Non-zero Degree Maps between 3-Manifolds*

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Abstract

First the title could be also understood as “3-manifolds related by non-zero degree maps” or “Degrees of maps between 3-manifolds” for some aspects in this survey talk.

The topology of surfaces was completely understood at the end of 19-th century, but maps between surfaces kept to be an active topic in the 20-th century and many important results just appeared in the last 25 years. The topology of 3-manifolds was well-understood only in the later 20-th century, and the topic of non-zero degree maps between 3-manifolds becomes active only rather recently.

We will survey questions and results in the topic indicated by the title, present its relations to 3-manifold topology and its applications to problems in geometry group theory, fixed point theory and dynamics.

There are four aspects addressed: (1) Results concerning the existence and finiteness about the maps of non-zero degree (in particular of degree one) between 3-manifolds and their suitable correspondence about epimorphisms on knot groups and 3-manifold groups. (2) A measurement of the topological complexity on 3-manifolds and knots given by “degree one map partial order”, and the interactions between the studies of non-zero degree map among 3-manifolds and of topology of 3-manifolds. (3) The standard forms of non-zero degree maps and automorphisms on 3-manifolds and applications to minimizing the fixed points in the isotopy class. (4) The uniqueness of the covering degrees between 3-manifolds and the uniqueness embedding indices (in particular the co-Hopfian property) between Kleinian groups.

The methods used are varied, and we try to describe them briefly.

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0. Introduction

The topology of surfaces was completely understood by the end of 19th century, but maps between surfaces keep to be an active topic in the 20th century, and some basic results just appeared in the last 25 years, among which are the Nielsen-Thurston classification of surface automorphisms [Th3], and Edmonds’ standard form for surface maps [E1]. Then fine results followed, say the realization of Nielsen number in the isotopy class of surface automorphisms by Jiang [J], and the simple loop theorem for surface maps by Gabai [Ga2].

The topology of 3-manifolds was well-understood only in the later 20th century due to many people’s deep results, in particular Thurston’s great contribution to the geometrization of 3-manifolds, and the topic of non-zero degree maps between 3-manifolds becomes active only rather recently.

We will survey the results and questions in the topic indicated by the title, present its relations to 3-manifold topology and its applications to problems in geometry group theory, fixed point theory and dynamics. The methods used are varied, and we try to describe them briefly.

For standard terminologies of 3-manifolds and knots, see the famous books of J. Hempel, W. Jaco and D. Rolfsen. For a proper map \( f : M \to N \) between oriented compact 3-manifolds, \( \text{deg}(f) \), the degree of \( f \), is defined in most books of algebraic topology. A closed orientable 3-manifold is said to be geometric if it admits one of the following geometries: \( H^3 \) (hyperbolic), \( \widetilde{PSL}_2 \mathbb{R} \), \( H^2 \times E^1 \), \( \text{Sol} \), \( \text{Nil} \), \( E^3 \) (Euclidean), \( S^2 \times E^1 \), and \( S^3 \) (spherical). A compact orientable 3-manifold \( M \) admits a geometric decomposition if each prime factor of \( M \) is either geometric or Haken. Thurston’s geometrization conjecture asserts that any closed orientable 3-manifold admits a geometric decomposition. Each Haken manifold \( M \) with \( \partial M \) a (possibly empty) union of tori has a Jaco-Shalen-Johannson (JSJ) torus decomposition, that is, it contains a minimal set of tori, unique up to isotopy, cutting \( M \) into pieces such that each piece is either a Seifert manifold or a simple manifold, which admits a complete hyperbolic structure with finite volume [Th2].

In the remainder of the paper, all manifolds are assumed to be compact and orientable, all automorphisms are orientation preserving, all knots are in \( S^3 \), all Kleinian groups are classical, and all maps are proper, unless otherwise specified.

Let \( M \) and \( N \) be 3-manifolds and \( d \geq 0 \) an integer. We say that \( M \) \( d \)-dominates (or simply dominates) \( N \) if there is a map \( f : M \to N \) of degree \( \pm d \). Denote by \( D(M, N) \) the set of all possible degrees of maps from \( M \) to \( N \). A 3-manifold \( M \) is small if each closed incompressible surface in \( M \) is boundary parallel.

Due to space limitation, quoted literature are only partly listed in the references; while the others are briefly indicated in the context.

