Cusps of lattices in rank 1 Lie groups over local fields

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Abstract. Let $G$ be the group of rational points of a semisimple algebraic group of rank 1 over a nonarchimedean local field. We improve upon Lubotzky’s analysis of graphs of groups describing the action of lattices in $G$ on its Bruhat–Tits tree assuming a condition on unipotents in $G$. The condition holds for all but a few types of rank 1 groups. A fairly straightforward simplification of Lubotzky’s definition of a cusp of a lattice is the key step to our results. We take the opportunity to reprove Lubotzky’s part in the analysis from this foundation.

Keywords: semisimple algebraic group, local field, lattice, structure theorem, graph of groups

MSC: 22E40 (20G25)
1. Introduction and basics

When properly formulated, many results true for semisimple real Lie groups of non-compact type like $SL_2(\mathbb{R})$ continue to hold, when we switch to (algebraic) semisimple groups over non-discrete totally disconnected locally compact fields (local fields for short).

Since the topology the field induces on the group is fairly weak, this is surprising at first. It is explained by the existence of a combinatorial analogue of the symmetric space of a semisimple Lie group of non-compact type, the Bruhat–Tits building of the group.

We are interested in lattices (discrete subgroups of finite covolume) in groups of rank 1. For groups of rank 1 the building is a tree. Lubotzky proved, in (Lubotzky, 1991), the existence of moduli spaces of lattices in rank 1 groups over local fields by giving recipes for their construction and a classification result, which describes the quotient graph of groups obtained from the action of the lattice on the Bruhat–Tits tree.

The latter form of classification is particularly satisfactory. (It gives the quotient space, and at each point the stabilizer of a representative of the orbit mapped to this point drawn from a chosen connected fundamental domain.) This gives a concrete (and as we will see finite) method to construct lattices. The quotient graphs of lattices in rank 1 groups look like stickman drawings of some hyperbolic 2–manifold. The behaviour of the small class of arithmetic lattices in $SL_2$ over a function field as illustrated by the pictures accompanying Theorem 9 in Chapter II.2 in (Serre, 1980) already gives you the general picture.

Lubotzky’s construction recipes for lattices are of this graph of groups flavour. We draw the readers attention to the method used in Theorem 5.1 in loc.cit. to construct nonuniform lattices from lattices in the unipotent radical of a minimal parabolic subgroup and a free group on finitely many generators.

Our main result, part (4) of Theorem 2.3 allows a converse of this to be proved. Lubotzky also offers a converse, Theorem 7.1 in his paper, but it is weaker, since the link back to lattices in appropriate unipotent radicals of parabolics is missing.

This article could be described as the reshaping of the classification part of Lubotzky’s paper using a simpler, yet equivalent, definition of a cusp. The cusps of a lattice are supposed to describe the behaviour at infinity of the quotient space. We introduce them at the beginning of our main section, number 2. In the same section, the Structure Theorem 2.3 is stated and its part (4) proved.

There is no need to prove the parts (1) and (3) which give already a quite precise description of the quotient graph of groups. Since our and Lubotzky’s cusp definition are equivalent this is immediate from Lubotzky’s work. However, many ingredients which go into Lubotzky’s proof turn out to be not that obvious. We therefore decided to provide a more detailed version of Lubotzky’s proof in our last section. Most of the ingredients needed will be proved in earlier sections.
Part (2) of the Structure Theorem is an easy, but perhaps non-obvious application of Citation 3.15 and is also deferred to the last section.

Most readers are expected to skip the rest of the paper, with the exception of Observation 3.9, which gives the list of types of groups who possibly do not satisfy the condition under which part (4) of the Structure Theorem is true. We already noted, that the arguments in the last section are essentially known. The penultimate section contains the (partly difficult) facts on unipotent elements used. Most notable is an assumption which goes into an important ingredient of the proof of Citation 3.15 which is central in proving Lubotzky’s results. I am grateful to Raghunathan for providing the necessary argument; see Section 3.4.

This leaves the rest of this section to discuss: We list the basic facts on the action of rank 1 groups on their Bruhat–Tits trees and of automorphism groups of trees we will need. In summary, the classification problem for lattices in rank 1 groups can be treated geometrically. This is only true up to a compact kernel. The reduction steps discussed in Section 1.3 take care of that problem.

Our notational conventions are standard except that we write $G_A$ for the fixator (pointwise stabilizer) of a subset with respect to the action of the group $G$ and $G_{\{A\}}$ for its stabilizer.

For constructions from the theory of algebraic groups we use the notation from (Margulis, 1989) whose first chapter nicely summarizes most of the results from that theory we will need.

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1.1. The geometric interpretation of a rank 1 group

This subsection lists the basic properties of the geometric approach to the analysis of lattices in the semisimple group $G(k)$. There is some overlap with the next section.

The group $G(k)$ acts by automorphisms on its Bruhat–Tits building $X(G, k) =: X_G$. It is a locally finite tree. The orbit of a single edge covers $X_G$, hence there are at most 2 orbits of vertices hence at most two orders of ramification, $q_0 + 1$ and $q_1 + 1$. As another consequence, the image of $G(k)$ is cocompact in the group of all automorphisms of $X_G$ with its natural topology (see below).

Each maximal compact subgroup is either the stabilizer of a vertex or the stabilizer of the midpoint of an edge of $X_G$. The latter possibility occurs iff $G(k)$ acts with inversion. This can only happen when $q_0$ equals $q_1$, but if this is the case, it will happen if $G$ is adjoint. Stabilizers of points are open. Each bounded subgroup of $G(k)$ fixes a point. Therefore the kernel of the action of
$G(k)$ on $X_G$ is compact. The natural topology on $\text{Aut}(X_G)$ makes the image of $G(k)$ in $\text{Aut}(X_G)$ into a closed subgroup.

From the above we derive that the image of a (uniform) lattice in $G(k)$ will be a (uniform) lattice in the automorphism group.

There is a combinatorial way to compute the covolume of a lattice $\Gamma$ in $G := G(k)$ (or $\text{Aut}(X_G)$): The group $G$ acts without inversion on the barycentric subdivision $X'$ of $X_G$. The intersection of the $\Gamma$–stabilizers of two neighboring vertices $y_0, y_1$ of $X'$ thus is the stabilizer of the connecting edge, $e_0$ say. Put $U := G_{e_0} = G_{y_0} \cap G_{y_1}$ and normalize the Haar measure $\mu$ on $G$ to give volume 1 to $U$ if $\Gamma$ does not invert edges of $X$ and $\frac{1}{2}$ if it does. Then the formula for the volume of the quotient of $G$ modulo $\Gamma$ on page 84 in (Serre, 1980) gives:

$$\mu(G/\Gamma) = \sum_{e \in E(Y)} \frac{1}{|\Gamma_e|},$$

where $Y$ is a fundamental $\Gamma$–transversal in $X'$, (cf. (Dicks and Dunwoody, 1989, I.(2.6)) for notation) and $E(Y)$ is its set of edges.

1.2. Groups of automorphisms of trees

Let $X$ be a locally finite tree. By making each edge isometric to the unit interval we turn $X$ into a locally compact metric space $(X, d)$. Spheres and balls in $X$ with center $x$ and radius $r$ will we denoted $S_x(r)$ and $B_x(r)$ respectively. By a line (a ray) in $X$ we mean a subspace isometric to the real line (the set of positive reals). We will only use rays emanating from a vertex. This allows us to identify rays and lines with the sequence of vertices lying on them.

Each automorphism $\alpha$ of $X$ either has a fixed point (not necessary a vertex) or a unique invariant line, called its axis, on which it induces a translation of nonzero amplitude $l(\alpha)$. It is called elliptic or hyperbolic accordingly.

The ends of the topological space $X$ can be described purely combinatorially, using rays as follows: An end is an equivalence class of rays. Two rays define the same end iff their intersection is a ray. For any vertex of $X$ and every end $\epsilon$ there is a unique ray, written $[x, \epsilon)$, emanating at $x$ and representing $\epsilon$. The unique point on $[x, \epsilon)$ at distance 1 to $x$ will be written $\varphi_\epsilon(x)$. This map, which moves vertices closer to the end $\epsilon$ is useful in describing other constructions.

Given two distinct ends, $\epsilon$ and $\epsilon'$, there is a unique line, called the line joining $\epsilon$ and $\epsilon'$ and written $[\epsilon, \epsilon']$, which contains rays representing $\epsilon$ and $\epsilon'$ respectively. The points on this line are the fixed points of $\varphi^k \circ \varphi_{\epsilon'}$, for arbitrary $k \geq 1$.

The ends $\epsilon$ and $\epsilon'$ of $X$ are also ends of the line $[\epsilon, \epsilon']$, and will be called the end points of the line. The only ends fixed by a hyperbolic element are the end points of its axis.

We make implicit use of the existence of a compact topology on the union of $X$ and its set of ends in Proposition 3.4. Lets therefore describe the end topology

\footnote{Aside: In case $X$ is a higher rank building we have to use the cone topology on the geodesic compactification instead.}
by describing a set of basic neighborhoods for any given end $\epsilon$. Remove an edge from $X$. Only one of the connected components left will contain a ray representing $\epsilon$. We choose this component to be an open neighborhood of $\epsilon$.

Each automorphism $\alpha$ of $X$ induces a homeomorphism of its end compactification, and we have $\alpha \circ \varphi_\epsilon = \varphi_{\alpha(\epsilon)} \circ \alpha$.

Given vertices $x$ and $y$ and an end $\epsilon$ we put

$$(x, y)_\epsilon := \lim_{k \to \infty} \left( d(x, \varphi^k_\epsilon(x)) - d(y, \varphi^k_\epsilon(x)) \right).$$

Take an arbitrary point $x$ from $X$ and define the oriented length function $l_\epsilon$ with respect to an end $\epsilon$ by $l_\epsilon(\alpha) := (x, \alpha(x))_\epsilon$. Restricted to the stabilizer of $\epsilon$ it is a homomorphism to $\mathbb{Z}$ with kernel the elliptic elements. If $h$ is hyperbolic, and $\epsilon$, $\epsilon'$ are the end points of its axis, then $l_\epsilon(h) = -l_{\epsilon'}(h)$ and their absolute value equals $l(h)$. We call the end whose oriented length function with respect to $\epsilon$ is positive (negative) at $h$ attracting (repelling) for $h$.

Let $\epsilon$ be an end, $x$ a vertex and $0 < \lambda \leq 1$ a real number. Denote the unique point at distance $t$ from $x$ on $[x, \epsilon]$ by $x(t)$. The set $B_\epsilon(x; \lambda) := \bigcup_{t \geq 0} B_x(t)(\lambda t)$ is called a horoellipse centered at $\epsilon$ with radius vertex $x$ and eccentricity $\lambda$. A horoellipse with eccentricity 1 is written $B_\epsilon(x)$ and called a horoball. The boundary of the horoball $B_\epsilon(x)$ is the set

$$S_\epsilon(x) = \{ y \in X : \exists k \in \mathbb{N} \, \varphi^k_\epsilon(y) = \varphi^k_\epsilon(x) \} = \{ y \in X : (x, y)_\epsilon = 0 \}.$$

It is called the horosphere with center $\epsilon$ and radius vertex $x$. For an automorphism $\alpha$ fixing $\epsilon$ the general formula $\alpha(S_\epsilon(x)) = S_{\alpha(\epsilon)}(\alpha(x))$ becomes $\alpha(S_\epsilon(x)) = S_\epsilon(\alpha(x))$, hence $\alpha$ leaves some (every) horosphere invariant iff $l_\epsilon(\alpha) = 0$, i.e., iff $\alpha$ is elliptic.

