Thermal state on a cylindrical spacetime

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Abstract

We proof that if we have a thermal equilibrium state on Minkowski spacetime in two dimensions then we have a thermal equilibrium state on the cylindrical spacetime obtained from this Minkowski spacetime by making $2\pi$-periodic the spatial direction. We perform this by using the algebraic approach to Quantum Field Theory.

1 Introduction

Quantum Field Theory on non simply connected spaces has been studied before by several authors [1], [2], [3]. However, as far as we know, none of them have used the algebraic approach to quantum field theory in the sense of Haag and Kastler [4]. Here we shall address this problem by using the generalization of Algebraic Quantum Field Theory (AQFT) [4] given by Brunetti, Fredenhagen and Verch [5]. We take as a non simply connected space a cylinder and a simply connected space a plane.

We will say that a state $\omega$ is a thermal equilibrium state at temperature $T$ if it satisfies the KMS condition

$$\omega(B(\alpha_t A)) = \omega((\alpha_{t-i\beta} A)B),$$

where $A$ and $B$ are two elements of the algebra on which $\omega$ is defined, $\alpha_t$ is the
automorphism on the algebra corresponding to translations in time and $\beta = \frac{1}{T}$.

In this work we shall prove that if this condition is satisfied by a state in two dimensional Minkowski spacetime then it is satisfied for a state defined on a cylindrical spacetime obtained from Minkowski spacetime by making the spatial direction $2\pi$-periodic. The relation between the two states will be specified below.

The organization of this paper is as follows. In section 2, we prove that if the KMS condition is satisfied for a state in Minkowski spacetime then it is satisfied for the corresponding state in the cylindrical spacetime. In section 3, we discuss this result just by using the formalism introduced by Haag and Kastler and we compare our result with the image method for analyzing the same problem.

2 Thermal state on a cylinder and on a plane

A natural mathematical concept we can use for our purposes is the concept of covering space. Let us spell out how this concept enters in our problem. If we consider $\mathbb{R}^1 \times \mathbb{S}^1$ as the covering space of $\mathbb{R}^1 \times \mathbb{S}^1$ then for all $x \in \mathbb{R}^1 \times \mathbb{S}^1$ there is a neighborhood $V$ of $x$ such that $\pi^{-1}(V)$ is a family $\{U_\alpha\}$ of open disjoint pairwise subsets of $\mathbb{R}^1 \times \mathbb{R}^1$ and $\pi : U_\alpha \to V$ is a homeomorphism of $U_\alpha$ to $V$. $\pi : \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1 \times \mathbb{S}^1$ is called the covering map.

In [5], which we will refer to as BFV, the starting point is to consider the category of all globally hyperbolic spacetimes, $\mathfrak{Man}$, with morphisms, $\psi$, the isometric embeddings between two of these spacetimes, the objects of the category. An isometric embedding is a map $\psi : \mathcal{M}_1 \to \mathcal{M}_2$, where $\mathcal{M}_1$ and $\mathcal{M}_2$ are globally hyperbolic spacetimes such that $\psi$ is a diffeomorphism onto its range and $\psi$ is an isometry, $\psi_* g_1 = g_2$ when $\psi$ is restricted to $\mathcal{M}_1$.

We can apply this concept to our problem as follows: Obviously $\mathbb{R}^1 \times \mathbb{S}^1$ and $\mathbb{R}^1 \times \mathbb{R}^1$ are not diffeomorphic but if we just consider a small diamond shaped region, $D_c$, on $\mathbb{R}^1 \times \mathbb{S}^1$ then under the covering map this region maps to an infinite denumerable family $\{D_i\}$, $i = 0, \pm 1, \pm 2...$ of diamond shaped regions in $\mathbb{R}^1 \times \mathbb{R}^1$. Clearly in each element, say $i = 0$, of this family the covering map induces an isometric embedding which pushforward the metric on $D_c$ to $D_p$ where $D_p$ is the diamond shaped region in $\mathbb{R}^1 \times \mathbb{R}^1$ which corresponds to $i = 0$. The way that the covering map induces an isometric embedding from $D_c$ to $D_p$ can be seen more
clearly if we introduce atlases \( \{(U_{\alpha}, u_{\alpha})\} \) and \( \{(V_{\beta}, v_{\beta})\} \) in these two manifolds. Then \( \pi^{-1} \) determines continuous maps

