A combinatorial approach for the state complexity of the Shuffle product

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Abstract. We investigate the state complexity of the shuffle operation on regular languages initiated by Câmpeanu et al. and studied subsequently by Brzozowski et al. We shift the problem into the combinatorics domain by turning the problem of state accessibility into a problem of intersection of partitions. This allows us to develop new tools and to reformulate the conjecture of Brzozowski et al. about the above-mentioned state complexity.

1 Introduction

Studies on state complexity have been going on for more than forty years now. The seminal work of Maslov [14] which gives values (without proofs) for the state complexity of some operations: square root, cyclic shift and proportional removal, paves the way. From these foundations, a very active field of research was open mainly initiated by Yu et al [18]. Lots and lots of papers were produced and different sub-domains appeared depending on whether the used automata are deterministic or not, whether the languages are finite or infinite, belong to some classes (codes, star-free, ...) and so on. We focus here on the (complete) deterministic case for any language.

The state complexity of a rational language is the size of its minimal (complete deterministic) automaton and the state complexity of a rational operation is the maximal one of those languages obtained by applying this operation onto languages of fixed state complexities.

The classical approach is to compute an upper bound and to provide a witness, that is a specific example reaching the bound which is then the desired state complexity.

In some cases, the classical method has to be enhanced by an algebraic approach consisting in building a witness for a certain class of rational operations by searching in a set of automata with as many transition functions as possible. This method has the advantage of being applied to a large class of operations and has been described independently by Caron et al. in [4] as the monster approach and by Davies in [6] as the OLPA approach but was implicitly present in older papers like [2], [7].

The shuffle product of two languages is the set of words obtained by riffle shuffling any word of the first language together with any word of the second one. The shuffle product is a regular operation. While it is easy to describe in terms of automata [8], its state complexity is notoriously difficult to establish [2, 3]. In [2], the authors use implicitly the notion of monsters which we explicit in this paper. In particular, Brzozowski et al. introduced a class of tableaux allowing us to describe, in a combinatoric way, the states of the minimal DFA recognizing the shuffle product of two regular languages. By investigating the monoid of transformations

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through the point of view of modifiers and monsters, we give a more precise combinatorial
description of the underlying mechanism. Our main result consists in describing the state
complexity as the cardinal of a class of combinatorial objects.

Proving the conjecture of Brzozowski et al. is equivalent to prove that our class of objects is
in bijection with the tableaux they consider. Although we do not achieve this goal, we provide
numerous new tools and results in that context: related enumeration results, generating
functions, and partial description of the bijection.

The paper is organized as follows. Section 2 gives definitions and notations about au-
tomata and combinatorics. In Section 3, we recall the definition of the shuffle product and we
drag it into the realm of monsters and modifiers.

Section 4 is devoted to the description of the paths of the shuffle automaton in a combi-
natoric way. Related enumeration formulæ are studied in Section 5. In Section 6, we give an
exact expression for the state complexity of the shuffle product.

2 Preliminaries

Let $\Sigma$ denote a finite alphabet. A word $w$ over $\Sigma$ is a finite sequence of symbols of $\Sigma$. The set
of all finite words over $\Sigma$ is denoted by $\Sigma^*$. The empty word is denoted by $\varepsilon$. A language is a
subset of $\Sigma^*$. The cardinality of a finite set $E$ is denoted by $|E|$, the set of subsets of $E$ is denoted by
$2^E$ and the set of mappings of $E$ into itself is denoted by $E^E$.

A finite automaton (FA) is a 5-tuple $A = (\Sigma, Q, I, F, \delta)$ where $\Sigma$ is the input alphabet, $Q$
is a finite set of states, $I \subset Q$ is the set of initial states, $F \subset Q$ is the set of final states and $\delta$
is the transition function from $Q \times \Sigma$ to $2^Q$ extended in a natural way from $2^Q \times \Sigma^*$ to $2^Q$.
A word $w \in \Sigma^*$ is recognized by an FA $A$ if $\delta(I, w) \cap F \neq \emptyset$. The language recognized by
an FA $A$ is the set $L(A)$ of words recognized by $A$. Two automata are said to be equivalent
if they recognize the same language. A state $q$ is accessible in an FA if there exists a word
$w \in \Sigma^*$ such that $q \in \delta(I, w)$. An FA is complete and deterministic (CDF A) if # $I$ = 1 and for
all $q \in Q$, for all $a \in \Sigma$, $\# \delta(q, a) = 1$. Let $D = (\Sigma, Q_D, i_D, F_D, \delta)$ be a CDF A. For any word $w$, we
denote by $\delta^w$ the function $q \rightarrow \delta(q, w)$. Two states $q_1, q_2$ of $D$ are equivalent if for any word $w$
of $\Sigma^*$, $\delta(q_1, w) \in F_D$ if and only if $\delta(q_2, w) \in F_D$. Such an equivalence is denoted by $q_1 \sim q_2$.
A CDF A is minimal if there does not exist any equivalent CDF A with less states and it is well
known that for any CDF A, there exists a unique minimal equivalent one [12]. Such a minimal
CDF A can be obtained from $D$ by computing the accessible part of the automaton
$D/\sim = (\Sigma, Q_D/\sim, [i_D], F_D/\sim, \delta_\sim)$ where for any $q \in Q_D$, $[q]$ is the $\sim$-class of the state $q$ and
satisfies the property $\delta_{\sim}([q], a) = [\delta(q, a)]$, for any $a \in \Sigma$. The number of its states is defined by
$\#_{\text{Min}}(D)$. In a minimal CDF A, any two distinct states are pairwise inequivalent. For any integer
$n$, let us denote $\llbracket n \rrbracket$ for $\{0, \ldots, n - 1\}$. The state complexity of a regular language $L$
denoted by $sc(L)$ is the number of states of its minimal CDF A. Let $L_n$ be the set of languages of state
complexity $n$. The state complexity of a binary operation $\otimes$ is the function $sc_\otimes$ associating
$\max \{sc(L_1 \otimes L_2) \mid L_1 \in L_{n_1}, L_2 \in L_{n_2} \}$ with any couple of integers $n_1, n_2$. A witness for the
binary operation $\otimes$ is a couple $(L_1, L_2) \in (L_{n_1} \times L_{n_2})$ such that $sc(L_1 \otimes L_2) = sc_\otimes(n_1, n_2)$.

We also need some background from finite transformation semigroup theory [11]. Let $n$
be an integer. A transformation $t$ is an element of $\llbracket n \rrbracket^n$. We denote by $it$ the image of $i$ under $t$. A transformation of $\llbracket n \rrbracket$ can be represented by $t = [i_0, i_1, \ldots, i_{n-1}]$ which means that $i_k = kt$
for each $k \in \llbracket n \rrbracket$ and $i_k \in \llbracket n \rrbracket$. A permutation is a bijective transformation on $\llbracket n \rrbracket$. A cycle of
length $\ell \leq n$ is a permutation $c$, denoted by $(i_0, i_1, \ldots, i_{\ell-1})$, on a subset $I = \{i_0, \ldots, i_{\ell-1}\}$ of $\llbracket n \rrbracket$.
where $i_kc = i_{k+1}$ for $0 \leq k < \ell - 1$ and $i_{\ell - 1}c = i_0$. A permutation is always a composition of disjoint cycles.

3 Shuffle, tableaux and monsters.

3.1 The shuffle product

The shuffle operation [9] on regular languages is classically implemented as follows [8]. Let $K$ and $L$ be regular languages over an alphabet $\Sigma$ recognized by DFAs $K = (Q_K, \Sigma, \delta_K, q_K, F_K)$ and $L = (Q_L, \Sigma, \delta_L, q_L, F_L)$, respectively. Then $K \shuffle L$ is recognized by the NFA $N = (Q_K \times Q_L, \Sigma, \delta, (q_K, q_L), F_K \times F_L)$, where $\delta((p, q), a) = (\delta_K(p, a), q), (p, \delta_L(q, a))$.

The state complexity of this operation was first studied by Câmpeanu, Salomaa and Yu [3]. Later on, Brzozowski et al. [2] completed this study.

Let $D = (2^{Q_K \times Q_L}, \Sigma, \delta', \{(q_K, q_L), F'\})$ be the subset automaton of $N$. If $|Q_K| = m$ and $|Q_L| = n$, then NFA $N$ has $mn$ states. It follows that DFA $D$ has at most $2^{mn}$ reachable and pairwise distinguishable states.

Since states of $D$ belongs to $2^{Q_K \times Q_L}$, they are associated in a natural way to boolean tableaux of size $m \times n$, each cell of them being either empty or marked. Brzozowski et al. prove that only tableaux with at least one marked cell on the first line and one marked cell on the first column can be reached. Such tableaux are called valid and their number,

$$f(m, n) = 2^{mn} - \sum_{j=1}^{m-n+1} 2^{m-j} 2^{n-j} - 1,$$

is an upper bound for the state complexity of the shuffle operation. The authors also produce a couple of ternary languages $K, L$ for which all pairs of valid states are distinguishable.

The main difficulty is to prove that all valid states can be reached for some couple of languages $K, L$. This question of reachability only depends on the transition functions of $K$ and $L$. First, observe that the finality of states does not matter. Next, to reach any valid state in the most easier way, it is relevant to consider automata $K$ and $L$ having a maximum of transitions. This is the idea of monsters detailed in [4, 6] and formalized in the next section. Brzozowski et al. implicitly use this notion to prove the result for any $n$ when $m \leq 5$. They also obtained the desired answer by computation when $m = n = 6$, but they are unable to extend the result for any values of $m, n$. Even if we are not able to solve the conjecture, we provide a new approach for the question of the reachability for valid states, which especially allows to compute the exact value of the state complexity for the shuffle operation.

3.2 Modifiers, monsters and state complexity

The work of Brzozowski et al. is implicitly based on the fact that the shuffle operation is a describable operation. Let us recall here the definition of a describable operation as described in [4].

