Abstract. The main purpose of the present paper is to study the blow-up problem of the wave equation with space-dependent damping in the scale-invariant case and time derivative nonlinearity with small initial data. Under appropriate initial data which are compactly supported, by using a test function method and taking into account the effect of the damping term \( \mu \sqrt{1+|x|^2} u_t \), we provide that in higher dimensions the blow-up region is given by \( p \in (1, p_G(N + \mu)] \) where \( p_G(N) \) is the Glassey exponent. Furthermore, we shall establish a blow-up region, independent of \( \mu \) given by \( p \in (1, 1 + \frac{2}{N}) \), for appropriate initial data in the energy space with noncompact support.

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1. Introduction

In this work, we consider the semilinear wave equations with a power-nonlinearity of derivative type

\[
\begin{aligned}
\partial^2_t u - \Delta u + V(x) \partial_t u &= |\partial_t u|^p & \text{in } [0, T) \times \mathbb{R}^N, \\
u(x, 0) &= \varepsilon f(x), & \partial_t u(x, 0) &= \varepsilon g(x) & x \in \mathbb{R}^N,
\end{aligned}
\]

where \( V(x) = \frac{\mu}{\sqrt{1+|x|^2}} \), \( N \geq 1 \), \( p > 1 \), and \( \mu \geq 0 \). Moreover, the parameter \( \varepsilon \) is a positive number describing the size of the initial data.

It is worth-mentioning that the presence of nonhomogeneous damping term in (1.1) has an interesting impact on the (global) existence or the nonexistence of the solution of (1.1) and its lifespan. Hence, it is natural to study the influence of the nonlinear terms on the behavior of the solution and see whether or not this may produce a kind of competition between the damping term \( V(x) \partial_t u \) and the time derivative nonlinearity \( |\partial_t u|^p \).

As noted in [6], equation (1.1) can modelize the wave travel in a nonhomogeneous gas with damping, and the space dependent coefficients represent the friction coefficients or...
potential.

The semilinear wave equation equation for classical semilinear wave equation with power nonlinearity namely

\[ \begin{align*}
\partial_t^2 u - \Delta u &= |u|^p & \text{in } [0, T) \times \mathbb{R}^N, \\
u(x, 0) &= \varepsilon f(x), \quad \partial_t u(x, 0) = \varepsilon g(x) & \text{in } \mathbb{R}^N.
\end{align*} \tag{1.2} \]

has been studied extensively. For small initial data, non-negative and compactly supported, the critical power is so-called the Strauss exponent is denoted by \( p_S \) and is given by

\[ p_S(N) = \frac{N + 1 + \sqrt{N^2 + 10N + 7}}{2(N-1)}. \]

Indeed, if \( p \leq p_S \) then there is a blow-up solution for \((1.2)\) and for \( p > p_S \) a global solution exists; see e.g. \([9, 19, 25, 26]\) among many other references.

Coming back to \((1.1)\) and by ignoring the damping term \( V(x) \partial_t u \), the problem is reduced to the classical semilinear wave equation namely

\[ \begin{align*}
\partial_t^2 u - \Delta u &= |\partial_t u|^p & \text{in } [0, T) \times \mathbb{R}^N, \\
u(x, 0) &= \varepsilon f(x), \quad \partial_t u(x, 0) = \varepsilon g(x) & \text{in } \mathbb{R}^N,
\end{align*} \tag{1.3} \]

for which we have the Glassey conjecture. This case is characterized by a critical power, denoted by \( p_G \), and given by

\[ p_G = p_G(N) := 1 + \frac{2}{N - 1}. \tag{1.4} \]

More precisely, if \( p \leq p_G \) then there is no global solution for \((1.3)\), for small initial data, non-negative and compactly supported, and for \( p > p_G \) a global solution exists for small initial data; see e.g. \([4, 5, 8, 17, 18, 23, 27]\).

In the case where the damping term is given by \( \frac{\mu}{1+t} \partial_t u \) instead of \( V(x) \partial_t u \), then the equation \((1.1)\) becomes

\[ \begin{align*}
\partial_t^2 u - \Delta u + \frac{\mu}{1+t} \partial_t u &= |\partial_t u|^p, & \text{in } \mathbb{R}^N \times [0, \infty), \\
u(x, 0) &= \varepsilon f(x), \quad \partial_t u(x, 0) = \varepsilon g(x), & \text{in } \mathbb{R}^N.
\end{align*} \tag{1.5} \]

Concerning the blow-up results and lifespan estimate of the solution of \((1.5)\), a first blow-up region was obtained in \([14]\) for \( p \in (1, p_G(N + 2\mu)] \). Later, an important refinement was performed in \([16]\), using the integral representation formula, where the new bound is given by \( p \in (1, p_G(N + \mu)] \), for \( \mu \in (0, 2) \). Recently, thanks to a better understanding of the corresponding linear problem to \((1.5)\), an improvement in \([2]\) shows that \( p \in (1, p_G(N + \mu)] \) is probably the new critical exponent for \( \mu > 0 \).

Turning to the analogous nonlinear problem \((1.1)\) with \( \mu/\sqrt{1+|x|^2} \) being changed by \( \mu/(1+|x|^2)^{\beta/2} \), for some \( \beta > 1 \). In this case, it is reasonable to expect that the blow-up region is similar to the case of pure wave equation and the scattering damping term has no
influence in the dynamics. The predicted blow up result was obtained in the case \( \beta > 2 \), by Lai and Tu \cite{LT1}. However, up to our knowledge, there is no result in the case \( \beta \in (1, 2] \).

The emphasis in the first part of the manuscript is to establish a blow-up results for solution of \((\text{I}.1)\), in the case where the initial data has compact support, and to determine a candidate as critical exponent. Clearly, in the scale-invariant case the situation is different. Therefore, the goal will be on the comprehension of the influence of the damping term \( V(x) \partial_t u \) on the blow-up result and the lifespan estimate. In fact, our target is to give the upper bound, denoted here by \( p_G(N, \mu) \), delimiting a blow-up region for the energy solution of equation \((\text{I}.1)\). First, as observed for the problem \((\text{I}.5)\), where the damping produces a shift in \( p_G \) in the dimensional parameter of magnitude \( \mu \), we expect that the same phenomenon holds for \((\text{I}.1)\). In other words, we predict that the upper bound is given by:

\[
(1.6) \quad p_G(N, \mu) := p_G(N + \mu) = 1 + \frac{2}{N + \mu - 1}, \quad \text{if} \quad N \geq 1.
\]

The argument which led to our blow up result obtained here in the case for some initial data where the support is compact, is by employing the test function method. In fact, we shall use a test function as product of a cut-off function and an explicit solution of the conjugate equation corresponding to the linear problem of \((\text{I}.1)\). Let us denote that this strategy is inspired by \cite{ISW, LS, LT3}.