The group $\text{Aut}(X)$ is a topological group in the compact–open topology. A base of neighborhoods of the identity is thus given by the sets of automorphisms fixing finitely many vertices. Stabilizers of vertices are seen to be pro-finite, hence $\text{Aut}(X)$ in this topology is locally compact and totally disconnected. A subgroup $G$ of $\text{Aut}(X)$ is closed iff the $G$–stabilizer of some (equivalently any) vertex is closed (equivalently compact). A subgroup $\Gamma$ of $\text{Aut}(X)$ is discrete iff the $\Gamma$–stabilizer of some (equivalently any) vertex is finite.

If a locally compact $\sigma$–compact group $G$ acts on $X$ with compact open vertex stabilizers, then the induced subgroup $\overline{G}$ of automorphisms of $X$ is closed, and the natural homomorphism $G \to \overline{G}$ is open with compact kernel.

If $\text{Aut}(X)$ acts with finite fundamental domain, then a subgroup $G$ of $\text{Aut}(X)$ is cocompact iff $G$ acts with a finite fundamental domain.

1.3. Reduction steps

To reduce the case of a general connected semisimple $k$–group of $k$–rank 1 to the absolutely simple case, we go through the following steps:

1. Replace $G$ by its adjoint group $\text{AdG}$.
2. Decompose $G = \text{Ad}G$ into its $k$–simple factors. Since $G$ is supposed to have rank 1, all but one, say $G^\text{is}$, are anisotropic. Replace $G$ by $G^\text{is}$. At this stage, $G$ is adjoint and $k$–simple.

3. Choose a finite separable extension $K|k$ and a $K$–group $G'$ such that $G'$ is absolutely simple and the restriction of scalars $R_{K|k}(G')$ equals $G$. Replace $k$ by $K$ and $G$ by $G'$. After this final replacement, the group will be absolutely simple.

(It looks like the last step should be unnecessary for groups of rank 1.)

What happens to the groups of rational points can be seen from the following list:

1. The map $\text{Ad}_k$ induced by $\text{Ad}$ on $k$–points has finite kernel, equal to $Z(G)(k) = Z(G(k))$ and compact (abelian) cokernel.

2. Since the group of rational points of a reductive group over a local field is compact if (and only if) the group is anisotropic over the field, the projection onto $G^\text{an}(k)$ has compact kernel, equal to $G^\text{an}(k)$, where $G^\text{an}$ is the (in general almost) direct product of the $k$–simple $k$–anisotropic factors. Besides, it is continuous and open.

3. Note, that $K$ is also local. The restriction of scalar functor induces a topological isomorphism between $G'(K)$ and $G(k)$ according to (Margulis, 1989, I.1.7).

We list the changes to the Bruhat–Tits buildings next. We remind the reader that the Bruhat–Tits building is obtained from a natural valuation of its canonical root datum.

1. For any central isogeny of reductive groups, the map induced on the rational points induces an isomorphism of canonical root data (attached to a maximal split torus and its image), which leads to an isomorphism of Bruhat–Tits buildings. The isomorphism is equivariant when the covering group is made to act on the building of the covered group via the isogeny.

2. The Bruhat–Tits building of an (almost) direct product is the product of the Bruhat–Tits buildings of the factors. The action is the product action. The Bruhat–Tits building of a $k$–anisotropic group is a single point (and vice versa).

3. The restriction of scalar functor induces an equivariant isomorphism of corresponding root data, hence of Bruhat–Tits buildings.

The corresponding results for the spherical buildings over the fields involved hold as well. We remind the reader that the natural apartment system of the Bruhat–Tits building of a reductive group over a complete field is complete. As a result, the building at infinity (with its simpicial structure) is isomorphic to the spherical building of the group over the field in question.
One can compute the kernel of the action of a connected semisimple $k$–group $G(k)$ on its Bruhat–Tits building explicitly. It is equal to $Z(G)(k) \cdot G^\text{an}(k)$. The kernel will therefore be trivial iff the group is adjoint and has no $k$–anisotropic factors.

We summarize:

Starting with a connected semisimple $k$–rank 1 group $G$, we may find a finite separable field extension $K|k$, an absolutely simple $K$–rank 1 group $G'$ and a canonical homomorphism $\pi: G(k) \to G'(K)$ with compact kernel (equal to $Z(G)(k) \cdot G^\text{an}(k)$) and cokernel such that $G(k)$ and $G'(K)$ have isomorphic Bruhat–Tits buildings, with $G(k)$ acting via $\pi$ and $G'(K)$ embedding into $\text{Aut}(X_G)$ as a closed cocompact subgroup.

1.4. ENDS AS PARABOLIC SUBGROUPS

We finally add some results linking algebraic information on $G$ with geometric information on its Bruhat–Tits tree.

The ends of $X_G$ can be reinterpreted as the points of its building at infinity. Since the field $k$ is complete, the natural apartment system of $X_G$ is complete, hence the building at infinity coincides (as a set) with the spherical building of $G$ over $k$. As a consequence, every end of $X_G$ is a proper $k$–parabolic subgroup $P$, whose stabilizer in $G(k)$ is $P(k)$. In addition, the group of rational points of the unipotent radical of $P$, $U(k)$ say, acts simply transitively on the set of parabolics opposite to $P$. In other words, it acts simply transitively on the ends different from $P$.

Let $S$ be a maximal $k$–split torus. The set of elliptic elements $Z$ in $Z_G(S)(k)$ is a compact open normal subgroup. It fixes the affine apartment corresponding to $S$ pointwise. It contains every other compact subgroup of $Z_G(S)(k)$, hence is its unique maximal compact subgroup. The quotient $Z_G(S)(k)/Z$ is the translation lattice of the affine Weyl group of $G$ with respect to $S$.

Let $P$ be a minimal $k$–parabolic subgroup of $G$ (an end of $X_G$). Then for any hyperbolic element $h$ in $P(k)$ of minimal translation length, we have

$$P(k) = (Z \rtimes h) \rtimes R_u(P)(k).$$

where $Z$ is the maximal compact subgroup of $P(k) \cap P'(k)$ and $P'$ is the end point of the axis of $h$ different from $P$. Instead of $h$ we may choose $P'$ freely among the ends of $X_G$ different from $P$.

The point of intersection of a horosphere centered at $\epsilon$ and a line $[\epsilon', \epsilon]$ with $\epsilon' \neq \epsilon$ is unique. Since $X_G$ has no leaves, any point on a horosphere can be represented in this way, giving rise to a surjection from the set of ends different from $\epsilon$ to the points on a particular horosphere centered at $\epsilon$. This map is compatible with the action of the group of elliptic elements in the stabilizer.
of $\epsilon$ on both sets. As a consequence any group acting transitively on the ends different from $\epsilon$ and by elliptic transformations will act transitively on each of the horospheres centered at $\epsilon$.

It can be shown, that unipotent elements always act by elliptic transformations (see the remarks following Theorem 3.7). As a consequence, the group $U(k)$ introduced above acts transitively on each horosphere around $P \supseteq U(k)$.

At times we make use of the fact, that the group $U(k)$ maps injectively into the group of automorphisms of $X_G$. This follows from two facts:
First, the map $Ad$ induces an isomorphism between the unipotent radical of a $k$-parabolic and the unipotent radical of its image (use (Borel and Tits, 1972, 2.15, 2.20 and 2.26)). Second, the kernel of the action of $G(k)$ on its Bruhat–Tits building is $\mathcal{Z}(G)(k) \cdot G^u(k)$ (as mentioned in the last section).
2. Structure of the quotient graph of groups

In this section we analyze the quotient graphs of groups for lattices in semisimple groups of rank 1. It turns out that an infinite quotient graph can be explained in terms of the cusps of the lattice. This is a notion surprisingly similar to the one introduced in hyperbolic geometry. As in this classical example the notion interweaves geometric and group theoretic aspects.

Just as in the case of a lattice $\Gamma$ in $G := SL_2(\mathbb{R})$, we expect the notion of a cusp to capture both geometrical properties of the quotient of the homogeneous space of $G$ modulo $\Gamma$ and group theoretic properties of the $\Gamma$–stabilizers of these cusps.

The geometric aspect involves points at infinity of the homogeneous space and its $\Gamma$–quotient. Since over a local field the homogeneous space is a tree, it is natural to assign this role to certain ends of the tree and all ends of the quotient graph respectively. For lattices in $SL_2(\mathbb{R})$ the group theoretic aspect involved is that $\Gamma$–stabilizers of points at infinity whose horoball neighborhoods map to neighborhoods of cusps are (maximal) unipotent groups. (Horoballs as introduced in Section 1.2 will play a prominent role in this paper as well.)

The first idea concerning the group theoretic aspect would therefore be maximal unipotent subgroups of $\Gamma$. Unfortunately, from a geometric point of view, unipotents in semisimple groups over local fields do not necessarily behave as one expects extrapolating from the $SL_2(\mathbb{R})$. Unipotent elements can be grouped into three classes defined algebraically (see Section 3) which can be distinguished by their action on the Bruhat–Tits building as illustrated by Proposition 3.4. This Proposition makes clear that we should restrict ourselves to consider good unipotent elements only. (An element/subgroup of $G(k)$ is good iff it is contained in the unipotent radical of a $k$–parabolic subgroup. The first part of Theorem 3.7 explains, why we did not encounter that problem for groups over the reals.)

We thus arrive at the following definition:

**Definition 2.1 (cusps)** Let $\Gamma$ be a lattice in $G(k)$.

- An end of the Bruhat–Tits tree $X_G$ of $G(k)$ is called $\Gamma$–cuspidal iff there is a nontrivial good unipotent element in $\Gamma$ fixing $\epsilon$.

- A geometric cusp of $\Gamma$ is an end of the quotient of the barycentric subdivision of $X_G$ modulo $\Gamma$.

- A cusp subgroup of $\Gamma$ is a maximal nontrivial good unipotent subgroup.

- A cusp of $\Gamma$ is a $\Gamma$–conjugacy class of a cusp subgroup of $\Gamma$.

There is an obvious bijection between $\Gamma$–cuspidal ends and cusp subgroups of $\Gamma$: Since we assume that $G$ is semisimple of rank 1, a nontrivial good unipotent is contained in a unique proper $k$–parabolic, hence fixes a unique end. Therefore every cusp subgroup of $\Gamma$ fixes a unique end as well, which is $\Gamma$–cuspidal by
definition. Conversely any $\Gamma$–cuspidal end (i.e., proper $k$–parabolic) $P$ defines the cusp subgroup $\Gamma \cap U(k)$, where $U$ is the unipotent radical of $P$. Part (2) of Theorem 2.3 will show, that this bijection can be pushed down to the quotient level.

We next note that our cuspidal ends are what Lubotzky calls the cusps of the lattice in (Lubotzky, 1991). Therefore we may use his results whenever convenient.

**Remark 2.2** Let $\Gamma$ be a lattice in $G(k)$. The $\Gamma$–cuspidal ends of $X_G$ are precisely the cusps of $\Gamma$ in the sense of Lubotzky ((Lubotzky, 1991, Definition 6.4) — see below).

**Proof:** Recall that Lubotzky calls an end $\epsilon$ of $X_G$ a cusp of $\Gamma$ if and only if there exists a vertex $x$, such that writing $x$ as $g.x_j$; $j = 0, 1$ (where $x_0$, $x_1$ are representatives) as we may, we find a nontrivial good unipotent in the group $\Gamma \cap gN_1g^{-1}$ fixing $\epsilon$. (The definition of $N_1$ can be found just before Citation 3.10; it depends on $j$, a natural number $n$, and a lattice $L$ in the Lie algebra of $G$. The dependence on $j$ is discussed at the beginning of Section 3.4. The existence of an appropriate $L$ is proved in the same section. The parameter $n$ must be chosen large enough to guarantee a geometric property (★) found at the beginning of Section 4.) Call such an end a Lubotzky–cusp for the sake of this proof.

