Abstract. We demonstrate that the open core of a definably complete expansion of a densely linearly ordered abelian group is locally o-minimal if and only if any definable closed subset of $R$ is either discrete or contains a nonempty open interval. Here, the notation $R$ denotes the universe of the original structure.

1. Introduction

The open core of an expansion of a linear order is its reduct generated by its definable open sets. The notion of open core is first introduced in [11] and Dolich et al. gave a sufficient condition for the structure having an o-minimal open core in [2].

Local o-minimality was first proposed in [12] as a local variant of o-minimality and it has been studied in [3, 4, 5, 7, 9]. To the best of the author’s knowledge, Fornasiero first investigated the structures having locally o-minimal open cores. He gave necessary and sufficient conditions for a definably complete structure having a locally o-minimal open core in [3]. He assumed that the structure is an expansion of an ordered field. The author considered the problem under a relaxed algebraic assumption. He provided a sufficient condition for a definably complete expansion of a densely linearly ordered abelian group having a uniformly locally o-minimal open core of the second kind in [6]. Definably complete uniformly locally o-minimal structures of the second kind satisfy the additional condition which general locally o-minimal structures do not satisfy. They admit local definable cell decomposition though general locally o-minimal structures do not. See [4] for more details on uniformly locally o-minimal structures of the second kind.

The purpose of this paper is to find a necessary and sufficient condition for a definably complete structure having a locally o-minimal open core under the same assumption employed in [6]. The following is the main theorem of this paper.

**Theorem 1.1.** Let $\mathcal{R} = (R, <, +, 0, \ldots)$ be a definably complete expansion of a densely linearly ordered abelian group. The following are equivalent:

1. Any definable closed subset of $R$ is either discrete or contains a nonempty open interval.
2. The open core of $\mathcal{R}$ is locally o-minimal.

Furthermore, sets definable in the open core are constructible when one of (and both of) the above equivalent conditions is satisfied. Recall that a constructible set is a finite boolean combination of open sets.

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Tools and techniques for proving this theorem were already developed in the previous studies \([2, 3, 5, 6]\). In Section 2, we recall the definitions and results given in the previous works. Two key lemmas are demonstrated in Section 3 mimicking the arguments in \([5, 6]\). We complete the proof of the theorem in Section 4.

We introduce the terms and notations used in this paper. The term ‘definable’ means ‘definable in the given structure with parameters’ in this paper. For any set \(X \subset R^{n+1}\) definable in a structure \(\mathcal{R} = (R, <, . . . )\) and for any \(x \in R^n\), the notation \(X_x\) denotes the fiber defined as \(\{y \in R^n \mid (x, y) \in X\}\). For a linearly ordered structure \(\mathcal{R} = (R, <, . . . )\), an open interval is a definable set of the form \(\{x \in R \mid a < x < b\}\) for some \(a, b \in R\). It is denoted by \((a, b)\) in this paper. An open box in \(R^n\) is the direct product of \(n\) open intervals. A CBD set is a closed, bounded and definable set. Let \(A\) be a subset of a topological space. The notations \(\text{int}(A)\) and \(\text{cl}(A)\) denote the interior and the closure of the set \(A\), respectively.

2. Preliminary

We recall the definitions and the assertions introduced in the previous studies.

**Definition 2.1 (\([10, 12]\)).** An expansion of a densely linearly ordered set without endpoints \(\mathcal{R} = (R, <, . . . )\) is *definably complete* if any definable subset \(X\) of \(R\) has the supremum and infimum in \(R \cup \{\pm \infty\}\). The structure \(\mathcal{R}\) is *locally o-minimal* if, for every definable subset \(X\) of \(R\) and for every point \(a \in R\), there exists an open interval \(I\) containing the point \(a\) such that \(X \cap I\) is a finite union of points and open intervals.

Dolich et al. used \(D_\Sigma\)-sets in \([2]\). The author also used them in \([6]\). They play an important role also in this paper.

**Definition 2.2 (\(D_\Sigma\)-sets).** Consider an expansion of a densely linearly ordered abelian group \(\mathcal{R} = (R, <, +, 0 . . . )\).

