A Proof of the Riemann Hypothesis

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Abstract

In this paper we prove the Riemann hypothesis, i.e., that all zeros of the Riemann zeta function in the critical strip $D := \{z \in \mathbb{C} : 0 < \Re z < 1\}$ are simple and located on the critical line $\{z \in \mathbb{C} : \Re z = 1/2\}$. The main vehicle of our proof is the use of the special function $G(z) := \int_0^\infty y^{z-1}(1 + \exp(y))^{-1} dy$, which has the same zeros in the critical strip as the Riemann zeta function. Our analysis uses the Fourier integral representation of $G$, i.e., $G(\sigma + it) := \int_{-\infty}^{\infty} e^{ixt} \kappa(\sigma, x) dx$, where $\kappa(\sigma, x) := \exp(\sigma x)/(1+\exp(e^x))$, to derive a functional equation and analyticity identities of $G$. We combine these results with Rolle’s theorem, Schwarz reflection, and the Cauchy–Riemann equations to achieve our proof.

Keywords: Riemann hypothesis, Fourier transforms, Schwarz reflection principle, Rolle’s theorem, Cauchy–Riemann equations

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1 Introduction and Summary

In this paper we prove the Riemann hypothesis, i.e., that all the zeros of the Riemann zeta function in the critical strip $D := \{z \in \mathbb{C} : 0 < \Re z < 1\}$ are simple and located on the critical line $\{z \in \mathbb{C} : \Re z = 1/2\}$.

We accomplish this by use of the function $G$, which is defined by the integral representation over $\mathbb{R}^+ = (0, \infty)$:

$$G(z) := \int_{\mathbb{R}^+} \frac{y^{z-1}}{e^y + 1} dy, \quad \Re z > 0.$$  (1.1)
Lemma 1.1 The function $G$ defined in Equation (1.1) is analytic on the right half plane, and positive on $\mathbb{R}^+$. 

Proof. The proof follows by inspection of Equation (1.1). ■

The function $G$ of Equation (1.1) is intimately related to the Riemann zeta function $\zeta$ and the Dirichlet eta function, $\eta$, as follows:

$$\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z}, \quad \Re z > 1,$$

$$\eta(z) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z}, \quad \Re z > 0,$$

$$G(z) = \Gamma(z) \eta(z) = (1 - 2^{1-z}) \Gamma(z) \zeta(z), \quad z \in \mathbb{C}.$$  

(1.2)

By setting $y = e^x$, in (1.1) and $z = \sigma + it$, we get a Fourier integral representation of $G$, namely,

$$G(\sigma + it) := \int_{\mathbb{R}} \kappa(\sigma, x) e^{ixt} dx,$$

where

$$\kappa(\sigma, x) := \frac{e^{\sigma x}}{1 + \exp(e^x)}. $$  

(1.4)

In §2 we derive a functional equation for $G$, as well as some properties of $G$ in $\mathbb{C}$, and we present some identities related to $G$ and $\kappa$. In §3 we present Rolle’s theorem and the Schwarz reflection principle, and in §4 we complete our proof of the Riemann hypothesis by using identities derived in §2 and §3.

2 Some Properties of $\kappa$, $G$ and $\zeta$. 

In this section we define multiplicity, we use the definition of $G$ given in Equation (1.3) and $\kappa$ given in Equation (1.4) to derive analyticity properties of $G$, a functional equation for $G$, and some other results involving zeros and poles of $G$ in $\mathbb{C} \setminus D$, asymptotic properties of $\kappa$, and identities of derivatives of $\Re G$ and $\Im G$ on the line $\{z \in \mathbb{C} : \Im z = 0 \}$.

Let us first define the left and right half of the complex plane, the critical strip(s), and the critical line.
Definition 2.1 Let $\Omega^- := \{z \in \mathbb{C} : \Re z < 0\}$, and let $\Omega^+ := \{z \in \mathbb{C} : \Re z > 0\}$. Let the critical strip be defined by $D = \{z \in \mathbb{C} : 0 < \Re z < 1\}$, and let the negative and positive critical strips $D^\pm$ be defined in terms of the critical strip $D$ as follows: the negative critical strip: $D^- := \{z \in D : \Im z \leq 0\}$, and the positive critical strip: $D^+ := \{z \in D : \Im z \geq 0\}$. The critical line is defined by $\{z \in D : \Re z = 1/2\}$.

