$N = 1$ SUPER–$P$–BRANES

in twistor-like Lorentz harmonic formulation

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Abstract

Unique twistor–like Lorentz harmonic formulation for all $N = 1$ supersymmetric extended objects (super–$p$–branes) moving in the space–time of arbitrary dimension $D$ (admissible for given $p$) are suggested. The equations of motion are derived, explicit form of the $\kappa$–symmetry transformations is presented and the classical equivalence to the standard formulation is proved. The cases with minimal world–sheet dimensions $p = 1, 2$, namely of $D = 10$ heterotic string and $D = 11$ supermembrane, are considered in details. In particular, the explicit form of irreducible $\kappa$–symmetry transformations for $D = 11$ supermembrane is derived.

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1 Introduction

The extension of different variants of the twistor approach [1] for the case of supersymmetric extended objects (super–$p$–branes) moving in high dimensional space-time became an actual task after the works [2, 3, 4] had been published. In [3, 4] the famous $\kappa$–symmetry of $D = 3, 4$ superparticle [3, 4] have been identified with the world-line supersymmetry. Moreover, it has been proved [3] that the problem of irreducible $\kappa$–symmetry description which troubled superparticle and superstring covariant quantization [7] are automatically solved in the framework of twistor-like formulation, i.e., when an appropriate auxiliary bosonic spinor variables are present in the configurational space of the theory and the action functional has twistor-like form [1].

One of the directions of the generalizations of the results from Refs. [2, 3, 4] consist in the construction of world-line (world-sheet) superfield or doubly supersymmetric formulation of the supersymmetric objects, where the $\kappa$–symmetry is completely realized as superconformal world-line (worldsheet) supersymmetry. Such formulation have been presented for superparticle in $D = 3, 4, 6, 10$ [11], heterotic [12] and $D = 3$ Green-Schwarz superstrings [13], as well as for the supermembrane in $D = 11$ [14].

At the same time, a component variant of twistor-like approach has been developed using the Lorentz harmonic variables as twistor-like ones [15, 16, 17, 18]. The works [15, 16, 17, 18] gave the bridge between the twistor approach and the well known works [19, 20], where the idea of Lorentz covariant quantization of $D = 10$ superparticle and superstring using the extension of the phase space by specially chosen set of vector [21] and spinor harmonics had been realized. Extended objects without tension (null super–$p$–branes) in $D = 4$ space-time have been covariantly quantized in the twistor-like Lorentz harmonic formulation [16]. This demonstrated the power of such approach and gave the first example of selfconsistent quantum theory for the extended objects with the world-volume dimension $d = p + 1 > 2$. The formulations of such type have been constructed for $D = 4, N = 1$ and $D = 10, N = IIB$ superstrings [17] as well as for $D = 11$ supermembrane [18].

Another super–$p$–branes [22] – [31] also are physically interesting objects, because they appears as supersymmetric solitons in some field theories [22, 26] as well as in some super–$p$–brane theories [28, 29]. So, the heterotic 5–brane [27] appears as soliton solution in $D = 10$ superstring theory [28] and vice versa [29].

Here we will suggest an universal twistor-like Lorentz-harmonic formulation for the all

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1See also the papers [8, 9, 10], where the twistor formulations of supersymmetric particles and strings are discussed
admissible set \([24]\) of \(N = 1\) super-\(p\)-branes in \(D\)-dimensional space-time. We will prove their classical equivalence to the standard formulations, derive the motion equations and present an explicit form of the irreducible \(\kappa\)-symmetry transformations.

For simplicity, we give an explicit calculations for the dimensions \(D = 2, 3, 4, 10, 11\) where the Majorana spinors exist. The intermediate formulas for the general case may be easily reproduced using slightly more complicated notations of Ref.\([24]\).

2 Twistor-like action, equations of motion and \(\kappa\)-symmetry for \(N = 1\) super-\(p\)-branes

2.1. Lorentz harmonic variables. Definitions and admissible variation concept

In this subsection we describe the necessary set of the Lorentz harmonic variables which are suitable for the construction of the twistor like formulations of \(N = 1\) super-\(p\)-branes living in \(D\) dimensions for all admissible values \(D\) and \(p\) \([23, 24]\). For simplicity, explicit expressions will be given for the cases \(D = 2, 3, 4(\mod 8)\), where the Majorana spinor representation exists. Of course, all the results may be easily extended to another values of \(D\) using the spinor conventions developed in \([24]\).

Hence we suggest that the used \(\gamma\)-matrices and \(\sigma\)-matrices are symmetric and real ones

\[
(CT^m)^T = (CT_m), \quad (\Gamma^n C^{-1})^T = (\Gamma^n C^{-1}) \tag{1}
\]

Moreover, for the all admissible values of \(D\) and \(p\) (i.e., for all \(D\) and \(p\) where the standard \(N = 1\) super-\(p\)-brane formulations exist \([25]\) )

\[
(\Gamma^{m_1} C^{-1})_{\{\alpha \beta}(\Gamma^{m_2...m_p} C^{-1})_{\gamma \delta\}} = 0 \tag{2}
\]

The Lorentz harmonic variables form the matrix \(v^a_\alpha\) taking its values in the spinor representation of the double covering of \(D\)-dimensional Lorentz group \(Spin(1, 9) \simeq SO(1, 9)\)

\[
v^a_\alpha \in Spin(1, D - 1) \tag{3}
\]

In \([3]\) \(\alpha = 1, \ldots, 2^\nu\) is \(Spin(1, D - 1)(\simeq SO(1, D - 1))\) spinor index ; \(a = 1, \ldots, 2^\nu\) is the composed spinor index of the right product of (pseudo)orthogonal groups \([SO(1, p) \times\)
SO(D−p−1)] and 2ν is the dimension of the considered spinor representation: ν = [D/2] for D = 3, 4(mod8) and ν = (D−2)/2 for D = 2(mod8) when the Majorana-Weyl spinors are used.

The key point of the discussed approach consists in the statement that the basic elements of the orthonormal vector repere \( u^{(n)}_m \)

\[
 u^{(n)}_m u^{(l)}_m = \eta^{(n)(l)} = \text{diag}(1, -1, \ldots, -1)
\]

are determined by means of the ”square root” type universal relations in terms of a harmonic variable matrix \( v^a_\alpha \)

\[
 u^{(n)}_m \equiv 2^{-\nu} v^a_\alpha (CT_m)^{\alpha\beta} v^b_\beta (\Gamma^{(n)} C^{-1})_{ab}
\]

For \( D = 2(mod8) \) and the Majorana-Weyl spinors the matrices \((CT_m)\) and \((\Gamma^n C^{-1})\) should be understood as the chiral \( \sigma \)-matrices \( \tilde{\sigma}_m \) and \( \sigma^n \)

\[
 (CT_m) \rightarrow \tilde{\sigma}_m, (\Gamma^n C^{-1}) \rightarrow \sigma^n
\]

Then, more strong relation is satisfied for \( D = 10, p = 1 \) instead of (2)

\[
 \tilde{\sigma}_m^{\alpha(\beta \gamma \delta)} m \equiv \frac{1}{3} (\tilde{\sigma}_m^{\alpha\beta} \tilde{\sigma}_m^{\gamma\delta} + \text{cyclic permutations} (\alpha, \beta, \gamma)) = 0
\]

Eq. (3) is realized as the requirement for the harmonic matrix to satisfy some algebraic restrictions which are called harmonicity conditions (see [32])

\[
 v^a_\alpha \in \text{Spin}(1, D − 1) \Leftrightarrow \Xi_M(v) = 0
\]

An explicit form of the harmonicity conditions have been presented in [13], [32], [34] for \( D = 3, 4, 6, 10 \) superparticles and superstrings (see also [35] for \( D = 10 \) case) and in [18] for \( D = 11 \) supermembrane. Here we shall present only the necessary set of general features of harmonic variables and will not write these harmonicity conditions explicitly.

