A Dissipativity Theory for Undiscounted Markov Decision Processes

Sébastien Gros\textsuperscript{a}, Mario Zanon\textsuperscript{b}

\textsuperscript{a}NTNU, Gløshaugen, Trondheim, Norway
\textsuperscript{b}IMT School for Advanced Studies Lucca, Piazza San Francesco 19, 55100, Lucca, Italy

Abstract

Dissipativity theory is central to discussing the stability of policies resulting from minimizing economic stage costs. In its current form, the dissipativity theory applies to problems based on deterministic dynamics, and does not readily extend to Markov Decision Processes, where the dynamics are stochastic. In this paper, we clarify the core reason for this difficulty, and propose a generalization of the dissipativity theory that circumvents it. This generalization is based on nonlinear stage cost functionals, allowing one to discuss the Lyapunov asymptotic stability of policies for Markov Decision Processes in the set of probability measures. This theory is illustrated in the stochastic Linear Quadratic Regulator case, for which a storage functional can be provided analytically. For the sake of brevity, we limit our discussion to undiscounted MDPs.

Key words: Markov Decision Processes, dissipativity, storage functions, economic costs

1 Introduction

The use of optimization-based policies is widespread in control. Such policies are usually collectively referred to as Model Predictive Control (MPC) by the control community. MPC schemes are often used to steer and stabilize a system to a given, feasible reference input and state. In that context, the stage cost minimized in the MPC scheme is typically convex, taking its minimum at the reference. Quadratic costs are the most common choice. This type of MPC scheme is commonly referred to as tracking MPC.

In a control context, the system stability in closed-loop with a policy is a crucial feature. In particular, an asymptotically stable policy ensures that the closed-loop system will be steered to a given steady-state reference.

The stability of tracking MPC schemes is fairly straightforward to establish, and—under a mild controllability assumption—simply requires the MPC stage cost to be lower-bounded by a class-$\mathcal{K}_\infty$ function, with the possible addition of a terminal cost and constraint set if the MPC horizon is finite. When these criteria are fulfilled, a stability argument can be easily constructed via the Lyapunov stability theory [9,15].

Tracking MPC schemes are, however, fairly restrictive as to which stage cost can be used, and using stage costs belonging to a broader class of functions can be beneficial. Indeed, the recent literature on MPC argues for the use of economic stage costs, representing directly the performance of the system with regard to the overall control goals, rather than the specific objective of steering the system to a given reference, see e.g. [5,16]. Such economic objectives often correspond to the energy or the time or the financial cost of performing a given task. It is commonly argued that a policy minimizing an economic stage cost is more conducive to maximizing the system performance than a tracking stage cost can be [14].

While appealing, economic stage costs typically do not satisfy the criteria required to conclude to the stability of the resulting MPC-based control policy. To address that issue, a new stability theory has been developed, commonly referred to as dissipativity theory. The key

Email address: sebastien.gros@ntnu.no (Sébastien Gros)
idea behind that theory is to transform the economic stage cost into one that is lower-bounded by a class-$\mathcal{K}_\infty$ function, while leaving the resulting policy unchanged. This transformation is often referred to as cost rotation, and performed via a so-called storage function. If this transformation is possible, then the stability of economic MPC can be analyzed via the Lyapunov stability theory [1,3,4,8,12,19].

The dissipativity theory applies to systems having deterministic dynamics, and is not yet extended to stochastic systems. MPC for stochastic systems is often treated within the Robust MPC (RMPC) or Stochastic MPC (SMPC) frameworks. The former is equipped with stability theories establishing conditions under which the system trajectories are steered to a set, while the stability of the latter is still less mature. SMPC often targets system trajectories are steered to a set, while the stabilization requires generalizing the concept of stage cost to a Gaussian process noise, for which a storage function is provided.

However, the extension of the classical dissipativity theory to MDPs appears to be challenging. An approach borrowing principles from the stability analysis of RMPC [2,10,18] is arguably possible, but would limit the results to showing the stability of the closed-loop Markov Chain. In section 5, these concepts are deployed in the stochastic LQR context, showing that the proposed approach is sensible.

The paper is organized as follows. Section 2 proposes a discussion on the difficulties of extending the classical dissipativity theory on MDPs. Section 3 proposes a generalization of the classical Lyapunov-based stability arguments for optimal policies to MDPs, using a functional approach. The resulting stability theory shows that the cost function used in MDPs, made of the expected sum of stage costs, does not satisfy the necessary criterion for stability, and that a cost rotation is always needed. Section 4 generalizes the concept of rotation, dissipativity and storage function, allowing for a discussion of MDP stability in a Lyapunov context. In section 5, these concepts are deployed in the stochastic LQR context, showing that the proposed approach is sensible.

2 Stability of Markov Decision Processes

Consider the stochastic dynamic provided by

$$\xi [s_{k+1} | s_k, a_k],$$  \hspace{1cm} (1)

defining the conditional probability measure of observing a transition from the state-action pair $s_k, a_k$ to the subsequent state $s_{k+1}$. Furthermore, consider a deterministic policy

$$a_k = \pi (s_k),$$  \hspace{1cm} (2)

generating a closed-loop Markov Chain with the underlying sequence of probability measures $\rho_0, \ldots, \infty$ defined by:

$$\rho_{k+1} (\cdot) = \int \xi [\cdot | s, \pi (s)] d\rho_k (s) := \mathcal{T} [\rho_k, \pi].$$  \hspace{1cm} (3)

Note that the sequence of probability measures $\rho_0, \ldots, \infty$ can also be interpreted as a sequence of probability densities without changing the discussion proposed in this paper, provided that the measures $\rho_0, \ldots, \infty$ are equipped with probability densities (i.e. if they have Radon-Nykodim derivatives). We consider undiscouted MDPs of the classic form:

$$J^* \left[ \rho_0 \right] = \min_{\pi} \sum_{k=0}^{\infty} \mathbb{E}_{s \sim \rho_k} [L (s, \pi (s)) - L_0]$$  \hspace{1cm} (4a)

s.t. $\rho_{k+1} = \mathcal{T} [\rho_k, \pi].$  \hspace{1cm} (4b)
with optimal policy \( \pi_* \), where \( \rho_0 \) is the initial probability measure, and \( L_0 \) is the solution of the optimal steady-state problem

\[
L_0 = \min_{\pi} \frac{1}{N} \sum_{k=0}^{N} \mathbb{E}_{s \sim \rho_k} [L(s, \pi(s))],
\]

subject to

\[
\rho = T[\rho, \pi],
\]

which yields the optimal steady-state measure \( \rho^* \). In order to frame Problem (4) in the general context of undiscounted MDPs, we observe that, in case \( L_0 \) is also the optimal average cost, given by

\[
L_0 = \min_{\pi} \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N} \mathbb{E}_{s \sim \rho_k} [L(s, \pi(s))],
\]

subject to

\[
\rho_{k+1} = T[\rho_k, \pi],
\]

then Problem (4) yields bias optimality [13].

