Energy-momentum for Randall-Sundrum models

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We investigate the conservation law of energy-momentum for Randall-Sundrum models by the general displacement transform. The energy-momentum current has a superpotential and are therefore identically conserved. It is shown that for Randall-Sundrum solution, the momentum vanishes and most of the bulk energy is localized near the Planck brane. The energy density is \( \varepsilon = \varepsilon_0 e^{-3k|y|} \).

Keywords: Energy-momentum; Randall-Sundrum models.

PACS Nos.: 04.20.Cv; 04.20.Fy; 04.50.+h.

1. Introduction

The conservation law of energy-momentum (or the definition of energy-momentum density) for the gravitational field is one of the most fundamental and controversial problems in general relativity. As a true field, it would be natural to expect that gravity should have its own local energy-momentum density. However, it is usually asserted that such a density can not be locally defined because of the equivalence principle. As a consequence, many attempts to identify an energy-momentum density for the gravitational field lead to complexes that are not true tensors. The first of such attempt was made by Einstein who proposed an expression for the energy-momentum density of the gravitational field. Bauer and Shrödinger pointed out

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that Einstein’s expression of the energy-momentum density has a serious short-
coming, i.e., it holds true only in the quasi-Galilean coordinate system and that 
the transformation of the description of flat space from rectangular coordinates to 
spherical coordinates results in a nonzero energy density which yields an infinite 
total energy. Indeed it was nothing but the canonical expression obtained from 
Nöether’s theorem, \(^4\) and this quantity is a pseudotensor, an object that depends 
on the coordinate system. Among the class of conserved quantities, the Landau-
Lifshitz \(^5\) quantity is the only symmetric one. The advantage of symmetry is that it 
is easy in this case to construct further conservation laws in invariant integral form, 
which in turn are related to the angular momentum. But this did not overcome 
the difficulty of Einstein’s complex and was also only admissible in quasi-Galilean 
coordinates. So Landau’s expression also has the same defects as Einstein’s.

One of the present authors (Duan) had proposed one form of the conservation law 
of energy-momentum in Riemann space time, \(^6\) in which the energy-momentum has 
a Lorentz index and a Riemann index, and hence is a covariant vector in Riemann 
space time. This conservation law is general covariant and it overcomes the flaw of 
the Einstein and Landau \(^5\), \(^7\) forms of the energy-momentum conservation law. It 
can explain the nonzero energy flux density and energy density for Bondi’s plane 
wave, and get the correct gravitational radiation formula, \(^8\) where one could not get 
the nonzero energy density for Bondi’s plane wave by Møller’s formula. \(^9\), \(^10\), \(^11\) This 
conservation law had been generalized to the general space time. \(^12\), \(^13\)

On the other hand, Theories with extra dimensions have recently attracted 
enormous attention. The possible existence of such dimensions got strong moti-
vation from theories that try to incorporate gravity and gauge interactions in a 
unique scheme, in a reliable manner. The idea dates back to the 1920’s, to the 
works of Kaluza and Klein \(^14\), \(^15\) who tried to unify electromagnetism with Ein-
stein gravity by assuming that the photon originates from the fifth component of 
the metric. In the course of the last several years, there has been active interest 
in the brane world scenarios \(^16\), \(^17\), \(^18\), \(^19\), \(^20\), \(^21\), \(^22\), \(^23\), \(^24\), \(^25\), \(^26\), \(^27\), \(^28\), \(^29\), \(^30\) and fermionic zero modes 
in Large dimensions. \(^31\), \(^32\), \(^33\), \(^34\), \(^35\), \(^36\), \(^37\), \(^38\), \(^39\), \(^40\), \(^41\), \(^42\), \(^43\) The pioneering work was done by Randall and 
Sundrum. \(^20\), \(^21\) In their works, they present the so called Randall-Sundrum (RS) 
models \(^20\), \(^21\) for warped backgrounds, with compact or even infinite extra 
dimensions. The RS I scenario provides a way to solve the hierarchy problem, and 
the RS II scenario gives Newton’s law of gravity on the brane of positive tension 
embedded in an infinite extra dimension.

