Matching heavy-light currents with NRQCD and HISQ quarks

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Outline

- B physics, CKM unitarity and lattice QCD
- HPQCD’s HISQ-NRQCD perturbative matching programme
- Calculating $f_B$, $f_{B_s}$: the need for a new matching calculation
- Matching HISQ and NRQCD: automated LPT
- $f_B$ and $f_{B_s}$ results
  - [H. Na, Weak Decays & Matrix Elements session, Mon. 14:50]
- Extension to mixed action heavy-heavy currents
- Summary
B physics, CKM unitarity and lattice QCD

$|V_{ub}|$ from

- Semileptonic decays, $B \rightarrow \pi \ell \nu$
  \[
  \frac{d\Gamma}{dq^2} \propto |V_{ub}|^2 |f_+(q^2)|^2
  \]

- Leptonic decays, $B \rightarrow \ell \nu$
  \[
  \Gamma \propto |V_{ub}|^2 f_B^2
  \]

\[
V = \begin{pmatrix}
  |V_{ud}| & |V_{us}| & |V_{ub}| \\
  |V_{cd}| & |V_{cs}| & |V_{cb}| \\
  |V_{td}| & |V_{ts}| & |V_{tb}|
\end{pmatrix}
\]

\[
\langle 0 | A_\mu | B(p) \rangle = i f_B p^\mu
\]

overconstrain parameters $\rightarrow$ tensions $\Rightarrow$ new physics?
HPQCD’s HISQ-NRQCD matching programme

*B* physics HISQ-NRQCD perturbative matching programme:

1. **massless HISQ-NRQCD**
   - \(B(s)\) leptonic decays: \(f_B\) and \(f_{B_s}\)
     - [H. Na, Weak Decays & Matrix Elements session, Mon. 14:50]
     - match to \(\mathcal{O}(\alpha_s, \alpha_s a \Lambda_{\text{QCD}}, \alpha_s/(a M_b), \alpha_s \Lambda_{\text{QCD}}/M_b)\)
   - \(B(s)\) semileptonic decays
     - [C. Bouchard, Weak Decays & Matrix Elements session, Wed. 08:50]
     - match to \(\mathcal{O}(\alpha_s, \alpha_s/(a M_b))\)

2. **massive HISQ-NRQCD**
   - HISQ renormalisation parameters: \(Z_M, Z_Q, \epsilon_1\)
   - scattering channel for \(B(s) \rightarrow D(s)\) semileptonic decays
   - annihilation channel for \(B_s^{(*)}\) decays
   - match to \(\mathcal{O}(\alpha_s, \alpha_s/(a M_b))\)

3. **HISQ-NRQCD four-fermion operator matching for** \(B(s)\) mixing
Calculating $f_B$ and $f_{Bs}$:
the need for a new matching calculation

$f_{Bq}$ parameterises decay $B_q \rightarrow \ell \nu$

$$\Gamma \propto |V_{ub}|^2 f_B^2$$

Previous results from HPQCD collaboration:

1. $f_{Bd}, f_{Bs}$: NRQCD $b$ and improved relativistic $d/s$ (ASQTex)
   - 2 lattice spacings
   - uncertainties of $6 - 7\%$ in $f_{Bd}, f_{Bs}$ and $\sim 2\%$ in $f_{Bs}/f_{Bd}$

2. preliminary studies with HISQ $b, s$
New results from HPQCD collaboration (since Lattice 2011):

- $f_{B_s}$: highly-improved relativistic $b, s$ (HISQ)  
  - 5 lattice spacings
  - $1/M_b$ expansion up to physical $b$ quark mass
  - uncertainties of $\sim 2\%$ in $f_{B_s}$

