DC Josephson effect in superconductor-quantum dot-superconductor junctions

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A quantum dot weakly coupled to two normal metal leads exhibits resonant transmission at energies equal to the energy levels of the isolated dot. A nonequilibrium current can be driven in a transport channel connected to superconducting reservoirs by a Josephson phase difference between the reservoirs. We would like to see the combined effect of the above two phenomena and study transport across a quantum dot weakly connected to two finite superconductors maintained at a phase difference. The quantum dot consists of two sites and can be connected to the superconductors in two geometrical configurations: (A) one where both the sites are in the transport channel and (B) the other where only one site is in the transport channel and second site sidecoupled. Both the configurations show resonant transmission for Josephson current and give qualitatively same result when the onsite energies of the two sites in the dot are equal. We show that the two configurations can be distinguished if the onsite energies of the two sites are equal in magnitude and opposite in sign.

I. INTRODUCTION

In 1962, Josephson made the phenomenal prediction that across two superconductors maintained at different phases separated by a thin insulating layer, a DC current proportional to the sine of the phase difference should pass. The Josephson current between two superconductors is carried by Andreev bound states whose energy lies within the superconducting gap. The setup in point contact geometry can be modeled by a one dimensional continuum (lattice) model with a tunnel barrier (hopping) between the two superconductors. A nonequilibrium current can also be driven across a channel by application of a bias or by application of time dependent potentials in the channel. Quantum dots are the smallest mesoscale systems where a current can be driven by the application of a bias. But driving a current across a quantum dot in a system where a quantum dot is coupled to two superconducting reservoirs maintained at two different phases is relatively less studied. Motivated by this, we undertake the study of a simple noninteracting quantum dot coupled to two superconductors maintained at a phase difference. Coulomb interactions are important in quantum dots sites with both the spins, but a carbon nanotube quantum dot where interactions can be neglected have been coupled to superconducting leads experimentally.

In this work, we study different configurations of quantum dot consisting of two sites of each spin connected to two superconductors. We model the superconductors on a finite lattice connected to the quantum dot sites and diagonalize the entire Hamiltonian numerically. From the eigenstates, we calculate the Josephson current. The dot can be arranged so that (i) both the sites are in the transport channel (configuration A) or (ii) only one site of the dot is in the transport channel coupled to both the reservoirs, while the second site of the dot is coupled to just the first site of the dot (configuration B) as shown in Fig. 1. We first study the dependence of Josephson current on superconducting phase difference $\phi$ and onsite energy $\epsilon_d$ on both sites of the dot for the two configurations. Since the two configurations give qualitatively similar results, we further tweak the two configurations by applying a differential onsite energies $\pm \epsilon_d$ on the two sites. This tweaking shows a qualitative difference in the results for the two configurations. These results on Josephson currents in different configurations are then compared with the zero bias conductance of the dot versus the dot onsite energy $\epsilon_d$ in order to understand the results.

The paper is organized as follows. The section “Model and Calculations” describes the models for the two configurations and outlines the calculations performed. The following section presents the results and the analysis. This is followed by a section that puts together a summary of the work.

II. MODEL AND CALCULATIONS

We study two configurations of the system shown in Fig. 1. The Hamiltonians $H_{A/B}$ for configurations $A/B$
can be written as:

\[ H_A = H_L + H_R + H_D + H_{TA}, \]
\[ H_B = H_L + H_R + H_D + H_{TB}, \]
\[
H_L = -t \sum_{n=1}^{N_S+1} (c_{n-1}^\dagger \tau_z c_n + \text{h.c.})
- \sum_{n=1}^{N_S} c_n^\dagger \left[ -\mu \tau_z + \Delta \cos \left( \frac{i\phi}{2} \right) \tau_x + \Delta \sin \left( \frac{i\phi}{2} \right) \tau_y \right] c_n,
\]
\[ H_R = -t \sum_{n=1}^{N_S} (c_{n+1}^\dagger \tau_z c_n + \text{h.c.})
+ \sum_{n=1}^{N_S} c_n^\dagger \left[ -\mu \tau_z + \Delta \cos \left( \frac{i\phi}{2} \right) \tau_x - \Delta \sin \left( \frac{i\phi}{2} \right) \tau_y \right] c_n,
\]
\[ H_D = \epsilon_a c_{0a}^\dagger \tau_z c_{0a} + \epsilon_b c_{0b}^\dagger \tau_z c_{0b} - t_d (c_{0a}^\dagger \tau_z c_{0b} + \text{h.c.}),
\]
\[ H_{TA} = -t' (c_{-1}^\dagger \tau_z c_{0a} + c_{1}^\dagger \tau_z c_{0b} + \text{h.c.}),
\]
\[ H_{TB} = -t' (c_{-1}^\dagger \tau_z c_{0a} + c_{1}^\dagger \tau_z c_{0b} + \text{h.c.}). \]

