SHARP CONSTANTS OF APPROXIMATION THEORY. VI. WEIGHTED POLYNOMIAL INEQUALITIES OF DIFFERENT METRICS ON THE MULTIDIMENSIONAL CUBE AND BALL

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Abstract. We prove limit equalities between the sharp constants in weighted Nikolskii-type inequalities for multivariate polynomials on an \( m \)-dimensional cube and ball and the corresponding constants for entire functions of exponential type.

1. Introduction

We continue the study of the sharp constants in multivariate inequalities of approximation theory that began in [20]–[24]. In this paper we prove asymptotic equalities between the sharp constants in the multivariate weighted Nikolskii-type inequalities for entire functions of exponential type and algebraic polynomials on an \( m \)-dimensional cube and ball.

Notation. Let \( \mathbb{R}^m \) be the Euclidean \( m \)-dimensional space with elements \( x = (x_1, \ldots, x_m), \ y = (y_1, \ldots, y_m), \ t = (t_1, \ldots, t_m), \ s = (s_1, \ldots, s_m), \ v = (v_1, \ldots, v_m) \), the inner product \( (t, x) := \sum_{j=1}^{m} t_j x_j \), and the norm \( |x| := \sqrt{(x,x)} \). Next, \( \mathbb{C}^m := \mathbb{R}^m + i\mathbb{R}^m \) is the \( m \)-dimensional complex space with elements \( z = (z_1, \ldots, z_m) = x + iy \) and the norm \( |z| := \sqrt{|x|^2 + |y|^2} \); \( \mathbb{Z}^m \) denotes the set of all integral lattice points in \( \mathbb{R}^m \); \( \mathbb{Z}_+^m \) is a subset of \( \mathbb{Z}^m \) of all points with nonnegative coordinates; and \( \mathbb{N} := \{1, 2, \ldots\} \). We also use multi-indices \( k = (k_1, \ldots, k_m) \in \mathbb{Z}_+^m \) with \( \langle k \rangle := \sum_{j=1}^{m} k_j \) and \( x^k := x_1^{k_1} \cdots x_m^{k_m} \).

Given \( \sigma = (\sigma_1, \ldots, \sigma_m) \in \mathbb{R}^m \setminus \{0\} \), let \( \Pi^m(\sigma) := \{t \in \mathbb{R}^m : |t_j| \leq |\sigma_j|, 1 \leq j \leq m\} \) be the \( l \)-dimensional parallelepiped in \( \mathbb{R}^m \), where \( l \geq 1 \) is the number of nonzero coordinates of \( \sigma \). Given \( M > 0 \), let \( Q^m(M) := \{t \in \mathbb{R}^m : |t| \leq M, 1 \leq j \leq m\} \), \( \mathfrak{B}^m(M) := \{t \in \mathbb{R}^m : |t| \leq M\} \), \( O^m(M) := \{t \in \mathbb{R}^m : \sum_{j=1}^{m} |t_j| \leq M\} \), and \( S^{m-1} := \{t \in \mathbb{R}^m : |t| = 1\} \) be the \( m \)-dimensional cube, ball, octahedron, and the \((m - 1)\)-dimensional unit sphere in \( \mathbb{R}^m \), respectively. Next, let \( Q^m := Q^m(1) \) and \( \mathfrak{B}^m := \mathfrak{B}^m(1) \). In addition, \( |\Omega|_l \) denotes the \( l \)-dimensional Lebesgue measure of a measurable set \( \Omega \subseteq \mathbb{R}^m \), \( 1 \leq l \leq m \). We set \( S^0 := \{-1, 1\} \) and \( |S^0|_0 := 2 \). We also use the floor function \( \lfloor a \rfloor \), the gamma function \( \Gamma(z) \), and the beta function \( B(z,w) \).

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Let $W : \Omega \to [0, \infty]$ be a locally integrable weight on a measurable subset $\Omega$ of $\mathbb{R}^m$, and let $L_{r,W}(\Omega)$ be a weighted space of all measurable complex-valued functions $F : \Omega \to \mathbb{C}$ with the finite quasinorm
\[
\|F\|_{L_{r,W}(\Omega)} = \left\| F \right\|_{L_{r,W}(\Omega)} := \left\{ \begin{array}{ll}
\left( \int_{\Omega} |F(x)|^r W(x) dx \right)^{1/r}, & 0 < r < \infty, \\
\text{ess sup}_{x \in \Omega} |F(x)|, & r = \infty.
\end{array} \right.
\]
In the nonweighted case ($W = 1$) and also in the case of $r = \infty$, we set
\[
\| \cdot \|_{L_r(\Omega)} := \| \cdot \|_{L_r,1(\Omega)}, \quad L_r(\Omega) := L_{r,1}(\Omega), \quad 0 < r < \infty,
\]
\[
\| \cdot \|_{L_{\infty}(\Omega)} := \| \cdot \|_{L_{\infty,W}(\Omega)}, \quad L_{\infty}(\Omega) := L_{\infty,W}(\Omega).
\]
The quasinorm $\| \cdot \|_{L_{r,W}(\Omega)}$ allows the following "triangle" inequality:
\[
\| F + G \|_{L_{r,W}(\Omega)}^\ast \leq \| F \|_{L_{r,W}(\Omega)}^\ast + \| G \|_{L_{r,W}(\Omega)}^\ast, \quad F, G \in L_{r,W}(\Omega),
\] where $\ast := \min\{1, r\}$ for $r \in (0, \infty]$.

In this paper we will need certain definitions and properties of convex bodies in $\mathbb{R}^m$. Throughout the paper $V$ is a centrally symmetric (with respect to the origin) closed convex body in $\mathbb{R}^m$ and $V^* := \{y \in \mathbb{R}^m : \forall t \in V, |(t, y)| \leq 1\}$ is the polar of $V$. It is well known that $V^*$ is a centrally symmetric (with respect to the origin) closed convex body in $\mathbb{R}^m$ and $V^{**} = V$ (see, e.g., [10 Sect. 14]). The set $V$ generates the dual norm on $\mathbb{C}^m$ by $\|z\|_V^\ast := \sup_{t \in V} \left| \sum_{j=1}^m t_j z_j \right|$, $z \in \mathbb{C}^m$.

**Definition 1.1.** A body $V \subset \mathbb{R}^m$ satisfies the parallelepiped condition ($\Pi$-condition) if for every vector $t \in V \setminus \{0\}$ the parallelepiped $\Pi^m(t)$ is a subset of $V$.

It is easy to verify that $V$ satisfies the $\Pi$-condition if and only if $V$ is symmetric about all coordinate hyperplanes, that is, for every $t \in V$ the vectors $(\pm |t_1|, \ldots, \pm |t_m|)$ belong to $V$. A slightly different version of the $\Pi$-condition, which is equivalent to Definition 1.1, was introduced in [23 Sect. 1]. In particular, given $\lambda \in [1, \infty]$ and $\sigma \in \mathbb{R}^m$, $\sigma_j > 0$, $1 \leq j \leq m$, the set $V_{\lambda,\sigma} := \left\{ t \in \mathbb{R}^m : \left( \sum_{j=1}^m |t_j/\sigma_j|^\lambda \right)^{1/\lambda} \leq 1 \right\}$, satisfies the $\Pi$-condition. Therefore, the sets $\Pi^m(\sigma)$ (for $\lambda = \infty$), $Q^m(M)$ (for $\lambda = \infty$ and $\sigma = (M, \ldots, M)$), $\mathcal{B}^m(M)$ (for $\lambda = 2$ and $\sigma = (M, \ldots, M)$), and $O^m(M)$ (for $\lambda = 1$ and $\sigma = (M, \ldots, M)$) satisfy the $\Pi$-condition as well.

Given $a \geq 0$, the set of all trigonometric polynomials $T(x) = \sum_{\eta \in aV} c_{\eta} \exp(i\eta, x)$ with complex coefficients is denoted by $\mathcal{T}_{aV}$. In the univariate case we use the notation $\mathcal{T}_n := \mathcal{T}_{[-a,a]} = \mathcal{T}_{[-n,n]}$, where $n = |a|$, $n \in \mathbb{Z}_\pm$.

**Definition 1.2.** We say that an entire function $f : \mathbb{C}^m \to \mathbb{C}^1$ has exponential type $V$ if for any $\varepsilon > 0$ there exists a constant $C_0(\varepsilon, f) > 0$ such that for all $z \in \mathbb{C}^m$, $|f(z)| \leq C_0(\varepsilon, f) \exp((1 + \varepsilon)\|z\|_V^\ast)$.
The set of all entire function of exponential type $V$ is denoted by $B_V$, and the set of even functions in each variable from $B_V$ is denoted by $B_{V,e}$. In the univariate case we use the notation $B_\lambda := B_{[-\lambda,\lambda]}$ and $B_{\lambda,e} := B_{[-\lambda,\lambda],e}$, $\lambda > 0$. In addition, note that if $V$ satisfies the II-condition and $f \in B_V$, then the function $f(\pm z_1, \ldots, \pm z_m)$ belongs to $B_V$ as well since $\|f(\pm z_1, \ldots, \pm z_m)\|^*_V = \|z\|_V^*$ by the definition of $\|z\|_V^*$, $z \in \mathbb{C}^m$.

Throughout the paper, if no confusion may occur, the same notation is applied to $f \in B_V$ and its restriction to $\mathbb{R}^m$ (e.g., in the form $f \in B_V \cap L_{p,W}(\mathbb{R}^m)$). The class $B_V$ was defined by Stein and Weiss [44 Sect. 3.4]. For $V = \Pi^m(\sigma)$, $V = Q^m(M)$, and $V = \Psi^m(M)$, similar classes were defined by Bernstein [4] and Nikolskii [38 Sects. 3.1, 3.2.6], see also [11 Definition 5.1]. In particular, $B_{\Psi^m(M)}$ is the set of all entire functions of spherical type $M$ (see [38 Sect. 3.2.6]). Certain properties of functions from $B_V$ are presented in Lemma 3.1.

Given a convex bounded set $\Omega \subset \mathbb{R}^m$, let $P_{\Omega}$ be the set of all polynomials $P(x) = \sum_{k \in \mathbb{N}^{\mathbb{Z}^m_+}} c_k x^k$ in $m$ variables with complex coefficients whose Newton polyhedra (see, e.g., [3 Sect. 3] for the definition) are subsets of $\Omega$. In this paper we use the set $P_{aV}$ for a given $a \geq 0$ and the set $P_{aV,e}$ of even polynomials in each variable from $P_{aV}$. In the case of $V = O^m(1)$, $P_{nV} = P_{O^m(n)}$ coincides with the set $P_{n,m}$ of all polynomials $P(x) = \sum_{k \leq n} c_k x^k$ in $m$ variables of total degree at most $n$, $n \in \mathbb{Z}^1_+$. In the univariate case we use the notation $P_n := P_{[-a,a]} = P_{n,1}$, where $n = \lfloor a \rfloor$, $n \in \mathbb{Z}_+^1$. Newton polyhedra and polynomial classes $P_{\Omega}$ associated with Newton polyhedra play an important role in algebra, geometry, and analysis (see, e.g., a survey [3 Sect. 3]). Note that if $V$ satisfies the II-condition, then the Newton polyhedron of a polynomial from $P_{nV}$ is a downward closed set (see, e.g., [35 Sect. 2]). It is easy to verify that if $V_1 \subseteq V_2$, then $B_{V_1} \subseteq B_{V_2}$ and $P_{aV_1} \subseteq P_{aV_2}$.

Throughout the paper $A_1$, $A_2$, $C$, $C_1, \ldots, C_{19}$ denote positive constants independent of essential parameters. Occasionally we indicate dependence on certain parameters. The same symbol $C$ does not necessarily denote the same constant in different occurrences, while $A_1$, $A_2$, and $C_l$, $1 \leq l \leq 19$, denote the same constants in different occurrences.

**Inequalities of Different Metrics.** We first define sharp constants in Nikolskii-type inequalities for algebraic and trigonometric polynomials and entire functions of exponential type and then briefly discuss their asymptotic behaviours.

Let $x_0 \in \Omega$ be a fixed point and let $B$ be a subspace of $L_{p,W}(\Omega)$, where $\Omega$ is a closed subset of $\mathbb{R}^m$. In the capacity of $x_0$ we shall use either the origin 0 or boundary points, and in the capacity of $B$ we shall use sets of algebraic and trigonometric polynomials and entire functions of exponential
type. Let us define two sharp constants of different metrics
\[ N_{x_0}(B, L_p, W(\Omega)) := \sup_{h \in B \setminus \{0\}} \frac{|h(x_0)|}{\|h\|_{L_p, W(\Omega)}}, \quad (1.2) \]
\[ N(B, L_p, W(\Omega)) := \sup_{h \in B \setminus \{0\}} \frac{\|h\|_{L_\infty, W(\Omega)}}{\|h\|_{L_p, W(\Omega)}}, \quad (1.3) \]
These constants usually coincide for all \( x_0 \in \Omega \) in case of invariant subspaces \( B, W = 1 \), and homogeneous spaces \( \Omega \) like \( \mathbb{R}^m \), the torus \( T^m \), and the sphere \( S^{m-1} \). However, it is not the case in other situations, in particular, for algebraic polynomials. Our major goal here is to find the asymptotic behaviour of \( N(B, L_p, W(\Omega)) \) in certain cases of multivariate polynomial subspaces \( B \), sets \( \Omega \), and Gegenbauer-type weights \( W \).

The following limit relations for multivariate trigonometric polynomials
\[
\lim_{a \to \infty} a^{-m/p} N(T_{aV}, L_p(Q^m(\pi))) = \lim_{a \to \infty} a^{-m/p} N_0(T_{aV}, L_p(Q^m(\pi))) = N_0(B_V \cap L_p(\mathbb{R}^m), L_p(\mathbb{R}^m)), \quad p \in (0, \infty),
\]
were proved by the author \[21, \text{Theorem 1.3}\]. In the univariate case of \( V = [-1, 1] \) and \( a \in \mathbb{N} \), \[1.4\] were proved by the author and Tikhonov \[25\]. In earlier publications \[31, 32\], Levin and Lubinsky established versions of \[1.4\] on the unit circle. Quantitative estimates of the remainder in asymptotic equalities of the Levin-Lubinsky type were found by Gorbachev and Martynanov \[27, \text{Corollary 1}\]. Certain extensions of the Levin-Lubinsky’s results to the \( m \)-dimensional unit sphere in \( \mathbb{R}^{m+1} \) were recently proved by Dai, Gorbachev, and Tikhonov \[7\] (see also \[20, \text{Corollary 4.5}\]).

The classic inequalities of different metrics for algebraic polynomials
\[
\|P\|_{L_\infty(\Omega)} \leq C a^\mu \|P\|_{L_p(\Omega)}, \quad P \in \mathcal{P}_a V, \quad p \in [1, \infty),
\]
where \( \Omega \subset \mathbb{R}^m \) is a bounded closed domain and \( C \) is independent of \( P \) and \( a \), have been studied since the 1960s. The exponent \( \mu \) in \[1.5\] essentially depends on \( \Omega \). For example, inequality in \[1.5\] holds true for \( \mu = 2m/p \) and any domain \( \Omega \), satisfying the cone condition, in particular for convex bodies (not necessarily symmetric), see Daugavet \[8, \text{Theorem 1}\], \[9, \text{Theorem 2}\] and the author \[14, \text{Theorem 2}\]. It is also valid for \( \mu = (m + 1)/p \) and any domain \( \Omega \) with the smooth boundary, see Daugavet \[8, \text{Theorem 2}\], \[9, \text{Theorem 5}\] and Kroó and Schmidt \[29, p. 433\]. In case of the cube \( \Omega = Q^m \) for \( \mu = 2m/p \) and the unit ball \( \Omega = B^m \) for \( \mu = (m + 1)/p \), the factor \( a^\mu \) in \[1.5\] cannot be replaced by a smaller one as \( a \to \infty \) (see \[9, \text{Theorem 7}\]). More examples and discussions are presented by Ditzian and Prymak \[11, 12\]. In addition, note that weighted versions of \[1.5\] with Gegenbauer-type weights were obtained in \[8, \text{Theorems 1 and 2}\], \[9, \text{Theorems 2 and 5}\] and with \( k \)-concave weights in \[18, \text{Theorem 2.3}\].
If \( \|P\|_{L_\infty(\Omega)} \) in (1.5) is replaced by \( |P(x_0)|, x_0 \in \Omega \), then unlike Nikolskii-type inequalities for trigonometric polynomials (cf. (1.1)), the exponent \( \mu \) in (1.5) also depends on the location of \( x_0 \). In particular, if \( x_0 \) is an interior point of \( V \), then \( \mu = m/p \) (see [23 Lemma 2.9]). The asymptotic behaviours of the corresponding sharp constants for \( x_0 = 0 \) were found by the author in the following forms:

\[
\lim_{a \to \infty} a^{-m/p} N_0(P_{aV}, L_p(Q^m)) = N_0(B_V \cap L_p(\mathbb{R}^m), L_p(\mathbb{R}^m)), \quad p \in (0, \infty),
\]

(1.6)

\[
\lim_{n \to \infty} n^{-m/p} N_0(P_{n,m}, L_p(V^m)) = N_0(B_V \cap L_p(\mathbb{R}^m), L_p(\mathbb{R}^m)), \quad p \in (0, \infty).
\]

(1.7)

Relation (1.6) for \( V \), satisfying the II-condition, was proved in [23 Theorem 1.2], and (1.7) was established in [22 Theorem 1.2] (see also [19 Theorem 1.1] for \( m = 1 \) and [20 Corollary 4.4] for \( V = B^m \) and \( p \in [1, \infty) \)).

