Approximating Quadratic 0-1 Programming via SOCP

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Abstract

We consider the problem of approximating Quadratic 0-1 Integer Programs with bounded number of constraints and non-negative constraint matrix entries, which we term as PIQP.

We describe and analyze a randomized algorithm based on a program with hyperbolic constraints (a Second-Order Cone Programming -SOCP- formulation) that achieves an approximation ratio of $O(a_{\text{max}} n^{\beta(n)})$, where $a_{\text{max}}$ is the maximum size of an entry in the constraint matrix and $\beta(n) \leq \min_i W_i$, where $W_i$ are the constant terms that define the constraint inequalities. We note that by appropriately choosing $\beta(n)$ the randomized algorithm, when combined with other algorithms that achieve good approximations for smaller values of $W_i$, allows better algorithms for the complete range of $W_i$. This, together with a greedy algorithm, provides a $O^*(a_{\text{max}} n^{1/2})$ factor approximation, where $O^*$ hides logarithmic terms. Our solution is achieved by a randomization of the optimal solution to the relaxed version of the hyperbolic program. We show that this solution provides the approximation bounds using concentration bounds provided by Chernoff-Hoeffding and Kim-Vu.

1 Introduction

In this paper, we study optimizing 0-1 integral quadratic programs. We consider the class of quadratic integer programs where $X \in \{0,1\}, X = (x_1, \ldots x_n)$:

$$\max X^T B X + c^T X$$

subject to

$$a_i^T X \leq W_i, \quad i = 1, \ldots p$$

$$X_i \in \{0,1\}$$

for bounded number of constraints $p$. We assume that the quadratic objective function is defined by a symmetric matrix $B$ with non-negative coefficients and the linear term is defined by a positive vector $c$. Furthermore, $a_{ij} \geq 0, i = 1 \ldots p, j = 1 \ldots n$. We term the above as a Positive 0-1 Quadratic Program (PIQP). The problem is NP-hard and our interest is in designing approximation schemes for this problem.

While Quadratic Integer programs have been considered in the operations research community, approximation algorithms for the general problem have not been actively considered. A recent

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result for positive semi-definite matrices by Guruswami and Sinop [7], provides an approximation scheme for quadratic integer programs with PSD objective for graph problems with a bound of \((1+\epsilon)/\min(1,\lambda_{max})\) and a running time of \(n^{O(\epsilon^2/r^2)}\), where \(r^*\) is the number of eigenvalues of the graph Laplacian smaller than \(1-\epsilon\). Unfortunately we do not see any way to bound \(r^*\) in general.

An important subclass of the PIQP is the generalized knapsack problem. The motivation is a resource allocation problem, where the items are proposed projects, the weight of each item is the cost of implementation, and the benefit measures the gains due to implementation of that project. Typically, this budgeting problem is implemented either as a 0-1 Knapsack problem which assumes that the projects are pairwise independent so their benefits are additive, or all possible combinations of projects are considered which increases the size of the problem exponentially.

However, in most applications the benefit of two projects can be less than or greater than the sum of the individual projects due to underlying dependencies among the projects. The current authors with collaborators have explored these ideas of interdependency between projects in the context of transportation resource allocation [17]. This generalized knapsack model was applied in a computational study using real-life data on travel demand, roadway network designs, and traffic operations, as well as six major projects proposed for possible investments to the Chicago downtown area. The computational study revealed that the overall benefits in terms of travel-time savings after considering project network-wide impacts and their interdependency relationships are much lower (by 38-64 percent) as compared with those established without considering project interdependency relationships, indicating an inflated benefits when interdependencies are not considered.

To model this situation, we define a problem that will be critical to developing our approximation algorithm. The Graph Knapsack Problem, \(GKP(G, b, w, W)\), where \(G = (V, E)\) is an undirected graph with \(n\) vertices, \(w: V \rightarrow \mathbb{Z}^+\) is a weight function, \(b: E \cup V \rightarrow \mathbb{Z}\) is a benefit function on vertices and edges, and \(W\) is a weight bound. The vertices correspond to the items in the Knapsack problem. The benefit of a subgraph \(H = (V_H, E_H)\) is \(b(H) = \sum_{v \in V_H} b(v) + \sum_{e \in E_H} b_e\) while its weight is \(w(H) = \sum_{v \in V_H} w(v)\). Note that the benefits can be negative; negative weight edges model the case where two projects’ benefits are literally less than the sum of their parts. Given a subset of vertices \(S\), we consider the subgraph induced by \(S\), termed \(G[S]\). The graph knapsack problem requires the determination of a subset of vertices \(S \subseteq V\) that maximizes the benefit of the induced subgraph, \(b(G[S])\) with the restriction that its weight \(w(G[S])\) is less than \(W\). Note that this reduces to the classical knapsack problem when there are no edges in the graph \(G\). Clearly GKP is \(NP\)-Hard.

From a graph theoretic point of view, GKP is related to the maximum clique problem. We can reduce the clique problem to the graph-knapsack problem. Given a graph \(G\), suppose we wish to determine if \(G\) contains a clique of size \(t\). We define an instance of GKP on \(G\) with \(W = t\), \(w_i = 1\), \(b_i = 0\), \(b_e = 1\) for \(e \in E(G)\). Graph \(G\) has a \(K_t\) iff GKP has benefit at least \((t^2)/2\).

We may note that, unless \(P = NP\), achieving an approximation ratio better than \(n^{1-\epsilon}\) is impossible for the clique problem [8][27]. However, it is not known whether there exist approximation-preserving transformations between GKP and Max-Clique.

In fact, GKP also generalizes the Dense k-Subgraph problem (k-DSP problem) (see [1][23]), which requires finding an \(k\)-vertex induced subgraph of an edge-weighted graph with maximum density. This corresponds to GKP with edges of benefit 1 while vertices have zero benefit, and the weight of each vertex is 1 with \(W = k\). The dense \(k\)-subgraph problem is well-researched problem, with the best approximation provided by an \(O(n^{1/4})\) approximation factor algorithm in [1]. Algorithms that use semi-definite programming have been proposed by Goemans (as mentioned in [4]) and in [23]. These algorithms promise an approximation ratio of \(O(n/k)\). Note that a PTAS for the
dense \( k \)-subgraph has been ruled out in \([13]\) under a certain complexity assumption.

