MODULAR ESTIMATES IN ORLICZ SPACES
AND HAMMERSTEIN OPERATOR EQUATIONS

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Certain inequalities between the values of the modular and the norm in the Orlicz spaces are established. These inequalities are applied then to the theory of solvability of nonlinear integral equations of Hammerstein type.

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The aim of the paper is to establish certain inequalities between the values of the modular and the norm in Orlicz spaces. These inequalities turn out to be of use in the theory of nonlinear integral equations.

1. Let us recall the basic definitions of the theory of Orlicz spaces (see, for example, [1]). Henceforth $(\Omega, \mathcal{A}, \mu)$ is the triple consisting of a set $\Omega$, $\sigma$-algebra $\mathcal{A}$ of its subsets and a $\sigma$-additive measure $\mu$ defined on $\mathcal{A}$. It is assumed that $\mu$ is continuous on $\Omega$ (that is any set of positive measure can be divided into two sets of equal measures) and is finite: $\mu(\Omega) < \infty$.

Let $M : [0, +\infty) \to [0, +\infty)$ be an $N$-function that is a convex function satisfying the conditions
\[
\lim_{u \to 0} \frac{M(u)}{u} = 0, \quad \lim_{u \to \infty} \frac{M(u)}{u} = \infty.
\]
On the set of measurable functions on $\Omega$ one can consider the functional
\[
\mathcal{M}(x) := \int_\Omega M(|x(\omega)|) \, d\mu(\omega).
\]
This functional is conventionally called the modular generated by the $N$-function $M(\cdot)$. The set $O_M$ of the functions $x(\cdot)$ that are measurable on $\Omega$ and satisfy the condition $\mathcal{M}(x) < \infty$ is called the Orlicz class. We shall denote by $L_M$ the Orlicz space that is the set consisting of the functions $x(\cdot)$ that are measurable on $\Omega$ and such that for any $x(\cdot)$ there exists $\lambda > 0$ such that the following inequality holds
\[
\mathcal{M}\left(\frac{x}{\lambda}\right) < \infty.
\]
Orlicz space is a Banach space with respect to each of the following two norms
\[
\|x\|_\mathcal{L} = \inf \left\{ \lambda : \mathcal{M}\left(\frac{x}{\lambda}\right) \leq 1 \right\}, \quad \|x\|_\mathcal{A} = \inf_{\lambda > 0} \frac{1 + \mathcal{M}(\lambda x)}{\lambda}.
\]
The first of these norms is commonly called the Luxemburg norm and the second one — the Orlicz norm (in fact both these norms have been introduced by Orlicz). These norms are equivalent:

\[ \|x\|_L \leq \|x\|_A \leq 2\|x\|_L. \]  

(1)

The norm calculation in Orlicz spaces even for the most simple functions is a nontrivial problem. We present here the important formulae for the norms of the characteristic functions \( \chi_D(\cdot) \) (\( D \in \mathfrak{A} \)):

\[ \|\chi_D\|_L = \frac{1}{M^{-1}\left(\frac{1}{\mu(D)}\right)}, \quad \|\chi_D\|_A = \mu(D)\left(M^*(\cdot)^{-1}\left(\frac{1}{\mu(D)}\right)\right), \]

where \( M^*(\cdot) \) is the \( N \)-function dual to the \( N \)-function \( M(\cdot) \), that is the function given by the equality \( M^*(u) := \sup \{uv - M(v) : 0 \leq v < \infty\} \).

An important role in the theory of Orlicz spaces is played by the subspace \( L^0_M \) of measurable functions \( x(\cdot) \) on \( \Omega \) such that for each \( \lambda > 0 \) the following inequality takes place

\[ M\left(\frac{x}{\lambda}\right) < \infty. \]

This subspace coincides with the closure of the set of bounded functions in Orlicz space. The following embeddings are true

\[ L^0_M \subseteq O_M \subseteq L_M. \]

Under the presupposed assumptions on \( (\Omega, \mathfrak{A}, \mu) \) each of the equalities \( L^0_M = O_M \) and \( O_M = L_M \) is equivalent to the \( \Delta_2 \)-condition:

\[ \lim_{u \to \infty} \frac{M(2u)}{M(u)} < \infty. \]

In the general case the following embeddings take place

\[ \{x \in L_M : d(x, L^0_M) < 1\} \subseteq O_M \subseteq \{x \in L_M : d(x, L^0_M) \leq 1\}, \]

where \( d(\cdot, L^0_M) \) is the distance from the corresponding element to \( L^0_M \) (this distance is the same in the both norms!).