The exact value of the sharp constant \( \mathcal{N}(P_n, L_{p,W}([-1,1])) \) is known in several cases. Geronimus [26 Theorem 1] found \( \mathcal{N}(P_n, L_{1,1-\alpha^2-1/2}([-1,1])) \). Simonov and Glazyrina [43 Theorem 1] generalized this result by using and developing the Geronimus method. Note that the constants found in [26] and [43] are not explicit, and the problem of finding their asymptotic behaviour as \( n \to \infty \) is still open. For certain weights, the sharp constant \( \mathcal{N}(P_n, L_{2,W}([-1,1])) \) can be found by using extremal properties of orthonormal polynomials. Using this approach, Lupas [34] (see also [36 Theorem 6.1.8.2]) found \( \mathcal{N}(P_n, L_{2,W}([-1,1])) \) for the Jacobi weight. For the Gegenbauer weight, his result can be reduced to the following formula (\( \lambda \geq 0 \)):

\[
\mathcal{N}(P_n, L_{2,1-\alpha^2-1/2}([-1,1])) = \left( \frac{(2\lambda + 2n + 1)\Gamma(2\lambda + n + 1)}{2^{2\lambda}(2\lambda + 1)\Gamma^2(\lambda + 1/2)n!} \right)^{1/2}
\]

(1.8)

(see [34] and [36 Eq. 6.1.8.8]). Two special cases of (1.8) are well-known. The first of them, \( \mathcal{N}(P_n, L_2([-1,1])) = 2^{-1/2}(n + 1) \), has been known since the 1920s (see Polya and Szegő [38 Problem 6.103] and Labelle [30]). The second one is

\[
\mathcal{N}(P_n, L_{2,1-\alpha^2-1/2}([-1,1])) = 2^{1/2} \mathcal{N}(\mathcal{T}_n, L_2([-\pi, \pi])) = ((2n + 1)/\pi)^{1/2}
\]

(see [35 Sect. 4.9.2]).

In addition, the following univariate relations are known (\( p \in [1, \infty), \lambda \geq 0 \)):

\[
\lim_{n \to \infty} n^{-(2\lambda+1)/p} \mathcal{N}(P_{n}, L_{p,(1-u^2)^{\lambda-1/2}}([-1,1])) = \lim_{n \to \infty} n^{-(2\lambda+1)/p} N_1(P_{n}, L_{p,(1-u^2)^{\lambda-1/2}}([-1,1])) = 2^{1/p} N_0\left(B_{1,e} \cap L_{p,|u|^{2\lambda}}(\mathbb{R}^1), L_{p,|u|^{2\lambda}}(\mathbb{R}^1)\right) = 2^{1/p} N_0\left(B_1 \cap L_{p,|u|^{2\lambda}}(\mathbb{R}^1), L_{p,|u|^{2\lambda}}(\mathbb{R}^1)\right).
\]

(1.9)
In addition to (1.9), there exists a function \( g_0 \in \left( B_{1,e} \cap L_{p,|u|^{2\lambda}(\mathbb{R}^1)} \right) \setminus \{0\}, p \in [1, \infty) \), such that
\[
\lim_{n \to \infty} n^{-(2\lambda+1)/p} N \left( P_n, L_{p,(1-u^2)^{\lambda-1/2}}([-1,1]) \right) = 2^{1/p} |g_0(0)| / \|g_0\|_{L_{p,|u|^{2\lambda}(\mathbb{R}^1)}}. \tag{1.10}
\]

The first relation in (1.9) immediately follows from the equality
\[
N \left( P_n, L_{p,(1-u^2)^{\lambda-1/2}}([-1,1]) \right) = N_1 \left( P_n, L_{p,(1-u^2)^{\lambda-1/2}}([-1,1]) \right) \tag{1.11}
\]
proved by Arestov and Deikalova [2, Theorem 1] (see also Example 2.3 of Section 2), while the second one in (1.9) and also (1.10) were proved in [20, Theorem 4.3]. The third equality in (1.9) immediately follows from the following well-known symmetrization: if \( f \in B_{1,e} \cap L_{p,|u|^{2\lambda}(\mathbb{R}^1)} \), \( p \in [1, \infty) \), then the function \( f^*(u) := (f(u) + f(-u))/2 \) satisfies the properties
\[
f^* \in B_{1,e} \cap L_{p,|u|^{2\lambda}(\mathbb{R}^1)}, \quad f^*(0) = f(0), \quad \|f^*\|_{L_{p,|u|^{2\lambda}(\mathbb{R}^1)}} \leq \|f\|_{L_{p,|u|^{2\lambda}(\mathbb{R}^1)}}.
\]

The asymptotic formula for the univariate sharp constant in (1.5) with \( \Omega = [-1,1] \) and \( p \in [1, \infty) \) is a consequence of (1.9) in the following form:
\[
\lim_{n \to \infty} n^{-2/p} N \left( P_n, L_p([-1,1]) \right) = 2^{1/p} N_0 \left( B_1, L_p([-1,1]) \right), \tag{1.12}
\]
(see [20 Corollary 4.6]). A different version of (1.12) was proved in [19 Theorem 1.4] (see also [19, p. 94]).

Note that relations (1.4), (1.6), (1.7), and the second relation in (1.9) for the sharp Nikolskii-type constants are special cases of more general limit relations between sharp Markov–Bernstein–Nikolskii type constants with \( |h(0)| \) and \( \|h\|_{L_{\infty}(\Omega)} \) in (1.2) and (1.3) replaced by \( |D_N(h)(0)| \) and \( \|D_N(h)\|_{L_{\infty}(\Omega)} \), respectively, for certain differential operators \( D_N \) (see [25, 19, 21, 22, 23]). In addition, note that the sharp nonweighted Bernstein–Nikolskii constants for entire functions of exponential type can be easily found only for \( p = 2 \) (see [21, Eq. (1.6)]).

No exact or asymptotic equalities for multivariate sharp constants in Nikolskii-type inequalities for algebraic polynomials are known. In this paper we extend (1.9) to asymptotic relations for multivariate sharp constants \( N \left( P_{aV}, L_{p,W}(Q^m) \right) \) and \( N \left( P_{n,m}, L_{p,W}(\mathbb{B}^m) \right) \) with Gegenbauer-type weights. In addition, we prove a weighted version of limit equality (1.6) and find certain multivariate sharp constants for \( p = 2 \). Note that we are unaware of any applications of these asymptotically sharp results.

**Main Results and Remarks.** We first discuss asymptotics of the sharp constants on the cube \( Q^m \).
Theorem 1.3. If $V \subset \mathbb{R}^m$ satisfies the $\Pi$-condition, then for $p \in [1, \infty)$, $m \geq 1$, and $\lambda_j \geq 0$, $1 \leq j \leq m$, the following limit relation holds true:

$$
\lim_{a \to \infty} a^{-\left(m+2\sum_{j=1}^{m} \lambda_j\right)/p} N_0 \left( \mathcal{P}_{aV}, L_{p,\Pi_{j=1}^{m}(1-x_j^2)^{\lambda_j-1/2}}(Q^m) \right) = 2^{m/p} \frac{1}{\mathcal{N}_0} \left( B_V \cap L_{p,\Pi_{j=1}^{m}|t_j|^{2\lambda_j}}(\mathbb{R}^m) \right).
$$

(1.13)

In addition, there exists a function $f_0 \in \left( B_V \cap L_{p,\Pi_{j=1}^{m}|t_j|^{2\lambda_j}}(\mathbb{R}^m) \right) \setminus \{0\}$ such that

$$
\lim_{a \to \infty} a^{-\left(m+2\sum_{j=1}^{m} \lambda_j\right)/p} N_0 \left( \mathcal{P}_{aV}, L_{p,\Pi_{j=1}^{m}(1-x_j^2)^{\lambda_j-1/2}}(Q^m) \right) = 2^{m/p} |f_0(0)| / \|f_0\|_{L_{p,\Pi_{j=1}^{m}|t_j|^{2\lambda_j}}(\mathbb{R}^m)}.
$$

(1.14)

One of the major ingredients of the proof of Theorem 1.3 is the following theorem of independent interest that presents a weighted version of relation (1.6).

Theorem 1.4. If $V \subset \mathbb{R}^m$ satisfies the $\Pi$-condition, then for $p \in (0, \infty)$, $m \geq 1$, and $\alpha_j \geq 0$, $\beta_j \geq -1/2$, $1 \leq j \leq m$, the following limit relation holds true:

$$
\lim_{a \to \infty} a^{-\left(m+\sum_{j=1}^{m} \alpha_j\right)/p} N_0 \left( \mathcal{P}_{aV}, L_{p,\Pi_{j=1}^{m}|t_j|^\alpha_j(1-t_j^2)^\beta_j}(Q^m) \right) = N_0 \left( B_V \cap L_{p,\Pi_{j=1}^{m}|t_j|^\alpha_j(1-t_j^2)^\beta_j}(\mathbb{R}^m) \right).
$$

(1.15)

In addition, there exists a function $f_0 \in \left( B_V \cap L_{p,\Pi_{j=1}^{m}|t_j|^\alpha_j}(\mathbb{R}^m) \right) \setminus \{0\}$ such that

$$
\lim_{a \to \infty} a^{-\left(m+\sum_{j=1}^{m} \alpha_j\right)/p} N_0 \left( \mathcal{P}_{aV}, L_{p,\Pi_{j=1}^{m}|t_j|^\alpha_j(1-t_j^2)^\beta_j}(Q^m) \right) = |f_0(0)| / \|f_0\|_{L_{p,\Pi_{j=1}^{m}|t_j|^\alpha_j}(\mathbb{R}^m)}.
$$

(1.16)

Next, the asymptotic of the sharp constant on the unit ball $\mathfrak{B}^m$ is discussed.

Theorem 1.5. For $p \in [1, \infty)$, $m \geq 1$, and $\lambda \geq 0$, the following equalities hold true:

$$
\lim_{n \to \infty} n^{-\left(m+2\lambda\right)/p} N \left( \mathcal{P}_{n,m}, L_{p,\Pi_{j=1}^{m}(1-|x_j|^2)^{\lambda-1/2}}(\mathfrak{B}^m) \right) = A_1 N_0 \left( B_{1,e} \cap L_{p,|u|^{m+2\lambda-1}}(\mathbb{R}^1), L_{p,|u|^{m+2\lambda-1}}(\mathbb{R}^1) \right)
$$

$$
= A_2 N_0 \left( B_{\mathfrak{B}^m} \cap L_{p,|t|^{2\lambda}}(\mathbb{R}^m), L_{p,|t|^{2\lambda}}(\mathfrak{B}^m) \right),
$$

where

$$
A_1 = A_1(m, p, \lambda) := \left( \frac{2\Gamma(\lambda + m/2)}{\pi^{(m-1)/2}\Gamma(\lambda + 1/2)} \right)^{1/p},
$$

(1.18)

$$
A_2 = A_2(m, p, \lambda) := \left( \frac{2\pi^{1/2}\Gamma(\lambda + m/2)}{\Gamma(m/2)\Gamma(\lambda + 1/2)} \right)^{1/p}.
$$

(1.19)
In addition, there exists a function $f_0 \in \left( B_{B^m} \cap \mathcal{L}_{p,|t|^{2\lambda}}(\mathbb{R}^m) \right) \setminus \{0\}$ such that
\[
\lim_{n \to \infty} n^{-(m+2\lambda)/p} \mathcal{N}\left( \mathcal{P}_{n,m}, L_{p,|1-x|^2}^{\lambda-1/2}(\mathfrak{B}^m) \right) = A_2 |f_0(0)|/\|f_0\|_{L_{p,|t|^{2\lambda}}(\mathbb{R}^m)}.
\] (1.20)

Finally, the sharp constants in (1.17) can be found for $p = 2$.

**Theorem 1.6.** For $m \geq 1$ and $\lambda \geq 0$, the following equalities hold true:
\[
\mathcal{N}\left( \mathcal{P}_{n,m}, L_{2,|1-x|^2}^{\lambda-1/2}(\mathfrak{B}^m) \right) = \left( \frac{(2\lambda + 2n + m)\Gamma(2\lambda + n + m)}{2^{2\lambda+m-1}\pi^{(m-1)/2}(2\lambda + m)\Gamma(\lambda+1/2)\Gamma(\lambda+m/2)n!} \right)^{1/2};
\] (1.21)
\[
\mathcal{N}_0\left( B_{B^m} \cap L_{2,|t|^{2\lambda}}(\mathbb{R}^m), L_{2,|t|^{2\lambda}}(\mathbb{R}^m) \right) = \left( \frac{\Gamma(m/2)}{2^{2\lambda+m-1}\pi^{m/2}(2\lambda + m)\Gamma^2(\lambda+m/2)} \right)^{1/2}.
\] (1.22)

**Remark 1.7.** Relations (1.15) and (1.16) show that the function $f_0 \in B_V \cap L_{p,\prod_{j=1}^m|t_j|^{\alpha_j}}(\mathbb{R}^m)$ from Theorem 1.4 is an extremal function for $\mathcal{N}_0\left( B_V \cap L_{p,\prod_{j=1}^m|t_j|^{\alpha_j}}(\mathbb{R}^m), L_{p,\prod_{j=1}^m|t_j|^{\alpha_j}}(\mathbb{R}^m) \right)$. Moreover, the extremal function $f_0$ in (1.16) can be chosen from $B_{V,e} \cap L_{p,\prod_{j=1}^m|t_j|^{\alpha_j}}(\mathbb{R}^m)$ for $p \in [1, \infty)$. This result can be proved by the following symmetrization trick: for $p \in [1, \infty)$, the function
\[
f_{0,e}(t) := 2^{-m} \sum_{|\delta_j|=1, 1 \leq j \leq m} f_0(\delta_1 t_1, \ldots, \delta_m t_m)
\]
belongs to $B_{V,e}$ and
\[
\frac{|f_0(0)|}{\|f_0\|_{L_{p,\prod_{j=1}^m|t_j|^{\alpha_j}}(\mathbb{R}^m)}} \leq \frac{|f_{0,e}(0)|}{\|f_{0,e}\|_{L_{p,\prod_{j=1}^m|t_j|^{\alpha_j}}(\mathbb{R}^m)}}.
\]

In addition, for $p \in [1, \infty)$, $\mathcal{P}_{aV}$ in (1.5) and (1.16) can be replaced by $\mathcal{P}_{aV,e}$, and also $B_V$ in (1.13) and (1.16) and $B_{B^m}$ in (1.17) and (1.22) can be replaced by $B_{V,e}$ and $B_{B^m,e}$, respectively.

These results can be proved by the same symmetrization (invariance) theorems in [20] Theorems 1.1 and 1.2.

**Remark 1.8.** Limit relations (1.13) and (1.17) coincide for $m = 1$, and their slightly different versions were established in [20] Theorem 4.3 (see also (1.9)). A special case of Theorem 1.4 for $\alpha_j = \beta_j = 0, 1 \leq j \leq m$, was proved in [23] Theorem 1.2 (see also (1.6)). Note that to prove Theorem 1.3 it suffices to prove (1.15) for $p \in [1, \infty)$ and $\alpha_j = 2\lambda_j, \beta_j = \lambda_j - 1/2$, $\lambda_j \geq 0, 1 \leq j \leq m$, but it is possible to prove Theorem 1.4 for the wider range of $p \in (0, \infty)$ and $\alpha_j \geq 0, \beta_j \geq -1/2, 1 \leq j \leq m$, without additional steps. In addition, note that equality (1.21) for $m = 1$ is reduced to (1.8) and equality (1.22) is known for $\lambda = 0$ (see, e.g., [21] Eq. (1.6)).
Remark 1.9. In definitions (1.2) and (1.3) of the sharp constants, we discuss only complex-valued functions $P$ and $f$. We can define similarly the “real” sharp constants if the suprema in (1.2) and (1.3) are taken over all real-valued functions on $\mathbb{R}^m$ from $\mathcal{P}_{aV} \setminus \{0\}$ and $(B_V \cap L_{p,W}(\mathbb{R}^m)) \setminus \{0\}$, respectively. It turns out that the “complex” and “real” sharp constants coincide. For $m = 1$ this fact was proved in [19, Sect. 1] (cf. [25, Theorem 1.1] and [22, Remark 1.5]), and the case of $m > 1$ can be proved similarly.