The problem can be formulated as a 0-1 Quadratic Program:

\[
\begin{align*}
\text{maximize} & \quad \sum_i b(v_i)x_i + \sum_{i,j; v_i, v_j \in E(G)} b(v_iv_j)x_ix_j \\
\text{such that} & \quad \sum_i w(v_i)x_i \leq W \\
& \quad x_i \in \{0, 1\}
\end{align*}
\]

Replacing the term \( x_i x_j \) by an integer variable \( x_{ij} \in \{0, 1\} \) and adding the constraints \( x_{ij} \leq \frac{x_i + x_j}{2} \) and \( x_{ij} \geq \frac{x_i + x_j - 1}{2} \) also gives an integer linear program (ILP) for the problem. Without loss of generality, we can assume that \( G = K_n \), since we can give non-edges benefit zero. This quadratic program is equivalent to the well-studied Quadratic Knapsack Problem (QKP) \([5]\) (see \([21]\) for a survey):

\[
\begin{align*}
\text{maximize} & \quad \sum_{i=1}^n \sum_{j=1}^n b_{ij}x_i x_j \\
\text{such that} & \quad \sum_{i=1}^n w_i x_i \leq W \\
& \quad x_i \in \{0, 1\}
\end{align*}
\]

(To account for \( b(v_i) \), add a variable \( x_{i}' \) with \( b_{i'i'} = b(v_i) \) and \( w_{i'} = 0 \).) However, the benefits in QKP are typically assumed to be non-negative \([21]\).

The QKP is an important problem which has been mostly studied from the LP-based exact algorithms point of view \([5, 21]\). Rader and Woeginger \([10]\) developed a FPTAS for the case when all benefits are non-negative and the underlying graph is series parallel. They also show that when QKP has both negative and non-negative benefits, it cannot have a constant factor approximation unless \( P = NP \).

The idea of using discrete structures like graphs, digraphs, or posets to generalize the classical knapsack problem by modeling some sort of dependency among the items is not a new one. However all such generalizations of the Knapsack problem (described below) restrict the choice of subset of items that can be picked.

The Knapsack Problem with Conflict Graph (KPCG) is a knapsack problem where each edge in the underlying conflict graph on the items introduces the constraint that at most one of those two items can be chosen. This can be modeled as the Graphical Knapsack problem by putting large negative benefit on the edges of the conflict graph and using that as the underlying graph for GKP. The KPCG was first studied by Yamada et al. \([26]\) who analyzed some greedy algorithms for KPCG. Pferschy and Schauer \([20]\) give exact dynamic programs which can be implemented as FPTAS when the underlying conflict graph is a chordal graph or has bounded tree width.

Borradaile, Heeringa, and Wilfong \([2]\) study two versions of the Constrained Knapsack Problem (CKP) in which dependencies between items are given by a graph. In the first version, an item can be selected only if at least one of its neighbors is also selected. In the second version, an item can be selected only when all its neighbors are also selected. They give upper and lower bounds on the approximation ratios for both these problems on undirected and directed graphs.

Similarly, the Precedence-Constrained Knapsack Problem (PCKP) introduces a directed acyclic graph (or, equivalently a poset) on the items so that a item can be chosen if and only if all of its predecessors have already been chosen (selected items should form an ideal in the poset). See Kolliopoulos and Steiner \([16]\), Samphaiboon and Yamada \([22]\), Kellerer, et. al \([11]\) for further discussion.
In the *Subset-Union Knapsack Problem* (SUKP), each item (with benefit) is a subset of a ground set of elements (each with a weight). SUKP asks to find a set of elements with maximum benefit that fits in the knapsack. This corresponds to HKP with hyperedges corresponding to an item in SUKP and vertices corresponding to the elements of SUKP. Goldschmidt, et. al [6] show that SUKP is NP-hard and give an exact algorithm that runs in exponential time.

1.1 Our Results and an Outline

In this paper we describe a polynomial-time randomized algorithm that approximates PIQP to within a factor $O(a_{\max} n^{1/2} \log^{2+\gamma} n)$ w.h.p, when the benefit function $b$ is non-negative, $\gamma$ is a very small constant and $a_{\max}$ is the maximum co-efficient in the constraint matrix.

The approximation to the PIQP problem results from a combination of two algorithms, one which works well for small values of $W = \max_i \{W_i\}$ with an $O(\min(n,W)/t)$-approximation bound (for any fixed $t > 0$), and the other a randomized algorithm which gives a $O(a_{\max} \beta(n) \log^{2+\gamma} n)$ approximation bound where $a_{\max}$ is the maximum size of an entry in the constraint matrix and $\beta(n) \leq \min_i W_i$, where $W_i$ are the constant terms that define the constraint inequalities and $\gamma$ is a small constant.

Our primary aim in this paper is to illustrate the use of the following main tools as compared to the typical application of Semi-definite Programs and Chernoff bounds in problems of this nature:

1. A second-order cone program (a convex program with hyperbolic constraints). While a similar result can be obtained via general semi-definite programming, SOCP has more efficient solutions (see the discussion in [18], where it is said that “solving SOCPs via SDP is not a good idea, interior-point methods that solve the SOCP directly have a much better worst-case complexity than a SDP method”).

2. A combination of the Kim-Vu bounds on concentration of polynomial random variables, and Chernoff-Hoeffding concentration bounds.

Section 2 presents a simplified version of PIQP, and some preliminaries for the analysis of the algorithm. Section 3 presents an analysis of a natural greedy method that gives an $O(\min(n,W)/t)$-approximation algorithm (for any fixed $t > 0$). In Section 4, we give the complete description and analysis of our main algorithm with $O(a_{\max} \beta(n) \log^{2+\gamma} n)$ approximation. This bound can be thought as a generalization of the $O(n/k)$-approximation algorithm for the $k$-DSP ( [4], [24]), although our methodology is different. In Section 5 we combine the algorithms from sections 3 and 4 to provide the general approximation algorithm.

