CRYSTALLINE AND SEMI-STABLE REPRESENTATIONS IN THE
IMPERFECT RESIDUE FIELD CASE∗

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Abstract. Let $K$ be a $p$-adic local field with residue field $k$ such that $[k : k^p] = p^e < \infty$ and $V$ be a $p$-adic representation of $\text{Gal}(\overline{K}/K)$. Then, by using the theory of $p$-adic differential modules, we show that $V$ is a potentially crystalline (resp. potentially semi-stable) representation of $\text{Gal}(\overline{K}/K)$ if and only if $V$ is a potentially crystalline (resp. potentially semi-stable) representation of $\text{Gal}(\overline{K}^{\text{pf}}/K^{\text{pf}})$ where $K^{\text{pf}}/K$ is a certain $p$-adic local field whose residue field is the smallest perfect field $k^{\text{pf}}$ containing $k$. As an application, we prove the $p$-adic monodromy theorem of Fontaine in the imperfect residue field case.

Key words. $p$-adic Galois representation, $p$-adic cohomology, $p$-adic differential equation.

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1. Introduction. Let $K$ be a complete discrete valuation field of characteristic 0 with residue field $k$ of characteristic $p > 0$ such that $[k : k^p] = p^e < \infty$. Choose an algebraic closure $\overline{K}$ of $K$ and put $G_K = \text{Gal}(\overline{K}/K)$. By a $p$-adic representation of $G_K$, we mean a finite dimensional vector space $V$ over $\mathbb{Q}_p$ endowed with a continuous action of $G_K$. As in the perfect residue field case, we can define the imperfect residue field versions of $B_{\text{cris}}$ and $B_{\text{st}}$ and, by using these rings, crystalline and semi-stable representations of $G_K$.

Now, we shall state the main results of this article. Let us fix some notations. Fix a lift $(b_i)_{1 \leq i \leq e}$ of a $p$-basis of $k$ in $\mathcal{O}_K$ (the ring of integers of $K$) and for each $m \geq 1$, fix a $p^m$-th root $b_i^{1/p^m}$ of $b_i$ in $\overline{K}$ satisfying $(b_i^{1/p^{m+1}})^p = b_i^{1/p^m}$. Put $K^{(p^m)} = \cup_{m \geq 1} K(b_i^{1/p^m}, 1 \leq i \leq e)$ and let $K^{\text{pf}}$ be the $p$-adic completion of $K^{(p^e)}$. These fields depend on the choice of the sequences $(b_i^{1/p^m})_{m \in \mathbb{N}}$. Note that, if $V$ is a $p$-adic representation of $G_K$, it can be restricted to a $p$-adic representation of $G_K^{\text{pf}} = \text{Gal}(\overline{K}^{\text{pf}}/K^{\text{pf}})$ where we choose an algebraic closure $\overline{K}^{\text{pf}}$ of $K^{\text{pf}}$ containing $\overline{K}$. Since $K^{\text{pf}}$ is a complete discrete valuation field with perfect residue field, we can apply the classical theory (i.e. in the perfect residue field case) to $p$-adic representations of $G_K^{\text{pf}}$. Our main results are the following.

Theorem 1.1. With notation as above, we have the following equivalences.
1. $V$ is a potentially crystalline representation of $G_K$ if and only if $V$ is a potentially crystalline representation of $G_K^{\text{pf}}$.
2. $V$ is a potentially semi-stable representation of $G_K$ if and only if $V$ is a potentially semi-stable representation of $G_K^{\text{pf}}$.

Corollary 1.2. Keep the notation as in Theorem 1.1. Then, $V$ is a de Rham representation of $G_K$ if and only if $V$ is a potentially semi-stable representation of $G_K$.

This paper is organized as follows. In Section 2, we shall review the definitions and basic known facts on crystalline and semi-stable representations, first in the perfect residue field case and then in the imperfect residue field case. In Section 3, first we

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shall review the theory of $p$-adic differential modules and then shall introduce some special elements which behave well under the action of $p$-adic differential operators. In Section 4, by using these elements, we shall prove the main theorem. In Section 5, as an application, we deduce the $p$-adic monodromy theorem of Fontaine in the imperfect residue field case (Corollary 1.2) by using results of Berger [Be] and author [M].

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2. Review of crystalline and semi-stable representations.

2.1. Crystalline and semi-stable representations in the perfect residue field case. (See [F1] for details.) Let $K$ be a complete discrete valuation field of characteristic 0 with perfect residue field $k$ of characteristic $p > 0$. Put $K_0 = \text{Frac}(W(k))$ where $W$ denotes the ring of Witt vectors with coefficients in $k$. Choose an algebraic closure $\bar{K}$ of $K$ and consider its $p$-adic completion $\mathbb{C}_p$. Put

$$\hat{\mathbb{E}} = \lim_{\longleftarrow} \mathbb{E} = \left\{ (x^{(0)}, x^{(1)}, \ldots) \mid (x^{(i+1)})^p = x^{(i)}, x^{(i)} \in \mathbb{C}_p \right\}.$$ 

For two elements $x = (x^{(i)})$ and $y = (y^{(i)})$ of $\hat{\mathbb{E}}$, define their sum and product by $(x + y)^{(i)} = \lim_{j \to \infty} (x^{(i+j)} + p^j y^{(i+j)})^p$ and $(xy)^{(i)} = x^{(i)} y^{(i)}$. Let $\epsilon = (\epsilon^{(n)})$ denote an element of $\hat{\mathbb{E}}$ such that $\epsilon^{(0)} = 1$ and $\epsilon^{(1)} \neq 1$. Then, $\hat{\mathbb{E}}$ is a perfect field of characteristic $p > 0$ and is the completion of an algebraic closure of $k((\epsilon - 1))$ for the valuation defined by $v_p(x) = v_p(x^{(0)})$ where $v_p$ denotes the $p$-adic valuation of $\mathbb{C}_p$ normalized by $v_p(p) = 1$. The field $\hat{\mathbb{E}}$ is equipped with an action of a Frobenius $\sigma$ and a continuous action of the Galois group $G_K = \text{Gal}(\bar{K}/K)$ with respect to the topology defined by the valuation $v_p$. Define $\hat{\mathbb{E}}^+$ to be the ring of integers for this valuation. Put $\hat{A}^+ = W(\hat{\mathbb{E}}^+)$ and $\hat{B}^+ = \hat{A}^+[1/p] = \{ \sum_{k \gg -\infty} p^k [x_k] \mid x_k \in \hat{\mathbb{E}}^+ \}$ where $[\ast]$ denotes the Teichmüller lift of $\ast \in \hat{\mathbb{E}}^+$. This ring $\hat{B}^+$ is equipped with a surjective homomorphism

$$\theta : \hat{B}^+ \twoheadrightarrow \mathbb{C}_p : \sum p^k [x_k] \mapsto \sum p^k x_k^{(0)}.$$ 

Let $\tilde{p}$ denote $(p^{(n)}) \in \hat{\mathbb{E}}^+$ such that $p^{(0)} = p$. Then, $\text{Ker} (\theta)$ is the principal ideal generated by $\omega = \left[ \tilde{p} \right] - p$. The ring $B_{\text{dr}, K}^+$ is defined to be the $\text{Ker} (\theta)$-adic completion of $\hat{B}^+$

$$B_{\text{dr}, K}^+ = \lim_{\longleftarrow} B^+ / (\text{Ker} (\theta))^n.$$ 

This is a discrete valuation ring and $t = \log([\epsilon])$ which converges in $B_{\text{dr}, K}^+$ is a generator of the maximal ideal. Put $B_{\text{dr}, K} = B_{\text{dr}, K}^+[1/t]$. The ring $B_{\text{dr}, K}$ becomes a
field and is equipped with an action of the Galois group $G_K$ and a filtration defined by $\text{Fil}^i B_{\text{dR}, K} = i^* B_{\text{dR}, K}$ ($i \in \mathbb{Z}$). Then, $(B_{\text{dR}, K})^{G_K}$ is canonically isomorphic to $K$. Thus, for a $p$-adic representation $V$ of $G_K$, $D_{\text{dR}, K}(V) = (B_{\text{dR}, K} \otimes_{\mathbb{Q}_p} V)^{G_K}$ is naturally a $K$-vector space. We say that a $p$-adic representation $V$ of $G_K$ is a de Rham representation of $G_K$ if we have

$$\dim_{\mathbb{Q}_p} V = \dim_K D_{\text{dR}, K}(V) \quad \text{(we always have } \dim_{\mathbb{Q}_p} V \geq \dim_K D_{\text{dR}, K}(V)).$$

Let $\theta : \hat{\mathbb{F}}^+ \rightarrow \mathcal{O}_{C_p}$ be the natural homomorphism where $\mathcal{O}_{C_p}$ denotes the ring of integers $C_p$. Define the ring $A_{\text{cris}, K}$ to be the $p$-adic completion of the PD-envelope of $K$. We say that a semi-stable representation of $G_K$ is a crystalline representation of $G_K$ and an action of Frobenius $\phi$ which commutes with the action of $G_K$. Furthermore, define the $\text{Fil}_{\text{cris}, K}$-derivation $N : B_{\text{st}, K} \rightarrow B_{\text{st}, K}$ by putting $N(u_s) = p^{s_0}$. The element $u_s$ is transcendental over $B_{\text{cris}, K}$ and we extend the Frobenius $\varphi$ on $B_{\text{cris}, K}$ to $B_{\text{st}, K}$ by putting $\varphi(u_s) = p u_s$. Furthermore, define the $B_{\text{cris}, K}$-derivation $N : B_{\text{st}, K} \rightarrow B_{\text{st}, K}$ by putting $N(u_s) = -1$. It is easy to verify $N\varphi = p \varphi N$. As in the case of $B_{\text{cris}, K}$, we have $(B_{\text{st}, K})^{G_K} = K$. Thus, for a $p$-adic representation $V$ of $G_K$, $D_{\text{st}, K}(V) = (B_{\text{st}, K} \otimes_{\mathbb{Q}_p} V)^{G_K}$ is naturally a $K_{t}$-vector space endowed with a Frobenius operator and a filtration after extending the scalars to $K$. We say that a $p$-adic representation $V$ of $G_K$ is a semi-stable representation of $G_K$ if we have

$$\dim_{\mathbb{Q}_p} V = \dim_{K_{t}} D_{\text{st}, K}(V) \quad \text{(we always have } \dim_{\mathbb{Q}_p} V \geq \dim_{K_{t}} D_{\text{st}, K}(V)).$$

Furthermore, we say that a $p$-adic representation $V$ of $G_K$ is a potentially crystalline representation of $G_K$ if there exists a finite field extension $L/K$ in $\overline{K}$ such that $V$ is a crystalline representation of $G_L$.