Remark 1.10. Note that the problem of finding the asymptotic behaviour of $\mathcal{N}(B, L_{p,W}(\Omega))$ for $B$, $W$, and $\Omega$ other than those ones in Theorems 1.3, 1.5, and 1.6 is still open. In addition, the following two open questions are raised by an anonymous referee: is it possible firstly, to replace the Gegenbauer-type weights of Theorems 1.3 and 1.4 by more general Jacobi-type weights and secondly, to prove a multivariate version of (1.12) for the cube $Q_m^m$ and $p \in (0, 1)$ as it was done in [27, Sect. 3.1]?

The proofs of Theorems 1.3–1.6 are presented in Sections 3–5. The proof of Theorem 1.4 is a weighted version of the proof of [23, Theorem 1.2], and it follows general ideas developed in [24, Corollary 7.1].

To prove Theorems 1.3 and 1.5 we first reduce the corresponding sharp constants (1.3) of these theorems to constants (1.2). This important step is accomplished by using multivariate analogues of equality (1.11) that are discussed in Section 2. In particular, Lemma 2.1 of independent interest discusses in general settings the equality $\mathcal{N}(B, L_{p,W}(\Omega)) = \mathcal{N}_{x_0}(B, L_{p,W}(\Omega))$ with a special point $x_0$ on the boundary of $\Omega$.

The next step of the proofs of Theorems 1.3 and 1.5 (and Theorem 1.6 as well) is based on equalities between the sharp constants on the cube $Q^m$ and the ball $B^m$ discussed in Sections 4 and 5. In particular, an important role in the proofs of such the equalities from Section 5 plays Proposition 5.1 of independent interest that discusses the representation of a polynomial that is invariant under certain rotation subgroups. Finally, Theorem 1.3 is reduced to Theorem 1.4 and Theorem 1.5 is reduced to known relations (1.9).

2. Extremal Polynomials and Generalized Translation Operators

Let $\Omega$ be a compact subset of $\mathbb{R}^m$, and let $B$ be a finite-dimensional subspace of continuous functions from $L_{p,W}(\Omega)$ whose elements are called polynomials. Recall that $\mathcal{N}(B, L_{p,W}(\Omega))$ is defined by (1.3).
We say that \( P^* \in B \setminus \{0\} \) is an extremal polynomial for \( \mathcal{N}(B, L_{p,W}(\Omega)) \) if
\[
\mathcal{N}(B, L_{p,W}(\Omega)) = \frac{\|P^*\|_{L_\infty(\Omega)}}{\|P^*\|_{L_{p,W}(\Omega)}}.
\]

In this section we discuss an important property of certain extremal polynomials for \( \mathcal{N}(B, L_{p,W}(\Omega)) \) that the uniform norm of these polynomials is attained at special points on the boundary of \( \Omega \). We shall use this result for the proofs of Theorems 1.3, 1.5, and 1.6 in the cases of \( B = P_{a,V}, a \geq 0, \Omega = Q^m \), and \( B = P_{n,m}, n \in \mathbb{Z}_+^1, \Omega = \Omega^m \).

**General Result.** Let \( \Omega \subset \mathbb{R}^m, X_0 \subset \Omega, \) and \( G \subset \mathbb{R}^l \) \((m \in \mathbb{N}, l \in \mathbb{N})\) be compact sets.

We assume that there exists a function (operation) \( y = O(t, x) : G \times \Omega \rightarrow \Omega \), either satisfying the strong surjective condition (SSC), that is, for any \( y_0 \in \Omega \) there exist \( t_0 \in G \) and \( x_0 \in X_0 \) such that \( O(t_0, x_0) = y_0 \), or satisfying the very strong surjective condition (VSSC), that is, for any \( y_0 \in \Omega \) and \( x_0 \in X_0 \) there exists \( t_0 \in G \) such that \( O(t_0, x_0) = y_0 \).

We also assume that there exists a family of operators \( \{T_t(P, \Omega)\}_{t \in G} \), satisfying the following conditions:
\[
T_t(P, \Omega) : B \rightarrow B, \quad t \in G; \tag{2.1}
\]
\[
T_t(P, \Omega)(x_0) = P(O(t, x_0)), \quad t \in G, \; x_0 \in X_0, \; P \in B; \tag{2.2}
\]
\[
\|T_t(P, \Omega)\|_{L_{p,W}(\Omega)} \leq \|P\|_{L_{p,W}(\Omega)}, \quad t \in G, \; P \in B, \; p \in (0, \infty). \tag{2.3}
\]

Various examples of \( B, \Omega, X_0, G, O(t, x) \), and \( T_t(P, \Omega) \) are discussed below in Examples 2.2, 2.6, 2.8, and 2.10. The following general result is valid:

**Lemma 2.1.** Let conditions (2.1), (2.2), and (2.3) on \( \{T_t(P, \Omega)\}_{t \in G} \) be satisfied.

(a) If an operation \( O(t, x) \) satisfies the SSC, then there exist \( x_0 \in X_0 \) and an extremal polynomial \( P^* \in B \setminus \{0\} \) for \( \mathcal{N}(B, L_{p,W}(\Omega)) \) such that \( \|P^*\|_{L_\infty(\Omega)} = |P^*(x_0)| \).

(b) If an operation \( O(t, x) \) satisfies the VSSC, then given \( x_0 \in X_0 \), there exists an extremal polynomial \( P^* \in B \setminus \{0\} \) for \( \mathcal{N}(B, L_{p,W}(\Omega)) \) such that \( \|P^*\|_{L_\infty(\Omega)} = |P^*(x_0)| \).

**Proof.** The set \( \Omega \) is compact and \( B \) is a finite-dimensional subspace of continuous functions from \( L_{p,W}(\Omega) \). Hence the existence of an extremal polynomial \( P_0 \) for \( \mathcal{N}(B, L_{p,W}(\Omega)) \) can be proved by the standard compactness argument. Indeed, given \( d \in \mathbb{N} \), let \( P_d \in B \) satisfy the following relations:
\[
\|P_d\|_{L_\infty(\Omega)} = 1 \quad \text{and} \quad \mathcal{N}(P_{a,V}, L_{p,W}(\Omega)) < \|P_d\|_{L_\infty(\Omega)}/\|P_d\|_{L_p(\Omega)} + 1/d.
\]
Then there exist a nontrivial polynomial \( P_0 \in B \) and a sequence of polynomials \( \{P_d\}_{d=1}^\infty \subseteq \mathbb{N} \) such that \( \lim_{d \to \infty} P_d(x) = P_0(x) \) uniformly on \( \Omega \). Thus \( P_0 \) is an extremal polynomial for \( \mathcal{N}(B, L_{p,W}(\Omega)) \).
Next, assume that
\[ \|P_0\|_{L^\infty(\Omega)} = |P_0(y_0)|, \quad y_0 \in \Omega. \]  
(2.4)

If \(O(t, x)\) satisfies the SSC and \(y_0 \notin X_0\), then there exist \(t_0 \in G\) and \(x_0 \in X_0\) such that \(O(t_0, x_0) = y_0\). If \(O(t, x)\) satisfies the VSSC and \(y_0 \neq x_0\) for a given \(x_0 \in X_0\), then there exists \(t_0 \in G\) such that \(O(t_0, x_0) = y_0\).

Therefore, the function \(P^* := T_{t_0}(P_0, \Omega)\) belongs to \(B\) by (2.1), and \(P^*\) satisfies the condition
\[ |P^*(x_0)| = \|P_0\|_{L^\infty(\Omega)} \]  
(2.5)
by (2.2) and (2.4). Furthermore, by (2.3) and (2.5), \(\|P_0\|_{L^\infty(\Omega)}/\|P_0\|_{L^p(\Omega)} \leq |P^*(x_0)|/\|P^*\|_{L^p(\Omega)}\). Therefore, \(P^*\) is an extremal polynomial for \(\mathcal{N}(B, L^p_{p,W}(\Omega))\) and \(\|P^*\|_{L^\infty(\Omega)} = |P^*(x_0)|\). Thus the lemma is established.

\[ \square \]

**Examples.** In the capacity of sets \(B, \Omega, X_0, G\) and the operation \(O(t, x)\) we use in this paper the following objects:

**Example 2.2.** \(B = \mathcal{P}_n, n \in \mathbb{Z}_+^1, \Omega = [-1, 1], X_0 = \{-1, 1\}, G = [-1, 1], l = m = 1, O(t, x) = tx, t \in [-1, 1], x \in [-1, 1]\). The operation \(O(t, x)\) satisfies the VSSC since given \(y_0, |y_0| \leq 1,\) and given \(x_0, |x_0| = 1\), one can choose \(t_0 = y_0 \text{ sgn} x_0\).

**Example 2.3.** \(B = \mathcal{P}_{aV}, a \geq 0, \Omega = Q^m, X_0 = \{x \in Q^m : |x_j| = 1, 1 \leq j \leq m\}, G = Q^m, l = m, O(t, x) = (t_1 x_1, \ldots, t_m x_m), t \in Q^m, x \in Q^m\). The operation \(O(t, x)\) satisfies the VSSC since given \(y_0 = \{y_{0,1}, \ldots, y_{0,m}\}, |y_{0,j}| \leq 1, 1 \leq j \leq m\) and given \(x_0 = \{x_{0,1}, \ldots, x_{0,m}\}, |x_{0,j}| = 1, 1 \leq j \leq m\), one can choose \(t_0 = \{y_{0,1} \text{ sgn} x_{0,1}, \ldots, y_{0,m} \text{ sgn} x_{0,m}\}\).

**Example 2.4.** \(B = \mathcal{P}_{a,m}, n \in \mathbb{Z}_+^1, \Omega = B^m, X_0 = S^{m-1}, G = [-1, 1], l = 1, O(t, x) = tx, t \in [-1, 1], x \in B^m\). The operation \(O(t, x)\) satisfies the SSC since given \(y_0, |y_0| \leq 1\), one can choose \(t_0 = \pm |y_0| \text{ and } x_0 = \pm y_0/|y_0|, \) if \(y_0 \neq 0\), and \(x_0 \in S^{m-1}\), if \(y_0 = 0\).

**Generalized Translation Operators.** In the capacity of the family of operators \(T_t(P, \Omega)\) from Lemma 2.1, we use **generalized translation operators** (GTOs). The GTOs are linear operators of the form
\[ T_t(f, \Omega)(x) = \int_{\Omega^*} f(\Phi(t, x, s)) d\mu(s), \quad t \in G \subset \mathbb{R}^l, \quad x \in \Omega \subset \mathbb{R}^m, \]
where \(\mu\) is a probability measure on \(\Omega^* \subset \mathbb{R}^\nu, \nu \in \mathbb{N}\). Note that mostly \(\Omega^* = \Omega\) but there are exceptions (see Examples 2.5 and 2.8). The GTOs \(T_t(\cdot, \Omega)\) are generated by product formulae for orthogonal polynomials on \(\Omega\) with respect to a weighted measure.
Since the 1960s the numerous GTOs have been defined, studied, and applied to univariate and multivariate approximation by algebraic polynomials on various domains and surfaces (see, e.g., [5] [6] [33] [10] [49] [50] and references therein). Applications of the GTOs to analysis of sharp constants in univariate inequalities of different metrics were initiated by Arestov, Deikalova, and Rogozina [10] [11] [2].

Examples of the Gegenbauer-type multivariate GTOs that are needed for the proofs of Theorems 1.3, 1.5 and 1.6 are presented in Examples 2.6, 2.8 and 2.10. Note that the GTO from Example 2.6 is a new one. We also discuss the univariate Gegenbauer GTO in Example 2.5 to illustrate the transition to multivariate ones. This GTO is a special case of more complicated Examples 2.6, 2.8 and 2.10.

**Example 2.5. Univariate Gegenbauer GTO.** We first define sets $\Omega = [-1, 1]$, $X_0 = \{-1, 1\}$, $G = [-1, 1]$ and the operation $O(t, x) = tx$, $t \in [-1, 1]$, $x \in [-1, 1]$, from Example 2.2 with $O(t, x)$, satisfying the VSSC. In addition, let $B = \mathcal{P}_n$, $n \in \mathbb{Z}_+$. Next, given $\lambda \geq 0$ we define the weight $W(x) := (1 - x^2)^{\lambda - 1/2}$, $x \in [-1, 1]$, and the probability measure $\mu_{1, \lambda}(s)$ on $[-1, 1]$ by the formula

$$d\mu_{1, \lambda}(s) := \begin{cases} (1/C_{1}(\lambda)) \left(1 - s^2\right)^{\lambda - 1} ds, & \lambda > 0, \\ (1/2)d(\delta_1(s) + \delta_{-1}(s)), & \lambda = 0, \end{cases}$$

(2.6)

where $C_{1}(\lambda) := \pi^{1/2}\Gamma(\lambda)/\Gamma(\lambda + 1/2)$ and $\delta_b$ is the Dirac measure centered at $b \in [-1, 1]$.

Furthermore, we define the GTO by the formula

$$T_t(f, [-1, 1])(x) := \begin{cases} \frac{1}{C_{1}(\lambda)} \int_{-1}^{1} f \left(tx + s\sqrt{1-t^2}\sqrt{1-x^2}\right) \left(1 - s^2\right)^{-1} ds, & \lambda > 0, \\ \frac{1}{2} \left(f \left(tx + \sqrt{1-t^2}\sqrt{1-x^2}\right) + f \left(tx - \sqrt{1-t^2}\sqrt{1-x^2}\right)\right), & \lambda = 0, \end{cases}$$

(2.7)

$$= \int_{-1}^{1} f \left(tx + s\sqrt{1-t^2}\sqrt{1-x^2}\right) d\mu_{1, \lambda}(s),$$

where $t \in [-1, 1]$ and $x \in [-1, 1]$. Note that GTO (2.7) is generated by the classic product formula for the Gegenbauer polynomials $C_{n}^\lambda$, $n \in \mathbb{Z}_+$, $\lambda \geq 0$,

$$T_t \left(C_{n}^\lambda, [-1, 1]\right)(x) = C_{n}^\lambda(t)C_{n}^\lambda(x)/C_{n}^\lambda(1),$$

(2.8)

see [46] Sect. 11.5 for $\lambda > 0$, while for $\lambda = 0$ product formula (2.8) for the Chebyshev polynomials is trivial.

Next, $T_t(\cdot, [-1, 1]) : L_{p,W}([-1, 1]) \to L_{p,W}([-1, 1])$, $1 \leq p \leq \infty$, is a linear operator whose norm is 1 for all $t \in [-1, 1]$ (see [2] Lemmas 5, 6). This fact immediately implies condition (2.2), while condition (2.2) follows from (2.7). In addition, condition (2.1) is satisfied as well since by (2.8), $T_t(P, [-1, 1]) \in \mathcal{P}_n$ if $P \in \mathcal{P}_n$, $n \in \mathbb{Z}_+$. 


Therefore, applying Lemma 2.1 (b), we arrive at the existence of an extremal polynomial for \( \mathcal{N} \left( \mathcal{P}_n, L_{p,1-x^2}^{-1/2}([-1,1]) \right) \), \( p \in [1,\infty) \), \( \lambda \geq 0 \), whose uniform norm is attained at a fixed endpoint of \([-1,1]\). This result was obtained by Arestov and Deikalova [2] Theorem 1 by the ingenious trick of using GTO (2.7).

**Example 2.6. Gegenbauer-type GTO on a Cube.** We first define sets \( \Omega = Q^m, X_0 = \{ x \in Q^m : \| x \| = 1, 1 \leq j \leq m \} \), \( G = Q^m \) and the operation \( O(t,x) = (t_1 x_1, \ldots, t_m x_m) \), \( t \in Q^m, x \in Q^m \), from Example 2.3 with \( O(t,x) \), satisfying the VSSC. In addition, let \( B = \mathcal{P}_{aV} \), \( a \geq 0 \), where \( V \) satisfies the II-condition. Next, given \( \lambda = (\lambda_1, \ldots, \lambda_m), \lambda_j \geq 0, 1 \leq j \leq m \), we define the weight \( W(x) := \prod_{j=1}^m (1 - x_j^2)^{\lambda_j - 1/2} \), \( x \in Q^m \), and the probability measure \( \mu_{m, \lambda}(s) \) on \( Q^m \) by the formula

\[
d\mu_{m, \lambda}(s) := \prod_{j=1}^m d\mu_{1, \lambda_j}(s_j), \quad 1 \leq j \leq m, \text{ is defined by (2.6)}.
\]

Furthermore, we define the GTO by the formula

\[
T_t(f, Q^m)(x) := \prod_{j=1}^m T_j \left( f (y_j, \ldots, y_j), [-1,1] \right) (x_j)
\]

\[
= \int_{Q^m} f \left( \sum_{j=1}^m t_j \left( \sum_{j=1}^m y_j \sqrt{1 - \frac{t_j^2}{1 - x_j^2}} \right) \right) d\mu_{m, \lambda}(s), \quad (2.9)
\]

where \( t \in Q^m, x \in Q^m \), and \( T_j \left( h (y_j), [-1,1] \right) (x_j) = T_j \left( h, [-1,1] \right) (x_j) \), \( 1 \leq j \leq m, \) is defined by (2.7). Note that GTO (2.9) is generated by the product formula for the multivariate Gegenbauer-type polynomials \( C_n^\lambda(y) := \prod_{j=1}^m C_n^{\lambda_j}(y_j), n_j \in \mathbb{Z}_+, \lambda_j \geq 0, 1 \leq j \leq m, \)

\[
T_t \left( C_n^\lambda, Q^m \right) (x) = C_n^\lambda(t) C_n^\lambda(x) / \prod_{j=1}^m C_n^{\lambda_j}(1), \quad (2.10)
\]

which follows from (2.7), (2.8), and (2.9).

