On Large-Lifting and Large-Supplemented Modules

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Abstract
In this paper, we introduce the concepts of Large-lifting and Large-supplemented modules as a generalization of lifting and supplemented modules. We also give some results and properties of this new kind of modules.

Keywords. L-small, L-lifting module, L-supplemented module.

1. Introduction
Throughout this paper, we assume that R is a commutative ring with identity. A submodule N of an R-module M is called Large (essential) submodule in M, (N ≤_g M), if for every nonzero submodule K of M, then N ∩ K ≠ 0 [1]. A proper submodule N of an R-module M is called small (N ≪ M), if for any submodule K of M such that N + K = M implies that K = M [1]. Assume that N and K are submodules of M, where M is R module, then N is called supplement of K in M, if N is minimal with respect to the property M = N + K. This is equivalent to M = N + K and N ∩ K ≪ N, if every submodule of M has a supplement in M, then M is called supplemented module [2]. An R-module M is called lifting, if for every submodule N of M there exists a submodule K of M such that M = K ⊕ H and N ∩ H ≪ H where H be a submodule of M, equivalently M is called lifting, if and only if for every submodule N of M there exists a submodule K of N such that M = K ⊕ H and N ∩ H ≪ M [2]. In [3], we give the concept of Large-small (L-small) submodule, it is given as follows; Let N be a proper submodule of M, then N is called L-small submodule of M (N ≪_L M), if N + K = M where K ≤ M, then K is essential submodule of M (K ≤_g M). In [4], we also give the concept of Large-coessential (L-coessential) submodule. It is given as follows; Let M be an R-module and K, N are submodules of M such that K ≤ N ≤ M, then K is said to be Large-coessential submodule, if N/k ≪_L M/k. This paper consists two sections, in section one we

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give the concept of Large-lifting (L-lifting) modules and some of its properties, such that an R-module $M$ is said to be L-lifting, if for every submodule $N$ of $M$ there exists a submodule $K$ of $N$ such that $M = K \oplus H$ and $N \cap H \ll_L M$ where $H$ is a submodule of $M$. In section two we introduce the concept of Large-supplemented (L-supplemented) modules, such that an R-module $M$ is called L-supplemented, if every submodule of $M$ has L-supplement in $M$, where a submodule $N$ is called L-supplement of $K$ in $M$, if $M = N + K$ and $N \cap K \ll_L N$. In Lemma(1.1), Lemma(1.2) and Lemma(1.3) we give some properties in [3] and [4] that we need it in this paper.

**Lemma 1.1[3]:** (1) Let $M$ be an R-module and $K, N$ be submodules of $M$ such that $K \leq N \leq M$, if $N \ll_L M$ then $K \ll_L M$.

(2) Let $f : M \to M'$ be an epimorphism where $M$ and $M'$ are R-modules such that $N \ll_L M'$ then $f^{-1}(N) \ll_L M$.

(3) Let $M$ be an R-module and $K, N$ be submodules of $M$ where $K$ is a closed in $M$ such that $K \leq N \leq M$, if $N \ll_L M$ then $K \ll_L M$ and $\frac{N}{K} \ll_L \frac{M}{K}$.

(4) Let $M$ be an R-module and $K, N$ be submodules of $M$ such that $K \leq N \leq M$, and $N$ is direct summand of $M$, if $K \ll_L M$, then $K \ll_L N$.

**Lemma 1.2[3]:** (1) Let $M = \bigoplus_{i \in I} M_i$ be a fully stable module, if $N_i \ll_L M_i$ then $\bigoplus_{i \in I} N_i \ll_L \bigoplus_{i \in I} M_i$.

(2) Let $M$ be an R-module such that $M$ is faithful, finitely generated and multiplication module and let $I$ be an ideal of $R$ then $I \ll_L R$ if and only if $IM \ll_L M$.

**Lemma 1.3[4]:** (1) Let $M$ be an R-module and $K, N$ be submodules of $M$ such that $K \leq N \leq M$, if $\frac{N}{K} \ll_L \frac{M}{K}$ then $N \ll_L M$.

