IMPROVED MOMENT ESTIMATES FOR INVARIANT MEASURES OF SEMILINEAR DIFFUSIONS IN HILBERT SPACES AND APPLICATIONS

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Abstract. We study regularity properties for invariant measures of semilinear diffusions in a separable Hilbert space. Based on a pathwise estimate for the underlying stochastic convolution, we prove a priori estimates on such invariant measures. As an application, we combine such estimates with a new technique to prove the $L^1$-uniqueness of the induced Kolmogorov operator, defined on a space of cylindrical functions. Finally, examples of stochastic Burgers equations and thin-film growth models are given to illustrate our abstract result.

1. Introduction

The aim of this work is to obtain improved moment estimates of invariant measures of semilinear stochastic evolution equations of the type

$$dX(t) = \left(AX(t) + B(X(t))\right)dt + \sqrt{Q}dW_t, \quad t \geq 0$$

(1.1)
defined on a separable real Hilbert space $H$. Here $A$ is a self-adjoint linear operator of negative type $\omega$ on $H$ having a compact resolvent, $B$ is a nonlinear function with subdomain $D(B) \subset H$. $Q$ is a symmetric positive definite operator and $(W_t)_{t \geq 0}$ is a cylindrical Wiener process in $H$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$.

Equation (1.1) can be read as an abstract formulation of many partial differential equations perturbed by random noise such as stochastic reaction diffusion, Allen-Cahn, Burgers and Navier-Stokes equations. Existence and uniqueness of solutions to such equations are well studied, we refer to the monographs by Da Prato, Zabczyk [8, 9], Cerrai [4] and the works [6, 15]. We will be in particular interested in the situation, where (1.1) has a mild solution $X(t), t \geq 0$, with a time-invariant distribution $\mu = \mathbb{P} \circ X(t)^{-1}$. Throughout this paper, we call such a solution a stationary mild solution and $\mu$ an invariant measure of (1.1). Given such a stationary mild solution, we will then derive in Section 3 moment estimates on its time-invariant distribution $\mu$ under appropriate assumptions on the coefficients of (1.1).

Moment estimates for invariant measures of stochastic partial differential equations have been studied quite intensively for some time. Recently, in the case where $B$ is locally Lipschitz, the authors proved in [12] existence and moment estimates of an invariant measure $\mu$ corresponding to (1.1) under a Lyapunov type assumption on the coefficients $A$ and $B$. These moment estimates have been the main tool to discuss well-posedness of the parabolic Cauchy problem corresponding to stochastic reaction diffusion or Allen-Cahn equations in $L^1(\mu)$. However, there are many important examples, e.g. the stochastic Burgers equation, that are still not covered by our analysis. The results in this paper can be seen as improved moment estimates on invariant measures to semilinear diffusions under weaker assumptions on its coefficients.

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The main ingredient, to obtain our moment estimates, is a pathwise control on the stochastic convolution arising in the mild formulation of (1.1). This idea is taken from the paper [14] by Flandoli and Gatarek on stochastic Navier-Stokes equations, see also the paper [5] by Da Prato and Debussche where the same idea has been applied to the stochastic Burgers equation. We have generalized this technique and found simplified proofs to apply the same technique in an abstract context. To illustrate this result we discussed at the end examples of stochastic Burgers equations and thin-film growth models. We shall remark that the same result can be proved for stationary solutions of stochastic Navier-Stokes equations in the spirit of Flandoli and Gatarek [14].

The existence of a stationary mild solution is a rather weak assumption on the equation (1.1) and in particular does not imply neither the existence of an associated full Markov process nor an associated transition semigroup \( (P_t)_{t \geq 0} \). The existence of \( (P_t)_{t \geq 0} \), however, can be obtained from the Hille-Yosida theory, in the case, where the Kolmogorov operator associated with (1.1) \( (L, D(L)) \) (resp. its closure on suitable test functions) generates a \( C_0 \)-semigroup in \( L^1(H, \mu) \).

Based on the improved moment estimates on \( \mu \) we will therefore study the existence (and uniqueness) of \( (P_t)_{t \geq 0} \) in Section 4. The method which we follow here is new and different to the one presented in [19] due the fact that the drift term \( B \) is not supposed to be dissipative and the coefficients of the finite dimensional realization of \( L \) are not bounded. Hence we can not use the classical theory by [17] to obtain uniform gradient estimates for the pseudo-resolvents associated with finite dimensional approximations of \( L \).

Let us now specify our precise assumptions:

\((\text{H}_0)\) \( A \) is selfadjoint, \( \| e^{tA} \| \leq e^{-\omega t} \) for certain \( \omega > 0 \) and its resolvent \( A^{-1} \) (which exists) is compact.

\((\text{H}_1)\) \( B : D(B) \subset H \to H \) is a measurable vector-field, defined on a measurable subset \( D(B) \subset H \). We will always consider \( B \) as everwhere defined, by setting \( B(h) = 0 \) for \( h \notin D(B) \).

\((\text{H}_2)\) \( Q \) is a bounded, nonnegative, symmetric operator such that \( A \) and \( Q \) are simultaneously diagonalizable and there exist \( \nu \in]0, \frac{1}{2}[ \) such that for all \( t > 0 \)

\[ \int_0^t s^{-2\nu} \| \sqrt{Q} e^{tA} \|_{HS}^2 ds < \infty. \]

\((\text{H}_3)\) There exists a mild solution

\[ X(t) = e^{tA} X_0 + \int_0^t e^{(t-s)A} B(X(s)) ds + \int_0^t e^{(t-s)A} \sqrt{Q} dW_s, \quad t \geq 0, \]

of (1.1) having a time-invariant distribution \( \mu = \mathbb{P} \circ X(t)^{-1} \).

We shall introduce the following interpolation spaces: For \( \theta \in \mathbb{R} \) let

\[ V_\theta := (D((-A)^\theta), \| \cdot \|_\theta), \quad \text{where} \quad \| x \|_\theta = \langle (-A)^\theta x, (-A)^\theta x \rangle \quad \text{for} \quad x \in V_\theta. \]

Hypotheses \((\text{H}_2)\) implies that the stochastic convolution \( W_A(t) \) defined by

\[ W_A(t) := \int_0^t e^{(t-s)A} \sqrt{Q} dW_s \]

is well defined and satisfies the uniform moment estimate

\[ M := \sup_{t \geq 0} \mathbb{E} (\| W_A(t) \|_{\gamma}^2) = \int_0^\infty \| (-A)^\gamma e^{tA} \sqrt{Q} \|_{HS}^2 dt < \infty, \quad 0 < \gamma < \nu. \quad (1.2) \]
2. Pathwise estimates for stochastic convolutions

The aim of this section is to prove a pathwise estimate for the stochastic convolution associated with the linear operator $A$. The estimate will be useful in the next section to obtain improved moment estimates on $\mu$. We start with the following 1-dimensional result:

**Proposition 2.1.** Let $(\beta(t))_{t \geq 0}$ be a 1-dimensional Brownian motion. For $t \geq 0$ set

$$W_{-\lambda}(t) = \int_0^t e^{-\lambda(t-s)} \, d\beta(s), \quad \lambda > 0. \tag{2.1}$$

Then for all $\delta \in (0, \frac{1}{2})$

$$\sup_{0 \leq t \leq T} |W_{-\lambda}(t)| \leq \lambda^{-\delta} \cdot C_\delta M(\delta, T) \tag{2.2}$$

with

$$C_\delta := \Gamma(\delta + 1) + \delta^\delta e^{-\delta}, \quad M(\delta, T) := \sup_{0 \leq s \leq t \leq T} \frac{|\beta(t) - \beta(s)|}{|t - s|^\delta}. \tag{2.3}$$

Moreover,

$$\mathbb{E}(M(\delta, T)^m) \leq M \cdot T^m(\frac{1}{2} - \delta) \quad \text{for all } m \geq 1. \tag{2.4}$$

for some constant $M$ that is independent of $\lambda$ and $T$.

**Proof.** Itô’s product rule implies that

$$W_{-\lambda}(t) = \beta(t) - \lambda \int_0^t e^{-\lambda(t-s)} \beta(s) \, ds \tag{2.4}$$

so that for $t \leq T$

$$|W_{-\lambda}(t)| \leq \lambda \int_0^t e^{-\lambda(t-s)} (t-s)^\delta \, ds \cdot M(\delta, T) + e^{-\lambda t} \cdot t^\delta \cdot M(\delta, T) \tag{2.5}$$

$$\leq \left( \lambda \int_0^{+\infty} e^{-s} s^\delta \, ds + \delta^\delta e^{-\delta} \cdot \lambda^{-\delta} \right) \cdot M(\delta, T).$$

The moment estimate (2.3) follows from Théorème 3 in [18] (see also [1]).

