Black holes and black strings in plane waves

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Abstract

We investigate the construction of black holes and black strings in vacuum plane wave spacetimes using the method of matched asymptotic expansions. We find solutions of the linearised equations of motion in the asymptotic region for a general source on a plane wave background. We observe that these solutions do not satisfy our previously defined conditions for being asymptotically plane wave. Hence, the space of asymptotically plane wave solutions is restricted. We consider the solution in the near region, treating the plane wave as a perturbation of a black object, and find that there is a regular black string solution but no regular black hole solution.

1 Introduction

Plane waves are of interest both from the point of view of classical gravity and in the context of considerations of holography in string theory. To a relativist, the plane waves are a rich class of exact solutions, which can be obtained as the result of applying the Penrose limit to an arbitrary spacetime. In string theory, they are of interest because the theory on the worldsheet admits a simple realisation, making explicit computations possible \cite{1}. Secondly, certain maximally supersymmetric plane wave backgrounds \cite{2} admit a dual field theory interpretation as a scaling limit of certain field theories \cite{3}. This correspondence is obtained by taking a Penrose limit of the AdS/CFT correspondence. As a result, our understanding of holography in this case is rather indirect, and no holographic dictionary has yet been constructed for this duality.

From both these points of view, the study of black hole solutions with plane wave asymptotics is clearly interesting. Black holes for other “simple” asymptotics, such as flat space or anti-de Sitter space, have long been known. The construction of black hole solutions with plane wave asymptotics would offer a new, rich family of black hole solutions, whose thermodynamics could exhibit interesting dependence on the asymptotic plane wave considered. For string theory, the black hole solutions in

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the maximally supersymmetric plane wave backgrounds of [2] would presumably be related to finite-temperature excitations of the corresponding field theory, as in the AdS/CFT correspondence [4]. Consideration of such solutions could therefore cast interesting light on this poorly-understood duality.

Some exact solutions describing black strings in plane wave backgrounds have been obtained by applying solution generating transformations [5, 6, 7, 8, 9]. A review of this work and the structure of horizons and plane waves can be found in [10]. However, such methods are available only in special cases, and a solution describing the simplest situation, a regular black hole or black string in a vacuum plane wave background, has not been obtained by these methods. Constructing solutions by directly solving the equations of motion is challenging.

In this paper, we adopt the method of matched asymptotic expansions to find approximate solutions when the horizon size $r_+$ of the black hole or black string is small compared to the curvature scale $\mu^{-1}$ of the plane wave. This gives a separation of scales, which can be exploited to solve the equations of motion in the linearised approximation in separate regions, matching the solutions in an overlap region. Such methods have been successfully applied to the construction of caged black holes in Kaluza-Klein theory [11, 12] and to construct black ring solutions in more than five spacetime dimensions [13] and in anti-de Sitter space [14]. These ideas have been further developed in [15, 16], where general extended black objects wrapping a sub-manifold in an arbitrary spacetime have been considered at leading order in the region far from the black object.

We proceed in a similar way to these previous examples, first finding the metric far from the source (for $r \gg r_+$) by studying the linearised approximation to gravity with an appropriate delta-function source. The wave equation in the plane wave background is rather complicated, so we focus on solving this problem in an intermediate region $r_+ \ll r \ll \mu^{-1}$ where the deviations from flat space due to both the source and the plane wave are small.

Solving the equation in this regime, we find that simple dimensional analysis indicates that the solutions will violate the asymptotic boundary conditions proposed in [17] as a definition of asymptotically plane wave spacetimes. In fact, the perturbation due to the delta-function source becomes large relative to the background metric at large distances. An explicit analysis in four and five dimensions shows that the terms violating these boundary conditions are indeed non-zero. These solutions thus appear not to be asymptotically plane wave; we will refer to them as black holes or black strings in plane wave backgrounds. The fact that the linearised solutions for a delta-function source violate the asymptotic boundary conditions suggests that as in AdS$_2$ [18] and the Kerr/CFT correspondence [19, 20, 21], the space of asymptotically plane wave spacetimes may be highly restricted.

We then obtain the near horizon metric in the region $r \ll \mu^{-1}$ by solving the linearised Einstein equations on the background of the black object, treating the plane wave as a perturbation. For a black hole, we find that there is no linearised solution which is regular on the horizon. For the black string, we obtain a regular solution in the near region, and verify that it matches on to the solution in the intermediate region.
When solving the equations, we focus on vacuum plane waves in the lowest possible
dimension for simplicity, but the method of matched asymptotic expansion is more
general, and a similar analysis could be applied to construct black string solutions
in any plane wave background of interest in arbitrary dimensions. We will remark
on the extension to other waves and higher dimensions at appropriate points in the
calculation. The calculation in the region \( r \gg r_+ \) is described in section 2, and the
calculation in the region \( r \ll \mu^{-1} \) is described in section 3. We conclude with some
remarks on the implications of our results in section 4.

2 Linearised solutions on a plane wave background

We want to construct solutions corresponding to a black hole or black string of radius
\( r_+ \) in a general vacuum plane wave background in \( D = d + 2 \) dimensions
\[
\text{ds}^2 = -dt^2 + dz^2 + \mu_{\alpha\beta}(t + z)x^\alpha x^\beta(dt + dz)^2 + \delta_{\alpha\beta}dx^\alpha dx^\beta,
\]
where \( x^\alpha, \alpha = 1, \ldots d \) are Cartesian coordinates on the transverse space. We will work
in the parameter range \( r_+ \ll \mu^{-1} \), where we take the matrix \( \mu_{\alpha\beta}(t + z) \) characterising
the wave to have a single characteristic scale \( \mu \) for simplicity. The black object can
then be treated as a small perturbation of the plane wave background for \( r \gg r_+ \). In
this region of the spacetime, the problem of constructing a black hole or black string
solution thus reduces to solving the linearised Einstein’s equations for a suitable source
\( T_{\mu\nu} \). In transverse gauge, the linearised equations are
\[
\Box h_{\mu\nu} = -16\pi G T_{\mu\nu}.
\]
For a pointlike source, the relevant stress tensor is simply \( T_{\mu\nu} = MV_\mu V_\nu\delta(x^\mu - x^\mu(\tau)) \),
where \( x^\mu(\tau) \) is the particle’s trajectory, \( V^\mu = dx^\mu/d\tau \) is the tangent to this trajectory,
and \( M \) is the proper mass. For a black string solution, the stress tensor can be
determined by linearising the vacuum black string solution in \( d + 2 \) dimensions,
\[
\text{ds}^2 = -\left(1 - \frac{r_+^{d-2}}{r^{d-2}}\right)dt^2 + dz^2 + \left(1 - \frac{r_+^{d-2}}{r^{d-2}}\right)^{-1}dr^2 + r^2d\Omega_{d-1}^2,
\]
which gives the stress tensor in these coordinates as
\[
T_{tt} = \left(\frac{d - 1}{16\pi G}\right)r_+^{d-2}\delta^d(r), \quad T_{zz} = -\left(\frac{r_+^{d-2}}{16\pi G}\right)\delta^d(r).
\]
The source is fixed to follow some appropriate trajectory in the plane wave back-
ground. For a pointlike source, the appropriate trajectory is a timelike geodesic of the
background spacetime. To obtain a stationary black hole solution, we should require
this geodesic to be the orbit of a timelike Killing vector in the spacetime. This forces
us to restrict to plane waves with a constant matrix \( \mu_{\alpha\beta}(t + z) = \mu_{\alpha\beta} \), so that the

\footnote{Note that in our actual calculations, we will not assume the transverse traceless gauge, as it is
more convenient to use the gauge freedom to fix particular components of the perturbation.}
solution has a timelike Killing vector, and to consider the geodesic $z = 0$.

$\delta^d(x^\alpha)$, which is the unique geodesic trajectory which is also an orbit of the Killing vector. The appropriate source is then $T_{tt} = M\delta(z)\delta^d(x^\alpha)$, and the size of the black hole is $r^{d-1} \propto M$.

For the black string, the equation of motion for a probe string is [22]

$$K_{\mu\nu}^\rho T^{\mu\nu} = 0,$$

where $K_{\mu\nu}^\rho$ is the second fundamental tensor of the submanifold defining the embedding of the string worldvolume, and $T_{\mu\nu}$ is the stress tensor of the source. We will consider embedding the black string along the submanifold $x^\alpha = 0$, which has $K_{\mu\nu}^\rho = 0$. As a result, there is no constraint on the form of the stress tensor. As for the black hole, we need to restrict to constant $\mu_{\alpha\beta}(t + z) = \mu_{\alpha\beta}$ so that this submanifold is an orbit of the spacetime isometries, so that we can expect to obtain a stationary uniform black string solution. We can then use boosts in the $t - z$ plane to choose the black string solution to be in its rest frame, setting $T_{tz} = 0$, without loss of generality. The appropriate source is thus [1]. We want to find a uniform black string solution, so the components of the stress tensor are assumed to be constants along the worldvolume. The blackfold equations of [16] are hence trivially satisfied.

In each case, the problem thus reduces in principle to solving [2] on the plane wave background for an appropriate source. However, we do not have the Green’s function for this differential equation in closed form, so we will content ourselves with studying this problem in the intermediate region $r_+ \ll r \ll \mu^{-1}$, where we can treat the plane wave itself as a small perturbation of flat space, and obtain the solution of [2] order by order in $\mu^2 r^2$.

### 2.1 Dimensional analysis

We first discuss the perturbation in general dimensions using a simple dimensional analysis argument. For the case of a point source, we find it convenient to rewrite the metric in spherical polar coordinates, introducing a radial coordinate

$$r^2 = z^2 + \delta_{\alpha\beta} x^\alpha x^\beta,$$

and defining coordinates $\theta^i$ on the $S^d$ at constant $r$. As in [23], we use $a, b$ to denote coordinates on the two dimensional space spanned by $r, t$. By dimensional analysis, the form of the perturbation to first order in $M$ and in $\mu^2$ will be

$$h_{ab} = \frac{M}{r^{D-3}} h_{ab}^{(0)} + \frac{M \mu^2}{r^{D-5}} h_{ab}^{(1)}(\theta^i),$$

$$h_{ai} = \frac{M}{r^{D-4}} h_{ai}^{(0)} + \frac{M \mu^2}{r^{D-6}} h_{ai}^{(1)}(\theta^i),$$

$$h_{ij} = \frac{M}{r^{D-5}} h_{ij}^{(0)} + \frac{M \mu^2}{r^{D-7}} h_{ij}^{(1)}(\theta^i),$$

(7)

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[2] We can make this choice without loss of generality by translation invariance in $z$. 

