ON THE INFLUENCE FUNCTION FOR THE THEIL-LIKE CLASS OF INEQUALITY MEASURES

† TCHILABALO ABOZOU KPANZOU, †† DIAM BA, ††† PAPE DJIBY MERGANE, AND †††† GANE SAMB LO

Abstract. On one hand, a large class of inequality measures, which includes the generalized entropy, the Atkinson, the Gini, etc., for example, has been introduced in Mergane and Lo (2013). On the other hand, the influence function of statistics is an important tool in the asymptotics of a nonparametric statistic. This function has been and is being determined and analysed in various aspects for a large number of statistics. We proceed to a unifying study of the IF of all the members of the so-called Theil-like family and regroup those IF’s in one formula. Comparative studies become easier.

† Tchilabalo Abozou Kpanzou (corresponding author).
Kara University, Kara, Togo
Email : kpanzout@gmail.com

†† Diam Ba
LERSTAD, Gaston Berger University, Saint-Louis, Sénégal.
Email : diamba79@gmail.com.

†† Pape Djiby Mergane
LERSTAD, Gaston Berger University, Saint-Louis, Sénégal.
Email : mergane@gmail.com.

††† Gane Samb Lo.
LERSTAD, Gaston Berger University, Saint-Louis, Sénégal (main affiliation).
LSTA, Pierre and Marie Curie University, Paris VI, France.
AUST - African University of Sciences and Technology, Abuja, Nigeria
gane-samb.lo@edu.ugb.sn, gslo@aust.edu.ng, ganesamblo@ganesamblo.net
Permanent address : 1178 Evanston Dr NW T3P 0J9, Calgary, Alberta, Canada.

keywords and phrases. Influence function, Measures of inequality, Lorenz curve, quantile function, Pareto law, Exponential law, Singh-Maddala law, lognormal law, .

AMS 2010 Mathematics Subject Classification : 62G35, 97K70.

1. Introduction

Over the years, a number of measures of inequality have been developed. Examples include the generalized entropy, the Atkinson, the Gini, the quintile share ratio and the Zenga measures (see e.g. Cowell and Flachaire (2007); Cowell et al. (2009); Hulliger and Schoch (2009); Zenga (1984).
and Zenga (1990)). Recently, Mergane and Lo (2013) gathered a significant number of inequality measures under the name of Theil-like family. Such inequality measures are very important in capturing inequality in income distributions. They also have applications in many other branches of Science, e.g. in ecology (see e.g. Magurran (1991)), Sociology (see e.g. Allison (1978)), Demography (see e.g. White (1986)) and information science (see e.g. Rousseau (1993)).

In order to make the above mentioned measures applicable, one often makes use of estimation. Classical methods unfortunately rely heavily on assumptions which are not always met in practice. For example, when there are outliers in the data, classical methods often have very poor performance. The idea in robust Statistics is to develop estimators that are not unduly affected by small departures from model assumptions, and so, in order to measure the sensitivity of estimators to outliers, the influence function (IF) was introduced (see Hampel (1974), Hampel et al. (1986)).

Let us begin by precising the objects and notation of our study, in particular the influence function. To make the reading of what follows easier, we suppose that we have a probability space \((\Omega, \mathcal{A}, \mathbb{P})\) holding a random variable \(X\) associated with the cumulative distribution function \((\text{cdf})\) \(F(x) = \mathbb{P}(X \leq x), \; x \in \mathbb{R}\), and a sequence of independent copies of \(X\): \(X_1, X_2, \ldots\). This random variable is considered as an income variable so that it is non-negative and \(F(0) = 0\). The absolute density distribution function \((\text{pdf})\), if it exists, is denoted by \(f\). Its mean, we suppose finite and non-zero, and moments of order \(\alpha \geq 1\) are denoted by

\[
\mu_F = \int_0^{+\infty} y \; dF(y) \in (0, \infty) \quad \text{and} \quad \mu_{F,\alpha} = \int_0^{+\infty} y^\alpha \; dF(y), \; \mu_{F,1} = \mu_F.
\]