As the consequence of (8) the \( \Gamma \)-matrices can be removed from one side of (4) to another. I.e. there are the following consequences of (3) for any \( D = 2, 3, 4(mod8) \)

\[
 \Xi_M(v) = 0 \Rightarrow
\]

\footnote{Such approach to harmonic variables was used in our previous works [18] as well as in the recent work [36]. Previously the similar approach to the spinor realization of the repere variables (which are identical with Lorentz harmonic variables as well as with generalized Newman–Penrose dyades [16, 17, 18]) was used in the interesting work of Wiegmann [37] devoted to the effective actions of spinning and heterotic strings.}
\[ u^{(n)}_m (\Gamma^m C^{-1})_{\alpha\beta} = v^a_{(n)} (\Gamma^m C^{-1})_{ab} \alpha \chi[5 v^b_{\beta} \]

\[ u^{(n)}_m (C \Gamma_{(n)})^{ab} = v^a_{(n)} (C \Gamma_m)^{\alpha\beta} v^b_{\beta} \quad (9) \]

\[ (\Gamma^m C^{-1})_{\alpha\beta} = v^a_{(n)} (\Gamma^m C^{-1})_{ab} v^b_{\beta} u^{(n)}_m , \]

\[ (C \Gamma_{(n)})^{ab} = v^a_{(n)} (C \Gamma_m)^{\alpha\beta} v^b_{\beta} u^{(n)}_m , \quad (10) \]

\[ u^{(n)}_m (\Gamma_{(n)} C^{-1})_{ab} = v^a_{(n)} (\Gamma_m C^{-1})_{\alpha\beta} v^b_{\beta} , \]

\[ u^{(n)}_m (C \Gamma_m)^{ab} = v^a_{(n)} (C \Gamma_{(n)})^{\alpha\beta} v^b_{\beta} \quad (11) \]

Here \( v^a_{\alpha} \equiv v^{-1}_{a} \), i.e. the relations

\[ \Xi^b_{a} \equiv v^a_{\alpha} v^b_{\beta} - \delta^b_{a} = 0 \quad (12) \]

should be satisfied. This is an independent harmonicity condition for \( D = 2(mod8) \).

However, the matrix \( v^a_{\alpha} \) can be constructed from \( v^b_{\alpha} \) one for \( D = 3, 4(mod8) \) due to harmonicity conditions

\[ \Xi^{ab} \equiv v^a_{\alpha} C^{\alpha\beta} v^b_{\beta} - C^{ab} = 0 \Rightarrow v^a_{\alpha} = C^{-1}_{ab} v^b_{\beta} C^{\beta\alpha} \quad (13) \]

Eq.(13) is the manifestation of the invariance of the charge conjugation matrix \( C^{ab} \) under \( SO(1, D - 1) \) rotations.

Moreover, the following relations

\[ Sp(v^T C \Gamma_{m_1...m_k} v \Gamma_{(n)} C^{-1}) = 0, \quad Sp(v^T C \Gamma_m v \Gamma_{(n_1)...(n_k)} C^{-1}) = 0, \quad (\text{when } k > 1) \quad (14) \]

are satisfied for the matrix \( v^a_{\alpha} \in Spin(1, D - 1) \) (3).

The relations (3), (9) – (11) are the basic ones of the twistor-like Lorentz harmonic approach to super-p-brane theories.

To solve the variational problem formulated in the configurational space which includes the Lorentz-harmonic variables the concept of admissible variations [21, 17] is very useful. These are the variations which do not violate the harmonicity conditions (8) or, equivalently, the relation (3)

\[ (v^a_{\alpha} + \delta v^a_{\alpha}) \in Spin(1, D - 1) \quad (15) \]

For the definition of the admissible variation it is convenient to discuss the variation of the (9)–(11) arising from the harmonicity conditions. So, varying the relation (3), we get
\[ \delta u_m^{(n)}(C\Gamma_m)^{ab} = \delta v_\alpha^{\alpha\beta} v_\beta^b + v_\alpha^{\alpha\beta} \delta v_\beta^b \equiv \]
\[ \equiv (v^{-1} \delta v)_a^a u^{m(k)}(C\Gamma_m)^{db} + (v^{-1} \delta v)_b^b u^{m(k)}(C\Gamma_m)^{da} \]

Use of the first relation (11) transforms Eq. (16) into the form

\[ u_m^{(k)} \delta u_m^{(n)}(C\Gamma_m)^{ab} = D(v^{-1} \delta v)^a_v + (C\Gamma_m)(v^{-1} \delta v)\Gamma_m C^{-1})^a \]

(17)

Taking into account the Fiertz identities (101) (see Appendix A) for the matrix \((v^{-1} \delta v)\), we can derive from (17) the following set of relations which defines the admissible variation

\[ Sp(v^{-1} \delta v) = 0 \]
\[ -2^{-(v-1)} Sp(v^{-1} \delta v) \Gamma^{(k)(l)} = u_m^{m(k)} \delta u_m^{(l)} \equiv \Omega^{(k)(l)}(\delta) \]

(18)

(19)

\[ Sp(v^{-1} \delta v) \Gamma_{m_1...m_q} = 0, \]

(20)

where \(q = 1, 3, 4, \ldots \) for \(D = 3, 4(mod 8)\), and \(q = 4, 6, \ldots \) for \(D = 2(mod 8)\).

Hence, the admissible variation of \(v_\alpha^a\) is one which may be reduced to \(SO(1, D - 1)\) rotation

\[ \delta v_\alpha^a = \frac{1}{4} \Omega^{(k)(l)}(\delta)(\Gamma_{k(l)})^a_b v_\alpha^b \quad \leftrightarrow \quad \frac{\delta v_\alpha^a}{4} = \frac{1}{4} \Omega^{(k)(l)}(\delta)(\Gamma_{k(l)})^a_b \]

(21)

with the Cartan form \(\Omega^{(k)(l)}(\delta)\) (19) as the parameter.

This result seems to be just evident when the definition (3) of the harmonic variables is taken into account.

2.2. Twistor-like action functional

for \(N = 1\) SUPER–P–BRANES in D-dimensions

The proposed in \[17, 18\] action functional for super-p-branes moving in space–time of any admissible \[25\] dimension \(D\) has the following form

\[ S_{D,N=1,p} = (\alpha')^{-\frac{1}{2}} \int d^{p+1}\xi \ e(\xi) \left( -e_{f}^{\mu} \omega_{m}^m u_m^{(f)} \right) + c(\alpha')^{\frac{1}{2}} + S_{D,N=1,p}^{W-Z} \]

(22)
\[ S_{D,N=1,p}^{W-Z} \equiv (\alpha')^{-\frac{1}{2}} \frac{ai}{p+1} \int d^{p+1}\xi \ e^{\mu_1 \mu_2 \ldots \mu_p} \sum_{k=0}^{p+1} [\omega_{\mu_p}^{m_k} \ldots \omega_{\mu_{k+1}}^{m_1}] \times \]
\[ \partial_{\mu_k} x^{m_k} \ldots \partial_{\mu_1} x^{m_1} (\partial_{\mu_0} \theta \Gamma_{m_1 \ldots m_p} C^{-1} \theta)]. \quad (23) \]

Here \( \alpha' \) is the Regge slope like parameter with the dimension equal the square of length, \( c \) is a dimensionless parameter \( (c(\alpha')^{1\frac{1}{2}}) \) is the worldsheet cosmological constant) and the value of the parameter \( a \)

\[ a = \pm i \frac{\mu_p}{p!} (c^2 \alpha')^{1\frac{1}{2}} \]

is defined (up to a sign factor) by the \( \kappa \)-symmetry requirement;

\[ \omega_{\mu}^m = \partial_{\mu} x^m - i \partial_{\mu} \theta \Gamma^m C^{-1} \theta \equiv \partial_{\mu} x^m - i \partial_{\mu} \theta^\alpha \left( \Gamma^m C^{-1} \right)_{\alpha\beta} \theta^\beta, \quad (25) \]

\( x^m (m = 0, 1, \ldots, D - 1) \) are the ordinary (flat) space-time coordinates and \( \theta^\alpha (\alpha = 1, \ldots, 2^\nu) \) are the fermionic (Grassmannian) coordinates of the \( N = 1 \) \( D \)-dimensional superspace which have the properties of the Majorana (Majorana-Weyl for \( D = 2, 10 \)) spinors with respect to \( SO(1, D - 1) \) group.

