A CLASS OF INTEGRAL GRAPHS CONSTRUCTED FROM
THE HYPERCUBE

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Abstract. In this paper, we determine the set of all distinct eigenvalues
of the line graph which is induced by the first and second layers of the
hypercube $Q_n$, $n > 3$. We show that this graph has precisely five distinct
eigenvalues and all of its eigenvalues are integers.

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1. Introduction

In this paper, a graph $\Gamma = (V, E)$ is considered as an undirected simple graph
where $V = V(\Gamma)$ is the vertex-set and $E = E(\Gamma)$ is the edge-set. For all the
terminology and notation not defined here, we follow [1,4,5].

Let $n \geq 1$ be an integer. The hypercube $Q_n$ is the graph whose vertex
set is $\{0, 1\}^n$, where two $n$-tuples are adjacent if they differ in precisely one
coordinates. In the graph $Q_n$, the layer $L_i$ is the set of vertices which contain
$i$ 1s, namely, vertices of weight $i$, $1 \leq i \leq n$. We denote by $Q_n(i, i + 1)$,
the subgraph of $Q_n$ induced by layers $L_i$ and $L_{i+1}$. In this paper, we want to
determine the set of all distinct eigenvalues of line graph of the graph $Q_n(1, 2)$.

We can consider the graph $Q_n$ from another point of view. The Boolean
lattice $BL_n$, $n \geq 1$, is the graph whose vertex set is the set of all subsets of $[n] = \{1, 2, ..., n\}$, where two subsets $x$ and $y$ are adjacent if their symmetric
difference has precisely one element. In the graph $BL_n$, the layer $L_i$ is the set of
$i$–subsets of $[n]$. We denote by $BL_n(i, i + 1)$, the subgraph of $BL_n$ induced by
layers $L_i$ and $L_{i+1}$. It is an easy task to show that the graph $Q_n$ is isomorphic
with the graph $BL_n$, by an isomorphism that induces an isomorphism from
$Q_n(i, i + 1)$ to $BL_n(i, i + 1)$.

In the sequel, we denote the graph $BL_n(1, 2)$ by $H(n)$. Therefore, the graph
$H(n)$ is a graph with vertex set

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\[ V = \{ v \mid v \subset [n], |v| \in \{1, 2\} \} \] and the edge set
\[ E = \{ \{ v, w \} \mid v, w \in V, v \subset w \text{ or } w \subset v \}. \]

It is clear that \( H(n) \) is a bipartite graph with cells \( V_1, V_2 \) where
\[ V_1 = \{ v \mid v \subset [n], |v| = 1 \} \quad \text{and} \quad V_2 = \{ v \mid v \subset [n], |v| = 2 \}. \]

Also, if \( v \in V_1 \), then \( \deg(v) = n - 1 \) whereas if \( v \in V_2 \), then \( \deg(v) = 2 \), hence \( H(n) \) is not a regular graph. Now, it is obvious that \( H(n) \) has \( n(n-1) \) edges. The following figure shows \( H(5) \) in the plane.

![H(5)](image)

We can see, by an easy argument that the graph \( H(n) \) is connected and its diameter is 4.

We now consider the line graph of \( H(n) \). We denote by \( L(n) \) the line graph of the graph \( H(n) \). Then each vertex of \( L(n) \) is of the form \( \{v, w\} = \{\{i\}, \{i, j\}\} \), where \( i, j \in [n], i \neq j \), and two vertices \( \{v, w\}, \{u, s\} \) are adjacent whenever the intersection of them is a set of order 1. In the sequel, we denote the vertex \( \{\{i\}, \{i, j\}\} \) by \( [i, ij] \). Thus, if \( [i, ij] \) is a vertex of \( L(n) \), then
\[ N([i, ij]) = \{[i, ik] \mid j, i \neq k \in [n]\} \cup \{[j, ij]\}. \]

Hence, \( \deg([i, ij]) = n - 1 \), in other words, \( L(n) \) is a regular graph of valency \( n - 1 \). By an easy argument, we can show that the graph \( L(n) \) is a connected graph with diameter 3. Also, its girth is 3 and hence it is not a bipartite graph. The following figure displays \( L(4) \) in the plane. Note that in the following figure the vertex \( [i, ij] \) is denoted by \( i, ij \).
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1,12 1,13

2,12 1,14

2,23 2,24 4,24

4,41 4,43

3,31 3,34

L(4)

The graph $L(n)$ has various interesting properties, amongst of them, we interested in its spectrum. In the present paper, we show that $L(n)$ has precisely 5 distinct eigenvalues. Also, we show that each eigenvalue of $L(n)$ is an integer.