The appendix contains an algorithm for the solving the generalized knapsack problem in the case of bounded treewidth. Since series-parallel graphs have tree-width at most 2, our algorithm generalizes the one from Rader and Woeginger [10].
2 Preliminaries

2.1 Preliminaries-A

We study the following class of quadratic integer programs where \( X \in \mathbb{Z}^n, X = (x_1, \ldots, x_n) \):

\[
\begin{align*}
\max & \quad X^T BX + c^T X \\
\text{subject to} & \quad a_i^T X \leq W_i, \quad i = 1, \ldots, p \\
& \quad x_i \in \{0, 1\}
\end{align*}
\]

for bounded number of constraints \( p \) and positive integer vectors \( a_i \) and \( c \) and the matrix \( B \in \mathbb{Z}^{n \times n} \). We term the above as a Positive Integer Quadratic Program (PIQP). W.l.o.g we will assume that \( B \) is a symmetric matrix.

The objective function has both quadratic and linear terms. We can interpret optimizing the linear term as the classical 0-1 Knapsack problem with multiple constraints for which there is a PTAS \([12]\). Separating the linear and quadratic terms will introduce a constant factor in the approximation bound. Hence we will restrict our attention to the quadratic objective function

\[
\sum_{i \neq j} b_{ij} x_i x_j.
\]

For simplicity we will transform the above inequalities by appropriate scaling such that \( \hat{a}_{ij}^T X \leq W, \quad i = 1, \ldots, p \), where \( W = \max_i \{W_i\} \) and \( \hat{a}_{ij} = \lceil a_{ij} W / W_i \rceil \). Note that \( \hat{X}_i \cdot \frac{a_{ij} W / W_i}{\lceil a_{ij} W / W_i \rceil} \) is a solution to the original problem if \( \hat{X} \) is a solution to the scaled problem. Let \( x^*_i \) and \( x^*_{ij} \) be the optimum solution to the scaled and original problems, respectively. Since \( \frac{\lceil a_{ij} W / W_i \rceil}{a_{ij} W / W_i} \leq 2 \), we get that \( x^*_{ij} / 2 \) is a solution to the scaled problem. We thus obtain that the optimum solution to the scaled problem is within a factor of 4 of the optimum solution to the original problem.

Furthermore, we can assume that \( a_{ij} + a_{ik} \leq W \) for every pair of indices \((j, k)\). Otherwise, the benefit \( b_{ij} \) can be set to 0 as it is not possible to set both variables \( x_i \) and \( x_j \) to 1. Thus we can assume that \( x_k \) and \( x_l \) are such that \((x_k, x_l) = \arg \max_{(x_i, x_j)} b_{ij}\) can be set to 1.

The transformed problem, which we term PIQP also, is:

\[
\begin{align*}
\max & \quad \sum_{i,j} b_{ij} x_i x_j \\
\text{subject to} & \quad a_i^T X \leq We, \quad i = 1, \ldots, p \\
& \quad x_i \in \{0, 1\}
\end{align*}
\]

where \( e \) is the vector \((1, 1, \ldots, 1)\). We also denote the matrix \([a_{ij}]\) by \( A \), with the \( i \)th row of the matrix \( A \) corresponding to the constraint specified by the vector \( a_i \).

We next define two restrictions of the PIQP problem. Let \( X_A = \{x_j | \exists i, \ a_{ij} > W/2\} \)

- PIQPS Problem: This problem is a version of the PIQP under the assumption that \( a_{ij} \leq W/2 \). This gives us a simplified version of the PIQP problem by assigning all variables in \( X_A \) to be of value 0.
Theorem 2.2. \[\text{Chernoff-Hoeffding Bound-1}\] Let 
\[
\max_{\{X\}} \text{subject to } x \in S
\]
variables. Denote 
\[
\mu \delta_2 \varepsilon_0 \leq \varepsilon_2
\]
\[
P[E[Y] - Y > \varepsilon_0] \leq 2e^{\varepsilon_0^2} \exp(-t/32(2\varepsilon_0')^{1/4} + \log n)
\]
where \(\varepsilon_0\) is maximum of \(\varepsilon_0, \varepsilon_1, \varepsilon_2\), and 
\[
\varepsilon' = \max\{\varepsilon_1, \varepsilon_2\}
\]
We will also use the following standard bounds of Chernoff and Hoeffding \[\text{[3,9]}\]:

\[\text{Theorem 2.2. [Chernoff-Hoeffding Bound-1]}\] Let \(X_1, X_2, \ldots, X_n\) be independent 0-1 random variables. Denote \(X = \sum_{i=1}^n a_iX_i\), where \(a_i \in [0,1]\), and let \(E(X) = \mu\). Then 
\[
P[X \geq (1 + \delta)\mu] \leq e^{-(\mu\delta)^2/3}
\]
when \(0 < \delta \leq 1\).
and

**Theorem 2.3.** [Chernoff-Hoeffding Bound-2] Let \(X_1, X_2, \ldots X_n\) be independent 0-1 random variables. Denote \(X = \sum_{i=1}^{n} X_i\), and let \(E(X) = \mu\). Then \(\Pr[X - \mu \geq t] \leq e^{-2nt^2}\).

For our purpose, as we will be dealing with weighted sums, the following form of Chernoff-Hoeffding bound is appropriate.

**Corollary 2.3.1.** Let \(X_1, X_2, \ldots X_n\) be independent 0-1 random variables. Denote \(X = \sum_{i=1}^{n} a_i X_i\), where \(a_i \geq 0\), and let \(E(X) = \mu\). Then, for \(a_{max} = \max\{a_i\}\), \(\Pr[X \geq (1 + \delta)\mu]\) \(\leq e^{-\frac{(\mu\delta^2)}{3a_{max}}}/(3a_{max})\) when \(0 < \delta \leq 1\).

### 3 A Greedy Approach

In this section we consider the two restricted problems \(PIQPS\) and \(PIQPR\), as defined in Section 2.

\[
\max \sum_{i,j} b_{ij} x_i x_j \\
\text{subject to} \\
a_i^T X \leq We, \ i = 1, \ldots p \\
X_i \in \{0, 1\}
\]

where \(e\) is the vector \((1, 1, \ldots, 1)\). In PIQPS, \(a_{ij} \leq W/2 \forall i, j\). And in PIQPR \(\forall i, \exists j \text{ s.t.} a_{ij} > W/2\).