\[
\pi_{\beta\alpha}^{-1} : u_{\alpha}(U_{\alpha} \cap \pi(V_{\beta})) \to v_{\beta}(V_{\beta}) \tag{1}
\]

where \( \pi_{\beta\alpha}^{-1} = v_{\beta} \circ \pi^{-1} \circ u_{\alpha}^{-1} \). In the present case we can cover \( D_c \) and \( D_p \) with the same single chart. Hence in this case the maps (1) are smooth and \( \pi^{-1} \) is smooth. So we have a diffeomorphism between \( D_c \) and \( D_p \). Clearly we can make it an isometric embedding by pushing forward the flat metric on the cylinder to the flat metric on the plane. This does not depend on the element we take in the family \( \{D_i\} \), \( i = 0, \pm 1, \pm 2, \ldots \). If we take other element in \( \{D_i\} \) then we can relate it to \( D_p \) by acting on \( D_p \) with an element \( \gamma_n \) of \( \Gamma \) where \( \Gamma \) is the discrete abelian group of spatial translations by \( 2\pi \) in \( \mathbb{R}^1 \times \mathbb{R}^1 \). In these circumstances the following diagram commute

\[
\begin{array}{ccc}
D_c & \xrightarrow{\pi^{-1}} & D_p \\
\downarrow_{\pi^{-1}} & & \uparrow_{\gamma_n} \\
D_n & & 
\end{array}
\tag{2}
\]

where \( D_n := \gamma_n D_p \).

At this stage we can apply the formalism given by BFV with the diamond shaped regions introduced above as the elements of \( \mathbf{Man} \). Let us write down explicitly the elements which are relevant for our purposes.

In addition to the category \( \mathbf{Man} \) above introduced we need to introduce the category \( \mathbf{Alg} \) whose objects are all the \( C^* \)-algebras, and the morphisms, \( \alpha \), are faithful unit-preserving \(*\)-homomorphisms. Then a locally covariant quantum field theory is a covariant functor \( \mathcal{A} \) between the categories \( \mathbf{Man} \) and \( \mathbf{Alg} \), in a diagram we have

\[
\begin{array}{ccc}
(M, g) & \xrightarrow{\psi} & (M', g') \\
\downarrow_{\mathcal{A}} & & \downarrow_{\mathcal{A}} \\
\mathcal{A}(M, g) & \xrightarrow{\alpha_{\psi}} & \mathcal{A}(M', g')
\end{array}
\tag{3}
\]

together with the covariance properties \( \alpha_{\psi'} \circ \alpha_{\psi} = \alpha_{\psi' \circ \psi} \) and \( \alpha_{\id M} = \id_{\mathcal{A}(M, g)} \) for all morphism \( \psi \in \text{hom}_{\mathbf{Man}}(M_1, g_1)(M_2, g_2) \), \( \psi' \in \text{hom}_{\mathbf{Man}}(M_1, g_1)(M_2, g_2) \) and all \( (M, g) \in \text{Obj}(\mathbf{Man}) \). There are two additional properties which are
satisfied by the functor $\mathcal{A}$, but for our purposes it is enough with the property just introduced. We should note that in our problem $M$ corresponds to the region $D_c$ or $D_i$.

Also we need to introduce one category more, the category of the set of states which we will denote as $\mathfrak{Sts}$. An object $S \in \text{Obj}(\mathfrak{Sts})$ is a set of states on a C*-algebra $\mathcal{A}$. Morphisms between members $S'$ and $S$ of $\text{Obj}(\mathfrak{Sts})$ are positive maps $\gamma^*: S' \to S$. $\gamma^*$ arises as the dual map of a faithful C*-algebraic endomorphism $\gamma: \mathcal{A} \to \mathcal{A}'$ via

$$\gamma^* \omega'(A) = \omega'(\gamma(A)), \quad \omega' \in S', \quad A \in \mathcal{A}. \quad (4)$$

Then a state space for $\mathcal{A}$ is a contravariant functor $S$ between $\text{Man}$ and $\mathfrak{Sts}$:

\[
\begin{align*}
(M, g) & \xrightarrow{S} (M', g') \\
\downarrow S & \downarrow S \\
S(M, g) & \xleftarrow{\alpha^*_\psi} S(M', g')
\end{align*}
\]

where $S(M, g)$ is a set of states on $\mathcal{A}(M, g)$ and $\alpha^*_\psi$ is the dual map of $\alpha_\psi$; the covariance property is $\alpha^*_\psi \circ \alpha_\psi = \alpha^*_\psi \circ \alpha^*_\psi$ together with the requirement that unit morphisms are mapped to unit morphisms.

