The chain relation in sofic subshifts
Alexandr Kazda*

Abstract
The paper gives a characterisation of the chain relation of a sofic subshift. Every sofic subshift Σ can be described by a labelled graph G. Factorising G in a suitable way we obtain the graph G/∞ that offers insight into some properties of the original subshift. Using G/∞ we describe first the chain relation in Σ, then characterise chain-transitive sofic subshifts, chain-mixing sofic subshifts and finally the attractors of the subshift dynamic system. At the end we present (straightforward) algorithms deciding chain-transitivity and chain-mixing properties of a sofic subshift and listing all the attractors of the subshift system.

Keywords: Sofic subshift, chain relation, chain-transitivity, chain-mixing, attractor.

1 Introduction
The properties of subshifts play a key role in symbolic dynamics and in the study of various dynamical systems (for example, cellular automata). The chain relation is an important characteristic of subshifts as well as of dynamical systems in general.

This paper addresses the question how, given a sofic subshift Σ (described by a labelled graph), one can deduce certain properties of this subshift: what does its chain relation look like, whether the subshift is chain-transitive or chain-mixing and what are the attractors of the system (Σ, σ). While the answer to this question was previously known in the special case of subshifts of finite type (a subshift Σ is of finite type when x ∈ Σ iff x does not contain any factor from a given finite set F of forbidden words.), it could not be directly applied to the more general sofic subshifts.

The core idea of this paper is to factorise the graph G describing the sofic subshift Σ. By identifying the pairs of vertices whose restricted follower sets have infinite intersections, we obtain the linking graph of G. We can then deduce certain properties of Σ from this linking graph. We shall formulate an explicit description of the chain relation (Theorem 14), obtaining the characterisation of chain-transitive sofic subshifts as a corollary (Corollary 15). Then we focus on the attractors of (Σ, σ), characterising them (Theorem 19), and use the description of the chain-transitive property for the explicit description of the chain-mixing property (Theorem 16). Finally, we show that we can check all these properties algorithmically.

The properties discussed have several applications. The chain relation tells us whether we can get from x ∈ Σ to y ∈ Σ in a certain way. Several theorems connect this relation and the attractors of (Σ, σ).

The chain-mixing property can be used in the study of cellular automata. In [2], it is shown that each subshift attractor (an attractor that is also a subshift) of a one-dimensional cellular automaton must be chain-mixing. Using the algorithms from this paper, it is now possible to conclude that many sofic subshifts can never be attractors of cellular automata because they are not chain-mixing. Unfortunately, being chain-mixing is not a sufficient condition to be an attractor (see examples in [2]).

2 Preliminaries
In this section we shall define several key terms from symbolic dynamics.

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Let \( A \) be a finite alphabet. Then \( A^* \) resp. \( A^\mathbb{Z} \) is the set of all finite resp. (two-sided) infinite words consisting of letters from \( A \). Given a (finite or infinite) word \( x \) denote by \( x_i \) the \( i \)-th letter of \( x \). Note that the set \( A^\bullet \) contains the empty word \( \lambda \).

We can write nonempty finite words as \( u = u_1 u_2 \cdots u_n \). Denote the length of \( u \) as \( |u| \) and define the concatenation operation as \( uv = u_1 \cdots u_n v_1 \cdots v_k \) where \( n \) is the length of \( u \) and \( k \) the length of \( v \). We adopt the notation \( u_{[k,l]} \) for the word \( u_k u_{k+1} \cdots u_l \).

Let \( u, v \) be words (\( v \) may be finite or (one- or two-sided) infinite, \( u \) is finite) in alphabet \( A \). We say that \( u \) is a factor of \( v \) and write \( u \subseteq v \) iff there exist \( k, l \in \mathbb{Z} \) such that \( u = v_{[k,l]} \) or if \( u = \lambda \).

The shift map is the mapping \( \sigma : A^\mathbb{Z} \to A^\mathbb{Z} \) given by \( \sigma(x)_i = x_{i+1} \).

The set \( A^\mathbb{Z} \) can be understood as a metric space: For \( x \neq y \in A^\mathbb{Z} \) let \( \rho(x,y) = 2^{-n} \), where \( n \) is the minimal nonnegative integer such that \( x_n \neq y_n \) or \( x_{-n} \neq y_{-n} \). If \( x = y \), set \( \rho(x,y) = 0 \). Then \( \rho \) is a metric on \( A^\mathbb{Z} \). The space \( A^\mathbb{Z} \) equipped with the topology induced by \( \rho \) is sometimes called the Cantor space. Topologically, \( A^\mathbb{Z} \) is a product of compact discrete spaces \( A \) and it is thus compact.

The set \( \Sigma \subseteq A^\mathbb{Z} \) is called a subshift iff it is topologically closed and strongly invariant under \( \sigma \), that is, \( \sigma(\Sigma) = \Sigma \).

Notice that a subshift is a closed subspace of the compact space \( A^\mathbb{Z} \) and is thus itself compact.

A dynamical system \((X,F)\) is a pair consisting of a compact metric space \( X \) and a continuous map \( F : X \to X \). An example of a dynamical system is the pair \((\Sigma,\sigma)\) where \( \Sigma \subseteq A^\mathbb{Z} \) is a subshift and \( \sigma \) is the shift map.

Let \( \Sigma \) be a subshift. The language of \( \Sigma \), denoted by \( L(\Sigma) \), is the set of all factors of words from \( \Sigma \): \( L(\Sigma) = \{ v \mid \exists x \in \Sigma, v \subseteq x \} \).

**Remark.** The language \( L(\Sigma) \) is central (some authors use the term extendable), that is:

- For every \( w \subseteq v \in L(\Sigma) \) we have \( w \in L(\Sigma) \).
- For every \( v \in L(\Sigma) \) there exist nonempty words \( u, w \) such that \( uvw \in L(\Sigma) \).

We say that a language is regular if it can be recognised by some finite automaton (a machine with finite memory). A subshift whose language is regular is called sofic.

A labelled graph \( G \) over alphabet \( A \) is an oriented multidigraph (a graph where we allow multiple parallel edges and loops) whose edges are labelled with the letters from alphabet \( A \). More precisely, the labelled graph is a quintuple \((V(G),E(G),s,t,l)\) where \( V(G) \) is the set of vertices, \( E(G) \) the set of edges and the mappings \( s,t : E(G) \to V(G) \) and \( l : E(G) \to A \) assign to each edge \( e \) its starting and ending vertex and its label, respectively. Both \( V(G) \) and \( E(G) \) must be finite.

