The Study of Global Stability of a Diffusive Michaelis-Menten and Tanner Predator-Prey Model

Demou Luo

Abstract. In this paper, we consider a parabolic predator-prey model of Michaelis-Menten and Tanner functional response with random diffusion:

\begin{align*}
    u_t &= d_1 \Delta u + au - bu^2 - \frac{\delta uv}{au + v}, \\
    v_t &= d_2 \Delta v + rv - \gamma \frac{v^2}{u},
\end{align*}

with \( d_1, d_2, a, b, r, a, \gamma, \delta > 0 \) under the no-flux boundary condition in a smooth bounded domain \( \Omega \subset \mathbb{R}^n (n = 1, 2, 3) \). By applying a new method, we establish much improved global asymptotic stability of the unique positive equilibrium solution than works in literature. We also show the result can be extended to more general type of systems with heterogeneous environment.

1. Introduction

The main purpose of this article is to consider the following parabolic predator-prey model with Michaelis-Menten and Tanner functional response

\begin{align}
    \left\{ \begin{array}{ll}
    u_t &= d_1 \Delta u + au - bu^2 - \frac{\delta uv}{au + v}, & x \in \Omega, \quad t \in (0, \infty), \\
    v_t &= d_2 \Delta v + rv - \gamma \frac{v^2}{u}, & x \in \Omega, \quad t \in (0, \infty), \\
    \frac{\partial u}{\partial v} - \frac{\partial v}{\partial v} &= 0, & x \in \partial \Omega, \quad t \in (0, \infty), \\
    u(x, 0) &= u_0(x) > 0, \quad v(x, 0) = v_0(x) \geq 0(\neq 0), & x \in \Omega,
    \end{array} \right. \tag{1}
\end{align}

where \( d_1, d_2, a, b, r, a, \gamma, \delta > 0 \), \( \Omega \) is a bounded domain in \( \mathbb{R}^n (n = 1, 2, 3) \) with smooth boundary \( \partial \Omega \), \( 0 < T \leq +\infty \), and \( u(x, t) \) and \( v(x, t) \) are the density of prey and predator, respectively. Throughout this article, we suppose that the two diffusion coefficients \( d_1 \) and \( d_2 \) are equal, but not necessarily constants. We shall apply
Definition 1.1 (Global stability). Let \((u', v')\) be a positive solution of model (1). We say that it is global asymptotically stable if any other positive solution \((u(x, t), v(x, t))\) of model (1) has the property
\[
\lim_{t \to \infty} (u(x, t), v(x, t)) = (u', v').
\]
Our main theorem is as follows.

**Theorem 1.2.** Suppose \( d = d(x, t) \) is strictly positive, bounded and continuous in \( \Omega \times [0, +\infty) \), \( a, b, r, \alpha, \gamma, \) and \( \delta \) are positive constants, \( r < a \), then the positive equilibrium solution \((u^*, v^*)\) is globally asymptotically stable in the sense that every solution \((u(x, t), v(x, t))\) of (1) satisfies

\[
\lim_{t \to \infty} (u(x, t), v(x, t)) = (u^*, v^*)
\]

uniformly in \( x \in \Omega \).

**Remark 1.3.** The approach we apply here is more powerful than that applied in [6] and more flexible than the Lyapunov function and linear analysis approaches, and the results cover more general settings such as when the Laplace operator is replaced by a uniform elliptic operator. It means we can cover cases with heterogeneous environment.

Let us denote by

\[
L_u = \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}
\]

a uniform elliptic operator in \( \Omega \) with continuous coefficients \( a_{ij}(x), i, j = 1, 2, \cdots, N \). Therefore, we can easily reveal a outcome similar to Theorem 1.2 for the following initial-boundary value problem:

\[
(II) \begin{cases}
    u_t = Lu + au - bu^2 - \frac{\delta uv}{\alpha u + v'}, & x \in \Omega, \quad t \in (0, \infty), \\
    v_t = Lv + rv - \gamma \frac{v^2}{u}, & x \in \Omega, \quad t \in (0, \infty), \\
    \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial \Omega, \quad t \in (0, \infty), \\
    u(x, 0) = u_0(x) > 0, \quad v(x, 0) = v_0(x) \geq 0(\neq 0), & x \in \Omega.
\end{cases}
\]