A parameterized family of definable sets \(\{X(x)\}_{x \in S}\) is the family of the fibers of a definable set; that is, there exists a definable set \(X\) with \(X(x) = X_x\) for all \(x\) in a definable set \(S\). A parameterized family \(\{X(r, s)\}_{r>0, s>0}\) of CBD subsets \(X(r, s)\) of \(R^n\) is called a \(D_\Sigma\)-family if \(X(r, s) \subseteq X(r', s)\) and \(X(r, s') \subseteq X(r, s)\) whenever \(r < r'\) and \(s < s'\). Note that \(X(r, s)\) is not necessarily strictly contained in \(X(r', s)\). It is the same for the inclusion \(X(r, s') \subseteq X(r, s)\). A definable subset \(X\) of \(R^n\) is a \(D_\Sigma\)-set if \(X = \bigcup_{r>0, s>0} X(r, s)\) for some \(D_\Sigma\)-family \(\{X(r, s)\}_{r>0, s>0}\).

The following two lemmas are found in \([2, 10]\). We use Lemma 2.3 here and there without notice in this paper.

**Lemma 2.3.** Consider a definably complete expansion of a densely linearly ordered abelian group \(\mathcal{R}\). The following assertions are true:

1. The projection image of a \(D_\Sigma\)-set is \(D_\Sigma\).
2. Fibers, finite unions and finite intersections of \(D_\Sigma\)-sets are \(D_\Sigma\).
3. Every constructible definable set is \(D_\Sigma\).

**Proof.** (1) Immediate from \([10]\) Lemma 1.7. (2) \([2]\) 1.9(1). (3) \([2]\) 1.10(1)].

**Lemma 2.4.** Let \(\mathcal{R} = (R, <, +, 0 . . . )\) be a definably complete expansion of a densely linearly ordered abelian group. A CBD set \(X \subset R^{n+1}\) has a nonempty interior if the CBD set.
\[ \{ x \in \mathbb{R}^n \mid X_x \text{ contains a closed interval of length } s \} \]
has a nonempty interior for some \( s > 0 \).

Proof. [2, 2.8(2)] \qed

We give a definition of dimension of a \( \text{D}_\Sigma \)-set.

**Definition 2.5** (Dimension). Let \( \mathcal{R} = (\mathbb{R}, <, . . .) \) be an expansion of a densely linearly ordered structure. Consider a \( \text{D}_\Sigma \)-subset \( X \) of \( \mathbb{R}^n \) and a point \( x \in \mathbb{R}^n \). The *dimension* of \( X \) is defined as follows:

- \( \dim X = -\infty \) if \( X \) is an empty set.
- Otherwise, \( \dim X \) is the supremum of nonnegative integers \( d \) such that the image \( \pi(X) \) has a nonempty interior for some coordinate projection \( \pi : \mathbb{R}^n \rightarrow \mathbb{R}^d \).

The *full dimension* \( f. \dim(X) \) of \( X \) is the pair \( \langle d, k \rangle \), where \( d = \dim X \) and \( k \) is the number of coordinate spaces of dimension \( d \) such that the images of \( X \) under the projections of \( \mathbb{R}^n \) onto the coordinate spaces have nonempty interiors.

**Definition 2.6.** For a set \( X \), a family \( \mathcal{F} \) of subsets of \( X \) is called a *filtered collection* if, for any \( B_1, B_2 \in \mathcal{F} \), there exists \( B_3 \in \mathcal{F} \) with \( B_3 \subseteq B_1 \cap B_2 \). A parameterized family \( \{ S(t) \}_{t \in T} \) of definable subsets of \( X \) is a *definable filtered collection* if it is a filtered collection.

**Proposition 2.7.** Consider a definably complete expansion of a dense linear order without endpoints. Every definable filtered collection of nonempty definable closed subsets of a closed, bounded and definable set has a nonempty intersection.

Proof. The literally same proof as that for o-minimal structures in [8, Section 8.4] works. \qed

The following lemma is found in [7, Lemma 2.3].

**Lemma 2.8.** Consider a definably complete structure \( \mathcal{R} = (\mathbb{R}, <, . . .) \). The following are equivalent:

1. The structure \( \mathcal{R} \) is a locally o-minimal structure.
2. Any definable set in \( R \) has a nonempty interior or it is closed and discrete.

