INTERLEAVING OF PATH SETS

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ABSTRACT. Path sets are spaces of one-sided infinite symbol sequences corresponding to the one-sided infinite walks beginning at a fixed initial vertex in a directed labeled graph \(G\). Path sets are a generalization of one-sided sofic shifts. This paper studies decimation operations \(\psi_{j,n}(\cdot)\) which extract symbol sequences in infinite arithmetic progressions \((\text{mod } n)\) starting with the symbol at position \(j\). It also studies a family of \(n\)-ary interleaving operations \(\ominus_n\), one for each \(n \geq 1\), which act on an ordered set \((X_0, X_1, \ldots, X_{n-1})\) of one-sided symbol sequences \(X_0 \ominus X_1 \ominus \cdots \ominus X_{n-1}\) on an alphabet \(A\), by interleaving the symbols of each \(X_i\) in arithmetic progressions \((\text{mod } n)\). It studies a set of closure operations relating interleaving and decimation. This paper gives basic algorithmic results on presentations of path sets and existence of a minimal right-resolving presentation. It gives an algorithm for computing presentations of decimations of path sets from presentations of path sets, showing the minimal right-resolving presentation of \(\psi_{j,n}(X)\) has at most one more vertex than a minimal right-resolving presentation of \(X\). It shows that a path set has only finitely many distinct decimations. It shows the class of path sets on a fixed alphabet is closed under interleavings and gives an algorithm to compute presentations of interleavings of path sets. It studies interleaving factorizations and classifies path sets that have infinite interleaving factorizations and gives an algorithm to recognize them. It shows the finiteness of a process of iterated interleaving factorizations, which “freezes” factors that have infinite interleavings.

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1. INTRODUCTION

Let $\mathcal{A}$ be a finite alphabet with at least two elements, and let $\mathcal{A}^\mathbb{N}$ denote the full one-sided shift on $\mathcal{A}$: that is, the space of one-sided infinite strings of symbols from $\mathcal{A}$. One-sided symbolic dynamics studies the action of the one-sided shift map $S : \mathcal{A}^\mathbb{N} \rightarrow \mathcal{A}^\mathbb{N}$ given by

$$S(a_0 a_1 a_2 \cdots) = a_1 a_2 a_3 \cdots$$
onumber

on subsets $X \subset \mathcal{A}^\mathbb{N}$.

This paper considers a class of such $X$ called path sets, studied in [2]. Consider a finite labeled graph $\mathcal{G} = (G, E)$ whose underlying graph $G = (V, E)$ is a directed graph with edge set $E$ (permitting loops and multiple edges), with labeling $E : E \rightarrow \mathcal{A}$ specifying a labeling of the graph edges by elements of $\mathcal{A}$, and some fixed initial vertex $v$ of $G$. From $(\mathcal{G}, v)$ we obtain a set $P = X(\mathcal{G}, v) \subset \mathcal{A}^\mathbb{N}$ of symbol sequences $x = a_0 a_1 a_2 \cdots$ giving the edge labels of all one-sided infinite walks in $\mathcal{G}$ that begin at the marked vertex $v$. We call any such $P$ a path set, and we call $(\mathcal{G}, v)$ a presentation of $P = X(\mathcal{G}, v)$. The name path set for these objects was given in [2], where previous work was reviewed. We let $\mathcal{C}(\mathcal{A})$ denote the class of all path sets on the alphabet $\mathcal{A}$. Path sets are a generalization of one-sided sofic shifts in symbolic dynamics; however in general they are not invariant under the one-sided shift map. The main feature of path sets is specification of a fixed initial vertex $v$, which encodes an initial condition which can break shift-invariance in the dynamics.

The path set concept has recently found application to number theory, fractal geometry and to neural networks. More specifically, Abram, Bolshakov, and Lagarias [11] used path sets to study Hausdorff dimension of intersections of multiplicative translates of $p$-adic Cantor sets, for application to a problem of Erdős on ternary expansions of powers of 2, see [3, 4, 22]. It turns out that these intersections are always presentable as $p$-adic path set fractals, a kind of geometric realization of path sets inside the $p$-adic integers $\mathbb{Z}_p$ viewed as the full shift on $\{0, 1, \ldots, p - 1\}$. In another direction Ban and Chang [10] show that the mosaic solution space of the initial value problem of a multi-layer cellular neural network is topologically conjugate to a path set. Thus, according to Ban and Chang [10], “The more properties we know about path sets, the more we
know about the topological structure of the solution spaces derived from differential equations with initial conditions, and vice versa.”

This paper studies the effect on path sets of two families of operations defined on one-sided symbol sequences $\mathcal{A}^\mathbb{N}$, which are decimations and interleavings. Decimation and interleaving operations are initially defined using individual elements of $\mathcal{A}^\mathbb{N}$, extend to act on arbitrary subsets $X \subseteq \mathcal{A}^\mathbb{N}$ by set union, and then are specialized to path sets $\mathcal{P} \subseteq \mathcal{A}^\mathbb{N}$ in this paper. The paper [5] studies these operations acting on general sets $X \subseteq \mathcal{A}^\mathbb{N}$.

Decimation operations on path sets were studied by the first two authors in [2].

**Definition 1.1.** Let $\mathcal{A}$ be a finite alphabet of symbols.

1. For $j \geq 0$ the $j$-th decimation operation at level $n$, $\psi_{j,n} : \mathcal{A}^\mathbb{N} \to \mathcal{A}^\mathbb{N}$, is defined on an individual sequence $x = x_0\cdot x_1\cdot x_2\cdot x_3 \cdots$ by

   \[ \psi_{j,n}(x) = y := x_jx_{j+n}x_{j+2n}x_{j+3n} \cdots, \]

2. The $j$-th decimation of level $n$, denoted $\psi_{j,n}(X)$, of the set $X \subseteq \mathcal{A}^\mathbb{N}$ is the set defined

   \[ \psi_{j,n}(X) := \bigcup_{x \in X} \psi_{j,n}(x) \]

The definition applies for all $j \geq 0$, but a special role is played by the operations with $0 \leq j \leq n - 1$, which we term principal decimations, see (1.1) below. The set of all level $n$ decimation operators for $j \geq n$ are obtained from principal decimations by iterating the one-sided shift operator $S$, noting that $\psi_{j+n,n}(x) = S \circ \psi_{j,n}(x)$.

The main emphasis of the paper is the family of interleaving operations, which comprise an infinite collection of $n$-ary operations ($n \geq 1$), one for each $n$, defined for arbitrary subsets $X$ of the shift space $\mathcal{A}^\mathbb{N}$.

**Definition 1.2.** Let $\mathcal{A}$ be a finite alphabet of symbols.

1. The $n$-fold interleaving operation $(\oplus)^{n-1}_{j=0} \mathcal{A}^\mathbb{N} \times \mathcal{A}^\mathbb{N} \times \cdots \times \mathcal{A}^\mathbb{N} \to \mathcal{A}^\mathbb{N}$ is defined pointwise on individual sequences: $x_j = a_{j,0}a_{j,1}a_{j,2} \cdots$ for $0 \leq j \leq n - 1$ by

   \[ (x_0, x_1, \ldots, x_{n-1}) \mapsto y := (\oplus)^{n-1}_{j=0} x_j = x_0 \oplus x_1 \oplus \cdots \oplus x_{n-1} = b_0b_1b_2 \cdots \]

   with symbol sequence

   \[ b_{ni+k} = a_{k,i} \text{ for } i \geq 0, \ 0 \leq k \leq n - 1. \]

2. The $n$-fold interleaving $X = (\oplus)^{n-1}_{i=0} X_i$ of the the sets $X_0, X_1, X_2, \ldots, X_{n-1} \subseteq \mathcal{A}^\mathbb{N}$ in the specified order is

   \[ (\oplus)^{n-1}_{i=0} X_i := X_0 \oplus X_1 \oplus X_2 \oplus \cdots \oplus X_{n-1} = \{(x_i)^\infty_{i=0} \in \mathcal{A}^\mathbb{N} : x_jx_{j+n}x_{j+2n} \cdots \in X_j \text{ for all } 0 \leq j \leq n - 1\}. \]

The notation $\oplus$ above does not indicate the arity of the $n$-fold interleaving operation $\oplus_n$; the arity of composed operations is to be inferred via groupings of parentheses. The $n$-fold interleaving operations of arity 2 are not commutative, in general $X_1 \oplus X_2 \neq X_2 \oplus X_1$, and not associative, in general $(X_1 \oplus X_2) \oplus X_3 \neq X_1 \oplus (X_2 \oplus X_3)$.

Interleaving and decimation operators are related by a pointwise identitiy stating that the $n$-fold interleaving of the ordered set of principal decimations at level $n$ gives the identity map:

\[ (\oplus)^{n-1}_{j=0} \psi_{j,n}(x) = x \text{ for } x \in \mathcal{A}^\mathbb{N}. \]  \hspace{1cm} (1.1)

At the set level it shows that the level $n$ principal decimations provides a right-inverse to recover the individual factors of an $n$-fold interleaving

\[ X_j = \psi_{j,n}(X_0 \oplus X_1 \oplus \cdots \oplus X_{n-1}). \]  \hspace{1cm} (1.2)
Symbolic dynamics. This paper studies decimation and interleaving operations on path sets from the viewpoint of symbolic dynamics and coding theory, with (one-sided) symbol sequences viewed inside the full one-sided shift $\mathcal{A}^\mathbb{N}$ with the shift topology, which is the product topology where each factor $\mathcal{A}$ has the discrete topology, so that $\mathcal{A}^\mathbb{N}$ is a compact set. Much of symbolic dynamics studies closed sets $X$ which are shift-invariant: $SX = X$.

The interleaving concept is useful in coding theory as a method for improving the burst error correction capability of a code, cf. [1], Section 7.5. The analogue of Definition 1.2 for finite codes is referred to by coding theorists as block interleaving at depth $n$. Each interleaving operation $\otimes_n$ respects the shift topology in the sense that if $\{X_i : 0 \leq i \leq n - 1\}$ are closed subsets of the one-sided shift space $\mathcal{A}^\mathbb{N}$ then $X = (\otimes_n)_{i=0}^{n-1}X_i = X_0 \otimes X_1 \otimes X_2 \otimes \cdots \otimes X_{n-1}$ is also a closed subset of $\mathcal{A}^\mathbb{N}$.

A great deal of study has been given to the subclasses of shifts of finite type, and of the generalization to sofic shifts, studied for two-sided infinite sequences in Lind and Marcus [23]. These are special cases of path sets. General path sets $\mathcal{P}$ are not shift-invariant, but we show in Theorem 1.4 below that they satisfy a weak form of shift-invariance.

1.2. Path sets. We recall the definition of path set from [2]. A pointed graph $(G, v)$ over a finite alphabet $\mathcal{A}$ comprises a finite edge-labeled directed graph $\mathcal{G} = (G, E)$ and a distinguished vertex $v$ of the underlying directed graph $G$. The directed graph $G = (V, E)$ is specified by its vertex set $V$ and (directed) edge set $E$ with edges $e = (v_1, v_2) \in V \times V$, and the data $E \subset G \times \mathcal{A}$ specifies the set of labeled edges $(e, a)$, with labels drawn from the alphabet $\mathcal{A}$. We allow loops and multiple edges, but require that all triples $(e, a) = (v_1, v_2, a)$ be distinct. We use interchangeably the terms vertex and state of $G$, as in Lind and Marcus [23]. Sect. 2.2. The results of this paper regard the finite alphabet $\mathcal{A}$ as fixed, unless specifically noted otherwise.

Definition 1.3. A path set $(\mathcal{G}, v)$ (or pointed follower set) $\mathcal{P} = X(\mathcal{G}, v)$ specified by a pointed labeled graph $(\mathcal{G}, v)$ and a distinguished vertex $v$ is the subset of $\mathcal{A}^\mathbb{N}$ made up of the symbol sequences of successive edge labels of all possible one-sided infinite walks in $\mathcal{G}$ issuing from the distinguished vertex $v$.

We let $\mathcal{C}(\mathcal{A})$ denote the collection of all path sets using labels from the alphabet $\mathcal{A}$. We call the data $(\mathcal{G}, v)$ a presentation of the path set $\mathcal{P} = X(\mathcal{G}, v)$. A path set $\mathcal{P}$ typically has many different presentations. The set $\mathcal{P}$ by definition includes every symbolic path sequence with multiplicity one, although the graph $\mathcal{G}$ could potentially contain many paths starting from $v$ with identical symbol sequences.

The paper [2] showed that the class $\mathcal{C}(\mathcal{A})$ of all path sets $\mathcal{P}$ with fixed (finite) alphabet $\mathcal{A}$ is closed under all decimation operations $\nu_{j,n}(\mathcal{P})$. We will give a second proof of this result in Section 3.

This paper shows in addition that the class $\mathcal{C}(\mathcal{A})$ of all path sets with fixed (finite) alphabet $\mathcal{A}$ is closed under all the interleaving operations $\otimes_n$. Conversely it shows that if a path set $\mathcal{P}$ has an $n$-fold interleaving factorization, then each factor in the factorization is necessarily a path set.

In contrast, the smaller classes of one-sided sofic shifts and of shifts of finite type are not closed under $n$-interleaving. Interleaving can break one-sided shift-invariance even for the most well-behaved shift spaces. Path sets therefore appear to be a natural level of generality at which to study interleaving operations.

1.3. Main results. In this paper we study the decimation and interleaving operations restricted to path sets.

1.3.1. Presentations of path sets. In Section 2 we prove basic properties of presentations of path sets, and discuss algorithms to test for these properties, in the language of symbolic dynamics.

The paper [2], Section 3] showed that every path set $\mathcal{P}$ has a presentation with several additional properties: right-resolving, reachable, and pruned. The right-resolving property guarantees uniqueness of symbolic

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1In [2] the notation $X(\mathcal{G},v)$ was used.

2A labeled directed graph $\mathcal{G}$ is called right-resolving if any two edges emanating from the same vertex have distinct labels. A pointed graph $(\mathcal{G},v)$ is called reachable if there is a directed path in $\mathcal{G}$ from the initial vertex $v$ to every other vertex of $\mathcal{G}$; for a finite automaton this property is called accessible. A reachable pointed graph $(\mathcal{G},v)$ is called pruned if in it has no sinks, meaning...
paths: any two distinct paths in the graph $G$ starting from the initial vertex $v$ have different symbol sequences; that is, for such a presentation the (finite or infinite) symbol sequence uniquely specifies the path.

In contrast in symbolic dynamics sofic systems need not have unique minimal right-resolving presentations (in the sense of sofic system presentations) if the sofic system is reducible, compare [23] Example 3.3.21.

1.3.2. Decimations and the shift. The paper [2] showed that all decimations $\psi_{j,n}(P)$ of a path set are path sets, giving a constructive algorithm to compute a presentation of $\psi_{j,n}(P)$ from that of $P$. Section 3 presents a different construction, the modified $n$-th higher power presentation, to obtain presentations $\psi_{j,n}(P)$ which require at most one vertex more than the number of vertices $m$ of the input presentation of $P$. (The $n$-th higher power construction is well known, cf. [?].)

As a first consequence of the modified higher power presentation we show that path sets satisfy a weak version of shift-invariance. We say that a set $X$ is weakly shift-invariant if there exist $k > j \geq 0$ with $S^kX = S^jX$.

**Theorem 1.4.** (Weak shift-invariance of path sets) For any path set $P$ there exist integers $k > j \geq 0$ giving the equality of iterated shifts $S^kP = S^jP$.

This result is proved as Theorem 3.3. It comes from the property that iterations of the shift operator $S$ are given by 1-decimations. We have in general $\psi_{j,n}(P) = \psi_{0,n}(S^jP)$ and particular $\psi_{j,1}(P) = S^jP$.

**Definition 1.5.** The full decimation set $D(X)$ of an arbitrary set $X \subset A^N$ is the set of all (principal and non-principal) decimations:

$$D(X) := \{\psi_{j,m}(X) : m \geq 1 \text{ and } j \geq 0\}.$$  

The modified higher power construction implies finiteness for the set of all decimations of a path set.

**Theorem 1.6.** (Full decimation set bound) For each path set $P$ its full decimation set $D(P)$ is a finite set. If $P$ has a presentation having $m$ vertices, then $D(P)$ has cardinality bounded by

$$|D(P)| \leq 2^{(m+1)^2|A|}.$$  

This result is proved as Theorem 3.7. In contrast, there are closed set $X \subset A^{NN}$ such that $D(X)$ is an infinite set, cf. [5] Section 7.

Theorem 1.6 answers a question raised in [2] page 113. That paper defined the $n$-kernel of a path set $P$ by

$$\text{Ker}_n(P) := \{\psi_{j,n}(P) : j \geq 0, k \geq 0\}.$$  

It called a path set $n$-automatic if $\text{Ker}_n(P)$ is finite. This notion was proposed in analogy with the definition of $n$-automatic sequences made by Allouche and Shallit [6], [7]. Since $\text{Ker}_n(P) \subset D(P)$, Theorem 1.6 implies that every path set is $n$-automatic in this sense for every $n \geq 1$.

The output presentation of the $n$-th higher power construction is not necessarily right-resolving, even if the input presentation is right-resolving. However a standard construction, the subset construction, shows that if $P$ has a presentation with $m$ vertices, then each decimation $\psi_{j,n}(P)$ has a right-resolving presentation with at most $2^{m+1} - 1$ vertices, an upper bound which is independent of $n$.

**Theorem 1.7.** (Right-resolving presentations of decimation sets of a path set.) Given a path set $P$ on alphabet $A$ with at least two letters, having a (not necessarily right-resolving) presentation $P = X(G, v)$ with $m$ vertices. Then for each $n \geq 1$ and each $j \geq 0$ the decimation set $\psi_{j,n}(P)$ has a right-resolving presentation having at most $2^{m+1} - 1$ vertices.

This result is proved as Theorem 3.9.

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every vertex has out-degree at least one; for finite automata this property is called coaccessible. A finite automaton is trim if it is both accessible and coaccessible, see [18], [27] page 27.
1.3.3. **Closure of \( \mathcal{C}(\mathcal{A}) \) under interleavings.** In Section 4 we start the study of interleaving operations on the class \( \mathcal{C}(\mathcal{A}) \) of all path sets on a fixed alphabet \( \mathcal{A} \). We first show that \( \mathcal{C}(\mathcal{A}) \) is closed under the interleaving operations.

**Theorem 1.8.** \( \mathcal{C}(\mathcal{A}) \) is closed under interleaving \( \psi \). If \( P_0, \ldots, P_{n-1} \) are path sets on the alphabet \( A \), then their \( n \)-fold interleaving

\[
X := (\otimes_n)_{i=0}^{n-1} P_i = P_0 \otimes P_1 \otimes \cdots \otimes P_{n-1}
\]

is a path set; i.e., \( X \in \mathcal{C}(\mathcal{A}) \).

This result is proved as Theorem 4.1. We establish Theorem 1.8 as a corollary of an effective algorithmic construction of the \( n \)-fold interleaving \( \mathcal{P} \) at the level of presentations of path sets.