Let $\epsilon$ be a Lubotzky–cusp of $\Gamma$. By Lubotzky’s definition there is even a nontrivial good unipotent in $\Gamma \cap gN_1g^{-1}$, so $\epsilon$ is evidently $\Gamma$–cuspidal. For the converse, let the end $\epsilon$ be cuspidal, i.e., assume there is a nontrivial good unipotent element $u$ in $\Gamma$ fixing $\epsilon$. Choose a sequence $(g_i)_{i \in \mathbb{N}}$ in $G(k)$ such that $g_i^{-1}ug_i$ converges to $e$ if $i$ tends to infinity (use Theorem 3.2). In addition let $r \in \mathbb{N}$ be large enough to guarantee $N := G(k)_{B_x(r)} \subseteq N_1$.

If we choose $k \in \mathbb{N}$ large enough to ensure $g_k^{-1}ug_k \in N$, we get $e \neq u \in \Gamma \cap g_kN_1g_k^{-1} \subseteq \Gamma \cap g_kN_1g_k^{-1}$. We may then choose the vertex $x$ to be $g_kx_j$ (where $j$ depends on $N_1$).

A further remark is in order here: The group $N_1$ is only defined for absolutely simple groups over fields of positive characteristic. For fields of characteristic 0 all lattices in $G(k)$ are uniform thanks to a result of Tamagawa; see page 84 in (Serre, 1980). But then a lattice $\Gamma$ can not contain any nontrivial good unipotent elements thanks to Corollary 3.3. So there are no $\Gamma$–cuspidal ends hence no Lubotzky–cusps unless the field has positive characteristic. If the group is not absolutely simple, Lubotzky’s definition has to be adapted by working backwards through the reduction steps of Section 1.3 to cover the general case.

The following theorem gives the best discrete analogue of the classical description of the structure of the quotient of the upper half plane by a lattice in $SL_2(\mathbb{R})$ one can hope for. To guarantee that $G(k)$ acts without inversion, we work with the barycentric subdivision $X'_G$ of $X_G$ in the following theorem. The function $q(\cdot) + 1$ will give the order of ramification at a vertex or any vertex in the $\Gamma$–orbit of a vertex in the quotient graph as appropriate.
Theorem 2.3 (structure of $\Gamma \backslash X'_G$) Let $G$ be a connected semisimple $k$–group of $k$–rank 1.

(1) For any lattice $\Gamma$ in $G(k)$ the quotient graph $\Gamma \backslash X'_G$ is the union of a finite connected graph $E$ with finitely many simplicial rays $r_i$; $1 \leq i \leq c$ attached to $E$ at their respective origin.

If $y$ and $y'$ are two neighboring vertices on one of these rays which are sufficiently far from $E$ and with $y$ nearer to $E$ as $y'$ then $\Gamma_y$ is a subgroup of $\Gamma_{y'}$ of Index $q(y')$.

(2) The map from the set of cusps of $\Gamma$ to the set of geometric cusps of $\Gamma$ induced by the map sending each maximal good unipotent subgroup to the unique end it fixes is bijective.

(3) The $\Gamma$–cuspidal ends of $X'_G$ are precisely the ends whose $\Gamma$–stabilizer is infinite and locally finite. They are maximal infinite locally finite subgroups of $\Gamma$. Every infinite and locally finite subgroup of $\Gamma$ fixes a unique end.

(4) Suppose that all bad unipotent elements of $G(k)$ contained in $\Gamma$ are anisotropic. Then each cusp subgroup of $\Gamma$ is a cocompact lattice in the group of elliptic elements fixing the $\Gamma$–cuspidal end $e$ fixed by the cusp subgroup. In other words, the cusp subgroup is of finite index in $\Gamma_e$.

Lubotzky (using his notion of cusps) proves parts (1) and (3) and states part (2). We will therefore confine ourselves here to report the proof of (4). A complete proof of the remaining parts can be found in Section 4.

The proof of (4) is an immediate Corollary to a technical result derived in the central steps 4.2–4.5 in Raghunathans proof of Citation 3.12. What makes it worth reporting, is that the condition we impose in (4) almost always holds, see Remark 3.9.

We first make sure, that it suffices to treat the absolutely simple case: The reduction steps are explained in Section 1.3. The facts listed in Proposition 3.1 make sure that the lattice obtained after going through the reduction steps still satisfies our precondition. On the other hand the last fact mentioned in Section 1.4 makes sure, that the conclusion will hold for the original lattice once it is proved for its “reduced” version. So we may indeed assume that $G$ is absolutely simple.

If the field $k$ has characteristic 0, we know (c.f. the remark following Remark 2.2) that there are then no nontrivial good unipotents in the lattice $\Gamma$ and therefore no cusp groups. It follows that the claim of (4) is trivially valid for fields of characteristic 0. So we may also assume that the field has positive characteristic. This will enable us to use the results of (Raghunathan, 1989).

Now, let $V \leq \Gamma$ be a cusp subgroup. Let $P$ be the unique end of $X_G$ it fixes, $U$ its unipotent radical, $P := P(k)$ and $U := U(k)$. We have to show that $V$ is a cocompact lattice in the group of elliptic elements in $P(k) = P$. It is evidently discrete.
By definition \( V \) is contained in \( U \). Since \( U \) is cocompact in the group of elliptic elements of \( P \) (see Section 1.4), it suffices to show that \( V \) is cocompact in \( U \). The unipotent group \( \Lambda \) defined in the first paragraph on page 142 of (Raghunathan, 1989) satisfies

\[
V = U \cap \Gamma \leq \Lambda \leq P \cap \Gamma.
\]

Raghunathan proves that \( \Lambda \) is cocompact in the group \( L \) of rational points of its Zariski–closure, and that \( L \) contains \( U \).

Our hypotheses on unipotents in \( G(k) \) enables us to prove that actually \( V = \Lambda \) holds and our claim will follow. All elements of \( \Lambda \) are unipotent, and none of them is anisotropic, since they are all contained in \( P = P(k) \). By our hypotheses then, \( \Lambda \) can not contain any bad unipotents, therefore \( \Lambda \subseteq U \cap \Gamma = V \). □

Assuming the additional assumption we made to derive part (4) of Theorem 2.3 we can improve upon Lubotzky’s converse to his receivee to construct nonuniform lattices as follows.

**Proposition 2.4** Let \( \Gamma \) be a lattice in a connected semisimple \( k \)-group of \( k \)-rank 1. Assume that all cusp stabilizers of \( \Gamma \) are residually finite and that all bad unipotent elements of \( \Gamma \) are anisotropic.

Then there is a sublattice \( \Gamma^* \) of \( \Gamma \) whose cusp stabilizers are cusp subgroups of \( \Gamma^* \) and which is the free product of the representatives \( \Delta_1^*, \ldots, \Delta_c^* \) of its (algebraic) cusps and a free group of rank \( \text{rank}_Z(H_1(\Gamma^* \setminus X^*_\Gamma)) \) generated by hyperbolic elements.

**Proof:** The method of proof applied to derive Lubotzky’s converse, Theorem 7.1 in (Lubotzky, 1991), can be reused. We supplement it with a geometric interpretation.

Part (1) of the Structure Theorem implies that a lattice in the group of rational points of a semisimple group of rank 1 over a local field is the fundamental group of a finite graph of groups, obtained by “contraction of cusps”: Along the simplicial rays \( r_i \) we eventually have an increasing chain of vertex groups. We may therefore replace an appropriate tail of each of these rays of groups by a single point whose attached vertex group equals the direct limit (the fundamental group) of the tail.

The fundamental group is unchanged by this modification. But the new graph of groups is finite with finite edge groups and residually finite vertex groups. This is obvious except for the vertex groups obtained by contraction of a tail of a geometric cusp. The latter vertex groups stabilize the cuspidal end covering the tail. They are therefore residually finite by assumption. (Indeed those vertex groups are the stabilizers of that cuspidal end, since cusp stabilizers consist of elliptic elements, compare Corollary 3.6.) By a result of Bass–Serre theory ((Serre, 1980, Proposition 12)), the group \( \Gamma \) is residually finite and the topology of subgroups with finite index in \( \Gamma \) induces the topology of subgroups with finite index on each of the vertex groups.

By our second assumption and part (4) of the Structure Theorem, the cusp subgroups of \( \Gamma \) have finite index in the cusp stabilizers. We may therefore choose
a normal subgroup $\Gamma^*$ of finite index in $\Gamma$ such that the intersection with $\Gamma^*$ of the vertex groups obtained by contraction consist of good unipotent elements and the intersection of $\Gamma^*$ with the other vertex groups is trivial. The group $\Gamma^*$ determines a covering of the modified graph of groups of $\Gamma$. Each of the vertex groups for $\Gamma^*$ above the contracted tails is conjugate to the intersection of $\Gamma^*$ with the fundamental group of the tail, hence consists of good unipotent elements. The other vertex groups for $\Gamma^*$ are trivial. Hence $\Gamma^*$ is the free product of its nontrivial vertex groups extended by the free group on the set of edges outside a maximal subtree. Since the graph of groups of $\Gamma^*$ is finite, its sets of vertex and edge groups are finite, and our claim is proved modulo the geometric interpretation.

To arrive at it, we interpret the universal covering tree $\overline{X}_G$ of the modified graph of groups of $\Gamma$ in terms of the original tree $X_G$. According to Lemma 4.8, if we choose the tails to be contracted small enough, $\overline{X}_G$ can be realized by contracting horoballs around cusps which are independent with respect to $\Gamma$. These horoballs are then independent with respect to the subgroup $\Gamma^*$ as well. The groups $\Gamma^*$ and $\Gamma$ are commensurable, hence have the same set of cuspidal ends (this is obvious when using their characterisation in part (3) of the Structure Theorem). The stabilizers of the independent horoballs (which are the vertices of infinite ramification index in $\overline{X}_G$) coincide thus with the stabilizers of the cuspidal end they contain. This shows that the nontrivial vertex groups of the graph of groups for $\Gamma^*$ we considered in the previous section represent the conjugacy classes of the stabilizers of the $\Gamma^*$–cuspidal ends. We are left to confirm that each of the edges of the quotient graph of groups of $\Gamma^*$ acts by a hyperbolic transformation and that there are exactly $\text{rank}_Z(H_1(\Gamma^* \setminus \overline{X}'_G))$ many of them. But this is obvious from the interpretation of the action of $\Gamma^*$ on $\overline{X}_G$ in terms of the original action. □

This stronger converse enters into the statement and proof of Theorem 4.1 in (Zalesskii, 1995). Dependence however seems not to be critical.

**Remark 2.5** Our second assumption almost always holds, for we are going to show that in most groups of rank 1 each bad unipotent element is anisotropic; compare Observation 3.9. On the other hand it is not known, whether cusp stabilizers of a lattice must be residually finite, equivalently, whether lattices in groups of rank 1 necessarily are residually finite. For the subclass of lattices whose cusp subgroups have finite index in the corresponding cusp stabilizer the question becomes whether a lattice in the group of rational points of the unipotent radical of a semisimple group of rank 1 over a local field of positive characteristic is residually finite. If the unipotent radical is abelian, it is the additive group of a vector space over the field and hence residually finite. I conjecture that each discrete unipotent subgroup of an algebraic group over a local field of positive characteristic is residually finite. (Discreteness is needed, since the unipotent radical of (any) minimal $k$–parabolic in the group $SU_3$ over an infinite field with respect to the standard hermitian form is not residually finite.)
3. Unipotent elements

3.1. Good, bad and ugly

Recall that an element $u$ in an (affine) algebraic group is called unipotent iff it is unipotent in some faithful rational linear representation. It will then be unipotent in every representation. It is important to note, that in positive characteristic $p$ unipotents are exactly the elements of $p$–power order.

