THE SYZYGIES OF SOME THICKENINGS OF DETERMINANTAL VARIETIES

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Abstract. The vector space of $m \times n$ complex matrices ($m \geq n$) admits a natural action of the group $\text{GL} = \text{GL}_m \times \text{GL}_n$ via row and column operations. For positive integers $a, b$, we consider the ideal $I_{a \times b}$ defined as the smallest $\text{GL}$–equivariant ideal containing the $b$-th powers of the $a \times a$ minors of the generic $m \times n$ matrix. We compute the syzygies of the ideals $I_{a \times b}$ for all $a, b$, together with their $\text{GL}$–equivariant structure, generalizing earlier results of Lascoux for the ideals of minors ($b = 1$), and of Akin–Buchsbaum–Weyman for the powers of the ideals of maximal minors ($a = n$).

1. Introduction

For positive integers $m \geq n$, we consider the ring $S = \text{Sym}(\mathbb{C}^m \otimes \mathbb{C}^n)(= \mathbb{C}[z_{ij}])$ of polynomial functions on the vector space of $m \times n$ matrices with entries in the complex numbers. The ring $S$ admits an action of the group $\text{GL} = \text{GL}_m(\mathbb{C}) \otimes \text{GL}_n(\mathbb{C})$, and it decomposes into irreducible $\text{GL}$–representations according to Cauchy’s formula:

$$S = \bigoplus_{\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)} S_{\lambda} \mathbb{C}^m \otimes S_{\lambda} \mathbb{C}^n,$$

where $S_{\lambda}$ denotes the Schur functor associated to a partition $\lambda$. For each $\lambda$, we let $I_{\lambda}$ denote the ideal in $S$ generated by the irreducible representation $S_{\lambda} \mathbb{C}^m \otimes S_{\lambda} \mathbb{C}^n$. Every ideal $I \subset S$ which is preserved by the $\text{GL}$-action is a sum of ideals $I_{\lambda}$; such ideals $I$ have been classified and their geometry has been studied by De Concini, Eisenbud and Procesi in the 80s [dCEP80]. Nevertheless, their syzygies are still mysterious, and in particular the following problem remains unsolved:

Problem 1.1. Describe the syzygies of the ideals $I_{\lambda}$, together with their $\text{GL}$-equivariant structure.

The goal of our paper is to solve this problem in the case when $\lambda$ is a rectangular partition, which means that there exist positive integers $a, b$ such that $\lambda_1 = \cdots = \lambda_a = b$ and $\lambda_i = 0$ for $i > a$ (alternatively, the Young diagram associated to $\lambda$ is the $a \times b$ rectangle). In this case we write $\lambda = a \times b$ and $I_{\lambda} = I_{a \times b}$. One can think of $I_{a \times b}$ as the smallest $\text{GL}$-equivariant ideal which contains the $b$-th powers of the $a \times a$ minors of the generic matrix of indeterminates $Z = (z_{ij})$. What distinguishes the ideals $I_{a \times b}$ among all the $I_{\lambda}$’s is that they define a scheme without embedded components, so from a geometric point of view they form the simplest class of $\text{GL}$-equivariant ideals after the reduced (and prime) ideals of minors. Examples of ideals $I_{a \times b}$ include:

- $I_{a \times 1} = I_{a}$, the ideal generated by the $a \times a$ minors of $Z$.
- $I_{n \times b} = I_{b}^{n}$, the $b$-th power of the ideal $I_{n}$ of maximal minors of $Z$.

