Testing fixed points in the 2D $O(3)$ non–linear $\sigma$–model.

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Using high statistic numerical results we investigate the properties of the $O(3)$ non–linear 2D $\sigma$–model. Our main concern is the detection of an hypothetical Kosterlitz–Thouless–like ($KT$) phase transition which would contradict the asymptotic freedom scenario. Our results do not support such a $KT$–like phase transition.

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I. INTRODUCTION

The bidimensional $O(N)$ non–linear sigma model is defined on the lattice by the action

$$S = -\beta \sum_{x,\mu} \phi(x) \cdot \phi(x + \mu).$$

Here $\phi(x)$ is a $N$–component vector constrained by $\phi^2(x) = 1$. Perturbation theory predicts that this model is asymptotically free if $N \geq 3$. Large distance observables must scale accordingly with the renormalization group equation. In particular we find for the correlation length

$$\xi = C_\xi \left( \frac{N - 2}{2\pi\beta} \right)^{\frac{N-2}{2}} \exp \left( \frac{2\pi\beta}{N - 2} \right) \left( 1 + \sum_{k=1}^{\infty} c_k \beta^k \right)$$

and for the magnetic susceptibility

$$\chi = C_\chi \left( \frac{N - 2}{2\pi\beta} \right)^{\frac{N+1}{2}} \exp \left( \frac{4\pi\beta}{N - 2} \right) \left( 1 + \sum_{k=1}^{\infty} b_k \beta^k \right).$$

The coefficients $a_k$ and $b_k$ can be evaluated perturbatively starting from 3–loop order. They describe non–universal (scheme dependent) corrections to asymptotic scaling. The constants $C_\xi$ and $C_\chi$ cannot be evaluated perturbatively. There are many possible definitions of correlation length. All of them scale in the same way, Eq. (2) with the same coefficients $a_k$. The constant $C_\xi$ is instead definition dependent. The exponential fall–off of the correlation function at large distances provides the so–called exponential definition whose constant $C_\xi$ can be obtained exactly (alas not rigorously!) with the Bethe–Ansatz technique

$$C_\xi = \left( \frac{e}{8} \right)^{\frac{N-2}{2}} \Gamma \left( \frac{N - 1}{N - 2} \right) 2^{-5/2} \exp \left( -\frac{\pi}{2(N - 2)} \right).$$

$C_\chi$ is known at second order in the $1/N$ expansion. In the ratio $\chi/\xi^2$ the exponential dependence on $\beta$ disappears, and the quantity

$$R_{PT} \equiv \frac{\chi}{\xi^2} \left( \frac{2\pi\beta}{N - 2} \right)^{\frac{N-1}{2}} \left( 1 + \sum_{k=1}^{\infty} c_k \beta^k \right)^{\beta \to \infty} \frac{C_\chi}{C_\xi}$$

should stay constant for large enough $\beta$. The coefficients $c_k$ can be easily calculated from $a_k$ and $b_k$.

According to the Mermin–Wagner theorem the continuous $O(N)$ symmetry cannot be broken in two dimensions. Perturbative expansions are constructed around a non–symmetric trivial vacuum, so their validity is not guaranteed. By analysing the percolation properties of the Fortuin–Kasteleyn clusters it has been argued that perturbative methods are inadequate to describe the model. The main conclusion from these analyses is that the non–linear sigma model in 2 dimensions has no mass–gap, contradicting the results of and that for any $N \geq 2$ it undergoes a $KT$–like phase transition at a finite inverse temperature $\beta_{KT}$. 

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If this is the case, then the ratio

\[ R_{KT} \equiv \frac{\chi}{\xi^{2-\eta}} (\beta_{KT} - \beta)^r \]  

should be asymptotically constant for \( \beta_{KT} - \beta \) positive and small. Here \( \eta \) is a critical exponent. For the O(2) model, where the KT transition is generally trusted, \( \eta = 1/4 \). On the other hand, renormalization group considerations and numerical evaluations of \( r \) yield the bound \( |r| \lesssim 0.1 \) [10–14].

