Performance of Curzon-Ahlborn engine along with its engine speed and compression ratio

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Abstract
We evaluated the ideal limits of performance (power and efficiency) of heat engines operated with external heat sources along with their engine speed and compression ratio, using the method of adjoint equations based on variational principle. It is known that the power and efficiency of heat engines are maximum when the finite-time heat-transfer from/to the heat sources occurs isothermally, and such an engine is called Curzon–Ahlborn (CA) engine, so we derived a formula to express the temperature in the isothermal process of the CA engine as functions of the rate constant (or time constant) of either expansion or compression of the volume of the working fluid during that process. Using this formula, we found that the CA engine has the maximum of compression ratio and the slowest limit of engine speed for each compression ratio while at each compression ratio the thermal efficiency becomes greater with increasing engine speed and at each engine speed the efficiency increases with increasing compression ratio. These characteristics indicate the ideal performance envelope of the heat engines operated with external heat sources.

Keywords: Power, Thermal efficiency, Curzon-Ahlborn engine, Engine speed, Compression ratio, Thermal cycle, Heat transfer, Adjoint equation, Ideal limit

1. Introduction

“Energy efficiency is the only energy resource possessed by all countries” (International Energy Agency, 2017). Improvement in the energy efficiency of heat devices is required for the suppression of global warming and climate change worldwide. However, to what extent can the thermal efficiency of a system be improved? It is well known that the Carnot efficiency is the theoretical maximum of thermal efficiency for heat engines, and it depends on the temperature of external hot and cold heat baths. However, to achieve Carnot efficiency, the heat transfer must occur at a thermal equilibrium between working fluid and each heat bath, which is impossible to realize unless an infinitely slow heat transfer is assumed.

To address this limitation, the Curzon–Ahlborn (CA) efficiency was proposed (Andresen et al., 1984; Hoffmann et al., 1997; Vaudrey et al., 2014). This is the thermal efficiency attainable at maximum power when an irreversible change is due to only finite-time heat transfer between each heat bath and working fluid, which are not in thermal equilibrium. The CA efficiency is favored as a realistic estimate of the lower bound of efficiency within a practical range of operation for a given class of heat engines (Chen et al., 2001) because of its simple form of expression (Andresen et al., 1984; Curzon and Ahlborn, 1975; Hoffmann et al., 1997; Rubin 1979a; Vaudrey et al., 2014):

\[ \eta_{\text{mp}} \equiv 1 - \frac{T_L}{T_H}, \]

where \( \eta_{\text{mp}} \) represents the thermal efficiency at the maximum power of the CA engine and known as the CA efficiency that only depends on the temperature ratio between the two heat baths \( T_L/T_H \) (similar to the Carnot efficiency, i.e., \( \eta_{\text{carnot}} \equiv 1 - T_L/T_H \)) where \( T_H \) and \( T_L \) denote the temperatures of the hot and cold heat baths, respectively.
Such a simple expression of $\eta_{\text{Carnot}}$ originates from the reversibility of the cycle, i.e.,

$$\frac{Q_H}{T_H} - \frac{Q_L}{T_L} = 0,$$

(2)

where $Q_H$ and $Q_L$ represent the heat transferred from the hot and to the cold heat baths, respectively, while $\eta_{\text{mp}}$ of the CA engine comes from the endoreversibility condition Eq. (A.4) that means internally no entropy production: The CA engine is an endoreversible cycle in which the heat addition/disposal process occurs owing to the law of irreversible heat transfer between the working fluid and each external heat bath. Hereafter, we call such a process CA process when the work done is maximized during that process.

In this article, we derive and discuss the explicit relationship that has yet to be derived between the power (or efficiency) and the cycle duration of the CA engine along with its compression ratio, although Gordon (1989) derived the relationship between the power and thermal efficiency along with only the cycle duration.

The rest of this paper is structured as follows. In Section 2, we describe the available but insufficient progress towards obtaining the relationship between cycle duration and thermal efficiency (or power) for the CA engine, and why it has not yet been obtained. In Section 3, we describe our variational problem to maximize the work done by the CA engine for each given cycle duration, and we derive the corresponding adjoint equations. In Section 4, we derive the solutions of the adjoint equations or the equations of thermodynamics for each process of the CA engine cycle. In Section 5, we discuss the influence of the compression ratio on the efficiency at maximum power (CA efficiency). In Section 6, we derive the relationship between cycle duration and thermal efficiency for the CA engine. In Section 7, we present sample calculation for the CA engine. The result indicates that the CA engine has a feasible range of compression ratio to achieve the CA efficiency $\eta_{\text{mp}}$, which is explained in Section 8. Finally, in Section 9, we summarize the findings of this work.