In the PIQPS problem we will assume that the trivial solution where every variable is set to 1 is not possible else we can trivially return the optimal solution.

We will interpret this quadratic optimization program with zero diagonal entries as a directed graph optimization problem, \(GKP M\) (Generalized Knapsack with Multiple Constraints), with each variable \(x_i\) corresponding to a vertex \(v_i\) and \(b_{ij}\) corresponding to the benefit on the edge \(v_i v_j\). The support of function \(b\) can be thought as the edges in the underlying graph \(G\). The coefficient \(a_{ij}\) is the weight associated with \(v_j\) in the \(i\)th constraint.

We define a combined weight for each vertex \(i\) as \(w_i = \sum_j a_{ji}\), the sum of all the weights in column \(i\) (associated with the variable \(x_i\)) in all the budget constraints. Let \(W\) be as specified in the problems PIQPS and PIQPR. Note that this gives us a multi-constraint version of an instance of \(GKP(G, b, w, W)\) as defined in the introduction, which we term \(GKP M\).

Define \(w(T) = \sum_{v_i \in T} w_i\) for every subset of vertices \(T\) for the rest of this discussion.

Now, using this definition of weight, the greedy algorithm can be defined naturally as: Fix an integer \(t \geq 2\).

1. Initialize \(S = \emptyset\).

2. Pick a subset \(T\) of \(V(G) - S\) of cardinality at most \(t\) such that its benefit (the sum of the benefits of the edges incident to vertices in \(T\) in \(S \cup T\)) to weight (sum of the combined weights, \(\sum_{i \in T} w_i\)) ratio is highest.

3. Update \(S = S \cup T\) if weight of \(S \cup T\) satisfies each of the \(p\) budget constraints, and then go to step 2. Otherwise pick whichever of \(S\) or \(T\) has larger benefit as the final solution as long as the set satisfies all the budget constraints.
Note that any final solution given by this algorithm will satisfy all the constraints. This follows since for any choice of $T$ it is easy to check the feasibility of $T$ and $S \cup T$.

The following is true from Lemma 2.1 and by the definition of PIQPR.

**Claim 3.1.** Every solution generated by the greedy algorithm for PIQPR and PIQPS uses at least $W/2$ of the budget in at least one the constraints.

For an instance $I$ of $GKPM$, let $A(I)$ be an approximate solution generated by the greedy method and let $C(I) = A(I) \cap O(I)$, where $O(I)$ is an optimal solution. W.L.O.G. we assume that $G$ is a complete graph by defining $b(e) = 0, \forall e \notin E(G)$.

Note that in general $A(I)$ and $O(I)$ will overlap and this intersection between the solutions makes it difficult to analyze the relative worth of the two solutions. To overcome this difficulty we use $A(I)$ and $O(I)$ to define a new instance of $GKPM$ which has disjoint greedy and optimal solutions with its greedy solution same as $A(I)$. This makes it easier to analyze and indirectly bound the benefits of the original $A(I)$ and $O(I)$.

We define a transformation to an instance $I' = (G', b', w', W)$ as follows: Define a new set of vertices $C' = \{ v' \mid v \in C(I) \}$, i.e., the vertex set $C'$ is a duplicate of $C(I)$. In instance $I'$, we let $A = A(I)$ and $O' = (O(I) \setminus C(I)) \cup C'$. Note that $A(I) \cup O' \setminus C' = A(I) \cup O(I)$.

1. $V(G') = V(G) \cup C'$
2. The benefits for the vertices are all zero (as illustrated above).
3. The benefits for the edges are

   $$b'(e) = \begin{cases} 
   b(e) & \text{if } e = uv \text{ s.t. } u, v \in A(I) \\
   b(e) & \text{if } e = uv \text{ s.t. } u \in O' \text{ and } v \in O' \setminus C', \text{ or } v \in O' \text{ and } u \in O' \setminus C' \\
   0 & \text{otherwise}
   \end{cases}$$

   Note that this means we are assigning zero benefit to the edges whose both vertices are in $C'$ to avoid duplication of benefits assigned to edges within $C(I)$. The total benefit of $E(G')$ is the same as the total benefit of $E(G)$.

4. $w'(v') = w(v), \forall v' \in C'$, i.e. the copies of the vertices in $C'$ have the same weight on each vertex as in $C(I)$. All other vertices $v$ have the same weight in $G'$ as in $G$.

We consider two cases:

*Case 1* $b(C(I)) \geq b(O(I))/2$. In this case $b(A(I)) \geq b(C(I)) \geq b(O(I))/2$. Thus $\frac{b(O(I))}{b(A(I))} \leq 2$.

*Case 2* $b(C(I)) < b(O(I))/2$. Here, $b'(O') = b(O(I)) - b(C(I)) > b(O(I))/2$. We consider this case in more detail below.

Let $A(I')$ be an approximate solution provided by the greedy algorithm in the instance $I'$. We show that there exists a sequence of choices made by the greedy such that $A(I') = A(I)$.

**Lemma 3.1.** For solution $A(I)$ provided by the greedy algorithm in instance $I$, the greedy algorithm can also produce the solution $A(I)$ on instance $I'$.
Proof. We show that at every step of the greedy algorithm on \( I' \), the partial solution produced is the same as that in \( I \). Consider the \( i \)th step of the greedy algorithm, assuming that the claim is true for prior steps. Prior to that step, let \( A_i(I') \) be the set of vertices chosen by the greedy in \( I' \). By assumption, prior to the \( i \)th step, the same set of vertices in \( A(I) \) will be chosen by the greedy algorithm on \( I \) also.

Let \( V_i \) be the vertices chosen at step \( i \) in \( I' \). Note that the Greedy Algorithm does not need to pick anything from \( O \setminus C' \) since there is as good or better choice in \( A(I') \) as per the greedy choice in the instance \( I \). Thus, \( V_i \subset A(I') \cup C' \).

Let \( C_i(I') \) be the set of vertices in \( C(I) \) corresponding to \( V_i \cap C' \), and let \( V'_i = (V_i - C') \cup C_i(I') \).

