A note on Hawking radiation via complex path analysis

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Abstract
As long as we neglect backreaction, the Hawking temperature of a given black hole would not depend upon the parameters of the particle species we are considering. In the semiclassical complex path analysis approach of Hawking radiation, this has been verified by taking scalar and Dirac spinors separately for different stationary spacetime metrics. Here we show in a coordinate-independent way that for an arbitrary spacetime with any number of dimensions, the equations of motion for a Dirac spinor, a vector, spin-2 and spin-$\frac{3}{2}$ fields reduce to Klein–Gordon equations in the WKB semiclassical limit. We then obtain, under some suitable assumptions, the complex solutions of those resulting scalar equations across the Killing horizon of a stationary spacetime to get a coordinate-independent expression for the emission probability identical for all particle species. Finally we consider some explicit examples to demonstrate the validity of that expression.

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1. Introduction
The semiclassical tunneling method [1–7] is an alternative approach to model particle creation by black holes [8]. The basic scheme of this method is to compute the imaginary part of the ‘particle’ action which gives the emission probability from the event horizon. From the expression of the emission probability one identifies the temperature of the radiation. The earliest works in this context can be found in [1, 2]. Following these works an approach called the null geodesic method was developed [3, 4]. There also exists another way to model black hole evaporation via tunneling called complex path analysis [5–7] which we discuss here. This method involves writing down, in the semiclassical limit $\hbar \to 0$, a Hamilton–Jacobi equation from the matter equations of motion, treating the horizon as a singularity in the complex plane.
(which is a simple pole for all known solutions) and then complex integrating the equation across that singularity to obtain an imaginary contribution for the particle action.

Both these two alternative approaches have received great attention during the last few years. It is noteworthy that since both these methods deal only with the near-horizon geometry, they can be very useful alternatives particularly when the spacetime has no well-defined asymptotic structure or infinities [9].

As far as we neglect the backreaction of the matter we are considering, the temperature of the radiation or the Hawking temperature would not depend upon the parameters (e.g. mass, spin and charge) of the particle species. The Smarr formula for black hole mechanics predicts that this temperature is proportional to the surface gravity of the event horizon for a stationary black hole with a Killing horizon.

The complex path analysis approach has been successfully applied to scalar emissions as well as to spinor emissions separately for a wide class of stationary black holes giving the expected expressions of Hawking temperatures that were predicted by the Smarr formula. To tackle the Dirac equation in this approach the usual method has been employed, i.e. finding a proper representation of the general \( \gamma \) matrices in terms of the Minkowskian \( \gamma \)'s and the metric functions and then making the variable separation. For an exhaustive review and list of references on this, see e.g. [10]. See also e.g. [11–20] for some recent issues concerning the tunneling approach.

Thus, the universality of the Hawking temperature has been proved case by case for a wide variety of black holes via the complex path method. Can we prove this universality from a more general point of view?

In particular, in this paper we show that for the Dirac spinors we do not need to work with any particular representation of the \( \gamma \) matrices in the semiclassical framework. In this work we point out, in a coordinate-independent way, that in any arbitrary spacetime with any number of dimensions, the equations of motion for a Dirac spinor, a vector, spin-2 meson and spin-\( \frac{3}{2} \) fields reduce to Klein–Gordon equations in the semiclassical limit \( \hbar \to 0 \) for the usual WKB ansatz. The equations for a charged Dirac spinor reduce to that of a charged scalar. This clearly shows that at the semiclassical level all those different equations of motion of various particle species are equivalent and it is sufficient to deal with the scalar equation only. We also present, for a stationary spacetime with some assumed geometrical properties, a general coordinate-independent expression for the emission probability and the Hawking temperature, which is characterized by the black hole parameters itself (equation (25)). We further consider some explicit examples to demonstrate that our formula indeed gives the expected Hawking temperature in terms of the horizon’s surface gravity.

Thus the semiclassical complex path method gives us a way in which we may treat the different spin fields on an identical footing, giving the same Hawking temperature and thereby proving the universality of the Hawking temperature for stationary black holes from a very general point of view.

The paper is organized as follows. In the next section we deal with Dirac spinors (neutral and then charged) to show that the equations reduce to those of scalars in the semiclassical limit for the WKB ansatz. In section 3, we explicitly expand the resultant scalar equation in a coordinate-independent way in the near-horizon limit for a stationary black hole with a Killing horizon and present a general expression that gives the emission or absorption probabilities. We illustrate the validity of this expression by taking a few explicit examples. In section 4, we also demonstrate that similar results also hold for the vector, massive spin-2 and spin-\( \frac{3}{2} \) fields. Finally we discuss our results.

We take \( G = 1 = c \), but retain \( \hbar \) throughout.
2. Reduction of the semiclassical Dirac equation into the Klein–Gordon equation

Let us then start by considering a spacetime of dimension \( n \) and a metric \( g_{ab} \) defined on it, at least in our region of interest. We consider the Dirac equation

\[
\left( i \gamma^a \nabla_a - \frac{m}{\hbar} \right) \Psi = 0.
\]

Here \( \nabla_a \) is the spin-covariant derivative defined by \( \nabla_a \Psi := (\partial_a + \Gamma_a) \Psi \), where \( \Gamma_a \) are the spin connection matrices. The matrices \( \gamma^a(x) \) are the curved space generalization of the Minkowskian \( \gamma^\mu \). We expand \( \gamma^a \) in an orthonormal basis, \( \gamma^a = \gamma^\mu e^a_\mu \), where \( e_\mu^a \) are the orthonormal basis vectors. The matrices \( \gamma^a(x) \) satisfy the well-known anti-commutation relation:

\[
\{ \gamma^a(x), \gamma^b(x) \} = 2 \eta^{ab} \mathbf{I},
\]

where \( \mathbf{I} \) denotes the identity matrix.

