Shadow of a charged rotating black hole in \( f(R) \) gravity

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Abstract A charged rotating black hole in \( f(R) \) gravity is characterized by mass, \( M \), spin, \( a \), electric charge, \( Q \), and \( R_0 \) which is proportional to cosmological constant. We analyze the image of the black hole shadow in four types (1) at \( r \to \infty \), (2) at \( r \to r_O \) in vacuum, (3) at \( r \to \infty \) and (4) at \( r \to r_O \) for an observer at the presence of plasma. Moreover, we investigate the effect of spin, charge, and modification of gravity on the shape of the shadow. In addition, we use two observable parameters, the radius \( R_s \) and the distortion parameter \( \delta_s \), characterizing the apparent shape. We show that the shadow becomes smaller with increasing electric charge for all cases. Also, by increasing the rotation parameters, the circular symmetry of the black hole’s shadow image will change. Furthermore, in the presence of plasma, the plasma parameter also affects the size of the shadow.

1 Introduction

Since the opening of the human eye to the wonders introduced by geniuses about a century ago, we have seen two significant discoveries in the field of gravity in less than five years. Detection of gravitational waves of colliding black holes (BHs) and neutron stars in 2016 [1, 2], which was the first reliable ones in the past decades, unlocked an overlooking new window to the cosmos. However, those brilliant successes have been concluded or possibly shaded by the first-ever image of a black hole [3]–[8], the objects which Einstein himself had thought they are not real and were just mathematical artifacts. Thanks to M87, its enormous mass and relative proximity make itself a perfect target for the Event Horizon Telescope (EHT), which has resolved angular scales of tens of microarcseconds of Very Long Baseline Interferometry (VLBI) observations, comparable to the scale of the event horizon [9, 10].

Referring to the recent viral picture of the observed black hole, the dark region at the center is called “shadow.” The question immediately comes to mind is the definition of a black hole’s shadow? General Relativity predicts that nothing can escape from the boundary of the event horizon due to the excessively curved space-time geodesics, so the event horizon has known precisely as a shell of nonreturn points, while photons circling just outside will escape to infinity. Shadow is a gravitationally lensed image of the event horizon that strongly depends on the closed orbits of photons in its outer boundary. Therefore, it appears as a relatively sharp boundary between bright and dark regions and arises from a deficit of those photons captured by the event horizon. Shadow and its geometrical properties correspond entirely to the transparent image of the photon capture sphere as seen from a distance and uniquely on the dynamical properties of the BH, such as its angular momentum. However, they do not depend on the photon’s energy.

Black holes intake surrounding matter because of their strong gravity via “accretion.” This surrounding matter accreting onto the hole heats up through viscous dissipation. It converts gravitational energy into radiation, radiating bright light at many frequencies, including radio waves that can be picked up by radio telescopes, such as what is received by VLBI. The bright accretion disk surrounding the black hole appears distorted due to the phenomenon of gravitational lensing. The accretion disk region behind the black hole also gets visible due to the bending of light by the black hole, which provides an excellent opportunity to understand the geometrical structure of the event horizon and assess the predictions of general relativity in the regime of strong gravity.

The hypothesis of EHT collaboration is based on the fact that the M87 contains a Kerr black hole. However, they analyzed their observational data with alternative theories to see if these theories matched with observational data or not. The alternatives are grouped into three categories: (1) GR black holes with additional fields, (2) Solutions of BH from alternative theories of gravity or including quantum effects, and (3) Compact objects in GR or in alternative theories whose properties are similar to BHs. The comparison between observational data with numerical simulation shows that the shadow of the second group of alternative theory’s BHs are very similar to those of Kerr black holes and are not distinguishable with present EHT capabilities. However, the polarization
of the emitted radiation’s degree or the variability of the accretion flow with higher-frequency observations can be used to assess their viability [7].

The theory of black hole shadows has been well-developed in literature for decades and is currently under investigation. Some notable results could be mentioned as Schwarzschild black hole surrounded by a Bach–Weyl ring [11]; Einstein–Dilaton–Gauss–Bonnets black hole [12]; Konoplya–Zhidenko or more general parameterized black holes [13, 14]; Einstein–Maxwell–Chern–Simons black hole [15]; Einstein–Maxwell–Dilaton–Axion black hole [16]; Kerr [17], Kerr–Newman [18], Kerr–NUT [19, 20], Kerr–Newman–Kasuya black hole [21]; Kerr–Perfect fluid dark matter black hole [22]; Kerr–de Sitter black hole [23]; Kerr–MOG black hole [24, 25]; regular black hole [26, 27], multi-black hole [28], black holes in extended Chern–Simons modified gravity [29], Randall–Sundrum braneworld [30] and Kerr black holes with scalar hair [31, 32]; Kerr–Sen black hole [33]; non-commutative black holes [34], Tomimatsu–Sato space-time [35], black hole surrounded by dark matter halo [36–38], shadow images of a rotating global monopole [39], testing the rotational nature of the supermassive object M87 and possible signals for extra dimensions in the shadow of M87 [40, 41], etc., and also naked singularities [42, 43]. Recently, there also are some interesting works about the curvature and topology of the geometry of the black hole shadows [44, 45], Shadows in Einstein–Æther Theory [46], and the shadow of BHs Surrounded by a Dark Matter [47].

On the other hand, properties of BHs in alternative theories of gravity such as braneworld cosmology [48], Lovelock gravity [49], scalar–tensor [50, 51], and $f(R)$ gravity have been considered eagerly [52, 53]. These theories are replaced and refined to justify some subjects like cosmic acceleration, dark matter, and cosmic inflation [54–73].

