The ability of senior high school students in comprehending mathematical proofs

H Herizal, S Suhendra and E Nurlaelah

1Program Studi Pendidikan Matematika, Sekolah Pascasarjana, Universitas Pendidikan Indonesia, Jl. Dr. Setiabudhi No. 229, Bandung 40154, Indonesia
2Departemen Pendidikan Matematika, Universitas Pendidikan Indonesia, Jl. Dr. Setiabudhi No. 229, Bandung 40154, Indonesia

*herizal_mathedu@student.upi.edu

Abstract. Mathematical proving ability is an ability which has function to improve mathematical high order thinking and reasoning skill. This ability consists of two sub-abilities, which are mathematical proof comprehension and mathematical proof construction. Both of the sub-abilities are related each other. To construct mathematical proofs, students need to have good mathematical proof comprehension. Due to the importance of mathematical proof comprehension, a qualitative descriptive study was conducted to describe senior high school students’ ability in mathematical proof comprehension. A test was given to 10 students of 11th grade in one of senior high schools at Bandung, West Java, Indonesia. The students’ works were analyzed for each proof comprehension indicator. Overall, the research found four results, the students: (i) were good in giving a reason for each step of proofs; (ii) got difficulties in validating a step of proof; (iii) were good in applying the steps of a proof to prove similar statement; and (iv) failed in determining and proving a notion based on pattern from the given statements. This study concludes that the senior high school students’ ability in comprehending mathematical proofs has not been satisfied. In several parts, it needs to be improved during mathematics learning process.

1. Introduction
Mathematical proof is an important component in mathematics. Mathematical proof is defined as an argument to prove theorem [1]. Mathematical proof is also defined as the arguments used in mathematical practice to justify the correctness of theorems [2]. In addition, another definition of mathematical proof is a convincing argument expressed in the mathematics language to determine that a statement is true [3]. In short, mathematical proof is a mathematics argument aimed to convince that a statement of mathematics e.g. theorem, lemma, and corollary is true.

Mathematical proof has several functions, they are: (1) Verification, (2) Explanation, (3) Communication, (4) Discovery, and (5) Systematization [4]. Meanwhile the aim of mathematical proof is to improve students’ critical thinking skill and mathematical high order thinking and reasoning [5]. In Indonesia’s curriculum, one of the goals in learning mathematics is to be expert in mathematical reasoning [6] and several mathematical reasoning indicators related to mathematical proof [7]. The arguments above indicate that mathematical proof is important for students. Therefore, students need to understand mathematical proof well.
The ability to do mathematical proof is called mathematical proving ability. This ability is a part of mathematical reasoning [8]. Mathematical proving ability consists of two sub-abilities which are mathematical proof comprehension and mathematical proof construction [9]. Both of the sub-abilities are related each other. To construct mathematical proofs, students need to have good mathematical proof comprehension. Unfortunately, the researches related to proof comprehension is limited. There are two reasons why conducting research about proof comprehension is important. First, in mathematics education literature, there were no many researches about it, the main concern of the researcher was in constructing mathematical proof and problems around it. This fact supported by a research found that from 131 articles about idea of proving and argumentation in mathematics, there was just 3 articles focused on students comprehension toward given mathematical proof [10]. Second, in constructing mathematical proof, one of the abilities needed by the students is the ability to comprehend mathematical proof [9].

Mathematical proof comprehension is an ability including all understanding of proof, logic behind the proof, why the author construct the proof in that particular way or decide whether a proof is valid or no [9]. It means that mathematical proof comprehension just concerns on understanding a proof which has already been constructed. All of skills in comprehending mathematical proof is useful for constructing mathematical proof, for example, validating a step of a proof. If the students can do it well, then when constructing mathematical proof they will not write wrong step since they have known that statement in the step is wrong. But, facts showed that students made errors and got difficulties in constructing mathematical proof, e.g. proved using concrete example and did not know how to start write a proof [11], [12]. These error and difficulties were not only faced by senior high school student, but also undergraduate students. A study also reported that the undergraduate students made errors and got difficulties in proving, i.e. proving general statements using specific example, misusing definition, not proving both condition in a biconditional statement, and complicating to start the proof [13]. As stated before, one of the causes of the errors and difficulties was lack of understanding about mathematical proof comprehension. Based on the fact, it is imperative to explore the ability of students in comprehending mathematical proofs. Thus, this study aims to describe senior high school students’ ability in comprehending mathematical proofs.