We can now apply the Proposition to obtain a pathwise estimate on the stochastic convolution

$$W_{A-\lambda}(t) := \int_0^t e^{(t-s)(A-\lambda)} \sqrt{Q} \, dW(s), \quad \lambda > 0. \tag{2.6}$$

To this end, denote by $(\lambda_k)_{k \geq 1}$ and $(q_k)_{k \geq 1}$ the eigenvalues of $-A$ and $Q$ respectively corresponding to the same eigenbasis $(e_k)_{k \geq 1}$ in $H$. Then the last Proposition implies

**Corollary 2.2.** Let $\delta \in (0, \frac{1}{2})$ and $\gamma \in \mathbb{R}$. Then

$$\sup_{0 \leq t \leq T} \|W_{A-\lambda}(t)\|_\gamma^2 \leq C_\delta^2 \sum_{k=1}^{+\infty} \lambda_k^{2\gamma} q_k \left( \frac{\lambda_k^2 q_k}{\lambda + \lambda_k} \right)^{2\delta} M_k(\delta, T)^2. \tag{2.7}$$

where

$$M_k(\delta, T) := \sup_{0 \leq s < t \leq T} \frac{|\beta_k(t) - \beta_k(s)|}{|t - s|^\delta}, \quad k \geq 1$$

See [11, 13] for more details.
are independent random variables satisfying the moment estimate (2.3). In particular, if there exists \( \varepsilon > 0 \) such that
\[
Z_{\gamma,\delta,\varepsilon} := \sum_{k=1}^{+\infty} \lambda_k^{-2(\delta-\gamma-\varepsilon)} q_k < +\infty, \tag{2.6}
\]
then
\[
\sup_{0 \leq t \leq T} \|W_{A-\lambda}(t)\|_\gamma^2 \leq \lambda^{-2\varepsilon} \cdot M_{\delta,\gamma,\varepsilon} \tag{2.7}
\]
for some random variable \( M_{\delta,\gamma,\varepsilon} \), independent of \( \lambda \), having finite moments of any order.

**Proof.** Clearly,
\[
\|W_{A-\lambda}(t)\|_\gamma^2 = \sum_{k=1}^{+\infty} \lambda_k^{2\gamma} \langle W_{A-\lambda}(t), e_k \rangle^2 = \sum_{k=1}^{+\infty} \lambda_k^{2\gamma} \left( \int_0^t e^{-(\lambda+\lambda_k)(t-s)} \sqrt{q_k} \, d\beta_k(s) \right)^2,
\]
where \( \beta_k, k \geq 1 \), are independent 1-dimensional Brownian motions. Proposition 2.1 now implies that
\[
\sup_{0 \leq t \leq T} \|W_{A-\lambda}(t)\|_\gamma^2 \leq \sum_{k=1}^{+\infty} \lambda_k^{2\gamma} q_k \sup_{0 \leq t \leq T} \|W^{(k)}_{-(\lambda+\lambda_k)}(t)\|^2 \leq C_\delta^2 \sum_{k=1}^{+\infty} \lambda_k^{2\gamma} q_k \cdot M_k(\delta, T)^2.
\]
If \( Z_{\gamma,\delta,\varepsilon} < +\infty \), then
\[
\sup_{0 \leq t \leq T} \|W_{A-\lambda}(t)\|_\gamma^2 \leq \lambda^{-2\varepsilon} \cdot M_{\delta,\gamma,\varepsilon} \tag{2.8}
\]
with
\[
M_{\delta,\gamma,\varepsilon} := C_\delta^2 \sum_{k=1}^{+\infty} \lambda_k^{-2(\delta-\gamma-\varepsilon)} q_k M_k(\delta, T)^2.
\]

For the proof of the last statement of the corollary, take \( m \geq 1 \). By Jensen’s inequality we can write
\[
M_{\delta,\gamma,\varepsilon}^m = Z_{\gamma,\delta,\varepsilon}^m \left( \frac{1}{Z_{\gamma,\delta,\varepsilon}} \sum_{k=1}^{+\infty} \lambda_k^{-2(\delta-\gamma-\varepsilon)} q_k M_k(\delta, T)^2 \right)^m \leq Z_{\gamma,\delta,\varepsilon}^{m-1} \sum_{k=1}^{+\infty} \lambda_k^{-2(\delta-\gamma-\varepsilon)} q_k M_k(\delta, T)^{2m}
\]
and using the moment estimate (2.3) we conclude that
\[
\mathbb{E} \left( M_{\delta,\gamma,\varepsilon}^m \right) \leq Z_{\gamma,\delta,\varepsilon}^{m-1} \sum_{k=1}^{+\infty} \lambda_k^{-2(\delta-\gamma-\varepsilon)} q_k M \cdot T^{m(1-2\delta)} = M \cdot Z_{\gamma,\delta,\varepsilon}^{m-1} \cdot T^{m(1-2\delta)} < \infty, \tag{2.9}
\]
where \( M \) is a universal constant.

### 3. A PRIORI ESTIMATES ON IN Variant MEASURES

In this section we will prove improved moment estimates on the invariant distribution \( \mu \) of a stationary mild solution of (1.1). The existence of a stationary mild solution is known in many important applications that are covered by our setting, especially for stochastic Burgers equations and thin-film growth models (see Section 5 below). For our analysis we need the following assumptions. Fix \( 0 \leq \gamma_1 \leq \gamma_2 \) and assume

\( (H_4) \) There exists \( \varepsilon > 0 \) such that
\[
Z_{\gamma_2,\delta,\varepsilon} := \sum_{k=1}^{+\infty} \lambda_k^{-2(\delta-\gamma_2-\varepsilon)} q_k < +\infty,
\]
There exist positive constants $\alpha, \beta, \gamma, \delta$ and $s \geq 2$ such that
\[
\langle Ay + B(y + w), y \rangle \leq -\alpha\|y\|_{\gamma_1}^2 + \beta\|w\|_{\gamma_2}^s \cdot \|y\|_{\gamma_1}^2 + \gamma\|w\|_{\gamma_2}^s + \delta
\]
for all $y \in D(A), w \in V_{\gamma_2}$.

For $\lambda > 0$ consider the following decomposition
\[
X(t) = Y_\lambda(t) + W_{A-\lambda}(t), \quad t \geq 0,
\]
of the mild solution. It is then easy to see that $Y_\lambda(t)$ satisfies the following semilinear evolution equation in the mild sense
\[
dY_\lambda(t) = \left(AY_\lambda(t) + \lambda W_{A-\lambda}(t)\right)dt + B(Y_\lambda(t) + W_{A-\lambda}(t))dt
\]
with the random time-dependent nonlinearity $B(\cdot + W_{A-\lambda}(t))$.

**Lemma 3.1.** For any positive increasing $C^1$-function $\Psi$ on $\mathbb{R}^+$ we have
\[
\frac{1}{2} \frac{d}{dt} \Psi(\|Y_\lambda(t)\|^2) \leq -\frac{\alpha}{4} \Psi'(\|Y_\lambda(t)\|^2)\|Y_\lambda(t)\|_{\gamma_1}^2 \Psi(\|Y_\lambda(t)\|^2) R_\lambda(t).
\]  

Where
\[
R_\lambda(t) = \delta + \gamma\|W_{A-\lambda}(t)\|_{\gamma_2}^s + \frac{\lambda^2}{2\alpha}\|W_{A-\lambda}(t)\|_{\gamma_1}^2,
\]
\[
\lambda = \left(\frac{4}{\alpha}(\beta M_T(\gamma_2, s) + 1)\right)^{\frac{1}{2s}}.
\]

**Proof.** We have for all $\lambda \geq 0$
\[
\frac{1}{2} \frac{d}{dt} \Psi(\|Y_\lambda(t)\|^2) \leq \Psi'(\|Y_\lambda(t)\|^2) \left(\frac{4}{\alpha}(\beta M_T(\gamma_2, s) + 1)\right)^{\frac{1}{2s}}.
\]

Hence by using $(H_4)$ and Corollary 2.2 we can write
\[
\|W_{A-\lambda}(t)\|_{\gamma_2}^s \leq \lambda^{-\varepsilon s} M_T(\gamma_2, s)
\]
with
\[
\mathbb{E}(M_T^m(\gamma_2, s)) < \infty \quad \text{for all } m \geq 1.
\]

Thus
\[
\frac{1}{2} \frac{d}{dt} \Psi(\|Y_\lambda(t)\|^2) \leq \Psi'(\|Y_\lambda(t)\|^2) \left(-\frac{\alpha}{2}\|Y_\lambda(t)\|_{\gamma_1}^2 + \beta \lambda^{-\varepsilon s} M_T(\gamma_2, s)\|Y_\lambda(t)\|_{\gamma_1}^2 + R_\lambda(t)\right).
\]

