Model-Free Algorithm and Regret Analysis for MDPs with Long-Term Constraints

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Abstract

In the optimization of dynamical systems, the variables typically have constraints. Such problems can be modeled as a constrained Markov Decision Process (CMDP). This paper considers a model-free approach to the problem, where the transition probabilities are not known. In the presence of long-term (or average) constraints, the agent has to choose a policy that maximizes the long-term average reward as well as satisfy the average constraints in each episode. The key challenge with the long-term constraints is that the optimal policy is not deterministic in general, and thus standard Q-learning approaches cannot be directly used. This paper uses concepts from constrained optimization and Q-learning to propose an algorithm for CMDP with long-term constraints. For any $\gamma \in (0, \frac{1}{2})$, the proposed algorithm is shown to achieve $O(T^{1/2+\gamma})$ regret bound for the obtained reward and $O(T^{1-\gamma/2})$ regret bound for the constraint violation, where $T$ is the total number of steps. We note that these are the first results on regret analysis for MDP with long-term constraints, where the transition probabilities are not known apriori.

1 Introduction

Optimization of dynamical systems typically have constraints, e.g., average power and delay constraints in communication systems [1], average resource constraints in finance [1], etc. The dynamical systems are typically modeled as a Markov Decision Process (MDP), while the transition probabilities may not be known apriori (or may be dynamic). In the absence of knowledge of transitions, the MDP is modeled as a Reinforcement Learning (RL) problem which aims to maximize the long-time reward by taking actions given the state of the process to be controlled. RL algorithm can be divided into model-based and model-free, where the model-based approaches estimate the transition probabilities, while model-free approaches do not. In this paper, we consider a model-free approach to RL in the presence of average (or long-term) constraints.

When the system model (the transition probability distribution, the reward function, and the constraint function) is known, the problem is generally considered as Constrained Markov Decision Process (CMDP). CMDP in the form of discounted and average reward has been studied in [3], It is well known that
CMDP problem is convex and can be converted into an equivalent unconstrained MDP problem by using the method of Lagrange multipliers. Thus, when the model is known, CMDP can be solved using linear programming (LP) or dynamic programming (DP). In addition to the LP method, [13] proposed three different algorithms, WeiMDP, AugMDP, and RecMDP, to solve CMDP in different settings. Recently, [27] considered online constrained MDP and proposed an algorithm with tight bound $O(\sqrt{T})$ regret and constraint violation. However, these works assumed a known system model, while we provide algorithms without the knowledge of the state transitions.

Recently, model-free approaches have been developed for CMDP with long-term constraints. Using Lagrange multipliers, [21, 8, 18, 25, 22] proposed policy gradient, actor-critic, Q-learning, or trust region policy optimization methods for CMDP or constrained risk-sensitive reinforcement learning. These algorithms either do not have convergence guarantees or are shown to converge to saddle-points of the Lagrangian using two-time-scale stochastic approximations [7]. However, due to the projection on the Lagrange multipliers, the saddle-point achieved by these approaches might not be the stationary point of the original CMDP problem. An iterative algorithm based on a novel construction of Lyapunov functions is proposed in [9], and is shown to converge to the optimal policy. [29] formulates policy optimization as a constrained optimization to prove almost surely convergence to a stationary point of original CMDP problem with large state and action space. The authors of [12] provided an algorithm using Markov-Bandit games for CMDP, which is shown to converge to the optimal solution. However, these works do not consider the regret analysis of the CMDP with long-term constraints. Besides, all the above algorithms are based on an infinite horizon setting, which is different from the finite horizon setting considered in this paper. We note that the optimal policy in finite horizon case is not stationary, which is an important difference from the infinite horizon case. For the finite horizon case, [10] proposed a Linear Programming formulation for the Constrained MDP. [19] proposed Multi-criteria based heuristic algorithm and Simultaneously Perturbation Stochastic Approximation (SPSA) algorithm to linearize the optimal solution online and prove its convergence. However, these algorithms are model-based, which means it is necessary to access the transition dynamics. Thus, there is no model-free algorithm for the MDP problem with average constraints on a finite horizon with sub-linear regret analysis.

Regret analysis for reinforcement learning has been considered for both the model-based approaches [14, 2, 3, 16] and the model-free approaches [17, 24, 15]. Our paper aims to use model-free approaches to come up with an efficient algorithm for episodic MDP with long-term constraints, with a provable regret bound. One of the key challenge in the use of Q-learning directly is that the optimal policy with long-term constraints is not deterministic, in general. To see this, we consider an example, where there is only one state and two actions are possible. We consider an episodic MDP with time-horizon $H = 1$. The reward is 1 for action 1 and -1 for action 2, while the constraint function is -1 for action 1 and 1 for action 2. We aim to find a policy (choice of actions) that maximizes the achievable reward such that the average constraint greater than or equal
to 0. Choosing deterministic policy of action 1 does not satisfy the constraint while the deterministic policy of action 2 provides reward of $-1$ for the problem. Thus, the optimal deterministic policy gives a reward of $-1$ while satisfying the average constraints. In constrast, randomly choosing the two actions uniformly achieves a reward of 0 while satisfying the average constraints. This indicates that algorithms for CMDP with average constraints must consider stochastic policies. The standard Q-learning policies provide deterministic policies. In this paper, we use the concepts of game theory and optimization theory to provide a Q-learning based algorithm which has stochastic policies, and provides sub-linear regret for both the objective and constraint violations. Different from average constraints, peak constraints have been studied for convergence and regret analysis in model-free settings [6]. However, the peak constraints do not require stochastic policies, which make the problem complex in the case of average constraints.

**Contributions:** In this paper, we propose a novel model-free algorithm, which extends Q-learning to provide a stochastic policy using the concepts of constrained optimization theory. Since the algorithm is model-free, the algorithm works in the scenario where the transition probability is unknown, the reward function is observed, and the constraint function can be queried but does not need to be known in closed form. The constraints are accounted using the method of Lagrange multipliers while the policy is found using a max-min between the policy and the Lagrange multipliers. The proposed algorithm is analyzed and found to have sub-linear regret for both the objective and constraint violations. More precisely, for any $\gamma \in (0, \frac{1}{2})$, our algorithm achieves $O(T^{1/2+\gamma})$ regret bound for the obtained reward, and $O(T^{1-\gamma/2})$ regret bound for the constraint violations, where $T$ is the total number of steps. Further, the proposed algorithm is evaluated on a discrete time single server queue studied in [3], and the theoretical results are validated on this system.

## 2 Problem Formulation and Assumptions

We consider an episodic setting of the Constrained Markov Decision Process with finite state and action space, defined by CMDP($\mathcal{S}, \mathcal{A}, H, \mathbb{P}, r$), where $\mathcal{S}$ is the state space with $|\mathcal{S}| = S$, $\mathcal{A}$ is the set of action with $|\mathcal{A}| = A > 1$, $H$ is the number of steps in each episode, $\mathbb{P}$ is the transition matrix so that $\mathbb{P}_h(\cdot|s, a)$ gives the probability distribution over next state based on the state and action pair $(s, a)$ at the step $h$. Further, $r(s, a, 0) : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$ is the deterministic reward function and $r(s, a, j) : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$, $j = 1, \cdots, J$ are constraint functions.

In the RL setting, both the reward function and the constraint functions are unknown to the agent but can be measured given a state action pair $(s, a)$. In this paper, we make the following two assumptions.

**Assumption 1.** *The absolute values of the reward function $r(\cdot, \cdot, 0)$ and constraint functions $r(\cdot, \cdot, j)$, $j = 1, \cdots, J$ are strictly bounded by a constant known to the agent. Without loss of generality, we let this constant be 1.*
Assumption 2. The value of the reward function $r(\cdot, \cdot, 0)$ is non-negative, i.e., $0 \leq r(s, a, 0) \leq 1, \forall (s, a) \in \mathcal{S} \times \mathcal{A}$.

These assumptions on reward function are typical in reinforcement learning [13, 28, 4], and the bound of reward function can be normalized. Further, the reward can be shifted up by adding a constant to make the reward function non-negative.

We define the policy as a function that maps a state $s \in \mathcal{S}$ to a probability distribution of the actions with a probability assigned to each action $a \in \mathcal{A}$. In episodic setting, the policy $\pi$ is a collection of $H$ policy functions $\pi_h$ at each step, that is $\pi_h(s) = a$ with probability $\Pr(a|s, h)$.

Constrained RL problem aims to find the optimal policy that achieve the highest total reward subject to a set of average constraints, which can be formally stated as

$$\max_{\pi} \mathbb{E} \left[ \sum_{h=1}^{H} r(s_h, \pi_h(s_h), 0) \right] \quad \text{s.t.} \quad \mathbb{E} \left[ \sum_{h=1}^{H} r(s_h, \pi_h(s_h), j) \right] \geq 0 \forall j \in [J],$$

(1)

where the expectation is taken with respect to the randomness introduced by the policy $\pi$ and the transition mapping $\mathbb{P}$. At the beginning of each episode of Constrained MDP, an initial state is chosen arbitrarily. Further, an action $a_h$ is taken by the agent at step $h$ using the policy $\pi_h(\cdot|s_h)$, and the MDP transits to another state $s_{h+1}$ with the probability $\mathbb{P}_h(\cdot|s_h, a_h)$ for each $h = 1, \cdots, H$.

