DEHORNOY-LIKE LEFT ORDERINGS AND ISOLATED LEFT ORDERINGS

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Abstract. We introduce a Dehornoy-like ordering of groups, which is a generalization of the Dehornoy ordering of the braid groups. Under a weak assumption which we call Property \( F \), we show that Dehornoy-like orderings have properties similar to the Dehornoy ordering, and produce isolated left orderings. We also construct new examples of Dehornoy-like ordering and isolated orderings and study their more precise properties.

1. Introduction

A left-ordering of a group \( G \) is a total ordering \(<_G\) of \( G \) preserved by the left action of \( G \) itself. That is, \( g <_G g' \) implies \( hg <_G hg' \) for all \( g, g', h \in G \). A group \( G \) is left-orderable if \( G \) has at least one left-ordering.

One of the most important left ordering is the Dehornoy ordering of the braid group \( B_n \). The Dehornoy ordering has a simple, but still mysterious definition which uses a special kind of word representatives called \( \sigma \)-positive words. The Dehornoy ordering can be regarded as the most natural left ordering of the braid groups, but its combinatorial structure is rather complicated.

In this paper we introduce a Dehornoy-like ordering of groups. This is a left-ordering defined in a similar way to the Dehornoy ordering. The aim of this paper is to study Dehornoy-like orderings and give new examples of Dehornoy-like orderings.

The study of the Dehornoy-like ordering produces another interesting family of left-orderings. Recall that the positive cone \( P = \{ g \in G \mid 1 <_G g \} \) of a left ordering \(<_G\) has the following two properties \( \text{LO1} \) and \( \text{LO2} \).

\[
\text{LO1: } P \cdot P \subset P,
\text{LO2: } G = P \coprod \{1\} \coprod P^{-1}.
\]

Conversely, for a subset \( P \) of \( G \) having the properties \( \text{LO1} \) and \( \text{LO2} \) one can obtain a left-ordering \(<_G\) by defining \( h <_G g \) if \( h^{-1}g \in P \). Thus, the set of all left-orderings of \( G \), which we denote by \( \text{LO}(G) \), is naturally regarded as a subset of the powerset \( 2^{G^{-1}} = \{+, -\}^{G^{-1}} \). We equip a discrete topology on \( 2 = \{+, -\} \) and equip the power set topology on \( 2^{G^{-1}} \). This induces a topology on \( \text{LO}(G) \) as the subspace topology. \( \text{LO}(G) \) is compact, totally disconnected, and metrizable [14]. It is known that for a countable group \( G \), \( \text{LO}(G) \) is either finite or uncountable [9]. So \( \text{LO}(G) \) is very similar to the Cantor set if \( G \) has infinitely many left orderings.

An isolated ordering is a left-ordering which corresponds to an isolated point of \( \text{LO}(G) \). Isolated orderings are easily characterized by their positive cones. Observe

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that by \( \text{LO1} \), the positive cone of a left ordering is a submonoid of \( G \). A left-ordering \(<\) is isolated if and only if its positive cone is finitely generated as a submonoid of \( G \).

We begin with a systematic study of Dehornoy-like orderings in Section 2. We introduce a property called \( \text{Property F} \) for an ordered finite generating set \( S \), which plays an important role in our study of Dehornoy-like orderings. We show that \( \text{Property F} \) allows us to relate Dehornoy-like orderings and isolated orderings in a very simple way. Moreover, using \( \text{Property F} \) we generalize known properties of the Dehornoy ordering of \( B_2 \) to Dehornoy-like orderings of a general group \( G \).

In section 3, we construct a new example of Dehornoy-like and isolated orderings and study their detailed properties. Our examples are generalization of Navas’ example of Dehornoy-like and isolated orderings given in [11].

We consider the groups of the form \( \mathbb{Z} \ast \mathbb{Z} \), the amalgamated free product of two infinite cyclic groups. Such a group is presented as

\[
G_{m,n} = \langle x, y \mid x^m = y^n \rangle \quad (m \geq n).
\]

The groups \( G_{m,n} \) appear in many contexts. Observe that \( G_{2,2} \) is the Klein bottle group and \( G_{3,2} \) is the 3-strand braid group \( B_3 \). For coprime \((m, n)\), \( G_{m,n} \) is nothing but the fundamental group of the complement of the \((m, n)\)-torus knot. The family of groups \( \{G_{m,2}\} \) are the central extension of the Hecke groups, studied by Navas in [11]. We always assume \((m, n) \neq (2, 2)\) because the Klein bottle group \( G_{2,2} \) is exceptional since it admits only finitely many left orderings. As we will see later, other groups \( G_{m,n} \) have infinitely many (hence uncountably many) left orderings.

To give a Dehornoy-like and an isolated ordering, we introduce generating sets \( S = \{s_1 = xyx^{-m+1}, s_2 = x^{m-1}y^{-1}\} \) and \( A = \{a = x, b = yx^{-m+1}\} \). Using the generator \( S \), the group \( G_{m,n} \) is presented as

\[
G_{m,n} = \langle s_1, s_2 \mid s_2s_1s_2 = ((s_1s_2)^{m-2}s_1)^{-1} \rangle
\]

Observe that for \((m, n) = (3, 2)\), this presentation agrees with the standard presentation of the 3-strand braid group \( B_3 \). Similarly, using the generators \( A \), the group \( G_{m,n} \) is presented as

\[
G_{m,n} = \langle a, b \mid (ba^{m-1})^{-1} = a \rangle.
\]

For \( n = 2 \), the above presentation coincide with Navas’ presentation of \( G_{m,2} \). We will show that \( S \) defines a Dehornoy-like ordering \(<_D\) of \( G_{m,n} \) and \( A \) defines an isolated ordering \(<_A\) of \( G_{m,n} \) in Theorem 3.

We will also give an alternative description of the Dehornoy-like ordering \(<_D\) of \( G_{m,n} \) in Theorem 5 by using the action on the Bass-Serre tree. Such an action is natural since \( G_{m,n} = \mathbb{Z} \ast \mathbb{Z} \) is an amalgamated free product. In this point of view, the Dehornoy-like ordering \(<_D\) can be regarded as a natural left ordering of \( G_{m,n} \) like the Dehornoy ordering of \( B_3 \), although the combinatorial definition seems to be quite strange.

The dynamics of ordering allows us to give more detailed properties of the Dehornoy-like ordering \(<_D\). In Theorem 6 we will show that one particular property of the Dehornoy ordering, called \( \text{Property S} \) (Subword Property), fails for the Dehornoy-like ordering of \( G_{m,n} \). However, we observe that the Dehornoy-like ordering of \( G_{m,n} \) have a slightly weaker property in Theorem 7.

As an application, by using the Dehornoy-like ordering of \( G_{m,n} \), we construct left orderings having an interesting property: a left-ordering which admits no non-trivial
proper convex subgroups. In fact, we observe that almost all normal subgroup of $G_{m,n}$ contains no non-trivial proper convex subgroup in Theorem 8.

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2. Dehornoy-like ordering

Throughout the paper, we always assume that $G$ is a finitely generated group.

2.1. Dehornoy-like ordering. Let $S = \{s_1, \ldots, s_n\}$ be an ordered finite generating set of $G$. We consider the two submonoids of $G$ defined by $S$, the $S$-word positive monoid and the $\sigma(S)$-positive monoid.

A ($S$-) positive word is a word on $S$. We say an element $g \in G$ is ($S$-) word positive if $g$ is represented by a $S$-positive word. The set of all $S$-word positive elements form a submonoid $P_S$ of $G$, which we call the ($S$)-word positive monoid. The $S$-word positive monoid is nothing but a submonoid of $G$ generated by $S$.