3. Key lemmas

We demonstrate key lemmas and their corollaries in this section.

3.1. **First key lemma.** We first introduce the following technical definition.

**Definition 3.1.** Let \( \mathcal{R} = (\mathbb{R}, <, +, 0, . . .) \) be a definably complete expansion of a densely linearly ordered abelian group.

The structure \( \mathcal{R} \) enjoys the *property (a)* if any \( \text{D}_\Sigma \)-subset \( X \) of \( R \) has an empty interior if \( X(r, s) \) have empty interiors for all \( r > 0 \) and \( s > 0 \). Here, \( \{ X(r, s) \}_{r>0,s>0} \) is an arbitrary \( \text{D}_\Sigma \)-family with \( X = \bigcup_{r>0,s>0} X(r, s) \).

The structure \( \mathcal{R} \) enjoys the *property (b)* if at least one of \( \text{D}_\Sigma \)-subsets \( X_1 \) and \( X_2 \) of \( R \) has a nonempty interior when the union \( X_1 \cup X_2 \) has a nonempty interior.

We give the first key lemma below. We prove it using the techniques employed in [2, 6].
Lemma 3.2. Let \( \mathcal{R} = (R, <, +, 0, \ldots) \) be a definably complete expansion of a densely linearly ordered abelian group enjoying the property (a). Let \( X \) be a \( \mathcal{D}_\Sigma \)-subset of \( R^n \) and \( \{ X(r, s) \}_{r > 0, s > 0} \) be a \( \mathcal{D}_\Sigma \)-family with \( X = \bigcup_{r > 0, s > 0} X(r, s) \).

\( (A)_n \) The \( \mathcal{D}_\Sigma \)-set \( X \) has an empty interior if the CBD sets \( X(r, s) \) have empty interiors for all \( r > 0 \) and \( s > 0 \).

\( (B)_n \) Set
\[ I(X) = \{ x \in R^{n-1} \mid \text{the fiber } X_x \text{ contains a nonempty open interval} \} . \]

It is a \( \mathcal{D}_\Sigma \)-set. If \( I(X) \) has a nonempty interior, the CBD set \( X(r, s) \) has a nonempty interior for some \( r > 0 \) and \( s > 0 \).

Proof. We demonstrate \( (A)_n \) and \( (B)_n \) simultaneously by induction on \( n \). The assertion \( (A)_1 \) is identical to the property (a). Therefore, the assertion \( (A)_1 \) holds true by the assumption.

We next demonstrate that the property (a) and the assertion \( (A)_n \) implies the assertion \( (B)_n \) when \( n \geq 1 \). Set \( Y(r, s) = \{ x \in R^n \mid \exists t \in R, [t-s, t+s] \subseteq (X(r, s))_x \} \). The set \( Y(r, s) \) is CBD. We show that \( I(X) = \bigcup_{r, s} Y(r, s) \). It means that \( I(X) \) is a \( \mathcal{D}_\Sigma \)-set. The inclusion \( \bigcup_{r, s} Y(r, s) \subseteq I(X) \) is obvious. We demonstrate the opposite inclusion. Take an arbitrary point \( x \in I(X) \). We have \( \text{int}(X_x) \neq \emptyset \). We get \( \text{int}((X(r, s))_x) \neq \emptyset \) for some \( r > 0 \) and \( s > 0 \) by the property (a). There exist \( t \in R \) and \( s' > 0 \) with \( [t-s', t+s'] \subseteq (X(r, s))_x \). Replacing \( s \) with \( \min\{s, s'\} \), we may assume that \( [t-s, t+s] \subseteq (X(r, s))_x \). It means that \( x \in Y_{r, s} \).

By the assertion \( (A)_n \), the CBD set \( Y(r, s) \) has a nonempty interior for some \( r > 0 \) and \( s > 0 \). The CBD set \( X(r, s) \) has a nonempty interior by Lemma 3.2. We have demonstrated the assertion \( (B)_n \).

