Modelling the vibrational field of a single-layered ground containing a buried object

G Duval\(^1\), B Laulagnet\(^2\)

\(^1\) PhD student, Laboratoire Vibrations Acoustique, INSA Lyon, France
\(^2\) Assistant professor - HDR, Laboratoire Vibrations Acoustique, INSA Lyon, France

E-mail: \(^1\) ghislain.duval@insa-lyon.fr, \(^2\) bernard.laulagnet@insa-lyon.fr

Abstract. Better understanding and accurate representation of the vibrational field propagated through the ground is a major challenge in the industrial field. In particular, interactions between vibrating surface objects and elements buried in the soil are poorly understood. This article presents an analytical model of the vibrational field of a single-layered soil solicited at its surface by a non deformable piston and containing a passive flat plate. It offers an original approach to the problem, using coupling between plates and ground to investigate how the surface velocity field is modified by the presence of an object in the soil. Examples of numerical simulations are provided, and possible extensions to the model are briefly discussed to underline its versatility and its use in potential future studies.

1. Introduction

Interactions between ground vibration and buried structures is a relatively recent field of studies. However potential applications are numerous, for example detection of buried pipes or service installations in urban areas from non-destructive surface measurements, or prevision of the vibrations transmitted to buildings foundations nearby a major source of vibrations.

Surface-wave methods are commonly used to determine the composition of the ground [1] [2]. In such methods, surface velocity field measurements are used as input data for an inversion process in order to find the mechanical properties of a predetermined number of layers of soil. As such, a buried obstacle is considered an anomaly in the measurements and is largely ignored. However, attempts have been made at using the SASW (Spectral Analysis of Surface Waves) method to locate buried objects [3] [4]. Recently, the Mapping the Underworld initiative in the UK has sought to address the inability to locate buried infrastructures in the ground, yielding valuable experimental results [5] [6]. The detection of objects and irregularities in the ground is currently an active field of investigations [7] [8].

This article builds on the analytical model developed at LVA (Laboratoire Vibrations Acoustique) at INSA Lyon [9] [10] to investigate the modification of the vibrational field of the ground by the presence of a buried object. This approach’s originality resides in accounting for the structural coupling between vibrating structures and propagating medium. An other advantage of this method is the ability to perform numerical simulations in relatively short spans of time. Examples of such simulations are provided to illustrate the modifications to the surface velocity field caused by including a plate in the ground.
2. Model development and resolution
In this section, the model adopted to represent the vibration of a single-layered ground containing a buried object is detailed and solved. The goal is to be able to compute the vibrational field at any point of the ground, and more specifically at its surface.

2.1. Governing equations and continuity conditions
As shown in Figure 1, the ground is solicited at its free surface by a rectangular vibrating plate (referred to as 'A') of mass $M_A$ and surface $a \times b = S_A$. An harmonic force of amplitude $F$ and pulsation $\omega$ is applied to this structure. A second rectangular plate 'B' (of mass $M_B$ and surface $c \times d = S_B$) is buried within the soil, parallel to the surface plate and at a depth $z = h$. The origin of the coordinates system is set in the middle of the surface plate and the offset between the two plates centers is $(\delta_x, \delta_y)$, as illustrated in Figure 2. Both plates are treated as rigid solid bodies, meaning their deformations are neglected.

![Figure 1. Side view of the modelled situation.](image1.png)

![Figure 2. Top view of the modelled situation.](image2.png)

The single-layered ground is considered infinite along the directions $x$ and $y$ and semi-infinite along the $z$ coordinate, with $z = 0$ being the free surface (coupling with the surrounding medium is neglected). The formulation requires to artificially separate the soil into two distinct layers at the depth at which the plate B is buried. Thus, the upper and lower layers (for which the subscripts 1 and 2 are respectively used) mechanical properties are described by their Lamé coefficients $\mu_1$, $\lambda_1$ and $\mu_2$, $\lambda_2$, and by their mass densities $\rho_1$ and $\rho_2$. Their respective motion fields $\vec{u}_1$ and $\vec{u}_2$ are described by the Navier equations (1) and (2); they are written here for a harmonic regime and the volume forces are neglected:

$$\mu_1 \tilde{\Delta} \vec{u}_1 + (\lambda_1 + \mu_1) \vec{\text{grad}}(\text{div} \vec{u}_1) + \rho_1 \omega^2 \vec{u}_1 = \vec{0}$$  \hspace{1cm} (1)

$$\mu_2 \tilde{\Delta} \vec{u}_2 + (\lambda_2 + \mu_2) \vec{\text{grad}}(\text{div} \vec{u}_2) + \rho_2 \omega^2 \vec{u}_2 = \vec{0}$$  \hspace{1cm} (2)