**Definition 1.** A 2-modifier $m$ is a 4-tuple of mappings $(\Sigma, \delta, i, f)$ acting on 2 CDFA $A_1, A_2$ with $A_j = (Q_j, \Sigma_j, i_j, F_j, \delta_j), j \in \{1, 2\}$ to build a CDFA $m(A_1, A_2)$ = $(Q, i, F, \delta)$, where

$$Q = \Sigma(Q_1, i_1, F_1, Q_2, i_2, F_2), \quad i = i(Q_1, i_1, F_1, Q_2, i_2, F_2), \quad F = f(Q_1, i_1, F_1, Q_2, i_2, F_2)$$

and

$$\forall a \in \Sigma, \delta^a = \delta(Q_1, i_1, \delta_1^a, F_1, Q_2, i_2, \delta_2^a, F_2).$$
**Definition 2.** We consider an operation $\otimes$ acting on a couple of languages defined on the same alphabet. The operation $\otimes$ is said to be describable if there exists a 2-modifier $m$ such that for any couple of CDFAs $(A_1, A_2)$, we have $L(m(A_1, A_2)) = L(A_1) \otimes L(A_2)$.

We are now able to define the $\mathcal{SHUF}$ modifier for the shuffle operation on automata. Only relevant parameters will appear in the definition.

\[
\mathcal{SHUF} = (\mathcal{S}, \mathcal{D}, i, f)
\]

where

- $\mathcal{S}(Q_1, Q_2) = 2^{Q_1 \times Q_2}$
- $i((i_1, i_2)) = (i_1, i_2)$
- $f(Q_1, Q_2, F_2) = \{ E \in \mathcal{S}(Q_1, Q_2) \mid E \cap (F_1 \times F_2) \neq \emptyset \}$
- $d(Q_1, \delta_1, Q_2, \delta_2) : 2^{Q_1 \times Q_2} \to 2^{Q_1 \times Q_2}$

\[
d(Q_1, \delta_1, Q_2, \delta_2) \mid (q_1, q_2) \in \{ (q_1, \delta_2(q_2)) \mid (q_1, q_2) \in E \}.
\]

The classical construction for an automaton recognizing the language $L_1 \shuffle L_2$ [8] for any pair $(L_1, L_2)$ of regular languages described, respectively, by two automata $A_1$ and $A_2$, is equivalent to the following statement:

\[
L(\mathcal{SHUF}(A_1, A_2)) = L_1 \shuffle L_2.
\]

In other words, the shuffle is a describable operation.

A 2-monster is a couple of DFAs of size $n_1, n_2$ having $n_1^{n_1} n_2^{n_2}$ letters representing couple of functions from $\lceil n_1 \rceil$ to $\lceil n_1 \rceil$ and from $\lceil n_2 \rceil$ to $\lceil n_2 \rceil$. There are $2^{n_1+n_2}$ different 2-monsters depending on the set of their final states.

**Definition 3.** A 2-monster is a couple of automata $(M_{F_1}^{n_1}, M_{F_2}^{n_2})$ where $M_{F_j}^{n_j} = (\Sigma, \lceil n_j \rceil, 0, F_j, \delta_j)$ for $j \in \{1, 2\}$ is defined by

- the common alphabet $\Sigma = \lceil n_1 \rceil^{\lceil n_1 \rceil} \times \lceil n_2 \rceil^{\lceil n_2 \rceil}$,
- the set of states $\lceil n_j \rceil$,
- the initial state 0,
- the set of final states $F_j$,
- the transition function $\delta_j$ defined for any $(a_1, a_2) \in \Sigma$ by $\delta_j(q, (a_1, a_2)) = a_j(q)$.

Notice that a symbol of the alphabet is assimilated to a single transition function from any state.

### 3.3 Using monsters to compute state complexity

The idea behind the notion of monster is to define kind of universal pairs of automata maximizing the state complexity for any describable binary operation. It implies a common alphabet for these automata.

If an operation is describable, it is sufficient to study the behavior of its modifiers over monsters to compute its state complexity. From Theorem 1 in [4] we obtain

\[
sc_{\shuffle}(n_1, n_2) = \max\{ \#M_{\text{Min}}(\mathcal{SHUF}(M_{F_1}^{n_1}, M_{F_2}^{n_2})) \mid F_j \subset \lceil n_j \rceil \}.
\]

It means that a witness belongs to the set of monsters.

Brzozowski et al. show the following results that are translated in terms of modifier as:
– any accessible state in $\mathcal{S}(M_{f_1, f_2}^{M_1, F_1}, M_{f_2}^{M_2, F_2})$ is valid,
– any couple of valid states in $\mathcal{S}(M_{f_1, f_2}^{M_1, F_1}, M_{f_2}^{M_2, F_2})$ can be distinguished by using successively many times the letters:
  • $a = ((0, \ldots, n_1 - 1), 0)$
  • $b = (0, (0, \ldots, n_2 - 1))$
  • $c = ((\alpha \to \delta_{0,\alpha}, n_2 - 1)$ where $\delta_{x,y} = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$ denotes the Kronecker delta.

4 The combinatorics of paths in the shuffle automaton

4.1 A first example

We illustrate the notions investigated in this section with the following example. Let $f_a$, $f_b$, and $f_c$ be three maps from $[4]$ to itself such that $f_b(0) = 0$, $f_a(0) = f_c(0) = 2$, and $f_b(2) = f_c(2) = f_c(3) = 3$. Let $g_a$, $g_b$, and $g_c$ be three maps from $[3]$ to itself such that $g_b(1) = 0$, $g_a(0) = g_c(2) = 1$, and $g_b(0) = g_c(0) = g_c(1) = 2$. We denote by $x$ the pair $(f_x, g_x)$ for $x = a, b, c$. These pairs of functions represent transition functions in a 2-monster. These transition functions are drawn in Figure 1 (missing transitions are not relevant for the example and are not drawn).

![Fig. 1. Two automata figuring the transitions $a = (f_a, g_a), b = (f_b, g_b)$, and $c = (f_c, g_c)$ in a 2-monster](image)

We investigate the transitions in $\mathcal{S}(M_{f_1}^4, M_{f_2}^3)$ drawn in figure 2. The tableaux represent subsets of $[4] \times [3]$; a symbol $\times$ is written in the cell $(i, j)$ if $(i, j)$ belongs to the subset. Following a transition symbol $(f, g)$, each tableau is sent to another one constructed by merging the two tableaux obtained from the original one by acting on the lines and on the columns respectively.

![Fig. 2. Some transitions in $\mathcal{S}(M_{f_1}^4, M_{f_2}^3)$](image)
by $f$ and $g$. An example is given in Figure 3. In the aim to capture the path used to access to a state, we fill the cells of the tableaux by sets of integers as in figure 4. More precisely, in a transition $T \xrightarrow{(f,g)} T'$ the tableau $T'$ is obtained from the tableau $T$ by moving the numbers according to the transformation (on lines) induced by $f$ and copying the numbers once shifted according to the transformation (on columns) induced by $g$. The non empty entries of a tableau obtained this way partition the set $\{1,\ldots,2^k\}$ for a given $k$. Such a configuration is completly described by a pair of vectors of disjoint parts $([\lambda_1,\ldots,\lambda_m],[\rho_1,\ldots,\rho_n])$ satisfying $\bigcup_i \lambda_i = \bigcup_j \rho_j = \{1,\ldots,2^k\}$. The left vector is obtained by partitioning $\{1,\ldots,2^k\}$ with respect to the lines of the tableau and the right vector is obtained by partitioning it with respect to the columns. Conversely, knowing the pair $([\lambda_1,\ldots,\lambda_m],[\rho_1,\ldots,\rho_n])$, we easily reconstitue the tableau $T$ by noting that $T[i,j] = \lambda_i \cap \rho_j$; see Figure 5 for an example. Hence, the process allowing to obtain a tableau from a path is easily translated in terms of pairs of vectors. For instance, in Figure 6, we follows the same path as in Figure 4 but we construct the associated pairs of vectors instead of the tableaux. It is easy to see that only some pairs can be constructed in such a way. For instance, the construction implies that 1 always belongs to the first entry of the right vector and $2^k$ in the first entry of the left vector. But in the aim to completely describe the valid pairs in a combinatorics way, we need more rules. Indeed, observe the
was obtained in the previous step. In the same way, the fact that 6 and 8 belong to a same 
bottom pair ($\Lambda$, 8) also belongs to a same set in $P$ for some $E$. Indeed, the set $\{5, 8\}$ is the (shifted) image of $\{1, 4\}$ which was obtained in the previous step. In the same way, the fact that 6 and 8 belongs to a same set in $\Lambda$ implies that 2 and 4 also belongs to a same set in $\Lambda$. These properties are captured by the notion of $U$-pair formally described in the next section. In what follows, we investigate the combinatorics of such objects and their relation with the state complexity of the shuffle product.

4.2 A combinatorial representation of the paths

We define the left domain (resp. right domain) of $E \subset [m] \times [n]$ as $D_L(E) = \{i \mid (i, j) \in E$ for some $j\}$ (resp. $D_R(E) = \{j \mid (i, j) \in E$ for some $i\}$). We associate with any transition $t = (E_1, (f, g), E_2)$ in the automaton $\mathcal{H}(M_{F_1}^{\mu_1}, M_{F_2}^{\mu_2})$ its useful step defined as the triplet $t^{\mu} = (E_1, (f|_{D_2(E_1)}, g|_{D_2(E_1)}), E_2)$ where the notation $f|_D$ means the restriction of the map $f$ to the subdomain $D$. Consider a path $\nu = (t_1, \ldots, t_k)$, its associated useful path is defined by $\nu^{\mu} = (t_1^{\mu}, \ldots, t_k^{\mu})$.