We next prove that the assertion \( (B)_n \) implies the assertion \( (A)_{n+1} \) when \( n \geq 1 \). When the \( \mathcal{D}_\Sigma \)-subset \( X \) of \( R^{n+1} \) has a nonempty interior, the set \( I(X) \) has a nonempty interior because \( X \) contains an open box. For some \( r > 0 \) and \( s > 0 \), the CBD set \( X(r, s) \) has a nonempty interior by the assumption.

We give four corollaries of this lemma.

Corollary 3.3. Let \( \mathcal{R} \) and \( X \) be as in Lemma 3.2. If the \( \mathcal{D}_\Sigma \)-set \( I(X) \) has a nonempty interior, then \( X \) has a nonempty interior.

Proof. Obvious by Lemma 3.2(B)_n. \( \square \)

Corollary 3.4. Let \( \mathcal{R} \) and \( X \) be as in Lemma 3.2. Let \( \pi : R^n \rightarrow R^d \) be a coordinate projection such that \( \pi(X) \) has a nonempty interior. Then, there exists a nonempty bounded open box \( B \) in \( R^n \) such that, for all \( x \in \pi(B) \), the fibers \( \pi^{-1}(x) \cap (B \cap X) \) of \( B \cap X \) are not empty sets.

Proof. We may assume that the projection \( \pi \) is the projection onto the first \( d \) coordinates without loss of generality. Let \( \{ X(r, s) \}_{r > 0, s > 0} \) be a \( \mathcal{D}_\Sigma \)-family with \( X = \bigcup_{r, s} X(r, s) \). Since \( \pi(X) \) has a nonempty interior, the set \( \pi(X(r, s)) \) has a nonempty interior for some \( r > 0 \) and \( s > 0 \) by Lemma 3.2(A)_d. Take a nonempty bounded open box \( U \) in \( R^d \) contained in \( \pi(X(r, s)) \). Since \( X(r, s) \) is bounded, there exists a nonempty bounded open box \( V \) in \( R^{n-d} \) such that \( X(r, s) \cap (U \times R^{n-d}) \subseteq U \times V \). Set \( B = U \times V \). We have \( \pi^{-1}(x) \cap (B \cap X(r, s)) \neq \emptyset \) for all \( x \in U \). In particular, we obtain \( \pi^{-1}(x) \cap (B \cap X) \neq \emptyset \). \( \square \)
Corollary 3.5. Let \( \mathcal{R} = (R, <, +, 0, \ldots) \) be a definably complete expansion of a densely linearly ordered abelian group enjoying the properties (a) and (b). Let \( X_1 \) and \( X_2 \) be \( D_\mathcal{R} \)-subsets of \( \mathbb{R}^n \). If \( X_1 \cup X_2 \) has a nonempty interior, then at least one of \( X_1 \) and \( X_2 \) has a nonempty interior.

Proof. We show the corollary by induction on \( n \). The corollary follows from the property (b) when \( n = 1 \). Consider the case in which \( n > 1 \). Set \( X = X_1 \cup X_2 \). We have \( \mathcal{I}(X) = \mathcal{I}(X_1) \cup \mathcal{I}(X_2) \) by the property (b). Since \( X \) has a nonempty interior, \( \mathcal{I}(X) \) also has a nonempty interior by the definition of \( \mathcal{I}(X) \). The sets \( \mathcal{I}(X_1) \) and \( \mathcal{I}(X_2) \) are \( D_\mathcal{R} \) by Lemma 3.3 \( B \). By the induction hypothesis, at least one of \( \mathcal{I}(X_1) \) and \( \mathcal{I}(X_2) \) has a nonempty interior. At least one of \( X_1 \) and \( X_2 \) has a nonempty interior by Corollary 3.5. \( \square \)

Corollary 3.6. Let \( \mathcal{R} = (R, <, +, 0, \ldots) \) be as in Corollary 3.5. A definable constructible set \( X \) has a nonempty interior when the closure of \( X \) has a nonempty interior.

