PARTIAL SUMS FOR CERTAIN CLASSES OF
MEROMORPHIC FUNCTIONS

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Abstract. In this paper, we define and study new classes of meromorphic functions in the punctured disk by using their partial sums.

1. Introduction

Let $\Sigma_\alpha$ denote the class of functions of the form

$$f(z) = \frac{1}{z^{1+\alpha}} + \sum_{n=1}^{\infty} a_n z^{n+\alpha}, \quad (0 \leq \alpha < 1),$$

which are analytic in the punctured unit disk $U := \{ z \in \mathbb{C}, 0 < |z| < 1 \}$.

A function $f \in \Sigma_\alpha$ belongs to the class $\mathcal{S}_\alpha(A, B)$, the class of meromorphically $\alpha$-valent starlike functions if and only if $f \neq 0$, and for $-1 \leq A < B \leq 1$,

$$-\frac{zf'(z)}{f(z)} < \frac{1 + Az}{1 + Bz}, \quad (z \in U).$$

A function $f \in \Sigma_\alpha$ belongs to the class $\mathcal{C}_\alpha(A, B)$, the class of meromorphically $\alpha$-valent convex functions if and only if $f' \neq 0$, and

$$-\frac{zf''(z)}{f'(z)} < \frac{1 + Az}{1 + Bz}, \quad (z \in U).$$

The class $\Sigma_0 \equiv \Sigma$, was studied by many authors (see [1, 2, 3, 4, 5]). Note that the authors defined and studied the class $\Sigma_\alpha$ for normalized analytic functions in an open disk (see [6, 7]).

In the present paper, we are motivated with the work done by Silverman [8], and will investigate in similar manner the ratio of a function of the form (1) to its sequence of partial sums

$$f_k(z) = \frac{1}{z^{1+\alpha}} + \sum_{n=1}^{k} a_n z^{n+\alpha}, \quad (0 \leq \alpha < 1),$$

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when the coefficients are sufficiently small. More precisely, we will determine sharp lower bounds for
\[ \Re \left\{ \frac{f(z)}{f_k(z)} \right\}, \Re \left\{ \frac{f_k(z)}{f(z)} \right\}, \Re \left\{ \frac{f'(z)}{f_k'(z)} \right\}, \text{ and } \Re \left\{ \frac{f_k'(z)}{f'(z)} \right\}. \]

2. Preliminary results

First we prove sufficient conditions for \( f \in \Sigma_\alpha \) to be in the classes \( \mathcal{S}_\alpha(A, B) \) and \( \mathcal{C}_\alpha(A, B) \).

**Theorem 2.1.** Let \( f \in \Sigma_\alpha \). If
\[ \sum_{n=1}^{\infty} \left[ (n + \alpha)(1 + B) + (A + 1) \right] |a_n| \leq (B - A) - \alpha(1 - B), \ (z \in U) \] holds and \( B(1 + \alpha) > A + \alpha \), then \( f \in \mathcal{S}_\alpha(A, B) \).

**Proof.** Assume that \( f \in \Sigma_\alpha \) and satisfies (3). It is sufficient to show that
\[ \left| \frac{1 + zf'(z)}{Bzf'(z) + A} \right| < 1, \]
that is
\[ \left| \frac{f(z) + zf'(z)}{Af(z) + Bzf'(z)} \right| < 1. \]

Consider
\[ \left| \frac{f(z) + zf'(z)}{Af(z) + Bzf'(z)} \right| = \left| \frac{-\frac{B}{z^{1+\alpha}} + \sum_{n=1}^{\infty} a_n(n + \alpha + 1)z^{n+\alpha}}{\frac{A}{z^{1+\alpha}} + \sum_{n=1}^{\infty} Aa_nz^{n+\alpha} - \frac{B(1+\alpha)}{z^{1+\alpha}} + \sum_{n=1}^{\infty} (n + \alpha)Ba_nz^{n+\alpha}} \right| \leq \frac{\alpha + \sum_{n=1}^{\infty} (n + \alpha + 1) |a_n|}{B(1+\alpha) - A} \]
\[ \Rightarrow \sum_{n=1}^{\infty} (n + \alpha + 1) |a_n| \leq \frac{B(1+\alpha) - A}{\alpha} - \sum_{n=1}^{\infty} (n + \alpha) B |a_n|. \]

