HOLOMORPHIC MORSE INEQUALITIES AND SYMPLECTIC REDUCTION

MAXIM BRAVERMAN

Abstract. We introduce Morse-type inequalities for a holomorphic circle action on a holomorphic vector bundle over a compact Kähler manifold. Our inequalities produce bounds on the multiplicities of weights occurring in the twisted Dolbeault cohomology in terms of the data of the fixed points and of the symplectic reduction. This result generalizes both Wu-Zhang extension of Witten's holomorphic Morse inequalities and Tian-Zhang Morse-type inequalities for symplectic reduction.

As an application we get a new proof of the Tian-Zhang relative index theorem for symplectic quotients.

1. Introduction

Let \((M, \omega)\) be a compact Kähler manifold of complex dimension \(n\) and let \(E\) be a holomorphic Hermitian vector bundle over \(M\) with the compatible holomorphic connection. We denote by \(H^\bullet(M, \mathcal{O}(E))\) the cohomology groups with coefficients in the sheaf of holomorphic sections of \(E\), calculated from the twisted Dolbeault complex \((\Omega^{0,*}(M, E), \bar{\partial}_E)\).

Suppose that the circle group \(S^1\) acts holomorphically and effectively on \(M\) preserving the Kähler structure and that the action can be lifted to \(E\) preserving the Hermitian structure on \(E\). Then, for each \(p = 0, 1, \ldots, n\), we obtain a representation of \(S^1\) on \(H^p(M, \mathcal{O}(E))\). The holomorphic inequalities we are going to discuss in this paper give an estimate on the characters of these representations.

More precisely, assume that \(\mu : M \to \mathbb{R}\) is a momentum map for the circle action on \(M\) (it always exists provided the set of fixed points is not empty \([7]\)) and that \(a \in \mathbb{R}\) is a regular value of \(\mu\). Our inequalities estimate the character of \(H^p(M, \mathcal{O}(E))\) in terms of the fixed-point data and the structure of the reduced space \(M_a = \mu^{-1}(a)/S^1\).

Our inequalities contain as special cases both the Wu-Zhang extension \([14]\) of the Witten holomorphic Morse inequalities \([13, 9]\) and the Tian-Zhang \([10]\) Morse type inequalities for geometric quantization. In fact, our inequalities may be considered as a combination of the results of \([14, 10]\) (note, however, that in \([10]\) the inequalities are obtained for a much more general case where the circle is replaced by an arbitrary compact Lie group).

The proof of our main theorem is based on Witten deformation of the Dolbeault operator \(\bar{\partial}\). Technically it is very simple since all the necessarily calculations are already
contained in $\mathbb{L}$. Moreover, we need only a very simple version of the calculations in
$\mathbb{L}$ since we work with a circle and not with an arbitrary compact Lie group.

As an application of our inequalities we get a new proof of the Tian-Zhang relative
index theorem for symplectic quotients $\mathbb{L}$, Theorem 5.7 (In $\mathbb{L}$, the result is obtained for
arbitrary symplectic manifolds. Though we prove the theorem only for Kähler manifolds,
our method can be easily extended to the case of an arbitrary symplectic manifold).

2. Main results

In this section we formulate our main result (Theorem 2.4) and discuss various appli-
cations. The proof of Theorem 2.4 is postponed to the next section.

2.1. Weights and formal characters. Irreducible representation of the circle group
$S^1 = \{e^{i\theta} : \theta \in \mathbb{R}\}$ are classified by integer weights (here we use the identification of the
Lie algebra of $S^1$ with $\mathbb{R}$ which takes the negative primitive lattice element,
$-2\pi i \in i \mathbb{R} = \text{Lie}(S^1)$, to 1). A representation of weight $k \in \mathbb{Z}$ is isomorphic to the complex line
$\mathbb{C}$ on which the element $e^{i\theta} \in S^1$ act by multiplications by $e^{-ik\theta}$.

