Synchronization of coupled nonidentical dynamical systems

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Abstract – We analyze the stability of synchronized state for coupled nearly identical dynamical systems on networks by deriving an approximate master stability function (MSF). Using this MSF we treat the problem of designing a network having the best synchronizability properties. We find that the edges which connect nodes with a larger relative parameter mismatch are preferred. Also, the nodes having values at one extreme of the parameter mismatch are preferred as hubs where the extreme is the one which gives a better stability according to the MSF curve.

Introduction. – When two or more dynamical systems are coupled or driven by a common signal the systems may synchronize under suitable conditions[1–4]. Recently, there is considerable interest in the synchronization of coupled dynamical systems on a network [5]. For coupled identical systems which give exact synchronization, Pecora and Carroll [6] have introduced a master stability function (MSF) which can be calculated from a simple set of master stability equations and then applied to the study of stability of the synchronous state of different networks. This general approach has become popular and has been used in various studies of synchronization on networks [7–11]. Several works on different networks have shown that small-world and scale-free networks show better synchronization properties [7].

For coupled nonidentical systems, in general it is difficult to obtain exact synchronization. However, one can get synchronization of some generalized type [12–18]. The parameter mismatch between different coupled systems can lead to desynchronization bursts and this is known as the bubbling transition [19–21]. After the desynchronization burst the system returns to the synchronized state. Sun et al. [13] have extended the master stability approach for nearly identical systems to calculate the deviation from the average trajectory and the deviation is shown to be bounded.

In this paper we address the question of the stability of synchronization of coupled nearly identical systems on networks. By noting that the homogeneous part of the linearized equations determines the exponential rates for the synchronized state and that in the homogeneous part the parameter mismatch appears in the quadratic term of the expansion, we derive a master stability equation for coupled nearly identical systems. This master stability equation uses the trajectory of a system with a typical parameter and two additional parameters, α for the network coupling and Δ for the mismatch in nonidentical systems. We calculate the Lyapunov exponents and define the MSF as the largest transverse Lyapunov exponent and study the stability properties of the synchronized state. We note that MSF as defined in refs. [13,14] is the deviation from the average trajectory and is different from the definition of MSF in ref. [6] as the largest transverse Lyapunov exponent which we follow.

When one considers identical coupled systems the important question is about the type of network which gives better synchronization properties. When one considers coupled nearly identical systems, additional interesting and important questions arise. Which nodes are better chosen as hubs? Which edges give better synchronization? Using our MSF we find that for better synchronization nodes on one extreme of parameter mismatch are preferred as hubs and nodes with larger relative parameter mismatch are preferred for constructing edges.

Master stability function. – Consider N coupled dynamical systems,

\[ \dot{x}_i = f(x_i, r_i) + \varepsilon \sum_{j=1}^{N} G_{ij} h(x_j); \quad i = 1, \ldots, N, \quad (1) \]
where, \( x^i \in R^m \) is a \( m \) dimensional state vector of the system \( i \), \( f : R^m \to R^m \) gives the dynamics of an isolated system, \( \varepsilon \) is a scalar coupling parameter and \( h : R^m \to R^m \) is a coupling function, \( G \) is the coupling matrix of the network, \( r_i \) is some parameter which depends on the node \( i \).

For the coupled identical systems, i.e., \( r_i = r, \forall i \), the synchronization manifold is defined by \( x^i = \ldots = x^N = x \) and is an invariant manifold provided that \( \sum_j G_{ij} = 0, \forall i \). This also ensures that \( G \) has one eigenvector \( e_1 = (1, \ldots, 1)^T \) with eigenvalue \( \gamma_1 = 0 \) and it defines the synchronization manifold. All the remaining eigenvectors are related to the transverse Lyapunov exponents. The synchronized state is stable provided all the transverse Lyapunov exponents are negative.

For nonidentical systems, let the parameter mismatch be \( \delta r_i = r_i - \bar{r} \), where \( \bar{r} \) is some typical value of the parameters \( r_i \). In general, for nonidentical systems we get a generalized synchronization where there is some functional relationship between variables of the systems. The generalized synchronization is stable provided the largest transverse Lyapunov exponent is negative.