We use the state value function $V^\pi_h(s, 0) : \mathcal{S} \rightarrow \mathbb{R}$ to denote the value function at step $h$ under policy $\pi$, where $V^\pi_h(s, 0)$ is given as

$$V^\pi_h(s, 0) := \mathbb{E} \left[ \sum_{h'=h}^{H} r(s_{h'}, \pi_{h'}(s_{h'}), 0) | s_h = s \right]$$

(2)

Similarly, we define the state value function for the $j^{th}$ constraint as $V^\pi_h(s, j)$, given as

$$V^\pi_h(s, j) := \mathbb{E} \left[ \sum_{h'=h}^{H} r(s_{h'}, \pi_{h'}(s_{h'}), j) | s_h = s \right]$$

(3)

Finally, we define the set $\Pi$ to be the constrained set for which $\pi \in \Pi$ satisfies the constraints in Eq. (1). We denote the optimal policy for the Constrained MDP problem as $\ast$, which gives the highest reward and satisfies the constraints, i.e.,

$$V^\pi_h(s, 0) = \sup_{\pi \in \Pi} V^\pi_h(s, 0).$$

(4)

We note that in the MDP problem with average constraints, the optimal policy $\pi^*$ can be stochastic as explained in the example in Section 1. Assuming the agent takes actions for $K$ episodes, we define the regret and constraint violations as
Regret(K) = \[ \sum_{k=1}^{K} [V_1^*(s_k^1, 0) - V_1^\pi_k(s_k^1, 0)] \]

Violation(K) = \[ \sum_{k=1}^{K} \left| V_1^\pi_k(s_k^h, j) - \right| \]

The notation \( \pi_k \) is the policy given by the algorithm. Moreover, the notation \( [x]_- \) is defined as \( [x]_- := \min\{0, x\} \). For \( K \) episodes, each with \( H \) steps, the total number of steps are \( T = KH \).

## 3 Proposed Algorithm

Define \( \eta = T \gamma \). For any fixed \((k, h)\), define a new reward function with Lagrange multiplier \( \lambda^k = (\lambda_1^k, \ldots, \lambda_J^k) \) with \( \lambda_j^k \geq 0 \) for all \( j = 1, \ldots, J \), and a second-order regularization term, which is given as

\[
R(s_h^k, a_h^k, \lambda^k) = r(s_h^k, a_h^k, 0) + \sum_{j=1}^{J} \left( \lambda_j^k r(s_h^k, a_h^k, j) + \frac{(\lambda_j^k)^2}{4\eta} \right) \]

(6)

Based on the modified reward function, we define a counterpart of the value function \( W_h(s_h^k) \) as

\[
W_h(s_h^k, \lambda^k) = V_h(s_h^k, 0) + \sum_{j=1}^{J} \left( \lambda_j^k V_h(s_h^k, j) + \sum_{h'=h}^{H} \frac{(\lambda_j^k)^2}{4\eta} \right) \]

and the state-action value function \( U_h(s_h^k, a_h^k, \lambda^k) \) as

\[
U_h(s_h^k, a_h^k, \lambda^k) = Q_h(s_h^k, a_h^k, 0) + \sum_{j=1}^{J} \left( \lambda_j^k Q_h(s_h^k, a_h^k, j) + \sum_{h'=h}^{H} \frac{(\lambda_j^k)^2}{4\eta} \right) \]

(8)

Similarly, we define the optimal functions for \( W \) and \( U \) as

\[
W_h^*(s, \lambda) = \sup_{\pi} W_h^\pi(s, \lambda) \]

\[
U_h^*(s, a, \lambda) = \sup_{\pi} U_h^\pi(s, a, \lambda) \]

(9)

Using the notation

\[
[P_h W_{h+1}]_{(s, a, \lambda)} := E_{s' \sim P_h(\cdot|s, a)} W_{h+1}(s', \lambda), \]

the Bellman equation can be written as

\[
U_h^\pi(s, a, \lambda) \triangleq R(s, a, \lambda) + \mathbb{E} \left[ \sum_{h' = h+1}^{H} R(s_h', \pi_h'(s_h'), \lambda) \right| s_h = s, a_h = a \]

\[
= (R + P_h W_{h+1}^\pi)(s, a, \lambda) \]

(11)
Then, the modified unconstrained problem can be defined as finding the optimal policy for \( W_1^{\pi_k}(s_k^1, \lambda^k) \) and the regret for it should be

\[
\sum_{k=1}^{K} W_1^{\pi_k}(s_k^1, \lambda^k) - W_1^{\pi_k}(s_k^1, \lambda^k)
\]  

(12)

We use the modified reward function to provide a Q-learning based algorithm as described in Algorithm 1. The standard Q-learning is adapted using the tools of optimization theory (through the use of Lagrange multipliers) and game theory.

**Algorithm 1 Constrained Q-Learning Algorithm**

1. Initialize \( Q_h(s,a,0) \leftarrow H, Q_h(s,a,j) \leftarrow H \) and \( N_h(s,a) \leftarrow 0 \) for all \((s,a,h) \in S \times A \times [H]\) and \( j \in [J] \)
2. for episode \( k = 1, ..., K \) do
3. Observe \( s_1 \)
4. \( \pi_k^1 = \arg\max_{\pi} \left\{ E \left[ Q_1(s_1, \pi(s_1), 0) + \sum_{j=1}^{J} \lambda_j Q_1(s_1, \pi(s_1), j) + \frac{H}{\eta} \lambda_j^2 \right] \right\} \)
5. \( \lambda_j^1 = -\frac{2\eta}{K} \left[ E[Q_1(s_1, \pi_k^1(s_1), j)] \right] \)
6. for step \( h = 1, ..., H \) do
7. if \( h = 1 \) then
8. Take action \( a_h \) from the probability distribution \( \pi_k^1 \) and observe \( s_{h+1} \)
9. else
10. \( a_h = \arg\max_{a \in A} \left[ Q_h(s_h, a, 0) + \sum_{j=1}^{J} \lambda_j^1 Q_h(s_h, a, j) \right] \) and observe \( s_{h+1} \)
11. end if
12. \( t = N_h(s_h, a_h) \leftarrow N_h(s_h, a_h) + 1 \)
13. \( b_t \leftarrow 4 \frac{1 + \sum_{j=1}^{J} (\lambda_j^1)^2}{1 + \sum_{j=1}^{J} (\lambda_j^1)^2} \sqrt{\frac{H \ell}{t}} \)
14. \( Q_h(s_h, a_h, 0) \leftarrow (1 - \alpha_t) Q_h(s_h, a_h, 0) + \alpha_t [r(s_h, a_h, 0) + V_{h+1}(s_{h+1}, 0) + b_t] \)
15. \( Q_h(s_h, a_h, j) \leftarrow (1 - \alpha_t) Q_h(s_h, a_h, j) + \alpha_t [r(s_h, a_h, j) + V_{h+1}(s_{h+1}, j) + b_t] \)
16. if \( h = 1 \) then
17. \( \pi_1^1 = \arg\max_{\pi} \left\{ E \left[ Q_1(s_1, \pi(s_1), 0) + \sum_{j=1}^{J} \lambda_j Q_1(s_1, \pi(s_1), j) + \frac{H}{\eta} \lambda_j^2 \right] \right\} \)
18. \( \lambda_j^1 = -\frac{2\eta}{K} \left[ E[Q_1(s_1, \pi_1^1(s_1), j)] \right] \)
19. end if
20. \( a' = \arg\max_{a \in A} \left[ Q_h(s_h, a, 0) + \sum_{j=1}^{J} \lambda_j^1 Q_h(s_h, a, j) \right] \)
21. \( V_h(s_h, 0) \leftarrow \min \{ H, Q_h(s_h, a', 0) \} \)
22. \( V_h(s_h, j) \leftarrow \min \{ H, Q_h(s_h, a', j) \} \)
23. end for
24. end for

The standard Q-learning is adapted using the tools of optimization theory (through the use of Lagrange multipliers) and game theory.
(where a game between policy and Lagrange multipliers). In line 1, the agent initializes the Q-table for both reward and constraint functions and $N_h(s, a)$, which indicates the number of times state-action pair $(s, a)$ is taken at step $h$. In line 3, the agent observes the initial state at the beginning of each episode. Then, in line 4, the agent takes the policy of the first step by a max-min approach. Note that this max-min step leads to a stochastic policy, in general. Line 5 provides the value of Lagrange multiplier using the stochastic policy found above. Lines 7-11 indicates the action taken at step $h$, where at first step, the policy $\pi^1_k$ is used while for $h > 1$, the policy that maximizes $Q_h(s_h, a, 0) + \sum_{j=1}^{J} \lambda^j_k Q_h(s_h, a, j)$ is used. We note that the policy at each step $h$ depends on the action at time 1 which is stochastic through the state evolution. $N_h(s, a)$ is updated in line 12. In line 13, we calculate the upper confidence bound $b_t$, where $\ell = \log(2SAT^p)$. It is used to update Q-table for reward and constraint functions in lines 14 and 15, respectively, where the learning rate $\alpha_t$ is given as

$$\alpha_t \triangleq \frac{H + 1}{H + t}. \quad (13)$$

Based on the updated Q-table, the value functions are updated in lines 16-22. We note that the step of max-min can be solved efficiently using the algorithms in [23, 26, 20].

Given a Markov Decision Problem with long-term constraints, this paper shows that Algorithm 1 converges to the optimal policy. The regret bound and constraint violations of the proposed algorithm will be analyzed in the next section.