Let us also note the fundamental equalities:

\[ ((L^0_M)_L)^* = (L^*_M)_A, \quad ((L^0_M)_A)^* = (L^*_M)_L, \]

where by the symbols \( L \) and \( A \) we mark the spaces equipped with the corresponding norms. These equalities mean in particular that under the passage to the dual spaces the Luxemburg and Orlicz norms interchange their places.

In what follows the notation \( \langle \cdot, \cdot \rangle \) will mean the standard coupling of the spaces \( L^*_M \) and \( L_M \):

\[ \langle y, x \rangle = \int_{\Omega} y(\omega)\overline{x(\omega)} \, d\mu(\omega). \]

One can verify that

\[ \langle y, x \rangle \leq \|y\|_{(L^*_M)_L} \|x\|_{(L^*_M)_A} \]

and

\[ \langle y, x \rangle \leq \|y\|_{(L^*_M)_A} \|x\|_{(L^*_M)_L}. \]
This along with (1) implies
\[ \langle y, x \rangle \leq \|y\|_{(L_{M^*})_A} \|x\|_{(L_M)_A} \]
and
\[ \langle y, x \rangle \leq 2\|y\|_{(L_{M^*})_L} \|x\|_{(L_M)_L}. \]

As the particular examples of Orlicz spaces we have the classical Lebesgue spaces \( L_p \) \((1 < p < \infty)\) that correspond to the \( N \)-functions \( M(u) = |u|^p \) \((1 < p < \infty)\).

The modular is linked with the Luxemburg norm by the relations
\[ \|x\|_L \leq 1 \Rightarrow M(x) \leq \|x\|_L, \quad \|x\|_L \geq 1 \Rightarrow M(x) \geq \|x\|_L; \quad (2) \]
and with the Orlicz norm by the inequality
\[ M(\lambda x) \geq \lambda \|x\|_A - 1, \quad 0 < \lambda < \infty. \quad (3) \]

It was observed in [3] that the next equality
\[ \lim_{\|x\| \to \infty} \frac{M(x)}{\|x\|} = \infty \quad (4) \]
plays a significant role in applications. As it was shown in [4] this equality is not always true. The necessary and sufficient condition for its validity is the equality
\[ \lim_{u \to \infty} \frac{M(ku)}{u(M^*)^{-1}(u)} = \infty, \quad 1 < k < \infty. \quad (5) \]

As the examples of \( N \)-functions that do not satisfy condition (5) one can take the functions \((1 + u) \ln(1 + u) - u\) and \(u \sqrt{\ln(1 + u)}\) (see [4]). These functions are characterized by the property that the corresponding dual functions satisfy the so called \( \Delta_3 \)-condition (that is for some \( k > 1 \) and large \( u \) the inequality \( uM(u) \leq M(ku) \) holds; see [4]). We also remark that a number of properties of Orlicz space associated with relation (4) were considered in [5].

However in the particular case \( M(u) = |u|^p \) \((1 < p < \infty)\) the Luxemburg norm coincides with the standard norm on \( L_p \) and satisfies the equality
\[ M(x) = \|x\|^p, \quad (6) \]
This equality is stronger than relations (2) and (4). Thus one naturally arrives at the problem of a possibility of refinement of these relations for arbitrary Orlicz spaces.

Henceforth we confine ourselves to the estimates in the Luxemburg norm. Therefore for the sake of brevity the sign \( _L \) in the notation of the norm \( \| \cdot \|_L \) is omitted.

2. In this section we discuss the problem of validity in Orlicz spaces of the estimates of the type
\[ M(x) \geq \phi(\|x\|), \quad \|x\| \geq R, \quad (7) \]
where \( \phi(\cdot) \) is a certain positive function and \( R \) is a sufficiently large number. Note that (2) implies that one can always take as the function \( \phi(\cdot) \) the function
\[ \phi(\lambda) = \begin{cases} 0 & 0 \leq \lambda < 1 \\ \lambda & 1 \leq \lambda < \infty \end{cases} \quad (8) \]
But since we are aimed at the functions $\phi(\cdot)$ of prime interest are those who grow faster than linear functions at infinity.

The next statement gives a partial answer to the question.