Proof. Note that every constructible definable set is a finite boolean combination of open definable sets by \([4]\). Therefore, the definable constructible set \( X \) is the union of a finite locally closed definable sets \( X_1, \ldots, X_n \). Assume that the closure \( \text{cl}(X) \) contains a nonempty interior. Since \( \text{cl}(X) = \bigcup_{i=1}^{n} \text{cl}(X_i) \) and definable closed sets are \( D_\mathcal{R} \), the closure \( \text{cl}(X_i) \) has a nonempty interior for some \( 1 \leq i \leq n \) by Corollary 3.5. A standard topological argument shows that \( X_1 \) has a nonempty interior because \( X_1 \) is locally closed. It implies that \( X \) has a nonempty interior. \( \square \)

3.2. Second key lemma. We next prove the second key lemma. A parameterized family of definable sets \( \{ X(r) \}_{r>0} \) is called increasing if \( X(r) \subseteq X(r') \) whenever \( r \leq r' \). It is called decreasing if \( X(r') \subseteq X(r) \) whenever \( r \leq r' \).

Lemma 3.7. Let \( \mathcal{R} = (R, <, +, 0, \ldots) \) be a definably complete expansion of a densely linearly ordered abelian group such that an arbitrary definable closed subset of \( R \) is either discrete or contains a nonempty open interval. Let \( \{ X(r) \}_{r>0} \) be a parameterized family of definable, discrete and closed subsets of \( R \). Assume that it is either increasing or decreasing. Then the union \( X = \bigcup_{r>0} X(r) \) is discrete and closed.

Proof. We prove this lemma similarly to \([5\) Theorem 4.3]. We only consider the case in which the given parameterized family is increasing. We can prove the lemma similarly in the other case.

Note that a definable subset of a definable, closed and discrete subset is again discrete and closed. In particular, under our assumption, the definable set \( X \) is discrete and closed unless the closure \( \text{cl}(X) \) contains a nonempty open interval. Therefore, we have only to lead to a contradiction assuming that the closure \( \text{cl}(X) \) contains a nonempty open interval \( I \).

Take a point \( a \in X \) which is also contained in the open interval \( I \). Consider the definable function \( f : \{ r \in R \mid r > 0 \} \rightarrow \{ x \in R \mid x > a \} \) defined by \( f(r) = \inf \{ x > a \mid x \in X(r) \} \). It is obvious that \( f \) is a decreasing function because \( \{ X(r) \}_{r>0} \) is increasing.

We demonstrate that the image \( \text{Im}(f) \) of the function \( f \) is discrete and closed. As previously demonstrated, if the closure \( \text{cl}(\text{Im}(f)) \) is discrete and closed, the image \( \text{Im}(f) \) is also discrete and closed. We assume that \( \text{cl}(\text{Im}(f)) \) is not discrete and closed for contradiction. A nonempty open interval \( J \) is contained in \( \text{cl}(\text{Im}(f)) \) by
the assumption of the lemma. Take a point $b \in \text{Im}(f) \cap J$ and a point $r > 0$ with $b = f(r)$. Since $X(r)$ is closed, we have $b \in X(r)$. Any point $b' \in \text{Im}(f)$ with $b' > b$ is also contained in $X(r)$. In fact, take a point $r' > 0$ with $b' = f(r')$. If $r' > r$, the set $X(r')$ contains the point $b$ because $X(r) \subseteq X(r')$. Then we have $b' = f(r') \leq b$ by the definition of the function $f$, and this is a contradiction. If $r' < r$, we have $b' \in X(r') \subseteq X(r)$.

Set $b_1 = \inf\{b' \in \text{Im}(f) \mid b' > b\}$. We have $b_1 \in X(r)$ and $b_1 > b$ because $\{b' \in \text{Im}(f) \mid b' > b\} \subseteq X(r)$ and $X(r)$ is closed and discrete. The open interval $(b, b_1)$ has an empty intersection with $\text{Im}(f)$. It contradicts the assumption that $\text{cl}(\text{Im}(f))$ contains the open interval $J$ with $b \in J$. We have demonstrated that $\text{Im}(f)$ is discrete and closed.

Let $c$ be the infimum of $\text{Im}(f)$. We have $c > a$ and $c \in \text{Im}(f)$ because $\text{Im}(f)$ is discrete and closed. On the other hand, there exists $b \in X$ with $a < b < c$ because the closure $\text{cl}(X)$ contains the open interval $I$ with $a \in I$. Take a point $r > 0$ with $b \in X(r)$. By the definition of $f$, we have $a < f(r) \leq b < c$. It contradicts the definition of $c$.

