Beyond $O(\sqrt{T})$ Regret for Constrained Online Optimization: Gradual Variations and Mirror Prox

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Abstract

We study constrained online convex optimization, where the constraints consist of a relatively simple constraint set (e.g. a Euclidean ball) and multiple functional constraints. Projections onto such decision sets are usually computationally challenging. So instead of enforcing all constraints over each slot, we allow decisions to violate these functional constraints but aim at achieving a low regret and a low cumulative constraint violation over a horizon of $T$ time slot. The best known bound for solving this problem is $O(\sqrt{T})$ regret and $O(1)$ constraint violation, whose algorithms and analysis are restricted to Euclidean spaces. In this paper, we propose a new online primal-dual mirror prox algorithm whose regret is measured via a total gradient variation $V^*(T)$ over a sequence of $T$ loss functions. Specifically, we show that the proposed algorithm can achieve an $O(\sqrt{V^*(T)})$ regret and $O(1)$ constraint violation simultaneously. Such a bound holds in general non-Euclidean spaces, is never worse than the previously known $(O(\sqrt{T}), O(1))$ result, and can be much better on regret when the variation is small. Furthermore, our algorithm is computationally efficient in that only two mirror descent steps are required during each slot instead of solving a general Lagrangian minimization problem. Along the way, our bounds also improve upon those of previous attempts using mirror-prox-type algorithms solving this problem, which yield a relatively worse $O(T^{2/3})$ regret and $O(T^{2/3})$ constraint violation.

1 Introduction

We study online convex optimization (OCO) with a sequence of arbitrarily varying loss functions $f^1, f^2, \cdots$. The decision maker needs to choose an action $x^t$ from a set $\mathcal{X}$ before observing the actual loss function $f^t$ to optimize during each time slot. The goal is to minimize the regret over $T$ time slots defined as

$$\text{Regret}(T) = \sum_{t=1}^{T} f^t(x^t) - \min_{x \in \mathcal{X}} \sum_{t=1}^{T} f^t(x),$$

where $x_t$ is the learned decision iterate for round $t$, and the minimization indicates we are comparing to a best fixed strategy in hindsight knowing all loss function over $T$ time slots. In this

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paper, we define \( \mathbf{x}^* \) as a solution in hindsight to the following static constrained optimization problem \( \min_{\mathbf{x} \in \mathcal{X}} \sum_{t=1}^{T} f_t(\mathbf{x}) \), i.e., \( \mathbf{x}^* \in \arg\min_{\mathbf{x} \in \mathcal{X}} \sum_{t=1}^{T} f_t(\mathbf{x}) \), such that \( \text{Regret}(T) = \sum_{t=1}^{T} f_t(\mathbf{x}_t) - \sum_{t=1}^{T} f_t(\mathbf{x}^*) \). Over the decades, this problem has been studied extensively in various works, e.g. Cesa-Bianchi et al. (1996); Gordon (1999); Zinkevich (2003); Hazan (2016). In particular, a commonly known algorithm solving this problem is online mirror descent (OMD), which achieves \( \mathcal{O}(\sqrt{T}) \) regret with a dimension dependency related to the chosen norm on the optimization space and logarithmic on the probability simplex. One could further improve upon this \( \mathcal{O}(\sqrt{T}) \) regret bound by considering the "path-lengths" or variations of the function gradient sequence (e.g. Hazan and Kale (2010); Chiang et al. (2012); Yang et al. (2014); Steinhardt and Liang (2014)). For example, the work Chiang et al. (2012) obtains a regret that scales \( (\sum_{t=1}^{T} \max_{x \in \mathcal{X}} \|\nabla f_t(x) - \nabla f_{t-1}(x)\|_2^2)^{1/2} \). It is never worse than \( \mathcal{O}(\sqrt{T}) \) and can be much better when the variation is small.

This paper considers a more challenging OCO problem where the constraint set \( \mathcal{X} \) consists of not only a simple compact set \( \mathcal{X}_0 \), but also \( K \) functional constraints:

\[
\mathcal{X} = \{ \mathbf{x} \in \mathbb{R}^d : \mathbf{x} \in \mathcal{X}_0 \text{ and } g_k(\mathbf{x}) \leq 0, \forall k = 1, \ldots, K \}.
\]

Here \( g_k(\mathbf{x}) \) denotes the \( k \)-th constraint function \( \forall k \in \{1, \ldots, K\} \) which is convex and differentiable. OMD does not work well on this problem as projecting onto such a complex constraint set on each round is usually computationally heavy. A common criterion for this problem, which has been adopted by a series of works, e.g. Mahdavi et al. (2012); Jenatton et al. (2015); Yu et al. (2017); Chen et al. (2017); Liakopoulos et al. (2019), is to instead allow functional constraints to be violated at each time slot, but to maintain sublinear regret and constraint violations at the same time. More specifically, apart from (1), we also look at the following constraint violation

\[
\text{Violation}(T, k) = \sum_{t=1}^{T} g_k(\mathbf{x}_t), \forall k = 1, \ldots, K,
\]

and aim at achieving an \( o(T) \) growth rate. The works Mahdavi et al. (2012); Jenatton et al. (2015); Yuan and Lamperski (2018) show one can get \( T^{\max\{\beta, 1-\beta\}} \) and \( T^{1-\beta/2} \) constraint violation via a primal dual gradient descent type algorithm where \( \beta \in (0, 1) \). This bound is later improved in the work Yu and Neely (2020) where a much better \( \mathcal{O}(\sqrt{T}) \) regret and \( \mathcal{O}(1) \) constraint violation is shown under Slater condition and only Euclidean spaces. To achieve this bound, one has to solve a general Lagrangian minimization problem during each round, which in general requires inner loops to obtain approximate solutions. Yu et al. (2017); Wei et al. (2019) show \( \mathcal{O}(\sqrt{T}) \) regret and constraint violation which also apply to more general scenarios with stochastic constraints.

In this paper, we treat constrained OCO from a new angle where the sequence of objective function \( f^1, \ldots, f^T \) might not be completely adversarial but changes gradually. We propose a new online primal-dual mirror prox method which exploits such a property in particular, and falls back to the best known \( (\mathcal{O}(\sqrt{T}), \mathcal{O}(1)) \) result when the objective function sequence is arbitrary. Mirror prox algorithm is introduced in the seminal work (Nemirovski, 2004), which has been shown to achieve \( \mathcal{O}(1/T) \) convergence rate minimizing smooth convex functions (Bubeck, 2014). Interestingly, our bound also falls back to \( \mathcal{O}(1/T) \) convergence rate (measured by \( \text{Regret}(T)/T \)) in the case of solving a deterministic smooth convex optimization by the mirror prox algorithm. It is also worth noting
that the previous work (Mahdavi et al., 2012) tries to apply mirror prox to constrained OCO with a relatively worse $O(T^{2/3})$ regret and constrained violation compared to our work.

### 1.1 Contributions

The main contributions of this paper are as follows: (1) We propose a new online primal-dual mirror prox method which achieves an $O(\max\{\sqrt{V_*(T)}, L_f\})$ regret, where $L_f$ is the Lipschitz constant for $\nabla f^t$, and $O(1)$ constraint violation simultaneously in a general normed space $(X_0, \| \cdot \|)$ under Slater condition. The value $V_*(T)$ is the gradient variation (Chiang et al., 2012) defined as

$$V_*(T) = \max_{t=1}^T \sum_{x \in X_0} \| \nabla f^t(x) - \nabla f^{t-1}(x) \|_\ast^2,$$

where $\| \cdot \|_\ast$ is the dual norm, i.e. $\|x\|_\ast := \sup_{\|y\|\leq 1} \langle x, y \rangle$. (2) Note that even in the worst case, $V_*(T)$ is in the level of $O(T)$ if we assume the gradients of loss functions are bounded in $X_0$. Thus, our bound is never worse than the best known ($O(\sqrt{T}), O(1)$) bound and can be much better when the variation is small. In particular, when the objective never changes, our bound matches the best known $O(1/T)$ convergence rate (measured by Regret($T$)/$T$) when solving a general smooth convex optimization by the mirror prox algorithm. (3) Our method applies to general normed spaces with efficient implementations in that we only compute two mirror descent steps during each round to obtain decision variables. This is in contrast to the previous work achieving the best known rate (Yu and Neely, 2020), which requires solving a general Lagrangian minimization problem involving entire constraint functions each round. For more detailed comparison of our work to previous works, see Table 1.

#### Table 1: Comparison with results from existing works. We omit $O(\cdot)$ notation for regret and constraint violation. We use const. to denote $O(1)$ bound. $\beta \in (0, 1)$. Gradient update means whether each round only involves gradient updates to compute decision variables (see Remark 3.1). Here $L_f$ is the Lipschitz constant for the gradient of the loss function, i.e. $\nabla f^t$, for any $t \geq 0$. We let $a \lor b$ denote $\max\{a, b\}$.

| Paper                        | Regret          | Constraint Violation | Gradient Update | Space       |
|------------------------------|-----------------|----------------------|-----------------|-------------|
| Mahdavi et al. (2012)        | $\sqrt{T}$      | $T^{3/4}$            | Yes             | Euclidean   |
| Jenatton et al. (2015)       | $T^{\max\{\beta, 1-\beta\}}$ | $T^{1-\beta/2}$       |                |             |
| Yu et al. (2017)             | $\sqrt{T}$      | $\sqrt{T}$           | Yes             | General     |
| Yuan and Lamperski (2018)    | $T^{\max\{\beta, 1-\beta\}}$ | $T^{1-\beta/2}$       |                |             |
| Wei et al. (2019)            | $\sqrt{T}$      | $\sqrt{T}$           | Yes             | General     |
| Yu and Neely (2020)          | $\sqrt{T}$      | const.                | No              | Euclidean   |
| This work                    | $\sqrt{V_*(T) \lor L_f}$ | const.                | Yes             | General     |

This work
2 Problem Setup

We introduce the problem setup in our paper. Suppose that the feasible set $\mathcal{X}_0 \subset \mathbb{R}^d$ is convex and compact, and there are $K$ long-term constraints $g_k(x) \leq 0, \forall k \in K$, which compose the set $\mathcal{X}$ defined as (2). At each round $t$, after generating an iterate $x_t$, the decision maker will observe a loss function $f^t : \mathcal{X}_0 \to \mathbb{R}$. Our goal is to propose primal-dual online mirror descent algorithms to generate a sequence of iterates $\{x_t\}_{t \geq 0}$ within $\mathcal{X}_0$, such that the regret $\text{Regret}(T)$ and constraint violation $\text{Violation}(T, k), \forall k \in [K]$, defined in (1) and (3) grow sublinearly w.r.t. the total number of rounds $T$. Hereafter, we denote $g(x) = [g_1(x), g_2(x), \ldots, g_K(x)]^T$ as a vector constructed by stacking all the constraint function values at the point $x$. We let $\| \cdot \|$ be a norm with $\| \cdot \|_\ast$ denoting its dual norm. We let $[n]$ denote the set $\{1, 2, \ldots, n\}$ for any $n \in \mathbb{Z}^+$. Then, for the functions $g_k$ and $f^t$, we make the following common assumptions for constrained online optimization.

Assumption 2.1. We make several assumptions on the properties of the set $\mathcal{X}_0$, the function $f^t$, and the function $g_k, \forall k \in [K]$:

(a). The set $\mathcal{X}_0$ is convex and compact.

(b). The gradient of the function $f^t$ is bounded, i.e., $\| \nabla f^t(x) \|_\ast \leq F$, $\forall x \in \mathcal{X}_0, \forall t \geq 0$. Moreover, $\nabla f^t$ is $L_f$-Lipschitz continuous, i.e. $\| \nabla f^t(x) - \nabla f^t(y) \|_\ast \leq L_f \| x - y \|, \forall x, y \in \mathcal{X}_0, \forall t \geq 0$ and some $L_f > 0$.

(c). The function $g_k$ is bounded, i.e., $\sum_{k=1}^{K} |g_k(x)| \leq G, \forall x \in \mathcal{X}_0$. Moreover, $g_k$ is $H_k$-Lipschitz continuous, i.e., $|g_k(x) - g_k(y)| \leq H_k \| x - y \|, \forall x, y \in \mathcal{X}_0, \forall k \in [K]$. We let $H := \sum_{k=1}^{K} H_k$.

(d). The gradient of the function $g_k$ is $L_g$-Lipschitz continuous, i.e., $\| \nabla g_k(x) - \nabla g_k(y) \|_\ast \leq L_g \| x - y \|, \forall x, y \in \mathcal{X}_0, \forall k \in [K]$ and some $L_g > 0$.

Assumption 2.2 (Slater Condition). There exist a constant $\varsigma > 0$ and a point $\bar{x} \in \mathcal{X}_0$ such that $g_k(\bar{x}) \leq -\varsigma$ holds for any $k \in [K]$.

Assumption 2.2 is a standard assumption for constrained optimization problem (Boyd et al., 2004) and is also commonly adopted in constrained OCO problem (Yu and Neely, 2017; Yu et al., 2017; Yu and Neely, 2020). We further define the Bregman divergence $D(x, y)$ for any $x, y \in \mathcal{X}_0$. Let $\omega : \mathcal{X}_0 \to \mathbb{R}$ be a strictly convex function, which is continuously differentiable in the interior of $\mathcal{X}_0$. The Bregman divergence is defined as $D(x, y) = \omega(x) - \omega(y) - \langle \nabla \omega(y), x - y \rangle$. We call $\omega$ a distance generating function with modulus $\rho$ w.r.t. the norm $\| \cdot \|$ if it is $\rho$-strongly convex, i.e. $\langle x - y, \nabla \omega(x) - \nabla \omega(y) \rangle \geq \rho \| x - y \|^2$. Then, the Bregman divergence defined by the distance generating function $\omega$ satisfies $D(x, y) \geq \frac{\rho}{2} \| x - y \|^2$. Some common examples for Bregman Divergence are: (1) If $\omega(x) = \frac{1}{2} \| x \|^2_2$ with modulus $\rho = 1$ w.r.t. $\ell_2$ norm $\| \cdot \|_2$, then the Bregman divergence is Euclidean distance on $\mathbb{R}^d$, i.e. $D(x, y) = \frac{1}{2} \| x - y \|^2_2$, where the dual norm is $\ell_2$ norm as well. (2) If $\omega(x) = -\sum_{i=1}^{d} x_i \log x_i$ for any $x \in \mathcal{X}_0$ with $\mathcal{X}_0$ being a probability simplex $\Delta$, i.e.

$$
\Delta := \{ x \in \mathbb{R}^d : \| x \|_1 = 1 \text{ and } x_i \geq 0, \forall i \in [d] \},
$$

(5)
Algorithm 1 Online Primal-Dual Mirror Prox Algorithm

1: **Initialize:** \( \gamma > 0; \ x_0, x_1, x_1 \in X_0; \ Q_k(0) = 0, \forall k = 1, \ldots, K. \)
2: **for** Round \( t = 1, \ldots, T \) **do**
3:   **Update the dual iterate** \( Q_k(t) \):
   \[
   Q_k(t) = \max \{-\gamma g_k(x_{t-1}), \ Q_k(t-1) + \gamma g_k(x_{t-1})\}.
   \]
4:   **Update the primal iterate** \( x_t \):
   \[
   x_t = \arg\min_{x \in X_0} \langle \nabla f^{t-1}(x_{t-1}), x \rangle + \sum_{k=1}^{K} [Q_k(t) + \gamma g_k(x_{t-1})] \langle \gamma \nabla g_k(x_{t-1}), x \rangle + \alpha_t D(x, \tilde{x}_t).
   \]
5:   **Play** \( x_t \).
6:   **Suffer loss** \( f^t(x_t) \) and compute \( \nabla f^t(x_t) \).
7:   **Update the intermediate iterate** \( \tilde{x}_{t+1} \) for the next round:
   \[
   \tilde{x}_{t+1} = \arg\min_{x \in X_0} \langle \nabla f^t(x_t), x \rangle + \sum_{k=1}^{K} [Q_k(t) + \gamma g_k(x_{t-1})] \langle \gamma \nabla g_k(x_{t-1}), x \rangle + \alpha_t D(x, \tilde{x}_t).
   \]
8: **end for

Then \( D(x, y) = D_{KL}(x, y) := \sum_{i=1}^{d} x_i \log(x_i/y_i) \) is the Kullback-Leibler (KL) divergence, where \( x \in \Delta \) and \( y \in \Delta^0 := \Delta \cap \text{relint}(\Delta) \) with \( \text{relint}(\Delta) \) denoting the relative interior of \( \Delta \). In this case, \( \omega \) has a modulus \( \rho = 1 \) w.r.t. \( \ell_1 \) norm \( \| \cdot \|_1 \) with the dual norm \( \| \cdot \|_\infty \).

For our proposed Algorithm 1 presented in the next section, we make the following assumption.

**Assumption 2.3.** There exists \( R > 0 \) such that \( D(x, y) \leq R^2, \forall x, y \in X_0, \) such that \( \| x - y \|^2 \leq 2R^2/\rho. \)

**Remark 2.4.** Note that Assumption 2.3 does not hold when \( D(x, y) \) is KL divergence with \( X_0 = \Delta \) being a probability simplex. If \( y \) is close to the boundary of the probability simplex, \( D_{KL}(x, y) \) can be arbitrarily large according to the definition of KL divergence such that \( D(x, y) \) is not bounded. We further analyze the probability simplex setting and complement our theory in Section 6 with presenting Algorithm 2 as an extension of Algorithm 1.

### 3 Algorithm

In this section, we introduce our proposed online primal-dual mirror prox algorithm for the constrained online convex optimization problem, which is illustrated in Algorithm 1.

At the \( t \)-th round of this algorithm, we let \( Q_k(t) \) be the dual iterates updated based on the \( k \)-th constraint function \( g_k(x) \), for all \( k \in [K] \). Hereafter, we denote \( Q(t) = [Q_1(t), \ldots, Q_K(t)]^\top \in \mathbb{R}^k \) as the vector constructed by the values of all \( Q_k(t) \). Intuitively, \( Q(t) \) can be viewed as a Lagrange multiplier vector where each entry’s value is guaranteed to be non-negative by induction with initializing \( Q_k(0) = 0 \). From another perspective, \( Q(t) \) is a virtual queue for backlogging
constraint violations. Similar ideas of employing the virtual queue are adopted in several recent works (Yu and Neely, 2017; Yu et al., 2017; Wei et al., 2019; Yu and Neely, 2020).

