Skeletons and Variation

Hermann Schulz

Institut für Theoretische Physik, Universität Hannover
Appelstraße 2, D-30167 Hannover, Germany

Well known from the sixties, the pressure of e.g. massless \( \phi^4 \) theory may be written as a series of 2PI-diagrams (skeletons) with the lines fully dressed. Varying the self–energy \( \Pi \) in this expression, it turns into a functional \( \mathcal{U}[Y] \) having a maximum in function space at \( Y = \Pi \). There is also the Feynman–Jensen thermal variational principle \( \mathcal{U}[S] \), a potentially non–perturbative tool. Here actions \( S \) are varied.

It is shown, through a few formal but exact steps, that the functional \( \mathcal{U} \) is covered by \( \mathcal{V} \). The corresponding special subset of trial actions is made explicit.

\section{I. INTRODUCTION}

Beyond perturbation theory, we are in search for some optimization calculus. The thermal variational principle, while well appreciated in non–relativistic quantum statistics, plays a merely minor role in field theory so far. At present, there are two apparently different such principles, which we shall call the Luttinger–Ward principle \( \mathcal{V} \) and the Feynman–Jensen principle \( \mathcal{U} \). For the study of the possible difference of these two principles we shall concentrate on scalar \( \phi^4 \) theory in the first two sections.

In 1960 Luttinger and Ward \( \mathcal{V} \) made a remark in parentheses, that the pressure \( p \) of their fermionic system, after the diagram lines were fully dressed, becomes minimal under variation of the self–energy. For the massless thermal \( \phi^4 \)–system with Lagrangian \( \mathcal{L} = (\partial \phi)^2/2 - g^2 \phi^4/24 \) this property may be recapitulated as follows. The skeleton version for the pressure \( p \), which is \(-1/V\) times the free energy \( \mathcal{F} \), reads

\[ p = -\frac{1}{2} \sum_P \ln (\Pi(P) - P^2) + \frac{1}{2} \sum_P \frac{\Pi(P)}{\Pi(P) - P^2} \]

\[ + \frac{1}{\beta V} \Gamma \] \quad \text{with} \quad \Gamma = \sum_{n=1}^{\infty} f_{2n}^{\text{PI}} \] \quad (1)

\( \Pi(P) \) is the exact self–energy, \( \sum_P = T \sum_n (2\pi)^{-3} \int d^3 p \) is the thermal sum–integral, and \( f_{2n}^{\text{PI}} \) are the 2–particle irreducible contributions of order \( g^{2n} \) to the logarithm \( \ln(Z_{\text{int}}) \) of the partition function.

\[ \Gamma = 3 \bigcirc + 12 \bigcirc \bigcirc + 2 \cdot 12^2 \bigcirc \bigcirc \]

\[ + \frac{3}{2} \cdot 12^3 \bigcirc \bigcirc \bigcirc + 6 \cdot 12^4 \bigcirc \bigcirc \bigcirc \bigcirc + \ldots \] \quad (2)

The result \( \mathcal{U} \) is found in \( \mathcal{V} \). It can be derived by a Legendre transformation \( \mathcal{V} \) or, equivalently, by minimizing the free energy \( \mathcal{F} \). It was recently taken up in \( \mathcal{V} \), see also the article by A. Pesher in these proceedings.

Three minor modifications make the above expression \( \mathcal{V} \) to become the functional \( \mathcal{U} \) of what we call the Luttinger–Ward variational principle. We multiply \( \mathcal{V} \) with \( \beta V \). We supply the volume \( V \) with periodic boundary conditions, hence the thermal sum turns into the "bare sum" \( \sum_P \equiv \sum_n \sum_P = \beta V \sum_P \). Third, we replace the self–energy \( \Pi(P) \) by some function \( Y(P) \) to be varied and introduce the notation \( G(P) = 1/(Y(P) - P^2) \). Then, as is seen shortly, the principle states that

\[ \mathcal{U}[Y] = \frac{1}{2} \sum_P \ln (G(P)) + \frac{1}{2} \sum_P Y(P) G(P) \]

\[ + \Gamma[G] \leq \beta V p \] \quad (3)

The fact that here the pressure is at maximum, in contrast to that of \( \mathcal{V} \), is due to the boson content of our system. The replacement \( Y \rightarrow Y \) had to be performed even in the last term, i.e. in the diagram lines of \( \mathcal{U} \). So, these lines, the bones of the skeleton, have become variable propagators \( G(P) \).

