On skewon modification of light cone structure

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Electromagnetic media with generic linear response provide a rich class of Lorentz violation models. In the framework of a general covariant metric-free approach, we study electromagnetic wave propagation in this media. We define the notion of an optic tensor and present its unique canonical irreducible decomposition into the principle and skewon parts. The skewon contribution to the Minkowski vacuum is a subject that is not embedded in the ordinary models of Lorentz violation based on a modified Lagrangian. We derive the covector parametrization of the skewon optic tensor and discuss its $U(1)$-gauge symmetry. This way, we obtain several compact expressions for the contribution of the principle and skewon optic tensor into dispersion relation. As an application of the technique proposed here, we consider the case of a generic skewon tensor contributed to a simple metric-type principle part. Our main result: Every solution of the skewon modified Minkowski dispersion relation is necessary spacelike or null. It proves a hard violation of the Lorentz symmetry. The case of the antisymmetric skewon is studied in detail and some new special cases (electric, magnetic, and degenerate) are find out. In the case of a skewon represented by symmetric matrix, we observe a fact of the parametric gap that has some similarity to the Higgs model. We worked out a set of specific examples that justify the generic properties of the skewon models and demonstrate the different types of the Lorentz violation phenomena.

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I. INTRODUCTION

As it is well known, the plain wave solution of Maxwell electrodynamics system yields the dispersion relation $\omega^2 - k^2 = 0$, that is the characteristic expression of Minkowski geometry. Propagation of electromagnetic waves on a curved manifold of GR is governed by the dispersion relation of a similar form $q^2 q_{ij} = 0$, where $q_i = (\omega, k)$ is a wave covector. From this basic expression of the pseudo-Riemann geometry one immediately derives that GR does not modify the point-wise light cone structure.

At the other hand, modern field theories usually predict crucial modifications of the light cone structure expressed by an anisotropic dispersion relation, a birefringent vacuum and a violation of local Lorentz and CPT invariance. Such theoretical phenomena emerge in loop quantum gravity [1], [2], string theory [3], [4] and the very special relativity models [5], [6]. Since these models are yet very far from their complete form and observational predictions, it is very important to have a phenomenological model that involves the indicated optics phenomena.

A well known achievement in this direction is the standard model extension (SME) construction due to Kostelecky and his collaborators [7], [8], [9], [10]. In the electromagnetic sector of this construction, one starts with a covariant extension of the electromagnetic Lagrangian involving a set of numerical parameters: the resulting field equations indeed yield solutions with breaking CPT symmetry and birefringence. However, the individual terms in the Lagrangian, apart from their covariance property, remain without a clear physical description.

Another approach is based on a generalization of Riemann geometry to a generic Finsler geometry, see [11], [12] for theoretical description, and [13], [14] for observable effects analysis. Since Finslerian geometry is anisotropic due to its very definition, the modified dispersion relation with CPT-violation and birefringence naturally emerges in this construction. What is unclear, however, is the physical reasons for the Finslerian modification of space-time structure. Moreover, Finsler geometry of the Lorentz type signature has serious problems in its very definition, that have not yet reached a final resolution, see [15], [16].

In this paper we study an alternative approach inspired by the ideas from solid state physics. It is well known that to a considerable extent the original formulation of Maxwell’s electrodynamics and GR uses the medium analogy. It is rather natural to expect that the solid state analogies may be useful for the modern modifications of GR and other field theories. It should be taken into account that the classical GR is dealing only with a relativity simple second order metric tensor while solid state physics (elasticity and electromagnetism) uses tensors of a higher order. In modified field theory models, the higher order tensors appear in a natural way.

Electromagnetic wave propagation in media is a classical issue of mathematical physics. The standard presentation of this subject, e.g., [17], [18], [19], [20], is essentially non-relativistic (3-dimensional). Moreover, the consideration is usually restricted to an isotropic medium characterized by two scalar parameters $\varepsilon$ and $\mu$ or to an anisotropic medium described by two symmetric $3 \times 3$ matrices $\varepsilon_{\alpha\beta}$ and $\mu_{\alpha\beta}$. It is clear, however, that a proper

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description of the wave propagation phenomena must be presented in terms of 4-dimensional tensorial quantities defined on space-time. This is due to the principal fact that the existence of the wave propagation is independent of a motion of an observer.

Furthermore, the propagation of waves in an anisotropic medium must be described by the medium parameters. It is not directly connected to the vacuum Minkowski metric. Thus the description of the electromagnetic wave phenomena must be independent of the metric structure of the space-time. It is well known, [21], [22], [23], [24], that such metric-free description of classical electromagnetism is admissible. In this generic formalism, vacuum electromodynamics on a flat space-time and even on a curved space-time of general relativity is presented as specially simple cases. Recent investigations, see [25] and the reference given therein, established the metric-free electromagnetics as a logically complete construction based on the conservation laws for the electric charge, the magnetic flux, and the energy-momentum current.

One of the valuable achievements of this construction is the proof that the most general linear electromagnetic media is characterized by a set of 36 independent parameters that form a 4-th order pseudo-tensor $\chi_{ijkl}$. This is instead of 12 material parameters coming from two 6-parametric sets, $\varepsilon_{\alpha\beta}$ and $\mu_{\alpha\beta}$, of the classical representation. Due to the standard group theory paradigm, the pseudo-tensor $\chi_{ijkl}$ is decomposed into three independent pieces: (i) The principal part $(1)\chi_{ijkl}^{(1)}$ of 20 independent components; (ii) The skewon part $(2)\chi_{ijkl}^{(2)}$ of 15 independent components; (iii) The axion part $(3)\chi_{ijkl}^{(3)}$ that has 1 component. These pieces are unique, canonical, and irreducible under the action of the general linear group $GL(4, \mathbb{R})$. The principle part, $(1)\chi_{ijkl}^{(1)}$, comes as a covariant extension of the classical set of 12 components of the 3-dimensional tensors $\varepsilon_{\alpha\beta}$ and $\mu_{\alpha\beta}$. Two other parts, the skewon $(2)\chi_{ijkl}^{(2)}$ and the axion $(3)\chi_{ijkl}^{(3)}$, do not have classical analogs.

The current paper is devoted to the theoretical description of the wave propagation in generic linear response media. Our aim to understand the possible new physical phenomena coming from the additional electromagnetic parameters. Note a considerable recent progress in this area, [26], [27], [28], [29], [31], [32], [33], [34], [36], [37], [40], [41], [43]. We are interested mostly in the skewon effects on the light cone structure. It must be noted that skewon does not contribute to the electromagnetic Lagrangian, thus such effects cannot be accounted in Kostelecký’s construction no in Finsler space schemes.

The organization of the paper is as follows: In Section 2, we introduce our basic notations and present the integral Maxwell equations and jump conditions in a metric-free form. We discuss the linear constitutive relation. The decomposition of the constitutive tensor is based on the Young diagrams technique. It proves that the decomposition is irreducible and unique (canonical). In Section 3, we give a generic definition of the wave-type solutions and present the dispersion relation for these solutions. The characteristic system is derived from the jump conditions. We show that it is completely described by a Christoffel-type tensor $M_{ij}$ which is a close analog of the acoustic Christoffel 3D-matrix. We provide a decomposition of $M_{ij}$ into two independent optic tensors $P_{ij}$ and $Q_{ij}$ defined by the principal and the skewon part correspondingly. Moreover, the show that skewon part $Q_{ij}$ is of the form $q_i Y_j - q_j Y_i$. Thus the covector $Y_i$ completely describes skewon effects. In Section 4, the dispersion relation in terms of the tensors $P_{ij}$ and $Q_{ij}$ is derived. The whole contribution of the skewon into the light cone structure is presented by a compact term, which allows a qualitative analysis. Section 5 is devoted to an analysis of the skewon effects on the wave propagation in a (pseudo)Riemannian vacuum. In particular, we prove that an arbitrary skewon extends the light cone and thus yields superluminal wave propagation. In section 6 we study the special case of an antisymmetric skewon. We reinstate the known fact that the Fresnel surface separates into two surfaces. We prove that both surfaces have the Lorentz signature and thus can serve as light cones. Moreover we show that for antisymmetric skewon these cones intersect along two optic axes. The variety of mutual arrangement is still rich. In particular, we identify the electric, magnetic and degenerate types of asymmetric skewon. In section 7, the results about symmetric skewon are presented. We show that, in this case, the wave propagation is forbidden for small values of skewon parameters and reinstated only for a sufficient big values. This behavior is similar to the Higgs model from particle physics. We describe different parametrizations of the symmetric skewon that can be characterized by the rank of a corresponding matrix. Explicit examples are provided. In Conclusion section, we discuss our results and their possible extensions. Some technical details are presented into Appendix.

II. COVARIANT DESCRIPTION OF ELECTROMAGNETISM IN ANISOTROPIC MEDIA

A. Maxwell’s system

Our goal is to study the wave propagation in an electromagnetic medium with a generic local linear response. The vacuum case can be considered as a specific type of a unique “medium” completely characterized by the metric tensor. In a contrast, different types of real anisotropic media, that characterized by their own sets of parameters, may present simultaneously in the same problem. For instance, often we must deal with two media separated by a thin boundary. In phenomenological models, such a boundary is described by a smooth 2-dimensional surface that is stationary or moving in the 3-dimensional space. In the 4-dimensional space-time representation, the stationary or moving boundaries are described in
the same fashion as smooth 3-dimensional hypersurfaces. In this situation, some components of the field necessary have a jump through the boundary surface. Thus, in general, the electromagnetic field in media is non-differentiable and the ordinarily used partial differential field equations are not directly applicable. Quite similarly, the non-differentiable fields emerge at the wave fronts in geometrical optics even in the vacuum case.

It is well known that a precise mathematical way to deal with such non-differentiable fields is to consider the integral field equations instead of the differential ones. The most natural objects for the integration on a differential manifold is differential forms. Due to their definition, differential forms are correlated with the dimension of the integration domain and thus provide a covariant formulation of physics laws. In terms of differential forms, the integral Maxwell equations are given in a compact invariant form

$$\int_{M_2} F = 0, \quad \int_{N_2} \mathcal{H} = \int_{N_3} \mathcal{J}. \quad (2.1)$$

Here $F$ is an even (untwisted) 2-form of the field strength, while $\mathcal{H}$ is an odd (twisted) 2-form of the excitation, $\mathcal{J}$ is an odd 3-form of the electric current. The integration domain $M_2$ is a smooth closed 2-dimensional surface, while $N_3$ is a smooth 3-dimensional oriented hypersurface with a smooth boundary $N_2 = \partial N_3$. The case of the non-smooth domains sometimes also useful for physics application. The extension of the formulation given in Eqs. (2.1) is provided by the limiting procedure. In this paper, we restrict to the case of smooth integration domains.

Eqs. (2.1) have a clear physical interpretation. In particular, the first equation represents the conservation law of the magnetic flux. The second equation of (2.1) may be treated as a consequence of the electric current conservation law. For the details of such an interpretation, see [25].

### B. Covariant jump conditions and wavefront

A useful consequence of the integral Maxwell equations (2.1) is the covariant jump conditions, [41]. Consider an arbitrary smooth non-degenerate hypersurface $\Sigma$ in the space-time. Let its implicit description be given by the equation

$$\varphi(x^i) = 0. \quad (2.2)$$

Thus, at an arbitrary point of $\Sigma$, the covector $q_i = \partial \varphi / \partial x^i$ is assumed to be well-defined and non-zero. Here and in the sequel, the Roman indices take values in the range $\{0, 1, 2, 3\}$. We define two limit values

$$F_\pm = \lim_{\varepsilon \to 0} F(x^i)\rvert_{\varphi(x^i) = \pm \varepsilon}. \quad (2.3)$$

The jump of the field strength $F$ at the hypersurface $\Sigma$ is defined as the difference of these two values

$$[F] = F_+ - F_- . \quad (2.4)$$

Similar definition is assumed for the jump of the excitation $\mathcal{H}$.

