow |s_m\rangle \rightarrow |s_n\rangle \rightarrow |a_k\rangle$. Correspondingly, we generalize Eq. (2) by introducing a quantity for such configuration,

$$W_{jmnk}^{ASSA} = Q_{jm}^{AS} R_{nk}^{SA} = (R_{mj}^{SA})^\dagger R_{nk}^{SA}.$$  

(5)

The second step utilizes Eq. (3). This quantity is interpreted as a weight associated with the potential measurement configuration $|a_j\rangle \rightarrow |s_m\rangle \rightarrow |s_n\rangle \rightarrow |a_k\rangle$. Suppose
we do not perform actual measurement and inference information is not available, the probability of finding \( S \) in a future measurement outcome can be calculated by summing \( W^{ASSA}_{jmnk} \) from all applicable alternatives of measurement configurations.

With this framework, the remaining task to calculate the probability is to correctly count the applicable alternatives of measurement configuration. This task depends on the expected measurement outcome. For instance, suppose the expected outcome of an ideal measurement is event \( |s_i\rangle \), i.e., measuring variable \( q \) gives eigenvalue \( q \). The probability of event \( |s_i\rangle \) occurs, \( p_i \), is proportional to the summation of \( W^{ASSA}_{jmnk} \) from all the possible configurations related to \( |s_i\rangle \). Mathematically, we select all \( W^{ASSA}_{jmnk} \) with \( m = n = i \), sum over index \( j \) and \( k \), and obtain the probability \( p_i \).

\[
p_i \propto \sum_{j,k=0}^M (R^{SA}_{ij})^* R^{SA}_{ik} = |\sum_j R^{SA}_{ij}|^2. \tag{6}
\]

This leads to the definition of wave function \( \varphi_i = \sum_j R_{ij} \), so that \( p_i = |\varphi_i|^2 \). The quantum state can be described either by the relational matrix \( R \), or by a set of variables \( \{\varphi_i\} \). The vector state of \( S \) relative to \( A \), is \( |\psi\rangle^A_S = (\varphi_0, \varphi_1, \ldots, \varphi_N)^T \) where superscript \( T \) is the transposition symbol. More specifically,

\[
|\psi\rangle^A_S = \sum_i \varphi_i |s_i\rangle \quad \text{where} \quad \varphi_i = \sum_j R_{ij}. \tag{7}
\]

The justification for the above definition is that the probability of finding \( S \) in eigenvector \( |s_i\rangle \) in future measurement can be calculated from it by defining a projection operator \( \hat{P}_i = |s_i\rangle\langle s_i| \). Noted that \( \{|s_i\rangle\} \) are orthogonal eigenbasis, the probability is rewritten as:

\[
p_i = \langle \psi | \hat{P}_i | \psi \rangle = |\varphi_i|^2 \tag{8}
\]

Eqs. (8) and (9) are introduced on the condition that there is no entanglement\(^2\) between quantum system \( S \) and \( A \). If there is entanglement between them, the summation in Eq. (8) over-counts the applicable alternatives of measurement configurations and should be modified accordingly. A more generic approach to describe the quantum state of \( S \) is the reduced density matrix formulation, which is defined as:

\[
\rho_S = RR^\dagger \tag{9}
\]

The probability \( p_i \) is calculated using the projection operator \( \hat{P}_i = |s_i\rangle\langle s_i| \)

\[
p_i = Tr_S(\hat{P}_i \rho_S) = \sum_j |R_{ij}|^2. \tag{10}
\]

The effect of a quantum operation on the relational probability amplitude matrix can be expressed through an operator. Defined an operator \( \hat{M} \) in Hilbert space \( \mathcal{H}_S \) as \( M_{ij} = \langle s_i|\hat{M}|s_k \rangle \). The new relational probability amplitude matrix is obtained by

\[
(R_{new}^{SA})_{ij} = \sum_k M_{ik}(R^{SA}_{init})_{kj}, \quad \text{or} \tag{11}
\]

\[
R_{new} = MR_{init}.
\]

Consequently, the reduced density becomes,

\[
\rho_{new} = R_{new}(R_{new})^\dagger = M \rho_{init} M^\dagger. \tag{12}
\]

C. Entanglement Measure

The description of \( S \) using the reduced density matrix \( \rho_S \) is valid regardless there is entanglement between \( S \) and \( A \). To determine whether there is entanglement between \( S \) and \( A \), a parameter to characterize the entanglement measure should be introduced. There are many forms of entanglement measure \([17, 18]\), the simplest one is the von Neumann entropy. Denote the eigenvalues of the reduced density matrix \( \rho_S \) as \( \{\lambda_i\}, i = 0, \ldots, N \), the von Neumann entropy is defined as

\[
H(\rho_S) = -\sum_i \lambda_i \ln \lambda_i. \tag{13}
\]

A change in \( H(\rho_S) \) implies there is change of entanglement between \( S \) and \( A \). Unless explicitly pointed out, we only consider the situation that \( S \) is described by a single relational matrix \( R \). In this case, the entanglement measure \( E = H(\rho_S) \). Since \( \rho_S = RR^\dagger \), the entanglement measure is sometimes expressed as \( H(R) \). Theorem 1 in Appendix A provides a simple criteria to determined whether \( H(R) = 0 \) based on the decomposition of \( R_{ij} \).

\( H(\rho_S) \) enables us to distinguish different quantum dynamics. Given a quantum system \( S \) and its referencing apparatus \( A \), there are two types of the dynamics between them. In the first type of dynamics, there is no physical interaction and no change in the entanglement measure between \( S \) and \( A \). \( S \) is not necessarily isolated in the sense that it can still be entangled with \( A \), but the entanglement measure remains unchanged. This type of dynamics is defined as **time evolution**. In the second type of dynamics, there is a physical interaction and correlation information exchange between \( S \) and \( A \), i.e., the von Neumann entropy \( H(\rho_S) \) changes. This type of dynamics is defined as **quantum operation**. **Quantum measurement** is a special type of quantum operation with a particular outcome. Whether the entanglement measure changes distinguishes a dynamic as either a time evolution or a quantum operation.

Ref. [12] has provided the formulation of time evolution. Particularly, when \( S \) is in an isolated state, its dynamics is governed by the Schrödinger Equation. The purpose of this paper is to provide the formulation of

\(^2\) See Section 1.3 for the definition of entanglement.
quantum operations when the entanglement measure between $S$ and $A$ changes.

III. QUANTUM MEASUREMENT

A. Goals of Measurement Theory

The entanglement measure defined in Section II.C characterizes the quantum correlation between the measured system $S$ and the apparatus system $A$. The correlation enables the inference of measurement outcome. A change in entanglement measure implies change in the quantum correlation, consequently, change in the capability of inference. The capability of inference can be described by the mutual information variable, which is defined as \[ I(S, A) = H(\rho_S) + H(\rho_A) - H(\rho_{SA}). \] (14)

where $H(\rho)$ is the von Neumann entropy of a density matrix.\(^3\) Mutual information is a quantity that measures the amount of information about $S$ through knowing information about $A$. For a pure bipartite state, $H(\rho_{SA}) = 0$ and $H(\rho_S) = H(\rho_A)$, thus $I(S, A) = 2H(\rho_S)$, only differs from the Von Neumann entropy of the reduced density matrix of $S$ by a factor of 2. Thus, in this case, it is equivalent to state that quantum operation is a process that alters the mutual information between $S$ and $A$. The term information exchange used in the following text strictly refers the changes of mutual information.

Although the cause of information exchange is the physical interaction, the measurement theory does not aim to explain the detailed physical process on how $A$ records a particular outcome. Instead, the measurement theory just describes how the mutual information is transferred from one system to another. In the context of this work, the goal is to describe how the relational probability amplitude matrix $R$ is transformed during measurement, and how mutual information is exchanged in the process.

Suppose the measurement is performed using apparatus $A$ and the initial correlation matrix is $R_0$. Although measurement dynamics involves information exchange between $S$ and $A$, the composite system $S + A$ is isolated, and can be described as a unitary process. This is Process 2 in the von Neumann measurement theory. The result is that the correlation matrix $R_0$ is mapped to $R'$, denoted as $R_0 \rightarrow R'$. $R'$ is then used by the intrinsic observer $O_T$ to calculate the probability of a particular measurement outcome. As pointed out in Ref. [10], if an external observer $O_E$ only knows there is a measurement process occurred, but does not know the measurement outcome, his description of the measurement process is limited to $R_0 \rightarrow R'$. On the other hand, the intrinsic observer, $O_T$ who reads the pointer variable of $A$, knows the measurement outcome after the measurement finishes. This additional information on the exact measurement outcome allows $O_T$ to infer the final quantum state of $S$. It results in another map $R' \rightarrow R''$. This is Process 1 in the von Neumann measurement theory.

