Iteration of Cox rings of klt singularities

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Abstract
In this article, we study the iteration of Cox rings of klt singularities (and Fano varieties) from a topological perspective. Given a klt singularity \((X, \Delta; x)\), we define the iteration of Cox rings of \((X, \Delta; x)\). The first result of this article is that the iteration of Cox rings \(\text{Cox}^{(k)}(X, \Delta; x)\) of a klt singularity stabilizes for \(k\) large enough. The second result is a boundedness one, we prove that for an \(n\)-dimensional klt singularity \((X, \Delta; x)\), the iteration of Cox rings stabilizes for \(k \geq c(n)\), where \(c(n)\) only depends on \(n\). Then, we use Cox rings to establish the existence of a simply connected factorial canonical (or scfc) cover of a klt singularity, with general fiber being an extension of a finite group by an algebraic torus. The scfc cover generalizes both the universal cover and the iteration of Cox rings. We prove that the scfc cover dominates any sequence of quasi-étale finite covers and reductive abelian quasi-torsors of the singularity. We characterize when the iteration of Cox rings is smooth and when the scfc cover is smooth. We also characterize when the spectrum of the iteration coincides with the scfc cover. Finally, we give a complete description of the regional fundamental group, the iteration of Cox rings, and the scfc cover of klt singularities of complexity one. Analogous versions of all our theorems are also proved for Fano-type morphisms. To extend the results to this setting, we show that the Jordan property holds for the regional fundamental group of Fano-type morphisms.
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INTRODUCTION

The Cox ring $\text{Cox}(X)$ of an algebraic variety $X$ captures the geometry of the variety and all the line bundles on it [8]. This ring is also known as universal torsor in arithmetic geometry [28, 29]. The Cox ring construction generalizes the classic description of toric varieties as quotients of (big open subsets of) affine spaces [30]. Whenever the $\mathbb{C}$-algebra $\text{Cox}(X)$ is finitely generated, it controls all birational models of $X$ via GIT [54]. In this case, we say that $X$ is a Mori dream space (MDS), since the variety $X$ behaves optimally with respect to the minimal model program (also known as Mori program). The use of Cox rings to study varieties has become a standard technique in algebraic geometry. An explicit presentation of the Cox ring in terms of generators and relations often enlightens the geometry of the variety. In this direction, there are several results on weak del Pezzo surfaces [11, 45]. Many results about Cox rings have been obtained for K3 surfaces [5, 6]. In higher dimensions, the Cox rings of some Fano manifolds have been described explicitly [32, 48, 49]. The Cox rings of certain Moduli spaces are considered in [26, 44, 62]. The study of Cox rings has also become a central topic in $\mathbb{T}$-varieties [2, 3], especially in the case of complexity one, that is, $n$-dimensional algebraic varieties with an effective action of a $(n-1)$-dimensional torus [4, 13, 50]. In this case, the Cox ring can be described combinatorially. More generally, horospherical varieties can be described using Cox rings [59, 72]. In many cases, computations of intersection theory can be carried out in the Cox ring of an algebraic variety. The most general setting in which Cox rings are known to be well defined is algebraic stacks [52, 53]. In the present work, we will mostly deal with the class of integral, noetherian, normal schemes [53, Section 2.3]. We refer the reader to [8, 58] for a systematic study of Cox rings.

Cox rings have also been used to study singularities. In this case, the definition of the Cox ring is often applied to a certain resolution of singularities. In [37], the authors study the Cox ring of the minimal resolution of a surface Du Val singularity. In [33], the author provides two different descriptions of the Cox ring of the minimal resolution of a quotient singularity. Further results have been obtained toward the computation of the Cox ring of some minimal (or crepant) resolution of singularities [34, 35, 42]. In [7], a slightly different approach to studying Kawamata log terminal (klt) singularities via Cox rings is proposed by the authors. Instead of looking at the Cox ring of a resolution of singularities, the authors use the definition of Cox ring on the germ itself. Then, if possible, this process is iterated to simplify the singularity (and possibly, increasing the dimension). This construction generalizes the presentation of surface klt singularities as quotients of factorial canonical singularities by solvable finite groups. In [7], the iteration of Cox rings is performed for singularities of complexity one. The iteration has at most four steps and can be read off directly from the first Cox ring. Furthermore, the last variety in this sequence, the so-called master Cox ring, is factorial and it can be listed explicitly. In [51], the authors characterize all varieties with a torus action of complexity one that admit a finite iteration of Cox rings. In [39], it is shown that for spherical varieties, the iteration of Cox rings has at most two steps. More generally, in the works [72, 73], Vezier considers the iteration of Cox rings for $G$-varieties of complexity one and determines bounds on the number of iterations.

In order to define an iteration of Cox rings, we must check that (the spectrum of) the Cox ring $\text{Cox}(X)$ of our variety $X$ is itself a MDS. It is known that Fano-type varieties are a special class of MDSs [15]. Furthermore, the Cox ring of a Fano type variety is an affine Gorenstein canonical quasi-cone [19, 22, 40]. In particular, it is an affine model of a klt singularity. A klt singularity, in turn, is a local version of a Fano-type variety. Indeed, a klt singularity is a relative MDS over itself, that is, when considering the identity as the structure morphism. Thus, it is natural to iterate the
Cox construction for Fano-type varieties, or more generally, for klt singularities. The first author made this observation in [19], where he proves the existence and termination of the iteration of Cox rings for Fano-type varieties and klt quasi-cones.

**Theorem 1** (Cf. [19]). Let $X$ be a Fano-type variety. Then, for each $k \geq 0,l$ the $k$th iteration of Cox rings $\text{Cox}^{(k)}(X)$ exists. Furthermore, the iteration stabilizes for $k$ large enough.

We recall that the iteration of Cox rings can lead to two different outcomes: it could stop with a factorial master Cox ring or an affine variety that is not a MDS. Furthermore, it could lead to an infinite sequence of Cox rings. In this article, we recover Theorem 1 for klt singularities in the general setting. This means that the iteration of Cox rings always exists for klt singularities and terminates after finitely many iterations. We also generalize the concept of Cox rings for log pairs, leading to our first result.

**Theorem 2.** Let $(X, \Delta; x)$ be a Kawamata log terminal singularity. Then, for each $k \geq 0$, the $k$th iteration of Cox rings $\text{Cox}^{(k)}(X, \Delta; x)$ exists. Furthermore, the iteration stabilizes for $k$ large enough.

The main tool used to prove the above theorem is the finiteness of the regional fundamental group of a klt singularity.

Two natural questions emanate from the two above theorems. First, we can ask how many times we need to iterate the Cox construction before it stabilizes. A natural way to study the iteration of Cox rings is to quotient (in each step) by the connected component of the solvable group acting on each model $\text{Cox}^{(k)}(X, \Delta; x)$. In this way, we obtain a sequence of finite solvable Galois covers of the starting singularity (or Fano variety). This method was initiated in [19]. Using the Jordan property for the regional fundamental group of klt singularities [20], we prove that the number of iterations is bounded from above by a constant that only depends on the dimension. This means that the iteration of Cox rings is controlled by the topology of the variety (or singularity). The following theorem has a projective and a local version. For simplicity of the exposition, we just write the local version in the introduction.

**Theorem 3.** There exists a constant $c(n)$, only depending on $n$, satisfying the following. Let $(X, \Delta; x)$ be an $n$-dimensional Kawamata log terminal singularity. Then, the $k$th iteration of Cox rings $\text{Cox}^{(k)}(X, \Delta; x)$ stabilizes for $k \geq c(n)$.

Second, we can ask how (if possible) to control the dimension of the iteration of Cox rings. For instance, we can ask if there is any invariant of the singularity that can give an upper bound for the dimension of the master Cox ring. Note that, in general, the iteration of Cox rings could have arbitrarily large dimension. Indeed, the spectrum of the Cox ring of an affine toric variety of dimension $n$ and rank $\rho$ of the class group is isomorphic to the affine space $\mathbb{A}^{n+\rho}$. On the other hand, even if $\rho$ is bounded for the singularity itself, it could happen that the Cox ring itself (or any of the higher iterated Cox rings) has unbounded $\rho$. Thus, in general, the rank of the class group of the initial germ does not control the dimension of the master Cox ring. This leads to our third result, answering the above question in terms of the second homotopy group of the smooth locus.

**Theorem 4.** Let $n$ and $o$ be positive integers. Let $(X, \Delta; x)$ be a $n$-dimensional Kawamata log terminal singularity. Assume that $\pi_2^{\text{reg}}(X, \Delta; x) \otimes \mathbb{Q}$ has rank $o$. Then, the master Cox ring $R$ of $(X, \Delta; x)$ has dimension at most $n + o$. If $\dim R = n + o$, then $\pi_2^{\text{reg}}(\text{Spec}R)$ is finite.
We will prove Theorem 2 in two different settings, for two different definitions of the iteration of Cox rings. We will define the iteration of Cox rings for the (Zariski) local ring and the Henselization of the local ring (i.e., the local ring in the étale topology) of a Kawamata log terminal singularity. The first one will be called the \textit{affine iteration}, whereas the second will be called the \textit{Henselian iteration}.

The advantage of the affine iteration is that the outcome of the iteration is an affine klt variety with a distinguished point. Thus, techniques of affine geometry can be applied to the master Cox ring in this case. On the other hand, the Henselian iteration captures the local topology of the singularity. This is the main property that we will use for our next theorem. We prove that klt singularities admit factorial canonical simply connected covers. This cover can be understood as a cover that encompasses all the good properties of the universal cover and the iteration of Cox rings.

\textbf{Theorem 5.} Let \((X, \Delta; x)\) be a Kawamata log terminal singularity. Let \(X^h\) be the spectrum of the Henselization of the local ring of \(X\) at \(x\). There exists a local Henselian ring \(R_Y\) so that \(Y = \text{Spec}(R_Y)\) satisfies:

1. \(Y\) is canonical factorial,
2. \(\pi_{1}^{\text{reg}}(Y, y)\) is trivial,
3. \(Y\) admits the action of a reductive group \(G\), and
4. we have an isomorphism \(Y / G \cong X^h\).

Furthermore, \(G\) is an extension of a solvable reductive group and \(\pi_{1}^{\text{reg}}(X, \Delta, x)\).

Throughout this article, reductive groups are not assumed to be connected. In particular, a solvable reductive group is an extension of a finite solvable group by a torus.

We call the germ \((Y, y)\) constructed in Theorem 5 the \textit{simply connected factorial canonical cover} of the klt singularity, or \textit{scfc} cover for short. Note that the name of this cover is idiosyncratic because the condition on the regional fundamental group is stronger than being simply connected. However, in the context of singularities, it is natural to consider the fundamental group of the smooth locus instead of the fundamental group of the germ itself. The scfc cover is a generalization of both; the universal cover and the iteration of Cox rings of a singularity. Our next result makes this precise by saying that the scfc cover of a Kawamata log terminal singularity dominates any sequence of pointed abelian covers and pointed finite covers.

\textbf{Theorem 6.} Let \((X, \Delta; x)\) be a Kawamata log terminal singularity. Let \(X^h\) be the spectrum of the Henselization of the local ring of \(X\) at \(x\). Let \((Y, y)\) be the scfc cover of \((X, \Delta; x)\). Let \((X^h, x) \leftarrow (X_1, x_1) \leftarrow (X_2, x_2) \leftarrow \cdots \leftarrow (X_n, x_n)\) be a sequence of pointed finite covers and pointed abelian covers. Let \(X^h_n\) be the spectrum of the Henselization of the local ring of \(X_n\) at \(x_n\). Then, there is a quotient morphism \(Y \to X^h_n\).

In view of the above theorem, the scfc cover of a Kawamata log terminal singularity can be regarded as the best singularity that can be obtained from \((X, \Delta; x)\) by taking sequences of finite covers and abelian covers, more generally, by taking solvable-finite covers, that is, covers by finite extensions of solvable reductive groups. So far, we have four different covers of klt singularities.
Corollary 1. Let $(X, \Delta; x)$ be a klt singularity. Then, the following objects are isomorphic:

1. the simply connected factorial canonical cover of $(X, \Delta; x)$,
2. the Cox space of the universal cover of $(X, \Delta; x)$, and
3. the Cox space of the universal cover of the spectrum of the iteration of Cox rings of $(X, \Delta; x)$.

In the previous statement, the Cox space is nothing else than the spectrum of the Cox ring, also known as total coordinate space.

The universal cover of a klt singularity is smooth if and only if the singularity is the quotient of $\mathbb{C}^n$ by a finite group acting linearly. Furthermore, we know that the Cox ring of a klt singularity is smooth if and only if the singularity is formally toric. The following theorem characterizes when the iteration of Cox rings of a klt singularity is smooth.

Theorem 7. Let $(X, \Delta; x)$ be a klt singularity. Then, the following statements are equivalent:

1. the spectrum of the iteration of Cox rings $\text{Cox}^{\text{it}}(X, \Delta; x)$ is smooth, and
2. $(X, \Delta; x)$ is a finite quasi-étale solvable quotient of a toric singularity.

The following theorem characterizes when the scfc cover of a klt singularity is smooth.

Theorem 8. Let $(X, \Delta; x)$ be a klt singularity. Then, the following statements are equivalent:

1. the simply connected factorial canonical cover of $(X, \Delta; x)$ is smooth, and
2. $(X, \Delta; x)$ is a finite quasi-étalequotient of a toric singularity.

In Appendix, we show a diagram with all the natural morphisms among the covers considered in this article (Table A.1). In this direction, it is also natural to compare the iteration of Cox rings with the scfc cover. We prove that they coincide as long as the regional fundamental group of the klt singularity is a solvable group.

Theorem 9. Let $(X, \Delta; x)$ be a klt singularity. Then, the following are equivalent.

1. The spectrum of the iteration of Cox rings coincides with the simply connected factorial canonical cover.
2. The regional fundamental group $\pi_1^{\text{reg}}(X, \Delta; x)$ is solvable.

All the theorems in this article are also proved for Fano-type varieties. In many cases, we also prove the statements for Fano-type morphisms. As mentioned above, the boundedness of iterations is a consequence of the Jordan property for the regional fundamental group of klt singularities [20]. To generalize to the relative setting, we will need the following relative version of the Jordan property.

Theorem 10. Let $n$ be a positive integer. There exists a constant $c(n)$, only depending on $n$, satisfying the following. Let $\phi : X \to Z$ be a projective contraction so that $X$ has dimension $n$. Let $(X, \Delta)$ be a log pair of Fano-type over $Z$. Let $z \in Z$ be a closed point. Then, the fundamental group $\pi_1^{\text{reg}}(X/Z, \Delta; z)$
is finite. Furthermore, there exists a normal abelian subgroup $A \leq \pi_{1}^{\text{reg}}(X/Z, \Delta; z)$ of rank at most $n$ and index at most $c(n)$.

Here, the group $\pi_{1}^{\text{reg}}(X/Z, \Delta; z)$ consists of loops over a small punctured analytic neighborhood of a closed point $z \in Z$. In this direction, we also prove an enhanced version of the Jordan property for klt $\mathbb{T}$-singularities. We recall that a $\mathbb{T}$-variety of complexity $r$ is an algebraic variety $X$ with a faithful action of an algebraic torus of dimension $\dim(X) - r$. In this direction, we prove the following theorem.

**Theorem 11.** Let $r$ be a positive integer. There exists a constant $c(r)$, only depending on $r$, satisfying the following. Let $(X, \Delta; x)$ be an $n$-dimensional klt $\mathbb{T}$-singularity of complexity $r$. Then, there exists a normal abelian subgroup $A \leq \pi_{1}^{\text{reg}}(X, \Delta; x)$ of rank at most $n$ and index at most $c(r)$.

In particular, the nonabelian quotient of the regional fundamental group of complexity one klt $\mathbb{T}$-singularities is bounded by a constant that is independent of the dimension. As a consequence of Theorem 11 and the proof of Theorem 3, we conclude the following statement about the iteration of Cox rings of klt $\mathbb{T}$-singularities.

**Theorem 12.** Let $r$ be a positive integer. There exists a constant $c(r)$, only depending on $r$, satisfying the following. Let $(X, \Delta; x)$ be a klt $\mathbb{T}$-singularity of complexity $r$. Then, the $k$th iteration of Cox rings $\text{Cox}^{(k)}(X, \Delta; x)$ stabilizes for $k \geq c(r)$.

Note that the constant $c(r)$ only depends on the complexity and not on the dimension of the germ. In Subsection 7.3, we will culminate the article with an extensive study of the regional fundamental group, the iteration of Cox rings, and the scfc covers of klt $\mathbb{T}$-singularities of complexity one. The iteration of Cox rings of these singularities has already been considered in the works [7, 51].

The present article gives a good understanding of the finite-solvable covers of klt singularities and Fano-type varieties. It is natural to try to extend the above results to general reductive groups. However, to do so, a better understanding of semisimple covers of Fano varieties is required. It is also interesting to consider the opposite question: whether the quotient of an affine klt singularity by a reductive group is klt type. The authors will settle this question in a forthcoming article.

**Structure of the paper**

The structure of the paper is as follows. In Section 2, we give some preliminaries about Cox rings, graded-local rings, and the minimal model program. In Section 3, we turn to define Cox rings in a variety of cases, for morphisms of log pairs, over local rings, and over Henselian rings. In Section 4, we turn to prove the existence and boundedness of the iteration of Cox rings for klt singularities. In Section 5, we prove the existence of the simply connected factorial canonical cover of a klt singularity. In Section 6, we give a characterization of Fano-type varieties with a smooth iteration of Cox rings and Fano-type varieties with smooth scfc cover. Finally, in Section 7, we give several examples including a complete classification of the iteration of Cox rings of klt complexity one $\mathbb{T}$-singularities.
2 | PRELIMINARIES

Throughout this article, we work over the field of complex numbers $\mathbb{C}$. The rank of a finite group is the least number of generators. As usual, we may denote by 1 (resp. 0) the trivial multiplicative (resp. additive) group.

In this section, we collect some preliminary results and definitions. In Subsection 2.1, we recall the concept of Cox rings and MDSs. In Subsection 2.2, we prove some properties about the class groups of gr-local rings. In Subsection 2.3, we recall the concept of gr-Henselian rings. Then, in Subsection 2.4, we define sheaves of gr-local rings. The Cox sheaves considered in this article will be sheaves of gr-local rings. In Subsection 2.5, we bring together the concepts of regional fundamental groups and Cox rings. Finally, in Subsection 2.6, we recollect some notions of singularities of the minimal model program.

2.1 | Cox rings and MDSs

In this subsection, we recall the concept of Cox rings and MDSs.

**Definition 2.1.** Let $X$ be a normal algebraic variety with free finitely generated class group $\text{Cl}(X)$. We can define the Cox ring of $X$ to be

$$\text{Cox}(X) := \bigoplus_{[D] \in \text{Cl}(X)} \Gamma(X, \mathcal{O}_X(D)).$$

Here, the multiplication of sections is computed in the field of fractions of $X$. We say that a normal algebraic variety $X$ is a MDS if its Cox ring is finitely generated over $\mathbb{C}$. In this case, we denote the affine variety $\widetilde{X} := \text{Spec} \text{Cox}(X)$ and call it the total coordinate space of $X$. We get $X$ back as a good quotient of the big open subset $\widetilde{X} \subseteq \widetilde{X}$ by the diagonalizable group (also called a quasi-torus) $H_X := \text{Spec} \mathbb{C}[\text{Cl}(X)]$. This big open subset $\widetilde{X}$ is called the characteristic space, whereas $H_X$ is called the characteristic quasi-torus of $X$.

The name MDS is given to these varieties because they behave optimally with respect to the minimal model program. For any divisor $D$ on an MDS $X$, we can run a $D$-MMP that will terminate with either a Mori fiber space or a good minimal model for $D$ (i.e., a model on which its strict transform is a semiample divisor). Toric varieties are known to be MDSs. It is known that the Cox ring of an $n$-dimensional smooth projective toric variety of Picard rank $\rho$ is a polynomial ring in $\rho + n$ variables (see, e.g., [54, Corollary 2.10]). Furthermore, in [15, Corollary 1.3.2], it is proved that smooth Fano varieties are MDSs.

The quotient morphism $\widetilde{X} \xrightarrow{H_X} X$ restricted to the preimage of the smooth locus of $X$ is a torsor, that is, a principal $H_X$-bundle. Following [8, Definition 1.6.4.1], we say that the action of an affine algebraic group $G$ on a variety $Y$ is strongly stable, if $Y$ allows an open subset $W$, such that

1. the complement $Y \setminus W$ is of codimension at least two in $Y$,
2. $G$ acts freely on $W$,
3. the orbit $G \cdot w$ is closed in $Y$ for every $w \in W$.

In particular, we see that the action of $H_X$ on $\widetilde{X}$ is strongly stable [8, Section 1.6.4].
2.2 \textbf{Graded local rings}

In this subsection, we recall the concept of gr-local rings and prove some preliminary results about their class groups.

Let $K$ be a finitely generated abelian group and $A$ be a $K$-algebra containing the field of complex numbers $\mathbb{C}$. In this subsection, we aim to find a suitable category in which certain generalizations of Cox rings fit in. Later on, we define the Cox ring for pair structures on projective varieties and quasi-cones, that is, affine varieties with a $\mathbb{C}^*$-action, such that all orbit closures meet in one distinguished point, the vertex. Obviously, quasi-cones share similarities with spectra of local rings. In particular, the Picard group is trivial. Thus, it is natural to extend the definition of Cox rings to spectra of local rings. This will be done in the next section. It turns out that the Cox rings of all these objects will be graded local rings in the sense of \cite[Definition 1.1.6]{41}. These rings are graded by a finitely generated abelian group $K$, such that the set of graded ideals has a unique maximal element. In the original definition, this ideal does not need to be a maximal ideal, but as we will see, in our context it is always so. The equivalent notion of $\mathbb{C}^*$-local rings is considered in \cite[Definition 1.5.13]{23}.

In \cite[Theorem 2.5]{60}, it is proved that a graded ring is graded-local if and only if its degree zero part is a local ring in the classical sense. In particular, the graded maximal ideal is generated by all homogeneous nonunits. As the graded maximal ideal is a maximal ideal in our context, it is straightforward to see that, in fact, $m = m_0 \oplus \bigoplus_{k \neq 0} A_k$. We will use the following definition, including the restriction that the ring is finitely generated over the degree zero part.

**Definition 2.2.** Let $K$ be a finitely generated abelian group. Let

$$A^{(K)} = \bigoplus_{k \in K} A^{(K)}_k$$

be a $K$-graded noetherian integral domain. Then, we call $A^{(K)}$ a gr-local ring if

1. the set of graded ideals of $A^{(K)}$ has a unique maximal element $m$, which is a maximal ideal in the usual sense, and
2. the degree-zero part $A^{(K)}_0$ is a local ring with maximal ideal $m_0 = m \cap A^{(K)}_0$ and $A^{(K)}$ is finitely generated as an algebra over $A^{(K)}_0$.

**Example 2.3.** Let $X$ be a quasi-cone. Then, $A := \mathcal{O}_X(X)$ is a gr-local ring with $A_0 = \mathbb{C}$. If in addition $X$ is an MDS, then the Cox ring $\text{Cox}(X)$ has a $(\text{Cl}(X) \times \mathbb{Z})$-grading that endows it with the structure of a gr-local ring. Note that the degree zero part $A = (\text{Cox}(X)^{\text{Cl}(X)})_0$ with respect to the coarsened grading by $\text{Cl}(X)$ is a gr-local but not a local ring, so $\text{Cox}(X)^{\text{Cl}(X)}$ is not gr-local. On the other hand, if $X$ is a projective MDS, then $\text{Cox}(X)^{\text{Cl}(X)}$ is a gr-local ring. Indeed, in this case, $(\text{Cox}(X)^{\text{Cl}(X)})_0$ is just the ground field and thus a local ring.

**Remark 2.4.** We remark that since $A$ is noetherian, every homogeneous component $A_k$ is a finite $A_0$-module. To see this, consider an ascending chain of finitely generated $A_0$-submodules $N_i \subseteq A_k$. By noetherianity of $A$, the chain of ideals $A \cdot N_i$ stabilizes after finitely many steps. Since the ideals $A \cdot N_i$ are homogeneous, we have the equality $N_i = A \cdot N_i \cap A_k$, and thus, the sequence $N_i$ stabilizes after finitely many steps as well.
Similar to the process of localizing at a prime ideal, we can graded-localize at a graded prime ideal. This process gives us a gr-local ring.

**Definition 2.5.** Let $A$ be a $K$-graded ring and $\mathfrak{p}$ be a graded prime ideal. Let $S$ be the set of homogeneous elements of $A \setminus \mathfrak{p}$. Then,

$$A_{(\mathfrak{p})} := S^{-1}A$$

is the gr-localization of $A$ at $\mathfrak{p}$. It is a graded-local ring. It is not necessarily finitely generated over $(A_{(\mathfrak{p})})_0$. Furthermore, the unique maximal graded ideal is not necessarily maximal in the usual sense.

It is straightforward to see that the unique graded maximal ideal of $A_{(\mathfrak{p})}$ is maximal if and only if $\mathfrak{p}$ is maximal.

**Example 2.6.** Consider $\mathbb{C}[x, y]$ with the $\mathbb{Z}$-grading given by $\deg(x) = 1$ and $\deg(y) = -1$. We consider gr-localizations at different graded prime ideals $\mathfrak{p}$. We study them by understanding which scheme points of $\mathbb{A}^2$ define scheme points of $\text{Spec} \mathbb{C}[x, y]_{(\mathfrak{p})}$.

When gr-localizing at $\mathfrak{p} = \langle x, y \rangle$, all closed points on the coordinate axes of $\mathbb{A}^2$ define scheme points of $\text{Spec} \mathbb{C}[x, y]_{(\mathfrak{p})}$. Among the curves, the coordinate axes are the only closures of $C^*$-orbits that define points on the GIT quotient.

If instead, we gr-localize at the coordinate axis $\langle x \rangle$, which is not even graded maximal since it is contained in the graded $\langle x, y \rangle$, only the closed points (different from the origin) on this axis survive. The other axis together with its points vanishes, as all other closures of $C^*$-orbits do. Moreover, those curves that do not meet the axis $\langle x \rangle$ become closed points.

If finally, we gr-localize at the orbit $\langle xy - 1 \rangle$, which is graded maximal but not maximal, the surviving closed points are exactly those that lie on this curve.

We remark that localization at a prime $\mathfrak{p}$ factors through gr-localization at $\mathfrak{p}$. In the following, we collect some useful properties that gr-local rings possess. First, we note that our definition of gr-local contains the finite-generation property over the degree-zero part $A_0$. Hence, if $A_0$ is essentially of finite type over $\mathbb{C}$, then so is $A$ (see, e.g., [43, Proposition 1.3.9]).

The second advantage of the finite-generation property is that the arguments of [8, Section 1.2] apply. Indeed, we have a surjection $A_0 \otimes \mathbb{C}[x_1, \ldots, x_n] \to A$. In particular, we have an equivalence of categories between gr-local rings and affine schemes of finite type over the spectrum of a local ring with a quasi-torus action.

**Example 2.7.** We consider $\mathbb{C}[x, y]$ with the $\mathbb{Z}$-grading given by the weight $(1, -1)$ and the maximal graded ideal $\mathfrak{m} = \langle x, y \rangle$. The degree zero part is $\mathbb{C}[xy]$ and the set of homogeneous elements of $\mathbb{C}[x, y] \setminus \mathfrak{m}$ is just $S := \mathbb{C}[xy] \setminus \langle xy \rangle$. On the other hand, graded localizing gives the gr-local ring $\mathbb{C}[x, y]_{(\mathfrak{m})}$, which has degree-zero elements of the form $\frac{f}{g}$, where $f \in \mathbb{C}[x, y]$, $g \in S$, and the degree of $f$ equals the degree of $g$. Since the elements of $S$ are of degree zero, both $f$ and $g$ are of degree zero. We conclude that there is an isomorphism

$$(\mathbb{C}[x, y]_{(\mathfrak{m})})_0 \cong \mathbb{C}[xy]_{(xy)}.$$
This means that we can see the local ring at the origin of the $\mathbb{C}^*$-quotient $\mathbb{C} = \text{Spec } \mathbb{C}[x, y]_0$ as a $\mathbb{C}^*$-quotient of the gr-local ring $\mathbb{C}[x, y]_{(m)}$. In particular, this quotient parametrizes orbits in the usual way good quotients do: closed points parametrize closed orbits.

We remark here, that since we work in characteristic zero, categorical GIT-quotients by reductive groups are universally categorical, that is, they behave well under base change, see [68, Theorem 1.1]. The following lemma shows what happens when the grading group is finite.

**Lemma 2.8.** Let $K$ be a finite abelian group and $A$ be a $K$-graded gr-local ring. Then $A$ is local.

**Proof.** We have to show that the unique maximal and graded ideal $m$ is the only maximal ideal. Assume that there is another maximal ideal, which is not graded by the uniqueness property. Consider the corresponding closed point $x \in \text{Spec } A$. Let $G \cong K$ be the finite abelian group acting on $\text{Spec } A$. The orbit $Gx = G\overline{x}$ is closed; hence, its image $y'$ under the quotient morphism $\text{Spec } A \to \text{Spec } A_0$ is a closed point. Since $A_0$ is local, the point $y$ coincides with the point corresponding to the maximal ideal $m_0 = m \cap A_0$. But since $Gx$ is closed, it coincides with the orbit $x_m$. This leads to a contradiction. □

In particular, we see that our gr-local rings as defined in Definition 2.2 satisfy the classical definition of graded local rings (see, e.g., [41, Definition 1.1.6]), where only free grading groups are allowed. Indeed, the degree-zero part — with respect to the free part of the grading group — of a gr-local ring is always local, as the above lemma shows.

We finish this subsection proving that the class group of the spectrum of a gr-local ring is concentrated at the unique maximal graded ideal. In particular, the following holds.

**Lemma 2.9.** Let $A$ be a gr-local ring, $X := \text{Spec } A$, and $x \in X$ be the closed point corresponding to the unique graded maximal ideal $m$. Then

$$\text{Cl}(X) \cong \text{Cl}(X_x) \quad \text{and} \quad \text{Pic}(X) \cong \text{Pic}(X_x) \cong 0.$$ 

**Proof.** By [71, Proposition 7.1], the class group $\text{Cl}(X)$ is isomorphic to the group of graded divisorial ideals modulo the subgroup of principal graded ideals. Thus, we only have to show that for an ideal $I \subseteq A$, if $IA_m$ is principal, then $I$ is already principal. Let $a \in A_m$ be a generator of $IA_m$, which we can assume to be a graded element of $A$. Then, for a graded $x \in I$, there are $p \in A$ and $q \in S = A \setminus m$, such that $x = \frac{p}{q} a$ in $A_m$. So, $xq = pa$ holds in $A$. Writing $q_1$ for the graded components of $q$, we know that $q_0$ is nonzero and a unit. Thus, there is a homogeneous component $p_k$ of $p$, such that $xq_0 = p_k a$. Hence, $I$ is generated by $a$ in $A$.

