On the BCJR Algorithm for Asynchronous Physical-layer Network Coding
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Abstract—In practical asynchronous bi-directional relaying, symbols transmitted by two source nodes cannot arrive at the relay with perfect symbol alignment and the symbol-asynchronous multiple-access channel (MAC) should be seriously considered. Recently, Lu et al. proposed a Tanner-graph representation of symbol-asynchronous MAC with rectangular-pulse shaping and further developed the message-passing algorithm for optimal decoding of the asynchronous physical-layer network coding. In this paper, we present a general channel model for the asynchronous multiple-access channel with arbitrary pulse-shaping. Then, the Bahl, Cocke, Jelinek, and Raviv (BCJR) algorithm is developed for optimal decoding of asynchronous MAC channel. This formulation can be well employed to develop various low-complexity algorithms, such as Log-MAP algorithm, Max-Log-MAP algorithm, which are favorable in practice.

Index Terms—asynchronous bi-directional relaying, network coding, synchronization, BCJR algorithm.

I. INTRODUCTION

NETWORK coding has shown its power for disseminating information over networks [1], [2]. For wireless cooperative networks, there are increased interests in employing the idea of network coding for improving the throughput of the network. Indeed, the gain is very impressive for the special bi-directional relaying scenarios with two-way or multi-way traffic as addressed in [3].

For bi-directional relaying, it was soon recognized that the superimposed signal received at the relay can be viewed as the physically-combined network coding form of the two source messages, which is further impaired by the channel noise. Hence, the so-called physical-layer network coding (PNC) can be well employed to improve the throughput of the bi-directional relaying.

For bi-directional relaying with PNC, it is assumed that communication takes place in two phases - a multiple access (MAC) phase and a broadcast phase. In the first phase, the two source nodes send signals simultaneously to the relay; in the second phase, the relay processes the superimposed of the simultaneous packets and maps them to a network-coded packet for broadcast back to the source nodes. Compared with the traditional relay system, PNC doubles the throughput of the two-way relay channel by reducing the time slots for the exchange of one packet from four to two.

A key issue in practical PNC is how to deal with the asynchrony between the signals transmitted by the two source nodes. That is, symbols transmitted by the two source nodes could arrive at the receiver with symbol misalignment.

In [5], Lu et al. proposed a Tanner-graph representation of symbol-asynchronous multiple-access channel (MAC) with rectangular-pulse shaping and further developed the message-passing algorithm for optimal decoding of asynchronous physical-layer network coding.

In this paper, we provide further insights into the optimal decoding for asynchronous physical-layer network coding. In particular, the general asynchronous MAC channel with arbitrary pulse-shaping is developed and its connection to the rectangular-pulse shaping [5] is discussed. Then, the BCJR formulation of the asynchronous MAC channel is proposed, which can shed lights for various practical algorithms suitable for implementation.

II. GENERAL CHANNEL MODEL FOR ASYNCHRONOUS PHYSICAL-LAYER NETWORK CODING

A. Asynchronous Multiple Access Channel Model

During the MAC phase, the source nodes A and B transmit the modulated signals $x_a(t)$ and $x_b(t)$ to the relay. For a general continuous-time multiple-access channel, the received signal at the relay can be expressed as

$$y(t) = h_\alpha x_a(t) + h_\beta x_b(t) + w(t)$$
$$= \sum_{k=1}^{\infty} h_\alpha c_\alpha(k) g_\alpha(t - kT - \tau_\alpha)$$
$$+ \sum_{k=1}^{\infty} h_\beta c_\beta(k) g_\beta(t - kT - \tau_\beta) + w(t),$$ (1)

where the delays $\tau_\alpha \in [0, T), \tau_\beta \in [0, T)$ account for the symbol asynchronism between source nodes A and B and known to the receiver, $w(t)$ is the complex white Gaussian noise with power spectral density equal to $\frac{2}{\Gamma_f}$, the channel coefficients $h_\alpha, h_\beta$ are complex channel gains keeping fixed during transmission, and $g_\alpha(t), g_\beta(t)$ are normalized pulse-shaping functions $\left(\frac{1}{T_0} \int_0^T |g_a(t)|^2 dt = 1\right)$ for source nodes A and B, respectively. Without loss of generality, we assume that $0 \leq \tau_\alpha \leq \tau_\beta < T$.