in which $\tilde{\Delta}$, $\vec{\text{grad}}$ and $\text{div}$ are respectively the vectorial Laplacian, gradient and divergence operators. Let’s consider that a compression solicitation is used, that is to say the surface plate motion is only vertical, entirely along the $z$ axis. The surface plate thus functions as an ideal non-deformable piston. The equation (3) describes its motion $w_{A_z}$ along the $z$ axis:

$$-\omega^2 M_A w_{A_z} = F + \int_{S_A} \sigma_{zz}^{(1)}(z = 0) dS$$  \hspace{1cm} (3)
in which $\sigma^{(1)}_{zz}(z = 0)$ is the stress applied to the plate through the coupling with the first layer of soil. As for the passive buried plate, the $x$, $y$ and $z$ components of its motion field have to be taken into consideration. $w_{Bx}$, $w_{By}$ and $w_{Bz}$ are thus described by the equation (4):

$$-\omega^2 M_B w_{Bi} = \int \int_{S_B} \left[ \sigma^{(2)}_{zi}(z = h^+) - \sigma^{(1)}_{zi}(z = h^-) \right] dS$$

in which $i = x, y, z$ is the spatial component and $\sigma^{(2)}_{zi}(z = h^+) - \sigma^{(1)}_{zi}(z = h^-)$ is the stress difference between the lower and upper side of the buried plate along the direction $i$.

Appropriate continuity conditions are required to solve the problem. The plates and soil displacements are considered continuous within the limits of the plates surfaces, as written in the equations (5) and (6):

$$u_{1z}(z = 0) = w_{Az} \quad (x, y) \in S_A$$
$$u_{1i}(z = h^-) = u_{2i}(z = h^+) = w_{Bi} \quad (x, y) \in S_B$$

The ground surface is free, meaning the stress is zero outside of the surface $S_A$. This translates to the following array of equations (7) for the first layer of soil:

$$\begin{align*}
\sigma^{(1)}_{xx}(z = 0) &= 0 \\
\sigma^{(1)}_{xy}(z = 0) &= 0 \\
\sigma^{(1)}_{zz}(z = 0) &= \begin{cases} 
0 \quad (x, y) \notin S_A \\
\Delta P_z \quad (x, y) \in S_A 
\end{cases}
\end{align*}$$

Similarly at the depth $h$ and within the surface $S_B$, the stresses differences between the lower and upper surface of the plate are (8):

$$\sigma^{(2)}_{zi}(z = h^+) - \sigma^{(1)}_{zi}(z = h^-) = \Delta P_i \quad (x, y) \in S_B$$

In the previous equations (7) and (8), $P_z$, $\Delta P_x$, $\Delta P_y$ and $\Delta P_z$ respectively denote uniform stress reacting on the excited piston and stress jumps along $x$, $y$ and $z$ when crossing the passive buried plate. Saying the stress is uniformly distributed is a necessary condition to render the problem consistent. These four unknowns need to be determined for the problem to be solved.

### 2.2 Solving the problem

In this subsection, the general approach to solving the problem is outlined step by step. A matrix formulation is used due to the length of the resulting mathematical expressions. References to the appendices containing the explicit expressions are provided.

A Helmholtz representation is used to decompose the soil motion fields $\vec{u}_1$ and $\vec{u}_2$ into sums of scalar and vector potentials, respectively $\phi_1$, $\phi_2$ and $\vec{\psi}_1$, $\vec{\psi}_2$, as shown below (9):

$$\vec{u}_1 = \vec{\text{grad}}\phi_1 + \vec{\text{rot}}\vec{\psi}_1 \quad \vec{u}_2 = \vec{\text{grad}}\phi_2 + \vec{\text{rot}}\vec{\psi}_2$$

in which $\vec{\text{rot}}$ is the rotational operator. These representations are applied to the Navier equations (1) and (2), leading to the following eight equations (10) on the potentials:

$$\begin{align*}
\Delta \phi_1 + k_{B1}^2 \phi_1 &= 0 \\
\Delta \psi_{1i} + k_{B1}^2 \psi_{1i} &= 0 \\
\Delta \phi_2 + k_{B2}^2 \phi_2 &= 0 \\
\Delta \psi_{2i} + k_{B2}^2 \psi_{2i} &= 0
\end{align*}$$

in which \( \Delta \) is the Laplacian scalar operator, \( \psi_{1i} \) and \( \psi_{2i} \) are the \( i = x, y, z \) components of the vector potentials, \( k_{d1}, k_{d2} \) are the compression-dilatation wave numbers, and \( k_{s1}, k_{s2} \) are the shear wave numbers. The wave numbers are functions of the dilatation and shear velocities \( c_d = \sqrt{\frac{2\lambda+\mu}{\rho}} \) and \( c_s = \sqrt{\frac{\mu}{\rho}} \) of the layers of soil, which depend on the mechanical properties of the considered layer.