Now let see how can we apply all this formalism to our problem. We assume there a thermal state on $\mathbb{R} \times \mathbb{R}$. We also assume it is invariant under the action of the isomorphism, $\alpha_t$, generated by translations in time, the usual time in Minkowski spacetime.

We would like to proof that when we make $x$, the spatial coordinate in Minkowski spacetime, $2\pi$-periodic we still have a thermal state on the resulting spacetime. Using the structure given in the diagram (5) we just need to proof that $\alpha_\psi$ and $\alpha_t$ commute, however as it stand now we do not know how elements of the algebras in $D_c$ and in $D_i$ are related to each other. Therefore, it is necessary to introduce more structure before we proof what we want. Fortunately this structure also has been given by BFV.

We introduce the concept of locally covariant quantum field. This concept needs the introduction of another category, the category $\text{Test}$ of smooth test functions with compact support, $C^\infty_0(M)$. The morphisms in this category are
the pushforwards of $\psi$ the morphisms in $\mathfrak{Man}$. Where $M$ stands for $D_c$ or $D_n$. We also introduce a family of quantum fields $\Phi_{M,g}$, indexed by all spacetimes in $\mathfrak{Man}$. For each spacetime this field is a map from $C^\infty_0(M)$ to $\mathcal{A}(M,g)$

$$\Phi_{(M,g)} : C^\infty_0(M) \to \mathcal{A}(M,g). \quad (6)$$

This structure can be put in a diagram as

$$\mathcal{D}(M,g) \xrightarrow{\Phi_{(M,g)}} \mathcal{A}(M,g) \xrightarrow{\psi_*} \mathcal{A}(M',g') \quad (7)$$

where the commutativity of the diagram expresses the covariance for fields

$$\alpha_\psi \circ \Phi_{(M,g)} = \Phi_{(M',g')} \circ \psi_* \quad (8)$$

Let us now go back to our problem and use the formalism we have just introduced. Let $f \in \mathcal{D}(M,g)$ and take $M$ as $D_c$ and $M'$ as $D_p$ then, from (8), we have

$$\alpha_{\pi^{-1}} \circ \Phi_{(D_c,g_c)}(f) = \Phi_{(D_p,g_p)} \circ \pi^{-1}_p(f). \quad (9)$$

Now, on $D_p$ acts $\Lambda(t)$, the usual translation in time in Minkowski spacetime. We define a transformation on $\mathbb{R}^1 \times S^1$ induced by $\Lambda$ in such a way that the following diagram commutes

$$\mathcal{D}_c \xrightarrow{\pi^{-1}} \mathcal{D}_p \\
\Lambda' \downarrow \quad \Lambda \\
\mathcal{D}_c' \xrightarrow{\pi^{-1}} \mathcal{D}_p' \quad (10)$$

Using $\Lambda$ and $\Lambda'$ we have two maps

$$\Lambda_* : \mathcal{D}(D_p,g_p) \to \mathcal{D}(D_p',g_p) \quad (11)$$

$$\Lambda'_* : \mathcal{D}(D_c,g_c) \to \mathcal{D}(D_c',g_c) \quad (12)$$

given by

$$\Lambda_* f_p := f'_p \quad (13)$$


and

$$\Lambda'_c f_c := f'_c$$  \hspace{1cm} (14)$$

where $f_p \in \mathcal{D}(D_p, g_p)$ and $f'_p \in \mathcal{D}(D_p, g_p)$, similarly for $f_c$ and $f'_c$. The pushforwards induced by $\pi^{-1}$, $\Lambda$ and $\Lambda'$ are given explicitly by

$$f_p(D_p) := f_c(\pi D_p) \quad f'_p(D'_p) := f_c(\Lambda'^{-1} D'_c)$$

$$f'_p(D'_p) := f'_c(\pi D'_c) \quad f'_p(D'_p) := f_p(\Lambda^{-1} D'_p)$$  \hspace{1cm} (15)$$

