Lorentz Transformations from Reflections: Some Applications

H.K. Urbantke
Institut für Theoretische Physik, Universität Wien
Boltzmanngasse 5, A-1090 Vienna, Austria
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Abstract

We point out, by exhibiting two examples and mentioning a third one, that it is sometimes useful to consider Lorentz transformations as generated from hyperplane or line reflections. One example concerns the construction of boosts linking two given 4-vectors, the other one concerns the Minkowski geometric understanding of V. Moretti’s polar decomposition of orthochronous Lorentz matrices.
1 Introduction

The Cartan-Dieudonné theorem\(^1\), to the effect that the orthogonal groups \(O(n,\mathbb{C})\), \(O(p, q)\) are generated by hyperplane reflections—each group element being a product of at most \(n = p+q\) of them—is usually invoked only to study the structure of those groups in general and to establish their relation to Clifford algebras. However, the possibility of generating orthogonal transformations from reflections may sometimes be of some practical value in solving simple concrete problems involving orthogonal transformations. In this note we want to illustrate this point by discussing two problems involving Lorentz boosts from a geometrical rather than matrix calculational point of view. One concerns a geometric understanding of the construction of the boost matrix that carries two given non-spacelike future-directed 4-vectors into each other. The other concerns the geometrical understanding of \(V\). Moretti’s observation [4] that the usual rotation-boost decomposition (=Cartan decomposition, in the sense of the theory of symmetric spaces [5]) of an orthochronous Lorentz matrix (cf. [6] for an elementary approach) is a polar decomposition in the sense of (real) Hilbert space theory. In both cases we find that the true understanding of the elementary matrix solution comes from an analysis of boosts in terms of reflections. Another instance of the application of reflections to Lorentz transformations, using Clifford algebra in addition, is considered, e.g., in [7].

2 Preparation: hyperplane and line reflections

We work in a Lorentzian vector space \(V\) of 4-vectors, i.e., a real 4-dimensional vector space with a symmetric bilinear form \(\eta\) of signature \((-+++)\). Since we want to maintain a geometric language, it will be preferable to consider Lorentz transformations in their active interpretation as linear maps \(L: V \to V\) satisfying \(\eta(Lx, Ly) = \eta(x, y)\) for all \(x, y \in V\). By the Cartan-Dieudonné theorem, each such \(L\) is a product of at most four hyperplane reflections. A hyperplane reflection (more precisely, a reflection in a hyperplane) is a Lorentz transformation \(S\) which is involutory (i.e., \(S^2 = \text{id}_V\) \(\Rightarrow\) eigenvalues \(\pm 1\)) and such that the \((+)\)eigenspace is a 3-plane and the \((-)\)eigenspace is 1-dimensional and not lightlike. Thus, if \(u\) with \(\eta(u, u) \neq 0\) spans the \((-)\)eigenspace, an arbitrary 4-vector \(x\) decomposes as \(x = x_\perp + x_\parallel\), where
\[
x_\perp := \frac{\eta(x, u)}{\eta(u, u)} u,
\]
and the reflection \(S_u\) in the hyperplane orthogonal to \(u\) is given by
\[
x = x_\parallel + x_\perp \mapsto S_u x := x_\parallel - x_\perp = x - 2x_\perp.
\]

\(^1\)See, e.g., [1]; Dieudonné [2] contains a shortcut to the proof; the shortcut presented in [3] seems to be mistaken.
Using (abstract) index notation, we find that $S_u$ is given by the tensor

$$\delta^\mu_\nu - 2u^\mu u_\nu/\eta(u, u),$$

and the above-mentioned properties of $S_u$ are immediate from this construction. It is essential that $S_u$ is improper, $\det S_u = -1$.

The decomposition of a given $L$ as a product of hyperplane reflections is not unique; however, the parities of the numbers of reflections in timelike and in spacelike hyperplanes involved are unique and define the four components of the Lorentz group $\mathcal{L}$ as (even, even) $\Leftrightarrow \mathcal{L}_\uparrow^\uparrow$, (odd, even) $\Leftrightarrow \mathcal{L}_\downarrow^\uparrow$, (even, odd) $\Leftrightarrow \mathcal{L}_\downarrow^\downarrow$, (odd, odd) $\Leftrightarrow \mathcal{L}_\uparrow^\downarrow$ in the usual Wigner notation.

For the generation of the Lorentz group in 4 dimensions, hyperplane reflections may be replaced by line reflections, which are slightly easier to visualize. The reflection in the line spanned by the non-lightlike vector $u$ is just given by $-S_u$. The essential feature of $S_u$ to be involutory and improper is preserved in this transition because of the assumed even dimensionality of $V$. Using this we immediately see the following facts which we note down for future use. 1) We have $S_u x = -x$ or $S_u x = +x$ for $x \parallel u$ or $x \perp u$. 2) If $u$ is timelike, $S_u$ is antichronous and is the time reversal with respect to an observer with 4-velocity $\parallel u$. 3) If $x$ lies in the 2-plane spanned by two non-lightlike vectors $a$, $b$, then $S_a x$, $S_b x$, $S_a S_b x$ also lie in that plane. 4) If $\eta(a, a) = \eta(b, b)$ (possibly $=0$) and $\eta(a + b, a + b) \neq 0$, then $-S_{a+b} = a$ (reflection in the bisector of the Lorentzian angle between $a$ and $b$). (Of course, these statements can be checked by direct calculation as well.)

