ON THE VARIATIONAL REPRESENTATION
OF MONOTONE OPERATORS

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ABSTRACT. Let $V$ be a Banach space, $z' \in V'$, and $\alpha : V \to \mathcal{P}(V')$ be a maximal monotone operator. A large number of phenomena can be modelled by inclusions of the form $\alpha(u) \ni z'$, or by the associated flow $D_t u + \alpha(u) \ni z'$. Fitzpatrick proved that there exists a lower semicontinuous, convex representative function $f_\alpha : V \times V' \to \mathbb{R} \cup \{+\infty\}$ such that

$$f_\alpha(v, v') \geq \langle v', v \rangle \quad \forall (v, v'), \quad f_\alpha(v, v') = \langle v', v \rangle \iff v' \in \alpha(v). \quad (0.1)$$

This provides a variational formulation for the above inclusions. Here we use this approach to prove two results of existence of a solution, without using the classical theory of maximal monotone operators. This is based on a minimax theorem, and on the duality theory of convex optimization.

1. Introduction. In this note we outline a variational formulation due to Fitzpatrick for the study of maximal monotone inclusions, and prove two results of existence of a solution without using the classical theory.

Let $V$ be a real Banach space, $\alpha : V \to \mathcal{P}(V')$ (the set of the parts of $V'$) be a (possibly multi-valued) operator, and $z' \in V'$. Several nonlinear problems can be formulated as inclusions of the form

$$\alpha(u) \ni z' \quad \text{in } V', \quad (1.1)$$

or

$$\begin{cases} D_t u + \alpha(u) \ni z' & \text{in } V', \text{ in } [0, T], \\ u(0) = u^0. \end{cases} \quad (1.2)$$

In several cases $\alpha$ is maximal monotone in the sense of Minty and Browder; see e.g. [5],[7],[22]. It is well-known that this may also represent variational inequalities with convex constraints, see e.g. [2],[5],[18].

In [14] Fitzpatrick proved that for any maximal monotone operator $\alpha : V \to \mathcal{P}(V')$ there exists a lower semicontinuous, convex function $f_\alpha : V \times V' \to \mathbb{R} \cup \{+\infty\}$ such that

$$f_\alpha(v, v') \geq \langle v', v \rangle \quad \forall (v, v'), \quad f_\alpha(v, v') = \langle v', v \rangle \iff v' \in \alpha(v). \quad (1.3)$$

This provides a variational formulation for both inclusions (1.1) and (1.2); this is at the basis of the present note.
Our plan is as follows. In Section 2 we briefly review some results of Fitzpatrick’s theory. In Section 3 we use that approach to prove existence of a solution via a minimax method, cf. with [33]. In Section 4 we prove another result of existence via the duality theory of convex optimization, along the lines of [15]. This also provides an answer to a question that Brezis and Ekeland raised in [6] for the problem (1.2).

2. Short review of Fitzpatrick theory. In this section we briefly review the variational representation of maximal monotone operators that S. Fitzpatrick pioneered in [14].

The Fitzpatrick Theorem. Fitzpatrick associated the following function to any operator $\alpha: V \to P(V')$:

$$f_\alpha(v, v') := \sup \left\{ \langle v', w \rangle - \langle w', w - v \rangle : (w, w') \in V \times V, w' \in \alpha(w) \right\}$$

\quad $\forall (v, v') \in V \times V'.$

(2.1)

Nowadays this is called the Fitzpatrick function of $\alpha$. Being a pointwise supremum of a family of continuous and linear functions, $f_\alpha$ is convex and lower semicontinuous.

The next result is the cornerstone of that theory.

Theorem 2.1. [14] If $\alpha: V \to P(V')$ is maximal monotone, then

$$f_\alpha(v, v') \geq \langle v', v \rangle \quad \forall (v, v') \in V \times V',$$

(2.2)

$$f_\alpha(v, v') = \langle v', v \rangle \iff v' \in \alpha(v).$$

(2.3)

Defining the function

$$J(v, v') := f_\alpha(v, v') - \langle v', v \rangle \quad \forall (v, v') \in V \times V',$$

(2.4)

(2.3) also reads

$$J(v, v') = \inf J = 0 \iff v' \in \alpha(v).$$

(2.5)

We shall label “$J(v, v') = \inf J = 0$” a problem of null-minimization.

The maximal monotone relation is thus tantamount to minimizing $J$ with respect to both variables. In this case the vanishing of the minimum value need not be prescribed, since a strictly positive infimum value is excluded by (2.3).