The primal iterate \( x_t \) is the decision at each round by the decision maker. To obtain the decision iterate, we propose a new online mirror-prox-type update: (1) Incorporating the dual iterates \( Q_k(t) \) and the gradient of constraints \( \nabla g_k(x_{t-1}) \), Line 4 runs a mirror descent step based on the last decision \( x_{t-1} \) and an intermediate iterate \( \tilde{x}_t \) generated in the last round. (2) After observing a new loss function \( f^t \), Line 7 further generates an intermediate iterate \( \tilde{x}_{t+1} \) for the next round. Technically, our proposed updating step can yield a lower regret without sacrificing the constraint violation bound, as shown in the following sections.

Note that Line 4 and Line 7 are just two mirror descent steps. More specifically, let \( \eta \) can be rewritten as

\[
\nabla \omega(y_t) = \nabla \omega(\tilde{x}_t) - \alpha_t^{-1} \eta_{t-1}, \quad x_t = \arg\min_{x \in X_0} D(x, y_t).
\]

When choosing the Bregman divergence as \( D(x, y) = \frac{1}{2} \| x - y \|_2^2 \), it further reduces to gradient descent based update with Euclidean projection on \( X_0 \), i.e., \( x_t = \text{Proj}_{X_0} \{ \tilde{x}_t - \alpha_t^{-1} \eta_{t-1} \} \).

**Remark 3.1.** The primal update of Yu and Neely (2020) is a minimization problem involving the exact constraint function \( g_k(x) \), which can be any complex form. Thus, its primal update cannot reduce to the projection of gradient descent based updates if \( g_k(x) \) is not in a linear form, which thus leads to a high computational cost. As opposed to Yu and Neely (2020), our proposed updates only rely on a local linearization of the constraint function \( g_k(x) \) by its gradient at \( x_{t-1} \), which is \( g_k(x_{t-1}) + \langle \nabla g_k(x_{t-1}), x - x_{t-1} \rangle \) (the constant term \( g_k(x_{t-1}) - \langle \nabla g_k(x_{t-1}), x_{t-1} \rangle \) does not affect the result of the update and is ignored in Algorithm 1). Thus, Algorithm 1 can enjoy the advantage of lower computational complexity resulting from the local linearization of \( g_k(x) \).

## 4 Main Results

In this section, we present the regret bound (Theorem 4.1) and the constraint violation bound (Theorem 4.2) for Algorithm 1.

**Theorem 4.1 (Regret).** Under the Assumptions 2.1, 2.2, and 2.3, setting \( \eta = \left[ \max \{ V_s(T), L_f^2 \} \right]^{-1/2} \), \( \gamma = \left[ \max \{ V_s(T), L_f^2 \} \right]^{1/4} \), and \( \alpha_t = \max \{ 2\rho^{-1}(\gamma^2 L_g G + \eta L_f^2 + \eta^{-1} + \xi_t), \alpha_{t-1} \} \) with initializing \( \alpha_0 = 0 \) and defining

\[
\xi_t := \gamma L_g \| Q(t) \|_1 + \gamma^2 (L_g G + H^2), \tag{6}
\]

then Algorithm 1 ensures the following regret

\[
\text{Regret}(T) \leq \left( C_1 + \frac{C_2}{\zeta} \right) \max \left\{ \sqrt{V_s(T)}, L_f \right\} + \frac{C_3}{\zeta} = \mathcal{O} \left( \max \{ \sqrt{V_s(T)}, L_f \} \right),
\]

where \( C_1, C_2, \text{ and } C_3 \) are constants \( \text{poly}(R, H, F, G, L_g, K, \rho) \). \(^1\)

\(^1\)Hereafter, we use \( \text{poly}(\cdots) \) to denote a polynomial term composed of the variables inside the parentheses.
As shown in this theorem, the setting of $\alpha_t$ guarantees that $\{\alpha_t\}_{t \geq 0}$ is a non-decreasing sequence such that $\alpha_{t+1} \geq \alpha_t$. Note that this setting is sensible since it in fact implies a non-increasing step size $\alpha_t^{-1}$. For a clear understanding, consider a simple example: if $D(x, y) = \frac{1}{2}\|x - y\|_2^2$ and all $g_k(x) = 0, \forall x \in X_0$ such that we have a regular constrained online optimization problem, then Line 4 becomes $x_t = \text{Proj}_{\mathcal{X}_0}[\tilde{x}_t - \alpha_t^{-1}\nabla f^{t-1}(x_{t-1})]$ with $\alpha_t^{-1}$ being a non-increasing step size.

**Theorem 4.2** (Constraint Violation). **Under the Assumptions** 2.1, 2.2, and 2.3, with the same settings of $\eta$, $\gamma$, and $\alpha_t$ as Theorem 4.1, Algorithm 1 ensures the following constraint violation bound

$$\text{Violation}(T, k) \leq C'_1 + \frac{C'_2}{\varsigma} = \mathcal{O}(1), \quad \forall k \in [K],$$

where $C'_1$ and $C'_2$ are constants $\text{poly}(R, H, F, G, L_f, L_g, K, \rho)$.

Theorem 4.1 shows that the regret is bounded by $\mathcal{O}(\max\{\sqrt{V_x(T)}, L_f\})$, which can be interpreted as: (1) Since $\mathcal{O}(\sqrt{V_x(T)})$ is never worse than $\mathcal{O}(\sqrt{T})$, we achieve a neutral to better regret bound compared to $\mathcal{O}(\sqrt{T})$ regret in Yu and Neely (2020) with the same $\mathcal{O}(1)$ constraint violation. (2) When $f^1 = f^2 = \cdots = f^T$ such that $V_x(T) = 0$, the $\mathcal{O}(1)$ regret and constraint violation is equivalent to the $\mathcal{O}(1/T)$ convergence rate (measured by Regret$(T)/T$) solving a general smooth convex optimization, matching the results of mirror-prox algorithms.

Moreover, our theorems hold in the general non-Euclidean space where the Euclidean space is a special case. Previous studies (Wei et al., 2019) for this problem in non-Euclidean space only obtain a worse $\mathcal{O}(\sqrt{T})$ bound for the regret and constraint violation. In addition, our result improves upon previous attempts using mirror-prox-type algorithms, which achieve a relatively worse $\mathcal{O}(T^{2/3})$ bound for both regret and constraint violation (Mahdavi et al., 2012). Please see Table 1 for the detailed comparisons.

### 5 Theoretical Analysis

Our analysis starts from the drift-plus-penalty expression with the drift-plus-penalty term defined as

$$\frac{1}{2}[\|Q(t + 1)\|_2^2 - \|Q(t)\|_2^2] + \langle \nabla f^{t-1}(x_{t-1}), x \rangle + \alpha_tD(x, \tilde{x}_t). \quad (7)$$

The drift term shows the one-step change of the vector $Q(t)$ which is the backlog queue of the constraint functions. The penalty term is associated with a mirror descent step when observing the gradient of the loss function $f^{t-1}$. The drift-plus-penalty expression is investigated in recent papers on constrained online optimization problems, e.g. Yu and Neely (2017); Yu et al. (2017); Wei et al. (2019); Yu and Neely (2020). However, the techniques in our analysis for this expression are largely distinct from the existing works. Our theoretical analysis also makes a step towards understanding the drift-plus-penalty expression under mirror-prox-type algorithms.

Thus, we give upper bound of the drift-plus-penalty term at $x = x_t$ in the following lemma.
Lemma 5.1. At the t-th round of Algorithm 1, for any \( \gamma > 0 \) and any \( z \in \mathcal{X}_0 \), letting \( \xi_t \) as in (6), the drift-plus-penalty term admits the following bound

\[
\frac{1}{2} [\|Q(t + 1)\|^2_2 - \|Q(t)\|^2_2] + \langle \nabla f_{t-1}(x_{t-1}), x_t \rangle + \alpha_t D(x_t, \bar{x}_t)
\]

\[
\leq \frac{\xi_t}{2} \|x_t - x_{t-1}\|^2 + \frac{\gamma^2}{2} [\|g(x_t)\|^2_2 - \|g(x_{t-1})\|^2_2] + \langle \nabla f_t(x_t), z \rangle + \alpha_t D(z, \bar{x}_t) - \alpha_t D(z, \bar{x}_{t+1})
\]

\[
+ \langle \nabla f_{t-1}(x_{t-1}) - \nabla f_t(x_t), \bar{x}_{t+1} \rangle - \alpha_t D(\bar{x}_{t+1}, x_t) + \gamma \langle Q(t) + \gamma g(x_{t-1}), g(z) \rangle.
\]

Please see Section A.2 in the supplementary material for the proof of Lemma 5.1. This lemma is one of the key lemmas in our proof, by which we will further obtain Lemma 5.2 and Lemma 5.4.

5.1 Proof Sketches

Within this subsection, all the lemmas and the proofs are under the Assumptions 2.1, 2.2, and 2.3. By Lemma 5.1, we further have the following lemma.

Lemma 5.2. At the t-th round of Algorithm 1, for any \( \eta, \gamma > 0 \) and any \( z \in \mathcal{X}_0 \), the following inequality holds

\[
\frac{1}{2} [\|Q(t + 1)\|^2_2 - \|Q(t)\|^2_2] + \langle \nabla f_t(x_t), x_t - z \rangle \leq \zeta_t (\|x_t - \bar{x}_t\|^2 + \|x_t - \bar{x}_{t+1}\|^2) + U_t - U_{t+1}
\]

\[
+ \frac{\eta}{2} \|\nabla f_{t-1}(x_t) - \nabla f_t(x_t)\|^2_2 + (\alpha_{t+1} - \alpha_t) D(z, \bar{x}_{t+1}) + \gamma \langle Q(t) + \gamma g(x_{t-1}), g(z) \rangle,
\]

where we define

\[
\zeta_t := \xi_t + \eta L_f^2 - \rho \alpha_t / 2 + 1 / \eta + \gamma^2 L_g G,
\]

\[
U_t := \left( \xi_t + \eta L_f^2 \right) \|x_{t-1} - \bar{x}_t\|^2 + \alpha_t D(z, \bar{x}_t) - \gamma^2 / 2 \cdot \|g(x_{t-1})\|^2_2.
\]

Please see Section A.3 in the supplementary material for the detailed proof. Based on Lemma 5.2, we further obtain the following lemma.

Lemma 5.3. With setting \( \eta, \gamma, \) and \( \alpha_t \) the same as in Theorem 4.1, Algorithm 1 ensures

\[
\alpha_{T+1} \leq \left( \widetilde{C}_1 + \frac{\widetilde{C}_2}{\zeta} \right) \max \{ \sqrt{V_s(T)}, L_f \} + \frac{\widetilde{C}_3}{\zeta},
\]

\[
\|Q(T + 1)\|_2 \leq \left( \widetilde{C}_1 + \frac{\widetilde{C}_2}{\zeta} \right) \left[ \max \{ V_s(T), L_f^2 \} \right]^{1/4} + \frac{\widetilde{C}_3}{\zeta L_f^{1/2}},
\]

where \( \widetilde{C}_1, \widetilde{C}_2, \widetilde{C}_3, \widetilde{C}_1', \widetilde{C}_2', \) and \( \widetilde{C}_3' \) are absolute constants \( \text{poly}(R, H, F, G, L_g, K, \rho) \).

Please see Section A.4 in the supplementary material for the detailed proof of Lemma 5.3. This is another key lemma in our proof. It shows that \( \alpha_{T+1} \leq \mathcal{O}(\sqrt{V_s(T)}) \) and \( \|Q(T + 1)\|_2 \leq \mathcal{O}(V_s(T)^{1/4}) \) after \( T + 1 \) rounds of Algorithm 1, which is sufficiently tight to yield the eventual regret lower than \( \mathcal{O}(\sqrt{T}) \) and \( \mathcal{O}(1) \) constraint violation.
**Lemma 5.4.** For any $\eta, \gamma \geq 0$, setting $\alpha_t$ the same as in Theorem 4.1, Algorithm 1 ensures

$$\text{Regret}(T) \leq \frac{\eta}{2} V_*(T) + \mathcal{C}_1' L_f^2 \eta + \mathcal{C}_2' \gamma^2 + \mathcal{C}_3' \alpha_{T+1},$$

where $\mathcal{C}_1'$, $\mathcal{C}_2'$, and $\mathcal{C}_3'$ are absolute constants that are $\text{poly}(R, H, G, L_g, \rho)$.

Please see Section A.5 in the supplementary material for the detailed proof. Lemma 5.4 is also deduced from Lemma 5.2. Combining this lemma with Lemma 5.3 consequently yields the regret bound in Theorem 4.1.

**Lemma 5.5.** For any $\eta > 0$, the updating rule of $Q_k(t)$ in Algorithm 1 ensures

$$\text{Violation}(T, k) \leq \frac{1}{\gamma} \|Q(T + 1)\|_2.$$

Please see Section A.6 in the supplementary material for the detailed proof of Lemma 5.5. Combining this lemma with the drift bound of $Q(T+1)$ in Lemma 5.3 can finally give the constraint violation bound in Theorem 4.2.

### 5.1.1 Regret Bound

**Proof of Theorem 4.1.** According to Lemma 5.4, by the settings of $\eta$ and $\gamma$, we have

$$\text{Regret}(T) \leq \left(\frac{1}{2} + \mathcal{C}_1''\right) \max \left\{ \sqrt{V_*(T)}, L_f \right\} + \mathcal{C}_2'' \max \left\{ \sqrt{V_*(T)}, L_f \right\} + \mathcal{C}_3'' \alpha_{T+1},$$

where the inequality is due to $\eta/2 \cdot V_*(T) + \mathcal{C}_1'' L_f^2 \eta \leq \eta (1/2 + \mathcal{C}_1'') \max \left\{ V_*(T), L_f^2 \right\} \leq (1/2 + \mathcal{C}_1'') \max \left\{ \sqrt{V_*(T)}, L_f \right\}$. Further combining the above inequality with the bound of $\alpha_{T+1}$ in Lemma 5.3 yields

$$\text{Regret}(T) \leq \left( C_1 + \frac{C_2}{\varsigma} \right) \max \left\{ \sqrt{V_*(T)}, L_f \right\} + \frac{C_3}{\varsigma},$$

where $C_1 = 1/2 + \mathcal{C}_1'' + \mathcal{C}_2'' + \mathcal{C}_3'' \mathcal{C}_1$, $C_2 = \mathcal{C}_3'' \mathcal{C}_2$, and $C_3 = \mathcal{C}_3'' \mathcal{C}_3$. This completes the proof. \(\square\)

### 5.1.2 Constraint Violations

**Proof of Theorem 4.2.** According to Lemma 5.5 and the drift bound of $Q(T+1)$ in Lemma 5.3, with the setting of $\gamma$, we have

$$\text{Violation}(T, k) \leq \frac{1}{\gamma} \|Q(T + 1)\|_2 \leq \mathcal{C}_1' + \frac{\mathcal{C}_2'}{\varsigma L_f} = C_1' + \frac{C_2'}{\varsigma},$$

where the second inequality is by $1/\gamma = \min \{ [V_*(T)]^{-1/4}, L_f^{-1/2} \} \leq L_f^{-1/2}$. Specifically, we set $C_1' = \mathcal{C}_1'$, $C_2' = \mathcal{C}_2' + \mathcal{C}_3'/L_f$. This completes the proof. \(\square\)
Algorithm 2 Online Primal-Dual Mirror Prox Algorithm on Probability Simplex

1: Initialize: $\gamma > 0; \nu \in (0, 1]; x_0 = x_1 = \bar{x} = 1/d; Q_k(0) = 0, \forall k = 1, \ldots, K.$

2: for Round $t = 1, \ldots, T$ do

3: Mix the iterates:

$$\bar{y}_t = (1 - \nu)\bar{x}_t + \frac{\nu}{d} 1.$$ 

4: Update the dual iterate $Q_k(t)$:

$$Q_k(t) = \max \{-\gamma g_k(x_{t-1}), Q_k(t-1) + \gamma g_k(x_{t-1})\}.$$ 

5: Update the primal iterate $x_t$:

$$x_t = \arg\min_{x \in \Delta} \langle \nabla f^{t-1}(x_{t-1}), x \rangle + \sum_{k=1}^{K} [Q_k(t) + \gamma g_k(x_{t-1})] \langle \gamma \nabla g_k(x_{t-1}), x \rangle + \alpha_t D(x, \bar{y}_t).$$

6: Play $x_t$.

7: Suffer loss $f^t(x_t)$ and compute $\nabla f^t(x_t)$.

8: Update the intermediate iterate $\bar{x}_{t+1}$ for the next round:

$$\bar{x}_{t+1} = \arg\min_{x \in \Delta} \langle \nabla f^t(x_t), x \rangle + \sum_{k=1}^{K} [Q_k(t) + \gamma g_k(x_{t-1})] \langle \gamma \nabla g_k(x_{t-1}), x \rangle + \alpha_t D(x, \bar{y}_t).$$

9: end for

6 Extension to the Probability Simplex Case

In the probability simplex case, we have $X_0 = \Delta$ where $\Delta$ denotes the probability simplex as in (5) and the Bregman divergence is the KL divergence, namely $D(x, y) = D_{KL}(x, y)$. Thus, the norm $\| \cdot \|$ defined in this space is $\ell_1$ norm $\| \cdot \|_1$ with the dual norm $\| \cdot \|_* = \| \cdot \|_\infty$ such that the gradient variation is measured by

$$V_\infty(T) = \sum_{t=1}^{T} \max_{x \in X_0} \| \nabla f^t(x) - \nabla f^{t-1}(x) \|_\infty^2.$$ 

Then, the results in this section is expressed in terms of $V_\infty(T)$. Note that the Assumption 2.3 is no longer valid, since $D_{KL}(x, y)$ can tend to infinity by its definition if there some entry $y_i \to 0$. Our algorithm for the probability simplex case is illustrated in Algorithm 2.