If $W$ is a finite dimensional representation of $S^1$ we denote by $\text{mult}_k(W)$ the multiplicity
of the weight $k \in \mathbb{Z}$ in $W$. Note that the multiplicity $\text{mult}_0(W)$ of the zero weight is equal
to the dimension of the space $W^{S^1}$ of vectors in $W$ which are invariant with respect to
the action of $S^1$. Let support of $W$ be $\text{supp}(W) = \{k \in \mathbb{Z} : \text{mult}_k(W) \neq 0\}$.

The formal character of $W$ is the formal sum

$$\text{char}(W) = \sum_{k \in \mathbb{Z}} \text{mult}_k(W)e^{-ik\theta}.$$ 

It lies in the ring $\mathbb{Z}[e^{i\theta}, e^{-i\theta}]$ of Laurent polynomials in $e^{i\theta}$ with integer coefficients. This
ring is called the ring of formal characters of the circle group. We will also consider the
completion $\mathcal{L} = \mathbb{Z}[[e^{i\theta}, e^{-i\theta}]]$ of this ring. The elements of $\mathcal{L}$ are formal infinite sums
$q(\theta) = \sum_{k \in \mathbb{Z}} q_k e^{-ik\theta}$ where $q_k \in \mathbb{Z}$.

2.2. Momentum map and symplectic reduction. Let $V$ denote the vector field on
$M$ that generates the $S^1$-action. We will assume that $S^1$-action is Hamiltonian, i.e. there
is a moment map $\mu : M \to \mathbb{R}$ such that $i_V \omega = d\mu$. Note (4) that it is always the case if
the fixed-point set of $S^1$ on $M$ is non-empty.

Assume that $a \in \mathbb{R}$ is a regular value of the momentum map $\mu$. Then $\mu^{-1}(a) \subset M$ is
a smooth submanifold endowed with a locally free action of $S^1$. For simplicity, we will
assume that this action is free. Then the quotient space $M_a = \mu^{-1}(a)/S^1$ is a smooth
Kähler manifold. The vector bundle $E$ descends to a holomorphic Hermitian vector bundle
$E_a$ over $M_a$. 


Let $\mathcal{F} = M \times \mathbb{C}$ denote the trivial line bundle over $M$ with $S^1$ action defined by the formula

$$e^{i\theta} : (x, z) \mapsto (e^{i\theta} \cdot x, e^{i\theta} z), \quad x \in M, \ z \in \mathbb{C}. \quad (2.1)$$

Denote by $q : \mu^{-1}(a) \to M_a = \mu^{-1}(a)/S^1$ the projection. The restriction $\mathcal{F}|_{\mu^{-1}(a)}$ of $\mathcal{F}$ on $\mu^{-1}(a)$ descends to a unique bundle $\mathcal{F}_a$ over $M_a$ such that $q^*\mathcal{F}_a = \mathcal{F}|_{\mu^{-1}(a)}$.

Let $\mathcal{F}^{-1}, \mathcal{F}_a^{-1}$ be the dual bundles to $\mathcal{F}$ and $\mathcal{F}_a$ respectively. If $k \geq 0$, we denote by $\mathcal{F}^\pm k$ (resp. $\mathcal{F}_a^\pm k$) the $k$-th tensor power of the bundle $\mathcal{F}^\pm 1$ (resp. $\mathcal{F}_a^\pm 1$). Note that $q^*\mathcal{F}^k_a = \mathcal{F}^k$. Obviously,

$$\text{mult}_m H^*(M, \mathcal{O}(E \otimes F^k)) = \text{mult}_{m+k} H^*(M, \mathcal{O}(E)) \quad (2.2)$$

for any $k, m \in \mathbb{Z}$.

We will be interested in cohomology $H^*(M_a, \mathcal{O}(E_a \otimes F^k_a))$ of $M_a$ with coefficients in the sheaf of holomorphic sections of $E_a \otimes F^k_a$.

There is another action of $S^1$ on $\mathcal{F}$ given by the formula

$$e^{i\theta} : (x, z) \mapsto (x, e^{i\theta} z), \quad x \in M, \ z \in \mathbb{C}. \quad (2.3)$$

This action commutes with $(2.1)$ and, hence, reduces to $\mathcal{F}_a$. Therefore we obtain induced actions of $S^1$ on $\mathcal{F}^k_a$. This action preserve the base points in $F$ and the weight of the representation on the fibers is equal to $-k$.