For taking the functional derivative of \( \mathcal{U} \) with respect to \( Y(Q) \) at an arbitrary "position" \( Q \), we appreciate the general functional relation \( \mathcal{V} \)

\[ 2 G_0^2 \delta g \left( \frac{\partial}{\partial g} \right) f = G \] \quad \left( f \equiv \ln (Z) \right) \quad , \quad (4)

between bare and dressed lines, which is valid separately in each loop order \( \mathcal{V} \). Hence

\[ 2G_0^2 \delta G \Gamma[G] = G^{2\text{PI}}[Y] \equiv \]

\[ = - G^2 \Pi^{2\text{PI}}[Y] \] \quad (5)

and

\[ 2 \delta_{Y(Q)} \mathcal{U} \equiv G^2 (Y - \Pi^{2\text{PI}}[Y]) \]

\[ \Rightarrow \quad Y(Q) = \Pi(Q) \] \quad , \quad (6)

\[ \text{Electronic adress : hschulz@itp.uni-hannover.de} \]
because the full $\Pi(Q)$ (up to a given order $n$) is reconstructed by iterative use of $Y_m = \Pi^{2\text{PT}}_m$ with $m < n$ \cite{2}. At any position $Q$, when $Y$ increases there, $\delta Y(Q)$ reaches the zero coming from positive values and turning into negative. So, the functional $\mathcal{U}$ has a true maximum at $Y = \Pi$.

Some warning is in order, if the sum $\sum f^{2\text{PT}}_{2n}$ in $\mathcal{U}$ is truncated. Let $2n'$ be the highest $g^r$-power contained in $\mathcal{U}$. Then, at best, the resulting $\Pi$ would be correct to order $g^{n'+3}$ only ($n' \geq 4$). $\Pi$ in $g^{12}$, for instance, needs $\mathcal{U}$ to be developed to $g^{2n'} = g^{18}$. This unfortunate fact is due to reduction of $g$-order by evaluation. Things are realized by power counting of soft scale contributions, or along the lines given by Braaten and Nieto \cite{11}. One might question for a better functional, which is free from this defect. For the following we stay with the exact expression \cite{3}. To introduce the Feynman–Jensen variational principle we must distinguish between the action $S$ of a trial theory and the action $S_\bullet$ of the true $\phi^4$-system under study. With or without a bullet index, we have by definition $S = - \int^\beta L$, $\int^\beta = \int^\beta_0 d\tau \int d^3r$ and $\beta = 1/T$. Then \cite{3}

$$Z_\bullet = \int D\phi \ e^{-S_\bullet} = \left( \int D\phi \ e^{-S} \right) \frac{\int D\phi \ e^{-S} \ e^{S-S_\bullet}}{\int D\phi \ e^{-S}}$$

$$= Z \langle e^{S-S_\bullet} \rangle \geq Z e^{(S-S_\bullet)} \ .$$

(7)

with $Z$, $Z_\bullet$ the partition functions of trial and studied theory, respectively. The inequality in \cite{3} is the familiar Jensen inequality \cite{12} applied to functional integrals. The functional measure is obviously irrelevant here. Note that the average refers to the trial action $S$, which usually might be the simpler theory. Taking the logarithm of \cite{3} we arrive at

$$\mathcal{U}[S] = \ln (Z) + \langle S - S_\bullet \rangle \leq \ln (Z_\bullet) \ .$$

(8)

Also the functional $\mathcal{U}$ has a true maximum, this time at $S = S_\bullet$. The principle \cite{3} was studied with detail in \cite{3} and applied to gauge fields. Working with free trial theories, several known results could be reproduced in \cite{3}. But to get more information from the principle, the free trial space has turned out being too poor.