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• \( I_{1 \times b} \), the ideal of \( b \times b \) permanents of \( Z \): here by a \( b \times b \) permanent of \( Z \) we mean the permanent of a \( b \times b \) matrix obtained by selecting \( b \) rows and \( b \) columns of \( Z \), not necessarily distinct; for instance, when \( m = n = 2 \) we have:

\[
I_{1 \times 2} = (z_{11}^2, z_{12}^2, z_{21}^2, z_{22}^2, z_{11}z_{12}, z_{11}z_{21}, z_{12}z_{22}, z_{21}z_{22}, z_{11}z_{22} + z_{12}z_{21}).
\] (1.1)

To state our main result, we need to introduce some notation. We write \( \text{Rep}_{GL} \) for the representation ring of the group GL, and for a given GL-representation \( M \), we let \([M] \in \text{Rep}_{GL}\) denote its class in the representation ring. We let

\[
B_{i,j}(I_{a \times b}) = \text{Tor}_i^S(I_{a \times b}, \mathbb{C})_j
\] (1.2)

denote the vector space of \( i \)-syzygies of degree \( j \) of \( I_{a \times b} \). We encode the syzygies of \( I_{a \times b} \) into the equivariant Betti polynomial

\[
B_{a \times b}(z, w) = \sum_{i,j \in \mathbb{Z}} [B_{i,j}(I_{a \times b})] \cdot w^i \cdot z^j \in \text{Rep}_{GL}[z, w],
\] (1.3)

so the variable \( z \) keeps track of the internal degree, while \( w \) keeps track of the homological degree.

If \( r, s \) are positive integers, \( \alpha \) is a partition with at most \( r \) parts \((\alpha_i = 0 \text{ for } i > r)\) and \( \beta \) is a partition with parts of size at most \( s \) \( (\beta_1 \leq s) \), we construct the partition

\[
\lambda(r, s; \alpha, \beta) = (r + \alpha_1, \ldots, r + \alpha_r, \beta_1, \beta_2, \ldots).
\] (1.4)

This is easiest to visualize in terms of Young diagrams: one starts with an \( r \times s \) rectangle, and attach \( \alpha \) to the right and \( \beta \) to the bottom of the rectangle. If \( r = 4, s = 5, \alpha = (4, 2, 1), \beta = (3, 2) \), then

\[
\lambda(r, s; \alpha, \beta) = \begin{array}{cccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\] (1.5)

We write \( \mu' \) for the conjugate partition to \( \mu \) (obtained by transposing the Young diagram of \( \mu \)) and consider the polynomials \( h_{r \times s} \in \text{Rep}_{GL}[z, w] \) given by

\[
h_{r \times s}(z, w) = \sum_{\alpha, \beta} [S_{\lambda(r, s; \alpha, \beta)} \mathbb{C}^m \otimes S_{\mu' \times (\lambda(r, s; \alpha, \beta)) \mathbb{C}^n}] \cdot z^{r-s+|\alpha|+|\beta|} \cdot w^{|\alpha|+|\beta|},
\] (1.6)

where the sum is taken over partitions \( \alpha, \beta \) such that \( \alpha \) is contained in the \( (m-r) \times (n-r) \) rectangle \( (\alpha_1 \leq n-r, \alpha'_1 \leq \min(r, s)) \) and \( \beta \) is contained in the \( (m-r) \times \min(r, s) \) rectangle \( (\beta_1 \leq \min(r, s) \text{ and } \beta'_1 \leq m-r) \). We also need to introduce the Gauss polynomial \( \binom{r+s}{r}_w \in \mathbb{Z}[w] \),

\[
\binom{r+s}{r}_w = \sum_{s \geq t_1 \geq \cdots \geq t_r \geq 0} w^{t_1 + \cdots + t_r},
\] (1.7)

which is the generating function for partitions contained inside the \( r \times s \) rectangle. Note that \( \binom{r+s}{r}_w \) is the Poincaré polynomial of the Grassmannian of \( r \)-dimensional subspaces of an \( (r+s) \)-dimensional vector space, and also that \( \binom{r+s}{1}_w = \binom{r+s}{r}_w \) is the usual binomial coefficient. Our main result is:

**Theorem 1.2.** The equivariant Betti polynomial of \( I_{a \times b} \) is

\[
B_{a \times b}(z, w) = \sum_{q=0}^{n-a} h_{(a+q) \times (b+q)} \cdot w^{q^2+2q} \cdot \binom{q + \min(a, b) - 1}{q} w^2
\]
When \( b = 1 \), this recovers the result of Lascoux on syzygies of determinantal varieties \([Las78]\). When \( a = n \), we obtain the syzygies of the powers of the ideals of maximal minors, as originally computed by Akin–Buchsbaum–Weyman \([ABW81]\).