Next, \( T_t(\cdot, Q^m) : L_{p,V}(Q^m) \to L_{p,V}(Q^m), 1 \leq p \leq \infty, \) is a linear operator whose norm is 1 for all \( t \in Q^m \). This result follows from the univariate one [2] Lemmas 5, 6] and (2.9) by standard induction on \( m \). Thus condition (2.3) is satisfied, while condition (2.2) immediately follows from (2.9).

In addition, \( T_t(P, Q^m) \in \mathcal{P}_{aV} \) if \( P \in \mathcal{P}_{aV}, a \geq 0 \). Indeed, let

\[
c_k y^k = \sum_{0 \leq n_j \leq k_j, 1 \leq j \leq m} d_n C_n^\lambda(y), \quad k \in aV \cap \mathbb{Z}_+^m,
\]

be a monomial of \( P \). Then using product formula (2.10), we see that \( T_t(c_k y^k, Q^m)(x) \) is a polynomial of degree at most \( k_j \) in variable \( x_j, 1 \leq j \leq m \). Since the convex set \( V \) satisfies the II-condition, this polynomial belongs to \( \mathcal{P}_{aV} \). Then \( T_t(P, Q^m) \in \mathcal{P}_{aV} \) because \( T_t(\cdot, Q^m) \) is a linear operator. Thus condition (2.1) is satisfied.

Therefore, applying Lemma 2.1 (b), we arrive at the following lemma:
Lemma 2.7. There exists an extremal polynomial $P^*$ for $\mathcal{N}(\mathcal{P}_{\alpha V}, L_{p,\prod_{j=1}^{m}(1-x_j^2)^{\lambda_j-1/2}}(Q^m)), p \in [1,\infty), \lambda_j \geq 0, 1 \leq j \leq m$, whose uniform norm is attained at a fixed vertex $x_0$ of $Q^m$, that is, $\|P^*\|_{L_\infty(Q^m)} = |P^*(x_0)|$ for a given $x_0 \in X_0$.

Example 2.8. Chebyshev-type GTO on the Unit Ball. We first define sets $\Omega = \mathfrak{B}^m, X_0 = S^{m-1}, G = [-1,1]$ and the operation $O(t, x) = tx, t \in [-1,1], x \in \mathfrak{B}^m$, from Example 2.4 with $O(t, x)$, satisfying the SSC. In addition, let $B = \mathcal{P}_{n,m}, n \in \mathbb{Z}_{+}^1$. Next, we define the weight $W(x) := (1 - |x|^2)^{-1/2}, x \in \mathfrak{B}^m$, and the probability measure $\mu^*_m(s)$ on $S^{m-1}$ by the formula $d\mu^*_m(s) := (1/C_2(m))ds$, where $ds$ is the $(m-1)$-dimensional surface element of $S^{m-1}$ and $C_2(m) := |S^{m-1}|_{m-1} = 2\pi^{m/2}/\Gamma(m/2)$.

Furthermore, we define the GTO by the formula
\[
T_t(f, \mathfrak{B}^m)(x) := \int_{S^{m-1}} f \left( tx + \sqrt{1-t^2} sD(x)H(x) \right) d\mu^*_m(s),
\]
where $t \in [-1,1]$ and $x \in \mathfrak{B}^m$. Here, $s$ and $x$ are $m$-dimensional row vectors, $D(x)$ is the $m \times m$ diagonal matrix whose diagonal elements are $\sqrt{1-|x|^2}, 1, \ldots, 1$, and $H(x)$ is an $m \times m$ orthogonal matrix whose first row is $x/|x|$.

GTO (2.11) was defined by the author [15, 16] and it is generated by the product formula for a system of polynomials (eigenfunctions of $T_t(f, \mathfrak{B}^m)$)
\[
\Phi_{l,N}(x) := P_l(x)C_{2N}^{(m-1)/2}(\sqrt{1-|x|^2}), x \in \mathfrak{B}^m, l \in \mathbb{Z}_{+}, N \in \mathbb{Z}_{+}^1,
\]
in the form
\[
T_t(\Phi_{l,N}, \mathfrak{B}^m)(x) = C_{n}^{(m-1)/2}(t)\Phi_{l,N}(x)/C_{n}^{(m-1)/2}(1), \quad n = l + 2N,
\]
(see [16] Lemma 2]). Here, $P_l$ is a spherical harmonic of degree $l, l \in \mathbb{Z}_{+}^1$. In addition, the system \{\Phi_{l,N}\}_{l+2N\leq n} forms an orthogonal basis for $\mathcal{P}_{n,m}$ on $\mathfrak{B}^m$ with respect to the weight $W$, that is,
\[
P(x) = \sum_{0 \leq l+2N \leq n} c_{l,N}\Phi_{l,N}(x), \quad P \in \mathcal{P}_{n,m},
\]
(see [16] Lemma 1)).

Note that formula (2.11) for the Chebyshev-type GTO on $\mathfrak{B}^m$ follows from the classic formula for the GTO on the sphere $S^m$ (see, e.g., [47, 42]) since
\[
T_t(f, \mathfrak{B}^m)(x) = T_t(F, S^m)(x) := (C_2(m))^{-1}(1-t^2)^{-(m-1)/2}\int_{(\tilde{x},\tilde{y})=t} F(\tilde{y}) d\tilde{y} = (C_2(m))^{-1}\int_{S^m} F(t\tilde{x} + \sqrt{1-t^2 s}) d\tilde{s},
\]
where \( \tilde{x} = (x, x_{m+1}) \in S^m \), \( \tilde{y} = (y, y_{m+1}) \in S^m \), \( \tilde{s} = (s, s_{m+1}) \in S^m \), \( F(\tilde{y}) = f(y) \), and \( S_x^{m-1} \) is the intersection of \( S^m \) and the equatorial hyperplane orthogonal to the vector \( \tilde{x} \). Here \( x \), \( y \), and \( s \) are points of \( \mathcal{B}^m \).

Roughly speaking, the GTO \( T_t(f, \mathcal{B}^m) \) is the "projection" of the spherical GTO \( T_t(F, S^m) \) on \( \mathcal{B}^m \). So all major properties of \( T_t(f, \mathcal{B}^m) \) follow from the corresponding well-studied properties of \( T_t(F, S^m) \).

In particular, \( T_t(\cdot, \mathcal{B}^m) : L_{p,W}(\mathcal{B}^m) \to L_{p,W}(\mathcal{B}^m) \), \( 1 \leq p \leq \infty \), is a linear operator whose norm is 1 for all \( t \in [-1, 1] \). Indeed, it is known [3] Lemma 4.2.2] that for \( F \in L_p(S^m) \), \( 1 \leq p \leq \infty \),

\[
\|T_t(F, S^m)\|_{L_p(S^m)} \leq \|F\|_{L_p(S^m)}. \tag{2.16}
\]

Choosing \( F(\tilde{y}) = f(y), \tilde{y} \in S^m, y \in \mathcal{B}^m \), we obtain from (2.15) and (2.16)

\[
\|T_t(f, \mathcal{B}^m)\|_{L_{p,W}(\mathcal{B}^m)} = \|T_t(F, S^m)\|_{L_p(S^m)} \leq \|F\|_{L_p(S^m)} = \|f\|_{L_{p,W}(\mathcal{B}^m)}. \tag{2.17}
\]

Equality in (2.17) holds true for a constant \( f \), so the norm of \( T_t(\cdot, \mathcal{B}^m) \) is 1. Then condition (2.3) is satisfied, while condition (2.2) immediately follows from (2.11).

In addition to (2.17), note that relations (2.13) and (2.14) for polynomials (2.12) follow from the corresponding properties of spherical harmonics (see [33, Sect. 2] and [44, Theorem 4.2.1], respectively). Thus by (2.13) and (2.14), \( T_t(P, \mathcal{B}^m) \in \mathcal{P}_{n,m} \) if \( P \in \mathcal{P}_{n,m}, n \in \mathbb{Z}_+^1 \). Then condition (2.1) is satisfied.

Therefore, applying Lemma 2.1 (a), we arrive at the following lemma:

**Lemma 2.9.** There exists an extremal polynomial \( P^* \) for \( \mathcal{N}(\mathcal{P}_{n,m}, L_{p,(1-|x|^2)^{-1/2}}(\mathcal{B}^m)) \), \( p \in [1, \infty) \), whose uniform norm is attained on the sphere \( S_{x}^{m-1} \), that is, there exists \( x_0 \in S_{x}^{m-1} \) such that \( \|P^*\|_{L_{\infty}(\mathcal{B}^m)} = |P^*(x_0)| \).

**Example 2.10. Gegenbauer-type GTO on the Unit Ball.** We first define sets \( \Omega = \mathcal{B}^m, X_0 = S_{x}^{m-1}, G = [-1, 1] \) and the operation \( O(t, x) = tx, t \in [-1, 1], x \in \mathcal{B}^m \), from Example 2.4 with \( O(t, x) \), satisfying the SSC. In addition, let \( B = \mathcal{P}_{n,m}, n \in \mathbb{Z}_+^1 \). Next, given \( \lambda > 0 \) we define the weight \( W(x) := (1 - |x|^2)^\lambda^{-1/2}, x \in \mathcal{B}^m \), and the probability measure \( \mu_{m,\lambda}^*(s) \) on \( \mathcal{B}^m \) by the formula \( d\mu_{m,\lambda}^*(s) := (1/C_3(m, \lambda))(1 - |s|^2)^\lambda^{-1}ds \), where \( C_3(m, \lambda) := \pi^m/\Gamma(\lambda)/\Gamma(\lambda + m/2) \).

Furthermore, we define the GTO by the formula

\[
T_t(f, \mathcal{B}^m)(x) := \int_{\mathcal{B}^m} f \left( tx + \sqrt{1-t^2} s D(x) H(x) \right) d\mu_{m,\lambda}^*(s), \tag{2.18}
\]

where \( t \in [-1, 1] \) and \( x \in \mathcal{B}^m \). Here, \( s \) and \( x \) are \( m \)-dimensional row vectors, and \( D(x) \) and \( H(x) \) are the same matrices as in (2.11).
GTO (2.18) was defined by Xu [50, Corollary 3.7], and it is generated by the product formula for the generalized Gegenbauer polynomials \( C_n^{\lambda+(m-1)/2}(v, y) \), \( v \in \mathfrak{B}^m \), \( y \in S^{m-1} \), \( n \in \mathbb{Z}_+\), \( \lambda > 0 \),

\[
T_t \left( C_n^{\lambda+(m-1)/2}(\cdot, y), \mathfrak{B}^m \right)(x) = C_n^{\lambda+(m-1)/2}(t)C_n^{\lambda+(m-1)/2}((x, y))/C_n^{\lambda+(m-1)/2}(1) \tag{2.19}
\]

for all \( y \in S^{m-1} \) (see [50, p. 500]).

Next, \( T_t(\cdot, \mathfrak{B}^m) : L_p, w(\mathfrak{B}^m) \to L_p, w(\mathfrak{B}^m) \), \( 1 \leq p \leq \infty \), is a linear operator whose norm is 1 for all \( t \in [-1, 1] \) (see [50, Proposition 3.4(5)]). This fact immediately implies condition (2.3), while condition (2.2) follows from (2.18). In addition, by (2.19), \( T_t(P, \mathfrak{B}^m) \in \mathcal{P}_{n,m} \) if \( P \in \mathcal{P}_{n,m} \), \( n \in \mathbb{Z}_+ \) (see [50, Proposition 3.4(3)]). Then condition (2.1) is satisfied.

Therefore, applying Lemma 2.1 (a), we arrive at the following lemma:

**Lemma 2.11.** There exists an extremal polynomial \( P^* \) for \( \mathcal{N} \left( \mathcal{P}_{n,m}, L_{p, (1-x^2)^{\lambda-1/2}}(\mathfrak{B}^m) \right) \), \( p \in [1, \infty) \), \( \lambda > 0 \), whose uniform norm is attained on the sphere \( S^{m-1} \), that is, there exists \( x_0 \in S^{m-1} \) such that \( \|P^*\|_{L_\infty(\mathfrak{B}^m)} = |P^*(x_0)| \).

**Remark 2.12.** Xu [49, 48] introduced a weighted version of the spherical GTO \( T_t(F, S^m) \) defined by (2.15). Note that GTO (2.18), similarly to the Chebyshev-type GTO (2.11), is the ”projection” of the weighted spherical GTO on \( \mathfrak{B}^m \). In addition, note that, in a sense, GTO (2.18) is a limit version of GTO (2.11) as \( \lambda \to 0^+ \) (cf. [50, Sect. 3.3]).

### 3. The \( m \)-dimensional Cube \( Q^m \). Proof of Theorem 1.4

Throughout the section we assume that \( V \subset \mathbb{R}^m \) satisfies the II-condition and \( \alpha_j \geq 0, \beta_j \geq -1/2, 1 \leq j \leq m \). The proof of Theorem 1.4 is based on four lemmas below. We start with three standard properties of multivariate entire functions of exponential type.

**Lemma 3.1.** (a) If \( f \in B_V \), then there exists \( M = M(V) > 0 \) such that \( f \in B_{Q^m(M)} \).

(b) The following crude Nikolskii-type inequalities hold true:

\[
\|f\|_{L_\infty(\mathbb{R}^m)} \leq C_4 \|f\|_{L_p(\mathbb{R}^m)}, \quad f \in B_V \cap L_p(\mathbb{R}^m), \quad p \in (0, \infty), \tag{3.1}
\]

\[
\|f\|_{L_\infty(\mathbb{R}^m)} \leq C_5 \|f\|_{L_p, \prod_{j=1}^m |t_j|^{\alpha_j}(\mathbb{R}^m)}, \quad f \in B_V \cap L_p, \prod_{j=1}^m |t_j|^{\alpha_j}(\mathbb{R}^m), \quad p \in (0, \infty), \tag{3.2}
\]

where \( C_4 \) and \( C_5 \) are independent of \( f \).

(c) For any sequence \( \{f_n\}_{n=1}^\infty \), \( f_n \in B_V \cap L_\infty(\mathbb{R}^m), n \in \mathbb{N} \), with \( \sup_{n \in \mathbb{N}} \|f_n\|_{L_\infty(\mathbb{R}^m)} = C \), there exists a subsequence \( \{f_{n_d}\}_{d=1}^\infty \) and a function \( f_0 \in B_V \cap L_\infty(\mathbb{R}^m) \) such that \( \lim_{d \to \infty} f_{n_d} = f_0 \) uniformly on any compact set in \( \mathbb{R}^m \).
Thus (3.2) is established for $E$. It remains to show that $E$ is \((L,\delta)\)-dense for certain $L$ and $\delta$. It is clear by contradiction that the following relation holds true:

\[
E^c := \left\{ t \in \mathbb{R}^m : \prod_{j=1}^m|t_j|^{\alpha_j} < 1 \right\} \subseteq \bigcup_{j=1}^m E_j, \tag{3.4}
\]

where $E_j := \{ t \in \mathbb{R}^m : |t_j| \leq 1 \}, 1 \leq j \leq m$. It follows from (3.4) that for any cube $Q_{y,L}$ with $L > 2m$ and $y \in \mathbb{R}^m$,

\[
|Q_{y,L} \cap E^c|_m \leq \sum_{j=1}^m |Q_{y,L} \cap E_j|_m \leq 2mL^{m-1}, \quad |Q_{y,L} \cap E|_m \geq (L - 2m)L^{m-1}.
\]

Thus $E$ is \((2m + 1, (2m + 1)^{m-1})\)-dense, and the lemma is established. \(\square\)

In the next lemma we discuss the error of polynomial approximation for functions from $B_V$.

