ON THE CHROMATIC NUMBERS OF SPHERES IN $\mathbb{R}^n$

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Let $\chi(S_{r}^{n-1})$ be the minimum number of colours needed to colour the points of a sphere $S_{r}^{n-1}$ of radius $r \geq \frac{1}{2}$ in $\mathbb{R}^n$ so that any two points at the distance 1 apart receive different colours. In 1981 P. Erdős conjectured that $\chi(S_{r}^{n-1}) \to \infty$ for all $r > \frac{1}{2}$. This conjecture was proved in 1983 by L. Lovász who showed in [11] that $\chi(S_{r}^{n-1}) \geq n$. In the same paper, Lovász claimed that if $r < \sqrt{\frac{n}{2n+2}}$, then $\chi(S_{r}^{n-1}) \leq n+1$, and he conjectured that $\chi(S_{r}^{n-1})$ grows exponentially, provided $r \geq \sqrt{\frac{n}{2n+2}}$. In this paper, we show that Lovász’ claim is wrong and his conjecture is true: actually we prove that the quantity $\chi(S_{r}^{n-1})$ grows exponentially for any $r > \frac{1}{2}$.

1. Introduction

In this paper, we study a classical problem going back to H. Hadwiger, E. Nelson, and P. Erdős. The original problem was to find the chromatic number of the Euclidean space $\mathbb{R}^n$, which is the quantity

$$\chi(\mathbb{R}^n) = \min \{ \chi : \mathbb{R}^n = X_1 \cup \ldots \cup X_\chi, \forall i \forall x, y \in X_i \ | x - y | \neq 1 \}.$$ 

In other words, $\chi(\mathbb{R}^n)$ is the minimum number of colours needed to colour the points in $\mathbb{R}^n$ so that any two points at the distance 1 apart receive different colours.

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The quantity $\chi(\mathbb{R}^n)$ has been thoroughly investigated. Numerous results concerning it can be found in the books [4], [20] and surveys [12], [21]. For example, in 1981 P. Frankl and R. M. Wilson proved in their seminal paper [8] that

$$\chi(\mathbb{R}^n) \geq (\eta_1 + o(1))^n, \quad \eta_1 = \frac{1 + \sqrt{2}}{2} = 1.207 \ldots$$

It was the first exponential lower bound for the chromatic number. Its slight improvement

$$\chi(\mathbb{R}^n) \geq (\eta_2 + o(1))^n, \quad \eta_2 = 1.239 \ldots,$$

was obtained in 2000 by this author (see [13]). The best known upper bound

$$\chi(\mathbb{R}^n) \leq (3 + o(1))^n$$

was provided in 1972 by D.G. Larman and C.A. Rogers (see [10]).

In 1981 P. Erdős proposed another important colouring problem (see [7]). He suggested to study the chromatic numbers $\chi(S^{n-1}_r)$ of the Euclidean spheres $S^{n-1}_r$ of radii $r \geq \frac{1}{2}$ in $\mathbb{R}^n$:

$$\chi(S^{n-1}_r) = \min \{ \chi: S^{n-1}_r = X_1 \cup \ldots \cup X_\chi, \quad \forall i \forall x, y \in X_i \ | x - y | \neq 1 \}.$$  

Erdős conjectured that $\chi(S^{n-1}_r) \to \infty$ for any fixed value of $r > \frac{1}{2}$. It is obvious that $\chi(S^{n-1}_{1/2})=2$, and in 1983 L. Lovász proved Erdős’ conjecture in [11] using topological tools. The exact assertion of Lovász is as follows: for any $r > \frac{1}{2}$ and $n \in \mathbb{N}$, the inequality $\chi(S^{n-1}_r) \geq n$ holds; if $r < \sqrt{\frac{n}{2n+2}} \sim \frac{1}{\sqrt{2}}$, i.e., the length of any side of a regular $n$-simplex inscribed into $S^{n-1}_r$ is smaller than 1, then $\chi(S^{n-1}_r) \leq n + 1$. Although this result is widely cited (see, e.g., [21]), its second part is completely wrong. Actually, for every $r > \frac{1}{2}$, the quantity $\chi(S^{n-1}_r)$ grows exponentially, not linearly.