To define a Dehornoy-like ordering, we introduce slightly different notions. A word $w$ on $S \cup S^{-1}$ is called $i$-positive (or, $i(S)$-positive, if we need to indicate the ordered finite generating set $S$) if $w$ contains at least one $s_i$ but contains no $s_1^{-1}, \ldots, s_{i-1}^{-1}, s_{i+1}, \ldots, s_n$. We say an element $g \in G$ is $i$-positive (or, $i(S)$-positive) if $g$ is represented by an $i$-positive word. An element $g \in G$ is called $\sigma$-positive ($\sigma(S)$-positive) if $g$ is $i$-positive for some $1 \leq i \leq n$. The notions of $i$-negative and $\sigma$-negative are defined in the similar way. The set of $\sigma(S)$-positive elements of $G$ forms a submonoid $\Sigma_S$ of $G$. We call the monoid $\Sigma_S$ the $\sigma$-positive monoid (or, $\sigma(S)$-positive monoid).

Definition 1 (Dehornoy-like ordering). A Dehornoy-like ordering is a left ordering $<_D$ whose positive cone is equal to the $\sigma(S)$-positive monoid $\Sigma_S$ for some ordered finite generating set $S$ of $G$. In this situation, we say $S$ defines a Dehornoy-like ordering.

To study Dehornoy-like orderings we introduce the following two properties.

Definition 2. Let $S$ be an ordered finite generating set of $G$.

1. We say $S$ has Property A (the Acyclic property) if no $\sigma(S)$-positive words represent the trivial element. That is, $\Sigma_S$ does not contains the identity element 1.

2. We say $S$ has Property C (the Comparison property) if every non-trivial element of $G$ admits either $\sigma(S)$-positive or $\sigma(S)$-negative word expression.

Proposition 1. Let $S$ be an ordered finite generating set of a group $G$. Then $S$ defines a Dehornoy-like ordering if and only if $S$ has both Property A and Property C.

Proof. Property C implies that $G = \Sigma_S \cup \Sigma_S^{-1} \cup \{1\}$, and the Property A implies that $\Sigma_S, \Sigma_S^{-1}$ and $\{1\}$ are disjoint. Thus, the $\sigma(S)$-positive monoid $\Sigma_S$ satisfies both LO1 and LO2. Converse is clear.

As we have already mentioned, the definition of Dehornoy-like orderings is motivated from the Dehornoy ordering of the braid groups.
Example 1 (Dehornoy ordering of \(B_n\)). Let us consider the standard presentation of the braid group \(B_n\),
\[ B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \mid |i-j| = 1 \rangle \]
and \(S = \{\sigma_1, \ldots, \sigma_{n-1}\}\) be the set of the standard generators. The seminal work of Dehornoy [3] shows that \(S\) has both Property A and Property C, hence \(S\) defines a left-ordering of \(B_n\). The ordering \(\prec_D\) defined by \(S\) is called the Dehornoy ordering.

Interestingly, there are various proof of Property A and Property C, and each proof gives a new characterization of the Dehornoy ordering. A proof of Property A or Property C provides new insights for not only the Dehornoy ordering, but also the braid group itself. See [5] for the theory of the Dehornoy orderings. Moreover, as the author showed, the Dehornoy orderings are also related to the knot theory [7,8].

Now we introduce an operation to construct new ordered finite generating sets from an ordered finite generating set, which connects a Dehornoy-like ordering and an isolated ordering.

The twisted generating set of \(S\) is an ordered finite generating set \(A = A_S = \{a_1, \ldots, a_n\}\) where each \(a_i\) is defined by
\[ a_i = (s_i \cdots s_{n-1})^{(-1)^{n-i+1}}. \]

An ordered finite generating set \(D = D_S = \{d_1, \ldots, d_n\}\) whose twisted generating set is equal to \(S\) is called the detwisted generating set of \(S\). The detwisted generating set \(D\) is explicitly given as
\[ d_i = \begin{cases} 
  s_{n-1}^{-1} & \text{if } i = n \\
  s_{i+1}^{-1} & \text{if } n - i \text{ is even} \\
  s_is_{i+1} & \text{if } n - i \text{ is odd}
\end{cases} \]

For each \(1 \leq i \leq n\), let \(S^{(i)} = \{s_i, s_{i+1}, \ldots, s_n\}\) and \(G^{(i)}_S\) be the subgroup of \(G\) generated by \(S^{(i)}\). Thus, \(S^{(i)}\) is an ordered finite generating set of \(G^{(i)}_S\). We denote the \(S^{(i)}\)-word positive monoid \(P^{(i)}_S\) and the \(\sigma(S^{(i)})\)-positive monoid \(\Sigma^{(i)}_S\) by \(P^{(i)}_S\), \(\Sigma^{(i)}_S\) respectively. They are naturally regarded as submonoids of \(G^{(i)}_S\). By definition of the twisted generating set, \(A_{S^{(i)}} = (A_S)^{(i)}\). Thus, \(G^{(i)}_S = G^{(i)}_A\) so we will often write \(G^{(i)}\) to represent \(G^{(i)}_S = G^{(i)}_A\).

There is an obvious inclusion for \(\sigma(S)\)-positive and \(A\)-word positive monoids.

Lemma 1. Let \(S\) be an ordered finite generating set and \(A = A_S\) be the twisted generating set of \(S\). Then \(\Sigma_S \cup \Sigma_S^{-1} \supset P_A \cup P_A^{-1}\).

Proof: We show \(P_A \subset \Sigma_S \cup \Sigma_S^{-1}\). The proof of \(P_A^{-1} \subset \Sigma_S \cup \Sigma_S^{-1}\) is similar. Let \(g \in P_A\) and \(w\) be an \(A\)-positive word expression of \(g\). Put
\[ i = \min \{ j \in \{1, 2, \ldots, n\} \mid w \text{ contains the letter } a_j \} \]
Since \(a_i = (s_is_{i+1} \cdots s_n)^{(-1)^{n-i+1}}\), \(g\) is \(\sigma(S)\)-positive if \((n-i)\) is odd and is \(\sigma(S)\)-negative if \((n-i)\) is even. \(\square\)

Now we introduce a key property called Property \(F\) (the Filtration property) which allows us to generalize various properties of the Dehornoy ordering for Dehornoy-like orderings.
**Definition 3.** Let \( S = \{s_1, \ldots, s_n\} \) be an ordered finite generating set of \( G \) and \( A = \{a_1, \ldots, a_n\} \) be the twisted generating set of \( S \). We say \( S \) has Property \( F \) (the Filtration property) if

\[
F: a_i \cdot (P_A^{(i+1)})^{-1} \cdot a_i^{-1} \subset P_A^{(i)}, a_i^{-1} \cdot (P_A^{(i+1)})^{-1} \cdot a_i \subset P_A^{(i)}
\]

holds.

We say a finite generating set \( A \) defines an isolated ordering if the \( A \)-word positive monoid is the positive cone of an isolated ordering \( <_A \). First we show a Dehornoy-like ordering and an isolated ordering are closely related if we assume Property \( F \).

**Theorem 1.** Let \( S \) be an ordered finite generating set of \( G \) having Property \( F \) and \( A \) be the twisted generating set of \( S \). Then \( S \) defines a Dehornoy-like ordering of \( G \) if and only if \( A \) defines an isolated left ordering of \( G \).

**Proof.** Let \( n \) be the cardinal of the generating set \( S \). We prove theorem by induction on \( n \). The case \( n = 1 \) is trivial. General cases follow from the following two claims.

**Claim 1.** \( S \) has Property \( C \) if and only if \( P_A \cup P_A^{-1} \cup \{1\} = G \) holds.

By Lemma \( \square \) \( G = P_A \cup P_A^{-1} \cup \{1\} \subset \Sigma_S \cup \Sigma_S^{-1} \cup \{1\} = G \).

To show converse, assume that \( S \) has Property \( C \). Let \( g \in G \) be a non-trivial element. We assume that \( g \) is \( \sigma(S) \)-positive. The case \( g \) is \( \sigma(S) \)-negative is proved in a similar way.

First of all, assume that \( g \) has a \( k(S) \)-positive word representative for \( k > 1 \). Then \( g \in G^{(2)} \) and \( g \) is \( \sigma(S^{(2)}) \)-positive. By inductive hypothesis, \( g \in P_A^{(2)} \cup (P_A^{(2)})^{-1} \cup \{1\} \subset P_A \cup P_A^{-1} \cup \{1\} \).

