On two conjectural series for $\pi$ and their $q$-analogues

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Abstract. In terms of the operator method, we prove two conjectural series for $\pi$ of Sun involving harmonic numbers of order two. Furthermore, we also give $q$-analogues of six $\pi$-formulas including the two ones just mentioned.

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1 Introduction

For a complex variable $x$, define the well-known Gamma function to be

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \text{ with } Re(x) > 0.$$ 

Three important properties of it can be stated as follows:

$$\Gamma(x+1) = x\Gamma(x), \quad \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}, \quad \lim_{n \to \infty} \frac{\Gamma(x+n)}{\Gamma(y+n)} \frac{n^{y-x}}{n!} = 1,$$

which will be used directly in this paper. For a nonnegative integer $n$, define the shifted-factorial as

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}.$$ 

Then we can provide the definition of the hypergeometric series

$$\,_{r}F_{s}\left[\begin{array}{c}a_{1}, a_{2}, \ldots, a_{r} \\ b_{1}, b_{2}, \ldots, b_{s}\end{array}; z\right] = \sum_{k=0}^{\infty} \frac{(a_{1})_{k}(a_{2})_{k} \cdots (a_{r})_{k} z^{k}}{(b_{1})_{k}(b_{2})_{k} \cdots (b_{s})_{k} k!}.$$ 

In 1914, Ramanujan [20] listed 17 series for $1/\pi$ without proof. Decades later, Borweins [3] proved all of them firstly. Three of Ramanujan’s formulas are expressed as

$$\sum_{k=0}^{\infty} (6k+1) \frac{(1/2)_k^3}{k!^3 4^k} = \frac{4}{\pi}, \quad (1.1)$$

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\[
\sum_{k=0}^{\infty} (8k + 1) \frac{\binom{1}{k} \binom{1}{k} \binom{3}{k}}{k!^3 9^k} = \frac{2\sqrt{3}}{\pi},
\]
\[
\sum_{k=0}^{\infty} (42k + 5) \frac{\binom{3}{k}^2}{k!^3 64^k} = \frac{16}{\pi}.
\]

There are a lot of different \(\pi\)-formulas in the literature. Two of them (cf. [27, Equation (23)] and [11, P. 221]) read

\[
\sum_{k=0}^{\infty} \frac{k!}{(2k+1)!!} = \frac{\pi}{2},
\]
\[
\sum_{k=0}^{\infty} (3k + 2) \frac{\binom{3}{k}^2}{\binom{4}{k} 4^k} = \frac{\pi^2}{4},
\]

where the double factorial has been defined by

\[
(1 + 2k)!! = \frac{(2k + 1)!}{2^k k!}.
\]

In 2021, Guo and Lian [14] conjectured the interesting double series for \(\pi\) related to (1.1):

\[
\sum_{k=0}^{\infty} (6k + 1) \frac{\binom{3}{k}^3}{k!^3 4^k} \sum_{j=1}^{k} \left\{ \frac{1}{(2j - 1)^2} - \frac{1}{16 j^2} \right\} = \frac{\pi}{12},
\]

which has been proved by the author [25]. Moreover, the author and Ruan [26] discovered the following double series for \(\pi\) associated with (1.2):

\[
\sum_{k=0}^{\infty} (8k + 1) \frac{\binom{1}{k} \binom{1}{k} \binom{3}{k}}{k!^3 9^k} \sum_{i=1}^{k} \left\{ \frac{1}{(2i - 1)^2} - \frac{1}{36 i^2} \right\} = \frac{\sqrt{3} \pi}{54}.
\]

For more known series on \(\pi\), we refer the reader to the papers [2, 4, 12, 19, 24, 28].