The jump conditions are given by the covariant and metric-free equations, that take in the source-free case the form

$$[F] \wedge d\varphi = 0, \quad [\mathcal{H}] \wedge d\varphi = 0. \quad (2.5)$$

These equations are straightforward consequences of the integral field equations (2.1) providing the sources do not enter the region. In the tensorial form, Eqs. (2.5) read as

$$\varepsilon^{ijkl}[F_{jk}] \varphi_{,i} = 0, \quad [\mathcal{H}^{ij}] \varphi_{,j} = 0 . \quad (2.6)$$

Note that both sides of (2.6) depend on a point lying on the surface $\varphi(x^i) = 0$.

Since the jump conditions (2.5, 2.6) are metric-free, they can be used for a wide class of applications. The hyper-surface $\Sigma$ may represent:

1. A Cauchy initial surface that is ordinarily required to be time-like.
2. A boundary surface that separates two different media. It is ordinarily represented by a space-like hypersurface.
3. A wavefront that emerges in the 4-dimensional geometrical optics as a light-like hypersurface.

The covariant boundary surface problem was studied in [41], see also [30]. In particular, when the space-time decomposition of the fields $F$ and $\mathcal{H}$ is applied and the fields are assumed to be independent of the time coordinate, Eqs. (2.5) result into the standard boundary conditions for the 3-dimensional fields $E, B$ and $D, H$. In the current paper, we are dealing with the electromagnetic waves propagation, i.e., with the wavefront problems.

### C. Constitutive pseudotensor

Eqs. (2.1) present only the formal structure of electromagnetism. They are filled with a physical content only when a constitutive relation between the fields $\mathcal{H}$ and $F$ is postulated. In general, such relation can be non-linear and even non-local (as in ferromagnetics). In this paper, we consider the simplest but a rather widespread case of a linear, local constitutive relation:

$$\mathcal{H}^{ij} = \frac{1}{2} \chi^{ijkl} F_{kl}. \quad (2.7)$$

Due to its definition, the constitutive pseudotensor $\chi^{ijkl}$ possess the symmetries

$$\chi^{ijkl} = -\chi^{jikl} = -\chi^{ijlk}. \quad (2.8)$$

Hence, in 4-dimensional space, $\chi^{ijkl}$ has 36 independent components. In general, the constitutive pseudotensor is a field that depends of the point on the manifold.
A useful way to deal with a multi-component tensor, such as $\chi_{ijkl}$, is to decompose it into the sum of simpler independent sub-tensors with less numbers of components. Naturally, these parts themselves must be tensors of the same rank. Moreover, it is preferable to have a unique irreducible decomposition. In this case, to the partial sub-tensors can have independent physical meaning. So, we are looking for a decomposition of a 4-rank tensor defined on a 4-dimensional space-time. Moreover, we must take into account that $\chi_{ijkl}$ is not a general 4-th order tensor, it carries its original symmetries (2.8). In other words, tensor $\chi_{ijkl}$ itself constitutes a subspace of a generic tensor space. Consequently we need a decomposition of an invariant subspace into the direct sum of its invariant (sub)subspaces. The known way to derive such a decomposition is by applying the Young diagram technique. A brief account of this subject is given in [55]. Note that, a generic 4-rank tensor (without additional symmetries) is canonically decomposed into five sub-tensors. An important fact that these partial tensors are yet reducible. Their successive decomposition is irreducible but not canonical and thus non-unique. The situation is changed drastically when the decomposition of a tensor with additional symmetries is considered. First, the Littlewood-Richardson rule restricts the number of the relevant Young’s diagrams. In our case, there are only three different types of diagrams

![Diagram](image)

The dimensions of the subspaces depicted by the diagrams (2.9) are 20, 15, and 1, correspondingly. It proves that there is a unique decomposition of the number of the components

$$36 = 20 + 15 + 1.$$ \hfill (2.10)

Inasmuch as there are not invariant subspaces of smaller dimensions, the canonical decomposition of the constitutive pseudotensor into three pieces is unique and irreducible.

Following [25], we denote the decomposition depicted at (2.9) as

$$\chi_{ijkl} = (1)\chi_{ijkl} + (2)\chi_{ijkl} + (3)\chi_{ijkl}. \hfill (2.11)$$

Here the axion part $(3)\chi_{ijkl}$ is represented by the third diagram of (2.9). It means the complete antisymmetrization of $\chi_{ijkl}$ in all its four indices. This pseudotensor has only one independent component. Consequently it can be represented by a pseudoscalar $\alpha$,

$$(3)\chi_{ijkl} = \chi_{[ijkl]} = \alpha\varepsilon_{ijkl}. \hfill (2.12)$$

Such pseudoscalar is a rather popular object in particle physics. Note that in the current setting axion appears rather naturally and is not involved by hands.

The skewon part $(2)\chi_{ijkl}$ of 15 independent components corresponds to the middle diagram of (2.9). It is expressed as

$$\langle 2 \rangle\chi_{ijkl} = \frac{1}{2}(\chi_{ijkl} - \chi_{klij}). \hfill (2.13)$$

The principal part of 20 independent components corresponds to the first diagram of (2.9). It is expressed as

$$\langle 1 \rangle\chi_{ijkl} = \frac{1}{6}(2\chi_{ijkl} + 2\chi_{klij} - \chi_{ijkl} - \chi_{ijlk} - \chi_{iljk} - \chi_{jilk}). \hfill (2.14)$$

It is straightforward to check now that the sum of the terms (2.12), (2.13), and (2.14) is equal to the initial tensor $\chi_{ijkl}$. A derivation of the expressions (2.12), (2.13), and (2.14) can be done by the Young diagram technique. The result is equivalent to the derivation based on the 6 x 6 representation [25]. The group theory consideration provides a proof that this decomposition is canonical, unique and irreducible under the action of the general linear group.

### III. OPTICS TENSORS

Our goal is to study the propagation of electromagnetic waves in a medium with a generic constitutive pseudotensor $\chi_{ijkl}$. Particularly, we are interested in the contributions of the irreducible parts $(1)\chi_{ijkl}$, $(2)\chi_{ijkl}$ and $(3)\chi_{ijkl}$ to the shape of the Fresnel surface and to the corresponding light cone structure. In order to have a description that is valid for an arbitrary observer, we are looking for a covariant formulation.

#### A. Generalized wavefront and characteristic system

The first notion we need is a covariant description of the electromagnetic wave in media. Usually a wave is represented by a plane wave ansatz or by its generalization in series, see [42], [43] for a modern review of this non-covariant approach. Instead, we will apply the following covariant description that is similar to what is given in [44] and [45].

**Definition 1:** Generalized electromagnetic wave is a set of the solutions, $F$ and $\mathcal{H}$, of Maxwell’s system that are non-zero on one side of some hypresurface $\varphi(x^i) = 0$ and zero on its other side.

With this definition, the generic jump conditions (2.5) take the form of the wavefront conditions

$$F \wedge d\varphi = 0, \quad \mathcal{H} \wedge d\varphi = 0. \quad (3.1)$$

In tensorial representation with the wave covector $q_i = \partial \varphi / \partial x^i$, they read

$$\varepsilon^{ijkl} F_{jklq} = 0, \quad \mathcal{H}^{ij} q_j = 0. \quad (3.2)$$
The hypersurface $\varphi(x^i) = 0$ appearing in Definition 1 is referred to as a wavefront.

An explicit expression of the wavefront is determined from the uniqueness and existence conditions for the solutions of (3.2). When the linear constitutive relation (2.7) is substituted here we obtain

$$\varepsilon^{ijkl} q_j F_{kl} = 0, \quad \chi^{ijkl} q_j F_{kl} = 0. \quad (3.3)$$

This is linear system of 8 equations for 6 independent variables, the components of the tensor $F_{kl}$. However it is overdetermined only formally. Indeed, there are two linear identities between the equations. When the right hand sides of (3.3) are multiplied by $q_l$ they vanish identically due to the symmetries (2.8).

How can we deal with this formally overdetermined system? One can try to substitute the constraints into (3.3) in order to remain with a well definite system of 6 independent equations for 6 independent variables. Note that the field $F_{kl}$ is a measurable quantity, thus it is unique and a type of an existence and uniqueness theorem must hold: (i) For existence, the $6 \times 6$ determinant of the matrix of the coefficients must be equal to zero. (ii) For uniqueness, this determinant must be non-trivial. Thus one comes to the $6 \times 6$ determinant and correspondingly to a characteristic equation that is of the six order in $q$. This straightforward procedure explicitly violates the covariance of the system. So the question is: How can we deal with the overdetermined system (3.3) without violating its covariance?

In order to have a covariant derivation, we start with the first equation of (3.3). Its most general solution is of the form

$$F_{kl} = \frac{1}{2} (a_k q_l - a_l q_k) \quad (3.4)$$

with an arbitrary covector $a_k$. In fact, $a_k$ can be viewed as an algebraic analog of the standard electromagnetic potential. Substituting (3.4) into the second equation of (3.3) we obtain the characteristic system

$$M^{ik} a_k = 0, \quad (3.5)$$

where the characteristic matrix is

$$M^{ik} = \chi^{ijkl} q_j q_l. \quad (3.6)$$

[Eq.(3.5) is a linear system of 4 covariant equations for 4 components of the covector $a_k$. Now we face with a new problem: The relations

$$M^{ik} q_k = 0 \quad \text{and} \quad M^{ik} q_i = 0 \quad (3.7)$$

hold identically for the matrix defined in (3.6). They can be treated as linear relations between the rows and columns of the matrix $M_{ik}$. Consequently the system (3.5) is undetermined. In fact, it is a very expected situation. Indeed, in contrast to $F_{ij}$, the potential $a_k$ is an unmeasurable quantity. Consequently the solutions of (3.5) even cannot be unique. The relations (3.7) correspond to the principle physical facts – the gauge invariance and the charge conservation of our system, see [36]. Thus, in order to preserve the gauge invariance and the covariance of the characteristic system, we must proceed with an undetermined characteristic system and with a singular characteristic matrix $M^{ik}$.

B. Two optic tensors

The electromagnetic wave propagation is described now by a linear characteristic system $M^{ik} q_k = 0$. In particular, it is completely determined by the matrix $M^{ik}$. This matrix is an analog of the acoustic (Christoffel) $3 \times 3$ matrix which is used to describe the acoustic wave propagation in linear elasticity. We will refere to $M^{ik}$ as the optic tensor. Substituting the irreducible decomposition (2.11) into Eq.(3.6), we obtain

$$M^{ik} = (1) \chi^{ijkl} q_j q_l + (2) \chi^{ijkl} q_j q_l + (3) \chi^{ijkl} q_j q_l. \quad (3.8)$$

The last term here is a contraction of symmetric and skew-symmetric matrices, thus it vanishes identically

$$(3) \chi^{ijkl} q_j q_l = 0. \quad (3.9)$$

Consequently the axion part $(3) \chi^{ijkl}$ does not contribute to the wave propagation. This fact is not a contradiction to the known results about the axion modification of the dispersion relation [46]. In fact, the axion contribution emerges only in the higher order approximation [47], [48], [49]. In this paper we restrict to the standard geometric optics approximation that does not accounts the axion contributions.

