On the equivalence between the Boltzmann equation
and classical field theory at large occupation numbers

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Abstract

We consider a system made up of excitations of a neutral scalar field, $\phi$, having a $\lambda \phi^4$ interaction term. Starting from an ensemble where the occupation number $f$ is large, but $\lambda f$ is small, we develop a classical field theory description of the evolution of the system toward equilibrium. A Boltzmann equation naturally emerges in this description and we show by explicit calculation that the collision term is the same as that coming from elastic scattering. This shows the equivalence of a Boltzmann equation description and a classical field theory description of the same system.
I. INTRODUCTION

It is believed that in the very early stages after a heavy ion collision, gluon occupation numbers are as large as $1/\alpha$, $f_g \sim 1/\alpha$. As the QCD quark-gluon plasma evolves toward equilibrium, occupation numbers decrease until $f_g \sim 1$ at equilibrium. Understanding the evolution of dense quark-gluon matter toward equilibrium is both practically important and theoretically challenging. So long as $f_g \ll 1/\alpha$ the Boltzmann equation, with elastic and inelastic collision terms and quantum-statistical factors, should furnish a systematic theoretical framework for this problem, although this equation cannot be expected to be useful at the earliest times after the collision when $f_g \sim 1/\alpha$. On the other hand, when occupation numbers are large one can expect classical field theory to apply, and there is currently an interesting program studying the early stages after a heavy ion collision using classical Yang-Mills equations with an initial condition fixed by the McLerran-Venugopalan saturation model. Thus one might expect that classical field theory should be a good theoretical framework for studying a dense gluon system up to times just before equilibration occurs. It would seem then that classical field theory and the Boltzmann equation are equivalent frameworks for studying dense non-equilibrium QCD in the period where $1 \ll f_g \ll 1/\alpha$ holds for the important phase space region of the system.

While this equivalence have been studied previously, most notably in connection with wave turbulence, it is less familiar in the context of the early stages of heavy ion collision. We thus explore this equivalence in detail in this paper. To simplify the discussion we deal with a $\phi^4$ field theory rather than with QCD, but we believe our argumentation should work equally well for Yang-Mills theories. Our goal is to derive the collision term in the Boltzmann equation using only classical field theory, and we do this explicitly at lowest order in the coupling for the elastic-scattering part of the collision term. This part of the collision term, illustrated in Fig. (5) below, has a contribution cubic in the occupation number and a contribution which is quadratic. Perhaps surprisingly we are able to reproduce both of these parts from the classical theory, and the reasons for this are discussed briefly at the end of Sec. IV. For inelastic parts of the collision term the classical field description cannot be expected to give the complete answer, even at the lowest nontrivial term in the coupling, but it should give these portions having the maximum and maximum minus one powers of the occupation number.
In Sec. II we review the description of a non-equilibrium system, described by a \( \phi^4 \) field theory including the usual doubling of the number of fields. When the fields are large we show how a classical field theory naturally replaces the quantum field theory description of the system, and we introduce combinations of the field variables to make the classical description transparent.

In Sec. III we introduce Green’s functions in the classical field theory and write equations of motions for the Green’s functions. If the medium does not vary too rapidly we show how a Boltzmann-like equation emerges with a “collision term” given by Green’s functions of the classical theory, depending of course on the initial ensemble defining the system.

In Sec. IV we identify the “collision” term derived in the classical theory with the usual collision term given by elastic scattering. The identification is done by explicit calculation at lowest nontrivial order in perturbation theory.

Our whole discussion is carried out under conditions where the occupation number \( f \) is large \( (f \gg 1) \) while \( \lambda f \) is small \( (\lambda f \ll 1) \) where \( \lambda \) is the usual \( \phi^4 \) coupling constant. We believe that higher order corrections in \( \lambda f \) can be done in the classical theory, but higher order in \( \lambda \) will, in general, be quantum.

II. DESCRIBING A DENSE NON-EQUILIBRIUM SYSTEM

The system which concerns us here is made up of excitations of a neutral scalar field of mass \( m \) and with interactions described by a \( -\frac{\lambda}{4!} \phi^4 \) interaction term in the Lagrangian density. Suppose at time \( t \) we wish to determine the expectation of some observable, \( O \), made of of field \( \phi \). For example, \( O \) might be \( \phi^2(\vec{x}, t), \phi(\vec{x}_1, t)\phi(\vec{x}_2, t), \phi^2(\vec{x}_1, t)\phi^2(\vec{x}_2, t) \) etc.