A subgraph \( H \) of a labelled graph \( G = (V(G),E(G),s,t,l) \) (over alphabet \( A \)) is a labelled graph \((V(H),E(H),s_H,t_H,l_H)\) such that \( V(H) \subseteq V(G), E(H) \subseteq E(G) \) and the mappings \( s_H,t_H \) and \( l_H \) are restrictions of \( s,t \) and \( l \), respectively. The subgraph of \( G \) induced by the set \( U \) is the subgraph \( H \) of \( G \) such that \( V(H) = U \) and the set \( E(H) \) contains all the edges \( e \in E(G) \) such that \( s(e),t(e) \in V(H) \).

We call a subgraph \( H \) of \( G \) terminal if there is no edge \( e \in E(G) \) starting in \( v \in V(H) \) and ending in \( u \in V(G) \setminus V(H) \). Similarly, we define initial subgraph \( H \) of \( G \) as a subgraph such that no edge \( e \in E(G) \) starts in \( v \in V(G) \setminus V(H) \) and ends in \( u \in V(H) \).

A walk of length \( l \) in a graph \( G \) is any sequence \( v_0 e_1 v_1 \cdots e_l v_l \) where \( v_i \in V(G), e_i \in E(G) \) are vertices and edges of \( G \) and the edge \( e_{i+1} \) leads from \( v_i \) to \( v_{i+1} \) in \( G \) for each \( i = 0,1, \ldots , l-1 \). We allow walks of length zero (a single vertex). We also define biinfinite walks as sequences \( \ldots v_{-2} e_{-1} v_{-1} e_0 v_0 e_1 v_1 e_2 v_2 \cdots \) such that the edge \( e_{i+1} \) leads from \( v_i \) to \( v_{i+1} \) in \( G \) for each \( i \in \mathbb{Z} \).

A walk of length \( 0 < l < \infty \) is closed if \( v_0 = v_l \).

As any walk of nonzero length is uniquely determined by its sequence of edges, we shall often use a shorthand description of walks, writing down just the edges.

A labelled graph \( G \) is called connected iff for all \( u, v \in V(G) \) there exists a walk from \( u \) to \( v \).

To be precise, this is the definition of a strongly connected graph. However, we do not consider weak connectivity in the article and so we can safely omit this adjective.) We allow walks of length zero, so a single vertex graph is considered connected in this paper.
A labelled graph $G$ is periodic iff $V(G)$ can be partitioned into $n > 1$ disjoint sets of vertices $V_0, V_1, \ldots, V_{n-1}$ such that every edge $e \in E(G)$ leads from some $v \in V_k$ to some $u \in V_{k+1}$ (here $V_n = V_0$) for a suitable $k$. A graph is aperiodic iff it is not periodic.

Given a graph $G$, we can partition $G$ into its maximal connected subgraphs $K_1, K_2, \ldots, K_n$. These graphs are obviously disjoint and for every vertex $v \in G$ there exists $i$ such that $v \in V(K_i)$ (we allow single-vertex components). Call $K_1, \ldots, K_n$ the components of $G$.

The subshift of a labelled graph $G$ is the set $\Sigma(G)$ of all $x \in A^\mathbb{Z}$ such that in $G$ there exists a biinfinite walk $\{e_i\}_{i \in \mathbb{Z}}$ such that $l(e_i) = x_i$. It is easy to verify that $\Sigma(G)$ is indeed a subshift of $A^\mathbb{Z}$. Call the language of $\Sigma(G)$ the language of the graph $G$ and denote it by $L(G)$.

A vertex $v$ of a graph $G$ is called stranded iff it does not have at least one outgoing and at least one ingoing edge (a loop counts as both).

A graph $G$ is called essential iff it does not contain stranded vertices.

It is easy to see that for an essential $G$ a (finite) word $u = u_1u_2 \ldots u_k$ belongs to the language $L(G)$ of $G$ iff there exists a walk $e_1e_2 \ldots e_k$ in $G$ such that the edge $e_i$ is labelled by the letter $u_i$. Call such a walk a presentation of $v$ in $G$. Because we allow empty walks, every $L(G)$ also contains the empty word $\lambda$. It can be shown that forgetting all stranded vertices of $G$ does not change the language $L(G)$, so we can safely limit ourselves to essential graphs. See [5, p. 37] for details.

**Proposition 1.** [4, p. 133] The central language $L$ is a language of a sofic subshift iff it can be obtained as the language of some labelled graph.

For the sake of providing context we shall now briefly discuss transitivity and the mixing property.

We say that the subshift $\Sigma$ is transitive, iff for every $\varepsilon > 0$ and every $u, v \in \Sigma$ there exists $k$ and $w \in \Sigma$ such that $\rho(u, w) < \varepsilon$ and $\rho(v, \sigma^k(w)) < \varepsilon$. We say that $\Sigma$ is mixing if for every $u, v \in \Sigma$ and every $\varepsilon > 0$ there exists $n$ such that for every $k > n$ there exists $w$ such that $\rho(u, w) < \varepsilon$ and $\rho(v, \sigma^k(w)) < \varepsilon$.

It can be shown (see [5, p. 80–82, 127–129], note that the authors of [5] use the term “irreducible”) that a sofic subshift $\Sigma$ is transitive iff there exists a connected labelled graph whose subshift is $\Sigma$ and it is mixing iff there exists a connected aperiodic labelled graph whose subshift is $\Sigma$.

The chain-transitivity and chain-mixing properties are weaker properties than transitivity and mixing, respectively.

An $\varepsilon$-chain (in $\Sigma$) of length $n > 0$ from $x^0$ to $x^n$ is a sequence of words $x^0, x^1, \ldots, x^n \in \Sigma$ such that $\rho(\sigma(x^i), x^{i+1}) < \varepsilon$ for all $i = 1, 2, \ldots, n - 1$.

Let $\Sigma$ be a subshift. We say that $\Sigma$ is chain-transitive iff for every words $x, y \in \Sigma$ and every $\varepsilon > 0$ there exists an $\varepsilon$-chain in $\Sigma$ from $x$ to $y$. We say that $\Sigma$ is chain-mixing if for every $x, y \in \Sigma$ and $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that for every $n > k$ there exists an $\varepsilon$-chain in $\Sigma$ of length $n$ from $x$ to $y$.