In this paper, we will do an in-depth analysis of the asymptotic behaviour of the value $\chi(S^{n-1}_r)$. We will study even some cases when $r$ may depend on $n$.

We will use the linear algebra method introduced by Frankl and Wilson in [8] (see also [1], [2], [14]). Note that Lovász conjectured in [11] that this method should work for $r \geq \sqrt{\frac{n}{2n+2}}$. Thus, we will prove Lovász’ conjecture for $r \geq \sqrt{\frac{n}{2n+2}}$ and disprove his assertion for $\frac{1}{2} < r \leq \sqrt{\frac{n}{2n+2}}$. 

2. Statements of the main results

The starting point for our investigation is the following assertion.

**Theorem 1.** For any \( r \in \left( \frac{1}{2}, \frac{1}{\sqrt{2}} \right) \), there exists a function \( \delta(n) = \delta(n, r) = o(1), n \to \infty \), such that for every \( n \in \mathbb{N} \), we have

\[
\chi(S^{n-1}_r) \geq \left(2 \left( \frac{1}{8r^2} \right)^{1/8r^2} \left(1 - \frac{1}{8r^2} \right)^{1/8r^2} \right)^n + \delta(n).
\]

As we know from Section 1, Lovász asserted in [11] that \( \chi(S^{n-1}_r) \leq n+1 \). However, it is easy to see that for any \( r \in \left( \frac{1}{2}, \frac{1}{\sqrt{2}} \right) \), the value

\[
\gamma(r) = 2 \left( \frac{1}{8r^2} \right)^{1/8r^2} \left(1 - \frac{1}{8r^2} \right)^{1/8r^2}
\]

is greater than 1. Thus, Theorem 1 says that the quantity \( \chi(S^{n-1}_r) \) grows essentially like an exponent. Moreover, if \( r' \geq \frac{1}{\sqrt{2}} \), then for any \( r < \frac{1}{\sqrt{2}} \), we have \( S^{n-1}_r \subset S^n_{r'} \) and therefore \( \chi(S^{n-1}_{r'}) \) also has exponential growth. Consequently, for any \( r \), there exists an \( n_0 \) such that for all \( n \geq n_0 \), we have \( \chi(S^{n-1}_r) > n+1 \) contradicting the assertion of Lovász.

Looking at Theorem 1, we see that if \( r \) becomes closer and closer to \( \frac{1}{\sqrt{2}} \), then the quantity \( \gamma(r) \) approaches the value \( \eta_3 = 1.139 \ldots \) Since \( S^{n-1}_r \subset \mathbb{R}^n \) leading to \( \chi(S^{n-1}_r) \leq \chi(\mathbb{R}^n) \), one may not expect that \( \eta_3 \) could be somehow replaced by anything greater than \( \eta_2 \) (cf. Section 1). However, there is some room to spare here, and one may apply an appropriate optimization procedure to replace \( \gamma(r) \) by a larger \( \gamma'(r) \). We do not want to dwell on it in this paper, since this is much more technical.

It is worth noting that some exponential bounds for the chromatic numbers of some spheres can be derived immediately from the paper [8] by Frankl and Wilson: that is what Lovász knew and why he made his conjecture concerning \( r \geq \sqrt{\frac{n}{2n+2}} \).

So in Theorem 1 we have refuted Lovász’ assertion from [11]. Also, we have substantially improved his linear lower bound, which is now exponential in the whole range \( r > \frac{1}{2} \). However, in low dimension, the value \( \delta(n) \) from Theorem 1 may become dominating. Hence, this work does not concern any fixed values of \( n \). It treats only of the asymptotic behaviour of the chromatic number. As for low dimension, many lower bounds were obtained in [9], [18], [19], but we do not want to dwell on it here.
The gap between exponents from Theorem 1 and the linear function from Lovász’ assertion is quite large. Thus, one may expect that superlinear lower bounds for \( \chi(S_{r-1}^n) \) also hold not only for a constant \( r > \frac{1}{2} \), but also for some sequences \( r_n \rightarrow \frac{1}{2} \). The most general assertion of this kind is in Theorem 2.

