Quantum signal splitting that avoids initialization of the targets

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ABSTRACT

The classical signal splitting and copying are not possible in quantum mechanics. Specifically, one cannot copy the basis up and down states of the input ($I$) two-state system (qubit, spin) into the copy ($C$) and duplicate-copy ($D$) two-state systems if the latter systems are initially in an arbitrary state. We consider instead a quantum evolution in which the basis states of $I$ at time $t$ are duplicated in at least two of the systems $I, C, D$, at time $t + \Delta t$. In essence, the restriction on the initial target states is exchanged for uncertainty as to which two of the three qubits retain copies of the initial source state.

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1. Introduction and Definition of the Model

The “classical” signal-copying process starts from the input value $I$ and after some time $\Delta t$ results in the same value at the copy $C$ and, if needed, duplicate-copy $D$. We assume that the value of $I$ is unchanged. This is the case when a signal is copied, for instance, by connecting wires and forcing the voltage in one of them to the value 0 or 1. This input-wire voltage, and the equilibrium state, will be established in all the connected wires, after a time $\Delta t$ determined by the relaxation processes of the charge-carrier distribution in the wires. The important point to note is that this “classical” copying/duplicating of a signal is not governed by reversible dynamics; there are inevitably some irreversible dissipation processes involved.

Quantum-mechanical copying from $I$ to $C$, and more complicated, multi-copy processes, have been discussed in the literature [1-6]. Generally, one cannot copy an arbitrary quantum state. However, one can duplicate a set of basis states of $I$, for instance, the qubit states up and down ($|1\rangle$ and $|0\rangle$). One can also discuss an approximate, optimized copying of a general linear combination of the basis states of $I$ [3-5]. A added limitation of these copying procedures has been that the initial state of $C$ (or more generally, of the systems which are imprinted with the
copies) must be *fixed*. This feature makes it unlikely that any interesting interference effects will be involved in the copying process.

Here we propose to explore those quantum-mechanical processes that do not involve any restriction on the initial state of the target system(s), even though the property of making copies will be meaningful only for the basis states of the input system \( I \). If we require that the basis states of \( I \) at time \( t \) be copied in such a way that both \( I \) and \( C \), and if needed, another copy \( D \), are all in that basis state at time \( t + \Delta t \) for an arbitrary initial state of \( C \) (and \( D \)), then one can easily verify that no unitary transformation can accomplish the desired mapping. Such quantum copying is not possible.

Our proposal is to consider instead the process in which an initial state of \( I \), from the basis set \(|1\rangle, |0\rangle\), is duplicated in *at least two* of the three final states \( I, C, D \). Thus, we consider three two-state systems. The initial state of \( I \), as long as it is one of the qubit states, will be “multiplied” in such a way that at time \( t + \Delta t \) two or three of the systems \( I, C, D \), are in that state, but we do not know if it is two or three, and in the case of two, which two are in that state. A unitary quantum evolution is possible that satisfies these conditions; we provide an explicit example. We note that the same unitary operator will also “evolve” an arbitrary linear combination of the basis states of \( I \). However, the resulting state
does not involve any exact copies of that linear combination.

Quantum copying has applications in quantum cryptography and signal transmission—a field in which presently theoretical and first experimental results are available [7-23]. It can also find uses in quantum computing, reviewed, e.g., in [24-33]. These fields deal with quantum dynamical processes that involve “binary” states constructed from the up and down states of two-state systems (qubits), such as photon polarization states or spin-$\frac{1}{2}$ quantum states. We will use the terms “qubit” or “spin.” Study of coherent quantum evolution is also of great “basic science” value.

The outline of the rest of this work is as follows. In the rest of this section, we define our “blind fanout” copying model. In Section 2, an explicit Hamiltonian is derived for the three-qubit system involved in the process. It turns out that the Hamiltonian involves three-spin interactions. Therefore, in Section 3, we also derive a reduction of the copying process in terms of a sequence of two-spin and one-spin “gates” in a formulation popular in the quantum-computing literature [24-27]. These gates must be applied in sequence by switching the interactions on and off. Sections 3 also includes a summarizing discussion of our results.