On the other hand, whenever $v'$ is prescribed, in order to determine $v$ such that $v' \in \alpha(v)$, the functional $J(\cdot, v')$ is only minimized with respect to the first variable. As it is illustrated in Section 7 of [33], in this second case a priori it is not obvious that the minimum value vanishes. This question is at the focus of this note: in the next two sections we prove two results that provide sufficient conditions for the vanishing of the minimum. This reduces the problem of null-minimization to one of ordinary minimization.

Representative functions. The notion of Fitzpatrick function was extended as follows. One says that a convex and lower semicontinuous function $\psi: V \times V' \to \mathbb{R} \cup \{+\infty\}$ (variationally) represents the operator $\alpha: V \to P(V')$ whenever it fulfills the following system, that we shall refer to as the Fitzpatrick system,

$$\psi(v, v') \geq \langle v', v \rangle \quad \forall (v, v') \in V \times V',$$

(2.6)

$$\psi(v, v') = \langle v', v \rangle \iff v' \in \alpha(v).$$

(2.7)

Accordingly, we shall say that $\psi$ is a representative function of $\alpha$, and that $\alpha$ is representable. Let us denote the class of these functions by $\mathcal{F}(V)$. 
For any convex and lower semicontinuous function \( \varphi : V \to \mathcal{P}(V') \), let us denote its convex conjugate function by \( \varphi^* \) and its subdifferential by \( \partial \varphi \). Besides the Fitzpatrick function, the classical Fenchel function [13]

\[
\psi(v, v') := \varphi(v) + \varphi^*(v')
\]

represents the operator \( \partial \varphi \). In this case the Fitzpatrick system (2.6), (2.7) is reduced to the Fenchel system

\[
\begin{align*}
\varphi(v) + \varphi^*(v') & \geq \langle v', v \rangle \quad \forall (v, v') \in V \times V', \\
\varphi(v) + \varphi^*(v') & = \langle v', v \rangle \quad \Leftrightarrow \quad v' \in \partial \varphi(v).
\end{align*}
\]

(2.9) (2.10)

For subdifferential operators this is a well-known result in convex analysis, see e.g. [11],[13],[28]. A basic contribution of Fitzpatrick was its extension to general maximal monotone operators. Several other examples of representable operators are displayed e.g. in [32].

Representable operators are monotone; but, at variance with subdifferentials of lower semicontinuous convex function, they need not be either cyclically monotone or even maximal monotone. For instance, the nonmaximal monotone operator with graph \( A = \{(0,0)\} \) is represented by the function \( I_{\{(0,0)\}} \in \mathcal{F}(V) \).

**Some results.** Let us next assume that the Banach space \( V \) is reflexive, although this is not really needed for some of the assertions that follow. Besides the duality between \( V \) an \( V' \), let us consider the duality between the product spaces \( V \times V' \) and \( V' \times V \), and the corresponding convex conjugation. More specifically, for any function \( g : V \times V' \to \mathbb{R} \cup \{+\infty\} \), let us define its convex conjugate function in the duality between \( V \times V' \) and \( V' \times V \):

\[
g^*(w', w) := \sup \{ \langle w', v \rangle + \langle v', w \rangle - g(v, v') : (v, v') \in V \times V' \} \quad \forall (w', w) \in V' \times V.
\]

(2.11)

Here are some results of this theory that may be useful for its application.

**Theorem 2.2.** [10],[30] A function \( \psi \in \mathcal{F}(V) \) represents a maximal monotone operator \( \alpha : V \to \mathcal{P}(V') \) if and only if \( \psi^* \in \mathcal{F}(V') \). In this case \( \psi^* \) represents the inverse operator \( \alpha^{-1} : V' \to \mathcal{P}(V) \).

The convex biconjugate function of \( f_{\alpha} \), denoted by \( (f_{\alpha})^{**} \), thus also represents \( \alpha \), whenever the operator \( \alpha \) is maximal monotone.

**Theorem 2.3.** [9],[14],[21],[24] Let \( \alpha : V \to \mathcal{P}(V') \) be a maximal monotone operator, \( f_{\alpha} \) be its Fitzpatrick function, and \( \psi : V \times V' \to \mathbb{R} \cup \{+\infty\} \) be a convex and lower semicontinuous function. Then

\[
\psi \in \mathcal{F}(V), \psi \text{ represents } \alpha \quad \Leftrightarrow \quad f_{\alpha} \leq \psi \leq (f_{\alpha})^{**}.
\]

(2.12)

**Corollary 1.** If two functions \( \psi_1, \psi_2 \in \mathcal{F}(V) \) represent a maximal monotone operator \( V \to \mathcal{P}(V') \), then \( \psi = \max\{\psi_1, \psi_2\} \in \mathcal{F}(V) \) represents the same operator.

