OPTIMAL DIVIDEND AND CAPITAL INJECTION STRATEGY WITH EXCESS-OF-LOSS REINSURANCE AND TRANSACTION COSTS

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ABSTRACT. This article deals with an optimal dividend, reinsurance and capital injection control problem in the diffusion risk model. Under the objective of maximizing the insurance company’s value, we aim at finding the joint optimal control strategy. We assume that there exist both the fixed and proportional costs in control processes and the excess-of-loss reinsurance is “expensive”. We derive the closed-form solutions of the value function and optimal strategy by using stochastic control methods. Some economic interpretations of the obtained results are also given.

1. Introduction. In mathematical insurance literature, the classical dividend problem consists in finding a dividend strategy that maximizes the total expected discounted dividends until the time of bankruptcy. Its origin can be traced to the work of [5]. Since then much research on this issue has been carried out for varieties of models. In practice, the company sometimes needs to raise new capital from market to continue the business. The expected present values of the dividends payout minus capital injections up to the time of ruin can be regarded as the company’s value. The company seeks to find optimal dividend and capital injection strategy for maximizing its value. Recently, the combinational optimization of dividend and capital injection has attracted many attentions. Sometimes, transaction costs generated by the control processes are added in the risk models. The literature on this issue includes [2],[7],[9],[11],[15],[17],[19] and so on. As shown in above references, transaction costs usually include two parts: the fixed costs and the proportional

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costs. The former are generated by the advisory and consulting as well as the latter are generated by the tax. In general, the fixed costs can generate more difficult impulse control problem. To manage its risks and explore profit opportunities, the company needs to determine the times and amounts of dividend payments and capital injections, respectively.

As everyone knows, the insurance company usually cedes risks by purchasing reinsurance. An appropriate use of reinsurance protects it against unexpected, potentially large losses, and hence reduces the insurer’s earnings volatility. In practice, the excess-of-loss reinsurance is one of most popular reinsurance policies. The optimal dividend problem in the presence of excess-of-loss reinsurance has attracted many interests for its potential application on insurance industry. For example, [11] studied the optimal excess-of-loss reinsurance and dividend strategies for maximizing the expected total discounted dividends received by shareholders until the time of bankruptcy. [3] further explored this problem by taking into account both the proportional and fixed costs incurred by dividends. [12] investigated the optimal dividend, capital injection and excess-of-loss reinsurance problem in a diffusion risk model. They also considered the influences of the fixed and proportional transaction costs. In discussed references, it was assumed that both insurer and reinsurer used the expectation premium principle with same safety loading to calculate premiums, i.e., the reinsurance is “cheap”. However, the reinsurer may use a higher safety loading in practice, which results in the cedent paying larger amount of premium than the amount which is reinsured, i.e., the reinsurance is “expensive”. In this case, the excess of premium is viewed as transaction costs. [10] studied the optimal dividend and capital injection problem with “expensive” excess-of-loss reinsurance. But the fixed transaction costs were not taken into account. With the exception of [10], very little work has considered such a joint optimal control problem. The literature on the “expensive” excess-of-loss reinsurance includes [8], [13], [16], [18] and so on.

Inspired by [10] and [12], this paper further studies the combined dividend, reinsurance and capital injection problem in the framework of diffusion risk model. Both the fixed and proportional transaction costs are included and the excess-of-loss reinsurance is assumed to be “expensive”. We will solve the problem in 11 cases, which depend on the relationships among the parameters. Some existing results are extended. The rest of the paper is organized as follows. In Section 2, we introduce the risk model and raise the combined dividend-reinsurance-capital injection problem. In Section 3, we analyze some properties of the value function and give the QVI (quasi-variational inequalities) associated with value function. Based on the costs of reinsurance, as measured by the safety loading, we establish the value function and associated optimal strategy in Section 4 and Section 5, respectively. Section 6 is a conclusion.

2. Model formulation and the optimal control problem. We work on a complete probability space \((\Omega, \mathcal{F}, P)\) on which all processes are well defined. The information at time \(t\) is given by \(\mathcal{F}_t\), in which \(\{\mathcal{F}_t : t \geq 0\}\) is the complete filtration generated by the claim, dividend, reinsurance and capital injection processes. Our results will be formulated under the controlled diffusion model. However, for the purpose of motivation it is convenient to start from the classical Cramér-Lundberg model. In this model, the uncontrolled surplus process \(\{U_t\}_{t \geq 0}\) of an insurance company follows that
\[ U_t = x + ct - \sum_{i=1}^{N_t} Y_i, \]  

where \( U_{0-} = x \geq 0 \) is the initial surplus, \( c > 0 \) is the rate of premiums, \( N_t \) is a Poisson process with constant intensity \( \lambda \), random variables \( Y_i \)'s are positive i.i.d. claims with a common distribution function \( F(\cdot) \). Define the quantity \( m = \inf\{y \geq 0: F(y) = 1\} < \infty \), the finite mean \( \mu_m = \mathbb{E}Y_1 < \infty \) and finite second moment \( \nu_m = \mathbb{E}(Y_1^2) < \infty \). Assume that the premium \( c \) in (2.1) is calculated via the expected value principle, i.e.,

\[
c = (1 + \theta_1)\mathbb{E}\left(\sum_{i=1}^{N_t} Y_i\right) = (1 + \theta_1)\lambda \mu_m,
\]

where \( \theta_1 > 0 \) is the safety loading of the insurer. Let \( a \) denote the excess-of-loss retention level and

\[
\mu(a) = \mathbb{E}(Y_1 \wedge a) = \int_0^a F(y)dy,
\]

\[
\nu(a) = \mathbb{E}((Y_1 \wedge a)^2) = \int_0^a 2yF(y)dy,
\]

where \( F(y) = P(Y_1 > y) = 1 - F(y) \) and \( y \wedge a = \min\{y, a\} \). Then both functions \( \mu(a) \) and \( \nu(a) \) are increasing on \([0, m]\), while on \([m, \infty)\) they are constants equal to \( \mu(m) = \mu_m \) and \( \nu(m) = \nu_m \). Assume that the reinsurer also employs the expected value principle but with a higher safety loading \( \theta_2 \in (\theta_1, \infty) \) to calculate premiums, then

\[
c^a = (1 + \theta_2)\mathbb{E}\left(\sum_{i=1}^{N_t} (Y_i - Y_i \wedge a)\right) = \lambda(1 + \theta_2)(\mu_m - \mu(a))
\]

represents the reinsurance premium that is payable by the insurer to the reinsurer. Then, the surplus process involving excess-of-loss reinsurance can be written as

\[
U_t^a = x + (c - c^a)t - \sum_{i=1}^{N_t} (Y_i \wedge a),
\]

with \( U_{0-}^a = x \). According to [6], we approximate model (2.4) by a pure diffusion model \( \{X_t^a\}_{t \geq 0} \) with the same drift and volatility. Namely, \( X_t^a \) satisfies the following stochastic process

\[
X_t^a = x + ((\theta_1 - \theta_2)\lambda \mu_m + \theta_2 \lambda \mu(a))t + \sqrt{\lambda \nu(a)}B_t,
\]

with \( X_{0-}^a = x \).

Suppose that \( a \in [0, m] \) can be adjusted dynamically to control the risk exposure. We use the process \( \{a_t\}_{t \geq 0} \) to describe a reinsurance strategy. Moreover, we incorporate dividend distribution and capital injection in model (2.4). Let \( D_t = \sum_{i=1}^{\infty} I_{(\zeta_i \leq t)} \zeta_i \) denote the cumulative amount of dividends paid from time 0 to time \( t \). It is determined by a sequence of increasing stopping times \( \{\tau_i, i = 1, 2, \cdots\} \) and a sequence of non-negative random variables \( \{\zeta_i, i = 1, 2, \cdots\} \), which represent the times and the sizes of dividends, respectively. Furthermore, let \( R_t = \sum_{i=1}^{\infty} I_{(\tau_i \leq t)} \eta_i \) denote the cumulative amount of capital injections from time 0 to time \( t \). It is described by a sequence of increasing stopping times
\{τ_i, i = 1, 2, \cdots \} and a sequence of random variables \{η_i, i = 1, 2, \cdots \}, which represent the times and the amounts of capital injections, respectively. Given a joint strategy \(π = (a^π, D^π, R^π)\), the controlled surplus process follows

\[
X^π_t = x + ((\theta_1 - \theta_2)\lambda m + \theta_2\lambda)(a) + \lambda \nu(a) B_t - D^π_t + R^π_t,
\]

with \(X^π_0 = x\). The insurance company selects a reinsurance strategy \(a^π\), a dividend strategy \(D^π\) and a capital injection strategy \(R^π\) at any time \(t\) based on information available up to and including time \(t\), say \(F^B_t = σ\{B_s : s ≤ t\}\). Mathematically, we give the following definition of admissible strategy that can be selected by the insurer.

**Definition 2.1.** A strategy \(π = (a^π, D^π, R^π)\) is said to be admissible if it satisfies the following conditions:

(i) The retention level \(a^π = \{a^π_t\}_{t≥0}\) is an \(F^B_t\)-adapted process with \(0 ≤ a^π_t ≤ m\) for all \(t ≥ 0\).

(ii) \(\{\varphi^π_t\}\) is a sequence of stopping times w.r.t. \(F^B_t\), and \(0 ≤ \varphi^π_t < \cdots < \varphi^π_i < \cdots\), a.s.

(iii) \(0 < ζ^π ≤ X_{\varphi^π_i} ≤ X_{\varphi^π_{i+1}}\), \(i = 1, 2, \cdots\) is measurable w.r.t. \(F^B_{\varphi^π_i}\).

(iv) \(\{τ^π_i\}\) is a sequence of stopping times w.r.t. \(F^B_t\), and \(0 ≤ τ^π_i < \cdots < τ^π_i < \cdots\), a.s.

(v) \(η^π_t > 0, i = 1, 2, \cdots\) is measurable w.r.t. \(F^B_{τ^π_i}\).

(vi) \(P(\lim_{i→∞} \varphi^π_t < t) = 0, P(\lim_{i→∞} τ^π_i < t) = 0, ∀ t > 0\).

(vii) \((D^π_t - D^π_{τ^π_i}) \cdot (R^π_t - R^π_{τ^π_i}) = 0\) for all \(t ≥ 0\), i.e., the insurance company cannot pay dividend and raise new capital at the same time.

We write \(Π\) for the space of these admissible strategies. For each \(π ∈ Π\), we define the time of bankruptcy as \(T^π = \inf\{t ≥ 0 : X^π_t < 0\}\), which is an \(F^B_t\)-stopping time. For each dividend payment, it incurs a fixed cost \(K_1 > 0\), which is independent of the amount of the payment. Let \(β_1 ∈ (0, 1)\) be a positive constant, where \(1 - β_1\) is the tax rate at which the dividends are taxed. Consequently, if the amount \(ζ\) of liquid surplus is withdrawn, the net amount of money that the shareholders receive after transaction costs have been paid is \(β_1ζ - K_1\). Similarly, the company needs to pay \(β_2 η + K_2\) to meet the capital injection \(η\), where \(β_2 > 1\) measures the proportional costs and \(K_2 > 0\) is the fixed cost. Then the value of the company is measured by the following performance function

\[
V(x; π) = E^x \left( ∑_{i=1}^{∞} e^{-δτ^π_i}(β_1ζ^π_i - K_1)1_{(\varphi^π_i < T^π)} - ∑_{i=1}^{∞} e^{-δτ^π_i}(β_2η^π_i + K_2)1_{(τ^π_i < T^π)} \right),
\]

which is the expected present values of the dividends payout minus capital injections up to the time of bankruptcy. \(E^x\) denotes the expectation conditional on \(X^π_0 = x\) and \(δ > 0\) is the discount factor.

**Problem 2.2** The optimization problem of the insurance company is to find the value function

\[
V(x) = \max_{π ∈ Π} V(x; π)
\]

and associated optimal strategy \(π^* ∈ Π\) such that \(V(x) = V(x; π^*)\).
3. Properties of the value function. In this section, we will establish the QVI associated with the optimization problem and derive some properties of the value function.