**Example 1.3.** When \( m = n = 2 \), the ideal \( I_{1 \times 2} \) from (1.1) has the equivariant Betti polynomial
\[
B_{1 \times 2}(z, w) = h_{1 \times 2} + h_{2 \times 3} \cdot w^3,
\]
where
\[
h_{1 \times 2} = ([\text{Sym}^2 \mathbb{C}^2 \otimes \text{Sym}^2 \mathbb{C}^2] \cdot z^2 + ([\text{Sym}^3 \mathbb{C}^2 \otimes S_{2,1} \mathbb{C}^2] + [S_{2,1} \mathbb{C}^2 \otimes \text{Sym}^3 \mathbb{C}^2]) \cdot z^3 \cdot w + [S_{3,1} \mathbb{C}^2 \otimes S_{3,1} \mathbb{C}^2] \cdot z^4 \cdot w^2,
\]
and
\[
h_{2 \times 3} = [S_{3,3} \mathbb{C}^2 \otimes S_{3,3} \mathbb{C}^2] \cdot z^6.
\]
The equivariant Betti table (where the \((i, j)\)-entry is \([B_{i,i+j}(I_{1 \times 2})] \in \text{Rep}_{\text{GL}}\), represented pictorially in terms of Young diagrams; as in (1.5) we use empty boxes for the \( r \times s \) rectangle inside \( \lambda(r, s; \alpha, \beta) \) and \( \lambda(r, s; \beta', \alpha') \), blue boxes for the partitions \( \alpha, \alpha' \) and green boxes for the partition \( \beta, \beta' \)) then looks like

```
\[
\begin{array}{cccc}
\boxempty & \boxtimes & \boxtimes & \boxtimes \\
\boxtimes & \boxtimes & \boxplus & \boxtimes \\
- & - & - & - \\
- & - & - & - \\
\end{array}
\]
```

Taking dimensions of representations (\( \dim(\text{Sym}^r \mathbb{C}^2) = r + 1 \), \( \dim(S_{r,1} \mathbb{C}^2) = r \), \( \dim(S_{r,r} \mathbb{C}^2) = 1 \)), we get the usual Betti table, which can be verified for instance using Macaulay2 \([GS]\):

```
\[
\begin{array}{cccc}
9 & 16 & 9 & - \\
- & - & - & - \\
\end{array}
\]
```

The proof of our main result is based on the following two ingredients:

- Joint work of the second author with Akin \([AW97, AW07]\): they introduce and study in the context of \( \mathfrak{gl}(m|n) \)-modules a family of linear complexes \( X^{r \times s}_* \), whose homology consists entirely of direct sums of ideals \( I_{(r+q) \times (s+q)} \). The polynomials \( h_{r \times s}(z, w) \) introduced in (1.6) precisely encode the terms of these linear complexes.
- The recent work of the authors on computing local cohomology with support in determinantal ideals: in \([RW14]\) we compute all the modules \( \text{Ext}^\bullet_S(I_{a \times b}, S) \), together with their GL-equivariant structure.

Based on these two ingredients, our strategy is as follows. We obtain a non-minimal resolution of \( I_{a \times b} \) via an iterated mapping cone construction involving the linear complexes \( X^{(a+q) \times (b+q)}_* \), \( q \geq 0 \). We then use the GL-equivariance to conclude that whenever cancellations occur for some of the terms of an \( X^{r \times s}_* \), they must in fact occur for all the terms of \( X^{r \times s}_* \). This implies that the minimal resolution of \( I_{a \times b} \) is also built out of copies of \( X^{(a+q) \times (b+q)}_* \), and it remains to determine the number of such copies, as well as their homological shifts. This is done by dualizing the minimal resolution and using the GL-equivariant description of \( \text{Ext}^\bullet_S(I_{a \times b}, S) \). We elaborate on this argument in Section 3 after we establish
some notational conventions in Section 2 and collect some preliminary results on functoriality of syzygies, on the complexes $X^{r \times s}$, and on the computation of Ext modules.