**Lemma 3.2.** For any $F \in B_V \cap L_\infty(\mathbb{R}^m)$, $\tau \in (0, 1)$, and $a \geq 1$, there is a polynomial $P_a \in P_{aV}$ such that for $r \in (0, \infty]$,

\[
\lim_{a \to \infty} \|F - P_a\|_{L_{r,\prod_{j=1}^m|t_j|^{\alpha_j}(1-(r_j/(a\tau))^2)^{\beta_j}(Q^m(\tau a))}} = 0. \tag{3.5}
\]

**Proof.** A nonweighted version of Lemma 3.2 was proved in [23, Lemma 2.7]. In particular, the following inequality holds true for any $F \in B_V \cap L_\infty(\mathbb{R}^m)$, $\tau \in (0, 1)$, and a certain $P_a \in P_{aV}$, $a \geq 1$ (see [23, Eq. (2.28))):

\[
\|F - P_a\|_{L_\infty(Q^m(\tau a))} \leq C_7(\tau, m, V) a^{\frac{2m}{a\tau}} \exp[-C_8(\tau, m, V) a] \|F\|_{L_\infty(\mathbb{R}^m)}. \tag{3.6}
\]
Using (3.6), we obtain
\[
\| F - P_a \|_{L^r_{\pi_{j=1}^m |t_j|^{\alpha_j}(1-(t_j/\alpha_j)^2)^{\beta_j}(Q^m(\alpha_j))}} \\
\leq C_7 \left( \prod_{j=1}^m B(\alpha_j/2 + 1/2, \beta_j + 1) \right)^{1/r} \frac{a}{m+1} + (m+\sum_{j=1}^m \alpha_j)^{1/r} \exp[-C_8 a] \| F \|_{L^\infty(\mathbb{R}^m)}. \tag{3.7}
\]

Then (3.5) follows from (3.7).

A certain inequality of different weighted metrics for multivariate polynomials is discussed in the following lemma.

**Lemma 3.3.** Given \( a \geq 1, M > 0, p \in (0, \infty), \tau \in (2/3, 1), \) and \( P \in P_a V, \) the following inequality holds true:
\[
\| P \|_{L^\infty(Q^m(M))} \leq C_9 (a/M)^{(m+\sum_{j=1}^m \alpha_j)/p} \| P \|_{L^p_p(\pi_{j=1}^m |t_j|^{\alpha_j}(1-(t_j/\alpha_j)^2)^{\beta_j}(Q^m(M)))}, \tag{3.8}
\]
where \( C_9 \) is independent of \( a, M, \) and \( P. \)

**Proof.** Inequality (3.8) for \( \alpha_j = \beta_j = 0, 1 \leq j \leq m, \) was proved in Lemma 2.9. Then setting \( \alpha_j = \beta_j = 0, 1 \leq j \leq m, \) and replacing \( M \) by \((1-\varepsilon/2)M\) and \( \tau \) by \((1-\varepsilon)/(1-\varepsilon/2), \) \( \varepsilon \in (0, 1/2), \) in (3.8), we obtain
\[
\| P \|_{L^\infty(Q^m((1-\varepsilon)/M))} \leq C_{10} (a/M)^{m/p} \| P \|_{L^p_p(Q^m((1-\varepsilon)/2)M)} \tag{3.9}
\]

Therefore, (3.8) follows from (3.9) and from the following inequalities:
\[
\| P \|_{L^p_p(Q^m((1-\varepsilon)/2)M))} \leq C_{11} (a/M)^{\sum_{j=1}^m \alpha_j/p} \| P \|_{L^p_p(\pi_{j=1}^m |t_j|^{\alpha_j}(Q^m((1-\varepsilon)/2)M))} \\
\leq C_{12} (a/M)^{\sum_{j=1}^m \alpha_j/p} \| P \|_{L^p_p(\pi_{j=1}^m |t_j|^{\alpha_j}(1-(t_j/\alpha_j)^2)^{\beta_j}(Q^m((1-\varepsilon)/2)M))}, \tag{3.10}
\]
where \( C_{10}, C_{11}, \) and \( C_{12} \) in (3.9) and (3.10) are independent of \( a, M, \) and \( P. \) The second inequality in (3.10) is all but trivial. To prove the first inequality in (3.10), we observe that
\[
\| P \|_{L^p_p(\pi_{j=1}^m |t_j|^{\alpha_j}(Q^m((1-\varepsilon)/2)M))} \geq \| P \|_{L^p_p(\pi_{j=1}^m |t_j|^{\alpha_j}(Q^m((1-\varepsilon)/2)M)\setminus Q^m(CM/a))} \\
\geq (CM/a)^{\sum_{j=1}^m \alpha_j/p} \| P \|_{L^p_p(Q^m((1-\varepsilon)/2)M)\setminus Q^m(CM/a))}, \tag{3.11}
\]
where \( C \in (0, 1/3) \) is a fixed number. Next, we note that \( 0 < C/a < 1/3 < 1-\varepsilon, \) so by (3.9),
\[
\| P \|_{L^p_p(Q^m(CM/a))} \leq (2CM/a)^{m/p} \| P \|_{L^\infty((Q^m(CM/a)))} \leq (2C)^{m/p} C_{10} \| P \|_{L^p_p(Q^m((1-\varepsilon)/2)M)}. \tag{3.12}
\]
Choosing now $C := \min\{1/3, C_{10}^{-p/m}, 2^{-1/m-1}\}$, we obtain from (3.12)

$$\|P\|_{L_p(Q^m((1-\varepsilon/2)M))} = \|P\|_{L_p(Q^m((1-\varepsilon/2)M))} - \|P\|_{L_p(Q^m(CM/a))} \geq (1 - (2C)^m C_{10}) \|P\|_{L_p(Q^m((1-\varepsilon/2)M))} \geq (1/2) \|P\|_{L_p(Q^m((1-\varepsilon/2)M))}. \quad (3.13)$$

Finally, combining (3.11) and (3.13), we arrive at the first inequality in (3.10). Thus (3.8) is established.

We also need the following technical lemma.

**Lemma 3.4.** The following estimates hold true ($\tau \in (0, 1)$, $v_j \in [-\pi/2, \pi/2]$, $1 \leq j \leq m)$:

$$0 \leq \prod_{j=1}^{m} |v_j|^{\alpha_j} - \prod_{j=1}^{m} \sin v_j^{\alpha_j} \left( \tau^2 \cos^2 v_j + 1 - \tau^2 \right)^{\beta_j} \cos v_j \leq C_{13} \prod_{j=1}^{m} |v_j|^{\alpha_j} \sum_{j=1}^{m} v_j^2, \quad (3.14)$$

where $C_{13} := \max\{1, \alpha_1, \ldots, \alpha_m, 2\beta_1 + 1, \ldots, 2\beta_m + 1\}$. 

**Proof.** The proof is based on the following elementary inequalities

$$1 - \theta^\gamma \leq \max\{1, \gamma\}(1 - \theta), \quad \theta \in [0, 1], \quad \gamma \geq 0; \quad 1 - \sin \theta/\theta \leq \theta^2/6, \quad \theta \in [0, \pi/2];$$

$$(\cos \theta)^{2\gamma_1 + 1} \leq \left( \tau^2 \cos^2 \theta + 1 - \tau^2 \right)^{\gamma} \cos \theta \leq (\cos \theta)^{2\gamma_2 + 1} \leq 1, \quad \theta \in [0, \pi/2], \tau \in (0, 1), \gamma \geq -1/2;$$

where $\gamma_1 := \max\{\gamma, 0\}$ and $\gamma_2 := \min\{\gamma, 0\}$. In particular, the left inequality in (3.14) immediately follows from the elementary inequality $\left( \tau^2 \cos^2 \theta + 1 - \tau^2 \right)^{\gamma} \cos \theta \leq 1, \theta \in [0, \pi/2]$, given above.

Next, for $v_j \in [0, \pi/2]$, $1 \leq j \leq m$, we obtain

$$\prod_{j=1}^{m} v_j^{\alpha_j} - \prod_{j=1}^{m} (\sin v_j)^{\alpha_j} \left( \tau^2 \cos^2 v_j + 1 - \tau^2 \right)^{\beta_j} \cos v_j \leq \prod_{j=1}^{m} v_j^{\alpha_j} \left( 1 - \prod_{j=1}^{m} \left( \frac{\sin v_j}{v_j} \right)^{\alpha_j} \right) + \prod_{j=1}^{m} \left( \tau^2 \cos^2 v_j + 1 - \tau^2 \right)^{\beta_j} \cos v_j. \quad (3.15)$$

Using now the easy identity

$$1 - \prod_{j=1}^{m} D_j = \sum_{j=1}^{m} (1 - D_j) \prod_{l=j+1}^{m} D_l, \quad D_j \in \mathbb{R}^1, \quad 1 \leq j \leq m,$$
where \( \prod_{l=j+1}^{m} : = 1 \) for \( m < j + 1 \), and, in addition, using the elementary inequalities given above, we obtain

\[
\begin{align*}
1 - \prod_{j=1}^{m} \left( \frac{\sin v_j}{v_j} \right)^{\alpha_j} &= \sum_{j=1}^{m} \left( 1 - \left( \frac{\sin v_j}{v_j} \right)^{\alpha_j} \right) \prod_{l=j+1}^{m} \left( \frac{\sin v_l}{v_l} \right)^{\alpha_l} \\
&\leq \sum_{j=1}^{m} \left( 1 - \left( \frac{\sin v_j}{v_j} \right)^{\alpha_j} \right) \leq (1/6) \max\{1, \alpha_1, \ldots, \alpha_m\} \sum_{j=1}^{m} v_j^2; \\
&= \sum_{j=1}^{m} \left( 1 - \left( \tau^2 \cos^2 v_j + 1 - \tau^2 \right)^{\beta_j} \cos v_j \right) \prod_{l=j+1}^{m} \left( \tau^2 \cos^2 v_l + 1 - \tau^2 \right)^{\beta_l} \cos v_l \\
&\leq \sum_{j=1}^{m} \left( 1 - \left( \tau^2 \cos^2 v_j + 1 - \tau^2 \right)^{\beta_j} \cos v_j \right) \leq \sum_{j=1}^{m} \left( 1 - \left( \cos v_j \right)^{2 \max\{\beta_j, 0\} + 1} \right) \\
&\leq (1/2) \max\{1, 2\beta_1 + 1, \ldots, 2\beta_m + 1\} \sum_{j=1}^{m} v_j^2.
\end{align*}
\]

Thus the right inequality in (3.14) follows from (3.15), (3.16), and (3.17).

**Proof of Theorem 1.4.** Throughout the proof we use the notation \( \tilde{p} = \min\{1, p\}, p \in (0, \infty) \), introduced in Section 4.

**Step 1.** We first prove the inequality

\[
\begin{align*}
\mathcal{N}_0 \left( B_V \cap L_{p, \prod_{j=1}^{m} |t_j|^{\alpha_j}}(\mathbb{R}^m), L_{p, \prod_{j=1}^{m} |t_j|^{\alpha_j}}(\mathbb{R}^m) \right) \\
&\leq \liminf_{a \to \infty} a^{-(m+\sum_{j=1}^{m} \alpha_j)/p} \mathcal{N}_0 \left( P a V, L_{p, \prod_{j=1}^{m} |t_j|^{\alpha_j}}(1-t_j)^{\beta_j}(Q^m) \right). 
\end{align*}
\]

(3.18)

Let \( f \) be any function from \( B_V \cap L_{p, \prod_{j=1}^{m} |t_j|^{\alpha_j}}(\mathbb{R}^m), p \in (0, \infty) \), and let \( \tau \in (0, 1) \) be a fixed number. Then \( f \in L_{\infty}(\mathbb{R}^m) \) by Nikolskii-type inequality (3.2).

In addition, we need one more function related to \( f \). Let \( \gamma = \gamma(V) > 0 \) be a smallest number such that \( Q^m \subseteq \gamma V \). Given \( \varepsilon \in (0, 1/(2\gamma)) \), we define an \( L_{p, \prod_{j=1}^{m} |t_j|^{\alpha_j}}(\mathbb{R}^m) \)- version of \( f \) by

\[
F(t) = F_\varepsilon(t) := f((1 - \gamma\varepsilon)t) \left( \prod_{j=1}^{m} \frac{\sin (\varepsilon t_j/d)}{\varepsilon t_j/d} \right)^d,
\]

(3.19)

where \( d := \lceil (m + \sum_{j=1}^{m} \alpha_j)/p \rceil + 1 \). Then \( F \) has exponential type \((1 - \gamma\varepsilon)V + \varepsilon Q^m \subseteq (1 - \gamma\varepsilon)V + \gamma\varepsilon V = V \). Therefore, \( F \in B_V \cap L_{\infty}(\mathbb{R}^m) \) and, in addition,

\[
\|F\|_{L_{p, \prod_{j=1}^{m} |t_j|^{\alpha_j}}(\mathbb{R}^m)} \leq (1 - \gamma\varepsilon)^{-(m+\sum_{j=1}^{m} \alpha_j)/p} \|f\|_{L_{p, \prod_{j=1}^{m} |t_j|^{\alpha_j}}(\mathbb{R}^m)}.
\]

(3.20)
Next, we use polynomials $P_a \in \mathcal{P}_{aV}$, $a \geq 1$, from Lemma 3.2 such that for $r = \infty$ or $r = p, p \in (0, \infty)$, the following relations hold true by (3.5):

$$
\lim_{a \to \infty} \| F - P_a \|_{L \infty(Q^m(\alpha r))} = 0, \quad \lim_{a \to \infty} \| F - P_a \|_{L^p \prod_{j=1}^m |t_j|^a_j (1 - (t_j/\alpha r)^2)^{\beta_j}}(Q^m(\alpha r)) = 0.
$$

(3.21)

Using the first limit equality of (3.21) and the definition of the sharp constant given by (1.2), we obtain

$$
|f(0)| = |F(0)| \leq \lim_{a \to \infty} |F(0) - P_a(0)| + \liminf_{a \to \infty} |P_a(0)| = \liminf_{a \to \infty} |P_a(0)|
$$

$$
\leq \liminf_{a \to \infty} N_0 \left( \mathcal{P}_{aV}, L_{p \prod_{j=1}^m |t_j|^a_j (1 - (t_j/\alpha r)^2)^{\beta_j}}(Q^m(\alpha r)) \right)
$$

$$
\times \limsup_{a \to \infty} \| P_a \|_{L^p \prod_{j=1}^m |t_j|^a_j (1 - (t_j/\alpha r)^2)^{\beta_j}}(Q^m(\alpha r))
$$

$$
= \liminf_{a \to \infty} (a + \sum_{j=1}^m \alpha_j)/p N_0 \left( \mathcal{P}_{aV}, L_{p \prod_{j=1}^m |t_j|^a_j (1 - (t_j/\alpha r)^2)^{\beta_j}}(Q^m) \right)
$$

$$
\times \limsup_{a \to \infty} \| P_a \|_{L^p \prod_{j=1}^m |t_j|^a_j (1 - (t_j/\alpha r)^2)^{\beta_j}}(Q^m(\alpha r))
$$

(3.22)

It remains to estimate the last line in (3.22). Using \textquoteleft{}triangle\textquoteright{} inequality (1.1) and the second relations of (3.21), we have

$$
\limsup_{a \to \infty} \| P_a \|_{L^\beta \prod_{j=1}^m |t_j|^a_j (1 - (t_j/\alpha r)^2)^{\beta_j}}(Q^m(\alpha r))
$$

$$
\leq \lim_{a \to \infty} \| F - P_a \|_{L^\beta \prod_{j=1}^m |t_j|^a_j (1 - (t_j/\alpha r)^2)^{\beta_j}}(Q^m(\alpha r))
$$

$$
+ \limsup_{a \to \infty} \| F \|_{L^\beta \prod_{j=1}^m |t_j|^a_j (1 - (t_j/\alpha r)^2)^{\beta_j}}(Q^m(\alpha r))
$$

$$
= \limsup_{a \to \infty} P^\beta(a) := \limsup_{a \to \infty} \| F \|_{L^\beta \prod_{j=1}^m |t_j|^a_j (1 - (t_j/\alpha r)^2)^{\beta_j}}(Q^m(\alpha r))
$$

(3.23)

Furthermore, we prove the estimate

$$
\limsup_{a \to \infty} I(a) \leq (1 - \gamma \varepsilon)^{- \sum_{j=1}^m \alpha_j} /p \| f \|_{L^p \prod_{j=1}^m |t_j|^a_j (\mathbb{R}^m)}.
$$

(3.24)

Indeed, setting $\Omega_\nu(A) := \{ t \in \mathbb{R}^m : |t_\nu| \geq A \}$, $A > 0$, and using the estimate

$$
|F(t)| \leq \| f \|_{L_{\infty}(\mathbb{R}^m)} (\varepsilon |t_\nu|/d)^{-d}, \quad t \in \Omega_\nu(A),
$$

that follows from (3.19), we obtain for any $\delta \in (0, 1)$

$$
\limsup_{a \to \infty} \| F \|_{L^p \prod_{j=1}^m |t_j|^a_j (1 - (t_j/\alpha r)^2)^{\beta_j}}(Q^m(\alpha r) \cap \Omega_\nu(\delta r))
$$

$$
\leq C_{14} \| f \|_{L_{\infty}(\mathbb{R}^m)} \limsup_{a \to \infty} a^{(m + \sum_{j=1}^m \alpha_j)/p - d} = 0, \quad 1 \leq \nu \leq m,
$$

(3.25)
where
\[ C_{14} \leq \left( \prod_{j=1}^{m} B(\alpha_j/2 + 1/2, \beta_j + 1) \right)^{1/p} \tau^{(m + \sum_{j=1}^{m} \alpha_j)/p} (\varepsilon \tau/d)^{-d}. \]

Next, without loss of generality we assume that there exists \( l \in \mathbb{Z}_+^1, 0 \leq l \leq m, \) such that \(-1/2 \leq \beta_j < 0, 1 \leq j \leq l,\) and \( \beta_j \geq 0, l + 1 \leq j \leq m.\) Then it follows from (3.25) and (3.20) that
\[
\limsup_{a \to \infty} \|F\|_{L_{p, \Pi_{j=1}^{m} |t_j|^{\alpha_j}}}^p \leq \limsup_{a \to \infty} \sum_{\nu=1}^{m} \|F\|_{L_{p, \Pi_{j=1}^{m} |t_j|^{\alpha_j}}}^p (1 - (t_{j}/(ar))^2)^{\frac{\beta_j}{p}} (Q^m(\delta a \tau)) \leq \|f\|_{L_{p, \Pi_{j=1}^{m} |t_j|^{\alpha_j}}}^p (Q^m(\delta a \tau)).
\]

Letting \( \delta \to 0^+ \) in the last line of (3.26), we arrive at (3.24) from (3.26). Combining (3.22), (3.23), and (3.24), we obtain
\[
\frac{|f(0)|}{\|f\|_{L_{p, \Pi_{j=1}^{m} |t_j|^{\alpha_j}}}^p (Q^m)} \leq \left[ (1 - \gamma \varepsilon) \tau \right]^{-\left( m + \sum_{j=1}^{m} \alpha_j \right)/p}
\times \liminf_{a \to \infty} a^{-\left( m + \sum_{j=1}^{m} \alpha_j \right)/p} \mathcal{N}_0\left( \mathcal{P}_{a \nu}, L_{p, \Pi_{j=1}^{m} |t_j|^{\alpha_j}} (Q^m) \right).
\]

Finally, letting \( \varepsilon \to 0^+ \) and \( \tau \to 1^- \) in (3.27), we complete the proof of (3.18).