### 2.3. SUPER-P-BRANE equations of motion

Using the admissible variation concept (24) we can write an arbitrary variation of the functional (22) in the form

\[ \begin{align*}
\delta S_{D,N=1,p} &= (\alpha')^{-1/2} \int d^{p+1}\xi \ [e \delta e^{\mu} \{-\omega_{\mu}^m u^{(f)}_m - e^{g} (-\omega_{\mu}^m u^{(g)}_m e^\nu + c(\alpha')^{1/2})\}] + \\
&+ \Omega^{g\{f\}}(\delta) e e^{j}_f u_{(g) m} \omega_{\mu}^m + \Omega^{i\{f\}}(\delta) e e^{j}_f \omega_{\mu}^m u_{(i) m} + \\
&+ \omega^m (\delta) \{ \partial_{\mu} \left( e e^{j}_f u_{m}^{(f)} \right) + i p a e^{\mu_1 \mu_2 \ldots \mu_p} \omega_{\mu_p}^m \ldots \omega_{\mu_2}^m \partial_{\mu_1} \theta \Gamma_{mm_2 \ldots m_p} C^{-1} \partial_{\mu_0} \theta \} + \\
&+ 2i \{ e e^{j}_f (\partial_{\mu} \theta \Gamma_{m} C^{-1})_\alpha u_{m}^{(f)} + a e^{\mu_1 \mu_2 \ldots \mu_p} \omega_{\mu_p}^m \ldots \omega_{\mu_1}^m (\partial_{\mu_0} \theta \Gamma_{m_1 \ldots m_p} C^{-1})_\alpha \} \delta^\alpha, \quad (26)
\end{align*} \]

Hence, the motion equations for the discussed super-p-brane formulation have the following form

\[^{3}\text{The use of the supersymmetric invariant variation } \omega^m = \delta x^m - i \partial \theta \Gamma^m C^{-1} \theta \text{ instead of } \delta x^m \text{ simplifies the derivation of the motion equations and symmetry transformations in the manifestly supersymmetric form.} \]
\[ \omega^m u_m^{(f)} = \frac{c}{p} (\alpha')^{1/2} c' \mu_f, \]  

(27)

\[ \omega^m u_m^{(i)} = 0, \]  

(28)

\[ \partial_\mu \left( ee_\mu f \{u_m\} \right) + ipae_{\mu_0 \mu_1 \mu_2} \omega_{\mu_0} \ldots \omega_{\mu_2} \partial_{\mu_1} \theta \Gamma_{\mu_0 \mu_2 \ldots \mu_p} C^{-1} \partial_{\mu_2} \theta = 0 \]  

(29)

\[ e^\mu_0 \partial_\mu \theta^2 v_\beta \left( \Gamma^{(f)} C^{-1} \right)_{bc} (\delta^c_a \pm (\Gamma')^c_a) = 0 \]  

(30)

In the derivation of Eq. (30) the explicit value of parameter \( a \) was substituted and the following identity was used

\[ \epsilon_{\mu_0 \ldots \mu_1 f} \left( \Gamma \{m_0\} \ldots \{m_p\} C^{-1} \right)_{\alpha}^{\beta} \]  

(31)

where the matrix \( \Gamma' \) is defined by the relation

\[ (\Gamma')^c_a \equiv \frac{1}{(p+1)!} \epsilon_{f_\mu \ldots f_1 f_0} \left( C \Gamma \{f_0\} \ldots \{f_p\} C^{-1} \right)^c_a \]  

(32)

and is the square root from the unity matrix

\[ (\Gamma')^2 = I \]  

(33)

So, \( \frac{1}{2} (1 \pm \Gamma') \) are the projectors, which are related to the known ones \( \frac{1}{2} (1 \pm \Gamma) \) building from the matrix \( [23] \)

\[ (\Gamma')^a_\beta \equiv \left( \frac{p^{(p-1)}}{(p+1)!} \right)^{p+1} \frac{1}{c(\alpha')^{1/2}} e_{\mu_0 \ldots \mu_1 f} \omega_{\mu_0} \ldots \omega_{\mu_1} \omega_{\alpha} \left( C \Gamma \{m_0\} \ldots \{m_p\} C^{-1} \right)^a_\beta \]  

(34)

on the mass shell \footnote{To reduce the expression \( (34) \) for the projector \( \Gamma \) to the form presented in [23] the value of the dimensionless constant \( c \) should be fixed to be \( p!/(p+1) \).}

However, \( \frac{1}{2} (1 \pm \Gamma') \) has the projection properties off–shell in distinction with \( \frac{1}{2} (1 \pm \Gamma) \). This is one of the advantages of the twistor–like approach.

After the exclusion of the Lorentz-harmonic variables using (27), (28), the equations of motion (29), (30) coincide with the standard ones [7, 23, 25]. This proves the classical equivalence of the discussed formulation with standard one on the level of motion equations. The proof on the level of action functionals is presented in the next subsection.

However, equations of the Lorentz-harmonic formulation have the simpler form. So, Eq. (29) has the \( \sigma \)-model-like form. The simplicity of Eq. (30) is evident when the \( SO(1,9)_L \times [SO(1,p) \times SO(D - p - 1)]_R \)-invariant splitting of the harmonic matrix

\[ v^a_\alpha = \left( v^q_{\alpha A}, v^q_{\alpha A} \right) \]  

(35)
is used. Here \( q \) is the spinor index of \( SO(1, p) \), \( A \) and \( \dot{A} \) are the indices of some (may be, the coinciding) spinor representations of \( SO(1, D - p - 1) \) group. Because of the projection character of the matrices \( \frac{1}{2}(1 \pm \Gamma') \) there exists a representation where the matrices 

\[
\left( \Gamma^{(f)}(C^{-1}) \right)_{bc}(\delta_a^b \pm (\Gamma')^c_a) \text{ have the diagonal form with only nonvanishing components}
\]

\[
\left( \left( \Gamma^{(f)}(C^{-1}) \right)(I \pm \Gamma') \right)_{AB} \propto \delta_{A\dot{B}}(\epsilon \gamma^f)^{q\dot{p}}
\]

\[
\left( \left( \Gamma^{(f)}(C^{-1}) \right)(I \mp \Gamma') \right)_{q\dot{A}\dot{B}} \propto \delta_{AB}(\gamma \gamma^{f-1})_{qp}
\]

where \( \gamma^f \) and \( \epsilon \) are \( d = (p + 1) \)-dimensional \( \gamma \)-matrix and charge–conjugation matrix respectively.

