Integrable Hamiltonian systems generated by antisymmetric matrices

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Abstract. We construct a family of integrable systems generated by the Casimir functions of Lie algebra of skew-symmetric matrices, where the Lie bracket is deformed by a symmetric matrix.

1. Introduction
Let $\mathcal{A}(n)$ be the vector space of antisymmetric $n \times n$ matrices and $\text{Sym}(n)$ be the vector space of symmetric $n \times n$ matrices. The $(\mathcal{A}(n), [\cdot, \cdot]_S)$ is a Lie algebra with the $S$–bracket defined by a deformed commutator

$$[X, Y]_S = XSY - YSX$$

for fixed $S \in \text{Sym}(n)$ and $X, Y \in \mathcal{A}(n)$, (see [4, 5]).

In this paper we construct the family of integrable systems — a hierarchy generated by the Casimir functions on the dual of Lie algebra $\mathcal{A}(n)$. We prove that the integrals of this family of Hamiltonian systems are in involution.

The idea of considering these systems comes from [1]. In this paper we present more general case, which reduces to the case considering in [1] if we put that the matrix $S = 1$. Also in [3] the authors studied similar systems in the complex setting and for matrices with a different internal structure.

2. Hierarchy generated by Casimir functions
We identify $\mathcal{A}(n)$ with its dual $\mathcal{A}^*(n) \cong \mathcal{A}(n)$ using natural non-degenerate pairing by trace of the product

$$\langle X, \rho \rangle = \text{Tr}(\rho X), \quad \rho \in \mathcal{A}^*(n), \quad X \in \mathcal{A}(n).$$

We shall write a general element $\mathcal{A}(n)$ as

$$X = \begin{pmatrix} A & B \\ -B^\top & C \end{pmatrix},$$

where $A \in \mathcal{A}(2), C \in \mathcal{A}(n-2)$ and $B \in \text{Mat}_{2 \times (n-2)}(\mathbb{R})$. Having Lie algebra $(\mathcal{A}(n), [\cdot, \cdot]_S)$ one defines the Lie-Poisson bracket on $C^\infty(\mathcal{A}(n))$ by

$$\{f, g\}_S = \text{Tr}\left(X \begin{pmatrix} \frac{\partial f}{\partial X} & \frac{\partial g}{\partial X} \end{pmatrix}_S\right), \quad f, g \in C^\infty(\mathcal{A}(n)),$$
where
\[
\frac{\partial f}{\partial X} = \left( \frac{\partial f}{\partial A}, \frac{\partial f}{\partial B} \right) \quad (5)
\]

In order to obtain the second Poisson bracket ("frozen" Poisson bracket) we fix the element \( X_0 \in \mathcal{A}(n) \) and put
\[
\{f, g\}_{FS} = \text{Tr} \left( X_0 \left[ \frac{\partial f}{\partial X}, \frac{\partial g}{\partial X} \right]_S \right), \quad f, g \in C^\infty(\mathcal{A}(n)). \quad (6)
\]

It is a general fact that the Lie-Poisson bracket and frozen bracket are compatible in the sense that their linear combination
\[
\alpha \{\cdot, \cdot\}_S + \beta \{\cdot, \cdot\}_{FS} \quad (7)
\]
is also a Poisson bracket.

We shall choose
\[
X_0 = \frac{1}{2} \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} S_1 & S_3 \\ S_3^\top & S_2 \end{pmatrix}, \quad (8)
\]
where \( A_0 \) is \( 2 \times 2 \) matrix defined by
\[
A_0 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (9)
\]

After simple calculation we show that the Poisson bracket (6) can be written in the form
\[
\{f, g\}_{FS} = \text{Tr} \left( \left( \frac{\partial g}{\partial A} A_0 \frac{\partial f}{\partial B} - \frac{\partial f}{\partial A} A_0 \frac{\partial g}{\partial B} \right) S_3^\top \right) + \text{Tr} \left( \frac{\partial f}{\partial B} A_0 \frac{\partial g}{\partial B} S_2 \right) . \quad (10)
\]

**Basic assumption.** From now we put the block \( S_3 \) equal to zero \((S_3 \equiv 0)\). After reducing to this case we obtain
\[
\{f, g\}_{FS} = \text{Tr} \left( \frac{\partial f}{\partial B} A_0 \frac{\partial g}{\partial B} S_2 \right). \quad (11)
\]
Thus, we can think of this bracket as being defined on \( C^\infty(\text{Mat}_{2 \times (n-2)}(\mathbb{R})) \) (thus \( \mathcal{A}(n) \to \text{Mat}_{2 \times (n-2)}(\mathbb{R}) \) is injective smooth Poisson map).