In short, a measurement theory should describe how the relational matrix $R$ and the mutual information are changed during the measurement process. We start the formulation with a simpler case that the $S + A$ composite system is initially in a product state.

B. Product Initial State

In Ref. [12], it is shown that when the composite system $S + A$ is described by a relational probability amplitude matrix $R$ and assuming $S + A$ is in an isolated environment, it is mathematically equivalent to describe the composite system with a wave function,

\[ |\Psi\rangle = \sum_{ij} R_{ij} |s_i\rangle |a_j\rangle. \] (15)

Suppose $S$ and $A$ initially are unentangled, they can be described as a product state, $|\Psi_0\rangle_{SA} = \sum_{ij} R_{ij} |s_i\rangle |a_j\rangle$ where $R_{ij} = c_i d_j$. This implies that $|\Psi_0\rangle_{SA}$ can be written as $|\psi_0\rangle_S |\phi_0\rangle_A$, where $|\psi_0\rangle_S = \sum_i c_i |s_i\rangle$ and $|\phi_0\rangle_A = \sum_j d_j |a_j\rangle$. $S + A$ as a whole follows the Schrödinger Equation. Since there is interaction between $S$ and $A$, the overall unitary operator cannot be decomposed to $\hat{U}_S \otimes \hat{U}_A$. Instead, $\hat{U}_{SA}$ can be decomposed such that it gives the following map

\[ |\Psi_1\rangle_{SA} = \hat{U}_{SA} |\Psi_0\rangle_{SA} = \hat{U}_{SA} |\psi_0\rangle_S |\phi_0\rangle_A = \sum_m \hat{M}_m |\psi_0\rangle_S |a_m\rangle \] (16)

where the set of operators $\hat{M}_m$ satisfies the completeness condition $\sum_m \hat{M}_m \hat{M}_m^\dagger = I$. Appendix A shows that such a decomposition always exists as long as the initial state is a product state. Substitute $|\psi_0\rangle_S = \sum_i c_i |s_i\rangle$ to Eq. (16)

\[ |\Psi_1\rangle_{SA} = \sum_m \left( \sum_i \left( \sum_k (\hat{M}_m)_{ik} c_k \right) |s_i\rangle |a_m\rangle \right) \] (17)

This gives the new correlation matrix $R'$ with element $R'_m = \sum_k (\hat{M}_m)_{ik} c_k$. $R'$ has now encoded the correlation between $S$ and $A$ and can be used to predict the probability of a possible measurement outcome. At the end of the measurement, $O_E$ who operates and reads the outcome of his apparatus $A$ knows the measurement outcome as $A$ ends up in a distinguishable pointer state $|a_m\rangle$. This allows $O_T$ to infer exactly the resulting state of $S$. Since there is no additional interaction between $S$ and $A$, the description of $S$ remains unchanged.

\(^3\) However, there is speculation that quantum mutual information should be defined as $I(S, A) = H(\rho_S) - H(\rho_{SA})$, see remark in Ref. [20].
and $A$, the process can be modeled as a local operator $I^S \otimes P^A_m$ where $P^A_m = \ket{a_m}\bra{a_m}$. According to Theorem 2 in Appendix C the relational matrix is updated to $R''_{m} = I^S R'(P^A_m)^T = R'(P^A_m)^T$. Substituting $R'$ obtained earlier, one has

$$
(R''_{m})_{ij} = \sum_n (R'_{m})_{ij} (P^T_m)_{nj} = \sum_n \sum_k (\hat{M}_n)_{ik} c_k \delta_{nm} \delta_{jm} \\
= \sum_k (\hat{M}_m)_{ik} c_k \delta_{jm}
$$

The last step shows that $(R''_{m})_{ij}$ can be written as $c'_i d'_j$ with $c'_i = \sum_k (\hat{M}_n)_{ik} c_k$ and $d'_j = \delta_{jm}$. From Theorem 1, $H(R''_{m}) = 0$. Therefore, we can use Eq. (17) to calculate the wave function of $S$ corresponding to $\ket{a_m}$,

$$
\varphi^m_i = \sum_j (R''_{m})_{ij} = \sum_k (\hat{M}_m)_{ik} c_k \sum_j \delta_{jm} = \sum_k (\hat{M}_m)_{ik} c_k
$$

Recall the initial state of $S$ is $\ket{\psi_0}_S = \sum_i c_i \ket{s_i}$, the resulting state vector for $S$, $\ket{\psi_m} = \sum_i \varphi^m_i \ket{s_i}$, can be written as $\ket{\psi_m} = \hat{M}_m \ket{\psi_0}_S$ without normalization. Applying the normalization factor, and omitting the subscript referring to $S$, one finally gets

$$
\ket{\psi_m} = \frac{\hat{M}_m \ket{\psi_0}}{\sqrt{\langle \psi_0 | \hat{M}_m^\dagger \hat{M}_m | \psi_0 \rangle}}
$$

The normalization factor is the probability of finding $A$ in the pointer state $\ket{a_m}$ after Process 2, i.e., the probability of measurement with outcome $m$. This can be verified by combining Eq. (22) in Appendix C and Eq. (16),

$$
p_m = \langle \Psi_1 | I^S \otimes P^A_m | \Psi_1 \rangle = \langle \psi_0 | \hat{M}_m^\dagger \hat{M}_m | \psi_0 \rangle
$$

Due to the correlation in Eq. (16), the probability of finding $A$ in $\ket{a_m}$ is exactly the probability of inferring $S$ in the resulting state $\ket{\psi_m}$. If $O_T$ repeats the same experiment many times, he shall find that the outcome $m$ occurs with a frequency of $p_m$, even though the outcome of a particular measurement is random.

Eqs. (20) and (21) typically appear in textbooks as a postulate for quantum measurement [17] [19]. In deriving these results, a mysterious ancillary system is introduced. The property of the ancillary system is traced out at the end to obtain Eq. (20) and (21). As shown in this section, the ancillary system is nothing but the apparatus $A$. Its property can be traced out because the initial state is a product state, and at the end of the measurement, $S$ and $A$ are still in a product state.

The last statement of the above paragraph needs more qualification. At the beginning of a measurement $H(R) = 0$. At the end of the measurement $H(\rho_m) = 0$ as well. The entanglement measure appears to be the same at the beginning and at the end of the measurement. However, during the measurement process, $H(R)$ does not stay as a constant. This can be seen from Eq. (16). The correlation matrix $R''_{m} = \sum_k (\hat{M}_m)_{ik} c_k$. It is not difficult to calculate $H(R'') = \sum p_m \ln(p_m)$. From Eq. (13) we can analyze the change of mutual information between $S$ and $A$. Initially $S$ and $A$ share no mutual information. In the initial phase of measurement, $S$ and $A$ interact and become entangled. Information from $S$ is encoded in $A$. The mutual information increases to $-2 \sum p_m \ln(p_m)$. This allows an observer to infer probability of measurement outcome through $A$, but without knowing the exact measurement outcome. At the later phase of an ideal projection measurement, $A$ becomes disentangled with $S$ again and converges into a particular pointer state $\ket{a_m}$ with a probability $p_m$, this allows $O_T$ to infer exactly which state $S$ is in. When the measurement ends, $S$ and $A$ share no mutual information again. During the measurement, the mutual information is changed as $0 \to -2 \sum p_m \ln(p_m) \to 0$.

The increase of mutual information in the first arrow is described as a unitary process of the composite system, and the decrease of mutual information in the second arrow is described by a projection operator. The update of the relational matrix from $R'$ to $R''$ was perceived as “wave function collapse” in the Copenhagen Interpretation. However, in this paper this update is not associated with a physical reality change. Instead, it is interpreted as change of description of the relational matrix due to the fact that $O_T$ knows the exact measurement outcome.

An external observer $O_E$ does not know the measurement outcome and therefore still describes $S$ with $R''$. $O_E$ can obtain the measurement outcome through communication with $O_T$. But this means there is a physical interaction between $A'$ and $A$. An interaction between $A'$ and $A$ disqualifies $O_E$ to describe the composite system $S + A$ as a unitary time evolution. Thus Process 1 cannot be described as a unitary process by either $O_T$ or $O_E$. In other words, Process 1 cannot be described by the Schrödinger Equation. One of the preconditions for applying Schrödinger Equation is that there should have no information exchange between the observed system and the reference apparatus. But for an observer to know the exact measurement outcome, such information exchange is unavoidable.