Then \( A_i(I') \cup V'_i \) has benefit greater or equal to the benefit of \( A_i(I') \cup V_i \) (since benefits are zero within \( C' \), and the same or less weight as \( A_i(I') \cup V_i \) (less if \( C_i(I') \) intersects \( V_i \cap C(I) \)). Therefore \( V'_i \) is another valid choice for the greedy algorithm on \( I' \) at the \( i \)th step. Since \( V'_i \subset A(I) \), the greedy algorithm on \( I' \) always has a choice within \( A(I) \).

\[ \Box \]

Given that \( b'(O') > \frac{b(O(I))}{2} \), by Lemma 3.1 and consequently \( b(A(I)) = b'(A(I')) \), it follows that

\[ \frac{b(A(I))}{b(O(I))} \geq \frac{b'(A(I'))}{2b'(O')} \]

We now bound the ratio \( \frac{b'(A(I'))}{b'(O')} \). Let \( V_i(I') \) be the set of vertices chosen at the \( i \)th step of the greedy algorithm. And let \( K_i = \frac{b'(V_i(I'))}{w'(V_i(I'))} \). Note that \( b'(A(I')) = \sum_{i=1}^t b'(V_i(I')) \) and \( w'(A(I')) = \sum_{i=1}^t w'(V_i(I')) \), i.e. \( b'(V_i(I')) \) is the sum of the benefits of edges incident to vertices in \( V_i(I') \) within \( \cup_{j=1}^t V_j(I') \). Further, let \( K_{\text{min}} = \min_i \{ K_i, i = 1 \ldots r \} \) when the greedy executes for \( r \) iterations.

Let \( K_{\text{min}} = \frac{b'(V_i(I'))}{w'(V_i(I'))} \). Then \( \frac{b'(V_i(I'))}{w'(V_i(I'))} \leq \frac{b'(V_i(I'))}{w'(V_i(I'))} \forall i \neq t \).

This means \( \sum_i w'(V_i(I')) b'(V_i(I')) \leq \sum_i w'(V_i(I')) b'(V_i(I')) \), that is \( \frac{b'(V_i(I'))}{w'(V_i(I'))} \leq \frac{\sum_i b'(V_i(I'))}{\sum_i w'(V_i(I'))} \).

Thus, we have

\[ K_{\text{min}} \leq \frac{b'(A(I'))}{w'(A(I'))} \]

By the choice of the greedy, and the fact that \( A(I') \cap O' = \emptyset \),

\[ b'(T) \leq K_{\text{min}}w'(T), \ \forall T \subseteq O', |T| = t \]

Note that \( b'(T) = \sum_{e \in T} b'(e) \). Adding up the inequalities for all such \( T \subseteq O' \) (Note that all such \( T \) are feasible) gives

\[ \left( \frac{|O'| - 2}{t - 2} \right) \sum_{e \in O'} b'(e) \leq K_{\text{min}} \left( \frac{|O'| - 1}{t - 1} \right) \sum_{v \in O'} w'(v) \]  \hspace{3em} (1)

Since \( b'(O') = \sum_{e \in O'} b'(e) \)

\[ \left( \frac{|O'| - 2}{t - 2} \right) b'(O') \leq K_{\text{min}} \left( \frac{|O'| - 1}{t - 1} \right) w'(O'), \]
which yields

\[ b'(O') < \frac{|O'|}{t-1} K_{\min} w'(O') \]

and

\[ \frac{b(O(I))}{2} < b'(O') < \frac{|O'| b(A(I'))}{(t-1) \cdot w(A(I'))} w(O') \]

Further, since \( w(O') \leq pW, b(A(I')) = b(A(I)) \) and \( w(A(I')) = w(A(I)) \geq W/2 \) we get

\[ \frac{b(O(I))}{b(A(I))} \leq \frac{2}{t-1} |O(I)| \cdot \frac{pW}{w(A(I))} \leq \frac{4p}{t-1} |O(I)| \]

Since the weight \( w(v) \) of each vertex is assumed to be positive (and hence at least 1 under our integrality assumptions), \( |O(I)| \leq W \) and \( n \), we get the two bounds \( \frac{4p}{t-1} \) and \( \frac{4pW}{t-1} \).

In the end, we obtain (by change of parameter from \( t-1 \) with \( t \geq 2 \) to \( t' \) with \( t' \geq 1 \)) and using lemma :

**Theorem 3.1.** For a fixed \( t \geq 1 \), the greedy algorithm gives a \((8p \min(n, W)/t)\)-factor polynomial time \((O(2^{t+1} (\frac{n}{t+1})^+))-running time\) approximation algorithm for PIQPR, PIQPS and hence for PIQP, where \( W = \max_i W_i \).

### 4 Randomized Algorithm

In this section we describe a randomized approximation method for the PIQP problem. We first describe a randomized algorithm to provide an approximation bound and then improve that bound by combining with the bound obtained via the greedy algorithm.

#### 4.1 A Randomized Method

Our main result in this section is:

**Theorem 4.1.** PIQP with at most logarithmic number of linear constraints \((p \leq \log n)\) and \( \min_i W_i \geq \beta(n) \) can be approximated to within a factor of \( O(a_{\max} \frac{n}{\beta(n)} \log^{2+\gamma} n) \) w.h.p. , where \( a_{\max} \) is the maximum size of an entry in the constraint matrix \( A \), and \( \gamma > 0 \) is an arbitrary small constant.

The randomized solution that we will use for PIQP will be the solution with maximum benefit among three different solutions, two deterministic and one randomized. We will use the Greedy Algorithm from Section 3, to get a solution within an approximation factor of \( O(\min\{n, W\}) \), as indicated by Theorem [3.1] to PIQP. Note that the approximation provided by the greedy implies that we can assume \( a_{\max} \leq \beta(n) \). If \( a_{\max} > \beta(n) \) then \( n < a_{\max} n/\beta(n) \) and hence the Greedy Algorithm gives a better approximation factor than claimed by Theorem 4.1.

