D = 10 Super-Yang-Mills Theory and Poincaré Duality in Supermanifolds

Pietro Freé $^{a,c,d,e,*}$ and Pietro Antonio Grassi $^{b,c,d,†}$

(a) Dipartimento di Fisica, Università di Torino, via P. Giuria, 1, 10125 Torino, Italy.
(b) Dipartimento di Scienze e Innovazione Tecnologica, Università del Piemonte Orientale, viale T. Michel, 11, 15121 Alessandria, Italy.
(c) INFN, Sezione di Torino, via P. Giuria 1, 10125 Torino.
(d) Arnold-Regge Center, via P. Giuria 1, 10125 Torino, Italy.
(e) National Research Nuclear University MEPhI (Moscow Engineering Physics Institute), Kashirskoye shosse 31, 115409 Moscow, Russia

Abstract

We consider super Yang-Mills theory on supermanifolds $\mathcal{M}^{(D|m)}$ using integral forms. The latter are used to define a geometric theory of integration and are essential for a consistent action principle. The construction relies on Picture Changing Operators $\mathcal{Y}^{(0|m)}$, analogous to those introduced in String Theory, that admit the geometric interpretation of Poincaré duals of closed submanifolds of superspace $\mathcal{S}^{(D|0)} \subset \mathcal{M}^{(D|m)}$ having maximal bosonic dimension $D$. We discuss the case of Super-Yang-Mills theory in $D = 10$ with $\mathcal{N} = 1$ supersymmetry and we show how to retrieve its pure-spinor formulation from the rheonomic lagrangian $\mathcal{L}_{\text{rheo}}$ of D’Auria, Fré and Da Silva, choosing a suitable $\mathcal{Y}^{(0|m)}_{\text{ps}}$. From the same lagrangian $\mathcal{L}_{\text{rheo}}$, with another choice $\mathcal{Y}^{(0|m)}_{\text{comp}}$ of the PCO, one retrieves the component form of the SYM action. Equivalence of the formulations is ensured when the corresponding PCO.s are cohomologous, which is true, in this case, of $\mathcal{Y}^{(0|m)}_{\text{ps}}$ and $\mathcal{Y}^{(0|m)}_{\text{comp}}$.
1 Introduction

In 10 dimensions, there are few examples of supersymmetric models which are consistent and well-defined. Among them, there are super-Yang-Mills theory with $\mathcal{N} = 1$ supersymmetry (sixteen supercharges), and three supergravity theories: $\mathcal{N} = 1$ type I, $\mathcal{N} = 2$ type IIA and $\mathcal{N} = 2$ type IIB. This is essentially due to the fact that, in 10 dimensions, the spinorial representation of the Lorentz group SO(1,9) admits a minimal realization in terms of Majorana-Weyl spinors with two different chiralities (A and B). Therefore only the massless physical multiplets with a spin-one vector field (super-Yang-Mills) or the spin-two graviton constitute closed multiplets with respect to the action of the supercharges [1]. From the string theory side, we have learnt that the lowest massless states of the open superstring theory and of the closed superstring theory are respectively described by those super gauge theories and supergravities [2,3].

Here we consider only the case of super-Yang-Mills in 10 dimensions with $\mathcal{N} = 1$ supersymmetry. That theory is characterized by a supermultiplet containing the gauge degrees of freedom (eight on-shell d.o.f.'s) and those of its supersymmetric partner, the gaugino (eight on-shell d.o.f.'s). The off-shell theory is described by a component action which is manifestly invariant under Lorentz and gauge symmetry and supersymmetric up to total derivatives [4]. Moreover, there is a superspace formulation of the theory [5,6] which is valid only for on-shell fields. Namely, in order to take into account the correct supersymmetry transformations and the quantum numbers, the gauge field and the gaugino are described by a single superconnection (which is a one-superform with a vectorial and a spinorial superfield). Unfortunately, in order to match the physical degrees of freedom one has to choose specific constraints on the superconnection and, at the end of the day, these latter imply also the equations of motion. Hence, there seems to be no off-shell superspace formulation that leads to a superspace action (some attempts and remarks can be read in [7]) or, put differently, there is only an on-shell superspace formulation [8].

The needed superspace constraints to be imposed on the superconnections can be derived as integrability conditions for the motion of a superparticle. This has been used in [5,9,10] and it has been successfully implemented at the string level in [11,12]. It has been shown that BRST cohomology restricted to the shell of pure spinor constraints coincides with the superstring spectrum. In the case of open superstrings, restricting the latter construction to the massless sector one obtains a pure-spinor formulation of D=10 $\mathcal{N} = 1$ SYM. The success of the pure spinor approach as a supersymmetric target space formulation of superstrings prompted the analysis of scattering amplitudes and quantum corrections to classical dynamics. Indeed such effects can be mastered only by means of an effective action. More significantly, an effective off-shell action is needed for any string field theory analysis. It has been shown that, upon
integration over the bosonic coordinates, over the fermionic coordinates and over the pure-spinor zero modes, a Chern-Simons type Lagrangian provides a convenient reformulation of super-Yang-Mills theory within a superspace framework. Such a formulation appears to be the so far known presentation of the theory that is the closest to a true superspace action of $D=10\mathcal{N}=1$ SYM.

From a completely different point of view, many years ago the Torino group established a formulation of supersymmetric field theories in a more geometrical setup [14]. In such an approach each individual field of the space-time component formulation is promoted to the status of a superfield and the large amount of non-physical components is suppressed by a collection of suitable constraints. Furthermore, a variational principle is provided that yields, just in one stroke, both the physical equations of motion and the suitable superspace constraints mentioned above. Unfortunately this variational principle is formulated in a very intrinsic way by specifying the immersion of the bosonic space-time into superspace; in the case of interest to us here, it is the immersion of a ten-dimensional bosonic manifold into the $(10|16)$ supermanifold that is at stake. In such a general setup, that is dubbed the rheonomic formulation, the variation of the immersion is compensated by a diffeomorphism of the action. In our case the action is a suitable 10-form that was derived many years ago by D’Auria, Fré and da Silva [15], generalizing similar constructions of SYM theories in $D=4$ pioneered by Fré. [16, 17]. One of the main advantages of this formulation is that it naturally provides the coupling of the rigid supersymmetric theory to supergravity or its localization on curved supermanifolds, such as coset superspaces.

Recently in [18,19], an alternative formulation, based on the rheonomic action, was discovered. The immersion of the bosonic submanifold into the supermanifold has been implemented introducing suitable Poincaré duals in superspace. This leads to an integral form suitable to be integrated over the supermanifold. The choice of the Poincaré dual amounts to choose a specific representation of the theory. In several papers [20,21,23,24], utilizing different choices of the Poincaré dual (in the text the form will be denoted by $PCO$), it has been shown how one can interpolate between the several formulations of the same supersymmetric model that range from the component action to the superspace action.

Here, we show that by the same token we are able to interpolate between the component action of $D=10\mathcal{N}=1$ super Yang-Mills and the pure spinor action of the same theory [11,12]. In all cases, the differential form that is paired with the Poincaré dual of a suitable 10-dimensional cycle in superspace $C \subset \mathcal{M}^{10|16}$ is the rheonomic Lagrangian of [15]. Schematically one has:

\[
\begin{align*}
\text{Action of formulation A} &= \int_{C_A} t^* [\mathcal{L}_{\text{rheonomic}}] = \int_{\mathcal{M}^{10|16}} \mathcal{L}_{\text{rheonomic}} \wedge Y_{A}^{0|16} \\
\text{Action of formulation B} &= \int_{C_B} t^* [\mathcal{L}_{\text{rheonomic}}] = \int_{\mathcal{M}^{10|16}} \mathcal{L}_{\text{rheonomic}} \wedge Y_{B}^{0|16}
\end{align*}
\] (1.1)
where

\[ \iota : C_{A,B} \hookrightarrow \mathcal{M}^{10|16} \]  \tag{1.2} \]

is the immersion map and \( \iota^* [... ] \) denotes its pull-back. The picture changing operators \( \Upsilon_{A,B}^{0|16} \) are the Poincaré duals of the corresponding cycles.

Since homology theory in superspace is conceptually and technically very hard and ill-defined, the analogue of a simplicial approach being missing, the dual cohomological formulation turns out to be much more promising and it is well founded on the theory of integral forms. Eventually integral forms \( \Upsilon_{A,B}^{0|16} \) define their Poincaré dual cycles and implicitly define the immersion maps (1.2). The various pull-backs \( \iota^* [\mathcal{L}_{\text{rheonomic}}] \) of the time-honored rheonomic lagrangian encode all possible formulations of the same supersymmetric theory that are either cohomologous or correspond to different cohomology classes.

In view of this in the next section we provide an ultra short review of integral forms.