We may write

\[
\langle O \rangle = \int \mathcal{D}[\phi] O[\phi] \int \mathcal{D}[\phi_0] \rho[\phi_0] U^*[\phi, \phi_0, t - t_0] U[\phi, \phi_0, t - t_0] \tag{1}
\]

with

\[
U[\phi, \phi_0, t - t_0] = \int \mathcal{D}[\phi(\tau)] e^{i \int_{t_0}^{t} L[\phi(\tau)] d\tau} \tag{2}
\]

where the functional integral on the right hand side of Eq. (2) goes between fields \( \phi_0(\vec{x}) \) at \( t_0 \) and \( \phi(\vec{x}) \) at \( t \). The functional \( \rho[\phi_0] \) gives the initial ensemble defining the system. We make no assumption of an equilibrium or near-equilibrium ensemble. Later on we shall state some general assumptions on \( \rho \).
The separate functional integrals between \( t_0 \) and \( t \) in \( U \) and \( U^* \) is characteristic of any real-time formalism describing the time evolution of a statistical system \([10]\). Rather than viewing \( U^*U \) as determined by two separate functional integrals over the single field \( \phi \) one can introduce two fields, \( \phi_- \) and \( \phi_+ \), with Lagrangian

\[
\mathcal{L} = \left[ \frac{1}{2} (\partial_\mu \phi_-)^2 - \frac{1}{2} m^2 \phi_-^2 - \frac{\lambda}{4!} \phi_-^4 \right] - \left[ \frac{1}{2} (\partial_\mu \phi_+)^2 - \frac{1}{2} m^2 \phi_+^2 - \frac{\lambda}{4!} \phi_+^4 \right] \tag{3}
\]

Now

\[
U^*U = \int \mathcal{D}[\phi_-] \mathcal{D}[\phi_+] e^{i \int_{t_0}^t L[\phi_-, \phi_+] \, dt} \tag{4}
\]

with both \( \phi_- \) and \( \phi_+ \) taking on values \( \phi(x) \) at \( t \) and \( \phi_0(x) \) at \( t_0 \).

It is convenient to rewrite \( \mathcal{L} \) in Eq. (3) in terms of new variables \( \phi \) and \( \pi \) defined by \([11]\)

\[
\phi = \frac{1}{2} (\phi_- + \phi_+), \quad \pi = \phi_- - \phi_+, \tag{5}
\]

\[
\phi_- = \phi + \frac{\pi}{2}, \quad \phi_+ = \phi - \frac{\pi}{2}. \tag{6}
\]

One easily finds

\[
\mathcal{L} = \partial_\mu \phi \partial_\mu \pi - m^2 \phi \pi - \frac{\lambda}{3!} (\phi^3 \pi + \frac{1}{4} \pi^3 \phi) \tag{7}
\]

where one should be careful not to confuse the \( \phi \) in Eqs. (3)–(7) with the original field appearing in Eqs. (1)–(2). In what follows \( \phi \) will always denote the variable defined in Eq. (5). Noting that \( \pi = 0 \) at times \( t_0 \) and \( t \), one can rewrite Eq. (1) as

\[
\langle O \rangle = \int \mathcal{D}[\phi] O[\phi] \int \mathcal{D}[\phi_0] \rho[\phi_0] \int \mathcal{D}[\phi(\tau)] \mathcal{D}[\pi(\tau)] e^{i \int_{t_0}^t L[\phi, \pi] \, d\tau}, \tag{8}
\]

with

\[
L[\phi, \pi] = \int d^3x \mathcal{L}[\phi, \pi]. \tag{9}
\]

Our focus is on systems where \( \phi_- \) and \( \phi_+ \) are large. As we shall see, \( \phi \) is naturally large while \( \pi \) is small, thus one can neglect the \( \pi^3 \phi \) term in Eq. (7), which gives

\[
\mathcal{L}_c = \partial_\mu \phi \partial_\mu \pi - m^2 \phi \pi - \frac{\lambda}{3!} \pi^3 \phi \tag{10}
\]