The quantile function associated to \(F\), also called generalized inverse function is defined by

\[
Q(p) \equiv F^{-1}(z) = \inf\{z \in \mathbb{R}, \; F(z) \leq x\}, \; p \in [0, 1]
\]

and the Lorentz curve of \(F\) is given by

\[
L(F, p) = \frac{q(p)}{\mu_F}, \quad \text{with} \quad q(p) = \int_0^p Q(s) \; ds, \quad 0 \leq s \leq 1.
\]
A nonparametric estimation $T(F)$ will studied as well as its plug-in nonparametric estimator of the form $T(F_n)$ which is based on the sample $X_1,...,X_n$, $n \geq 1$.

The influence function $IF(o,T(F))$ of $T(F)$ is the Gateaux derivative of $T$ at $F$ in the direction of Dirac measures in the form

\begin{equation}
IF(z,T(F)) = \lim_{\epsilon \to 0} \frac{T(F^{(z)}_{\epsilon}) - T(F)}{\epsilon} = \frac{\partial}{\partial \epsilon} T(F^{(z)}_{\epsilon})|_{\epsilon=0},
\end{equation}

where

$$F^{(z)}_{\epsilon}(u) = (1 - \epsilon)F(u) + \epsilon \Delta z(u), \epsilon \in [0; 1],$$

$\Delta_z$ is the cdf of the $\delta_z$, the Dirac measure with mass one at $z$ and $z$ is in the value domain of $F$.

It is known that the asymptotic variance of the plug-in estimator $T(F_n)$ of statistic $T(F)$ is of the form $\sigma^2 = \int IF(x,T(F))^2dF(x)$ under specific condition, among them the Hadamard differentiability (see Wasserman (2000), Theorem 2.27, page 19). So the influence function gives an idea of what might be the variance of the Gaussian limit of the estimator if it exists. At the same time, the behavior of its tails (lower and upper) give indications on how lower extreme and/or upper extreme values impact on the quality of the estimation. For example, recently, the sensitivity of a statistic $T(F)$ and the impact of extreme observations of some influence functions have been studied by, e.g., Cowell and Flachaire (2007).

Another interesting fact is that the influence function behaves in nonparametric estimation as the score function does in the parametric setting (see Wasserman (2000), page 19).

An area of application of the influence function is that of measures of inequality (see, e.g., Van Praag et al. (1983), Victoria-Feser (2000) and Kpanzou (2015)). Due to the importance of that key element in nonparametric estimation in Econometric and welfare studies, a collection of inequality measures is being actively made. To cite a few, the $IF$’s of the following measures are given in the Appendix section: the generalized entropy class of measures of inequality $GE(\alpha)$, where $\alpha > 0$, the mean logarithmic deviation (MDL), the Theil Measure,
the Atkinson Class of Inequality Measures of parameter $\alpha \in (0, 1]$, the Gini Coefficient, the Quintile Share Ratio Measure of Inequality (QSR).

Fortunately, Mergane and Lo (2013) introduced the so-called Theil-like family, in which are gathered the Generalized Entropy Measure, the Mean Logarithmic Deviation (Cowell (2003), Theil (1967), Cowell (1980a)), the different inequality measures of Atkinson (Atkinson (1970)), Champernowne (Champernowne and Cowell (1998)) and Kolm (Kolm (1976a)) in the following form:

\begin{equation}
T(F) = \tau \left( \frac{1}{h_1(\mu_n)} \frac{1}{n} \sum_{j=1}^{n} h(X_j) - h_2(\mu_n) \right),
\end{equation}

where $\mu_n = \frac{1}{n} \sum_{j=1}^{n} X_j$ denotes the empirical mean while $h, h_1, h_2,$ and $\tau$ are measurable functions.