**Theorem 1.** Let an $N$-function $M(\cdot)$ satisfy the inequality

$$M(\lambda u) \geq \phi(\lambda) \cdot M(u), \quad \lambda \geq 1,$$

where $\phi(\cdot)$ is a positive function. Then

$$\mathcal{M}(x) \geq \phi(\|x\|), \quad \|x\| \geq 1. \quad (10)$$

□ The proof is quite simple. Indeed, let $\|x\| \geq 1$. Then we have by (9) and (2)

$$\mathcal{M}(x) = M\left(\|x\| \frac{x}{\|x\|}\right) \geq \phi(\|x\|) \cdot M\left(\frac{x}{\|x\|}\right) \geq \phi(\|x\|).$$

As the examples of application of this theorem one can consider not only the $N$-functions $M(u) = u^p$ ($1 < p < \infty$) corresponding to the classical Lebesgue spaces $L_p$ (for these functions one can take $\phi(\lambda) = \lambda^p$), but also the $N$-functions $e^u - u - 1$ and $e^{u^2} - 1$; for these functions inequality (9) is satisfied with $\phi(\lambda) = \lambda^2$. Thus Theorem 1 implies that in the corresponding spaces the Luxemburg norm satisfies the inequality

$$\mathcal{M}(x) \geq \|x\|^2, \quad \|x\| \geq 1.$$

For the functions $e^u - u - 1$ and $e^{u^2} - 1$ mentioned above it is natural to expect that the inequality of the form (7) for large $\|x\|$ is satisfied with a function $\phi(\cdot)$ that grows at infinity essentially faster than $\lambda^2$. It turns out that this is really true but Theorem 1 is not enough to prove this fact: by means of this theorem the function $\lambda^2$ for both the functions $e^u - u - 1$ and $e^{u^2} - 1$ cannot be changed for the one that grows faster.

**Theorem 2.** Let an $N$-function $M(\cdot)$ satisfy the inequality

$$M(\lambda u) \geq \phi(\lambda, u) \cdot M(u), \quad \lambda \geq 1,$$

where $\phi(\cdot, \cdot)$ is a positive and nondecreasing with respect to the first and the second argument function. Then the following inequality holds

$$\mathcal{M}(x) \geq \phi\left(\|x\|, \frac{R - 1}{R\|1\|}\right) \quad \text{for} \quad \|x\| \geq R > 1, \quad (12)$$

where $1$ is the function identically equal to 1.

□ Let a function $x \in L_M$ satisfy the condition $\|x\| \geq 1$ and in addition for each point $\omega \in \Omega$ where $x(\omega) \neq 0$ we have $|x(\omega)| \geq h\|x\|$, where $h$ is a certain positive number. Repeating word by word the argument of the proof of Theorem 1 and using (11) instead of (9) and the monotone property of $\phi$ with respect to $u$ we get

$$\mathcal{M}(x) \geq \phi(\|x\|, h).$$

Now let $x \in L_M$ be a function for which $\|x\| \geq R > 1$. Set

$$h := \frac{R - 1}{R\|1\|}$$
and
\[ x_h(\omega) := \begin{cases} x(\omega), & |x(\omega)| \geq h\|x\|, \\
0, & |x(\omega)| < h\|x\|. \end{cases} \]

Then
\[ \|x_h\| \geq \|x\| - \|x - x_h\| \geq \|x\| - h\|1\| \cdot \|x\| = \frac{\|x\|}{R} \geq 1. \]

Now from the inequality \(|x(\omega)| \geq |x_h(\omega)||\), \((13)\) and the monotone property of \(\phi\) with respect to \(\lambda\) we get
\[ \mathcal{M}(x) \geq \mathcal{M}(x_h) \geq \phi \left( \frac{\|x\|}{R}, h \right) \text{ for } \|x\| \geq R > 1. \]

**Remark.** Clearly if we additionally presuppose that \(\phi\) is continuous then in \((12)\) we can also take \(R = 1\).