As mentioned earlier, measurement theory is to develop a physical model that describes how mutual information is exchanged during measurement. The detailed physical process of interaction is not explained here. For
example, after the S and A become entangled, one must assume there is no further interaction between S and A in order to model the process with the local projection operator \( I^S \otimes P_m^A \). It may be just an approximation. Ref. 21 provides tremendous amount of physical details to describe this process. The measurement process goes through several sub-processes such as registration, truncation, decoherence, and the emergence of a unique outcome that is interpreted using quantum statistics mechanics 21. It is of great interest to find out the physical details on the measurement process, but the primary interest of the measurement theory developed here is how the relational matrix \( R \) and the mutual information are changed during the measurement process.

C. Entangled Initial State

When S and A are initially entangled, A already has some level of correlation with S. In a sense that A has already measured S since the information of A can be used to infer information of S. One may ask what the goal of subsequent measurement is in this case. In the situation that S and A are initially in product state, an operation involving interaction between S and A increases the mutual information, thus allowing A to infer information of S. Similarly, in the case when S and A are initially entangled, the goal of the measurement can be further increasing the mutual information. After more mutual information is encoded in A, a subsequent projection operation can be applied so that A evolves to a unique distinguishable pointer state. Since the mutual information is defined as \( I(S, A) = H(\rho_S) + H(\rho_A) - H(\rho_{SA}) \), the maximum mutual information for a pure bipartite state is \( I_{\text{max}} = 2lnN \) where \( N \) is the rank of matrix \( R \). We can define the amount of unmeasured mutual information as

\[
I_u(S, A) = 2lnN - I(S, A).
\]

(22)

Thus, the goal of measurement is to minimize \( I_u(S, A) \).

Alternatively, the goal of measurement is to alter the probability distribution \( \{ p_m \} \) such that the probability to find S in a particular state is adjusted as desired, or such that the expectation value of an observable of S matches a desired value. We will discuss both cases in this section.

Denote the initial entangled state for \( S + A \) as \( |\Psi_0\rangle_{SA} = \sum_{ij} R_{ij} |s_i\rangle |a_j\rangle \). The interaction between S and A is still described as a unitary operation over the whole \( S + A \) composite system. \( |\Psi_1\rangle_{SA} = U_{SA} |\Psi_0\rangle_{SA} \). The relational matrix \( R' \) is

\[
R'_{ij} = \langle s_i | (a_j | U_{SA} | \Psi_0 \rangle = \sum_{kl} R_{kl} \langle s_i | (a_j | U_{SA} | s_k \rangle |a_l\rangle.
\]

(23)

Then A is projected to a particular state \( |a_m\rangle \). Similar to the approach in deriving Eq. [13], the relational matrix is updated to \( R''_m = I^S R'(P^A_m)^T \) where \( P^A_m = |\phi_m\rangle \langle \phi_m | \), we get

\[
(R''_m)_{ij} = \sum_n (R'_{in})(P^T_m)_{nj} = \sum_n (\langle s_i | (a_n | U_{SA} | \Psi_0 \rangle \langle a_j | \phi_m \rangle |\phi_m \rangle |a_n\rangle)
\]

(24)

\[
= \sum_n \langle \phi_m | (a_n | s_i | U_{SA} | \Psi_0 \rangle \langle a_j | \phi_m \rangle |\phi_m \rangle |a_n\rangle
\]

\[
= \langle s_i | (\phi_m | U_{SA} | \Psi_0 \rangle \langle a_j | \phi_m \rangle |\phi_m \rangle |a_n\rangle
\]

where the property \( \sum_n |a_n\rangle \langle a_n | = I \) is applied in the third step. Since \( (R''_m)_{ij} \) can be written as product of two terms with index \( i \) and \( j \) separated, according to Theorem 1, \( H(R) = 0 \). We can use Eq. [7] to calculate the wave function of S associated with outcome \( |a_m\rangle \)

\[
\varphi^m_i = \sum_j (R''_m)_{ij} = \sum_j \langle a_j | \phi_m \rangle |\phi_m \rangle |a_m\rangle
\]

(25)

\[
= \sum_j \langle s_i | \langle a_j | \phi_m \rangle | U_{SA} | \Psi_0 \rangle |\phi_m \rangle |a_m\rangle
\]

\[
d_m \langle s_i | \langle a_j | \phi_m \rangle | U_{SA} | \Psi_0 \rangle |\phi_m \rangle |a_m\rangle
\]

where \( d_m = \sum_j \langle a_j | \phi_m \rangle \) is a normalization constant. The probability of finding measurement outcome associated with \( |a_m\rangle \) is given by Eq. [12] in Appendix C,

\[
p'_m = \langle |\Psi_1 \rangle | I^S \otimes P^A_m | \Psi_1 \rangle = \sum_i |\langle s_i | \langle a_j | \phi_m \rangle | U_{SA} | \Psi_0 \rangle |^2
\]

(26)

\[
= \langle |\Psi_0 \rangle | U^+ | \phi_m \rangle \langle \phi_m | U^+ | \Psi_0 \rangle.
\]

The resulting state vector of S before normalization is

\[
|\psi_m\rangle = \sum_i \varphi^m_i |s_i\rangle
\]

(27)

\[
= d_m \sum_i |s_i\rangle \langle s_i | \langle a_j | \phi_m \rangle | U_{SA} | \Psi_0 \rangle |\phi_m \rangle |a_m\rangle
\]

\[
= d_m |\phi_m \rangle | U_{SA} | \Psi_0 \rangle |\phi_m \rangle |a_m\rangle
\]

Normalizes requires that \( d_m = 1/\sqrt{p'_m} \). In order to simplify Eq. [27], we rewrite the initial entangled bipartite state using the Schmidt decomposition \( |\Psi_0\rangle = (U_S \otimes V_A) \sum_i \lambda_i |\tilde{s}_i\rangle |\tilde{a}_i\rangle \) where \( U_S \otimes V_A \) is a local unitary transformation. \( \lambda_i \) is the Schmidt coefficient, which essentially is the eigenvalue of the correlation matrix \( R \). This gives

\[
|\tilde{\psi}_m\rangle = \frac{1}{\sqrt{p'_m}} \sum_i \langle \phi_m | U_{SA} (U_S \otimes V_A) |\tilde{s}_i\rangle |\tilde{a}_i\rangle
\]

(28)

\[
= \frac{1}{\sqrt{p'_m}} \sum_i \tilde{M}_{mi} |\tilde{s}_i\rangle
\]

where \( \tilde{M}_{mi} = \lambda_i |\phi_m \rangle \langle \phi_m | U_{SA} (U_S \otimes V_A) |\tilde{a}_i\rangle \). Note that \( \tilde{M}_{mi} \) depends on the initial state itself, Eq. [28] is not a simple
form. If $|\Psi_0\rangle$ is a product state, $\lambda_0 = 1$ and $\lambda_i = 0$ for $i > 0$, Eq. (28) is reduced to Eq. (29). Given that $\sum_m \hat{p}_m = 1$, it is easy to verify the completeness property of $\hat{M}_{m,i}$

\[
\sum_m \hat{M}_{m,i}^{\dagger} \hat{M}_{m,j} = \delta_{ij} \lambda_i^2 I_S
\]

\[
\sum_{m} \sum_{ij} \hat{M}_{m,i}^{\dagger} \hat{M}_{m,j} = I_S.
\]  

(29)

Since $\langle \psi_m | \psi_m \rangle = 1$, from Eq. (28) one gets $p'_m = \sum_{ij} \langle s_i | \hat{M}_{m,i}^{\dagger} \hat{M}_{m,j} | s_j \rangle$. It follows from Eq. (29) that $\sum_m p'_m = \sum_{ij} \lambda_i^2 = 1$. From the expression for $p'_m$, the mutual information after the unitary operation can be calculated as $I'(S,A) = -2 \sum_m p'_m \ln(p'_m)$. On the other hand, the initial mutual information $I(S,A) = -2 \sum_i |\lambda_i|^2 \ln(|\lambda_i|^2)$. If the goal of measurement is to increase the mutual information, one wishes to find a unitary operator $\hat{U}$ such that $I'(\hat{S},A) > I(S,A)$, that is,

\[
\sum_m p'_m \ln(p'_m) < \sum_i |\lambda_i|^2 \ln(|\lambda_i|^2).
\]  

(30)

On the other hand, if the goal of measurement is not necessarily to increase the mutual information, but to increase the probability that $S$ is in a state inferred by $A$ being in the pointer state $|a_m \rangle$. The initial probability before measurement operation is $p_m = \sum_i |R_{im}|^2$. After measurement operation, we want $p'_m > p_m$. This means the goal of measurement is to find a unitary operator $\hat{U}_{SA}$ such that

\[
\sum_{ij} \langle s_i | \hat{M}_{m,i}^{\dagger} \hat{M}_{m,j} | s_j \rangle > \sum_i |R_{im}|^2.
\]  

(31)

Neither Eq. (30) nor Eq. (31) is simple to solve. It is not clear that for a given initial correlation matrix $R$, a unitary operator $\hat{U}_{SA}$ that satisfies either Eq. (30) or Eq. (31) always exists. This is an open topic for future research.