We recall that in a local ring $R$ with maximal ideal $m$, $I$ being locally free of rank one implies $I = I_m \cong R_m = R$, that is, $I$ being free of rank 1. The argument for triviality of the Picard group is thus the same as in the proof of [69, Lemma 5.1]. □

The advantage of the notion of gr-local rings is that not only it will encompass the local Cox rings of singularities, but also it stresses the grading. Note that this provides us with a meaningful notion of finite generation for Cox rings of (spectra of) local rings: the Cox ring should be finitely generated (as an algebra) over the local ring itself.
Remark 2.10. If the variety $X$ is an MDS, then the Cox ring $\text{Cox}(X)$ may have no grading that makes it a gr-local ring. However, we will see later that the Cox sheaf

$$R_X = \bigoplus_{[D] \in \text{Cl}(X)} \mathcal{O}_X(D)$$

is always a sheaf of gr-local rings (see Definition 2.16).

2.3 Graded-Henselian rings

In this subsection, we recall the concept of gr-Henselian rings and prove some preliminary results about their class groups.

The local rings in the Zariski topology are too coarse to capture the local topology at a singularity well. Thus, in the following, we consider also the local rings in the étale topology, which are Henselian local rings. The resulting Cox rings are gr-local rings with a Henselian degree-zero part. Such rings were studied in [24] and are called gr-Henselian rings.

Definition 2.11 (Cf. [24]). Let $A$ be a gr-local ring. Then, we say that $A$ is gr-Henselian, if it satisfies one of the following equivalent conditions:

1. $A_0$ is Henselian and
2. every graded $A$-algebra is a direct sum of gr-local rings.

In fact, [24, Theorem 4.6] contains several more equivalent characterizations analogous to those for Henselian rings. For us, maybe the most important property of gr-Henselian rings is the following.

Theorem 2.12. Let $A$ be an excellent rational $\mathbb{Z}^k$-graded gr-Henselian ring. Let $A^h_m$ be the Henselization of the local ring at the unique maximal graded ideal $m$ and $\hat{A}$ be the $m$-adic completion. Assume that the graded prime ideals $\mathfrak{p}$ of height one in $A$ are in one-to-one-correspondence with the height one prime ideals in $A_0$ via $\mathfrak{p} = \mathfrak{p}_0 A$. Furthermore, we assume that the same property holds for the base change $\hat{\tilde{A}} := A \otimes_{A_0} \hat{A}_0$. Then, we have isomorphisms

$$\text{Cl}(A) \cong \text{Cl}(A^h_m) \cong \text{Cl}(\hat{A}).$$

The assumptions on the height one prime ideals are not as restrictive as they may seem. They are fulfilled for gr-local rings if the morphisms $\text{Spec}(A) \to \text{Spec}(A_0)$ and $\text{Spec}(\hat{A}) \to \text{Spec}(\hat{A}_0)$ are locally trivial fiber bundles in codimension 1.

To prove the theorem, we follow the line of arguments of [38, Section 1 & 2], where an analogous result is proved for $\mathbb{N}$-graded rational rings. Before we can use the results from [38], we have to prove the following lemma.

Lemma 2.13. Let $A$ be a $\mathbb{Z}^k$-graded gr-local ring with maximal graded ideal $m = m_0 \oplus \bigoplus_{k \neq 0} A_k$. Then, the degree $k$ piece of the $m$-adic completion $\hat{A} = \prod_{k \in \mathbb{Z}^k} \varprojlim A_k/(m^i)_k$ is isomorphic to the
m₀-adic completion of the A₀-module A_k. This means that we have isomorphisms

\[ \lim A_k / (m^l)_k \cong \lim A_k / (m_0)_k A_k =: \hat{A}_k. \]

Proof. Let \( g_{01}, \ldots, g_{0n_0}, g_{11}, \ldots, g_{1n_1}, \ldots, g_{1n_1}, \ldots, g_{mn_m} \) be a finite set of homogeneous A-module generators of \( \mathfrak{m} \) that is also a set of A₀-module generators of A, where \( g_{ij} \in A_j \) for \( 1 \leq j \leq n_j \). In the following, by degree, we mean the standard degree of a monomial \( m(g_{ij}) \). Otherwise, we speak of the \( \mathbb{Z}_K \)-degree.

Since \( \mathfrak{m}^l \) is generated as an A-module by all monomials in the \( g_{ij} \) of degree \( l \), we know that \( (\mathfrak{m}^l)_k \) is generated as an A₀-module by all monomials in the \( g_{ij} \) of \( \mathbb{Z}_K \)-degree \( k \) and degree at least \( l \). Indeed, a monomial of degree \( l \) and nonzero \( \mathbb{Z}_K \)-degree \( k \) needs an A-coefficient in \( A_{-k} \) in order to lie in A₀, and expanding this coefficient in A₀-module generators of \( A_k \) that are monomials in the \( g_{ij} \) leads to monomials in the \( g_{ij} \) of degree at least \( l \).

On the other hand, by Gordan’s lemma, we know that there are finitely many monomials \( m_1, \ldots, m_M \) of \( \mathbb{Z}_k \)-degree 0 in the \( g_{ij} \) such that any other monomial in the \( g_{ij} \) of \( \mathbb{Z}_k \)-degree 0 is a monomial in the \( m_1, \ldots, m_M \). We set \( \mu := \max(\deg(m_i))_{i=1,\ldots,M} \). So, if a monomial in the \( g_{ij} \) of \( \mathbb{Z}_k \)-degree 0 is of degree \( \geq \nu \mu \), then we know that it is a monomial in the \( m_i \) of degree \( \geq \nu \). Thus, we get

\[ (m_0)_{0}^{\nu \mu} \subseteq (m_0)_{0}^{\nu} \]

for \( \nu \geq 1 \). So, the claim follows for \( k = 0 \). The argument for the A₀-modules \( A_k \) is similar. There are finitely many monomials \( m_{k,1}, \ldots, m_{k,M_k} \) in the \( g_{ij} \) of \( \mathbb{Z}_K \)-degree \( k \), which generate \( A_k \) as an A₀-algebra, cf. Remark 2.4. So, all other monomials in the \( g_{ij} \) of \( \mathbb{Z}_k \)-degree \( 0 \) are products of some \( m_{k,j} \) with a monomial in the \( g_{ij} \) of \( \mathbb{Z}_k \)-degree 0. Set \( \mu_k \) to be the maximal degree of these finitely many monomials. Then, we get

\[ (m_0)_{0}^{\nu \mu_k} A_k \subseteq (m_0)_{0}^{\nu} A_k, \]

and the claim is proved.

We need two additional lemmas and use the following definitions. Let \( B \) be the \( \mathfrak{m} \)-adic completion of \( A[t_1, t_1^{-1}, \ldots, t_k, t_k^{-1}] \). Denote \( A \llbracket x \rrbracket := A \llbracket x_1, \ldots, x_k \rrbracket \) and let \( p : \hat{A} \to A \llbracket x \rrbracket \) and \( p_0 : \hat{A} \to B \) be the canonical injections. Let \( q : \hat{A} \to A \llbracket x \rrbracket \) be the homomorphism defined by

\[ A(z_1, \ldots, z_k) \ni f \mapsto f \cdot (x_1 + 1)^z_1 \cdots (x_k + 1)^z_k, \]

where \( (x_i + 1)^{-1} = \sum_{j=1}^{\infty} (-x_i)^j \). Further, let \( q_0 : \hat{A} \to B \) be the homomorphism defined by

\[ A(z_1, \ldots, z_k) \ni f \mapsto f \cdot t_1^{z_1} \cdots t_k^{z_k}, \]

and \( g : B \to A \llbracket x \rrbracket \) the \( \hat{A} \)-homomorphism mapping \( t_i \) to \( x_i + 1 \). Observe that the equalities \( g \circ p_0 = p \) and \( g \circ q_0 = q \) hold. We prove the following lemma.

Lemma 2.14. The map \( g_* : \text{Cl}(B) \to \text{Cl}(A \llbracket x \rrbracket) \) is injective.
Proof. Let \( b \) be a divisorial ideal of \( B \) in the kernel of \( g \). We note that \( \hat{A}[x] \) is the \( \langle t_1 - 1, \ldots, t_k - 1 \rangle \)-adic completion of \( B \). For any prime \( p \subseteq \hat{A} \), the ring extension \( B_p \to \hat{A}[x]_{p} \) is faithfully flat. Thus, due to principality of \( b \otimes_p \hat{A}[x] \), also \( b \cdot B_p \) is principal for any prime \( p \subseteq \hat{A} \).

We want to show that \( b \) is locally principal, that is, \( b_p \) is principal for any prime \( p \). So, let \( n \subseteq B \) be maximal and \( \mathfrak{m} = \hat{A} \cap n \) be the unique maximal ideal of \( \hat{A} \). Now, we have an isomorphism \( \hat{A}/\mathfrak{m} \cong B/n \) and the local homomorphism \( \hat{A} \to B_n \) of local rings is formally smooth. Then, \([16, II, Corollaire 9.8]\) implies that \( B_{q} \) is parafactorial for any prime \( q \subseteq n \) with \( q \nsubseteq \mathfrak{m} B \) and \( \dim(B_{q}) \geq 2 \). Due to normality of \( B_n \) and by induction on \( \dim(B_{q}) \), we get that \( b_q \) is principal. Thus, \( b \) is locally principal. But since \( B \) is \( \mathfrak{m} B \)-adically complete and \( A/\mathfrak{m} \) is a field, we get

\[
\text{Pic}(B) \cong \text{Pic}(B/\mathfrak{m} B) \cong \text{Pic}((A/\mathfrak{m})[t_1, t_1^{-1}, \ldots, t_k, t_k^{-1}]) \cong 0.
\]

So, \( b \) is principal. \( \square \)

Lemma 2.15. Under the assumptions of Theorem 2.12, the sequence

\[
0 \to \text{Cl}(A) \to \text{Cl}(\hat{A}) \xrightarrow{p_0^* - q_0^*} \text{Cl}(B)
\]

is exact.

Proof. We define

\[
\hat{A} := A \otimes_{A_0} \hat{A}_0 = \bigoplus_{k \in \mathbb{Z}^K} \left( A_k \otimes_{A_0} \hat{A}_0 \right) = \bigoplus_{k \in \mathbb{Z}^K} \hat{A}_k \cong \bigoplus_{k \in \mathbb{Z}^K} \lim_{\rightarrow} A_k / (\mathfrak{m}^k)_k,
\]

where hats denote \( \mathfrak{m}_0 \)-adic completion, the second identity is due to the fact that the \( A_k \) are finitely generated \( A_0 \)-modules and the isomorphism is due to Lemma 2.13. The \( A_0 \)-algebra-homomorphism \( A \to \hat{A} \) factors through \( A \to \hat{A} \) and \( \hat{A} \to \hat{A} \). By \([38, \text{Section 2}]\), it follows that for any height one prime ideal \( \hat{a} \) of \( \hat{A} \) such that \( q_0^*(\hat{a}) = p_0^*(\hat{a}) \) in \( \text{Cl}(B) \), there is a graded height one ideal \( \hat{a} \subseteq \hat{A} \) such that \( \hat{a} = \hat{a} \). Then, \( \hat{a}_0 \) is an ideal of \( \hat{A}_0 \) of height one. In particular, \( \hat{a}_0, \hat{A} = \hat{a} \) by the assumptions of the theorem. Since \( A_0 \) is rational, by \([14, \text{Theorem (6.2)}]\), there is a height one prime ideal \( a_0 \) of \( A_0 \), such that \( \hat{a}_0 = \hat{a}_0 \). Then, \( a_0 A \) is a height one prime of \( A \) such that \( a_0 A \otimes_{A_0} \hat{A}_0 = \hat{a} \). So, the equalizer of \( q_0^* \) and \( p_0^* \) indeed equals the image of \( \text{Cl}(A) \) in \( \text{Cl}(\hat{A}) \). This concludes the proof of the lemma. \( \square \)

Proof of Theorem 2.12. Since \( A \) is excellent and rational, the completion \( \hat{A} \) has the DCG property. This means that \( \pi^* : \text{Cl}(\hat{A}[x]) \to \text{Cl}(\hat{A}) \) induced by \( \pi : \hat{A}[x] \to \hat{A} \) mapping \( x \) to 0 is a bijection, see \([38, \text{p. 128}]\). But then \( \hat{A}[x_1, \ldots, x_k] \) has the DCG property. Moreover, \( \omega : \hat{A}[x_1, \ldots, x_k] \to \hat{A} \) mapping all the \( x_i \) to 0 induces a bijection \( \omega_* \) between the divisor class groups by induction. But since \( \omega \circ \pi = \omega \circ \omega \circ \pi \), \( p = g \circ p_0 \), and \( q = g \circ q_0 \), by Lemma 2.14 and Lemma 2.15, we get \( p_0 = q_0 \) and \( \text{Cl}(A) \to \text{Cl}(\hat{A}) \) is surjective and hence bijective. Since this map factors through \( \text{Cl}(A) \to \text{Cl}(A^h_{\mathfrak{m}_0}) \), which is injective, bijectivity of the three divisor class groups follows as claimed. \( \square \)
2.4 Sheaves of gr-local rings

In this subsection, we define sheaves of gr-local rings on algebraic varieties.

Throughout this subsection, we consider the case that $X$ is only locally an MDS, that is, its Cox sheaf $R_X$ is locally of finite type in the sense of [8, Constr. 1.3.2.1]. This means that every $x \in X$ has an open affine neighbourhood $U$, such that $R_X(U)$ is a finitely generated $\mathbb{C}$-algebra. This makes it possible to define the relative spectrum $\hat{X} := \text{Spec}_X R_X$ of the Cox sheaf, the so-called characteristic space of $X$. However, it may happen that the ring of global sections $\text{Cox}(X)$ is not a finitely generated $\mathbb{C}$-algebra.

In this subsection, we define sheaves of gr-local rings, and we show that a Cox sheaf is locally of finite type in the aforementioned sense if and only if it is a sheaf of gr-local rings. This means, in particular, that this property has to be checked only locally at the singularities whenever the divisor class group $\text{Cl}(X)$ is finitely generated.

**Definition 2.16.** Let $X$ be a normal variety. Let $\mathcal{S}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. If the stalk $\mathcal{S}_x$ of $\mathcal{S}$ at any point of $X$ is a gr-local ring, then we call $\mathcal{S}$ a sheaf of gr-local rings.

We recall from [8, Def 1.3.1.1], that the sheaf of divisorial algebras associated to a finitely generated subgroup $K \subseteq \text{WDiv}(X)$ is the quasi-coherent sheaf $\mathcal{S} := \bigoplus_{D \in K} \mathcal{S}_D$, $\mathcal{S}_D := \mathcal{O}_X(D)$.

**Definition 2.17.** Let $X$ be a normal variety. Let $\mathcal{S}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{S}$ is locally of finite type if for every point $x \in X$, there is an open affine neighborhood $x \in U$ with $\mathcal{S}(U)$ a finitely generated $\mathbb{C}$-algebra.

**Lemma 2.18.** Let $X$ be a normal algebraic variety and $\mathcal{S}$ a sheaf of divisorial algebras associated to the finitely generated subgroup $K \subseteq \text{WDiv}(X)$. Then, the stalk $\mathcal{S}_x$ is a finitely generated $\mathcal{O}_{X,x}$-algebra for any $x \in X$ if and only if $\mathcal{S}$ is locally of finite type.

**Proof.** First, let $\mathcal{S}$ be a sheaf of divisorial algebras locally of finite type and $U \subseteq X$ be affine. It follows that for some $k \in \mathbb{N}$, there is a surjection $\mathcal{O}_X(U)[x_1, \ldots, x_k] \to \mathcal{S}(U)$. Since surjectivity of $R$-modules is local, this induces a surjection $\mathcal{O}_{X,x}[x_1, \ldots, x_k] \to \mathcal{S}_x$, and thus, $\mathcal{S}_x$ is a finitely generated $\mathcal{O}_{X,x}$-algebra for every $x \in U$.

Now, fix $x \in X$ and assume that $\mathcal{S}_x$ is a finitely generated $\mathcal{O}_{X,x}$-algebra. We fix a set of generators $D_1, \ldots, D_m$ of $K$. Then, there is an open affine neighborhood $x \in U \subseteq X$, such that $x$ lies in every irreducible component of $D_i \cap U$ for any $i$. Let $f_1, \ldots, f_k \in \mathcal{S}_x$ be a finite set of $K$-homogeneous $\mathcal{O}_{X,x}$-algebra-generators of the stalk $\mathcal{S}_x$. By shrinking $U$ if necessary, we can lift these germs to sections $f_1, \ldots, f_k \in \mathcal{S}(U)$, such that $x$ lies in every irreducible component of $\text{supp}(f_i)$ for any $i$.

Now let $D \in K$. We have a primary decomposition $S_D(U) = q_1 \cdots q_r$ of the divisorial ideal $S_D(U) = \Gamma(\mathcal{O}_X(D), U)$, such that the associated primes $p_i$ all lie in $m_x$. In particular, $\text{sat}_{m_x}(S_D(U)) = S_D(U)$, see, for example, [10, Proposition 4.9]. The localization $S_{x,D}$ of $S_D(U)$ is generated as an $\mathcal{O}_{X,x}$-module by monomials $p_1, \ldots, p_i$ in the $f_i$. In particular, $x$ lies in every irreducible component of $\text{supp}(p_i)$ for any $i$. So, the $\mathcal{O}(U)$-module $J := \sum_{i=1}^r \mathcal{O}(U)p_i$ has localization $S_{x,D} = \sum_{i=1}^r \mathcal{O}_U x p_i$ and saturation $\text{sat}_{m_x}(J) = J$. Thus, $J = S_D(U)$ and $\mathcal{S}(U)$ is generated as an $\mathcal{O}(U)$-algebra by the $f_i$. The proof is finished.
**Corollary 2.19.** Let $X$ be a normal algebraic variety such that $\text{Cl}(X)$ is finitely generated. Then, $R_X$ is a sheaf of gr-local rings if and only if it is locally of finite type.

**Example 2.20.** If $X$ is a point, then a sheaf of gr-local rings over $X$ is a gr-local ring.

### 2.5 Coverings of gr-local rings

In this subsection, we bring together the concepts of fundamental group and Cox rings.

There are different notions for the regional fundamental groups of singularities. In the case of a klt singularity, they all agree. Let $x \in (X, \Delta)$ be a klt singularity. Then, the regional fundamental group $\pi_1^{\text{reg}}(X, \Delta; x)$ is the inverse limit of the orbifold fundamental groups $\pi_1(U_{\text{reg}}, \Delta; x)$, where $U$ runs through analytic open neighborhoods of $x$. See [18, Definition 5.1] for the definition of orbifold fundamental group. The regional fundamental group is computed by some neighborhood $U$, which can be chosen to be the intersection of $X$ with a small euclidean ball around $x$ in some complex manifold $M \supseteq X$. It equals the fundamental group of the regional link of $x$, the intersection of $X_{\text{reg}}$ with a small euclidean sphere, which is just a deformation retract of $U_{\text{reg}}$.

However, when we work in the algebraic category, we deal with étale neighborhoods of local rings. In the case of klt singularities, this makes no difference. This follows from the fact that the regional fundamental group is finite by [18, Theorem 1]. In particular, $\pi_1^{\text{reg}}(X, \Delta; x)$ equals the étale fundamental group of the smooth locus of the spectrum of the holomorphic local ring $\mathcal{O}_{X, x}^{\text{hol}}$. Since this ring is Henselian, by [36, Corollary, p. 579], we have

$$\pi_1^{\text{reg}}(X, \Delta; x) \cong \pi_1^{\text{et}}(X_{x, \text{reg}}^h, \Delta_{\text{reg}}^h) \cong \pi_1^{\text{et}}(\hat{X}_{x, \text{reg}}, \hat{\Delta}_{\text{reg}}).$$

Here, $X_{x}^h$ and $\hat{X}_{x}$ denote the spectra of the étale and complete local rings. The subscript (or superscript) reg means that we consider the regular locus. Furthermore, the divisor $\Delta_{\text{reg}}^h$ (resp. $\hat{\Delta}_{\text{reg}}$) is the pull-back of $\Delta$ to $X_{x, \text{reg}}^h$ (resp. $\hat{X}_{x, \text{reg}}$). Since $\pi_1^{\text{reg}}(X, \Delta; x)$ is finite, it is computed by an affine étale neighborhood $V_x \to X$ of $x$. Moreover, by [14, Section 6], we know that $\text{Cl}(X_{x}^h)$ and $\text{Cl}(\hat{X}_{x})$ are finitely generated and isomorphic. Thus, we can find an affine étale neighborhood $\psi : U_x \to X$ of $x$ that computes both the regional fundamental group and the local divisor class group, that is, there are isomorphisms $\pi_1(U_{x}, \psi^{-1}(\Delta)) \to \pi_1^{\text{reg}}(X, \Delta; x)$ and $\text{Cl}(U_x) \to \text{Cl}(X_x)$ induced by $\psi$.

We will use these facts often throughout the article.

### 2.6 Minimal model program

In this subsection, we recall the definition of the singularities of the minimal model program. We also recall some basic constructions as the purely log terminal blow-up.

**Definition 2.21.** A projective morphism $f : X \to Z$ is called a contraction if $f_*\mathcal{O}_X = \mathcal{O}_Z$. In particular, if $X$ is normal and $X \to Z$ is a contraction, then $Z$ is normal as well.

On the other hand, if $g : X \to Z$ is an affine morphism, then $X$ is isomorphic to the relative spectrum over $Z$ of the direct image sheaf $f_*\mathcal{O}_X$, that is, $X \cong \text{Spec}_Z f_*\mathcal{O}_X$. So, if $h = f \circ g : X \to Y \to Z$ is an affine morphism $g$ composed with a contraction $f$, then $h_*\mathcal{O}_X$ is a quasi-coherent...
sheaf of $\mathcal{O}_Z$-modules that is locally of finite type. Morphisms of this kind will become important in the following.

**Definition 2.22.** A morphism $h : X \to Z$ that factors through an affine morphism $g : X \to Y$ and a contraction $f : Y \to Z$ is called an *aff-contraction*.

**Example 2.23.** Let $X$ be a projective MDS with structure morphism $\phi : X \to \text{Spec}(\mathbb{C})$. Let $\psi : \hat{X} = \text{Spec}_X R_X \to X$ be its characteristic space. Then, $h := \phi \circ \psi$ is an aff-contraction.

**Definition 2.24.** Let $X$ be a normal quasi-projective variety. A *log pair* $(X, \Delta)$ consists of $X$ and an effective divisor $\Delta \geq 0$ so that $K_X + \Delta$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor.

**Definition 2.25.** Let $(X, \Delta)$ be a log pair. A *primed divisor* over $X$ is a primed divisor on a normal quasi-projective variety $Y$ that admits a projective birational morphism to $X$. This means that there exists a projective birational morphism $\pi : Y \to X$ and $E \subset Y$ a prime divisor. The *log discrepancy* of $(X, \Delta)$ at $E$ is defined to be

$$a_E(X, \Delta) := 1 + \text{coeff }_E (K_Y - \pi^*(K_X + \Delta)).$$

A log resolution of a log pair $(X, \Delta)$ is a projective birational morphism $\pi : Y \to X$ so that $Y$ is a regular variety, the exceptional divisor $E$ is purely divisorial, and $E_{\text{red}} + \pi_*^{-1} \Delta$ has simple normal crossings support. Any log pair admits a log resolution by Hironaka’s resolution of singularities.

**Definition 2.26.** A pair $(X, \Delta)$ is said to be *Kawamata log terminal* (or *klt* for short) if all its log discrepancies are positive. This means that $a_E(X, \Delta) > 0$ for every prime divisor $E$ over $X$. A pair $(X, \Delta)$ is said to be *log canonical* (or *lc* for short) if all its log discrepancies are nonnegative. This means that $a_E(X, \Delta) \geq 0$ for every prime divisor $E$ over $X$. In both cases, it suffices to check all the prime divisors that appear on an arbitrary log resolution of the pair. A *non-klt* center of a pair $(X, \Delta)$ is the image on $X$ of a prime divisor $E$ over $X$ for which $a_E(X, \Delta) \leq 0$. In particular, if $(X, \Delta)$ is a log canonical pair, a non-klt center is the image on $X$ of a divisor with log discrepancy zero.

**Definition 2.27.** A variety $X$ is said to be *klt type* if there exists a boundary $\Delta$ so that $(X, \Delta)$ is a klt pair. Analogously, we say that a germ $(X, x)$ is *klt type* if there exists $\Delta$ through $x$ so that $(X, \Delta)$ is a klt germ.

**Definition 2.28.** A pair $(X, \Delta)$ is called *divisorially log terminal* or *dlt* if there exists an open subset $U \subset X$ satisfying the following conditions:

1. $U$ is smooth and $\Delta|_U$ has simple normal crossing support,
2. the coefficients of $\Delta$ are at most one,
3. all the nonklt centers of $(X, \Delta)$ intersect $U$ and are given by strata of the divisor $\lfloor \Delta \rfloor$.

A pair $(X, \Delta)$ is said to be *purely log terminal* or *plt* if it is dlt and it has at most one nonklt center.

**Definition 2.29.** Let $X \to Z$ be a contraction and $(X, \Delta)$ be a log pair. We say that $(X, \Delta)$ is of *Fano type* over $Z$ if there exists a boundary $\Delta'$ on $X$ that is big over $Z$, so that $(X, \Delta + \Delta')$ is klt and $K_X + \Delta + \Delta' \sim_{\mathbb{Q}, Z} 0$. 
**Definition 2.30.** Let \((X, \Delta; x)\) be a klt singularity. A purely log terminal blow-up of \((X, \Delta)\) at \(x\) (or a plt blow-up for short) is a projective birational morphism \(\pi : Y \to X\) satisfying the following conditions:

1. \(\pi\) is an isomorphism on the complement of \(x\),
2. the preimage of \(x\) on \(Y\) is a unique prime divisor \(E\),
3. the pair \((Y, E + \Delta_Y)\) is plt, where \(\Delta_Y := \pi_*^{-1}(\Delta)\), and
4. \(-E\) is ample over \(X\).

In particular, the log pair \((E, \Delta_E)\) obtained by adjunction of \((Y, E + \Delta_Y)\) to \(E\) is of Fano type.

In this article, we will be concerned with orbifold structures on Fano-type varieties and klt singularities. Therefore, we will need the following definitions.

**Definition 2.31.** We say that the coefficients of \(\Delta\) are standard if they have the form \(1 - \frac{1}{n}\), where \(n\) is a positive integer. Given a boundary \(\Delta\) on \(X\), we define its standard approximation to be the effective divisor \(\Delta_s\) on \(X\) with largest standard coefficients such that \(\Delta \geq \Delta_s\). Note that if \((X, \Delta)\) is of Fano type over \(Z\), then \((X, \Delta_s)\) is of Fano type over \(Z\) as well.

The following is the definition of one of the main kind of covers that we will consider in this article.

**Definition 2.32.** Let \((X, \Delta; x)\) be a klt singularity. We say that \(\phi : (Y, \Delta_Y) \to (X, \Delta)\) is a finite Galois log quasi-étale cover if the following conditions are satisfied.

1. There exists a finite group \(G\) acting on \(Y\).
2. \(X\) is the quotient of \(Y\) by \(G\).
3. The pull-back of \(K_X + \Delta\) equals \(K_Y + \Delta_Y\), where \(\Delta_Y\) is effective.

Note that \(Y \to X\) may not be unramified in codimension 1. However, it is unramified in codimension 1 when considering \((X^{\text{reg}}, \Delta^{\text{reg}})\) as an orbifold. This justifies the log quasi-étale property. If \(\Delta = 0\), then we drop the “log” from the notation and we simply say that \(\phi\) is a finite Galois quasi-étale morphism. We say that \((Y, y)\) is a pointed finite Galois log quasi-étale cover of \((X, \Delta; x)\) if \(y \in Y\) is a point whose image on \(X\) is \(x\). To shorten the notation, we may say that \(Y \to X\) is a pointed finite cover of \((X, \Delta; x)\).

### 3 GENERALIZED COX RINGS

In this section, we generalize the concept of Cox rings to different settings and prove some basic properties. In Subsection 3.1, we will define the Cox ring of a log pair and study its properties. In Subsections 3.2 and 3.3, we introduce the relative Cox ring and the local Cox ring, respectively. In Subsection 3.4, we prove some properties of the above generalizations. For instance, we prove that the Cox ring of a relatively Fano-type variety admits the structure of a klt-type singularity (Theorem 3.26). Finally, in Subsection 3.5, we will define the local Henselian Cox ring of a singularity. This is one of the main objects considered in this article.
3.1 The Cox ring of a log pair

In this subsection, we generalize the Cox ring and related notions to log pairs \((X, \Delta)\), where \(X\) is a normal algebraic variety and \(\Delta\) is an effective divisor on \(X\). We prove some basic properties of the Cox ring of a log pair and show that these objects become interesting even for log pair structures on \(\mathbb{P}^1\).

In the case that \(\Delta\) has standard coefficients, such pairs can be viewed as geometric orbifolds in the sense of Campana [25]. We proceed to define the class group \(\text{Cl}(X, \Delta)\) of a log pair \((X, \Delta)\). We will define the class group \(\text{Cl}(X, \Delta)\) to be \(\text{Cl}(X, \Delta_s)\), where \(\Delta_s\) is the standard approximation. Hence, it suffices to define the class group for standard pairs.

**Definition 3.1.** Let \((X, \Delta)\) be a log pair and \(\Delta_s\) be the standard approximation of \(\Delta\). For a prime divisor \(P \subset X\), we write \(n_P\) for the positive integer for which \(\text{coeff}_P(\Delta_s) = 1 - \frac{1}{n_P}\). We define \(\text{WDiv}(X, \Delta)\) to be the group generated by \(\mathbb{Q}\)-divisors \(D\) on \(X\) so that for every prime divisor \(P \subset X\), the denominator of \(\text{coeff}_P(D)\) divides \(n_P\). The group \(\text{Cl}(X, \Delta)\) is defined to be the group of orbifold Weil divisors \(\text{WDiv}(X, \Delta)\) quotiented by linear equivalence of \(\mathbb{Q}\)-divisors. Here, two \(\mathbb{Q}\)-divisors \(D_1\) and \(D_2\) are said to be linearly equivalent, denoted by \(D_1 \sim D_2\) if \(D_1 - D_2\) is a principal Weil divisor.

We give an example of the orbifold class group.

**Example 3.2.** We consider the log pair 

\[
(X, \Delta) := \left(\mathbb{P}^1, \frac{n-1}{n}\{0\} + \frac{m-1}{m}\{\infty\}\right).
\]

Then, the orbifold Weil divisor group of \((X, \Delta)\) is generated by all points in \(\mathbb{P}^1\), \(D_0 := \frac{1}{n}\{0\}\), and \(D_\infty := \frac{1}{m}\{\infty\}\). Hence, we have that 

\[
\text{Cl}(X, \Delta) := \langle D_0, D_\infty \mid nD_0 = mD_\infty \rangle.
\]

Now, we define the sheaves of sections \(\mathcal{O}_{(X, \Delta)}(D)\) for orbifold Weil divisors \(D\) on pairs \((X, \Delta)\).