Hence (4) is bounded by 1, if
\[ \alpha + \sum_{n=1}^{\infty} (n + \alpha + 1) |a_n| \leq \frac{B(1+\alpha) - A}{\alpha} - \sum_{n=1}^{\infty} (n + \alpha) B |a_n| \]
\[ \Rightarrow \sum_{n=1}^{\infty} (n + \alpha + 1) |a_n| + \sum_{n=1}^{\infty} [(n + \alpha) B + A] |a_n| \leq \frac{B(1+\alpha) - A}{\alpha} - \alpha \]
\[ \Rightarrow \sum_{n=1}^{\infty} [(n + \alpha)(1 + B) + (1 + A)] |a_n| \leq (B - A) - \alpha(1 - B), \]
where \( B(1 + \alpha) > A + \alpha \). This completes the proof.

In a similar manner, we can prove the following result.
Theorem 2.2. Let \( f \in \Sigma_\alpha \). If

\[
\sum_{n=1}^{\infty} (n + \alpha) \left[ (n + \alpha)(1 + B) + (A + 1) \right] |a_n| \leq (1 + \alpha) \left[ (B - A) - \alpha(1 - B) \right], \quad (z \in U) \tag{5}
\]

holds and \( B(1 + \alpha) > A + \alpha \), then \( f \in C_\alpha(A, B) \).

Note that when \( \alpha = 0 \), Theorem 2.1 and Theorem 2.2 reduce to Theorem 2.2 and Theorem 2.1 in [5] respectively. Further, we note that these sufficient conditions are also necessary for functions of the form (1) when \( \alpha = 0, A = 2\mu - 1, B = 1 \) with positive or negative coefficients ([1, 2, 3]).

3. Main results

We consider in this section partial sums of functions in the classes \( S_\alpha(A, B) \) and \( C_\alpha(A, B) \) and obtain the sharp lower bounds for the ratio of real part of \( f(z) \) to \( f_k(z) \) and \( f'(z) \) to \( f'_k(z) \). In the sequel, we will make use of the generalized result such that \( \Re\{((1 + w_\alpha(z)))/(1 - w_\alpha(z))\} > 0 \), \( (z \in U) \) if and only if \( w_\alpha(z) = \sum_{n=1}^{\infty} c_n z^{n+\alpha} \) satisfies the inequality \(|w_\alpha(z)| < |z|\).

Theorem 3.1. Let \( f \) be given by (1) and satisfies (3) then

\[
\Re\left\{ \frac{f(z)}{f_k(z)} \right\} \geq \frac{2(1 + k + \alpha + A)}{2(k + \alpha) + (2 + A + B)}, \quad (z \in U). \tag{6}
\]

The result is sharp for every \( k \) with extremal function

\[
f(z) = \frac{1}{z^{1+\alpha}} + \frac{(B - A) - \alpha(1 - B)}{2(k + \alpha) + (2 + A + B)} z^{k+1+\alpha}, \quad k \geq 0. \tag{7}
\]

Proof. Assume that \( f \in \Sigma_\alpha \) and satisfies (3). Consider

\[
\frac{2(k + \alpha) + (2 + A + B)}{(B - A) - \alpha(1 - B)} \left[ \frac{f(z)}{f_k(z)} - \frac{2(1 + k + \alpha + A)}{2(k + \alpha) + (2 + A + B)} \right] \left[ \frac{f(z)}{f_k(z)} - \frac{2(k + \alpha)}{2(k + \alpha) + (2 + A + B)} \right] = 1 + \sum_{n=1}^{k} a_n z^{n+2\alpha+1} + \frac{2(k+\alpha)(2+A+B)}{(B-A)-\alpha(1-B)} \sum_{n=k+1}^{\infty} a_n z^{n+2\alpha+1}
\]

\[
= 1 + \sum_{n=1}^{k} a_n z^{n+2\alpha+1} = \frac{1 + w_\alpha(z)}{1 - w_\alpha(z)}
\]

where

\[
w_\alpha(z) = \frac{2(k+\alpha)(2+A+B)}{(B-A)-\alpha(1-B)} \sum_{n=k+1}^{\infty} a_n z^{n+2\alpha+1}
\]

\[
+ \frac{2(k+\alpha)(2+A+B)}{(B-A)-\alpha(1-B)} \sum_{n=k+1}^{\infty} a_n z^{n+2\alpha+1}
\]
and
\[ |w_\alpha(z)| \leq \frac{2(k+\alpha) + (2 + A + B)}{(B - A) - \alpha(1 - B)} \sum_{n=k+1}^\infty |a_n|. \]