As in the (3+1)-dimensional case, we should have some conservation laws in 
order to understand high dimensional gravity well. In this paper, we would like to 
study the simple but typical RS models. The purpose of this paper is to present 
the relationship between conservation theorems and invariance properties of physical 
systems in the models. It is the extension of previous works and includes the original 
general relativity formula of Einstein. In the present paper we pay more attention to 
the energy-momentum conservation law based on the Lagrangian which includes the 
contribution of the branes in RS models, and prove that there exists a superpotential
with respect to the Lagrangian.

The paper is arranged as follows. In section 2, we give a general description of the scheme for establishing general covariant conservation laws in general relativity. In section 3, we first give a simple review of the general displacement transform and the RS models, then use the general displacement transform and the scheme to obtain a general covariant conservation law of energy-momentum for RS models. At the last of this section, we calculate the energy density and the total energy and momentum of the bulk for Randall-Sundrum solution by the superpotential. Section 4 is devoted to some remarks and discussions.

2. Conservation laws in general relativity

The conservation law is one of the important problems in gravitational theory. It is due to the invariance of the action corresponding to some transforms. In order to study the covariant energy-momentum law of more complicated systems, it is necessary to discuss conservation laws by Nöether theorem in the general case.\[\text{Suppose that the space-time manifold } M \text{ is of dimension } n = 1 + d \text{ and the Lagrangian density is in the first order formalism, i.e.}\]

\[I = \int_M d^n x L(\phi^A, \partial_\mu \phi^A),\]  

(1)

where \(\phi^A\) denotes the general fields. If the action is invariant under the infinitesimal transformations

\[x^\mu = x^\mu + \delta x^\mu,\]  

(2)

\[\phi'^A(x') = \phi^A(x) + \delta \phi^A(x),\]  

(3)

and \(\delta \phi^A\) vanishes on the boundary of \(M\), \(\partial M\), then following relation holds

\[\partial_\mu \left( L \delta x^\mu + \frac{\partial L}{\partial \partial_\mu \phi^A} \delta_0 \phi^A \right) + [L]_{\phi^A} \delta_0 \phi^A = 0,\]  

(4)

where

\[[L]_{\phi^A} = \frac{\partial L}{\partial \phi^A} - \phi^A \frac{\partial L}{\partial \partial_\mu \phi^A},\]  

(5)

and \(\delta_0 \phi^A\) is the Lie derivative of \(\phi^A\)

\[\delta_0 \phi^A = \phi'^A(x) - \phi^A(x) = \delta \phi^A(x) - \partial_\mu \phi^A \delta x^\mu.\]  

(6)

If \(L\) is the total Lagrangian density of the system, the field equation of \(\phi^A\) is just \([L]_{\phi^A} = 0\). Hence from Eq. (4), we can obtain the conservation equation corresponding to transformations (2) and (3)

\[\partial_\mu \left( L \delta x^\mu + \frac{\partial L}{\partial \partial_\mu \phi^A} \delta_0 \phi^A \right) = 0.\]  

(7)
It is important to recognize that if $L$ is not the total Lagrangian, e.g. the gravitational part $L_g$, then so long as the action of $L_g$ remains invariant under transformations (2) and (3), Eq. (4) is still valid yet Eq. (7) is no longer admissible because of $[L_g]_{\phi^A} \neq 0$.

In a gravitational theory with the vierbein as elementary fields, we can separate $\phi^A$ as $\phi^A = (e^\nu_a, \psi^B)$, where $\psi^B$ is an arbitrary tensor under general coordinate transformations. Suppose that $L_g$ does not contain $\psi^B$, then Eq. (4) reads

$$\partial_\mu \left( L_g \delta x^\mu + \frac{\partial L_g}{\partial e^\nu_a} \delta e^\nu_a + [L_g]_{e^\nu_a} \delta e^\nu_a \right) + [L_g]_{\epsilon^\nu_a} \delta e^\nu_a = 0. \tag{8}$$