Now we square equation (1) by acting with \( i \gamma^b \nabla_b \) on both sides from the left, producing

\[
\frac{1}{2} (\gamma^b \gamma^a + \gamma^a \gamma^b) \nabla_b \nabla_a \Psi + \frac{1}{4} (\gamma^b \gamma^a - \gamma^a \gamma^b) (\nabla_b \nabla_a - \nabla_a \nabla_b) \Psi + (\gamma^b \nabla_b \gamma^a) \nabla_a \Psi = -\frac{m^2}{\hbar^2} \Psi.
\]

But the commutator of two covariant derivatives acting on \( \Psi \) is proportional to the Riemann tensor \( \gamma^b \gamma^a - \gamma^a \gamma^b \nabla_b \nabla_a - \nabla_a \nabla_b) \Psi = (\gamma^a \gamma^b - \gamma^b \gamma^a) R_{abcd} (\gamma^c \gamma^d - \gamma^d \gamma^c) \Psi. \)

Using this fact and the anti-commutation relation for \( \gamma^a \) (equation (2)), equation (3) becomes

\[
\nabla_a \nabla_a \Psi + \frac{1}{4} (\gamma^a, \gamma^b) R_{abcd} (\gamma^c, \gamma^d) \Psi + (\gamma^b \nabla_b \gamma^a) \nabla_a \Psi = -\frac{m^2}{\hbar^2} \Psi.
\]

We look at equation (4) semiclassically. We choose the usual WKB ansatz for a spin-’up’ particle

\[
\Psi = \begin{bmatrix} A(x) \\ 0 \\ B(x) \end{bmatrix} e^{i\frac{m}{\hbar} \frac{1}{2} \int A(x) dx},
\]

and substitute into equation (4). Since we are neglecting backreaction, the components of the Riemann tensor are independent of \( \hbar \). Then it is clear that in the semiclassical limit \( \hbar \to 0 \), on the left-hand side only the first term survives because only this one contains some double derivatives of \( \mathcal{O}(\hbar^{-2}) \). The single-derivative terms coming from the Laplacian will certainly not survive in the semiclassical limit (which is also true for an actual scalar equation), but we will formally keep the Laplacian \( \nabla_a \nabla_a \Psi \) intact till later when we discuss its expansion explicitly. Thus in the semiclassical limit, the WKB ansatz (5) implies that equation (4) can be effectively represented by two Klein–Gordon equations for spin-’up’ particles

\[
\nabla_a \nabla_a \Psi + \frac{m^2}{\hbar^2} \Psi = 0.
\]

A similar result holds for a spin-’down’ particle also.

If we consider a Dirac particle with a charge \( e \) coupled to a gauge field \( A_a \), the spin-covariant derivative \( \nabla_a \) in equation (1) is replaced by the gauge-covariant derivative \( \nabla_a \equiv \nabla_a - \frac{e}{\hbar} A_a \) such that the equation of motion becomes

\[
\left( i \gamma^a \nabla_a - \frac{m}{\hbar} \right) \Psi = 0.
\]
We now apply from the left \((i\gamma^b \nabla_b + \xi \gamma^b A_b)\) on both sides of this equation. Using equations (2) and (4) we obtain
\[
\nabla_a \nabla^a \Psi + \frac{1}{4} [\gamma^a, \gamma^b] R_{abcd} [\gamma^c, \gamma^d] \Psi + (\gamma^b \nabla_b \gamma^a) \nabla_a \Psi - \frac{e^2}{\hbar^2} A_b A^b \Psi + \frac{2ie}{\hbar} A^a \nabla_a \Psi \\
- \frac{ie}{\hbar} \left( (\gamma^b \nabla_b \gamma^a) A_a + \frac{1}{4} [\gamma^a, \gamma^b] F_{ab} + (\nabla_a A^b) \right) \Psi = - \frac{m^2}{\hbar^2} \Psi,
\]
where \(F_{ab} = \nabla_a A_b - \nabla_b A_a\). We now substitute the ansatz (equation (5)) into equation (8) and take the semiclassical limit \(\hbar \to 0\). We see that in this limit equation (8) can formally be represented by
\[
\nabla_a \nabla^a \Psi - \frac{e^2}{\hbar^2} A_b A^b \Psi + \frac{2ie}{\hbar} A^a \nabla_a \Psi + \frac{m^2}{\hbar^2} \Psi = 0,
\]
each of which effectively has the form of the equation of motion of a charged scalar.

What have we seen so far? We have dealt with neutral and charged Dirac spinors and have explicitly shown in a coordinate-independent way that for the semiclassical WKB ansatz all those equations of motion are equivalent to those of scalars in any arbitrary spacetime of dimension \(n\). So it is clear that the single-particle Hawking radiation will be identical for Dirac spinors and scalars for any given black hole.

We also show explicitly in section 4 that similar conclusions hold for Proca, massive spin-\(\frac{3}{2}\) and spin-\(\frac{1}{2}\) fields. But before that we discuss the explicit expansions and the near-horizon limits of equations (6) and (9) in a stationary spacetime containing a black hole. We address only the charged Dirac spinor (or equivalently, charged scalar, equation (9)). The other case will be equivalent to setting \(e = 0\) in equation (9).

3. Hawking temperature for a stationary black hole with a Killing horizon

We present in the following a general coordinate-independent expression for the emission or absorption probability from a stationary black hole with some assumed geometrical properties. Let us first list the definitions and assumptions we make.