1. Existence and finiteness

A fundamental question in this area (and in 3-manifold theory) is the following.
Question 1.1. Given a pair of closed 3-manifolds $M$ and $N$, can one decide if $M$ $d$-dominates $N$? In particular, can one decide if $M$ 1-dominates $N$?

The following two natural problems concerning finiteness can be considered as testing cases of Question 1.1.

Question 1.2 [Ki, Problem 3.100 (Y. Rong)]. Let $M$ be a closed 3-manifold. Does $M$ 1-dominate at most finitely many closed 3-manifolds?

Question 1.3. Let $N$ be a closed 3-manifold. When is $|D(M, N)|$ finite for any closed 3-manifold $M$?

An important progress towards the solution of Question 1.2 is the following:

**Theorem 1.1** ([So2], [WZh2], [HWZ3]). Any closed 3-manifold $1$-dominates at most finitely many geometric 3-manifolds.

Theorem 1.1 was proved by Soma when the target manifolds $N$ admit hyperbolic geometry [So2]. The proof is based on the argument of Thurston’s original approach to the deformation of acylindrical manifolds. Porti and Reznikov had a quick proof of Soma’s result, based on the volume of representations [Re2]. However Soma’s approach deserves attention as it proves that the topological types of all hyperbolic pieces in closed Haken manifolds 1-dominated by $M$ are finite [So3].

Theorem 1.1 was proved in [WZh2] when the target manifolds admit geometries of $H^2 \times E^1$, $\widetilde{PSL}_2(R)$, Sol or Nil. The proof for the case of $H^2 \times E^1$ geometry invokes Gabai’s result that embedded Thurston Norm and singular Thurston Norm are equal [Ga1], and the proof for case of $\widetilde{PSL}_2(R)$ geometry uses Brooks and Goldman’s work on Seifert Volume [BG]. Theorem 1.1 was proved in [HWZ3] when the target manifolds admit $S^3$ geometry, using the linking pairing of 3-manifolds. Note that only finitely many 3-manifolds admit the remaining two geometries.

For maps between 3-manifolds which are not necessarily orientable, there is a notion of geometric degree (See D. Epstein, Proc. London Math. Soc. 1969). It is worth mentioning that if $d$-dominating maps are defined in terms of geometric degree, then Rong constructs a non-orientable 3-manifold which 1-dominates infinitely many lens spaces [Ro3]. Actually there is a non-orientable hyperbolic 3-manifold which 1-dominates infinitely many hyperbolic 3-manifolds [BW1]. Such examples do not exist in dimension $n > 3$ due to Gromov’s work on simplicial volume and H.C. Wang’s theorem that, for any $V > 0$, there are at most finitely many closed hyperbolic $n$-manifolds of volume $< V$.

The answer to Question 1.2 is still unknown for closed irreducible 3-manifolds admitting geometric decomposition. The following result is related.

**Theorem 1.2** ([Ro1], [So3]). For any 3-manifold $M$ there exists an integer $N_M$, such that if $M = M_0 \rightarrow M_1 \rightarrow \ldots \rightarrow M_k$ is a sequence of degree one maps with $k > N_M$, and each $M_i$ admits a geometric decomposition, then the sequence contains a homotopy equivalence.

The situation for Question 1.3 can be summarized in the following theorem.
Theorem 1.3 ([Gr1], [BG], [W2]). Suppose $N$ is a closed 3-manifold admitting geometric decomposition. Then

(1) $|D(M, N)|$ is finite if either a prime factor of $N$ contains a hyperbolic piece in its JSJ decomposition, or $N$ itself admits the geometry of $\widetilde{\text{PSL}_2}(R)$.

(2) $|D(N, N)|$ is infinite if and only if either (i) $N$ is covered by a torus bundle over the circle or a surface $\times S^1$, or (ii) each prime factor of $N$ has a cyclic or finite fundamental group.

Part (1) of Theorem 1.3 follows from the work of Gromov [Gr1] and Brooks-Goldman [BG]. Part (2) can be found in [W2]. Note that if $|D(N, N)|$ is infinite and $D(M, N)$ contains non-zero integers, then $|D(M, N)|$ is also infinite. I suspect that Theorem 1.3 (2) indicates a general solution to Question 1.3.