(2) Let $M$ be an R-module and $K, N, and U$ be submodules of $M$ such that $K \leq N \leq U \leq M$, then $N \ll_L U$ in $M$ if and only if $\frac{N}{K} \leq_L \frac{U}{K}$ in $\frac{M}{K}$.

Now, we need to prove the following lemma.

**Lemma 1.4:** Let $M = M_1 \oplus M_2$ then $N_1 \ll_L M_1$ and $N_2 \ll_L M_2$ if and only if, $N_1 \cap N_2 \ll_L M_1 \oplus M_2$.

**Proof:** ($\Leftarrow$) Let $U_1 \oplus U_2$ be a submodule of $M_1 \oplus M_2$ such that $N_1 \oplus N_2 + U_1 \oplus U_2 = M_1 \oplus M_2$. So that $(N_1 + U_1) \oplus (N_2 + U_2) = M_1 \oplus M_2$ and hence $N_1 + U_1 = M_1$ and $N_2 + U_2 = M_2$. Since $N_1 \ll_L M_1$ and $N_2 \ll_L M_2$, then $U_1 \leq_L M_1$ and $U_2 \leq_L M_2$, this implies that $U_1 \oplus U_2 \leq_L M_1 \oplus M_2$. By [1], and therefore $N_1 \oplus N_2 \ll_L M_1 \oplus M_2$.

($\Rightarrow$) Let $N_1 \oplus N_2 \ll_L M_1 \oplus M_2$. Since $N_1 \leq N_1 \oplus N_2 \ll_L M_1 \oplus M_2 = M$ then by Lemma(1.1), we have $N_1 \ll_L M$ and since $N_1 \leq M_1 \leq M$ and $M_1$ is direct summand of $M$ then by Lemma(1.1) we get $N_1 \ll_L M_1$. Similarly we have $N_2 \ll_L M_2$.

2. Large-Lifting modules.

In this section we introduce the concept of Large-lifting modules and some properties of it are considered.

**Definition 2.1:** An R-module $M$ is called Large-lifting (L-lifting), if for every submodule $N$ of $M$ there exists a submodule $K$ of $N$ such that $M = K \oplus H$ and $N \cap H \ll_L M$ where $H$ is a submodule of $M$.

**Remarks and Examples 2.2:**

(1) Every lifting is L-lifting.

**Proof:** Let $M$ be a lifting module and $N \leq M$, then $M = K \oplus H$ where $K \leq N$ and $N \cap H \ll_L M$ so $N \cap H \ll_L M$ where $H \leq M$ by [3].

(2) The following example shows that the converse of (1) is not true.

Example: $Z$ as $Z$-module is L-lifting since for $N = nZ \leq Z$, there exists $\{0\}$ direct summand of $nZ$ such that $M = Z = \{0\} + Z$ and $nZ \cap Z = nZ \ll_L Z$ by [3], also if $N = Z \leq Z$, let $K = Z \leq N$ such that $Z = Z \oplus \{0\}$ and $Z \cap \{0\} = \{0\} \ll_L Z$, but $Z$ is not lifting since $nZ$ no
(3) \( Z_{24} \) as \( Z \)-module is not L-lifting since, Let \( N = Z_{24} \), the only direct summand of \( Z_{24} \) are \( \{0\} \) and \( 3Z_{24} \), such that \( Z_{24} = K \oplus H \). If \( K = \{0\} \) thus \( H = Z_{24} \) and \( N \cap H = Z_{24} \cap Z_{24} = Z_{24} \) which is not L-small in \( Z_{24} \) and if \( K = 3Z_{24} \) thus \( H = 8Z_{24} \) and \( Z_{24} \cap 8Z_{24} = 8Z_{24} \) which is not L-small in \( Z_{24} \) and if \( K = 8Z_{24} \) thus \( H = 3Z_{24} \) and \( \exists Z_{24} \cap 3Z_{24} = 3Z_{24} \) which is not L-small in \( Z_{24} \).

(4) Every semisimple module is lifting [2], hence L-lifting by (1). Thus \( Z_6 \) as \( Z \)-module is L-lifting.

(5) Let \( M \) be a semisimple module, then \( M \) is lifting if and only if \( M \) is L-lifting.

(6) Every hollow module is lifting [2], hence L-lifting by (1). Thus \( Z_4 \) as \( Z \)-module is hollow, so it is L-lifting.