The inequality measures mentioned above are derived from (1.2) with the particular values of $\alpha, \tau, h, h_1$ and $h_2$ as described below for all $s > 0$:

(a) Generalized Entropy

$\alpha \neq 0, \alpha \neq 1, \tau(s) = \frac{s - 1}{\alpha (\alpha - 1)}, h(s) = h_1(s) = s^\alpha, h_2(s) \equiv 0$;

(b) Theil’s measure

$\tau(s) = s, h(s) = s \log(s), h_1(s) = s, h_2(s) = \log(s)$;

(c) Mean Logarithmic Deviation

$\tau(s) = s, h(s) = h_2(s) = \log(s^{-1}), h_1(s) \equiv 1$;

(d) Atkinson’s measure

$\alpha < 1$ and $\alpha \neq 0, \tau(s) = 1 - s^{1/\alpha}, h(s) = h_1(s) = s^\alpha, h_2(s) \equiv 0$;

(e) Champernowne’s measure

$\tau(s) = 1 - \exp(s), h(s) = h_2(s) = \log(s), h_1(s) \equiv 1$;

(f) Kolm’s measure

$\alpha > 0, \tau(s) = \frac{1}{\alpha} \log(s), h(s) = h_1(s) = \exp(-\alpha s), h_2(s) \equiv 0$. 
This is simply the plug-in estimator of

\[ T(F) = \tau \left( \frac{\mathbb{E} h(X)}{h_1(\mu_F)} - h_2(\mu_F) \right) = \tau(I). \]

The following conditions are required for the asymptotic theory.

**B1.** The functions \( \tau \) admits a derivative \( \tau' \) which is continuous at \( I \) and \( \tau'(I) \neq 0 \).

**B2.** The functions \( h_1 \) and \( h_2 \) admit derivatives \( h_1' \) and \( h_2' \) which are continuous at \( \mu_F \) with \( h_1(\mu_F) \neq 0 \).

**B3.** \( \mathbb{E} h_j(X) < +\infty, j = 1, 2. \)

This offers an opportunity to present a significant number of IF’s in a unified approach. This may be an asset for inequality measures comparison. By the way, it constitutes the main goal of this paper.

Let us add more notation. The lower endpoint and upper endpoint of cdf \( F \) are denoted by

\[ \text{lep}(F) = \inf\{y \in \mathbb{F}, F(x) > 0\} \quad \text{and} \quad \text{uep}(F) = \sup\{y \in \mathbb{F}, F(x) < 1\}. \]

So the domain of admissible values for \( X \), denoted by \( \mathcal{V}_X \), satisfies \( \mathcal{V}_X \subset \mathcal{R}_X = [\text{lep}(F), \text{uep}(F)] \), the latter being the range of \( F \).

The layout of this paper is as follows. In the next section we state our main result on the influence function of the TLIM family members and some particularized forms related to each known members. For member whose IF’s are already given, we will make a comparison. In Section 3, we give the complete proofs. In Section 4 we provide a conclusion and some perspectives. Section 5 is an appendix gathering IF’s expressions of some members of the TLIM available in the literature.

2. **Main results**

(A) - The main theorem.

**Theorem 1.** If conditions \((B1) - (B2)\) hold, then the Influence function of the TLIM index is given by

\[ IF(z, F) = \tau'(I) \left( - \left( \frac{h_1'(\mu_F) \mathbb{E} h(X)}{h_1(\mu_F)^2} + h_2'(\mu_F) \right) (z - \mu_F) + \frac{h(X) - \mathbb{E} h(X)}{h_1(\mu_F)} \right), \]
for \( z \in \mathcal{V}_X \).

**Remark on the asymptotic variance.** It was said earlier that the plug-in estimator should give the asymptotic variance of the limiting Gaussian variable, if it exists, as

\[
\sigma^2 = \int_{\mathcal{V}_X} IF(X)^2 \, d\mathbb{P} = \mathbb{E} IF(X)^2.
\]

This is exactly the case from the asymptotic normality of the plug-in estimator as established in Theorem 2 in Mergane *et al.* (2018).

Let us move to the illustrations of our results for particular cases.

**(B) - Particular forms.**

Let us proceed to the study of particular members of the TLIM class. We will have to compare our results with existent ones if any in the appendix. When the computation are simple, we only give the result without further details.