Let us consider as an example of \(N\)-function the function \(e^u - u - 1\). Evidently the function
\[ \phi(\lambda, u) = \frac{e^{\lambda u} - \lambda u - 1}{e^u - u - 1} \quad (14) \]

is nondecreasing with respect to \(\lambda\). Observe that it is also nondecreasing with respect to \(u\) on \([0, \infty)\) for \(\lambda \geq 1\). Indeed, it is enough to verify the nonnegativity of its derivative; but the latter is equivalent to the inequality
\[ (\lambda e^{\lambda u} - \lambda)(e^u - u - 1) - (e^{\lambda u} - \lambda u - 1)(e^u - 1) \geq 0 \]

which is equivalent to
\[ \left( \sum_{j=2}^{\infty} \frac{j\lambda^j u^{j-1}}{j!} \right) \left( \sum_{k=2}^{\infty} \frac{u^k}{k!} \right) - \left( \sum_{j=2}^{\infty} \frac{\lambda^j u^j}{j!} \right) \left( \sum_{k=2}^{\infty} \frac{k u^{k-1}}{k!} \right) \geq 0. \]

Collecting similar terms in the left hand part of this inequality we obtain
\[ \sum_{2 \leq k < j < \infty} \frac{(j - k)(\lambda^j - \lambda^k)u^{j+k-1}}{j!k!} \geq 0. \]

The latter inequality is evident since \(\lambda \geq 1\).

In addition we have
\[ e^{\lambda u} - \lambda u - 1 = \phi(\lambda, u) (e^u - u - 1). \]

Now Theorem 2 implies the validity of estimate \((12)\) with function \((14)\).

In an analogous way one can consider the \(N\)-function \(e^{u^2} - 1\) taking here
\[ \phi(\lambda, u) := \frac{e^{\lambda^2 u^2} - 1}{e^{u^2} - 1}. \]

It can be shown that Theorems 1 and 2 contain the results of Ja. B. Rutitzky from \([4]\). Note that the statement of the main theorem of the latter paper contains an inexactness: condition \((7)\) in the proof of sufficiency is used not for large \(u\) but for all \(u\).
3. In the paper [7] there was investigated in Orlicz spaces the nonlinear integral Hammerstein equation of the form

\[ x = Sf(x) + g, \]  

(15)

where \( f \) is a nonlinear superposition operator

\[ f(x) = f(\omega, x(\omega)), \]

where \( f(\omega, u) : \Omega \times \mathbb{R} \to \mathbb{R} \) is a function satisfying the Caratheodory conditions, \( S \) is a linear operator and \( g \) is a known function. As examples of (15) one can consider the nonlinear singular integral equations (see, for example, [6, 7]); in these examples \( \Omega \subseteq \mathbb{R}^2 \) is a certain (open or closed) sufficiently smooth curve and \( S \) is the linear integral operator with the Cauchy type kernel.

Unfortunately in [7] the signs in a number of inequalities happened to be mixed up and as a result the statements of Lemma 4 and Theorem 2 in [7] on the conditions of existence and uniqueness of the solution to equation (15) are false. Hereafter we give the general scheme of the investigation suggested in [7] with the necessary refinement. A number of new important additional observations are presented as well.

To start with we describe the general scheme of the proof of existence theorems. Henceforth in this Section we shall consider the situation \( L_M \subset L_2 \subset L_M^* \) that implies in particular the inequality

\[ \gamma \|x\| \geq \|x\|_2, \]

(16)

where \( \|x\|_2 \) is the \( L_2 \)-norm of \( x \) and \( \gamma > 0 \) is a certain constant. Let the superposition operator \( f \) act from an Orlicz space \( L_M \) into its dual Orlicz space \( L_M^* \) (recall that \( M^*(\cdot) \) is the \( N \)-function dual to the \( N \)-function \( M(\cdot) \)). Further, let the operator \( S \) be defined on \( L_M^* \) and take values in the space of measurable functions, in addition let there exist a linear operator \( T \) acting from \( L_M \) into \( L_M^* \) such that

\[ STx = x, \quad x \in L_M. \]

Finally, let the function \( g \) also belong to \( L_M \). Under these assumptions to prove the solvability of equation (15) one can consider an auxiliary equation \( \Phi x = 0 \), where

\[ \Phi x = Tx - f(x) - Tg. \]

Indeed, applying the operator \( S \) to this equation we arrive at equation (15), which means that every solution \( x \in L_M \) to the equation \( \Phi x = 0 \) is also the solution in \( L_M \) to equation (15).