In the case when \( \det S \neq 0 \) the Casimir functions for the Lie–Poisson bracket (4) are given by
\[
C_k(X) = \frac{1}{2k} \text{Tr}(X S^{-1})^{2k}, \quad k = 1, 2, \ldots \quad (12)
\]
see [5]. For the degenerate case when \( S_1 \equiv 0 \) we know only some Casimir functions of the following form
\[
C_k(X) = \frac{1}{k} \text{Tr} \left( B^\top B S_2^{-1} \right)^k, \quad k = 1, 2, \ldots, \quad (13)
\]
(see [2] for the case \( S_2 = 1 \)). In this case the Lie-Poisson bracket (4) can be rewritten in the form
\[
\{f, g\}_S = 2 \text{Tr} \left( \frac{\partial f}{\partial B} A_0 \frac{\partial g}{\partial B} S_2 + \frac{\partial g}{\partial C} C \frac{\partial f}{\partial C} S_2 \right) + 2 \text{Tr} \left( \left( \frac{\partial f}{\partial B} B \frac{\partial g}{\partial C} - \frac{\partial g}{\partial B} B \frac{\partial f}{\partial C} \right) S_2 \right). \quad (14)
\]
Now, we show that the functions given by (13) are Casimir functions for the bracket (14). Since the derivative of $C_k$ is

$$\frac{\partial C_k}{\partial B} = 2BS_2^{-1} \left( B^\top BS_2^{-1} \right)^{k-1},$$

$$\frac{\partial C_k}{\partial B^\top} = 2S_2^{-1} \left( B^\top BS_2^{-1} \right)^{k-1} B^\top,$$

$$\frac{\partial C_k}{\partial C} = 0,$$

we have

$$\{C_k, C_l\}_\mathcal{S} = \text{Tr} \left( \frac{\partial C_k}{\partial B^\top} A \frac{\partial C_l}{\partial B} S_2 \right) = 4 \text{Tr} \left( S_2^{-1} \left( B^\top BS_2^{-1} \right)^{k-1} B^\top ABS_2^{-1} \left( B^\top BS_2^{-1} \right)^{l-1} S_2 \right)$$

$$= 4 \text{Tr} \left( B^\top ABS_2^{-1} \left( B^\top BS_2^{-1} \right)^{k+l-2} \right) = 0 = \{C_k, C_l\}_{FS},$$

because the matrix $B^\top AB$ is antisymmetric and $S_2^{-1} \left( B^\top BS_2^{-1} \right)^{k+l-2}$ is symmetric. Moreover, we have the following proposition.

**Proposition 1** The Casimir functions $C_k$ defined by (12) or (13) for the Lie-Poisson bracket (4) considered as functions of $B$ are in involution with respect to the frozen bracket (11).

**Proof 1** Since the derivative of $C_k$ given by (12) is

$$\frac{\partial C_k}{\partial B} = -2P_+ (XS^{-1})^{2k-1} P_-,$$

$$\frac{\partial C_k}{\partial B^\top} = 2P_- (S^{-1}X)^{2k-1} P_+,$$

where $P_+, P_-$ are the orthogonal projectors given, in block matrix notation, by

$$P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$ (21)

After a direct calculation we obtain

$$\{C_k, C_l\}_{FS} = \text{Tr} \left( \frac{\partial C_k}{\partial B^\top} A \frac{\partial C_l}{\partial B} S_2 \right) = 4 \text{Tr} \left( P_- (S^{-1}X)^{2k-1} P_+ (XS^{-1})^{2l-1} P_- S_2 \right)$$

$$= -4 \text{Tr} \left( P_- (XS^{-1})^{2k-2} XP_+ A_0 P_+ (XS^{-1})^{2l-1} P_- \right)$$

$$= -4 \text{Tr} \left( (XS^{-1})^{2k-2} XP_+ A_0 P_+ (XS^{-1})^{2l-1} P_- \right) + 4 \text{Tr} \left( P_+ (XS^{-1})^{2k-2} XP_+ A_0 P_+ (XS^{-1})^{2l-1} P_+ \right)$$

$$= -4 \text{Tr} \left( (XS^{-1})^{k+l-2} XP_+ A_0 P_+ X (S^{-1}X)^{k+l-2} S^{-1} \right) = 0.$$ (22)

Above vanishes because in the first term we have a product of three antisymmetric $2 \times 2$ matrices which is also antisymmetric and in the second term we have a product of an antisymmetric matrix $(XS^{-1})^{k+l-2} XP_+ A_0 P_+ X (S^{-1}X)^{k+l-2}$ and symmetric matrix $S^{-1}$.

