ON THE BOUNDEDNESS OF THE MAXIMAL AND FRACTIONAL MAXIMAL, POTENTIAL OPERATORS IN THE GLOBAL MORREY-TYPE SPACES WITH VARIABLE EXPONENTS

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Abstract. We consider the global Morrey-type spaces $GM_{p(.),\theta(.),w(\cdot)}(\Omega)$ with variable exponents $p(x)$, $\theta(x)$ and general function $w(x, r)$ defining these spaces. In the case of unbounded sets $\Omega \subset \mathbb{R}^n$, we prove boundedness of the Hardy–Littlewood maximal operator, potential type operator in these spaces.

1 Introduction

In this paper we consider the global Morrey-type spaces $GM_{p(.),\theta(.),w(\cdot)}(\Omega)$ with variable exponents $p(.), \theta(.)$ and a general function $w(x, r)$ defining a Morrey-type norm. The Morrey spaces $M_{p,\lambda}$ are introduced in [1] in relation to the study of partial differential equations. Many classical operators of harmonic analysis (for example, maximal, fractional maximal, potential operators) were studied in the Morrey-type spaces with constant exponents $p, \theta$ [2, 3, 4]. The Morrey spaces also attracted attention of researchers in the area of variable exponent analysis; see [5, 6, 7, 8, 9, 10]. The Morrey spaces $L_{p(.),\lambda(\cdot)}(\Omega)$ with variable exponent $p(.), \lambda(.)$ were introduced and studied in [5]. The general version $M_{p(.),w(\cdot)}(\Omega)$, $\Omega \subset \mathbb{R}^n$ were introduced and studied in [11] in the case of bounded sets $\Omega \subset \mathbb{R}^n$, and in [12] in the case of unbounded sets $\Omega \subset \mathbb{R}^n$. The boundedness of maximal and potential type operators in the generalized Morrey-type spaces with a variable exponent were considered in [11] in the case of bounded sets $\Omega \subset \mathbb{R}^n$, in [12] in the case of unbounded sets $\Omega \subset \mathbb{R}^n$.

Let $f \in L_{loc}(\mathbb{R}^n).$ The Hardy–Littlewood maximal operator is defined as

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy$$

where $B(x,r)$ is the ball in $\mathbb{R}^n$ centered at the point $x \in \mathbb{R}^n$ and of the radius $r$, $B(x,r) = B(x,r) \cap \Omega$, $\Omega \subset \mathbb{R}^n$.

The fractional maximal operator of variable order $\alpha(x)$ is defined as

$$M^{\alpha(\cdot)} f(x) = \sup_{r>0} |B(x,r)|^{-1+\frac{\alpha(x)}{n}} \int_{B(x,r)} |f(y)|dy, 0 \leq \alpha(x) < n.$$ 

In the case of $\alpha(x) = \alpha = \text{const}$, this operator coincides with the classical fractional maximal operator $M^{\alpha}$. If $\alpha(x) = 0$, then $M^{\alpha(\cdot)}$ coincides with the operator $M$. 

The Riesz potential $I^{\alpha(x)}$ of variable order $\alpha(x)$ is defined by the following equality:

$$I^{\alpha(x)} f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha(x)}} dy, \ 0 < \alpha(x) < n.$$ 

In the case of $\alpha(x) = \alpha = \text{const}$, this operator coincides with the classical Riesz potential $I^\alpha$.

2 Preliminaries. Variable Exponent Lebesgue Spaces $L_{p(.)}$.

Generalized variable exponent Morrey spaces $M_{p(.)},w(.)$

Let $p(x)$ be a measurable function on an open set $\Omega \subset \mathbb{R}^n$ with values $(1, \infty)$. Let

$$1 < p_-- p(x) \leq p_+ < \infty \quad (2.1)$$

where $p_- = p_-(\Omega) = \text{essinf}_{x \in \Omega} p(x)$, $p_+ = p_+(\Omega) = \text{ess sup}_{x \in \Omega} p(x)$. We denote by $L_{p(.)}(\Omega)$ the space of all measurable functions $f(x)$ on $\Omega$ such that

$$J_{p(.)}(f) = \int_{\Omega} [f(x)]^{p(x)} dx < \infty,$$

where the norm is defined as follows

$$||f||_{p(.)} = \inf \{ \eta > 0, J_{p(.)} \left( \frac{f}{\eta} \right) \leq 1 \}.$$

This is a Banach space. The conjugate exponent $p'$ is defined by the formula

$$p'(x) = \frac{p(x)}{p(x) - 1}.$$

Hölder inequality for the variable exponents $p(.)$, $p'(.)$ is

$$\int_{\Omega} f(x)g(x)dx \leq C(p) ||f||_{L_{p(.)}(\Omega)} ||g||_{L_{p'(.)}(\Omega)},$$

where $C(p) = \frac{1}{p_-} + \frac{1}{p_+}$.