Let $\Sigma$ be a subshift, $x, y \in \Sigma$. We say that $x$ and $y$ (in this order) are in the chain relation, writing $(x, y) \in C$ if for every $\varepsilon > 0$ there exists an $\varepsilon$-chain from $x$ to $y$.

Obviously, $\Sigma$ is chain-transitive iff $C = \Sigma \times \Sigma$.

As we have a one-to-one correspondence between subshifts and central languages, we may call a language chain-transitive or chain-mixing meaning that its subshift has this property. We characterise these properties in terms of the language $L(\Sigma)$.

**Definition 2.** Let $L$ be a language, $u, v \in L$, $|u| = |v| = m$. The chain of length $n$ from $u$ to $v$ in $L$ is a word $w, |w| = n + m$ with prefix $u$ and suffix $v$ (more precisely, $w_{[1, m]} = u$ and $w_{[m + 1, |w|]} = v$) such that if $z \subseteq w, |z| \leq m$ then $z \in L$.

Note that while $\varepsilon$-chains can not have zero length, we do allow chains of length zero.

**Proposition 3.** The pair $(x, y)$ is in $C$ iff for every $l \in \mathbb{N}$ there exists a chain of nonzero length from $x_{[-l,l]}$ to $y_{[-l,l]}$. 

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Definition 5. Let $G$ be a labelled graph and $v$ be a vertex in $G$. The follower set of $v$ in $G$ is the language $F_G(v)$ of all finite words that have presentations in $G$ that start at vertex $v$. 

Figure 1: The correspondence between $\varepsilon$-chains and chains.

Proof. Let first $(x, y)$ lie in $C$, and let $l \in \mathbb{N}$. Consider $u = x_{[-l,l]}, v = y_{[-l,l]}, \varepsilon = 2^{-l-1}$. There exists an $\varepsilon$-chain $x$ of nonzero length from $u$ to $v$. Now consider the letters $a_i = z^i_{n+1}$. As $\rho(\sigma(z^i), z^{i+1}) < 2^{-l-1}$, it must be $z^i_{[n+1, n+2]} = z^i_{[n+1, n+2]}$.

Let $w = u_{a_1} \ldots a_{n-1}$. Then the word $w_{[i+1, i+2l+1]}$ is equal to $z^i_{[-l,l]}$ (see Figure 1) for every $i = 0, 1, \ldots, n$ and so all the factors of $w$ of length $2l + 1$ belong to $L$. Because $v = z^0_{[-l,l]}$, the word $ua_1 \ldots a_{n-1}$ is a chain from $u$ to $v$. Notice that the length $n > 0$ of the original $\varepsilon$-chain from $x$ to $y$ is equal to the length of the constructed chain from $u$ to $v$.

On the other hand, let without loss of generality $\varepsilon = 2^{-l+1}$ for some $l$ and assume that there exists a chain of nonzero length from $u = x_{[-l,l]}$ to $v = y_{[-l,l]}$. We shall prove that there exists an $\varepsilon$-chain from $x$ to $y$.

Let $w$ be the chain from $u$ to $v$ of length $n > 0$. Set $z^0 = x, z^n = y$. Denote by $z^i, i = 1, 2, \ldots, n - 1, \varepsilon$ an arbitrary infinite extension of the word $w_{[i+1, i+2l+1]}$ in $\Sigma$ (such an extension exists because $w_{[i+1, i+2l]} \in L, L$ is central and $\Sigma$ is compact) with $w_{[i+1, i+2l+1]} = z^i_{[-l,l]}$. Note that $x$ and $y$ are extensions of $u$ and $v$ respectively. We want to check that $x = z^0, z^1, \ldots, z^n = y$ is an $2^{l+1}$-chain:

$z^i_{[-l+2l]} = w_{[i+3, i+2l+1]} = z^{i+1}_{[-l+2l]}$ and thus $\rho(\sigma(z^i), z^{i+1}) < 2^{l+1}$. Again note that the length of the produced $\varepsilon$-chain is equal to the length of the chain $w$.

Corollary 4. Let $L = \mathcal{L}(\Sigma)$.

1. The subshift $\Sigma$ is chain-transitive iff for every $u, v \in L, |u| = |v| = m$ there exists a chain of nonzero length from $u$ to $v$

2. The subshift $\Sigma$ is chain-mixing iff for every $u, v \in L, |u| = |v| = m$ exists $k \in \mathbb{N}$ such that for all $n > k$ there is a chain of length $n$ from $u$ to $v$.

Proof. The first claim directly follows from Proposition 3. To prove the second claim it is sufficient to notice that the length of $\varepsilon$-chain and of the corresponding chain construed in the proof of Proposition 3 is the same. 

3 The Linking Graph

The purpose of this section is to define the linking graph which provides useful tool for the study of the chain relation.
If $v$ is a vertex of $G$ belonging to a component $K$, we shall call the set $F_K(v)$ the restricted follower set and denote it by $F(v)$.

**Definition 6.** Let $G$ be a graph and $v, w \in V(G)$. We say that the vertices $v, w$ are **linked** iff $|F(v) \cap F(w)| = \infty$ (note that we consider restricted follower sets here) and write $v \sim w$. Denote by $\sim$ the transitive and reflexive closure of $\sim$.

**Lemma 7.** The vertices $v$ and $w$ are linked iff for any positive integer $n$ there exists a word $z \in F(v) \cap F(w), |z| = n$.

**Proof.** First observe that if $yz \in F(v)$ then $y \in F(v)$ as we can simply forget the ending of the presentation of $yz$ to obtain a presentation of $y$.

If now $|F(v) \cap F(w)| = \infty$ then $F(v) \cap F(w)$ must contain arbitrarily large words as $A$ is finite. For a given $n$, consider $y \in F(v) \cap F(w), |y| > n$. There exist $z_1, z_2$ such that $|z_1| = n$ and $z_1z_2 = y$.

Thanks to the above remark we have that $z_1 \in F(v) \cap F(w)$ and we are done.

On the other hand, if $F(v) \cap F(w)$ contains word $z_1$ of length 1, $z_2$ of length 2 and so on, then there is an infinite subset $\{z_1, z_2, \ldots \}$ of $F(v) \cap F(w)$ and thus $|F(v) \cap F(w)| = \infty$.