**Nomenclature**

$A$: Heat-transfer area of the working fluid, assumed to be equal to the cross-sectional area of the cylinder that is perpendicular to the piston movement

$C_0$: Coefficient in the equation of state, Eq. (28)

$g$: Defined by Eq. (10)

$h$: Heat-transfer coefficient

$f$: Function to be maximized, as defined by Eq. (9)

$K_R$: Defined by Eqs. (4) or (5)

$l$: Stroke length

$m$: Mass of the working fluid

$n_1$: Exponent of temperature in the equation of state, Eq. (28)

$n_2$: Exponent of volume in the equation of state, Eq. (28)

$P$: Pressure of the working fluid

$Q$: Heat gained by the working fluid (may be negative or zero)

$Q_H$: Heat gained by the working fluid in the heat-addition CA process (always $Q_H \geq 0$)

$Q_L$: Heat lost by the working fluid in the heat-disposal CA process (always $Q_L \geq 0$)

$R_g$: Gas constant per mass of the working fluid

$t$: Time

$t_0$: Start time of the expansion/compression stage

$t_1$: Defined in Table 1 in Section 3

$t_2$: End time of the expansion/compression stage

$t_3$: Duration of the CA process with heat addition in the expansion stage

$t_4$: Duration of the CA process with heat disposal in the compression stage

$T$: Temperature of the working fluid

$T_H$: Temperature of the hot heat bath or reservoir

$T_L$: Temperature of the cold heat bath or reservoir

$T_R$: Temperature of the heat bath or reservoir

$V$: Volume of the working fluid
Greek symbols

\( \varepsilon \): Compression ratio
\( \eta \): Thermal efficiency
\( \lambda \): Lagrange multiplier
\( \xi \): Defined by Eqs. (31) or (33)
\( \Pi \): Power
\( \tau \): Ideally effective cycle duration defined by Eq. (46)
\( \omega_{CA} \): Defined by Eqs. (32) or (34)

Suffixes

\( BDC \): Bottom dead center
\( c \): Heat disposal CA process (“c” is an abbreviation of “cool” or “cold”)
\( CA \): Curzon–Ahlborn engine or its isothermal process
\( Carnot \): Carnot cycle
\( comp \): Compression stage
\( cr \): Critical value of the compression ratio of Curzon–Ahlborn engine
\( exp \): Expansion stage
\( H \): Heat bath with a high temperature (“H” is an abbreviation of “high”)
\( h \): Heat addition CA process (“h” is an abbreviation of “hot”)
\( L \): Heat bath with a low temperature (“L” is an abbreviation of “low”)
\( mp \): At the maximum power
\( R \): Heat bath or reservoir (“R” may be replaced by “H” or “L”)
\( TDC \): Top dead center

2. Posing the question

The relationship between the thermal efficiency \( \eta \) and the power \( \Pi \) of the CA engine can be expressed by the following formula (Gordon, 1989; Hoffmann et al., 1997):

\[
\Pi = \frac{1}{K_H/(1+1/T_H) + 1/(1+1/T_L)} T_H^{\eta_{Carnot}/(1-\eta) \eta},
\]

where \( t_H \) and \( t_L \) denote the durations of heat addition and heat disposal of the CA processes, respectively. Meanwhile, \( K_H \) and \( K_L \) are respectively defined by:

\[
K_H \equiv A_H h_H \quad \text{and} \quad K_L \equiv A_L h_L,
\]

where \( A \) denotes the boundary area between the fluid and the heat bath through which the heat transfer occurs, \( A_H \) and \( A_L \) represent the areas for the hot and cold heat baths or reservoirs, respectively, and \( h_H \) and \( h_L \) are the corresponding heat-transfer coefficients.

To plot the \( \eta-\Pi \) diagram of the CA engine, Eq. (3) requires the relationship between \( t_H \) and \( t_L \), which is known to be (Gordon, 1989; Hoffmann et al., 1997):

\[
t_H/t_L = \sqrt{K_L/K_H}.
\]

Substituting Eq. (6) into Eq. (3) leads to:

\[
\Pi = \frac{1}{\left(\frac{1}{K_H} + \frac{1}{K_L}\right)} T_H^{\eta_{Carnot}/(1-\eta) \eta},
\]
by which we can plot the $\eta - \Pi$ diagram of the CA engine as shown in Fig. 1 for the sample calculation in Section 7. However, Eq. (7) does not give any information about the relationship between $t_H$ (or $t_L$) and $\eta$ (or $\Pi$). Furthermore, Eq. (6) does not determine $t_H$ and $t_L$, but only their ratio $t_H/t_L$. The reason for this lack of information about $t_H$ and $t_L$ is that any formulas obtained for the CA engine thus far, such as Eqs. (3) and (6), come from only two premises: endoreversibility and the first principle of thermodynamics. This lean and economical premise gives some universality and theoretical attractiveness to the CA engine. To obtain the relationship between $t_H$ (or $t_L$) and $\eta$ (or $\Pi$) for the CA engine, we need to specify the working fluid, i.e., the equation of state, while Appendix A shows that the following relationship holds between $Q_H/t_H$ and $\eta$ for the CA engine:

$$1 - \eta = \frac{t_L}{t_H - \frac{Q_H}{t_H R}} \left(1 \right).$$

Eq. (8) indicates that $Q_H$ and $t_H$ are not separable from each other at this stage of formulation to determine $\eta$.

Fig. 1  $\Pi$ vs. $\eta$ for the CA engine. The plotted values of the power $\Pi$ are calculated for the sample calculation in Section 7. The power normalized by its maximum value (at $\eta = \eta_{mp}$) is common for all CA engines.