Thus we assume that \( g \) is \( 1(S) \)-positive. We also assume that \( n \) is even. The case \( n \) is odd is similar. Since \( s_1 = a_1a_2 \), by rewriting a \( 1(S) \)-positive word representative of \( g \) by using the twisted generating set \( A \), we write \( g \) as

\[
g = V_0a_1V_1 \cdots a_1V_m
\]

where \( V_i \) is a word on \( A^{(2)} \cup (A^{(2)})^{-1} \subset G^{(2)} \). By inductive hypothesis, we may assume that either \( V_i \in P_A^{(2)} \) or \( V_i \in (P_A^{(2)})^{-1} \). If all \( V_i \) belong to \( P_A^{(2)} \), then \( g \in P_A \). Assume that some \( V_i \) belongs to \( (P_A^{(2)})^{-1} \). By Property \( F \), \( a_1V_i \subset P_A \cdot a_1 \) and \( V_i a_1 \subset a_1 \cdot P_A \), so we can rewrite \( g \) so that it belongs to \( P_A \).

**Claim 2.** \( S \) has Property \( A \) if and only if \( 1 \notin P_A \).

Assume that \( 1 \notin P_A \) and let \( g \in G \) be a \( \sigma(S) \)-positive element. If \( g \) has a \( k(S) \)-positive word representative for \( k > 1 \), then \( g \in G^{(2)} \) so inductive hypothesis shows \( g \neq 1 \). Thus we assume \( g \) is \( 1(S) \)-positive. Assume that \( n \) is even. Then as we have seen in the proof of Claim \( \square \) \( g \in P_A \) so we conclude \( g \neq 1 \). The case \( n \) is odd, and the case \( g \) is \( \sigma(S) \)-negative are proved in a similar way. Converse is obvious from Lemma \( \square \). \[ \square \]

Thus, we obtain a new criterion for an existence of isolated orderings.

**Corollary 1.** Let \( G \) be a left-orderable group. If \( G \) has a Dehornoy-like ordering having Property \( F \), then \( G \) also has an isolated left ordering.

Theorem \( \square \) is motivated from the construction of the Dubrovina-Dubrovin orderings, which are left-ordering obtained by modifying the Dehornoy ordering.

\( \square \)
Example 2 (Dubrovina-Dubrovin ordering). Let $A = \{a_1, \ldots, a_{n-1}\}$ be the twisted generating set of the standard generating set $S = \{\sigma_1, \ldots, \sigma_{n-1}\}$ of the braid group $B_n$. $S$ has the property $F$, so, the submonoid $A$ defines an isolated left-ordering which is known as the Dubrovina-Dubrovin ordering $<_{DD}$ [6].

2.2. Property of Dehornoy-like orderings. In this section we study fundamental properties of Dehornoy-like orderings and isolated orderings derived from the Dehornoy-like orderings.

Let $S = \{s_1, \ldots, s_n\} (n > 1)$ be an ordered finite generating set of a group $G$ which defines a Dehornoy-like ordering $<_D$ and $A$ be the twisted generating set of $S$.

We begin with recalling standard notions of left orderable groups. A left-ordering $<_G$ of $G$ is called discrete if there is the $<_G$-minimal positive element. Otherwise, $<_G$ is called dense. $<_G$ is called a Conradian ordering if $fg^k >_G g$ holds for all $<_G$-positive $f, g \in G$ and $k \geq 2$. It is known that in the definition of Conradian orderings it is sufficient to consider the case $k = 2$. That is, $<_G$ is Conradian if and only of $fg^2 >_G g$ holds for all $<_G$-positive elements $f, g \in G$ (See [10]).

A subgroup $H$ of $G$ is $<_G$-convex if $h <_G g <_G h'$ for $h, h' \in H$ and $g \in G$, then $g \in H$ holds. $<_G$-convex subgroups form a chain. That is, for $<_G$-convex subgroups $H, H'$, either $H < H'$ or $H' < H$ holds. The $<_G$-Conradian soul is the maximal (with respect to inclusions) $<_G$-convex subgroup of $G$ such that the restriction of $<_G$ is Conradian.

First of all, we observe that a Dehornoy-like ordering have the following good properties with respect to the restrictions.

**Proposition 2.** Let $S = \{s_1, \ldots, s_n\}$ be an ordered finite generating set of $G$ which defines a Dehornoy-like ordering $<_D$.

1. For $1 \leq i \leq n$, $S^{(i)}$ defines a Dehornoy-like ordering $<_D$ of $G^{(i)}$. Moreover, the restriction of the Dehornoy-like ordering $<_D$ to $G^{(i)}$ is equal to the Dehornoy-like ordering $<_D$.

2. For $1 \leq i \leq n$, the subgroup $G^{(i)}$ is $<_D$-convex. In particular, $<_D$ is discrete, and the minimal $<_D$-positive element is $s_n$.

3. If $H$ is a $<_D$-convex subgroups of $G$, then $H = G^{(i)}$ for some $1 \leq i \leq n$.

**Proof.** Since $S$ has Property $A$, $S^{(i)}$ has Property $A$. Assume that $S^{(i)}$ does not have Property $C$, so there is an element $g \in G^{(i)} - \{1\}$ which is neither $\sigma(S^{(i)})$-positive nor $\sigma(S^{(i)})$-negative. Assume that $1 <_D g$, so $g$ is represented by a $\sigma(S)$-positive word $W$. The case $1 >_D g$ is similar. Since $g \in G^{(i)}$ we may find a word representative $V$ of $g$ which consists of the alphabets in $S^{(i)} \cup S^{(i)}$. Then $V^{-1}W$ is $\sigma(S)$-positive word which represents the trivial element, so this contradicts the fact that $S$ has Property $A$. Thus, $S^{(i)}$ has Property $C$, hence $S^{(i)}$ defines a Dehornoy-like ordering $<_D$. Now the Property $A$ and Property $C$ of $S^{(i)}$ implies $\Sigma_{S^{(i)}} = \Sigma_S \cap G^{(i)}_S$, so $<_D$ is equal to the restriction of $<_D$ to $G^{(i)}_S$.

Next we show $G^{(i)}$ is $<_D$-convex. Assume that $1 <_D h <_D g$ hold for $g \in G^{(i)}$ and $h \in G$. If $h$ is $j(S)$-positive for $j < i$, then $g^{-1}h$ is also $j(S)$-positive, so $g <_D h$. This contradicts the assumption, so $h$ must be $j(S)$-positive for $j \geq i$. This implies $h \in G^{(i)}$, so we conclude $G^{(i)}$ is $<_D$-convex.

To show there are no $<_G$-convex subgroups other than $G^{(i)}$, it is sufficient to show if $H \supset G^{(2)}$ then $H = G^{(2)}$ or $G^{(1)} = G$. Assume that $H \neq G^{(2)}$, hence $H$
contains an element $g$ in $G - G^{(2)}$. Let us take such $g$ so that $1 <_D g$. Then $g$ must be $1(S)$-positive, hence we may write $g = hs_1 P$ where $h \in G^{(2)}$ and $P >_D 1$. Then we have

$$1 <_D hs_1 \leq_D hs_1 P = g$$

Since $H$ is convex, this implies $hs_1 \in H$. Since $h \in G^{(2)} \subset H$, we conclude $s_1 \in H$ hence $H = G$. 

From now on, we will always assume that $S$ has Property $F$, hence $A$ defines an isolated left ordering $<_A$. First of all we observe that $<_A$ also has the same properties as we have seen in Proposition

Proposition 3. Let $A = \{a_1, \ldots, a_n\}$ be the twisted generating set of $S$ which defines an isolated left ordering $<_A$.

1. For $1 \leq i \leq n$, $A^{(i)}$ defines an isolated ordering $<_A^{(i)}$ of $G^{(i)}$. Moreover, the restriction of the isolated ordering $<_A$ to $G^{(i)}$ is equal to the isolated ordering $<_A^{(i)}$.

2. For $1 \leq i \leq n$, the subgroup $G^{(i)}$ is $<_A$-convex. In particular, $<_A$ is discrete, and the minimal $<_A$-positive element is $a_n$.