In such representation Eq.(36) acquires the following simple particle–like form

\[
e_f^\mu \partial_\mu \theta^\beta v_{\beta q} \dot{A} (\epsilon \gamma^f)^{q\dot{p}} = 0
\]

Note that Eq.(37) can be presented in the form of the Dirac equation

\[
(\epsilon \gamma^f)^{pq} D_f \theta_{q\dot{A}} = 0
\]

for the variable \( \theta_{q\dot{A}} = \theta^\beta v_{\beta q} \dot{A} \) which appears as a covariant piece of the Lorentz invariant Grassmann field \( \theta^a = \theta^\beta v^a_\beta = (\theta^\mu_A, \theta^\mu_{\dot{A}}) \). The covariant derivative \( D_f \) involved into Eq.(38) is defined by the relation

\[
D_f \theta^a \equiv e^\mu_f \left( \partial_\mu \theta^a - \frac{1}{4} \Omega_{(\mu)}^{(k)(l)} \theta^b (\Gamma^{(k)(l)})^a_b \right)
\]

with \( \Omega_{(\mu)}^{(k)(l)} \) to be the components of the \( SO(1, D - 1) \) covariant Cartan form (19)

2.4. Classical equivalence with standard formulations

Eqs.(27) and (28) means that the vectors \( u^{(f)}_m \) are tangent to the worldsheet and the vectors \( u^{(i)}_m \) are orthogonal to the worldsheet

\[
\omega^m_\mu = \frac{c}{p} (\alpha')^{\frac{1}{2}} e^\mu_f u^{(f)m},
\]

From the other hand, Eq.(27) results in the fact that the first term in the Lagrangian (22) is proportional to the second one

\[
e e^\mu_f \omega^m_\mu u^{(f)}_m = \frac{1}{p} e c (\alpha')^{\frac{1}{2}},
\]
Further, take into account, that $e \equiv \det(e^f_\mu)$ may be rewritten in terms of induced metric $g_{\mu\nu}$

$$g_{\mu\nu} \equiv e^f_\mu e^f_\nu = \frac{p^2}{c^2}\alpha'\omega^m_\mu \omega^m_\nu$$

in the form

$$e \equiv \det(e^f_\mu) \equiv \det^{1/2}(g_{\mu\nu})$$

Hence,

$$e = \frac{p^{2(p+1)}}{c^{2(p+1)}(\alpha')^{p+1}}\det^{1/2}(\omega^m_\mu \omega^m_\nu) \tag{42}$$

Substituting Eq. (42) into the action functional (22) we get the standard action in the Dirac–Nambu–Goto–like form [23] (up to numerical constant)

$$S_{\text{Dirac–Nambu–Goto}} = \frac{p^{2(p+1)}(p-1)}{c^{2(p+1)}(\alpha')^{p+1}}\int d^{p+1}\xi \det^{1/2}(\omega^m_\mu \omega^m_\nu) + S_{W-Z}$$

where $S_{W-Z}$ has the form (23).

This concludes the proof of the classical equivalence of the discussed twistor–like formulation of super–$p$–branes with the standard ones [23].

### 2.5. Irreducible $\kappa$–symmetry transformations

To derive the form of the $\kappa$–symmetry transformations it is useful to rewrite the action variation by extracting the blocks proportional to the left hand sides of motion equations (27) – (29). First of all we transform the terms containing the auxiliary fields variations $\delta e$ and $\delta v \propto \Omega(\delta)$ (see (21)). As a result the first two terms in Eq. (26) turn into

$$\int d^{p+1}\xi \{ -e\delta e^f_\mu + e\delta \tilde{e}^f_\nu e^f_\mu + e\delta \eta^f_\mu \Omega_{gf}(\delta) \} \{(\alpha')^{-1/2}\omega^m_\mu u^f_m - \frac{1}{p}e^f_\mu \} +$$

$$\Omega^{(f)}(\delta)(\alpha')^{-1/2}e^f_\mu \omega^m_\nu u^{(i)}_m \tag{43}$$

To transform in the similar way the terms involving $\omega^m(\delta)$ and $\delta \Theta$ the following relations should be used

$$e^{\mu_\nu...\mu_{p+1}} \omega^{m_{p+1}}_\mu \omega^{m_1}_{\mu_1}(\partial_{\mu_0}\theta \Gamma_{m_1...m_p}C^{-1})_\alpha =$$

$$= e^{\mu_\nu...\mu_{p+1}}(\omega^{m_{p+1}}_\mu \omega^{m_1}_{\mu_1})\partial_{\mu_0}\theta \gamma^b \gamma^\gamma \Omega_{gf}(\gamma^{(f)}_1...\gamma^{(f)_p})C^{-1})ba\nu^a_\alpha =$$

$$= (c(\alpha')^{1/2}/p)^p e^f_\mu \partial_{\mu_0}\theta \gamma^b \gamma^\gamma e^{f_\mu...f_\nu}(\Gamma_{(f_1)...(f_n)}C^{-1})ba\nu^a_\alpha +$$

$$+ (\alpha')^{-1/2}\omega^m_\mu u^{(f)}_m - \frac{e^{f_\mu}}{p}H^\mu_\alpha \nu^a_\alpha + \omega^m_\mu u^{(i)}_m C^{mi}v^a_\alpha \tag{44}$$
In Eq. (44) $H$ and $G$ are defined as follows

$$H_{(f)}^\mu_a = - \sum_{k=1}^{p} \left( \frac{c(\alpha')^{1/2}}{p} \right)^{p-k} \epsilon^{\mu \ldots \nu_1 \mu \nu_p} \epsilon_{f_k} \ldots \epsilon_{f_{k+1}}$$

$$\{ (\alpha')^{-1/2} \omega^m_{\mu_k} u^i_m \} - \frac{c}{p} \epsilon_{f_k} \ldots \{ (\alpha')^{-1/2} \omega^m_{\mu_2} u^i_2 \} \times$$

$$\partial_{\mu_i} \theta^\beta v^b_{\beta} \left( \Gamma_{(f)} (f_k) \ldots (f_p) C^{-1} \right)_{ba},$$

(45)

$$G^\mu_a = \sum_{k=1}^{p} (-1)^k \epsilon^{\mu \ldots \nu_1 \mu \nu_p} u^i_m \omega^m_{\mu_k} u^j_{m_k} \omega^m_{\mu_{k+1}} u^j_{m_{k+1}} \omega^m_{\mu_2} u^j_2 \times$$

$$(\Gamma_{(i)} (i_k) \ldots (i_p) \Gamma (f_k) \ldots (f_p) C^{-1})_{ba}$$

(46)

Substituting Eqs. (13), (14), (11) and the explicit value of constant $a$ (24) into Eq. (26) we get the following expression for action variation

$$\delta S_{D,N=1,p} = (\alpha')^{-1/2} \int d^{p+1} \xi [ e^{-\delta e_f} + \delta e_g e^\nu e^\mu + e^\mu \Omega_g (\delta) - 2ia(\alpha')^{1/2} e^{-1} H_{(f)}^\mu_a v^a_\alpha \delta^\alpha \{ \omega^m_{\mu_k} u^i_m \} - \frac{c}{p} (\alpha')^{1/2} e_f^i ]$$

$$+ (e e_f^\mu \Omega^{i}) (f) (\delta) - 2ia G^\mu_a v^a_\alpha \delta^\alpha \omega^m_{\mu_k} u^i_m +$$

$$+ \omega^m (\delta) \{ \partial_{\mu} (e e_f^\mu u^i_m) \} + ipa e^{\mu \ldots \nu_1 \mu_0 \mu \nu_p} \omega^m_{\mu_p} \ldots \omega^m_{\mu_2} \partial_{\mu_1} \theta \Gamma_{mm_2 \ldots m_p} C^{-1} \partial_{\mu_0} \theta \} +$$

$$+ 2ie^\mu \partial_{\mu} \theta^\beta v^b_{\beta} \left( \Gamma^{(f)} C^{-1} \right)_{ba} \left( \delta^\alpha_a - i \frac{p(p-1)}{2} a \left( \frac{c(\alpha')^{1/2}}{p} \right)^p p! (\Gamma')^\alpha_a \right) v^a_\alpha \delta^\alpha$$

(47)

The demand that $\delta S_{D,N=1,p}$ (47) vanish permits to find the explicit form of the $\kappa$-symmetry transformations and the values (24) of the coefficient $a$ of the Wess-Zumino term (22) for which the action (22) is invariant under these transformations.

