Subset Space Public Announcement Logic
Revisited

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Abstract
By a modification of the central definition given in [1], we provide semantics for public announcements in subset spaces. We argue that these revised semantics improve on the original, and we provide a simple sound and complete axiomatization of the resulting logic.

1 Introduction
The purpose of this note is to offer an interpretation of public announcements in subset structures that differs in a small but important way from the account given by Wáng and Agotnes (hereafter WA) in [1], from which the primary inspiration for this work is drawn.

1.1 Review
To begin, let us review some of the key notions introduced in [1]. The language of (single-agent) public announcement logic, denoted $\mathcal{PAL}$, is recursively generated by

$$\varphi ::= p | \neg \varphi | \varphi \land \psi | K\varphi | [\varphi]\psi,$$

where $p \in \text{PROP}$, the set of primitive propositions. We also denote by $\mathcal{EL}$ the purely epistemic fragment of $\mathcal{PAL}$ obtained by omitting those formulas involving public announcements.

WA observe that the interpretation of the formula $[\varphi]\psi$ in public announcement logic is of the general form

$$\mathcal{M}, m \models [\varphi]\psi \iff \mathcal{M}, m \models \text{pre}(\varphi) \Rightarrow \mathcal{M}', m \models \psi,$$

where $\text{pre}(\varphi)$ is the precondition for announcing $\varphi$, while $\mathcal{M}'$ is the result of updating the model $\mathcal{M}$ based on the (successful) public announcement of $\varphi$. 

Classically, $\text{pre}(\varphi)$ is simply $\varphi$ itself and $\mathfrak{M}'$ is formed by an update procedure that deletes non-$\varphi$ worlds from $\mathfrak{M}$. However, WA point out that subset structures have a model-internal mechanism for representing states of knowledge that obtain “after some effort”, and it would be nice to utilize this mechanism to obtain a representation of $\mathfrak{M}'$ not based on update semantics, that is, without altering the model itself.

Formally, a subset space is a pair $(X, O)$ where $X \neq \emptyset$ and $O \subseteq 2^X$, and a subset model is a subset space equipped with a function $v : \text{PROP} \to 2^X$ specifying the truth sets of the primitive propositions in the usual way. Truth of formulas is evaluated with respect to pairs of the form $(x, O)$ where $x \in O \in O$; such pairs are called epistemic scenarios. The set $O$ in an epistemic scenario $(x, O)$ is called the epistemic range; it functions like an information set in the sense that knowledge statements at $(x, O)$ are evaluated by quantification over $O$. More precisely, given a subset model $\mathcal{X} = (X, O, v)$, we can interpret $\mathcal{E} \mathcal{L}$ in $\mathcal{X}$ as follows:

\[
\begin{align*}
\mathcal{X}, x, O \models p & \iff x \in v(p) \\
\mathcal{X}, x, O \models \neg \varphi & \iff \mathcal{X}, x, O \not\models \varphi \\
\mathcal{X}, x, O \models \varphi \land \psi & \iff \mathcal{X}, x, O \models \varphi \text{ and } \mathcal{X}, x, O \models \psi \\
\mathcal{X}, x, O \models K\varphi & \iff (\forall y \in O)(\mathcal{X}, y, O \models \varphi).
\end{align*}
\]

Typically, subset models interpret a richer language containing a second modality representing “effort” which, intuitively, works by shrinking the epistemic range. We return to this point in Section 3.

We are interested in interpreting $\mathcal{PACL}$ in subset space semantics. Given a subset model $\mathcal{X} = (X, O, v)$, WA propose to define the “updated model” $\mathfrak{M}'$ of $\{1\}$ that arises in the interpretation of public announcements by simply restricting the epistemic range:

\[
\mathcal{X}, x, O \models [\varphi]\psi \iff \mathcal{X}, x, O \models \text{pre}(\varphi) \Rightarrow \mathcal{X}, x, (\downarrow \varphi)O \models \psi,
\]

where

\[
(\downarrow \varphi)O := \{y \in O : \mathcal{X}, y, O \models \varphi\},
\]

called the truth set of $\varphi$ under $O$. Of course, in order for $\mathcal{X}, x, (\downarrow \varphi)O \models \psi$ to make sense, we must have $x \in (\downarrow \varphi)O \in O$, and this is precisely what is taken to be the precondition for the announcement:

\[
\mathcal{X}, x, O \models \text{pre}(\varphi) \iff x \in (\downarrow \varphi)O \in O.
\]