Fix a prime element $\varphi$ of $\mathcal{O}_K$ (the ring of integers of $K$) and an element $s = \langle s^{(n)} \rangle \in \mathbb{Z}^+$ such that $s^{(0)} = \varphi$. Then, the series $\log(s_\varphi^{-1})$ converges to an element $u_s$ in $B_{\text{dR}, K}$ and the subring $B_{\text{dR}, K}/u_s$ of $B_{\text{dR}, K}$ depends only on the choice of $\varphi$. We denote this ring by $B_{\text{st}, K}$. Since we have the inclusion $K \otimes_{K_0} B_{\text{cris}, K} \hookrightarrow B_{\text{dR}, K}$, the ring $K \otimes_{K_0} B_{\text{cris}, K}$ is equipped with the action of Frobenius $\varphi$ and the filtration induced by that of $B_{\text{dR}, K}$. Then, $(B_{\text{cris}, K})^{G_K}$ is canonically isomorphic to $K_0$. Thus, for a $p$-adic representation $V$ of $G_K$, $D_{\text{cris}, K}(V) = (B_{\text{cris}, K} \otimes_{\mathbb{Q}_p} V)^{G_K}$ is naturally a $K_0$-vector space endowed with a Frobenius operator and a filtration after extending the scalars to $K$. We say that a $p$-adic representation $V$ of $G_K$ is a crystalline representation of $G_K$ if we have

$$\dim_{\mathbb{Q}_p} V = \dim_{K_0} D_{\text{cris}, K}(V) \quad \text{(we always have } \dim_{\mathbb{Q}_p} V \geq \dim_{K_0} D_{\text{cris}, K}(V)).$$

Furthermore, we say that a $p$-adic representation $V$ of $G_K$ is a potentially crystalline representation of $G_K$ if there exists a finite field extension $L/K$ in $\overline{K}$ such that $V$ is a crystalline representation of $G_L$.  

2.2. Crystalline and semi-stable representations in the imperfect residue field case. Let $K$ be a complete discrete valuation field of characteristic $0$ with residue field $k$ of characteristic $p > 0$ such that $[k : k^p] = p^e < \infty$. Here, we do not assume that the residue field $k$ is perfect. Choose an algebraic closure $\overline{K}$ of $K$ and put $G_K = \text{Gal}(\overline{K}/K)$. As in Introduction, fix a lift $(b_i)_{1 \leq i \leq e}$ of a $p$-basis of $k$ in $\mathcal{O}_K$ (the ring of integers of $K$) and for each $m \geq 1$, fix a $p^m$-th root $b_i^{1/p^m}$ of $b_i$ in $\overline{K}$ satisfying $(b_i^{1/p^m})^p = b_i^{1/p^m}$. Put

$$K^{(p^e)} = \bigcup_{m \geq 0} k(b_i^{1/p^m}, 1 \leq i \leq e) \quad \text{and} \quad K^{pf} = p\text{-adic completion of } K^{(p^e)}.$$
These fields depend on the choice of a lift of a $p$-basis of $k$ in $\mathcal{O}_K$. Let $k^{pf}$ denote the perfect residue field of $K^{pf}$ and put $K^{pf}_0 = \text{Frac}(W(k^{pf}))$. Define $K_0$ to be $K_0 = K \cap K^{pf}_0$. Then, $K_0$ has the residue field $k$ and the extension $K/K_0$ is finite. If $k$ is perfect (that is $e = 0$), the field $K_0$ coincides with $K^{pf}_0$. Furthermore, since $K_0$ is a complete $p$-ring, it is isomorphic to the field $\text{Frac}(W(k))$ and thus is endowed with an action of Frobenius $\sigma$. Since $K^{pf}$ is a Henselian discrete valuation field, we have an isomorphism $G_{K^{pf}} = \text{Gal}(K^{pf}/K^{pf}) \simeq G_{K^{(pf)}} = \text{Gal}(\mathcal{K}/K^{pf}) \subset G_K$ where we choose an algebraic closure $K^{pf}$ of $K^{pf}$ containing $\mathcal{K}$. With this isomorphism, we identify $G_{K^{pf}}$ with a subgroup of $G_K$. We have a bijective map from the set of finite extensions of $K^{pf}$ contained in $\mathcal{K}$ to the set of finite extensions of $K^{pf}$ contained in $\mathcal{K}$. Define $\mathcal{L} \rightarrow L \rightarrow \mathcal{K}^{pf}$. Furthermore, $\mathcal{L}^{pf}$ is the $p$-adic completion of $L$. Hence, we have an isomorphism of rings

$$\mathcal{O}_L/p^n\mathcal{O}_L \simeq \mathcal{O}_{\mathcal{L}^{pf}}/p^n\mathcal{O}_{\mathcal{L}^{pf}}$$

where $\mathcal{O}_L$ and $\mathcal{O}_{\mathcal{L}^{pf}}$ denote the rings of integers of $L$ and $\mathcal{L}^{pf}$. Thus, the $p$-adic completion of $L$ is isomorphic to the $p$-adic completion of $\mathcal{L}^{pf}$ which we will write $\mathcal{C}_p$. As in Subsection 2.1, construct the rings $\mathcal{E}^{+}$ and $\mathcal{A}^{+} = W(\mathcal{E}^{+})$ from this $\mathcal{C}_p$. Put $\mathcal{O}_K = \mathcal{O}_L \cap W(k^{pf})$. Thus, $\mathcal{O}_K$ is a $\mathcal{O}_L$-algebra equipped with an action of the Galois group $G_K$. Let $\alpha : \mathcal{O}_K \otimes_{\mathcal{O}_L} \mathcal{A}^{+} \rightarrow \mathcal{O}_L/(\mathcal{O}_K \otimes_{\mathcal{O}_L} \mathcal{A}^{+})$ be the natural surjection and define $\mathcal{A}_i(K)$ to be $\mathcal{A}_i(K) = \lim_{\longleftarrow n \geq 0} (\mathcal{O}_K \otimes_{\mathcal{O}_L} \mathcal{A}^{+})/(\mathcal{O}_K \otimes_{\mathcal{O}_L} \mathcal{A}^{+})/(\mathcal{O}_K \otimes_{\mathcal{O}_L} \mathcal{A}^{+})$. Let $\theta_K : \mathcal{A}_i(K) \otimes_{\mathcal{O}_p} \mathcal{Q}_p \rightarrow \mathcal{C}_p$ be the natural extension of $\theta : \mathcal{A}^{+}[1/p] \rightarrow \mathcal{C}_p$. Define $B_{dr,K}$ to be the $\mathcal{K}$-adic completion of $\mathcal{A}_i(K) \otimes_{\mathcal{O}_p} \mathcal{Q}_p$.

This is a $\mathcal{K}$-algebra equipped with an action of the Galois group $G_K$. Let $\mathcal{L}^{pf}$ denote $(b_i^{(n)}) \in \mathcal{E}^{+}$ such that $b_i^{(0)} = b_i$ and then the series which defines log divides $b_i^{(n)}$ to an element $t_i$ in $B_{dr,K}$. Then, the ring $B_{dr,K}$ becomes a local ring with the maximal ideal $m_{dr} = (t_1, t_2, \ldots, t_e)$. Define a filtration on $B_{dr,K}^{+}$ by $\text{fil}^n B_{dr,K}^{+} = m_{dr}^n$. Then, the homomorphism

$$f : B_{dr,K}^{+}[[t_1, \ldots, t_e]] \rightarrow B_{dr,K}^{+}$$

is an isomorphism of filtered algebras (see [Br2], Proposition 2.9). From this isomorphism, it follows that

$$i : B_{dr,K}^{+} \rightarrow B_{dr,K}$$

are $G_{K^{pf}}$-equivariant homomorphisms and the composition

$$p \circ i : B_{dr,K}^{+} \rightarrow B_{dr,K}^{+} \rightarrow B_{dr,K}^{+}$$

is an identity. Put $B_{dr,K} = B_{dr,K}^{+}[1/t]$. Then, $K$ is canonically embedded in $B_{dr,K}$ and we have a canonical isomorphism $(B_{dr,K})^{G_K} = K$. Thus, for a $p$-adic representation $V$ of $G_K$, $D_{dr,K}(V) = (B_{dr,K} \otimes_{\mathcal{Q}_p} V)^{G_K} = K$ is naturally a $K$-vector space. We say that a $p$-adic representation $V$ of $G_K$ is a de Rham representation of $G_K$ if we have