**Theorem 2.** Let \( \mathbb{P} \) be the set of prime numbers. Let \( f(x) \) be such a function that for any \( x \in \mathbb{R}, x \geq 0 \),

\[
f(x) = \min\{p \in \mathbb{P}: p > x\} - x.
\]

Let

\[
m(x) = \max\{m < x: m \equiv 0 \pmod{4}\}.
\]

Consider a sequence \( \{r_n\}_{n=1}^{\infty} \), where \( r_n > \frac{1}{2} \) for each \( n \in \mathbb{N} \). Set

\[
p(n) = \frac{m(n)}{8r_n^2} + f\left(\frac{m(n)}{8r_n^2}\right).
\]

If

\[
\frac{m(n)}{4} < p(n) \leq \frac{m(n)}{2}, \quad n \in \mathbb{N},
\]

then

\[
\chi(S_{r_n}^{n-1}) \geq \left(\frac{m(n)}{p(n)}\right)^{\left(\frac{m(n)}{m(n)/2}\right)}.
\]

Comparing a rather implicit result of Theorem 2 with Lovász’ assertion we get the following theorem.

**Theorem 3.** Consider a sequence \( \{r_n\}_{n=1}^{\infty} \), where \( r_n > \frac{1}{2} \) for each \( n \in \mathbb{N} \). Let \( \kappa < 2 \), and let \( p(n) \) be the same as in Theorem 2. If

\[
\frac{m(n)}{4} < p(n) < \frac{m(n)}{2} - \sqrt{\frac{m(n) \ln m(n)}{\kappa}}, \quad n \in \mathbb{N},
\]

then

\[
\chi(S_{r_n}^{n-1}) > n + 1, \quad \forall n \geq n_0.
\]

The bound given by Theorem 3 depends on the estimates for the function \( f(x) \). Determining the exact asymptotic behaviour of \( f(x) \) is a very hard problem of analytical number theory (see [6]). As far as we know, the best upper bound is \( f(x) = O(x^{0.525-\varepsilon}) \) with a very small \( \varepsilon > 0 \) (see [3]). However, it is conjectured that \( f(x) = O(\ln^2 x) \) (see [5]). The tightest lower bound is given in [15] and [17], but it is sublogarithmic and conjectured to be far from the truth. Using this information, we may derive
Theorem 4. Assume that $c_0 > 0$ is such that $f(x) \leq c_0 x^{0.525}$ for every $x$. Then, there exists a constant $c'_0 > 0$ such that for any sequence of radii $r_n$ satisfying the inequality

$$r_n \geq \frac{1}{2} + \frac{c'_0}{n^{0.475}},$$

we have the bound

$$\chi(S_{r_n}^{n-1}) > n + 1, \quad \forall n \geq n_0.$$

Theorem 5. Assume that $c_1 > 0$ is such that $f(x) \leq c_1 \ln^2 x$ for every $x$. Then, there exists a constant $c'_1 > 0$ such that for any sequence of radii $r_n$ satisfying the inequality

$$r_n \geq \frac{1}{2} + c'_1 \sqrt{\frac{\ln n}{n}},$$

we have the bound

$$\chi(S_{r_n}^{n-1}) > n + 1, \quad \forall n \geq n_0.$$

So $r_n > \frac{1}{2}$ may be quite close to the value $\frac{1}{2}$, and, nevertheless, the chromatic numbers will exceed the Lovász upper bound. Finally, it is of interest for which sequences of $r_n$, we do really have the bound $\chi(S_{r_n}^{n-1}) \leq n + 1$.

Theorem 6. There exists a constant $c_2 > 0$ such that for any sequence of radii $r_n$ satisfying the inequality

$$r_n \leq \frac{1}{2} + \frac{c_2}{n},$$

we have the bound

$$\chi(S_{r_n}^{n-1}) \leq n + 1, \quad \forall n \geq n_0.$$

Further structure of the paper is as follows. In Section 3, we shall give proofs of Theorems 1 and 2. Section 4 will be devoted to proving Theorems 3–5. In Section 5, we shall discuss Theorem 6. In Section 6, some more comments and suggestions will be given. In particular, we shall exhibit more general upper bounds for $\chi(S_{r_n}^{n-1})$ than those in Theorem 6.
3. Proof of Theorems 1 and 2

We start by proving Theorem 1.

