Connecting the geometric measure of entanglement and entanglement witnesses

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The geometric measure of entanglement is an approach to quantifying entanglement that is based on the Hilbert-space distance (or, equivalently, angle) between pure states and their best unentangled approximants. An entanglement witness is an operator that reveals entanglement for a given entangled state. A connection is identified between entanglement witnesses and the geometric measure of entanglement. This offers a new interpretation of the geometric measure of entanglement of a state, and renders it experimentally verifiable, doing so most readily for states that are pure.

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Introduction: Entanglement is now recognized as a resource central to much of quantum information processing [1]. Thus, characterizing and quantifying entanglement has emerged as a prominent theme in quantum information theory. Achievements in quantifying mixed-state entanglement lie primarily in bipartite settings [2]. For multipartite mixed states the issue of entanglement evidently presents even greater challenges.

Recent research on quantifying multipartite entanglement has explored a geometric approach. First introduced by Shimony [3] in the setting of bipartite pure states, this geometric approach has been generalized to multipartite settings by Barnum and Linden [4], and further developed in Ref. [5]. In the present Paper, our aim is to identify a connection between two apparently distinct aspects of entanglement: entanglement witnesses [6] and the geometric approach to entanglement. As entanglement witnesses are observables, and can, in principle, be measured, the geometric measure of entanglement thus becomes verifiable experimentally.

Geometric measure of entanglement (GME): We begin by briefly reviewing the formulation of the GME in the pure-state setting. Let us start by analyzing a multipartite system comprising $n$ parts, each of which may have a distinct Hilbert space. Consider the general $n$-partite pure state, expanded in the local bases\[{\psi} = \sum_{p_1\cdots p_n} \chi_{p_1\cdots p_n} |e_{p_1}\rangle \cdots |e_{p_n}\rangle\]. As shown in Ref. [5], the closest separable pure state

\[|\phi\rangle \equiv \otimes_{i=1}^{n} |\phi^{(i)}\rangle = \otimes_{i=1}^{n} \sum_{p_i} c_{p_i}^{(i)} |e_{p_i}\rangle \] (1)

obeys the equations

\[\sum_{p_1\cdots p_n} \chi_{p_1\cdots p_n}^{*} c_{p_1}^{(1)} \cdots c_{p_n}^{(n)} = \Lambda c_{p_i}^{(i)*}, \] (2a)

\[\sum_{p_1\cdots p_n} \chi_{p_1\cdots p_n}^{*} c_{p_1}^{(1)*} \cdots c_{p_i}^{(i)*} \cdots c_{p_n}^{(n)*} = \Lambda c_{p_i}^{(i)}, \] (2b)

where the eigenvalue $\Lambda \in [-1,1]$ is associated with the Lagrange multiplier enforcing the constraint $\langle \phi | \phi \rangle = 1$, and $\sim$ denotes exclusion. As discussed in Ref. [5], the spectrum of eigenvalues $\{\Lambda\}$ can be interpreted as the cosine of the angle between $|\psi\rangle$ and stationary states $\{|\phi\rangle\}$. Furthermore, the largest eigenvalue $\Lambda_{\text{max}}$, which we call the entanglement eigenvalue, corresponds to the separable state closest to $|\psi\rangle$. An equivalent way to view $\Lambda_{\text{max}}$ is via

\[\Lambda_{\text{max}}^2 (|\psi\rangle) = \max_{\text{separable } \phi} ||\langle \phi | \psi \rangle||^2 = \max_{\text{separable } \phi} \text{Tr} (|\phi\rangle \langle \psi | \psi \rangle). \] (3)

The precise measure of the entanglement of $|\psi\rangle$ adopted in Ref. [5] is $E_{\text{min}} = 1 - \Lambda_{\text{max}}^2$.

Entanglement witness (EW): The entanglement witness $W$ for an entangled state $\rho$ is defined to be an operator that is Hermitian and obeys the following conditions:

(i) $\text{Tr}(W \sigma) \geq 0$ for all separable states $\sigma$, and

(ii) $\text{Tr}(W \rho) < 0$.