**Theorem 2.4.** [4] Let \( \alpha : V \to \mathcal{P}(V') \) be a maximal monotone operator, \( f_{\alpha} \) be its Fitzpatrick function, and set

\[
F_{\alpha}(v, v', w, w') := f_{\alpha}(v + w, v' + w') + f_{\alpha}(v - w, v' - w') + \|w\|_{V}^{2} + \|w'\|_{V'}^{2}.
\]

\( \forall (v, v'), (w, w') \in V \times V' \);

\[
\phi_{\alpha}(v, v') := \frac{1}{2} \inf \{ F_{\alpha}(v, v', w, w') : (w, w') \in V \times V' \} \quad \forall (v, v') \in V \times V'.
\]

(2.13) (2.14)
Then
\[ \phi^*(v', v) = \phi(v, v') \quad \forall (v, v') \in V. \quad (2.15) \]

Any function of \( F(V) \) that fulfills this identity is called a \textit{self-dual representative}.

Further results of this theory are briefly reviewed e.g. in [27],[32],[34].

**Existence methods.** If \( \alpha : V \to P(V') \) is a maximal monotone operator, a subdifferential flow of the form
\[
\begin{aligned}
D_t u + \alpha(u) \ni z' & \quad \text{in } V', \text{ a.e. in } ]0, T[,

u(0) = u^0
\end{aligned}
\quad (2.16)
\]
can be studied via the above theory. We briefly illustrate some basic techniques.

(i) If \( \alpha = \partial \phi \), with \( \phi : V \to P(V') \) convex and lower semicontinuous, (2.16) was reformulated as a null-minimization problem by Brezis and Ekeland [6] and Nayroles [23]. Let us assume that \( V, H \) are Hilbert spaces, and that \( V \subset H = H' \subset V' \) with dense inclusions. On the basis of the Fenchel system (2.9), these authors showed that the inclusion
\[
D_t u + \partial \phi(u) \ni z' \quad \text{is tantamount to the null-minimization of the functional}
\]
\[ \Phi(v, z') = \int_0^T \left[ \phi(v) + \phi^*(z' - D_tv) - \langle z', v \rangle \right] dt + \frac{1}{2} \left( \|v(T)\|_H^2 - \|v(0)\|_H^2 \right), \]
\[ (2.17) \]
as \( v \) ranges in \( H^1(0, T; V') \cap L^2(0, T; V) \subset C^0([0, T]; H) \). In [1] Auchmuty provided a direct proof of existence of the null-minimizer via a saddle-point reformulation. (Incidentally, a different saddle-point approach is also at the basis of Theorem 3.2 ahead.)

On the basis of the Fitzpatrick theory, this approach is easily extended to general representable operators \( \alpha \). In this case (2.17) is simply replaced by
\[ \Phi(v, z') = \int_0^T \left[ \psi(v, z' - D_tv) - \langle z', v \rangle \right] dt + \frac{1}{2} \left( \|v(T)\|_H^2 - \|v(0)\|_H^2 \right), \]
\[ (2.18) \]
for any representative function \( \psi \) of \( \alpha \), see [31].

(ii) An inclusion like (2.16) may be approximated by a sequence of equations for which existence of a solution is already known; uniform estimates may then be derived. This approximated problem may be represented as an equivalent null-minimization problem, and the limit may be taken in this formulation. If in this procedure the functional is also approximated, its \( \Gamma \)-convergence must also be proved, see e.g. [32],[34],[35].

(iii) Along the lines of [33], here in Section 3 existence of a solution of an inclusion like (2.16) is proved, first by reformulating the problem via a representative function, and then applying an extension of the classical minimax theorem of Ky Fan; see Theorem 3.2 ahead.

**A look at the literature.** After the pioneering work of Fitzpatrick [14] and its rediscovery by Martinez-Legaz and Théra [19] and also by Burachik and Svaiter [9], in the last fifteen years a rapidly expanding literature has been devoted to this theory; see e.g. [4],[10],[17],[20],[21],[24],[25]. This extended the classical theory of maximal monotone operators, see e.g. [3],[5],[7]. An extensive research has also been devoted to the abovementioned idea of Brezis, Ekeland and Nayroles, see e.g. [1],[26],[29]. The use of self-duality may be compared with the approach that was

\[ \text{These two works predate [14], and somehow contain some elements of the Fitzpatrick theory. This connection however was fully realized just several years later.} \]
developed in the monograph [15] and in several other works of the same author, see e.g. [16],[17]. See also the notion of bipotential of Buliga, de Saxc´e and Vall´ee, see e.g. [8].