3. If $H$ is an $<_A$-convex subgroup of $G$, then $H = G^{(i)}$ for some $1 \leq i \leq n$.

Proof. The proofs of (1) and (3) are similar to the case of Dehornoy-like orderings. To show (2), assume that $1 <_A h <_A g$ hold for $g \in G^{(i)}$ and $h \in G$. If $h \neq P^{(j)}$, then we may write $h$ as $h = h'a_j w$ where $h' \in P^{(j+1)}$ and $w \in P^{(j)} \cup \{1\}$ for $j < i$. If $g^{-1}h' \in P^{(j+1)}$, then $g^{-1}h = (g^{-1}h')a_jw >_A 1$. If $g^{-1}h' \in (P^{(j+1)})^{-1}$, then by Property $F$, $(g^{-1}h')a_j >_A 1$ so $g^{-1}h = [(g^{-1}h')a_j]w >_A 1$. Therefore in both cases, $g^{-1}h >_A 1$, it is a contradiction.

To deduce more precise properties, we observe the following simple lemma.

Lemma 2. If $a_n^{-1}a_n a_n^{-1}a_n \neq a_n$, then $a_n^{-1}a_n a_n^{-1}a_n a_n^{-1} \in P^{(n-1)} - P^{(n)}$.

Proof. To make notation simple, we put $p = a_n^{-1}a_n$ and $q = a_n$.

By Property $F$, $pq^{-1}p^{-1}, p^{-1}q^{-1}p \in P^{(n-1)}$. We show $pq^{-1}p^{-1} \neq P^{(n)}$. The proof of $p^{-1}q^{-1}p \neq P^{(n)}$ is similar. Assume that $pq^{-1}p^{-1} = q^k$ for $k > 0$. Since we have assumed that $pq \neq q$, $k > 1$. Then

$$q = p^{-1}(pq^{-1}p^{-1})^{-1}p = p^{-1}q^{-k}p = (p^{-1}q^{-1}p)^k,$$

so we have $1 <_A p^{-1}q^{-1}p <_A q$. This contradicts Proposition 3 (2), the fact that $q = a_n$ is the minimal $<_A$-positive element.

Next we show that in most cases the Dehornoy-like ordering $<_D$ is not isolated, so it makes a contrast to the isolated ordering $<_A$. For $g \in G$ and a left ordering $<_A$ of $G$ whose positive cone is $P$, we define $<_g = <_A g$ as the left ordering defined by the positive cone $P \cdot g$. Thus, $x <_g x'$ if and only if $xg < x'g$. This defines a right action of $G$ on $\text{LO}(G)$. Two left orderings are said to be conjugate if they belong to the same $G$-orbit.

Theorem 2. If $a_n^{-1}a_n a_n^{-1}a_n \neq a_n$, then the Dehornoy-like ordering $<_D$ is an accumulation point of the set of its conjugates $\{<_D \cdot g\}_{g \in G}$. Thus, $<_D$ is not isolated in $\text{LO}(G)$, and the $\sigma(S)$-positive monoid is not finitely generated.
Proof. Our argument is generalization of Navas-Wiest’s criterion [12]. As in the proof of Lemma 2 we put \( p = a_{n-1} \) and \( q = a_n \) to make notation simple.

We construct a sequence of left orderings \( \{<_n\} \) so that \( \{<_n\} \) non-trivially converge to \( <_D \) and that each \( <_n \) is conjugate to \( <_D \). Here the word non-trivially means that \( <_n \neq <_D \) for sufficiently large \( n > 0 \).

Let \( <_n =<_D \cdot (q^n p) \). Thus, \( 1 <_n g \) if and only if \( 1 <_D (q^n p)^{-1}g(q^n p) \). First we show the orderings \( <_n \) converge to \( <_D \) for \( n \to \infty \). By definition of the topology of \( LO(G) \), it is sufficient to show that for an arbitrary finite set of \( <_D \)-positive elements \( c_1, \ldots, c_r \), \( 1 <_N c_i \) holds for sufficiently large \( N > 0 \).

If \( c_i \not\in G^{(n-1)} \), then \( 1 <_n c_i \) for all \( n \). Thus, we assume \( c_i \in G^{(n-1)} \).

First we consider the case \( c_i \not\in G^{(n)} \). Then \( c_i \) is \( (n-1)(S) \)-positive, hence by using generators \( \{p, q\} \), \( c_i \) is written as \( c_i = q^mpw \) where \( w \in S_S^{(n-1)} \) and \( m \in \mathbb{Z} \). For \( k > m \), by Property \( F \) \( p^{-1}q^{m-k}p \in P^{(n-1)}_A \). So, if we take \( k > m \), then
\[
(q^k p)^{-1}c_i(q^k p) = (p^{-1}q^{m-k}p)wq^k
\]
is \( (n-1)(S) \)-positive. So \( 1 <_k c_i \) for \( k > m \).

Next assume that \( c_i \in G^{(n)} \), so \( c_i = q^{-m} (m > 0) \). Then \( (q^k p)^{-1}c_i(q^k p) = p^{-1}q^{-m}p \). By Lemma 2 \( p^{-1}q^{-m}p \in P^{(n-1)}_A \), so \( p^{-1}q^{-m}p \) is \( (n-1)(S) \)-positive. Therefore \( 1 <_D p^{-1}q^{-m}p \), hence \( 1 <_k c_i \) for all \( k > 1 \).

To show the convergent sequence \( \{<_n\} \) is non-trivial, we observe that the minimal positive element of the ordering \( P_n \) is \( (q^k p)^{-1}q(q^k p) = p^{-1}qp \). From the assumption, \( p^{-1}qp \) is not identical with \( q \), the minimal positive element of the ordering \( <_D \). Thus, \( <_n \) are different from the ordering \( <_D \).

It should be mentioned that our hypothesis \( a_{n-1}a_na_{n-1} \neq a_n \) is really needed. Let us consider the Klein bottle group \( G = \langle s_1, s_2 \mid s_2s_1s_2 = s_1 \rangle \). It is known that \( S = \{s_1, s_2\} \) defines a Dehornoy-like ordering \( <_D \) of \( G \) (See [11] or the proof of Theorem 4 in Section 3.1, which is valid for the Klein bottle group case, \((m, n) = (2, 2)\)). However, since \( G \) has only finitely many left orderings, \( <_D \) must be isolated. Observe that for the twisted generating set \( A = \{a_1, a_2\} \) of \( S \), the Klein bottle group has the same presentation \( G = \langle a_1, a_2 \mid a_2a_1a_2 = a_1 \rangle \).

Finally we determine the Conradian soul of \( <_D \) and \( <_A \).

**Theorem 3** (Conradian properties of Dehornoy-like and isolated orderings). Let \( S = \{s_1, \ldots, s_n\} (n > 1) \) be an ordered finite generating set of a group \( G \) which defines a Dehornoy-like ordering \( <_D \). Assume that \( S \) has Property \( F \) and let \( A = \{a_1, \ldots, a_n\} \) be the twisted generating set of \( S \). Let \( <_A \) be the isolated ordering defined by \( A \). If \( a_{n-1}a_na_{n-1} \neq a_n \), then two orderings \( <_D \) and \( <_A \) have the following properties.

1. \( <_D \) is not Conradian. Thus, the \( <_D \)-Conradian soul is \( G^{(n)} \), the infinite cyclic subgroup generated by \( s_n \).
2. \( <_A \) is not Conradian. Thus, the \( <_A \)-Conradian soul is \( G^{(n)} \), the infinite cyclic subgroup generated by \( a_n \).

