On the normal approximation for random fields via martingale methods

Magda Peligrad and Na Zhang

Department of Mathematical Sciences, University of Cincinnati, PO Box 210025, Cincinnati, Oh 45221-0025, USA.
email: peligrm@ucmail.uc.edu
email: zhangn4@mail.uc.edu

Abstract

We prove a central limit theorem for strictly stationary random fields under a sharp projective condition. The assumption was introduced in the setting of random sequences by Maxwell and Woodroofe. Our approach is based on new results for triangular arrays of martingale differences, which have interest in themselves. We provide as applications new results for linear random fields and nonlinear random fields of Volterra-type.

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1 Introduction

Martingale methods are very important for establishing limit theorems for sequences of random variables. The theory of martingale approximation, initiated by Gordin (1969), was perfected in many subsequent papers. A random field consists of multi-indexed random variables \((X_u)_{u \in \mathbb{Z}^d}\). The main difficulty when analyzing the asymptotic properties of random fields, is the fact that the future and the past do not have a unique interpretation. Nevertheless, it is still natural to try to exploit the richness of the martingale techniques. The main problem consists of the construction of meaningful filtrations. In order to overcome this difficulty mathematicians either used the lexicographic order or introduced the notion of commuting filtration. The lexicographic order appears in early papers, such as in Rosenblatt (1972), who pioneered the field of martingale approximation in the context of random fields. An important result was obtained by Dedecker (1998) who pointed out an interesting projective criteria for random fields, also based on the lexicographic order. The lexicographic order leads to normal approximation under projective conditions with respect to rather large, half-plane indexed sigma algebras. In order to reduce the size of the filtration used in projective conditions, mathematicians introduced the so-called commuting filtrations. The traditional way for constructing commuting filtrations is to
consider random fields which are functions of independent random variables. We would like to mention several remarkable recent contributions in this direction by Gordin (2009), El Machkouri et al. (2013), Volný and Wang (2014), and Cuny et al. (2015), who provided interesting martingale approximations in the context of random fields. It is remarkable that Volný (2015) imposed the ergodicity conditions to only one direction of the stationary random field. Other recent results involve interesting mixing conditions such as in the recent paper by Bradley and Tone (2017).

In this paper we obtain a central limit theorem for random fields, for the situation when the variables satisfy a generalized Maxwell-Woodroofe condition. This is an interesting projective condition which defines a class of random variables satisfying the central limit theorem and its invariance principle, even in its quenched form. This condition is in some sense minimal for this type of behavior as shown in Peligrad and Utev (2005). Its importance was pointed out, for example, in papers by Maxwell and Woodroofe (2000), who obtained a central limit theorem (CLT); Peligrad and Utev (2005) obtained a maximal inequality and the functional form of the CLT; Cuny and Merlevède (2014) obtained the quenched form of this invariance principle. The Maxwell-Woodroofe condition for random fields was formulated in Wang and Woodroofe (2013), who also pointed out a variance inequality in the context of commuting filtrations.

Compared to the paper of Wang and Woodroofe (2013), our paper has double scope. First, to provide a central limit theorem under generalized Maxwell-Woodroofe condition that extends the original result of Maxwell and Woodroofe (2000) to random fields. Second, to use more general random fields than Bernoulli fields. Our results are relevant for analyzing some statistics based on repeated independent samples from a stationary process.

The tools for proving these results will consist of new theorems for triangular arrays of martingales differences which have interest in themselves. We present applications of our result to linear random fields and nonlinear random fields, which provide new limit theorems for these structures.

Our results could also be formulated in the language of dynamical systems, leading to new results in this field.

2 Results

Everywhere in this paper we shall denote by $\| \cdot \|$ the norm in $L^2$. By $\Rightarrow$ we denote the convergence in distribution. In the sequel $|x|$ denotes the integer part of $x$. As usual, $a \wedge b$ stands for the minimum of $a$ and $b$.

Maxwell and Woodroofe (2000) introduced the following condition for a stationary processes $(X_i)_{i \in \mathbb{Z}}$, adapted to a stationary filtration $(\mathcal{F}_i)_{i \in \mathbb{Z}}$:

$$\sum_{k \geq 1} \frac{1}{k^{3/2}} \| E(S_k | \mathcal{F}_1) \| < \infty, \quad S_k = \sum_{i=1}^k X_i, \tag{1}$$

and proved a central limit theorem for $S_n/\sqrt{n}$. In this paper we extend this result to random fields.
For the sake of clarity we shall explain first the extension to random fields with double indexes and, at the end, we shall formulate the results for general random fields.