3. Variational treatment

We use the method of adjoint equations that was employed in our previous works (Kojima, 2017a, 2017b) where we obtained the maximum work of free-piston engine generators with various types of heat cycles, Stirling (with ideal regeneration, i.e., perfect regeneration with no entropy production), Carnot (irrespective of reversibility), Otto, and Brayton ones, but here the working fluid is not limited to thermally and calorically perfect gas. This implies that the equation of state of the working fluid is more generalized than in the previous works.

We maximize the following function $J$ to optimize the trajectory of volume $V$ of the working fluid with respect to time $t$:

$$\max J = \int_{t_0}^{t_f} \left( P \dot{V} - \lambda g \right) dt,$$

where:

$$g = \dot{U} + P \dot{V} - \dot{Q} = 0,$$  

$$\dot{Q} = K_R(T_R - T),$$  

$$P = P(V, T),$$  and

$$U = U(V, T).$$

Here, $t_0$ and $t_f$ denote the start and end times, respectively, of the expansion or compression stage of the engines and $P$, $V$, $T$, and $U$ respectively represent the pressure, volume, temperature, and internal energy of the working fluid. The parameter $\lambda$ is the Lagrange multiplier, $Q$ is the amount of heat added to the fluid, and the suffix $R$ denotes the heat baths/reservoirs. The characteristics of the fluid are not specified, except that $P$ and $U$ are functions of $V$ and
\( T \), i.e., Eqs. (12) and (13).

Equation (9) represents the maximization of the work done during the period \( t_0 \) to \( t_f \), i.e., the maximization of the power for the given duration. In the following, we analyze \( J \) under the assumption that \( \dot{Q} \neq 0 \) only from \( t_0 \) to \( t_1 \) (a CA process) and \( \dot{Q} = 0 \) from \( t_1 \) to \( t_f \) (an adiabatic process) (see Table 1).

In the following argument, we focus only on the expansion stage, but it also holds for the compression stage (except the specific arguments about expansion and compression stages). In general, the word “heat addition” can be interpreted as “heat disposal” if \( \dot{Q} < 0 \). The work done in a whole cycle is simply the sum of the work done in each expansion and compression stage.

As described by Kojima (2017a), Pontryagin’s maximum principle requires that \( PV \) be as large as possible, \( \dot{Q} \geq 0 \) for the expansion stage, and \( \dot{Q} \leq 0 \) for the compression stage, when the control variables are \( Q \) and \( V \) and the state variables are \( P \) and \( T \).

From Eqs. (9)–(13) and with integration by parts, we obtain the following variation of \( J \):

\[
\delta J = \left[ \left( (1-\lambda)P - \frac{\partial u}{\partial T} \right) \delta V - \lambda \frac{\partial u}{\partial T} \delta T \right]_{t=t_0}^{t=t_f} + \int_{t_0}^{t_f} \left\{ \left( (\lambda-1)\dot{P} + \dot{\lambda} P + \dot{\lambda} \frac{\partial u}{\partial V} \right) \delta V + (1-\lambda)\dot{V} \delta P + \left( \dot{\lambda} \frac{\partial u}{\partial T} - \lambda K_R \right) \delta T \right\} dt. \tag{14}
\]

Using Eq. (12) (the equation of state), we express \( \delta P \) by \( \delta T \) and \( \delta V \), and \( \dot{P} \) by \( \dot{T} \) and \( \dot{V} \) for ease of analysis. Equation (14) then becomes:

\[
\delta J = \left[ \left( (1-\lambda)P - \frac{\partial u}{\partial T} \right) \delta V - \lambda \frac{\partial u}{\partial T} \delta T \right]_{t=t_0}^{t=t_f} + \int_{t_0}^{t_f} \left\{ \left( \frac{\partial u}{\partial V} - \dot{\lambda} K_R \right) \dot{V} + (1-\lambda)\dot{V} \delta P + \left( \frac{\partial u}{\partial T} - \lambda K_R \right) \dot{T} \right\} dt. \tag{15}
\]

From this equation, we obtain the following stationary conditions of \( J \), i.e., the adjoint equations:

\[
\frac{\delta J}{\delta T(t_f)} = \dot{\lambda} \frac{\partial u}{\partial T} - \lambda K_R + (1-\lambda)\dot{\lambda} \frac{\partial u}{\partial V} = 0, \tag{16}
\]

\[
\frac{\delta J}{\delta V(t)} = -(1-\lambda) \frac{\partial u}{\partial T} \dot{\lambda} + \dot{\lambda} \left( P + \frac{\partial u}{\partial V} \right) = 0, \tag{17}
\]

\[
\lambda(t_f) = 0. \tag{18}
\]

4. Solution for each process

The CA engine cycle consists of two adiabatic processes (\( K_R = 0 \)) and two CA processes (\( K_R \neq 0 \)). The sequence of each process is given in Table 1. Substituting Eq. (13) into Eq. (10), we obtain:

\[
\left( \frac{\partial u}{\partial V} + P \right) \dot{V} + \frac{\partial u}{\partial T} \dot{T} - \dot{Q} = 0. \tag{19}
\]

4.1 Adiabatic process

For the adiabatic processes (\( \dot{Q} = 0 \)), Eq. (19) becomes:
\[
\left(\frac{\partial u}{\partial V} + P\right)\dot{V} + \frac{\partial u}{\partial T} \dot{T} = 0, \tag{20}
\]

which generally gives the relationship between \(T\), \(P\), and \(V\) using Eqs. (12) and (13). Then, we can determine \(T\) and \(P\) if \(V\) is given.