Under transformations (2) and (3), the Lie variations are

$$\delta e^\nu_a = e^\nu_a \delta x^\nu - e^\nu_a \delta x^\nu, \tag{9}$$

where "," denotes partial derivative. Substituting Eq. (9) into Eq. (8) gives

$$\partial_\mu \left( L_g \delta x^\mu + \frac{\partial L_g}{\partial e^\nu_a} e^\nu_a \delta x^\nu - \frac{\partial L_g}{\partial e^\nu_a} \epsilon^\nu_a \delta x^\nu + [L_g]_{e^\nu_a} e^\nu_a \delta x^\nu \right) + [L_g]_{\epsilon^\nu_a} e^\nu_a = 0. \tag{10}$$

Comparing the coefficients of $\delta x^\mu$, $\delta x^\nu$, and $\delta x^\nu$, we can obtain an identity

$$[L_g]_{e^\nu_a} e^\nu_a + \partial_\nu ([L_g]_{e^\nu_a} e^\nu_a) = 0. \tag{11}$$

Then Eq. (10) can be written as

$$\partial_\mu \left( L_g \delta x^\mu - \frac{\partial L_g}{\partial e^\nu_a} e^\nu_a \delta x^\nu + [L_g]_{e^\nu_a} e^\nu_a \right) \delta x^\sigma + \frac{\partial L_g}{\partial e^\nu_a} e^\nu_a \delta x^\nu = 0. \tag{12}$$

This is the general conservation law in the vierbein formalism of general relativity. By definition, we introduce

$$I^\mu_\sigma = L_g \delta x^\mu - \frac{\partial L_g}{\partial e^\nu_a} e^\nu_a \delta x^\nu + [L_g]_{e^\nu_a} e^\nu_a, \tag{13}$$

$$\tilde{V}^\mu_\sigma = \frac{\partial L_g}{\partial e^\nu_a} e^\nu_a. \tag{14}$$

Then Eq. (12) gives

$$\partial_\mu (I^\mu_\sigma \delta x^\sigma + \tilde{V}^\mu_\sigma \delta x^\sigma) = 0. \tag{15}$$

Eq. (15) is tenable under arbitrary infinitesimal transformations, so we can compare the coefficients of $\delta x^\sigma$, $\delta x^\nu$, and $\delta x^\nu$ and obtain

$$\partial_\mu I^\mu_\sigma = 0, \quad I^\mu_\sigma = -\partial_\nu \tilde{V}^\nu_\mu, \quad \tilde{V}^\mu_\sigma = -\tilde{V}^\nu_\nu. \tag{16}$$

Eqs. (10)-(18) are fundamental to the establishing of conservation law of energy-momentum.
3. Conservation law of energy-momentum for RS models

3.1. General displacement transformations

In 3+1 dimension, the conservation of energy-momentum in special relativity is a consequence of the invariant property of the action under the infinitesimal translation of the Lorentz coordinates

\[ x^\bar{a} = x^{\bar{a}} + b^{\bar{a}}, \quad b^{\bar{a}} = \text{const.} \quad (\bar{a} = 0, 1, 2, 3) \] (19)

The corresponding transformation of arbitrary coordinates in flat spacetime is

\[ x^{\bar{\mu}} = x^{\bar{\mu}} + \delta x^{\bar{\mu}}, \quad (\bar{\mu} = 0, 1, 2, 3) \] (20)

where

\[ \delta x^{\bar{\mu}} = \frac{\partial x^{\bar{\mu}}}{\partial x^{\bar{a}}} \delta x^{\bar{a}} \delta x^{\bar{a}} = \frac{\partial x^{\bar{\mu}}}{\partial x^{\bar{a}}} b^{\bar{a}}. \] (21)

If we extend this idea to general relativity, we have the generalized translation transformation

\[ x^{\bar{\mu}} = x^{\bar{\mu}} + \delta x^{\bar{\mu}}, \quad \delta x^{\bar{\mu}} = e^{\bar{\mu}}_{\bar{a}} b^{\bar{a}}. \] (22)

Using the invariance of the action with respect to general displacement transformations and Einstein equations, we get the following general covariant conservation law of energy-momentum

\[ \nabla_{\bar{\mu}} (T^{\bar{\mu}}_{\bar{a}} + t^{\bar{\mu}}_{\bar{a}}) = 0. \] (23)