$$\dim_{\mathcal{Q}_p} V = \dim_K D_{dr,K}(V)$$

(we always have $\dim_{\mathcal{Q}_p} V \geq \dim_K D_{dr,K}(V)$).
Let \( \theta_{K_0} : \mathcal{O}_{K_0} \otimes \hat{\mathbb{A}}^+ \to \mathcal{O}_{\hat{C}_p} \) denote the natural extension of \( \theta : \hat{\mathbb{A}}^+ \to \mathcal{O}_{\hat{C}_p} \) where \( \mathcal{O}_{K_0} \) (resp. \( \mathcal{O}_{C_0} \)) denotes the ring of integers of \( K_0 \) (resp. \( C_0 \)). Define \( A_{\text{cris},K} \) to be the \( p \)-adic completion of the PD-envelope of \( \ker (\theta_{K_0}) \) compatible with the canonical PD-envelope over the ideal generated by \( p \). Put \( B_{\text{cris},K}^+ = A_{\text{cris},K}[1/p] \) and \( B_{\text{cris},K} = B_{\text{cris},K}^+ [1/t] \). The ring \( B_{\text{cris},K} \) is the \( K_0 \)-algebra endowed with an action of \( G_K \) and an action of Frobenius \( \varphi \) which commutes with the action of \( G_K \). Furthermore, since we have the inclusion \( K \otimes_{K_0} B_{\text{cris},K} \hookrightarrow B_{\text{dR},K} \) (see [Br2, Proposition 2.47.]), the ring \( K \otimes_{K_0} B_{\text{cris},K} \) is endowed with the filtration induced by that of \( B_{\text{dR},K} \). For \( 1 \leq i \leq e \), put \( r_i = [b_i] - b_i \in \mathcal{O}_{K_0} \otimes \hat{\mathbb{A}}^+ \). Then, we have \( r_i \in \ker (\theta_{K_0}) \) for \( 1 \leq i \leq e \) and an isomorphism

\[
f : (p-\text{adic completion of } A_{\text{cris},K^p}(r_1, \ldots, r_e)) \to A_{\text{cris},K}\]

where \((*)\) denotes PD-polynomial (see [Br2, Proposition 2.39.]). From this isomorphism, it follows that

\[
i : B_{\text{cris},K^p} \hookrightarrow B_{\text{cris},K} \quad \text{and} \quad p : B_{\text{cris},K} \twoheadrightarrow B_{\text{cris},K^p} : r_1 \mapsto 0\]

are \( G_{K^p} \)-equivariant homomorphisms and the composition

\[p \circ i : B_{\text{cris},K^p} \twoheadrightarrow B_{\text{cris},K} \to B_{\text{cris},K^p}\]

is identity. By [Br2, Proposition 2.50.], we have a canonical isomorphism \( (B_{\text{cris},K})^{G_K} = K_0 \). Thus, for a \( p \)-adic representation \( V \) of \( G_K \), \( D_{\text{cris},K}(V) = (B_{\text{cris},K} \otimes_{\mathbb{Q}_p} V)^{G_K} \) is naturally a \( K_0 \)-vector space endowed with a Frobenius operator and a filtration after extending the scalars to \( K \). We say that a \( p \)-adic representation \( V \) of \( G_K \) is a crystalline representation of \( G_K \) if we have

\[
\dim_{\mathbb{Q}_p} V = \dim_{K_0} D_{\text{cris},K}(V).
\]

Note that, for a \( p \)-adic representation \( V \) of \( G_K \), we always have \( \dim_{\mathbb{Q}_p} V \geq \dim_{K_0} D_{\text{cris},K}(V) \) by [Br2, Proposition 3.22.]. Furthermore, we say that a \( p \)-adic representation \( V \) of \( G_K \) is a potentially crystalline representation of \( G_K \) if there exists a finite field extension \( L/K \) in \( \overline{K} \) such that \( V \) is a crystalline representation of \( G_L \).

Fix a prime element \( \wp \) of \( \mathcal{O}_K \) and an element \( s = (s^{(n)}) \in \hat{\mathbb{A}}^+ \) such that \( s^{(0)} = \wp \). Then, the series \( \log (s^{(n)})^{-1} \) converges to an element \( u_s \) in \( B_{\text{dR},K}^+ \) and the subring \( B_{\text{cris},K}[u_s] \) of \( B_{\text{dR},K} \) depends only on the choice of \( \wp \). We denote this ring by \( B_{\st,K} \). We can prove that the element \( u_s \) is transcendental over \( B_{\text{cris},K} \) (see [F1, 4.3.]). Since we have the inclusion \( K \otimes_{K_0} B_{\st,K} \hookrightarrow B_{\text{dR},K} \), the ring \( K \otimes_{K_0} B_{\st,K} \) is endowed with the action of \( G_K \) and the filtration induced by that of \( B_{\text{dR},K} \). We extend the Frobenius \( \varphi \) on \( B_{\text{cris},K} \) to \( B_{\st,K} \) by putting \( \varphi(u_s) = pu_s \). Furthermore, define the \( B_{\text{cris},K} \)-derivation \( N : B_{\st,K} \to B_{\st,K} \) by \( N(u_s) = -1 \). It is easy to verify \( N \varphi = p\varphi N \). As in the case of \( A_{\text{cris},K} \), we have an isomorphism

\[
f : (p-\text{adic completion of } A_{\text{cris},K^p}(r_1, \ldots, r_e))[1/p, u_s, 1/t]) \to B_{\st,K}\]

where \((*)\) denotes PD-polynomial. From this isomorphism, it follows that

\[
i : B_{\st,K^p} \hookrightarrow B_{\st,K} \quad \text{and} \quad p : B_{\st,K} \twoheadrightarrow B_{\st,K^p} : r_1 \mapsto 0\]

are \( G_{K^p} \)-equivariant homomorphisms and the composition

\[p \circ i : B_{\st,K^p} \twoheadrightarrow B_{\st,K} \to B_{\st,K^p}\]
is identity. By imitating the result [Br2, Proposition 2.50.], we can show that we have a canonical isomorphism \((B_{st,K})^{G_K} = K_0\) as follows.

**Lemma 2.1.** We have \((\text{Frac} B_{st,K})^{G_K} = K_0\).

**Proof.** From the map \(K \otimes_{K_0} B_{st,K} \hookrightarrow B_{\text{dR},K}\), we obtain a \(G_K\)-equivariant injection \(K \otimes_{K_0} \text{Frac} B_{st,K} \hookrightarrow \text{Frac} B_{\text{dR},K}\) by localization. It follows that we have an injection \(K \otimes_{K_0} (\text{Frac} B_{st,K})^{G_K} \hookrightarrow (\text{Frac} B_{\text{dR},K})^{G_K}\). Since we have \((\text{Frac} B_{\text{dR},K})^{G_K} = K\), we get \(\dim_{K_0} (\text{Frac} B_{st,K})^{G_K} \leq 1\) and thus \((\text{Frac} B_{st,K})^{G_K} = K_0\). \(\Box\)

**Proposition 2.2.** We have \((B_{st,K})^{G_K} = K_0\).

**Proof.** We have \(K_0 \subset (B_{st,K})^{G_K} \subset (\text{Frac} B_{st,K})^{G_K} = K_0\). \(\Box\)

Thus, for a \(p\)-adic representation \(V\) of \(G_K\), \(D_{st,K}(V) = (B_{st,K} \otimes_{\mathbb{Q}_p} V)^{G_K}\) is naturally a \(K_0\)-vector space endowed with a Frobenius operator and a filtration after extending the scalars to \(K\). We say that a \(p\)-adic representation \(V\) of \(G_K\) is a semi-stable representation of \(G_K\) if we have

\[
\dim_{\mathbb{Q}_p} V = \dim_{K_0} D_{st,K}(V).
\]

Since \((B_{st,K})^{G_K}\) is the field \(K_0\) (Proposition 2.2.) and we have \((\text{Frac} B_{st,K})^{G_K} = K_0\) (Lemma 2.1.), it follows from [Br2, Proposition 3.3.] that we always have \(\dim_{\mathbb{Q}_p} V \geq \dim_{K_0} D_{st,K}(V)\). Furthermore, we say that a \(p\)-adic representation \(V\) of \(G_K\) is a potentially semi-stable representation of \(G_K\) if there exists a finite field extension \(L/K\) such that \(V\) is a semi-stable representation of \(G_L\).

3. The theory of \(p\)-adic differential modules. In this section, we shall review the theory of \(p\)-adic differential modules which plays an important role in this article. First, let us fix the notations. Let \(K\) be a complete discrete valuation field of characteristic 0 with residue field \(k\) of characteristic \(p > 0\) such that \([k : k^p] = p^e < \infty\) and \(V\) be a \(p\)-adic representation of \(G_K\). Define \(K^{(p)}\) and \(K^{\text{pf}}\) as in Introduction and Subsection 2.2. Put \(K^{(p)}_\infty = \cup_{m \geq 0} K^{(p)}(\zeta_{p^m})\) (resp. \(K^{\text{pf}}_\infty = \cup_{m \geq 0} K^{\text{pf}}(\zeta_{p^m})\)) where \(\zeta_{p^m}\) denotes a primitive \(p^m\)-th root of unity in \(K\) (resp. \(\overline{K}^{\text{pf}}\)) such that \((\zeta_{p^m+1})^p = \zeta_{p^m}\).