We consider an \(n\)-dimensional stationary spacetime containing a black hole with a Killing horizon \(\mathcal{H}\). We assume that the spacetime can be foliated into a family of hypersurfaces \(\Sigma\), orthogonal to a vector field \(\chi^a\). The hypersurface is spacelike everywhere except at the horizon \(\mathcal{H}\), which is defined to be an \((n - 1)\)-dimensional null hypersurface. So, \(\chi^a\) is orthogonal to a null hypersurface over \(\mathcal{H}\) and hence \(\chi^a\) is itself null over \(\mathcal{H}\). Everywhere else, \(\chi^a\) is timelike.

Since \(\mathcal{H}\) is a Killing horizon, the vector field \(\chi^a\) becomes a null Killing vector field, say \(\chi^a_{\mathcal{H}}\), over \(\mathcal{H}\). \(\chi^a\) is not necessarily a Killing field everywhere, but it is Killing at least over \(\mathcal{H}\):
\[
\chi^a|_{\mathcal{H}} = \chi^a_{\mathcal{H}} : \nabla_{(a} \chi_{b)} = 0, \quad \chi^a_{\mathcal{H}} \chi_{\mathcal{H}a} = - \beta^2 |_{\mathcal{H}} = 0.
\]
We now write the spacetime metric \(g_{ab}\) as
\[
g_{ab} = - \beta^{-2} \xi_a \xi_b + \lambda^{-2} R_a R_b + \gamma_{ab},
\]
where \(R^a\) is a spacelike vector field orthogonal to \(\chi^a\), and \(\lambda^2\) is the norm of \(R^a\). \(\gamma_{ab}\) is the non-null spacelike portion of the metric perfectly well behaved on or in an infinitesimal neighborhood of the horizon.

Let us denote the Killing fields of this spacetime by \((\xi_i, \{\phi_i^a\})\), where \(i = 1, 2, \ldots, m\). Let \(\xi_a\) be the timelike Killing field and \(\{\phi_i^a\}\) be the spacelike Killing field(s). We assume that the hypersurface orthogonal vector field \(\chi^a\) (which is orthogonal to \(\{\phi_i^a\}\) and any other spacelike field) can be written as a linear combination of all the Killing fields:
\[
\chi_a = \xi_a + \alpha^i(x) \phi_i^a.
\]
where repeated indices are summed over and \( \{ \alpha'(x) \} \) are smooth functions. Then, using Killing’s equation, we have \( \nabla_{(a} X_{b)} = \phi'_{(a} \nabla_{b)} \alpha'(x) \). Thus, we have
\[
\chi^a \chi^b \nabla_a X_b = - \frac{1}{2} \chi^a \nabla_b \beta^2 = \chi^a \chi^b \phi'_{(a} \nabla_{b)} \alpha'(x) = 0. \tag{13}
\]
Equation (13) shows that \( \nabla_a \beta^2 \) is everywhere orthogonal to \( \chi^a \) and hence it is spacelike when \( \chi^a \) is timelike. So, we may choose \( R_a = \nabla_a \beta^2 \) in equation (11).

To look at the behavior of \( \nabla_a \beta^2 \) over the horizon, we recall that over the Killing horizon \( \mathcal{H} \) \cite{21, 22}
\[
\nabla_a \beta^2 = - 2 \kappa \chi_{1a}. \tag{14}
\]
where \( \kappa \) is a function. Since by definition \( \chi_{1a} \) is the null hypersurface orthogonal at the horizon, it turns out that \( \kappa \) is a constant over the horizon \cite{21}. Equation (14) shows that \( \nabla_a \beta^2 \) is null over \( \mathcal{H} \). However, the choice \( R_a = \nabla_a \beta^2 \) is not unique, we could have multiplied \( \nabla_a \beta^2 \) by some non-diverging function over \( \mathcal{H} \), even some positive power of \( \beta \). But we will retain this choice for convenience.

Let \( R \) be the parameter along \( R^a \). Then using equation (14), we have over \( \mathcal{H} \)
\[
R^a \nabla_a \beta^2 = \frac{d \beta^2}{dR} = -4 \kappa^2 \beta^2, \tag{15}
\]
which implies over \( \mathcal{H} \)
\[
\beta^2 = e^{-4 \kappa^2 R}. \tag{16}
\]
With the choice of \( R^a \) we have made, it is clear that metric (11) becomes doubly degenerate over \( \mathcal{H} \). Note that equation (11) can readily be realized in its doubly degenerate form for a static spherically symmetric black hole by employing the usual \((t, r^\star)\) coordinates, where \( r^\star \) is the Tortoise coordinate. We will be more explicit about \( R \), when we go into specific examples.

The assumption of stationarity and Killing horizon would help us to provide a meaningful notion of the ‘particle’ energy \cite{21}.

For \( n > 4 \), the uniqueness and other general properties of black holes are not very well understood and there may exist more general stationary black holes. However, we show below that for known stationary exact solutions, those assumptions will be sufficient.

Let us now expand equation (9) with decomposition (11). The single-derivative terms do not contribute in the \( \hbar \to 0 \) limit we are concerned with and the equation explicitly becomes
\[
\lambda^2 (\chi^a \partial_a I - ef)^2 - \beta^2 (R^a \partial_a I + eg)^2
= \beta^2 (R^a \partial_a I + eg)^2 - (\beta \lambda)^2 [\gamma_{ab} \partial^a I \partial^b I + e^2 \gamma_{ab} \gamma^a \partial^b I + m^2] = 0, \tag{17}
\]
where \( f = - \chi^a A_a \) and \( g = R_a A^a \). Here it is clear that had we multiplied \( R^a \) by a function \( h(x) \) non-diverging over \( \mathcal{H} \), we would have multiplied equation (17) only by an overall factor \( h^2(x) \).

