ON THE CENTRALIZERS OF MINIMAL APERIODIC ACTIONS ON THE CANTOR SET.

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Abstract. In this article we study the centralizer of a minimal aperiodic action of a finitely generated group on the Cantor set (an aperiodic minimal Cantor system). We show that this centralizer is always an extension of some LEF groups, having as a consequence that the Thompson group $T$ cannot be a subgroup of the centralizer of any aperiodic minimal Cantor system. On the other hand we show that any countable residually finite group is the subgroup of the centralizer of some minimal $\mathbb{Z}$ action on the Cantor set, and that any countable group is the subgroup of the normalizer of a minimal aperiodic action of an abelian countable free group on the Cantor set.

1. Introduction

An automorphism of the topological dynamical system $(X, T, \Gamma)$ given by the continuous action $T : \Gamma \times X \rightarrow X$ of a countable group $\Gamma$ on the (compact) topological space $X$, is a self-homeomorphism of $X$ commuting with each transformation $T(g, \cdot)$. A classical question in dynamics is to understand the algebraic properties of the group of all the automorphisms, also called centralizer, of a prescribed dynamical system and their relationships with the dynamical properties of the system. In this paper we focus on the case when the space $X$ is a Cantor set. One of the reasons for this choice is the topology of the Cantor set does not restrict the algebraic properties of the groups of homeomorphisms because any countable group acts faithfully on the Cantor set. This situation is very different in other spaces like manifolds. For instance, only extensions of orderable group can act faithfully on the circle [26]. For higher dimensional compact manifolds, the restrictions fall in the scope of R. J. Zimmer’s conjectures.

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The dynamical properties determine important restrictions on the groups of automorphisms that can be realized. For instance, Hedlund observed that the automorphism group of the dynamical systems given by an expansive action is always countable. The subsets which are invariant by the dynamics can also restrict the automorphism group [24]. For instance, in the case of mixing $\mathbb{Z}$-subshift of finite type, every automorphism preserves the finite set of periodic points of a given period and since the periodic points are dense, their restrictions separate the automorphisms implying that the group of automorphisms is residually finite [3]. To avoid such limitations, we will focus on dynamical systems which are minimal, i.e. systems without proper invariant closed subsets. In this case, none of the former restrictions appears: the centralizer can be uncountable as in the odometer action, or not residually finite as it was shown in [3], where the authors construct a minimal subshift whose centralizer contains a group isomorphic to the rationals $\mathbb{Q}$. However it appears from recent works that the centralizers of zero entropy minimal subshifts are very limited [2, 6, 7, 8, 9, 10, 11, 12].

Another motivation to focus our attention on the centralizer of a minimal Cantor system comes from the study of (topological) full groups. From the work of Juschenko and Monod [19], it is known that the topological full group of a $\mathbb{Z}$ Cantor minimal system is a countable amenable group. Thanks to this result together with those shown in [22] by Matui, the commutator of the topological full groups of minimal $\mathbb{Z}$-subshifts become the first known examples of infinite groups which are amenable, simple and finitely generated. On the other hand, Giordano, Putnam, and Skau [16], and Medynets [23] prove that abstract isomorphisms between full groups of Cantor minimal systems have a topological realization. More precisely, they show the outer automorphism group of a topological full group (of a $\mathbb{Z}$ Cantor minimal system $(X, T, \Gamma)$) is isomorphic to the normalizer of transformations $T(n, \cdot), n \in \mathbb{Z}$. Since the centralizer is a normal subgroup of the normalizer, the study of automorphisms provides informations about the outer automorphism group of full groups.

In this paper we study the algebraic properties of the centralizer of aperiodic Cantor minimal system $(X, T, \Gamma)$, i.e. when $X$ is a Cantor set and the action of the group $\Gamma$ on $X$ is continuous, free and minimal (any countable group $\Gamma$ admits such an action, see [18] or [1] for an expansive action).

We prove the centralizer group has some algebraic restrictions (Corollary 20): it is the extension of two groups that are Locally Embeddable into Finite groups (LEF). The notion of LEF group has been introduced by Gordon and Vershik [25], it includes the one of residually
finite group and is a restricted notion of sofic group. As a consequence, we deduce that the Thompson group $T$ can not be in the centralizer of any aperiodic minimal Cantor system (Corollary 21). We also obtain restrictions for the normalizer of an aperiodic minimal Cantor system (Corollary 22). Despite this restriction, it is possible to realize any residually finite group, eventually infinitely generated, as a subgroup of the centralizer of a $\mathbb{Z}$ Cantor minimal system (Proposition 7). We do not known if any extension of LEF groups can be in the centralizer of a $\mathbb{Z}$ Cantor minimal system.

After recalling necessary background on automorphisms and Cantor minimal systems that we call generalized subshift, we prove in Section 3, at the opposite of Corollary 20, that any countable group may appear in the normalizer of a Cantor minimal aperiodic action of a free abelian group. The main result of Section 4 shows that the centralizer of any aperiodic Cantor minimal system given by the action of a finitely generated group is a subgroup of the centralizer of a $\mathbb{Z}$ minimal Cantor system. We deduce that the property “to be a subgroup of the group of automorphisms of a $\mathbb{Z}$ Cantor minimal system” is stable by direct product and by the wreath product with any finite group. In the same section, we show for $\mathbb{Z}$ minimal Cantor system that the quotient group of the centralizer by the group generated by the transformation may be any arbitrary residually finite group (Corollary 15). We recall the necessary notions of LEF groups and dimension groups in order to prove Corollary 20 in the last section.

2. Definitions and background.

We say that $(X, T, \Gamma)$ is a Cantor system if $T : \Gamma \times X \to \Gamma$ is a continuous action on the Cantor set $X$. For every $\gamma \in \Gamma$, we let $T^\gamma : X \to X$ denote the homeomorphism given by $T^\gamma(x) = T(\gamma, x)$, for every $x \in X$. The action is said faithful when the map $\gamma \mapsto T^\gamma$ is injective. We say that the Cantor system is aperiodic if the action $T$ is free, i.e., $T^\gamma(x) = x$ implies $\gamma = 1_\Gamma$ for any $x \in X$. The Cantor system is minimal if for every $x \in X$, its orbit $\sigma_T(x) = \{T^\gamma(x) : \gamma \in \Gamma\}$ is dense in $X$. The group generated by a collection of homeomorphisms $\{T_i\}_{i \in I}$ is denoted $\langle T_i : i \in I \rangle$ and the one generated by the homeomorphisms of a $T$ action is simply denoted $\langle T \rangle$.

The topological full group associated to the Cantor system $(X, T, \Gamma)$ is denoted by $[\langle T \rangle]$ and is defined as the subgroup of all homeomorphisms $f$ on $X$ such that for every $x \in X$ there exist a clopen neighborhood $V$ of $x$ and $\gamma \in \Gamma$ such that the restriction of $f$ on $V$ is equal to $T^\gamma$. 