Two other terms on the right hand side of Eq.(3.8) are non-zero in general. Consequently, the optic matrix is irreducibly decomposed into the sum of two terms

$$M^{ik} = P^{ik} + Q^{ik}. \quad (3.10)$$

Here, the principle optic tensor $P^{ik}$ and the skewon optic tensor $Q^{ik}$ are defined as

$$P^{ik} = (1) \chi^{ijkl} q_j q_l, \quad Q^{ik} = (2) \chi^{ijkl} q_j q_l. \quad (3.11)$$

Observe some basic properties of these two tensors:

1. Symmetry: The principle optic tensor $P^{ik}$ is symmetric while the skewon optic tensor $Q^{ik}$ is antisymmetric:

$$P^{ik} = P^{ki}, \quad Q^{ik} = -Q^{ki}. \quad (3.12)$$

2. Linear relations: Since the partial pseudo-tensors $(1) \chi^{ijkl}$ and $(2) \chi^{ijkl}$ preserve the symmetries of the original pseudo-tensor $\chi^{ijkl}$, the matrices $P^{ik}$ and $Q^{ik}$ satisfy the linear relations

$$P^{ik} q_k = 0, \quad \text{and} \quad Q^{ik} q_k = 0. \quad (3.13)$$
(3) **Determinants**: In an addition to the relation \( \det(M) = 0 \), we have
\[
\det(P) = 0, \quad \text{and} \quad \det(Q) = 0. \tag{3.14}
\]

(4) **Adjoint of the skewon optics tensor**: In order to have a non-trivial (non-zero) expression for the adjoint matrix, the rank of the original 4 \( \times \) 4-matrix must be equal to 3. But the rank of an arbitrary antisymmetric matrix is even. Thus our skewon matrix satisfies
\[
\text{adj}(Q) = 0. \tag{3.15}
\]

We will see in the sequel, that this fact yields that the skewon part alone does not emerge a non-trivial dispersion relation. Thus skewon can serve only as a supplement to the principle part.

**C. Skewon optics covector**

Since the skewon part of the constitutive tensor contributes to the wave propagation only via the antisymmetric tensor \( Q_{ij} \), it allows a simpler representation. We start with the relation
\[
Q^{ij}q_j = 0. \tag{3.16}
\]

It is convenient to define an auxiliary tensor
\[
\hat{Q}_{ij} = \frac{1}{2} \varepsilon_{ijmk} Q^{mk}, \quad \text{thus} \quad Q^{pq} = -\frac{1}{2} \varepsilon^{ijpq} \hat{Q}_{ij}. \tag{3.17}
\]

Equation (3.16) turns out to be
\[
\varepsilon^{ijpq} \hat{Q}_{ij} q_p = 0. \tag{3.18}
\]

This equation is somewhat simpler for a treatment because it has exactly the same form as the first equation of the Maxwell system (3.3). Hence the most general solution of (3.18) can be written in the form of (3.4),
\[
\hat{Q}_{ij} = Y_i q_j - Y_j q_i. \tag{3.19}
\]

with an arbitrary covector \( Y_i \). We will refer to \( Y_i \) as the skewon optic covector. Observe its basic properties:

1. Since \( \hat{Q}_{ij} \) is quadratic in \( q_i \), the components of \( Y_i \) are the first order homogeneous functions of \( q_i \);
2. \( Y_i \) is a covector density, because \( \hat{Q}_{ij} \) is a tensor density;
3. Due to (3.19), \( Y_i \) is defined only up to an arbitrary addition of the wave covector \( q_i \). So we have here a type of a gauge symmetry which is similar to the ordinary gauge symmetry of the Maxwell system;
4. An additional gauge fixing condition on \( Y_i \) can be applied.

Substituting (3.19) into Eq.(3.17) we obtain
\[
Q^{ij} = -\frac{1}{2} \varepsilon^{ijrs} (Y_r q_s - Y_s q_r). \tag{3.20}
\]

Using the skew-symmetry, this expression can be rewritten finally as
\[
Q^{ij} = \varepsilon^{ijkl} q_k Y_l. \tag{3.21}
\]

**D. Skewon in matrix representation**

Due to the gauge symmetry, the covector \( Y_i \) can be expressed as
\[
Y_i = S_i^j q_j + \alpha q_i \tag{3.22}
\]

with an arbitrary scalar \( \alpha \) that represents the gauge freedom. We observe that in this representation the tensor \( S_i^j \) and the scalar \( \alpha \) must be zero order homogeneous functions of the wave covector \( q_k \). Without loss of generality we can assume the tensor \( S_i^j \) to be traceless, \( S_i^i = 0 \), because its trace can be absorbed into the scalar \( \alpha \). Recall that the skewon has 15 independent components exactly as a traceless quadratic matrix. A representation of the skewon by a traceless mixed tensor was derived in [29],
\[
\chi_{ijkl} = \varepsilon_{ijkl} q_m S_m^i - \varepsilon_{ijkl} q_k S_k^i. \tag{3.23}
\]

Let us check that this expression meets exactly our definition (3.22). We substitute (3.23) into Eq.(3.11), thus the skewon optic tensor is represented as
\[
Q^{ik} = \chi_{ijkl} q_j q_l = \varepsilon_{ijkl} q_m S_m^i q_l. \tag{3.24}
\]

Rearranging the indices we rewrite it as
\[
Q^{ij} = \varepsilon^{ijkl} q_k (S_i^m q_m). \tag{3.25}
\]

Compare it now with the definition of the skewon covector (3.21). In the brackets of (3.25), we recognize (up to the \( \alpha \)-factor) the covector \( Y_i \) as it appears in Eq.(3.22). Thus the tensor \( S_i^j \) turns out to be independent of \( q \). Alternatively, the scalar \( \alpha \) remains an arbitrary zero order homogeneous functions of \( q_k \). Different choices of \( \alpha \) represent different gauge conditions. In the sequel, we will see that the Lorenz-type gauge condition for the covector \( Y_i \) requires a non-zero and even non-regular scalar function \( \alpha \).

**IV. DISPERSION RELATION**

**A. Covariant dispersion relation**

In order to have a non-trivial solution \( a_k \) of the characteristic system
\[
M^{ik}(q) a_k = 0. \tag{4.1}
\]
the components of the wave covector \( q_i \) must satisfy certain consistent relation. This dispersion relation is a polynomial algebraic equation with the coefficients depending on the media parameters. Usually an equation of the form (4.1) is consistent (has non-trivial solutions) when the relation \( \det(M) = 0 \) holds. The electromagnetic system, however, is a singular one, so its determinant vanishes identically. This fact is due to the gauge invariance of the system. Indeed, \( a_i \sim q_i \) is a solution of Eq.(4.1) that does not contribute to the electromagnetic field strength, so it is unphysical. Hence, the system (4.1) accepts a physical meaning, only if it has at least two linearly independent solutions — one for gauge freedom and one for physics. We recall a known fact from linear algebra: A linear system has two (or more) linearly independent solutions if and only if the rank of its matrix \( M^{ij} \) is of 2 (or less). It means that the adjoint of the matrix \( M^{ij} \) must be equal to zero. Recall that the adjoint matrix is constructed from the cofactors of \( M^{ij} \).

Thus, in electromagnetism, as well as in an arbitrary \( U(1) \)-gauge invariant system, existence of physically meaningful solutions of a system \( M^{ik}(q)\alpha_k = 0 \) requires

\[
\text{adj}(M) = 0. \tag{4.2}
\]

The standard expression of the 4-th order adjoint matrix \( A_{ij} = \text{adj}(M) \) is of the form

\[
A_{ij} = \frac{1}{3!} \varepsilon_{ijlms} \varepsilon_{jklms} M^{i1j1} M^{i2j2} M^{i3j3}. \tag{4.3}
\]

Consequently the dispersion relation reads

\[
\varepsilon_{ijlms} \varepsilon_{jklms} M^{i1j1} M^{i2j2} M^{i3j3} = 0. \tag{4.4}
\]

The condition (4.4) is somewhat unusual because it is given in a matrix form. However, we have here only one independent condition. Indeed, we can observe the following algebraic fact, see [36] for a proof. Let an \( n \times n \) matrix \( M^{ij} \) satisfies \( M^{ij} q_i = 0 \) and \( M^{ij} q_j = 0 \) for some \( n \)-covector \( q_i \neq 0 \). Then its adjoint matrix \( A_{ij} = \text{adj}(M) \) is proportional to the tensor product of the covectors \( q_i \),

\[
A_{ij} = \lambda(q) q_i q_j, \tag{4.5}
\]

where \( \lambda(q) \) is a polynomial of \( q \).

Thus (4.4) reads as

\[
\lambda(q) q_i q_j = 0. \tag{4.6}
\]

Since \( q_i \) is non-zero, the dispersion relation takes the ordinary scalar form

\[
\lambda(q) = 0. \tag{4.7}
\]

From (4.3) it follows that the scalar function \( \lambda(q) \) is a 4-th order homogeneous polynomial of the wave covector \( q_i \). It means that, for an arbitrary linear response medium, there are at most two independent quadratic wave cones at every points of the space. This non-trivial physical fact was previously observed in [25].

Observe that in a widely used non-covariant treatment of the characteristic system (4.1), see, for instance, [42], one obtains instead of (4.7) the equation of the form \( \omega^2 \lambda(q) = 0 \). In an addition to two light cones mentioned above, this equation has a degenerate solution \( q_i = (0, k_1, k_2, k_3) \), called zero frequency electromagnetic wave. In our covariant description, as well as in [25], this unpleasant solutions are absent. Thus the problem how to interpret or how to remove the zero frequency waves is not emerges in our approach at all.

For different explicit forms of the coefficients of the quartic form \( \lambda(q) \) called Tamm-Rubilar tensor, see [25], [26], [35], [36]. To our opinion, it is more convenient to work with the generic form \( \text{adj}(M) = 0 \) for qualitative analysis and even for explicit calculations. We will demonstrate it in the sequel.

### B. Skewon part of the dispersion relation

We discuss now how the skewon part contributes to the dispersion relation. Substituting the decomposition of the optic tensor \( M^{ij} = P^{ij} + Q^{ij} \) into Eq.(4.3) we have

\[
A_{ij} = \frac{1}{3!} \varepsilon_{ii_{12}i_{12}i_{12}i_{12}} (P^{i1j1} P^{i2j2} P^{i3j3} + 3P^{i1j1} Q^{i2j2} Q^{i3j3} + 3P^{i1j1} P^{i2j2} Q^{i3j3} + Q^{i1j1} Q^{i2j2} Q^{i3j3}). \tag{4.8}
\]

The left hand side of this equation is a symmetric matrix, thus the antisymmetric matrices in its right hand side must vanish. Indeed, we can check straightforward that the identities

\[
\varepsilon_{ii_{12}i_{12}i_{12}i_{12}} Q^{i1j1} Q^{i2j2} Q^{i3j3} = 0, \tag{4.9}
\]

\[
\varepsilon_{ii_{12}i_{12}i_{12}i_{12}} P^{i1j1} P^{i2j2} Q^{i3j3} = 0 \tag{4.10}
\]

hold for an arbitrary symmetric matrix \( P \) and antisymmetric matrix \( Q \). Note that Eq.(4.9) represents the mentioned fact that the adjoint of an antisymmetric matrix \( Q \) vanishes identically. Thus we remain in (4.8) with the relation

\[
\lambda(P, Q) q_i q_j = \frac{1}{3!} \varepsilon_{ii_{12}i_{12}i_{12}i_{12}} (P^{i1j1} P^{i2j2} P^{i3j3} + 3P^{i1j1} Q^{i2j2} Q^{i3j3} = 0. \tag{4.11}
\]

The first term in the right hand side is the adjoint of the symmetric matrix \( P^{ij} \). Consequently, the dispersion relation takes the form

\[
\lambda(P) q_i q_k + \frac{1}{2!} \varepsilon_{ii_{12}i_{12}i_{12}i_{12}} P^{i1j1} Q^{i2j2} Q^{i3j3} = 0, \tag{4.12}
\]

where \( \lambda(P) \) is the quartic form evaluated on the principle part only. The second term of (4.12) is calculated in Appendix. The result is surprisingly simple:
Proposition 1: For a most generic linear constitutive pseudo-tensor $\chi^{ijkl}$, the dispersion relation reads

$$\lambda(P) + P^{ij} Y_i Y_j = 0. \quad (4.13)$$

Observe some immediate consequences of this formula:

1. The skewon modification of the light cone is provided by a quadratic form $P^{ij} Y_i Y_j$. 

2. Due to the identity $P^{ij} q_i = 0$, Eq.(4.13) is invariant under a gauge transformation of the skewon covector $Y_i \rightarrow Y_i + a q_i$. 