**Theorem 1.4.** Suppose \( r < a \) and \( a, b, r, \alpha, \gamma, \delta \) are positive constants satisfying the assumption in Theorem 1.2 and \( L \) is a uniform elliptic operator in \( \Omega \) with continuous coefficients. Then, the unique positive equilibrium \((u^*, v^*)\) of (II) is globally asymptotically stable.

It is known via a direct and simple computation that (1) possesses a unique positive equilibrium \((u^*, v^*)\), where

\[
\begin{align*}
    u^* &= \frac{a \left( \alpha + \frac{r}{\gamma} \right) - \delta + \sqrt{a \left( \alpha + \frac{r}{\gamma} \right) - \delta^2}}{2b \left( \alpha + \frac{r}{\gamma} \right)}, \\
    v^* &= \frac{\gamma}{a} u^*.
\end{align*}
\]

**Remark 1.5.** To guarantee that the ecosystem (1) has a unique positive equilibrium \((u^*, v^*)\), the condition, \( a \left( \alpha + \frac{r}{\gamma} \right) - \delta > 0 \), is imposed according to the expressions of \( u^* \) and \( v^* \).

The rest of the paper is organized as follow. In Sect. 2, we prove our main result. We shall argue how to generalize our results to more general setting in Sect. 3, such as different functional responses and time delay.
2. Proof of the main result

We define \( w = \frac{u}{v} \), then we obtain

\[
\begin{align*}
\nu_{w} &= \nabla v \cdot u - \frac{v u}{u^2}, \\
\Delta w &= \frac{\Delta v u^3 + u^2 \nabla u \cdot \nabla v - u^2 v \Delta u - u^2 \nabla v \cdot (\nabla v \cdot u - \nabla u \cdot v) 2u \nabla u}{u^4},
\end{align*}
\]

Therefore the equation satisfied by \( w(x,t) \) is

\[
\begin{align*}
\nu_{w} - d \Delta w &= \frac{(\nu_{u} - \frac{u \nu_{v}}{u^2}) - d \left( \frac{\nu_{v} \Delta u}{u} - \frac{\nu u \cdot \nabla v}{u^2} - \frac{2 \nabla u \cdot \nabla v}{u^3} + \frac{2 \nu |\nabla u|^2}{u^3} \right)}{u} \\
&= \frac{\nu_{u} - d \Delta u - \nu (u_{1} - d \Delta u)}{u^2} + 2d \frac{\nabla u}{u} \left( \frac{\nu v}{u} - \frac{\nu u}{u^2} \right) \\
&= \frac{\nu (r - a)}{u} - \frac{\nu (a u - b u^2 - \frac{\alpha \gamma}{\alpha u + v})}{u^2} + 2d \frac{\nabla u}{u} \cdot \nabla w \tag{4}
\end{align*}
\]

Proposition 2.1. Suppose \( r < a \) and \( \epsilon_1 > 0 \) small. There exists a sufficiently large constant \( T > 0 \) such that the solution \( u \) of (1) satisfies

\[
uw \leq \nu_{u}(\epsilon_1) \equiv \frac{ax - b \frac{\epsilon_1}{2} u_{1} b + \sqrt{(ax - b \frac{\epsilon_1}{2} u_{1} b)^2 + 4ab\alpha \gamma^{2} u_{1}^2}}{2b\alpha} + O(\epsilon_1),
\]

for \( x \in \Omega \) and \( t \geq T \), where

\[
\begin{align*}
u_{u_{1}} &= \frac{a(x + \nu_{1}(\epsilon_1)) - \delta \nu_{1}(\epsilon_1)}{2b(x + \nu_{1}(\epsilon_1))}, \\
u_{w_{1}} &= \frac{(r + \delta)\nu_{1} + b\nu_{1}^2 - (a + \alpha \gamma)\nu_{1}}{2\nu_{1}}, \\
&\quad + \sqrt{(a + \alpha \gamma)\nu_{1} - (r + \delta)\nu_{1} - b\nu_{1}^2 + 4\gamma \nu_{1} [(r - a\alpha + b\nu_{1})\nu_{1}]} \frac{2\nu_{1}}{2\nu_{1}}.
\end{align*}
\]

and \( \nu_{1} \equiv \nu_{1}. \)