2. Preliminaries

2.1. Representation Theory \cite{FH91}, \cite[Ch. 2]{Wey03}. If $W$ is a complex vector space of dimension $\dim(W) = n$, a choice of basis determines an isomorphism between $\text{GL}(W)$ and the group $\text{GL}_n(\mathbb{C})$ of $n \times n$ invertible matrices. We will refer to $n$–tuples $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n$ as weights of the corresponding maximal torus of diagonal matrices. We say that $\lambda$ is a dominant weight if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Irreducible representations of $\text{GL}(W)$ are in one-to-one correspondence with dominant weights $\lambda$. We denote by $S_{\lambda} W$ the irreducible representation associated to $\lambda$. We write $|\lambda|$ for the total size $\lambda_1 + \cdots + \lambda_n$ of $\lambda$.

When $\lambda$ is a dominant weight with $\lambda_n \geq 0$, we say that $\lambda$ is a partition of $r = |\lambda|$. We will often represent a partition via its associated Young diagram which consists of left–justified rows of boxes, with $\lambda_i$ boxes in the $i$–th row: for example, the Young diagram associated to $\lambda = (5, 2, 1)$ is

\[
\begin{array}{cccccc}
& & & & & \\
& & & & \\
& & & \\
& & \\
& \\
\end{array}
\]

Note that when we’re dealing with partitions we often omit the trailing zeros. We define the length of a partition $\lambda$ to be the number of its non-zero parts, and denote it by $l(\lambda)$. If $l(\lambda) > \dim(W)$ then $S_{\lambda} W = 0$. The transpose $\lambda'$ of a partition $\lambda$ is obtained by transposing the corresponding Young diagram. For the example above, $\lambda' = (3, 2, 1, 1, 1)$, $l(\lambda) = 3$ and $l(\lambda') = 5$. If $\mu$ is another partition, we write $\mu \subset \lambda$ to indicate that $\mu_i \leq \lambda_i$ for all $i$, and say that $\mu$ is contained in $\lambda$.

For a pair of finite dimensional vector spaces $F, G$, we write $\text{GL}(F, G)$ (or simply $\text{GL}$ when $F, G$ are understood) for the group $\text{GL}(F) \times \text{GL}(G)$. If $M$ is a $\text{GL}(F, G)$-representation, we write

\[\langle S_{\lambda} F \otimes S_{\mu} G, M \rangle\]

for the multiplicity of the irreducible $\text{GL}$-representation $S_{\lambda} F \otimes S_{\mu} G$ inside $M$. If $M^\bullet$ is a cohomologically graded module, then we record the occurrences of $S_{\lambda} F \otimes S_{\mu} G$ inside the graded components of $M^\bullet$ by

\[\langle S_{\lambda} F \otimes S_{\mu} G, M^i \rangle = \sum_{i \in \mathbb{Z}} \langle S_{\lambda} F \otimes S_{\mu} G, M^i \rangle \cdot w^i, \tag{2.1}\]

where the variable $w$ encodes the cohomological degree (note a slight difference from (1.3), where $w$ was used for homological degree).

2.2. Functoriality of syzygies. It will be useful to think of the polynomial ring $S = \text{Sym}(\mathbb{C}^m \otimes \mathbb{C}^n)$ as a functor $S$ which assigns to a pair $(F, G)$ of finite dimensional vector spaces the polynomial ring $S(F, G) = \text{Sym}(F \otimes G)$. For each $a, b$ we obtain functors $I_{a \times b}$ which assign to $(F, G)$ the corresponding ideal $I_{a \times b}(F, G) \subset S(F, G)$. The syzygy modules in (1.2) become functors $B_{i,j}^{a \times b}(-, -)$, defined by

\[B_{i,j}^{a \times b}(F, G) = \text{Tor}_i^{S(F, G)}(I_{a \times b}(F, G), \mathbb{C})_j.\]