**Theorem 1.9.** (Interleaving pointed graph product construction) Let \( n \geq 2 \) and suppose \( P_0, \ldots, P_{n-1} \) are path sets with given presentations \( (\mathcal{G}_0, v_0), \ldots, (\mathcal{G}_{n-1}, v_{n-1}) \), respectively. There exists a construction taking as inputs these presentations and giving as output a presentation \( (\mathcal{H}, v) \) of the \( n \)-fold interleaving \( X := P_0 \otimes P_1 \otimes \cdots \otimes P_{n-1} \). In particular \( X = X(\mathcal{H}, v) \) is a path set. This construction has the following properties:

(i) If \( \mathcal{G}_i \) has \( k_i \) vertices for each \( 0 \leq i \leq n - 1 \), then \( \mathcal{H} \) will have at most \( \prod_{i=0}^{n-1} k_i \) vertices.

(ii) If the pointed graphs \( (\mathcal{G}_i, v_i) \) are right-resolving for all \( 0 \leq i \leq n - 1 \), then the output pointed graph \( \mathcal{H} \) will also be right-resolving.

(iii) If the pointed graphs \( (\mathcal{G}_i, v_i) \) are pruned for all \( 0 \leq i \leq n - 1 \), then the output pointed graph \( \mathcal{H} \) will also be pruned.

This result is proved as Theorem 4.2. Theorem 1.9 constructs \( \mathcal{H} \) by a graph-theoretic construction, termed here the \( n \)-fold interleaved (pointed) graph product, which takes as input pointed graphs \( (\mathcal{G}_i, v_i) \), for \( 0 \leq i \leq n - 1 \) and produces as output a pointed graph

\[
(\mathcal{H}, v) := (\otimes_n)_{i=0}^{n-1} (\mathcal{G}_i, v_i),
\]

such that the underlying path set \( \mathcal{P} = X(\mathcal{H}, v) \) is the \( n \)-fold interleaving of the path sets \( \mathcal{P}_i = X(\mathcal{G}_i, v_i) \). The presentation \( (\mathcal{H}, v) \) found by the construction depends on the input presentations of \( \mathcal{P}_i \). We provide examples showing that minimality of right-resolving presentations is not always preserved under the interleaved pointed graph product.

1.3.4. **Decimations and Interleaving Factorizations.** Decimations and Interleaving operations together define a set of closure operations on symbol sets. We define the \( n \)-fold interleaving closure \( X^{[n]} \) of a general set \( X \subseteq A^N \) by

\[
X^{[n]} = \psi_{0,n}(X) \oplus \psi_{1,n}(X) \oplus \cdots \oplus \psi_{n-1,n}(X).
\]  

(1.3)

These closure operations were studied in [5]. One always has \( X \subseteq X^{[n]} \), the operation is idempotent: \( (X^{[n]})^{[n]} = X^{[n]} \), and the operation takes closed sets to closed sets.

We say a set has an \( n \)-fold interleaving factorization if \( X = X^{[n]} \). In that case its associated interleaving factors are given by \( X_{j,n} = \psi_{j,n}(X) \) for \( 0 \leq j \leq n - 1 \). These interleaving factors are unique, i.e. \( X = X_0 \oplus X_1 \oplus \cdots \oplus X_{n-1} \) and \( X = Y_0 \oplus Y_1 \oplus \cdots \oplus Y_{n-1} \) then \( X_j = Y_j \) for \( 0 \leq j \leq n - 1 \) as sets.

**Theorem 1.10.** \( \mathcal{C}(\mathcal{A}) \) is stable under \( n \)-fold interleaving closure operations. If \( \mathcal{P} \) is a path set, then for each \( n \geq 1 \) the \( n \)-fold interleaving closure \( \mathcal{P}^{[n]} \) is a path set. In addition, if \( \mathcal{P} \) is \( n \)-factorizable then each of its \( n \)-fold interleaving factors \( \mathcal{P}_i = \psi_{i,n}(\mathcal{P}) \) for \( 0 \leq i \leq n - 1 \) are path sets.

This result is proved as Theorem 5.4. All interleaving factors are decimations \( \psi_{j,n}(\mathcal{P}) \) with \( 0 \leq j \leq n - 1 \). We show that if a path set \( \mathcal{P} \) has an \( n \)-fold interleaving presentation, then there is an improved upper bound on the size of the associated decimations \( \psi_{j,n}(\mathcal{P}) \) relative to that given in Theorem 1.7.
Theorem 1.11. (Upper bound on minimal presentation size of n-fold interleaving factors) Let \( P \) be a path set having \( m \) vertices in its minimal right-resolving presentation. Suppose that \( P \) has an \( n \)-fold interleaving factorization \( P = (\ast)^{n-1}_{j=0}P_j \). Then each \( n \)-fold interleaving factor \( P_j = \psi_{j,n}(P) \) has a minimal right-resolving presentation having at most \( m \) vertices.

This result is proved as Theorem 7.5.

1.3.5. Classification of interleaving factorizations of path sets. The factorization problem is the problem of finding all the possible interleaving factorizations of a path set \( P \) under \( n \)-fold interleaving.

This problem is interesting and complicated due to the fact that some general sets \( X \subset \mathcal{A}^N \) may have factorizations for infinitely many \( n \geq 1 \), which we call infinitely factorizable sets. The simplest example is the one-sided shift \( \mathcal{A}^N \), which factors for all \( n \geq 1 \) as

\[ \mathcal{A}^N = (\mathcal{A}^N)^{(\ast n)}, \]

It is a path set, and Section 7 gives a structural characterization of infinitely factorizable path sets.

The paper [5] gave a classification of the possible pattern of interleaving factorizations for a general set \( X \), and a separate classification for a general closed set \( X \subset \mathcal{A}^N \), see Section 6.1. For a closed set \( X \subset \mathcal{A}^N \), exactly one of the following holds.

(i) \( X \) is factorizable for all \( n \geq 1 \).
(ii) \( X \) is \( n \)-factorizable for a finite set of \( n \), which are all the divisors of an integer \( f = f(X) \geq 1 \).

All path sets are closed sets so this classification applies to them. One can easily show that all allowed patterns of interleaving factorizations in this dichotomy occur for path sets.

The main results of this paper about interleaving factorizations concern the structure of factorizations of a path set \( P \). They are given in Sections 7 and 8.

(2) We characterize infinitely factorizable path sets in terms of the form of their minimal right-resolving presentation, which we call “leveled.” (Theorem 7.2)

(2) For the remaining finitely factorizable path sets, we obtain an upper bound on the size of \( n \) for which \( n \)-factorability can occur, in terms of the size of their minimal right-resolving presentation: one has \( n \geq m - 1 \). Thus \( f(P) \leq m - 1 \). (Theorem 8.1).

(3) We show that the process of iterated interleaving factorization of any path set always terminates in finitely many steps if we agree to “freeze” any infinitely factorizable path set encountered in the process. (Theorem 8.5).

The bound in (2) leads to an effective algorithm for determining if a path set is infinitely factorizable or not. The finiteness result in (3) also follows from (2). For general closed sets \( X \), the paper [5] gave examples having an infinite depth tree of recursively refined iterated factorizations.

1.4. Related work. There is a large literature of related work in automata theory, semigroups and symbolic dynamics; see the discussion in [2, Sect. 1.2].

The path set concept has previously been studied in automata theory and formal language theory, given in different terminology. In [2] we observed that path sets are characterized as those \( \omega \)-sets recognizable by a finite (deterministic) Büchi automaton having one initial state and having every state be a terminal state. These sets were characterized in Perrin and Pin [27, Chapter III, Proposition 3.9] as the set of (Büchi) recognizable sets that are closed in the product topology on \( \mathcal{A}^N \). The name path set is consistent with the term “path” in a finite automaton used in Eilenberg [18, page 13].

The set of finite initial blocks of a path set \( P \) forms a rational language (also called a regular language) \( L(P) \) in \( \mathcal{A}^* \), the set of all finite words in the alphabet \( \mathcal{A} \). The formal language \( L(P) \) uniquely characterizes the path set. We may call the set of formal languages

\[ \mathcal{L}(\mathcal{A}) := \{ L(P) : P \text{ a path set on alphabet } \mathcal{A} \} \]
the set of path set languages. These languages may be characterized as being the prefix-closed regular languages, see Appendix A. The set \( \mathcal{L}(\mathcal{A}) \) forms a strict subset of all rational languages on the finite alphabet \( \mathcal{A} \).

Special cases of interleaving operations have been considered in automata theory and formal languages. Eilenberg [18, Chapter II.3, page 20] introduced a notion of internal shuffle product \( A \sqcup B \) of two recognizable sets (= regular language) which corresponds to 2-interleaving. In Proposition 2.5 in Chapter II of that book, Eilenberg proved that the collection of recognizable sets is closed under internal shuffle product.

Interleaving operations have been used in coding theory in various code constructions, see for example Vanstone and van Oorschot [31], Chapters 5 and 7.

One-sided path sets also appear in studies in aperiodic order. The 1989 paper of de Bruijn [16] Sect. 5 and 6] deals mainly with two-sided infinite connector sequences, but has one-sided “singular” examples as well. He studied these sequences in connection with rewriting rules describing inflation and deflation for aperiodic tilings, in particular the Penrose tilings, topics which he previously studied in [13], [14], [15].

One may consider extensions of interleaving to infinite alphabets. The notion of full one-sided shift based on the product topology does not give a compact space for infinite alphabets. In 2014 Ott, Tomforde, and Willis [26] formulated a definition of full shifts on infinite alphabets which gives a compact shift space, which may be useful for this purpose.

1.5. Contents. The contents of the paper are as follows:

1. Section 2 collects together preliminary results about path sets. In particular, it shows they are uniquely determined by their set of allowed finite initial blocks. It gives an effective algorithm to tell whether two presentations \((G_1, v_1)\) and \((G_2, v_2)\) give the same path set. It shows the uniqueness (up to isomorphism) of minimal right-resolving presentations of a path set \(\mathcal{P}\), a fact which does not parallel the theory for sofic shifts (whose definition of right-resolving does not require the presentation to be pointed.)

2. Section 3 presents an algorithm for finding a presentation of a decimation \(\psi_{j,n}(\mathcal{P})\) from a presentation of \(\mathcal{P}\). It proves that all path sets are weakly shift-invariant. It shows that the full set of decimations \(\mathcal{D}(\mathcal{P})\) of a path set \(\mathcal{P}\) is a finite set. It gives a upper bound on the size of right resolving presentation of all decimations \(\psi_{j,n}(\mathcal{P})\) that depends only on the size of the presentation of \(\mathcal{P}\).

3. Section 4 shows that any \(n\)-fold interleaving of path sets is a path set. It gives a construction, the (pointed) interleaving graph product, which when given presentations of the individual \(\mathcal{P}_i\), yields a presentation of the \(n\)-fold interleaving of these path sets. It gives examples showing the output presentations need not be right-resolving, even when the presentations of the \(\mathcal{P}_i\) are minimal right-resolving.

4. Section 5 first reviews results for decimation and interleaving operations acting on general sets \(X \subset \mathcal{A}^\mathbb{N}\) which were shown in [5]. That paper defines a hierarchy of closure operations \(X \mapsto X^{[n]}\), the \(n\)-th interleaving closure of \(X\), and shows that a set \(X\) is \(n\)-factorizable if and only if \(X^{[n]} = X\). Second, it restricts to path sets, and proves that if \(\mathcal{P}\) is a path set, then so is \(\mathcal{P}^{[n]}\) for all \(n \geq 1\).

5. Section 6 recalls results from [5] on the structure of the allowed sets of possible \(n\)-factorizations that a general set \(X \subset \mathcal{A}^\mathbb{N}\) may have. It deduces that the set \(C^\infty(\mathcal{A})\) of infinitely factorizable path sets is stable under all decimation and interleaving operations. It proves that if a path set \(\mathcal{P}\) is \(n\)-factorizable, the minimal right resolution presentation of such \(\psi_{j,n}(\mathcal{P})\) requires no more nodes than that of a minimal right-resolving presentation of \(\mathcal{P}\).

6. Section 7 determines the structure of infinitely factorizable closed subsets \(X \subset \mathcal{A}^\mathbb{N}\). It characterizes infinitely factorizable path sets in two ways: the first is a syntactic property of the infinite words in \(\mathcal{P}\), and the second characterizes the form of their minimal right-resolving presentations.

7. Section 8 analyzes the structure of factorizations of finitely factorizable path sets. An iterated interleaving factorization is complete if all of its factors are either infinitely factorizable or indecomposable. It proves that every finitely factorizable path set has at least one complete factorization.

8. Section 9 discusses open questions and further work.
Appendix A discusses path sets from the viewpoint of automata theory.

Appendix B gives a sufficient condition on a presentation of a path set \( P \) for all of its interleaving factorizations to be self-interleaving factorizations. An \( n \)-fold self-interleaving factorization is one having all factors \( X_i = Z \) identical, for \( 0 \leq i \leq n - 1 \), where \( Z \) may depend on \( n \).

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2. Structure and Presentations of Path Sets

In this section we establish basic structural results about path sets, formulated in the terminology of symbolic dynamics in Lind and Marcus [23].

We fix a finite alphabet \( \mathcal{A} \), and let \( C(\mathcal{A}) \) denote the collection of all path sets on \( \mathcal{A} \). We use interchangeably the term word or block to mean a finite string of consecutive symbols from \( \mathcal{A} \), often viewed inside an infinite word.

In this section we show that path sets have a minimal right-resolving presentation, unique up to isomorphism of path set presentations. Equivalent results can be found in the automata theory literature, which uses different terminology, see Appendix B. We have included proofs to provide a self-contained treatment in the language of symbolic dynamics. In particular, we introduce two notions of finite and infinite follower sets and characterize path sets in terms of finiteness properties of both finite and infinite follower sets. These notions are needed for later proofs.

Path sets also have an important characterization in terms of automata theory, which uses different terminology. In automata theory path sets \( P \) are characterized as those recognizable sets in \( \mathcal{A}^\mathbb{N} \) for non-deterministic Büchi automata which are closed sets in its product topology. We discuss automata theory results in Appendix A.

2.1. Closure properties of path sets. We recall basic results on the closure of path sets in the symbol topology, and under set operations, shown in [2].

Theorem 2.1. ([2, Theorem 1.2])

1. Each path set \( P \) in \( C(\mathcal{A}) \) is a closed subset in the product topology on \( \mathcal{A}^\mathbb{N} \).
2. If \( P_1 \) and \( P_2 \) are path sets, then so is \( P_1 \cap P_2 \).
3. If \( P_1 \) and \( P_2 \) are path sets, then so is \( P_1 \cup P_2 \).

Remark 2.2. The collection \( C(\mathcal{A}) \) of all path sets in \( \mathcal{A}^\mathbb{N} \) is not closed under complementation inside \( \mathcal{A}^\mathbb{N} \). See [2, Example 2.3].

2.2. Presentations of path sets. Each path set has infinitely many presentations. We recall the following properties such a presentation may have.

Definition 2.3. (1) A labeled directed graph \( G \) is called right-resolving if any two edges emanating from the same vertex have distinct labels.

(2) A pointed graph \((G, v)\) is called reachable if there is a directed path in \( G \) from the initial vertex \( v \) to every other vertex of \( G \).

(3) A pointed graph \((G, v)\) is called pruned if it has no sinks, meaning every vertex has an exiting edge; i.e., it has out-degree at least one.

Proposition 2.4. Every path set \( P \) has a presentation that is right-resolving, pruned and reachable.

Proof. Theorem 3.2 of [2] gives (in its proof) an algorithm which when given an arbitrary presentation \( P = X(G, v) \) of a path set will compute another presentation \( (G', v') \) for \( P \) which is right-resolving. The pruning operation given in Section 3 of [2] then iteratively removes stranded vertices while retaining the right-resolving property. A vertex is stranded if it either has no entering edges or no exit edges, or both. When a stranded vertex is removed, new vertices may become stranded, and the operation repeats until no stranded vertices remain. Finally a pruned, right-resolving presentation can be converted to a right-resolving, pruned and reachable presentation by further removing all vertices not reachable from \( v' \).
Remark 2.5. A presentation $(G, v)$ specifies a finite automaton in the sense of Eilenberg [18, Chapter II], having a single initial state $I = \{v\}$, where we impose the requirement that all states be terminal states: i.e., $T = V(G)$. (The terminal states are not specified in the path set definition.) In automata theory right-resolving is equivalent to the automaton being deterministic. For a single initial state automaton reachable is equivalent to being accessible. For an automaton having all states being terminal states, pruned is equivalent to being co-accessible. Therefore the presentation produced by Proposition 2.4 is a trim automaton, i.e. deterministic, accessible and co-accessible.

2.3. **Word follower sets and vertex follower sets.** The internal structure of presentations of path sets is determined by their patterns of initial words over all paths. We formulate two notions of follower set which capture this internal structure. These definitions are adapted from definitions in Lind and Marcus [23] for two-sided infinite sequences.

The first of these notions applies to general subsets $X$ of the one-sided shift, parallel to [23] Defn. 3.2.4.

**Definition 2.6.** (Word follower set) Let $X$ be a subset of the one-sided shift $\mathcal{A}^\mathbb{N}$, and let $w = b_0b_1 \cdots b_{k-1}$ be a finite word of length $|w| = k$, allowing the empty word $\emptyset$ of length 0.

1. The **word follower set** $F_X(b)$ of an initial finite word $b$ of $X$ is the set of all finite blocks

$$F_X(b) = \{a \mid a \text{ is a finite block such that } ba \text{ is an initial block of some } x = bax' \in X\}.$$

2. A set $F \subseteq \mathcal{A}^\mathbb{N}$ is a word follower set of $X$ if there exists some initial block $b$ of $\mathcal{P}$ such that $F = F_X(b)$.

A closed set $X$ in $\mathcal{A}^\mathbb{N}$ may possess infinitely many different word follower sets $F_X(b)$, as $b$ varies. We show below that path sets $\mathcal{P} = X(G, v)$ have only finitely many different word follower sets as $b$ varies. Note that the initial block set $\mathcal{B}^I(\mathcal{P}) = F_X(\emptyset)$.

The second of these notions applies to presentations $(G, v)$ of path sets $\mathcal{P}$, parallel to [23] Defn. 3.3.7.

**Definition 2.7.** (Vertex follower set) The **vertex follower set** $F(G, v')$ of a vertex $v'$ in a labeled directed graph $G$ is the set of all finite words $b = b_0b_1 \cdots b_{k-1}$ that can be presented by labels of paths on $G$ beginning at vertex $v'$.

The next result shows for a right-resolving presentation $(G, v)$ of a path set $\mathcal{P}$ that all its possible word follower sets occur as vertex follower sets of the directed labeled graph $G$.

**Proposition 2.8.** Let $(G, v)$ be a right-resolving, pruned presentation of a path set $\mathcal{P}$, with $G$ having $m$ vertices.

1. For each finite initial word $b \in \mathcal{B}^I(\mathcal{P})$, the word follower set $F_{\mathcal{P}}(b)$ equals the vertex follower set $F(G, v')$ for some reachable vertex $v'$ of $G$. In particular, there are at most $m$ different word follower sets.

2. Conversely, each vertex follower set $F(G, v')$ of a reachable vertex occurs as the word follower set $F_{\mathcal{P}}(b)$ of some initial word $b \in \mathcal{B}^I(\mathcal{P})$. The word $b$ can be chosen to be of length at most $m - 1$. If $v' = v$ we choose $b = \emptyset$.

