A family of separability criteria and lower bounds of concurrence

Xian Shi · Yashuai Sun

Received: 9 November 2022 / Accepted: 6 February 2023 / Published online: 27 February 2023 © The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2023

Abstract
Background and Aim  The problem on detecting the entanglement of a bipartite state is significant in quantum information theory.
Methods  In this article, we apply the Ky Fan norm to the revised realignment matrix of a bipartite state.
Results  We consider a family of separable criteria for bipartite states and present when the density matrix corresponds to a state is real, the criterion is equivalent to the enhanced realignment criterion. Moreover, we present analytical lower bounds of concurrence and the convex-roof extended negativity for arbitrary dimensional systems.

Keywords  Entanglement · Realignment matrix · Concurrence

1 Introduction
Entanglement is one of the essential features in quantum mechanics when comparing with the classical physics [1, 2]. It plays key roles in quantum information processing, such as quantum cryptography [3], teleportation [4] and superdense coding [5].

One of the most important problems in quantum information theory is how to distinguish separable and entangled states. If a quantum state \( \rho_{AB} \) of a bipartite system can be written as a convex combination of product states,

\[
\rho_{AB} = \sum_i p_i \rho^i_A \otimes \rho^i_B,
\]

where \( \{p_i\} \) is a probability distribution, \( \rho^i_A \) and \( \rho^i_B \) are states of subsystems \( A \) and \( B \), respectively, then it is separable. Otherwise, it is entangled. The above problem is
completely solved for $2 \otimes 2$ and $2 \otimes 3$ systems by the Peres-Horodecki criterion: A bipartite state $\rho_{AB}$ is separable if and only if it is positive partial transpose (PPT), i.e., $(id \otimes T)(\rho_{AB}) \succeq 0$ [6]. However, the problem is NP-hard for arbitrary dimensional systems [7]. In the past twenty years, there are several other prominent criteria. The computable cross norm or realignment criterion (CCNR) criterion is proposed by Rudolph [8] and Chen and Wu [9]. In 2006, the authors proposed the local uncertainty relations (LURs) and showed that the LURs is stronger than the CCNR criterion [10]. In 2007, the author proposed a criterion which is based on Bloch representations [11]. Then, Zhang et al. presented the enhanced realignment criterion [12]. In 2015, the authors proposed an improved CCNR criterion where they showed it is stronger than the CCNR criterion [13]. In 2018, Shang et al. presented a sufficient condition for the separability of a bipartite state, which is called ESIC criterion [14]. Recently, Sarbicki et al. proposed a family of separability criteria which are based on the bloch representation of a state [15]. Subsequently, they showed that the detection power of the criteria is equivalent to the enhanced realignment criterion [16]. Jivulescu et al. proposed a class of entanglement criteria via projective tensor norms [17]. In 2022, Yan et al. proposed several entanglement detection criteria using quantum designs [18].

To quantify the entanglement is the other important problem in quantum entanglement theory, various entanglement measures are proposed in the past years [19–25]. Concurrence [19] and the convex-roof extended negativity (CREN) [26] are two of the most used among the measures under the convex-roof extended method. However, both of them are difficult to compute for higher dimensional systems [27]. Lots of results have been made on the lower bounds of concurrence and CREN [28–32]. Moreover, better lower bounds of concurrence and CREN can be used to distinguish entangled states and separable states. In [30], de Vicente presented analytical lower bounds of concurrence in terms of LUR and correlation matrix separability criteria [30]. Recently, Li et al. improved the lower bounds of concurrence and CREN based on Bloch representations [32].

In this work, we propose a class of separability criteria based on the Ky Fan norm of the realignment matrix of a state. Moreover, we show when a bipartite state corresponds to a real matrix, the criterion is as strong as the enhanced realignment criterion. At last, we present lower bounds for both concurrence and CREN under the methods here.

2 Separability criterion for bipartite states

In this section, we first introduce some knowledge needed. Assume $A = [a_{ij}] \in M_{m \times n}(\mathbb{C})$ is a matrix, $\text{vec}(A)$ is defined as

$$\text{vec}(A) = (a_{11}, a_{21}, \cdots, a_{m1}, a_{12}, \cdots, a_{m2}, \cdots, a_{mn})^T,$$

where $T$ means transposition. Let $Z$ be an $m \times m$ block matrix with block size $n \times n$. Then, the realignment operator $\mathcal{R}$ changes $Z$ into a new matrix with size $m^2 \times n^2$. 
\[ R(Z) \equiv \begin{pmatrix} \text{vec}(Z_{1,1})^T \\ \vdots \\ \text{vec}(Z_{m,1})^T \\ \vdots \\ \text{vec}(Z_{1,m})^T \\ \vdots \\ \text{vec}(Z_{m,m})^T \end{pmatrix}. \]

The CCNR criterion [8, 9] shows that any separable state \( \rho_{AB} \) satisfies
\[
||R(\rho)||_1 \leq 1. \tag{2}
\]