**Proof.** As in the proof of Lemma 2 we put \( p = a_{n-1} \) and \( q = a_n \). To prove theorem it is sufficient to show for \( n > 2 \), \( <_D \) and \( <_A \) are not Conradian, since by Proposition 2 and Proposition 3 if \( H \) is a \( <_D \)- or \( <_A \)- convex subgroup, then \( H = G^{(i)} \) for some \( i \). First we show \( <_A \) is not Conradian. By Lemma 2 every \( A \)-word positive representative of \( p^{-1}q^{-1}p \) contains at least one \( p \), so we put \( p^{-1}qp = Np^{-1}q^{-k} \).
where $N \leq A$ and $k \geq 0$. Then we obtain an inequality

$$(q^k p)^{-1} q (q^k p)^2 = (p^{-1} qp) q^k p = N \leq A$$

hence $<_A$ is not Conradian.

To show $<_D$ is not Conradian, we observe

$$(pq^{k+1} pq^{k+1})^{-1} (pq^{k+2})(pq^{k+1} pq^{k+1})^2 = q^{-(k+1)} (p^{-1} qp) q^{k+1} pq^{k+1} pq^{k+1}$$

$$= q^{-(k+1)} N (p^{-1} q) pq^{k+1} pq^{k+1} pq^{k+1} = \ldots$$

$$= q^{-(k+1)} N^4 p^{-1} q$$

$$<_D 1.$$

Question 1. In our study of Dehornoy-like ordering, we assumed somewhat artificial conditions, such as Property $F$ or $a_n^{-1} a_n a_{n-1} \neq a_n$. Are such assumptions really needed? That is,

1. If $S$ defines a Dehornoy-like ordering of $G$, then does $S$ have Property $F$?
2. If $G$ has infinitely many left orderings and $S$ defines a Dehornoy-like ordering, then does $a_{n-1} a_n a_{n-1} = a_n$ hold for $a_{n-1}, a_n$ in the twisted generating set $A$ of $S$?

It is likely that the above questions have affirmative answers, hence the relationships between Dehornoy-like orderings and isolated orderings are quite stronger than stated in this paper.

3. Isolated and Dehornoy-like ordering on $\mathbb{Z} \ast_{\mathbb{Z}} \mathbb{Z}$

In this section we construct explicit examples of Dehornoy-like and isolated left orderings of the group $\mathbb{Z} \ast_{\mathbb{Z}} \mathbb{Z}$ and study more detailed properties.

3.1. Construction of orderings. First we review the notations. Let $G = \mathbb{Z} \ast_{\mathbb{Z}} \mathbb{Z}$ be the amalgamated free product of two infinite cyclic groups. As we mentioned such a groups are presented as

$$G_{m,n} = \langle x, y \mid x^m = y^n \rangle. \quad (m \geq n)$$

and we will always assume $(m, n) \neq (2, 2)$.

We consider an ordered generating set $S = \{s_1 = x y^{-m+1}, s_2 = x^{-m} y^{-1}\}$ and its twisted generating set $A = \{a = x, b = y x^{-m+1}\}$. Using $S, A$, the group $G_{m,n}$ is presented as

$$G_{m,n} = \langle s_1, s_2 \mid s_2 s_1 s_2 = ((s_1 s_2)^{m-2} s_1)^{n-1} \rangle$$

$$= \langle a, b \mid (ba^{n-1})^{n-1} b = a \rangle$$

respectively. In this section we present a new family of Dehornoy-like and isolated left orderings.

Theorem 4. Let $S, A$ be the ordered finite generating sets of $G_{m,n}$ as above.

1. $S$ defines a Dehornoy-like ordering $<_D$.
2. $A$ defines an isolated left ordering $<_A$.

By the presentation of $G$, it is easy to see that $S$ have Property $F$ and $bab \neq b$ if $(m, n) \neq (2, 2)$. Thus from general theories developed in Section 2, we obtain various properties of $<_D$ and $<_A$. 
Corollary 2. Let $<_D$ be the Dehornoy-like ordering of $G = G_{m,n}$ and $<_A$ be the isolated ordering in Theorem \[4\]

1. If $H$ is a $<_D$-convex subgroup, then $H = \{1\}$ or $\langle s_2 \rangle$ or $G$.
2. If $H$ is a $<_A$-convex subgroup, then $H = \{1\}$ or $\langle b \rangle$ or $G$.
3. The Conradian soul of $<_D$ is $G(2) = \langle s_2 \rangle$.
4. The Conradian soul of $<_A$ is $G(2) = \langle b \rangle$.
5. $<_D$ is an accumulation point of the set of its conjugate $\{<_D \cdot g \}_{g \in G}$. Thus, $<_D$ is not isolated in $LO(G)$ and the $\sigma(S)$-positive monoid is not finitely generated.

Before giving a proof, first we recall the structure of the group $G_{m,n}$. Let $Z_{m,n} = \mathbb{Z}_m * \mathbb{Z}_n = \langle X, Y \mid X^m = Y^n = 1 \rangle$, where $\mathbb{Z}_m$ is the cyclic group of order $m$ and let $\pi : G_{m,n} \to Z_{m,n}$ be a homomorphism defined by $\pi(x) = X$, $\pi(y) = Y$. The kernel of $\pi$ is an infinite cyclic group generated by $x^m = y^n$ which is the center of $G_{m,n}$. Thus, we have a central extension

$$1 \to \mathbb{Z} \to G_{m,n} \to Z_{m,n} \to 1.$$

We describe an action of $Z_{m,n}$ on $S^1$ which is used to prove Property A. Let $T = T_{m,n}$ be the Bass-Serre tree for the free product $Z_{m,n} = \mathbb{Z}_m * \mathbb{Z}_n$. That is, $T_{m,n}$ is a tree whose vertices are disjoint union of cosets $Z_{m,n} / \mathbb{Z}_m \sqcup \mathbb{Z}_{m,n} / \mathbb{Z}_n$ and edges are $Z_{m,n}$. Here an edge $g \in Z_{m,n}$ connects vertices $g\mathbb{Z}_m$ and $g\mathbb{Z}_n$. (See Figure 1 left for example the case $(m,n) = (4,3)$).

In our situation, the Bass-Serre tree $T$ is naturally regarded as a planer graph. More precisely, we regard $T$ as embedded into the hyperbolic plane $\mathbb{H}^2$. Now $X$ acts on $T_{m,n}$ as a rotation centered on $P$, and $Y$ acts on $T$ as a rotation centered on $Q$. This defines an faithful action of $Z_{m,n}$ on $T$. Let $E(T)$ be the set of the ends of $T$, which is identified with the set of infinite rays emanating from a fixed base point of $T$. The end of tree $E(T)$ is a Cantor set, and naturally regarded as a subset of the points at infinity $S^1_\infty$ of $\mathbb{H}^2$. The action of $Z_{m,n}$ induces a faithful action on $E(T)$, and this action extends an action on $S^1$. We call this action the standard action of $Z_{m,n}$.

The standard action is easy to describe since $X$ and $Y$ act as rotations of the tree $T$. $X$ acts on $S^1$ so that it sends an interval $[p_i, p_{i+1}]$ to the adjacent interval $[p_{i+1}, p_{i+2}]$ (here indices are taken modulo $m$), and $Y$ acts on $S^1$ so that it sends an interval $[q_i, q_{i+1}]$ to $[q_{i+1}, q_{i+2}]$ (here indices are taken modulo $n$). See Figure 1 right. More detailed description of the set of ends $E(T)$ and the standard action will be given in next section.

Using the standard action, we show the Property A for $S$, which is equivalent to the following statement by Claim 2.

Lemma 3. If $g \in P_A$, then $g \neq 1$.