An important identity of the Riemann zeta function is the well known functional equation, which can be written for all $z \in \mathbb{C}$ as follows:

$$
\pi^{-(1-z)/2} \Gamma((1-z)/2) \zeta(1-z) = \pi^{-z/2} \Gamma(z/2) \zeta(z). \quad (2.1)
$$

This equation is valid for all $z \in \mathbb{C}$. It has many important uses, including, e.g., the analytic continuation of the zeta function to all of $\mathbb{C}$. The function $G$ also possesses a functional equation with the same uses. It is gotten by substitution of the right-hand-side of the third equation of (1.2) into (2.1) and by use of the duplication formula for the Gamma function: $\Gamma(2z) = (2\pi)^{-1/2} 2^{2z-1/2} \Gamma(z+1/2) \Gamma(z)$ (see [A], Chapter 6). We thus get

Lemma 2.2 Let $z \in \mathbb{C}$, and let $G$ be defined as in (1.1) or (1.3). Then, a functional equation for the function $G$ is:

$$
\frac{(4\pi)^{-z/2}}{(2^{1-z} - 1) \Gamma((1+z)/2)} G(z) = \frac{(4\pi)^{-(1-z)/2}}{(2^z - 1) \Gamma(1-z/2)} G(1-z) \quad (2.2)
$$

By Lemma 1.1 $G(z)$ is analytic and bounded for $z \in \Omega^+$, so that $G(1-z)$ is analytic and bounded for $\Re z - 1 \in \Omega^-$.

Definition 2.3 Let $m$ denote a non-zero integer, let $r > 0$ and let $\mathcal{N}(z_0,r) := (i.) \{z \in \mathbb{R} : 0 < |z - z_0| < r\}$ or (ii.) $\{z \in \mathbb{C} : 0 < |z - z_0| < r\}$. Let all derivatives of $f$ exist in $\mathcal{N}(z_0,r)$. We shall say that $f$ has multiplicity $m$ at $z_0$ if the following limit exists for some $c \neq 0 \in \mathbb{C}$:

$$
\lim_{z \to z_0} \frac{f(z)}{(z-z_0)^m} = c. \quad (2.3)
$$

Thus $z_0$ is: resp. a zero of $f$ of multiplicity $m$ if $m > 0$) or, resp., a pole of $f$, of multiplicity $m$ if $m < 0$. In particular, if $m = 1$, (resp., if $m = -1$) then $z_0$ is resp. a simple zero (resp., a simple pole) of $f$. If $z_0$ is a zero
of multiplicity \( m \geq 1 \) of \( f(z_0) \), then \( z_0 \) is a zero of multiplicity \( m - 1 \) of \( f'(z_0) \). If there exist \( m > 1 \) distinct points \( z_j \in D' \subset D \), for \( j = 1, \ldots, m \) such that \( f(z_j) \) vanishes with multiplicity \( \mu_j > 0 \), then we shall say that \( f \) vanishes in \( D' \) with total multiplicity \( \sum_{j=1}^{m} \mu_j \).

The functional equation \( \text{(2.2)} \) provides an analytic continuation of \( G \) from its known region of analyticity, \( \Omega^+ \) to all of \( C \). Indeed, additional detailed information about \( G \) can be derived by means of the functional equation. To this end, let us start by listing some known properties of the functions in Equation \( \text{(2.2)} \) which will assist us in this derivation.

**Lemma 2.4**\(^([A])\) (i.) The function \( 1/\Gamma \) is an entire function. It has zeros at \( z = -k \), for all \( k \in \mathbb{Z} \); these zeros are simple, and the only zeros in \( \mathbb{C} \) of \( 1/\Gamma \); and (ii.) If \( x \in \mathbb{R} \), then

- \( |\Gamma(1/2 + i x)| \in (0, 1) \), by Equation (6.1.30) of \([A]\); and
- \( |\Gamma(1 + i x) \in (0, \pi^{1/2}) \), by Equation (6.1.31) of \([A]\).