If the underlying group $G$ is semisimple (or even reductive) and defined over $k$, the most obvious unipotents are those lying in the unipotent radical of a $k$–parabolic subgroup. An element or group contained in the unipotent radical of a $k$–parabolic subgroup is called $k$–good or more concisely good. (Unipotent) elements which are not good will be named bad. An element will be called anisotropic or very bad iff the only $k$–parabolic subgroup containing that element is the whole group. In a semisimple group nontrivial anisotropic unipotents are necessarily bad. We will call bad elements which are not anisotropic ugly. For the reduction steps we need the know, how these concepts behave under central isogeny, direct products and application of the restriction of scalars functor.

**Proposition 3.1** Let $\pi: \tilde{G} \rightarrow G$ be a central isogeny between reductive $k$–groups, which is defined over $k$.

(i) $\pi$ induces a bijection between the set of all good unipotent elements in $\tilde{G}(k)$ and the set of all good unipotent elements in $G(k)$ ((Tits, 1987), 2.2).

(ii) An image of an anisotropic element is anisotropic, and an image of a bad element is bad (obvious from (i)).

(iii) An element in an (almost) direct product is good/anisotropic, iff all its components are good/anisotropic. In an anisotropic group any element is anisotropic.

(iv) The restriction of scalars functor induces bijections between the good, bad and anisotropic elements of both groups respectively over the respective fields. (This follows again from the properties stated in (Margulis, 1989, I.1.7).)

It is possible to picture the different kinds of unipotent elements in a connected semisimple group $G$ over a local field geometrically. This characterization rests on the following well known result:

**Theorem 3.2** Let $G$ be a connected semisimple group defined over a local field $k$. An element of $G(k)$ is a good unipotent element iff the closure of its $G(k)$–conjugacy class contains $e$.

We note the following important consequence. It hints at a link between good unipotent elements in a lattice and a noncompact fundamental domain (as made precise by the Structure Theorem in case of rank 1 groups).
Corollary 3.3 Let $G$ be a connected semisimple group defined over a local field $k$. A uniform lattice in $G(k)$ does not contain any nontrivial good unipotent element.

Proof: Suppose that $\gamma$ is a nontrivial good unipotent element of the lattice $\Gamma$. By Theorem 3.2 there is a sequence $(x_n)_{n \in \mathbb{N}}$ in $G(k)$ such that $x_n \gamma x_n^{-1}$ converges to $e$ as $n$ converges to infinity. Let $(\gamma_n)_{n \in \mathbb{N}}$ be the sequence constantly $\gamma$. Put $G := G(k)$. Then these objects satisfy the conditions on the objects with the same name in Theorem 1.12 from (Ragunathan, 1972) and we conclude that the images of the points $x_n$ in the space $G/\Gamma$ do not have a convergent subsequence. It follows that $\Gamma$ can not be cocompact in $G(k)$, a contradiction. □

We now derive the geometrical distinction between the different types of unipotent elements. Since we are not using this result, we only state it for adjoint groups without anisotropic factors. As is easily seen from their definition, good unipotent elements fix a simplex at infinity in the spherical building of $G(k)$. On the other hand, anisotropic elements do not and it is in this respect that they are evident (that is very) bad elements. Using the action of the group on its Bruhat–Tits building we derive:

Proposition 3.4 Let $G$ an adjoint connected semisimple group defined over a local field $k$ without $k$–anisotropic factors. Then $G(k)$ may be considered a closed subgroup of the groups of automorphisms of its Bruhat–Tits building (use Section 1.3) and we have:

- An element of $G(k)$ is good iff its fixed point set contains balls of arbitrary large diameter.

- An element of $G(k)$ is anisotropic iff its fixed point set is bounded. (As will be obvious from the first part of the next Theorem it will be unipotent as well iff it has order a power of the characteristic of the field.)

- An element of $G(k)$ is ugly iff its fixed point set is unbounded but it does not contain balls of arbitrary large diameter. It then obviously fixes points at infinity as well. (The remark in parentheses of the previous item applies.)

- An element of $G(k)$ is bad iff its fixed point set does not contain balls of arbitrary large diameter. (The remark in parentheses of the previous item applies.)

Proof: By Theorem 3.2 an element is a good unipotent iff its fixed point set contains a $G(k)$–translate of any ball in the Bruhat–Tits building. Since $G(k)$ acts cocompactly this will hold already if the fixed point set contains arbitrary large balls. The characterization of good unipotents claimed follows and so does the characterization of the bad ones.

In geometric terms, to say that an element is anisotropic means nothing else than that it has no fixed points in the spherical building of $G$ over $k$. Stated
equivalently, it has no fixed points at infinity. This will be true iff the fixed point set is bounded. This gives the characterization of the anisotropic elements. Applying the characterization of the bad elements, the claim on the ugly elements is established. We are done.

(If the group \( G \) is not adjoint, or has anisotropic factors, the above statements have to be modified. Still, a good unipotent has a fixed point set containing balls of arbitrary large diameter. The converse is not true, but one can always find an element in any given covering, which maps to the given element and is good.)

We will need the following sharper statement on fixed point sets of good elements available in the case of relative rank 1. There is a rather obvious analogue for groups of higher rank, which we leave to the reader to formulate.

**Lemma 3.5 (Fixed point sets of good unipotent elements)** Let \( G \) be a connected semisimple group of rank 1 over the local field \( k \). The set of fixed points of a \( k \)-good unipotent element \( u \in G(k) \) in \( X_G \) contains a horoellipse \( B_\epsilon(x; \frac{1}{3}) \) for a suitable chosen vertex \( x \) and an end \( \epsilon \) (which is unique, if \( u \neq e \)). In particular, if \( h \) is hyperbolic with repelling fixed point \( \epsilon \), the sequence of automorphisms of \( X_G \) defined by \( h^iuh^{-i} \) tends to \( e \) as \( i \) tends to infinity.

**Proof:** We begin by proving the second assertion from the first:

Note that a horoellipse \( B_\epsilon(x; \frac{1}{3}) \) contains with each interior point \( y \) the whole horoellipse \( B_\epsilon(y; \frac{1}{3}) \). We may therefore assume, replacing \( x \) if necessary, that \( x \) is a vertex on the axis of \( h \). The automorphism of \( X_G \) corresponding to \( h^iuh^{-i} \) will then fix all points in the ball \( B_\epsilon(x; \frac{1}{3}) \). This shows, that the automorphisms \( h^iuh^{-i} \) converge to the identity for \( i \) to infinity as desired. It remains to prove, that \( u \) fixes all points of a horoellipse. To show this, assume \( u \neq e \) and let \( P \) be the unique minimal parabolic \( k \)-subgroup in \( G \) containing \( u \). \( P \) will then be the unique end fixed by \( u \). Let \( U \) denote the unipotent radical of \( P \), and identify \( U \) with a root group \( U_a \) of \( \Phi(S,G) \) with respect to a maximal \( k \)-split torus \( S \) of \( G \) contained in \( P \).

One of the groups \( U_{a,m} \) of the canonical filtration of \( U_a := U(a)(k) \) will contain \( u \). Let \( \alpha_{a,m} \) be the corresponding closed halfappartment (a ray in \( X_G \)). Choose \( x \in \alpha_{a,m} \) and let \( D \) be a vector cone (\( chambre vectorielle \) in the terminology of (Bruhat and Tits, 1972)) with \( x + D \subseteq \alpha_{a,m} \). The group \( U_{x+D} \) (notation of (Bruhat and Tits, 1972, (7.1.1))) contains \( U_{a,m} \), therefore \( u \). According to (Bruhat and Tits, 1972, 7.4.33) \( U_{x+D} \) will then fix the horoellipse \( B_\epsilon(x; \frac{1}{3}) \) pointwise (as noted in (Bruhat and Tits, 1984, E8) the scalar \( \frac{1}{2} \) must be changed to \( \frac{1}{3} \)).

As an immediate Corollary we obtain:

**Corollary 3.6** Let \( G \) be a connected semisimple group of rank 1 over the local field \( k \). Let \( \Gamma \) be a lattice in \( G(k) \). If \( \epsilon \) is a \( \Gamma \)-cuspidal end, then all elements of \( \Gamma_\epsilon \) are elliptic.
Proof: By definition of $\Gamma$–cuspidal, $\Gamma$ contains a nontrivial good unipotent element, $u$ say. Suppose that $\Gamma$ contains a hyperbolic element $h$ as well. We may suppose then, that $\epsilon$ is a repelling fixed point for $h$ and apply Lemma 3.5. From the second statement listed there we conclude, that the automorphisms corresponding to $h^i u h^{-i}$ converge to $e$ for $i$ to infinity. Since the image of $\Gamma$ in the group of automorphisms of $X_G$ is discrete, this implies that $h^i u h^{-i}$ acts as the identity for large $i$, hence $u$ does, which is impossible, since it is known to fix exactly one end. □

The following result tells us that often all unipotent subgroups are good. However this is not always the case, see (Tits, 1987, 3.5) (note that this example works with any local field of the correct characteristic!).

**Theorem 3.7**

Let $G$ be a reductive group defined over a perfect field $k$ (e.g. one of characteristic 0). Then every unipotent subgroup is good [Corollaire 3.7 in (Borel and Tits, 1971)].

Let $G$ be a simply connected semisimple group defined over a field $k$ of characteristic $p$. If $|k : k^p| \leq p$, then every unipotent subgroup of $G(k)$ is good [Theorem 1 in (Gille, 1999)].

Thanks to Corollary 1 to Proposition 4 in Chapter I, § 4 in (Weil, 1995) all local fields satisfy one of the conditions on the field $k$ in the above Theorem.

Note that this implies that unipotent elements always act by elliptic transformations, i.e., they have fixed points. For either the field is of characteristic 0, any unipotent is good and fixes arbitrary large balls or the field has prime characteristic, in which case any unipotent has finite order and therefore fixed points.

As a further consequence we get the following result, which acquires central importance in the article (Raghunathan, 1989). (The reference to (Borel and Tits, 1971) has to be replaced by a reference to Theorem 3.7.)

**Lemma 3.8 (Lemma 3.4 in (Raghunathan, 1989))** If $G$ is a connected semisimple group over a local field $k$, then every unipotent element in $[G(k), G(k)]$ is good.

We finally use classification to produce a fairly short list of groups of relative rank 1 which may contain bad unipotent elements which are not anisotropic. (Along similar lines one can compile a slightly longer list of groups of relative rank 1 which may contain bad unipotents at all.)

Suppose that $u$ is not anisotropic, hence contained in a proper $k$–parabolic subgroup $P$. Since we assume that the rank is 1, it must be minimal, in particular minimal with the property $u \in P$. Proposition 3.2(B) from (Tits, 1987) produces an anisotropic $u'$ in a $k$–Levi-factor $L$ of $P$. Since $P$ is a proper parabolic, $L$ is a reductive group of strictly smaller $k$–rank than $G$. In our case this means $k$–rank($L$) = 0 and $L$ is the so called anisotropic kernel of $G$ whose type (necessarily $A$) is readily computed by removing the distinguished vertices...
form the index of $G$. Thanks to Proposition 3.2(A) of loc. cit. $u$ will be good in $G$ iff $u'$ is good in $L$.