In the second part, we consider the solution of \((\text{I}.1)\) in the case where the initial data is decaying slowly at infinity.

It is well-known that the solution of \((\text{I}.2)\) blow up for any \( p > 1 \), for suitable initial data decaying slowly at infinity. Indeed, if \((f, g)\) satisfies that

\[
f(x) \equiv 0 \quad \text{and} \quad g(x) \geq \frac{\Pi_0(x)}{(1 + |x|)^{1+\kappa}},
\]

where \( \Pi_0 \equiv C \), if \( 0 < \kappa < \kappa_0 := \frac{2}{p-1} \), and \( \Pi \) is positive, monotonously increasing, \( \lim_{r \to \infty} \Pi_0(r) = \infty \) if \( \kappa = \kappa_0 \), then the system \((\text{I}.2)\) has a blow up solution for any \( \kappa \in (0, \kappa_0] \). We mention the pioneering results on non-compactly supported initial data by Asakura \cite{A} and also \cite{20, 21}. Furthermore, in the supercritcal range \( p > p_S(N) \), a global (in time) result for solution to \((\text{I}.2)\) in \cite{3}, for small initial data in the weighted space \( L^\infty(\mathbb{R}^N, (1+r^\kappa)) \), if \( \kappa > \kappa_0 \). Therefore, the critical (in the sense of interface between blow-up and global existence in the case \( p > p_S(N) \)) decay of the initial data is \( \kappa_0 \). Let us denote that if \( u \) is a solution of \((\text{I}.2)\), then for all \( \lambda > 0 \), \( u_\lambda(x, t) = \lambda^{\kappa_0} u(\lambda x, \lambda t) \) is also a solution. Therefore, the critical value of \( \kappa_0 \) is somehow related to the scaling of the equation \((\text{I}.2)\).
In the same way, it is proven, in \cite{10}, when \( N = 2 \) and \( N = 3 \) that a solution of (1.3) blows up in finite time, in the case where the initial data satisfies:

\[
    f(x) \equiv 0 \quad \text{and} \quad g(x) \geq \frac{M}{(1 + |x|)^{\kappa}},
\]

for some positive constant \( M \) and \( 0 < \kappa < \kappa_1 := \frac{1}{p-1} \). Moreover, the lifespan \( T_\varepsilon \) satisfies \( T_\varepsilon \leq C \varepsilon^{-\frac{1}{r-\kappa(p-1)}} \). This result was improved in \cite{22}, for any \( N \geq 2 \). (see also \cite{24} for more general initial data). On the other hand, in the case \( N = 3 \), a global result for solution to (1.3) in \( H^1(\mathbb{R}^3) \), for radial initial data small in the weighted space \( L^\infty(\mathbb{R}^N, (1 + r^\kappa)) \), if \( \kappa > \kappa_1 \) and \( p > p_G(N) \). In addition, we remark that if \( u \) is a solution of (1.3), then for all \( \lambda > 0 \),

\[
    u_\lambda(x, t) = \lambda^{\kappa_1} u(\lambda x, \lambda t)
\]

is also a solution. Therefore, the expected critical decay of the initial data is \( \kappa_1 \).

As we said before, we are interested in this part in studying the blow-up result of the solution of (1.1) in the case where the support of the initial data is decaying slowly at infinity. By applying the test function method for a cut-off function, we derive a blow up result for weak solution of (1.1). In particular, we deduce a blow up result for an energy solution to (1.1), if \( p \in (1, p_0(N)) \), where

\[
    p_0(N) := 1 + \frac{2}{N}, \quad \text{for all} \quad N \geq 1.
\]

This paper is organized as follows: First, Section 2 is devoted to the definition of the weak formulation of (1.1), in the energy space and the definition of weak solution, together with the statement of the main theorems of our work. Then, in Section 3, we get a blow-up result in higher dimensions as stated in Theorem 1. Finally, in Section 4, we establish a new blow up result for weak solutions of the problem (1.1) with some initial data as stated in Theorem 2.

2. Main Result

This section is devoted to the statement of the main results. However, before that we start by giving the definition of energy solution for our problem (1.1).

**Definition 1.** Let \( N \geq 1 \), \( f \in H^1(\mathbb{R}^N) \), \( g \in L^2(\mathbb{R}^N) \) and \( T > 0 \). Let \( u \) be such that \( u \in C([0, T), H^1(\mathbb{R}^N)) \cap C^1([0, T), L^2(\mathbb{R}^N)) \) and \( \partial_t u \in L^p_{\text{loc}}((0, T) \times \mathbb{R}^N) \), verifies, for any \( \varphi \in C^1_0\left((0, T) \times \mathbb{R}^N\right) \cap C^\infty((0, T) \times \mathbb{R}^N) \), the following identity:

\[
    \varepsilon \int_{\mathbb{R}^N} g(x) \varphi(x, 0) dx + \int_0^T \int_{\mathbb{R}^N} |\partial_t u|^p \varphi(x, t) dx dt
    = \int_0^T \int_{\mathbb{R}^N} -\partial_t u(t, x) \partial_t \varphi(x, t) dx dt + \int_0^T \int_{\mathbb{R}^N} \nabla u(t, x) \cdot \nabla \varphi(x, t) dx dt
    + \int_0^T \int_{\mathbb{R}^N} V(x) \partial_t u(t, x) \varphi(x, t) dx dt
\]
and the condition \( u(x,0) = \varepsilon f(x) \) is satisfied. Then, \( u \) is called an **energy solution of** \((A)\) on \([0,T)\).

We denote the lifespan for the energy solution by:

\[
T_\varepsilon(f,g) := \sup\{T \in (0, \infty) ; \text{there exists a unique energy solution } u \text{ of } (A)\}.
\]

Moreover, if \( T > 0 \) can be arbitrary chosen, i.e. \( T_\varepsilon(f,g) = \infty \), then \( u \) is called a **global energy solution of** \((A)\).

Furthermore, we shall write the definition of weak solutions for the problem \((A)\).

**Definition 2.** Let \( \mathbb{N} \geq 1 \), \( f \in L^1_{\text{loc}}(\mathbb{R}^N) \), \( g \in L^1_{\text{loc}}(\mathbb{R}^N) \) and \( T > 0 \). Let \( u \) be such that \( u \in L^1_{\text{loc}}((0,T) \times \mathbb{R}^N) \) and \( \partial_t u \in L^p_{\text{loc}}((0,T) \times \mathbb{R}^N) \), verifies, for all \( \varphi \in C^1_c([0,T) \times \mathbb{R}^N) \cap C^\infty((0,T) \times \mathbb{R}^N) \), the following:

\[
\varepsilon \int_{\mathbb{R}^N} g(x) \varphi(x,0) dx + \int_0^T \int_{\mathbb{R}^N} |\partial_t u|^p \varphi(x,t) \, dx \, dt \\
= \int_0^T \int_{\mathbb{R}^N} -\partial_t u(t,x) \partial_t \varphi(x,t) \, dx \, dt - \int_0^T \int_{\mathbb{R}^N} u(t,x) \Delta \varphi(x,t) \, dx \, dt \\
+ \int_0^T \int_{\mathbb{R}^N} V(x) \partial_t u(t,x) \varphi(x,t) \, dx \, dt
\]

and the condition \( u(x,0) = \varepsilon f(x) \) is fulfilled. Then, \( u \) is called a **weak solution of** \((A)\) on \([0,T)\).