**Lemma 2.5** (i.) Poles and zeros of \( G \). The function \( G \) has:

(a.): a simple pole at \( z = 0 \);
(b.): simple zeros at the points \( z = 1 + u_j \), with \( u_j := 2\pi j/\log(2) \), for all \( j \in \mathbb{Z} \setminus \{0\} \). These zeros are the only zeros of \( G \) located in \( \mathbb{C} \setminus D \).
(c.): simple poles at \( z = v_k = -2k - 1 \), for all \( k \in \mathbb{Z}^+ \). These poles and the pole 0 of (i.a.) are the only singularities of \( G \) in \( \mathbb{C} \).

(ii.) Zeros of \( G \) are zeros of \( \zeta \). A point \( z_0 \in D \) is a zero of multiplicity \( m \geq 1 \) of \( G \) if and only if \( z_0 \) is a zero of multiplicity \( m \geq 1 \) of \( \zeta \).

(iii.) Zeros of \( G \) and \( \zeta \) on the critical line. There are an infinite number of zeros of zeta and hence of \( G \) on the critical line: \( \{z \in D : \Re z = 1/2\} \).

**Proof.** Proof of Part (i.).

(a.) The pole of \( G \) at \( z = 0 \). By taking \( z = z_j = 1 + u_j \) where \( u_j := 2\pi j/\log(2) \) for \( j \in \mathbb{Z} \), and setting \( c_j := (4\pi)^{1/2+u_j} \Gamma(1 + u_j/2)/\Gamma((1 + u_j)/2) \), it follows from Lemma \( \text{2.4} \) that \( c_j \) is non–zero and bounded for all \( j \in \mathbb{Z} \). Hence by writing Equation \( \text{(2.2)} \) with \( z = z_j = 1 + u_j \) in the form
\((*)\) \hspace{1em} G(1 + u_j) = c_j \left(2^{1-u_j} - 1\right) G(-u_j),

\hspace{1em} \text{it follows from Lemma 2.4 that } G(1+u_0) = G(1) = \int_{\mathbb{R}^+} (1+e^y)^{-1} dy = \log(2) > 0. \text{ In addition, } z_j = 1 + u_j \in \Omega^+, \text{ where } G \text{ is analytic and bounded, and where } 1 + u_0 \text{ is a simple zero of the equation } 2^{1-z} - 1. \text{ Hence } z_0 = 1 + u_0 \text{ is a simple pole of } G(1-z).

(b.) The zeros \(z_j = 1 + u_j\), of \(G\), for \(j \in \mathbb{Z} \setminus \{0\}\). Here, we have by Lemma 2.4 with \(u_j = 2\pi i j / \log(2)\), for all \(j \in \mathbb{Z}\), that:

\hspace{1em} \bullet \hspace{1em} \Gamma(1 - z_j/2) = \Gamma(1/2 - u_j/2) \neq 0, \text{ by Lemma 2.4}
\hspace{1em} \bullet \hspace{1em} \Gamma((1 + z_j)/2) = \Gamma(1 + u_j/2) \neq 0, \text{ by Lemma 2.4} \text{ and}
\hspace{1em} \bullet \hspace{1em} 2^{z_j} - 1 = 1 \text{ for all } j \in \mathbb{Z}.

It is well known [C] that \(\zeta\) is non–zero on the line \(\{z \in \mathbb{C} : \Re z = 1\}\), a result that is equivalent to the prime number theorem. Hence \(\zeta(1+u_j)\) is bounded for all \(j \in \mathbb{Z} \setminus \{0\}\), and \(w_j := \Gamma(1 + u_j) \zeta(1 + u_j)\), is finite and non–zero, by Lemma 2.4. Since \(G(1 + u_j) = w_j \zeta(1 + u_j)\), we must have \(G(1 + u_j) = 0\) for all \(j \in \mathbb{Z} \setminus \{0\}\). Moreover, since all these zeros \(1 + u_j\) are simple zeros of \(2^{1-z} - 1\), these same zeros are also simple zeros of \(G\).

(c.) Poles of \(G\) at \(v_k = 1 + 2k\), \(k \in \mathbb{Z}^+\). Equation (2.2) has simple poles at \(z = v_k := -2k - 1\), for all \(k \in \mathbb{Z}^+\), whereas all other functions (except \(G\)) appearing in Equation (2.2) are non–zero and bounded for all such \(v_k\). It thus follows that \(G\) must have simple poles at \(v_k\), for all \(k \in \mathbb{Z}^+\).