Then, based on the realignment of \( \rho_{AB} - \rho_A \otimes \rho_B \), Zhang et al. showed the enhanced realignment criterion: for any separable state \( \rho_{AB} \), the following inequality is valid,
\[
||R(\rho_{AB} - \rho_A \otimes \rho_B)||_1 \leq \sqrt{1 - tr \rho_A^2} \sqrt{1 - tr \rho_B^2}, \tag{3}
\]

it is stronger than the CCNR criterion (2) [12]. By using some parameters and the reduced density matrices of a bipartite state \( \rho_{AB} \), Shen et al. [13] constructed
\[
N_{\beta,l}(\rho) = \begin{pmatrix} G \beta \omega_l(\rho_A)^T \\ \beta \omega_l(\rho_A) \end{pmatrix}, \tag{4}
\]

where \( G - \beta^2 E_{l \times l} \) is positive semidefinite, \( \beta \in \mathbb{R} \), and \( \omega_l(X) \) means
\[
\omega_l(X) = (\text{vec}(X), \ldots, \text{vec}(X)). \tag{5}
\]

There they showed that when \( G - \beta^2 E_{l \times l} \geq 0 \), then a separable state \( \rho \) satisfies
\[
||N_{\beta,l}(\rho)||_1 \leq 1 + tr(G). \tag{6}
\]

In this manuscript, we denote \( \mathcal{M}_{\alpha,\beta}(\rho_{AB}) \) on a bipartite state \( \rho_{AB} \) as
\[
\mathcal{M}_{\alpha,\beta}(\rho_{AB}) = \begin{pmatrix} \alpha \beta \\alpha \text{vec}(\rho_B)^T \\ \beta \text{vec}(\rho_A) \\mathcal{R}(\rho_{AB}) \end{pmatrix}, \tag{7}
\]

where \( \alpha, \beta \in \mathbb{R} \), \( \rho_A \) and \( \rho_B \) are reduced density matrices of the A and B system, respectively. As Ky Fan norm is commonly used in matrix analysis [33], we use Ky Fan norm in this manuscript. The Ky Fan norm of a matrix \( A_{m \times n} \) is defined as the sum of all singular values \( d_i \), that is,
\[
\|A\|_{KF} = \sum_{i=1}^{\min(m,n)} d_i = \text{tr} \sqrt{A^\dagger A}. \]
Theorem 1 Assume $\rho_{AB}$ is a separable state, when $\alpha, \beta \in \mathbb{R}$,

$$\|M_{\alpha,\beta}(\rho_{AB})\|_{KF} \leq \sqrt{(\alpha^2 + 1)(\beta^2 + 1)}$$

Proof As $\rho_{AB}$ is separable, it can be written as

$$\rho_{AB} = \sum_i p_i \rho^i_A \otimes \rho^i_B,$$

(8)

where $p_i \in [0, 1]$, $\sum_i p_i = 1$, $\rho^i_A$ and $\rho^i_B$ are pure states of $A$ and $B$ systems, respectively. Next

$$\|M_{\alpha,\beta}(\rho_{AB})\|_{KF} = \|\sum_i p_i \left( \begin{array}{cc} \alpha \beta & \alpha \text{vec}(\rho^i_B)^T \\ \beta \text{vec}(\rho^i_A) & \text{vec}(\rho^i_A) \text{vec}(\rho^i_B)^T \end{array} \right) \|_{KF}$$

$$= \|\sum_i p_i \left( \begin{array}{c} \alpha \\ \text{vec}(\rho^i_A) \end{array} \right) \left( \begin{array}{c} \beta \\ \text{vec}(\rho^i_B)^T \end{array} \right) \|_{KF}$$

$$\leq \sum_i p_i \sqrt{(\alpha^2 + 1)(\beta^2 + 1)}$$

$$= \sqrt{(\alpha^2 + 1)(\beta^2 + 1)}.$$

Here, we use $\rho^i_A$ and $\rho^i_B$ are pure states. Hence, we finish the proof. \qed

Obviously, when $\alpha = \beta = 0$, the formula (8) is the CCNR criterion. Hence, the criterion is not weaker than the CCNR criterion.

Then, we present $\|M_{\alpha,\beta}(\rho_{AB})\|_{KF}$ is convex and invariant under local unitary operations.

Theorem 2 Assume $\rho_{AB}$ and $\sigma_{AB}$ are two bipartite states, then

1. When $p \in (0, 1)$,

$$\|M_{\alpha,\beta}(p\rho_{AB} + (1 - p)\sigma_{AB})\|_{KF} \leq p\|M_{\alpha,\beta}(\rho_{AB})\|_{KF} + (1 - p)\|M_{\alpha,\beta}(\sigma_{AB})\|_{KF}.$$

2. When $U$ and $V$ are unitary matrices on the $A$ and $B$ systems, respectively,

$$\|M_{\alpha,\beta}((U \otimes V)\rho_{AB}(U \otimes V)^\dagger)\|_{KF} = \|M_{\alpha,\beta}(\rho_{AB})\|_{KF}.$$

Proof (1) Due to (7), we have

$$\|M_{\alpha,\beta}(p\rho_{AB} + (1 - p)\sigma_{AB})\|_{KF}$$

$$= \|pM_{\alpha,\beta}(\rho_{AB}) + (1 - p)M_{\alpha,\beta}(\sigma_{AB})\|_{KF}$$

$$\leq p\|M_{\alpha,\beta}(\rho_{AB})\|_{KF} + (1 - p)\|M_{\alpha,\beta}(\sigma_{AB})\|_{KF}.$$

\qed Springer
that is, \( \|M_{\alpha,\beta}(\cdot)\| \) is convex.