**Definition 3.3.** Let \((X, \Delta)\) be a log pair and let \(D \in \text{WDiv}(X, \Delta)\). Then, we define the sheaf \(\mathcal{O}_{(X, \Delta)}(D)\) by 

\[
\Gamma(U, \mathcal{O}_{(X, \Delta)}(D)) := \langle f \mid f \in \mathbb{C}(X), f^n \in \mathbb{C}(X), \text{div}(f^n)/n \in \text{WDiv}(X, \Delta), \text{ and } (\text{div}(f^n)/(n + D)|_U \geq 0) \rangle
\]

for any open \(U \subseteq X\). Here, \(\text{PDiv}(U)\) is the group of principal divisors on \(U\) (without the orbifold structure). In particular, \(\mathcal{O}_{(X, \Delta)}(D)\) is a coherent sheaf of \(\mathcal{O}_X\)-modules for any \(D \in \text{WDiv}(X, \Delta)\).

Proceeding as in [8, Section 3.1 and 3.2], we first define the sheaves of divisorial algebras for subgroups \(N \leq \text{Cl}(X, \Delta)\), before defining Cox sheaves and Cox rings.
Definition 3.4. Let $(X, \Delta)$ be a log pair. Let $N \subseteq \text{WDiv}(X, \Delta)$ be a subgroup. Then the sheaf of divisorial algebras associated to $N$ is

$$S^{(N)} := \bigoplus_{D \in N} S_D^{(N)}$$

where $S_D^{(N)} := \mathcal{O}_{(X, \Delta)}(D)$.

Now, if $\text{Cl}(X, \Delta)$ is torsion free, we can define the Cox sheaf to be the sheaf of divisorial algebras associated to any $N \subseteq \text{WDiv}(X, \Delta)$ such that $N \to \text{Cl}(X, \Delta)$ is an isomorphism. If $\text{Cl}(X, \Delta)$ has torsion, we proceed similarly to [8, Constr. 1.4.2.1] in the case of ordinary Cox rings. The idea is to take the sheaf of divisorial algebras $N \subseteq \text{WDiv}(X, \Delta)$ projecting onto $\text{Cl}(X, \Delta)$, and then quotient by a certain ideal sheaf identifying homogeneous components $S_D^{(N)}$ and $S_{D'}^{(N)}$ whenever $D$ and $D'$ are linearly equivalent.

Definition 3.5. Let $(X, \Delta)$ be a log pair with finitely generated log divisor class group $\text{Cl}(X, \Delta)$. Let $N \subseteq \text{WDiv}(X, \Delta)$ be a finitely generated subgroup. Assume that

$$c : N \to \text{Cl}(X, \Delta), \quad D \mapsto [D]$$

is onto and denote its kernel by $N^0$. Let $S$ be the sheaf of divisorial algebras associated to $N$. Let $\chi : N^0 \to \mathbb{C}(X)^*$ be a group homomorphism yielding

$$\text{div}(\chi(E)) = E$$

(3.1)

for all $E \in N^0$. Denote by $I$ the sheaf of ideals of $S$ locally generated by the sections $1 - \chi(E)$, where $E$ runs through $N^0$. We define the log Cox sheaf of $(X, \Delta)$ to be the quotient sheaf $\mathcal{R}_{(X, \Delta)} := S / I$, graded by

$$\mathcal{R}_{(X, \Delta)} := \bigoplus_{[D] \in \text{Cl}(X, \Delta)} (\mathcal{R}_{(X, \Delta)})_{[D]}, \quad \text{where} \quad (\mathcal{R}_{(X, \Delta)})_{[D]} := \pi \left( \bigoplus_{D' \in c^{-1}([D])} S_{D'} \right),$$

and $\pi : S \to \mathcal{R}_{(X, \Delta)}$ is the projection. The ring of global sections

$$\text{Cox}(X, \Delta) := \Gamma(X, \mathcal{R}_{(X, \Delta)})$$

of this sheaf is called the log Cox ring of $(X, \Delta)$.

Remark 3.6. It is clear from the construction that $N^0$ is always a subgroup of $\text{PDiv}(X)$, so $\chi$ is a group homomorphism $\chi : N^0 \to \mathbb{C}(X)^*$ to the field of rational functions on $X$. Thus, the assertions from [8, Section 1.4.2] hold. In particular, if $\Gamma(X, \mathcal{O}^*) = \mathbb{C}^*$, then the above definition of the log Cox sheaf and log Cox ring does not depend on the choice of $N$ and $\chi$ up to isomorphism, see [8, Proposition 1.4.2.2]. Note that the requirement $\Gamma(X, \mathcal{O}^*) = \mathbb{C}^*$ is fulfilled for projective varieties and quasi-cones.

In what follows, we may need to consider the Cox ring with respect to a finitely generated subgroup $N \subseteq \text{WDiv}(X, \Delta)$ that may not surject onto $\text{Cl}(X, \Delta)$. Analogously, in this case, we have a homomorphism $N \to \text{Cl}(X, \Delta)$ with kernel $N^0$ and we choose a group homomorphism $\chi : N^0 \to \mathbb{C}^*$. 

C(X)* satisfying the equality (3.1). In this case, we denote the respective Cox ring by

$$\text{Cox}(X, \Delta)_{N, \chi}.$$ 

The definition of Cox(X, Δ)_{N, \chi} does depend on the choice of N and \chi, even if we consider Δ = 0 (see, e.g., [53, Theorem A]).

**Proposition 3.7.** Let (X, Δ) be a log pair. The Cox ring Cox(X, Δ) is finitely generated if and only if Cox(X) is finitely generated.

**Proof.** Note that we have an inclusion of groups Cl(X) ≤ Cl(X, Δ) of finite index. Furthermore, we have a monomorphism of graded rings Cox(X) ↪ Cox(X, Δ) obtained by coarsening the grading. By [8, Corollary 1.1.2.6], we conclude that $R_{(X, \Delta)}$ is finitely generated over $\mathbb{C}$ if and only if $R_X$ is finitely generated over $\mathbb{C}$. □

**Corollary 3.8.** Let X be an MDS. For any log pair structure (X, Δ) the Cox ring $R(X, \Delta)$ is finitely generated.

The following proposition says that the only case in which the Cox ring of a log pair (X, Δ) may be nonisomorphic to the Cox ring of X is when there is at least one coefficient of Δ that is equal to or larger than one-half.

**Proposition 3.9.** Let (X, Δ) be a log pair so that coeff_Δ(P) < $\frac{1}{2}$ for every prime divisor P on X. Then, Cox(X, Δ) ≅ Cox(X).

**Proof.** Note that coeff_Δ(P) < $\frac{1}{2}$ if and only if Δ_0, the standard approximation of Δ, equals the zero divisor. The above condition is equivalent to WDiv(X, Δ) ≅ WDiv(X). Furthermore, any section of an orbifold Weil divisor of (X, Δ) is just a section of a Weil divisor on X. Hence, we have that $R_{(X, \Delta)}$ ≅ $R_X$. This implies the desired isomorphism. □

We are interested in the universal abelian covering space that the log Cox ring provides us. We can also study other abelian covers of X. In analogy to the case of the ordinary Cox ring, they should correspond to quotients of Cox(X, Δ) by subgroups of the characteristic quasi-torus, see [8, Theorem 4.2.1.4]. We explore the Cox ring with and without a log structure in the following basic example.

**Example 3.10.** Consider the $A_1$ toric singularity X given by the equation \{x_1x_2 - x_3^2 = 0\} in $\mathbb{A}^3$. Then

$$\text{Cl}(X) = \langle D_1 \mid 2D_1 = 0 \rangle \cong \mathbb{Z}/2\mathbb{Z},$$

where $D_1$ is the $x_1$-axis. Note that $D_1$ is linearly equivalent to the $x_2$-axis $D_2$. Then, the classical Cox ring Cox(X) is $\mathbb{C}[y_1, y_2]$ with the Cl(X)-grading given by $\text{deg}(y_1) = \text{deg}(y_2) = 1$. The generating invariants of the corresponding $\mathbb{Z}/2\mathbb{Z}$-action on the Cox space $\hat{X} = \overline{X} = \mathbb{A}^2$ are

$$x_1 := y_1^2, \quad x_2 := y_2^2, \quad x_3 := y_1y_2,$$
which gives us back our singularity \(X\). Now we consider pair structures \((X, \Delta)\) where
\[
\Delta = \left(1 - \frac{1}{m_1}\right)D_1 + \left(1 - \frac{1}{m_2}\right)D_2.
\]
Then
\[
\mathsf{Cl}(X, \Delta) = \left\langle \frac{1}{m_1}D_1, \frac{1}{m_2}D_2 \mid D_1 = D_2 = 0 \right\rangle \cong \mathbb{Z} \left/ \frac{2m_1m_2}{\text{gcd}(m_1, m_2)} \mathbb{Z} \right. \oplus \mathbb{Z} / \text{gcd}(m_1, m_2) \mathbb{Z}.
\]

The log Cox ring \(\text{Cox}(X, \Delta)\) again is \(\mathbb{C}[z_1, z_2]\), but the grading by \(\mathsf{Cl}(X, \Delta)\) obviously differs from that by \(\mathsf{Cl}(X)\). In the case that \(m_1\) and \(m_2\) are coprime (otherwise it is more difficult), it is given by \(\deg(z_1) = m_2, \deg(z_2) = m_1\). Thus, the generators of the invariant ring now are
\[
x_1 := z_2^{m_1}, \quad x_2 := z_2^{m_2}, \quad x_3 := y_1^{m_1}y_2^{m_2},
\]
which satisfy our initial relation once again. But the corresponding quotient
\[
\overline{X}_\Delta := \text{Spec} \text{Cox}(X, \Delta) \xrightarrow{\left(\mathbb{Z}/2m_1m_2\mathbb{Z}\right)} X
\]
ramifies in the right way over \(D_1\) and \(D_2\), so the log pullback of the pair \((X, \Delta)\) is \((\overline{X}_\Delta, 0)\). We also note that this log Cox space factors through the classical Cox space from above.

Remark 3.11. In Section 8, we explain how log Cox rings of toric log pairs (and more general, torus invariant pair structures on \(\mathbb{T}\)-varieties of complexity one) can be viewed and computed combinatorially in terms of the ordinary toric fan.

To finish this subsection, we show the Cox ring of a log Fano structure on \(\mathbb{P}^1\). In this case, the standard approximation has at most three nontrivial coefficients. In the case that there are two nontrivial coefficients, the Cox ring is isomorphic to \(\mathbb{A}^2\) with a characteristic quasi-torus action. In the case that there are three nontrivial coefficients, the Cox ring may not be isomorphic to \(\mathbb{A}^2\).

Example 3.12. Let \(\Delta\) be an effective divisor on \(\mathbb{P}^1\) so that \(-(K_{\mathbb{P}^1} + \Delta)\) is ample. Assume that \(\Delta_s\) has two nontrivial coefficients. Then \(\Delta_s = (1 - \frac{1}{n})p + (1 - \frac{1}{m})q\) for some positive integers \(n\) and \(m\). In this case, we have that \(\mathsf{Cl}(\mathbb{P}^1, \Delta) = \left\langle \frac{1}{n}p, \frac{1}{m}q \mid p = q \right\rangle\). The above group is isomorphic to \(\mathbb{Z} \oplus \mathbb{Z} / \text{gcd}(n, m) \mathbb{Z}\). Let \(g = \text{gcd}(n, m)\). We conclude that the Cox ring is isomorphic to \(\mathbb{A}^2\) with the Picard action
\[
t \cdot (x, y) \mapsto \left( t^{-\frac{m}{\text{gcd}(n,m)}}x, t^{-\frac{n}{\text{gcd}(n,m)}}y \right)
\]
and
\[
\mu \cdot (x, y) \mapsto \left( \mu^{\frac{n}{\text{gcd}(n,m)}}x, \mu^{-\frac{m}{\text{gcd}(n,m)}}y \right),
\]
where \(\mu\) is a \(g\)th root of unity.

Example 3.13. Let \(\Delta\) be an effective divisor on \(\mathbb{P}^1\) so that \(-(K_{\mathbb{P}^1} + \Delta)\) is ample. Assume that \(\Delta_s\) has three nontrivial coefficients. Thus, the coefficients of \(\Delta_s\) correspond to platonic triples (see, e.g., [57, 61]). In this case, we have that \(\mathsf{Cl}(\mathbb{P}^1, \Delta) = \left\langle \frac{1}{n}p, \frac{1}{m}q, \frac{1}{r} \mid p = q = r \right\rangle\). Let
\( g = \gcd(ms, ns, nm) \). We may assume that the points \( p, q, \) and \( r \) are \( 0, \{\infty\} \) and 1, respectively. The class group is isomorphic to \( \mathbb{Z} \oplus T_{n,m,s} \), where \( T_{n,m,s} \) is the roots system of the fork Dynkin diagram with three branches of length \( n, m, \) and \( s \). The Cox ring is isomorphic to

\[
\mathbb{C}[x, y, z] / (x^n + y^m + z^s).
\]

The characteristic quasi-torus action is given by

\[
t \cdot (x, y, z) = \left( t^{\frac{ms}{g}} x, t^{\frac{ns}{g}} y, t^{\frac{nm}{g}} z \right)
\]

and \( T_{n,m,s} \) acts on \((x, y, z)\) in the usual way (see, e.g., [67]).

**Remark 3.14.** By taking all the possible Cox rings of log Fano pairs on \( \mathbb{P}^1 \) and quotient by the finite part of the characteristic quasi-torus action, we obtain surface klt singularities. These singularities are quotients of smooth points by finite groups. For the classification of surface klt singularities, see, for example, [1].

### 3.2 The relative Cox ring of a log pair

In this subsection, we define the relative Cox ring of a log pair, prove some basic properties, and give some examples.

**Definition 3.15.** Let \((X, \Delta)\) be a log pair and \( \phi : X \to Z \) be a contraction. We define the relative log Cox sheaf of \( X/Z \) to be the direct image sheaf

\[
\mathcal{R}_{(X/Z, \Delta)} := \phi^* \mathcal{R}_{(X, \Delta)},
\]

where \( \mathcal{R}_{(X, \Delta)} \) is the log Cox sheaf of \((X, \Delta)\) as in Definition 3.5. If \( \mathcal{R}_{(X/Z, \Delta)} \) is a sheaf of finitely generated \( \mathcal{O}_Z \)-algebras, we say that \( X \to Z \) is a relative MDS for the log pair \((X, \Delta)\). The relative affine log Cox ring is defined to be

\[
\text{Cox}^{\text{aff}}(X/Z, \Delta) := \Gamma(Z, \mathcal{R}_{(X/Z, \Delta)}).
\]

We write \( \text{aff} \) on top of the relative Cox ring to stress that, in this case, we are working with an affine base \( Z \). Later on, we will be interested in the local behavior around some special point of the base. The relative Cox ring is graded by \( \text{Cl}(X/Z, \Delta) \), the group of orbifold Weil divisors on \( X \) modulo linear equivalence over \( Z \).

More generally, we can make the above definitions if \( h : X \to Z \) is an aff-contraction, see Definition 2.22.

**Remark 3.16.** If \( Z \) is a point, we can identify the relative log Cox sheaf \( \mathcal{R}_{(X/Z, \Delta)} \) with the log Cox ring \( \text{Cox}(X, \Delta) \). More generally, when \( Z \) is affine, then \( \mathcal{R}_{(X/Z, \Delta)} \) is a sheaf of finitely generated \( \mathcal{O}_Z \)-algebras if and only if the algebra of global sections \( \text{Cox}^{\text{aff}}(X/Z, \Delta) \) is finitely generated over \( \mathcal{O}_Z(Z) \) and thus over \( \mathbb{C} \) by the same argument as in [8, Proposition 4.3.1.3].
Again more generally, if \( \mathcal{R}_{(X/Z, \Delta)} \) is a sheaf of finitely generated \( \mathcal{O}_Z \)-algebras, then any fiber \( X_z := \phi^{-1}(z) \) of \( \phi : X \to Z \) has an open neighborhood \( X_z \subseteq U \subseteq X \), such that \( \mathcal{R}_{(X, \Delta)}(U) \) is a finitely generated \( \mathcal{O}_X(U) \)-algebra.

### 3.3 The local Cox ring

In this subsection, we define the local Cox ring for germs \((X, \Delta; x)\), where \((X, \Delta)\) is a pair and \(x \in X\) is a closed point. More generally, when \(\phi : X \to Z\) is a contraction, we define the relative local Cox ring for closed points \(z \in Z\).

Here, it makes sense to consider different local models depending on the needs. We consider an algebraic variety \(X\) (or pair \((X, \Delta)\)) and a closed point \(x \in X\). The local definitions of Cox rings come in two flavors: The local (resp. Henselian) that is defined over the localization (resp. Henselization) of \(X\) at \(x\). On the other hand, we have the affine-local Cox ring, that is, we consider \(X\) affine and take the affine Cox ring with respect to a group of Weil divisors that surject onto \(\text{Cl}(X_x)\). This object allows to create affine covers of \(X\) on which Weil divisors become Cartier over \(x\).

In what follows, we write \(\text{Cl}(X, \Delta, x)\) for the orbifold class group \(\text{Cl}(X_x, \Delta_x)\) of the localization at \(x\), where \(\Delta_x\) is the pull-back of the boundary to the localization. The approach is to define the local Cox ring at \(x \in X\) to be roughly

\[
\bigoplus_{[D] \in \text{Cl}(X, \Delta, x)} \Gamma(X, \mathcal{O}_{(X, \Delta)}(D)).
\]

This definition amounts to choosing a subgroup \(N\) of the orbifold Weil divisors of \((X, \Delta)\) surjecting onto \(\text{Cl}(X, \Delta, x)\) with kernel \(N^0\) and a character \(\chi : N^0 \to \mathbb{C}(X)^*\). Note that by \([53, \text{Theorem 2.3}]\), the set of isomorphism classes of Cox rings defined in this way is in bijection to

\[
\text{Ext}^1(\text{Cl}(X, \Delta, x), \mathcal{O}(X)^*).
\]

This construction only makes sense if \(X\) is affine, so we will assume this in the following. Moreover, we assume that the group \(N\) consists of Weil divisors going through \(x\) and we fix a character \(\chi : N^0 \to \mathbb{C}(X)^*\). Then, we can define the affine local Cox ring (or aff-local Cox ring for short) as above. If it is finitely generated over \(\mathcal{O}(X)\), then its spectrum is an affine scheme of finite type.

We denote by \(X_x\) the spectrum of the local ring of \(X\) at \(x\). We have \(\text{Cl}(X_x, \Delta_x) \cong \text{Cl}(X, \Delta, x)\), and we can identify uniquely the group \(N\) with a subgroup of \(\text{WDiv}(X_x, \Delta_x)\). Here, \(\Delta_x\) is the pull-back of \(\Delta\) to \(X_x\). Moreover, since \(X\) and \(X_x\) are birational, we can use the character \(\chi\) from above in order to define the Cox ring

\[
\bigoplus_{[D] \in \text{Cl}(X_x, \Delta_x)} \Gamma(X_x, \mathcal{O}_{(X_x, \Delta_x)}(D)).
\]

This is a gr-local ring, finitely generated over the degree-zero part \(\mathcal{O}_{X_x,x}\), which is why we call it the gr-local Cox ring of \(x \in X\). By localizing at the unique graded maximal ideal, we get a local ring.

Now, we turn to define (three variants of) the local Cox ring. In a few words, the definition is the same as for the log Cox ring up to restricting to Weil divisors through \(x\) and localization (either at \(x\) or at the unique preimage of \(x\) in the Cox space).
Definition 3.17. Let $X$ be an affine variety, $(X, \Delta)$ a log pair, and $x \in X$ a closed point. Fix a subgroup $N \subseteq \text{WDiv}(X, \Delta)$ of orbifold Weil divisors going through $x$ such that the induced homomorphism $\varphi: N \to \text{Cl}(X, x)$ is surjective. Fix a character $\chi: \ker(\varphi) \to \mathbb{C}(X)^\ast$. Let $S$ be the sheaf of divisorial algebras on $X$ associated to $N$ and $I$ the ideal subsheaf generated by sections $1 - \chi(E)$, where $E \in \ker(\varphi)$. Then, we define the **aff-local Cox ring** of $x \in (X, \Delta)$ to be

$$\text{Cox}(X, \Delta; x)_{\text{aff}} := \bigoplus_{[D] \in \text{Cl}(X, \Delta, x)} \left( \bigoplus_{D' \in \varphi^{-1}([D])} S_{D', x}(X) \right) \mathfrak{I}(X).$$

Similarly, where $X_x := \text{Spec} \mathcal{O}_{X, x}$, we define the **gr-local Cox ring** of $x \in (X, \Delta)$ to be

$$\text{Cox}(X, \Delta; x)_{\text{gr}} := \text{Cox}(X_x, \Delta_x)_{N, \chi} := \bigoplus_{[D] \in \text{Cl}(X_x, \Delta_x)} \left( \bigoplus_{D' \in \varphi^{-1}([D])} S_{D', x} \right) \mathfrak{I}_x.$$

Finally, we define the **local Cox ring** of $x \in (X, \Delta)$ to be the localization

$$\text{Cox}(X, \Delta; x)_{\text{loc}} := \left( \text{Cox}(X, \Delta; x)_{\text{gr}} \right)_{m}$$

at the unique homogeneous maximal ideal of the gr-local Cox ring. We denote the spectra of these rings by

$$\overline{X}_{N, \chi}^{\text{aff}}, \overline{X}_{N, \chi}^{\text{gr}}$$

and

$$\overline{X}_{N, \chi}^{\text{loc}}$$

respectively. The isomorphism classes of the Cox rings just defined depend on the choice of $N$ and $\chi$, but having made such a choice, we will usually omit them in the notation.

Example 3.18. To illustrate the difference between the three spaces $\overline{X}^{\text{aff}}, \overline{X}^{\text{gr}}$, and $\overline{X}^{\text{loc}}$ just defined, we need to consider an example with *infinite* class group (otherwise by Lemma 2.8, the gr-local and the local Cox ring would be isomorphic). The three-dimensional toric singularity coming as the origin $0$ of the cone $X := V(xy - zw) \subseteq \mathbb{A}^4$ has $\text{Cl}(X) = \mathbb{Z}$ and $\overline{X}^{\text{aff}} = \mathbb{A}^4$, endowed with the $\mathbb{C}^\ast$-action given by

$$t \cdot (x_1, x_2, x_3, x_4) = (t^{-1} x_1, t^{-1} x_2, tx_3, tx_4).$$

The unique $\mathbb{C}^\ast$-closed orbit in the preimage $V(x_1, x_2) \cup V(x_3, x_4)$ of $0$ in $\overline{X}^{\text{aff}}$ is just the origin, which we denote by $\overline{0}$ here. Now gr-localizing $\mathbb{C}[\mathbb{A}^4] = \text{Cox}(X, x)^{\text{aff}}$ at $\overline{0}$ with respect to the $\text{Cl}(X)$-grading preserves all maximal ideals that lie on orbits having $\overline{x}$ in their closure (e.g., those on the coordinate axes). On the other hand, $\overline{X}^{\text{loc}}$ is the spectrum of $\text{Cox}(X, x)^{\text{aff}}_{\overline{0}}$, there is only one maximal ideal and no $\mathbb{C}^\ast$-action.

In particular, $\text{Cox}(X, \Delta; x)^{\text{gr}}$ is the stalk of the quotient sheaf $S/I$ at $x$ or, equivalently, the gr-localization at the unique $\text{Spec} \mathbb{C}'[\text{Cl}(X_x, \Delta_x)]$-closed orbit in the preimage of $x \in X$ in $\overline{X}^{\text{aff}}$.

In the following diagram, by $\overline{X}^{\text{gr}}_{\text{fin}}$ and $\overline{X}^{\text{aff}}_{\text{fin}}$, we mean the quotients by the torus part of $\text{Cl}(X, x)$ of $\overline{X}^{\text{gr}}$ and $\overline{X}^{\text{aff}}$, respectively, that is, the corresponding finite abelian covers of $X_x$ and $X$. Since
localization factors through gr-localization, the diagram is indeed commutative. Moreover, since a gr-local ring with respect to a finite group is already local (see Lemma 2.8), we also have a “finite local Cox space” $X_{f_{\text{fin}}}^{\text{loc}} := X_{f_{\text{fin}}}^{\text{gr}}$. The morphism $X^{\text{loc}} \to X_{f_{\text{fin}}}^{\text{loc}}$, however, is not a quotient in general.

In particular, there is still a morphism $X^{\text{loc}} \to X_x$, but it may not be a quotient by the characteristic quasi-torus.

Note that $\text{Cl}(X_{\text{gr}}) \cong \text{Cl}(X_{\text{loc}})$, since the class group of the gr-local ring $\text{Cox}(X, \Delta; x)^{\text{gr}}$ is concentrated at the unique graded maximal ideal. So, the identity

$$\text{Cox}(X^{\text{loc}}) = \text{Cox}(X^{\text{gr}}) \otimes_{\text{Cox}(X, \Delta; x)^{\text{gr}}} \text{Cox}(X, \Delta; x)^{\text{loc}}$$

holds. This essentially means that we can iterate Cox rings in a unique way not depending on if we prefer gr-local or local Cox rings. The iteration of Cox rings is defined in Definition 4.5.

**Definition 3.19.** Let $(X, \Delta)$ be a log pair and $\phi : X \to Z$ be a contraction. Let $z \in Z$ be a closed point. Let $Z_z$ be the spectrum of the local ring $O_{Z, z}$. Let $X_z \to Z_z$ be the projective morphism obtained by the base change $Z_z \to Z$. We denote by $\Delta_z$ the pull-back of $\Delta$ to $X_z$. Analogously to the local case, we can define the relative gr-local Cox ring at $z \in Z$ to be

$$\text{Cox}(X/Z, \Delta; z)^{\text{gr}} := \bigoplus_{[D] \in \text{Cl}(X_z, \Delta_z)} \sum_{D' \in \phi^{-1}([D])} S_{D', \phi^{-1}(z)} I_{\phi^{-1}(z)}.$$ 

Note that the relative gr-local Cox ring comes with a natural maximal graded ideal $m$, that is, the ideal generated by homogeneous regular functions in the Cox ring that correspond to Weil divisors on $X$ that intersect the fiber $\phi^{-1}(z)$ nontrivially. The relative local Cox ring of $(X/Z, \Delta)$ at $z$ is then defined to be

$$\text{Cox}(X/Z, \Delta; z)^{\text{loc}} := (\text{Cox}(X/Z, \Delta; z)^{\text{gr}})_m.$$ 

The local Cox ring comes as the special case where $X \to X$ is the identity and $x \in X$:

$$\text{Cox}(X, \Delta; x)^{\text{loc}} \cong \text{Cox}(X_x/X_x, \Delta)^{\text{loc}}.$$
3.4 Properties of generalized Cox rings

In this subsection, we prove some properties of the Cox rings defined in the previous subsections. First, we prove two general statements concerning relative MDSs, and then we focus on the case of klt pairs.

**Proposition 3.20.** Let \( X \to Z \) be a relative MDS. Then \( R_{(X/Z,\Delta)} \) is a sheaf of gr-local rings. In particular, if \( Z \) is the spectrum of a local ring essentially of finite type, then the local Cox ring \( \text{Cox}^{\text{loc}}(X/Z,\Delta;z) \) is a local ring essentially of finite type.

**Proof.** Since \( X \to Z \) is a relative MDS, we know that the stalks \( (R_{(X/Z,\Delta)})_z \) are graded rings, with zero-graded piece \( \mathcal{O}_{Z,z} \), which is a local ring. Finite generation over \( \mathcal{O}_{Z,z} \) follows as in Corollary 2.19. The last assertion follows from the definition. \( \square \)

The following set of statements shows that klt singularities and weakly Fano pairs behave optimally with respect to the Cox construction. This is known for the classical Cox ring of weakly Fano pairs and klt quasi-cones (see, e.g., [40]).

**Theorem 3.21.** Let \((X,\Delta)\) be of Fano type over \(Z\). Assume that \(Z\) is either:
- affine,
- projective,
- the spectrum of a local ring essentially of finite type.

Then \(X \to Z\) is a relative MDS for the log pair \((X,\Delta)\).

**Proof.** By Remark 3.16, it suffices to check that for every point \(z \in Z\) the stalk \((R_{(X/Z,\Delta)})_z\) is a finitely generated \(\mathcal{O}_{Z,z}\)-algebra. To do so, we will use the fact that \((X,\Delta)\) is of Fano type over the base. Since the problem is local on \(Z\), the projective case follows from the affine case.

Assume that \(Z\) is affine. We denote by \(\phi_X : X \to Z\) the contraction morphism. By [15, Corollary 1.4.3], we can take a small \(\mathbb{Q}\)-factorialization \(\psi : Y \to X\) of \(X\). Let \(K_Y + \Delta_Y = \pi^*(K_X + \Delta)\). The pair \((Y,\Delta_Y)\) is still of Fano type over \(Z\) (see, e.g., [40, Lemma 3.1]). We denote by \(\phi_Y : Y \to Z\) the contraction morphism. By the Fano-type condition, there exists \(B_Y\) big over \(Z\) such that \((Y,\Delta_Y + B_Y)\) is klt and \(K_Y + \Delta_Y + B_Y \sim_{\mathbb{Q},Z} 0\). Since \(B_Y\) is big over \(Z\), we may write \(B_Y \sim_{\mathbb{Q},Z} A_Y + E_Y\) where \(A_Y\) is effective and ample over \(Z\) and \(E_Y\) is effective. By possibly replacing \(B_Y\) with \((1 - \varepsilon)B_Y + \varepsilon A_Y + \varepsilon E_Y\), we may assume that \(B_Y \geq \varepsilon A_Y \geq 0\). Note that the pair \((Y,\Delta_Y + B_Y - \varepsilon A_Y)\) is klt and \(-(K_Y + \Delta_Y + B_Y - \varepsilon A_Y)\) is ample over \(Z\). By [15, Corollary 1.3.2] and Lemma 2.18, we conclude that \((R_{(Y/Z)})_z\) is a finitely generated algebra over \(\mathcal{O}_{Z,z}\). Since \(\psi\) is small, we conclude that \((R_{(X/Z)})_z\) is a finitely generated algebra over \(\mathcal{O}_{Z,z}\). By Proposition 3.7, we deduce that \((R_{(X/Z,\Delta)})_z\) is a finitely generated algebra over \(\mathcal{O}_{Z,z}\). Hence, \(R_{(X/Z,\Delta)}\) is a sheaf of finitely generated \(\mathcal{O}_{Z}\)-algebras. Then, \(X \to Z\) is a relative MDS for the log pair \((X,\Delta)\).

Now, assume that \(z \in Z\) is the spectrum of a local ring essentially of finite type. Then, there exists an affine variety \(\overline{Z}\) such that \(Z \to \overline{Z}\) is a localizing immersion. Hence, \(X \to \overline{Z}\) is an essentially of finite type morphism. By [70, Theorem 3.6], there is a localizing immersion \(X \to \overline{X}\) and a finite-type morphism \(\overline{X} \to \overline{Z}\) that factorize \(X \to \overline{Z}\). The boundary \(\Delta\) on \(X\) extends to a boundary \(\overline{\Delta}\) on \(\overline{X}\). As \((X,\Delta)\) is of Fano type over \(Z\), then \((\overline{X},\overline{\Delta})\) is of Fano type over \(\overline{Z}\) up to possibly shrinking \(\overline{Z}\).
around $z$. Indeed, both the klt condition and relative bigness are open in the Zariski topology. As the stalks of $\mathcal{R}_{(X/Z, \Delta)}$ and $\mathcal{R}_{(X/Z, \Delta)}$ are isomorphic, the statement follows from the affine case. □

**Corollary 3.22.** Let $x \in (X, \Delta)$ be a klt singularity. Then $(X_x, \Delta_x)$ is an MDS at $x \in X$, that is, $\text{Cox}^\text{gr}(X, \Delta; x)$ is a gr-local ring, finitely generated as an algebra over $\mathcal{O}_{X, x}$.