Proof. Since \( u > 0, v \geq 0 \), it is easily to verify by a direct calculation that \( u \) satisfies

\[
u_{u} - d \Delta u \leq u(a - bu), \quad x \in \Omega \times (0, \infty).
\]

By the well established fact and a simple comparison that any positive solution of

\[
\begin{align*}
u_{u} - d \Delta u &\leq u(a - bu), \quad x \in \Omega, \quad t \in (0, \infty), \\
\frac{\partial u}{\partial v} &= 0, \quad x \in \partial \Omega, \quad t \in (0, \infty),
\end{align*}
\]

...
converges to the asymptotic stable equilibrium $\bar{u}$ uniformly as $t \to \infty$, i.e. $\lim_{t \to \infty} u = \bar{u}$, we can obtain that $\forall \epsilon_1 > 0, \exists t_1 > 0$ such that
\[
u(x, t) < \bar{u}_1(\epsilon_1) \equiv \frac{a}{b} + \frac{\epsilon_1}{5}
\]
for $x \in \Omega$ and $t \geq t_1$. Therefore, for $t \geq t_1$,
\[
w_t - d\Delta w \leq w \left( r - a + b\bar{u}_1(\epsilon_1) \right) + \omega \left( \frac{\delta \bar{u}_1(\epsilon_1)}{\bar{u}^2(\epsilon_1) + \bar{u}(\epsilon_1)w} - \gamma \right) + 2d\frac{\nabla u}{u} \cdot \nabla w.
\]

It is obvious that the following ordinary differential equation (ODE) about $W(t)$
\[
W_t = W \left( r - a + b\bar{u}_1(\epsilon_1) \right) + W \left( \frac{\delta \bar{u}_1(\epsilon_1)}{\bar{u}^2(\epsilon_1) + \bar{u}(\epsilon_1)W} - \gamma \right)
\]
possesses three solutions:
\[
W_0 = 0,
\]
\[
W_{1,2} = \frac{(r + \delta)\bar{u}_1(\epsilon_1) + b\bar{u}^2_1(\epsilon_1) - (a + \alpha\gamma)\bar{u}_1(\epsilon_1)}{2\gamma\bar{u}_1(\epsilon_1)} \pm \frac{\sqrt{[(a + \gamma\gamma)\bar{u}_1(\epsilon_1) - (r + \delta)\bar{u}_1(\epsilon_1) - b\bar{u}^2_1(\epsilon_1)]^2 + 4\gamma^2\bar{u}_1(\epsilon_1)(r - a + b\bar{u}_1(\epsilon_1)\delta\bar{u}_1(\epsilon_1))}}{2\gamma\bar{u}_1(\epsilon_1)}.
\]