Now we look at equation (17) in the near-horizon limit. By our assumption the metric functions \( \gamma_{ab} \) are well behaved over the horizon. So, \( \gamma_{ab} A^a A^b \) is non-divergent over \( \mathcal{H} \). Also, examples with \( g \neq 0 \) seem to be unknown in the literature. So, we set \( g = 0 \) in equation (17) and write equation (17) in the near-horizon limit as
\[
\lambda^2 (\chi^a \partial_a I - ef)^2 - \beta^2 (R^a \partial_a I)^2 - (\beta \lambda)^2 [\gamma_{ab} \partial^a I \partial^b I - 2 e \gamma_{ab} \gamma^a \partial^b I] = 0. \tag{18}
\]
To further simplify equation (18), let us choose an orthogonal basis \( \{ m^n_i \}_{i=1}^{n-2} \) for \( \gamma_{ab} \). Let \( \theta_i \) be the parameter along each \( m^n_i \). Let us consider the first term within the square brackets. This is basically a sum of the squares of \((n - 2)\) Lie derivatives: \( \frac{1}{m_1^n} (\varepsilon_n I)^2 + \frac{1}{m_2^n} (\varepsilon_m I)^2 \cdots \) where \( m^n_2 \) is the norm of each \( m^n_i \). By our definition, those norms are non-zero finite over \( \mathcal{H} \). Since \( I \) is a scalar, those Lie derivatives are basically partial derivatives: \( \varepsilon_n I = \partial_n I \).
We now check whether the terms within the square bracket in equation (18) are divergent over $\mathcal{H}$. Let us suppose that close to $\mathcal{H}$, if possible, the following divergence occurs:

$$\gamma_{ab} \partial^a I \partial^b I = D(x) \beta^2,$$

(19)

where $D(x)$ is bounded over or close to $\mathcal{H}$ and independent of $\beta$ at leading order. Then equation (15) implies that $D(x)$ is also independent of $R$ over $\mathcal{H}$:

$$\mathcal{L}_R D(x)|_{\mathcal{H}} = 0.$$

(20)

Also by our choice, $R_a = \nabla_a \beta^2$, whose norm is $\lambda^2$, vanishes over $\mathcal{H}$ as $O(\beta^2)$ (equation (14)). So the function $D(x)$ is also independent of $\lambda$ in the leading order over $\mathcal{H}$. Since the metric functions $\gamma_{ab}$ are well behaved over $\mathcal{H}$, the divergence of $\gamma_{ab} \partial^a I \partial^b I$ arises from the Lie derivatives $(\partial^2 I)$ which, by equation (21), is $O(\beta^{-1})$. Hence comparing equations (19) and (21), we have $D(x) = \frac{C^2(x)}{m_i}$.

Using equation (15), we obtain from equation (21) the following divergence over $\mathcal{H}$:

$$\frac{\partial^2 I}{\partial R \partial \theta_i} = \pm \frac{2 \lambda \beta}{\kappa} C_i(x).$$

(22)

On the other hand, we can write equation (18) near $\mathcal{H}$ now as

$$(\partial_R I)^2 = \lambda^2 \left[ (\chi^a \partial_a I - ef)^2 - D(x) \right].$$

(23)

We take the Lie derivative of equation (23) with respect to $m_i^a$ over $\mathcal{H}$. By our choice $R_a R^a = \lambda^2 = \nabla_a \beta^2 \nabla^a \beta^2$. Also, the function $\kappa$ in equation (14) is a constant over $\mathcal{H}$. This means that $\partial_0 \kappa = 0$ over $\mathcal{H}$. So we can write the vector field $\chi_{H}^a$ is Killing over $\mathcal{H}$, the term $(\chi^a \partial_a I - ef)$ is a conserved quantity, i.e. a constant [21]. We regard this term to be the conserved effective energy ($E$) of the particle. So, using equations (14) and (22), the Lie derivative of equation (23) with respect to $m_i^a$ gives the following $O(\beta^{-1})$ divergence over $\mathcal{H}$:

$$\partial_\theta D(x) = \pm \frac{\lambda}{\beta^2} C_i(x) [E^2 - D(x)]^\frac{1}{2}.$$

(24)

Equation (24) contradicts the fact that $D(x)$ is independent of $\beta$, $\lambda$ or $R$ in the leading order over $\mathcal{H}$. So, equation (19) cannot be true. Similarly we can show that the term $\gamma_{ab} \partial^a I \partial^b I$ cannot be divergent as $O(\beta^{-n})$ for any $n > 2$. Thus, $\beta^2 \gamma_{ab} \partial^a I \partial^b I = 0$ over the horizon.

With all these, we now integrate equation (18) across the horizon along a complex path

$$I_{\pm} = \pm \int_{\mathcal{H}} \frac{\lambda}{\beta} \left( \chi_{H}^a \partial_a I - ef \right) dR,$$

(25)

where complex integration is understood. The $+$ ($-$) sign stands for outgoing (incoming) solution. Equation (25) gives the emission (absorption) probability for a stationary black hole satisfying the assumptions we have made.
In order to verify the validity of equation (25), at this point we need some particular metrics. We find the vector fields $\chi^a_H$ and $R^a$, and then compute $I_{\pm}$ from equation (25).