2. Preliminaries

Let $\Gamma$ be a graph with vertex set $V = \{v_1, v_2, ..., v_n\}$ and edge set $E(\Gamma)$. The adjacency matrix $A = A(\Gamma) = [a_{ij}]$ of $\Gamma$ is an $n \times n$ symmetric matrix of 0's and 1's with $a_{ij} = 1$ if and only if $v_i$ and $v_j$ are adjacent. The characteristic polynomial of $\Gamma$ is the polynomial $P(G) = P(G, x) = det(xI_n - A)$, where $I_n$ denotes the $n \times n$ identity matrix. The spectrum of $A(\Gamma)$ is also called the spectrum of $\Gamma$. If the eigenvalues of $\Gamma$ are ordered by $\lambda_1 > \lambda_2 > ... > \lambda_r$, and their multiplicities are $m_1, m_2, ..., m_r$, respectively, then we write

$$Spec(\Gamma) = (\lambda_1, \lambda_2, ..., \lambda_r) \text{ or } Spec(\Gamma) = \{\lambda_1^{m_1}, \lambda_2^{m_2}, ..., \lambda_r^{m_r}\}$$

If all the eigenvalues of the adjacency matrix of the graph $\Gamma$ are integers, then we say that $\Gamma$ is an integral graph. The notion of integral graphs was first introduced by F. Harary and A.J. Schwenk in 1974 (see [6]). In 1976 Bussemaker and Cvetkovic [3], proved that there are exactly 13 connected cubic integral graphs. In general, the problem of characterizing integral graphs seems to be very difficult. There are good surveys in this area (for example [2]).

Let $X$ be a graph with vertex set $V$. We have the following definitions and facts [5].
Definition 2.1. A partition $\pi$ of $V$ with cells $C_1, \ldots, C_r$ is equitable if the number of neighbours in $C_j$ of a vertex $v \in C_i$ is a constant and depends only on the choice of $C_i$ and $C_j$. In this case, we denote the number of neighbors in $C_j$ of any vertex in $C_i$ by $p_{ij}$.

It is clear that if $\pi$ is an equitable partition with cells $C_1, \ldots, C_r$ then every vertex in $C_i$ has the same valency.

Definition 2.2. Let $X$ be a graph. If $H \leq Aut(X)$ is a group of automorphisms of $X$, then $H$ partition the vertex set of $X$ into orbits. The partition of $X$ consisting of the set of orbits which are constructed by $H$, is called an orbit partition of $X$.

Definition 2.3. Let $X$ be a graph with equitable partition $\pi = \{C_1, \ldots, C_r\}$. The directed graph with vertex set $\pi$ with $b_{ij}$ arcs from $C_i$ to $C_j$ is called the quotient of $X$ over $\pi$ and is denoted by $X/\pi$.

Therefore, if $P = (p_{ij})$ is the adjacency matrix of the directed graph $X/\pi$, then $p_{ij}$ is the number of neighbours in $C_j$ of any vertex in $C_i$.

Theorem 2.4. Let $X$ be a graph with equitable partition $\pi$. Let $P$ be the adjacency matrix of the directed graph $X/\pi$ and $A$ be the adjacency matrix of $X$. Then each eigenvalue of the matrix $P$ is an eigenvalue of the matrix $A$.

Theorem 2.5. Let $X$ be a vertex-transitive graph and $H \leq Aut(X)$ is a group of automorphisms of $X$. If $\pi$, the orbit partition of $H$, has a singleton cell $\{u\}$, then every eigenvalue of $X$ is an eigenvalue of $X/\pi$.

Proposition 2.6. Every orbit partition is an equitable partition.