**Proof.** (1) Let $b = b_1b_2 \cdots b_k$ be a finite initial word of $\mathcal{P}$, and let $F_{\mathcal{P}}(b)$ be its word follower set. Since $G$ is right-resolving, there is exactly one path on $G$ beginning at $v$ presenting label set $b$. Let $v'$ be the (reachable) vertex where this path terminates. A finite word $w$ will be in the word follower set of $b$ if and only if there is a directed path beginning at vertex $v'$ that presents the label set $w$, and since $(G, v)$ is pruned, there exists an infinite word $x = bwx' \in \mathcal{P}$. Consequently the word follower set $F_{\mathcal{P}}(b)$ coincides with the vertex follower set $F(G, v')$.

(2) Let $v'$ be a reachable vertex in $(G, v)$, so that there exists a directed path $\pi$ from $v$ to $v'$, which can be chosen to have length at most $m - 1$, since there are $n$ vertices in $G$. Let $b$ be the block labeling this path, which uniquely determines $v'$ since the presentation is right-resolving. As above, since $G$ is pruned, the vertex follower set $F(G, v')$ equals the word follower set $F_{\mathcal{P}}(b)$. \qed
2.4. Finiteness of follower sets for path sets. Proposition \(\ref{prop:finite_follower_sets} \) implies the finiteness of the number of distinct word follower sets for any path set.

**Theorem 2.9.** (\(\mathcal{C}(\mathcal{A})\) characterized by finiteness of word follower sets) A set \(X \subseteq \mathcal{A}^\mathbb{N}\) is a path set if and only if it is closed and has a finite number of distinct word follower sets.

**Proof.** Suppose first that \(X = \mathcal{P}\) is a path set. Then \(\mathcal{P}\) is closed by Theorem \(\ref{thm:closed_sets} \). Now \(\mathcal{P}\) has a right-resolving, pruned presentation \((\mathcal{G},v)\) by Theorem \(\ref{thm:presentation_step} \). Let \(n\) be the number of its vertices. Proposition \(\ref{prop:finite_follower_sets} \) implies it has at most \(n\) distinct word follower sets \(F_X(b)\) as \(b\) varies. Thus \(X\) is closed and has a finite number of distinct word follower sets.

Conversely, suppose \(X\) is closed and has a finite number \(n\) of distinct word follower sets. We construct a right-resolving presentation \(X(\mathcal{G},v)\) of a path set having \(n\) vertices, and check that \(X = X(\mathcal{G},v)\). This construction will yield a minimal right-resolving presentation of \(X\). We give a name to each follower set in the finite list by assigning to it the minimal prefix \(b\) defining it as \(F_P(b)\). (We may put a total order on the alphabet \(\mathcal{A}\) and use the lexicographic order on prefixes to define “minimal”.) These word follower sets \(F_P(b)\) will name the states of \(\mathcal{G}\). For \(b = \emptyset\) we select \(v = F_P(\emptyset) = B^I(\mathcal{P})\) to be the initial vertex of \(\mathcal{G}\).

For each vertex \(v_i := F_P(b_i)\) and each letter \(a\) that occurs as an initial letter of some word of the follower set \(F(P,b_i)\), we add an exit edge with label \(a\) which goes from \(v_i\) to the follower set \(F_P(b_ia)\). Since the vertices enumerate the complete list of possible word follower sets, it will be some vertex \(v_j = F_P(b_j)\) for some \(b_j\). (We have \(F_P(b_ia) = F_P(b_j)\) but we may have \(b_ia \neq b_j\).) We have constructed \((\mathcal{G},v)\), and it is right-resolving as there is at most one exit edge with a given symbol from each vertex. It is also pruned and reachable by construction.

It remains to show that \(X = X(\mathcal{G},v)\), which will verify that it is a path set.

1. We show the inclusion \(X \subseteq X(\mathcal{G},v)\). Take \(x = a_0a_1a_2\cdots \in X\) and construct a path in \((\mathcal{G},v)\) realizing this symbol sequence. Given a finite sequence \(b_k = a_0a_1\cdots a_k \in B^I(X)\), we prove by induction on \(k \geq 0\), that at the \(k\)-th step finds a path in \((\mathcal{G},v)\) moving from vertex labeled \(F_X(b_{k-1})\) to vertex labeled \(F_X(b_k)\) with edge symbol \(a_k\). Here \(b_{-1} = \emptyset\) is the base case, and both the base case and the induction step follow by the definition of edges in \(\mathcal{G}\).

2. We show the reverse inclusion \(X(\mathcal{G},v) \subseteq X\) using the fact that \(X\) is closed. Each infinite symbol sequence \(y = a_0a_1a_2\cdots \in X(\mathcal{G},v)\), starting at vertex labeled \(F_X(\emptyset)\), appears on a unique path (by right-resolving property) which at step \(k\) is at vertex corresponding to \(F_X(a_0a_1\cdots a_k)\) of \(\mathcal{G}\). We may prove by induction on \(k\) that the finite path \(b_k = a_0a_1\cdots a_k\) is an initial block in \(X\). The word follower set property of \(F_X(a_0a_1\cdots a_k)\) permits inductively adding the symbol \(a_{k+1}\), for both the base case \(k = -1\) and the induction step. Since \(X\) is closed by hypothesis, the infinite word \(y\) belongs to \(X\). \(\square\)

**Corollary 2.10.** (Path sets are characterized by initial words) A path set \(\mathcal{P}\) is characterized by its set \(B^I(\mathcal{P})\) of all finite initial words. That is, if two path sets have the same set of initial words, then they are identical.

**Proof.** The limit set of the set of initial words \(B^I(X)\) of an arbitrary set \(X \subseteq \mathcal{A}^\mathbb{N}\) is its topological closure \(\overline{X}\). Since path sets \(\mathcal{P}\) are closed sets by Theorem \(\ref{thm:closed_sets} \), they are determined by \(B^I(\mathcal{P})\). \(\square\)

**Remark 2.11.** The characterization of Theorem \(\ref{thm:characterization} \) parallels a characterization of sofic shifts found by Ashley, Kitchens and Stafford \([8]\) (cf. \([2]\) Theorem B.1)) in 1992, which assumes the extra condition of one-sided shift-invariance: Any shift-invariant subset \(X\) of \(\mathcal{A}^\mathbb{N}\) is a sofic shift if and only if it has only finitely many different word follower sets \(F(X,b)\).

2.5. Minimal presentations of path sets.

**Definition 2.12.** (1) A minimal presentation for a path set \(\mathcal{P}\) is a presentation with a minimal number of vertices.

(2) A minimal right-resolving presentation is a right-resolving presentation having a minimal number of vertices among all right-resolving presentations.

A minimal presentation is always pruned and reachable, but it need not be right-resolving. Minimal right-resolving presentations are sometimes not minimal presentations.
**Theorem 2.13.** Let \( P \) be a path set having a minimal presentation having \( m \) vertices. Then \( P \) has a minimal right-resolving presentation having at most \( 2^m - 1 \) vertices.

**Proof.** This result is proved by the well-known subset construction in the automata theory literature, for obtaining a minimal deterministic finite state automaton matching a finite state deterministic automaton. It appears [18, Chapter III, Sect. 5, Theorem 5.2]. □

Below we characterize minimal right-resolving presentations, showing uniqueness in the process. We first recall a definition from symbolic dynamics, which is a one-sided shift version of a definition in Lind and Marcus [23, page 71, page 78]

**Definition 2.14.** A directed labeled graph \( G \) is called follower-separated if all vertices have distinct vertex follower sets.

In characterizing the number of states in a minimal right-resolving presentation, we formulate two definitions that view path sets as “infinite follower sets”.

**Definition 2.15.** (1) Given a path set \( P \) and a finite word \( w \in A^* \), the word path set \( P^w \) of \( P \) is the set of all infinite words \( x \) such that \( wx \in P \).

(2) Given a presentation \( (\mathcal{G}, v) \) of a path set \( P \), an associated vertex path set is any path set \( X(\mathcal{G}, v') \) where \( v' \) is a vertex of \( G \).

A nonempty word path set \( P^w \) is a path set. For a right-resolving presentation \( (\mathcal{G}, v) \) of \( P \) an argument parallel to Proposition 2.8(1) shows that \( P^w \) equals the vertex path set \( X(\mathcal{G}, v') \) for the vertex \( v' \) of \( G \) that is the final vertex on the unique path with symbol labels \( w \) from the initial vertex \( v \).

**Theorem 2.16.** (Minimal right-resolving presentation)

1. A path set \( P \) has a minimal right-resolving presentation \( (\mathcal{G}, v) \), which is unique up to isomorphism of pointed labeled graphs. This presentation is pruned, reachable and follower-separated.

2. Conversely, if a right-resolving presentation of \( P \) is pruned, reachable and follower-separated, then it is minimal.

3. The number \( m \) of vertices in a minimal right-resolving presentation of \( P \) is the number of distinct word follower sets \( F_P(b) \) of \( P \). It also equals the number of distinct vertex follower sets \( F(\mathcal{G}, v) \) of \( P \) in any right-resolving, reachable presentation \( \mathcal{P} = X(\mathcal{G}, v) \).

4. The number \( m \) of vertices in a minimal right-resolving presentation of \( P \) is the number of distinct nonempty word path sets \( P^w \) of \( P \). It also equals the number of distinct nonempty vertex path sets \( X(\mathcal{G}, v') \) in any right-resolving presentation \( (\mathcal{G}, v) \) of \( P \).

**Proof.** (1) Proposition 2.8 implies that the number of vertices of any right-resolving presentation of \( P \) must equal or exceed the number of distinct word follower sets \( F_P(b) \). The proof of Theorem 2.9 constructed a right-resolving presentation \( (\mathcal{G}, v) \) for \( P \), which has one vertex for each distinct word follower set, which must therefore be minimal. By construction it is pruned and the vertex follower sets in this presentation are distinct, so it is follower-separated.

It remains to show uniqueness. We know that any minimal right-resolving presentation necessarily has vertices labeled by all of the possible word follower sets. The exit edges from the pointed vertex \( v \) have different labels \( a' \) (from right-resolving property) and the edge labeled \( a' \) must go to the vertex labeled by the word follower set \( F_P(a') \). This assignment is the only way to permit the initial word follower set \( F_P(\emptyset) \) to reach all words for \( a' \) that begin with prefix \( a' \). Similarly the exit edges from each vertex \( F_P(b) \) must take the allowed prefix labels \( a' \) of words in \( F_P(b) \) and for each \( a' \) must map to vertex with follower set label \( F_P(ba') \). Every labeled edge is forced, so the construction is unique.

(2) Suppose that a right-resolving presentation \( (\mathcal{G}, v) \) of \( P \) is pruned, follower-separated and reachable. By Proposition 2.8(2) each vertex follower sets is a symbolic follower set. By Proposition 2.8(1) the vertex follower sets includes all distinct word follower sets. The follower-separation property implies each distinct follower set occurs exactly once in \( G \) as a vertex follower set, so \( (\mathcal{G}, v) \) is minimal.
(3) The two assertions are a consequence of Proposition 2.8 saying that every vertex follower set of every right-resolving presentation is some word follower set, and that all distinct word follower sets appear as vertex follower sets in every right-resolving presentation.

(4) The two assertions follow from (3) using Theorem 2.10 which implies that word path sets are uniquely determined by their word follower sets, and vice-versa. Similarly vertex path sets are uniquely determined by their vertex follower sets, and vice-versa.

Remark 2.17. For any minimal presentation of $P$ that has fewer vertices than in a minimal right-resolving presentation, there will necessarily be word follower sets $P^w$ that are not vertex follower sets of such a presentation.

Remark 2.18. For sofic shifts minimal right-resolving presentations (in the two-sided sofic shift sense, which is not the path set sense) are not necessarily unique. A counterexample is given in [23] Example 3.3.21.

2.6 Recognizing and distinguishing path sets. We describe algorithms for testing identity of path sets and for finding minimal right-resolving presentations. They are based on the following effective bound for telling when two given vertex follower sets in a (possibly disconnected) presentation are equal.

Proposition 2.19. Let $G$ be a right-resolving pruned labeled graph, not necessarily connected, that has $m$ vertices. If two vertices, $v_1$ and $v_2$, have distinct vertex follower sets $F(G, v_1)$, then there is some word $w = a_1a_2\cdots a_r$ of length $m$ in $A$ which belongs to exactly one of the two follower sets $F(G, v_1)$ and $F(G, v_2)$.

Proof. The proof of this theorem is outlined in Exercise 3.4.10 of [23]. Similar results appear in Conway [12] Chapter 1, Theorems 6 and 7.

Proposition 2.20. Let $P_1 = X(G_1, v_1)$ and $P_2 = X(G_2, v_2)$ be path sets with right-resolving presentations, where $G_1$ has $m_1$ vertices and $G_2$ has $m_2$ vertices. Then $P_1 = P_2$ if and only if $P_1$ and $P_2$ share the same set of initial $(m_1 + m_2)$-blocks; i.e., $B^I_{m_1+m_2}(G_1, v_1) = B^I_{m_1+m_2}(G_2, v_2)$.

Proof. This result follows directly from Proposition 2.19. We form a (disconnected) graph $G = G_1 \sqcup G_2$, which has $m_1 + m_2$ vertices. The contrapositive of Proposition 2.19 says that the follower sets $F(G, v_1)$ and $F(G, v_2)$ are equal if and only if they contain the same set of words of length $m_1 + m_2$; i.e., if $F_{m_1+m_2}(G, v_1) = F_{m_1+m_2}(G, v_2)$. Because the graph $G$ is disconnected in two pieces, these follower sets are $F(G, v_1) = B^I(G_1, v_1)$ and $F(G, v_2) = B^I(G_2, v_2)$. For the same reason, we have that the length $m_1$ follower sets are $F_{m_1+m_2}(G_1, v_1) = B^I_{m_1+m_2}(G_1, v_1)$ and $F_{m_1+m_2}(G_2, v_2) = B^I_{m_1+m_2}(G_2, v_2)$, whence $B^I_{m_1+m_2}(G_1, v_1) = B^I_{m_1+m_2}(G_2, v_2)$. Therefore we conclude that the latter equality implies equality of the initial follower sets $B^I(G_1, v_1) = B^I(G_2, v_2)$. By Theorem 2.10 we conclude $P_1 = P_2$.

Proposition 2.21. (Testing Identity of Path Sets) There is an effective algorithm which when given two pointed graphs $(G_1, v_1)$ and $(G_2, v_2)$ determines whether the path sets $P_1 = X(G_1, v_1)$ and $P_2 = X(G_2, v_2)$ are identical.

Proof. Proposition 2.20 yields an effective algorithm to tell if two path sets $P_1 = X(G_1, v_1)$ and $P_2 = X(G_2, v_2)$ are equivalent. We first use the method of [2, Theorem 3.2] to convert the given presentations to pointed graphs $(G_1', v_1')$ and $(G_2', v_2')$ that are right-resolving and reachable. Suppose these two graphs $G_1'$ and $G_2'$ have $m_1$ and $m_2$ vertices respectively. It now suffices to exhaustively determine all members of the finite sets $B^I_{m_1+m_2}(G_1', v_1')$ and $B^I_{m_1+m_2}(G_2', v_2')$ of initial blocks of length $m_1 + m_2$ by tracing paths through the graphs, and to check whether these sets are identical.

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3One can improve the length bound on $w$ to $m - 1$, and there exist examples showing that the upper bound $m - 1$ is the best possible.
3. Decimations of path sets

We study the effect of decimation operations on path sets. The following result was originally established as Theorem 1.5 in [2].

**Theorem 3.1.** (C(A) is closed under decimation) If $\mathcal{P} \in C(A)$ is a path set, then for any $(j, n)$ with $j \geq 0$ and $n \geq 1$, the $j$-th decimation set $\mathcal{P}_{j,n}$ of $\mathcal{P}$ at depth $n$, given by

$$\mathcal{P}_{j,n} = \psi_{j,n}(\mathcal{P}) := \bigcup_{x \in \mathcal{P}} \{\psi_{j,n}(x)\},$$

is a path set.

The proof given in [2] formulated an algorithm which when given as input a right-resolving presentation $(G, v)$ of a path set $\mathcal{P}$, and the values $(j, n)$ produced as output a (not necessarily right-resolving) presentation $(\mathcal{G}_{j,n}, \psi_{j,n})$ for the $j$-th decimation at level $n$ of $\mathcal{P}$, $\psi_{j,n}(\mathcal{P})$, for $0 \leq j \leq n - 1$. The algorithm was outlined in the discussion in [2, Section 7]. It has the feature that number of vertices of the output presentation it produces can be much larger than the number of vertices in the input presentation.

Here we present algorithms which produce presentations of $\psi_{j,n}(\mathcal{P})$ which are smaller: they increase the number of vertices of the input presentations by at most 1.

The paper [2] showed that any iterated shift $S^j(\mathcal{P})$ of a path set is a path set. In Section 3.1 we give an algorithm which shows that from any presentation of a path set with $m$ vertices one can constructively find a presentation of any $S^j(\mathcal{P})$ having at most $m + 1$ vertices. Note that $S^j(\mathcal{P}) = \psi_{j,1}(\mathcal{P})$. In Section 3.2 we present a second constructive algorithm that finds a presentation of $\psi_{j,n}(\mathcal{P})$ for $0 \leq j \leq n - 1$, the higher power presentation, having no more than $m$ vertices. Combining it with the algorithm of Section 3.1 we obtain presentation for each $\psi_{j,n}(\mathcal{P})$ with $j \geq n$. In Section 3.3 we use this result to prove finiteness of the set of all decimations $\psi_{j,n}(\mathcal{P})$ of a path set $\mathcal{P}$.

For general sets $X \subseteq A^N$, we will term the decimations $\psi_{j,n}(X)$ with $0 \leq j \leq n - 1$ principal decimations and call the remaining $\psi_{j,n}(X)$ with $j \geq n$ subsidiary decimations. This terminology reflects the fact that, when acting on a single word $a_0a_1 \cdots$, the principal decimations at level $n$ supply enough information to reconstruct $X$ word by word, using the identity (1.1).

### 3.1. Iterated shift operators on path sets

We have $\psi_{0,1}(X) = X$ and $\psi_{j,1}(X) = S^j(X)$, where $S^j$ is the $j$-fold iteration of the left shift operator, which operates on individual symbol sequences $a_0a_1a_2 \cdots \in A^N$ by

$$S^j(a_0a_1a_2a_3 \cdots) = a_ja_{j+1}a_{j+2}a_{j+3} \cdots.$$

**Theorem 3.2.** (Iterated shift operator) Given a path set with presentation $\mathcal{P} = X(\mathcal{G}, v)$ having $m$ vertices, which is reachable. Then for each $j \geq 1$ there exists a presentation of the $j$-th iterated shift

$$S^j\mathcal{P} := \psi_{j,1}(\mathcal{P}) = X(\mathcal{G}_{j,1}, v)$$

which has at most $m + 1$ vertices.