The proof of the involution of the functions (13) with respect to the frozen Poisson bracket (11), was given before this proposition.
Proposition 2 The smooth functions $\delta_k : \text{Mat}_{2 \times (n-2)}(\mathbb{R}) \rightarrow \mathbb{R}$ defined by

\[
\delta_k(B) = \text{Tr} \left( B S_2^{-1} (CS_2^{-1})^{2k-1} B^\top A_0 \right) \tag{23}
\]

are in involution with respect to the frozen Poisson bracket (11)

\[
\{\delta_k, \delta_l\}_{FS} = 0. \tag{24}
\]

Proof 2 Since the derivative of $\delta_k$ is

\[
\frac{\partial \delta_k}{\partial B} = 2 A_0 B S_2^{-1} (CS_2^{-1})^{2k-1}, \tag{25}
\]

\[
\frac{\partial \delta_k}{\partial B^\top} = 2 S_2^{-1} (CS_2^{-1})^{2k-1} B^\top A_0, \tag{26}
\]

we have

\[
\{\delta_k, \delta_l\}_{FS} = \text{Tr} \left( \frac{\partial \delta_k}{\partial B^\top} A_0 \frac{\partial \delta_l}{\partial B} S_2 \right) = \tag{27}
\]

\[
= 4 \text{Tr} \left( S_2^{-1} (CS_2^{-1})^{2k-1} B^\top A_0 A_0 B S_2^{-1} (CS_2^{-1})^{2l-1} S_2 \right) = \tag{28}
\]

\[
= -4 \text{Tr} \left( B^\top A_0 B S_2^{-1} (CS_2^{-1})^{2(k+l)-2} \right) = 0,
\]

because the matrix $B^\top A_0 B$ is antisymmetric and $S_2^{-1} (CS_2^{-1})^{2(k+l)-2}$ is symmetric.

Proposition 3 Assume that $S_1 = 1$. Then the functions $\delta_k$ and $C_l$ (given by (12)), $k,l = 1,2,\ldots$, are in involution with respect to the frozen Poisson bracket (11)

\[
\{\delta_k, C_l\}_{FS} = 0. \tag{29}
\]

Proof 3 First, we show that $\delta_1$ commutes with $C_k$ given by (12)

\[
\{\delta_1, C_k\}_{FS} = \text{Tr} \left( \frac{\partial \delta_1}{\partial B^\top} A_0 \frac{\partial C_k}{\partial B} \right) = \tag{30}
\]

\[
= -4 \text{Tr} \left( S_2^{-1} C S_2^{-1} B^\top A_0 A_0 P_+ (XS^{-1})^{2k-1} P_- S_2 \right) = \tag{31}
\]

\[
= 4 \text{Tr} \left( C S_2^{-1} B^\top P_+ (XS^{-1})^{2k-1} P_- \right) = \tag{32}
\]

\[
= -4 \text{Tr} \left( P_- (XS^{-1}) P_- X P_+ (XS^{-1})^{2k-1} P_- \right) = \tag{33}
\]

\[
= -4 \text{Tr} \left( (XS^{-1}) P_- X P_+ (XS^{-1})^{2k-1} \right) + \tag{34}
\]

\[
+ 4 \text{Tr} \left( P_+ (XS^{-1}) P_- X P_+ (XS^{-1})^{2k-1} P_+ \right) = \tag{35}
\]

\[
= -4 \text{Tr} \left( P_- X P_+ (XS^{-1})^{2k} \right) + \tag{36}
\]

\[
- 4 \text{Tr} \left( P_- X P_+ (XS^{-1})^{2k-1} (P_- X P_+)^\top S^{-1} \right) = \tag{37}
\]

\[
= -4 \text{Tr} \left( X P_+ (XS^{-1})^{2k} \right) + \tag{38}
\]

\[
+ 4 \text{Tr} \left( P_+ X P_+ (XS^{-1})^{2k} P_+ \right) = \tag{39}
\]

\[
= -4 \text{Tr} \left( P_+ (XS^{-1})^{2k} X P_+ \right) = 0,
\]
because $P_+ (X S^{-1})^{2k} XP_+$ is antisymmetric, $P_+ X P_+$ is antisymmetric and $P_+ (X S^{-1})^{2k} P_+$ is symmetric. Second, the functions $\delta_k$ and $C_l$ satisfy the following recursion formula

$$\{\delta_k, C_l\}_{FS} = \{\delta_{k-1}, C_{l+1}\}_{FS}. \quad (30)$$

Thus the relation (28) is valid for any $k$.