The present author dealt with a variational approach for equations of the form (1.1) and (1.2) (including quasilinear problems) also in other works. In [32] the method (iii) was used, and in particular quasilinear maximal monotone equations and first-order flows were formulated as null-minimization problems. The structural stability, namely, the dependence of the solution on data and operators, was then studied via De Giorgi’s notion of Γ-convergence. In [33] the Fitzpatrick method was extended to nonmonotone operators.

3. Existence via minimax. In this section we assume that \( \alpha : V \to \mathcal{P}(V') \) is maximal monotone, and deal with the inclusion

\[
\alpha(u) \ni z' \quad (z' \in V' \text{ prescribed}).
\]  

If \( V \) is reflexive and \( \alpha \) is coercive, it is well known that this inclusion has a solution. Here we reformulate this inclusion in terms of a representative function of \( \alpha \), and prove existence of a solution of the associated null-minimization problem via a minimax method, along the lines of Theorem 5.1 of [33].

In order to perform this program, we use a simple extension of the classical minimax theorem of Ky Fan, that next we recall.

**Lemma 3.1.** [12] Let \( C \) be a convex subset of a real Hausdorff topological vector space \( X \), and \( \Phi : C \times C \to \mathbb{R} \) be such that

\[
\Phi(\cdot, y) \text{ is lower semicontinuous, } \forall y \in C, \tag{3.2}
\]

\[
\Phi(x, \cdot) \text{ is quasi-concave, } \forall x \in C, \tag{3.3}
\]

\[
\Phi(x, x) \leq 0, \forall x \in C, \tag{3.4}
\]

\[
\exists \text{ compact convex set } K \subset X, \exists y_0 \in C \cap K : \Phi(x, y_0) > 0 \quad \forall x \in C \setminus K. \tag{3.5}
\]

Then there exists \( \bar{x} \in C \cap K \) such that

\[
\sup_{y \in C} \Phi(\bar{x}, y) = \inf_{x \in C} \sup_{y \in C} \Phi(x, y) \leq 0. \tag{3.6}
\]

**Corollary 2.** Let \( X \) be the dual of a real Banach space equipped with the weak star topology, \( C \) be a convex subset of \( X \), and \( \Phi \) be as above. Lemma 3.1 then holds under the assumption

\[
\exists M > 0 \text{ such that } \sup_{\|x\| \leq M} \inf_{\|y\| > M} \Phi(x, y) > 0, \tag{3.7}
\]

in place of the condition (3.5).

**Proof.** As the set \( K = \{ x \in X : \|x\| \leq M \} \) is weakly star compact, (3.7) yields (3.5) for this topology. \( \square \)

Next we deal with the inclusion (3.1) assuming that

\[
V \text{ is real reflexive Banach space, } z' \in V', \tag{3.8}
\]

\[
\alpha : V \to \mathcal{P}(V') \text{ is maximal monotone.}
\]

We prove existence of a solution via an associated representative function.
Theorem 3.2. Let a mapping \( \psi \in \mathcal{F}(V) \) represent \( \alpha \), and be such that
\[
\inf_{v'} \frac{\psi(v,v')}{\|v\|_V} \to +\infty \quad \text{as} \quad \|v\|_V \to +\infty.
\] (3.9)

Then there exists \( u \in V \) such that
\[
\psi(u,z') = \langle z', u \rangle.
\] (3.10)

The condition (3.9) entails the coerciveness of the operator \( \alpha \), and by (2.7) the equality (3.10) is equivalent to the inclusion \( \alpha(u) \ni z' \). We thus retrieve a classical result, namely, the surjectivity of coercive maximal monotone operators acting on a reflexive Banach space; see e.g. [3], [5], [7].

Proof. This argument is based on reformulating the equation (3.10) as a minimax problem, and then applying the classical Fan theorem. This follows the lines of the more general argument of Section 5 of [33], where this result is extended to nonmonotone operators. For the reader’s convenience, we split this proof into three steps.