The problem is then transposed into the wave numbers domain \( (k_x, k_y) \) using a two dimensions spatial Fourier transform along the \( x \) and \( y \) axis. The convention used for this transform can be found in Appendix A. The Helmholtz equations (10) thus lead to eight differential equations (11) on the potentials:

\[
\begin{align*}
\frac{\partial^2 \tilde{\phi}_1}{\partial z^2} + k_{11}^2 \tilde{\phi}_1 &= 0 \\
\frac{\partial^2 \tilde{\psi}_{1i}}{\partial z^2} + k_{21}^2 \tilde{\psi}_{1i} &= 0 \\
\frac{\partial^2 \tilde{\phi}_2}{\partial z^2} + k_{12}^2 \tilde{\phi}_2 &= 0 \\
\frac{\partial^2 \tilde{\psi}_{2i}}{\partial z^2} + k_{22}^2 \tilde{\psi}_{2i} &= 0
\end{align*}
\]  

in which \( \tilde{\cdot} \) indicates the Fourier transformed function and \( k_{11}, k_{21} \) and \( k_{12}, k_{22} \) are reduced wave numbers related respectively to the first and second layers of soil (12):

\[
\begin{align*}
k_{11}^2 &= k_{d1}^2 - k_x^2 - k_y^2 \\
k_{21}^2 &= k_{d1}^2 - k_x^2 - k_y^2 \\
k_{12}^2 &= k_{d2}^2 - k_x^2 - k_y^2 \\
k_{22}^2 &= k_{d2}^2 - k_x^2 - k_y^2
\end{align*}
\]  

The general solutions of these differential equations (11) are (13):

\[
\begin{align*}
\tilde{\phi}_1 &= A e^{-j\omega k_{11}z} + A^* e^{j\omega k_{11}z} \\
\tilde{\psi}_{1x} &= B e^{-j\omega k_{21}z} + B^* e^{j\omega k_{21}z} \\
\tilde{\psi}_{1y} &= C e^{-j\omega k_{21}z} + C^* e^{j\omega k_{21}z} \\
\tilde{\psi}_{1z} &= D e^{-j\omega k_{21}z} + D^* e^{j\omega k_{21}z} \\
\tilde{\phi}_2 &= E e^{-j\omega k_{12}z} \\
\tilde{\psi}_{2x} &= F e^{-j\omega k_{22}z} \\
\tilde{\psi}_{2y} &= G e^{-j\omega k_{22}z} \\
\tilde{\psi}_{2z} &= H e^{-j\omega k_{22}z}
\end{align*}
\]  

in which \( A...H \) are twelve unknowns which need to be determined for the problem to be solved. Note that since the second (lower) layer of soil is considered semi-infinite along \( z \), the backward traveling wave need not be taken into account.

Computation of those twelve unknowns is performed using the continuity conditions related to stress (7) and (8). For a small displacements hypothesis, Hooke’s law is used to relate the stresses to the strains, and then to the components of the displacements fields \( \vec{u}_1 \) and \( \vec{u}_2 \). A two dimensions Fourier transform is then performed to write the stresses \( \tilde{\sigma}_{zj} \) as functions of the twelve unknowns \( A...H \) in the wave number domain \( (k_x, k_y) \); these manipulations are detailed in Appendix B. Six equations on the unknowns \( A...H \) are thus established (see Appendix C).

The continuity condition (6) related to the displacement of the layers of soil \( u_{1i}(z = h^-) = u_{2i}(z = h^+) \) is also transposed in the Fourier domain and provides three more equations for the unknowns \( A...H \); they are explicitly written in Appendix D.

Finally, the Lorenz gauge condition provides the three last equations needed to compute these twelve unknowns. This condition states that the divergence of the vector potentials \( \tilde{\psi}_1 \) and \( \tilde{\psi}_2 \) must be zero whatever the depth \( z \) and is necessary to ensure the solution’s unicity. Details on this computation are provided in Appendix E.