Now \(|w_\alpha(z)| \leq 1\) if and only if
\[ 2 \left( \frac{2(k+\alpha) + (2 + A + B)}{(B - A) - \alpha(1 - B)} \right) \sum_{n=k+1}^\infty |a_n| \leq 2 - \frac{2k}{\sum_{n=1}^{\infty} |a_n|}. \]

which is equivalent to
\[ \sum_{n=1}^{k} |a_n| + \left( \frac{2(k+\alpha) + (2 + A + B)}{(B - A) - \alpha(1 - B)} \right) \sum_{n=k+1}^\infty |a_n| \leq 1. \]  (8)

It is suffices to show that the left hand side of (8) is bounded above by
\[ \sum_{n=1}^\infty \left( \frac{2(n+\alpha) + A + B}{(B - A) - \alpha(1 - B)} \right) |a_n| \]

which is equivalent
\[ \sum_{n=1}^{k} \left( \frac{2(n+\alpha) + A + B}{(B - A) - \alpha(1 - B)} \right) |a_n| + \sum_{n=k+1}^\infty \left( \frac{2(n-k-1+\alpha)}{(B - A) - \alpha(1 - B)} \right) |a_n| \geq 0. \]

To show that the function \(f\) given by (7) gives the sharp result, we observe that for
\[ z = re^{\pi i (k+2+2\alpha)}, \]

\[ \frac{f(z)}{f_k(z)} = 1 + \frac{(B - A) - \alpha(1 - B)}{2(k+\alpha) + (2 + A + B)} z^{k+2+2\alpha} \]
\[ \rightarrow 1 - \frac{(B - A) - \alpha(1 - B)}{2(k+\alpha) + (2 + A + B)} = \frac{2(k+\alpha) + (2 + A + B) - (B - A) + \alpha(1 - B)}{2(k+\alpha) + (2 + A + B)} = \frac{2(1 + k + A + \alpha)}{2(k+\alpha) + (2 + A + B)} \]

when \(r \to 1^{-}\). Therefore we complete the proof of Theorem 3.1.

Next result can be found in [5].

**Corollary 3.1.** Let \(f\) be given by (1) and satisfies (3) then
\[ \Re \left\{ \frac{f(z)}{f_k(z)} \right\} \geq \frac{2(1 + k + A)}{2k + 2 + A + B}, \quad (z \in U). \]  (9)
The result is sharp for every $k$ with extremal function
\[ f(z) = \frac{1}{z} + \frac{(B - A)}{2k + 2 + A + B} z^{k+1}, \; k \geq 0. \] (10)

**Proof.** Assume that $\alpha = 0$.

Moreover, the following result can be found in [8].

**Corollary 3.2.** Let the assumptions of Theorem 3.1 hold. Then for $f$ of the form (1) satisfies condition
\[ \sum_{n=1}^{\infty} (n + \mu) |a_n| \leq 1 - \mu, \; (z \in U), \]
\[ \Re \left\{ \frac{f(z)}{f_k(z)} \right\} \geq \frac{k + 2 \mu}{k + 1 + \mu}, \; (z \in U). \] (11)

The result is sharp for every $k$ with extremal function
\[ f(z) = \frac{1}{z} + \frac{1 - \mu}{k + 1 + \mu} z^{k+1}, \; k \geq 0. \] (12)

**Proof.** Assume that $\alpha = 0$, $A = 2\mu - 1$, $B = 1$.

**Theorem 3.2.** Let $f \in \Sigma_\alpha$ and
\[ \sum_{n=1}^{\infty} (n + \alpha) \left[ (n + \alpha)(1 + B) + (A + 1) \right] |a_n| \leq (1 + \alpha) \left[ (B - A) - \alpha(1 - B) \right], \; (z \in U) \]
holds, then
\[ \Re \left\{ \frac{f(z)}{f_k(z)} \right\} \geq \frac{(k+2)(2k+A+B)+\alpha[(2k+2+\alpha)(1+B)+2\alpha(1-B)-B+2A+1]}{(k+1)(2k+2+A+B)+\alpha[(2k+2+\alpha)(1+B)+(A+1)]}. \] (13)