(2) Due to [8, 9], \( \rho_{AB} \) can be written as \( \rho_{AB} = \sum_i \tau_i \otimes \delta_i \). Next, let \( \gamma_i = (U \otimes V)(\tau_i \otimes \delta_i)(U^\dagger \otimes V^\dagger) \), we have

\[
M_{\alpha,\beta}(\gamma_i) = \left( \begin{array}{cc} \alpha \beta & atr \tau_i (V \otimes \overline{V} vec(\delta_i))^T \\ \beta tr \delta_i (U \otimes \overline{U}) vec(\tau_i) & R(\gamma_i) \end{array} \right) = \left( \begin{array}{cc} 1 & 0 \\ 0 & (U \otimes \overline{U}) \end{array} \right) \left( \begin{array}{cc} \alpha \beta & atr \tau_i (V \otimes \overline{V} vec(\delta_i))^T \\ \beta tr \delta_i vec(\tau_i) & R(\gamma_i) \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & (U \otimes \overline{U}) \end{array} \right)^\dagger, \tag{9}
\]

where \( \overline{A} \) is an operator whose matrix representation has entries that is complex conjugate to the matrix representation of \( A \), i.e., \( \overline{A}_{a,b} = A^*_{a,b} \). As \( \tilde{U} = \left( \begin{array}{cc} 1 & 0 \\ 0 & (U \otimes \overline{U}) \end{array} \right) \) and \( \tilde{V} = \left( \begin{array}{cc} 1 & 0 \\ 0 & (U \otimes \overline{U}) \end{array} \right)^\dagger \) are both unitary, and \( M_{\alpha,\beta}(\sum_i \gamma_i) = \tilde{U} M_{\alpha,\beta}(\rho_{AB}) \tilde{V} \), then we finish the proof. \( \Box \)

Next, we present two examples on states in \( 3 \otimes 3 \) system, which shows the detection power of our theorem.

**Example 3** Here, we consider the separability on the mixture of the chessboard state [34] with white noise.

The chessboard states are defined as follows,

\[
\rho = \frac{1}{N} \sum_i |V_i\rangle\langle V_i|,
\]

\[
|V_1\rangle = |m, 0, s; 0, n, 0; 0, 0, 0\rangle,
\]

\[
|V_2\rangle = |0, a, 0; b, 0, c; 0, 0, 0\rangle
\]

\[
|V_3\rangle = |n^*, 0, 0; 0, -m^*, 0, t, 0, 0\rangle,
\]

\[
|V_4\rangle = |0, b^*, 0; -a^*, 0, 0; d, 0, 0\rangle.
\]

Here, \( N \) is a normalized factor. To make this class of states be PPT, here we assume \( m, s, n, a, b, c, t \) and \( d \) are real parameters, \( s = \frac{ac}{n} \), \( t = \frac{ad}{m} \). The mixture of the chessboard state with the white noise is

\[
\rho_p = p \rho + (1 - p) \frac{I_3 \otimes I_3}{9}.
\]

Next, we randomly choose

\[
\begin{align*}
a &= 0.3346; b = -0.109; c = -0.645, p = 0.9; \\
m &= 0.469; n = -0.3161; d = 0.8560.
\end{align*}
\]

Through computation, we have \( \rho_p \) is PPT, and this state cannot be detected by CCNR criterion[8, 9]. Next, when we take \( \alpha = 250 \) and \( \beta = 240 \), \( \|M_{\alpha,\beta}(\rho_{AB})\|_{KF} - \)
Fig. 1 Entanglement detection of $\sigma$. States above the blue line are detected by CCNR criterion, and states above the orange line and green line are detected by Theorem 1 when $\alpha = \beta = 2$ and $\alpha = \beta = 5$, respectively (Color figure online)

$$\sqrt{\left(\alpha^2 + 1\right)\left(\beta^2 + 1\right)} > 0,$$

due to the Theorem 1, $\rho_p$ is entangled. When $p = 0.8$, $\rho_p$ cannot be detected by ESIC. When we take $\alpha = \beta = 5.9$, $\|M_{\alpha,\beta}(\rho_{AB})\|_F - \sqrt{\left(\alpha^2 + 1\right)\left(\beta^2 + 1\right)} > 0$, $\rho_p$ is entangled.

**Example 4** Here, we consider the family of bound entangled states proposed by Horodecki [35],

$$\rho_x = \frac{1}{1 + 8x} \begin{pmatrix} x & 0 & 0 & 0 & x & 0 & 0 & 0 & x \\ 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & 0 & 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 & x & 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1+x}{2} & 0 & \sqrt{\frac{1-x^2}{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & \frac{1+x}{2} \\ x & 0 & 0 & 0 & x & \sqrt{\frac{1-x^2}{2}} & 0 & \frac{1+x}{2} \end{pmatrix},$$

let $\sigma(x, p) = p\rho_x + \frac{(1-p)I}{9}$. In Fig. 1, we plot the range when the states $\sigma(x, p)$ are detected by the CCNR criterion and Theorem 1. It is clear from the figure that the power of Theorem 1 is stronger than the CCNR criterion, and the detected area by Theorem 1 when $\alpha = \beta = 5$ is bigger than that when $\alpha = \beta = 2$.

Recently, the authors showed that the criterion proposed in [15, 16] is as strong as the enhanced realignment criterion (3), here we show that the detection power of Theorem 1 is the same as the power of the enhanced realignment criterion when the bipartite state $\rho_{AB}$ corresponds to a real matrix.