The total energy-momentum is

\[ P_{\bar{a}} = \frac{1}{c} \int_{\Sigma} d\Sigma_{\bar{\mu}} \sqrt{-g} (T^{\bar{\mu}}_{\bar{a}} + t^{\bar{\mu}}_{\bar{a}}) = \frac{1}{c} \int_{S} dS_{\bar{\mu}\bar{\nu}} V^{\bar{\mu}\bar{\nu}}_{\bar{a}}, \] (24)

where \( V^{\bar{\mu}\bar{\nu}}_{\bar{a}} \) is the superpotential

\[ V^{\bar{\mu}\bar{\nu}}_{\bar{a}} = \frac{c^4}{8\pi G} \left[ e^{\bar{\mu}}_{\bar{b}} e^{\bar{\nu}}_{\bar{c}} \omega^{\bar{a}}_{\bar{b} \bar{c}} + (e^{\bar{\mu}}_{\bar{b}} e^{\bar{\nu}}_{\bar{b}} - e^{\bar{\mu}}_{\bar{b}} e^{\bar{\nu}}_{\bar{a}}) \omega^{\bar{a}} \right] \sqrt{-g}. \] (25)

This definition of energy-momentum has the following main properties:

1). It is a covariant definition with respect to general coordinate transformations. But the energy-momentum tensor is not covariant under local Lorentz transformations, this is reasonable because of the equivalence principle.

2). For a closed system, the total energy-momentum does not depend on the choice of Riemann coordinates and transformations in the covariant way

\[ P'_{\bar{a}} = L_{\bar{a}}^{\bar{b}} P_{\bar{b}} \] (26)

under local Lorentz transformation \( L_{\bar{a}}^{\bar{b}} \) which is constant \( L_{\bar{a}}^{\bar{b}} \) at spatial infinity.

3). For a closed system with static mass center, the total energy-momentum is \( P_{\bar{a}} = (Mc, 0, 0, 0) \), i.e., the total energy \( E = Mc^2 \).
4). For a rather concentrated matter system, the gravitational energy radiation is
\[ -\frac{\partial E}{\partial t} = \frac{G}{45c^5}(D_{ij})^2, \] (27)
where \( D_{ij} \) is the mass quadrupole moment. This quadrupole radiation formula is in good agreement with the observational data from the synthetic analysis of the gravitational radiation damping for the pulsar PSR 1913+16 in a binary star system. 36

5). For Bondi’s plane wave, the energy current is
\[ t_0^0 = \left( \frac{1}{4\pi} \beta^2, \frac{1}{4\pi} \beta^2, 0, 0 \right), \] (28)
where \( \beta \) is a function of \( (t - x^1) \). From the above expression we can see that the energy density is determined by \( \beta^2 \) which always has positive values. While using Møller’s expression, Kuchar and Langer calculated the energy density for Bondi’s plane wave, the result obtained is zero.

6). For the solution of gravitational solitons, we can obtain finite energy while the Landau-Lifshitz definition leads to infinite energy. 37

7). In Ashtekar’s complex formalism of general relativity, the energy-momentum and angular momentum constitute a 3-Poincare algebra and the energy coincides with the ADM energy. 35

In the next subsection, we first give a brief introduction of the RS models. Then, with these foundations above, we use (4+1)-dimensional transformations to obtain conservative energy-momentum for RS models.

3.2. RS models

Let us consider the following setup. A five dimensional spacetime with an orbifolded fifth dimension of radius \( r \) and coordinate \( y \) which takes values in the interval \([0, \pi r]\). Consider two branes at the fixed (end) points \( y = 0, \pi r \); with tensions \( \tau \) and \(-\tau\) respectively. The brane at \( y = 0 \) (\( y = \pi r \)) is usually called the hidden (visible) or Planck (SM) brane. We will also assign to the bulk a negative cosmological constant \(-\Lambda\). Here we shall assume that all parameters are of the order the Planck scale.