But now \( \pi \) is simply a variable of constraint so that the functional integral over \( \pi \) in Eq. (8), with \( L \) replaced by \( \mathcal{L}_c \), gives

\[
\prod_{\vec{x}, \tau} 2\pi \delta \left[ (\Box + m^2) \phi(\vec{x}, \tau) + \frac{\lambda}{3!} \phi^3(\vec{x}, \tau) \right], \tag{11}
\]
which means that only fields satisfying the classical equations of motion
\[(\Box + m^2)\phi = -\frac{\lambda}{3!}\phi^3\]  \hspace{1cm} (12)
contribute to the functional integral in Eq. (8). In other words, if one defines the evolution kernel
\[K[\phi, \phi_0, t - t_0] = \int D[\phi(\tau)] D[\pi(\tau)] e^{i \int_{t_0}^{t} L_{e}[\phi, \pi] d\tau},\]  \hspace{1cm} (13)
where, as usual, \(\phi_0\) and \(\phi\) are the fields between which the functional integral is taken, then \(K\) is nonzero only for \(\phi\) satisfying the field equation with the initial condition \(\phi(t_0, \vec{x}) = \phi_0(\vec{x})\), i.e.,
\[\left[(\Box + m^2)\phi(\vec{x}, t) + \frac{\lambda}{3!}\phi^3(\vec{x}, t)\right] K[\phi, \phi_0, t - t_0] = 0,\]  \hspace{1cm} (14)
and
\[\lim_{t \to t_0} K[\phi, \phi_0, t - t_0] = \prod_{\vec{x}} \delta[\phi(\vec{x}) - \phi_0(\vec{x})],\]  \hspace{1cm} (15)
The classical nature of \(K\) is manifested in Eqs. (14) and (15). In terms of \(K\) one can write \(\langle O \rangle\) as
\[\langle O \rangle = \int D[\phi] D[\phi_0] O[\phi] K[\phi, \phi_0, t - t_0] \rho[\phi_0].\]  \hspace{1cm} (16)
In fact, the functional integral (13) with the Lagrangian of the type (10) provides a convenient starting point to a diagrammatic approach for classical statistical systems [12].

**III. EQUATIONS FOR THE GREEN’S FUNCTIONS**

In this section we derive equations for the Green’s functions of a system with Lagrangian given by Eq. (10). We begin our discussion with the Green’s functions (propagators) of the free theory corresponding to
\[L_0 = \partial_{\mu} \phi \partial^{\mu} \pi - m^2 \phi \pi,\]  \hspace{1cm} (17)
then we go on to the equations for the full (classical) theory governed by the Lagrangian (10). We limit ourselves in this section to vacuum propagators, leaving the generalization to the medium case to Sec. [IV].
A. The free propagators

Perhaps the easiest way to get the propagators for the Lagrangian (17) is to go back to the \( \phi^- \) and \( \phi^+ \) variables, in which case

\[
L_0 = \frac{1}{2} (\partial_\mu \phi^-)^2 - \frac{1}{2} m^2 \phi^-^2 - \frac{1}{2} (\partial_\mu \phi^+)^2 + \frac{1}{2} m^2 \phi^+^2
\]

where \( \phi^- \), \( \phi^+ \), \( \phi \), and \( \pi \) are related according to Eqs. (5) and (6). Then

\[
G^{(0)}_{-\cdots}(x) = \langle 0 | T \phi^-(x) \phi^- (0) | 0 \rangle = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik \cdot x}
\]

is the usual Feynman propagator. In momentum space

\[
G^{(0)}_{-\cdots}(p) = \frac{i}{p^2 - m^2 + i\epsilon}.
\]

The propagator for the \( \phi_+ \) field is given by

\[
G^{(0)}_{++}(p) = \frac{-i}{p^2 - m^2 - i\epsilon},
\]

with the \( i\epsilon \) choice reflecting the fact that the \( \phi_+ \) originally come from complex conjugate amplitudes. While the Lagrangian (18) does not by itself require a \( G^{(0)}_{-\cdots} \) or a \( G^{(0)}_{+\cdots} \) propagator, the functional integral does naturally give such propagators because of our identification \( \phi_+ = \phi^- \) at the largest time to which we evolve the system. Such propagators also naturally occur as Feynman lines which pass from an amplitude to a complex conjugate amplitude in the two-time formulation of a calculation of an expectation value or transition probability. Clearly,

\[
G^{(0)}_{++}(p) = 2\pi \theta(p_0) \delta(p^2 - m^2), \quad G^{(0)}_{-\cdots}(p) = 2\pi \theta(-p_0) \delta(p^2 - m^2).
\]