**Theorem 5** Assume $\rho_{AB}$ corresponds to a real matrix, it satisfies the enhanced realignment criterion if and only if it satisfies Theorem 1 for all $\alpha, \beta \in \mathbb{R}^+$. 
**Proof** \(\implies\): As

\[
\mathcal{M}_{\alpha,\beta}(\rho_{AB}) = \mathcal{M}_{\alpha,\beta}(\rho_A \otimes \rho_B) + \mathcal{L}_{\alpha,\beta}(\rho_{AB}),
\]

(10)

then if \(\rho_{AB}\) satisfies the enhanced realignment criterion, then

\[
\|\mathcal{M}_{\alpha,\beta}(\rho_{AB})\|_{K_F} \leq \|\mathcal{M}_{\alpha,\beta}(\rho_A \otimes \rho_B)\|_{K_F} + \|\mathcal{R}(\rho_{AB} - \rho_A \otimes \rho_B)\|_{K_F}
\]

\[
\leq \sqrt{(\alpha^2 + tr\rho_A^2)(\beta^2 + tr\rho_B^2) + (1 - tr\rho_A^2)(1 - tr\rho_B^2)}
\]

\[
\leq \sqrt{(1 + \alpha^2)(1 + \beta^2)}.
\]

(11)

Here, the last inequality is due to the Cauchy-Schwarz inequality.

Next, we prove the converse \(\iff\): First, let us recall that when \(A \in \mathcal{M}_{m \times n}(\mathbb{R})\),

\[
\|A_{m \times n}\|_{K_F} = \max_{P \in \mathcal{U}_{m \times n}(\mathbb{R})} |Tr A^\dagger P|,
\]

(12)

where the maximum takes over all the unitary matrices \(P \in \mathcal{U}_{m \times n}(\mathbb{R})\). Assume \(\rho_{AB}\) is a separable state, according to the Theorem 1, we have for any given \(\alpha \geq 0, \beta \geq 0\),

\[
\|\mathcal{M}_{\alpha,\beta}(\rho_{AB})\|_{K_F} \leq \sqrt{(\alpha^2 + 1)(\beta^2 + 1)},
\]

then let

\[
f(\alpha, \beta, P) = \sqrt{(\alpha^2 + 1)(\beta^2 + 1)} + \min_{P \in \mathcal{U}_{d_A^2 \times d_B^2}} tr P^\dagger \mathcal{M}_{\alpha,\beta}(\rho_{AB}),
\]

we have

\[
f(\alpha, \beta, P) = \sqrt{(\alpha^2 + 1)(\beta^2 + 1)} + \min_{P \in \mathcal{U}_{d_A^2 \times d_B^2}} tr P^\dagger \mathcal{M}_{\alpha,\beta}(\rho_{AB})
\]

\[
= \sqrt{(\alpha^2 + 1)(\beta^2 + 1)} - \max_{P \in \mathcal{U}_{d_A^2 \times d_B^2}} tr P^\dagger \mathcal{M}_{\alpha,\beta}(\rho_{AB})
\]

\[
= \sqrt{(\alpha^2 + 1)(\beta^2 + 1)} - \|\mathcal{M}_{\alpha,\beta}(\rho_{AB})\|_{K_F} \geq 0,
\]

(13)
where $P$ takes over all the unitary matrices. As $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta$ can be written as

$$\alpha = r \cos \theta, \quad \beta = r \sin \theta,$$

(14)

where $r \in \mathbb{R}^+$ and $\theta \in (0, \pi/2)$. Next, assume $P$ takes the following unitary matrix,

$$P = \begin{pmatrix} -\sqrt{1 - \eta^2} & \frac{\eta}{r} u^T \\ \frac{\eta}{r} v & \sqrt{1 - \eta^2} O \end{pmatrix},$$

where $r$ tends to the infinity, $\eta$ is positive and infinitesimal with respect to $r$, $O$ is a $d_A^2 \times d_B^2$ matrix. As $P$ is unitary, $\|u\| = \|v\| = 1$, $v = Ou$, and $OO^T$ and $O^T O$ are projectors. Then, the left hand side of (13) can be written as

$$f(\alpha, \beta, P) = \sqrt{(1 + r^2 \cos^2 \theta)(1 + r^2 \sin^2 \theta) + \eta(\cos \theta \operatorname{vec}(\rho_B)^T u + \sin \theta v^T \operatorname{vec}(\rho_A))}$$

$$+ \sqrt{1 - \eta^2 (tr R(\rho_{AB}) O^T - r^2 \cos \theta \sin \theta)}$$

$$= \sqrt{(1 + r^2 \cos^2 \theta)(1 + r^2 \sin^2 \theta) + tr R(\rho_{AB}) O^T - r^2 \cos \theta \sin \theta}$$

$$+ \frac{\eta^2}{2} \cos \theta \sin \theta + \eta(\sin \theta \operatorname{vec}(\rho_A)^T O + \cos \theta \operatorname{vec}(\rho_B)^T) u + o(1),$$

$$\geq \sqrt{(1 + r^2 \cos^2 \theta)(1 + r^2 \sin^2 \theta) - r^2 \cos \theta \sin \theta}$$

$$+ \frac{\eta^2}{2} \cos \theta \sin \theta - \eta(\sin \theta \operatorname{vec}(\rho_A)^T O + \cos \theta \operatorname{vec}(\rho_B)^T) u + tr R(\rho_{AB}) O^T$$

$$= \sqrt{(1 + r^2 \cos^2 \theta)(1 + r^2 \sin^2 \theta) - r^2 \cos \theta \sin \theta} + tr R(\rho_{AB}) O^T$$

$$+ \frac{\cos \theta \sin \theta}{2} [(\eta - m)^2 - m^2],$$

(15)

the second equality is due to that $\eta$ is infinitesimal relative to $r$. In the first inequality, when $u$ is antiparallel to $-(\sin \theta \operatorname{vec}(\rho_A)^T O + \cos \theta \operatorname{vec}(\rho_B)^T) u$, the equality is valid.