We shall introduce a stationary random field adapted to a stationary filtration. For constructing a flexible filtration it is customary to start with a stationary random field \((\xi_{n,m})_{n,m \in \mathbb{Z}}\) and to introduce another stationary random field \((X_{n,m})_{n,m \in \mathbb{Z}}\) defined by

\[ X_{n,m} = f(\xi_{i,j}, i \leq n, j \leq m), \]

where \(f\) is a measurable function. Note that \(X_{n,m}\) is adapted to the filtration \(F_{n,m} = \sigma(\xi_{i,j}, i \leq n, j \leq m)\).

We raise the question of normal approximation for stationary random fields under projection conditions with respect to the filtration \((F_{n,m})_{n,m \in \mathbb{Z}}\). In several previous results involving various types of projective conditions, the methods take advantage of the existence of commuting filtrations, i.e.

\[ E(E(X|F_{a,b})|F_{u,v}) = E(X|F_{u,b} \land v). \]

This type of filtration is induced by an initial random field \((\xi_{n,m})_{n,m \in \mathbb{Z}}\) of independent random variables, or, more generally can be induced by stationary random fields \((\xi_{n,m})_{n,m \in \mathbb{Z}}\) where only the columns are independent, i.e. \(\bar{\eta}_m = (\xi_{n,m})_{n \in \mathbb{Z}}\) are independent. This model often appears in statistical applications when one deals with repeated realizations of a stationary sequence. We prove this property in Lemma 17 in the Appendix.

It is interesting to point out that commuting filtrations can be described by the equivalent formulation: for \(a \geq u\) we have

\[ E(E(X|F_{a,b})|F_{u,v}) = E(X|F_{u,b} \land v). \]  

This follows from this Markovian-type property, see for instance Problem 34.11 in Billingsley (1995).

Our main result is the following theorem which is an extension of the CLT in Maxwell and Woodroofe (2000) to random fields. Below we use the notation

\[ S_{k,j} = \sum_{u,v=1}^{k,j} X_{u,v}. \]

**Theorem 1** Define \((X_{n,m})_{n,m \in \mathbb{Z}}\) by (2) and assume that (3) holds. Assume that the following projective condition is satisfied

\[ \sum_{j,k \geq 1} \frac{1}{j^{3/2}k^{3/2}} ||E(S_{j,k}|F_{1,1})|| < \infty. \]

In addition assume that the vertical shift \(T\) is ergodic. Then there is a constant \(c\) such that

\[ \frac{1}{n_1 n_2} E(S_{n_1,n_2}^2) \to c^2 \text{ as } \min(n_1, n_2) \to \infty. \]
By simple calculations involving the properties of conditional expectation we obtain the following corollary.

**Corollary 2** Assume the following projective condition is satisfied

\[
\sum_{j,k \geq 1} \frac{1}{j^{1/2}k^{1/2}} ||E(X_{j,k} | F_1, 1)|| < \infty,
\]

and \( T \) is ergodic. Then there is a constant \( c \) such that the CLT in (5) holds.

The results are easy to extend to general random fields \( (X_u)_{u \in \mathbb{Z}^d} \) introduced in the following way. We start with a stationary random field \( (\xi_n)_{n \in \mathbb{Z}^d} \) and introduce another stationary random field \( (X_n)_{n \in \mathbb{Z}^d} \) defined by \( X_k = f(\xi_j, j \leq k) \), where \( f \) is a measurable function and \( j \leq k \) denotes \( j_i \leq k_i \) for all \( i \). Note that \( X_k \) is adapted to the filtration \( F_k = \sigma(\xi_u, u \leq k) \). As a matter of fact \( Y_k = T_1T_2...T_d(Y_0) \) where \( T_i \) are the shift operators.

In the next theorem we shall consider commuting filtrations in the sense that for \( a \geq u \in \mathbb{R}^1, b, v \in \mathbb{R}^{d-1} \) we have

\[
E(E(X | F_{a,b}) | F_{u,v}) = E(X | F_{u,b \land v}).
\]

For example, this kind of filtration is induced by stationary random fields \( (\xi_{n,m})_{n \in \mathbb{Z}, m \in \mathbb{Z}^d} \) such that the variables \( \eta_m = (\xi_{n,m})_{n \in \mathbb{Z}^d} \) are independent, \( m \in \mathbb{Z}^{d-1} \). All the results extend in this context via mathematical induction.

Below, \( |n| = n_1 \cdot ... \cdot n_d \).