By Theorem 2.3 of loc. cit. $u'$ will be very good hence good in $L$ if the type of this group has no connected component of the form $A_{kp-1}$, where $p$ is the characteristic of $k$. Using the classification table of reductive groups over local fields in (Tits, 1979) together with the list of indices from (Tits, 1966) we see that this will always be the case, except we hit on one of the groups $G$ of one of the types listed below. Since we already ruled out the possibility of bad unipotents in simply connected groups we get:

**Observation 3.9** In an absolutely almost simple $k$–group $G$ of $k$–rank 1 all bad unipotent elements will actually be anisotropic except possibly in one of the following cases ($p$ denotes the characteristic of the field as usual and the type names are those of the first column in the classification tables of (Tits, 1979)):

- $p|d$, with $d \geq 2$, $G$ not simply connected of type $^dA_{2d-1}$ (with index $^1A_{2d-1,1}$)
- $p = 2$, $G$ either
  - not simply connected of type $^2A_3$ (index $^2A_{3,1}$)
  - adjoint of type $^2C_2$ or $^2C_3$ (index $^2C_{2,1}$ and $^2C_{3,1}$ respectively)
  - not simply connected of absolute type $D$, i.e., $^2C–B_3$, $^4D_4$ (both with index $^2D_{4,1}$), $^4D_5$ (with index $^1D_{5,1}$) or $^2C–B_2$ (with index $^2D_{5,2}$).

Remember that our interest for this list stems from statement (4) of Theorem 2.3. One may not bother about the possibility that for certain groups over fields of characteristic 2 cusp subgroups may not be large in the stabilizer of the end they fix. However, the type listed under the first item potentially leads to infinitely many exceptional characteristics. It would therefore be worth to explore whether ugly elements will really turn up for groups of this type. The groups in this type are strictly isogeneous to the group $SL_2$ of a skew field $D$ with degree $d^2$ (hence index $d$) over its center. (If $d \geq 5$, there are $\frac{1}{2} \varphi(d)$ groups in each strict isogeny class. The adjoint group of each type would be the most interesting to treat.)

### 3.2. The role of good unipotents in Raghunathan’s paper

As already emphasized, we draw heavily on the results of Raghunathans paper (Raghunathan, 1989). Unfortunately, we have to clarify a subtle point (see next Section) involving a technical construction used there. We therefore have to go into some detail.

We agree that in this subsection we assume that $G$ is absolutely simple and that the characteristic of the field $k$ is prime. Denote the valuation ring of $k$ by $\mathcal{O}$. 
Raghunathan works with a specific base of neighborhoods for the topology of $G(k)$, which is obtained from a full $O$–lattice $L$ in its Lie algebra. The construction runs as follows:

Fix a maximal $k$–split torus $S$ of $G$ and consider the decomposition of the Lie algebra of $G$ into eigenspaces

$$\text{Lie}(G) = v \oplus z \oplus u$$

with respect to the action of $S$. Here $v$ and $u$ are the sum of the root spaces with respect to negative respectively positive roots of $S$ and $z$ is the Lie algebra of its centralizer.

Denote by $Z$ the unique maximal compact subgroup of the centralizer; for existence see Section 1.4. Choose some generator $\nu$ for the spherical Weyl group in the normalizer of $S$ and put $N := Z \cup \nu Z$. This is a maximal compact subgroup of the normalizer of $S$.

Choose $L$ now to be any full $O$–lattice in $\text{Lie}(G)$ such that

(a) $L$ is stable under the adjoint action of $N$.

(b) $L = v \cap L \oplus z \cap L \oplus u \cap L$.

In Section 3.4 we will show that such an $L$ exists, which is even stable under a maximal compact subgroup.

Since $G$ is supposed to be adjoint, we may identify it with its image under the adjoint representation. We put

$$G := G(0) := \{ x \in G(k) : x(L) = L \} = G(k) \cap GL(L)$$

and

$$G(i) := \{ x \in G(k) : (x - \text{id})(L) \subseteq \pi^i L \}$$

for $i > 0$. It is evident, that the family of all $G(i)$ for $i > 0$ consists of compact open subgroups, which are pro-$p$, define the Hausdorff topology of $G(k)$ and are normal in $G$.

We now give a slightly weaker version of some of the main results from (Raghunathan, 1989). We use the abbreviations $N_1$ and $N_2$ for the groups $G(2n + l)$ and $G(n)$ respectively. The parameter $n$ will be chosen later.

**Citation 3.10** ((Raghunathan, 1989), Theorem 3.15) Let $G$ be an absolutely simple algebraic group of relative rank 1 over the local field $k$ of positive characteristic. For any lattice $\Gamma$ in $G(k)$, there exist (integers $l$, $N_0$ and) a finite set $\Delta_1 \subseteq \Gamma$ of nontrivial good unipotent elements, such that for all $g \in G(k)$ (and $n \geq N_0$) the following holds:

$\Gamma \cap gN_1 g^{-1} \neq 1 \Rightarrow$ there are $\delta \in \Delta_1$ and $\gamma \in \Gamma$ such that $\theta := \gamma \delta \gamma^{-1} \in gN_2 g^{-1}$.

The following result is an easy Corollary.
Citation 3.11 ((Raghunathan, 1989), Corollary 3.16) Let $G$, $k$ and $\Gamma$ be as in the last result. If $P$ is a $k$–parabolic subgroup of $G$ which is $\Gamma$–cuspidal, then there is a $\delta$ in $\Delta_1$ such that $P$ is $\Gamma$–conjugate to the unique $\Gamma$–cuspidal $k$–parabolic containing $\delta$.

We may reformulate (Raghunathan, 1989, theorem 4.1) into a first general result on the structure of cusp stabilizers:

Citation 3.12 ((Raghunathan, 1989), Theorem 4.1) Denote by $^\circ P(k)$ the kernel of the modular function of $R_u(P)(k)$ restricted to conjugation by elements of $P(k)$.

If the $k$–parabolic $P$ is $\Gamma$–cuspidal then $\Gamma \cap ^\circ P(k)$ is a cocompact lattice in $^\circ P(k)$.

3.3. Implications and reinterpretations

Reading Corollary 3.11 differently, we get for free:

Proposition 3.13 The elements of $\Delta_1$ represent all conjugacy classes of cuspidal ends. Hence every lattice has only finitely many cusps.

To make use of Citation 3.12, we have to reinterpret $^\circ P(k)$ geometrically. Write $\text{mod}_G$ for the modular function of a locally compact group. We have:

Proposition 3.14 If $P$ is a minimal $k$–parabolic subgroup of $G$ then $^\circ P(k)$ is the group of elliptic elements of $P(k)$; in fact: $\text{mod}_{U(k)}(\text{int}(h)) = (q_0q_1)^{\frac{1}{2}}^\circ P(h)$ for any $h \in P(k)$.

Proof: Since the canonical map $G(k) \to \text{Aut}(X_G)$ induces an isomorphism $U(k)$ to it’s image $U$ (c.f. Section 1.4), we may compute the module of inner conjugation by $h \in P(k)$ ”geometrically“ inside $\text{Aut}(X_G)$ : Let $\mu$ be a left invariant Haar measure on $U$. We first show that all elliptics in $P(k)$ are contained in $^\circ P(k)$, i.e. $\text{mod}(\text{int}(g)|_{U(k)}) = 1$ for any elliptic $g \in P(k)$. Chose a vertex $x$ fixed by $g$. Using the fact that $U_x$ is an open compact subgroup of $U$ we get:

$$\text{mod}(\text{int}(h)|_{U}) = \frac{\mu(\text{int}(g)(U_x))}{\mu(U_x)} = 1,$$

To compute $\text{mod}(\text{int}(h)|_{U})$ for hyperbolic $h \in P(k)$ it is sufficient to stick to the case where $P$ is attracting for $h$. Let $x$ denote a vertex on the axis of $h$. We have $hU_xh^{-1} = U_{hx} \supseteq U_x$ and

$$\text{mod}(\text{int}(h)|_{U}) = |U_{hx} : U_x| = #(S_{hx}(l(h)) \cap S_P(x)).$$

Only the second equality requires proof. $U_{hx}$ acts on $S_{hx}(l(h)) \cap S_P(x)$, the stabilizer of $x \in S_{hx}(l(h))$ being $U_x$. We are therefore obliged to show that this action is transitive. The group $U$ acts transitively on each horosphere with
center \(P\) (Section 1.4), hence in particular on \(S\mathbf{p}(x) \supseteq S_{h,x}(l(h)) \cap S\mathbf{p}(x)\). Any element \(u \in U\) mapping a point in the latter set to another one must fix \(h.x\), which proves equality. Since the cardinality of \(S_{h,x}(l(h)) \cap S\mathbf{p}(x)\) is easily seen to be \((q_0 q_1)^{\frac{1}{2}} l(h)\) we are done. \(\square\)

3.4. INTERPRETING RAGHUNATHAN’S NEIGHBORHOOD BASIS

We need to comment on one last result needed in the proof of parts (1) to (3) of Theorem 2.3. It asserts that whenever a nontrivial good unipotent element \(\theta\) of a lattice fixes a sufficiently large ball around some point \(x\), then the whole stabilizer of \(x\) will fix the end fixed by \(\theta\). (This will be combined with Citation 3.10 which states that whenever there is a nontrivial element of the lattice, which fixes a huge ball around a point \(x\), then there is a nontrivial good unipotent element fixing a large ball around \(x\).)

In (Lubotzky, 1991), where this result was proved, two points were overlooked. For one, the statement depends on the point \(x\) obviously, but the condition imposed varies only with the \(G(k)\)–orbit of \(x\). There is an easy fix for this problem; only vertices and probably midpoints of edges are of interest as choices for \(x\) and we simply make the data dependent on the corresponding \(G(k)\)–orbit. The second problem is serious. Since it is visible only on close inspection, we proceed to the precise form of the statement in question.

Let \(x_0\) be a fixed vertex, and \(x_1\) some point closest to but distinct from \(x_0\), whose stabilizer is also maximal compact. Note that \(x_1\) will be the midpoint of an edge incident with \(x_0\) if the tree is regular, not a vertex. (In that case the adjoint group acts with inversion.)

Recall Raghunathans neighborhood base introduced at the beginning of Section 3.2. The fix to the second problem consists in proving:

(*) There exist \(O\)–lattices \(L[0]\) and \(L[1]\) such that the corresponding group \(G[0]\) respectively \(G[1]\) is the full stabilizer of \(x_0\) and \(x_1\) respectively.

This amounts to a geometrical interpretation of Raghunathans neighborhood basis. (The derived groups \(G(i)\), \(N_1\) and \(N_2\) introduced in Section 3.2 also acquire dependence on \(x_j\). They should therefore be written accordingly, but we only do this when dependence on \(j\) is important.)

The precise version of Lubotzky’s Lemma 6.5 is then is follows: (Remark: It is obvious from the proof, that it is unnecessary to nail down \(N\) to be \(N_2\).)

Citation 3.15 (Lubotzky, 1991, Lemma 6.5) Let \(G\) be an absolutely simple algebraic group of relative rank 1 over the local field \(k\) of positive characteristic and let \(\Gamma\) be a lattice in \(G(k)\). Let \(x = g.x_j\) with \(j \in \{0, 1\}\) and \(g \in G(k)\). Let \(N = N[j]\) be (one of the groups \(G(i)[j]\) with \(i > 0\), e.g.) the group \(N_2[j]\). If \(\Gamma \cap gNg^{-1}\) contains a nontrivial good unipotent, \(\theta\) say, then writing \(\epsilon\) for the unique end fixed by \(\theta\) we’ll have \(\Gamma_x \subseteq \Gamma_\epsilon\).

