TOEPLITZ DETERMINANTS WHOSE ELEMENTS ARE THE COEFFICIENTS OF UNIVALENT FUNCTIONS

MD FIROZ ALI, D. K. THOMAS, AND A. VASUDEVARAO

Abstract. Let $S$ denote the class of analytic and univalent functions in $D := \{z \in \mathbb{C} : |z| < 1 \}$ of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. In this paper, we determine sharp estimates for the Toeplitz determinants whose elements are the Taylor coefficients of functions in $S$ and its certain subclasses. We also discuss similar problems for typically real functions.

1. Introduction and Preliminaries

Let $H$ denote the space of analytic functions in the unit disk $D := \{z \in \mathbb{C} : |z| < 1 \}$ and $A$ denote the class of functions $f$ in $H$ with Taylor series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$ (1.1)

The subclass $S$ of $A$, consisting of univalent (i.e., one-to-one) functions has attracted much interest for over a century, and is a central area of research in Complex Analysis. A function $f \in A$ is called starlike if $f(D)$ is starlike with respect to the origin i.e., $t f(z) \in f(D)$ for every $0 \leq t \leq 1$. Let $S^*$ denote the class of starlike functions in $S$. It is well-known that a function $f \in A$ is starlike if, and only if,

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > 0, \; z \in D.$$ 

An important member of the class $S^*$ as well as of the class $S$ is the Koebe function $k$ defined by $k(z) = z/(1 - z)^2$. This function plays the role of extremal function in most of the problems for the classes $S^*$ and $S$.

A function $f \in A$ is called convex if $f(D)$ is a convex domain. Let $C$ denote the class of convex functions in $S$. It is well-known that a function $f \in A$ is in $C$ if, and only if,

$$\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0, \; z \in D.$$ 

From the above it is easy to see that $f \in C$ if, and only if, $zf' \in S^*$.

A function $f \in A$ is said to be close-to-convex if there exists a starlike function $g \in S^*$ and a real number $\alpha \in (-\pi/2, \pi/2)$, such that

$$\text{Re} \left( e^{i\alpha} \frac{zf'(z)}{g(z)} \right) > 0, \; z \in D.$$ (1.2)

2010 Mathematics Subject Classification. Primary 30C45, 30C55.

Key words and phrases. univalent functions, starlike functions, convex functions, close-to-convex function, typically real function, Toeplitz determinant.
Let $\mathcal{K}$ denote the class of all close-to-convex functions. It is well-known that every close-to-convex function is univalent in $\mathbb{D}$ (see [11]). Geometrically, $f \in \mathcal{K}$ means that the complement of the image-domain $f(\mathbb{D})$ is the union of rays that are disjoint (except that the origin of one ray may lie on another one of the rays).

Let $\mathcal{R}$ denote class of functions $f$ in $\mathcal{A}$ satisfying $\text{Re} f'(z) > 0$ in $\mathbb{D}$. It is well-known that functions in $\mathcal{R}$ are close-to-convex, and hence univalent. Functions in $\mathcal{R}$ are sometimes called functions of bounded boundary rotation.

A function $f$ satisfying the condition $(\text{Im} z)(\text{Im} f(z)) \geq 0$ for $z \in \mathbb{D}$ is called a typically real. Let $\mathcal{T}$ denote the class of all typically real functions. Robertson [7] proved that $f \in \mathcal{T}$ if, and only if, there exists a probability measure $\mu$ on $[-1, 1]$ such that

$$f(z) = \int_{-1}^{1} k(z, t) \, d\mu(t),$$

where

$$k(z, t) = \frac{z}{1 - 2tz + z^2}, \quad z \in \mathbb{D}, \quad t \in [-1, 1].$$

Hankel matrices and determinants play an important role in several branches of mathematics, and have many applications [10]. The Toeplitz determinants are closely related to Hankel determinants. Hankel matrices have constant entries along the reverse diagonal, whereas Toeplitz matrices have constant entries along the diagonal. For a good summary of the applications of Toeplitz matrices to the wide range of areas of pure and applied mathematics, we refer to [10]. Recently, Thomas and Halim [9] introduced the concept of the symmetric Toeplitz determinant for analytic functions $f$ of the form (1.1), and defined the symmetric Toeplitz determinant $T_q(n)$ as follows

$$T_q(n) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_n \end{vmatrix}$$

where $n, q = 1, 2, 3 \ldots$ with $a_1 = 1$. In particular,

$$T_2(2) = \begin{bmatrix} a_2 & a_3 \\ a_3 & a_2 \end{bmatrix}, \quad T_2(3) = \begin{bmatrix} a_3 & a_4 \\ a_4 & a_3 \end{bmatrix}, \quad T_3(1) = \begin{bmatrix} 1 & a_2 & a_3 \\ a_2 & 1 & a_2 \\ a_3 & a_2 & 1 \end{bmatrix}, \quad T_3(2) = \begin{bmatrix} a_2 & a_3 & a_4 \\ a_3 & a_2 & a_3 \\ a_4 & a_3 & a_2 \end{bmatrix}.$$