3. If there is a solution $q_i$ of Eq.(4.13) for which $Y_i(q) \sim q_i$, then $\lambda(P) = 0$, i.e., the wave vector $q_i$ lies on the non-modified light cone.

4. If $Y_i(q) \sim q_i$ for all solutions of Eq.(4.13), then the corresponding skewon does not modify the light cone structure.

5. If $\lambda(P) = 0$ for all solutions of Eq.(4.13), then simultaneously $P^{ij} Y_i Y_j(q) = 0$ holds. In this case we remain with a subset of the original (non-skewon) light cone.

6. For a non-trivial solution of the dispersion relation, two scalar terms $\lambda(P)$ and $P^{ij} Y_i Y_j(q)$ must be of opposite signs.

In order to analyze the contributions of the skewon part to wave propagation, we restrict in the next section two media with a simplest principal part.

V. SKEWON CONTRIBUTIONS TO THE (PSEUDO) RIEMANNIAN VACUUM

Let the principal part be constructed from a generic metric tensor of Euclidean or Minkowski signature. The skewon part, however, let be of the most general form. How, in this case, the skewon modifies the standard light cone structure? An extensive analysis of this problem was given by Obukhov and Hehl in [29]. Recently Ni [52] provided an analysis of skewon effects on cosmic wave propagation in the approximation of small skewon. In [53], [54], some new generic facts about skewon contribution to Minkowski vacuum are presented. We will give a detailed derivation analysis of these results.

A. Skewon field on a metric space

Consider a manifold endowed with a metric tensor $g^{ij}$. For generality, we will not restrict to the Lorentz signature, even the Euclidean case does not have a direct physical application. Let the principal part of the constitutive pseudo-tensor be presented in the metric-type form [25]

$$\chi^{ijkl} = \sqrt{|g|} \left( g^{ik} g^{jl} - g^{il} g^{jk} \right). \quad (5.1)$$

For simplicity, we assume the units for which the dimension factor in (5.1) is neglected, i.e., the constitutive tensor $\chi^{ijkl}$ is dimensionless. Recall that in SI-system the left hand side of (5.1) comes with an additional factor $\lambda_0 = \sqrt{\varepsilon_0/\mu_0}$, see [25] for the details.

From (5.1), the principal optic tensor reads

$$P^{ik} = \sqrt{|g|} \left( g^{ik} q^2 - q^i q^k \right). \quad (5.2)$$

Here the indices are raised by the metric $g^{ik}$, i.e.,

$$q^i = g^{ij} q_j, \quad q^2 = g^{ij} q_i q_j. \quad (5.3)$$

First we calculate the adjoint (4.7) of the symmetric matrix (5.2). It takes the form

$$\epsilon(Y) q_i q_j = \frac{1}{3!} \varepsilon_{i_1 i_2 i_3} \varepsilon_{j_1 j_2 j_3} q^{i_1 i_2 i_3} q^{j_1 j_2 j_3} \left( q^2 g^{i_1 j_1} - 3 q^{i_1} q^{j_1} \right). \quad (5.4)$$

Using the standard identities for contraction of the $\varepsilon$-symbols with the metric tensor, we obtain in (5.4)

$$\lambda(P) q_i q_j q_k = \text{sgn}(g) \sqrt{|g|} q^4 q_i q_j q_k. \quad (5.5)$$

Thus the principle part contribution to the dispersion function reads

$$\lambda(P) = \text{sgn}(g) \sqrt{|g|} q^4. \quad (5.6)$$

The contribution of the skewon part follows straightforward from (5.1)

$$P^{ij} Y_i Y_j = \frac{1}{\sqrt{|g|}} \left( Y^2 q^2 - (Y, q)^2 \right). \quad (5.7)$$

where the scalar product is defined by the use of the metric, $(q, Y) = g^{ij} Y_i q_j$. Substituting into (4.13) we remain with

$$\lambda = \sqrt{|g|} \left( \text{sgn}(g) q^4 + Y^2 q^2 - (Y, q)^2 \right). \quad (5.8)$$

Thus we proved the following

Proposition 2: The dispersion relation for the most generic skewon media in the (pseudo) Riemannian vacuum is represented as

$$\text{sgn}(g) q^4 + Y^2 q^2 - (Y, q)^2 = 0. \quad (5.9)$$

B. Lorenz-type gauge

The dispersion relation (5.9) may be given even in a more simpler form. Since the covector $Y_i$ is defined only up to an arbitrary addition of the covector $q_i$, we can apply an arbitrary scalar gauge condition. On a space endowed with a metric it can be used in a form similar to the Lorenz gauge condition

$$(Y, q) = g^{ij} Y_i q_j = 0. \quad (5.10)$$

Note that this condition is applicable for an arbitrary signature of the metric tensor, even it is usually used for
the Lorentz metric. With this expression at hand, the dispersion relation takes a form of a system

\[ \text{sgn}(g)q^4 + q^2Y^2 = 0 \quad \text{and} \quad (q, Y) = 0. \]  

(5.11)

Let us analyze how the gauge condition (5.10) can be realized on a metric space. If a solution \( q_i \) of Eq.(5.9) satisfies the relation \( q^2 = 0 \), then relation (5.10) is a direct consequence of Eq.(5.9) for an arbitrary covector \( Y \). Let now \( q_i \) be a non-light solution of Eq.(5.9), i.e., \( q^2 \neq 0 \). We consider a generic expression of the covector \( Y \) in terms of the tensor \( S^{ij}q_iq_j + \alpha q_i \).

(5.12)

Recall that \( S^{ij} \) is independent of \( q_i \), while the scalar \( \alpha \) can be an arbitrary function of \( q_i \). It value corresponds to different choices of the gauge condition. Multiplying both sides of (5.12) by \( q \) and requiring (5.10) we obtain

\[ S^{ij}q_iq_j + \alpha q_i^2 = 0. \]  

(5.13)

Consequently, for \( q^2 \neq 0 \),

\[ \alpha = -\frac{S^{mn}q_mq_n}{q^2}. \]  

(5.14)

Thus for \( q^2 \neq 0 \), the optic covector is expressed in the Lorenz-type gauge as

\[ Y_i = S_i^jq_j - \frac{S^{mn}q_mq_n}{q^2}q_i. \]  

(5.15)

Consequently, the Lorenz-type condition \((Yq) = 0\) may be applied for an arbitrary tensor \( S_i^j \) and for an arbitrary wave covector \( q_i \).

C. Euclidean signature

For a positive signature of the metric, the dispersion relation takes the form

\[ q^4 + Y^2q^2 = 0. \]  

(5.16)

Both terms in (5.16) are nonnegative. Consequently any solution of (5.16) must satisfy \( q^2 = 0 \). For Euclidean metric, it yields a unique trivial solution, \( q_i = 0 \). The gauge condition \((q, Y) = 0\) also holds. Thus we come to a conclusion:

**Proposition 3:** For a Euclidean signature metric space endowed with an arbitrary skewon, the dispersion relation has a unique trivial solution, \( q_i = 0 \).

Thus an arbitrary skewon part cannot modify the elliptic character of the Euclidean metric. The wave propagation was forbidden in an Euclidean signature model and it remains forbidden even when an arbitrary skewon field is "switched on".

D. Lorentz signature

Also for a negative signature of the metric, it is more convenient to use the dispersion relation in the Lorenz gauge. Thus we are dealing with the system

\[ q^4 = Y^2q^2, \quad \text{and} \quad (Y, q) = 0. \]  

(5.17)

In order to analyze its solutions, we first observe that the left hand side is nonnegative, \( q^4 \geq 0 \). If \( q^2 \neq 0 \), then from Eq.(5.17) it follows that \( Y^2q^2 > 0 \). Thus the terms \( q^2 \) and \( Y^2 \) have the same sign. It means that the covectors \( Y_i \) and \( q_i \) must be both timelike or both spacelike. But two timelike covectors cannot be orthonormal, as it is require by the second equation of (5.17). Consequently, if \( q_i \neq 0 \), the covectors \( Y_i \) and \( q_i \) must be both spacelike. So we proved the following

**Proposition 4:** In the Lorentz signature metric space, the solution of the skewon modified dispersion relation can be spacelike or null,

\[ q^2 \leq 0. \]  

(5.18)

In other words, skewon cannot restrict the light cone. Thus the skewon contribution to a generic pseudo-Riemannian vacuum media increases the light velocity.

VI. ANTISYMMETRIC SKEWON

Our qualitative consideration does not involve any specific form of skewon. We will consider now some explicit examples of electromagnetic waves propagation in skewon media. For certain special cases, the detailed analysis was provided in [29]. We will study how these results are embedded in our formalism and how they can be extended. We will also discuss the results of the approximated analysis provided recently by Ni, [52].

A simplest way to construct a skewon is to start with an arbitrary traceless tensor \( S_i^j \) that is completely equivalent to \( (2)\chi^{ijkl} \). On a space endowed with a metric tensor \( g_{ij} \), the mixed tensor \( S_i^j \) can be replaced by the covariant and contravariant tensors

\[ S^{ij} = g_{ik}S_k^j, \quad \text{and} \quad S_{ij} = g_{ik}S^k_i. \]  

(6.1)

Note that these two tensors depend not only of the skewon itself but also of the metric. An advantage of use the tensors (6.1) is that they can be invariantly decomposed into symmetric and skew-symmetric parts. Since these two parts do not mix under coordinate transformations, they can be studied separately.

In this section we consider a case of a pure antisymmetric skewon tensor

\[ S_{ij} = -S_{ji}. \]  

(6.2)
A. Dispersion relation

We observe first that the antisymmetric skewon is, in fact, simplest type of a skewon. Indeed, the traceless condition for such tensor holds identically,

\[ S_{ij}g^{ij} = 0. \]  (6.3)

Moreover,

\[ S_{ij}q^iq^j = 0. \]  (6.4)

Then when the antisymmetric tensor is substituted into Eq.(5.15), the non-regular term is canceled. Consequently we can realize the Lorentz gauge condition that is linear in the wave covector \( q \),

\[ Y_i = S_{ij}q^j. \]  (6.5)

We observe that the Lorentz gauge condition is satisfied automatically. In the dispersion relation

\[ q^4 = q^2Y^2 \]  (6.6)

\( Y(q) \) is a polynomial now. Hence this equation is decomposed into two independent equations

\[ q^2 = 0, \quad \text{and} \quad q^2 = Y^2. \]  (6.7)

The later equation is given in an implicit form (recall that \( Y \) is linear in \( q \)). Since \( Y_i \) is the first order homogeneous (even linear) function of \( q_i \), this equation represents an algebraic cone.