Now it is straightforward to go to propagators in terms of \( \phi \) and \( \pi \) fields. For example,

\[
G^{(0)}_{11} = G^{(0)}_{\phi\phi} = \frac{1}{4} \left[ G^{(0)}_{-\cdots} + G^{(0)}_{++} + G^{(0)}_{-\cdots} + G^{(0)}_{++} \right]
\]

and

\[
G^{(0)}_{12} = G^{(0)}_{\phi\pi} = \frac{1}{2} \left[ G^{(0)}_{-\cdots} - G^{(0)}_{++} + G^{(0)}_{-\cdots} + G^{(0)}_{++} \right]
\]

etc. One easily finds

\[
G^{(0)}_{11}(p) = \pi \delta(p^2 - m^2), \quad G^{(0)}_{22}(p) = 0,
\]

\[
G^{(0)}_{12}(p) = \frac{i}{p^2 - m^2 + i\epsilon p_0}, \quad G^{(0)}_{21}(p) = \frac{i}{p^2 - m^2 - i\epsilon p_0}.
\]
B. Equations for the full propagators

It will be useful to have a pictorial notation for the various propagators. For the free propagators this notation is given in Fig. 1. For the full propagators our pictorial notation is given in Fig. 2. $G_{22} = 0$ even in the presence of radiative corrections because of the fact that $G_{12}$ and $G_{21}$ are causal and anticausal, respectively. (More precisely, one can follow a $G_{21}^{(0)}$ propagator through a graph much as one can follow an electron through a graph in QED. In $G_{22}$ we can follow $G_{21}^{(0)}$ propagators through the graph and they must form at least one closed loop which is impossible because of the anticausal property.)

\[
\begin{align*}
G_{11}^{(0)} &= G \\
G_{12}^{(0)} &= G \rightarrow \Sigma \leftarrow G \\
G_{21}^{(0)} &= \Sigma \rightarrow G
\end{align*}
\]

FIG. 1: The free propagators

\[
\begin{align*}
G_{11} &= G \\
G_{12} &= G \rightarrow \Sigma \leftarrow G \\
G_{21} &= \Sigma \rightarrow G
\end{align*}
\]

FIG. 2: The full propagators

One can write equations for $G$ in terms of one-particle-irreducible parts, $\Sigma$’s, which are useful in applying perturbation theory and in deriving a Boltzmann equation. For $G_{11}$ this equation takes either the form

\[
G_{11}(x, y) = -i \int dw \, dz \left\{ G_{11}^{(0)}(x, w) \Sigma_{12}(w, z) G_{21}(z, y) + G_{12}^{(0)}(x, w) \Sigma_{22}(w, z) G_{21}(z, y) + G_{12}^{(0)}(x, w) \Sigma_{21}(w, z) G_{11}(z, y) \right\} + G_{11}^{(0)}(x, y)
\]

(25a)

or

\[
G_{11}(x, y) = -i \int dw \, dz \left\{ G_{12}(x, w) \Sigma_{21}(w, z) G_{11}^{(0)}(z, y) + G_{12}(x, w) \Sigma_{22}(w, z) G_{21}^{(0)}(z, y) + G_{11}(x, w) \Sigma_{12}(w, z) G_{21}^{(0)}(z, y) \right\} + G_{11}^{(0)}(x, y)
\]

(25b)
\[ i \begin{array}{c} G \end{array} = \begin{array}{c} \Sigma \end{array} \begin{array}{c} G \end{array} + \begin{array}{c} \Sigma \end{array} \begin{array}{c} G \end{array} + i \begin{array}{c} \Sigma \end{array} \begin{array}{c} G \end{array} \]

(a)

\[ i \begin{array}{c} G \end{array} = \begin{array}{c} G \end{array} \begin{array}{c} \Sigma \end{array} + \begin{array}{c} G \end{array} \begin{array}{c} \Sigma \end{array} + \begin{array}{c} G \end{array} \begin{array}{c} \Sigma \end{array} + i \begin{array}{c} \Sigma \end{array} \begin{array}{c} G \end{array} \]

(b)

FIG. 3: Diagrammatic representation of Eq. (25) as illustrated in Figs. 3a and Fig. 3b, respectively.