For small values of $n$ and $q$, estimates of the Toeplitz determinant $|T_q(n)|$ for functions in $\mathcal{S}$ and $\mathcal{K}$ have been studied in [9]. Similarly, estimates of the Toeplitz determinant $|T_q(n)|$ for functions in $\mathcal{R}$ have been studied in [6], when $n$ and $q$ are small. Apart from [6] and [9], there appears to be little in the literature concerning estimates of Toeplitz determinants. In both [6] and [9] we observe an invalid assumption in the proofs. It is the purpose of this paper to give estimates for Toeplitz determinants $T_q(n)$ for functions in $\mathcal{S}$, $\mathcal{S}^*$, $\mathcal{C}$, $\mathcal{K}$, $\mathcal{R}$, and $\mathcal{T}$, when $n$ and $q$ are small.
Let $\mathcal{P}$ denote the class of analytic functions $p$ in $\mathbb{D}$ of the form

$$(1.3) \quad p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

such that $\text{Re} p(z) > 0$ in $\mathbb{D}$. Functions in $\mathcal{P}$ are sometimes called Carathéodory functions. To prove our main results, we need some preliminary results for functions in $\mathcal{P}$.

**Lemma 1.1.** ([1], p. 41) For a function $p \in \mathcal{P}$ of the form (1.3), the sharp inequality $|c_n| \leq 2$ holds for each $n \geq 1$. Equality holds for the function $p(z) = (1 + z)/(1 - z)$.

**Lemma 1.2.** ([2], Theorem 1) Let $p \in \mathcal{P}$ be of the form (1.3) and $\mu \in \mathbb{C}$. Then

$$|c_n - \mu c_k c_{n-k}| \leq 2 \max \{1, |2\mu - 1|\}, \quad 1 \leq k \leq n - 1.$$  

If $|2\mu - 1| \geq 1$ then the inequality is sharp for the function $p(z) = (1 + z)/(1 - z)$ or its rotations. If $|2\mu - 1| < 1$ then the inequality is sharp for the function $p(z) = (1 + z^n)/(1 - z^n)$ or its rotations.

### 2. Main Results

**Theorem 2.1.** Let $f \in \mathcal{S}$ be of the form (1.1). Then

(i) $|T_2(n)| = |a_n^2 - a_{n+1}^2| \leq 2n^2 + 2n + 1$ for $n \geq 2$,

(ii) $|T_3(1)| \leq 24$.

Both inequalities are sharp.

**Proof.** Let $f \in \mathcal{S}$ be of the form (1.1). Then clearly

$$(2.1) \quad |T_2(n)| = |a_n^2 - a_{n+1}^2| \leq |a_n^2| + |a_{n+1}^2| \leq n^2 + (n + 1)^2 = 2n^2 + 2n + 1.$$  

Equality holds in (2.1) for the function $f$ defined by

$$(2.2) \quad f(z) := \frac{z}{(1 - iz)^2} = z + 2iz^2 - 3z^3 - 4iz^4 + 5z^5 + \cdots.$$  

Again, if $f \in \mathcal{S}$ is of the form (1.1) then by the Fekete-Szegö inequality for functions in $\mathcal{S}$, we have

$$(2.3) \quad |T_3(1)| = |1 - 2a_2^2 + 2a_2a_3 - a_3^2|$$

$$\leq 1 + 2|a_2^2| + |a_3||a_3 - 2a_2^2|$$

$$\leq 1 + 8 + (3)(5)$$

$$= 24.$$  

Equality holds in (2.3) for the function $f$ defined by (2.2). \qed

**Remark 2.1.** Since the function $f$ defined by (2.2) belongs to $\mathcal{S}^*$, and $\mathcal{S}^* \subset \mathcal{K} \subset \mathcal{S}$, the sharp inequalities in Theorem 2.1 also hold for functions in $\mathcal{S}^*$ and $\mathcal{K}$. In particular, the sharp inequalities $|T_2(2)| \leq 13$ and $|T_2(3)| \leq 25$ hold for functions in $\mathcal{S}^*$, $\mathcal{K}$ and $\mathcal{S}$.

**Theorem 2.2.** Let $f \in \mathcal{S}^*$ be of the form (1.1). Then $|T_3(2)| \leq 84$.

The inequality is sharp.
Proof. Let \( f \in \mathcal{S}' \) be of the form (1.1). Then there exists a function \( p \in \mathcal{P} \) of the form (1.3) such that \( z f'(z) = f(z) p(z) \). Equating coefficients, we obtain

\[
(2.4) \quad a_2 = c_1, \quad a_3 = \frac{1}{2}(c_2 + c_1^2) \quad \text{and} \quad a_4 = \frac{1}{6}c_1^3 + \frac{1}{2}c_1c_2 + \frac{1}{3}c_3.
\]