Proof of Theorem 1. Fix an \( r \in \left( \frac{1}{2}, \frac{1}{\sqrt{2}} \right) \) and an \( n \in \mathbb{N} \). Let \( m < n \) be the maximum natural number which is divisible by 4. Let us find \( a' \) from the relation

\[
\frac{\sqrt{m}}{\sqrt{2m - 2a'}} = r, \quad \text{i.e.,} \quad a' = \frac{m(2r^2 - 1)}{2r^2}.
\]

Let \( p \) be the smallest prime number satisfying the inequality

\[
p > \frac{m - a'}{4} = \frac{m}{8r^2}.
\]

Set

\[
a = m - 4p < a'.
\]

Consider the following graph \( G(V, E) \):

\[
V = \left\{ \mathbf{x} = (x_1, \ldots, x_m) : x_i \in \left\{ -\frac{1}{\sqrt{2m - 2a}}, \frac{1}{\sqrt{2m - 2a}} \right\}, x_1 + \ldots + x_m = 0 \right\},
\]

\[
E = \left\{ \{\mathbf{x}, \mathbf{y}\} : \mathbf{x}, \mathbf{y} \in V, |\mathbf{x} - \mathbf{y}| = 1 \right\}.
\]

Obviously \( V \subset S^m_{r'} \), where

\[
r' = \frac{\sqrt{m}}{\sqrt{2m - 2a}} < \frac{\sqrt{m}}{\sqrt{2m - 2a'}} = r.
\]

If we use the standard notation \( \chi(G) \) for the chromatic number of \( G \) and \( \alpha(G) \) for its independence number, then we get

\[
\chi(S^{m-1}_r) \geq \chi(S^{m-1}_{r'}) \geq \chi(G) \geq \frac{|V|}{\alpha(G)} = \frac{(m/2)}{(m/2)} = \frac{\alpha(G)}{2}.
\]

So we are led to estimate \( \alpha(G) \) from above. It is convenient to take another representation of \( G(V, E) \). Let

\[
W = \left\{ \mathbf{x} \cdot \sqrt{2m - 2a} : \mathbf{x} \in V \right\},
\]

\[
F = \left\{ \{\mathbf{x}, \mathbf{y}\} : \mathbf{x}, \mathbf{y} \in W, |\mathbf{x} - \mathbf{y}| = \sqrt{2m - 2a} \right\}.
\]

Clearly \( G(V, E) \) is isomorphic to \( G(W, F) \). Let us denote by \( (\mathbf{x}, \mathbf{y}) \) the Euclidean scalar product of \( \mathbf{x} \) and \( \mathbf{y} \). Since for any \( \mathbf{x} \in W \), \( (\mathbf{x}, \mathbf{x}) = m \), we may rewrite \( F \) as follows:

\[
F = \left\{ \{\mathbf{x}, \mathbf{y}\} : \mathbf{x}, \mathbf{y} \in W, (\mathbf{x}, \mathbf{y}) = a \right\}.
\]
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Notice that for $x, y \in W$, the quantity $(x, y)$ lies in the interval $[-m, m]$ and is congruent to zero modulo 4. The last observation is due to the fact that $m \equiv 0 \pmod{4}$ and every vector $x \in W$ contains an even number of negative coordinates. Also,

$$m - 8p < m - \frac{m}{r^2} < -m.$$  

Thus, for every $x, y \in W$,

$$\forall x, y \in W \quad (x, y) \equiv m \pmod{p} \iff (x, y) = m \text{ or } (x, y) = a.$$  

Now, we are about to prove that $\alpha(G) \leq \binom{m}{p}$. Take an arbitrary

$$Q = \{x_1, \ldots, x_s\} \subset W, \quad \forall i \neq j, \quad (x_i, x_j) \neq a.$$  

In other words, $Q$ is an independent set in $G$. We have to show that $s \leq \binom{m}{p}$. For this purpose, we use the linear algebra method (see [1], [2], [8], [14]).