By passing the observations through two matched filters for signals $(x_a(t))$ and $(x_b(t))$, respectively, one can get the
as able to find a physically realizable, stable discrete-time filter into a white random process. This procedure often matched filter (WMF) is often employed for transforming the some estimators, including the maximum-likelihood sequence where

\[ \mathbf{F} \equiv \left[ \begin{array}{c} f_{oa} \\ f_{ob} \end{array} \right] \] (12)

Consequently, passage of the received vector sequence \( \{ y(k), y_b(k) \} \) through the digital filter \( \mathbf{F}^{-1}(z^{-1}) \) results into an output vector sequence \( \{ r(k) \} \) that can be expressed at the top of the next page. Now, the discrete random process \( \{ n(k), n_b(k) \} \) is zero-mean white Gaussian process with covariance of \( \sigma^2 \mathbf{I} \).

Let \( c_{ab}(k) = \left[ \begin{array}{c} c_a(k) \\ c_b(k) \end{array} \right] \) and \( r(k) = \left[ \begin{array}{c} r_a(k) \\ r_b(k) \end{array} \right] \). Then, the formulation (13) can be elegantly expressed as

\[ r(k) = \Psi (c_{ab}(k), c_{ab}(k-1)) + n(k). \] (14)

It is clear that the function \( \Psi(\cdot, \cdot) \) is linear. By assuming the ideal knowledge on \( \Psi(\cdot, \cdot), \sigma^2 \), the asynchronous MAC channel can be modeled as the vector inter-symbol interference (ISI) channel. To estimate the a posteriori probability (APP) \( \Pr (c_{ab}(k)|y_0^N) \), the BCJR algorithm can be naturally employed.

**B. Rectangular-pulse shaping**

Let \( \delta = \frac{n_0 - n_a}{T} \) denote the relative delay between source nodes A and B. For the rectangular pulse-shaping functions \( g_a(t), g_b(t) \), i.e., \( g_a(t) = g_b(t) = u(t) - u(t-T) \) with \( u(t) \) denoting the unit step function, the authors in [5] proposed to consider the following discrete-time samples

\[ y_c(k) = \frac{1}{1 - \delta T} \int_{kT + \tau_a}^{(k+1)T + \tau_a} g(t) y(t) dt \]

\[ y_o(k) = \frac{1}{1 - \delta T} \int_{kT + \tau_b}^{(k+1)T + \tau_b} g(t) y(t) dt. \] (15)

It is clear that \( y_c(k) = \delta y_c(k) + (1 - \delta) y_o(k) \) and \( y_o(k) = (1 - \delta) y_c(k) + \delta y_o(k) \). Hence, the samples \( \{ y_c(k), y_o(k) \} \) are also the sufficient statistics for MAP detection. By combining (1) and (15), it follows that

\[ y_c(k) = h_a c_a(k) + h_o c_b(k-1) \]

\[ y_o(k) = h_a c_b(k) + h_o c_a(k) + w_o(k). \] (16)

where \( w_c(k) \) and \( w_o(k) \) are independent zero-mean complex Gaussian variables with variance of \( \frac{1}{1-\delta} \sigma^2 \) and \( \frac{1}{1-\delta} \sigma^2 \). Hence, one can write (14) as the following matrix form

\[ \left[ \begin{array}{c} y_c(k) \\ y_o(k) \end{array} \right] = \left[ \begin{array}{cc} h_b & c_a(k-1) \\ 0 & c_b(k-1) \end{array} \right] + \left[ \begin{array}{c} h_a \\ h_b \end{array} \right] \left[ \begin{array}{c} c_a(k) \\ c_b(k) \end{array} \right] + \left[ \begin{array}{c} w_c(k) \\ w_o(k) \end{array} \right]. \] (17)

Hence, the equivalent ISI channel model (14) is still valid.