Let $S(\mathcal{F}_a, \mathcal{F}_a^{-1})$ denote the direct sum

$$S(\mathcal{F}_a, \mathcal{F}_a^{-1}) = \bigoplus_{k \in \mathbb{Z}} \mathcal{F}_a^k.$$ 

One should think about $S(\mathcal{F}_a, \mathcal{F}_a^{-1})$ as about symmetric algebra in $\mathcal{F}_a, \mathcal{F}_a^{-1}$. This is an infinite-dimensional $S^1$-equivariant bundle over $M_a$. However the cohomology groups $H^*(M_a, \mathcal{O}(E_a \otimes S(\mathcal{F}_a, \mathcal{F}_a^{-1})))$ are sums of those with coefficients in $\mathcal{F}_a^k$. Therefore the multiplicity of any weight $k$ in $H^*(M_a, \mathcal{O}(E_a \otimes S(\mathcal{F}_a, \mathcal{F}_a^{-1})))$ is finite and the character

$$\text{char} H^*(M_a, \mathcal{O}(E_a \otimes S(\mathcal{F}_a, \mathcal{F}_a^{-1}))) = \sum_{k \in \mathbb{Z}} e^{ik\theta} \dim_{\mathbb{C}} H^*(M_a, \mathcal{O}(E_a \otimes F^k_a)). \quad (2.4)$$

is well defined as an element of $\mathcal{L}$.

2.3. **Fixed-point set.** Suppose that $F$ is a connected component of the fixed-point set of the $S^1$-action on $M$. Then $F$ is a compact Kähler manifold. Let $n - n_F$ be the complex dimension of $F$. The complexification of the normal bundle $N_F \to F$ in $M$ has the decomposition $N_F^\mathbb{C} = N_F^{(1,0)} \oplus N_F^{(0,1)}$, where $N_F^{(1,0)}$ is a holomorphic vector bundle over $F$ of rank $n_F$. The circle $S^1$ acts on $N_F$ preserving the base points in $F$. Moreover, the weights of the isotropy representations on the normal fiber are constant along $F$. 

Let $\lambda_k$ ($1 \leq k \leq n - n_F$) be the isotropy weights on $N_F^{(1,0)}$. Let $N_F^{\pm,(1,0)}$ be the direct sum of the sub-bundles corresponding to the weights $\lambda_k > 0$ and $\lambda_k < 0$ respectively. We denote by $\nu F$ the rank of the holomorphic vector bundle $N_F^{-(1,0)}$.

The polarized symmetric tensor products (cf. [5, 14]) are the vector bundles

\begin{align*}
K_F^+ &= S((N_F^{+(1,0)})^*) \otimes S(N_F^{-(1,0)}) \otimes \det(N_F^{-(1,0)}), \\
K_F^- &= S((N_F^{-(1,0)})^*) \otimes S(N_F^{+(1,0)}) \otimes \det(N_F^{+(1,0)}).
\end{align*}

(2.5)

Here $S((N_F^{\pm,(1,0)})^*)$, $S(N_F^{\pm,(1,0)})$ denote the sums of all symmetric powers of the bundles $(N_F^{\pm,(1,0)})^*$ and $N_F^{\pm,(1,0)}$ respectively and $\det(N_F^{\pm,(1,0)})$ denotes the top exterior power of $N_F^{\pm,(1,0)}$.

The fiber $E_p$ over each fixed point $p \in F$ is a representation of $S^1$, and $\text{char}(E_p)$ is independent on $p \in F$. Consider infinite dimensional holomorphic bundles $K_F^\pm \otimes E|_F$. The circle acts on the total space while preserving the base points in $F$. A sub-bundle of any given weight is a holomorphic vector bundle of finite rank, i.e.,

\begin{equation}
K_F^\pm \otimes E|_F = \oplus_{k \in \mathbb{Z}} E_{F,k}^\pm,
\end{equation}

(2.6)

where $E_{F,k}^\pm$ is a $S^1$-invariant sub-bundle of finite rank on which the circle acts with weight $k$. The cohomology groups $H^*(F, \mathcal{O}(K_F^\pm \otimes E|_F))$ are the sum of those with coefficients in $E_{F,k}^\pm$, each equipped with an induced $S^1$-action. Therefore, for any $k \in \mathbb{Z}$ the multiplicities

\begin{equation}
\text{mult}_k H^*(F, \mathcal{O}(K_F^\pm \otimes E|_F)) = \dim \mathbb{C} H^*(F, \mathcal{O}(E_{F,k}^\pm))
\end{equation}

(2.7)

are finite.