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4
where $h_{\mu\nu}^{(0)}$ and $h_{\mu\nu}^{(1)}$ are dimensionless functions depending only on the angles $\theta^i$. In fact, since the spherical symmetry is only broken by the plane wave, $h_{ab}^{(0)}$ are constants, and the component on the sphere $h_{ij}^{(0)}$ will be proportional to the metric on the sphere $\gamma_{ij}$. We will always work in a gauge where $h_{ij}^{(0)}$ vanishes. Each addition of an $i$ index raises the power of $r$ by one because the coordinates on the sphere are written in terms of dimensionless angles.

This simple dimensional analysis already indicates a significant issue: this perturbation does not satisfy the boundary conditions introduced in [17]. There, it was assumed that components of the perturbation in the directions transverse to the wave would fall off at least as $1/r^{D-4}$ (corresponding to $h_{ij} \propto 1/r^{D-6}$, because of the extra factors of $r$ from writing the perturbation in polar coordinates), characteristic of a localised source in a flat spacetime. However, we find that the term resulting from the interaction with the wave must grow more quickly than this on dimensional grounds. When we think of the plane wave as a perturbation around flat space, the plane wave background introduces corrections which grow more quickly with $r$ than the original leading-order response.

Similarly, when we consider a black string source, it is convenient to write the metric in the directions transverse to the wave in polar coordinates, introducing a radial coordinate

$$r^2 = \delta_{\alpha\beta} x^\alpha x^\beta,$$

and introducing coordinates $\theta^i$ on the $S^{d-1}$ at constant $r, z$. In the string source case, $a, b$ will denote coordinates in the three dimensional space spanned by $t, r, z$. Then to leading order in $r_+ \text{ and } \mu^2$, the perturbation sourced by a black string will have the form

$$h_{ab} = \frac{r_+^{D-4}}{r^{D-4}} h_{ab}^{(0)} + \frac{r_+^{D-4} \mu^2}{r^{D-6}} h_{ab}^{(1)}(\theta^i),$$

$$h_{ai} = \frac{r_+^{D-4}}{r^{D-5}} h_{ai}^{(0)} + \frac{r_+^{D-4} \mu^2}{r^{D-7}} h_{ai}^{(1)}(\theta^i),$$

$$h_{ij} = \frac{r_+^{D-4}}{r^{D-6}} h_{ij}^{(0)} + \frac{r_+^{D-4} \mu^2}{r^{D-8}} h_{ij}^{(1)}(\theta^i),$$

where $h_{ab}^{(0)}$ are constants and $h_{\mu\nu}^{(1)}$ are functions of the coordinates $\theta^i$ on the sphere only. Thus, as in the black hole case, the perturbation does not satisfy the boundary conditions introduced in [17].

This is a significant issue because at least in low spacetime dimensions, the resulting perturbation actually grows more quickly with $r$ than the background metric. In $D = 4$ for the black hole and $D = 5$ for the black string, the perturbation of the angular metric $h_{ij}$ has a contribution that goes like $r_+ \mu^2 r^3$, which is growing faster than the background metric on the sphere, which goes like $r^2$. Furthermore, what we have discussed so far is just the leading order correction in $\mu^2$. Higher order terms in $\mu^2$ will come with additional powers of $r$. One might hope that when the problem is solved to all orders in $\mu^2$, the resulting behaviour could be under better control, but it is hard to see how such a cancellation between different orders could be arranged. We will see later in a particular example that this does not occur.
Thus, we are faced with the odd situation that the linearised field of a point source may become more important than the background, signalling a breakdown of the linearised approximation, far from the source itself. Thus, the solutions we construct should not be thought of as “asymptotically plane wave” black holes/strings, as the metric in the asymptotic regime is not close to the original plane wave metric. As a result, the analysis of [17] will not apply to these spacetimes, and in particular we do not expect that they will have finite action with respect to the action principle discussed there.

One might hope that the terms which violate those boundary conditions which are allowed by dimensional analysis may actually vanish. This hope would be encouraged by the fact that the specific examples of plane wave black strings constructed in [5, 6, 7] satisfied the asymptotic boundary conditions of [17]. However, the examples of [5, 6, 7] are special cases in that they are constructed by the Garfinkle-Vachaspati solution-generating transformation [24], and by construction can only differ from the seed solution in the metric components along the null direction. By contrast, the solution constructed in [9], which was obtained by a different method, has precisely the kinds of corrections that are predicted by this dimensional analysis argument.

In the next two subsections, we will consider the solution of the linearised equations of motion for the perturbation in detail for the lowest possible dimension for black hole and black string sources, and see in these particular examples that the terms which violate the asymptotic boundary conditions of [17] do indeed appear. Thus, the approximate solutions we obtain for black holes and black strings in plane wave backgrounds are not asymptotically plane wave in the sense defined in [17]. Given the above dimensional analysis arguments and the results below, it seems reasonable to expect that this is the generic case, so that the space of asymptotically plane wave solutions is very limited. We will comment on this in the conclusions.

### 2.2 Black hole

Let us consider the perturbation sourced by a point source in the lowest possible dimension, \( D = 4 \), in detail. By a choice of coordinates, the most general four dimensional vacuum plane wave can be written as

\[
ds_{\text{wave}}^2 = -dt^2 + dx^2 + dy^2 + dz^2 - \mu^2 (x^2 - y^2) (dt + dz)^2.\]

We rewrite this in spherical polars by defining

\[
z = r \cos \theta, \quad x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi,
\]

so

\[
ds_{\text{wave}}^2 = -dt^2 + dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) - \mu^2 r^2 \sin^2 \theta (\cos^2 \phi - \sin^2 \phi) (dt + \cos \theta dr - r \sin \theta d\theta)^2.
\]

As in the above subsection, we can use dimensional analysis to fix the dependence of the perturbation on \( r \). We can in fact determine the perturbation to zeroth order in
μ² by simply linearising the Schwarzschild solution, which gives \( h_{rt} = h_{rr} = \frac{2M}{r} \). This satisfies the linearised equations of motion for a delta-function point source, but not in the transverse traceless gauge which was assumed in writing (2). In what follows, we will not assume the transverse traceless gauge, as it is more convenient to use the gauge freedom to fix some components of the perturbation.

For the terms of first order in \( μ² \), we can use the freedom to choose a gauge for the perturbation to set \( h^{(1)}_{φφ} \) and \( h^{(1)}_{φφ} \) to zero. Note that we have four gauge degrees of freedom but have only eliminated three components, hence we have one remaining degree of freedom which we will use later. We then make an ansatz for the \( φ \) dependence of the perturbation, and write our perturbation as

\[
\begin{align*}
    h_{ab} &= \frac{M}{r} h^{(0)}_{ab} + \mu² r (\cos² φ - \sin² φ) h^{(1)}_{ab}(θ), \\
    h_{aθ} &= \mu² r² (\cos² φ - \sin² φ) h^{(1)}_{aθ}(θ), \\
    h_{ij} &= \mu² r³ (\cos² φ - \sin² φ) h^{(1)}_{ij}(θ),
\end{align*}
\]

where the non-zero components of \( h^{(0)}_{ab} \) are \( h^{(0)}_{tt} = 2, h^{(0)}_{rr} = 2 \), and the non-zero components of \( h^{(1)}_{μν}(θ) \) are \( h^{(1)}_{tt}(θ), h^{(1)}_{tr}(θ), h^{(1)}_{tθ}(θ), h^{(1)}_{rθ}(θ), h^{(1)}_{θθ}(θ) \) and \( h^{(1)}_{φφ}(θ) \).

We now want to substitute this ansatz into the linearised Einstein equations and solve for the undetermined functions \( h^{(1)}_{μν}(θ) \), requiring regularity on the sphere. In an arbitrary gauge, the linearised Einstein equations for \( r \neq 0 \) are

\[
R^{(1)}_{μν} = \frac{1}{2} g^{ρσ} (\nabla_ρ \nabla_μ h_{νσ} + \nabla_ρ \nabla_ν h_{μσ} - \nabla_μ \nabla_ν h_{ρσ} - \nabla_ρ \nabla_σ h_{μν}) = 0. \tag{14}
\]

Substituting our ansatz, these equations become (where primes denote derivatives with respect to \( θ \))

\[
\begin{align*}
- \sin² θ h^{(1)′′}_{tt}(θ) - \sin θ \cos θ h^{(1)′}_t(θ) + 2(\cos² θ + 1) h^{(1)}_t(θ) \\
- 6 \cos⁶ θ - 2 \cos⁴ θ + 22 \cos² θ - 14 = 0, \tag{15}
\end{align*}
\]

\[
\begin{align*}
- \sin² θ h^{(1)′′}_{tr}(θ) - \sin θ \cos θ h^{(1)′}_t(θ) + 2 \sin² θ h^{(1)}_t(θ) + 4 h^{(1)}_t(θ) \\
+ 2 \sin θ \cos θ h^{(1)}_t(θ) - 8 \cos⁵ θ + 16 \cos³ θ - 8 \cos θ = 0, \tag{16}
\end{align*}
\]

\[
\begin{align*}
- \sin² θ h^{(1)′′}_r(θ) - \sin θ \cos θ h^{(1)′}_r(θ) + 4 \sin² θ h^{(1)}_r(θ) + 2(3 - \cos² θ) h^{(1)}_r(θ) \\
- 2 \sin² θ h^{(1)}_{θθ}(θ) - 2 \sin² θ h^{(1)}_r(θ) + 10 \cos⁶ θ - 26 \cos⁴ θ + 22 \cos² θ - 6 = 0, \tag{17}
\end{align*}
\]