Recall that an \( R \)-module \( M \) is called L-hollow module if every proper submodule of \( M \) is L-small submodule in \( M \) [3].

**Remark 2.3:** Every L-hollow module is L-lifting.

**Proof:** Let \( M \) be an L-hollow module and \( N \) be a proper submodule of \( M \) and let \( M = \{0\} \oplus M \) and \( N \cap M = N \ll_L M \), so that \( M \) is L-lifting.

The converse of previous remark is not true, the following example: \( Z_6 \) as \( Z \)-module is L-lifting by (4) but not L-hollow by [3].

**Remark 2.4:** Every Local module is hollow so L-hollow [3], hence it is L-lifting by Remark(2.3), where an \( R \)-module \( M \) is called local if it is hollow and has a unique maximal submodule [5].

**Proposition 2.5:** Let \( M \) be an indecomposable, then \( M \) is L-hollow if and only if \( M \) is L-lifting.

**Proof:** Clear from Remark (2.3).

\( \Rightarrow \) Let \( M \) be L-lifting and \( N \) be a proper submodule of \( M \) and let \( K \leq N \) such that \( M = K \oplus H \) where \( H \leq M \) and \( N \cap H \ll_L M \), since \( M \) is indecomposable, then either \( K = 0 \) or \( K = M \). If \( K = M \) then \( N = M \) and this is a contradiction, so that \( K = 0 \), and hence \( M = H \), so \( N = N \cap M = N \cap H \ll_L M \) hence \( N \ll_L M \). Therefore \( M \) is L-hollow.

The characterization of L-lifting module is given by the next theorem.

**Theorem 2.6:** Let \( M \) be an \( R \)-module, then the following statements are equivalent:

1- \( M \) is L-lifting module .

2- Every submodule \( N \) of \( M \) can be written as \( N = V \oplus W \) where \( V \) direct summand of \( M \) and \( W \ll_L M \).

3- Every submodule \( N \) of \( M \) there exists a direct summand \( K \) of \( M \) such that \( K \leq N \) and \( \frac{N}{K} \ll_L \frac{M}{K} \).

**Proof:** (1) \( \Rightarrow \) (2) Let \( N \) be a submodule of \( M \) then there exists a submodule \( K \) of \( N \) such that \( M = K \oplus H \) and \( N \cap H \ll_L M \) where \( H \) is a submodule of \( M \). Now \( N = N \cap M = N \cap (K \oplus H) = K \oplus (N \cap H) \) by modular law. Let \( V = K \) and \( W = N \cap H \), so \( N = V \oplus W \) where \( V \) direct summand of \( M \) and \( W \ll_L M \).

(2) \( \Rightarrow \) (3) Let \( N \) be a submodule of \( M \) and \( N = V \oplus W \) where \( V \) direct summand of \( M \) and \( W \ll_L M \). It is enough to show that \( \frac{N}{V} \ll_L \frac{M}{V} \). Let \( \frac{U}{V} \leq \frac{M}{V} \) such that \( \frac{N}{V} + \frac{U}{V} = \frac{M}{V} \) so \( \frac{V}{V} = \frac{M}{V} \), hence \( M = V + W + U = W + U \). Since \( W \ll_L M \), then \( U \leq_L M \), and since \( V \) direct summand of \( M \) then \( V \) is closed in \( M \), from [6-10], we have \( \frac{U}{V} \leq_L \frac{M}{V} \), so that \( \frac{N}{V} \ll_L \frac{M}{V} \).

(3) \( \Rightarrow \) (1) Let \( N \) be a submodule of \( M \) then there exists a submodule \( K \) of \( N \) such that \( M = K \oplus H \) and \( \frac{N}{K} \ll_L \frac{M}{K} \). By Lemma(1.3), we have \( N \ll_L M \) by and since \( N \cap H \leq N \leq M \) so we get \( N \cap H \ll_L M \) by Lemma(1.1).
Proposition 2.7: Let $M$ be an indecomposable module, then $M$ is not $L$-lifting for every nontrivial submodule $N$ of $M$.