We know that the effect of $f(R)$ is to renormalize the gravitational constant, and the solution that we have used here is similar to Kerr–Newman–de Sitter [74]. However, a change in the action affects the dynamics of the universe and dynamics of galactic or solar system scales. The earlier studies of shadow had shown that the image of Schwarzschild’s black hole’s shadow is circular and has a photon sphere [75]. In contrast, the Kerr black hole has a photon region, and it does not have a circular shadow image, which means that it can take into account the deviation from circular symmetry [76]. Also, deviation from circular symmetry and changes in the shadow’s size is defined as $\delta_l$ and $R_0$, by Hioki and Maeda [77]. However, in most investigations on the study of a black hole’s shadow, the observer’s location is at infinity. In this work, we study the case of an observer at infinity and at a limited distance, the same situations considered in [78]. Also, the shadow has been analyzed in the absence and the presence of plasma for an observer at infinity and a limited distance. Here we consider no light source close to the black hole, which means we use the light like geodesic as the path of the incident light rays. This paper is organized as follows, In Sects. 2 and 3, we summarize the properties of gravity and its geodesics. In Sect. 4, we calculate shadow for an observer at $r = \infty$. In Sect. 5, we obtain an analytical formula for an observer in $r = r_0$. In Sects. 6 and 7, we analyze these situations in the presence of plasma and our results conclude in Sect. 8.

2 Field equations in $f(R)$ modified gravity

This section studies the field equations and the metric in $f(R)$ gravity. The action with Maxwell term is

$$S = S_g + S_M,$$

(1)

where $S_g$ and $S_M$ are the gravitational action and the electromagnetic action, respectively, as

$$S_g = \frac{1}{16\pi} \int d^Dx \sqrt{|g|} (R + f(R)),$$

(2)

$$S_M = -\frac{1}{16\pi} \int d^4x \sqrt{-g} [F_{\mu\nu}F^{\mu\nu}],$$

(3)

where $R$ is the scalar curvature, $R + f(R)$ is the function defining the theory under consideration, and $g$ is the determinant of the metric. The Maxwell equations and the field equations are

$$\nabla_\mu F^{\mu\nu} = 0,$$

(4)

$$R_{\mu\nu}(1 + f'(R)) - \frac{1}{2}(R + f(R)) g_{\mu\nu} + (g_{\mu\nu} \nabla^2 - \nabla_\mu \nabla_\nu) f'(R) = 2T_{\mu\nu},$$

(5)

where $\nabla$ is the usual covariant derivative, $R_{\mu\nu}$ is the Ricci tensor and the stress–energy tensor of the electromagnetic field is given by

$$T_{\mu\nu} = F_{\mu\rho} F^{\rho\nu} - \frac{g_{\mu\nu}}{4} F_{\rho\sigma} F^{\rho\sigma},$$

(6)

with

$$T^\mu_\mu = 0.$$  

(7)

the constant curvature scalar $R = R_0$ and the trace of Eq. (5) leads to

$$R_0(1 + f'(R_0)) - 2(R_0 + f(R_0)) = 0,$$  

(8)
which introduces the negative constant curvature scalar as

$$ R_0 = \frac{2f(R_0)}{f'(R_0) - 1}. $$

(9)

Using this relation in Eq. (5) gives the Ricci tensor

$$ R_{\mu\nu} = \frac{1}{2} \left( \frac{f(R_0)}{(1 + f'(R_0))^2} \right) g_{\mu\nu} + \frac{2}{(1 + f'(R_0))^3} T_{\mu\nu}. $$

(10)

Finally, Alexis Larrañaga [79] introduced the axisymmetric ansatz in Boyer–Lindquist–type coordinates $(t, r, \theta, \phi)$ inspired by the Kerr–Newman–AdS black hole solution as

$$ d s^2 = -\Delta_r \left[ dt - \frac{a \sin^2 \theta \ d \psi}{\Xi} \right]^2 + \frac{\rho^2}{\Delta_r} dr^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2 + \frac{\Delta_\theta \sin^2 \theta}{\rho^2} \left[ \frac{\rho^2 \ d\phi}{\Xi} \right]^2. $$

(11)

where

$$ \Delta_r = (r^2 + a^2) \left( 1 + \frac{R_0}{12} r^2 \right) - 2 M r + \frac{Q^2}{(1 + f'(R_0))}, $$

$$ \Xi = 1 - \frac{R_0}{12} a^2, \quad \rho^2 = r^2 + a^2 \cos^2 \theta, \quad \Delta_\theta = 1 - \frac{R_0}{12} a^2 \cos^2 \theta, $$

(12)

(13)

which, $a$ is the angular momentum per mass of the black hole, $R_0$ is a constant proper to cosmological constant $(R_0 = -4 \Lambda$ and it is constant), and $Q$ is the electric charge.

### 3 The geodesic equations

This section studies the geodesic equations and introduces effective potential. The Hamilton–Jacobi equation is

$$ \frac{\partial S}{\partial \tau} + \frac{1}{2} g_{ij} \frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x^j} = 0, $$

(14)

Eq. (14) can be solved with an ansatz for the action

$$ S = \frac{1}{2} \epsilon \tau - E t + L_\phi \phi + S_\theta (\theta) + S_r (r). $$

(15)

The angular momentum $L$ and the energy $E$, are constants of motion as

$$ g_{ti} + g_{t\psi} \psi = -E, \quad g_{t\phi} \phi + g_{t\psi} \psi = L. $$

(16)