## 4 Regret Bound Analysis

The following analysis consists of two parts. First, we show that the modified unconstrained problem has a sub-linear regret bound. However, we note that the analysis does not directly extend from the unconstrained Q-learning results as in [12], since their framework considers a deterministic policy, while the policy in our paper is stochastic. Then, we reveal the connection between the original constrained problem and the modified problem to derive the main result that both the regret for objective and constraint violations are sub-linear.

Assume the state-action pair $(s^k_h, a^k_h)$ is visited at the step $h$ in episode $k$, and define the $Q^k_h, V^k_h, N^k_h$ are the $Q_h, V_h, N_h$ functions at the beginning of episode $k$, respectively. Recall the notation in (10), and define its empirical counterpart for episode $k$ as

$$[\hat{P}_h W_{h+1}] (s^k_h, a^k_h, \lambda) := W_{h+1} (s^k_{h+1}, \lambda). \quad (14)$$

The next result finds the guarantees on the modified unconstrained problem.

**Lemma 1.** For any $p \in (0, 1)$ and $\gamma \in (0, \frac{1}{2})$, using the Lagrange multiplier $\lambda^k$ selected in the Algorithm 1 with probability at least $1 - p$, the total regret of Q-learning with UCB Hoeffding in the modified unconstrained MDP problem is
bounded as
\[ \sum_{k=1}^{K} [W_1^*(s_1^k, \lambda^k) - W_1^{\pi_k}(s_1^k, \lambda^k)] \leq O(\eta \sqrt{H^4 SAT \ell}), \tag{15} \]

where \( \ell = \log\left(\frac{2SAT}{\eta p}\right) \) and \( \eta = T^\gamma \).

**Proof Sketch.** To prove a sub-linear regret for the modified unconstrained MDP problem, we first find the recursive form for \( U_h^* (s, a, \lambda^k) \) by the update rule in line 14,15 in Algorithm 1 and the definition for the function \( U \) in Eq. (3). Then, combining with the Bellman equation in Eq. (11), we find the difference in line 14,15 in Algorithm 1 and the definition for the function \( U \) in Lemma 2. Next, we bound \( U_h^* - U_h^* \) from above and below in Lemma 6 in Appendix A.2 by using the martingale difference sequence and Azuma-Hoeffding inequality. With these results, it is sufficient to bound the regret of reward function for the modified unconstrained MDP \( \sum_{k=1}^{K} (W_1^* - W_1^{\pi_k})(s_1^k, \lambda^k) \) by bounding \( \sum_{k=1}^{K} (W_1^* - W_1^{\pi_k})(s_1^k, \lambda^k) \) with respect to step \( h \) in order to calculate the difference between \( W_1^* - W_1^{\pi_k} \). Note that these steps are different from that in [15] because the proposed algorithm gives a stochastic policy when \( h = 1 \) in each episode, which requires deeper analysis. Finally, we bound each term in \( \sum_{k=1}^{K} (W_1^* - W_1^{\pi_k})(s_1^k, \lambda^k) \) to obtain the sub-linear result in the statement of the Lemma. The detailed proof is provided in Appendix A.

Now, equipped with the bound for the regret of modified unconstrained problem, it’s natural to analyze them in the form of original value function and the constraint violations. The two following results, Lemma 2 and Lemma 3, describe the relationship between the optimal policy and the policy \( \pi^k \) given by the proposed algorithm, respectively.

**Lemma 2.** If the problem is feasible, for any episode \( k \) and the Lagrange multiplier \( \lambda^k \) given by the Algorithm 1, the optimal value function \( V^*(s_1^k, 0) \) for the original CMDP is always less than or equal to the optimal modified value function \( W^*(s_1^k, \lambda^k) \).

**Proof Sketch.** To prove this lemma, we note that \( V^*(s_1^k, 0) \) is the optimal solution for the primal problem and the function \( W^*(s_1^k, \lambda^k) \) is the optimal solution for the dual problem. We use the relationship between primal and dual problem to prove this result. The detailed proof is provided in Appendix B.

**Lemma 3.** The value function using policy \( \pi^k \), \( W_1^{\pi_k} \), for the modified problem can be expressed by the value function for the original CMDP, \( V_1^{\pi_k} \), and terms stating the violation of constraints, which can be expressed as
\[
W_1^{\pi_k}(s_1^k, \lambda^k) \leq V_1^{\pi_k}(s_1^k, 0) - \frac{\eta}{p} \sum_{j=1}^{J} \left[ V_1^{\pi_k}(s_1^k, j) \right]^2 - E \left[ U_1^{k,s_1}(s_1^k, \pi_1^k(s_1))(\lambda^k)' - U_1^{\pi_k}(s_1^k, \pi_1^k(s_1))(\lambda^k)' \right] \tag{16} \]
where \( (\lambda_k^j)' = -\frac{2H}{T} E[Q_1^\pi(s_1^k, \pi_1^k(s_1^k), j)] \).

**Proof Sketch.** In order to see the relationship between \( W_1^\pi \) and \( V_1^\pi \), we first notice that \( \lambda_k^j \) is the minimizer of \( E[U_1^k(s_1^k, \pi_1^k(s_1^k), (\lambda_k^j)')] \). Combining the result that \( U_k \geq U_1^\pi \) in Lemma 6, it is sufficient to bound \( W_1^\pi(s_1^k, \lambda_k^j) \) by \( E[U_1^k(s_1^k, \pi_1^k(s_1^k), (\lambda_k^j)')] \).

Then, by adding and subtracting the same term \( E[U_1^\pi(s_1^k, \lambda_k^j')] \) we obtain the term \( E[U_1^\pi] \) and an extra term \( E[U_k - U_1^\pi] \), but with a different \( (\lambda_k^j)' \).

Finally, expanding \( E[U_1^\pi] \) by the definition and \( (\lambda_k^j)' \) as defined in the statement of this lemma. The detailed proof is provided in the Appendix C.

Now, equipped with Lemmas 1, 2, and 3, we reach the main result of regret bound in the following theorem.

**Theorem 1.** For any \( p \in (0, 1) \) and \( \gamma \in (0, \frac{1}{2}) \), with probability at least \( 1 - p \), the regret bound for the value function in original CMDP and violation of the constraints are both sub-linear. Specifically,

\[
\sum_{k=1}^{K} V_1^*(s_1^k, 0) - V_1^\pi(s_1^k, 0) \leq O(T^{1/2} + H^4 \sqrt{SA\ell})
\]

\[
\sum_{k=1}^{K} \left| V_1^*(s_1^k, j) - V_1^\pi(s_1^k, j) \right| \leq O(T^{1/2} - \gamma / 2) \quad j = 1, 2, \ldots, J,
\]

where \( \ell = \log(\frac{2SAp}{p}) \).

**Proof Sketch.** In order to prove the main result, firstly, by using the relationship between \( W \) and \( V \) functions for optimal policy and policy \( \pi_k \) given by the algorithm, we obtain the sub-linear bound on the regret for original problem plus constraint violations and an extra term \( E[U_1^k(s_1^k, \pi_1^k(s_1^k), (\lambda_k^j)') - U_1^\pi(s_1^k, \pi_1^k(s_1^k), (\lambda_k^j)')] \). Then, following the ideas in the proof of Lemma 1, this extra term can be removed so that it does not influence the sub-linear bound. Thus, we can directly get the sub-linear bound for the reward function. Finally, by Cauchy-Schwartz inequality, we obtain the bound \( O(T^{1/2}) \) for the constrained violation.

### 5 Simulation Result

In this section, we evaluate the proposed algorithm on a queuing system with a single server in discrete time. Such a model has been studied in [3]. In this model, we assume there is a buffer of finite size \( L \). A possible arrival is assumed to occur at the beginning of the time slot. The state of the system is the number of customers waiting in the queue at the beginning of time slot such that \( |S| = L + 1 \). We assume there are two kinds of actions, service action and
flow action. The service action space is a finite subset $A$ of $[a_{\min}, a_{\max}]$ and $0 < a_{\min} \leq a_{\max} < 1$. With a service action $a$, we assume that a service of a customer is successfully completed with probability $a$. If the service succeeds, the length of the queue will reduce by one, otherwise there is no change of the queue. The flow is a finite subset $B$ of $[b_{\min}, b_{\max}]$ and $0 \leq b_{\min} \leq b_{\max} < 1$. Given a flow action $b$, a customer arrives during the time slot with probability $b$.

Let the state at time $t$ be $x_t$. We assume that no customer arrives when state $x_t = L$ and thus can model this by the state update not increasing on customer arrival when $x_t = L$. Finally, the overall action space is the product of service action space and flow action space, i.e., $A \times B$. Given an action pair $(a, b)$ and current state $x_t$, the transition of this system $P(x_{t+1} | x_t, a_t = a, b_t = b)$ is shown in Table 1.

Given this transition probability matrix, it is clear that the next state is only decided by the current state and current action, which means it is a Markov Decision Process. Moreover, the reward function $r(s, a, b, 0)$ is assumed to be only related to the length of the queue and is a decreasing linear function with respect to the state. It is reasonable because the reward can be seen as the expected waiting time by the Little’s law. Besides, there are two constraint functions, related to the service action and flow action, respectively. The less time customers wait, the higher the reward is. The service constraint function $r(s, a, b, 1)$ is assumed to be only related to $a$ and decreasing with the service action $a$, while the flow constraint function $r(s, a, b, 2)$ is assumed to be only related to $b$ and increasing with the service action $b$.