For a complex variable \(x\) and two positive integers \(\ell, n\), define the generalized harmonic number of order \(\ell\) to be

\[
H_{n}^{(\ell)}(x) = \sum_{k=1}^{n} \frac{1}{(x + k)^\ell}.
\]

When \(x = 0\), it becomes the harmonic number of order \(\ell\):

\[
H_{n}^{(\ell)} = \sum_{k=1}^{n} \frac{1}{k^\ell}.
\]
Taking $\ell = 1$ in $H_n^{(\ell)}(x)$, we have the generalized harmonic number:

$$H_n(x) = \sum_{k=1}^{n} \frac{1}{x + k}.$$  

The $x = 0$ case of it is the classical harmonic number:

$$H_n = \sum_{k=1}^{n} \frac{1}{k}.$$  

In 2015, Sun [21] proved a nice series for $\pi^3$ containing harmonic number of order two related to (1.4):

$$\sum_{k=0}^{\infty} \frac{k!}{(2k + 1)!!} H_k^{(2)} = \frac{\pi^3}{48}. \quad (1.8)$$  

In a recent paper [23], he rewrote (1.6) and (1.7) as

$$\sum_{k=0}^{\infty} (6k + 1) \left( \frac{1}{2} \right)^{\frac{3}{2}} \frac{k!}{k! 3^{k} 4k} \left\{ H_{2k}^{(2)} - \frac{5}{16} H_{k}^{(2)} \right\} = \frac{\pi}{12},$$

$$\sum_{k=0}^{\infty} (8k + 1) \left( \frac{1}{2} \right)^{\frac{3}{2}} \frac{k!}{k! 9^{k} 12k} \left\{ H_{2k}^{(2)} - \frac{5}{18} H_{k}^{(2)} \right\} = \frac{\sqrt{3} \pi}{54},$$

and proposed the following two conjectures associated with (1.5) and (1.3) (cf. [23 Equations (3.67) and (3.13)]).

**Theorem 1.1.**

$$\sum_{k=0}^{\infty} (3k + 2) \left( \frac{1}{2} \right)^{\frac{3}{2}} \frac{k!}{k! 3^{k} 4k} \left\{ H_{2k+1}^{(2)} - \frac{5}{4} H_{k}^{(2)} \right\} = \frac{\pi^4}{48}. \quad (1.9)$$

**Theorem 1.2.**

$$\sum_{k=0}^{\infty} (42k + 5) \left( \frac{1}{2} \right)^{\frac{3}{2}} \frac{k!}{k! 6^{k} 64k} \left\{ H_{2k}^{(2)} - \frac{25}{92} H_{k}^{(2)} \right\} = \frac{2\pi}{69}. \quad (1.10)$$

For an integer $n$ and two complex numbers $x, q$ with $|q| < 1$, define the $q$-shifted factorial as

$$(x; q)_n = \prod_{i=0}^{n-1} (1 - xq^i), \quad (x; q)_\infty = \frac{(x; q)_n}{(xq^n; q)_\infty}.$$

For convenience, we sometimes utilize the compact notation:

$$(x_1, x_2, \ldots, x_r; q)_m = (x_1; q)_m (x_2; q)_m \cdots (x_r; q)_m.$$
where \( r \in \mathbb{Z}^+ \) and \( m \in \mathbb{Z}^+ \cup \{0, \infty\} \). Then following Gasper and Rahman \([10]\), the basic hypergeometric series can be defined by

\[

r \phi_s \left[ \frac{a_1, a_2, \ldots, a_r}{b_1, b_2, \ldots, b_s} ; q, z \right] = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r ; q)_k}{(b_1, b_2, \ldots, b_s ; q)_k} (-1)^k q^k z^k.

\]

Let \([n] = 1 + q + \cdots + q^{n-1}\) be the \(q\)-integer. Recently, Guo and Liu \([15]\) and Guo and Zudilin \([16]\) obtained the following \(q\)-analogues of (1.1) and (1.2):

\[

\sum_{k=0}^{\infty} q^{k^2}[6k + 1] \frac{(q; q^2)_k^2(q^2; q^4)_k}{(q^4; q^4)_k^3} = \frac{(1 + q)(q^2, q^6; q^4)_{\infty}}{(q^4; q^4)_{\infty}^2},

\sum_{k=0}^{\infty} q^{2k^2}[8k + 1] \frac{(q; q^2)_k^2(q; q^2)_{2k}}{(q^2; q^2)_{2k}(q^6; q^6)_k^2} = \frac{(q^3, q^3; q^6)_{\infty}}{(q^2; q^2)_{\infty}(q^6; q^6)_{\infty}}.