(i) First we set
\[
K(v,t) := \sup_{t' \in V'} \{ \langle v, t' \rangle - \psi^*(t', t) \} \quad \forall v, t \in V.
\] (3.11)

By a standard procedure,
\[
K(v,t) = \sup_{t' \in V'} \inf_{w \in V} \{ \langle v - w, t' \rangle + \langle w, t \rangle - \psi(w, w') \}
= \inf_{w \in V} \sup_{t' \in V'} \{ -\langle w', t \rangle + \psi(w, w') \} \quad \forall v, t \in V.
\] (3.12)

By (3.11) and (3.12) we infer that
\[
K(\cdot, t) \text{ is convex and lower semicontinuous } \forall t \in V,
\]
\[
K(v, \cdot) \text{ is concave and upper semicontinuous } \forall v \in V.
\] (3.13)

By (3.8) and Theorem 2.2, \( \psi \) and \( \psi^* \) are both representative functions. The Fitzpatrick system (2.6), (2.7) thus yields
\[
K(t,t) = \inf_{w' \in V'} \{ -\langle w', t \rangle + \psi(t, w') \} \geq 0 \quad \forall t \in V,
\] (3.14)
\[
K(t,t) = \sup_{t' \in V'} \{ \langle t, t' \rangle - \psi^*(t', t) \} \leq 0 \quad \forall t \in V,
\] (3.15)
whence
\[
K(t,t) = 0 \quad \forall t \in V.
\] (3.16)

Thus \( (t,t) \) is a saddle point of \( K \) for any \( t \in V \).

(ii) Next we set
\[
\Phi(v, t) := K(v, t) + \langle z', t - v \rangle \quad \forall v, t \in V,
\] (3.17)
whence
\[
\Phi(v, v) = K(v, v) \overset{(3.16)}{=} 0 \quad \forall v \in V.
\] (3.18)
Because of (3.11),
$$\sup_{t \in V} \Phi(v, t) = \sup_{(t', t) \in V \times V'} \left( (z', t) + \langle v, t' \rangle - \psi^*(t', t) \right) - \langle z', v \rangle \tag{3.19}$$
$$v \mapsto \Phi(v, t)$$ is concave and weakly lower semicontinuous, \(\forall t \in V.\)\tag{3.20}

Moreover,
$$\Phi(v, 0) \overset{(3.17)}{=} K(v, 0) - \langle z', v \rangle \geq \inf_{w' \in V'} \psi(v, w') - \|z'\|_{V'} \|v\|_V \quad \forall v \in V; \tag{3.21}$$
by (3.9) then
$$\exists M > 0 : \|v\| > M \Rightarrow \Phi(v, 0) > 0. \tag{3.22}$$

(iii) By (3.18), (3.20), (3.22), we can apply Fan’s Theorem via Corollary 2, selecting \(X = V\) equipped with the weak topology and \(C = V.\) Therefore there exists \(u \in V\) such that \(\|u\| \leq M\) and
$$\sup_{t \in V} \Phi(u, t) = \inf_{v \in V} \sup_{t \in V} \Phi(v, t) \leq 0. \tag{3.23}$$

Hence, recalling that \(\psi \in F(V),\)
$$0 \overset{(2.6)}{\leq} \psi(u, z') - \langle z', u \rangle \overset{(3.19)}{=} \sup_{t \in V} \Phi(u, t) \leq 0. \tag{3.24}$$
Thus \(\psi(u, z') = \langle z', u \rangle.\)

4. **Existence via self-duality.** In this section we assume that the maximal monotone operator \(\alpha\) is represented by a self-dual function \(\psi \in F(V),\) with no loss of generality because of Bauschke and Wang’s Theorem 2.4. We then reduce the equation \(\psi(u, z') = \langle z', u \rangle\) (for a prescribed \(z' \in V')\) to a problem of ordinary minimization, and prove existence of a solution via the duality theory of convex optimization, along the lines of Chapter 6 of [15].

**Theorem 4.1.** Let \(V\) be a reflexive Banach space and \(z' \in V'.\) Let \(\psi \in F(V)\) be self-dual, i.e.,
$$\psi^*(v', v) = \psi(v, v') \quad \forall (v, v') \in V \times V', \tag{4.1}$$
and fulfill the classical Slater condition:
$$\exists v_0 \in V : \psi(v_0, \cdot) \text{ is bounded in a neighbourhood of } z'. \tag{4.2}$$

Then there exists \(u \in V\) such that
$$\psi(u, z') = \langle z', u \rangle. \tag{4.3}$$

More precisely, setting
$$h(v') = \inf_{v \in V} \left\{ \psi(v, v' + z') - \langle z', v \rangle \right\} \quad \forall v' \in V, \tag{4.4}$$

\(\partial h(0) \neq \emptyset,\) and any \(u \in \partial h(0)\) fulfills (4.3).