Let us label the states of the combined system $I + C + D$ by $|111\rangle$, $|110\rangle$, $|101\rangle$, $|100\rangle$, $|011\rangle$, $|010\rangle$, $|001\rangle$, $|000\rangle$, where the order of the sys-
tems is $|ICD\rangle$. One can then argue that unitary $8 \times 8$ matrices can be found that accomplish the desired transformation. The requirement is that any linear combination of the states $|1CD\rangle$ is mapped onto a linear combination of $|111\rangle$, $|110\rangle$, $|101\rangle$ and $|011\rangle$, while any linear combination of the states $|0CD\rangle$ is mapped onto a linear combination of $|100\rangle$, $|010\rangle$, $|001\rangle$ and $|000\rangle$. The general unitary transformation actually has many free parameters; it is by no means limited or special. Many different quantum evolutions accomplish the task.

For our explicit calculations we choose the simplest root to the desired copying: we consider a unitary transformation that flips (and possibly changes phases of) the basis states only in the subspace of $|100\rangle$, $|011\rangle$. The $8 \times 8$ unitary evolution matrix $U$ can then be represented as follows:

$$U = \begin{pmatrix} I_{3 \times 3} & U_{2 \times 2} \\ I_{3 \times 3} & I_{3 \times 3} \end{pmatrix}.$$  \hspace{1cm} (1)

Here $I$ are unit matrices. The subscripts indicate matrix dimensions while all the undisplayed elements are zero. The most general form of the matrix $U$ is

$$U = \begin{pmatrix} 0 & e^{i\beta} \\ e^{i\alpha} & 0 \end{pmatrix}.$$  \hspace{1cm} (2)
2. Derivation of the Hamiltonian

Our aim is to calculate the Hamiltonian $H$ according to

$$U = e^{-iH\Delta t/\hbar}.$$  \hfill (3)

We adopt the usual approach in the quantum-computing literature [24-32] of assuming that the (constant) Hamiltonian $H$ “acts” during the time interval $\Delta t$, i.e., we only consider evolution from $t$ to $t + \Delta t$. The dynamics can be externally timed, with $H$ being switched on at $t$ and off at $t + \Delta t$. The time interval $\Delta t$ is then related to the strength of couplings in $H$ which are of order $\hbar/\Delta t$. One can replace the constant Hamiltonian $H$ by $f(t)H$ provided the shape or “protocol” function $f(t)$ averages to 1 over the time interval $\Delta t$. This allows for a smoother time dependence [33] without the need to introduce time-ordering in (3).

To obtain an expression for $H$, we calculate the “logarithm” of $U$ in its diagonal representation. One can verify that the diagonalizing matrix $T$, such that $T^\dagger U T$ is diagonal, is of the same structure as $U$ in (1), with the nontrivial part $U$ replaced by $T$, where
\[ T = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\beta/2} & e^{i\beta/2} \\ e^{i\alpha/2} & -e^{i\alpha/2} \end{pmatrix}. \] (4)

In the diagonal representation, the Hamiltonian is the diagonal $8 \times 8$ matrix $-\hbar A/\Delta t$, where $A$ has diagonal elements $2\pi N_1$, $2\pi N_2$, $2\pi N_3$, $\frac{1}{2}(\alpha + \beta) + 2\pi N_4$, $\frac{1}{2}(\alpha + \beta) + \pi + 2\pi N_5$, $2\pi N_6$, $2\pi N_7$, $2\pi N_8$. Here $N_j$ are arbitrary integers.

The Hamiltonian is then obtained as $H = -\hbar T A T^\dagger /\Delta t$, and it depends on the two (real) parameters $\alpha$ and $\beta$ and on the integers $N_j$. We restrict the number of parameters to obtain a specific example. In fact, we seek a Hamiltonian with few energy gaps [33]. However, we would also like to have a symmetric energy level structure. The following choice leads to a particularly elegant result for $H$. We put $N_j = 0$ for $j = 1, 2, 3, 6, 7, 8$, and also $\alpha + \beta + \pi + 2\pi (N_4 + N_5) = 0$ and $N_5 - N_4 = N$. This corresponds to the following energies: $E_{1,2,3} = 0$, $E_4 = \pi \hbar (N + \frac{1}{2}) /\Delta t$, $E_5 = -E_4$, $E_{6,7,8} = 0$.