Proof. Let $g \in P_A$ and put $A = \pi(a) = X$ and $B = \pi(b) = YX^{-m+1} = YX$. If $g \in \text{Ker} \ \pi$, that is, $g = a^{mN}$ for $N > 0$, then $g \neq 1$ is obvious so we assume $g \neq a^{mN}$. Let us put

$$\pi(g) = A^{s_k} B^{r_k} \cdots A^{s_1} B^{r_1},$$

where $0 < s_i < m \ (i < k)$, $0 \leq s_k < m$ and $0 < b_i \ (i > 1)$, $0 \leq b_1$. 
First observe that the dynamics of $B$, $A$ and $(BA^{m-1})^i B$ are given by the following formulas.

\[
B[p_m, p_{m-1}] = YX[p_m, p_{m-1}] = Y[p_1, p_m] = Y[q_1, q_2] = [q_2, q_3] \subset [p_m, p_1]
\]

\[
A^{i}[p_m, p_1] \subset [p_m, p_{m-1}] \quad (i \neq m - 1)
\]

\[
(BA^{m-1})^i B[p_m, p_{m-1}] = (YX)^i YX[p_m, p_{m-1}] = Y^{i+1}[p_1, p_m] \subset Y^{i+1}[q_1, q_2] \subset [q_2+i, q_3+i] \subset [p_m, p_1].
\]

Here $[p_m, p_{m-1}]$ represents the interval $[p_m, p_1] \cup [p_1, p_2] \cup \ldots \cup [p_{m-2}, p_{m-1}]$. Since $(BA^{m-1})^{n-1} B = A$, we can assume that the above word expression does not contains a subword of the form $(BA^{m-1})^{n-1} B$. Thus, by using of the above formulas repeatedly, we conclude

\[
\begin{cases}
\pi(g)[p_{r_k+1}, p_{r_k}] = [p_{r_k}, p_{r_k+1}] & (s_1 \neq 0, r_k \neq 0) \\
\pi(g)[p_{m}, p_1] = [p_{r_k}, p_{r_k+1}] & (s_1 = 0, r_k \neq 0) \\
\pi(g)[p_1, p_2] = [p_m, p_1] & (s_1 \neq 0, r_k = 0)
\end{cases}
\]

So we conclude $\pi(g) \neq 1$ if $s_1 \neq 0$ or $r_1 \neq 0$. If $s_1 = r_k = 0$, we need more careful argument to treat the case the word $\pi(g)$ contains a subword of the form $(BA^{m-1})^i$. Let us write

\[
\pi(g) = B^{s_k-1}(BA^{m-1})^i B \cdots A^{r_1}
\]

where we take $i$ the maximal among such description of the word $\pi(g)$. That is, the prefix of the word $\pi$ is not $BA^{m-1}$. Then for $i \neq 0$,

\[
\pi(g)[q_n, q_{n+1}] \subset B^{s_k-1}(BA^{m-1})^i[p_m, p_1] \subset B^{s_k-1}[p_m, p_1].
\]

Thus if $s_k \neq 1$, then $\pi(g)[q_n, q_1] \subset [q_2, q_3]$. and if $s_k = 1$, then $\pi(g)[q_2, q_3] = [q_2+i, q_3+i]$. Thus we proved that in all cases $\pi(g)$ acts on $S^1$ non trivially, hence $g \neq 1$.

Next we show Property $C$, which is equivalent to the following statement according to Claim 1:

**Lemma 4.** $P_A \cup \{1\} \cup P_A^{-1} = G$.

**Proof.** Let $g \in G$ be a non-trivial element.

Since $a^m = (ba^{m-1})^n = (a^{m-1}b)^n$ is central, we may write $g$ as

\[
g = a^{nM} a_{s_k} b^{r_k} \cdots a_{s_1} b^{r_1},
\]

where $0 < s_i < m$ ($i < k$), $0 \leq s_k < m$ and $0 < b_i$ ($i > 1$), $0 \leq b_1$. 

---

**Figure 1.** Bass-Serre Tree and action of $Z_{m,n}$ on $S^1$
Among such word expressions, we choose the word expression $w$ so that $k$ is minimal. If $mM + s_k \geq 0$, then $g \in P_A$. So we assume that $mM + s_k < 0$ and we prove $g \in P_A^{-1}$ by induction on $k$. The case $k = 1$ is a direct consequence of Property $F$.

First observe that from a relation $a^{-1}b = (a^m b)^{-n}$, we get a relation
\[ a^{-1}b^r = [(a^m b)^{(n-1)}b^{-1}a^{-m+1}]^{-1}(a^m b)^{-n} \]
for all $r > 0$. Thus, by applying this relation, $g$ is written as
\[ g = X(a^m b)^{-n} \cdot a^s b^r \cdots \]
for some $X \in P_A^{-1}$. Unless $s_{k-1} = s_{k-2} = \cdots = s_{k-n} = m$ and $r_{k-1} = r_{k-2} = \cdots = r_{k-n} = 1$, by reducing this word expression, we obtain a word expression of the form
\[ g = X(a^{-1}b^r a^s \cdots) \]
where $X' \in P_A^{-1}$ and $i < k$. By inductive hypothesis, $a^{-1}b^r a^s \cdots \in P_A^{-1}$, hence we conclude $g \in P_A^{-1}$.

Now assume $s_{k-1} = s_{k-2} = \cdots = s_{k-n} = m$ and $r_{k-1} = r_{k-2} = \cdots = r_{k-n} = 1$. Since $(a^m b)^n = b^{-1}a$, by replacing the subword $(a^m b)^n$ with $b^{-1}a$ and canceling $b^{-1}$ we obtain another word representative
\[ g = a^{(m+1)M} a^{sk b^{r_1}a^{s_{k-1}} \cdots} \]
which contradicts the assumption that we have chosen the first word representative of $g$ so that $k$ is minimal. \qed

These two lemmas and Theorem 4 prove Theorem 4.

**Question 2.** Theorem 4 provides a negative answer to the Main question raised by Navas in [11], which concerns the characterization of groups having an isolated ordering defined by two elements. Now we ask the following refined version of the question:

1. If a group $G$ has a Dehornoy-like ordering defined by an ordered generating set of cardinal two, then $G = G_{m,n}$ for some $m, n > 1$?

1'. If a group $G$ has a Dehornoy-like ordering defined by an ordered generating set of cardinal two having Property $F$, then $G = G_{m,n}$ for some $m, n > 1$?

2. If a group $G$ has an isolated ordering whose positive cone is generated by two elements, then $G = G_{m,n}$ for some $m, n > 1$?

These questions also concern the Question 1, since an affirmative answer for (1) yields an affirmative answer of Question 1 for the case $n = 2$.

We also mention that to find a group $G \neq B_n$ with a Dehornoy-like ordering defined by an ordered generating set of cardinal more than two is an open problem. In particular, does a natural candidate, a group of the form $(\cdots(Z*Z)Z*Z\cdots)*Z$ have a Dehornoy-like ordering?

### 3.2. Dynamics of the Dehornoy-like ordering of $\mathbb{Z} *_\mathbb{Z} \mathbb{Z}$

As is well-known, if $G$ faithfully acts on the real line $\mathbb{R}$ as orientation preserving homeomorphisms, then one can construct a left-ordering of $G$ as follows. Let $\Theta : G \rightarrow \text{Homeo}_+(\mathbb{R})$ be a faithful left action of $G$ and choose a dense countable sequence $I = \{x_i\}_{i=1,2,\ldots}$ of $\mathbb{R}$. For $g, g' \in G$ we define $g <_I g'$ if there exists $j$ such that $g(x_i) = g'(x_i)$ for $i < j$ and $g(x_j) <_\mathbb{R} g'(x_j)$, where $<_\mathbb{R}$ be the standard ordering of $\mathbb{R}$ induced by the orientation of $\mathbb{R}$. We say the ordering $<_I$ is defined by the action $\Theta$ and the
sequence $I$. Assume that the stabilizer of a finite initial subsequence $\{x_1, \ldots, x_k\}$ is trivial. Then to define the ordering, we do not need to consider the rest of sequence $\{x_{k+1}, x_{k+2}, \ldots\}$. In such case, we denote the ordering $<_I$ by $<_{\{x_1, \ldots, x_k\}}$ and call the \textit{ordering defined by} $\{x_1, \ldots, x_k\}$ (and the action $\Theta$).

Conversely, one can construct an orientation-preserving faithful action of $G_m,n$ on the real line $\mathbb{R}$ from a left-ordering of $G$ if $G$ is countable. See [10] for details.

In this section we give an alternative definition of the Dehornoy-like ordering $<_D$ of $G_{m,n}$ by using the dynamics of $G_{m,n}$. Recall that $G_{m,n}$ is a central extension of $Z_{m,n}$ by $Z$. By lifting the standard action of $Z_{m,n}$ on $S^1$, we obtain a faithful orientation-preserving action of $G_{m,n}$ on the real line. We call this action the \textit{standard action} of $G_{m,n}$ and denote by $\Theta : G_{m,n} \to \text{Homeo}_+(\mathbb{R})$.