\[
\begin{align*}
\sin² θ h^{(1)′}_r(θ) - \sin² θ h^{(1)}_{φφ}(θ) - \sin θ \cos θ h^{(1)}_{φφ}(θ) + \sin θ \cos θ h^{(1)}_{θθ}(θ) \\
+ 2(\cos² θ + 1) h^{(1)}_{rθ}(θ) - 4 \sin θ (\cos⁵ θ - 2 \cos³ θ + \cos θ) = 0, \tag{18}
\end{align*}
\]

\[
\begin{align*}
\sin² θ h^{(1)′}_r(θ) + 2(\cos² θ + 1) h^{(1)}_t(θ) + 2 \sin θ (\cos⁴ θ - 4 \cos² θ + 3) = 0, \tag{19}
\end{align*}
\]

\[
\begin{align*}
h^{(1)}_{rθ}(θ) - \cot θ h^{(1)}_{rθ}(θ) - h^{(1)}_{θθ}(θ) + h^{(1)}_{rr}(θ) - \cos⁴ θ + 2 \cos² θ - 1 = 0, \tag{20}
\end{align*}
\]
\[ h_{t\theta}^{(1)}(\theta) - \cot \theta h_{tt}^{(1)}(\theta) + h_{tt}^{(1)}(\theta) + 2 \cos \theta (1 - \cos^2 \theta) = 0, \quad (21) \]

\[ -\sin \theta (h_{rr}^{(1)}(\theta) - h_{tt}^{(1)}(\theta)) + \cos \theta (h_{r\theta}^{(1)}(\theta) - h_{tt}^{(1)}(\theta)) + 2 \sin \theta h_{\phi\phi}^{(1)}(\theta) + 2 \cos \theta (\cos^4 \theta - 1) = 0, \quad (22) \]

\[ - \sin^2 \theta h_{\phi\phi}^{(1)}(\theta) + \sin^2 \theta h_{tt}^{(1)}(\theta) + \sin \theta \cos \theta (h_{\phi\phi}^{(1)}(\theta) - 2 h_{\phi\phi}^{(1)}(\theta)) \]

\[ - \sin^2 \theta h_{rr}^{(1)}(\theta) 6 \sin^2 \theta h_{t\theta}^{(1)}(\theta) + 5 \sin^2 \theta h_{\theta\theta}^{(1)}(\theta) + 3 \sin^2 \theta h_{tt}^{(1)}(\theta) \quad (23) \]

\[ - \sin^2 \theta h_{tt}^{(1)}(\theta) + \sin^2 \theta h_{tt}^{(1)}(\theta) + 2 \cos^2 \theta (\cos^4 \theta + 3 \cos^2 \theta - 5) + 2 = 0, \]

\[ \sin^2 \theta h_{\phi\phi}^{(1)}(\theta) + \sin \theta \cos \theta (h_{rr}^{(1)}(\theta) - h_{tt}^{(1)}(\theta)) + 2 \sin^2 \theta h_{\phi\phi}^{(1)}(\theta) - h_{\theta\theta}^{(1)}(\theta) \]

\[ + \cos^2 \theta (3 h_{rr}^{(1)}(\theta) - 3 h_{\theta\theta}^{(1)}(\theta) + h_{tt}^{(1)}(\theta)) - 2 \sin^2 \theta h_{r\theta}^{(1)}(\theta) + 3 \sin^2 \theta h_{\phi\phi}^{(1)}(\theta) \]

\[ - 7 h_{rr}^{(1)}(\theta) + 3 h_{tt}^{(1)}(\theta) + 2 \cos^2 \theta (3 \cos^4 \theta - 7 \cos^2 \theta + 9) - 10 = 0. \quad (24) \]

We have a system of ten equations in seven unknown functions (in fact, there will be only six unknown functions once we have made use of the one remaining degree of gauge freedom) and so it seems our system is over-constrained. We find, however, that there are only six independent equations and hence that our system is in fact well defined. It is convenient to subtract a multiple of \( (20) \) from \( (17) \) to simplify it to

\[- \sin^2 \theta h_{rr}^{(1)}(\theta) - \sin \theta \cos \theta (h_{rr}^{(1)}(\theta) - h_{tt}^{(1)}(\theta)) + 2 (1 + \cos^2 \theta) h_{rr}^{(1)}(\theta) + 8 \sin \theta \cos \theta h_{r\theta}^{(1)}(\theta) \]

\[ + 2 \sin^2 \theta h_{\theta\theta}^{(1)}(\theta) - 2 \sin^2 \theta h_{\phi\phi}^{(1)}(\theta) + 10 \cos^2 \theta - 14 \cos^4 \theta + 6 \cos^6 \theta - 2 = 0. \quad (25) \]

By using combinations of \( (20), (22), (25) \) and their derivatives it is possible to reduce \( (15) \) to an algebraic equation

\[ 2 \sin \theta \cos \theta h_{r\theta}^{(1)}(\theta) + 2 \sin^2 \theta h_{\phi\phi}^{(1)}(\theta) + 3 h_{tt}^{(1)}(\theta) - 5 h_{rr}^{(1)}(\theta) \]

\[ + 2 \cos^2 \theta h_{r\theta}^{(1)}(\theta) + 2 (- \cos^6 \theta + 4 \cos^4 \theta + \cos^2 \theta - 4) = 0. \quad (26) \]

We find we can write \( (16), (18), (23) \) and \( (24) \) as linear combinations of \( (19), (20), (21), (22), (25) \) and \( (26) \) and hence that these equations are not independent. We now see that a convenient choice of gauge is one in which \( h_{r\theta}^{(1)}(\theta) = 0 \). In this gauge, the solution which is regular on the sphere is

\[ h_{\mu\nu} dx^\mu dx^\nu = \frac{2M}{r} dt^2 + \frac{2M}{r} dr^2 + M \mu^2 r \sin^2 \theta (\cos^2 \phi - \sin^2 \phi) \times \]

\[ \left[ (4 - \frac{1}{3} \sin^2 \theta) dt^2 + 4 \cos \theta dt dr - 4r \sin \theta dt d\theta \right. \]

\[ + \frac{1}{3} \sin^2 \theta dr^2 - \frac{2}{3} r^2 \sin^2 \theta (d\theta^2 + \sin^2 \theta d\phi^2) \right]. \quad (27) \]

We note that as stated earlier, the regular solution for the terms of first order in \( \mu^2 \) has non-zero components on the sphere which grow faster than the background metric on the sphere. These solutions are hence not asymptotically plane wave. While this leading order term would not grow faster than the background metric in higher dimensions, higher order terms in \( \mu^2 \) will in principle do so.
2.3 Black string

We now consider the perturbation for a black string source in the lowest possible dimension, which is $D = 5$ for the black string. The most general vacuum plane wave solution in five dimensions is

$$ds^2_{\text{wave}} = -dt^2 + dx^2 + dy^2 + dz^2 + dw^2 - \mu^2(\alpha(x^2 + y^2 - 2w^2) + \beta(x^2 - y^2))(dt + dz)^2; \quad (28)$$

note that there is a two-parameter family of plane wave solutions here. We rewrite this in spherical polars in the directions transverse to the wave by writing

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad w = r \cos \theta, \quad (29)$$

so

$$ds^2_{\text{wave}} = -dt^2 + dz^2 + dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) - \mu^2(r^2(1 - 3 \cos^2 \theta) + \beta r^2 \sin^2 \theta(\cos^2 \phi - \sin^2 \phi))(dt + dz)^2. \quad (30)$$

As in the previous subsection, we can determine the perturbation to zeroth order in $\mu^2$ by simply linearising the Schwarzschild black string solution (3), which gives $h_{tt} = h_{rr} = 2M/r^2$. We will again find it convenient to fix the gauge by choosing some components of the perturbation to vanish at each order in $\mu^2$. For the terms of first order in $\mu^2$, we note that the background has an invariance under $t \rightarrow -t$, $z \rightarrow -z$ which is not broken by the source, so the $h_{t\mu}$, $h_{z\mu}$ components for $\mu \neq t, z$ will automatically vanish.

At first order in $\mu^2$, we can treat the two different components of the plane wave separately. We first consider the first-order terms in the perturbation associated to $\alpha$. Let us therefore set $\alpha = 1$ and $\beta = 0$ in the plane wave background (28). There is then a translation invariance in $\phi$ and a symmetry under $\phi \rightarrow -\phi$, which imply that $h_{\phi\mu}$ vanish for $\mu \neq \phi$. We will make a choice of gauge to set $h_{t\phi}^{(1)}$ and $h_{r\phi}^{(1)}$ to zero. This gauge choice proves to be convenient for comparing to the solution in the near region to be obtained later. The form of the perturbation is then

$$h_{ab} = \frac{M}{r}h_{ab}^{(0)} + M\mu^2 r h_{ab}^{(1)}(\theta),$$

$$h_{a\theta} = M\mu^2 r^2 h_{a\theta}^{(1)}(\theta),$$

$$h_{ij} = M\mu^2 r^3 h_{ij}^{(1)}(\theta), \quad (31)$$

where the non-zero components of $h_{ab}^{(0)}$ are $h_{tt}^{(0)} = 2, h_{rr}^{(0)} = 2$, and the non-zero components of $h_{\mu\nu}^{(1)}(\theta)$ are $h_{tt}^{(1)}(\theta), h_{r\theta}^{(1)}(\theta), h_{t\phi}^{(1)}(\theta), h_{\theta\theta}^{(1)}(\theta)$ and $h_{\phi\phi}^{(1)}(\theta)$.