**Proof.** This follows by setting $X = Z = X_x$ in the statement of Theorem 3.21. □

**Corollary 3.23.** Let $(X, \Delta)$ be a klt pair with finitely generated log divisor class group $\text{Cl}(X, \Delta)$. Then, the Cox sheaf $\mathcal{R}_{(X, \Delta)}$ is a sheaf of gr-local rings. In particular, it is a sheaf of finitely generated $\mathcal{O}_X$-algebras.

**Proof.** Since $(X, \Delta)$ is klt, Corollary 3.22 tells us that for any $x \in X$, the gr-local Cox ring $\text{Cox}^\text{gr}(X, \Delta; x)$ is finitely generated over $\mathcal{O}_{X, x}$.

Let $D_1, \ldots, D_n$ be a finite set of generators of $\text{Cl}(X, \Delta)$, such that for $k < n$,

- the restrictions of divisors $D_{k+1}, \ldots, D_n$ generate $\text{Cl}(X_x, \Delta_x)$, and
- for any $1 \leq i \leq k$, the divisor $D_i$ becomes principal on some open neighborhood $U_i$ of $x$.

We can find a sequence of neighborhoods $(U_j)_{1 \leq j \leq \infty}$ of $x$, such that $X_x$ is the inverse limit of the $U_j$ and $D_1, \ldots, D_k$ are principal on each $U_j$. Let $N$ be the subgroup of $\text{WDiv}(X, \Delta)$ generated by $D_{k+1}, \ldots, D_n$ and $S^{(N)}$ be the associated divisorial sheaf (see Definition 3.4).

Analogous to [8, Remark 1.3.1.4], for each $1 \leq j \leq \infty$ and $D_i = V(f_{ji})$ on $U_j$, we have an isomorphism

$$\Gamma(U_j, S^{(N)})[x_1^{\pm 1}, \ldots, x_k^{\pm 1}] \rightarrow \Gamma(U_j, \mathcal{R}_{(X, \Delta)}); \quad g T_1^{\nu_1} \cdots T_k^{\nu_k} \mapsto g f_{j1}^{-\nu_1} \cdots f_{jk}^{-\nu_k}.$$

So, we get an isomorphism for the inverse limits:

$$\text{Cox}^\text{gr}(X, \Delta; x)[x_1^{\pm 1}, \ldots, x_k^{\pm 1}] \rightarrow (\mathcal{R}_{(X, \Delta)})_x.$$

Note that we have an isomorphism $\text{Cl}(X, \Delta, x) \simeq \text{Cl}(X, \Delta)/\text{PDiv}(X, \Delta; x)$, where $\text{PDiv}(X, \Delta; x)$ is the group of divisors on $(X, \Delta)$ that are principal around $x$. Moreover, the group $\text{PDiv}(X, \Delta; x)$ has $k$ generators. Hence, we have that

$$(\mathcal{R}_{(X, \Delta)})_x$$

is a gr-local ring, finitely generated as an algebra over $\mathcal{O}_{X, x}$. The last assertion follows from Lemma 2.18. □

In view of [8, Proposition 4.3.1.4], we have the stronger result that affine klt varieties with finitely generated class group are MDSs.

**Corollary 3.24.** Let $(X, \Delta)$ be an affine klt pair with finitely generated log divisor class group $\text{Cl}(X, \Delta)$. Then, the Cox ring $\text{Cox}^\text{aff}(X, \Delta)$ is finitely generated.

**Proof.** This follows directly from Corollary 3.23 and [8, Proposition 4.3.1.3]. □
**Lemma 3.25.** Let $X$ be a $\mathbb{Q}$-factorial variety. Let $X \rightarrow Z$ be a projective contraction. Let $(X,\Delta)$ be a log pair that is of Fano type over $Z$. Let $D$ be a $\mathbb{Q}$-divisor on $X$. Assume that for each prime $P \subset X$, we have that

$$\text{coeff}_P(\Delta) \geq 1 - \frac{1}{i_P(D)}, \quad (3.2)$$

where $i_P(D)$ is the Cartier index of $D$ at the generic point of $P$. Then, the spectrum $Y$ of the ring

$$\bigoplus_{m \in \mathbb{Z}} \Gamma(X/Z, \mathcal{O}_X(mD))$$

is of klt type.

**Proof.** Note that if neither $-D$ or $D$ is effective over the base $Z$, then there is nothing to prove. Without loss of generality, we may assume that $D$ is effective. We run a $D$-MMP over the base $X \rightarrow X'$. This $D$-MMP terminates since $X$ is of Fano type over the base. Furthermore, $X'$ is also of Fano type over the base. Let $X''$ be the ample model of $D'$ over $X$. Hence, $X''$ is of Fano type over $Z$, being the image of a Fano-type variety over $Z$. Let $X^{(3)}$ be a small $\mathbb{Q}$-factorialization of $X''$. Replacing $X$ by $X^{(3)}$, we may assume that $D$ is semiample and big over $Z$.

Let $\phi: \tilde{X} \rightarrow X$ be the relative spectrum of the divisorial sheaf $\bigoplus_{m \in \mathbb{Z}} \mathcal{O}_X(mD)$. Hence, we have an equivariant birational projective morphism $r: \tilde{X} \rightarrow Y$ that contracts at most one horizontal divisor over $X$. Indeed, there are only two invariant horizontal divisors over $X$, and at most one of them has negative normal bundle. If $r$ contracts a divisor, we denote it by $F$. By the Fano-type condition, we can find $\Delta'$ so that $(X, \Delta_s + \Delta')$ is klt and log Calabi–Yau over $Z$. Since $\Delta'$ is big over the base, we can find a general ample divisor $A$ and $E \geq 0$ so that $\Delta' \sim_{\mathbb{Q},Z} A + E$. We can assume that $(X, \Delta_s + A + E)$ is a klt pair that is trivial over $Z$. Let $\epsilon > 0$ be small enough so that $A - \epsilon D$ is an ample divisor over $Z$. We can write $A - \epsilon D \sim_{\mathbb{Q},Z} \Gamma \geq 0$ general enough so that the pair $(X, \Delta_s + E + \Gamma)$ remains klt. Note that we have a $\mathbb{Q}$-linear equivalence over the base

$$-(K_X + \Delta_s + E + \Gamma) \sim_{\mathbb{Q},Z} \epsilon D.$$ 

By (3.2), we have that the coefficient of $\Delta_s + E + \Gamma$ at each prime divisor $P$ of $X$ is at least $1 - i_P(D)^{-1}$. The positive integer $i_P(D)$ equals the isotropy of the $\mathbb{C}^*$-action on $\tilde{X}$ over the generic point of $P$ (see, e.g., [8, Proposition 1.3.4.7]). Indeed, $i_P(D)$ is the Cartier index of $D$ at the generic point of $D$. Hence, we have that $\phi^*(K_X + \Delta_s + E + \Gamma) = K_{\tilde{X}} + \Gamma_{\tilde{X}}$ satisfies that $(\tilde{X}, \Gamma_{\tilde{X}} + (1 - \epsilon)F)$ is a log pair. Furthermore, this pair is klt and $K_{\tilde{X}} + \Gamma_{\tilde{X}} + (1 - \epsilon)F$ is $\mathbb{Q}$-trivial over $Y$. Define $\Gamma := r_* \Gamma_{\tilde{X}}$. Then, we have that $(Y, \Gamma)$ is crepant equivalent to $(\tilde{X}, \Gamma_{\tilde{X}} + (1 - \epsilon)F)$, so it is a klt pair. \hfill $\square$

**Theorem 3.26.** Let $\phi: X \rightarrow Z$ be a contraction. Assume that $(X, \Delta)$ is a log pair which is of Fano type over $Z$. Let $N \leq \text{WDiv}(X,\Delta)$ be a finitely generated subgroup. Consider $\pi: N \rightarrow \text{Cl}(X/Z,\Delta)$ the induced homomorphism and $N_0$ its kernel. Let $\chi: N_0 \rightarrow \mathbb{C}(X)^*$ be a character. Then, the spectrum of the affine Cox ring $\text{Cox}^{\text{aff}}(X/Z,\Delta)_{N,\chi}$ is of klt type.

**Proof.** Observe that replacing $X$ with a small $\mathbb{Q}$-factorialization does not change the Cox ring, so we may assume that $X$ is $\mathbb{Q}$-factorial. Let $D_1, \ldots, D_s, D_{s+1}, \ldots, D_r$ be a finite set of Weil divisors so that $\langle D_1, \ldots, D_s \rangle$ maps isomorphically to $\pi(N)_{\text{free}}$ and $\langle D_{s+1}, \ldots, D_r \rangle$ surjects onto $\pi(N)_{\text{tor}}$. We denote the spectrum of the Cox ring $\text{Cox}^{\text{aff}}(X/Z,\Delta)_{N,\chi}$ by $Y'$. Note that we have a natural split
\[ \mathbb{T} \cong \mathbb{T}_0 \times A, \] where \( A \) is a finite abelian group and \( \mathbb{T}_0 \) is a torus. Let \( Y \) be the quotient of \( Y' \) by \( A \) and \( X' \) the quotient of \( Y' \) by \( \mathbb{T}_0 \). Then, we have a commutative diagram as follows:

\[
\begin{array}{ccc}
Y' & \xrightarrow{A} & Y \\
\downarrow{\mathbb{T}_0} & & \downarrow{\mathbb{T}_0} \\
X' & \xrightarrow{A} & X
\end{array}
\]

We denote the finite Galois log quasi-étale morphism \( X' \to X \) by \( p \). We have a natural isomorphism

\[
\text{Cox}^{\text{aff}}(X/Z, \Delta)_{N,X} \cong \bigoplus_{(m_1, \ldots, m_s) \in \mathbb{Z}^s} \Gamma(X'/Z, \mathcal{O}_{X'}(m_1 p^*(D_1) + \cdots + m_s p^*(D_s))).
\]

The above isomorphism is induced by the isomorphism

\[
p_* \mathcal{O}_{X'} \cong \bigoplus_{D \in \pi(N)_{tor}} \mathcal{O}_X(D).
\]

The finite morphism \( X' \to X \) ramifies with multiplicity at most \( m \) at prime divisors of \( X \) with coefficient at least \( 1 - \frac{1}{m} \). Then, the log pull-back of \( K_X + \Delta \) is a klt pair \( K_{X'} + \Delta' \). Since \((X, \Delta)\) is of Fano type over \( Z \), we can find a boundary \( B \) on \( X \) so that \( K_X + \Delta + B \sim_{\mathbb{Q}, Z} 0 \) is klt and \( B \) is big over \( Z \). We may assume that \( B \) contains no component of the branch locus of \( p \). Then, the pullback \( B' := p^*(B) \) satisfies that \( K_{X'} + \Delta' + B' \sim_{\mathbb{Q}, Z} 0 \) is klt and \( B' \) is big over \( Z \). Hence, \((X', \Delta')\) is of Fano type over \( Z \). Hence, it suffices to prove the statement for \( \pi(N) \) free. Thus, we may replace \( X \) with \( X' \) and assume \( s = r \).

We reduce to the case in which each \( D_i \sim_{\mathbb{Q}, Z} K_X + B_i \), where \((X, B_i)\) is klt and \( B_i \geq A \) for some fixed effective divisor \( A \) ample over \( Z \). Without loss of generality, we may assume that each \( D_i \) is effective. Furthermore, we may assume that \((X, B' + D_i)\) is klt, where \( B' := \Delta + B \) is big over \( Z \). For this purpose, it suffices to replace \( D_i \) with \( D_i/n_i \) with \( n_i \) large enough. Note that \( K_X + B' + D_i \sim_{\mathbb{Q}, Z} D_i \). Since \( B' \) is big over \( Z \), we can write \( B' \sim_{\mathbb{Q}, Z} A + E \) where \( A \) is ample over \( Z \) and \( E \) is effective. By choosing a very general section of \( A \), we may replace \( B' \) with \( A + E \) and set \( B_i = D_i + A + E \).

In this step, we reduce to the case in which there is a single divisor \( D_1 \). For each \( D_i \), we can find \( k_i \geq 0 \) so that \( k_i D_i \) is Cartier. Consider the orbifold projective bundle

\[
X_1 := \mathbb{P}_X(\mathcal{O}_X(D_1) \oplus \cdots \oplus \mathcal{O}_X(D_s))
\]

over \( X \), and the projective bundle

\[
X_2 := \mathbb{P}_X(\mathcal{O}_X(k_1 D_1) \oplus \cdots \oplus \mathcal{O}_X(k_s D_s))
\]

over \( X \). Note that we have a finite morphism \( X_1 \to X_2 \). We denote by \( \pi_1 : X_1 \to X \) and \( \pi_2 : X_2 \to X \) the corresponding morphisms. We claim that \( X_2 \) is of Fano type over \( Z \). Let \( H_1, \ldots, H_{s+1} \) be the hyperplane sections of \( X_2 \) over \( X \) and \( H := H_1 + \cdots + H_{s+1} \). Let \( \Delta_{X_2} = \pi_2^*(B') \). By inversion of adjunction, we conclude that the pair \((X_2, H + \Delta_{X_2})\) is dlt. Indeed, adjunction to \( H_1 \cap \cdots \cap H_{s+1} \simeq X_2 \) gives us \((X_2, \Delta_{X_2})\) which is klt. Further, \( K_{X_2} + H + \Delta_{X_2} \) is \( \mathbb{Q} \)-trivial over \( Z \). Observe that the
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boundary $H + \Delta_X$ is big over $Z$. For $\varepsilon > 0$ small enough, the pair $\pi^*_2(A) + \varepsilon H$ is ample over $Z$. Let $A_{X_2} \sim_{Q,Z} \pi^*_2(A) + \varepsilon H$ be a general effective divisor. We conclude that

$$K_{X_2} + (1 - \varepsilon)H + \Delta_X + (1 - \varepsilon)\pi^*_2A + \pi^*_2H + A_{X_2} \sim_{Q,Z} 0$$

is klt and its boundary is effective. We conclude that $X_2$ is of Fano type over $Z$. By taking $\varepsilon$ small enough, we can make sure that the log pull-back of the above pair to $X_1$ is a klt pair. Thus, we conclude that $X_1$ is of Fano type over $Z$ as well. Note that the section ring of the tautological $Q$-line bundle of $X_1$ coincides with the multisection ring generated by the $D_i$’s on $X$. However, the grading given by the ring of sections of the tautological line bundle is coarser. Thus, replacing $X$ with $X_1$, we reduced the statement to the $\mathbb{T}$-equivariant case with $s = 1$. Then, the statement follows from Lemma 3.25.

**Corollary 3.27.** Let $(X, \Delta)$ be a log pair and $\phi : X \to Z$ be a contraction. Assume that $(X, \Delta)$ is of Fano type over $Z$. Let $z \in Z$ be a closed point. Then, the spectrum of the Cox ring $\text{Cox}^{\text{aff}}(X/Z, \Delta)$ (resp. $\text{Cox}^{\text{gr}}(X/Z, \Delta; z)$ or $\text{Cox}^{\text{loc}}(X/Z, \Delta; z)$) is of klt type.

**Proof.** The statement for $\text{Cox}^{\text{aff}}(X/Z, \Delta; z)$ is a direct consequence of Theorem 3.26. It suffices to show the statement for $\text{Cox}^{\text{gr}}(X/Z, \Delta; z)$. Indeed, the spectrum of the localization at the maximal graded ideal will have klt-type singularities, provided that the spectrum of $\text{Cox}^{\text{gr}}(X/Z, \Delta; z)$ satisfies that property. Let $W'_1, \ldots, W'_k$ be Weil divisors on $X_z$ whose classes generate $\text{Cl}(X_z/Z_z, \Delta_z)$. Up to shrinking $Z$ around $z$, we may find Weil divisors $W_1, \ldots, W_k$ on $X$ whose pull-backs to $X_z$ coincide with the $W'_i$’s. Up to shrinking $Z$ around $z$ again, we may assume that the group $K$ generated by the $W_i$’s in $\text{Cl}(X/Z, \Delta)$ is isomorphic to the group $N'$ generated by the $W_i$’s in $\text{Cl}(X_z/Z_z, \Delta_z)$. We can consider the Cox ring construction with respect to the subgroup $N \leq \text{Cl}(X/Z, \Delta)$. We denote this ring by

$$\text{Cox}^{\text{aff}}(X/Z, \Delta)(W_1, \ldots, W_k).$$

By Theorem 3.26, we know that the spectrum of the ring (3.3) is of klt type. Note that the spectrum of the relative gr-local Cox ring $\text{Cox}^{\text{gr}}(X/Z, \Delta; z)$ is induced by the base change $Z_z \to Z$ from the spectrum of the ring (3.3). Hence, we conclude that the spectrum of $\text{Cox}^{\text{gr}}(X/Z, \Delta; z)$ is of klt type.

**3.5 The local Henselian Cox ring**

In this section, we define the local Henselian Cox ring for germs $(X, \Delta; x)$, where $(X, \Delta)$ is a pair and $x \in X$ is a closed point. More generally, when $\phi : X \to Z$ is a contraction, we define the relative local Henselian Cox ring for closed points $z \in Z$.

**Definition 3.28.** Let $(X, \Delta)$ be a log pair and $\phi : X \to Z$ be a contraction. Let $z \in Z$ be a closed point. As usual, we denote by $Z^h$ the spectrum of the Henselization of the local ring of $Z$ at $z$. We obtain a morphism by base change $X^h \to Z^h$. The relative gr-Henselian Cox ring at $z \in Z$ is defined to be

$$\text{Cox}^{\text{gr}-h}(X/Z, \Delta; z) := \text{Cox}(X^h/Z^h, \Delta^h),$$

where $\Delta^h := \Delta \circ \phi_!^h$. This provides a natural way to study the extremal rays of the Cox ring through the log pull-back to $X^h$.
where the right side is the relative Cox ring of the associated morphism as defined in Definition 3.15. Then, the relative local Henselian Cox ring at $z \in Z$ is defined to be

$$\text{Cox}^h(X/Z; \Delta; z) := (\text{Cox}^{gr-h}(X/Z; \Delta; z)_m)^h.$$ 

In the right-hand side, we are taking the localization at the maximal ideal $m$ defined by homogeneous regular functions on the Cox ring that correspond to Weil divisors on $X^h$ that intersect $\Delta^h$ nontrivially, followed by Henselization. The boundary $\Delta^h$ is the boundary induced by $\Delta$ on $X^h$. We can also define the gr-Henselian and local Henselian Cox ring, which come as the special cases where $X \to X$ is the identity and $x \in X$ is a closed point. We denote them, respectively, by

$$\text{Cox}^{\text{gr-h}}(X; \Delta; x) \quad \text{and} \quad \text{Cox}^h(X; \Delta; x).$$

The following example shows that the Henselian local Cox ring often differs from the local Cox ring. The former captures the local topology of the singularity, whereas the latter does not (see Subsection 2.5).

**Example 3.29.** This example shows that the Henselian local Cox ring can be different from the local Cox ring. Consider the factorial quasi-cone threefold singularity $X$ defined by

$$\{(x, y, z, w) \in \mathbb{A}^4 \mid x^2 + y^3 + z^4w = 0\}.$$ 

There is a singular stratum $C$ given by $\{x = y = z = 0\}$. Around a general point $c$ of $C$, étale locally, $X$ is isomorphic to $\mathbb{C}$ times the $E_6$-singularity. Thus, the regional fundamental group at $c$ is the binary tetrahedral group, which has abelianization $\mathbb{Z}/3\mathbb{Z}$. Note that $X$ admits a $\mathbb{C}^*$-action, so the class groups $\text{Cl}(X)$ and $\text{Cl}(X_c)$ are generated by torus invariant divisors. In particular, $\text{Cl}(X)$ has been explicitly computed in [7] — the singularity $X$ is number $18$ in [7, Thm. 1.8] and is factorial due to [7, Thm. 1.9]. So $\text{Cl}(X)$ and thus also $\text{Cl}(X_c)$ is trivial.

On the other hand, since the regional fundamental group of the singularity is not perfect, the abelianization $\text{Cl}(X_c^h)$ is nontrivial, so the local Henselian Cox ring at $c$ is nontrivial. Similar examples are given in [14, Ex 5.5].

The following theorem shows that the relative local Henselian Cox ring is well behaved for Fano-type morphisms.

**Theorem 3.30.** Let $X \to Z$ be a Fano-type morphism, where $(Z, z)$ is the spectrum of a local ring essentially of finite type over $\mathbb{C}$. Let $X^h \to Z^h$ be the base change to the Henselization of the local ring. Then, the following statements hold.

1. The class group $\text{Cl}(X^h/Z^h)$ is finitely generated,
2. The relative gr-Henselian Cox ring $\text{Cox}^{\text{gr-h}}(X/Z; \Delta; z)$ is finitely generated over $\mathcal{O}_{Z^h}$.
3. The spectra $\text{Spec} \left( \text{Cox}^{\text{gr-h}}(X/Z; \Delta; z) \right)$ and $\text{Spec} \left( \text{Cox}^h(X/Z; \Delta; z) \right)$ are klt type.

**Proof.** Note that $(Z, z)$ is a klt-type singularity. Indeed, since $X \to Z$ is a Fano-type morphism, we can find a boundary $B$ on $X$ so that $(X, B)$ is klt and $K_X + B \sim_{\mathbb{Q}, Z} 0$. By the canonical bundle formula, we can find a boundary $B_Z$ on $Z$ so that $(Z, B_Z)$ is klt.
We prove the first statement. The subgroup of \( Cl(X^h/Z^h) \) generated by the class of the effective Weil divisors contracted by \( X^h \to Z^h \) and the class group of the generic fiber is finitely generated. Hence, it suffices to show that \( Cl(Z^h) \) is finitely generated. Let \( Y_0 \to Z \) be a purely log terminal blow-up of the klt-type singularity \((Z, z)\). By base change, we obtain a plt blow-up \( Y_0^h \to Z^h \) of the local Henselian klt singularity. Let \( E \) be the exceptional divisor. Then, we have that

\[
\text{rank}_\mathbb{Q}(Cl(Z^h)_\mathbb{Q}) \leq \text{rank}_\mathbb{Q}Cl(\mathbb{Q})(E) + 1.
\]

On the other hand, the torsion subgroup of \( Cl(Z^h) \) is finite. Indeed, its order is bounded by the order of the regional fundamental group of \( Z \) at \( z \), which is finite by [18, Theorem 2]. We conclude that \( Cl(Z^h) \) is finitely generated, so \( Cl(X^h/Z^h) \) is finitely generated as claimed. This proves the first statement.

We prove the second statement. Since \( Cl(X^h/Z^h) \) is finitely generated, we can find a finite set of Weil divisors \( W_1, \ldots, W_r \) on \( X^h \) that generate this group. Recall that \( Z^h \to Z \) is a colimit of étale morphisms. Hence, there exists a pointed étale cover \((Z', z') \to (Z, z)\) a base change \( X' \to Z' \), and divisors \( W'_1, \ldots, W'_r \) on \( X' \) which pull-back to \( W_1, \ldots, W_r \), respectively. Since \( Z' \to Z \) is of finite type, we conclude that \( Z' \) is essentially of finite type over \( \mathbb{C} \). Hence, \( X' \to Z' \) is a projective morphism to the spectrum of a local ring essentially of finite type over \( \mathbb{C} \). We can find a projective morphism \( X'' \to Z'' \) of Fano type over a pointed affine algebraic variety \((Z'', z'')\) so that the base change of \( X'' \to Z'' \) to the localization of \( Z'' \) at \( z'' \) is isomorphic to \( X' \to Z' \). Let \( W''_1, \ldots, W''_r \) be Weil divisors on \( X'' \) that restrict to the divisors \( W'_1, \ldots, W'_r \). By Theorem 3.21, we conclude that the multigraded ring

\[
\text{Cox}(X''/Z'', \Delta)(W''_1, \ldots, W''_r) \tag{3.4}
\]

is finitely generated over \( \mathbb{C} \). By faithfully flat base change, we conclude that the ring

\[
\text{Cox}^{\text{gr-h}}(X/Z, \Delta; z)
\]

is finitely generated over \( \mathcal{O}_{Z^h} \). This proves the second statement.

By the proof of Corollary 3.27, we have that the spectrum of the ring (3.4) is klt type. Hence, the same statement holds when we take the base change with respect to \( Z^h \to Z'' \). We conclude that the spectrum of \( \text{Cox}^{\text{gr-h}}(X/Z, \Delta; z) \) is klt type. Since the spectrum of \( \text{Cox}^h(X/Z, \Delta; z) \) is obtained from this ring by localization and Henselization, it is klt type as well.

\( \square \)

**Remark 3.31.** Finite generation of the class group \( Cl(\mathcal{O}_{X,x}^h) \) of the étale local ring more generally holds for rational singularities [14, Theorem 6.1]. Moreover, it is isomorphic to the class group of the completion, that is, \( Cl(\mathcal{O}_{X,x}^h) \cong Cl(\widehat{\mathcal{O}_{X,x}}) \), by [14, Theorem 6.2]. This complements the statement of Theorem 2.12, that for gr-Henselian rational rings, all local class groups are isomorphic.

**Remark 3.32.** We remark at this point that due to the above considerations, it would also be possible to define Cox rings and iteration of Cox rings for complete local rings. We omit the complete local case in order not to overload the notation. One may feel free to pass to a completion or also to base change to a \textit{gr-complete Cox ring} anytime. Graded rings over complete local rings are also considered in [24].
4 | BOUNDEDNESS OF ITERATION OF COX RINGS

In this section, we aim to prove that the iteration of the Cox ring of a relatively log Fano variety is bounded in terms of the dimension. In Subsection 4.1, we will define the iteration of Cox. We prove that for a Fano-type morphism, the iteration stops after finitely many steps. In particular, the iteration stabilizes for klt singularities. In Subsection 4.2, we prove the Jordan property for the relative regional fundamental group of a relative Fano-type variety. Finally, in Subsection 4.3, we use the Jordan property to prove the boundedness of the iteration of Cox rings. This means that there exists an upper bound for the number of iterations that only depends on the dimension.

4.1 | Iteration of Cox rings for relative MDSs

In this subsection, we define the iteration of Cox rings and generalize some results from [19] to the case of relative MDSs. An important property of Cox rings is that their spectra dominate all quotient presentations in the sense of [8, Section 4.2.1]. The quotient presentations in this sense are the good quasi-torus quotients $Y \to X = Y / H$, such that the action of $H$ is strongly stable, see [8, Definition 1.6.4.1]. This property is essential in the following, so the first thing to do is to define it in the log setting.

Definition 4.1. Let $(X, \Delta)$ be a log pair and $G$ be an affine algebraic group acting on it. We say that the $G$-action is log-strongly stable if there is an open $G$-invariant subset $X' \subseteq X$ with the following properties:

1. the complement $X \setminus X'$ is of codimension at least two in $X$,
2. the group $G$ acts log-freely on $(X', \Delta)$, that is, it acts freely on $X' \setminus \Delta$ and with at most finite isotropy groups along $\Delta$,
3. for every $x \in X'$, the orbit $G \cdot x$ is closed in $X'$.

For the rest of this section, let $(X, \Delta)$ be a log pair and $\phi : X \to Z$ be a contraction, so that $(X, \Delta)$ becomes a relative MDS over $Z$. Here, $Z$ will either be affine, the spectrum of a local ring essentially of finite type, or the Henselization of such a ring. We denote by $T_X$ the characteristic quasi-torus of $X$ over $Z$, which is a direct product of a torus $T_0$ and a finite abelian group $A$. The next crucial statement is a generalization of [19, Lemma 1].

Lemma 4.2. Let $\phi : X \to Z$ be a contraction and let $(X, \Delta)$ be a log pair. Assume that $(X, \Delta)$ is a relative MDS over $Z$. Denote by $\overline{X} := \text{Spec} \text{Cox}(X / Z, \Delta)_{N, \chi}$ the total coordinate space of the relative log Cox ring with respect to $N \subseteq W\text{Div}(X, \Delta)$ and $\chi$. Denote by $X_1$ the finite Galois log quasi-étale cover of $X$ corresponding to the abelian group $A$, by $\Delta_1$ the divisor such that $K_X + \Delta$ equals the log-pullback of $K_X + \Delta$, and by $Y$ the quotient of $\overline{X}$ by $A$. Then, the following statements hold:

1. $Y$ is $\mathbb{Q}$-factorial over $Z$ and there exists a boundary $\Delta_Y$ on $Y$, such that the aff-contraction $Y \to Z$ is a relative MDS for $(Y, \Delta_Y)$ with characteristic quasi-torus $A$. In particular, there are $N_Y \subseteq W\text{Div}(Y, \Delta)$ and $\chi_Y$, such that

$$\text{Cox}(X / Z, \Delta)_{N, \chi} \cong \text{Cox}(Y / Z, \Delta_Y)_{N_Y, \chi_Y}.$$
There exists a boundary \( \Delta \) on \( X \), so that \( (X_1, \Delta_1) \) is a relative MDS over \( Z \) if and only if \( (\overline{X}, \overline{\Delta}) \) is a relative MDS over \( Z \). If this is the case, then \( \mathbb{T}_{X_1} \cong \mathbb{T}_{\overline{X}} \times \mathbb{T}_0 \) and there exist \( N_{X_1}, \chi_{X_1}, N_X, \) and \( \chi_X \), such that

\[
\text{Cox}(X_1/Z, \Delta_1)_{N_{X_1}, \chi_{X_1}} \cong \text{Cox}(\overline{X}/Z, \overline{\Delta})_{N_X, \chi_X}.
\]

In particular, if (2) holds, we have a commutative diagram, where dashed arrows denote good quasitorus quotients of big open subsets

```
\begin{tikzcd}
\overline{X}_1 \arrow{r}{/Y_0} \arrow{dr}{/Y_z} & \overline{X} \arrow{r}{/A} \arrow{d}{/Y_0} \arrow{dl}{/Y_z} & Y \\
\overline{X} \arrow{r}{/A} \arrow{d}{/Y_0} \arrow{dl}{/Y_z} & X_1 \arrow{r}{/A} \arrow{d}{/Y_0} \arrow{dl}{/Y_z} & X \\
& Z.
\end{tikzcd}
```

**Corollary 4.3.** Under the assumptions of Lemma 4.2, assume further that \((X, \Delta)\) is of Fano type over \( Z \). Then:

1. \((X_1, \Delta_1)\) is of Fano type over \( Z \).
2. \( \overline{X} \) has Gorenstein canonical singularities.
3. There is a boundary \( \overline{\Delta} \) on \( \overline{X} \), such that \( (\overline{X}, \overline{\Delta}) \) is a relative MDS over \( Z \) and its Cox ring coincides with the Cox ring of \((X_1, \Delta_1)\).