Let \(K^{\text{pf}}_\infty\) denote the \(p\)-adic completion of \(K^{\text{pf}}_\infty\). These fields \(K^{(p)}_\infty, K^{\text{pf}}_\infty\) and \(K^{\text{pf}}_\infty\) depend on the choice of a lift of a \(p\)-basis of \(k\) in \(\mathcal{O}_K\). Then, we have the following inclusions

\[
K^{(p)}_\infty \subset K^{\text{pf}}_\infty \subset \hat{K}^{\text{pf}}_\infty.
\]

Let \(H\) denote the kernel of the cyclotomic character \(\chi : G_{K^{(p)}} \to \mathbb{Z}_p^\times\). Then, the Galois group \(H\) is isomorphic to the subgroup \(\text{Gal}(\overline{K}/K^{(p)}_\infty)\) of \(G_K\). Define \(\Gamma_K = G_K/H\). Let \(\Gamma_0\) denote the subgroup \(\text{Gal}(\overline{K}^{(p)}_\infty/K^{(p)}_\infty)\) (\(\simeq G_{K^{(p)}}/H\)) of \(\Gamma_K\). Let \(\Gamma_i (1 \leq i \leq e)|\) be the subgroup of \(\Gamma_K\) such that actions of \(\beta_i \in \Gamma_i (1 \leq i \leq e)\) satisfy \(\beta_i(\zeta_{p^m}) = \zeta_{p^m}\) and \(\beta_i(b_j^{1/p^m}) = b_j^{1/p^m} (i \neq j)\) and define the homomorphism \(c_i : \Gamma_i \to \mathbb{Z}_p\) such that we have \(\beta_i(b_j^{1/p^m}) = b_j^{1/p^m} c_i(\beta_i)\). Then, the homomorphism \(c_i\) defines an isomorphism \(\Gamma_i \simeq \mathbb{Z}_p\) of profinite groups. With this, we can see that there exist isomorphisms of profinite groups

\[
\Gamma_K \simeq \Gamma_0 \times \prod_{i=1}^e \Gamma_i \simeq \Gamma_0 \times \mathbb{Z}_p^\infty.
\]
3.1. Review of the classical theory. In this subsection, we will give the definitions of $p$-adic differential modules $D_{\text{Sen}}(V)$, $D_{\text{Br}}(V)$, $D_{\text{dif}}^+(V)$ and $D_{\text{c-dif}}^+(V)$ which are obtained by Sen, Brinon, Fontaine and Andreata-Brinon ([S], [Br1], [F2], [A-B]). The modules $D_{\text{Sen}}(V)$ and $D_{\text{dif}}(V)$ are obtained when $V$ is a $p$-adic representation of $\text{Gal}(\overline{L}/L)$ where $L$ is a complete discrete valuation field of characteristic $0$ with perfect residue field of characteristic $p > 0$ and we choose an algebraic closure $\overline{L}$ of $L$. However, for simplicity, we will state the results in the case $L = K_{\text{pf}}$.

3.1.1. The module $D_{\text{Sen}}(V)$. In the article [S], Sen shows that, for a $p$-adic representation $V$ of $G_K$, the $K_{\text{pf}}$-vector space $(C_p \otimes_{Q_p} V)^H$ has dimension $d = \dim_{Q_p} V$ and the union of the finite dimensional $K_{\text{pf}}$-subspaces of $(C_p \otimes_{Q_p} V)^H$ stable under $\Gamma_0$ ($\simeq G_{K_{\text{p}}} / H$) is a $K_{\text{pf}}$-vector space of dimension $d$ stable under $\Gamma_0$ (called $D_{\text{Sen}}(V)$). We have $C_p \otimes_{K_{\text{pf}}} D_{\text{Sen}}(V) = C_p \otimes_{Q_p} V$ and the natural map $K_{\text{pf}} \otimes_{K_{\text{pf}}} D_{\text{Sen}}(V) \rightarrow (C_p \otimes_{Q_p} V)^H$ is an isomorphism. Furthermore, if $\gamma \in \Gamma_0$ is close enough to $1$, then the series of operators on $D_{\text{Sen}}(V)$

$$\frac{\log(\gamma)}{\log(\chi(\gamma))} = - \frac{1}{\log(\chi(\gamma))} \sum_{k \geq 1} \frac{(1 - \gamma)^k}{k}$$

converges to a $K_{\text{pf}}$-linear operator $\nabla^{(0)} : D_{\text{Sen}}(V) \rightarrow D_{\text{Sen}}(V)$ and does not depend on the choice of $\gamma$.

3.1.2. The module $D_{\text{Br}}(V)$. In the article [Br1], Brinon generalizes Sen’s work above. For a $p$-adic representation $V$ of $G_K$, he shows that the union of the finite dimensional $K_{\text{pf}}$-subspaces of $(C_p \otimes_{Q_p} V)^H$ stable under $\Gamma_K$ is a $K_{\text{pf}}$-vector space of dimension $d$ stable under $\Gamma_K$ (we call it $D_{\text{Br}}(V)$). We have $C_p \otimes_{K_{\text{pf}}} D_{\text{Br}}(V) = C_p \otimes_{Q_p} V$ and the natural map $K_{\text{pf}} \otimes_{K_{\text{pf}}} D_{\text{Br}}(V) \rightarrow (C_p \otimes_{Q_p} V)^H$ is an isomorphism. As in the case of $D_{\text{Sen}}(V)$, the $K_{\text{pf}}$-vector space $D_{\text{Br}}(V)$ is endowed with the action of the $K_{\text{pf}}$-linear operator $\nabla^{(0)} = \frac{\log(\gamma)}{\log(\chi(\gamma))}$ if $\gamma \in \Gamma_0$ is close enough to $1$. In addition to this operator $\nabla^{(0)}$, if $\beta_i \in \Gamma_i$ is close enough to $1$, then the series of operators on $D_{\text{Br}}(V)$

$$\frac{\log(\beta_i)}{c_i(\beta_i)} = - \frac{1}{c_i(\beta_i)} \sum_{k \geq 1} \frac{(1 - \beta_i)^k}{k}$$

converges to a $K_{\text{pf}}$-linear operator $\nabla^{(i)} : D_{\text{Br}}(V) \rightarrow D_{\text{Br}}(V)$ and does not depend on the choice of $\beta_i$.

3.1.3. The module $D_{\text{dif}}^+(V)$. In the article [F2], by using Sen’s theory, Fontaine shows that, for a $p$-adic representation $V$ of $G_{K_{\text{p}}}$, the union of $K_{\text{pf}}[[t]]$-submodules of finite type of $(B_{\text{dif},K_{\text{pf}}} \otimes_{Q_p} V)^H$ stable under $\Gamma_0$ ($\simeq G_{K_{\text{p}}} / H$) is a free $K_{\text{pf}}[[t]]$-module of rank $d$ stable under $\Gamma_0$ (called $D_{\text{dif}}^+(V)$). We have $B_{\text{dif},K_{\text{pf}}} \otimes_{K_{\text{pf}}[\![t]\!] } D_{\text{dif}}^+(V) = B_{\text{dif},K_{\text{pf}}} \otimes_{Q_p} V$ and the natural map $(B_{\text{dif},K_{\text{pf}}})^H \otimes_{K_{\text{pf}}[\![t]\!]} D_{\text{dif}}^+(V) \rightarrow (B_{\text{dif},K_{\text{pf}}} \otimes_{Q_p} V)^H$ is an isomorphism. Furthermore, if $\gamma \in \Gamma_0$ is close enough to $1$, then the series of operators on $D_{\text{dif}}^+(V)$

$$\frac{\log(\gamma)}{\log(\chi(\gamma))} = - \frac{1}{\log(\chi(\gamma))} \sum_{k \geq 1} \frac{(1 - \gamma)^k}{k}$$

converges to a $K_{\text{pf}}$-linear operator $\nabla^{(0)} : D_{\text{dif}}^+(V) \rightarrow D_{\text{dif}}^+(V)$.
converges to a $K^p$-linear derivation $\nabla(0) : D^+_{\text{dif}}(V) \to D^+_{\text{dif}}(V)$ and does not depend on the choice of $\gamma$. Note that this $D^+_{\text{dif}}(V)$ is a little different from one which the author used by the same symbol in the article [M].

3.1.4. The module $D^+_{\text{dif}}(V)$. In the article [A-B], Andreattia and Brinon generalize Fontaine’s work above. For a $p$-adic representation $V$ of $G_K$, they show that the union of $K^p[[t, t_1, \ldots, t_e]]$-submodules of finite type of $(B^+_{\text{dif}} \otimes_{\mathbb{Q}_p} V)^H$ stable under $\Gamma_K$ is a free $K^p[[t, t_1, \ldots, t_e]]$-module of rank $d$ stable under $\Gamma_K$ (we call it $D^+_{\text{dif}}(V)$). We have $B^+_{\text{dif}} \otimes_{K[[t,t_1,\ldots, t_e]]} D^+_{\text{dif}}(V) = B^+_{\text{dif}} \otimes_{\mathbb{Q}_p} V$ and the natural map $(B^+_{\text{dif}} \otimes_{K[[t,t_1,\ldots, t_e]]} D^+_{\text{dif}}(V))$ to $(B^+_{\text{dif}} \otimes_{\mathbb{Q}_p} V)^H$ is an isomorphism. As in the case of $D^+_{\text{dif}}(V)$, the $K^p[[t, t_1, \ldots, t_e]]$-module $D^+_{\text{dif}}(V)$ is endowed with the $K^p$-linear derivation $\nabla(0) = \log(\gamma)$ if $\gamma \in \Gamma_0$ is close enough to $1$. In addition to this operator $\nabla(0)$, if $\beta_i \in \Gamma_i$ is close enough to $1$, then the series of operators on $D^+_{\text{dif}}(V)$

$$\log(\beta_i) = -\frac{1}{c_i(\beta_i)} \sum_{k \geq 1} \frac{(1 - \beta_i)^k}{k}$$

converges to a $K\infty^p$-linear derivation $\nabla(i) : D^+_{\text{dif}}(V) \to D^+_{\text{dif}}(V)$ and does not depend on the choice of $\beta_i$.