The result is sharp for every $k$ with extremal function
\[ f(z) = \frac{1}{z^{1+\alpha}} + \frac{(1 + \alpha) \left[ (B - A) - \alpha(1 - B) \right]}{(k+1)(2k+2+A+B)+\alpha[(2k+2+\alpha)(1+B)+(A+1)]} z^{k+\alpha}, \; k \geq 0. \] (14)
Proof. Let \( f \in \Sigma_\alpha \). Then we obtain

\[
(k + 1)(2k + 2 + A + B) + \alpha[(2k + 2 + \alpha)(1 + B) + (A + 1)] \times \frac{B - A - \alpha(1 - B)}{(1 + \alpha)[(B - A) - \alpha(1 - B)]}
\]

\[
\left[ f(z) - \frac{(k + 2)(2k + A + B) + \alpha[(2k + 2 + \alpha)(1 + B) + 2\alpha(1 - B) - B + 2A + 1]}{(k + 1)(2k + 2 + A + B) + \alpha[(2k + 2 + \alpha)(1 + B) + (A + 1)]}ight]
\]

\[
1 + \frac{\sum_{n=1}^{\infty} a_n z^{n+2\alpha+1}}{1 + \sum_{n=1}^{\infty} a_n z^{n+2\alpha+1}}
\]

\[
:= \frac{1 + w_\alpha(z)}{1 - w_\alpha(z)},
\]

where

\[
w_\alpha(z) = \frac{(k+1)(2k+2+A+B) + \alpha[(2k+2+\alpha)(1+B)+(A+1)]}{(1+\alpha)[(B-A)-\alpha(1-B)]} \sum_{n=k+1}^{\infty} a_n z^{n+2\alpha+1}
\]

\[
2 + 2 \sum_{n=1}^{\infty} a_n z^{n+2\alpha+1} + \frac{(k+1)(2k+2+A+B) + \alpha[(2k+2+\alpha)(1+B)+(A+1)]}{(1+\alpha)[(B-A)-\alpha(1-B)]} \sum_{n=k+1}^{\infty} a_n z^{n+2\alpha+1}
\]

and

\[
|w_\alpha(z)| \leq \frac{2 - 2 \sum_{n=1}^{\infty} |a_n|}{|a_n|} - \frac{(k+1)(2k+2+A+B) + \alpha[(2k+2+\alpha)(1+B)+(A+1)]}{(1+\alpha)[(B-A)-\alpha(1-B)]} \sum_{n=k+1}^{\infty} |a_n|.
\]

Since \( |w_\alpha(z)| \leq 1 \) if and only if

\[
2 \left[ (k+1)(2k+2+A+B) + \alpha[(2k+2+\alpha)(1+B)+(A+1)] \right] \sum_{n=k+1}^{\infty} |a_n| \leq 2 - 2 \sum_{n=1}^{k} |a_n|,
\]

this means

\[
\sum_{n=1}^{k} |a_n| + \frac{(k+1)(2k+2+A+B) + \alpha[(2k+2+\alpha)(1+B)+(A+1)]}{(1+\alpha)[(B-A)-\alpha(1-B)]} \sum_{n=k+1}^{\infty} |a_n| \leq 1.
\]

Thus by the assumption of the theorem, the left hand side of (15) is bounded above by

\[
\sum_{n=1}^{\infty} \left[ \frac{n(2n + A + B) + \alpha[(2n + \alpha)(1 + B) + (A + 1)]}{(1+\alpha)[(B-A)-\alpha(1-B)]} \right] |a_n|.
\]
if

\[
\frac{1}{(1 + \alpha)[(B - A) - \alpha(1 - B)]} \left\{ \sum_{n=1}^{k} \left[ n(2n + A + B) + \alpha[(2n + \alpha)(1 + B) + (A + 1)] \right] - (1 + \alpha)[(B - A) - \alpha(1 - B)] \right\} |a_n| + \sum_{n=k+1}^{\infty} \left[ n(2n + A + B) + \alpha[(2n + \alpha)(1 + B) + (A + 1)] \right] |a_n| \geq 0.
\]

Which completes the proof of Theorem 3.2.

The following result can be found in [5].

**Corollary 3.3.** Let \( f \) be given by (1) and satisfies (5) then

\[
\Re \left\{ \frac{f(z)}{f_k(z)} \right\} \geq \frac{(k + 2)(2k + A + B)}{(k + 1)(2k + 2 + A + B)}, \quad (z \in U). \tag{16}
\]

The result is sharp for every \( k \) with extremal function

\[
f(z) = \frac{1}{z} + \frac{(B - A)}{(k + 1)(2k + 2 + A + B)} z^{k+1}, \quad k \geq 0. \tag{17}
\]

**Proof.** Assume that \( \alpha = 0 \).

Further, the next result can be found in [8].