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For a group \( G \), we let \( \text{Aut}(G) \) denote the group of automorphisms of \( G \). If \( \Gamma \) is another group, \( \Gamma \leq G \) means that \( \Gamma \) is a subgroup of \( G \) or isomorphic to a subgroup of \( G \). Recall that a group \( \Gamma \) is residually finite if for any \( \gamma \in \Gamma \setminus \{1\} \) there exists an homomorphism \( \pi \) from \( \Gamma \) to a finite group \( H \) such that \( \pi(\gamma) \neq 1_H \). A result of Mal’cev ensures that any finitely generated subgroup of \( \text{GL}(k, \mathbb{C}) \) are residually finite.

2.1. Group of automorphisms. Let \((X, T, \Gamma)\) be an aperiodic Cantor system. The normalizer group of \((X, T, \Gamma)\), denoted \( \text{Norm}(T, \Gamma) \), is defined as the subgroup of all self-homeomorphism \( h \) of \( X \) such that \( h(T)h^{-1} = \langle T \rangle \), or equivalently, there exists \( \alpha_h \in \text{Aut}(\Gamma) \) such that \( h \circ T^g = T^{\alpha_h(g)} \circ h \), for every \( g \in \Gamma \).

By the aperiodicity of the action, for any element \( h \in \text{Norm}(T, \Gamma) \) the associated automorphism \( \alpha_h \in \text{Aut}(\Gamma) \) is unique. Thus we can define

\[
\text{Aut}(T, \Gamma) = \{ h \in \text{Norm}(T, \Gamma) : \alpha_h = id \}.
\]

It is direct to check that \( \text{Aut}(T, \Gamma) \) is a normal subgroup of \( \text{Norm}(T, \Gamma) \) and is the set of automorphisms of \((X, T, \Gamma)\).

**Lemma 1.** For an aperiodic minimal Cantor system \((X, T, \Gamma)\) we have the following exact sequence

\[
\{1\} \longrightarrow \text{Aut}(T, \Gamma) \overset{\text{Id}}{\longrightarrow} \text{Norm}(T, \Gamma) \overset{\alpha}{\longrightarrow} \text{Aut}(\Gamma),
\]

where \( \alpha \) is the map \( h \mapsto \alpha_h \).

**Proof.** It is enough and straightforward to check that the aperiodicity of the action implies that the map \( \alpha \) is a group morphism. \( \square \)

It follows that the quotient group \( \text{Norm}(T, \Gamma)/\text{Aut}(T, \Gamma) \) is isomorphic to a subgroup of \( \text{Aut}(\Gamma) \). Thus, since \( \text{Aut}(\mathbb{Z}) \) is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \), for a minimal Cantor system \((X, T, \mathbb{Z})\), the group \( \text{Norm}(T, \mathbb{Z}) \) is either isomorphic to \( \text{Aut}(T, \mathbb{Z}) \) or to a semi-direct product \( \text{Aut}(T, \mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z} \).

**Lemma 2.** Let \((X, T, \Gamma)\) be a minimal system. Then the natural action of \( \text{Aut}(T, \Gamma) \) on \( X \) is free.

**Proof.** It is enough to observe that for an element \( \phi \in \text{Aut}(T, \Gamma) \), its set of fixed points is a closed \( T \)-invariant subset of \( X \). By minimality, if not empty, this set is all \( X \) and \( \phi \) is the identity. \( \square \)

We will use this lemma to identify the automorphisms of a given minimal action.
2.2. Generalized subshifts. We introduce some notations and system coming from symbolic dynamics, we will use several times in this paper. An alphabet \( A \) is a compact (not necessarily finite) space with a metric \( \text{dist}_A \). A word of length \( \ell \) is a sequence \( x_1 \ldots x_\ell \) of \( \ell \) letters in \( A \), and the set of such words is denoted \( A^\ell \). The length of a word \( u \) is denoted by \( |u| \). Any word can be interpreted as an element of the free monoid \( A^* \) endowed with the operation of concatenation. For a word \( w = u.v \) that is the concatenation of two words \( u \) and \( v \), the words \( u \) and \( v \) are respectively a prefix and a suffix of \( w \). For an integer \( \ell \in \mathbb{N} \) and a word \( u \), the concatenation of \( \ell \) times the word \( u \) is denoted \( u^\ell \) and \( u^\omega \) is the bi infinite sequence \((x_n)_{n\in\mathbb{Z}}\) such that \( x_k|u| \ldots x_{(k+1)|u|-1} = u \) for each integer \( k \in \mathbb{Z} \). The set \( A^\mathbb{Z} \) is the collection of two sided infinite sequences \((x_n)_{n\in\mathbb{Z}}\). This last set is a compact space for the product topology endowed with a distance
\[
\text{Dist}((x_n)_n,(y_n)_n) := \sum_{n\in\mathbb{Z}} 2^{-|n|} \text{dist}_A(x_n,y_n).
\]
Note that if \( A \) is a Cantor set then, \( A^\mathbb{Z} \) also is a Cantor set. For a sequence \( x = (x_n)_{n\in\mathbb{Z}} \) in \( A^\mathbb{Z} \), we will use the notation \( x[i,j] \) to denote the word \( x_ix_{i+1} \ldots x_j \) belonging to \( A^{j-i+1} \).

We let \( \sigma \) denote the shift map, that is the self-homeomorphism of \( A^\mathbb{Z} \) such that \( \sigma((x_n)_{n\in\mathbb{Z}}) = (x_{n+1})_{n\in\mathbb{Z}} \). A generalized subshift is a topological dynamical system \((X, \sigma)\) where \( X \) is a closed \( \sigma \)-invariant subset of \( A^\mathbb{Z} \).

3. Realization of countable groups as subgroups of a normalizer

We show in this section that any countable group can be realized as a subgroup of the centralizer of a minimal aperiodic system given by the action of a countable free abelian group on the Cantor set.

**Lemma 3.** Let \( \Gamma \) be a countable group. There exist an aperiodic Cantor system \((X,T,\Gamma)\) and \( f \in \text{Aut}(T,\Gamma) \) such that \((X,f)\) is also aperiodic.

**Proof.** Let \( \Gamma \) be a countable group. Since \( \Gamma \times \mathbb{Z} \) is still a countable group, there exists an aperiodic Cantor system \((X,\phi,\Gamma \oplus \mathbb{Z})\) (see [1, LS]).

Let \( f : X \to X \) be the homeomorphism induced by the action of \((1_\Gamma,1) \in \Gamma \oplus \mathbb{Z} \) on \( X \), i.e., \( f = \phi^{(1_\Gamma,1)} \). For every \( g \in \Gamma \), let \( T^g : X \to X \) be the homeomorphism induced by the action of \((g,0) \) on \( X \), i.e., \( T^g = \phi^{(g,0)} \). The new Cantor system \((X,T,\Gamma)\) is also aperiodic, and since \((1_\Gamma,1) \) is in the center of \( \Gamma \oplus \mathbb{Z} \), we have \( f \in \text{Aut}(T,\Gamma) \). Furthermore, since \( \phi \) is aperiodic, we get \( f^n(x) = x \) implies \( n = 0 \). \( \square \)
Proposition 4. Let $G$ be a countable group. Then there exist a countable subgroup $\Gamma \leq \bigoplus_{\mathbb{N}} \mathbb{Z}$ (a countable free abelian group) and an aperiodic minimal Cantor system $(X, S, \Gamma)$, such that $G \leq \text{Norm}(S, \Gamma)$.