**Proof.** The path set $S^j(\mathcal{P})$ is exactly the set of infinite words in $\mathcal{G}$ emanating from the set $V^{(j)}(\mathcal{G}, v)$ of vertices of $\mathcal{G}$ that can be reached from the initial vertex $v$ after traversing a path with $j$ edges. We create a new graph $G_{j,1}$ from $\mathcal{G}$ by adding a new vertex $w$, so that $V(G_{j,1}) = V(\mathcal{G}) \cup \{w\}$. The directed labeled graph $G_{j,1}$ has the same directed labeled edges as $\mathcal{G}$ on the vertices $V(\mathcal{G})$, the new vertex $w$ has no entering edges and is defined to have labeled exit edges from $w$ to vertex $v_2 \in V(\mathcal{G})$ whenever there is an entering edge $v_1 \rightarrow v_2$ from some $v_1 \in V(\mathcal{G})$ in having the given label. Any duplicate labeled edges obtained this way are to be discarded. The new vertex will be the marked vertex in the presentation $X(\mathcal{G}', w)$. We claim that the presentation $X(\mathcal{G}', w)$ is $\psi_{j,1}(\mathcal{P})$. Indeed, after one step each path from $w$ enters $\mathcal{G}$ and stays there forever after. \qed

**Remark 3.3.** The presentation $S^j(\mathcal{P}) = X(\mathcal{G}_{j,1}, w)$ obtained in this construction need not be right-resolving; there may be multiple edges with the same label emanating from $w$. 


Definition 3.4. (Shift-invariance, weak shift-invariance, weak shift-stability)

(1) A set $X \subseteq \mathcal{A}^n$ is shift-invariant if $SX = X$.

(2) A set $X$ is weakly shift-invariant if there are integers $k > j \geq 0$ such that the iterated shifts $S^kX = S^jX$.

(3) A set $X \subseteq \mathcal{A}^n$ is weakly shift-stable if there are integers $k > j \geq 0$ such that $S^kX \subseteq S^jX$.

The concept of weak shift-stability was introduced and studied in [5]. Weak shift-invariance implies weak shift-stability of a set $X$.

Theorem 3.5. (Weak shift-invariance of path sets) For any path set $\mathcal{P}$ there exist integers $k > j \geq 0$ giving the equality of iterated shifts $S^k\mathcal{P} = S^j\mathcal{P}$. That is, all path sets $\mathcal{P}$ are weakly shift-invariant.

Proof. Given $\mathcal{P}$, take a reachable presentation for it, letting $m$ be its number of vertices. According to Theorem 3.2, all iterated shifts $S^j(\mathcal{P})$ for $j \geq 1$ have reachable presentations (on the same alphabet) having at most $m+1$ vertices. The number of such presentations is finite. (There are at most $(m+1)(2^{\mathcal{A}(m+1)(m+2)}$ of them, noting that presentations do not allow multiple directed edges with the same label between two vertices.) By the pigeonhole principle, there must exist two integers $0 \leq j < k$ giving the same presentation, so $S^k\mathcal{P} = S^j\mathcal{P}$.

3.2. Higher power presentation for decimations of path sets. We present an algorithm which constructs from a given presentation of $\mathcal{P} = X(\mathcal{G}, v)$ a presentation $\mathcal{G}_{j,n}$ of $\psi_{j,n}(\mathcal{P})$ for principal decimations, called here the modified $n$-th higher power presentation, which has the vertex bound $|V(\mathcal{G}_{j,n})| \leq |V(\mathcal{G})|$. It is based on the well known $n$-th higher power construction, cf. [23] Sect. 1.4, which presents $\mathcal{P}$ in blocks using labels from a larger symbol alphabet $\mathcal{A}^n$. The modified algorithm replaces the $n$-block word labels produced by this construction to labels using the original alphabet $\mathcal{A}$ in such a way as to obtain a presentations $\mathcal{G}_{j,n}$ of all principal decimations $\psi_{j,n}(\mathcal{P})$, for $0 \leq j \leq n - 1$. We then apply the shift construction in Theorem 3.2 to get a presentation of each $\psi_{j,n}(\mathcal{P})$ for $j \geq n$ having at most one extra vertex: $|V(\mathcal{G}_{j,n})| \leq |V(\mathcal{G})| + 1$.

Theorem 3.6. (Higher powers of a path set) Given a path set with presentation $\mathcal{P} = X(\mathcal{G}, v)$ on alphabet $\mathcal{A}$. For $n \geq 2$ there exists a presentation $\psi_{j,n}(\mathcal{P}) = X(\mathcal{G}_{j,n}, v)$ of the $(j, n)$-th decimation of $\mathcal{P}$, for $0 \leq j \leq n - 1$, such that each $\mathcal{G}_{j,n}$ has the same vertex set as $\mathcal{G}$ and has the same marked vertex $v$. For $j \geq n$ there exists a presentation $\psi_{j,n}(\mathcal{P}) = X(\mathcal{G}_{j,n}, w)$, where $\mathcal{G}_{j,n}$ has the same vertex set of $\mathcal{G}$, plus one extra vertex $w$, which will be the marked vertex.

Proof. We associate to the presentation $(\mathcal{G}, v)$ a construction $(\mathcal{G}_{n}, v)$ called the $n$-th higher power presentation, in which $\mathcal{G}_{n}$ has the same vertex set as $\mathcal{G}$ and the same initial vertex $v$, but its edges are labeled by the product alphabet $\mathcal{A}^n$. (This construction parallels that in [23], Defn. 2.3.10.) In $\mathcal{G}_{n}$ we draw a directed edge between vertices $v_1$ and $v_2$ with edge label $b_0b_1 \cdots b_{n-1} \in \mathcal{A}^n$ if there is a directed path of length $n$ in $\mathcal{G}$ starting at $v_1$ and ending at $v_2$, having successive edge labels $b_0, b_1, \cdots, b_{n-1}$. It is straightforward to see that $\mathcal{P} = X(\mathcal{G}_{n}, v)$, viewed in the enlarged alphabet $\mathcal{A}^n$, generates the output infinite words in blocks of $n$ symbols.

We now obtain a presentation $(\mathcal{G}_{j,n}, v)$ from $(\mathcal{G}_{n}, v)$ by relabeling edges, replacing each edge symbol $b_0b_1 \cdots b_{n-1} \in \mathcal{A}^n$ by a single symbol $b_j \in \mathcal{A}$, its $j$-th symbol. After this is done, there may exist pairs of vertices $v_1$ and $v_2$ being connected by multiple edges labeled with the same symbol $b_j$; we delete duplicate edges. In addition, the resulting graph might be disconnected; we retain the induced subgraph having the set of vertices reachable starting from $v$ using this set of edges.

We claim that $\psi_{j,n}(\mathcal{P}) = X(\mathcal{G}_{j,n}, v)$. To prove the claim we show inclusions hold in both directions. Suppose $x = x_0x_1 \cdots \in \psi_{j,n}(\mathcal{P})$. Then there is some word $y = y_0y_1 \cdots \in \mathcal{P}$ such that $x_i = y_{i+j+n}$ for all $i$. Since $y$ is presented by $(\mathcal{G}, v)$, there is an infinite path in $\mathcal{G}$ starting at $v$, presenting $y$. Therefore, there is an edge in $\mathcal{G}_{n}$ from $v$ to a vertex $v'$ of $\mathcal{G}$ labeled with the first $n$ letters of $y$, another edge from $v'$ to another vertex $v''$ labeled with the next $n$ letters, and so on. Take a corresponding path in $\mathcal{G}_{j,n}$. The word presented will begin with the $j$th letter of the first block of $n$ letters from $y$, followed by the $j$th letter of
the second block of \( n \) letters, and so on. Thus, the word presented will be \( y_j y_{j+n} y_{j+2n} \cdots \). This word is \( x \) so \( x \in X(G_{j,n}, v) \), whence \( \psi_{j,n}(\mathcal{P}) \subseteq X(G_{j,n}, v) \). For the other inclusion, suppose \( x \in X(G_{j,n}, v) \). Then there is a path on \( G_{j,n} \) starting at \( v \) presenting \( x \). A corresponding path on \( G_n \) will present a word whose \((j,n)\)th decimation is \( x \). Thus, \( x \in \psi_{j,n}(\mathcal{P}) \), and we have \( \psi_{j,n}(\mathcal{P}) \supseteq X(G_{j,n}, v) \).

We have completed the construction for principal decimations. For the remaining decimations \( \psi_{j,n}(\mathcal{P}) \) with \( j \geq n \),

we apply the higher power construction to the presentation obtained in Theorem 3.2 in which \( \psi_{j,n}(\mathcal{P}) \) becomes the initial principal decimation \( \psi_{0,n}(S^j\mathcal{P}) \) of \( S^j\mathcal{P} \).

3.3. Finiteness of full decimation set. Definition 1.5 states that the full decimation set \( \mathcal{D}(\mathcal{P}) \) of a path set is defined by

\[ \mathcal{D}(\mathcal{P}) := \{ \psi_{j,n}(\mathcal{P}) : \text{ all } n \geq 1 \text{ and all } j \geq 0 \} \]

For the special case of path sets we show finiteness of the full decimation set.

**Theorem 3.7.** (Full decimation set bound) For each path set \( \mathcal{P} \) its full decimation set \( \mathcal{D}(\mathcal{P}) \) is a finite set. If \( \mathcal{P} \) has a presentation having \( m \) vertices, then \( \mathcal{D}(\mathcal{P}) \) has cardinality bounded by

\[ |\mathcal{D}(\mathcal{P})| \leq 2^{m^2|\mathcal{A}|} \]

**Proof.** Theorem 3.2 implies a upper bound for \( |\mathcal{D}(\mathcal{P})| \) given by the number of distinct labeled directed graphs on \( m \) vertices having the property that there is at most one directed edge going from one given vertex \( v_1 \) to another given vertex \( v_2 \), with a given symbol \( a \in \mathcal{A} \). The number of such directed vertex pairs is \( m^2 \) and the number of possible directed edge patterns from a fixed vertex \( v_1 \) to another fixed vertex \( v_2 \) is exactly \( 2^{|\mathcal{A}|} \), so we obtain \( |\mathcal{D}(\mathcal{P})| \leq 2^{m^2|\mathcal{A}|} \).

**Remark 3.8.** In contrast to Theorem 1.6 there exist closed \( X \subseteq \mathcal{A}^\mathbb{N} \) for which all members of the infinite collection \( \{ \psi_{j,n}(X) \} \) for \( j \geq 0 \) are distinct, see [5, Example 6.5].

3.4. Right resolving presentations for decimations of path sets. The output presentation of \( \psi_{j,n}(\mathcal{P}) \) produced by Theorem 3.2 need not be right-resolving, even if the given input presentation of \( \mathcal{P} \) were right-resolving. Using the subset construction for obtaining a right-resolving presentation from a general presentation, we obtain the following result.

**Theorem 3.9.** (Right-resolving presentations of decimation sets of a path set) Given a path set \( \mathcal{P} \) on alphabet \( \mathcal{A} \) with at least two letters, having a (not necessarily right-resolving) presentation \( \mathcal{P} = X(G, v) \) with \( m \) vertices. Then for each \( n \geq 1 \) and each \( j \geq 0 \) the decimation set \( \psi_{j,n}(\mathcal{P}) \) has a right-resolving presentation having at most \( 2^{m+1} - 1 \) vertices.

This bound on the number of vertices implies a finiteness result for the number of distinct decimation sets; see Theorem 3.7.

4. INTERLEAVING OF PATH SETS

Our object is to constructively prove the following result.

**Theorem 4.1.** (\( \mathcal{C}(\mathcal{A}) \) is closed under interleaving) If \( \mathcal{P}_0, \ldots, \mathcal{P}_{n-1} \) are path sets on the alphabet \( \mathcal{A} \), then their \( n \)-fold interleaving

\[ X := (\otimes_n)_{i=0}^{n-1} \mathcal{P}_i = \mathcal{P}_0 \otimes \mathcal{P}_1 \otimes \cdots \otimes \mathcal{P}_{n-1} \]

is a path set; i.e., \( X \in \mathcal{C}(\mathcal{A}) \).

To do this, we give an effective procedure for computing a presentation \( (G, v) \) of the \( n \)-interleaving set \( X := \mathcal{P}_0 \otimes \mathcal{P}_1 \otimes \cdots \otimes \mathcal{P}_{n-1} \), given presentations of each input factor \( \mathcal{P}_i = (G_i, v_i) \). This presentation certifies that \( X \) is a path set. We give examples. We also prove the converse result that every interleaving factor of a path set \( \mathcal{P} \) is a path set given by some decimation of \( \mathcal{P} \).
4.1. \( n \)-fold interleaving construction.

**Theorem 4.2.** (Interleaving pointed graph product construction) Let \( n \geq 2 \) and suppose that \( \mathcal{P}_0, \ldots, \mathcal{P}_{n-1} \) are path sets with given presentations \((G_0, v_0), \ldots, (G_{n-1}, v_{n-1})\), respectively. There exists a construction taking as inputs these presentations and giving as output a presentation \((H, v)\) of the \( n \)-fold interleaving \( X := \mathcal{P}_0 \circ \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_{n-1} \). In particular \( X = X(\mathcal{H}, v) \) is a path set. This construction has the following properties:

(i) If \( G_i \) has \( k_i \) vertices for each \( 0 \leq i \leq n-1 \), then \( \mathcal{H} \) will have at most \( n \prod_{i=0}^{n-1} k_i \) vertices.

(ii) If the pointed graphs \((G_i, v_i)\) are right-resolving for all \( 0 \leq i \leq n-1 \), then the output pointed graph \( \mathcal{H} \) will also be right-resolving.

(iii) If the pointed graphs \((G_i, v_i)\) are pruned for all \( 0 \leq i \leq n-1 \), then the output pointed graph \( \mathcal{H} \) will also be pruned.

**Proof of Theorem 4.2** Suppose \( \mathcal{P}_i \) is presented by the pointed graph \((G_i, v_i^0)\), which has vertex set \( V_i \) having \( k_i \) vertices, for all \( 0 \leq i \leq n-1 \). We construct a new pointed labeled graph \((H, v)\), which we term the \( n \)-fold interleaving pointed graph product of \( \mathcal{G}_i := (G_i, v_i^0) \).

The underlying directed labeled graph \( \mathcal{H} \) is the \( n \)-fold interleaving graph product of the labeled graphs \( \mathcal{G}_i \):

\[
\mathcal{H} := \mathcal{G}_0 \circ \mathcal{G}_1 \circ \cdots \circ \mathcal{G}_{n-1},
\]

using as input an ordered set of \( n \) directed labeled graphs \( \mathcal{G}_i \).

The vertices of \( \mathcal{H} \) consist of a union of products of the vertices of the \( \mathcal{G}_i \), for \( n \) cyclically rotated copies of the \( \mathcal{G}_i \). To begin, choose (for convenience) a numbering to the vertices \( V_i \) from each \( \mathcal{G}_i \), and let \( v_i^j \) be the \( j \)th element of \( V_i \), with \( v_i^0 \) the marked vertex of \( V_i \). We define the \( i \)-th vertex set in \( \mathcal{H} \) to be

\[
\mathcal{V}^i := V_i \times V_{i+1} \times \cdots \times V_{n-1} \times V_0 \times V_1 \times \cdots \times V_{i-1}.
\]

and let \( \mathcal{V}(H) = \bigcup_{i=0}^{n-1} \mathcal{V}^i \) be the vertex set of \( \mathcal{H} \). Here a vertex in \( \mathcal{V}^i \) is a vector,

\[
(v_i^0, v_i^1, v_{i+1}^1, \ldots, v_{n-1}^1, v_0^0, v_1^1, \ldots, v_{i-1}^1) : \text{where each } 0 \leq j_m \leq k_m - 1.
\]

The labeled edges of \( \mathcal{H} \) all connect vertices of \( \mathcal{V}^i \) to vertices of the next set \( \mathcal{V}^{i+1} \), with indices taken modulo \( n \). Whenever there is an edge from \( v_i^j \) to \( v_i^{j+1} \) in \( \mathcal{G}_i \) which has label \( a \), draw edges in \( \mathcal{H} \) labeled \( a \) from each vertex in \( \mathcal{V}^i \) that is of the form

\[
(v_i^0, v_i^1, v_{i+1}^1, \ldots, v_{n-2}^1, v_{n-1}^1, v_0^0, v_1^1, \ldots, v_{i-1}^1).
\]

to that vertex in \( \mathcal{V}^{i+1} \) given by

\[
(v_{i+1}^0, v_{i+1}^1, \ldots, v_{n-2}^1, v_{n-1}^1, v_0^0, v_1^1, \ldots, v_{i-1}^1).\]

We use the cyclic ordering for superscripts \( i \) (mod \( n \)) of \( \mathcal{V}^i \), so that \( \mathcal{V}^n \equiv \mathcal{V}^0 \).

For the pointed graph version of this construction, we add as the pointed vertex of \( \mathcal{H} \) vertex of \( \mathcal{V}^0 \) given by \( \mathcal{V} = (v_0^0, v_1^0, \ldots, v_0^{n-2}, v_{n-1}^0) \) determined by the pointed vertices \( v_i^0 \) of the individual \( \mathcal{G}_i \).

Now define \( \overline{\mathcal{P}} \) to be the path set presented by \((\mathcal{H}, v^0)\).

**Claim.** \( \overline{\mathcal{P}} = (\oplus n)^{n-1} \mathcal{P}_i := \mathcal{P}_0 \circ \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_{n-1} \).

To prove the claim, we first show the inclusion \((\oplus n)^{n-1} \mathcal{P}_i \subseteq \overline{\mathcal{P}}\). Let \((x_i)^{\infty} \in (\oplus n)^{n-1} \mathcal{P}_i \). By definition there exist edge label elements separately in each factor \( \mathcal{P}_i \) that traverse an infinite edge path, \((e_i, t)_{t=0}^\infty \) in \( \mathcal{G}_i \) which has associated symbol sequence

\[
(x_i, t)^{\infty} \in \mathcal{P}_i \text{ for all } 0 \leq i \leq n-1,
\]

\[4\text{There are } \prod_{j=0}^{n-1} k_j \text{ such edges.}\]
which visits an associated sequence of vertices
\[(v_{i,t})_{t=0}^{\infty}\]
in the graph \(G_i\). We call \((v_{i,t})_{t=0}^{\infty}\) a vertex path associated to the edge path \((x_{i,t})_{t=0}^{\infty}\). (A vertex path is uniquely determined by the edge path, requiring that the initial vertex be the marked vertex. There could be several edge paths giving the same marked vertex, if there are multiple edges.)

The edge symbol sequences \((x_{i,t})_{t=0}^{\infty}\) interleave to reconstruct the sequence \((x_{i})_{t=0}^{\infty}\) via
\[x_{i,t} = x_{i+n}t \quad \text{for all } 0 \leq t < \infty.\]

We show these elements give a sequence of update edges for an edge path in \(H\) realizing this symbol sequence \(x_t\). We start at \(t = 0\) at initial vertex \(v^0 = (v_0^0, v_1^0, \ldots, v_{n-1}^0) \in \mathcal{V}^0\). We proceed in "rounds" of \(n\) steps. At the beginning of "round" \(k\) at \(t = kn\) we will be at a vertex \(v^k = (v_0^{j_0,k}, v_1^{j_1,k}, \ldots, v_{n-1}^{j_{n-1,k}}) \in \mathcal{V}^0\). During the round, with steps numbered \(0 \to n - 1\), the vertices cyclically rotate (to the left) and at the start of \(i\)-th step the vector is initially in \(\mathcal{V}^i\), the leftmost vertex \(v_{i}^{j_{i,k}}\) is updated to \(v_{i}^{j_{i+1,k+1}}\), by moving on an edge on \(G_i\) between these two vertices, which has edge label \(x_{nk+i}\), and the new vertex in \(G_i\) is moved all the way to the right, to get a vector in \(\mathcal{V}^{i+1}\). At the end of the round we back in \(\mathcal{V}^{0}\) at a new vertex vector \(v^{k+1}\). We proceed by induction on the number of rounds \(k\). The base case \(k = 0\) starts with all \(j_{i,0} = 0\).