Proof. Let $X$ be a graph and $H$ be a subgroup of the automorphism group of $X$. Let $v, w \in V = V(X)$ and $O_1 = H(v), O_2 = H(w)$ are orbits of $H$. Let $v$ has $r$ neighbours in $O_2$, and $u$ has $s$ neighbours in $O_2$. Let $N(v) \cap O_2 = \{w_1, \ldots, w_r\}$. Since $u$ is a vertex in $O_1$, then there is some $h \in H$ such that $u = h(v)$, and thus we have:

$$N(u) \cap O_2 = N(h(v)) \cap O_2 = h(N(v)) \cap O_2 = h(\{w_1, \ldots, w_r\}) \cap O_2 = \{h(w_1), \ldots, h(w_r)\} \cap O_2 = \{h(w_1), \ldots, h(w_r)\}$$

Note that since $w_i \in H(w) = O_2, 1 \leq i \leq r$, then $h(w_i) \in hH(w) = H(w) = O_2$, because $H$ is a group. Now, Our argument shows that $r = s$. 

□
3. Main results

Proposition 3.1. The graph $L(n)$ is a vertex-transitive graph.

Proof. We let $\Gamma = L(n)$. Note that the group $\text{Sym}([n])$ is a subgroup of the group $\text{Aut}(\Gamma)$. If $\theta \in \text{Sym}([n])$, then $f_\theta$ defined by the rule $f_\theta[i, ij] = [\theta(i), \theta(i)\theta(j)]$ is an automorphism of the graph $\Gamma$. In fact, if $G = \{f_\theta \mid \theta \in \text{Sym}(n)\}$, then $G$ is isomorphic with $\text{Sym}([n])$ and $G \subseteq \text{Aut}(\Gamma)$. Let $[i, ij]$ and $[l, lk]$ are arbitrary vertices of $L(n)$. Then for $\theta = (i, l)(j, k) \in \text{Sym}([n])$, we have $f_\theta[i, ij] = [\theta(i), \theta(i)\theta(j) = [l, lk]$.

Let $\Gamma = L(n)$. We can determine some of the eigenvalues of $\Gamma$. For example since $\Gamma$ is $(n - 1)$-regular, then $n - 1$ is the largest eigenvalue of $\Gamma$ [1, chap 3]. On the other hand, since $\Gamma$ is a line graph, then for each eigenvalue $e$ of $\Gamma$ we have $-2 \leq e$ [5, chap 8]. We show that $-2$ is an eigenvalue of $\Gamma$.

Proposition 3.2. Let $\Gamma = L(n)$. Then $-2$ is an eigenvalue of $\Gamma$.

Proof. Let $S$ be the subgraph of $\Gamma$ induced by the vertex set;

$$B = \{[1, 12], [1, 13], [3, 13], [3, 32], [2, 32], [2, 21]\}$$

It is clear that $S$ is a 6-cycle, and hence $-2$ is an eigenvalue of the subgraph $S$ [1, chap 3]. Let $\theta_{\min}$ be the minimum of the eigenvalues of $\Gamma$. Therefore, by [5, chap 8] we have $\theta_{\min} \leq -2$. Now by what is stated above, we conclude that $\theta_{\min} = -2$.

We now try to find some other eigenvalues of the graph $L(n)$, by constructing a suitable partition of the vertex set of this graph.

Proposition 3.3. Let $\Gamma = L(n)$. Then $-1$ and $n - 2$ are eigenvalues of the graph $\Gamma$.

Proof. Let $H = \{f_\alpha \mid \alpha \in \text{Sym}([n]), \alpha(1) = 1\}$. Then $H$ is a subgroup of $\text{Aut}(\Gamma)$, the automorphism group of $\Gamma$. Let $H_1 = \{\alpha \mid f_\alpha \in H\}$. Note that $H_1$ is a subgroup of $\text{Sym}([n])$ isomorphic with $\text{Sym}([n - 1])$. In the sequel, we try to find the orbit partition of $H$. The group $H$ has three orbits in its action on the vertex set of $\Gamma$. In fact, we have the following orbits;

$O_1 = H([1, 12]) = \{h([1, 12]) \mid h \in H\} = \{[\alpha(1), \alpha(1)\alpha(2)] \mid \alpha \in H_1\} = \{[1, li] \mid i \in \{2, 3, ..., n\}\}$. 