In particular for $\lambda := \left(\frac{4}{\alpha}(\beta M_T(\gamma_2, s) + 1)\right)^{\frac{1}{2s}}$ we have
\[
\frac{1}{2} \frac{d}{dt} \Psi(\|Y_\lambda(t)\|^2) \leq \Psi'(\|Y_\lambda(t)\|^2) \left(-\frac{\alpha}{4}\|Y_\lambda(t)\|_{\gamma_1}^2 + R_\lambda(t)\right),
\]
which yields the proof of the lemma.
Proposition 3.2. Let $0 \leq \gamma_1 \leq \gamma_2$ and let $\mu$ be the distribution of any stationary mild solution of (1.1). Then
\[
\int \|x\|^p \mu(dx) < \infty \quad \forall p \geq 0.
\]

Proof. First note that for any $q > 0$ there exist positive constants $D_1, D_2$ and $D_3$ such that for
\[
E \left( R_\lambda(t)^q \right) \leq D_1 + D_2 E \left( \|W_{A-\lambda}(t)\|_{\gamma_2}^q \right) + D_3 E \left( MT(\gamma_2, s)^q \|W_{A-\lambda}(t)\|^2 \right)^{\gamma_1}.
\]

Then
\[
E \left( \|W_{A-\lambda}(t)\|^q \right) \leq E \left( \|W_A(t)\|^q \right) < \infty,
\]

inequality (3.3) now implies that $E \left( R_\lambda(t)^q \right)$ is locally integrable w.r.t. $t$.

For the proof of the moment estimate let us first consider $p \in [0, 1]$ and define $\Psi(t) := (1 + t)^{\frac{p}{2}}$. Then Lemma 3.1 implies that
\[
\frac{d}{dt} \left( 1 + \|Y_\lambda(t)\|^2 \right)^{\frac{p}{2}} \leq -C_1 \|Y_\lambda(t)\|^2 (1 + \|Y_\lambda(t)\|^2)^{\frac{p}{2}} + C_2 R_\lambda(t)
\]
for finite strictly positive constants $C_1, C_2$. Fix $K > 0$ and define $\Psi_K(t) := (1 + t)^{\frac{p}{2}} \wedge K, \Phi_K(t) := 1_{\{1+(t)^{\frac{p}{2}} \leq K\}}(1 + t)^{\frac{p}{2} - 1}$. Then
\[
\frac{d}{dt} \Psi_K(\|Y_\lambda(t)\|^2) \leq -C_1 \Phi_K(\|Y_\lambda(t)\|^2) + C_2 R_\lambda(t)
\]
again, hence
\[
\Psi_K(\|Y_\lambda(t)\|^2) + C_1 \int_0^t \Phi_K(\|Y_\lambda(s)\|^2) ds
\leq \Psi_K(\|X(0)\|^2) + C_2 \int_0^t R_\lambda(s) ds.
\]

Since for $0 \leq p \leq 1$ we have $(1 + (s + t)^2)^{\frac{p}{2}} \leq (1 + s^2)^{\frac{p}{2}} + t^p$ for all $s, t \geq 0$, we conclude that
\[
\Psi_K(\|X(t)\|^2) + C_1 \int_0^t \Phi_K(\|Y_\lambda(s)\|^2) ds
\leq \left( (1 + \|Y_\lambda(t)\|^2)^{\frac{p}{2}} + \|W_{A-\lambda}(t)\|^p \right) \wedge K + C_1 \int_0^t \Phi_K(\|Y_\lambda(s)\|^2) ds
\leq (1 + \|Y_\lambda(t)\|^2)^{\frac{p}{2}} \wedge K + C_1 \int_0^t \Phi_K(\|Y_\lambda(s)\|^2) ds + \|W_{A-\lambda}(t)\|^p
\leq \Psi_K(\|X(0)\|^2) + C_2 \int_0^t R_\lambda(s) ds + \|W_{A-\lambda}(t)\|^p.
\]
Taking expectations and using stationarity of $(X(t))_{t \geq 0}$ yields the inequality
\[
C_1 \int_0^t E(\Phi_K(\|Y_\lambda(s)\|^2)) ds \leq C_2 \int_0^t E(R_\lambda(s)) ds + E(\|W_{A-\lambda}(t)\|^p) < \infty.
\]

Since the right hand side does not depend on $K$, we can now take the limit $K \to \infty$ to conclude that
\[
\int_0^t E \left( \|Y_\lambda(s)\|^2 \left( 1 + \|Y_\lambda(s)\|^2 \right)^{\frac{p}{2} - 1} \right) ds < \infty.
\]
hence
\[ \int_0^t \mathbb{E}(\|Y_\lambda(s)\|^p) \, ds < \infty \]
too, so that
\[ t \int \|x\|^p \mu(dx) = \int_0^t \mathbb{E}(\|X(s)\|^p) \, ds \]
\[ \leq 2^p \int_0^t \mathbb{E}(\|Y_\lambda(s)\|^p) \, ds + 2^p \int_0^t \mathbb{E}(\|W_{A-\lambda}(s)\|^p) \, ds \]
\[ < \infty. \]

For the general case \( p > 1 \) we proceed by induction. Suppose the assumption is proven for \( p \) with \( 2p \leq n + 1 \) and consider now \( p > 1 \) with \( 2p \leq n + 1 \). Lemma 3.1 now implies that for finite strictly positive constants \( C_1, C_2 \) and \( C_p \)
\[ \frac{d}{dt} \left( 1 + \|Y_\lambda(t)\|^2 \right)^{\frac{p}{2}} \leq -C_1 \|Y_\lambda(t)\|^2 \left( 1 + \|Y_\lambda(t)\|^2 \right)^{\frac{p-1}{2}} + C_2 \left( 1 + \|Y_\lambda(t)\|^2 \right)^{\frac{p-1}{2}} R_\lambda(t) \]
\[ \leq -C_1 \|Y_\lambda(t)\|^2 \left( 1 + \|Y_\lambda(t)\|^2 \right)^{\frac{p-1}{2}} + C_p \left( \|Y_\lambda(t)\|^{p-1} + R_\lambda(t)^{p-1} + 1 \right). \]

Fix \( K > 0 \) and let \( \Psi_K \) and \( \Phi_K \) be as above, the last inequality now implies that
\[ \Psi_K(\|Y_\lambda(t)\|^2) + C_1 \int_0^t \Phi_K(\|Y_\lambda(s)\|^2) \, ds \]
\[ \leq \Psi_K(\|X(t)\|^2) + C_p \int_0^t \left( \|Y_\lambda(s)\|^{p-1} + R_\lambda(s)^{p-1} + 1 \right) \, ds. \]

Note that for \( p > 1 \) there exists a finite positive constant \( C_3 \) such that
\[ (1 + (s+t)^2)^{\frac{p}{2}} \leq (1 + s^2)^{\frac{p}{2}} + C_3 (s^{p-\frac{1}{2}} + t^{2p-1} + 1) \]
for all \( s, t \geq 0 \), so that the last inequality now implies that
\[ \Psi_K(\|X(t)\|^2) + C_1 \int_0^t \Phi_K(\|Y_\lambda(s)\|^2) \, ds \]
\[ \leq (1 + \|Y_\lambda(t)\|^2)^{\frac{p}{2}} + K + C_1 \int_0^t \Phi_K(\|Y_\lambda(s)\|^2) \, ds \]
\[ + C_3 \left( \|Y_\lambda(t)\|^{p-\frac{1}{2}} + ||W_{A-\lambda}(t)||^{2p-1} + 1 \right) \]
\[ \leq \Psi_K(\|X(0)\|^2) + C_p \int_0^t \left( \|Y_\lambda(s)\|^{p-1} + R_\lambda(s)^{p-1} + 1 \right) \, ds \]
\[ + C_3 \left( \|Y_\lambda(t)\|^{p-\frac{1}{2}} + ||W_{A-\lambda}(t)||^{2p-1} + 1 \right). \]

Taking expectations, using stationarity of \( (X(t))_{t \geq 0} \) and the fact that
\[ \mathbb{E} \left( \|Y_\lambda(t)\|^{p-\frac{1}{2}} \right) + \int_0^t \mathbb{E}(\|Y_\lambda(s)\|^{p-1}) \, ds < \infty \]
by assumption on \( p \), we conclude that
\[ C_1 \int_0^t \mathbb{E}(\Phi_K(\|Y_\lambda(s)\|^2)) \, ds \leq C_p \int_0^t \mathbb{E}(\|Y_\lambda(s)\|^{p-1} + R_\lambda(s)^{p-1} + 1) \, ds \]
\[ + C_3 \left( \mathbb{E}(\|Y_\lambda(t)\|^{p-\frac{1}{2}}) + \mathbb{E}(\|W_{A-\lambda}(t)\|^{2p-1}) + 1 \right) < \infty. \]
Again, the right hand side does not depend on \( K \), hence taking the limit \( K \to \infty \) we conclude that
\[
\int_0^t \mathbb{E} (|Y_\lambda(s)|^2 (1 + |Y_\lambda(s)|^2)^{\frac{\sigma}{2}}) ds < \infty,
\]
hence
\[
\int_0^t \mathbb{E} (|Y_\lambda(s)|^p) ds < \infty
\]
and thus \( \int |x|^p \mu(dx) < \infty \) too.