The \(-i\) in Eqs. (25) reflects the usual convention that \(-i\Sigma\) be the sum of one-particle irreducible graphs. Applying \(\Box_x + m^2\) to Eq. (25a), \(\Box_y + m^2\) to Eq. (25b) and substracting, one obtains

\[
(\Box_x - \Box_y)G_{11}(x, y) = \int dz \left\{ G_{11}(x, z)\Sigma_{12}(z, y) - \Sigma_{21}(x, z)G_{11}(z, y) + G_{12}(x, z)\Sigma_{22}(z, y) - \Sigma_{22}(x, z)G_{21}(z, y) \right\}.
\]

(26)

Now it is convenient to write

\[
G(x, y) = \int \frac{d^4p}{(2\pi)^4} G\left(\frac{x + y}{2}, p\right) e^{-ip \cdot (x - y)}
\]

(27)

with similar formalae for \(\Sigma(x, y)\) in terms of \(\Sigma(\frac{x + y}{2}, p)\). We now assume that the dependence of \(G\) and \(\Sigma\) on \(\frac{x + y}{2}\) is slow enough that one can replace \(\frac{x + y}{2}\) by \(x\) in Eq. (27). Then using

\[
\Box_x - \Box_y = \frac{\partial}{\partial (\frac{x + y}{2})} \cdot \frac{\partial}{\partial (\frac{x + y}{2})}.
\]

(28)

one gets from Eq. (26)

\[
2ip \cdot \frac{\partial}{\partial x} G_{11}(x, p) = G_{11}(x, p)[\Sigma_{21}(x, p) - \Sigma_{12}(x, p)] + \Sigma_{22}(x, p)[G_{21}(x, p) - G_{12}(x, p)].
\]

(29)

It is important to emphasize that Eq. (29) is an equation for a classical field theory. We are dealing with the Lagrangian (10) which is classical. In the quantum case, with \(L\) given by Eq. (6), \(G_{22}\) and \(\Sigma_{11}\) are still zero, but when using the equations derived in this
section one must verify that $G_{11} \gg G_{12}, G_{21}$ so that the approximation $\phi \gg \pi$ which leads to the classical theory is valid. This will, of course, depend on the initial ensemble, the $\rho$ in Eq. (14), as well as on how far from the initial ensemble the system has evolved. In particular we suppose the initial ensemble, $\rho$, is dominated by large $\phi_0$ contributions. Of course a system near kinetic equilibrium cannot be described by a classical field theory.

Turn now to $G_{12}$. Analogously to Eqs. (25) one may write

$$G_{12}(x, y) = -i \int dw dz G_{12}(0)(x, w) \Sigma_{21}(w, z) G_{12}(z, x) + G_{12}(0)(x, y)$$

and

$$G_{12}(x, y) = -i \int dw dz G_{12}(x, w) \Sigma_{21}(w, z) G_{12}(0)(z, x) + G_{12}(0)(x, y)$$

illustrated in Fig. 4. It is now straightforward to get

$$2ip \cdot \frac{\partial}{\partial x} G_{12}(x, p) = 0$$

with an identical equation for $G_{21}$. Equation (31) does not mean that the graphs shown in Fig. 4 do not change $G_{12}$ from the value given in Eq. (24) for $G_{12}(0)$, but it does mean that the evolution of the system toward equilibrium is controlled by the equation for $G_{11}$ where the right hand side of Eq. (29) will shortly be identified with the Boltzmann collision term.