We are interested in characterizing conditions on the MDP stage cost \( L \) such that the optimal policy \( \pi^* \) solution of (4) is stabilizing the closed-loop Markov Chain to the optimal steady-state solution of (5), i.e., such that

\[
\lim_{k \to \infty} \rho_k = \rho^*,
\]

in some sense that we will specify. We observe that such a limit can fail to exist if, e.g., the sequence \( \rho_0, \rho_1, \ldots, \rho_N \) enters a limit cycle over time rather than converging to a specific measure.

We recall that in the special case where (4) is deterministic, such that \( \rho_k \) reduces to a sequence of Dirac measures, then the stability of (4) can be discussed in the framework of the classic dissipativity theory, using the concept of storage function. In that context, a storage function \( L(s, \pi(s)) \) is sought such that

\[
L(s_k, \pi(s_k)) - \lambda(s_{k+1}) + \lambda(s_k) \geq \varphi(\|s_k - s_*\|) \tag{8}
\]

holds over the system trajectories for a steady state \( s_* \) of the system and a class-\( \mathcal{K}_\infty \) function \( \varphi \). Under the condition that the storage function remains bounded over the prediction horizon then the value function resulting from the rotated cost

\[
L^R(s_k, \pi(s_k)) = L(s_k, \pi(s_k)) - \lambda(s_{k+1}) + \lambda(s_k) \tag{9}
\]

is a Lyapunov function for the system. Hence, the general philosophy of the dissipativity theory is to transform the stage cost \( L \) of an optimal control problem (4) into a new stage cost \( L^R \) that yields the same optimal policy, while resulting in a value function \( J \) that is a Lyapunov function for the closed-loop trajectories. Note that, since we subtract \( L_0 \) from the stage cost, we do not need to subtract from \( L^R \) the steady-state cost

\[
L(s_*, \pi_*(s_*)) = L_0. \tag{10}
\]

A direct extension of this philosophy to treat the stability of MDPs in the form (4) is appealing. The rotated cost would be given by

\[
\mathbb{E}_{s \sim \rho_k} [L^R(s, \pi(s))] = \mathbb{E}_{s \sim \rho_k} [L(s, \pi(s))] + \mathbb{E}_{s \sim \rho_k} [\lambda(s_*) - \lambda(s)]. \tag{11}
\]

and the stage cost \( L^R - L_0 \) would be used in (4). Note that if (5) is formulated by replacing \( L \) with \( L^R \), the solution is still \( L_0 \). In order to prove that \( J_* \) is non-increasing using the standard approach, one would then need

\[
\mathbb{E}_{s \sim \rho_k} [L^R(s_k, \pi(s_k)) - L_0] \geq \varphi(\|s_k - s_*\|). \tag{12}
\]

However, \( L^R - L_0 \) cannot be non-negative everywhere, such that (12) cannot hold by construction. This statement is supported in a more formal way by the following Lemma.

**Lemma 1** The following statements cannot hold together:

1. \( L^R(s, \pi(s)) - L_0 \geq 0 \) for all \( s \), and is zero on a set of zero measure.
2. \( J_*[\rho_0] \) exists and is bounded for a non-empty set of initial probability measures \( \rho_0 \).
3. \( \lim_{k \to \infty} \rho_k \) is a measure equipped with a probability density function.

**Proof.** By contradiction. In order for \( J_*[\rho_0] \) to exists and be bounded for certain densities \( \rho_0 \), the limit

\[
\lim_{k \to \infty} \mathbb{E}_{s \sim \rho_k} [L^R(s, \pi(s)) - L_0] = 0 \tag{13}
\]

must hold. If \( L^R - L_0 \geq 0 \) and \( L^R - L_0 = 0 \) on a set of zero measure, then (13) requires that \( \lim_{k \to \infty} \rho_k \) converges to a Dirac measure centred on the set where \( L^R - L_0 = 0 \). This is, however, in contradiction with 3. \( \square \)

The consequences of that Lemma is that the stage cost of an MDP can be rotated such that the resulting value function is a Lyapunov function in some specific cases only, i.e.:

- If the MDP converges to a deterministic steady-state, such that \( \rho_k \) becomes a Dirac measure. This typically requires a “vanishing perturbations” scenario.
• If the rotation makes \(L^k = 0\) on a measurable set. The stability discussion is then limited to a convergence of the support of the probability measure to that set, and requires a bounded support. The discussion is then similar to the one used in RMPC.

Unfortunately, the classic dissipativity theory cannot apply to a more general context. In this paper, we will show that the issues detailed above do not stem from the philosophy underlying the classic dissipativity theory, but simply from considering a cost rotation (11) that is restricted to being linear in the probability measures \(\rho_k\).

### 3 Lyapunov Stability for MDPs

In order to address the issues described in the previous section, we propose to generalize MDP (4) into the following functional optimal control problem (FOCP):

\[
V_*[\rho_0] = \min_\pi \sum_{k=0}^{\infty} \mathcal{L}[\rho_k, \pi],
\]

s.t. \(\rho_{k+1} = \mathcal{T}[\rho_k, \pi]\),

(14a)

where \(\mathcal{L}\) is a (possibly nonlinear) functional over the probability measures \(\rho_k\), and the policy \(\pi\). We assume here that \(V_*[\rho_0]\) is finite for a non-empty set of measures \(\rho_0\). One can readily observe that problem (4) can be cast as (14) by selecting

\[
\mathcal{L}[\rho_k, \pi] = \mathbb{E}_{s \sim \rho_k}[L(s, \pi(s))].
\]

(15)

However, the choice (15) is a special case of (14) as it restricts the cost functional \(\mathcal{L}\) to linear functionals of \(\rho_k\). We will see in this paper that the freedom of manipulating nonlinear functionals in \(\mathcal{L}\) is the key to generalizing the dissipativity theory to MDPs.