**Proof.** The first item follows analogously as in the proof of Theorem 3.26. Concretely, the finite morphism \( p : X_1 \to X \) ramifies with multiplicity at most \( m \) at prime divisors of \( X \) appearing with coefficient at least \( 1 - \frac{1}{m} \) in \( \Delta \). Then, the log pull-back of \( K_X + \Delta \), that is \( K_{X_1} + \Delta_1 \), is a klt pair. Since \((X, \Delta)\) is of Fano type over \( Z \), we can find a boundary \( B \) on \( X \) so that \( K_X + \Delta + B \sim_{Q,Z} 0 \) is klt and \( B \) is big over \( Z \). We may assume that \( B \) contains no component of the branch locus of \( p \). Then, the pull-back \( B_1 := p^*(B) \) satisfies that \( K_{X_1} + \Delta_1 + B_1 \sim_{Q,Z} 0 \) is klt and \( B_1 \) is big over \( Z \). Hence, \((X_1, \Delta_1)\) is of Fano type over \( Z \).

The second item follows by the same considerations as in the proof of [19, Theorem 1], with the following two differences. First, \( Y \) is in general only \( Q \)-factorial over \( Z \), that is, every class in \( \text{Cl}(Y/Z, \Delta) \) has a multiple that is Cartier, so we only get Gorensteinness locally.

Second, Cox rings are not unique but involve a choice of subgroup \( N \subseteq \text{WDiv} \) and \( \chi \). However, index-one covers, as defined, for example, in [56, Definition 5.19] involve a choice as well, namely, a choice of a section \( s \) of \( rK_Y \), where \( r \) is the Gorenstein index. This means having chosen a Cox ring \( \text{Cox}(Y/Z, \Delta_Y)_{N_Y, \chi_Y} \), we get (a choice of) an index one cover for free, namely, the Veronese subalgebra \( A \) of the Cox ring with respect to the subgroup \( \langle K_Y \rangle \subseteq \text{Cl}(Y/Z, \Delta_Y) \).
By [56, Definition 2.49 and 2.50], \(\text{Spec } A\) is indeed Gorenstein over \(Z\). Thus, \(\overline{Y}\) is Gorenstein over \(Z\). Since it is of klt type by Theorem 3.26, it has canonical singularities. The third item follows from Lemma 4.2, (2) and the fact that the relative Fano type \((X_1, \Delta_1)\) is a relative MDS over \(Z\)

\[\square\]

Remark 4.4. Note that in order to ensure that \(\overline{X}\) is Mori Dream, in general, it does not suffice that it is Gorenstein canonical. The essential property is that \((X_1, \Delta_1)\) is of Fano type relative over the base.

Proof of Lemma 4.2. We start by defining the boundaries on \(Y\) and \(\overline{X}\). Since \(\overline{X} \to X_1\) and \(Y \to X\) are locally trivial torus bundles in codimension 1, we can uniquely pullback Weil divisors by first restricting to the smooth locus, pulling back via usual pullback of Cartier divisors and finally taking the closure (see, e.g., [8, Rem 1.3.4.1]). Hence, we can define \(\Delta_Y\) and \(\overline{\Delta}_{\overline{X}}\) to be the pullbacks of \(\Delta\) and \(\overline{\Delta}\), respectively. Moreover, the divisor \(\overline{\Delta}\) is the unique divisor so that \(K_{\overline{X}} + \overline{\Delta}\) is the log pull-back of \(K_Y + \Delta_Y\).

Now, we argue that \(\text{Cox}(X/\overline{Z}, \Delta)_{N,Y, \overline{\chi}}\) together with the coarsened \(A\)-grading is the Cox ring of \((Y, \Delta_Y)\) over \(Z\). First, since \(\text{Cox}(X/\overline{Z}, \Delta)_{N,Y, \overline{\chi}}\) is the Cox ring of \((X, \Delta)\) over \(Z\), it is factorially \(\text{Cl}(X/\overline{Z}, \Delta)\)-graded by [8, Theorem 1.5.3.7]. Here, factoriality of a \((K\text{-})\)grading means that every \(K\)-homogeneous nonzero nonunit is a product of \(K\text{-}primes\), that is, homogeneous nonzero nonunits \(f\) with the property that if \(f \mid gh\) with \(g, h\) homogeneous, then \(f \mid g\) or \(f \mid h\), see [8, Definition 1.5.3.1]. This is one of the defining properties of Cox rings, see [8, Theorem 1.6.4.3, Corollary 1.6.4.4].

Now by [12, Theorem 1.5], the ring \(\text{Cox}(X/\overline{Z}, \Delta)_{N,Y, \overline{\chi}}\) is also factorially \(A\)-graded. Since \(\overline{X} \to X\) is the characteristic space of \((X, \Delta)\) over \(Z\), the characteristic quasi-torus \(T_X\) acts log-strongly stably on \((\overline{X}, \overline{\Delta})\); thus, the subgroup \(A\) acts log-strongly stably as well. Altogether, by Theorem 1.6.4.3 and Corollary 1.6.4.4 of [8], we get that \(\text{Cl}(Y/Z, \Delta_Y) \cong A\) and \(\text{Cox}(X/\overline{Z}, \Delta)_{N,Y, \overline{\chi}}\) is a Cox ring for \((Y, \Delta_Y)\) over \(Z\). The choice of \(N_Y \subseteq \text{WDiv}(Y, \Delta)\) and \(\chi_Y\) is as follows: we can identify \(N \subseteq \text{WDiv}(X, \Delta)\) with a subgroup mapping isomorphically to the free part of \(\text{Cl}(X/\overline{Z}, \overline{\Delta})\) and a subgroup mapping to its torsion part. Second, for a choice of \(\chi_{\overline{X}}\), we can assume that \(\chi_{\overline{X}}\) maps to \(\text{C}(X_1)^* \subseteq \text{C}(\overline{X})^*\). So, the first item from the lemma follows.

We prove the second item. We already defined the boundary \(\overline{\Delta}\) on \(\overline{X}\). First assume that \((\overline{X}, \overline{\Delta})\) is a relative MDS over \(Z\). Then, for a choice \(N_{\overline{X}} \subseteq \text{WDiv}(\overline{X}, \overline{\Delta})\), we can assume that \(N_{\overline{X}}\) is a subgroup of \(\text{WDiv}(\overline{X}, \overline{\Delta})\). Moreover, it is the direct product of a subgroup mapping isomorphically to the free part of \(\text{Cl}(\overline{X}/\overline{Z}, \overline{\Delta})\) and a subgroup mapping to its torsion part. Second, for a choice of \(\chi_{\overline{X}}\), we can assume that \(\chi_{\overline{X}}\) maps to \(\text{C}(X_1)^* \subseteq \text{C}(\overline{X})^*\).

Then, the Cox ring \(\text{Cox}(\overline{X}/\overline{Z}, \overline{\Delta})_{N_{\overline{X}}, \chi_{\overline{X}}}\) is factorially \(\text{Cl}(\overline{X}/\overline{Z}, \overline{\Delta})\)-graded. Invoking [12, Theorem 1.5] as above, we see that it also factorially \(\text{Cl}(\overline{X}/\overline{Z}, \overline{\Delta}) \times \mathbb{Z}^{\text{dim}(T_0)}\)-graded. Let \(T_{\overline{X}, 0}\) be the connected component of 0 of the characteristic quasi-torus of \(\overline{X}\). Moreover, the action of \(T_{\overline{X}, 0} \times T_0\) on \(\overline{X}\) is strongly stable, since the actions of \(T_{\overline{X}, 0}\) and \(T_0\) on \(\overline{X}\) and \(\overline{X}\), respectively, are so. Thus, \(\text{Cox}(\overline{X}/\overline{Z}, \overline{\Delta})_{N_{\overline{X}}, \chi_{\overline{X}}}\) is indeed a Cox ring for \((X_1, \Delta_1)\) over \(Z\). In particular,

\[\text{Cl}(X_1/Z, \Delta_1) \cong \text{Cl}(\overline{X}/\overline{Z}, \overline{\Delta}) \times \mathbb{Z}^{\text{dim}(T_0)}\].
The choice of $N_{X_1} \subseteq \mathrm{Cl}(X_1/Z, \Delta_1)$ and $\chi_{X_1}$ is as follows: for $N_{X_1}$, take the direct product of $N_{\overline{X}}$ as a subgroup of $\mathrm{WDiv}(X_1/Z, \Delta_1)$ and an arbitrary subgroup mapping isomorphically to the $\mathbb{Z}^{\dim(T_0)}$-part of $\mathrm{Cl}(X_1/Z, \Delta_1)$. Hence, we can identify the kernel of $N_{X_1} \to \mathrm{Cl}(X_1/Z, \Delta_1)$ with the kernel of $N_{\overline{X}} \to \mathrm{Cl}(\overline{X}/Z, \overline{\Delta})$. Thus, $\chi_{\overline{X}}$ from above can be taken to define $\chi_{X_1}$.

The arguing in the other direction, that is, when $(X_1, \Delta_1)$ is a MDS over $Z$, is analogous to the proof of the first item. This concludes the proof. \[ \square \]

**Definition 4.5.** Let $\phi : X \to Z$ be a contraction (or an aff-contraction) and $(X, \Delta)$ a relative MDS over $Z$. We denote $\mathbb{T}^1 := \mathbb{T}_X$ with torus part $\mathbb{T}_0^1$ and finite abelian part $A_1^1$, respectively. We define

$$\mathrm{Cox}^{(1)}(X/Z, \Delta) := \mathrm{Cox}(X/Z, \Delta), \quad \overline{X}_1 := \overline{X} = \mathrm{Spec} \left( \mathrm{Cox}(X/Z, \Delta) \right), \quad \text{and} \quad \overline{\Delta}_1 := \overline{\Delta}.$$ 

We iteratively define $\mathrm{Cox}^{(i)}(X/Z, \Delta)$ as follows. Assume that $(\overline{X}_{i-1}/\overline{\Delta}_{i-1})$ is a relative MDS over $Z$. Then, we set

$$\mathrm{Cox}^{(i)}(X/Z, \Delta) := \mathrm{Cox}(\overline{X}_{i-1}/Z, \overline{\Delta}_{i-1}), \quad \overline{X}_i := \mathrm{Spec} \left( \mathrm{Cox}^{(i)}(X/Z, \Delta) \right), \quad \text{and} \quad \mathbb{T}^i := \mathbb{T}_{X_{i-1}} = \mathbb{T}_0^i \times A_i^i.$$ 

We let $\overline{\Delta}_i$ be the log-pullback of $\Delta_{i-1}$. Then, we call the (possibly infinite) chain

$$\cdots \to (\overline{X}_3, \overline{\Delta}_3) \to (\overline{X}_2, \overline{\Delta}_2) \to (\overline{X}_1, \overline{\Delta}_1) \to (X, \Delta)$$

the *iteration of Cox spaces of $(X, \Delta)$ over $Z$. If $\mathrm{Cl}(\overline{X}_i/Z, \overline{\Delta}_i)$ is trivial for some $i \geq 1$, we say that $(X, \Delta)$ has finite iteration of Cox rings over $Z$. If the iteration stabilizes for some $k$, that is, the ring is eventually factorial, then we denote by $\mathrm{Cox}^{(k)}(X/Z, \Delta)$ the isomorphism class over $Z$ of this ring. The ring $\mathrm{Cox}^{(k)}(X/Z, \Delta)$ is called the *iteration of Cox rings or the master Cox ring.*

**Remark 4.6.** In the case that $Z$ is local, essentially of finite type or Henselian, in the above definition, we iterate the gr-local or gr-Henselian Cox rings. In each step, we can also localize (and Henselize, respectively) $\overline{X}_{i-1}$ at the unique graded maximal ideal and take the Cox ring of such spectrum. However, by Lemma 2.9 and Theorem 2.12, the class groups of these spaces agree. Hence, the iteration defined in such way is compatible to the iteration defined above. Indeed, the localization (and Henselization respectively) of $\overline{X}_i$ will always yield the same spectrum.

**Remark 4.7.** By Lemma 4.2, $(\overline{X}, \overline{\Delta})$ is Mori Dream over $Z$ if and only if $(X_1, \Delta_1)$ is so. Thus, the iteration of Cox rings induces a chain of finite abelian Galois covers $(X_1, \Delta_1) \to (X_{i-1}, \Delta_{i-1})$, where $\Delta_i$ is the log-pullback of $\Delta_{i-1}$. In particular, the characteristic quasi-tori satisfy

$$\mathbb{T}_{X_i} \cong \mathbb{T}^i \times \mathbb{T}_{X_{i-1}}^0.$$
We get the following commutative diagram:

\[ \begin{array}{cccc}
\mathcal{X}_1 & \mathcal{X}_2 \\
\downarrow & \downarrow \\
\mathcal{X} & Z
\end{array} \]

In particular, Corollary 4.3 tells us that if \((X, \Delta)\) is of Fano type over \(Z\), so is \((X_i, \Delta_i)\) for any \(i \geq 1\). Hence, the \(i\)th iterated Cox ring and \((\mathcal{X}_i, \mathcal{\Delta}_i)\) is defined for any \(i \geq 1\). The question remains if the iteration stabilizes or not. And, if yes, if there is any bound on the number of iteration steps. We answer these questions in Section 4.3.

We finish the present subsection by showing that the actions of the characteristic quasi-tori can be lifted to the iterated total coordinate spaces. Moreover, they induce an action of a solvable reductive group. This generalizes observations made in [7, 19]. We start with the following lemma slightly generalizing [9, Theorem 5.1]. This lemma covers the lifting of automorphisms to the Cox ring of relative MDSs, affine over the base.

**Lemma 4.8.** Let \((X, \Delta)\) be a relative MDS, affine over \(Z\) and \(\text{Aut}_Z(X)\) the automorphism group of \(X\) over \(Z\). Denote by \(\text{Aut}^\tau(\mathcal{X})\) the normalizer of the characteristic quasi-torus \(\mathbb{T}\) in the automorphism group \(\text{Aut}_Z(\mathcal{X}) := \text{Spec} \text{Cox}(X/Z, \Delta)_{N, \chi}\). Then there is a short exact sequence:

\[ 1 \longrightarrow \mathbb{T} \longrightarrow \text{Aut}^\tau(\mathcal{X}) \longrightarrow \text{Aut}_Z(X) \longrightarrow 1. \]

The above short exact sequence is called a lifting of \(\text{Aut}_Z(X)\) to \(\text{Aut}_Z(\mathcal{X})\).

**Proof.** The proof is analogous to the one of [9, Theorem 5.1]; note that the assumption \(\kappa^* = \kappa[X]^*\) in *loc. cit.* is the standard assumption in the classical theory of Cox rings to make sure that they are unique in the presence of torsion in the class group. See [8, Proposition 1.4.2.2] and also [53, Theorem 2.3]. For the convenience of the reader, we recapitulate the proof and indicate, where we have to argue in addition due to the absence of this assumption in our case. Each automorphism \(\phi \in \text{Aut}^\tau(\mathcal{X})\) induces an automorphism of \(\text{Aut}_Z(X)\). Now, due to the absence of the assumption \(\kappa^* = \kappa[X]^*\), we need to prove the surjectivity of the induced map \(\text{Aut}^\tau(\mathcal{X}) \to \text{Aut}_Z(X)\) because we have nonisomorphic Cox rings depending on the choice of \(N \subseteq \text{WDiv}(X, \Delta)\) and \(\chi\). An automorphism \(\psi \in \text{Aut}_Z(\mathcal{X})\) induces an automorphism \(\psi\) of \(\text{WDiv}(X, \Delta)\). Observe that \(\psi\) maps the kernel of the surjective map \(N \to \text{Cl}(X/Z, \Delta)\) to the kernel of the surjective map \(\varphi : \psi(N) \to \text{Cl}(X/Z, \Delta)\) and induces a character \(\psi^*(\chi) : \ker(\varphi) \to \text{C}(X)^*\). Hence, \(\text{Cox}(X/Z, \Delta)_{N, \chi}\) and \(\text{Cox}(X/Z, \Delta)_{\psi(N), \psi^*(\chi)}\)
are isomorphic. Moreover, fixing an isomorphism \( \tau \) between them, \( \psi^* \circ \tau \) is an element of \( \text{Aut}^\mathbb{T}(\widehat{X}) \) mapping to \( \psi \). So, the map \( \text{Aut}^\mathbb{T}(\widehat{X}) \rightarrow \text{Aut}_Z(X) \) is surjective. On the other hand, the quasi-torus \( \mathbb{T} \) is a subgroup of \( \text{Aut}^\mathbb{T}(\widehat{X}) \) by definition, and it acts by homogeneous automorphisms, which are trivial on \( X \). This finishes the proof of the lemma.

The following proposition explains how to lift automorphisms to the Cox ring of a relative MDS that is projective over the base. Here, we have to distinguish between automorphisms on the total coordinated space \( \widehat{X} \) and its big open subset \( \widehat{X} \). Following [8, Section 4.2.4], we denote by \( \text{Bir}_{Z,Z}(X) \) the weak automorphisms of \( X \) over \( Z \), namely, birational maps \( X \rightarrow X \) being regular isomorphisms in codimension one over \( Z \).

**Proposition 4.9.** Let \( (X, \Delta) \) be a relative MDS, projective over \( Z \). Let \( \text{Aut}_Z(X) \) be the automorphism group of \( X \) over \( Z \). Then, the following statements hold.

1. \( \text{Aut}_Z(X) \) is a linear algebraic group.
2. There is a divisor \( L \) on \( X \), ample over \( Z \), with relative section ring \( R_L \) and spectrum \( \widehat{X}^* := \text{Spec} R_L \), such that there is a short exact sequence
   
   \[
   1 \longrightarrow \mathbb{C}^* \longrightarrow \text{Aut}^\mathbb{C}^*(\widehat{X}) \longrightarrow \text{Aut}_Z(X) \longrightarrow 1.
   \]

3. There is a commutative diagram with exact sequences as rows and vertical inclusions of finite index:

\[
\begin{array}{c}
1 \longrightarrow \mathbb{T} \longrightarrow \text{Aut}^\mathbb{T}(\widehat{X}) \longrightarrow \text{Bir}_{Z,Z}(X) \longrightarrow 1 \\
| & | & | \\
1 \longrightarrow \mathbb{T} \longrightarrow \text{Aut}^\mathbb{T}(X) \longrightarrow \text{Aut}_Z(X) \longrightarrow 1.
\end{array}
\]

**Proof.** Since \( (X, \Delta) \) is relative Mori Dream over \( Z \), the cone of nef divisors relative to \( Z \) is rational polyhedral. Its rays are permuted by the group of components of \( \text{Aut}_Z(X) \). Taking the sum of the ray generators, we get the relatively ample \( \text{Aut}_Z(X) \)-invariant class \([L]\). By [21, Theorem 2.16], \( \text{Aut}_Z(X) \) is a linear algebraic group.

Now, \( \text{Aut}_Z(X) \) may not stabilize the divisor \( L \), but only its class. We argue as in the proof of Lemma 4.8. Since \( L \) is ample, its relative stable base locus is empty (see, e.g., [21, Section 2.3]). Thus, we have isomorphisms \( L^\vee \setminus L_0^\vee \cong \widehat{X} \setminus \{0\} \) and \( \text{Aut}^\mathbb{C}^*(L) \cong \text{Aut}^\mathbb{C}^*(L^\vee) \) (where \( \{0\} \) is the unique \( \mathbb{C}^* \)-fixed point on \( \widehat{X} \)), so we obtain an isomorphism \( \text{Aut}^\mathbb{C}^*(L) \cong \text{Aut}^\mathbb{C}^*(\widehat{X}) \). Hence, we get the short exact sequence of the second item by [21, Proposition 2.13]. The proof of the last item is analogous to the one of [8, Theorem 4.2.4.1].

In what follows, we aim to lift the whole characteristic quasi-torus action to the iterated Cox rings. In this way, we will produce an action of a solvable reductive group. The derived normal series of this solvable reductive group reflects the iteration of Cox rings. We denote the \( k \)th derived subgroup of a group \( G \) by \( D_k^G := [D_{k-1}^G, D_{k-1}^G] \), where \( D_0^G := G \).
Now we have a closer look to the quotient presentations from [8, Section 4.2.1] we already encountered. In our setting, in analogy to the notion of quasi-étale covers, we will call them 
abelian quasi-torsors in the following.

In the classical setting, where \( X \) is assumed to have only constant invertible global functions, the quasi-torsors are assumed to have only constant invertible \( H \)-homogeneous functions. Thus, in our setting, we have two major differences: first, \textit{a priori}, we have invertible nonconstant functions on \( X \), which means that we have nonisomorphic Cox rings depending on the choice of \( N \subset \ WD\text{Div}(X) \) and the character \( \chi \). This means that an abelian quasi-torsor \( Y \to X \) is dominated by \( X_{N,\chi} \to X \) for some choice of \( N \) and \( \chi \). Second, what we have to impose is not that invertible \( H \)-homogeneous functions are constant, but that they descend to \( X \). This property is fulfilled, for example, if there is at least one maximal homogeneous ideal in \( \mathcal{O}(Y) \). This is the case in all situations relevant for us, for example, finite coverings of relative Fano type \((X,\Delta)\) or (iterated) Cox rings. However, even if \( X \) has only constant invertible functions, there may be nonconstant non-homogeneous invertible functions in the Cox ring (see, e.g., [8, Ex 1.4.4.2]). The precise definition of an abelian quasi-torsor in our setting is the following.

**Definition 4.10.** Let \((X,\Delta)\) be a relative MDS over \( Z \). Let \( Y \to X = Y/H \) be a good quotient by a quasi-torus \( H \). We call \( \varphi : Y \to X \) an \textit{abelian quasi-torsor}, if the following are satisfied.

- \((1)\) Let \( H_0 \) be the identity component and \( A \) be the group of components of \( H \). Then, the finite abelian cover \( Y' := Y/H_0 \to X \) is finite Galois log quasi-étale over \((X,\Delta)\) with log-pullback \( \Delta_{Y'} \) of \( \Delta \).
- \((2)\) There are big open subsets \( U_{Y'} \subseteq Y' \) and \( U_Y := \varphi^{-1}(U_{Y'}) \subseteq Y \) such that the restriction
  \[
  \varphi|_{U_Y} : U_Y / H_0 \to U_{Y'}
  \]
  is an étale locally trivial \( H_0 \)-bundle. In particular, the action of \( H_0 \) on \( Y \) is log strongly-stable.
- \((3)\) Global invertible homogeneous functions on \( Y \) descend to \( X \) via the induced homomorphism \( \mathcal{O}(Y)^H \cong \mathcal{O}(X) \hookrightarrow \mathcal{O}(Y) \).

In the case that \( H \) is a torus, then we may say that it is a \textit{torus quasi-torsor}. Whenever the quasi-torus \( H \) is clear from the context, we may just say that \( Y \to X \) is a \textit{quasi-torsor}.

Let \((X,\Delta;x)\) be a klt singularity. We say that \((Y,y)\) is a \textit{pointed abelian quasi-torsor} of \((X,\Delta;x)\) if there exists an abelian quasi-torsor \( Y \to X \) so that the image of \( y \) in \( X \) equals \( x \). To shorten notation, we may say that \( Y \to X \) is an \textit{abelian pointed cover}. Observe that if \( Y \to X \) is an abelian pointed cover, then the corresponding finite morphism \( Y' \to X \) is a finite pointed cover.

**Proposition 4.11.** Let \((X,\Delta)\) be a relative MDS over \( Z \). Let \( Y \to X = Y/H \) be a quasi-torsor. Then, there exists

- a monomorphism \( \chi(H) \to Cl(X/Z, \Delta) \),
- a subgroup \( N_Y \leq N \leq WD\text{iv}(X) \),
- surjections \( \varphi : N \to Cl(X/Z, \Delta) \) and \( \varphi|_{N_Y} : N_Y \to \chi(H) \), and
- a character \( \chi : \ker(\varphi) \to \mathbb{C}(X)^* \),

such that the following statements are satisfied:

- \((1)\) \( Y \cong \text{Spec}_X R_{(X/Z,\Delta)} \).
(2) There is a commutative diagram

\[ \begin{array}{ccc}
\hat{X}_{N,X} & \xrightarrow{H'} & Y \\
/ & & / \\
/ & & /
\end{array} \]

where the quasi-torus $H'$ is defined by the exact sequence $1 \to H' \to \mathbb{T}_X \to H \to 1$.

Proof. The proof is analogous to the one of [8, Theorem 4.2.1.4], with the two differences mentioned above. In particular, invoking [8, Proposition 1.6.4.5] and the notation therein, we get the following. Let $M := \mathbb{X}(H)$ and $E(Y)$ be the multiplicative group of nonzero $M$-homogeneous rational functions on $Y$. Then, we have the following diagram of group homomorphisms:

\[ \begin{array}{ccc}
E(Y) & \xrightarrow{f_{\ast} = \text{div}(f)} & \text{WDiv}(Y)^H \\
& & \downarrow q^* \\
& & \text{WDiv}(X)
\end{array} \]

As $Y \to X$ is an étale locally trivial $H$-bundle in codimension one, the homomorphisms $q_\ast$ and $q^*$ are inverse to each other. As in [8, Proposition 1.6.4.5], but using item (3) of Definition 4.10, the homomorphism $E(Y) \to \text{WDiv}(X)$ induces a monomorphism $M \to \text{Cl}(X/Z, \Delta)$. Thus, we can choose a subgroup $N_Y$ of $\text{WDiv}(X) \cong \text{WDiv}(Y)^H$ surjecting onto $M$ and enlarge $N_Y \subseteq N$ such that $\varphi : N \to \text{Cl}(X/Z, \Delta)$ is onto. Choosing a character $\chi : \ker(\varphi) \to \mathbb{C}(X)^\ast$ yields the desired statements together with the rest of the proof of [8, Theorem 4.2.1.4]. \qed

Corollary 4.12. Let $(X, \Delta)$ be a relative MDS over $Z$. Assume that the $k$th iterated Cox ring $\text{Cox}^{(k)}(X/Z, \Delta)$ exists and is of finite type over $Z$. Then $\hat{X}_k$ allows an action of a solvable reductive group $S$ with maximal torus $\mathbb{T} := \mathbb{T}_{X_k}$ and an $S$-invariant big open subset $\hat{X}_k^S$, such that:

1. $X_k \cong \hat{X}_k^S / \mathbb{T}$ and $X \cong \hat{X}_k^S / S$,
2. $\hat{X}_j \cong \hat{X}_k^S / D_j^S$ and $\mathbb{T}_j \cong D_{j-1}^S / D_j^S$ for $j \leq k$,
3. for the finite solvable group $S_{\text{fin}} := S / \mathbb{T}$ and the finite covers $X_j$, the assertions hold analogously, that is,

\[ X_j \cong X_k^S / D_j^{S_{\text{fin}}}, \quad \text{and} \quad A_j \cong D_{j-1}^{S_{\text{fin}}} / D_j^{S_{\text{fin}}}. \]

Proof. The arguing is analogous to the proof of [7, Theorem 1.6] in the case that $X$ is affine over $Z$, where we use Proposition 4.11 instead of [9, Proposition 3.5]. If $X$ is projective over $Z$, then we choose a divisor $L$ on $X$ ample over $Z$. By the same argument as in the proof of Lemma 4.2 (1), we have $\text{Cox}(X/Z, \Delta) \cong \text{Cox}(\hat{X}/Z, \hat{\Delta})$, and thus, we can reduce to the relatively affine case. \qed

Remark 4.13. In Definition 4.10 (2), it is essential that not only $U_{Y'} \subseteq Y'$ but also $U_Y \subseteq Y$ is a big open subset. Otherwise $\varphi : Y \to Y/H$ may contract divisors. In particular, the existence of the monomorphism $\mathbb{X}(H) \to \text{Cl}(X)$ from Proposition 4.11 would not hold true in this more general setting. As an example, consider the blowup of $Y^n := \text{Bl}_0(\mathbb{A}^n) \to \mathbb{A}^n$ at the origin, which has
relative Cox ring \( \text{Cox}(Y^n / \mathbb{A}^n) \cong \mathbb{C}[x_1, \ldots, x_{n+1}] \). Then the induced \( \mathbb{C}^\ast \)-quotient \( \mathbb{A}^{n+1} / \mathbb{A}^n \), given by the weights \((1, \ldots, 1, -1)\), is not a quotient presentation. Indeed, the divisor \( \{x_{n+1} = 0\} \) maps to the origin. Observe that we have an infinite sequence of \( \mathbb{C}^\ast \)-quotients \( \mathbb{A}^1 \hookrightarrow \mathbb{A}^2 \hookrightarrow \mathbb{A}^3 \hookrightarrow \ldots \). This sequence does not contradict Theorem 6, because this covers are not pointed abelian covers in the sense of Definition 4.10.

4.2 | Regional fundamental group of a relative Fano-type variety

In this subsection, we prove that the regional fundamental group of a relative Fano-type variety is finite and it satisfies the Jordan property.

**Definition 4.14.** Let \( \phi : X \to Z \) be a projective contraction. Let \((X, \Delta)\) be a log pair. Let \( z \in Z \) be a closed point. Let \( U \) be a path-connected analytic neighborhood of \( z \in Z \). The smooth locus of \( \phi^{-1}(U) \) is path connected as \( U^\text{reg} \) is path connected and the fibers are the smooth locus of algebraic varieties. Let \( \Delta_U \) be the restriction of \( \Delta \) to \( \phi^{-1}(U) \). Then, the fundamental group \( \pi^\text{reg}_1(\phi^{-1}(U), \Delta_U; z) \) is independent of the chosen base point, up to isomorphism. We define the fundamental group \( \pi^\text{reg}_1(X / Z, \Delta; z) \) to be the inverse limit of the fundamental groups \( \pi^\text{reg}_1(\phi^{-1}(U), \Delta_U) \) where the limit runs through all the path-connected analytic neighborhoods \( U \) of \( Z \) that contain \( z \).

**Theorem 4.15.** Let \( n \) be a positive integer. There exists a constant \( c(n) \), only depending on \( n \), satisfying the following. Let \( \phi : X \to Z \) be a projective contraction so that \( X \) has dimension \( n \). Let \((X, \Delta)\) be a log pair of Fano type over \( Z \). Let \( z \in Z \) be a closed point. Then, the fundamental group \( \pi^\text{reg}_1(X / Z, \Delta; z) \) is finite. Furthermore, there exists a normal abelian subgroup \( A \leq \pi^\text{reg}_1(X / Z, \Delta; z) \) of rank at most \( n \) and index at most \( c(n) \).