In matrix form, this set of twelve equations is written as \( [M] \{ X \} = \{ \tilde{P} \} \), in which \( \{ X \} \) is the column vector containing the unknowns \( A...H \) and \( \{ \tilde{P} \} \) is the column vector (14):

\[
\{ \tilde{P} \} = \{ 0 \ 0 \ \tilde{P}_x \ \tilde{\Delta} P_x \ \tilde{\Delta} P_y \ \tilde{\Delta} P_z \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \}^T
\]
In which \( \tilde{P}_z \) and \( \Delta \tilde{P}_i \) are the Fourier transforms of the four stress unknowns \( P_z \) and \( \Delta P_i \); explicit expressions for these transforms are provided in Appendix C. \([M]\) is a 12 \( \times \) 12 matrix generated by the set of twelve equations linking the unknowns \( A...H \) and \( P_z \) and \( \Delta P_i \).

A numerical solution to this matrix equation is computed by summing four particular solutions; these are found by alternatively setting the value of one of the stress unknowns to 1 whilst the others’ values are set to 0. The column vector \( \{X\} \), solution to the problem about the unknowns \( A...H \), is thus written as a sum of the unknowns \( \tilde{P}_z \) and \( \tilde{\Delta}P_i \) (15):

\[
\{X\} = \{X_1\} \tilde{P}_z + \{X_2\} \Delta \tilde{P}_x + \{X_3\} \Delta \tilde{P}_y + \{X_4\} \Delta \tilde{P}_z
\]  

(15)

in which \( \{X_1\}, \{X_2\}, \{X_3\} \) and \( \{X_4\} \) are the particular solutions computed numerically. The last step to solving the problem is to determine the four unknowns \( P_z \) and \( \Delta P_i \); this is done by considering the mechanical coupling between the soil and the plates. First, the three components \( u_{1i} \) of the displacement field of the upper layer of soil are written as functions of \( P_z \) and \( \Delta P_i \) and of the particular solutions vectors \( \{X_1\} \) to \( \{X_4\} \) using (15). These expressions, particularly lengthy, are written in condensed form in Appendix F. The continuity conditions related to displacement (5) and (6) are then integrated over the surfaces of the plates \( S_A \) and \( S_B \) using those explicit expressions. This leads to the matrix equation (16) relating the plates displacements \( w_{1z} \) and \( w_{2i} \) to the four stress unknowns via the 4 \( \times \) 4 matrix \([I]\):

\[
[S] \{w\} = \frac{j}{4\pi^2} [I] \{P\}
\]  

(16)

in which :

\[
[S] = \begin{bmatrix}
S_A & 0 & 0 & 0 \\
0 & S_B & 0 & 0 \\
0 & 0 & S_B & 0 \\
0 & 0 & 0 & S_B
\end{bmatrix}
\]

\[
\{w\} = \begin{bmatrix}
w_{1z} \\
w_{2x} \\
w_{2y} \\
w_{2z}
\end{bmatrix}
\]

\[
\{P\} = \begin{bmatrix}
P_z \\
\Delta P_x \\
\Delta P_y \\
\Delta P_z
\end{bmatrix}
\]

(17)

The matrix \([I]\), which is made of sixteen integrals over the wave number domain \((k_x, k_y)\), is explicated in Appendix G. Finally, by substitution in the equations of motion (3) and (4), the matrix equation (18) allowing for the computation of the stress unknowns is written:

\[
\left[-j\omega^2 [M] [S]^{-1} [I] - [S]\right] \{P\} = \{F\}
\]  

(18)

in which \( f = \frac{\omega}{2\pi} \) is the frequency and:

\[
[M] = \begin{bmatrix}
M_A & 0 & 0 & 0 \\
0 & M_B & 0 & 0 \\
0 & 0 & M_B & 0 \\
0 & 0 & 0 & M_B
\end{bmatrix}
\]

\[
\{F\} = \begin{bmatrix}
F \\
0 \\
0 \\
0
\end{bmatrix}
\]

(19)

The problem is solved, as the knowledge of the four stress unknowns \( P_z \) and \( \Delta P_i \) and of the twelve unknowns \( A...H \) makes it possible to compute any component of the displacement fields at any space coordinate for a given force amplitude \( F \).