\]

The author \([25]\) and the author and Ruan \([26]\) got the following \(q\)-analogues of (1.6) and (1.7):

\[

\sum_{k=0}^{\infty} q^{k^2}[6k + 1] \frac{(q; q^2)_k^2(q^2; q^4)_k}{(q^4; q^4)_k^3} \sum_{j=1}^{k} \left\{ \frac{q^{2j^2}}{[2j - 1]^2} - \frac{q^{4j}}{[4j]^2} \right\} \]

\[

= \frac{(q^2; q^4)_{\infty}^2(q^3; q^4)_{\infty}}{(q; q^4)_{\infty}(q^4; q^4)_{\infty}^2} \sum_{i=1}^{\infty} (-1)^{i-1} \frac{q^{2i}}{[2i]^2},

\sum_{k=0}^{\infty} q^{2k^2}[8k + 1] \frac{(q; q^2)_k^2(q; q^2)_{2k}}{(q^2; q^2)_{2k}(q^6; q^6)_k^2} \sum_{j=1}^{k} \left\{ \frac{q^{2j-1}}{[2j - 1]^2} - \frac{q^{6j}}{[6j]^2} \right\} \]

\[

= \frac{(q^3, q^3; q^6)_{\infty}}{(q^2; q^2)_{\infty}(q^6; q^6)_{\infty}} \sum_{j=1}^{\infty} (-1)^{j-1} \frac{q^{2j}}{[3j]^2}.

\]

More \(q\)-analogues of \(\pi\)-formulas can be seen in the papers \([13, 17, 18, 22]\).

Inspired by the works just mentioned, we shall establish \(q\)-analogues of (1.4), (1.5), and (1.3) in the following theorem.

**Theorem 1.3.**

\[

\sum_{k=0}^{\infty} q^{k+1} \frac{(q; q)_k}{(q^3; q^2)_k} = \frac{(q^2; q^2)_{\infty}^2}{(q; q^2)_{\infty}},

\sum_{k=0}^{\infty} q^{k+1} [3k + 2] \frac{(q; q)_k^2(q^2; q^2)_k}{(q^3; q^2)^3_k} = \frac{(q^2; q^2)_{\infty}^4}{(q^3; q^2)_{\infty}^2(q^3; q^2)_{\infty}},

\sum_{k=0}^{\infty} q^{6k^2} \frac{(q; q^2)^3_k}{(q^2; q^2)^3_k} (1 + q^{1+2k})^3 (1 - q^{1+6k}) - q^{1+6k} (1 - q^{3+6k}) = \frac{(q^3, q^5; q^2)_{\infty}}{(q^4; q^2)_{\infty}}.

\]
Further, we shall furnish $q$-analogues of (1.8)-(1.10) in the following three theorems.

Theorem 1.4.

$$
\sum_{k=0}^{\infty} q^{(k+1)} \frac{(q; q)_k}{(q^3; q^2)_k} \sum_{i=1}^{k} \frac{q^i}{[i]^2} = \frac{(q^2; q^2)_\infty}{(q, q^3; q^2)_\infty} \sum_{j=1}^{\infty} \frac{q^{2j}}{[2j]^2} \tag{1.14}
$$

Theorem 1.5.

$$
\sum_{k=0}^{\infty} q^{(k+1)} [3k + 2] \frac{(q; q)_k^2 (q^2; q^2)_k}{(q^3; q^2)_k^3} \left\{ \sum_{i=1}^{k} \frac{q^i}{[i]^2} - \sum_{i=1}^{k+1} \frac{q^{2i-1}}{[2i-1]^2} \right\} = \frac{(q^2; q^2)_\infty^2}{(q, q^3; q^2)_\infty^2} \sum_{j=1}^{\infty} (-1)^j \frac{q^j}{[j]^2} \tag{1.15}
$$

Theorem 1.6.