We detail next how (14) makes it possible to derive a fairly straightforward and classic Lyapunov stability result on the set of probability measures. To that end, let us introduce the following key concepts.

**Definition 1 (Dissimilarity measure)** Let us define a dissimilarity measure \(D(\cdot \| \cdot) : \mathcal{P} \times \mathcal{P} \rightarrow \mathbb{R}\) as an application from the set of probability measures to the real positive numbers, such that:

\[
D(\rho \| \rho') \geq 0 \quad \text{and} \quad D(\rho \| \rho) = 0, \quad \forall \rho, \rho' \in \mathcal{P},
\]

(16)

where \(\mathcal{P}\) is (possibly) a subset of the set of probability measures.

Useful examples of dissimilarity measures are the Kullback-Leibler divergence, the Wasserstein metric, and the total variation distance. The notion of stability on the set of probability measures can then be formalized as follows.

**Definition 2 (D-Stability)** A Markov Chain is D-stable with respect to probability measure \(\rho^*\) and dissimilarity measure \(D\) if, for each \(\epsilon > 0\) there exists a \(\delta(\epsilon) > 0\) such that \(D(\rho_0 \| \rho^*) < \delta\) implies \(D(\rho_k \| \rho^*) < \epsilon\). If, moreover, the probability measure \(\rho^*\) is D-attractive, i.e.,

\[
\lim_{k \to \infty} D(\rho_k \| \rho^*) = 0,
\]

(17)

holds almost everywhere, then the Markov Chain is D-asymptotically stable.

The next theorem formalizes the stability of FOCP (14) on the set of probability measures, following the same arguments as classic Lyapunov stability for optimal policies over deterministic problems.

**Theorem 1** Assume that the inequalities

\[
\mathcal{L}[\rho, \pi_\star] \geq g_1(D(\rho \| \rho^*)) \quad (18a)
\]

\[
V_*[\rho] \leq g_2(D(\rho \| \rho^*)) \quad (18b)
\]

hold for some class-\(K_\infty\) functions \(g_1, g_2\) and for all \(\rho \in \Xi \subseteq \mathcal{P}\), where set \(\Xi\) is a non-empty set such that \(V_* < \infty\) on \(\Xi\). Then the Markov chain is D-asymptotically stable.

**Remark 1** Note that assumption (18b) corresponds to standard assumption in the context of MPC—often referred to as a form of weak controllability [15]—which requires that the value function is upper-bounded by a class-\(K_\infty\) function of a norm of \(s - s_\star\).

**Proof.** We first observe that because \(V_*\) is bounded on \(\Xi\), \(\Xi\) is positive invariant and

\[
V_*[\rho_{k+1}] - V_*[\rho_k] = -\mathcal{L}[\rho_k, \pi_\star] \leq -g_1(D(\rho_k \| \rho^*))
\]

(19)

holds on \(\Xi\). Furthermore, we observe that from (18a), the bound:

\[
V_*[\rho_k] \geq \mathcal{L}[\rho_k, \pi_\star] \geq g_1(D(\rho_k \| \rho^*)) \geq 0
\]

(20)

holds for any \(\rho_k \in \Xi\). Hence \(V_*[\rho_k] \geq 0\) is bounded and monotonically decreasing on \(\Xi\), such that it must converge to a finite positive value \(\bar{V}\) as \(k \to \infty\). We then
need to prove that $V = 0$. To that end, consider $\delta, \epsilon > 0$ selected as

$$D(\rho_0 \parallel \rho^*) \leq \delta, \quad \epsilon = g_1^{-1}(g_2(\delta)), \quad (21)$$

such that

$$V_*[\rho_0] \leq g_2(\delta) = g_1(\epsilon). \quad (22)$$

Using (19) and (20), we observe that for all $k$:

$$D(\rho_k \parallel \rho^*) \leq g_1^{-1}(V_*[\rho_k]) \leq g_1^{-1}(V_*[\rho_0])$$

$$\leq g_1^{-1}(g_1(\epsilon)) = \epsilon, \quad (23)$$

which proves stability. In order to prove attractivity, we proceed by contradiction. Assume that

$$\lim_{k \rightarrow \infty} V_*[\rho_k] = \bar{V} > 0, \quad (24)$$

then using (18b) and (23), the inequalities

$$g_2^{-1}(\bar{V}) \leq \lim_{k \rightarrow \infty} D(\rho_k \parallel \rho^*) \leq g_1^{-1}(\bar{V}) \quad (25)$$

hold. Using (19) we obtain:

$$V_*[\rho_k] \leq V_*[\rho_0] - \sum_{j=0}^{k} g_1(D(\rho_j \parallel \rho^*)). \quad (26)$$

Since $D(\rho_k \parallel \rho^*)$ converges to the interval $[g_2^{-1}(\bar{V}), g_1^{-1}(\bar{V})]$, then

$$\lim_{k \rightarrow \infty} g_1(D(\rho_k \parallel \rho^*)) \geq g_1(g_2^{-1}(\bar{V})) > 0, \quad (27)$$

such that $V_*[\rho_k] \rightarrow -\infty$ as $k \rightarrow \infty$, which is in contradiction with (20). Consequently, $\bar{V} = 0$, and (17) must hold. 

Note that the stability result of this theorem can carry several meanings, depending on the properties of $D$. If, e.g., $D(\rho_1 \parallel \rho_2) = \|E_{s \sim \rho_1} [s] - E_{s \sim \rho_2} [s]\|$, then only the expected value of the state is guaranteed to converge. In case the selected dissimilarity measure carries stronger properties, stronger stability results arise. Let us detail a useful special case in the next Corollary.

**Corollary 1** Assume that the assumptions of Theorem 1 hold, and the dissimilarity measure $D(\rho \parallel \rho^*)$ is such that $D(\rho \parallel \rho^*) = 0$ implies that $\rho = \rho^*$ almost everywhere. Then

$$\lim_{k \rightarrow \infty} \rho_k(\cdot) = \rho^*(\cdot) \quad (28)$$

holds almost everywhere.