Lets pause for a moment, to see why we need a result like (*): The proof of Lemma 6.5 in (Lubotzky, 1991) is completed by the observation that \(Q\)
is normal in $\Gamma_x$. This follows from $N_2 \triangleleft R$, where $R$ denotes the stabilizer of a point of the appropriate type (red). The last claim is stated as a fact in (Lubotzky, 1991, 6.2). The reference given is (Raghunathan, 1989, 3.15) cited herein as Citation 3.10. As the reader can check, within (Raghunathan, 1989) (and (Lubotzky, 1991)) $N_2 = G(n)[j]$ is only shown to be normal in $\mathcal{G}$.

But we are safe, once we show that the lattice $L$ on which $\mathcal{G}$ depends can be chosen as to guarantee that $\mathcal{G}$ equals any maximal compact subgroup we prescribe. That’s what we are going to do now.

Set $M$ equal to the stabilizer of $x_j$ for $j = 1, 2$. Take any maximal split torus $S$ guaranteed to exist by the Corollary below as the split torus needed for the construction of $\mathcal{G}$, starting on page 19. This torus is provided by the Proposition. From the proof of the latter it is clear, that the affine apartment of any torus which qualifies will contain $x_j$. (Alternatively, if we assume that the residue field of $k$ has at least 4 elements, we can argue as follows: Maximality of $M$ implies that $M$ contains the group of units of the torus $S$. But under our assumption, the only fixed points of the group of units of $S$ are the points of its affine apartment thanks to (Tits, 1979, 3.6.1).)

Statement (B) below then clearly implies condition (b) for $L$ (to be found on page 19). Statement (A) translates into $M \subseteq \mathcal{G}$. Since $M$ is maximal, this implies $M = \mathcal{G}$.

The group $N$ which condition (a) on the same page refers to is a compact subgroup of the normalizer of $S$. We will show that it can be chosen to lie in $M = \mathcal{G}$. Since $x_j$ lies in the apartment corresponding to $S$, we know that $Z$ is contained in $M$. Now choose $\nu$ in such a way, that $x_j$ is its only fixed point in the affine apartment for $S$. Then we’ll have $N \subseteq M$.

We thus are reduced to showing the Proposition and its Corollary. We assume that $\mathcal{G}$ is a connected semisimple algebraic group and $p: \widetilde{\mathcal{G}} \to \mathcal{G}$ its universal covering. The weight space corresponding to a root $\alpha$ will be denoted $\text{Lie } \mathcal{G}(\alpha)$. The set of rational points, $\text{Lie } \mathcal{G}(\alpha)(k)$ will be denoted $\text{Lie } \mathcal{G}(\alpha)$. As before we denote by $\mathfrak{z} := \mathfrak{z}(S)$ the Lie Algebra of the centralizer of the torus $S$.

**Corollary 3.16** Let $\mathcal{G}$ be a connected semisimple algebraic group defined over the local field $k$. For any maximal compact subgroup $M$ of $\mathcal{G}(k)$ there is a full $\mathcal{O}$-lattice $L$ in $\text{Lie}(\mathcal{G})(k)$ and a maximal $k$-split torus $S \subseteq \mathcal{G}$ such that

(A) $L$ is stable under the adjoint action of $M$.

(B) $L = L \cap \mathfrak{z}(\widetilde{S}) \oplus \bigoplus_{\alpha \in \Phi(S, \mathcal{G})} (\text{Lie } \mathcal{G}(\alpha) \cap L)$.

In the Proposition we will make use of several field extensions $k', \bar{k}$ and $\hat{k}$. The corresponding rings of integers and residue fields will be written $\mathcal{O}', \mathcal{O}, \bar{\mathcal{O}}$ and $F', \bar{F}, \hat{F}$ respectively. The Galois group of a field extension $K|k$ will be written $\mathcal{G}(K|k)$.
Proposition 3.17 Let $M$ be any compact subgroup of $G = G(k)$ and $\tilde{k}|k$ a finite unramified extension. Then there is a maximal $k$–split torus $\tilde{S} \subset \tilde{G}$ and a parahoric subgroup $\tilde{P}$ of $\tilde{G} = G(k)$ with the following properties.

(i) $p(\tilde{P}(\tilde{O}))$ and $M$ generate a compact subgroup of $G(\tilde{k})$. (Here $\tilde{P}$ is the parabolic group scheme over $\tilde{O}$ defined as $\text{spec } R_{\tilde{P}}$ with

\[ R_{\tilde{P}} = \{ f \in k[G] \mid f(\tilde{P}) \subseteq \tilde{O} \} \]

$k[\tilde{G}]$ being the coordinate ring of $\tilde{G}$).

(ii) The maximal compact subgroup of $\tilde{S}(\tilde{k})$ is contained in $\tilde{P}$.

Proof (of Corollary): Let $\tilde{k}$ be an unramified extension chosen such that for any $k$–split maximal torus $\tilde{S}$ of $G$, there is an element $t \in \tilde{S}(\tilde{k})$ such that $\alpha(t), \alpha(t) - 1$ are units for every $\alpha \in \Phi(\tilde{S}, \tilde{G})$ and $\alpha(t) - \beta(t)$ is a unit for every pair of distinct roots $\alpha, \beta$ in $\Phi(\tilde{S}, \tilde{G})$ (the residue field of $\tilde{k}$ needs to be sufficiently large to secure this). Choose $\tilde{S}$ and a parahoric subgroup $\tilde{P}$ as in the proposition (for the compact group $M$). Since $M$ and $p(\tilde{P}(\tilde{O}))$ generate a compact subgroup $M'$ of $G(k)$, there is $\tilde{O}$–lattice $L$ in $\text{Lie } G(k)$ which is stable under $M'$. The element $t$ necessarily belongs to the maximal compact subgroup of $\tilde{S}(\tilde{k})$ (all its eigenvalues in the adjoint representations are assumed to be units). By our choice of $t$, the eigen spaces of $\text{Ad}(t)$ are the same as those of $\tilde{S}(\tilde{k})$ and it follows from elementary linear algebra that

\[ L = L \cap \tilde{S}(\tilde{S}) \oplus \bigoplus_{\alpha \in \Phi(\tilde{S}, \tilde{G})} (\text{Lie } G(\alpha) \cap L) . \]

Hence the corollary. \qed

Proof (of Proposition): To prove the proposition we need to use Bruhat Tits theory. Fix a $k$–split torus $\tilde{S}$ in $\tilde{G}$ and a maximal torus $\tilde{S}_1$ of $\tilde{G}$ defined over $k$ and containing $\tilde{S}$ and such that $\tilde{S}_1$ contains a maximal split torus over the maximal unramified extension $\tilde{k}$ of $k$. Suppose now that $k'|k$ is an unramified extension such that the maximal split torus over $k'$ contained in $\tilde{S}_1$ is also maximal split over $\tilde{k}$. Let $B$ be the Bruhat–Tits building associated to $G(k')$ and $A \subset B$ the $U_1$ fixed points in $B$ where $U_1$ is the maximal compact subgroup of $\tilde{S}_1(k')$. Then $A$ is “an apartment” in $B$ and is stable under the action of the Galois group $G(k'/k)$; and by the fixed point theorem of Bruhat–Tits, there is a point $b \in B$ fixed by $G(k'/k)$. Let $\tilde{P}'$ be the subgroup of $G(k')$ that fixes the point $b$. Then the $\tilde{O}$–algebra $R_{\tilde{P}} = \{ f \in k[\tilde{G}] \mid f(\tilde{P}') \subseteq \tilde{O} \}$ defines a parahoric group scheme $\tilde{P} = \text{spec } R_{\tilde{P}}$ over $\tilde{O}$ and $\tilde{P}(\tilde{O}) = \tilde{P}'$. Let $F$ be the residue field of $\tilde{O}$ and $\pi = \tilde{P}(\tilde{O}) \to \tilde{P}(F)$ the natural map. $\tilde{P} \otimes_{\tilde{O}} F$ is a connected group scheme over the finite field and hence admits a Borel subgroup $B$ over $F$; then $\pi^{-1}(B(F))$ (resp $\pi^{-1}(B(F')) : \pi$ also denote the map $\tilde{P}(\tilde{O}') \to \tilde{P}(F')$, is an Iwahori subgroup $I$ of $\tilde{G}(k)$ (resp $\tilde{P}$ of $\tilde{G}(k')$). If $R_{\tilde{I}} = \{ f \in k[\tilde{G}] \mid f(\tilde{I}) \subseteq \tilde{O} \}$,
then \( \text{spec } \mathcal{R}_{\mathcal{I}} = \mathcal{I} \) is an Iwahori group scheme with \( \mathcal{I}(\mathcal{O}) = \mathcal{I} \) and \( \mathcal{I}(\mathcal{O}') = \mathcal{I}' \). The torus \( \mathcal{S} \) being split has a natural definition over \( \mathbb{Z} \) and hence over \( \mathcal{O} \). Moreover it is easy to see that the restriction map of functions in \( \mathcal{R}_{\mathcal{P}} \) to \( \mathcal{S} \) gives a closed immersion of this split torus over \( \mathcal{O} \) in \( \mathcal{P} \). From the considerations it is easy to see that \( B \) above can be so chosen that it contains the reduction modulo the maximal ideal of the split torus \( \mathcal{S} \) (over \( \mathcal{O} \)). This means that \( \mathcal{I}' = \mathcal{I}(\mathcal{O}') \) contains the maximal compact subgroup \( (= \mathcal{S}(\mathcal{O}')) \) of \( \mathcal{S}(k') \). This leads us to the conclusion that any Iwahori subgroup of \( \mathcal{G}(k') \) stable under \( \mathcal{G}(k'|k) \) necessarily contains the maximal compact subgroup of \( \mathcal{S}(k') \) with \( \mathcal{S} \) a maximal \( k \)-split torus in \( \mathcal{G} \).

Suppose now that \( M \) is as in the Proposition. We assume, as we may, that \( M \) is a maximal compact subgroup of \( \mathcal{G} \). The group \( M \) as well as the group \( \mathcal{G}(\kbar/k) \) (\( \kbar \) as in the proposition) act as isometries of the Bruhat–Tits bundle \( \mathcal{B}_0 \) associated to \( \mathcal{G}(\kbar) \). Since their actions on \( \mathcal{B}_0 \) commute with each other, they generate together a compact group of isometries of \( \mathcal{B}_0 \) and hence by the Bruhat–Tits fixed point theorem they have a common fixed point \( b \). Such a fixed point determines a parabolic subgroup \( \mathcal{P} \) over \( k \) which necessarily contains an Iwahori subgroup \( \mathcal{I} \) over \( k \). Let \( \mathcal{I} \) (resp \( \mathcal{P} \)) be the group scheme over \( \mathcal{O} \) associated to \( \mathcal{I} \) (resp \( \mathcal{P} \)). Then the isotropy group in \( \mathcal{G}(\kbar) \) of the point \( b \in \mathcal{B} \) contains \( M \) as well as \( p(\mathcal{P}(\mathcal{O})) \) hence \( p(\mathcal{I}(\mathcal{O})) \); and from the preceding paragraph, we know that \( \mathcal{I}(\mathcal{O}) \supset \) maximal compact of some \( \mathcal{S}(\kbar) \) with \( \mathcal{S} \) maximal split torus over \( k \). This proves the proposition. \( \square \)
4. The details

In this last section, we supply a complete proof of parts (1) to (3) of “our” main result, Theorem 2.3. The reasoning follows closely Lubotzky’s original proof. At some places we have to inject facts we piled up in earlier sections for which (Lubotzky, 1990) provides an easier proof in the special case of $G = \text{SL}_2$.