With a finite horizon $H$, we want to optimize the total reward collected during horizon $H$ and satisfies two constraints with respect to service and flow simultaneously. Thus, the overall optimization problem is given as

$$
\max_{\pi^a_h, \pi^b_h} \mathbb{E} \left[ \sum_{h=1}^{H} r(s_h, \pi^a_h(s_h), \pi^b_h(s_h), 0) \right]
$$

subject to:

$$
\mathbb{E} \left[ \sum_{h=1}^{H} r(s_h, \pi^a_h(s_h), \pi^b_h(s_h), 1) \right] \geq 0
$$

$$
\mathbb{E} \left[ \sum_{h=1}^{H} r(s_h, \pi^a_h(s_h), \pi^b_h(s_h), 2) \right] \geq 0
$$

where $\pi^a_h$ and $\pi^b_h$ are the policies for the service and flow at time slot $h$, respectively. We note that the expectation in the above is with respect to both the stochastic policies and the transition probability. Above all, the problem is
formulated as the problem we propose in the section 3.

In the setting of the simulation, we choose the length of the queue $L = 5$ and the horizon $H = 5$. We let the service action space be $A = [0.3, 0.4, 0.5, 0.6, 0.7]$ and the flow action space be $B = [0, 0.2, 0.4, 0.6]$ for all states besides the state $s = L$. Moreover, the reward function is set to be $r(s, a, b, 0) = -s + 10$, the constraint function for the service is defined as $r(s, a, b, 1) = -10a + 5$ and the constraint function for the flow is $r^2(s, a, b, 2) = -10(1 - b)^2 + 2$.

![Figure 1: Evaluation of proposed algorithm on the queueing system. We see that the rewards achieved by the proposed algorithm achieves the optimal offline solution with the knowledge of the model (Linear Programming). The constraints also converge to be boundary of the region of constraints.](image)

The simulation result is shown in Fig. 1. Due to the influence of the stochastic policy and random transitions, we run the simulation 100 times and take the average along the episode. It can be seen that the algorithm violates both constraints at the beginning of the simulation. During the learning process, the collected reward reduces in order to be able to satisfy the constraints. It can be seen that around episode 3000, the value of both constraint violations oscillate around 0, which shows the convergence of the queueing system. The green line in the figure is the maximum reward given by the offline Linear Programming algorithm with finite horizon $[10]$. It can be seen that the collected reward given by the proposed algorithm is close to the optimal value. Rather than using the optimal algorithm for max-min (line 4 of Algorithm 1 in [20]), we ran the algorithm by discretizing $\lambda$, and the small gap in the achieved reward is likely due to the loss of the accuracy of discretization of $\lambda$. Thus, the proposed algorithm is able to obtain maximum reward while satisfying the long-term constraints to have at least a certain average arrival flow and at most a certain service flow on an average.
6 Conclusion

In this paper, we formulate a constrained MDP problem with a set of long-term (or average) constraints. By using the tools from optimization theory and game theory, a model-free algorithm for the constrained MDP is provided. The key challenge in the constrained MDP is that the optimal policy is stochastic while most Q-learning based policies are deterministic in nature. For any $\gamma \in (0, \frac{1}{2})$, the proposed algorithm achieves a regret of $O(T^{1/2 + \gamma})$ on the objective reward and $O(T^{1-\gamma/2})$ on the constraint violations. We note that this is the first result on the regret analysis of CMDP with average constraints when the state evolution and the constraint functions are unknown. The results are applied to a single server queue, and the proposed algorithm is shown to converge to the optimal solution (obtained by LP using the entire system model).

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A Proof of Lemma 1

For the convenience of the analysis, we define the related quantities $\alpha_0^t$ and $\alpha_i^t$.

\[
\alpha_0^t = \prod_{j=1}^{t} (1 - \alpha_j), \quad \alpha_i^t = \alpha_i \prod_{j=i+1}^{t} (1 - \alpha_j)
\] (19)

In the following proofs, we will use the basic properties of the $\alpha_0^t$ and $\alpha_i^t$ several times. Thus, we summarize them in the following lemma.

**Lemma 4.** The following properties hold for $\alpha_0^t$ and $\alpha_i^t$

(a) $\alpha_0^0 = 0$ for $t \geq 1$, $\alpha_0^0 = 1$ for $t = 0$

(b) $\sum_{i=1}^{t} \alpha_i^t = 1$ for $t \geq 1$, $\sum_{i=1}^{t} \alpha_i^t = 0$ for $t = 0$

(c) $\frac{1}{\sqrt{t}} \leq \sum_{i=1}^{t} \alpha_i^t \leq \sqrt{\frac{2}{t}}$

(d) $\max_{i \in [t]} \alpha_i^t \leq \frac{2H}{t}$ and $\sum_{i=1}^{t} (\alpha_i^t)^2 \leq \frac{2H}{t}$ for every $t \geq 1$

(e) $\sum_{i=1}^{\infty} \alpha_i^t = 1 + \frac{1}{\ell}$ for every $i \geq 1$

**Proof.** These properties have been derived in [15] (See (4.2) in [15] for (a)-(b), Lemma 4.1 in [15] for proof of (c)-(e)), and hence the proof is omitted. \qed

In order to prove the result in the Lemma, we will first derive a recursive form of function $U$ in Appendix A.1. This will then be used to obtain lower and upper bound on $U_h^k - U^*_h$ in Appendix A.2. This will then be used to prove the main result in Appendix A.3.

A.1 Recursive form of function $U$

In the following Lemma, we will derive a recursive form of function $U$, given as follows

**Lemma 5.** For any $(s, a, h, k) \in S \times A \times \{H\} \times \{K\}$ and the Lagrange multiplier $\lambda^k$ selected in Algorithm 7, let $t = N_h^k(s, a)$, denote $c_h^k = 1 + \sum_{j=1}^{t} \left( \lambda_j^k + \sum_{h'=h}^{H} \left( \frac{\lambda_j^k}{4qH} \right)^2 \right)$, and suppose that $(s, a)$ was previously taken at step $h$ of episodes $k_1, \ldots, k_t < k$. Then:

\[
(U_h^k - U_h^*(s, a, \lambda^k)) = \alpha_0^t \left[ c_h^k H - U_h^*(s, a, \lambda^k) \right] + \sum_{i=1}^{t} \alpha_i^t \left[ (W_{h+1}^k - W_{h+1}^*(s, a, \lambda^k)) \right. \\
+ \left. (\hat{P}_{h+1}^k - P_h^*) W_{h+1}^*(s, a, \lambda^k) \right] + b'_i.
\] (20)

where $b'_i = (1 + \sum_{j=1}^{i} \lambda_j^k) b_i = 4 c_h^k \sqrt{\frac{H \ell}{i}}$
Proof. By the update rule in line 9 and 10 of the algorithm, we know the value of \( U^{k+1}_h(s,a,\lambda^k) \) will be updated if and only if \((s,a) = (s^k_h,a^k_h)\) and is updated as

\[
U^{k+1}_h(s,a,\lambda^k) = Q^{k+1}_h(s^k_h,a^k_h,0) + \sum_{j=1}^J \left( \lambda_j^k Q^{k+1}_h(s^k_h,a^k_h,j) + \frac{1}{4\eta} (\lambda_j^k)^2 \right)
\]

Thus, we have

\[
U^{k+1}_h(s,a,\lambda^k) = (1 - \alpha_t)Q^{k}_h(s^k_h,a^k_h,0) + \alpha_t \left[ r(s^k_h,a^k_h,0) + V^{k+1}_{h+1}(s^k_{h+1},0) + b_t \right]
\]

Under the assumption that \((s,a)\) was previously taken at step \(h\) of episodes \(k_1,...,k_t < k\), recursively using the update rule Eq. (21) and the notation
defined in Eq. (19), we have

\[
U_{h}^{i}(s, a, \lambda^{k}) = (1 - \alpha_{i})U_{h}^{i-1}(s, a, \lambda^{k}) + \alpha_{i}[R(s, a, \lambda^{k}) + W_{h+1}^{k}(s_{h+1}, \lambda^{k}) + b_{i}']
\]

\[
= (1 - \alpha_{i}) \left[ 1 - \alpha_{i-1} \right] U_{h}^{i-1}(s, a, \lambda^{k}) + \alpha_{i-1}[R(s, a, \lambda^{k}) + W_{h+1}^{k-1}(s_{h+1}, \lambda^{k}) + b_{i-1}'] \]

\[+ \alpha_{i}[R(s, a, \lambda^{k}) + W_{h+1}^{k}(s_{h+1}, \lambda^{k}) + b_{i}'] \]

\[= \prod_{i=t-1}^{t} \left[ 1 - \alpha_{i} \right] U_{h}^{i-1}(s, a, \lambda^{k}) + \sum_{i=t}^{t} \alpha_{i} \prod_{j=i+1}^{t} \left[ 1 - \alpha_{j} \right] [R(s, a, \lambda^{k}) + W_{h+1}^{k}(s_{h+1}, \lambda^{k}) + b_{i}'] \]

\[= \prod_{i=1}^{t} \left[ 1 - \alpha_{i} \right] U_{h}^{i}(s, a, \lambda^{k}) + \sum_{i=1}^{t} \alpha_{i} [R(s, a, \lambda^{k}) + W_{h+1}^{k}(s_{h+1}, \lambda^{k}) + b_{i}'] \]

\[= \alpha_{i}^{0} U_{h}^{k}(s, a, \lambda^{k}) + \sum_{i=1}^{t} \alpha_{i} [R(s, a, \lambda^{k}) + W_{h+1}^{k}(s_{h+1}, \lambda^{k}) + b_{i}'] \]  