All this structure can be put in the following commuting diagram

$$\mathcal{D}(D_c, g_c) \xrightarrow{\pi^{-1}_*} \mathcal{D}(D_p, g_p)$$

$$\Lambda'_c \downarrow \quad \downarrow \Lambda_c$$

$$\mathcal{D}(D'_c, g_c) \xrightarrow{\pi^{-1}_*} \mathcal{D}(D'_p, g_p)$$  \hspace{1cm} (16)$$

If also we define the field as a map $\Phi(D_c, g_c) : C_0^\infty(D_c) \to \mathcal{A}(D_c, g_c)$, then we have the following commuting diagram

$$\mathcal{D}(D_c, g_c) \xrightarrow{\Phi(D_c, g_c)} \mathcal{A}(D_c, g_c)$$

$$\Lambda'_c \downarrow \quad \downarrow \alpha_{\Lambda'}$$

$$\mathcal{D}(D'_c, g_c) \xrightarrow{\Phi(D_c, g_c)} \mathcal{A}(D'_c, g_c)$$  \hspace{1cm} (17)$$

Let $f \in \mathcal{D}(D_c, g_c)$. Then

$$\alpha_{\pi^{-1}} \circ \alpha_{\Lambda'} \Phi(f) = \alpha_{\pi^{-1}} \circ \Phi(\Lambda'_c f) = \Phi(\pi^{-1}_* \circ \Lambda'_c f)$$  \hspace{1cm} (18)$$

but

$$\Phi(\pi^{-1}_* \circ \Lambda'_c f) = \Phi(\Lambda_c \circ \pi^{-1}_* f)$$  \hspace{1cm} (19)$$

because diagrams (16) and (18). But

$$\Phi(\Lambda_c \circ \pi^{-1}_* f) = \alpha_{\Lambda} \circ \alpha_{\pi^{-1}} \Phi(f).$$  \hspace{1cm} (20)$$

Hence from (18), (19) and (20) we have

$$\alpha_{\pi^{-1}} \circ \alpha_{\Lambda'} \Phi(f) = \alpha_{\Lambda} \circ \alpha_{\pi^{-1}} \Phi(f).$$  \hspace{1cm} (21)$$
From diagram (5) we see that a positive state on $\mathcal{A}(D_p, g_p)$ is mapped to a positive state on $\mathcal{A}(D_c, g_c)$. Also from diagram (7), with $M = D_c$, $M' = D_p$ and $\psi = \pi^{-1}$ we have for $f \in \mathcal{D}(D_c, g_c)$

$$\Phi(\pi^{-1}_* f) = \alpha_{\pi^{-1}} \Phi(f).$$

(22)

If we denote the state on $\mathcal{A}(D_p, g_p)$ as $\omega_p$ and the state on $\mathcal{A}(D_c, g_c)$ as $\omega_c \equiv \alpha_{\pi^{-1}}^* \omega_p$ then from (4) we have

$$\omega_c(\Phi(f)) = \omega_p(\Phi(\pi^{-1}_* f)) = \omega_p(\alpha_{\pi^{-1}} \Phi(f)).$$

(23)

We are assuming that $\omega_p$ satisfies the KMS condition, i.e.,

$$\omega_p(\Phi(\pi^{-1}_* f) \alpha_t(\Phi'(\pi^{-1}_* g))) = \omega_p((\alpha_{t-i\beta} \Phi'(\pi^{-1}_* g)) \Phi(\pi^{-1}_* f))$$

(24)

where $f$ and $g$ are in $\mathcal{D}(D_c, g_c)$. Using (21), (22) and (23) in (24) we have

$$\omega_c(\Phi(f)(\alpha_t \Phi'(g))) = \omega_c((\alpha_{t-i\beta} \Phi'(g)) \Phi(f))$$

(25)

Thus the state $\omega_c$ satisfies the KMS condition too.

### 3 Discussion

It is clear that Haag-Kastler formalism can be applied to a cylindrical spacetime by replacing Poincaré symmetry for just translation symmetry in time and space plus spatial $2\pi$-periodicity. Then we can consider both quantum field theories, on the cylinder and on the plane, on the same footing. The principal differences are the symmetries of the field as consequences of the manifold symmetries.