The fact of separation of the dispersion relation for the antisymmetric skewon was derived in [29]. In this publication, the decomposition of dispersion relation into two independent equations was interpreted as the birefringence effect known from crystal optics. In fact, this statement requires a bit more precise analysis. In particular, it is not clear if the equation \( q^2 = Y^2 \) has real solutions and if these solutions represent an observable light cone.

B. Light-cone condition

In order to represent a light cone, Eq.(6.7) must have non-zero real solutions. We will prove now that this is indeed a case for almost arbitrary antisymmetric skewon.

First, we rewrite the equation \( q^2 - Y^2 = 0 \) in terms of an effective metric \( \tilde{g}_{ij} \)

\[ \tilde{g}_{ij}q^iq^j = 0. \]  (6.8)

Substituting (6.5) we derive

\[ (g_{ij} - S_{im}S_{jn}g^{mn}) q^iq^j = 0. \]  (6.9)

Consequently, the effective metric is given by

\[ \tilde{g}_{ij} = g_{ij} - S_{im}S_{jn}g^{mn}. \]  (6.10)

On a four dimensional manifold, the metric \( \tilde{g}_{ij} \) can be:

(i) Euclidean with the signature (++,++),

(ii) Lorentzian with the signature (++--),

(iii) of a mixed type with the signature (++--),

(iv) degenerate with \( \det g = 0 \).

Only the Lorentz case provides an additional light cone. On the 4-dimensional manifold, the metric has Lorentz signature if and only if its determinant is negative. Consequently we have a necessary condition for birefringence

\[ \det (g_{ij} - S_{im}S_{jn}g^{mn}) < 0. \]  (6.11)

This relation can be modified being multiplied by \( \det g^{ik} \). Since \( g \) is a Lorentz signature metric, we obtain an equivalent condition

\[ \det (\delta_j^i + S^i_ms^m_j) > 0. \]  (6.12)

Note that, in this form, the condition is independent of the metric tensor and applicable even for a more generic case.

Since the inequality (6.11) is pure algebraic, i.e., pointwise, we can apply coordinate transformations to replace the metric \( g_{ij} \) by the flat Minkowski metric \( g_{ij} = \text{diag}(1, -1, -1, -1) \). We use the standard "electromagnetic parametrization" of the antisymmetric matrix

\[ S_{01} = \alpha_1, \quad S_{02} = \alpha_2, \quad S_{03} = \alpha_3, \]  (6.13)

and

\[ S_{23} = \beta_1, \quad S_{13} = -\beta_2, \quad S_{12} = \beta_3. \]  (6.14)

With this parametrization, the effective metric takes the form

\[
\tilde{g}_{ij} = \begin{pmatrix}
1 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 & \alpha_2\beta_3 - \alpha_3\beta_2 & \alpha_3\beta_1 - \alpha_1\beta_3 & \alpha_1\beta_2 - \alpha_2\beta_1 \\
\alpha_2\beta_3 - \alpha_3\beta_2 & -1 + \beta_2^2 + \beta_3^2 - \alpha_1^2 & -\alpha_1\alpha_2 - \beta_1\beta_2 & -\alpha_1\alpha_3 - \beta_1\beta_3 \\
\alpha_3\beta_1 - \alpha_1\beta_3 & -\alpha_1\alpha_2 - \beta_1\beta_2 & 1 + \beta_1^2 + \beta_2^2 - \alpha_3^2 & -\alpha_2\alpha_3 - \beta_2\beta_3 \\
\alpha_1\beta_2 - \alpha_2\beta_1 & -\alpha_1\alpha_3 - \beta_1\beta_3 & -\alpha_2\alpha_3 - \beta_2\beta_3 & 1 + \beta_1^2 + \beta_2^2 - \alpha_3^2
\end{pmatrix}
\]  (6.15)
We calculate the determinant of this matrix by use the the computer algebra package Reduce-Excalc, see [51]. The result is surprisingly simple

\[
\det(\hat{g}_{ij}) = -(\alpha \cdot \beta)^2 + (\beta^2 - \alpha^2 - 1)^2.
\] (6.16)

Here the standard scalar product notations are used, \((\alpha \cdot \beta) = \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3\), and \(\alpha^2 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2\), and similarly for \(\beta^2\). Eq.(6.16) results in the following statement.

**Proposition 5:** For an arbitrary antisymmetric skewon, the determinant of the effective metric is non-positive. Thus, the equation \(q^2 = Y^2\) has non-zero real solutions. In the case \((\alpha \cdot \beta)^2 \neq \alpha^2 - \beta^2 + 1\), the effective metric has Lorentz signature.

C. The form of the modified light cone

The solution \(q^2 = 0\) represents a perfect light cone. Using the standard notation \(q_i = (\omega, k_1, k_2, k_3)\) we observe its basic properties:

1. The unique vertex is localized at the origin \(q_i = 0\);
2. The symmetry axis is directed along the \(\omega\)-axis;
3. The timelike sections \(\omega = \text{const}\) is a sphere with the full \(SO(3)\) symmetry.

What can be said about the second cone \(q^2 = Y^2\)?

The origin point \(q_i = 0\) is evidently a solution of this equation, because the fact that \(Y_i\) is a 1-st order homogeneous polynomial in \(q_i\). This fact yields also the algebraic cone structure: For every solution \(q_i\), the scaled covector \(C q_i\) is also a solution. An only point that remains fixed under rescaling transformation is the origin \(q_i = 0\). Thus the hypersurface is an algebraic cone with a unique vertex localized at the origin.

In order to understand the position of the cone axis we need the equation \(q^2 = Y^2\) in the space-time decomposed form. Using the effective metric (6.15) we derive

\[
(1 + \alpha^2) \omega^2 + 2\omega (k \cdot [\alpha \times \beta]) - k^2 - (\alpha \cdot k)^2 + (\beta \cdot k)^2 = 0,
\] (6.17)

where the vector (cross) and scalar (dot) products of 3-vectors are used. We restrict here our analysis of this algebraic equation only for two special cases.

For \(\beta = 0\), we remain in (6.17) with

\[
(1 + \alpha^2) \omega^2 = k^2 + (\alpha \cdot k)^2.
\] (6.18)

We observe that the right hand side of this equation is non-negative. Consequently, for constant values of \(\omega\), we have in (6.18) a compact second order polynomial hypersurface. Thus it is an ellipsoid. Moreover, if \((\omega, k)\) is a solution than \((\omega, -k)\) also a solution. Thus the line \(k = 0\) is a symmetry axis of the cone.

For \(\alpha = 0\), Eq.(6.17) takes the form

\[
\omega^2 - k^2 + (\beta \times k)^2 = 0.
\] (6.19)

We have here two possibilities:

For \(||b|| \leq 1\), we rewrite Eq.(6.18) as

\[
\omega^2 = (1 - \beta^2)k^2 + (\beta \cdot k)^2.
\] (6.20)

The right hand side here is non-negative. In this case, all sections of the surface with the constant \(\omega\) are ellipsoids. Here the symmetry axis of the cone is the line \(k = 0\) as usual.

For \(||b|| > 1\), we rewrite Eq. (6.20) as

\[
\omega^2 + (\beta^2 - 1)k^2 = (\beta \cdot k)^2.
\] (6.21)

Now the left hand side is non-negative. Thus the sections with the constant values of \((\beta \cdot k)\) are ellipsoids. Moreover, the symmetry axis of the 3 dimensional cone is lying in hyperplane \((\beta \cdot k)\), i.e., it is normal to the \(\omega\)-axis.

In other words, we identify a new feature of this model: The time and spatial axes are interchanged when the parameter \(\beta^2\) crosses the value 1. It has some similarity to the Schwarzschild solution in ordinal coordinates when the time and radial axes interchange at the critical radius.

D. Degenerate skewon

We bypassed some special sets of skewon parameters which bring the degenerate behavior. Let us observe what can we have in these special cases. The effective metric is singular in the case

\[
\det(\delta^i_j + S^{im}S_{mj}) = 0.
\] (6.22)

In the parametric form, this equation takes the form

\[
(\alpha \beta)^2 + \beta^2 - \alpha^2 - 1 = 0.
\] (6.23)

Due to the Cauchy-Schwarz inequality, \((\alpha \beta)^2 \leq \alpha^2 \beta^2\), this equation has real solution only if

\[
\beta^2 \geq 1.
\] (6.24)

We consider, for instance, a special solution of Eq.(6.23) of the form

\[
\alpha = 0, \quad \beta^2 = 1.
\] (6.25)

In this case, the effective metric (6.15) is given by

\[
\hat{g}_{ij} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -\beta_1^2 & -\beta_1 \beta_2 & -\beta_1 \beta_3 \\
0 & -\beta_1 \beta_2 & -\beta_2^2 & -\beta_2 \beta_3 \\
0 & -\beta_1 \beta_3 & -\beta_2 \beta_3 & -\beta_3^2
\end{pmatrix}.
\] (6.26)

Consequently, the dispersion relation is represented by

\[
\omega^2 - (\beta_1 k_1 + \beta_2 k_2 + \beta_3 k_3)^2 = 0,
\] (6.27)
or, equivalently, by two linear equations

\[
w = \pm(\beta_1 k_1 + \beta_2 k_2 + \beta_3 k_3).
\] (6.28)
This linear dispersion relation must not be mixed with the ordinary linear dispersion in vacuum, that is of the form \( w = k = \sqrt{k_1^2 + k_2^2 + k_3^2} \). In Eq.(6.28), we remain with two hyperplanes instead of a hypercone. The effective velocity of the wave is not bounded on such a "wavefront." It means that this special case provides a very hard violation of the special relativity principles.

\[ q^2 = 0, \quad \text{and} \quad Y^2 = q^2. \tag{6.29} \]

Due to homogeneity, every solution describes a 2-dimensional plane in 4-dimensional space-time. Its spatial projection represents the optic axis. This is a special direction in which a wave propagates without birefringence. The system (6.23) is equivalent to

\[ q^2 = 0, \quad \text{and} \quad Y^2 = 0. \tag{6.30} \]

With these simple equations at hand, we can prove the following:

**Proposition 6:** In a pseudo-Riemannian manifold endowed with an arbitrary antisymmetric skewon, the wavefront have two separated optic axes.

**Proof:** In order to have a non-trivial solution of system (6.29), both covectors \( Y_i \) and \( q_i \) must be light-like relative to the basic metric \( g_{ij} \). We recall that these two covectors also must satisfy the gauge condition

\[ (qY) = 0, \tag{6.31} \]

i.e., they must be pseudo-orthonormal. It is well known that two light-like vectors are pseudo-orthonormal if and only if they are proportional. Thus the optic axis covector \( q \) must satisfy the system

\[ q^2 = 0, \quad \text{and} \quad Y_i(q) = mq_i. \tag{6.32} \]

with some real parameter \( m \). We derive now the values of the parameter \( m \). In terms of the skewon tensor \( S_{ij} \), the equation \( Y = mq \) takes the form of an eigenvector problem

\[ (S_{ij} - mg_{ij})q^2 = 0. \tag{6.33} \]

This system has a non-trivial solution if and only if

\[ \det(S_{ij} - mg_{ij}) = 0. \tag{6.34} \]

Every real eigenvalue \( m \) produces a real eigenvector \( q \) of Eq.(6.33). Moreover, distinct real eigenvalues correspond to linear independent real eigenvectors. We observe also that, due to the gauge relation \( (q,Y) = 0 \), every covector corresponded to a non-zero eigenvalue \( m \) is necessary light-like. Indeed, equation \( Y = mq \) results into \( mq^2 = 0 \). Note that it is not correct, in general, for \( m = 0 \). Thus we must give a special treatment of the case of zero eigenvalues.