$$
\sum_{k=0}^{\infty} q^{6k^2} \frac{(q; q)_k^6}{(q^2; q^2)_k^3} \left\{ \lambda_q(k) \sum_{i=1}^{2k} \frac{q^{2i}}{[2i]^2} - \mu_q(k) \sum_{i=1}^{k} \frac{q^{2i-1}}{[2i-1]^2} - \nu_q(k)(1 - q)q^{1+6k} \right\} = \frac{(q, q^3; q^2)_\infty^3}{(q^2; q^2)_\infty^3} \left\{ \sum_{j=1}^{\infty} \frac{q^{2j}}{[2j]^2} - \frac{3(1 + q)^3}{64} \sum_{j=1}^{\infty} \frac{q^{2j-1}}{[2j-1]^2} \right\}, \tag{1.16}
$$

where

$$
\lambda_q(k) = \frac{1 + 2q^{1+2k} - q^{1+6k}(2 + 2q^2 + q^{1+2k} + q^{3+2k} - 3q^{3+6k})}{(1 - q)(1 - q^{1+2k})(1 + q^{1+2k})^3},
$$

$$
\mu_q(k) = \frac{1 + 3q^{1+2k} + 3q^{2+4k} - 2q^{1+6k} + q^{3+6k} - 3q^{2+8k} - 3q^{3+10k}}{64(1 - q)(1 + q^{1+2k})^3(137q + 27q^2 + 9q^3)},
$$

$$
\nu_q(k) = \frac{3(1 + q)^3(1 + 2q^{1+2k} + 3q^{2+4k})}{64(1 - q^{1+2k})(1 + q^{1+2k})^3} - \frac{q^{1+2k}(1 + q^{1+2k} + q^{2+4k})^2}{(1 - q^{1+2k})(1 + q^{1+2k})^3}. 
$$

For a multivariable function $f(x_1, x_2, \ldots, x_m)$, define the partial derivative operator $D_x$ by

$$
D_x f(x_1, x_2, \ldots, x_m) = \frac{d}{dx_i} f(x_1, x_2, \ldots, x_m) \quad \text{with} \quad 1 \leq i \leq m.
$$

Then there are the following two relations:

$$
D_x(x + y)_n = (x + y)_n H_n(x + y - 1),
$$

$$
D_x(xy; q)_n = -(xy; q)_n \sum_{i=1}^{n} yq^{i-1} \frac{1}{1 - x y q^{i-1}}.
$$

The rest of the paper is arranged as follows. We shall verify Theorems 1.1 and 1.2 via the partial derivative operator and some summation and transformation formulas for hypergeometric series in Section 2. Theorems 1.3-1.6 will be certified through the partial derivative operator and several summation and transformation formulas for basic hypergeometric series in Section 3.
2 Proof of Theorems 1.1 and 1.2

Above all, we shall prove Theorem 1.1.

Proof of Theorem 1.1. In order to achieve the goal, we need the summation formula for hypergeometric series due to Gosper (1977)(cf. [5] Equation (5.1e)):

\[
\begin{aligned}
7F_6 \left[ a - \frac{1}{2}, \frac{2a+3}{3}, 2b-1, 2c-1, 1 + a - c, b + c - \frac{1}{2}, 2a + 2n, -2n \right] ; 1 \\
= \frac{(\frac{1}{2} + a)n(b)n(c)n(a - b - c + \frac{3}{2})n}{(\frac{1}{2})n(1 + a - b)n(1 + a - c)n(b + c - \frac{1}{2})n}.
\end{aligned}
\]  

Apply the operator $\mathcal{D}_b$ on both sides of the $c = 2 - b$ case of (2.1) to obtain

\[
\begin{aligned}
\sum_{k=0}^{n} \frac{(a - \frac{1}{2})_k(\frac{2a+2}{3})_k(2a - 2)_k(2b - 1)_k(3 - 2b)_k(a + n)_k(-n)_k}{(1)_k(\frac{2a-1}{3})_k(\frac{2}{3})_k(1 + a - b)_k(a + b - 1)_k(2a + 2n)_k(-2n)_k} \\
\times \left\{ 2H_k(2b - 2) - 2H_k(2 - 2b) + H_k(a - b) - H_k(a + b - 2) \right\} \\
\times \frac{(a + \frac{1}{2})_n(a - \frac{1}{2})_n(b)_n(2 - b)_n}{(\frac{1}{2})_n(\frac{3}{2})_n(1 + a - b)_n(a + b - 1)_n} \\
\times \left\{ H_n(b - 1) - H_n(1 - b) + H_n(a - b) - H_n(a + b - 2) \right\}.
\end{aligned}
\]  