### 4.2 CA process

By substituting Eq. (11) into Eq. (19), we obtain:

\[
\left(\frac{\partial u}{\partial V} + P\right)\dot{V} + \frac{\partial u}{\partial T} \dot{T} - K_R(T_R - T) = 0. \tag{21}
\]

We solve Eqs. (21), (17), and (16) simultaneously to determine the three variables \(T\), \(V\), and \(\lambda\) that depend on \(t\), as follows. First, Eqs. (21) and (17) can be rewritten in the following forms, respectively, by applying a universal thermodynamic relation \(\frac{\partial U}{\partial V} + P = T\frac{\partial P}{\partial T}\) (Moore, 1963):

\[
T \frac{\partial \dot{V}}{\partial T} + \frac{\partial u}{\partial T} \dot{T} - K_R(T_R - T) = 0 \quad \text{and} \quad -(1 - \lambda) \frac{\partial \dot{T}}{\partial T} + \dot{\lambda} T \frac{\partial \dot{p}}{\partial T} = 0. \tag{22}
\]

It is possible to solve the simultaneous equations, Eqs. (22), (23), and (16) and deduce the solution \(\dot{T} = 0\) obtained by different methods (Rubin 1979a, 1979b) or presumed in early research (Curzon and Ahlborn, 1975), but here for brevity we use the known result \(\dot{T} = 0\) to obtain the solution for the variables other than \(T\) as follows. Substituting \(\dot{T} = 0\) into Eqs. (22) and (23), we obtain:

\[
T_{CA} \frac{\partial \dot{V}}{\partial T} - K_R(T_R - T_{CA}) = 0 \quad \text{and} \quad \frac{\partial \dot{T}_{CA}}{\partial T} = 0. \tag{24}
\]

where \(T_{CA}\) denotes the constant temperature during the CA process. Since \(\dot{\lambda} = 0\) as per Eq. (25), Eq. (16) becomes:

\[
-\lambda_{CA} K_R + (1 - \lambda_{CA}) \frac{\partial \dot{V}}{\partial T} = 0, \tag{26}
\]

where \(\lambda_{CA}\) is the constant value of \(\lambda\) in the CA process. Eliminating \((\partial P/\partial T)\dot{V}\) between Eqs. (24) and (26) gives the following solution of \(\lambda_{CA}\):

\[
\lambda_{CA} = 1 - \frac{T_{CA}}{T_R}. \tag{27}
\]

We can determine \(V\) by substituting \(T_{CA}\) into Eq. (24) if \(T_{CA}\) is obtained. However, we find that \(T_{CA}\) is determined by \(\dot{V}\), as follows. First, assuming Eq. (28) is the equation of state of the working fluid, we can determine \(V\) from Eq. (24) because \(\partial P/\partial T\) can be expressed by \(V\) and \(T\):

\[
P = P(V, T) = C_0 T^{n_1} V^{n_2}. \tag{28}
\]

Substituting Eq. (28) into Eq. (24), we obtain \(K_R(T_R - T_{CA}) = n_1 C_0 T_{CA}^{n_1} V^{n_2} \frac{\partial V}{\partial t}\). Then:

\[
V_{CA}(t) = V_{CA}(t_0) \exp[\xi(t - t_0)] \quad \text{for } n_2 = -1 \tag{29}
\]

or:

\[
V_{CA}(t)^{n_2+1} = V_{CA}(t_0)^{n_2+1} + (n_2 + 1) \xi(t - t_0) \quad \text{for } n_2 \neq -1. \tag{30}
\]

where:
\[ \xi \equiv \omega_{CA} \frac{T_R - T_{CA}}{T_{CA}^{n_1}}, \quad (31) \]
\[ \omega_{CA} \equiv \frac{k_R}{n_1 C_0}, \quad (32) \]

and the suffix \( CA \) denotes the CA process (which occurs from \( t_0 \) to \( t_1 \), as in Table 1). Equation (29) is applicable to a thermally perfect gas (irrespective of caloric perfectness), with constants \( C_0, n_1, \) and \( n_2 \) as given in Table 2. Then, Eqs. (31) and (32) respectively become:

\[ \xi = \omega_{CA} \left( \frac{T_R}{T_{CA}} - 1 \right) \quad \text{and} \]
\[ \omega_{CA} = \frac{k_R}{m R_g}. \quad (34) \]

Table 2  Parameters and their values for the equation of state.

| Parameter | \( C_0 \) | \( n_1 \) | \( n_2 \) |
|-----------|-----------|-----------|-----------|
| Thermally perfect gas \( ^a \) | \( m R_g \) | 1 | -1 |

\( ^a \) \( m \) is the mass of the working fluid and \( R_g \) is the gas constant per mass of working fluid.