Proposition 1. The value function defined by (2.8) satisfies that

$$\beta_1(x - y) - K_1 \leq V(x) - V(y) \leq \beta_2(x - y) + K_2, \quad 0 \leq y \leq x,$$

with the boundedness condition

$$0 \leq V(x) \leq \beta_1(x + \lambda \theta_1 \mu_m / \delta).$$

Proof. Consider an admissible strategy $\pi_1$ with $V(y; \pi_1) \geq V(y) - \varepsilon$ for any $\varepsilon > 0$. For $x \geq y \geq 0$, we define a new admissible strategy as follows: $x - y$ is paid immediately as dividend and then the strategy $\pi_1$ with initial surplus $y$ is followed. Then for $\varepsilon > 0$, it holds that

$$V(x) \geq \beta_1(x - y) - K_1 + V(y; \pi_1) \geq \beta_1(x - y) - K_1 + V(y) - \varepsilon.$$

Because $\varepsilon$ is arbitrary, $V(x) - V(y) \geq \beta_1(x - y) - K_1$. The second inequality in (3.1) can be proved similarly. The surplus process $\{X_t^\pi\}_{t \geq 0}$ with only reinsurance is expressed by (2.5), then

$$E^x \left( \int_0^{T^\pi} e^{-\delta \tau} dX_s^a \right) = E^x \left( \int_0^{T^\pi} e^{-\delta \tau} ((\theta_1 - \theta_2) \lambda \mu_m + \theta_2 \lambda \mu(\tau)) d\tau \right) \leq \lambda \theta_1 \mu_m / \delta.$$

By Itô’s formula, we have

$$e^{-\delta T^\pi} X^\pi_T = x - \delta \int_0^{T^\pi} e^{-\delta \tau} X^\pi_\tau d\tau + \int_0^{T^\pi} e^{-\delta \tau} dX^a_\tau.$$

Since $X^\pi_T = 0$ and $X^\pi_t \geq 0$, for $t \leq T^\pi$, taking expectation on both sides yields

$$-E^x \left( \int_0^{T^\pi} e^{-\delta \tau} dX^a_\tau \right) = x - E^x \left( \delta \int_0^{T^\pi} e^{-\delta \tau} X^\pi_\tau d\tau \right) \leq x.$$

Then, it has

$$V(x; \pi) = E^x \left( \sum_{i=1}^{\infty} e^{-\delta \tau^\pi_i} (\beta_1 \zeta^\pi_i - K_1) I(\tau^\pi_i \leq T^\pi) - \sum_{i=1}^{\infty} e^{-\delta \tau^\pi_i} (\beta_2 \eta^\pi_i + K_2) I(\tau^\pi_i \leq T^\pi) \right)$$

$$\leq E^x \left( \sum_{i=1}^{\infty} e^{-\delta \tau^\pi_i} (\beta_1 \zeta^\pi_i) I(\tau^\pi_i \leq T^\pi) - \sum_{i=1}^{\infty} e^{-\delta \tau^\pi_i} (\beta_2 \eta^\pi_i) I(\tau^\pi_i \leq T^\pi) \right)$$

$$\leq \beta_1 E^x \left( \sum_{i=1}^{\infty} e^{-\delta \tau^\pi_i} \zeta^\pi_i I(\tau^\pi_i \leq T^\pi) - \sum_{i=1}^{\infty} e^{-\delta \tau^\pi_i} \eta^\pi_i I(\tau^\pi_i \leq T^\pi) \right)$$

$$= \beta_1 \left[ E^x \left( \int_0^{T^\pi} e^{-\delta \tau} dX^a_\tau \right) - E^x \left( \int_0^{T^\pi} e^{-\delta \tau} dX^\pi_\tau \right) \right]$$

$$\leq \beta_1 (x + \lambda \theta_1 \mu_m / \delta).$$

The last inequality is confirmed by (3.3) and (3.4). In addition, $V(x) \geq 0$ is obvious. Therefore, (3.1) and (3.2) are proven.
To proceed with our work, let’s define some operators for a function $v$:

(i) The dividend payment operator: $\mathcal{D} v(x) = \max_{\zeta \geq 0} \{ v(x - \zeta) + \beta \zeta - K_1 \} ;$

(ii) The capital injection operator: $\mathcal{C} v(x) = \max_{\eta \geq 0} \{ v(x + \eta) - \beta_2 \eta - K_2 \} ;$

(iii) The differential operator: $\mathcal{A} v(x) = \frac{1}{2} \lambda v'(x) + [(\theta_1 - \theta_2) \lambda \mu_m + \theta_2 \lambda \mu(a)] v'(x) - \delta v(x).$

Similar to [4], using a standard application of the dynamic programming principle, we can give a characterization of the value function by the following definition.

**Definition 3.1.** We say that a function $v(x) : [0, \infty) \to \mathbb{R}_+$ satisfies QVI of Problem 2.2 if for every $x \in [0, \infty)$$$
\max_{0 \leq a \leq m} \{ \mathcal{A}^a v(x) \} \leq 0, \quad (3.5)
\mathcal{D} v(x) - v(x) \leq 0, \quad (3.6)
\mathcal{C} v(x) - v(x) \leq 0, \quad (3.7)
(\max_{0 \leq a \leq m} \{ \mathcal{A}^a v(x) \})(\mathcal{D} v(x) - v(x))(\mathcal{C} v(x) - v(x)) = 0, \quad (3.8)
\max\{ \mathcal{C} v(0) - v(0), -v(0) \} = 0. \quad (3.9)

**Remark 1.** It is optimal to postpone refinancing as long as possible, i.e., they may happen only at the moments when the surplus process hits the barrier 0. The result can be established by repeating a similar procedure to that in Lemma 3.2 of [14].

It is easy to understand that the insurer should buy less reinsurance with the increasing cost of reinsurance, which is measured by the safety loading $\theta_2$ for reinsurer. It is expected that full retention will be taken once $\theta_2$ exceeds some critical level. The following analysis shows that the critical level is $\frac{2m \mu_m}{\nu_m} \theta_1 \in (\theta_1, \infty)$. Consequently, we should solve the optimization problem in two different cases.

4. The case of $\theta_2 \in (\theta_1, \frac{2m \mu_m}{\nu_m} \theta_1)$. In this section, let’s consider the first case with

$$
\theta_1 < \theta_2 < \frac{2m \mu_m}{\nu_m} \theta_1. \quad (4.1)
$$

In view of the structure of (3.9), we shall discuss the solution of QVI according to different boundary conditions in the next two subsections. Following the approach in optimal control theory, we assume that (3.5)-(3.9) have a appropriate solution, which is continuously differentiable on $(0, \infty)$, twice continuously differentiable on $(0, u_2]$ and linear on $[u_2, \infty)$ for some parameter $u_2$. This assumption will be verified later.

4.1. The case without capital injection. Suppose that it is optimal to get out of the business whenever the surplus is null, then the corresponding boundary conditions are $v(0) = 0$ and $\mathcal{C} v(0) - v(0) \leq 0$. Following the approach in stochastic control theory, the candidate solution $f(x)$ for $v(x)$ in this case should satisfy that

$$
\max_{0 \leq a \leq m} \{ \mathcal{A}^a f(x) \} = 0, \quad 0 < x < u_2, \quad (4.2)
\mathcal{D} f(x) - f(x) = 0, \quad x \geq u_2, \quad (4.3)
f(0) = 0, \quad (4.4)
\mathcal{C} f(0) - f(0) \leq 0, \quad (4.5)
$$
with some unknown parameters \(0 < u_2 < \infty\).

Differentiating (4.2) with respect to \(a\) and setting the derivative to zero yield

\[
a(x) = -\theta_2 f'(x) \quad \text{if } f''(x) \neq 0,
\]

Plugging (4.6) in (4.2) yields

\[
(\lambda \theta_2 w(a) + (\theta_1 - \theta_2) \mu_m) f'(x) - \delta f(x) = 0,
\]

where \(w(a) = \mu(a) - \frac{\nu(a)}{2m} \in \mathcal{C}^1(0, \infty)\). It is easy to show that \(w(0+) = 0, w(\infty) = \mu_m\) and \(w'(a) = \frac{\nu(a)}{2ma} > 0\). The inverse function \(w^{-1}(z)\) of \(w(z)\) exists. Thus, (4.4) and (4.7) lead to

\[
a(0) = w^{-1}\left(\frac{\theta_2 - \theta_1}{\theta_2} \mu_m\right) > 0.
\]

Note that \(a\) is a function of \(x\). Taking derivative with respect to \(x\) on both sides of (4.7) and using (4.6) again result in

\[
a'(x) = \frac{\lambda \theta_2^2 w(a(x)) + \lambda \theta_2 (\theta_1 - \theta_2) \mu_m + \delta a(x)}{\lambda \theta_2 a(x) w'(a(x))}, \quad x > 0.
\]

Let’s define that, for \(x \in [w^{-1}\left(\frac{\theta_2 - \theta_1}{\theta_2} \mu_m\right), \infty)\),

\[
Q(x) = \int_{w^{-1}\left(\frac{\theta_2 - \theta_1}{\theta_2} \mu_m\right)}^{x} \frac{\lambda \theta_2 y w'(y)}{\lambda \theta_2^2 w(y) + \lambda \theta_2 (\theta_1 - \theta_2) \mu_m + \delta y} \, dy
\]

\[
= \int_{w^{-1}\left(\frac{\theta_2 - \theta_1}{\theta_2} \mu_m\right)}^{x} \frac{\lambda \theta_2^2 \nu(y)}{\lambda \theta_2^2 (2y \mu(y) - \nu(y)) + 2 \lambda \theta_2 (\theta_1 - \theta_2) \mu_m y + 2 \delta y^2} \, dy.
\]

It is not difficult to prove that the integrand is positive, so \(Q(x)\) is increasing strictly and \(Q(\infty) < \infty\). Consequently, the inverse \(Q^{-1}(x)\) of the function \(Q(x)\) exists. Since the function \(Q^{-1}(x)\) satisfies the same 1st order ordinary differential equation for \(x > 0\) as the function \(a(x)\) (see (4.9)) and \(Q^{-1}(0) = w^{-1}\left(\frac{\theta_2 - \theta_1}{\theta_2} \mu_m\right) = a(0)\), we conclude that

\[
a(x) = Q^{-1}(x),
\]

for \(x \geq 0\). We conjecture that there exists \(x_0 = Q(m) \leq u_2\) such that the insurer will not buy reinsurance once the surplus exceeds \(x_0\). Due to the strictly increasing property of \(Q(x)\), the inequality \(Q(m) > 0\) is equivalent to \(m > Q^{-1}(0) = a(0)\) where the last equality follows by (4.10). It then follows by (4.8) that \(w^{-1}\left(\frac{2\theta_1 - \theta_2}{\theta_2} \mu_m\right) = a(0) < m\). Due to the strictly increasing property of \(w\), we then obtain \(\frac{\theta_2 - \theta_1}{\theta_2} \mu_m < w(m) = \mu_m - \frac{\theta_1}{\theta_2} \mu_m\), where the first inequality is equivalent to (4.1) and the last equality follows by the definition of \(w\). Under condition (4.1), in view of (4.4) and (4.6), we can express \(f(x)\) through \(a(x)\) by

\[
f(x) = k \int_{0}^{x} e^{\int_{y}^{\infty} \frac{\theta_2}{\mu_m} \, dy} \, dy, \quad 0 \leq x < x_0,
\]

in which \(k > 0\) is the free coefficient . Moreover, since \(a(x) \equiv m\) for \(x \in [x_0, u_2]\), (4.2) turns to be a second-order ordinary differential equation

\[
\frac{1}{2} \lambda \mu_m f''(x) + \theta_1 \lambda \mu_m f'(x) - \delta f(x) = 0.
\]