We now want to substitute this ansatz into the linearised Einstein equations and solve for the undetermined functions $h_{\mu\nu}^{(1)}(\theta)$, requiring regularity on the sphere. In an arbitrary gauge, the linearised Einstein equations for $r \neq 0$ are

$$R_{\mu\nu}^{(1)} = \frac{1}{2}g^{\rho\sigma}(\nabla_{\rho}\nabla_{\mu}h_{\nu\sigma} + \nabla_{\rho}\nabla_{\nu}h_{\mu\sigma} - \nabla_{\mu}\nabla_{\nu}h_{\rho\sigma} - \nabla_{\rho}\nabla_{\sigma}h_{\mu\nu}) = 0. \quad (32)$$
Substituting our ansatz, these equations become

\begin{align}
\partial^2_{\theta\theta} h_{tt}^{(1)}(\theta) + \cot \theta \partial_\theta h_{tt}^{(1)}(\theta) + 2h_{tt}^{(1)}(\theta) + 16(1 - 3 \cos^2 \theta) &= 0, \\
\partial^2_\theta h_{tz}^{(1)}(\theta) + \cot \theta \partial_\theta h_{tz}^{(1)}(\theta) + 2h_{tz}^{(1)}(\theta) + 12(1 - 3 \cos^2 \theta) &= 0, \\
\partial^2_\theta h_{zz}^{(1)}(\theta) + \cot \theta \partial_\theta h_{zz}^{(1)}(\theta) + 2h_{zz}^{(1)}(\theta) + 8(1 - 3 \cos^2 \theta) &= 0, \\
h_{\theta\theta}^{(1)}(\theta) + h_{\phi\phi}^{(1)}(\theta) + 2(1 - 3 \cos^2 \theta) &= 0, \\
\tan \theta \partial_\theta h_{\phi\phi}^{(1)}(\theta) - h_{\theta\theta}^{(1)}(\theta) + h_{\phi\phi}^{(1)}(\theta) + 6 \sin^2 \theta &= 0, \\
\partial^2_\theta h_{tt}^{(1)}(\theta) - \partial^2_\theta h_{\phi\phi}^{(1)}(\theta) - \partial^2_\theta h_{zz}^{(1)}(\theta) + \cot \theta (\partial_\theta h_{\theta\theta}^{(1)}(\theta) - 2\partial_\theta h_{\phi\phi}^{(1)}(\theta)) \\
+ h_{tt}^{(1)}(\theta) - h_{zz}^{(1)}(\theta) - 5h_{\theta\theta}^{(1)}(\theta) - h_{\phi\phi}^{(1)}(\theta) + 12 \sin^2 \theta - 2(1 - 3 \cos^2 \theta) &= 0,
\end{align}

\begin{align}
\partial^2_\theta h_{\phi\phi}^{(1)}(\theta) + \cot \theta (\partial_\theta h_{\phi\phi}^{(1)}(\theta) - \partial_\theta h_{\phi\phi}^{(1)}(\theta) - \partial_\theta h_{tt}^{(1)}(\theta)) \\
+ 3h_{\theta\theta}^{(1)}(\theta) + 3h_{\phi\phi}^{(1)}(\theta) - h_{tt}^{(1)}(\theta) + h_{zz}^{(1)}(\theta) + 2(1 - 3 \cos^2 \theta) &= 0.
\end{align}

We first solve equations (33), (34) and (35) for $h_{tt}^{(1)}(\theta), h_{zt}^{(1)}(\theta)$ and $h_{zz}^{(1)}(\theta)$ respectively. We then solve for $h_{\theta\theta}^{(1)}(\theta)$ and $h_{\phi\phi}^{(1)}(\theta)$ using equations (37) and (39). It is easy to verify that these solutions satisfy (38) and (39). Keeping only the regular part of the solution, we find

\begin{align}
h_{tt}^{(1)}(\theta) &= 4(1 - 3 \cos^2 \theta), \\
h_{zt}^{(1)}(\theta) &= 3(1 - 3 \cos^2 \theta), \\
h_{zz}^{(1)}(\theta) &= 2(1 - 3 \cos^2 \theta), \\
h_{\theta\theta}^{(1)}(\theta) &= -(1 - 3 \cos^2 \theta), \\
h_{\phi\phi}^{(1)}(\theta) &= -\sin^2 \theta(1 - 3 \cos^2 \theta).
\end{align}

As in the black hole case, we see that terms that grow faster than the background metric at large $r$ do indeed occur.

It turns out that for this background, the linearised equations of motion can be solved exactly by including one further term at next order in $\mu^2$. If we take

\begin{align}
h_{ab} &= \frac{M}{r} h_{ab}^{(0)} + M \mu^2 r h_{ab}^{(1)}(\theta) + M \mu^4 r^3 h_{ab}^{(2)}(\theta), \\
h_{a\theta} &= M \mu^2 r^2 h_{a\theta}^{(1)}(\theta), \\
h_{ij} &= M \mu^2 r^3 h_{ij}^{(1)}(\theta),
\end{align}

with $h_{ab}^{(0)}$ and $h_{ab}^{(1)}$ as given above, and

\begin{equation}
h_{tt}^{(2)} = h_{tz}^{(2)} = h_{zz}^{(2)} = \frac{1}{2} (3 - 30 \cos^2 \theta + 27 \cos^4 \theta),
\end{equation}

this will solve the equations to linear order in $M$ but to all orders in $\mu^2$. This gives an approximation valid in the full far region $r \gg M$, demonstrating that the bad asymptotic behaviour of this solution is not resolved at higher order in $\mu^2$. 


We now consider briefly the similar analysis for the other independent component, setting $\alpha = 0$ and $\beta = 1$ in the plane wave background. The $\phi$ dependence in this background restricts our ability to simplify the form of the solution by general arguments, but the results from the previous case suggest we take an ansatz of the form

\[
\begin{align*}
    h_{ab} &= \frac{M}{r} h^{(0)}_{ab} + M \mu^2 r \sin^2 \theta (\cos^2 \phi - \sin^2 \phi) h^{(1)}_{ab}, \\
    h_{a\theta} &= M \mu^2 r^2 \sin^2 \theta (\cos^2 \phi - \sin^2 \phi) h^{(1)}_{a\theta}, \\
    h_{\theta\theta} &= M \mu^2 r^3 \sin^2 \theta (\cos^2 \phi - \sin^2 \phi) h^{(1)}_{\theta\theta}, \\
    h_{\phi\phi} &= M \mu^2 r^3 \sin^2 \theta (\cos^2 \phi - \sin^2 \phi) h^{(1)}_{\phi\phi},
\end{align*}
\]

assuming the angular dependence at first order in $\mu^2$ will reproduce the angular dependence of the background plane wave. The non-zero components of $h^{(0)}_{ab}$ are $h^{(0)}_{tt} = 2, h^{(0)}_{rr} = 2$, and we assume the $h^{(1)}_{\mu\nu}$ above are constants. We find that we can solve the linearised equations of motion to first order in $\mu^2$ for this ansatz by setting $h^{(1)}_{tt} = 4, h^{(1)}_{tz} = 3, h^{(1)}_{zz} = 2, h^{(1)}_{\theta\theta} = -1, h^{(1)}_{\phi\phi} = -1$.

We can summarise these results in a more invariant fashion by saying that for a plane wave background of the form

\[
ds^2_{\text{wave}} = -dt^2 + dz^2 + dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) - \mu^2 f(\theta, \phi) (dt + dz)^2,
\]

a solution of the linearized equations of motion for a black string source, to linear order in $\mu^2$, is

\[
\begin{align*}
    h_{tt} &= \frac{2M}{r} + 4M \mu^2 r f(\theta, \phi), \\
    h_{tz} &= 3M \mu^2 r f(\theta, \phi), \\
    h_{zz} &= 2M \mu^2 r f(\theta, \phi), \\
    h_{rr} &= \frac{2M}{r}, \\
    h_{\theta\theta} &= -M \mu^2 r^3 f(\theta, \phi), \\
    h_{\phi\phi} &= -M \mu^2 r^3 \sin^2 \theta f(\theta, \phi).
\end{align*}
\]

We would expect that this generalises straightforwardly to higher dimensions. As in the black hole case, this demonstrates that these solutions are not asymptotically plane wave, as the perturbation is large compared to the background metric far from the source.

3 Near region analysis

Having explored the behaviour in the intermediate region, where we can use a linearised approximation about the plane wave background, we now turn to the analysis in the region $r \ll \mu^{-1}$ near the black hole or black string. In this region we can treat the plane wave as a small perturbation of the black object, and the problem
reduces to linearised perturbations on the black hole or black string background, with boundary conditions at large distances determined from the previous solution in the intermediate region and a boundary condition at the horizon determined by requiring regularity of the perturbed solution there. We will find that there is no regular solution in the black hole case. For the black string, we find a regular solution which matches on to the solution we discussed above in the intermediate region. We will focus on the analysis for the black hole in four dimensions and the black string in five dimensions, as in the previous section, but the same techniques can easily be applied in higher dimensions. We will comment briefly on the extension of the analysis to higher dimensions for the black hole case.

3.1 Black hole

We first study the near region of the black hole, treating the plane wave as a perturbation. We will do the analysis in the lowest possible dimension, $D = 4$, even though there is a simple symmetry argument that no regular solution exists in this case. The calculation is simplest in this dimension, and it serves to illustrate the method of calculation, which will be very similar in higher dimensions.

Take the Schwarzschild black hole solution in four dimensions,

$$\frac{ds^2}{f(r)} = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

(47)

with $f(r) = 1 - 2M/r$. We want to find a solution of the source-free linearised vacuum equations on this background which asymptotically approaches the four-dimensional plane wave (10). This implies that we want a perturbation $h_{\mu\nu}$ with asymptotic boundary conditions

$$\lim_{r \to \infty} h_{\mu\nu} dx^\mu dx^\nu = -\frac{\mu^2 r^2}{2} \sin^2 \theta (e^{2i\phi} - e^{-2i\phi}) (dt + \cos \theta dr - r \sin \theta d\theta)^2 + \ldots,$$

(48)

where the $\ldots$ denotes terms going like $\mu^2 M^n$ for $n \neq 0$. These terms are suppressed relative to the leading term because dimensional analysis tells us the mass will always appear in the combination $M/r$. At linear order in $M$ the subleading terms at large $r$ should match onto the results of the analysis in the intermediate region obtained in the previous section.