We denote the lifespan for the weak solution by:

\[
T_w(f,g) := \sup\{T \in (0, \infty) ; \text{there exists a unique weak solution } u \text{ of } (A)\}.
\]

Moreover, if \( T > 0 \) can be arbitrary chosen, i.e. \( T_w(f,g) = \infty \), then \( u \) is called a **global weak solution of** \((A)\).

Let us mention that, by integrating by parts, an energy solution to \((A)\) is also a weak solution to \((A)\).

The following theorems state the main results of this article.

**Theorem 1.** Let \( R > 0 \), \( \mathbb{N} \geq 2 \), \( \mu \geq 0 \) and \( 1 < p \leq p_G(N+\mu) \). Assume that \( f \in H^1(\mathbb{R}^N) \), \( g \in L^2(\mathbb{R}^N) \) are compactly supported functions on \( B_{\mathbb{R}^N}(0,R) \) and satisfy

\[
\int_{\mathbb{R}^N} (\Delta f(x) + g(x)) \phi(x) dx > 0.
\]

where \( \phi(x) \) is a solution of the elliptic problem \((3.3)\).

Suppose that \( u \) is an energy solution of \((A)\) with compact support

\[
\text{supp } u \in \{(x,t) \in \mathbb{R}^N \times [0,T) : |x| \leq R + t\}.
\]
Then, there exists a constant $\varepsilon_0 = \varepsilon_0(f, g, \mu, N, p, R) > 0$ such that the lifespan $T_\varepsilon$ verifies

\begin{equation}
T_\varepsilon \leq \begin{cases} 
C \varepsilon^{-\frac{2(p-1)}{2(N+\mu-1)(p-1)}} & \text{for } 1 < p < p_G(N + \mu), \\
\exp(C\varepsilon^{-(p-1)}) & \text{for } p = p_G(N + \mu),
\end{cases}
\end{equation}

for $0 < \varepsilon \leq \varepsilon_0$ and some constant $C$ independent of $\varepsilon$.

To state our second result, we define, $\Pi_1 \equiv C_1$, if $0 < \kappa < \frac{1}{p-1}$, and $\Pi_1$ is positive, monotonously increasing, $\lim_{r \to \infty} \Pi_1(r) = \infty$, if $\kappa = \frac{1}{p-1}$.

Here is the statement of second theorem in this paper:

**Theorem 2.** Let $N \geq 1$, $\mu \geq 0$ and $p > 1$. Assume that $f \in L^1_{loc}(\mathbb{R}^N)$ and $g \in L^1_{loc}(\mathbb{R}^N)$. Suppose that $u$ is a weak solution of (1.1). Therefore,

\begin{itemize}
  \item[i)] If $f \equiv 0$ and $g$ satisfies
  \begin{equation}
g(x) \geq \frac{\Pi_1(|x|)}{(1 + |x|)^\kappa},
\end{equation}
  for some $\kappa \leq \frac{1}{p-1}$, then the solution of (1.1) blows-up in finite time. Moreover, if $\kappa < \frac{1}{p-1}$, there exists a constant $\varepsilon_0 = \varepsilon_0(g, \mu, p) > 0$ such that the lifespan $T_w$ verifies
  \begin{equation}
  T_w \leq C \varepsilon^{-\frac{p-1}{1-\kappa(p-1)}}, \quad \forall \varepsilon \in (0, \varepsilon_0].
  \end{equation}
  \item[ii)] If $g \geq 0$ and $f$ satisfies
  \begin{equation}
  \Delta f(x) \geq \frac{\Pi_1(|x|)}{(1 + |x|)^{\kappa+1}},
  \end{equation}
  for some $\kappa \leq \frac{1}{p-1}$, then the solution of (1.1) blows-up in finite time. Moreover, if $\kappa < \frac{1}{p-1}$, there exists a constant $\varepsilon_0 = \varepsilon_0(f, g, \mu, p) > 0$ such that the lifespan $T_w$ verifies (2.7).
  \item[iii)] If $f \in L^1(\mathbb{R}^N)$ and $g$ satisfies (2.6), for some $\kappa < \min(N + 1, \frac{1}{p-1})$, then the solution of (1.1) blows-up in finite time. Moreover, there exists a constant $\varepsilon_0 = \varepsilon_0(f, g, \mu, p) > 0$ such that the lifespan $T_w$ verifies (2.7).
\end{itemize}

By exploiting the fact that, if $f \in H^1(\mathbb{R}^N)$ and $g \in L^2(\mathbb{R}^N)$ and by integrating by parts, we conclude that a weak solution of (1.1) with initial data in $H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ is also an energy solution of (1.1). Therefore, an important consequence of Theorem 2 is the following result:

**Corollary 1.** Let $\mu \geq 0$ and $1 < p < 1 + \frac{2}{N}$. Assume that $f \in H^1(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ and $g \in L^2(\mathbb{R}^N)$ and $g$ satisfying (2.6), for some $\kappa \in \left(\frac{N}{2}, \frac{1}{p-1}\right)$ and $\kappa < N + 1$, then the weak solution $u$ of (1.1) is an energy solution. Therefore the solution of (1.1) blows-up in finite time.
time. Moreover, there exists a constant \( \varepsilon_0 = \varepsilon_0(f,g,\mu,p) > 0 \) such that the lifespan \( T_\varepsilon \) verifies

\[
T_\varepsilon \leq C \varepsilon^{-\frac{\mu-1}{r-k(p-1)}},
\]

for \( 0 < \varepsilon \leq \varepsilon_0 \) and some constant \( C \) independent of \( \varepsilon \).

**Remark 2.1.** The result obtained in Theorem 1 holds only in high space dimensions, \( N \geq 2 \). Although, we expect the result is also true in the one-dimensional case, this has to be confirmed.

**Remark 2.2.** The blow up result stated in Corollary 1 shows that there exist a blow-up region not depending on the parameter \( \mu \). In addition, we deduce that, in the case \( \mu > 1 \), the result obtained in Corollary 1 implies that, for some initial data in the energy space with non compact support, there is a new region comparing to the result obtained in Theorem 1.