(22)

Notice that there is no update for \(U_{h}^{k}\), and thus \(U_{h}^{k}\) is the initial value, which is given as

\[
U_{h}^{k}(s, a, \lambda^{k}) = Q_{h}^{k}(s, a, 0) + \sum_{j=1}^{J} \left( \lambda_{j}^{k} Q_{h}^{k}(s, a, j) + \sum_{h'=h}^{H} \frac{(\lambda_{j}^{k})^{2}}{4\eta} \right)
\]

\[= \left[ 1 + \sum_{j=1}^{J} \left( \lambda_{j}^{k} + \sum_{h'=h}^{H} \frac{(\lambda_{j}^{k})^{2}}{4\eta H} \right) \right] H = c_{h}^{k} H \]  

(23)

Substituting \(U_{h}^{k}(s, a, \lambda^{k})\) in (22), we have

\[
U_{h}^{k}(s, a, \lambda^{k}) = \alpha_{i}^{0} c_{h}^{k} H + \sum_{i=1}^{t} \alpha_{i} [R(s, a, \lambda^{k}) + W_{h+1}^{k}(s_{h+1}, \lambda^{k}) + b_{i}'] \]  

(24)

Note that the Bellman equations hold for the optimal policy \(U_{h}^{*}(s, a, \lambda^{k})\) due to \(U_{h}^{*}\) and \(W_{h}^{*}\) defined in Eq. (9), since they do not have constraints. Recall the notation \([\mathbb{P}_{h}^{*}W_{h+1}](s, a, \lambda) := \mathbb{E}_{s' \sim \pi_{h}(s, a)} W_{h+1}(s', \lambda)\). For \(t \geq 1\), we have

\[
U_{h}^{*}(s, a, \lambda^{k}) = (R + [\mathbb{P}_{h}^{*}W_{h+1}](s, a, \lambda^{k}))
\]

(25)

\[\overset{(a)}{=} \sum_{i=1}^{t} \alpha_{i}^{*} \left[ (R + [\mathbb{P}_{h}^{*}W_{h+1}](s, a, \lambda^{k})) \right]
\]

(26)

\[\overset{(b)}{=} \sum_{i=1}^{t} \alpha_{i}^{*} \left[ R(s, a, \lambda^{k}) + ([\mathbb{P}_{h}^{*} - \hat{W}_{h}^{k}](s, a, \lambda^{k})) + W_{h+1}^{*}(s_{h+1}, \lambda^{k}) \right]
\]

where step (a) holds due to Lemma 4 (a) and step (b) holds by the definition of \([\hat{W}_{h}^{k}]W_{h+1}](s, a, \lambda) := W_{h+1}(s_{h+1}, \lambda)\). For \(t = 0\), we have \(U_{h}^{*}(s, a, \lambda) =
Thus, we have
\[
U^*_h(s, a, \lambda^k) = \alpha^0_t U^*_h(s, a, \lambda^k) + \sum_{i=1}^{t} \alpha^i_t \left[ R(s, a, \lambda^k) + (\mathbb{P}_h - \mathbb{P}_h^k) W^*_{h+1}(s, a, \lambda^k) + W^*_{h+1}(s_{h+1}, \lambda^k) \right]
\]  
(28)

Combining Eq. (24) and Eq. (28), we have
\[
(U^*_h - U^*_h)(s, a, \lambda^k) = \alpha^0_t \left[ c^k H - U^*_h(s, a, \lambda^k) \right] + \sum_{i=1}^{t} \alpha^i_t \left[ (W^k_{h+1} - W^*_h(s_{h+1}, \lambda^k) ) + (\mathbb{P}_h - \mathbb{P}_h^k) W^*_{h+1}(s, a, \lambda^k) + b'_t \right]
\]  
(29)

This proves the result as in the statement of the Lemma.

A.2 Bounded difference between \( U^k \) and \( U^* \)

We next provide a result to bound the difference between \( U^k \) and \( U^* \).

**Lemma 6.** For any \( p \in (0, 1) \) and the Lagrange multiplier \( \lambda^k \) selected in the algorithm \( \mathcal{A} \) denote \( c^k = 1 + \sum_{j=1}^{J} (\lambda^k_j + (\lambda^k_j)^2) \), and, \( c_{\max} = \max_k c^k \). Further, let \( b'_t = 4 \sqrt{H^3/\ell} \) and \( \beta_t = 12 c_{\max} \sqrt{H^3/\ell} \). With probability at least \( 1 - p \), the following holds simultaneously for all \((s, a, h, k) \in S \times A \times [H] \times [K] \):

\[
0 \leq (U^k_h - U^*_h)(s, a, \lambda^k) \leq \alpha^0_t (1 + 4 J) H + \sum_{i=1}^{t} \alpha^i_t (W^k_{h+1} - W^*_h(s_{h+1}, \lambda^k) ) + \beta_t
\]  
(30)

**Proof.** For each fixed \((s, a, h) \in S \times A \times [H] \), any fixed \( k \in [K] \), and the selected Lagrange multiplier \( \lambda^k = (\lambda^k_1, ..., \lambda^k_J) \), let \( t = N^k_h(s, a) \), and suppose that \((s, a)\) was previously taken at step \( h \) of episodes \( k_1, ..., k_t < k \). Let \( \mathcal{F}_i \) be the sigma field generated by all the random variables until episode \( k_i \), step \( h \). Then, \( (\alpha^i_t \cdot [(\mathbb{P}_h^k - \mathbb{P}_h) W^*_{h+1})(s, a, \lambda^k) )_{i=1}^{t} \) is clearly a martingale difference sequence w.r.t the filtration \( \mathcal{F}_i \). According to the assumption of the reward and constrain functions, the \( i \)th term in the martingale difference sequence is bounded by

\[
a_i = 2c^k H \cdot \alpha^i_t.
\]  
(31)

Let

\[
E = 2 \sqrt{2c^k H} \sqrt{\sum_{i=1}^{t} (\alpha^i_t)^2 } \cdot \ell.
\]  
(32)
Using Azuma-Hoeffding’s inequality, we have

$$\mathbb{P}\left[ \left| \sum_{i=1}^{t} \alpha_i \cdot [(\hat{p}_{h_i} - p_h)W_{h+1}^{*}](s, a, \lambda^k) \right| \leq E \right] \geq 1 - 2 \exp\left( \frac{-E^2}{2 \sum_{i=1}^{t} \alpha_i^2} \right)$$

$$= 1 - 2 \exp\left( \frac{-8(c_k)^2 H^2 \sum_{i=1}^{t} (\alpha_i)^2 \cdot \ell}{8(c_k)^2 H^2 \sum_{i=1}^{t} (\alpha_i)^2} \right)$$

$$= 1 - \frac{p}{SAT}$$

(33)

By union bound, the following holds for all \((s, a, h, k) \in S \times A \times [H] \times [K]\) with probability at least \(1 - p\):

$$\left| \sum_{i=1}^{t} \alpha_i \cdot [(\hat{p}_{h_i} - p_h)W_{h+1}^{*}](s, a, \lambda^k) \right| \leq E \leq 4c^k \sqrt{\frac{H^3 \ell}{t}}.$$  \hspace{1cm} (34)

where the last step comes from the result in Lemma 3(d).

We consider three cases for the proof of the lower bound, which include all possible scenarios:

1. We assume for all \(h\) that

$$\min\{H, Q_{h+1}^{k_i}(s_{h+1}^{k_i}, a', 0)\} = Q_{h+1}^{k_i}(s_{h+1}^{k_i}, a', 0)$$

$$\min\{H, Q_{h+1}^{*}(s_{h+1}^{k_i}, a', j)\} = Q_{h+1}^{*}(s_{h+1}^{k_i}, a', j)$$

(35)

where \(a'_{h+1} = \arg\max_{a \in A} \left[ Q_{h+1}^{k_i}(s_{h+1}^{k_i}, a, 0) + \sum_{j=1}^{J} \lambda_j Q_{h+1}^{k_i}(s_{h+1}^{k_i}, a, j) \right].\)
Then, we have

\[(U_h^k - U_h^*) (s, a, \lambda^k)\]

\[= \alpha_i^k \left[ c_h^k H - U_h^* (s, a, \lambda^k) \right]
+ \sum_{i=1}^t \alpha_i^k \left[ (W_{h+1}^{k_i} - W_{h+1}^*) (s_{h+1}^{k_i}, \lambda^k) + \left[ (\hat{P}_h^{k_i} - \hat{P}_h) W_{h+1}^* (s, a, \lambda^k) + b_i \right] \right] \tag{36}\]

\[\geq (a) \sum_{i=1}^t \alpha_i^k (W_{h+1}^{k_i} - W_{h+1}^*) (s_{h+1}^{k_i}, \lambda^k) - 4c \sqrt{H^3} \ell \sum_{i=1}^t \alpha_i^k \sum_{i=1}^t \lambda_i^k \min \{H, Q_{h+1}^{k_i} (s_{h+1}^{k_i}, a_{h+1}^*, j) \}
- U_{h+1}^* (s_{h+1}^{k_i}, a^*, \lambda^k) \right] \tag{37}\]