Proof. Let $(X, T, G)$ be a Cantor aperiodic system and $f \in \text{Aut}(T, G)$ as in Lemma 3. Since $(X, f)$ is aperiodic, if $Y \subseteq X$ is a minimal component of $(X, f)$ then $Y$ is a Cantor set. Observe that $T^g(Y)$ is also a minimal component of $(X, f)$, for every $g \in G$. Consider the group $Est_G(Y) = \{ g \in G : T^g(Y) = Y \}$ and a collection $\{ g_i : i \in I \}$ of elements of $G$ containing one and only one representative element of each class in $G/Est_G(Y)$. We set

$$\tilde{Y} = \prod_{i \in I} T^{g_i}(Y).$$

With the product topology $\tilde{Y}$ is a Cantor set.

Set $\Gamma$ to be the group $\bigoplus_I \mathbb{Z}$, namely

$$\Gamma = \{(n_i)_{i \in I} \in \mathbb{Z}^I : n_i = 0, \text{ for all but a finite number of index } i \in I\}.$$  

Notice that $\Gamma$ is a countable free abelian group (if $Est_G(Y)$ is of finite index in $G$ then $\Gamma$ is finitely generated).

Given $n = (n_i)_{i \in I} \in \Gamma$ and $y = (y_i)_{i \in I} \in \tilde{Y}$, we define

$$S^n(y) = (f^{n_i}(y_i))_{i \in I}.$$  

Since each $T^{g_i}(Y)$ is invariant by $f$, we have that $S^n : \tilde{Y} \to \tilde{Y}$ is well defined and is an homeomorphism. We call $S$ the action of $\Gamma$ on $\tilde{Y}$ induced by the $S^n$ maps. It is straightforward to show that $(\tilde{Y}, S, \Gamma)$ is aperiodic. Since every $T^{g_i}(Y)$ is a minimal component of $(X, f)$, the system $(\tilde{Y}, S, \Gamma)$ is also minimal.

For every, $g \in G$ let define $\sigma_g : I \to I$ such that $gg_i \in g_{\sigma_g(i)}Est_G(Y)$. We have that $\sigma_g$ is a permutation. Moreover, $\sigma_g$ induces the isomorphism $\alpha_g : \Gamma \to \Gamma$ given by

$$\alpha_g((n_i)_{i \in I}) = (n_{\sigma_g(i)})_{i \in I}.$$  

For $g \in G$ we set

$$\tilde{T}^g(y) = (T^{g_{\sigma^{-1}_g(i)}}(y_{\sigma_g^{-1}(i)}))_{i \in I}.$$
We have

\[ S^n \circ \tilde{T}^g((y_i)_{i \in I}) = S^n((T^g(y_{\sigma^{-1}(i)})_{i \in I}) \]

\[ = (f^{n_i}(T^g(y_{\sigma^{-1}(i)})))_{i \in I} \]

\[ = (T^g(f^{n_i}(y_{\sigma^{-1}(i)})))_{i \in I} \]

\[ = \tilde{T}^g((f^{n_{\sigma^{-1}(i)}}(y_i))_{i \in I}) \]

\[ = \tilde{T}^g \circ S^{\sigma^{-1}(n)}((y_i)_{i \in I}). \]

This shows that \( \tilde{T}^g \in \text{Norm}(S, \Gamma) \), and since \( g \mapsto \tilde{T}^g \) is an injective homomorphism, we get \( G \leq \text{Norm}(S, \Gamma) \) and \( Est_G(Y) \leq \text{Norm}(S, \Gamma) \cap \text{Norm}(f, Z) \).

\[ \square \]

4. **Subgroups of the centralizer of a \( Z \) minimal Cantor system**

We study the class of subgroups of automorphisms of a \( Z \) Cantor minimal system. We start showing that this class contains the centralizer of any aperiodic Cantor minimal system of a finitely generated group. From this we deduce the stability of this class by direct product and by the wreath product with a finite group. Finally we show in Section 4.2 that any countable group in this class appears as a quotient \( \text{Aut}(T, Z)/\langle T \rangle \) of a \( Z \) Cantor minimal system \((X, T, Z)\).

4.1. **Realization of subgroups of the centralizer.**

**Proposition 5.** Let \((X, T, \Gamma)\) be a Cantor aperiodic minimal system where \( \Gamma \) is a finitely generated group. Then there exists a Cantor minimal system \((Y, S, Z)\) such that \( \text{Aut}(T, \Gamma) \leq \text{Aut}(S, Z) \).

As we will see in Remark 23, there is actually no analogue of this result for normalizer groups.

**Proof.** Let \( \Gamma \) be a group generated by a finite and symmetric set of elements \( S = S^{-1} = \{s_1, \ldots, s_d\} \), and let \((X, T, \Gamma)\) be a Cantor aperiodic minimal system.

Let us recall the notion of generalized substitution introduced in [14]. Consider the notion of generalized subshift introduced in Section 2.2 (in this case the alphabet will be \( X \)). We will consider a substitutive subshift over the infinite alphabet \( X \).

In order to define a substitution reasonably on an alphabet space \( X \), we need to deal with several topological considerations that are trivial in the case where the alphabet is finite. For a word \( w \in X^* \) and \( 1 \leq j \leq |w| \), let \( \pi_j(w) \) denote the \( j \)th letter of \( w \). We say that \( \tau : X \to X^* \) is a generalized substitution on \( X \) if \( a \mapsto |\tau(a)| \) is continuous and the
projection map \( \pi_j \circ \tau \) is continuous on the set \( \{ a \in X : |\tau(a)| \geq j \} \). The words \( \tau(z) \), \( z \in X \), are called \( \tau \)-words.

Consider the substitution \( \tau \) on \( X \) defined by

\[
\tau : x \mapsto T^{s_1}(x) T^{s_2}(x) \ldots T^{s_d}(x)
\]

where we recall that \( \{ s_1, \ldots, s_d \} = S \).

We shall consider \( L(\tau) \) the language generated by \( \tau \), again a trickier notion to define than in the classical case. Fix a letter \( a \) in the alphabet space \( X \). By the language generated by \( a \), denoted \( L(\tau, a) \), we mean the set of words \( w \in X^* \) such that \( w \) is a subword of \( \tau^j(a) \) for some \( j \in \mathbb{N} \), or \( w \) is the limit in \( X^n \) of such words. We set \( L(\tau) = \bigcup_a L(\tau, a) \).

**Claim 1:** The substitution \( \tau \) is primitive, i.e., given any non-empty open set \( V \subset X \), there is an \( j \in \mathbb{N} \) such that for any letter \( a \in X \) and any \( k \geq j \), one of the letters of \( \tau^k(a) \) is in the set \( V \).