## 2 Integral Forms

Integral forms are the crucial ingredients in order to define a geometric integration theory on supermanifolds that inherits all the good properties of differential form integration theory in conventional geometry.

We consider a supermanifold with \( n \) bosonic dimensions and \( m \) fermionic dimensions denoted by \( \mathcal{M}^{(n|m)} \). We denote the local coordinates in an open set as \( (x^a, \theta^\alpha) \).

A \((p|q)\) integral form \( \omega^{(p|q)} \) has the following structure

\[ \omega^{(p|q)} = \omega(x, \theta)dx^{a_1}\cdots dx^{a_r}d\theta^{\alpha_1}\cdots d\theta^{\alpha_s}\delta^{(b_1)}(d\theta^\beta_1)\cdots \delta^{(b_q)}(d\theta^\beta_q) \]  \tag{2.1} \]

where the \( d\theta^{\alpha} \) appearing in the product are independent of those appearing in the delta’s \( \delta(d\theta^\beta) \). By \( \omega(x, \theta) \) we denote the set of superfields provided by the collection of the components \( \omega_{[a_1\ldots a_r][\alpha_1\ldots \alpha_s][\beta_1\ldots \beta_q]}(x, \theta) \).

The two quantum numbers \( p \) and \( q \) correspond to the form number and the picture number and they range from \(-\infty \) to \(+\infty \) for \( p \) and \( 0 \leq q \leq m \). The index on the delta \( \delta^{(a)}(d\theta^\alpha) \) denotes the degree of the derivative of the delta function. The total picture of \( \omega^{(p|q)} \) corresponds to the number of delta functions (we call it a superform if \( q = 0 \), an integral form if \( q = m \), otherwise it is dubbed a pseudoform). The total form degree is given by \( p = r + s - \sum_{i=1}^{i=q} b_i \) since the derivatives act effectively as negative forms and the delta functions do not carry any form degree. We recall the following properties

\[ d\delta^{(a)}(d\theta^\alpha) = 0, \quad d\theta^\alpha \delta^{(a)}(d\theta^\alpha) = -a\delta^{(a-1)}(d\theta^\alpha), \quad a > 0, \quad d\theta^\alpha \delta(d\theta^\alpha) = 0 \]  \tag{2.2} \]
The index $\alpha$ is not summed. The indices $a_1 \ldots a_r$ and $\beta_1 \ldots \beta_q$ are anti-symmetrized, the indices $\alpha_1 \ldots \alpha_s$ are symmetrized because of the rules

$$dx^a dx^b = -dx^b dx^a, \quad dx^a d\theta^\alpha = d\theta^\alpha dx^a, \quad d\theta^\alpha d\theta^\beta = d\theta^\beta d\theta^\alpha,$$

$$\delta(d\theta^\alpha)\delta(d\theta^\beta) = -\delta(d\theta^\beta)\delta(d\theta^\alpha), \quad dx^a\delta(d\theta^\beta) = \delta(d\theta^\alpha)dx^a, \quad d\theta^\alpha \delta(d\theta^\beta) = \delta(d\theta^\beta)d\theta^\alpha.$$

One can calculate the integral of an integral form $\omega^{(n|m)}$, it is a top form, namely an element of the line bundle known as the Berezinian bundle (the transition functions are represented by the superdeterminant of the Jacobian) and it can be locally expressed as

$$\omega^{(n|m)} = \omega(x, \theta)dx^1 \ldots dx^n\delta(d\theta^1) \ldots \delta(d\theta^m). \quad (2.3)$$

where $\omega(x, \theta)$ is a superfield. By replacing the 1-forms $dx^a, d\theta^\alpha$ as it follows $dx^a \to E^a = E^a_m dx^m + E^a_\mu d\theta^\mu$ and $d\theta^\alpha \to E^\alpha = E^\alpha_m dx^m + E^\alpha_\mu d\theta^\mu$, we get

$$\omega \to \text{sdet}(E) \omega(x, \theta)dx^1 \ldots dx^n\delta(d\theta^1) \ldots \delta(d\theta^m) \quad (2.4)$$

where $\text{sdet}(E)$ is the superdeterminant of the supervielbein $(E^a, E^\alpha)$.

As in conventional geometry, the integral form $\omega^{(n|m)}$ is also viewed as a section of the cotangent bundle $T^*\mathcal{M}^{(n|m)}$ and we perform the integral as follows

$$I[\omega] = \int_{\mathcal{M}^{(n|m)}} \omega^{(n|m)} = \int_{T^*\mathcal{M}^{(n|m)}} \omega(x, \theta, dx, d\theta)[dx d\theta d(dx)d(d\theta)] \quad (2.5)$$

where the order of the integration variables is kept fixed. The symbol $[dx d\theta d(dx)d(d\theta)]$ denotes the integration variables and it is invariant under any coordinate transformations. The integrations over $d\theta$ and $d(dx)$ are Berezin integrals, and $dx$ and $d(d\theta)$ are usual Lebesgue integrals. See Witten [25] for a complete discussion on the symbol $[dx d\theta d(dx)d(d\theta)]$.

Let us consider a superform $\omega^{(n|0)}$ with degree $n$ equal to the bosonic dimension of the bosonic submanifold $\mathcal{M}^{(n)} \subset \mathcal{M}^{(n|m)}$. The superform is obtained in the usual manner using the supervielbeins $E$, superconnections $\Omega$ and the covariant derivatives of differential superforms, and formally it can be integrated over the submanifold $\mathcal{M}^{(n)}$. However, the transformation properties become manifest if the integral can be converted into an integral form integrated over the entire supermanifold. That can be achieved by constructing the Poincaré dual form $\gamma^{(0|m)}$ of the immersion of $\iota : \mathcal{M}^{(n)} \to \mathcal{M}^{(n|m)}$. Then, we can write the integral as follows

$$I[\omega] = \int_{\mathcal{M}^{(n)}} \omega^{(n|0)} = \int_{\mathcal{M}^{(n|m)}} \omega^{(n|0)} \wedge \gamma^{(0|m)} \quad (2.6)$$

where $\omega^{(n|0)} = \iota^* \omega^{(n|0)}$ is the pull-back of $\omega^{(n|0)}$ on $\mathcal{M}^{(n)}$. Any variation of the embedding of $\mathcal{M}^{(n)}$ into $\mathcal{M}^{(n|m)}$ is compensated by a diffeomorphism. The r.h.s. is the integral of an integral form for which the usual rules of Cartan calculus do apply. Notice that we can modify
the embedding by changing $\gamma^{(0|m)}$ by exact terms if $\omega^{(m|0)}$ is a closed form. The Poincaré dual $\gamma^{(0|m)}$ is closed and not exact. Any variation of the embedding is exact: $\delta \gamma^{(0|m)} = d\eta^{(-1|m)}$.

For rigid supersymmetric models, the closed form $\omega^{(n|0)}$ is represented by the Lagrangian of the model $\mathcal{L}^{(n|0)}(\Phi, V, \psi)$ built using the rheonomic rules and it contains the dynamical fields $\Phi$ (each dynamical field is promoted to a superfield) and the rigid supervielbeins $V^a = dx^a + \theta \gamma^a d\theta$, $\psi^\alpha = d\theta^\alpha$ satisfying the Maurer-Cartan equations

$$dV^a = \frac{i}{2} \psi \wedge \gamma^a \psi, \quad d\psi^\alpha = 0.$$  \hfill (2.7)

In the present formula, we have used real Majorana spinors; the notation can be made more precise only upon the choice of the dimensions $(n|m)$ of the supermanifold $\mathcal{SM}$.

On the other hand the Poincaré dual form $\gamma^{(0|m)}$ (a.k.a. Picture Changing Operator PCO in string theory literature) contains only geometric data (for instance the supervielbein or just the coordinates). One can choose a different PCO which has some manifest symmetries.

For rigid supersymmetric models we have

$$S_{\text{rig}} = \int_{\mathcal{M}^{(n|m)}} \mathcal{L}^{(n|0)}(\Phi, V, \psi) \wedge \gamma^{(0|m)}(V, \psi)$$  \hfill (2.8)

with $d\mathcal{L}^{(n|0)}(\Phi, V, \psi) = 0$ in order to change the PCO by exact terms. The action $\mathcal{L}^{(n|0)}(\Phi, V, \psi)$ depends upon the dynamical fields $\Phi$ of the theory and the PCO depends upon the geometrical data of the supermanifold.

In the case of supergravity, the supervielbein $V^a$ and $\psi^\alpha$ are promoted to dynamical fields $(E^a, E^\alpha)$ so that the action becomes:

$$S_{\text{sugra}} = \int_{\mathcal{M}^{(n|m)}} \mathcal{L}^{(n|0)}(\Phi, E) \wedge \gamma^{(0|m)}(E).$$  \hfill (2.9)

The closure of the action and the closure of the PCO implies the conventional constraints of supergravity that reduce the independent fields to the physical ones.