3. Surface velocity mapping

In this section are presented results of numerical simulations performed using the previously detailed model. The goal is to highlight how the presence of a buried object can disrupt the vibrational field. The surface velocity map is computed using the expressions (F.1) and (F.2) with \( z = 0 \) and for an amplitude \( F = 1 \text{ N} \). The map is limited to the quarter-space \( x = [0 \ 3] \text{ m} \cup y = [0 \ 3] \text{ m} \).
Throughout this section, the ground is supposed single-layered. Although the formulation requires a separation into two layers, and this separation has been preserved throughout the mathematical resolution, a single-layered soil can be numerically simulated by considering that the two layers share the same mechanical properties. The mechanical properties used for the following numerical simulations can be found in Table 1. The damping is introduced by adding an imaginary part to the velocities.

| Property                   | Symbol | Value |
|----------------------------|--------|-------|
| Dilatation velocity        | $c_d$  | 600 m/s |
| Shear velocity             | $c_s$  | 160 m/s |
| Volumetric mass density    | $\rho$ | 1200 kg/m$^3$ |
| Damping                    | $\eta$ | 2%    |

### 3.1. Free soil vibration

First let’s consider a free soil, meaning not containing any buried object, in order to provide a reference case. Numerically, this is done by setting the mass of the buried plate $M_B = 0$ kg. The surface is solicited by compression along the $z$ axis at 30 Hz, the surface plate has a surface of 1 m$^2$ and a mass of 125 kg (this last value is chosen considering roughly a 5 cm-thick aluminium or concrete plate). The Figure 3 shows the amplitude (left) and the unwrapped phase (right) of the $z$ component of the surface velocity field. Phase unwrapping is performed along the $x$ and $y$ axis successively; the amplitude dB scale is computed using a reference velocity of $5 \times 10^{-8}$ m/s.

**Figure 3.** Amplitude $v_z$ (left) in dB and unwrapped phase $\phi_z$ (right) in radians of the $z$ component of the surface velocity field. Free soil solicited along $z$ by a 1 m$^2$, 125 kg surface plate (full black line) at 30 Hz.

The resulting velocity field is axisymmetric around the $z$ axis, except very close to the plate. The surface plate has little influence over the resulting field at this frequency: the far field is reached after a small distance. This is apparent when plotting the unwrapped phase along the $x$ axis (Figure 4). After $x = 1$ m the phase decay is regular and a linear regression yields a phase
velocity of 148.5 m/s, which is the expected velocity of a Rayleigh surface wave (roughly \(0.9 \times c_s\) for a single-layered soil). Similarly, the amplitude \(v_z\) along the \(x\) axis exhibits a behaviour typical of a \(\frac{1}{\sqrt{r}}\) decay after 1 m, which is also characteristic of a Rayleigh surface wave. The far field is thus reached around 1 m and the simulation is coherent with a Rayleigh surface wave.

Figure 4. •: Unwrapped phase \(\phi_z\) in radians of the \(z\) component of the surface velocity field along \(x\). – – – : linear regression between \(x = 1\) m and \(x = 3\) m. Free soil solicited along \(z\) by a \(1\) m\(^2\), 125 kg surface plate at 30 Hz.

Since the vibrational field is axisymmetric except in the near field of the surface plate, let’s consider the radial velocity rather than the \(x\) and \(y\) components of the velocity field (Figure 5). The radial velocity is the component of the velocity field, in a polar representation, which is in the direction of the radius connecting the origin \(O\) to the observation point. It is deduced from the \(x\) and \(y\) components of the velocity field and the angle formed by \((Ox)\) and said radius.

Figure 5. Amplitude \(v_r\) (left) in dB and unwrapped phase \(\phi_r\) (right) in radians of the radial component of the surface velocity field. Free soil solicited along \(z\) by a \(1\) m\(^2\), 125 kg surface plate (full black line) at 30 Hz.

Note that the amplitude of the radial velocity slightly increases with the distance from the origin, whereas one would expect the amplitude to decrease with the distance. This is a near field effect: since the surface plate’s motion is entirely along \(z\), it hinders soil radial displacement.
in its near field. Computations at a greater distance from the origin show that the amplitude of the radial velocity decreases in the far field.

3.2. Soil with buried plate vibration
Let’s now investigate how the surface velocity field is modified by the presence of a buried plate. The Figure 6 shows the amplitude and phase of the \( z \) component of the surface velocity field for a soil containing a 4 m\(^2\) and 1570 kg plate buried at 0.5 m and offset by 1 m along \( x \); the surface solicitation is at 30 Hz. It shows that the vibrational field is influenced by the buried plate, as it is no longer axisymmetric.

![Figure 6](image_url). Amplitude \( v_z \) (left) in dB and unwrapped phase \( \phi_z \) (right) in radians of the \( z \) component of the surface velocity field. Soil solicited along \( z \) by a 1 m\(^2\), 125 kg surface plate (full black line) at 30 Hz with a 4 m\(^2\), 1570 kg plate buried at \( z = 0.5 \) m and offset by \( \delta_x = 1 \) m (black broken line).

The loss of axisymmetry is also visible when representing the radial surface velocity field, as can be seen in Figure 7. From these two figures, it can be said that the buried plate acts as a guide for the vibrational field along the direction of the offset.