**Step 2.** Furthermore, we will prove the inequality
\[
\limsup_{a \to \infty} a^{-\left( m + \sum_{j=1}^{m} \alpha_j \right)/p} \mathcal{N}_0\left( \mathcal{P}_{a \nu}, L_{p, \Pi_{j=1}^{m} |t_j|^{\alpha_j}} (Q^m) \right) \leq \mathcal{N}_0\left( B_V \cap L_{p, \Pi_{j=1}^{m} |t_j|^{\alpha_j}} (Q^m), L_{p, \Pi_{j=1}^{m} |t_j|^{\alpha_j}} (Q^m) \right),
\]
by constructing a nontrivial function \( f_0 \in B_V \cap L_{p, \Pi_{j=1}^{m} |t_j|^{\alpha_j}} (Q^m) \) such that
\[
\limsup_{a \to \infty} a^{-\left( m + \sum_{j=1}^{m} \alpha_j \right)/p} \mathcal{N}_0\left( \mathcal{P}_{a \nu}, L_{p, \Pi_{j=1}^{m} |t_j|^{\alpha_j}} (Q^m) \right) \leq \frac{|f_0(0)|}{\|f_0\|_{L_{p, \Pi_{j=1}^{m} |t_j|^{\alpha_j}}} (Q^m)} \leq \mathcal{N}_0\left( B_V \cap L_{p, \Pi_{j=1}^{m} |t_j|^{\alpha_j}} (Q^m), L_{p, \Pi_{j=1}^{m} |t_j|^{\alpha_j}} (Q^m) \right).
\]

Then inequalities (3.18) and (3.28) imply (1.15). In addition, \( f_0 \) is an extremal function for \( \mathcal{N}_0\left( B_V \cap L_{p, \Pi_{j=1}^{m} |t_j|^{\alpha_j}} (Q^m), L_{p, \Pi_{j=1}^{m} |t_j|^{\alpha_j}} (Q^m) \right), \) that is, (1.16) is valid.
It remains to construct a nontrivial function \( f_0 \), satisfying \((3.29)\). We first note that
\[
\inf_{a \geq 1} a^{-\left(m+\sum_{j=1}^m \alpha_j\right)/p} \mathcal{N}_0 \left( \mathcal{P}_{aV}, L_{p,\prod_{j=1}^m |t_j|^{\alpha_j}(1-t_j^2)^{\beta_j}(Q^m)} \right) \geq C_{15},
\] (3.30)
where \( C_{15} \) is independent of \( a \). This inequality follows immediately from \((3.18)\).

Let \( U_a \in \mathcal{P}_{aV} \) be a polynomial, satisfying the equality
\[
\mathcal{N}_0 \left( \mathcal{P}_{aV}, L_{p,\prod_{j=1}^m |t_j|^{\alpha_j}(1-t_j^2)^{\beta_j}(Q^m)} \right) = \frac{|U_a(0)|}{\|U_a\|_{L_{p,\prod_{j=1}^m |t_j|^{\alpha_j}(1-t_j^2)^{\beta_j}(Q^m)}}}, \quad a \geq 1.
\] (3.31)
The existence of an extremal polynomial \( U_a \) in \((3.31)\) can be proved by the standard compactness argument (see, e.g., [25, Proof of Theorem 1.5] and [21, Proof of Theorem 1.3]). Next, setting \( P_a(t) := U_a\left(t/a\right) \), we have from \((3.31)\) that
\[
\inf_{a \geq 1} a^{-\left(m+\sum_{j=1}^m \alpha_j\right)/p} \mathcal{N}_0 \left( \mathcal{P}_{aV}, L_{p,\prod_{j=1}^m |t_j|^{\alpha_j}(1-t_j^2)^{\beta_j}(Q^m)} \right) \geq \frac{|P_a(0)|}{\|P_a\|_{L_{p,\prod_{j=1}^m |t_j|^{\alpha_j}(1-t_j^2)^{\beta_j}(Q^m)}}},
\] (3.32)
since we can assume that \( |P_a(0)| = 1 \).

Then it follows from \((3.32)\) and \((3.30)\) that
\[
\sup_{a \geq 1} \|P_a\|_{L_{p,\prod_{j=1}^m |t_j|^{\alpha_j}(1-t_j^2)^{\beta_j}(Q^m)}} \leq 1/C_{15}.
\] (3.33)

Hence using Lemma \(3.3\) for \( M = a \) and \( \tau \in (2/3, 1) \), we obtain the estimate
\[
\sup_{a \geq 1} \|P_a\|_{L_{\infty}(Q^m(a\tau))} \leq C_9/C_{15} = C_{16}.
\] (3.34)

Furthermore, given \( a \geq 1 \) and \( \tau \in (2/3, 1) \) we define the trigonometric polynomial
\[
R_{a,\tau}(y) := P_a(a\tau \sin(y_1/(a\tau)), \ldots, a\tau \sin(y_m/(a\tau))), \quad y \in \mathbb{R}^m,
\] (3.35)
which is a periodic version of \( P_a \). Then \( R_{a,\tau} \) satisfies the following properties:

(P1) \( R_{a,\tau} \in B_{(1/\tau)V} \).

(P2) The following relations hold true:
\[
\sup_{a \geq 1} \|R_{a,\tau}\|_{L_{\infty}(Q^m(a\tau\pi/2))} = \sup_{a \geq 1} \|R_{a,\tau}\|_{L_{\infty}(\mathbb{R}^m)} \leq C_{16}.
\] (3.36)

(P3) \( R_{a,\tau}(0) = P_a(0) \).
Thus inequality (3.37) of property (P4) for $a, \tau \in (2/3, 1)$, $p \in (0, \infty)$, and $M \in (0, a\tau/\sqrt{mC_{13}})$,

$$
\|P_a\|_{L_p,\Pi_j^m}|_{t_j}^{\alpha_j}, (1-(t_j/\alpha)^2)^{\alpha_j}(Q^m(a)) \\
\geq (1 - C_{13}mM^2(\alpha)^{-2})^{1/p} \|R_{a,\tau}\|_{L_p,\Pi_j^m}|_{v_j}^{\alpha_j}(Q^m(M)),
$$

(3.37)

where the constant $C_{13} \geq 1$ is defined in Lemma 3.1. Indeed, property (P3) is trivial, and property (P2) is an immediate consequence of (3.34) and property (P1). To prove (P4), we first discuss estimates for the weight $W_{a,\beta}(y) := (a\tau)^{\sum_{j=1}^m \alpha_j} \sum_{j=1}^m |\sin(y_j/(a\tau))|^{\alpha_j} (\tau^2 \cos^2(y_j/(a\tau)) + 1 - \tau^2)^{\beta_j} \cos(y_j/(a\tau)).$

Then $R_{a,\tau}(b) \in T_{aV}$, since $V$ satisfies the II-condition, and therefore, $R_{a,\tau}(\cdot) \in B_{(a/b)V}$. This proves property (P1). To prove (P4), we first discuss estimates for the weight

$$
W_{a,\beta}(y) := (a\tau)^{\sum_{j=1}^m \alpha_j} \sum_{j=1}^m |\sin(y_j/(a\tau))|^{\alpha_j} (\tau^2 \cos^2(y_j/(a\tau)) + 1 - \tau^2)^{\beta_j} \cos(y_j/(a\tau)).
$$
on $Q^m(a\tau/2)$. Using Lemma 3.4 for $v_j = y_j/(a\tau), 1 \leq j \leq m, and y \in Q^m(M), M \in (0, a\tau/2), we obtain

$$
0 \leq \prod_{j=1}^m |y_j|^{\alpha_j} - W_{a,\beta}(y) \leq C_{13}mM^2(\alpha)^{-2} \prod_{j=1}^m |y_j|^{\alpha_j}.
$$

Next, it follows from (3.35) and the left inequality in (3.38) that for $a \geq 1, \tau \in (2/3, 1), p \in (0, \infty)$, and $M \in (0, a\tau/2)$,

$$
\|P_a\|_{L_p,\Pi_j^m}|_{t_j}^{\alpha_j}, (1-(t_j/\alpha)^2)^{\alpha_j}(Q^m(a)) \geq \|P_a\|_{L_p,\Pi_j^m}|_{t_j}^{\alpha_j}, (1-(t_j/\alpha)^2)^{\alpha_j}(Q^m(a)) \\
\geq \int_{Q^m(a\tau/2)} |R_{a,\tau}(y)|^p W_{a,\beta}(y) dy \geq \int_{Q^m(M)} |R_{a,\tau}(y)|^p W_{a,\beta}(y) dy \\
\geq \|R_{a,\tau}\|_{L_p,\Pi_j^m}|_{v_j}^{\alpha_j}(Q^m(M)) - \|R_{a,\tau}\|_{L_p,\Pi_j^m}|_{v_j}^{\alpha_j}(Q^m(M)).
$$

(3.39)

Thus inequality (3.37) of property (P4) for $M \in (0, a\tau/\sqrt{mC_{13}})$ follows from (3.39) and the right inequality in (3.38).

Let $\{a_n\}_{n=1}^{\infty}$ be an increasing sequence of numbers such that $\inf_{n \in \mathbb{N}} a_n \geq 1, \lim_{n \to \infty} a_n = \infty$, and

$$
\limsup_{n \to \infty} a_n^{-(m+\sum_{j=1}^m \alpha_j)/p} N_0 \left( P_{a_n V}, L_{p,\Pi_j^m}|_{t_j}^{\alpha_j}, (1-t_j^2)^{\alpha_j}(Q^m) \right) = \lim_{n \to \infty} a_n^{-(m+\sum_{j=1}^m \alpha_j)/p} N_0 \left( P_{a_n V}, L_{p,\Pi_j^m}|_{t_j}^{\alpha_j}, (1-t_j^2)^{\alpha_j}(Q^m) \right).
$$

(3.40)
Property (P1) and relation (3.33) of property (P2) show that the sequence of trigonometric polynomials \( \{R_{a_n, r}\}_{n=1}^{\infty} = \{f_n\}_{n=1}^{\infty} \) satisfies the conditions of Lemma 3.1 (c) with \( B_V \) replaced by \( B_{(1/r)}V \). Therefore, there exist a subsequence \( \{R_{a_n, r}\}_{d=1}^{\infty} \) and a function \( f_{0, r} \in B_{(1/r)}V \) such that

\[
\lim_{d \to \infty} R_{a_n, r} = f_{0, r} \tag{3.41}
\]

uniformly on any cube \( Q^m(M), M > 0 \).

Moreover, it follows from (3.41), property (P3), and (3.33) that

\[
|f_{0, r}(0)| = \lim_{d \to \infty} \left| R_{a_n, r}(0) \right| = \lim_{d \to \infty} \left| P_{a_n}(0) \right| = 1. \tag{3.42}
\]

In addition, using (1.1), (3.41), (3.37), (3.32), and (3.40), we obtain for any cube \( Q^m(M), M > 0 \),

\[
\|f_{0, r}\|_{L_{p, \Pi_j=1}^{m} |y_j|^{\alpha_j}(Q^m(M))} = \lim_{d \to \infty} \left( \pm \|f_{0, r} - R_{a_n, r}\|_{L_{p, \Pi_j=1}^{m} |y_j|^{\alpha_j}(Q^m(M))} + \|R_{a_n, r}\|_{L_{p, \Pi_j=1}^{m} |y_j|^{\alpha_j}(Q^m(M))} \right)^{1/p}
\]

\[
= \lim_{d \to \infty} \left( \|P_{a_n, r}\|_{L_{p, \Pi_j=1}^{m} |y_j|^{\alpha_j}(Q^m(M))} \right)
\]

\[
\leq \lim_{d \to \infty} \left( \|P_{a_n, r}\|_{L_{p, \Pi_j=1}^{m} |y_j|^{\alpha_j}(Q^m(M))} \right)
\]

\[
= 1/\lim_{d \to \infty} a_n^{-(m+\sum_{j=1}^{m} \alpha_j)/p} N_0 \left( \mathcal{P}_{a_n} V, L_{p, \Pi_j=1}^{m} |y_j|^{\alpha_j}(Q^m) \right). \tag{3.43}
\]

Next using (3.30) and (3.43), we see that

\[
\|f_{0, r}\|_{L_{p, \Pi_j=1}^{m} |y_j|^{\alpha_j}(Q^m(M))} \leq 1/C_{15}. \tag{3.44}
\]

Therefore, \( f_{0, r} \) is a nontrivial function from \( B_{(1/r)}V \cap L_{p, \Pi_j=1}^{m} |y_j|^{\alpha_j}(\mathbb{R}^m) \) by (3.41) and (3.42). Thus for any cube \( Q^m(M), M > 0 \), we obtain from (3.40), (3.32), (3.37), (3.41), and (3.42)

\[
\limsup_{a \to \infty} a^{-(m+\sum_{j=1}^{m} \alpha_j)/p} N_0 \left( \mathcal{P}_{a} V, L_{p, \Pi_j=1}^{m} |y_j|^{\alpha_j}(Q^m) \right)
\]

\[
= \lim_{d \to \infty} \left( \|P_{a_n, r}\|_{L_{p, \Pi_j=1}^{m} |y_j|^{\alpha_j}(Q^m(M))} \right)^{-1}
\]

\[
\leq \lim_{d \to \infty} \left( \|R_{a_n, r}\|_{L_{p, \Pi_j=1}^{m} |y_j|^{\alpha_j}(Q^m(M))} \right)^{-1}
\]

\[
= |f_{0, r}(0)| / \|f_{0, r}\|_{L_{p, \Pi_j=1}^{m} |y_j|^{\alpha_j}(Q^m(M))}. \tag{3.45}
\]
It follows from (3.45) that for \( \tau \in (2/3, 1) \),

\[
\limsup_{a \to \infty} a^{-(m + \sum_{j=1}^{m} \alpha_j)/p} N_0 \left( \mathcal{P}_{aV}, L_{p, \|j\|^{\alpha_j}(1-t_j^2)^{\beta_j}}(Q^m) \right) \\
\leq N_0 \left( B(1/\tau)V \cap L_{p, \|j\|^{\alpha_j}(\mathbb{R}^m)}, L_{p, \|j\|^{\alpha_j}(\mathbb{R}^m)} \right) \\
= \tau^{-(m + \sum_{j=1}^{m} \alpha_j)/p} N_0 \left( B_V \cap L_{p, \|j\|^{\alpha_j}(\mathbb{R}^m)}, L_{p, \|j\|^{\alpha_j}(\mathbb{R}^m)} \right),
\]

(3.46)

Then letting \( \tau \to 1^- \) in (3.46), we arrive at (3.28). However, we also need to prove stronger relations (3.29).

