On the supersolubility of a finite group with NS-supplemented Sylow subgroups

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Abstract

A subgroup $A$ of a group $G$ is said to be NS-supplemented in $G$, if there exists a subgroup $B$ of $G$ such that $G = AB$ and whenever $X$ is a normal subgroup of $A$ and $p \in \pi(B)$, there exists a Sylow $p$-subgroup $B_p$ of $B$ such that $XB_p = B_pX$. In this paper, we proved the supersolubility of a group with NS-supplemented non-cyclic Sylow subgroups. The solubility of a group with NS-supplemented maximal subgroups is obtained.

1 Introduction

Throughout this paper, all groups are finite and $G$ always denotes a finite group. We use the standard notations and terminology of [1]. The set of all prime divisors of the order of $G$ is denoted by $\pi(G)$. The notation $Y \leq X$ means that $Y$ is a subgroup of a group $X$. The semidirect product of a normal subgroup $A$ and a subgroup $B$ is denoted by $[A]B$.

By the Zassenhaus Theorem ([1, IV.2.11]), a group $G$ with cyclic Sylow subgroups has a cyclic Hall subgroup $H$ such that the quotient $G/H$ is also cyclic. In particular, $G$ is supersoluble.

A group $G$ with abelian Sylow subgroups may be non-soluble (for example, $PSL(2, 5)$) and the compositional factors of $G$ are known [2].

In some papers, the sufficient conditions of solubility and supersolubility of a group in which Sylow subgroups permute with some subgroups are established. For example, the supersolubility of a group $G$ such that every Sylow subgroup $P$ of $G$ permutes with subgroups of some supplement of $P$ in $G$ is obtained in works [3–4].

The following concept is introduced in [5].

Definition 1.1. Two subgroups $A$ and $B$ of a group $G$ are said to be NS-permutable, if they satisfy the following conditions:

1. Whenever $X$ is a normal subgroup of $A$ and $p \in \pi(B)$, there exists a Sylow $p$-subgroup $B_p$ of $B$ such that $XB_p = B_pX$;
2. Whenever $Y$ is a normal subgroup of $B$ and $p \in \pi(A)$, there exists a Sylow $p$-subgroup $A_p$ of $A$ such that $YA_p = A_pY$.

Moreover, if $G = AB$, we say that $G$ is an NS-permutable product of the subgroups $A$ and $B$.

The totally permutable [6] and totally c-permutable [7] subgroups are NS-permutable [5, Lemma 2]. The supersolubility of a group $G = AB$ which is the NS-permutable product of supersoluble subgroups $A$ and $B$ is obtained in [5].

We introduce the following
Definition 1.2. A subgroup $A$ of a group $G$ is said to be NS-supplemented in $G$, if there exists a subgroup $B$ of $G$ such that:

1. $G = AB$;
2. whenever $X$ is a normal subgroup of $A$ and $p \in \pi(B)$, there exists a Sylow $p$-subgroup $B_p$ of $B$ such that $XB_p = B_pX$.

In this case we say that $B$ is a NS-supplement of $A$ in $G$.

In this paper, we proved the supersolubility of a group in which every non-cyclic Sylow subgroup is NS-supplemented. The solubility of a group with NS-supplemented maximal subgroups is obtained.

2 Preliminaries

Definition 1.2 implies the following result for $X = A$.

Lemma 2.1. Let $A$ be an NS-supplemented subgroup of $G$ and $B$ is its NS-supplement in $G$. Then for every $p \in \pi(B)$ there exists a Sylow $p$-subgroup $B_p$ of $B$ such that $AB_p = B_pA$.

Lemma 2.2. Let $K$ be a normal subgroup of $G$. If $A$ is NS-supplemented in $G$ and $B$ is its NS-supplement in $G$, then $AK/K$ is NS-supplemented in $G/K$ and $BK/K$ is its NS-supplement in $G/K$.

Proof. It’s obvious that $G/N = (AK/K)(BK/K)$. Let $X/K$ be a normal subgroup of $AK/K$ and $p \in \pi(BK/K)$. Then $X = (A \cap X)K$ and $A \cap X$ is normal in $A$. By the hypothesis, for every $p \in \pi(B)$ there exists a Sylow $p$-subgroup $B_p$ of $B$ such that $(A \cap X)B_p = B_p(A \cap X)$. Hence $((A \cap X)K)B_p = B_p((A \cap X)K)$ and $X/K$ permutes with Sylow $p$-subgroup $B_pK/K = (BK/K)_p$ of $BK/K$. \hfill $\Box$

Lemma 2.3. ( [8, Theorem 2]) Let $G$ be a group with $p \in \pi(G)$ and $p \neq 3$. If $G$ has a Hall $\{p, r\}$-subgroup for every $r \in \pi(G)$, then $G$ is $p$-soluble.