**Proof.** The limit (17) follows from Theorem 1. By the properties assumed on the dissimilarity measure, this directly entails (28).

It may be useful here to discuss what form of stability is established in Theorem 1. Stability proofs in the context of Classic MPC and Economic MPC discuss the behavior of single trajectories, starting from arbitrary initial conditions in a set, and proves the convergence to an optimal steady-state. In RMPC one discusses the behavior of singular stochastic trajectories, and proves the convergence to a set, without describing the behavior inside that set. In contrast, Theorem 1 discusses the behavior of trajectories by showing that their asymptotic behavior is to be distributed according to a distribution with zero dissimilarity with respect to the optimal steady-state measure of the MDP. For suitably selected dissimilarity measures (see, e.g., Corollary 1), this entails that these two distributions must coincide.

We now turn to discussing how the stability of the MDP resulting from a generic stage cost functional $\mathcal{L}$ can be discussed in terms of (18a) via functional cost rotations.

### 4 Functional Cost Rotations

Making a Lyapunov stability argument on FOCP (14) requires the cost functional $\mathcal{L}[\rho_k, \pi]$ to satisfy (18a). Following the arguments of Lemma 1, one can readily observe that, in general, for MDP (4) to be well-posed, the MDP stage cost $L - L_0$ cannot be strictly positive.

As a result, when recasting a given MDP (4) in its equivalent functional form (14) using identity (15), the resulting functional stage cost $\mathcal{L}[\rho, \pi]$ cannot be positive for all probability measures $\rho$, such that (18a) cannot hold. This challenges by construction the application of classical Lyapunov stability to general MDPs. As detailed in Lemma 1, a “classic rotation” in the form (11) does not address that issue in general.

Fortunately, it is possible to tackle these difficulties by adopting a more general cost rotation than (11). More specifically, we will consider functional cost rotations in the form:

$$\mathcal{L}^R[\rho_k, \pi] = \mathcal{L}[\rho_k, \pi] - \lambda[\rho_{k+1}] + \lambda[\rho_k], \quad (29)$$

where $\lambda$ is a nonlinear functional. Rotation (11) is then a special case of (29), where the form

$$\lambda[\rho_k] = E_{s \sim \rho_k}[\lambda(s)] \quad (30)$$
Similarly to classical cost rotations, we observe that (29) leaves the policy solution of (14) unchanged, as long as \( \lambda[\rho_k] \) is bounded for all \( k \). A generalized dissipativity criterion then requires that

\[
\mathcal{L}[\rho_k, \pi] - \lambda[\rho_{k+1}] + \lambda[\rho_k] \geq \varrho(D(\rho \parallel \rho^*)) \tag{31}
\]

holds for all \( \rho_k \) such that \( V_*(\rho_k) \) is finite. The dissipativity criterion (31) then yields \( D \)-asymptotic stability. Indeed, let us define a rotated FOCP as:

\[
V^R_*(\rho_0) = \min_\pi \lim_{N \to \infty} \sum_{k=0}^{N-1} \mathcal{L}^R[\rho_k, \pi] + \lambda[\rho_N] \tag{32a}
\]

\[
\text{s.t. } \rho_{k+1} = T[\rho_k, \pi], \tag{32b}
\]

and establish \( D \)-asymptotic stability in the next theorem.

**Theorem 2** Assume that there exists a bounded storage functional \( \lambda \) satisfying (31). Assume moreover that

\[
V_*(\rho) \leq g_2(D(\rho \parallel \rho^*)). \tag{33}
\]

Then, the rotated FOCP (32) and the original FOCP (14) deliver the same primal solution. Moreover, the Markov chain is \( D \)-asymptotically stable with respect to the probability measure \( \rho^* \) and dissimilarity measure \( D \).

**Proof.** The first claim follows from standard arguments, since boundedness of \( \lambda[\rho_k] \) entails

\[
\sum_{k=0}^{N-1} \mathcal{L}^R[\rho_k, \pi] + \lambda[\rho_N] = \sum_{k=0}^{N-1} \mathcal{L}[\rho_k, \pi] + \lambda[\rho_0]. \tag{34}
\]

By taking the limit \( N \to \infty \), the cost of the rotated and original FOCP only differ by the constant \( \lambda[\rho_0] \), for any evolution of the density \( \rho_k \) which satisfies the Markov chain (1). Consequently, we obtain \( V^R_*(\rho_0) = V_*(\rho_0) + \lambda[\rho_0] \).

We now observe that \( \mathcal{L}^R \) and \( V^R_* \) satisfy the assumptions of Theorem 1. Consequently, the rotated FOCP yields a feedback policy guaranteeing that the closed-loop system satisfies \( D \)-asymptotic stability with respect to density \( \rho^* \). Because the primal solution of the rotated and original FOCP coincide, this proves the second claim.

The existence of a functional \( \lambda \) satisfying (31) entails a stability of (14) in the set of probability measures. We will show next that such a storage function exists in the LQR case, hence giving credence to this concept. In line with Lemma 1, the resulting modified cost functional does not take a form (30). It is not trivial, and its derivation is fairly technical.

### 5 The LQR Case

In this section, we will develop a storage functional \( \lambda[\rho] \) satisfying (31), for the specific choice \( D = D_{KL} \), where \( D_{KL} \) denotes the Kullback-Leibler divergence. Due to the technicality of the material presented here, most proofs are provided in the Appendix. We consider the LQR problem:

\[
s_{+} = As + Ba + w, \tag{35}
\]

with \( w \sim \mathcal{N}(0, W) \) i.i.d., \( E[ws^\top] = 0 \), \( E[wa^\top] = 0 \) and stage cost

\[
L(s, a) = \begin{bmatrix} s \\ a \end{bmatrix}^\top H \begin{bmatrix} s \\ a \end{bmatrix} - L_0, \quad H = \begin{bmatrix} T & U^\top \\ U & R \end{bmatrix} > 0. \tag{36}
\]