3 Basics of D=10 $\mathcal{N} = 1$ SYM

We list some of the ingredients for D=10, $\mathcal{N} = 1$ super-Yang-Mills theory. The theory is a maximally supersymmetric model in D=10 and its field content consists of a gauge field $A_a(x)$ and of its superpartner $\chi_\alpha(x)$. In ten dimensions, they are respectively assigned to the vector and to the spinor representation of the Lorentz group. This amounts to $10 + 16$ off-shell degrees of freedom. The gauge field is defined up to a gauge transformation which removes one degree of freedom, but still 7 bosonic dof.s are missing for an off-shell matching. On the other hand, imposing the equations of motion, the gauge field reduces to 8 effective dof.s (since one additional constraint is removed by the Hamiltonian constraint) while the gaugino
reduces to 8 dof's since the Dirac equation halves the off-shell ones. Hence there is a physical
supersymmetric on-shell multiplet [1].

We assume an unspecified, compact, non-abelian gauge group \( G \) and both the gauge field
and the gaugino are assigned to the adjoint representation of \( G \). The consistent equations of
motion are \([4,6]\]

\[
\nabla^a F_{ab} = [\chi, \gamma_b \chi], \quad \gamma^a \nabla_a \chi = 0 \tag{3.1}
\]

where the bracket \([\cdot, \cdot]\) is with respect to the Lie algebra of \( G \) while \( \nabla_a \) denotes the covariant
derivative with respect to the gauge field in the adjoint representation. These equations of
motion are gauge and supersymmetric covariant. They can be derived from a component
action which can be written as follows:

\[
S = \int d^{10}x \text{ Tr} \left( -\frac{1}{4} F_{ab} F^{ab} + \chi \gamma^a \nabla_a \chi \right) \tag{3.2}
\]

the trace being taken in the adjoint representation of the gauge group. The supersymmetry
transformations \([4]\) are the following ones:

\[
\delta A_a = \epsilon \gamma_a \chi, \quad \delta \chi^\alpha = \frac{1}{4} (\gamma^{ab} \epsilon)_{\alpha} F_{ab} \tag{3.3}
\]

Their anticommutator closes on the translations, on the gauge transformations and on the
fermionic equations of motion (3.1). The supersymmetry algebra does not close off-shell since
there is no consistent auxiliary field set in the present formulation.

We remark that in the case of \( D=10, \mathcal{N}=1 \) SYM, there is not a finite set of auxiliary fields
that can be used to construct an off-shell formulation and therefore a superspace description.
There have been several attempts (see [7] for a clear discussion on this point) to build a super-
space action for this theory. Nonetheless, in [15] the existence has been shown of a rheonomic
action, whose associated equations of motion in superspace lead to the supersymmetry trans-
formations rules and to the field equations of the component fields. That action, as it will be
explained in forthcoming sections, contains additional off-shell degrees of freedom, precisely
the 0-form superfields \( Z_{ab}(x, \theta), W^\alpha(x, \theta) \). The first component of the former \( Z_{ab}(x, \theta = 0) \) is
identified, by its own field equation, with the field strength \( F_{ab}(x) \) of the gauge field \( A_a(x) \)
while the first component of the latter is similarly identified with the gaugino field strength.
Indeed the rheonomic formulation is intrinsically a first order action – no Hodge dual being
used – and therefore it needs these additional superfields. As explained in [15] the presence of
these first order superfields circumvents the usual no-go theorems of Siegel and Roček [26].

As it is well-known, reducing the superspace from \( \mathcal{N} = 1, D = 10 \) down to \( \mathcal{N} = 4, D = 4 \)
one finds the maximally extended supersymmetric Yang–Mills theory. Equivalently, also in the
case of that theory no off-shell formulation is known that makes the full \( \mathcal{N} = 4 \) supersymmetry
manifest. A superspace action in the usual sense is still lacking and maybe it does not exist at all. Once more, in [15] the dimensional reduction of the rheonomic action was obtained, yielding a long expression which can be utilized as starting point in our new integral form constructions.

4 Superspace D=10 $\mathcal{N} = 1$ SYM

Even though there is no off-shell closure of the supersymmetry algebra and the superspace action is absent, still we can give a superspace formulation of the equations of motion.

We start from a super 1-form $A^{(1|0)} = A_a V^a + A_\alpha \psi^\alpha$, (where the superfields $A_a(x, \theta)$ and $A_\alpha(x, \theta)$ take value in the adjoint representation of the gauge group) and we define the field strength

$$F^{(2|0)} = dA^{(1|0)} + A^{(1|0)} \wedge A^{(1|0)}$$

$$= F_{ab} V^a \wedge V^b + F_{a\alpha} V^a \wedge \psi^\alpha + F_{\alpha\beta} \psi^\alpha \wedge \psi^\beta,$$

(4.1)

where we have introduced the following field strengths (we recall that $A_a$ is a bosonic superfield while $A_\alpha$ is a fermionic superfield, therefore the notation $[A_a, A_b]$ denotes the Lie commutator, while $\{A_\alpha, A_\beta\}$ denotes the anticommutator to take into account the statistic of the superfields)

$$F_{ab} = \partial_a A_b - \partial_b A_a + [A_a, A_b],$$

$$F_{a\alpha} = \partial_a A_\alpha - D_\alpha A_a + [A_\alpha, A_a],$$

$$F_{\alpha\beta} = D_{(\alpha} A_{\beta)} + \gamma^a_{\alpha\beta} A_a + \{A_\alpha, A_\beta\}.$$

(4.2)

In order to reduce the redundancy of degrees of freedom contained in the two components $A_a$ and $A_\alpha$ of the $(1|0)$ connection, one imposes (by hand) the conventional constraint

$$\iota_\alpha \iota_\beta F^{(2|0)} = 0 \iff F_{\alpha\beta} = \nabla_{(\alpha} A_{\beta)} + \gamma^a_{\alpha\beta} A_a = 0,$$

(4.3)

from which it follows that

$$A_a = -\frac{1}{8} \gamma^a_{\alpha\beta} \nabla_\alpha A_\beta, \quad W^\alpha = \nabla^\beta \nabla^\alpha A_\beta, \quad F_{a\alpha} = \gamma_{a,\alpha\beta} W^\beta, \quad F_{ab} = (\gamma_{ab})_{\alpha}^{\beta} \nabla_\beta W^\alpha,$$

(4.4)

and the dynamical equation

$$\gamma^\alpha_{\alpha\beta} \nabla_{(\alpha} A_{\beta)} = 0,$$

(4.5)

(where $\gamma^\alpha_{\alpha\beta}$ is the anti-symmetrized product of five gamma matrices) which implies the field equations $\mathcal{N} = 1, D = 10$ super-Yang-Mills theory

$$\nabla^a F_{ab} = 0, \quad \gamma^a_{\alpha\beta} \nabla_\alpha W^\beta = 0.$$

(4.6)
The gaugino field strength $W^\alpha$ is gauge invariant under the non-abelian transformations $\delta A_\alpha = \nabla_\alpha \Lambda$. These transformations follow from $\delta A = \nabla \Lambda$ where $\Lambda$ is a $(0|0)$-form. The field strengths satisfy the following Bianchi’s identities

\begin{align*}
\nabla_a F_{bc} &= 0, \\
\nabla_\alpha F_{ab} + (\gamma_a \nabla_b)W_\alpha &= 0, \\
F_{ab} + \frac{1}{2} (\gamma_{ab})^\alpha_{\beta} \nabla_\alpha W^\beta &= 0, \\
\nabla W^\alpha &= 0.
\end{align*}

(4.7)

and, by expanding the superfields $A_a, A_\alpha$ and $W^\alpha$ at the first level, we have

\begin{align*}
A_\alpha &= (\gamma^a \theta)_\alpha A_a(x) + \chi_\alpha \theta^2, \\
A_a &= A_a(x) + \chi \gamma_a \theta + \ldots, \\
W^\alpha &= \chi^\alpha + F^\alpha_\beta \theta^\beta + \ldots
\end{align*}

(4.8)

where $A_a(x)$ is the gauge field, $\chi_\alpha(x)$ is the gaugino and $F_{\alpha \beta} = \gamma_{ab}^\alpha F_{ab}$ is the gauge field strength with $F_{ab} = \partial_a A_b - \partial_b A_a$.

The Bianchi identities, together with the constraint $F_{\alpha \beta} = 0$ fix all components of the superfield $A_\alpha$ in terms of the gauge field $A_a(x)$ and of the gaugino $\chi^\alpha(x)$ with the requirement that they are on-shell. Therefore, the higher components of the superfield $A_\alpha$ are completely fixed. This is equivalent to say that there is no superspace off-shell formulation.