In the third equality, $m = \frac{|\sin \theta \operatorname{vec}(\rho_A)^T O + \cos \theta \operatorname{vec}(\rho_B)^T|}{\cos \theta \sin \theta}$. When $\eta = m$, (15) gets the minimum. Then,

$$f(\alpha, \beta, P) \geq \sqrt{(1 + r^2 \cos^2 \theta)(1 + r^2 \sin^2 \theta) - r^2 \cos \theta \sin \theta + tr R(\rho_{AB}) O^T}$$

$$- \frac{|\sin \theta \operatorname{vec}(\rho_A)^T O + \cos \theta \operatorname{vec}(\rho_B)^T|^2}{2 \cos \theta \sin \theta}$$

$$= \frac{1}{2 \sin \theta \cos \theta}$$

$$- \frac{\sin^2 \theta r_A^2 + \cos^2 \theta r_B^2 + \cos \theta \sin \theta (\operatorname{vec}(\rho_A)^T O \operatorname{vec}(\rho_B) + \operatorname{vec}(\rho_B)^T O^T \operatorname{vec}(\rho_A))}{2 \cos \theta \sin \theta}.$$
\[ + \text{tr} R(\rho_{AB})O^T \]
\[ = \frac{1}{2 \sin \theta \cos \theta} - \frac{\tan \theta \text{tr} \rho_A^2}{2} + \frac{\cot \theta \text{tr} \rho_B^2}{2} + \text{tr} R(\rho_{AB})O^T \]
\[ - \frac{\text{vec}(\rho_A)^T \text{vec}(\rho_B)}{2} + \text{vec}(\rho_B)^T O \text{vec}(\rho_A) \]
\[ \geq \sqrt{(1 - \text{tr} \rho_A^2)(1 - \text{tr} \rho_B^2) + \text{tr} R(\rho_{AB})O^T - \text{vec}(\rho_B)^T O \text{vec}(\rho_A)}, \]
\[ = \sqrt{(1 - \text{tr} \rho_A^2)(1 - \text{tr} \rho_B^2) + \text{tr}(R(\rho_{AB} - \rho_A \otimes \rho_B))O^T}. \]

Next as
\[ \text{min}_O \text{tr}(R(\rho_{AB} - \rho_A \otimes \rho_B))O^T = - \text{max}_O \text{tr}(R(\rho_{AB} - \rho_A \otimes \rho_B))O^T = - \|R(\rho_{AB} - \rho_A \otimes \rho_B)\|_{KF}. \]

Hence, there exists a \( P \in M(\mathbb{R}) \) such that \( f(\alpha, \beta, P) = \sqrt{(1 - \text{tr} \rho_A^2)(1 - \text{tr} \rho_B^2) - \|R(\rho_{AB} - \rho_A \otimes \rho_B)\|_{KF}} \), that is, the detection power of the enhanced realignment criterion and Theorem 1 is equivalent for the bipartite states that correspond to real matrices.

3 Lower bounds of concurrence and CREN

In the bipartite entanglement theory, entanglement monotones are useful to distinguish the entanglement states and separable states. However, it is very hard to compute almost all entanglement monotones for arbitrary dimensional systems. In the last section, we first recall two popular entanglement monotones, concurrence and CREN, for mixed states of arbitrary bipartite systems, then we present a family of lower bounds of the two entanglement monotones. At last, by means of an example, our results can be used as entanglement criteria and can present better bounds than the results in [30, 32].

Assume \( |\psi\rangle_{AB} \) is a pure state in \( \mathcal{H}_{AB} \), its concurrence is defined as

\[ C(|\psi\rangle_{AB}) = \sqrt{2(1 - \text{tr} \rho_A^2)}, \]

where \( \rho_A = \text{tr}_B |\psi\rangle_{AB} \langle \psi| \). The concurrence for a mixed state \( \rho_{AB} \) is defined as

\[ C(\rho_{AB}) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle_{AB}), \]

where the minimum takes over all the decompositions of \( \rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB} \langle \psi_i| \), \( p_i \geq 0 \) and \( \sum_i p_i = 1 \). Next, for a pure state \( |\psi\rangle_{AB} \), its CREN [26] is defined as

\[ \mathcal{N}(|\psi\rangle) = \frac{\|(|\psi\rangle \langle \psi|)^{Tr} \| - 1}{k - 1}, \]
where $k = \min(dim(H_A), dim(H_B))$, $(|\psi\rangle\langle\psi|)^T_B$ denotes the partial transpose of $|\psi\rangle\langle\psi|$. For a mixed state $\rho_{AB}$, its CREN is defined as

$$\mathcal{N}(\rho_{AB}) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i \mathcal{N}(|\psi_i\rangle_{AB}),$$

where the minimum takes over all the decompositions of $\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB}\langle\psi_i|$, $p_i \geq 0$ and $\sum_i p_i = 1$.