Therefore,

\[
f(x) = k_1 e^{\gamma_+ (x-x_0)} + k_2 e^{\gamma_- (x-x_0)}, \quad x_0 \leq x \leq u_2,
\]

where \(\gamma_{\pm} = \frac{1}{2} \lambda \mu_m \pm \theta_1 \lambda \mu_m\),
with
\[ r_+ = \frac{1}{\lambda m} \left( -\lambda \theta_1 \mu_m + \sqrt{(\lambda \theta_1 \mu_m)^2 + 2 \lambda \delta \nu_m} \right) > 0, \]
\[ r_- = \frac{1}{\lambda m} \left( -\lambda \theta_1 \mu_m - \sqrt{(\lambda \theta_1 \mu_m)^2 + 2 \lambda \delta \nu_m} \right) < 0. \]
The continuity of \( f'(x) \) and \( f''(x) \) at point \( x_0 \) yields that
\[ k_1 r_+ + k_2 r_- = k, \]
\[ k_1 (r_+)^2 + k_2 (r_-)^2 = -\frac{\theta_2}{m}. \]
Above pair of equations gives
\[ k_1 = k c_1, \quad k_2 = k c_2 \] (4.14)
with
\[ c_1 = \frac{(r_- + \frac{\theta_2}{m})}{r_+ (r_- - r_+)} > 0, \] (4.15)
\[ c_2 = \frac{(r_- + \frac{\theta_2}{m})}{r_- (r_+ - r_-)} < 0. \] (4.16)
The inequality in (4.15) is confirmed by (4.1), see Appendix A. For \( x \geq u_2 \), we conjecture that there exists \( u_1 \in (0, u_2) \) such that
\[ f(x) = Q f(x) = f(u_1) + \beta_1 (x - u_1) - K_1 \] (4.17)
and \( a(x) \equiv m \) is optimal. Then the suggested solution to (4.2)-(4.5) is of the form
\[ f(x) = \begin{cases} 
  k \int_0^x \frac{d_0}{x_0} f(y) \, dy, & 0 \leq x < x_0, \\
  \left\{ \begin{array}{ll}
  k (c_1 e^{r_+ (x-x_0)} + c_2 e^{r_- (x-x_0)}), & x_0 \leq x \leq u_2, \\
  f(u_1) + \beta_1 (x - u_1) - K_1, & x \geq u_2,
\end{array} \right. 
\end{cases} \] (4.18)
with \( a(z) = Q^{-1}(z) \). The optimal reinsurance strategy is described by
\[ a^\pi(x) = \begin{cases} 
  Q^{-1}(x), & 0 \leq x < x_0, \\
  m, & x \geq x_0.
\end{cases} \] (4.19)
It now remains to determine \( k, u_1 \) and \( u_2 \). Inspired by [4], we start with constructing a function \( \psi \) by
\[ \psi(x) = \begin{cases} 
  e^{\int_0^x} \frac{d_0}{x_0} \, dx, & 0 < x < x_0, \\
  e^{c_1 r_+ (x-x_0)} + c_2 e^{r_- (x-x_0)}, & x \geq x_0.
\end{cases} \] (4.20)
We can prove that \( \psi(x) \in \mathbb{C}^2 \) is convex and has the following useful properties
\[ \psi(x_1) := \alpha = \inf_{0 \leq x < \infty} \psi(x) > 0, \quad \psi'(x_1) = 0, \]
\[ \alpha \psi(x_1) < \rho := \psi(0) < \infty, \quad \lim_{x \to \infty} \psi(x) = \infty. \]
where
\[ x_1 = x_0 + \frac{1}{r_+ - r_-} \ln \left( \frac{r_- (r_+ + \frac{\theta_2}{m})}{r_+ (r_- + \frac{\theta_2}{m})} \right) > x_0. \] (4.21)
The inequality in (4.21) holds iff \( r_- + \frac{\theta_2}{m} < 0 \), which is confirmed by (4.1). See the proof process in Appendix A.
The function \( \psi(x) \) plays an important role in this paper. Next, we shall determine the values of \( k, u_1 \) and \( u_2 \) in different cases.

**Case A.** The case of \( 0 < K_1 < J_1(\beta_1/\rho) \), here the integral is defined by

\[
J_1(k) := \int_{u_2^k}^{u_1^k} (\beta_1 - k\psi(x))dx,
\]

where \( k \in [\beta_1/\rho, \beta_1/\alpha] \) and \( 0 \leq u_1^k \leq x_1 \leq u_2^k \) satisfy

\[
k\psi(u_1^k) = k\psi(u_2^k) = \beta_1. \tag{4.22}
\]

In view of the structure of \( \psi(x) \), we know that \( u_1^k \) is increasing, \( u_2^k \) is decreasing with respect to \( k \). In particular, if \( k = \beta_1/\rho \) then \( u_1^k = 0 \) and if \( k = \beta_1/\alpha \) then \( u_1^k = x_1 = u_2^k \). \( J_1(k) \) is a strict decreasing function of \( k \) on \([\beta_1/\rho, \beta_1/\alpha]\). Its maximum on \([\beta_1/\rho, \beta_1/\alpha]\) is \( J_1(\beta_1/\rho) > 0 \) and the minimum is \( J_1(\beta_1/\alpha) = 0 \). Therefore, there exists \( k_1^* \in (\beta_1/\rho, \beta_1/\alpha) \) such that

\[
J_1(k_1^*) = \int_{u_2^{k_1^*}}^{u_1^{k_1^*}} (\beta_1 - k_1^*\psi(x))dx = K_1. \tag{4.23}
\]

Obviously, letting \( k = k_1^*, u_1 = u_1^{k_1^*} > 0 \) and \( u_2 = u_2^{k_1^*} > 0 \), then \( f(x) \) given by (4.18) and \( a^+(x) \) given by (4.19) satisfy (4.2) and (4.4). Moreover, we write that

\[
f'(x) = \begin{cases} k_1^*\psi(x), & 0 < x \leq u_2^{k_1^*}, \\ \beta_1, & x \geq u_2^{k_1^*}. \end{cases} \tag{4.24}
\]

For \( x \geq u_2^{k_1^*} \), we calculate that

\[
\mathcal{D}f(x) - f(x) = \max_{\zeta \geq 0} \{ f(x) - (\beta_1 - \psi) \zeta - K_1 \} - f(x)
\]

\[
= \max_{\zeta \geq 0} \left\{ \int_{x-\zeta}^{x} (\beta_1 - f'(y)) dy \right\} - K_1
\]

\[
= \int_{u_1^{k_1^*}}^{u_2^{k_1^*}} (\beta_1 - k_1^*\psi(y)) dy - K_1 = 0, \tag{4.25}
\]

so (4.3) follows. In view of

\[
f'(0) = k_1^*\psi(0) = k_1^*e^{0} \frac{a_2}{\alpha} \int_{u_1^{k_1^*}}^{u_2^{k_1^*}} dx < \infty, \tag{4.26}
\]

we need to check the inequality (4.5) in following subcases.

(A1) : \( k_1^* \leq \beta_2 < k_1^* \rho \).

In this case, we know that \( f'(0) = k_1^*\psi(0) = k_1^*\rho > \beta_2 \) and \( f'(x_0) = k_1^* \leq \beta_2 \). Therefore, there exists a unique number \( \eta^{k_1^*} \in (0, \min\{x_0, u_1^{k_1^*}\}) \) such that \( f'(\eta^{k_1^*}) = k_1^*\psi(\eta^{k_1^*}) = \beta_2 \). Define the integral

\[
I_1(k_1^*) := \int_{0}^{\eta^{k_1^*}} (k_1^*\psi(x) - \beta_2) dx
\]

\[
= f(\eta^{k_1^*}) - f(0) - \beta_2\eta^{k_1^*}
\]

\[
= k_1^* \int_{0}^{\eta^{k_1^*}} e^{f(y)} \frac{a_2}{\alpha} dy - \beta_2\eta^{k_1^*} > 0. \tag{4.27}
\]

Obviously, (4.5) holds if and only if

\[
K_2 \geq I_1(k_1^*). \tag{4.28}
\]
(A2): $k_1^* > \beta_2$.
In this case, we get that $f'(0) = k_1^*\psi(0) = k_1^*\rho > \beta_2$ and $f'(x_0) = k_1^* > \beta_2$. Then there exists a unique number $\eta^k \in (x_0, u_1^k)$ such that $f'(\eta^k) = k_1^*\psi(\eta^k) = \beta_2$. Define the integral
\[
I_1(k_1^*) := \int_0^{\eta^k} (k_1^*\psi(x) - \beta_2)dx
= f(\eta^k) - f(0) - \beta_2\eta^k
= k_1^* (e_1 e^r (\eta^k - x_0) + c_2 e^r (\eta^k - x_0)) - \beta_2\eta^k > 0. \quad (4.29)
\]
Obviously, (4.5) holds if and only if
\[
K_2 \geq I_1(k_1^*). \quad (4.30)
\]
In addition, the following integral will be used later:
\[
I_2(k_1^*) := \int_{x_0}^{\eta^k} (k_1^*\psi(x) - \beta_2)dx
= f(\eta^k) - f(x_0) - \beta_2(\eta^k - x_0)
= k_1^* (e_1 e^r (\eta^k - x_0) + c_2 e^r (\eta^k - x_0)) - \beta_2(\eta^k - x_0) > 0. \quad (4.31)
\]

(A3): $\beta_2 \geq k_1^*\rho = f'(0)$.
In this case, we know that $f'(x) \leq \beta_2$ holds for all $x \geq 0$. Hence
\[
\mathcal{C}(f(0) - f(0)) = \max_{\zeta \geq 0} \{f(\zeta) - \beta_2\zeta - K_2 - f(0)
= \max_{\zeta \geq 0} \{\int_{0}^{\zeta} (f'(x) - \beta_2)dx\} - K_2
= -K_2 < 0, \quad (4.32)
\]
so (4.5) follows.

Case B. $K_1 \geq J_1(\beta_1/\rho)$.
Let us define the integral
\[
J_2(k) = \int_0^{u_2^k} (\beta_1 - k\psi(x))dx,
\]
where $k \in (0, \beta_1/\rho]$ and $u_2^k \in (x_1, \infty)$ is the unique solution to
\[
k\psi(u_2^k) = \beta_1. \quad (4.33)
\]
Then $J_2(k)$ is a strictly decreasing function of $k$ with $\lim_{k \to 0} J_2(k) = \infty$ and $J_2(\beta_1/\rho) = J_1(\beta_1/\rho) \leq K_1$. Therefore, there exists $k_2^* \in (0, \beta_1/\rho]$ such that
\[
J_2(k_2^*) = \int_0^{u_2^k} (\beta_1 - k_2^*\psi(x))dx = K_1. \quad (4.34)
\]
Obviously, letting $k = k_2^*$, $u_1 = u_1^{k_2^*} = 0$ and $u_2 = u_2^{k_2^*}$, then $f(x)$ given by (4.18) and $a^*(x)$ given by (4.19) can solve (4.2) and (4.4). Certainly, we can also show that (4.3) and (4.5) hold.

The foregoing analysis shows that the conjecture $0 < x_0 < u_2^k, i = 1, 2$ is true and all forms of $f(x)$ and $a^*(x)$ can solve Eqs. (4.2)-(4.5). In addition, $f(x)$ is increasing and continuously differentiable on $(0, \infty)$. Logically, we need to further
check that they satisfy QVI except for a single point, but we omit it here, please see a similar proof process for $\tilde{f}_1(x)$ in Appendix B.