**Remark 2.3.** Note that the result in Theorem 1 holds true after replacing the linear damping term in (1.1) \( V(x)\partial_t u \) by \( b(x)\partial_t u \) with \( (b(x) - V(x)) \) belongs to \( L^1(\mathbb{R}^N) \). The proof of this generalized damping case can be obtained by following the same steps as in the proofs of Theorem 1 with the necessary modifications related the test functions.

**Remark 2.4.** The techniques used in this article can be easily adapted in other contexts. More precisely, the case of the equation (1.1) with mass term \( \mu^2 
abla |x|^2 u \), for suitable values of \( \mu \) and \( \nu \). Furthermore, we can use the aforementioned techniques to study the associated system of (1.1).

Throughout this article, we will denote by \( C \) a generic positive constant which may depend on the data \( (p,\mu,N,f,g) \) but not on \( \varepsilon \) and whose the value may change from line to line. Nevertheless, we will precise the dependence of the constant \( C \) on the parameters of the problem when it is necessary.

### 3. Blow-up results in higher dimensions

This section is devoted to the proof of Theorem 1 which is somehow related to determine the critical exponent associated with the nonlinear term in the problem (1.1) in the higher dimensional space.

It is well known that the choice of the test function that will be introduced later is crucial. In fact, we construct a particular positive solution \( \psi(x,t) \) with separated variables and satisfies the conjugate equation corresponding to the linear problem, namely \( \psi(x,t) \) satisfies

\[
\partial_t^2 \psi(x,t) - \Delta \psi(x,t) - V(x) \partial_t \psi(x,t) = 0.
\]
More precisely, we choose the function $\psi$ given by:

$$\psi(x, t) := \rho(t) \phi(x); \quad \rho(t) := e^{-t}$$

where $\phi(x)$ is a solution of the elliptic problem

$$\Delta \phi(x) = (1 + V(x)) \phi(x), \quad \forall x \in \mathbb{R}^N.$$ 

Note that the existence of a positive solution for the elliptic problem (3.3) is studied in [12] in the case where $N \geq 2$. In fact, from Lemma 2.4 in [12], we know that there exists a $C^2(\mathbb{R}^N)$ function $\phi$ solution of (3.3) which satisfies

$$0 < \phi(x) \leq C_0 (1 + |x|)^{-\frac{N-1-\mu}{2}} e^{|x|}, \quad \forall x \in \mathbb{R}^N,$$

for some $C_0 > 0$.

Now, we are in a position to state and prove the following:

**Lemma 3.1.** There exists a constant $C = C(N, R, \mu) > 0$ such that

$$\int_{|x| \leq R+t} \psi(x, t) \, dx \leq C (1 + t)^{\frac{N-1+\mu}{2}}, \quad \forall t \geq 0.$$ 

*Proof.* Integrating (3.4) over the set $\{x, |x| \leq R+t\}$, implies

$$\int_{|x| \leq R+t} \phi(x) \, dx \leq C \int_0^{R+t} r^{N-1}(1 + r)^{-\frac{N-1-\mu}{2}} e^r \, dr, \quad \forall t \geq 0.$$ 

In addition, it is easy to get

$$\int_0^{R+t} r^{N-1}(1 + r)^{-\frac{N-1-\mu}{2}} e^r \, dr \leq C (1 + t)^{\frac{N-1+\mu}{2}} e^t, \quad \forall t \geq 0.$$ 

Thus, combining (3.6) with (3.7), we derive that

$$\int_{|x| \leq R+t} \phi(x) \, dx \leq C (1 + t)^{\frac{N-1+\mu}{2}} e^t, \quad \forall t \geq 0.$$ 

Employing the estimate (3.8) and the expression of $\psi$ given by (3.2), we deduce (3.5). This concludes the proof of Lemma 3.1. \qed

Now, we are ready to give the proof of Theorem 1.

**Proof of Theorem 1.** For the strategy of proof, we basically follow the test function method. Let $\eta$ be a non-increasing cut-off function such that $\eta(r) \in C^\infty([0, +\infty))$ and satisfies

$$\eta(r) := \begin{cases} 
1 & \text{for } r \leq \frac{1}{2}, \\
\text{decreasing} & \text{for } \frac{1}{2} < r < 1, \\
0 & \text{for } r \geq 1.
\end{cases}$$

(3.9)
Let $T > 0$. Now, we introduce the following test function:

$$\Phi(x,t) := \begin{cases} -\partial_t \left( \eta_M^{2p'}(t) \psi(x,t) \right) \chi(x,t), & \text{for } t > 0, \\ \phi(x) \chi(x,0), & \text{for } t = 0, \end{cases}$$

where $M \in (1,T)$, $p' = \frac{p}{p-1}$, the function $\psi$ is given by (3.2) and

$$\eta_M(t) := \eta \left( \frac{t}{M} \right), \quad \chi(x,t) := \eta \left( \frac{|x|}{2(R+t)} \right).$$

Using Definition and performing an integration by parts in space in the second term in the right-hand side of (2.1), we write

$$\varepsilon \int_{\mathbb{R}^N} g(x)\varphi(x,0)dx + \int_0^T \int_{\mathbb{R}^N} |\partial_t u(x,t)|^p \varphi(x,t) \, dx \, dt$$

$$= - \int_0^T \int_{\mathbb{R}^N} \partial_t u(x,t) \partial_t \varphi(x,t) \, dx \, dt - \int_0^T \int_{\mathbb{R}^N} u(x,t) \Delta \varphi(x,t) \, dx \, dt$$

$$+ \int_0^T \int_{\mathbb{R}^N} V(x) \partial_t u(x,t) \varphi(x,t) \, dx \, dt.$$ 

It is worth mentioning that, $\Phi(x,t) \in C^1([0, +\infty) \times \mathbb{R}^N) \cap C^\infty((0, +\infty) \times \mathbb{R}^N)$. Now, substituting in (3.12) $\varphi(x,t)$ by $\Phi(x,t)$, exploiting the compact support condition (2.4) on $u$, we get

$$\varepsilon \int_{\mathbb{R}^N} g(x)\varphi(x,0)dx - \int_0^T \int_{\mathbb{R}^N} |\partial_t u(x,t)|^p \partial_t \left( \eta_M^{2p'}(t) \psi(x,t) \right) \, dx \, dt$$

$$= \int_0^T \int_{\mathbb{R}^N} \partial_t u(x,t) \partial_t^2 \left( \eta_M^{2p'}(t) \psi(x,t) \right) \, dx \, dt + \int_0^T \int_{\mathbb{R}^N} u(x,t) \partial_t \left( \eta_M^{2p'}(t) \Delta \psi(x,t) \right) \, dx \, dt$$

$$- \int_0^T \int_{\mathbb{R}^N} V(x) \partial_t u(x,t) \partial_t \left( \eta_M^{2p'}(t) \psi(x,t) \right) \, dx \, dt.$$ 