Before presenting our results, we consider $M_{\alpha,\beta}(|\psi\rangle)$ when $|\psi\rangle = \sum_{i=0}^{k-1} \sqrt{\lambda_i} |ii\rangle$ is a pure state,

$$M_{\alpha,\beta}(|\psi\rangle_{AB}\langle\psi|) = M_1 + M_2,$$

$$M_1 = \begin{pmatrix} \alpha \beta & \alpha \text{vec}(\rho_B)^T \\ \beta \text{vec}(\rho_A) & \Lambda_1 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 0 & 0 \\ 0 & \Lambda_2 \end{pmatrix},$$

$$\text{vec}(\rho_A) = (\lambda_0, 0, \ldots, 0, \lambda_1, 0, \ldots, 0, \ldots, \lambda_{k-1})^T,$$

$$\text{vec}(\rho_B) = (\lambda_0, 0, \ldots, 0, \lambda_1, 0, \ldots, 0, \ldots, 0, \ldots, \lambda_{k-1})^T,$$

$$\Lambda_1 = \text{diag}(\lambda_0, 0, \ldots, 0, \lambda_1, \ldots, 0, \ldots, 0, \ldots, \lambda_{k-1}),$$

$$\Lambda_2 = \text{diag}(0, \sqrt{\lambda_0 \lambda_1}, \ldots, \sqrt{\lambda_0 \lambda_{k-1}}, \sqrt{\lambda_1 \lambda_0}, \ldots, 0).$$

Then, we have

$$\|M_{\alpha,\beta}(|\psi\rangle_{AB}\langle\psi|)\|_{KF} = \|M_1\|_{KF} + \|M_2\|_{KF},$$

$$= \|M_1\|_{KF} + 2 \sum_{i<j} \sqrt{\lambda_i \lambda_j}, \quad (16)$$

In the following, we denote $\|\cdot\|$ as $\|\cdot\|_{KF}$. Next due to the definition of $M_{\alpha,\beta}(\cdot)$, $M_1 = M_{\alpha,\beta}(\sigma_{AB})$, $\sigma = \sum_i \lambda_i |ii\rangle\langle ii|$ is a separable state. According to Theorem 1,

$$\|M_{\alpha,\beta}(|\psi\rangle_{AB}\langle\psi|)\| \leq \sqrt{(1 + \alpha^2)(1 + \beta^2) + 2 \sum_{i<j} \sqrt{\lambda_i \lambda_j}}, \quad (17)$$

Next, in [28], the authors showed that

$$C^2(|\psi\rangle) \geq \frac{8}{k(k-1)} \left(\sum_{i<j} \sqrt{\lambda_i \lambda_j}\right)^2,$$
then combining (17), we have
\[
C(|\psi\rangle) \geq \frac{\sqrt{2}}{\sqrt{k(k-1)}} (\|M_{\alpha,\beta}(|\psi\rangle_{AB}\langle\psi|)\| - \sqrt{(1+\alpha^2)(1+\beta^2)}).
\]

For a mixed state $\rho_{AB}$, assume $\{p_i, |\psi_i\rangle\}$ is the optimal decomposition for $\rho_{AB}$ such that $C(\rho_{AB}) = \sum_i p_i C(|\psi_i\rangle_{AB})$, then
\[
C(\rho_{AB}) = \sum_i p_i C(|\psi_i\rangle_{AB}) \geq \frac{\sqrt{2}}{\sqrt{k(k-1)}} \sum_i p_i (\|M_{\alpha,\beta}(|\psi_i\rangle_{AB}\langle\psi|)\| - \sqrt{(1+\alpha^2)(1+\beta^2)})
\]
\[
\geq \frac{\sqrt{2}}{\sqrt{k(k-1)}} (\|M_{\alpha,\beta}(\rho_{AB})\| - \sqrt{(1+\alpha^2)(1+\beta^2)}).
\]

Then, we have the following theorem,

**Theorem 6** Assume $\rho_{AB}$ is a mixed state, for any $\alpha, \beta \in \mathbb{R}^+$, we have
\[
C(\rho_{AB}) \geq \frac{\sqrt{2}}{\sqrt{k(k-1)}} (\|M_{\alpha,\beta}(\rho_{AB})\| - \sqrt{(1+\alpha^2)(1+\beta^2)}).
\]

Next we consider the lower bound for CREN of a mixed state $\rho_{AB}$. Assume $|\psi\rangle_{AB} = \sum_{i=0}^{k-1} \sqrt{\lambda_i} |ii\rangle$ is a pure state, then
\[
N(|\psi\rangle_{AB}) = \frac{2(\sum_{j<i} \sqrt{\lambda_j \lambda_i})}{k-1},
\]
(19)
based on (17),
\[
2 \sum_{i<j} \sqrt{\lambda_i \lambda_j} \geq \|M_{\alpha,\beta}(|\psi\rangle_{AB}\langle\psi|)\| - \sqrt{(1+\alpha^2)(1+\beta^2)},
\]
let $\sum_i p_i |\psi_i\rangle\langle\psi_i|$ be the optimal decomposition for $\rho$ such that $N(\rho) = \sum_i p_i N(|\psi_i\rangle\langle\psi_i|)$ be the optimal for $\rho$ such that $N(\rho) = \sum_i p_i N(|\psi_i\rangle\langle\psi_i|)$, we have
\[
N(\rho_{AB}) = \sum_i p_i N(|\psi_i\rangle\langle\psi_i|) \geq \sum_i p_i \frac{\|M_{\alpha,\beta}(|\psi_i\rangle\langle\psi_i|)\| - \sqrt{(1+\alpha^2)(1+\beta^2)}}{k-1}
\]
\[
\geq \|M_{\alpha,\beta}(\rho_{AB})\| - \sqrt{(1 + \alpha^2)(1 + \beta^2)} \frac{k}{k-1}.
\] (20)