To tackle the challenge of the unbounded KL divergence, we propose to mix the iterate $\bar{x}_t$ with an all-one vector $1 \in \mathbb{R}^d$, as shown in Line 3 of Algorithm 2. Intuitively, the mixing step is to push the iterates $\bar{x}_t$ slightly away from the boundary of $\Delta$ in a controllable way with a weight $\nu$. By setting a proper mixing weight $\nu = 1/T$, we obtain theoretical results for the regret and constraint violation.
Theorem 6.1 (Regret). Under the Assumptions 2.1 and 2.2, setting $\eta = \max\{V_\infty(T), L_g^2}\}^{-1/2}$, $\gamma = \max\{V_\infty(T), L_g^2}\}^{1/4}$, $\nu = 1/T$, and $\alpha_t = \max\{3(\eta L_g^2 + \gamma^2 L_g G) + 2/\eta + 3\xi_t, \alpha_{t-1}\}$ with $\alpha_0 = 0$, for $T > 2$ and $d \geq 1$, Algorithm 1 ensures the following regret

$$\text{Regret}(T) \leq \left(\tilde{C}_1 + \tilde{C}_2\right) \max\left\{\sqrt{V_\infty(T), L_f}\right\} \log^2(Td) + \frac{\tilde{C}_3 \log(Td)}{\xi} = \tilde{O}\left(\max\left\{\sqrt{V_\infty(T), L_f}\right\}\right),$$

where $\tilde{C}_1$, $\tilde{C}_2$, and $\tilde{C}_3$ are constants $\text{poly}(H, F, G, L_g, K)$, and $\tilde{O}$ hides the logarithmic factor.

Theorem 6.2 (Constraint Violation). Under the Assumptions 2.1 and 2.2, with the same settings of $\eta$, $\gamma$, $\nu$, and $\alpha_t$ as Theorem 4.1, Algorithm 1 ensures the following constraint violation bound

$$\text{Violation}(T, k) \leq \left(\tilde{C}_1 + \tilde{C}_2\right) \log(Td) = \mathcal{O}(\log T), \quad \forall k \in [K],$$

where $C'_1$ and $C'_2$ are constants $\text{poly}(H, F, G, L_f, L_g, K)$.

The results in Theorem 6.1 and Theorem 6.2 show that the iterate mixing step of Algorithm 2 only introduce extra logarithmic factors $\log T$, which almost remain the results in Theorem 4.1 and Theorem 4.2. Please see detailed proofs in Section B of the supplementary material.

7 Conclusion

In this paper, we propose a new online primal-dual mirror prox method for constrained OCO problems, which achieves an $\mathcal{O}(\max\{\sqrt{V_\ast(T), L_f}\})$ regret and $\mathcal{O}(1)$ constraint violation simultaneously in a general normed space $(X_0, \|\cdot\|)$ under Slater condition. In particular, our bound is never worse than the best known ($\mathcal{O}(\sqrt{T}), \mathcal{O}(1)$) bound and can be much better when the variation $\sqrt{V_\ast(T)}$ is small.

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Supplementary Material

A Proofs for Section 5

A.1 Preliminary Lemmas

Lemma A.1. Supposing that the updating rule for $Q_k(t)$ is $Q_k(t+1) = \max\{-\gamma g_k(x_t), Q_k(t) + \gamma g_k(x_t)\}, \forall k = 1, \ldots, K$ with $\gamma > 0$, then we have

(a). $Q_k(t) \geq 0$,

(b). $\frac{1}{2}\|Q(t+1)\|_2^2 - \|Q(t)\|_2^2 \leq \gamma\langle Q(t), g(x_t) \rangle + \gamma^2\|g(x_t)\|_2^2$,

(c). $\|Q(t+1)\|_2 \leq \|Q(t)\|_2 + \gamma\|g(x_t)\|_2$,

(d). $\|Q(t+1)\|_1 - \|Q(t)\|_1 \leq \gamma\|g(x_t)\|_1$,

where we let $Q(t) = [Q_1(t), \ldots, Q_K(t)]^T$ and $g(x_t) = [g_1(x_t), \ldots, g_K(x_t)]^T$.

Proof. The inequalities (a), (b), and (c) are immediately obtained from Yu and Neely (2020). For the third inequality, we prove it in the following way. According to the updating rule, we know $Q_k(t+1) = \max\{-\gamma g_k(x_t), Q_k(t) + \gamma g_k(x_t)\} \geq Q_k(t) + \gamma g_k(x_t)$, which thus implies that $Q_k(t+1) - Q_k(t) \geq \gamma g_k(x_t)$. On the other hand, the updating rule also implies that $Q_k(t+1) = \max\{-\gamma g_k(x_t), Q_k(t) + \gamma g_k(x_t)\} \leq |Q_k(t)| + |\gamma g_k(x_t)| = Q_k(t) + \gamma g_k(x_t)$, where we also use the fact that $Q_k(t) \geq 0, \forall t$. Thus, we have $\gamma g_k(x_t) \leq Q_k(t+1) - Q_k(t) \leq |\gamma g_k(x_t)|$, which leads to $\gamma\sum_{k=1}^K g_k(x_t) \leq \|Q(t+1)\|_1 - \|Q(t)\|_1 \leq \gamma\|g(x_t)\|_1$ since $Q_k(t) \geq 0, \forall t$. Then, we have $\|Q(t+1)\|_1 - \|Q(t)\|_1 \leq \max\{\gamma\|g(x_t)\|_1, \gamma|\sum_{k=1}^K g_k(x_t)|\} \leq \gamma\|g(x_t)\|_1$. This completes the proof.

Lemma A.2 (Wei et al. (2019)). Suppose that $h : C \mapsto \mathbb{R}$ is a convex function with $C$ being a convex closed set. Let $D(\cdot, \cdot)$ be the Bregman divergence defined on the set $C$, and $M \subseteq C$ be a convex closed set. For any $y \in M \cap \text{relint}(C)$ where $\text{relint}(C)$ is the relative interior of $C$, let $x_{opt} := \arg\min_{x \in M}\{h(x) + \eta D(x, y)\}$, then we have

$h(x_{opt}) + \eta D(x_{opt}, y) \leq h(z) + \eta D(z, y) - \eta D(z, x_{opt}), \forall z \in M$.

Lemma A.3. For any function $h : C \mapsto \mathbb{R}$, if the gradient of $h(x)$ is $L$-Lipschitz continuous, i.e., $\|\nabla h(x) - \nabla h(y)\|_* \leq L\|x - y\|$, then we have

$h(x) \leq h(y) + \langle \nabla h(y), x - y \rangle + \frac{L}{2}\|x - y\|^2, \forall x, y \in C$.

A.2 Proof of Lemma 5.1

Proof. At the $t$-th round of Algorithm 1, the drift-plus-penalty term is

$$\frac{1}{2}\|Q(t+1)\|_2^2 - \|Q(t)\|_2^2 + \langle \nabla f^{t-1}(x_{t-1}), x_t \rangle + \alpha_t D(x_t, \bar{x}_t),$$
by setting \( x = x_t \) in (7). Thus, we start our proof from bounding the above term.

\[
\frac{1}{2} \left[ \| Q(t+1) \|_2^2 - \| Q(t) \|_2^2 \right] + \langle \nabla f^{t-1}(x_{t-1}), x_t \rangle + \alpha_t D(x_t, \bar{x}_t) \\
\leq \gamma \langle Q(t), g(x_t) \rangle + \gamma^2 \| g(x_t) \|_2^2 + \langle \nabla f^{t-1}(x_{t-1}), x_t \rangle + \alpha_t D(x_t, \bar{x}_t),
\]

(8)

where the inequality is by Lemma A.1. To further bound the term \( \langle \nabla f^{t-1}(x_{t-1}), x_t \rangle + \alpha_t D(x_t, \bar{x}_t) \) on the right-hand side of (8), recall the updating rule of \( x_t \) in Algorithm 1, and then apply Lemma A.2 by letting \( x^{opt} = x_t, \ y = \bar{x}_t, \ z = \bar{x}_{t+1}, \ \eta = \alpha_t, \) and \( h(x) = \langle \nabla f^{t-1}(x_{t-1}), x \rangle + \sum_{k=1}^{K} [Q_k(t) + \gamma g_k(x_{t-1})] \langle \gamma g_k(x_{t-1}), x \rangle. \) Thus, we have

\[
\langle \nabla f^{t-1}(x_{t-1}), x_t \rangle + \sum_{k=1}^{K} [Q_k(t) + \gamma g_k(x_{t-1})] \langle \gamma g_k(x_{t-1}), x_t \rangle + \alpha_t D(x_t, \bar{x}_t) \\
\leq \langle \nabla f^{t-1}(x_{t-1}), \bar{x}_{t+1} \rangle + \sum_{k=1}^{K} [Q_k(t) + \gamma g_k(x_{t-1})] \langle \gamma g_k(x_{t-1}), \bar{x}_{t+1} \rangle \\
+ \alpha_t D(\bar{x}_{t+1}, \bar{x}_t) - \alpha_t D(\bar{x}_{t+1}, x_t),
\]

which leads to

\[
\langle \nabla f^{t-1}(x_{t-1}), x_t \rangle + \alpha_t D(x_t, \bar{x}_t) \\
\leq \langle \nabla f^{t-1}(x_{t-1}), \bar{x}_{t+1} \rangle + \sum_{k=1}^{K} [Q_k(t) + \gamma g_k(x_{t-1})] \langle \gamma g_k(x_{t-1}), \bar{x}_{t+1} - x_t \rangle \\
+ \alpha_t D(\bar{x}_{t+1}, \bar{x}_t) - \alpha_t D(\bar{x}_{t+1}, x_t).
\]

Combining the above inequality with (8), we have

\[
\frac{1}{2} \left[ \| Q(t+1) \|_2^2 - \| Q(t) \|_2^2 \right] + \langle \nabla f^{t-1}(x_{t-1}), x_t \rangle + \alpha_t D(x_t, \bar{x}_t) \\
\leq \gamma \langle Q(t), g(x_t) \rangle + \gamma^2 \| g(x_t) \|_2^2 + \langle \nabla f^{t-1}(x_{t-1}), \bar{x}_{t+1} \rangle + \alpha_t D(\bar{x}_{t+1}, \bar{x}_t) \\
- \alpha_t D(\bar{x}_{t+1}, x_t) + \sum_{k=1}^{K} [Q_k(t) + \gamma g_k(x_{t-1})] \langle \gamma g_k(x_{t-1}), \bar{x}_{t+1} - x_t \rangle.
\]

(9)

For the term \( \langle \nabla f^{t-1}(x_{t-1}), \bar{x}_{t+1} \rangle + \alpha_t D(\bar{x}_{t+1}, \bar{x}_t) \) in (9), we decompose it as

\[
\langle \nabla f^{t-1}(x_{t-1}), \bar{x}_{t+1} \rangle + \alpha_t D(\bar{x}_{t+1}, \bar{x}_t) \\
= \langle \nabla f^{t-1}(x_{t-1}) - \nabla f^t(x_t), \bar{x}_{t+1} \rangle + \langle \nabla f^t(x_t), \bar{x}_{t+1} \rangle + \alpha_t D(\bar{x}_{t+1}, \bar{x}_t).
\]

(10)

We will bound the last two terms on the right-hand side of (10). Recall the updating rule for \( \bar{x}_{t+1} \) in Algorithm 1, and further employ Lemma A.2 with setting \( x^{opt} = \bar{x}_{t+1}, \ y = x_t, \) any \( z \in X_0, \)
\( \eta = \alpha_t \), and \( h(x) = \langle \nabla f^t(x_t), x \rangle + \sum_{k=1}^{K} [Q_k(t) + \gamma g_k(x_{t-1})]\langle \nabla g_k(x_{t-1}), x \rangle \). Then, we have

\[
\langle \nabla f^t(x_t), \bar{x}_{t+1} \rangle + \sum_{k=1}^{K} [Q_k(t) + \gamma g_k(x_{t-1})]\langle \nabla g_k(x_{t-1}), \bar{x}_{t+1} \rangle + \alpha_t D(\bar{x}_{t+1}, \bar{x}_t)
\]

\[
\leq \langle \nabla f^t(x_t), z \rangle + \sum_{k=1}^{K} [Q_k(t) + \gamma g_k(x_{t-1})]\langle \nabla g_k(x_{t-1}), z - \bar{x}_{t+1} \rangle + \alpha_t D(z, \bar{x}_t) - \alpha_t D(z, \bar{x}_{t+1}),
\]

rearranging whose terms yields

\[
\langle \nabla f^t(x_t), \bar{x}_{t+1} \rangle + \alpha_t D(\bar{x}_{t+1}, \bar{x}_t)
\]

\[
\leq \langle \nabla f^t(x_t), z \rangle + \sum_{k=1}^{K} [Q_k(t) + \gamma g_k(x_{t-1})]\langle \nabla g_k(x_{t-1}), z - \bar{x}_{t+1} \rangle + \alpha_t D(z, \bar{x}_t) - \alpha_t D(z, \bar{x}_{t+1}).
\]

Then, combining the above inequality with (9) and (10) gives

\[
\frac{1}{2} \left[ \|Q(t+1)\|^2 - \|Q(t)\|^2 \right] + \langle \nabla f^{t-1}(x_{t-1}), x_t \rangle + \alpha_t D(x_t, \bar{x}_t)
\]

\[
\leq \gamma \langle Q(t), g(x_t) \rangle + \gamma^2 \|g(x_t)\|^2 + \langle \nabla f^{t-1}(x_{t-1}) - \nabla f^t(x_t), \bar{x}_{t+1} \rangle - \alpha_t D(\bar{x}_{t+1}, x_t) + \sum_{k=1}^{K} [Q_k(t) + \gamma g_k(x_{t-1})]\langle \nabla g_k(x_{t-1}), z - x_t \rangle
\]

\[
+ \alpha_t D(z, \bar{x}_t) - \alpha_t D(z, \bar{x}_{t+1}) + \langle \nabla f^t(x_t), z \rangle.
\]

The term \( \gamma \langle Q(t), g(x_t) \rangle \) in (11) can be further bounded as

\[
\gamma \langle Q(t), g(x_t) \rangle = \sum_{k=1}^{K} \gamma Q_k(t) g_k(x_t)
\]

\[
= \sum_{k=1}^{K} \gamma [Q_k(t) + \gamma g_k(x_{t-1})] g_k(x_t) - \sum_{k=1}^{K} \gamma^2 g_k(x_{t-1}) g_k(x_t)
\]

\[
\leq \sum_{k=1}^{K} \gamma [Q_k(t) + \gamma g_k(x_{t-1})] \left[ g_k(x_{t-1}) + \langle \nabla g_k(x_{t-1}), x_t - x_{t-1} \rangle \right]
\]

\[
+ \frac{L_g}{2} \|x_t - x_{t-1}\|^2 - \sum_{k=1}^{K} \gamma^2 g_k(x_{t-1}) g_k(x_t),
\]

where the inequality is by

\[
Q_k(t) + \gamma g_k(x_{k-1}) = \max\{-\gamma g_k(x_{t-1}), Q_k(t-1) + \gamma g_k(x_{t-1})\} + \gamma g_k(x_{k-1}) \geq 0,
\]

and also by the gradient Lipstchitz assumption of \( g_k \) in Assumption 2.1 and Lemma A.3 such that

\[
g_k(x_k) \leq g_k(x_{k-1}) + \langle \nabla g_k(x_{k-1}), x_k - x_{k-1} \rangle + \frac{L_g}{2} \|x_k - x_{k-1}\|^2.
\]
Combining (12) and (11) and then rearranging the terms lead to

\[
\frac{1}{2}||Q(t + 1)||^2_2 - ||Q(t)||^2_2 + \langle \nabla f^{t-1}(x_{t-1}), x_t \rangle + \alpha_t D(x_t, \tilde{x}_t) \\
\leq \sum_{k=1}^{K} \gamma [Q_k(t) + \gamma g_k(x_{t-1})][g_k(x_{t-1}) + \langle \nabla g_k(x_{t-1}), z - x_{t-1} \rangle] + \langle \nabla f^t(x_t), z \rangle \\
+ \sum_{k=1}^{K} \frac{\gamma L_0}{2}[Q_k(t) + \gamma g_k(x_{t-1})][x_t - x_{t-1}]^2 - \sum_{k=1}^{K} \gamma^2 g_k(x_{t-1})g_k(x_t) + \gamma^2 ||g(x_t)||^2_2 \\
+ \langle \nabla f^{t-1}(x_{t-1}) - \nabla f^t(x_t), \tilde{x}_{t+1} \rangle - \alpha_t D(\tilde{x}_{t+1}, x_t) + \alpha_t D(z, x_t) - \alpha_t D(z, \tilde{x}_{t+1}).
\]

Due to \(Q_k(t) + \gamma g_k(x_{t-1}) \geq 0\) as discussed above and also by the convexity of \(g_k\) such that \(g_k(x_{t-1}) + \langle \nabla g_k(x_{t-1}), z - x_{t-1} \rangle \leq g_k(z)\), we bound the first term on the right-hand side of (14) as

\[
\sum_{k=1}^{K} \gamma [Q_k(t) + \gamma g_k(x_{t-1})][g_k(x_{t-1}) + \langle \nabla g_k(x_{t-1}), z - x_{t-1} \rangle] \\
\leq \sum_{k=1}^{K} \gamma [Q_k(t) + \gamma g_k(x_{t-1})]g_k(z) = \gamma (Q(t) + \gamma g(x_{t-1}), g(z)).
\]

Next, we bound the term \(\sum_{k=1}^{K} \frac{\gamma L_0}{2}[Q_k(t) + \gamma g_k(x_{t-1})][x_t - x_{t-1}]^2\) in (14) as follows

\[
\sum_{k=1}^{K} \frac{\gamma L_0}{2}[Q_k(t) + \gamma g_k(x_{t-1})][x_t - x_{t-1}]^2 \overset{1}{\leq} \sum_{k=1}^{K} \frac{\gamma L_0}{2}[Q_k(t) + \gamma g_k][x_t - x_{t-1}]^2 \\
\overset{2}{=} \left( \frac{\gamma L_0}{2} \sum_{k=1}^{K} [Q_k(t) + \frac{\gamma^2 L_0 G}{2}] \right) \|x_t - x_{t-1}\|^2,
\]

where \(1\) is due to the boundedness of \(g_k\) such that \(g_k(x_{t-1}) \leq |g_k(x_{t-1})| \leq G_k\) and \(2\) is by \(G := \sum_{k=1}^{K} G_k\) according to Assumption 2.1.