![Diagram](image)

(FIG. 4: Diagrammatic representation of Eq. (30))

IV. EQUVALENCE BETWEEN THE BOLTZMANN EQUATION AND CLASSICAL FIELD EQUATIONS

We now turn to the task of showing that the right hand side of Eq. (29) is exactly the Boltzmann collision term. We do this explicitly only at lowest order in perturbation theory for the collision term, after which the higher order corrections should be straightforward. We begin by identifying a part of $G_{11}$ with the Boltzmann phase space particle density.
A. The phase space density

The generalization of Eqs. (21)–(22b) to a medium is straightforward. The results are, in form, exactly like those familiar finite temperature field theory. One has

\[
G_{--}(x, p) = \frac{i}{p^2 - m^2 - i\epsilon} + 2\pi\delta(p^2 - m^2)f, \quad (32a)
\]

\[
G_{++}(x, p) = \frac{-i}{p^2 - m^2 + i\epsilon} + 2\pi\delta(p^2 - m^2)f, \quad (32b)
\]

\[
G_{-+}(x, p) = 2\pi\delta(p^2 - m^2)[\theta(-p_0) + f], \quad (32c)
\]

\[
G_{--}(x, p) = 2\pi\delta(p^2 - m^2)[\theta(p_0) + f], \quad (32d)
\]

but in contrast to finite temperature field theory \( f \) is not the thermal phase space distribution but rather

\[
f = f(\vec{x}, \vec{p}, t) \quad (32e)
\]

is a time-dependent phase space density of particles characterizing the medium. Equations (32) are appropriate to free particles with a phase space density given by \( f \). They are not compatible in detail with Eqs. (25) and (30). For example, space and momentum dependent mass effects are not included. Nevertheless, when \( f \ll 1/\lambda \) the one-loop correction to the mass square is small compared to either the bare mass square or the square of the typical particle momentum. Hence we believe Eqs. (32) are adequate for our purposes of demonstrating the equivalence between classical field theory and the Boltzmann equation and match well with the level of accuracy of our Boltzmann equations, Eqs. (35) and (36).\(^1\)

Using equations identical to Eqs. (23a), (23b) etc., one easily finds

\[
G_{11}(x, p) = 2\pi\delta(p^2 - m^2) \left[ f + \frac{1}{2} \right], \quad (33a)
\]

\[
G_{12}(x, p) = \frac{i}{p^2 - m^2 + i\epsilon_0}, \quad (33b)
\]

\[
G_{21}(x, p) = \frac{i}{p^2 - m^2 - i\epsilon_0}, \quad (33c)
\]

\[
G_{22}(x, p) = 0, \quad (33d)
\]

\(^1\) For quantities sensitive to soft modes, like the bulk viscosity, medium corrections to the mass are important.\(\Box\)
and we note that only $G_{11}$ has changed from Eq. (24). When $f$ is large only $G_{11}$ is large, which means that $\phi$ is large while $\pi$ is small, as we have assumed in the discussion before Eq. (10). Now that we have found the relationship between the phase space density of particles, $f$, and $G_{11}$, we can view Eq. (29) as an equation for $f$. Indeed we can rewrite Eq. (29) as

$$\left(\frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}\right) f(\vec{x}, \vec{p}, t) = \frac{-i}{2\omega(p)}(\Sigma_{21} - \Sigma_{12}) \left(f + \frac{1}{2}\right) + \frac{i}{2\omega(p)}\Sigma_{22},$$

(34)

where $p_0 = \omega(p) = \sqrt{\vec{p}^2 + m^2}$ is to be taken in the $\Sigma$’s in Eq. (34). Equation (34) has the form of a Boltzmann equation. Our task now is to show that the right hand side of Eq. (34) agrees with the collision term in $\phi^4$ theory, at least when $f \gg 1$.

### B. The collision term in $\phi^4$ theory

We are now going to give the lowest order contribution to the collision term calculated directly from the elastic scattering cross section from the graph shown in Fig. 5. The first term on the right hand side of Fig. 5 is the gain term for a particle of momentum $\vec{p}$ while the second term is the loss term. With the notation

$$\left(\frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}\right) f(\vec{x}, \vec{p}, t) = C(\vec{x}, \vec{p}, t)$$

(35)

one has

$$C = \frac{1}{2}\lambda^2 \int \frac{d^3P d^3k}{(2\pi)^62\omega(P-k)2\omega(P)2\omega(p)2\omega(P)} \times$$

$$\times \left\{ f(p-k)f(P)[1 + f(p)][1 + f(P-k)] - f(p)f(P-k)[1 + f(p-k)][1 + f(P)] \right\}$$

(36)

$$C(p) = \begin{vmatrix} p-k & 2 \\ P & - \end{vmatrix} - \begin{vmatrix} p-k & 2 \\ P-k & - \end{vmatrix}$$