4.2. The case with forced capital injection. Then, it remains to discuss the solution of (4.2)-(4.5) when

$$0 < K_1 < J_1(\beta_1/\rho), \quad k_1^* \rho > \beta_2 \quad \text{and} \quad 0 < K_2 < I_1(k_1^*). \quad \text{(4.35)}$$

Above analysis shows that $\mathcal{C} f(0) - f(0) \leq 0$ does not hold in the case of (4.34), $f(x)$ in (4.18) can not solve QVI now. It implies that it is no longer optimal to withdraw from the market when the surplus is null. The company should inject new capital to continue the business. Then the associated boundary conditions become $v(0) \geq 0$ and $\mathcal{C} v(0) - v(0) = 0$. Then the candidate solution $\tilde{f}(x)$ for $v(x)$ should satisfy that

$$\max_{0 \leq x \leq m} \{ \mathcal{C} \tilde{f}(x) \} = 0, \quad 0 < x < b_2, \quad \text{(4.36)}$$
$$\mathcal{C} \tilde{f}(x) - \tilde{f}(x) = 0, \quad x \geq b_2, \quad \text{(4.37)}$$
$$\tilde{f}(0) \geq 0, \quad \text{(4.38)}$$
$$\mathcal{C} \tilde{f}(0) - \tilde{f}(0) = 0, \quad \text{(4.39)}$$

with some parameter $0 < b_2 < \infty$. The solution will be given in following cases.

**Case A.** $k_1^* \leq \beta_2 < k_1^* \rho$ and $0 < K_2 < I_1(k_1^*)$, where $I_1(k_1^*)$ is given in (4.26).

Define a function $\hat{f}_1(x) = f(x + p_1^*)$ as below

$$\hat{f}_1(x) = \begin{cases} k_1^* \int_0^{x+p_1^*} e^{\int_y^{x+p_1^*} \frac{p_1^*}{\eta} dz} dz, & 0 \leq x < x_0 - p_1^*, \\ k_1^* (c_1 e^{r_+(x+p_1^*-x_0)} + c_2 e^{r_-(x+p_1^*-x_0)}), & x_0 - p_1^* \leq x \leq u_2^{k_1^*} - p_1^*, \\ f(u_1^{k_1^*}) + \beta_1(x + p_1^* - u_1^{k_1^*}) - K_1, & x \geq u_2^{k_1^*} - p_1^*, \end{cases} \quad \text{(4.40)}$$

with $p_1^* \geq 0$ and $a(z) = Q^{-1}(z)$. This means $\hat{f}_1(x)$ can be obtained by shifting $f(x)$ to the left $p_1^*$ units. Correspondingly, define a reinsurance policy as

$$\hat{a}_1^+(x) = a^+(x + p_1^*) = \left\{ \begin{array}{ll} Q^{-1}(x + p_1^*), & 0 \leq x < x_0 - p_1^*, \\ m, & x \geq x_0 - p_1^*. \end{array} \right. \quad \text{(4.41)}$$

Letting $b_1 = b_1^{k_1^*} := u_1^{k_1^*} - p_1^*$ and $b_2 = b_2^{k_1^*} := u_2^{k_1^*} - p_1^*$, we can verify easily that $\hat{f}_1(x)$ satisfies (4.35) and (4.37), the verification process is omitted here. For convenience, we write

$$\tilde{f}_1(x) = \begin{cases} k_1^* \psi(x + p_1^*), & 0 < x < b_2^{k_1^*}, \\ \beta_1, & x \geq b_2^{k_1^*}. \end{cases} \quad \text{(4.42)}$$

By repeating a similar proof process in (4.24), we can also establish that $\tilde{f}_1(x)$ satisfies (4.36), i.e.,

$$\tilde{f}_1(x) = \mathcal{C} \tilde{f}_1(x) = \hat{f}_1(b_1) + \beta_1(x - b_1) - K_1.$$ 

Then we are in a position to determine $p_1^*$ and check equality (4.38). Define a function $\phi(p)$ in $p$ as following:

$$\phi(p) := f(y^{k_1^*}) - f(p) - \beta_2(\eta^{k_1^*} - p) - K_2, \quad 0 \leq p \leq \eta^{k_1^*}.$$
It is easy to see that
\[
\phi(0) = f(\eta^k_1) - f(0) - \beta_2\eta^k_1 - \xi = I_1(k_1^*) - K_2 > 0, \quad (4.42)
\]
\[
\phi'(p) = \beta_2 - f'(p) < 0, \quad 0 \leq p \leq \eta^k_1, \quad (4.43)
\]
\[
\phi(\eta^k_1) = -K_2 < 0. \quad (4.44)
\]
Consequently, there exists a unique \( p_1^* \in (0, \eta^k_1) \) such that \( \phi(p_1^*) = 0 \), that is
\[
f(\eta^k_1) - f(p_1^*) - \beta_2(\eta^k_1 - p_1^*) - K_2 = 0, \quad (4.45)
\]
or, equivalently,
\[
\tilde{f}_1(\eta_1^*) - \tilde{f}_1(0) - \beta_2\eta_1^* - K_2 = 0, \quad (4.46)
\]
with \( \eta_1^* := \eta^k_1 - p_1^* \). Note that \( \tilde{f}_1'(\eta_1^*) = f'(\eta^k_1) = \beta_2 \). Then,
\[
\max_{\eta^k \geq \eta_1^*} \{ \tilde{f}_1(\eta) - \beta_2\eta - K_2 \} = \tilde{f}_1(\eta_1^*) - \beta_2\eta_1^* - K_2 = \tilde{f}_1(0) \quad (4.47)
\]
is true, (4.38) follows.

It proves that \( \tilde{f}_1(x) \) and \( \tilde{a}_2^*(x) \) satisfy (4.35)-(4.38) under condition (4.34). Finally, we need to further check that they also solve QVI except for a single point \( b_2^{k_1} \). Please see the detailed proof procedure in Appendix B.

Using a similar method as above, we can give the solutions in other cases. The detailed proof processes are omitted.

**Case B.** \( k_1^* > \beta_2 \) and \( I_2(k_1^*) < K_2 < I_1(k_1^*) \), where \( I_1(k_1^*) \) is given in (4.28).

Define a function \( \tilde{f}_2(x) = f(x + p_2^*) \) as below:
\[
\tilde{f}_2(x) = \begin{cases} 
 k_1^* \int_0^{x+p_2^*} e^{z} \frac{dz}{(z-x)^{\alpha} + 1} dz, & 0 \leq x \leq x_0 - p_2^*, \\
 k_1^* (c_1 e^{z-x_0} + c_2 e^{-(x+p_2^*-x_0)}), & x_0 - p_2^* \leq x \leq u_2^k - p_2^*, \\
f(u_2^k) + \beta_1 (x + p_2^* - u_2^k) - K_1, & x \geq u_2^k - p_2^*,
\end{cases} \quad (4.48)
\]
where \( a(z) = Q^{-1}(z) \) and \( p_2^* \in (0, x_0) \) is the unique solution that satisfies
\[
f(\eta^k_1) - f(p_2^*) - \beta_2(\eta^k_1 - p_2^*) - K_2 = 0, \quad (4.49)
\]
or, equivalently,
\[
\tilde{f}_2(\eta_2^*) - \tilde{f}_2(0) - \beta_2\eta_2^* - K_2 = 0 \quad (4.50)
\]
with \( \eta_2^* := \eta^k_2 - p_2^* \). Define a reinsurance policy by
\[
\tilde{a}_2^*(x) = a^*(x + p_2^*) = \begin{cases} 
 Q^{-1}(x + p_2^*), & 0 \leq x < x_0 - p_2^*, \\
 m, & x \geq x_0 - p_2^*,
\end{cases} \quad (4.51)
\]
Letting \( b_1 := u_1^k - p_2^* \) and \( b_2 := u_2^k - p_2^* \), then \( \tilde{f}_2(x) \) and \( \tilde{a}_2^*(x) \) satisfy (4.35)-(4.38) as well as QVI except for a single point \( b_2^{k_1} \).

**Case C.** \( k_1^* > \beta_2 \) and \( 0 < K_2 \leq I_2(k_1^*) \).

Define a function \( \tilde{f}_3(x) = f(x + p_3^*) \) as below:
\[
\tilde{f}_3(x) = \begin{cases} 
 k_1^* (c_1 e^{z-x_0} + c_2 e^{-(x+p_3^*-x_0)}), & 0 \leq x \leq u_2^k - p_3^*, \\
f(u_2^k) + \beta_1 (x + p_3^* - u_2^k) - K_1, & x \geq u_2^k - p_3^*,
\end{cases} \quad (4.52)
\]
where \( p_3^* \in [x_0, \eta^k_1] \) is the unique solution such that
\[
f(\eta^k_1) - f(p_3^*) - \beta_2(\eta^k_1 - p_3^*) - K_2 = 0, \quad (4.53)
\]
or, equivalently,
\[ \tilde{f}_3(\eta_3^*) - \tilde{f}_3(0) - \beta_2 \eta_2^* - K_2 = 0, \quad (4.54) \]
with \( \eta_3^* := \eta^k_i - p_3^* \). Define a reinsurance policy by
\[ \tilde{a}_3^*(x) = a_\pi^*(x + p_3^*) \equiv m, \quad x \geq 0, \quad (4.55) \]
namely, the insurer never buys reinsurance. Letting \( b_1 = b_1^k_i := u_1^k_i - p_3^* \) and \( b_2 = b_2^k_i := u_2^k_i - p_3^* \), then \( \tilde{f}_3(x) \) and \( \tilde{a}_3^*(x) \) satisfy (4.35)-(4.38) as well as QVI except for a single point \( b_2^k_i \).

### 4.3. The value function and associated optimal strategy

Based on above analysis, we can identify the explicit solutions of the value function and associated optimal strategy in this subsection.

**Theorem 4.1.** Under condition (4.1), the value function \( V(x) \) and associated optimal strategy \( \pi^* \) can be obtained in following 7 cases, which exhaust all of the possibilities:

1. \( 0 < K_1 < J_1(\beta_1/\rho), k_1^* < \beta_2 \leq k_1^* \rho \) and \( K_2 \geq I_1(k_1^*) \), where \( I_1(k_1^*) \) is given in (4.26).

   In this case, the value function \( V(x) \) is identical with
   \[
   f(x) = \begin{cases} 
   k_1 \int_0^x e^{\int_0^y \frac{a(s)}{r(s)} \, ds} \, dy, & 0 \leq x < x_0, \\
   k_1^*(c_1 e^{r^-(x-x_0)} + c_2 e^{r^-(x-x_0)}), & x_0 \leq x < u_2^k_i, \\
   f(u_2^k_i) + \beta_1(x - u_2^k_i) - K_1, & x \geq u_2^k_i,
   \end{cases}
   \]
   \[
   (4.56)
   \]
   where \( a(z) = Q^{-1}(z) \). The optimal dividend strategy \( D^{\pi^*} \) is characterized by
   \[
   \int_0^{T^{a,\pi^*}} I_{\{t: X_t^{a,\pi^*} < u_2^k_i \}} \, dD_t^{\pi^*} = 0, \\
   \rho_1^{a,\pi^*} = \inf \{ t \geq 0 : X_t^{a,\pi^*} \geq u_2^k_i \},
   \]
   \[
   \rho_i^{a,\pi^*} = \inf \{ t > \rho_{i-1}^{a,\pi^*} : X_t^{a,\pi^*} \geq u_2^k_i \}, \quad i = 2, 3, \ldots,
   \]
   \[
   \zeta_i^{a,\pi^*} = u_2^k_i - u_1^k_i, \quad \zeta_1^{a,\pi^*} = x - u_1^k_i, \quad \zeta_i^{a,\pi^*} = u_2^k_i - u_1^k_i, \quad i = 2, 3, \ldots.
   \]
   It is unprofitable to injection new capital, so
   \[
   R_t^{a,\pi^*} = 0.
   \]
   \[
   (4.58)
   \]
   The optimal reinsurance policy is characterized by (4.19). The surplus process controlled by \( \pi^* = (a^{\pi^*}, D^{\pi^*}, R^{\pi^*}) \) satisfies that
   \[
   \begin{cases} 
   X_t^{a,\pi^*} = x + \int_0^t \left( (\theta_1 - \theta_3) \mu_m + \theta_2 \lambda \mu(a^{\pi^*}(X_s^{\pi^*})) \right) ds \\
   + \int_0^t \sqrt{\lambda \nu(a^{\pi^*}(X_s^{\pi^*}))} dB_s - \sum_{i=1}^{\infty} I_{\{\varphi_i^{\pi^*} \leq t \}} \xi_i^{a,\pi^*},
   \end{cases}
   \]
   \[
   (4.59)
   \]
   (2) \( 0 < K_1 < J_1(\beta_1/\rho), \beta_2 \leq k_1^* \) and \( K_2 \geq I_1(k_1^*) \), where \( I_1(k_1^*) \) is given in (4.28).