At the horizon, the boundary condition is that the solution be regular there. Since the background metric is not regular at the horizon in the Schwarzschild coordinate system we are using, this condition is most easily applied by writing the perturbation in an orthonormal frame. A suitable frame is $e^{(0)} = \sqrt{f(r)} dt$, $e^{(1)} = f^{-1/2} dr$, $e^{(2)} = rd\theta$, $e^{(3)} = r \sin \theta d\phi$. Requiring that the components of the perturbation in the orthonormal frame are regular at the horizon implies that we must require that as $r \to 2M$,

$$h_{tt} \sim (r - 2M), \quad h_{r\mu} \sim (r - 2M)^{1/2} \text{ for } \mu \neq t,$$

(49)

$$h_{rr} \sim (r - 2M)^{-1}, \quad h_{r\mu} \sim (r - 2M)^{-1/2} \text{ for } \mu \neq r,$$

(50)

$$h_{ij} \sim (r - 2M)^{0}.$$  

(51)
These conditions can also be derived by requiring finiteness of \( h_{\mu\nu} \) in a coordinate system which is well-behaved at \( r = 2M \), such as Kruskal coordinates.

Matching the leading term written in (48) and imposing regularity at the horizon should determine the solution of the perturbation equations uniquely. In fact, as we mentioned above, we will find that there is no solution of the linearised perturbation equations that satisfies these two boundary conditions.

For the black hole case, the analysis of the components on the sphere is sufficiently complicated that it is useful to exploit the results of [23] on the spherical harmonic decomposition for perturbations of Schwarzschild and rewrite the linearised equations of motion in terms of gauge-invariant variables with respect to coordinate transformations on the sphere. We therefore want to convert (48) into boundary conditions for their gauge-invariant perturbations. Let \( a, b = t, r \) and \( i, j = \theta, \phi \). Then we have boundary conditions which are scalars \( h_{ab} \), vectors \( h_{ai} \), and a tensor \( h_{ij} \), for which the boundary condition only has an \( h_{\theta\theta} \) component. Following [23] we expand the perturbation in terms of harmonics on \( S^2 \): the scalar harmonics

\[
\Box S = -l(l+1)S, \quad l = 0, 1, 2, \ldots ,
\]

the vector harmonics

\[
\Box V_i = -(l(l+1)+1)V_i, \quad l = 1, 2, 3, \ldots ,
\]

with \( D_i V^i = 0 \), and the transverse traceless tensor harmonics

\[
\Box T_{ij} = -(l(l+1)+2)T_{ij}, \quad l = 2, 3, 4, \ldots ,
\]

with \( D_i T^i_j = 0, T^i_i = 0 \). There are, however, no pure tensor harmonics \( T_{ij} \) on \( S^2 \). We use the notation \( \Box = D_i D^i \) for the d’Alembertian operator on \( S^2 \), where \( D_i \) is the covariant derivative with respect to the metric \( \gamma_{ij} \) on the unit two-sphere.

In terms of these harmonics, the scalar components of the perturbation are

\[
h_{ab} = \sum_{l,m} f_{ab} S_l^m.
\]

Note that here and hereafter we will omit the \( l, m \) indices on the coefficients \( f_{ab} \) or equivalent in the general relations like this for brevity. The vector perturbations are decomposed into their scalar derived and pure vector components \( h_{ai} = h_{ai}^S + h_{ai}^V \), where

\[
h_{ai}^S = r \sum_{l,m} f_a (-\frac{1}{k^2} D_i S_l^m),
\]

where \( k^2 = l(l+1) \), and

\[
h_{ai}^V = r \sum_{l,m} f_a^V (V_l^m)_i.
\]

Similarly, the tensor part of the perturbation is decomposed into scalar derived, vector derived and pure tensor components \( h_{ij} = h_{ij}^S + h_{ij}^V + h_{ij}^T \), where

\[
h_{ij}^S = 2r^2 \sum_{l,m} (H_L \gamma_{ij} S_l^m + H_T S_{ij}),
\]

13
where \( S_{ij} = \frac{1}{k^2} D_i D_j S_{lm}^m + \frac{\gamma_{ij}}{2} S_{lm}^m \),
\[
h_{ij}^V = 2r^2 \sum_{l,m} H_L^V V_{ij},
\] (59)
where \( V_{ij} = -\frac{1}{2k^2} (D_i V_j + D_j V_i) \) with \( k^2 = l(l+1) - 1 \), and
\[
h_{ij}^T = 2r^2 \sum_{l,m} H_L^T T_{ij}.
\] (60)

There are, however, no pure tensor harmonics \( T_{ij} \) on \( S^2 \).

Thus, to determine the boundary conditions for the gauge invariant variables, we must apply this expansion to (48) and find the asymptotic values for the unknown expansion coefficients. For scalar perturbations this is straightforward. Substituting (48) into (55) we are able to read off that
\[
\lim_{r \to \infty} (f_{tt})^{\pm 2} = -\frac{\mu^2 r^2}{2},
\]
\[
\lim_{r \to \infty} (f_{rr})^{\pm 2} = -\frac{\mu^2 r^2}{14},
\] (61)
\[
\lim_{r \to \infty} (f_{tr})^{\pm 3} = -\frac{\mu^2 r^2}{2},
\]
\[
\lim_{r \to \infty} (f_{rr})^{\pm 4} = -\frac{\mu^2 r^2}{14}.
\] (62)

We now turn our attention to the vector perturbations. Since \( D_i V_i = 0 \) we have
\[
D_i h_{ai} = D_i h_{ai}^S = r \sum_{l,m} f_a S_{lm}^m,
\] (63)
where we have used \( D_i D_i S = -k^2 S \). Explicit computation gives us the boundary conditions for the scalar derived vector coefficients,
\[
\lim_{r \to \infty} (f_t)^{2} = -\frac{\mu^2 r^2}{7},
\]
\[
\lim_{r \to \infty} (f_t)^{3} = 2\mu^2 r^2, \quad \lim_{r \to \infty} (f_t)^{4} = \frac{5\mu^2 r^2}{14}.
\] (64)

To find the pure vector coefficients we write
\[
h_{ai}^V = h_{ai} - h_{ai}^S = h_{ai} + r \sum_{l,m} f_a D_i S_{lm}^m = r \sum_{l,m} f_a^V (V_{lm}^m)_i.
\] (65)

Again, by explicit computation we find,
\[
\lim_{r \to \infty} (f_t^V)^{2} = \frac{\mu^2 r^2}{3}, \quad \lim_{r \to \infty} (f_t^V)^{3} = \frac{\mu^2 r^2}{6}.
\] (66)

Finally we consider the tensor perturbations. We can write
\[
h_i^i = (h^S)_i^i = 4r^2 \sum_{l,m} H_L S_{lm}^m,
\] (67)
where we have used \( D_i V_i = 0, T_i^i = 0, S_i^i = 0 \) and \( \gamma_i^i = 2 \). This allows us to easily show that
\[
\lim_{r \to \infty} (H_L)_i^{\pm 2} = -\frac{3\mu^2 r^2}{28}, \quad \lim_{r \to \infty} (H_L)_i^{\pm 2} = \frac{\mu^2 r^2}{56}.
\] (68)
To find the scalar derived transverse modes we will need the following results,

\[ D^iD^jV_{ij} = 0, \]  
\[ D^iD^jS_{ij} = \frac{(k^2 - 2)}{2} S, \]

which are proved in an appendix. Using the above results along with \( D^iT_{ij} = 0 \), we find

\[ D^iD^jh_{ij} = D^iD^j h^S_{ij} = 2r^2 \sum_{l,m} (-k^2 H_L S^m_l + H_T D^j D^i S_{ij}) \]
\[ = 2r^2 \sum_{l,m} (-k^2 H_T + H_T \frac{(k^2 - 2)}{2} S). \]

We can now show that

\[ \lim_{r \to \infty} (H_T)_{\pm} = -\frac{\mu^2 r^2}{7}, \quad \lim_{r \to \infty} (H_T)^{-2} = -\frac{5\mu^2 r^2}{12 \cdot 7}. \]  

To find the vector derived transverse modes we will use the identities

\[ D^iS_{ij} = -\frac{1}{2k^2}(k^2 - 2)D_jS, \]
\[ D^iV_{ij} = \frac{1}{2k^V}(k^2 V_j - 1)V_j, \]

which we also prove in an appendix. Since \( D^iT_{ij} = 0 \), we have

\[ D^ih_{ij} = D^i h^S_{ij} + D^i h^V_{ij}, \]

and using the results above we can write this as

\[ D^i h_{ij} = 2r^2 \sum_{l,m} (H_L - \frac{1}{2k^2}(k^2 - 2)H_T) D_j S + 2r^2 \sum_{l,m} H_T^V \frac{1}{2k^V}(k^2 V_j - 1)V_j. \]

We are now able to show that

\[ \lim_{r \to \infty} (H_T^V)_{\pm} = -\frac{11\mu^2 r^2}{12}. \]

Using Maple we find that \( h_{ij} = h^S_{ij} + h^V_{ij} \), so there are no pure tensor perturbations as expected.

We now want to translate this into boundary conditions for the gauge-invariant variables introduced in [23]. For vector perturbations the gauge-invariant variable is

\[ F^V_a = f^V_a + \frac{r}{k^V} D_a H^V_T. \]
For \( l = 2 \), \( \lim_{r \to \infty} (f^V_t)^{\pm2} = \frac{\mu^2 r^2}{2} \), so \( \lim_{r \to \infty} F_t = \frac{\mu^2 r^2}{4} \), and we have \( F_a = r^{-1} \epsilon_{ab} D^b(r\Phi) \) \[23\], so the boundary condition for the vector master function \( \Phi^V_{l=2} \) is

\[
\lim_{r \to \infty} \Phi^V_{l=2} = \frac{\mu^2 r^3}{12}.
\] (79)

For \( l = 3 \), \( \lim_{r \to \infty} (f^V_r)^{\pm2} = \frac{\mu^2 r^2}{6} \) and \( \lim_{r \to \infty} (H^V_T)^{\pm2} = -\frac{11 \mu^2 r^2}{12} \) so \( F_a = 0 \): this mode is pure gauge. This is as we might expect: the \( r^2 \) behaviour of the plane wave is typical of an \( l = 2 \) spherical harmonic, so the higher \( l \) modes that seem to appear in our decomposition of the mode in terms of spherical harmonics ought to be pure gauge.