Substituting Eqs. (15) and (16) in Eq. (14), obtains

$$ \Delta_\theta \left( \frac{d s}{d \theta} \right)^2 + \frac{\epsilon a^2 \cos^2 \theta}{\Delta_\theta} - \frac{2 a E L \Xi}{\Delta_\theta} E^2 - \frac{2a^2 \sin^2 \theta}{\Delta_\theta} - \frac{L^2 \Xi^2}{\Delta_\theta} \sin^2 \theta = -\Delta_r \left( \frac{d s}{d r} \right)^2, $$

$$ -\epsilon \rho^2 + \frac{(a^2 + r^2) E^2 + a^2 L^2 \Xi^2 - 2a E L \Xi (r^2 + a^2)}{\Delta_r}. $$

(17)

where each side depends on $r$ or $\theta$ only. The separation ansatz equation (15) derives the equations of motion with the help of the Carter constant [80]

$$ \rho^4 \left( \frac{d r}{d \tau} \right)^2 = -\Delta_r (K + \epsilon r^2) + \left[ (a^2 + r^2) E - a L \Xi \right]^2 = R(r), $$

(18)

$$ \rho^4 \left( \frac{d \theta}{d \tau} \right)^2 = \Delta_\theta (K - \epsilon a^2 \cos^2 \theta) - \frac{1}{\sin^2 \theta} (a E \sin^2 \theta - L \Xi \sin^2 \theta) = \Theta(\theta), $$

(19)

$$ \rho^2 \left( \frac{d \phi}{d \tau} \right) = \frac{a E \Xi (a^2 + r^2) - a^2 L^2 \Xi^2}{\Delta_r} - \frac{1}{\Delta_\theta} \sin^2 \theta \left( a E L \Xi \sin^2 \theta - \Xi^2 L \right), $$

(20)

$$ \rho^2 \left( \frac{d \psi}{d \tau} \right) = \frac{E (r^2 + a^2)^2 - a L \Xi (r^2 + a^2)}{\Delta_r} - \frac{\sin^2 \theta}{\Delta_\theta} \left( E a^2 - L \Xi a \sin^2 \theta \right). $$

(21)

In the following, we investigate null geodesic, so $\epsilon = 0$, then we have

$$ \rho^4 \left( \frac{d r}{d \tau} \right)^2 = -\Delta_r K + \left[ (a^2 + r^2) E - a L \Xi \right]^2 = R(r), $$

(22)
\[ \rho^4 \left( \frac{d \theta}{d \tau} \right)^2 = \Delta_\phi(K) - \frac{1}{\sin^2 \theta} (a E \sin^2 \theta - L \Xi)^2 = \Theta(\theta), \]  
\[ \rho^2 \left( \frac{d \varphi}{d \tau} \right) = \frac{a E \Xi(a^2 + r^2) - a^2 \Xi L}{\Delta_\varphi} - \frac{1}{\Delta_\varphi} \left( \frac{a \Xi E}{a} \sin^2 \theta - \Xi^2 L \right), \]  
\[ \rho^2 \left( \frac{d t}{d \tau} \right) = \frac{E(r^2 + a^2)^2 - a L \Xi(r^2 + a^2)}{\Delta_t} - \frac{\sin^2 \theta}{\Delta_\theta} \left( \frac{Ea^2 - L \Xi a}{\sin^2 \theta} \right). \]

Now, we introduce dimensionless quantities such that \( \xi = \frac{L}{E} \) and \( \eta = \frac{K}{E^2} \), which are constant along the geodesics, so Eq. (22) becomes
\[ \rho^4 \left( \frac{d r}{d \tau} \right)^2 = -\Delta_r(E^2 \eta) + \left[ (a^2 + r^2)E - a(E \xi) \Xi \right]^2 = R(r). \]

An effective potential for the radial motion of particles is an important tool that can be obtained using Eq. [81]
\[ \rho^4 \left( \frac{d r}{d \tau} \right)^2 + V_{\text{eff}} = 0. \]  
So,
\[ V_{\text{eff}} = \Delta_r(E^2 \eta) - \left[ (a^2 + r^2)E - a(E \xi) \Xi \right]^2. \]

Circular orbits of the photons are important to find out \( \xi \) and \( \eta \) [82]
\[ V_{\text{eff}} = 0, \quad \frac{d V_{\text{eff}}}{d r} = 0. \]  
The condition in Eq. (29) is equal to \( R(r) = 0 \) and \( \frac{d R(r)}{d r} = 0 \). Using Eqs. (28) and (29), we can obtain the parameters \( \eta \) and \( \xi \),
\[ \eta = \frac{16r^2 \Delta_r}{\Delta_r^2}, \]  
and
\[ \xi = \frac{(a^2 + r^2)}{\Xi a} - \frac{4r \Delta_r}{\Xi a \Delta_r}, \]
where \( \Delta_r \) represents the derivative of \( \Delta_r \) respect to \( r \).

### 4 The shadow for an observer in \( r = \infty \)

In this section, we want to analyze the shadow of black holes for an observer in \( r = \infty \), so we introduce the celestial coordinate \( \alpha \) and \( \beta \), which are [83]
\[ \alpha = \lim_{r_O \to \infty} \left( r_O^2 \sin(\theta_O) \frac{d \varphi}{d r} \right). \]  
and
\[ \beta = \lim_{r_O \to \infty} r_O^2 \frac{d \theta}{d r}. \]
where \( \theta_O \) is the inclination angle between the rotation axis of the black hole and the line of sight of the observer, also considering an observer far away from the black hole, we have \( r_O \to \infty \). Using Eqs. (22)–(24), and taking the limit of a faraway observer, the celestial coordinates take the forms
\[ \alpha = -\xi \csc \theta_O, \]  
and
\[ \beta = \pm \sqrt{\eta + a^2 \cos^2 \theta_O - \xi^2 \cot^2 \theta_O}. \]
For an observer located in the equatorial plane of the black hole, i.e., \( \theta_O = \frac{\pi}{2} \), the gravitational effects are maximum, and \( \alpha \) and \( \beta \) become

\[
\alpha = -\xi, \tag{36}
\]

and

\[
\beta = \pm \sqrt{\eta}. \tag{37}
\]