1. **Mean Logarithmic Deviation.** We have
   \[
   \tau(s) = s, \quad h(s) = h_2(s) = \log(s^{-1}), \quad h_1(s) \equiv 1
   \]
   and next \( \tau'(s) \equiv 1, \ h'(s) = h_2'(s) = -1/s \) and \( h_1'(s) \equiv 0 \). The application of Theorem 1 gives
   \[
   IF(z, DLM) = \mu_F^{-1}(z - \mu_F) + (\log z - \mathbb{E} \log X), \ z \in \mathcal{R}_F.
   \]

2. **Theil’s Index.** We have
   \[
   \tau(s) = s, \quad h(s) = s \log s, \quad h_1(s) = s, \quad h_2(s) = \log s,
   \]
   and next \( \tau'(s) \equiv 1, \ h_1'(s) \equiv 1 \) and \( h_2'(s) = 1/s \). The application of Theorem 1 gives
   \[
   IF(z, DLM) = \mu_F^{-1}(z \log z - \mathbb{E} X \log X) - \mu_F^{-2}(\mu_F + \mathbb{E} \log X), \ z \in \mathcal{R}_F.
   \]

3. **Class of Generalized Entropy Measures of parameter \( \alpha \), \( \alpha \notin \{0, 1\} \).** We have
   \[
   \tau(s) = \frac{s - 1}{\alpha(\alpha - 1)}, \quad \tau'(s) = \frac{1}{\alpha(\alpha - 1)}, \quad h(s) = h_1(s) = s^\alpha, \quad h_2(s) \equiv 0.
   \]
   The application of Theorem 1 gives
   \[
   IF(z, GE(\alpha)) = \frac{z^\alpha - \mu_{F,\alpha} - \mu_{F,\alpha}(\alpha - 1)\mu_F^{\alpha+1}(z - \mu_F)}{\alpha(\alpha - 1)\mu_F^\alpha}, \ z \in \mathcal{R}_F.
   \]
(4) Class of Atkinson measures with parameter $\beta \in (0, 1)$. We have 
\[ \tau(s) = 1 - s^{1/\beta}, \quad h(s) = h_1(s) = s^{\beta}, \quad h_2(s) \equiv 0. \]
If we denote $\|X\|_\beta = (\mathbb{E}|X|^{\beta})^{1/\beta}$, the application of Theorem 1 yields 
\[ IF(z, At(\beta)) = \frac{\|X\|_\beta}{\mu_F} \left( \frac{z - \mu_F}{\mu_F} - \frac{z^\alpha - \mu_F}{\beta \mu_F \mu_{F,\beta}} \right), \quad z \in \mathcal{R}_F. \]

(5) Champernowne's index. We have 
\[ \tau(s) = 1 - \exp(s), \quad h(s) = h_2(s) = \log(s), \quad h_1(s) \equiv 1. \]
The application of Theorem 1 implies that 
\[ IF(z, Champ) = \frac{\exp(\mathbb{E}\log X)}{\mu_F} \left( \frac{1}{\mu_F} (z - \mu_F) - (\log z - \mathbb{E}\log X) \right), \quad z \in \mathcal{R}_F. \]

(6) Kolm's Familily of inequality measure of parameter $\alpha \neq 0$. We have 
\[ \tau(s) = \frac{1}{\alpha} \log(s), \quad h(s) = h_1(s) = \exp(-\alpha s), \quad h_2(s) \equiv 0. \]
By Theorem 1, we have 
\[ IF(z, Kolm(\alpha)) = \frac{1}{\alpha_F} \left( (z - \mu_F) - \left( \frac{\exp(-\alpha \mu_F)}{\mathbb{E}\exp(-\alpha X)} - 1 \right) \right), \quad z \in \mathcal{R}_F. \]