\[\geq (b) \sum_{i=1}^t \alpha_i^k (W_{h+1}^{k_i} - W_{h+1}^*) (s_{h+1}^{k_i}, \lambda^k) \tag{38}\]

\[\geq (c) \sum_{i=1}^t \alpha_i^k \left[ \min \{H, Q_{h+1}^{k_i} (s_{h+1}^{k_i}, a_{h+1}^*, 0) \} + \sum_{j=1}^J \lambda_i^k \min \{H, Q_{h+1}^{k_i} (s_{h+1}^{k_i}, a_{h+1}^*, j) \}
- U_{h+1}^* (s_{h+1}^{k_i}, a^*, \lambda^k) \right] \tag{39}\]

\[\geq \sum_{i=1}^t \alpha_i^k \left[ Q_{h+1}^{k_i} (s_{h+1}^{k_i}, a_{h+1}^*, 0) + \sum_{j=1}^J \lambda_i^k Q_{h+1}^{k_i} (s_{h+1}^{k_i}, a_{h+1}^*, j)
- Q_{h+1}^* (s_{h+1}^{k_i}, a_{h+1}^*, 0) - \sum_{j=1}^J \lambda_i^k Q_{h+1}^* (s_{h+1}^{k_i}, a_{h+1}^*, j) \right] \tag{40}\]

\[= \sum_{i=1}^t \alpha_i^k (U_{h+1}^{k_i} - U_{h+1}^*) (s_{h+1}^{k_i}, a_{h+1}^*, \lambda^k), \tag{41}\]

where step (a) holds because of Eq. (34) and \(U_h^* \leq c_h^k H\) holds. Step (b) follows from Lemma 4(c). In step (c), the notation \(a^* = \text{arg max}_{a \in A} U_{h}^* (s_{h+1}^{k_i}, a, \lambda^k)\).

Recursively using the above equation for \(h = H\), on the right hand side of Eq. (41), we obtain the summation of \(U_{H+1}\). However, \(U_{h+1}^k (s, a, \lambda) = 0\) for all \((s, a) \in S \times A\), all \(k \in [K]\), and any Lagrange multiplier \(\lambda\) because \(Q_{h+1}^* (s, a, 0) = 0\) and \(Q_{h+1}^* (s, a, j) = 0\) by definition. Thus,

\[(U_h^k - U_h^*) (s, a, \lambda^k) \geq 0 \tag{42}\]

2. There is a \(h'\) such that

\[
\min \{H, Q_{h'+1}^{k_i} (s_{h+1}^{k_i}, a_{h'+1}^*, 0) \} = H
\]

\[
\min \{H, Q_{h'+1}^* (s_{h+1}^{k_i}, a_{h'+1}^*, j) \} = H
\]

In this case, \((U_h^k - U_h^*) (s, a, \lambda^k) \geq 0\) can be proved easily because \(Q_{h'+1}^* (s, a, 0)\) and \(Q_{h'+1}^* (s, a, 0)\) are both less than or equal to 0.
3. There is a $h'$ such that

$$
\begin{align*}
\min \{ H, Q^k_{h' + 1}(s^k_{h' + 1}, a^j_{h' + 1}, 0) \} &= Q^k_{h' + 1}(s^k_{h' + 1}, a^j_{h' + 1}, 0) \\
\min \{ H, Q^*_{h' + 1}(s^k_{h' + 1}, a^j_{h' + 1}, 0) \} &= H
\end{align*}
$$

(44)

or there is a $h'$ such that

$$
\begin{align*}
\min \{ H, Q^*_{h' + 1}(s^k_{h' + 1}, a^j_{h' + 1}, 0) \} &= H \\
\min \{ H, Q^k_{h' + 1}(s^k_{h' + 1}, a^j_{h' + 1}, 0) \} &= Q^k_{h' + 1}(s^k_{h' + 1}, a^j_{h' + 1}, 0)
\end{align*}
$$

(45)

Then, we have

$$
(U^k_{h'} - U^*_{h'}) (s, a, \lambda^k) \geq \sum_{i=1}^{t} a_i^j (Q^k_{h' + 1} - Q^*_{h' + 1})(s^k_{h' + 1}, a^j_{h' + 1}, 0)
$$

(46)

or

$$
(U^k_{h'} - U^*_{h'}) (s, a, \lambda^k) \geq \sum_{i=1}^{t} a_i^j (Q^*_{h' + 1} - Q^k_{h' + 1})(s^k_{h' + 1}, a^j_{h' + 1}, 0)
$$

(47)

Using a recursive approach for $Q^k_{h' + 1} - Q^*_{h' + 1}$ using the same approach for $U^k_{h'} - U^*_{h'}$, it follows that $(U^k_{h'} - U^*_{h'}) (s, a, \lambda^k) \geq 0$. Moreover, we can also see from this approach that $(Q^k_{h'} - Q^*_{h'})(s, a, j) \geq 0$.

This finishes the proof for the lower bound.

Further, we have a bound for $c_{max}$

$$
c_{max} = \max_h \left( 1 - \frac{\eta}{H} \sum_{j=1}^{J} \left[ \mathbb{E}[Q^k(s^k_1, \pi^k_j(s^k_1), j)] \right] \right)
$$

(48)

$$
\leq \max_h \left( 1 + \frac{\eta}{H} \sum_{j=1}^{J} \left[ \mathbb{E}[Q^*_k(s^k_1, \pi^*_j(s^k_1), j)] \right] \right) \leq 1 + \eta J
$$

(49)

where step (a) follows since $Q^*_h \geq Q^*_h \geq Q^*_h$. Similarly, for the term $U^*_h (s, a, \lambda^k)$, we have the bound

$$
U^*_h (s, a, \lambda^k) = \sup_{\pi_k} \left[ Q^*_h (s, a, 0) + \sum_{j=1}^{J} \left( \lambda^k_j Q^*_h (s, a, j) + \sum_{h'=h}^{H} \frac{(\lambda^k_j)^2}{4\eta} \right) \right]
$$

$$
\geq \eta (H - h + 1) \sum_{j=1}^{J} (\lambda^k_j)^2 + \frac{3\eta (H - h + 1)}{4\eta} \sum_{j=1}^{J} \left[ \mathbb{E}[Q^k(s^k_1, \pi^k_j(s^k_1), j)] \right]
$$

$$
\geq -3\eta J H
$$

(50)
Based on the result of Lemma 5, we have

\[(U_h^k - U_h^*) (s, a, \lambda^k)\]
\[= \alpha_t^0 c_h^k H - U_h^k (s, a, \lambda^k)\]
\[+ \sum_{j=1}^{t} \alpha_t^j (W_{h+1}^k - W_{h+1}^*) (s_{h+1}^k, \lambda^k) + [\hat{v}_h - v_h] W_{h+1}^* (s, a, \lambda^k) + b_t^i\]  
\[\leq \alpha_t^0 c_{\max} H - U_h^k (s, a, \lambda^k)\]
\[+ \sum_{j=1}^{t} \alpha_t^j (W_{h+1}^k - W_{h+1}^*) (s_{h+1}^k, \lambda^k) + \sum_{j=1}^{t} \alpha_t^j b_t^i\]
\[\leq \alpha_t^0 (1 + 4\eta J) H + \sum_{j=1}^{t} \alpha_t^j (W_{h+1}^k - W_{h+1}^*) (s_{h+1}^k, \lambda^k) + \beta_t,\]

where step (a) follows from Eq. (51), step (b) holds due to Eq. (19) and (50), and step (c) comes from Lemma 4(c) that \(\sum_{i=1}^{t} \alpha_t^i b_t^i \leq 8c_k \sqrt{H^3/\ell}\) and \(c_k \leq c_{\max}\). This finishes the proof for right hand side of the Lemma 6.

\[\square\]

A.3 Proof of the Lemma 1

Proof. Recall the definition that \(c_k = \left(1 + \sum_{j=1}^{J} (\lambda_j^k)^2 + \frac{(\lambda_j^k)^2}{d_j}\right), c_{\max} = \max_k c_k\) and \(\beta_t = 12c_{\max} \sqrt{H^3/\ell}\). Let

\[\delta_h^k := (W_h^k - W_{h+1}^*) (s_h^k, \lambda^k) \quad \text{and} \quad \phi_h^k := (W_h^k - W^*) (s_h^k, \lambda^k)\]

By Lemma 5, we know with probability \(1 - p\), \((U_h^k - U_h^*) (s, a, \lambda^k) \geq 0\) for all \((s, a, h, k) \in S \times A \times [H] \times [K]\). Thus, \(W_h^k (s_h^k, \lambda^k) \geq W^* (s_h^k, \lambda^k)\). The total regret can be bounded as

\[\text{Regret}(K) = \sum_{k=1}^{K} (W_1^k - W_1^*) (s_1^k, \lambda^k) \leq \sum_{k=1}^{K} (W_1^k - W_1^*) (s_1^k, \lambda^k) = \sum_{k=1}^{K} \delta_1^k\]