We will derive now the solutions of Eq.(6.34). Using the "electromagnetic parametrization" (6.13) we have

\[ S_{ij} - mg_{ij} = \begin{pmatrix} m & \alpha_1 & \alpha_2 & \alpha_3 \\ -\alpha_1 & -m & \beta_3 & -\beta_2 \\ -\alpha_2 & -\beta_3 & -m & \beta_1 \\ -\alpha_3 & \beta_2 & -\beta_1 & -m \end{pmatrix}. \tag{6.35} \]

Calculating the determinant of this matrix we obtain the characteristic equation

\[ \det(S_{ij} - mg_{ij}) = m^4 + m^2(\beta^2 - \alpha^2) - (\alpha\beta)^4 = 0. \tag{6.36} \]

The discriminant of this biquadratic equation is non-negative for all values of the parameters

\[ (\alpha^2 - \beta^2)^2 + 4(\alpha\beta)^4 \geq 0. \tag{6.37} \]

Also we observe that the free term in Eq.(6.36) is non-positive, \(-(\alpha\beta)^4 \leq 0\). It means:

1) In the case \((\alpha\beta) \neq 0\), Eq.(6.36) has two real and two complex conjugated solutions. The real solutions are non-zero and expressed as

\[ m = \pm \left( \frac{\sqrt{(\alpha^2 - \beta^2)^2 + 4(\alpha\beta)^4} + \alpha^2 - \beta^2}{2} \right)^{1/2}. \tag{6.38} \]

With these two real distinct values of the parameter \( m \neq 0 \), we have two optic axes.

2) We consider now the case

\[ (\alpha\beta) = 0, \quad \alpha^2 > \beta^2. \tag{6.39} \]

Then there are four real solutions – two non-zero solutions

\[ m = \pm \sqrt{\alpha^2 - \beta^2}, \tag{6.40} \]

and a zero solution \( m = 0 \) of the multiplicity two. As we already observed, every non-zero real eigenvalue \( m \) corresponds to an optic axis. The zero eigenvalue \( m = 0 \) does not correspond to any optic axis. It can be checked by elementary explicit calculations that the conditions (6.39) together with (6.31) have only a trivial solution.

So we remain with two optic axes corresponding to the eigenvalues given in Eq.(6.40).

3) We consider now the opposite case:

\[ (\alpha\beta) = 0, \quad \alpha^2 < \beta^2. \tag{6.41} \]

Two eigenvalues (6.40) are imaginary now, and we remain with \( m = 0 \) of the multiplicity two. Comparing with the
previous case, we get now two real light-like eigenvectors corresponding to the eigenvalue \( m = 0 \). Namely, it is enough to chose \( q = (\beta, 0, \sqrt{\beta^2 - \alpha^2}, \pm \alpha) \). Once more we have two optic axes.

4) Consider the last case,

\[
(\alpha, \beta) = 0, \quad \alpha^2 = \beta^2. \tag{6.42}
\]

Now there is only one eigenvalue \( m = 0 \) of the multiplicity four. It corresponds to two eigenvectors \( q = (\beta, 0, 0, \pm \alpha) \), and consequently two optic axes.

Thus our proposition is proved. \( \blacksquare \)

F. Illustrative examples of antisymmetric skewon

For almost arbitrary parameters of the skewon we derived two wavefront surfaces, that are tangential one to the other along two optic axes. The mutual dispositions of these surfaces may of topologically different type. In order to identify these topological types, we proceed with special examples.

1. Electric type skewon

Consider a skewon of a special pure "electric type"

\[
\alpha = (\alpha_1, \alpha_2, \alpha_3), \quad \text{and} \quad \beta = (0, 0, 0). \tag{6.43}
\]

The effective metric (6.15) related to the extraordinary light cone takes now the form

\[
\hat{g}_{ij} = \begin{pmatrix}
1 + \alpha^2 & 0 & 0 & 0 \\
0 & -1 - \alpha_1^2 & -\alpha_1 \alpha_2 & -\alpha_1 \alpha_3 \\
0 & -\alpha_1 \alpha_2 & -1 - \alpha_2^2 & -\alpha_2 \alpha_3 \\
0 & -\alpha_1 \alpha_3 & -\alpha_2 \alpha_3 & -1 - \alpha_3^2
\end{pmatrix}. \tag{6.44}
\]

The determinant of this metric,

\[
\det(\hat{g}_{ij}) = -(\alpha^2 + 1)^2, \tag{6.45}
\]

is negative and non-zero for all values of the parameters \( \alpha \).

The dispersion relation for the extraordinary wave \( \hat{g}_{ij} q_i q_j = 0 \) is expressed as

\[
(1 + \alpha^2)\omega^2 = k^2 + (\alpha \cdot k)^2. \tag{6.46}
\]

Since all terms in Eq.(6.46) are positive, all sections of this 3-dimensional cone with the constant values of \( \omega \) are 2-dimensional ellipsoids.

The optic covector reads

\[
Y_i = -((\alpha \cdot k), \alpha_1 \omega, \alpha_2 \omega, \alpha_3 \omega). \tag{6.47}
\]

Two optic axes are straightforward calculated now from (6.32) with \( m = \pm ||\alpha|| = \pm \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2} \). We have

\[
q_i = (\pm ||\alpha||, \alpha_1, \alpha_2, \alpha_3). \tag{6.48}
\]

On Fig. 1, we present different sections of two 3-dimensional light cones.

Consequently, the antisymmetric skewon of the pure "electric type" provides a model of birefringent medium.

2. Magnetic-type skewon

We consider now a skewon of a pure "magnetic type" parametrized by the parameters

\[
\alpha = (0, 0, 0), \quad \text{and} \quad \beta = (\beta_1, \beta_2, \beta_3). \tag{6.49}
\]

In this case, the effective metrics (6.15) takes the form

\[
\hat{g}_{ij} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 + \beta_1^2 & -\beta_1 \beta_2 & -\beta_1 \beta_3 \\
0 & -\beta_1 \beta_2 & -1 + \beta_2^2 & -\beta_2 \beta_3 \\
0 & -\beta_1 \beta_3 & -\beta_2 \beta_3 & -1 + \beta_3^2 + \beta_2^2
\end{pmatrix}. \tag{6.50}
\]

The determinant of this metric takes the form

\[
\det(\hat{g}_{ij}) = -(\beta^2 - 1)^2. \tag{6.51}
\]

It is non-positive for all values of the parameter \( \beta \) and vanishes for \( |\beta| = 1 \). Consequently, for \( |\beta| \neq 1 \), Eq.(6.51) represents a hypercone. For \( \beta^2 = 1 \) we have a special case with the hypercone degenerated into two hyperplanes.

The dispersion relation for the extraordinary wave takes the form

\[
\omega^2 = (1 - \beta^2)k^2 + (\beta \cdot k)^2. \tag{6.52}
\]

From this equation, we conclude:

(1) For \( |\beta| < 1 \), the right hand side of this equation is positive, thus a section with a constant value of \( \omega \)
FIG. 1: Sections of the light cones generated by electric-type skewon. Plane images represent the sections \( w = 1 \) and \( k_3 = 0 \) for the skewons parametrized with \( \alpha = (1,0,0), \alpha = (0,1,0), \alpha = (1,1,0), \) and \( \alpha = (1,-1,0), \) respectively. Space images of the ordinary and extraordinary light cones and wavefront ellipsoids are generated by the skewon with \( \alpha = (1,0,0). \)

is a 2-dimensional ellipsoid. This is the ordinary case of birefringence. The extraordinary wave has a velocity greater than speed of light and depends on the spatial direction.

(2) For \( |\beta| = 1, \) Eq.(6.52) takes the form \( w = \pm (\beta \cdot k). \) This is a pair of hyperplans. This is the degenerated case with a non-limiting velocity of the extraordinary wave.

(3) For \( |\beta| > 1, \) Eq.(6.52) returns to be a hypercone. Its central axis, however, lies in a spatial direction, instead of the time direction.

In all cases, there are two optic axes given by the equations

\[
q = (\pm ||\beta||, \beta_1, \beta_2, \beta_3), \tag{6.53}
\]

where \( ||\beta|| = \sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2}. \)

Consequently, only for small values of the magnetic type skewon parameters \( \beta < 1 \) the ordinary type of birefringence emerges. Higher values of \( \beta \) bring rather pathological violation of Lorentz symmetry.

FIG. 2: Sections of the light cones generated by the magnetic-type skewon. On the plan graphs, sections \( w = 1, k_3 = 0 \) correspond to \( \beta_i = (1/2,0,0), \beta_i = (1,0,0), \beta_i = (2,0,0), \) and \( \beta_i = (1,1,0), \) respectively. Space images with \( k_3 = 0 \) are generated by the skewons \( \beta_i = (1/2,0,0), \beta_i = (1,0,0), \beta_i = (2,0,0), \) respectively.
G. Results

We derived that for an arbitrary choice of the antisymmetric skewon the wave front is decomposed into two hypersurfaces. One of them is the non-modified light cone $q^2 = 0$. For almost all entries of the skewon, the second cone is exterior to the ordinary one. Two cones are tangential one to another along two optic axes. The degenerate case, when the extraordinary cone turns into two hyperplanes is identified. For higher values of the skewon parameter $\beta > 1$, the hypersurface returns to be a hypercone that, however, directed into the space.

VII. SYMMETRIC SKEWON

When skewon is considered on a space endowed with a metric tensor, a special symmetric traceless skewon can be extracted. In term of the tensor $S_{ij}$, it satisfies the relations $S_{ij} = S_{ji}$ and $S_{ii} = 0$. In a 4-dimensional space, such a tensor has 9 independent components.

A. Parametric gap

Consider the dispersion relation for a generic skewon in the form

$$q^4 - q^2 Y^2 + (q, Y)^2 = 0. \quad (7.1)$$

Under the gauge condition $\alpha = 0$, every term in this equation is polynomial in the wave covector $q$. Consequently we can consider a solution of Eq.(7.1) in the form

$$q^2 = \frac{1}{2} \left( Y^2 \pm \sqrt{Y^4 - (q, Y)^2} \right). \quad (7.2)$$

The Lorentz square $q^2$ must be real, consequently the expression under the square root must be non-negative. Recall that in the case of an antisymmetric matrix $S_{ij}$, the term $(q, Y)$ equal zero also in the case of the gauge $\alpha = 0$. Thus the term under the square root remains non-negative, $Y^4 \geq 0$, and there are two real solutions for the variable $q^2$. In the case of a symmetric skewon, the situation changes crucially. Now both terms under the square root are non-zero and the inequality

$$Y^4 - (q, Y)^2 \geq 0 \quad (7.3)$$

must hold. Recall that the covector $Y$ is linear in the skewon matrix $S_{ij}$. If we rescale this matrix, $S_{ij} \rightarrow CS_{ij}$, Eq.(7.3) takes the form

$$C^2 Y^4 - (q, Y)^2 \geq 0. \quad (7.4)$$

We observe that for small values of the parameter $C$, the first term approaches zero and the inequality is broken. In fact, the same is true even for a generic skewon since the term $(q, Y)$ is proportional only to the symmetric part of the skewon. Consequently, we proved

Proposition 7: Let the skewon have a non-zero symmetric part. For sufficient small magnitudes of the skewon parameters there are no solutions of the dispersion relation.