There are many partial results for Question 1.1: When both $M$ and $N$ are Seifert manifolds with infinite fundamental groups Rong has an algorithm to determine if $M$ 1-dominates $N$ [Ro3]. When $N$ is the Poincare homology sphere and a Heegaard diagram of $M$ is given, Hayat-Legrand, Matveev and Zieschang have an algorithm to decide if $M$ d-dominates $N$ [HMZ]. There are simple answers to Question 1.1 in the following cases: (1) $M$ and $N$ are prism spaces and $d = 1$ [HWZ2]; (2) $M = N$ admit geometry of $S^3$ and $f_*$ an automorphism on $\pi_1$ [HKWZ]; (3) $N$ is a lens space. I will state (3) as a theorem, since both its statement and proof are short, and since it has rich connections with previous results and with different topics.

Theorem 1.4 ([HWZ1], [HWZ3]). A closed 3-manifold $M$ d-dominates the lens space $L(p, q)$ if and only if there is an element $\alpha$ in the torsion part of $H_1(M, \mathbb{Z})$ such that $\alpha \circ \alpha = \frac{dq}{p}$ in $\mathbb{Q}/\mathbb{Z}$, where $\alpha \circ \alpha$ is the self-linking number of $\alpha$.

A direct consequence of Theorem 1.4 is the known fact that $L(p, q)$ 1-dominates $L(m, n)$ if and only if $p = km$ and $n = kqc^2 \mod m$. This fact has at least four different proofs: using equivariant maps between spheres by de Rham (J. Math. 1931) and by Olum (Ann. of Math. 1953), using Whitehead torsion by Cohen (GTM 10, 1972), using pinch in [RoW] and using linking pair in [HWZ1].

Degree one maps from general 3-manifolds to some lens spaces, in particular the $\mathbb{R}P^3$, have been studied by Bredon-Wood (Invent. Math. 1969) and by Rubinstein (Pacific J. Math. 1976) to find one-sided incompressible surfaces, by Luft-Sjerve (Topo. Appl. 1990) to study cyclic group actions on 3-manifolds, by Shastri-Williams-Zvengrowski [SWZ] in theoretical physics, by Taylor (Topo. Appl. 1984) to define normal bordism classes of degree one maps, and by Kirby-Melvin (Invent. Math. 1991) to connect with new 3-manifold invariants.

Degree one maps induce epimorphisms on $\pi_1$. There are easy examples indicate that Question 1.2 does not have direct correspondence in the level of 3-manifold groups [BW1], [RWZh]. However the following related question was raised in 1970's.

Question 1.4 [Ki, Problem 1.12 (J. Simon)]. Conjectures:
(1) Given a knot group $G$, there is a number $N_G$ such that any sequence of epimorphisms of knot groups $G \to G_1 \to \ldots \to G_n$ with $n \geq N_G$ contains an isomorphism.

(2) Given a knot group $G$, there are only finitely many knot groups $H$ for which there is an epimorphism $G \to H$.

According to a conversation with Gonzalez-Acuna, who discussed Question 1.4 with Simon before it was posed, the epimorphisms in Question 1.4 are peripheral preserving in their minds.

**Theorem 1.5 ([So5], [RW]).** The conjecture in Question 1.4 (1) holds if the knot complements involved are small. The conjecture in Question 1.4 (2) holds if the knot complements are small and the epimorphisms are peripheral preserving.

The first claim is due to Soma [So5] and the second claim is in [RW]. Both of them invoke Culler-Shalen’s work on the representation varieties of knot groups. It is also proved that any infinite sequence of epimorphisms among 3-manifold groups contains an isomorphism if all manifolds are either hyperbolic [So5] or Seifert fibered [RWZh]. In [RWZh], the proof uses the fact that epimorphisms between aspherical Seifert manifolds with the same $\pi_1$ rank are realized by maps of non-zero degree. Both this fact and Question 1.4 (1) are variations of the Hopfian property.

We end this section by mention that there are results about $D(M, N)$ in [DW] for $(n-1)$-connected $2n$-manifolds, $n > 1$, which are quite explicit and of interest from both topological and number-theoretic point of view.

## 2. Uniqueness

The following question is raised in 1970’s.

**Question 2.1 ([Ki, Problem 3.16 (W. Thurston)].** Suppose a 3-manifold $M$ is not covered by (surface)$\times S^1$ or a torus bundle over $S^1$. Let $f, g : M \to N$ be two coverings, must $\text{deg}(f) = \text{deg}(g)$?