The classical action describing the above setup is given by
\[ S_g = S_0 + S_h + S_v, \] (29)
here
\[ S_0 = \int d^4x \sqrt{g} \left( \frac{1}{2k^2} R + \Lambda \right) \] (30)
gives the bulk contribution, whereas the visible and hidden brane parts are given by
\[ S_{v,h} = \pm \tau \int d^4x \sqrt{-g_{v,h}}, \] (31)
where \( g_{v,h} \) stands for the induced metric at the visible and hidden branes, respectively. And \( 2k^2_* = 8\pi G_* = M^3_* \). Five dimensional Einstein equations for the given action are

\[
G_{\mu \nu} = -k^2_\star \Lambda g_{\mu \nu} + k^2_\star \tau \sqrt{-g} \delta^\bar{\mu} \delta^\bar{\nu} g_{\bar{\mu} \bar{\nu}} \delta(y) - k^2_\star \tau \sqrt{-g} \delta^\mu \delta^\nu g_{\mu \nu} \delta(y - \pi r),
\]

(32)

where the Einstein tensor

\[
G_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R \asusual,
\]

Greek indices without bar \( \mu, \nu = 0, \ldots, 4 \) and the others with bar \( \bar{\mu}, \bar{\nu} = 0, \ldots, 3 \). The solution that gives a flat induced metric on the branes is

\[
ds^2 = g_{\mu \nu} dx^\mu dx^\nu = e^{-2k|y|} \eta^\bar{\mu} dx^\bar{\mu} - dy^2,
\]

(33)

in which \( x^\bar{\mu} \) are coordinates for the familiar four dimensions, \( k \) is a scale of order the Planck scale \( k^2 \star = \frac{k^2 \star \Lambda}{6M^3_*} \).

(34)

The effective Planck scale in the theory is given by \( M^2_\star = \frac{M^3_*}{k} (1 - e^{-2k\pi r}) \).

(35)

Notice that for large \( r \), the exponential piece becomes negligible, and above expression has the familiar form given in ADD models for one extra dimension of (effective) size \( R_{ADD} = 1/k \):

\[
M^2_\star = M^2_* + n R^2_{ADD}.
\]

(36)

### 3.3. The energy-momentum for RS models

The Lagrangian density for Randall-Sundrum background can be written as

\[
\mathcal{L} = \mathcal{L}_g + \mathcal{L}_m,
\]

(37)

where \( \mathcal{L}_m \) denotes the matter part and

\[
\mathcal{L}_g = \sqrt{g} \left( \frac{1}{2k^2_*} R + \Lambda \right) - \tau \sqrt{-g_v} \delta(y) + \tau \sqrt{-g_h} \delta(y - \pi r),
\]

(38)

\[
R = \omega_a \omega^a - \omega_{abc} \omega^{cba} - \frac{2}{\sqrt{g}} \partial_\mu (\sqrt{g} e^\mu_b \omega^a),
\]

(39)

\[
\omega_{abc} = \frac{1}{2}(\Omega_{abc} - \Omega_{bac} + \Omega_{cab}),
\]

(40)

\[
\Omega_{abc} = e^\mu_a e^\nu_b (\partial_\mu e_{cv} - \partial_v e_{cm}),
\]

(41)

\[
\omega_a = \eta^{bc} \omega_{bac} = \omega^c_{ac}.
\]

(42)

Eliminating the divergence expression, we rewrite the \( \mathcal{L}_g \) as

\[
\mathcal{L}_g = \frac{1}{2k^2_*} (\omega_a \omega^a - \omega_{abc} \omega^{cba}) \sqrt{g} + \Lambda \sqrt{g} - \tau \sqrt{-g_v} \delta(y) + \tau \sqrt{-g_h} \delta(y - \pi r).
\]

(43)
For transformations Eq. (22), Eq. (15) implies
\[ \partial_{\mu} (\tilde{I}_{\mu}^{\sigma} e_{\alpha}^{\sigma} + \tilde{V}_{\mu}^{\nu} e_{\alpha,\nu}^{\sigma}) = 0. \] (44)

From Einstein equations \( \sqrt{g} T_{\alpha}^{\mu} = [L_{g}] e_{\alpha}^{\mu} \) and Eq. (13), we can express \( \tilde{I}_{\mu}^{\nu} e_{\alpha}^{\nu} \) as
\[ \tilde{I}_{\mu}^{\nu} e_{\alpha}^{\nu} = \left( L_{g} \delta_{\mu}^{\nu} - \frac{\partial L_{g}}{\partial e_{\lambda,\mu}^{a}} e_{\alpha}^{\lambda} + \sqrt{g} T_{\alpha}^{\mu} \right). \] (45)