It is well known that in the O(3) model there is not a good numerical evidence of the onset of the asymptotic scaling regime. In [15] the authors find that in this model the numerical data for the ratio \( R_{KT} \) with \( \eta = 1/4 \) can be well fitted with a constant, while the ratio \( R_{PT} \) is clearly decreasing. They make use of data obtained from a simulation with the standard action Eq. (1).

In [16] it has been shown however that \( R_{KT} \) is not constant. To do this, the 0–loop Symanzik improved action was numerically simulated at high statistics \((O(10^7)) \) measurements, including corrections to asymptotic scaling up to 3 loops. In [16] we claimed that the discrepancy on the constancy of \( R_{KT} \) between [16] and [15] originates from the higher resolution of the data in [16]. Arguments against these conclusions has been raised based on possible distorting effects of the antiferromagnetic coupling present in the improved action. Here we want to perform an analogous study, at high statistics too, on the O(3) model with standard action where such a coupling is absent.

For completeness we will show the results for \( C_\xi \), \( C_\chi \) and the \( R_{PT} \) ratio obtained from our data.

II. MONTE CARLO SIMULATION

We have simulated the standard action Eq. (1) on lattice sizes \( L \) by using the Wolff algorithm [17]. The wall–wall correlation function

\[ G(t) \equiv \frac{1}{L} \sum_x \langle \phi(0) \cdot \phi(t, x) \rangle \]  

has been measured with improved estimators [18]. We have always chosen \( L \) big enough to avoid sizable finite volume effects: \( L/\xi \gtrsim 7 \) [13,16].

Using these data we can evaluate the correlation function at momentum \( p \)

\[ \tilde{G}(p) \equiv L \sum_t e^{ipt} G(t), \]  

the magnetic susceptibility \( \chi \equiv \tilde{G}(0) \) and the correlation length from the exponential fall–off of Eq. (8)

\[ \xi = -\lim_{t \to \infty} \frac{t}{\ln G(t)}. \]  

The data for these quantities are listed in Table I, together with the energy density defined as

\[ E \equiv \langle \phi(0) \cdot \phi(0 + \hat{\mu}) \rangle \]  

where \( \hat{\mu} \) is a generic direction and there is no a sum over that \( \mu \).

III. RESULTS

Although the present note is mainly concerned on the existence of a phase transition at a finite temperature, we give for completeness also the results for \( C_\xi \), \( C_\chi \) and \( R_{PT} \).

By using Eq. (2) and our Monte Carlo data we obtain a prediction about \( C_\xi \), which we call \( C^{MC}_\xi \). In Fig. 1 we plot with empty symbols the ratio \( C^{MC}_\xi / C_\xi \) versus \( \beta \), adding scaling corrections up to four loops [21] (we have taken into account the corrections of [21]). It is apparent that we are not in an asymptotic scaling regime, although the non–universal scaling corrections provide a systematic improvement. In the best (4–loop) case the lack of asymptotic scaling is around 15%.

We have also used an effective scheme [22] defined by the perturbative expansion of the energy Eq. (10)
\[ E = 1 - \frac{w_1}{\beta} - \frac{w_2}{\beta^2} - \frac{w_3}{\beta^3} - \frac{w_4}{\beta^4} - O\left(\frac{1}{\beta^5}\right), \]
\[ \beta_E \equiv \frac{w_1}{1 - E} = \beta - \frac{w_2}{w_1} + \frac{w_2^2 - w_1 w_3}{\beta} - \frac{w_1^2 - 2 w_1 w_2 w_3 + w_1^2 w_4}{w_1^2} + O\left(\frac{1}{\beta^2}\right). \]

The results in this scheme are displayed as full symbols in Fig. [3]. Here the lack of asymptotic scaling is still large: about 7–8% in the best case.

Starting from Eq. (3) we can study in the same way the ratio \( C^{MC}/C_x \), using for \( C_x \) the \( O(1/N^2) \) result \[ C^{(1/N^2)} = 0.0127. \] The percentage of discrepancy in Fig. 3 between the Monte Carlo result and the \( 1/N^2 \) approximation is similar to that of the correlation length in Fig. 3. It is interesting to note that the 3–loop correction is irrelevant, while the 4–loop one is not.