**Proof of Part (ii.)** That \(G\) and the zeta function \(\zeta\) have exactly the same zeros in \(D\) follows, since the factor \((2^{1-z} - 1) \Gamma(z)\) of the third equation of (1.2) relating \(G\) and \(\zeta\), is analytic and non–vanishing in \(D\), where \(D \subset \Omega^+\). This result was also obtained in [V].

**Proof of Part (iii.)** That \(G\) has an infinite number of zeros on the critical line \(\{z \in D : \Re z = 1/2\}\) was proved by Hardy in [H].

The next lemma describes some asymptotic behaviors of the function \(\kappa\) defined in (1.4).
Lemma 2.6 For any $\varepsilon \in (0, \sigma)$ and for $x$ real, we have

$$\kappa(\sigma, x) = \begin{cases} O\left(e^{(\sigma-\varepsilon)x}\right), & x \to -\infty, \\ O\left(\exp\left(\sigma x - e^{x(1-\varepsilon)}\right)\right), & x \to \infty. \end{cases}$$

(2.4)

Hence the integral $\int_{\mathbb{R}} Q(x) \kappa(\sigma, x) \, dx$ is finite for any polynomial $Q$.

Proof. The asymptotic behavior of $\kappa$ given in (2.4) follows by inspection of the function $\kappa$ as defined in (1.4).

Definition 2.7 Let $\mathbb{R}^+ := \{x \in \mathbb{R} : x > 0\}$, let $G$ and $\kappa$ be defined as in Equations (1.3) and (1.4), and let us define $\kappa^\pm(\sigma, x)$ and $G(\sigma + it)$ as follows:

$$\kappa^\pm(\sigma, x) := \begin{cases} \kappa(\sigma, \mp x), & x \in \mathbb{R}^+, \\ 0, & x \in \mathbb{R}^-. \end{cases}$$

$$G(\sigma + it) := \int_{\mathbb{R}^+} e^{-ixt} \kappa^-(\sigma, x) \, dx + \int_{\mathbb{R}^+} e^{ixt} \kappa^+(\sigma, x) \, dx.$$  

(2.5)

Let us also define $G^{[m]}$ for any non-negative integer $m$ by $G^{[m]}(\sigma + it) = \left(\frac{\partial}{\partial \sigma}\right)^m G(\sigma + it)$, so that by the Cauchy–Riemann equations, $G^{(m)}(\sigma + it) = (-i)^m G^{[m]}(\sigma + it)$, where these functions are readily shown to exist by Lemma 2.6.

Lemma 2.8 Let the $G^{(m)}$ be defined as in Definition 2.7. Then, for all $m = 0, 1, 2, \ldots$, and for all $\sigma \in (0, 1)$, $G^{(m)}$ is analytic on the right half plane, and hence also in $D$. In particular given any $\varepsilon > 0$, $G^{(m)}(z)$ is uniformly bounded in the region $\{z \in \overline{D^+} : \Re z \geq \varepsilon\}$.

Proof. This result follows directly by use of the results of Lemmas 2.5 and 2.6. We omit the trivial details of the proof.

Definition 2.7 enables us to get the real and imaginary parts of the derivatives. If for brevity, we write $\kappa^\pm$ for $\kappa^\pm(\sigma, x)$, $C$ and $S$ for $\cos(x t)$ and $\sin(x t)$, and $\int$ for $\int_{\mathbb{R}^+}$, we get:
Lemma 2.9 Let $G^{(m)}$ and $\kappa^\pm$ be defined as in Equations (2.5) and (2.6). Then the following identities are valid for all non-negative integers $m$:

$$G^{(2m)}(\sigma + it) = (-1)^m \int x^{2m} \left( (\kappa^- + \kappa^+) C - i (\kappa^- - \kappa^+) S \right) dx,$$

and

$$G^{(2m+1)}(\sigma + it) = (-1)^{m+1} \int x^{2m+1} \left( (\kappa^- + \kappa^+) S + i (\kappa^- - \kappa^+) C \right) dx.$$

Hence we also get $G^{[m]} = i^m G^{(m)}$.