To each vector $x \in W$ we assign a polynomial $P_x \in \mathbb{Z}/p\mathbb{Z}[y_1, \ldots, y_m]$. First, we take

$$P'_x(y) = \prod_{i \in I} (i - (x, y)),$$

where

$$I = \{0, 1, \ldots, p - 1\} \setminus \{m \pmod{p}\}, \quad y = (y_1, \ldots, y_m),$$

and so $P'_x \in \mathbb{Z}/p\mathbb{Z}[y_1, \ldots, y_m]$. Obviously,

$$\forall x, y \in W \quad P'_x(y) \equiv 0 \pmod{p} \iff (x, y) \neq m \pmod{p}.$$  

Second, we represent $P'_x$ as a sum of monomials. If a monomial has the form

$$y_{i_1}^{\alpha_{i_1}} \cdot \ldots \cdot y_{i_q}^{\alpha_{i_q}}, \quad \alpha_{i_1} > 0, \ldots, \alpha_{i_q} > 0,$$

then we replace it by

$$y_{i_1}^{\beta_{i_1}} \cdot \ldots \cdot y_{i_q}^{\beta_{i_q}},$$

where $\beta_{i_\nu} = 1$, provided $\alpha_{i_\nu}$ is odd, and $\beta_{i_\nu} = 0$, provided $\alpha_{i_\nu}$ is even. Eventually, we get a polynomial $P_x$. It is worth noting that this polynomial does also satisfy property (3).

It follows from properties (1), (2), and (3) that the polynomials

$$P_{x_1}, \ldots, P_{x_s}$$
assigned to the vectors of the set \( Q \) are linearly independent over \( \mathbb{Z}/p\mathbb{Z} \). It is also easy to see that the dimension of the space generated by
\[
P_{x_1}, \ldots, P_{x_s}
\]
does not exceed \( \binom{m}{p} \). Thus, \( s = |Q| \leq \binom{m}{p} \) and, therefore,
\[
\chi(S_r^{n-1}) \geq \left( \frac{m}{\binom{m}{p}} \right)^n.
\]

Standard analytical tools (like Stirling’s formula) together with \( p \sim \frac{m}{8r^2} \) give us, finally, the expected bound
\[
\chi(S_r^{n-1}) \geq \left( 2 \left( \frac{1}{8r^2} \right)^{\frac{1}{8r^2}} \left( 1 - \frac{1}{8r^2} \right)^{1-\frac{1}{8r^2}} + \delta(n) \right)^n,
\]
which completes the proof of Theorem 1.

**Proof of Theorem 2.** The proof of Theorem 2 is now clear. We just re-produce the above argument with \( r_n \) instead of \( r \). The only thing one has to explain here is why we impose additional conditions on the value of a prime. Indeed, the inequality \( p(n) > \frac{m(n)}{4} \) is quite important, since property (1) becomes false without it. As for the inequality \( p(n) \leq \frac{m(n)}{2} \), it is necessary to correctly estimate the independence number of our graph \( G \) by the quantity \( \binom{m}{p} \). Moreover, \( \chi(G) = 1 \), provided \( p(n) > \frac{m(n)}{2} \), and the result is trivial. Theorem 2 is proved.

4. Proofs of Theorems 3–5

**Proof of Theorem 3.** Set \( m = m(n) \), \( p = p(n) \). Since the function \( \left( \frac{m}{p} \right) \) is decreasing in \( p \), we just have to show that for
\[
p = \left[ \frac{m}{2} - \sqrt{\frac{m \ln m}{\kappa}} \right],
\]
the inequality \( \frac{\binom{m}{p}}{\binom{m/2}{p}} > n + 1 \) is true for large values of \( n \). We have
\[
\frac{\binom{m}{p}}{\binom{m/2}{p}} = \frac{(m/2 + 1) \cdot (m/2 + 2) \cdot \ldots \cdot (m/2 + \frac{m}{2} - p)}{\frac{m}{2} \cdot (\frac{m}{2} - 1) \cdot \ldots \cdot (\frac{m}{2} - \left( \frac{m}{2} - p - 1 \right))} =
\]
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\[ (1 + \frac{2}{m}) \cdot (1 + \frac{4}{m}) \cdot \ldots \cdot (1 + \frac{m-2p}{m}) \sim e^{\frac{(m-2p)^2}{2m}} \geq e^{\frac{2\ln m}{\kappa}} = m^{\frac{2}{\kappa}}. \]