Indeed for this (and only for this) values of parameter $a$ the matrix

$$\left( \delta^\alpha_a - i \frac{p(p-1)}{2} a \left( \frac{c(\alpha')^{1/2}}{p} \right)^p p! (\Gamma')^\alpha_a \right),$$

involved into a second brackets, becomes the projection operator

$$(\delta^\alpha_a + (\Gamma')^\alpha_a)$$
The same projector appears in the motion equation (30). Hence, the multiplication of (30) on the second projector \((\delta^c_a \mp (\Gamma')^c_a)\) from the right hand side results in the identity. In accordance with the Second Noether theorem, this means the presence of symmetry in the theory, which is just the \(\kappa\)–symmetry.

Hence, the general form of the \(\kappa\)–symmetry transformations for \(D\)-dimensional super-p–branes in twistor-like Lorentz–harmonic formulation is characterized by the relation

\[
v^a_\alpha \delta \theta^\alpha = (\delta^a_b \mp (\Gamma')^a_b) \epsilon^b
\]  


These transformations involves the \(\kappa\)–symmetry parameter \(\epsilon^b\) only in the contraction with projector \((\delta^a_b \mp (\Gamma')^a_b) \epsilon^b\) which kills half of \(2^\nu\) components of \(\epsilon^b\). Hence, the \(\kappa\)–symmetry has only \(2^\nu - 1\) parameters.

Indeed, in the representation (38) Eq.(48) simplifies essentially and takes the form

\[
v^a_\alpha \delta \theta^\alpha = \epsilon^q_A v^\alpha_{q \dot{A}}
\]  


where \(\epsilon^q_A\) is the \(2^\nu - 1\)-component parameter of the irreducible \(\kappa\)–symmetry and \(v^\alpha_{q \dot{A}}\) is the component of inverse harmonic matrix (see (35)).

The transformation rule for \(x^m\) is defined by the (target space) supersymmetric condition

\[
\omega^m(\delta) = \delta x^m - i \delta \theta \Gamma^m C^{-1} \theta = 0 \quad \Rightarrow \quad \delta x^m = \delta \theta \Gamma^m C^{-1} \theta,
\]  

and the transformations for the auxiliary fields \(e^i_f\) and \(v^a_\alpha\) are defined by the relations

\[
\delta \left( ee^i_f \right) = e e^\mu g \Omega_{(g)(f)}(\delta) - 2ia(\alpha')^{1/2} H^\mu_{(f)a} v^a_\alpha \delta \theta^\alpha
\]  

\[
\Omega^{(i)(f)}(\delta) = 2ia(\alpha')^{1/2} e^{-1} G^\mu i_{(f)a} v^a_\alpha \delta \theta^\alpha
\]  

where \(H^\mu_{(f)a}\) and \(G^\mu i_{(f)a}\) are defined by (45), (46).

Of course, the \(SO(1, p)\) Cartan form \(\Omega^{(f)(g)}(\delta)\) is not determined by the requirement of the action invariance. This fact means the \(SO(1, p)\) gauge invariance of the discussed action. For the \(\kappa\)–symmetry transformations we may set

\[
\Omega^{(f)(g)}(\delta) = 0
\]  

Hence, we present the action functional, the form of the irreducible \(\kappa\)–symmetry transformations and the equations of motion for the twistor–like formulation of any \(N = 1\)
super-\(p\)-brane moving in space-time of arbitrary admissible (see [23]) dimension. In the next section we discuss the most interesting cases of Heterotic superstring in D=10 and supermembrane in D=11.

3 Example 1: \(D = 10\) Heterotic superstring

The action functional for the heterotic superstring in the twistor-like Lorentz-harmonic formulation (22) may be specified as follows

\[
S_{10,N=1,1} = S_1 + S_{W-Z},
\]

\[
S_1 = (\alpha')^{-1/2} \int d\tau d\sigma e(c(\alpha')^{\frac{1}{2}} - (e^{\mu[+2]}u_m^{-2} + e^{\mu[-2]}u_m^{+2})\omega^m_\mu),
\]

\[
\equiv (\alpha')^{-1/2} \int d\tau d\sigma e(c(\alpha')^{+1/2} - \frac{1}{16}e^{\mu[+2]}\omega^m_\mu (v^-_A\bar{\sigma}_m v^-_A) - \frac{1}{16}e^{\mu[-2]}\omega^m_\mu (v^+_A\bar{\sigma}_m v^+_A))
\]

\[
S_{W-Z} \equiv -(c\alpha')^{-1} \int d\tau d\sigma e^{\mu} i\omega^m_\mu (\partial_\mu \theta \sigma_m \theta)
\]

Here \(\omega^m_\mu\) is defined by the relation (25) with evident replacements of the \(\Gamma\)-matrices by the symmetric \(16 \times 16\) Pauli matrices for \(D = 10\) space-time \(\sigma_{\alpha\beta}\) (see [19] for the notations):

\[
\omega^m_\mu = \partial_\mu x^m - i\partial_\mu \theta \sigma_m \theta \equiv \partial_\mu x^m - i\partial_\mu \theta^\alpha \sigma^m_{\alpha\beta} \theta^\beta,
\]

\(x^m(m = 0, 1, \ldots, 9)\) are the ordinary (flat) space-time coordinates and \(\theta^\alpha(\alpha = 1, \ldots, 16)\) are the Grassmannian spinor coordinates of the \(D = 10, N = 1\) superspace. In Eq.(56) the basic directions tangent to the string worldsheet are chosen to be light-like and the world-sheet zweinbein is parametrized as follows

\[
e^\mu_f = \left(\frac{1}{2} e^{\mu[+2]} + \frac{1}{2} e^{\mu[-2]}\right), \quad e^\mu_g = \left(\frac{1}{2} e^{\mu[+2]} - \frac{1}{2} e^{\mu[-2]}\right)
\]

\[
e^\mu_f e^\mu_g = \delta^f_g, \quad e^\mu_f e^\mu_g = \delta^f_g
\]

\(^5\)Heterotic fermion term is omitted here. This form of the action follows from the twistor–like formulation for \(D = 10, N = IIB\) Green–Schwarz superstring [17] after the trivial reduction to the \(N = 1\) case: \(\theta^2 = 0\).
are presented in Appendix B in the light-like notations.

The light–like notations are also convenient for the parametrization of the components of the composed moving repere \( u_m^{(l)} \) of the target space tangent to the worldsheet

\[
u_m^{(l)} \equiv \frac{1}{16}Sp(v^T \sigma_m v^{(l)}) \equiv \frac{1}{16}v^a_\alpha \bar{\sigma}^{\alpha \beta} v^b_\beta \sigma_{ab} \equiv (u_m^{(0)}, u_m^{(1)}, \ldots, u_m^{(9)}) \equiv (u_m^{(f)}, u_m^{(i)}),
\]

\[
u_m^{(f)} = (u_m^{(0)}, u_m^{(9)}) = \left( \frac{1}{2}(u_m^{[+2]} + u_m^{[-2]}), \frac{1}{2}(u_m^{[+2]} - u_m^{[-2]}) \right),
\]

(62)

To get the simple expressions for composed moving frame vectors \((62)\) we will use the following \(SO(1, 9)_{\text{Global}} \times (SO(1, 1) \times SO(8))_{\text{Local}}\) invariant splitting of the spinor moving frame matrix variable (harmonic matrix) \(v^a_\alpha\) \((3)\)

\[
v^a_\alpha = (v^{+}_{\alpha A}, v^{-}_{\alpha A}) \in Spin(1, 9)
\]

(63)

The vectors \(u_m^{[\pm 2]}, u_m^{(i)}\) are defined in terms of the Lorentz harmonic variables \(v^{+}_{\alpha A}, v^{-}_{\alpha A}\) by the relations

\[
u_m^{[\pm 2]} = \frac{1}{8}(v^{+}_{\alpha A} \bar{\sigma}_m v^{\pm}_{\alpha A}) \equiv \frac{1}{8}v^{+}_{\alpha A} \bar{\sigma}_m v^{\pm}_{\alpha A};
\]

(64)

\[
u_m^{[-2]} = \frac{1}{8}(v^{-}_{\alpha A} \bar{\sigma}_m v^{-}_{\alpha A}),
\]

(65)

\[
u_m^{(i)} = \frac{1}{8} \left( v^{+}_{\alpha A} \bar{\sigma}_m \bar{\gamma}^i_{A A} v^{-}_{\alpha A} \right),
\]

(66)

The contracted \(SO(1, 9)\) spinor indices in Eqs.(65), (66) and in the following formulas are omitted.