To construct \( f_0 \), recall first that \( f_{0, \tau}(\cdot) \in B(1/\tau)V, \tau \in (2/3, 1) \), and by (3.44) and (3.2),

\[
\sup_{\tau \in (2/3, 1)} \|f_{0, \tau}(\cdot)\|_{L_{\infty}(\mathbb{R}^m)} = \sup_{\tau \in (2/3, 1)} \|f_{0, \tau}(\tau)\|_{L_{\infty}(\mathbb{R}^m)} \\
\leq C_5 \sup_{\tau \in (2/3, 1)} \tau^{-(m + \sum_{j=1}^{m} \alpha_j)/p} \|f_{0, \tau}\|_{L_{p, \|j\|^{\alpha_j}(\mathbb{R}^m)}} \leq (3/2)^{(m + \sum_{j=1}^{m} \alpha_j)/p} C_5 / C_{15} = C,
\]

where \( C_5, C_{15} \), and \( C \) are independent of \( \tau \) because \( f_{0, \tau}(\tau) \in B_V. \) Therefore, by Lemma 3.1 (c) applied to a sequence \( \{f_{0, \tau_n}\}_{n=1}^{\infty} = \{f_n\}_{n=1}^{\infty} \), where \( \tau_n \in (2/3, 1), \ n \in \mathbb{N}, \) and \( \lim_{n \to \infty} \tau_n = 1 \), there exist a subsequence \( \{f_{0, \tau_{n_d}}\}_{d=1}^{\infty} \) and a function \( f_0 \in B_V \cap L_{\infty}(\mathbb{R}^m) = \bigcap_{d=1}^{\infty} \left( B(1/\tau_{n_d})V \cap L_{\infty}(\mathbb{R}^m) \right) \) such that \( \lim_{d \to \infty} f_{0, \tau_{n_d}} = f_0 \) uniformly on any compact set in \( \mathbb{C}^m. \)

Note that by (3.44) and (3.42), \( f_0 \) is a nontrivial function from \( B_V \cap L_{p, \|j\|^{\alpha_j}(\mathbb{R}^m)}. \) Then using (3.45), we obtain

\[
\limsup_{a \to \infty} a^{-(m + \sum_{j=1}^{m} \alpha_j)/p} N_0 \left( \mathcal{P}_{aV}, L_{p, \|j\|^{\alpha_j}(1-t_j^2)^{\beta_j}}(Q^m) \right) \\
\leq \lim_{M \to \infty} \lim_{n \to \infty} |f_{0, \tau_n}(0)| / \|f_{0, \tau_n}\|_{L_{p, \|j\|^{\alpha_j}(\mathbb{R}^m)}} (Q^m(M)) \\
= |f_0(0)| / \|f_0\|_{L_{p, \|j\|^{\alpha_j}(\mathbb{R}^m)}} \\
\leq N_0 \left( B_V \cap L_{p, \|j\|^{\alpha_j}(\mathbb{R}^m)}, L_{p, \|j\|^{\alpha_j}(\mathbb{R}^m)} \right).
\]

Thus (3.29) holds true, and this completes the proof of the theorem. \( \square \)

4. The \( m \)-dimensional Cube \( Q^m \). Proof of Theorem 1.3

Recall that \( \mathcal{P}_{aV, e} \) is the set of all even polynomials in each variable from \( \mathcal{P}_{aV}. \) Throughout the section we assume that \( V \subset \mathbb{R}^m \) satisfies the II-condition, \( \lambda_j \geq 0, 1 \leq j \leq m, \) and \( p \in [1, \infty). \) The proof reduces Theorem 1.3 to Theorem 1.4 by using Lemma 2.7 and equalities between the sharp constants on the cube \( Q^m \) discussed in the lemma below.
Lemma 4.1. The following equalities hold true:

\[ N(1, \ldots, 1) \left( \mathcal{P}_{aV}, L, \Pi_{j=1}^{m} (1-x_j^2)^{\lambda_j-1/2} (Q^m) \right) \]

\[ = 4^{-(\sum_{j=1}^{m} \lambda_j)/p} N_0 \left( \mathcal{P}_{2aV,e}, L, \Pi_{j=1}^{m} |t_j|^{2\lambda_j} (1-t_j^2)^{\lambda_j-1/2} (Q^m) \right) \]

\[ = 4^{-(\sum_{j=1}^{m} \lambda_j)/p} N_0 \left( \mathcal{P}_{2aV}, L, \Pi_{j=1}^{m} |t_j|^{2\lambda_j} (1-t_j^2)^{\lambda_j-1/2} (Q^m) \right). \quad (4.1) \]

Proof. We first prove that

\[ \mathcal{P}_{2aV,e} = S := \{ P_{2a}(t) = R_a (1 - 2t_1^2, \ldots, 1 - 2t_m^2) : R_a \in \mathcal{P}_{aV} \}. \quad (4.2) \]

Indeed, if \( k \in aV \cap \mathbb{Z}_+^m \), that is, \( R_a(x) := x^k \) is a monomial from \( \mathcal{P}_{aV} \), then the polynomial \( P_{2a}(t) := (1 - 2t_1^2)^{k_1} \cdots (1 - 2t_m^2)^{k_m} \) from \( S \) belongs to \( \mathcal{P}_{\Pi(2k),e} \subseteq \mathcal{P}_{2aV,e} \), by the II-condition. Therefore, \( S \subseteq \mathcal{P}_{2aV,e} \). Conversely, if \( 2k \in aV \cap \mathbb{Z}_+^m \), that is, \( P_{2a}(t) := t^{2k} \) is a monomial from \( \mathcal{P}_{2aV,e} \), then the polynomial \( R_a(x) := (\frac{1-x_1}{2})^{k_1} \cdots (\frac{1-x_m}{2})^{k_m} \) belongs to \( \mathcal{P}_{\Pi(2k),e} \subseteq \mathcal{P}_{aV} \), by the II-condition; hence \( P_{2a} \in S \). Therefore, \( \mathcal{P}_{2aV,e} \subseteq S \), and (4.2) holds true.

Next, making the substitution \( x = (1 - 2t_1^2, \ldots, 1 - 2t_m^2) : [0,1]^m \to Q^m \) and setting \( P_{2a}(t) := R_a (1 - 2t_1^2, \ldots, 1 - 2t_m^2) \) for any \( R_a \in \mathcal{P}_{aV} \), we have

\[
\begin{align*}
\int_{Q^m} |R_a(x)|^p \prod_{j=1}^{m} \left( 1 - x_j^2 \right)^{\lambda_j - 1/2} dx &= |P_{2a}(0)|^p \\
&= 4^{m/2 + \sum_{j=1}^{m} \lambda_j} \int_{[0,1]^m} |P_{2a}(t)|^p \prod_{j=1}^{m} t_j^{2\lambda_j} \left( 1 - t_j^2 \right)^{\lambda_j - 1/2} dt \\
&= 4^{\sum_{j=1}^{m} \lambda_j} \left\| P_{2a} \right\|_{L, \Pi_{j=1}^{m} t_j^{2\lambda_j} (1-t_j^2)^{\lambda_j-1/2}(Q^m)}^p.
\end{align*}
\]

Then the first equality in (4.1) follows from (4.2) and (4.3). Furthermore, using the symmetrization trick from Remark 1.7, we see that for any \( P_{2a} \in \mathcal{P}_{2aV} \) the polynomial

\[ P_{2a,e}(t) := 2^{-m} \sum_{|\delta_j|=1, 1 \leq j \leq m} P_{2a}(\delta_1 t_1, \ldots, \delta_m t_m) \]

belongs to \( \mathcal{P}_{2aV,e} \) and

\[
\frac{|P_{2a}(0)|}{\left\| P_{2a} \right\|_{L, \Pi_{j=1}^{m} t_j^{2\lambda_j} (1-t_j^2)^{\lambda_j-1/2}(Q^m)}} \leq \frac{|P_{2a,e}(0)|}{\left\| P_{2a,e} \right\|_{L, \Pi_{j=1}^{m} t_j^{2\lambda_j} (1-t_j^2)^{\lambda_j-1/2}(Q^m)}}.
\]

Hence the second equality in (4.1) holds true. \( \square \)
Proof of Theorem 1.3. An extremal polynomial \( P^* \) for \( \mathcal{N} \left( \mathcal{P}_{aV}, L_{p, \prod_{j=1}^{m} (1-t_j^2)^{\lambda_j-1/2}}(Q^m) \right) \) whose uniform norm is attained at the vertex \( x_0 = (1, \ldots, 1) \) of \( Q^m \), that is, \( \|P^*\|_{L_{\infty}(Q^m)} = \|P^*((1, \ldots, 1))\| \) exists by Lemma 2.7. Then Lemma 4.1 implies the equality

\[
\mathcal{N} \left( \mathcal{P}_{aV}, L_{p, \prod_{j=1}^{m} (1-t_j^2)^{\lambda_j-1/2}}(Q^m) \right) = 4^{-(\sum_{j=1}^{m} \lambda_j)/p} N_0 \left( \mathcal{P}_{2aV}, L_{p, \prod_{j=1}^{m} |t_j|^{2\lambda_j} (1-t_j^2)^{\lambda_j-1/2}}(Q^m) \right). \tag{4.4}
\]

Next, setting \( \alpha_j = 2\lambda_j \), \( \beta_j = \lambda_j - 1/2 \), \( 1 \leq j \leq m \), and replacing \( a \) with \( 2a \), we obtain from limit relations (1.15) and (1.16) of Theorem 1.4

\[
\lim_{a \to \infty} (2a)^{(m+2\sum_{j=1}^{m} \lambda_j)/p} N_0 \left( \mathcal{P}_{2aV}, L_{p, \prod_{j=1}^{m} |t_j|^{2\lambda_j} (1-t_j^2)^{\lambda_j-1/2}}(Q^m) \right) = \mathcal{N}_0 \left( \mathcal{B}_V \cap L_{p, \prod_{j=1}^{m} |t_j|^{2\lambda_j}}(\mathbb{R}^m), L_{p, \prod_{j=1}^{m} |t_j|^{2\lambda_j}}(\mathbb{R}^m) \right) \]

\[
= |f_0(0)|/\|f_0\|_L \left( p, \prod_{j=1}^{m} |t_j|^{2\lambda_j}(\mathbb{R}^m) \right), \tag{4.5}
\]

where \( f_0 \in \left( \mathcal{B}_V \cap L_{p, \prod_{j=1}^{m} |t_j|^{2\lambda_j}}(\mathbb{R}^m) \right) \setminus \{0\} \). Thus equalities (1.13) and (1.14) of Theorem 1.3 immediately follow from relations (4.4) and (4.5). \( \square \)

5. The \( m \)-dimensional Ball \( B^m \). Proofs of Theorems 1.5 and 1.6

Throughout the section we assume that \( \lambda \geq 0 \) and \( p \in [1, \infty) \). The proofs of Theorems 1.5 and 1.6 are based on Lemmas 2.9 and 2.11 and on four propositions of independent interest below. Three of these propositions discuss equalities between the sharp constants on \( \mathfrak{B}^m \) or \( \mathbb{R}^m \).

We first need three invariance theorems (this term was introduced in [17, 20]). Let \( D(m) \), \( m \geq 2 \), be the group of all proper and improper rotations \( \rho = \rho_m \) (about the origin) of \( \mathbb{R}^m \). We identify \( D(m) \) with the group \( D^*(m) \) of all \( m \times m \) orthogonal matrices \( A = A_m \) that is isomorphic to \( D(m) \) since \( \rho \in D(m) \) if and only if \( \rho x = A(\rho)x^T \), where \( A(\rho) \) is an \( m \times m \) orthogonal matrix with \( |\det A(\rho)| = 1, x \in \mathbb{R}^m \), and \( x^T \) is a column vector.

Let \( x_0 = x_0(m) := (1,0,\ldots,0) \in \mathbb{R}^m \), and let \( x_0^T \) be the column version of \( x_0 \). Next, let \( D(m, x_0) \), \( m \geq 2 \), be the subgroup of \( D(m) \) of all proper and improper rotations \( \rho \) around the \( x_1 \)-axis of \( \mathbb{R}^m \) (i.e., \( \rho x_0 = x_0 \)) that is isomorphic to the group \( D^*(m, x_0) \) of all \( m \times m \) orthogonal matrices \( A_m \), satisfying the condition \( A_m x_0^T = x_0^T \). The group \( D^*(m, x_0) \) can be characterized as the subgroup of \( D^*(m) \) of all \( m \times m \) orthogonal matrices \( A_m \) of the following block form:

\[
A_m = \begin{bmatrix} 1 & 0 \\ 0 & A_{m-1} \end{bmatrix}, \tag{5.1}
\]
where $A_{m-1} \in D^*(m-1)$. Representation (5.1) immediately follows from the condition $A_m x_0^T = x_0^T$ and from the fact that the transpose of $A_m \in D^*(m, x_0)$ belongs to $D^*(m, x_0)$ as well. For example, $D^*(2, x_0) = \{I_2, A_2\}$, where $I_2$ is the $2 \times 2$ identity matrix and

$$A_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$ 

So the only nontrivial rotation from $D(2, x_0)$ is the reflection around the $x_1$-axis.

In terms of rotations, representation (5.1) of $A_m \in D^*(m, x_0)$ is equivalent to the criterion

$$\rho_m \in D(m, x_0) \iff \rho_m x = (x_1, \rho_{m-1} x'), \quad x \in \mathbb{R}^m,$$

(5.2)

where $x' := (x_2, \ldots, x_m)$ and $\rho_{m-1} \in D(m-1)$ is a rotation about the origin of the $(m-1)$-dimensional subspace \{\(x \in \mathbb{R}^m : x_1 = 0\)\} of $\mathbb{R}^m$.

We say that $f : \mathbb{R}^m \to \mathbb{R}^m$ is invariant under a subgroup $D$ of $D(m)$ if $f(\rho x) = f(x)$ for all $\rho \in D$ and $x \in \mathbb{R}^m$. The set of all polynomials $P \in \mathcal{P}_{n,m}$ that are invariant under $D$ is denoted by $\mathcal{P}^D_{n,m}$, and the set of all entire functions $f \in B_{\mathbb{R}^m}$ that are invariant under $D$ is denoted by $B^D_{\mathbb{R}^m}$.

In addition, let $\mathcal{P}_{n,2,e}$ be the set of all polynomials $P_2(u, v)$ of two variables from $\mathcal{P}_{n,2}$ that are even with respect to the second variable. We first find the representation of a polynomial that is invariant under $D(m, x_0)$. Note that the proof of the following proposition is the refinement of the proof of Lemma 4.2.12 in [44].

**Proposition 5.1.** A polynomial $P$ belongs to $\mathcal{P}^D_{n,m}(x_0)$, $m \geq 2$, if and only if there exists a polynomial $P_2 \in \mathcal{P}_{n,2,e}$ such that $P(x) = P_2 \left( x_1, \left( \sum_{j=2}^m x_j^2 \right)^{1/2} \right)$.

**Proof.** It follows from (5.2) that for any $P_2 \in \mathcal{P}_{n,2,e}$ the polynomial $P(x) = P_2 \left( x_1, \left( \sum_{j=2}^m x_j^2 \right)^{1/2} \right)$ is invariant under $D(m, x_0)$.

Conversely, let a polynomial $P(x) = \sum_{l=0}^n x_1^l P_{l,n}(x_2, \ldots, x_m)$ from $\mathcal{P}_{n,m}$ be invariant under $D(m, x_0)$. Here, $P_{l,n} \in \mathcal{P}_{n-l,m-1}$, $0 \leq l \leq n$. Then the polynomials $\frac{1}{l!} \frac{\partial^l P(x)}{\partial x_1^l}$, $0 \leq l \leq n$, are invariant under $D(m, x_0)$ as well because by (5.2), $Q(x_1 + \delta, \rho_{m-1} x') = Q(x_1 + \delta, x')$ for any $\delta \in \mathbb{R}^1$ and every $Q \in \mathcal{P}^D_{n-l,m}$, $0 \leq \nu \leq n$. Therefore, the polynomials

$$P_{l,n}(x') = P_{l,n}(x_2, \ldots, x_m) = \frac{1}{l!} \frac{\partial^l P(x)}{\partial x_1^l} \bigg|_{x_1=0}, \quad 0 \leq l \leq n,$$

are invariant under $D(m, x_0)$. Hence taking into account that $P_{l,n}$ are independent of $x_1$, we conclude from (5.2) that $P_{l,n}(\rho_{m-1} x') = P_{l,n}(x')$, $0 \leq l \leq n$, for every rotation $\rho_{m-1}$ of the subspace \{\(x \in \mathbb{R}^m : x_1 = 0\)\} of $\mathbb{R}^m$. Therefore, $P_{l,n}$ are invariant under $D(m-1)$ and by Lemma 4.2.11 from
Thus
\[ P(x) = \sum_{0 \leq l + 2\nu \leq n} c_{l,\nu} x_1^l (x_2^2 + \ldots + x_m^2)^\nu = P_2 \left( x_1, \left( \sum_{j=2}^m x_j^2 \right)^{1/2} \right), \]
where \( P_2 \in \mathcal{P}_{n,2,e} \), and the proposition is established.

A different version of Proposition 5.1 was discussed in [17, Proposition 5.1].