Our first main result in this paper now is the following:

**Theorem 3.3.** Let \( \gamma_1 \leq \gamma_2 \) and assume hypotheses \((H_0)-(H_5)\) hold. Then the invariant distribution \( \mu \) of any stationary mild solution \( (X(t))_{t \geq 0} \) of (1.1) satisfies the following moment estimates:

(i) \( \int |x|^{2p} \mu(dx) < \infty \) for \( p \geq 0 \).

(ii) \( \int |x|_{2}^p |x|^{2p} \mu(dx) < \infty \) for \( p \geq 0, \sigma < \gamma_1 \).

**Proof.** Clearly, (i) follows from the previous Proposition. For the proof of (ii) note that Lemma 3.1 implies that for \( \Psi(t) = t^p \) where \( p \geq 1 \)
\[
\|Y_\lambda(t)\|^{2p} + \frac{\alpha}{2} p \int_0^t \|Y_\lambda(s)\|^{2(p-1)} \|Y_\lambda(s)\|^{\frac{\sigma}{2} \gamma_1} ds \leq \|x\|^{2p} + 2p \int_0^t \|Y_\lambda(s)\|^{2(p-1)} R_\lambda(s) ds
\]
\[
\leq \|x\|^{2p} + 2(p - 1) \int_0^t \|Y_\lambda(s)\|^{2p} ds + 2 \int_0^t R_\lambda(s)^p ds. \tag{3.4}
\]
From the interpolation inequality
\[
\|x\|_\sigma \leq C \|x\|_0^{\frac{\gamma_1 - \sigma}{\gamma_1}} \|x\|_{\gamma_1}^{\frac{\sigma}{\gamma_1}}
\]
and Young’s inequality, there exist positive constants \( C, C_1, C_2 \) such that
\[
\int_0^t \|W_{A-\lambda}(s)\|^{2(p-1)} \|Y_\lambda(s)\|^{\frac{\sigma}{\gamma_1}} ds \leq C \int_0^t \|W_{A-\lambda}(s)\|^{2(p-1)} \|Y_\lambda(s)\|^{\frac{2\gamma_1 - \sigma}{\gamma_1}} \|Y_\lambda(s)\|^{\frac{\sigma}{\gamma_1}} ds
\]
\[
\leq C_1 \int_0^t \|Y_\lambda(s)\|^{\frac{2\gamma_1 - \sigma}{\gamma_1}} \|Y_\lambda(s)\|_\sigma^{\frac{\sigma}{\gamma_1}} ds + C_2 \int_0^t \|W_{A-\lambda}(s)\|^{2(p-1)} \|Y_\lambda(s)\|^{\frac{\gamma_1}{2}} ds \tag{3.5}
\]
and
\[
\int_0^t \|Y_\lambda(s)\|^{2(p-1)} \|W_{A-\lambda}(s)\|^{\frac{\gamma_1}{2}} ds \leq \frac{1}{2} \int_0^t \|Y_\lambda(s)\|^{4(p-1)} ds + \frac{1}{2} \int_0^t \|W_{A-\lambda}(s)\|^{\frac{4}{\gamma_1}} ds. \tag{3.6}
\]
Putting this together with (3.4) and (3.5) yields
\[
\int_0^t \|X(s)\|^{2(p-1)} \|X(s)\|_{\sigma}^{\frac{\sigma}{\gamma_1}} ds \leq C_1 \|X(0)\|^{2p} + C_2 \int_0^t \|X(s)\|^{2p_1} ds + C_3 \int_0^t R_\lambda(s)^{p_2} ds
\]
\[
+ C_4 \int_0^t \|W_{A-\lambda}(s)\|_{\gamma_2}^{2p_3} ds,
\]
for some constants \( p_i \) and \( C_i \).

Taking expectations we obtain that
\[
t \int_{\mathbb{H}} |x|^{2(p-1)} |x|_{\sigma}^{\frac{\sigma}{2}} \mu(dx) = \mathbb{E} \left( \int_0^t \|X(s)\|^{2(p-1)} \|X(s)\|_{\sigma}^{\frac{\sigma}{2}} ds \right) < \infty.
\]
hence the assertion.

\[\blacksquare\]
In the previous section we discussed a priori estimates of invariant measures $\mu$ for the equation (1.1). Suppose for the moment that (1.1) has a unique mild solution $X(t, x)$, $t \geq 0$, for any initial condition $x \in H$, that $x \mapsto X(t, x)$ is measurable for any $t$ and that the stationary solution $X(t)$, $t \geq 0$, of (1.1) can be represented as $X(t) = X(t, X_0)$, $t \geq 0$. Furthermore we take $Q = (-A)^{2\gamma_0}$ for some $\gamma_0 < \frac{1}{2}$. It is then easy to see that in this case, the associated transition semigroup

$$P_t \varphi(x) = E(\varphi(X(t, x))) , \varphi \in B_b(H),$$

induces a $C_0$-semigroup of Markovian contractions $(\tilde{P}_t)_{t \geq 0}$ on $L^1(H, \mu)$, in fact on any $L^p(H, \mu)$ for $p \in [1, \infty[$. In the case where

$$(B(x), h), (x, h) \in L^1(\mu) \quad \text{for any} \quad h \in D(A),$$

the corresponding infinitesimal generator $L$ has the expression

$$L \varphi(x) = \frac{1}{2} \text{Tr}_H \left( \sqrt{Q} D^2 \varphi(x) \sqrt{Q} \right) + \langle x, AD \varphi(x) \rangle + \langle B(x), D \varphi(x) \rangle \quad \varphi \in FC^2_b(D(A)).$$

Here,

$$FC^2_b(D(A)) := \left\{ \varphi \in C^2_b(H) \mid \varphi(x) = f(\langle x, h_1 \rangle, \ldots, \langle x, h_m \rangle), f \in C^2_b(\mathbb{R}^m), m \geq 1,
\quad h_1, \ldots, h_m \in D(A) \right\}$$

denotes the space of suitable cylindrical test functions (see Proposition 3.1 in [12] for a proof).

As an application of the improved moment estimates on $\mu$, obtained in the last section, we shall discuss in this section whether $(\tilde{P}_t)_{t \geq 0}$ is the only $C_0$-semigroup in $L^1(H, \mu)$ whose infinitesimal generator extends $(L, FC^2_b(D(A)))$. In this case we say that $L$ is $L^1$-unique.

In the general case, the mere existence of a stationary solution of (1.1) neither ensures the existence of the associated transition semigroup $(P_t)_{t \geq 0}$ nor the existence of its $L^1$-counterpart $(\tilde{P}_t)_{t \geq 0}$, but only implies that the measure $\mu$ is infinitesimally invariant for $L$, i.e.,

$$\int_H L \varphi(x) \mu(dx) = 0$$

for all $\varphi \in FC^2_b(D(A))$ with $L \varphi \in L^1(H, \mu)$.

However, in this case, $(L, FC^2_b(D(A)))$ is dissipative, in particular closable, in $L^1(H, \mu)$ (see [12]). Therefore, to obtain the existence (and also the uniqueness) of $(\tilde{P}_t)_{t \geq 0}$, it is sufficient to prove that the closure of $L$ in $L^1(H, \mu)$ generates a $C_0$-semigroup. The $L^1$-counterpart $(\tilde{P}_t)_{t \geq 0}$ will be Markovian and its existence can therefore be regarded as a first necessary step in the construction of a full Markov process associated with (1.1).

For our analysis in this section we need the following assumptions:

(A0) The measure $\mu$ is infinitesimally invariant for $L$.

(A1) $\|B\| \in L^1(H, \mu)$, where the vector field $B : D(B) \subset H \to H$ is considered as a vector field on all of $H$ by setting $B(x) = 0$ if $x \in H \setminus D(B)$.