It is obvious that $W_1(t)$ is the unique asymptotically stable positive equilibrium point of (6), and $W_0(t) = 0$ is unstable. Since the trajectories of (6) cannot cross the $x$-axis, then all positive solutions $W(t)$ of (6) will converge to the unique positive asymptotically stable equilibrium point $W_1(t)$. By a simple comparison argument, we obtain that there possesses a positive constant $t_2 \geq t_1$ satisfies
\[
0 < v(x, t) = \frac{u(x, t)}{\bar{u}(x, t)} \leq \bar{u}_1(\epsilon_1) \equiv W_1 + \frac{\epsilon_1}{5}
\]
for all $x \in \Omega$ and $t \geq t_2$. Therefore, $v \leq \bar{v}(\epsilon_1)u$, and
\[
u_t - d\Delta u \geq u(a - bu) - \frac{\delta \bar{v}_1(\epsilon_1)u}{\alpha + \bar{v}_1(\epsilon_1)} = u \left[ \frac{(a - bu)(\alpha + \bar{v}_1(\epsilon_1)) - \delta \bar{u}_1(\epsilon_1)}{\alpha + \bar{v}_1(\epsilon_1)} \right]
\]
for all $x \in \Omega$ and $t \geq t_2$. Let
\[
(a - bu)(\alpha + \bar{v}_1(\epsilon_1)) - \delta \bar{u}_1(\epsilon_1) = 0,
\]
then we obtain only one positive solution
\[
u_R = \frac{a\alpha + (a - \delta)\bar{u}_1(\epsilon_1)}{b(\alpha + \bar{v}_1(\epsilon_1))}
\]
which is a stable equilibrium point of the corresponding ordinary differential equation
\[
u_t = \frac{u \left[ (a - bu)(\alpha + \bar{v}_1(\epsilon_1)) - \delta \bar{u}_1(\epsilon_1) \right]}{\alpha + \bar{v}_1(\epsilon_1)}.
\]
Hence, all positive solution of (10) will converge to $\nu_R$, which means that there exists $t_3 > t_2$ such that
\[
u \geq \nu_1(\epsilon_1) \equiv \nu_R - \frac{\epsilon_1}{5} \equiv \frac{a\alpha + (a - \delta)\bar{u}_1(\epsilon_1)}{b(\alpha + \bar{v}_1(\epsilon_1))} - \frac{\epsilon_1}{5},
\]
By comparison principle, we can draw a conclusion that there exists a constant $t_4 > t_3$ such that
\[ v \geq u_1(\varepsilon_1) = \frac{ru_1(\varepsilon_1)}{\gamma} - \frac{\varepsilon_1}{5} \]
for all $x \in \Omega$ and $t \geq t_4$. Therefore, there exists a constant $t_5 > t_3$ such that
\[ v \geq u_1(\varepsilon_1) = \frac{ru_1(\varepsilon_1)}{\gamma} - \frac{\varepsilon_1}{5} \]
for all $x \in \Omega$ and $t \geq t_5$. Setting the estimate $v \geq u_1(\varepsilon_1)$ into the first equation of (1), we have
\[ u_t - d\Delta u \leq au - bu^2 - \frac{\delta u v_1(\varepsilon_1)}{au + v_1(\varepsilon_1)} \]
\[ = u \left[ (a - bu)(au + v_1(\varepsilon_1)) - \delta v_1(\varepsilon_1) \right] \]
The quadratic equation of one variable
\[ (a - bu)(au + v_1(\varepsilon_1)) - \delta v_1(\varepsilon_1) = 0 \]
possesses only one positive solution
\[ u^R = \frac{a\alpha - v_1(\varepsilon_1)b + \sqrt{(a\alpha - v_1(\varepsilon_1)b)^2 + 4(\alpha v_1(\varepsilon_1) - \delta v_1(\varepsilon_1))ba}}{2ba} \]
(13)
By comparison principle, we can draw a conclusion that there exists $t_5 > t_4$ such that if $t \geq t_5$,
\[ u \leq \overline{u}_2(\varepsilon_1) \equiv u^R + \frac{\varepsilon_1}{5} \]
\[ = \frac{a\alpha - v_1(\varepsilon_1)b + \sqrt{(a\alpha - v_1(\varepsilon_1)b)^2 + 4(\alpha v_1(\varepsilon_1) - \delta v_1(\varepsilon_1))ba}}{2ba} + \frac{\varepsilon_1}{5} \]
(14)