4. Conclusion
An analytical approach to computing the vibrational field at any point of a single-layered soil containing a buried object has been presented in this article. This formulation’s originality resides in the fact that it considers the problem in terms of coupling between vibrating objects (here, rectangular non deformable plates) and propagating medium. The numerical simulations provided in Section 3 show that the surface velocity field can be significantly disrupted by the presence of a buried object. The buried plate seems to act as a waveguide as it renders the surface field asymmetric.

This approach to the problem shows great versatility, as it is possible to model a number of different situations. For example, the formulation can be adapted to model a shear surface solicitation instead of a compression one. Multiple layers can be added to the soil; it is also possible to include a layer of fluid instead of soil. The surface and buried plates can be circular instead of rectangular, or behave as flexible objects rather than non deformable ones. However, the problem always has to be composed of parallel layers for the resolution technique in the Fourier space domain to remain valid.

Future uses of this model could include inverse problems in order to provide a basis for the determination of the surface injected force based on measurements of the surface velocity field.
Figure 7. Amplitude $v_r$ (left) in dB and unwrapped phase $\phi_r$ (right) in radians of the radial component of the surface velocity field. Soil solicited along $z$ by a 1 m$^2$, 125 kg surface plate (full black line) at 30 Hz with a 4 m$^2$, 1570 kg plate buried at $z = 0.5$ m and offset by $\delta_x = 1$ m (black broken line).

Appendix A. Convention for the spatial Fourier transform
The two dimensions spatial Fourier transform along the coordinates $x$ and $y$ of a function of space $\zeta(x,y,z)$ is $\tilde{\zeta}(k_x,k_y,z)$ and is computed using the following convention (A.1):

$$\tilde{\zeta}(k_x,k_y,z) = \int_{x=-\infty}^{+\infty} \int_{y=-\infty}^{+\infty} \zeta(x,y,z)e^{-j k_x x} e^{-j k_y y} dxdy$$  \hspace{1cm} (A.1)

The inverse transform is thus computed with (A.2):

$$\zeta(x,y,z) = \frac{1}{4\pi^2} \int_{k_x=-\infty}^{+\infty} \int_{k_y=-\infty}^{+\infty} \tilde{\zeta}(k_x,k_y,z) e^{j k_x x} e^{j k_y y} dk_x dk_y$$  \hspace{1cm} (A.2)

Appendix B. Writing the stresses as functions of $A...H$ in $(k_x,k_y)$
In the following, in order to simplify the notations, the computations are only presented for the upper layer of soil, which allows to omit the subscript 1 relative to the layer. The resulting mathematical expressions can be adapted to fit the second layer of soil.

In order to write the stresses as functions of the components of the motion field of the soil, Hooke’s law (B.1) is used for a small displacements hypothesis:

$$\begin{cases} 
\sigma_{ii} = (\lambda + 2\mu)\epsilon_{ii} + \lambda(\epsilon_{jj} + \epsilon_{kk}) \\
\sigma_{ij} = 2\mu\epsilon_{ij} & i \neq j \end{cases}$$  \hspace{1cm} (B.1)

in which $i,j,k$ are the spatial components $x,y,z$. The strains $\epsilon_{ij}$ are related to the components of the motion field $\vec{u}$ with (B.2):

$$\epsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial j} + \frac{\partial u_j}{\partial i} \right)$$  \hspace{1cm} (B.2)

Using (9), (13), (A.1) and the previous two equations, the components of the stresses in the domain $(k_x,k_y)$ are written as functions of the unknowns $A...D'$ (B.3):
\[
\begin{align*}
\tilde{\sigma}_{xx} &= \mu \left[ 2k_x k_1 \left( A e^{-j k_1 z} - A' e^{j k_1 z} \right) + k_x k_y \left( B e^{-j k_2 z} + B' e^{j k_2 z} \right) \right. \\
&\quad \left. + \left( k_2^2 - k_x^2 \right) \left( C e^{-j k_2 z} + C' e^{j k_2 z} \right) + k_y k_2 \left( D e^{-j k_2 z} - D' e^{j k_2 z} \right) \right] \\
\tilde{\sigma}_{yy} &= \mu \left[ 2k_y k_1 \left( A e^{-j k_1 z} - A' e^{j k_1 z} \right) + \left( k_2^2 - k_x^2 \right) \left( B e^{-j k_2 z} + B' e^{j k_2 z} \right) \right. \\
&\quad \left. - k_x k_y \left( C e^{-j k_2 z} + C' e^{j k_2 z} \right) - k_y k_2 \left( D e^{-j k_2 z} - D' e^{j k_2 z} \right) \right] \\
\tilde{\sigma}_{zz} &= -(\lambda k_3^2 + 2\mu k_1^2) \left( A e^{-j k_1 z} + A' e^{j k_1 z} \right) + 2\mu k_2 \left[ -k_y \left( B e^{-j k_2 z} - B' e^{j k_2 z} \right) + k_x \left( C e^{-j k_2 z} - C' e^{j k_2 z} \right) \right]
\end{align*}
\]