Example 1. In Figure 2, we consider the transition labeled by $c$ given by the triplet $t = (((0, 0), (0, 1), (2, 2), (3, 0)), (f_c, g_c), ((0, 2), (2, 0), (2, 1), (3, 0), (3, 2)))$. Since $D_L(((0, 0), (0, 1), (2, 2), (3, 0))) = \{0, 2, 3\}$ and $D_R(((0, 0), (0, 1), (2, 2), (3, 0))) = \{0, 1, 2\}$, we have $t = (((0, 0), (0, 1), (2, 2), (3, 0)), (f_c|_{[0,2,3]}, g_c|_{[0,1,2]}), ((0, 2), (2, 0), (2, 1), (3, 0), (3, 2)))$. The useful path associated to the path drawn in Figure 2 is

$$
\left( \begin{array}{c}
((0, 0), (f_a|_{[0]}, g_a|_{[0]}), (0, 1), (2, 0)), \\
((0, 1), (2, 0)), (f_a|_{[0,2]}, g_a|_{[0,1]}), ((0, 1), (0, 2), (2, 2), (3, 0)), \\
((0, 0), (0, 1), (2, 2), (3, 0)), (f_a|_{[0,2,3]}, g_a|_{[0,1,2]}), ((0, 2), (2, 0), (2, 1), (3, 0), (3, 2)))
\end{array} \right).
$$

For any accessible state $E$ in $\mathcal{H}(M_{F_1}^{\mu_1}, M_{F_2}^{\mu_2})$ and any path $\nu = (t_1, \ldots, t_k)$ from $((0, 0))$ to $E$, we define recursively a pair of vectors of sets $\mathcal{P}(\nu) = ([\lambda_0, \ldots, \lambda_{m-1}], [\rho_0, \ldots, \rho_{n-1}])$.
If $k = 0$ (this means that $p = ()$ and $E = \{(0, 0)\}$) then $\mathcal{P}(p) = \{([1], \emptyset, \ldots, \emptyset), ([1], \emptyset, \ldots, \emptyset)\}$.

If $k > 0$ then denotes $\mathcal{P}(t_1, \ldots, t_{k-1}) = (\Lambda, P)$ and $t_k = (E_{k-1}, (f, g), E_k)$. We set

$$\mathcal{P}(p) = [(\Lambda \cdot f) \cup \Lambda^\dagger, P \cup (P \cdot g)^\dagger]$$

with the notation

$$[\pi_{\ell_0}, \ldots, \pi_{\ell_{-1}}] \cdot h = [\bigcup_{\ell = 0} \pi_{q_\ell}, \ldots, \bigcup_{\ell = \ell - 1} \pi_q],$$

$$[\pi_{\ell_0}, \ldots, \pi_{\ell_{-1}}] \cup [\pi_{\ell_0}', \ldots, \pi_{\ell_{-1}}'] = [\pi_{\ell_0} \cup \pi_{\ell_0}', \ldots, \pi_{\ell_{-1}} \cup \pi_{\ell_{-1}}'],$$

and

$$[\pi_{\ell_0}, \ldots, \pi_{\ell_{-1}}]^\dagger = [(q + r \mid q \in \pi_{\ell_0}), \ldots, (q + r \mid q \in \pi_{\ell_{-1}})],$$

where $r = \max \bigcup_q \pi_q$.

Such an object is called a U-pair and for a given $(m, n)$ the set of the U-pairs is denoted by $\mathcal{U}_{m,n}$.

**Example 2.** Consider again the path $p = (t_a, t_b, t_c)$ drawn in Figure 2. We compute successively

$\mathcal{P}((i)) = \{([1], \emptyset, \emptyset, \emptyset), ([1], \emptyset, \emptyset)\},$

$\mathcal{P}((t_a)) = \{([0, 0, 1, 3], ([1], \emptyset, \emptyset) \cup [0, 1])\} = \{([2], [0, 1, \emptyset], ([1], [2], \emptyset))\},$

$\mathcal{P}((t_a, t_b)) = \{([2], [0, 1, \emptyset], ([1], [1, 2], [1]))\} = \{([2, 4], [0, 3], [1, 3])\},$

$\mathcal{P}(p) = \{([6], [2, 4, 7, [1, 3, 5]], ([1, 4], [2, 3], [1, 3, 3, 4]))\}.$

This is exactly the process described in Figure 6.

The following proposition compiles some basic facts about U-pairs.

**Proposition 1.** Let $\mathcal{P}((t_1, \ldots, t_k)) = ([\lambda_{\ell_0}, \ldots, \lambda_{\ell_{m-1}}], [\rho_{\ell_0}, \ldots, \rho_{\ell_{n-1}}])$. We have

1. $\bigcup_q \lambda_q = \bigcup_q \rho_q = \{1, \ldots, 2^k\}$,
2. $\lambda_i \cap \lambda_j = \emptyset$ for all $i \neq j$,
3. $\rho_i \cap \rho_j = \emptyset$ for all $i \neq j$,
4. $\mathcal{P}((t_1, \ldots, t_k)) = \mathcal{P}((t_1^u, \ldots, t_k^w))$.

We notice that $\mathcal{U}_{m,n}$ is a graded set $\mathcal{U}_{m,n} = \bigcup_{k \geq 0} \mathcal{U}_{m,n}^{(k)}$, where

$$\mathcal{U}_{m,n}^{(k)} = \left\{([\lambda_{\ell_0}, \ldots, \lambda_{\ell_{m-1}}], [\rho_{\ell_0}, \ldots, \rho_{\ell_{n-1}}]) \in \mathcal{U}_{m,n} \mid \bigcup_{\ell} \lambda_q = \bigcup_{\ell} \rho_q = \{1, \ldots, 2^k\}\right\},$$

that is the set of images by $\mathcal{P}$ of paths of length $k$ and source $\{(0, 0)\}$.
Example 3.

\[ U_{0}^{(0)} = \{(\{1\}, \emptyset), (\{1\}, \emptyset)\}, \]

\[ U_{2,2}^{(2)} = \{(\{1, 2\}, \emptyset), (\{1, 2\}, \emptyset), (\{1, 2\}, \emptyset), (\{2\}, \{1\}, \{1, 2\}, \emptyset), (\{2\}, \{1\}, \{1, 2\}, \emptyset)\}, \]

Proposition 2. Each graded component \( U_{m,n}^{(k)} \) is in a one to one correspondence with the set of useful paths of length \( k \) and source \((0,0)\).

Proof. From the last point of proposition 1 it suffices to prove that if \( \psi_{1} \neq \psi_{2} \) then \( \mathcal{P}(\psi_{1}) \neq \mathcal{P}(\psi_{2}) \). We show this statement by induction on \( k \). For \( k = 0 \) the result is obvious. Suppose \( \psi = (t_{1}, \ldots, t_{k}) \) and \( \psi' = (t'_{1}, \ldots, t'_{k}) \) are two paths of length \( k > 0 \) such that \( \psi_{1} \neq \psi_{2} \). Let us denote \( (\Lambda, P) = \mathcal{P}((\psi_{1}, \ldots, t_{k-1})) \) and \( (\Lambda', P') = \mathcal{P}((\psi'_{1}, \ldots, t'_{k-1})) \). If \( (t_{1}, \ldots, t_{k-1}) \neq (t'_{1}, \ldots, t'_{k-1}) \) then by induction \( (\Lambda, P) \neq (\Lambda', P') \). For any \( f, f' \in [m] \) and \( g, g' \in [n] \), we have \( \{(\Lambda, f) \cup \Lambda', P \cup (P' \cdot g')\} \neq \{(\Lambda', f') \cup \Lambda, P' \cup (P \cdot g')\} \) because \( \Lambda \neq \Lambda' \) or \( P \neq P' \) and thus \( \mathcal{P}(\psi) \neq \mathcal{P}(\psi') \). If \( (t_{1}, \ldots, t_{k-1}) = (t'_{1}, \ldots, t'_{k-1}) \) then \( t_{k} \neq t'_{k} \). There exist \( f, g \neq (f', g') \) such that \( \mathcal{P}(\psi) = \{(\Lambda, f) \cup \Lambda', P \cup (P' \cdot g')\} \) and \( \mathcal{P}(\psi') = \{(\Lambda', f') \cup \Lambda, P' \cup (P \cdot g')\} \) and there exists \( i \) with \( \lambda_{i} \neq 0 \) and \( f(i) \neq f'(i) \) or there exists \( j \) with \( \rho_{j} = 0 \) and \( g(j) \neq g'(j) \). Hence \( \Lambda \cdot f \neq \Lambda' \cdot f' \) or \( P \cdot g \neq P' \cdot g' \). \( \square \)

A k-EXPComposition of size \( n \) is a vector of sets \( \{\pi_{1}, \ldots, \pi_{n}\} \) satisfying \( \bigcup_{q} \pi_{q} = \{1, \ldots, 2^{k}\} \) and \( \pi_{q} \cap \pi_{q'} = \emptyset \) implies \( q = q' \). A k-Valid vector of size \( n \) is a k-EXPComposition \( \{\pi_{1}, \ldots, \pi_{n}\} \) such that \( 1 \in \rho_{1} \) and for any \( k' < k \) and any \( i, j \leq 2^{k} \), if \( i, j \in \rho_{q} \) for some \( a \) then \( i + 2^{k}, j + 2^{k} \in \rho_{\beta} \) for some \( \beta \). The set of k-Valid vector of size \( n \) is denoted by \( \mathcal{R}_{n}^{(k)} \). We define also a k-Valid vector of size \( n \) as a k-EXPComposition \( \{\lambda_{1}, \ldots, \lambda_{n}\} \) such that \( 2^{k} \in \lambda_{1} \) and for any \( k' < k \) and any \( i, j \leq 2^{k} \), if \( 2^{k} + 1 - i, 2^{k} + 1 - j \in \rho_{a} \) for some \( a \) then \( 2^{k} + 1 - i, 2^{k} + 1 - j \in \rho_{\beta} \) for some \( \beta \). The set of k-Valid vector of size \( m \) is denoted by \( \mathcal{L}_{m}^{(k)} \). Remark that we can obtain any element of \( \mathcal{L}_{m}^{(k)} \) from an element of \( \mathcal{R}_{n}^{(k)} \) by replacing the only occurrence of \( i \) by \( 2^{k} + 1 - i \) for any \( 1 \leq i \leq 2^{k} \).