FIG. 5: The lowest order collision term

The first factor on the right hand side of Eq. (36), the $\frac{1}{2}$, is a symmetry factor. The term quartic in the $f$’s cancels in Eq. (36) and one is left with cubic and quadratic terms. The
cubic term is

\[ \{ \}_3 = f(p-k)f(P)[f(p) + f(P-k)] - f(p)f(P-k)[f(p-k) + f(P)] \]  

(37)

with the first two terms on the right hand side of Eq. (37) coming from the gain term and the second two terms coming from the loss term. The quadratic term is

\[ \{ \}_2 = f(p-k)f(P) - f(p)f(P-k) \]  

(38)

We shall see that the classical field evolution reproduces the \( \{ \}_3 \) part of the collision term and, perhaps surprisingly, even the \( \{ \}_2 \) part.

C. Evaluating the collision term from classical field theory

Now we turn to evaluating the right hand side of Eq. (34) at lowest order, order \( \lambda^2 \), in perturbation theory. We begin with the \( \Sigma_{22} \)-term whose lowest order contribution is illustrated in Fig. 3. One has

\[ \frac{i\Sigma_{22}}{2\omega(p)} = \frac{i}{2\omega(p)} \frac{(-i\lambda)^2}{3!} \int \frac{d^4k_1 d^4k_2 d^4k_3}{(2\pi)^8} \delta^4(k_1 + k_2 + k_3 - p) G_{11}(k_1)G_{11}(k_2)G_{11}(k_3), \]

(39)

where the second factor of \( i \) on the right hand side of Eq. (39) comes because \(-i\Sigma_{22}\) is given by the usual Feynman rules, and the \(1/3!\) is the symmetry factor for the graph. In one of the three factors of \( G_{11} \) we take \( k_0 < 0 \) while in the remaining two factors \( k_0 > 0 \). (This is the only way to satisfy the \( \delta \)-function constraint. The choice of one of the lines to have \( k_0 < 0 \) also introduces a counting factor of 3.) Using Eq. (33a), it is now straightforward to get

\[ \frac{i\Sigma_{22}}{2\omega(p)} = \frac{\lambda^2}{2} \int \delta[\omega(p-k)+\omega(P)-\omega(p)-\omega(P-k)] \frac{d^3P d^3k}{(2\pi)^6 2\omega(p-k) 2\omega(P) 2\omega(P-k) 2\omega(p)} \]

\[ \left[ f(p-k) + \frac{1}{2} \right] \left[ f(P) + \frac{1}{2} \right] \left[ f(P-k) + \frac{1}{2} \right], \]

(40)

where we have chosen, say, \( k_1 = P, k_2 = p-k, k_3 = -(P-k) \), to match the picture of incoming and outgoing lines in Fig. 4. One easily sees that the cubic term in \( f \) agrees with one of the gain terms in Eq. (37).

Now we turn to the remaining term on the right hand side of Eq. (34), with \( \Sigma_{21} - \Sigma_{12} \) illustrated in Fig. 4 at order \( \lambda^2 \). One has

\[ \frac{-i}{2\omega(p)} \left[ f(p) + \frac{1}{2} \right] \left( \Sigma_{21} - \Sigma_{12} \right) = \frac{-i}{2\omega(p)} \frac{(-i\lambda)^2}{2} \int \frac{d^4k_1 d^4k_2 d^4k_3}{(2\pi)^8} \delta^4(k_1 + k_2 + k_3 - p) \]

\[ 2\pi\epsilon(k_{30}) \delta(k_{3}^2 - m^2) G_{11}(k_1)G_{11}(k_2) \]

(41)
When \( k_{30} > 0 \) there are two (identical) terms, one having \( k_{10} > 0 \) and \( k_{20} < 0 \) and the other having \( k_{10} < 0 \) and \( k_{20} > 0 \). When \( k_{30} < 0 \) it is necessary that \( k_{10} \) and \( k_{20} \) both be positive. Thus one finds

\[
-\frac{i}{2\omega(p)} \left[ f(p) + \frac{1}{2} \right] (\Sigma_{21} - \Sigma_{12}) = -\frac{\lambda^2}{2} \int \delta[\omega(P-k) + \omega(P) - \omega(p) - \omega(P-k)] d^3P d^3k
\]