In fact, each $B_{i,j}^{a \times b}$ is a polynomial functor in the sense of \cite[Ch. I, Appendix A]{Mac95}. As such they decompose into a (usually infinite) direct sum indexed by pairs of partitions

\[B_{i,j}^{a \times b}(-, -) = \bigoplus_{\lambda, \mu} (S_{\lambda}(-) \otimes S_{\mu}(-)) \otimes \mathbb{C}^{m_{\lambda, \mu}}. \tag{2.2}\]
When evaluating $B_{i,j}^{s\times b}$ on a pair of vector spaces $(F,G)$, only finitely many terms on the right hand side of (2.2) survive, namely the ones for which $l(\lambda) \leq \dim(F)$ and $l(\mu) \leq \dim(G)$. The multiplicities $m_{\lambda,\mu}$ for such pairs $(\lambda, \mu)$ are then determined by the GL$(F,G)$-equivariant structure of $B_{i,j}^{s\times b}(F,G)$. In particular, knowing the GL-equivariant structure for the syzygies of $I_{a\times b}(\mathbb{C}^m, \mathbb{C}^n)$ determines the syzygies of $I_{a\times b}(F,G)$ for all pairs of vector spaces $(F,G)$ with $\dim(F) \leq m$, $\dim(G) \leq n$.

2.3. The linear complexes $X_r^{\times s}$ of Akin and Weyman. In [AW97, AW07], Akin and the second author construct linear complexes $X_r^{\times s} = X_r^{\times s}(F,G)$ which depend functorially on a pair of finite dimensional vector spaces $(F,G)$. The terms in the complex are given (using notation (1.4)) by

$$X_r^{\times s}(F,G) = \left( \bigoplus_{|\alpha|+|\beta|=i} S_{\lambda(r,s;\alpha,\beta)} F \otimes S_{\lambda(r,s;\beta',\alpha')} G \right) \otimes S(F,G).$$

(2.3)

Note that since $S_j W = 0$ when $l(\lambda) > \dim(W)$, only finitely many of the terms $X_r^{\times s}(F,G)$ in (2.3) are non-zero for a given pair $(F,G)$. More precisely, we must have $\alpha_1 \leq \dim(G) - r$, $\beta'_1 \leq \dim(F) - r$, so $|\alpha| \leq \min(r,s) \cdot (\dim(G) - r)$, $|\beta| \leq \min(r,s) \cdot (\dim(F) - r)$, $i \leq \min(r,s) \cdot (\dim(F) + \dim(G) - 2r)$. We can rewrite (1.6) as

$$h_{r^{\times s}}(z,w) = \sum_{i=0}^{\min(r,s) \cdot (m+n-2r)} [X_r^{\times s}(\mathbb{C}^m, \mathbb{C}^n)_{r,s+i}] \cdot z^{r-s+i} \cdot w^i,$$

where $X_r^{\times s}(\mathbb{C}^m, \mathbb{C}^n)_{r,s+i}$ is the vector space of minimal generators of the free module $X_r^{\times s}(\mathbb{C}^m, \mathbb{C}^n)$. The complex $X_r^{(a+q) \times (1+q)}(\mathbb{C}^m, \mathbb{C}^n)$ can be identified with the $q$-th linear strand of the Lascoux resolution of the ideal of $a \times a$ minors of the generic $m \times n$ matrix. In this paper we'll see that more generally, the complexes $X_r^{(a+q) \times (b+q)}$, $q \geq 0$, form the building blocks of the minimal resolutions of the ideals $I_{a\times b}$.

In [AW97] it was shown that $X_r^{\times s} = X_r^{\times s}(\mathbb{C}^m, \mathbb{C}^n)$ is the irreducible $\mathfrak{gl}(m|n)$-module of highest weight $(s^r, 0^{m-r}|0^{n-r}, -s^s)$. In [AW07] the homology of the complexes $X_r^{\times s}$ is shown to consist of direct sums of the rectangular ideals $I_{(r+q)\times (s+q)}$. To state this more precisely, we need to introduce some notation. We denote by $P(r,s;i)$ the number of partitions of $i$ contained in the $r \times s$ rectangle. The Gauss polynomial defined in (1.7) is then

$$\left( \begin{array}{c} r+s \\ r \end{array} \right) w = \sum_{i=0}^{r,s} P(r,s;i) w^i.$$

**Theorem 2.1** ([AW07 Thm. 2]). With the above notation, we have

1. $H_{2j+1}(X_r^{\times s}) = 0$;

2. $H_{2j}(X_r^{\times s}) = \bigoplus_{q=0}^{j} P(q, \min(r,s) - 1; j-q)$.