- $f_B$: ideally calculate with relativistic $b, s$  
  - but fine lattices and light masses $\Rightarrow$ expensive
  - (currently) more efficient to update NRQCD calculation
  - ASQTad $\rightarrow$ HISQ valence $u/d$ and $s$
  - taste-breaking discretisation errors reduced by factor of $\sim 3$
  - uncertainty in $f_B$ and $f_{B_s}$ largely cancels in $f_{B_s}/f_B$
  - calculate $f_B/f_{B_s}^{(NRQCD)} \times f_{B_s}^{(HISQ)}$
  - result has errors $\sim 2\%$

require new operator matching calculation
Operator matching

$f_{Bq}$ defined through

$$\langle 0 | A_\mu | B(p) \rangle = i f_B p^\mu$$

Simulations carried out with effective lattice operators

$$J_0^{(0)} = \Psi_q \Gamma_0 \Psi_Q$$
$$J_0^{(1)}(x) = -\frac{1}{2M_b} \Psi_q \Gamma_0 \gamma \cdot \nabla \Psi_Q$$
$$J_0^{(2)}(x) = -\frac{1}{2M_b} \Psi_q \gamma \cdot \nabla_0 \Gamma_0 \Psi_Q$$

Need to match lattice operators to continuum

match perturbatively $\Rightarrow$ Lattice perturbation theory (LPT)
Matching relation:

\[ \langle A_0 \rangle_{QCD} = (1 + \alpha_s \rho_0) \langle J_0^{(0)} \rangle + (1 + \alpha_s \rho_1) \langle \tilde{J}_0^{(1)} \rangle + \alpha_s \rho_2 \langle \tilde{J}_0^{(2)} \rangle \]

Use improved currents with better power law behaviour.

\[ \tilde{J}_0^{(i)} = J_0^{(i)} - \alpha_s \zeta_{10} J_0^{(0)} \]

Matching coefficients:

\[ \rho_0 = B_0 - \frac{1}{2} (Z_H + Z_q) - \zeta_{00} \]

\[ \rho_1 = B_1 - \frac{1}{2} (Z_H + Z_q) - Z_M - \zeta_{10} - \zeta_{01} \]

\[ \rho_2 = B_2 - \zeta_{02} - \zeta_{12} \]
Calculating matching coefficients

Diagrams for continuum $\langle A_0 \rangle_{QCD}$:

$\rho_0 = B_0 - \frac{1}{2} (Z_H + Z_q) - \zeta_{00}$

$\rho_1 = B_1 - \frac{1}{2} (Z_H + Z_q) - Z_M - \zeta_{01} - \zeta_{11}$

$\rho_2 = B_2 - \zeta_{02} - \zeta_{12}$
Diagrams for lattice $\langle J_0^{(i)} \rangle$:

\[ \rho_0 = B_0 - \frac{1}{2} (Z_H + Z_q) - \zeta_{00} \]
\[ \rho_1 = B_1 - \frac{1}{2} (Z_H + Z_q) - Z_M - \zeta_{01} - \zeta_{11} \]
\[ \rho_2 = B_2 - \zeta_{02} - \zeta_{12} \]
Independent determinations of lattice quantities:

1. automated lattice perturbation theory: HiPPy and HPsrc
   - HiPPy – python routines produce Feynman rules encoded as “vertex files” Hart, von Hippel, Horgan arXiv:0904.0375
   - HPsrc – Fortran 90 routines reconstruct diagrams and evaluate integrals with VEGAS

2. “by hand” calculation
   - Mathematica file handles Dirac algebra
   - Fortran suite extracts Feynman rules via iterated convolution
   - integrals evaluated numerically with VEGAS
Cross checks for the matching results

Consistency checks for the matching calculation

1. Both methods agree!
2. Reproduce expected infrared behaviour:
   - $Z_q$, $\zeta_{00}$ and $\zeta_{11}$ logarithmically divergent
   - fit to gluon mass regulator
   - control divergence with subtraction function
3. Run in Feynman and Landau gauges to confirm $Z_M$ and e.g. $\zeta_{10}$ gauge independent
4. Reproduce previous matching calculation results with massless ASQTad-NRQCD results in PRD 69 (2004) 074501

advantage of automated LPT: simply change input files
We find

\[ f_B = 0.191(9) \, \text{GeV} \quad \text{and} \quad f_{B_s} = 0.228(10) \, \text{GeV} \]

so

\[ \frac{f_{B_s}}{f_B} = 1.188(18) \]