**Corollary 3.4.** Let the assumptions of Theorem 3.2 hold. Then for \( f(z) \) of the form (1) satisfies condition

\[
\sum_{n=1}^{\infty} n(n + \mu) |a_n| \leq 1 - \mu, \quad (z \in U),
\]

\[
\Re \left\{ \frac{f(z)}{f_k(z)} \right\} \geq \frac{(k + 2)(k + \mu)}{(k + 1)(k + 1 + \mu)}, \quad (z \in U). \tag{18}
\]

The result is sharp for every \( k \) with extremal function

\[
f(z) = \frac{1}{z} + \frac{1 - \mu}{(k + 1)(k + 1 + \mu)} z^{k+1}, \quad k \geq 0. \tag{19}
\]

**Proof.** Assume that \( \alpha = 0, A = 2\mu - 1, B = 1 \).

We next determine the bounds for \( \Re \{ f_k(z) / f(z) \} \) of functions in the classes \( S_\alpha(A, B) \) and \( C_\alpha(A, B) \).
Theorem 3.3. Let \( f \in \Sigma_\alpha \) such that
\[
\sum_{n=1}^{\infty} \left[ (n + \alpha)(1 + B) + (A + 1) \right]|a_n| \leq (B - A) - \alpha(1 - B), \quad (z \in U)
\]
holds. Then
\[
\Re \left\{ \frac{f_k(z)}{f(z)} \right\} \geq \frac{2(k + 1) + A + B + \alpha(1 + B)}{k + 2 + 2\alpha B}, \quad (z \in U).
\]
Equalities hold for the functions given by (7).

Proof. Let \( f \in \Sigma_\alpha \), then we have
\[
\frac{k + 2 + 2\alpha B}{(B - A) - \alpha(1 - B)} \left\{ \frac{f_k(z)}{f(z)} - \frac{2(k + 1) + A + B + \alpha(1 + B)}{k + 2 + 2\alpha B} \right\}
\]
\[
= 1 + \sum_{n=1}^{k} a_n z^{n+2\alpha+1} + \sum_{n=k+1}^{\infty} \frac{2(k+1)+A+B+\alpha(1+B)}{(B-A)-\alpha(1-B)} \sum_{n=k+1}^{\infty} a_n z^{n+2\alpha+1}
\]
\[
1 + \sum_{n=1}^{k} a_n z^{n+2\alpha+1}
\]
\[:= 1 + w_\alpha(z) \]
1 - w_\alpha(z),
where
\[
w_\alpha(z) = \frac{2(k+1)+A+B+\alpha(1+B)}{(B-A)-\alpha(1-B)} \sum_{n=k+1}^{\infty} a_n z^{n+2\alpha+1}
\]
\[+ 2 + 2\sum_{n=1}^{k} a_n z^{n+2\alpha+1} + \sum_{n=k+1}^{\infty} \frac{2(k+1)+A+B+\alpha(1+B)}{(B-A)-\alpha(1-B)} \sum_{n=k+1}^{\infty} a_n z^{n+2\alpha+1}
\]\[\text{with}\]
\[
|w_\alpha(z)| \leq \frac{2(k+1)+A+B+\alpha(1+B)}{(B-A)-\alpha(1-B)} \sum_{n=k+1}^{\infty} |a_n| \sum_{n=1}^{k} |a_n| - \alpha(1-B) \sum_{n=k+1}^{\infty} |a_n|.
\]
Note that \(|w_\alpha(z)| \leq 1\) if and only if
\[
2 \left[ \frac{2(k + 1) + A + B + \alpha(1 + B)}{(B - A) - \alpha(1 - B)} \right] \sum_{n=k+1}^{\infty} |a_n| \leq 2 - 2 \sum_{n=1}^{k} |a_n|
\]
which implies
\[
\sum_{n=1}^{k} |a_n| + \left[ \frac{2(k + 1) + A + B + \alpha(1 + B)}{(B - A) - \alpha(1 - B)} \right] \sum_{n=k+1}^{\infty} |a_n| \leq 1.
\]
From the assumption of the theorem, we can observe that the left hand side of (21) is bounded above by
\[
\sum_{n=1}^{\infty} \left[ \frac{2n + A + B + \alpha(1 + B)}{(B - A) - \alpha(1 - B)} \right]|a_n|.
\]
Hence the proof.