Employing the operator $\mathcal{D}_b$ on both sides of it, we have

\[
\begin{aligned}
\sum_{k=0}^{n} \frac{(a - \frac{1}{2})_k(\frac{2a+2}{3})_k(2a - 2)_k(2b - 1)_k(3 - 2b)_k(a + n)_k(-n)_k}{(1)_k(\frac{2a-1}{3})_k(\frac{2}{3})_k(1 + a - b)_k(a + b - 1)_k(2a + 2n)_k(-2n)_k} \\
\times \left\{ \left[ 2H_k(2b - 2) - 2H_k(2 - 2b) + H_k(a - b) - H_k(a + b - 2) \right]^2 \\
- 4H_k^{(2)}(2b - 2) + 4H_k^{(2)}(2 - 2b) - H_k^{(2)}(a - b) - H_k^{(2)}(a + b - 2) \right\} \\
\times \frac{(a + \frac{1}{2})_n(a - \frac{1}{2})_n(b)_n(2 - b)_n}{(\frac{1}{2})_n(\frac{3}{2})_n(1 + a - b)_n(a + b - 1)_n} \\
\times \left\{ H_n(b - 1) - H_n(1 - b) + H_n(a - b) - H_n(a + b - 2) \right\}^2 \\
- \left[ H_n^{(2)}(b - 1) + H_n^{(2)}(1 - b) - H_n^{(2)}(a - b) - H_n^{(2)}(a + b - 2) \right] \right\}.
\end{aligned}
\]  

The $(a, b) = (\frac{3}{2}, 1)$ case of (2.2) engenders

\[
\begin{aligned}
\sum_{k=0}^{n} \frac{(\frac{5}{2})_k(1)_k(\frac{3}{2} + n)_k(-n)_k}{(\frac{3}{2})_k(\frac{2}{3})_k(3 + 2n)_k(-2n)_k} \left\{ 4H_{2k+1}^{(2)} - 5H_k^{(2)} - 4 \right\} \\
= \frac{\Gamma(2 + n)\Gamma(1 + n)^3 \Gamma(\frac{3}{2})\Gamma(\frac{3}{2})^3}{\Gamma(\frac{1}{2} + n)\Gamma(\frac{3}{2} + n)^3 \Gamma(2)\Gamma(1)^3} \left\{ 4H_{2n+1}^{(2)} - 2H_n^{(2)} - 4 \right\}.
\end{aligned}
\]
Since that the \((a, b, c) = (\frac{2}{3}, 1, 1)\) case of (2.1) reads
\[
\sum_{k=0}^{n} \frac{(\frac{2}{3})_k (\frac{3}{2} + n)_k (-n)_k}{(\frac{3}{2})_k (\frac{3}{2})_k (3 + 2n)_k (-2n)_k} = \frac{\Gamma(2 + n)\Gamma(1 + n)^3 \Gamma(\frac{1}{2})\Gamma(\frac{3}{2})^3}{\Gamma(\frac{1}{2} + n)\Gamma(\frac{3}{2} + n)^3 \Gamma(2)\Gamma(1)^3},
\]
(2.4)
the linear combination of (2.3) and (2.4) gives
\[
\sum_{k=0}^{n} \frac{(\frac{2}{3})_k (\frac{3}{2} + n)_k (-n)_k}{(\frac{3}{2})_k (\frac{3}{2})_k (3 + 2n)_k (-2n)_k} \left\{ 4H_{2k+1}^{(2)} - 5H_k^{(2)} \right\} = \frac{\Gamma(2 + n)\Gamma(1 + n)^3 \Gamma(\frac{1}{2})\Gamma(\frac{3}{2})^3}{\Gamma(\frac{1}{2} + n)\Gamma(\frac{3}{2} + n)^3 \Gamma(2)\Gamma(1)^3} \left\{ 4H_{2n+1}^{(2)} - 2H_n^{(2)} \right\}.
\]
Letting \((x, n) \to (\frac{1}{2}, \infty)\) and making use of Euer’s formula:
\[
\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6},
\]
(2.5)
we catch hold of (1.9).