B. Vector parametrization

It is convenient to represent the matrix $S_{ij}$ in term of vector fields [29]. We call such a representation as vector parametrization. Let us consider a vector field $v_i$. Thus we can write a symmetric traceless tensor in the form

$$S_{ij} = v_i v_j - \frac{1}{4} v^2 g_{ij}. \quad (7.5)$$

Here the square of the vector $v$ is defined by the metric tensor, $v^2 = g^{ij} v_i v_j$. Certainly one cannot expect that a generic 9-component tensor can be represented by 4 components of the vector $v_i$. Thus (7.5) can be used only as an example of a special symmetric skewon.

In [29], a symmetric skewon parametrized by two vector fields was considered.

$$S_{ij} = \frac{1}{2} (u_i v_j + u_j v_i) - \frac{1}{4} (u, v) g_{ij}. \quad (7.6)$$

This skewon was treated as a one that has 8 independent components. In fact, the rescaling of the vector fields $u \rightarrow Cu, v \rightarrow C^{-1} v$ does not change the left hand side of Eq.(7.6), thus the number of independent components is at least equal to 7. The form of the ansatz (7.6) can be modified by a reparametrization $u = k + \ell$, and $v = k - \ell$. This way we come to an equivalent ansatz

$$S_{ij} = \left( k_i k_j - \frac{1}{4} k^2 g_{ij} \right) - \left( \ell_i \ell_j - \frac{1}{4} \ell^2 g_{ij} \right), \quad (7.7)$$

that is more convenient for a generalization.

The ansatz (7.6) includes less independent components than it is need for a generic symmetric skewon. Let us try to consider an ansatz constructed from three null vector fields $k, \ell, m$.

$$S_{ij} = A k_i k_j + B \ell_i \ell_j + C m_i m_j, \quad (7.8)$$

such that

$$k^2 = \ell^2 = m^2 = 0. \quad (7.9)$$

We observe that the tensor $S_{ij}$ in Eq.(7.8) is traceless. Moreover, every null vectors can be considered as having 3 independent components. Consequently, we can expect to have all together 9 independent components as it must be for a generic traceless symmetric matrix $S_{ij}$. Unfortunately, this skewon is not a most generic one. Indeed, we immediately observe the determinant of the tensor (7.8) is equal to zero. In fact, for linearly independent vectors $k, \ell, m$, the rank of the matrix in Eq.(7.8) is equal to three. It means that the ansatz (7.8) is not
most generic because it cannot describe a matrix \( S_{ij} \) of the fourth rank.

Consequently we come to our final vector ansatz

\[
S_{ij} = A_k k_j + B_\ell \ell_j + C m_i m_j + D n_i n_j ,
\]

(7.10)

with 4 linear independent null vectors \( k, \ell, m, n \) such that

\[
k^2 = \ell^2 = m^2 = n^2 = 0 .
\]

(7.11)

Again, due to these relations, the matrix \( S_{ij} \) in Eq.(7.10) is traceless. For the linear independent vectors \( k, \ell, m, n \), the determinant of the tensor in Eq.(7.10) is non-zero

\[
det S_{ij} = ABCD \text{vol}^2 ,
\]

(7.12)

where the volume of the 4-dimensional parallelepiped determined by the vectors \( k, \ell, m, n \) is involved. Although the positive values of the coefficients \( A, B, C, D \) can be absorbed into the vectors, it is convenient to preserve them in order to control the signs and the magnitude of different contributions to the skewon field.

We assume that the vectors \( k, \ell, m, n \) are linearly independent. In the case of the non-zero coefficients \( A, B, C, D \), the rank of the matrix \( S_{ij} \) is equal to 4. When one of the coefficients is zero the rank is equal to 3, as in the ansatz (7.8). In the case, we have two coefficients, say \( C \) and \( D \), equal to zero, we remain with the matrix of the rank equal to 2. This is the case given in Eqs. (7.6) and (7.7). When there are three zero coefficients, say \( B, C, D \), the rank of the matrix \( S_{ij} \) is equal to 1.

Consequently, with our ansatz (7.10) we are able to describe a traceless symmetric matrix of an arbitrary rank.

### C. Dispersion relation

In order to derive the dispersion relation for the skewon (7.10), we start with the skewon covector. In a generic gauge, it is written as

\[
Y_i = Ak_i (k, q) + B\ell_i (\ell, q) + Cm_i (m, q) + Dn_i (n, q) + \alpha q_i .
\]

(7.13)

We choose the Lorenz-type gauge, \((Y, q) = 0\) and derive the value of the parameter \( \alpha \)

\[
Y_i = A \frac{(k, q)}{q^2} (q^2 k_i - (k, q) q_i) + B \frac{(\ell, q)}{q^2} (q^2 \ell_i - (\ell, q) q_i) + C \frac{(m, q)}{q^2} (q^2 m_i - (m, q) q_i) + D \frac{(n, q)}{q^2} (q^2 n_i - (n, q) q_i)
\]

(7.14)

The square of this covector reads

\[
Y^2 = - \frac{1}{q^2} \left[ A(k, q)^2 + B(\ell, q)^2 + C(m, q)^2 + D(n, q)^2 \right]^2 + 2 \left[ AB(k, q)(\ell, q)(k, \ell) + AC(k, q)(m, q)(k, m) + AD(k, q)(n, q)(k, n) + BC(\ell, q)(m, q)(\ell, n) + BD(\ell, q)(n, q)(\ell, n) + CD(m, q)(n, q)(m, n) \right] .
\]

(7.15)

The dispersion relation \( q^4 = Y^2 q^2 \) takes now the form

\[
q^4 = - \left[ A(k, q)^2 + B(\ell, q)^2 + C(m, q)^2 + D(n, q)^2 \right] + 2q^2 \left[ AB(k, q)(\ell, q)(k, \ell) + AC(k, q)(m, q)(k, m) + AD(k, q)(n, q)(k, n) + BC(\ell, q)(m, q)(\ell, n) + BD(\ell, q)(n, q)(\ell, n) + CD(m, q)(n, q)(m, n) \right] .
\]

(7.16)

In order to study certain conclusions of these formulas, we consider some specific examples.

### D. Examples

1. **Skewon of rank 1**

In the case \( B = C = D = 0 \), we have a symmetric skewon of the lowest rank 1. The dispersion relation (7.16) takes the form

\[
q^4 + A^2(k, q)^4 = 0
\]

(7.17)

Since both terms in the left hand side are positive, the unique solution must satisfy

\[
q^2 = 0 , \quad (k, q) = 0
\]

(7.18)

Geometrically it is a low dimensional intersection of the light cone with a 3 dimensional hyperspace. It can be a point \( q_i = 0 \) or two second-dimensional lines lying on the light cone. It is very questionable if some physically meaningful situation can be related to this solution.

2. **Diagonal skewon of rank 2**

A generic symmetric skewon of the rank 2 can be generated by a choice of parameters \( C = D = 0 \) in Eq.(7.13). In this case, the dispersion relation takes the form

\[
q^4 = - \left[ A(k, q)^2 + B(\ell, q)^2 \right] + 2ABq^2(k, q)(\ell, q)(k, \ell).
\]

(7.19)

This relation can be considered as a quadratic equation relative to the variable \( q^2 \). In order to have real solutions, the following inequality must hold

\[
A^2 B^2 (k, q)^2 (\ell, q)^2 (k, \ell)^2 \geq \left[ A(k, q)^2 + B(\ell, q)^2 \right]^2 .
\]

(7.20)

Let us consider, for example, a simplest symmetric traceless skewon of rank 2 with two non-zero components

\[
S_{00} = S_{11} = A .
\]

(7.21)

It is convenient to proceed now with \( Y_i = S_{ij} q^j \), i.e. in a gauge \( \alpha = 0 \) instead of the Lorenz-type gauge. The skewon covector takes the form

\[
Y_i = (A \omega , -Ak_1, 0, 0) .
\]

(7.22)
We calculate
\[(Y, q) = A(\omega^2 + k_1^2), \quad Y^2 = A^2(\omega^2 - k_1^2). \quad (7.23)\]

Thus the dispersion relation in Eq.(5.12) takes the form
\[q^4 - A^2(\omega^2 - k_1^2)q^2 + A^2(\omega^2 + k_1^2)^2 = 0. \quad (7.24)\]

From this equation, we derive that the solution cannot be null. Moreover, every solution \(q_i = (\omega, k)\) must satisfy the inequality \(|\omega| < |k_1|\). It is with the correspondence with the inequality \(q^2 < 0\).

Eq.(7.24) can be rewritten as
\[q^2 = A^2/2(\omega^2 - k_1^2) \pm A^2/2\sqrt{(A^2 - 4)(\omega^2 - k_1^2)^2 - 16\omega^2k_1^2}. \quad (7.25)\]

Let us analyze the solutions of these equations for different values of the skewon parameter.

(1) For \(A = 0\), we return here to the non-modified light cone \(q^2 = 0\).

(2) For \(0 < |A| \leq 2\) there are no real solutions at all.

(3) For \(|A| > 2\), there are two real solutions.

For the images of these algebraic cones, see Fig. 3 and Fig. 4. We observe that the light cones intersect only at the origin, thus optic axes absent.

3. Non-diagonal skewon of rank 2

We consider another example of a symmetric skewon of rank 2 of a non-diagonal type. Let the non-zero components of the symmetric traceless matrix \(S_{ij}\) be
\[S_{01} = S_{10} = B. \quad (7.26)\]

Let us analyze the solutions of these equations for different values of the skewon parameter.

(1) For \(A = 0\), we return here to the non-modified light cone \(q^2 = 0\).

(2) For \(0 < |A| \leq 2\) there are no real solutions at all.

(3) For \(|A| > 2\), there are two real solutions.

For the images of these algebraic cones, see Fig. 3 and Fig. 4. We observe that the light cones intersect only at the origin, thus optic axes absent.

4. Skewon of rank 3

A generic symmetric skewon of the rank 3 can be derived from Eq.(7.13) by a choice of parameters \(D = 0\). Instead, we consider a simplest diagonal symmetric traceless skewon of rank 3 with three non-zero components
\[S_{00} = 2A, \quad S_{11} = S_{22} = A. \quad (7.31)\]
Using the solution of this biquadratic equation we rewrite
\[ Y = 4. \]
The component \( B_{WZ} \) diagonal rank 2 symmetric skewon (7.24). The parameter \( A = 4 \). The component \( w \) is directed as \( z \)-axis, \( k_1, k_2 \) are directed as \( x \) and \( y \) axes respectively.