The variable exponent Lebesgue spaces $L_{p(.)}$ were introduced in [13], and were investigated in [14, 15].

$P(\Omega)$ is the set of measurable functions $p : \Omega \to [1, \infty)$, $P^{\log}(\Omega)$ is the set of measurable functions $p(x)$ satisfying the local log-condition

$$|p(x) - p(y)| \leq \frac{A_p}{-\ln |x - y|}, \ : |x - y| \leq \frac{1}{2}, \ x, y \in \Omega$$

where $A_p$ is independent of $x$ и $y$. $P^{\log}(\Omega)$ is the set of measurable functions $p(x)$ satisfying (2.1) and the log-condition. In the case of $\Omega$ is an unbounded set, we denote
by \( P^\log(\Omega) \) the set of exponents which is a subset of the set of \( \mathbb{P}^\log(\Omega) \) and satisfying the decay condition

\[
|p(x) - p(\infty)| \leq A_\infty \ln(2 + |x|), \ x \in \mathbb{R}^n.
\]

Let \( A_\log^\log(\Omega) \) be the set of bounded exponents \( \alpha : \Omega \to \mathbb{R} \) satisfying the log-condition.

Let \( \Omega \) be an open bounded set, \( p \in P^\log(\Omega) \) and \( \lambda(x) \) be a measurable function on \( \Omega \) with values in \([0, \infty]\). The variable Morrey space \( \mathcal{L}_{p(\cdot), \lambda(\cdot)}(\Omega) \) is introduced in [3] with the norm

\[
\|f\|_{\mathcal{L}_{p(\cdot), \lambda(\cdot)}(\Omega)} = \sup_{x \in \Omega, t > 0} \frac{t^{\lambda(x)}}{\frac{1}{p(x)}} \|f\|_{L_{p(x)}(\tilde{B}(x,t))}.
\]

Let \( w(x, r) \) be nonnegative measurable function on \( \Omega \), where \( \Omega \subset \mathbb{R}^n \) is a open bounded set. The generalized Morrey type space \( M_{p(\cdot), w(\cdot)}(\Omega) \) with variable exponent is defined in [11] with the norm

\[
\|f\|_{M_{p(\cdot), w(\cdot)}(\Omega)} = \sup_{x \in \Omega, r > 0} \frac{r^{-\frac{n}{p(x)}}}{w(x, r)} \|f\|_{L_{p(x)}(\tilde{B}(x,r))}.
\]

Let \( w(x, r) \) be nonnegative measurable function on \( \Omega \), where \( \Omega \subset \mathbb{R}^n \) is a open unbounded set. The generalized Morrey type space \( M_{p(\cdot), w(\cdot)}(\Omega) \) with variable exponent is defined in [12] with the norm

\[
\|f\|_{M_{p(\cdot), w(\cdot)}(\Omega)} = \sup_{x \in \Omega, r > 0} \frac{\|f\|_{L_{p(x)}(\tilde{B}(x,r))}}{w(x, r)}.
\]

Let

\[
\eta_p(x, r) = \begin{cases} \frac{n}{p(x)}, & \text{if } r \leq 1; \\ \frac{r}{p(\infty)}, & \text{if } r > 1. \end{cases}
\]

**Definition 1.** Let \( p \in P^\log(\Omega) \), \( w(x, r) \) be a positive function on \( \Omega \times [0, \infty] \), where \( \Omega \in \mathbb{R}^n \). Global Morrey-type space with variable exponent \( GM_{p(\cdot), \theta(\cdot), w(\cdot)}(\Omega) \) is defined as the set of functions \( f \in L_{loc}^{p(\cdot)}(\Omega) \) with finite norm

\[
\|f\|_{GM_{p(\cdot), \theta(\cdot), w(\cdot)}(\Omega)} = \sup_{x \in \Omega} \|w(x, r)^{-\eta_p(x, r)} \|f\|_{L_{p(x)}(\tilde{B}(x,r))}\|_{L_{\theta(x)}(0, \infty)}.
\]

We assume that the positive measurable function \( w(x, r) \) satisfies the condition

\[
\sup_{x \in \Omega} \|w(x, r)\|_{L_{\theta(x)}(0, \infty)} < \infty.
\]