The induction hypothesis is that at the end of the \(r\)-th round we have followed a path on \(H\) that incremented motion on each of the \(G_i\) by one symbol of \(x_{i,r}\) and has moved in the \(i\)-th vector coordinate from vertex \(v_{i,r}\) to the vertex \(v_{i,r+1}\), corresponding to \(G_i\). That is, at step \(nr + i\) we produced symbol \(x_{i,r+1}\) and at time \(t = n(r + 1)\) the \(i\)-th vector component of \(v^{r+1} = (v_{0}^{j_{0},r+1}, \ldots, v_{n-1}^{j_{n-1},r+1})\), has entry \(v_{i,r+1}\) in \(G_i\), for \(0 \leq i \leq n - 1\). The induction step is completed using the fact that each \((e_{i,t}: t \geq 0) \in \mathcal{P}_i\) is a legal edge path in \(G_i\) that permits taking the next step at \(G_i\) in the next round. We conclude that there is an infinite path in \(H\) originating at \(v^0\) with edge labels \((x_0, x_1, x_2, \ldots)\), producing \((x_i)_{t=0}^{\infty}\) an element of \(\mathcal{P}\). Thus \((\otimes n)_{i=0}^{n-1} \mathcal{P}_i \subseteq \mathcal{P}\).

It remains to show the reverse inclusion \(\widehat{\mathcal{P}} \subseteq (\otimes n)_{i=0}^{n-1} \mathcal{P}_i\). Suppose given \((x_{k})_{k=0}^{\infty} \in \widehat{\mathcal{P}}\). Then in the first \(n\) steps there is a vertex path
\[(v_0^0, v_1^0, \ldots, v_{n-1}^0, v_0^{0})\; ; \; (v_0^0, v_2^0, \ldots, v_{n-1}^0, v_0^{j_{0,1}})\; ; \; (v_0^0, v_3^0, \ldots, v_{n-1}^0, v_0^{j_{0,1}}, v_1^{j_{1,1}})\; ; \; \ldots\; ; \; (v_{n-1}^0, v_0^{j_{0,1}}, v_1^{j_{1,1}}, \ldots, v_{n-2}^{j_{n-2,1}}, v_{n-1}^{j_{n-1,1}})\; ; \; (v_{n-1}^{j_{n-1,1}}, v_0^{j_{0,1}}, v_1^{j_{1,1}}, \ldots, v_{n-2}^{j_{n-2,1}}, v_{n-1}^{j_{n-1,1}})\; ; \; \ldots\; ;
\]
in \(H\) which can be traversed by edges labeled \(x_0, x_1, \ldots, x_{n-1}\). Notice that the first coordinate of a vertex path will be the last coordinate of the vertex that follows after \(n - 1\) steps. Since the initial vertex is \((v_0^0, v_1^0, \ldots, v_{n-1}^0, v_0^{0})\), we know that for each \(0 \leq i \leq n - 1\), there is a matching edge in \(G_i\) from \(v_0^i\) to \(v_1^{j_{i,1}}\).

For any \(k < \infty\), an edge in \(H\) with edge label \(x_k\) from vertex
\[(v_i^{j_i,r}, \ldots, v_{n-1}^{j_{n-1,r}}, v_0^{j_{0,r+1}}, v_1^{j_{1,r+1}}, \ldots, v_{n-1}^{j_{n-1,r+1}}) \in \mathcal{V}_i\]
to vertex
\[(v_i^{j_i+1,r}, \ldots, v_{n-1}^{j_{n-1,r}}, v_0^{j_{0,r+1}}, v_1^{j_{1,r+1}}, \ldots, v_{n-1}^{j_{n-1,r+1}}) \in \mathcal{V}_{i+1}\]
corresponds to a directed edge in \(G_i\) from \(v_i^{j_i,r}\) to \(v_i^{j_i+1,r}\) that has label \(x_k\). Following our given vertex path in \(H\) for \(n - 1\) more steps gets us to a vertex in \(H\) whose last coordinate is \(v_i^{j_i,r+1}\). There is an edge in \(H\) labeled \(x_{k+n}\) emanating from this vertex which corresponds to an edge in \(G_i\) labeled \(x_{k+n}\) emanating from \(v_i^{j_i+1,r+1}\) and going to \(v_i^{j_i,r+2}\). Thus, for each \(0 \leq i \leq n - 1\), the labels \(x_{i}, x_{i+n}, x_{i+2n}, \ldots\) are the labels of an infinite path in \(G\) originating at \(v_i\), so \((x_{k})_{k=0}^{\infty} \in (\otimes n)_{i=0}^{n-1} \mathcal{P}_i\). We conclude that \(\widehat{\mathcal{P}} \subseteq (\otimes n)_{i=0}^{n-1} \mathcal{P}_i\).

The claim follows.

The claim shows that the interleaving \((\otimes n)_{i=0}^{n-1} \mathcal{P}_i\) is the path set \(\mathcal{P}\), having a presentation \((H, v^0)\). Since \(H\) has \(n \prod_{i=0}^{n-1} k_i\) vertices, this proves (i). For (ii), if each of the \(G_i\) is right-resolving, then it is evident from the interleaving graph product construction that \(H\) is right-resolving. Each vertex \(v\) of \(\mathcal{V}\) has at most one
exit edge having a given label \( a \), inherited from \( G_i \). For (iii), if the graph \( G_i \) is pruned, then each vertex has at least one exit edge. The construction of edges for \( H \) then shows that each vertex in \( V \) has an exit edge if and only if each vertex of \( G_i \) has an exit edge.

**Proof of Theorem 1.8**. Theorem 1.8 is an immediate consequence of Theorem 1.9.

**Remark 4.3.** (1) The \( n \)-fold interleaving graph product operation does not always produce minimal right-resolving presentations, even when all the input presentations are minimal right-resolving, see Example 4.4.

(2) The \( n \)-fold interleaving graph product operation does not always produce reachable presentations when all the input presentations are reachable.

### 4.2. Examples

We present examples showing that the \( n \)-fold interleaving pointed graph product, given minimal right-resolving presentations \( (G_i, v_i) \) as input, may not produce a right-resolving presentation of their interleaving as output.

**Example 4.4.** (Non-preservation of minimal right-resolving property: extra automorphisms) Let \( P_0 = X(G_0, v_0) \) and \( P_1 = X(G_1, v_1) \), where \( G_0 \) and \( G_1 \) are the graphs given in Figure 4.1. Evidently \( P_0 = P_1 = \{0, 1\}^\mathbb{N} \), the full shift on two letters, and \( (G_0, v_0), (G_1, v_1) \) are (isomorphic) minimal right-resolving presentations. It is easy to see that \( P_0 \bowtie P_1 = \{0, 1\}^\mathbb{N} \) as well. Figure 4.2 shows the presentation of \( P_0 \bowtie P_1 \) given by our algorithm. This presentation is right-resolving but non-minimal; it is a double-covering of the minimal right-resolving representation.

![FIGURE 4.1. Presentations of \( P_0 \) and \( P_1 \) of Example 4.4](image)

The non-minimality of the presentation constructed in Example 4.4 is a result of the fact that the \( n \)-fold interleaving graph product construction keeps track of which input path set each digit comes from. If all the input path sets have the same presentation, then the graph product has a cyclic automorphism of order \( n \). For any path set \( P \), the presentation of the \( n \)-fold self-interleaving \( P \bowtie P \bowtie \cdots \bowtie P \) given by the construction of Theorem 1.9 is an \( n \)-fold covering of another presentation, the one constructed in [1, Proposition 3.4].

**Example 4.5.** (Non-preservation of minimal right-resolving property: failure of follower-separation) Consider the path sets \( Q_0 = \{(0^\infty)\} \cup \{(0^n12^n)^\infty \mid n \in \mathbb{N}\} \) and \( Q_1 = \{(32^\infty)\} \). Figure 4.3 gives minimal right-resolving presentations \( (H_0, v_0) \) of \( Q_0 \) and \( (H_1, v_2) \) of \( Q_1 \). The presentation of \( Q_0 \bowtie Q_1 \) given by Theorem 1.9 is shown in Figure 4.4. This presentation is not minimal, since \( v_1v_3 \) and \( v_3v_1 \) have the same follower sets. However, identifying the vertices \( v_1v_3 \) and \( v_3v_1 \) and replacing the edges between them with a single self-loop labeled 2 will give a minimal right-resolving presentation. This presentation is shown in Figure 4.5.
5. INTERLEAVING CLOSURE OPERATIONS

The paper [5] shows that interleavings of principal $n$-decimations define a series of closure operations $X \mapsto X^{[n]}$ for arbitrary subsets $X \subset \mathcal{A}^N$. We recall properties of these operations established in [5] which relate them to $n$-fold decimations. Then we show that the class $\mathcal{C}(\mathcal{A})$ of path sets is stable under these closure operations.

5.1. Interleaving closure operations. Decimations combined with $n$-interleavings define a series of closure operations on path sets. The closure operations are defined for arbitrary subsets $X \subset \mathcal{A}^N$, as described in [5].

Definition 5.1. (Interleaving closure operations) Given a subset $X$ of $\mathcal{A}^N$ the $n$-fold interleaving closure $X^{[n]}$ of $X$ is given by the $n$-fold interleaving

$$X^{[n]} := \psi_{0,n}(X) \star \psi_{1,n}(X) \star \cdots \star \psi_{n-1,n}(X).$$

We recall some results from [5]. The following result parts (1) and (2) are consequences of Theorem 4.2 of [5], and parts (3) and (4) are consequences of Theorem 4.12 of [5].

Theorem 5.2. ($n$-fold interleaving closure) Given a subset $X$ of $\mathcal{A}^N$ one has the set inclusion

$$X \subseteq X^{[n]},$$

where $X^{[n]}$ is the $n$-fold interleaving closure of $X$. If $X \subseteq Y$ then $X^{[n]} \subseteq Y^{[n]}$. In addition:
(1) The operation \( X \mapsto X^{[n]} \) is idempotent; i.e., \((X^{[n]})^{[n]} = X^{[n]}\) for all \( X \subseteq A^N \).

(2) The \( n \)-fold interleaving closure \( X^{[n]} \) has the property that it is the maximal set \( Y \) such that \( X \subseteq Y \) and

\[
\psi_{j,n}(Y) = \psi_{j,n}(X) \quad \text{for} \quad 0 \leq j \leq n - 1.
\]

(3) If \( X \) is a closed set in \( A^N \) then each decimation set \( X_{j,n} = \psi_{j,n}(X) \) is a closed set. The \( n \)-th interleaving closure \( X^{[n]} \) is a closed set in \( A^N \).

(4) The \( n \)-fold interleaving closure operation commutes with the closure operation on the product topology on \( A^N \), in the sense that

\[
(\overline{X})^{[n]} = \overline{X^{[n]}}.
\]

The following result is a consequence of Theorem 2.8 of [5].

**Theorem 5.3.** (Decimations and interleaving factorizations) A general subset \( X \) of \( A^N \) has an \( n \)-fold interleaving factorization \( X = X_0 \oplus X_1 \oplus \cdots \oplus X_{n-1} \) if and only if \( X = X^{[n]} \). In this case, each

\[
X_i = \psi_{j,n}(X) \quad \text{for} \quad 0 \leq j \leq n - 1,
\]

so when they exist, \( n \)-fold interleaving factorizations are unique.

### 5.2. Interleaving closures of path sets

We specialize to path sets, and show all the \( n \)-fold interleaving closures \( P^{[n]} \), \((n \geq 1)\), of a path set \( P \) are path sets. (Theorem 1.10).

**Theorem 5.4.** (\( C(A) \) is stable under \( n \)-fold interleaving closure operations) If \( P \) is a path set, then for each \( n \geq 1 \) the \( n \)-fold interleaving closure \( P^{[n]} \) is a path set. In addition, if \( P \) is \( n \)-factorizable then each of its \( n \)-fold interleaving factors \( P_j = \psi_{j,n}(P) \) for \( 0 \leq j \leq n - 1 \) are path sets.

**Proof.** If \( P \) is a path set then by Theorem 3.1 each \( \psi_{j,n}(P) \) is a path set. Thus

\[
P^{[n]} := \psi_{0,n}(P) \oplus \psi_{1,n}(P) \oplus \cdots \oplus \psi_{n-1,n}(P),
\]

is a path set by Theorem 3.1.

Now suppose \( P \) has an \( n \)-fold interleaving factorization

\[
P = X_{0,n} \oplus X_{1,n} \oplus \cdots \oplus X_{n-1,n}.
\]

Then by Theorem 5.3 \( X_{j,n} = \psi_{j,n}(P) \) for \( 0 \leq j \leq n - 1 \). But \( \psi_{j,n}(P) \) is a path set by Theorem 3.1. \( \square \)

### 6. Interleaving factorizations

We recall results on interleaving factorizations of general sets \( X \subseteq A^N \) from [5], relating to closure operations. We deduce that the set \( C^\infty(X) \) of infinitely factorizable path sets is stable under \( n \)-fold interleavings of its members, for all \( n \geq 1 \). Finally we obtain a bound on the size of minimal right-resolving presentations of interleaving factors of path sets, which are necessarily principal decimations, which improves on the bound of Theorem 3.2 for general decimations of path sets.

#### 6.1. Structure of interleaving factors: arbitrary sets \( X \).

We recall from [5] results on the structure of possible interleaving factorizations for a general set \( X \subseteq A^N \), and derive corollaries of one of them. The following result is a consequence of Theorem 2.12 of [5].

**Theorem 6.1.** (Divisibility for interleaving factorizations)

(1) Let \( X \subseteq A^N \) have an \( n \)-fold interleaving factorization. If \( d \) divides \( n \), then \( X \) also has an \( d \)-fold interleaving factorization.

(2) Let \( X \) have \( m \)-fold and \( n \)-fold interleaving factorizations. Then \( X \) has an \( \ell \)-fold interleaving factorization, where \( \ell = \text{lcm}(m,n) \) is the least common multiple of \( m \) and \( n \).

An immediate consequence of this result is a dichotomy. For a general set \( X \subseteq A^N \), exactly one of the following hold.
(1) \( X \) is factorizable for infinitely many \( n \),
(2) \( X \) is \( n \)-factorizable for a finite set of \( n \), which are exactly the divisors of a fixed integer \( f = f(X) \).

The following result shows that if \( X \) is a closed set, then infinite factorizability implies \( n \)-factorizability for all \( n \geq 1 \). It is a consequence of Theorem 2.13 of [5].

**Theorem 6.2.** (Classification of infinitely factorizable closed \( X \)) For a closed set \( X \subseteq A^N \) where \( A \) is a finite alphabet, the following properties are equivalent.

(i) \( X \) is infinitely factorizable; i.e., \( X \) has an \( n \)-interleaving factorization for infinitely many \( n \geq 1 \).

(ii) \( X \) has an \( n \)-interleaving factorization for all \( n \geq 1 \).

(iii) For each \( k \geq 0 \) there are nonempty subsets \( A_k \subseteq A \) such that \( X = \prod_{k=0}^\infty A_k \) is a countable product of finite sets with the product topology.

**Remark 6.3.** (1) If \(|A| \geq 2\), then there are uncountably many infinitely factorizable closed sets \( X \subseteq A^N \), while there are only countably many path sets.

(2) For \( k, \ell \geq 1 \) and a finite set of consecutive \( A_k, A_{k+1}, A_{k+2}, \ldots, A_{k+\ell} \) such that there is a block \( a_k a_{k+1} \cdots a_{k+\ell} \) with each \( a_{k+i} \in A_{k+i} \) for \( 0 \leq i \leq \ell \) that does not occur in any element of \( X \), we say that \( X \) has a \((k, \ell)\)-missing-configuration. The proof shows that existence of a \((k, \ell)\)-missing-configuration certifies that \( X \) has no \( n \)-fold interleaving factorization with \( n \geq k+\ell+1 \). The proof of Theorem 6.2 given in [5] shows that if \( X \) is not infinitely factorizable then it has a \((k, \ell)\)-missing configuration for some finite \( k, \ell \geq 1 \).

**Corollary 6.4.** Let \( X \) be an infinitely factorizable closed subset of \( A^N \). Then it is factorizable for all \( n \geq 1 \), so its factor set \( \mathfrak{F}(X) \) contains all decimations \( \psi_{j,n}(X) \) for \( n \geq 1 \) and \( 0 \leq j \leq n-1 \). Each decimated set \( \psi_{j,n}(X) \) is also infinitely factorizable.

**Proof.** By property (ii) of Theorem 6.2, \( X \) is factorizable for each \( n \geq 1 \), and its \( n \)-fold factors are \( \psi_{j,n}(X) \) for \( 0 \leq j \leq n-1 \). Now the property (iii) is preserved under decimations of all orders, hence all \( \psi_{j,n}(X) \) must be infinitely factorizable. \( \square \)

For an infinitely factorizable \( X \) it is possible that all decimations \( \psi_{j,n}(X) \) are pairwise distinct. In such cases the factor set \( \mathfrak{F}(X) \) would be infinite.

**Corollary 6.5.** The set \( \mathcal{Y}(A) \) of all infinitely factorizable closed subsets \( X \subseteq A^N \) is closed under the operation of \( n \)-fold interleaving for all \( n \geq 1 \). That is, if \( X_0, X_1, \ldots, X_{n-1} \in \mathcal{Y}(A) \), then

\[ X_0 \mathcal{O} X_1 \mathcal{O} \cdots \mathcal{O} X_{n-1} \in \mathcal{Y}(A). \]

**Proof.** This fact follows using the characterization of infinitely factorizable by property (iii) of Theorem 6.2. This property is inherited under \( n \)-fold interleaving of sets \( X_i \) that have it. \( \square \)

Combining Theorem 6.1 and Theorem 6.2 yields the following result.

**Theorem 6.6.** ([5] Theorem 2.10) (Dichotomy theorem) Let \( X \) be a closed subset of \( A^N \). Then exactly one of the following holds for \( X \).

(1) (Infinitely factorizable) The set of \( n \) where \( X \) has an \( n \)-interleaving factorization is the set of all positive integers \( \mathbb{N}^+ \).

(2) (Finitely factorizable) The set of \( n \) where \( X \) has an \( n \)-interleaving factorization is the set of all divisors of some integer \( f = f(X) \).

6.2. **Infinitely factorizable path sets.** We deduce that the collection \( C^\infty(A) \) of all infinitely factorizable path sets on \( A \) is closed under all interleaving operations.