Lemma 1. For any \( k > 0 \), we have \( \mathcal{L}_{m}^{(k)} = \{(\Lambda, f) \cup \Lambda' \mid \Lambda \in \mathcal{L}_{m}^{(k-1)}, f \in \{m\}^{\{m\}} \} \) and \( \mathcal{R}_{n}^{(k)} = \{P \cup (P' \cdot g') \mid P \in \mathcal{R}_{n}^{(k-1)}, g \in \{n\}^{\{n\}} \} \).

Proof. We remark that the elements of \( \mathcal{R}_{n}^{(k)} \) are in a one to one correspondence with the elements of \( \mathcal{L}_{m}^{(k)} \). Indeed, a bijection sends any element of \( \mathcal{R}_{n}^{(k)} \) to an element \( \mathcal{L}_{m}^{(k)} \) by replacing the only occurrence of \( i \) by \( 2^{k} + 1 - i \) for any \( 1 \leq i \leq 2^{k} \). So it suffices to prove the result for \( \mathcal{R}_{n}^{(k)} \).
Let \( P \in \mathcal{R}^{(k)}_n \) and consider \( P' = [\rho'_1, \ldots, \rho'_m] \) obtained from \( P \) by erasing all the numbers strictly greater than \( 2^{k-1} \). We check that \( P' \in \mathcal{R}^{(k-1)}_n \) and we set \( P'' = [\rho''_1, \ldots, \rho''_m] \) such that \( P = P' \cup P'' \). Since \( P \in \mathcal{R}^{(k)}_n \), for any \( 1 \leq i \leq m \) one has \( \rho'_i \subset \rho''_i \) for some integer \( \alpha_i \). Setting \( i \cdot g = \alpha_i \), one obtains \( P' \cdot g = P'' \) and so \( \mathcal{R}^{(k)}(1) \in \{ P \cup (P \cdot g) \mid P \in \mathcal{R}^{(k-1)}_n, g \in [n]^{[n]} \} \).

Conversely, consider \( P' = P \cup (P \cdot f) \) with \( P = [\rho_1, \ldots, \rho_n] \in \mathcal{R}^{(k-1)}_n \) and \( g \in [n]^{[n]} \). Straightforwardly, one obtains \( \bigcup \rho'_q = [1, \ldots, 2^{k}] \), \( 1 \in \rho'_{q} \) and \( \rho'_q \cap \rho'_q \neq \emptyset \) implies \( q = q' \). Furthermore, since \( P \in \mathcal{R}^{(k-1)}_n \), one has for any \( k' < k-1 \) and any \( i, j \leq 2^{k'} \) if \( i, j \in \rho'_a \) for some \( a \) then \( i + 2^{k'}, j + 2^{k'} \in \rho'_b \) for some \( b \). Let \( i, j \leq 2^{k-1} \) such that \( i, j \in \lambda_a \) for some \( a \) one has \( i + 2^{k-1}, j + 2^{k-1} \in \rho'_{a,g} \) and so we deduce that \( P' \in \mathcal{R}^{(k)}_n \) and \( \{ P \cup (P \cdot g) \mid P \in \mathcal{R}^{(k-1)}_n, g \in [n]^{[n]} \} \in \mathcal{R}^{(k)}_n \).

As a direct consequence, one obtains the following result.

**Proposition 3.** Each graded component \( \mathcal{U}^{(k)}_{m,n} \) splits into the cartesian product

\[
\mathcal{U}^{(k)}_{m,n} = \mathcal{L}^{(k)} \times \mathcal{R}^{(k)}_n.
\]

**Proof.** By induction on \( k \) from the definition of \( \mathcal{U}^{(k)}_{m,n} \) and the previous lemma. \( \square \)

**Example 4.** We have

\[
\mathcal{L}^{(2)}_2 = \{ [(1, 2, 3, 4), 0], [(3, 4), (1, 2)], [(1, 2, 4), (3)], [(1, 4), (2, 3)], [(2, 4), (1, 3)], [(4), (1, 2, 3)] \}
\]

and

\[
\mathcal{R}^{(2)}_2 = \{ [(1, 2, 3, 4), 0], [(1, 2), (3, 4)], [(1, 3, 4), (2)], [(1, 4), (2, 3)], [(1, 3), (2, 4)], [(1), (2, 3, 4)] \}
\]

Compare \( \mathcal{L}^{(2)}_2 \times \mathcal{R}^{(2)}_2 \) to \( \mathcal{U}^{(2)}_2 \) as described in Example 3.

**Example 5.** In Figure 6, the pair obtained by the last transition satisfies \( [[6, 8], 0, [2, 4, 7], [1, 3, 5]] = \{ (\{2, 4\}, 0, (3), 1\} \} \) and \( [[1, 4], [2, 7], [3, 5, 6, 8]] = [[1, 4], [2], [3]] \cup \{ (1), (2), (3) \} \) for \( g \). We get \( f_1(0) = 2, f_1(2) = f_1(3) = 3, g_1(0) = g_1(1) = 2 \), and \( g_1(2) = 1 \).

# 5 Enumeration formulae for U-pairs

## 5.1 Counting the successors of U-pairs

Let \( \mathcal{L}^{(k)}_{m,\ell} = \{ [\lambda_1, \ldots, \lambda_{m-1}] \in \mathcal{L}^{(k)}_m \# \{ i \mid \lambda_i \neq 0 \} = \ell \} \) and \( \mathcal{R}^{(k)}_{n,\ell,\ell'} = \{ [\rho_0, \ldots, \rho_{n-1}] \in \mathcal{R}^{(k)}_n \# \{ i \mid \rho_i \neq 0 \} = \ell \}. \) Obviously \( \# \mathcal{L}^{(k)}_{m,\ell} = \# \mathcal{R}^{(k)}_{n,\ell,\ell'} \). So we only investigate \( \# \mathcal{R}^{(k)}_{m,\ell} \). Let \( P \in \mathcal{R}^{(k)} \). We define \( \text{succ}(P) = \{ P \cup (P \cdot g) \mid g \in [n]^{[n]} \} \) and \( \text{succ}(P) = \{ P \cup (P \cdot g) \mid g \in [n]^{[n]} \} \) is \( \subseteq \) \( \# \mathcal{R}^{(k+1)}_{n,\ell,\ell'} \). We notice that the cardinal of \( \text{succ}(P) \) depends only on three parameters: \( n, \ell, \ell' \); we denote by \( s_{n,\ell,\ell'} \) this number. We remark that if \( \ell' < \ell \) then \( s_{n,\ell,\ell'} = 0 \).

**Example 6.** Let \( P = [[1], [2], [3, 4], 0, 0] \). Then any \( P' \in \text{succ}(P) \) can be written as \( P' = P \cup (P \cdot g) \) where \( \# ((0, 1, 2) \cdot g) \cap [3, 4] = 1 \). In the aim to compute the number of elements of \( \text{succ}(P) \), one has only to consider the case where \( (0, 1, 2) \cdot g \cap [3, 4] = [3] \) and multiply this number by \( 2 \); in the general case, one has to multiply by a binomial number. The set of functions \( g \) satisfying \( ((0, 1, 2) \cdot g) \cap [3, 4] = [3] \) splits into several disjoint subsets according to the values of the number \( \# \{ i \in [0, 1, 2] \mid i \cdot g = 3 \} \).
1. For \( \# \{ i \in \{0, 1, 2 \} \mid i \cdot g = 3 \} = 3 \), there exists only one function \( g \) satisfying this condition and this function corresponds to the vector \([1], [2], [3, 4], [5, 6, 7, 8], \emptyset \).

2. For \( \# \{ i \in \{0, 1, 2 \} \mid i \cdot g = 3 \} = 2 \), there are \( 3 = \binom{3}{2} \) possibilities considering that \( \{ i \in \{0, 1, 2 \} \mid i \cdot g = 3 \} = \{0, 1 \} \) or \( \{0, 2 \} \) or \( \{1, 2 \} \). For each of these possibilities one obtains 3 vectors that correspond to the possible images of the only element which does not belong to \( \{ i \in \{0, 1, 2 \} \mid i \cdot g = 3 \} \). So we obtain the following 9 vectors:

\[
[[1, 7, 8], [2], [3, 4], [5, 6], \emptyset], [[1, 2, 7, 8], [3, 4], [5, 6], \emptyset], [[1, 2], [3, 4, 7, 8], [5, 6], \emptyset],
[[1, 6], [2], [3, 4], [5, 7, 8], \emptyset], [[1, 2, 6], [3, 4], [5, 7, 8], \emptyset], [[1, 2], [3, 4, 6], [5, 7, 8], \emptyset],
[[1, 7], [2], [3, 4, 6], [5, 7, 8], \emptyset], [[1, 2, 7], [3, 4, 6], [5, 7, 8], \emptyset], [[1, 2], [3, 4, 6, 7, 8], [5, 7, 8], \emptyset].
\]

3. For \( \# \{ i \in \{0, 1, 2 \} \mid i \cdot g = 3 \} = 1 \), there are \( 3 = \binom{3}{1} \) possibilities considering that \( \{ i \in \{0, 1, 2 \} \mid i \cdot g = 3 \} = \{0, 1 \} \) or \( \{1, 2 \} \). For each of these possibilities one obtains 9 = \( 3^2 \) vectors that correspond to the possible images of the two elements which do not belong to \( \{ i \in \{0, 1, 2 \} \mid i \cdot g = 3 \} \). So we obtain the following 27 vectors:

\[
[[1, 6, 7, 8], [2], [3, 4], [5, 0], [[1, 6], [2, 7, 8], [3, 4], [5, 0], [[1, 6, 2], [3, 4, 7, 8], [5, 0],
[[1, 7, 8], [2], [3, 4], [5, 0], [[1, 2, 6, 7, 8], [3, 4], [5, 0], [[1, 2], [3, 4, 7, 8], [5, 0],
[[1, 7], [2], [3, 4], [5, 0], [[1, 2, 7, 8], [3, 4], [5, 0], [[1, 2], [3, 4, 6], [5, 0],
[[1, 5], [2], [3, 4], [5, 0], [[1, 2, 5, 7, 8], [3, 4], [5, 0], [[1, 2], [3, 4, 6, 7, 8], [5, 0],
[[1, 7], [2], [3, 4], [5, 0], [[1, 2, 7, 8], [3, 4], [5, 0], [[1, 2], [3, 4, 5, 7, 8], [5, 0],
[[1, 5], [2], [3, 4], [7, 8], [0], [[1, 5], [2], [3, 4], [7, 8], [0], [[1, 5], [2], [3, 4], [7, 8], [0],
[[1, 6], [2], [3, 4], [7, 8], [0], [[1, 2, 5, 6], [3, 4], [7, 8], [0], [[1, 2, 5], [3, 4], [7, 8], [0],
[[1, 6], [2], [3, 4], [7, 8], [0], [[1, 2, 6], [3, 4], [7, 8], [0], [[1, 2], [3, 4, 5, 6], [7, 8], [0].
\]

This gives \( s_5^{(3,4)} = 1 + 9 + 27 = 37 \).