Table 3  Correspondence of the usual variables to variables indicating expansion and compression.

| Stage          | \( V(t_0) \) | \( V(t_f) \) | \( T_R \) | \( T_{CA} \) | \( \omega_{CA} \) | \( \xi \) | Duration of CA process |
|----------------|--------------|--------------|-----------|-------------|-----------------|-------|------------------------|
| Expansion      | \( V_{TDC} \) | \( V_{BDC} \) | \( T_H \) | \( T_H \) | \( \omega_H \) | \( \xi_H \) | \( t_H \)               |
| Compression    | \( V_{BDC} \) | \( V_{TDC} \) | \( T_L \) | \( T_L \) | \( \omega_L \) | \( \xi_L \) | \( t_L \)               |

Hereafter, we use the new variable symbols to discriminate between the expansion and compression stages. Table 3 indicates the correspondence of the new symbols to the common symbols that do not discriminate between the two stages.

The temperature in the CA process, \( T_{CA} \), can be determined by Eq. (33) as a function of \( \xi \) as \( T_{CA} = \frac{T_R}{1 + \frac{\omega_{CA}}{\xi}} \). Then, we have the temperature in the heat-addition CA process:

\[ T_H = \frac{T_R}{1 + \frac{\omega_H}{\xi_H}} \quad (35) \]

and the temperature in the heat-disposal CA process:

\[ T_L = \frac{T_L}{1 + \frac{\omega_L}{\xi_L}} \quad (36) \]

Meanwhile, \( \xi_L \) must be negative and \( 1 + \frac{\xi_L}{\omega_L} > 0 \) because \( T_L > T_H \) and \( T_L > 0 \), respectively.

The volume at the beginning of the CA process \( V_{CA}(t_0) \) must equal \( V_{TDC} \) or \( V_{BDC} \) (depending on the stage, as given in Table 4), where \( V_{TDC} \) and \( V_{BDC} \) represent the volumes at the TDC and BDC, respectively.

Table 4  The volume in Eqs. (29) and (30).

| Stage          | \( V_{CA}(t_0) \) |
|----------------|------------------|
| Expansion stage | \( V_{TDC} \)   |
| Compression stage | \( V_{BDC} \)   |
The volume at the end of the adiabatic process, \( V(t_f) \) is \( V_{BDC} \) or \( V_{TDC} \) (depending on the stage). Then we obtain the following relation using Eqs. (20) and (28) with Table 2, and the function \( U = U(T) \), for a thermally and calorically perfect gas of the working fluid:

\[
V(t_{\text{exp}}) = \left( \frac{T_h}{T_c} \right)^{1/(y-1)} V_{BDC} \quad \text{and} \quad V(t_{\text{comp}}) = \left( \frac{T_h}{T_c} \right)^{1/(y-1)} V_{TDC},
\]

where the suffixes \( \text{exp} \) and \( \text{comp} \) respectively indicate the expansion and compression stages, and \( y \) denotes the specific heat ratio of the working fluid. On the other hand, Table 4 and Eqs. (29) and (33) lead to:

\[
V(t_{\text{exp}}) = V_{TDC} \exp \left[ \omega_H \left( \frac{T_h}{T_h} - 1 \right) t_H \right] \quad \text{and} \quad V(t_{\text{comp}}) = V_{BDC} \exp \left[ \omega_L \left( \frac{T_k}{T_c} - 1 \right) t_L \right],
\]

where \( t_H \) and \( t_L \) are the respective durations of the CA process in the expansion and compression stages.

Substituting Eqs. (37) and (38) into Eqs. (39) and (40), respectively, we obtain:

\[
\epsilon = \left( \frac{T_h}{T_c} \right)^{1/(y-1)} \exp \left[ \omega_H \left( \frac{T_h}{T_h} - 1 \right) t_H \right] \quad \text{and} \quad \epsilon = \left( \frac{T_h}{T_c} \right)^{1/(y-1)} \exp \left[ - \omega_L \left( \frac{T_k}{T_c} - 1 \right) t_L \right],
\]

where \( \epsilon \) is the compression ratio \( \epsilon \equiv V_{BDC}/V_{TDC} \). By equating the right-hand sides of Eqs. (41) and (42), we obtain the condition of endoreversibility, Eq. (A.4). Thus, Eqs. (41) and (42) are identical under the condition of endoreversibility. By substituting:

\[
T_h = \Gamma \sqrt{T_H},
\]

which holds at the maximum power (Curzon and Ahlborn, 1975; Rubin 1979a), into Eq. (41), we obtain:

\[
\epsilon = \left( \frac{T_h}{T_c} \right)^{1/(y-1)} \exp \left[ \omega_H \left( \frac{1 - \frac{T_h}{T_H}}{\sqrt{K_H}} \right) t_H \right],
\]

where:

\[
\Gamma = \sqrt{K_H \sqrt{T_H} + K_L \sqrt{T_L}}
\]

This indicates that the compression ratio depends on the values of \( T_h, T_L, \omega_H, \omega_L, \) and \( t_H \) (or \( t_L \) due to the endoreversibility condition). That is, we cannot select an arbitrary value of the compression ratio to achieve the CA efficiency \( \eta_{mp} \) (thermal efficiency at the maximum power).