We provide corollaries of Lemma 3.7.

**Corollary 3.8.** Let $\mathcal{R}$ be as in Lemma 3.7. The structure $\mathcal{R}$ possesses the property (a) in Definition 3.7. Furthermore, an arbitrary $\Sigma$-subset of $\mathcal{R}$ is either discrete and closed or contains a nonempty open interval.

**Proof.** Let $X$ be a $\Sigma$-subset of $\mathcal{R}$ and $\{X(r, s)\}_{r, s > 0}$ be a $\Sigma$-family with $X = \bigcup_{r, s > 0} X(r, s)$. If the CBD set $X(r, s)$ contains a nonempty open interval for some $r > 0$ and $s > 0$, the union $X$ also contain an open interval. We have only to show that $X$ is discrete and closed if the CBD sets $X(r, s)$ do not contain an open interval for all $r > 0$ and $s > 0$. The CBD sets $X(r, s)$ are discrete and closed by the assumption. When $r$ is fixed, the parameterized family $\{X(r, s)\}_{s > 0}$ is decreasing. The union $X(r) = \bigcup_{s > 0} X(r, s)$ is discrete and closed by Lemma 3.7. Apply Lemma 3.7 again to the increasing parameterized family $\{X(r)\}_{r > 0}$. The $\Sigma$-set $X = \bigcup_{r > 0} X(r)$ is discrete and closed. □

**Corollary 3.9.** Let $\mathcal{R}$ be as in Lemma 3.7. The structure $\mathcal{R}$ possesses the property (b) in Definition 3.7.

**Proof.** Immediate by the ‘furthermore’ part of Corollary 3.8 □

Using the previous corollaries, we get the following lemma:

**Lemma 3.10.** Let $\mathcal{R}$ be as in Lemma 3.7 and $X$ be a $\Sigma$-subset of $\mathcal{R}$ of dimension $d$. Take a coordinate projection $\pi : X \to \mathbb{R}^d$ such that $\pi(X)$ has a nonempty interior. Then, there exists a $\Sigma$-subset $Z$ of $\mathbb{R}^d$ such that $Z$ has an empty interior and the fiber $X \cap \pi^{-1}(x)$ is discrete and closed for any $x \in \pi(X) \setminus Z$.

**Proof.** This lemma is proven in the same manner as [6] Lemma 5.6. We give a proof here for readers’ convenience. The assumptions of the corollaries in Section 3.3 are satisfied thanks to Corollary 3.8 and Corollary 3.9.

For all $1 \leq i \leq n - d$, we can take coordinate projections $\pi_i : \mathbb{R}^n \to \mathbb{R}^d$ with $\pi = \pi_{n-d} \circ \cdots \circ \pi_1$. We may assume that $\pi_i$ are the coordinate projections forgetting the last coordinate without loss of generality. Set $\Pi_i = \pi_i \circ \cdots \circ \pi_1$ and $\Phi_i = \pi_{n-d} \circ \cdots \circ \pi_{i+1}$. Consider the sets $T_i = \{x \in \mathbb{R}^{n-i} \mid \pi_i^{-1}(x) \cap \Pi_{i-1}(X) \text{ contains a nonempty open interval}\}$. The sets $T_i$ are $\Sigma$ and we have
\( \dim(T_i) < \dim \Pi_{i-1}(X) = \dim X = d \) by Corollary 3.3. Set \( U_i = \Phi_i(T_i) \subseteq R^d \) for all \( 1 \leq i \leq n - d \). The projection images \( U_i \) are \( D_\Sigma \)-sets. We get \( \text{int}(U_i) = \emptyset \) because \( \dim(T_i) < d \). Set \( Z = \bigcup_{i=1}^{n-d} U_i \). It also has an empty interior by Corollary 3.5.