In the proofs of the next two propositions, we need the following simple observation: if \( W(x) = \psi(|x|) \) is a radial weight, then for \( \Omega = \mathfrak{B}^m \) or \( \Omega = \mathbb{R}^m \) the norm in \( L_{p,W}(\Omega) \) is invariant under rotation, that is, for any \( F \in L_{p,W}(\mathbb{R}^m) \) and any rotation \( \rho \in D(m) \),
\[ \|F(\rho \cdot)\|_{L_{p,W}(\mathbb{R}^m)} = \|F(\cdot)\|_{L_{p,W}(\mathbb{R}^m)}, \quad p \in [1, \infty). \]

Next, we reduce the sharp constant for the \( m \)-dimensional ball \( \mathfrak{B}^m \), \( m \geq 2 \), to the one on the disk \( \mathfrak{B}^2 \). Recall that \( x_0 = x_0(m) = (1, 0, \ldots, 0) \in \mathbb{R}^m \).

**Proposition 5.2.** The following equalities hold true for \( m \geq 2 \):
\[ N_{x_0(m)} \left( \mathcal{P}_{n,m}, L_{p,(1-|x|^2)^{\lambda-1/2}}(\mathfrak{B}^m) \right) = N_{x_0} \left( \mathcal{P}^{D(m,x_0(m))}_{n,m}, L_{p,(1-|x|^2)^{\lambda-1/2}}(\mathfrak{B}^m) \right) = C_{17} N_{x_0(2)} \left( \mathcal{P}_{n,2,e}, L_{p,(1-|x|^2)^{\lambda-1/2}}(\mathfrak{B}^2) \right), \]
where
\[ C_{17} = \frac{\Gamma((m-1)/2)}{\pi^{m-1}/2} \left( \Gamma((m-1)/2) \right)^{1/p}. \]

**Proof.** The proof of the first equality in (5.3) follows general ideas developed in [20, Theorems 2.1 and 2.2]. Given \( P \in \mathcal{P}_{n,m} \), we define the Haar integral
\[ P^*(x) := \int_{D(m,x_0)} P(\rho x) d\mu(\rho), \]
where \( \mu \) is the invariant Haar measure on \( D(m,x_0) \) (see, e.g., [31, Theorem 5.14]). Since \( \rho : \mathfrak{B}^m \to \mathfrak{B}^m \) is a linear map, \( P(\rho \cdot) \in \mathcal{P}_{n,m} \), \( \rho \in D(m,x_0) \), and \( P^* \in \mathcal{P}_{n,m} \). In addition, for every \( \rho^* \in D(m,x_0) \),
\[ P^*(\rho^* x) = \int_{D(m,x_0)} P(\rho \rho^* x) d\mu(\rho) = \int_{D(m,x_0)} P(\rho x) d\mu(\rho) = P^*(x). \]
Therefore, \( P^* \in \mathcal{P}_{n,m}^{	ext{D}(m,x_0)} \). Using now the generalized Minkowski inequality (see, e.g., [13, Lemma 3.2.15]) and taking into account the fact that the norm \( \| \cdot \|_{\mathcal{L}_p(\mathbb{R}^m)} \) is invariant under rotation, we obtain

\[
\| P^* \|_{\mathcal{L}_p(1-|x|^2)^{\lambda-1/2}(\mathbb{B}^m)} \leq \int_{D(m,x_0)} \| P(\rho) \|_{\mathcal{L}_p(1-|x|^2)^{\lambda-1/2}(\mathbb{B}^m)} d\mu(\rho) = \| P \|_{\mathcal{L}_p(1-|x|^2)^{\lambda-1/2}(\mathbb{B}^m)}.
\]

(5.6)

Furthermore, since \( \rho \in D(m,x_0) \) in (5.5), we have

\[
P^*(x_0) = P(x_0).
\]

(5.7)

Therefore, by (5.3) and (5.7),

\[
\frac{|P(x_0)|}{\| P \|_{\mathcal{L}_p(1-|x|^2)^{\lambda-1/2}(\mathbb{B}^m)}} \leq \frac{|P^*(x_0)|}{\| P^* \|_{\mathcal{L}_p(1-|x|^2)^{\lambda-1/2}(\mathbb{B}^m)}},
\]

which proves the first equality in (5.3).

Next, by Proposition 5.1,

\[
\mathcal{N}_{x_0}(m) \left( \mathcal{P}_{n,m}^{	ext{D}(m,x_0)(m)}, L_p(1-|x|^2)^{\lambda-1/2}(\mathbb{B}^m) \right) = \sup_{P_2 \in \mathcal{P}_{n,2e} \setminus \{0\}} \left( \int_{\mathbb{B}^m} |P_2(x_1, (\sum_{j=2}^m x_j^2)^{1/2})|^{p/2} (1-|x|^2)^{\lambda-1/2} dx \right)^{1/p}.
\]

(5.8)

Using the spherical coordinate system in \( \mathbb{R}^m \), we see that \( x_1 = r \cos \varphi \) and \( (\sum_{j=2}^m x_j^2)^{1/2} = r |\sin \varphi| \), where \( \varphi \in [0,2\pi) \) if \( m = 2 \) and \( \varphi \in [0,\pi] \) if \( m > 2 \). Since \( P_2 \in \mathcal{P}_{n,2e} \) is even with respect to the second variable, we obtain

\[
\int_{\mathbb{B}^m} |P_2(x_1, (\sum_{j=2}^m x_j^2)^{1/2})|^{p/2} (1-|x|^2)^{\lambda-1/2} dx = (1/C_{17})^p \int_0^1 \int_{\varphi=0}^{2\pi} |P_2(r \cos \varphi, r \sin \varphi)|^{p/2} (1-r^2)^{\lambda-1/2} r^{m-1} |\sin \varphi|^{m-2} d\varphi dr
\]

\[
= (1/C_{17})^p \int_{(u,v) \in \mathbb{B}^2} |P_2(u,v)|^{p/2} (1-u^2-v^2)^{\lambda-1/2} |v|^{m-2} du dv,
\]

(5.9)

where

\[
C_{17} = \left( \frac{2 \int_0^{\pi} \sin^{m-2} \varphi d\varphi}{|S^{m-1}|^{m-1}} \right)^{1/p} = \left( \frac{\Gamma((m-1)/2)}{\pi^{(m-1)/2}} \right)^{1/p}.
\]

Thus the second equality in (5.3) follows from (5.8) and (5.9). \( \square \)
In the next proposition, we reduce the multivariate sharp constant for $\mathbb{R}^m$ to the univariate one for $\mathbb{R}^1$.

**Proposition 5.3.** The following equalities hold true for $m \geq 1$:

$$
\mathcal{N}_0 \left( B_{2m} \cap L_{p,|x|^2} (\mathbb{R}^m) , L_{p,|x|^2} (\mathbb{R}^m) \right) = \mathcal{N}_0 \left( B_{2m}^D(m) \cap L_{p,|x|^2} (\mathbb{R}^m) , L_{p,|x|^2} (\mathbb{R}^m) \right) = C_{18} \mathcal{N}_0 \left( B_{1,e} \cap L_{p,|x|^{m+2\lambda-1}} (\mathbb{R}^1) , L_{p,|x|^{m+2\lambda-1}} (\mathbb{R}^1) \right),
$$

(5.10)

where

$$
C_{18} = C_{18} (m,p) := \left( \frac{2}{S_{m-1}} \right)^{1/p} = \left( \frac{\Gamma(m/2)}{\pi^{m/2}} \right)^{1/p}.
$$

(5.11)

**Proof.** Equalities (5.10) for $\lambda = 0$ were proved in [20, Corollary 3.11]. The proof was based on [20, Theorem 2.2] and several propositions from [20], and it was comparatively long. The proof of Proposition 5.3 can be copied from the aforementioned one if we take into account the fact that the norm in $L_{p,|x|^2} (\mathbb{R}^m)$ is invariant under rotation. \qed

Next, we reduce the multivariate and bivariate sharp constants in (5.3) to the univariate one.

**Proposition 5.4.** The following equality holds true for $m \geq 1$:

$$
\mathcal{N}_{x_0(m)} \left( \mathcal{P}_{n,m} , L_{p,(1-|x|^2)^{\lambda-1/2}} (\mathbb{B}^m) \right) = C_{19} \mathcal{N}_1 \left( \mathcal{P}_n , L_{p,(1-u^2)^{m/2+\lambda-1}}([-1,1]) \right),
$$

(5.12)

where

$$
C_{19} = C_{19} (m,p,\lambda) := \left( \frac{\Gamma(\lambda + m/2)}{\pi^{(m-1)/2} \Gamma(\lambda + 1/2)} \right)^{1/p}.
$$

(5.13)

**Proof.** Since (5.12) is trivial for $m = 1$, we assume that $m \geq 2$. Then using Proposition 5.2 and making the substitution $v = \tau \sqrt{1-u^2}$, $\tau \in [-1,1]$, we obtain

$$
\mathcal{N}_{x_0(m)}^p \left( \mathcal{P}_{n,m} , L_{p,(1-|x|^2)^{\lambda-1/2}} (\mathbb{B}^m) \right) = C_{17}^p \sup_{P_2 \in \mathcal{P}_{n,2,e} \setminus \{0\}} \left[ \int_{-1}^{1} \left( \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} |P_2(u,v)|^p (1-u^2-v^2)^{\lambda-1/2} |v|^{m-2} dv \right) du \right]^{1/p} = C_{17}^p \sup_{P_2 \in \mathcal{P}_{n,2,e} \setminus \{0\}} \left[ \int_{-1}^{1} \int_{\tau = -1}^{\tau = 1} |P_2(u,\tau \sqrt{1-u^2})|^p (1-u^2)^{m/2+\lambda-1} du \right]^{1/p} \tau^{m-2} (1-\tau^2)^{\lambda-1/2} d\tau.
$$

(5.14)

Since the function

$$
Q_v(u) := P_2 \left( u, \tau \sqrt{1-u^2} \right), \quad \tau \in [-1,1], \quad P_2 \in \mathcal{P}_{n,2,e} \setminus \{0\},
$$
is a polynomial in \( u \) of degree at most \( n \) and \( Q_\tau(1) = P_2(1, 0) \), we have for each fixed \( \tau \in [-1, 1] \) and every fixed \( P_2 \in \mathcal{P}_{n,2,e} \setminus \{0\} \),

\[
\int_{-1}^{1} \left| P_2 \left( u, \tau \sqrt{1 - u^2} \right) \right|^p (1 - u^2)^{m/2 + \lambda - 1} du \\
\geq \left| P_2(1, 0) \right|^p \inf_{Q \in \mathcal{P}_{n}\setminus\{0\}} \frac{\int_{-1}^{1} |Q(u)|^p (1 - u^2)^{m/2 + \lambda - 1} du}{|Q(1)|^p}.
\]

(5.15)

Then combining (5.14) and (5.15), we obtain

\[
N^p_{x_0(m)} \left( \mathcal{P}_{n,m}, L_{p,(1-|x|^2)^{\lambda-1/2}} (\mathfrak{B}^m) \right) \\
\leq C_{17} \left( \int_{-1}^{1} \tau^{m-2}(1 - \tau^2)^{\lambda-1/2} d\tau \right)^{-1} \mathcal{N}^p_1 \left( \mathcal{P}_n, L_{p,(1-u^2)^{m/2 + \lambda - 1}([1,1])} \right) \\
= C_{19} N^p_1 \left( \mathcal{P}_n, L_{p,(1-u^2)^{m/2 + \lambda - 1}([1,1])} \right),
\]

(5.16)

where \( C_{17} \) and \( C_{19} \) are defined by (5.4) and (5.13), respectively. On the other hand, repeating calculations (5.14) for polynomials \( P_2(u, v) = Q(u) \) that are independent of \( v \), we have

\[
N^p_{x_0(m)} \left( \mathcal{P}_{n,m}, L_{p,(1-|x|^2)^{\lambda-1/2}} (\mathfrak{B}^m) \right) \\
\geq C_{17} \sup_{Q \in \mathcal{P}_{n}\setminus\{0\}} \int_{-1}^{1} \left( \int_{-1}^{1} \frac{|Q(u)|^p}{|Q(1)|^p} (1 - u^2 - v^2)^{\lambda-1/2} |v|^{m-2} dv \right) du \\
= C_{19} N^p_1 \left( \mathcal{P}_n, L_{p,(1-u^2)^{m/2 + \lambda - 1}([1,1])} \right).
\]

(5.17)

Thus (5.12) follows from (5.16) and (5.17).

\( \square \)

**Proof of Theorem 1.5** Recall that the constants \( A_1 \) and \( A_2 \) are defined by (1.18) and (1.19), respectively.

Next, by Lemma 2.9 for \( \lambda = 0 \) and Lemma 2.11 for \( \lambda > 0 \), there exists an extremal polynomial \( P^* \) for \( \mathcal{N} \left( \mathcal{P}_{n,m}, L_{p,(1-|x|^2)^{\lambda-1/2}} (\mathfrak{B}^m) \right) \), \( p \in [1, \infty) \), \( \lambda \geq 0 \), and there exists \( x_0 \in S^{m-1} \) such that \( \| P^* \|_{L_\infty(\mathfrak{B}^m)} = |P^*(x_0)| \).

Without loss of generality we can assume that \( x_0 = x_0(m) = (1, 0, \ldots, 0) \) for \( m \geq 2 \). Indeed, if \( x_0 \neq x_0(m) \), then there exists the rotation \( \rho_0 \in D(m) \) such that \( \rho_0 x_0 = x_0(m) \). Then the polynomial \( P^{**}(x) := P^*(\rho_0 x) \) is an extremal polynomial for \( \mathcal{N} \left( \mathcal{P}_{n,m}, L_{p,(1-|x|^2)^{\lambda-1/2}} (\mathfrak{B}^m) \right) \) and \( \| P^{**} \|_{L_\infty(\mathfrak{B}^m)} = |P^{**}(1, 0, \ldots, 0)| \).

Therefore, for \( p \in [1, \infty) \), \( m \geq 1 \), and \( \lambda \geq 0 \),

\[
\mathcal{N} \left( \mathcal{P}_{n,m}, L_{p,(1-|x|^2)^{\lambda-1/2}} (\mathfrak{B}^m) \right) = N_{x_0(m)} \left( \mathcal{P}_{n,m}, L_{p,(1-|x|^2)^{\lambda-1/2}} (\mathfrak{B}^m) \right).
\]

(5.18)

Note that for \( m = 1 \) (5.18) follows from (1.11). Thus the first relation in (1.17) of Theorem 1.5 with \( A_1 = 2^{1/p} C_{19} \) in (1.18), where \( C_{19} \) is defined by (5.13), follows from equalities (5.18) and (5.12).
and limit relation (1.9). The second equality in (1.17) of Theorem 1.5 with
\( A_2 = A_1/C_{18} \) in (1.19), where \( C_{18} \) is defined by (5.11), is an immediate consequence of Proposition 5.3.

Finally, to prove (1.20), we choose \( f_0(t) := g_0(|t|) \), where \( g_0 \in (B_{1,\varepsilon} \cap L_{p,|u|m+2\lambda-1}(\mathbb{R}^1)) \setminus \{0\} \) and

\[
\mathcal{N}_0 \left( B_{1,\varepsilon} \cap L_{p,|u|2\lambda}(\mathbb{R}^1), L_{p,|u|m+2\lambda-1}(\mathbb{R}^1) \right) = \frac{|g_0(0)|}{\|g_0\|_{L_{p,|u|m+2\lambda-1}(\mathbb{R}^1)}}.
\] (5.19)

The existence of \( g_0 \) was proved in [20, Theorem 4.3] (see also relations (1.9) and (1.10) with \( \lambda \geq 0 \), replaced by \( \lambda + (m - 1)/2 \)). Then \( f_0 \in (B_{2m} \cap L_{p,|t|2\lambda}(\mathbb{R}^m)) \setminus \{0\} \) and by (5.19) and by Proposition 5.3,

\[
\mathcal{N}_0 \left( B_{2m} \cap L_{p,|t|2\lambda}(\mathbb{R}^m), L_{p,|t|2\lambda}(\mathbb{R}^m) \right) = \frac{|f_0(0)|}{\|f_0\|_{L_{p,|t|2\lambda}(\mathbb{R}^m)}}.
\] (5.20)

Thus (1.16) follows from (1.14) and (5.20). This completes the proof of the theorem. \( \Box \)

**Proof of Theorem 1.6.** Using Proposition 5.4 and formula (1.8) with \( \lambda \) replaced by \( \lambda + (m - 1)/2 \), we arrive at (1.21). Next, the asymptotic

\[
\mathcal{N} \left( \mathcal{P}_{n,m}, L_{2,(1-|x|^2)^{\lambda-1/2}}(\mathfrak{B}^m) \right)
\]

\[
= \frac{n^{\lambda+m/2}(1 + o(1))}{(2^{2\lambda+m-2}\pi^{(m-1)/2}(2\lambda + m)^{1/2}\Gamma(\lambda + 1/2)\Gamma(\lambda + m/2))^{1/2}}, \quad n \to \infty,
\] (5.21)

immediately follows from (1.21). Therefore, formula (1.22) is a direct consequence of (1.17) and (5.21). \( \Box \)

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