**Proposition 4** The functions $\delta_k$ and $C_l$ (given by (13)), $k, l = 1, 2, \ldots$, are in involution with respect to the frozen Poisson bracket (11)

$$\{\delta_k, C_l\}_{FS} = 0. \quad (31)$$

**Proof 4** For the functions $C_k$ given by (13) we have

$$\{\delta_k, C_l\}_{FS} = \text{Tr} \left( \frac{\partial \delta_k}{\partial B^\top} A_0 \frac{\partial C_l}{\partial B} \right) = 4 \text{Tr} \left( S_2^{-1} (C S_2^{-1})^{2k-1} B^\top A_0 A_0 B S_2^{-1} (B^\top B S_2^{-1})^{l-1} S_2 \right) = -4 \text{Tr} \left( (C S_2^{-1})^{2k} C S_2^{-1} (B^\top B S_2^{-1})^{l} \right) = 0, \quad (33)$$

because the matrix $(C S_2^{-1})^{2k}$ is antisymmetric and $S_2^{-1} (B^\top B S_2^{-1})^{l}$ is symmetric.

We obtain a hierarchy of Hamilton’s equations generated by Hamiltonians $C_k$ given by (12) or (13) with respect the frozen Poisson bracket (11)

$$\frac{\partial B}{\partial t_k} = A_0 \frac{\partial C_k}{\partial B} S_2, \quad k = 1, 2, \ldots \quad (34)$$

**Example 1** In this example we consider the case when $X$ is $5 \times 5$-matrix which we denote

$$X = \begin{pmatrix} 0 & a & p_1 & p_2 & p_3 \\ -a & 0 & q_1 & q_2 & q_3 \\ -p_1 & -q_1 & 0 & -c_3 & c_2 \\ -p_2 & -q_2 & c_3 & 0 & -c_1 \\ -p_3 & -q_3 & -c_2 & c_1 & 0 \end{pmatrix} \quad (35)$$

and matrix $S$ is degenerate, that mean $S_1 = 0$ and

$$S_2 = \begin{pmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix}. \quad (36)$$

The frozen Poisson bracket in this case is

$$\{f, g\}_{FS}(p_1, q_i) = e_1 \left( \frac{\partial f}{\partial p_1} \frac{\partial g}{\partial q_1} - \frac{\partial f}{\partial q_1} \frac{\partial g}{\partial p_1} \right) + e_2 \left( \frac{\partial f}{\partial p_2} \frac{\partial g}{\partial q_2} - \frac{\partial f}{\partial q_2} \frac{\partial g}{\partial p_2} \right) + e_3 \left( \frac{\partial f}{\partial p_3} \frac{\partial g}{\partial q_3} - \frac{\partial f}{\partial q_3} \frac{\partial g}{\partial p_3} \right). \quad (37)$$
The integrals in involution are

\[ C_1 = (S_2^{-1} \tilde{p}) \cdot \tilde{p} + (S_2^{-1} \tilde{q}) \cdot \tilde{q}, \]  
\[ C_2 = \frac{1}{2} C_1^2 - \left( (S_2^{-1} \tilde{q}) \times (S_2^{-1} \tilde{p}) \right) \cdot (\tilde{q} \times \tilde{p}), \]  
\[ \delta_1 = -2 C_1 \cdot \left( (S_2^{-1} \tilde{q}) \times (S_2^{-1} \tilde{p}) \right). \]  

Hamilton’s equations for the Hamiltonian \( C_1 \) are

\[ \frac{\partial \tilde{p}}{\partial t} = 2 \tilde{q}, \]  
\[ \frac{\partial \tilde{q}}{\partial t} = -2 \tilde{p}. \]  

Hamilton’s equations for the Hamiltonian \( C_2 \) are

\[ \frac{\partial \tilde{p}}{\partial t} = 2 \left( C_1 \tilde{q} + (S_2^{-1} \tilde{p}) \times (\tilde{p} \times \tilde{q}) \right), \]  
\[ \frac{\partial \tilde{q}}{\partial t} = 2 \left( -C_1 \tilde{p} + (S_2^{-1} \tilde{q}) \times (\tilde{p} \times \tilde{q}) \right). \]  

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