3. Proof of the main theorem

In the following proof, we will use the method of finding the $IF$ following argument as given in Kahn (2015). Suppose that we are interested in estimating $T(\mathbb{P}_X)$, where $\mathbb{P}_X$ the image measure is $d\mathbb{P}$ defined by $d\mathbb{P}_X(B) = d\mathbb{P}(X \in B)$ for $B \in \mathcal{B}(\mathbb{R})$ and is also Lebesgue-Stieljes probability law associated $F$, that is $\mathbb{P}_X([a, b]) = F(b) - F(a)$ for all $-\infty \leq a \leq b \leq +\infty$. Here we use integrals based on measures and thus integrals in $d\mathbb{P}$ are integrals in $d\mathbb{P}_X$ in the following sense: for any non-negative and measurable function $\ell : \mathbb{R} \to \mathbb{R}$, we have 
\[ \int \ell(X)d\mathbb{P} = \inf h(y)d\mathbb{P}_X \equiv \int h(y)dF(y). \]
Suppose that $T(\mathbb{P})$ is defined on a family of probability measures $\mathbb{P}_\lambda$, $\mathbb{P}_\lambda$ being associated with the random variable $X_\lambda$ with $X = X_{\lambda_0}$ and $F = F_{\lambda_0}$. Suppose that $T$ is independent of $\lambda$. If we have 
\[ \frac{\partial}{\partial \lambda} T(\mathbb{P}_\lambda) = \int (\ell(y) \frac{\partial}{\partial \lambda} \mathbb{P}_\lambda), \]
where $\ell$ is measurable and $\mathbb{P}_X$-integrable. Then the IF at $T(F_{\lambda_0}) = T(F)$ is given by

$$IF(z, F) = \ell(z) - \int \ell(y) \, dF(y) = \ell(z) - \mathbb{E}\ell(X).$$

Actually, the rule uses Gâteaux differentiations properties and constitutes one of the fastest methods of finding the IF. We are going to apply it.

**Proof of Theorem 1.**

We remind the notation.

$$I = \frac{\mathbb{E}h(X)}{h_1(\mu_F)} - h_2(\mu_F).$$

We have

$$\frac{\partial}{\partial \lambda} TLIM(\mathbb{P}_X) = \frac{\partial}{\partial \lambda} \tau \left( \left( \frac{1}{h_1(\int Xd\mathbb{P})} \right) \int h(X)d\mathbb{P} - h_1 \left( \int Xd\mathbb{P} \right) \right).$$

We get

$$\frac{1}{\tau'(I)} TLIM(\mathbb{P}_X) = -\frac{h_1'(\mu_F)\mathbb{E}h(X)}{h_1(\mu_F)^2} \int X \frac{\partial}{\partial \lambda} d\mathbb{P}$$

$$+ \frac{1}{h_1(\mu_F)} \int h(X) \frac{\partial}{\partial \lambda} d\mathbb{P}$$

$$- h_2'(\mu_F) \int X \frac{\partial}{\partial \lambda} d\mathbb{P}$$

$$= \int \left( - \left( \frac{h_1'(\mu_F)\mathbb{E}h(X)}{h_1(\mu_F)^2} + h_2'(\mu_F) \right) X + \frac{h(X)}{h_1(\mu_F)} \right) \frac{\partial}{\partial \lambda} d\mathbb{P}.$$ 

By centering at expectations, we have

$$IF(z, F) = \tau'(I) \left( - \left( \frac{h_1'(\mu_F)\mathbb{E}h(X)}{h_1(\mu_F)^2} + h_2'(\mu_F) \right) (z - \mu_F) + \frac{h(X) - \mathbb{E}h(X)}{h_1(\mu_F)} \right), \ z \in V_X.$$ 

□
4. Conclusion and Perspectives

In this paper, we studied the Theil-like family of inequality measures introduced in Mergane et al. (2018). Following the paper on the asymptotic finite-distribution normality, we focus on the influence function of that family. Results are compared with those of some authors in particular. We think that this unified and compact approach will serve as general tools for comparison purpose. In addition, in computation packages, it allows more compact programs resulting in more efficiency. A paper on computational aspects will follow soon.
5. APPENDIX: A LIST OF SOME INFLUENCE FUNCTIONS

Here, we list a number of inequality measures and the corresponding influence functions.