**Proof:** Suppose that $M$ is $L$-lifting and by theorem (2.6), let $N = K + H$ where $K$ direct summand of $M$ and $H \ll_L M$, since $M$ be an indecomposable then $K = 0$, hence $N = H \ll_L M$ and this is contradiction, so $M$ is not $L$-lifting for every nontrivial submodule $N$ of $M$.

**Proposition 2.8:** Any direct summand of $L$-lifting module is $L$-lifting.

**Proof:** Let $M$ be $L$-lifting and assume that $M = M_1 \oplus M_2$. In order to show $M_1$ is $L$-lifting, let $N \leq M_1$ so that $N \leq M$ and by theorem (2.6), let $N = V \oplus W$ where $V$ direct summand of $M$ and $W \ll_L M$ hence $W \ll_L M_1$ by Lemma(1.1). Now, $M = V \oplus H$ where $H \leq M$, since $V$ direct summand of $M$, then we get the result if we prove $V$ direct summand of $M_1$, so $M_1 = M_1 \cap M = M_1 \cap (V \oplus H) = V \oplus (M_1 \cap H)$ by modular law, hence $V$ direct summand of $M_1$, so $M_1$ is $L$-lifting.

**Theorem 2.9:** Let $M$ be an $R$-module, then the following statements are equivalent:
1. $M$ is $L$-lifting module.
2. For each submodule $N$ of $M$, there exists $\varphi \in \text{End}(M)$ such that $\varphi^2 = \varphi$, $\varphi(M) \leq N$ and $(1 - \varphi)(N) \ll_L M$.

**Proof:** (1) $\Rightarrow$ (2) Let $N$ be a submodule of $M$ then there exists a submodule $K$ of $N$ such that $M = K \oplus H$ and $N \cap H \ll_L M$ where $H$ be a submodule of $M$. Let $\varphi: M \to K$ be a projection map clearly $\varphi^2 = \varphi$ and $M = K \oplus H = \varphi(M) \oplus (1 - \varphi)(M)$, $\varphi(M) \leq N$. Now $(1 - \varphi)(N) = N \cap (1 - \varphi)(M) = N \cap H \ll_L M$, so $(1 - \varphi)(N) \ll_L M$.

(2) $\Rightarrow$ (1) Let $N$ be a submodule of $M$ then there exists $\varphi \in \text{End}(M)$ such that $\varphi^2 = \varphi$, $\varphi(M) \leq N$ and $(1 - \varphi)(N) \ll_L M$. Clearly that $M = \varphi(M) \oplus (1 - \varphi)(M)$, let $K = \varphi(M)$ and $H = (1 - \varphi)(M)$, hence $N \cap H = N \cap (1 - \varphi)(M)$. To show that $N \cap (1 - \varphi)(M) = (1 - \varphi)(N)$, let $u = (1 - \varphi)(v) \in N \cap (1 - \varphi)(M)$, since $(1 - \varphi)^2 = (1 - \varphi)$ so $u = (1 - \varphi)^2(v) = (1 - \varphi)(v) \in (1 - \varphi)(N)$. Now let $u = (1 - \varphi)(v) \in (1 - \varphi)(N)$; $v \in N$, then $u \in (1 - \varphi)(M)$, $u = (1 - \varphi)(v) \in N$, hence $u \in N \cap (1 - \varphi)(M)$ so $N \cap H = N \cap (1 - \varphi)(M) = (1 - \varphi)(N) \ll_L M$, hence $N \cap H \ll_L M$, so $M$ is $L$-lifting module.

**Remark 2.10:** The following example shows that if $M$ is $L$-lifting module and $N$ is a submodule of $M$, then $N$ need not to be $L$-lifting module.

Example: Let $Z$ be $L$-lifting module and $24Z \leq Z$ but $\frac{Z}{24Z} \approx Z_{24}$ which is not $L$-lifting by (2.2).

Now, we introduce the following proposition in which $\frac{M}{N}$ be $L$-lifting module.

**Proposition 2.11:** Let $M$ be $L$-lifting module and $W$ be a submodule of $M$ such that for every direct summand $K$ of $M$, $\frac{K + W}{W}$ direct summand of $\frac{M}{W}$, then $\frac{M}{W}$ is $L$-lifting.