Now, using the fact $\partial_t \psi = -\psi$, we can write

$$- \partial_t \left( \eta_M^{2p'}(t) \psi(x,t) \right) = \eta_M^{2p'}(t) \psi(x,t) - \partial_t \left( \eta_M^{2p'}(t) \right) \psi(x,t), \quad \forall t \geq 0.$$
Therefore, by exploiting the compact support condition \( (2.4) \) on \( u \), integrating by parts, (3.14) and (3.13), we deduce that

\[
\varepsilon \int_{\mathbb{R}^N} (\Delta f(x) + g(x)) \phi(x) dx + \int_0^T \int_{\mathbb{R}^N} |\partial_t u(x,t)|^p \eta_M^{2p'}(t) \psi(x,t) \, dx \, dt
\]

\( (3.15) \)

\[
- \int_0^T \int_{\mathbb{R}^N} |\partial_t u(x,t)|^p \partial_t \left( \eta_M^{2p'}(t) \right) \psi(x,t) \, dx \, dt
\]

\[
= \int_0^T \int_{\mathbb{R}^N} \partial_t u(x,t) \psi(x,t) \left( \partial_t^2 \left( \eta_M^{2p'}(t) \right) - 2 \partial_t \left( \eta_M^{2p'}(t) \right) \right) \, dx \, dt
\]

\[
+ \int_0^T \int_{\mathbb{R}^N} \partial_t u(x,t) \eta_M^{2p'}(t) \left( \partial_t^2 \psi(x,t) - \Delta \psi(x,t) - V(x) \partial_t \psi(x,t) \right) \, dx \, dt
\]

\[
- \int_0^T \int_{\mathbb{R}^N} V(x) \partial_t u(x,t) \psi(x,t) \partial_t \left( \eta_M^{2p'}(t) \right) \, dx \, dt.
\]

By exploiting the fact \( \partial_t \left( \eta_M^{2p'}(t) \right) \leq 0 \), and taking into account that \( \psi \) verify \( (3.1) \), we obtain

\[
\varepsilon \, C_0 + \int_0^T \int_{\mathbb{R}^N} |\partial_t u(x,t)|^p \eta_M^{2p'}(t) \psi(x,t) \, dx \, dt \leq \int_0^T \int_{\mathbb{R}^N} \partial_t u(x,t) \psi(x,t) \partial_t^2 \left( \eta_M^{2p'}(t) \right) \, dx \, dt
\]

\[
- 2 \int_0^T \int_{\mathbb{R}^N} \partial_t u(x,t) \psi(x,t) \partial_t \left( \eta_M^{2p'}(t) \right) \, dx \, dt
\]

\[
- \int_0^T \int_{\mathbb{R}^N} V(x) \partial_t u(x,t) \psi(x,t) \partial_t \left( \eta_M^{2p'}(t) \right) \, dx \, dt
\]

\( =: I_1 + I_2 + I_3, \)

where

\[
C_0 \equiv C_0(f,g) := \int_{\mathbb{R}^N} (\Delta f(x) + g(x)) \phi(x) dx > 0
\]

is a positive constant thanks to \( (2.3) \).

Now, let us define the functions

\[
\theta(t) := \begin{cases} 0 & \text{for } t \leq \frac{t}{2}, \\ \eta(t) & \text{for } t > \frac{t}{2}, \end{cases}
\]

\( (3.17) \)

and

\[
\theta_M(t) := \theta \left( \frac{t}{M} \right).
\]

\( (3.18) \)

A straightforward computation implies the following inequalities:

\[
|\partial_t \left( \eta_M^{2p'}(t) \right)| \leq \frac{C}{M^2} \theta_M^{2p'}(t),
\]

\( E1 \)

\[
|\partial_t^2 \left( \eta_M^{2p'}(t) \right)| \leq \frac{C}{M^{p'} \theta_M^{2p'}}(t).
\]

\( E2 \)

Again here thanks to the fact that the compact support condition \( (2.4) \) on \( u \), \( (3.5) \), \( (3.20) \), and Hölder’s inequality, we deduce that
\[ I_1 \leq \frac{C}{M^2} \left( \int_0^M \int_{|x| \leq t+R} \psi(x,t) \, dx \, dt \right) \frac{1}{p} \left( \int_0^T \int_{\mathbb{R}^N} |\partial_t u(x,t)|^{p \theta_M^2(t)} \psi(x,t) \, dx \, dt \right)^\frac{1}{p} \]

**D2** (3.21) \[ \leq \frac{C}{M^2} \left( \int_0^M (1 + t)^{\frac{N-1+\rho}{2}} \, dt \right) \frac{1}{p} \left( \int_0^T \int_{\mathbb{R}^N} |\partial_t u(x,t)|^{p \theta_M^2(t)} \psi(x,t) \, dx \, dt \right)^\frac{1}{p} \]

\[ \leq CM^{-1+\frac{(N+1+\rho)(p-1)-2}{2p}} \left( \int_0^T \int_{\mathbb{R}^N} |\partial_t u(x,t)|^{p \theta_M^2(t)} \psi(x,t) \, dx \, dt \right)^\frac{1}{p}. \]

Similarly, by (3.20) we obtain

**D3** (3.22) \[ I_2 \leq CM^{-1+\frac{(N+1+\rho)(p-1)-2}{2p}} \left( \int_0^T \int_{\mathbb{R}^N} |\partial_t u(x,t)|^{p \theta_M^2(t)} \psi(x,t) \, dx \, dt \right)^\frac{1}{p}. \]

Also, similar estimations yield to

**D4** (3.23) \[ I_3 \leq CM^{-1+\frac{(N+1+\rho)(p-1)-2}{2p}} \left( \int_0^T \int_{\mathbb{R}^N} |\partial_t u(x,t)|^{p \theta_M^2(t)} \psi(x,t) \, dx \, dt \right)^\frac{1}{p}. \]

Gathering (3.16), (3.21), (3.22) and (3.23), we infer

**f1** (3.24) \[ \varepsilon C_0 + \int_0^T \int_{\mathbb{R}^N} |\partial_t u(x,t)|^{p \eta_M^{2\rho}}(t) \psi(x,t) \, dx \, dt \leq CM^{-1+\frac{(N+1+\rho)(p-1)-2}{2p}} \]

\[ \times \left( \int_0^T \int_{\mathbb{R}^N} |\partial_t u(x,t)|^{p \theta_M^{2\rho}}(t) \psi(x,t) \, dx \, dt \right)^\frac{1}{p}. \]

Now, we introduce the following:

**ff** (3.25) \[ F(M) = \int_1^M \left( \int_0^T \int_{\mathbb{R}^N} |\partial_t u(x,t)|^{p \psi(x,t) \theta_M^{2\rho}}(t) \, dx \, dt \right)^\frac{1}{p} \, d\rho, \quad \forall M \in [1,T). \]