**Proof.** The divisor \( -(K_X + \Delta) \) is nef and big over \( Z \). Hence, it is semiample and big over \( Z \), given that \( X \) is a relative MDS over \( Z \). Let \( g : X \to X' \) be an ample model of \( -(K_X + \Delta) \) over \( Z \). Let \( \Delta' \) be the push-forward of \( \Delta \) to \( X' \). Then, we have that \((X', \Delta')\) has klt singularities. Indeed, \( g^*(K_X' + \Delta') = K_X + \Delta \), so all the log discrepancies of \((X', \Delta')\) are positive as this holds for \((X, \Delta)\). Furthermore, \( -(K_{X'} + \Delta') \) is ample over \( Z \). We will prove the statement for \( \pi^\text{reg}_1(X'/Z, \Delta'; z) \). Let \( Y \) be the orbifold cone with respect to the \( \mathbb{Q} \)-polarization \( -(K_{X'} + \Delta') \) over \( Z \), that is,

\[
Y := \operatorname{Spec} \left( \bigoplus_{m \geq 0} \Gamma(Y/X, \mathcal{O}_X(-m(K_{X'} + \Delta'))) \right).
\]

Note that \( \dim(Y) = \dim(X) + 1 = n + 1 \). Without loss of generality, we may assume \( \dim(Y) \geq 2 \), otherwise \( \dim(X) = 1 \) and the statement is trivial. We have a rational map \( \pi : Y \to X' \), which is defined outside a codimension 2 subset of \( Y \). Let \( \Delta_Y \) be the effective divisor so that \( \pi^*(K_{X'} + \Delta') = K_Y + \Delta_Y \). Then, the pair \((Y, \Delta_Y)\) is klt. The variety \( Y \) is endowed with a \( \mathbb{G}_m \)-action induced by the \( Z_{\geq 0} \) grading on its defining ring. The blow-up of \( Y \) at the fixed locus of the torus action is a variety \( Y \) that admits a good quotient to \( X' \). The exceptional locus of \( Y \to Y \) is isomorphic to \( X' \) and its image in \( Y \) is isomorphic to \( Z \). Hence, under this isomorphism, we can consider an embedding \( Z \hookrightarrow Y \). So, we can consider \( z \in Y \). By [18, Theorem 1], we know that the regional fundamental group \( \pi^\text{reg}_1(Y, \Delta_Y; z) \) is finite. By [20, Theorem 2], we know that there exists an abelian normal...
subgroup $A_Y$ of $\pi^\reg_1(Y, \Delta_Y; z)$ of rank at most $n + 1$ and index at most $k(n + 1)$. Here, $c_0(n) := k(n + 1)$ is a constant that only depends on $n + 1$, and hence, it only depends on $n$.

Let $U_Z$ be an arbitrary open neighborhood of $z$ in $Z$. Since $X'$ is an ample model over $Z$, we have an associated projective morphism $\phi^t : X' \to Z$. We define $U_{X', Z} := \phi^t(U_Z)$. For every such $U_Z$, we have a short exact sequence:

$$1 \to \mathbb{Z}_m \to \pi^\reg_1(U_Y, \Delta_{U_Y}) \to \pi^\reg_1(U_{X', Z}, \Delta_{U_{X', Z}}) \to 1.$$ 

This is the exact sequence associated to a plt blow-up (see, e.g., [20, Proposition 4.9]). As usual, $\Delta_{U_Y}$ (resp. $\Delta_{U_{X', Z}}$) is the restriction of $\Delta_Y$ (resp. $\Delta_{X'}$) to the open set $U_Y$ (resp. $U_{X', Z}$). We claim that for a certain neighborhood $U$ of $z$ in $Z$, there is an isomorphism

$$\pi^\reg_1(U_Y, \Delta_{U_Y}) \cong \pi^\reg_1(U_Y, \Delta_Y; z). \quad (4.1)$$

Let $U_0$ be an open neighborhood of $z$ in $Y$ that computes the regional fundamental group of the pair $(Y, \Delta_Y)$ at $z$, that is, there is an isomorphism

$$\pi^\reg_1(Y, \Delta_Y; z) \cong \pi^\reg_1(U_0, \Delta_{U_0}).$$

Let $U_{Z, 0}$ be the inverse image of $U_0$ under the embedding $Z \hookrightarrow Y$. We define $U_{X', Z, 0} := \phi^t(U_{Z, 0})$ and $U_{Y, 0} := \pi^{-1}(U_{X', Z, 0})$. Note that $U_{Y, 0}$ is homotopic to an analytic open subset $V$ that is contained in $U_0$. This homotopy is given by the torus action, so it preserves the divisor $\Delta_Y$. Hence, the composition

$$\pi^\reg_1(V, \Delta_Y) \to \pi^\reg_1(U_0, \Delta_{U_0}) \to \pi^\reg_1(U_{Y, 0}, \Delta_{U_{Y, 0}})$$

is an isomorphism. On the other hand, the first homomorphism of the previous sequence is surjective. Indeed, since $U_0$ computes the regional fundamental group, all the loops in $U_0$ can be homotoped inside $V$. Hence, all the homomorphisms in the previous composition are isomorphisms. Thus, we conclude that $U_{Y, 0}$ satisfies the isomorphism in equation (4.1). Hence, we have an exact sequence

$$1 \to \mathbb{Z}_m \to \pi^\reg_1(Y, \Delta_Y; z) \to \pi^\reg_1(U_{X', Z, 0}, \Delta_{U_{X', Z, 0}}) \to 1.$$ 

Passing to the inverse limit, we have an exact sequence

$$1 \to \mathbb{Z}_m \to \pi^\reg_1(Y, \Delta_Y; z) \to \pi^\reg_1(X'/Z, \Delta; z) \to 1.$$ 

Hence, we conclude that $\pi^\reg_1(X'/Z, \Delta; z)$ is finite and satisfies the Jordan property of rank $n + 1$, that is, it contains a normal abelian subgroup of rank at most $n + 1$ and index at most $c_0(n)$.

Now, we turn to prove that $\pi^\reg_1(X/Z, \Delta; z)$ is finite and admits a normal abelian subgroup of rank at most $n + 1$ and index at most $c_0(n)$. It suffices to prove that there is a surjection

$$\pi^\reg_1(X'/Z, \Delta; z) \to \pi^\reg_1(X/Z, \Delta; z). \quad (4.2)$$
Indeed, let $U$ be an arbitrary open neighborhood of $z$ in $Z$. Let $U_X$ be its preimage in $X$ and $U_{X'}$ be its preimage in $X'$. Then, there is a natural surjection

$$\pi_1^{\text{reg}}(U_{X'}, \Delta_{U_{X'}}) \rightarrow \pi_1^{\text{reg}}(U_X, \Delta_{U_X}).$$

Indeed, the image of the exceptional locus of $U_X \rightarrow U_{X'}$ is a union of closed subsets that are either contained in the singular locus of $U_X$ or codimension two subsets of the smooth locus. Since $\pi_1^{\text{reg}}(X'/Z, \Delta'; z)$ surjects into each of the fundamental groups $\pi_1^{\text{reg}}(U_{X'}, \Delta_{U_{X'}})$, we conclude that for each $U$, there is a surjective homomorphism

$$\pi_1^{\text{reg}}(X'/Z, \Delta; z) \rightarrow \pi_1^{\text{reg}}(U_X, \Delta_{U_X}).$$

Taking inverse limit, we conclude that the surjection (4.2) holds. Hence, $\pi_1^{\text{reg}}(X/Z, \Delta; z)$ is finite and contains a normal abelian subgroup $A_0$ of rank at most $n+1$ and index at most $c_0(n)$.

Since $\pi_1^{\text{reg}}(X/Z, \Delta; z)$ is finite, we may take the universal cover of $X$ over $Z$ and obtain a commutative diagram as follows:

\[
\begin{array}{c}
(Y, \Delta_Y) \rightarrow (X, \Delta) \\
\downarrow \quad \downarrow \\
(Z', z') \rightarrow (Z, z). 
\end{array}
\]

We argue that $(Y, \Delta_Y)$ is a log pair of Fano type over $Z'$. Let $p : (Y, \Delta_Y) \rightarrow (X, \Delta)$ be the finite Galois log quasi-étale cover. Since $(X, \Delta)$ is of Fano type over $Z$, we know that there exists $\Delta' \geq 0$ on $X$ such that $\Delta'$ is big over $Z$, $(X, \Delta + \Delta')$ is klt, and $K_X + \Delta + \Delta' \sim_{Q,Z} 0$. By pulling back the previous $Q$-linear relation via $p$, we get $K_Y + \Delta_Y + p^*(\Delta') \sim_{Q,Z'} 0$. Note that $(Y, \Delta_Y + p^*(\Delta'))$ is klt as is the pull-back of a klt pair via a finite Galois morphism (see, e.g., [66, Proposition 2.11]). On the other hand $p^*(\Delta')$ is big over $Z'$ as is the pull-back of a relatively big divisor via a finite morphism. We conclude that $(Y, \Delta_Y)$ is a log Fano pair over $Z'$. Hence, $(Z', z')$ is a klt-type singularity, that is, there exists $B'$ through $z'$ so that $(Z', B'; z')$ is klt. Let $F$ be a general fiber on a small euclidean neighborhood of $z'$ in $Z'$. Let $\Delta_F$ be the restriction of $\Delta_Y$ to the general fiber. Then, $(F, \Delta_F)$ is a projective Fano-type pair. Note that $A_0$ acts on $(Y, \Delta_Y)$. Then, we have an exact sequence

$$1 \rightarrow A_{F,0} \rightarrow A_0 \rightarrow A_{Z',0} \rightarrow 1,$$

so that $A_{F,0}$ acts on $(F, \Delta_F)$ and $A_{Z',0}$ acts on $Z'$. By [20, Theorem 3], there exists an abelian subgroup $A_F \leq A_{F,0}$ of rank at most $\dim(F)$ and index at most $k(\dim(F))$. By [20, Theorem 2], there exists an abelian subgroup $A_{Z'}$ of rank at most $\dim(Z)$ and index at most $k(\dim(Z))$. We conclude that $A_0$ admits a subgroup $A$ of rank at most $\dim(F) + \dim(Z) = n$ and index at most $k(\dim(F)) + k(\dim(Z))$. Hence, $\pi_1^{\text{reg}}(X/Z, \Delta; z)$ admits a normal abelian subgroup $A$ of rank at most $n$ and index at most

$$c(n) := c_0(n) + k(\dim(F)) + k(\dim(Z)).$$

This completes the proof. \qed
The following corollary should be compared to [38, Satz (6.1)], which together with the universal coefficient theorem [46, Corollary 3.3] — stating, in particular, that the torsion of second cohomology equals the torsion of first homology — is used in the proof of [20, Corollary 3] to show a similar statement in the restricted setting of Zariski and holomorphic local ring of klt singularities. However, the following proof does not depend on these statements (at least not directly).

**Corollary 4.16.** Let \((X, \Delta)\) be of relative Fano type over \(Z\). Let \(A := \mathbb{T}_X / \mathbb{T}_X^0 \cong \text{Cl}(X/Z, \Delta; z)_{\text{tor}}\) be the group of components of the characteristic quasi-torus of \((X, \Delta)\) over \(Z\) at \(z\). Then, \(A\) is the abelianization of \(\pi_{1,\text{reg}}^\text{reg}(X/Z, \Delta; z)\), that is,

\[
A \cong \pi_{1,\text{reg}}^\text{reg}(X/Z, \Delta; z) / [\pi_{1,\text{reg}}^\text{reg}(X/Z, \Delta; z), \pi_{1,\text{reg}}^\text{reg}(X/Z, \Delta; z)],
\]

where \([\pi_{1,\text{reg}}^\text{reg}(X/Z, \Delta), \pi_{1,\text{reg}}^\text{reg}(X/Z, \Delta)]\) is the commutator subgroup of \(\pi_{1,\text{reg}}^\text{reg}(X/Z, \Delta)\).

**Proof.** By Theorem 4.15, we know that \(\pi_{1,\text{reg}}^\text{reg}(X/Z, \Delta; z)\) is finite and isomorphic to the étale fundamental group of \(X^h_{\text{reg}}\). Here, \(X^h \to Z^h\) is the base change to the Henselization of \(Z\) at \(z\). Thus, we can assume that \(Z\) is local Henselian. Then, \(G := \pi_{1,\text{reg}}^\text{reg}(X/Z, \Delta; z)\) induces a finite Galois log quasi-étale cover of \((X, \Delta)\), which by abuse of notation, we denote by

\[
(X, \Delta) \xrightarrow{G} (X, \Delta).
\]

Then, \([G, G]\) is a normal subgroup of \(G\) and induces a finite abelian Galois log quasi-étale cover

\[
(Y, \Delta_Y) \xrightarrow{[G, G]} (X, \Delta),
\]

which is a quasi-torsor in the sense of Definition 4.10. Indeed, \(G/[G, G]\) is finite and \((Y, \Delta_Y)\) is of Fano type over \(Z\) by the proof of Theorem 3.26. In particular, invertible functions of \(Y\) and \(X\) are invertible functions of \(Z\). Thus, by Proposition 4.11, \(G/[G, G]\) is a subgroup of \(\text{Cl}(X/Z, \Delta)\). In particular, it is a subgroup of \(A\).

But since \(A\) induces a quasi-étale abelian Galois cover of \((X, \Delta)\) as well, we have \(A = G/[G, G]\). Otherwise, there would be a normal subgroup of \(G\) smaller than \([G, G]\) with abelian quotient, which is a contradiction. This finishes the proof of the corollary.

\(\square\)

### 4.3 | Boundedness of the iteration of Cox rings

In this subsection, we prove the main theorem of this article, the boundedness of the iteration of Cox rings for Fano-type varieties.

**Theorem 4.17.** There exists a constant \(k(n)\), only depending on \(n\), satisfying the following. Let \(\phi : X \to Z\) be a projective contraction so that \(X\) has dimension \(n\). Let \((X, \Delta)\) be a log pair of Fano type over \(Z\). Then, \(\text{Cox}^{(k)}(X/Z, \Delta)\) stabilizes for \(k \geq k(n)\).

---

---
Proof. First, we prove that the iteration of Cox rings

\[ \text{Cox}^{(k)}(X/Z, \Delta) \]

stabilizes for \( k \) large enough. It suffices to show that \( \text{Cl}(\bar{X}_k/Z, \bar{\Delta}_k) \) is torsion-free for some \( k \in \mathbb{N} \), since then \( (\bar{X}_{k+1}, \bar{\Delta}_{k+1}) \) is factorial over \( Z \), see, for example, [8, 1.4.1.5]. By Remark 4.7, torsion-freeness of \( \text{Cl}(\bar{X}_k/Z, \bar{\Delta}_k) \) is equivalent to torsion-freeness of the divisor class group of the finite abelian covering space \( (X_k/Z, \Delta_k) \).

Now, we assume that

\[ \text{Cl}(X_k/Z, \Delta_k)_{\text{tor}} \neq 1 \]

for any \( k \in \mathbb{N} \). Then, by Corollary 4.12 (3), there is an infinite chain of log quasi-étale finite solvable Galois covers

\[ (X_k, \Delta_k) \xrightarrow{S_k} (X, \Delta), \]

where \( |S_{k+1}| > |S_k| \). This is a contradiction to the finiteness of \( \pi_1^{\text{reg}}(X/Z, \Delta) \). Thus, \( \text{Cox}^{(k)}(X/Z, \Delta) \) stabilizes for \( k \) large enough.

Now, we show that such bound at which the iteration of Cox rings stabilizes admits an upper bound that only depends on the dimension of \( X \). As before, we denote \( D_0 := \pi_1^{\text{reg}}(X/Z, \Delta) \), and inductively, we define \( D_i := [D_{i-1}, D_{i-1}] \). By Theorem 4.15, we know that there is an exact sequence \( 1 \to A_0 \to D_0 \to N_0 \to 1 \), where \( A_0 \) is an abelian normal subgroup of rank at most \( n \) and \( N_0 \) has order at most \( c(n) \). We denote \( A_i := D_i \cap A_0 \). Now, we have a commutative diagram

\[
\begin{array}{ccccccccc}
1 & & 1 & & 1 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
1 & \to & A_{i+1} & \to & A_i & \to & B_i & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & \\
1 & \to & D_{i+1} & \to & D_i & \to & \text{Cl}(X_i/Z, \Delta_i)_{\text{tor}} & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & \\
1 & \to & N_{i+1} & \to & N_i & \to & M_i & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & \\
1 & & 1 & & 1 & & \\
\end{array}
\]

with exact rows and columns for each \( i \geq 0 \). In particular, we get a chain of normal subgroups

\[ \cdots \leq N_2 \leq N_1 \leq N_0 \]

with \( M_i := N_{i+1}/N_i \). If in such a chain, no two consecutive \( M_i \) are trivial, then the length \( k \) of the chain is bounded by \( 2 \log_2(c(n)) \). Hence, we know that there is some
\( j \leq 2 \log_2(c(n)) + 1 \) with \( M_j \cong M_{j+1} \cong 1 \). We have a commutative diagram

\[
\begin{array}{ccccccc}
1 & 1 & 1 & & & & \\
& & & 1 & A_{j+2} & A_j & C & 1 \\
1 & & & & D_{j+2} & D_j & S & 1 \\
& & & & & N_{j+2} & N_j & L & 1 \\
& & & & & & 1 & 1 & 1 \\
\end{array}
\]

with exact rows and columns similar to the one from above. But here \( L \) is trivial, since \( N_{j+2} \cong N_{j+1} \cong N_j \). The group \( C \) is abelian, since \( A_j \) and \( A_{j+2} \) are abelian. So, \( S \cong C \) is abelian. Thus, \( D_{j+2} \) equals \( D_{j+1} \), the derived subgroup of \( D_j \). But then \( \text{Cl}(X_{j+1}/Z, \Delta X_{j+1})_{\text{tor}} \cong D_{j+1}/D_{j+2} \) is trivial and the iteration of Cox rings stabilizes for \( k \geq j + 2 \). Since \( j \leq 2 \log_2(c(n)) + 1 \), we can set \( k(n) := 2 \log_2(c(n)) + 3 \). Since \( c(n) \) only depends on \( n \), the proof is finished. 

\[ \square \]

## 5 Simply Connected Factorial Canonical Cover

In this section, we aim to prove the existence of a simply connected factorial canonical cover for klt singularities. In Subsection 5.1, we prove the existence of the scfc cover. In Subsection 5.2, we prove that the scfc cover dominates any sequence of finite covers and abelian covers. In Subsection 5.3, we give an upper bound for the dimension of the iteration of Cox rings of the singularity.

### 5.1 Existence of the simply connected factorial canonical cover

In this subsection, we prove the existence of a simply connected factorial canonical cover for a klt singularity. The following proposition is the cornerstone for the construction.

**Proposition 5.1.** Let \((X, \Delta)\) be a log pair. Let \( \phi : X \to Z \) be an aff-contraction where \( Z \) is local Henselian. Assume that \((X, \Delta)\) is of relative Fano type over \( Z \). Let \((X', \Delta') \to (X, \Delta)\) be a torus quasitorsor in the sense of Definition 4.10. Let \( \mathbb{T} \) be the acting torus on \( X' \). Let \((\tilde{X}, \tilde{\Delta}) \to (X, \Delta)\) be the finite log quasi-étale Galois cover associated to \( \pi := \pi_{\text{reg}}^1(X/Z, \Delta) \).
Define $\tilde{X}' := \tilde{X} \times_X X'$. Then, there is a commutative diagram

$$
\begin{array}{ccc}
(\tilde{X}', \tilde{\Delta}') & \xrightarrow{\pi} & (X', \Delta') \\
\downarrow /\Gamma & & \downarrow /\Gamma \\
(\tilde{X}, \tilde{\Delta}) & \xrightarrow{\pi} & (X, \Delta)
\end{array}
$$

where $\pi_1^{\text{reg}}(X'/Z, \Delta') \cong \pi_1^{\text{reg}}(X/Z, \Delta)$ and $X'$ admits a $\mathbb{T} \times \pi_1^{\text{reg}}(X'/Z, \Delta')$-action satisfying the following conditions.

1. $(\tilde{X}' \to X')$ is the finite Galois log quasi-étale cover associated to $\pi_1^{\text{reg}}(X'/Z, \Delta')$ and $\tilde{\Delta}'$ is the log-pullback of $\Delta'$.
2. $(\tilde{X}', \tilde{\Delta}') \to (\tilde{X}, \tilde{\Delta})$ is a $\mathbb{T}$-quasi-torsor.
3. The $\mathbb{T} \times \pi_1^{\text{reg}}(X'/Z, \Delta')$-action is log-free in codimension 1.
4. $\mathbb{T}$ is the same torus as in the quasi-torsor $X' \to X$.

Proof. First, we note that both $\tilde{X} \to X$ is log quasi-étale and $X' \to X$ is étale locally trivial over the smooth locus. All homotopy groups are considered relatively over $Z$. Thus, we have an exact sequence of homotopy groups

$$
\pi_2^{\text{reg}}(\mathbb{T}) \to \pi_2^{\text{reg}}(X' \setminus \Delta') \to \pi_2^{\text{reg}}(X \setminus \Delta) \to \pi_1^{\text{reg}}(\mathbb{T}) \to \pi_1^{\text{reg}}(X' \setminus \Delta') \to \pi_1^{\text{reg}}(X \setminus \Delta) \to \pi_0^{\text{reg}}(\mathbb{T}).
$$

Note that $\pi_2^{\text{reg}}(\mathbb{T}) \cong \pi_0^{\text{reg}}(\mathbb{T}) \cong 1$ and $\pi_1^{\text{reg}}(\mathbb{T}) \cong \mathbb{Z}^{\dim(\mathbb{T})}$. Hence, we have the following exact sequence:

$$
1 \to \pi_2^{\text{reg}}(X' \setminus \Delta') \to \pi_2^{\text{reg}}(X \setminus \Delta) \to \mathbb{Z}^{\dim(\mathbb{T})} \to \pi_1^{\text{reg}}(X' \setminus \Delta') \to \pi_1^{\text{reg}}(X \setminus \Delta) \to 1.
$$

Above, by abuse of notation, we denote by $X \setminus \Delta$ the complement of the support of $\Delta$. Now, we recall that the orbifold fundamental group $\pi_1^{\text{reg}}(X, \Delta)$ is by definition $\pi_1^{\text{reg}}(X \setminus \Delta)/\langle \gamma_i^{m_i} \rangle$, where $\gamma_i$ is a small loop around a general component of $\Delta_i$ and $m_i$ is the coefficient of $\Delta_i$ in the standard approximation of $\Delta$, cf. Definition 2.31. But since $X' \to X$ is étale locally trivial over the smooth locus, a small loop around a general point of $\Delta_i$ is also a small loop around a general point of the pullback $\Delta'_i$. Of course, the coefficient $m_i$ in the standard approximation of $\Delta'$ is $m_i/m_i$ again.

By the above exact sequence, setting $A := \text{im}(\mathbb{Z}^{\dim(\mathbb{T})} \to \pi_1^{\text{reg}}(X' \setminus \Delta'))$, we have $\pi_1^{\text{reg}}(X \setminus \Delta) = \pi_1^{\text{reg}}(X' \setminus \Delta')/A$. So, with the above considerations, we get

$$
\pi_1^{\text{reg}}(X, \Delta) := \pi_1^{\text{reg}}(X \setminus \Delta)/\langle \gamma_i^{m_i} \rangle = (\pi_1^{\text{reg}}(X' \setminus \Delta')/A)/\langle \gamma_i^{m_i} \rangle = (\pi_1^{\text{reg}}(X' \setminus \Delta')/\langle \gamma_i^{m_i} \rangle)/A = \pi_1^{\text{reg}}(X', \Delta')/A.
$$

Since $\pi_1^{\text{reg}}(X', \Delta')$ is finite by Theorem 4.15, in fact, $A$ is a finite abelian group. Recall that arbitrary base change preserves étaleness, finiteness and GIT-quotients. Hence, the cover $\tilde{X}' \to X'$ is indeed a finite Galois log quasi-étale cover with Galois group $\pi_1^{\text{reg}}(X, \Delta)$. So, $X' \to X$ is the Galois cover.
associated to the normal subgroup $A$. In particular, there is an $A$-quasi-torsor

$$Y \overset{A}{\longrightarrow} \tilde{X}'.$$

With the same arguments as above, $\tilde{X}' \rightarrow \tilde{X}$ is a $\mathbb{T}$-quasi-torsor.

Since $\mathbb{T}$ is connected, by [8, Theorem 4.2.3.2], we can lift the $\mathbb{T}$-action on $\tilde{X}'$ to a $\mathbb{T} \times A$ action on $Y$, such that $Y \rightarrow \tilde{X}$ becomes a $\mathbb{T} \times A$-quasi-torsor.

Furthermore, $\text{Cl}(\tilde{X}/Z, \tilde{\Delta})$ is torsion free. Otherwise, by Lemma 4.2, the torsion part would induce a log quasi-étale Galois cover of $(\tilde{X}, \tilde{\Delta})$, which contradicts the log-simply-connectedness of the smooth locus. By Proposition 4.11, this means that there is a monomorphism from $A$ to a torsion free group, which implies that $A$ is trivial. □

Now, we turn to prove the existence of the scfc cover.

**Theorem 5.2.** Let $(X, \Delta)$ be a log pair. Let $\phi : X \rightarrow Z$ be an aff-contraction so that $Z$ is local Henselian. Assume that $(X, \Delta)$ is relatively Fano over $Z$. Let $(\tilde{X}, \tilde{\Delta}) \rightarrow (X, \Delta)$ be the finite log quasi-étale Galois cover associated to $\pi := \pi^{\text{reg}}_1(X/Z, \Delta)$. Denote by $(\tilde{X}, \tilde{\Delta})$ the total coordinate space of $(\tilde{X}, \tilde{\Delta})$ over $Z$. Then, we have a commutative diagram

$$
\begin{array}{ccc}
(\tilde{X}, \tilde{\Delta}) & \overset{\mathbb{T}}{\longrightarrow} & (\tilde{X}, \tilde{\Delta}) \\
 & \downarrow{\pi} & \\
(\tilde{X}, \tilde{\Delta}) & \overset{\sigma}{\longrightarrow} & (X, \Delta),
\end{array}
$$

where the following conditions hold.

1. The characteristic quasi-torus $\mathbb{T}$ is connected, that is, a torus.
2. $G$ is a reductive group acting freely in log-codimension 1 on $(\tilde{X}, \tilde{\Delta})$ and fitting in the short exact sequence

$$
1 \longrightarrow \mathbb{T} \longrightarrow G \longrightarrow \pi \longrightarrow 1.
$$

3. $(\tilde{X}, \tilde{\Delta})$ is factorial over $Z$ and has canonical singularities.
4. $(\tilde{X}, \tilde{\Delta})$ is log-simply connected in codimension 1.

**Proof.** By Proposition 5.1, we know that $\text{Cl}(\tilde{X}/Z, \tilde{\Delta})$ is torsion free. Thus, $\mathbb{T}$ is a torus, and $(\tilde{X}, \tilde{\Delta})$ is factorial over $Z$. In particular, it is locally factorial, and since it is klt type by Theorem 3.30, it has canonical singularities, yielding items (1) and (3). In fact, by Corollary 4.3, we already know that the Cox ring of a relative Fano pair is Gorenstein and has canonical singularities, whereas, in general, it is, of course, not factorial. Proposition 5.1 shows that

$$
\pi^{\text{reg}}_1(\tilde{X}/Z, \tilde{\Delta}) \cong \pi^{\text{reg}}_1(X/Z, \Delta) \cong 1,
$$
yielding (4). Lastly, (2) follows from Item (3) of Proposition 4.9 by setting $G$ to be the preimage of $\pi \subseteq \text{Aut}_Z(\breve{X})$ under the map $\text{Aut}(\breve{X}) \to \text{Aut}_Z(\breve{X})$.

\[ \square \]

Remark 5.3. The construction of the scfc cover, as usual, depends on the choice of a subgroup $N \subseteq \text{Cl}(\breve{X}/Z, \breve{\Delta})$ and character $\chi$.

5.2 Universality of the simply connected factorial canonical cover

In this subsection, we prove a universality property for the scfc cover. This means that the scfc cover dominates any sequence of finite covers and abelian quasi-torsors over the singularity.

Theorem 5.4. Let $(X, \Delta)$ be a log pair. Let $\phi : X \to Z$ be an affine-contraction, where $X$ is of dimension $n$ and $Z$ is local Henselian. Assume that $(X, \Delta)$ is of relative Fano type over $Z$. Let

\[ \cdots \longrightarrow (X_{(2)}, \Delta_{(2)}) \longrightarrow (X_{(1)}, \Delta_{(1)}) \longrightarrow (X_{(0)}, \Delta_{(0)}) := (X, \Delta) \]

be a (possibly infinite) sequence of finite Galois log quasi-étale covers and abelian quasi-torsors. Then, $(X_{(j)}, \Delta_{(j)})$ stabilizes after finitely many steps and the scfc covers of $(X_{(j)}, \Delta_{(j)})$ coincide for all $j \geq 0$. If in addition all abelian (finite or quasi-torsor-) covers in the sequence are given by the respective Cox rings, then there is a constant $j(n)$ only depending on $n$, such that $(X_{(j)}, \Delta_{(j)})$ stabilizes for $j \geq j(n)$.

Remark 5.5. As mentioned before, the construction of the scfc cover depends on the choice of a subgroup $N \subseteq \text{Cl}(\breve{X}/Z, \breve{\Delta})$ and a character $\chi$. So, the equality of the scfc covers of $(X_{(j)}, \Delta_{(j)})$ means that for any $j_1 \neq j_2 \geq 0$, any scfc cover of $(X_{(j_1)}, \Delta_{(j_1)})$ is also a scfc cover of $(X_{(j_2)}, \Delta_{(j_2)})$ and vice versa.

Proof of Theorem 5.4. We show by induction that all the scfc covers coincide. So, let first $(X_{(j+1)}, \Delta_{(j+1)}) \to (X_{(j)}, \Delta_{(j)})$ be finite. Then the covers $(X_{(j+1)}, \breve{\Delta}_{(j+1)})$ and $(X_{(j)}, \breve{\Delta}_{(j)})$ associated to the respective regional fundamental groups obviously coincide, so the scfc covers coincide as well.

Now let $(X_{(j+1)}, \breve{\Delta}_{(j+1)}) \to (X_{(j)}, \breve{\Delta}_{(j)})$ be an $H$-quotient presentation. Then taking the quotient by the identity component $H^0$ yields a commutative diagram

\[ \begin{array}{ccc}
(X_{(j+1)}, \Delta_{(j+1)}) & \xrightarrow{H} & (X_{(j)}, \Delta_{(j)}) \\
\downarrow H^0 & & \downarrow (H/H^0) \\
(X'_{(j+1)}, \Delta'_{(j+1)}) & \xrightarrow{(H/H^0)} & (X_{(j)}, \Delta_{(j)})
\end{array} \]

where $(X'_{(j+1)}, \breve{\Delta}'_{(j+1)}) \to (X_{(j)}, \breve{\Delta}_{(j)})$ is finite log quasi-étale abelian. Thus, the scfc covers of $(X'_{(j+1)}, \breve{\Delta}'_{(j+1)})$ and $(X_{(j)}, \breve{\Delta}_{(j)})$ coincide. By Proposition 5.1, we can extend the diagram in the
following way:

\[
\begin{array}{ccc}
(X_{(j+1)}, \Delta_{(j+1)}) & \overset{\pi}{\longrightarrow} & (X_{(j)}, \Delta_{(j)}) \\
\downarrow / H^0 & & \downarrow / H^0 \\
(\tilde{X}_{(j)}, \tilde{\Delta}_{(j)}) & \overset{\pi}{\longrightarrow} & (X'_{(j+1)}, \Delta'_{(j+1)}) \\
\downarrow / H^0 & & \downarrow / (H/H^0) \\
(\tilde{X}_{(j)}, \tilde{\Delta}_{(j)}) & \overset{\pi}{\longrightarrow} & (X'_{(j)}, \Delta'_{(j)}) \\
\end{array}
\]

Observe that \((\tilde{X}_{(j+1)}, \tilde{\Delta}_{(j+1)}) \rightarrow (\tilde{X}_{(j)}, \tilde{\Delta}_{(j)})\) is a quasi-torsor. We conclude that the scfc cover of \((X_{(j+1)}, \Delta_{(j+1)})\) and the scfc cover of \((X_{(j)}, \Delta_{(j)})\) coincide.