Then the space contains bounded functions and thereby is nonempty. In the case of \( w(x, r) = r^{-\frac{\lambda(x)}{p(x)} + \eta_p(x, r)} \), the corresponding space is denoted by \( GM_{p(\cdot), \theta(\cdot)}^{\lambda(\cdot)} \):\n
\[
GM_{p(\cdot), \theta(\cdot)}^{\lambda(\cdot)}(\Omega) = GM_{p(\cdot), \theta(\cdot), w(\cdot)} |_{w(x, r) = r^{-\frac{\lambda(x)}{p(x)} + \eta_p(x, r)}},
\]

\[
\|f\|_{GM_{p(\cdot), \theta(\cdot)}^{\lambda(\cdot)}(\Omega)} = \sup_{x \in \Omega} \|w(x, r)^{-\frac{\lambda(x)}{p(x)} + \eta_p(x, r)} \|f\|_{L_{p(x)}(\tilde{B}(x,r))}\|_{L_{\theta(x)}(0, \infty)}.
\]
In the case of $\theta = \infty$, the space $GM(p,\infty,w)(\Omega)$ coincides the generalized Morrey space with variable exponent $M_{p,w}(\Omega)$ with finite quasi-norm

$$
||f||_{M_{p,w}(\Omega)} = \sup_{x \in \Omega} w(x,r) r^{-\eta_p(x,r)} ||f||_{L_{p}\left(\mathcal{B}(x,r)\right)}.
$$

If $p(\cdot) = p = \text{const}$, $\theta(x) = \theta = \text{const}$ then the space $GM_{p,\theta,w}(\Omega)$ coincides with the ordinary global Morrey space $GM_{p,\theta,w}(\Omega)$, considered in the works by V.I. Brenenkov, V. Guliev and others \cite{2,3,4}.

The Spanne and The Adams type theorems were proved in \cite{5} for bounded sets $\Omega$.

**Theorem 2.1.** Suppose that $p \in \mathbb{P}^{\log}(\Omega)$ and

$$
\sup_{t > r} \frac{\text{essinf}_{t < s < \infty} w_1(x,s)}{t^p(x,t)} \leq C \frac{w_2(x,r)}{r^p(x,r)}
$$

where $C$ is independent of $x$ and $r$. Then the maximal operator $M$ from $M_{p,w_1}(\Omega)$ to $M_{p,w_2}(\Omega)$ is bounded.

**Theorem 2.2** (Spanne type result with $\alpha = \text{const}$). Let $p \in \mathbb{P}^{\log}(\Omega)$, $\alpha$ and $p(\cdot),q(\cdot)$ satisfy $0 < \alpha < n$, $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}$, the functions $w_1$ and $w_2$ satisfy the condition

$$
\int_{r}^{\infty} \text{essinf}_{t < s < \infty} w_1(x,s) dt \leq C \frac{w_2(x,r)}{r^p(x,r)}
$$

where $C$ is independent of $x$ and $r$. Then the operators $M_\alpha$ and $I_\alpha$ from $M_{p,w_1}(\Omega)$ to $M_{q,w_2}(\Omega)$ are bounded.

The next theorem was proved in \cite{16}.

**Theorem 2.3.** Suppose that $p \in \mathbb{P}^{\log}(\Omega)$, $\alpha \in A^{\log}(\Omega)$ and $\alpha_- = \inf_{x \in \Omega} \alpha(x) > 0$, $(\alpha p)_+ = \sup_{x \in \Omega} \alpha(x)p(x) < \infty$. Then

$$
\sup_{x \in \Omega} \frac{1}{(1 + |x|)^{\gamma(x)} I^{\alpha(\cdot)} f} \leq C \|f\|_{L_{p}(\mathcal{B}(x,r))},
$$

where

$$
\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n}, \gamma(x) = A_\infty \alpha(x) \left[1 - \frac{\alpha(x)}{n}\right] \leq \frac{n}{4} A_\infty
$$

with $A_\infty$ comes from \cite{2,1}.

The following results were obtained in \cite{12}.

**Theorem 2.4.** Suppose that $p \in \mathbb{P}^{\log}(\Omega)$ satisfies \cite{2,1}, and let $1 < \theta_1^- \leq \theta_1(t) \leq \theta_1^+ < \infty$, $0 < t < l$, $1 < \theta_2^- \leq \theta_2(t) \leq \theta_2^+ < \infty$, $0 < t < l$. Assume that there exists $\delta > 0$ such that $\theta_1(t) \leq \theta_2(t)$, $t \in (0, \delta)$, $(\theta_1, w_1) \in \mathcal{W}(\delta,l)$. If

$$
\sup_{x \in \Omega, 0 < t < \delta} \int_0^t (w_2(x,\xi))^{\theta_2(\xi)} \left( \int_t^\delta \frac{1}{r w_1(x,r)} d\xi \right)^{\frac{\theta_2(\xi)}{\theta_1(\xi)}} d\xi < \infty,
$$