Here, we wish to establish a relationship between $\Lambda_{\text{max}}$ for the entangled pure state $|\psi\rangle$ and the optimal element of the set of entanglement witnesses $W$ for $|\psi\rangle$ that have the specific form

\[W = \lambda^2 \mathbb{1} - |\psi\rangle \langle \psi |, \] (4)

this set being parametrized by the real, non-negative number $\lambda^2$. By optimal we mean that, for this specific form of witnesses, the value of the “detector” $\text{Tr} (W |\psi\rangle \langle \psi |)$ is as negative as can be.

In order to satisfy condition (i) we must ensure that, for any separable state $\sigma$, we have $\text{Tr}(W \sigma) \geq 0$. As the density matrix for any separable state can be decomposed into a mixture of separable pure states [i.e., $\sigma = \sum_i |\phi_i\rangle \langle \phi_i |$ where $\{|\phi_i\rangle\}$ are separable pure states], condition (i) will be satisfied as long as $\text{Tr}(W |\psi\rangle \langle \psi |) \geq 0$ for all separable pure states $|\phi\rangle$. This condition is equivalent to

\[\lambda^2 - ||\langle \psi | \phi \rangle||^2 \geq 0 \text{ (for all separable } |\phi\rangle), \] (5)

which leads to

\[\lambda^2 \geq \max_{|\phi\rangle} ||\langle \psi | \phi \rangle||^2 = \Lambda_{\text{max}}^2 (|\psi\rangle). \] (6)
Similarly, for the states $\ket{\psi}$, this gives $\lambda^2 - 1 < 0$. Thus, we have established the range of $\lambda$ for which $\lambda^2 \mathbb{I} - \ket{\psi}\bra{\psi}$ is a valid EW for $\ket{\psi}$:

$$\Lambda_{\text{max}}^2(\ket{\psi}) \leq \lambda^2 < 1.$$ (7)

With these preliminaries in place, we can now establish the connection we have been seeking. Of the specific family $\mathcal{W}$ of entanglement witnesses for $\ket{\psi}$ that we have been considering, the one of the form $\mathcal{W}_{\text{opt}} = \Lambda_{\text{max}}^2(\ket{\psi})\mathbb{I} - \ket{\psi}\bra{\psi}$ is optimal, in the sense that it achieves the most negative value for the detector $\text{Tr}(\mathcal{W}_{\text{opt}}|\psi\rangle\langle\psi|)$:

$$\min_{\mathcal{W}} \text{Tr}(\mathcal{W}|\psi\rangle\langle\psi|) = \text{Tr}(\mathcal{W}_{\text{opt}}|\psi\rangle\langle\psi|) = -E_{\text{sin}^2}(\ket{\psi}),$$ (8)

where $\mathcal{W}$ runs over the class $\mathcal{H}$ of witnesses.

Some illustrative examples: For the state $\ket{\text{GHZ}} \equiv (\ket{000} + \ket{111})/\sqrt{2}$ the optimal witness is

$$\mathcal{W}_{\text{GHZ}} = \frac{1}{2} \mathbb{I} - \ket{\text{GHZ}}\bra{\text{GHZ}}$$ (9)

and $\text{Tr}(\mathcal{W}_{\text{GHZ}}|\text{GHZ}\rangle\langle\text{GHZ}|) = -E_{\text{sin}^2}(\ket{\text{GHZ}}) = -1/2$.