These semantics represent public announcements without the need to consider alternative models, thus elegantly realizing the central insight of WA in [1]. Clearly, this interpretation of $\text{pre}(\varphi)$ strengthens the classical precondition (that $\varphi$ merely be true at the state in question). However, we will see that this precondition is, in a sense, too strong, and moreover that the “updated model” thereby produced is, in a sense, too weak.
1.2 Motivation

For $\text{pre}(\varphi)$ to hold in an epistemic scenario $(x, O)$ two conditions must be satisfied. First, $x$ must be in $\langle \varphi \rangle^O$, which amounts simply to the subset space analogue of the classical precondition, namely that $(x, O) \models \varphi$. Second, we must have $\langle \varphi \rangle^O \in O$. Informally speaking, this condition has nothing to do with the state $x$ itself, but rather is concerned with the overall “shape” of $\varphi$ under $O$.

Consider, for example, Figure 1. Think of $X$ as a subset of the Euclidean plane, and let $O$ consist of those subsets of $X$ that are open in the inherited topology. This yields the subset space $(X, O)$. Dashed lines in this diagram represent open sets; in particular, we are interested in the epistemic scenario $(x, O)$.

First suppose that $\varphi$ is some formula such that $\langle \varphi \rangle^O = U$, where $U$ denotes the shaded region. Since $U \in O$, we can deduce that $(x, O) \models \text{pre}(\varphi)$. On the other hand, if $\varphi$ also happened to be true (under $O$) at the point $y$, so that $\langle \varphi \rangle^O = U \cup \{y\}$, then since $U \cup \{y\} \notin O$ we must conclude that $\varphi$ cannot be announced at $x$; that is, $(x, O) \not\models \text{pre}(\varphi)$. Thus, although the precondition for announcing $\varphi$ is in some sense locally satisfied at $x$ (by the set $U$), it can be ruined by non-local properties of $\varphi$, such as whether or not $\varphi$ holds at the point $y$. This makes $\text{pre}(\varphi)$ a strong requirement indeed.

For a more concrete example, consider Figure 2. Once again, we view $X$ as a subset of the plane to form the subset space $(X, O)$. Think of $X$ now as a room into which you have launched a probe, but you don’t know exactly where the probe has landed. There is a target in the room as well as a wall. From certain vantage points, the wall blocks the target; $A$ denotes the set of points where
this is so. Notice that $A$ is a \textit{closed} subset of $X$. We might think of $A$ as the truth set of a primitive proposition $a$ that says “the wall is blocking the target”.

Now suppose that the probe in fact landed at the point $x$, but you consider it possible that it is anywhere in the room. This corresponds to the epistemic scenario $(x, X)$. At the point $x$, the wall is in fact blocking the target. However, according to the theory of public announcements we have reviewed, this fact cannot be announced. Despite the fact that $a$ is true at $x$, and that $a$ could even be known with sufficient effort\footnote{For example, effort resulting in restricting the epistemic range to the open set $U$. Perhaps the probe can return measurements of its distance from the south and east walls, albeit with some margin of error.}, we must have $(x, X) \not\prec \pre(a)$ simply because $(a)^X = A \notin O$.

But once again, intuitively, the fact that $A$ is not open is not “local” to the point $x$, in the sense that there exists a neighbourhood about $x$ (e.g. $U$) on which $A$ “looks” open. The same cannot be said, for example, about the point $y$. All this can be formalized quite succinctly: $x$ lies in the \textit{interior} of $A$, while $y$ does not.

Before moving on to the formal details of the revised subset space semantics for public announcements given in Section 2.1, we consider one final example of a rather different sort. Consider the following scenario. You have learned (from ancient historical records) of the existence of a secret tomb within which was supposedly enshrined a priceless jewel. Now, you have no idea whether a priceless jewel was, in fact, placed within this tomb before it was sealed—perhaps that part of the historical record was simply an embellishment. You are also unsure as to whether this tomb is still lost or has been rediscovered in
modern times and its contents catalogued. So we can represent your uncertainty with a four state model \( X = \{s_{JD}, s_{J\bar{D}}, s_{\bar{J}D}, s_{\bar{J}\bar{D}}\} \) encoding whether the tomb actually contains a jewel \((J)\) or not \((\bar{J})\), and whether it has been rediscovered in modern times \((D)\) or not \((\bar{D})\).