Let us start with four dimensions by considering the charged Kerr black hole

$$\text{dr}^2 = -\frac{\Delta - a^2 \sin^2 \theta \, \text{d}t^2 - 2a \sin^2 \theta \, \text{d}r \, \text{d}\phi}{\Sigma} + \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}{\Sigma} \sin^2 \theta \, \text{d}\phi^2 + 2 \frac{\Sigma}{\Delta} \text{d}r^2 + \Sigma \, \text{d}\theta^2,$$

(26)

where $\Sigma = r^2 + a^2 \cos^2 \theta$, $\Delta = r^2 + a^2 + Q^2 - 2Mr$; here $a$ and $Q$ are the parameters specifying rotation and charge respectively. $\Delta = 0$ defines the horizon ($r_H$). The gauge field of this solution is $A_a = -\frac{Q_n}{\Sigma} [(\text{d}t)_a - a \sin^2 \theta (\text{d}\phi)_a]$. We first define $\chi^a = (\partial_t)^a - \frac{Q_n}{\Sigma_0} (\partial_r)^a$, such that $\chi_a (\partial_b)^a = 0$ everywhere. Near the horizon we have $\chi_a \chi^a = -\beta^2 \approx -\frac{\Delta \Sigma}{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta} \leq 0$. So, $\beta^2 = 0$ over the horizon which implies $\chi^a$ becomes null over the horizon and timelike outside it.

Over the horizon $\chi^a$ becomes $\chi^a_H = (\partial_t)^a - \frac{Q_n}{\Sigma_0} (r_H)(\partial_r)^a = (\partial_t)^a + \frac{a}{r_H} \chi^a (\partial_b)^a$, which is Killing and null. Thus we have specified the required vector field $\chi^a$ which becomes null and Killing over the horizon.

Next we need to find out $R^a$ and the parameter $R$ along it. Using the expression $\chi^a = (\partial_t)^a - \frac{Q_n}{\Sigma_0} (\partial_r)^a$, we have $\chi^a \nabla_b \beta^2 = 0$ everywhere. So we can let $R_a = \nabla_a \beta^2$. Then using the expressions for $\beta^2$ and the metric functions (equation (26)) we have near the horizon

$$R^a \nabla_a \beta^2 = \frac{\text{d}\beta^2}{\text{d}r} = \nabla_a \beta^2 \nabla^a \beta^2 = \frac{\Delta \Sigma}{(r^2 + a^2)^2} \Delta^2 + O(\Delta^2),$$

(27)

where the prime denotes derivative with respect to $r$. Thus, we have found the norm $\lambda^2(= \nabla_a \beta^2 \nabla^a \beta^2)$ of the vector field $R^a$ which becomes null over the horizon. Also, equation (27) gives near the horizon

$$R = \int \frac{(r^2 + a^2)^2 \, \text{d}\Delta}{\Delta \Delta^2}.$$  \hspace{1cm} (28)

Thus, we have specified the coordinate or the parameter $R$ along $R_a$. Note that equation (28) implies that near the horizon, choosing the vector field $R_a = \nabla_a \beta^2$ means a coordinate transformation $r \to R$ in metric (26).

The components of the gauge field $A_a$ on the horizon are given by $A_a \chi^a_H = -\frac{Q_n}{r_H \Sigma_0}$ and $A_a (\partial_b)^a = \frac{Q_n \Sigma \sin^2 \theta}{(r_H \Sigma_0)^2 \cos \theta (\Sigma_0)}$. The near-horizon contribution comes only from the first one.

Substituting the near-horizon norms $\chi_a \chi^a = -\beta^2 \approx -\frac{\Delta \Sigma}{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}$, $R^a R_a = \lambda^2 \approx \frac{\Delta \Sigma}{(r^2 + a^2)^2} \Delta^2$ and $\text{d}R = \frac{(r^2 + a^2) \, \text{d}\Delta}{\Delta \Delta^2}$ into equation (25), we have

$$I_{\pm} = \pm \int_H \left( \chi^a_H \partial_a I - ef \right) R^2 + a^2 \frac{\Delta}{\lambda} \, \text{d}r,$$  \hspace{1cm} (29)

where $f = -A_a \chi^a_H = -\frac{Q_n}{r_H \Sigma_0}$. Equation (29) was first obtained in [23, 24] by explicitly solving the semiclassical Dirac equation by the method of separation of variables.

The emission (absorption) probabilities are given by $\sim e^{-\frac{2\pi \lambda^2}{\kappa^2}}$ [5]. We shall not go into the details of the complexification of the 'path', the choice of contours and explicit evaluation of equation (29). We refer the reader to [5, 23, 24] for this. Explicit evaluation of equation (29) and the emission ($P_E$) or absorption ($P_A$) probabilities give the desired temperature of the emission from the exponential behavior of $\frac{\partial \chi}{\partial t}$. The Hawking temperature is found to be $T_H = \frac{\lambda^2}{2\pi}$, where $\kappa^2$ is the surface gravity of the event horizon.
After this, we consider some examples from higher dimensions. First, we consider a non-extremal rotating charged black hole solution of five-dimensional minimal supergravity with two different rotation parameters in the Boyer–Lindquist coordinates [25]:

\[
ds^2 = -\frac{\Delta_0(1 + q^2 r^2)}{\Sigma_a \Sigma_b} - \frac{\Delta_2(2m\rho^2 - q^2 + 2abq^2\rho^2)}{\rho^4 \Sigma_a \Sigma_b} \, dt^2 + \frac{\rho^2}{\Delta_0} \, dr^2 + \frac{\rho^2}{\Delta_0} \, d\theta^2
\]

\[
+ \left[ \frac{(r^2 + a^2) \sin^2 \theta}{\Sigma_a} + \frac{a^2(2m\rho^2 - q^2) \sin^2 \theta + 2abq^2 \sin^2 \theta}{\rho^4 \Sigma_a} \right] \, d\phi^2
\]