Subsequently, we shall display the proof of Theorem 1.2.

**Proof of Theorem 1.2** Recall a transformation formula for hypergeometric series (cf. [9 Theorem 31]):
\[
\sum_{k=0}^{\infty} (-1)^k \frac{(b)_k (c)_k (d)_k (e)_k}{(1 + a - b - c)_k (1 + a - b - d)_k (1 + e)_k} \frac{(1 + a - c - d)_k (1 + a - c - e)_k (1 + a - d - e)_k}{(1 + 2a - b - c - d - e)_k} \sigma_k(a, b, c, d, e)
\]
\[
= \sum_{k=0}^{\infty} \frac{(a + 2k)(b)_k (c)_k (d)_k (e)_k}{(1 + a - b)_k (1 + a - c)_k (1 + a - d)_k (1 + a - e)_k},
\]
where
\[
\sigma_k(a, b, c, d, e)
\]
\[
= \frac{(1 + 2a - b - c - d + 3k)(a - e + 2k)}{(1 + 2a - b - c - d - e + 2k)} + \frac{(e + k)(1 + a - b - c + k)}{(1 + a - b + 2k)(1 + a - d + 2k)}
\]
\[
\times \frac{(1 + a - b - d + k)(1 + a - c - d + k)(2 + 2a - b - d - e + 3k)}{(1 + 2a - b - c - d - e + 2k)(2 + 2a - b - c - d - e + 2k)}
\]
\[
+ \frac{(c + k)(e + k)(1 + a - b - c + k)(1 + a - b - d + k)}{(1 + a - b + 2k)(1 + a - c + 2k)(1 + a - d + 2k)(1 + a - e + 2k)}
\]
\[
\times \frac{(1 + a - b - e + k)(1 + a - c - d + k)(1 + a - d - e + k)}{(1 + 2a - b - c - d - e + 2k)(2 + 2a - b - c - d - e + 2k)}.
\]
Choosing \((a, b, c, d, e) = (\frac{1}{2}, \frac{1}{2}, x, 1 - x, -n)\) in the last equation and calculating the series on the right-hand side by Dougall’s \(5F_4\) summation formula (cf. [1, P. 71]):

\[
5F_4 \left[ \frac{a, 1 + \frac{a}{2}, b, c, -n}{\frac{a}{2}, 1 + a - b, 1 + a - c, 1 + a + n + 1} \right] = \frac{(1 + a)_n (1 + a - b - c)_n}{(1 + a - b)_n (1 + a - c)_n},
\]

we arrive at

\[
\sum_{k=0}^{n} (-1)^k \frac{\left(\frac{1}{2}\right)_k (1 + n)_k (-n)_k}{(1)_2k(\frac{1}{2} + n)_{2k}(\frac{3}{2} + n)_{2k}} \frac{(x)_{2k}^2 (1 - x)_{2k} (\frac{1}{2} + x + n)_k (\frac{3}{2} - x + n)_k}{(\frac{1}{2} + x)_{2k} (\frac{3}{2} - x)_{2k}}
\times \Omega_k(x; n) = \frac{(\frac{1}{2})_n (\frac{3}{2})_n}{(\frac{1}{2} + x)_n (\frac{3}{2} - x)_n},
\]

where

\[
\Omega_k(x; n) = (1 + 6k) + \frac{4(x + k)(1 - x + k)(3 + 2x + 2n + 6k)(k - n)}{(1 + 2x + 4k)(1 + 2n + 4k)(3 + 2n + 4k)} + \frac{16(x + k)^2(1 - x + k)(1 + 2x + 2n + 2k)(1 + n + k)(k - n)}{(1 + 2x + 4k)(3 - 2x + 4k)(1 + 2n + 4k)(3 + 2n + 4k)^2}.
\]