Defining
\[ \sqrt{g} t_{\alpha}^{\mu} = \left( L_{g} \delta_{\mu}^{\nu} - \frac{\partial L_{g}}{\partial e_{\lambda,\mu}^{a}} e_{\alpha}^{\lambda} + \sqrt{g} T_{\alpha}^{\mu} + \frac{\partial L_{g}}{\partial e_{\nu,\mu}^{b}} e_{\beta}^{b} e_{\sigma}^{\nu} \right), \] (46)
and considering Eq. (14), we then have
\[ \tilde{I}_{\mu}^{\nu} e_{\alpha}^{\nu} + \tilde{V}_{\mu}^{\nu} e_{\alpha,\nu}^{\sigma} = \sqrt{g} (T_{\alpha}^{\mu} + t_{\alpha}^{\mu}). \] (47)

So Eq. (44) can be written as
\[ \partial_{\mu} \left[ \sqrt{g} (T_{\alpha}^{\mu} + t_{\alpha}^{\mu}) \right] = 0. \] (48)

This equation is the desired general covariant conservation law of energy-momentum for Randall-Sundrum system. \( t_{\alpha}^{\mu} \) defined in Eq. (46) is the energy-momentum density of gravity field, and \( T_{\alpha}^{\mu} \) to that of matter part. By virtue of Eq. (17), the expression on the LHS of Eq. (47) can be expressed as divergence of superpotential
\[ \sqrt{g}(T_{\alpha}^{\mu} + t_{\alpha}^{\mu}) = \partial_{\nu} V_{\alpha}^{\nu\mu}, \] (49)
where
\[ V_{\alpha}^{\nu\mu} = \tilde{V}_{\sigma}^{\nu\mu} e_{\alpha}^{\sigma} = \frac{\partial L_{g}}{\partial e_{\nu,\mu}^{b}} e_{\beta}^{b} e_{\sigma}^{\nu}. \] (50)

Eq. (49) shows that the total energy-momentum density of a gravity system always can be expressed as divergence of superpotential. The total energy-momentum is
\[ P_{\alpha} = \int_{\Sigma} d\Sigma_{\mu} \sqrt{g} (T_{\alpha}^{\mu} + t_{\alpha}^{\mu}) = \int_{S} dS_{\mu\nu} V_{\alpha}^{\mu\nu}, \] (51)
where \( dS_{\mu\nu} = (1/3!){\varepsilon}_{\mu\nu\beta\gamma} dx^{\alpha} \wedge dx^{\beta} \wedge dx^{\gamma}. \)