   In this case, the value function \( V(x) \) and associated optimal strategy \( \pi^* \) take the same forms as those in (1).

(3) \( 0 < K_1 < J_1(\beta_1/\rho) \) and \( k_1^* \rho \leq \beta_2 \).

   In this case, the value function \( V(x) \) and associated optimal strategy \( \pi^* \) take
the same forms as those in (1).

(4) \[ K_1 \geq J_1(\beta_1/\rho). \]

The value function \( V(x) \) is identical with

\[
f(x) = \begin{cases} 
  k_2^2 \int_0^x e^{\int_0^y \frac{u_k}{y^{\rho+1}} dy}, & 0 \leq x < x_0, \\
  k_2^2 (c_1 e^{\theta_0} (x - x_0) + c_2 e^{\theta_0} (x - x_0)), & x_0 \leq x < u_2^k, \\
  \beta_1 x - K_1, & x \geq u_2^k, 
\end{cases}
\]

(4.60)

with \( a(z) = Q^{-1}(z) \). The optimal dividend strategy \( D^{\pi^*} \) is characterized by

\[
\begin{align*}
  \int_0^{T^{\pi^*}} I_{\{x_{T^{\pi^*}} < u_2^k\}} dD^{\pi^*}_t &= 0, \\
  \theta_1^{\pi^*} &= \inf\{t \geq 0 : X_{T^{\pi^*}}^\pi \geq u_2^k\}, \\
  \theta_i^{\pi^*} &= \infty, \quad i = 2, 3, \ldots, \\
  \zeta_i^{\pi^*} &= u_2^k, \quad \text{if } 0 \leq x \leq u_2^k, \\
  \zeta_i^{\pi^*} &= x, \quad \text{if } x \geq u_2^k, \\
  \zeta_i^{\pi^*} &= 0, \quad \text{if } i = 2, 3, \ldots. 
\end{align*}
\]

(4.61)

This implies that it is optimal to distribute all surplus \( x \) as dividend and declare bankruptcy immediately once \( x \geq u_2^k \). Whenever the surplus reaches the lower barrier \( 0 \), the company should declare bankruptcy at once. There is at most one lump sum dividend. It is unprofitable to raise new money, so

\[ R^{\pi^*}_t = 0. \]

(4.62)

The optimal reinsurance policy is characterized by (4.19). The surplus process controlled \( \pi^* = (\pi^*, D^{\pi^*}, R^{\pi^*}) \) satisfies that

\[
\begin{align*}
  X_{t}^{\pi^*} &= x + \int_0^t ((\theta_1 - \theta_2)\lambda u_0 + \theta_2 \lambda u_0 (X_{s}^{\pi^*})) ds \\
  &+ \int_0^t \sqrt{\lambda u_0 (X_{s}^{\pi^*})} dB_s - \sum_{i=1}^\infty I\{\theta_i^{\pi^*} \leq t\} \zeta_i^{\pi^*}, \\
  X_t^{\pi^*} &\leq u_2^k, \quad t > 0.
\end{align*}
\]

(4.63)

(5) \[ 0 < K_1 \leq J_1(\beta_1/\rho), k_1^* < \beta_2 \leq k_1^* \rho \quad \text{and} \quad 0 < K_2 < I_1(k_1^*), \]

where \( I_1(k_1^*) \) is given in (4.26).

The value function \( V(x) \) is identical with \( \tilde{f}_1(x) \) in (4.39). The optimal dividend strategy \( D^{\pi^*} \) is characterized by

\[
\begin{align*}
  \int_0^{T^{\pi^*}} I_{\{x_{T^{\pi^*}} < u_2^k - p_1\}} dD^{\pi^*}_t &= 0, \\
  \theta_1^{\pi^*} &= \inf\{t \geq 0 : X_{T^{\pi^*}}^\pi \geq u_2^k - p_1\}, \\
  \theta_i^{\pi^*} &= \inf\{t > \theta_{i-1}^{\pi^*} : X_{t}^{\pi^*} = u_2^k - p_1\}, \quad i = 2, 3, \ldots, \\
  \zeta_1^{\pi^*} &= u_2^k - u_1^k, \quad \text{if } 0 \leq x \leq u_2^k - p_1, \\
  \zeta_i^{\pi^*} &= x + p_1 - u_2^k, \quad \text{if } x \geq u_2^k - p_1, \\
  \zeta_i^{\pi^*} &= u_2^k - u_1^k, \quad \text{if } i = 2, 3, \ldots.
\end{align*}
\]

(4.64)

It is profitable to inject new capital when and only when the surplus is null, the bankruptcy should be avoided, and \( R^{\pi^*} \) is characterized by
The optimal reinsurance policy $\hat{\pi}_1^{\pi^*}$ is given by (4.40). The surplus process controlled by $\pi^* = (\tilde{\alpha}_1^{\pi^*}, D^{\pi^*}, R^{\pi^*})$ satisfies that

$$\begin{align*}
&\left\{ \int_0^{\infty} I_{\{t: X_t^{\pi^*} > 0\}} \, dR_t^{\pi^*} = 0, \\
&\tau_t^{\pi^*} = \inf\{t \geq 0 : X_t^{\pi^*} = 0\}, \\
&\tau_t^{\pi^*} = \inf\{t > \tau_t^{\pi^*} \leq 1 : X_t^{\pi^*} = 0\}, \\
&\eta_t^{\pi^*} = \eta_t^1 = \eta_t^1 - p_t^1, \\ &i = 1, 2, \ldots \end{align*} \tag{4.65}$$

The value function $V(x)$ is identical with $\hat{f}_2(x)$ in (4.47). The optimal dividend strategy $D^{\pi^*}$ is characterized by

$$\begin{align*}
&\left\{ \int_0^{T^{\pi^*}} I_{\{t: X_t^{\pi^*} < u_t^{k^*_1} - p_t^2\}} \, dD_t^{\pi^*} = 0, \\
&\varphi_t^{\pi^*} = \inf\{t \geq 0 : X_t^{\pi^*} \geq u_t^{k^*_1} - p_t^2\}, \\
&\varphi_t^{\pi^*} = \inf\{t > \varphi_t^{\pi^*} \geq 1 : X_t^{\pi^*} = u_t^{k^*_1} - p_t^2\}, \\
&\zeta_t^{\pi^*} = u_t^{k^*_1} - v_t^1, \\
&\zeta_t^{\pi^*} = x + p_t^2 - u_t^{k^*_1}, \\
&\zeta_t^{\pi^*} = u_t^{k^*_1} - u_t^1, \\ &i = 2, 3, \ldots \end{align*} \tag{4.67}$$

It is profitable to raise new capital when and only when the surplus is null, the bankruptcy should be avoided for ever, and $R^{\pi^*}$ is characterized by

$$\begin{align*}
&\left\{ \int_0^{\infty} I_{\{t: X_t^{\pi^*} > 0\}} \, dR_t^{\pi^*} = 0, \\
&\tau_t^{\pi^*} = \inf\{t \geq 0 : X_t^{\pi^*} = 0\}, \\
&\tau_t^{\pi^*} = \inf\{t > \tau_t^{\pi^*} \leq 1 : X_t^{\pi^*} = 0\}, \\
&\eta_t^{\pi^*} = \eta_t^2 = \eta_t^2 - p_t^2, \\ &i = 1, 2, \ldots \end{align*} \tag{4.68}$$

The optimal reinsurance policy $\hat{\pi}_2^{\pi^*}$ is given by (4.50). The surplus process controlled by $\pi^* = (\tilde{\alpha}_2^{\pi^*}, D^{\pi^*}, R^{\pi^*})$ satisfies that

$$\begin{align*}
&\left\{ X_t^{\pi^*} = x + \int_0^t \left[ \left( (\theta_1 - \theta_2) \lambda_m + \theta_2 \lambda \mu(\tilde{\alpha}_2^{\pi^*}(X_s^{\pi^*})) \right) \right] \, ds \\
&\quad + \int_0^t \sqrt{\lambda \nu(\tilde{\alpha}_2^{\pi^*}(X_s^{\pi^*}))} \, dB_s + \sum_{i=1}^{\infty} \int_{(\tau_t^{\pi^*} \leq 1)} \eta_t^{\pi^*} - \sum_{i=1}^{\infty} \int_{(\varphi_t^{\pi^*} \geq 1)} \zeta_t^{\pi^*} \\
&\quad \left. \right| 0 \leq X_t^{\pi^*} \leq u_t^{k^*_1} - p_t^2, \quad t > 0. \end{align*} \tag{4.69}$$

(7) $0 < K_1 < J_1(\beta_1/\rho), k_1^* > \beta_2$ and $0 < K_2 \leq I_2(k_1^*)$.

The value function $V(x)$ is identical with $\hat{f}_3(x)$ in (4.51). The optimal dividend
strategy $D^{π^∗}$ is characterized by
\[
\begin{align*}
    &\int_0^{T^{π^∗}_t} \mathbb{I}_{\{t : X^{π^∗}_t < u_2^i - p_3\}} dD^{π^∗}_t = 0, \\
    &\theta^{π^∗}_t = \inf\{t \geq 0 : X^{π^∗}_t \geq u_2^i - p_3\}, \\
    &\theta^{π^∗}_t = \inf\{t > \theta^{π^∗}_{i-1} : X^{π^∗}_t = u_2^i - p_3\}, \quad i = 2, 3, \ldots , \\
    &\zeta^{π^∗}_t = u_2^i - u_1^i, \quad \text{if} \quad 0 \leq x \leq u_2^i - p_3, \\
    &\zeta^{π^∗}_t = x + p_3 - u_1^i, \quad \text{if} \quad x \geq u_2^i - p_3, \\
    &\zeta^{π^∗}_t = u_2^i - u_1^i, \quad \text{if} \quad i = 2, 3, \ldots .
\end{align*}
\]  

(4.70)

Capital injection is optimal when and only when the surplus is null. Mathematically, $R^{π^∗}$ is characterized by
\[
\begin{align*}
    &\int_0^\infty \mathbb{I}_{\{t : X^{π^∗}_t > 0\}} dR^{π^∗}_t = 0, \\
    &\tau^{π^∗}_t = \inf\{t \geq 0 : X^{π^∗}_t = 0\}, \\
    &\tau^{π^∗}_t = \inf\{t > \tau^{π^∗}_{i-1} : X^{π^∗}_t = 0\}, \quad i = 2, 3, \ldots , \\
    &\eta^{π^∗}_t = \eta^{π^∗}_i = \zeta^{π^∗}_i - \zeta^{π^∗}_1, \quad i = 1, 2, \ldots .
\end{align*}
\]  

(4.71)

The optimal reinsurance policy $\hat{a}_3^{π^∗}$ is given by (4.54), namely, it is optimal choice to buy no reinsurance all the time. The surplus process controlled by $π^∗ = (a_3^0, D^{π^∗}, R^{π^∗})$ satisfies that
\[
\begin{align*}
    &X^{π^∗}_t = x + \theta_1 \lambda \nu t + \sqrt{\lambda \nu m} B_t + \sum_{i=1}^\infty \mathbb{I}_{\{\tau^{π^∗}_t \leq \zeta^{π^∗}_i\}} \eta^{π^∗}_i - \sum_{i=1}^\infty \mathbb{I}_{\{\zeta^{π^∗}_t \leq \zeta^{π^∗}_i\}} \zeta^{π^∗}_i, \\
    &0 \leq X^{π^∗}_t \leq u_2^i - p_3, \quad t > 0.
\end{align*}
\]  