From a well known result of Auslander, for the minimal \( \Gamma \) action and for any \( x \in X \), the set \( \{ g \in \Gamma : T^g(x) \in V \} \) is syndetic (or relatively dense). Also observe that for any finite set \( K \subset \Gamma \), there exists an integer \( n \) such that for any \( x \in X \), the letters \( T^g(x) \), \( g \in K \) occur in \( \tau^n(x) \). The primitivity of the substitution follows.

Proposition 27 in [14] implies that the \( L(\tau, a) \) does not depends on the letter \( a \). So we denote it \( L(\tau) \). We also define \( X_\tau \subset X^Z \) to be set of sequences \( \mathbf{x} \in X^Z \) such that \( \mathbf{x}[n, n] \in L(\tau) \) for all \( n \geq 0 \). It follows that \( X_\tau \) is a generalized subshift, i.e., a closed, \( \sigma \)-invariant subset of \( X^Z \). Observe that the system \( (X_\tau, \sigma) \) is aperiodic because, \( \tau \) is primitive and the language \( L(\tau) \) is infinite (there are uncountably many \( \Gamma \)-orbits).

**Claim 2:** The substitution \( \tau \) is recognizable: for every \( z \in X_\tau \), there is a unique set of integers \( \{ n_k : k \in \mathbb{Z} \} \) and unique \( \mathbf{x} \in X_\tau \) such that \( \tau(x_k) = z[n_k, n_{k+1} - 1] \) for all \( k \in \mathbb{Z} \).

Observe that the substitution \( \tau \) is of constant length and is injective on the letters (recall that the \( \Gamma \)-action is aperiodic). The proof of the claim is the same as the standard one for constant length, injective substitution on a finite alphabet.

Finally we get from Theorem 32 in [14] that the system \( (X_\tau, \sigma) \) is a minimal Cantor \( \mathbb{Z} \) system.

To conclude, let \( \phi \) be in \( \text{Aut}(T, \Gamma) \), that is \( \phi : X \rightarrow X \) continuous and \( \phi \circ T^\gamma(x) = T^\gamma \circ \phi(x) \) for all \( x \in X \), \( \gamma \in \Gamma \). We can associate to \( \phi \) a transformation \( \bar{\phi} \) on \( X^n \), by coordinate wise composition

\[
\bar{\phi}( (x_i)_k ) := (\phi(x_i))_i.
\]

This define by concatenation a continuous bijective map on the whole space \( X^Z \), still denoted \( \bar{\phi} \), that commutes with the shift. Observe
moreover that, for every $x \in X$,
\[(4.2) \quad \tau(\phi(x)) = \bar{\phi}(\tau(x)).\]
It follows that the map $\bar{\phi}$ preserves the subshift $X_\tau$. It is then straightforward to check, with Lemma 2, the map $\bar{\phi} : \phi \in Aut(T, \Gamma) \mapsto \bar{\phi} \in Aut(\sigma, \mathbb{Z})$ is an injective endomorphism. □

A first consequence of Proposition 5 is that the automorphisms group of a Cantor minimal $\mathbb{Z}$ system may be uncountable.

**Corollary 6.** Let $G$ be a topological group homeomorphic to a Cantor set with a dense finitely generated subgroup. Then there exists a Cantor minimal $\mathbb{Z}$ system $(X, S, \mathbb{Z})$ such that $G \leq Aut(S, \mathbb{Z})$.

**Proof.** Consider $\Gamma \subset G$ a dense finitely generated subgroup. For every $\gamma \in \Gamma$, and $g \in G$, we set $T^\gamma(g) = \gamma g$. Thus $(G, T, \Gamma)$ is the minimal aperiodic Cantor system induced by the left translations of $\Gamma$ on $G$. The group $G$ acts by right translations on itself. This action is transitive and commutes with the one of $\Gamma$ so $Aut(T, \Gamma)$ is isomorphic to $G$. Finally Proposition 5 provides the result. □

If $\Gamma$ is a finitely generated residually finite group, it is isomorphic to a dense subgroup of any $\Gamma$-odometers (see [5]), which is a topological group homeomorphic to the Cantor set. The application of Corollary 6 to a $\Gamma$-odometer gives a $\mathbb{Z}$ Cantor minimal system with $\Gamma$ a subgroup of its centralizer. Actually the same result is true for countable, eventually infinitely generated, residually finite group. We provide here a direct proof based on a Lidenstrauss-Weiss idea [21, Proposition 3.5].

**Proposition 7.** Let $\Gamma$ be a countable residually finite group. Then there exists a Cantor minimal system $(X, S, \mathbb{Z})$ such that $\Gamma \leq Aut(S, \mathbb{Z})$.

**Proof.** Recall that a countable residually finite group $\Gamma$ always admits a faithful action on the Cantor set $X$ where the set of points with a finite orbit is dense [4, Theorem 2.7.1]. We will construct a generalized subshift on the alphabet $X$ (see Section 2.2).

For each $n \geq 1$ and $\gamma \in \Gamma$ let $\gamma_n : X^n \to X^n$ be the homeomorphism defined as $\gamma_n(x_1, \ldots, x_n) = (\gamma(x_1), \ldots, \gamma(x_n))$, and let $\gamma_\omega : X^\mathbb{Z} \to X^\mathbb{Z}$ be the homeomorphism defined as $\gamma_\omega((x_n)_n) = (\gamma(x_n))_n$. This provides $\Gamma$-actions on $X^n$ and $X^\mathbb{Z}$.

It follows that the shift map $\sigma : X^\mathbb{Z} \to X^\mathbb{Z}$ and $\gamma_\omega$ commutes. We have to construct now the specific minimal generalized subshift.

Set $B_1$ as the collection of all the words of length 1, in the alphabet $X$. Fix $n \geq 2$ and suppose that at the step $n - 1$ we have defined a collection of words $B_{n-1}$ such that:
• All of its elements have the same length $\ell_{n-1}$.
• The set $B_{n-1}$ is closed on $X^{\ell_{n-1}}$ and preserved by each map $\gamma^{\ell_{n-1}}$.
• The set of points with a finite $\Gamma$-orbit is dense in $B_{n-1}$.

Let $\{x_{1,n}, \ldots, x_{k_n,n}\} \subset B_{n-1}B_{n-1}$ be a finite $\Gamma$-invariant set that is $1/n$-dense in $B_{n-1}B_{n-1} \subset X^{2\ell_{n-1}}$. Of course, each $x_{i,n}$ has a finite $\Gamma$-orbit. Set $B_n$ to be the collection of all the words $w$ that are a concatenation of words in $B_{n-1}$ of the form

$$w = w_1 \ldots w_n x_{s(1),n} \ldots x_{s(k_n),n},$$

where $w_1, \ldots, w_n$ are words in $B_{n-1}$ and $s$ is a permutation of $\{1, \ldots, k_n\}$.

It is clear that each word in $B_n$ is of length $(n + 2k_n)\ell_{n-1} =: \ell_n$, each map $\gamma^{\ell_n}$, with $\gamma \in \Gamma$, preserves the set $B_n$ and that the points with a finite $\Gamma$-orbit are dense in the closed set $B_n \subset X^{\ell_{n-1}}$.