Under a number of natural constraints on the operators \( S \) and \( f \) the operator \( \Phi \) as the operator from \( L_M \) into \( L_M^* \) turns out to be monotone in the sense of Minty:

\[ \text{Re} \langle \Phi x_1 - \Phi x_2, x_1 - x_2 \rangle \geq 0, \quad x_1, x_2 \in L_M, \]

and on the balls \( \{ x : \|x\| \leq R \} \) of the space \( L_M \) it satisfies the Rothe condition

\[ \text{Re} \langle \Phi x, x \rangle \geq 0, \quad \|x\| = R. \]

We recall that the Minty monotonicity along with the Rothe condition imply the existence of a solution to the equation \( \Phi x = 0 \) (see [7]) and therefore the existence of a solution to (15) as well.
Now we shall describe the corresponding constrains. Let the linear operator $T$ satisfies the inequality
\[
\text{Re} \langle Tx, x \rangle \geq \sigma \|x\|_2^2, \quad x \in L_M.
\] (18)
Then, if the function $f(\omega, u)$ satisfies with respect to the argument $u$ the inequality
\[
\text{Re} \langle f(\omega, u_1) - f(\omega, u_2) \rangle (u_1 - u_2) \leq \delta |u_1 - u_2|^2, \quad \omega \in \Omega, \ u_1, u_2 \in \mathbb{C}
\] (19)
(this inequality means that the function $\delta u - f(\omega, u)$ is monotone with respect to the argument $u$) then the operator $\Phi$ is evidently monotone in the sense of Minty on $L_M$ provided $\sigma \geq \delta$.

Further, if the function $f(\omega, u)$ satisfies the inequality
\[
- \text{Re} \langle f(x, x) \rangle \geq a \phi(b\|x\|) - c, \quad \|x\| \geq R; \quad c = \|c(\cdot)\|_1, \quad c \in \mathcal{L}^1(\Omega, \mathbb{R}.
\] (20)
where $a, b > 0$ and $c(\cdot)$ is an integrable on $\Omega$ function then for a certain function $\phi(\cdot)$ (see [2], [8] and Theorems 1 and 2) the operator $f$ satisfies the inequality of the form
\[
- \text{Re} \langle f(x, x) \rangle \geq a \phi(b\|x\|) - c, \quad \|x\| \geq R;
\] (21)
here $c = \|c(\cdot)\|_1$ is the $L_1$-norm of $c(\cdot)$. Therefore the operator $\Phi$ satisfies the estimate
\[
\text{Re} \langle \Phi x, x \rangle \geq \sigma \|x\|_2^2 + a \phi(b\|x\|) - 2\|Tg\|_{L_{M^*}} \|x\| - c, \quad \|x\| \geq R.
\] (22)
Observe that
\[
\|x\|^{-1} (\sigma \|x\|_2^2 + a \phi(b\|x\|) - 2\|Tg\|_{L_{M^*}} \|x\| - c) = \frac{\sigma \|x\|_2^2 + a \phi(b\|x\|) - 2\|Tg\|_{L_{M^*}} \|x\| - c}{\|x\|} \quad \|x\| \geq R.
\] (22)

Therefore we get the following statement: let $\sigma \geq 0$ and $r^{-1}\phi(r) \to \infty$ as $r \to \infty$ then the Rothe condition is satisfied for $R$ large enough.

In other words in this case it should be fulfilled condition (11) considered in [3] and [4], that is in this case an arbitrary Orlicz space can not be used (however if the norm \[\|Tg\|_{L_{M^*}}\] is sufficiently small one can take as the function $\phi(\cdot)$ in [22] function $\phi_0$, see the left hand inequality in (11); this means that equation (15) with $g$ of this type can be considered in any of the Orlicz spaces).

Now let $\sigma < 0$. Recalling that $\gamma \|x\| \geq \|x\|_2$ (see [16]) we conclude that equality (22) implies the inequality
\[
\|x\|^{-1} (\sigma \|x\|_2^2 + a \phi(b\|x\|) - 2\|Tg\|_{L_{M^*}} \|x\| - c) \geq \sigma \gamma^2 \|x\| + ab \phi(b\|x\|) \|x\| - 2\|Tg\|_{L_{M^*}} \|x\|^{-1} c \quad \|x\| \geq R. 
\] (23)

Thus in this situation we get the following statement: let $\sigma < 0$ then the Rothe condition is satisfied for $R$ large enough if $\phi(\cdot)$ has greater than quadric rate of growth at infinity.