**Proof:** Let $\frac{N}{W} \leq \frac{M}{W}$, since $M$ is $L$-lifting, then by theorem (2.6), there exists $K \leq N$ such that $M = K \oplus H$; $H \leq M$ and $\frac{N}{K} \ll_L \frac{M}{K}$, because of $K + W$ is direct summand of $M$, we have $\frac{N}{K + W} \ll_L \frac{M}{K + W}$ so $K + W \leq_{Lce} N$ in $M$ and by Lemma(1.3), we get $\frac{K + W}{W} \leq_{Lce} \frac{N}{W}$ in $\frac{M}{W}$, hence $\frac{N/W}{(K+W)/W} \ll_L \frac{M/W}{(K+W)/W}$, therefore $\frac{M}{W}$ is $L$-lifting.

An $R$-module is called distributive, if for all submodules $K$, $N$ and $U$ of $M$, then $K \cap (N + U) = (K \cap N) + (K \cap U)$ [9].

**Corollary 2.12:** Let $M$ be $L$-lifting and distributive module and let $W$ be a submodule of $M$ then $\frac{M}{W}$ is $L$-lifting.

**Proof:** Let $K$ be a direct summand of $M$, such that $M = K \oplus U$ for some submodule $U$ of $M$, hence $\frac{M}{W} = \frac{K \oplus U}{W} = \frac{K + W}{W} + \frac{U + W}{W}$ and since $M$ is distribution module, then $(K + W) \cap
\[(U + W) = ((K + W) \cap U) + ((K + W) \cap W) = (K \cap U) + (W \cap U) + (K \cap W) + W = W, \text{ hence } \frac{M}{W} = \frac{K + W}{W} \oplus \frac{U + W}{W} \text{ and by proposition (2.11), we get } \frac{M}{W} \text{ is L-lifting.}\]

**Lemma 2.13 [6]:** Let \( M = M_1 \bigoplus M_2 \) be an R-module, then \( \frac{M}{A} = \frac{A + M_1}{A} \bigoplus \frac{A + M_2}{A} \) for every fully invariant submodule \( A \) of \( M \).

**Corollary 2.14:** Let \( M \) be L-lifting module if \( W \) is fully invariant submodule of \( M \) then \( \frac{M}{W} \) is L-lifting.

**Proof:** It directly comes from Lemma (2.13) and proposition (2.11).

### 3. Large-Supplemented modules

In this section we introduce the concept of Large-supplemented modules. Some results are also given.

**Definition 3.1:** Let \( M \) be an R-module and \( N, K \) are submodules of \( M \), then \( N \) is called Large-supplement (L-supplement) of \( K \) in \( M \), if \( M = N + K \) and \( N \cap K \ll_k N \). If every submodule of \( M \) has L-supplement, then \( M \) is called L-supplemented module.

**Remarks and Examples 3.2:**

1. Every supplemented module is L-supplemented.
   **Proof:** Let \( M \) be a supplemented and \( N \) be a submodule of \( M \), then \( N \) is a supplement of \( K \) in \( M \), so \( M = N + K \) and \( N \cap K \ll_k N \) by [3], so \( N \) is L-supplement of \( K \) in \( M \), hence \( M \) is L-supplemented.

2. Next example indicates that the converse of (1) is not true.
   **Example:** \( Z \) as \( Z \)-module is L-supplemented since let \( n, m \in N \), \( nZ \) is L-supplement of \( mZ \) since \( Z = nZ + mZ \) and \( nZ \cap mZ = (nm)Z \ll_k nZ \), but \( Z \) is not supplemented since \( nZ \) is not supplement in \( Z \) since \( Z = nZ + mZ \) and \( nZ \cap mZ = (nm)Z \) but \( (nm)Z \) is not small in \( nZ \), since \( \{0\} \) is the only small submodule.