Now we calculate the expressions of \( V_{\alpha}^{\mu\nu} \) and \( t_{\alpha}^{\mu} \) by using the gravity Lagrangian density \( L_{g} \) of RS models. The explicit expressions are
\[ V_{\alpha}^{\mu\nu} = \frac{1}{K_{2}^{2}} \left[ e_{\mu}^{b} e_{\nu}^{c} \omega_{\alpha}^{bc} + (e_{\alpha}^{b} e_{\mu}^{\nu} - e_{\alpha}^{\nu} e_{\mu}^{b}) \omega^{b} \right] \sqrt{g}, \] (52)
\[ t_{\alpha}^{\mu} = \frac{1}{2 K_{2}^{2}} \left\{ e_{\alpha}^{b} \left[ (\omega_{\mu}^{b} - \omega_{c,\mu}^{b,\nu} \omega^{b}) - 2 e_{\alpha}^{b} (\omega_{\nu}^{b} - \omega_{c,\nu}^{b,\mu} \omega^{b}) \right] \right. \]
\[ -2 e_{\alpha}^{b} (\omega_{\nu}^{b} \omega_{\mu}^{a} - \omega_{\nu}^{a} \omega_{\mu}^{b}) + 2 e_{\alpha}^{b} \omega_{c,\nu}^{b,\mu} \omega^{bc} \]
\[ + e_{\alpha}^{b} \left\{ \Lambda - \tau \sqrt{-g_{\mu}} \delta(y) + \tau \sqrt{-g_{\mu}} \delta(y - \pi r) \right\}. \] (53)
For the solution (33), we can obtain the following vierbein
\[ e^a_\mu = (e^{-k|y|\delta^\mu_\bar{a}}, \delta^4_\mu), \quad (\bar{a} = 0, 1, 2, 3) \] (54)
The non-vanishing components of \( \omega_{abc} \) and \( \omega_0 \) are
\[ \omega_{004} = -k, \quad \omega_{114} = \omega_{224} = \omega_{334} = k, \quad \omega_4 = 4k. \] (55)
For superpotential \( V^\mu_\nu \), the calculated result is
\[ V^0_0^4 = V^1_1^4 = V^2_2^4 = V^3_3^4 = -3 \frac{k}{k^2} e^{-3k|y|}. \] (56)
Substituting Eq. (56) into integral expression (51) gives
\[ P_a = \left( k \frac{M^6}{k^2_p} (1 - e^{-3k\pi r})v, 0, 0, 0, 0 \right), \] (57)
where \( v \) stands for the volume of usual three dimensional space \( M_3 \)
\[ v = \int_{M_3} d^3x. \] (58)
Substituting \( k^2 = \frac{1}{2} M_k^{-3} \) and Eq. (35) into Eq. (57) yields
\[ P_a = \left( \frac{6 M^6}{M_p^2} (1 - e^{-3k\pi r})(1 - e^{-2k\pi r})v, 0, 0, 0, 0 \right). \] (59)
So the momentum of RS system is vanishing, but the energy
\[ E = \frac{6 M^6}{M_p^2} (1 - e^{-3k\pi r})(1 - e^{-2k\pi r})v \] (60)
is infinite, which is caused by the gravity on the warped extra dimension. When the radius \( r \) of the extra dimension is taken the limit \( r \to 0 \), i.e., the extra dimension disappears, the bulk is just degenerates into our flat universe and the energy of it vanishes. This result agrees with others.

From Eqs. (49) and (56), we can get the energy density \( \varepsilon \):
\[ \varepsilon = \varepsilon_0 e^{-3k|y|}, \quad \varepsilon_0 = \frac{9 M^9}{2 M_p^2} (1 - e^{-2k\pi r}). \] (61)
Clearly, from this expression, it can be concluded that the energy of the bulk distributes mainly near the Planck brane and force lines are denser near this brane. This conclusion can be used to explain why the gravitation on the SM brane (is our universe) is very weak.
4. Discussions

To summarize, by the use of general Nöether theorem, we have obtained the conservation law of energy-momentum for the RS models with respect to the general displacement transform. The energy-momentum current has a superpotential and are therefore identically conserved. General covariance is a fundamental demand for conservation laws in general relativity, and our definition Eqs. (49) and (51) of energy-momentum is coordinate independent. This conservation law is a covariant theory with respect to the generalized coordinate transformations, but the energy-momentum tensor is not covariant under the local Lorentz transformation which, due to the equivalent principle, is reasonable to require.

The conservative energy-momentum current and the corresponding superpotential for the RS models are the same with those in (3+1)- and (2+1)-dimensional Einstein theories, the Lagrangian density $L_{h,v}$ corresponding to the hidden and visible brane parts do not play a role in the conservation law, but they have an influence to the energy of gravity. Both energy-momentum current and the superpotential are determined only by vierbein field.

It is shown that, for the solution that gives a flat induced metric on the branes, the momentum vanishes but the total energy of bulk is infinite. This is quite different from the solution of an isolated system in 3+1 dimension. The reason is that, though the branes are flat, the extra dimension is warped even at the infinite of them. When the extra dimension disappears, the bulk is just degenerated into our four dimensional flat spacetime and the energy vanishes. From the energy density formula (61), one can see that most of the bulk energy is localized near the Planck brane, so the gravity on the SM brane is very weak.

We think a conservation law of angular momentum is also important in order to understand the conservative quantities for the RS models. This conservation law has been obtained using the approach in Ref. 39.

Acknowledgement

It is a pleasure to thank Dr Liming Cao, Zhenhua Zhao and Zhenbin Cao for interesting discussions. This work was supported by the National Natural Science Foundation of the People’s Republic of China and the Fundamental Research Fund for Physics and Mathematic of Lanzhou University.

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