5 Pure Spinor Formulation

In the pure spinor formulation \cite{11,12,27} for superstrings and superparticles, one starts from two–dimensional worldsheet fields or from one–dimensional worldline fields $(x^a, \theta^\alpha, p_\alpha)$, representing the coordinates of the target space, together with a pair of commuting spinor ghost fields $(\lambda^{\alpha}, w^{\alpha})$. Then, one defines a BRST differential operator $Q$ which restricted to zero modes reads

$$Q = \lambda^\alpha D_\alpha$$

(5.1)

with $D_\alpha = \frac{\partial}{\partial \theta^\alpha} - (\gamma^a \theta)_{\alpha} \partial_a$ the superderivative. \footnote{To be precise, $Q = \oint d\lambda^\alpha(z) d_\alpha(z)$, where $\lambda^\alpha(z)$ is a commuting holomorphic worldsheet field and $d_\alpha(z)$ is the holomorphic generator of fermionic constraints of Green-Schwarz formulation of superstring. Acting on zero modes, $d_\alpha(z) \sim D_\alpha$.} The latter satisfies the commutation relations

$$\{ D_\alpha, D_\beta \} = -2 \gamma_{\alpha \beta} \partial_a$$

and therefore $Q$ is nilpotent if and only if \cite{9-12}

$$\lambda^\alpha \gamma_{\alpha \beta}^a \lambda^\beta = 0, \quad a = 0, \ldots, 9.$$ 

(5.2)

These are the celebrated pure spinor constraints which can be solved in terms of 11 independent complex degrees of freedom. The pure spinor constraints (5.2) are first class constraints and generate the gauge symmetry $\delta w_\alpha = \eta_\alpha (\gamma^a \lambda_\alpha)$ on the conjugated fields $w_\alpha$ and $\eta^\alpha$ is the gauge parameter associated to the first class constraints (5.2).
In the superstring/superparticle formulation, the target space gauge field and its superpartner are states of the appropriate Hilbert space and they are described by a vertex operator $U^{(1)}$ at ghost number one. Since only the pure spinor $\lambda^\alpha$ carries positive ghost charge, a Lorentz invariant vertex operator has the generic form

$$U^{(1)} = \lambda^\alpha A_\alpha(x, \theta)$$

(5.3)

where $A_\alpha(x, \theta)$ is a superfield. No vectorial partner of $A_\alpha$ has been introduced and $A_\alpha$ is identified with the spinoral part of the superconnection $A^{(1|0)}$. Acting with $Q$ on $U^{(1)}$, using the algebras of superderivatives and imposing BRST closure we get

$$\{Q, U^{(1)}\} + \{U^{(1)}, U^{(1)}\} = \frac{1}{2} \lambda^\alpha \lambda^\beta (D_\alpha A_\beta + D_\beta A_\alpha + \{A_\alpha, A_\beta\}) = 0.$$  

(5.4)

In the above equation we added the second term in order to implement the non-abelian gauge symmetry where the product $\{\cdot, \cdot\}$ is the Lie algebra anticommutator between two anticommuting vertex operators $U^{(1)}$. In the following, this term is justified as derived from an action of Chern-Simons type.

Projecting the product $\lambda^\alpha \lambda^\beta$ and using the pure spinor constraints along the 5-form $\lambda[\gamma^{a_1 \ldots a_5}] \lambda$ yields the superspace equations of motion

$$\gamma^{a_1 \ldots a_5}_{a_1 \ldots a_5} (D_\alpha A_\beta + D_\beta A_\alpha + \{A_\alpha, A_\beta\}) = 0.$$  

(5.5)

If the vector part of $A^{(1|0)}$ is defined as

$$A_a = -\frac{1}{8} \gamma^{\alpha\beta}_{a} D_\alpha A_\beta,$$  

(5.6)

the Bianchi identities imply all the identities derived in the previous section and the correct equations of motion.

Since $Q$ is a nilpotent differential, the vertex operator is a physical quantity if not BRST exact, or equivalently, if it is defined up to gauge transformations

$$\delta U^{(1)} = QA^{(0)} + [U^{(1)}, \Lambda^{(0)}],$$  

(5.7)

where $\Lambda^{(0)}$ is a superfield with vanishing ghost number. Imposing a convenient gauge fixing for the connection $A_\alpha$ (e.g. the Wess-Zumino gauge), with an iterative procedure, one reconstructs all superfields from the on-shell target space fields $A_a(x), \chi_\alpha(x)$ (see [8] for a complete discussion).

The equations of motion (5.5) suggest a variational principle of the form

$$S = \int d^{10}x d^{16}\theta d^{11}\lambda \mu(\theta, \lambda) \text{Tr} \left( \frac{1}{2} U^{(1)} Q U^{(1)} + \frac{1}{3} U^{(1)} U^{(1)} U^{(1)} \right)$$  

(5.8)
where the trace is taken over the gauge group representation of the connection $A_\alpha$ which appears in the vertex operator $U^{(1)}$ (5.3) and the multiplication of the three vertex operators in (5.8) is matrix multiplication of the vertices at the same point in superspace. The integration over the pure spinor is done with a suitable measure $\mu(\theta, \lambda)$ given by

$$
\mu(\theta, \lambda) = \theta^{\alpha_1} \cdots \theta^{\alpha_{11}} \epsilon_{\alpha_1 \cdots \alpha_{16}} (\gamma^{abc})_{\alpha_1 \cdots \alpha_{16}} \gamma^a_{\alpha_1} \gamma^b_{\alpha_2} \gamma^c_{\alpha_3} \lambda_{\alpha_4} \lambda_{\alpha_5} \lambda_{\alpha_6} \lambda_{\alpha_7} \lambda_{\alpha_8} \lambda_{\alpha_9} \lambda_{\alpha_{10}} \lambda_{\alpha_{11}} \delta^{11}(\lambda)
$$

(5.9)

where $\iota_\beta = \partial/\partial \lambda^\beta$. The tensor $T^{(a_1 \cdots a_5)(\beta_1 \cdots \beta_3)}$ is the only invariant tensor constructed from Dirac matrices with five antisymmetrized spinorial indices and three symmetrized spinorial indices. This measure involves both the pure spinors $\lambda^\alpha$ and the fermionic coordinates $\theta^a$.

Since in $\mu(\theta, \lambda)$ there are already 11 $\theta$'s, the Berezin integration in (5.8) picks up 5 additional $\theta$'s from the expression in the bracket. The Dirac delta functions $\delta^{11}(\lambda)$ are needed to integrate over the pure spinors: they are bosonic variables and the integral has to be convergent. The measure $\mu(\theta, \lambda)$ is BRST invariant since

$$
Q \mu(\theta, \lambda) = 11 \lambda^{\alpha_1} \theta^{\alpha_2} \cdots \theta^{\alpha_{11}} \epsilon_{\alpha_1 \cdots \alpha_{16}} T^{(a_1 \cdots a_5)(\beta_1 \cdots \beta_3)} \iota_{\beta_1} \iota_{\beta_2} \iota_{\beta_3} \delta^{11}(\lambda)
$$

(5.10)

The last equality follows from the properties of $T^{(a_1 \cdots a_5)(\beta_1 \cdots \beta_3)}$, where a further antisymmetrization of one of its spinorial indices $\beta_1$, $\beta_2$ or $\beta_3$ makes it vanishing. The second line is due to the fact that the $\lambda^{\alpha_1}$ produced by the BRST variation of $\theta^{\alpha_1}$ has to be annihilated by one of the derivatives $\iota_\beta$ acting on $\delta^{11}(\lambda)$ otherwise it vanishes (integration-by-parts).

Integrating over the pure spinor space we get

$$
S = \int d^{10}x d^{16} \theta (\epsilon \theta^{11})_{a_1 \cdots a_5} (\gamma^{abc})_{a_1 \cdots a_5} \gamma^a_{\alpha_1} \gamma^b_{\alpha_2} \gamma^c_{\alpha_3} \text{Tr} \left( \frac{1}{2} A_{\beta_1} A_{\beta_2} A_{\beta_3} + \frac{1}{3} A_{\beta_1} A_{\beta_2} A_{\beta_3} \right)
$$

(5.11)

where we have used integration by parts for the pure spinors, removing them from the vertex operators, and the Dirac delta’s $\delta^{11}(\lambda)$ to integrate over them. The Berezin integral over the $\theta$'s requires extraction of the five $\theta$'s terms from the bracket. Collecting all the pieces, one gets the action (3.2), namely the component action. The expression (5.11) can be regarded as a superspace action.

### 5.1 Pure Spinor Volume form and PCO’s

As shown in [11], in the pure spinor formalism it exists a ghost-number 3 cohomology class admitting the following representative

$$
\omega^{(3|0)} = \lambda^a \theta \lambda^b \theta \lambda^c \theta \lambda_{abc} \theta
$$

(5.12)
which is BRST invariant modulo pure spinor constraints and it is not exact. The expression is not supersymmetric invariant since it explicitly depends upon the $\theta$'s, nonetheless its variation is BRST exact: $\delta_\omega^{(3|0)} = d\Sigma^{(2|0)}$ where $\delta_\theta^a = \epsilon^a$ and $\delta_\lambda^a = 0$.