Furthermore, we give the bound of the term \(-\sum_{k=1}^{K} \gamma^2 g_k(x_{t-1})g_k(x_t) + \gamma^2 ||g(x_t)||^2_2\) in (14) as

\[
- \sum_{k=1}^{K} \gamma^2 g_k(x_{t-1})g_k(x_t) + \gamma^2 ||g(x_t)||^2_2 \\
= \sum_{k=1}^{K} \gamma^2 \left( - g_k(x_{t-1})g_k(x_t) + [g_k(x_t)]^2 \right) \\
\overset{1}{=} \sum_{k=1}^{K} \gamma^2 \left( - \frac{1}{2}[g_k(x_{t-1})]^2 - \frac{1}{2}[g_k(x_t)]^2 + \frac{1}{2}[g_k(x_t) - g_k(x_{t-1})]^2 + [g_k(x_t)]^2 \right) \\
\overset{2}{=} \sum_{k=1}^{K} \gamma^2 \left( \frac{1}{2}[g_k(x_{t-1})]^2 - \frac{1}{2}[g_k(x_t)]^2 + \frac{H_k^2}{2}\|x_t - x_{t-1}\|^2 \right) \\
\overset{3}{=} \frac{\gamma^2}{2}||g(x_t)||^2_2 - \frac{\gamma^2}{2}||g(x_{t-1})||^2_2 + \frac{\gamma^2 H_k^2}{2}\|x_t - x_{t-1}\|^2,
\]

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where (1) is due to \( ab = (a^2 + b^2 - (a-b)^2)/2 \), (2) is due to the Lipschitz continuity assumption of the function \( g_k \) in Assumption 2.1 such that \( |g_k(x_t) - g_k(x_{t-1})| \leq H_k \|x_t - x_{t-1}\| \), and (3) is by \( \|g(x)\|^2 = \sum_{k=1}^{K} [g_k(x)]^2 \) and the definition of \( H := \sum_{k=1}^{K} H_k \) in Assumption 2.1 such that \( \sum_{k=1}^{K} H_k^2 \leq (\sum_{k=1}^{K} H_k)^2 = H^2 \).

Therefore, plugging (15), (16), (17) into (14), we have

\[
\frac{1}{2} [\|Q(t+1)\|_2^2 - \|Q(t)\|_2^2] + \langle \nabla f^{t-1}(x_{t-1}), x_t \rangle + \alpha_t D(x_t, \tilde{x}_t) \\
\leq \frac{1}{2} [\gamma L_g \|Q(t)\|_1 + \gamma^2 (L_g G + H^2)] \|x_t - x_{t-1}\|^2 + \frac{\gamma^2}{2} [\|g(x_t)\|_2^2 - \|g(x_{t-1})\|_2^2] \\
+ \langle \nabla f^t(x_t), z \rangle + \langle \nabla f^{t-1}(x_{t-1}) - \nabla f^t(x_t), \tilde{x}_{t+1} \rangle - \alpha_t D(\tilde{x}_{t+1}, x_t) \\
+ \alpha_t D(z, \tilde{x}_t) - \alpha_t D(z, \tilde{x}_{t+1}) + \gamma \langle Q(t) + g(x_{t-1}), g(z) \rangle,
\]

where we further use \( \sum_{k=1}^{K} Q_k(t) = \langle Q(t), 1 \rangle = \|Q(t)\|_1 \) according to the fact that \( Q_k(t) \geq 0 \) as shown in Lemma A.1. This completes the proof.

A.3 Proof of Lemma 5.2

Proof. Recall that Lemma 5.1 gives the inequality

\[
\frac{1}{2} [\|Q(t+1)\|_2^2 - \|Q(t)\|_2^2] + \langle \nabla f^t(x_t), x_t - z \rangle \\
\leq \frac{\xi_t}{2} \|x_t - x_{t-1}\|^2 + \frac{\gamma^2}{2} \|g(x_t)\|_2^2 - \|g(x_{t-1})\|_2^2] + \langle \nabla f^{t-1}(x_{t-1}) - \nabla f^t(x_t), \tilde{x}_{t+1} - x_t \rangle \\
- \alpha_t D(\tilde{x}_{t+1}, x_t) - \alpha_t D(x_t, \tilde{x}_t) + \alpha_t D(z, \tilde{x}_t) - \alpha_t D(z, \tilde{x}_{t+1}) + \gamma \langle Q(t) + g(x_{t-1}), g(z) \rangle.
\]

Moreover, due to \( D(\tilde{x}_{t+1}, x_t) \geq \rho \|x_t - \tilde{x}_{t+1}\|^2/2 \) and \( D(x_t, \tilde{x}_t) \geq \rho \|x_t - \tilde{x}_t\|^2/2 \), and also by \( \|x_t - x_{t-1}\|^2 \leq (\|x_t - \tilde{x}_t\| + \|x_{t-1} - \tilde{x}_t\|)^2 \leq 2 \|x_t - \tilde{x}_t\|^2 + 2 \|x_{t-1} - \tilde{x}_t\|^2 \), we have

\[
\frac{1}{2} [\|Q(t+1)\|_2^2 - \|Q(t)\|_2^2] + \langle \nabla f^t(x_t), x_t - z \rangle \\
\leq \left( \xi_t - \frac{\rho \alpha_t}{2} \right) \|x_t - \tilde{x}_t\|^2 + \frac{\gamma^2}{2} \|g(x_t)\|_2^2 - \|g(x_{t-1})\|_2^2] + \alpha_t D(z, \tilde{x}_t) \\
- \alpha_t D(z, \tilde{x}_{t+1}) + \epsilon_t \|x_t - \tilde{x}_{t+1}\|^2 - \frac{\rho \alpha_t}{2} \|x_t - \tilde{x}_{t+1}\|^2 \\
+ \langle \nabla f^{t-1}(x_{t-1}) - \nabla f^t(x_t), \tilde{x}_{t+1} - x_t \rangle + \gamma \langle Q(t) + g(x_{t-1}), g(z) \rangle.
\]
We can decompose the term $\alpha_t D(z, \tilde{x}_t) - \alpha_t D(z, \tilde{x}_{t+1})$ on the right-hand side of (18) as

$$\alpha_t D(z, \tilde{x}_t) - \alpha_t D(z, \tilde{x}_{t+1}) = \alpha_t D(z, \tilde{x}_t) - \alpha_{t+1} D(z, \tilde{x}_{t+1}) + (\alpha_{t+1} - \alpha_t) D(z, \tilde{x}_{t+1}).$$

Next, we bound the last term in (18) as follows

$$\langle \nabla f^{t-1}(x_{t-1}), \tilde{x}_{t+1} - x_t \rangle = \langle \nabla f^{t-1}(x_{t-1}) - \nabla f^t(x_t), \tilde{x}_{t+1} - x_t \rangle + \langle \nabla f^t(x_t), \tilde{x}_{t+1} - x_t \rangle$$

$$\leq \| \nabla f^{t-1}(x_{t-1}) - \nabla f^t(x_t) \|_* \| x_t - \tilde{x}_{t+1} \| + \| \nabla f^t(x_t) \|_* \| x_t - \tilde{x}_{t+1} \|$$

$$\leq L_f \| x_{t-1} - x_t \| \| x_t - \tilde{x}_{t+1} \| + \| \nabla f^{t-1}(x_{t-1}) - \nabla f^t(x_t) \|_* \| x_t - \tilde{x}_{t+1} \|$$

$$\leq \eta L_f^2 (\| x_{t-1} - \tilde{x}_{t+1} \|^2 + \| x_t - \tilde{x}_{t+1} \|^2) + \frac{\rho \alpha_t}{2} \| x_t - \tilde{x}_{t+1} \|^2 + \frac{\eta}{2} \| \nabla f^{t-1}(x_{t-1}) - \nabla f^t(x_t) \|^2.$$

where (1) is by Cauchy-Schwarz inequality for dual norm, (2) is by the gradient Lipschitz of $f^{t-1}$, (3) is by the triangular inequality for the norm $\| \cdot \|$, and (4) is by $ab \leq \theta/2 \cdot a^2 + 1/(2\theta) \cdot b^2, \forall \theta > 0$. Combining the above inequalities with (18) gives

$$\frac{1}{2} [\| Q(t + 1) \|^2 - \| Q(t) \|^2] + \langle \nabla f^t(x_t), x_t - z \rangle$$

$$\leq \left( \xi_t - \frac{\rho \alpha_t}{2} + \eta L_f^2 \right) \| x_t - \tilde{x}_t \|^2 + \frac{\gamma^2}{2} \| g(x_t) \|^2 - \| g(x_{t-1}) \|^2$$

$$+ \alpha_t D(z, \tilde{x}_t) - \alpha_{t+1} D(z, \tilde{x}_{t+1}) + (\alpha_{t+1} - \alpha_t) D(z, \tilde{x}_{t+1}) + \left( \xi_t + \eta L_f^2 \right) \| x_{t-1} - \tilde{x}_t \|^2$$

$$- \left( \frac{\rho \alpha_t}{2} - \frac{1}{\eta} \right) \| x_t - \tilde{x}_{t+1} \|^2 + \frac{\eta}{2} \| \nabla f^{t-1}(x_{t-1}) - \nabla f^t(x_t) \|^2 + \gamma \langle Q(t) + \gamma g(x_{t-1}), g(z) \rangle.$$

Also note that we have

$$\left( \xi_t + \eta L_f^2 \right) \| x_{t-1} - \tilde{x}_t \|^2 - \left( \frac{\rho \alpha_t}{2} - \frac{1}{\eta} \right) \| x_t - \tilde{x}_{t+1} \|^2$$

$$= \left( \xi_t + \eta L_f^2 \right) \| x_{t-1} - \tilde{x}_t \|^2 - \left( \xi_{t+1} + \eta L_f^2 \right) \| x_t - \tilde{x}_{t+1} \|^2$$

$$+ \left( \xi_{t+1} + \eta L_f^2 - \frac{\rho \alpha_t}{2} - \frac{1}{\eta} \right) \| x_t - \tilde{x}_{t+1} \|^2.$$

Thus, defining $U_t := \left( \xi_t + \eta L_f^2 \right) \| x_{t-1} - \tilde{x}_t \|^2 + \alpha_t D(z, \tilde{x}_t) - \frac{\gamma^2}{2} \| g(x_{t-1}) \|^2$, we eventually have

$$\frac{1}{2} [\| Q(t + 1) \|^2 - \| Q(t) \|^2] + \langle \nabla f^t(x_t), x_t - z \rangle$$

$$\leq \left( \xi_t - \frac{\rho \alpha_t}{2} + \eta L_f^2 \right) \| x_t - \tilde{x}_t \|^2 + \left( \xi_{t+1} + \eta L_f^2 - \frac{\rho \alpha_t}{2} + \frac{1}{\eta} \right) \| x_t - \tilde{x}_{t+1} \|^2 + U_t - U_{t+1}$$

$$+ \frac{\eta}{2} \| \nabla f^{t-1}(x_t) - \nabla f^t(x_t) \|^2 + \left( \alpha_{t+1} - \alpha_t \right) D(z, \tilde{x}_{t+1}) + \gamma \langle Q(t) + \gamma g(x_{t-1}), g(z) \rangle.$$
Here we also have

\[
\xi_{t+1} + \eta L_f^2 - \frac{\rho \alpha_t}{2} + \frac{1}{\eta} = \gamma L_g \|Q(t+1)\|_1 + \gamma^2 (L_g G + H^2) + \eta L_f^2 - \frac{\rho \alpha_t}{2} + \frac{1}{\eta}
\]

\[
\leq \gamma L_g (\|Q(t)\|_1 + \|g(t)\|_1) + \gamma^2 (L_g G + H^2) + \eta L_f^2 - \frac{\rho \alpha_t}{2} + \frac{1}{\eta}
\]

\[
\leq \gamma L_g \|Q(t)\|_1 + \gamma^2 (2L_g G + H^2) + \eta L_f^2 - \frac{\rho \alpha_t}{2} + \frac{1}{\eta}
\]

where (1) is due to \(\|Q(t+1)\|_1 - \|Q(t)\|_1 \leq \gamma\|g(x_t)\|_1\) as in Lemma A.1, and (2) is by \(\|g(x_t)\|_1 \leq G\) as in Assumption 2.1. Thus, we have

\[
\frac{1}{2} \left[ \|Q(t+1)\|_2^2 - \|Q(t)\|_2^2 \right] + \langle \nabla f^t(x_t), x_t - z \rangle \leq \left( \xi_t + \eta L_f^2 - \frac{\rho \alpha_t}{2} + \frac{1}{\eta} + \gamma^2 L_g G \right) \left( \|x_t - \bar{x}_t\|^2 + \|x_t - \bar{x}_{t+1}\|^2 \right) + U_t - U_{t+1} + \frac{\eta}{2} \|
\]

\[
\nabla f^{t-1}(x_t) - \nabla f^t(x_t) \|_z^2 + (\alpha_{t+1} - \alpha_t) D(\bar{x}, \bar{x}_{t+1}) + \gamma (Q(t) + \gamma g(x_{t-1}), g(z)).
\]

This completes the proof. \(\square\)

A.4 Proof of Lemma 5.3

Proof. We first consider the case that \(\delta \leq T\). For any \(t \geq 0, \delta \geq 1\) satisfying \(t + \delta \in [T+1]\), taking summation on both sides of the resulting inequality in Lemma 5.3 for \(\delta\) slots and letting \(z = \bar{x}\) as defined in Assumption 2.2 give

\[
\frac{1}{2} \left[ \|Q(t+\delta)\|_2^2 - \|Q(t)\|_2^2 \right] + \sum_{\tau=t}^{t+\delta-1} \langle \nabla f^\tau(x_\tau), x_\tau - \bar{x} \rangle \leq \sum_{\tau=t}^{t+\delta-1} \zeta_\tau (\|x_\tau - \bar{x}_\tau\|^2 + \|x_\tau - \bar{x}_{\tau+1}\|^2) + \frac{\eta}{2} \sum_{\tau=t}^{t+\delta-1} \|\nabla f^{\tau-1}(x_\tau) - \nabla f^\tau(x_\tau)\|_z^2
\]

\[
+ \sum_{\tau=t}^{t+\delta-1} (\alpha_{\tau+1} - \alpha_\tau) D(\bar{x}, \bar{x}_{\tau+1}) + U_t - U_{t+\delta} + \gamma \sum_{\tau=t}^{t+\delta-1} \langle Q(\tau) + \gamma g(x_{\tau-1}), g(\bar{x}) \rangle.
\]

where we have

\[
\xi_\tau := \gamma L_g \|Q(\tau)\|_1 + \gamma^2 (L_g G + H^2), \quad \zeta_\tau := \xi_\tau + \eta L_f^2 - \frac{\rho \alpha_\tau}{2} + \frac{1}{\eta} + \gamma^2 L_g G,
\]

\[
V_\tau := \left( \xi_\tau + \eta L_f^2 \right) \|x_{\tau-1} - \bar{x}_\tau\|^2 + \alpha_\tau D(\bar{x}, \bar{x}_\tau) - \frac{\gamma^2}{2} \|g(x_{\tau-1})\|_2^2.
\]
We bound the last term in (19) as
\[
\gamma \sum_{\tau=t}^{t+\delta-1} (Q(\tau) + \gamma g(x_{\tau-1}), g(\bar{x}))
\]
\[
= \gamma \sum_{\tau=t}^{t+\delta-1} \sum_{k=1}^{K} [Q_k(\tau) + \gamma g_k(x_{\tau-1})]g_k(\bar{x})
\]
\[
\leq -\varsigma \gamma \sum_{\tau=t}^{t+\delta-1} K [Q_k(\tau) + \gamma g_k(x_{\tau-1})]
\]
\[
= -\varsigma \gamma \delta \|Q(t)\|_1 - \varsigma \gamma \sum_{\tau=t}^{t+\delta-2} (t + \delta - \tau - 1) [\|Q(\tau + 1)\|_1 - \|Q(\tau)\|_1]
\]
\[
= -\varsigma \gamma \delta \|Q(t)\|_1 + \varsigma \gamma^2 \sum_{\tau=t}^{t+\delta-2} (t + \delta - \tau - 1) G + \gamma^2 \varsigma \delta G
\]
\[
\leq -\varsigma \gamma \delta \|Q(t)\|_1 + \frac{1}{2} \gamma^2 \varsigma \delta^2 G + \gamma^2 \varsigma \delta G,
\]
where ① is due to $g_k(\bar{x}) \leq -\varsigma$ and $Q_k(\tau) + \gamma g_k(x_{\tau-1}) \geq 0$ as shown in (13), ② is by $Q_k(t) \geq 0$ for any $t$ as in Lemma A.1, and ③ is due to Lemma A.1 and Assumption 2.1.