\[
+ \left[ \frac{(r^2 + b^2) \cos^2 \theta}{\Sigma_b} + \frac{b^2(2m\rho^2 - q^2) \cos^2 \theta + 2abq^2 \cos^2 \theta}{\rho^4 \Sigma_b} \right] \, d\psi^2
\]

\[
- \frac{2\Delta_0 \sin^2 \theta [a(2m\rho^2 - q^2) + bq^2(1 + a^2g^2)]}{\rho^4 \Sigma_a \Sigma_b} \, dt \, d\phi
\]

\[
- \frac{2\Delta_0 \cos^2 \theta [b(2m\rho^2 - q^2) + aq^2(1 + b^2g^2)]}{\rho^4 \Sigma_a \Sigma_b} \, dt \, d\psi
\]

\[
+ \frac{2 \sin^2 \theta \cos^2 \theta \omega (4m^2 b^2 + q^2 + 4abq)}{\rho^4 \Sigma_a \Sigma_b} \, dt \, d\psi,
\]

(30)

where \( \rho^2 = (r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta) \), \( \Delta_0 = (1 - a^2 g^2 \cos^2 \theta - b^2 g^2 \sin^2 \theta) \), \( \Sigma_a = (1 - a^2 g^2) \), \( \Sigma_b = (1 - b^2 g^2) \) and \( \Delta_r = \left[\frac{r^2 + a^2 \sin^2 \theta + b^2 \sin^2 \theta}{\Delta_0} + 2M \right] \). The black hole event horizon \((r_H)\) is given by \( \Delta_r(r_H) = 0 \). The parameters \((M, a, b, q)\) specify respectively the mass, angular momenta and the charge of the black hole. \( g \) is a real positive constant. The gauge field corresponding to the charge \( q \) is given by \( A_a = \frac{\sqrt{g}}{\rho^2} \left( \frac{\Delta_0}{\Sigma_a \Sigma_b} (dt)_a - \frac{\omega}{\Sigma_a \Sigma_b} (d\phi)_a - \frac{\omega}{\Sigma_a \Sigma_b} (d\psi)_a \right) \).

The angular velocities of the comoving observers on the horizon are given by [26]

\[
\Omega_\phi = -\left[ \frac{\delta_\phi \delta_\psi - \delta_\psi \delta_\phi}{\delta_\psi (\delta_\phi \delta_\phi)^2} \right]_{r=r_H} = \frac{a(r_H^2 + b^2)(1 + g^2r_H^2) + bq}{(r_H^2 + a^2)(r_H^2 + b^2) + abq}
\]

\[
\Omega_\psi = -\left[ \frac{\delta_\phi \delta_\psi - \delta_\psi \delta_\phi}{\delta_\psi (\delta_\phi \delta_\phi)^2} \right]_{r=r_H} = \frac{2b(r_H^2 + a^2)(1 + g^2r_H^2) + aq}{(r_H^2 + a^2)(r_H^2 + b^2) + abq}
\]

(31)

We note that the vector field

\[
\chi^a = (\partial_a)^a - \left[ \frac{\delta_\phi \delta_\psi - \delta_\psi \delta_\phi}{\delta_\psi (\delta_\phi \delta_\phi)^2} \right] (\partial_\phi)^a - \left[ \frac{\delta_\psi \delta_\phi - \delta_\phi \delta_\psi}{\delta_\phi (\delta_\phi \delta_\phi)^2} \right] (\partial_\psi)^a
\]

(32)

is orthogonal to \((\partial_\phi)^a\) and \((\partial_\psi)^a\) everywhere. Also, the near-horizon norm of \(\chi^a\) is

\[
\chi^a \chi_a = -\beta^2 = -\frac{\Delta_r}{\rho^2 (r^2 + a^2)(r^2 + b^2) + abq} + O(\Delta_r^2).
\]

Thus, \(\chi^a\) becomes null over the horizon.

Also, equation (31) shows that \(\chi^a\) becomes a Killing field \(\chi_H^a\) over the horizon, where

\[
\chi_H^a = (\partial_\phi)^a + \Omega_\phi (\partial_\phi)^a + \Omega_\psi (\partial_\psi)^a.
\]

(33)

So, we have specified the required vector field \(\chi^a\) which becomes null and Killing over the horizon.

Also, exactly through the same manner as in the Kerr–Newman metric, we can specify the other null vector field \(R^a\), its norm \(\lambda^2\) and the coordinate \(R\) for metric (30). Choosing \(R_a = \nabla_a \beta^2\), we have near the horizon

\[
R_a R^a = \lambda^2 = \nabla_a \beta^2 \nabla^a \beta^2 = R^a \nabla_a \beta^2 = \frac{\Delta_r \rho^2 \lambda^4}{(r^2 + a^2)(r^2 + b^2) + abq} \left( \frac{\Delta_r}{r^2 \Delta_r} \right)^2,
\]

which becomes null over the horizon. Also, near the horizon the coordinate \(R\) along \(R^a\) is given by

\[
R = \int \left[ \frac{\Delta_r}{r^2 \Delta_r} \right] d(r^2 \Delta_r).
\]

(35)
The gauge field $A_a$ has three components $(A_a \chi^a_H, A_a(\partial_\phi)^a, A_a(\partial_\psi)^a)$ of which the near-horizon contribution comes only from $A_a \chi^a_H$.