6. Cycle duration

In this section, we seek the relationship between the thermal efficiency \( \eta \) (and consequently the power \( P \)) and the ideally effective cycle duration \( \tau \equiv t_H + t_L \) (the duration of the adiabatic process is ideally set 0).

Eq. (6) leads to:

\[
\tau \equiv t_H + t_L = t_H \left( 1 + \frac{\sqrt{K_H}}{\sqrt{K_L}} \right).
\]

Therefore, we can find the relationship between \( \tau \) and \( \eta \) if \( \eta \) can be expressed as a function of \( t_H \). Then, the
relationship between $\tau$ and $\Pi$ is easily found as the relationship between $\eta$ and $\Pi$ is given in Eq. (7).

Substituting the relationship $Q_H/t_H = K_H(T_H - T_h)$ derived from Eq. (11) into Eq. (8), we obtain the following formula that expresses $T_h$ as a function of $\eta$:

$$T_h = \frac{\tau_L + T_H \sqrt{\frac{K_H}{K_L}}}{1 + \sqrt{\frac{K_H}{K_L}}}$$

(47)

In addition, $T_h$ and $T_c$ depend on each other as follows:

$$T_c = \frac{\tau_L}{1 + \sqrt{\frac{K_H}{K_L}}}$$

(48)

which is obtained by eliminating $t_L$ between Eqs. (6) and (A.4) (the endoreversibility condition). When $\eta = 0$, i.e., $T_h = T_c$, Eq. (48) or (47) gives the following value:

$$T_h = T_c = \frac{\tau_L + T_H \sqrt{\frac{K_H}{K_L}}}{1 + \sqrt{\frac{K_H}{K_L}}}$$

(49)

For the CA engine, it is impossible to find an algebraic solution of $T_h$ and $T_c$ for the set of Eqs. (41) and (42). That is, one cannot algebraically express $T_h$ and $T_c$, as a function of $t_H$ and $t_L$, and numerical calculation is not so easy due to high nonlinearity. However, it is possible and easier to determine $t_H$ from Eqs. (41) or (42) by substituting the values of $T_h$ and $T_c$ for a given $\varepsilon$. Furthermore, the values of $T_h$ and $T_c$ can be easily obtained from Eqs. (47) and (48), respectively, for a given $\eta$. Therefore, we determine $t_H$ for a given $\varepsilon$ after determining $T_h$ or $T_c$ for a given $\eta$, instead of determining $T_h$ and $T_c$ for given $t_H$ and $\varepsilon$.

Substituting Eqs. (47) and (48) into Equation (41), we obtain:

$$t_H = \frac{1 - \eta_{\text{Carnot}} \tau_H + (1 - \eta) \sqrt{\frac{K_H}{K_L}} \ln \left\{ \sqrt{\frac{K_H}{K_L}} \right\}}{(1 - \eta_{\text{Carnot}}) \omega_H}$$

and

$$\frac{dt_H}{d\eta} = \frac{1}{(1 - \eta_{\text{Carnot}}) \omega_H} \left\{ \frac{1 - \eta_{\text{Carnot}}}{(1 + \sqrt{\frac{K_H}{K_L}} \ln \left\{ \sqrt{\frac{K_H}{K_L}} \right\}} - \frac{1 - \eta}{1 - \eta} + \sqrt{\frac{K_H}{K_L} \frac{1}{\omega_H} \ln \left\{ \sqrt{\frac{K_H}{K_L}} \right\}} \right\}$$

(51)

Since $\eta_{\text{Carnot}} - \eta > 0$, $1 - \eta_{\text{Carnot}} > 0$, $1 - \eta > 0$, and $\gamma - 1 > 0$, the signs of $dt_H/d\eta$ and $d\eta/dt_H$ are determined by the sign and magnitude of $\ln \left\{ \sqrt{\frac{K_H}{K_L}} \right\}$ and $d\eta/dt_H$ may be negative.

Meanwhile, the following formulas can be obtained from Eq. (50):

$$t_{H,H=0} = \left( \frac{1 - \eta_{\text{Carnot}} + \sqrt{\frac{K_H}{K_L}}}{\eta_{\text{Carnot}}} \right) \ln \left\{ \frac{\tau_H}{\eta_{\text{Carnot}}} \right\} \omega_H$$

(52)

and

$$t_{H,mp} = \left( \frac{\sqrt{\frac{K_H}{K_L} + 1 - \eta_{mp}}}{\eta_{mp}} \right) \ln \left\{ \left( 1 - \eta_{mp} \right)^{\gamma - 1} \right\} \omega_H$$

(53)

where we used the formula $0 = \eta_{mp}^2 - 2\eta_{mp} + \eta_{\text{Carnot}}$, which can be derived from the definitions, Eq. (1) and $\eta_{\text{Carnot}} = 1 - T_L/T_H$.