According the setting of $\alpha_t$, we have
\[
\alpha_t = \max \left\{ \frac{2\gamma^2 L_g G}{\rho} + \frac{2\eta L_f^2}{\rho} + \frac{2\varsigma}{\rho}, \alpha_{t-1} \right\} \text{ with } \alpha_0 = 0,
\]
which, by recursion, is equivalent to
\[
\alpha_t = \frac{2\gamma^2 L_g G}{\rho} + \frac{2\eta L_f^2}{\rho} + \frac{2\varsigma}{\rho} + \frac{2}{\rho \eta} \max_{t' \in [\tau]} \xi_{t'}
\]
\[
= \frac{2\eta L_f^2 + 2\gamma^2 (2L_g G + H^2)}{\rho} + \frac{2\gamma L_g}{\rho} \max_{t' \in [\tau]} \|Q(t')\|_1.
\]
This setting guarantees that $\alpha_{\tau+1} \geq \alpha_{\tau}$ and also
\[
\zeta_{\tau} (\|x_{\tau} - \bar{x}_{\tau}\|^2 + \|x_{\tau-1} - \bar{x}_{\tau}\|^2)
\]
\[
= \gamma L_g \left( \|Q(\tau)\|_1 - \max_{t' \in [\tau]} \|Q(t')\|_1 \right) (\|x_{\tau} - \bar{x}_{\tau}\|^2 + \|x_{\tau-1} - \bar{x}_{\tau}\|^2) \leq 0.
\]
Since $\alpha_{\tau+1} \geq \alpha_{\tau}$, thus we have
\[
\sum_{\tau=t}^{t+\delta-1} (\alpha_{\tau+1} - \alpha_{\tau}) D(\bar{x}, \bar{x}_{\tau+1}) \leq \sum_{\tau=t}^{t+\delta-1} (\alpha_{\tau+1} - \alpha_{\tau}) R^2 = \alpha_{t+\delta} R^2 - \alpha_t R^2,
\]
where the inequality is by Assumption 2.3. Moreover, we have

$$- \sum_{\tau=t}^{t+\delta-1} \langle \nabla f^\tau(x_{\tau}), x_{\tau} - \bar{x} \rangle \overset{1}{\leq} \sum_{\tau=t}^{t+\delta-1} \| \nabla f^\tau(x_{\tau}) \|_2 \| x_{\tau} - \bar{x} \| \overset{2}{\leq} \sqrt{2} FR\delta,$$

(24)

where \(1\) is by Cauchy-Schwarz inequality for the dual norm, and \(2\) is by Assumption 2.3. In addition, due to \(1 \leq t + \delta \leq T + 1\), we also have

$$\frac{\eta}{2} \sum_{\tau=t}^{t+\delta-1} \| \nabla f^{\tau-1}(x_{\tau}) - \nabla f^{\tau}(x_{\tau}) \|_2^2 \leq \frac{\eta}{2} \sum_{\tau=1}^{T} \| \nabla f^{\tau-1}(x_{\tau}) - \nabla f^{\tau}(x_{\tau}) \|_2^2 \leq \frac{\eta}{2} V_*(T),$$

(25)

where the second inequality is by the definition of \(V_*(T)\). Next, we bound the term \(U_t - U_{t+\delta}\) as

$$U_t - U_{t+\delta} \overset{1}{\leq} \left( \xi_t + \eta L_f^2 \right) \| x_{t-1} - \bar{x}_t \|^2 + \alpha_t D(\bar{x}, \bar{x}_t) + \frac{\gamma^2}{2} \| g(x_{t+\delta-1}) \|_2^2$$

$$\overset{2}{\leq} \frac{2}{\rho} \left( \xi_t + \eta L_f^2 \right) R^2 + \alpha_t R^2 + \frac{\gamma^2}{2} G^2$$

$$= \frac{2}{\rho} \left[ \gamma L_g \| Q(t) \|_1 + \gamma^2 (L_g G + H^2) + \eta L_f^2 \right] R^2 + \alpha_t R^2 + \frac{\gamma^2}{2} G^2,$$

(26)

where \(1\) is by removing the negative terms, and \(2\) is due to Assumption 2.3.

Now, we combine (20) (22) (23) (24) (25) (26) with (19) and then obtain

$$\frac{1}{2} \left[ \| Q(t + \delta) \|_2^2 - \| Q(t) \|_2^2 \right]$$

$$\leq \frac{2}{\rho} \left[ \gamma L_g \| Q(t) \|_1 + \gamma^2 (L_g G + H^2) + \eta L_f^2 \right] R^2 + \frac{\gamma^2}{2} G^2 + \frac{\eta}{2} V_*(T)$$

$$+ \frac{\sqrt{2} FR\delta + \alpha_{t+\delta} R^2 - \varsigma \delta \| Q(t) \|_1 + \frac{1}{2} \varsigma^2 \delta^2 G + \gamma^2 \varsigma \delta G}{\rho} \| Q(t) \|_1,$$

$$\leq \overline{C} + \alpha_{t+\delta} R^2 - \gamma \left( \varsigma \delta - \frac{2}{\rho} L_g R^2 \right) \| Q(t) \|_1,$$

(27)

where we let

$$\overline{C} = \left( \frac{2(L_g G + H^2) R^2}{\rho} + \frac{G^2 + \varsigma^2 \delta^2 G}{2} + \varsigma \delta G \right) \gamma^2 + \left( \frac{2L_f^2 R^2}{\rho} + \frac{V_*(T)}{2} \right) \frac{\eta}{\sqrt{2}} + \frac{\sqrt{2} FR\delta}{\rho}.$$

(28)

Consider a time interval of \([1, T + 1 - \delta]\). Since \(\alpha_{t+\delta} \leq \alpha_{T+1}\) for any \(t \in [1, T + 1 - \delta]\) due to non-decrease of \(\alpha_t\), and letting

$$\delta \geq 2L_g R^2 / (\rho \varsigma)$$

(29)

in (27), then we have for any \(t \in [1, T + 1 - \delta]\)

$$\| Q(t + \delta) \|_2^2 - \| Q(t) \|_2^2 \leq 2\overline{C} + 2\alpha_{T+1} R^2 - \varsigma \gamma \| Q(t) \|_1 \leq 2\overline{C} + 2\alpha_{T+1} R^2 - \varsigma \gamma \| Q(t) \|_2,$$

(30)
where the second inequality is due to \( \|Q(t)\|_1 \geq \|Q(t)\|_2 \).

If \( \zeta \delta \gamma \|Q(t)\|_2 / 2 \geq 2C + 2\alpha_{T+1}R^2 \), namely \( \|Q(t)\|_2 \geq 4(\overline{C} + \alpha_{T+1}R^2)/(\zeta \delta \gamma) \), by (30), we have

\[
\|Q(t + \delta)\|_2^2 \leq \|Q(t)\|_2^2 - \zeta \delta \gamma \|Q(t)\|_2 + 2C + 2\alpha_{T+1}R^2
\]

\[
\leq \|Q(t)\|_2^2 - \frac{\zeta \delta \gamma}{2} \|Q(t)\|_2
\]

\[
\leq \left( \|Q(t)\|_2 - \frac{\zeta \delta \gamma}{4} \right)^2.
\]

From the second inequality above, we know that \( \|Q(t)\|_2 \geq \zeta \delta \gamma / 2 \) holds such that \( \|Q(t)\|_2 \geq \zeta \delta \gamma / 4 \). Thus, the above inequality leads to

\[
\|Q(t + \delta)\|_2 \leq \|Q(t)\|_2 - \frac{\zeta \delta \gamma}{4}.
\]  

(31)

Next, we prove that \( \|Q(t)\|_2 \leq 4(\overline{C} + \alpha_{T+1}R^2)/(\zeta \delta \gamma) + 2\delta \gamma G \) for any \( t \in [1, T + 1 - \delta] \) when \( \delta \geq 2L_g R^2 / (\rho \varsigma) \).

For any \( t \in [1, \delta] \), by (c) of Lemma A.1, we can see that \( \|Q(t)\|_2 \leq \|Q(0)\|_2 + \gamma \sum_{\tau=1}^{t} \|g(x_{\tau})\|_2 \leq \gamma \sum_{\tau=1}^{t} \|g(x_0)\|_1 \leq \gamma \delta G < 4(\overline{C} + \alpha_{t+1}R^2)/(\zeta \delta \gamma) + 2\delta \gamma G \) since we initialize \( Q(0) = 0 \) in the proposed algorithm. Furthermore, we need to prove for any \( t \in [\delta + 1, T + 1 - \delta] \) that \( \|Q(t)\|_2 \leq 4(\overline{C} + \alpha_{T+1}R^2)/(\zeta \delta \gamma) + 2\delta \gamma G \) by contradiction. Here we assume that there exists \( t_0 \in [\delta + 1, T + 1 - \delta] \) being the first round that

\[
\|Q(t_0)\|_2 > \frac{4(\overline{C} + \alpha_{T+1}R^2)}{\zeta \delta \gamma} + 2\delta \gamma G,
\]  

(32)

which also implies \( \|Q(t_0 - \delta)\|_2 \leq 4(\overline{C} + \alpha_{T+1}R^2)/(\zeta \delta \gamma) + 2\delta \gamma G. \) Then, we make the analysis from two aspects:

- If \( \|Q(t_0 - \delta)\|_2 < 4(\overline{C} + \alpha_{T+1}R^2)/(\zeta \delta \gamma) \), then by (c) of Lemma A.1, we have \( \|Q(t_0)\|_2 \leq \|Q(t_0 - \delta)\|_2 + \delta \gamma G \leq 4(\overline{C} + \alpha_{T+1}R^2)/(\zeta \delta \gamma) + \delta \gamma G \), which contradicts the assumption in (32).

- If \( 4(\overline{C} + \alpha_{T+1}R^2)/(\zeta \delta \gamma) + 2\delta \gamma G \geq \|Q(t_0 - \delta)\|_2 \geq 4(\overline{C} + \alpha_{T+1}R^2)/(\zeta \delta \gamma) \), according to (31), we know

\[
\|Q(t_0)\|_2 \leq \|Q(t_0 - \delta)\|_2 - \frac{\zeta \delta \gamma}{4} \leq \frac{4(\overline{C} + \alpha_{T+1}R^2)}{\zeta \delta \gamma} + 2\delta \gamma G - \frac{\zeta \delta \gamma}{4}
\]

\[
< \frac{4(\overline{C} + \alpha_{T+1}R^2)}{\zeta \delta \gamma} + 2\delta \gamma G,
\]

which also contradicts (32).

Thus, we know that there does not exist such \( t_0 \) and for any \( t \in [1, T + 1 - \delta] \), the following inequality always holds

\[
\|Q(t)\|_2 \leq \frac{4(\overline{C} + \alpha_{T+1}R^2)}{\zeta \delta \gamma} + 2\delta \gamma G.
\]
Moreover, further by (c) of Lemma A.1, we know that for any \( t \in [1, T + 1] \), we have

\[
\|\mathbf{Q}(t)\|_2 \leq \frac{4(\mathcal{C} + \alpha_{T+1}R^2)}{\zeta \delta \gamma} + 3\delta \gamma G. \tag{33}
\]

The value of \( \alpha_{T+1} \) remains to be determined, by which we can give the exact value of the bound in (33). By plugging (33) into the setting of \( \alpha_{T+1} \), we have

\[
\begin{align*}
\alpha_{T+1} &= \frac{2\eta L_f^2 + 2\gamma^2(2L_gG + H^2)}{\rho} + \frac{2}{\rho \eta} + \frac{2\gamma L_g}{\rho} \max_{t' \in [T+1]} \|\mathbf{Q}(t')\|_1 \\
&\leq \frac{2\eta L_f^2 + 2\gamma^2(2L_gG + H^2)}{\rho} + \frac{2}{\rho \eta} + \frac{2\sqrt{K} \gamma L_g}{\rho} \left[ \frac{4(\mathcal{C} + \alpha_{T+1}R^2)}{\zeta \delta \gamma} + 3\delta \gamma G \right] \\
&= \mathcal{C}' + \frac{2\sqrt{K} L_g R^2}{\rho \delta} \alpha_{T+1},
\end{align*}
\]

where the inequality is due to \( \|\mathbf{Q}(t)\|_1 \leq \sqrt{K} \|\mathbf{Q}(t)\|_2 \), and the constant \( \mathcal{C}' \) is defined as

\[
\mathcal{C}' := \frac{2\eta L_f^2 + 2\gamma^2(2L_gG + H^2)}{\rho} + \frac{2}{\rho \eta} + \frac{2\sqrt{K} \gamma L_g}{\rho} \left( \frac{4\mathcal{C}}{\zeta \delta \gamma} + 3\delta \gamma G \right). \tag{35}
\]

Further letting \( 1 - \frac{2\sqrt{K} L_g R^2}{(\rho \delta)} \geq 1/2 \), namely,

\[
\delta \geq \frac{4\sqrt{K} L_g R^2}{\rho \varsigma}, \tag{36}
\]

with (34), we have

\[
\alpha_{T+1} \leq 2\mathcal{C}'. \tag{37}
\]

Substituting (37) back into (33) gives

\[
\|\mathbf{Q}(t)\|_2 \leq \frac{4(\mathcal{C} + 2\mathcal{C}' R^2)}{\zeta \delta \gamma} + 3\delta \gamma G, \quad \forall t \in [T + 1]. \tag{38}
\]

Next, we need to give tight bounds of \( \alpha_{T+1} \) and \( \|\mathbf{Q}(T + 1)\|_2 \) by setting the value of \( \delta \). Comparing (36) with (29), we can see that \( \delta \) should be in the range of \([4\sqrt{K} L_g R^2/(\rho \varsigma), +\infty)\). Note that by the definitions of \( \mathcal{C} \) in (28) and \( \mathcal{C}' \) in (35), we observe that the dependence of \( \mathcal{C} \) on \( \delta \) is \( \mathcal{O}(\delta^2 + \delta + 1) \) such that \( \mathcal{C}' = \mathcal{O}(\delta + \delta^{-1} + 1) \). Therefore, the dependence of \( \alpha_{T+1} \) on \( \delta \) is \( \mathcal{O}(\delta + \delta^{-1} + 1) \) and \( \|\mathbf{Q}(T + 1)\|_2 = \mathcal{O}(\delta + \delta^{-1} + \delta^{-2}) \). Thus, both \( \alpha_{T+1} \) and \( \|\mathbf{Q}(T + 1)\|_2 \) are convex functions w.r.t. \( \delta \) in \([4\sqrt{K} L_g R^2/(\rho \varsigma), +\infty)\). Then, for the upper bounds of \( \alpha_{T+1} \) and \( \|\mathbf{Q}(T + 1)\|_2 \), we simply set

\[
\delta = \frac{4\sqrt{K} L_g R^2}{\rho \varsigma}, \tag{39}
\]

such that the tightest upper bounds of them must be no larger than the values with setting \( \delta \) as in
(39). Therefore, for any $T \geq \delta = 4\sqrt{KL}gR^2/(\rho\varsigma)$, the results (37) and (38) hold.

Then, we consider the case where $T < \delta = 4\sqrt{KL}gR^2/(\rho\varsigma)$. Specifically, due to (c) of Lemma A.1 and Assumption 2.1, we have

\[
\|Q(t)\|_2 \leq \gamma(T + 1)G \leq \gamma\delta G, \quad \forall t \in [T + 1],
\]

which satisfies (38). Therefore, we further have

\[
\alpha_{T+1} = \frac{2\eta L_f^2 + 2\gamma^2(2L_gG + H^2)}{\rho} + \frac{2}{\rho \eta} + \frac{2\gamma L_g}{\rho} \max_{t' \in [T+1]} \|Q(t')\|_1 \\
\leq \frac{2\eta L_f^2 + 2\gamma^2(2L_gG + H^2)}{\rho} + \frac{2}{\rho \eta} + \frac{2\sqrt{K}\gamma^2L_g\delta G}{\rho} \leq \mathcal{C}',
\]

which satisfies (37). Thus, we have that (37) and (38) hold for any $T > 0$ and $\delta = 4\sqrt{KL}gR^2/(\rho\varsigma)$.

Now, we let $\eta = [\max\{V_*(T), L_f^2\}]^{-1/2}$ and $\gamma = [\max\{V_*(T), L_f^2\}]^{1/4}$. We have

\[
\mathcal{C} \leq \left(c_1 + \frac{c_2}{\varsigma}\right) \max\{\sqrt{V_*(T)}, L_f\} + \frac{c_3}{\varsigma}, \\
\mathcal{C}' \leq \left(c_4 + \frac{c_5}{\varsigma}\right) \max\{\sqrt{V_*(T)}, L_f\} + \frac{c_6}{\varsigma},
\]

where we use the facts that $2L_f^2R^2/\rho + V_*(T)/2 \leq (2R^2/\rho + 1/2)\max\{V_*(T), L_f^2\}$ for $\mathcal{C}$ and $2\eta L_f^2/\rho + 2/(\rho \eta) \leq 4/\rho\cdot\max\{\sqrt{V_*(T)}, L_f\}$ for $\mathcal{C}'$. In particular, $c_1, c_2, c_3, c_4, c_5,$ and $c_6$ are constants $\text{poly}(R, H, F, G, L_g, K, \rho)$ that can be decided by (28), (35), and (39). Thus, we have that

\[
\alpha_{T+1} \leq \left(\overline{C}_1 + \frac{\overline{C}_2}{\varsigma}\right) \max\{\sqrt{V_*(T)}, L_f\} + \frac{\overline{C}_3}{\varsigma}, \\
\|Q(T + 1)\|_2 \leq \left(\overline{C}_1 + \frac{\overline{C}_2}{\varsigma}\right) [\max\{V_*(T), L_f^2\}]^{1/4} + \frac{\overline{C}_3}{\varsigma L_f^{1/2}},
\]

where we also use the fact that $1/\max\{V_*(T), L_f^2\}]^{1/4} \leq L_f^{-1/2}$, and $\overline{C}_1, \overline{C}_2, \overline{C}_3, \overline{C}_1', \overline{C}_2'$, and $\overline{C}_3'$ are constants $\text{poly}(R, H, F, G, L_g, K, \rho)$ that can be decided by $c_1, c_2, c_3, c_4, c_5,$ and $c_6$ as well as (37) and (38). This completes the proof. \hfill \square

A.5 Proof of Lemma 5.4

Proof. We start the proof by bounding the regret

\[
\text{Regret}(T) = \sum_{t=1}^{T} f^t(x_t) - \sum_{t=1}^{T} f^t(x^\star) \leq \sum_{t=1}^{T} \langle \nabla f^t(x_t), x_t - x^\star \rangle,
\]

where the equality is by the definition $x^\star := \arg\min_{x \in \mathcal{X}} \sum_{t=1}^{T} f^t(x)$, and the inequality is due to the convexity of function $f^t$ such that $f^t(x_t) - f^t(x^\star) \leq \langle \nabla f^t(x_t), x_t - x^\star \rangle$. 