(A2) For some $\beta \in (\gamma_0, \frac{1}{2})$, there exists $C : V_{\beta} \to V_{\beta}$ with $\int \|C(x)\|_{\beta}^2 \mu(dx) < +\infty$ such that

$$\langle B(x) - B(y), x - y \rangle \leq \|x - y\|_{\beta}^2 + \langle C(x) - C(y), x - y \rangle \quad \forall x, y \in V_{\frac{1}{2}}$$

(4.1)
In the following, let us define finite dimensional Galerkin approximations for $L$. To this end let

$$i_n : \mathbb{R}^n \rightarrow H, \quad (x_1, \ldots, x_n) \mapsto \sum_{k=1}^{n} x_k e_k$$

be the natural injection of $\mathbb{R}^n$ into $H$ and

$$\pi_n : H \rightarrow \mathbb{R}^n, \quad x \mapsto ((x, e_1), \ldots, (x, e_n))$$

be the natural projection of $H$ on $\mathbb{R}^n$. Let

$$A^n := \pi_n \circ A \circ i_n : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

and

$$B^n := \pi_n \circ B \circ i_n : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad C^n := \pi_n \circ C \circ i_n : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

be the corresponding operator and vector-fields induced by $A$, $B$ and $C$ on $\mathbb{R}^n$ and consider the Kolmogorov operator

$$L^n \varphi(x) := \frac{1}{2} \sum_{k=1}^{n} \langle (-A^n)^{-2\gamma} e_k, e_k \rangle \varphi_{x_k x_k}(x) + \sum_{k=1}^{n} \langle A^n x + B^n(x) - C^n(x), e_k \rangle \varphi_{x_k}(x), \quad \varphi \in C^2_b(\mathbb{R}^n).$$

We now make the following additional assumption on $L^n$.

(A3) For $n \geq 1$, $B^n$ and $C^n$ are smooth, polynomially bounded vector-fields.

Note that (A3) now implies the one-sided Lipschitz condition

$$\langle (A^n x + B^n(x) - C^n(x)) - (A^n y + B^n(y) - C^n(y)), x - y \rangle \leq 0 \quad x, y \in \mathbb{R}^n, \quad (4.2)$$

for the finite-dimensional approximations of $Ax + B(x) - C(x)$.

Next, let $U : H \rightarrow V_\beta$ be a smooth vector field that is Lipschitz continuous w.r.t the $H$-norm with Lipschitz constant $\text{Lip}_U$, and denote by $L^n_U$ the Kolmogorov operator

$$L^n_U \varphi(x) = L^n \varphi(x) + \sum_{k=1}^{n} \langle U^n(x), e_k \rangle \varphi_{x_k}(x), \quad \varphi \in C^2_b(\mathbb{R}^n),$$

where $U^n = \pi_n \circ U \circ i_n : \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 1$. (4.2) now implies the one-sided Lipschitz condition

$$\langle (A^n x + B^n(x) - C^n(x) + U^n(x)) - (A^n y + B^n(y) - C^n(y) + U^n(y)), x - y \rangle \leq \text{Lip}_U \|x - y\|^2, \quad x, y \in \mathbb{R}^n, \quad (4.3)$$

Since the coefficients of $L^n_U$ are smooth there exists for any $f \in C^2_b(\mathbb{R}^n)$ a solution $C([0, +\infty) \times \mathbb{R}^n) \cup C_{\text{loc}}^{1,2}(0, +\infty) \times \mathbb{R}^n)$ of the Cauchy-problem

$$\begin{cases}
du(t, x) = L^n_U u(t, x) dt, & (t, x) \in (0, +\infty) \times \mathbb{R}^n \\
u(0, x) = f(x), & x \in \mathbb{R}^n,
\end{cases} \quad (4.4)$$

satisfying $\|u\|_\infty \leq \|f\|_\infty$ (and $u \geq 0$ if $f \geq 0$). In addition, there exists a semigroup of linear operators $(T^n_U t)_{t \geq 0}$ on $C^1_b(\mathbb{R}^n)$ such that for $f \in C^1_b(\mathbb{R}^n)$ the solution of (4.4) is represented as

$$u(t, x) = T^n_U t f(x), \quad t \geq 0, \quad x \in \mathbb{R}^n$$

(see Theorem 2.2.5 in [2]). According to Theorem 6.1.7 in [2] we also have the norm-estimates

$$\|T^n_U t \|_{C^1_b(\mathbb{R}^n)} \leq C \|f\|_{C^1_b(\mathbb{R}^n)}, \quad f \in C^1_b(\mathbb{R}^n) \quad (4.5)$$
for some uniform constant $C > 0$. A simple coupling argument shows that the constant in (4.5) may be chosen to be $e^{\text{Lip}_U t}$, taking into account (4.3). Note that this constant is independent of $n, n \geq 1$.

In the following we will use the notation “$\bar{\varphi}$” for $\varphi \in B(\mathbb{R}^n)$ to denote the function $\bar{\varphi} = \varphi \circ \pi_n$. Then

$$
\langle D\bar{\varphi}(x), e_k \rangle = \begin{cases} 
\varphi_{x_k}(x) & \text{if } k = 1, \ldots, n \\
0 & \text{otherwise}
\end{cases}
$$

and $\bar{\varphi} \in FC^2_b(D(A))$ if $\varphi \in C^2_b(\mathbb{R}^n)$. In particular $\overline{T^U_t f} \in FC^2_b(D(A))$ for $t \geq 0, f \in C^2_b(\mathbb{R}^n)$.

We will also use the notation

$$
\|x\|_\alpha := \|i_n x\|_\alpha, x \in \mathbb{R}^n, \alpha \in \mathbb{R}.
$$

The following a priori estimate is crucial.

**Lemma 4.1.** Let $f \in C^2_b(\mathbb{R}^n)$ and $\lambda > 0$. Then for $n \geq n_0$ we have

$$
\int_0^t e^{-\lambda s} \int_H \|D\bar{T}^U_t f\|^2_{-\gamma_0} \, d\mu \, ds \leq 4 \frac{e^{\text{Lip}_U t}}{\lambda} \int_H \|B - B^n\| \, d\mu \cdot \|f\|^2_{C^3_b(\mathbb{R}^n)} + 2\|f\|^2_{\infty} + \frac{4}{\lambda} \int_H \|C^n - U^n\|^2_{\gamma_0} \, d\mu \cdot \|f\|^2_{\infty}.
$$

(4.6)

**Proof.** Clearly, invariance of $\mu$ implies for $\varphi \in FC^2_b(D(A))$ that

$$
\frac{1}{2} \int_H \|D\varphi\|^2_{\gamma_0} \, d\mu = -\int_H L\varphi \varphi \, d\mu
$$

$$
= -\int_H L_U^2 \varphi \varphi \, d\mu - \int_H \langle B - B^n + C^n - U^n, D\varphi \rangle \varphi \, d\mu
$$

$$
\leq -\int_H L_U^2 \varphi \varphi \, d\mu + \|D\varphi\|_{\infty} \|\varphi\|_{\infty} \int_H \|B - B^n\| \, d\mu
$$

$$
+ \left( \int_H \|C^n - U^n\|^2_{\gamma_0} \, d\mu \right)^{\frac{1}{2}} \left( \int_H \|D\varphi\|^2_{\gamma_0} \, d\mu \right)^{\frac{1}{2}} \cdot \|\varphi\|_{\infty}
$$

and thus

$$
\int_H \|D\varphi\|^2_{\gamma_0} \, d\mu \leq -4 \int_H L_U^2 \varphi \varphi \, d\mu + 4\|D\varphi\|_{\infty} \|\varphi\|_{\infty} \int_H \|B - B^n\| \, d\mu
$$

$$
+ 4\|\varphi\|_{\infty} \int_H \|C^n - U^n\|^2_{\gamma_0} \, d\mu.
$$

(4.7)

Inserting $\overline{T^U_t f}$ in (4.7), using $\|D\overline{T^U_t f}\|_{\infty} = \|DT^U_t f\|_{\infty} \leq e^{\text{Lip}_U t} \|f\|_{C^3_b(\mathbb{R}^n)}$ and $L_U^2 T^U_t f = \frac{d}{ds} T^U_t f$, we obtain that

$$
\int_H \|D\overline{T^U_t f}\|^2_{\gamma_0} \, d\mu \leq 4 \int_H \|B - B^n\| \, d\mu \cdot e^{\text{Lip}_U t} \|f\|^2_{C^3_b(\mathbb{R}^n)} + 2 \int_H \frac{d}{ds} (\overline{T^U_t f})^2 \, d\mu + 4\|f\|^2_{\infty} \int_H \|C^n - U^n\|^2_{\gamma_0} \, d\mu.
$$

(4.8)

Multiplying both sides of the above inequality by $e^{-\lambda s}$ and using

$$
\frac{d}{ds} \left( e^{-\lambda s} (\overline{T^U_t f})^2 \right) \leq e^{-\lambda s} \frac{d}{ds} (T^U_t f)^2
$$
we conclude for \( s \leq t \) that

\[
e^{-\lambda s} \int_H \|DT_s^{U_n} f\|^2 d\mu \leq -2 \int_H \frac{d}{ds} \left( e^{-\lambda s} \|T_s^{U_n} f\|^2 \right) d\mu + 4e^{-\lambda s} \int_H \|B - B^n\| d\mu \cdot e^{\text{Lip}_U t} \|f\|^2_{C^1_b(\mathbb{R}^n)} + 4e^{-\lambda s} \|f\|^2_\infty \int_H \|C^n - U^n\|^2_{\gamma_0} d\mu .
\]

Integrating the last inequality with respect to \( ds \) yields inequality (4.6).