then the maximal operator $M$ from $M_{p,\theta_1,w_1}(\Omega)$ to $M_{p,\theta_2,w_2}(\Omega)$ is bounded.
Theorem 2.5. Suppose that \( p, \alpha \in \mathbb{P}^{\log}(\Omega) \) satisfy (2.1), and let \( \alpha > 0 \), \( (\alpha p(\cdot))_+ = \sup_{x \in \Omega} \alpha p(x) < n \), \( \frac{1}{p_2(x)} = \frac{1}{p_1(x)} - \frac{1}{n} \), \( 1 < \theta_1^- \leq \theta_1(t) \leq \theta_1^+ < \infty \), \( 0 < t < l \), \( 1 < \theta_2^- \leq \theta_2(t) \leq \theta_2^+ < \infty \), \( 0 < t < l \). Assume that there exists \( \delta > 0 \) such that \( \theta_1(t) \leq \theta_2(t) \), \( t \in (0, \delta) \), \((\theta_1, w_1) \in \mathcal{W}(\delta, l) \). If
\[
\sup_{x \in \Omega, 0 < t < \delta} \int_0^t (w_2(x, \xi))^{\theta_2(\xi)} \left( \int_t^d \frac{[\theta_1(\xi)]'}{w_1(x, r)} r \, dr \right) \, d\xi < \infty,
\]
then the operators \( I_{\alpha} \) \( u M_{\alpha} \) from \( M_{p_1(\cdot), \theta_1(\cdot), w_1(\cdot)}(\Omega) \) to \( M_{p_2(\cdot), \theta_2(\cdot), w_2(\cdot)}(\Omega) \) are bounded.

In [12] it was proved that, if \( \alpha(x) \) is a variable, under the conditions of Theorem 2.2, the operators \( \frac{1}{(1 + |x|)^{\gamma(x)}} M_{\alpha(x)}^\cdot \) and \( \frac{1}{(1 + |x|)^{\gamma(x)}} I_{\alpha(x)}^\cdot \) from \( M_{p, w_1}(\cdot) \) to \( M_{q, w_2}(\cdot) \) are bounded, where \( \gamma(x) = A_\infty \alpha(x) \left[ 1 - \frac{\alpha(x)}{n} \right] \leq \frac{1}{n} A_\infty \).

The next lemma was proved in [12].

Lemma 2.1. Assume that \( p \in \mathbb{P}_c^{\log}(\Omega) \) and \( f \in L_{loc}^{p(\cdot)}(\mathbb{R}^n) \). Then
\[
\|f\|_{L_{p(\cdot)}(B(x, t))} \leq Ct^\eta_p(x,t) \int_t^\infty r^{-\eta_p(x,r)-1} \|f\|_{L_{p(\cdot)}(B(x,r))} \, dr.
\]

In the same paper, the next theorem was proved.

Theorem 2.6. Suppose that \( p \in \mathbb{P}_c^{\log}(\Omega) \). Then
\[
\|Mf\|_{L_{p(\cdot)}(B(x, t))} \leq Ct^\eta_p(x,t) \sup_{r > 2t} r^{-\eta_p(x,r)} \|f\|_{L_{p(\cdot)}(B(x,r))},
\]
for every \( f \in L_{p(\cdot)}(\Omega) \), where \( C \) is independent of \( f, x \in \Omega \) and \( t > 0 \).

We prove the next necessary inequality.

Theorem 2.7. Let \( p \in \mathbb{P}_c^{\log}(\Omega) \). Then
\[
\|Mf\|_{L_{p(\cdot)}(B(x, t))} \leq Ct^\eta_p(x,t) \int_t^\infty s^{-\eta_p(x,s)-1} \|f\|_{L_{p(\cdot)}(B(x,s))} \, ds,
\]
where \( C \) is independent of \( f, x, t \).

Proof. Using Theorem 2.6 and Lemma 2.1 we have
\[
\|Mf\|_{L_{p(\cdot)}(B(x, t))} \leq Ct^\eta_p(x,t) \sup_{r > 2t} r^{-\eta_p(x,r)} \|f\|_{L_{p(\cdot)}(B(x,r))} \leq Ct^\eta_p(x,t) \sup_{r > t} r^{-\eta_p(x,r)} \|f\|_{L_{p(\cdot)}(B(x,r))}
\]
\leq Ct^\eta_p(x,t) \sup_{r > t} \int_r^\infty s^{-\eta_p(x,s)-1} \|f\|_{L_{p(\cdot)}(B(x,s))} \, ds = Ct^\eta_p(x,t) \int_t^\infty s^{-\eta_p(x,s)-1} \|f\|_{L_{p(\cdot)}(B(x,s))} \, ds.
\]

The next inequality was proved in [12].
Theorem 2.8. Let $p \in \mathbb{P}^\log(\Omega)$ and $\alpha$, $q$ satisfy conditions $0 < \alpha < n$, $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}$. Then the next inequality holds

$$
||I^\alpha f||_{L^q(\tilde{P}(x,t))} \leq C_{\nu}(x,t) \int_t^\infty r^{-\eta(x,r)-1} ||f||_{L^p(\tilde{P}(x,r))} dr, \quad t > 0
$$

(2.3)

where $C$ is independent of $x$ and $t$.

The inequality (2.3) holds, if we put $\frac{1}{(1+|y|)^{\gamma(y)}} I^\alpha(f(y))$ instead of $I^\alpha f(x)$. Namely, the following is true.