By a simple computation \( T_3(2) \) can be written as \( T_3(2) = (a_2-a_4)(a_3^2-2a_3^2+a_2a_4) \). If \( f \in \mathcal{S}' \) then clearly, \( |a_2-a_4| \leq |a_2| + |a_4| \leq 6 \). Thus we need to maximize \( |a_2^2-2a_3^2+a_2a_4| \) for functions in \( \mathcal{S}' \), and so writing \( a_2, a_3 \) and \( a_4 \) in terms of \( c_1, c_2 \) and \( c_3 \) with the help of (2.4), we obtain

\[
|a_2^2-2a_3^2+a_2a_4| = \left| c_1^2 - \frac{1}{3}c_1^4 - \frac{1}{2}c_1^2c_2 - \frac{1}{2}c_2^2 + \frac{1}{3}c_3c_1 \right|
\]

\[
\leq |c_1|^2 + \frac{1}{3}|c_1|^4 + \frac{1}{2}|c_2|^2 + \frac{1}{3}|c_1||c_3| - \frac{3}{2}c_1c_2.
\]

From Lemma 1.1 and Lemma 1.2 it easily follows that

\[
(2.5) \quad |a_2^2-2a_3^2+a_2a_4| \leq 4 + \frac{16}{3} + \frac{4}{2} + \frac{2}{3}(4) = 14.
\]

Therefore, \( |T_3(2)| \leq 84 \), and the inequality is sharp for the function \( f \) defined by (2.2). \( \square \)

Remark 2.2. In [9], it was claimed that \( |T_2(2)| \leq 5 \), \( |T_2(3)| \leq 7 \), \( |T_3(1)| \leq 8 \) and \( |T_3(2)| \leq 12 \) hold for functions in \( \mathcal{S}' \), and these estimates are sharp. Similar results were also obtained for certain close-to-convex functions. For the function \( f \) defined by (2.2), a simple computation gives \( |T_3(2)| = 13 \) and \( |T_3(3)| = 25 \), \( |T_3(1)| = 24 \) and \( |T_3(2)| = 84 \) which shows that these estimates are not correct. In proving these estimates the authors assumed that \( c_1 > 0 \) which is not justified, since the functional \( |T_q(n)| \) \( (n \geq 1, q \geq 2) \) is not rotationally invariant.

To prove our next result we need the following results for functions in \( \mathcal{S}' \).

Lemma 2.1. [3] Theorem 3.1] Let \( g \in \mathcal{S}' \) and be of the form \( g(z) = z + \sum_{n=2}^{\infty} b_n z^n \). Then \( |b_2b_4-b_3^2| \leq 1 \), and the inequality is sharp for the Koebe function \( k(z) = z/(1-z)^2 \), or its rotations.

Lemma 2.2. [4] Lemma 3] Let \( g \in \mathcal{S}' \) be of the form \( g(z) = z + \sum_{n=2}^{\infty} b_n z^n \). Then for any \( \lambda \in \mathbb{C} \),

\[
|b_3 - \lambda b_2^2| \leq \max\{1, |3-4\lambda|\}.
\]

The inequality is sharp for \( k(z) = z/(1-z)^2 \), or its rotations if \( |3-4\lambda| \geq 1 \), and for \( (k(z^2))^{1/2} \), or its rotations if \( |3-4\lambda| < 1 \).

Lemma 2.3. [5] Theorem 2.2] Let \( g \in \mathcal{S}' \) be of the form \( g(z) = z + \sum_{n=2}^{\infty} b_n z^n \). Then

\[
|\lambda b_n b_{m-n-1}| \leq \lambda nm - (n + m - 1) \quad \text{for} \quad \lambda \geq \frac{2(n + m - 1)}{nm},
\]

where \( n, m = 2, 3, \ldots \). The inequality is sharp for the Koebe function \( k(z) = z/(1-z)^2 \), or its rotations.