Substituting the near-horizon norms $\beta^2 = \frac{(r^2 + a^2)(r^2 + \lambda^2)}{(r^2 + a^2 + \lambda^2)} r^2 \Delta_r$ and $dR = \frac{r^2 \Delta_r}{(r^2 + a^2 + \lambda^2)} dr$ into equation (25), we have

$$I_\pm = \pm \int \left[ \chi_H \partial_\phi e^a - ef \right] \left[ \frac{(r^2 + a^2)(r^2 + b^2) + abq}{r^2 \Delta_r} \right] dr,$$

where $f = -A_a \chi^a_H = \frac{\sqrt{a q}}{r + a h}$, $\chi_H$ is the charge of the black hole, and $e$ is the electric field of the black hole. Equation (36) was first obtained in [26] by the explicit solution of the semiclassical Dirac equation by the method of separation of variables. Complex integration of equation (36) across the horizon and computation of the emission (absorption) probabilities give the expected Hawking temperature in terms of the Killing horizon’s surface gravity [25, 26].

It can be easily verified using the same methods as above that equation (25) also applies well and recovers the desired results for the five-dimensional stationary solutions with Killing horizons like the Kerr–Gödel black hole [27], squashed Kaluza–Klein black hole [28, 29], a black string [28, 30], black hole solutions of $z = 4$, Horava–Lifshitz gravity [31, 32] and the toroidal black hole solutions like in [33].

Our scheme also applies very easily to an $n$-dimensional generalization of the Kerr black hole with a single rotation parameter [34]

$$ds^2 = -dr^2 + \left( r^2 + a^2 \right) \sin^2 \theta d\phi^2 + \frac{\mu}{r^2 - 5} \left( dr - a \sin^2 \theta d\phi \right)^2 + \frac{r^n - 5}{r^n - a^2} d\Omega^{n-4},$$

where the parameters $\mu$ and $a$ represent the mass and the angular momentum of the black hole, respectively. $\Sigma = r^2 + a^2 \cos^2 \theta$ and $d\Omega^{n-4}$ represent the metrics over an $(n - 4)$ sphere.

Equation (25) also applies to a de Sitter horizon, provided the assumptions stated at the beginning of this section are true for that case. Such an example is the Kerr–de Sitter spacetime. The de Sitter horizon for this spacetime is a Killing horizon [35]. One can show, following exactly the similar way as before, that all the other assumptions are valid for this case. Explicit evaluation of equation (25) gives the expected thermal character of the incoming radiation.

### 4. Vector, spin-2 and spin-$\frac{3}{2}$ fields

Now we show that all the approaches and conclusions made in the preceding sections also hold for the Proca, massive spin-2 and spin-$\frac{3}{2}$ fields. Let us first consider the equation of motion for a Proca field $A^\mu$:

$$\nabla_\mu F^{\mu\nu} = -\frac{m^2}{\hbar^2} A^\nu,$$

where $F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Equation (38) can be written as

$$\nabla_\mu \nabla^\mu A_\nu - R_\nu^{\quad\mu} A_\mu - \nabla_\mu (\nabla_\nu A^\mu) = -\frac{m^2}{\hbar^2} A_\nu.$$

But equation (38) implies that $\nabla_\mu A^\mu = 0$ identically. Now let us choose a set of orthonormal basis $\{e^{(\mu)}\}$. We expand the vector field $A_\mu$ in this basis, $A_\mu = e^{(\mu)}_{(a)} A^{(a)}$. With this expansion and the fact that $\nabla_\mu A^\mu = 0$, equation (38) becomes

$$e^{(\mu)}_{(a)} \nabla_\mu \nabla^\mu A^{(a)} + A^{(a)} \nabla_\mu e^{(\mu)}_{(a)} + 2 \nabla_\mu A^{(a)} \nabla^\mu e^{(\mu)}_{(a)} - R_\nu^{(\mu)} A^{(a)} = -\frac{m^2}{\hbar^2} A^{(a)} e^{(\mu)}_{(a)}.$$

9
which, after contracting both sides by \( e_b^{(\nu)} \), reduces to

\[
\nabla_a \nabla^a A^{(\nu)} + A_{(\mu)} e_b^{(\nu)} \nabla_a e_b^{(\mu)} + 2 e_b^{(\nu)} \nabla_a A_{(\mu)} \nabla^a e_b^{(\mu)} - R^{(\nu)} e_b^{(\mu)} A_{(\mu)} = -\frac{m^2}{\hbar^2} A^{(\nu)}. \tag{41}
\]

We choose the usual WKB ansatz for each \( A^{(\nu)} \):

\[ A^{(\nu)} = f^{(\nu)}(x) e^{iI^{(\nu)}(x)/\hbar}, \]

substitute into equation (41) and take the semiclassical limit \( \hbar \to 0 \).

Then it immediately turns out that in the semiclassical limit equation (41) can be effectively represented by \( n \) Klein–Gordon equations for the scalars \( A^{(\nu)} \):

\[
\nabla_a \nabla^a A^{(\nu)} + \frac{m^2}{\hbar^2} A^{(\nu)} = 0, \tag{42}
\]

with \( \nu = 0, 1, 2, \ldots, (n - 1) \). When each of equations (42) is explicitly expanded and the near-horizon limit is taken, we get back equation (25) with \( e = 0 \).