**Theorem 3** Assume that \( (X_u)_{u \in \mathbb{Z}^d} \) and \( (F_u)_{u \in \mathbb{Z}^d} \) are as above and assume that the following projective condition is satisfied

\[
\sum_{u \geq 1} \frac{1}{|u|^{9/2}} ||E(S_u | F_1)|| < \infty.
\]

In addition assume that \( T_1 \) is ergodic. Then there is a constant \( c \) such that

\[
\frac{1}{|n|} E(S_n^2) \rightarrow c^2 \quad \text{as} \quad \min(n_1, ..., n_d) \rightarrow \infty
\]

and

\[
\frac{1}{\sqrt{|n|}} S_n \Rightarrow N(0, c^2) \quad \text{as} \quad \min(n_1, ..., n_d) \rightarrow \infty.
\]

**Corollary 4** Assume that

\[
\sum_{u \geq 1} \frac{1}{|u|^{1/2}} ||E(X_u | F_1)|| < \infty
\]

and \( T_1 \) is ergodic. Then the CLT in (7) holds.
Corollary 4 above shows that Theorem 1.1 in Wang and Woodroofe (2013) holds for functions of random fields which are not necessarily functions of i.i.d.

We shall give examples providing new results for linear and Volterra random fields. For simplicity, they are formulated in the context of functions of i.i.d.

**Example 5** (Linear field) Let \((\xi_n)_{n \in \mathbb{Z}^d}\) be a random field of independent, identically distributed random variables which are centered and have finite second moment. Define

\[
X_k = \sum_{j \geq 0} a_j \xi_{k-j}.
\]

Assume that \(\sum_{j \geq 0} a_j^2 < \infty\) and

\[
\sum_{j \geq 1} \frac{|b_j|}{|j|^{3/2}} < \infty \quad \text{where} \quad b_j^2 = \sum_{i=0}^{j} (\sum_{u=1}^{i} a_{u+i})^2.
\]

Then the CLT in (7) holds.

Let us mention how this example differs from other results available in the literature. Example 1 in El Machkouri et al. (2013) contains a CLT under the condition \(\sum_{u \geq 0} |a_u| < \infty\). If we take for instance for \(u_i\) positive integers

\[
a_{u_1, u_2, \ldots, u_d} = \prod_{i=1}^{d} (-1)^{u_i} \frac{1}{\sqrt{u_i \log u_i}},
\]

then \(\sum_{u \in \mathbb{Z}^2} |a_u| = \infty\). Furthermore, condition (8), which was used in this context by Wang and Woodroofe (2013), is not satisfied but condition (9) holds.

Another class of nonlinear random fields are the Volterra processes, which plays an important role in the nonlinear system theory.

**Example 6** (Volterra field) Let \((\xi_n)_{n \in \mathbb{Z}^d}\) be a random field of independent random variables identically distributed centered and with finite second moment. Define

\[
X_k = \sum_{(u,v) \geq (0,0)} a_{u,v} \xi_{k-u} \xi_{k-v},
\]

where \(a_{u,v}\) are real coefficients with \(a_{u,u} = 0\) and \(\sum_{u,v \geq 0} a_{u,v}^2 < \infty\). Denote

\[
c_{u,v}(j) = \sum_{k=1}^{j} \theta_{k+u,k+v}
\]

and assume

\[
\sum_{j \geq 1} \frac{|b_j|}{|j|^{3/2}} < \infty \quad \text{where} \quad b_j^2 = \sum_{u \geq 0, v \geq 0, u \neq v} (c_{u,v}(j) + c_{u,v}(j) c_{v,u}(j)).
\]

Then the CLT in (7) holds.
Remark 7 In examples 5 and 6 the fields are Bernoulli. However, we can take as innovations the random field \((\xi_{n,m})_{n,m \in \mathbb{Z}}\) having as columns independent copies of a stationary and ergodic martingale differences sequence.

3 Proofs

In this section we gather the proofs. They are based on a new result for a random field consisting of triangular arrays of row-wise stationary martingale differences, which allows us to find its asymptotic behavior by analyzing the limiting distribution of its columns.

Theorem 8 Assume that for each \(n\) fixed \((D_{n,k})_{k \in \mathbb{Z}}\) forms a stationary martingale difference sequence adapted to the stationary nested filtration \((\mathcal{F}_{n,k})_{k \in \mathbb{Z}}\) and the family \((D_{n,1}^2)_{n \geq 1}\) is uniformly integrable. In addition assume that for all \(m \geq 1\) fixed, \((D_{n,1}, \ldots, D_{n,m})_{n \geq 1}\) converges in distribution to \((L_1, L_2, \ldots, L_m)\), and

\[
\frac{1}{m} \sum_{j=1}^{m} L_j^2 \to c^2 \text{ in } L^1 \text{ as } m \to \infty.
\]  

(10)

Then

\[
\frac{1}{\sqrt{n}} \sum_{k=1}^{n} D_{n,k} \Rightarrow cZ \text{ as } n \to \infty,
\]

where \(Z\) is a standard normal variable.