Let $X_n$ be the subshift whose any element is a concatenation of words in $B_n$ and let $X_\infty$ be the subshift $X_\infty = \bigcap_{n \geq 1} X_n$. The minimality follows from the next claim.

**Claim.** For each $n \geq 1$, for any $x \in X_{n+1}$ and $y \in X_n$, there exists an integer $|\ell| \leq \ell_n$ such that $\text{Dist}(\sigma^\ell(x), y) \leq 1/n + 3/2^{\ell_{n-1}}$.

For two words $w_1, w_2 \in B_n$, there exists a $x_{i,n+1}$ such that the word $w_1w_2$ is at distance less than $1/(n + 1)$ from $x_{i,n+1}$ in $X^{2\ell_n}$. Since $y$ is a concatenation of words in $B_n$ and since any word $x_{i,n+1}$ appears in each word of $B_{n+1}$, a direct computation provides the claim.

Clearly $X_\infty$ is invariant by every $\gamma^w$. It rests to check that $\Gamma$ acts faithfully on $X_\infty$. Let $\gamma \in \Gamma$ be such that $\gamma^w$ has a fixed point $x \in X_\infty$, then $\gamma$ fixes each letter of $x$. A direct induction proves that the set of letters occurring in the words $x_{i,n}$, $n \geq 1$ is dense in $X$. Hence in particular, $\gamma$ fixes a dense set of points in $X$, so $\gamma$ is the identity. □

Of independent interest, we also deduce that the property “to be a subgroup of automorphism of an aperiodic minimal Cantor system” is stable by direct product and by the wreath product with a finite group. Recall that for two groups $G$ and $K$ and an action $G \curvearrowright \Omega$ on a set $\Omega$, the wreath product $K \wr \Omega G$ is the semi direct product

$$\left( \prod_{\omega \in \Omega} K_\omega \right) \rtimes G,$$

where for each $\omega \in \Omega$, $K_\omega := K$ and the action of $G$ on $\Omega$ extends on an action on $\prod_{\omega \in \Omega} K_\omega$ by $g.(k_\omega)_\omega := (k_{g^{-1}\omega})_\omega$ for $g \in G$. We simply denote $K \wr G$ for $K \wr G$. It is well known (see [20]) that any group that is an extension of $G$ by $K$ is a subgroup of $K \wr G$. 

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Corollary 8. Let \((X, T, G)\) and \((X, S, H)\) be two aperiodic Cantor minimal systems for two finitely generated groups \(G\) and \(H\). Then

1. there exists a \(Z\) Cantor minimal system \((X, R, Z)\) such that \(\text{Aut}(T, G) \oplus \text{Aut}(S, H) \leq \text{Aut}(R, Z)\).

2. For any finite group \(K\), there exists a \(Z\) Cantor minimal system \((X, R, Z)\) such that \(K \wr \text{Aut}(T, G) \leq \text{Aut}(R, Z)\).

Proof. Item 1. Observe that the direct product \(\text{Aut}(T, G) \oplus \text{Aut}(S, H)\) is a subgroup of the centralizer of the product system on the space \(X \times X\) for the product action of \(G \oplus H\). This action is aperiodic and minimal so a direct application of Proposition 5 gives the conclusion.

Item 2. We need first a lemma.

Lemma 9. Let \(H\) be a subgroup of \(\text{Aut}(T, G)\). Then there exist an homeomorphism \(R\) of the Cantor set acting minimally and a subgroup \(H'\) of \(\text{Aut}(R, Z)\) isomorphic to \(H\) such that \(H' \cap \langle R \rangle\) is trivial.

Proof. By Proposition 5, let us assume that \(H\) is a subgroup of automorphisms of some \(Z\) Cantor minimal system \((X, R_1, Z)\). For any integer \(n \geq 2\), let \(H^n \times \mathfrak{S}_n\) be the semi direct product of the group of the abelian product of \(n\) copies of \(H\) denoted \(H^n\), with the group of permutations \(\mathfrak{S}_n\) of \(n\) elements where every permutation acts by permuting the coordinates in \(H^n\). It is standard to check that the group \(H^n \times \mathfrak{S}_n\) is a subgroup of automorphisms of the product system \((X^n, R_1 \times \cdots \times R_1, Z^n)\). This system is a \(Z^n\) aperiodic Cantor minimal system, so Proposition 5 provides a \(Z\) Cantor minimal system \((X, R, Z)\) containing \(H^n \times \mathfrak{S}_n\) as a subgroup of automorphisms. The monomorphism \(i\) defined by \(i(h) = (h, 1_H, \ldots, 1_H, 1_H)\) gives an embedding of \(H\) that is “transverse” to the center of the group, more precisely \(i(H) \cap Z(H^n \times \mathfrak{S}_n)\) is trivial. It follows that \(i(H) \cap \langle R \rangle\) is trivial. 

From this lemma, it is enough to prove that for any subgroup \(H\) of automorphism of a \(Z\) Cantor minimal system \((X, T, Z)\) with \(H \cap \langle T \rangle\) trivial, there exists a \(Z\) Cantor minimal system \((X, R, Z)\) such that \(K \wr H \leq \text{Aut}(R, Z)\).

We claim that \(K \wr X H\) acts on \(K \times X\) by the following way:

\[
((k_x)_{x \in X}, h) \cdot (k, y) := (k_x y, h, y).
\]

Since the action of \(\text{Aut}(T, Z)\) on \(X\) is free, this action is faithful.

Let \(x_0 \in X\) be a point with a trivial stabilizer for the \(H \oplus T\)-action, then it is straightforward to check that one has an injective homomorphism \(i: K \wr H \to K \wr X H\) defined by

\[
((k_t)_{t \in H}, g) \mapsto (\overline{(k_t)}_{t}, g),
\]

(4.4)
where \( \overline{k_x} = 1_K \) when \( x \notin \text{Orb}_{H \oplus T}(x_0) \) and otherwise \( \overline{k_hT^n(x_0)} = k_h \) for \( h \in H, \ n \in \mathbb{Z} \). Observe it is well defined because of the hypothesis in \( x_0 \).

It follows that \( K \wr H \) also acts on \( K \times X \). This action commutes with the \( K \times T \) product action (where \( K \) acts on itself by right multiplication) because the \( H \)-actions also commutes with \( T \) and the very definition of the embedding \( K \wr H \to K \wr X \).

It is straightforward to check that the stabilizer of the point \((k, x_0)\) with \( k \in K \) for the \( K \wr X \) action is the set 
\[
\text{Stab}(k, \omega_0) := \{(k_\ell, 1_H) : k_{x_0} = 1_K\}.
\]
It follows that \( i(K \wr H) \cap \text{Stab}(k, x_0) = \{(1_K, 1_H)\} \) and the \( K \wr H \) action is free. Hence \( K \wr H \) is a subgroup of \( \text{Aut}(K \times T, K \oplus \mathbb{Z}) \).

\textbf{Proposition 5} provides the conclusion. \( \square \)

### 4.2. Realization of groups as \( \text{Aut}(T, \mathbb{Z})/\langle T \rangle \).

We prove in this section that any countable subgroup of the centralizer of a \( \mathbb{Z} \) Cantor minimal system \((X, T, \mathbb{Z})\) appears as a quotient \( \text{Aut}(S, \mathbb{Z})/\langle S \rangle \) for a specific minimal almost 1-to-1 extension of \((X, T, \mathbb{Z})\).