For \( \rho_0 \sim \mathcal{N}(\mu_0, \Sigma_0) \), the dynamics of the system are given by \( \rho_k \sim \mathcal{N}(\mu_k, \Sigma_k) \) where the mean and covariances read as:

\[
\mu_{k+1} = A_c \mu_k \tag{36a}
\]

\[
\Sigma_{k+1} = A_c \Sigma_k A_c^\top + \Sigma_w \tag{36b}
\]

where \( A_c = A - BK \), and \( K \) is the regular LQR matrix gain associated to \( A, B, H \). Furthermore, we have

\[
E_\rho[L(s, \pi(s))] = \text{Tr}(W\Sigma_k) + \mu_k^\top W \mu_k - L_0 \tag{37}
\]

\[
D_{KL}(\mu \parallel \mu^*) = \frac{1}{2} \left( \text{Tr}(\Sigma^{-1}_k) + \mu_k^\top \Sigma^{-1}_k \mu_k - n + \log \det(\Sigma_k) - \log \det(\Sigma^\infty) \right) \tag{38}
\]

where \( n \) is the state-space dimension of the problem,

\[
W = \begin{bmatrix} I & -K^\top \\ K^\top & I \end{bmatrix}
\]

and \( \mu^* \) is given by \( \mu^\infty = 0 \) and by the solution of the Lyapunov equation:

\[
\Sigma^\infty = A_c \Sigma^\infty A_c^\top + \Sigma_w \tag{39}
\]

We observe that for the MDP to be well posed

\[
L_0 = \text{Tr}(W \Sigma^\infty) \tag{40}
\]

must hold. We then observe that

\[
\mathcal{L}[\rho_k, \pi] = \frac{1}{2} \mu_k^\top W \mu_k + \text{Tr}(W(\Sigma_k - \Sigma^\infty)) \tag{41}
\]
Before delivering a storage function, it will be useful to show that the densities $\rho_k$ converge to $\rho^*$ in the $D_{\text{KL}}$ sense.

### 5.1 Convergence under $D_{\text{KL}}$

Before delivering a storage function for the LQR case, we will first show that $D_{\text{KL}} (\rho_k \| \rho^*)$ is monotonically decreasing. This will require several technical results that will be needed for building a storage function. In order to proceed, let us introduce first a first useful technical Lemma.

**Lemma 2** Consider a full-rank matrix $M \in \mathbb{R}^{n \times n}$ such that its maximum singular value, i.e., $\sigma_{\max} (M)$ is less than 1, and consider a symmetric (not necessarily definite) matrix $\Delta \in \mathbb{R}^{n \times n}$. Consider $\Lambda(\cdot)$ the ordered eigenvalues of a matrix. Then the following holds:

$$\Lambda_i (M \Delta M^\top) = \alpha_i \Lambda_i (\Delta), \quad i = 1, \ldots, n$$

(42)

for a sequence $\alpha_1, \ldots, \alpha_n > 0$ with $\alpha_i \leq \sigma_{\max} (M)$.

This Lemma will be instrumental in showing the convergence of the LQR problem in several senses, and in particular in the $D_{\text{KL}}$ sense, i.e., we will show that:

$$D_{\text{KL}} (\rho_{k+1} \| \rho^*) < D_{\text{KL}} (\rho_k \| \rho^*).$$

(43)

In order to obtain this result, the following lemma will be useful, and follows fairly directly from Lemma 2.

**Lemma 3** Under dynamics (36b), the ordered eigenvalues of $\Sigma_{\infty}^{-1} \Sigma_k$, i.e., $\Lambda(\Sigma_{\infty}^{-1} \Sigma_k)$ converge monotonically to 1 without changing sign, i.e.,

$$\alpha_i (\Sigma_{\infty}^{-1} \Sigma_{k+1}) - 1 = \alpha_i (\Sigma_{\infty}^{-1} \Sigma_k) - 1,$$

(44)

for a sequence $\alpha_3, \ldots, \alpha_n > 0$ with $\alpha_i \leq \sigma_{\max} (M)$.

Using Lemma 3, the monotonic decreasing of $D_{\text{KL}}$ (and other similarity measures) under the dynamics (36) can be established. Let us deliver that result next.

**Theorem 3** Consider any dissimilarity measure $D (\rho_k \| \rho^*)$ that can be expressed in the form:

$$D (\rho_k \| \rho^*) = c + \mu^\top_k \Sigma_{\infty}^{-1} \mu_k + \sum_{i=1}^n \zeta (\Lambda_i (\Sigma_{\infty}^{-1} \Sigma_k))$$

(45)

for some function $\zeta$ that is strictly increasing away from 1, and some constant $c$. Then $D (\rho_k \| \rho^*)$ is strictly decreasing under dynamics (36).

Theorem 3 applies to several dissimilarity measures. In particular it applies to $D_{\text{KL}}$. We detail that in the following Corollary.

**Corollary 2** The dynamics (36) converge monotonically under $D_{\text{KL}}$, i.e.,

$$D_{\text{KL}} (\rho_{k+1} \| \rho^*) \leq D_{\text{KL}} (\rho_k \| \rho^*),$$

(46)

and the inequalities are strict for $\rho_k \neq \rho^*$.

**Proof.** We observe that

$$D_{\text{KL}} (\rho \| \rho^*) = \frac{1}{2} \mu_k^\top \Sigma_{\infty}^{-1} \mu_k$$

(47)

$$+ \frac{1}{2} \sum_{i=1}^n \alpha_i (\Sigma_{\infty}^{-1} \Sigma_k) - \log \Lambda_i (\Sigma_{\infty}^{-1} \Sigma_k) - 1,$$

and we observe that the scalar function

$$\zeta (x) = x - \log x - 1$$

(48)

is monotonically increasing away from 1.

The same results hold for, e.g., the $D_{\text{AB}}^{(\alpha, \beta)}$ family of dissimilarity measures. We omit them here for the sake of brevity.

### 5.2 A Storage Functional for the $D_{\text{KL}}$ Dissimilarity Measure

We propose next a storage functional for the LQR case. We need to start with the following technical Lemma.

**Lemma 4** Consider the function:

$$\zeta (\Delta) = \text{Tr} (\Delta) - \log \det (\Delta + I).$$

(49)

For any symmetric matrix $\Delta$ and matrix $M$ such that $\sigma_{\max} (M) < 1$, the following inequality holds:

$$(1 - \beta) \zeta (\Delta) - \zeta (M \Delta M^\top) \geq 0,$$

(50)

for any $\beta \leq 1 - \sigma_{\max} (M)$.