Lemma 2.4. Let \( f \in \mathcal{K} \) be of the form (1.1). Then \( |a_2a_4 - 2a_3^2| \leq 21/2 \).
Proof. Let \( f \in \mathcal{K} \) be of the form (1.1). Then there exists a starlike function \( g \) of the form \( g(z) = z + \sum_{n=2}^{\infty} b_n z^n \), and a real number \( \alpha \in (-\pi/2, \pi/2) \), such that (1.2) holds. This implies there exists a Carathéodory function \( p \in \mathcal{P} \) of the form (1.3) such that
\[
e^{ia}z f'(z) = p(z) \cos \alpha + i \sin \alpha.
\]
Comparing coefficients we obtain
\[
2a_2 = b_2 + c_1 e^{-ia} \cos \alpha
\]
\[
3a_3 = b_3 + b_2 c_1 e^{-ia} \cos \alpha + c_2 e^{-ia} \cos \alpha
\]
\[
4a_4 = b_4 + b_3 c_1 e^{-ia} \cos \alpha + b_2 c_2 e^{-ia} \cos \alpha + c_3 e^{-ia} \cos \alpha,
\]
and a simple computation gives
\[
72(a_2 a_4 - 2a_3^2) = (9b_2 b_4 - 16b_3^2) + (9b_4 - 23b_2 b_3) c_1 e^{-ia} \cos \alpha
\]
\[
+ (9b_3 - 16b_2^2) c_2^2 e^{-2ia} \cos^2 \alpha + (9b_2^2 - 32b_3) c_2 e^{-ia} \cos \alpha
\]
\[
+ (9c_3 - 23c_2 e^{-ia} \cos \alpha) b_2 e^{-ia} \cos \alpha + (9c_1 c_3 - 16c_2^2) e^{-2ia} \cos^2 \alpha.
\]
Consequently using the triangle inequality, we obtain
\[
(2.6)
72|a_2 a_4 - 2a_3^2| \leq |9b_2 b_4 - 16b_3^2| + |9b_4 - 23b_2 b_3||c_1| + |9b_3 - 16b_2^2||c_2^2|
\]
\[
+ |9b_2^2 - 32b_3||c_2| + |9c_3 - 23c_1 e^{-ia} \cos \alpha||b_2| + |9c_1 c_3 - 16c_2^2|.
\]
By Lemma 2.1, Lemma 2.2 and Lemma 2.3 it easily follows that
\[
(2.7)
|9b_2 b_4 - 16b_3^2| \leq 9|b_2 b_4 - b_3^2| + 7|b_3|^2 \leq 9 + 63 = 72,
\]
\[
(2.8)
|9b_4 - 23b_2 b_3| = 9\left|b_4 - \frac{23}{9}b_2 b_3\right| \leq 9\left(\frac{46}{9} - 4\right) = 102,
\]
\[
(2.9)
|9b_3 - 16b_2^2| = 9\left|b_3 - \frac{16}{9}b_2^2\right| \leq 9\left(\frac{64}{9} - 3\right) = 37,
\]
\[
(2.10)
|9b_2^2 - 32b_3| = 32\left|b_3 - \frac{9}{32}b_2^2\right| \leq 32\left(3 - \frac{9}{8}\right) = 60.
\]
Again, by Lemma 1.2 it easily follows that
\[
|9c_3 - 23c_1 e^{-ia} \cos \alpha| = 9|c_3 - \mu c_1 c_2| \leq 18 \max\{1,|2\mu - 1|\}
\]
where \( \mu = \frac{23}{9} e^{-ia} \cos \alpha \). Now note that
\[
|2\mu - 1|^2 = \left(\frac{23}{9} \cos 2\alpha + \frac{14}{9}\right)^2 + \left(\frac{23}{9} \sin 2\alpha\right)^2
\]
\[
= \left(\frac{23}{9}\right)^2 + \left(\frac{14}{9}\right)^2 + 2 \left(\frac{23}{9}\right) \left(\frac{14}{9}\right) \cos 2\alpha,
\]
and so
\[
1 \leq |2\mu - 1| \leq \frac{37}{9}
\]
Therefore
\[(2.11) \quad |9c_3 - 23c_1c_2e^{-i\alpha} \cos \alpha| \leq 74.\]

Again by Lemma 1.2 it easily follows that
\[(2.12) \quad |9c_1c_3 - 16c_2^2| \leq 9|c_1c_3 - c_4| + 9 \left| c_4 - \frac{16}{9}c_2^2 \right| \leq 18 + 46 = 64.\]

By Lemma 1.1 and using the inequalities (2.7), (2.8), (2.9), (2.10), (2.11) and (2.12) in (2.6), we obtain
\[|a_2a_4 - 2a_3^2| \leq \frac{1}{12}(72 + 204 + 148 + 120 + 148 + 64) = \frac{21}{2}.\]

**Theorem 2.3.** Let \(f \in \mathcal{K}\) be of the form (1.1). Then \(|T_3(2)| \leq 86\).

**Proof.** Let \(f \in \mathcal{K}\) be of the form (1.1). Then by Lemma 2.4 we have
\[|T_3(2)| = |a_2^3 - 2a_2a_3^2 - a_2a_4^2 + 2a_4a_3| \leq |a_2|^3 + 2|a_2||a_3|^2 + |a_4||a_2a_4 - 2a_3^2| \leq 8 + 36 + 42 = 86.\]

**Remark 2.3.** In Theorem 2.2, we have proved that \(|T_3(2)| \leq 84\) for functions in \(S^*\), and the inequality is sharp for the function \(f\) defined by (2.2). Therefore it is natural to conjecture that \(|T_3(2)| \leq 84\) holds for functions in \(\mathcal{K}\) and that equality holds for the function \(f\) defined by (2.2).

**Theorem 2.4.** Let \(f \in \mathcal{C}\) be of the form (1.1). Then
\[(i) \quad |T_2(n)| \leq 2 \text{ for } n \geq 2.
(ii) \quad |T_3(1)| \leq 4.
(iii) \quad |T_3(2)| \leq 4.

All the inequalities are sharp.

**Proof.** Let \(f \in \mathcal{C}\) be of the form (1.1). Then there exists a function \(p \in \mathcal{P}\) of the form (1.3) such that \(f'(z) + zf''(z) = f'(z)p(z)\). Equating coefficients, we obtain
\[(2.13) \quad 2a_2 = c_1, \quad 3a_3 = \frac{1}{2}(c_2 + c_1^2) \quad \text{and} \quad 4a_4 = \frac{1}{6}c_1^3 + \frac{1}{2}c_1c_2 + \frac{1}{3}c_3.\]

Clearly
\[(2.14) \quad |T_2(n)| = |a_n^2 - a_{n+1}^2| \leq |a_n^2| + |a_{n+1}^2| \leq 1 + 1 = 2.