Proof of Theorem 8 For the triangular array \((D_{n,k}/\sqrt{n})_{k \geq 1}\), we shall verify the conditions of Theorem 13, given for convenience in the Appendix. Note that for \(\varepsilon > 0\) we have

\[
\frac{1}{n} \mathbb{E}(\max_{1 \leq k \leq n} D_{n,k}^2) \leq \varepsilon^2 + \mathbb{E}(D_{n,1}^2 I(|D_{n,1}| > \varepsilon \sqrt{n}))
\]  

(11)

and, by the uniformly integrability of \((D_{n,1}^2)_{n \geq 1}\), we obtain:

\[
\lim_{n \to \infty} \mathbb{E}(D_{n,1}^2 I(|D_{n,1}| > \varepsilon \sqrt{n})) = 0.
\]

Therefore, by passing to the limit in inequality (11), first with \(n \to \infty\) and then with \(\varepsilon \to 0\), the first condition of Theorem 13 is satisfied. The result will follow from Theorem 13 if we can show that

\[
\frac{1}{n} \sum_{j=1}^{n} D_{n,j}^2 \to L^1 c^2 \text{ as } n \to \infty.
\]

To prove it, we shall apply the following lemma to the sequence \((D_{n,k}^2)_{k \in \mathbb{Z}}\) after noticing that, under our assumptions, for all \(m \geq 1\) fixed, \((D_{n,1}^2, \ldots, D_{n,m}^2)_{n \geq 1}\) converges in distribution to \((L_1^2, L_2^2, \ldots, L_m^2)\).
Lemma 9 Assume that the triangular array of random variables \((X_{n,k})_{k \in \mathbb{Z}}\) is row-wise stationary and \((X_{n,1})_{n \geq 1}\) is a uniformly integrable family. For all \(m \geq 1\) fixed, \((X_{n,1}, \ldots, X_{n,m})_{n \geq 1}\) converges in distribution to \((X_1, X_2, \ldots, X_m)\) and

\[
\frac{1}{m} \sum_{u=1}^{m} X_u \to c \text{ in } L^1 \text{ as } m \to \infty. \tag{12}
\]

Then

\[
\frac{1}{n} \sum_{u=1}^{n} X_{n,u} \to c \text{ in } L^1 \text{ as } n \to \infty.
\]

Proof of Lemma 9. Let \(m \geq 1\) be a fixed integer and define consecutive blocks of indexes of size \(m\), \(I_j(m) = \{(j-1)m+1, \ldots, jm\}\). In the set of integers from 1 to \(n\) we have \(k_n = k_n(m) = \lfloor n/m \rfloor\) such blocks of integers and a last one containing less than \(m\) indexes. Practically, by the stationarity of the rows and by the triangle inequality, we write

\[
\frac{1}{n} \sum_{j=1}^{k_n} E|\sum_{k \in I_j(m)} (X_{n,k} - c)| \leq
\]

\[
\frac{1}{n} \sum_{j=1}^{k_n} E |\sum_{k \in I_j(m)} (X_{n,k} - c)| + \frac{1}{n} E \sum_{u=k_n+1}^{n} (X_{n,u} - c)| \leq
\]

\[
\frac{1}{m} E \sum_{u=1}^{m} (X_{n,u} - c) + \frac{m}{n} E|X_{n,1} - c|. \tag{13}
\]

Note that, by the uniform integrability of \((X_{n,1})_{n \geq 1}\), we have

\[
\limsup_{n \to \infty} \frac{m}{n} E|X_{n,1} - c| \leq \limsup_{n \to \infty} \frac{m}{n} (E|X_{n,1}| + |c|) = 0.
\]

Now, by the continuous function theorem and by our conditions, for \(m\) fixed, we have the following convergence in distribution:

\[
\frac{1}{m} \sum_{u=1}^{m} (X_{n,u} - c) \Rightarrow \frac{1}{m} \sum_{u=1}^{m} (X_{u} - c).
\]

In addition, by the uniform integrability of \((X_{n,k})_{n}\) and by the convergence of moments theorem associated to convergence in distribution, we have

\[
\lim_{n \to \infty} \frac{1}{m} E|\sum_{u=1}^{m} (X_{n,u} - c)| = \frac{1}{m} E|\sum_{u=1}^{m} (X_{u} - c)|,
\]

and by assumption (12) we obtain

\[
E \frac{1}{m} \sum_{u=1}^{m} (X_{u} - c) \to 0 \text{ as } m \to \infty.
\]

The result follows by passing to the limit in (13), letting first \(n \to \infty\) followed by \(m \to \infty\). □