The harmonicity conditions \((8)\) which realize the statement \((3)\) take the following form in the discussed case \(\Xi\)

\[
\Xi_{m_1...m_4} = u^{m(n)} \eta_{(n)(l)} \Xi_{m_1...m_4 m} = 0,
\]

(67)

\[
\Xi_0 \equiv u_m^{[-2]} u_m^{[+2]} - 2 = 0,
\]

(68)

The expressions

\[
\Xi_{m_1...m_2} \equiv Sp(v^T \sigma_{m_1...m_2} v^{(n)}) \equiv v^a_\alpha (\bar{\sigma}_{m_1...m_2})^{\alpha \beta} v^b_\beta (\sigma^{(n)}_{ab}) = 0
\]

(69)

\(^6\) Such form of the harmonicity conditions for D=10 space-time have been proposed in the papers \([33, 34]\) where superparticle case have been discussed; the conditions \((3)\) for \(SO(10)\) group had been discussed in the earlier work \([35]\) and used for the discussion of the twistor transform for the superfields in the recent work \([36]\).
vanish as the consequence of Eqs. (67). The last expression of the type (20) vanishes identically because of the antisymmetric property of the matrix \((\tilde{\sigma}_{m_1...m_3})^{\alpha\beta}\) under the spinor index permutations.

The repere orthogonality conditions are satisfied here as the common consequence of the expressions (64)–(66), the conditions (67) and the famous identity (7). The normalization conditions for the composed repere (64)–(66) are satisfied due to the harmonicity conditions (67), (68) and due to the identity (7). The useful representation for \(D = 10\) \(\sigma\)-matrices is given in the Appendix C.

Now we specify the general form of the equations of motion (27)–(30) and irreducible \(\kappa\)-symmetry–transformations (48)–(49) for \(D = 10\) Heterotic string in the twistor-like Lorentz-harmonic formulation.

The equation of motion for auxiliary fields has the form

\[
\omega^{m' \mu}_{m} u^{[\pm 2]}_{m} = c (\alpha')^{1/2} e^{[\pm 2]}_{\mu},
\]

\[
\omega^{m}_{\mu} u^{(i)}_{m} = 0
\]

Thus the light–like vectors \(u^{m[\pm 2]}\) are tangent to the superstring world-sheet on the shell, defined by the motion equations. Contrary, the vectors \(u^{m(i)}\) are orthogonal to the world-sheet on this shell.

Using Eqs.(70), (71), the classical equivalence of the discussed \(D = 10\) superstring formulation with the standard Green-Schwarz one [7] can be justified easily (see [17] for \(N = 2\) Green–Schwarz superstring case).

The equations of motion for the \(x^m(\xi)\) and \(\theta(\xi)\) fields:

\[
\frac{\delta S}{\delta x^m(\xi)} = 0 \quad \text{and} \quad \frac{\delta S}{\delta \theta^I(\xi)} = 0
\]

have the form

\[
\partial_\mu (e \sum_{\pm} (e^{\mu[\pm 2]} u_{m}^{[\pm 2]}) ) - 2 i e^{\mu\nu} \partial_\mu \partial_\nu \theta / c (\alpha')^{1/2} = 0,
\]

\[
(\partial_\mu \theta^m)_{\alpha} (\sum_{\pm} e^{\mu[\pm 2]} u^{m[\pm 2]} - 2 e^{\mu\nu} \omega^{m}_{\nu}) = 0,
\]

which may be reduced to the standard one [7] in a same manner, as have been done for the general case. Excluding the fields \(\omega^{m}_{\nu}\) from the equation (73), we derive the following particle–like form of the equation for grassmannian field \(\theta\)

\[
e^{\mu[-2]} \partial_\mu \theta^a v_{aA}^+ = 0,
\]
4 Example 2: Supermembranes in $D = 11$

In this section we shall use the following $SO(1,2) \times SO(8)$ invariant representation for the charge conjugation matrix and the $\gamma$-matrices in $D = 11$.

$$C_{ab} = -C_{ba} = \text{diag} \left( \epsilon_{\hat{a} \hat{b}} \delta_{AB}, -\epsilon_{\check{a} \check{b}} \delta_{\check{A}\check{B}} \right),$$

$$C_{ab}^{-1} = \text{diag} \left( \epsilon_{\check{a} \check{b}} \delta_{\check{A}\check{B}}, -\epsilon_{\hat{a} \hat{b}} \delta_{AB} \right),$$

$$\Gamma^{(m)} \equiv \left( \Gamma(f), \Gamma(i) \right),$$

$$\Gamma^{(f)} \equiv \left( \Gamma^{(0)}, \Gamma^{(1)}, \Gamma^{(2)} \right) \equiv \left( \Gamma^0, \Gamma^9, \Gamma^{10} \right) = \text{diag} \left( \gamma^f_{\check{a} \check{b}} \delta_{\check{A}\check{B}}, -\gamma^f_{\hat{a} \hat{b}} \delta_{AB} \right),$$

$$\Gamma^{(i)} \equiv \left( \Gamma^1, \ldots, \Gamma^8 \right) = \left[ \begin{array}{c} \epsilon_{\check{a} \check{b}} \gamma^{(i)}_{\check{A}\check{B}} \\ -\epsilon_{\hat{a} \hat{b}} \gamma^{(i)}_{\hat{A}\hat{B}} \end{array} \right]$$ (75)

where

$$a = \left( \hat{a}, \check{b}, \hat{A}, \check{A} \right)$$ (76)

is the composed spinor (upper) index of $SO(1,2) \times SO(8)$, $\gamma^{(i)}$ are $d = 8$ $\gamma$-matrices which are similar those for $D = 10$; $\gamma^{(i)}_{\check{A}\check{B}} \equiv \gamma^{(i)}_{\check{B}\check{A}}$, $\gamma^{(i)}_{\hat{b} \hat{a}}$ are $d = 3$ $\gamma$-matrices, $\epsilon_{\hat{a} \hat{b}} = -\epsilon_{\check{a} \check{b}}$ ($\epsilon^{12} = -\epsilon_{12} = 1$) represents $d = 3$ charge conjugation matrix.

The Lorentz harmonics (3), (8) parametrize the coset $SO(1,10)/(SO(1,2) \times SO(8))$ and form $32 \times 32$ matrix $v^a_\alpha$

$$v^a_\alpha = \left( v^\hat{a}_{\hat{A}} , v^{\check{a} \check{A}} \right)$$ (77)

where $\alpha = 1, \ldots, 32$ are the spinor indices of the group $SO(1,10)$; $\hat{a}, \check{b} = 1,2$ belong to the spinor indices of $SO(1,2)$; $A, B = 1, \ldots, 8$; $\hat{A}, \check{B} = 1, \ldots, 8$ are $s$- and $c$- spinor indices of $SO(8)$ respectively. The matrix (75) takes its values in the group $Spin(1,10)$ which is a double-covering group for the Lorentz group $SO(1,10)$ because it satisfies the following harmonicity conditions

$$\Xi \equiv v^a_\alpha C^{\alpha \beta} v^b_\beta - C^{ab} = 0,$$ (78)

$$\Xi^{(n)}_{m_1 m_2} \equiv v^a_\alpha (C T_{m_1 m_2})^{\alpha \beta} v^b_\beta \left( \Gamma^{(n)} C^{-1} \right)_{ab} = 0,$$ (79)

$$\Xi^{(n)}_{m_1 \ldots m_5} \equiv v^a_\alpha (C T_{m_1 \ldots m_5})^{\alpha \beta} v^b_\beta \left( \Gamma^{(n)} C^{-1} \right)_{ab} = 0,$$ (80)

which exclude $496 + 11 + 462 = 969 (= 1024 - 55)$ degrees of freedom.