**II. THE WAY \mathcal{U} IS CONTAINED IN \mathcal{U}**

In three steps the principle $\mathcal{U}$ will be reformulated until it has the structure of $\mathcal{U}$.

Step one. To get rid of the infinite sum $\Gamma$ in \cite{3}, we exploit the fact that the l.h.s. of \cite{3} regains a physical meaning by the replacements $YG \rightarrow \Pi G$ and $\mathcal{U} \rightarrow \ln (Z)$. Here, $Z$ is the partition function of a $\phi^4$ system with some momentum-dependent mass $G^{-1}_0 + P^2$ such that a given exact propagator $G = 1/(Y - P^2)$ turns out perturbatively. Note that $\Pi = G^{-1} - G_0^{-1}$ differs from $Y = G^{-1} + P^2$. Since $\Gamma(G)$ is a functional of $G$ only, all the bare lines, which constitute the partition function $Z$ diagrammatically, must be re-expressed by $G$:

$$\sum_{n=1}^\infty f^{2\text{PT}}_{n} = \Gamma(G)$$

$$= \left[ \ln (Z) - \frac{1}{2} \sum \ln (G) - \frac{1}{2} \sum \Pi G \right]_\bullet$$

$$= \ln (Z_\bullet) - \frac{1}{2} \sum \ln (G) - \frac{1}{2} \sum \left( 1 - \frac{G}{G_0} \right)_\bullet \ .$$

(9)

Without star index, the inner line of \cite{3} is nothing but the skeleton formula \cite{3}, but this time for the massive case. The inequality refers to variation of $S_\bullet$. This unfortunate fact would be correct to

$$\Phi = \ln (Z) + \langle S - S_\bullet \rangle = \ln (Z_\bullet) \ .$$

(10)

The inequality refers to variation of $Y$. While still $G = 1/(Y - P^2)$, the objects $G_0 \ast$ and $Z_\ast$ are some nontrivial functionals of $G$, hence of $Y$.

Step two. We transcribe the inequality \cite{3} into functional integral language. Exponentiating we have

$$e^{\frac{1}{2} \sum (Y G - G_0)} Z_\ast \leq Z_\bullet \ ,$$

$$\mathcal{U}[Y] = \frac{1}{2} \sum f^{(2\text{PT})} \mathcal{U} \ ,$$

$$\mathcal{U}[G] = \frac{1}{2} \sum \ln (G) - \frac{1}{2} \sum \left( 1 - \frac{G}{G_0} \right) \ .$$

(11)

where $S_\bullet = - \int^\beta (\mathcal{L}_0 + \mathcal{L}_{\text{int}})$, but $S_\ast = - \int^\beta (\mathcal{L}_0 \ast + \mathcal{L}_{\text{int}} \ast)$ with

$$\mathcal{L}_0 \ast = \mathcal{L}_0 |_{G_0 \rightarrow G_0 \ast} \ ,$$

$$- \int^\beta \mathcal{L}_0 \ast = \frac{1}{2} \sum \tilde{\phi}(-P) \frac{1}{G_0 \ast} \tilde{\phi}(P) \equiv S_0 \ast \ .$$

(12)

We turn to the exponential in \cite{3}, which might be some functional average over two fields $\tilde{\phi}$. In fact, if we first separate a factor $G$ by


\[ YG - 1 + \frac{G}{G_{0\ast}} = G \left( \frac{1}{G_{0\ast}} - \frac{1}{G_{00}} \right) \quad , \]  

where \( G_{00} \equiv -1/P^2 \), and write this factor as

\[ G(P)2G_{0\ast}^2 \delta_{G_{0\ast}} \ln (Z_* ) = \]

\[ = \frac{2G_{0\ast}^2}{Z_*} \delta_{G_{0\ast}} \int D\phi\, e^{-\frac{1}{\beta V} \sum \overline{\phi}(Q) \frac{\partial}{\partial \phi(Q)} e^{-S_{\text{int}}}} \]

\[ = \frac{1}{\beta V} \left\langle \overline{\phi}(P) \phi(P) \right\rangle_* \quad , \]

then the exponential becomes

\[ \frac{1}{2} \sum_n \left( (YG - 1 + \frac{G}{G_{0\ast}}) = \right. \]

\[ = \frac{1}{2} \sum_p \left\langle \overline{\phi}(P) \left( \frac{1}{G_{0\ast}} - \frac{1}{G_{00}} \right) \phi(P) \right\rangle_* \]

\[ = (S_0 - S_0 \ast)_* = (S_* - S_\ast)_* \quad . \]