For an observer at infinity, we put \( \Lambda = 0 \) because when \( \Lambda = 0 \), \( \Delta_r \) convert to second order equation in terms of \( r \)

\[
\Delta_r = (r^2 + a^2) - 2Mr + \frac{Q^2}{(1 + f'(R_0))}, \tag{38}
\]

and the horizons can be obtained as

\[
r_{\pm} = M \pm \sqrt{M^2 - \frac{Q^2}{(1 + f'(R_0))} - a^2}, \tag{39}
\]

where \( a^2 \leq a_{\text{max}}^2 \), in which \( a_{\text{max}}^2 = M^2 - \frac{Q^2}{(1 + f'(R_0))} \). In the case of \( a^2 > a_{\text{max}}^2 \), instead of a black hole, we have a naked singularity, and the case \( a = a_{\text{max}} \) is called an extremal black hole. Since outside of the event horizon (\( \Delta_r \)) is greater than zero, \( \delta_r \) is spacelike, so communication is possible here. This region is called the domain of external communication, and our observer is located in this region. When \( \Lambda \neq 0 \) the space-time is not flat asymptotically. Therefore, the observer in the domain of outer communication is apart from an observer placed at \( \infty \), by cosmological horizon when, \( \Lambda > 0 \) [78]. In this situation, the space-time is a family of Kerr–Newman space-time analyzed in [84]. However, \( f'(R_0) \) is a different part of this space-time from Kerr–Newman. In following, considering the above conditions, we plot \( \beta \) in terms of \( \alpha \) to obtain the contour of the black hole’s shadow. These plots are represented for different values of rotation parameter \((a = 0, a = 0.5, a = 0.7 \text{ and } a = 1)\) and electric charge \( Q \), in Fig.1. By increasing the electric charge, the shadows become smaller; also, increasing of spin parameter, \( a \), leads to a change in symmetry of the image of shadows.

In addition, for studying the size and deviation of black hole’s shadow, following [85], we use two observable definitions \( R_s \) and \( \delta_s \) parameters. \( R_s \) indicates the size of shadow, and \( \delta_s \) explains the deviation of the shadow from circular. We consider three points top, bottom and rightmost of the shadow (see Fig.2), which expressed, respectively, by \((\alpha_0, \beta_0), (\alpha_b, \beta_b)\) and \((\alpha_r, 0)\), so we have

\[
R_s = \frac{(\alpha_t - \alpha_r)^2 + \beta_t^2}{2(\alpha_t - \alpha_r)}. \tag{40}
\]

Furthermore, \( \delta_s \) is indicated by \((\bar{\alpha}_p, 0)\) and \((\alpha_p, 0)\) as

\[
\delta_s = \frac{(\bar{\alpha}_p - \alpha_p)}{R_s}, \tag{41}
\]

where \((\alpha_p, 0)\) and \((\bar{\alpha}_p, 0)\) are the points, where the contour of the shadow and reference circle cut the horizontal axis at opposite side of \((\alpha_r, 0)\). In Fig. 3, the observable \( R_s \) and \( \delta_s \) for different values of \( f'(R_0) \) is plotted. We can see that, by increasing \( f'(R_0) \), \( R_s \) increases and \( \delta_s \) decreases. Thus, the larger value of parameter \( f'(R_0) \) leads to increasing in the size and decreasing in the distortion of the shadow.

### 5 Shadow of black hole for an observer at \( r = r_O \)

In this section, we study the shadow of a black hole for an observer at \( r = r_O \). We are interested in spherical lightlike geodesics, i.e., lightlike geodesics that stay on a sphere \( r = \) constant. The region which is filled by these geodesics is called the photon region. The Photon region around the black hole is essential for building shadow. We can say that the shadow is an image of the photon region. We put our observer at \((r_O, \theta_O)\), in the domain of outer communication, in this case \( \Lambda > 0 \) and we consider a light source at \( r = r_L \), where \( r_L \geq r_O \). In the introduction, we explain how the shadow forms. Now, we have an observer at \((r_O, \theta_O)\), and our purpose is to calculate the shadow of a black hole in this situation. At first, we choose an orthonormal tetrad [86] at the observer’s sky [78]

\[
\begin{align*}
  e_0 &= \frac{\alpha^2 r^2 + \beta^2 \sin^2 \theta}{\sqrt{\Delta_0}} |(r_0, \theta_0)>, \\
  e_1 &= \frac{\sqrt{\Delta_0}}{\rho} \partial_\rho |(r_0, \theta_0)>, \\
  e_2 &= -\left(\frac{\alpha^2 r^2 + \beta^2 \sin^2 \theta}{\sqrt{\Delta_0} \rho} \right) |(r_0, \theta_0)>, \\
  e_3 &= -\frac{\Delta_0}{\rho} \partial_\rho |(r_0, \theta_0)>. 
\end{align*} \tag{42}
\]
Fig. 1 The image of black hole’s shadow in $f(R)$ gravity, $a = 0$, $a = 0.5$, $a = 0.7$ and $a = 1$ for (a), (b), (c) and (d), respectively. For each value of $a$, $Q = 0$ (blue dash-dot line), $Q = \frac{Q_c}{2}$ (red dash line) and $Q = Q_c$ (green filled solid line) for (a), (b), (c) but for (d) only, $Q = 0$ is considered. The brown solid line is reference circle which is the edge of the Schwarzschild shadow. The detail of parameters is shown in Table 1 (see Appendix).