Eq. (27) can be derived using the reduced density matrix approach. The changes of the composite state of $S+A$ during the measurement process are $|\Psi_0\rangle \rightarrow |\Psi_1\rangle = \hat{U}_{SA}|\Psi_0\rangle \rightarrow |\Psi_2\rangle = (I_S \otimes \hat{P}_m) \hat{U}_{SA}|\Psi_0\rangle$. At the end of the measurement, the reduced density matrix of $S$ is

\[
\rho_S = Tr_A |\Psi_2\rangle \langle \Psi_2| = \langle \phi_m | \hat{U}_{SA} | \Psi_0 \rangle \langle \Psi_0 | \hat{U}_{SA}^{\dagger} | \phi_m \rangle
\]

(32)

This shows $\rho_S$ is a pure state and the wave function is $|\psi_m\rangle = \langle \phi_m | \hat{U}_{SA} | \Psi_0 \rangle$ up to a normalization factor. It gives the same result as Eq. (27). The probability $p'_m$ has been given by Eq. (26).

**IV. GENERAL QUANTUM OPERATION**

In Section III we only consider the selective measurement. At the end of the selective measurement operations, the apparatus $A$ is in a definite state, and $S$ and $A$ are in a product composite state. There is other type of quantum operation where at the end of the operation, $S$ and $A$ are in an entangled state and there are still multiple possible outcomes. This is the non-selective measurement. For instance, the composite system $S+A$ can go through the interaction characterized by an operator $\Lambda_{SA} \neq \hat{U}_S \otimes \hat{U}_A$ and there is no further projection operation. $S$ and $A$ are entangled at the end of the operation.

A more general global linear map on a bipartite system can be decomposed to $\Lambda_{SA} = \sum_k \alpha_k \hat{B}_k \otimes \hat{C}_k$ where $\hat{B}_k$ is local operator to $S$ and $\hat{C}_k$ is local operator to $A$ [17]. $\Lambda_{SA}$ is a general operation in the sense that $\hat{B}_k$ or $\hat{C}_k$ are not necessarily project operators, and the resulting $S+A$ can be in a product state or an entangled state. It is convenient to re-express the relational matrix by introduce a linear operator $\hat{R} = \sum_i R_{ij} |s_i \rangle \langle a_j|$.

According to Theorem 2, the operation of $\Lambda_{SA}$ on $S+A$ transforms the initial relational operator $\hat{R}_0$ to

\[
\hat{R} = \Lambda_{SA}(\hat{R}_0) = \sum_k \alpha_k \hat{B}_k \hat{R}_0 \hat{C}_k^T
\]

(33)

The reduced density operator for $S$ after the general quantum operation is

\[
\hat{\rho}_S = \hat{R} \hat{R}^\dagger = \sum_{kl} \alpha_k \alpha_l^* \hat{B}_k \hat{R}_0 \hat{C}_l^T \hat{C}_k \hat{R}_0^\dagger \hat{B}_l^T
\]

(34)

Suppose the initial composite state of $S+A$ is $|\Psi_0\rangle = \sum_i \lambda_i |s_i \rangle \langle a_i|$, the relational operator $\hat{R}_0$ can be expressed as $\hat{R}_0 = \sum_i \lambda_i |s_i \rangle \langle a_i|$. Substitute this into Eq. (34),

\[
\hat{\rho}_S = \hat{R} \hat{R}^\dagger = \sum_{ijkl} \lambda_i \lambda_j \alpha_k \alpha_l^* \hat{B}_k |s_i \rangle \langle a_i| \hat{C}_l^T \hat{C}_k |s_j \rangle \langle a_j| \hat{B}_l^T
\]

(35)

If the local operator $\hat{C}$ is a unit operator, $\Lambda_{SA} = \sum_k \alpha_k \hat{B}_k \otimes I_A = \Lambda_{S} \otimes I_A$. This means $\Lambda_{SA}$ only operates on $S$ and has no impact on $A$. Eq. (35) becomes

\[
\hat{\rho}_S = \Lambda_{S} \hat{\rho}_0 \Lambda_{S}^\dagger
\]

where $\hat{\rho}_0 = \sum_i \lambda_i^2 |s_i \rangle \langle s_i|$ is the initial density operator for $S$. However, if $\Lambda_{SA} \neq \Lambda_{S} \otimes I_A$, Eq. (35) does not hold in general.

Eq. (35) can be derived through the partial trace approach as well. The initial density operator of the composite system is $\rho_{SA} = |\Psi_0\rangle \langle \Psi_0| = \sum_{ij} \lambda_i \lambda_j |s_i \rangle \langle a_i| \hat{C}_l^T \hat{C}_k |s_j \rangle \langle a_j|$. After applying the general op-
eration $\Lambda_{SA}$, the reduced density operator of $S$ is

\[
\hat{\rho}_S = Tr_A(\Lambda_{SA}\rho_{SA}\Lambda_{SA}^\dagger)
= Tr_A\left(\sum_{ijkl} \lambda_i \lambda_j \alpha_k \alpha_l \, \hat{B}_k |\tilde{s}_i\rangle \langle \tilde{s}_j | \hat{B}_l^\dagger \otimes \hat{C}_k |\tilde{a}_i\rangle \langle \tilde{a}_j | \hat{C}_l^\dagger \right)
= \sum_{ijkl} \lambda_i \lambda_j \alpha_k \alpha_l \langle (\tilde{a}_j | \hat{C}_l^\dagger | \tilde{a}_i\rangle \hat{B}_k |\tilde{s}_i\rangle \langle \tilde{s}_j | \hat{B}_l^\dagger \}
= \sum_{ijkl} \lambda_i \lambda_j \alpha_k \alpha_l \langle (\tilde{a}_j | \hat{C}_l^\dagger | \tilde{a}_i\rangle \hat{B}_k |\tilde{s}_i\rangle \langle \tilde{s}_j | \hat{B}_l^\dagger \}
\]

(37)

which is the same as Eq. (35).

An application of Eq. (35) is briefly described as following. In Section III, we have been assuming the measuring apparatus is $A$. The initial interaction between $S$ and $A$ is described as a unitary operation and can be decomposed according to $\text{Tr}$. However, the measurement of $S$ can be performed using another apparatus $A'$. In this case, $A'$ shall interact with either $S$ or the $S + A$ composite system. The general map $\Lambda_{SA}$ is not a unitary operator anymore. Instead, it can be considered as a quantum operation decomposed from an unitary operator for the $S + A + A'$ composite system. If $A'$ interacts with both $S$ and $A$, the resulting reduced density operator of $S$ is given by Eq. (35). If the apparatus $A'$ only interacts with $S$ and has no impact on $A$, $C$ is a unit operator, and the resulting reduced density operator of $S$ is given by Eq. (36).

Eq. (35) is the most general form of equation describing different types of dynamics between $S$ and $A$, depending on how the map $\Lambda_{SA}$ is decomposed. If $\Lambda_{SA} = \hat{U}_S \otimes \hat{U}_A$, it results in the Schrödinger Equation [12]. If $\Lambda_{SA}$ is an unitary operator but decomposed according to Eq. (10), it describes process 2 of the von-Neumann measurement process. If $\Lambda_{SA} = I^S \otimes P^A_m$ where $P^A_m = |a_m\rangle \langle a_m|$, it describes the process 1 of the measurement process. Lastly, the most general decomposition of $\Lambda_{SA}$ gives Eq. (33). One logical conclusion is that Schrödinger Equation cannot describe all these quantum dynamics, particularly, cannot describe the process 1 in the measurement process, as discussed in the previous section. Attempt to interpret quantum mechanics just based on the Schrödinger Equation is not a plausible effort.