Agreement with previous HPQCD HISQ result a non-trivial consistency check:

\[ f_{B_s}^{(\text{HISQ})} = 0.225(4) \, \text{GeV} \]

Combining NRQCD-HISQ ratio with HISQ \( f_{B_s}^{(\text{HISQ})} \)

\[ f_B = \frac{f_B}{f_{B_s}} \times f_{B_s}^{(\text{HISQ})} = 0.189(4) \, \text{GeV} \]
## Error budget

| Source                                      | $f_B$ | $f_{Bs}$ | $f_{Bs}/f_B$ |
|---------------------------------------------|-------|----------|--------------|
| statistical                                 | 1.2   | 0.6      | 1.0          |
| $\mathcal{O}(\alpha_s)$ operator matching  | 4.1   | 0.1      |              |
| relativistic                                | 1.0   | 0.0      |              |
| $r_1$ scale                                 | 1.1   | -        |              |
| continuum extrapolation                     | 0.9   | 0.9      |              |
| chiral extrapolation                        | 0.2   | 0.5      | 0.6          |
| mass tuning                                 | 0.2   | 0.1      | 0.2          |
| finite volume                               | 0.1   | 0.3      | 0.36         |
| **total**                                   | 4.7   | 4.4      | 1.6          |
$f_B$ and $f_{B_s}$ results

- HPQCD this work ’12
- HPQCD HISQ b ’11
- HPQCD NRQCD b ’09
- Fermilab/MILC ’11
- ETMC ’11 (Nf=2)

B Meson Decay Constant in MeV

B$_s$ Meson Decay Constant in MeV
$f_{B_s}/f_B$ and $f_B^{(\text{fit})}$ results

- HPQCD this work ’12
- HPQCD ’09
- Fermilab/MILC ’11
- ETMC ’11 (Nf=2)

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B Meson Decay Constant in MeV

$\overline{f}_B$ from Lattice QCD:
- HPQCD this work, ’12

$f_B^{\text{fit}}$ from global fits:
- (Lunghi and Soni ’11)
  - no [$\sin(2\beta)$] input
  - no [$B \to \tau,\nu$] input
Mixed action heavy-heavy currents

Massless HISQ action:
  crossing symmetry $\Rightarrow$ scattering and annihilation equivalent
Mixed action heavy-heavy currents

**Massless** HISQ action:
   crossing symmetry $\Rightarrow$ scattering and annihilation equivalent

**Massive** HISQ action:
   crossing symmetry $\Rightarrow$ scattering and annihilation equivalent
Mixed action heavy-heavy currents

**Massless HISQ action:**
- crossing symmetry $\Rightarrow$ scattering and annihilation equivalent

**Massive HISQ action:**
- crossing symmetry $\Rightarrow$ scattering and annihilation equivalent $\times$

Scattering channel a simple extension of massless case:
1. set quarks onshell: $(G_0^{\text{HISQ}})^{-1} = 0$ and $(i\not p + m)u(p) = 0$
2. update vertex files ...
3. ... then just rerun code!

Annihilation channel (slightly) more involved:
1. replace quarks with antiquarks
2. isolate linear infrared divergence with subtraction function

For both cases recalculate HISQ renormalisation parameters
Massive HISQ renormalisation parameters

Relate $m_Q$, $Z_Q$ to parameters in action via quark propagator.