Based on the above analytics, we have

**Theorem 7** Assume \(\rho_{AB}\) is a mixed state, for any \(\alpha, \beta \in \mathbb{R}^+\), we have

\[
\mathcal{N}(\rho_{AB}) \geq \|M_{\alpha,\beta}(\rho_{AB})\| - \sqrt{(1 + \alpha^2)(1 + \beta^2)} \frac{k}{k-1}.
\] (21)

Then, we will present an example which shows our results are better than [30].

**Example 8** In this example, we consider the \(3 \otimes 3\) PPT entangled state

\[
\rho = \frac{1}{4} \left( I - \sum_i |\psi_i\rangle\langle \psi_i| \right),
\]

\[
|\psi_0\rangle = \frac{|0\rangle(|0\rangle - |1\rangle)}{\sqrt{2}},
\]

\[
|\psi_1\rangle = \frac{|0\rangle - |1\rangle)|2\rangle}{\sqrt{2}},
\]

\[
|\psi_2\rangle = \frac{|2\rangle(|1\rangle - |2\rangle)}{\sqrt{2}},
\]

\[
|\psi_3\rangle = \frac{|1\rangle - |2\rangle)|0\rangle}{\sqrt{2}},
\]

\[
|\psi_4\rangle = \frac{(|0\rangle + |1\rangle + |2\rangle)(|0\rangle + |1\rangle + |2\rangle)}{3}.
\]

When choosing \(\alpha = \beta = 1\), according to Theorem 6, \(C(\rho_{AB}) \geq 0.053999\). By using the theorem 1 in [30], we have \(C(\rho) \geq 0.052\), hence our bound is better than it. In [32], the authors showed that \(C(\rho) \geq 0.05554\), here when we take \(\alpha = \beta = 100\), the lower bound of \(C(\rho)\) is 0.055549. Next we consider a state by mixing \(\rho\) with the white noise,

\[
\rho_p = p I/9 + (1 - p)\rho,
\]

where \(I\) is the identity, \(p \in [0, 1]\). In Fig. 2, the orange line is the lower bound of \(C(\rho_p)\) obtained by Theorem 2 in [30]. From this bound, \(\rho_p\) is entangled when \(p \in [0, 0.0507]\). By Theorem 6, when taking \(\alpha = \beta = 5\), we obtain a lower bound of \(C(\rho_p)\) and plot it in Fig. 2 as a blue line. There we have \(\rho_p\) is entangled when \(p \in [0, 0.1177]\).

In Fig. 3, we plot the bound of \(\mathcal{N}(\rho_p)\) obtained by Theorem 7, there the orange line is on the bound of \(\mathcal{N}(\rho_p)\) with \(\alpha = \beta = 7\), the blue line is on the bound of \(\mathcal{N}(\rho_p)\) with \(\alpha = \beta = 1\). From the figure, we can see when taking \(\alpha = \beta = 7\), the lower bound is better.
Fig. 2 Lower bound of $C(\rho_p)$. The blue line is the bound given by Theorem 6, while the orange line is the bound obtained by Theorem 2 in [30] (Color figure online)

Fig. 3 Lower bound of CREN for the state $\rho_p$. The orange line is the bound of $N(\rho_p)$ with $\alpha = \beta = 7$, and the blue line is the bound with $\alpha = \beta = 1$. (Color figure online)

4 Conclusion

To detect the entanglement of a bipartite state is essential in quantum entanglement theory, in this paper, we presented a class of separability criteria which are better than CCNR criterion. Moreover, we proved that the detection power of our criteria is as strong as the enhanced realignment criterion for bipartite states corresponding to real density matrices. At last, we derived analytical lower bounds of the concurrence and CREN. Moreover, our methods here can also be used to obtain the lower bounds of
some multipartite entanglement measures. We hope our work could shed some light on related studies.

**Data availability** Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

**Declarations**

**Conflict of interest** We declare that we have no financial and personal relationships with other people or organizations that can inappropriately influence our work, there is no professional or other personal interest of any nature or kind in any product, service and/or company that could be construed as influencing the position presented in, or the review of, the manuscript entitled.