3. Let \( M \) be a semisimple module, then \( M \) is supplemented if and only if, \( M \) is L-supplemented.

4. Next example shows that if \( N \) and \( K \) are submodules of \( M \), and \( N \) is L-supplement of \( K \) in \( M \), then it is not necessary that \( K \) is L-supplement of \( N \) in \( M \).
   **Example:** In \( Z_4 \) as \( Z \)-module, \( Z_4 \) is L-supplement of \( \{0, 2\} \) in \( Z_4 \) since \( Z_4 = Z_4 + \{0, 2\} \) and \( Z_4 \cap \{0, 2\} = \{0, 2\} \ll_k Z_4 \) but \( \{0, 2\} \) is not L-supplement of \( Z_4 \) in \( Z_4 \) since \( Z_4 = \{0, 2\} + Z_4 \) and \( \{0, 2\} \cap Z_4 = \{0, 2\} \) but \( \{0, 2\} \) is not L-small in \( \{0, 2\} \).

5. In \( Z_6 \) as \( Z \)-module where \( Z_6 = \{0, 3\} \bigoplus \{0, \bar{2}, \bar{4}\} \) then \( \{0, 3\} \) is L-supplement of \( \{0, \bar{2}, \bar{4}\} \) since \( Z_6 = \{0, 3\} + \{0, 2, 4\} \) and \( \{0, 3\} \cap \{0, 2, 4\} = \{0\} \ll_k \{0, 3\} \) also \( \{0, 2, 4\} \) is L-supplement of \( \{0, 3\} \).

6. Every semisimple module is L-supplemented.

7. In [2], authors proved that every direct summand of \( M \) is supplement submodule of \( M \), hence it is L-supplement by (1).

8. Let \( M \) be an R-module and \( N \) be L-hollow of \( M \), then \( N \) is L-supplement of each proper submodule \( K \) of \( M \) such that \( M = N + K \).
   **Proof:** Let \( K \) be a proper submodule of \( M \) such that \( M = N + K \). It is clear that \( N \cap K \neq N \), since if \( N \cap K = N \), then \( N \leq K \) hence \( K = M \) and this is a contradiction. Since \( N \) is L-hollow then \( N \cap K \ll_k N \), so \( N \) is L-supplement of \( K \) in \( M \).

9. Let \( M \) be an R-module, then every L-small submodule of \( M \) has L-supplement in \( M \).
   **Proof:** Let \( N \) be L-small submodule of \( M \), so that \( M = N + M \) and \( N \cap M = N \ll_k M \), therefore \( M \) is L-supplement of \( N \) in \( M \).

10. The converse of (9) is not true, for example \( Z_6 \) as \( Z \)-module.

**Proposition 3.3:** Let \( M \) be an R-module and \( N, K \) be submodules of \( M \) such that \( N \leq K \leq M \) and \( N \) is closed in \( K \), if \( K \) is L-supplement of \( H \) in \( M \) then \( \frac{K}{N} \) is L-supplement of \( \frac{H + N}{N} \) in \( \frac{M}{N} \).
   **Proof:** Since \( K \) is L-supplement of \( H \) in \( M \), then we have \( M = K + H \) and \( K \cap H \ll_k K \). Now
\[
\frac{M}{N} = \frac{K+H}{N} = \frac{K}{N} + \frac{H+N}{N}, \text{ we have to show that } \frac{K}{N} \cap \frac{H+N}{N} \ll_{L} \frac{K}{N}, \text{ so that } \frac{K}{N} \cap \frac{H+N}{N} = \frac{K \cap (H+N)}{N} = \frac{K \cap H + N + U}{N} \]

by modular law. Let \( \frac{U}{N} \leq \frac{K}{N} \) where \( U \leq K \) and \( N \leq U \) such that \( \frac{K \cap H + N + U}{N} = \frac{K}{N} \), hence \( (K \cap H) + N + U = K \) and since \( N \leq U \) we have \( (K \cap H) + U = K \), since \( K \cap H \ll_{L} K \) then \( U \leq e \), but \( N \leq U \leq K \) and \( N \) is closed in \( K \). from \([10-15]\), we get \( \frac{U}{N} \leq e \), therefore \( \frac{K}{N} \) is \( L \)-supplement in \( M \).

**Proposition 3.4:** Let \( f: M \to M' \) be an epimorphism, if \( M' \) is \( L \)-supplemented module then \( M \) is \( L \)-supplemented.