Crucially, however, we would also like this model to encode the fact that the only way to learn about the jewel is to discover the tomb (all other records of the jewel’s existence, or the lack thereof, having been irrevocably lost to time). As such, we define the set of legal information sets \( \mathcal{O} \subseteq 2^X \) with this in mind. You could conceivably know whether or not the tomb has been discovered in modern times, and, if you know it has been, you might also know whether or not a priceless jewel was found inside. But that’s all:

\[
\mathcal{O} := \{X, \{s_{JD}, s_{J\bar{D}}\}, \{s_{\bar{J}D}, s_{\bar{J}\bar{D}}\}, \{s_{JD}\}, \{s_{\bar{J}D}\}, \emptyset\}.
\]

Let \( j \) and \( d \) be primitive propositions standing for “the jewel is in the tomb” and “the tomb has been discovered”, respectively, and let \( v : \{j,d\} \to 2^X \) be defined in the obvious way. Then it is easy to see that for all epistemic scenarios \((x, O)\), we have \((x, O) \not\models \text{pre}(\neg j \land \neg d)\), simply because \(\{s_{JD}\} \notin \mathcal{O}\). Indeed, this corresponds to the intuition that one could not in principle know that the jewel is not in the tomb without also knowing that the tomb has been discovered (and that the jewel was not inside!). In particular, such a state of affairs should not be announceable.

However, it is also easy to see that \((s_{JD}, X) \not\models \text{pre}(j)\); this follows from the fact that \(\{s_{JD}, s_{J\bar{D}}\} \notin \mathcal{O}\). Thus, “the jewel is in the tomb” is not announceable. This seems wrong—rather than denying the announceability of \(j\) altogether, intuitively, an announcement of \(j\) should carry with it the implication that the tomb has indeed been discovered. In other words, we should expect \([j]d\) to be valid. The semantics WA provide cannot give credence to this intuition: announcements act by restricting the epistemic range to the truth set of the announced formula, and so no inference beyond the truth of that formula (and all the logical consequences thereof) is supported.

## 2 Revised semantics

### 2.1 Definition and intuition

Given a topological space \(\mathcal{X} = (X, \mathcal{O})\), let \(\text{int}_X : 2^X \to \mathcal{O}\) denote the topological interior operator; namely, \(\text{int}_X(A)\) returns the largest open set contained in \(A\). To ease notational clutter, we often drop the subscript and sometimes omit the parentheses.

\(^2\)A subset space \((X, \mathcal{O})\) is a topological space provided \(\mathcal{O}\) is closed under unions and finite intersections.
The central intuition of this paper is that the precondition for an announcement should not depend on the “global shape” (i.e. the truth set) of the announced formula, but rather on its “local shape”. This intuition is borne out in the following reformulated semantics for public announcements:

\[ \mathcal{X}, x, O \models \lbrack \varphi \rbrack \psi \text{ iff } \mathcal{X}, x, O \models \text{int}(\varphi) \Rightarrow \mathcal{X}, x, \text{int}(\lbrack \varphi \rbrack)_O \models \psi, \]

where \( \mathcal{X}, x, O \models \text{int}(\varphi) \text{ iff } x \in \text{int}(\lbrack \varphi \rbrack)_O. \)

To distinguish subset semantics for \( \mathcal{PA}_L \) as presented in \([1]\) and outlined in Section \([1]\) from that given here, we refer to them as “pre-subset semantics” and “int-subset semantics”, respectively. It is easy to see that \( \text{int}(\varphi) \) is a strictly weaker condition than \( \text{pre}(\varphi) \) and a strictly stronger condition than \( \varphi. \)

Returning to the example of Figure \([1]\) we can see that \( \text{int}(\varphi) \) holds at \( x \) (and not at \( y \)) regardless of whether \( (\lbrack \varphi \rbrack)_O \) is equal to \( U \) or to \( U \cup \{y\} \). Similarly, in the target and wall example, it is easy to see that \( \varphi \) can be announced at \( x \), though not at \( y \). These observations are direct consequences of the fact that the precondition for announcing \( \varphi \) in the epistemic scenario \((x, O)\) depends now only on the local behaviour of the set \((\lbrack \varphi \rbrack)_O\).