Equality holds in (2.14) for the function \(f\) defined by
\[(2.15) \quad f(z) := \frac{z}{1 - iz} = z + iz^2 - z^3 - iz^4 + z^5 + \cdots.\]
Again if \( f \in \mathcal{C} \) is of the form (1.1) then from Lemma 1.2 and (2.13), we obtain

\[
|T_3(1)| = |1 - 2a_2^2 + 2a_2^2a_3 - a_3^2| \\
\leq 1 + 2|a_2^2| + |a_3||a_3 - 2a_2^2| \\
\leq 1 + 2 + \frac{1}{6}|c_2 - 2c_1^2| \\
\leq 4.
\]

It is easy to see that equality holds in (2.16) for the function \( f \) defined by (2.15).

Next note that \( T_3(2) = (a_2^2 - a_4)(a_4^2 - 2a_3^2 + a_2a_4) \). If \( f \in \mathcal{C} \) then clearly \(|a_2 - a_4| \leq |a_2| + |a_4| \leq 2\). Thus we need to maximize \(|a_2^2 - 2a_3^2 + a_2a_4|\) for functions in \( \mathcal{C} \).

Writing \( a_2, a_3 \) and \( a_4 \) in terms of \( c_1, c_2 \) and \( c_3 \) with the help of (2.13), we obtain

\[
|a_2^2 - 2a_3^2 + a_2a_4| = \frac{1}{144} \left| 5c_1^4 - 36c_1^2 + 7c_2^2c_2 + 8c_1^2 - 6c_1c_3 \right| \\
\leq \frac{1}{144} \left( 5|c_1|^4 + 36|c_1|^2 + 8|c_2|^2 + 6|c_1||c_3 - \frac{7}{6}c_1c_2| \right).
\]

From Lemma 1.1 and Lemma 1.2 it easily follows that

\[
|a_2^2 - 2a_3^2 + a_2a_4| \leq \frac{1}{144}(80 + 144 + 32 + 32) = 2.
\]

Therefore, \(|T_3(2)| \leq 4\), and the inequality is sharp for the function \( f \) defined by (2.15).

**Theorem 2.5.** Let \( f \in \mathcal{R} \) be of the form (1.1). Then

(i) \(|T_2(n)| \leq \frac{4}{n^2} + \frac{4}{(n+1)^2}\) for \( n \geq 2 \).

(ii) \(|T_3(1)| \leq \frac{35}{9}\).

(iii) \(|T_3(2)| \leq \frac{7}{3}\).

The inequalities in (i) and (ii) are sharp.

**Proof.** Let \( f \in \mathcal{R} \) be of the form (1.1). Then there exists a function \( p \in \mathcal{P} \) of the form (1.3) such that \( f'(z) = p(z) \). Equating coefficients we obtain \( na_n = c_{n-1} \), and so

\[
|a_n| = \frac{1}{n}|c_{n-1}| \leq \frac{2}{n}, \quad n \geq 2.
\]

The inequality is sharp for the function \( f \) defined by \( f'(z) = (1 + z)/(1 - z) \), or its rotations. Thus

\[
|T_2(n)| = |a_n^2 - a_{n+1}^2| \leq |a_n^2| + |a_{n+1}^2| \leq \frac{4}{n^2} + \frac{4}{(n+1)^2}.
\]

Equality holds in (2.18) for the function \( f \) defined by

\[
f'(z) := \frac{1 + iz}{1 - iz}.
\]
Next, if $f \in \mathcal{R}$ is of the form (1.1) then

\begin{align}
|T_3(1)| &= |1 - 2a_2^2 + 2a_2^2a_3 - a_3^2| \\
&\leq 1 + 2|a_2^2| + |a_3||a_3 - 2a_2^2| \\
&\leq 1 + 2 + \frac{2}{3}\left|\frac{1}{3}c_2 - \frac{1}{2}c_1^2\right| \\
&\leq 3 + \frac{2}{9}\left|c_2 - \frac{3}{2}c_1^2\right| \\
&\leq 3 + \frac{8}{9} = \frac{35}{9}.
\end{align}

It is easy to see that equality in (2.20) holds for the function $f$ defined by (2.19).