The following result can be found in [5].
Corollary 3.5. Let the assumptions of Corollary 3.1 hold. Then
\[ \Re \left\{ \frac{f_k(z)}{f(z)} \right\} \geq \frac{2(k + 1) + A + B}{k + 2}, \quad (z \in U). \] (22)

Proof. Assume that \( \alpha = 0 \).

Further, the next result can be found in [8].

Corollary 3.6. Let the assumptions of Corollary 3.2 hold. Then
\[ \Re \left\{ \frac{f_k(z)}{f(z)} \right\} \geq \frac{k + 1 + \mu}{k + 2}, \quad (z \in U). \] (23)

Proof. Assume that \( \alpha = 0, A = 2\mu - 1, B = 1 \).

In the same manner, we can prove the following result

Theorem 3.4. Let \( f \) be given by (1) and satisfies (5) then
\[ \Re \left\{ \frac{f_k(z)}{f(z)} \right\} \geq \frac{(k + 1)2(2k + 2 + A + B) + \alpha \nu}{2(k + 1)(k + 2) - (B - A) + \alpha \omega}, \quad (z \in U). \] (24)

where
\[ \nu := \left[ (k + 1 + \alpha)(1 + B) + (A + 1) + (k + 1)(B + 1) \right] \]
and
\[ \omega := \left[ (k + 1 + \alpha)(1 + B) + (A + 1) + (k + 1)(B + 1) + (B - A) - (\alpha + 1)(1 - B) \right]. \]

Equalities hold for the function given by (14).

Corollary 3.7. Let the assumptions of Corollary 3.3 hold. Then
\[ \Re \left\{ \frac{f_k(z)}{f(z)} \right\} \geq \frac{(k + 1)2(2k + 2 + A + B)}{2(k + 1)(k + 2) - (B - A)}, \quad (z \in U). \] (25)

Proof. Assume that \( \alpha = 0 \).

Further, the next result can be found in [8].

Corollary 3.8. Let the assumptions of Corollary 3.4 hold. Then
\[ \Re \left\{ \frac{f_k(z)}{f(z)} \right\} \geq \frac{(k + 1)(k + 1 + \mu)}{(k + 1)(k + 2) - k(1 - \mu)}, \quad (z \in U). \] (26)

Proof. Assume that \( \alpha = 0, A = 2\mu - 1, B = 1 \).

We turn to ratios involving derivatives (see [9]). In the similar manner, we can prove the following results and so the details may be omitted.
**Theorem 3.5.** Let $f$ be given by (1) and satisfies (3) with $A = -B$. Then
\[ \Re \left\{ \frac{f'(z)}{f(z)} \right\} \geq 0, \quad (z \in U), \tag{27} \]
\[ \Re \left\{ \frac{f_k'(z)}{f'(z)} \right\} \geq \frac{1 + 2\alpha}{2(1 + \alpha)}, \quad (z \in U). \tag{28} \]
In both cases, the extremal function is given by (7) with $\alpha = 0$, $A = -B$.

**Theorem 3.6.** Let $f$ be given by (1) and satisfies (5). Then
\[ \Re \left\{ \frac{f'(z)}{f'(z_k)} \right\} \geq \frac{2(k + A + B) + \phi - \alpha[(k + 1)(B - A - \alpha(1 - B))]}{(2 + 2k + A + B) + \phi}, \quad (z \in U), \tag{29} \]
\[ \Re \left\{ \frac{f_k'(z)}{f'(z)} \right\} \geq \frac{2k + 2 + A + B + \phi}{2(k + 2) + \phi + \alpha[(k + \alpha)(B - A - (\alpha + 1)(1 - B)) - (k + 1)(1 - B)]} , \quad (z \in U). \tag{30} \]
where
\[ \phi := \alpha[(k + 1 + \alpha)(1 + B) + (A + 1) + (1 + B)(k + 1)]. \]
In both cases, the extremal function is given by (14).

**Proof.** The proof comes immediately from Theorems 3.1 and 3.3 respectively.

**Remark 3.1.** We note that $\alpha = 0$ in Theorems 3.5 and 3.6 coincide with the results obtained in [5].

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