First, we analyze the case \(h \geq 2\). For any fixed \(k \in [K]\), let \(t = N_h^k (s_h^k, a_h^k)\), and suppose \((s_h^k, a_h^k)\) was previously taken at step \(h\) of episodes \(k_1, \ldots, k_t < k\), then
we have,

\[
\delta_h^k = (W_h^k - W_h^{\pi_h^k}) \leq (U_h^k - U_h^{\pi_h^k})(s_h^k, a_h^k, \lambda^k) \\
= (U_h^k - U_h^{\pi_h^k})(s_h^k, a_h^k, \lambda^k) + (U_h^{\pi_h^k} - U_h^{\pi_h^k})(s_h^k, a_h^k, \lambda^k) \\
\leq \alpha_t^0 c_k H + \sum_{i=1}^t \alpha_t^i (W_h^{k_{i+1}} - W_h^{k_i})(s_h^{k_i}, \lambda^k) + \beta_t + \left[ \mathbb{P}_h(W_h^{k_{i+1}} - W_h^{k_i}) \right](s_h^k, a_h^k, \lambda^k)
\]

\[
\leq \alpha_t^0 c_k H + \sum_{i=1}^t \alpha_t^i \phi_{h+1}^i + \beta_t - \phi_{h+1}^i + \xi_{h+1},
\]

where \( \xi_{h+1} := [(\mathbb{P}_h - \hat{\mathbb{P}}_h)(W_h^{k_{i+1}} - W_h^{k_i})](s_h^k, a_h^k, \lambda^k) \) is also a martingale difference sequence. Inequality (a) holds due to the update rule line 10, 11 and 12 in Algorithm 1 that

\[
W_h^k(s_h^k, \lambda^k) \leq \min\{c_k H, \max_{a' \in \mathcal{A}} U_h^k(s_h^k, a', \lambda^k)\} \leq \max_{a' \in \mathcal{A}} U_h^k(s_h^k, a', \lambda^k) = U_h^k(s_h^k, a_h^k, \lambda^k)
\]

Besides, \( W_h^{\pi_h^k}(s_h^k, \lambda^k) = U_h^{\pi_h^k}(s_h^k, a_h^k, \lambda^k) \) because \( \pi_h^k \) select the action \( a_h^k \) at episode \( k \) step \( h \) when \( h \geq 2 \). Step (b) holds due to Lemma 10 and the Bellman equation. Inequality (c) holds due to the definition of \( \delta, \phi, \) and \( \xi \).

However, when \( h = 1 \), \( W_1^{\pi_h^k}(s_1^k, \lambda^k) \neq U_1^{\pi_h^k}(s_1^k, a_1^k, \lambda^k) \) because \( \pi_1^k \) is a stochastic policy. Moreover, \( W_1^k(s_1^k, \lambda^k) \) is not less than or equal to \( U_1^k(s_1^k, a_1^k, \lambda^k) \) because \( a_1^k \neq a' \). However, notice that

\[
W_1^k(s_1^k, \lambda^k) \leq \mathbb{E}[U_1^k(s_1^k, \pi_1^k(s_1^k), \lambda^k)]
\]

Thus, for \( h = 1 \), we have the following result,

\[
\delta_1^k \leq \alpha_t^0 c_k H + \sum_{i=1}^t \alpha_t^i \phi_{h+1}^i + \beta_t - \phi_{h+1}^i + \xi_{h+1} + U_1^{\pi_h^k}(s_1^k, a_1^k, \lambda^k) - W_1^{\pi_h^k}(s_1^k, \lambda^k) + \mathbb{E}[U_1^k(s_1^k, \pi_1^k(s_1^k), \lambda^k)] - U_1^k(s_1^k, a_1^k, \lambda^k)
\]

Denote \( n_h^k = N_h^k(s_h^k, a_h^k) = t \). It’s easy to bound the first term as

\[
\sum_{k=1}^K \alpha_{n_h^k}^0 c_k H \leq c_{\max} \sum_{k=1}^K \|a_h^k = 0\| \leq O(\eta \text{SAH})
\]

To bound the second term, we rearrange the summation as

\[
\sum_{k=1}^K \sum_{i=1}^{n_h^k} \alpha_{n_h^k}^i \phi_{h+1}^i(s_h^k, a_h^k) \leq \sum_{k'=1}^K \phi_{h+1}^k \sum_{t=n_h^k + 1}^{\infty} \alpha_t^k \phi_h^t \leq (1 + \frac{1}{H}) \sum_{k=1}^K \phi_h^k,
\]
where the last inequality uses Lemma 4(c). Plugging Eq. (61) and Eq. (62) back into Eq. (57), we have for $h \geq 2$

$$
\sum_{k=1}^{K} \delta_h^k \leq cSAH + (1 + \frac{1}{H}) \sum_{k=1}^{K} \phi_{h+1}^k - \sum_{k=1}^{K} \phi_{h+1}^k + \sum_{k=1}^{K} \delta_{h+1}^k + \sum_{k=1}^{K} (\beta_{n_h^k}^k + \xi_{h+1}^k)
$$

$$
\leq cSAH + (1 + \frac{1}{H}) \sum_{k=1}^{K} \delta_{h+1}^k + \sum_{k=1}^{K} (\beta_{n_h^k}^k + \xi_{h+1}^k),
$$

where the last inequality uses $\phi_{h+1}^k \leq \delta_{h+1}^k$ (the fact that $W^* \geq W^\pi_s$). Combining the result for $h = 1, 2, \ldots, H$ (notice that for $h = 1$, we need to add the extra term in the second line of Eq. (60)) and using the fact $\delta_{H+1}^k = 0$, we have

$$
\sum_{k=1}^{K} \delta_1^k \leq O(\eta H^2 SA) + \sum_{h=1}^{K} \sum_{k=1}^{K} (\beta_{n_h^k}^k + \xi_{h+1}^k)
$$

$$
+ \sum_{k=1}^{K} U_1^\pi(s_1^k, a_1^k, \lambda^k) - W_1^\pi_s(s_1^k, \lambda^k) + \sum_{k=1}^{K} \mathbb{E}[U_1^k(s_1^k, \pi_1^k(s_1^k), \lambda^k) - U_1^k(s_1^k, a_1^k, \lambda^k)]
$$

For the second term, using pigeonhole principle, for any $h \in [H]$, we have

$$
\sum_{k=1}^{K} \beta_{n_h^k}^k \leq O(\eta H^3 M) = O(\eta H^3 M)
$$

$$
\leq O(\eta \sqrt{H^3 \ell SA}) = O(\eta \sqrt{H^3 \ell SA})
$$

(65)

where inequality (a) holds because $\sum_{s,a} N_h^K(s,a) = K$ and the left hand side of (a) is maximized when $N_h^K(s,a) = K$. For the third term, using Azuma-Hoeffding inequality, we know that $\xi_{h+1}$ is bounded by $4cH$ for any $k$ and $h$. Thus, with probability at least $1 - p$, we have

$$
\sum_{h=1}^{H} \sum_{k=1}^{K} \xi_{h+1}^k = \left| \sum_{h=1}^{H} \sum_{k=1}^{K} (P_h - \hat{P}_h)(W_{h+1}^* - W_{h+1}^\pi_s)(s_h^k, a_h^k, \lambda^k) \right| \leq O(\eta H \sqrt{\ell} H)
$$

(67)

Finally, for the second line, notice that

$$
W_1^\pi(s_1^k, \lambda) = \mathbb{E}[U_1^\pi(s_1^k, \pi_1^k(s_1^k), \lambda^k)] - \mathbb{E}_{a_1^k}[U_1^k(s_1^k, a_1^k, \lambda^k)]
$$

(68)

By the Hoeffding’s Inequality, each of the following holds with probability at least $1 - p$:

$$
\sum_{k=1}^{K} \left| U_1^\pi(s_1^k, a_1^k, \lambda^k) - W_1^\pi(s_1^k, \lambda^k) \right| \leq O(\eta \sqrt{HT\ell})
$$

(69)

$$
\sum_{k=1}^{K} \left| \mathbb{E}[U_1^k(s_1^k, \pi_1^k(s_1^k), \lambda^k)] - U_1^k(s_1^k, a_1^k, \lambda^k) \right| \leq O(\eta \sqrt{HT\ell})
$$
Combining Eq. (64), (67), (65), and (69), we have $\sum_{k=1}^{K} \delta_k^1 \leq O(\eta H^2 SA + \eta \sqrt{H^4 SAT\ell})$ with probability at least $1 - 4p$. When $T \geq \sqrt{H^4 SAT\ell}$, then $\eta \sqrt{H^4 SAT\ell} \geq \eta H^2 SA \geq \eta H^2 SA$ since $H \geq 1$ and $\ell = \log_e (2SAT/p) \geq \log_e (4ST/p) > 1$ for $A > 1$. When $T \leq \eta \sqrt{H^4 SAT\ell}$, we have $\sum_{k=1}^{K} \delta_k^1 \leq c_{\text{max}} KH \leq (1 + \eta J) T \leq (1 + \eta J) \sqrt{H^4 SAT\ell}$. Therefore, we may remove the term $H^2 SA$ in the regret bound. Re-scaling the probability $p$ to $\frac{p}{4}$ completes the proof.