Fig. 2 The black hole shadow and reference circle. $d_\ell$ is the distance between the left point of the shadow and the reference circle.

Note that the different positions of an observer can change these orthonormal tetrads. Since we put our observer in the domain of external communication, $\Delta r$ is positive, and the coefficients in Eq. (42) are real. The vector $e_3$ gives the spatial direction to the center.
Fig. 3 Changes of $R_s$ and $\delta_s$ for an observer at $r \to \infty$. $a = 0.3$, $a = 0.6$ and $a = 0.7$ for the purple (dot line), the blue (dash line) and the red (dash-dot line), respectively.

of the black hole. In addition, each light ray $\lambda(s)$ has the coordinate $r(s)$, $\theta(s)$, $\varphi(s)$, $t(s)$, so the tangent vector at the position of the observer can be written as

$$\dot{\lambda} = \partial_r + \dot{\theta} \partial_\theta + \dot{\varphi} \partial_\varphi + \dot{t} \partial_t,$$

(43)

or

$$\dot{\lambda} = \Omega(-e_0 + \sin \omega \cos \psi e_1 + \sin \omega \sin \psi e_2 + \cos \omega e_3),$$

(44)

where $\Omega$ is a scalar factor and it can be obtained from Eqs. (11), (42) and (43) as

$$\Omega = g(\dot{\lambda}, e_0) = \frac{aL - \frac{(a^2+r^2)E}{6}}{\sqrt{\Delta_r}} |_{(r_0,\theta_0)}.$$  

(45)

With comparison coefficients of $\partial_\varphi$ and $\partial_r$ in Eqs. (43) and (44), we have

$$\sin \psi = -\frac{\rho \sqrt{\Delta_r}}{\sin \omega} \frac{\sin \theta}{\frac{\rho \sqrt{\Delta_r}}{\sin \omega}} \left(\frac{\psi}{\Omega} + \frac{a}{\rho \sqrt{\Delta_r}}\right) |_{(r_0,\theta_0)},$$

(46)

$$\cos \omega = -\frac{\rho \sqrt{\Delta_r}}{\Omega} |_{(r_0,\theta_0)},$$

(47)

where $\psi$ and $\omega$ are the celestial coordinates. Also, as previous section, the constants of motion for this light ray are $\xi (30)$ and $\eta (31)$. Using Eqs. (22), (24), (45), (30) and (31), we can calculate $\sin \psi$ and $\sin \omega$.

Using Eqs. (46)–(31) and stereographic projection from the celestial sphere onto a plane [78], we plot the images of the black hole’s shadow. Eqs. (46) and (47) give the contour of the shadow of the black hole. The contour of the shadow demonstrated the light rays which approach the spherical light like geodesic with radius $r_p$. Moreover, the Cartesian coordinate is obtained by

$$x(r_p) = -2 \tan \frac{\omega(r_p)}{2} \sin \psi(r_p),$$

$$y(r_p) = -2 \tan \frac{\omega(r_p)}{2} \cos \psi(r_p).$$

(48)

So, these equations are used for plotting the image of the black hole’s shadow. Some examples of these plots are shown in Figs. 4, 5, and 6. In these figures, $\theta_0 = \frac{\pi}{2}$ and the observer is located in the domain of external communication.

It can be seen from Figs. 4, 5, and 6, by increasing the spin parameter, the shadow of black holes becomes more symmetric into the vertical axis, and the size of the shadow decreases by increasing the electric charge from left to right.

Also, in Fig. 7, the effect of $\theta_0$ on the image of the black hole’s shadow is demonstrated. It can be observed from this figure that by decreasing $\theta_0$ from $\frac{\pi}{2}$ to a limit of 0, the asymmetric decreases.
Moreover, the effect of $\Lambda$ on the size and shape of the shadow is shown in Fig. 8. It is obvious from Fig. 8 that for $\Lambda$ from 0 to $6 \times 10^{-2}$ and the observer placed at $\theta_O = \pi/2$ and $r_O = 5M$, the size of shadow become smaller. But in Fig. 8d, since $a = 0$, these changes do not appear.

6 Shadow of black hole in the presence of plasma at $r \rightarrow \infty$

This section will discuss the shadow of a black hole in the presence of plasma at $r \rightarrow \infty$. In 1975, Bicak and Hadrava [87] had investigated the travel of radiation in a dispersive and isotropic environment in General Relativity. In addition, the shadow of a black hole in the presence of plasma has been discussed in Refs. [88–90]. In this section, we analyze the effect of the plasma around the
black holes in $f(R)$ gravity. The plasma has the refraction index equal to $n = n(x^i, \omega)$. This refraction index is attached to the photon four-momentum as

$$ n^2 = 1 + \frac{p_\alpha p^\alpha}{(p_\mu u^\mu)^2}, \quad (49) $$

where $u^\mu$ is the observer velocity. Note that, for the vacuum environment, $n = 1$. Introducing the specific form of the plasma frequency for analytical results, we have

$$ n^2 = 1 - \frac{\omega_e^2}{\omega_p^2}, \quad (50) $$

where $\omega_p$ is the photon frequency and $\omega_e$ is the plasma frequency. Using the Hamilton–Jacobi equation for this geometry $[93]$ 

$$ \frac{\partial S}{\partial \tau} = -\frac{1}{2} \left[ g^{ij} p_i p_j - (n^2 - 1) \left( p_0 \sqrt{-g^{00}} \right)^2 \right]. \quad (51) $$