This last example describes the strategy we use to obtain the following result.

**Proposition 4.** We have

\[
s_n^{(\ell, \ell+b)} = \delta(\frac{n - \ell}{\delta}) \sum_{\alpha=b}^{\ell} \binom{\ell}{\alpha} \alpha^{\ell - \alpha} \left( \frac{\alpha}{\delta} \right),
\]

where \( \binom{a}{b} \) denotes the Stirling number of second kind counting the number of partitions of \( \{1, 2, \ldots, a\} \) into \( b \) non empty sets.

**Proof.** Let

\[
P = [\rho_0, \ldots, \rho_{n-1}] \in \mathcal{R}_{n, \ell}^{(k)}
I = \{i_1, \ldots, i_\ell\} = \{i \mid Pi \neq \emptyset\}
\]

and

\[
f = \{0, \ldots, n-1\} \setminus I.
\]

The set \( \text{succ}_{\ell+b}(P) \) splits as the following disjoint union

\[
\text{succ}_{\ell+b}(P) = \bigcup_{\alpha=\delta}^{\ell} \text{succ}_{\ell+b,\alpha}(P).
\]
where
\[
\text{succ}_{\ell+\delta,\alpha}(P) = \{ P \cup (P \cdot g) \mid \text{#}(i \in I \mid i \cdot g \in J) = \alpha \}. \tag{5}
\]
But we have
\[
\#\text{succ}_{\ell+\delta,\alpha}(P) = \#\{ g \in [n]^I \mid \text{#}(i \in I \mid i \cdot g \in J) = \alpha \}. \tag{6}
\]
It remains to compute the cardinal of \( \chi^{n,\ell,\delta}_{\alpha} = \{ g \in [n]^I \mid \text{#}(i \in I \mid i \cdot g \in J) = \alpha \} \).
Let \( g \in \chi^{n,\ell,\delta}_{\alpha} \), then \( I \) splits as the partition \( I = I' \cup I'' \) such that \((I' \cdot g) \subset I, (I'' \cdot g) \subset J, \text{and } \#(I'' \cdot g) = \delta \), and \( \#I'' = \alpha \). To construct such a map, one has first to choose the sets \( I'' \) and \((I'' \cdot g)\); we have \( \binom{n}{\delta} \binom{n-\delta}{\alpha} \) possibilities to do that. Hence, we have to construct the image \((I'' \cdot g)\) and this is equivalent to give an ordered partition of \( I'' \) into \( \delta \) sets. We have \( \delta! \binom{\alpha}{\delta} \) possibilities to do that.
Finally, to complete the description of \( g \), we have to construct the restriction \( g|_{I'} \) and this gives \( \ell^{\ell-\delta} \) possibilities. In conclusion, we have shown that \( s^{(\ell,\ell+\delta)}_n = \sum_{\alpha=0}^{\ell} \#\chi^{n,\ell,\delta}_{\alpha} \) and \( \#\chi^{n,\ell,\delta}_{\alpha} = \delta! \binom{\alpha}{\delta} \ell^{\ell-\delta} \binom{n}{\delta} \). This is equivalent to our statement. \( \square \)

**Example 7.** Let us illustrate where the Stirling numbers in formula (3) come from. In fact, one has to compute the number of surjective functions from a set having \( \alpha \) elements onto a set having \( \delta \) elements. For our example, we suppose that we want to enumerate the surjective functions from \{0,1,2,3,4\} to \{0,1,2\}. Let \( g \) be such a map. The image of the elements by \( g \) partitions the initial set into 3 disjoint parts \( \{0,1,2,3,4\} = \bigcup_{i=0}^{2} \{ j \mid f(j) = i \} \).
The number of such set partitions equals to the Stirling number \( \binom{5}{3} = 25 \). But a partition does not characterize completely a surjective function and we need also to take into account the image associated to each part. So we have to multiply the number \( \binom{5}{3} \) by \( \delta! = 6 \) to obtain the number of surjections (here 150). This gives the factors \( \delta! \binom{\alpha}{\delta} \) in each term of (3).

**Example 8.** The numbers \( s^{(i,j)}_n \) have some special values. Let us list some of them.

- \( s^{(i,i)}_n = i! \) for \( 1 \leq i \leq n \),
- \( s^{(i+1,i)}_n = (n-i)(i+1)! - (i)! \) for \( 1 \leq i \leq n-1 \),
- \( s^{(2,2)}_n = i! \binom{n}{i} \) for \( 1 \leq i \leq \frac{n}{2} \), in particular \( s^{(n,2n)}_n = n! \).

Recall that the Hadamard product of two matrices of the same dimension \( M = (m_{ij})_{ij} \) and \( N = (n_{ij})_{ij} \) is the matrix \( M \cdot N = (m_{ij}n_{ij})_{ij} \). Another way to state Proposition 4 is to write that the coefficient \( s^{(i,j)}_n \) is an entry of an infinite matrix
\[
S_n = B \cdot A_n \tag{7}
\]
that is the Hadamard product of the matrix
\[
B = \left( \sum_{k=j-i}^{i} \binom{i}{k} \binom{n-i}{j-k} \right)_{ij} \tag{8}
\]
which does not depend on \( n \) with the matrix \( A_n = \left( (j-i)! \binom{n-i}{j-i} \right)_{ij} \) depending on \( n \).
Example 9.

\[
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & \ldots \\
0 & 4 & 5 & 1 & 0 & 0 & 0 & \ldots \\
0 & 0 & 27 & 37 & 12 & 1 & 0 & \ldots \\
0 & 0 & 0 & 256 & 369 & 151 & 22 & \ldots \\
0 & 0 & 0 & 0 & 3125 & 4651 & 2190 & \ldots \\
0 & 0 & 0 & 0 & 0 & 46656 & 70993 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}
= \begin{bmatrix}
1 & 4 & 12 & 24 & 24 & 0 & 0 & \ldots \\
0 & 1 & 3 & 6 & 6 & 0 & 0 & \ldots \\
0 & 0 & 1 & 2 & 2 & 0 & 0 & \ldots \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}
\]

These matrices have combinatorial interpretation. First the entry \((i, j)\) of \(A_n\) is nothing but the number \(a_{n-i,j-i} = \frac{(n-i)!}{(j-i)!}\) of ways of obtaining an ordered subset of \(j - i\) elements from a set of \(n - i\) elements (in the case where \(n - i < 0, j - i < 0\) or \(n < j\) we assume \(a_{n-i,j-i} = 0\) by convention).

The entries of the matrix \(B\) are interpreted in terms of \(r\)-Stirling numbers. The \(r\)-Stirling number \(\left\{\begin{array}{c} n \\ k \end{array}\right\}_r\) is the number of partitions of a set of \(n\) elements into \(k\) nonempty disjoints subsets such that the first \(r\) elements are in distinct subsets [1]. The \(r\)-Bell polynomials [15] are defined by

\[
B_{n,r}(x) = \sum_{k=0}^{n} \binom{n+r}{k+r}_r x^r. \tag{9}
\]

Their exponential generating function is (see [15] Theorem 3.1)

\[
\sum_n B_{n,r} \frac{x^n}{n!} = e^{x(r-1)+rz}, \tag{10}
\]

and they satisfy (see [15] Corollary 3.2)

\[
B_{n,r}(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}(x), \tag{11}
\]

where \(B_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k\) is the usual Bell polynomial. Comparing (8) and (11), we find

\[
B_{i,j} = \left\{ \begin{array}{c} 2i \\ j \end{array} \right\}_i. \tag{12}
\]

For instance, the numbers of the line \(i = 2\) are interpreted as follows:

- There are 4 partitions of \(\{1, 2, 3, 4\}\) into two parts such that the numbers 1 and 2 are in two distinct parts: \(\{\{1, 3, 4\}, \{2\}\}, \{\{1, 4\}, \{2, 3\}\}, \{\{1, 3\}, \{2, 4\}\},\) and \(\{\{1\}, \{2, 3, 4\}\}\).
- There are 5 partitions of \(\{1, 2, 3, 4\}\) into three parts such that the numbers 1, 2, and 3 are in three distinct parts: \(\{\{1, 4\}, \{2\}, \{3\}\}, \{\{1\}, \{2, 4\}, \{3\}\}, \{\{1\}, \{2\}, \{3, 4\}\}, \{\{1, 3\}, \{2\}, \{4\}\},\) and \(\{\{1\}, \{2, 3\}, \{4\}\}\).
- There is only 1 partition of \(\{1, 2, 3, 4\}\) into four parts such that the numbers 1, 2, 3, and 4 are in four distinct parts.

So we have the following result.