We start with a reduction to the case of an absolutely simple group, using the steps listed in Section 1.3. Statements (1) and (3) will follow in general, once they are proven in the absolutely simple case for the simple reason that the restriction of the map $G(k) \to G'(K) \leq \text{Aut}(X'_G)$ to $\Gamma$ has finite kernel. Claims (2) and (4), dealing with good unipotents, need in addition that the group of rational points the unipotent radical of a $k$–parabolic injects into $G'(K) \leq \text{Aut}(X'_G)$, as stated in Section 1.4. The reader may wish to have a glance at the detailed description of the map from algebraic to geometric cusps given in the final step of the proof of Corollary 4.5 on page 29 to verify that statement for claim (4).

Further, all claims in Theorem 2.3 are trivial for cocompact lattices. By Tama-gawas result, already quoted on page 10, we may therefore assume that the field $k$ has positive characteristic. In sum, all the results of Raghunathans paper (Raghunathan, 1989) will be applicable. The results to follow however will usually be true in general, as will be obvious a posteriori.

Recall how the points $x_0$ and $x_1$ were chosen (beginning of Section 3.4): $x_0$ is some fixed vertex, and $x_1$ closest to but distinct from $x_0$, with the property that it has a maximal compact stabilizer as well. We now need to choose the free parameters $n = n[0], n[1]$ with $n \geq N_0$ which determines the groups $N_1$ and $N_2$ (of type [0] and [1]) and a radius $\rho$ large enough such that we have

$$(\star) \quad G_{B_{x_0}(\rho)} \cup G_{B_{x_1}(\rho)} \subseteq N_1[0] \cap N_1[1] \text{ and } N_2[0] \cup N_2[1] \subseteq G_{B_{x_0}(1)} \cap G_{B_{x_1}(1)}$$

This is clearly possible, since the family of groups $G(i); i \in \mathbb{N}$ is a system of neighborhoods of $e$ in $G(k)$ defining the Hausdorff–topology and since the (adjoint, absolutely almost simple) group $G(k)$ can be identified with a topological subgroup of $\text{Aut}(X'_G)$.

The following fundamental Lemma describes what happens near a $\Gamma$–cuspidal end. We will make the barycentric subdivision $X'_G$ into a metric subspace of the metric space $X_G$ by assigning length $\frac{1}{2}$ to the edges of $X'_G$. When considered as a function on $X'_G$ $\varphi_\epsilon$ will translate by the distance $\frac{1}{2}$ accordingly.

**Lemma 4.1** Let $\epsilon$ be a $\Gamma$–cuspidal end. Then the following assertions hold:

(i) If a vertex $y$ is chosen to lie sufficiently close to $\epsilon$, then

1. the action of $\Gamma_\epsilon$ on $S_\epsilon(y)$ and therefore also on $S_\epsilon(\varphi_k^\epsilon(y)); k \geq 0$ will be transitive.

2. For any $y$ as in 1. $\#\Gamma_{\varphi_k^\epsilon(y)} \geq (q_0 q_1)^{\frac{1}{k}}.$
(ii) Given $r \in \mathbb{R}$ we can find a vertex $y := y(r)$ sufficiently close to $\epsilon$ such that

1. $\Gamma_\epsilon$ contains a non trivial good unipotent $\theta_y$ fixing $B_{y_k}(r)$ pointwise with $y_k := \varphi^k_\epsilon(y) \in [y, \epsilon[; k \in \mathbb{N}$. 
2. if we chose $r \geq \rho$, then $\varphi^k_\epsilon(y) \subseteq \Gamma \varphi^{k+1}_\epsilon(y) \subseteq \Gamma_\epsilon$ for all $k \geq 0$.

(iii) If we chose $y \in X'_G \mathfrak{g}$ to have both properties (i).1 and (ii).1 with $r \geq \rho$, then

1. any vertex $x \in B_x(y) \mathfrak{g}$ will also have property (ii).1 with respect to $r$, i.e. there exists a non trivial good unipotent $\theta_x \in \Gamma_\epsilon$ fixing $B_{x}(r)$ pointwise. Furthermore for any such $x$: $\Gamma_x \subseteq \Gamma \varphi_\epsilon(x) \subseteq \Gamma_\epsilon$.
2. Any vertex $x \in B_x(y)$ has the property

$$\forall k \geq 0 \quad |\Gamma \varphi^{k+1}_\epsilon(x) : \Gamma \varphi^k_\epsilon(x)| = |\varphi^k_\epsilon(x)|.$$ 

Proof: We may suppose without loss of generality that $G$ is absolutely almost simple and adjoint (and for trivial reasons that furthermore char $k \neq 0$). We begin by proving (i). $\Gamma_\epsilon$ acts on the horospheres with center $\epsilon$ according to Corollary 3.6. Choose a ray $(y_i)_{i \in \mathbb{N}}$ representing $\epsilon$. Regarding $\epsilon$ as a parabolic $k$-subgroup $P$ we find from proposition 3.14 that $^0P(k)$ also acts on the horospheres with center $P$. This action is transitive, since the same already holds for the subgroup $R_q(P)(k)$. $^0P(k)$ is covered by the increasing sequence of compact open subgroups $(^0P(k), y_i)_{i \in \mathbb{N}}$. Any compact subset of $^0P(k)$ in particular a compact system of representatives of $\Gamma \cap ^0P(k)$ in $^0P(k)$ (which exists according to 3.12) is contained in one of these say in $^0P(k), y_i$.

From $(^0P(k) = (\Gamma \cap ^0P(k)) \cdot ^0P(k), y_i)$ we conclude that $\Gamma \cap ^0P(k)$ acts transitively on $Sp(y_i) = S_\epsilon(y_i)$. The same will then be true for the horospheres $S_\epsilon(\varphi^k_\epsilon(y_i)) = S_\epsilon(y_{i+k})$ for all $k \geq 0$. (i).1 will then follow with $y := y_i$.

Choose a vertex $y$ which has property (i).1. An element of $\Gamma_\epsilon$ permuting the points of $S_k := S_\epsilon(y) \cap S_\epsilon(\varphi^k_\epsilon(y))(k)$ necessarily fixes $\varphi^k_\epsilon(y)$. This implies

$$\#\Gamma \varphi^k_\epsilon(y) \geq \#S_k.$$ 

To prove (ii).2 it therefore remains to bound the size of $S_k$ from below by $(q_0 q_1)^{\frac{1}{2}}$ which is easy.

We now turn to assertions (ii) and (iii): Choose a nontrivial good unipotent $\theta \in \Gamma_\epsilon$. It fixes some horoellipse $B_{\epsilon}(y_0; \frac{1}{2})$; c.f. Lemma 3.5. If $y = \varphi^k_\epsilon(y_0)$ is a vertex which so close to $\epsilon$ to assure that on the one hand $\Gamma_\epsilon$ acts transitively on $S_\epsilon(y)$ and on the other hand $k \geq 6r$ then we find that $\theta := \theta_y$ fixes $\epsilon$ and all points of the ball $B_{y}(r)$. All $y' \in [y, \epsilon[ \epsilon \epsilon \epsilon$ will have the same property. We have shown (ii).1.

Before turning to the second part of (ii) we consider now (iii).1: Let $x \in B_\epsilon(y)$ be a vertex on the same horosphere as $y$. Choose $y' \in \Gamma_\epsilon$ mapping $y$ to $x$ and put $\theta_x := \gamma \theta_y \gamma^{-1} \in \Gamma_\epsilon$. This is evidently a good unipotent element suitable
Lemma 4.2 Suppose along some (equivalently any) ray defining an end of $X_G$ the order of the $\Gamma$–stabilizers of vertices increase to infinity. Then that end is $\Gamma$–cuspidal.

Proof: Denote the vertex sequence along the ray in question by $(y_i)_{i \in \mathbb{N}}$. The stabilizer of the edge connecting $y_i$ to $y_{i+1}$ equals $\Gamma_{y_i} \cap \Gamma_{y_{i+1}}$ which is of index at most $q(y_i) + 1$ in $\Gamma_{y_i}$. We conclude that the order of $\Gamma_{y_i} \cap \Gamma_{y_{i+1}}$ tends to infinity as $i$ tends to infinity as well.

Choose an index $i_0$ such that for all $i \geq i_0$

$$\#\Gamma_{y_i} > \max\{\#\text{Aut}(B_{x_0}(\rho)), \#\text{Aut}(B_{x_1}(\rho))\}$$

Then the restriction map $\Gamma_{y_i} \to \text{Aut}(B_{y_i}(\rho))$ can not be injective, therefore using property (★) we see that

$$1 \neq \Gamma_{y_i} \cap \Gamma_{B_{y_i}(\rho)} \subseteq \Gamma \cap gN_1g^{-1}.$$
We claim that $\epsilon = \epsilon'$. Suppose this is not the case. Let $y_{i'}$ with $i' \geq i_0$ be the first vertex of the ray which lies on the line $[\epsilon, \epsilon']$. The group $\Gamma_{y_{i'}}$ for $i \geq i'$ fixes $\epsilon'$ (and $y_{i'}$), therefore $y_{i'}$ as well. This is a contradiction, since $\Gamma_{y_{i'}}$ is a finite group while the order of the groups $\Gamma_{y_i}$ tends to infinity. We conclude that $\epsilon = \epsilon'$.

Thus $\theta_{i_0} \in \Gamma_{y_{i_0}} \subseteq \Gamma_{\epsilon}$. We have shown that $\epsilon$ is indeed $\Gamma$–cuspidal. □

The two preceding Lemmata together with the classification of groups of elliptic automorphisms of trees give us part (3) of our main Theorem:

**Corollary 4.3** An end of $X_G$ is $\Gamma$–cuspidal iff its $\Gamma$–stabilizer is infinite and locally finite; it is then maximal with this property. Any discrete infinite and locally finite group of automorphisms of a locally finite tree fixes a unique end. In other words claim (3) of Theorem 2.3 is true.

**Proof:** Suppose first that $\epsilon$ is an end whose $\Gamma$–stabilizer is infinite and locally finite. We know that a discrete group of tree automorphisms has finite point stabilizers (see 1.2). Since $\Gamma_\epsilon$ is infinite, it can not have a common fixed point. It can not contain hyperbolic elements either, since these have infinite order. We conclude from Proposition 2 in (Bass, 1993) that there is a unique end $\epsilon'$ fixed by $\Gamma_\epsilon$, and that $\Gamma_\epsilon$ is the increasing union of the stabilizers of the vertices on any ray defining $\epsilon'$ and the inclusions are proper infinitely often. Uniqueness of the end gives us $\epsilon' = \epsilon$. This also proves that discrete infinite locally finite groups of automorphisms fix a unique end, hence the second claim is true as well.

We conclude that the order of the $\Gamma$–stabilizers along any ray defining $\epsilon$ tend to infinity. According to the previous Lemma $\epsilon$ is $\Gamma$–cuspidal as claimed. Obviously, any infinite locally finite group is contained in a maximal one, which fixes a unique end by what we already know. Therefore $\Gamma_\epsilon$ is indeed a maximal infinite and locally finite subgroup.

Finally, we prove the converse of the first claim. Let $\epsilon$ be a $\Gamma$–cuspidal end. We use again Proposition 2 from (Bass, 1993). This result is applicable, since $\Gamma_\epsilon$ does not contain any hyperbolic elements thanks to Corollary 3.6. We wish to prove that $\Gamma_\epsilon$ satisfies the third condition listed in Proposition 2 of loc.cit.; it will then be the (infinitely often strictly) increasing union of stabilizers of vertices. Since stabilizers of points in $\Gamma$ are finite groups, it will then follow that $\Gamma_\epsilon$ is infinite and locally finite.

We show that $\Gamma_\epsilon$ does not satisfy the two other conditions listed. The stabilizer of an end can not contain inversions, therefore the second condition can not hold. To exclude the first possibility it suffices to show, again involving discreteness, that $\Gamma_\epsilon$ is an infinite group. But this easily follows from parts (iii) and (ii) of Lemma 4.1.