Recall that an \emph{almost 1-1 extension} of a minimal \( \mathbb{Z} \) topological system \((X, T, \mathbb{Z})\) is topological dynamical system \((Y, S, \mathbb{Z})\) such that there exists a factor map \( \pi : Y \to X \) commuting with the action, i.e., \( S \circ \pi = \pi \circ T \), surjective and having a singleton fiber, i.e., there exists a point \( x \in X \) with \( \#\pi^{-1}(\{x\}) = 1 \). To control the automorphism group of an almost 1-1 extension let us recall a result of \cite{[11]} (Lemmas 2.1, 2.4)

\textbf{Lemma 10} \cite{[11]}. Let \( \pi : (X, T, \mathbb{Z}) \to (Y, D, \mathbb{Z}) \) be an almost 1-1 extension of minimal systems. Then the map
\[
\text{Aut}(T, \mathbb{Z}) \to \text{Aut}(D, \mathbb{Z}) \\
\phi \mapsto z \mapsto \pi(\phi(\pi^{-1}(z)))
\]
is a well defined injective morphism.

We derive from the notion of semicycle in \cite{[13]} a way to produce almost 1-1 extension of a given minimal Cantor system \((X, T, \mathbb{Z})\). We adapt it on our context.

A \emph{semicycle} \( f : X \to K \) is a map to a compact metric space \((K, d_K)\), which is continuous on a dense \( G_\delta \) set of \( X \) (i.e., on a countable intersection of dense open subsets). Let \( C_f \) denote the set of continuity points and its complementary \( D_f := C_f^c \). Observe that \( D_f \) and \( \bigcup_{n \in \mathbb{Z}} T^n(D_f) \) are Baire meager (i.e. the countable union of closed subsets with empty interiors).
Example 11. Let us fix a point $x_0 \in X$ and $(U_n)_{n \geq 0} \subseteq X$ a nested sequence of clopen sets such that $\bigcap_n U_n = \{x_0\}$ and $U_0 = X$. Let $g$ be the function defined by $g(x_0) = 1$ and for $x \neq x_0$,

$$g(x) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{otherwise} \end{cases},$$

where $n$ is index such that $x \in U_n \setminus U_{n+1}$. Hence the function $f : x \in X \mapsto (x, g(x)) \in X \times \{0, 1\}$ is a semicocycle where $D_f = \{x_0\}$.

Let $y \in X \setminus \bigcup_{n \in \mathbb{Z}} T^n(D_f)$ be a point. The generalized subshift $(X_f, \sigma)$ associated to $f$ is the orbit closure for the shift action $\sigma : (x(n))_{n \in \mathbb{Z}} \mapsto (x(n+1))_{n \in \mathbb{Z}}$ in $K^\mathbb{Z}$, with respect to the Tychonov topology, for the sequence $(f(T^n y))_{n \in \mathbb{Z}}$.

**Lemma 12.** If $(X, T, \mathbb{Z})$ is a minimal system, then $(X_f, \sigma, \mathbb{Z})$ is also minimal.

**Proof.** By a classical result of Auslander, it is enough to show the sequence $(f(T^n y))_n$ is almost periodic for the shift, i.e. for any $\epsilon > 0$ and a finite set of coordinates $C \subseteq \mathbb{Z}$, the set $\{n \in \mathbb{Z} : d_K(f(T^n y), f(T^{n+c} y)) < \epsilon, \forall c \in C\}$ is a syndetic subset of $\mathbb{Z}$. Since $y$ and all the points in its $T$-orbit are continuity points of $f$, there is a neighborhood $U$ of $y$ such that for any $z \in U$, $d_K(f(T^n y), f(T^{n+c} y)) < \epsilon \forall c \in C$. We conclude by the minimality of the system $(X, T, \mathbb{Z})$.

Let $F$ be the closure of the graph of $f \{(x, f(x)) : x \in X\} \subseteq X \times K$, and for $x \in X$, let $F(x)$ be the section $\{k \in K : (x, k) \in F\}$. It follows for any continuity point $x \in C_f$, $F(x)$ is a singleton.

We say that $f$ is separating if $F(T^n x) \cap F(T^n z) \neq \emptyset$ for all integer $n$, implies $x = z$.

It is simple to check that the semicocycle $f$ in Example 11 is separating.

**Proposition 13.** If $(X, T, \mathbb{Z})$ is a minimal system and $f : X \to K$ a separating semicocycle. Then the system $(X_f, \sigma, \mathbb{Z})$ is a minimal almost 1-1 extension of $(X, T, \mathbb{Z})$. The set of points having several pre-images is $igcup_{n \in \mathbb{Z}} T^n(D_f)$.

**Proof.** We first prove that the system is an extension of $(X, T, \mathbb{Z})$. For any $x = (x(n))_{n \in \mathbb{Z}} \in X_f$, there exists a sequence $(n_i)_{i \geq 0} \subseteq \mathbb{Z}$ such that $x(n) = \lim_{i \to \infty} f(T^{n+n_i}(y))$ for each $n \in \mathbb{Z}$. It follows that for all $n \in \mathbb{Z}$, $x(n) \in F(T^n x)$ where $x \in X$ is an accumulation point of the sequence.

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1 this notion is a bit stronger than the one of cocycle invariant under no rotations in [13].
(T^n(y))_i. Any other point x' such that x(n) ∈ F(T^n x') for all n ∈ ℤ, satisfies F(T^n x) ∩ F(T^n x') ≠ ∅ for all n. The semicocycle being separating, this implies that x = x'. It follows the map (f(T^{k+n}y))_{k∈Z} ↦ T^n y can be continuously extended onto X_f. Let π: X_f → X be this map. Clearly it is a factor map between the systems.

Moreover it satisfies π((f(T^n y)_n) = y and since F(T^n y) is a singleton for each n, the fiber π⁻¹(y) also is a singleton and π defines an almost 1-1 extension.

Conversely a point x having several pre-images #{π⁻¹(x)} ≥ 2, is such that #{F(T^n x)} ≥ 2 for some integer n. Also observe that the set \{F(T^n y), f(T^n y)_n : n ∈ ℤ\} is dense in F by the denseness of the continuity points. Hence the points x ∈ X having several preimages by π are those with #{F(T^n x)} ≥ 2 for some integer n.□

**Proposition 14.** Let (X, T, ℤ) be Cantor minimal system and Γ ⊆ Aut(T, ℤ) be a countable subgroup such that Γ ∩ {Id} = \{Id\}. Then there exists a Cantor minimal system (Y, S, ℤ) which is an almost 1-1 extension of (X, T, ℤ) and Aut(S, ℤ) ≅ Γ ⊕ ⟨S⟩.