It is known [WWu2] that Question 2.1 has positive answer if $M$ admits geometric decomposition and is not a graph manifold ($M$ is a graph manifold if each piece of its JSJ decomposition is Seifert fibered.) For graph manifolds there are four different covering invariants introduced in middle 1990’s by [WWu2], Luecke and Wu [LWu], Neumann [N] and Reznikov [Re1]. Unfortunately all those four covering invariants are either vanishing or not well-defined for some non-trivial graph manifolds. It is also known that covering degree is uniquely determined if the graph manifold in the target is either a knot complement [LWu] or its corresponding graph is simple [WWu2, N]. The positive answer to Question 2.1 for graph manifolds was finally obtained in [YW], using the matrix invariant defined in [WWu2] and an elegant application of matrix theory due to Yu.

**Theorem 2.1 ([WWu2], [YW]).** For 3-manifolds admitting geometric decomposition and not covered by either (surface)$\times S^1$ or a torus bundle over $S^1$, covering degrees are uniquely determined by the manifolds involved.
It is worth mentioning an interesting fact that any knot complement is non-trivially covered by at most two knot complements and any knot complement non-trivially covers at most one knot complement. The first claim follows from the cyclic surgery theorem of Culler-Gordon-Luecke-Shalen and the positive answer to the Smith Conjecture. The second claim is in [WWu1].

Question 2.1 is equivalent to asking the uniqueness of indices of finite index embeddings between 3-manifold groups. Recently there are also some discussions on the uniqueness of indices of self-embeddings of groups. A group $G$ is said to be co-Hopf if each self-monomorphism of $G$ is an isomorphism.

**Question 2.2.** Let $G$ be either a 3-manifold group, or a Kleinian group, or a word hyperbolic group. When is $G$ co-Hopf?

The cohopficity of groups were first considered by Baer (Bull. AMS 1944). For word hyperbolic groups it was first considered by Gromov in 1987 [Gr2, p.157], and subsequently by Rips-Sela (GAFA, 1994), Sela [Se], and Kapovich-Wise (Israel J. Math. 2001). Cohopficity of 3-manifold groups was first studied in 1989 by Gonzalez-Acuna and Whitten [GWh], and then in [WWu2] and [PW]. The answer for 3-manifolds admitting geometric decomposition with boundary either empty set or a union of tori is known [GWh], [WWu2], and partial results for 3-manifolds with boundary of high genus surfaces are in [PW]. Cohopficity of Kleinian groups was first considered in 1992 in an early version of [PW], then in 1994 in an early version of [WZh1], also by Ohshika-Potyagailo (Ann. Sci. Ecole Norm. Sup. 1998) and Delzant-Potyagailo (MPI Preprint, 2000) for high dimensional Kleinian groups.

**Theorem 2.2.** Suppose $K$ is a non-elementary, freely indecomposable, geometrically finite Kleinian group and $K$ contains no $\mathbb{Z} \oplus \mathbb{Z}$ subgroup. Then

1. [Se], [PW], [WZh1] $K$ is co-Hopf if $K$ is a group of one end.
2. [WZh1] If the singular locus of the hyperbolic 3-orbifold $H^3/K$ is a 1-manifold, then $K$ is co-Hopf if and only if no circle component of singular locus meets a minimal splitting system of hyperbolic cone planes.

The proof of Theorem 2.2 (1) in [WZh1], influenced by that of torsion free case in [PW], use a generalization of Thurston-Gromov’s finiteness theorem on the conjugacy classes of group embeddings (Delzant, Comm. Math. Helv. 1995) and a proper conjugation theorem of Kleinian groups (Wang-Zhou, Geometriae Dedicata, 1995). Theorem 2.2 (2) is proved by using 3-dimensional hyperbolic orbifold structures and orbifold maps, which turn out to be useful geometric tools.