**Proposition 5.**

\[
s_n^{(i,j)} = a_{n-i,j-i} \left\{ \begin{array}{c} 2i \\ j \end{array} \right\}_i. \tag{13}
\]
### 5.2 Generating functions

From (10), we deduce that the coefficient of \( x^i y^j z^l \) in the Taylor expansion of \( \frac{e^{(e^{(e-1)})}}{z-xye^z} \) equals \( \frac{1}{n!} B_{i,j} \). Equivalently, consider the function \( \frac{e^{(e^{(e-1)})}}{z-xye^z} \) as a Taylor series in \( y \). Each monomial \( y^j \) has a coefficient that is a Laurent series in \( z \). Hence, the double generating function of \( \frac{1}{n!} B_{i,j} \) is obtained by computing the constant term in \( z \) in this expansion. In terms of residue, we summarize this result as

\[
\sum_{i,j} B_{i,j} x^i y^j \frac{y^j}{i!} = \text{Res}_{z=0} \left( \frac{e^{(e^{(e-1)})}}{z-xye^z} \right),
\]

(14)

where \( \text{Res}_{z=0} \) denotes the residue in \( z = 0 \) acting on each terms of the Taylor expansion in \( y \) of its argument.

**Example 10.** We have

\[
\frac{e^{(e^{(e-1)})}}{z-xye^z} = \sum_{i=0}^{\infty} z^{-i-1} e^{(e^{(e-1)})} e^i y^j.
\]

The coefficient of \( y^4 \) in the previous expression admits the following Laurent expansion

\[
z^{-5} e^{(e^{(e-1)})} e^{4z} = x^4 z^{-5} + x^4 (x + 4) z^{-4} + \frac{1}{2} x^4 (16 + 9x + x^2) z^{-3} + \frac{1}{6} x^4 (64 + 61x + 15x^2 + x^3) z^{-2} + \frac{1}{24} x^4 (256 + 369x + 151x^2 + 22x^3 + x^4) z^{-1} + \frac{1}{120} (1024 + 2101x + 1275x^2 + 3053x^3 + 30x^4 + x^5) + \cdots
\]

The residue is nothing but the coefficient of \( z^{-1} \) in this expression, that is \( \frac{1}{24} x^4 (256 + 369x + 151x^2 + 22x^3 + x^4) \) as expected.

But the residue Theorem implies

\[
\text{Res}_{z=0} \left( \frac{e^{(e^{(e-1)})}}{z-xye^z} \right) = \frac{1}{2i\pi} \oint_{\gamma} \frac{e^{(e^{(e-1)})}}{z-xye^z} dz
\]

(15)

where \( \gamma \) is a counterclockwise path around a Jordan curve enclosing 0 and \( i^2 = -1 \). In terms of symbolic computations, all works as if the curve encloses all the poles of \( \frac{e^{(e^{(e-1)})}}{z-xye^z} \) now considered as a function of \( z \). The function \( z \rightarrow z-xye^z \) admits only one (simple) zero at the value \( z = -W(-xy) \), where \( W \) denotes the Lambert \( W \) function [13, 10]. We recall that \( W(z) \) is the inverse function of \( z \rightarrow ze^z \) and its Taylor expansion (see eg [5]) is

\[
W(z) = \sum_{i>0} (-i)^{i-1} \frac{z^i}{i!}.
\]

(16)

Hence,

\[
\sum_{i,j} B_{i,j} x^i y^j \frac{y^j}{i!} = \text{Res}_{z=-W(-xy)} \left( \frac{e^{(e^{(e-1)})}}{z-xye^z} \right).
\]

(17)

Since \( z = -W(-xy) \) is the only pole of \( \frac{e^{(e^{(e-1)})}}{z-xye^z} \) of order 1, the series \( \sum_{i,j} B_{i,j} x^i y^j \frac{y^j}{i!} \) is the constant term in the Taylor expansion of \( (z + W(-xy))e^{(e^{(e-1)})} \) at \( z = -W(-xy) \). In other words,

\[
\sum_{i,j} B_{i,j} x^i y^j \frac{y^j}{i!} = \lim_{z \rightarrow -W(-xy)} (z + W(-xy)) \frac{e^{(e^{(e-1)})}}{z-xye^z}.
\]

(18)
But noticing that $W(a) = ae^{-W(a)}$ and $W'(a) = \frac{W(a)}{a(1+W(a))}$, we obtain

$$
\lim_{a \to W(a)} \frac{a - ae^{-a}}{a - W(a)} = \lim_{\beta \to a} \frac{W(\beta) - ae^{-W(\beta)}}{W(\beta) - W(a)} = \lim_{\beta \to a} \frac{(\beta - a)W(a)}{(W(\beta) - W(a))\beta} = \frac{W(a)}{aW'(a)} = 1 + W(a). \tag{19}
$$

Using this equality in (18) we find the following result.

**Proposition 6.** The generating series of the coefficients $B_{i,j}$ is

$$
\sum_{i,j} B_{i,j} x^j y^i \frac{y^i}{i!} = \frac{e^{-(W(-xy)+x)}}{1 + W(-xy)}. \tag{20}
$$

**Example 11.** By applying the previous proposition, we find:

$$
\sum_{i,j} B_{i,j} x^j y^i \frac{y^i}{i!} = 1 + x(1 + x) y + \frac{1}{2} x^2 (x + 4) (1 + x) y^2 + \frac{1}{6} x^3 \left(27 + 37 x + 12 x^2 + x^3\right) y^3 + \frac{1}{24} x^4 \left(256 + 369 x + 151 x^2 + 22 x^3 + x^4\right) y^4 + \cdots. \tag{21}
$$

### 5.3 Closed expressions for the number of successors

Define the matrix $S_n$ constituted with the $n$ first lines and the $n$ first columns in $S$, that is $S_n = (s_{i,j}^{(n,j)})_{1 \leq i,j \leq n}$.

**Example 12.** The first matrices $S_n$ follow:

$$
S_2 = \begin{bmatrix} 1 & 1 \\ 0 & 4 \end{bmatrix}, \quad S_3 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 5 \\ 0 & 0 & 27 \end{bmatrix}, \quad S_4 = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 4 & 10 & 2 \\ 0 & 0 & 27 & 37 \\ 0 & 0 & 0 & 256 \end{bmatrix}, \quad S_5 = \begin{bmatrix} 1 & 4 & 0 & 0 & 0 \\ 0 & 4 & 15 & 6 & 0 \\ 0 & 0 & 27 & 74 & 24 \\ 0 & 0 & 0 & 256 & 369 \\ 0 & 0 & 0 & 0 & 3125 \end{bmatrix}, \quad S_6 = \begin{bmatrix} 1 & 5 & 0 & 0 & 0 & 0 \\ 0 & 4 & 20 & 12 & 0 & 0 \\ 0 & 0 & 27 & 111 & 72 & 6 \\ 0 & 0 & 0 & 256 & 738 & 302 \\ 0 & 0 & 0 & 0 & 3125 & 4651 \\ 0 & 0 & 0 & 0 & 0 & 46656 \end{bmatrix}, \quad S_7 = \begin{bmatrix} 1 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 25 & 20 & 0 & 0 & 0 \\ 0 & 0 & 27 & 148 & 144 & 24 & 0 \\ 0 & 0 & 0 & 256 & 1107 & 906 & 132 \\ 0 & 0 & 0 & 0 & 3125 & 9302 & 4830 \\ 0 & 0 & 0 & 0 & 0 & 46656 & 70993 \\ 0 & 0 & 0 & 0 & 0 & 0 & 823543 \end{bmatrix}, \quad \ldots \tag{22, 23}
$$

Let us denote by $s_{n,k}^{(i,j)}$ the entries of the $k$th power $(S_n)^k$. From its definition, the number $s_{n,k}^{(i,j)}$ is the cardinal of the set $\text{succ}^k(P) \cap R_{n,i}^{(k,k')}$ for any $P \in R_{n,i}^{(k)}$. Hence, the entries $s_{n,k}^{(1,0)}$ of the first line of $(S_n)^k$ are the cardinal of the sets of $R_{n,i}^{(k)}$ obtained by applying $k$ times the map $\text{succ}$ to the vector $[[1], \emptyset, \ldots, \emptyset]$. Equivalently, we have the following statement.
Proposition 7. We have

\[ \#R_{n,k}^{(k)} = s_{n,k}^{(1,\ell)}, \]  
(24)

\[ \#R_m^{(k)} = \sum_\ell s_{n,k}^{(1,\ell)}, \]  
(25)

and

\[ \#U_{m,n}^{(k)} = \sum_{\ell',\ell''} s_{n,k}^{(1,\ell')} s_{n,k}^{(1,\ell'')}. \]  
(26)

Example 13. Examine the first line of \( S_3 \):

\[
\begin{bmatrix}
1 & 10 & 10 \\
0 & 16 & 155 \\
0 & 0 & 729
\end{bmatrix}
\]

This means that \( R_3^{(2)} \) contains

1. One vector having exactly one non empty entry: \([1, 2, 3, 4, 0, 0] \);
2. Ten vectors having exactly two non empty entries: \([1, 2, 3, 4, 0, 0], [1, 2, 0, 3, 4, 0], [1, 3, 0, 2, 4, 0], [1, 4, 0, 2, 3, 0], [1, 2, 3, 4, 0, 0], [1, 3, 0, 2, 4, 0], [1, 4, 0, 2, 3, 0], [1, 0, 2, 3, 4, 0] \);
3. Ten vectors having exactly three non empty entries: \([1, 2, 3, 4, 0, 0], [1, 3, 0, 2, 4, 0], [1, 4, 0, 2, 3, 0], [1, 0, 2, 3, 4, 0], [1, 2, 3, 4, 0, 0], [1, 3, 0, 2, 4, 0], [1, 4, 0, 2, 3, 0], [1, 0, 2, 3, 4, 0] \).