**Proof.** Let us enumerate the elements of Γ = {γ_n} n≥0 ⊆ Aut(T, ℤ). Set ℤ_2 = \{∑_{k≥0} x_k 2^k : x_k ∈ \{0, 1\} ∀ k ≥ 0\} to be the set of 2-adic integers endowed with their classical metric. Let x_0 ∈ X and g be the map defined in Example 13, we define a new semicocycle f: X → X × ℤ_2 by

\[ f(x) = (x, \sum_{k≥0} g(γ^{-1}(x)) 2^k). \]

It is direct to check that its set of discontinuity points is D_f = \{γ(x_0) : γ ∈ Γ\} and that f is separating. It follows from Proposition 13 the system (X_f, σ, ℤ) is a minimal almost 1-1 extension of (X, T).

We have to check first that Γ is a subgroup of the centralizer of (X_f, σ, ℤ). Let us fix a γ ∈ Γ. Recall for any x = (x(n))_{n≥0} ∈ X_f, there exists a sequence (n_i)_{i≥0} ⊆ ℤ such that x(n) = lim_i f(T^{n+n_i}(y)) for each n ∈ ℤ. This mean that the sequence (T^{n_i}y_i) converges to a point x ∈ X and for each n ∈ ℤ, γ ∈ Γ, the sequence (g(γ^{-1}(T^{n+n_i}y_i))) converges to a value g_{γ, n}. Moreover, both x and g_{γ, n} are independent of the choice of the sequence (n_i)_i.

Observe that the sequence (g(γ^{-1}(T^{n+n_i}γ(y))))_i = (g((γ^{-1}γ)^{-1}(T^{n+n_i}y))))_i converges to g^{-1}_{γ, n}. So the limit lim_i f(T^{n+n_i}γ(y)) exists for each n. We denote it [γ](x(n)). Since it is independent of the sequence (n_i)_i, we define the map [γ]: (x(n))_{n} ∈ X_f → ([γ](x(n)))_{n} ∈ X_f. It is straightforward to check this defines an homeomorphism commuting with the shift.
Moreover, if \( \pi \) denotes the factor map constructed in Proposition 13, it satisfies by construction \( \pi([\tilde{\gamma}]x) = \tilde{\gamma}(\pi(x)) \) for any \( x \in X_f \). So by Lemma 10 the map \( \tilde{\gamma} \in \Gamma \mapsto [\tilde{\gamma}] \in \text{Aut}(\sigma, Z) \) is an injection.

If \( \phi \in \text{Aut}(\sigma, Z) \) is an automorphism of \( (X_f, \sigma, Z) \), it induces, via the map \( \pi \), an automorphism \( \tilde{\phi} \) of \( (X, T, Z) \) such that \( \pi \circ \phi = \tilde{\phi} \circ \pi \) (Lemma 10), with, of course, \( \sigma = T \). Therefore \( \tilde{\phi} \) preserves the set of points with several pre-images. Proposition 13 provides \( \tilde{\phi}(\bigcup_n T^n D_f) = \bigcup_n T^n D_f \). In particular, we get \( \phi(x_0) = T^n \gamma(x_0) \) for some \( \gamma \in \Gamma \), \( n \in \mathbb{Z} \), meaning \( \phi \in \Gamma \oplus \langle T \rangle \) by the freeness of the action of the automorphisms. We conclude with Lemma 10 that \( \text{Aut}(\sigma, Z) \simeq \Gamma \oplus \mathbb{Z} \).

A direct application of this property with Lemma 9 provides

**Corollary 15.** Let \( \Gamma \) be countable subgroup of the centralizer of a \( \mathbb{Z} \) Cantor minimal system. Then there exists a Cantor minimal system \( (X, T, Z) \) such that \( \text{Aut}(T, Z) = \Gamma \oplus \langle T \rangle \).

This results particularly applies to any countable residually finite group \( \Gamma \) by Proposition 7.

### 5. Restrictions on the subgroups of centralizer of minimal Cantor systems

#### 5.1. LEF groups.

**Definition 16.** A group \( G \) is Locally Embeddable into Finite groups (LEF) if for every finite set \( K \subset G \), there exists a finite group \( H \) and a map \( \varphi : G \to H \) such that

1. \( \varphi(k_1k_2) = \varphi(k_1)\varphi(k_2) \) for all \( k_1, k_2 \in K \)
2. the restriction of \( \varphi \) to \( K \) is injective.

This notion was introduced by Vershik and Gordon (see [25] and [4, Chapter 7]). This notion is closely related to the one of residually finite group: any residually finite group is a LEF group (take \( H \) a finite quotient) and conversely, any finitely presented LEF group is always residually finite [25].

For instance the non residually finite Baumslag-Solitar group \( BS(n, m) = \langle a, b; ba^n b^{-1} = a^m \rangle \), \( n > m \geq 2 \) is not LEF. Analogously the Thompson groups \( T \) and \( V \) are finitely presented and simple (hence not residually finite) so they are not LEF. It is also shown in [23] that a finitely presented group with undecidable word problem is not a LEF-group. There also exist LEF groups that are not residually finite like the group of rationals number \( \mathbb{Q} \) or the topological full group of a \( \mathbb{Z} \) Cantor minimal system [17].
5.2. Dimension groups and centralizers of $\mathbb{Z}$ minimal Cantor systems. We recall the notions of dimension group associated with a $\mathbb{Z}$ minimal Cantor system. We show the automorphisms of a dimension group form a LEF group. We deduce that the automorphism group of a Cantor minimal system is the extension of a LEF group by a LEF group. We do not know if the automorphism group is itself a LEF group because the semi direct product of LEF groups is not necessarily a LEF group [25]. Notice however that the subgroup of a LEF group is also a LEF group.

5.2.1. Dimension group and its automorphisms. An important invariant in topological orbit equivalence is the dimension group $K_0(X, T)$. For a $\mathbb{Z}$ Cantor minimal system $(X, T, \mathbb{Z})$, the dimension group $K_0(X, T)$ is the quotient of the set of continuous integer valued functions by the group of coboundaries

$$\beta C(X, \mathbb{Z}) := \{ f - f \circ T; f \in C(X, \mathbb{Z}) \},$$

$$K_0(X, T) := C(X, \mathbb{Z})/\beta C(X, \mathbb{Z}).$$

Define the positive cone

$$K_0^+(X, T) := \{ [f] \in K_0(X, T) : f \in C(X, \mathbb{N}) \},$$

where $[f]$ denotes the class of the function $f$. The dimension group $K_0(X, T)$ is an abelian ordered group with this positive cone and the equivalences class $[1]$ of the constant function one is called an ordered unit of $K_0(X, T)$. Giordano, Putnam, and Skau proved [15] that $K_0(X, T)$ as an ordered group with an ordered unit, is a complete invariant of strong orbit equivalence class of $(X, T, \mathbb{Z})$. Since each automorphism $\phi \in \text{Aut}^0(T, \mathbb{Z})$ satisfies $f \circ \phi^{-1} \in \beta C(X, \mathbb{Z})$ for every $f \in \beta C(X, \mathbb{Z})$, each automorphism $\phi$ induces an isomorphism $\text{mod}(\phi)$ of $K_0^+(X, T)$ by

$$\text{mod}(\phi)([f]) := [f \circ \phi^{-1}].$$

Each isomorphism $\text{mod}(\phi)$ preserves the order and the order unit. It can be easily checked that the mod map is a group homomorphism from $\text{Aut}(T, \mathbb{Z})$ to $\text{Aut}(K_0^+(X, T))$ the group of automorphisms of $K_0^+(X, T)$ preserving the order and the unit.