**Theorem 6.7.** \( C^\infty(A) \) is closed under interleaving.

(1) If \( P \) is a path set on the alphabet \( A \) having an \( n \)-fold interleaving factorization for all \( n \geq 1 \), then each interleaving factor \( \psi_{j,n}(P) \) is itself infinitely factorizable.
(2) Conversely, if the $n$ path sets $\{P_i : 0 \leq i \leq n-1\}$ are each infinitely factorizable then the $n$-fold interleaving $P := P_0 \circ P_1 \circ \cdots \circ P_{n-1}$ is infinitely factorizable.

Proof. Statement (1) of Theorem 6.7 follows from Corollary 6.4 combining it with the fact that all interleaving factors $\psi_{j,n}(P)$ are path sets. Statement (2) follows from Corollary 6.5 combining it with Theorem 1.8 to infer that $P$ is a path set. 

6.3. Size of minimal right-resolving presentations for interleaving factors of path sets. We now suppose that a path set $P$ has an $n$-fold interleaving factorization. The following bound on the size of minimal right-resolving presentations of interleaving factors (which are necessarily principal decimations) improves on the upper bound of Theorem 3.2 for general $n$-level decimations.

Theorem 6.8. (Upper bound on minimal presentation size of $n$-fold interleaving factors) Let $P$ be a path set having $m$ vertices in its minimal right-resolving presentation. Suppose that $P$ has an $n$-fold interleaving factorization $P = (\circ)^{n-1}_{j=0}P_j$. Then each $n$-fold interleaving factor $P_j = \psi_{j,n}(P)$ has a minimal right-resolving presentation having at most $m$ vertices.

Proof. According to the equivalences in Theorem 2.16 it suffices to show that for each $j$, the number of distinct word path sets of the form $P^w$, where $w$ is an initial word of $P_j$, is no larger than the number of distinct word path sets of the form $P^w'$, where $w'$ is an initial word of $P$. For an initial word $w = w_0w_1 \ldots w_{k-1}$ of $P_j$, let $z \in P$, and say $z = z^0 \circ z^1 \circ \cdots \circ z^{n-1}$, with $w$ the initial $k$-length word of $z^j$. Let $w'$ be the initial word of $z$ of length $nk + j$. Then the letter in $z$ immediately following $w'$ is from $z^j$.

We assert that $\psi_{0,n}(P^{w'}) = P^w_j$. We show both inclusions hold. The set $P^{w'}$ is the set of all infinite words $x$ such that $w'x \in P$. Note that if $w'x \in P$, then $\psi_{j,n}(w'x) = wy$, where $y = \psi_{0,n}(x)$. Thus, the $(0,n)$ decimation of any infinite word following $w'$ in $P$ is an infinite word following $w$ in $\psi_{j,n}(P) = P^w_j$, so we have the inclusion $\psi_{0,n}(P^{w'}) \subseteq P^w_j$. For the other inclusion, note that for any $y \in P^w_j$, we have $wy \in P_j$. Hence if we define $\bar{z} = z^0 \circ \cdots \circ z^{j-1} \circ (wy) \circ z^j+1 \circ \cdots \circ z^{n-1}$, then $\bar{z} \in P$, and the initial word of $\bar{z}$ of length $nk + j$ is $w'$, since the choice of $y$ does not affect the first $nk + j$ letters of $\bar{z}$. Thus $\bar{z} = w'x$ for some $x \in P^{w'}$, and $\psi_{0,n}(x) = y$. Hence $y \in \psi_{0,n}(P^{w'})$, and we get $P^w_j \subseteq \psi_{0,n}(P^{w'})$, proving the assertion.

Thus, every path set $P^w_j$ is the $(0,n)$-decimation of a path set $P^{w'}$. It follows that there are at least as many distinct path sets of the form $P^{w'}$ as there are of the form $P^w_j$.

Remark 6.9. There is nothing special about the use of $(0,n)$-decimations in this proof. If, after choosing $j$, $w$, and $z$, we had chosen the word $w'$ to be of length $nk + j - i$, for any $0 \leq i \leq n - 1$, then the letter in $z$ occurring $(i+1)$ steps after the last letter of $w'$ would be from $z^j$, and we could have shown that every path set $P^{w'}$ is the $(i,n)$-decimation of a path set $P^w$.

7. Structure of Infinitely Factorizable Path Sets

This section classifies all infinitely factorizable path sets, in terms of the structure of their minimal right-resolving presentation. It deduces an improved upper bound on the size of minimal right-resolving presentations of interleaving factors of a general path set (which are decimations) than that derived for decimations in Theorem 3.6.

7.1. Characterization of infinitely factorizable path sets. We characterize the path sets $P$ that are infinitely factorizable as having a minimal right-resolving presentation $(G,v)$ of a particularly simple kind.

Definition 7.1. (Leveled presentation) A presentation $(G,v)$ of a path set $P$ is leveled if it is right-resolving and all infinite paths in $G$ from the marked vertex $v$ visit exactly the same set of vertices in the same order; i.e., all exit edges of $G$ from a vertex $v'$ necessarily go the same target vertex $v''$ (depending on $v'$). There may be multiple edges (with different symbol labels) between $v'$ and $v''$. We say that a path set $P$ is leveled if it has such a presentation; otherwise it is non-leveled.
Figure 7.1 Leveled presentation of a path set \( \mathcal{P} \). The marked vertex is \( v_0 \).

Theorem 7.2. A path set \( \mathcal{P} \) is infinitely factorizable if and only if it has a minimal right-resolving presentation \((\mathcal{G}, v_0)\) that is leveled.

Proof. To prove necessity, we must show that if \( \mathcal{P} \) is infinitely factorizable, then its minimal right-resolving presentation is leveled. We prove the assertion by induction on the number of vertices reached from the initial vertex in \( G \), starting from the marked vertex \( v_0 \). Using Theorem 6.2 condition (iii) for being infinitely factorizable says that \( \mathcal{P} = \prod_{k=0}^{\infty} A_k \), where each \( A_k \) is a subset of the (finite) alphabet \( \mathcal{A} \). Each exit edge from the marked vertex \( v_0 \) of \((\mathcal{G}, v_0)\) goes to a vertex \( v' \) whose vertex path set \( X(\mathcal{G}, v') \) must be \( \mathcal{P}' = \prod_{k=1}^{\infty} A_k \). The (finite) vertex follower set \( F(\mathcal{G}, v') \) is then

\[
F(\mathcal{G}, v') = \bigcup_{m=1}^{\infty} \prod_{i=1}^{m} A_k.
\]

By Theorem 2.16 all vertex follower sets in \( \mathcal{G} \) are distinct. Consequently there can be only one choice for \( v'' \), and all exit edges from \( v_0 \) go to it. If \( v'' = v_0 \) we have a self-loop at \( v \) and are done. Otherwise \( v'' \) is a new vertex; call it \( v_1 \). There may be multiple edges (with different labels) from \( v_0 \) to \( v_1 \); the labels are exactly the letters in \( A_0 \).

The induction hypothesis on \( v_j \) supposes that the vertex \( v_j \) has associated vertex path set \( \mathcal{P}_j = \prod_{k=j}^{\infty} A_k \). We next study exit edges from \( v_j \). They necessarily go to a vertex \( v'' \) whose vertex path \( X(\mathcal{G}, v'') \) is \( \mathcal{P}'_{j+1} = \prod_{k=2}^{\infty} A_k \) and whose (finite) vertex follower set is

\[
F(\mathcal{G}, v'') = \bigcup_{m=2}^{\infty} \prod_{i=1}^{m} A_k.
\]

By uniqueness of a vertex in the minimal presentation having a particular finite follower set, all exit edges must go to the same vertex \( v'' \). If \( v'' \) is one of the previously found vertices, we are done. Otherwise we are at a new vertex \( v_{j+1} \). Since \( \mathcal{P} \) is a path set it has a presentation with finitely many vertices so the process must terminate. The induction is complete, so \((\mathcal{G}, v_0)\) is leveled.

Suppose such the leveled presentation \( \mathcal{G} \) has \( m \) vertices. The directed graph \( \mathcal{G} \) either has a unique vertex path that is an \( m \)-cycle or else this graph has the appearance of a Greek letter \( \rho \), with the unique vertex path having a preperiodic part \( v_0 \to v_1 \to v_2 \to \cdots \to v_{s-1} \), with \( s \) vertices, followed by moving around a periodic part \( v_s \to v_{s+1} \to \cdots \to v_{s+p-1} \to v_s \), a period \( p \) cycle, with \( p = m - s \).

To prove sufficiency, we must show that every path set \( \mathcal{P} \) having a leveled presentation \((\mathcal{G}, v_0)\) is infinitely factorizable. A leveled presentation is right-resolving, since the labels \( A_k \) exiting from vertex \( v_k \) are distinct. It is clear from the internal structure of a leveled presentation (as a rho-graph) that the associated path set \( \mathcal{P} = (\mathcal{G}, v_0) \) necessarily has the form \( \mathcal{P} = \bigcup_{k=0}^{\infty} A_k \) where for the first \( m \) steps \( A_k \) are the set of edge labels for vertices \( v_0, v_1, \ldots, v_{m-1} \). After this point the edge labels repeat periodically with a period \( p = m - s \), where \( \ell \) is the length of the pre-period, having the equality of sets \( A_{m+j} = A_{k+j} \) with \( 0 \leq k \leq p - 1 \) determined by the congruence \( k \equiv j(\text{mod}p) \). Since this presentation satisfies Theorem...
vertex) is not an

Remark 7.4

which proves (1).

7.2. Bounds for the number of distinct factors of infinitely factorizable path sets. We upper bound

the number of distinct interleaving factors \( \mathfrak{F}(P) \) of a infinitely factorizable path set in terms of the size of

its minimal right-resolving presentation. Since all factors are decimations we know by Theorem 1.6 that

\( \mathfrak{F}(P) \subseteq \mathcal{D}(P) \) is finite.

Theorem 7.3. Suppose that \( P \) is an infinitely factorizable path set that has a right-resolving presentation

\((G, v)\) with \( m \) vertices.

(1) Each possible distinct factor occurs in some \( n \)-fold factorization having \( n \leq 2m - 1 \).

(2) The cardinality \( |\mathfrak{F}(P)| \) of the factor set \( \mathfrak{F}(P) \) is at most \( m^2 \).

Proof. It suffices to consider the minimal right-resolving presentation \((G, v_0)\) of \( P \), which must be a leveled

presentation, and which has at most \( m \) vertices. Let \( p \) be the period of the graph. There is a unique vertex

path on \( G \) starting from the initial vertex, which we consider to be the 0th vertex. Call the vertex reachable

in 1 step from the 0th vertex the 1st vertex, and so on. Now \( A_k(P) \) is the set of symbols available at the

\( k \)-th vertex, and by Theorem 6.2 \( P = \prod_{k=0}^{\infty} A_k(P) \). Then \( \psi_{j,n}(P) = \prod_{k=0}^{\infty} A_{kn+j}(P) \). Now, since there

are only \( m \) vertices (the 0th vertex through the \( m \)-th vertex), we may choose \( j' \) so that the \( j' \)-th vertex is

the \( j \)-th vertex, which also gives us that the \((j' + 1)\)-st vertex is the \((j + 1)\)-st vertex, and so on. Hence:

\[
\psi_{j,n}(P) = \prod_{k=0}^{\infty} A_{kn+j}(P) = \prod_{k=0}^{\infty} A_{kn+j'}(P) = \psi_{j,n}(P)
\]

Likewise, because there are only \( m \) vertices, we may choose \( n' < m \) so that the \( j' + n' \)-th vertex is the

\( j' + n \)-th vertex. We wish to show that this will also imply that the \((j' + kn')\)-th vertex is the \((j' + kn)\)-th

vertex for all \( k \). If \( n < m \), we may take \( n' = n \). If \( n \geq m \), then the \((j' + n)\)-th vertex is in the periodic part

of the graph, so the fact that the \((j' + n')\)-th vertex is the \((j' + n)\)-th vertex implies that \( n' = n \pmod{p} \).

This in turn implies that the \((j' + kn)\)-th vertex is the \((j' + kn')\)-th vertex for all \( k \). Hence we have:

\[
\psi_{j,n}(P) = \prod_{k=0}^{\infty} A_{kn+j}(P) = \prod_{k=0}^{\infty} A_{kn'+j'}(P)
\]

Since \( j' \) and \( n' \) are both chosen between 0 and \( m - 1 \), there are at most \( m^2 \) distinct sets of this form,

whence the cardinality of the factor set \( \mathfrak{F}(P) \) is at most \( m^2 \), proving (2).

However, we have not guaranteed that \( j' < n' \), and so the set \( \prod_{k=0}^{\infty} A_{kn'+j'}(P) \) is not guaranteed to be

one of the \( n' \)-fold interleaving factors of \( P \). If \( n < m \), then we have \( j' \leq j < n = n' \), and so we are done.

If \( n \geq m \), then because the \((j' + n')\)-th vertex is in the periodic part of the graph, we may take \( n'' = n' + pr \)

for any \( r \geq 1 \) and get that the \((j' + kn'')\)-th vertex is the \((j' + kn)\)-th vertex for all \( k \geq 0 \). Since \( j' < m \) and

\( p \leq m \), It is always possible to choose \( r \) such that \( j' < n'' \leq 2m - 1 \). We then have:

\[
\psi_{j,n}(P) = \prod_{k=0}^{\infty} A_{kn'+j'}(P) = \prod_{k=0}^{\infty} A_{kn''+j'}(P) = \psi_{j',n''}(P)
\]

which proves (1). \( \square \)

Remark 7.4. The bound \( 2m - 1 \) in Theorem 7.3 is sharp. Consider a circular graph with \( m \) vertices, where

the available alphabets at all vertices are distinct. Mark one of the vertices. If \( P \) is the path set presented,

then it can be shown that \( \psi_{m-1,2m-1}(P) \) (which is the full shift over the alphabet available at the \((m - 1)\)-st

vertex) is not an \( n \)-fold interleaving factor for any \( n < 2m - 1 \).
7.3 Minimal right-resolving presentations for interleaving factors-part 2. Theorem 7.5 showed that any path set $\mathcal{P}$ having a minimal right-resolving presentation with $m$ vertices that has an $n$-fold interleaving factorization $\mathcal{P} = (\otimes)_{i=0}^{n-1} \mathcal{P}_j$, necessarily has every factor $\mathcal{P}_j = \psi_{j,n}(\mathcal{P})$ has a minimal right-resolving presentation having $m$ or fewer vertices. We now show that the equality case implies all these sets are leveled.

**Theorem 7.5.** Let $\mathcal{P}$ be a path set having a minimal right-resolving presentation with $m$ vertices. Suppose that $\mathcal{P}$ has an $n$-fold interleaving factorization $\mathcal{P} = (\otimes)_{i=0}^{n-1} \mathcal{P}_j$ such that at least one factor $\mathcal{P}_j = \psi_{j,n}(\mathcal{P})$ has a minimal right-resolving presentation with $m$ vertices. Then the path set $\mathcal{P}$ must be leveled. and all of the factors $\mathcal{P}_j$ for $0 \leq j \leq n - 1$ are leveled.

**Proof.** Suppose $\mathcal{P} = X(\mathcal{G}, v)$ and $\mathcal{P}_i = X(\mathcal{G}_i, v_i)$ are minimal right-resolving presentations, where $\mathcal{G}$ and $\mathcal{G}_i$ both have $m$ vertices. We begin by showing that the hypotheses of the theorem imply the following:

**Claim.** All word path sets $\mathcal{P}_i^w$ are determined by their $(0, n)$-decimations $\psi_{0,n}(\mathcal{P}_i^w)$

Since $\mathcal{G}$ and $\mathcal{G}_i$ both have $m$ vertices with distinct follower sets, they both have $m$ distinct vertex path sets $X(\mathcal{G}, \psi')$ and $X(\mathcal{G}_i, \psi'_i)$ that can be presented by choosing initial vertices. The proof of Theorem 6.8 established that every word path set $\mathcal{P}_i^w$ is a set of the form $\psi_{0,n}(\mathcal{P}_i^w')$, for another word $w'$. Our claim follows immediately from the fact that there are $m$ such distinct path sets, but also only $m$ distinct path sets $\mathcal{P}_i^w$.

Now, we will show that $\mathcal{P}_j$ is leveled, for all $0 \leq j \leq n - 1$, using only the key fact stated in our claim (in particular, we no longer need to use any special information about $\mathcal{P}_i$, and what follows works for $j = i$ as well as for $j \neq i$). Let $w^{j,1}$ and $w^{j,2}$ be initial words of $\mathcal{P}_j$, both of length $k$. Say they are the initial blocks of the infinite words $x^{j,1}$ and $x^{j,2}$, respectively. It will suffice to show the equality of word path sets $\mathcal{P}_j^{w^{j,1}} = \mathcal{P}_j^{w^{j,2}}$, since this equality is equivalent to the equality of the word follower sets $F_{\mathcal{P}_j}(w^{j,1})$ and $F_{\mathcal{P}_j}(w^{j,2})$ of $\mathcal{P}_j$, using Theorem 2.10 because these word follower sets are the initial words $B_1^1(\mathcal{P}_j^{w^{j,1}})$ and $B_1^1(\mathcal{P}_j^{w^{j,2}})$, respectively. This equality of the word follower sets then implies the equality of the vertex follower sets $F(\mathcal{P}_j, w^{j,1})$ and $F(\mathcal{P}_j, w^{j,2})$, which since the presentation of $\mathcal{P}_j$ is minimal right-resolving means that we arrived at the same vertex of $\mathcal{P}_j$ following the symbol paths for $w^{j,1}$ and $w^{j,2}$ from the initial vertex $v_j$ of this presentation. The property of being at the same vertex is exactly the desired leveling property of $\mathcal{P}_j$.

It remains to show that $\mathcal{P}_j^{w^{j,1}} = \mathcal{P}_j^{w^{j,2}}$. Now for all $l \neq j$, let $x^l \in \mathcal{P}_l$ be chosen arbitrarily. Then let

\[ y = x^0 \otimes x^1 \otimes \ldots \otimes x^{j,1} \otimes x^{j+1} \otimes \ldots \otimes x^{n-1} \text{ and } z = x^0 \otimes x^1 \otimes \ldots \otimes x^{j,2} \otimes x^{j+1} \otimes \ldots \otimes x^{n-1}. \]

Let $b^1$ and $b^2$ be the words made up of the first $(n(k - 1) + j + 1)$ entries of $y$ and $z$, respectively. In particular, the last entry of $b^1$ is the last entry of $w^{j,1}$, and the last entry of $b^2$ is the last entry of $w^{j,2}$.