We now can see that the eigenvalues of the matrix $P$ are $n-1, n-2$ and $-1$. Since by Theorem 2.4. and Theorem 2.6., every eigenvalue of $P$ is an eigenvalue of the graph $\Gamma$, thus $-1$ and $n-2$ are also eigenvalues of $\Gamma$.  \( \square \)
We now try to find an orbit partition \( \pi \) such that this partition has a singleton cell. If we construct such a partition for the vertex set of \( L(n) \), then by Theorem 2.5. and Theorem 2.6. every eigenvalue of the graph \( L(n) \) is an eigenvalue of the matrix \( P \), the adjacency matrix of the directed graph \( L(n)/\pi \).

Let \( K = \{ f_\alpha \mid \alpha \in \text{Sym}([n]), \alpha(1) = 1, \alpha(2) = 2 \} \). Then \( K \) is a subgroup of \( \text{Aut}(\Gamma) \), the automorphism group of \( \Gamma \). Let \( H_2 = \{ \alpha \mid f_\alpha \in H \} \). Note that \( H_2 \) is a subgroup of \( \text{Sym}([n]) \) isomorphic with \( \text{Sym}([n-2]) \). In the sequel, we want to determine the orbit partition of the subgroup \( K \). In fact, \( K \) generates the following orbits:

\[
\begin{align*}
O_1 &= K([1, 12]) = \{ k([1, 12]) \mid k \in K \} = \{ [\alpha(1), \alpha(1)\alpha(2)] \mid \alpha \in H_2 \} = [1, 12], \\
O_2 &= K([1, 13]) = \{ k([1, 13]) \mid k \in K \} = \{ [\alpha(1), \alpha(1)\alpha(3)] \mid \alpha \in H_2 \} = \{ [1, 1i] \mid 3 \leq i \leq n \}, \\
O_3 &= K([2, 12]) = \{ k([2, 12]) \mid k \in K \} = \{ [\alpha(2), \alpha(1)\alpha(2)] \mid \alpha \in H_2 \} = [2, 12], \\
O_4 &= K([2, 23]) = \{ k([2, 23]) \mid k \in K \} = \{ [\alpha(2), \alpha(2)\alpha(3)] \mid \alpha \in H_2 \} = \{ [2, 2i] \mid 3 \leq i \leq n \}, \\
O_5 &= K([3, 13]) = \{ k([3, 13]) \mid k \in K \} = \{ [\alpha(3), \alpha(1)\alpha(3)] \mid \alpha \in H_2 \} = \{ [i, 1i] \mid 3 \leq i \leq n \}, \\
O_6 &= K([3, 23]) = \{ k([3, 23]) \mid k \in K \} = \{ [\alpha(3), \alpha(2)\alpha(3)] \mid \alpha \in H_2 \} = \{ [i, 2i] \mid 3 \leq i \leq n \}, \\
O_7 &= K([3, 34]) = \{ k([3, 34]) \mid k \in K \} = \{ [\alpha(3), \alpha(3)\alpha(4)] \mid \alpha \in H_2 \} = \{ [i, ij] \mid 3 \leq i, j \leq n, i \neq j \}.
\end{align*}
\]

If we let \( \pi = \{ O_1, O_2, O_3, ..., O_7 \} \) and \( p_{ij} \) is the number of arcs from \( O_i \) to \( O_j \), then \( O_1 \cup O_2 \cup ... \cup O_7 = V = V(\Gamma) \), and the following hold.

\[ p_{12} = n - 2, \quad p_{13} = 1, \quad \text{and} \quad p_{1j} = 0, j \neq 2, 3, \] because the vertex \([1, 12] \in O_1\) is adjacent to all the \( n - 2 \) vertices in \( O_2 \), and 1 vertex in \( O_3 \), namely, \([2, 12]\) and therefore 0 vertex in other orbits.