Here, \( H_L \) and \( H_R \) are the Hamiltonians for the superconductors on left side and right side of the dot respectively. Each superconductor has \( N_S \) sites and is modeled by meanfield BdG Hamiltonian. The left (right) superconductor has a superconducting phase \( \phi/2 \) (\( -\phi/2 \)), maintaining a phase difference of \( \phi \) between the two. \( c_n = [c_{n,\uparrow}, -c_{n,\downarrow}, c_{n,\uparrow}^\dagger, c_{n,\downarrow}^\dagger]^T \), where \( c_{n,\sigma} \) is annihilation operator at site \( n \) with spin \( \sigma \). \( \tau_x,y,z \) are the Pauli matrices acting in the particle-hole sector. The quantum dot has two sites labeled by \( 0a \) and \( 0b \). The two sites have onsite energies \( \epsilon_{a,b} \) and are connected by a hopping amplitude \( t_d \). In configuration-A, the dot site \( 0a \) is connected to the left superconductor and the dot site \( 0b \) is connected to the right superconductor, while in configuration-B, both the superconductors are connected to the dot site \( 0a \) as can be seen from Eq. (1) and Fig. 1.

Charge current is not conserved in the superconductors as the Hamiltonians \( H_{L/R} \) do not commute with charge operator. Physically, the Josephson current that flows in the superconductor comes from the electron reservoir (which is not explicitly shown in the Hamiltonian) connected to the superconductor to maintain its chemical potential. But the charge is conserved in the quantum dot and the charge current between the superconductor and quantum dot is the Josephson current. The eigenfunctions of the Hamiltonian Eq. (1) are four-spinors at each site having the form \( [\psi_{i,e,n,\uparrow}, \psi_{i,e,n,\downarrow}, \psi_{i,h,n,\uparrow}, \psi_{i,h,n,\downarrow}]^T \), where \( i \) is the index denoting different eigenfunctions, the index \( e/h \) denotes electron/hole component and \( n \) is the site index. The eigenspectrum is centered around zero energy. All states below zero energy are filled and all states above zero energy are empty. The Josephson current calculated at the bond \(-1-0a\) is given by the formula:

\[ I(\phi) = \frac{2e t'}{h} \sum_{E_i<0} \sum_{p=e,h} \sum_{\sigma=\uparrow,\downarrow} \text{Im}[\psi^*_{i,p,-1,\sigma} \psi_{i,p,0a,\sigma}] \]

III. RESULTS AND ANALYSIS

For demonstrating our primary results, we choose the superconductors of size \( N_S = 10 \). We maintain a superconducting pair potential of magnitude \( \Delta = 0.1t \), where \( t \) is the hopping strength in the superconductor. We would like the quantum dot to be weakly coupled to the superconductors. So, we choose \( t' = 0.1t \). The hopping between the two sites within the quantum dot and the chemical potential in the superconductor take values \( t_d = 0.1t \) and \( \mu = 0 \) respectively. First we take the onsite potentials on two sites of the dot to be equal \( \epsilon_a = \epsilon_b = \epsilon_d \) and plot the Josephson current \( I(\phi) \) as a
function of the phase difference $\phi$ and $\epsilon_d$ for the two configurations in Fig. 2. The plots look similar for the two configurations except that the peak in Josephson current takes a value of about $3 \times 10^{-2} e \tau / h$ for configuration A and a value of about $8 \times 10^{-2} e \tau / h$ for configuration B. The peaks in Josephson current occur at $\epsilon_d = \pm t_d$. This is because, the isolated dot hosts zero energy states when $\epsilon_a = \epsilon_b = \epsilon_d = \pm t_d$. Hence the peaks in $I(\phi)$ as a function of $\epsilon_d$ is a resonance effect. The plot of $I(\phi)$ versus $\phi$ for $\epsilon_d = t_d$ is shown in Fig. 3 for the configurations A and B.