Using (13) in (12), the result

\[ Z_\ast e^{(S_\ast - S_\ast)_\ast} \leq Z_\ast \leq Z_\ast e^{(S_\ast - S_\ast)_\ast} \]

formally agrees with (13) and is a Jensen inequality.

Step three is merely one in mind rather than in formulation. So far we varied functions \( Y \) around \( \Pi \) or, equivalently, \( G' \)'s around \( 1/(II - P^2) \). But this is not the variation in the Feynman–Jensen functional. The latter depends on actions. So, we now change the philosophy and consider \( G_{0\ast} \) to be the variable function. Hence (14) becomes the Feynman–Jensen principle, indeed. But the class of trial actions \( S_\ast \) is very restricted. The full interaction \( S_{\text{int}} \) is part of \( S_\ast \) and remains untouched. Due to (12) \( S_\ast \) differs from \( S_0 \) only in some variable momentum–dependent mass term. Moreover, the solution to the optimization problem has become trivial: the equality sign in (16) is simply reached at vanishing mass term. The true value of using the equality sign in (16) is simply reached at vanishing mass term.

From the fact, that we are on a very safe ground with the special Feynman–Jensen variational principle (16), we are led to reverse the order of the above three steps. Then they guide a possible derivation of the skeleton formula (1) including its variational property (3). This idea is followed up in the next section.

### III. THE SKELETONS OF YANG AND MILLS

We derive the skeleton formula for the pure gluon system by following the above equations in backward direction. In covariant gauges the action includes a gauge fixing term, a term arising from the Faddeev–Popov determinant and the 3– and 4–point interactions,

\[ S_\ast = S_0 + S_{\text{g.f.}} + S_{\text{FP}}[A] + S_{\text{int}} \quad . \]

For the functional inequality (16) to make sense (but of no relevance in the sequel), \( S_{\text{FP}} \) must be viewed as a functional of the fields \( A_\mu \). The trial action \( S_\ast \) and the action \( S_\ast \) have the same gauge fixing parameter \( \alpha \) (see § II.B of [3]). Then, the only difference between the two actions is a momentum–dependent mass term, which we immediately write as

\[ S_\ast - S_\ast = \]

\[ = \frac{1}{2} \sum_p \overline{A}_\mu(-P) \left[ \frac{1}{G_{0\ast}} - \frac{1}{G_0} \right]^{\mu\nu} \overline{A}_\nu(P) \quad . \]

The fully dressed Greens function \( G_{\mu\nu} \) of the \( \ast \)-theory enters when averaging (18)

\[ \langle S_\ast - S_\ast \rangle_\ast = \]

\[ = \frac{n}{2} \beta V \sum_p \left[ \frac{1}{G_{0\ast}(P)} - \frac{1}{G_0(P)} \right]^{\mu\nu} G_{\mu\nu}(P) \]

with \( n = C_A = N^2 - 1 \). Introducing a variable matrix self–energy \( Y^{\mu\nu}(P) \) we are led to

\[ \left[ \frac{1}{G_{0\ast}} - \frac{1}{G_0} \right] G = \left[ \frac{1}{G_{0\ast}} - \frac{1}{G_0} \right] \left( \frac{1}{G_0 - Y} \right) \]

\[ = \frac{1}{G_{0\ast}} G - 1 - YG \quad . \]

with all products being Lorentz matrix multiplications. Some signs differ compared to (13) due to Minkowski metrics \(+---\) and the conventions of [3]. Changing from variable \( G_{0\ast} \) to variable \( Y \), the logarithm of (16) reads

\[ \mathcal{U}[Y] = n \sum_p X[G] - \frac{n}{2} \sum_p YG + \Gamma[G] \]

\[ \leq \ln (Z_\ast) \quad (21) \]

with

\[ \Gamma[G] = \left[ \ln (Z) - n \sum_p X[G] + \frac{n}{2} \sum_p \left( \frac{1}{G_0} - 1 \right) \right]_* \]

\[ = \sum_{n=1}^{\infty} \langle f_n^{2\pi i} \rangle_* \quad . \]