Utilizing the definition of \( \theta_M \) (given by (3.18)), we easily write

\[ F(M) = \int_0^T \int_{\mathbb{R}^N} |\partial_t u(x,t)|^{p \psi(x,t)} \int_1^M \theta_M^{2\rho} \left( \frac{t}{M} \right) \frac{1}{\rho} \, d\rho \, dx \, dt \]

**sigma0** (3.26) \[ = \int_0^T \int_{\mathbb{R}^N} |\partial_t u(x,t)|^{p \psi(x,t)} \int_{\frac{1}{M}}^t \theta_M^{2\rho} \left( \frac{t}{M} \right) \frac{1}{\rho} \, d\rho \, dx \, dt. \]

Let us recall from the expressions of \( \theta, \theta_M, \eta \) and \( \eta_M \) defined in (3.17), (3.18), (3.9) and (3.11) that we have

**sigma** (3.27) \[ F(M) \leq \int_0^T \int_{\mathbb{R}^N} |\partial_t u(x,t)|^{p \psi(x,t) \eta_M^{2\rho}} \left( \frac{t}{M} \right) \int_{\frac{1}{M}}^1 \frac{1}{\rho} \, d\rho \, dx \, dt \]

\[ = \ln 2 \int_0^T \int_{\mathbb{R}^N} |\partial_t u(x,t)|^{p \psi(x,t) \eta_M^{2\rho}}(t) \, dx \, dt. \]
A differentiation in \( M \) of the equation \( (3.25) \) gives

\[
\text{diff} \quad F'(M) = \frac{1}{M} \int_0^T \int_{\mathbb{R}^n} |\partial_t u(x,t)|^p \psi(x,t) \theta_M^{2p'}(t) \, dx \, dt, \quad \forall M \in [1, T).
\]

Combining \( (3.24) \) together with \( (3.27) \) and \( (3.28) \), we get

\[
M^{\frac{N-1+\mu(p-1)}{2}} F'(M) \geq C(C_0 \varepsilon + F(M))^p, \quad \forall M \in [1, T).
\]

Therefore, we easily obtain the blowup in finite time for the functional \( F(M) \). This follows \( (2.5) \) and we complete the proof of Theorem 1.

\[\square\]

4. Blow-up results in the case of weak solutions

In this section, we prove Theorem 2 here.

4.1. Proof of Theorem 2.

**Proof of Theorem 2.** For the strategy of proof, we basically follow the test function method.

Let \( \xi \) be a cut-off function such that \( \xi(r) \in C^\infty([0, +\infty)), \, 0 \leq \xi \leq 1 \), and satisfies

\[
\xi(r) := \begin{cases} 
0 & \text{for } 0 \leq r \leq 1, \\
\text{increasing} & \text{for } 1 \leq r \leq 2, \\
1 & \text{for } 2 \leq r \leq 3, \\
\text{decreasing} & \text{for } 3 \leq r \leq 4, \\
0 & \text{for } r \geq 4.
\end{cases}
\]

Let \( \eta \) be a cut-off function such that \( \eta(r) \in C^\infty([0, +\infty)) \) and satisfies

\[
\eta(r) := \begin{cases} 
1 & \text{for } r \leq \frac{1}{2}, \\
\text{decreasing} & \text{for } \frac{1}{2} < r < 1, \\
0 & \text{for } r \geq 1.
\end{cases}
\]

If \( T_\varepsilon \leq 1 \), then the assertion is trivial by choosing \( \varepsilon \) small enough. Assume that \( T_\varepsilon \geq 1 \) and let \( T \in (1, T_\varepsilon) \). Now, we introduce the following test function:

\[
\Phi(x,t) := \eta_T^k(t) \phi_T^\ell(x),
\]

where \( k, \ell \geq 2p' \), and

\[
\eta_T(t) := \eta \left( \frac{t}{T} \right), \quad \phi_T(x) := \xi \left( \frac{|x|}{T} \right).
\]

Let us define an additional function \( \zeta_T = \zeta_T(t) \) such that

\[
\zeta_T(t) := \int_t^\infty \eta_T^k(\tau) \, d\tau.
\]

From \( (4.5) \), we write \( \zeta_T'(t) = -\eta_T^k(t) \), and \( \text{supp } \zeta_T \subseteq [0, T] \).
Substituting in (2.2) \( \Phi(x, t) \) by \( \Phi(x, t) \), we get

\[
(4.6) \quad \varepsilon \int_{\mathbb{R}^N} g(x) \phi_T^\varepsilon(x) dx + \int_0^T \int_{\mathbb{R}^N} |\partial_t u(x, t)|^p \Phi(x, t) \, dx \, dt
\]

\[
= - k \int_0^T \int_{\mathbb{R}^N} \partial_t u(x, t) \eta_t^{k-1} \eta_t' \phi_T^\varepsilon(x) \, dx \, dt + \int_0^T \int_{\mathbb{R}^N} \frac{d}{dt} \zeta_T(t) u(x, t) \Delta(\phi_T^\varepsilon(x)) \, dx \, dt
\]

\[
+ \int_0^T \int_{\mathbb{R}^N} V(x) \partial_t u(x, t) \eta_t^k \phi_T^\varepsilon(x) \, dx \, dt.
\]

Performing an integration by parts for the second term in the second line yields

\[
(4.7) \quad \varepsilon \int_{\mathbb{R}^N} g(x) \phi_T^\varepsilon(x) dx + \int_0^T \int_{\mathbb{R}^N} |\partial_t u(x, t)|^p \Phi(x, t) \, dx \, dt
\]

\[
= - k \int_0^T \int_{\mathbb{R}^N} \partial_t u(x, t) \eta_t^{k-1} \eta_t' \phi_T^\varepsilon(x) \, dx \, dt - \int_0^T \int_{\mathbb{R}^N} \partial_t u(x, t) \zeta_T(t) \Delta(\phi_T^\varepsilon(x)) \, dx \, dt
\]

\[
+ \int_0^T \int_{\mathbb{R}^N} V(x) \partial_t u(x, t) \eta_t^k \phi_T^\varepsilon(x) \, dx \, dt - \varepsilon \zeta_T(0) \int_{\mathbb{R}^N} f(x) \Delta(\phi_T^\varepsilon(x)) \, dx.
\]