Thus the harmonics $v^a_\alpha A, v^a_\alpha A$ describe $55 = \text{dim} SO(1,10)(= 1024 - 969)$ independent degrees of freedom. Among the latter $31 = 3 + 28 = \text{dim} SO(1,2) + \text{dim} SO(8)$ degrees of freedom are pure gauge ones due to $SO(1,2) \times SO(8)$ local symmetry of $S_{11,1,2}$. 

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The harmonicity conditions (78) are independent, but the relations (79), (80) contain only 11 and 462 independent conditions on harmonics, respectively (see [18] for details).

The relations (78) allow to construct the matrix inverse to $v^a_\alpha$ using the same variables $v^\hat{a}_{\alpha A}, v^{\hat{a} A}_{\alpha}$:

$$ (V^{-1})^\beta_b = \left(-C^\beta_\alpha v_{abA}, C^\alpha_b v^\beta_{a\hat{A}} \right) $$

(81)

Here and further the spinor indices belonging to SO(1,2) group are lifted and lowered

$$ v_{abA} = \epsilon_{ba} v^a_{\alpha A}, \quad v^b_{\hat{a} A} = \epsilon_{ba} v_{\hat{a} \alpha } $$

(82)

using the $d = 3$ charge conjugation matrix. Now it is possible to specialize the expressions (5) for the case of supermembrane in $D = 11$ using the representation (75)

$$ u^{(i)}_m = -\frac{1}{16} v^a_{\alpha A} (C \Gamma_m) v^\beta_{\hat{a} \alpha \hat{A}} $$

(83)

The converted SO(1,10) spinor indices are omitted in (83), (84). The use of Eqs. (9)-(11) allows to present the expressions (83), (84) in the form

$$ u^{(f)}_m (\Gamma^m C^{-1})_{\alpha \beta} = \left( v^\hat{a}_{\alpha A} v^\beta_{\hat{b} a \hat{A}} + v^{\hat{a} A}_{\alpha} v^\beta_{\hat{b} \alpha \hat{A}} \right) \gamma^j_\hat{a} $$

(85)

$$ u^{(i)}_m (\Gamma^m C^{-1})_{\alpha \beta} = 2 v^a_{\alpha A} v^\beta_{\alpha \hat{A} \hat{A}} $$

(86)

Note that the left and right parts of Eq. (85) are symmetric under permutation of SO(1,10) spinor indices owing to the well-known $\gamma$-matrix identities for $\gamma$-matrices in $D = 11$ and $d = 3$

$$ (\Gamma^m C^{-1})^T = (\Gamma^m C^{-1}) $$

$$ (\epsilon \gamma^f)^T = (\epsilon \gamma^f) $$

(87)

The matrix (31) for $D = 11$ supermembrane ($p = 2$) has the form

$$ \Gamma' = \text{diag}(\delta^\hat{a} \delta_{AB}, -\delta^\hat{a} \delta_{\hat{A} \hat{B}}) $$

(88)

The twistor–like action for the supermembrane in $D = 11$ has the form (22) with the Wess–Zumino term (23) which may be presented in the form

$$ S_{11,1,2}^{WZ} = \frac{2}{\alpha' \sqrt{\alpha' c^2}} \int d^3 \xi \epsilon^{\alpha \rho \sigma} \left[ \partial_\mu x^m \partial_\nu x^n + i \partial_\mu \theta^a (\Gamma^m C^{-1})_{\alpha \beta} \theta^\beta - \frac{1}{3} \partial_\mu \theta (\Gamma^m C^{-1}) \partial_\nu \theta (\Gamma^m C^{-1}) \partial_\rho \theta (\Gamma^m C^{-1}) \theta \right] $$

(89)
Equations of motion (27)–(30) may be specialized as follows for \( D = 11 \) supermembrane:

\[
\omega^m u_m^f = \frac{c(\alpha'^{1/2})}{2} e^f, \tag{90}
\]

\[
\omega^m u_m^{(i)} = 0, \tag{91}
\]

\[
\partial_\mu (ee^f u_m^f) + \frac{2}{\epsilon(\alpha')^{1/2}} \epsilon^{\mu
u\rho} \epsilon_{\mu f} u^m u^f d_\nu \theta \Gamma_{mn} C^{-1} \partial_\rho \theta = 0 \tag{92}
\]

\[
e^f_\mu \partial_\mu \theta^\alpha v_{\beta \hat{b} \hat{A}} \left( \epsilon_\gamma v^\beta \right)^{\hat{b} \hat{a}} = 0 \tag{93}
\]

The expressions for the \( \kappa \)-symmetry transformations have the form

\[
\delta \theta^\alpha v^a_\alpha = e^a_\alpha, \quad \delta \theta^\alpha v_{\alpha \beta} = 0, \quad \iff \delta \theta^\alpha = e^a_\alpha v^a_\alpha \tag{94}
\]

\[
\omega^m(\delta) = \delta x^m - i \delta \theta \Gamma^m C^{-1} \theta = 0 \quad \Rightarrow \quad \delta x^m = \delta \theta \Gamma^m C^{-1} \theta, \tag{95}
\]

\[
\Omega^{(i)}(\delta) = 2 i \alpha(\alpha')^{1/2} e^{-1} G^a_\alpha v^a_\alpha \delta \theta^\alpha, \tag{96}
\]

\[
\delta(ee^f) = ee^{\mu \rho} \Omega_{(\theta)}(\delta) - 2 i \alpha(\alpha')^{1/2} H^\mu_{(f)} v^a_\alpha \delta \theta^\alpha, \tag{97}
\]

where

\[
G^a_\alpha v^a_\alpha = e^{\mu_1 \mu_2} \omega^m_{\mu_1} \partial_\mu_2 \theta^\beta \left( v^m_\alpha \left( v^i_\alpha \gamma_ii_1 v_\beta_{\hat{b} \hat{a}} - v_{\beta \tilde{b}} \tilde{\gamma}_{ii_1} v_{\alpha \hat{a}} \right) + v^f_\alpha \left( v^i_\alpha \gamma_{ii} v_{\alpha \hat{b}} + v_{\beta \tilde{b}} \tilde{\gamma}_{ii} v_{\alpha \hat{a}} \right) \right) \tag{98}
\]

and

\[
H^\mu_{(f)} v^a_\alpha = i e^{\mu_1 \mu_2} \epsilon_{\mu \nu \rho} \omega^m_{\mu_1} u^m_\alpha \partial_\mu_2 \theta^\beta \left( v^i_\beta \gamma_{ii_1} v_\alpha_{\hat{b} \hat{a}} - v_{\beta \tilde{b}} \tilde{\gamma}_{ii_1} v_{\alpha \hat{a}} \right) \tag{99}
\]

As a result of \( SO(1,2) \) gauge invariance of the action, Cartan form \( \Omega^{(f)}(\delta) \) is not determined and we may set

\[
\Omega^{(f)}(\delta) = 0 \tag{100}
\]

for the \( \kappa \)-symmetry transformations.
5 Conclusion

Therefore, the twistor–like formulations for the all known $N = 1$ super–p–brane theories \cite{24} are presented here. The possible application of these formulations is the investigation of the nonlinear equations of motion for super–$p$–brane with $p \geq 2$ using the generalized twistor methods.