The proof will rest upon the following two lemmas.

**Lemma 4.2.** Let \(V, \psi, z'\) be as above. The function
$$\Phi(v, v') = \psi(v, v' + z') - \langle z', v \rangle \quad \forall (v, v') \in V \times V' \tag{4.5}$$
is self-dual, i.e.,
$$\Phi^*(v', v) = \Phi(v, v') \quad \forall (v, v') \in V \times V'. \tag{4.6}$$
Proof. For any $(v, v') \in V \times V'$,
\[
\Phi^*(v', v) = \sup_{(u, u') \in V \times V'} \left\{ \langle v', u \rangle + \langle u', v \rangle - \Phi(u, u') \right\}
\]
\[
= \sup_{(u, u') \in V \times V'} \left\{ \langle v' + z', u \rangle + \langle u' + z', v \rangle - \psi(u, u' + z') - \langle z', v \rangle \right\}
\]
\[
= \sup_{(u, u') \in V \times V'} \left\{ \langle v' + z', u \rangle + \langle w', v \rangle - \psi(u, w') \right\} - \langle z', v \rangle
\]
\[
= \psi^*(v' + z', v) - \langle z', v \rangle \quad \text{(4.1)}
\]
\[
\psi^*(v, v' + z') - \langle z', v \rangle \quad \text{(4.5)}
\]
\[
\equiv \Phi^*(v, v'). \tag{4.7}
\]

Lemma 4.3. Let $V, \Phi, h$ be as above. Then
\[
h^*(u) = \Phi^*(0, u) \quad \forall u \in V. \tag{4.8}
\]

Proof. For any $u \in V$,
\[
h^*(u) = \sup_{v' \in V'} \left\{ \langle v', u \rangle - h(v') \right\}
\]
\[
= \sup_{v' \in V'} \left\{ \langle v', u \rangle - \inf_{v \in V} \Phi(v, v') \right\}
\]
\[
= \sup_{(v, v') \in V \times V'} \left\{ \langle v', u \rangle - \Phi(v, v') \right\} = \Phi^*(0, u). \tag{4.9}
\]

Proof of Theorem 4.1. The function $h$ is convex and nonnegative, as it is the infimum of a family of convex nonnegative functions, see (4.4). By (4.2), $h$ is also bounded in a neighbourhood of 0. Hence $\partial h(0) \neq \emptyset$, and by (2.10)
\[
h(0) + h^*(u) = 0 \quad \forall u \in \partial h(0). \tag{4.10}
\]
Therefore
\[
0 \leq \Phi(u, 0) \quad \Phi^*(0, u) \quad h^*(u) \quad -h(0) \leq 0 \quad \forall u \in \partial h(0). \tag{4.11}
\]
Thus $\Phi(u, 0) = 0$ for any $u \in \partial h(0)$. \hfill \Box

By Theorem 2.3, Theorem 4.1 can be extended as follows.

Corollary 3. Let $V$ be a reflexive Banach space, $z' \in V'$, $\psi \in F(V)$ be self-dual and fulfill the Slater condition (4.2). Let $\phi \in F(V)$ be such that $\phi \leq \psi$ in $V \times V'$. Then there exists $u \in V$ such that
\[
\phi(u, z') = \langle z', u \rangle, \tag{4.12}
\]
and this equation is equivalent to (4.3).

In particular this corollary applies to the Fitzpatrick function $f_\alpha$ of any maximal monotone operator $\alpha : V \to P(V')$.

Remarks. (i) By Theorem 2.4, any maximal monotone operator $\alpha : V \to P(V')$ has a self-dual representative. Whenever the Slater condition (4.2) is fulfilled, Theorem 4.1 thus entails the existence of a solution of the inclusion $\alpha(u) \ni z'$.

(ii) For any convex and lower semicontinuous function $\varphi : V \to P(V')$, as we pointed out the Fenchel function (2.8) is a self-dual representative of the subdifferential operator $\partial \varphi$. Theorem 4.1 thus provides an answer to the question that Brezis and Ekeland raised in [6]: any minimizer of their functional solves the first-order flow
$D_t u + \partial \varphi(u) \ni z'$, with no need of prescribing the minimization value. This issue was also discussed by other authors by other means, see e.g. [1],[23],[26],[29]. The same conclusion applies if in that inclusion $\partial \varphi$ is replaced by a maximal monotone operator.

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