**Definition 8.** Let $G$ be a labelled graph. The graph $G/\approx$, called the linking graph of $G$, is a graph obtained from $G$ by joining all pairs of linked vertices. More precisely: The set $V(G/\approx)$ is the set of all equivalence classes of $\approx$ on $V(G)$. An edge $e$ (labelled by the letter $a$) goes from a vertex $x$ to a vertex $y$ in $G/\approx$ iff there exist vertices $u \in x, v \in y$ such that an edge $f$ labelled by $a$ goes from $u$ to $v$ in $G$.

In a labelled graph, a word might have multiple presentations. Define the projection $\pi : V(G) \to V(G/\approx)$ that assigns to every $v \in V(G)$ the equivalence class of $v$ in $\approx$. We want to show that, for $v$ long enough, the images under $\pi$ of every pair of presentations of $v$ intersect in a nice way.

In the following, let $c$ be the number of components of $G$.

**Lemma 9.** Let $t$ be a positive integer. Let the (nonempty) graph $G$ consist of $c$ components and let $k \geq (t+1)c^2 + c$. Let $e_1, \ldots, e_k$ and $f_1, \ldots, f_k$ be two walks in $G$. Then there exist two components $M_e, M_f$ of $G$ and a positive integer $r \leq k - t + 1$ such that $e_{i+r} \in M_e$ and $f_{i+r} \in M_f$ for every $i \in \{0, 1, 2, \ldots, t-1\}$.

**Proof.** To prove the lemma, we shall use the pigeonhole principle. Without loss of generality, assume $k = (t+1)c^2 + c$. Since there are $c$ components, at most $c - 1$ edges are spent traversing between components. Thus there exists an interval $e_s, e_{s+1}, \ldots, e_{s+c(t+1)-1}$ of the walk $e_1, \ldots, e_k$ of length $c(t+1)$ that passes through only one component $M_e$. 

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Figure 2: An example of a linking graph.
Consider the walk \( f_s, f_{s+1}, \ldots, f_{s+c(t+1)-1} \). This is a walk of length \( tc + c \) and thus, using the above argument again, there exists an interval \( f_r, f_{r+1}, \ldots, f_{r+t-1} \) going through only one component \( M_f \). But then \( e_{r+i} \in E(M_r) \) and \( f_{r+i} \in E(M_f) \) for \( i = 0, 1, \ldots, t-1 \).

**Corollary 10.** For every (nonempty) labelled graph \( G \) there exists a length \( l \) such that if \( w \in \mathcal{L}(G) \) is of length at least \( l \) and \( v_1e_1v_2e_2\ldots e_{|w|}v_{|w|+1} \) and \( v_1'e_1'v_2'e_2'\ldots e_{|w|}'v_{|w|+1}' \) are two presentations of \( w \) in \( G \), then there exists \( r \) such that \( \pi(v_r) = \pi(v_r') \) (recall that \( \pi \) is the projection from \( G \) to \( G/\sim \)).

**Proof.** If \( w_1, w_2 \in V(G) \) are not linked then there exists a finite \( m_{w_1, w_2} \) such that \( |F_K(w_1) \cap F_L(w_2)| = m_{w_1, w_2} \). Take \( m = 1 + \max\{m_{w_1, w_2} : w_1, w_2 \in V(G) \text{ not linked} \} \). It is then easy to see that any two vertices \( u, v \in V(G) \) are linked iff the intersection of their restricted follower sets contains a word of length \( m \).

Take \( l = (m+1)c^2 + c \), and use Lemma 9 for the two presentations of \( w \). We obtain that there exists \( r \) such that \( v_r \) and \( v_r' \) both contain the word \( w|_{r+r+m-1} \) in their restricted follower sets. Thus \( v_r \) and \( v_r' \) must be linked and so \( \pi(v_r) = \pi(v_r') \).

The components of \( G/\sim \) are partially ordered by the relation “the component \( K \) can be reached from the component \( L \)”. Denote this relation by \( K \geq L \).

**Definition 11.** We call a sequence \( z_0, \ldots, z_n \) of vertices of \( G \) a generalised walk if there exist edges \( e_1, e_2, \ldots, e_{n-1} \in E(G) \) such that every \( e_i \) leads from \( z_i \) to some \( z_{i+1} \approx z_{i+1} \).

Notice that every walk in \( G/\sim \) corresponds to a generalised walk in \( G \). Informally, a generalised walk is a sequence of ordinary walks interleaved by occasional “jumps” to equivalent (under \( \sim \)) vertices. Crucial to our proof will be the following lemma describing a way to perform these “jumps”.

**Lemma 12.** Let \( u \approx u' \) be vertices of a labelled graph \( G \). Let \( w \) be a word whose presentation ends in \( u \). Then there exists a word \( w' \) whose presentation ends in \( u' \) such that \( |w| = |w'| \) and there exists a chain from \( w \) to \( w' \).

**Proof.** As \( u \approx u' \), there exists a sequence of linked vertices \( u = v_0 \sim v_1 \sim \cdots \sim v_k = u' \). Let us proceed by induction on \( k \):

1. If \( k = 0 \) then we are done, as it suffices to take \( w = w' \).
2. Assume that the claim is true when $k < n$ for some $n$. Let us have $u = v_0 \sim \cdots \sim v_{n-1} \sim v_n = u'$. By the induction hypothesis, there exists a chain $q$ from $w$ to some $w''$ such that one presentation of $w''$ ends in the vertex $s = v_{n-1}$.

It is $s \sim u'$, so there exists $z \in F(s) \cap F(u'), |z| = |w''|$. Because $z \in F(u')$, there exists a presentation of $z$ beginning in $u'$ that does not leave the component of $u'$. Call $t$ the ending vertex of this presentation. There exists a walk from $t$ back to $u'$ which presents some word $r$ (see Figure 4). Consider now the sequence $w''zr$. It is easy to verify that this is a chain from $w''$ to $w' = (w''zr)(|w''zr| - |w''| + 1, |w''zr|)$ and that $w'$ has a presentation ending in $u'$. But then $qzr$ is a chain from $w$ to $w'$, ending the proof.

\[\square\]

**Remark.** If $x$ is an infinite word and $p, q$ two presentations of $x$ then the images $\pi(p)$ and $\pi(q)$ of $p$ and $q$ under the projection $\pi$ both begin in the same component $\alpha(x)$ and end in the same component $\omega(x)$ of $G/\approx$.