The expression of $\overline{u}_2(\varepsilon_1)$ and that of $u_n(\varepsilon_1)$ and $\overline{w}(\varepsilon_1)$ are valid by a simple computation using (5), (7), (8) and (11)-(14). The proof is complete. □
By repeating the above step, there exists a sufficiently large $T$ such that when $t \geq T$,
\[ u \geq \underline{u}_n(\varepsilon_1) \equiv \frac{a(\alpha + \overline{w}_n(\varepsilon_1)) - \delta \overline{w}_n(\varepsilon_1) + \sqrt{[a(\alpha + \overline{w}_n(\varepsilon_1)) - \delta \overline{w}_n(\varepsilon_1)]^2 - 4\overline{w}_n(\varepsilon_1)}}{2b\alpha} - \frac{\varepsilon_1}{5} \]
uniformly in $\Omega$ for any positive integer $n$, where
\[ \underline{u}_n(\varepsilon_1) = \frac{ru_n(\varepsilon_1)}{\gamma} - \frac{\varepsilon_1}{5} \]
\[ \overline{w}_n = \frac{(r + \alpha)\overline{w}_n + b\overline{w}_n^2 - (a + \alpha\gamma)\overline{w}_n}{2\gamma\overline{w}_n} \]
\[ + \sqrt{[(r + \alpha\gamma)\overline{w}_n - (r + \delta)\overline{w}_n - b\overline{w}_n^2]^2 + 4\gamma\overline{w}_n [(r\alpha - aa + ba\overline{w}_n)\overline{w}_n]} \]
When setting $\varepsilon_1 = 0$, we obtain
\[
\bar{u}_{n+1} = \frac{a\alpha - \frac{r}{\gamma} u_n b + \sqrt{(a\alpha - \frac{r}{\gamma} u_n b)^2 + 4 \left( a \frac{r}{\gamma} u_n - \delta \frac{r}{\gamma} u_n \right) b\alpha}}{2b\alpha},
\]
\[
\bar{u}_n = \frac{a(\alpha + \bar{w}_n) - \delta \bar{w}_n + \sqrt{(a(\alpha + \bar{w}_n) - \delta \bar{w}_n)^2}}{2b(\alpha + \bar{w}_n)},
\]
and $\bar{w}_1 = \frac{r}{\gamma}, \bar{u}_1 > u', u_1 < u'$. It is known by a direct calculation with the first equality proposed above that
\[
\left( a\alpha - \frac{r}{\gamma} u_1 b \right)^2 + 4 \left( a \frac{r}{\gamma} u_1 - \delta \frac{r}{\gamma} u_1 \right) b\alpha = (a\alpha)^2 + \frac{r^2 b^2 u_1^2}{\gamma^2} + 2(a\alpha) - \frac{r}{\gamma} u_1 - \frac{4br\gamma u_1}{\gamma} < \left( a + \frac{b}{\gamma} u_1 \right)^2.
\]
Therefore,
\[
\bar{u}_2 = \frac{a\alpha - \frac{r}{\gamma} u_1 b + \sqrt{(a\alpha - \frac{r}{\gamma} u_1 b)^2 + 4 \left( a \frac{r}{\gamma} u_1 - \delta \frac{r}{\gamma} u_1 \right) b\alpha}}{2b\alpha} < \frac{2a\alpha}{2b\alpha} = \bar{u}_1.
\]
Then, by induction, we can obtain that the sequence $\{\bar{u}_n\}$ is decreasing as $n \to \infty$. Similarly, since
\[
\bar{w}_n = \frac{r + \delta}{2\gamma} + \frac{b\bar{w}_n}{2\gamma} - \frac{a + \alpha\gamma}{2\gamma} + \frac{\sqrt{\left( \frac{r + \delta}{2\gamma} + \frac{b\bar{w}_n}{2\gamma} - \frac{a + \alpha\gamma}{2\gamma} \right)^2 + \frac{1}{\gamma} (r\alpha - a\alpha + b\alpha\bar{w}_n)}}{2\gamma},
\]
and
\[
\bar{u}_n = \frac{1}{2} \left[ \left( a - \frac{\delta\bar{w}_n}{b(\alpha + \bar{w}_n)} \right) + \sqrt{\left( a - \frac{\delta\bar{w}_n}{b(\alpha + \bar{w}_n)} \right)^2 + \frac{4\alpha\delta\bar{w}_n}{b^2(\alpha + \bar{w}_n)}} \right],
\]
where $r < a$, we obtain that the sequence $\{\bar{w}_n\}$ is decreasing and the sequence $\{u_n\}$ is increasing. Hence, we obtain
\[
\lim_{n \to \infty} \bar{u}_n = \lim_{n \to \infty} u_n = u'
\]
under the assumption of Theorem 1.2. Thus, we obtain
\[
\lim_{n \to \infty} \bar{w}_n = \lim_{n \to \infty} v_n = v'.
\]
Now, we prove $\lim_{l \to \infty} (u(x, t), v(x, t)) = (u', v')$, uniformly in $x \in \Omega$. 