Now, let us describe the three potential solutions. For any solution \( x \in \{0,1\}^n \), we define the functions values \( F(x) \) and \( G_i(x) \) as the objective function value and the constraint usage values, respectively, of \( x \). That is, \( F(x) = \sum_{i,j} b_{ij} x_i x_j \) and \( G_i(x) = \sum_j a_{ij} x_j \). As in Section 3, we will interpret as an instance of GKPM, the multiconstraint version of GKP, with each variable \( x_i \) corresponding
to a vertex \( v_i \) and \( b_{ij} \) corresponding to the benefit on the edge \( v_i v_j \). The coefficient \( a_{ij} \) is the weight (cost) associated with \( v_j \) in the \( i \)th constraint. To simplify the notation, sometimes we will use vertex notation \( u \) in the subscripts, as in \( b_{uv} \) (the benefit on the edge \( uv \)) and \( a_{i,u} \) (weight in the constraint \( i \) associated with vertex \( u \)), since each such subscript \( i \) corresponds to both a variable and a vertex.

**Solution 1:** Note that by the form of \( \text{PIQP} \) we can assume that \( a_{ij} + a_{ik} \leq W \) for every pair of indices \((j,k)\). We can pick a solution \( x_i = x_j = 1 \) corresponding to the edge \( v_i v_j \) with maximum value of \( b_{ij} \) over all possible edges \( v_i v_j \). This will be out first solution.

**Solution 2:** For the remaining two solutions, we need to use a “hyperbolic” program \((P^*)\):

\[
\begin{align*}
\text{maximize} & \quad \sum_{uv \in E(G)} b_{uv} x_{uv} \\
\text{subject to} & \quad AX \leq We \\
& \quad x_u x_v \geq \sqrt{x_{uv}} \\
& \quad x_u, x_{uw} \in [0,1]
\end{align*}
\]

**Solution 2(a) This optimization problem can be solved in polynomial time [19] to get an optimal solution \( x_u^* \in [0,1] \). Let \( Y \) be a random 0-1 solution generated by \( Y_u = 1 \) with probability \( \sqrt{x_u^*} / \lambda \), where \( \lambda = 2\sqrt{a_{max} n} / \beta(n) \) is a scaling factor.

**Lemma 4.1.** Let \( W = \beta(n) \). Then for each \( i \), \( \sum_u a_{i,u} \sqrt{x_u} \leq 2W \sqrt{a_{max} n} / \beta(n) \), where \( a_{max} = \max\{a_{ij}\} \).

**Proof.** Fix an \( i \). Let \( I = \{u | x_u > \beta(n)/(a_{iu} n)\} \) and \( J = \{u | x_u \leq \beta(n)/(a_{iu} n)\} \). Then, using the fact that \( \sqrt{x_u} < x_u \sqrt{a_{iu} n} / \beta(n) \) when \( u \in I \), we get

\[
\sum_u a_{i,u} \sqrt{x_u} = \sum_{u \in I} a_{i,u} \sqrt{x_u} + \sum_{u \in J} a_{i,u} \sqrt{x_u} \\
\leq \sum_{u \in I} a_{i,u} x_u + \sum_{u \in J} a_{i,u} \sqrt{a_{iu} n} / \beta(n) \cdot 1 \\
\leq \sqrt{a_{max} n} / \beta(n) \sum_{u \in I} a_{i,u} x_u + \sum_{u \in J} \sqrt{a_{iu} n} / \beta(n) \cdot 1 \\
\leq \sqrt{a_{max} n} / \beta(n) W + \sqrt{a_{max} n} / \beta(n) W \\
\leq 2 \sqrt{a_{max} n} / \beta(n) W
\]

Taking \( \lambda = 2\sqrt{a_{max} n} / \beta(n) \), gives us that \( E[G_i(Y)] = \sum u a_{i,u} \sqrt{x_u} / \lambda \leq 2 \sqrt{a_{max} n} / \beta(n) W / \lambda \leq W \), using the lemma. Note that \( E[F(Y)] = \sum b_{uv} \sqrt{x_u} \sqrt{x_v} / \lambda^2 \) equals the optimal value of the hyperbolic program \((P^*)\) divided by \( \lambda^2 \), since \( x_u^* x_v^* = x_{uv}^* \) at optimality.

**Solution 2(b):** Let \( Z \) be a 0-1 solution generated by the following procedure. First find \( v \in V(G) \) such that \( v = \arg \max_w \sum_{u \in N_G(w)} b_{uw} \sqrt{x_u} / \lambda \). Then, for this fixed \( v \), define \( Z_v = 1 \), \( Z_w = 0 \) at every vertex, \( w \notin N_G(v) \cup \{v\} \) (\( v \) and its neighbors in \( G \)), and \( Z_u \) for \( u \in N_G(v) \) is determined by solving a “local” 0-1 Knapsack problem with multiple constraints, whose items are the neighbors of \( v \) and
benefit of each such item equals the benefit of the edge incident to it and \( v \), and is solvable by a PTAS \[12\]:

\[
\max \sum_{e \in E_u} b_e z_u \\
\text{such that} \sum_{e \in E_u} a_{iu} z_u \leq W - a_{iv}, \ i = 1, \ldots, p \\
z_u \in \{0, 1\}
\]

Thus, \( G_i(Z) = \sum_{e \in E_u} a_{iu} z_u + w(v) \leq W \), that is \( Z \) is a feasible solution (recall all other variables are 0). Also note that \( F(Z) = \sum_{e \in E_u} b_e z_u \).

We claim that picking the best of the solutions obtained will give a solution with the stated approximation factor.

We have to show that the function \( F \) evaluated at this 0-1 solution is not too far from the optimal solution for \textbf{PIQP scaled} with high probability, and similarly each of the functions \( G_i \) evaluated at this 0-1 solution satisfy the budget bound with high probability.

4.2 Analyzing the Objective Function

Let us first study the function \( F(x) = \sum_{u \in E(G)} b_{uv} x_u x_v \).

For \( 0 < \alpha < 1 \), with \( P \) denoting the 0-1 hyperbolic program for solving QKP, we have that

\[
\begin{align*}
\mathbb{P}[F(Y) < (1 - \alpha)OPT(P)/\lambda^2] &\leq \mathbb{P}[F(Y) < (1 - \alpha)OPT(P^*)/\lambda^2] \\
&= \mathbb{P}[F(Y) < (1 - \alpha)\mathbb{E}[F(Y)]] \\
&= \mathbb{P}[\mathbb{E}[F(Y)] - F(Y) > \alpha \mathbb{E}[F(Y)]]
\end{align*}
\]

If we can show that this probability is small, then that would prove that \( Y \) is within a factor \( \lambda^2/(1 - \alpha) \) of the optimal (as long as the budget constraints are satisfied).