In [AW07] the projective dimension of the ideals $I_{a\times b}$ is calculated. The calculation of Ext modules in [RW11] Thm. 4.3 in fact allows one to compute the projective dimension and regularity for all the ideals $I_\lambda$, i.e the shape of their minimal resolution. More work is however necessary in order to completely determine the syzygies.
2.4. The Ext modules $\operatorname{Ext}^*_S(I_{a \times b}, S)$. In [RW14] Theorem 4.3 we determined the decomposition into irreducible $\operatorname{GL}$-representations for all the modules $\operatorname{Ext}^*_S(I_x, S)$. In the case when $\lambda$ is a rectangular partition, we obtain the following consequence which will be useful for our calculation of syzygies.

Theorem 2.2. Assume that $m = n$ and write $q = n - a$, $S = S(\mathbb{C}^n, \mathbb{C}^n)$, $I_{a \times b} = I_{a \times b}(\mathbb{C}^n, \mathbb{C}^n)$, $\operatorname{GL} = \operatorname{GL}(\mathbb{C}^n, \mathbb{C}^n)$. The occurrences of the irreducible $\operatorname{GL}$-representation $S_{(-b-q)^n}\mathbb{C}^n \otimes S_{(-b-q)^n}\mathbb{C}^n$ inside $\operatorname{Ext}^*_S(I_{a \times b}, S)$ (see (2.1)) are encoded as

$$\langle S_{(-b-q)^n}\mathbb{C}^n \otimes S_{(-b-q)^n}\mathbb{C}^n, \operatorname{Ext}^*_S(I_{a \times b}, S) \rangle = w^{q^2+2q} \left( q + \min(a, b) - 1 \right) w^q.$$ 

3. The syzygies of the ideals $I_{a \times b}$

We now proceed to state and prove the main result of our paper:

Theorem 3.1. The equivariant Betti polynomial of $I_{a \times b} \subset \operatorname{Sym}(\mathbb{C}^m \otimes \mathbb{C}^n)$, $m \geq n$, is

$$B_{a \times b}(z, w) = \sum_{q=0}^{n-a} h_{(a+q) \times (b+q)} \cdot w^{q^2+2q} \left( q + \min(a, b) - 1 \right) w^q,$$

where $h_{r \times s} = h_{r \times s}(z, w)$ is as defined in (1.6).

We prove Theorem 3.1 in a few stages. We first note that by functoriality (Section 2.2) it is enough to prove Theorem 3.1 in the case $m = n$, which we assume for the remainder of this section. We begin by constructing a non-minimal resolution of $I_{a \times b}$:

Proposition 3.2. The ideal $I_{a \times b}$ has a (not necessarily minimal) free $\operatorname{GL}$-equivariant resolution over $S$ which is filtered by the complexes $X^{(a+q) \times (b+q)}$.

Proof. We prove by descending induction on $q$ that $I_{(a+q) \times (b+q)}$ admits a (not necessarily minimal) resolution $Y^{(a+q) \times (b+q)}$ which is filtered by complexes $X^{(a+q') \times (b+q')}$ with $q' \geq q$. If $q = n - a$ then $I_{(a+q) \times (b+q)} = I_{a \times (b+n-a)}$ coincides with $X^{n \times (b+n-a)}$: they are both isomorphic to a free module of rank one, generated by the $(b + n - a)$-th power of the determinant of the generic $n \times n$ matrix. Assuming now that the result is true for the ideals $I_{(a+q) \times (b+q)}$ with $q > q_0$, we’ll prove it for $q = q_0$ to finish the inductive argument. By Theorem 2.1 the higher homology of the linear complex $X^{(a+q_0) \times (b+q_0)}$ consists of direct sums of ideals $I_{(a+q_0) \times (b+q_0)}$, $q > q_0$, and $H_0(X^{(a+q_0) \times (b+q_0)}) = I_{(a+q_0) \times (b+q_0)}$. We can therefore construct a resolution $Y^{(a+q_0) \times (b+q_0)}$ of $I_{(a+q_0) \times (b+q_0)}$ as a mapping cone of the maps from the complexes $Y^{(a+q) \times (b+q)}$, $q > q_0$, to the complex $X^{(a+q_0) \times (b+q_0)}$ that cancel its higher homology. □