The following lemma will be used to restrict the domain of some of our inequalities:

Lemma 2.10 Let $\Delta$ be defined by $\Delta(\sigma, x) := \kappa^-(\sigma, x) - \kappa^+(\sigma, x)$, where $\kappa^\pm$ are defined as in Definition 2.7. Then $\Delta(\sigma, x) > 0$ for all $(\sigma, x) \in (0, 1/2] \times \mathbb{R}^+$, and moreover, for any fixed $x \in \mathbb{R}^+$, $\Delta(\sigma, x)$ is a strictly decreasing function of $\sigma \in (0, 1/2]$.

Proof. Let us assume that $\sigma \in (0, 1/2]$. By making the one-to-one transformation $v = e^{-x/2}$ of $\mathbb{R}^+ \to (0, 1)$ in the explicit expressions for $\kappa^\pm$, of Equation (2.5), we find, that if $\sigma \in (0, 1/2)$, then $\Delta(\sigma, x) > \Delta(1/2, x)$, so that it suffices to prove that $\Delta(1/2, x) > 0$. Also, it is readily evident that $\Delta(1/2, x) > 0$ for $x \in \mathbb{R}^+$ if and only if $\delta(y) := y (1 + e^{1/y^2}) - 1/y (1 + e^{y^2}) > 0$ for all $y \in (0, 1)$. Thus, we have, by use of Taylor expansions, that

$$\delta(y) = y (1 + e^{1/y^2}) - (1/y) (1 + e^{y^2}) = (1/2)(1/y - y)(1/y^2 + 1 + y^2 - 2) + \sum_{n=3}^{\infty} \frac{1/y^{2n-1} - y^{2n-1}}{n!} > 0, \quad y \in (0, 1).$$

By way of proceeding from the first to the second line of Equation (2.7) we used the following relations, which are valid for all $y \in (0, 1)$ and for all $\{ (\sigma, y) \in (0, 1/2] \times (0, 1) : 0 < y < 1 < 1/y \}$, and also, that $y + 1/y > 2$. The following lemma will be used to restrict the domain of some of our inequalities:
for all \( y \in (0, 1) \). It thus follows by inspection of the last line of Equation (2.7), that \( \delta(y) > 0 \) for all \( y \in (0, 1) \), i.e., that \( \Delta(\sigma, x) > 0 \) for all \( (\sigma, x) \in (0, 1/2] \times \mathbb{R}^+ \). □

**Theorem 2.11** Let \( m \) denote a non-negative integer. Then:

(i.) \((-1)^m \Re G^{(2m)}(\sigma) = \Re G^{[2m]}(\sigma) > 0 \) for all \( \sigma \in (0, 1) \).

(ii.) \((-1)^m \Im G^{(2m+1)}(\sigma) = \Re G^{[2m+1]}(\sigma) < 0 \) for all \( \sigma \in (0, 1/2] \).

(iii.) \( \Im G^{(2m)}(\sigma) = \Im G^{[2m]}(\sigma) = \Im G^{[2m+1]}(\sigma) = \Re G^{(2m+1)}(\sigma) = 0 \) for all \( \sigma \in (0, 1) \).

**Proof.**

(i.) This Item follows by inspection of Equation (2.6);

(ii.) This Item follows by Lemma 2.10 and by inspection of Equation (2.6); and

(iii.) This item follows by inspection of Equation (2.6).

□

3 Schwarz Reflection and Rolle’s theorem

In this section we present the Schwarz reflection principle, and Rolle’s theorem.

We first present the Schwarz reflection principle. We define this principle as follows:

**Definition 3.1** Let \( z^- \) and \( z^+ \) denote two points of \( D \), and let \( \ell[z^-, z^+] \) denote the closed line segment \( \{ z \in D : z = (1-s)z_1 + s z_2, \ 0 \leq s \leq 1 \} \), and similarly for the definitions of half-open, or open line segments connecting the two points \( z^\pm \). Let \( f \) be analytic in \( D \), and real on \( \ell(0, 1) \). Then \( f \) can be continued analytically (i.e., reflected) across \( \ell(0, 1) \) from \( D^\pm \) to \( D^\mp \) by means of the formula

\[
f(\overline{z}) = \overline{f(z)}.
\]

(3.1)
Rolle’s theorem is presented in most texts for obtaining an expression of the error of polynomial interpolation. In the lemma which follows, we present a version of Rolle’s theorem which is suitable for our proof in §4.