As is obvious from the last part of the previous proof we may reformulate again, getting the converse to Lemma 4.2:

**Corollary 4.4** An end of $X_G$ is $\Gamma$–cuspidal iff the orders of the $\Gamma$–stabilizers of vertices along some (any) ray representing that end increase to infinity.
Piecing together what we got so far, we get part (1) and half of part (2) of our main Theorem:

**Corollary 4.5** The quotient graph of groups defined by the \( \Gamma \)-action on \( X'_G \) looks like described in part (1) of Theorem 2.3. The map described in part (2) is a surjection. The geometric cusps correspond bijectively to the ends of any lift of a maximal subtree of the quotient graph.

**Proof:** Choose a fundamental \( \Gamma \)-transversal \( Y \) with maximal subtree \( Y_0 \) in \( X'_G \) (cf. (Dicks and Dunwoody, 1989, I.(2.6)) for notation).

**First step.** All ends of \( Y_0 \) are \( \Gamma \)-cuspidal. There are only finitely many of them. The vertices along any ray in \( Y_0 \) run through vertices which are all inequivalent modulo \( \Gamma \). The combinatorial volume formula at the end of Section 1.1 shows that the orders of their \( \Gamma \)-stabilizers must converge to infinity. Thanks to Lemma 4.2 (or Corollary 4.4) that means that the end defined by the ray is \( \Gamma \)-cuspidal. This proves the first statement. According to Proposition 3.13 \( \Gamma \) acts with finitely many orbits on the \( \Gamma \)-cuspidal ends. It will then map appropriate tails of rays defining ends lying in the same \( \Gamma \)-orbit onto each other. It follows that \( Y_0 \) does only contain finitely many inequivalent rays, hence has only finitely many ends.

**Second step.** On each ray representing an end of \( Y_0 \) the index condition on stabilizers stated in Theorem 2.3 eventually holds. This is immediate from the first step, using Lemma 4.1, part (iii).

**Third step.** Deleting appropriate rays representing all the ends of \( Y_0 \) leaves a finite graph. The geometric cusps correspond bijectively to the ends of any lift of a maximal subtree of the quotient graph.

The first claim is already obvious from the first step. By the result of the second step, the deletion alluded to can be done in such a way that even the remainder of \( Y \) is finite. This shows that all rays of \( Y \) actually define inequivalent ends of the quotient graph. This observation is independent of \( Y \), and the second claim follows.

It remains to observe:

**Final step.** The map described in part (2) of Theorem 2.3 is a surjection.

It may seem, that the second statement of the third step already takes care of part (2) of Theorem 2.3 altogether, but this is not the case. Lets prove the statement above. Given a cusp of \( \Gamma \), we choose a representative \( \Lambda \). It fixes a unique end \( \epsilon_\Lambda \) of \( X_G \), which is evidently \( \Gamma \)-cuspidal. Choose a ray \( r \) representing that end and let \( \overline{r} \) be its image in the quotient graph. We claim that some subray of \( r \) maps to a ray in \( \Gamma \backslash X'_G \). Due to the structure of the quotient graph, which we have already established, any ray there will define an end. This end will be the image of the \( \Gamma \)-conjugacy class of \( \Lambda \). This spells out the definition of the map between cusps and geometric cusps in detail.

The vertex stabilizers along the ray \( r \) have orders going to infinity. The stabilizers attached to any bounded subgraph of \( \Gamma \backslash X'_G \) have bounded orders.
Therefore the images of the vertices on the ray \( r \) must eventually leave any bounded subgraph forever. Thanks to the structure of the quotient graph, this means that these vertices converge to an end of \( \Gamma \backslash X'_G \). Looking more closely, we see that the images of the vertices on a tail of the ray \( r \) will actually define a ray, thanks to the index condition established in the second step.

The map will obviously be independent of the choice of \( r \). To see that it is independent of the choice of \( \Lambda \), we must prove that \( \Gamma \)-conjugacy of \( \Gamma_\epsilon \) and \( \Gamma_{\epsilon'} \) implies that \( \epsilon \) and \( \epsilon' \) are in the same \( \Gamma \)-orbit. Both \( \gamma \Gamma_\epsilon \gamma^{-1} \) and \( \Gamma_{\epsilon'} \) are infinite, locally finite groups thanks to Corollary 4.3 without fixed points. Invoking the Proposition form (Bass, 1993) used in the proof of that Corollary again, we see, that the end they fix is unique. If they are equal, we therefore conclude that \( \gamma . \epsilon = \epsilon' \). The map described is therefore well defined.

Finally, to show surjectivity, take any end \( \partial \) of \( \Gamma \backslash X'_G \). Extend a ray representing that end to a maximal subtree, lift it, and extend it to a \( \Gamma \)-transversal \((Y, Y_0)\). The end \( \epsilon \) of \( Y_0 \) mapping to \( \partial \) is cuspidal thanks to the first step. One easily checks that the \( \Gamma \)-conjugacy class of the cusp subgroup of \( \Gamma_\epsilon \) maps to \( \partial \) under the map just described. We are done.

We next attempt to prove part (2) of Theorem 2.3. It amounts to showing that, whenever two vertex sequences \((x_i)\) and \((y_i)\) define \( \Gamma \)-cuspidal rays, such that there is a sequence of elements \((\gamma_i)\) in \( \Gamma \) with \( \gamma_i . x_i = y_i \), there is actually a single element \( \gamma \) in \( \Gamma \) mapping the corresponding ends on each other.

We will show that this is the case using a concept, which allows the geometric interpretation of the contraction of cusp construction we promised earlier. We are going to build a quotient space of the tree \( X'_G \) by identifying horoballs to points. These will correspond to the \( \Gamma \)-orbits of the contracted rays. When forming a quotient of a space acted upon by a group, one may lose control over stabilizers, unless further conditions are imposed. One will be able to avoid this problem if the following conditions are met:

**Definition 4.6** Let \( \Gamma \) be a group acting on the set \( X \). A subset \( Y \) of \( X \) is called precisely invariant under the subgroup \( \Lambda \) of \( \Gamma \) iff the conditions

\[
\begin{align*}
(1) \quad & \Lambda = \Gamma \{ Y \} \quad \text{and} \\
(2) \quad & g(Y) \cap Y = \emptyset \quad \text{for all } g \in \Gamma \setminus \Lambda \quad \text{hold.}
\end{align*}
\]

An \( n \)-tuple \((Y_1, \ldots, Y_n)\) of subsets of \( X \) is called precisely invariant under an \( n \)-tuple \((\Lambda_1, \ldots, \Lambda_n)\) of subgroups iff

\[
\begin{align*}
(1) \quad & Y_i \text{ is precisely invariant under } \Lambda_i \text{ for all } i \\
(2) \quad & \text{for all } g \in \Gamma \text{ and indices } m \neq k \text{ we have } g(Y_m) \cap Y_k = \emptyset.
\end{align*}
\]

Borrowing a notion from hyperbolic geometry, we will call a horoball \( B_\epsilon (x) \) in a tree independent (with respect to a group of automorphisms \( \Gamma \)) iff it is precisely invariant under \( \Gamma_\epsilon \) in \( \Gamma \).
The following result extends Lemma 2.5 from (Lubotzky, 1990) to all semisimple group of rank 1. Via Lemma 4.1 this result depends again crucially on Citation 3.15.

**Lemma 4.7 (Existence of independent horoballs)** Let $\Gamma$ be a lattice in $G(k)$, where $G$ is connected semisimple $k$–group of $k$–rank 1.

(i) Let $x_\epsilon$ be a vertex of $X'_G$, such that any vertex $x$ in $B_\epsilon(x_\epsilon)$ contains a nontrivial good unipotent $\theta_x \in \Gamma_\epsilon$ fixing $B_\epsilon(\rho)$ pointwise (; existence of $x_\epsilon$ is assured by part (iii).1 of Lemma 4.1). Then $B_\epsilon(x_\epsilon)$ is independent with respect to $\Gamma$.

(ii) If $\epsilon$ and $\epsilon'$ are $\Gamma$–inequivalent $\Gamma$–cuspidal ends, then $(B_\epsilon(x_\epsilon), B_{\epsilon'}(x_{\epsilon'}))$ is precisely invariant under $(\Gamma_\epsilon, \Gamma_{\epsilon'})$.

**Proof:** We defer the proof of (i). Modulo (i) claim (ii) and the verification of the property (2) for claim (i) come down to the following: Given $y \in B_\epsilon(x_\epsilon)$ and $y' \in B_{\epsilon'}(x_{\epsilon'})$ and further an element $\gamma \in \Gamma$ with $\gamma. y = y'$, we will have $\gamma. \epsilon = \epsilon'$. To see this, choose a nontrivial good unipotent $\theta_y \in \Gamma_y$, whose conjugate under $\gamma$ lies in $\Gamma_y' \subseteq \Gamma_{\epsilon'}$. This is seen to be possible by applying part (iii).1 of Lemma 4.1 twice. We have $\gamma. \theta_y \gamma^{-1}. \epsilon' = \epsilon'$. Since a nontrivial good unipotent element fixes a unique end, we infer $\gamma. \epsilon = \epsilon'$, showing the claim.

We still have to prove that property (1) holds in claim (i). It suffices to show that $\Gamma_\epsilon$ leaves the horoball $B_\epsilon(x_\epsilon)$ invariant. This is the case, since we know from Corollary 3.6, that all elements of $\Gamma_\epsilon$ are elliptic. 

The whole point of this notion is that it allows the following result to be proved:

**Lemma 4.8** Let $\Gamma$ be a nonuniform lattice in $G(k)$, where $G$ is connected semisimple $k$–group of $k$–rank 1 over the local field $k$. Let $\{\epsilon_1, \ldots, \epsilon_c\}$ be a system of pairwise inequivalent representatives of the $\Gamma$–orbits of $\Gamma$–cuspidal ends. For each $\epsilon_i$ choose an independent horoball $B_i$ around $\epsilon_i$ in $X'_G$. Identify each of the $\Gamma$–translates of any $B_i$ to a point, i.e., form the quotient graph $X_G := X'_G / \Gamma B_1 \cup \cdots \cup \Gamma B_c$ (for notation cf. (Serre, 1980, p.80)). It is a tree thanks to Proposition 13 from (Serre, 1980, p.22). We call $X_G$ a tree obtained by contracting horoballs around cusps.

Denote the vertex, which is the image of $B_i$ in $X_G$ by $\overline{B}_i$. We have $\Gamma_{\overline{B}_i} = \Gamma_{\epsilon_i}$. The contraction map $X'_G \rightarrow X_G$ induces a map $\Gamma \backslash X'_G \rightarrow \Gamma \backslash X_G$, which is just the contraction of cusps construction described earlier. The tail of a ray $r(i)$ (representing a geometric cusps of $\Gamma$) which gets contracted to $\Gamma_{\overline{B}_i}$ is determined by the condition that some ($\iff$ any) lift of its initial point lies on the boundary horosphere of a $\Gamma$–translate of $B_i$.

Since the points $\Gamma B_i$ are different for different indices, we get as a Corollary that the canonical map from the cusps of $\Gamma$ to the geometric cusps of $\Gamma$ is injective.

**Proof:** Digesting the definitions and constructions involved, the only nontrivial claim to prove is disjointness of horoballs of the form $yB_i$ and $y'B_{i'}$ whenever
\( i \neq i' \text{ or } i = i' \text{ but } y^{-1}y' \notin \Gamma_{\epsilon_i} = \Gamma_{\epsilon_{i'}}. \) But this has been proved in the above Lemma. \( \square \)
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