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Recalling the Lemma 5.2 and setting $z = x^*$ in the lemma give

$$\frac{1}{2} \left[ \|Q(t+1)\|^2_2 - \|Q(t)\|^2_2 \right] + \langle \nabla f^i(x_t), x_t - x^* \rangle$$

$$\leq \zeta_t (\|x_t - \bar{x}_t\|^2 + \|x_t - \bar{x}_{t-1}\|^2) + U_t - U_{t+1} + \frac{\eta}{2} \|\nabla f^{-1}(x_t) - \nabla f^i(x_t)\|^2$$

$$+ (\alpha_{t+1} - \alpha_t) D(x^*, \bar{x}_{t+1}) + \gamma \langle Q(t) + \gamma g(x_{t-1}), g(x^*) \rangle,$$

where we define

$$\xi_t := \gamma L_g \|Q(t)\|_1 + \gamma^2 (L_g G + H^2), \quad \zeta_t := \xi_t - \frac{\rho \alpha_t}{2} + \frac{1}{\eta},$$

$$U_t := \left( \xi_t + \frac{\eta}{2} \right) \|x_{t-1} - \bar{x}_t\|^2 + \alpha_t D(x^*, \bar{x}_t) - \gamma^2 / 2 \cdot \|g(x_{t-1})\|^2.$$

Since $x^*$ is the solution that satisfies all the constraints, i.e., $g_k(x^*) \leq 0$ and also $Q_k(x_t) + \gamma g_k(x_{t-1}) \geq 0$ as shown in (13), thus the last term in (41) can be bounded as

$$\gamma \langle Q(t) + \gamma g(x_{t-1}), g(x^*) \rangle = \sum_{k=1}^K [Q_k(x_t) + \gamma g_k(x_{t-1})] g_k(x^*) \leq 0. \quad (42)$$

Combining (41) (42) with (40) yields

$$\text{Regret}(T) \leq \sum_{t=1}^T \langle \nabla f^i(x_t), x_t - x^* \rangle$$

$$\leq \sum_{t=1}^T \left( \zeta_t (\|x_t - \bar{x}_t\|^2 + \|x_{t-1} - \bar{x}_t\|^2) + \frac{\eta}{2} \|\nabla f^{-1}(x_t) - \nabla f^i(x_t)\|^2$$

$$+ U_t - U_{t+1} + (\alpha_{t+1} - \alpha_t) D(x^*, \bar{x}_{t+1}) + \frac{1}{2} [\|Q(t)\|^2_2 - \|Q(t+1)\|^2_2] \right) \quad (43)$$

$$= \sum_{t=1}^T \zeta_t (\|x_t - \bar{x}_t\|^2 + \|x_{t-1} - \bar{x}_t\|^2) + \frac{\eta}{2} \sum_{t=1}^T \|\nabla f^{-1}(x_t) - \nabla f^i(x_t)\|^2$$

$$+ \sum_{t=1}^T (\alpha_{t+1} - \alpha_t) D(x^*, \bar{x}_{t+1}) + U_1 - U_{T+1} + \frac{1}{2} [\|Q(1)\|^2_2 - \|Q(T+1)\|^2_2].$$

We analyze the terms in (43) in the following way. By the setting of $\alpha_t$

$$\alpha_t = \max \left\{ \frac{2 \gamma^2 L_g G}{\rho} + \frac{2 \eta L_f^2}{\rho}, \frac{2 \xi_t}{\rho}, \alpha_{t-1} \right\} \quad \text{with} \quad \alpha_0 = 0,$$

which, similar to (21), is equivalent to

$$\alpha_t = \frac{2 \eta L_f^2 + 2 \gamma^2 (2 L_g G + H^2)}{\rho} + \frac{2 \xi_t}{\rho} + \frac{2 \gamma L_g}{\rho} \max_{t' \in [t]} \|Q(t')\|_1. \quad (44)$$
which guarantees that $\alpha_{t+1} \geq \alpha_t, \forall t \geq 0$ and also

$$
\sum_{t=1}^{T} \zeta_t (\|x_t - \tilde{x}_t\|^2 + \|x_{t-1} - \tilde{x}_t\|^2)
= \sum_{t=1}^{T} \gamma L_g \left( \|Q(t)\|_1 - \max_{t' \in [t]} \|Q(t')\|_1 \right) (\|x_t - \tilde{x}_t\|^2 + \|x_{t-1} - \tilde{x}_t\|^2) \leq 0.
$$

(45)

Since $\alpha_{t+1} \geq \alpha_t, \forall t$, thus we have

$$
\sum_{t=1}^{T} (\alpha_{t+1} - \alpha_t) D(x^*, \tilde{x}_{t+1}) \leq \sum_{t=1}^{T} (\alpha_{t+1} - \alpha_t) R^2 = \alpha_{T+1} R^2 - \alpha_1 R^2,
$$

(46)

where the inequality is by Assumption 2.3.

Moreover, by the definition of $V_*(T) = \sum_{t=1}^{T} \|\nabla f^{t-1}(x) - \nabla f^t(x)\|_2$, we have

$$
\frac{\eta}{2} \sum_{t=1}^{T} \|\nabla f^{t-1}(x_t) - \nabla f^t(x_t)\|^2 \leq \frac{\eta}{2} V_*(T).
$$

(47)

In addition, we bound the term $U_1 - U_{T+1}$ by

$$
U_1 - U_{T+1} = \left( \xi_1 + \eta L_f^2 \right) \|x_0 - \tilde{x}_1\|^2 + \alpha_1 D(x^*, \tilde{x}_1) - \frac{\gamma^2}{2} \|g(x_0)\|^2
- \left( \xi_{T+1} + \eta L_f^2 \right) \|x_T - \tilde{x}_{T+1}\|^2 - \alpha_{T+1} D(x^*, \tilde{x}_{T+1}) + \frac{\gamma^2}{2} \|g(x_T)\|^2
\leq \left( \xi_1 + \eta L_f^2 \right) \|x_0 - \tilde{x}_1\|^2 + \alpha_1 D(x^*, \tilde{x}_1) + \frac{\gamma^2}{2} \|g(x_T)\|^2
\leq \left( \xi_{T+1} + \eta L_f^2 \right) \|x_T - \tilde{x}_{T+1}\|^2 - \alpha_{T+1} D(x^*, \tilde{x}_{T+1}) + \frac{\gamma^2}{2} \|g(x_T)\|^2
\leq \frac{2}{\rho} [\gamma L_g \|Q(1)\|_1 + \gamma^2 (L_g G + H^2) + \eta L_f^2] R^2 + \alpha_1 R^2 + \frac{\gamma^2}{2} G^2
\leq \frac{2}{\rho} [\eta L_f^2 + \gamma^2 (2L_g G + H^2)] R^2 + \alpha_1 R^2 + \frac{\gamma^2}{2} G^2,
$$

(48)

where ① is by removing negative terms, ② is by $\|x_0 - \tilde{x}_1\|^2 \leq 2R^2/\rho$ and $D(x^*, \tilde{x}_1) \leq R^2$ as well as $\|g(x_T)\|^2 \leq \|g(x_T)\|^2 \leq G^2$ according to Assumptions 2.1 and 2.3, and ③ is by (4) of Lemma A.1 and $Q(0) = 0$.

Therefore, combining (45) (46) (47) (48) and (43), further by $\|Q(1)\|^2 = \gamma^2 \|g(x_0)\|^2 \leq \gamma^2 \|g(x_0)\|^2 \leq \gamma^2 G^2$, we have

$$
\text{Regret}(T) \leq \frac{\eta}{2} V_*(T) + \frac{2}{\rho} [\eta L_f^2 + \gamma^2 (2L_g G + H^2)] R^2 + \alpha_{T+1} R^2 + \frac{3\gamma^2}{2} G^2
\leq \frac{\eta}{2} V_*(T) + \frac{2R^2}{\rho} L_f^2 \eta + \left( \frac{2(2L_g G + H^2) R^2}{\rho} + \frac{3G^2}{2} \right) \gamma^2 + \alpha_{T+1} R^2.
$$

This completes the proof. □
A.6 Proof of Lemma 5.5

Proof. According to the updating rule of $Q_k(t)$ in Algorithm 1, we have

$$Q_k(t + 1) = \max \{-\gamma g_k(x_t), \ Q_k(t) + \gamma g_k(x_t)\}$$

$$\geq Q_k(t) + \gamma g_k(x_t),$$

which thus leads to

$$\gamma g_k(x_t) \leq Q_k(t + 1) - Q_k(t).$$

Taking summation over $t = 1 \ldots T$ and multiplying $\gamma^{-1}$ on both sides yields

$$\sum_{t=1}^{T} g_k(x_t) \leq \frac{1}{\gamma} [Q_k(T + 1) - Q_k(1)] \leq \frac{1}{\gamma} Q_k(T + 1) \leq \frac{1}{\gamma} \|Q(T + 1)\|_2,$$

where the second inequality is due to $Q_k(1) \geq 0$ as shown in Lemma A.1. This completes the proof.

B Proofs for Section 6

B.1 Lemmas for Section 6

Lemma B.1. The mixing step in Algorithm 2, i.e., $\bar{y}_t = (1 - \nu)\bar{x}_t + \nu/d \cdot 1$ with $\nu \in (0, 1]$, ensures the following inequalities

$$D_{KL}(z, \bar{y}_t) - D_{KL}(z, \bar{x}_t) \leq \nu \log d, \quad \forall z \in \Delta,$$

$$D_{KL}(z, \bar{y}_t) \leq \log \frac{d}{\nu}, \quad \forall z \in \Delta,$$

$$\|\bar{y}_t - \bar{x}_t\|_1 \leq 2\nu.$$

Proof. The proofs of the first two inequalities are immediate following Lemma 31 in Wei et al. (2019). For the third inequality, we prove it as follows

$$\|\bar{y}_t - \bar{x}_t\|_1 = \nu\|\bar{x}_t - 1/d\|_1 \leq \nu(\|\bar{x}_t\|_1 + \|1/d\|_1) = 2\nu,$$

where the last equality is due to $\bar{x}_t \in \Delta$. This completes the proof.

Lemma B.2. At the $t$-th round of Algorithm 2, for any $\gamma > 0$, $\nu \in (0, 1]$ and any $z \in \Delta$, letting $\xi_t = \gamma L_g \|Q(t)\|_1 + \gamma^2 (L_g G + H^2)$, we have the following inequality

$$\frac{1}{2} [\|Q(t + 1)\|^2_2 - \|Q(t)\|^2_2] + \langle \nabla f^{t-1}(x_{t-1}), x_t \rangle + \alpha_t D_{KL}(x_t, \bar{y}_t)$$

$$\leq \frac{\xi_t}{2} \|x_t - x_{t-1}\|^2 + \frac{\gamma^2}{2} [\|g(x_t)\|^2_2 - \|g(x_{t-1})\|^2_2] + \langle \nabla f^t(x_t), z \rangle + \alpha_t D_{KL}(z, \bar{y}_t) - \alpha_t D_{KL}(z, \bar{y}_{t+1})$$

$$+ \langle \nabla f^{t-1}(x_{t-1}) - \nabla f^t(x_t), \bar{x}_{t+1} \rangle - \alpha_t D_{KL}(\bar{x}_{t+1}, x_t) + \gamma \langle Q(t) + \gamma g(x_{t-1}), g(z) \rangle + \alpha_t \nu \log d.$$
Proof. This lemma is proved by modifying the proof of Lemma 5.1 in Section A.2. More specifically, when applying Lemma A.2, we need to replace \( D(\cdot, \bar{x}_t) \) in Section A.2 with \( D_{KL}(\cdot, \bar{y}_t) \). Then, the rest proof is similar to the proof of Lemma 5.1. Therefore, we have

\[
\frac{1}{2} [\|Q(t + 1)\|_2^2 - \|Q(t)\|_2^2] + \langle \nabla f^{t-1}(x_t), x_t \rangle + \alpha_t D_{KL}(x_t, \bar{y}_t)
\]

\[
\leq \frac{\xi_t}{2} \|x_t - x_{t-1}\|^2 + \frac{\gamma^2}{2} [\|g(x_t)\|_2^2 - \|g(x_{t-1})\|_2^2] + \langle \nabla f^t(x_t), z \rangle + \alpha_t D_{KL}(z, \bar{y}_t) - \alpha_t D_{KL}(z, \bar{x}_{t+1})
\]

\[
+ \langle \nabla f^{t-1}(x_{t-1}) - \nabla f^t(x_t), \bar{x}_{t+1} \rangle - \alpha_t D_{KL}(\bar{x}_{t+1}, x_t) + \gamma \langle Q(t) + \gamma g(x_{t-1}), g(z) \rangle.
\]

Furthermore, due to Lemma B.1, we know

\[-\alpha_t D_{KL}(z, \bar{x}_{t+1}) \leq -\alpha_t D_{KL}(z, \bar{y}_{t+1}) + \alpha_t \nu \log d.
\]

Thus, we have

\[
\frac{1}{2} [\|Q(t + 1)\|_2^2 - \|Q(t)\|_2^2] + \langle \nabla f^{t-1}(x_t), x_t \rangle + \alpha_t D_{KL}(x_t, \bar{y}_t)
\]

\[
\leq \frac{\xi_t}{2} \|x_t - x_{t-1}\|^2 + \frac{\gamma^2}{2} [\|g(x_t)\|_2^2 - \|g(x_{t-1})\|_2^2] + \langle \nabla f^t(x_t), z \rangle + \alpha_t D_{KL}(z, \bar{y}_t) - \alpha_t D_{KL}(z, \bar{y}_{t+1})
\]

\[
+ \langle \nabla f^{t-1}(x_{t-1}) - \nabla f^t(x_t), \bar{x}_{t+1} \rangle - \alpha_t D_{KL}(\bar{x}_{t+1}, x_t) + \gamma \langle Q(t) + \gamma g(x_{t-1}), g(z) \rangle + \alpha_t \nu \log d.
\]

This completes the proof. \(\square\)

Lemma B.2 further leads to the following lemma.

**Lemma B.3.** At the t-th round of Algorithm 2, for any \( \eta, \gamma > 0, \nu \in (0, 1] \), and any \( z \in \Delta \), the following inequality holds

\[
\frac{1}{2} [\|Q(t + 1)\|_2^2 - \|Q(t)\|_2^2] + \langle \nabla f^t(x_t), x_t - z \rangle
\]

\[
\leq \psi_t [\|x_t - \bar{y}_t\|^2 + \|x_t - \bar{x}_{t+1}\|^2] + \frac{\eta}{2} \|\nabla f^{t-1}(x_t) - \nabla f^t(x_t)\|_\infty^2
\]

\[
+ (\alpha_{t+1} - \alpha_t) D_{KL}(z, \bar{y}_{t+1}) + \gamma \langle Q(t) + \gamma g(x_{t-1}), g(z) \rangle
\]

\[
+ 6 \left( \xi_t + \eta L_f^2 \right) \nu^2 + \alpha_t \nu \log d + W_t - W_{t+1},
\]

where we define

\[
\psi_t := 3/2 \cdot (\xi_t + \eta L_f^2 + \gamma^2 L_g G) - \alpha_t/2 + 1/\eta,
\]

\[
W_t := 3/2 \cdot (\xi_t + \eta L_f^2) \|x_{t-1} - \bar{x}_t\|^2 + \alpha_t D_{KL}(z, \bar{y}_t) - \gamma^2/2 \cdot \|g(x_{t-1})\|_2^2.
\]

Proof. The proof mainly follows the proof of Lemma 5.2 in Section A.3. We start our proof with
the result of Lemma B.2

\[
\frac{1}{2} \left[ \|Q(t + 1)\|_2^2 - \|Q(t)\|_2^2 \right] + \langle \nabla f^{t-1}(x_{t-1}), x_t \rangle + \alpha_t D_{KL}(x_t, \bar{y}_t) \\
\leq \frac{\xi_t}{2} \|x_t - x_{t-1}\|^2 + \frac{\gamma^2}{2} \left[ \|g(x_t)\|_2^2 - \|g(x_{t-1})\|_2^2 \right] + \langle \nabla f^t(x_t), z \rangle \\
+ \alpha_t D_{KL}(z, \bar{y}_t) - \alpha_t D_{KL}(z, \bar{y}_{t+1}) + \langle \nabla f^{t-1}(x_{t-1}) - \nabla f^t(x_t), \bar{x}_{t+1} \rangle \\
- \alpha_t D_{KL}(\bar{x}_{t+1}, x_t) + \gamma \langle Q(t) + \gamma g(x_{t-1}), g(z) \rangle + \alpha_t \nu \log d.
\]

Due to \(D(\bar{x}_{t+1}, x_t) \geq \|x_t - \bar{x}_{t+1}\|^2/2\) and \(D(x_t, \bar{y}_t) \geq \|x_t - \bar{y}_t\|^2/2\), we have

\[
\frac{1}{2} \left[ \|Q(t + 1)\|_2^2 - \|Q(t)\|_2^2 \right] + \langle \nabla f^t(x_t), x_t - z \rangle \\
\leq \frac{\xi_t}{2} \|x_t - x_{t-1}\|^2 + \frac{\gamma^2}{2} \left[ \|g(x_t)\|_2^2 - \|g(x_{t-1})\|_2^2 \right] + \alpha_t D_{KL}(z, \bar{y}_t) \\
- \alpha_t D_{KL}(z, \bar{y}_{t+1}) - \frac{\alpha_t}{2} \|x_t - \bar{y}_t\|^2_t - \frac{\alpha_t}{2} \|x_t - \bar{x}_{t+1}\|^2_t + \alpha_t \nu \log d \\
+ \langle \nabla f^{t-1}(x_{t-1}) - \nabla f^t(x_t), \bar{x}_{t+1} - x_t \rangle + \alpha_t \nu \log d.
\]

We bound the term \(\alpha_t D_{KL}(z, \bar{y}_t) - \alpha_t D_{KL}(z, \bar{y}_{t+1})\) on the right-hand side of (49) as follows

\[
\alpha_t D_{KL}(z, \bar{y}_t) - \alpha_t D_{KL}(z, \bar{y}_{t+1}) = \alpha_t D_{KL}(z, \bar{y}_t) - \alpha_t D_{KL}(z, \bar{y}_{t+1}) + (\alpha_{t+1} - \alpha_t) D_{KL}(z, \bar{y}_{t+1}).
\]