**Proof.** Let $K, L$ be the components of $G/\approx$ where $\pi(p), \pi(q)$, respectively, end. These components are guaranteed to exist because $G/\approx$ is finite; they are the maximal (under the ordering $\leq$) components visited by $\pi(p)$ resp. $\pi(q)$. There exists $n$ such that the intervals $[n, \infty)$ of $\pi(p), \pi(q)$ both stay in $K, L$. But due to Corollary 10 there must exist some $v \in p_{[n, \infty)}$ and $v' \in q_{[n, \infty)}$ such that $\pi(v) = \pi(v')$ and so $K = L$. Similar argument holds for the beginnings of $p, q$. \[\square\]

**Remark.** Let $G$ be a nonempty labelled graph. Let $H$ be a subgraph of $G/\approx$ and let $K = \pi^{-1}(H)$. Denote by $L$ the subgraph of $G$ induced by the vertices not in $K$ (see Figure 5). Then there exists $m$ such that no word of length at least $m$ may simultaneously have a presentation in $K$ and in $L$.

**Proof.** Let $m$ be equal to the constant from Corollary 10. This Corollary tells us that if some $u$ of length $m$ had a presentation in both $K$ and $L$ then there would exist two vertices $u \in K, v \in L$ such that $\pi(u) = \pi(v)$. But the whole set $\pi^{-1}(u)$ must lie either in $K$ or in $L$, a contradiction. \[\square\]

**Lemma 13.** Let $H$ be a terminal subgraph of $G/\approx$, let $K, L$ and $m$ be as in Remark 8. Let $u$ be a word of length $l \geq 2m + 1$ with a presentation in $K$ and let there lead a chain from $u$ to some $v$. Then $v_{[m+1,l]}$ has a presentation in $K$.

**Proof.** Take the chain $w$ of length $n$ from $u$ to $v$. We prove this lemma by mathematical induction on $n$.

1. For $n = 0$ the claim is trivial.

![Figure 4: Getting from $w''$ to $w'$.](image-url)
2. Let all lengths smaller than \( n \) satisfy the condition. Observe that \( w^{[1,n+m]} \) is a chain from \( u \) to \( t = w^{[n+m+1-l,n+m]} \) of length \( n + m - l < n \) (because \( l > m \)). Thanks to the induction hypothesis we know that \( t^{[m+1,l]} \) has a presentation in \( K \). It is \( v = w^{[n+1,n+l]} \) and so \( v^{[1,m]} = w^{[n+1,n+m]} = t^{[n-m+l]} \) has a presentation in \( K \) (here we use that \( n-m+1 > m+1 \)).

Denote \( v^{[1,m]} \) by \( z \). Because \( |z| = m \) and \( z \) has a presentation in \( K \) then \( z \) must not have a presentation in \( L \). But then all presentations of \( z \) must end in some vertex of \( K \): Any walk in \( G \) which enters the terminal subgraph \( K \) is not be able to leave \( K \). (Had \( z \) a presentation beginning and ending in \( L \) then the whole presentation would lie in \( L \)).

Because it is \( v = zv^{[m+1,l]} \) we see that any presentation of \( v \) must enter \( K \) after at most \( m \) edges. Taking any presentation of \( v \) in \( G \) then gives us a presentation of \( v^{[m+1,l]} \) in \( K \).

Now comes the core theorem of this section that uses all the above results to describe the relation \( C \).

**Theorem 14.** Let \( \Sigma \) be a sofic subshift, \( G \) a labelled graph, \( \Sigma = \Sigma(G) \). Let \( x, y \in \Sigma \). Then \( (x, y) \in C \) iff \( \omega(x) \leq \alpha(y) \) or \( y = \sigma^n(x) \) for some \( n > 0 \).

**Proof.** Obviously, if \( y = \sigma^n(x) \) for \( n > 0 \) then \( (x, y) \in C \). Assume now that \( \omega(x) \leq \alpha(y) \). Using Proposition 5 it suffices to prove that there exists a chain from \( w = x^{[1,l]} \) to \( w' = y^{[1,l]} \) for any \( l \in \mathbb{N} \).

Let \( u \) be the ending vertex of some presentation of \( w \) and \( u' \) be the starting vertex of some presentation of \( w' \). As \( \omega(x) \leq \alpha(y) \), there exists a generalised walk \( u = z_0, z_1, \ldots, z_{n-1}, z_n = u' \) in \( G \). We want to take this walk and turn it into a chain. More precisely, we want to find a chain from \( w \) to some \( w'' \) such that one presentation of \( w'' \) ends in \( u' \). As \( w''w' \) is a chain from \( w'' \) to \( w' \), by composing both chains we get a chain from \( w \) to \( w' \) of nonzero length (see Figure 7).
exists a chain of nonzero length from prefix and suffix, \(\{w\}\). Since a chain for \(\ell_1\) if \(n_2\). Let the theorem hold for \(n\). We want to prove that then \(\alpha\) belongs to \(\Sigma^+\). Here we use that \(\ell_2\) to get that the whole word \(M\) only one component \(\alpha\) to \(y\). The word \(v\) has a presentation that ends in \(z'\) and, using Lemma 12 again, we get that there exists a chain from \(v'\) to some \(w''\) whose presentation ends in \(z_n = u'\).

In the other direction, let \((x, y) \in C\). Using Proposition 3 we get that for every \(l > 0\) there exists a chain of nonzero length from \(x_{[-t, l]}\) to \(y_{[-t, l]}\). Denote by \(n_l\) the minimum (nonzero) length of such a chain. Since a chain for \(l\) can be easily obtained from a chain for \(k > l\) by forgetting the prefix and suffix, \(\{n_l\}_{l=1}^{\infty}\) is a nondecreasing sequence. We shall distinguish two cases:

1. Let the sequence be bounded and let \(n = \max\{n_l | l \in \mathbb{N}\}\). We claim that then \(y = \sigma^n(x)\). Indeed, for all but finitely many values of \(l\) we have \(n_l = n\) and if a chain of length \(n\) leads from \(x_{[-t, l]}\) to \(y_{[-t, l]}\) then \(\rho(\sigma^n(x), y) < 2^{n-l}\) (see Figure 8). Thus \(\rho(\sigma^n(x), y) = 0\) and \(y = \sigma^n(x)\).