3 Boosts and space rotations from reflections

Lorentz boosts are characterized as proper orthochronous Lorentz transformations that leave invariant a timelike 2-plane as a whole and leave invariant vectorwise the orthogonal (spacelike) 2-plane. More specifically, a boost relative to an observer with 4-velocity $u$—or $u$-boost, for short—is a Lorentz transformation of this kind such that $u$ is contained in the invariant timelike 2-plane.

We shall now argue that all $u$-boosts can be written as $S_u S_u$, where $w$ is some other timelike 4-vector, determined up to proportionality and becoming unique by requiring it to be normalized and future-directed. Indeed, since $S_u$ and $S_w$ both are improper and antichronous, their product is proper and orthochronous; if $x$ is in the timelike 2-plane spanned by $u$ and $w$, so is $S_w S_u x$, and if $x$ is orthogonal to this plane, $S_u x = x$, $S_w x = x$, so $x$ is left fixed by $S_w S_u$: hence $S_w S_u$ is a $u$-boost. Conversely, if $B$ is a $u$-boost, carrying $u$ to $u' := Bu$, we can, as noted above, take $w$ to be a multiple of $u + u'$: then $S_w S_u u = S_w (-u) = u' = Bu$. So $S_{u+u} S_u$ does the same job as $B$ and is therefore identical to $B$. Note that $u' + u$ is timelike and future-directed just as $u$, $u'$ are.
Similarly, a proper space rotation *relative to an observer* with 4-velocity $u$—or $u$-rotation, for short—is a Lorentz transformation leaving invariant as a whole a spacelike 2-plane and leaving invariant vectorwise the orthogonal timelike 2-plane which must contain $u$. The direction in the latter orthogonal to $u$ gives the axis of rotation relative to $u$. We can generate such $u$-rotations simply as products $S_a S_b$, where $a$, $b$ are orthogonal to $u$ and span the invariant spacelike 2-plane. If they are normalized, $\eta(a, b)$ will give the cosine of half the rotation angle, as is well known from elementary geometry. Improper $u$-rotations are similarly generated by an odd number of hyperplane reflections with normals orthogonal to $u$.

As a first application we mention the calculation of the *Thomas angle*: the product of a $u$-boost $B$ sending $u$ to $u'$, followed by a $u'$-boost $B'$ sending $u'$ to $u''$ and a $u''$-boost $B''$ sending $u''$ back to $u$ leaves $u$ invariant and is proper, so is a proper $u$-rotation; the angle of rotation involved is the Thomas angle. One can calculate it as $S_u + S_{u''} S_{u'} S_u$, using Clifford multiplication for further evaluation and comparison with an expression $S_a S_b$ for $u$-rotations (see, e.g., [7]).

## 4 Boosts linking two given 4-vectors

In the preceding section, we constructed from reflections a $u$-boost that carries $u$ to another given 4-velocity $u'$. Here we want to find, more generally, a $u$-boost $B$ that carries a given 4-vector $x$ to another given 4-vector $x'$. Naturally, since $B$ is orthochronous, we must assume $x$, $x'$ not only to have the same non-positive 4-square $\eta(x, x) = \eta(x', x') \leq 0$ but also to have the same time orientation to make the problem well-posed.

It is, of course, just a clumsy exercise to find $B$ using the ordinary matrix form for boosts, i.e., to find the matrix $B$ for $B$ in the rest frame of $u$, in particular to find the components $v$ of the relative velocity involved in $B$. Even more complicated is the algebra involved when the problem is related to relativistic velocity addition: a clear way of disentangling it has been developed only relatively recently (see [8] for exposition). However, using reflections we can give a surprisingly simple solution. Namely, as above we can write all $u$-boosts as $B = S_w S_u$, where $w$ is a timelike 4-vector to be determined. Requiring $x' = B x = S_w S_u x$ we see that $w$ must be chosen such that $S_w$ carry $S_u x$ into $x'$. By the last remark of section 2, we see that under the assumptions made on $x$, $x'$ we have $w \propto x' - S_u x$, and the unique solution is