Further discussion is given in Sections 7 and 8, where $d\eta/dt_H$ is negative for a sample calculation and the CA limit $\eta_{mp}$ is not achievable for a CA engine with the working fluid of a thermally and calorically perfect gas unless $\varepsilon \geq \left( \sqrt{T_H/T_L} \right)^{1/(\gamma - 1)}$, and the highest compression ratio of the CA engine is $(T_H/T_L)^{1/(\gamma - 1)}$.

7. Sample calculation

In this section, we show the calculation results for the CA engine with the working fluid of a thermally and
calorically perfect gas. The results are easily obtained by substituting the values given in Table 5 into the solution obtained earlier.

### Table 5 Specifications for the sample calculation.

| Specifications                        | Value          |
|---------------------------------------|----------------|
| Heat transfer area [cm²]              | 64             |
| Heat transfer coefficient [W/m²/K]    | 10             |
| Hot heat bath temperature [K]         | 900            |
| Cold heat bath temperature [K]        | 300            |
| Mixture mass [kg]                     | 7.44 × 10⁻⁴    |
| Pressure at BDC [Pa]                  | 10⁵            |
| Gas constant [J/kg/K]                 | 2.869 × 10²    |

- a: The cross-sectional area of the piston.
- b: Fixed at the value of the air in a chamber with a compression ratio of 10, a stroke length of 90 mm, a Tc of 300 K, and 0.1 MPa at BDC. The variable volume of the working fluid at BDC and the variable stroke length are derived in Appendix B as functions of Tc and the compression ratio.
- c: For the working fluid of air (McAllister et al., 2011).
- d: Fixed at the value for air at 1 atm and 1000 K (McAllister et al., 2011).

Table 6 indicates the equations used for calculations with the results presented in Figs. 2–5 along with the cycle duration τ.

### Table 6 Equations used in the sample calculation.

| Figure 2 a | Eqs. (6), (34), (41), (46), (47), and (48) |
| Figure 3   | As for Figure 2                             |
| Figure 4   | Eqs. (6), (34), (41), (46), (47), and (A.1) |
| Figure 5 a | Eq. (7) and those for Figure 2              |

- a: The formula η = 1 - Tc/Tₜ is also used.

Figures 2 shows the variation of η for the CA engine.

1. η becomes greater for the CA engine when τ becomes shorter at each compression ratio. Also, at each engine speed (reciprocal of τ) the efficiency of the CA engine increases with increasing compression ratio.
2. The CA engine has the maximum of ε, i.e., ε_max and a trivial minimum, as discussed later in Section 8.
3. There is a range of ε that cannot achieve the CA efficiency η_mp shown in Fig. 1. This range is ε < ε_cr, which is discussed in Section 8. For the sample case, ε_cr ≅ 5.1.

Fig. 2  The thermal efficiency vs. cycle duration.

Fig. 3  The temperature in each CA process vs. cycle duration. Orange and blue lines represent Tₜ and T_c, respectively.
Figure 3 shows $T_h$ and $T_c$. The former increases and the latter decreases when $\varepsilon$ becomes larger, which is the consequence of the equations listed in Table 6. It is notable that the maximum of $\tau$ exists for each $\varepsilon$ because of Eq. (49), which corresponds to $\eta = 0$.

Figure 4 shows $Q_H$, which decreases at each $\tau$ when $\varepsilon$ increases because $T_h$ increases along with $\varepsilon$, as shown in Fig. 3.

Figure 5 shows the power $\Pi$. The maximum of $\Pi$ exists and its value is common for all $\varepsilon$, with the exception that the maximum power cannot be achieved when $\varepsilon < \varepsilon_{cr}^2$, which is a consequence of the mapping of Fig. 2 through Fig. 1.

**8. Discussion**

Here, we clarify the reasons for the following two findings of Section 7 (Figure 2):

1. The CA engine has the maximum of $\varepsilon$ in addition to the minimum.
2. There is a range of $\varepsilon$ that cannot achieve the CA efficiency $\eta_{mp}$ shown in Figure 1. This range is $\varepsilon < \varepsilon_{cr}$.

First, when $\varepsilon = 1$, Eq. (52) gives $t_{h,\eta=0} = 0$. When $\varepsilon < 1$, $t_{h,\eta=0}$ is negative and, consequently, $\tau < 0$ due to Eq. (46). Therefore, the value 1 is the minimum of $\varepsilon$, although the condition $\varepsilon \geq 1$ is trivial and we may assume there is no minimum of $\varepsilon$.

The reason for the existence of the maximum of $\varepsilon$ is a little bit complex and we think about it according to the following two steps:

i. Physically speaking, a too-small compression ratio $\varepsilon$ cannot attain a given value of the ratio $T_h/T_c$ by the adiabatic process. In other words, $t_H$ cannot be positive when $\varepsilon \leq (T_h/T_c)^{1/(y-1)}$ for the working fluid of a thermally and calorically perfect gas since the CA process requires a change in the volume of the working fluid, i.e., the volume expands in the CA heat addition and it erodes the extent of volume expansion available to the adiabatic process for a given $\varepsilon$, as clearly expressed by Eq. (41). Note that Eq. (41) indicates that the ratio $T_h/T_c$ (and $\eta$) decreases with increasing $t_H$ (and $\tau$) for a given $\varepsilon$ (Figure 3). Therefore, $t_{h,\eta=0}$ (the value of $t_h$ at the minimum of $\eta$, $\eta = 0$) is the maximum of $t_H$ and, consequently, it gives the maximum of $\tau$ (due to Eq. (46)) for a given $\varepsilon$ of the CA engine.

ii. Although $t_{h,\eta=0}$ increases for a larger $\varepsilon$, owing to Eq. (52), the Carnot limit determines the maximum of $T_h/T_c$.