The resulting Hamiltonian depends only on one real parameter,

\[ \gamma = (\alpha - \beta)/2, \] (5)

and on one arbitrary integer, $N$. All the diagonal elements of the Hamiltonian will be zero with these choices of parameters. Indeed, calculation
of $H$ yields the result that this $8 \times 8$ matrix with elements $H_{mn}$, where $m$ labels the rows and $n$ the columns, has only two nonzero entries,

$$
H_{45} = \frac{\pi \hbar}{\Delta t} \left( N + \frac{1}{2} \right) e^{-i\gamma} \quad \text{and} \quad H_{54} = \frac{\pi \hbar}{\Delta t} \left( N + \frac{1}{2} \right) e^{i\gamma} .
$$

Any matrix in a space with a multiple-qubit basis can be expanded in terms of the direct products of the four “basis” $2 \times 2$ matrices for each of the two-level systems involved: the unit matrix $I$, and the standard Pauli matrices $\sigma_x$, $\sigma_y$, $\sigma_z$. The latter are proportional to spin components for two-state systems which are the spin states of spin-$\frac{1}{2}$ particles. We will use the spin-component nomenclature, and their representation in terms of the Pauli matrices. We report here the result of such an expansion for the Hamiltonian $H$. While its matrix form is simple and only contains two nonzero elements, the spin-component representation is surprisingly complicated,

$$
H = \frac{\pi \hbar}{4\Delta t} \left( N + \frac{1}{2} \right) \\
\times \left[ (\cos \gamma)(\sigma_I\sigma_x\sigma_C\sigma_{xD} - \sigma_I\sigma_y\sigma_C\sigma_{yD} + \sigma_I\sigma_x\sigma_C\sigma_{yD} + \sigma_I\sigma_y\sigma_C\sigma_{xD}) \\
- (\sin \gamma)(\sigma_y\sigma_I\sigma_C\sigma_{yD} - \sigma_y\sigma_I\sigma_x\sigma_{xD} + \sigma_x\sigma_I\sigma_y\sigma_{xD} + \sigma_x\sigma_I\sigma_x\sigma_{yD}) \right] .
$$

(7)
3. Reduction in Terms of Quantum Gates, and Discussion

We note that the Hamiltonian (7) involves three-spin interactions. The triplet $x, y$-component products are essential in the GHZ-paradox in quantum mechanics [34,35]. However, in that case these operators are measured. In fact, the need for multispin interactions in the Hamiltonian is a shortcoming as far as actual realizations, for instance, in the field of quantum computing, are concerned. Indeed, two-spin interactions are much more common and better understood theoretically and experimentally in solid-state and other systems, than three-spin interactions.

As mentioned earlier, our choice of the Hamiltonian is not unique. Its simplicity in the matrix form has allowed exact analytical result (7) be obtained. We have also explored certain unitary transformation choices more general than (1). However, presently we cannot offer a quantum signal splitting process of the type proposed in this work that can be accomplished “in one shot” with two-spin interactions only.

There are results in the quantum-computing literature [36-39] that establish that any unitary transformation in a multiqubit space can in principle be represented with arbitrary high accuracy by a sequence of two-spin and one-spin “quantum gates” which implies at most two-spin interactions; these interactions must be switched on and off sequentially.
However, generally the number of such gates involved may be quite large, and no systematic “reduction” procedure seems to follow from the existence-type proofs [36-39]. For our copying process, though, we managed to obtained a reduction, basically by guessing the gate sequence.

For simplicity, we put $\alpha = \beta = 0$ so that our unitary matrix defined in (1) and (2) only contains elements 1 or 0. A quantum-gate sequence that generates this unitary transformation is shown in Figure 1. It involves the standard quantum-computing NOT and controlled-controlled-NOT (CCNOT) gates [24-27]. The CCNOT gate is also known as Toffoli gate. It corresponds to the binary function whereby the NOT is applied on the “controlled” qubit (denoted by $\oplus$ in the figure) only when both “controlling” qubits (denoted by $\bullet$) are 1. Its quantum-computing version is still a three-spin gate. However, it can be expressed in terms of the two-spin controlled-NOT (CNOT) and single-spin-rotation quantum gates, e.g., [39]. We point out that explicit Hamiltonians for single-spin rotations and for CNOT are, respectively, one-spin and two-spin, and they have been considered in the literature, e.g., [6,33,40].

In summary, we proposed a variant of the quantum copying/signal splitting in which the initial state is multiplied but there is uncertainty in which of the two-state systems involved is the multiple copy stored. In our scheme the initial copy-system states are not fixed.
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Figure 1: Reduction of the unitary transformation (1)-(2), with $\alpha = \beta = 0$, to a sequence of NOT and CCNOT gates; see text for details.
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