Our main result is the following

**Theorem 2.4.** For any $k \in \mathbb{Z}$ there exists a polynomial $Q_k(t)$ with non-negative integer coefficients such that

\begin{align*}
\sum_{p=0}^{n-1} t^p \dim \mathbb{C} H^p(M_{a, \mathcal{O}(E_a \otimes F^a)}) + \sum_{\mu | F > a} t^{\mu F - \nu F} \sum_{p=0}^{n_F} t^p \text{mult}_k H^p(F, \mathcal{O}(K_F^- \otimes E|_F)) \\
+ \sum_{\mu | F < a} t^{\mu F} \sum_{p=0}^{n_F} t^p \text{mult}_k H^p(F, \mathcal{O}(K_F^+ \otimes E|_F)) \\
= \sum_{p=0}^{n} t^p \text{mult}_k H^p(M, \mathcal{O}(E)) + (1 + t)Q_k(t)
\end{align*}

(2.8)

where the second and third sums in the left hand side are taken over connected components of the fixed-point set.

Theorem 2.4 is proven in Section 3.
The rest of this section is devoted to discussion of different applications and reformulation of Theorem 2.4.

2.5. Reformulation in the language of characters. It follows from (2.7) that the character
\[
\text{char } H^p(F, \mathcal{O}(K_F^\pm \otimes E|_F)) = \sum_{k \in \mathbb{Z}} e^{-ik\theta} \dim_{\mathbb{C}} H^p(F, \mathcal{O}(E_{F,k}^\pm))
\]
(2.9)
of the infinite dimensional representation \(H^p(F, \mathcal{O}(K_F^\pm \otimes E|_F))\) is well defined as an element of \(L\).

Definition 2.6. Let \(q(\theta) = \sum_{k \in \mathbb{Z}} q_k e^{-ik\theta}\) be a formal character of \(S^1\), we say \(q(\theta) \geq 0\) if \(q_k \geq 0\) for all \(k \in \mathbb{Z}\). Let \(Q(\theta, t) = \sum_{m=0}^n q_m(\theta)t^m\) be a polynomial of degree \(n\) with coefficients in \(L\), we say \(Q(\theta, t) \geq 0\) if \(q_m(\theta) \geq 0\) for all \(m\).

For two such polynomials \(P(\theta, t)\) and \(q(\theta, t)\), we say \(P(\theta, t) \leq Q(\theta, t)\) if the exists a polynomial \(Q(\theta, t) - P(\theta, t) \geq 0\).

Using (2.4) we can reformulate Theorem 2.4 in the language of characters.

Theorem 2.7. There exists a polynomial \(Q(\theta, t) \in L[t]\), such that \(Q \geq 0\) and
\[
\sum_{p=0}^{n-1} t^p \text{char } H^p(M_a, \mathcal{O}(E_a \otimes S(F_a, F_a^{-1}))) + \sum_{\mu|_F > a} t^{n_F - \nu_F} \sum_{p=0}^{n_F} t^p \text{char } H^p(F, \mathcal{O}(K_F^+ \otimes E|_F))
\]
\[
+ \sum_{\mu|_F < a} t^{\nu_F} \sum_{p=0}^{n_F} t^p \text{char } H^p(F, \mathcal{O}(K_F^- \otimes E|_F))
\]
\[
= \sum_{p=0}^n t^p \text{char } H^p(M, \mathcal{O}(E)) + (1 + t)Q(\theta, t)
\]
(2.10)