So far, the term with \( X[G] \) has been only added and subtracted. But now we require the square bracket to be the sum over \( 2\pi i \) contributions to \( \ln (Z_\ast) \). With the
Yang–Mills counterparts of (11) and (14), as listed in (23), this condition turns into the matrix differential equation in the third line:

\[ 2G_0(\delta G_0 f)G_0 = nG, \]

\[ \frac{2}{n} \delta G \Gamma[G] = \frac{1}{G_0} - \frac{1}{G}, \]

\[ \delta G \sum X[G] = \frac{1}{G}. \quad (23) \]

The solution to the \( X \) equation needs (24) and is given (apart from some \( Y \)–independent constant) in (25) below: first term. Herewith, one might drop the question mark in (22).

To supply the first two terms of (21) with detail, we need the structure of the exact gluon propagator. Due to Weldon [13] this propagator and the self energy may be written as

\[ G^{\mu\nu} = \Delta_{\mu\nu} + \Delta_{\mu\nu}B^{\mu\nu} + \frac{\alpha}{P^2} F^{\mu\nu}, \]

\[ Y^{\mu\nu} = \Pi^{\mu\nu} + \frac{\alpha}{P^2} F^{\mu\nu}, \quad \quad (24) \]

where \( \Delta_{\mu\nu} = 1/(P^2 - \Pi_{\mu\nu}) \). Apart from \( A = g - B - D \) all the Lorentz matrices in (24) are dyadic products of the four–vectors \( P = (P_0, \vec{p}) \) and \( \Pi = (P_0, \vec{\Pi}/P) \), or of \( R = P - y\vec{P} \) and \( \Pi = P_0 - yP_0 \), respectively. To be specific: \( B = -P \circ \Pi/P^2, \quad D = P \circ \Pi/P^2, \quad \Pi \circ \Pi = -R \circ R/P^2, \quad \Pi = R \circ R/P^2 \). We emphasize that even the Weldon coefficient \( y \) is varied. With (24), and after preparing the trace over all pairs of the above matrices, one verifies \( Y^{\mu\nu}G_{\rho\sigma} = 2Y_{\nu}\Delta_{\rho\sigma} + Y_{\sigma}\Delta_{\rho\nu} - \alpha y^2 \) with ease. The Luttinger–Ward functional \( U \) for the gluon system then finally reads

\[ U[Y] = \frac{n}{2} \sum_p \ln \left( \frac{1}{y_\nu - P^2} \right) \frac{1}{y_\nu - P^2} \]

\[ + \frac{n}{2} \sum_p \left( \frac{2y_\nu}{y_\nu - P^2} + \frac{y_\nu}{y_\nu - P^2} + \alpha y^2 \right) \Gamma. \quad (25) \]

The maximum is reached at \( Y_{\nu} = Y_{\nu}, Y_{\mu} = Y_{\mu} \) and \( y = b \), where the coefficient \( b \) is defined by (24) at \( Y^{\mu\nu} = \Pi^{\mu\nu} \) and is given by \( \sqrt{2}\Pi_{\mu\nu}/\Pi_{\mu\nu} \) in earlier notations [14]. At maximum \( \frac{U}{y} \) becomes the pressure of the hot gluon medium.

IV. CONCLUSIONS

The competition between the Luttinger–Ward and the Feynman–Jensen variational principle is won by the latter. It contains the former as a special case, whose space of trial actions is reduced to a rather trivial variation of mass terms. Nevertheless, the machinery relating the two principles can be used to derive the skeleton formula for the pressure of both, the massless scalar \( \phi^4 \) theory and the gluon system.

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