**References**

1. Horodecki, R., Horodecki, P., Horodecki, M., Horodecki, K.: Quantum entanglement. Rev. Mod. Phys. 81(2), 865 (2009)
2. Plenio, M.B., Virmani, S.S.: An introduction to entanglement theory. In: Quantum Information and Coherence, pp. 173–209. Springer (2014)
3. Ekert, A.K.: Quantum cryptography based on Bell’s theorem. Phys. Rev. Lett. 67(6), 661 (1991)
4. Bennett, C.H., Brassard, G., Crépeau, C., Jozsa, R., Peres, A., Wootters, W.K.: Teleporting an unknown quantum state via dual classical and Einstein–Podolsky–Rosen channels. Phys. Rev. Lett. 70(13), 1895 (1993)
5. Bennett, C.H., Wiesner, S.J.: Communication via one-and two-particle operators on Einstein–Podolsky–Rosen states. Phys. Rev. Lett. 69(20), 2881 (1992)
6. Peres, A.: Separability criterion for density matrices. Phys. Rev. Lett. 77(8), 1413 (1996)
7. Gurvits, L.: Classical deterministic complexity of Edmonds’ problem and quantum entanglement. In: Proceedings of the Thirty-Fifth Annual ACM Symposium on Theory of Computing, pp. 10–19 (2003)
8. Rudolph, O.: Further results on the cross norm criterion for separability. Quantum Inf. Process. 4(3), 219–239 (2005)
9. Chen, K., Wu, L.-A.: A matrix realignment method for recognizing entanglement. Quantum Inf. Comput. 6, 66 (2022)
10. Gühne, O., Mechler, M., Tóth, G., Adam, P.: Entanglement criteria based on local uncertainty relations are strictly stronger than the computable cross norm criterion. Phys. Rev. A 74(1), 010301 (2006)
11. De Vicente, J.I.: Separability criteria based on the Bloch representation of density matrices. Quantum Inf. Comput. 6, 66 (2022)
12. Zhang, C.-J., Zhang, Y.-S., Zhang, S., Guo, G.-C.: Entanglement detection beyond the computable cross-norm or realignment criterion. Phys. Rev. A 77(6), 060301 (2008)
13. Shen, S.-Q., Wang, M.-Y., Li, M., Fei, S.-M.: Separability criteria based on the realignment of density matrices and reduced density matrices. Phys. Rev. A 92(4), 042332 (2015)
14. Shang, J., Asadian, A., Zhu, H., Gühne, O.: Enhanced entanglement criterion via symmetric informationally complete measurements. Phys. Rev. A 98(2), 022309 (2018)
15. Sarbicki, G., Scala, G., Chruściński, D.: Family of multipartite separability criteria based on a correlation tensor. Phys. Rev. A 101(1), 012341 (2020)
16. Sarbicki, G., Scala, G., Chruściński, D.: Enhanced realignment criterion vs linear entanglement witnesses. J. Phys. A Math. Theor. 53(45), 455302 (2020)
17. Jivulescu, M.A., Lancien, C., Nechita, I.: Multipartite entanglement detection via projective tensor norms. Annales Henri Poincaré 23(11), 3791–3838 (2022)
18. Yan, X., Liu, Y.-C., Shang, J.: Operational detection of entanglement via quantum designs. Annalen der Physik 534(5), 2100594 (2022)
19. Wootters, W.K.: Entanglement of formation of an arbitrary state of two qubits. Phys. Rev. Lett. 80(10), 2245 (1998)
20. Vedral, V., Plenio, M.B.: Entanglement measures and purification procedures. Phys. Rev. A 57(3), 1619 (1998)
21. Vidal, G., Tarrach, R.: Robustness of entanglement. Phys. Rev. A 59(1), 141 (1999)
22. Terhal, B.M., Horodecki, P.: Schmidt number for density matrices. Phys. Rev. A 61(4), 040301 (2000)
23. Vidal, G.: Entanglement monotones. J. Mod. Opt. 47(2–3), 355–376 (2000)
24. Wei, T.-C., Goldbart, P.M.: Geometric measure of entanglement and applications to bipartite and multipartite quantum states. Phys. Rev. A 68(4), 042307 (2003)
25. Christandl, M., Winter, A.: “Squashed entanglement”: an additive entanglement measure. J. Math. Phys. 45(3), 829–840 (2004)
26. Lee, S., Chi, D.P., Oh, S.D., Kim, J.: Convex-roof extended negativity as an entanglement measure for bipartite quantum systems. Phys. Rev. A 68(6), 062304 (2003)
27. Huang, Y.: Computing quantum discord is np-complete. New J. Phys. 16(3), 033027 (2014)
28. Chen, K., Albeverio, S., Fei, S.-M.: Concurrence of arbitrary dimensional bipartite quantum states. Phys. Rev. Lett. 95(4), 040504 (2005)
29. Brandao, F.G.: Quantifying entanglement with witness operators. Phys. Rev. A 72(2), 022310 (2005)
30. de Vicente, J.I.: Lower bounds on concurrence and separability conditions. Phys. Rev. A 75(5), 052320 (2007)
31. Chen, Z.-H., Ma, Z.-H., Gühne, O., Severini, S.: Estimating entanglement monotones with a generalization of the Wooters formula. Phys. Rev. Lett. 109(20), 200503 (2012)
32. Li, M., Wang, Z., Wang, J., Shen, S., Fei, S.-M.: Improved lower bounds of concurrence and convex-roof extended negativity based on Bloch representations. Quantum Inf. Process. 19(4), 1–11 (2020)
33. Bhatia, R.: Matrix Analysis, vol. 169. Springer (2013)
34. Bruß, D., Peres, A.: Construction of quantum states with bound entanglement. Phys. Rev. A 61(3), 030301 (2000)
35. Horodecki, P.: Separability criterion and inseparable mixed states with positive partial transposition. Phys. Lett. A 232(5), 333–339 (1997)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.