What about the volume form? As pointed out in [13], by introducing the delta functions of the pure spinors $\delta(\lambda)$, the volume form is an integral form (restricted to $\theta$'s and $\lambda$'s space) reads

$$\text{Vol}^{(0|11)} = \epsilon_{\alpha_1...\alpha_{16}} \theta^{\alpha_1} \ldots \theta^{\alpha_{16}} \delta^{11}(\lambda).$$  (5.13)

where we defined the expression

$$\delta^{11}(\lambda) = \epsilon_{\alpha_1...\alpha_{11}} \delta(\lambda^{\alpha_1}) d\lambda^{\alpha_1} \wedge \ldots \wedge \delta(\lambda^{\alpha_{11}}) d\lambda^{\alpha_{11}}$$  (5.14)

taking into account only the independent degrees of freedom. Another way to write the same quantity is by introducing a set of commuting spinors $C_{\alpha,i}$ with $i = 1, \ldots, 11$ and writing

$$\bigwedge_{i=1}^{11} \delta(C_{\alpha,i} \lambda^{\alpha}) d(C_{\alpha,i} \lambda^{\alpha}) = \delta^{11}(\lambda)$$  (5.15)

since the determinant coming from the differentials cancels against that coming from the Dirac delta functions. This is normalized as

$$\int_{\mathcal{M}^{(0|16)}} \text{Vol}^{(0|11)} = 1.$$  (5.16)

The volume form (5.13) is BRST closed and it is not exact. It has ghost number zero and maximal picture.

According to the theory of integral forms [18, 19], we can construct the PCO, Hodge dual to $\omega^{(3|0)}$ ($\star : H^{(3|0)} \rightarrow H^{(-3|11)}$) [22]. It should belong to the space $H^{(-3|11)}$, such that locally it satisfies

$$\Psi^{(-3|11)} \wedge \omega^{(3|0)} = \text{Vol}^{(0|11)},$$  (5.17)

The closure and non-exactness of $\omega^{(3|0)}$ and of $\text{Vol}^{(0|11)}$ imply that $Q\Psi^{(-3|11)} = 0$.

The result is the following

$$\Psi^{(-3|11)} = \epsilon_{\alpha_1...\alpha_{16}} \theta^{\alpha_1} \ldots \theta^{\alpha_{11}} \gamma_{a}^{\alpha_{12} \beta_1} \gamma_{b}^{\alpha_{13} \beta_2} \gamma_{c}^{\alpha_{14} \beta_3} (\gamma^{abc}) \alpha_{15} \alpha_{16} \frac{\partial}{\partial \lambda^{\beta_1}} \frac{\partial}{\partial \lambda^{\beta_2}} \frac{\partial}{\partial \lambda^{\beta_3}} \delta^{11}(\lambda).$$  (5.18)

By using the pure spinor properties and gamma matrix algebra, this PCO is indeed $Q$-closed and not $Q$-exact. Eq. (5.17) can be easily checked: one can note that the fermionic coordinates match the total number and the derivatives on the delta’s act by integration-by-parts on $\omega^{(3|0)}$ absorbing the three $\lambda$’s.
As announced in the introduction, one can write the action of $D = 10$ super Yang–Mills theory in the geometrical language of rheonomy. This was done in [15]. The independent fields are $\mathfrak{F}_{ab}(x, \theta), W^\alpha(x, \theta)$ and the connection $A^{(1|0)} = A_a(x, \theta)V^a + A_\alpha(x, \theta)$, while the supervielbein $E^A = (V^a, \psi^\alpha)$ is kept constant (when coupling this action to supergravity, the vielbein $E^A$ becomes dynamical). The starting point to build the rheonomic action is provided by the component action (3.2), by the weights of the different fields, and by Lorentz invariance. The action $L^{(10|0)}$ is a $(10|0)$ superform and it is built avoiding the Hodge dual product. As constructed in [15] and reviewed in [14] it reads

$$L^{(10|0)} = -\frac{1}{90} \mathfrak{F}_{ab} \mathfrak{F}^{ab} V^{a_1} \wedge \ldots \wedge V^{a_{10}} + \mathfrak{F}^{a_1 a_2} F^{(2|0)} \wedge V^{a_3} \ldots \wedge V^{a_{10}} + 2i \mathfrak{F}^{a_1 a_2} W \gamma_\alpha \psi^\alpha V^{a_3} \ldots \wedge V^{a_{10}} + \frac{4}{9} i W \gamma_\alpha \nabla W \wedge V^{a_2} \ldots \wedge V^{a_{10}} \nonumber$$

$$+ \frac{8}{3} i W \gamma_{a_1 \ldots a_3} \psi^\alpha F^{(2|0)} \wedge V^{a_4} \ldots \wedge V^{a_{10}} + \left(1 + \frac{3}{8} a\right) W \gamma_{a_1 \ldots a_3} W \psi^\alpha \gamma_\alpha \psi^\beta V^{a_4} \ldots \wedge V^{a_{10}} + 4i W \gamma_{a_1 a_2} W \psi^\alpha \gamma_\alpha \psi^\beta V^{a_3} \ldots \wedge V^{a_{10}} \epsilon_{a_1 \ldots a_{10}}$$

$$- 84i \left(A^{(1|0)} \wedge F^{(2|0)} - \frac{1}{3} A^{(1|0)} \wedge A^{(1|0)} \wedge A^{(1|0)} \wedge \psi^\alpha \wedge \gamma_{a_1 \ldots a_5} \psi^\beta V^{a_5} \ldots \wedge V^{a_{10}} \right) \epsilon_{a_1 \ldots a_{10}}$$

with $F^{(2|0)} = dA^{(1|0)} + A^{(1|0)} \wedge A^{(1|0)}$, satisfying the Bianchi identity $dF^{(2|0)} + A^{(1|0)} \wedge F^{(2|0)} = 0$. The variation of the action with respect to the $(0|0)$-forms $\mathfrak{F}_{ab}, W^\alpha$ yields the following constraints:

$$F^{(2|0)} = \mathfrak{F}_{ab} V^a \wedge V^b - 2i W \gamma_\alpha \psi^\alpha V^a, \quad \nabla W = V^a \nabla_a W - \frac{1}{4} \gamma^{ab} \psi^\beta \mathfrak{F}_{ab} \epsilon,$$

which imply the equations of motion

$$\nabla^a \mathfrak{F}_{ab} = 0, \quad \gamma^a \nabla_a W = 0,$$

Comparing eq.s (6.2) with eq.s (4.1), we see that the field equation of the superfield $\mathfrak{F}_{ab}$ is algebraic and implies:

$$F_{ab} = \mathfrak{F}_{ab}$$

and

$$F_{a\beta} = -2i(W_{\gamma_\alpha})_{\beta}.$$

Hence on the mass–shell the superfield $\mathfrak{F}_{ab} = \mathcal{F}_{ab}(x) + \mathcal{O}(\theta)$ starts with the bosonic field strength while $W^\alpha = \chi^\alpha(x) + \mathcal{O}(\theta)$ starts with the gaugino.
The action depends upon a free parameter $a$ which parameterizes the two different spinorial structures. It is easy to see that by setting $\psi = 0$, the action reduces to the first order formalism for the component action of SYM as given in (3.2).

The truly interesting term is the last one. It has the form $Chern-Simons \times a$ Chevalley-Eilenberg cohomology class, which is common to all rheonomic formulations of supersymmetric theories and it plays a crucial role here. Indeed, using Fierz identities it can be proved that

$$\omega^{(7|0)} = \psi \gamma^{a_1 \ldots a_5} \psi V_{a_1} \wedge \ldots \wedge V_{a_5},$$

is an element of Chevalley-Eilenberg cohomology for the $N = 1$ super-Poincaré Lie algebra in $D = 10$. The superform $\omega^{(7|0)}$ is closed but not exact. From the point of view of supergravity and of Free Differential Algebras², this Chevalley-Eilenberg cohomology class is responsible for the extension of the $N = 1$, $D = 10$ super Poincaré Lie Algebra to an FDA including a 6-form $B^{[6]}$ which can be regarded as the magnetic dual of the Kalb-Ramond 2-form $B^{[2]}$ (see [29] about this point).