Appendix C. Continuity conditions relative to stress written in \((k_x, k_y)\)

The continuity conditions relative to stress (7) and (8) are transposed in the \((k_x, k_y)\) domain, and thus yield six equations linking the unknowns \(A...H\) to the unknowns \(P_z\) and \(\tilde{\Delta}P_i\):

\[
\begin{aligned}
\tilde{\sigma}_{zz}^{(1)} (z = 0) &= 0 \\
\tilde{\sigma}_{yy}^{(1)} (z = 0) &= 0 \\
\tilde{\sigma}_{xx}^{(1)} (z = 0) &= \tilde{P}_z \\
\tilde{\sigma}_{zz}^{(2)} (z = h^+) - \tilde{\sigma}_{zz}^{(1)} (z = h^-) &= \tilde{\Delta}P_i
\end{aligned}
\]

Computation of \(\tilde{P}_z\) and \(\tilde{\Delta}P_i\) yields (C.3) and (C.4):

\[
\begin{aligned}
\tilde{P}_z &= P_z \frac{4}{k_x k_y} \sin \left( \frac{k_x a}{2} \right) \sin \left( \frac{k_y b}{2} \right) = P_z g_1 \\
\tilde{\Delta}P_i &= \Delta P_i \frac{4}{k_x k_y} \sin \left( \frac{k_x c}{2} \right) \sin \left( \frac{k_y d}{2} \right) e^{-j(k_x \delta_x + k_y \delta_y)} = \Delta P_i g_2
\end{aligned}
\]

in which \(g_1\) and \(g_2\) are functions of \(k_x\) and \(k_y\); they are introduced to shorten the expressions.

Appendix D. Continuity condition relative to displacement

From (9) and (13), the displacements fields \(\vec{u}_1\) and \(\vec{u}_2\) are written as functions of the unknowns \(A...H\). The continuity condition (6) becomes (D.1) in the Fourier domain:

\[
\begin{aligned}
k_x \left( A e^{-j k_1 z} + A' e^{j k_1 z} \right) + k_{21} \left( C e^{-j k_2 z} - C' e^{j k_2 z} \right) + k_y \left( D e^{-j k_2 z} + D' e^{j k_2 z} \right) \\
&= k_x E e^{-j k_1 z} + k_{22} G e^{-j k_2 z} + k_y H e^{-j k_2 z} \\
k_y \left( A e^{-j k_1 z} + A' e^{j k_1 z} \right) - k_{21} \left( B e^{-j k_2 z} + B' e^{j k_2 z} \right) - k_x \left( D e^{-j k_2 z} + D' e^{j k_2 z} \right) \\
&= k_y E e^{-j k_1 z} - k_{22} G e^{-j k_2 z} - k_x H e^{-j k_2 z} \\
-k_{11} \left( A e^{-j k_1 z} + A' e^{j k_1 z} \right) - k_y \left( B e^{-j k_2 z} + B' e^{j k_2 z} \right) + k_x \left( C e^{-j k_2 z} + C' e^{j k_2 z} \right) \\
&= -k_{12} E e^{-j k_1 z} - k_{22} F e^{-j k_2 z} + k_x G e^{-j k_2 z}
\end{aligned}
\]

Appendix E. Lorenz gauge condition

The Lorenz gauge condition for the upper layer of soil, in the Fourier domain, is \(\text{div} \vec{\psi}_1 = 0\). In order for it to be verified whatever the depth \(z\), the two following equations (E.1) must be true:

\[
\begin{aligned}
k_x B + k_y C - k_{21} D &= 0 \\
k_x B' + k_y C' + k_{21} D' &= 0
\end{aligned}
\]

Similarly, the gauge condition for the lower layer of soil leads to (E.2):

\[
k_x F + k_y G - k_{22} H = 0
\]
Appendix F. Condensed expression of the motion field of the upper layer of soil