Theorem 2.9. Let $p \in \mathbb{P}^\log(\Omega)$ and the function $\alpha(x)$, $q(x)$ satisfy the condition $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n}$. Then for each $x \in \mathbb{R}^n, t > 0$ the following inequality holds

$$
|| \frac{1}{(1 + |y|)^{\gamma(y)}} I^\alpha f(y) ||_{L^q(\tilde{P}(x,t))} \leq C_{\nu}(x,t) \int_t^\infty r^{-\eta(x,r)-1} ||f||_{L^p(\tilde{P}(x,r))} dr
$$

(2.4)

Proof. We represent the function $f$ as $f(x) = f_1(x) + f_2(x)$, $f_1(x) = f(x) \chi_{B(x,2t)}$, $f_2(x) = f(x) \chi_{(\Omega \setminus B(x,2t))}$. Then

$$
\frac{1}{(1 + |y|)^{\gamma(y)}} I^\alpha f(y) = \frac{1}{(1 + |y|)^{\gamma(y)}} I^\alpha f_1(y) + \frac{1}{(1 + |y|)^{\gamma(y)}} I^\alpha f_2(y).
$$

By Theorem 2.3,

$$
|| \frac{1}{(1 + |y|)^{\gamma(y)}} I^\alpha f_1 ||_{L^q(\tilde{P}(x,t))} \leq || \frac{1}{(1 + |y|)^{\gamma(y)}} I^\alpha f_1 ||_{L^q(\mathbb{R}^n)} \leq C ||f_1||_{L^p(\mathbb{R}^n)} = C ||f||_{L^p(\tilde{P}(x,2t))}.
$$

By Lemma 2.1,

$$
|| \frac{1}{(1 + |y|)^{\gamma(y)}} I^\alpha f_2 ||_{L^q(\tilde{P}(x,t))} \leq C_{\nu}(x,t) \int_{2t}^\infty r^{-\eta(x,r)-1} ||f||_{L^p(\tilde{P}(x,r))} dr.
$$

(2.5)

If $|x - z| \leq t$ and $|z - y| \geq 2t$, we have $\frac{1}{2} |z - y| \leq |x - y| \leq \frac{3}{2} |z - y|$. Using the inequality $\frac{1}{(1+|y|)^{\gamma(y)}} \leq 1$, we infer

$$
|| \frac{1}{(1 + |y|)^{\gamma(y)}} I^\alpha f_1 ||_{L^q(\tilde{P}(x,t))} \leq \int_{\mathbb{R}^n \setminus B(x,2t)} |z - y|^{\alpha-n} f(y) dy ||_{L^q(\tilde{P}(x,t))}
$$

$$
\leq C \int_{\mathbb{R}^n \setminus B(x,2t)} |x - y|^{\alpha-n} f(y) \chi_{B(x,t)} dy ||_{L^q(\mathbb{R}^n)}.
$$

Choosing $\beta > \frac{n}{q}$, we obtain

$$
\int_{\mathbb{R}^n \setminus B(x,2t)} |x - y|^{\alpha-n} f(y) dy = \beta \int_{\mathbb{R}^n \setminus B(x,2t)} |x - y|^{\alpha-n+\beta} f(y) \left( \int_{|x-y|}^\infty s^{-\beta-1} ds \right) dy.
$$
Therefore 
\[ \frac{1}{(1 + |y|)^\gamma(y)} \| f \|_{L^{q}(x,t)} \leq C t\eta_{q}(x,t) \int_{2t}^\infty \beta^{-\gamma(y)} \| f \|_{L^{q}(x,t)} \, ds. \]
which, together with \((2.3)\), yields \((2.4)\).

Let \( u \) and \( v \) be a positive measurable functions. The dual Hardy operator is defined by the identity
\[ \tilde{H}_{v,u}f(x) = v(x) \int_{x}^\infty f(t)u(t) \, dt, \quad x \in \mathbb{R}^n. \]

Suppose that \( a \) is a positive fixed number. Let \( \theta_{1,a}(x) = \text{essinf}_{y \in [x,a]} \theta_{1}(y) \),
\[ \tilde{\theta}_{1}(x) = \begin{cases} \theta_{1,a}(x) & \text{if } x \in [0,a]; \\ \bar{\theta}_{1} = \text{const} & \text{if } x \in [a,\infty); \end{cases} \]
\[ \theta_{1} = \text{essinf}_{x \in \mathbb{R}^n} \theta_{1}(x), \quad \Theta_{2} = \text{esssup}_{x \in \mathbb{R}^n} \theta_{2}(x). \]

The next theorem was proved in \([17]\).