(4.72)

Proof. We only give the proof of (5) as an example. The method is applicable to prove other results. We fix an arbitrary admissible strategy $π = (a^0, D^{π}, R^{π}) \in \Pi$. By the previous statement, the function $f_1(x)$ is continuously differentiable on $(0, \infty)$ and is twice continuously differentiable on $(0, b_2^k) \cup (b_2^k, \infty)$. However, for $x = b_2^k$, the continuity of $f_1(x)$ fails. In spite of this, since $\{0 \leq t \leq T^π : X^π_t = b_2^k\}$ has zero Lebesgue measure almost surely under $P$, Itô’s formula is applicable to such a $f_1(x)$ as well. Thus, we have
\[
\begin{align*}
    e^{-\delta(t \wedge T^π)} f_1(X^π_{t \wedge T^π}) - f_1(x) &= \int_0^{t \wedge T^π} e^{-\delta s} \left(\frac{\partial f_1}{\partial x}(X^π_s)\right) ds + \int_0^{t \wedge T^π} e^{-\delta s} \sqrt{\lambda \nu (a^π)} f_1(X^π_s) dB_s \\
    &\quad + \sum_{X^π_s \neq X^π_{s-}, 0 \leq s \leq t \wedge T^π} e^{-\delta s} \left(f_1(X^π_s) - f_1(X^π_{s-})\right) \\
    &\leq \int_0^{t \wedge T^π} e^{-\delta s} \sqrt{\lambda \nu (a^π)} f_1(X^π_s) dB_s \\
    &\quad + \sum_{X^π_s \neq X^π_{s-}, 0 \leq s \leq t \wedge T^π} e^{-\delta s} \left(f_1(X^π_s) - f_1(X^π_{s-})\right), 
\end{align*}
\]  

where the inequality follows from (3.5) on $(0, b_2^k) \cup (b_2^k, \infty)$. In view of the boundedness of $f_1(x) \in [k_1^1, \psi(p_1^1)]$ on $[0, \infty)$, we know that
\[
\int_0^{t \wedge T^π} e^{-\delta s} \sqrt{\lambda \nu (a^π)} f_1(X^π_s) dB_s
\]
is a martingale. Taking expectations on both sides of (4.72) yields

$$E^x \left( e^{-\delta (t \wedge T^\pi)} \tilde{f}_1(X^\pi_{t \wedge T^\pi}) \right) - \tilde{f}_1(x) \leq E^x \left( \sum_{X^\pi_\tau \neq X^\pi_0, 0 \leq s \leq t \wedge T^\pi} e^{-\delta s} \left( \tilde{f}_1(X^\pi_s) - \tilde{f}_1(X^\pi_{s-}) \right) \right). \quad (4.74)$$

Because of \( \{ s : X^\pi_s \neq X^\pi_0 \} = \{ \theta_1^\pi, \cdots \theta_i^\pi, \cdots \}, \) it has

$$\sum_{X^\pi_\tau \neq X^\pi_0, 0 \leq s \leq t \wedge T^\pi} e^{-\delta s} \left( \tilde{f}_1(X^\pi_s) - \tilde{f}_1(X^\pi_{s-}) \right)$$

$$= \sum_{i=1}^{\infty} e^{-\delta \theta_i^\pi} \left( \tilde{f}_1(X^\pi_{\theta_i^\pi}) - \tilde{f}_1(X^\pi_{\theta_i^\pi-}) \right) I_{\{ \theta_i^\pi \leq t \wedge T^\pi \}}$$

$$+ \sum_{i=1}^{\infty} e^{-\delta \tau_i^\pi} \left( \tilde{f}_1(X^\pi_{\tau_i^\pi}) - \tilde{f}_1(X^\pi_{\tau_i^\pi-}) \right) I_{\{ \tau_i^\pi \leq t \wedge T^\pi \}}. \quad (4.75)$$

Since the function \( \tilde{f}_1(x) \) satisfies (3.6) and (3.7), we have

$$e^{-\delta \theta_i^\pi} \left( \tilde{f}_1(X^\pi_{\theta_i^\pi}) - \tilde{f}_1(X^\pi_{\theta_i^\pi-}) \right) \leq -e^{-\delta \theta_i^\pi} (\beta_1 \zeta_i^\pi - K_1), \quad \text{if } \theta_i^\pi \leq t \wedge T^\pi, \quad (4.76)$$

$$e^{-\delta \tau_i^\pi} \left( \tilde{f}_1(X^\pi_{\tau_i^\pi}) - \tilde{f}_1(X^\pi_{\tau_i^\pi-}) \right) \leq e^{-\delta \tau_i^\pi} (\beta_2 \eta_i^\pi + K_2), \quad \text{if } \tau_i^\pi \leq t \wedge T^\pi. \quad (4.77)$$

(4.73)-(4.76) lead to

$$E^x \left[ e^{-\delta (t \wedge T^\pi)} \tilde{f}_1(X^\pi_{t \wedge T^\pi}) \right] - \tilde{f}_1(x)$$

$$\leq -E^x \left( \sum_{i=1}^{\infty} e^{-\delta \theta_i^\pi} (\beta_1 \zeta_i^\pi - K_1) I_{\{ \theta_i^\pi \leq t \wedge T^\pi \}} \right)$$

$$- \sum_{i=1}^{\infty} e^{-\delta \tau_i^\pi} (\beta_2 \eta_i^\pi + K_2) I_{\{ \tau_i^\pi \leq t \wedge T^\pi \}} \right). \quad (4.78)$$

The first term of the left side is positive due to \( \tilde{f}_1(x) = f(x + p^\pi_1) > 0. \) Thus

$$- \tilde{f}_1(x) \leq -E^x \left( \sum_{i=1}^{\infty} e^{-\delta \theta_i^\pi} (\beta_1 \zeta_i^\pi - K_1) I_{\{ \theta_i^\pi \leq t \wedge T^\pi \}} \right)$$

$$- \sum_{i=1}^{\infty} e^{-\delta \tau_i^\pi} (\beta_2 \eta_i^\pi + K_2) I_{\{ \tau_i^\pi \leq t \wedge T^\pi \}} \right). \quad (4.79)$$

Letting \( t \to \infty \) in (4.78) and rearranging the inequality, we get

$$\tilde{f}_1(x) \geq E^x \left( \sum_{i=1}^{\infty} e^{-\delta \theta_i^\pi} (\beta_1 \zeta_i^\pi - K_1) I_{\{ \theta_i^\pi \leq T^\pi \}} - \sum_{i=1}^{\infty} e^{-\delta \tau_i^\pi} (\beta_2 \eta_i^\pi + K_2) I_{\{ \tau_i^\pi \leq T^\pi \}} \right)$$

$$= V(x; \pi), \quad (4.80)$$

which indicates that \( \tilde{f}_1(x) \geq V(x). \) When the strategy \( \pi^* \in \Pi \) described in case (5) is applied, all inequalities become equalities, that is

$$\tilde{f}_1(x) = V(x; \pi^*),$$

which implies \( \tilde{f}_1(x) \leq V(x). \) In summary, we obtain \( \tilde{f}_1(x) = V(x) = V(x; \pi^*). \quad \Box \)
5. The case of $\theta_2 \in \left[ \frac{2m\mu_m}{\nu_m} - \theta_1, \infty \right)$. In this section, let’s consider the other case with

$$\theta_2 \geq \frac{2m\mu_m}{\nu_m} \theta_1. \quad (5.1)$$

Similar to analysis in Section 4, we need to consider two suboptimal problems each corresponding to different boundary conditions. In particular, the proofs resemble those of Section 4, so we present the main results without giving the verification processes.

5.1. The case without refinancing. Similar to Subsection 4.1, we consider the first case with $v(0) = 0$ and $\mathcal{C}v(0) - v(0) \leq 0$, which means that it is optimal to get out of the business whenever the surplus is null. With the same argument, the candidate solution $g(x)$ for $v(x)$ should satisfy that

$$\max_{0 \leq o \leq m} \{ \mathcal{A}^o g(x) \} = 0, \quad 0 < x < h_2, \quad (5.2)$$

$$\mathcal{D} g(x) - g(x) = 0, \quad x \geq h_2, \quad (5.3)$$

$$g(0) = 0, \quad (5.4)$$

$$\mathcal{C} g(0) - g(0) \leq 0, \quad (5.5)$$

with some parameter $0 < h_2 < \infty$.

In the case of (5.1), we expect that the optimal reinsurance strategy to take full retention, that is $a(x) \equiv m$ for all $x \geq 0$. Then, solving (5.2) with boundary condition (5.4) yields

$$g(x) = l(e^{r+x} - e^{r-x}), \quad 0 \leq x \leq h_2. \quad (5.6)$$

To prove that $a(x) \equiv m$ is optimal, referring to (4.6), it suffices to verify that

$$\chi(x) := -\theta_2 \frac{g'(x)}{g''(x)} = -\theta_2 \frac{r+e^{r+x} - r_e^x}{r^2} e^{r+x} - \frac{r^2 e^{r-x}}{r^2} \geq m, \quad (5.7)$$

for all $0 < x < h_2$. In fact, under condition (5.1), it is easy to verify that $\chi(0) \geq m$ and $\chi'(x) > 0$, then (5.7) follows.

For $x \geq h_2$, there exists some level $h_1 \in [0, h_2)$ such that

$$g(x) = \mathcal{D} g(x) = g(h_1) + \beta_1(x - h_1) - K_1. \quad (5.8)$$

Then the suggested solution to (5.2)-(5.5) takes the form as

$$g(x) = \begin{cases} 
    l(e^{r+x} - e^{r-x}), & 0 \leq x \leq h_2, \\
    g(h_1) + \beta_1(x - h_1) - K_1, & x \geq h_2. 
\end{cases} \quad (5.9)$$

The optimal reinsurance strategy is $a^*(x) \equiv m$. Define a function as

$$\varphi(x) := r_+ e^{r+x} - r_- e^{r-x}, \quad x \geq 0. \quad (5.10)$$

Note that the convex function $\varphi(x) \in \mathbb{C}^2$ satisfies

$$\varphi(x_2) = r_+ \left( \frac{r_e - r}{r_e} \right)^{2r_+} - r_- \left( \frac{r_e - r}{r_e} \right)^{2r_-} = \inf_{0 < x < \infty} \varphi(x) := \vartheta > 0, \quad (5.11)$$

$$\varphi'(x_2) = 0, \quad \varphi(x_2) < \varphi(0) = r_+ - r_- := \gamma < \infty, \quad \lim_{x \to \infty} \varphi(x) = \infty,$$

where

$$x_2 = \frac{1}{r_+ - r_-} \ln \left( \frac{r_-}{r_+} \right)^2 > 0. \quad (5.11)$$
Next, we shall determine the values of \( l, h_1 \) and \( h_2 \) in all cases.

**Case A.** If \( 0 < K_1 < H_1(\beta_1/\gamma) \), here the integral is defined by

\[
H_1(l) := \int_{h_1^1}^{h_2^1} (\beta_1 - l\varphi(x))dx,
\]

where \( l \in [\beta_1/\gamma, \beta_1/\vartheta] \) and \( 0 < h_1^1 < x_2 < h_2^1 \) satisfy that

\[
l\varphi(h_1^1) = l\varphi(h_2^1) = \beta_1.
\]  \hspace{1cm} (5.12)

There exists \( l_1^* \in (\beta_1/\gamma, \beta_1/\vartheta) \) such that

\[
H_1(l_1^*) = \int_{h_1^1}^{h_2^1} (\beta_1 - l_1^*\varphi(x))dx = K_1.
\]

Obviously, letting \( l = l_1^*, h_1 = h_1^1 > 0 \) and \( h_2 = h_2^1 > 0 \), then \( g(x) \) given by (5.9) and \( a(x) \equiv m \) solve (5.2)-(5.4). In view of

\[
g'(0) = l_1^*\varphi(0) = l_1^*(r_+ - r_-) \leq \infty,
\]  \hspace{1cm} (5.13)

we need to check inequality (5.5) in two different subcases.