The equations of motion of photons in the presence of plasma can be obtained as

$$ \rho^4 \left( \frac{d r}{d \tau} \right)^2 = - \Delta_r (K + \varepsilon r^2) + \left[ (a^2 + r^2) E - a L \Xi \right]^2 + (r^2 + a^2)^2 (n^2 - 1) E^2 = R(r), \quad (52) $$

$$ \rho^4 \left( \frac{d \theta}{d \tau} \right)^2 = \Delta_\theta (K - \varepsilon a^2 \cos^2 \theta) - \frac{1}{\sin^2 \theta} (a E \sin^2 \theta - L \Xi)^2 - (n^2 - 1) a^2 E^2 \sin^2 \theta = \Theta(\theta), \quad (53) $$

$$ \rho^2 \left( \frac{d \varphi}{d \tau} \right) = \frac{a E (a^2 + r^2) - a^2 \Xi^2 L}{\Delta_r} - \frac{1}{\Delta_\theta} \sin^2 \theta (a \Xi E \sin^2 \theta - \Xi^2 L), \quad (54) $$

$$ \rho^2 \left( \frac{d t}{d \tau} \right) = \frac{n^2 E (r^2 + a^2)^2 - a L \Xi (r^2 + a^2)}{\Delta_r} - \frac{1}{\Delta_\theta} \sin^2 \theta \left( n^2 E a^2 - \frac{L \Xi a}{\sin^2 \theta} \right). \quad (55) $$

Now, we consider the plasma frequency as

$$ \omega_e^2 = \frac{4 \pi e^2 N(r)}{m_e}, \quad (56) $$

where $m$ and $e$ are the mass and the electron charge, respectively. Also, in Eq. (56), $N(r)$, is the plasma number density, which is considered as below

$$ N(r) = \frac{N_0}{r^\kappa}, \quad (57) $$
So, we have

$$\omega_r^2 = \frac{4\pi e^2 N_0}{m_e r^h} = \frac{k}{r^h},$$

in which $h \geq 0$. In following, for this case, we consider $h = 1$ \cite{95}, and $n$ is equal to $\sqrt{1 - \frac{k}{r^2}}$. Therefore, we obtain the constants of motion (i.e., $\eta$ and $\xi$), using $R(r) = 0$ and $\dot{R}(r) = 0$ conditions in Eq. (52). Then, for an observer in $\theta_o = \pi/2$, the celestial coordinates (32)--(33) will take the forms

$$\alpha = -\frac{\xi}{n},$$

$$\beta = \frac{\sqrt{\eta + a^2 - n^2 a^2}}{n}.$$  \hspace{1cm} (59)

Now, using $\alpha$ and $\beta$, we plot some examples of black holes’ shadows in the presence of plasma, which are shown in Fig. 9. It can be seen from Fig. 9 that the shape and size of the shadow are dependent on values of $a$, $Q$, and plasma parameters. In addition, again, we put $\Lambda = 0$ for this situation because our observer is placed at infinity.

In Fig. 10, the effects of different plasma parameter $k$ on the shape of shadow are investigated. One can see that the size of the black hole’s shadow decreases in the presence of plasma; in other words, the size of the shadow decreases when $k$ increases.

Moreover, the influence of charge and rotation parameters in the presence of plasma is similar to the vacuum state (i.e., by increasing electric charge, the size of the shadow becomes smaller, and, by increasing spin parameter, the symmetry of black hole’s shadow decreases). On the other hand, the radius of shadow in the presence of plasma is always less than or equal to the vacuum state.
Fig. 8 Shadow of black holes for $a = \frac{20}{100} a_{\text{max}}$. $Q = 0$, $Q = 0.75$, $Q = 1.35$ and $Q = 1.49$ in (a), (b), (c) and (d), respectively. Each figure is shown for different values of $\Lambda$. The blue dash-dot line is for $\Lambda = 0$, in the dashed red line $\Lambda = 10^{-2}$ and in the green solid filled line $\Lambda = 6 \times 10^{-2}$, the detail of parameters is shown in Table 3.

7 The shadow for an observer in $(r_O, \theta_O)$ in the presence of plasma

In this section, we plot the shadow of black hole in $f(R)$ gravity for an observer in $r=r_O$ in the presence of plasma. For this purpose, by the conditions $R(r) = 0$ and $\frac{dR(r)}{dr} = 0$ for Eq. (52), we can obtain the constant of motion, i.e., $(\xi$ and $\eta$) as

$$\xi = \frac{a^2 r \Delta_r' + r^3 \Delta_r'}{r \Delta_r' a \Xi} + 2 r^2 \Delta_r + \sqrt{\mathcal{N}},$$ (60)

$$\mathcal{N} = -a^4 n^2 r^2 \Delta_r'^2 - 2 a^2 n^2 r^4 \Delta_r'^2 - n^2 r^6 \Delta_r'^2 - a^4 n^2 r \Delta_r' \Delta_r' + 2 a^2 n^2 r^3 \Delta_r \Delta_r' \Delta_r' + 3 n^2 r^5 \Delta_r \Delta_r' \Delta_r' + a^4 r^2 \Delta_r'^2 + 2 a^2 r^4 \Delta_r'^2 + r^6 \Delta_r'^2 + a^4 r \Delta_r \Delta_r' \Delta_r' - 2 a^2 r^3 \Delta_r \Delta_r' \Delta_r' - 3 r^5 \Delta_r \Delta_r' + 4 r^4 \Delta_r'^2.$$ (61)

and

$$\eta = \frac{\Xi a^2 \xi^2 + a^4 n^2 + 2 a^2 n^2 r^2 + n^2 r^4 - 2 \Xi a^3 \xi - 2 \Xi a r^2 \xi}{\Delta_r}.$$ (62)