Quark two-point function

\[ \langle \psi(t, p') \bar{\psi}(0, p) \rangle = (2\pi)^3 \delta(p - p') G(t, p), \]

Starting point: quark field creates single- and multi-particle states

\[ G(t, p) = Z_2(p) e^{-E(p)t} \Gamma_{\text{proj}} + \cdots. \]

Define:

- mass renormalisation $m_Q \equiv Z_m m_0 \equiv E(p = 0)$
- wavefunction renormalisation $Z_Q \equiv Z_2(p = 0)$

\[ G(t, 0) = Z_Q e^{-Et} \frac{1 + \gamma_0}{2} + \cdots. \]
Introduce self energy via

\[ G^{-1}(p) = G_0^{-1}(p) - \Sigma(p), \]

For the HISQ action

\[ G_0^{-1}(p) = \sum_\mu i \gamma_\mu \sin(p_\mu) \left( 1 + \frac{1+\epsilon}{6} (\sin p_\mu)^2 \right) + m_0 \]

Write one loop self energy as

\[ \Sigma^{(1)}(p) = \sum_\mu i \gamma_\mu \sin(p_\mu) \Sigma^{(a)}_\mu(p) + \Sigma^{(b)}(p) \]
At $p = (iE, 0)$ pole condition is

$$\sinh(E) \left(1 - \frac{1 + \epsilon}{6} \left(\sinh(E)\right)^2 - \alpha_s \Sigma_0^{(a)}\right) = m_0 - \alpha_s \Sigma^{(b)},$$

- Tree level

$$\epsilon = \epsilon_{\text{tree}} + \alpha_s \epsilon_1$$

$$E = m_{\text{tree}} + \alpha_s m_1$$

$$\sinh(m_{\text{tree}}) \left[1 - \frac{1 + \epsilon_{\text{tree}}}{6} \left(\sinh(m_{\text{tree}})\right)^2\right] = m_0.$$ 

Fix $\epsilon_{\text{tree}}$: set pole mass = kinetic mass

$$\epsilon_{\text{tree}} = -1 + \frac{2}{(\sinh(m_{\text{tree}}))^2} \left[2 - \sqrt{1 + \frac{3m_{\text{tree}}}{\cosh(m_{\text{tree}}) \sinh(m_{\text{tree}})}}\right].$$

Solve simultaneously for $m_{\text{tree}}$

- Repeat at one loop . . .
Match expressions for propagator

\[ \int_{-\pi/a}^{\pi/a} \frac{dp_0}{2\pi} e^{-ip_0 t} G(p_0, 0) = Z_Q e^{-Et} \frac{1 + \gamma_0}{2} + \ldots. \]

Re-express as

\[ -i \int \frac{dz}{2\pi} z^{t-1} G(z, 0) \equiv -i \int \frac{dz}{2\pi} \frac{g_1(z)}{g_2(z)} \]

\( Z_Q \) defined as residue at \( p = 0 \) or \( z = z_1 = e^{-E} \)

\[ \text{Res}_{z=z_1} G(z, 0) = z_1^t \frac{g_1(z_1)}{z_1 g_2'(z_1)} = e^{-Et} \frac{1 + \gamma_0}{2} f(E) \]

Compare equations \( \Rightarrow \quad z_Q = f(E) \)

\[ Z_Q^{-1} = \cosh(E) \left[ 1 - \frac{1 + \epsilon}{2} (\sinh(E))^2 \right] + i\alpha_s \frac{d}{dp_0} \left[ i \sin(p_0) \Sigma_0^{(a)} + \Sigma^{(b)} \right] \]

\[ \epsilon = \epsilon_{\text{tree}} + \alpha_s \epsilon_1 \]

\[ E = m_{\text{tree}} + \alpha_s m_1 \]
Summary

- HISQ-NRQCD operator matching for $f_B$ and $f_{B_s}$
  - combined results give uncertainties of $\sim 2\%$ for $f_B$
- $B$ physics HISQ-NRQCD perturbative matching programme:
  1. massless HISQ-NRQCD for $B_{(s)}$ decays ✓
  2. massive HISQ-NRQCD
    - new determination of HISQ renormalisation parameters ✓
    - scattering channel for $B_{(s)} \rightarrow D_{(s)}$ semileptonic decays ✓
    - annihilation channel for $B_{c(\ast)}$ decays
  3. HISQ-NRQCD four-fermion operator matching for $B_{(s)}$ mixing
Thank you!
Fits and correlators

- Delta function and Gaussian smearing used at both source and sink for meson correlators
- Random wall sources in operator-meson correlators
- Correlators fitted between
  - $t_{\text{min}} = 2 \sim 4$ and $t_{\text{max}} = 16$ on coarse ensembles
  - $t_{\text{min}} = 4 \sim 8$ and $t_{\text{max}} = 24$ on fine ensembles
- Bayesian multiexponential fits with $t_{\text{min}}$, $t_{\text{max}}$ fixed and no. exponentials increased until saturation in results
Chiral and lattice spacing fits

Fit to lattice spacing dependence as described by Rachel Dowdall, but include chiral fit.