3.1.5. Some properties of differential operators. We shall describe the actions of operators $\{\nabla(i)\}_{i=0}^{e}$ on $D_{\text{Bri}}(V)$ and $D^+_{\text{dif}}(V)$. First, by a standard argument, we can show that, if $x \in D_{\text{Bri}}(V)$ (resp. $D^+_{\text{dif}}(V)$), we have

$$\nabla(0)(x) = \lim_{\gamma \rightarrow 1} \frac{\gamma(x) - x}{\log(\gamma)} \text{ and } \nabla(i)(x) = \lim_{\beta_i \rightarrow 1} \frac{\beta_i(x) - x}{c_i(\beta_i)}.$$  

With this, we can describe the actions of $K\infty^p$-linear derivations $\{\nabla(i)\}_{i=0}^{e}$ on the ring $K\infty^p[[t, t_1, \ldots, t_e]] = D^+_{\text{dif}}(\mathbb{Q}_p)$ (here $\mathbb{Q}_p$ is equipped with the structure of $p$-adic representations of $G_K$ induced by the trivial action of $G_K$) as

$$\nabla(0) = \frac{d}{dt} \text{ and } \nabla(i) = \frac{d}{dt_i}(1 \leq i \leq e).$$

We extend naturally actions of $K\infty^p$-linear derivations $\{\nabla(i)\}_{i=0}^{e}$ on $K\infty^p[[t, t_1, \ldots, t_e]]$ to $K\infty^p[[t, t_1, \ldots, t_e]][x^{-1}] \subset B_{\text{dif}, K}$ by putting $\nabla(0)(t^{-1}) = -t^{-1}$ and $\nabla(i)(t^{-1}) = 0 (1 \leq i \leq e)$. Furthermore, the bracket $[,]$ of operators $\{\nabla(i)\}_{i=0}^{e}$ on $D_{\text{Bri}}(V)$ (resp. $D^+_{\text{dif}}(V)$) satisfies (see [M, Proposition 3.3.1])

$$[\nabla(0), \nabla(i)] = \nabla(i) (i \neq 0) \text{ and } [\nabla(i), \nabla(j)] = 0 (i, j \neq 0).$$

3.2. Construction of special elements. In this subsection, we shall introduce some special elements which behave well under the action of $p$-adic differential operators.

3.2.1. A special basis of $D^+_{\text{dif}}(V)$. We shall construct a special basis of $D^+_{\text{dif}}(V)$ over $K\infty^p[[t, t_1, \ldots, t_e]]$ which bridges the gap between $D^+_{\text{dif}}(V)$ and $D^+_{\text{dif}}(V)$ and behaves well under the action of $\nabla(0)$. Note that there is no $G_K$-equivariant injection $K \hookrightarrow B^+_{\text{dif}, K}$: we will sometimes write $L^+_{\text{dif}}$ instead of the misleading $K^p[[t]]$. First, let us recall the following result.
Proposition 3.1. [M, Proposition 4.8.] Let $V$ be a $p$-adic representation of $G_K$. If $V$ is a de Rham representation of $G_{K_{ur}}$, there exists a $\nabla^{(0)}$-equivariant isomorphism of $K_{\inf}^{\lfloor \{[t, t_1, \ldots, t_e]\}}$-modules

$$D_{e, \text{dif}}^+(V) \simeq \nabla^{(0)} \oplus \bigoplus_{j=1}^d K_{\inf}^{\lfloor \{[t, t_1, \ldots, t_e]\}}(n_j) \quad (d = \dim_{Q_p} V, n_j \in \mathbb{Z}).$$

Next, let us define the $K_{\inf}^{\lfloor \{[t, t_1, \ldots, t_e]\}}$-submodule $X$ of $(B_{\text{dR}, K}^+ \otimes_{Q_p} V)^H$ by $X = K_{\inf}^{\lfloor \{[t, t_1, \ldots, t_e]\}} \otimes_{K_{\inf}^{\lfloor \{[t, t_1, \ldots, t_e]\}}} D_{e, \text{dif}}^+(V)$. If we put $D_{e, \text{dif}}^+(V) = D_{e, \text{dif}}^+(V)/(t, t_1, \ldots, t_e)^r D_{e, \text{dif}}^+(V)$, the inclusion $K_{\inf}^{\lfloor \{[t, t_1, \ldots, t_e]\}} \otimes_{K_{\inf}^{\lfloor \{[t, t_1, \ldots, t_e]\}}} D_{e, \text{dif}}^+(V) \hookrightarrow (t, t_1, \ldots, t_e)^r \otimes_{L_{\text{dif}}^+} D_{e, \text{dif}}^+(V)$ by the theory of Sen. Since both sides have the same dimension over $K_{\inf}^{\lfloor \{[t, t_1, \ldots, t_e]\}}$, the inclusion above actually gives an isomorphism. By taking the projective limit with respect to $r$, we obtain a $\Gamma_0$-equivariant isomorphism $X \simeq L_{\text{dif}}^+[[t_1, \ldots, t_e]] \otimes_{L_{\text{dif}}^+} D_{e, \text{dif}}^+(V)$.

Proposition 3.2. Let $V$ be a $p$-adic representation of $G_K$. If $V$ is a de Rham representation of $G_{K_{ur}}$, there exists a basis $\{f_j\}_{j=1}^d$ of $D_{e, \text{dif}}^+(V)$ over $L_{\text{dif}}^+$ such that

1. $\{1 \otimes f_j\}_{j=1}^d$ forms a basis of $D_{e, \text{dif}}^+(V)$ over $L_{\text{dif}}^+[[t_1, \ldots, t_e]]$,
2. the action of $\nabla^{(0)}$ on $\{1 \otimes f_j\}_{j=1}^d$ is given by $\nabla^{(0)}(1 \otimes f_j) = n_j(1 \otimes f_j)$ where the integers $n_j$ are those of Proposition 3.1.

Proof. Let $\{G_j\}_{j=1}^d$ denote a basis of $D_{e, \text{dif}}^+(V)$ over $K_{\inf}^{\lfloor \{[t]\}}$. Since $D_{e, \text{dif}}^+(V)$ is a submodule of $X = L_{\text{dif}}^+[[t_1, \ldots, t_e]] \otimes_{L_{\text{dif}}^+} D_{e, \text{dif}}^+(V)$, any element of $D_{e, \text{dif}}^+(V)$ can be written as linear combinations of $\{1 \otimes G_j\}_{j=1}^d$ over $L_{\text{dif}}^+[[t_1, \ldots, t_e]]$. On the other hand, fix a basis $\{F_j\}_{j=1}^d$ of $D_{e, \text{dif}}^+(V)$ over $K_{\inf}^{\lfloor \{[t, t_1, \ldots, t_e]\}}$ that gives the isomorphism of Proposition 3.1, that is, $\nabla^{(0)}(F_j) = n_j F_j$ with $n_j \in \mathbb{Z}$. Then, we can write

\begin{equation}
1 \otimes F_j = \sum_{(m_1, \ldots, m_d) \in \mathbb{N}^d} t_1^{m_1} \cdots t_e^{m_e} \otimes \left( \sum_{k=1}^d a_{j,k}^{(m_1, \ldots, m_d)} G_k \right)
\end{equation}

where the $a_{j,k}^{(m_1, \ldots, m_d)}$ are elements of $L_{\text{dif}}^+$. Put $f_j = \sum_{k=1}^d a_{j,k}^{(0, \ldots, 0)} G_k \in D_{\text{dif}}^+$. Then, it follows that we have $\nabla^{(0)}(f_j) = n_j f_j$. On the other hand, we have $\{f_j\}_{j=1}^d$ forms a basis of $D_{\text{dif}}^+(V)$ over $K_{\inf}^{\lfloor \{[t]\}}$ and the lift $\{1 \otimes f_j\}_{j=1}^d$ of $\{f_j\}_{j=1}^d$ in $X$ forms a basis of $X$ over $K_{\inf}^{\lfloor \{[t, t_1, \ldots, t_e]\}}$. Furthermore, since $\{f_j\}_{j=1}^d$ are elements of $D_{e, \text{dif}}^+(V)$, it follows that $\{f_j\}_{j=1}^d$ also forms a basis of $D_{e, \text{dif}}^+(V)$ over $K_{\inf}^{\lfloor \{[t]\}}$. Thus, it remains to show that $\{1 \otimes f_j\}_{j=1}^d$ forms a basis of $D_{e, \text{dif}}^+(V)$ over $K_{\inf}^{\lfloor \{[t, t_1, \ldots, t_e]\}}$. Put $X_r = X/(t, t_1, \ldots, t_e)^r X$. Let $Y_r$ denote the $K_{\inf}^{\lfloor \{[t, t_1, \ldots, t_e]\}}$-submodule of $X_r$ generated by the finite set $\{\sum_{k=1}^d a_{j,k}^{(m_1, \ldots, m_d)} G_k\}_{j=1}^{\lfloor \{[t, t_1, \ldots, t_e]\}}(m_1 + \cdots + m_d < r} \subset (B_{\text{dR}, K_{ur}}^+ \otimes_{Q_p} V)^H$. Then, it follows that this finitely generated $K_{\inf}^{\lfloor \{[t, t_1, \ldots, t_e]\}}$-module $Y_r$ is stable under the action of $\Gamma_K$ by (3.1) and thus is contained in $D_{e, \text{dif}}^+(V)$ by definition. On the other hand, $Y_r$ contains the elements $\{1 \otimes f_j\}_{j=1}^d$ which are linearly independent over $K_{\inf}^{\lfloor \{[t, t_1, \ldots, t_e]\}}/(t, t_1, \ldots, t_e)^r$. Thus, both of $Y_r$ and $D_{e, \text{dif}}^+(V)$ have the same dimension over $K_{\inf}^{\lfloor \{[t, t_1, \ldots, t_e]\}}$ and we get the equality $Y_r = D_{e, \text{dif}}^+(V)$. Therefore, by taking
the projective limit with respect to $r$, we conclude that \( \{1 \otimes f_j\}_{j=1}^d \subset \lim_{\to r} Y_r \) forms a basis of \( D^{+}_{\text{dR}}(V) \) over \( K_{\text{pf}}[[t, t_1, \ldots, t_e]] \). \hfill \Box

**Lemma 3.3.** By restricting \( \nabla^{(i)} : D^+_{\text{dR}}(V) \to D^+_{\text{dR}}(V) \) \((0 \leq i \leq e)\), we obtain \( \nabla^{(i)} : D^+_{\text{dR}}(V) \cap (B^+_{\text{dR}, K^p} \otimes_{Q_p} V)^H \to D^+_{\text{dR}}(V) \cap (B^+_{\text{dR,K^p}} \otimes_{Q_p} V)^H \) in \( (B^+_{\text{dR,K^p}} \otimes_{Q_p} V)^H \).