Recall that \( \varepsilon = \max\{\varepsilon_0, \varepsilon_1, \varepsilon_2\} \), and \( \varepsilon' = \max\{\varepsilon_1, \varepsilon_2\} \).
Here \( \varepsilon_0 = \mathbb{E}[F(Y)] = OPT(P^*)/\lambda^2 \), \( \varepsilon_1 = \max_v (\sum_{u \in N_G(v)} b_{uv} \mathbb{P}[Y_u = 1]) = \max_v (\sum_{u \in N_G(v)} b_{uv} \sqrt{x_u} / \lambda), \varepsilon_2 = \max_{uv \in E(G)} b_{uv} \).

We split the analysis into four cases depending on the relative worth of the expected value of the solution \( Y \) with regard to \( Z \) and the edge with maximum benefit (Solution 1). Let \( \gamma > 0 \) be a fixed small constant.

\textbf{Case (i)} When \( \varepsilon_2 > \varepsilon_1 \) and \( \varepsilon_0 < \varepsilon_2 \log^{2+\gamma} n \), Solution 1, \( \max_{uv} b_{uv} \) provides a good solution.

By assumption, \( \max_{uv \in E(G)} b_{uv} \) is greater than \( \max_v (\sum_{u \in N_G(v)} \sqrt{x_u} / \lambda \) as well as \( \sum b_{uv} \sqrt{x_u} / \lambda^2 \log^{2+\gamma} n \), i.e., \( \max_{uv} b_{uv} > OPT(P) / \lambda^2 \). So \( \max_{uv} b_{uv} \) is within a factor \( \lambda^2 \log^{2+\gamma} n \) of the optimum.

\textbf{Case (ii)} When \( \varepsilon_2 > \varepsilon_1 \) and \( \varepsilon_0 > \varepsilon_2 \log^{2+\gamma} n \), the concentration bounds provided by the results in Kim-Vu \[14, 25\] bound the error in the randomized solution \( Y \).
From the Kim-Vu bound, 

$$P[E[F(Y)] - F(Y) > t^2] < 2e^2 \exp \left( \frac{-\sqrt{\alpha}}{32} \sqrt{\frac{t}{2(\varepsilon')^{1/4}}} + \log n \right).$$

Taking 

$$t^2 = \alpha E[F(Y)],$$

we get

$$P[E[F(Y)] - F(Y) > \alpha E[F(Y)]] \leq 2e^2 \exp \left( \frac{-\sqrt{\alpha}}{32} \sqrt{\frac{\alpha}{\varepsilon_0/(\varepsilon')^{1/4}}} + \log n \right).$$

Under the given assumptions, 

$$\varepsilon = \varepsilon_0/\log^{2+\gamma} n$$

and 

$$\varepsilon' = \varepsilon_2,$$

the above bound reduces to

$$2e^2 \exp \left( \frac{-\sqrt{\alpha}}{32} \frac{(\varepsilon_0/\varepsilon_2) \log^{2+\gamma} n}{\log^{1+\gamma} n} + \log n \right) \leq 2e^2 \exp \left( \frac{-\sqrt{\alpha}}{32} (\log^{2+\gamma} n \log^{2+\gamma} n)^{1/4} + \log n \right) = 2e^2 \exp \left( \frac{-\sqrt{\alpha}}{32} \log^{1+\frac{1}{2} \gamma} n + \log n \right)$$

which is \(o(1).\)

Case (iii)  When \(\varepsilon_1 > \varepsilon_2\) and \(\varepsilon_0 > \varepsilon_1 \log^{2+\gamma} n\), the bounds provided by Kim-Vu \cite{14,25} can again be applied to bound the error in \(Y\), the randomized solution.

The application of the concentration bounds from Kim-Vu to \(F(Y)\) is identical to the previous case with the roles of \(\varepsilon_1\) and \(\varepsilon_2\) interchanged.

Case (iv)  When \(\varepsilon_1 > \varepsilon_2\) and \(\varepsilon_0 < \varepsilon_1 \log^{2+\gamma} n\), we show that the solution \(Z\), obtained from the local multi-constrained knapsack problem works as a good solution.

By the definition of \(Z\), \(F(Z)\), the objective function value obtained from the integer solution to the knapsack problem by rounding the fractional solution is within a factor of \(p + 1\) of the optimal fractional solution provided by 

\[
\max \sum_{v : (v, u) \in E} x'_u \quad \text{subject to} \quad \sum_{v : (v, u) \in E} a_{iu} z^i_u \leq W - w(v), \quad \forall i, z^i_u \in [0, 1].
\]

This bound is proved in the appendix.

Since \(w(v) \leq W/2\), this fractional solution is within a factor of 2 of the solution to the relaxation, \(RKP\):

\[
\max \sum_{v : (v, u) \in E} b_v z^i_u \\
\text{s. t.} \quad \sum_{v : (v, u) \in E} a_{iu} z^i_u \leq W, \quad \forall i \\
z^i_u \in [0, 1]
\]

Since \(\varepsilon_1\) is the objective function value evaluated at the neighborhood of \(v\), \(\sum_{u \in N_G(v)} b_u W^u/\lambda\), at a particular solution \(x^*_u\) of \(P^*\), and by Lemma 3.1, such a solution also satisfies the constraint \(\sum_{u} a_{iu} W^u/\lambda \leq W\), we conclude that 

\[
\sum_{v : (v, u) \in E} b_v z^{opt}_u \geq \varepsilon_1,
\]

where \(z^{opt}\) is the optimal solution to the relaxation \(RKP\).