In order to show that the sequence stabilizes, we show the following claim by induction.

**Claim:** The first \(k\) covers in the sequence induce a sequence of \(k\) finite covers

\[
(X'_{(j+1)}, \Delta'_{(j+1)}) \rightarrow (X'_{(j)}, \Delta'_{(j)}),
\]

where

\[
(X_{(j)}, \Delta_{(j)}) \rightarrow (X'_{(j)}, \Delta'_{(j)})
\]

is a torus-quasi-torsor for every \(j\).

**Proof of the Claim.** We aim to construct a diagram of the form

\[
\begin{array}{c}
\cdots \quad (X_{(3)}, \Delta_{(3)}) \quad \cdots \\
\downarrow \\
\cdots \quad (X'_{(3)}, \Delta'_{(3)}) \quad (X_{(2)}, \Delta_{(2)}) \quad \cdots \\
\downarrow \\
\cdots \quad (X'_{(3)}, \Delta'_{(3)}) \quad (X_{(2)}, \Delta_{(2)}) \quad (X_{(1)}, \Delta_{(1)}) \quad \cdots \\
\downarrow \\
\cdots \quad (X'_{(3)}, \Delta'_{(3)}) \quad (X'_{(2)}, \Delta'_{(2)}) \quad (X'_{(1)}, \Delta'_{(1)}) \quad (X_{(0)}, \Delta_{(0)}) \quad \cdots
\end{array}
\]

where

1. vertical arrows from the \(k\)th to the \((k-1)\)th row counting from below are quotients by the identity component \(H^0_k\) of the general fiber of \((X_{(k)}, \Delta_{(k)}) \rightarrow (X_{(k-1)}, \Delta_{(k-1)})\), which is trivial in case the cover is finite,
2. horizontal arrows from the \(k\)th to the \((k-1)\)th column counting from the right are quotients by the group of connected components of the general fiber of \((X_{(k)}, \Delta_{(k)}) \rightarrow (X_{(k-1)}, \Delta_{(k-1)})\).

For \(k = 1\), either the cover is finite or it is an \(H\)-quasi-torsor. In the latter case, we take the finite cover given by the group of components \(H/H^0\). Assume that we have constructed the above diagram up to index \(k\). Then, as we have seen before, \((X_{(k+1)}, \Delta_{(k+1)}) \rightarrow (X_{(k)}, \Delta_{(k)})\) yields a finite Galois log quasi-étale cover \((X_{(k+1)}, \Delta_{(k+1)}) \rightarrow (X_{(k)}, \Delta_{(k)})\). Due to Proposition 5.1 applied
multiple times going down in the diagram from \((X_{(k)}, \Delta_{(k)})\) to \((X'_{(k)}, \Delta'_{(k)})\), we know that the regional fundamental groups of \((X_{(k)}), \Delta_{(k)}\), all \((X^{(j)}_{(k)}, \Delta^{(j)}_{(k)})\) for \(j \leq k\) and, in particular, the \(k\)th finite cover \((X'_{(k)}, \Delta'_{(k)})\) coincide. So, \((X^{(k+1)}_{(k+1)}, \Delta^{(k+1)}_{(k+1)}) \rightarrow (X_{(k)}, \Delta_{(k)})\) induces finite Galois log quasi-étale covers \((X^{(j)}_{(k+1)}, \Delta^{(j)}_{(k+1)}) \rightarrow (X^{(j)}_{(k)}, \Delta^{(j)}_{(k)})\) for \(j \leq k\). Here,

\[
\left( X^{(j)}_{(k+1)}, \Delta^{(j)}_{(k+1)} \right) \rightarrow \left( X^{(j-1)}_{(k+1)}, \Delta^{(j-1)}_{(k+1)} \right)
\]

are \(H^0\)-quasi-torsors by Proposition 5.1. Thus, we showed the existence of the diagram up to index \(k\) and in particular — in the bottom row of the diagram — the sequence of \(k\) finite covers as claimed. This finishes the proof of the claim. □

By the claim, we have already that the scfc cover \((\tilde{X}, \tilde{\Delta})\) of \((X, \Delta)\) dominates the original sequence. It follows that at most

\[
x := \dim(\tilde{X}) - \dim(X)
\]

domination of the original covers can induce a trivial finite cover. Thus, by finiteness of \(\pi^\text{reg}_1(X/Z, \Delta)\), the original sequence stabilizes for \(j\) large enough. In the case that all quasi-torsors in the original sequence are Cox covers, the bound \(j(n)\) on the number of nontrivial covers follows as in the proof of Theorem 4.17. □

Remark 5.6. If we do not assume the quasi-torsors to be Cox covers, there is no bound depending only on the dimension. This already happens in dimension two. We can construct sequences of arbitrary length of nontrivial abelian quasi-étale covers over two-dimensional \(A_n\)-singularities, if we do not fix \(n\).

We conclude this subsection by proving Corollary 1. For simplicity, we write the proof for klt singularities. The statement is still valid for Fano 1-type varieties.

Proof of Corollary 1. Let \((X, \Delta; x)\) be a klt singularity. By Theorem 5.2 the scfc cover is isomorphic to the Cox space of the universal cover of \((X, \Delta; x)\). Now, let \((\tilde{X}, \tilde{\Delta}, \tilde{x})\) be the spectrum of the iteration of Cox rings of \((X, \Delta; x)\), that is, the iteration of Cox spaces of \((X, \Delta; x)\). Let \((\tilde{X}, \tilde{\Delta}, \tilde{x})\) be the universal cover of the spectrum of the iteration of Cox rings of \((X, \Delta; x)\). Let \((Y, \Delta_Y; y)\) be the Cox space of \((\tilde{X}, \tilde{\Delta}, \tilde{x})\). By Theorem 5, we conclude that \((Y, \Delta_Y; y)\) is isomorphic to the scfc cover. □

5.3 Upper bound for the dimension of the iteration of Cox rings

In this subsection, we give an upper bound for the dimension of the iteration of Cox rings in terms of homotopy groups. For orbifolds (and more general, orbispaces), similarly to the fundamental group, one may define higher homotopy groups \(\pi_k\), and for orbispace fibrations, these groups satisfy the same long exact sequence as ordinary homotopy groups, cf. [27, Theorem 4.5]. The precise statement is the following.
Theorem 5.7. Let $(X, \Delta)$ be a log pair. Let $\phi : X \to Z$ be an aff-contraction so that $Z$ is local Henselian. Assume that $(X, \Delta)$ is relatively Fano over $Z$. Let $(\widetilde{X}, \widetilde{\Delta})$ be the scfc cover of $(X, \Delta)$. Then
$$\dim(\widetilde{X}) \leq \dim(X) + \text{rk}(\pi_2^{\text{reg}}(X/Z, \Delta) \otimes \mathbb{Q}).$$

If equality holds in the above formula, then $\pi_2^{\text{reg}}(\widetilde{X}, \widetilde{\Delta})$ is finite.

Proof. As in the proof of Proposition 5.1, we use the fact that $(\widetilde{X}, \widetilde{\Delta}) \to^G (X, \Delta)$ is an étale locally trivial $G$-bundle over $(X_{\text{reg}}, \Delta_{\text{reg}})$. The complement of the preimage of $(X_{\text{reg}}, \Delta_{\text{reg}})$ is (in general) a subset of $(\widetilde{X}_{\text{reg}}, \widetilde{\Delta}_{\text{reg}})$ of complex codimension 2, that is, real codimension 4. In particular, not only the fundamental group but also the second homotopy group of these two smooth spaces agree. Thus, we have an exact sequence of orbifold homotopy groups
$$\pi_2^{\text{reg}}(G) \to \pi_2^{\text{reg}}(\widetilde{X}/Z, \widetilde{\Delta}) \to \pi_2^{\text{reg}}(X/Z, \Delta) \to \pi_1^{\text{reg}}(G) \to \pi_1^{\text{reg}}(\widetilde{X}/Z, \widetilde{\Delta}) \to \pi_1^{\text{reg}}(X/Z, \Delta) \to \pi_0^{\text{reg}}(G).$$

Note that the orbifold structure on the fiber $G$ is trivial and, moreover, $\pi_2^{\text{reg}}(G) \cong 1$, $\pi_1^{\text{reg}}(G) \cong \mathbb{Z}^{\dim(\widetilde{X}) - \dim(X)}$, and $\pi_0^{\text{reg}}(G) \cong \pi_1^{\text{reg}}(X/Z, \Delta)$. Since $\mathbb{Q}$ is a flat $\mathbb{Z}$-module, tensoring the above exact sequence with $\mathbb{Q}$ yields an exact sequence of $\mathbb{Q}$-vector spaces
$$\pi_2^{\text{reg}}(X/Z, \Delta) \otimes \mathbb{Q} \to \mathbb{Q}^{\dim(\widetilde{X}) - \dim(X)} \to 0.$$

The last statement follows immediately.

Remark 5.8. If we consider the scfc cover with respect only to $X$, without the orbifold structure, then the second homotopy group in Theorem 5.7 is the ordinary second homotopy group $\pi_2^{\text{reg}}(X/Z)$ of the smooth locus.

6 FANO-TYPE VARIETIES WITH SMOOTH ITERATION OF COX RINGS

Throughout this paper, we introduced some special covers of Fano-type varieties and klt singularities. The aim of this section is to explain when such coverings are smooth. In Subsection 6.1, we give a characterization of Fano-type varieties with smooth iteration of Cox rings. In Subsection 6.2, we give a characterization of Fano-type varieties with smooth simply connected factorial canonical cover. Analogous theorems hold for klt singularities. In such case, we will just enunciate the results without a proof because these are verbatim from the projective case. Finally, in Subsection 6.3, we will give a characterization of Fano type varieties for which the spectrum of the iteration coincides with the scfc cover.

6.1 Smoothness of the iteration of Cox rings

In this subsection, we give a characterization of the smoothness of the iteration of Cox rings.
Theorem 6.1. Let \((X, \Delta)\) be a Fano-type pair. Then, the following statements are equivalent.

(1) The spectrum of the iteration of Cox rings \(\text{Cox}^n(X, \Delta)\) is smooth.
(2) \((X, \Delta)\) is a finite quasi-étale solvable quotient of a projective toric pair.

Furthermore, if any of the above conditions holds, then we have that

(1') the simply connected factorial canonical cover coincides with the spectrum of the iteration of Cox rings.

Proof. Assume that \((X, \Delta)\) is a finite quasi-étale solvable quotient of a projective toric pair \((T, \Delta_T)\). Then, we can find:

- a sequence of finite abelian groups \(A_1, \ldots, A_n\),
- projective Fano-type pairs \((X_i, \Delta_i)\) with \((X_n, \Delta_n) = (T, \Delta_T)\), and
- \(A_i\) acts on \((X_i, \Delta_i)\) so that \((X_{i-1}, \Delta_{i-1})\) is the quotient by this action.

Moreover, we may assume \((X_0, \Delta_0) = (X, \Delta)\). Let \(\text{Cox}^{(k)}(X, \Delta)\) be the \(k\)th iteration of the Cox ring of \((X, \Delta)\). We denote by \(T_k\) the connected component of the reductive solvable group acting on the \(k\)-th iteration \(\text{Cox}^{(k)}(X, \Delta)\). Since the Cox ring dominates all quasi-étale finite abelian covers, we have a diagram as follows:

\[
\begin{array}{c}
(T, \Delta_T) \\ \downarrow \text{inclusion} \\
(X_{n-1}, \Delta_{n-1}) & \text{left map} & \text{right map} & \text{right map} & \text{right map} \\
\downarrow \text{inclusion} & \downarrow \text{inclusion} & \downarrow \text{inclusion} & \downarrow \text{inclusion} \\
(X_1, \Delta_1) & \text{left map} & \text{right map} & \text{right map} & \text{right map} \\
\downarrow \text{inclusion} & \downarrow \text{inclusion} & \downarrow \text{inclusion} & \downarrow \text{inclusion} \\
(X, \Delta).
\end{array}
\]

In the above diagram, each map \((Y_1, B_1) \rightarrow (X_1, \Delta_1)\) is a finite quotient. Since \(T\) is toric and \(Y_n \rightarrow T\) only branches along the support of \(\Delta_T\), we conclude that \(Y_n \rightarrow T\) does not branch along the torus. Hence, \((Y_n, B_n)\) is a projective toric pair. Thus, we have that \(\text{Cox}(Y_n, B_n) \cong A_{\rho(Y_n) + \dim(Y_n)}\).

Since the morphism \(\text{Cox}^{(n)}(X, \Delta) \rightarrow Y_n\) is a quasi-étale abelian cover, we conclude that there is a commutative diagram:

\[
\begin{array}{c}
\text{Cox}^{(n)}(X, \Delta) \\ \downarrow \text{inclusion} \\
\text{Cox}^{(n)}(X, \Delta) \rightarrow (Y_n, \Delta_n).
\end{array}
\]
Here, $Q$ is a quasi-torus acting on the affine space. Thus, $\mathbb{A}^{\rho(Y_n) + \dim(Y_n)} \to \text{Cox}^{(n)}(X, \Delta)$ is a quasi-étale abelian cover. We conclude that

$$\text{Cox}^{(n+1)}(X, \Delta) \cong \mathbb{A}^{\rho(Y_n) + \dim(Y_n)}.$$ 

This concludes that (2) implies (1).

Now, we prove that (1) implies (2) and we argue that (1’) holds in this case. We assume that the spectrum of the iteration of Cox rings is smooth. By Corollary 1, we know that the simply connected factorial canonical cover is the Cox ring of the universal cover of the spectrum of the iteration of Cox rings. This implies that (1’) holds in this case. Let $G$ be the reductive solvable group acting on the spectrum of the iteration of Cox rings. Let $G_0 \leq G$ be the connected component at the identity. Then, $G_0$ is a connected reductive solvable group, so it is a torus. By construction, $G_0$ acts on spectrum of the iteration of Cox rings with a smooth, fixed, attractive point. By [55, Theorem 2.5], we conclude that the spectrum of the iteration of Cox rings is isomorphic to $\mathbb{A}^n$ and $G_0$ acts linearly on it. Hence, the quotient $(T, \Delta_T) := \mathbb{A}^n / G_0$ is a projective toric variety that admits a finite solvable Galois log quasi-étale morphism to $(X, \Delta)$. This finishes the proof. □

We have the following corresponding statement for klt singularities.

**Theorem 6.2.** Let $(X, \Delta; x)$ be a klt singularity. Then, the following statements are equivalent.

1. The spectrum of the iteration of local Cox rings $\text{Cox}^{\text{lt}}(X, \Delta; x)$ is smooth.
2. $(X, \Delta; x)$ is a finite quasi-étale solvable quotient of a toric singularity.

Furthermore, if any of the above conditions hold, then we have that

1’ the simply connected factorial canonical cover coincides with the spectrum of the iteration of Cox rings.

### 6.2 Smoothness of the scfc cover

In this subsection, we give a characterization of Fano-type varieties with smooth scfc cover.

**Theorem 6.3.** Let $(X, \Delta)$ be a Fano-type pair. Then, the following statements are equivalent.

1. The simply connected factorial canonical cover of $(X, \Delta)$ is smooth.
2. $(X, \Delta)$ is a finite quasi-étale quotient of a projective toric pair.

**Proof.** Assume that $(X, \Delta)$ is a finite quasi-étale quotient of a projective toric pair $(T, \Delta_T)$. Let $F$ be the finite group acting on $T$. Let $S$ be the perfect core of $F$. Then, the quotient $H := F / S$ is solvable. Let $(T', \Delta_{T'})$ be the quotient of $(T, \Delta_T)$ by the perfect core $S$. Then, we have a commutative diagram:

$$
\begin{array}{ccc}
(T, \Delta_T) & \leftarrow & (T, \Delta_T) \\
\downarrow /F & & \downarrow /S \\
(X, \Delta) & \leftarrow & (T', \Delta_{T'}). \\
\end{array}
$$
Here, the top vertical arrow is the identity. Note that \((X, \Delta)\) and \((T', \Delta_{T'})\) differ by a solvable quotient. Then, they have the same iteration of Cox ring. We may assume that the spectrum of the iteration of Cox ring \(\text{Cox}^{\text{it}}(X, \Delta)\) dominates \((T', \Delta_{T'})\) with a solvable quotient by \(H_0\). We may take the normalization of the main component of the fiber product of \(T \to T' \leftarrow \text{Spec}(\text{Cox}^{\text{it}}(X, \Delta))\) to obtain a diagram as follows:

\[
\begin{array}{c}
(T, \Delta_T) \xleftarrow{(T, \Delta_Y)} (Y, \Delta_Y) \\
\downarrow \quad \downarrow /S_i \\
(X, \Delta) \xleftarrow{(T', \Delta_{T'})} \text{Spec}(\text{Cox}^{\text{it}}(X, \Delta)).
\end{array}
\]

Here, \(S_1\) is finite and \(H_1\) is solvable. Note that \((Y, \Delta_Y) \to (T, \Delta_T)\) is a solvable cover of a toric pair. Hence, by Proposition 4.11, \((Y, \Delta_Y)\) is itself toric. We argue that the universal cover of \(\text{Spec}(\text{Cox}^{\text{it}}(X, \Delta))\) is a torus quotient of the Cox ring of \((T, \Delta_T)\). Since \(S_1\) is finite, both \((Y, \Delta_Y)\) and \(\text{Spec}(\text{Cox}^{\text{it}}(X, \Delta))\) have the same universal cover. On the other hand, as \((Y, \Delta_Y)\) is toric, its universal cover is a torus quotient of the spectrum of the Cox ring of \((Y, \Delta_Y)\), which agrees with the Cox ring of \((T, \Delta_T)\). The Cox ring of \((T, \Delta_T)\) is the affine space \(\mathbb{A}^{\rho(T)+\dim(T)}\). Hence, the simply connected factorial canonical cover of \((X, \Delta)\) is the affine space \(\mathbb{A}^{\rho(T)+\dim(T)}\) itself. We conclude that (2) implies (1).

Now, we prove that (1) implies (2). We assume that the simply connected factorial canonical cover of \((X, \Delta)\) is smooth. Let \(G\) be the reductive group acting on the simply connected factorial canonical cover. Let \(G_0 \leq G\) be the connected component at the identity. Then, \(G_0\) is a connected reductive solvable group, so it is a torus. By construction, \(G_0\) acts on the scfc cover with a fixed attractive point. By assumption, that point is smooth. Thus, by [55, Theorem 2.5], we conclude that the scfc cover is isomorphic to \(\mathbb{A}^n\) and \(G_0\) acts linearly on it. We conclude that \((T, \Delta_T) := \mathbb{A}^n / G_0\) is a projective toric variety that admits a finite Galois log quasi-étale morphism to \((X, \Delta)\). This finishes the proof.

We have the following corresponding statement for klt singularities.

**Theorem 6.4.** Let \((X, \Delta; x)\) be a klt singularity. Then, the following statements are equivalent.

1. The simply connected factorial canonical cover of \((X, \Delta)\) is smooth.
2. \((X, \Delta)\) is a finite quasi-étale quotient of a toric singularity.

**Remark 6.5.** The singularities that appear in Theorem 6.4 are considered by the second author in [63, 64], where they are called toric quotient singularities. In [63, 64], it is shown that toric quotient singularities are the prototypes of klt singularities with large fundamental group. Moreover, the minimal log discrepancies of these singularities are described in [65].

### 6.3 Iteration of Cox rings and scfc cover

In this subsection, we characterize when the iteration of Cox rings is isomorphic to the scfc cover.

**Theorem 6.6.** Let \((X, \Delta)\) be a Fano-type variety. Then, the following are equivalent.
The spectrum of the iteration of Cox rings has trivial regional fundamental group.

2. The spectrum of the iteration of Cox rings coincides with the simply connected factorial canonical cover.

3. The fundamental group $\pi_1^{reg}(X, \Delta)$ is solvable.

Proof. If the spectrum of the iteration of Cox rings has trivial regional fundamental group, then it is factorial and simply connected. Thus, we have that (1) implies (2). The condition (2) trivially implies (1).

Assume that the spectrum of the iteration of Cox rings has trivial regional fundamental group. Let $G$ be the solvable reductive group acting on the iteration of Cox rings $\text{Cox}^{\text{II}}(X, \Delta)$. We know that $X$ is the quotient of $\text{Cox}^{\text{II}}(X, \Delta)$ by $G$. Let $G^0$ be the connected component at the identity of $G$. The quotient $X' := \text{Cox}^{\text{II}}(X, \Delta)/G^0$ is a finite solvable cover of $(X, \Delta)$. Furthermore, the pull-back of $K_X + \Delta$ to $X'$ equals $K_{X'} + \Delta'$ where $\Delta'$ is an effective divisor. Assume that $\pi_1^{reg}(X, \Delta)$ is not solvable. Then, $X' \to X$ is not the regional universal cover of $(X, \Delta)$. Thus, we can take a nontrivial finite log quasi-étale Galois cover of $(X', \Delta')$. We call this finite cover $Y \to X$. By Proposition 5.1, this cover induces a nontrivial finite log quasi-étale Galois cover of the spectrum of the iteration of Cox rings. This contradicts the fact that the spectrum of the iteration of Cox rings has trivial regional fundamental group. We conclude that $(X', \Delta')$ is the universal cover of $(X, \Delta)$. Thus, $\pi_1^{reg}(X, \Delta)$ is a solvable group. We conclude that (2) implies (3).

Now, assume that the regional fundamental group $\pi_1^{reg}(X, \Delta)$ is solvable. We proceed as in the previous paragraph. Let $G$ be the solvable reductive group acting on the iteration of Cox rings $\text{Cox}^{\text{II}}(X, \Delta)$. Let $G^0$ be the connected component at the identity of $G$. The quotient $X' := \text{Cox}^{\text{II}}(X, \Delta)/G^0$ is a finite solvable cover of $(X, \Delta)$. Since the regional fundamental group of $(X, \Delta)$ is solvable, the regional fundamental group of $(X', \Delta')$ is solvable. If $(X', \Delta')$ has simply connected log smooth locus, then by the proof of Theorem 5.2, we conclude that $\text{Cox}^{\text{II}}(X, \Delta)$ equals the scfc cover. Indeed, in this case, the Cox ring of $(X', \Delta')$ equals $\text{Cox}^{\text{II}}(X, \Delta)$. On the other hand, assume that $(X', \Delta')$ has nontrivial regional fundamental group $S$. By assumption, $S$ is solvable. In particular, its commutator is a proper subgroup. By Proposition 5.1, there exists a finite log quasi-étale Galois cover of $\text{Cox}^{\text{II}}(X, \Delta)$ with acting group isomorphic to $S$. Since $[S, S] \subseteq S$ is proper, by Corollary 4.16, we conclude that $\text{Cox}^{\text{II}}(X, \Delta)$ is not factorial. This leads to a contradiction. We conclude that $(X', \Delta')$ is simply connected and its Cox ring is isomorphic to $\text{Cox}^{\text{II}}(X, \Delta)$. Thus, the spectrum of the iteration equals the scfc of $(X, \Delta)$. We have that (3) implies (2). Hence, all the statements are equivalent.

The following local version of the above theorem is proved analogously.

Theorem 6.7. Let $(X, \Delta; x)$ be a klt singularity. Then, the following are equivalent:

1. the spectrum of the iteration of Cox rings has trivial regional fundamental group,
2. the spectrum of the iteration of Cox rings coincides with the simply connected factorial canonical cover, and
3. the regional fundamental group $\pi_1^{reg}(X, \Delta; x)$ is solvable.

7 EXAMPLES AND PROOFS OF THE THEOREMS

In this section, we collect some examples that enlighten the techniques of the paper. Then, we explain how the theorems of the introduction are implied by the theorems proved throughout
the manuscript. In Subsection 7.1, we will give an example of a Fano-type variety with nonsolvable regional fundamental group. We describe its scfc and iteration of Cox rings explicitly. In Subsection 7.2, we prove a special Jordan property for singularities with torus action. Finally, in Subsection 7.3, we give a detailed study of the iteration of Cox rings, regional fundamental groups, and scfc covers of klt singularities of complexity one.

7.1 Examples with $\pi_1(X^{\text{reg}})$ nonsolvable

In this subsection, we give an example of a Fano-type variety $X$ so that its regional fundamental group $\pi_1^{\text{reg}}(X)$ is nonsolvable. We also explain how to obtain the simply connected factorial canonical cover of $X$.

Example 7.1. Let $X$ be the variety obtained from $(\mathbb{P}^1)^n$ quotiented by the action of $A_n$ permuting the coordinates, for $n \geq 5$. We denote the quotient by $\rho : (\mathbb{P}^1)^n \to X$. Then, $X$ is a projective variety of Fano type, which is not toric. Furthermore, we have that $X$ satisfies that $\pi_1^{\text{reg}}(X) \cong A_n$. Indeed, we have a natural étale Galois morphism $\rho : \rho^{-1}(X^{\text{reg}}) \to X^{\text{reg}}$ with Galois group $A_n$. Moreover, the complement of $\rho^{-1}(X^{\text{reg}})$ has codimension at least two in $(\mathbb{P}^1)^n$ from which we conclude that $\pi_1(\rho^{-1}(X^{\text{reg}})) = 0$. This implies the claim.

We proceed to compute the iteration of Cox rings of $X$. Note that $\rho(X) = 1$ and $-K_X$ is 2-divisible. Hence, its first Cox ring is just the ring over some Weil $\mathbb{Q}$-Cartier divisor $H$ with $2H \sim -K_X$. This gives us a klt cone singularity $$(\text{Cox}(X), x_0)$$ where $x_0$ is the vertex for the action. The singularity $\text{Cox}(X)$ is factorial at $x_0$. Hence, the Iteration of Cox rings coincides with the first Cox ring. Furthermore, the regional fundamental group of $\text{Cox}(X)$ at $x_0$ is isomorphic to $A_n$ and its universal cover is isomorphic to the cone over $(\mathbb{P}^1)^n$ with respect to $\rho^*H$. We denote this variety by $\text{Cone}((\mathbb{P}^1)^n)$. Note that the regional fundamental group of $\text{Cone}((\mathbb{P}^1)^n)$ is trivial and its Cox ring is the affine space $\mathbb{A}^{2n}$. Hence, the simply connected factorial canonical cover of $X$ is the $2n$-dimensional affine space. We obtain the following commutative diagram:

\[
\begin{array}{ccc}
\mathbb{A}^{2n} & \xrightarrow{\mathbb{T}_0} & \text{Cone}((\mathbb{P}^1)^n) \\
/\mathbb{T}_1 & & /\mathbb{A}_n \\
& & /\mathbb{C}^* \\
(\mathbb{P}^1)^n & \xrightarrow{\rho^*} & X \\
/\mathbb{C}^* & & /\mathbb{A}_n \\
& & /\mathbb{C}^*
\end{array}
\]

Here, the torus $\mathbb{T}_1$ acts on the affine space $\mathbb{A}^{2n}$ by

\[(t_1, \ldots, t_n) \cdot (x_1, \ldots, x_{2n}) = (t_1x_1, t_1x_2, t_2x_3, t_2x_4, \ldots, t_nx_{2n-1}, t_nx_{2n}).\]

On the other hand, the torus $\mathbb{T}_0$ acts on the affine space $\mathbb{A}^{2n}$ by

\[(t_1, \ldots, t_{n-1}) \cdot (x_1, \ldots, x_{2n}) = (tx_1, tx_2, t_1x_3, t_1x_4, t_2x_5, t_2x_6, \ldots, t_{n-1}x_{2n-1}, t_{n-1}x_{2n}).\]
where \( t = (t_1 \ldots t_{n-1})^{-1} \). Thus, we obtain a representation \( X \cong \mathbb{A}^{2n}/(A_n \rtimes \mathbb{T}_0) \), where \( A_n \) is acting as permutation on the components of \( \mathbb{A}^{2n} \cong (\mathbb{A}^2)^n \) and \( \mathbb{T}_0 \) is acting as above. The above example reflects two different ways in which the simply connected factorial canonical ring can be obtained: As the iteration of Cox rings of the universal cover, or as the Cox ring of the universal cover of the iteration of Cox ring. This example is a particular case of Theorem 6.3.

### 7.2 Jordan property for \( \mathbb{T} \)-varieties

In this subsection, we prove a strengthened version of the Jordan property for the regional fundamental group of affine klt \( \mathbb{T} \)-varieties of complexity \( k \). Then, we specialize this statement for \( \mathbb{T} \)-varieties of complexity one.

**Theorem 7.2.** Let \( k \) be a positive integer. There exists a constant \( c(k) \), only depending on \( k \), satisfying the following. Let \( (X, \Delta; x) \) be an \( n \)-dimensional klt \( \mathbb{T} \)-singularity of complexity \( k \). Then, there exists an exact sequence

\[
1 \to A \to \pi_1^{\text{reg}}(X, \Delta; x) \to N \to 1,
\]

where \( A \) is an abelian group of rank at most \( n \) and index at most \( c(k) \).

**Proof.** Let \( (Y, B_Y) \) be the normalized Chow quotient of \( (X, \Delta) \). Then \( (X, \Delta) \) is defined by a polyhedral divisor \( D \) on \( Y \) with tailcone \( \sigma^\vee \in M_\mathbb{Q} \) (see, e.g., [2, Theorem]). By [61, Theorem 4.9], we know that \( (Y, B_Y) \) is a Fano-type pair. We may replace \( Y \) with a small \( \mathbb{Q} \)-factorialization to assume that \( Y \) is \( \mathbb{Q} \)-factorial. Let \( \tilde{X} \) be the relative spectrum over \( Y \) of \( \bigoplus_{u \in \sigma^\vee \cap M} \mathcal{O}_Y(D(u)) \). Then, \( q : \tilde{X} \to Y \) is an orbifold toric bundle. In the complement of the support of the polyhedral divisor \( D \), the toric bundle \( q \) is trivial. Let \( (\tilde{X}, \Delta_{\tilde{X}}) \) be the log pull-back of \( (X, \Delta) \) to \( \tilde{X} \). Let \( \Gamma \) be the boundary obtained from \( \Delta_{\tilde{X}} \) by increasing to one the coefficients of all the torus invariant divisors that are horizontal over \( Y \). Let \( U \subset Y \) be an open set on which the toric bundle trivializes, so \( q^{-1}(U) \cong \mathbb{G}_m^{n-k} \times U \). Then, we have a commutative diagram

\[
\begin{array}{ccc}
1 & \longrightarrow & \mathbb{Z}^{n-k} \\
\downarrow & & \downarrow \\
1 & \longrightarrow & \pi_1^{\text{reg}}(\tilde{X}, \Gamma) \\
\downarrow & & \downarrow \\
1 & \longrightarrow & \pi_1^{\text{reg}}(X, \Delta; x) \\
\end{array}
\]

where the vertical arrows are surjective and the top horizontal sequence splits. By the commutativity of the diagram, \( \mathbb{Z}^{n-k} \) lies in the center of \( \pi_1^{\text{reg}}(\tilde{X}, \Gamma) \). Furthermore, we have a surjection \( \pi_1^{\text{reg}}(\tilde{X}, \Gamma) \to \pi_1^{\text{reg}}(X, \Delta; x) \). Indeed, the birational map is a purely log terminal blow-up (see, e.g., [20, Proof of Theorem 4.14]). Observe that \( Y \) has dimension \( k \). By [20, Theorem 3], we can find an abelian normal subgroup \( A_Y \leq \pi_1^{\text{reg}}(Y, B_Y) \) of rank at most \( k \) and index at most \( c(k) \), where \( c(k) \) is a constant that only depends on \( k \). Hence, the preimage \( A_{\tilde{X}} \) of \( A_Y \) in \( \pi_1^{\text{reg}}(\tilde{X}, \Gamma) \) is a free finitely generated abelian group of rank at most \( n \) and index at most \( c(k) \). Let \( A \) be the image of \( A_{\tilde{X}} \) in \( \pi_1^{\text{reg}}(X, \Delta; x) \). Since \( \pi_1^{\text{reg}}(X, \Delta; x) \) is finite, we conclude that \( A \) is a finite abelian group of rank at most \( n \) and index at most \( c(k) \). \( \square \)
Remark 7.3. Note that the size of the nonabelian part $N$ of the regional fundamental group only depends on the complexity and not on the dimension of the germ as in [20]. This, of course, happens because the $(n-k)$-dimensional torus action cannot contribute to the nonabelian part of the regional fundamental group. If the complexity is zero, we can simply take $c(0) = 0$ because the regional fundamental group of a toric pair is always abelian. The following corollary gives an explicit bound for $c(1)$.