The non-zero components of the skewon covector are
\[
Y_0 = 2Aw, \quad Y_1 = -Ak_1, \quad Y_2 = -Ak_2.
\]

Consequently
\[
(Y, q) = A (2\omega^2 + z^2), \quad Y^2 = A^2 (4\omega^2 - z^2),
\]
where the notation \( z^2 = k_1^2 + k_2^2 \) is used. The dispersion relation taken in the form \( q^4 = q^2Y^2 - (Y, q)^2 \) reads
\[
q^4 - A^2 (4\omega^2 - z^2) q^2 + A^2 (2\omega^2 + z^2)^2 = 0.
\]
Using the solution of this biquadratic equation we rewrite Eq.(7.34) as
\[
q^2 = \frac{A^2}{2} (4\omega^2 - z^2) \pm \frac{A}{2} \sqrt{A^2 (4\omega^2 - z^2)^2 - 4 (2\omega^2 + z^2)^2}.
\]
Rearranging the terms under square root we get
\[
q^2 = \frac{A^2}{2} (4\omega^2 - z^2) \pm \frac{A}{2} \sqrt{f_1(\omega, z) f_2(\omega, z)},
\]
where
\[
f_1(\omega, z) = 4(A - 1)\omega^2 - (A + 2)z^2,
\]
and
\[
f_2(\omega, z) = 4(A + 1)\omega^2 - (A - 2)z^2.
\]

For \( A = 0 \), Eq.(7.36) represents the non-modified light cone \( q^2 = 0 \).
For \( 0 < |A| \leq 1 \), the term \( f_1(w, z) \) is non-positive and the term \( f_2(w, z) \) is non-negative. Moreover, if one of these terms vanishes, the unique solution of Eq.(7.34) is the zero vector \( q_i = 0 \). Consequently, for \( 0 < |A| \leq 1 \) there are no non-zero solutions of the dispersion relation.
For \( 1 < |A| \leq 2 \), the discriminant expression can be positive in some directions. However, the whole equation does not have real solutions. It can be easily observed taking into account the fact that \( q^2 \leq 0 \).

\section{5. Skewon of rank 4}

In order to complete our consideration, we recall the known example of the diagonal symmetric skewon of the rank equal to 4. In the form used in [29], the non-zero components of its matrix \( S_{ij} \) read
\[
S_{00} = 3A, \quad S_{11} = S_{22} = S_{33} = A.
\]
We will use the gauge \( \alpha = 0 \), so the skewon covector is given by \( Y_i = S_{ij}q^j \). It has the components \( (\alpha = 1, 2, 3) \)
\[
Y_0 = 3Aw, \quad Y_\alpha = -Ak_\alpha.
\]
Consequently,
\[
Y^2 = A^2 (9\omega^2 - k^2), \quad (Y, q) = A(3\omega^2 + k^2).
\]
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FIG. 8: Section \( w = 0 \) of the algebraic cones of the non-diagonal rank 2 symmetric skewon (7.24). The parameter \( A = 10 \). The component \( k_1, k_2, k_3 \) are directed as \( x, y \) and \( z \) axes respectively.

The dispersion relation (5.9) takes now the form

\[
\omega^4 - 2\omega^2 k^2 (1 - 8A^2) + k^4 = 0. \tag{7.42}
\]

It is rewritten as

\[
\frac{\omega^2}{k^2} = (1 - 8A^2) \pm 4\sqrt{4A^4 - A^2}. \tag{7.43}
\]

Consequently we have the following cases:

1. For \( |A| = 0 \), Eq.(7.43) has a unique solution – the ordinary light cone \( \omega^2 = k^2 \).

2. For \( 0 < |A| \leq 1/2 \), Eq.(7.43) does not have real solutions at all.

3. For \( 1/2 < |A| \), the right hand side of Eq.(7.43) is real and negative. Thus the wave propagation absents also in this case.

Consequently, in the isotropic symmetric skewon model the wave propagation absents as it was observed in [29].

E. Results

Using the skewon covector formalism, we studied some new properties of the symmetric skewon models. In particular we observed the parametric gap that appears in most models, but no in all of them. We extended the vector parametrization, particularly by use of the null vectors. In order to distinguish the different types of the symmetric skewon we used the notion of rank of the corresponding matrix. Dispersion relations for the skewon models of different rank are derived. We worked out some specific examples that present different types of the modified light cone structures.

VIII. CONCLUSIONS AND DISCUSSION

In this paper, we study an electromagnetic system with a generic local linear response. We presented here a new formalism based on the notion of the optic tensor. This tensor completely describes the wave propagation; in particular, dispersion relation is represented as the adjoint of the optic tensor. The decomposition of the optic tensor into symmetric and skew symmetric parts is directly related to the irreducible decomposition of the constitutive tensor into principle part and skewon part, respectively. Moreover, we show that the skewon optic tensor is represented by the skewon covector, that is defined only up to addition of a term proportional to the wave covector. Consequently, the Lorenz-type gauge condition is introduced. These notations simplify the dispersion relation considerably.

We apply the optic tensor formalism to the problem of wave propagation in the media with the pseudo-Riemannian principle part and a generic skewon part. We proved that independently of a specific type of the skewon field, the wave covector in this media is necessary space-like or null.

Special types of the antisymmetric and symmetric skewon are considered. For the antisymmetric skewon, the light cone is separated into two light cones – the ordinary one \( q^2 = 0 \) and the extraordinary with \( q^2 > 0 \). This is a case of birefringence. We derived a degenerated case when the extraordinary light cone turns into a hyperplane. Moreover, for a special magnetic type, the extraordinary light cone has a space-like axis. This behavior can be interpreted as the situation when two cones do not observable both together, i.e., as absence of the birefringence phenomena. For all types of the antisymmetric skewon, we proved that there are exactly two optic axes and derive their explicit form.

In the case of the symmetric skewon, we observed a parametric gap that appears in most of special cases. The vector parametrization of the symmetric skewon is discussed in details. We classified the symmetric skewon using the rank of the corresponded matrix. Specific examples are worked out.

Our analysis shows that skewon field is able to describe certain new types of Lorentz violation in electrodynamics that were not accounted in models based on a modified Lagrangian.

To our opinion, the skewon model provides a rich subject of interest for algebraical analysis and the physical consideration of the modified light cone structure.

It is straightforward to observe that the unusual features of the wave propagation in the skewon media is mostly related to the real skewon. The situation changes crucially when we turn to the imaginary skewon field. Such situation is even may be viable in the problems of crystal physics. The corresponding treatment will be presented in a separated publication.
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Appendix A: Calculation of the second adjoint

For an arbitrary matrix $M^{ij}$ second adjoint is defined as

$$B_{ijkl} = 1 + 2\epsilon_{ijmn} \varepsilon_{klrs} M^{mr} M^{ns}. \quad (A1)$$

Let us calculate this expression in the term of the skewon optic covector. Substituting (3.21) into (B1) we write it as

$$B_{ijkl}(Q) = \frac{1}{2!} \left( \epsilon_{ijmn} \varepsilon^{mrab} q_a Y_b \right) \cdot \left( \epsilon_{klsd} \varepsilon^{qscd} q_c Y_d \right). \quad (A2)$$

Using the standard formulas for the product of two permutation tensors we have

$$\epsilon_{ijmn} \varepsilon^{mrab} q_a Y_b = \begin{vmatrix} \delta^n_i & \delta^n_j & \delta^n_k & \delta^n_l \\ q_n & q_m & q_l & q_i \\ \delta^n_l & \delta^n_m & \delta^n_k & \delta^n_q \\ q_n & q_m & q_l & q_i \end{vmatrix}.$$

Similarly,

$$\epsilon_{klsd} \varepsilon^{qscd} q_c Y_d = \begin{vmatrix} \delta^n_k & \delta^n_l & \delta^n_m & \delta^n_n \\ q_k & q_l & q_m & q_n \\ \delta^n_m & \delta^n_n & \delta^n_k & \delta^n_q \\ q_k & q_l & q_m & q_n \end{vmatrix}.$$

Thus

$$B_{ijkl} = \frac{1}{2!} \left| \begin{array}{cccc} \delta^n_i & q_i & Y_j \\ \delta^n_j & q_j & Y_i \\ \delta^n_k & q_k & Y_l \\ \delta^n_l & q_l & Y_k \end{array} \right| \cdot \left| \begin{array}{cccc} \delta^n_i & \delta^n_j & \delta^n_k & \delta^n_l \\ q_i & q_j & q_k & q_l \\ q_l & q_m & q_k & q_i \\ q_i & q_j & q_k & q_l \end{array} \right|. \quad (A3)$$

Expanding the third order determinants we have

$$B_{ijkl} = \frac{1}{2} \left( \delta^n_i q_j Y_j - \delta^n_j q_i Y_i + \delta^n_k q_l Y_l - \delta^n_l q_k Y_k \right). \quad (A4)$$

Expanding these expressions we come to a compact formula

$$B_{ijkl} = (q_i Y_j - q_j Y_i)(q_k Y_l - q_l Y_k). \quad (A8)$$

It can be also written in a matrix form

$$B_{ijkl} = \begin{vmatrix} q_i & Y_j \\ q_j & Y_i \end{vmatrix} \begin{vmatrix} q_k & Y_l \\ q_l & Y_k \end{vmatrix}. \quad (A9)$$

Note that the symmetry properties

$$B_{ijkl} = -B_{jikl} = -B_{klji}. \quad (A10)$$

and also

$$B_{ijkl} = B_{klji}. \quad (A11)$$

evidently hold for these representations.

Appendix B: Proof of Proposition 2

It is useful to involve the second adjoint notion. For a square matrix $M$, the second adjoint $^{(2)adj}(M)$ is defined by removing two rows and two columns. So it is represented as a 4-th order tensor

$$B_{ijkl} = 1 + 2\epsilon_{ijmn} \varepsilon_{klrs} M^{mr} M^{ns}. \quad (B1)$$

Evidently,

$$B_{ijkl} = -B_{jikl} = -B_{klji}. \quad (B2)$$

For a symmetric or a skew-symmetric matrix $M$, but not for a generic matrix,

$$B_{ijkl} = B_{klji}. \quad (B3)$$

In term of the second adjoint, the dispersion relation (4.11) takes the form

$$A_{ik}(P) + B_{ijkl}(Q) P^{jl} = 0, \quad (B4)$$

In our formalism, the skewon contributions to the dispersion relation is completely represented by the second term of (B4). Let us calculate this expression in the term of the skewon optic covector. Substituting (3.21) into (B1) we have

$$B_{ijkl}(Q) = \frac{1}{2!} \epsilon_{ijmn} \varepsilon_{klrs} \varepsilon^{mrab} \varepsilon^{qscd} q_a q_c Y_b Y_d. \quad (B5)$$

Using the standard formulas for the product of permutation tensors, we calculate

$$\epsilon_{ijmn} \varepsilon^{mrab} q_a Y_b = \begin{vmatrix} \delta^n_i & q_i & Y_j \\ \delta^n_j & q_j & Y_i \\ \delta^n_k & q_k & Y_l \\ \delta^n_l & q_l & Y_k \end{vmatrix}. \quad (B6)$$

Consequently,

$$B_{ijkl}(Q) = \frac{1}{2!} \begin{vmatrix} \delta^n_i & q_i & Y_j \\ \delta^n_j & q_j & Y_i \\ \delta^n_k & q_k & Y_l \\ \delta^n_l & q_l & Y_k \end{vmatrix}. \quad (B7)$$
Evaluating the determinants we obtain a compact formula, see Appendix,
\[
B_{ijkl}(Q) = \begin{vmatrix} q_i & Y_j \\ q_j & Y_i \end{vmatrix} \cdot \begin{vmatrix} q_k & Y_l \\ q_l & Y_k \end{vmatrix}.
\]
Consequently, the contribution of the skewon part into the dispersion relation is given by
\[
P^{ij}B_{ijkl}(Q) = P^{ij} \begin{vmatrix} q_i & Y_j \\ q_j & Y_i \end{vmatrix} \cdot \begin{vmatrix} q_k & Y_l \\ q_l & Y_k \end{vmatrix} = q_i q_k (P^{ij} Y_j Y_i) .
\]

(8)

(9)

Observe that this expression is given by a scalar multiplied by \( q_i q_k \). It is in a correspondence with Proposition 1. Thus we come to Proposition 2.

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