Furthermore, int-subset semantics not only weakens the precondition for an announcement, but also narrows the resulting epistemic range; an announcement of \( \varphi \) in the epistemic scenario \((x, O)\) has the effect of restricting the epistemic range to \( \text{int}(\lbrack \varphi \rbrack)_O \) rather than just \((\lbrack \varphi \rbrack)_O\). Thus, in int-subset semantics, but not pre-subset semantics, the “updated” epistemic range of a successful announcement of \( \varphi \) may be strictly smaller than the truth set of \( \varphi. \)

This allows a representation of situations where, intuitively, an announcement conveys more information than just the announced formula. Consider again the example of the jewel in the tomb. As with pre-subset semantics, int-subset semantics determines that \( \neg j \wedge \neg d \) is not announceable in any epistemic scenario; this follows from the fact that \( \text{int}(\{s_{JD}\}) = \emptyset \). By contrast, \( j \) is announceable in the epistemic scenario \((s_{JD}, X)\) because \( \text{int}(\{s_{JD}, s_{JD}\}) = \{s_{JD}\} \); moreover, it is easy to see that \((s_{JD}, X) \models \lbrack j \rbrack_d \) for precisely the same reason.

In an arbitrary subset space \((X, O)\), of course, there is no guarantee that for a subset \( A \subset X \) there is a largest set \( O \in O \) contained in \( A \). For the interior operator to be well-defined, we must make this assumption, which amounts to assuming that \((X, O)\) is a topological space (provided we want the interior operator to behave nicely). Henceforth, whenever we consider int-subset semantics, we implicitly assume that the subset models are in fact topological spaces.

2.2 Technical results

As in \([1]\), it is straightforward to check that \( \text{int}(\varphi) \) is definable in \( \mathcal{PA}_L \) by \( \neg \lbrack \varphi \rbrack \bot \). The properties of a modalized interior operator have been thoroughly
investigated (see, e.g., [2]), so much of the following proposition should come as little surprise.

**Proposition 2.1:** For all $\varphi, \psi \in \mathcal{PAC}$, the following hold:

(a) $\models \text{int}(\varphi) \rightarrow \varphi$
(b) $\models \text{int}(\varphi) \rightarrow \text{int}(\text{int}(\varphi))$
(c) $\models \text{int}(\varphi \rightarrow \psi) \rightarrow (\text{int}(\varphi) \rightarrow \text{int}(\psi))$
(d) $\models \varphi \implies \models \text{int}(\varphi)$
(e) $\models K\varphi \rightarrow \text{int}(\varphi)$
(f) $\not\models \text{int}(\varphi) \rightarrow K(\varphi \rightarrow \text{int}(\varphi))$
(g) $\not\models \neg(\varphi \rightarrow \text{int}(\varphi)) \rightarrow K\neg\text{int}(\varphi)$.

**Proof:** (a) through (d) constitute a standard $S4$ axiomatization of the interior operator, and the proof that they hold in this setting is analogous to the classical proof. (e) follows from the fact that $\mathcal{X}, x, O \models K\varphi \Rightarrow (\models \text{int}(\varphi))^O = O \Rightarrow \text{int}(\models \varphi)^O = O$.

(f) and (g) are included to exhibit some of the differences between $\text{pre}(\varphi)$ (for which these two schemas are valid) and $\text{int}(\varphi)$. A counterexample demonstrating both (f) and (g) can be found in Figure 1; take $(\models \varphi)^O = U \cup \{y\}$ and consider the epistemic scenarios $(x, O)$ and $(y, O)$, respectively.

Like $\text{pre}$-subset semantics, $\text{int}$-subset semantics does not collapse $\mathcal{PAC}$ and $\mathcal{EL}$.