Remark 1. Noticing that the diagonal entries in the matrices \( S_n \) are \( 1^1, 2^2, \ldots, m^m \). It results that the number of elements \( R_n^{(k)} \) is a linear combination

\[ \#R_n^{(k)} = \sum_{i=1}^n a_i^{(n)} (i^i)_k, \]  
(27)

where \( a_i^{(n)} \) are rational numbers that are independent of \( k \). While we do not know a closed form for the values of the \( a_i^{(n)} \) coefficients, we can easily compute them by solving a system of linear equations or, alternatively, use some known formulas as those described in [16].
The first formulas are:

\[ \#R_2^{(k)} = \frac{2}{3} + \frac{1}{3} 4^k, \]
\[ \#R_3^{(k)} = \frac{6}{13} + \frac{12}{23} 4^k + \frac{5}{299} 27^k, \]
\[ \#R_4^{(k)} = \frac{372}{1105} + \frac{100}{161} 4^k + \frac{2880}{68471} 27^k + \frac{23}{136255} 256^k, \]
\[ \#R_5^{(k)} = \frac{135040}{517803} + \frac{1006400}{1507443} 4^k + \frac{7530000}{106061579} 27^k + \frac{46000}{78183119} 256^k + \frac{4150701}{10832451881581} 3125^k, \]
\[ \#R_6^{(k)} = \frac{344810430}{1610539931} + \frac{4008890625}{5860435903} 4^k + \frac{18418610000}{18316834693} 27^k + \frac{5833053}{4534620902} 256^k + \frac{47154746285712511}{806896274400} 4665^k, \]
\[ \#R_7^{(k)} = \frac{5818082250876}{31579697044181} + \frac{15729243009924}{229823691577177} 4^k + \frac{4868360340900}{37710516098219107} 27^k + \frac{200723945058}{88887962822497} 256^k + \frac{47452318871846138}{888824849603838210} 4665^k \]
\[ + \frac{193433013191149163934799}{32920001103738912355} 3125^k + \frac{26332206893852752878029}{823543} 4665^k, \ldots \]

All the coefficients seem to be positive but their combinatorial interpretation remains to be investigated. It is also interesting to note that the coefficients \(a_n^{(k)}\) of the dominant power \((n^n)^k\) in \(\#R_n^{(k)}\) decreases very quickly to zero. For instance, \(a_{35}^{(35)} < 5.267408697 \cdot 10^{-238}\).

The first values of \(\#R_n^{(k)}\) for \(n = 2\) are 1, 2, 6, 22, 86, 342, 1366, 5462, 21846, 87382, 349526, \ldots. This series of number appears in other contexts. The sequence A047849 of [17] lists some of these interpretations. For instance, these numbers also count the closed walks of length \(2n\) at a vertex of the cyclic graph on 6 nodes, the permutations of length \(n\) avoiding 4321 and 4123, the closed walks of length \(n\) at a vertex of a triangle with two loops at each vertex etc. The sequences for other value of \(n\) are not referenced in [17]. It should be interesting to investigate if some of these interpretations naturally extends for \(n > 3\). By construction, one notices that, for any \(k \geq 1\), \(n\) divides \(\#R_n^{(k)}\).

6 The state complexity of the shuffle operation revisited

Let \((\Lambda, P) \in \mathcal{U}_{m,n}\) with \(\Lambda = [\lambda_0, \ldots, \lambda_{m-1}]\) and \(P = [\rho_0, \ldots, \rho_{n-1}]\). We define \(s(\Lambda, P) = \{(i, j) \mid \lambda_i \cap \rho_j \neq \emptyset\}\).

Proposition 8. The set of accessible states of \(\exists \text{shuffle}(M_{F_1}^{(i)}, M_{F_2}^{(i)})\) is

\[ \{s(\Lambda, P) \mid \exists k \leq f(m, n), (\Lambda, P) \in \mathcal{U}_{m,n}^{(k)}\}. \]

Proof. Let us prove by induction that \(\exists \text{shuffle}(M_{F_1}^{(i)}, M_{F_2}^{(i)}) \subseteq \{s(\Lambda, P) \mid \exists k \leq f(m, n), (\Lambda, P) \in \mathcal{U}_{m,n}^{(k)}\}\).

We have \(s(P(0)) = \{(0, 0)\}\). Suppose that \(v = (t_1, \ldots, t_k)\) is a path from \(\{(0, 0)\}\) to \(E\) and assume, as an induction hypothesis, that \(s(P(v)) = E\). Let \(t_{k+1} = (E, (f, g), E')\) be a transition and set...
\( v' = (t_1, \ldots, t_{k+1}) \). Then \( E' = \{(i \cdot f, j) \mid (i, j) \in E \} \cup \{(i, j \cdot g) \mid (i, j) \in E \} \). If we set \( \mathcal{P}(v) = (\Lambda, P) \) with \( \Lambda = [\lambda_0, \ldots, \lambda_{m-1}] \) and \( P = [\rho_0, \ldots, \rho_{n-1}] \), then we have

\[
\mathcal{P}(v') = ((\Lambda \cdot f) \cup \Lambda^\top, P \cup (P \cdot g)^\top).
\]  

(34)

Hence, if we set \( (\Lambda \cdot f) \cup \Lambda^\top = [\lambda'_0, \ldots, \lambda'_{m-1}] \) and \( P \cup (P \cdot g)^\top = [\rho'_0, \ldots, \rho'_{n-1}] \) we have

\[
s(\mathcal{P}(v')) = \{(i, j) \mid \lambda'_i \cap \rho'_j \neq \emptyset \}
\]

\[
= \{(i \cdot f, j) \mid \lambda_i \cap \rho_j \neq \emptyset \} \cup \{(i, j \cdot g) \mid \lambda_i \cap \rho_j \neq \emptyset \}
\]

\[
= \{(i \cdot f, j) \mid (i, j) \in E \} \cup \{(i, j \cdot g) \mid (i, j) \in E \} = E',
\]

as expected.

Conversely, by construction, for any state \( E \) there exists \( (\Lambda, P) \) such that \( s(\Lambda, P) = E \). Since the state complexity is bounded by \( f(m, n) \), any state is reached in at most \( f(m, n) \) transitions in \( \mathcal{I} \mathcal{U}(M_{F_1}^{m}, M_{F_2}^{n}) \). This means that there exists \( k \leq f(m, n) \) such that there exists \( (\Lambda, P) \in \mathcal{U}_{m,n}^{(k)} \) with \( s(\Lambda, P) = E \).

From all the results of this section, one deduces

**Theorem 1.**

\[
s_{cl}(m, n) = \#\left\{ s(\Lambda, P) \mid (\Lambda, P) \in \bigcup_{k=0}^{f(m,n)} \mathcal{U}_{m,n}^{(k)} \right\}.
\]

Although Formula (36) is an exact expression for the state complexity of the shuffle product, this is not a number easy to manipulate because the set \( \bigcup_{k \leq f(m,n)} \mathcal{U}_{m,n}^{(k)} \) is very huge. In fact, we would hope to find \( \#\left\{ s(\Lambda, P) \mid (\Lambda, P) \in \bigcup_{k=0}^{f(m,n)} \mathcal{U}_{m,n}^{(k)} \right\} = f(m, n) \). Equivalently the conjecture of Brzozowski et al. reduces to

**Conjecture 1**

\[
s(\mathcal{U}_{m,n}) = \{ E \subseteq \mathbb{I} \times \mathbb{I} \mid E \cap (\{0\} \times \mathbb{I}) \neq \emptyset \text{ and } E \cap (\mathbb{I} \times \{0\}) \neq \emptyset \}
\]

(37)

**Example 14.** For \( m = n = 2 \), the \( f(2, 2) = 10 \) states are recovered from \( \mathcal{U}_{2,2} \) as follows

\[
\begin{align*}
\times & = s\left(\begin{array}{c}
1
\end{array}\right), \\
\times x & = s\left(\begin{array}{c}
1 2
\end{array}\right),
\end{align*}
\]

\[
\begin{align*}
\times x & = s\left(\begin{array}{c}
2
\end{array}\right),
\end{align*}
\]

\[
\begin{align*}
\times x & = s\left(\begin{array}{c}
2 3
\end{array}\right),
\end{align*}
\]

\[
\begin{align*}
\times x & = s\left(\begin{array}{c}
3
\end{array}\right),
\end{align*}
\]

\[
\begin{align*}
\times x & = s\left(\begin{array}{c}
4
\end{array}\right),
\end{align*}
\]

Although we have no general algorithm allowing us to compute a reverse function to \( s \), we know how to handle a few families of tableaux.
Example 15. Suppose that \( n = m \) and define \( E_\sigma = \{(i, i \cdot \sigma) \mid i \in [n]\} \) for any permutation \( \sigma \) of \([n]\). First suppose that \( 0 \cdot \sigma = 0 \) and let \( k \) be such that \( 2^{k-1} \leq n < 2^k \). We consider the vector \( \Pi_n = \{[1, 2^k], [2, 2^k - 1], \ldots, [2^k - n + 1], [2^k - n + 2], \ldots, [n]\} \). We check that \( \Pi_n \in \mathcal{L}_n^{(k)} \cap \mathcal{R}_n^{(k)} \) and \( s(\Pi_n, \Pi_n) = \{(i, i) \mid i \in [n]\} \). For instance for \( n = 5 \), one has

\[
\begin{pmatrix}
1 & 8 \\
2 & 7 \\
3 & 6 \\
4 & 5 \\
\end{pmatrix} = 
\begin{pmatrix}
1 & 8 \\
2 & 7 \\
3 & 6 \\
4 & 5 \\
\end{pmatrix}
\]

To construct any other set \( E_\sigma \) with \( 0 \cdot \sigma = 0 \) it suffices to consider the pair \( [\Pi_n \cdot \sigma, \Pi_n] \) because we have \( (\Pi_n \cdot \sigma)[i] = \Pi_n[i \cdot \sigma] \). For instance we have,

\[
\begin{pmatrix}
1 & 8 \\
2 & 7 \\
3 & 6 \\
4 & 5 \\
\end{pmatrix} = 
\begin{pmatrix}
1 & 8 \\
2 & 7 \\
3 & 6 \\
4 & 5 \\
\end{pmatrix}
\]