**Proof.** Equipped with this Lemma, we are now ready to provide a storage functional for the LQR case.
Theorem 4 The choice:

\[
\lambda [\rho_k] = \kappa (\text{Tr} (\Sigma_k^{-1} \Sigma_k) + \log \det (\Sigma_k) - n) \\
- \text{Tr} \left( \Omega \left( \Sigma_k^{-\frac{1}{2}} \Sigma_k^{-\frac{1}{2}} - I \right) \right)
\]

(51)

satisfies the functional dissipativity functional criterion (31) where matrix \( \Omega \) is solution of the discrete Lyapunov equation:

\[
M \Omega M^T - \Omega + \Sigma W^\frac{1}{2} \Sigma = 0
\]

(52)

for \( M = \Sigma A_c \Sigma \) and where \( \kappa, \varrho \) are constants satisfying:

\[
\kappa \geq \frac{1}{2 (1 - \sigma_{\text{max}}(M))}, \quad \varrho \leq \sigma_{\text{min}}(W) \sigma_{\text{min}}(\Sigma)
\]

(53)

5.3 Illustration

We illustrate next Theorem 4. We chose a case with \( n = 2 \) states and a single input having the dynamics

\[
s_{k+1} = \frac{1}{10} \begin{bmatrix} 8 & 5 \\ -5 & 7 \end{bmatrix} s_k + \frac{1}{10} \begin{bmatrix} 0 \\ 5 \end{bmatrix} \alpha + w,
\]

(54)

where \( w \sim \mathcal{N}(0, \Sigma_w) \) and

\[
\Sigma_w = \begin{bmatrix} 2 & -1 \\ -1 & 1.6 \end{bmatrix}.
\]

(55)

The stage cost is based on the weighting matrices \( Q = I, R = 1 \). The corresponding constant \( L_0 \) reads as:

\[
L_0 = 8.92.
\]

(56)

The initial density \( \rho_0 = \mathcal{N}(\mu_0, \Sigma_0) \) was selected with

\[
\mu_0 = \frac{1}{2} \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \quad \Sigma_0 = \frac{1}{10} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.
\]

(57)

The optimal steady-state density \( \rho^* = \mathcal{N}(0, \Sigma_\infty) \) has the covariance

\[
\Sigma_\infty = \begin{bmatrix} 3.73 & -1.76 \\ -1.76 & 3 \end{bmatrix}.
\]

(58)

The constants \( \kappa = 1.72, \varrho = 1.56 \) then satisfy (53). Figure 1 provides a graphical illustration of this case, showing the trajectories \( \rho_k = \mathcal{N}(\mu_k, \Sigma_k) \), the stage cost \( L [\rho_k, \pi] \) given by (41), the rotated stage cost functional (29) using the storage functional (51), and the KL functional. One can readily observe that the stage cost is not necessarily positive, while the rotated stage cost is lower-bounded by \( D_{\text{KL}} \), and therefore yields a value function that can serve as a Lyapunov function for the corresponding MDP.

5.4 A Storage Functional for the Wasserstein Metric

In this last part, we show that the proposed approach applies to the 2-Wasserstein metric, which is a dissimilarity measure that is popular in control. It can be useful to specify here that, unlike \( D_{\text{KL}} \), when choosing the 2-Wasserstein metric, the limit case where \( \rho_k \to \infty \) tends to a sequence of Dirac measures, the proposed dissipativity theory matches the classical dissipativity theory.

The reasoning presented below will follow the same lines as for the \( D_{\text{KL}} \) case, and will be kept brief.

We will consider the 2-Wasserstein metric for Normal densities, applied to the state-space transformation:

\[
\tilde{s} = \Sigma_\infty^{-\frac{1}{2}} s
\]

(59)

having \( \rho^* = \mathcal{N}(0, I) \) as optimal steady-state density.
For $\rho_k, \rho$ with bounded variance, the metric reads as:

$$W_2(\hat{\rho}_k \| \hat{\rho}^*) = \mu^T \Sigma_\infty^{-1} \mu + \text{Tr} \left( \Psi_k \right) + n - 2\text{Tr} \left( \Psi_k \right)$$

$$= \mu^T \Sigma_\infty^{-1} \mu + \sum_{i=1}^{n} \Lambda_i \left( \Psi_k \right) + 1 - 2\Lambda_i \left( \Psi_k \right)^{1/2}, \quad (60)$$

where $\Psi_k = \Sigma_\infty^{-1/2} \Sigma_k \Sigma_\infty^{-1/2}$. We can then propose the following two results.

**Corollary 3** $W_2(\hat{\rho}_k \| \hat{\rho}^*)$ defined in (60) is strictly decreasing under dynamics (36).

**Proof.** We observe that the function

$$\zeta(x) = x + 1 - 2x^{1/2} \quad (61)$$

is convex on $x \in [0, \infty)$, and that

$$W_2^\Sigma(\hat{\rho}_k \| \hat{\rho}^*) = \zeta \left( \Lambda_i \left( \Psi_k \right) \right). \quad (62)$$

The result then follows directly from Theorem 3.

A storage functional is provided next.

**Corollary 4** The functional

$$\lambda_W [\rho_k] = \kappa W_2^\Sigma(\rho_k, \rho^*) - \text{Tr} \left( \Omega \left( \Sigma_\infty^{-1/2} \Sigma_k \Sigma_\infty^{-1/2} - I \right) \right) \quad (63)$$

is a valid storage functional for the LQR case and the 2-Wasserstein metric (60) for some constant $\kappa > 0$, where matrix $\Omega$ is solution of the discrete Lyapunov equation (52).

6 Conclusion

In this paper, we proposed a generalization of the classic dissipativity theory to Markov Decision Processes. We showed that this generalization is not straightforward, but can be done by extending the notion of storage functions to nonlinear storage functionals. A classic Lyapunov argument can then be used to discuss the asymptotic stability of the probability measures underlying the Markov Decision Processes to the steady-state optimal probability measure. The asymptotic stability can be expressed in terms of dissimilarity measures such as the Kullback-Leibler divergence, the Wasserstein metric, or the total variation distance. The theory is illustrated on the LQR case with Normal process noise, for which a storage functional can be provided. Future work will consider the use of this theory for discussing the stability of Stochastic MPC, the construction of stability-constrained learning using MPC. An extension to policies based on finite-horizon MPC schemes will be considered.