**Theorem 2.10.** Let \( \theta_{1}(x) \) and \( \theta_{2}(x) \) measurable functions on \( R_{+} \). Suppose that there exists a positive number \( a \) for all \( x > a \) holds \( \theta_{1}(x) = \bar{\theta}_{1} = \text{const}, \quad \theta_{2}(x) = \bar{\theta}_{2} = \text{const} \) and \( 1 < \theta_{1} \leq \tilde{\theta}_{1}(x) \leq \theta_{2}(x) \leq \Theta_{2} < \infty \) for a.a. If
\[ G = \sup_{t \geq 0} \int_{t}^{\infty} [v(x)]^{\theta_{2}(x)} (\int_{t}^{\infty} u^{\tilde{\theta}_{1}(\tau) \overline{\theta_{1}}(\tau)} \overline{\theta_{1}}(\tau) d\tau) d\tau < \infty \]
then the operator \( \tilde{H}_{v,u} \) is bounded from \( L_{\theta_{1}(\cdot)}(R_{+}) \) to \( L_{\theta_{2}(\cdot)}(R_{+}) \).

### 3 The Main Results

**Theorem 3.1.** Assume that \( p(.) \in \mathbb{P}^{\log}(\Omega) \), and \( \theta_{1}(x) \) and \( \theta_{2}(x) \) are measurable functions on \( R_{+} \). Suppose that there exists a positive number \( a \) such that for all \( t > a \) we have \( \theta_{1}(x) = \bar{\theta}_{1} = \text{const}, \quad \theta_{2}(x) = \bar{\theta}_{2} = \text{const} \) and \( 1 < \theta_{1} \leq \tilde{\theta}_{1}(x) \leq \theta_{2}(x) \leq \Theta_{2} < \infty \) for a.a., the positive measurable functions \( w_{1} \) \( u \) \( w_{2} \) satisfy the condition
\[ A = \sup_{x \in \Omega, t > 0} \int_{t}^{\infty} \left( w_{2}(x,r) \right)^{\theta_{2}(r)} \left( \int_{t}^{\infty} \left( \frac{1}{w_{1}(x,s)} \right)^{\theta_{1}(s)} ds \right) \, dr < \infty \quad (3.1) \]
Then the maximal operator \( M \) from \( GM_{p(.)},\theta_{1}(\cdot),u_{1}(\cdot)(\Omega) \) to \( GM_{p(.)},\theta_{2}(\cdot),u_{2}(\cdot)(\Omega) \) is bounded.
Corollary 3.1. Let \( p(\cdot) \in \mathbb{P}^\log_\infty(\Omega) \), \( w_1(x, r) = w_2(x, r) = r^\beta(x) \). If
\[
\inf_{x \in \Omega, r > 0} (\beta(x) + 1)|\tilde{\theta}_1(r)|' > 1,
\] (3.2)
\[
sup_{x \in \Omega, t > 0} \int_0^t r^{\theta_2(r)\beta(x)} \frac{[-(\beta(x)+1)|\tilde{\theta}_1(r)|'+1]|\tilde{\theta}_2(r)|}{|\tilde{\theta}_1(r)|} dr < \infty.
\] (3.3)
Then the maximal operator \( M \) from \( GM_{p_1(\cdot),\theta_1(\cdot),\alpha(\cdot)}(\Omega) \) to \( GM_{p_2(\cdot),\theta_2(\cdot),\beta(\cdot)}(\Omega) \) is bounded.

The following theorems give Spanne-type results on the boundedness of the Riesz potential \( f^\alpha \) in global Morrey-type spaces with variable exponent \( GM_{p(\cdot),\theta(\cdot),w(\cdot)}(\Omega) \). In the following theorem \( \alpha = \text{const} \).

Theorem 3.2. Assume that \( p(\cdot) \in \mathbb{P}^\log_\infty(\Omega) \), the constant number \( \alpha \) satisfies the condition \( \alpha > 0 \), \( (\alpha p(\cdot))_+ = \sup_{x \in \Omega} \alpha p(x) < n \). Let \( \theta_1(x) \) and \( \theta_2(x) \) be measurable functions on \( R_+ \). Suppose that there exists a positive number \( a \) such that for all \( x > a \) we have \( \theta_1(x) = \overline{\theta}_1 = \text{const} \), \( \theta_2(x) = \overline{\theta}_2 = \text{const} \) and \( 1 < \theta_1(x) \leq \theta_2(x) \leq \Theta_2 < \infty \) for a.a., the functions \( p_1(x) \) and \( p_2(x) \) satisfy \( \frac{1}{p_2(x)} = \frac{1}{p_1(x)} - \frac{\alpha(x)}{n} \), and the functions \( w_1 \) and \( w_2 \) satisfy the condition
\[
T = \sup_{x \in \Omega, t > 0} \int_0^t (w_2(x, r))^{\theta_2(r)} \left( \int_t^\infty \left( \frac{s^{\alpha-1}}{w_1(x, s)} \right)^{\theta_1(r)} ds \right)^{\frac{\theta_2(r)}{\theta_1(r)}} dr < \infty.
\] (3.4)
Then the operators \( I_\alpha \) and \( M_\alpha \) from \( GM_{p_1(\cdot),\theta_1(\cdot),w_1(\cdot)}(\Omega) \) to \( GM_{p_2(\cdot),\theta_2(\cdot),w_2(\cdot)}(\Omega) \) are bounded.