The data in the effective energy–scheme for the Fig. [3] are displayed as full symbols. Here we see an excellent agreement between the Monte Carlo data at 4–loops and the \( 1/N^2 \) approximation. We do not know how good is the \( 1/N^2 \) approximation to \( C_x \). Therefore we avoid drawing optimistic conclusions about the approach to the asymptotic scaling regime in this case.

In Fig. 4 we plot the logarithm of the ratio \( R_{PT} \) versus the logarithm of the correlation length. Again the full (open) symbols correspond to the effective (standard) scheme. We see that data cannot be considered constant, although corrective terms give a systematic improvement. We stress the fact that further corrections and further statistics do not worsen the results. From the larger \( O \) to functional forms similar to Eq. (12) but with other powers of \( \beta \). This value however must be regarded as an upper bound because data in Fig. 3 are still descending.

In calculating this figure we have omitted the power \((1/N^2)\), see Eq. (4). Had the model to undergo a phase transition at finite coupling \( \beta_{KT} \), the correlation length \( \xi \) should behave like

\[ \xi = A \exp\left(\frac{B}{\sqrt{\beta_{KT} - \beta}}\right). \]

If we try to fit our data to such functional form we obtain very unstable results (including more or less data points in the fit, the results for \( A \), \( B \) and \( \beta_{KT} \) vary strongly). The fit to the form Eq. (2) is much more stable. This phenomenon has a simple interpretation: \( \beta_{KT} \) is so large that Eq. (12) can be approximated to

\[ \xi \approx A' \exp(B' \beta) \]

\[ A' = A \exp\left(\frac{B}{\sqrt{\beta_{KT}}}ight) \]

\[ B' = \frac{B}{2\beta_{KT}^{3/2}} \]

and therefore we are fitting actually the combination \( B/\beta_{KT}^{3/2} \). If \( \beta_{KT} \) is so large and recalling that \(|r| \lesssim 0.1 \), then the power \((\beta_{KT} - \beta)\) is almost constant within the narrow interval of our working \( \beta \)'s. This is why we omitted such a factor in the calculation of \( R_{KT} \).

Notice that the shape of the curve for \( R_{KT} \) can be obtained by using the perturbative expressions in Eqs. (3). Owing to the incomplete asymptotic scaling, we obtain a 20% difference in the overall scale in the best case (4 loops and energy scheme). However the position of the minimum is well predicted: the perturbative expansion at 4 loops and energy scheme display a minimum at \( \ln \xi = 3.2 \) while the curve in Fig. 4 has minimum at \( \ln \xi = 3.5(4) \) (the error has been estimated by interpolating the top and bottom values of the error bars in the data of Fig. 4).

The prediction of \( \beta_{KT} \) is that the \( O(N) \) model behaves like the \( O(2) \) model for \( N \geq 3 \). Therefore the set of critical exponents of \( O(2) \) should also apply for \( O(3) \). However one could imagine a set of slightly different exponents (after all, the \( O(3) \) and \( O(2) \) models have different numbers of degrees of freedom). Following this idea, we have made further fits on the data for \( \xi \) to functional forms similar to Eq. (12) but with other powers of \((\beta_{KT} - \beta)\). In all cases the only sensible conclusion was that \( \beta_{KT} \) is very large. One can also try other versions of the ratio \( R_{KT} \) by varying \( \eta \). The tendency is that for larger (smaller) \( \eta \) the curve acquires a positive (negative) slope for all points. We could not find a value of \( \eta \) for which the whole set of points in Fig. 4 flattens out.

**IV. CONCLUSIONS**

We have done a Monte Carlo simulation for the standard action of the \( O(3) \) non–linear \( \sigma \)–model in 2-dimensions. We have improved the statistics with respect to previous work taking advantage of the recently calculated corrections
to scaling \[20,21\] and energy \[16\]. We have tested the perturbation theory predictions in both the standard and effective schemes.