Again, if $f \in \mathcal{R}$ is of the form (1.1) then

\begin{align}
|T_3(2)| &= |a_2^3 - 2a_2a_3^2 - a_2a_4^2 + 2a_3^2a_4| \\
&\leq |a_2|^3 + 2|a_2||a_3^2| + |a_4||a_2a_4 - 2a_3^2| \\
&\leq 1 + \frac{8}{9} + \frac{1}{2}|a_2a_4 - 2a_3^2| \\
&\leq \frac{17}{9} + \frac{1}{2}|a_2a_4 - 2a_3^2|.
\end{align}

Thus we need to find the maximum value of $|a_2a_4 - 2a_3^2|$ for functions in $\mathcal{R}$. By (2.12), it easily follows that

$$|a_2a_4 - 2a_3^2| = \frac{1}{72} |9c_1c_3 - 16c_2^2| \leq \frac{64}{72} = \frac{8}{9}.$$  

Therefore

$$|T_3(2)| \leq \frac{17}{9} + \frac{4}{9} = \frac{7}{3}.$$

**Remark 2.4.** The above theorem shows that for $f \in \mathcal{R}$, the sharp inequalities $|T_2(2)| \leq 13/9$ and $|T_2(3)| \leq 17/36$ hold. In [5], it was claimed that $|T_2(2)| \leq 5/9$, $|T_2(3)| \leq 4/9$, $|T_3(1)| \leq 13/9$ and $|T_3(2)| \leq 4/9$ hold for functions in $\mathcal{R}$ and these estimates are sharp. For the function $f$ defined by (2.19), a simple computation gives $|T_3(2)| = 13/9$, $|T_2(3)| = 17/36$, $|T_3(1)| = 35/9$ and $|T_3(2)| \leq 25/12$, showing that theses estimates are not correct. As explained above, the authors assumed that $c_1 > 0$, which is not justified, since the functional $|T_q(n)|$ ($n \geq 1, q \geq 2$) is not rotationally invariant.

If $f \in \mathcal{T}$ is given by (1.1), then the coefficients of $f$ can be expressed by

$$a_n = \int_{-1}^{1} \frac{\sin(n \arccos t)}{\sin(\arccos t)} d\mu(t) = \int_{-1}^{1} U_{n-1}(t) d\mu(t), \quad n \geq 1$$

where $U_n(t)$ are Chebyshev polynomials of degree $n$ of the second kind.

Let $A_{n,m}$ denote the region of variability of the point $(a_n, a_m)$, where $a_n$ and $a_m$ are coefficients of a given function $f \in \mathcal{T}$ with the series expansion (1.1), i.e.,
Lemma 2.6. The boundary of $F_p$ curve $F$ functions $F$ functions $F$ functions $X$ solutions. According to Theorem 2.6, the boundary of the convex hull of the functions $A_{n,m} := \{(a_n(f), a_m(f)) : f \in T\}$. Therefore, $A_{n,m}$ is the closed convex hull of the curve $\gamma_{n,m} : [-1, 1] \ni t \mapsto (U_{n-1}(t), U_{m-1}(t))$.

By the Caratheodory theorem we conclude that it is sufficient to discuss only functions

$$F(z, \alpha, t_1, t_2) := \alpha k(z, t_1) + (1 - \alpha) k(z, t_2),$$

where $0 \leq \alpha \leq 1$ and $-1 \leq t_1 \leq t_2 \leq 1$.

Let $X$ be a compact Hausdorff space, and $J_\mu = \int_X J(t) d\mu(t)$. Szapiel [8] proved the following theorem.

**Theorem 2.6.** Let $J : [\alpha, \beta] \to \mathbb{R}^n$ be continuous. Suppose that there exists a positive integer $k$, such that for each non-zero $\overrightarrow{p}$ in $\mathbb{R}^n$ the number of solutions of any equation $\langle \overrightarrow{J(t)}, \overrightarrow{p} \rangle = \text{const}$, $\alpha \leq t \leq \beta$ is not greater than $k$. Then, for every $\mu \in P_{[\alpha, \beta]}$ such that $J_\mu$ belongs to the boundary of the convex hull of $J([\alpha, \beta])$, the following statements are true:

1. if $k = 2m$, then
   a. $|\text{supp}(\mu)| \leq m$, or
   b. $|\text{supp}(\mu)| = m + 1$ and $\{\alpha, \beta\} \subset \text{supp}(\mu)$.

2. if $k = 2m + 1$, then
   a. $|\text{supp}(\mu)| \leq m$, or
   b. $|\text{supp}(\mu)| = m + 1$ and one of the points $\alpha$ and $\beta$ belongs to supp(\mu).

In the above, the symbol $\langle \overrightarrow{u}, \overrightarrow{v} \rangle$ means the scalar product of vectors $\overrightarrow{u}$ and $\overrightarrow{v}$, whereas the symbols $P_X$ and $|\text{supp}(\mu)|$ describe the set of probability measures on $X$, and the cardinality of the support of $\mu$, respectively.

Putting $J(t) = (U_1(t), U_2(t))$, $t \in [-1, 1]$ and $\overrightarrow{p} = (p_1, p_2)$, we can see that any equation of the form $p_1 U_1(t) + p_2 U_2(t) = \text{const}$, $t \in [-1, 1]$ has at most 2 solutions. According to Theorem 2.6, the boundary of the convex hull of $J([-1, 1])$ is determined by atomic measures $\mu$ for which support consists of at most 2 points. Thus we have the following:

**Lemma 2.5.** The boundary of $A_{2,3}$ consists of points $(a_2, a_3)$ that correspond to the functions $F(z, 1, t, 0) = k(z, t)$ or $F(z, \alpha, 1, -1)$ with $0 \leq \alpha \leq 1$ and $-1 \leq t \leq 1$ where $F(z, \alpha, t_1, t_2)$ is defined by (2.21).