(A1) : \( g'(0) = l_1^*\varphi(0) = l_1^*\gamma > \beta_2 \).

In this case, there exists a unique number \( \eta_1^* \in (0, h_1^1) \) such that \( g'(\eta_1^*) = l_1^*\varphi(\eta_1^*) = \beta_2 \). Define the integral

\[
G(l_1^*) := \int_{\eta_1^*}^{h_1^1} (l_1^*\varphi(x) - \beta_2)dx = g(\eta_1^*) - g(0) - \beta_2\eta_1^* > 0.
\]  \hspace{1cm} (5.14)

Obviously, (5.5) holds if and only if

\[
K_2 \geq G(l_1^*).
\]  \hspace{1cm} (5.15)

The case of \( K_2 < G(l_1^*) \) will be considered in Subsection 5.2.

(A2) : \( g'(0) = l_1^*\varphi(0) = l_1^*\gamma \leq \beta_2 \).

In this case, (5.5) holds naturally.

**Case B.** If \( K_1 \geq H_1(\beta_1/\gamma) \), let us define the integral

\[
H_2(l) = \int_0^{h_2^1} (\beta_1 - l\varphi(x))dx,
\]  \hspace{1cm} (5.16)

where \( l \in (0, \beta_1/\gamma] \) and \( h_2^1 > x_2 \) is the unique solution to

\[
l\varphi(h_2^1) = \beta_1.
\]  \hspace{1cm} (5.17)

There exists a unique number \( l_2^* \in (0, \beta_1/\gamma] \) such that

\[
H_2(l_2^*) = \int_0^{h_2^1} (\beta_1 - l_2^*\varphi(x))dx = K_1.
\]  \hspace{1cm} (5.18)

Then (5.5) follows automatically. Letting \( l = l_2^*, h_1 = h_1^1 = 0 \) and \( h_2 = h_2^1 \), then \( g(x) \) given by (5.9) and \( a^*(x) \equiv m \) can solve (5.2)-(5.5) as well as QVI except for a single point \( h_2^1 \).
5.2. The case with forced refinancing. Then, it remains to determine the solution of (5.2)-(5.5) when

$$0 < K_1 < H_1(\beta_1/\gamma), \ l_1^* \gamma > \beta_2 \quad \text{and} \quad K_2 < G(l_1^*). \quad (5.19)$$

Now $g(x)$ in (5.9) can no longer solve QVI. It suggests that it is optimal to collect new capital from market whenever the surplus is null. Then the associated boundary conditions are $v(0) \geq 0$ and $\Phi v(0) - v(0) = 0$. We conjecture that there is a candidate solution $\tilde{g}(x)$ for $v(x)$ that satisfies

$$\max_{0 \leq s \leq m} \{ \partial_s \tilde{g}(x) \} = 0, \quad 0 < x < d_2, \quad (5.20)$$

$$\mathcal{D}\tilde{g}(x) - \tilde{g}(x) = 0, \quad x \geq d_2, \quad (5.21)$$

$$\tilde{g}(0) \geq 0, \quad (5.22)$$

$$\mathcal{C}\tilde{g}(0) - \tilde{g}(0) = 0, \quad (5.23)$$

with some parameter $0 < d_2 < \infty$. Some calculations indicate that $\tilde{g}(x) = g(x + q^*)$ takes the following form

$$\tilde{g}(x) = \begin{cases} l_1^*(e^{r_s(x+q^*)} - e^{r_s(x+q^*)}), & 0 \leq x \leq h_2^* - q^*, \\ g(h_1^*) + \beta_1(x + q^* - h_1^*) - K_1, & x \geq h_2^* - q^*, \end{cases} \quad (5.24)$$

where $q^* \in (0, \eta_l^*)$ is the unique solution of the equation $g(x^*) - g(q^*) - \beta_2(x^* - q^*) - K_2 = 0$. Correspondingly, $a(x) \equiv m$ is optimal. Letting $d_1 = d_1^* := h_1^* - q^*$, $d_2 = d_2^* := h_2^* - q^*$ and $\eta^* = h_1^* - q^*$ then $\tilde{g}(x)$ satisfies (5.20)-(5.23) as well as QVI except for a single point $d_2^*$.

5.3. The solutions of value function and optimal strategy. Finally, we will identify the explicit solutions to the value function and associated optimal strategy in this subsection.

**Theorem 5.1.** Under condition (5.1), the value function $V(x)$ and associated optimal strategy $\pi^*$ can be established in following 4 cases, which explore all of the possibilities:

1. $0 < K_1 < H_1(\beta_1/\gamma), l_1^* \gamma > \beta_2$ and $K_2 \geq G(l_1^*)$.

   In this case, the value function $V(x)$ is identical with

   $$g(x) = \begin{cases} l_1^*(e^{r_s-x} - e^{-r_s-x}), & 0 \leq x \leq h_2^*, \\ g(h_1^*) + \beta_1(x - h_1^*) - K_1, & x \geq h_2^*. \end{cases} \quad (5.25)$$

   The optimal dividend strategy $D^{\pi^*}$ is characterized by

   $$\int_0^{\tau^{\pi^*}} I_{\{t, X_t^\pi < h_2\}} \, dD^{\pi^*} = 0,$$

   $$\tau^{\pi^*} = \inf\{t \geq 0 : X_t^\pi \geq h_2\},$$

   $$\vartheta_i^{\pi^*} = \inf\{t \geq \tau^{\pi^*_i} : X_t^\pi = h_1\}, \quad i = 2, 3, \cdots,$$

   $$h_i = h_2^* - h_1^*,$$

   $$\varsigma_i^* = \begin{cases} x - h_1^*, & 0 \leq x \leq h_2^*, \\ h_2^* - h_1^*, & x \geq h_2^*, \end{cases} \quad (5.26)$$
Moreover, $R^*_t \equiv 0$ and $a^*_m(x) \equiv m$ for all $x \geq 0$. Then the surplus process controlled by $\pi^* = (a^*, D^*, R^*)$ follows that

\[
\begin{align*}
X^*_t &= x + \theta_1 \lambda \mu_m t + \sqrt{\lambda \nu_m} B_t - \sum_{i=1}^{\infty} I_{(a^*_m \leq i)} \xi^*_i, \\
X^*_t &\geq h_2^i, \quad t > 0.
\end{align*}
\]  

(2) $0 < K_1 < H_1(\beta_1/\gamma)$ and $l_1^* \gamma < \beta_2$.

In this case, the value function $V(x)$ and associated optimal strategy $\pi^*$ take the same forms as those in (1).

(3) $K_1 \geq H_1(\beta_1/\gamma)$.

In this case, the value function $V(x)$ is identical with

\[
g(x) = \begin{cases} 
I_2(e^x - e^{-x} - x^2), & 0 \leq x < h_2^i \\
\beta_1 x - K_1, & x \geq h_2^i.
\end{cases}
\]  

(5.28)

The optimal dividend strategy $D^*$ is characterized by

\[
\begin{align*}
\left\{ \begin{array}{l}
\int_0^T \mathbb{I}_{\{t : X^*_t < h_2^i \}} dD^*_t = 0, \\
\varphi^*_i = \inf \{t \geq 0 : X^*_t \geq h_2^i \}, \\
\varphi^*_i = \infty, & i = 2, 3, \ldots, \\
\xi^*_i = h_2^i, & \text{if } 0 \leq x < h_2^i, \\
\xi^*_i = x, & \text{if } x \geq h_2^i, \\
\xi^*_i = 0, & i = 2, 3, \ldots.
\end{array} \right.
\]  

(5.29)

Moreover, $R^*_t \equiv 0$ and $a^*_m(x) \equiv m$. Then the surplus process controlled by $\pi^* = (a^*, D^*, R^*)$ follows that

\[
\begin{align*}
X^*_t &= x + \theta_1 \lambda \mu_m t + \sqrt{\lambda \nu_m} B_t - \sum_{i=1}^{\infty} I_{(a^*_m \leq i)} \xi^*_i, \\
X^*_t &\leq h_2^i, \quad t > 0.
\end{align*}
\]  

(5.30)

(4) $0 < K_1 < H_1(\beta_1/\gamma), l_1^* \gamma < \beta_2$ and $K_2 < G(l_1^*)$.

In this case, the value function $V(x)$ is identical with $\tilde{g}(x)$ in (5.24). The optimal dividend strategy $D^*$ is characterized by

\[
\begin{align*}
\left\{ \begin{array}{l}
\int_0^T \mathbb{I}_{\{t : X^*_t < h_2^i - q^* \}} dD^*_t = 0, \\
\varphi^*_i = \inf \{t \geq 0 : X^*_t \geq h_2^i - q^* \}, \\
\varphi^*_i = \inf \{t > \varphi^*_1 : X^*_t = h_2^i - q^* \}, & i = 2, 3, \ldots, \\
\varphi^*_i = h_2^i - h_1^i, & \text{if } 0 \leq x \leq h_2^i - q^*, \\
\xi^*_i = x + q^* - h_2^i, & \text{if } x \geq h_2^i - q^*, \\
\xi^*_i = h_2^i - h_1^i, & i = 2, 3, \ldots.
\end{array} \right.
\]  

(5.31)

The optimal capital injection strategy $R^*$ is depicted by

\[
\begin{align*}
\left\{ \begin{array}{l}
\int_0^\infty \mathbb{I}_{\{t : X^*_t > 0 \}} dR^*_t = 0, \\
\tau^*_1 = \inf \{t \geq 0 : X^*_t = 0 \}, \\
\tau^*_i = \inf \{t > \tau^*_1 : X^*_t = 0 \}, & i = 2, 3, \ldots, \\
\eta^*_i = \eta^*_1 - q^*, & i = 1, 2, \ldots.
\end{array} \right.
\]  

(5.32)
The optimal reinsurance policy is to buy no reinsurance, i.e., \( a^\pi(x) \equiv m \) for all \( x \geq 0 \). Then the surplus process controlled by \( \pi^* = (\tilde{a}^\pi, D^\pi, R^\pi) \) follows that

\[
\begin{cases}
X_t^{\pi^*} = x + \theta_1 \lambda m t + \sqrt{\lambda \delta m} B_t + \sum_{i=1}^{\infty} I(\tau_i^{\pi^*} \leq t) \eta_i^{\pi^*} - \sum_{i=1}^{\infty} I(\varrho_i^{\pi^*} \leq t) \xi_i^{\pi^*}, \\
0 \leq X_t^{\pi^*} \leq h_{l_1}^{m^*} - q^*, \quad t > 0.
\end{cases}
\]  

(5.33)

Proof. The proof procedure is similar to Appendix B. So it is omitted here.

Remark 2. \([12]\) has studied a similar optimization problem when the insurer and reinsurer used same safety loading, i.e., \( \theta_1 = \theta_2 \). They solved the problem by using the stochastic control method in a diffusion model. In this paper, we generalize the model by assuming that the reinsurer uses a higher safety loading, i.e., \( \theta_2 > \theta_1 \). In other words, the reinsurance is the “expensive” style, which incurs the transaction cost of reinsurance contract. All results in \([12]\) can be obtained by letting \( \theta_2 \rightarrow \theta_1 \) in this paper. Obviously, the assumption of “expensive” reinsurance seems more reasonable and the problem becomes more complex.

\([10]\) has investigated the optimal dividend and capital injection problem with “expensive” excess-of-loss reinsurance in a diffusion model. They solved the optimal control problem and identify the value function and associated optimal strategy. Comparing with the work of \([10]\), we added the fixed costs in the model, which lead to that dividends and capital injections can not occur continuously. The existences of fixed costs generate more difficult impulse control problem. To manage its risks and explore profit opportunities, the company needs to determine the times and amounts of dividend payments and capital injections, respectively. Certainly, all results in \([10]\) can be obtained by letting \( K_1, K_2 \rightarrow 0 \) and \( \beta_1 \rightarrow 1 \) in this paper.