2.8. Witten-Wu-Zhang inequalities. Theorem 2.7 provides estimates on the character of \(H^*(M, \mathcal{O}(E))\) for any regular value \(a \in \mathbb{R}\) of the momentum map. Let as choose \(a < \min\{\mu(x) : x \in M\}\). Then the reduced space \(M_a\) is empty and the first and the third summands in the left hand side of (2.10) vanish. Hence, (2.10) reduces to
\[
\sum_{F} t^{n_F - \nu_F} \sum_{p=0}^{n_F} t^p \text{char } H^p(F, \mathcal{O}(K_F^\pm \otimes E|_F)) = \sum_{p=0}^n t^p \text{char } H^p(M, \mathcal{O}(E)) + (1 + t)Q(\theta, t)
\]
(2.11)
where the sum in the left is taken over all connected components of the fixed-point set. This is precisely the Wu-Zhang extension of the Witten holomorphic Morse inequalities for a circle action [14, Theorem 2.4].
Note that choosing $a > \max\{\mu(x) : x \in M\}$ leads to inequalities which are similar but different from (2.11). It is shown in [13] that combination of those inequalities with (2.11) gives much better estimates than (2.11) alone. Even more information about $H^*(M, \mathcal{O}(E))$ may be obtained by considering (2.10) with all possible values of $a$.

2.9. The Tian-Zhang inequalities for symplectic reduction. Let $F$ be a connected component of the fixed-point set. As we have already mentioned in Section 2.3, the circle acts on the normal bundle $N_F$ to $F$ preserving the base points in $F$. Also the weights of the isotropy representation on the normal fiber are constant along $F$.

Let $\lambda^+_F$ (resp. $\lambda^-_F$) denote the sum of the positive (resp. negative) weights. One easily checks that $\text{supp} \dim H^p(F, \mathcal{O}(K^+_F \otimes E|_F)) \subset (-\infty, k_2 - |\lambda^-_F|)$ and $\text{supp} \dim H^p(F, \mathcal{O}(K^-_F \otimes E|_F)) \subset [k_1 + \lambda^+_F, \infty)$. It follows that, if $\text{supp} E|_F \subset [k_1, k_2]$ then

$$\text{supp} \dim H^p(F, \mathcal{O}(K^+_F \otimes E|_F)) \subset (-\infty, k_2 - |\lambda^-_F|);$$
$$\text{supp} \dim H^p(F, \mathcal{O}(K^-_F \otimes E|_F)) \subset [k_1 + \lambda^+_F, \infty).$$

If there exists an integer $k$ which is greater than $k_2 - |\lambda^-_F|$ for any $F$ with $\mu(F) < a$ and which is smaller than $k_1 + \lambda^+_F$ for any $F$ with $\mu(F) > a$ then (2.8), (2.12) imply that (cf. [13, §4])

$$\sum_{p=0}^{n-1} t^p \dim \text{dim}_C H^p(M, \mathcal{O}(E_a \otimes \mathcal{F}_a)) = \sum_{p=0}^{n} t^p \text{mult}_k H^p(M, \mathcal{O}(E)) + (1 + t)Q_k(t).$$

Consider the case when $E$ is a pre-quantum line bundle. That means that the Kähler form $\omega$ represents the Chern class of $E$ in the cohomology $H^2(M)$. For this case there is a natural choice of the momentum map $\mu$ given by the Kostant formula: the infinitesimal generator of the action of $S^1$ on the space of sections of $E$ is given by $-\nabla^E_V + 2\pi i \mu$, where $\nabla^E_V$ is the covariant derivative along the vector field $V$ which generates the action of $S^1$ on $M$.

With this choice of $\mu$ one easily checks that the weight of the representation of $S^1$ on $E|_F$ is positive (resp. negative) if $\mu(F) > 0$ (resp. $\mu(F) < 0$). It follows that (2.13) holds for $k = 0, a = 0$. In other words, there exists a polynomial $Q(t)$ with non-negative coefficients such that

$$\sum_{p=0}^{n-1} t^p \dim \text{dim}_C H^p(M_0, \mathcal{O}(E_0)) = \sum_{p=0}^{n} t^p \dim H^p(M, \mathcal{O}(E))^{S^1} + (1 + t)Q(t).$$