**Theorem 2.2:** $\mathcal{PAC}$ is strictly more expressive than $\mathcal{EL}$ in $\text{int}$-subset semantics.

**Proof:** As in [1, Theorem 10], it suffices to show that $\text{int}(p)$ can distinguish two subset models that are partially bisimilar. Consider two subset models

$\mathcal{X}' = (\{x, y\}, 2^\{x,y\}, v)$

and

$\mathcal{Y}' = (\{x, y\}, 2^\{x,y\} - \{x\}, v)$

with $v(p) = \{x\}$. Thus $\mathcal{X}'$ is a discrete space while in $\mathcal{Y}'$ the singleton set $\{y\}$ is open but $\{x\}$ is not. The relation

$(x, \{x, y\}) \models p (x, \{x, y\})$

$(y, \{x, y\}) \models p (y, \{x, y\})$
Proof: The first four equivalences are straightforward to prove. To see that
\[ \varphi \Leftrightarrow \psi \]
where the last line follows from the fact that
\[ \text{Theorem 2.3:} \]
The following \( \mathcal{PAC} \)-formulas are valid (where \( p \in \text{PROP} \)):

\[
\begin{align*}
[\varphi] \bot & \iff \lnot \text{int}(\varphi) \\
[\varphi] p & \iff \text{int}(\varphi) \rightarrow p \\
[\varphi] \lnot \psi & \iff \text{int}(\varphi) \rightarrow \lnot [\varphi] \psi \\
[\varphi] \psi & \iff \text{int}(\varphi) \rightarrow [\varphi] \psi
\end{align*}
\]

Hence, \( \text{int}(p) \) cannot be equivalent to any \( \mathcal{E}\mathcal{L} \)-formula. ■

We also obtain an analogue to the reduction of [11, 11] Theorem 111.

**Theorem 2.3:** The following \( \mathcal{PAC} \)-formulas are valid (where \( p \in \text{PROP} \)):

\[
\begin{align*}
[\varphi] \bot & \iff \lnot \text{int}(\varphi) \\
[\varphi] p & \iff \text{int}(\varphi) \rightarrow p \\
[\varphi] \lnot \psi & \iff \text{int}(\varphi) \rightarrow \lnot [\varphi] \psi \\
[\varphi] \psi & \iff \text{int}(\varphi) \rightarrow [\varphi] \psi
\end{align*}
\]

**Proof:** The first four equivalences are straightforward to prove. To see that
\[ [\varphi] K \psi \iff (\text{int}(\varphi) \rightarrow [\varphi] \psi) \] valid, first note that if \( x, O \not\models \text{int}(\varphi) \) then this equivalence holds trivially at \( (x, O) \). Otherwise, assuming that \( x, O \models \text{int}(\varphi) \), we have:

\[ x, O \models [\varphi] K \psi \iff x, \text{int}(\varphi)^O \models \psi \]
\[ \iff (\forall y \in \text{int}(\varphi)^O)(x, \text{int}(\varphi)^O \models \psi), \]

whereas

\[ x, O \models \text{int}(\varphi) \rightarrow K[\varphi] \psi \iff x, O \models K[\varphi] \psi \]
\[ \iff (\forall z \in O)(z, O \models [\varphi] \psi) \]
\[ \iff (\forall z \in O)(z \in \text{int}(\varphi)^O \Rightarrow z, \text{int}(\varphi)^O \models \psi) \]
\[ \iff (\forall z \in \text{int}(\varphi)^O)(z, \text{int}(\varphi)^O \models \psi). \]

To see that \( [\varphi] \psi \chi \iff [\text{int}(\varphi) \wedge [\varphi] \text{int}(\psi)] \chi \) is valid, first observe that

\[ x, O \models [\varphi] \psi \chi \iff x \in \text{int}(\varphi)^O \Rightarrow x, \text{int}(\varphi)^O \models [\psi] \chi \]
\[ \iff x \in \text{int}(\varphi)^O \Rightarrow (x \in \text{int}(\psi) \text{int}(\varphi)^O \Rightarrow x, \text{int}(\psi) \text{int}(\varphi)^O \models \chi) \]
\[ \iff x \in \text{int}(\psi) \text{int}(\varphi)^O \Rightarrow x, \text{int}(\psi) \text{int}(\varphi)^O \models \chi, \]

where the last line follows from the fact that

\[ \text{int}(\psi) \text{int}(\varphi)^O \subseteq (\psi) \text{int}(\varphi)^O \subseteq \text{int}(\varphi)^O. \]