Note that groups in Theorem 2.2 are word hyperbolic groups. According to Sela ([Se] and his MSRI preprints in 1994), people once expected that a non-elementary word hyperbolic group is co-Hopf if and only if it has one end. Sela proved this expectation for the torsion free case [Se]. Theorem 2.2 (2) and examples in [WZh1] show that cohopficity phenomenon is very complicated in the torsion case. In particular there are co-Hopf word hyperbolic groups which have infinitely many ends.
Inspired by Questions 2.1 and 2.2 it is natural to ask

**Question 2.3.** Are the indices (including the infinity) of embeddings \( H \to G \) between co-Hopf groups unique?

### 3. Interactions with 3-manifold topology

Degree one maps define a partial order on Haken manifolds and hyperbolic 3-manifolds. By Gordon-Luecke’s theorem knots are determined by their complements [GL]. We say that a knot \( K \) 1-dominates a knot \( K' \) if the complement of \( K \) 1-dominates the complement of \( K' \). 1-domination among knots also gives a partial order on knots. This partial order seems to provide a good measurement of complexity of 3-manifolds and knots. The reactions of non-zero degree maps between 3-manifolds and 3-manifold topology are reflected in the following very flexible

**Question 3.1.** Suppose \( M \) and \( N \) are 3-manifolds (knots) and \( M \) 1-dominates \( (d\)-dominates) \( N \).

1. Is \( \sigma(M) \) not “smaller” than \( \sigma(N) \) for a topological invariant \( \sigma \)?
2. If \( M \) and \( N \) are quite “close”, are they homeomorphic? do they admit the same topological structure?

Positive answers to Question 3.1 (1) are known in many cases. Suppose \( M \) 1-dominates \( N \). Then \( \sigma(M) \geq \sigma(N) \) when \( \sigma \) is either the rank of \( \pi_1 \), or Gromov’s simplicial volume, or Haken number (of incompressible surfaces), or genus of knots; \( \sigma(N) \) is a direct summand of \( \sigma(M) \) when \( \sigma \) is the homology group, and \( \sigma(N) \) is a factor of \( \sigma(M) \) if \( \sigma \) is the Alexander polynomial of knots. The answer to Question 3.1 (1) is still unknown for many invariants of knots and 3-manifolds, for example crossing number, unknotting number, Jones polynomial, knot energy, and tunnel number, etc. Li and Rubinstein are specially interested in Question 3.1 (1) for Casson invariant in order to prove it is a homotopy invariant [LRu].

There are both positive and negative answers to Question 3.1 (2), depending on the interpretation of the problem. On the negative side, Kawauchii has constructed, using the imitation method invented by himself, degree one maps between non-homeomorphic 3-manifolds \( M \) and \( N \) with many topological invariants identical, see his survey paper [Ka]. On the positive side, there are many results. An easy one is that if \( M \) \( d\)-dominates \( N \) and both \( M \) and \( N \) are aspherical Seifert manifolds, then the Euler number of \( M \) is zero if and only if that of \( N \) is zero [W1]. A deeper result is Gromov-Thurston’s Rigidity theorem, which says that a degree one map between hyperbolic 3-manifolds of the same volume is homotopic to an isometry [Th2]. The following are some recent results in this direction.

**Theorem 3.1** ([So4], [So1]). (1) For any \( V > 0 \), suppose \( f : M \to N \) is a degree one map between closed hyperbolic 3-manifolds with \( \text{Vol}(M) < V \). Then there is a constant \( c = c(V) \) such that \( (1 - c) \text{Vol}(M) \leq \text{Vol}(N) \) implies that \( f \) is homotopic to an isometry.
(2) If $M \to N$ is a map of degree $d$ between Haken manifolds such that $||M|| = d||N||$, then $f$ can be homotoped to send $H(M)$ to $H(N)$ by a covering, where $||*||$ is the Gromov norm and $H(*)$ is the hyperbolic part under the JSJ decomposition.

**Theorem 3.2 ([BW1], [BW2]).** (1) Let $M$ and $N$ be two closed irreducible 3-manifolds with the same first Betti number and suppose $M$ is a surface bundle. If $f : M \to N$ is a map of degree $d$, then $N$ is also a surface bundle.

(2) Let $M$ and $N$ be two closed, small hyperbolic 3-manifolds. If there is a degree one map $f : M \to N$ which is a homeomorphism outside a submanifold $H \subset N$ of genus smaller than that of $N$, then $M$ and $N$ are homeomorphic.

(1) and (2) of Theorem 3.1 provide a stronger version and a generalization of Gromov-Thurston’s Rigidity theorem, respectively. In respect of Theorems 3.2, the following examples should be mentioned: There are degree one maps between two non-homeomorphic hyperbolic surface bundles with the same first Betti number and between two non-homeomorphic small hyperbolic 3-manifolds [BW2]. The constructions of those maps are quite non-trivial. There are many applications of Theorem 3.2. We list two of them which are applications of Theorem 3.2 to Thurston’s surface bundle conjecture and to Dehn surgery respectively, where degree one maps constructed by surgery on null-homotopic knots are involved.