V. OPEN QUANTUM SYSTEM

There is similarity between the measurement theory described in this paper and the open quantum system (OQS) theory. OQS studies the dynamics when a quantum system interacts with its environment $E$ [17, 22, 23]. Such interaction can result in entanglement and information exchange between the quantum system $S$ and the environment system $E$. Recall the definition of apparatus in Section III A includes the interacting environment as one type of apparatus, if we replace the environment system $E$ with the apparatus system $A$, the OQS theory gives the same formulations as shown in Section III. First, we give a brief review of the OQS theory. Suppose the initial composite state for the quantum system and environment is described by a density matrix $\rho_{SE}$, the interaction between $S$ and $E$ changes the density matrix $\rho_{SE} \rightarrow \hat{U} \rho_{SE} \hat{U}^\dagger$. The resulting density matrix of $S$ is

$\rho_S = Tr_E(\hat{U} \rho_{SE} \hat{U}^\dagger)$. Denote the orthogonal eigenbasis of the environment $\{ |e_k\rangle \}$. Since the orthogonal eigenbasis of the environment is not necessary the same eigenbasis that diagonalizes $\rho_E$, we further denote the spectral decomposition of $\rho_E$ as $\rho_E = \sum_m \lambda_m |e_m\rangle \langle e_m|$. Assumed the initial state of the quantum system plus environment is a product state, i.e., $\rho_{SE} = \rho_S \otimes \rho_E$, the density operator of $S$ after the interaction with the environment is

\[
\hat{\rho}_S = \Lambda(\rho_S)
= \sum_{mk} |\lambda_m|^2 |e_k\rangle \langle e_k| \rho_S \hat{U} |e_m\rangle \langle e_m| \hat{U}^\dagger |e_k\rangle
= \sum_{mk} E_{mk} \rho_S E_{mk}^\dagger.
\]

(38)

where $E_{mk} = \lambda_m (e_k \hat{U} |e_m\rangle)$ and satisfies the completeness condition $\sum_{mk} E_{mk} E_{mk}^\dagger = I$. Eq. (35) is the Kraus representation of the linear map $\Lambda$. It is proved that $\Lambda$ can be a Kraus representation if and only if it can be induced from an extended system with initial condition $\rho_{SE} = \rho_S \otimes \rho_E$ [23]. If $\rho_E = |e_0\rangle \langle e_0|$ is a pure state, the linear map is further simplified to $\Lambda(\rho_S) = \sum_k E_k \rho_S E_k^\dagger$ and $E_k = \langle e_k| \hat{U} |e_0\rangle$. The operator set $\{ E_k \}$ forms a POVM, and $\Lambda(\rho_S)$ is a Complete Positive Trace Preserving (CPTP) map [19]. To connect to the measurement theory, suppose the measurement outcome $m$ corresponds to an orthogonal state $|\phi_m\rangle$ of $E$, and represented by a projection operator $P_m = |\phi_m\rangle \langle \phi_m|$, $\rho_{S}^m = \Lambda(\rho_S)
= \sum_k |e_k\rangle \langle e_k| \rho_S \hat{U} |\phi_m\rangle \langle \phi_m| \hat{U}^\dagger |e_k\rangle
= |\phi_m\rangle \rho_S |\phi_m\rangle \langle \phi_m|
= \hat{M}_m \rho_S \hat{M}_m^\dagger
\]

where $\hat{M}_m = |\phi_m\rangle \hat{U} |\phi_0\rangle$ is the operator defined on $\mathcal{H}_S$. The probability of finding outcome $m$ is $p_m = Tr(\Lambda(\rho_S)) = Tr(\hat{M}_m \hat{M}_m^\dagger \rho_S)$. It is evident that if we replace the environment system $E$ with the apparatus system $A$, the OQS theory gives the same formulations as shown in Section III. In the case of initial product state, Eqs. (20) versus (39) are effectively the same. Let’s consider the case of initial entangled state in the OQS context. Denote the initial system plus environment state as pure bipartite state $|\Psi_0\rangle$. After the global unitary operation $\hat{U}$ and subsequent projection $\hat{P}_m = I^S \otimes |\phi_m\rangle \langle \phi_m|$, the composite state becomes
$|\Psi_1\rangle = (I^S \otimes \hat{P}_m^E)\hat{U}|\Psi_0\rangle$, take the partial trace over $E$, we get

$$\rho_m^S = TR_E(|\Psi_1\rangle\langle \Psi_1|)$$

$$= \langle \phi_m|\hat{U}_S^A|\Psi_0\rangle\langle \Psi_0|\hat{U}^\dagger_S A|\phi_m\rangle$$

$$= |\psi_m\rangle\langle \psi_m|.$$  

This implies the resulting state for $S$ is $|\psi_m\rangle = \langle \phi_m|\hat{U}_S^A|\Psi_0\rangle$, which is equivalent to Eq. (27). Taking a similar approach in deriving Eq. (28), we can express $|\psi_m\rangle$ in the eigenbasis derived from the Schmidt decomposition of $|\Psi_0\rangle$. The result is $|\psi_m\rangle = 1/\sqrt{p_m} \sum_i M_{mi}|\phi_i\rangle$, where the definitions of $M_{mi}$ and $p_m$ are the same as those in Eq. (28) except replacing the apparatus system $A$ with the environment system $E$. The reduced density operator for $S$, $\rho_m^S$, is given by

$$\rho_m^S = \frac{1}{p_m} \sum_{ij} M_{mi}|\phi_i\rangle\langle \phi_i|M_{mj}^\dagger$$  

Eq. (41) can be considered as a generalization of Eq. (39) when the initial state is entangled. There is no simple form of map $\Lambda_s$ such that $\rho_m^S = \Lambda_s(\rho_S)$ where $\rho_S$ is the initial density matrix of $S$. A different representation of $\rho_m^S$ can be derived by rewriting the initial state as $\rho_S = \rho_S \otimes \rho_E + \rho_{corr}$ where $\rho_{corr}$ is a correlation term [23].

VI. EPR

A. Hidden Assumptions

In the relational formulation of quantum mechanics, even though a quantum system $S$ should be described relative to a reference system $A$, there are mathematical tools that provide equivalent descriptions without explicitly calling out the reference system $A$. When $S$ and $A$ are unentangled, $S$ can be described by a wave function defined in Eq. (17). When $S$ and $A$ are entangled, $S$ is described by a reduced density matrix that traces out the information of $A$. Since the reference system is not typically called out in traditional quantum mechanics, there are assumptions on the reference apparatus that are not obvious. Two of such hidden assumptions are:

1. An unentangled reference apparatus always exists regardless the composition of the observed system $S$. For instance, $S$ can be as large as the Universe.

2. Suppose $S$ comprises multiple subsystems and these subsystems are space-like separated. When $A$ measures a subsystem of $S$, the observer knows the measurement result instantaneously regardless where the observer locates.

Let us call such an observer who knows the measurement result instantaneously as a Super Observer $O_S$. Because $O_S$ always exists and knows the changes of $R$ instantaneously, one can choose the apparatus associated with $O_S$ as an absolute reference. A quantum state can then be described as an absolute state. The assumption that there exists a Super Observer enables the notion of absolute state for a quantum system [1]. In most of physical situations where $S$ is an isolated system and the locations of its subsystems are sufficiently close, an observer-independent state will not lead to paradox. Mathematically it is more convenient and elegant to describe a quantum state as observer-independent. However, when a quantum system comprises two entangled subsystems and the two subsystems are remotely separated, the view of $O_S$ can lead to the paradox described in the EPR paper [13]. Ref. [11] had already provided a thorough analysis of the EPR paradox from the RQM perspective. This section is intended to reaffirm the argument in Ref. [11] by analyzing the role of a hidden Super Observer in the EPR argument.

B. EPR Argument

The EPR argument is briefly reviewed as following. Assuming two systems $\alpha$ and $\beta$ are initially in the same physical location and interact for a period of time. They become entangled and then move away from each other with a space-like separation. We will adopt Bohm’s version of the EPR argument by assuming $\alpha$ and $\beta$ are two spin half particles. The quantum state of the composite system can be decomposed based on the up and down eigenstates along the $z$ direction, or left and right eigenstates along the $x$ direction.

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\alpha_u\rangle|\beta_u\rangle - |\alpha_d\rangle|\beta_d\rangle)$$

$$= \frac{1}{\sqrt{2}}(|\alpha_l\rangle|\beta_l\rangle - |\alpha_r\rangle|\beta_r\rangle).$$  

In the context of RQM, Eq. (42) assumes there is an observer, Alice, with apparatus $A$ that can measure $\alpha$ and $\beta$, and $A$ is unentangled with the composite system. The issue here is that after $\alpha$ and $\beta$ are remotely separated, there is no apparatus that can perform measurement on $\alpha$ and $\beta$ at the same time. Alice needs to perform measurement on $\alpha$ or $\beta$ once at a time. Alternatively, there can be two local observers, Alice with apparatus $A$ and located with particle $\alpha$, and Bob with apparatus $B$ and located with particle $\beta$, to perform the measurements at the same time.

The EPR paper then proposed a definition of realism as following,
If, without in any way disturbing a system, we can predict with certainty the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.