Therefore in this case an arbitrary Orlicz space is not suitable. Theorems 1 and 2 give us a possibility to indicate the conditions on the $N$-function $M(\cdot)$ under which the function $\phi(\cdot)$ of this sort does exist.
Now let us describe the general scheme of the proof of uniqueness theorems for equation (15). Suppose that for the operator $S$ there exists an operator $T$ having the property

$$TSz = z, \quad z \in L_{M^*}.$$ 

The equalities $x_1 = Sfx_1 + g$ and $x_2 = Sfx_2 + g$ for $x_1, x_2 \in L_M$ imply $x_1 - x_2 = S(fx_1 - fx_2)$. Applying the operator $T$ to this equality we get $T(x_1 - x_2) = (fx_1 - fx_2)$.

From this under the fulfilment of (18) and (19) we obtain

$$0 = \Re \langle (T(x_1 - x_2) - (fx_1 - fx_2)), (x_1 - x_2) \rangle \geq (\sigma - \delta)\|x_1 - x_2\|_2^2,$$

and thus if $\sigma - \delta > 0$ we have $x_1 = x_2$.

4. Estimates (7) deduced in Section 2 were true only for sufficiently large $\|x\|$. For small $\|x\|$ the similar inequalities are not valid in the general case. In particular one can easily show that for $N$-functions $M(\cdot)$ satisfying the relation

$$\lim_{u \to \infty} M(\lambda u) M(u) = 0, \quad 0 < \lambda < 1,$$

the next equality is true:

$$\inf \{M(x) : \|x\| \geq R\} = 0, \quad 0 < R < 1$$

This equality means that the inequalities of the form (7) with small $\|x\|$ and positive functions $\phi(\cdot)$ are impossible for $N$-functions satisfying (24). This remark is true in particular for the functions $e^u - u - 1$ and $e^{u^2} - 1$ considered above.

The foregoing observation implies that the estimate we are interested in is possible only if

$$\lim_{u \to \infty} \frac{M(\lambda u)}{M(u)} > 0, \quad 0 < \lambda < 1.$$

It is easy to see that the latter condition is equivalent to the condition that the $N$-function $M(u)$ satisfies $\Delta_2$-condition. It turns out that $\Delta_2$-condition guarantees the satisfaction of estimates (7) for $\|x\| \leq 1$ and $\phi$ taken from (5).

**Theorem 3.** Let an $N$-function $M(\cdot)$ satisfy the inequality

$$M(\lambda u) \geq \phi(\lambda) \cdot M(u), \quad 0 < \lambda \leq 1,$$

where $\phi(\cdot)$ is a positive function. Then the next inequality is true

$$M(x) \geq \phi(\|x\|), \quad \|x\| \leq 1.$$

□ The proof is the word by word repetition of the proof of Theorem 1. ■

As the examples of $N$-functions satisfying inequality (23) one can consider the functions $M_1 = (1 + u) \ln(1 + u) - u$ and $M_2 = u^p \ln(1 + u) \int (1 < p < \infty)$. For the first of these functions the corresponding function $\phi(\cdot)$ is defined by the equality $\phi(\lambda) = \lambda^2$.

Indeed, for each $u > 0$ and $0 < \lambda < 1$ by the Cauchy theorem we have

$$\frac{M_1(\lambda u)}{M_1(u)} = \frac{\lambda M_1'(\lambda u_1)}{M_1'(u_1)} = \frac{\lambda^2 M_1''(\lambda u_2)}{M_1''(u_2)}, \quad 0 < u_2 < u_1 < u.$$
Since \( M_1'(u) = \frac{1}{1+u} \) is monotone decreasing function on \((0, \infty)\) it follows from the latter equality that \( M_1(\lambda u) \geq \lambda^2 M_1(u) \). This inequality cannot be refined since by the L’Hospital theorem we have

\[
\lim_{u \to 0} \frac{M_1(\lambda u)}{M_1(u)} = \lim_{u \to 0} \frac{\lambda \ln(1 + \lambda u)}{\ln(1 + u)} = \frac{\lambda^2(1 + u)}{1 + \lambda u} = \lambda^2.
\]

Observe also that the equalities

\[
\lim_{u \to \infty} \frac{M_1(\lambda u)}{M_1(u)} = \lim_{u \to \infty} \frac{\lambda \ln(1 + \lambda u)}{\ln(1 + u)} = \frac{\lambda^2(1 + u)}{1 + \lambda u} = \lambda
\]

and the monotone property of the function

\[
\frac{\lambda^2(1 + u)}{1 + \lambda u}, \quad u \in (0, \infty)
\]

show that the ratio between \( M_1(\lambda u) \) and \( M_1(u) \) for \( 0 < \lambda < 1 \) is contained in the interval \((\lambda^2, \lambda)\). Thus \( M_1 \) satisfies the following inequalities

\[
\lambda^2 M_1(u) \leq M_1(\lambda u) \leq \lambda M_1(u), \quad 0 < \lambda < 1.
\] \hspace{1cm} (27)