By varying the action with respect to the independent fields $\mathfrak{F}_{ab}, W^a, A^{(1|0)}$, we get the following equations of motion in superspace:

$$- \frac{1}{45} \varepsilon_{a_1 \ldots a_{10}} \mathfrak{F}_{ab} V^{a_1} \wedge \ldots \wedge V^{a_{10}} + \varepsilon_{a_3 a_4 a_5 a_10} F^{(2|0)} \wedge V^{a_3} \wedge \ldots \wedge V^{a_{10}}$$

$$+ 2i \varepsilon_{a_3 a_4 a_5 a_10} W_{\gamma^c} \psi \wedge V^c \wedge V^{a_3} \wedge \ldots \wedge V^{a_{10}} = 0,$$

$$(6.7)$$

$$+ \frac{8i}{9} \varepsilon_{a_1 a_2 a_3 a_4 a_5 a_10} (\gamma^{a_1} \nabla W) \wedge V^{a_2} \wedge \ldots \wedge V^{a_{10}}$$

$$+ 2i \varepsilon_{a_1 a_2 a_3 a_4 a_5 a_10} B^{a_1 a_2 a_3} (\gamma^{a_1} \psi) \wedge V^{a_3} \wedge \ldots \wedge V^{a_{10}}$$

$$- 2 \varepsilon_{a_1 a_2 a_3 a_4 a_5 a_10} (\gamma^{a_1} W)(\psi \wedge \gamma^{a_2} \psi) \wedge V^{a_3} \wedge \ldots \wedge V^{a_{10}}$$

$$+ \frac{8i}{3} \varepsilon_{a_1 a_2 a_3 a_4 a_5 a_10} (\gamma^{a_1 a_2 a_3} \psi) \wedge F^{(2|0)} \wedge V^{a_4} \wedge \ldots \wedge V^{a_{10}}$$

$$+ 2 \left(1 + \frac{8}{3} a\right) \varepsilon_{a_1 a_2 a_3 a_4 a_5 a_10} (\gamma^{a_1 a_2 a_3} W)(\psi \wedge \gamma^{a_4} \psi) \wedge V^{a_4} \wedge \ldots \wedge V^{a_{10}}$$

$$+ 2 a \varepsilon_{a_1 a_2 a_3 a_4 a_5 a_10} (\gamma^{a_1 a_2 a_3} W)(\psi \wedge \gamma^{a_4} \psi) \wedge V^{a_4} \wedge \ldots \wedge V^{a_{10}} = 0,$$

$$(6.8)$$

$$+ \varepsilon_{a_1 a_2 a_3 a_4 a_5 a_10} \nabla \delta^{a_1 a_2} \wedge V^{a_3} \wedge \ldots \wedge V^{a_{10}}$$

$$+ 4i \varepsilon_{a_1 a_2 a_3 a_4 a_5 a_10} B^{a_1 a_2} (\psi \wedge \gamma^{a_3} \psi) \wedge V^{a_4} \wedge \ldots \wedge V^{a_{10}}$$

$$+ \frac{8i}{3} \varepsilon_{a_1 a_2 a_3 a_4 a_5 a_10} (\nabla W \wedge \gamma^{a_1 \ldots a_3} \psi) \wedge V^{a_4} \wedge \ldots \wedge V^{a_{10}}$$

$$+ \frac{8i}{3} \varepsilon_{a_1 a_2 a_3 a_4 a_5 a_10} (W \gamma^{a_1 \ldots a_3} \psi) \wedge (\psi \wedge \gamma^{a_4} \psi) \wedge V^{a_5} \wedge \ldots \wedge V^{a_{10}}$$

$$- 84i F^{(2|0)} \wedge \psi \wedge \gamma^{a_1 \ldots a_5} \psi \wedge V_{a_1} \wedge \ldots \wedge V_{a_5} = 0.$$  

$$(6.9)$$

²For a modern review of the theory of FDA's in relation with Sullivan's theorems and Chevalley-Eilenberg cohomology of super Lie Algebras see sections 6.3 and 6.4 (pages 227-238) in the second volume of [28].
Notice that the first two equations are \((10|0)\) superforms and the last is a \((9|0)\) superform. By expanding \(F^{(2|0)}\) into \((2|0)\) components and \(\nabla W\) into \((1|0)\) components one finds the already anticipated constraints (6.2) which determine the supersymmetry transformation rules. Notice that these constraints reproduce only the on-shell supersymmetry transformations and from those one recovers also the dynamical equations of motion

\[
\nabla^a F_{ab} = 0, \quad \gamma^a \nabla_a W = 0,
\]

The above field equations implied from the superspace constraints follow also from the projection of the field equations on the maximal vielbein sector \((\psi = 0\) projection). This is the fundamental consistency check that guarantees the supersymmetry of the component lagrangian as extracted from the rheonomic one. These equations are indeed necessary in order to satisfy the set of equations (6.7)-(6.9). Notice that expanding the \((10|0)\) forms into the different components, at the highest order in the gravitino field \(\psi\) ( the \(\psi^3 V^7\) sector, in this case), one sees that only the algebraic properties of gamma matrices are needed to solve such equations. Note that there are several redundancies among the equations. This is due to the fact that at each order in the gravitino expansion, the action encodes the same amount of information. This allows us to extract different superspace actions from the rheonomic action (6.1).

7 PCO’s and interpolating actions

Here we start from the rheonomic action (6.1) and we show how to interpolate between the spacetime action (3.2) and the superspace action of the form (5.8).

In order to derive the component action, we have to integrate it over the supermanifold \(\mathcal{M}^{(10|16)}\) and for that we need a PCO which multiplies \(L^{(10|0)}\) of (6.1). This is achieved by constructing the expression

\[
\mathcal{Y}^{(0|16)} = \theta^{16} \delta^{16}(\psi)
\]

where we recall that \(\psi = d\theta\) for flat superspace. This projects to the space with \(\theta^\alpha = 0\) and \(\psi^\alpha = 0\).

Then, we obtain

\[
S = \int_{\mathcal{M}^{(10|16)}} L^{(10|0)} \wedge \mathcal{Y}^{(0|16)} = \int \left( -\frac{1}{90} \delta_{ab} \delta^{ab} V^{a_1} \wedge \cdots \wedge V^{a_{10}} 
\right.
\]

\[
+ \delta^{a_1 a_2} F^{(2|0)} \wedge V^{a_3} \cdots \wedge V^{a_{10}} + \frac{4}{9} i W \gamma^{a_1} \nabla W \wedge V^{a_2} \cdots \wedge V^{a_{10}} \epsilon_{a_1 \cdots a_{10}} \wedge \mathcal{Y}^{(0|16)}
\]

\[
= \int d^{10} x \left( -\frac{1}{4} F_{ab} F^{ab} + \chi \gamma^a \nabla_a \chi \right)
\]

(7.2)
where $F_{ab}$ depends only on $x$, as already stated and $\chi^\alpha = W^\alpha(x, \theta = 0)$. Note that all the vielbeins $V^\alpha$ reduce to $dx^a$ since $\theta = d\theta = 0$. Furthermore, in the last step we have substituted back into the lagrangian the algebraic field equation of the auxiliary 0-form $\tilde{s}_{ab} \big|_{\theta = 0} = F_{ab}$.

In this way we obtain the standard second order form of the component lagrangian.

Let us now change the above PCO and let us introduce a new choice

$$
\mathcal{Y}^{(0|16)} = V^{a_1} \wedge V^{a_2} \wedge V^{a_3} \wedge V^{a_4} \wedge V^{a_5} \epsilon_{\beta_1 \ldots \beta_{16}} \theta^{\beta_1} \ldots \theta^{\beta_{11}} (\gamma_{\alpha_1 t})^{\beta_{12}} \ldots (\gamma_{\alpha_5 t})^{\beta_{16}} \delta^{16}(\psi) \quad (7.3)
$$

Notice that the above PCO has zero form degree (since it contains 5 vielbeins and 5 contractions $\iota_\alpha$) and picture equal to sixteen. Notice that it contains only eleven $\theta$’s. They carry an upper index $\alpha$ while the contraction $\iota_\alpha$ has a lower index. Therefore, the combination $(\gamma_{\alpha t})^\alpha$ cannot be contracted with $\theta^\alpha$. Then, one needs the Levi-Civita tensor $\epsilon_{\beta_1 \ldots \beta_{16}}$ which leaves eleven anti-symmetric indices to be contracted with the $\theta$’s. Notice that $\iota_\alpha$ is a commuting differential operator, the combination $(\gamma_{\alpha t})^\beta$ works as an anticommuting quantity because of the factor $V^{a_1} \wedge V^{a_2} \wedge V^{a_3} \wedge V^{a_4} \wedge V^{a_5}$ which is antisymmetric in the vector indices $a_1 \ldots a_5$.