Now, we tweak the two configurations by choosing the onsite potentials on the two sites of the dot to be opposite in sign, but equal in magnitude (i.e., $\epsilon_a = -\epsilon_b = -\epsilon_d$). The Josephson currents for this are plotted in Fig. 4. We see a striking contrast between results for configuration A and configuration B. The maximum variation of Josephson current happens for $\epsilon_d = 0$ for configuration A, while the same happens for $\epsilon_d = \pm t_d$ for configuration B. Further, the peaks in the Josephson current versus $\epsilon_d$ are not sharp as in the case $\epsilon_a = \epsilon_b = \epsilon_d$. Furthermore, the peaks in the Josephson current $I(\phi)$ versus $\phi$ occur at $\phi = \pi/2$ as can be seen in Fig. 2 which is very much unlike the case $\epsilon_a = \epsilon_b = \epsilon_d$ where the peak occurs close to $\phi = \pi$ (see Fig. 3). In the case $\epsilon_a = -\epsilon_b = -\epsilon_d$, the energy levels of the isolated dot are at $\pm \sqrt{\epsilon_d^2 + t_d^2}$. So, as $|\epsilon_d|$ increases from zero, the dot energy levels move far away from the Fermi energy. Hence, the Josephson current has peaks when $\epsilon_d = 0$, for configuration A though the magnitude of the peak current is order of magnitude smaller than that for the case $\epsilon_a = \epsilon_b = \epsilon_d$. The result for configuration B is however puzzling. The puzzle can be resolved if we look at zero-energy transmission probability of the dot attached to normal metal leads on either sides in configuration B as a function of $\epsilon_d$ when $\epsilon_a = -\epsilon_b = -\epsilon_d$. The zero energy transmission probability is given by the expression

$$T(E = 0) = \frac{4t'^4 \epsilon_d^2}{4t'^4 \epsilon_d^2 + t'^2 (\epsilon_d^2 + t_d^2)^2}. \quad (3)$$

In arriving at the above expression, the leads on either sides of the dot are taken to be semi-infinite one dimensional lattice with chemical potential $\mu = 0$ and hopping amplitude $t$. When $T(E = 0)$ is plotted versus $\epsilon_d$ for the same parameters as in Fig. 4 it shows not-so-sharp peaks at $\epsilon_d = \pm t_d$ as can be seen in Fig. 5. These peaks in $T(E = 0)$ at $\epsilon_d = \pm t_d$ imply a maximum variation of $I(\phi)$ at $\epsilon_d = \pm t_d$ in Fig. 4.

We find that all our results on the Josephson current remain qualitatively same in the limit of large sized superconductors (i.e., $1/N_S \to 0$).
IV. SUMMARY

To summarize, we have studied a lattice model of two superconductors maintained at a phase difference coupled to each other through a quantum dot which has two energy levels for each spin. The two superconductors can be coupled to each other through the dot in two geometrical ways which we term as configurations. We first study the case when onsite energies of both the energy levels of the dot are equal. We examine at the dependence of the Josephson current as a function of the onsite energy of the dot and the phase difference. We find resonance effect in Josephson current results in both the configurations and the results in the two configurations are very similar, except for the values of the current. We then tweak the two configurations by applying onsite energies on the two sites of the dot that are equal in magnitude and opposite in sign. We find that the two configurations give different results for the Josephson current. Hence, we show that it is possible to distinguish the two configurations by choosing the onsite energy profile appropriately. We understand our results in terms of resonant transmission effect and the expression for transmission probability of the dot coupled to normal metal leads. Our results are within the reach of current experiments where onsite energies of the two sites of the dot can be controlled by applying separate gate voltages to the two sites.

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