2. Let the sequence be unbounded. We want to prove that then \(\alpha(y) \geq \omega(x)\).

Let \(p\) be a presentation of \(x \in G\) and let \(h\) be an index such that the walk \(p_{[h, \infty[}\) belongs to only one component \(M\) of \(G\). Without loss of generality assume that \(h > 0\). Take an arbitrary \(l > 0\). As the sequence \(\{n_k\}_{k=1}^{\infty}\) is not bounded, there exists \(k \geq h + 2l\) such that \(n_k > l + h\). The chain from \(x_{[-k, k]}\) to \(y_{[-k, k]}\) contains as a factor the chain from \(x_{[h, h+2l]}\) to \(y_{[-l, l]}\). Here we use that \(k \geq h + 2l\) to ensure that \(x_{[h, h+2l]} \subseteq x_{[-k, k]}\) and \(n_k > l + h\) to ensure that the length of the chain is positive. See Figure 8.

For \(l\) sufficiently large, we can use Lemma 13 with \(u = x_{[h, h+2l]}\), \(v = y_{[-l, l]}\) and \(H\) equal to the minimal terminal subgraph of \(G/\omega\) containing \(\omega(x)\), obtaining that \(y_{[m-l, l]}\) has a presentation in the graph \(K = \pi^{-1}(H)\). Because \(m\) is a constant and \(\Sigma(K)\) is compact we get that the whole word \(y\) belongs to \(\Sigma(K)\).
Using Theorem 14, we see that if \( G \) is chain-transitive iff \( \alpha(y) \geq \omega(x) \) as one presentation of \( y \) begins in \( K \).

**Corollary 15.** Let \( L = \mathcal{L}(G) \) be a central regular language, \( G \) an essential labelled graph. Then, \( L \) is chain-transitive iff \( G/\approx \) is connected.

**Proof.** Using Theorem 13 we see that if \( G/\approx \) is connected then \( L \) must be chain-transitive.

In the other direction, assume by contradiction that \( K \) is a terminal component of \( G/\approx \), \( L \) is an initial component of \( G/\approx \) and \( K \neq L \). The preimages \( \pi^{-1}(K) \) resp. \( \pi^{-1}(L) \) must contain at least one terminal resp. initial component of \( G \). As \( G \) is essential, there must exist \( x \in \Sigma(\pi^{-1}(K)) \) and \( y \in \Sigma(\pi^{-1}(L)) \). Were \( y = \sigma^n(x) \) we would get a contradiction with Remark 3 and so, using Theorem 13 we get \( K \leq L \), a contradiction. Thus \( K = L \) and \( G \) is connected. \( \square \)

### 4 Chain-Mixing Sofic Subshifts

We have seen that (for \( G \) essential) we can translate the question of chain-transitivity to a question about the structure of \( G/\approx \). In this section we characterise the chain-mixing property using the structure of \( G/\approx \).

**Theorem 16.** Let \( L = \mathcal{L}(G) \) be a central regular language, \( G \) essential. Then, \( L \) is the language of a chain-mixing subshift iff \( G/\approx \) is connected and aperiodic.

**Proof.** First let us prove the necessity of the conditions. Let \( L \) be chain-mixing. Then \( L \) is also chain-transitive and, as follows from Corollary 15, \( G/\approx \) must be connected.

Let us now assume that \( G/\approx \) is periodic with a period \( n > 1 \). Let \( w \in \mathcal{L}(G) \) be a word that is long enough to satisfy the conditions of Corollary 10. We claim that then all ending vertices of all presentations of \( w \) must be projected to the same partition \( V_i \) of \( G/\approx \). Let \( v_1v_2\ldots v_{|w|} \) and \( v_1'v_2'\ldots v_{|w|}' \) be two walks presenting \( w \). Then Corollary 10 tells us that there exists \( r \) such that \( v_r \equiv v_r' \). But then \( \pi(v_r) = \pi(v_r') \in V_j \) for some \( j \) and thus \( \pi(v_{r+1}), \pi(v'_{r+1}) \in V_{j+1} \) and \( \pi(v_{r+2}), \pi(v'_{r+2}) \in V_{j+2} \) and so on, ending with \( \pi(v_{|w|}), \pi(v'_{|w|}) \in V_{(j+|w|-r) \mod n} \).

Take some \( w \) such that \( |w| = l + 1 \) where \( l \) is the constant from Corollary 10. Let \( z \) be a chain from \( w \) to \( w \) of length \( m \). We claim that then \( n|m \). This will be a contradiction to the chain-mixing property (via Corollary 3). Without loss of generality, let \( V_0 \) contain all the end vertices of all presentations of \( w \). Then \( z_{[i,i+l]} \) and \( z_{[i+1,i+1+l]} \) share a factor of length \( l \) and so, due to the above remark, if presentations of \( z_{[i,i+l]} \) all end in \( V_i \) then presentations of \( z_{[i+1,i+1+l]} \) all end in \( V_{(i+1) \mod n} \).

In the opposite direction, assume we have a labelled graph \( G \) such that \( G/\approx \) is connected and aperiodic and \( L = \mathcal{L}(G) \). We want to show that \( L \) is chain-mixing. We begin by showing that for any word \( w \) there exists a \( k \) such that we can find chains of any length \( n > k \) from \( w \) to \( w \).

Take the least common divisor \( d \) of the lengths of all chains from \( w \) to \( w \). Notice that all closed walks in \( G \) must have lengths divisible by \( d \), otherwise we could easily produce chains from \( w \) to \( w \) of lengths not divisible by \( d \) (\( L \) is chain-transitive).

\[\text{Figure 9: The chain from } x_{[-k,k]} \text{ to } y_{[-k,k]} \text{ and the chain from } x_{[h,h+2l]} \text{ to } y_{[-l,l]}\]
Assume that \( d > 1 \). Let \( v \) be the ending vertex of one presentation of \( w \). As \( G / \sim \) is aperiodic and connected, there exists a generalised walk in \( G \) of some length \( l \) not divisible by \( d \) from \( v \) back to \( v \). Using the same algorithm as in the first part of proof of Theorem 13 we obtain a chain from \( w \) to \( w \). Each closed walk in \( G \) has length divisible by \( d \) and in the proof of Theorem 13 we have produced the chain from the generalised walk by adding only words presented by closed walks. This means that the length of our chain gives the same remainder when divided by \( d \) as the length \( l \) of the corresponding generalised walk. Thus we get a chain from \( w \) to \( w \) of length not divisible by \( d \), a contradiction.

If \( d = 1 \) then there must exist chains from \( w \) to \( w \) of lengths \( l_1, l_2, \ldots, l_p \) such that the least common divisor of \( l_1, \ldots, l_p \) is 1.