6. Conclusion. In this paper, we consider a combined optimal dividend, capital injection and excess-of-loss reinsurance problem in the framework of diffusion model. To reflect the reality more, the fixed and proportional transaction costs generated by the control processes are included in the risk model and the reinsurance is assumed to be “expensive”. Using methodologies from stochastic control theory, we derive the value function and associated optimal strategy. We find that the retention level of direct insurer is increasing with respect to the surplus \( x \) and takes value on the interval \( (w^{-1} \left( \frac{\theta_2}{\theta_1} \mu_m \right), m] \). The insurer should take all risks if the reinsurance cost is too expensive, not choose refinancing if its costs are too high. The insurance company does not distribute dividends until the surplus exceeds the switch level and the excess is paid out immediately as dividends. In brief, the optimal strategies depend on the relationships among the parameters.

Appendix

Appendix A. The proof of the inequality \( r_- + \frac{\theta_2}{m} < 0 \) in the case of (4.1).

Proof. Recall the definition of \( r_- \). We rewrite the inequality \( r_- + \frac{\theta_2}{m} < 0 \) as

\[
\frac{1}{\lambda \mu_m} \left( -\lambda \theta_1 \mu_m - \sqrt{(\lambda \theta_1 \mu_m)^2 + 2 \lambda \delta m} \right) + \frac{\theta_2}{m} < 0,
\]

(A.1)
or, equivalently,

\[
\sqrt{(\lambda \theta_1 \mu_m)^2 + 2 \lambda \delta m} > \frac{\lambda \theta_2 \mu_m}{m} - \lambda \theta_1 \mu_m.
\]

(A.2)
(i) If \( \theta_2 \leq \frac{m\mu_m}{\nu_m} \theta_1 \), then \( \lambda \theta_2 \nu_m - \lambda \theta_1 \mu_m \leq 0 \) follows. In this case, (A.2) holds automatically.

(ii) If \( \frac{m\mu_m}{\nu_m} \theta_1 < \theta_2 < \frac{2m\mu_m}{\nu_m} \theta_1 \), we calculate that

\[
\sqrt{(\lambda \theta_1 \mu_m)^2 + 2 \lambda \theta_1 \nu_m} > \frac{\lambda \theta_2 \nu_m}{\nu_m} - \lambda \theta_1 \mu_m.
\]  

(A.3)

Squaring both sides of (A.3) and rearranging the expression yield

\[
2 \lambda \theta_1 \nu_m > \frac{\theta_2 (\lambda \nu_m)^2}{m^2} \left( \theta_2 - \frac{2m\mu_m}{\nu_m} \theta_1 \right).
\]  

(A.4)

Obviously, (A.4) holds since the right-hand side of the inequality is negative. \( \square \)

Appendix B. The verification process of QVI.

Proof. We only prove that \( \tilde{f}_1(x) \) and \( \tilde{\alpha}_1^* \) satisfy QVI as an example, even though the method is applicable to other cases. The outline of the proof is given as following.

- **Step 1.** To show max_{0 \leq a \leq m} \{ \mathcal{A} f_1(x) \} \leq 0 on \( (0, b_2^*) \cup (b_2^*, \infty) \).

(i) If \( 0 \leq x < x_0 - \alpha_1^* \), by construction, \( \tilde{f}_1(x) \) and \( \tilde{\alpha}_1^* \) satisfy (4.25). That is max_{0 \leq a \leq m} \{ \mathcal{A} f_1(x) \} = \mathcal{A} \tilde{\alpha}_1^* \tilde{f}_1(x) = 0.

(ii) If \( x_0 - \alpha_1^* \leq x < x_1 - \alpha_1^* \), it is easy to prove that \( \tilde{f}_1(x) > 0, \tilde{f}_1''(x) < 0 \) and \( m \tilde{f}_1''(x) + \theta_2 \tilde{f}_1''(x) \geq 0 \). Then, for each \( a \in [0, m] \), it has

\[
\frac{d}{da} \left( \mathcal{A} \tilde{f}_1(x) \right) = \lambda F(a)(a \tilde{f}_1''(x) + \theta_2 \tilde{f}_1''(x)) \geq 0.
\]  

(B.1)

Consequently, max_{0 \leq a \leq m} \{ \mathcal{A} f_1(x) \} = \mathcal{A} m \tilde{f}_1(x) = \mathcal{A} \tilde{\alpha}_1^* \tilde{f}_1(x) = 0.

(iii) If \( x_1 - \alpha_1^* \leq x < b_2^* \), it has \( \tilde{f}_1(x) > 0 \) and \( \tilde{f}_1''(x) > 0 \). From the structure of \( \mathcal{A} f_1(x) \), we know it reaches the maximum at \( a = m \), i.e., max_{0 \leq a \leq m} \{ \mathcal{A} f_1(x) \} = \mathcal{A} m \tilde{f}_1(x) = \mathcal{A} \tilde{\alpha}_1^* \tilde{f}_1(x) = 0.

(iv) If \( x > b_2^* \), it has \( \tilde{f}_1(x) \geq \tilde{f}_1(b_2^*) \), \( \tilde{f}_1(x) = \beta_1 \) and \( \tilde{f}_1'(x) = 0 \). Therefore, for arbitrary \( a \geq 0 \), we derive that

\[
\mathcal{A} \tilde{f}_1(x) = \frac{1}{2} \lambda \nu m (a \tilde{f}_1''(x) + ((\theta_1 - \theta_2) \lambda \mu_m + \theta_2 \lambda \mu(a)) \tilde{f}_1'(x) - \delta \tilde{f}_1(x)
\]

\[
= ((\theta_1 - \theta_2) \lambda \mu_m + \theta_2 \lambda \mu(a)) \tilde{f}_1'(x) - \delta \tilde{f}_1(x)
\]

\[
\leq \theta_1 \lambda \mu_m \tilde{f}_1'(b_2^*) - \delta \tilde{f}_1(b_2^*).
\]  

(B.2)

Note that

\[
\max_{0 \leq a \leq m} \{ \mathcal{A} \tilde{f}_1(b_2^* -) \} = \mathcal{A} m \tilde{f}_1(b_2^* -)
\]

\[
= \frac{1}{2} \lambda \nu m \tilde{f}_1''(b_2^* -) + \theta_1 \lambda \mu_m \tilde{f}_1'(b_2^* -) - \delta \tilde{f}_1(b_2^* -)
\]

\[
= 0.
\]  

(B.3)

Because of \( \tilde{f}_1''(b_2^* -) > 0 \), we have

\[
\theta_1 \lambda \mu_m \tilde{f}_1'(b_2^* -) - \delta \tilde{f}_1(b_2^* -) < 0.
\]  

(B.4)

Then, together with the continuity of \( \tilde{f}_1 \) at \( b_2^* \), (B.3) and (B.4) imply that max_{0 \leq a \leq m} \{ \mathcal{A} \tilde{f}_1(x) \} < 0 holds for \( x > b_2^* \).
• **Step 2.** To show $\mathcal{D}\tilde{f}_1(x) - \tilde{f}_1(x) \leq 0$ on $(0, b_1^{k_1^+}) \cup (b_2^{k_1^+}, \infty)$. We write that
\[
\mathcal{D}\tilde{f}_1(x) - \tilde{f}_1(x) = \max_{\zeta \geq 0} \left\{ \int_{x-\zeta}^{x} (\beta_1 - \tilde{f}_1'(y))dy \right\} - K_1.
\] (B.5)

(i) If $x \leq b_1^{k_1^+}$, it has
\[
\mathcal{D}\tilde{f}_1(x) - \tilde{f}_1(x) = \max_{\zeta \geq 0} \left\{ \int_{x-\zeta}^{x} (\beta_1 - \tilde{f}_1'(y))dy \right\} - K_1 = \int_{x-0}^{x} (\beta_1 - \tilde{f}_1'(y))dy - K_1 = -K_1 < 0.
\] (B.6)

(ii) If $b_1^{k_1^+} < x \leq b_2^{k_1^+}$, it has
\[
\mathcal{D}\tilde{f}_1(x) - \tilde{f}_1(x) = \max_{\zeta \geq 0} \left\{ \int_{x-\zeta}^{x} (\beta_1 - \tilde{f}_1'(y))dy \right\} - K_1 = \int_{x-(x-b_1^{k_1^+})}^{x} (\beta_1 - \tilde{f}_1'(y))dy - K_1 < 0.
\] (B.7)

(iii) If $x > b_2^{k_1^+}$, it has
\[
\mathcal{D}\tilde{f}_1(x) - \tilde{f}_1(x) = \max_{\zeta > 0} \left\{ \int_{x-\zeta}^{x} (\beta_1 - \tilde{f}_1'(y))dy \right\} - K_1 = \int_{b_2^{k_1^+}}^{b_2^{k_1^+}} (\beta_1 - \tilde{f}_1'(y))dy - K_1 = 0.
\] (B.8)

• **Step 3.** To show $\mathcal{C}\tilde{f}_1(x) - \tilde{f}_1(x) \leq 0$ on $(0, b_1^{k_1^+}) \cup (b_2^{k_1^+}, \infty)$. We write that
\[
\mathcal{C}\tilde{f}_1(x) - \tilde{f}_1(x) = \max_{\eta \geq 0} \left\{ \tilde{f}_1(x + \eta) - \beta_2 \eta - K_2 \right\} - \tilde{f}_1(x)
\]
\[
= \max_{\eta \geq 0} \left\{ \int_{x}^{x+\eta} (\tilde{f}_1'(y) - \beta_2)dy \right\} - K_2.
\]

(i) If $0 < x < \eta_1^+$, it has $\tilde{f}_1'(x) - \beta_2 > 0$. Conversely, $\tilde{f}_1'(x) - \beta_2 \leq 0$ holds for $x \geq \eta_1^+$. Thus,
\[
\mathcal{C}\tilde{f}_1(x) - \tilde{f}_1(x) = \max_{\eta \geq 0} \left\{ \int_{x}^{x+\eta} (\tilde{f}_1'(y) - \beta_2)dy \right\} - K_2
\]
\[
\leq \int_{0}^{\eta_1^+} (\tilde{f}_1'(y) - \beta_2)dy - K_2 = 0.
\] (B.9)

(ii) If $x \in [\eta_1^+, \infty)$, the inequality $\tilde{f}_1'(x) - \beta_2 \leq 0$ holds true, then
\[
\mathcal{C}\tilde{f}_1(x) - \tilde{f}_1(x) = \max_{\eta \geq 0} \left\{ \int_{x}^{x+\eta} (\tilde{f}_1'(y) - \beta_2)dy \right\} - K_2 = -K_2 < 0.
\] (B.10)

• **Step 4.** To show $\max_{0 \leq a \leq m} (\mathcal{A}^a \tilde{f}_1(x)) (\mathcal{D} \tilde{f}_1(x) - \tilde{f}_1(x)) (\mathcal{C} \tilde{f}_1(x) - \tilde{f}_1(x)) = 0$.

(i) If $x = 0$, by (4.38), $\mathcal{C} \tilde{f}_1(x) - \tilde{f}_1(x) = 0$. 
(ii) If \( 0 < x < b^1 \), by (4.35), \( \max_{0 \leq a \leq m} \{a \mathcal{A} \hat{f}_1(x)\} = 0 \).

(iii) If \( x \geq b^1 \), by (4.36), \( \mathcal{A} \hat{f}_1(x) = 0 \).

Step 5. To show \( \max \{\mathcal{C} \hat{f}_1(0) - \hat{f}_1(0), -\hat{f}_1(0)\} = 0 \).

\( \mathcal{C} \hat{f}_1(0) - \hat{f}_1(0) = 0 \) is true by (4.38). In addition, the inequality \( \hat{f}_1(0) = f(p^*_1) > f(0) = 0 \) holds since \( f(x) \) is an increasing function.

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