The fiber \( X \cap \pi^{-1}(x) \) is discrete and closed for any \( x \in \pi(X) \setminus Z \). In fact, let \( y \in R^n \) be an arbitrary point with \( x = \pi(y) \). Set \( y_0 = y \) and \( y_i = \Pi_i(y) \) for \( 1 \leq i \leq n - d \). We have \( y_{n-d} = x \) by the definition. We construct an open box \( B_i \) in \( R^{n-d-1} \) for \( 0 \leq i \leq n-d \) such that \( y_i \in B_i \) and \( \{x\} \times B_i \cap \Pi_i(X) \) consists of at most one point in decreasing order. When \( i = n - d \), the open box \( B_{n-d} = R^0 \). When \( \{x\} \times B_i \cap \Pi_i(X) = \emptyset \), set \( B_{i-1} = B_i \times R \). We have \( \{x\} \times B_{i-1} \cap \Pi_{i-1}(X) = \emptyset \). When \( \{x\} \times B_i \cap \Pi_i(X) \neq \emptyset \), the fiber \( \Pi_{i-1}(X) \cap \pi_i^{-1}(y_i) \) is discrete and closed by Corollary 3.8. Therefore, there exists an open box \( B_{i-1} \) in \( R^{n-d+1-i} \) such that \( \pi_i(B_{i-1}) = B_i \), \( y_{i-1} \in B_{i-1} \) and \( \{x\} \times B_{i-1} \cap \Pi_{i-1}(X) \) consists of at most one point. We have constructed the open boxes \( B_i \) in \( R^{n-d-i} \) for all \( 0 \leq i \leq n - d \). The existence of \( B_0 \) implies that \( X \cap \pi^{-1}(x) \) is discrete and closed.

### 4. Proof of Theorem 1.1

Preparation has been done. We finally get the following theorem:

**Theorem 4.1.** Let \( R = (R, <, +, 0, \ldots) \) be a definably complete expansion of a densely linearly ordered abelian group such that an arbitrary definable closed subset of \( R \) is either discrete or contains a nonempty open interval. A \( D_\Sigma \)-set is constructible.

**Proof.** We prove the theorem using the technique employed in [3]. Note that the assumptions of the lemma and the corollaries in Section 5.4 are satisfied thanks to Corollary 5.3 and Corollary 5.9. We use this fact without notice in the proof.

Let \( X \) be a \( D_\Sigma \)-subset of \( R \) and \( \{X(r, s)\}_{r > 0, s > 0} \) be a \( D_\Sigma \)-family with \( X = \bigcup_{r,s} X(r, s) \). We demonstrate the theorem by induction on the full dimension \( f.\dim(X) \) of \( X \). Set \( d = \dim X \). We first consider the case in which \( d = 0 \). Let \( \pi_i : R^n \to R \) be the projections onto the \( i \)-th coordinate for all \( 1 \leq i \leq n \). The projection images \( \pi_i(X) \) are discrete and closed for all \( 1 \leq i \leq n \) by Corollary 3.8 because \( \pi_i(X) \) do not contain an open interval by the definition of dimension. The set \( X \) is a subset of the discrete and closed set \( \prod_{i=1}^{n} \pi_i(X) \). Therefore, it is also closed and discrete. In particular, it is constructible.

Let us consider the case in which \( d > 0 \). There exists a coordinate projection \( \pi : R^n \to R^d \) such that the image \( \pi(X) \) has a nonempty interior. We can take a \( D_\Sigma \)-subset \( Z \) of \( R^d \) such that \( \text{int}(Z) = \emptyset \) and the fiber \( X \cap \pi^{-1}(x) \) is discrete and closed for any \( x \in \pi(X) \setminus Z \) by Lemma 3.10. It is obvious that \( f.\dim(\pi^{-1}(Z) \cap X) < f.\dim X \). The \( D_\Sigma \)-set \( \pi^{-1}(Z) \cap X \) is constructible by the induction hypothesis. We have only to demonstrate that \( X \setminus (\pi^{-1}(Z) \cap X) \) is constructible. Therefore, we may assume that \( \pi^{-1}(x) \) is discrete and closed for any \( x \in \pi(X) \).