On the other hand, \( x, O \models [\text{int}(\varphi) \wedge [\varphi] \text{int}(\psi)] \chi \) iff

\[ x \in \text{int}(\psi) \text{int}(\varphi)^O \Rightarrow x, \text{int}(\psi) \text{int}(\varphi)^O \models \chi; \]
thus, to complete the proof it suffices to show that
\[ \text{int}(\text{int}(\varphi) \land [\varphi]\text{int}(\psi))^{O} = \text{int}(\psi)^{\text{int}(\varphi)^{O}}. \] (3)

By definition,
\[
\begin{align*}
(\text{int}(\varphi) \land [\varphi]\text{int}(\psi))^{O} & = \{ y \in O : y, O \vDash \text{int}(\varphi) \land [\varphi]\text{int}(\psi) \} \\
& = \{ y \in O : y \in \text{int}(\varphi)^{O} \land (y \in \text{int}(\varphi)^{O} \Rightarrow y, \text{int}(\varphi)^{O} \vDash \text{int}(\psi)) \} \\
& = \{ y \in \text{int}(\varphi)^{O} : y, \text{int}(\varphi)^{O} \vDash \text{int}(\psi) \} \\
& = \{ y \in \text{int}(\varphi)^{O} : y \in \text{int}(\psi)^{\text{int}(\varphi)^{O}} \} \\
& = \text{int}(\psi)^{\text{int}(\varphi)^{O}},
\end{align*}
\]
where the third line follows from the fact that $\text{int}(\varphi)^{O} \subseteq O$, and the last line follows from (2). Since $\text{int}^{2} = \text{int}$, this establishes (3).

Finally, we turn our attention to a sound and complete axiomatization of $\mathcal{PAC}$ in int-subset semantics. As in [1], we first focus on a restricted language and then extend our results to $\mathcal{PAC}$ by means of Theorem 2.3.

Let $\mathcal{L}_{\text{int}}$ be the language recursively generated by the grammar
\[
\varphi ::= p \mid \neg \varphi \mid \varphi \land \varphi \mid K \varphi \mid \text{int}(\varphi),
\]
where $p \in \text{PROP}$. Let CPL denote the axioms and rules of classical propositional logic, let $\text{S4}_{\text{int}}$ denote the S4 axioms and rules for the int modality, and let $\text{KD45}_{K}$ denote the KD45 axioms and rules for the $K$ modality. Let (KI) denote the axiom scheme $K \varphi \rightarrow \text{int}(\varphi)$, and set
\[
\mathcal{L}_{\text{int}} := \text{CPL} + \text{S4}_{\text{int}} + \text{KD45}_{K} + (\text{KI}).
\]

**Theorem 2.4:** $\mathcal{L}_{\text{int}}$ is a sound and complete axiomatization of $\mathcal{L}_{\text{int}}$ with respect to int-subset semantics.

**Proof:** Soundness of $\text{CPL} + \text{KD45}_{K}$ is easy to show in the usual way, while soundness of $\text{S4}_{\text{int}} + (K \varphi \rightarrow \text{int}(\varphi))$ follows from Proposition 2.1.

Completeness can be proved by a relatively straightforward canonical model construction. Let $X$ denote the set of all maximal consistent subsets of $\mathcal{L}_{\text{int}}$. Define a relation $\sim$ on $X$ by
\[
x \sim y \iff (\forall \varphi \in \mathcal{L}_{\text{int}})(K \varphi \in x \iff K \varphi \in y).
\]
Clearly $\sim$ is an equivalence relation: let $[x]$ denote the equivalence class of $x$ under $\sim$. These equivalence classes partition $X$ according to what is known, but we cannot simply take the set of epistemic ranges to be $\{[x] : x \in X\}$,
since we require this set to be a topology on $X$ and to interact with the int modality in the appropriate way, that is, to support its interpretation as the interior operator. So we need to do a bit more work to define $O$.