**Proof.** For simplicity, put \( L_{\text{dR}}^i = (B^+_{\text{dR,K^p}})^H, L_{\text{dR}}^i(V) = (B^+_{\text{dR,K^p}} \otimes_{Q_p} V)^H \) and \( Z = (B^+_{\text{dR,K^p}} \otimes_{Q_p} V)^H \). Let \( m_{\text{dR}} \) denote the maximal ideal \((t, t_1, \ldots, t_e)\) of \((B^+_{\text{dR,K^p}})^H\).

Then, we have
\[
Z = \lim_{\to r} Z, m_{\text{dR}}^k Z \supset L_{\text{dR}}^i(V) = \lim_{\to r} L_{\text{dR}}^i(V)/(m_{\text{dR}}^k Z \cap L_{\text{dR}}^i(V)) \cup D^+_{\text{dR}}(V) = \lim_{\to r} D^+_{\text{dR}}(V)/(m_{\text{dR}}^k Z \cap D^+_{\text{dR}}(V)).
\]

Define \( W \) as the \( L_{\text{dR}}^i \cap K_{\text{pf}}[[t, t_1, \ldots, t_e]] \)-submodule of \( Z \) generated by \( L_{\text{dR}}^i(V) \cap D^+_{\text{dR}}(V) \). If we put \( \tilde{W} = \lim_{\to r} W_r \) where \( W_r \) denotes \( W/(m_{\text{dR}}^k Z \cap W) \), we have \( L_{\text{dR}}^i(V) \supset \tilde{W} \) and \( D^+_{\text{dR}}(V) \supset \tilde{W} \). Thus, we obtain \( \tilde{W} = W \) by definition. Therefore, it suffices to show that \( W_r \) is stable under the actions of \( \{\nabla^{(i)}\}_{i=0}^e \). Fix a basis \( \{g_j\}_{j=1}^d \) of \( D^+_{\text{dR}}(V)/(m_{\text{dR}}^k Z \cap D^+_{\text{dR}}(V)) \) over \( K_{\text{pf}} \). Then, there exists a finite field extension \( L/K \) in \( K_{\text{pf}} \) such that \( \otimes_{L=1}^d g_j \) is stable by the action of \( \Gamma = G_{K} \cdot H = \text{Gal}(K_{\text{pf}}/K) \). Thus, there exists an open subgroup \( \Gamma_r \) of \( \Gamma_r \) such that, for all \( \gamma \in \Gamma_r \) (resp. \( \beta \in \Gamma_r \)), the action of \( \gamma \) (resp. \( \beta \)) on \( \otimes_{L=1}^d g_j \) is linear. Then, the series
\[
\log(\gamma) = -\sum_{k=1}^{\infty} \frac{(\gamma - 1)^k}{k} \quad \text{(resp. } \log(\beta) = -\sum_{k=1}^{\infty} \frac{(\beta - 1)^k}{k})
\]
converges to an endomorphism of \( \otimes_{L=1}^d L \cdot g_j \). These actions on \( \otimes_{L=1}^d L \cdot g_j \) can be extended to those on \( D^+_{\text{dR}}(V)/(m_{\text{dR}}^k Z \cap D^+_{\text{dR}}(V)) \) by \( K_{\text{pf}} \)-linearity. Since \( W_r \) is contained in \( D^+_{\text{dR}}(V)/(m_{\text{dR}}^k Z \cap D^+_{\text{dR}}(V)) \) and stable under the action of \( \Gamma_r \), it follows that \( W_r \) is equipped with actions of \( \nabla^{(i)} = \frac{\log(\gamma)}{\log(K^{(i)})} \) and \( \nabla^{(i)} = \frac{\log(\beta)}{\log(K^{(i)})} \). \hfill \Box

**3.2.2.** \( \mathcal{D}_{\text{cris,K}(V)} \) and \( \mathcal{D}_{\text{st,K}(V)} \). In this subsection, for simplicity, we shall denote \( B_{\text{cris,K}} = (B_{\text{cris,K}}^+)^H \) and \( D_{\text{cris,K}}^+ = (B_{\text{cris,K}}^+ \otimes_{Q_p} V)^H \) (resp. \( B_{\text{st,K}}^+ = (B_{\text{st,K}}^+)^H \) and \( D_{\text{st,K}}^+ = (B_{\text{st,K}}^+ \otimes_{Q_p} V)^H \)).

**Proposition 3.4.** (cf. Proposition 3.2.) Let \( V \) be a \( p \)-adic representation of \( G_K \). If \( V \) is a crystalline (resp. semi-stable) representation of \( G_{K^p} \), there exists a basis \( \{g_j\}_{j=1}^d \) of \( \mathcal{D}_{\text{cris,K}}(V) \) over \( B_{\text{cris,K}} \) (resp. \( \mathcal{D}_{\text{st,K}}(V) \) over \( B_{\text{st,K}} \)) such that

1. \( \{g_j\}_{j=1}^d \) forms a basis of \( D^+_{\text{dR}}(V)[1/t] \) over \( K_{\text{pf}}[[t, t_1, \ldots, t_e]][1/t] \),
2. \( \{g_j\}_{j=1}^d \) is contained in \( \text{Ker}(\nabla^{(0)})(\subset D^+_{\text{dR}}(V)[1/t]) \).

**Proof.** Since the semi-stable representation case is similar, we shall consider only the crystalline representation case. Since \( V \) is also a de Rham representation of \( G_{K^p} \), by Proposition 3.2, there exists a basis \( \{f_j\}_{j=1}^d \) of \( D^+_{\text{dR}}(V) \) over \( K_{\text{pf}}[[t]] \) such that (1) \( \{1 \otimes f_j\}_{j=1}^d \) forms a basis of \( D^+_{\text{dR}}(V) \) over \( K_{\text{pf}}[[t, t_1, \ldots, t_e]] \) and (2) \( \nabla^{(0)}(1 \otimes f_j) = n_j(1 \otimes f_j) \) with \( n_j \in \mathbb{Z} \). Then, since the action of \( \nabla^{(0)} \) on \( \{f_j t^{-n_j}\}_{j=1}^d \) is trivial and
\{f_j t^{-n_j} \}_{j=1}^d \text{ is contained in } D_{\text{diff}}^+(V)[1/t] \subset (B_{\text{dR}, K^{\text{ur}}} \otimes V)^H, \text{ there exists a finite field extension } L^{\text{pf}}/K^{\text{pf}} \text{ in } K^{\text{ur}} \text{ such that } \{f_j t^{-n_j} \}_{j=1}^d \text{ forms a basis of } D_{\text{dR}, L^{\text{pf}}}(V) \text{ over } L^{\text{pf}}. \\
If } K = K_0(\alpha) \text{ and } L^{\text{pf}} = K^{\text{pf}}(\beta) \text{ for some } \beta = \zeta_0 \in K^{\text{ur}}, \text{ there exists an element } a \in K^{\text{ur}} \text{ such that } K_0(\alpha, \beta) = K_0(a). \text{ Then, we have } L^{\text{pf}} = K_0^{\text{pf}}(\alpha, \beta) = L_0^{\text{pf}}(a). \text{ Since } V \text{ is also a crystalline representation of } G_{L^{\text{pf}}}, \text{ we have } D_{\text{dR}, L^{\text{pf}}}(V) = L_0^{\text{pf}}(a) \otimes_{L_0^{\text{pf}}} D_{\text{cris}, L^{\text{pf}}}(V). \text{ Thus, we can write }

\[ (3.2) \quad f_j t^{-n_j} = \sum_{k=0}^{\delta-1} a^k \otimes g_{jk} \quad (g_{jk} \in D_{\text{cris}, L^{\text{pf}}}(V), \ \delta = |L_0^{\text{pf}}(a) : L_0^{\text{pf}}|). \]

We can extract a basis of } D_{\text{cris}, L^{\text{pf}}}(V) \text{ over } L_0^{\text{pf}} \text{ from the family } \{g_{jk}\}_{j,k}: \text{ denote it by } \{g_{j,k}\}_{j,k}. \text{ Since we have } B_{\text{cris}, K^{\text{ur}}} \otimes_{L_0^{\text{pf}}} D_{\text{cris}, L^{\text{pf}}}(V) \simeq B_{\text{cris}, K^{\text{ur}}} \otimes_{Q_p} V, \text{ by taking the invariant part by } H, \text{ it follows that } \{g_{j,k}\}_{j,k} \text{ forms a basis of } \tilde{D}_{\text{cris}, K^{\text{ur}}}(V) \text{ over } \tilde{B}_{\text{cris}, K^{\text{ur}}}. \text{ Furthermore, by } (3.2), \text{ the action of } \nabla(0) \text{ on } \{g_{j,k}\}_{j,k} \text{ is trivial. Thus, it remains to show that } \{g_{j,k}\}_{j,k} \text{ forms a basis of } D_{\text{dR}, L^{\text{pf}}}(V)[1/t] \text{ over } K_0^{\text{pf}}[[t, t_1, \ldots, t_e]][[1/t]].\

First, let } Z_e \text{ denote the union of } K_0^{\text{pf}}[[t, t_1, \ldots, t_e]]-\text{submodules of finite type of } (B_{\text{dR}, K} \otimes_{Q_p} V)^H/(t, t_1, \ldots, t_e)^\infty((B_{\text{dR}, K} \otimes_{Q_p} V)^H \text{ that are stable under the action of an open subgroup } \Gamma \text{ of } \Gamma_K. \text{ Since we have the inclusion } D_{\text{dR}, L^{\text{pf}}}(V) \subset Z_e \text{ by definition and both sides have the same dimension over } K_0^{\text{pf}}, \text{ we have } D_{\text{dR}, L^{\text{pf}}}(V) = Z_e. \text{ Thus, by taking the projective limit with respect to } r, \text{ we obtain } D_{\text{dR}, L^{\text{pf}}}(V) = \lim_{\rightarrow \chi} Z_e. \text{ Choose integers } \{m_j\}_{1 \leq j \leq d, 0 \leq k \leq \delta-1} \subset Z \text{ such that we have }

\[ \{t^{m_j} a^k \otimes g_{jk}\}_{1 \leq j \leq d, 0 \leq k \leq \delta-1} \subset (B_{\text{dR}, K} \otimes_{Q_p} V)^H. \]