**Corollary 7.4.** Let $(X, x)$ be a $n$-dimensional klt $\mathbb{T}$-singularity of complexity one. Then, there exists an exact sequence

$$1 \to A \to \pi_1^{\text{reg}}(X, x) \to N \to 1,$$

where $A$ is an abelian group of rank at most $n$ and index at most 60.

**Proof.** In the notation of the proof of Theorem 7.2, we have an exact sequence

$$1 \to \mathbb{Z}^{n-1} \to \pi_1^{\text{reg}}(\tilde{X}, \Gamma) \to \pi_1^{\text{reg}}(\mathbb{P}^1, B_{\mathbb{P}^1}) \to 1$$

and a surjection

$$\pi_1^{\text{reg}}(\tilde{X}, \Gamma) \to \pi_1^{\text{reg}}(X, x).$$

Here, $(\mathbb{P}^1, B_{\mathbb{P}^1})$ is a pair with standard coefficients that is log Fano. The image of $\mathbb{Z}^{n-1}$ on $\pi_1^{\text{reg}}(X, x)$ is a normal abelian subgroup $A_0$ of rank at most $n - 1$. The elements of $A_0$ lie in the center of $\pi_1^{\text{reg}}(X, x)$ (see, e.g., [20, Corollary 4.13]). By [57, Example 5.1], we know that there is an exact sequence

$$1 \to \mathbb{Z} \to \pi_1^{\text{reg}}(\mathbb{P}^1, B_{\mathbb{P}^1}) \to N_0 \to 1,$$

where $N_0$ has order at most 60. Let $\gamma$ be the generator of $\mathbb{Z} \leq \pi_1^{\text{reg}}(\mathbb{P}^1, B_{\mathbb{P}^1})$. Then, the homomorphic image of $\gamma$ and $A_0$ generate a normal abelian subgroup $\tilde{A}$ of $\pi_1^{\text{reg}}(X, x)$ of rank at most $n$. Hence, the quotient of $\pi_1^{\text{reg}}(X, x)$ by $A$ is $N$ a homomorphic image of $N_0$ that has order at most 60. □

Remark 7.5. The bound 60 is obtained by the binary icosahedral group of order 120 with the center $\mathbb{Z}_2$ being the only normal abelian subgroup, that is, when $B_{\mathbb{P}^1}$ has coefficients $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{4}{5}$.

### 7.3 Complexity one klt singularities

In the case of klt-type singularities with a torus action of complexity one, we are able to explicitly determine all invariants defined so far: Cox rings, iterated Cox rings, regional fundamental groups, associated universal covers, and the simply connected factorial canonical covers. We start by recalling the construction of affine rational complexity one $\mathbb{T}$-varieties.
Definition 7.6. Let $N$ be a free finitely generated abelian group of rank $r$. Let $M$ be the dual of $N$. We denote by $N_Q$ and $M_Q$ the corresponding $Q$-vector spaces. Given a polyhedron $\Delta \subset N_Q$, we denote its recession cone to be the set of $v \in N_Q$ so that $v + \Delta \subset \Delta$. The recession cone of a polyhedron is a strongly convex polyhedral cone. It is denoted by $\text{rec}(\Delta)$. Let $\sigma$ be a strongly convex polyhedral cone in $N_Q$. We denote by $\text{Pol}_Q(N, \sigma)$ the semigroup of polyhedra $\Delta$ of $N_Q$ for which $\text{rec}(\Delta) = \sigma$. The additive structure of this semigroup is the Minkowski sum. The elements of this group are called $\sigma$-polyhedra.

We denote by $\text{CaDiv}_{\geq 0}(\mathbb{P}^1)$ the semigroup of effective Cartier divisors on $\mathbb{P}^1$. A polyhedral divisor on $(\mathbb{P}^1, N)$ with recession cone $\sigma$ is an element of

$$\text{Pol}_Q(N, \sigma) \otimes_{\mathbb{Z}_{\geq 0}} \text{CaDiv}_{\geq 0}(\mathbb{P}^1).$$

Note that a polyhedral cone can be written as a formal finite sum

$$D = \sum_{i=1}^{s} \Delta_i \otimes \{p_i\},$$

for a finite set of points $p_1, \ldots, p_s$ in $\mathbb{P}^1$ and $\sigma$-polyhedra $\Delta_1, \ldots, \Delta_s$. If we don’t fix $N$ or the recession cone, then we just say that $D$ is a polyhedral divisor on $\mathbb{P}^1$.

Let $D$ be a polyhedral divisor on $\mathbb{P}^1$. We have a homomorphism of semigroups, called the evaluation homomorphism, defined as follows:

$$D : \sigma^\vee \to \text{CaDiv}_Q(\mathbb{P}^1)$$

$$D(u) = \sum_{i=1}^{s} \min \langle \Delta_i, u \rangle p_i.$$

By abuse of notation, we are denoting the polyhedral divisor and the evaluation homomorphism by $D$.

Definition 7.7. A polyhedral divisor in $\mathbb{P}^1$ is said to be a proper polyhedral divisor if $D(u)$ is has nonnegative degree for $u \in \sigma^\vee$ and $D(u)$ has positive degree for $u \in \text{relint}(\sigma^\vee)$. For a proper polyhedral divisor $D$ on $\mathbb{P}^1$, we can define its degree polyhedron to be

$$\deg(D) = \sum_{i=1}^{s} \Delta_i \subset \sigma.$$

Given a proper polyhedral divisor $D$, we can associate to it a normal rational affine variety of dimension $r + 1$ with an effective action of a $r$-dimensional torus. We have a sheaf of $\mathcal{O}_{\mathbb{P}^1}$-algebras

$$\mathcal{A}(D) = \bigoplus_{u \in \sigma^\vee \cap M} \mathcal{O}_{\mathbb{P}^1}(D(u)) \chi^u.$$

We denote by $\tilde{X}(D)$ the relative spectrum of $\mathcal{A}(D)$ over $\mathbb{P}^1$. We denote by $X(D)$ spectrum of the ring of sections of $\mathcal{A}(D)$. The variety $X(D)$ is a normal rational affine variety of dimension $r + 1$ with an effective action of a $r$-dimensional torus. Indeed, it admits an effective action of $\mathbb{T} := \text{Spec}(\mathbb{C}[M])$. 
This means that $X(D)$ is a rational $\mathbb{T}$-variety of complexity one. It is known that every rational $\mathbb{T}$-variety of complexity one is isomorphic to $X(D)$ for some polyhedral divisor $D$ on $\mathbb{P}^1$ (see, e.g., [2, Theorem on p. 559]).

**Notation 7.8.** Let $D$ be a proper polyhedral divisor on $\mathbb{P}^1$, with recess cone $\sigma$, and let $p \in \mathbb{P}^1$. We denote $\Delta_p = \Delta_p$ if $p = p_i$ or $\Delta_p = \sigma$ otherwise. For every vertex $v \in \Delta_p$, we denote by $\mu(v)$ the smallest positive integer so that $\mu(v)v \in N$. For every $p \in \mathbb{P}^1$, we define

$$\mu_p := \max\{\mu(v) \mid v \in \Delta_p\}.$$ 

For every $p \in \mathbb{P}^1$, we define $b_p := (1 - \mu_p^{-1})p$. We define $B(D) := \sum_{p \in \mathbb{P}^1} b_p$. Note that $B(D)$ is a divisor on $\mathbb{P}^1$ with standard coefficients, that is, $(\mathbb{P}^1, B(D))$ is a log pair with standard coefficients.

From now on, we focus on complexity one $\mathbb{T}$-singularities. This means that affine $\mathbb{T}$-varieties of complexity one $X$ with a distinguished point $x \in X$ that is a klt singularity. We have the following theorem that characterizes the klt-ness of the complexity one $\mathbb{T}$-singularity.

**Theorem 7.9** (Cf. [61]). Let $D$ be a polyhedral divisor on $\mathbb{P}^1$. Then, $(X(D), x)$ is klt if and only if $(\mathbb{P}^1, B(D))$ is a log Fano pair.

Note that this happens if and only if $\mu_p$ is nontrivial for at most three points in $\mathbb{P}^1$, and, in addition, for these three points, the corresponding $\mu_p$ must form a platonic triple in the sense of [57, Example 4.1].

From now on, we turn to describe the regional fundamental group of $X(D)$ at $x$. To do so, first, we need to understand the $\mathbb{T}$-equivariant birational contraction $r : \tilde{X}(D) \to X(D)$. We proceed to explain which divisors are contracted by this birational contraction. There are two types of $\mathbb{T}$-invariant divisors in $\tilde{X}(D)$; the divisors that are mapped to points in $\mathbb{P}^1$ via the projection $\tilde{X}(D) \to \mathbb{P}^1$, which are called vertical invariant divisors. Vertical invariant divisors are in bijection with pairs $(p, v)$ where $p \in \mathbb{P}^1$ and $v$ is a vertex of the polyhedron $\Delta_p$. Hence, we will denote the corresponding vertical divisor by $\Delta_{(p, v)}$. The invariant divisors that dominate $\mathbb{P}^1$ are called horizontal divisors. Horizontal divisors are in bijection with rays of the recession cone $\sigma$. The contraction $r$ contracts exactly those horizontal divisors corresponding to rays of $\sigma$ that intersect $\text{deg}(D)$ nontrivially (see, e.g., [2, §10]).

**Notation 7.10.** Let $D$ be a proper polyhedral divisor on $\mathbb{P}^1$ with recession cone $\sigma$. Let $N_D \subset N$ be the sublattice generated by elements of $N$ that belong to a regular subcone of $\sigma$ that does not intersect $\text{deg}(D)$. We introduce variables $t_1, \ldots, t_r$ corresponding to a basis of $N$. For every $n \in N_D$, we let $t^n := t_1^{n_1} \ldots t_r^{n_r}$.

Let $p \in \mathbb{P}^1$ and $\Delta_p$ the corresponding $\sigma$-polyhedra. Consider the cone $\sigma(D, p)$ in $N_\mathbb{Q} \times \mathbb{Q}$ spanned by $\{0\} \times \sigma$ and $\{1\} \times \Delta_p$. Let $N_{\sigma(D, p)} \subset N$ be the sublattice generated by elements of $N \times \mathbb{Z}$ that belong to a regular subcone of $\sigma(D, p)$ which does not intersect $\text{deg}(D)$. We denote by $B(D, p)$ a basis of $N_{\sigma(D, p)}$. For every $v \in B(D, p)$, we denote by $\pi_1(v)$ the projection in $N$ and by $\pi_2(v)$ the projection in $\mathbb{Z}_{\geq 1}$.
Theorem 7.11. Let $D$ be a polyhedral divisor on $(\mathbb{P}^1, N)$. Write $D = \sum_{i=1}^s \Delta_i \otimes \{p_i\}$. Let $x \in X(D)$ be the vertex of the torus action. Then, $\pi_1^{\text{reg}}(X(D), x)$ is isomorphic to the group generated by

$$t_1, \ldots, t_r, b_1, \ldots, b_s$$

with the relations

- $b_1 \cdots b_s$,
- $[t_i, t_j]$ for every $1 \leq i \leq j \leq r$,
- $[t_i, b_j]$ for every $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, s\}$,
- $t^n$ for every $n \in N_D$, and
- $t^{\pi_1(v)} b^{\pi_2(v)}$ for every $v \in B(D, p)$.

Proof. We have a good quotient $\widetilde{X}(D) \to \mathbb{P}^1$. This good quotient is trivial with fiber $X(\sigma) = \text{Spec}(\mathbb{C}[\sigma^\vee \cap M])$ over $\mathbb{P}^1 \setminus \{p_1, \ldots, p_s\}$. Around each $p_i$, the variety $\widetilde{X}(D)$ has an analytic neighborhood diffeomorphic to an analytic neighborhood of the fiber of $X(\sigma(D, p)) \to \mathbb{A}^1$ around zero (see, e.g., [61, Example 2.5]). The equivariant birational contraction $r : \widetilde{X}(D) \to X(D)$ contracts the closure of the $\mathbb{T}$-invariant cycles of the form $X(\tau) \times (\mathbb{P}^1 \setminus \{p_1, \ldots, p_s\})$, where $\tau \leq \sigma$ is a cone intersecting $\text{deg}(D)$ (see, e.g., [3, §5]). In order to compute the regional fundamental group of $X(D)$ at $x$, it suffices to compute the regional fundamental group of $\widetilde{X}(D) \setminus \text{Ex}(r)$. Indeed, the image of every prime component of $\text{Ex}(r)$ has codimension at least two in $X(D)$. If the image of such component is contained in the singular locus of $X(D)$, then it does not contribute to the regional fundamental group. Thus, it suffices to compute $\pi_1^{\text{reg}}(\widetilde{X}(D) \setminus \text{Ex}(r))$.

A general fiber of $X(D)^{\text{reg}} \setminus \text{Ex}(r) \to \mathbb{P}^1$ is isomorphic to the open subvariety of $X(\sigma)$ that corresponds to the regular subcones of $\sigma$ that does not intersect $\text{deg}(D)$. Around each $p_i$, the variety $\widetilde{X}(D)^{\text{reg}} \setminus \text{Ex}(r)$ is diffeomorphic to the open subvariety of $X(\sigma(D, p))$ corresponding to the regular subcones of $\sigma(D, p)$ not intersecting $\text{deg}(D)$. Thus, we have a formally toric description of $\widetilde{X}(D)^{\text{reg}} \setminus \text{Ex}(r)$ over $\mathbb{P}^1$. Then, the rest of the description follows from applying Van Kampen theorem to glue the fundamental group of $X(\sigma)^{\text{reg}} \times (\mathbb{P}^1 \setminus \{p_1, \ldots, p_s\})$ with those of the analytic neighborhoods of the fibers of the $p_i$’s. The proof proceeds similarly as in [57, Theorem 3.4].

In the above theorem, the loop $t_i$ corresponds to a loop around the $i$th factor of the $r$-dimensional torus $\mathbb{T} \cong (\mathbb{C}^*)^r$ of a general fiber of $\widetilde{X}(D) \to \mathbb{P}^1$. On the other hand, the loops $b_j$ correspond to liftings to $\mathcal{X}(D)$ of the loops around the points $p_j$ in $\mathbb{P}^1$.

Note that the above description gives an explicit version of Corollary 7.4. Indeed, for the group $A$, we can consider the normal abelian group generated by the $t_i$’s. Since we have at most three points for which $\mu_p$ is nontrivial, we conclude that the quotient $\pi_1^{\text{reg}}(X(D), x)$ has order at most 60. Indeed, the quotient $\pi_1^{\text{reg}}(X(D), x)/A$ admits a surjection from $\pi_1^{\text{reg}}(\mathbb{P}^1, B(D))$.

Theorem 7.11 gives a simple way to construct the universal cover of a complexity-one $\mathbb{T}$-singularity. Let $D$ be a proper polyhedral divisor on $(\mathbb{P}^1, N)$. Let $(\mathbb{P}^1, B(D))$ be the associated log Fano pair. Let $p : (\mathbb{P}^1, B') \to (\mathbb{P}^1, B(D))$ be the universal cover of $\pi_1^{\text{reg}}(\mathbb{P}^1, B(D))$. Then, $p^* D$ is a proper polyhedral divisor on $\mathbb{P}^1$ and we have a finite quasi-étale Galois morphism

$$p : (X(p^*D), x') \to (X(D), x).$$

We denote it by $p$ by abuse of notation. Here, $x'$ is the unique preimage of $x$. We are considering the pull-back of proper polyhedral divisors as defined in [2, §8]. By Theorem 7.11, the regional
fundamental group of $(X(p^*D), x')$ is generated by the loops $t_i$. In particular, it is abelian. Hence, its universal cover is nothing else than an isogeny of the torus given by a lattice extension $N \hookrightarrow N'$.

Now, we turn to describe the Cox ring of an affine $\mathbb{T}$-variety of complexity one. We restrict ourselves to the klt case, so the singularities will impose some restriction on the structure of the Cox ring.

**Definition 7.12** (Cf. [7]). Let $D$ be a proper polyhedral divisor on $(\mathbb{P}^1, N)$ that defines a klt complexity one affine variety $X(D)$.

Fix integers $m \geq 0$, $n, r > 0$, and a partition $n = n_0 + \cdots + n_r$. For every $i = 0, \ldots, r$, let $l_i = (l_{i1}, \ldots, l_{in_i}) \in \mathbb{Z}^{n_i}$ with $l_{i1} \geq \cdots \geq l_{in_i} > 0$ and $l_{i1} \geq \cdots \geq l_{r1}$. Define monomials $T^l_i := T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}}$ in the polynomial ring

$$\mathbb{C}[T_{ij}, S_k] := \mathbb{C}[T_{ij}, S_k; i = 1, \ldots, r, j = 1, \ldots, n_i, k = 1, \ldots, m].$$

Now, define pairwise different scalars $\vartheta_0 = 1, \vartheta_1, \ldots, \vartheta_{r-a} \in \mathbb{C}^*$ and for $i = 0, \ldots, r - 2$ a trinomial

$$g_i := \vartheta_i T^l_i + T^l_{i+1} + T^l_{i+2}.$$

If the $l_i := \max(l_{i1}, \ldots, l_{in_i})$ are platonic tuples, that is, of the form

$$(5, 3, 2, 1 \ldots, 1), (4, 3, 2, 1 \ldots, 1), (3, 3, 2, 1 \ldots, 1), (k, 2, 2, 1 \ldots, 1), (k, l, 1, 1 \ldots, 1),$$

then we call the factor ring $R := \mathbb{C}[T_{ij}, S_k]/\langle g_0, \ldots, g_{r-2} \rangle$ a platonic ring.

Now, we have the following slight generalization of [7, Theorem 1.3] that was first stated in [17, Theorem 5].

**Theorem 7.13.** Let $D$ be a proper polyhedral divisor on $(\mathbb{P}^1, N)$ that defines a klt complexity one affine variety $X(D)$. Let $x \in X(D)$ be the vertex of the torus action. Write $D = \sum_{i=1}^s \Delta_i \otimes \{p_i\}$ and assume $\mu(p_i) = 1$ for $i \geq 4$. Then, the Cox ring of $(X(D), x)$ is the platonic ring with associated tuple $(\mu(p_1), \mu(p_2), \mu(p_3))$.

Now, we turn to explicitly describe the possible Cox ring iterations in terms of the platonic Cox ring of a klt singularity of complexity one. The original reference is [7, Rem 6.7].

**Theorem 7.14** (Cf. [51]). Let $D$ be a proper polyhedral divisor on $(\mathbb{P}^1, N)$ that defines a klt complexity one affine variety $X(D)$. Then, the possible sequences of platonic triples arising from Cox ring iterations of $X(D)$ are the following:

- $(1, 1, 1) \rightarrow (2, 2, 2) \rightarrow (3, 3, 2) \rightarrow (4, 3, 2)$,
- $(1, 1, 1) \rightarrow (x, x, 1) \rightarrow (2x, 2, 2)$,
- $(1, 1, 1) \rightarrow (x, x, 1) \rightarrow (x, 2, 2)$, and
- $(l^{-1}l_0, l^{-1}l_1, 1) \rightarrow (l_0, l_1, 1)$ where $l := \gcd(l_0, l_1) > 1$.

Finally, the following theorem describes the simply connected factorial canonical cover of a klt singularity of complexity one.
Theorem 7.15. Let $D$ be a proper polyhedral divisor on $(\mathbb{P}^1, N)$ that defines a klt complexity one affine variety $X(D)$. Let $p : \mathbb{P}^1 \to \mathbb{P}^1$ be the universal cover of $(\mathbb{P}^1, B(D))$. Then, the scfc cover is $\text{Cox}(X(p^*D))$.

Proof. We have a Galois quasi-étale finite morphism $X(p^*D) \to X(D)$. By Theorem 7.11, we know that the regional fundamental group of $X(p^*D)$ is abelian and generated by the loops $t_1, \ldots, t_r$. Furthermore, the platonic triple of $p^*D$ is of the form $(1,1,1)$. Otherwise, the regional fundamental group is nontrivial by Theorem 7.11. By Theorem 6.7, we conclude that the scfc cover of $X(p^*D)$ agrees with its iteration of Cox rings. On the other hand, by Theorem 7.14, we conclude that the Cox ring of $X(p^*D)$ is factorial. We conclude that the scfc cover of $X(p^*D)$, which coincides with the scfc cover of $X(D)$, is isomorphic to $\text{Cox}(X(p^*D))$. □

7.4 Proof of the theorems

In this subsection, we explain how the theorems in the introduction follow from the theorems proved throughout the manuscript.

Proof of Theorem 2. Follows from Theorem 4.17. □

Proof of Theorem 3. Follows from Theorem 4.17. □

Proof of Theorem 4. Follows from Theorem 5.7. □

Proof of Theorem 5. Follows from Theorem 5.2. □

Proof of Theorem 6. Follows from Theorem 5.4. □

Proof of Theorem 7. Follows from Theorem 6.2. □

Proof of Theorem 8. Follows from Theorem 6.4. □

Proof of Theorem 9. Follows from Theorem 6.7. □

Proof of Theorem 10. Follows from Theorem 4.15. □

Proof of Theorem 11. Follows from Theorem 7.2. □

Proof of Theorem 12. Follows from Theorem 7.2 and the proof of Theorem 4.17. □

8 COMBINATORICS AND COMPUTATIONS OF TORIC AND COMPLEXITY ONE LOG COXRINGS

In this short section, we provide a combinatorial description of the log Cox ring $\text{Cox}(X, \Delta)$ of torus-invariant pair structures on toric and, more generally, complexity one varieties in terms
of fans. This can be especially useful for explicit computations of log Cox rings of such pairs, because the combinatorics are implemented in various computer algebra programs, such as in the MDS-PACKAGE [47] for Maple.

We refer to [31] for definitions related to toric geometry. Recall that a toric variety $X = X_\Sigma$ can be described by a rational polyhedral fan $\Sigma \subseteq N$ in a lattice $N \cong \mathbb{Z}^{\dim(X)}$. The one-dimensional cones of such a fan are denoted by $\rho_i$ and correspond to torus-invariant divisors, denoted by $D_{\rho_i}$. Essential are also the primitive ray generators $u_{\rho_i}$ of these rays. We note that the collection $\Sigma(1)$ of these rays determines a toric variety up to birational equivalence. In particular, it determines the Cox ring together with the grading.

We shortly recall from [31, Section 5.1] how this works. Let $M$ be the dual lattice of $N$ (it is isomorphic to the character group of the torus). We have a short exact sequence

$$0 \rightarrow M \rightarrow \bigotimes_{\rho} \mathbb{Z}D_{\rho} \rightarrow \text{Cl}(X_\Sigma) \rightarrow 0,$$

where $m \in M$ is mapped to $\text{div}(\chi^m) = \sum_{\rho} \langle m, u_{\rho} \rangle D_{\rho}$. The toric morphism associated to the map $\mathbb{Z}^{\Sigma(1)} \rightarrow N$ given by the matrix $P$ with columns $u_{\rho_1}, \ldots, u_{\rho_{\Sigma(1)}}$ is the morphism $\mathbb{A}^{\Sigma(1)} = X \rightarrow X_\Sigma$ from the total coordinate space to $X$ itself. The $\text{Cl}(X_\Sigma)$-degrees of the variables (or $\mathbb{C}$-algebra generators of $\mathbb{C}[\mathbb{A}^{\Sigma(1)}]$) $x_{\rho_1}, \ldots, x_{\rho_{\Sigma(1)}}$ are given by the Gale-dual matrix $Q$ of $P$, see [8, Chapter 2.2], which can be computed by bringing $P$ into Smith normal form (cf. [7, Section 8]).

The above short exact sequence immediately shows the adaption that has to be made for the log Cox ring, namely, filling the rows of the matrix $P$ with appropriate multiples of the primitive ray generators. This yields the following description of the log Cox ring of a toric pair.

**Proposition 8.1.** Let $X_\Sigma$ be a toric variety associated to the fan $\Sigma \subseteq N$ with rays $\rho_1, \ldots, \rho_k$. Let $\Delta := \sum_{\rho} (1 - \frac{1}{n_{\rho}})D_{\rho}$ for $n_{\rho} \in \mathbb{N}_{\geq 1}$. Then the log class group $\text{Cl}(X_\Sigma, \Delta)$ is given by the short exact sequence

$$0 \rightarrow M \rightarrow \bigotimes_{\rho} \mathbb{Z}D_{\rho} \rightarrow \text{Cl}(X_\Sigma, \Delta) \rightarrow 0,$$

where $m \in M$ is mapped to $\sum_{\rho} \langle m, n_{\rho} u_{\rho} \rangle D_{\rho}$. The action of $\text{Cl}(X_\Sigma, \Delta)$ on the Cox space is given via the correspondence of quasi-torus actions and graded algebras. Here, the grading of the Cox ring $\text{Cox}(X_\Sigma, \Delta) = \mathbb{C}[x_{\rho_1}, \ldots, x_{\rho_k}]$ is given by the Gale-dual $Q_{\Delta}$ of the matrix

$$P_{\Delta} := [n_{\rho_1} u_{\rho_1}, \ldots, n_{\rho_k} u_{\rho_k}].$$

**Remark 8.2.** The Gale-dual can, for example, be computed with the function “AGHP2Q” in the MDS-PACKAGE [47] for Maple. Moreover, the function “GRveronese” of the same package computes generators of $\mathbb{C}[X_\Sigma]$ inside $\text{Cox}(X_\Sigma, \Delta)$ in case $X_\Sigma$ is affine.

We have seen in Theorem 7.13 how the Cox ring of a klt complexity one singularity is defined in terms of platonic tuples. The grading of the Cox ring of a complexity one $\mathbb{T}$-variety $X$ can be encoded via the matrix $P$ containing the primitive ray generators of the fan of a canonical toric variety into which $X$ can be embedded in such a way that the embedding induces an isomorphism of class groups [8, Chapter 3.4]. This can be done in such a way that certain entries of the matrix $P$ also define the exponents of the defining trinomials $g_i$ of the Cox ring (cf. Definition 7.12). In
the basic case of the two-dimensional ADE-singularities, the matrix $P_X$ is of the form

$$P_X = \begin{bmatrix} -l_0 & l_1 & 0 \\ -l_0 & 0 & l_2 \\ 1 - l_0 & 1 & 1 \end{bmatrix},$$

with a platonic triple $(l_0, l_1, l_2)$, where $T_0^{l_0} + T_1^{l_1} + T_2^{l_2}$ is the single relation of the Cox ring and the Gale-dual of $P_X$ determines the $\text{Cl}(X)$-grading.

So, replacing columns of $P_X$ with integer multiples yields not only a different $\text{Cl}(X)$-grading, but also a different relation in the Cox ring. In particular, the resulting triple of exponents may not be platonic anymore, yielding a non-klt log Cox ring. We explore this phenomenon in the case of the $D_4$-singularity exemplarily.

**Example 8.3.** The $D_4$-singularity is given by the matrix

$$P = \begin{bmatrix} -2 & 2 & 0 \\ -2 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix},$$

that is, it has classical Cox ring $\mathbb{C}[T_0, T_1, T_2] / \langle T_0^2 + T_1^2 + T_2^2 \rangle$ and the grading by the class group $(\mathbb{Z}/\mathbb{Z})^2$ is given by the Gale-dual

$$Q = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Thus, the ring of invariants is generated by $x_0 := T_0^2$, $x_1 := T_1^2$, $x_2 := T_2^2$, and $x_3 = T_1 T_2 T_3$, satisfying the relation $x_0 x_1 x_2 - x_3^2$. The relation in the Cox ring gives $x_0 + x_1 + x_2$ and plugging this into the first one yields $x_3^2 + x_1^2 x_2 + x_1 x_2^2$, which is the defining relation of $D_4$. There are three torus-invariant divisors $\Delta_i$ $(i = 0, 1, 2)$ on $D_4$, given by $T_i = 0$ on the Cox ring. These correspond to the rays through the $i$th column of $P$. Thus, for the pair $(D_4, \Delta)$ with $\Delta = \sum (1 - \frac{1}{m_i}) \Delta_i$, the log Cox ring has defining equation

$$T_0^{2m_0} + T_1^{2m_1} + T_2^{2m_2}.$$ 

We see that the exponent triple is platonic if and only if $m_0 = m_1 = 1$ and $m_2$ arbitrary up to permutation of indices. In this case, the spectrum of the Cox ring is an $A_{2m_2-1}$-singularity.

**Remark 8.4.** As in the toric case, imposing the respective relations in the Cox ring, the MDS-Package [47] for Maple is able to perform computations also for complexity-one $\mathbb{T}$-varieties.

**APPENDIX: TABLE OF COVERS**

In this appendix, we summarize all the different categories of covers of klt singularities (or Fanotype varieties) that we consider throughout this article. We describe the category of covers over $X$, the group that acts on such covers, the inverse limit, and the main property of the inverse limit.

Note that all the group isomorphism classes considered in the above table are closed under extensions. The following diagram shows the natural morphisms between the different covers in
TABLE A.1 Covers of klt singularities.

| Category | Acting group | Inverse limit | Main property |
|----------|--------------|---------------|---------------|
| Finite Galois log quasi-étale covers | Finite group | Universal cover | Simply connectedness |
| Abelian reductive quasi-étale covers | Quasi-torus | Cox ring | $T$-factoriality |
| Solvable reductive quasi-étale covers | Solvable reductive group | Iteration of Cox ring | Factoriality |
| Finite-solvable quasi-étale covers | Finite extensions of solvable reductive group | Simply connected factorial canonical cover (scfc cover) | Simply connectedness and factoriality |

the above table.

Here, $(\tilde{X}, \tilde{\Delta}; \tilde{x})$ is the universal cover of $(X, \Delta; x)$ and $(Y, \Delta_Y; y)$ is the scfc cover of $(X, \Delta; x)$. We finish the Appendix explaining when the morphisms in the above diagram are isomorphisms:

1. By Theorem 6.7, $\phi_1$ is an isomorphism if and only if $\pi_1^{\text{reg}}(X, \Delta; x)$ is solvable.
2. $\phi_i$ is an isomorphism if and only if the target is factorial for $i \in \{2, 3, 5\}$.
3. $\phi_4$ is an isomorphism if and only if $\pi_1^{\text{reg}}(X, \Delta; x)$ is trivial.

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