By a condition of Theorem 3, $\kappa < 2$. Thus,

\[ m^{\frac{2}{\kappa}}(1 + o(1)) > n + 1, \quad \forall n \geq n_0. \]

Theorem 3 is proved.

**Proof of Theorem 4.** We just have to show that for our choice of $r_n$,

\[ p = \frac{m}{8r_n^2} + f \left( \frac{m}{8r_n^2} \right) < \frac{m}{2} - \sqrt{\frac{m \ln m}{\kappa}}, \]

provided $\kappa < 2$ is a constant and $n$ is large enough.

Indeed, assume that $c'_0$ is large (say, $c'_0 > c_0$). Then,

\[ p \leq \frac{m}{8 \left( \frac{1}{2} + \frac{c'_0}{n^{0.475}} \right)^2} + c_0 \left( \frac{m}{8 \left( \frac{1}{2} + \frac{c'_0}{n^{0.475}} \right)^2} \right)^{0.525} < \frac{m}{2} - \sqrt{\frac{m \ln m}{\kappa}}, \]

\[ \left( \frac{m}{8 \left( \frac{1}{4} + \frac{c'_0}{n^{0.475}} \right)} \right)^{0.525} = \frac{m}{2} \left( 1 - \frac{4c'_0}{n^{0.475}} + o \left( \frac{1}{n^{0.475}} \right) \right) + c_0 \left( \frac{m}{2} \left( 1 - \frac{4c'_0}{n^{0.475}} + o \left( \frac{1}{n^{0.475}} \right) \right) \right)^{0.525}. \]

For any sufficiently large value of $n$, the last quantity is bounded from above by

\[ \frac{m}{2} - c'_0 m^{0.525} + c_0 m^{0.525} = \frac{m}{2} - c''_0 m^{0.525}, \quad c''_0 > 0. \]

Obviously, for any $n \geq n_0$,

\[ \frac{m}{2} - c''_0 m^{0.525} < \frac{m}{2} - \sqrt{\frac{m \ln m}{\kappa}}. \]

Theorem 4 is proved.
Proof of Theorem 5. Let us briefly write down a sequence of inequalities similar to those in 4.2:

\[ p \leq \frac{m}{8 \left( \frac{1}{2} + c_1' \sqrt{\ln n} \right)^2} + c_1 \ln \left( \frac{m}{8 \left( \frac{1}{2} + c_1' \sqrt{\ln n} \right)^2} \right) < \]

\[ < \frac{m}{2} \left( 1 - 4c_1' \sqrt{\ln n} + o \left( \frac{1}{n^{3/2}} \right) \right) \]

\[ + c_1 \ln \left( \frac{m}{2} \left( 1 - 4c_1' \sqrt{\ln n} + o \left( \frac{1}{n^{3/2}} \right) \right) \right) < \]

\[ < \frac{m}{2} - c_1' \sqrt{m \ln m}, \]

and we are done.

5. Proof of Theorem 6

Proof. Let us take \( S_{1/2}^{n-1} \) and divide it into \( n + 1 \) parts of smallest possible diameters. To this end, we inscribe a regular \( n \)-simplex \( \Delta^n \) into \( S_{1/2}^{n-1} \) and consider multidimensional polyhedral cones \( C_1, \ldots, C_{n+1} \) with common vertex at the center of \( S_{1/2}^{n-1} \) and coming through the \((n-1)\)-faces of \( \Delta^n \). Obviously,

\[ (4) \quad S_{1/2}^{n-1} = (S_{1/2}^{n-1} \cap C_1) \cup \ldots \cup (S_{1/2}^{n-1} \cap C_{n+1}). \]