By using the identity (4.7), we get that

\[
(4.8) \quad \varepsilon \zeta_T(0) \int_{\mathbb{R}^N} f(x) \Delta(\phi_T^\varepsilon(x)) \, dx + \varepsilon \int_{\mathbb{R}^N} g(x) \phi_T^\varepsilon(x) dx + \int_0^T \int_{\mathbb{R}^N} |\partial_t u(x, t)|^p \Phi(x, t) \, dx \, dt \leq J_1 + J_2 + J_3,
\]

where

\[
J_1 = : k \int_0^T \int_{\mathbb{R}^N} |\partial_t u(x, t)| \eta_t^{k-1} \eta_t' \phi_T^\varepsilon(x) \, dx \, dt,
\]

\[
J_2 = : \int_0^T \int_{\mathbb{R}^N} |\partial_t u(x, t)| \zeta_T(t) \Delta(\phi_T^\varepsilon(x)) \, dx \, dt,
\]

\[
J_3 = : \int_0^T \int_{\mathbb{R}^N} V(x) |\partial_t u(x, t)| \eta_t^k \phi_T^\varepsilon(x) \, dx \, dt.
\]

Let \( \nu > 0 \). By applying \( \nu \)-Young’s inequality

\[
(4.9) \quad AB \leq \nu A^p + C(\nu, p) B^{p'}, \quad A \geq 0, \quad B \geq 0, \quad p + p' = pp', \quad C(\nu, p) = (\nu p^p)^{-1/(p-1)}(p-1),
\]

we get

\[
(4.10) \quad J_1 \leq \nu \int_0^T \int_{\mathbb{R}^N} |\partial_t u(x, t)|^p \Phi(x, t) \, dx \, dt + C(\nu) \int_0^T \int_{\mathbb{R}^N} \eta_t^{k-p'} \eta_t' \phi_T^\varepsilon(x) \, dx \, dt.
\]

By (1.10), the fact that \( k \geq p' \) and taking into account the expression of \( \eta_T \) and \( \phi_T \) given by (4.4), we conclude

\[
(4.11) \quad J_1 \leq \nu \int_0^T \int_{\mathbb{R}^N} |\partial_t u(x, t)|^p \Phi(x, t) \, dx \, dt + C(\nu) T^{N+1-p'}.
\]
Similarly, we obtain

\[
J_2 \leq \nu \int_0^T \int_{\mathbb{R}^N} |\partial_t u(x, t)|^p \Phi(x, t) \, dx \, dt + C(\nu) \int_0^T \int_{\mathbb{R}^N} \eta_T^{-k} \eta_T^{-\nu}(t) \int_t^T \eta_T^{-k}(\tau) \, d\tau \leq T^{\nu} \eta_T(t) \leq T^{\nu}.
\]

Using the fact that \(\eta_T\) is decreasing and \(\text{supp} \, \eta_T \subseteq [0, T]\), we obtain

\[
\eta_T^{-k} \eta_T^{-\nu}(t) \leq \eta_T^{-k} \eta_T^{-\nu}(t) \left( \int_t^T \eta_T^{-k}(\tau) \, d\tau \right) \leq T^{\nu} \eta_T(t) \leq T^{\nu}.
\]

In addition, by exploiting the identity \(\Delta(\phi_T(x)) = \ell \phi_T^{-1}(x) \Delta \phi_T(x) + \ell (\ell - 1) \phi_T^{-2}(x) |\nabla \phi_T(x)|^2\), we deduce

\[
\phi_T^{-\nu}(x) \Delta \phi_T(x) \leq C \phi_T^{-\nu}(x) |\Delta \phi_T(x)|^{\nu} + C \phi_T^{-2\nu}(x) |\nabla \phi_T(x)|^{2\nu} \leq C \phi_T^{-2\nu}(x) T^{-2\nu}.
\]

Plugging the above inequality, (4.13) and the fact that \(\ell - 2\nu \geq 0\) into (4.12), we get

\[
J_2 \leq \nu \int_0^T \int_{\mathbb{R}^N} |\partial_t u(x, t)|^p \Phi(x, t) \, dx \, dt + C(\nu) T^{N+1-p}.
\]

In the same way, thanks to (4.9), we infer

\[
J_3 \leq \nu \int_0^T \int_{\mathbb{R}^N} |\partial_t u(x, t)|^p \Phi(x, t) \, dx \, dt + C(\nu) \int_0^T \int_{\mathbb{R}^N} \eta_T^{k}(t) \phi_T(x) |V(x)|^{\nu} \, dx \, dt.
\]

To estimate the second term on the right-hand side, we have

\[
\int_{\mathbb{R}^N} \phi_T(x) |V(x)|^{\nu} \, dx = \int_{T \leq |x| \leq 4T} \phi_T(x) |V(x)|^{\nu} \, dx \leq C T^{-\nu} \int_{|x| \leq 4T} \phi_T(x) \, dx \leq C T^{N-\nu}.
\]

Consequently, we derive

\[
J_3 \leq \nu \int_0^T \int_{\mathbb{R}^N} |\partial_t u(x, t)|^p \Phi(x, t) \, dx \, dt + C(\nu) T^{N+1-p}.
\]

Gathering (4.8), (4.11), (4.14), and (4.17) and choosing \(\nu\) small enough, we deduce

\[
\varepsilon \zeta_T(0) \int_{\mathbb{R}^N} f(x) \Delta(\phi_T^p(x)) \, dx + \varepsilon \int_{\mathbb{R}^N} g(x) \phi_T^p(x) \, dx + \int_0^T \int_{\mathbb{R}^N} |\partial_t u(x, t)|^p \Phi(x, t) \, dx \, dt \leq C T^{N+1-p}.
\]

Now, we distinguish three cases:

**Case I:** Let us denote in this case \(f \equiv 0\) and \(g\) satisfies

\[
g(x) \geq \Pi_1(|x|) \frac{1}{(1 + |x|)^\kappa},
\]

\[4.19\]
where $\Pi_1 \equiv C$ if $0 < \kappa < \kappa_1 := \frac{1}{p-1}$, and $\Pi_1$ is positive, monotonously increasing, $\lim_{r \to \infty} \Pi_1(r) = \infty$ if $\kappa = \kappa_1$. In this case, the inequality (4.18) becomes

$$\varepsilon \int_{\mathbb{R}^N} g(x)\phi_T^f(x)dx + \int_0^T \int_{\mathbb{R}^N} |\psi u(x,t)|^p \Phi(x,t) \, dx \, dt \leq C T^{N+1-p'}.$$  

By exploiting (4.1), (4.4) and (4.19), we conclude that we have

$$\varepsilon \int_{\mathbb{R}^N} g(x)\phi_T^f(x)dx \geq \varepsilon \int_{2T \leq |x| \leq 3T} g(x)dx \geq \varepsilon \int_{2T \leq |x| \leq 3T} \frac{\Pi_1(|x|)}{(1 + |x|)^\kappa} dx \geq C \Pi_1(2T) \varepsilon T^{N-\kappa},$$

for any $T > 1$. By combining (4.20) and (4.21), we obtain

$$C \Pi_1(2T) \varepsilon \leq T^{\kappa+1-p'}, \quad \text{for all } T > 1,$$

which leads, using $\kappa \leq p' - 1$, to a contradiction by letting $T \to \infty$. In addition, when $\kappa < p' - 1$, it is easy to derive there exists a constant $\varepsilon_0 = \varepsilon_0(g, N, p, \mu) > 0$ such that $T_w$ satisfies

$$T_w \leq C \varepsilon^{-\frac{p-1}{\kappa(p'-1)}}, \quad \text{for all } \varepsilon \leq \varepsilon_0.$$