Now, as we have seen under the covering map $\pi$ a diamond shaped region in $\mathbb{R}^1 \times \mathbb{S}^1$ maps to a denumerable infinite number of diamond shaped regions in $\mathbb{R}^1 \times \mathbb{R}^1$. Taking into account that an observable is associated with a local region of spacetime then to each observable localized in $\mathbb{R}^1 \times \mathbb{S}^1$, say in $D_c$, correspond a denumerable infinite number of observables localized in $\{D_i\}$. The observables in $D_i$ are related by an $\ast$-isomorphism between the algebras associate to the family $\{D_i\}$. Invoking locality these observables form an equivalence class given
by the equivalence relation $a_j \sim a_i$ if $D_j = \gamma_j D_i$ with $\gamma_j$ the action of the abelian translation group. In these circumstances we can relate a positive state on $\mathbb{R}^1 \times S^1$ to a positive state on $\mathbb{R}^1 \times \mathbb{R}^1$ as follows

$$\omega_{\text{BTZ}}(a) \approx \omega_{\text{AdS}}([a]), \quad (26)$$

where $[a]$ is the equivalence class associated with the observable $a$ in $\mathbb{R}^1 \times S^1$, where $\approx$ means approximately. We have seen that the formalism introduced by BFV tell us more precisely how the relation between states on the cylinder and on the plane should be.

The idea of studying quantum field theory on a multiple connected spacetime by studying quantum field theory on the covering spacetime of it is known as automorphic fields [1]. In the previous section we have used the BFV formalism to address this problem for a simple case, a cylindrical and flat spacetime. Now we are going to compare it with the image method [3] and will show that both formalisms are equivalent for this case.

Let us consider a scalar quantum field on the flat two dimensional cylindrical spacetime $\mathbb{R}^1 \times S^1$. We can address this problem at least by two procedures. For instance, we can consider the quantum field on two dimensional Minkowski spacetime and imposing $2\pi$-periodic boundary conditions. Let us first consider a $L$-periodic field and later take the particular case $2\pi$. In this case the two point function turns out to be

$$\langle 0_c | \hat{\phi}(U,V) \hat{\phi}(U',V') | 0_c \rangle = -\frac{1}{4\pi} \ln \{(1 - e^{-\frac{2\pi}{L}(U - U' - i\epsilon)})(1 - e^{-\frac{2\pi}{L}(V - V' - i\epsilon)})\}, \quad (27)$$

where $U = t - x$ and $V = t + x$ are null coordinates and $\epsilon > 0$. Other procedure is to calculate the two point function in Minkowski spacetime and later use the images sum prescription. The two point function in Minkowski spacetime is

$$\langle 0 | \hat{\phi}(U,V) \hat{\phi}(U',V') | 0 \rangle = -\frac{1}{4\pi} \ln \{(U - U' - i\epsilon)(V - V' - i\epsilon)\}. \quad (28)$$

Let us denote the images sum as $F(U,V; U',V')$. Then we have

$$F(U,V; U',V') = \sum_{n \in \mathbb{N}_0} \langle 0 | \hat{\phi}(U,V) \hat{\phi}(U' - Ln, V' + Ln) | 0 \rangle. \quad (29)$$

8
With the help of the identity \[ \cot z = \frac{1}{z} + 2z \sum_{k=1}^{\infty} \frac{1}{z^2 - k^2 \pi^2}, \quad z \neq 0, \pm \pi, \pm 2\pi, \ldots \]
we obtain

\[
F(U, V; U', V') = -\frac{1}{4\pi} \ln \left\{ \sin \frac{\pi}{L} (U - U' - i\epsilon) \sin \frac{\pi}{L} (V - V' - i\epsilon) \right\}.
\] (30)

The expression (28) can be written as (30) plus terms which are linear in \( t \) and \( t' \) and in \( \epsilon \). Hence the two procedures give the same answer module these terms. However in a massless two dimensional field theory what really matters is the two times differentiated two point function [S], hence both procedures give the same answer. These calculations show the vacuum \( |0_c\rangle \) state is different from the state \( |0\rangle \). This has been pointed out long time ago in [9]. Going back to our problem we can see that by addressing it with the formalism given by BFV is equivalent to make the \( L \)-periodic the field in Minkowski spacetime. Then we have shown that in this simple case the BFV formalism and automorphic fields are equivalent.

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