**Lemma 4.2.** Let \( \lambda > 0 \) and \( h \in B_b(H) \) be such that

\[
\int_H (\lambda - L) \varphi h d\mu = 0 \quad \text{for all } \varphi \in FC^2_b(D(A)) , \varphi = f \circ \pi_{n_0}, f \in C^2_b(\mathbb{R}^n) .
\]

Then for \( n \geq n_0 \)

\[
\left| \int_H \varphi h d\mu \right| \leq e^{-\lambda t} \|\varphi\|_\infty \|h\|_\infty + \|f\|_{C^1_b(\mathbb{R}^n)} \|h\|_\infty e^{\text{Lip}_U t} \left( \int_H \|B - B^n\| d\mu \right) \frac{1}{\lambda} \left( \int_0^t e^{-\lambda s} \int_H \|DT_s^{U_n} f\|^2_{\gamma_0} d\mu ds \right)^{\frac{1}{2}} \left( \int_H \|C^n - U^n\|^2_{\gamma_0} d\mu \right)^{\frac{1}{2}} .
\]

**Proof.** Since

\[
\frac{d}{ds} T_s^{U_n} f = L_n^U T_s^{U_n} f , \quad s > 0 ,
\]

it follows for \( s \leq t \) that

\[
\frac{d}{ds} e^{-\lambda s} \int_H T_s^{U_n} f h d\mu = e^{-\lambda s} \int_H (L_n^U - \lambda) T_s^{U_n} f h d\mu
\]

\[
= e^{-\lambda s} \int_H \langle B^n - B - C^n + U^n, DT_s^{U_n} f \rangle h d\mu
\]

\[
\leq e^{-\lambda s} e^{\text{Lip}_U t} \|f\|_{C^1_b(\mathbb{R}^n)} \|h\|_\infty \int_H \|B^n - B\| d\mu
\]

\[
+ e^{-\lambda s} \|h\|_\infty \left( \int_H \|C^n - U^n\|^2_{\gamma_0} d\mu \right)^{\frac{1}{2}} \left( \int_H \|DT_s^{U_n} f\|^2_{\gamma_0} d\mu \right)^{\frac{1}{2}} .
\]

Integrating the last inequality with respect to \( s \) and applying Hölder’s inequality to the second term yields the assertion (4.9).

We are now ready to prove the main result

**Proposition 4.3.** Let \( \lambda > 0 \) and suppose \( h \in B_b(H) \) is such that

\[
\int_H (\lambda - L) \varphi h d\mu = 0 \quad \text{for all } \varphi \in FC^2_b(D(A)) .
\]

Then \( h = 0 \) \( \mu \text{-a.e.} \)

**Proof.** Suppose on the contrary that \( h \neq 0 \). Then there exists \( \varphi = f \circ \pi_{n} \) for some \( f \in C^2_b(\mathbb{R}^n) \) with

\[
\varepsilon := \left| \int_H \varphi h d\mu \right| > 0 .
\]

We may suppose that \( \varepsilon \leq 1. \)
Let $U : H \to V_\beta$ be such that $U$ is Lipschitz w.r.t. the $H$-norm and
\[
\left( \int_H \|U - U^n\|_\beta^2 \, d\mu \right)^{\frac{1}{2}} \leq \frac{\varepsilon}{8 \left( 1 + \frac{\|f\|_\infty^2}{\lambda} \right) \left( 1 + \frac{\|f\|_\infty^2}{\lambda} \right)^{\frac{1}{2}} + \frac{\varepsilon}{\lambda} \|f\|_\infty^2}.
\]

Since
\[
\lim_{n \to +\infty} \int_H \|U - U^n\|_\beta^2 \, d\mu + \int_H \|C - C^n\|_\beta^2 \, d\mu = 0,
\]
and using the fact that $\gamma_0 < \beta$ we can find $n_\varepsilon \geq n_0$ such that
\[
\sup_{n \geq n_\varepsilon} \left( \int_H \|U - U^n\|_{\gamma_0}^2 \, d\mu \right)^{\frac{1}{2}} \leq \frac{\varepsilon}{4 \left( \frac{\|h\|_\infty^2}{\lambda} \left( 2 \|f\|_\infty^2 + 1 \right)^{\frac{1}{2}} + \frac{\varepsilon}{\lambda} \|f\|_\infty^2 \right)}.
\]

In particular,
\[
\sup_{n \geq n_\varepsilon} \left( \int_H \|C^n - U^n\|_{\gamma_0}^2 \, d\mu \right)^{\frac{1}{2}} \leq \frac{\varepsilon}{2 \left( \frac{\|h\|_\infty^2}{\lambda} \left( 2 \|f\|_\infty^2 + 1 \right)^{\frac{1}{2}} + \frac{\varepsilon}{\lambda} \|f\|_\infty^2 \right)}.
\]

Let $t_\varepsilon > 0$ be such that $e^{-\lambda t_\varepsilon} \|\varphi\|_\infty \|h\|_\infty < \frac{\varepsilon}{4}$. Since $\lim_{n \to \infty} \int_H \|B - B^n\| \, d\mu = 0$, we can find by Lemma 4.1 and (4.10) $\bar{n}_\varepsilon \geq n_\varepsilon$ such that
\[
\sup_{n \geq \bar{n}_\varepsilon} \int_0^{t_\varepsilon} e^{-\lambda s} \int_H \left\| DT_s^{U^n} f(x) \right\|_{-\gamma_0}^2 \mu(dx) \, ds \leq 2 \|f\|_\infty^2 + 1.
\]

Inserting (4.11) into (4.9) we obtain for $n \geq \bar{n}_\varepsilon$ the estimate
\[
\left| \int_H \varphi \, h \, d\mu \right| \leq e^{-\lambda t_\varepsilon} \|\varphi\|_\infty \|h\|_\infty
\]
\[
+ \|f\|_{C_b^1(\mathbb{R}^n_0)} \|h\|_\infty \frac{e^{\text{Lip}_U \cdot t_\varepsilon}}{\lambda} \int_H \|B - B^n\| \, d\mu
\]
\[
+ \frac{\|h\|_\infty^2}{\lambda} \left( 2 \|f\|_\infty^2 + 1 \right)^{\frac{1}{2}} \left( \int_H \|U^n - C^n\|_{\gamma_0}^2 \, d\mu \right)^{\frac{1}{2}}
\]
\[
\leq \frac{3\varepsilon}{4} + \|f\|_{C_b^1(\mathbb{R}^n_0)} \|h\|_\infty \frac{e^{\text{Lip}_U \cdot t_\varepsilon}}{\lambda} \int_H \|B - B^n\| \, d\mu
\]
where the last inequality follows from (4.10). Consequently,
\[
\left| \int_H \varphi \, h \, d\mu \right| \leq \lim_{n \to \infty} \sup_{n \geq \bar{n}_\varepsilon} \frac{3\varepsilon}{4} + \|f\|_{C_b^1(\mathbb{R}^n_0)} \|h\|_\infty \frac{e^{\text{Lip}_U \cdot t_\varepsilon}}{\lambda} \int_H \|B - B^n\| \, d\mu = \frac{3\varepsilon}{4},
\]
which is a contradiction to our assumption. Thus $h = 0 \mu$-a.e. and the proof is complete.

We have thus proven the following

**Theorem 4.4.** Let $(\bar{L}, D(\bar{L}))$ be the closure of $(L, FC_0^2(D(A)))$ in $L^1(H, \mu)$. Then $(\bar{L}, D(\bar{L}))$ generates a $C_0$-semigroup of contractions $(\bar{P}_t)_{t \geq 0}$ on $L^1(H, \mu)$, $(\bar{P}_t)_{t \geq 0}$ is Markovian and the measure $\mu$ is $(\bar{P}_t)_{t \geq 0}$-invariant.
Stochastic Burgers equation.