Let } Z \text{ denote the } K_0^{\text{pf}}[[t, t_1, \ldots, t_e]]-\text{submodule of } (B_{\text{dR}, K} \otimes_{Q_p} V)^H \text{ generated by the finite set } \{t^{m_j} a^k \otimes g_{jk}\}_{1 \leq j \leq d, 0 \leq k \leq \delta-1}. \text{ Take an open subgroup } \Gamma \text{ of } \Gamma_K \text{ such that the action of } \Gamma \text{ on the finite set } \{a^k \}_{k=0}^{\delta-1} \text{ is trivial. Then it follows from } (3.2) \text{ that this finitely generated } K_0^{\text{pf}}[[t, t_1, \ldots, t_e]]-\text{module } Z \text{ is stable under the action of } \Gamma \text{ and thus is contained in } D_{\text{dR}, L^{\text{pf}}}(V) \text{ by the preceding argument. In particular, it follows that the elements } \{g_{j,k}\}_{j,k} \text{ are contained in } D_{\text{dR}, L^{\text{pf}}}(V)[1/t]. \text{ Furthermore, since } \{g_{j,k}\}_{j,k} \text{ forms a basis of } B_{\text{dR}, K} \otimes_{Q_p} V \text{ over } B_{\text{dR}, K}, \text{ it is, in particular, linearly independent over } B_{\text{dR}, K} \text{ in } B_{\text{dR}, K} \otimes_{Q_p} V. \text{ Take } m_j \in Z \text{ such that we have }

\[ \{g_{j,k} = t^{m_j} a^j \}_{j,k} \subset D_{\text{dR}, L^{\text{pf}}}(V). \]

Let } L(V) \text{ be the submodule of } B_{\text{dR}, K} \otimes_{Q_p} V \text{ generated by } \{g_{j,k}\}_{j,k} \text{ over } B_{\text{dR}, K}, \text{ and let } \mathbb{D}(V) \text{ denote the union of } K_0^{\text{pf}}[[t, t_1, \ldots, t_e]]-\text{submodules of finite type of } L(V)^H \text{ stable under } \Gamma_K. \text{ Since } \{g_{j,k}\}_{j,k} \text{ forms a basis of } L(V) \text{ over } B_{\text{dR}, K}, \text{ it follows that } \{g_{j,k}\}_{j,k} \text{ also forms a basis of } \mathbb{D}(V) \text{ over } K_0^{\text{pf}}[[t, t_1, \ldots, t_e]] \text{ (see [A-B, Lemma 5.10]). For any element } x \in D_{\text{dR}, L^{\text{pf}}}(V)[1/t], \text{ one can see that there exists an integer } m \in Z \text{ such that we have } t^m x \in \mathbb{D}(V). \text{ Thus, } t^m x \text{ can be written as linear combinations of } \{g_{j,k}\}_{j,k} \text{ over } K_0^{\text{pf}}[[t, t_1, \ldots, t_e]][1/t]. \]

From now on, we shall keep the notation and assumptions of Proposition 3.4. The following result is proved in Proposition 3.5 and Corollary 3.6 of [M].
PROP 3.5. The action of \( \{\nabla^{(i)}\}_{i=1}^{e} \) on the basis \( \{g_j\}_{j=1}^{d} \) is given by
\[
\nabla^{(i)}(g_j) = \sum_{k=1}^{d} \frac{c_j(k_1, \ldots, k_d)}{k_1! \cdots k_d!} g_{k_1+\cdots+k_d} \mod (t^e, t^e),
\]
where \( c_j(k_1, \ldots, k_d) \) are elements of \( K^{[pd]}[[t, t_1, \ldots, t_d]] \) such that \( \nabla^{(0)}(g_j) \mod (t^e, t^e) = 0 \).

PROP 3.6. Let \( V \) be a \( p \)-adic representation of \( G_K \). If \( V \) is a crystalline (resp. semi-stable) representation of \( G_{K^{ur}} \), we have
\[
\nabla^{(i)}(g_1) \in \tilde{D}_{cris,K^{ur}}(V) \quad \text{(resp. } \in \tilde{D}_{st,K^{ur}}(V))
\]
for all \( (k_i)_{1 \leq i \leq e} \in \mathbb{N}^e \) and \( 1 \leq j \leq d \).

Proof. Since the semi-stable representation case is similar, we shall consider only the crystalline representation case. It is enough to prove that if \( g \in D_{c-diff}(V)[1/t] \) is such that \( g \in \tilde{D}_{cris,K^{ur}}(V) \) and \( \nabla^{(0)}(g) = 0 \), then \( \nabla^{(i)}(g_1) \in \tilde{D}_{cris,K^{ur}}(V) \) for all \( (k_i)_{1 \leq i \leq e} \in \mathbb{N}^e \). Since the proof of the general case is exactly the same (only with heavier notations), we just show that \( \nabla^{(i)}(g) \in \tilde{D}_{cris,K^{ur}}(V) \) for \( 1 \leq i \leq e \). First, for \( r \in \mathbb{N}_{>0} \) and \( h \in D_{c-diff}(V) \), there exists an open subgroup \( \Gamma_{\ell,r} \) of \( \Gamma_1 \) such that we have \( \beta_i(h) = \exp(c_i(b_i)\nabla^{(i)})(h) \mod (t, t, \ldots, t) \). Thus, if we take \( M \in \mathbb{N} \) such that \( L^M g \in D_{c-diff}(V) \), we obtain
\[
\beta_i(t^M g) = t^M g + \sum_{j=0}^{L} \frac{c_i(b_i)^j}{j!} (t^M g) + \sum_{j=0}^{L} \frac{c_i(b_i)^j}{j!} (t^M g) + \cdots
\]
mod \( (t, t_1, \ldots, t_d) D_{c-diff}(V) \) for all \( \beta_i \in \Gamma_{\ell,r} \). Note that this series is a finite sum mod \( (t, t_1, \ldots, t_d) D_{c-diff}(V) \) by Proposition 3.5. Thus, there exists \( L \in \mathbb{N} \) such that we have \( \nabla^{(i)}(L^M g) \neq 0 \) and \( \nabla^{(i)}(L+1)^M g \) \( \mod (t, t_1, \ldots, t_d) D_{c-diff}(V) \). On the other hand, since we have \( \nabla^{(i)}(g) \in (B_{\text{diff},K^{ur}} \otimes_{\mathbb{Q}_p} V)^{L} \) by Lemma 3.3 and \( \nabla^{(0)}(L^M g) \neq 0 \) by Proposition 3.5, there exists a finite field extension \( M^{pd}/K^{pd} \) in \( K^{pd} \) such that \( \nabla^{(i)}(g) \in (B_{\text{diff},K^{ur}} \otimes_{\mathbb{Q}_p} V)^{L} \). Write \( M^{pd} = M^{pd}_0 \). Then, since \( V \) is also a crystalline representation of \( G_{M^{pd}} \), we have the equality \( D_{\text{diff},M^{pd}}(V) = M^{pd}_0 \otimes_{M^{pd}} D_{\text{cris},M^{pd}}(V) \). Thus, we can write
\[
1/L \nabla^{(i)}(g) = \sum m_n b_m \otimes a_{ijmn} g_n \quad (a_{ijmn} \in \tilde{B}_{cris,K^{ur}}).
\]
By (3.3) and (3.4), we obtain
\[
\beta_i(t^M g) = t^M g \sum m_n \sum_{j=0}^{L} \frac{c_i(b_i)^j}{j!} a_{ijmn} t^j g_n \quad (\mod (t, t_1, \ldots, t_d)).
\]

On the other hand, since \( \tilde{D}_{cris,K^{ur}}(V) \) is stable under the action of \( \Gamma^{t^M g,\ell} \) and \( \{b_m \}_{m=0}^{\delta} \) (\( \delta = [M^{pd} : M^{pd}_0] \)) is linearly independent over \( \tilde{D}_{cris,K^{ur}} \), the terms of the RHS of (3.5) have to be 0 for \( m \neq 0 \). Then, for \( m \neq 0 \), we have \( \sum_{j=0}^{L} \lambda^j a_{ijmn} t^j = 0 \) for \( \lambda \) in an open subgroup of \( \mathbb{Z}_p \): this implies that \( a_{ijmn} = 0 \) for \( m \neq 0 \). In particular, we obtain \( \nabla^{(i)}(g) \in \tilde{D}_{cris,K^{ur}}(V) (i \neq 0) \) by (3.4).
4. Proof of the main theorem. In this section, we will give proofs only in the crystalline representation case since the semi-stable representation case is similar.

Proposition 4.1. We have the following implications.