**Lemma 17.** Let \( \{l_1, \ldots, l_p\} \) be a set of positive integers whose greatest common divisor is 1. Then there exists \( k \) such that every \( n > k \) can be written as \( n = r_1l_1 + r_2l_2 + \cdots + r_pl_p \) where \( r_i \) are positive integers or zeroes.

**Proof.** First notice that for some integers \( s_i \) it is \( 1 = s_1l_1 + s_2l_2 + \cdots + s_pl_p \) because \( \mathbb{Z} \) is a principal ideal domain. Set \( m = |s_1|l_1 + |s_2|l_2 + \cdots + |s_p|l_p \).

Let \( k = ml_1 \). If now \( n = q \cdot l_1 + t \) where \( t \in \{0, 1, 2, \ldots, l_1 - 1\} \) and \( q \geq m \) then it is

\[
\begin{align*}
n &= (q-m)l_1 + ml_1 + ts_l_1 + s_2l_2 + \cdots + s_pl_p \\
n &= (q-m)l_1 + \sum_{i=1}^{p}(ts_i + |s_i|l_1l_i)
\end{align*}
\]

Letting \( r_1 = q - m + ts_1 + l_1|s_1| \) and \( r_i = ts_i + l_i|s_i| \) for \( i = 2, 3, \ldots, p \) we obtain \( r_i \geq 0 \) such that

\[n = \sum_{i=1}^{p} r_il_i.\]

Using Lemma 17 we see that for any \( n > k \) we can compose the chains from \( w \) to \( w \) to obtain a chain of length \( n \).

Having found chains from \( w \) to \( w \) of any length \( n > k \) we want to find chains from \( w \) to \( w \) to some arbitrary \( z \). As \( L \) is chain-transitive, there exists a chain from \( w \) to \( z \) of length \( k' \). By composing this chain with a suitable chain from \( w \) to \( w \) of length \( n > k \) we can obtain a chain from \( w \) to \( z \) of any length \( n' = n + k' > k + k' \). As we can do this (with different \( k, k' \)) for all \( w, z \), the language \( L \) must be chain-mixing. \( \Box \)

## 5 Attractors of Sofic Subshifts

In this section we use the linking graph to characterise all the attractors of the dynamical system \((\Sigma, \sigma)\) when \( \Sigma \) is a sofic subshift. There are several slightly different definitions of attractor. We shall use the following one (from [3]):

**Definition 18.** Denote by \( d(x, Y) \) the distance of the point \( x \in X \) from the set \( Y \subset X \) in \( X \). An **attractor** \( Y \) of a dynamical system \((X, F)\) is a nonempty closed subset of \( X \) that satisfies the following:

1. \( F(Y) = Y \)
2. \( \forall \epsilon > 0, \exists \delta > 0, \forall x \in X, d(x, Y) < \delta \Rightarrow \forall n > 0, d(F^n(x), Y) < \epsilon \)
3. \( \exists \delta > 0, \forall x \in X, d(x, Y) < \delta \Rightarrow \lim_{n \to \infty} d(F^n(x), Y) = 0. \)

There are several theorems that show the correspondence between attractors and the chain relation. We use the following theorem.

**Theorem 19.** [4, p. 82] Let \( \Omega \) be an attractor. Then \( \Omega \) is chain-invariant, i.e. \( \forall z \in \Omega, (z, y) \in C \Rightarrow y \in \Omega. \)

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Theorem 20. Let $G$ be an essential labelled graph. Let $H$ be a nonempty terminal subgraph of $G/\approx$. Then the set $\Omega = \Sigma(\pi^{-1}(H))$ is an attractor of $(\Sigma(G), \sigma)$. Moreover, all attractors of $(\Sigma(G), \sigma)$ are of this type.

Proof. We shall first prove that $\Omega$ is indeed an attractor. Obviously, it is a subshift and so it is shift-invariant and closed. As $\pi^{-1}(H)$ is nonempty and essential, $\Omega$ is also nonempty.

Because $H$ is a terminal subgraph of $G/\approx$, we can utilise Remark 3 and obtain $m$ such that if $|v| \geq m$ then $v$ may not lie both in the language of $\pi^{-1}(H)$ and the language of $G \setminus \pi^{-1}(H)$.

Without loss of generality, let $\varepsilon = 2^{-k}, k \geq 0$. Let $\delta = 2^{-m-k}$. If $d(x, \Omega) < \delta$ then $x_{[-m-k,-k-1]}$ does not belong to the language of $G \setminus \pi^{-1}(H)$. Thus all presentations of $x_{[-m-k,-k-1]}$ end in a vertex of $\pi^{-1}(H)$ and $x_{[-k,\infty]}$ has a presentation in $\pi^{-1}(H)$. Then $\sigma^n(x)_{[-k,k]} = x_{[-k+n,k+n]}$ belongs to the language $L(\pi^{-1}(H))$ and so $d(\sigma^n(x), \Omega) < \varepsilon$ for all $n > 0$.

Similarly, letting $\delta = 2^{-m}$ yields that if $d(x, \Omega) < \delta$ then $\sigma^n(x)_{[-n,n]} = x_{[0,2n]}$ belongs to $L(\pi^{-1}(H))$ and so $d(\sigma^n(x), \Omega) < 2^{-n}$ for all $n$. As $n$ tends to infinity then $d(\sigma^n(x), \Omega)$ tends to zero, concluding the proof that $\Omega$ is an attractor.

Let now $\Omega$ be an attractor of the subshift system. Let $x \in \Omega$ and take a presentation $p$ of $x$ in $G$. Let $K$ be the component of $G$ where $p$ starts. Let $H$ be the terminal subgraph of $G/\approx$ generated by $\pi(K)$ (subgraph induced by all vertices that can be reached from $\pi(K)$). We claim that $\Sigma(\pi^{-1}(H)) \subseteq \Omega$.

To prove this claim, we just have to prove that there exists $z \in \Sigma(K) \cap \Omega$ and use Theorem 14 and the chain-invariance of attractors (Theorem 9). Then $\Sigma(\pi^{-1}(H))$ is precisely the set of words of $\Sigma$ that $z$ is in the chain relation with and so $\Sigma(\pi^{-1}(H)) \subseteq \Omega$.