7 Appendix

The proofs of Lemma 2, Lemma 3, Theorem 3, Lemma 4, Theorem 4 and Corollary 4 are provided hereafter. Note that we will denote the element-wise product of vectors as $\odot$, the ordered eigenvalues of matrix $A$ as $\Lambda(A)$ and its ordered singular values as $\sigma(A)$. We will denote the trace of $A$ as $\text{Tr}(A)$. We will further use the following properties

$$\Lambda(AB) = \Lambda(BA), \quad \Lambda(I + cA) = 1 + c\Lambda(A),$$

$$\Lambda(AB) = \Lambda(A) \odot \Lambda(B), \quad \sigma_i(AB) \leq \max(\sigma(A))\sigma_i(B),$$

$$\text{Tr}(ABC) = \text{Tr}(CAB), \quad \text{det}(AB) = \text{det}(A) \text{det}(B),$$

$$\text{det}(ABC) = \text{det}(CAB),$$

and all such permutations.

7.1 Proof of Lemma 2

First, we observe that the following inequality holds:

$$\sigma \left( M\Delta M^T \right) \leq \sigma_{\max} \left( M \right)^2 \sigma \left( \Delta \right), \quad (64)$$

where $\sigma(\cdot)$ is the vector of singular values. Hence since $\Delta$ is symmetric:

$$\Lambda_i \left( M\Delta M^T \right)^2 \leq \sigma_{\max} \left( M \right)^2 \Lambda_i \left( \Delta \right)^2, \quad (65)$$

such that

$$|\Lambda_i \left( M\Delta M^T \right)| = |\alpha_i| |\Lambda_i \left( \Delta \right)| \quad (66)$$

holds for some sequence $|\alpha_1|, \ldots, |\alpha_n| \leq \sigma_{\max} \left( M \right)$. We then need to show that $\alpha_1, \ldots, \alpha_n > 0$. We observe that:

$$\Lambda_i \left( M\Delta M^T \right) = \Lambda_i \left( \Phi \Delta \right), \quad (67)$$

where $\Phi = M^T M$ is symmetric, positive definite. Let us define:

$$\Gamma(t) = e^{t \log \Phi \Delta}, \quad (68)$$

where we use the matrix exponential and logarithms. Then

$$\Gamma(0) = \Delta \quad \text{and} \quad \Gamma(1) = M^T M \Delta \quad (69)$$
trivially hold. We further observe that
\[
\det (\Gamma(t)) = \det (e^{t \log \Phi}) \det (\Delta) = e^{t \text{Tr}(\log \Phi)} \det (\Delta)
\]
(70)
\[
= \left( e^{t \text{Tr}(\log \Phi)} \right)^4 \det (\Delta) = \det (\log \Phi)^t \det (\Delta).
\]
Since \( \det (\Gamma(1)) = \det (M \Delta M^\top) \neq 0 \) by assumption, it follows that \( \det (\Gamma(t)) \neq 0 \) for all \( t \in [0, 1] \). We can then conclude that the eigenvalues
\[
\Lambda_i (\Gamma(t)) \neq 0, \quad \forall t,
\]
(71)
such that \( \Lambda_i (\Gamma(t)) \) does change sign over \( t \in [0, 1] \). This establishes that \( \alpha_1, \ldots, n > 0 \) and hence (42).

7.2 Proof of Lemma 3

We first observe that the dynamics for \( \Sigma_k \) can be reformulated as:
\[
\Psi_{k+1} = M \Psi_k M^\top + N,
\]
(72)
where \( \Psi_k = \Sigma_{\infty}^{-\frac{3}{2}} \Sigma_k \Sigma_{\infty}^{-\frac{1}{2}} \) and
\[
M = \Sigma_{\infty}^{-\frac{3}{2}} \Theta \Sigma_{\infty}^{-\frac{1}{2}}, \quad N = \Sigma_{\infty}^{-\frac{3}{2}} \Sigma_w \Sigma_{\infty}^{-\frac{1}{2}},
\]
(73)
such that \( \lim_{k \to \infty} \Psi_k = I \) and
\[
MM^\top + N = I.
\]
(74)
We can then observe that
\[
\sigma(M)^2 = \Lambda (M^\top M) = \Lambda (MM^\top) = \Lambda (I - N)
\]
(75)
\[
= 1 - \Lambda (N) \geq 0,
\]
(76)
and that
\[
\Lambda(N) = \Lambda \left( \Sigma_{\infty}^{-\frac{3}{2}} \Sigma_w \Sigma_{\infty}^{-\frac{1}{2}} \right) = \Lambda \left( \Sigma_{\infty}^{-1} \Sigma_w \right) > 0,
\]
(77)
since \( \Sigma_{\infty}, \Sigma_w \) are positive definite. It follows that
\[
\sigma(M) \in [0, 1).
\]
(78)
Let us label
\[
\Delta_k = \Psi_k - I,
\]
(79)
and observe that
\[
\Delta_{k+1} = M \Delta_k M^\top \quad \text{and} \quad \Lambda(\Delta_k) = \Lambda(\Psi_k) - 1.
\]
(80)
Using Lemma 2, we observe that (42) applies, i.e.
\[
\Lambda_i (\Delta_{k+1}) = \alpha_i \Lambda_i (\Delta_k), \quad i = 1, \ldots, n,
\]
(81)
for a sequence \( \alpha_1, \ldots, n > 0 \) with \( \alpha_i \leq \sigma_{\max}(M) < 1 \).

Finally, we observe that:
\[
\Lambda_i (\Delta_k) = \Lambda_i \left( \Sigma_{\infty}^{-\frac{3}{2}} \Sigma_k \Sigma_{\infty}^{-\frac{1}{2}} - I \right) = \Lambda_i \left( \Sigma_{\infty}^{-1} \Sigma_k \right) - 1,
\]
(82)
and conclude that (44) holds.