1. If $V$ is a crystalline representation of $G_K$, then it is a crystalline representation of $G_{K^{p^e}}$.

2. If $V$ is a semi-stable representation of $G_K$, then it is a semi-stable representation of $G_{K^{p^e}}$.

Proof. Since $V$ is a crystalline representation of $G_K$, there exists a $G_K$-equivariant isomorphism of $B_{\text{cris}, K^p}$-modules

\[(4.1) \quad B_{\text{cris}, K} \otimes_{Q_p} V \simeq (B_{\text{cris}, K})^d \quad (d = \dim_{Q_p} V).\]

By tensoring (4.1) by $B_{\text{cris}, K^{p^e}}$ over $B_{\text{cris}, K}$ (which is induced by the $G_{K^{p^e}}$-equivariant surjection $p : B_{\text{cris}, K} \twoheadrightarrow B_{\text{cris}, K^{p^e}} : t_i \mapsto 0$), we obtain a $G_{K^{p^e}}$-equivariant isomorphism

\[B_{\text{cris}, K^{p^e}} \otimes_{Q_p} V \simeq (B_{\text{cris}, K^{p^e}})^d.\]

This means that $V$ is a crystalline representation of $G_{K^{p^e}}$. \[\]

Proof of Theorem 1.1. It remains to show that, if $V$ is a $p$-adic representation of $G_K$ whose restriction to $G_{K^{p^e}}$ is crystalline, then $V$ is a potentially crystalline representation of $G_K$. Since $V$ is a crystalline representation of $G_{K^{p^e}}$, there exists a basis $\{g_i\}_{i=1}^d$ of $\hat{B}_{\text{cris}, K^{p^e}}(V)$ over $B_{\text{cris}, K^{p^e}}$ which satisfies the properties in Proposition 3.4. From this $\{g_i\}_{i=1}^d$, for all finite extension $L/K$ in $\mathcal{K}$, we shall construct $L_0^{p^e}$-linearly independent elements $\{f_j\}_{j=1}^d$ in $B_{\text{cris}, K} \otimes_{Q_p} V$ such that $\nabla^{(i)}(f_j) = 0 \ (0 \leq \forall i \leq e)$ and $1 \leq \forall j \leq d$.

(A) Construction of $\{f_j\}_{j=1}^d$ in $B_{\text{cris}, K} \otimes_{Q_p} V$. By Propositions 3.5 and 3.6, we have $(\nabla^{(1)})^{k_1} \cdots (\nabla^{(e)})^{k_e}(g_j) = t_1^{k_1} \cdots t_e^{k_e} \sum_{j=1}^d c_{j,k} g_j$ where the $c_{j,k} (k = (k_1, \ldots, k_e))$ are elements of $B_{\text{cris}, K^{p^e}}$ such that $\nabla^{(0)}(c_{j,k}) = 0$. On the other hand, for $N \in \mathbb{N}$, we obtain,

\[(4.2) \quad \varphi^{N+1}((\nabla^{(1)})^{k_1} \cdots (\nabla^{(e)})^{k_e}(g_j)) = (pt)^{k_1 + \cdots + k_e} \sum_{i=1}^d p^{N(k_1 + \cdots + k_e)} \varphi^{N+1}(c_{j,k} g_j)\]

where the $\varphi^{N+1}(c_{j,k})$ are elements of $B_{\text{cris}, K^{p^e}}$ such that $\nabla^{(0)}(\varphi^{N+1}(c_{j,k})) = 0$. Let $U_i$ denote the matrix which represents the action of $\nabla^{(i)}/t$ ($1 \leq i \leq e$) with respect to the basis $\{g_j\}_{j=1}^d$ and take $N$ large enough such that we have $p^N U_i \in M_d(A_{\text{cris}, K^{p^e}})$ for all $1 \leq i \leq e$. On the other hand, by applying the same method as in Proposition 3.6 to the entries of $U_i$, we have $\sum_{t} (p^N U_i) \in M_d(\hat{B}_{\text{cris}, K^{p^e}})$. Since we have $\nabla^{(0)}(\sum_{t} (p^N U_i)) = 0$, this means that we obtain $\sum_{t} (p^N U_i) \in M_d(L_0^{p^e})$ for a finite extension $L/K$ in $\mathcal{K}$ and, in particular, $\sum_{t} (p^N U_i) \in M_d(B_{\text{cris}, K^{p^e}})$ ($1 \leq i, j \leq e$). Furthermore, since $\nabla^{(j)}$ is the form $t \frac{d}{dt}$ on $K_\infty^{(p)}[[t, t_1, \ldots, t_e]]$ and $\sum_{t} = \frac{d}{dt}$ does not decrease the $p$-adic valuation of an element of $K_\infty^{(p)}[[t, t_1, \ldots, t_e]] \cap A_{\text{cris}, K^{p^e}} (\subset B_{\text{DR}, K})$, we obtain $\sum_{t} (p^N U_i) \in M_d(A_{\text{cris}, K^{p^e}})$ ($1 \leq i, j \leq e$). Thus, it follows that we
have \( p^{N(k_1+\cdots+k_
u)}c_{ij\ell} \in A_{\text{cris},K^p} \) and \( p^{N(k_1+\cdots+k_
u)}\varphi^{N+1}(c_{ij\ell}) \in A_{\text{cris},K^p} \). Define \( \{f_j\}^d_{j=1} \subset B_{\text{cris},K} \otimes_{Q_p} V \) by

\[
f_j = \sum_{0 \leq k_1, \ldots, k_d} (-1)^{k_1+\cdots+k_d} \frac{t_{i_1}^{k_1} \cdots t_{i_e}^{k_e}}{k_1! \cdots k_d!} \varphi^{N+1}(\nabla^{(1)}k_1, \ldots, \nabla^{(e)}k_e)(g_j)
\]

where \( t_i = \log([b_i]/b_i) \) denotes the element of \( \ker(\theta_{K_{0,\ell}}) \subset A_{\text{cris},K} \). Note that this series converges in \( B_{\text{cris},K} \otimes_{Q_p} V \) for the \( p \)-adic topology by (4.2) and thus \( f_j \) actually defines an element of \( B_{\text{cris},K} \otimes_{Q_p} V \). Then, it is easy to verify that we have \( \nabla^{(i)}(f_j) = 0 \) for all \( 1 \leq i \leq e \) and \( 1 \leq j \leq d \) by using the Leibniz rule. Furthermore, by using (4.2) and the fact \( \nabla^{(0)}(\varphi^{N+1}(g_j)) = 0 \), we can deduce that we have \( \nabla^{(0)}(f_j) = 0 \) for all \( 1 \leq j \leq d \).

(B) \( \{f_j\}_{j=1}^d \subset B_{\text{cris},K} \otimes_{Q_p} V \) is linearly independent over \( L_0^{\text{pf}} \). The homomorphism \( p : B_{\text{cris},K} \to B_{\text{cris},K^{pf}} \) induces

\[
B_{\text{cris},K} \otimes_{Q_p} V \to B_{\text{cris},K^{pf}} \otimes_{Q_p} V : f_j \mapsto \varphi^{N+1}(g_j)
\]

Since \( \{g_j\}_{j=1}^d \) forms a basis of \( \tilde{D}_{\text{cris},K^{pf}}(V) \) over \( B_{\text{cris},K^{pf}} \) and satisfies \( \nabla^{(0)}(g_j) = 0 \), there exists a finite field extension \( M/K \) in \( \bar{K} \) such that \( \{g_j\}_{j=1}^d \) forms a basis of \( D_{\text{cris},M^{pf}}(V) \) over \( M_0^{pf} \). Furthermore, since \( \varphi : D_{\text{cris},M^{pf}}(V) \to D_{\text{cris},M^{pf}}(V) \) is bijective, \( \varphi^{N+1}(g_j)_{j=1}^d \) also forms a basis of \( D_{\text{cris},M^{pf}}(V) \) over \( M_0^{pf} \). Thus, it follows that \( \{f_j\}_{j=1}^d \) is linearly independent over \( L_0^{\text{pf}} \) in \( B_{\text{cris},K} \otimes_{Q_p} V \) for all finite extension \( L/K \) in \( \bar{K} \).

(C) Conclusion. The maps \( \log(\gamma) \) and \( \log(\beta_i) \) (1 \( \leq i \leq e \)) act trivially on the \( K_0 \)-vector space generated by \( \{f_j\}_{j=1}^d \) (because \( \nabla^{(0)}(f_j) = 0 \) and \( \nabla^{(i)}(f_j) = 0 \) for all \( 1 \leq i \leq e \) and \( 1 \leq j \leq d \)). This means that \( \Gamma_K \) acts on this \( K_0 \)-vector space via finite quotient and there exists a finite field extension \( L/K \) in \( \bar{K} \) such that \( \{f_j\}_{j=1}^d \) forms a basis of \( D_{\text{cris},L}(V) \) over \( L_0 \subset L_0^{\text{pf}} \). \( \square \)

Remark 4.2. Since the proof is carried out by using the differential operators, it is not obvious whether we can get rid of the potentiality from the statement of the main theorem.

5. The \( p \)-adic monodromy theorem of Fontaine in the imperfect residue field case. In this section, we generalize the \( p \)-adic monodromy theorem of Fontaine to the imperfect residue field case. Now, we recall the results of [Be] and [M].

Theorem 5.1. [Be, Corollary 5.22.] Let \( L \) be a complete discrete valuation field of characteristic 0 with perfect residue field \( k \) of characteristic 0 and \( V \) be a \( p \)-adic representation of \( G_L \). Then, \( V \) is a de Rham representation of \( G_L \) if and only if \( V \) is an admissible semi-stable representation of \( G_L \).

Theorem 5.2. [M, Theorem 4.10.] Let \( K \) be a complete discrete valuation field of characteristic 0 with residue field \( k \) of characteristic 0 such that \([k:k^p] = p^s < \infty\) and \( V \) be a \( p \)-adic representation of \( G_K \). Let \( K^{pf} \) be the field extension of \( K \) defined as before. Then, \( V \) is a de Rham representation of \( G_K \) if and only if \( V \) is a de Rham representation of \( G_{K^{pf}} \).
Since $K^{pf}$ has perfect residue field, we can apply Theorem 5.1 to the restriction of $V$ to $G_{K^{pf}}$.

**Proof of Corollary 1.2.** $V$ is a de Rham representation of $G_K$ if and only if $V$ is a de Rham representation of $G_{K^{pf}}$ by Theorem 5.2. Next, $V$ is a de Rham representation of $G_{K^{pf}}$ if and only if $V$ is a potentially semi-stable representation of $G_{K^{pf}}$ by Theorem 5.1. Finally, $V$ is a potentially semi-stable representation of $G_{K^{pf}}$ if and only if $V$ is a potentially semi-stable representation of $G_K$ by Theorem 1.1. 

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