We give a detailed description of the action of $G_{m,n}$ on $\mathbb{R}$. First of all, we give a combinatorial description of the end of the tree $T = T_{m,n}$.

Let us take a basepoint $\ast$ of $T$ as the midpoint of $P$ and $Q$. For each edge of $T$, we assign a label as in Figure 2. Let $e$ be a point of $E(T)$, which is represented by an infinite ray $\gamma_e$ emanating from $\ast$. Then by reading a label on edge along the infinite path $\gamma_e$, $\gamma_e$ is encoded by an infinite sequence of integers $\pm l_1 l_2 \cdots$.

![Figure 2. Labeling of edge](image)

Now let $p : \mathbb{R} \to S^1$ be the universal cover, and $E(T) = p^{-1}(E(T)) \subset \mathbb{R}$. Then the standard action $G_{m,n}$ preserves $E(T)$. A point of $E(T)$ is given as the sequence of integers $(N; \pm l_1 l_2 \cdots)$ where $N \in \mathbb{Z}$. Observe that the set of such a sequence of integers has a natural lexicographical ordering. This ordering of $E(T)$ is identical with the ordering induced by the standard ordering $<_\mathbb{R}$ of $\mathbb{R}$, so we denote the ordering by the same symbol $<_\mathbb{R}$.

The action of $G_{m,n}$ on $E(T)$ is easy to describe, since $X$ and $Y$ act on $T_{m,n}$ as rotations of the tree.

$$x : \begin{cases} 
(N; + i \cdots) & \mapsto (N; +(i+1) \cdots) \quad (i \neq m-1) \\
(N; +(m-1) \cdots) & \mapsto (N+1; - \cdots) \\
(N; - i \cdots) & \mapsto (N; +1i \cdots)
\end{cases}$$

$$y : \begin{cases} 
(N; + i \cdots) & \mapsto ((N+1); -1i \cdots) \\
(N; - i \cdots) & \mapsto (N; -(i+1) \cdots) \quad (i \neq n-1) \\
(N; -(n-1) \cdots) & \mapsto (N; + \cdots)
\end{cases}$$
Therefore, the action of $s_1$, $s_2$ and $s_2^{-1}$ are given by the formula:

\[
\begin{align*}
(s_1)_m &= \begin{cases} 
(N;+i \cdots) & \mapsto (N;+11(i+1) \cdots) \quad (i \neq m-1) \\
(N;+(m-1)i \cdots) & \mapsto (N;+1(i+1) \cdots) \quad (i \neq n-1) \\
(N;+(m-1)(n-1)i \cdots) & \mapsto (N;+(i+1) \cdots) \quad (i \neq m-1) \\
(N;+(m-1)(n-1)(m-1) \cdots) & \mapsto ((N+1); \cdots) \\
(N;+ \cdots) & \mapsto (N;111 \cdots) \\
(N;+(m-1) \cdots) & \mapsto (N;m+1 \cdots) \\
(N;+(m-1)(n-1) \cdots) & \mapsto ((N+1); \cdots) \\
(N;+i \cdots) & \mapsto (N;+11(i+1) \cdots) \quad (i \neq m-1) \\
(N;+(m-1)i \cdots) & \mapsto (N;+(m-1)(i-1) \cdots) \quad (i \neq n-1) \\
(N;+(m-1)(i-1) \cdots) & \mapsto (N;+(i-1) \cdots) \quad (i \neq m-1) \\
(N;+ \cdots) & \mapsto (N;+11 \cdots) \\
(N;+ \cdots) & \mapsto (N;11 \cdots) \\
\end{cases}
\end{align*}
\]

Let $E = (0; -111 \cdots)$ and $F = (0; +1111 \cdots)$ be the point of $\tilde{E}(T)$ and let $<_{\{E,F\}}$ be a left-ordering of $G_{m,n}$ defined by the sequence $\{E, F\}$ and the standard action $\Theta$. The following theorem gives an alternative definition of $<_D$.

**Theorem 5.** The left ordering $<_{\{E,F\}}$ is identical with the Dehornoy-like ordering $<_D$ defined by $S$.

**Proof.** By the formula of the action of $G_{m,n}$ on $\tilde{E}(T_{m,n})$ given above, it is easy to see that for $1(S)$-positive element $g \in G_{m,n}$, $E \preceq g(E)$. Thus by Property $C$ of $S$, $g(E) = E$ if and only if $g = s_2^{k}$ for $q \in \mathbb{Z}$. Similarly, $s_2^{q}(F) = (0; +(m-1)(n-1) \cdots) > F$ if $q > 0$. Thus, we conclude $1 <_D g$ then $1 <_{\{E,F\}} g$, hence two orderings are identical. 

We remark that a more direct proof is possible. That is, we can prove that $<_{\{E,F\}}$ is a Dehornoy-like ordering without using Theorem 4.

In fact, the proof of Theorem 5 provides an alternative (but essentially equivalent since it uses the standard action of $G_{m,n}$) proof of the fact that $S$ has Property $A$. On the other hand, using the description of the standard action given here, we can give an completely different proof of Property $C$, as we give an outline here. Let $g \in G$. If $g(E) = E$, then $g = s_2^{k}$ for $q \in \mathbb{Z}$. So assume $g(E) <_R E$, so $g(E) = (N; l_1 l_2 l_3 \cdots)$ where $N \leq 0$, $l_1 \in \{+, -\}$. Let $c(g)$ be the minimal integer such that $l_j = 1$ for all $j' > j$. We define the complexity of $g$ by $C(g) = (|N|, c(g))$. Now we can find $1(S)$-positive element $p_g$ such that $C(p_g) < C(g)$. The construction of $p_g$ is not difficult but requires complex case-by-case studies, so we omit the details. Here we compare the complexity by the lexicographical ordering of $\mathbb{Z} \times \mathbb{Z}$. Since $C(g) = (0,0)$ implies $g(E) = E$, by induction of the complexity we prove $g$ is $\sigma(S)$-negative.

**Remark 1.** Regard $G_{3,2} = B_3$ as the mapping class group of 3-punctured disc $D_3$ having a hyperbolic metric and let $\tilde{D}_3 \subset \mathbb{H}^2$ be the universal cover of $D_3$. By considering the action on the set of points at infinity of $\tilde{D}_3$, we obtain an action of $G_{3,2}$ on $\mathbb{R}$ which we call the Nielsen-Thurston action. The Dehornoy ordering $<_D$ of $B_3 = G_{3,2}$ is defined by the Nielsen-Thurston action. See [15]. In the case $(m,n) = (3,2)$, the standard action $\Theta$ derived from Bass-Serre tree is conjugate to the Nielsen-Thurston action. Thus, the Dehornoy-like ordering of $G_{m,n}$ is also...
regarded as a generalization of the Dehornoy ordering of $B_3$, from the dynamical point of view.  

We say a Dehornoy-like ordering $<_D$ defined by an ordered finite generating set $S$ has Property $S$ (the Subword property) if $g <_D wg$ holds for all $S$-word positive element $w$ and for all $g \in G$. The Dehornoy ordering of the braid group $B_n$ has Property $S$ \cite{D}.  

One remarkable fact is that our Dehornoy-like ordering $<_D$ of $G_{m,n}$ does not have Property $S$ except the braid group case.  

**Theorem 6.** The Dehornoy-like ordering $<_D$ of $G_{m,n}$ does not have Property $S$ unless $(m,n) = (3,2)$. 

*Proof.* We use the dynamical description of $<_D$ given in Theorem\cite{D}. If $m > 2$ and $n \neq 2$, then 

$$[s_1(s_2s_1)]E = (0; +211\cdots) <_R (0; +(m-1)(n-1)1\cdots) = [s_2s_1]E$$  

hence $s_1(s_2s_1) <_D (s_2s_1)$.  