For scalar perturbations, the gauge-invariant variables are \[23\]

\[
F = H_L + \frac{1}{2} H_T + \frac{1}{r} (D^a r) X_a
\] (80)

\[
F_{ab} = f_{ab} + D_a X_b + D_b X_a
\] (81)

with

\[
X_a = \frac{r}{k^2} (f_a + r D_a H_T).
\] (82)

The master variable \( \Phi \) is

\[
\Phi = \frac{2\tilde{Z} - r(X + Y)}{4},
\] (83)

with

\[
X = F^t - 2F
\] (84)

\[
Y = F^r - 2F
\] (85)

\[
\tilde{Z} = 0.
\] (86)

For \( l = 2 \) perturbations direct substitution gives us \( \lim_{r \to \infty} X = \mu^2 r^2, Y = 0, \tilde{Z} = 0 \), hence the boundary condition on \( \Phi \) is

\[
\lim_{r \to \infty} \Phi^S_{l=2} = -\frac{\mu^2 r^3}{4}.
\] (87)

For the \( l = 3 \) and \( l = 4 \) modes we find the gauge-invariant variables \( F \) and \( F_{ab} \) are zero, so these modes are pure gauge as expected. Thus, we are left with two non-trivial modes, the \( l = 2 \) scalar and the \( l = 2 \) vector modes.

Having established which modes are non-zero and their boundary conditions we consider the bulk solution. For the vector mode the equation for the master field is \[23\]

\[
\partial_r( (1 - \frac{2M}{r}) \partial_r \Phi) - \frac{1}{r^2} [l(l+1) - 3 \cdot \frac{2M}{r^2}] \Phi = 0.
\] (88)

The boundary condition is \( \lim_{r \to \infty} \Phi^V_{l=2} = \frac{\mu^2 r^3}{12} \), therefore we set \( \Phi = r^3 \psi \). This allows us to reduce the master equation \[88\] to

\[
\partial_r(r^6(1 - \frac{2M}{r}) \partial_r \psi) = 0.
\] (89)
which has solution
\[
\psi = a \left( \frac{1}{8Mr^4} + \frac{1}{12M^2r^3} + \frac{1}{16M^3r^2} + \frac{1}{16M^4r} + \frac{1}{32M^5} \ln(1 - \frac{2M}{r}) \right) + b. \tag{90}
\]

Solutions with \( a \neq 0 \) are clearly not regular at \( r = 2M \), therefore the solution for the vector master field is \( \Phi^V = br^3 \). The boundary condition at large \( r \) then requires \( b = \frac{\mu^2}{12} \). However, the boundary condition at the horizon (49) requires that \( h_{tt} \) and \( h_{ti} \) vanish at the horizon. This implies that \( f_t^V \) and hence \( F_t^V \) also vanish at the horizon. Finally \( F^t = r^{-1}D_r(r\Phi) \) implies that \( \Phi \) too must vanish at the horizon, which would require \( b = 0 \). Hence, there is no solution which satisfies the boundary conditions at both the horizon and infinity.

Thus, there is no regular solution describing a four-dimensional black hole in the plane wave background (10). In fact, this is not a surprising result in four dimensions; the rigidity theorem [25] shows that regular black holes must be static or stationary axisymmetric, and the plane wave (10) is not static and does not preserve a \( U(1) \) symmetry. Thus, the plane wave perturbation breaks too many of the symmetries of the black hole for a regular deformed black hole solution to be possible.

One might hope to avoid this problem by considering a non-vacuum plane wave solution. We can for example consider in four dimensions the electromagnetic plane wave
\[
ds^2_{\text{wave}} = -dt^2 + dx^2 + dy^2 + dz^2 - \mu^2(x^2 + y^2)(dt + dz)^2 \tag{91}
\]
supported by the electric flux
\[
F = 2\mu(dt + dz) \wedge dx. \tag{92}
\]
This is also interesting as a simplified model of the maximally supersymmetric plane wave of [26]. Here, the metric perturbation preserves a \( U(1) \) symmetry, but this is broken by the gauge field, and as a result, we again do not expect to find a regular black hole solution. In this case, the problem is that the equation of motion for the gauge field on the Schwarzschild black hole background has no solution which is regular on the horizon and satisfies the boundary condition at large \( r \).

If we consider the situation in higher dimensions, the above rigidity argument does not apply, but there is still no regular solution. Take for example a six-dimensional Schwarzschild black hole and add as a perturbation the six-dimensional vacuum plane wave
\[
ds^2_{\text{wave}} = -dt^2 + dv^2 + dw^2 + dx^2 + dy^2 + dz^2 - \mu^2(v^2 + w^2 - x^2 - y^2)(dt + dz)^2. \tag{93}
\]
This clearly preserves two \( U(1) \) isometries, in the \( x - y \) and \( v - w \) planes. However, if we rewrite this in spherical polars, there is again an \( l = 2 \) vector part to the perturbation in the decomposition into spherical harmonics. The analysis is very similar to the above four-dimensional case, and it is not possible to find a solution for the vector part of the perturbation that satisfies the plane wave boundary conditions at large distances and the regularity condition on the event horizon. In this case, the plane wave preserves two \( U(1) \) isometries on the \( S^4 \) surrounding the black hole, so
the above argument does not apply; a regular deformed black hole solution would not violate the conditions of [27]. This problem seems to be very general. In all cases we have explored in the vacuum Einstein equations, the plane wave has a vector part in the spherical harmonic decomposition, and it is not possible to find a regular perturbation of the black hole which satisfies the plane wave boundary condition. It would be interesting to understand the physical origins of this restriction further.

3.2 Black string

We next study the near horizon region of the black string, treating the plane wave as a perturbation. The background is the five-dimensional black string solution

\[ ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) + dz^2, \]

(94)

with \( f(r) = 1 - 2M/r \). We want to find a solution of the source-free linearised vacuum equations on this background which asymptotically approaches the five-dimensional plane wave \([28]\). This implies that we want a perturbation \( h_{\mu\nu} \) with asymptotic boundary conditions

\[
\lim_{r \to \infty} h_{\mu\nu} dx^\mu dx^\nu = -\mu^2 r^2 [\alpha(1 - 3 \cos^2 \theta) + \beta \sin^2 \theta (\cos^2 \phi - \sin^2 \phi)](dt + dz)^2 + \ldots, \tag{95}
\]

where the \( \ldots \) denotes terms going like \( \mu^2 M^n \) for \( n \neq 0 \). These terms are suppressed relative to the leading term because dimensional analysis tells us the mass will always appear in the combination \( M/r \).

As in the analysis in the intermediate region, we will deal with the \( \alpha \) and \( \beta \) components separately. It will turn out that the analysis is identical in these two cases. In terms of the spherical harmonic analysis on the two-sphere, these are scalar-type perturbations, which excite the \( l = 2, m = 0 \) and \( l = 2, m = 2 \) harmonic modes respectively. In the linearised theory, we can assume that the perturbation has only these modes turned on. Since only scalar-type modes are excited, the analysis on the sphere is fairly simple, and we will follow the similar analysis by Emparan et al [13].

The boundary conditions, and hence the perturbation, are invariant under simultaneously taking \( t \to -t \), \( z \to -z \), so the only modes we need to consider are \( h_{tt}, h_{tz}, h_{zz}, h_{rr}, \) and the longitudinal and transverse scalar-derived perturbations on the sphere.

We first consider only the \( l = 2, m = 0 \) perturbation (we set \( \beta = 0 \)). Assuming that only this spherical harmonic is excited, we can write the perturbation as

\[
h_{tt} = \alpha(1 - 3 \cos^2 \theta)a(r), \quad h_{tz} = \alpha(1 - 3 \cos^2 \theta)b(r), \quad h_{zz} = \alpha(1 - 3 \cos^2 \theta)c(r), \tag{96}
\]

\[
h_{rr} = \alpha \frac{(1 - 3 \cos^2 \theta)}{(1 - 2M/r)} f(r), \tag{97}
\]

\[
h_{\theta\theta} = \alpha r^2 [(1 - 3 \cos^2 \theta)g(r) - 3 \sin^2 \theta h(r)], \tag{98}
\]

\[
h_{\phi\phi} = \alpha r^2 \sin^2 \theta [(1 - 3 \cos^2 \theta)g(r) + 3 \sin^2 \theta h(r)]. \tag{99}
\]
Note that \( g(r) \) is the coefficient of the longitudinal mode on the sphere, and \( h(r) \) is the coefficient of the transverse mode on the sphere. As in [13], there is a remaining coordinate freedom, under

\[
  r \rightarrow r + \gamma(r)(1 - 3 \cos^2 \theta), \quad \theta \rightarrow \theta + 6\beta(r) \cos \theta \sin \theta, \quad (100)
\]

with

\[
  \beta'(r) = -\frac{\gamma(r)}{r(r - 2M)}, \quad \gamma(2M) = 0. \quad (101)
\]

Similarly, for the \( l = 2, m = 2 \) perturbation (obtained by setting \( \alpha = 0 \)), we define

\[
  h_{tt} = \beta \sin^2 \theta (\cos^2 \phi - \sin^2 \phi) a(r), \quad h_{tz} = \beta \sin^2 \theta (\cos^2 \phi - \sin^2 \phi) b(r), \quad (102)
\]

\[
  h_{zz} = \beta \sin^2 \theta (\cos^2 \phi - \sin^2 \phi) c(r), \quad (103)
\]

\[
  h_{rr} = \beta \frac{\sin^2 \theta (\cos^2 \phi - \sin^2 \phi)}{(1 - 2M/r)} f(r), \quad (104)
\]

\[
  h_{\theta\phi} = \beta r^2 \sin \theta \cos \phi \sin \phi h(r), \quad (105)
\]

\[
  h_{\theta\theta} = \beta r^2 [\sin^2 \theta (\cos^2 \phi - \sin^2 \phi) g(r) - (\cos^2 \theta + 1)(\cos^2 \phi - \sin^2 \phi) h(r)], \quad (106)
\]

\[
  h_{\phi\phi} = \beta r^2 \sin^2 \theta [\sin^2 \theta (\cos^2 \phi - \sin^2 \phi) g(r) + (\cos^2 \theta + 1)(\cos^2 \phi - \sin^2 \phi) h(r)]. \quad (107)
\]

Now we have remaining coordinate freedom under

\[
  r \rightarrow r + \gamma(r) \sin^2 \theta (\cos^2 \phi - \sin^2 \phi), \quad (108)
\]

\[
  \theta \rightarrow \theta + 2\beta(r) \sin \theta \cos \theta (\cos^2 \phi - \sin^2 \phi), \quad (109)
\]

\[
  \phi \rightarrow \phi - 4\beta(r) \sin^2 \theta \cos \phi \sin \phi, \quad (110)
\]

with

\[
  \beta'(r) = -\frac{\gamma(r)}{r(r - 2M)}, \quad \gamma(2M) = 0. \quad (111)
\]

We find both coordinate transformations produce identical shifts

\[
  a(r) \rightarrow a(r) - \frac{2M}{r^2} \gamma(r), \quad f(r) \rightarrow f(r) + \left(2\gamma' - \frac{2M}{r} \frac{\gamma(r)}{r - 2M}\right), \quad (112)
\]

\[
  g(r) \rightarrow g(r) + \frac{2}{r} \gamma(r) - 6\beta(r), \quad h(r) \rightarrow h(r) + 2\beta(r), \quad (113)
\]

while \( b(r) \) and \( c(r) \) are unchanged.