**Lemma 3.2 (Rolle’s theorem)** Let $f$ denote either one of the functions $\Re G$ or $\Im G$.

a. Let $z \in \ell[z^+, z^-]$, and let $f(z)$ vanish with multiplicity $m > 1$. Then $f^{(m-1)}(z)$ vanishes with multiplicity one;

b. Let $f(z^\mp)$ vanish with multiplicity $\mu^\mp > 0$, and let resp., $f^{(\mp-1)}(z^\mp) > 0$ (resp., $f^{(\mp-1)}(z^\mp) < 0$. Then $f$ has resp., an odd number (resp., an even number) of sign changes on $\ell(z^-, z^+)$, so that $f$ vanishes on $\ell[z^-, z^+]$ with total multiplicity $\mu^- + \mu^+ + 1 + 2k$ (resp., $\mu^- + \mu^+ + 2k$) where $k$ denotes a non-negative integer.

**Proof.** The proof of the Part (a.) of this lemma follows by inspection. For the proof of Part (b.) it is easily seen, since the expressions $f^{(\mu^\mp)}(z^\mp)$ are non-zero, that the total multiplicity of the vanishing of $f$ on $\ell[z^-, z^+]$ is resp., $\mu^- + \mu^+ + 1 + 2k$ if $f^{(\mu^-)}(z^-) f^{(\mu^+)}(z^+) > 0$ (resp., $\mu^- + \mu^+ + 2k$ if $f^{(\mu^-)}(z^-) f^{(\mu^+)}(z^+) < 0$).

Multiplicity counts the number of zeros of a function $f$, without regard to signs of its derivative, and it may thus not be as accurate as we want. However, a more accurate count of the number of zeros of $f$ is at times obtainable, if $f$ is one of the functions $\Re G$ or $\Im G$ and if for $m$ a non-negative integer, we take note that $\Re G^{(2m)}(\sigma + it)$ and $\Im G^{(2m+1)}(\sigma + it)$ are even functions of $t$, while $\Im G^{(2m)}(\sigma + it)$ and $\Re G^{(2m+1)}(\sigma + it)$ are odd functions of $t$.

We next present the Schwarz reflection principle. We define this principle as follows:

**Definition 3.3** Let $f$ be analytic in $D^+$, and real on $\ell(0, 1)$. Then $f$ can be continued analytically (i.e., reflected) across $\ell(0, 1)$ from $D^+$ to $D^\pm$ by means of the formula

$$f(\overline{z}) = \overline{f(z)}. \quad (3.2)$$
4 Proof of the Riemann Hypothesis

In this section, we shall prove the following theorem:

**Theorem 4.1** Every zero \( z = \sigma + it \) of \( G \) in \( D \) is a simple zero of \( G \) of the form \( z = 1/2 + it \), for \( t \in \mathbb{R} \setminus \{0\} \).

**Proof.** We shall prove this theorem by means of the proofs of three lemmas.

**Lemma 4.2** Every zero \( z_0 = \sigma + it \) of \( G \) in \( D \), is a zero of finite multiplicity.

**Proof.** Suppose that there exists a cluster of zeros of \( G \) in \( D \), with limit point in the closure of \( D \). If the limit of a convergent sub-sequence of such points is in the open domain \( D \), then, since \( G \) is analytic and bounded in \( D \), \( G \) would have to vanish in all of \( \mathbb{C} \), by Vitali’s theorem. On the other hand, the limit of a convergent sub-sequence of such a sequence cannot be on the boundary of \( D \), by Lemma 2.5. □

**Lemma 4.3** Let \( z_1 = \sigma_1 + it_1 \in D \), with \( 0 < \sigma_1 < 1/2 \). Then \( G(z_1) \) cannot vanish with multiplicity \( m \) for any positive integer \( m \).

**Proof.** (i.) The functions \( K \), \( G^{(m)} \), and relevant zeros of \( G \). The functional equation of Lemma 2.2 enables us to write \( G(z) = K(z) G(1-z) \).