7.3 Proof of Theorem 3

We first observe that the monotonic convergence of the second term in (45)
\[
\sum_{i=1}^n \zeta(\Lambda_i (\Sigma_{\infty}^{-1} \Sigma_k))
\]
(83)
follows directly from Lemma 3. The convergence of the first term follows classic system dynamic theory. We recall the argument for completeness. Consider the state space transformation:
\[
\nu_k = \Sigma_{\infty}^{-\frac{3}{2}} \mu_k,
\]
(84)
following the dynamics:
\[
\nu_{k+1} = M \nu_k,
\]
(85)
with
\[
\Lambda_i (M^\top M) \leq \sigma_{\max}(M)^2 < 1.
\]
(86)
It follows that
\[
\mu_{k+1} \Sigma_{\infty}^{-1} \mu_{k+1} = ||\nu_{k+1}||^2 = \nu_k^\top M^\top M \nu_k < ||\nu_k||^2 = \mu_k^\top \Sigma_{\infty}^{-1} \mu_k.
\]
(87)

7.4 Proof of Lemma 4

We first observe that for any \( a < 1 \) the inequality:
\[
\vartheta_{a,b}(x) := (1 - b) (x - \log (x + 1)) - ax + \log (ax + 1) \geq 0
\]
holds on \( x \in (-1, \infty) \) for \( 0 < b \leq 1 - a < 1 \). Indeed, we observe that \( \vartheta_{a,b}(0) = 0 \) and that on the interval \( x \in (-1, \infty) \)
\[
\frac{d\vartheta_{a,b}}{dx} = -ax ((a + b - 1) x - 1 + b) \frac{1}{ax + 1} (x + 1) = 0
\]
(88)
has the unique solution \( x = 0 \). Furthermore, the sign of \( \frac{d\vartheta_{a,b}}{dx} \) entails that \( \vartheta_{a,b} \) is monotonically increasing away
from $x = 0$, which establishes (88). Using Lemma 2, we then observe that for all $i$:

$$\Lambda_i (M \Delta M^T) = \Lambda_i (M^T M \Delta) = \alpha_i \Lambda_i (\Delta)$$  

(90)

holds for some sequence $\alpha_1, \ldots, \alpha_n \leq \sigma_{\max} (M)$. Then

$$- \Lambda_i (M \Delta M^T) + \log (\Lambda_i (M \Delta M^T) + 1) = - \alpha_i \Lambda_i (\Delta) + \log (\alpha_i \Lambda_i (\Delta) + 1).$$

(91)

Hence, using

$$\zeta (\Delta) = \sum_{i=1}^n \Lambda_i (\Delta) - \log (\Lambda_i (\Delta) + 1)$$

(92)

$$\zeta (M \Delta M^T) = \sum_{i=1}^n \Lambda_i (M \Delta M^T) - \log (\Lambda_i (M \Delta M^T) + 1),$$

we observe that

$$(1 - \beta) \zeta (\Delta) - \zeta (M \Delta M^T) = \sum_{i=1}^n \beta_i (\Lambda_i (\Delta)),$$

such that the choice

$$\beta \leq \min_i 1 - \alpha_i \leq 1 - \sigma_{\max} (M) < 1$$

(93)

ensures that (50) holds.

### 7.5 Proof of Theorem 4

We first observe that $\lambda [\rho^*] = 0$ holds by construction. We further observe that:

$$\text{Tr} \left( \Omega \left( \Sigma_\infty ^{\frac{1}{2}} \Sigma_k \Sigma_\infty ^{\frac{1}{2}} - I \right) \right) = \text{Tr} (\Omega \Delta_k),$$

(94)

and, using (80) and (52), we obtain:

$$\text{Tr} (\Omega \Delta_k) - \text{Tr} (\Omega \Delta_{k+1}) = - \text{Tr} (W (\Sigma_k - \Sigma_\infty)).$$

(95)

Using (95) and (49) we then observe that:

$$\lambda [\rho_k] - \lambda [\rho_{k+1}] = \kappa \left( \zeta (\Delta_k) - \zeta (\Delta_{k+1}) \right) - \text{Tr} (W (\Sigma_k - \Sigma_\infty)).$$

(96)

Using (41) and (96) and Lemma 4, it follows that

$$\mathcal{L} [\rho_k, \pi [\rho_k]] - \lambda [\rho_{k+1}] + \lambda [\rho_k]$$

(97)

$$= \frac{1}{2} \mu_k^T W \mu_k + \kappa \zeta (\Delta_k) - \zeta (\Delta_{k+1})$$

$$\geq \frac{1}{2} \mu_k^T W \mu_k + \kappa \beta \zeta (\Delta_k).$$

We further observe that

$$D_{KL} (\rho_k \mid \mid \rho^*) = \frac{1}{2} \mu_k^T \Sigma_\infty^{-1} \mu_k + \frac{1}{2} \zeta (\Delta_k).$$

(98)

It follows that for (53), $\kappa \geq 1 / 2\beta$, and the inequality

$$\mathcal{L} [\rho_k, \pi [\rho_k]] - \lambda [\rho_{k+1}] + \lambda [\rho_k] \geq \vartheta D_{KL} (\rho_k \mid \mid \rho^*)$$

(99)

holds.

### 7.6 Proof of Corollary 4

Let us define the scalar function

$$g(x) = (1 - \beta) \left( x - 2 (x + 1)^\frac{1}{2} + 2 \right) - \left( \alpha x - 2 (\alpha x + 1)^\frac{1}{2} + 2 \right),$$

(100)

which satisfies $g(0) = 0$. One can verify that for $0 < \alpha < 1$ and $0 \leq \beta \leq 1 - \alpha^\frac{1}{2}$, $g$ is convex on $x \in (-1, \infty)$ and takes its minimum at $x = 0$. One can then verify that

$$(1 - \beta) W^\Sigma_2 (\rho_k, \rho^*) - W^\Sigma_2 (\rho_{k+1}, \rho^*) = g (\Delta_k) \geq 0.$$  

(101)

Using (63) and (101) one can then verify that

$$\mathcal{L} [\rho_k, \pi] - \lambda [\rho_{k+1}] + \lambda [\rho_k] \geq \frac{1}{2} \mu_k^T W \mu_k + \kappa W^\Sigma_2 (\rho_k, \rho^*)$$

$$\geq \vartheta (\mu^T \Sigma_\infty^{-1} \mu + W^\Sigma_2 (\rho_k \mid \mid \rho^*))$$

(102)

holds for $\kappa \geq \beta^{-1}$ and $\vartheta$ small enough.

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