(here $H^p(M, \mathcal{O}(E))^{S^1}$ denotes the space of $S^1$ invariant vectors in $H^p(M, \mathcal{O}(E)))$. Equation (2.14) is precisely the Tian-Zhang Morse-type inequalities for symplectic reduction on a Kähler manifold [10, Theorem 5.1] (note, however, that in [10] the inequalities are obtained for a much more general case where the circle is replaced by an arbitrary compact Lie group).
2.10. **Index theorem.** An interesting corollary of Theorem 2.4 may be obtained by setting $t = -1$ in (2.8). Then the last summand in the right hand side of (2.8) vanishes and we obtain a combination of the Atiyah-Bott fixed point theorem [1, 2] and the Guillemin-Sternberg “quantization commutes with reduction” theorem [3, Theorem 5.2]. We will now explain this in more details.

For a connected component $F$ of the fixed-point set, define

$$
\text{ind}_k(F; K_F^+ \otimes E|_F) = \sum_{p=0}^{n_F} t^{p+\nu_F} \text{mult}_k H^p(F, \mathcal{O}(K_F^+ \otimes E|_F)),
$$

$$
\text{ind}_k(F; K_F^- \otimes E|_F) = \sum_{p=0}^{n_F} t^{p+\nu_F-\nu_a} \text{mult}_k H^p(F, \mathcal{O}(K_F^- \otimes E|_F)).
$$

(2.15)

Recall that bundles $E_{F,k}^\pm$ are introduced in (2.6). By the Riemann-Roch-Hirzebruch theorem [3] (see also [4, Theorem 4.9]) we have

$$
\text{ind}_k(F; K_F^\pm \otimes E|_F) = \int_F Td(F) \ ch(E_{F,k}^\pm),
$$

(2.16)

where $Td$ and $ch$ stand for the Todd class and Chern character respectively.

Setting $t = -1$ in (2.8) and taking into account (2.15), we obtain

$$
\sum_{p=0}^{n-1} (-1)^p \dim_\mathbb{C} H^p(M_0, \mathcal{O}(E_0 \otimes F_a^k)) + \sum_{\mu|_F < a} \text{ind}_k(F; K_F^- \otimes E|_F)
$$

$$
+ \sum_{\mu|_F > a} \text{ind}_k(F; K_F^+ \otimes E|_F) = \sum_{p=0}^{n} (-1)^p \text{mult}_k H^p(M, \mathcal{O}(E)).
$$

(2.17)

If in (2.17) we choose $a < \min\{\mu(x) : x \in M\}$, then the first term in the left hand side of (2.17) vanishes and, in view of (2.10), we get the Atiyah-Bott fixed-point theorem. From the other side, if $E$ is a pre-quantum line bundle $a = 0$ and $k = 0$, then (cf. Section 2.9) the second and the third terms in the left hand side of (2.17) vanish and we obtain the Guillemin-Sternberg “quantization commutes with reduction” theorem [3, Theorem 5.2]:

$$
\sum_{p=0}^{n-1} (-1)^p \dim_\mathbb{C} H^p(M_0, \mathcal{O}(E_0)) = \sum_{p=0}^{n} (-1)^p \dim_\mathbb{C} H^p(M, \mathcal{O}(E))^{S^1}.
$$

2.11. **The Tian-Zhang relative index theorem for symplectic quotients.** Let $a < b$ be two regular values of the momentum map $\mu$ and suppose that $S^1$ acts freely on $\mu^{-1}(a), \mu^{-1}(b)$. Then we can form smooth manifolds $M_a$ and $M_b$ as in Section 2.2 and
then apply (2.17) to each one of them. Comparing the results we obtain

\[
\sum_{p=0}^{n-1} (-1)^p \dim C H^p (M_b, \mathcal{O}(E_b \otimes \mathcal{F}_b^k)) - \sum_{p=0}^{n-1} (-1)^p \dim C H^p (M_a, \mathcal{O}(E_a \otimes \mathcal{F}_a^k)) = \sum_{a<\mu|F<b} \text{ind}_k (F; K^-_F \otimes E|_F) - \sum_{a<\mu|F<b} \text{ind}_k (F; K^+_F \otimes E|_F). \tag{2.18}
\]

This formula was first obtained by Tian and Zhang [11, Theorem 5.7] using rather sophisticated study of the spectral flow of a family of Dirac operators with the Atiyah-Patodi-Singer boundary conditions on a symplectic manifold with boundary. (In fact, Tian and Zhang considered only the case \(k = 0\). However, (2.18) follows easily from their result).