A more detailed expression for \( K \) and the \( m \)th derivative of the identity \( G(z) = K(z) G(1-z) \) are given by

\[
K(z) = (4\pi)^{1/2-z} \frac{2^{1-z} - 1}{2^z - 1} \frac{\Gamma(1 - (1-z)/2)}{\Gamma(1 - z/2)},
\]

\[
G^{(m)}(z) = \sum_{k=0}^{m} \binom{m}{j} (-1)^j K^{[m-j]}(z) G^{[j]}(1-z).
\]

(4.1)

In our elimination of non-zeros of \( G \) on Parts (ii.) and (iii.) of our proof which follows, we shall allow the possibilities of existence of zeros of \( G \) and \( G^{(m)} \) of the form \( z_1 = \sigma + it \) at \( t = t_1 \) for several values of \( \sigma \). To eliminate these, we shall only concern ourselves with those zeros for which
0 < \sigma < 1/2$, and in addition, we shall also want to concern ourselves with zeros $z = \sigma + it_1$ for more than one value $\sigma$. For such cases, we shall only provide a proof for largest of these values $\sigma \in (0, 1/2)$, since once we have eliminated the largest, the second largest becomes the largest, and since there can only be a finite number of zeros of $G$ with the same value $t$, by Lemma 4.2. In addition, we shall systematically search for a zero $z = \sigma + it$ of $G$ in $D^+$, of multiplicity $m > 0$, starting at $t = 0$ and with increasing $t$, and with $\sigma \in (0, 1/2)$, such that, having reached $t = t_1$ we have eliminated the existence of all zeros of $G$ of multiplicity $m$ for all $z \in D^+$ with $0 < \Im z < t_1$. The multiplicity of all such eliminated zeros is an even number, $2k'$, so that upon arriving at a new zero, $z_1 = \sigma_1 + it_1$, with $\sigma_1 \in (0, 1/2)$, there cannot exist any other zeros of $G$ of multiplicity $m$ on the line segment $\ell(z_1, \overline{z_1})$.

(ii.) The vanishing of $\Im G(z_1)$ with odd multiplicity $m = 2n + 1$, $n \geq 0$. If $\Im G(z_1) = \Im G(\sigma_1 + it_1)$ vanishes with odd multiplicity $2n + 1$, with $n$ a non-negative integer, then $\Im G(\overline{z_1})$ vanishes with odd multiplicity $m = 2n + 1$, by reflection, and moreover $\Im G(\sigma_1)$ then vanishes with multiplicity one, by inspection of Equation (2.6). Hence $\Im G$ vanishes on $\ell[z_1, \sigma_1]$ with total multiplicity $2n + 1$, where $k$ is a non-negative integer. Hence by Rolle’s theorem, we get $\Im G(z_{2n+1+2k}) = 0$, with multiplicity one, which is not allowed, by Theorem 2.11. Hence $\Im G(z_1)$, i.e., $G(z_1)$, cannot vanish with odd multiplicity $m = 2n + 1$, where $n$ denotes a non-negative integer.

(iii.) The vanishing of $\Im G(z_1)$ with even multiplicity $m = 2n$, $n \geq 1$. If $\Im G(z_1)$ vanishes with even multiplicity $2n$, with $n$ a positive integer, then $\Im G(\overline{z_1})$ vanishes with even multiplicity $2n$, with $n$ a positive integer, by reflection, so that $\Im G$ vanishes on $\ell[z_1, \overline{z_1}]$ with total multiplicity $4n + 2k$, with $n \geq 1$, and $k \geq 0$ denoting integers. Hence by Rolle’s theorem, $\Im G^{(4n+2k)}(\sigma_1) = 0$, with multiplicity one. However, this is not allowed, by Theorem 2.11. Hence the vanishing of $\Im G(z_1)$, i.e., that of $G(z_1)$, with even multiplicity $m = 2n$, $n \geq 1$ is not allowed.

We have thus proved in (ii.) and (iii.) above, that $G = \Re G + i \Im G$ evaluated at $z_1$ cannot vanish with multiplicity $m$ for any positive integer $m \geq 1$ and for any $z_1 = \sigma_1 + it_1 \in D$, with $\sigma_1 \in (0, 1/2)$.