- Fit to

\[ \Phi_q = f_{Bq} \sqrt{M_{Bq}} = \Phi_0 (1 + \delta f_q + [{\text{analytic}}]) (1 + [{\text{disc.}}]) \]

- \( \delta f_q \) includes chiral logs using one-loop \( \chi \)PT and lowest order in \( 1/M \)
- [{\text{analytic}}] - powers of \( m_{\text{val}}/m_c \) and \( m_{\text{sea}}/m_c \), with \( m_c \) scale chosen for convenience
- [{\text{disc.}}] - powers of \( (a/r_1)^2 \) with expansion coefficient functions of \( aM_b \) or \( am_q \)
ASQTad action correct to $O(a^2)$, strongly reduced $O(\alpha_S a^2)$ errors:

$$S_{\text{ASQTad}} = \sum_x \bar{\psi}(x) \left( \gamma^\mu \Delta_{\mu}^{\text{ASQTad}} + m \right) \psi(x)$$

where

$$\Delta_{\mu}^{\text{ASQTad}} = \Delta_{\mu}^F - \frac{1}{6} (\Delta_{\mu})^3.$$ 

$F$ indicates

$$U_\mu \rightarrow \mathcal{F}_\mu \tilde{U}_\mu = u^{-1}_0 \left[ \prod_{\nu \neq \mu} \left( 1 + \frac{\Delta_{\nu}^{(2)}}{4} \right)_{\text{symm}} - \sum_{\nu \neq \mu} \frac{(\Delta_{\nu})^2}{4} \right] U_\mu$$
HISQ action correct to $O(a^4)$, $O(\alpha_s a^2)$ with reduced taste=changing:

$$S_{\text{HISQ}} = \sum_x \bar{\psi}(x) \left( \gamma^\mu \Delta^\text{HISQ}_\mu + m \right) \psi(x)$$

where

$$\Delta^\text{HISQ}_\mu = \Delta_\mu \left[ \mathcal{F}^\text{HISQ}_\mu U_\mu(x) \right] - \frac{1 + \epsilon}{6} (\Delta_\mu)^3 \left[ U \mathcal{F}^\text{HISQ}_\mu U_\mu(x) \right].$$

and

$$\mathcal{F}^\text{HISQ}_\mu = \mathcal{F}_{\text{ASQTad}} U_\mu \mathcal{F}_{\text{ASQTad}}$$
Lattice NRQCD action

\[ S_{\text{NRQCD}} = \sum_{\mathbf{x}, \tau} \psi^+(\mathbf{x}, \tau) \left[ \psi(\mathbf{x}, \tau) - \kappa(\tau) \psi(\mathbf{x}, \tau - 1) \right] \]

with

\[ \kappa(\tau) = \left( 1 - \frac{\delta H}{2} \right) \left( 1 - \frac{H_0}{2n} \right)^n U_4^\dagger \left( 1 - \frac{H_0}{2n} \right)^n \left( 1 - \frac{\delta H}{2} \right) \]

- Link variable in temporal direction: \( U_4^\dagger \)
- Leading nonrelativistic kinetic energy: \( H_0 = -\Delta^{(2)}/2M \)
- Higher order terms in \( \delta H \):
  - Chromoelectric and chromomagnetic interactions
  - Leading relativistic kinetic energy correction
  - Discretisation error corrections
HiPPy generates Feynman rules, encoded as “vertex files”

To generate vertex files:

- Expand link variables

\[ U_{\mu>0}(x) = \exp \left( gA_{\mu} \left( x + \frac{\hat{\mu}}{2} \right) \right) = \sum_{r=0}^{\infty} \frac{1}{r!} \left( gA_{\mu} \left( x + \frac{\hat{\mu}}{2} \right) \right)^r \]

with \( U_{-\mu} \equiv U_{\mu}^\dagger(x - \hat{\mu}) \)

- Actions built from products of link variables - Wilson lines

\[
L(x, y; U) = \sum_r \left( \frac{g^r}{r!} \right) \sum_{k_1, \mu_1, a_1} \cdots \sum_{k_r, \mu_r, a_r} \tilde{A}_{\mu_1}^{a_1}(k_1) \cdots \tilde{A}_{\mu_r}^{a_r}(k_r) \times V_r(k_1, \mu_1, a_1; \ldots; k_r, \mu_r, a_r)
\]

where the \( V_r \) are “vertex functions”
• Vertex functions decomposed into colour structure matrix, $C_r$ and "reduced vertex", $Y_r$

$$V_r(k_1, \mu_1, a_1; \ldots; k_r, \mu_r, a_r) = C_r(a_1; \ldots; a_r) Y_r(k_1, \mu_1; \ldots; k_r, \mu_r)$$

• Reduced vertices are products of exponentials

$$Y_r(k_1, \mu_1; \ldots; k_r, \mu_r) = \sum_{n=1}^{n_r} f_n \exp \left( \frac{i}{2} \left( k_1 \cdot v_1^{(n)} + \cdots + k_r \cdot v_r^{(n)} \right) \right)$$

where the $f_n$ are amplitudes and the $v^{(n)}$ the locations of each of the $r$ factors of the gauge potential

• Feynman rules encoded as ordered lists

$$E = (\mu_1, \ldots, \mu_r; x, y; v_1, \ldots, v_r; f)$$
For example, the product of two links, \( L(0, 2x, U) = U_x(0)U_x(x) \), is

\[
U_x(0)U_x(x) = \left[ \sum_{r_1=0}^{\infty} \frac{1}{r_1!} \left( gA_x \left( \frac{x}{2} \right) \right)^{r_1} \right] \left[ \sum_{r_2=0}^{\infty} \frac{1}{r_2!} \left( gA_x \left( \frac{3x}{2} \right) \right)^{r_2} \right]
\]

\[
= 1 + g \sum_{k_1} \tilde{A}_x(k_1)e^{ik_1 \cdot x/2} + g \sum_{k_2} \tilde{A}_x(k_2)e^{i2k_1 \cdot 3x/2} + \ldots
\]

\[
= 1 + g \sum_{k_1} \sum_{a_1} \tilde{A}_x^{a_1}(k_1) T^{a_1} \left( e^{ik_1 \cdot x/2} + e^{ik_1 \cdot 3x/2} \right)
\]

Vertex function

\[
V_1(k_1, x, a_1) \equiv C_1(a_1) Y_1(k_1, x) = T^{a_1} \left( e^{ik_1 \cdot x/2} + e^{ik_1 \cdot 3x/2} \right)
\]

Reduced vertex

\[ Y_1(k_1, x) = \left( e^{ik_1 \cdot x/2} + e^{ik_1 \cdot 3x/2} \right) \]
Reduced vertex

\[ Y_1(k_1, x) = \sum_{n=1}^{n_1=2} f_n \exp \left( \frac{i}{2} \left( k_1 \cdot v_1^{(n)} \right) \right) \]

So in this case

\[ f_1 = f_2 = 1; \quad v_1^{(1)} = (1, 0, 0, 0), \quad v_1^{(2)} = (3, 0, 0, 0) \]

We store this information as the list

\[ E = (\mu_1; x, y; v_1^{(1)}, v_1^{(2)}; f) \]
\[ = (x; (0, 0, 0, 0), (2, 0, 0, 0); (1, 0, 0, 0), (3, 0, 0, 0); (1, 1)) \]