Mathematically speaking, by substituting $T_h = T_H$ and $T_c = T_L$ (the values at the Carnot limit that give the maximum of the ratio $T_h/T_c$ and $\eta$) into Eqs. (41) or (42), we obtain the following maximum value of $\varepsilon$ for the CA engine:

$$\varepsilon_{max} = \left(\frac{T_H}{T_L}\right)^{1/(y-1)}. \quad (54)$$

For point 2 about $\eta_{mp}$, the value of $\varepsilon_{cr}$ can be found as follows. From Eq. (41), we obtain the following for $t_H = 0$:
\[ \varepsilon = \left( \frac{T_h}{T_c} \right)^{1/(\gamma-1)}. \]  (55)

Furthermore, we obtain the following from the relationship \( \eta = 1 - T_c/T_h \):

\[ \frac{T_h}{T_c} = \frac{1}{1-\eta}. \]  (56)

Substituting Eq. (56) into Eq. (55), we obtain:

\[ \eta_{CA,T_H=0} = 1 - \varepsilon^{1-\gamma}, \]  (57)

which indicates that \( \eta_{CA,T_H=0} \) (i.e., \( \eta_{CA,T=0} \)) monotonically increases with \( \varepsilon \) (Figure 2). When \( \eta_{CA,T_H=0} = \eta_{mp} \), \( \varepsilon \) becomes \( \varepsilon_{cr} \). Therefore, using Eq. (57), we obtain:

\[ \varepsilon_{cr} = \left( 1 - \eta_{mp} \right)^{1/(1-\gamma)} = \left( \frac{T_h}{T_c} \right)^{1/(\gamma-1)}. \]  (58)

Meanwhile, substituting Eq. (54) into Eq. (57), we obtain \( \eta_{CA,T_H=0} = \eta_{Carnot} \) for \( \varepsilon = \varepsilon_{max} \), which is consistent with Figure 2.

9. Summary and Conclusion

Using the method of adjoint equations described in our previous works (Kojima, 2017a, 2017b), we derived a formula to express the temperature in the Curzon–Ahlborn (CA) process as a function of the rate constant (or time constant) of either expansion or compression of the volume of the working fluid. Using this formula, we can show the power and thermal efficiency of the CA engine along with its engine speed and compression ratio, i.e., an ideal performance of heat engines with external heat sources. We summarize the main results of this work as follows:

1. At each engine speed the efficiency of the CA engine increases with increasing compression ratio while at each compression ratio the efficiency increases with increasing engine speed.
2. The CA engine has its own limit of cycle duration (or engine speed). For each feasible \( \varepsilon \), the CA engine has the maximum of \( \tau \) (slowest limit of engine speed) where the power and efficiency disappear.
3. The CA engine has the maximum of \( \varepsilon \), which corresponds to the Carnot limit.
4. For the CA engine, there is a range of \( \varepsilon \) that cannot achieve the CA efficiency (\( \eta_{mp} \equiv 1 - \sqrt{T_c/T_H} \)). This range is \( \varepsilon < \left( 1 - \eta_{mp} \right)^{1/(1-\gamma)} \).

Appendix A: Derivation of Eq. (8)

The heats transferred from/into the heat baths in the CA processes are:

\[ Q_H = K_H(T_H - T_h)t_H \text{ and } Q_L = K_L(T_c - T_L)t_L. \]  (A.1, A.2)

From the above, we can obtain:

\[ 1 - \eta = \frac{T_c}{T_h} = \frac{Q_L + Q_L}{Q_H - K_Ht_H}. \]  (A.3)

Then, the endoreversibility condition:

\[ \frac{Q_H}{T_h} - \frac{Q_L}{T_c} = 0 \]  (A.4)

becomes the following equation:

\[ Q_L = (1 - \eta)Q_H. \]  (A.5)
The substitution of Eqs. (A.5) and (6) into (A.3) leads to Eq. (8).

**Appendix B: Variable volume of the working fluid at BDC and variable stroke length**

According to the equation of state, the volume, mass, pressure, and temperature of the working fluid are related to each other. Thus, we fix the mass \( m \) and pressure \( P_{BDC} \) of the working fluid at the BDC.

The equation of state for a thermally perfect gas leads to:

\[
V_{BDC} = \frac{mR_T}{P_{BDC}},
\]

i.e., \( V_{BDC} \) is a function of \( T_c \).

Meanwhile, the stroke length \( l \) is a function of \( \epsilon \) and \( V_{BDC} \), owing to the geometric properties of the chamber:

\[
l = V_{BDC} \frac{\epsilon-1}{\epsilon}.
\]

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