We will show:

\[ \mathcal{P}_j^{w^{j,1}} = \psi_{n-1,n}(\mathcal{P}^{b^1}) \text{ and } \mathcal{P}_j^{w^{j,2}} = \psi_{n-1,n}(\mathcal{P}^{b^2}). \]

by reasoning along similar lines as in the proof of Theorem 6.8. Specifically, for the first equality, $\mathcal{P}^{b^1}$ is the set of all infinite words $x$ such that $b^1 x \in \mathcal{P}$, and if $b^1 x \in \mathcal{P}$, then $\psi_{j,n}(b^1 x) = w^{j,1} y$, where $y = \psi_{n-1,n}(x)$. (The reason that we have $(n - 1, n)$-decimations here instead of $(0, n)$-decimations is that the words $b^1$ and $b^2$ have length $n - 1$ less than $kn + j$, the length of the word used in the proof of Theorem 6.8.) Thus, the $(n - 1, n)$ decimation of any infinite word following $b^1$ in $\mathcal{P}$ is an infinite word following $w^{j,1}$ in $\psi_{j,n}(\mathcal{P}) = \mathcal{P}_j$, and we have $\psi_{n-1,n}(\mathcal{P}^{w^1}) \subseteq \mathcal{P}_j^{w^{j,1}}$. On the other hand, for any $y \in \mathcal{P}_j^{w^{j,1}}$, we have $w^{j,1} y \in \mathcal{P}_j$. Hence $\tilde{z} = x^0 \otimes \ldots \otimes x^{j,1} \otimes (w^{j,1} y) \otimes x^{j+1} \otimes \ldots \otimes x^{n-1} \in \mathcal{P}$, and the initial word of $\tilde{z}$ of length $(n(k - 1) + j + 1)$ is $b^1$, since the choice of $y$ does not affect the first $(n(k - 1) + j + 1)$ letters of $\tilde{z}$. Thus $\tilde{z} = w^x' x$ for some $x \in \mathcal{P}^{w^1}$, and $\psi_{n-1,n}(x) = y$. Hence $y \in \psi_{n-1,n}(\mathcal{P}^{w^1})$, and we get $\mathcal{P}_j^{w} \subseteq \psi_{n-1,n}(\mathcal{P}^{w})$, proving that $\mathcal{P}_j^{w^{j,1}} = \psi_{n-1,n}(\mathcal{P}^{b^1})$. By the same argument (replacing $b^1$ with $b^2$ and $w^{j,1}$ with $w^{j,2}$), we get $\mathcal{P}_j^{w^{j,2}} = \psi_{n-1,n}(\mathcal{P}^{b^2})$. 

Thus, if we can show that \( P^{b_1} = P^{b_2} \), then we get the desired equality \( P^{n+1}_j = P^{n+2}_j \).

But our earlier claim tells us that \( P^{b_1} \) and \( P^{b_2} \) are determined by their \((0, n)\)-decimations. Their \((0, n)\)-decimations are determined by the choice of (the first \( k - 1 \) letters of) \( y^{j+1} \), which is the same for \( b_1 \) and \( b_2 \) by construction.

Thus, we have shown that all the interleaving factors of \( P \) are leveled, and so \( P \) is leveled.

\[ \square \]

8. Finitely Factorizable Path Sets

Finitely factorizable path sets coincide with non-leveled path sets, so they are algorithmically recognizable.

8.1. Bounds for number of distinct \( n \)-fold interleaving factorizations.

**Theorem 8.1.** Let \( P \) be a path set having a right-resolving presentation \((G, v)\) having \( m \) vertices. If \( P \) is finitely factorizable, i.e., non-leveled, and has an \( n \)-fold interleaving factorization, then \( n \leq m - 1 \).

**Proof.** To prove the bound, we will assume \( P \) has an \( n \)-fold interleaving factorization, and show that a minimal right-resolving presentation of \( P \) must have at least \( n + 1 \) vertices. If so we must have \( m \geq n + 1 \), giving the result.

We are given the interleaving factorization \( P = P_1 \circ P_2 \circ \cdots \circ P_{n-1} \). We let \((G_i, v_i^0)\) for \( 0 \leq i \leq n - 1 \) be minimal right-resolving presentations for each \( P_i \). In particular, by Theorem 2.16, each labeled graph \( G_i \) has the property that all its vertices have distinct vertex follower sets.

One of the \( P_i \) must be finitely factorizable. For if all of the \( P_i \) were infinitely factorizable, then by Corollary 6.5, their \( n \)-fold interleaving \( P := P_1 \circ P_2 \circ \cdots \circ P_{n-1} \) would be infinitely factorizable, a contradiction. For definiteness suppose \( P_{i_0} \) is finitely factorizable. Then \( P_{i_0} \) is non-leveled, by Proposition 7.2.

Because \( P_{i_0} = X(G_{i_0}, v_{i_0}^0) \) is non-leveled, the graph \( G_{i_0} \) has a reachable vertex \( w \) which has exit edges to two different vertices—call them \( w_1 \) and \( w_2 \)—which have distinct vertex follower sets \( F(G_{i_0}, w_1) \) and \( F(G_{i_0}, w_2) \) by minimality of the presentation \( X(G_{i_0}, v_{i_0}^0) \).

We now study the \( n \)-fold interleaving graph product \((H, v^0) := (\otimes_{j=0}^{n-1} (G_i, v_i^0)) \) studied in Theorem 1.9. We will show that \((H, v^0)\) has at least \( n + 1 \) different vertex follower sets, over all vertices reachable from \( v^0 \). If so, by Proposition 2.8 \( P = X(H, v^0) \) will have at least \( n + 1 \) distinct word follower sets, and Theorem 2.16 then implies that the minimal right-resolving presentation \((G, v)\) of \( P \) has at least \( n + 1 \) vertices. Consequently \( m \geq n + 1 \) and we are done.

Recall that the vertex set \( V(H) = \bigcup_{i=1}^{m-1} V^i \), but that the graph \( H \) need not be connected. Our argument must establish that the \( n + 1 \) vertex follower sets constructed are in the reachable component of \( v \). We first consider a shortest directed path in \( G_{i_0} \) from the initial vertex \( v_{i_0} \) to the vertex \( v \)—call its length \( k \)—and denote it \( v_{i_0}^0, v_{i_0}^1, v_{i_0}^2, \ldots, v_{i_0}^{k-1} \), with \( v_{i_0}^{k-1} = w \). Then let \( v_{i_0}^0, v_{i_0}^1, v_{i_0}^2, \ldots \) denote an arbitrary vertex path from the initial vertex in \( G_{i_0} \).

We know that the initial vertex of \( H \) is \( v^0 = (v_0^0, v_1^0, \ldots, v_{n-1}^0) \in V^0 \). By the construction in the proof of Theorem 1.9, there is an edge from this vertex to the vertex \((v_1^0, v_2^0, \ldots, v_{n-1}^0, v_0^1) \in V^1 \), from here to \((v_2^0, v_3^0, \ldots, v_{n-1}^0, v_0^1, v_1^1) \in V^2 \), eventually reaching after \( kn + i_0 \) steps the vertex

\[ v_{i_0} := (v_{i_0}^{k-1}, v_{i_0}^{k-1}, \ldots, v_{i_0}^{k-1}, v_{i_0}^{k-1}) \in V^{i_0} \]

where \( V^{i_0} = V_{i_0} \times V_{i_0+1} \times \cdots \times V_{n-1} \times V_0 \times V_1 \times \cdots \times V_{i_0-1} \). For the next step, we have two choices, moving \( w \to w_\ell \) for \( \ell = 1, 2 \), and we can then reach, in sequence, the following \( n \) vertices in \( H \):

\[ v_1(\ell) := (v_{i_0}^{k-1}, v_{i_0}^{k-1}, \ldots, v_{i_0}^{k-1}, v_1^0, v_1^1, \ldots, v_{i_0}^{k-1}, w_\ell, v_{i_0}^{k-1}, \ldots, v_{i_0}^{k-1}) \in V_{i_0+1} \]

\[ v_2(\ell) := (v_{i_0}^{k-1}, v_{i_0}^{k-1}, \ldots, v_{i_0}^{k-1}, v_1^0, v_1^1, \ldots, v_{i_0}^{k-1}, w_\ell, v_{i_0}^{k-1}, \ldots, v_{i_0}^{k-1}) \in V_{i_0+2} \]

\[ \vdots \]

\[ v_n(\ell) := (w_\ell, v_{i_0}^{k-1}, \ldots, v_{i_0}^{k-1}, v_1^0, v_1^1, \ldots, v_{i_0}^{k-1}, v_{i_0}^{k-1}, \ldots, v_{i_0}^{k-1}) \in V_{i_0} \]
The last \( n - 1 \) of these steps will be chosen to travel edges in \( \mathcal{H} \) corresponding to identical edges for \( \ell = 1, 2 \) in the graphs \( G_{i0+j} \) for \( 1 \leq j \neq n - 1 \). We have now specified a list of \( 2n \) vertices in \( \mathcal{H} \) that are reachable from its initial vertex \( v^0 \).

We will show that among these \( 2n \) vertices there are at least \( n + 1 \) distinct vertex follower sets in \( \mathcal{H} \). To tell vertex follower sets apart we use the following test.

**Claim.** Given two vertex follower sets \( F(\mathcal{H}, y^1) \) and \( F(\mathcal{H}, y^2) \) with vertices in \( \mathcal{H} \) in vertex subsets \( y^1 \in \mathcal{V}^i \) and \( y^2 \in \mathcal{V}^j \), respectively, a necessary condition for the vertex follower set equality

\[
F(\mathcal{H}, y^1) = F(\mathcal{H}, y^2)
\]

is that each of their projected vertices in the individual graphs \( G_i \) for \( 0 \leq i \leq n - 1 \) must have identical vertex follower sets,

\[
F(G_{i+j}, y^1(j)) = F(G_{i+j}, y^2(j)), \quad \text{for} \quad 0 \leq j \leq n - 1.
\]

These conditions are equivalent, for each \( j \), to the path set equalities

\[
X(G_{i+j}, y^1(j)) = \psi_{j,n}(X(\mathcal{H}, y^1)) = \psi_{j,n}(X(\mathcal{H}, y^2)) = X(G_{i+j}, y^2(j)).
\]

To prove the claim, note that following an infinite path in the graph \((\mathcal{H}, v^0)\) is the same thing as following separate independent paths in each of the \( n \) graphs \((G_i, v_i)\), starting from their initial vertices. Write \( v^j = (v^j(0), v^j(1), \ldots, v^j(n-1)) \). The finite follower set equality (8.1) implies the path set equality \( X(\mathcal{H}, y^1) = X(\mathcal{H}, y^2) \). Note that \( \psi_{j,n}(X(\mathcal{H}, y^j)) = X(G_{i+j}, v^j(j)) \), for all \( j \). The vertex follower sets of the projections are completely determined by the property that they are initial languages of the path sets \( \psi_{j,n}(X(\mathcal{H}, y^j)) \). Therefore the equality \( F(G_{i+j}, v_i(1)) = F(G_{i+j}, v_i(2)) \) holds for all \( 0 \leq j \leq n - 1 \), proving the claim.

Thus, for example, for the vertex follower set of vertex

\[
(v_{i0+1}^{k-1}, \ldots, v_{n-1}^{k-1}, v_0^k, y_1^k, \ldots, v_{i0-1}^k, w_2) \in \mathcal{V}^{i0+1}
\]

to be equal to the vertex follower set of

\[
(v_{i0+2}^{k-1}, \ldots, v_{n-1}^{k-1}, v_0^k, y_1^k, \ldots, w_1, v_{i0+1}^k) \in \mathcal{V}^{i0+2},
\]

we would need to have the row of path sets

\[
\left( X(G_{i0+1}, v_{i0+1}^{k-1}), X(G_{i0+2}, v_{i0+2}^{k-1}), \ldots, X(G_{i0-1}, v_{i0-1}^{k-1}), X(G_{i0}, w_2) \right)
\]

to be identical with the row of path sets

\[
\left( X(G_{i0+2}, v_{i0+2}^{k-1}), X(G_{i0+3}, v_{i0+3}^{k-1}), \ldots, X(G_{i0}, w_1), X(G_{i0+1}, v_{i0+1}^{k-1}) \right).
\]

We want to make sure that not too many coincidences of vertex follower sets \( \mathcal{H} \) can occur in this manner. For each of these \( 2n \) vertices, consider the associated row of path sets as above. Considering all of these rows together gives us path sets filling in two arrays of the following schematic shape, where the entries are path sets. Here \( \bar{a} = X(G_{i0}, w_1) \) and \( \bar{b} = X(G_{i0}, w_2) \).

\[
A = \begin{pmatrix}
  x_{i+1} & x_{i+2} & \cdots & x_{i-1} & \bar{a} \\
  x_{i+2} & x_{i+3} & \cdots & \bar{a} & y_{i+1} \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  x_{i-1} & \bar{a} & \cdots & y_{i-3} & y_{i-2} \\
  \bar{a} & y_{i+1} & \cdots & y_{i-2} & y_{i-1}
\end{pmatrix}
\]
\[
B = \begin{pmatrix}
  x_{i+1} & x_{i+2} & \cdots & x_{i-1} & \bar{b} \\
  x_{i+2} & x_{i+3} & \cdots & \bar{b} & y_{i+1} \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  x_{i-1} & \bar{b} & \cdots & y_{i-3} & y_{i-2} \\
  \bar{b} & y_{i+1} & \cdots & y_{i-2} & y_{i-1}
\end{pmatrix}
\]

We show that matrices of this schematic shape always have at least \( n + 1 \) distinct rows, as a special case of the following combinatorial lemma.
Lemma 8.2. For any set $X$, let $A$ and $B$ be $n \times n$ matrices with coefficients from $X$, and let $R_A$ be the set of distinct rows of $A$, $R_B$ be the set of distinct rows of $B$. If $A$ and $B$ have constant skew-diagonal entries $a_{i,n+1-i} = \bar{a}$ and $b_{i,n+1-i} = \bar{b}$, with $\bar{a} \neq \bar{b}$, and if in addition $a_{ij} = b_{ij}$ holds for all entries not on the skew-diagonal ($i+j \neq n+1$), then $|R_A \cup R_B| \geq n+1$.

The assertion of Lemma 8.2 completes the proof of the theorem; we prove it below. □

Proof of Lemma 8.2. We prove the result by induction on $n$. It holds for the base case $n = 1$, for $A$ has one entry: $\bar{a}$, and $B$ has a distinct entry: $\bar{b}$.

Now suppose the theorem holds for $n$, and let $A, B \in M^{(n+1)\times(n+1)}(X)$. Then we have two matrices of the following form:

$$
A = \begin{pmatrix}
    x_{1,1} & x_{1,2} & \cdots & x_{1,n} & \bar{a} \\
    x_{2,1} & x_{2,2} & \cdots & \bar{a} & x_{2,n+1} \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    x_{n,1} & \bar{a} & \cdots & x_{n,n} & x_{n,n+1} \\
    \bar{a} & x_{n+1,2} & \cdots & x_{n+1,n} & x_{n+1,n+1}
\end{pmatrix},
B = \begin{pmatrix}
    x_{1,1} & \cdots & x_{1,n} & \bar{b} \\
    x_{2,1} & \cdots & \bar{b} & x_{2,n+1} \\
    \vdots & \vdots & \ddots & \vdots \\
    x_{n,1} & \cdots & x_{n,n} & x_{n,n+1} \\
    \bar{b} & \cdots & x_{n+1,n} & x_{n+1,n+1}
\end{pmatrix}.
$$

Let $R_A$ be the set of distinct rows of $A$, and $R_B$ be the set of distinct rows of $B$, and let $R = R_A \cup R_B$. We need to show $|R| \geq n + 2$. Consider $A', B' \in M^{n\times n}(X)$ defined as the upper right $n \times n$ corner of $A$ and $B$.

$$
A' = \begin{pmatrix}
    x_{1,1} & \cdots & x_{1,n} & \bar{a} \\
    x_{2,1} & \cdots & \bar{a} & x_{2,n+1} \\
    \vdots & \vdots & \ddots & \vdots \\
    x_{n-1,1} & \cdots & x_{n-1,n} & x_{n-1,n+1} \\
    \bar{a} & \cdots & x_{n,n} & x_{n,n+1}
\end{pmatrix},
B' = \begin{pmatrix}
    x_{1,1} & \cdots & x_{1,n} & \bar{b} \\
    x_{2,1} & \cdots & \bar{b} & x_{2,n+1} \\
    \vdots & \vdots & \ddots & \vdots \\
    x_{n-1,1} & \cdots & x_{n-1,n} & x_{n-1,n+1} \\
    \bar{b} & \cdots & x_{n,n} & x_{n,n+1}
\end{pmatrix}.
$$

By the induction hypothesis $A'$ and $B'$ have between them at least $n + 1$ distinct rows. These rows will all remain distinct in $A$ and $B$ when we include the first coordinate position. The $(n+1)$-st rows of $A$ and $B$ are identical in their last $n$ entries:

$$
v_{n+1} = (x_{n+1,1} \cdots x_{n+1,n} x_{n+1,n+1})
$$

If this row differs from all rows in $A'$ and $B'$, then we can include the last row of $A$ with the rows of $A$ and $B$ corresponding to the distinct rows of $A$ and $B$, to obtain at least $n + 1$ distinct rows in $R(A) \cup R(B)$. On the other hand if $v_{n+1}$ already occurs in the set $R(A') \cup R(B')$ then remove it from the list for $R(A') \cup R(B')$, obtaining at least $n$ row positions in $A$ and $B$ corresponding to distinct rows in $R(A') \cup R(B')$ not equal to $v$. Combine these row positions in $A, B$ with the last row of both $A$ and $B$ (which differ in their $(n+1,1)$ entry) to obtain $n + 2$ distinct rows in $R(A) \cup R(B)$, completing the induction step. □

Remark 8.3. The lower bound $n + 1$ of Lemma 8.2 is tight, on taking all $a_{i,j} = b_{i,j} = \bar{a}$ off the skew-diagonal.

8.2. Iterated interleaving and complete factorizations. We next consider iterated interleaving factorizations, in which we may continue to factorize a finitely factorizable path set $P$ at finitely factorizable factors $Q$ if they are decomposable. An iterated factorization is said to be complete if all factors are either indecomposable or are infinitely factorizable.

We associate to any (finite) iterated factorization a rooted tree with root node $P$, and with leaf nodes corresponding to the factors in the iterated factorization, and with internal nodes labeled by factors in some intermediate factorization in the iteration process. The depth of a node in the tree is the number of edges traversed in the unique path to the root node. The root node has depth 0. The branching factor at a node is the number of edges from it to nodes at the next higher depth.
The iterated interleaving process grows the tree, starting with a single root node \( P \). Each iteration step replaces one factor \( Q \) in the current factorization by an \( n \)-fold interleaving factorization of it (for some \( n \geq 2 \)). This step corresponds to adding to this current factor node \( Q \) (which is a leaf node of the current tree) \( n \) new branches with the \( n \)-fold interleaving factors \( Q_j = \psi_{j,n}(Q) \) as new leaves of \( Q \), so that \( Q \) becomes an internal node of the new tree. We treat any infinitely factorizable factors \( Q \) encountered in the process as “frozen” and do not factor them further. Figure 8.1 exhibits such a tree.

![Figure 8.1. Iterated interleaving tree for \( P = P_{0,4} \star (R_{0,3} \star R_{1,3} \star R_{2,3} \star Q_{1,2}) \star P_{2,4} \star P_{3,4} \).](image)

A factorization is said to be complete if each individual factor is either infinitely factorizable or cannot be further factorized.

In the case of general closed sets \( X \subset A^N \) the paper [5, Theorem 7.1] constructed examples where the recursive factorization procedure above can go on forever (with no infinitely factorizable factors), leading to an infinite tree of factors of \( X \). Furthermore these examples have no complete factorizations. The situation for path sets is different: any sequence of factorizations always terminates in a complete factorization, see Theorem 8.5 below.

The following result will be used to establish termination of the iterated interleaving process for path sets.