Similar to the proof of Lemma 5.2, we bound the last term in (49) as follows

\[
\langle \nabla f^{t-1}(x_{t-1}) - \nabla f^t(x_t), \bar{x}_{t+1} - x_t \rangle \\
= \langle \nabla f^{t-1}(x_{t-1}) - \nabla f^{t-1}(x_t), \bar{x}_{t+1} - x_t \rangle + \langle \nabla f^{t-1}(x_t) - \nabla f^t(x_t), \bar{x}_{t+1} - x_t \rangle \\
\leq \|\nabla f^{t-1}(x_{t-1}) - \nabla f^{t-1}(x_t)\|_\infty \|x_t - \bar{x}_{t+1}\|_1 + \|\nabla f^{t-1}(x_t) - \nabla f^t(x_t)\|_\infty \|x_t - \bar{x}_{t+1}\|_1 \\
\leq L_f \|x_{t-1} - x_t\|_1 \|x_t - \bar{x}_{t+1}\|_1 + \|\nabla f^{t-1}(x_t) - \nabla f^t(x_t)\|_\infty \|x_t - \bar{x}_{t+1}\|_1 \\
\leq \frac{\eta L_f^2}{2} \|x_{t-1} - x_t\|_1^2 + \frac{1}{\eta} \|x_t - \bar{x}_{t+1}\|_1^2 + \frac{\eta}{2} \|\nabla f^{t-1}(x_t) - \nabla f^t(x_t)\|_\infty^2.
\]

Combining the above inequalities with (49) gives

\[
\frac{1}{2} \left[ \|Q(t + 1)\|_2^2 - \|Q(t)\|_2^2 \right] + \langle \nabla f^t(x_t), x_t - z \rangle \\
\leq \frac{1}{2} \left( \xi_t + \eta L_f^2 \right) \|x_t - x_{t-1}\|^2 + \frac{\gamma^2}{2} \left[ \|g(x_t)\|_2^2 - \|g(x_{t-1})\|_2^2 \right] + \alpha_t D_{KL}(z, \bar{y}_t) \\
- \alpha_t D_{KL}(z, \bar{y}_{t+1}) + (\alpha_{t+1} - \alpha_t) D_{KL}(z, \bar{y}_{t+1}) - \frac{\alpha_t}{2} \|x_t - \bar{y}_t\|^2_t - \left( \frac{\alpha_t}{2} - \frac{1}{\eta} \right) \|x_t - \bar{x}_{t+1}\|^2_t \\
+ \frac{\eta}{2} \|\nabla f^{t-1}(x_t) - \nabla f^t(x_t)\|_\infty^2 + \gamma \langle Q(t) + \gamma g(x_{t-1}), g(z) \rangle.
\]

Further, we bound the term \(\|x_t - x_{t-1}\|_1^2\) in the above inequality as follows

\[
\|x_t - x_{t-1}\|_1^2 \leq 3(\|x_t - \bar{y}_t\|_1^2 + \|\bar{y}_t - \bar{x}_t\|_1^2 + \|\bar{x}_t - x_{t-1}\|_1^2) \leq 3\|x_t - \bar{y}_t\|_1^2 + 12\nu^2 + 3\|\bar{x}_t - x_{t-1}\|_1^2.
\]
where the first inequality is due to \((a + b + c)^2 \leq 3a^2 + 3b^2 + 3c^2\), and the second inequality is due to Lemma B.1. Thus, we have

\[
\frac{1}{2} \|Q(t+1)\|^2 - \|Q(t)\|^2 + \langle \nabla f^t(x_t), x_t - z \rangle \\
\leq \left[ \frac{3}{2} (\xi_t + \eta L_f^2) - \frac{\alpha_t}{2} \right] \|x_t - \tilde{y}_t\|^2 + \frac{3}{2} (\xi_t + \eta L_f^2) \|\tilde{x}_t - x_{t-1}\|^2 + 6 \left( \xi_t + \eta L_f^2 \right) \nu^2 + \alpha_t \nu \log d \\
+ \frac{\gamma^2}{2} \|g(x_t)\|^2 + \|g(x_{t-1})\|^2 + \alpha_t D_{KL}(z, \tilde{y}_t) - \alpha_t D_{KL}(z, \tilde{y}_{t+1}) + (\alpha_{t+1} - \alpha_t) D_{KL}(z, \tilde{y}_{t+1}) \\
- \left( \frac{\alpha_t}{2} - \frac{1}{\eta} \right) \|x_t - \tilde{x}_{t+1}\|^2 + \eta \|f^{t-1}(x_t) - \nabla f^t(x_t)\|^2 + \gamma \langle Q(t) + \gamma g(x_{t-1}), g(z) \rangle.
\]

Also note that we have

\[
\frac{3}{2} (\xi_t + \eta L_f^2) \|\tilde{x}_t - x_{t-1}\|^2 = \frac{3}{2} (\xi_t + \eta L_f^2) \|\tilde{x}_t - x_{t-1}\|^2 - \frac{3}{2} (\xi_{t+1} + \eta L_f^2) \|x_t - \tilde{x}_{t+1}\|^2 \\
+ \left( \frac{3}{2} (\xi_{t+1} + \eta L_f^2) - \frac{\alpha_t}{2} + \frac{1}{\eta} \right) \|x_t - \tilde{x}_{t+1}\|^2.
\]

Thus, defining

\[
W_t := \frac{3}{2} (\xi_t + \eta L_f^2) \|x_{t-1} - \tilde{x}_t\|^2 + \alpha_t D_{KL}(z, \tilde{y}_t) - \frac{\gamma^2}{2} \|g(x_{t-1})\|^2,
\]

we eventually have

\[
\frac{1}{2} \|Q(t+1)\|^2 - \|Q(t)\|^2 + \langle \nabla f^t(x_t), x_t - z \rangle \\
\leq \left[ \frac{3}{2} (\xi_t + \eta L_f^2) - \frac{\alpha_t}{2} \right] \|x_t - \tilde{y}_t\|^2 + \left[ \frac{3}{2} (\xi_{t+1} + \eta L_f^2) - \frac{\alpha_t}{2} + \frac{1}{\eta} \right] \|x_t - \tilde{x}_{t+1}\|^2 \\
+ \frac{\eta}{2} \|f^{t-1}(x_t) - \nabla f^t(x_t)\|^2 + \gamma \langle Q(t) + \gamma g(x_{t-1}), g(z) \rangle \\
+ 6 \left( \xi_t + \eta L_f^2 \right) \nu^2 + \alpha_t \nu \log d + W_t - W_{t+1}.
\]

Here we also have

\[
\frac{3}{2} (\xi_{t+1} + \eta L_f^2) - \frac{\alpha_t}{2} + \frac{1}{\eta} = \frac{3}{2} \gamma L_g \|Q(t+1)\|_1 + \frac{3}{2} \gamma^2 (L_g G + H^2) + \frac{3}{2} \eta L_f^2 - \frac{\alpha_t}{2} + \frac{1}{\eta} \\
\leq \frac{3}{2} \gamma L_g (\|Q(t)\|_1 + \|g(t)\|_1 + \|Q(t)\|_1 + \gamma \|g(t)\|_1) + \frac{3}{2} \gamma^2 (L_g G + H^2) + \frac{3}{2} \eta L_f^2 - \frac{\alpha_t}{2} + \frac{1}{\eta} \\
\leq \frac{3}{2} \gamma L_g \|Q(t)\|_1 + \frac{3}{2} \gamma^2 L_g G + \frac{3}{2} \gamma^2 (L_g G + H^2) + \frac{3}{2} \eta L_f^2 - \frac{\alpha_t}{2} + \frac{1}{\eta} \\
= \frac{3}{2} \xi_t + \frac{3}{2} \eta L_f^2 - \frac{\alpha_t}{2} + \frac{1}{\eta} + \frac{3}{4} \gamma^2 L_g G,
\]

where \(1\) is due to \(\|Q(t+1)\|_1 - \|Q(t)\|_1 \leq \gamma \|g(x_t)\|_1\) as in Lemma A.1, and \(2\) is by \(\|g(x_t)\|_1 \leq G\).
as in Assumption 2.1. Thus, we have

\[
\frac{1}{2} \left[ \|Q(t+1)\|_2^2 - \|Q(t)\|_2^2 \right] + \langle \nabla f^t(x_t), x_t - z \rangle \\
\leq \left[ \frac{3}{2} (\xi_t + \eta L_J^2 + \gamma^2 L_G G) - \frac{\alpha_t}{2} + \frac{1}{\eta} \right] (\|x_t - \bar{y}_t\|^2 + \|x_t - \bar{x}_{t+1}\|^2) \\
+ \frac{\eta}{2} \|\nabla f^{t-1}(x_t) - \nabla f^t(x_t)\|_\infty^2 + (\alpha_{t+1} - \alpha_t) D_{KL}(z, \bar{y}_{t+1}) + \gamma \langle Q(t) + \gamma g(x_{t-1}), g(z) \rangle \\
+ 6 \left( \xi_t + \eta L_J^2 \right) \nu^2 + \alpha_t \nu \log d + W_t - W_{t+1}.
\]

This completes the proof. \(\square\)

Based on Lemma B.3, we obtain the following lemma.

**Lemma B.4.** With setting \(\eta, \gamma, \nu,\) and \(\alpha_t\) the same as in Theorem 6.1, for \(T > 2\) and \(d \geq 1,\) Algorithm 2 ensures

\[
\alpha_{T+1} \leq \left( \tilde{C}_1 + \frac{\tilde{C}_2}{\varsigma} \right) \max \{ \sqrt{V_\infty(T)}, L_f \} \log(Td) + \frac{\tilde{C}_3}{\varsigma},
\]

\[
\|Q(T+1)\|_2 \leq \left( \tilde{C}_1' + \frac{\tilde{C}_2'}{\varsigma} \right) \left[ \max \{ V_\infty(T), L_f^2 \} \right]^{1/4} \log(Td) + \frac{\tilde{C}_3'}{\varsigma L_f^{1/2}},
\]

where \(\tilde{C}_1, \tilde{C}_2, \tilde{C}_3, \tilde{C}_1', \tilde{C}_2',\) and \(\tilde{C}_3'\) are absolute constants that are \(\text{poly}(H, F, G, L_G, K)\).

**Proof.** We prove this lemma following the proof of Lemma 5.3 in Section A.4. We first consider the case that \(\delta \leq T.\) For any \(t \geq 1,\) \(\delta \geq 1\) satisfying \(t + \delta \in [T + 1],\) taking summation on both sides of the resulting inequality in Lemma B.4 for \(\delta\) slots and letting \(z = \bar{x}\) as defined in Assumption 2.2 give

\[
\frac{1}{2} \left[ \|Q(t+\delta)\|_2^2 - \|Q(t)\|_2^2 \right] + \sum_{\tau=t}^{t+\delta-1} \langle \nabla f^\tau(x_\tau), x_\tau - \bar{x} \rangle \\
\leq \sum_{\tau=t}^{t+\delta-1} \psi_\tau (\|x_\tau - \bar{y}_\tau\|^2 + \|x_\tau - \bar{x}_{\tau+1}\|^2) + \frac{\eta}{2} \sum_{\tau=t}^{t+\delta-1} \|\nabla f^{\tau-1}(x_\tau) - \nabla f^\tau(x_\tau)\|_\infty^2 \\
+ \sum_{\tau=t}^{t+\delta-1} (\alpha_{\tau+1} - \alpha_\tau) D_{KL}(\bar{x}, \bar{y}_{\tau+1}) + W_t - W_{t+\delta} + \gamma \sum_{\tau=t}^{t+\delta-1} \langle Q(\tau) + \gamma g(x_{\tau-1}), g(\bar{x}) \rangle \\
+ \sum_{\tau=t}^{t+\delta-1} 6 \left( \xi_\tau + \eta L_J^2 \right) \nu^2 + \sum_{\tau=t}^{t+\delta-1} \alpha_\tau \nu \log d,
\]

where \(\psi_\tau := 3/2 \cdot (\xi_\tau + \eta L_J^2 + \gamma^2 L_G G) - \alpha_\tau/2 + 1/\eta\) and \(W_t := 3/2 \cdot (\xi_t + \eta L_J^2) \|x_{t-1} - \bar{x}_t\|_1^2 + \alpha_t D_{KL}(z, \bar{y}_t) - \gamma^2/2 \cdot \|g(x_{t-1})\|_2^2.\) We bound the term in (50) and obtain

\[
\gamma \sum_{\tau=t}^{t+\delta-1} \langle Q(\tau) + \gamma g(x_{\tau-1}), g(\bar{x}) \rangle \leq -\varsigma \gamma \delta \|Q(t)\|_1 + \frac{1}{2} \gamma^2 \varsigma \delta^2 G + \gamma^2 \varsigma \delta G,
\]

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whose proof is the same as (20).

According the setting of \( \alpha_t \), we have

\[
\alpha_t = \max \left\{ 3(\xi_t + \eta L_f^2 + \gamma^2 L_g G) + \frac{2}{\eta} \alpha_{t-1} \right\} \quad \text{with} \quad \alpha_0 = 0,
\]

which, by recursion, is equivalent to

\[
\alpha_t = 3\eta L_f^2 + 3\gamma^2 L_g G + \frac{2}{\eta} + 3 \max_{t' \in [T]} \xi_{t'}
\]

\[
= 3\eta L_f^2 + 3\gamma^2 (2L_g G + H^2) + \frac{2}{\eta} + 3\gamma L_g \max_{t' \in [T]} \|Q(t')\|_1,
\]

which guarantees \( \alpha_{t+1} \geq \alpha_t \) and

\[
\psi_t(\|x_t - \bar{x}_t\|_1^2 + \|x_{t-1} - \bar{x}_t\|_1^2)
\]

\[
= 3/2 \cdot \gamma L_g \left( \|Q(t)\|_1 - \max_{t' \in [T]} \|Q(t')\|_1 \right) \left( \|x_t - \bar{x}_t\|_1^2 + \|x_{t-1} - \bar{x}_t\|_1^2 \right) \leq 0.
\]

Since \( \alpha_{t+1} \geq \alpha_t \), we have

\[
\sum_{\tau=t}^{t+\delta-1} (\alpha_{\tau+1} - \alpha_{\tau}) D(\bar{x}_\tau, \bar{y}_{\tau+1}) \leq \sum_{\tau=t}^{t+\delta-1} (\alpha_{\tau+1} - \alpha_{\tau}) \log \frac{d}{\nu} = \alpha_{t+\delta} \log \frac{d}{\nu} - \alpha_t \log \frac{d}{\nu},
\]

where the inequality is by Lemma B.1. Moreover, we have

\[
- \sum_{\tau=t}^{t+\delta-1} \left( \nabla f^\tau(x_\tau), x_\tau - \bar{x} \right) \leq \sum_{\tau=t}^{t+\delta-1} \|\nabla f^\tau(x_\tau)\|_\infty \|x_\tau - \bar{x}\|_1 \leq 2F\delta,
\]

where \( \odot \) is by Cauchy-Schwarz inequality for the dual norm, and \( \oslash \) is by \( \|x_\tau - \bar{x}\|_1 \leq \|x_\tau\|_1 + \|\bar{x}\|_1 = 2 \) since \( x_\tau, \bar{x} \in \Delta \). In addition, due to \( 1 \leq t + \delta \leq T + 1 \), we also have

\[
\frac{\eta}{2} \sum_{\tau=t}^{t+\delta-1} \|\nabla f^{\tau-1}(x_\tau) - \nabla f^{\tau}(x_\tau)\|_\infty^2 \leq \frac{\eta}{2} \sum_{\tau=1}^{T} \|\nabla f^{\tau-1}(x_\tau) - \nabla f^{\tau}(x_\tau)\|_\infty^2 \leq \frac{\eta}{2} V_\infty(T).
\]

Then, we bound the term \( W_t - W_{t+\delta} \). We have

\[
W_t - W_{t+\delta} \overset{\odot}{\leq} 3 \left( \xi_t + \eta L_f^2 \right) \|x_{t-1} - \bar{x}_t\|_1^2 + \alpha_t D_{KL}(\bar{x}, \bar{y}_t) + \frac{\gamma^2}{2} \|g(x_{t+\delta-1})\|_2^2
\]

\[
\overset{\oslash}{\leq} 6 \left( \xi_t + \eta L_f^2 \right) + \alpha_t \log \frac{d}{\nu} + \frac{\gamma^2}{2} G^2
\]

\[
= 6[\gamma L_g \|Q(t)\|_1 + \gamma^2 (L_g G + H^2) + \eta L_f^2] + \alpha_t \log \frac{d}{\nu} + \frac{\gamma^2}{2} G^2,
\]

where \( \odot \) is by removing the negative terms, and \( \oslash \) is due to Lemma B.1 and \( \|x_{t-1} - \bar{x}_t\|_1^2 \leq \frac{\eta}{2} V_\infty(T) \).
\[(\|\mathbf{x}_{t-1}\|_1 + \|\bar{\mathbf{x}}_t\|_1)^2 = 4.\] In addition, we have

\[
\sum_{\tau=t}^{t+\delta-1} 6 \left( \xi_\tau + \eta L^2_1 \right) \nu^2 + \sum_{\tau=t}^{t+\delta-1} \alpha_\tau \log d \leq \delta \alpha_{t+\delta} \nu \log d + 6 \delta \left( \max_{\tau \in [t+\delta]} \xi_\tau + \eta L^2_f \right) \nu^2
\]

\[
\leq \delta \alpha_{t+\delta} \nu \log d + 2\nu^2 \delta \alpha_{t+\delta}
\]

\[
= \delta \nu (\log d + 2\nu) \alpha_{t+\delta}.
\]

We combine (51) (53) (54) (55) (56) (57) (58) with (50) and then obtain

\[
\frac{1}{2} \| Q(t + \delta) \|^2 - \| Q(t) \|^2
\]

\[
\leq 6[\gamma L_g \| Q(t) \|_1 + \gamma^2 (L_g G + H^2) + \eta L^2_1] + \frac{\gamma^2}{2} G^2 + \frac{\eta}{2} V_\infty(T)
\]

\[
+ 2F \delta + \alpha_{t+\delta} \log d \nu - \gamma \delta \| Q(t) \|_1 + \frac{1}{2} \gamma^2 \delta^2 G + \gamma^2 \delta G
\]

\[
\leq \bar{C} + \left[ \log d \nu + \delta \nu (\log d + 2\nu) \right] \alpha_{t+\delta} - \gamma (\delta - 6L_g) \| Q(t) \|_1,
\]

where we let

\[
\bar{C} = 6(L_g G + H^2) + \frac{G^2 + \delta^2 G}{2} + \delta G
\]

\[
\gamma^2 + \left( \frac{L^2_1}{2} + \frac{V_\infty(T)}{2} \right) \eta + 2F \delta.
\]

The following discussion is similar to Section A.4 after (60). Thus, we omit some details for a more clear description. We consider a time interval of \([1, T + 1 - \delta]\). Since \(\alpha_{t+\delta} \leq \alpha_{T+1}\) for any \(t \in [1, T + 1 - \delta]\) due to non-decrease of \(\alpha_t\), and letting

\[
\delta \geq \frac{12L_g}{\varsigma}
\]

in (59), then we have for any \(t \in [1, T + 1 - \delta]\)

\[
\| Q(t + \delta) \|^2 - \| Q(t) \|^2 \leq 2\bar{C} + 2 \left[ \log d \nu + \delta \nu (\log d + 2\nu) \right] \alpha_{T+1} - \varsigma \delta \gamma \| Q(t) \|_2,
\]

by the inequality \(\| Q(t) \|_1 \geq \| Q(t) \|_2\).