For coupled identical systems the stability of the synchronized state \( x^i = \ldots = x^N = x \) can be determined by expanding around this state and this state is a solution of the uncoupled dynamics. For coupled nonidentical systems, in the synchronized state, the attractors for the uncoupled dynamics. For coupled nonidentical systems the stability of the synchronized state \( x^i = \ldots = x^N = x \) is to includethe quadratic terms in \( z^i = x^i - \bar{x} \) and \( \delta r_i \) since the effect of parameter mismatch is not seen in the linear terms. Equation (3) can be put in a matrix form as [22]

\[
\dot{z}^i = D_x f \ z^i + \varepsilon \sum_{j=1}^{N} G_{ij} D_x h \ z^j + D_x D_z f \ z^i \delta r_i. \tag{3}
\]

We note that in the homogeneous equation, it is necessary to include the quadratic terms in \( z^i = x^i - \bar{x} \) and \( \delta r_i \), where, \( Z = (z^1, \ldots, z^N) \) is an \( m \times N \) matrix and \( R = \text{diag}(\delta r_1, \ldots, \delta r_N) \) is an \( N \times N \) diagonal matrix.

Let \( \gamma_j, \epsilon^R_j, j = 1, \ldots, N \) be the eigenvalues and right eigenvectors of \( G^T \). Multiplying eq. (4) by \( e_j^R \) from the right and using the \( m \) dimensional vectors \( \phi_j = Ze_j^R \), we get

\[
\dot{\phi}_j = [D_x f + \varepsilon \gamma_j D_x h] \phi_j + D_x D_z f R e_j^R. \tag{5}
\]

In general, \( e_j^R \) are not eigenvectors of \( R \) and hence eq. (5) is not easy to treat. To solve eq. (5) we use the first-order perturbation theory and write eq. (5) as

\[
\dot{\phi}_j = [D_x f + \varepsilon \gamma_j D_x h + \nu_j D_z f] \phi_j, \tag{6}
\]

where \( \nu_j = e_j^R \text{Re}_j \) is the first-order correction and \( e_j^R \) is the left eigenvector of \( G^T \).

Since both \( \gamma_j \) and \( \nu_j \) can be complex and observing that \( j \) is only an index, we treat them as complex parameters \( \alpha = \varepsilon \gamma_j \) and \( \Delta = \nu_j \). Thus, we can construct the master stability equation as

\[
\phi = [D_x f + \alpha D_x h + \Delta D_z f] \phi. \tag{7}
\]

For the coupled identical systems, the above equation reduces to the master stability equation given by Pecora and Carroll [6]. The MSF is defined as the largest Lyapunov exponent of eq. (7). Thus, in the complex space defined by \( \alpha \) and \( \Delta \), the MSF is a surface. For a given network, for each of the eigenvalues and eigenvectors of

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We now present numerical evidence to support our above derivation leading to eq. (7). Let us first consider the expansion (2) around some typical value of the parameter. For coupled identical systems one expands around the synchronized solution which corresponds to the solution of uncoupled dynamics, i.e., the dynamics without the coupling term. Taking hint from this, for nearly identical systems, we consider the properties of synchronized dynamics by omitting the coupling term. Consider \( N \) coupled chaotic Rössler systems with different frequencies,

\[
\begin{align*}
\dot{x}_i &= -\omega_i y_i - z_i + \varepsilon \sum_{j=1}^{N} L_{ij} (x_j - x_i) \\
\dot{y}_i &= \omega_i x_i + ay_i, \\
\dot{z}_i &= b + z_i (x_i - c),
\end{align*}
\]