Lemma 17. Let $(X, T, \mathbb{Z})$ be a minimal Cantor system. Then $\ker(\text{mod})$ is a LEF normal subgroup of $\text{Aut}(T, \mathbb{Z})$.

Proof. It follows from standard facts that any element in $\ker \text{mod}$ can be approximated by elements of $[[T]]$ [16, Proposition 2.11] for the uniform convergence topology. Since the property of LEF group is a closed property among groups we conclude by the main result in [17].
Lemma 18. Let \((X, T, Z)\) be a minimal Cantor system. Then \(\text{Aut}(K^0(X, T))\) is a LEF group.

**Proof.** Recall that the dimension group \(K^0(X, T)\) has no torsion element. Moreover the dimension group can be described as a direct limit of free abelian groups [15]. Analogously, it is an increasing union of finitely generated group. To fix the notations, \(K^0(X, T) = \bigcup_{n \geq 0} S_n\), where \(S_n \subset S_{n+1}\) and each \(S_n\) is an abelian group generated by a finite family \(\{a_i^{(n)}\}_{i=1}^{k_n}\), so that each \(S_n\) is isomorphic to some \(\mathbb{Z}^{k_n}\).

To show the group \(\text{Aut}(K^0(X, T))\) is LEF, notice this notion is local in the sense that it is enough to prove it for each finitely generated subgroup. Let \(K \subset \text{Aut}(K^0(X, T))\) be a finite subset, and we will restrict ourselves on the group generated by \(K\). Since the groups \(S_k\) are nested there exists an integer \(n > 0\) such that the restrictions of all the maps \(\phi \in K\) restricted to \(S_n\) differ. Let \(p > n\) be an integer such that \(\phi(S_n) \subset S_p\) for each \(\phi \in K\). Each map \(\phi \in \langle K \rangle\) induces an injective homomorphism \(\phi' : \mathbb{Z}^{k_n} \to \mathbb{Z}^{k_p}\), so it defines an injective linear map \(\mathbb{R}^{k_n} \to \mathbb{R}^{k_p}\). By extending the base, such a map can be eventually extended by the identity to an isomorphism \(\bar{\phi} : \mathbb{R}^{k_p} \to \mathbb{R}^{k_p}\). The very definition of \(n\) insures that the map \(\phi \mapsto \bar{\phi}\) is injective when restricted to \(K\).

It follows the maps \(\bar{\phi}\), for \(\phi \in K\), generate a subgroup of \(\text{GL}(k_p, \mathbb{R})\). By a result of Mal’cev this subgroup is residually finite, so that it admits a homomorphism \(\pi\) onto a finite group and is injective in restriction to \(\{\bar{\phi} : \phi \in K\}\). It is straightforward to check that the map \(\pi \circ \bar{\phi}\) gives the conditions 1 and 2 in the definition of a LEF group. \(\square\)

5.2.2. **Applications to the centralizer of a Cantor minimal system.** The next result summarizes the results of lemmas 17 and 18.

**Theorem 19.** Let \((X, T, Z)\) be a minimal Cantor system. Then \(\text{Aut}(T, Z)\) is the extension of a LEF group by a LEF group by the exact sequence

\[
\begin{array}{c}
\{1\} \longrightarrow \ker \text{mod} \longrightarrow \text{Aut}(T, Z) \overset{\text{mod}}{\longrightarrow} \text{Aut}(K^0(X, T)) \end{array}
\]

Concerning the automorphism group of Cantor minimal system for a finitely generated group action \(\Gamma\), Proposition 5 enables us to give a similar property.

**Corollary 20.** Let \((X, T, \Gamma)\) be a Cantor aperiodic minimal system where \(\Gamma\) is finitely generated group. Then \(\text{Aut}(T, \Gamma)\) is the extension of a LEF group by a LEF group.

This property restricts the automorphism group of minimal Cantor system as shows the following result.
Corollary 21. Let $G$ be be a simple group which is not LEF. Then for every aperiodic minimal Cantor system $(X, T, \Gamma)$ of a finitely generated group $\Gamma$, each homomorphism $\rho: G \to \text{Aut}(T, \Gamma)$ is trivial.

It follows, for instance, that the Thompson groups $T$ and $V$ can not be embedded into the automorphism group of an aperiodic minimal Cantor system. However, Corollary 20 is not true for the Thompson group $F$, because its abelianized group $F/[F, F]$ is isomorphic to $\mathbb{Z}^2$ and there exists an automorphism group of Cantor minimal systems containing $\mathbb{Z}^2$ (see Corollary 15).

Proof. Assume there exists a non trivial homomorphism $\rho: G \to \text{Aut}(T, \Gamma)$ for some aperiodic minimal Cantor system $(X, T, \Gamma)$. Since $G$ is simple, this homomorphism is injective. By Corollary 20 the group $G$ is the extension $1 \to G_1 \to G_2$ for some LEF groups $G_1, G_2$. Since $G$ is not LEF, the homomorphism $\pi_2$ is not injective. Since $G$ is simple, $G$ equals $\ker \pi_2 = G_1$ leading to a contradiction. □

Corollary 21 can be extended to the normalizer group $\text{Norm}(T, \Gamma)$ of an aperiodic minimal Cantor system.

Corollary 22. Let $\Gamma$ be a finitely generated group and let $G$ be a simple non LEF group which is not a subgroup of $\text{Aut}(\Gamma)$. Then for every aperiodic minimal Cantor system $(X, T, \Gamma)$, each homomorphism $\rho: G \to \text{Norm}(T, \Gamma)$ is trivial.

Proof. As in the proof of Corollary 21 one can suppose that $G$ is a subgroup of $\text{Norm}(T, \Gamma)$. Since $G$ is simple and by Corollary 21 the group $\text{Aut}(X, T)\cap G$ is trivial. It follows from Lemma 11 that $G$ is a subgroup of $\text{Aut}(\Gamma)$, a contradiction. □

Remark 23. For the Thompson group $V$, Let $\alpha: V \to \text{Aut}(V)$ be the monomorphism given by the inner automorphism $\alpha_\gamma: g \mapsto \gamma g \gamma^{-1}$ (recall that $V$ has a trivial centralizer, so that $\alpha$ is injective). Given an aperiodic minimal Cantor system $(X,T,V)$, each map $T^\gamma$, $\gamma \in V$ defines an element in $\text{Aut}(T,V)$ associated to $\alpha_\gamma$, so that $V$ is a subgroup of $\text{Norm}(T,V)$. Corollary 22 implies that there is no minimal Cantor $\mathbb{Z}$-system $(Y,S,Z)$ such that $\text{Norm}(T,V) \leq \text{Norm}(S,Z)$. Hence there is no analogue of Proposition 5 for the normalizer group $\text{Norm}(T,V)$.

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