The Generalized Entropy Measures of Inequality $GE(\alpha)$, which depends on a parameter $\alpha > 0$ and defined by

$$
I_{E}^{\alpha} = \int_{0}^{\infty} \frac{1}{\alpha(\alpha - 1)} \left[ \left( \frac{y}{\mu_{F}} \right)^{\alpha} - 1 \right] dF(y)
$$

$$
= \frac{1}{\alpha(\alpha - 1)} \left( \frac{\mu_{F,\alpha}}{\mu_{F}^{\alpha}} - 1 \right), \alpha > 0, \alpha \notin \{0, 1\},
$$

has the IF (see e.g. Cowell and Flachaire (2007))

$$(5.1)$$

$$
IF(z; I_{E}^{\alpha}) = \frac{1}{\alpha(\alpha - 1)\mu_{F}^{\alpha}}(z^{\alpha} - \mu_{\alpha}) - \frac{\mu_{\alpha}}{(\alpha - 1)\mu_{F}^{\alpha+1}}[z - \mu_{F}], \alpha \notin \{0, 1\}.
$$

**Important remark.** Our result on the IF of the $GE(\alpha)$ is different from that of Cowell and Flachaire (2007) by the multiplicative coefficient $\frac{1}{\alpha(\alpha - 1)\mu_{F}^{\alpha}}$. In other words, that coefficient is missing in Cowell and Flachaire (2007). We also find the same result by the computations below which is a direct proof.

$$
\frac{\partial}{\partial \lambda} \frac{1}{\alpha(\alpha - 1)} \left( \int X^{\alpha} dP \right)^{\alpha - 1} = \frac{1}{\alpha(\alpha - 1)} \int \frac{\mu_{F}^{\alpha}X^{\alpha} - \alpha \mu_{F}^{\alpha-1}X}{\mu_{F}^{2\alpha}} \frac{\partial}{\partial \lambda} dP.
$$

By the method described in the proof, we may center the integrand to get

$$
IF(X, GE(\alpha)) = \frac{1}{\alpha(\alpha - 1)} \frac{\mu_{F}^{\alpha}(X^{\alpha} - \mathbb{E}X^{\alpha}) - \alpha \mu_{F}^{\alpha-1}(X - \mathbb{E}X)}{\mu_{F}^{2\alpha}}.
$$

which again gives the result.
The Mean Logarithmic Deviation (MDL), which is a special case of the GE class where $\alpha = 0$, defined by

\begin{equation}
I_0^E = - \int_0^\infty \log \left( \frac{y}{\mu_F} \right) dF(y) = \log \mu_1 - \nu, \quad \nu = \mathbb{E} \log X,
\end{equation}

is associated to the IF

\begin{equation}
IF(z; I_0^E) = -[\log z - \nu] + \frac{1}{\mu_1} [z - \mu_F].
\end{equation}

The Theil Measure, which also is a special case of the GE class for $\alpha = 1$,

\begin{equation}
I_1^E = \int_0^\infty \frac{y}{\mu_F} \log \left( \frac{y}{\mu_F} \right) dF(y) = \frac{\nu}{\mu_F} - \log \mu_F, \quad \nu = \mathbb{E} X \log X,
\end{equation}

has the IF

\begin{equation}
IF(z; I_1^E) = \frac{1}{\mu_F} [z \log z - \nu] - \frac{\nu + \mu_F}{\mu_1^2} [z - \mu_F].
\end{equation}

The Atkinson Class of Inequality Measures of parameter $\alpha \in (0, 1]$, defined by (see Cowell and Flachaire (2007))

\begin{equation}
I_\alpha^A = 1 - \left[ \int_0^\infty \left( \frac{y}{\mu_F} \right)^{1-\alpha} dF(y) \right]^{1/(1-\alpha)} = 1 - \frac{\mu_{F,1-\alpha}^{1/(1-\alpha)}}{\mu_F}, \quad \alpha > 0, \alpha \neq 1,
\end{equation}

and its influence function is given by

\begin{equation}
IF(z; I_\alpha^A) = -\frac{\nu^{1/(1-\alpha) - 1}}{(1 - \epsilon)\mu_F} (z^{1-\epsilon} - \nu) + \frac{\nu^{1/(1-\epsilon)}}{\mu_F^2} (z - \mu_F),
\end{equation}

where $\nu = \mathbb{E} X^{1-\epsilon}$.