2. Research method
This study was descriptive qualitative research. The participants were ten students of 11th grade in one of private Senior High Schools in Bandung, West Java, Indonesia. To measure the students’ mathematical proof comprehension ability, four essay questions were given to them. The topic of the questions was trigonometry especially about sine rules, cosine rules and area of triangle. In this study, four indicators of mathematical proof comprehension were measured, they were: (1) applying steps of a proof of a statement in proving another similar statements; (2) giving reason for each step in a proof; and (3) validating the statements in each proving step; if there was an error then the students can correct it (4) Determining a notion based on pattern from the given statements and proving it.

The questions were developed based on the indicators above. Before giving to the students, the questions was consulted and validated by two mathematics experts and two mathematics teachers. Data from the students’ work were analyzed for each proof comprehension indicator according to the goal of the research. It included the general ability of the students in each indicator and both errors and difficulties made by the students in solving each problem.

3. Result and discussion
The senior high school students’ ability in comprehending mathematical proof was obtained from the test. There were four questions to measure the ability of mathematical proof comprehension. The result could be seen in Table 1. Data in the table show that students’ ability in giving reason for each step in a mathematical proof was good since the attaining of indicator was 72,85%. The similar result was also got by the students in applying steps of a proof of a statement in proving another similar statements where their attaining in the indicator was 73,33%. The medium result was got by the
students in validating the statements in each mathematical proving step, i.e. 53.12%. But the students’ ability was not satisfied in determining a notion based on pattern from the given statements and proving it, the attaining of the indicator was just 18.57%.

Table 1. The quantitative result of mathematical proof comprehension test.

| No | Indicators                                           | Average Score | Ideal Score |
|----|------------------------------------------------------|---------------|-------------|
| 1  | Giving reason for each step in a mathematical proof | 5.10          | 7           |
| 2  | Validating the statements of each mathematical proving steps | 8.50          | 16          |
| 3  | Applying steps of a proof of a statement in proving another similar statements | 4.40          | 6           |
| 4  | Determining a notion based on pattern from the given statements and prove it | 1.30          | 7           |

3.1. The description of students’ ability in indicator 1
For the first indicator, students were asked to give reason for each step of a mathematical proof. To solve the question, students need to have knowledge about some concepts in mathematics. The concepts needed were sine rules, cosine rules, area of triangle and some topics related to triangle e.g. sum of interior angle in a triangle and also about algebraic process. In the beginning of the test, they got difficulty in understanding the meaning of the question. They needed more explanation from the researchers about the question. They confessed that they never met the question like the question in this test before. In learning sine rules, cosine rules and area of triangle, their teacher just gave them routine problems e.g. determining an angle where one angle and two sides were given. This case indicates that experience is important in solving problem. This is in line with a research stated that lack of experience would cause error and difficulties in solving mathematics problem [14]. After researchers explained what the question asked to, then they can solve the question. Students’ work displayed that they could give right reason for each step of a mathematical proof given. In giving reason, some students did not write the reason in simple sentence, but the meaning of the sentence was right. For the example, the student wrote “sine rules, \(1 \frac{\sqrt{3}}{2} \text{ from } \sin 60^\circ \) and \(1 \frac{\sqrt{2}}{2} \text{ from } \sin 45^\circ\)” whereas the student could write it simply i.e. “substitute the value of sine”.

In this question, the problems faced by the students were they forgot the concepts that they had learned. For step 5, the researchers gave a statement i.e. \(45^\circ + 60^\circ + m\angle B = 180^\circ\) and asked the students to write reason. Some students forgot that the reason were “theorem of interior angle in a triangle” while the topic was learned when junior high school. In addition, for the statement above, there was also a student written “triangle theory” for the reason. It is still in general and not accepted for the right reason. Generally, the students’ ability in the first indicator is good with several notes that had explained above.

3.2. The description of students’ ability in indicator 2
Question number 2 measured the ability of students in validating the statements in each mathematical proving steps. In detail, the students were asked to determine whether a statement in a proof was right or wrong, if the statement was right then the students must give reason. On the contrary if the statement was wrong then the students needed to correct it. Result in Table 1 indicates that student’s ability for this indicator was not as good as indicator 1. In the question, there were eight steps consisted of five right steps and three wrong steps.
For validating right step, the students could do it well, but they could not give right reason. It is contrary with the previous question where they could give good reason for each step of proof. Furthermore, in validating wrong step, the students did error. Researchers gave an equation looked like sine rule (see Figure 1 step 6). Most of the students decided it as a right step. It indicates that their knowledge about form of sine rules was not comprehensive. They just memorize that if a side divided by sine of an angle equals another side over sine of another angle, then it is a sine rules without knowing whether the side is opposite the angle or no. It is important to determine the corresponding sides and angles but the students failed in it. This case means that weak ability in understanding a concept affected students’ ability in solving a problem. It is supported by a study which found that one of the factors caused error was lack of understanding about a concept thoroughly [14]. As a consequence of error in step 6, the students were also wrong in determining the correctness of step 7. They stated that step 7 was wrong while the answer is right. It was caused by considered true for the previous step (Step 6). The contrary result of validating wrong statement was shown in step 8. This step asked the students to validate a statement about the value of sine. Most of them could determine that the statement was wrong and could correct it into the right statement. Thus, for this indicator, the students still got difficulties in giving the reason for right step and made error when determine the right form of sine rules.