Eqs. (60) and (62) convert to (30) and (31) when $n$ is equal to 1 (vacuum case). Using Eqs. (42), (46), (47), the geodesic equations in the presence of plasma (52)–(55) and constant of motion, (60) and (62), we obtain the Cartesian coordinates as

$$x(r_p) = -2 \tan \frac{\omega(r_p)}{2} \sin \psi(r_p),$$
$$y(r_p) = -2 \tan \frac{\omega(r_p)}{2} \cos \psi(r_p).$$ (63)
Fig. 9 Shadow of the black hole in the presence of plasma for different values of the rotation parameter $a = 0, a = 0.5, a = 0.7$ and $a = 1$ in (a), (b), (c) and (d), respectively. In each figure, $Q = 0$, $Q = \frac{Q_{\text{crit}}}{2}$ and $Q = Q_{\text{crit}}$ is shown by blue dash-dot line, red dash line and green solid filled line, respectively. The solid brown circle is reference circle which is the edge of the Schwarzschild shadow. The detail of parameters is shown in Table 4 (see Appendix).

Fig. 10 Plot showing the influence of $k$ for $a = 0.7$ and $Q = 0$. The blue dash-dot line indicate $k = 0$, the dash red line show $k = 0.15$ and the green filled line show $k = 0.25$

Now, we use of $x(r_p)$ and $y(r_p)$ parameters to plot some examples of black hole’s shadow. In this situation, the observer is in $(r_0, \theta_0)$ in the domain of outer communication, so we can put $\Lambda > 0$. The shadows for different values of spin $a$, are shown in Fig.11.
Fig. 11 Shadow of black hole in the presence of plasma for specific observer. In this figure $Q = 0.75$, $A = 0$, $\Lambda = 0.01$ and $\Lambda = 0.06$ in (a), (b) and (c), respectively. In (a) and (b), shows $\frac{1}{5}a_{\text{max}}$, $\frac{2}{5}a_{\text{max}}$ and $\frac{3}{5}a_{\text{max}}$ by the blue dash-dot line, the red dash line and the green filled shape, respectively. The solid brown circle is reference circle $r_O = 5M$. The detail of parameter are shown in Table 5 (see Appendix).

Fig. 12 Shadow of a black hole for an observer at $r_O = 5M$ and different inclination angles $\theta_O$, with fixed $Q = 0.75, A = 0.01$ and $a = \frac{45}{54}a_{\text{max}}$. The detail of parameter are shown in Table 6 (see Appendix).
One can see that by increasing $a$, the shape of the shadow deviates into the horizontal axis. The effect of a spin parameter in this situation is similar to the vacuum case for an observer in a specific coordinate $(r_O, \theta_O)$. However, it is essential to note that in similar circumstances, the radius of the shadow is smaller than or equal to the radius of the black hole’s shadow in a vacuum.

In addition, we analyze the effect of inclination $\theta_O$ on the shape of the black hole’s shadow in Fig. 12. We see that the symmetry in the vertical axis becomes better if the observer approaches the axis, in the limit of $\theta_O \to 0$, which is similar to the vacuum situation.

## 8 Conclusion

This paper investigated charged rotating black holes in $f(R)$ gravity. First, we obtained a geodesic equation for this space-time. Then, using geodesic equations, we studied the image of a black hole’s shadow in the absence and the presence of plasma for an observer at infinity and limited distance. The results show that space-time parameters affect the size and symmetry of the black hole’s shadow. It can be seen that for all cases, increasing the rotation parameter reduces the circular symmetry of the shadow image of the black hole. Also, it is shown that by increasing the electric charge $Q$, the size of the shadow decreases. Furthermore, the effect of modified gravity parameter $f'(R_0)$ was investigated for an observer who was placed at infinity in a vacuum. We observed that, by increasing $f'(R_0)$, the size of the shadow image increases, and the symmetry of the black hole’s shadow improves. In addition, for an observer at infinity, the shadow’s size in the presence of plasma is less than or equal to the size of shadow in a vacuum. Also, changes in inclination angle $\theta_O$ and $\Lambda$ parameter were investigated for an observer at a limited distance. We show that, by changing $\theta_O$ from $\frac{\pi}{2}$ to a limit of zero, the deviation of the shadow image decreases in a vacuum and the presence of plasma. In addition, we show that for an observer at a limited distance in a vacuum, increasing $\Lambda$ will affect the size of the shadow, and it becomes small.

Since the question is considered in this paper is categorized as a form of alternatives theories to General Relativity and so belongs to the second group of theories that have been assessed in [7], the image of shadows are qualitatively very similar to those of Kerr black holes and are not distinguishable with present EHT capabilities. However, higher-frequency observations, together with the degree of polarization of the emitted radiation or the variability of the accretion flow, must be taken to assess its viability. All detail data of figures are shown in Tables 1–6 (see Appendix).

### Appendix

#### Table 1

| Constant parameters | Fig | $a$ | $Q_{\text{crit}}$ | $\Delta r$ ( roots ) |
|---------------------|-----|-----|------------------|----------------------|
| $\Lambda = 0$       | 1a  | 0   | 1.49             | 1.11, 0.88           |
| $\theta_O = \frac{\pi}{2}$ | 1b  | 0.5 | 1.29             | 1.10, 0.89           |
| $M = 1$             | 1c  | 0.7 | 1.07             | 1.03, 0.96           |
| $f'(R_0) = 1.25$    | 1d  | 1   | 0                | 1                    |