When we have additional information about the type of the limiting distribution for the columns the result simplifies.
Corollary 10 If in Theorem 8 the limiting vector \((L_1, L_2, ..., L_m)\) is stationary Gaussian, then condition (10) holds and

\[
\frac{1}{\sqrt{n}} \sum_{k=1}^{n} D_{n,k} \Rightarrow cZ \text{ as } n \to \infty,
\]

where \(Z\) is a standard normal variable and \(c\) can be identified by

\[
c^2 = \lim_{n \to \infty} E(D_{n,1}^2).
\]

**Proof.** We shall verify the conditions of Theorem 8. Note that, by the martingale property, we have that \(\text{cov}(D_{n,1}, D_{n,k}) = 0\). Next, by the condition of uniform integrability, by passing to the limit we obtain \(\text{cov}(L_1, L_k) = 0\). Therefore, the sequence \((L_m)_m\) is an i.i.d. Gaussian sequence of random variables and condition (10) holds. □

In order to prove Theorem 1 we start by pointing out an upper bound for the variance given in Corollary 7.2 in Wang and Woodroofe (2013). It should be noticed that to prove it, the assumption that the random field is Bernoulli is not needed.

**Lemma 11** Define \((X_{n,m})_{n,m \in \mathbb{Z}}\) by (2) and assume that (3) holds. Then, there is a universal constant \(C\) such that

\[
\frac{1}{\sqrt{nm}} ||S_{n,m}|| \leq C \sum_{i,j \geq 1} \frac{1}{(ji)^{3/2}} ||E(S_{j,i}|F_{1,1})||.
\]

By applying the triangle inequality, the contractivity property of the conditional expectation and changing the order of summations we easily obtain the following corollary.

**Corollary 12** Under the conditions of Lemma 11 there is a universal constant \(C\) such that

\[
\frac{1}{\sqrt{nm}} ||S_{n,m}|| \leq C \sum_{i,j \geq 1} \frac{1}{(ji)^{1/2}} ||E(X_{j,i}|F_{1,1})||.
\]

**Proof of Theorem 1**

We shall develop the "small martingale method" in the context of random fields. To construct a row-wise stationary martingale approximation we shall introduce a parameter. Let \(\ell\) be a fixed positive integer and denote \(k = \lfloor n_2/\ell \rfloor\).

We start the proof by dividing the variables in each line in blocks of size \(\ell\) and making the sums in each block. Define

\[
X_{j,i}^{(\ell)} = \frac{1}{\ell^{1/2}} \sum_{u=(i-1)\ell+1}^{i\ell} X_{j,u}, i \geq 1.
\]
Then, for each line $j$ we construct the stationary sequence of martingale differences $(Y_{j,i}^{(\ell)})_{i \in \mathbb{Z}}$ defined by

$$Y_{j,i}^{(\ell)} = X_{j,i}^{(\ell)} - E(X_{j,i}^{(\ell)} | \mathcal{F}_{j,i-1}^{(\ell)}),$$

where $\mathcal{F}_{j,k}^{(\ell)} = \mathcal{F}_{j,kt}$. Also, we consider the triangular array of martingale differences $(D_{n,1,i}^{(\ell)})_{i \geq 1}$ defined by

$$D_{n,1,i}^{(\ell)} = \frac{1}{\sqrt{n_1}} \sum_{j=1}^{n_1} Y_{j,i}^{(\ell)}.$$

In order to find the limiting distribution of $(\sum_{i=1}^{k} D_{n,1,i}^{(\ell)} / \sqrt{k})_k$ when $\min(n_1, k) \to \infty$, we shall apply Corollary 10. It is enough to show that

$$(D_{n,1,1}^{(\ell)}, ..., D_{n,1,N}^{(\ell)}) \Rightarrow (L_1, ..., L_N),$$

where $(L_1, ..., L_N)$ is stationary Gaussian and $[\{(D_{n,1,1}^{(\ell)})^2\}]_{n_1}$ is uniformly integrable. Both these conditions will be satisfied if we are able to verify the conditions of Theorem 14 in the Appendix, for the sequence $(a_1 Y_{n,1}^{(\ell)} + \ldots + a_N Y_{n,N}^{(\ell)})_n$, where $a_1, ..., a_N$ are arbitrary, fixed real numbers. We have to show that, for $\ell$ fixed

$$\sum_{k \geq 1} \frac{1}{k^{3/2}} \| \sum_{j=1}^{k} E(a_1 Y_{j,1}^{(\ell)} + \ldots + a_N Y_{j,N}^{(\ell)} | \mathcal{F}_{1,N}^{(\ell)})) \| < \infty. \quad (14)$$

By the triangle inequality it is enough to treat each sum separately and to show that for all $1 \leq v \leq N$ we have

$$\sum_{k \geq 1} \frac{1}{k^{3/2}} \| \sum_{j=1}^{k} E(Y_{j,v}^{(\ell)} | \mathcal{F}_{1,N}^{(\ell)}) \| < \infty.$$