**Theorem 8.4.** Let \( P \) be a finitely factorizable path set having \( m \) vertices in its minimal right-resolving presentation. Then for \( n \geq 2 \) any interleaving factor \( P_j = \psi_{j,n}(P) \) in an \( n \)-fold interleaving factorization of \( P \) has at most \( m - 1 \) vertices in its minimal right-resolving presentation.

**Proof.** Theorem 6.8 shows that if \( P \) has an \( n \)-fold factorization \( P = (\star)_{i=0}^{n-1} P_j \) then each \( P_j \) has at most \( m \) vertices in its minimal right-resolving presentation. Theorem 7.5 shows that if in addition some factor \( P_j \) has exactly \( m \) vertices in its minimal right-resolving factorization, then \( P \) and all \( P_j \) are leveled, hence infinitely factorizable, contradicting the hypothesis that \( P \) is finitely factorizable. It follows that all factors \( P_j \) must have at most \( m - 1 \) vertices in their minimal right-resolving presentation.

Theorem 8.4 implies the existence of complete factorizations of finitely factorizable path sets.

**Theorem 8.5.** (Complete Factorizations of Path Sets)

1. Every path set \( P \) has at least one iterated interleaving presentation that is complete.
2. Every path set \( P \) has finitely many iterated interleaving factorizations (with “frozen” infinitely factorizable factors) that are complete.
3. If \( P \) has \( m \) vertices in its minimal right-resolving presentation, then each such complete factorization has at most \( (m - 1)! \) factors. In addition the associated tree has depth at most \( m - 1 \), and has branching factor at internal nodes of depth \( j \) of at most \( m - j - 1 \).

**Proof.** The path set \( P \) has \( m \) vertices in its minimal right-resolving presentation. We take any nontrivial \( n \)-fold interleaving factorization of \( P \), and we know at least one factor in it will be finitely factorizable. If any finitely factorizable \( Q = P_i \) appearing in it is decomposable, we may further factorize \( iP_i \) (as an iterated
interleaving factorization), each new factorization having at least one finitely factorizable factor. Theorem 8.1 says that any \( n \)-fold factorization of \( Q \) necessarily has \( n < m(Q) \), where \( m(Q) \) is the number of vertices in a minimal right-resolving presentation of \( Q \). Let \( Q = (\otimes)^{n-1}_{i=0} Q_{j,n} \). Then says the number of vertices \( m(Q_{j,n}) \) in the minimal right-resolving presentation of \( Q_{j,n} \) has \( m(Q_{j,n}) < m(Q) \). Call the level of a node its distance from the root node, where the root node is assigned level 0. Then any level \( k \) node in the tree that is finitely factorizable corresponds to a factor \( Q \) that has at most \( m - k \) vertices in its minimal right-resolving presentation. It follows from this fact that the maximal possible level of any node in the tree is \( m - 1 \), bounding the maximum nesting level of parentheses in the iterated interleaving factorization. In addition the maximal branching factor possible at level \( k \) is \( m - k - 1 \), applying Theorem 8.1. We conclude that all possible such trees are finite, that they can have one node at level 0, at most \( m - 1 \) nodes at level 1, at most \( (m - 1)(m - 2) \) nodes at level 2, up to \( (m - 1)! \) nodes at level \( (m - 1) \). The total number of leaves possible in such a tree, terminating this process on any level, is \( (m - 1)! \). The total number of nodes, counting internal nodes, is \((m - 1)! (1 + \frac{1}{2} + \frac{1}{3!} + \cdots + \frac{1}{(m-1)!} \leq e(m-1)!\)

\[\text{Remark 8.6. We do not settle the question whether there is uniqueness of the indecomposable factors in a complete factorization. The infinitely factorizable factors are non-unique.}\]

9. Concluding Remarks

9.1. Automatic sequences associated to path sets. Theorem 1.6 has the consequence that every path set is \( n \)-automatic for all \( n \geq 1 \) in the sense given in [2]. One may ask whether the are integer counting statistics of an arbitrary path set \( \mathcal{P} \) are \( n \)-automatic sequences in the sense of Allouche and Shallit [3, 6]. A statistic of particular interest is the function \( f(k) = N^L_k(\mathcal{P}) \) that counts the number of initial words of length \( k \) in the path set \( \mathcal{P} \).

Decimations and interleavings together define an infinite collection of closure operations \( X \mapsto X^{[n]} \) on \( \mathcal{A}^n \). These are idempotent operations that preserve the property of being closed sets in the symbol topology. One may define \( C(\mathcal{A})^{[n]} \) to be the subclass of path sets that is invariant under the \( n \)-th closure operation. (It is not clear whether these classes of sets are closed under union or intersection, however.) These closure operations are effectively computable on path sets, using presentations. One can ask if there are automata-theoretic characterizations of the class of path sets that is invariant under a given closure operation.

Appendix A. Path Sets in Automata Theory

Path sets have an important characterizations in automata theory. We recall basic definitions and terminology in automata theory, following Eilenberg [18] and, for infinite words, Perrin and Pin [27]. Recall that \( \mathcal{A}^* \) denotes the set of all finite words in the alphabet \( \mathcal{A} \), including the empty word \( \emptyset \). We let \( \mathcal{A}^\omega \) denote all infinite words \( a_0a_1a_2 \cdots \) in the language with alphabet \( \mathcal{A} \) (rather than \( \mathcal{A}^n \), which we used in the main text.)

A.1. Automata and languages.

**Definition A.1.** A finite automaton on an alphabet \( \mathcal{A} \) is denoted \( \mathcal{A} := (Q, I, T) \) (in full \( (Q, \mathcal{A}, E, I, T) \)) which has a finite directed labeled graph in which \( Q \) denotes the (finite) set of its states, with specified subsets \( I \) of initial states and \( T \) of terminal states (or final states). (The sets \( I \) and \( T \) may overlap.) Additional data specifying the automaton consists of labeled edge data \( E \subset Q \times \mathcal{A} \times Q \), writing \( e = (v_1, a, v_2) \), for the directed edge from state \( v_1 \) to state \( v_2 \), carrying a label \( a \in \mathcal{A} \).

The alphabet \( \mathcal{A} \) and the labeled edge data \( E \) are traditionally omitted from the notation for \( \mathcal{A} \).

**Definition A.2.** A finite automaton is deterministic if \( I \) contains one element and each state \( v \in V \) has for each symbol \( a \in \mathcal{A} \) at most one exit edge labeled with this symbol. Otherwise it is nondeterministic.

A deterministic automaton is characterized by the property that for each finite path the symbolic path label data plus the initial state on the path uniquely determine the path; i.e., the sequence of states visited in following the path.
Definition A.3. The formal language $L(A) \subseteq A^*$ associated to a finite automaton $A$ is the set of all finite symbol sequences obtained as labels following some directed path starting from an initial vertex $v \in I$ and ending at some terminal vertex $w \in T$. The empty sequence (denoted $1$) is included in $L(A)$ if some initial state is a terminal state.

Definition A.4. A finite Büchi automaton has the same automaton $A$, but the (Büchi) language $L^\omega(A)$ recognized by such an automation is the set of all infinite words $a_0a_1a_2\ldots \in A^\mathbb{N}$ which are labels of an infinite directed path starting at an initial state and passing through terminal states at infinitely many times.

The theory of Büchi [11] as originally developed allowed automata having a countably infinite number of states.

Definition A.5. For fixed $A$ the set of languages recognized by some finite Büchi automaton, allowing nondeterminism and arbitrary sets of initial and terminal vertices, are called recognizable languages on alphabet $A$. ([27, page 25].)

The class of recognizable languages are characterized as coinciding with the class of $\omega$-rational languages. ([27, Chapter I. Theorem 5.4]).

A.2. Automata-theoretic characterization of path sets. To a presentation $(G, v)$ of a path set $\mathcal{P}$ on a finite alphabet $A$ we canonically associate the finite automaton $A_G = (Q, I, T)$ where $Q = V(G)$, the initial state set $I = \{v\}$ and the terminal state set $T = V(G)$ consisting every state of $G$, and with edge label data specified by $E$. A path set $\mathcal{P} = X(G, v)$ then coincides with the Büchi language $L^\omega(A)$ recognized by the associated Büchi automaton $A$ associated to $(G, v)$. We have the following characterization.

Theorem A.6. (Automata Characterization of Path Sets) The following properties are equivalent, for a finite alphabet $A$.

1. $\mathcal{P}$ is a path set with alphabet $A$.
2. $\mathcal{P}$ is a recognizable language that is a closed set in $A^\mathbb{N}$.
3. $\mathcal{P}$ is recognized by a finite Büchi automaton in which every state is terminal.
4. $\mathcal{P}$ is recognized by a finite deterministic Büchi automaton in which every state is terminal.

Proof. The path set definition (1) is equivalent to (3). The equivalence of (2), (3), (4) and (5) is shown in Proposition 3.9 in Chapter III of Perrin and Pin [27]. The reduction from (3) to (4) uses the fact that every state of the automaton is a terminal state. □

It is known that the set of languages recognizable languages is strictly larger that those recognized by deterministic Büchi automata. The set of recognizable languages are closed under complement, while the languages recognized by deterministic Büchi automata (allowing arbitrary subsets of $Q$ of terminal states) are not closed under complement. However the set of determinisitic Büchi languages has a nice characterization, given in [27, Chapter III, Corollary 6.3]

The class $\mathcal{C}(A)$ of path sets forms a strict subset of the languages recognized by some deterministic Büchi automaton; i.e., there are non-closed sets of $A^\mathbb{N}$ recognized by some deterministic Büchi automaton, as shown by the following example.

Example A.7. Consider the directed labeled graph with two states pictured, on alphabet $A = \{0, 1\}$.

![Graph for Example A.7](image)

FIGURE A.1. Graph for Example A.7. The marked initial vertex is $v_0$. 

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Let $I = \{v_0\}$ be the initial state and let $A_1$ denote the automaton with every state terminal $T = \{v_0, v_1\}$ and let $A_2$ denote the automaton with $T = \{v_1\}$. Then

$$L^\omega(A_1) = \{1^k01^\infty : k \geq 0\} \cup \{1^\infty\}$$

while $L^\omega(A_2) = \{1^k01^\infty : k \geq 0\}$. The set $L^\omega(A_1)$ is a closed set in $A^\omega$, and is a path set. The set $L^\omega(A_2)$ is not closed in $A^\omega$ because it omits the limit point $\{1^\infty\}$.

A.3. **Minimal deterministic automata.** A finite automaton $A$ is deterministic if the initial state plus the symbols sequence of some legal path uniquely determine the set of states that path passes through. This concept is equivalent to right-resolving in symbolic-dynamics terminology.

**Definition A.8.** (1) An automaton $(Q, I, T)$ is accessible if every state can be reached by a directed path from some initial state.

(2) An automaton is co-accessible if every state has a directed path to a terminal state.

(3) An automaton that is accessible and co-accessible is called trim.

For a finite automaton $A$ with one initial state $v$, the property of reachability is equivalent to it being accessible. For a finite automaton $A$ in which every state is terminal, the property of being pruned is equivalent to being co-accessible. For finite automaton the property of right-resolving is equivalent to being deterministic. A deterministic automaton is complete if each state has an exit edge with label for each letter of the label alphabet $A$.

**Theorem A.9.** Every finite (non-deterministic) automaton $A = (Q, I, T)$ has an equivalent deterministic automaton $A^* = (\bar{Q}^*, I^*, T^*)$ that has $L(A) = L(A^*)$.

**Proof.** This is proved using the subset construction [18 Chapter III, Sect. 2].

**Theorem A.10.** (1) Every finite automaton $A$ has a minimal deterministic presentation $A_{\text{min}}$ giving the same language $L(A)$. This presentation is unique up to isomorphism of automata. (2) The minimal automaton is deterministic and accessible. The behavior of accepting paths from each state of the minimal automaton is different. The minimal automaton is trim unless there is a state which has no accepting exit paths (”null state.”) (3) The minimal automaton is not complete unless $L(A) = A^1$ is the full shift.

**Proof.** See Eilenberg [18 Chapter III, Sect. 5]. Theorem 5.2 gives (1) and (2), and (3) is mentioned in this proof. The proof is by construction.

Given a finite automaton, there is an effectively computable procedure to obtain the minimal deterministic automaton, see the discussion in Eilenberg [18 Chapter III]. Some authors study complete minimal automata, which are obtained by adding an extra “null state” to the automaton which is not co-accessible, e.g. Sakarovich [29] Chapter I.4.2].

**Remark A.11.** Every finite Büchi automaton recognizes a language recognizable by a finite trim Büchi automaton (27 Chapter I, Proposition 5.2). If the automaton is deterministic then the corresponding trim automaton is deterministic. But for general automata the set of terminal states may be altered under the transformation.

A.4. **Path set languages and closures of rational languages.**

**Definition A.12.** A path set language $L^I(P) \subset A^*$ of a path set $P$ is the set of finite initial prefixes of words in the path set.

Eilenberg [18 Chapter XIV, page 379] defines a closure operation on a language $L \subset A^*$, which defines $\bar{L} \subset A^\omega$ to be the set of all infinite words $a_0a_1a_2 \ldots$ which has infinitely many prefixes $a_0a_1 \ldots a_n$ being a word in $L$.

**Definition A.13.** A language $L \subset A^*$ is called prefix-closed if all prefixes of each word in $L$ belongs to $L$.

A path set language $C = L(P)$ is prefix-closed. One also has $\bar{L(P)} = P$. 

Theorem A.14. (Path Set Language Characterization) The following properties are equivalent.
1. $L$ is a path set language $L^i(\mathcal{P})$ for some $\mathcal{P}$ using the finite alphabet $A$.
2. $L$ is a rational language which is prefix-closed in the sense that all the prefixes of each word $w \in L$ also belong to $L$.
3. $L$ is recognizable by a trim finite deterministic automaton, in which all states are terminal.

For any such language the associated path set $\mathcal{P} = \overline{L}$ consists of all infinite words which have infinitely many prefixes belonging to $L$. Any other rational languages $L_1$ which have $\mathcal{P}$ as their closure have $L_1 \subseteq L$.

Proof. The equivalence of (2) and (3) appears as Exercise 5.1 in [18], page 48.

One can define a closure operation on regular languages $L$, as follows. One first passes to the infinite language $\overline{L}$, which is a path set. Then one passes to the initial prefix language $L^0(\overline{L})$ of this path set, which is a prefix-closed language. This relation is idempotent because the closure of $L^i(\overline{L})$ is again $\overline{L}$. If $L$ is recognized by a co-accessible automaton then $L \subseteq L^i(\overline{L})$.

APPENDIX B. SELF-INTERLEAVING CRITERION

We give a sufficient condition on a presentation of a path set to guarantee that all interleaving factorizations of it are self-interleavings (possibly of other path sets).

Theorem B.1. (Sufficient condition for only self-interleaving factorizations) Let $\mathcal{P} = X(\mathcal{G}, v)$ be a path set having a right-resolving presentation in which the graph $\mathcal{G}$ has a self-loop at the initial vertex $v$. Then for all $n \geq 1$ any $n$-fold interleaving factorization $\mathcal{P} = \mathcal{P}_0 \otimes \mathcal{P}_1 \otimes \cdots \otimes \mathcal{P}_{n-1}$ must be a self-interleaving $\mathcal{P} = \mathcal{Q}^{[\leq n]}$ of a single factor $\mathcal{Q}$, where $\mathcal{Q}$ depends on $n$.

Proof. Suppose that the presentation $(\mathcal{G}, v)$ has a self-loop at vertex $v$ with label $a_0 \in A$. Then $x \in \mathcal{P}$ implies that $(a_0)^k x \in \mathcal{P}$ for each $k \geq 1$. In consequence:

1. We have $\mathcal{P} \subseteq S(\mathcal{P})$, where $S$ denotes the left-shift operator, because $S(a_0x) = x$.
2. The $n$-decimation sets $\mathcal{P}_j = \psi_{j,n}(\mathcal{P})$ satisfy the inclusions
   $$\psi_{j,n}(\mathcal{P}) \subseteq \psi_{j+1,n}(\mathcal{P})$$
   for all $j \geq 0$. Indeed if $y \in \psi_{j,n}(\mathcal{P})$ then there exists $x \in \mathcal{P}$ with $\psi_{j,n}(x) = y$. But now $a_0x \in \mathcal{P}$ and $y = \psi_{j+1,n}(a_0x)$ so $y \in \psi_{j+1,n}(\mathcal{P})$.

By Theorem 2.10 a path set is characterized by its set $B^l(\mathcal{P})$ of initial blocks. If we can show that $\mathcal{P}_j$ has the same set of initial blocks as $\mathcal{P}_0$ for all $1 \leq j \leq n - 1$, then we will have proven the result, taking $\mathcal{Q} = \mathcal{P}_0$. By hypothesis there is a right-resolving presentation $(\mathcal{G}, v)$ of $\mathcal{P}$ having a self-loop labeled $a_0$ at $v$. Let $x = x_0x_1x_2 \ldots x_m$ be an initial block in $\mathcal{P}_0$. Then there are words $w_0, w_1, \ldots w_m$, each of length $n - 1$, such that $x_0w_1x_1w_2x_2 \ldots w_mx_m$ is an initial block in $\mathcal{P}$. Then for all $0 \leq i \leq n - 1$, $a^i x_0w_1x_1w_2x_2 \ldots w_mx_m$ is an initial block in $\mathcal{P}$. By the definition of interleaving, $x$ is an initial block of $\mathcal{P}_j$ for each $0 \leq i \leq n - 1$. Thus, any initial block of $\mathcal{P}_0$ is an initial block of all other $\mathcal{P}_j$ with $1 \leq j \leq n - 1$.

Now, for some $0 \leq i \leq n - 1$, let $y = y_0y_1 \ldots y_m$ be an initial block of $\mathcal{P}_i$. Since $\mathcal{G}$ has a self-loop labeled $a_0$, a path traversing the loop infinitely many times presents the point $(a_0)\infty \in \mathcal{P}$. Thus, each $\mathcal{P}_j$, $0 \leq j \leq n - 1$, contains the point $(a_0)\infty$. Using this point from each $\mathcal{P}_j$, except $\mathcal{P}_i$, we can form a point in $\mathcal{P}$ with initial word $(a_0)^{i-1}y_0(a_0)^{n-1}y_1(a_0)^{n-1}y_2 \ldots (a_0)^{n-1}y_m$. Since $\mathcal{G}$ is right-resolving, the path presenting this point begins by traversing the $a_0$-labeled self-loop $n - 1$ times, thus ending at the initial vertex. Therefore, there is a path beginning at the initial vertex and presenting $y_0(a_0)^{n-1}y_1(a_0)^{n-1}y_2 \ldots (a_0)^{n-1}y_m$. Therefore, $y$ is an initial block of $\mathcal{P}_0$. Therefore $\mathcal{P}_0 = \mathcal{P}_i$ for all $i$ since they have the same initial block set.

Remark B.2. Certain examples of path sets (denoted $X(1, M)$) studied in Abram et al [11], Section 3.4 and Proposition 5.1] exhibited interleaving factorizations that were self-interleavings. The presentations of such $X(1, M)$ have self-loops at the initial vertex, and Theorem B.1 applies to give this result. These path sets
arise in the study of intersections of $p$-adic path set fractals, as defined and studied in [3], and arose in consideration of a problem of Erdős, c.f. [22].

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