We want to consider combinations which are invariant under these coordinate transformations. \( B = b(r) \) and \( C = c(r) \) are already invariant. We define in addition

\[
  A = a(r) + \frac{M}{r} (g(r) + 3h(r)), \quad (114)
\]

\[
  F = f(r) - \frac{d}{dr} (r(g(r) + 3h(r))) + \frac{M(g(r) + 3h(r))}{r - 2M}, \quad (115)
\]
Note that in this section, primes denote derivatives with respect to \( r \). As in [13], the constant part of \( h(r) \) can be fixed using the constant part of \( \beta(r) \). Using the gauge-invariant combinations basically amounts to setting \( g(r) = -3h(r) \), which can be achieved for \( r \neq 2M \) by an appropriate choice of gauge. Because of the boundary condition in (111), \( g(2M) + 3h(2M) \) is gauge-invariant. It will however not be determined by solving the equations of motion for the above gauge-invariant variables, and will have to be separately specified. It will turn out to be determined by requiring regularity of the solution at the horizon.

For either \( \alpha = 0 \) or \( \beta = 0 \), substituting into the linearised Einstein equations gives the same system of equations for the unknown functions \( A, B, C, F, H' \) (keeping terms up to \( \mathcal{O}(\mu^2) \)),

\[
R_{tt}^{(1)} \propto r^2(r - 2M)^2 A'' + r(r - 2M)(2r - 5M)A' - M(r - 2M)^2 C' - (6r(r - 2M) - 2M^2)A + M(r - 2M)^2 F' + 6M(r - 2M)^2 H',
\]

\[
R_{iz}^{(1)} \propto r(r - 2M)B'' + 2(r - 2M)B' - 6B,
\]

\[
R_{zz}^{(1)} \propto r(r - 2M)C'' + 2(r - M)C' - 6C,
\]

\[
R_{rr}^{(1)} \propto r^2(r - 2M)^2 A'' - rM(r - 2M)A' + 2M(2r - 3M)A - r(r - 2M)^3 C'' - M(r - 2M)^2 C' + (2r - 3M)(r - 2M)^2 F' + 6(r - 2M)^2 F + 6r(r - 2M)^3 H'' + 6(2r - 3M)(r - 2M)^2 H',
\]

\[
R_{\theta\theta}^{(1)} \propto -r^2(r - 2M)A' + r(r - M)A + r(r - 2M)^2 C' - (r - 2M)^2 C - (r - 2M)(r - M)F - r(r - 2M)^2 H',
\]

\[
R_{\phi\phi}^{(1)} + \frac{1}{\sin^2 \theta} R_{\phi\phi}^{(1)} \propto r(r - 2M)A' - (3r + 2M)A - (r - 2M)^2 C' + 3(r - 2M)C
+ (r - 2M)^2 F' + 5(r - 2M)F + 3r(r - 2M)^2 H'' + 6(2r - 3M)(r - 2M)H',
\]

\[
R_{\theta\theta}^{(1)} - \frac{1}{\sin^2 \theta} R_{\phi\phi}^{(1)} \propto -rA + (r - 2M)C + (r - 2M)F + r(r - 2M)^2 H'' + 2(r - M)(r - 2M)H'.
\]

In fact it is easy to show that the linearised Einstein equations must be the same for both modes. The perturbation involves some \( l = 2 \) scalar harmonic, let’s call it \( S \), so

\[
h_{ab} = f_{ab}(r)S, \quad h_{ai} = f_i(r)\nabla_i S, \quad h_{ij} = f(r)Sg_{ij} + f'(r)\nabla_i \nabla_j S,
\]

where \( i, j \) are coordinates on the two-sphere and \( a, b = t, r, z \). Then the first order Ricci tensor constructed from the second covariant derivatives of \( h_{\mu\nu} \) will also depend on angular coordinates only through \( S \) and its derivatives. Using \( \nabla_i \nabla^i S = -6S \) and
the fact that the sphere is an Einstein space, so $R_{ij} = g_{ij}$, one can eliminate extra derivatives of $S$, to leave us with

$$
R_{ab}^{(1)} = \epsilon_{ab}(r)S, \quad R_{ai}^{(1)} = \epsilon_a(r)\nabla_i S, \quad R_{ij}^{(1)} = \epsilon(r)Sg_{ij} + \epsilon'(r)\nabla_i \nabla_j S.
$$

(125)

Hence, the resulting equations $\epsilon_{ab}(r) = \epsilon_a(r) = \epsilon(r) = \epsilon'(r) = 0$ are independent of whether $S$ is in the $m = 0$ or $m = 2$ mode. Thus, solving the equations (117-123) will give us the general solution for the perturbation in the near-horizon region for both modes.

The boundary conditions at large $r$ imply that at order $M^0$, $a(r), b(r), c(r) \to -\mu^2 r^2$, and $f(r), g(r), h(r)$ have no $\mu^2 M^0$ term. This implies that

$$
A, B, C \to -\mu^2 r^2,
$$

(126)

and $F$ and $H'$ have no $\mu^2 M^0$ term. Regularity at the horizon requires $a(r) \propto (r-2M)$, $b(r) \propto \sqrt{r-2M}$, and the other functions $c(r), f(r), g(r)$ and $h(r)$ are required to be finite there. In terms of the gauge-invariant combinations, these boundary conditions are best expressed in terms of the alternative combinations

$$
A = A - \frac{M}{r}(r-2M)H', \quad F = F - MH'.
$$

(127)

The conditions for regularity at the horizon are then that $\tilde{A} \to 0$, $\tilde{F}$ is finite, and $H'$ is allowed to diverge like $(r-2M)^{-1}$.

We now want to solve this system of equations. We see that there are two decoupled equations, (118) and (119). The solutions of these satisfying our boundary conditions are

$$
B(r) = -\mu^2(r-M)(r-2M)
$$

(128)

and

$$
C(r) = -\mu^2(r^2 - 2Mr + \frac{2}{3}M^2).
$$

(129)

It is also convenient to subtract a multiple of (119) from (120) to simplify it to

$$
0 = r^2(r-2M)^2A'' - rM(r-2M)A' + 2M(2r-3M)A
+(2r-5M)(r-2M)^2C' - 6(r-2M)^2C + (2r-3M)(r-2M)^2F'
+6(r-2M)^2F + 6r(r-2M)^3H'' + 6(2r-3M)(r-2M)^2H'.
$$

(130)

We first solve (123) for $A$,

$$
A = \frac{(r-2M)}{r}[C + F + r(r-2M)H'' + 2(r-2M)H'],
$$

(131)

and then solve $R_{tt}^{(1)} - (r-2M)^2R_{zz}^{(1)} - R_{rr}^{(1)}$ for $F$,

$$
F = \frac{1}{6} \left[ r(r-2M)^2H'' - 2(r-2M)(r+2M)H'' - 2(5r-7M)H' - MC' \right].
$$

(132)
The remaining equations then need to be solved for \( H' \). By combining equations, we can obtain a second-order inhomogeneous equation for \( H' \),

\[
-2r(r + M)(r - 2M)^2H''' - 2(4r^2 + 3rM - 4M^2)(r - 2M)H'' \\
+ 2(4r^2 - 13rM + 4M^2)H' = M[(r - 2M)C' + 6C].
\] (133)

It’s useful to note at this point that if \( M = 0 \), we have a solution with \( F = H' = 0 \) and \( A = C = -\mu^2 r^2 \), which is precisely our original plane wave.