Let $Y^{a \times b}$ be a non-minimal $\operatorname{GL}$-equivariant resolution of the ideal $I_{a \times b}$ as in Proposition 3.2. We can minimize $Y^{a \times b}$ by making appropriate cancellations. Notice that since the generators of the free modules appearing in $X^{(a+q) \times (b+q)}$ and $X^{(a+q') \times (b+q')}$ don’t share isomorphic irreducible $\operatorname{GL}$-subrepresentations for $q \neq q'$, the only cancellations that can occur are between the terms in various copies of the same $X^{(a+q) \times (b+q)}$.

Lemma 3.3. Any $\operatorname{GL}(F, G)$-equivariant degree preserving endomorphism of the linear complex $X^s_{a \times s}(F, G)$ is a multiple of the identity.
Proof. Let \( \psi \) denote a GL-equivariant degree preserving endomorphism of \( X^r \times s \), and write \( \psi_i \) for its component in homological degree \( i \). By GL-equivariance and using the decomposition (2.3), we have
\[
\psi_i = \bigoplus_{\alpha, \beta} \psi_{\alpha, \beta},
\]
where \( \psi_{\alpha, \beta} \) is the restriction of \( \psi_i \) to the free submodule \( X^r \times s \) generated by the irreducible representation \( S_{\lambda(r,s,\alpha,\beta)} F \otimes S_{\lambda(r,s,\beta',\alpha')} G \). Such an endomorphism is necessarily a multiple of the identity. Writing \( \psi_{\alpha, \beta} = c_{\alpha, \beta} \), it suffices to show that all \( c_{\alpha, \beta} \) are the same. We prove this by induction on \( i = |\alpha| + |\beta| \).

Consider \( (\alpha, \beta) \) with \( i = |\alpha| + |\beta| > 0 \), and consider a pair \( (\overline{\alpha}, \overline{\beta}) \) with \( |\overline{\alpha}| + |\overline{\beta}| = i - 1 \), such that the restriction of the differential \( \partial_i : X^r_i \times s \times n \rightarrow X^r_{i-1} \times s \times n \)
is non-zero: such a pair exists since otherwise \( S_{\lambda(r,s,\alpha,\beta)} F \otimes S_{\lambda(r,s,\beta',\alpha')} G \) would contribute to the homology of \( X^r \times s \), which would contradict Theorem 2.1. Since \( \psi \) commutes with the differentials, we have a commutative diagram
\[
\begin{array}{ccc}
X^r_{\alpha, \beta} & \xrightarrow{\partial_i} & X^r_{\overline{\alpha}, \overline{\beta}} \\
\downarrow c_{\alpha, \beta} & & \downarrow c_{\overline{\alpha}, \overline{\beta}} \\
X^r_{\alpha, \beta} & \xrightarrow{\partial_i} & X^r_{\overline{\alpha}, \overline{\beta}}
\end{array}
\]
Since \( \partial_i \neq 0 \), it follows that \( c_{\alpha, \beta} = c_{\overline{\alpha}, \overline{\beta}} \), and we conclude by induction.