By (3) we have that $E(Y_{j,v}^{(\ell)} | \mathcal{F}_{1,N}^{(\ell)}) = E(Y_{j,v}^{(\ell)} | \mathcal{F}_{1,1}^{(\ell)})$. Therefore, by stationarity, the latter condition is satisfied if we can prove that

$$\sum_{k \geq 1} \frac{1}{k^{3/2}} \| \sum_{j=1}^{k} E(Y_{j,1}^{(\ell)} | \mathcal{F}_{1,1}^{(\ell)}) \| < \infty.$$ 

Now, by using once again (3), we deduce

$$E(Y_{j,1}^{(\ell)} | \mathcal{F}_{1,1}^{(\ell)}) = E(X_{j,1}^{(\ell)} - E(X_{j,1}^{(\ell)} | \mathcal{F}_{1,1}^{(\ell)}) | \mathcal{F}_{1,1}^{(\ell)}) = E(X_{j,1}^{(\ell)} | \mathcal{F}_{1,1}^{(\ell)}) - E(X_{j,1}^{(\ell)} | \mathcal{F}_{1,1}^{(\ell)}).$$

So, by the triangle inequality and the monotonicity of the $L^2$--norm of the conditional expectation with respect to increasing random fields, we obtain

$$\| \sum_{j=1}^{k} E(Y_{j,1}^{(\ell)} | \mathcal{F}_{1,1}^{(\ell)}) \| \leq 2 \| \sum_{j=1}^{k} E(X_{j,1}^{(\ell)} | \mathcal{F}_{1,1}^{(\ell)}) \| = 2 \frac{1}{\ell^{1/2}} \| E(S_{k,1}^{(\ell)} | \mathcal{F}_{1,1}^{(\ell)}) \|. \quad 9$$
Furthermore, since the filtration is commuting, by the triangle inequality we obtain

\[ \|E(S_{k,\ell}|\mathcal{F}_{1,\ell})\| = \| \sum_{u=1}^{k} \sum_{v=1}^{\ell} E(X_{u,v}|\mathcal{F}_{1,v}) \| \leq \ell \| \sum_{u=1}^{k} E(X_{u,1}|\mathcal{F}_{1,1}) \|. \]

By taking into account condition (4), it follows that we have

\[ \sum_{k \geq 1} \frac{1}{k^{3/2}} \left( \sum_{j=1}^{k} E(Y^{(\ell)}_{j,v}|\mathcal{F}_{1,N}) \right) \leq 2^{1/2} \sum_{k \geq 1} \frac{1}{k^{3/2}} \| E(S_{k,1}|\mathcal{F}_{1,1}) \| < \infty, \]

showing that condition (13) is satisfied, which implies that the conditions of Corollary [10] are satisfied. The conclusion is that

\[ \frac{1}{\sqrt{n_1 k}} \sum_{j=1}^{n_1} \sum_{i=1}^{k} Y_{j,i}^{(\ell)} \Rightarrow N(0, \sigma^2) \text{ as } \min(n_1, k) \to \infty, \]

where \( \sigma^2 \) is defined, in accordance with Theorem [14], by

\[ \sigma^2 = \lim_{n \to \infty} \frac{1}{n} E \left( \sum_{j=1}^{n} Y_{j,1}^{(\ell)} \right)^2. \]

According to Theorem 3.2 in Billingsley (1999), in order to prove convergence and to find the limiting distribution of \( S_{n_1,n_2}/\sqrt{n_1 n_2} \) we have to show that

\[ \lim_{\ell \to \infty} \limsup_{n_1,k \to \infty} \left[ \frac{1}{\sqrt{n_1 n_2}} S_{n_1,n_2} - \frac{1}{\sqrt{n_1 n_2}} \sum_{j=1}^{n_1} \sum_{i=1}^{k} Y_{j,i}^{(\ell)} \right] = 0 \] (15)

and \( N(0, \sigma^2) \Rightarrow N(0, \sigma^2) \), which is equivalent to

\[ \sigma^2 \Rightarrow \sigma^2 \text{ as } \ell \to \infty. \] (16)

The conclusion will be that \( S_{n_1,n_2}/\sqrt{n_1 n_2} \Rightarrow N(0, \sigma^2) \) as \( \min(n_1, n_2) \to \infty. \)

Let us first prove (15). By the triangle inequality we shall decompose the difference in (15) into two parts. Relation (15) will be established if we show both

\[ \lim_{\ell \to \infty} \limsup_{n_1,k \to \infty} \left[ \frac{1}{\sqrt{n_1 n_2}} \sum_{j=1}^{n_1} \sum_{i=1}^{k} E(X_{j,1}^{(\ell)}|\mathcal{F}_{j,i-1}) \right] = 0. \] (17)

and

\[ \lim_{n_1,k \to \infty} \left[ \frac{1}{\sqrt{n_1 n_2}} S_{n_1,n_2} - \frac{1}{\sqrt{n_1 n_2}} S_{n_1,k\ell} \right] = 0. \] (18)