Now Alice performs a measurement on $\alpha$ and supposed the outcome is spin up. Traditional quantum mechanics states that wave function vector $|\Psi\rangle$ collapses to $|\alpha_u\rangle|\beta_u\rangle$, thus $\beta$ is deterministically in the spin up state after Alice’s measurement. If instead Alice performs a measurement along $x$ axis and finds $\alpha$ is in the spin left eigenstate, $\beta$ is deterministically in the left eigenstate after the measurement. A measurement in the location where $\alpha$ is does not cause a state change for $\beta$ that is space-like separated, otherwise it violates the principle of locality demanded by special relativity. Since one can predict the spin of $\beta$ in both $z$ and $x$ directions without disturbing it, by the above definition of realism, $\beta$ can simultaneously have elements of reality for the spin properties in both $z$ and $x$ directions. Denoting these properties as eigenvalues of operators $\sigma_x$ and $\sigma_z$. However, $\sigma_x$ and $\sigma_z$ are non-commutative. According to Heisenberg Uncertainty Principle, $\beta$ cannot simultaneously have definite eigenvalues for $\sigma_x$ and $\sigma_z$. Therefore, there are elements of physical reality of $\beta$ that the quantum mechanics cannot describe. This leads to the conclusion that quantum mechanics is an incomplete theory.

The issue here is that the definition of realism assumes the element of physical reality is observer-independent. It assumes the measurement of Alice on $\alpha$ reveals a physical reality that is observer-independent, and Bob at a remote location knows the same physical reality instantaneously. But both Alice and Bob are local observers, such definition is not operational to them, unless faster that light interaction is permitted. If, however, there is another observer, Charles, who always know the state of $\alpha$ and $\beta$ at the same time, any measurement on either $\alpha$ or $\beta$ is known to Charles instantaneously. With the help of Charles, the definition of absolute physical reality is operational. However, Charles is a super observer according to our definition. Such an observer is imaginary, although we unintentionally assume he always exists and we build physical concepts with such assumption. It is the assumption that there exists a Super Observer that allows the definition of absolute element of physical realism.

C. Resolution

Since the original definition of an element of physical realism depends on a Super Observer and is not operational, it should be modified as following:

If, without in any way disturbing a system, a local observer can predict with certainty the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity, relative this local observer.

With this modified definition, let’s proceed the EPR reasoning to see whether it leads to the conclusion of incompleteness of quantum mechanics. If Alice performs a measurement on $\alpha$ along the $z$ direction and the outcome is spin up, the wave function after measurement is updated to $|\alpha_u\rangle|\beta_u\rangle$. This just means that $\beta$ is in the spin up state according to Alice. The element of physical reality is true only relative to Alice. From Bob’s perspective, before he knows the Alice’s measurement result, he still views the composite system in the original state, no quantum event happened yet. In other words, Bob still predicts that future measurement on $\beta$ will find it is in spin up state with fifty percentage of chance. We can apply here the measuring theory in Section III. Since $\alpha$ and $\beta$ are entangled, $\alpha$ can be considered $\beta$ already, i.e., $\alpha$ is a measuring apparatus to $\beta$. Since Alice performed a measurement on $\alpha$, she effectively reads the measuring apparatus. Therefore, she is the intrinsic observer $O_\alpha$, and Bob is an external observer $O_\beta$. At this point, both observers are out of synchronization on the relational information of the two particles, thus give different descriptions of particle $\beta$. To verify the physical description Alice obtained on particle $\beta$ after measuring particle $\alpha$, Alice can travel to Bob’s location to perform a measurement, or can send the measurement result to Bob and ask Bob to perform a measurement. Suppose Alice sends the measurement outcome to Bob. Bob updates the wave function accordingly to $|\alpha_u\rangle|\beta_u\rangle$, same as the wave function relative to Alice. He now can confirm the physical reality that $\beta$ is in spin up state $|\beta_u\rangle$ with unit probability. However, in this state, he cannot predict deterministically that $\beta$ is in spin left or right, since $|\beta_u\rangle = \frac{1}{\sqrt{2}}(|\beta_l\rangle + |\beta_r\rangle)$. Similarly, if Alice performs a measurement on $\alpha$ along the $x$ direction and the outcome is spin left, $\beta$ is deterministically in the spin left state relative to Alice, but nothing happened from Bob’s view. If Alice sends the measurement result to Bob, Bob updates the wave function accordingly to $|\alpha_l\rangle|\beta_l\rangle$. He now can confirm the physical reality that $\beta$ is in spin up or down. Since Alice cannot perform measurement on $\alpha$ along $z$ and $x$ directions in the same time, Bob cannot confirm $\beta$ has spin values in both $z$ and $x$ directions simultaneously. The reality that $\beta$ simultaneously have definite values for $\sigma_x$ and $\sigma_z$ cannot be verified. This is consistent with the Heisenberg Uncertainty Principle. There is no incompleteness issue for quantum mechanics. Hence the original EPR argument no longer holds with the modification on the definition of physical realism.
D. Non-causal Correlation

However, there is still a puzzle here. It appears Bob’s measurement outcome on $\beta$ “depends” on which direction Alice chooses to measure $\alpha$. Since Alice’s measurement does not impact the physical property of particle $\beta$, exactly what spin state $\beta$ is in before Alice’s measurement? To answer this subtle question, we first note that it is Alice’s new knowledge on $\beta$, not the physical reality of $\beta$, that depends on the axis along which the measurement is performed. One cannot assume there exists an absolute reality for $\beta$. To confirm the new-found reality of $\beta$ relative to Alice, Alice sends the measurement result to Bob who performs a subsequent measurement. There is no faster-than-light action here. Secondly, it is true that Bob’s measurement outcome correlates to the Alice’s measurement result. But this is an informational correlation, not a causal relation. This correlation is encoded in the entangled state of the composite system $\alpha + \beta$ described in (42). Since the entanglement is preserved even when both particles are space-like separated, the correlation is preserved. Such entangled quantum state contains not only the classical correlation, but also the coherence information of the composite system. When Alice measures particle $\alpha$, she effectively measures the composite system, because she obtains information not only about $\alpha$, but also about the correlation between $\alpha$ and $\beta$. In addition, the measurement induces decoherence of the $\alpha + \beta$ composite system. Before Alice performs the measurement, it is meaningless to speculate what spin state particle $\beta$ is in. When Alice measures $\alpha$ along $z$ direction and obtains result of spin up, she knows that in this condition, $\beta$ is also in spin up and later this is confirmed by Bob. If instead, she measures $\alpha$ along $x$ direction and obtains result of spin left, she knows that in this new condition, $\beta$ is in spin left and later confirmed by Bob. But such correlation is not a causal relation. To better understand this non-causal relation, supposed there are many identical copies of the entangled pairs described by Eq. (42). Alice measures the $\alpha$ particles sequentially along the $z$ direction and she does not send measurement results to Bob. Bob independently measures the $\beta$ particles along $z$ direction as well. Both of them observe their own measurement results for $\sigma_z$ as randomly spin up or spin down, but with fifty percent of chance for each. When later they meet and compare measurement results, they find the sequence of $\sigma_z$ values are exactly the same. They can even choose a random sequence of $z$ or $x$ direction but both follow the exact sequence in their independent measurements. When later they meet and compare measurement results, they still find their measured values are the same sequentially.

In summary, as pointed out in Ref. [10], Special Relativity forces us to abandon the concept of absolute time. Measurement of time is observer-dependent. Similarly, in RQM, the idea of observer-independent quantum state should be abandoned. Space-like separated observers, however, can reconcile the different descriptions of the same quantum system through classical communication of information obtained from local measurements.

VII. DISCUSSION AND CONCLUSION

A. Two Aspects of Relational Formulation

The starting point of the relational formulation of quantum mechanics is that a quantum system should be described relative to a reference system. This implies the relational properties between two quantum systems are more basic than the properties of one system. This is the first aspect of the relational formulation. This reference system is not arbitrary. It is the apparatus, or environment, $A$, that the system $S$ has previously interacted with. Although the reference system $A$ is unique and objectively selected, it is possible that another observer does not have complete information of the interaction (or, measurement) results between $A$ and $S$. In such case she can describe $S$ differently using a different set of relational properties between $S$ and $A$. It is in this sense that we say the relational properties themselves are observer-dependent. This is the second aspect of relational formulation, and is indeed the main thesis of Ref. [10]. In the example of ideal measurement described by Eq. (1), supposed the measurement outcome is correspondent to eigenvector $|s_n\rangle$. For an observer that operates and reads the pointer variable of $A$, she knows the measurement outcome. At the end of the measurement, her relational description is given by $|s_n\rangle |a_n\rangle$. On the other hand, for another observer who only knows there is interaction between $S$ and $A$, but does not know the measurement outcome, the relational description is given by $\sum_i c_i |s_i\rangle |a_i\rangle$. Both descriptions are based on relational properties, and they are observer-dependent.