\[ p_{21} = 1, \quad p_{22} = n - 3, \quad p_{25} = 1 \quad \text{and} \quad p_{2,j} = 0, j \neq 1, 2, 5, \] because the vertex \([1, 13] \in O_2\) is adjacent to 1 vertex in \( O_1 \), say, \([1, 12]\), and \( n - 3 \) other vertices in \( O_2 \), and 1 vertex in \( O_5 \), namely, \([3, 31] \), and therefore 0 vertex in other orbits.
\[ p_{31} = 1, \; p_{34} = n - 2, \; \text{and} \; p_{3,j} = 0, j \neq 1, 4, \] because the vertex \([2, 21] \in O_3\)

is adjacent to 1 vertex in \(O_1\), namely, \([1, 12]\), and all of the \(n - 2\) vertices in

\(O_4\), and therefore 0 vertex in other orbits.

\[ p_{43} = 1, \; p_{44} = n - 3, \; p_{46} = 1 \; \text{and} \; p_{4,j} = 0, j \neq 3, 4, 6, \] because the vertex

\([2, 23] \in O_4\) is adjacent to 1 vertex in \(O_3\), say, \([2, 21]\), and \(n - 3\) other vertices

in \(O_4\) and 1 vertex in \(O_6\), namely, \([3, 32]\), and therefore 0 vertex in other orbits.

\[ p_{52} = 1, \; p_{56} = 1, \; p_{57} = n - 3 \; \text{and} \; p_{5,j} = 0, j \neq 2, 6, 7, \] because the vertex

\([3, 31] \in O_5\) is adjacent to 1 vertex in \(O_2\), namely, \([1, 13]\), and 1 vertex in \(O_6\),

say, \([3, 32]\), and \(n - 3\) vertices in \(O_7\), namely, every vertex in \(O_7\) of the form

\([3, 3,j]\), \(4 \leq j \leq n\), and therefore 0 vertex in other orbits.

\[ p_{64} = 1, \; p_{65} = 1, \; p_{67} = n - 3 \; \text{and} \; p_{6,j} = 0, j \neq 4, 5, 7, \] because the vertex

\([3, 23] \in O_6\) is adjacent to 1 vertex in \(O_4\), say, \([2, 23]\), and 1 vertex in \(O_5\),

namely, \([3, 31]\), and \(n - 3\) vertices in \(O_7\), namely, every vertex in \(O_7\) of the form

\([3, 3,j]\), \(4 \leq j \leq n\) and therefore 0 vertex in other orbits.

\[ p_{75} = 1, \; p_{76} = 1, \; p_{77} = n - 3, \; \text{and} \; p_{7,j} = 0, j \neq 5, 6, 7, \] because the vertex

\([3, 34] \in O_7\) is adjacent to 1 vertex in \(O_5\), say, \([3, 31]\), and 1 vertex in \(O_6\),

namely \([3, 32]\), and \(n - 3\) vertices in \(O_7\), namely, every vertex in \(O_7\) of the form

\([3, 3,j]\), \(4 \leq j \leq n\) and therefore 0 vertex in other orbits.

Therefore, the following matrix \(P\) is an adjacency matrix for the directed graph \(\Gamma/\pi\).

\[
P = \begin{bmatrix}
0 & n - 2 & 1 & 0 & 0 & 0 & 0 \\
1 & n - 3 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & n - 2 & 0 & 0 & 0 \\
0 & 0 & 1 & n - 3 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & n - 3 \\
0 & 0 & 0 & 1 & 1 & 0 & n - 3 \\
0 & 0 & 0 & 1 & 1 & 0 & n - 3 
\end{bmatrix}
\]

We can use the Mathematica program for finding the eigenvalues of the matrix \(P\). By the Mathematica program, we have;
Since by Theorem 2.4, Theorem 2.5, and Theorem 2.6, the set of distinct eigenvalues of the graph $L(n)$ is equal to the set of distinct eigenvalues of the matrix $P$, hence $\{-2, -1, 0, n - 2, n - 1\}$ is the set of all distinct eigenvalues of the graph $L(n)$. We now have the following result.

**Theorem 3.4.** Let $n > 3$ be an integer. Then the graph $L(n)$ is vertex-transitive integral graph with distinct eigenvalues $-2, -1, 0, n - 2, n - 1$.

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