3.3. The description of students’ ability in indicator 3
This indicator measured students’ ability in applying steps of a proof of a statement in proving another similar statements. To answer the question related to this indicator, the students need skill of reading a proof. They have to understand each step of a proof given, then adapt the idea of each step to prove the similar statement. The other skills are choosing suitable sine rules, using the right formula of triangle’s area, and doing algebraic process well. The students’ score for question number 3 (see Figure 2) was good. The percentage of average score was 73.33%.

![Figure 1. Part of question number 2.](image1.png)

![Figure 2. Question number 3.](image2.png)
In question number 3, most of the students could adapt the ideas of the proof given to prove the other statements. They could choose sine rules that head for the statement that would be proved. They also could use the right formula of triangle’s area (choose one from three forms of the formula) and could do algebraic processes well which were substituting an equation from step 1 into the formula in step 2 and simplifying the new equation into the form asked. Meanwhile, the errors made by the student that failed in solving the question were: (1) they failed in determining the right sine rules that will be used in proving process and (2) they could not choose the suitable form of triangle’s formula. From all of students’ work, there was one answer that different from the others. The student tried to prove the statement backward (see Figure 3). Unfortunately, she made error in writing symbol i.e. written “s”, in fact no “s” in the triangle and in the last line she wrote “r sin α” where the right answer was “r sin α”. Overall, in indicator 3, students showed good ability in applying steps of a proof of a statement in proving another similar statements. As addition, students must be accustomed in choosing the suitable formula/rules if there are several forms of formulas/rules.

![Figure 3. “backward process”, Student’s respond for number 3](image)

3.4. The description of students’ ability in indicator 4

The result shown in Table 1 indicates that senior high school students were low in indicator 4, it means that they are weak in determining a notion based on pattern from the given statements and proving it. The students’ average score was 18.57%. Question number 4 was constructed to measure students’ ability for indicator 4. The question was “A triangle ABC has length of sides respectively a cm, b cm, and c cm. 1st Statement: If each side of triangle ABC is extended 4 times which are 4a cm, 4b cm 4c cm, then the new triangle has area 16 times larger than area triangle ABC. 2nd statement: If each side of triangle ABC is extended 5 times which are 5a cm, 5b cm and 5c cm, then the new triangle has area 25 times larger than area triangle ABC. Based on the 1st and 2nd statements, it is guessed that if each side of triangle ABC is extended m times which are ma cm, mb cm, and mc cm, then the new triangle has area ____ times larger than area triangle ABC. Prove your answer!” To answer the question, students need skill of reading a pattern from given statements. Also, they must have knowledge about area of triangle since it is needed when proving a notion made.

Students’ work showed that it just three students could guess the right answer, i.e. m², but no one could prove it well. Two of three students who could guess the right answer wrote “because the relation between 1st and 2nd statements are square, there are 4 times to be 16 times and 5 times to be 25 times, then it can be concluded that m times to be m²” as proof. Another student started the proof by writing area of triangle’s formula then substituted the length of triangle’ sides into the formula. He knew how to start prove, but he was wrong in calculating multiplication (i.e. m x m = m), it caused he failed to get the form that wanted to be proved. Meanwhile, the others left it blank or wrote the wrong answer. They did not guess the answer and did not know how to start a proving. About difficulty in starting a proof, it is consistent with two studies reported that the students did not know how to begin to write a proof [12], [13].
4. Conclusion
Mathematical proof comprehension is a part of mathematical proving ability and it needs in constructing mathematical proof. From four indicators measured, the senior high school students’ ability in comprehending mathematical proof is still need to be improved especially in validating whether a step of proof is correct or incorrect and in both determining and proving a notion based on pattern from the given statements. During learning process, teachers have to teach about proving (mathematical proof comprehension and mathematical proof construction) particularly in mathematics topics that proof included in, e.g. geometry and trigonometry. The students also must be accustomed with the mathematical proof comprehension question as one of the efforts to improve students’ ability in comprehending mathematical proof. The further research is needed especially to develop strategies or learning methods to improve the skill of mathematical proof comprehension.

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