Since \(\varepsilon_1\) is at least \(OPT(P)/\lambda^2 \log^{2+\gamma} n\), \(Z\) is within a factor \(O(\lambda^2 \log^{2+\gamma} n)\) of the optimum. Note that \(Z\) satisfies the budget constraint.
4.3 Analyzing the Budget Function

We consider the function \( G_i(Y) \). We know by the choice of \( \lambda \) that
\[
E[G_i(Y)] \leq W
\]

**Claim 4.1.** \( P[G_i(Y) > (1 + \delta)W] = o(1) \forall i \)

**Case 1:** \( E[G_i(Y)] \geq W/\lambda \)

By Chernoff-Hoeffding Bound-1 we obtain:
\[
P[G_i(Y) > (1 + \delta)W] \leq \frac{\Phi(G_i(Y) - (1 + \delta)E(G_i(Y)))}{\lambda W} \quad \text{since } \frac{\lambda}{\lambda a_{max}}
\]
\[
\leq e^{-\frac{(\beta(n)\delta^2)/(\lambda a_{max})}{(\lambda a_{max})}} \quad \text{since } E(G_i(Y)) \leq W/\lambda \geq \beta(n)/\lambda
\]
\[
\leq e^{-\frac{((\delta^2/6\sqrt{n})(\beta(n)/a_{max}))^{3/2}}{\lambda a_{max}}}
\]
using the value of \( \lambda \)

**Case 2:** \( E[G_i(Y)] \leq W/\lambda \)

By Chernoff-Hoeffding Bound-2 we obtain:
\[
P[G_i(Y) > (1 + \delta)W] \leq \frac{\Phi(G_i(Y) - \frac{\lambda}{W} \delta W)}{\lambda W} \quad \text{since } \lambda > 0
\]
\[
\leq e^{-\frac{(2\delta^2 W^2/\lambda^2)/(\lambda a_{max})^3}{\lambda a_{max}}}
\]
\[
= e^{-\frac{2\lambda^2}{\lambda W^2}(\lambda^2 a_{max})^3}
\]
\[
\leq e^{-\frac{2\lambda^2}{\lambda W^2}(\beta(n)/a_{max})^{3/2}} \quad \text{using the value of } \lambda
\]
\[
\leq e^{-\frac{2\lambda^2}{\lambda W^2}(\beta(n)/a_{max})^{3/2}} \quad \text{since } W \geq \beta(n)
\]
\[
\leq e^{-\frac{2\lambda^2}{\lambda W^2}(\beta(n)/a_{max})^{3/2}} \quad \text{since } \beta(n) \geq a_{max}
\]

We note that to obtain a solution that satisfies the \( p+1 \) conditions:

1. \( P[E[F(Y)] - F(Y) > \alpha E[F(Y)] = o(1) \)

and

2. \( P[G_i(Y) > (1 + \delta)W] = o(1) \forall i \)

we compute the probability that any one of these conditions is not satisfied. We will simply concentrate on the second set of probabilities, since the first inequality has already been shown to be \( o(1) \).

If we repeat the randomized algorithm \( k \) times this probability becomes \( e^{-n(\delta^2 k/6\sqrt{n})} \) and adding up all these probabilities for the \( p \) constraints simply makes it \( p e^{-n(\delta^2 k/6\sqrt{n})} \) which is \( o(1) \) when \( k = n^{2+\epsilon} \), when \( p \) is a fixed number. In fact, when \( p = O(\log n) \), \( k = O(n^c) \), iterations, for a constant \( c \), suffices.

This gives us the approximation bound w.h.p.
4.4 Approximating PIQP

We use the result of the previous section to provide an approximation for the PIQP problem. The greedy algorithm (Theorem 3.1) provides an $O(\min(n, W))$ approximation factor method. Note that if $W < n^{1/2}$, the greedy algorithm provides a $O(\sqrt{n})$ factor algorithm.

Theorem 4.1 proved in the previous section gives a $O(\max n/\beta(n))$ factor algorithm. Choosing $\beta(n) = W \geq n^{1/2}$, Theorem 4.1 and Theorem 3.1 give us the following result by picking the best of the two algorithms:

**Theorem 4.2.** PIQP can be approximated to within a factor of $O(\max n/\beta(n))$ w.h.p. in polynomial time, where $\gamma$ is an arbitrary small constant.

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5 Appendix

Consider the multi-constrained knapsack problem, \( MKP \):

\[
\max B(X) = \sum_j b_j x_j \\
\text{s.t. } \sum_j a_{ij} x_j \leq W, \ i = 1, \ldots p \\
x_j \in [0, 1], j = 1, \ldots n
\]

We will also refer to the budget constraints by the inequality system

\[
AX \leq W
\]

where \( A \in \mathbb{R}^{p \times n} \) and \( X \in \mathbb{R}^n \). W.l.o.g. we will assume that \( a_{ij} \leq W \). An approximate solution to this optimization problem can be obtained by solving the fractional linear program obtained by relaxing the integrality constraints on \( x_j \), to be simply \( 0 \leq x_j \leq 1 \). Let the optimum benefit obtained be \( OPT_R(MKP) \). And let the optimum of \( MKP \) be denoted by \( OPT(MKP) \). Clearly, \( OPT(MKP) \leq OPT_R(MKP) \)

Note that in the relaxed LP, the polytope defining the feasible region has vertices defined by \( n \) inequalities, that must be satisfied at the vertex. At most \( p \) of these are from the budget constraints, \( AX \leq W \). The other inequalities that are satisfied specify integral values for the variables \( x_j \). Thus at most \( p \) components in the vector \( X \) are non-integral. Let us denote the corresponding variables by \( X_p \). Let \( x_m \) be the variable such that \( x_m = \arg \max_{x_j \in X_p} b_j x_j \). Construct an integral solution, \( X_I \), by setting the variables in \( X - X_p \) according to the integral solution in the relaxed LP. Let \( B(X_I) \) be the benefit obtained from the integral solution. Furthermore, consider the solution, \( X_M \), where \( x_m = 1 \) and all other variables in \( X_p \) are assigned the value 0. And let \( B(X_M) \) be its benefit. The benefit obtained by \( \max\{B(X_M), B(X_I)\} \geq OPT_R(MKP)/(p+1) \geq OPT(MKP)/(p+1) \).

This gives us the following result:

**Theorem 5.1.**

There exists an approximate solution, \( A \), to the multi-constrained knapsack problem such that the benefit of the optimal solution is within a factor \( p + 1 \).