In order for computing the standard deviation of the double sum involved, before taking the limit in (17), we shall apply Lemma [11] and a multivariate version of Remark [13] in the Appendix. This expression is dominated by a universal constant times

\[ \sum_{i,j \geq 1} \frac{1}{(ij)^{3/2}} \sum_{u=1}^{j} \sum_{v=1}^{i} E(E(X_{u,v}|\mathcal{F}_{u,v-1})|\mathcal{F}_{1,0})|. \]
Now,
\[ \sum_{u=1}^{j} \sum_{v=1}^{i} E(E(X_{u,v}^{(\ell)} | F_{u,v-1}^{(\ell)}) | F_{1,0}^{(\ell)}) = \frac{1}{\ell^{1/2}} E(S_{j,i\ell} | F_{1,0}). \]

So, the quantity in (17) is bounded above by a universal constant times
\[ \frac{1}{\ell^{1/2}} \sum_{i,j \geq 1} \frac{1}{(ij)^{3/2}} |E(S_{j,i\ell} | F_{1,0})|, \]
which converges to 0 as \( \ell \to \infty \) under our condition (4), by Lemmas 2.7 and 2.8 in Peligrad and Utev (2005), applied in the second coordinate.

As far as the limit (18) is concerned, since by Lemma 11 and condition (4) the array \( \sum_{j=1}^{n_1} \sum_{i=1}^{n_2} X_{j,i} / \sqrt{n_1 n_2} \) is bounded in \( L^2 \), it is enough to show that, for \( k \ell < n_2 < (k+1) \ell \), we have
\[ \lim_{n_1,n_2 \to \infty} \left| \frac{1}{\sqrt{n_1 n_2}} \sum_{j=1}^{n_1} \sum_{i=k \ell+1}^{n_2} X_{j,i} \right| = 0. \]

We just have to note that, again by Lemma 11, condition (4) and stationarity, there is a constant \( K \) such that
\[ \left| \sum_{j=1}^{n_1} \sum_{i=k \ell+1}^{n_2} X_{j,i} \right| \leq K \sqrt{n_1 \ell} \]
and \( \ell/n_2 \to 0 \) as \( n_2 \to \infty \).

We turn now to prove (16). By (15) and the orthogonality of martingale differences,
\[ \lim_{\ell \to \infty} \limsup_{n_1,n_2 \to \infty} \left| \frac{1}{\sqrt{n_1 n_2}} S_{n_1,n_2} - \frac{1}{\sqrt{n_1}} \sum_{j=1}^{n_1} Y_{j,0}^{(\ell)} \right| = 0. \]

So
\[ \lim_{\ell \to \infty} \limsup_{n_1,n_2 \to \infty} \left| \frac{1}{\sqrt{n_1 n_2}} S_{n_1,n_2} - \sigma_{\ell} \right| = 0. \]

By the triangle inequality, this shows that \( \sigma_{\ell} \) is a Cauchy sequence, therefore convergent to a constant \( \sigma \) and also
\[ \lim_{n_1,n_2 \to \infty} \left| \frac{1}{\sqrt{n_1 n_2}} S_{n_1,n_2} \right| = \sigma. \]

The proof is now complete. \( \square \)

The extensions to random fields indexed by \( Z^d \), for \( d > 2 \), are straightforward following the same lines of proofs as for a two-indexed random field. We shall point out the differences. To extend Lemma 11 we first apply a result of Peligrad and Utev (2005) (see Theorem 14 in the Appendix) to the stationary sequence \( Y_j(m) = \sum_{i=1}^{n_1} X_{j,i} \) with \( j \in Z^{d-1} \) and then we apply induction.

In order to prove Theorem 3 we partition the variables according to the last index. Let \( \ell \) be a fixed positive integer, denote \( k = [n_d/\ell] \) and define
\[ X_{j,i}^{(\ell)} = \frac{1}{\ell^{1/2}} \sum_{u=(i-1)\ell+1}^{i\ell} X_{j,u}, \quad i \geq 1. \]
Then, for each \( j \) we construct the stationary sequence of martingale differences \((Y_{j,i}^{(l)})_{i \in \mathbb{Z}}\) defined by \( Y_{j,i}^{(l)} = X_{j,i}^{(l)} - E(X_{j,i}^{(l)} | F_{j,i-1}) \) and

\[
P_{n',i}^{(l)} = \frac{1}{\sqrt{n'}} \sum_{j=1}^{n'} Y_{j,i}^{(l)}.
\]

For showing that \((D_{n',1}^{(l)}, ..., D_{n',N}^{(l)}) \Rightarrow (L_1, ..., L_N)\), we apply the induction hypothesis. □

**Proof of Example 5.**

Let us note first that the variables are square integrable and well defined.