$$B = S_{x'} S_u$$

whenever $x' - S_u x$ is not lightlike. The latter case arises only if $x'$ and $-S_u x$ are lightlike and parallel, i.e., if $x'$ and $x$ are lightlike and spatially antipodal with respect to $u$; in this case no $u$-boost exists that would do what is required. Otherwise, we see from the fact that $w$ bisects the Lorentzian angle between $u$ and $Bu = u'$ that the relative velocity $v$ involved in the matrix $B$ is just the
"relativistic double" of $w/w^0$ in the rest frame of $u$. Thus, writing in this frame the column matrices of components

$$\mathbf{x} = \begin{pmatrix} x^0 \\ \mathbf{x} \end{pmatrix}, \quad \mathbf{x}' = \begin{pmatrix} x'^0 \\ \mathbf{x}' \end{pmatrix},$$

we have $S_u \mathbf{x} = \begin{pmatrix} -x^0 \\ \mathbf{x} \end{pmatrix}$, $w = \begin{pmatrix} x'^0 + x^0 \\ \mathbf{x}' - \mathbf{x} \end{pmatrix}$, whence

$$\mathbf{v} = \frac{2(x'^0 + x^0)(\mathbf{x}' - \mathbf{x})}{(x'^0 + x^0)^2 + (\mathbf{x}' - \mathbf{x})^2},$$

where the speed of light has been set to $c = 1$.

We remark that the occurrence of the exceptional case above is a symptom of the following fact which plays a role in the massless helicity representations of the Poincaré group: if $x, x'$ are lightlike and future-oriented, one cannot find Lorentz transformations carrying a fixed $x$ to a variable $x'$ such that the transformation depends on $x'$ in a fashion continuous on the entire future light cone (see [6], [9] for discussion).

5 Symmetry and positivity for boosts

Recently, Moretti [4] observed that the usual polar decomposition of real matrices when applied to orthochronous Lorentz matrices gives just their rotation-boost decomposition. Indeed, as remarked in [10], this follows already from the uniqueness of the polar decomposition, since Lorentz matrices describing (proper or improper) spatial rotations are trivially orthogonal (in the 4-dimensional Euclidean sense!) and boost matrices $B$ are symmetric and positive definite (in the same sense), the latter because of the identity

$$(t \mathbf{x}^\top)B \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \equiv \gamma(t - \mathbf{v}x)^2 + \frac{1}{\gamma} \frac{(\mathbf{v}x)^2}{\mathbf{v}^2} + (x - \frac{\mathbf{v}x}{\mathbf{v}^2}\mathbf{v})^2.$$

(For the active interpretation, the sign of the boost velocity $\mathbf{v}$ must be reversed.)

We here want to provide a geometrical understanding of the latter properties, which are not immediate in a Lorentzian context, because the only geometric structure around in $V$ seems to be $\eta$ (and space+time orientation). What then is the geometric meaning of the "4-dimensional Euclidean sense" here, and how does the identity above look geometrically? It turns out that again reflections are helpful in clearing this up.

The "hidden" unit matrix defining the Hilbert space scalar product to which the terms "symmetric" and "positive-definite" refer in the polar decomposition context is a basis-dependent object, and the basis in $V$ to which it refers contains the observer 4-velocity $u$ as one of its constituents. This remark allows an observer-dependent Hilbert space scalar product to be geometrically defined as $U(x, y) = U_{\mu\nu}x^\mu y^\nu$, where the tensor $U$ is given by

$$U_{\mu\nu} := \eta_{\mu\nu} - 2u_\mu u_\nu/\eta(u, u).$$
in the rest frame of $u$ its components constitute the unit matrix. Then what we want to see is that every $u$-boost $B$ satisfies

\[
U(x, By) = U(y, Bx) \text{ for all } x, y \in V
\]
\[
U(x, Bx) > 0 \text{ for all } x \neq 0.
\]

We invoke the possibility of writing $B = S_w S_u$ for some timelike 4-vector $w$ which we may also assume normalized and future-directed. Interpreting $\eta, U$ as maps from $V$ to its dual and looking at the explicit form for $S_u$ written before we obviously have $U = \eta S_u$; and similarly we have that $W := \eta S_w$ relates $w$ and $S_w$ to a symmetric positive-definite bilinear form $W$. Using $S_u^T \eta S_u = \eta$, $S_u^2 = \text{id}_V$ (which says that $S_u$ is an involutory Lorentz transformation) we now calculate

\[
UB = \eta S_u S_w S_u = (S_u^T)^{-1} \eta S_w S_u = S_u^T WS_u,
\]

implying

\[
U(x, By) = W(S_u x, S_u y)
\]

for all $x, y$. The stated properties of $W$ thus imply the properties in question.

For $u$-rotations, proper or improper, $U$-orthogonality follows from $S_u^T U S_u = U$ whenever $\eta(a, u) = 0$; but this is less surprising.

\section{Conclusion}

We mentioned one and gave two more applications of the fact that Lorentz transformations, and boosts in particular, can be generated from reflections. There are probably many more instances where this point of view might be useful in clearing up the geometrical structure and/or replacing heavier matrix calculations. Our aim was to bring this point to fore.

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