From (9) and (13), and by injecting the numerically computed solution for \( \{X\}\) (15), the motion field components of the upper layer of soil write, after inverse Fourier transform:

\[
\begin{align*}
U_{1i} &= \frac{j}{4\pi^2} \left[ \int P_x \int g_1 U_{1i}^1 e^{j(k_x x + k_y y)} dk_x dk_y + \Delta P_x \int g_2 U_{1i}^2 e^{j(k_x x + k_y y)} dk_x dk_y 
+ \Delta P_y \int g_2 U_{1i}^3 e^{j(k_x x + k_y y)} dk_x dk_y + \Delta P_z \int g_2 U_{1i}^4 e^{j(k_x x + k_y y)} dk_x dk_y \right] \\
&= i_1 U_{1i}^1 + i_2 U_{1i}^2 + i_3 U_{1i}^3 + i_4 U_{1i}^4
\end{align*}
\]

in which \(i = x, y, z\) and \(U_{1i}^n\) \((n = 1, 2, 3, 4)\) are \(X_n(m)\) is the \(m\)-th row of the vector \(\{X_n\}\):

\[
\begin{align*}
U_x^n &= k_x (X_n(1)e^{-jk_{11}z} + X_n(2)e^{jk_{11}z}) + k_{21} (X_n(5)e^{-jk_{21}z} - X_n(6)e^{jk_{21}z}) \\
&+ k_y (X_n(7)e^{-jk_{21}z} + X_n(8)e^{jk_{21}z}) \\
U_y^n &= k_y (X_n(1)e^{-jk_{11}z} + X_n(2)e^{jk_{11}z}) - k_{21} (X_n(3)e^{-jk_{21}z} - X_n(4)e^{jk_{21}z}) \\
&- k_x (X_n(7)e^{-jk_{21}z} + X_n(8)e^{jk_{21}z}) \\
U_z^n &= -k_{11} (X_n(1)e^{-jk_{11}z} - X_n(2)e^{jk_{11}z}) - k_y (X_n(3)e^{-jk_{21}z} + X_n(4)e^{jk_{21}z}) \\
&+ k_x (X_n(5)e^{-jk_{21}z} + X_n(6)e^{jk_{21}z})
\end{align*}
\]

Appendix G. The matrix \([I]\)

The matrix \([I]\) is composed of sixteen integral over \((k_x, k_y);\) they all can be written using the following condensed form (G.1) (* is the complex conjugate):

\[
\int g_p g_q^* U_{1i}^n(z) dk_x dk_y
\]

in which \(p, q, i, n\) and \(z\) are, depending on the row and the column of the item:

- \(p = 1\) for the first column, otherwise \(p = 2;\)
- \(q = 1\) for the first row, otherwise \(q = 2;\)
- \(i = z\) for the first and fourth rows, \(i = x\) for the second row, \(i = y\) for the third row;
- \(n\) is the same as the rank of the column;
- \(z = 0\) for the first line, otherwise \(z = h.\)

References

[1] Foti S 2000 Multistation methods for geotechnical characterization using surface waves, PhD Dissertation (Politecnico di Torino, www.soilmech.polito.it/content/download/117/592/version/1/file/SF_Phd_diss.pdf)
[2] Strobbia C 2003 Surface wave methods: acquisition, processing and inversion, PhD Dissertation (Politecnico di Torino)
[3] Gucunski N, Ganji V and Maher M 1996 Effects of obstacles on Rayleigh wave dispersion obtained from the SASW test Soil Dynamics and Earthquake Engineering 15 223-231
[4] Ganji V, Gucunski N and Maher A 1997 Detection of underground obstacles by SASW method - numerical aspects Journal of Geotechnical and Geoenvironmental Engineering 123(3) 212-219
[5] Muggleton J, Brennan M and Gao Y 2011 Determining the location of buried plastic water pipes from measurements of ground surface vibration Journal of Applied Geophysics 75 54-61
[6] Muggleton J and Rustighi E 2013 Mapping the Underworld: recent developments in vibro-acoustic techniques to locate buried infrastructure Geotechnique Letters 3(3) 137-141
[7] Iodice M, Muggleton J and Rustighi E 2016 The detection of vertical cracks in asphalt using seismic surface wave methods *Journal of Physics: Conference Series* **744**(1)

[8] Kalkowski M, Muggleton J and Rustighi E 2018 Tree root detection from ground surface vibration measurements *MATEC Web of Conferences* **148**

[9] Grau L 2015 Approche analytique modale pour la prévision vibratoire de plaques couplées à des sols: applications ferroviaires, PhD Dissertation (INSA Lyon, theses.insa-lyon.fr/publication/2015ISAL0113/these.pdf)

[10] Grau L and Laulagnet B 2017 Semi-analytical modeling of ground/plate interaction for general elastic boundary conditions *The Journal of the Acoustical Society of America* **141**(6)