Let us check the closure of $\mathcal{Y}^{(0|16)}$. By using $\psi = d\theta$, we have

$$
d\mathcal{Y}^{(0|16)}_{\text{p.s.}} = 5(\psi \gamma^{a_1} \psi) V^{a_2} \wedge V^{a_3} \wedge V^{a_4} \wedge V^{a_5} \epsilon_{\beta_1 \ldots \beta_{16}} \theta^{\beta_1} \ldots \theta^{\beta_{11}} (\gamma_{\alpha_1 t})^{\beta_{12}} \ldots (\gamma_{\alpha_5 t})^{\beta_{16}} \delta^{16}(\psi) \\
+ 11 V^{a_1} \wedge V^{a_2} \wedge V^{a_3} \wedge V^{a_4} \wedge V^{a_5} \epsilon_{\beta_1 \ldots \beta_{16}} \theta^{\beta_1} \ldots \theta^{\beta_{11}} (\gamma_{\alpha_1 t})^{\beta_{12}} \ldots (\gamma_{\alpha_5 t})^{\beta_{16}} \delta^{16}(\psi) \\
+ 10 V^{a_2} \wedge V^{a_3} \wedge V^{a_4} \wedge V^{a_5} (\gamma_{\alpha_1}^{\alpha_1} \gamma_{\alpha_2}^{\alpha_2})^{\beta_{12} \beta_{13}} (\theta^{11} \epsilon)_{\beta_{12} \ldots \beta_{16}} (\gamma_{\alpha_2 t})^{\beta_{14}} \ldots (\gamma_{\alpha_5 t})^{\beta_{16}} \delta^{16}(\psi) \\
+ 55 V^{a_1} \wedge V^{a_2} \wedge V^{a_3} \wedge V^{a_4} \wedge V^{a_5} \gamma_{\alpha_1}^{\beta_{11} \beta_{12}} (\theta^{10} \epsilon)_{\beta_{11} \ldots \beta_{16}} (\gamma_{\alpha_2 t})^{\beta_{13}} \ldots (\gamma_{\alpha_5 t})^{\beta_{16}} \delta^{16}(\psi),
$$

$$
= 0. \quad (7.4)
$$

where $(\theta^{11} \epsilon)_{\beta_{12} \ldots \beta_{16}} = \epsilon_{\beta_1 \ldots \beta_{11} \beta_{12} \ldots \beta_{16}} \theta^{\beta_1} \ldots \theta^{\beta_{11}}$.

In eq. (7.4), the first line vanishes since the matrix $(\gamma_{\alpha_1}^{\alpha_1} \gamma_{\alpha_2}^{\alpha_2})^{\beta_{12} \beta_{13}}$ has to be antisymmetric in the spinorial indices and this implies that it should be proportional to the gamma matrix $\gamma_{\alpha_1 \alpha_2 \alpha_3}$ which is totally antisymmetric in the vectorial indices. However, the contractions between Lorentz indices in the formula imply that this vanishes. The second line vanishes since the indices of the symmetric gamma matrix $\gamma_{\alpha_1}$ are contracted with the $\epsilon$-tensor in spinorial space. Therefore this implies that the PCO is indeed closed.

Before completing the discussion about the action, we observe the following relation

$$
\omega^{(7|16)} = \omega^{(7|0)} \wedge \mathcal{Y}^{(0|16)}_{\text{p.s.}} \\
= V^{10}(\theta^{11} \epsilon)_{\alpha_1 \ldots \alpha_5} \mathcal{T}^{[\alpha_1 \ldots \alpha_5]}(\beta_1^{12} \beta_2^{13}) \epsilon_{\beta_1 \ldots \beta_5}^{\beta_1 \ldots \beta_5} \delta^{16}(\psi)
$$

(7.5)

where $\mathcal{T}^{[\alpha_1 \ldots \alpha_5]}(\beta_1^{12} \beta_2^{13}) = (\gamma_{abc})^{[\alpha_1 \alpha_2} (\gamma_{\alpha_3}^{\alpha_4} \alpha_5]^{[\beta_1} \gamma_{\beta_2}^{\beta_3]} \gamma_{\beta_4}^{\beta_5]} \gamma_{\beta_5]}^{\beta_5]}$ which is the invariant spinorial numerical tensor discussed in (3.2) and $V^{10} = e_{a_1 \ldots a_9} V^{a_9} \wedge \ldots \wedge V^{a_9}$ is the bosonic volume form. Notice that the new integral form $\omega^{(7|16)}$ has all desired good properties. It is Lorentz invariant and it is closed since both $\omega^{(7|0)}$ and $\mathcal{Y}^{(0|16)}_{\text{p.s.}}$ are closed.
Now, we apply the PCO to the action. Due to the presence of five vielbeins $V^a$, all terms except the last one drop out and we are left with

$$S = \int_{\mathcal{M}^{(10|16)}} \left[ \mathcal{A}_\lambda \mathcal{F} - \frac{1}{3} \mathcal{A}_\lambda \mathcal{A}_\lambda \mathcal{A} \right] \wedge \psi_\gamma a_1 \ldots a_5 \psi \wedge V_{a_1} \wedge \ldots \wedge V_{a_5} \wedge \psi^{(0|16)}_{p.s.} \tag{7.6}$$

Notice that the five contractions $\iota_\alpha$ appearing in $\psi_{p.s.}$ absorb five $\psi$’s both from $\omega^{(7|0)}$ and from the Chern-Simons term. In particular, the two $\psi$’s from $\omega^{(7|0)}$ and three $\psi$’s are removed from the Chern-Simons action. Since the latter is a three form, this means that it selects $L^{(3|0)} = \psi^\alpha \psi^\beta \psi^\gamma A_\alpha D_\beta A_\gamma + \ldots$ (as in the pure spinor action). The vielbeins $V^a$ are ten in total, hence they arrange themselves into a scalar. The Dirac delta’s allow for the integration over the $\psi$’s so that we are finally left with the counting of $\theta$’s. Notice that since we have already eleven $\theta$’s in the PCO, we need to take 5 $D$-derivatives of the action. This finally yields the SYM action in components.

In conclusion, the action admits the following very elegant writing:

$$S = \int_{\mathcal{M}^{(10|16)}} \left[ \text{Tr} \left( A^{(1|0)} \wedge F^{(2|0)} - \frac{1}{3} A^{(1|0)} \wedge A^{(1|0)} \wedge A^{(1|0)} \right) \wedge \omega^{(7|0)} \wedge \psi^{(0|16)}_{p.s.} \right] \wedge \psi^{(7|0)} \wedge \psi^{(0|16)}_{p.s.} \tag{7.7}$$

The combination $\omega^{(7|0)} \wedge \psi^{(0|16)}_{p.s.}$ yields

$$\omega^{(7|0)} \wedge \psi^{(0|16)}_{p.s.} = V^1 \wedge \ldots \wedge V^{10} \wedge \psi^{(-3|11)} \tag{7.8}$$

where $\psi^{(-3|11)}$ is given in (5.18).

We succeeded to show that the pure spinor formulation and the rheonomic formulation of SYM produce the same superspace action. The rheonomic formulation has its origin into the theory of integral forms which shares common features with pure spinor superstring by the introduction of the target-space PCO’s. We hope that this new point of view might be used for a fruitful re-formulation of type IIB supergravity along the same lines [30,31].

8 Conclusions

We add some considerations to the above presented discussion and we single out some interesting problems for future investigations.

a) We have illustrated the similarities between the pure spinor formulation and the geometrical formulation based on the rheonomic action plus the crucial ingredient of the PCO’s (i.e. Poincaré duals of top bosonic cycles in superspace). It would be desirable to establish a dictionary between the two frameworks. We are tempted to identify the
pure spinor $\lambda^\alpha$ (in D=10), satisfying the constraints $\lambda^\alpha \gamma^\alpha_{\alpha\beta} \lambda^\beta = 0$ with the differential $\psi^\alpha = d\theta^\alpha$, although the latter are not constrained.

Thus, a “trial” dictionary might be

$$\theta^\alpha \leftrightarrow \theta^\alpha, \quad \lambda^\alpha \leftrightarrow d\theta^\alpha, \quad Q \leftrightarrow d,$$  \hspace{1cm} (8.1)

The form number is replaced by the ghost number.