The result of Tian and Zhang is valid for a more general case where \(M\) is an arbitrary symplectic manifold. Note that our proof may be easily extended to that case.

3. Proof of Theorem 2.4

First of all, note that it is enough to prove Theorem 2.4 for \(k = 0\). Indeed, suppose that the theorem is proven for \(k = 0\) and recall that bundles \(\mathcal{F}^k\) are defined in Section 2.2. Applying Theorem 2.4 with \(k = 0\) to the tensor product \(E \otimes \mathcal{F}^m\) and using (2.2) we obtain the statement of the theorem for \(k = m\).

Let us prove Theorem 2.4 for \(k = 0\). In other words, we will be interested in \(S^1\)-invariant elements of \(H^* (M, \mathcal{O}(E))\) and \(H^* (F, \mathcal{O}(K^\pm_F \otimes E|_F))\). Also, without loss of generality, we assume that \(a = 0\).

Recall that \(\mu : M \to \mathbb{R}\) is a momentum map for the circle action on \(M\). Following [10], we consider a one parameter family of differentials \(\bar{\partial}_t : \Omega^{0,*} (M, E) \to \Omega^{0,*+1} (M, E)\) defined by

\[
\bar{\partial}_t \alpha = e^{-t|\mu|^2} \bar{\partial} e^{t|\mu|^2} \alpha = \bar{\partial} \alpha + 2\mu \bar{\partial} \mu \wedge \alpha.
\]

Let \(\bar{\partial}_t^*\) denote the formal adjoint to \(\bar{\partial}_t\) and consider the corresponding Laplacian

\[
\square_t = \bar{\partial}_t^* \bar{\partial}_t + \bar{\partial}_t \bar{\partial}_t^*.
\]

Clearly, for each \(t \in \mathbb{R}\) the cohomology \(H^* (M, \mathcal{O}(E))\) is isomorphic to the kernel \(\text{Ker} \square_t\) of \(\square_t\). Moreover, the \(S^1\) invariant part of \(H^* (M, \mathcal{O}(E))\) is isomorphic to the kernel of the restriction of \(\square_t\) on the space \((\Omega^{0,*} (M, E))^{S^1}\) of \(S^1\)-invariant anti-holomorphic differential forms. The later operator is calculated in [10]. It is shown in [10] that, for \(t \to \infty\), the calculation of the kernel may be localized to small neighborhoods of \(\mu^{-1}(0)\) and of fixed-point set of the action of \(S^1\). Such a localization, by standard techniques of [12, 13, 10, 14] leads to Morse-type inequalities. The contribution of the \(\mu^{-1}(0)\) to these inequalities is calculated in [10] and is precisely equal to the first summand in the left hand side of (2.8).
The contribution of the fixed-point set to the inequalities may be calculated using the technique of [14]. Let $F$ be a connected component of the fixed-point set. Then the restriction of $\mu$ on $F$ is a constant. Moreover, $\mu(F) \neq 0$ since 0 is a regular value of $\mu$.

In [14], Zhang and Wu considered a one parameter deformation of $\bar{\partial}$ given by

$$\bar{\partial}'_s \alpha = \bar{\partial}\alpha + s\bar{\partial}\mu \wedge \alpha.$$ 

Near $F$ our operator $\bar{\partial}_t$ looks like $\bar{\partial}'_s$ with $\mu(F)s = t$. Hence, if $\mu(F) > 0$, the asymptotic behavior for $t \to \infty$ of the eigenforms of $\Box_t$ which concentrate near $F$ is the same as in [14]. In particular, the contribution of $F$ to the inequalities is the same as in [14]. This leads to the second summand in the left hand side of (2.8).

If $\mu(F) < 0$, then, as $t \to \infty$, the operator $\Box_t$ behaves as the corresponding operator in [14] behaves for $s \to -\infty$. This leads to the last term in the left hand side of (2.8).

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