Next, we turn our attention to the massive spin-2 field \( \pi^{ab} \) satisfying the Pauli–Fierz equation [36]

\[
\nabla_c \nabla^c \pi^{ab} + \frac{m^2}{\hbar^2} \pi^{ab} = 0, \tag{43}
\]

where \( \pi^{ab} \) are the symmetric tensor fields. As before we expand \( \pi^{ab} \) in orthonormal basis,

\[ \pi^{ab} = e^{(\mu)} a e^{(\nu)} b \pi^{(\mu)(\nu)}. \]

In the semiclassical limit and for the WKB ansatz, equation (43) can effectively be represented by \( \frac{n(n + 1)}{2} \) Klein–Gordon equations for the scalars \( \pi^{(\mu)(\nu)} \):

\[
\nabla_c \nabla^c \pi^{(\mu)(\nu)} + \frac{m^2}{\hbar^2} \pi^{(\mu)(\nu)} = 0, \tag{44}
\]

and thus similar conclusions hold for this case also.

Finally, we briefly address the spin-3/2 fields satisfying the Rarita–Schwinger equation [37]. The tunneling phenomenon for this field was addressed in [38] for the Kerr black hole by explicitly solving the equations of motion in the near-horizon limit.

The Rarita–Schwinger equation in a curved spacetime reads

\[
\gamma^a \nabla_a \Psi_b = \frac{m}{\hbar} \Psi_b, \tag{45}
\]

where \( \Psi_b \equiv \Psi_b^{(s)} \) is a spinor with \( s \) being the spin index. The \( \gamma \)'s are the matrices (with matrix indices suppressed) satisfying the anti-commutation relation similar to the Dirac \( \gamma \)'s:

\[ \{ \gamma^a, \gamma^b \} = 2g^{ab} I. \]

The spin-covariant derivative \( \nabla \) is defined as \( \nabla_a \Psi_b := (\partial_a + \Gamma_a) \Psi_b \), where \( \Gamma_a \) are the spin connection matrices (with suppressed matrix indices). Also, \( \Psi_b \) satisfies an additional constraint \( \gamma^a \Psi_a = 0 \).

Due to the similarity of the spin-3/2 fields with the Dirac spinors discussed in section 2, we apply the same method here to show that \( \Psi_b \) satisfies the Klein–Gordon equation in the semiclassical WKB framework. So, we square equation (45) by applying \( i\gamma^a \nabla_a \) from the left. A little computation, using the definition of the spin-covariant derivative \( \nabla_a \), the anti-commutation relation satisfied by \( \gamma \)'s and also the commutativity of the partial derivatives yields

\[
\nabla_a \nabla^a \Psi_b + \frac{1}{4} [\gamma^a, \gamma^c] (\partial_a \Gamma_{c1} + \Gamma_{a1} \Gamma_{c1}) \Psi_b + (\gamma^a \nabla_c \gamma^c) \nabla_a \Psi_b = -\frac{m^2}{\hbar^2} \Psi_b. \tag{46}
\]

So, as in the previous cases, it then immediately follows for the usual ansatz

\[
\Psi_a = \begin{bmatrix}
A_a(x) e^{i\phi_a(x)} \\
B_a(x) e^{i\phi_a(x)} \\
C_a(x) e^{i\phi_a(x)} \\
D_a(x) e^{i\phi_a(x)}
\end{bmatrix}, \tag{47}
\]

where \( A_a(x) e^{i\phi_a(x)} ) \) is the spinor with \( s \) being the spin index. The \( \gamma \)'s are the matrices (with matrix indices suppressed) satisfying the anti-commutation relation similar to the Dirac \( \gamma \)'s:

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\[
\nabla_a \nabla^a \Psi_b + \frac{1}{4} [\gamma^a, \gamma^c] (\partial_a \Gamma_{c1} + \Gamma_{a1} \Gamma_{c1}) \Psi_b + (\gamma^a \nabla_c \gamma^c) \nabla_a \Psi_b = -\frac{m^2}{\hbar^2} \Psi_b. \tag{46}
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A_a(x) e^{i\phi_a(x)} \\
B_a(x) e^{i\phi_a(x)} \\
C_a(x) e^{i\phi_a(x)} \\
D_a(x) e^{i\phi_a(x)}
\end{bmatrix}, \tag{47}
\]
Equation (46) reduces to the Klein–Gordon equations in the semiclassical limit. We can easily generalize this result for a charged spin-$\frac{3}{2}$ particle coupled to a gauge field by replacing the spin-covariant derivative by the gauge spin-covariant derivative. This gives charged Klein–Gordon equations.

5. Discussions

We now summarize our results. The objective of this work was to put the complex path approach for stationary black holes in a general framework. To do this, we have dealt with some well-known physical matter equations and shown for any arbitrary spacetime in a coordinate-independent way that in the semiclassical limit the WKB ansatz implies that all those equations of motion are equivalent to the Klein–Gordon equation. We have done this without choosing any particular basis of the vector fields or the $\gamma$ matrices. We need to assume only that a metric $g_{ab}$ can be defined on the spacetime which guarantees the existence of the orthonormal basis $\{e^{(a)}(\bar{e})\}$ [21]. So it is clear that as far as the semiclassical level is concerned, it is sufficient to work only with scalars for any arbitrary black hole. It also becomes clear that the Hawking temperature is indeed independent of the particle species we are concerned with.

We further presented a general coordinate-independent expression for the emission probability from an arbitrary stationary black hole with some assumed geometrical properties (equation (25)). We showed that finding the emission probability or the Hawking temperature for such black holes reduces to merely finding a null coordinate or a null vector field (which is spacelike outside the horizon), and the norm of the timelike vector field which is orthogonal to the horizon and becomes null and Killing over the horizon. At this point we can use any specific metric for explicit computation and we illustrated the validity of equation (25) by taking several examples.

The principal message of this work is the following. The semiclassical method provides us a way in which we can treat the equations of motions of different spin fields and compute the single-particle emission probability or the Hawking temperature for a stationary black hole on an identical footing or manner.

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