As $A^2$ is a compact space and $\Omega$ is a closed subspace of $A^2$, $\Omega$ is a compact set. Thus any sequence in $\Omega$ has an accumulation point. Because $\Omega$ is $\sigma$-invariant it contains all the words $x_n = \sigma^{-n}(x), n \in \mathbb{N}$. This sequence has an accumulation point $z \in \Omega$. For any $m > 0$ there exists $k > 0$ such that $x_{[-\infty,-k+m]}$ has a presentation in $K$ and $x_{[-m,m]} = x_{[-k-m,-k+m]}$, thus $x_{[-m,m]}$ has a presentation in $K$ for any $m$. This means that $z \in \Sigma(K)$. But then $\Omega$ is a union of subshifts of the form $\Sigma(\pi^{-1}(H_z))$ where $H_z$ are terminal subgraphs of $G/\approx$ and $x \in \Omega$. While it is not in general true that $\Sigma(G) \cup \Sigma(H) = \Sigma(G \cup H)$, in this special case we can use to our advantage the fact that all $H_z$ are terminal and so all walks in $H$ are contained in at least one $H_z$. Thus by taking $H = \bigcup H_z$ we get $\Omega = \Sigma(\pi^{-1}(H))$ where $H$ is a terminal subgraph of $G/\approx$.

6 Algorithmic checking of properties

In this section we describe the algorithms that, given an essential labelled graph $G$, construct the graph $G/\approx$ and check whether $\Sigma(G)$ is chain-transitive or chain-mixing. As there is presently little need for practical implementations of such algorithms, we provide only very brief descriptions.

The construction of $G/\approx$ The proposed algorithm is quite straightforward: It first finds all pairs of linked vertices and then joins such pairs of vertices together. We shall use the double depth-first search algorithm from [1] p. 489 that finds all components of a given graph in time $O(|V| + |E|)$.

Definition 21. Given two labelled graphs $G, H$ we can construct their label product $G \ast H$: a graph with the vertex set $V(G) \times V(H)$ and edge set $\{(e, f) : e \in E(G), f \in E(H), l(e) = l(f)\}$. That is, an edge leads from $(u, v)$ to $(u', v')$ iff the edges from $u$ to $u'$ and from $v$ to $v'$ exist and have the same labels.

Algorithm 6.1. Given $G$ construct $G/\approx$.

1. Find all components $C_1, \ldots, C_k$ of $G$.

2. Construct the label products $C_i \ast C_j$ of all pairs of components.
3. For each \(i \leq j\) find all components of \(C_i \ast C_j\) that contain an oriented cycle. Paint these components red. Let \(G_{ij}\) be the subgraph of \(C_i \ast C_j\) induced by the set of vertices

\[
V(G_{ij}) = \{v \in V(C_i \ast C_j) | \text{there exists a walk from } v \text{ to some red component}\}.
\]

4. It is \(u \sim v\) iff \((u, v) \in V(G_{ij})\) for a suitable \(i, j\).

5. Join together all pairs of linked vertices in \(G\) to obtain \(G/\approx\).

To prove the correctness of the algorithm, it suffices to show that \((u, v) \in V(G_{ij})\) iff \(u \sim v\) for \(u \in V(C_i), v \in V(C_j)\).

Proof. If \(u \sim v\) then there exists a word \(w\) of length \(n = |V(G)|^2 + 1\) with a presentation \(e_1 \ldots e_n\) in \(C_i\) starting in \(u\) and a presentation \(e'_1 \ldots e'_n\) in \(C_j\) starting in \(v\). Then \((e_1, e'_1)(e_2, e'_2) \ldots (e_n, e'_n)\) is a walk in \(C_i \ast C_j\). Because \(C_i \ast C_j\) has at most \(|V(G)|^2\) vertices, this walk must return to an already visited vertex at least once and that is only possible in a red component. Thus \((u, v) \in V(G_{ij})\).

On the other hand, if \((u, v) \in V(G_{ij})\) then we can find arbitrarily long walks that start at \((u, v)\). Let \((e_1, e'_1) \ldots (e_n, e'_n)\) be such a walk of length \(n\). Then it is \(l(e_i) = l(e'_i)\) for all \(i\) and so \(e_1 \ldots e_n\) and \(e'_1 \ldots e'_n\) are two presentations of the same word starting in \(u\) and \(v\). We can do this for any \(n\) and so \(u \sim v\).

Checking for chain transitivity Using the algorithm from [4] p. 489] again we can easily check whether \(G/\approx\) is connected.

Checking the chain-mixing property We need to check that \(G/\approx\) is connected and find the period of \(G/\approx\). This can be done in linear time using breadth-first search as described in [3].

Attractors It suffices to write down all nonempty terminal subgraphs of \(G/\approx\). Since the number of such subgraphs may in general be exponential in the number of components of \(G/\approx\), there can be no fast algorithm. A backtracking algorithm can be used here.

Complexity of the algorithms The first three algorithms run in polynomial time. The creation of \(G/\approx\) demands the most time while the checks of connectivity and aperiodicity both run in time \(O(|V| + |E|)\).

Unfortunately, the number of terminal subgraphs may be exponential to the size of the input graph so outputting all the attractors of the shift dynamic system is in general not very practical for graphs with many components.

7 Conclusions

In this paper, we have introduced and used the notion of linking graph to better understand sofic subshifts of \(A^Z\). It turns out that we can characterise the relation \(\mathcal{C}\) and all attractors of \(\Sigma\) using the properties of \(G/\approx\).

We have proposed straightforward algorithms to decide in polynomial time (to the size of graph \(G\) describing \(\Sigma\)) whether a sofic subshift \(\Sigma\) is chain-mixing or chain-transitive and an algorithm that, given an essential graph \(G\), lists (generally not in polynomial time) all the attractors of \((\Sigma(G), \sigma)\).

It is interesting that while chain-transitivity can be decided in polynomial time, deciding transitivity is co-NP hard, as was recently shown in [6]. The cause of this contrast seems to be that when deciding transitivity, we have (explicitly or implicitly) to decide whether for \(H, H'\) graphs it is \(\Sigma(H) \subseteq \Sigma(H')\), a hard question, while deciding chain-transitivity requires merely that we decide if \(\Sigma(H) \cap \Sigma(H') \neq \emptyset\), an easy question.

Linking graphs have proven useful in describing the properties of \(\Sigma\) that depend mainly on \(\mathcal{C}\). Other properties of linking graphs might be a nice subject for further research.
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