Note that

\[
E(S_u | F_0) = \sum_{1 \leq k \leq u} \sum_{j \leq 0} a_{k-j} \xi_j
\]

and therefore

\[
E(E^2(S_u | F_0)) = \sum_{i \geq 0} (\sum_{1 \leq k \leq u} a_{k+i})^2 E(\xi_i^2).
\]

The result follows by applying Theorem 3 (see Remark 15 and consider a multivariate analog of it). □

**Proof of Example 6.**

Note that

\[
E(S_j | F_0) = \sum_{k=1}^{j} \sum_{(u,v) \geq (k,k)} a_{u,v} \xi_k - u \xi_{k-v}
\]

= \[ \sum_{(u,v) \geq (0,0)} (a_{k+u,k+v} \xi_{k-v} - u \xi_{k-v}) = \sum_{(u,v) \geq (0,0)} c_{u,v}(j) \xi_{u} - v \xi_{v}.\]

Since by our conditions \( c_{u,u} = 0 \) we obtain

\[
E(E^2(S_j | F_0)) = \sum_{u \geq 0, v \geq 0, u \neq v} (c_{u,v}(j) + c_{u,v}(j)c_{v,u}(j)) E(\xi_u \xi_v)^2.
\]

□

### 4 Appendix.

For convenience we mention a classical result of McLeish which can be found on pp. 237-238 Gänssler and Häusler (1979).

**Theorem 13** Assume \((D_{n,i})_{1 \leq i \leq n}\) is an array of square integrable martingale differences adapted to an array \((F_{n,i})_{1 \leq i \leq n}\) of nested sigma fields. Suppose that

\[
\max_{1 \leq j \leq n} |D_{n,j}| \rightarrow L^2 0 \text{ as } n \rightarrow \infty.
\]
\[
\sum_{j=1}^{n} D_{n,j}^2 \to^P c^2 \quad \text{as } n \to \infty.
\]
Then \( \sum_{j=1}^{n} D_{n,j} \) converges in distribution to \( N(0, c^2) \).

The following is a Corollary of Theorem 1.1 in Peligrad and Utev (2005). This central limit theorem was obtained by Maxwell and Woodroofe (2000).

**Theorem 14** Assume that \((X_i)_{i\in\mathbb{Z}}\) is a stationary sequence adapted to a stationary filtration \((\mathcal{F}_i)_{i\in\mathbb{Z}}\). Then there is a universal constant \( C_1 \) such that

\[
||S_n|| \leq C_1 n^{1/2} \sum_{k=1}^{\infty} \frac{1}{k^{3/2}} ||E(S_k|\mathcal{F}_1)||.
\]

If

\[
\sum_{k=1}^{\infty} \frac{1}{k^{3/2}} ||E(S_k|\mathcal{F}_1)|| < \infty,
\]
then \((S_n^2/n)_n\) is uniformly integrable and there is a positive constant \( c \) such that

\[
\frac{1}{n} E(S_n)^2 \to c^2 \quad \text{as } n \to \infty.
\]

If in addition the sequence is ergodic we have

\[
\frac{1}{\sqrt{n}} S_n \Rightarrow c\mathcal{N}(0, 1) \quad \text{as } n \to \infty.
\]

**Remark 15** Note that we have the following equivalence:

\[
\sum_{k=1}^{\infty} \frac{1}{k^{3/2}} ||E(S_k|\mathcal{F}_1)|| < \infty \iff \sum_{k=1}^{\infty} \frac{1}{k^{3/2}} ||E(S_k|\mathcal{F}_0)|| < \infty.
\]

**Remark 16** The condition \( (I) \) is implied by

\[
\sum_{k=1}^{\infty} \frac{1}{k^{1/2}} ||E(X_k|\mathcal{F}_1)|| < \infty.
\]

**Lemma 17** Assume that \(X, Y, Z\) are integrable random variables such that \((X, Y)\) and \( Z \) are independent. Assume that \( g(X, Y) \) is integrable. Then

\[
E(g(X, Y)|\sigma(Y, Z)) = E(g(X, Y)|Y) \quad \text{a.s.}
\]

and

\[
E(g(Z, Y)|\sigma(X, Y)) = E(g(Z, Y)|Y) \quad \text{a.s.}
\]

Proof. Since \((X, Y)\) and \( Z \) are independent, it is easy to see that \( X \) and \( Z \) are conditionally independent given \( Y \). The result follows from this observation by Problem 34.11 in Billingsley (1995). □
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