Corollary 3.2. Let \( p(\cdot) \in \mathbb{P}^\log_\infty(\Omega) \), \( w_1(x, r) = w_2(x, r) = r^\beta(x) \). If
\[
\sup_{x \in \Omega, r > 0} (\alpha - \beta(x) - 1)|\tilde{\theta}_1(r)|' < -1,
\] (3.5)
\[
\sup_{x \in \Omega, t > 0} \int_0^t r^{\theta_2(r)\beta(x)} \frac{[-(\alpha-\beta(x)-1)|\tilde{\theta}_1(r)|'+1]|\tilde{\theta}_2(r)|}{|\tilde{\theta}_1(r)|} dr < \infty,
\] (3.6)
then the operators \( I_\alpha \) and \( M_\alpha \) from \( GM_{p_1(\cdot),\theta_1(\cdot),\alpha(\cdot)}(\Omega) \) to \( GM_{p_2(\cdot),\theta_2(\cdot),\beta(\cdot)}(\Omega) \) are bounded.

In the following theorem \( \alpha(\cdot) \) is a variable exponent.

Theorem 3.3. Assume that \( p(\cdot) \in \mathbb{P}^\log_\infty(\Omega) \), the function \( \alpha(x) \) satisfies the condition \( \alpha(x) > 0 \), \( (\alpha(\cdot)p(\cdot))_+ = \sup_{x \in \Omega} \alpha(x)p(x) < n \). Let \( \theta_1(x) \) and \( \theta_2(x) \) be measurable functions on \( R_+ \). Suppose that there exists a positive number \( a \) such that for all \( x > a \) we have \( \theta_1(x) = \overline{\theta}_1 = \text{const} \), \( \theta_2(x) = \overline{\theta}_2 = \text{const} \) and \( 1 < \theta_1(x) \leq \theta_2(x) \leq \Theta_2 < \infty \) for a.a., the functions \( p_1(x) \) and \( p_2(x) \) satisfy \( \frac{1}{p_2(x)} = \frac{1}{p_1(x)} - \frac{\alpha(x)}{n} \), the functions \( w_1 \) and \( w_2 \) satisfy the condition
\[
T = \sup_{x \in \Omega, t > 0} \int_0^t (w_2(x, r))^{\theta_2(r)} \left( \int_t^\infty \left( \frac{s^{\alpha(x)-1}}{w_1(x, s)} \right)^{\theta_1(r)} ds \right)^{\frac{\theta_2(r)}{\theta_1(r)}} dr < \infty.
\]
Then the operators \( \frac{1}{(1+|x|)^{\gamma(x)}} I^\alpha(x) \) and \( \frac{1}{(1+|x|)^{\gamma(x)}} M^\alpha(x) \) from \( GM_{p_1(.)\alpha_1(.)w_1(.)}(\Omega) \) to \( GM_{p_2(.)\alpha_2(.)w_2(.)}(\Omega) \) are bounded.

In the following theorem \( \alpha = \text{const.} \).

**Theorem 3.4.** Assume that \( p(.) \in P^\log(\Omega), \) bounded measurable functions \( \theta_1(x), \theta_2(x) \) satisfy \( 1 < \theta_1(.) \leq \theta_2(.) \leq \theta_1 (+), \) \( 1 < \theta_2(.) \leq \theta_2 (+), \) and there exists a such that for all \( t > a \) we have \( \theta_1(t) = \text{const}, \theta_2(t) = \text{const}. \) Let the positive measurable functions \( w_1(x,r), w_2(x,r) \) satisfy the condition

\[
\sup_{x \in \Omega} \left\| w_2(x,r) \right\| \frac{1}{tw_1(x,r)} \left\| Mf \right\|_{L^p(B(x,r))} \leq \infty.
\]

Then the singular integral operator \( T \) from \( GM_{p(.)\alpha_1(.)w_1(.)} \) to \( GM_{p(.)\alpha_2(.)w_2(.)} \) is bounded.