The general solution of (133) is

\[
H' = \frac{\mu^2}{3}(r-M)+c_1\frac{r^2 - 2M^2}{r - 2M} + c_2\frac{[-6rM(r + M) + 4M^3 + (6rM^2 - 3R^3)\ln(1 - 2M/r)]}{r(r - 2M)}
\] (134)

This then satisfies all of the equations. To get a solution which is both regular and has the correct asymptotics, i.e. has \( A \to -\mu^2 r^2 \) at large \( r \), we need to take \( c_1 = -\frac{1}{3}\mu^2 \) and \( c_2 = 0 \). We find

\[
H' = -\frac{\mu^2 M}{3} \frac{3r - 4M}{r - 2M},
\] (135)

and

\[
A = -\mu^2 \left[ r^2 - 4rM + \frac{16}{3} M^2 - \frac{2 M^3}{r} \right], \quad F = \frac{2\mu^2 M}{3} \frac{3r^2 - 9rM + 5M^2}{r - 2M}.
\] (136)

In terms of the alternative combinations \( \bar{A}, \bar{F} \),

\[
\bar{A} = -\mu^2 (r - 2M) \left[ r - 2M + \frac{M^2}{3r} \right], \quad \bar{F} = \mu^2 M (2r - M).
\] (137)

Thus, this solution satisfies the regularity conditions at the horizon. Regularity of the original functions \( a(r), f(r), g(r), h(r) \) at \( r = 2M \) further requires us to choose

\[
g(2M) + 3h(2M) = -\frac{2\mu^2 M^2}{3}.
\] (138)

We now match the near horizon and intermediate region solutions in the intermediate region \( \mu^{-1} \gg r \gg M \), where both approximations are valid. The contribution from the black string background is

\[
ds_{NR,BG}^2 \approx -(1 - \frac{2M}{r})dt^2 + (1 + \frac{2M}{r})dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) + dz^2.
\] (139)

We must now find the unknown functions \( a(r), b(r), c(r), f(r), g(r), h(r) \) in this region to obtain the contribution from the perturbation. In addition to the solutions (128), (129) and (136) we must make a choice of gauge. We choose \( g + 3h = -M\mu^2 r \) in order to make the \( rr \)-component of the perturbation vanish, matching our gauge choice in the intermediate region solution. We find, keeping just the terms up \( \mathcal{O}(M) \) and \( \mathcal{O}(\mu^2) \),

\[
a(r) \approx -\mu^2 (r^2 - 4Mr), \quad b(r) \approx -\mu^2 (r^2 - 3Mr), \quad c(r) \approx -\mu^2 (r^2 - 2Mr),
\] (140)
\[ f(r) \approx 0, \quad g(r) \approx -M\mu^2 r, \quad h(r) \approx 0. \]  \hfill (141)

Hence the near region perturbation is

\[ ds^2_{\text{NR,P}} \approx (\alpha(1 - 3\cos^2 \theta) + \beta \sin^2 \theta(\cos^2 \phi - \sin^2 \phi)) \times \]
\[ (-\mu^2 r^2(dt + dz)^2 + M\mu^2 r(4dt^2 + 6dtdz + 2dz^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2))). \]  \hfill (142)

In the intermediate region the plane wave background is,

\[ ds^2_{\text{IR,BG}} = -dt^2 + dz^2 + dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \]
\[ -\mu^2 r^2(\alpha(1 - 3\cos^2 \theta) + \beta \sin^2 \theta(\cos^2 \phi - \sin^2 \phi))(dt + dz)^2. \]  \hfill (143)

From section [2.3] the perturbation due to the black string is

\[ ds^2_{\text{IR,P}} = \frac{2M}{r} dt^2 + \frac{2M}{r} dr^2 + M\mu^2 r(\alpha(1 - 3\cos^2 \theta) + \beta \sin^2 \theta(\cos^2 \phi - \sin^2 \phi)) \times \]
\[ (4dt^2 + 6dtdz + 2dz^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2)). \]  \hfill (144)

Thus the solution constructed in the near region \( ds^2_{\text{NR}} = ds^2_{\text{NR,BG}} + ds^2_{\text{NR,P}} \) agrees with the solution constructed in the intermediate region \( ds^2_{\text{IR}} = ds^2_{\text{IR,BG}} + ds^2_{\text{IR,P}} \) to the relevant order. This gives us an approximate solution describing a black string in a plane wave, valid when the size of the black string is small compared to the curvature scale of the wave, \( r_+ \ll \mu^{-1} \).

As in [13], the perturbation does not affect the thermodynamic properties of the black hole at this order. The area of the horizon cannot be affected at this order because the perturbation is entirely in an \( l = 2 \) mode, which deforms the shape of the \( S^2 \) but does not change its area. The temperature cannot be affected because it is constant over the horizon. Since the perturbation is an \( l = 2 \) mode, it will vanish at some point on the horizon, whence the temperature at that point must be unaffected, and since it is constant, it must be unchanged over the whole horizon.

4 Conclusions

In this paper, we have attempted to construct solutions describing black holes and black strings in plane wave backgrounds using the matched asymptotic expansion method. We have found that it is not possible to construct a regular black hole solution. In the approximation where the wave is thought of as a linearised perturbation on the black hole solution, we need a non-zero vector part in the spherical harmonic decomposition on the sphere, and it is not possible to make this vector part regular on the horizon. It would be interesting to have a deeper physical understanding of this failure of regularity. One might think that this is simply saying that the plane wave is exerting a force on the black hole, so no stationary solution exists. However, we do not believe this is the correct interpretation of our result. The black hole was chosen to follow a geodesic in the plane wave background, so there is no force on it at leading order. Finite size effects can be analysed in the asymptotic region using the classical effective field theory approach of [28, 29, 30]. In this approach, the work
done by such finite size terms involves derivatives of the long wavelength background fields along the black hole world-line. Since our world-line is chosen to be an orbit of the isometries of the background, the work done will vanish. Thus, we would have expected the background to simply produce some deformation of the horizon.

The regularity problem seems to be simply an inconsistency between the symmetry structure of the black hole and the plane wave. In four dimensions, the problem is that the solution will not be axisymmetric, so there cannot be a regular black hole solution as all stationary four-dimensional black holes are required to be axisymmetric [25]. In higher dimensions, however, stationary axisymmetric solutions describing black holes in plane waves could in principle exist, and the fact that our solutions are never regular is somewhat mysterious. Further exploration of this issue is an interesting project for the future.

The importance of this problem is reinforced by the fact that the failure of regularity here is a counter-example to the assumption in [16] that satisfying the blackfold equations implies horizon regularity. Understanding this issue in a more general context is clearly important for the blackfolds program [15, 16]; in considering the embedding of black branes in arbitrary backgrounds, we need to understand when the resulting deformation of the near-horizon region will preserve the regularity of the event horizon. Clearly we must require that the embedding of the blackfold in the background spacetime preserves enough symmetry to satisfy the rigidity theorems of [25, 27]. Our higher-dimensional examples indicate that this is a necessary but not a sufficient condition. Identifying sufficient conditions is an important general problem.

We successfully constructed an approximate solution describing a black string in an asymptotically vacuum plane wave background in 5 dimensions. It would clearly be interesting to extend this work to find black string solutions in backgrounds which asymptote to maximally supersymmetric plane waves. It should be straightforward to extend our calculation to this case.

Our analysis has also led to an interesting general result; the effect of localised objects in a plane wave background is not small, even far from the source. The usual $1/r^{d-1}$ falloff associated with a localised object in $d+1$ spatial dimensions is offset by the $\mu^2 r^2$ factors coming from the plane wave background. As a result, we find that the “perturbation” due to the source is larger than the background metric at sufficiently large $r$. This leads us to believe that these solutions should not be thought of as “asymptotically plane wave” spacetimes.

A definition of “asymptotically plane wave” was proposed in our earlier work [17], which allows the construction of a well-behaved action principle. This still seems a useful definition. However, from the present results it seems that the phase space associated with those boundary conditions will not include solutions describing localised sources in a plane wave background, so it may not admit many physically interesting solutions. Understanding the space of asymptotically plane wave spacetimes is clearly important for attempts to construct a direct holographic duality directly for plane waves, so we would like to understand this issue better.

Similar problems have arisen in AdS$_2$ spacetimes [18], where there are no finite-energy asymptotically AdS$_2$ geometries, and in the study of near-horizon extremal Kerr solutions (NHEK) [19, 20, 21], where the space of metrics which are asym-
totically NHEK consists only of the NHEK solution and solutions obtained from it by diffeomorphisms. It is interesting to note that the plane waves, like AdS$_2$, have a one-dimensional boundary [31, 32]. Perhaps the problem is that there is in some sense “not enough space” near infinity to have interesting asymptotically plane wave solutions. It would be interesting to carry out a general analysis for asymptotically plane wave solutions along the lines of that in [20, 21]. We leave this as a project for the future.

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A Appendix

In this appendix we prove some harmonic identities needed for our analysis of black holes in the near horizon region. Definitions are given in section 3.1. We want to show that:

- $D^i D^j V_{ij} = 0,$

Proof:

$$D^i D^j V_{ij} \propto D^i D^j D_i V_j + D^i D^j D_j V_i$$

$$= [D^i, D^j] D_i V_j + 2 D^j D^i D_i V_j$$

$$= - R^{k,ij} D_k V_j - R^{k,ij} D_i V_k - 2 k^2 D^j V_j$$

$$= R^{kij} D_k V_j - R^{kij} D_i V_k$$

$$= 0.$$

- $D^i D^j S_{ij} = \left(\frac{k^2 - 2}{2}\right) S,$

Proof:

$$D^i D^j S_{ij} = \frac{1}{k^2} D^i D^j D_i D_j S + \frac{1}{2} D^i D_j S$$

$$= \frac{1}{k^2} D^i [D^j, D_i] D_j S + \frac{1}{k^2} D^i D_i D^j D_j S + \frac{1}{2} D^j D_j S$$

$$= - \frac{1}{k^2} D^i (R^{k,ij} S) + \frac{k^2}{2} S$$

$$= \frac{1}{k^2} D^i (R^{k,ij} D_k S) + \frac{k^2}{2} S$$

for $S^2, R_{ij} = \gamma_{ij}$ so

$$D^i D^j S_{ij} = \frac{1}{k^2} D^i D_i S + \frac{k^2}{2} S$$

$$= \left(\frac{k^2 - 2}{2}\right) S.$$
• $D^i S_{ij} = -\frac{1}{2k^2}(k^2 - 2) D_j S,$

Proof:

$$D^i S_{ij} = \frac{1}{k^2} D^i D_i D_j S + \frac{1}{2} D_j S$$  \hspace{1cm} (148)

$$= \frac{1}{k^2} [D^i, D_j] D_i S + \frac{1}{k^2} D_j D^i D_i S + \frac{1}{2} D_j S$$

$$= -\frac{1}{k^2} R_{i j}^i D_l S - \frac{1}{2} D_j S$$

$$= -\frac{1}{2k^2}(k^2 - 2) D_j S$$

• $D^i V_{ij} = \frac{1}{2k^2}(k^2 - 1) V_j,$

Proof:

$$D^i V_{ij} = -\frac{1}{2k^2}(D^i D_i V_j + D^i D_j V_i)$$  \hspace{1cm} (149)

$$= \frac{1}{2} V_j - \frac{1}{2k^2} [D^i, D_j] V_i - \frac{1}{2k^2} D_j D^i V_i$$

$$= \frac{1}{2} V_j + \frac{1}{2k^2} R_{i j}^k V_k$$

$$= \frac{1}{2k^2}(k^2 - 1) V_j.$$

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