We remark that the coefficients of (5.1) satisfy the following Lyapunov-condition

\[ \langle A_y + \partial_x (y + w)^2, y \rangle \leq -\frac{1}{2} \| y \|_2^2 + \alpha \| w \|_2^2 \| y \|_2^2 + \beta \| w \|_2^2, \quad y \in D(A), w \in V_2. \]

\textbf{Corollary 4.5.} Suppose that (1.1) has a unique mild solution \( X(t, x), t \geq 0 \), for any initial condition \( x \in H \), and that \( x \mapsto X(t, x) \) is measurable, \( t \geq 0 \). If the measure \( \mu \) is subinvariant for the associated transition semigroup

\[ P_t \varphi(x) = E(\varphi(X(t, x))), \varphi \in B_b(H), \]

i.e., \( \int P_t \varphi d\mu \leq \int \varphi d\mu \) for all \( \varphi \in B_b(H), \varphi \geq 0 \), then \( P_t \varphi \) is a \( \mu \)-version of \( \bar{P}_t \varphi \) for all \( \varphi \in B_b(H) \).

For the proof of the Corollary, it is sufficient to note that under the assumptions made, the transition semigroup \( (P_t)_{t \geq 0} \) induces a \( C_0 \)-semigroup \( (\bar{P}_t)_{t \geq 0} \) on \( L^1(H, \mu) \) whose infinitesimal generator extends \( (L, \mathcal{F}C^2_b(D(A))) \). Since the latter is \( L^1 \)-unique, we conclude that \( \bar{P}_t = \bar{P}_t, t \geq 0 \).

\section{5. Application}

\subsection{Stochastic Burgers equation.}

Let \( I = [0, 1] \subset \mathbb{R} \) and \( A = \frac{d^2}{dx^2} \) be the Laplacian with Dirichlet boundary conditions and consider the stochastic partial differential equation

\[ dX(t, x) = \left( \frac{d^2 X}{dx^2}(t, x) + \partial_x (X^2(t, x)) \right) dt + \eta(t, x), \quad (t, x) \in \mathbb{R}_+ \times I, \]

where \( \eta(t, x) = dW_t(x) \) and \( (W_t) \) is a cylindrical Wiener process on \( L^2(I) \) with covariance operator \( Q = (-A)^{-2\gamma_0} \) for some \( \gamma_0 \in (0, \frac{1}{3}) \) fixed. This implies in particular that the stochastic convolution corresponding to (5.1) has a continuous version in \( V_{\gamma_0} := D((-A)^{\gamma_0}) \). Therefore by using a similar argument as in [9, Chap 14] (see also [16]) one can prove the existence of a unique mild solution \( X(t), t \geq 0 \) of (5.1). Existence of an invariant probability measure \( \mu \) for (5.1) has been shown in [7], [9]. We shall mention that in the sequel we will consider \( X(t), t \geq 0 \) as a stationary solution for (5.1) (see section 1).

It is clear that the nonlinear part of the drift term \( B(u) := \partial_x (u^2) \) is neither Lipschitz nor one-sided Lipschitz. However, it is straightforward to check that

\[ \langle B(u) - B(v), u - v \rangle \leq \frac{1}{2} \| u - v \|_2^2 + \langle u^3 - v^3, u - v \rangle, \quad u, v \in V_2, \]

using the elementary inequality

\[ \frac{1}{2} (a + b)^2(a - b)^2 \leq (a^3 - b^3)(a - b), \quad a, b \in \mathbb{R}. \]

We remark that the coefficients of (5.1) satisfy the following Lyapunov-condition

\[ \langle Ay + \partial_x (y + w)^2, y \rangle \leq -\frac{1}{2} \| y \|_2^2 + \alpha \| w \|_2^2 \| y \|_2^2 + \beta \| w \|_2^2, \quad y \in D(A), w \in V_2. \]
Indeed, for $y, w \in V_{\frac{1}{2}}$, it follows that
\[
\int \partial_x (y + w)^2 \, y \, dx = -2 \int \partial_x y \cdot y w \, dx - \int \partial_x y \cdot w^2 \, dx \\
\leq \frac{1}{4} \|y\|_{\frac{1}{2}}^2 + \int y^2 w^2 \, dx + \frac{1}{4} \|y\|_{\frac{1}{2}}^2 + ||w||_{L^4(I)}^4 \\
\leq \frac{1}{2} \|y\|_{\frac{1}{2}}^2 + C \|w\|_{L^4(I)}^2 \|y\|_{\frac{1}{2}}^2 + ||w||_{L^4(I)}^4.
\]
Using the Sobolev embedding $W^{\frac{1}{2}, 2}(I) \hookrightarrow L^4(I)$ and the fact that the norm of $W^{\frac{1}{2}, 2}(I)$ and $V_{\frac{1}{2}}$ are equivalent, we conclude that
\[
\int \partial_x (y + w)^2 \, y \, dx \leq \frac{1}{2} \|y\|_{\frac{1}{2}}^2 + \alpha \|w\|_{\frac{1}{2}}^2 \|y\|_{\frac{1}{2}}^2 + \beta \|w||_{L^4(I)}^4
\]
for suitable constants $\alpha, \beta$, hence (5.3) follows. The eigenvalues of $-A$ are given by $\lambda_k = +\pi^2 k^2$, $k \geq 1$. It follows from Corollary 2.2 that for $T > 0$ we have
\[
sup_{0 \leq t \leq T} \|W_{A-\lambda(t)}\|_{L^2} \leq \lambda^{-2\varepsilon} \cdot M_{\delta, \gamma, \varepsilon}, \quad \lambda > 0,
\]
for some random variable $M_{\delta, \gamma, \varepsilon}$ with finite moments of any order, if
\[
\kappa := (\delta + \gamma_0) - (\gamma + \varepsilon) > \frac{1}{4}, \quad \delta \in [0, \frac{1}{2}],
\]
because then
\[
Z_{\gamma, \delta, \varepsilon} = \sum_{k=1}^{\infty} \lambda_k^{-2(\delta - \gamma - \varepsilon)} q_k = \sum_{k=1}^{\infty} \lambda_k^{-2(\delta + \gamma_0 - \gamma - \varepsilon)} = 2^{-4\kappa} \sum_{k=1}^{\infty} k^{-4\kappa} < \infty.
\]
Theorem 3.3 now implies the following moment estimates
\[
\int \|u\|_{L^p_{\mu}}^{2p} \mu(du) < +\infty, \quad p \geq 0
\]
\[
\int \|u\|_{H^\sigma_{\mu}}^{2p} \mu(du) < +\infty, \quad p \geq 0, \quad \sigma \leq \frac{1}{2}.
\]
The following Proposition will be crucial for the uniqueness of the Kolmogorov operator associated with (5.1).

**Proposition 5.1.** Let $\beta \in (\frac{1}{4}, \gamma_0 + \frac{1}{4})$. Then
\begin{enumerate}[(i)]
\item $\|B(u)\| \in L^1(\mu)$.
\item $\|u^3\|_{L^\beta} \in L^2(\mu)$.
\end{enumerate}

The proof is accomplished in the following three Lemmata.

**Lemma 5.2.** We have $\int \|B(u)\| \mu(du) < +\infty$.

**Proof.** First note that $\|u\|_{\infty} \leq C_{\frac{1}{4}+\varepsilon} \|u\|_{\frac{1}{4}+\varepsilon}$ for any $\varepsilon > 0$, so that
\[
\|B(u)\| \leq \|u\|_{\frac{1}{2}} \|u\|_{\infty} \leq C_{\frac{1}{4}} \|u\|_{\frac{1}{2}}^2
\]
and now the moment estimate (5.6) implies the assertion.

**Lemma 5.3.** Let $\beta \in (\frac{1}{4}, \gamma_0 + \frac{1}{4})$. Then for $p = 1, 2, 3$ we have
\[
\int \|u^2\|_{L^\beta}^{2p} \mu(du) + \int \|u^p\|^{2p} \mu(du) < +\infty.
\]
Let us now prove so that (5.8) implies that

\[ \int \|u\|_{L^p(I)}^p \mu(du) < +\infty \] if \( \frac{p-2}{1+2\theta} \leq 2 \iff p \leq 4(1+2\theta). \)

Since \( \theta > \frac{3}{8} \) implies \( 4(1+2\theta) > 7 \), we thus obtain that

\[ \int \|u^3\|_{L^2(\beta)}^2 \mu(du) < +\infty. \]

Let us now prove \( \int \|u^3\|_{L^2(\beta)}^2 \mu(du) < +\infty. \) To this end we consider again the decomposition

\[ X_t = Y_t + W_A(t), \quad t \geq 0 \]

of the mild solution of (5.1). Then for \( p \geq 1 \)

\[
\frac{1}{2p} \left( \frac{d}{dt} \|Y_t\|_{L^{2p}(I)}^{2p} \right) = \int \frac{d^2Y_t}{dx^2} Y_t^{2p-1} \, dx + \int \frac{dY_t}{dx} \left( Y_t + W_A(t) \right)^2 Y_t^{2p-1} \, dx \\
- \frac{1}{2} \left( 2p - 1 \right) \int \frac{dY_t}{dx} Y_t^{2(p-1)} \, dx - 2 \left( 2p - 1 \right) \int W_A(t) Y_t^{2p-1} \, dx \\
\leq \frac{1}{2} \left( 2p - 1 \right) \int \frac{dY_t}{dx} Y_t^{2(p-1)} \, dx + C \left( \|W_A(t)\|_{L^p(I)}^8 + \|Y_t\|_{L^{2p}(I)}^{2p} \right). 
\]