The preceding discussion implies the following

**Corollary 3.4.** The minimal resolution of \( I_{a \times b} \) is filtered by the complexes \( X^{(a+q) \times (b+q)} \), \( q \geq 0 \). In particular, there exist polynomials \( M^q_{a \times b}(w) \) which account for the multiplicities of the complexes \( X^{(a+q) \times (b+q)} \) in the minimal resolution of \( I_{a \times b} \), as well as for their homological shifts, i.e.
\[
B_{a \times b}(z, w) = \sum_{q=0}^{n-a} h_{(a+q) \times (b+q)} \cdot M^q_{a \times b}(w).
\]

We are now ready to prove the main result of the paper:

**Proof of Theorem 2.1.** It remains to calculate the polynomials \( M^q_{a \times b}(w) \). We fix \( q \) and shrink \( n \) if necessary to assume that \( n = a + q \) (see Section 2.2), so \( X^{(a+q) \times (b+q)} = X^{n \times (b+q)} \) consists of a single free module, generated by the irreducible GL-representation \( S_{(b+q)^n} \mathbb{C}^n \otimes S_{(b+q)^n} \mathbb{C}^n \). Dualizing the minimal resolution \( Y \) of \( I_{a \times b} \) and computing the cohomology \( \text{Ext}_S^*(I_{a \times b}, S) \) of the resulting complex \( Y^\vee \), we get
(a) each occurrence of \( X^{n \times (b+q)} \) in \( Y \) yields a copy of \( S_{(-b-q)^n} \mathbb{C}^n \otimes S_{(-b-q)^n} \mathbb{C}^n \) in \( \text{Ext}_S^*(I_{a \times b}, S) \);
(b) the only occurrences of \( S_{(-b-q)^n} \mathbb{C}^n \otimes S_{(-b-q)^n} \mathbb{C}^n \) inside \( \text{Ext}_S^*(I_{a \times b}, S) \) arise in this way.

To prove (a), note that there are no non-zero maps going into the free module \( X^{n \times (b+q)} \), so its dual \( \text{Hom}_S(X^{n \times (b+q)}, S) \) will consists entirely of cocycles in \( Y^\vee \). Since \( Y^\vee \) is minimal, the space \( S_{(-b-q)^n} \mathbb{C}^n \otimes S_{(-b-q)^n} \mathbb{C}^n \) of minimal generators of \( (X^{n \times (b+q)})^\vee = \text{Hom}_S(X^{n \times (b+q)}, S) \) contains no coboundaries, so (a) follows. If (b) failed, one could find a free submodule \( M^* \otimes S \) in \( Y^\vee \), containing \( S_{(-b-q)^n} \mathbb{C}^n \otimes S_{(-b-q)^n} \mathbb{C}^n \), where \( M \) is an irreducible GL-representation appearing as a subspace of minimal generators in some complex \( X^{(a+q') \times (b+q')} \), \( q' < q \). The condition \( S_{(-b-q)^n} \mathbb{C}^n \otimes S_{(-b-q)^n} \mathbb{C}^n \subset M^* \otimes S \) implies that \( M \)
appears as a subrepresentation of $S_{(b+q)}^n \mathbb{C}^n \otimes S_{(b+q)}^n \mathbb{C}^n \otimes S$. This can only happen if $M = S_\lambda \mathbb{C}^n \otimes S_\mu \mathbb{C}^n$, where $\lambda, \mu$ are partitions containing the $n \times (b+q)$ rectangle. By (1.4), $M$ can only occur inside $X^{n \times (b+q)}$.

It follows from (a) and (b) that there is a one-to-one correspondence between occurrences of $X^{n \times (b+q)}$ inside $Y$ and those of $S_{(b-q)}^n \mathbb{C}^n \otimes S_{(b-q)}^n \mathbb{C}^n$ inside $\text{Ext}^*_S(I_{ab}, S)$, and moreover this correspondence replaces homological shifts with cohomological shifts. We get (see (2.1) and the remark following it) that

$$M_{ab}(w) = \langle S_{(b-q)}^n \mathbb{C}^n \otimes S_{(b-q)}^n \mathbb{C}^n, \text{Ext}^*_S(I_{ab}, S) \rangle \text{Thm. } 2.2 = w^{q^2+q}. \left( q + \min(a, b) - 1 \right) \frac{1}{q} w^2.$$

This concludes the proof of Theorem 3.1. □

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