where \( \omega_i \) is the frequencies of the \( i \)-th oscillator and \( L_{ij} = 1 \) if the nodes \( i \) and \( j \) are coupled and zero otherwise and \( L_{ii} = -\sum_{j \neq i} L_{ij} \). We choose the couplings \( L_{ij} \) randomly. The different frequencies \( \omega_i \) are chosen randomly in an interval \((\bar{\omega} - \delta \omega / 2, \bar{\omega} + \delta \omega / 2)\). Now, we evolve the coupled Rössler systems. To see how the attractors of different systems are related to each other, we define subsystem Lyapunov exponents as the exponents calculated by the following procedure. We evolve the Rössler equations as above with the coupling term, and then using the variables from this evolution, we calculate subsystem Lyapunov exponents by the usual procedure of calculating Lyapunov exponents with the coupling term omitted. Thus, for each system \( i \) we get three subsystem Lyapunov exponents, say \( \lambda_i^{\alpha_1}, \lambda_i^{\alpha_2} \) and \( \lambda_i^{\alpha_3} \). For coupled identical systems these subsystem Lyapunov exponents will be the same for all the systems, i.e., they will be independent of \( i \).

Figure 1(a) shows the subsystem Lyapunov exponents \( \lambda_1^{\alpha_1}, \lambda_2^{\alpha_2} \) and \( \lambda_3^{\alpha_3} \) as a function of the frequency \( \omega \) for a system of 32 coupled nearly identical Rössler oscillators in the synchronized state, with frequencies chosen randomly in the range \((0.999, 1.001)\). We see that the three Lyapunov exponents for the subsystems vary linearly with \( \delta \omega \). The linear variation is in agreement with eq. (3) omitting the coupling term. We find that the subsystem Lyapunov exponents calculated from eq. (3) match with those shown

\footnote{We note that the systems can synchronize even when they are outside the linear region.}

in fig. 1. The linear variation of the subsystem Lyapunov exponents for different systems, supports the conjecture that the different attractors are not significantly deformed in the synchronized state.

To see the range of \( \omega \) values for which the linear variation holds, in fig. 1(b) we plot the subsystem Lyapunov exponents \( \lambda_1^{\alpha_1}, \lambda_2^{\alpha_2} \) and \( \lambda_3^{\alpha_3} \) as a function of the frequency \( \omega \) for a system of 32 coupled nearly identical Rössler oscillators in the synchronized state, with frequencies chosen randomly in a wider range \((0.98, 1.02)\). We see a linear variation in the range \((0.99, 1.01)\) while nonlinearity can be seen outside this range. The linear region defines the range of validity of our theory\(^1\).

We now consider the choice of the typical value of the parameter \( \bar{\omega} \). One choice is the average value \( \bar{\omega} = 1 \). The three Lyapunov exponents for the uncoupled trajectory for \( \bar{\omega} \), say \( \lambda_j \), are shown in fig. 1 by stars and they lie almost on the three lines. In table 1 we give the three values \( \lambda_j \) and also the three values \( \lambda_{ij}^{\alpha_1} \) for \( \bar{\omega} \) obtained from the linear fits in fig. 1. There is a good agreement between the two sets of Lyapunov exponents. We find that the average value gives good results for all the

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Table 1: The table shows the three subsystem Lyapunov exponents $\lambda^s_{ij}$, $j = 1, 2, 3$ for the coupled Rössler system, eq. (8), obtained for $\omega = \bar{\omega} = 1$ from linear fits in fig. 1 and the corresponding values of the Lyapunov exponents $\lambda_j$, $j = 1, 2, 3$ for the attractor at the average frequency $\bar{\omega}$.

|      | $j = 1$ | $j = 2$ | $j = 3$ |
|------|---------|---------|---------|
| $\lambda^s_{ij}$ | 0.1130  | 0.0001  | −9.6771 |
| $\lambda_j$      | 0.1130  | 0.0000  | −9.6765 |

Fig. 2: (Colour on-line) The figure shows the three largest Lyapunov exponents $\lambda_i$, $i = 1, 2, 3$ (red, green and blue lines) and their estimated values $\lambda^MS_i$ obtained from the master stability equation (eq. (7)) (pink, cyan and black lines) as a function of $\varepsilon$ for two coupled Rössler systems with frequencies $\omega_1 = 1.05$ and $\omega_2 = 1.07$. Taking $\bar{\omega} = 1.0$ we get $\Delta_1 = \Delta_2 = 0.06$ which are used in eq. (7). Rössler parameters are $a = b = 0.2, c = 7.0$. The synchronous state is stable in the region given by $\varepsilon_1 < \varepsilon < \varepsilon_2$ indicated by the arrows.