#### Table 2

| Constant parameters | Fig | $Q$ | $a_{\text{max}}$ | $\Delta r$ ( roots ) |
|---------------------|-----|-----|------------------|----------------------|
| $\theta_O = \frac{\pi}{2}$ | 4a  | 0   | 1                | 1                    |
| $\Lambda = 0$       | 4b  | 0.75| 0.86             | 1.10, 0.89           |
| $M = 1$             | 4c  | 1.35| 0.43             | 1.07, 0.92           |
| $\Lambda = 0.01$    | 5a  | 0   | 1                | 0.92, 1.09, 16.22, −18.24 |
| $M = 1$             | 5b  | 0.75| 0.86             | 0.88, 1.13, 16.23, −18.24 |
| $\theta_O = \frac{\pi}{2}$ | 5c  | 1.35| 0.44             | 0.98, 1.02, 16.24, −18.26 |
| $\Lambda = 0.06$    | 6a  | 0   | 1                | 0.99, 1.14, 5.75, −7.90 |
| $M = 1$             | 6b  | 0.75| 0.88             | 0.93, 1.19, 5.78, −7.91 |
| $\Lambda = 0.06$    | 6c  | 1.35| 0.46             | 0.97, 1.12, 5.84, −7.94 |

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The details of parameter in Fig. 8, for an observer at limited distance

| Constant parameters | Fig | $\Lambda$ | $Q$ | $a_{\text{max}}$ | $a_{\text{max}}$ | $\Delta_{r}$ (roots) |
|---------------------|-----|----------|------|-----------------|-----------------|-------------------|
| $\theta_0 = \frac{\pi}{2}$ | 0 | 0 | 70/10 | 1 | 1.71, 0.29 |
| $f'(R_0) = 1.25$ | 8a | 0.01 | 0 | 70/10 | 1 | 0.28, 1.73, 16.21, −18.24 |
| $M = 1$ | 0.06 | 0.28, 1.89, 5.72, −7.90 |
| $\theta_0 = \frac{\pi}{2}$ | 0 | 1.62, 0.37 |
| $f'(R_0) = 1.25$ | 8b | 0.01 | 0.75 | 70/10 | 0.86 | 0.37, 1.64, 16.22, −18.25 |
| $M = 1$ | 0.06 | 0.37, 1.78, 5.76, −7.92 |
| $\theta_0 = \frac{\pi}{2}$ | 0 | 1.29, 0.70 |
| $f'(R_0) = 1.25$ | 8c | 0.01 | 1.35 | 70/10 | 0.46 | 0.70, 1.31, 16.24, −18.26 |
| $M = 1$ | 0.06 | 0.69, 1.41, 5.84, −7.94 |
| $\theta_0 = \frac{\pi}{2}$ | 0 | 1.10, 0.89 |
| $f'(R_0) = 1.25$ | 8d | 0.01 | 1.49 | 70/10 | 0 | 0.88, 1.13, 16.25, −18.26 |
| $M = 1$ | 0.06 | 0.85, 1.24, 5.86, −7.95 |

The effect of $\Lambda$ was investigated in this figure in vacuum.

The details of parameter in Fig. 9 for an observer at $\infty$, the effect of $a$ and $Q$, were investigated in this figure in the presence of plasma.

Plasma parameter is $k = 0.1$

| Constant parameters | Fig | $Q_{\text{crit}}$ | $a$ | $\Delta_{r}$ (roots) |
|---------------------|-----|-----------------|------|-------------------|
| $\Lambda = 0$ | 9a | 1.48 | 0 | 1.08, 0.89 |
| $\theta_0 = \frac{\pi}{2}$ | 9b | 1.26 | 0.5 | 1.21, 0.78 |
| $M = 1$ | 1 | 0.7 | 1.24, 0.75 |
| $f'(R_0) = 1.25$ | 9c | 0 | 1.34, 0.65 |
| $\theta_0 = \frac{\pi}{2}$ | 0 | 1.62, 0.37 |
| $f'(R_0) = 1.25$ | 11a | 0 | 0.86 | 1.62, 0.37 |
| $M = 1$ | 11b | 0.01 | 0.75 | 0.86 | 1.64, 0.37, 16.22, −18.25 |
| $\theta_0 = \frac{\pi}{2}$ | 11c | 0.06 | 0.88 | 1.78, 0.37, 5.76, −7.92 |

The plasma parameter is $k = 0.1$

The details of parameter in Fig. 11, the effect of spin parameter, $a$ for an observer at limited distance in the presence of plasma.

| Constant parameters | Fig | $\Lambda$ | $Q_{\text{crit}}$ | $a_{\text{max}}$ | $\Delta_{r}$ (roots) |
|---------------------|-----|----------|------|-----------------|-------------------|
| $\theta_0 = \frac{\pi}{2}$ | 11a | 0 | 0.86 | 1.62, 0.37 |
| $f'(R_0) = 1.25$ | 11b | 0.01 | 0.75 | 0.86 | 1.64, 0.37, 16.22, −18.25 |
| $M = 1$ | 11c | 0.06 | 0.88 | 1.78, 0.37, 5.76, −7.92 |

The plasma parameter is $k = 0.1$

The details of parameter in Fig. 12, the effect of electric charge parameter, $Q$ for an observer at limited distance in the presence of plasma.

| Constant parameters | Fig | $\Lambda$ | $Q_{\text{crit}}$ | $a_{\text{max}}$ | $\Delta_{r}$ (roots) |
|---------------------|-----|----------|------|-----------------|-------------------|
| $\theta_0 = \frac{\pi}{2}$ | 12a | 0 | 0.559 | 1.66, 0.33 |
| $f'(R_0) = 1.25$ | 12b | 0.01 | 0.75 | 0.559 | 1.85, 0.17, 16.21, −18.24 |
| $M = 1$ | 12c | 0.06 | 0.344 | 1.97, 0.20, 5.74, −7.92 |

The plasma parameter is $k = 0.1$

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