Integrating (5.7) with respect to \( t \) we conclude that

\[
\int_0^T \left( \frac{dY_t}{dx} \right)^p \, dt \leq C_1 \|Y_0\|_{L^{2p}(I)}^{2p} + C_2 \int_0^T \|W_A(t)\|_{L^p(I)}^8 + \|Y_t\|_{L^{2p}(I)}^{2p} \, dt. 
\]  

Clearly, for some constant \( C > 0 \) we have

\[ \mathbb{E} \left( \|W_A(t)\|_{L^p(I)}^8 \right) \leq C \left( \sum_{k=1}^{+\infty} \frac{1}{\lambda_k^{1+\gamma_0}} \right)^4 < +\infty, \]

so that (5.8) implies that

\[
\int_0^T \mathbb{E} \left( \left( \frac{dY_t}{dx} \right)^p \right) \, dt \leq C_1 \mathbb{E} \left( \|Y_0\|_{L^{2p}(I)}^{2p} \right) + \tilde{C}_2 + C_2 \int_0^T \mathbb{E} \left( \|Y_t\|_{L^{2p}(I)}^{2p} \right) \, dt 
\]

for uniform constants \( C_1, \tilde{C}_2 \) and \( C_2 \). Next, observe that for \( \beta < \frac{4}{1+\gamma_0} \) we have

\[ \sup_{t \geq 0} \mathbb{E} \left( \|W_A(t)\|_{L^p(I)}^{2p} \right) < +\infty, \quad p \geq 1, \]
hence for $T \geq 0$ we have
\[
\int_0^T \mathbb{E}\left(\|X_t^3\|_3^2\right) dt \leq C \left( \int_0^T \mathbb{E}\left(\|Y_t^3\|_3^2\right) dt + \int_0^T \mathbb{E}\left(\|W_A(t)^3\|_3^2\right) dt \right)
\leq C \left( \int_0^T \mathbb{E}\left(\|Y_t^3\|_3^2\right) dt + \int_0^T \mathbb{E}\left(\|W_A(t)^3\|_3^2\right) dt \right)
\leq C \left( T + \mathbb{E}\left(\|Y_0\|_{L^6(I)}^6\right) + \int_0^T \mathbb{E}\left(\|Y_t(x)\|_{L^6(I)}^6\right) dt \right)
\leq C \left( T + \mathbb{E}\left(\|X_0\|_{L^6(I)}^6\right) + \int_0^T \mathbb{E}\left(\|X_t(x)\|_{L^6(I)}^6\right) dt \right)
= C \left( T + \mathbb{E}\left(\|X_0\|_{L^6(I)}^6\right) + \int_0^T \mathbb{E}\left(\|X_t(x)\|_{L^6(I)}^6\right) dt \right),
\]
where the constant $C$ may change from line to line. Thus integrating with respect to $\mu$ and using the fact that $(X_t)_{t \geq 0}$ is a stationary solution for (5.1) with invariant distribution $\mu$ we get
\[
T \int_H \|x^3\|_3^2 \mu(dx) \leq \tilde{C} \left( T + \int_H \|x\|_{L^6(I)}^6 \mu(dx) + T \int_H \|x^3\|_3^2 \mu(dx) \right).
\]
This yields the statement of the lemma.

If we denote by $L$ the Kolomogorov operator associated with (5.1), we have by Theorem 4.4 the closure $(\bar{L}, D(\bar{L}))$ of $L$ in $L^1(H, \mu)$ generates a $C_0$-semigroup of contractions $(\bar{P}_t)_{t \geq 0}$ on $L^1(H, \mu)$, $(\bar{P}_t)_{t \geq 0}$ is Markovian and the measure $\mu$ is $(\bar{P}_t)_{t \geq 0}$-invariant.

5.2. **Stochastic equations modeling thin-film growth.** Let us consider the following stochastic partial differential equation
\[
du(t, x) = \left( -\partial_x^{(4)} u(t, x) + \nu \partial_x^{(2)} u(t, x) - \partial_x^{(2)} \left( \partial_x u(t, x) \right)^2 \right) dt + \eta(t, x), \quad (t, x) \in \mathbb{R}_+ \times I,
\]
(5.10)
where $\nu \geq 0$, $\eta(t, x) = dW_t(x)$ is a Wiener process on $L^2([0, 1])$ with covariance operator satisfying
\[
Qe_k = q_k e_k, \quad k \geq 1, \quad (q_k)_{k \geq 1} \subseteq L^\infty(N),
\]
and $(e_k)_{k \geq 1}$ denotes the orthonormal basis of $H := L^2([0, 1]) = \{ f \in L^2([0, 1]) : \int_0^1 f(r) dr = 0 \}$ consisting of eigenvectors of the self-adjoint extension of
\[
A := -\partial_x^{(4)} u + \nu \partial_x^{(2)} u
\]
in $H$ with periodic or Neumann boundary conditions.
Blömker and Hairer proved in [3] the existence of a stationary solution of (5.10). In particular, they showed the existence of an invariant measure $\mu$ satisfying the moment estimate
\[
\int_H \log(1 + \|x\|^2) \mu(dx) < +\infty.
\]
The purpose of the example is to demonstrate how to obtain improved a priori moment estimates on $\mu$, using Theorem 3.3. To this end note that the coefficients of (5.10) satisfy the inequality
\[
\langle Ay + B(y + w), y \rangle \leq -\frac{1}{4} \|y\|_2^2 + \beta \|w\|_8^8 \|y\|^2 + \gamma \|w\|^2_2.
\]
(5.11)
for suitable constants $\beta$ and $\gamma$, since
\[
\int_0^1 \partial_x^2 \left( \partial_x (y + w) \right)^2 (y) \, dx = 2 \int_0^1 \partial_x^2 y(x) \cdot \partial_x y(x) \cdot \partial_x w(x) \, dx + \int_0^1 \partial_x^2 y(x) \left( \partial w(x) \right)^2 \, dx \\
\leq \frac{1}{4} \| y \|_2 \| \partial_y (y(x)) \|_2^2 + \frac{1}{4} \| y \|_2 \| \partial_x w \|_{L^4([0,1])}^2 \\
\leq \frac{1}{2} \| y \|_2 \| \partial_x w \|^2_{L^2([0,1])} + \| \partial_x y \|^2_{L^2([0,1])}.
\]

Using the fact that $W^{2,2}_2(0,1) \subset W^{1,4}(0,1)$ and $\| u \|_{W^{1,4}(0,1)} \leq C \| u \|_{\frac{3}{2}}$, we conclude that
\[
\int_0^1 \partial_x (y + w) \right)^2 (y) \, dx \leq \frac{3}{4} \| y \|_2^2 + \beta \| \partial_x w \|_{\frac{5}{2}}^2 \| y \|_2^2 + \gamma \| w \|_2^2
\]
for suitable constants $\beta$, $\gamma$, hence (5.11) follows.

We can arrange the eigenvalues $(\lambda_k)_{k \geq 1}$ of $-A$ in such a way that $\lambda_k = 4\pi^2 k^2 (4\pi^2 k^2 + \nu)$ with multiplicity 1 (in the case of Neumann boundary conditions) or 2 (in the case of periodic boundary conditions). Then (H4) is satisfied for $\kappa = -\gamma - \varepsilon > \frac{1}{8}$, $\delta \in (0, \frac{1}{2})$, because then
\[
Z_{\gamma,\delta,\varepsilon} \leq 2 \sum_{k=1}^{+\infty} \left( 4\pi^2 k^2 (4\pi^2 k^2 + \nu) \right)^{-2(\delta - \gamma - \varepsilon)} \| q \|_{\infty} \leq C \sum_{k=1}^{+\infty} \kappa^{-8\kappa} \text{ if } \kappa > \frac{1}{8}.
\]

Theorem 3.3 now implies the following improved a priori moment estimates

**Corollary 5.4.** For any invariant measure $\mu$ of (5.10) we have

(i) $\int \| x \|_2^p \mu(dx) < \infty$ for $p \geq 0$.

(ii) $\int \| x \|_2^\sigma \| x \|_2^p \mu(dx) < \infty$ for $p \geq 0$, $\sigma \leq \frac{1}{2}$.

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