Given $\varphi \in \mathcal{EL}_{\text{int}}$, let $\widehat{\varphi} := \{x \in X : \varphi \in x\}$. Roughly speaking, the collection

$$ \{\text{int}(\varphi) : \varphi \in \mathcal{EL}_{\text{int}} \} $$

are the sets that int “thinks are open”. Thus, in order to respect both int and $K$, we define

$$ B := \{\text{int}(\varphi) \cap [x] : \varphi \in \mathcal{EL}_{\text{int}} \text{ and } x \in X \}, $$

and let $O$ be the topology generated by $B$. In fact, it is not difficult to show (using $S4_{\text{int}}$) that $B$ is a basis for $O$.

Finally, for $p \in \text{PROP}$, set $v(p) := \widehat{p}$. Let $\mathcal{X} = (X, O, v)$.

**Lemma 2.5:** For every $\varphi \in \mathcal{EL}_{\text{int}}$, for all $x \in X$, $\varphi \in x$ iff $\mathcal{X}, x, [x] \models \varphi$.

**Proof:** First we note that $\text{int}(\top) = X$, and thus for all $x \in X$ we have $[x] \in O$, so $(x, [x])$ is indeed an epistemic scenario in $\mathcal{X}$.

As usual, the proof proceeds by induction on the complexity of $\varphi$. The base case holds by definition of $v$, and the inductive steps for the Boolean connectives are straightforward.

Suppose the result holds for $\varphi$; we first show that it also holds for $K\varphi$. If $K\varphi \in x$, then by definition of $\sim$ we know that $(\forall y \in [x])(K\varphi \in y)$. But $K\varphi \in y \Rightarrow \varphi \in y$, so $(\forall y \in [x])(\varphi \in y)$, which by the inductive hypothesis implies that $(\forall y \in [x])(y, [y] \models \varphi)$. Since $[y] = [x]$, this is equivalent to $(\forall y \in [x])(y, [x] \models \varphi)$, which yields $(x, [x] \models K\varphi$.

For the converse, suppose that $K\varphi \notin x$. Then $\{K\psi : K\psi \in x\} \cup \{\neg \varphi \}$ is consistent, for if not there is a finite subset $\Gamma \subseteq \{K\psi : K\psi \in x\}$ such that

$$ \vdash \bigwedge_{\chi \in \Gamma} \chi \rightarrow \varphi, $$

from which it follows (using $\text{KD45}_K$) that

$$ \vdash \bigwedge_{\chi \in \Gamma} \chi \rightarrow K\varphi, $$

which implies $K\varphi \in x$, a contradiction. Therefore, we can extend $\{K\psi : K\psi \in x\} \cup \{\neg \varphi \}$ to some $y \in X$; by construction, we have $y \in [x]$ and $\varphi \notin y$. This latter fact, by the inductive hypothesis, yields $(y, [y]) \not\models \varphi$ and thus $(y, [x]) \not\models \varphi$ (since $[x] = [y]$), whence $(x, [x]) \not\models K\varphi$.

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3$B$ is a basis for topology $O$ if every element of $O$ is a union of elements of $B$. 

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Suppose again that the result holds for $\varphi$; we show now that it also holds for $\text{int}(\varphi)$. If $\text{int}(\varphi) \in x$, then observe that
\[
x \in \widehat{\text{int}(\varphi)} \cap [x] \subseteq \{ y \in [x] : \varphi \in y \};
\]
this is an easy consequence of the fact that $\vdash \text{int}(\varphi) \rightarrow \varphi$. Since $\widehat{\text{int}(\varphi)} \cap [x]$ is open, it follows that
\[
x \in \text{int}(\{ y \in [x] : \varphi \in y \}).
\]
Now by the inductive hypothesis we have
\[
\{ y \in [x] : \varphi \in y \} = \{ y \in [x] : (y, [y]) \models \varphi \} = \{ y \in [x] : (y, [x]) \models \varphi \} = (\varphi)[x],
\]
which by (4) yields $x \in \text{int}((\varphi)[x])$, hence $(x, [x]) \models \varphi$.