We first consider two coupled Rössler oscillators. Figure 2 plots the three largest Lyapunov exponents, $\lambda_i$, $i = 1, 2, 3$, as a function of the coupling strength $\varepsilon$ and their estimated values $\lambda^MS_i$ from eq. (7). Figure 3(a) plots the difference $\delta \lambda_i = \lambda_i - \lambda^MS_i$ as a function of $\varepsilon$ for these Lyapunov exponents. The region when the third largest Lyapunov exponent $\lambda_3 < 0$, corresponds to the synchronization region and in this region it is the largest transverse Lyapunov exponent. From fig. 3(a), we find that the differences $\delta \lambda_i$ are small in the synchronization region and very close to it. In the figure we plot only three largest Lyapunov exponents, but we find that the differences are small for the other Lyapunov exponents as well. Figure 3(b) plots the difference $\delta \lambda_1$ as a function of $\varepsilon$ for the three largest Lyapunov exponents for a random network of sixteen nodes. Again we observe that the errors are small in the synchronization region. Thus, we find that the master stability equation (7) can estimate the actual Lyapunov exponents for the synchronized state reasonably well².

Next, to examine how well eq. (7) allows the estimation of Lyapunov exponents, we calculate the Lyapunov exponents for the coupled Rössler systems and compare them with those obtained from eq. (7). For simplicity we restrict ourselves to symmetric coupling matrices $L$ so that the eigenvalues and hence $\alpha$ and $\Delta$ are real.

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Fig. 3: (a) The figure shows the difference $\delta \lambda_1 = \lambda_1 - \lambda^MS_1$ for the three largest Lyapunov exponents as a function of the coupling constant $\varepsilon$ for two coupled Rössler systems with parameters as in fig. 1. (b) The figure shows the difference $\delta \lambda_3$ for the three largest Lyapunov exponents as a function of $\varepsilon$ for sixteen randomly coupled Rössler systems having different internal frequencies $\omega$. We find that the differences are small in the synchronization region.

The conclusion that stability range changes with parameter mismatch is confirmed by numerical simulations.

We generally found that the error is somewhat larger for larger no. of coupled equations. But we could not determine any systematic behaviour.

²This conclusion differs from that of ref. [14] which states that the stability range is not affected by parameter mismatch. This may be due to the use of deviation from average trajectory as criteria for deciding synchronization in ref. [14] and this deviation does not provide a good criteria since it remains finite in the synchronized state that is obtained with parameter mismatch. Note that our conclusion that stability range changes with parameter mismatch is confirmed by numerical simulations.
Fig. 4: (Colour online) The master stability function $\lambda_{max}$ for Rössler system is plotted as a contour plot in the parameter plane ($\alpha, \Delta_\omega$). The stability region is given by the “V” shape region bordered by the 0 contours from both sides.

**Synchronized optimized networks.** – We now demonstrate the utility of the master stability function by considering the problem of construction of an optimized network which gives best synchronization properties. To construct the optimized network we adapt Monte Carlo optimization method [23] and rewire the edges of the network to construct a network that shows best synchronization, i.e., the largest interval $\Delta_{\epsilon}$ of the coupling constant $\epsilon$ which shows synchronization.

We start with a system of nearly identical coupled Rössler oscillators as in eq. (8) on a connected network of $N$ nodes and $E$ randomly chosen edges. In each Monte Carlo step we rewire one edge. If the rewired network increases the stability interval $\Delta_{\epsilon}$ of the synchronized state, then it is chosen with probability one, otherwise it is accepted with probability $e^{\beta(\omega_{\text{new}} - \omega_{\text{old}})}$ where $\beta$ is the inverse temperature.

We now investigate two questions. In the optimized network, which edges are more preferable and which nodes have larger number of connection or act as hubs?