Lemma 2.4. ( [9, Corollary 3]) Let $G$ be a group such that every maximal subgroup has prime power index. Then $G = S(G)$ or $G/S(G) \simeq \text{PSL}(2, 7)$.

Here $S(G)$ is the maximal normal soluble subgroup of $G$.

3 Groups with NS-supplemented subgroups

Theorem 3.1. If a Sylow $p$-subgroup $P$ of $G$ is NS-supplemented in $G$, then $G$ is $p$-supersoluble in each of the following cases:

1. $p \neq 3$;
2. $p = 3$ and $G$ is 3-soluble.

Proof. Let $B$ be an NS-supplement of $P$ in $G$. By Lemma 2.1 for any $q \in \pi(B) \setminus \{p\}$ there exists a Sylow $q$-subgroup $Q$ of $B$ such that $PQ = QP$. The subgroup $PQ$ is a Hall $\{p, q\}$-subgroup of $G$. Since $q$ is an arbitrary prime of $\pi(G) \setminus \{p\}$, it follows that by Lemma 2.3 $G$ is $p$-soluble for $p \neq 3$ and by the hypothesis, $G$ is 3-soluble for $p = 3$.

We use induction on the order of $G$. Let $N$ be an arbitrary non-trivial normal subgroup in $G$. Then by Lemma 2.2 a Sylow $p$-subgroup $PN/N$ is NS-supplemented in $G/N$. By induction, $G/N$ is $p$-supersoluble, $O_p'(G) = 1$ and $N = O_p(G) \neq 1$.\hfill $\Box$
We choose a subgroup $X$ of $G$ such that $X \leq N \cap Z(P)$ and $|X| = p$. Since $X$ is normal in $P$, it follows that for every $r \in \pi(B)$ there exists a Sylow $r$-subgroup $R$ of $B$ such that $XR = RX$. If $p \neq r$, then the subgroup $N \cap XR = X(N \cap R) = X$ is normal in $XR$. This is true for any prime $r$, hence $X$ is normal in $G$. Since the quotient $G/X$ is $p$-supersoluble by induction, $G$ is $p$-supersoluble.

\begin{proof}
Let $p$ be the smallest prime of $\pi(G)$ and $P$ be a Sylow $p$-subgroup of $G$. If $P$ is cyclic, then $G$ is $p$-nilpotent [11, IV.2.8]. If $P$ is non-cyclic, then $P$ is NS-supplemented in $G$ and by Theorem 3.1, $G$ is $p$-nilpotent. In particular, $G$ is soluble and we apply Theorem 3.1 for each $r \in \pi(G)$. Let $R$ be a Sylow $r$-subgroup of $G$. If $R$ is cyclic, then $R$ is $r$-supersoluble. If $R$ is non-cyclic, then $R$ is NS-supplemented in $G$ and $G$ is $r$-supersoluble by Theorem 3.1. Thus, $G$ is $r$-supersoluble for any $r \in \pi(G)$. Consequently, $G$ is supersoluble.
\end{proof}

Example. The group $PSL(2,7)$ is an NS-supplement of its Sylow 3-subgroup. Hence we can not omit the condition $\ll$group is 3-soluble$\gg$ in Theorem 3.1.

\begin{proof}
We use induction on the order of $G$. By Lemma 2.2, all non-trivial quotients are soluble, hence $S(G) = 1$. Let $M$ be a maximal subgroup of $G$ and $B$ is its NS-supplement in $G$. By the hypothesis, for every $p \in \pi(B)$ there exists a Sylow $p$-subgroup $B_p$ of $B$ such that $B_pM = MB_p$. Since $M \neq G$, there exists $r \in \pi(B)$ and a Sylow $r$-subgroup $B_r$ such that $MB_r = G$. Hence $|G : M| = r^b$. Consequently, every maximal subgroup of $G$ has prime power index. By Lemma 2.3 $G$ is either soluble, or $G \simeq PSL(2,7)$. The group $PSL(2,7)$ has a maximal subgroup $H \simeq [Z_7]Z_3$ and it has not a subgroup of the order $7 \cdot 2^3$. Hence $H$ is not NS-supplemented in $PSL(2,7)$. Consequently, $G$ is soluble.
\end{proof}

The following example shows that the group that satisfies the hypotheses of Theorem 3.3 can be non-supersoluble.

\begin{example}
In the group (10, IdGroup=[72,39])
\[ G = \langle a, b, c \mid a^3 = b^3 = c^8, \ ab = ba, \ a^c = b, \ b^c = ab \rangle \]
all maximal subgroups are the following subgroups:
\[ M_1 = [(\langle a \rangle \times \langle b \rangle)]\langle c^2 \rangle, \ M_4 = \langle c^x \rangle, \ x \in \langle a \rangle \times \langle b \rangle. \]
Moreover, all maximal subgroups are NS-supplemented in $G$ and the subgroup $M_1$ is non-supersoluble.
\end{example}

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