**Proof:** Let \( H \leq M \), then \( f(H) \leq M' \), since \( M' \) is \( L \)-supplemented then there exists \( K \) is \( L \)-supplement of \( f(H) \) in \( M' \), so \( M' = K + f(H) \) and \( K \cap f(H) \ll_{L} K \). Now \( f^{-1}(K + f(H)) = f^{-1}(M') \) hence \( f^{-1}(K) + H = M \) and since \( K \cap f(H) \ll_{L} K \) then \( f^{-1}(K \cap f(H)) \ll_{L} f^{-1}(K) \) by Lemma(1.1), hence \( f^{-1}(K) \cap H \ll_{L} f^{-1}(K) \) so, \( f^{-1}(K) \) is \( L \)-supplement of \( H \) in \( M \), hence \( M \) is \( L \)-supplemented.

**Proposition 3.5:** Let \( M \) be an \( R \)-module and \( N, K \) are submodules of \( M \) such that \( K \) is \( L \)-supplement of \( N \) in \( M \), if \( M = H + K \) for some submodule \( H \) of \( N \), then \( K \) is \( L \)-supplement of \( H \) in \( M \).

**Proof:** Suppose \( M = H + K \) for some submodule \( H \) of \( N \) and \( K \) is \( L \)-supplement of \( N \) in \( M \), so we have \( M = N + K \) and \( N \cap K \ll_{L} K \), and since \( H \cap K \leq N \cap K \ll_{L} K \), then \( H \cap K \ll_{L} K \) by Lemma(1.1), hence \( K \) is \( L \)-supplement of \( H \) in \( M \).

**Proposition 3.6:** Let \( M \) be an \( R \)-module and \( N, K \) and \( U \) are submodules of \( M \) such that \( N \leq K \), if \( N \) is \( L \)-supplement of \( U \) in \( M \) then \( N \) is \( L \)-supplement of \( U \cap K \) in \( K \).

**Proof:** Since \( N \) is \( L \)-supplement of \( U \) in \( M \) then we have \( M = N + U \) and \( N \cap U \ll_{L} N \). Now \( K = M \cap K = (N + U) \cap K = N + (U \cap K) \) by modular law, and since \( N \cap (U \cap K) \leq N \cap U \ll_{L} N \), so we get \( N \cap (U \cap K) \ll_{L} N \) by Lemma(1.1), hence \( N \) is \( L \)-supplement of \( U \cap K \) in \( K \).

**Proposition 3.7:** Let \( M = M_1 \oplus M_2 \), if \( N_1 \) is \( L \)-supplement of \( N_2 \) in \( M_1 \) and \( K_1 \) is \( L \)-supplement of \( K_2 \) in \( M_2 \), then \( N_1 \oplus K_1 \) is \( L \)-supplement of \( N_2 \oplus K_2 \) in \( M \).

**Proof:** Since \( N_1 \) is \( L \)-supplement of \( N_2 \) in \( M_1 \) and \( K_1 \) is \( L \)-supplement of \( K_2 \) in \( M_2 \), then we have \( M_1 = N_1 + N_2 \) and \( N_1 \cap N_2 \ll_{L} N_1 \). Also we have \( M_2 = K_1 + K_2 \) and \( K_1 \cap K_2 \ll_{L} K_1 \), so \( M = M_1 \oplus M_2 = (N_1 + N_2) \oplus (K_1 + K_2) = (N_1 \oplus K_1) + (N_2 \oplus K_2) \). Since \( N_1 \cap N_2 \ll_{L} N_1 \) and \( K_1 \cap K_2 \ll_{L} K_1 \), then by Lemma(1.4), we have \( (N_1 \cap N_2) \oplus (K_1 \cap K_2) \ll_{L} N_1 \oplus K_1 \). Clearly \((N_1 \oplus K_1) \cap (N_2 \oplus K_2) = (N_1 \cap N_2) \oplus (K_1 \cap K_2) \ll_{L} N_1 \oplus K_1 \), hence \( N_1 \oplus K_1 \) is \( L \)-supplement of \( N_2 \oplus K_2 \) in \( M \).

**Proposition 3.8:** Let \( M \) be faithful, finitely generated and multiplication module over commutative ring \( R \) and \( N \) be a submodule of \( M \), if \( N \) is \( L \)-supplement of \( IM \) in \( M \), then \( J \) is \( L \)-supplement of \( I \) in \( R \), where \( I \), \( J \) are ideals of \( R \).