Since the pure spinor components are treated as bosonic commuting coordinates, their Dirac delta functions correspond to usual Dirac delta’s. Then, in order to identify $\delta(\psi)$ with the pure spinor Dirac delta, we set

$$\delta(\psi^\alpha) = \delta(\lambda^\alpha)d\lambda^\alpha$$  \hspace{1cm} (8.2)

where the index $\alpha$ is not summed. Notice that in this way, the l.h.s. has no form degree, while the $\delta(\lambda^\alpha)$ has negative form degree $(-1)$ to compensate the form degree of $d\lambda^\alpha$. In addition, since the usual Dirac delta function $\delta(\lambda^\alpha)$ is a commuting quantity, the resulting $\delta(\psi^\alpha)$ is anticommuting, as it should be for our purposes. The combination $\delta(\lambda^\alpha)d\lambda^\alpha$ is trivially BRST closed.

b) The second remark is to point out that the rheonomic action is not $d$ closed (without the use of the superspace constraints that reduce the theory on shell). According to common lore and to actual calculations, this is intimately related to the absence of an auxiliary field formulation for an off-shell extension of the algebra. The present derivation does not solve such an issue, yet we emphasize that starting from a completely different point of view we arrive at the same result (7.7) in both formalisms.

c) In relation with point b) let us also stress that in [15] Bianchi identities of $D = 10$ super Yang-Mills theory in the anholonomic setup, proper to the rheonomic approach, were analyzed up to the third order in the spinor derivatives in search for a possible set of auxiliary fields. No match was found between bosons and fermions up to that order. The conjecture therefore was put forward that the only off-shell representation is that containing $819200$ bosons $+$ $819200$ fermions which corresponds to no constraint whatsoever on the superform $A^{(1|0)}$. Indeed starting from the most general parametrization of the curvature superform in terms of coefficient superfields arranged into irreducible $\text{SO}(1,9)$ representations:

$$ F^{(2|0)} = F_{ab} V^a \wedge V^b - 2\psi \left( \xi_{a2}^{(144)} - i \gamma^a \zeta^{(16)} \right) \wedge V^a + i E^{(10)}_a \psi \wedge \gamma^a \psi + i E^{(126)}_{a_1...a_5} \psi \wedge \gamma^{a_1...a_5} \psi$$  \hspace{1cm} (8.3)

\[3\text{see [14] and [28] for reviews.}\]

\[4\text{The affix (n) appended to each coefficient superfield denotes the dimension of the corresponding Lorentz representation.}\]
it was shown that any Lorentz covariant constraint, namely the suppression of any of
the four representations $144, 16, 10$ or $126$ immediately reduces the space-time fields
to the mass shell. This left the possibility that the Lorentz covariant constraint could
be imposed by suppressing some of the Lorentz irreducible representation in the spinor
derivatives of the above or in the derivatives of the derivatives. Yet, up to third deriv-
atives, as we recalled, no viable constraint was found and this motivated the quoted
conjecture. Now, in view of the identification in eq. (8.1) a new line of thinking is
brought to attention. That no Lorentz covariant off-shell representation does exist, ex-
cept the largest one, is probably a correct conclusion in unconstrained superspace, yet
what about constrained superspace in line with the ideas utilized in [32]? Let us observe
that if the theta’s satisfy the Lorentz covariant constraint $d \theta \wedge \gamma^a d \theta = 0$, analogous
to the pure spinor constraint, then we also have $\psi \wedge \gamma^a \psi = 0$ and in eq. (8.3) the
superfield $B_a^{(10)}$ naturally disappears. What happens in the subsequent orders is matter
to be analyzed, but it is not inconceivable that a consistent set of auxiliary fields can now
be found. This is a matter for future analysis.

d) The two PCO’s (7.1) and (7.3) are two possible choices; there are several other that are
cohomologous to them and it remains to be explored which kind of action they might
lead to.

Acknowledgement

We thank C. Maccaferri, L. Castellani and R. Catenacci for fruitful discussions.
References

[1] W. Nahm, “Supersymmetries and their Representations,” *Nucl. Phys.*, vol. B135, p. 149, 1978.

[2] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory Vol. 1*. Cambridge University Press, 2012.

[3] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory Vol. 2*. Cambridge University Press, 2012.

[4] L. Brink, J. H. Schwarz, and J. Scherk, “Supersymmetric Yang-Mills Theories,” *Nucl. Phys.*, vol. B121, pp. 77–92, 1977.

[5] E. Witten, “Twistor - Like Transform in Ten-Dimensions,” *Nucl. Phys.*, vol. B266, pp. 245–264, 1986.

[6] S. J. Gates, Jr. and H. Nishino, “On D = 10, N=1 Supersymmetry, Superspace Geometry and Superstring Effects. 2.,” *Nucl. Phys.*, vol. B291, p. 205, 1987.

[7] N. Berkovits and C. M. Hull, “Manifestly covariant actions for D = 4 selfdual Yang-Mills and D = 10 superYang-Mills,” *JHEP*, vol. 02, p. 012, 1998.

[8] J. P. Harnad and S. Shnider, “Constraints and field equations for ten-dimensional super Yang-Mills theory,” *Commun. Math. Phys.*, vol. 106, p. 183, 1986.

[9] P. S. Howe, “Pure spinors lines in superspace and ten-dimensional supersymmetric theories,” *Phys. Lett.*, vol. B258, pp. 141–144, 1991. [Addendum: Phys. Lett.B259,511(1991)].

[10] P. S. Howe, “Pure spinors, function superspaces and supergravity theories in ten-dimensions and eleven-dimensions,” *Phys. Lett.*, vol. B273, pp. 90–94, 1991.

[11] N. Berkovits, “Super Poincare covariant quantization of the superstring,” *JHEP*, vol. 04, p. 018, 2000.

[12] N. Berkovits, “Covariant quantization of the superparticle using pure spinors,” *JHEP*, vol. 09, p. 016, 2001.

[13] N. Berkovits, “Multiloop amplitudes and vanishing theorems using the pure spinor formalism for the superstring,” *JHEP* **0409**, 047 (2004) doi:10.1088/1126-6708/2004/09/047 [hep-th/0406055].

[14] L. Castellani, R. D’Auria, and P. Fré, *Supergravity and superstrings: A Geometric perspective. Vol. 1,2,3*. 1991.
[15] R. D’Auria, P. Fré, and A. J. da Silva, “Geometric Structure of $N = 1$, $D = 10$ and $N = 4$, $D = 4$ Superyang-mills Theory,” Nucl. Phys., vol. B196, pp. 205–239, 1982.

[16] P. Fré, “$N=1$ supersymmetric Yang-Mills theory on the supergroup manifold and the supercurrent,” Lett. Nuovo Cim., vol. 30, p. 507, 1981.

[17] P. Fré, “Remarks on the Supergroup Manifold Approach to Supersymmetric Yang-Mills Theories and the Explicit Construction of the $N = 2$ Case,” Nucl. Phys., vol. B187, pp. 376–388, 1981.

[18] L. Castellani, R. Catenacci, and P. A. Grassi, “Supergravity Actions with Integral Forms,” Nucl. Phys., vol. B889, pp. 419–442, 2014.

[19] L. Castellani, R. Catenacci, and P. A. Grassi, “The Geometry of Supermanifolds and New Supersymmetric Actions,” Nucl. Phys., vol. B899, pp. 112–148, 2015.

[20] L. Castellani, R. Catenacci, and P. A. Grassi, “Super Quantum Mechanics in the Integral Form Formalism,” 2017.

[21] P. Fré and P. A. Grassi, “The Integral Form of $D=3$ Chern-Simons Theories Probing $\mathbb{C}^n/\Gamma$ Singularities.” arXiv/hep-th1705.00752, 2017.

[22] L. Castellani, R. Catenacci and P. A. Grassi, “Integral representations on supermanifolds: super Hodge duals, PCOs and Liouville forms,” Lett. Math. Phys. 107, no. 1, 167 (2017) doi:10.1007/s11005-016-0895-x [arXiv:1603.01092 [hep-th]].

[23] L. Castellani, R. Catenacci, and P. A. Grassi, “The Integral Form of Supergravity,” JHEP, vol. 10, p. 049, 2016.

[24] P. A. Grassi and C. Maccaferri, “Chern-Simons Theory on Supermanifolds,” JHEP, vol. 09, p. 170, 2016.

[25] E. Witten, “Notes on Supermanifolds and Integration,” 2012.

[26] W. Siegel and M. Rocek, “On off-shell supermultiplets,” Phys. Lett., vol. 105B, pp. 275–277, 1981.

[27] N. Berkovits, “Cohomology in the pure spinor formalism for the superstring,” JHEP, vol. 09, p. 046, 2000.

[28] P. G. Fré, Gravity, a Geometrical Course, vol. 1.2. Springer Science & Business Media, 2012.
[29] R. D’Auria and P. Fre, “Duality in Superspace and Anomaly Free Supergravity: Some Remarks,” *Mod. Phys. Lett.*, vol. A3, p. 673, 1988.

[30] L. Castellani and I. Pesando, “The Complete superspace action of chiral D = 10, N=2 supergravity,” *Int. J. Mod. Phys. A* 8, 1125 (1993). doi:10.1142/S0217751X9300045X

[31] A. Sen, “Covariant Action for Type IIB Supergravity,” *JHEP* 1607, 017 (2016) doi:10.1007/JHEP07(2016)017 [arXiv:1511.08220 [hep-th]].

[32] P. Fre and P. A. Grassi, “Constrained Supermanifolds for AdS M-Theory Backgrounds,” *JHEP* 0801 (2008) 036 doi:10.1088/1126-6708/2008/01/036 [arXiv:0704.3413 [hep-th]].