**Proof of Theorem 3.1** According to the definition and to Theorem 2.10, using the Holder inequality with variable exponents \( \theta, \theta' \), we infer

\[
\left\| Mf \right\|_{GM_{p(.)\alpha_2(.)w_2(.)}(\Omega)} = \sup_{x \in \Omega} \left\| \frac{w_2(x,r)}{t^{\theta_2(x)}} \right\| \frac{Mf}{\theta_2(x)} ||f||_{L^p(B(x,r))} \leq C \sup_{x \in \Omega} \left\| w_2(x,r) \right\| \int_r^\infty t^{-\theta_2(x)-1} ||f||_{L^p(B(x,t))} dt \leq \infty,
\]

we denote

\[
\tilde{H}_{v,u} f(r) = v(r) \int_r^\infty g(t) u(t) dt,
\]

here \( v(r) = w_2(x,r), \) \( g(t) = \frac{w_1(x,t)}{t^{\theta_2(x)}} ||f||_{L^p(B(x,t))}, \) \( u(t) = \frac{1}{w_1(x,t)} \) for every fixed \( x \in \Omega. \)

Then the condition (2.2) has the form (3.1), from which boundedness of the operator \( \tilde{H}_{v,u} f(r) \) from \( L^\theta_1(.) \) to \( L^\theta_2(.) \) follows. Consequently, we have

\[
\left\| Mf \right\|_{GM_{p(.)\alpha_2(.)w_2(.)}(\Omega)} \leq A \cdot \sup_{x \in \Omega} \left\| w_1(x,r) t^{-\theta_2(x)-1} f \right\|_{L^p(B(x,r))} \leq \infty = A \cdot \left\| f \right\|_{GM_{p(.)\alpha_1(.)w_1(.)}} \]

this means that the operator \( M \) from \( GM_{p(.)\alpha_1(.)w_1(.)} \) to \( GM_{q(.)\alpha_2(.)w_2(.)} \) is bounded.

**Proof of Corollary 3.1** The condition (3.1) has the form

\[
\sup_{x \in \Omega, t > 0} \int_0^t t^{\theta_2(r)\beta(x)} \left( \int_t^\infty s^{-\theta_2(r)\beta(x)} d\tilde{s} \right)^{\frac{\theta_2(r)}{\theta_1(r)}} dr < \infty.
\]

By the convergence of the inner integral, we obtain the conditions (3.2) and (3.3).
Proof of Theorem 3.2. Using the definition and the Theorem 2.8 we have

\[ \|I^\alpha f\|_{\mathcal{G}M_{q(\cdot),\theta_2(\cdot),w_2(\cdot)}(\Omega)} = \sup_{x \in \Omega} \|w_2(x,r)r^{-\eta_0(x,r)}\|L_0f\|_{L_{q(\cdot)}(B(x,r))}\|L_{\theta_2(\cdot)(0,\infty)} \leq \]

\[ \leq C \sup_{x \in \Omega} \int_{r}^{\infty} t^{-\eta_0(x,t)} - 1 ||f||_{L_{p(\cdot)}(B(x,t))} dt \]

We denote

\[ \tilde{H}_{v,u} f(r) = v(r) \int_{r}^{\infty} g(t) u(t) dt, \]

we denote by \( v(r) = w_2(x,r), g(t) = \frac{w_2(x,t)}{v_0(x,t)} \|f\|_{L_{p(\cdot)}(B(x,t))}, u(t) = \frac{t^{\eta_0(x,t)-\eta_0(x,r)-1}}{w_2(x,t)} \) for every fixed \( x \in \Omega \). Then the condition (2.2) has the form (3.4), from which boundedness of the operator \( \tilde{H}_{v,u} f(r) \) from \( L_{\theta_1(\cdot)}(0, \infty) \) to \( L_{\theta_2(\cdot)}(0, \infty) \) follows. Consequently, we have

\[ \|I^\alpha f\|_{\mathcal{G}M_{p_1(\cdot),\theta_1(\cdot),w_1(\cdot)}(\Omega)} \leq T \cdot \sup_{x \in \Omega} \|w_1(x,t) t^{-\eta_0(x,t)}\|f\|_{L_{p(\cdot)}(B(x,t))}\|L_{\theta_1(\cdot)(0,\infty)} = \]

\[ = T \cdot \|f\|_{\mathcal{G}M_{p_1(\cdot),\theta_1(\cdot),w_1(\cdot)}}, \]

this means that the operator \( I^\alpha \) from \( \mathcal{G}M_{p_1(\cdot),\theta_1(\cdot),w_1(\cdot)} \) to \( \mathcal{G}M_{p_2(\cdot),\theta_2(\cdot),w_2(\cdot)} \) is bounded. \( \square \)

Proof of Corollary 3.2. The condition (3.4) takes the form

\[ \sup_{x \in \Omega, t > 0} \int_{0}^{t} r^{\theta_2(\cdot)(\cdot)} \left( \int_{t}^{\infty} \frac{r^{\theta_2(\cdot)(\cdot)}}{r^{\eta_0(\cdot)(\cdot)}} ds \right)^{\frac{\theta_1(\cdot)}{\theta_1(\cdot)}} dr < \infty. \]

By the convergence of the inner integral, we deduce the conditions (3.5) and (3.6). \( \square \)

Proof of Theorem 3.3. The proof of this theorem the same as the proof of Theorem 3.2, it is sufficient to put \( \frac{1}{1+|x|^2} I^\alpha(\cdot) f(x) \) instead of \( I^\alpha f(x) \). \( \square \)

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