**Proof:** Since \( N \) is \( L \)-supplement of \( IM \) in \( M \), then we have \( M = N + IM \) and \( N \cap IM \ll_{L} N \), since \( M \) is multiplication then \( N = JM \). Now \( M = RM = IM + JM = (I + J)M \), and since \( M \) is faithful, finitely generated and multiplication, then \( M \) is cancellation by \([8]\), so \( R = I + J \) also we have \( IM \cap N = IM \cap JM = (I \cap J)M \ll_{L} N = JM \), hence \( I \cap J \) is \( L \)-supplement of \( J \) in \( R \). To show \( I \cap J \ll_{L} J \), let \( H \) be an ideal of \( R \) such that \((I \cap J) + H = J \), so \((I \cap J)M + HM = JM \) and since \((I \cap J)M \ll_{L} IM \), then \( HM \leq e JM \) so \( H \leq e J \) so we get the result, and hence \( J \) is \( L \)-supplement of \( I \) in \( R \).

The characterization of \( L \)-supplement modules is given in the next theorem.

**Theorem 3.9:** Let \( M \) be an \( R \)-module and \( N, K \) are submodules of \( M \), then the following statements are equivalent:
1. \( K \) is \( L \)-supplement of \( N \) in \( M \).
2. \( M = N + K \) and for every non-essential submodule \( H \) of \( K \), then \( M \neq N + H \).
Proof: (1) $\Rightarrow$ (2) Assume $K$ is $L$-supplement of $N$ in $M$, so we have $M = N + K$ and $N \cap K \ll L K$ and suppose $M = N + H$ where $H$ is non-essential submodule of $K$, so $K = K \cap M = K \cap (N + H) = H + (N \cap K)$ by modular law, and since $N \cap K \ll L K$ so we have $H \leq e K$ and this is a contradiction, so that $M \neq N + H$.

(2) $\Rightarrow$ (1) From (2) $M = N + K$, we must show $N \cap K \ll L K$. Let $U \leq K$ such that $(N \cap K) + U = K$, if $U$ is non-essential submodule of $K$, then by assumption $M \neq N + U$, so $M = N + K = N + (N \cap K) + U = N + U$ and this is a contradiction, so that $U \leq e K$, hence $N \cap K \ll L K$, and we get $K$ is $L$-supplement of $N$ in $M$.

**Proposition 3.10:** Let $M$ be an $R$-module and $M_1$, $H$ are submodules of $M$, such that $M_1$ is $L$-supplemented module, if $M_1 + H$ has $L$-supplement in $M$ then $H$ has $L$-supplement in $M$.

**Proof:** By assumption $M_1 + H$ has $L$-supplement in $M$, so there exists $U \leq M$ such that $M_1 + H + U = M$ and $(M_1 + H) \cap U \ll L U$, since $M_1$ is $L$-supplemented then $(H + U) \cap M_1 \leq M_2$ has $L$-supplement in $M_1$, so there exists $V \leq M_1$ such that $(H + U) \cap M_1 \leq V$ and $(H + U) \cap V \ll L V$. Now $M = M_1 + H + U = (H + U) \cap M_1 + V + H + U = H + (V + U)$. One can easily show $H \cap (V + U) \leq (H + U) \cap V \ll L U$ so we have $H \cap (V + U) \ll L U + V$ and $V + U$ is $L$-supplement of $H$ in $M$, hence $H$ has $L$-supplement in $M$.

**Proposition 3.11:** Let $M = M_1 \oplus M_2$ such that $M_1$ and $M_2$ are $L$-supplemented modules then $M$ is $L$-supplemented module.

**Proof:** Let $H \leq M$ and since $M_1 + M_2 + H = M$, so it is trivial has $L$-supplement in $M$. By proposition (3.10) and since $M_1$ is $L$-supplemented, then $M_2 + H$ has $L$-supplement in $M$, again by proposition (3.10) and since $M_2$ is $L$-supplemented, then $H$ has $L$-supplement in $M$, and hence $M$ is $L$-supplemented module.

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