**Theorem 3.3 ([BW1], [BW2]).** (1) There are closed hyperbolic 3-manifolds $M$ such that any tower of abelian covering of $M$ contains no surface bundle.

(2) Suppose $M$ is a small hyperbolic 3-manifold and that $k \subset M$ is a null-homotopic knot, which is not in a 3-ball. If the unknotting number of $k$ is smaller than the Heegaard genus of $M$, then every closed 3-manifold obtained by a non-trivial Dehn surgery on $k$ contains an incompressible surface.

### 4. Standard forms

**Question 4.1.** What are standard forms of non-zero degree maps and of automorphisms of 3-manifolds?

Sample answers to analogs of Question 4.1 in dimension 2 are that each map of non-zero degree between closed surfaces is homotopic to a pinch followed by a branched covering [E1], and each automorphism on surfaces can be isotoped to a map which is either pseudo Anosov (Anosov), or periodic, or reducible [Th3].

**Theorem 4.1 (Haken, Waldhausen, [E2], [Ro2]).** (1) A degree one map between closed 3-manifolds is homotopic to a pinch.

(2) A map of degree at least three between closed 3-manifolds is homotopic to a branched covering.

(3) A non-zero degree map between Seifert manifolds with infinite $\pi_1$ is homotopic to a fiber preserving pinch followed by a fiber preserving branched covering.

(1) is proved by Haken (Illinois J. Math. 1966), also by Waldhausen, and a quick proof using differential topology is in [RoW]. (2) is proved by Edmonds [E2]
quickly after Hilden-Montesinos’s result that each 3-manifold is a 3-fold branched covering of 3-sphere. (3) is due to Rong [Ro2], which invokes [E1]. According to conversations with D. Gabai and with M. Freedman, people are still wondering if each map of degree 2 between closed 3-manifolds is homotopic to a pinch followed by a double branched covering.

For non-prime 3-manifolds, Cesar de Sa and Rourke claim that every automorphism is a composition of those preserving and permuting prime factors (Bull. AMS, 1979), and those so-called sliding maps. A proof is given by Hendricks and Laudenbach [HL], and by McCullough [Mc].

Standard forms of automorphisms on prime 3-manifolds admitting geometric decomposition have been studied in [JWW]. The orbifold version of Nielsen-Thurston’s classification of surface automorphisms is established, i.e., each orbifold automorphism is orbifold-isotopic to a map which is either (pseudo) Anosov, or periodic, or reducible. We then have the following theorem.

**Theorem 4.2 ([JWW])**. Let $M$ be a closed prime 3-manifold admitting geometric decomposition. Let $f : M \to M$ be an automorphism. Let $T$ be the product neighborhood of the JSJ tori. Then

1. $f$ is isotopic to an affine map if $M$ is a 3-torus.
2. $f$ is isotopic to an isometry if $M$ is the Euclidean manifold having a Seifert fibration over $\mathbb{RP}^2$ with two singular points of index 2.
3. $f$ is isotopic to a map which preserves the torus bundle structure over 1-orbifold if $M$ admits the geometry of Sol.
4. For all the remaining cases, $f$ can be isotoped so that $T$ is invariant under $f$, and for each $f$-orbit $O$ of the components in $\{T, M - T\}$, $f|O$ is an isometry if $O$ is hyperbolic, $f|O$ is affine if $O$ belongs to $T$, otherwise there is a Seifert fibration on $O$ so that $f$ is fiber preserving and the induced map on the orbifold is either periodic, or (pseudo) Anosov, or reducible.

As in dimension 2, standard forms in Theorems 4.2 are useful in the study of fixed point theory and dynamics of 3-manifold automorphisms. The following is a result in this direction.

**Theorem 4.3 ([JWW])**. Suppose $M$ is a closed prime 3-manifold admitting geometric decomposition and $f : M \to M$ is an automorphism. Then

1. The Nielsen number $N(f)$ is realized in the isotopy class of $f$.
2. $f$ is isotopic to a fixed point free automorphism unless some component of the JSJ decomposition of $M$ is a Seifert manifold whose orbifold is neither a 2-sphere with a total of at most three holes or cone points nor a projective plane with a total of at most two holes or cone points.

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