In a similar way, one can obtain the following:

**Lemma 2.6.** The boundary of $A_{3,4}$ consists of points $(a_3, a_4)$ that correspond to the functions $F(z, \alpha, t, -1)$ or $F(z, \alpha, t, 1)$ with $0 \leq \alpha \leq 1$ and $-1 \leq t \leq 1$ where $F(z, \alpha, t_1, t_2)$ is defined by (2.21).

Before we proceed further, we give some example of typically real functions.
Example 2.1. For each $t \in [-1, 1]$, the function $k(z, t) = z/(1 - 2tz + z^2)$ is a typically real function. For the function $k(z, 1) = z/(1 - z)^2$, we have $T_2(n) = n^2 - (n+1)^2 = -(2n+1)$, and $T_3(n) = a_n^3 - 2a_{n+1}a_n - a_{n+2}^2a_n + 2a_{n+1}^2a_n = 4(n+1)$.

Example 2.2. The function $f(z) = -\log(1 - z) = z + \sum_{n=2}^{\infty} (1/n)z^n$ is a typically real function. For this function, we have $T_2(n) = 1/n^2 - 1/(n+1)^2$ and $T_3(n) = 4(n^2 + 3n + 1)/(n^3(n+1)^2(n+2)^2)$.

Lemma 2.7. If $f \in T$ then $T_2(n)$ attains its extreme values on the boundary of $A_{n,n+1}$.

Proof. Let $\phi(x, y) = x^2 - y^2$, where $x = a_n$ and $y = a_{n+1}$. The only critical point of $\phi$ is $(0, 0)$ and $\phi(0,0) = 0$. Since $\phi$ may be positive as well as negative for $(x, y) \in A_{n,n+1}$ (see Example 2.1 and Example 2.2), the extreme values of $\phi$ are attained on the boundary of $A_{n,n+1}$. □

In a similar way, we can prove the following:

Lemma 2.8. If $f \in T$ then $T_3(1)$ attains its extreme values on the boundary of $A_{2,3}$.

Since all coefficients of $f \in T$ are real, we look for the lower and the upper bounds of $T_q(n)$ instead of the bound of $|T_q(n)|$. The proof of the following theorem is obvious.

Theorem 2.7. For every function $f \in T$ of the form (1.1), we have $-(n+1)^2 \leq T_2(n) \leq n^2$.

In particular

(i) if $n$ is odd then $\max\{T_2(n) : f \in T\} = n^2$ and equality attained for the function $F(z, 1/2, 1, -1)$.

(ii) if $n$ is even then $\min\{T_2(n) : f \in T\} = -(n+1)^2$ and equality attained for the function $F(z, 1/2, 1, -1)$.

Theorem 2.8. For $f \in T$, $\max\{T_2(2) : f \in T\} = 5/4$.

Proof. By Lemma 2.5 it is enough to consider the functions $F(z, 1, t, 0) = k(z, t)$ and $F(z, \alpha, 1, -1)$ with $0 \leq \alpha \leq 1$ and $-1 \leq t \leq 1$.

Case 1. For the function $F(z, 1, t, 0) = k(z, t) = z + 2tz^2 + (4t^2 - 1)z^3 + (8t^3 - 4t)z^4 + \cdots$, we have $a_2^2 - a_3^2 = -16t^4 + 12t^2 - 1 \leq 5/4$. 

Case 2. For the function $F(z, \alpha, 1, -1) = z + (4\alpha - 2)z^2 + 3z^3 + (8\alpha - 4)z^4 + \cdots$, we have $a_2^2 - a_3^2 = (2 - 4\alpha)^2 - 9 \leq -5$.

The conclusion follows from Cases 1 and 2, with the maximum attained for the function $F(z, 1, t, 0) = k(z, t)$ with $t = \frac{\sqrt{3}}{2\sqrt{2}}$. □

Corollary 2.1. For $f \in T$, we have the sharp inequality $-9 \leq T_2(2) \leq 5/4$. 

Theorem 2.9. For $f \in T$, we have $\min \{ T_2(3) : f \in T \} = -7$.

Proof. By Lemma 2.6, it is enough to consider the functions $F(z, \alpha, t, -1)$ and $F(z, \alpha, t, -1)$ with $0 \leq \alpha \leq 1$ and $-1 \leq t \leq 1$.

Case 1. For the function $F(z, \alpha, t, -1) = z + 2(\alpha + \alpha t - 1)z^2 + (4\alpha t^2 - 4\alpha + 3)z^3 + (4\alpha + 8\alpha t^3 - 4\alpha t - 4)z^4 + \cdots$, we have

$$T_2(3) = a_3^2 - a_4^2 = (4\alpha t^2 - 4\alpha + 3)^2 - (4\alpha + 8\alpha t^3 - 4\alpha t - 4)^2 : = \phi(\alpha, t).$$

By elementary calculus, one can verify that

$$\min_{0 \leq \alpha \leq 1, -1 \leq t \leq 1} \phi(\alpha, t) = \phi(0, 0) = -7.$$

Case 2. For the function $F(z, \alpha, t, 1) = z + 2(1 - \alpha + \alpha t)z^2 + (3 - 4\alpha + 4\alpha t^2)z^3 + (4 - 4\alpha - 4\alpha t + 8\alpha t^3)z^4 + \cdots$, we have $a_2^2 - a_3^2 = \phi(\alpha, -t)$, and so

$$\min_{0 \leq \alpha \leq 1, -1 \leq t \leq 1} \phi(\alpha, -t) = \phi(0, 0) = -7.$$

The conclusion follows from Cases 1 and 2, and the maximum is attained for the function $F(z, 0, 0, 1)$ or $F(z, 0, 0, -1)$.