\[
\frac{(2\pi)^5 2\omega(p) 2\omega(p-k) 2\omega(P) 2\omega(P-k)}{[f(p) + \frac{1}{2}] \times \left\{ 2 \left[ f(P-k) + \frac{1}{2} \right] \left[ f(P) + \frac{1}{2} \right] - \left[ f(p-k) + \frac{1}{2} \right] \left[ f(P) + \frac{1}{2} \right] \right\}}
\]

where in the first term in \{ \} in Eq. (42) we have taken \( k_3 = p-k \), \( k_1 = -(P-k) \), \( k_2 = P \) along with \( k_1 \leftrightarrow k_2 \), while in the second term we have taken \( k_1 = (p-k) \), \( k_2 = P \), and \( k_3 = -(P-k) \) so that the variables match those in Eq. (37). We note that the two loss terms in Eq. (37) are in fact identical.

Now, taking the terms cubic in \( f \) in Eqs. (40) and (41) exactly reproduces the cubic term, (37), in Eq. (36). For the quadratic terms we find, from Eqs. (40) and (42), the result that replaces \{ \} in Eq. (38) is

\[
\{ \} = \frac{1}{2} \left\{ f(p-k)f(P) + f(p-k)f(P-k) + f(P)f(P-k) - 2f(p)f(P-k) \\
- 2f(p)f(P) - 2f(P-k)f(P) + f(p)f(p-k) + f(p)f(P) + f(p-k)f(P) \right\}
\]

where in the first term in \{ \} in Eq. (42) we have taken \( k_3 = p-k \), \( k_1 = -(P-k) \), \( k_2 = P \) along with \( k_1 \leftrightarrow k_2 \), while in the second term we have taken \( k_1 = (p-k) \), \( k_2 = P \), and \( k_3 = -(P-k) \) so that the variables match those in Eq. (37). We note that the two loss terms in Eq. (37) are in fact identical.

Using the fact that \( f(p-k)f(P-k) \) and \( f(P)f(P-k) \) are equivalent expressions, after integration in Eq. (40) or (42), as are \( f(p)f(P) \) and \( f(p)f(p-k) \) one finds \{ \} = \{ \} in Eq. (38). Finally
there are the linear terms in the $f$’s coming from Eqs. (40) and (42) for which there are no counterparts in Eq. (36). In fact the linear terms do not cancel in Eq. (40) and (42) so that the classical field theory does not exactly reproduce the Boltzmann collision term.

Finally, a comment on the level of accuracy at which one can expect the classical field theory to reproduce the Boltzmann equation. We always suppose that $f \gg 1$ but that $\lambda f \ll 1$ so that a perturbative discussion makes sense. The size of our fields then are $\phi \sim \sqrt{f}$ while $\pi \sim 1/\sqrt{f}$. In going from the full Lagrangian, given in Eq. (7), to the classical Lagrangian, given in Eq. (10), we have dropped a term $-\frac{\lambda}{3!} \pi^3 \phi$ compared to the interaction term $-\frac{\lambda}{3!} \pi \phi^3$ which has been kept. Now an estimate of the size of these two terms is

$$\lambda \pi \phi^3 \sim \lambda f$$  \hspace{1cm} (44)

and

$$\lambda \pi^3 \phi \sim \frac{\lambda}{f}$$  \hspace{1cm} (45)

Thus, the quantum corrections are naturally down by a factor of $1/f^2$ as compared to a classical evaluation at a corresponding order of $\lambda$. Since the maximal possible terms in the collision term are of size $\lambda^2 f^3$, one expects quantum corrections to occur at a level of $\lambda^2 f$ leaving the $\lambda^2 f^2$ terms still in the classical field theory domain. So in retrospect, our ability to get both the $\lambda^2 f^3$ and $\lambda^2 f^2$ terms in Eq. (36) correctly was to be expected. This means that the $2 \to 2$ parts of a collision term can be obtained from a classical theory calculation. This cannot be expected to be the case for $2 \to 4$ processes where terms of size $\lambda^4 f^5$, $\lambda^4 f^4$, $\lambda^4 f^3$, and $\lambda^4 f^2$ in the collision terms will occur at leading order in $\lambda$. One can here expect to get the $\lambda^4 f^5$ and $\lambda^4 f^4$ terms correctly in a classical field theory, but not the $\lambda^4 f^3$ and $\lambda^4 f^2$ terms.

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