Ref. [12] focuses on the first aspect of relational formulation. In this paper, the observer-dependent aspect of the formulation is manifested in the measurement theory.

B. Relativity versus Objectivity of a Quantum State

Although a quantum system should be described relative to a reference system, and the relational probability amplitude matrix $R$ is considered as the most basic variable, there are mathematical tools that allow a quantum system $S$ to be described without explicitly calling out the reference system $A$. When $S$ and $A$ are unentangled, $S$ is described by a wave function that sums out the information of the reference system. When $S$ and $A$ are entangled, $S$ is described by a reduced density matrix that traces out the information of $A$.

These mathematical tools enable us to develop the formulations for time evolution and measurement theory that are equivalent to those in the traditional quantum mechanics. However, the conceptual different be-
tween the relational formulation and traditional formulation is clearly demonstrated in the EPR argument, as shown in Section VII. Without explicitly calling of the reference system of a quantum state may lead to the assumption that there exist a Super Observer and an observer-independent quantum state, and that results in the EPR paradox. By removing the assumption of the Super Observer and emphasizing the relational nature of a quantum state, the EPR paradox can be resolved.

However, the relational nature of a quantum state does not imply a quantum state is subjective. If an external observer $O_E$ does obtain the same information of the relational matrix $R$ as to the intrinsic observer $O_T$, $O_E$ then comes to an equivalent description of $S$ as of $O_T$. This is significant since it gives the meaning of objectivity of a quantum state. Objectivity can be defined as the ability of different observers coming to a consensus independently. If $O_E$ is out of synchronization on the latest information of $R$, for instance, there is update on $R$ due to measurement and not known to $O_E$, $O_E$ can have different descriptions of $S$.

This synchronization of latest information is operational, but it is necessary for consistent descriptions of the same quantum system from different observers. If the observed quantum system consists entangled subsystems that are space-like separated, classical communication can be used among local observers to avoid potential paradox caused by assuming the existing of the Super Observer. The technique of Local Operation and Classical Communication (LOCC) has been widely used in quantum information theory such as entanglement concentration and entanglement dilution [17, 19].

One interesting consequence is that it will be a challenge to construct a quantum mechanics description of the Universe. Suppose an observer is able to construct a wave function of the Universe at a given time. Then, a space event occurs, for example, one remote galaxy merges with another galaxy. If the observer is not aware of such event, the initial wave function becomes inaccurate. Since we abandon the existence of Super Observer, it appears impractical to construct an operational quantum description of the Universe. New methodology similar to LOCC is needed to resolve such a limitation.

### C. Mutual Information versus Entanglement Measure

Entanglement between the two systems is measured by the parameter $E(\rho)$ as defined by Eq. (13). In Section III we also use the mutual information variable to measure the information exchange between the measuring system and the measured system. The mutual information between $S$ and $A$ is defined as $I = H(\rho_S) + H(\rho_A) - H(\rho_{SA})$, where $H(\rho_{SA})$ is the von Neumann entropy of the composite system $S + A$. For a composite system $S + A$ that is described by a single relational matrix $R$, these two variables differ only by a factor of two. However, for a composite system of $S + A$ that is described by an ensemble of relational matrices, the two variables can be very different. This is illustrated by two examples described below.

**Case 1.** $S + A$ is in an entangled pure state described by $|\Psi\rangle_{SA} = \sum_i \lambda_i |s_i\rangle |a_i\rangle$ in Schmidt decomposition, where $\{\lambda_i\}$ are the Schmidt coefficients. Subsystem $S$ is in a mixed state. The entanglement measure is $E(\rho_S) = -\sum_i \lambda_i^2 \ln(\lambda_i^2)$ and the mutual information is $I = -2 \sum_i \lambda_i^2 \ln(\lambda_i^2)$.

**Case 2.** $S + A$ is in a mixed state described by $\rho_{SA} = \sum_i \lambda_i^2 |s_i\rangle \langle a_i| |a_i\rangle$. In the case, $H(\rho_S) = H(\rho_{SA}) = H(\rho_{\text{diag}}) = -\sum_i \lambda_i^2 \ln(\lambda_i^2)$. $\rho_{SA}$ is a separable bipartite state [13, 19]. There is no entanglement but there is mutual information since $I = -\sum_i \lambda_i^2 \ln(\lambda_i^2)$. Essentially the composite system is a mixed ensemble of product states $\{\lambda_i^2, |s_i\rangle |a_i\rangle\}$. One can infer that $S$ is in $|s_i\rangle$ from knowing $A$ is in $|a_i\rangle$, however such mutual information is due to classical correlation. The probability of finding $S$ in an eigenvector $|s_i\rangle$ is just the classical probability $\lambda_i^2$.

Although the reduced density operator for $S$, $\rho_S = \sum_i \lambda_i^2 |s_i\rangle \langle s_i|$, is the same in Case 1 and Case 2, the mutual information is different. More information is encoded in the pure bipartite state in Case 1. When $S + A$ is described by $|\Psi\rangle_{SA} = \sum_i \lambda_i |s_i\rangle |a_i\rangle$, besides the inference information between $S$ and $A$, there is additional indeterminacy due to the superposition at the composite system level. For instance, one cannot determine the composite system $S + A$ is in $|s_0\rangle |a_0\rangle$ or $|s_1\rangle |a_1\rangle$ before measurement. More indeterminacy before measurement means more information can be gained from measurement. On the other hand, such indeterminacy does not exist when $S + A$ is described by a mixture of product state as in Case 2. Since such indeterminacy is for the composite system as a whole, the reduced density operator for a subsystem $S$, $\rho_S$, cannot reflect the difference, therefore it appears the same in Case 1 and Case 2.

These two examples show that mutual information variable can substitute the entanglement measurement only when the composite system $S + A$ is described by a single relational matrix $R$. To quantify change of quantum correlation during a measurement, the entanglement measurement is a more appropriate parameter.

### D. Conclusions

As expressed philosophically in Ref. [24], the physical world is made of processes instead of objects, and the properties are described in terms of relationships between events. Ref. [12] and this paper together show that quantum mechanics can be constructed by shifting the starting point from the independent properties of a quantum system to the relational properties among quantum systems. In essence, quantum mechanics demands a new set of rules to calculate probability of a potential outcome from the physical interaction in quantum measurement.
Based on the basic formulations in Ref. [12], this paper further develops the formulations for quantum measurement and quantum operation. The formulation is compatible with the traditional quantum mechanics mathematically. For instance, it is equivalent to the Open Quantum System theory if we replace the environment system in OQS with the reference apparatus system in this formulation. More importantly, the formulation presented here gives additional insight on how mutual information is exchanged during measurement.

At the conceptual level, the difference between the relational formulation and traditional formulation results in fundamental consequence in some special scenario. This is demonstrated in the EPR paradox. The paradox is seemingly inevitable in traditional quantum mechanics but can be resolved by removing the assumption of the Super Observer. The completeness of quantum mechanics and locality can coexist by redefining the element of physical reality to be observer-dependent. Not having an absolute physical reality might be difficult to accept. However, the objectivity of a quantum state is still preserved in the sense that an observer always comes to the same description of a quantum system if she has the latest information of the relational matrix. The conceptual subtlety of the relativity and objectivity of a quantum state is not obvious to recognize in traditional quantum mechanics. The reformulation presented here provides more clarity to these subtle concepts. We hope it can be one step towards a better understanding of quantum mechanics.