We notice that for $\alpha = 1$, we have

\begin{equation}
I_1^A = 1 - e^{\int_0^\infty (\log y) dy} = 1 - e^{-I_0^E},
\end{equation}
The Gini Coefficient, defined by (see e.g. Cowell and Flachaire (2007)):

\begin{equation}
I_G = 1 - 2 \int_0^1 L(F, p) dp,
\end{equation}

has the IF

\begin{equation}
IF(z, I_G) = 2 \left[ R(F) - C(F, F(z)) + \frac{z}{\mu_F} (R(F) - (1 - F(z))) \right],
\end{equation}

where

\begin{equation}
R(F) = \int_0^1 L(F, p) dp
\end{equation}

and \( C \) is is the cumulative functional defined by

\begin{equation}
C(F, p) = \int_0^{Q(p)} x dF(x), \ 0 \leq p \leq 1.
\end{equation}

The Quintile Share Ratio Measure of Inequality (QSR), defined by

\begin{equation}
\eta = \frac{\int_{Q(0.8)}^{\infty} y dF(y)}{\int_0^{Q(0.2)} y dF(y)} = \frac{EX 1_{\{X > Q(0.8)\}}}{EX 1_{\{X \leq Q(0.2)\}}},
\end{equation}

where \( 1_A \) is an indicator function of a set \( A \), is associated with the IF described below (see Kpanzou (2015)). Let

\begin{equation}
N(F) = \int_{Q(0.8)}^{\infty} x dF(x)
\end{equation}

and

\begin{equation}
D(F) = \int_0^{Q(0.2)} x dF(x).
\end{equation}

and define the subdivision of \( \mathbb{R}_+ : A_1 = [0, Q(0.2)], A_2 = (Q(0.2), Q(0.8)), A_3 = (Q(0.8), 1] \) and set

\begin{align*}
I_1(z, \eta) &= -z N(F) + 0.2 Q(0.8) D(F) + 0.8 Q(0.2) N(F)] / D^2(F); \\
I_2(z, \eta) &= 0.2 Q(0.8) D(F) - 0.2 Q(0.2) N(F)] / D^2(F); \\
I_3(z, \eta) &= z D(F) - 0.8 Q(0.8) D(F) - 0.2 Q(0.2) N(F)] / D^2(F).
\end{align*}
The $SQR$ influence function is defined by

$$I_1(z, \eta) = I_1(z, \eta)1_{A_1}(z) + I_2(z, \eta)1_{A_2}(z) + I_3(z, \eta)1_{A_3}(z).$$

References

Allison, P. D. (1978). Measures of inequality. *American Sociological Review*, vol. 43, pp. 478-484.

Atkinson, A.B. (1970). On the Measurement of Inequality, *Journal of Economic Theory*, 2, 244-263.

Champernowne, D.G. and Cowell, F. A. (1998). *Economic inequality and income distribution*, Cambridge: Cambridge University Press.

Cowell, Frank A. (2003). *Theil, Inequality and the Structure of Income Distribution*. London School of Economics and Political Sciences. available at: http://eprints.lse.ac.uk/2288/.

Cowell, F.A. (1980a). Generalized entropy and the measurement of distributional change. *European Economic Review*, 13, 147-159.

Cowell, F. A. and Flachaire, E. (2007). Income Distribution and Inequality Measurement: The Problem of Extreme Values. *Journal of Econometrics*, vol. 141, pp. 1044-1072.

Cowell, F.A., Flachaire, E. and Bandyopadhyay, S. (2009). Goodness-of-Fit: An Economic Approach. *Distributional Analysis Research Programme (DARP 101)*. Discussion Paper. Department of Economics (University of Oxford).