Now suppose that \( 0 \cdot \sigma \neq 0 \). Let \( k \) such that \( 2^{k-1} < n \leq 2^k \) and \( r = 2n - 2^k \). If \( n \) is even we set \( q = \frac{2^k - 1}{2} \) and \( \Pi_n = \{[1, 2^k - 1], \ldots, [q, 2^k - q], [2^k - q, 2^k], [2^k - 1, 2^k - 1], \ldots, [2^k - q + 2, 2^k - q + 2], [q + 1, \ldots, [2^k - 1 - q + 1], [2^k - q + 1], \ldots, [2^k - q], [2^k - q + 1], \ldots, [2^k - q + 1], [2^k - q + 1], \ldots, [2^k - q + 1]] \}. This vector partitions the set \{1, \ldots, 2^k\} into \( n = 2q + r \) parts with \( q \) entries of size \( 2 \) and \( 2(2^k - 2q) = r \) entries of size 1. If \( n \) is odd we set \( q = \frac{2^k - 1}{2} - \frac{n-1}{2} \) and \( \Pi_n = \{[1, 2^k - 1], \ldots, [q, 2^k - q], [2^k - q, 2^k], [2^k - 1, 2^k - 1], \ldots, [2^k - q + 2, 2^k - q + 2], [q + 1, \ldots, [2^k - 1 - q + 1], [2^k - q + 1], \ldots, [2^k - q + 1], [2^k - q + 1], \ldots, [2^k - q + 1], [2^k - q + 1], \ldots, [2^k - q + 1]] \}. This vector partitions the set \{1, \ldots, 2^k\} into \( n = 2q - 1 + r \) parts with \( 2q - 1 \) entries of size \( 2 \) and \( 2(2^k - q + 1 - (q + 1) + 1) = r \) entries of size 1. In both cases \( (n \) even or \( n \) odd), a permutation \( \Lambda \) (resp. \( P \)) of \( \Pi_n \) belongs to \( \mathcal{L}_n^{(k)} \) (resp. \( \mathcal{R}_n^{(k)} \)) if and only if \( \Lambda[0] = \{2^{k-1}, 2^k\} \) (resp. \( P[0] = \{1, 2^k - 1 + 1\} \)). To construct a pair \([\Lambda, P]\) such that \( s(\Lambda, P) = E_\sigma \) we proceed as follows

1. We set \( \Lambda[0] = P[0 \cdot \sigma] = \{2^{k-1}, 2^k\} \) and \( \Lambda[0 \cdot \sigma^{-1}] = P[0] = \{1, 2^k - 1 + 1\} \).
2. We choose randomly the remaining entries in \( \Lambda \) in such a way that \( \Lambda \) is a permutation of \( \Pi_n \).
3. If \( j \) is the index of an entries filled in the previous step, we set \( P[j \cdot \sigma] = \Lambda[j] \).

For instance, consider the following tableau for \( n = 5 \)

\[
E_{(132)(45)} = 
\begin{pmatrix}
\times & \times \\
\times & \times \\
\times & \times \\
\times & \times \\
\times & \times \\
\end{pmatrix}
\]

We have \( \Pi_5 = \{[1, 5], [2, 6], [4, 8], [3], [7]\} \). Two values in each vector are set in the first step

\[
([4, 8], [1, 5], ?, ?, ?), ([1, 5], ?, [4, 8], ?, ?)).
\]
We complete randomly the first vectors
\[ (\{4,8\}, \{1,5\}, \{3\}, \{2,6\}, \{7\}), \{1,5\}, \{4,8\}, \{?, ?\}). \] (42)

The remaining values of the right vector are deduced using step 3.
\[ (\{4,8\}, \{1,5\}, \{3\}, \{2,6\}, \{7\}), \{1,5\}, \{4,8\}, \{7, 2, 6\}). \] (43)

Hence, we check that
\[
\begin{pmatrix}
4 & 8 \\
1 & 5 \\
3 & \ \\
2 & 6 \\
7 & \\
\end{pmatrix}
= 
\begin{pmatrix}
\times & \times \\
\times & \times \\
\times & \ \\
\times & \\
\times & \\
\end{pmatrix}
\] (44)
as expected.

Observe that the inequality \(2^{k-1} \leq n < 2^k\) when \(\sigma(0) = 0\) and \(2^{k-1} < n \leq 2^k\) when \(\sigma(0) \neq 0\).

As a consequence, if \(\sigma \neq 0\), the state \(E_\sigma\) is accessible from \((0,0)\) by a path of length \([\log_2(n)]\). But when \(\sigma(0) = 0\), we need a path of length \([\log_2(n+1)]\). This makes a difference only when \(n\) is a power of 2. For instance, for \(n = 2\) we can not access the state \((0,0), (1,1)\) in less than 2 steps while we need only one step to access to \((0,1), (1,0)\).

**Example 16.** Let \(k\) such that \(2^{k-1} \leq m < 2^k\). We set
\[ PP_n = [(1,\ldots,2^k), (2^k+1,\ldots,2^{k+1}), \ldots, (2^{k+n-2} + 1,\ldots,2^{k+n-1})],\] and \(PL_m = [p\lambda_0, \ldots, p\lambda_{m-1}] \in L^{n+k-1}_m\) with
\[ p\lambda_i = \{m - i + 2^k \alpha \mid \alpha \in \{2^n\}\} \cup \{2^k(\alpha + 1) - i \mid \alpha \in \{2^n\}\}, \] for \(i \in \{2^k - m\}\), and
\[ p\lambda_j = \{m - j + 2^k \alpha \mid \alpha \in \{2^n\}\}, \] for \(j \in \{2^k - m, \ldots, m - 1\}\). We let the reader check that \(PL_m \in L^{n+k-1}_m\) and \(PP_n \in R^{n+k-1}_m\). We have
\[ s(PL_m, PP_n) = [m] \times [n], \] because \(m - i + 2^k(2^j - 1) \in p\lambda_i \cap p\rho_j\) for any \((i, j) \in [m] \times [n]\). For instance,
\[
\begin{bmatrix}
3 & 4 & 7 & 8 & 11 & 12 & 15 & 16 & 19 & 20 & 23 & 24 & 27 & 28 & 31 & 32 & 35 & 36 & 39 & 40 & 43 & 44 & 47 & 48 \\
51 & 52 & 55 & 56 & 59 & 60 & 63 & 64 & 34 & 38 & 42 & 46 & 50 & 54 & 58 & 62 & 33 & 37 & 41 & 45 & 49 & 53 & 57 & 61 \\
\end{bmatrix}
\] (49)

The underlined numbers correspond to the elements \(m - i + 2^k(2^j - 1) = 2^{k+j} - (i + 1)\), for \(i = 0 \ldots 2\) and \(j = 0 \ldots 4\).

This last example allows us to prove that many other states are reachable in \(\exists \exists \exists(\Pi_{\Pi_{\Pi}}, \Pi_{\Pi_{\Pi}})\).

Let us illustrate this point. Let \(E\) be a state such that there exist \(i_1, i_2 \in [m], j_1, j_2 \in [n]\) satisfying
\{(i_1, j) \in E \mid j \in \llbracket n \rrbracket \} \subset \{(i_2, j) \in E \mid j \in \llbracket n \rrbracket \}$ and $\{(i, j_1) \in E \mid i \in \llbracket m \rrbracket \} \subset \{(i, j_2) \in E \mid i \in \llbracket m \rrbracket \}$. If we denote by $\binom{b}{a}$ the map sending $a$ to $b$ and letting unchanged the other numbers, we have

\[
E \cdot \binom{(i_2, j_2)}{(i_1, j_1)} = \{(i, j) \in E \mid i \neq i_1\} \cup \{(i_2, j) \mid (i_1, j) \in E\}
\]

\[
\cup \{(i, j) \in E \mid j \neq j_1\} \cup \{(i, j_2) \mid (i, j_1) \in E\}
\]

\[
= E \setminus \{(i_1, i_2)\}.
\]

Applying successively many times this property from $\llbracket m \rrbracket \times \llbracket n \rrbracket$ (which is reachable from Example 16), we find that any state $E$ such that $\llbracket m \rrbracket \times \llbracket i \rrbracket \cup \llbracket j \rrbracket \times \llbracket n \rrbracket \subset E$, for some $(i, j) \in \llbracket m \rrbracket \times \llbracket n \rrbracket$ is also reachable.

**Example 17.**

As a consequence, by applying the inclusion-exclusion principle one obtains

\[
sc_m(m, n) \geq \sum_{k=1}^{m} \sum_{\ell=1}^{n} (-1)^{k+\ell} \binom{m}{k} \binom{n}{\ell} 2^{(m-k)(n-\ell)} \geq 2^{(m-1)(n-1)}.
\]

Let us call dense the states $E$ such that $\{(i_1, j) \in E \mid j \in \llbracket n \rrbracket \} \subset \{(i_2, j) \in E \mid j \in \llbracket n \rrbracket \}$ implies $i_1 = i_2$ and $\{(i, j_1) \in E \mid i \in \llbracket m \rrbracket \} \subset \{(i, j_2) \in E \mid i \in \llbracket m \rrbracket \}$ implies $j_1 = j_2$. We denote by $D_{m,n}$ the set of dense $(m, n)$-states.

Conjecture 1 reduces to the following one

**Conjecture 2**

\[
D_{m,n} \subset s(U_{m,n}).
\]

**Example 18.** There are two kinds of dense states in $D_{3,3}$ : those that have exactly one cross by line and column and those that have exactly two crosses by line and by column. The states of first kind are clearly in $s(U_{3,3})$ by using Example 15. There are 6 states of second kinds that can be obtained the ones from the others by permuting the lines or the columns. They also belong in $s(U_{3,3})$ since we have

\[
\begin{bmatrix}
XX & s(583) \\
XX & \ \ 146
\end{bmatrix}
= \begin{bmatrix}
XX & 583 \\
\ \ 72 & \ \ 146
\end{bmatrix}
\]

\[
\begin{bmatrix}
XX & 583 \\
XX & 146
\end{bmatrix}
= \begin{bmatrix}
XX & 583 \\
XX & \ \ 27
\end{bmatrix}
\]

\[
\begin{bmatrix}
XX & 683 \\
XX & 524
\end{bmatrix}
= \begin{bmatrix}
XX & 368 \\
XX & 524
\end{bmatrix}
\]
7 Conclusion

We give a new approach to study the state complexity of the shuffle operation. Unfortunately, we did not succeed in proving the bound given by Brzozowski et al., but as we have translated the problem into a combinatorial one, independent from its language theoretical definition, we hope we have open up new research opportunities for this problem.

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