For the converse, suppose that $(x, [x]) \models \text{int}(\varphi)$. Then $x \in \text{int}((\varphi)[x])$ which, as above, is equivalent to $x \in \text{int}(\{ y \in [x] : \varphi \in y \})$. It follows that there is some basic open set $\widehat{\text{int}(\psi)} \cap [z]$ such that
\[
x \in \widehat{\text{int}(\psi)} \cap [z] \subseteq \{ y \in [x] : \varphi \in y \};
\]
of course, in this case it must be that $[z] = [x]$. This implies that for all $y \in [x]$, if $\text{int}(\psi) \in y$ then $\varphi \in y$. From this we can deduce that
\[
\{ K\psi' : K\psi' \in x \} \vdash \text{int}(\psi) \rightarrow \varphi,
\]
for if not, then $\{ K\psi' : K\psi' \in x \} \cup \{ \neg (\text{int}(\psi) \rightarrow \varphi) \}$ is consistent, in which case it can be extended to a $y \in [x]$ with $\text{int}(\psi) \in y$ but $\varphi \notin y$, a contradiction.

By (5), we can find a finite subset $\Gamma \subseteq \{ K\psi' : K\psi' \in x \}$ such that
\[
\vdash \bigwedge_{\chi \in \Gamma} \chi \rightarrow (\text{int}(\psi) \rightarrow \varphi),
\]
which implies (using KD45) that
\[
\vdash \bigwedge_{\chi \in \Gamma} \chi \rightarrow K(\text{int}(\psi) \rightarrow \varphi).
\]

Thus $K(\text{int}(\psi) \rightarrow \varphi) \in x$, so by (KI) we know also that $\text{int}(\text{int}(\psi) \rightarrow \varphi) \in x$, from which it follows (using S4int) that $\text{int}(\psi) \rightarrow \text{int}(\varphi) \in x$. Since $x \in \widehat{\text{int}(\psi)}$, we conclude that $\text{int}(\varphi) \in x$, as desired.

Completeness, of course, is an easy consequence: if $\varphi$ is not a theorem of $\text{EL}_{\text{int}}$, then $\{ \neg \varphi \}$ is consistent and can be extended to some $x \in X$, in which case by Lemma 2.5 we have $X, x, [x] \not\models \varphi$. \[\square\]
Let \( \text{PAL}_{\text{int}} \) denote \( \text{EL}_{\text{int}} \) together with the six reduction schemas given in Theorem 2.3. That these reduction axioms are not circular follows directly from [1, Proposition 23].

**Corollary 2.6:** \( \text{PAL}_{\text{int}} \) is a sound and complete axiomatization of \( \text{PAL} \) with respect to int-subset semantics.

### 3 Discussion

By an appeal to topology, we obtain an interpretation of public announcements in subset spaces that is less restrictive than the account provided by WA in [1]. In particular, more formulas are announceable, and as we have seen by example, some of these formulas really ought to be announceable.

This framework also allows us to capture epistemic implications of announcements that are not themselves encoded as logical tautologies. Subset spaces already encode such implications in the set \( \mathcal{O} \) of legal epistemic ranges; the presence of the interior operator in the definition of a post-announcement update is precisely what allows these implications to affect post-announcement truth.

It is interesting to note that the interior operator in subset spaces plays a role very similar to the notion of “knowability”. Let \( \text{SSL} \) be the language recursively generated by the grammar

\[
\phi ::= p \mid \neg \phi \mid \phi \land \phi \mid K\phi \mid \diamond \phi,
\]

where \( p \in \text{PROP} \). The diamond modality is interpreted as the “effort” alluded to in Section 1.1:

\[
\mathcal{X}, x, O \models \diamond \phi \text{ iff } (\exists O' \in \mathcal{O})(x \in O' \subseteq O \text{ and } \mathcal{X}, x, O' \models \phi).
\]

In particular, the formula \( \diamond K\phi \) can be read “\( \phi \) is knowable”, and the corresponding semantic operation is similar to taking the interior. An investigation of the interaction between \( \diamond K \) and \( \text{int} \) in the context of interpreting public announcements in subset spaces is of interest in this regard, and the topic of current research. It is worth noting that the canonical model constructed in Theorem 2.4, which trivializes the Boolean structure of \( \mathcal{O} \), will not work in the context of a modality like \( \diamond \).

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