However we show that the Dehornoy-like ordering $<_D$ of $G_{m,n}$ has a slightly weaker property which can be regarded as a partial subword property.  

**Theorem 7.** Let $<_D$ be the Dehornoy-like ordering of $G_{m,n}$. Then $g <_D s_2g$ holds for all $g \in G$.  

*Proof.* Observe that the standard action of $s_2$ on $\bar{E}(T)$ is monotone increasing. That is, for any $p \in \bar{E}(T)$, we have $p \leq_R s_2(p)$. Thus, $g(E) \leq s_2g(E)$. The equality holds only if $g(E) = E$, so in this case $g = s_2^k (k \in \mathbb{Z})$. So in this case we also have a strict inequality $g <_D s_2g$.  

**Remark 2.** A direct proof of Theorem\cite{D} which does not use the dynamics is easy once we found a counter example. If $(m,n) \neq (3,2)$, then $s_2s_1s_2 = s_1s_2s_1W$ for an $S$-positive word $W$, so 

$$s_1^{-1}s_2^{-1}s_1s_2s_1 = s_1^{-1}s_2^{-1}s_1(s_2s_1s_2)s_1^{-1} = s_1^{-1}s_2^{-1}s_1^{-1}(s_1s_2s_1W)s_2^{-1} = Ws_2^{-1}$$  

The last word is $1(S)$-positive hence $s_1(s_2s_1) <_D (s_2s_1)$.  

Using dynamics we can easily find a lot of other counter examples. The main point is that, as we can easily see, the action $s_2$ is not monotone increasing unlike the action of $s_2$. That is, there are many points $p \in \bar{E}(T)$ such that $s_2(p) <_R p$.  

**Remark 3.** Another good property of the standard generator $S = \{\sigma_1, \sigma_2\}$ of $B_3$ is that the $S$-word-positive monoid $P_S = B_3^+$ is a Garside monoid, so $B_3^+$ has various nice lattice-theoretical properties. See \cite{D, E} for a definition and basic properties of the Garside monoid and Garside groups. In particular, the monoid $B_3^+$ is atomic. That is, if we define the partial ordering $\prec$ on $B_3^+$ by $g \prec g'$ if $g^{-1}g' \in B_3^+$, then for every $g \in G$, the length of a strict chain 

$$1 \prec g_1 \prec \cdots \prec g_k = g$$  

is finite. However, if $m > 3$, then the $S$-word positive monoid $P_S$ is not atomic. If $m > 3$, then $s_2s_1s_2 = s_1s_2s_1s_2W$ holds for $W \in P_S$ so we have a chain 

$$\cdots \prec s_1^2s_2s_1s_2 \prec s_1s_2s_1s_2s_1s_2W = s_1s_2s_1s_2 \prec s_1s_2s_1s_2s_1s_2W = s_2s_1s_2$$
having infinite length. Thus for \( m > 3 \), \( P_S \) is not a Garside monoid.

On the other hand, the groups \( G_{m,n} \) have a lot of Garside group structures. For example, let \( \mathcal{X} = \{x,y\} \) be a generating set of \( G_{m,n} \). Then the \( \mathcal{X} \)-word positive monoid \( P_\mathcal{X} \) is a Garside monoid. Moreover, if \( m \) and \( n \) are coprime, that is, if \( G_{m,n} \) is a torus knot group, then there are other Garside group structures due to Picantin [13]. Thus, unlike the Dehornoy ordering of \( B_n \), a relationship between general Dehornoy-like orderings and Garside structures of groups seem to be very weak. Thus, it is interesting problem to find other family of left-orderings which is more related to Garside group structure.

In the remaining case \((m,n) = (3,3)\), the author could not determine whether \( P_S \) is a Garside monoid or not.

3.3. Exotic orderings: left orderings with no non-trivial proper convex subgroups. In [2], Clay constructed left orderings of free groups which has no non-trivial proper convex subgroups by using the Dehornoy ordering of \( B_3 \). Such an ordering is interesting, because many known constructions of left orderings, such as a method to use group extensions, produce an ordering having proper non-trivial convex subgroups.

In this section we construct such orderings by using a Dehornoy-like ordering of \( G_{m,n} \). By using the dynamics, we prove a stronger result even for the 3-strand braid group case. Let \( H \) be a normal subgroup of \( G_{m,n} \). By abuse of notation, we also use \( <_D \) to represent both the Dehornoy like ordering of \( G_{m,n} \) and its restriction to \( H \).

First we observe the following lemma, where the partial subword property plays an important role.

**Lemma 5.** Let \( C \) be a non-trivial \( <_D \)-convex subgroup of \( H \). If \( G^{(2)} \cap H = \{1\} \), then \( s_2^k c s_2^{-k} \in C \) for all \( k \in \mathbb{Z} \) and \( c \in C \).

**Proof.** Let \( c \in H \) be a \( <_D \)-positive element. Since \( G^{(2)} \cap H = \{1\} \), \( c \) must be \( 1(S) \)-positive. So \( s_2^k c^{-1} s_2^{-k} \) is \( 1(S) \)-negative, hence \( c s_2^k c^{-1} s_2^{-k} <_D c \). On the other hand, by Theorem 7, \( c s_2^k c^{-1} >_D 1 \). Thus \( c s_2^k c^{-1}(E) = c s_2^k c^{-1} s_2^{-k}(E) \geq E \), so \( c s_2^k c^{-1} s_2^{-k} \geq _D 1 \). \( H \) is a normal subgroup, so \( c s_2^k c^{-1} s_2^{-k} \in H \). Since \( C \) is \( <_D \)-convex subgroup, \( c s_2^k c^{-1} s_2^{-k} \in C \). Hence we conclude \( s_2^k c s_2^{-k} \in C \).

Now we show that in most cases, the restriction of the Dehornoy-like ordering to a normal subgroup of \( G_{m,n} \) yields a left-ordering having no non-trivial proper convex subgroups.

**Theorem 8.** Let \( H \) be a normal subgroup of \( G_{m,n} \) such that \( G^{(2)} \cap H = \{1\} \). Then the restriction of the Dehornoy-like ordering \( <_D \) to \( H \) contains no non-trivial proper convex subgroup.

**Proof.** Let \( C \) be a non-trivial \( <_D \)-convex subgroup of \( H \) and \( c \in C \) be \( <_D \)-positive element. Since \( c \) must be 1-positive, by Lemma 5 we may assume that \( c = s_2 s_1 s_2 w \) where \( w \) is a 1-positive element, by taking a power of \( c \) and conjugate by \( s_2 \) if necessary. Similarly, we also obtain \( c' \in C \) such that \( c' = w' s_2 s_1 s_2 \) where \( w' \) is a 1-positive element.

By computing the standard action of \( c' \), then we obtain
\[
c' c(E) = w' s_2 s_1 s_2^2 s_1 s_2 w(E) >_{\mathbb{R}} w' s_2 s_1 s_2^2 s_1 s_2 s_2(E) = w'(1; -11(n-1)11 \cdots) >_{\mathbb{R}} (1; -1111 \cdots).
\]
Thus, for any \( h \in H \), \((c'c)^N(E) \geq_R (N; -1111 \cdots) \geq_R h(E) \geq_R E \) holds for sufficiently large \( N > 0 \). Since \( C \) is convex and \( c'c \in C \), this implies \( h \in C \) so we conclude \( C = H \). 

We remark that the assumption that \( G^{(2)} \cap H = \{1\} \) is necessary, since \( H \cap G^{(2)} \) yields a \( <_D \) convex subgroup of \( H \). We also remark that the hypothesis \( G^{(2)} \cap H = \{1\} \) implies that \( H \) is a free group, since \( G_{m,n} = \mathbb{Z} \ast \mathbb{Z} \).

Theorem \ref{thm:example} provides an example of a left-ordering of the free group of infinite rank which does not have any non-trivial proper convex subgroups. For example, take \( F = [B_3', B_3'] \), where \( B_3' = [B_3, B_3] \cong F_2 \) be the commutator subgroup of \( B_3 \), which is isomorphic to the rank 2 free group.

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