[1] N. Bohr, Quantum Mechanics and Physical Reality, Nature 136, 65 (1935)
[2] N. Bohr, Can Quantum Mechanical Description of Physical Reality Be Considered Completed? Phys. Rev., 48, 696-702 (1935)
[3] M. Jammer, The Philosophy of Quantum Mechanics: The Interpretations of Quantum Mechanics in Historical Perspective, Chapter 6. New York: Wiley-Interscience, (1974)
[4] H. Everett, Relative State Formulation of Quantum Mechanics, Rev of Mod Phys 29, 454 (1957)
[5] J. A. Wheeler, Assessment of Everett’s ”Relative State” Formulation of Quantum Theory, Rev of Mod Phys 29, 463 (1957)
[6] B. S. DeWitt, Quantum mechanics and reality, Physics Today 23, 30 (1970)
[7] W. H. Zurek, Environment-induced Superselection Rules, Phys. Rev. D 26, 1862 (1982)
[8] W. H. Zurek, Decoherence, Einselection, and the Quantum Origins of the Classical, Rev. of Mod. Phys. 75, 715 (2003)
[9] M. Schlosshauer, Decoherence, The Measurement Problem, and Interpretation of Quantum Mechanics, Rev. Mod. Phys. 76, 1267-1305 (2004)
[10] C. Rovelli, Relational Quantum Mechanics, Int. J. of Theo. Phys., 35, 1637-1678 (1996)
[11] M. Smerlak and C. Rovelli, Relational EPR, Found. Phys., 37, 427-445 (2007)
[12] J. M. Yang, Quantum Mechanics from Relational Properties – Part I: Formulation. arXiv:1706.01317
[13] A. Einstein, B. Podolsky, and N. Rosen, Can Quantum-Mechanical Description of Physical Reality Be Considered Complete? Phys. Rev. 47, 777-780 (1935)
[14] J. Von Neumann, Mathematical Foundations of Quantum Mechanics, Chap. VI. Princeton University Press, Princeton Translated by Robert T. Beyer (1932/1955)
[15] C. A. Fuchs, Quantum Mechanics as Quantum Information (and only a little more). arXiv:quant-ph/0205039 (2002)
[16] C. A. Fuchs and R. Schack, Quantum-Bayesian Coherence: The No-Nonsense Version, Rev. Mod. Phys. 85, 1693-1715 (2013)
[17] M. A. Nielsen and I. L. Chuang, Quantum computation and quantum information. Cambridge University Press, Cambridge (2000)
[18] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum Entanglement, Rev. Mod. Phys., 81, 865-942 (2009)
[19] M. Hayashi, S. Ishizaka, A. Kawachi, G. Kimura, and T. Ogawa, Introduction to Quantum Information Science, page 90, 150, 152, 197. Springer-Verlag, Berlin Heidelberg (2015)
[20] Nielsen, M. A., Chuang, I. L., Quantum computation and quantum information, page 366, 564. Cambridge University Press, Cambridge (2000)
[21] Allahverdyan, A. E., Roger Balian, R., Nieuwenhuizen, T. M., Understanding quantum measurement from the solution of dynamical models, Phys. Rep. 525, 1-166 (2013)
[22] Breuer, H.-P., Petruccione, F., The Theory of Open Quantum Systems. Oxford University Press, Oxford New York (2007)
[23] Rivas, A., Huelga, S. F., Open Quantum System, An Introduction. Springer-Verlag, Berlin Heidelberg (2012)
[24] L. Smolin, Three Roads to Quantum Gravity. Basic Books, New York (2017)

Appendix A: Theorem 1

**Theorem 1** $H(R) = 0$ if and only if the matrix element $R_{ij}$ can be decomposed as $R_{ij} = c_i d_j$, where $c_i$ and $d_j$ are complex numbers.

**Proof:** According to the singular value decomposition, the relational matrix $R$ can be decomposed to $R = UV^\dagger$, where $D$ is rectangular diagonal and both $U$ and $V$ are $N \times N$ and $M \times M$ unitary matrix, respectively. This gives $\rho = R R^\dagger = U (D D^\dagger) U^\dagger$. If $H(R) = 0$, matrix $\rho$ is a rank one matrix, therefore $D D^\dagger$ is $\text{diag}(1, 0, 0...)$.

This means $D$ is a rectangular diagonal matrix with with only one eigenvalue $e^{i\phi}$. Expanding the matrix product...
\[ R = UDV \] gives

\[ R_{ij} = \sum_{nm} U_{in} D_{nm} V_{mj} = U_{i1} e^{i\phi} V_{1j}. \]  \hspace{1cm} (A1)

We just choose \( c_i = U_{i1} \) and \( d_j = e^{i\phi} V_{1j} \) to get \( R_{ij} = c_i d_j \). Conversely, if \( R_{ij} = c_i d_j \), \( R \) can be written as outer product of two vectors,

\[ R = (c_1 \ c_2 \ \cdots \ c_n)^T \times (d_1 \ d_2 \ \cdots \ d_m). \]  \hspace{1cm} (A2)

Considering vector \( C_1 = \{c_1, c_2, \ldots, c_n\} \) as an eigenvector in Hilbert space \( \mathcal{H}_S \), one can use the Gram-Schmidt procedure \cite{17} to find orthogonal basis set \( C_2, \ldots, C_n \). Similarly, considering vector \( D_1 = \{d_1, d_2, \ldots, d_m\} \) as an eigenvector in Hilbert space \( \mathcal{H}_A \), one can find orthogonal basis set \( D_2, \ldots, D_m \). Under the new orthogonal eigenbasis, \( R \) becomes a rectangular diagonal matrix \( D = \text{diag}\{1, 0, 0\ldots\} \). Therefore \( R = UDV \) where \( U \) and \( V \) are two unitary matrices associated with the eigenbasis transformations. Then \( \rho = RR^\dagger = U(DD^\dagger)U^\dagger \), and \( DD^\dagger = \text{diag}\{1, 0, 0\ldots\} \) is a square diagonal matrix. Since the eigenvalues of similar matrices are the same, the eigenvalues of \( \rho \) are \( (1, 0, \ldots) \), thus \( H(R) = 0 \).

**Appendix B: Decomposition of the Unitary Operator of a Bipartite System**

If there is interaction between \( S \) and \( A \), and the initial state of \( S + A \) is a product state, the global unitary operator can be decomposed into a set of measurement operators that satisfies Eq. (16). The proof shown here closely follows idea from Ref. [19]. Denote the initial state as product state, \( |\Psi_0\rangle = |\psi_0\rangle_A |\phi_0\rangle_s \). First we change the eigenbasis for \( A \) through a local unitary operator \( \hat{U}_A \) that satisfies Eq. (16). Define a linear operator \( \hat{M}_m \) as what we are looking for, since we can verify it satisfies Eq. (16),

\[ \hat{U}_A |\psi_0\rangle |\phi_0\rangle = \hat{U}_A |\psi_0\rangle |\phi_0\rangle = \sum_m \hat{M}_m |\psi_0\rangle |\phi_0\rangle. \]  \hspace{1cm} (B1)

The completeness condition can also be verified,

\[ \sum_m \hat{M}_m^\dagger \hat{M}_m = \sum_m \langle a_0 | \hat{U}_S^\dagger | a_m \rangle \langle a_m | \hat{U}_S | a_0 \rangle = \langle a_0 | \hat{U}_S^\dagger \hat{U}_S | a_0 \rangle = I_S. \]  \hspace{1cm} (B2)

**Appendix C: Theorem 2**

**Theorem 2** Applying operator \( Q \otimes O \) over the composite system \( S + A \) is equivalent to change the relational matrix \( R \) to \( R' = QRO^T \), where the superscript \( T \) represents a transposition.

**Proof:** Denote the initial state vector of the composite system as \( |\Psi_0\rangle = \sum_i R_{ij}|s_i\rangle|a_j\rangle \). Apply the composite operator \( \hat{Q}(t) \otimes \hat{O}(t) \) to the initial state,

\[ |\Psi_1\rangle = (\hat{Q} \otimes \hat{O}) \sum_{ij} R_{ij}|s_i\rangle \otimes |a_j\rangle \]

\[ = \sum_{ij} R_{ij} \hat{Q}|s_i\rangle \otimes \hat{O}|a_j\rangle \]

\[ = \sum_{ij} \sum_{mn} R_{ij} Q_{mn} |s_m\rangle \otimes |a_n\rangle \]

\[ = \sum_{mn} \sum_{ij} (QRO^T)_{mn} |s_m\rangle \otimes |a_n\rangle \]

where \( T \) represents the transposition of matrix. Compared the above equation to Eq. (1) for the definition of \( |\Psi_1\rangle \), it is clear that the relational matrix is changed to \( R' = QRO^T \).

**Appendix D: Probability in Selective Measurement**

Given the composite system \( S + A \) described by Eq. (15), the reduced density matrix of \( S \) can be defined

\[ \hat{\rho}_S = Tr_A |\Psi\rangle \langle \Psi| = \sum_{ii'} \sum_k \langle R_{ik} R^*_{ik}|s_i\rangle \langle s_{i'}| \]

\[ = \sum_{ii'} (RR^\dagger)_{ii'} |s_i\rangle \langle s_{i'}| \]  \hspace{1cm} (D1)

and the probability of finding event \( |s_i\rangle \) occurred to \( S \) is calculated by Eq. (10). Similarly, the probability of event \( |a_j\rangle \) occurred to \( A \) is \( p_j^A = \sum_i p_{ij} = \sum_i |R_{ij}|^2 \). This can be more elegantly written by introducing a partial projection operator \( P^S \otimes P^A \) where \( P^A = |a_j\rangle \langle a_j| \). It is easy to verify that

\[ p_j^A = \langle \Psi | P^S \otimes P^A | \Psi \rangle \]

\[ = \langle \Psi | a_j \langle a_j | \Psi \rangle = \sum_i |R_{ij}|^2. \]  \hspace{1cm} (D2)