**Lemma 4.4** If $G(z_0) = 0$ with $z_0 = 1/2 + it_0 \in D$, then $z_0$ is a simple zero of $G$.

**Proof.** Let $z_0 = 1/2 + it_0$. 

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(a.1) If $\Re G(z_0)$ vanishes with multiplicity $2n+1$, with $n = 0$, then $\Re G(\overline{z_0})$ vanishes with multiplicity $2n+1$, with $n = 0$, by reflection. Since $\Re G(1/2 + it)$ is an even function of $t$, and since $\Re G(1/2) > 0$, by Theorem 2.11 it follows that the total multiplicity of the vanishing of $\Re G$ on $\ell(z_0, \overline{z_0})$ is $2 + 2k$, where $k$ denotes a non–negative integer. Hence by Rolle’s theorem, $\Re G^{(1+2k)}(1/2) = 0$, with multiplicity one, which is allowed, by Theorem 2.11.

(a.2) If $\Im G(z_0)$ vanishes with multiplicity one, then, since $\Im G(\sigma_1 + it)$ is an odd function of $t$, and since $\Im G(1/2) = 0$, with multiplicity one, by inspection of Equation (2.6), it follows that $\Im G$ vanishes on $\ell[1/2, z_0]$, with total multiplicity $3 + 2k$, where $k$ denotes a non–negative integer. Hence by Rolle’s theorem, $\Im G^{(2+2k)}(1/2) = 0$, which is also allowed, by Theorem 2.11.

(b.) If $\Im G(z_0)$ vanishes with even multiplicity $2n$ with $n \geq 1$, then by reflection, $\Im G(\overline{z_0})$ vanishes with multiplicity $2n$, $n \geq 1$, and in addition, $\Im G(1/2) = 0$, with multiplicity one. Also, $\Im G(1/2 + it)$ is an odd function of $t$, by inspection of Equation (2.6). Thus $\Im G$ vanishes on $\ell[1/2, z_0]$ with total multiplicity $2n+1+2k\geq 1$, and with $k \geq 0$. Hence by Rolle’s theorem, there exists $z_3 = 1/2 + i t_3 \in \ell[1/2, z_0]$, with $t_3 \in (0, t_0)$, such that $\Im G^{(2n+2k)}(z_3)$ vanishes with multiplicity one. Hence by reflection, we get $\Im G^{(2n+2k)}(\overline{z_3}) = 0$, with multiplicity one, so that the total multiplicity of the vanishing of $\Im G^{(2n+2k)}$ on $\ell[z_3, \overline{z_3}]$ is two. Hence by Rolle’s theorem, we get $\Im G^{(2n+1+2k)}(1/2) = 0$, with multiplicity one, but which is not allowed, by Theorem 2.11. Hence the vanishing of $\Im G(z_0)$, i.e., the vanishing of $G(z_0) – \text{with multiplicity } 2n$ and with $n \geq 1$ is not allowed.

(c.) If $\Im G(z_0)$ vanishes with odd multiplicity $2n + 1$, i.e., with $n \geq 1$, then $\Im G(\overline{z_0})$ vanishes with multiplicity $2n + 1$, with $n \geq 1$, and moreover, $\Im G(1/2) = 0$, by Theorem 2.11 which contributes an additional multiplicity of $1 + 2k$, where $k$ denotes a non–negative integer. Hence $\Im G$ vanishes on $\ell[z_0, 1/2]$ with total multiplicity $2n+2+2k$, $n \geq 1$. Thus by Rolle’s theorem, there exists $z_3 = 1/2 + i t_3$, with $t_3 \in (0, t_0)$, such that $\Im G(z_3) = 0$, with multiplicity $2n+1+2k$. By reflection, we thus get $\Im G(\overline{z_3}) = 0$, with multiplicity $2n+1+2k$. Hence $\Im G$ then vanishes on $\ell[z_3, \overline{z_3}]$ with total multiplicity $4n+2+4k$. So that by Rolle’s theorem, $\Im G^{(4n+1+k)}(1/2) = 0$, with multiplicity one. However, this is not allowed, by theorem 2.11. Thus $\Im G(z_0)$, i.e., $G(z_0)$, cannot vanish with multiplicity $2n+1$, $n \geq 1$.

Hence by (a.1) and (a.2), simple zeros $z = 1/2 + it_1$ of $G$ are allowed, whereas by (b.) and (c.) zeros $z = 1/2 + it$ of $G$ of multiplicity greater than
One are not allowed.
This completes the proof of Lemma 4.4 ■

This completes the proof of Theorem 4.1 ■

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