In Figs. 1–3 we have plotted our results for \(C_\xi\), \(C_\chi\) and \(R_{PT}\) as a function of \(\beta\). We see that the use of an effective scheme improves the asymptotic scaling in a sensible way. However, in the best case, we are still far from asymptotic scaling by roughly 10\%. Much closer approaches to the scaling region were obtained by using the Symanzik action \[16\].

It is remarkable that in all cases, the 4–loop correction in the effective scheme is negligible and that if we trust the \(O(1/N^2)\) estimate of \(C_\chi\) then our data have reached the scaling regime after including 3 or 4 loops.

As for our main result, Fig. 4, it does not give any support to the KT–scenario. The plot has the same shape than for the Symanzik action \[16\] and this indicates that distorting effects due to the antiferromagnetic coupling are quite unlikely. In fact, in \[16\] we worked with \(\xi > 16\) in order to avoid any strong coupling effect.

As already shown in \[16\], the ratio \(R_{PT}\) is not constant but further corrections straighten it. This tendency is not satisfied for the \(R_{KT}\) ratio if we increase the statistics. In conclusion, we think that the main claim in \[15\] cannot be maintained within the present status of numerical simulations because either \(\beta_{KT}\) is too large to see any effect of the KT–transition or even it is infinity.

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VI. NOTE ADDED

To obtain the results displayed in Figs. 1–3, we have used the corrected values for the finite integrals published in \[21\] needed for the 4–loop beta function. We notice that these corrected values do not change appreciably the results shown in \[16\]. For instance, for the \(O(8)\) model with standard action and energy scheme it was shown in \[16\] that \(C_{MC}^M\) differs from Eq. (4) by 0.5\%; after the corrections of \[21\] this difference becomes 0.4–0.5\%.

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FIG. 1. The ratio between non–perturbative constants $C_{\xi MC}/C_{\xi}$ for the model. Empty circles (squares, diamonds) stand for the standard scheme 2–loop (3–loop,4–loop) results; filled circles (squares, diamonds) stand for the energy scheme 2–loop (3–loop,4–loop) results.

FIG. 2. The ratio between non–perturbative constants $C_{\chi MC}/C_{\chi}^{(1/N^2)}$ for the model. The meaning of the symbols is the same of Fig. 1. Some data have been horizontally shifted to render the figure clearer.
FIG. 3. The $P_T$ ratio for the model. Empty circles (squares, diamonds, triangles) stand for the 1–loop (2–loop, 3–loop, 4–loop) results in the standard scheme; filled circles (squares, diamonds, triangles) stand for the 1–loop (2–loop, 3–loop, 4–loop) results in the energy scheme. Some data have been horizontally shifted to render the figure clearer.

FIG. 4. The $K_T$ ratio for the model.
TABLE I. Results of Monte Carlo data for the $O(3)$ nonlinear $\sigma$–model with standard action.

| $\beta$ | $L$  | stat  | $\chi$     | $\xi$     | $E$       |
|---------|------|-------|------------|-----------|-----------|
| 1.50    | 100  | $5 \times 10^6$ | 176.20(0.16) | 10.99(1)  | 0.6015813(19) |
| 1.60    | 150  | $5 \times 10^6$ | 446.91(0.42)  | 18.89(2)  | 0.6357033(11) |
| 1.70    | 260  | $5 \times 10^6$ | 1264.4(1.00)  | 34.36(3)  | 0.6642223(6)  |
| 1.75    | 340  | $5 \times 10^6$ | 2189.5(2.2)   | 46.97(5)  | 0.6766299(4)  |
| 1.80    | 450  | $5 \times 10^6$ | 3827.4(3.9)   | 64.43(7)  | 0.6879333(3)  |
| 1.85    | 640  | $5 \times 10^6$ | 6717.6(7.1)   | 88.53(10) | 0.6983241(2)  |
| 1.90    | 860  | $5 \times 10^6$ | 11850.(13.)   | 121.70(10)| 0.7079167(1)  |