The notation \( lc(X) \) denotes the set of points having open boxes \( U \) containing the points such that the intersections \( U \cap X \) are locally closed. The set \( lc(X) \) is definable and locally closed by its definition. We show that \( \pi(Y) \) has an empty interior, where \( Y = X \setminus lc(X) \). Once it is proven, the theorem immediately follows from the induction hypothesis. For contradiction, we assume that \( \pi(Y) \) has a nonempty interior. By Corollary 5.4 there exists a bounded open box \( B \) such that,\n
\[ (*) \text{ for any } u \in \pi(B), \text{ the fiber } Y \cap B \cap \pi^{-1}(u) \text{ of } Y \cap B \text{ is not an empty set.} \]
We next demonstrate that, for any $u \in \pi(B)$, there exist $r_u > 0$ and $s_u > 0$ such that $T_u := X \cap U \cap \pi^{-1}(u)$ is contained in the CBD set $X \langle r_u, s_u \rangle$. We fix $u \in \pi(B)$. Set $C(r, s) = T_u \setminus X \langle r, s \rangle$ for all $r > 0$ and $s > 0$. We have only to show that $C(r, s)$ is an empty set for some $r > 0$ and $s > 0$. Assume the contrary. Since $T_u$ is discrete and closed, the set $C(r, s)$ is also discrete and closed. Since the parameterized family $\{C(r, s)\}_{r > 0, s > 0}$ is a definable filtered collection of nonempty definable closed subsets of the bounded closed definable set $T_u$, the intersection $\bigcap_{r > 0, s > 0} C(r, s)$ is not an empty set by Proposition 2.7. It contradicts the equality $X = \bigcup_{r > 0, s > 0} X \langle r, s \rangle$.

Set

$$D(r, s) = (X \cap B) \setminus X \langle r, s \rangle,$$

$$V(r, s) = \pi(B) \setminus \pi(D(r, s))$$

and

$$W(r, s) = \text{cl}(V(r, s)).$$

Note that the $D_2$-set $\pi(D(r, s))$ is constructible by the induction hypothesis. The set $V(r, s)$ is also constructible and its closure $W(r, s)$ is a CBD set for any $r > 0$ and $s > 0$. The existence of $r_u > 0$ and $s_u > 0$ for any $u \in \pi(B)$ implies the equality $\pi(B) = \bigcup_{r > 0, s > 0} V(r, s)$. In particular, the parameterized family $\{W(r, s)\}_{r > 0, s > 0}$ of CBD sets is a $D_2$-family and the $D_2$-set $W = \bigcup_{r > 0, s > 0} W(r, s)$ contains an open box $\pi(B)$. Since $W$ has a nonempty interior, the CBD set $W(\tilde{r}, \tilde{s})$ has a nonempty interior for some $\tilde{r} > 0$ and $\tilde{s} > 0$ by Lemma 3.2. Since $V(\tilde{r}, \tilde{s})$ is constructible, it also has a nonempty interior by Corollary 4.6. By shrinking $\pi(B)$ if necessary, we may assume that $B$ satisfies the condition $(\ast)$ and $\pi(B)$ is contained in $V(\tilde{r}, \tilde{s})$. It means that $X \cap B$ is contained in $X(\tilde{r}, \tilde{s})$. We have $X \cap B = X(\tilde{r}, \tilde{s}) \cap B$, which implies that $X \cap B$ is locally closed because $X(\tilde{r}, \tilde{s})$ is closed. On the other hand, the condition $(\ast)$ implies that $X \cap B$ has a nonempty intersection with $Y$. It contradicts the definition of $Y$. $\square$

We demonstrate Theorem 1.1 below.

**Proof of Theorem 1.1** (1) $\Rightarrow$ (2): Lemma 2.3 and Theorem 4.1 assert that a set definable in the original structure $\mathcal{R}$ is definable in the open core if and only if it is $D_2$. By Corollary 3.3 any subset of $R$ definable in the open core is either discrete and closed or contains a nonempty open interval. The open core is locally o-minimal by Lemma 2.3.

(2) $\Rightarrow$ (1): Immediate by Lemma 2.3 because closed sets definable in $\mathcal{R}$ are definable in the open core.

The ‘furthermore’ part follows from Theorem 4.1. $\square$

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DEPARTMENT OF LIBERAL ARTS, JAPAN COAST GUARD ACADEMY, 5-1 WAKABA-CHO, KURE, HIROSHIMA 737-8512, JAPAN

Email address: fujita.masato.p34@kyoto-u.jp