B Proof of Lemma 2

Proof. Given a fixed episode $k$, consider playing the optimal policy $\pi^*$ for the original Average Constrained MDP problem in the modified MDP problem with the Lagrange multiplier selected in Algorithm 1, we have

$$V_1^*(s_1^k, 0) = \mathbb{E} \left[ \sum_{h=1}^{H} r(s_h^k, \pi_h^*(s_h^k), 0) \right] \quad (70)$$

$$\leq \mathbb{E} \left\{ \sum_{h=1}^{H} \left[ r(s_h^k, \pi_h^*(s_h^k), 0) + \sum_{j=1}^{J} (\lambda_j^k r(s_h^k, \pi_h^*(s_h^k), j) + \frac{1}{4} (\lambda_j^k)^2) \right] \right\} \quad (71)$$

$$= \sum_{h=1}^{H} \mathbb{E} \left[ R(s_h^k, \pi_h^*(s_h^k), \lambda^k) \right] \leq W_1^*(s_1^k, \lambda^k) \quad (72)$$

The first inequality holds because with feasible optimal policy in average Constrained MDP, for any fixed $j, k$ and $\lambda_j^k \geq 0$, we have

$$\mathbb{E} \left[ \sum_{h=1}^{H} (\lambda_j^k r(s_h^k, \pi_h^*(s_h^k), j) + \frac{1}{4} (\lambda_j^k)^2) \right] \geq \mathbb{E} \left[ \sum_{h=1}^{H} \lambda_j^k r(s_h^k, \pi_h^*(s_h^k), j) \right] \quad (73)$$

$$= \lambda_j^k \mathbb{E} \left[ \sum_{h=1}^{H} r(s_h^k, \pi_h^*(s_h^k), j) \right] \geq (74)$$

The last inequality holds because we notice that the optimal policy for the original policy may not be the optimal policy for the modified unconstrained MDP. The summation over $R$ over optimal policy in original problem should be always less than or equal to the optimal value function $W_1^*(s_1^k, \lambda^k)$. Thus, we conclude that $V_1^*(s_1^k, 0) \leq W_1^*(s_1^k, \lambda^k)$ no matter whether the optimal policy in both problems are the same or not.

C Proof of Lemma 3

Proof. Define $[x]_+ = \min\{0, x\}$. According to line 4 in Algorithm 1, we know that the value of the Lagrange multiplier $\lambda_j^k$ is

$$\lambda_j^k = -\frac{2\eta}{H} \mathbb{E} \left[ Q_1^k(s_1^k, \pi_1^k(s_1^k), j) \right]_+ \quad (75)$$
We define a new value for \((\lambda^k_j)'
\)

\[
(\lambda^k_j)' = -\frac{2\eta}{HG} \left[ \mathbb{E}[Q^{\pi_k}(s^k_1, \pi^k_1(s^k_1), j)] \right],
\]

(76)

and define \((\lambda^k)' = (\lambda^k_1)', ..., (\lambda^k_N)'
\). Then, notice that \(\lambda^k_j\) is the minimizer of

\[
\mathbb{E} \left[ U^k(s^k_1, \pi^k_1(s^k_1), \lambda^k) \right],
\]

which means that

\[
\mathbb{E} \left[ U^k(s^k_1, \pi^k_1(s^k_1), \lambda^k) \right] \leq \mathbb{E} \left[ U^k(s^k_1, \pi^k_1(s^k_1), (\lambda^k)') \right]
\]

(77)

By using this property, we can bound the term \(W_1^{\pi_k}\) as

\[
W_1^{\pi_k}(s^k_1, \lambda^k) = \mathbb{E} \left[ U^k(s^k_1, \pi^k_1(s^k_1), \lambda^k) - U^k(s^k_1, \pi^k_1(s^k_1), \lambda^k) \right] + \mathbb{E} \left[ U^k(s^k_1, \pi^k_1(s^k_1), \lambda^k) \right]
\]

(78)

\[
\leq \mathbb{E} \left[ U^k(s^k_1, \pi^k_1(s^k_1), \lambda^k) - U^k(s^k_1, \pi^k_1(s^k_1), \lambda^k) \right] + \mathbb{E} \left[ U^k(s^k_1, \pi^k_1(s^k_1), (\lambda^k)') \right]
\]

(79)

\[
\leq \mathbb{E} \left[ U^k(s^k_1, \pi^k_1(s^k_1), (\lambda^k)') \right] + \mathbb{E} \left[ U^k(s^k_1, \pi^k_1(s^k_1), (\lambda^k)') - U^k(s^k_1, \pi^k_1(s^k_1), (\lambda^k)') \right]
\]

(80)

\[
= V_1^{\pi_k}(s^k_1, 0) + \sum_{j=1}^{F} \sum_{h=1}^{H} (\lambda^k_j)^r(s_h, \pi^k_1(s_h), j) + \frac{(\lambda^k_j)^2}{4\eta}
\]

\[
+E \left[ U^k(s^k_1, \pi^k_1(s^k_1), (\lambda^k)') - U^k(s^k_1, \pi^k_1(s^k_1), (\lambda^k)') \right]
\]

(82)

\[
= V_1^{\pi_k}(s^k_1, 0) + \sum_{j=1}^{F} \frac{(\lambda^k_j)^2}{4\eta}
\]

\[
+ \mathbb{E} \left[ U^k(s^k_1, \pi^k_1(s^k_1), (\lambda^k)') - U^k(s^k_1, \pi^k_1(s^k_1), (\lambda^k)') \right]
\]

(83)

\[
= V_1^{\pi_k}(s^k_1, 0) - \frac{\eta}{H} \sum_{j=1}^{F} \left[ V_1^{\pi_k}(s^k_1, j) \right]^2
\]

\[
+ \mathbb{E} \left[ U^k(s^k_1, \pi^k_1(s^k_1), (\lambda^k)') - U^k(s^k_1, \pi^k_1(s^k_1), (\lambda^k)') \right],
\]

(84)

where step (a) holds because we know \(U^k(s, a, \lambda^k) \geq U^1(s, a, \lambda^k) \geq U^1(s, a, \lambda^k)\) in Lemma 5. Besides, note that \(\mathbb{E}[Q^{\pi_k}(s^k_1, \pi^k_1(s^k_1), j)] = V_1^{\pi_k}(s^k_1, j)\) is used in multiple steps.

\[\Box\]
D Proof of Theorem 1

Proof. Combining the result in Lemma 2 and the sub-linear result for modified MDP in Lemma 1, we have

\[ \sum_{k=1}^{K} \left[ V^*_1(s^k_1, 0) - V^*_1(s^k_1, 0) \right] + \frac{\eta}{H} \sum_{j=1}^{J} \sum_{k=1}^{K} \left[ V^*_1(s^k_1, j) \right]^2 \]

\[ - \sum_{k=1}^{K} \mathbb{E} \left[ U^k_1(s^k_1, \pi^k_1(s^k_1), (\lambda^k)' - U^\pi_1(s^k_1, \pi^k_1(s^k_1), (\lambda^k)' \right] \]

\[ \leq \sum_{k=1}^{K} W^*_1(s^k_1, \lambda^k) - W^\pi_1(s^k_1, \lambda^k) \leq O(\eta \sqrt{H^4SAT}) \] (85)

In the proof of Lemma 1, from the second line of Eq. (57) to the end of the lemma, we have that

\[ \sum_{k=1}^{K} \left[ U^k_1(s^k_1, a^k_1, \lambda^k) - U^\pi_1(s^k_1, a^k_1, \lambda^k) \right] \leq O(\eta \sqrt{H^4SAT}) \] (86)

Following the same procedure, this would also hold for \((\lambda^k)'\). Notice that there is no difference for \(h = 1\) or \(h > 1\) because we only need exact \(a_1\) here. Thus, we also don’t need Eq. (58), (59), (60) and (69). Then, combining this result with the Eq. (86), we have

\[ \sum_{k=1}^{K} \left[ V^*_1(s^k_1, 0) - V^*_1(s^k_1, 0) \right] + \frac{\eta}{H} \sum_{j=1}^{J} \sum_{k=1}^{K} \left[ V^*_1(s^k_1, j) \right]^2 \leq O(\eta \sqrt{H^4SAT}) \] (87)

Thus, we obtain

\[ \sum_{k=1}^{K} [V^*_1(s^k_1) - V^*_1(s^k_1)] \leq O(\eta \sqrt{H^4SAT}) \] (88)

Since \(\eta = T^\gamma\), this proves the desired regret bound for the objective.

Furthermore, due to the original reward \(0 \leq r \leq 1\) by the Assumptions 1 and 2, we have a lower bound that

\[ \sum_{k=1}^{K} [V^*_1(s^k_1, 0) - V^*_1(s^k_1, 0)] \geq -O(KH) = -O(T) \] (90)

Using Eq. (88) and Eq. (90), the following inequality holds:

\[ \frac{\eta}{H} \sum_{j=1}^{J} \sum_{k=1}^{K} \left[ V^*_1(s^k_1, j) \right]^2 \leq O(\eta \sqrt{H^4SAT}) + O(T) \leq O(T) \] (91)
Thus, we have
\[
\sum_{k=1}^{K} \left[ V_{1}^{\pi_k}(s^k_1, j) \right]^2 \leq O\left( \frac{HT}{\eta} \right) \tag{92}
\]

Using Cauchy-Schwartz inequality, we have
\[
\frac{1}{K} \sum_{k=1}^{K} \left| V_{1}^{\pi_k}(s^k_1, j) \right| \leq \sqrt{\frac{1}{K} \sum_{k=1}^{K} \left[ V_{1}^{\pi_k}(s^k_1, j) \right]^2} = O\left( \frac{H}{\sqrt{\eta}} \right) \tag{93}
\]

Finally, we have the sub-linear bound for the average constraint violation, which is
\[
\sum_{k=1}^{K} \left| V_{1}^{\pi_k}(s^k_1, j) \right| \leq O(T^{1-\frac{\gamma}{2}}) \tag{94}
\]

This proves the bound on the violations of the constraints. \(\square\)