Proof. [Proof of Theorem 1.2] For any $\varepsilon > 0$, there exists $N_1 \in \mathbb{Z}^+$ such that when $n > N_1$,
\[
|\bar{u}_n - u'| + |u_n - u'| < \frac{\varepsilon}{4}.
\] (15)
We can choose a sufficiently small positive number $\varepsilon_1 > 0$ such that
\[
|\bar{u}_{N_1}(\varepsilon_1) - \bar{u}_{N_1}| + |\bar{u}_{N_1}(\varepsilon_1) - u_{N_1}| < \frac{\varepsilon}{4}.
\] (16)
For any $\varepsilon > 0$, there exists $N_2 \in \mathbb{Z}^+$ such that when $n > N_2$,
\[
|\bar{v}_n - v'| + |v_n - v'| < \frac{\varepsilon}{4}.
\] (17)
We can choose a sufficiently small positive number $\varepsilon_2 > 0$ such that
\[
|\bar{v}_{N_2}(\varepsilon_2) - \bar{v}_{N_2}| + |\bar{v}_{N_2}(\varepsilon_2) - v_{N_2}| < \frac{\varepsilon}{4}.
\] (18)
Furthermore, there exists $t_{M_1}, t_{M_2} \gg a$ such that when $t \geq t_{M_1}$ and $t \geq t_{M_2}$, we have
\[
\bar{u}_{N_1}(\varepsilon_1) \leq u(x, t) \leq \bar{u}_{N_1}(\varepsilon_1) \quad \text{in } \Omega
\]
\[
\bar{v}_{N_2}(\varepsilon_2) \leq v(x, t) \leq \bar{v}_{N_2}(\varepsilon_2) \quad \text{in } \Omega
\]
respectively.
Let $N = \max\{N_1, N_2\}$, $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ and $t_M = \max\{t_{M_1}, t_{M_2}\}$. Hence, by (15)-(18), when $t \geq t_M$, we obtain
\[
|u(x, t) - u'| < \varepsilon \quad \text{in } \Omega
\]
and
\[
|v(x, t) - v'| < \varepsilon \quad \text{in } \Omega
\]
This proves $\lim_{t \to \infty} u(x, t) = u'$ and $\lim_{t \to \infty} v(x, t) = v'$ uniformly in $x \in \Omega$. This completes the proof of Theorem 1.2. □

3. Generalization and future works

It is effortless to verify that the proof of Theorem 1.4 follows exactly the same way of argument as in Theorem 1.2. Hence, it is omitted.

The approach we propose in this paper is novel and can be used to many interesting reaction-diffusion type models where the stability of a unique positive equilibrium solution is an essential problem to be considered. For instance, the famous Gierer-Meinhardt model [19],
\[
\begin{align*}
\frac{\partial u}{\partial t} &= \varepsilon^2 \Delta u - u + \frac{uv}{v^2}, & x \in \Omega, t \in (0, \infty), \\
\tau \frac{\partial v}{\partial t} &= \Delta v - v + \frac{um}{v}, & x \in \Omega, t \in (0, \infty), \\
\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, & x \in \partial \Omega, t \in (0, \infty), \\
u(x, 0) = u_0(x) > 0, & v(x, 0) = v_0(x) \geq 0, \quad x \in \overline{\Omega},
\end{align*}
\] (19)
is an interesting system worth of looking into.

It will be interesting to see how can we incorporate other interesting features such as time delay into our model.
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