Moreover, it was proved that the $\kappa$–symmetry (as well as all another gauge symmetries) are present in the irreducible form in these formulations (see Subsection 2.5). So, the covariant Hamiltonian formalism for all set of $N = 1$ super–$p$–branes can be build using the results of this paper. (Such formalism for $D = 10$ $N = IIB$ superstring have been developed in Ref. \cite{17}).

We prove the classical equivalence of the discussed super–$p$–brane formulations with the known ones. However such equivalence can be destroyed by quantum effects as well as by the coupling to the gauge fields. Does this destruction indeed appears or no? This is the interesting question for further investigation.

One more point noteworthy is connected with the recently discovered type $IIA$ and $IIB$ super–$p$–branes previously thought not to exist for $p \geq 2$ \cite{30}. These type $II$ super–$p$–branes emerge as solitons of either type $IIA$ or $IIB$ supergravity and involve additional worldsheet vector or tensor fields. As it has been recently shown such super–$p$–branes exist in $D = 10$ only \cite{31}. Unfortunately at the present time supersymmetric and $\kappa$–symmetric formulations of these type $p$–brane actions are absent. It could be suggested that the twistor–like harmonic approach developed here may be useful for solving this problem.

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7 Appendix A

The Fiertz identities for \( D = 2, 3, 4 \pmod{8} \)

\[
F^\beta_\alpha = 2^{-\nu} \delta^\beta_\alpha \text{Sp}(F) - 2^{-(\nu+1)} \left( \Gamma_{m_1 m_2}^m \right)_\alpha^\beta \text{Sp}(\Gamma_{m_1 m_2} F) + \\
+ \sum_{k \geq 1} B_{2k}(\Gamma_{m_1 \ldots m_{2k}}) \alpha^\beta \text{Sp}(\Gamma_{m_1 \ldots m_{2k}} F) + \\
+ (1 - \delta_D(\pmod{8},2)) [2^{-\nu} \left( \Gamma_{m}^m \right)_\alpha^\beta \text{Sp}(\Gamma_{m} F) + \\
+ \sum_{k \geq 1} B_{2k+1}(\Gamma_{m_1 \ldots m_{2k+1}}) \alpha^\beta \text{Sp}(\Gamma_{m_1 \ldots m_{2k+1}} F)]
\]

(101)

where the last term is absent for \( D = 2(\pmod{8}) \),

\[
U^{\alpha\beta} = 2^{-\nu} (C \Gamma^m) \alpha^\beta \text{Sp}(U \Gamma_{m} C^{-1}) + \\
+ \sum_{k \geq 1} B_{2k+1}(C \Gamma_{m_1 \ldots m_{2k+1}}) \alpha^\beta \text{Sp}(U \Gamma_{m_1 \ldots m_{2k+1}} C^{-1}) + \\
+ (1 - \delta_D(\pmod{8},2)) [2^{-\nu} \left( \Gamma_{m}^m \right)_\alpha^\beta \text{Sp}(UC^{-1}) - \\
- 2^{-(\nu+1)} (C \Gamma_{m_1 m_2}^m) \alpha^\beta \text{Sp}(U \Gamma_{m_1 m_2} C^{-1}) + \\
+ \sum_{k \geq 2} B_{2k}(C \Gamma_{m_1 \ldots m_{2k}}) \alpha^\beta \text{Sp}(U \Gamma_{m_1 \ldots m_{2k}} C^{-1})],
\]

(102)

\[
W_{\alpha\beta} = 2^{-\nu} (\Gamma^m C^{-1}) \alpha^\beta \text{Sp}(C \Gamma_{m} W) + \\
+ \sum_{k \geq 1} B_{2k+1}(\Gamma_{m_1 \ldots m_{2k+1}} C^{-1}) \alpha^\beta \text{Sp}(C \Gamma_{m_1 \ldots m_{2k+1}} W) + \\
+ (1 - \delta_D(\pmod{8},2)) [2^{-\nu} C_{\alpha\beta}^{-1} \text{Sp}(CW) - 2^{-(\nu+1)} (\Gamma_{m_1 m_2} C^{-1}) \alpha^\beta \text{Sp}(C \Gamma_{m_1 m_2} W) + \\
\sum_{k \geq 2} B_{2k}(\Gamma_{m_1 \ldots m_{2k}} C^{-1}) \alpha^\beta \text{Sp}(C \Gamma_{m_1 \ldots m_{2k}} W)]
\]

(103)

We do not need in the expressions for the numerical coefficients \( B_k \) except for the first three ones

\[
B_0 = B_1 = 2^{-\nu}, \\
B_2 = -2^{-(\nu+1)}
\]
8 Appendix B

World-sheet repere \((p=1, d=2)\).

The orthonormality conditions in "light-like" notations

Repere variables of the (super)string worldsheet

\[
e_{\mu}^{[\pm 2]}, e_{\mu}^{\pm 2}
\]
satisfy the following conditions

\[
e_{\mu}^{[\pm 2]} e_{\nu}^{[\pm 2]} = 0 = e_{\mu}^{-2} e_{\nu}^{[-2]}, \quad e_{\mu}^{[+2]} e_{\nu}^{-2} = 2 = e_{\mu}^{[-2]} e_{\nu}^{[+2]},
\]

\[
e^{\mu \nu} = \frac{1}{2} e(e_{\mu}^{[+2]} e_{\nu}^{-2} - e_{\mu}^{-2} e_{\nu}^{[+2]}), \quad (e^{01} = -e_{01} = 1),
\]

\[
g^{\mu \nu} = \frac{1}{2} (e_{\mu}^{[+2]} e_{\nu}^{-2} + e_{\mu}^{-2} e_{\nu}^{[+2]}), \quad \sqrt{-g} \equiv e,
\]

\[
\delta_{\mu}^{\nu} = \frac{1}{2} (e_{\mu}^{[+2]} e_{\nu}^{-2} + e_{\mu}^{-2} e_{\nu}^{[+2]}), \quad e_{\mu \nu} e_{\mu}^{-[2]} e_{\nu}^{[+2]} = 2/e,
\]

9 Appendix C

The following representation for the \(\sigma\)-matrices should be used for an explicit calculations in the case of 10–dimensional space-time:

\[
\sigma^{0}_{ab} = \text{diag}(\delta_{AB}, \delta_{\dot{A}\dot{B}}) = \tilde{\sigma}^{0}_{ab},
\]

\[
\sigma^{9}_{ab} = \text{diag}(\delta_{AB}, -\delta_{\dot{A}\dot{B}}) = -\tilde{\sigma}^{9}_{ab},
\]

\[
\sigma^{(i)}_{ab} = \begin{pmatrix} 0 & \gamma^{i}_{AB} \\ \bar{\gamma}^{\dot{i}}_{\dot{A}\dot{B}} & 0 \end{pmatrix} = -\tilde{\sigma}^{(i)}_{ab},
\]

\[
\sigma^{[+2]}_{ab} = \sigma^{0} + \sigma^{9} = \text{diag}(2\delta_{AB}, 0) = -(\tilde{\sigma}^{0} - \tilde{\sigma}^{9})_{ab} = \tilde{\sigma}^{-[2]}_{ab},
\]

\[
\sigma^{[-2]}_{ab} = \sigma^{0} - \sigma^{9} = \text{diag}(0, 2\delta_{\dot{A}\dot{B}}) = (\tilde{\sigma}^{0} + \tilde{\sigma}^{9})_{ab} = \tilde{\sigma}^{[+2]}_{ab},
\]

Here \(\gamma^{i}_{AB}\) are the \(\sigma\) -matrices for \(SO(8)\) group, \(\tilde{\gamma}^{\dot{i}}_{\dot{A}\dot{B}} \equiv \gamma^{\dot{i}}_{\dot{B}\dot{A}}\).

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