Similarly, for the states $\ket{W} \equiv ((001) + |010) + |100)/\sqrt{3}$ and $\tilde{W} \equiv ((110) + |101) + |011)/\sqrt{3}$ we have

$$\mathcal{W}_W = \frac{4}{9} \mathbb{I} - \ket{W}\bra{W} \quad \text{and} \quad \tilde{W}_W = \frac{4}{9} \mathbb{I} - \tilde{W}|\tilde{W}\rangle$$ (10)

and $\text{Tr}(\mathcal{W}_W|\ket{W}\bra{W}|) = -E_{\text{sin}^2}(\ket{W}) = -5/9$, and similarly for $\tilde{W}_W$. For the four-qubit state $\ket{\Psi} \equiv (0011) + (0101)+|1010)+|1001)+|1010)+|1100))/\sqrt{6}$ the optimal witness is

$$\mathcal{W}_\Psi = \frac{3}{8} \mathbb{I} - \ket{\Psi}\bra{\Psi}$$ (11)

and $\text{Tr}(\mathcal{W}_\Psi|\ket{\Psi}\bra{\Psi}|) = -E_{\text{sin}^2}(\ket{\Psi}) = -5/8$. In passing, we note that linear combinations of witnesses—preferably optimal ones—can be used to detect entanglement for mixed states, as we shall illustrate later. We also note that the non-optimal witnesses can also be of use, e.g., in classifying and detecting distinct types of entangled states; see Ref. [5]. Furthermore, as entanglement witnesses are Hermitian operators, they can, at least in principle, be realized experimentally.

Mixed states: We conclude by briefly commenting on the EW/GME connection for mixed states. The GME can be generalized to mixed states $\rho$ via the convex hull construction (indicated by “co”). The essence of this construction is a minimization over all decompositions $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ into pure states:

$$E(\rho) \equiv (\text{co} E_{\text{sin}^2})(\rho) = \min_{\{p_i,\psi_i\}} \sum_i p_i E_{\text{sin}^2}(\ket{\psi_i}).$$ (12)

Condition (ii) requires that $\text{Tr}(\mathcal{W}|\psi\rangle\langle\psi|) < 0$, in order for $\mathcal{W}$ to be a valid EW for $\ket{\psi}$; this gives $\lambda^2 - 1 < 0$. Thus, we have established the range of $\lambda$ for which $\lambda^2 \mathbb{I} - \ket{\psi}\bra{\psi}$ is a valid EW for $\ket{\psi}$:

$$\Lambda_{\text{max}}^2(\ket{\psi}) \leq \lambda^2 < 1.$$ (7)

Before ending our discussion of mixed states, we elaborate on a point made earlier, i.e., that linear combinations (with non-negative coefficients) of optimal entanglement witnesses can be used to establish entanglement of mixed states. For illustration, consider the following family of mixed states:

$$\rho(x) \equiv x|W\rangle\langle W| + (1-x)|\tilde{W}\rangle\langle\tilde{W}|,$$ (14)

the GME of which is calculated analytically in Ref. [6] and which is entangled for all values of $x \in [0,1]$. We can actually construct EW’s that establish the entanglement of $\rho(x)$. Consider a linear combination of optimal witnesses of $|W\rangle$ and $|\tilde{W}\rangle$:

$$\mathcal{W}(y) \equiv y\mathcal{W}_W + (1-y)\mathcal{W}_{\tilde{W}}.$$ (15)
with $y \in [0, 1]$. If, for any given $x$, there exists a value of $y \in [0, 1]$ such that $\text{Tr}(W(y)\rho(x)) < 0$ then $\rho(x)$ is evidently entangled. Figure 1 shows that this is indeed the case (see captions for details). This illustrates the usefulness of linear combinations of pure-state optimal witnesses. It would be interesting to know whether witnesses for bound entangled states be constructed by this approach.

Concluding remarks: Although the observations we have made are, from a technical standpoint, elementary, we nevertheless find it intriguing that two distinct aspects of entanglement—the geometric measure of entanglement and entanglement witnesses—are so closely related. Furthermore, this connection sheds new light on the content of the geometric measure of entanglement. In particular, as entanglement witnesses are Hermitian operators, they can, at least in principle, be realized experimentally. Their connection with the geometric measure of entanglement ensures that the geometric measure of entanglement can, at least in principle, be verified experimentally.

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