**III. BCJR ALGORITHM**

In this section, we formulate the BCJR algorithm [8], which is known to be optimal in implementing the MAP symbol detection for channels with finite memory.
Let us define, at time epoch $k$, the state $s_k$ as

$$s_k = (c_{ab}(k-1)) = (c_a(k-1), c_b(k-1))$$  \hspace{1cm} (18)

and the branch metric function as

$$
\gamma_k(s_k, c_{ab}(k)) \\
\propto \exp \left( \frac{\|r(k) - \Psi(c_{ab}, c_{ab}(k-1))\|^2}{2\sigma^2} \right).$$  \hspace{1cm} (19)

The BCJR algorithm is characterized by the following forward and backward recursions:

$$
\alpha_{k+1}(s_{k+1}) = \sum_{c_{ab}(k)} \sum_{s_k} \mathcal{T}(c_{ab}(k), s_k, s_{k+1}) \cdot \alpha_k(s_k) \gamma_k(s_k, c_{ab}(k)),
$$  \hspace{1cm} (20)

where $\mathcal{T}(c_{ab}(k), s_k, s_{k+1})$ is the trellis indicator function, which is equal to 1 if $c_{ab}(k), s_k, s_{k+1}$ satisfy the trellis constraint and 0 otherwise;

$$\beta_k(s_k) = \sum_{c_{ab}(k)} \sum_{s_{k+1}} \mathcal{T}(c_{ab}(k), s_k, s_{k+1}) \cdot \beta_{k+1}(s_{k+1}) \gamma_k(s_k, c_{ab}(k)).$$  \hspace{1cm} (21)

Then, the joint APPs $\Pr(c_{ab}(k)|y_0^{N-1})$ can be calculated as

$$
\Pr(c_{ab}(k)|y_0^{N-1}) = \sum_{s_{k+1}} \mathcal{T}(c_{ab}(k), s_{k+1}) \alpha_{k+1}(s_{k+1}) \beta_{k+1}(s_{k+1}),
$$  \hspace{1cm} (22)

where the indicator function $\mathcal{T}(c_{ab}(k), s_{k+1})$ is equal to 1 if $s_{k+1}$ is compatible with $c_{ab}(k)$ and 0 otherwise.

It should be pointed out that the value of $\Psi(c_{ab}, c_{ab}(k-1))$ is independent of $c_a(k-1)$, hence the state $s_k$ can be further simplified as $s_k = (c_b(k-1))$.

Just like in [9], the proposed BCJR algorithm can be implemented efficiently in Log-domain, i.e., Log-MAP algorithm. The further simplification to the Max-Log-MAP algorithm is also straightforward, with some potential performance loss.

With the joint APPs $\Pr(c_{ab}(k)|y_0^{N-1})$, one can calculate the APPs of the XOR codeword $\Pr(c_a(k) \oplus c_b(k)|y_0^{N-1})$ for physical network coding. If both source A and B assumes the same linear channel coding, the relay node can make use of $\Pr(c_a(k) \oplus c_b(k)|y_0^{N-1})$ to perform channel decoding to obtain the pairwise XOR of the source symbols. However, this disjoint channel-decoding and network-coding scheme performs worse than the joint channel-decoding and network-coding scheme [5, 10].

IV. CONCLUSION AND FUTURE WORK

We have presented a general channel model for the asynchronous multiple-access channel with arbitrary pulse-shaping, typically encountered in bi-directional relaying. By evoking the WMF technique, one can arrive at an equivalent vector ISI channel, which can be employed to develop the well-known BCJR algorithm for getting the optimal APPs. This formulation can be well employed to develop various low-complexity algorithms, such as Log-MAP algorithm, Max-Log-MAP algorithm, which are favorable in practice.

Channel coding can be well employed to improve the system performance. It has been reported in [5, 10] that the joint channel-decoding and network-coding scheme can perform better than the disjoint channel-decoding and network-coding scheme. For joint network and LDPC coding over the asynchronous bi-directional relaying, it is interesting to find more efficient log-domain decoding algorithms suitable for practical implementation.

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