In principle, it is a good exercise in multidimensional geometry to prove that for any \( i \),

\[ \text{diam}(S_{1/2}^{n-1} \cap C_i) = 1 - \Theta \left( \frac{1}{n} \right). \]

It follows immediately from this observation that we may inflate \( S_{1/2}^{n-1} \) at most

\[ \frac{1}{1 - \Theta \left( \frac{1}{n} \right)} = 1 + \Theta \left( \frac{1}{n} \right) \]

times in order to get a partition of the resulting sphere into parts of diameter not exceeding 1. Thus, for a constant \( c_2 > 0 \), we have an appropriate coloring of \( S_{1/2 + c_2/n}^{n-1} \), which completes the proof of Theorem 6.
Apparently, in [11], the same construction was proposed. However, the author assumed that the diameter of any part in the corresponding partition is attained on the sides of a regular \( n \)-simplex \( \Delta^n \). This is true only for \( n = 2 \).

Already in \( \mathbb{R}^3 \), the diameter of a part is \( \sqrt{rac{3 + \sqrt{3}}{6}} = 0.888\ldots \), which is not \( \sqrt{rac{2}{3}} = 0.816\ldots \) – the length of a side of a tetrahedron inscribed into \( S^2_{1/2} \).

6. Comments and upper bounds

First of all, it is worth noting that there is still a certain gap between the estimates

\[
(5) \quad r_n \geq \frac{1}{2} + c_1' \sqrt{\frac{\ln n}{n}}
\]

and

\[
(6) \quad r_n \leq \frac{1}{2} + \frac{c_2}{n}.
\]

Removing this gap could be a good problem. As for (6), it cannot be improved by any refinement of the techniques of the previous section. The point is that the partition (4) is best possible: for any other decomposition of \( S^m_{1/2} \) into \( n + 1 \) parts, there exists a part whose diameter is not less than each of the diameters \( \text{diam}(S^m_{1/2} \cap C_i) \). Of course it is not necessary to divide a sphere into parts with diameters strictly smaller than 1; we just need to cut it in such a way that no part would contain a pair of points at the unit distance. However, we do not know such a partition. Perhaps it is easier to improve (5). One should combine linear algebra of Section 3 with some additional ideas.

Let us say a few words about general upper estimates for \( \chi(S^{m-1}_{r_n}) \). The simplest observation here is that

\[
(7) \quad \chi(S^{m-1}_{r_n}) \leq \chi(\mathbb{R}^n) \leq (3 + o(1))^n \quad (\text{cf. Section 1}).
\]

Thus, for constant values \( r > \frac{1}{2} \) (as in Theorem 1), we already get the order of magnitude for any quantity \( \log \chi(S^{m-1}_{r}) \).

In [16], C.A. Rogers proved that any sphere of radius \( r \) in \( \mathbb{R}^n \) can be covered by \( \left( \frac{r}{\rho} + o(1) \right)^n \) spheres of radius \( \rho < r \). In our case, this means that

\[
\chi(S^{m-1}_{r_n}) \leq (2r_n + o(1))^n.
\]

If \( r_n < 3/2 \), then this bound is better than that in (7).
More precisely, Rogers’ estimate is as follows: there is an absolute constant $c > 0$ such that, if $r > \frac{1}{2}$ and $n \geq 9$, any $n$-dimensional spheres of radius $r$ can be covered by less than $cn^{5/2}(2r)^n$ spheres of radius $\frac{1}{2}$. As $r_n \to \frac{1}{2}$, we may carefully apply this statement in order to obtain upper bounds like

$$\chi(S_{r_n}^{n-1}) \leq 2cn^{5/2}(2r_n)^n = \Theta \left(n^{5/2}(2r_n)^n \right).$$

Here the factor 2 is due to the fact that $\chi(S_{1/2}^{n-1}) = 2$. One should not forget that if, for example, $r_n = \frac{1}{2} + \Theta \left(\frac{1}{n}\right)$, then $(2r_n)^n = \Theta(1)$, so that estimate (8) is very good.

It is possible to evaluate even more sophisticated bounds for $\chi(S_{r_n}^{n-1})$, but this is not so interesting.

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