If we let \(\| Q(t) \|_2 \geq 4\{\bar{C} + \log(d/\nu) + \delta \nu (\log d + 2\nu)\alpha_{T+1}\}/(\varsigma \delta \gamma)\), by (62), we have \(\| Q(t + \delta) \|^2 \leq (\| Q(t) \|^2 - \varsigma \delta \gamma/4)^2\) and also \(\| Q(t) \|^2 \geq \varsigma \delta \gamma/4\). Thus, we further have

\[
\| Q(t + \delta) \|_2 \leq \| Q(t) \|_2 - \frac{\varsigma \delta \gamma}{4}.
\]

Thus, along the same analysis as from (31) to (33) in Section A.4, we can prove that for any \(t \in [1, T + 1]\), we have

\[
\| Q(t) \|_2 \leq \frac{4(C + B\alpha_{T+1})}{\varsigma \delta \gamma} + 3\delta \gamma G,
\]

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where we let
\[ B = \log \frac{d}{\nu} + \delta \nu (\log d + 2\nu). \] (65)

Note that in our setting, we let \( \nu = 1/T \). Here, we also assume \( \delta \leq T \). Thus, we have
\[ B = \log(Td) + \frac{\delta}{T} \log d + \frac{2}{T^2} \leq 3 \log(Td), \]

since we assume \( T > 2 \) and \( d \geq 1 \).

Then we determine the value of \( \alpha_{T+1} \). By plugging (64) into the setting of \( \alpha_{T+1} \), we have
\[ \alpha_{T+1} = 3\eta L_g^2 + 3 \gamma^2 (2L_g G + H^2) + \frac{2}{\eta} + 3 \gamma L_g \max_{t' \in [T+1]} \| Q(t') \|_1 \]
\[ \leq 3\eta L_g^2 + 3 \gamma^2 (2L_g G + H^2) + \frac{2}{\eta} + 3 \sqrt{K} \gamma L_g \left[ \frac{4(\bar{C} + B\alpha_{T+1})}{\zeta \delta \gamma} + 3\delta \gamma G \right] \] (66)
\[ = \bar{C}' + \frac{12 \sqrt{K} L_g B}{\zeta \delta} \alpha_{T+1}, \]

with \( \bar{C}' \) defined as
\[ \bar{C}' := 3\eta L_g^2 + 3 \gamma^2 (2L_g G + H^2) + \frac{2}{\eta} + 3 \sqrt{K} \gamma L_g \left( \frac{4\bar{C}}{\zeta \delta \gamma} + 3\delta \gamma G \right). \] (67)

Furthermore, by the definition of \( B \) in (65), when
\[ \delta \geq \frac{72 \sqrt{K} L_g \log(Td)}{\zeta} \geq \frac{24 \sqrt{K} L_g B}{\zeta}, \] (68)
we have \( 1 - 12 \sqrt{K} L_g B/(\zeta \delta) \geq 1/2 \). Then, with (66), we obtain
\[ \alpha_{T+1} \leq 2\bar{C}'. \] (69)

Substituting (69) back into (64) gives
\[ \| Q(t) \|_2 \leq \frac{4[\bar{C} + 6 \log(Td)\bar{C}']}{\zeta \delta \gamma} + 3\delta \gamma G, \quad \forall t \in [T + 1]. \] (70)

Next, we need to set the value of \( \delta \). Note that the condition (61) always holds if (68) holds. Then, should choose the value from \( \delta \geq 72 \sqrt{K} L_g \log(Td)/\zeta \). The dependence of \( \alpha_{T+1} \) on \( \delta \) is \( O(\delta + \delta^{-1} + 1) \) and \( \| Q(T+1) \|_2 = O(\delta + \delta^{-1} + \delta^{-2}) \). Thus, both \( \alpha_{T+1} \) and \( \| Q(T+1) \|_2 \) are convex functions w.r.t. \( \delta \) in \( [72 \sqrt{K} L_g \log(Td)/\zeta, +\infty) \). Then, for the upper bounds of \( \alpha_{T+1} \) and \( \| Q(T+1) \|_2 \), we simply set
\[ \delta = \frac{72 \sqrt{K} L_g \log(Td)}{\zeta}, \] (71)
such that the tightest upper bounds of them must be no larger than the values with setting \( \delta \) as in (71). Therefore, for any \( T \geq \delta = 72\sqrt{KL_g \log(Td)}/\varsigma \), the results (69) and (70) hold.

Then, similar to the discussion in Section A.4, considering the case where \( T < \delta = 72\sqrt{KL_g \log(Td)}/\varsigma \), the inequalities (69) and (70) also hold. Thus, we know that (69) and (70) hold for any \( T > 0 \) and \( \delta = 72\sqrt{KL_g \log(Td)}/\varsigma \).

Thus, letting \( \eta = [\max \{V_\infty(T), L_f^2\}]^{-1/2} \) and \( \gamma = [\max \{V_\infty(T), L_f^2\}]^{1/4} \), there exist absolute constants \( \tilde{c}_1, \tilde{c}_2, \tilde{c}_3, \tilde{c}_4, \tilde{c}_5 \), and \( \tilde{c}_6 \) that are \( \text{poly}(H, F, G, L_g, K) \) such that

\[
\tilde{C} \leq \left( \tilde{c}_1 + \frac{\tilde{c}_2}{\varsigma} \right) \max \left\{ \sqrt{V_\infty(T)}, L_f \right\} \log^2(Td) + \frac{\tilde{c}_3 \log(Td)}{\varsigma},
\]

\[
\tilde{C'} \leq \left( \tilde{c}_4 + \frac{\tilde{c}_5}{\varsigma} \right) \max \left\{ \sqrt{V_\infty(T)}, L_f \right\} \log(Td) + \frac{\tilde{c}_6}{\varsigma},
\]

This further leads to

\[
\alpha_{T+1} \leq \left( \tilde{C}_1 + \frac{\tilde{C}_2}{\varsigma} \right) \max \left\{ \sqrt{V_\infty(T)}, L_f \right\} \log(Td) + \frac{\tilde{C}_3}{\varsigma},
\]

\[
\| Q(T+1) \|_2 \leq \left( \tilde{C}'_1 + \frac{\tilde{C}'_2}{\varsigma} \right) \left[ \max \{V_\infty(T), L_f^2\} \right]^{1/4} \log(Td) + \frac{\tilde{C}'_3}{\varsigma L_f^{1/2}},
\]

for some constants \( \tilde{C}_1, \tilde{C}_2, \tilde{C}_3, \tilde{C}'_1, \tilde{C}'_2, \) and \( \tilde{C}'_3 \) which are \( \text{poly}(H, F, G, L_g, K) \). This completes the proof. \( \square \)

Furthermore, also by Lemma B.3, we give Lemma B.5

**Lemma B.5.** For any \( \eta, \gamma \geq 0 \), setting \( \nu \) and \( \alpha_t \) the same as in Theorem 6.1, Algorithm 2 ensures

\[
\text{Regret}(T) \leq \frac{\eta}{2} V_\infty(T) + \tilde{C}_1'' L_f^2 \eta + \tilde{C}_2'' \gamma^2 + 3 \log(Td) \cdot \alpha_{T+1}.
\]

where \( \tilde{C}_1'' \) and \( \tilde{C}_2'' \) are absolute constants that are \( \text{poly}(H, G, L_g) \).

**Proof.** According to (40), we have

\[
\text{Regret}(T) = \sum_{t=1}^{T} f_t'(x_t) - \sum_{t=1}^{T} f'(x^*) \leq \sum_{t=1}^{T} \langle \nabla f_t'(x_t), x_t - x^* \rangle.
\]

Setting \( z = x^* \) in Lemma B.3 gives

\[
\frac{1}{2} \left[ \| Q(t+1) \|_2^2 - \| Q(t) \|_2^2 \right] + \langle \nabla f_t'(x_t), x_t - x^* \rangle
\]

\[
\leq \psi_t(\| x_t - \tilde{y}_t \|_1^2 + \| x_t - \tilde{x}_{t+1} \|_1^2) + \frac{\eta}{2} \| \nabla f_{t-1}'(x_t) - \nabla f_t'(x_t) \|_2^2
\]

\[
+ (\alpha_{t+1} - \alpha_t) D_{\text{KL}}(x^*, \tilde{y}_{t+1}) + \gamma( Q(t) + \gamma g(x_{t-1}), g(x^*) )
\]

\[
+ 6 \left( \xi_t + \eta L_f^2 \right) \nu^2 + \alpha_t \nu \log d + W_t - W_{t+1},
\]

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where \( \psi_t := 3/2 \cdot (\xi_t + \eta L_f^2 + \gamma^2 L_g G) - \alpha_t/2 + 1/\eta \) and \( W_t := 3/2 \cdot \left( \xi_t + \eta L_f^2 \right) \| x_{t-1} - \overline{x}_t \|_2^2 + \alpha_t D_{KL}(x^*, \overline{y}_t) - \gamma^2/2 \cdot \| g(x_{t-1}) \|_2^2 \).

Since \( g_k(x^*) \leq 0 \) and also \( Q_k(x_t) + \gamma g_k(x_{t-1}) \geq 0 \) shown in (13), then the last term in (73) is bounded as

\[
\gamma \langle Q(t) + \gamma g(x_{t-1}), g(x^*) \rangle = \sum_{k=1}^K [Q_k(x_t) + \gamma g_k(x_{t-1})] g_k(x^*) \leq 0. \tag{74}
\]

Combining (73) (74) with (72) yields

\[
\text{Regret}(T) \leq \sum_{t=1}^T \langle \nabla f^t(x_t), x_t - x^* \rangle
\]

\[
= \sum_{t=1}^T \psi_t(\| x_t - \overline{y}_t \|_1^2 + \| x_t - \overline{x}_{t+1} \|_1^2) + \sum_{t=1}^T \frac{\eta}{2} \| \nabla f^t-1(x_t) - \nabla f^t(x_t) \|_2^2
\]

\[
+ \sum_{t=1}^T (\alpha_{t+1} - \alpha_t) D_{KL}(x^*, \overline{y}_{t+1}) + \sum_{t=1}^T 6 \left( \xi_t + \eta L_f^2 \right) \nu^2 + \sum_{t=1}^T \alpha_t \nu \log d
\]

\[
+ W_1 - W_{T+1} + \frac{1}{2} \left[ \| Q(1) \|_2^2 - \| Q(T+1) \|_2^2 \right]. \tag{75}
\]

We analyze the terms in (75) in the following way. By the setting of \( \alpha_t \)

\[
\alpha_t = \max \left\{ 3(\xi_t + \eta L_f^2 + \gamma^2 L_g G) + \frac{2}{\eta} \alpha_{t-1} \right\} \quad \text{with} \quad \alpha_0 = 0,
\]

which, by recursion, is equivalent to

\[
\alpha_t = 3\eta L_f^2 + 3\gamma^2 L_g G + \frac{2}{\eta} + 3 \max_{t' \in [t]} \xi_{t'}
\]

\[
= 3\eta L_f^2 + 3\gamma^2 (2L_g G + H^2) + \frac{2}{\eta} + 3\gamma L_g \max_{t' \in [t]} \| Q(t') \|_1. \tag{76}
\]

This setting guarantees that

\[
\alpha_{t+1} \geq \alpha_t, \quad \forall t \geq 0,
\]

and also

\[
\sum_{t=1}^T \psi_t(\| x_t - \overline{x}_t \|_1^2 + \| x_{t-1} - \overline{x}_t \|_1^2)
\]

\[
= \sum_{t=1}^T 3\gamma L_g \left( \| Q(t) \|_1 - \max_{t' \in [t]} \| Q(t') \|_1 \right) (\| x_t - \overline{x}_t \|_2^2 + \| x_{t-1} - \overline{x}_t \|_2^2) \leq 0. \tag{77}
\]
Since $\alpha_{t+1} \geq \alpha_t, \forall t$, thus we have
\[
\sum_{t=1}^{T} (\alpha_{t+1} - \alpha_t) D_{KL}(x^*, \tilde{y}_{t+1}) \leq \sum_{t=1}^{T} (\alpha_{t+1} - \alpha_t) \log \frac{d}{\nu} = \alpha_{T+1} \log \frac{d}{\nu} - \alpha_1 \log \frac{d}{\nu}. \tag{78}
\]
where the inequality is by Lemma B.1. Moreover, by the definition of $V_\infty(T)$, we have
\[
\frac{\eta}{2} \sum_{t=1}^{T} \|\nabla f^{t-1}(x_t) - \nabla f^t(x_t)\|_2^2 \leq \frac{\eta}{2} V_\infty(T). \tag{79}
\]
In addition, we bound the term $W_1 - W_{T+1}$ by
\[
W_1 - W_{T+1} \leq \frac{3}{2} \left( \xi_1 + \eta L_f^2 \right) \|x_0 - \bar{x}_1\|_1^2 + \alpha_1 D_{KL}(x^*, \tilde{y}_1) + \frac{\gamma^2}{2} \|g(x_T)\|_2^2 \tag{80}
\]
where $\xi_1$ is by removing negative terms, $\eta L_f^2 \|Q(1)\|_1 \leq \gamma^2 (L_g G + H^2) + \eta L_f^2$ as well as $\|g(x_T)\|_2 \leq \gamma L g + H^2$ according to Assumptions 2.1 and 2.3, and $\eta L_f^2 \|Q(0)\|_2 = 0$.

Moreover, we have
\[
\sum_{t=1}^{T} 6 \left( \xi_t + \eta L_f^2 \right) \nu^2 + \sum_{t=1}^{T} \alpha_t v \log d
\]
\[
\leq T \alpha_{T+1} \nu \log d + 6T \left( \max_{\tau \in [T]} \xi_\tau + \eta L_f^2 \right) \nu^2 \tag{81}
\]
\[
\leq T \alpha_{T+1} \nu \log d + 2\nu^2 T \alpha_{T+1}
\]
\[
= \left( \log d + \frac{2}{T} \right) \alpha_{T+1}.
\]

Therefore, combining (77) (78) (79) (80) (81) with (75), further by $\|Q(1)\|_2^2 = \gamma^2 \|g(x_0)\|_2^2 \leq \gamma^2 \|g(x_T)\|_2^2 \leq \gamma^2 G^2$, we have
\[
\text{Regret}(T) \leq \frac{\eta}{2} V_\infty(T) + 6[\eta L_f^2 + \gamma^2 (2L_g G + H^2)] + \alpha_{T+1} R^2 + \frac{3\gamma^2 G^2}{2}
\]
\[
\leq \frac{\eta}{2} V_\infty(T) + 6L_f^2 \eta + \left[ 6(2L_g G + H^2) + \frac{3G^2}{2} \right] \gamma^2 + \left( \log \frac{d}{\nu} + \log d + \frac{2}{T} \right) \alpha_{T+1}
\]
\[
\leq \frac{\eta}{2} V_\infty(T) + 6L_f^2 \eta + \left( 12L_g G + 6H^2 + \frac{3G^2}{2} \right) \gamma^2 + 3 \log(Td) \cdot \alpha_{T+1}.
\]

This completes the proof.
B.2 Proof of Theorem 6.1

Proof. According to Lemma B.5, by the settings of \( \eta \) and \( \gamma \), we have

\[
\text{Regret}(T) \leq (1/2 + \tilde{C}_1') \max \left\{ \sqrt{V_\infty(T)}, L_f \right\} + \tilde{C}_2'' \max \left\{ \sqrt{V_\infty(T)}, L_f \right\} + 3 \log(Td) \cdot \alpha_{T+1},
\]

where the inequality is due to \( \eta/2 \cdot V_\infty(T) + \tilde{C}_1'' L_f^2 \eta \leq \eta(1/2 + \tilde{C}_1'') \max\{V_\infty(T), L_f^2\} \leq (1/2 + \tilde{C}_1'') \max\{\sqrt{V_\infty(T)}, L_f\} \). Further combining the above inequality with the bound of \( \alpha_{T+1} \) in Lemma B.4 yields

\[
\text{Regret}(T) = \left( \tilde{C}_1 + \frac{\tilde{C}_2}{\varsigma} \right) \max \left\{ \sqrt{V_\infty(T)}, L_f \right\} \log(Td) + \frac{\tilde{C}_3 \log(Td)}{\varsigma},
\]

for some constants \( \tilde{C}_1, \tilde{C}_2, \tilde{C}_3 \) determined by \( \tilde{C}_1', \tilde{C}_2', \tilde{C}_3, \tilde{C}_1''', \tilde{C}_2''', \tilde{C}_3''' \). This completes the proof. \( \square \)

B.3 Proof of Theorem 6.2

Proof. According to Lemma 5.5 and the drift bound of \( \mathbf{Q}(T+1) \) in Lemma B.4, with the setting of \( \gamma \), we have

\[
\text{Violation}(T, k) \leq \frac{1}{\gamma} \left\| \mathbf{Q}(T + 1) \right\|_2 \leq \left( \tilde{C}_1' + \frac{\tilde{C}_2'}{\varsigma} \right) \log(Td) + \frac{\tilde{C}_3'}{\varsigma L_f} \leq \left( \tilde{C}_1 + \frac{\tilde{C}_2}{\varsigma} \right) \log(Td),
\]

where the second inequality is by \( 1/\gamma = \min\{[V_\infty(T)]^{-1/4}, L_f^{-1/2}\} \leq L_f^{-1/2} \). Specifically, we set \( \tilde{C}_1 = \tilde{C}_1', \tilde{C}_2 = \tilde{C}_2' + \tilde{C}_3''/L_f \). This completes the proof. \( \square \)