To investigate the question of which nodes act as hubs, i.e., whether the degree of a node and the frequency of that node have any relation in the optimized network, we define the correlation coefficient between the frequency and the degree of a node as $\rho_{\omega,k} = \frac{\langle(k_i - \langle k \rangle)(\omega_i - \langle \omega \rangle)\rangle}{\sqrt{\langle(k_i - \langle k \rangle)^2\rangle\langle(\omega_i - \langle \omega \rangle)^2\rangle}}$, where $k_i = -L_{ii}$ is the degree of node $i$. Figure 6(a) shows $\rho_{\omega,k}$ as a function of Monte Carlo steps. For the random network $\rho_{\omega,k} = 0$. We find that $\rho_{\omega,k}$ increases and saturates to a positive value. Thus, in the synchronized optimized network the nodes which have larger frequencies have more connections and are preferred as hubs. The reason for this is the “V” shape of the stability region in fig. 4, i.e., the stability range increases as $\Delta_{\omega}$ increases. We have also investigated a case where an opposite behavior is obtained. If instead of the frequency, we make the parameter $a$ in eq. (8) node dependent, then the stability region in the plot of MSF similar to fig. 4, has an inverted “V” shape (see fig. 5(b)). Figure 5 shows the stability region for mismatch in different parameters of Rössler system. In this case in the optimized network, nodes which have smaller values of $a$ have more connections and are preferred as hubs.

To understand the above behavior of the hubs and parameter mismatch in the optimized network, we look at the eigenvectors of the coupling matrix $G^T$ in the optimized network. As noted in the beginning $G^T$ has one eigenvector $e_1 = (1, \ldots, 1)^T$ with zero eigenvalue which defines the synchronization manifold. Of the remaining eigenvectors, the two, $e_2$ and $e_N$, which give the two extreme eigenvalues are the most important for stability of synchronization. We find that in the optimized network they are dominated by one or two large components. In particular, in $e_N$ these large components are the hubs.
In fig. 4 the stability curve (zero Lyapunov exponent) for the smaller $\alpha$ is decided by $e_2$ while the one for larger $\alpha$ is decided by $e_N$ [6]. If $e_i = (v_{i1}, v_{i2}, \ldots, v_{iN})^T$ is any eigenvector, then the corresponding $\Delta_\omega = \sum_j v_{ij}^2 \delta\omega_j$. By looking at fig. 4, we see that the larger the value of $\Delta_\omega$ for $e_N$, the better stability it will give. $e_N$ will give a maximum value of $\Delta_\omega$, if the dominant components corresponding to the hubs have a large positive value of $\delta\omega$. This is the reason that in the optimized network the hubs have a large positive frequency mismatch. A similar explanation can be given for the correlation between the hubs and the mismatch in parameter $\alpha$.

To investigate the question of which edges are preferred, we define the correlation coefficient between the absolute frequency differences between two nodes and the edges as

$$\rho_{\omega} = \frac{\langle (\omega_i - \omega_j - \delta\omega_i)\rangle}{\sqrt{\langle (\omega_i - \omega_j)^2 \rangle \langle (\omega_i - \delta\omega_i)^2 \rangle}},$$

where $A_{ij} = 1$ if nodes $i$ and $j$ are connected and 0 otherwise. Figure 6(b) shows $\rho_{\omega}$ as a function of Monte Carlo steps. We find that $\rho_{\omega}$ increases from 0 (the value for the random network) and saturates. Thus, in the synchronized optimized network the pair of nodes which have a larger relative frequency mismatch are preferred as edges for the optimized network. Again, the reason for this preference of edges is probably the conical shape of the stability region in fig. 4. The edges are to be chosen so that the parameter $\Delta_\omega$ increases and the stability region increases.

**Conclusion.** To conclude we have developed the MSF approach for coupled nonidentical systems. We use the property of differential equations that the homogeneous part is mainly responsible for the exponential dependence of the variables in the synchronized state. Our MSF uses the trajectory of a system with typical parameter, and it still allows us to study the stability properties of generalized synchronization for nonidentical systems. Using MSF, we construct optimized networks with better synchronization properties by rewiring the network keeping the number of edges constant. We find that in the optimized network the nodes having parameter mismatch at one extreme depending on the shape of stability region in MSF plot, have more edges and are preferred as hubs and the pair of nodes which have a larger relative parameter mismatch are preferred for constructing edges.

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