Corollary 2.2. For $f \in T$, we have the sharp inequality $-7 \leq T_2(3) \leq 9$.

Theorem 2.10. For $f \in T$, we have $\max \{ T_3(1) : f \in T \} = 8$, and $\min \{ T_3(1) : f \in T \} = -8$.

Proof. By Lemma 2.6, it is enough to consider the functions $F(z, 1, t, 0) = k(z, t)$ and $F(z, \alpha, 1, -1)$ with $0 \leq \alpha \leq 1$ and $-1 \leq t \leq 1$.

Case 1. For the function $F(z, 1, t, 0) = k(z, t) = z + 2tz^2 + (4t^2 - 1)z^3 + (8t^3 - 4t)z^4 + \cdots$, we have $T_3(1) = 1 - 2a_d^2 + 2a_d^2a_3 - a_3^2 = 8t^2(2t^2 - 1) : = \phi_1(t)$, and it is easy to verify that

$$\max_{-1 \leq t \leq 1} \phi_1(t) = \phi_1(-1) = 8 \quad \text{and} \quad \min_{-1 \leq t \leq 1} \phi_1(t) = \phi_1(-1/2) = -1.$$

Case 2. For the function $F(z, \alpha, 1, -1) = z + (4\alpha - 2)z^2 + 3z^3 + (8\alpha - 4)z^4 + \cdots$, we have $T_3(1) = 8(8\alpha^2 - 8\alpha + 1) : = \psi_1(\alpha)$,

and it is again easy to verify that

$$\max_{0 \leq \alpha \leq 1} \psi_1(\alpha) = \psi_1(0) = 8 \quad \text{and} \quad \min_{0 \leq \alpha \leq 1} \psi_1(\alpha) = \psi_1(1/2) = -8.$$

The conclusion follows from Cases 1 and 2, and the maximum is attained for the function $F(z, 1, -1, 0) = k(z, -1)$, and the minimum is attained for the function $F(z, 1/2, 1, -1)$.

Acknowledgement: The authors thank Prof. K.-J. Wirths for useful discussion and suggestions.
REFERENCES

[1] P. L. Duren, *Univalent functions* (Grundlehren der mathematischen Wissenschaften 259, New York, Berlin, Heidelberg, Tokyo), Springer-Verlag, 1983.

[2] I. Eframidis, A generalization of Livingston’s coefficient inequalities for functions with positive real part, *J. Math. Anal. Appl.* 435 (2016) (1), 369–379.

[3] A. Janteng, S. A. Halim and M. Darus, Hankel determinant for starlike and convex functions, *Int. J. Math. Anal.* 1 (2007), 619–625.

[4] W. Koepf, On the Fekete-Szegö problem for close-to-convex functions, *Proc. Amer. Math. Soc.* 101 (1987), 89–95.

[5] W. Ma, Generalized Zalcman conjecture for starlike and typically real functions, *J. Math. Anal. Appl.* 234 (1999)(1), 328–339.

[6] V. Radhika, S. Sivasubramanian, G. Murugusundaramoorthy and J. M. Jahangiri, Toeplitz matrices whose elements are the coefficients of functions with bounded boundary rotation, *J. Complex Anal.*, vol. 2016 (2016), Article ID 4960704, 4 pages.

[7] M. S. Robertson, On the coefficients of a typically-real function, *Bull. Amer. Math. Soc.*, 41, (1935), 565–572.

[8] W. Szapiel, Extremal problems for convex sets. Applications to holomorphic functions, Dissertation Ann. Univ. Mariae Curie-Sklodowska, Sect. A. 37 (1986).

[9] D. K. Thomas and S. A. Halim, Toeplitz matrices whose elements are the coefficients of starlike and close-to-convex functions, *Bull. Malays. Math. Sci. Soc.*, DOI 10.1007/s40840-016-0385-4, 10 pages.

[10] K. Ye and L.-H. Lim, Every matrix is a product of Toeplitz matrices, *Found. Comput. Math.* 16 (2016), 577–598.

Md Firoz Ali, Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur-721 302, West Bengal, India.

E-mail address: ali.firoz89@gmail.com

D. K. Thomas, Department of Mathematics, Swansea University, Singleton Park, Swansea, SA2 8PP, United Kingdom.

E-mail address: d.k.thomas@swansea.ac.uk

A. Vasudevarao, Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur-721 302, West Bengal, India.

E-mail address: alluvasu@maths.iitkgp.ernet.in