For the function \( M_2 \) we take \( \phi_2(\lambda) = \lambda^{p+1} \). Since for \( u > 0 \) we have

\[
\frac{(\lambda u)^p \ln(1 + \lambda u)}{u^p \ln(1 + u)} = \lambda^p \frac{\ln(1 + \lambda u)}{\ln(1 + u)} = \lambda^{p+1} \frac{1 + u_1}{1 + \lambda u_1} \geq \lambda^{p+1},
\] \hspace{1cm} (28)

where by the Cauchy theorem \( 0 < u_1 < u \). The equalities

\[
\lim_{u \to 0} \frac{M_2(\lambda u)}{M_2(u)} = \lim_{u \to 0} \lambda^p \frac{\ln(1 + \lambda u)}{\ln(1 + u)} = \lambda^{p+1} \lim_{u \to 0} \frac{1 + u}{1 + \lambda u} = \lambda^{p+1}
\]

show that the inequality obtained cannot be refined.

Observe also that relations \((28)\) and the inequality

\[
\frac{(\lambda u)^p \ln(1 + \lambda u)}{u^p \ln(1 + u)} \leq \lambda^p, \quad 0 < \lambda < 1,
\]

show that for this function the ratio between \( M(\lambda u) \) and \( M(u) \) for \( 0 < \lambda < 1 \) is contained in the interval \((\lambda^{p+1}, \lambda^p)\).

Thus \( M_2 \) satisfies the following inequalities

\[
\lambda^{p+1} M_2(u) \leq M_2(\lambda u) \leq \lambda^p M_2(u), \quad 0 < \lambda < 1.
\] \hspace{1cm} (29)

Inequalities \((29)\) and \((27)\) show in particular that if we take \( M_3 = M_1 + M_2 \) then we get the \( N \)-function satisfying the inequalities

\[
\lambda^{p+1} M_3(u) \leq M_3(\lambda u) \leq \lambda M_3(u), \quad 0 < \lambda < 1.
\] \hspace{1cm} (30)

Just as in the case of Theorem 1 inequality \((29)\) (the analogue to inequality \((9)\)) is too restrictive. Let us present the analogue to Theorem 2.
Theorem 4. Let an $N$-function $M(\cdot)$ satisfy the inequality
\[ M(\lambda u) \geq \phi(\lambda, u) \cdot M(u), \quad 0 < \lambda < 1, \] (31)
where $\phi(\cdot, \cdot)$ is a positive and nondecreasing with respect to the first and the second argument function. Then the following inequalities hold
\[ M(x) \geq \phi((1 - h\|1\|\|x\|), h), \quad 0 \neq \|x\| \leq 1, \quad 0 < h < \frac{1}{\|1\|}. \] (32)

□ Let a non-zero function $x \in L_M$ satisfy the condition $\|x\| \leq 1$ and in addition for each point $\omega \in \Omega$ where $x(\omega) \neq 0$ the inequality $|x(\omega)| \geq h\|x\|$ hold, where $h$ is a certain positive number. Then repeating the argument of the proof of Theorem 1 and applying (31) instead of (9) and the monotone property of $\phi$ with respect to $u$ we get
\[ M(x) \geq \phi(\|x\|, h). \] (33)

Now let $x \in L_M$ be a non-zero function such that $\|x\| \leq 1$. Let $h$ be any number from the interval $\left(0, \frac{1}{\|1\|}\right)$ and
\[ x_h(\omega) := \begin{cases} x(\omega), & |x(\omega)| \geq h\|x\|, \\ 0, & |x(\omega)| < h\|x\|. \end{cases} \]
We have
\[ \|x_h\| \geq \|x\| - \|x - x_h\| \geq \|x\| - h\|1\| \cdot \|x\| = \|x\| (1 - h\|1\|) > 0. \]
Since $|x(\omega)| \geq |x_h(\omega)|$ it follows that (33) and the monotone property of $\phi$ with respect to $\lambda$ imply
\[ M(x) \geq M(x_h) \geq \phi((1 - h\|1\|\|x\|), h). \]

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