**Case II:** First, we recall here that $f$ and $g$ satisfy

$$\nabla f(x) \geq \frac{\Pi_1(|x|)}{(1 + |x|)^{\kappa+1}} \quad \text{and} \quad g(x) \geq 0,$$

where $\Pi_1 \equiv C$, if $0 < \kappa < \frac{1}{p-1}$, and $\Pi_1$ is positive, monotonously increasing, $\lim_{r \to \infty} \Pi_1(r) = \infty$ if $\kappa = \frac{1}{p-1}$. In this case, the inequality (4.18) implies

$$\varepsilon \zeta_T(0) \int_{\mathbb{R}^N} \Delta f(x)\phi_T^f(x) dx \leq C T^{N+1-p'}, \quad \text{for all } T \geq 1.$$

On the other hand, using (4.1), (4.4) and (4.22), we get

$$\varepsilon \zeta_T(0) \int_{\mathbb{R}^N} \Delta f(x)\phi_T^f(x) dx \geq \varepsilon \zeta_T(0) \int_{2T \leq |x| \leq 3T} \frac{\Pi_1(|x|)}{(1 + |x|)^{\kappa+1}} dx.$$

Therefore,

$$\varepsilon \zeta_T(0) \int_{\mathbb{R}^N} \Delta f(x)\phi_T^f(x) dx \geq C \varepsilon \zeta_T(0) \Pi_1(2T) T^{N-\kappa-1}, \quad \text{for all } T \geq 1.$$

Furthermore, taking account of

$$\zeta_T(0) = \int_0^T \eta^k \left(\frac{T}{\tau}\right) d\tau \geq \int_0^{T/2} d\tau \geq \frac{T}{2},$$

we infer

$$\varepsilon \zeta_T(0) \int_{\mathbb{R}^N} \Delta f(x)\phi_T^f(x) dx \geq C \varepsilon \Pi_1(2T) T^{N-\kappa}, \quad \text{for all } T \geq 1.$$  

Now, combining (4.23) and (4.26), we obtain

$$\varepsilon \zeta_T(0) \int_{\mathbb{R}^N} \Delta f(x)\phi_T^f(x) dx \geq C \varepsilon \Pi_1(2T) T^{N-\kappa}, \quad \text{for all } T \geq 1.$$
which leads, using $\kappa \leq p' - 1$, to a contradiction by letting $T \to \infty$. In addition, when $\kappa < p' - 1$, it is easy to derive a constant $\varepsilon_0 = \varepsilon_0(g, N, p, \mu) > 0$ such that $T_w$ satisfies

$$T_w \leq C\varepsilon^{\frac{p}{1 - (p - 1)}},$$

for all $\varepsilon \leq \varepsilon_0$.

**Case III:** Let us recall here $f \in L^1(\mathbb{R}^N)$ and $g$ satisfies

$$g(x) \geq \frac{\Pi_1(|x|)}{(1 + |x|)^\kappa},$$

where $\Pi_1 \equiv C$, if $0 < \kappa < \frac{1}{p'}$, and $\Pi_1$ is positive, monotonously increasing, $\lim_{r \to \infty} \Pi_1(r) = \infty$ if $\kappa = \frac{1}{p' - 1}$. In this case, the inequality (4.18) implies

$$\varepsilon \int_{\mathbb{R}^N} g(x)\phi_T^\ell(x)dx \leq CT^{N+1-p'} + \varepsilon\zeta_T(0) \int_{\mathbb{R}^N} |f(x)||\Delta(\phi_T^\ell(x))| dx.$$

Taking account of the fact that $f \in L^1(\mathbb{R})$, the inequality $\|\Delta(\phi_T^\ell(x))\|_{L^\infty} \leq CT^{-2}$ and

$$\zeta_T(0) = \int_0^T \eta_T^k(\tau) d\tau \leq T,$$

we write

$$J_4 \leq C\varepsilon T^{-1} \int_{\mathbb{R}^N} |f(x)| dx = C\|f\|_{L^1(\mathbb{R})}T^{-1}.$$

By exploiting (4.11), (4.4) and (4.28), we conclude that we have

$$\varepsilon \int_{\mathbb{R}^N} g(x)\phi_T^\ell(x)dx \geq \varepsilon \int_{2T \leq |x| \leq 3T} \frac{\Pi_1(|x|)}{(1 + |x|)^\kappa} dx \geq C\Pi_1(2T)\varepsilon T^{N-\kappa},$$

for all $T \geq 1$.

By combining (4.29), (4.30) and (4.31), we obtain

$$C_0\varepsilon \Pi_1(2T)T^{N-\kappa} \leq C_1 T^{N+1-p'} + C_1\|f\|_{L^1(\mathbb{R})}T^{-1},$$

for all $T \geq 1$, which leads, using $\kappa \leq \min(N+1, \frac{1}{p'-1})$, to a contradiction by letting $T \to \infty$.

Moreover, when $\kappa < \min(N+1, \frac{1}{p'-1})$, (4.32) yields

$$C_0\varepsilon T^{N-\kappa} \leq C_1 T^{N+1-p'} + C_1\|f\|_{L^1(\mathbb{R})}T^{-1},$$

for all $T \geq 1$. Therefore,
If $T \geq \left( \frac{2C_1 \parallel f \parallel_1}{C_0} \right)^{1/(N+1-\kappa)} := T_0$, then
\[
C_1 \parallel f \parallel_1 T^{-1} \leq \frac{C_0 \varepsilon}{2} T^{N-\kappa}
\]
for all $T \geq \max(T_0, 1)$.

Hence, the inequality (4.33) becomes
\[
(4.34) \quad C_0 \varepsilon \leq 2C_1 T^{\kappa+1-p'}, \quad \text{for all } T \geq \max(T_0, 1).
\]

which leads, $T_w$ satisfies
\[
T_w \leq C \varepsilon^{-\frac{p-1}{p-1/p}} \varepsilon \leq \varepsilon_0,
\]
for all $\varepsilon \leq \varepsilon_0$,

where
\[
\varepsilon_0 := C \left( \frac{C_0}{2C_1 \parallel f \parallel_1} \right)^{\frac{1-\kappa(p-1)}{p-1} \frac{1}{(N+1-\kappa)} \left( \frac{p-1}{p-1/p} \right)}.
\]

This achieves the proof of Theorem 2. $\square$

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