Greselin, F., Pasquazzi, L. and Zitikis, R. (2010a). Zenga’s new index of economic inequality, its estimation, and analysis of incomes in Italy. *Journal of Probability and Statistics*, vol. 2010, pp. 1-26.

Hampel, F.R. (1974). The Influence Curve and its Role in Robust Estimation. *Journal of the American Statistical Association*, vol. 69, pp. 383-393.

Hampel, F.R., Ronchetti, E.M., Rousseeuw, P.J. and Stahel, W.A. (1986). *Robust Statistics: The Approach Based on Influence Functions*. New York: John Wiley.

Hulliger, B. and Schoch, T. (2009). Robust estimation of the quintile share ratio with bias reduction. Presented at the Swiss Statistics Meeting, October 30, 2009, Geneva.

Kahn J.(2015) Influence functions for fun and profit. University of Michigan, School of Business. Working paper. http://j-kahn.com/files/influencefunctions.pdf (visited 2018/07/08)

Kolm S. (1976a): Unequal Inequalities I, *Journal of Economic Theory*, 12, 416-442.

Koenker R. (2005). *Quantile Regression*. Cambridge University Press.
Kpanzou, T.A. (2015). On the Influence Function of the Quintile Share Ratio. *Communications in Statistics - Simulation and Computation*, vol. 44, pp. 2492-2499.

Langel, M. and Tillé, Y. (2011). Statistical inference for the quintile share ratio. *Journal of Statistical Planning and Inference*, vol. 141, pp. 2976-2985.

Langel, M. and Tillé, Y. (2012). Inference by linearization for the Zenga’s new inequality index: a comparison with the Gini index. *Metrika* 75, pp. 1093-1110.

Mergane P.D. and Lo G.S. (2013) On the Functional Empirical Process and Its Application to the Mutual Influence of the Theil-Like Inequality Measure and the Growth. *Applied Mathematics*. Vol. 4, 986-1000. http://dx.doi.org/10.4236/am.2013.47136. (http://www.scirp.org/journal/am)

Mergane P.D., Kpanzou T.A., Diam B. and Lo G.S. (2018) A Theil-like class of inequality measures, its asymptotic normality Theory and applications. *Afrika Statistika*, Vol. 13 (3), 2018, 1699 – 1717.

Magurran, A. E. (1991). *Ecological diversity and its measurement*, Chapman and Hall.

Polisicchio, M. and Porro, F. (2009). A comparison between Lorenz \( L(p) \) curve and Zenga \( I(p) \) curve. *Statistica Applicata - Italian Journal of Applied Statistics*, vol. 21, pp. 289-301.

Rousseau, R. (1993). Measuring concentration: sampling design issues as illustrated by the case of perfectly stratified samples. *Scientometrics*, vol. 28, pp. 3-14.

Theil, H. (1967). Economics and Information Theory, Amsterdam, North Holland.

Van Praag, B., Hagenaars, A. and Van Eck, W. (1983). The Influence of Classification and Observation Errors on the Measurement of Income Inequality. *Econometrica*, vol. 51, pp. 1093-1108.

Victoria-Feser, M.-P. (2000). Robust Methods for the Analysis of Income Distributions, Inequality and Poverty. *International Statistical Review*, vol. 68, pp. 277-293.

Wasserman L.(2006) *All of Nonparametric Statistics*. Springer Science+Business Media, Inc.

White, M. J. (1986). Segregation and diversity measures in population distribution. *Population Index*, vol. 52, pp. 198-221.

Zenga, M. (1984). Proposta per un Indice di Concentrazione Basato sui Rapporti tra Quantili di Popolazione e Quantili di Reddito. *Giornale degli Economisti e Annali di Economia*, vol. 43, pp. 301-326.

Zenga, M. (1990). Concentration Curves And Concentration Indexes Derived From Them. In Dagum C., Zenga M. (eds.), Income and
Wealth Distribution, Inequality and Poverty. Springer-Verlag, New York. Proceedings of the Second International Conference on Income Distribution by Size: Generation, Distribution, Measurements and Applications, held at the University of Pavia, Italy, September 28-30, 1989.