Apply the operator \(D_x\) on both sides of (2.6) to get

\[
\sum_{k=0}^{n} (-1)^k \frac{\left(\frac{1}{2}\right)_k (1 + n)_k (-n)_k}{(1)_2k(\frac{1}{2} + n)_{2k}(\frac{3}{2} + n)_{2k}} \frac{(x)_{2k}^2 (1 - x)_{2k} (\frac{1}{2} + x + n)_k (\frac{3}{2} - x + n)_k}{(\frac{1}{2} + x)_{2k} (\frac{3}{2} - x)_{2k}}
\times \left\{ 2H_k(x - 1) - 2H_k(-x) - H_{2k}(x - \frac{1}{2}) + H_{2k}(\frac{1}{2} - x) + H_k(x - \frac{1}{2} + n) - H_k(\frac{1}{2} - x + n) \right\} \Omega_k(x; n)
\]

\[
+ \sum_{k=0}^{n} (-1)^k \frac{\left(\frac{1}{2}\right)_k (1 + n)_k (-n)_k}{(1)_2k(\frac{1}{2} + n)_{2k}(\frac{3}{2} + n)_{2k}} \frac{(x)_{2k}^2 (1 - x)_{2k} (\frac{1}{2} + x + n)_k (\frac{3}{2} - x + n)_k}{(\frac{1}{2} + x)_{2k} (\frac{3}{2} - x)_{2k}}
\times D_x \Omega_k(x; n) = \frac{(\frac{1}{2})_n (\frac{3}{2})_n}{(\frac{1}{2} + x)_n (\frac{3}{2} - x)_n} \left\{ H_n(\frac{1}{2} - x) - H_n(x - \frac{1}{2}) \right\}.
\]

Dividing both sides by \(1 - 2x\) and utilizing the relation

\[
\frac{1}{v - u - 2x} \left\{ H_m(x + u) - H_m(v - x) \right\} = \sum_{i=1}^{m} \frac{1}{(x + u + i)(v - x + i)},
\]

Equation (2.7) can be manipulated as
\[
\sum_{k=0}^{n} (-1)^k \frac{\left(\frac{1}{2}\right)_k^2 (1+n)_k (-n)_k}{(1)_{2k} \left(\frac{1}{2} + n\right)_{2k} \left(\frac{3}{2} + n\right)_{2k}} \frac{(x)_{k}^2 (1-x)_{k}^2 \left(\frac{1}{2} + x + n\right)_k \left(\frac{3}{2} - x + n\right)_k}{(\frac{1}{2} + x)_{2k} \left(\frac{3}{2} - x\right)_{2k}} \times \left\{ 2 \sum_{i=1}^{k} \frac{1}{(x - 1 + i)(-x + i)} - \sum_{i=1}^{2k} \frac{1}{(x - \frac{1}{2} + i)(\frac{1}{2} - x + i)} + \sum_{i=1}^{k} \frac{1}{(x - \frac{1}{2} + n + i)(\frac{1}{2} - x + n + i)} \right\} \Omega_k(x; n)
\]

\[
+ \sum_{k=0}^{n} (-1)^k \frac{\left(\frac{1}{2}\right)_k^2 (1+n)_k (-n)_k}{(1)_{2k} \left(\frac{1}{2} + n\right)_{2k} \left(\frac{3}{2} + n\right)_{2k}} \frac{(x)_{k}^2 (1-x)_{k}^2 \left(\frac{1}{2} + x + n\right)_k \left(\frac{3}{2} - x + n\right)_k}{(\frac{1}{2} + x)_{2k} \left(\frac{3}{2} - x\right)_{2k}} \times \mathcal{D}_x \Omega_k(x; n) = \frac{1}{1 - 2x}
\]

Letting \((x, n) \to \left(\frac{1}{2}, \infty\right)\) and drawing upon Euler’s formula (2.5), there is
\[
\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^3}{k! 3^{64k}} \left\{ (42k + 5) \left[ 2\mathcal{H}_k^{(2)} - \mathcal{H}_{2k}^{(2)} \right] + \frac{9}{1 + 2k} \right\} = \frac{8\pi}{3}.
\] (2.9)

Recollect a summation formula for hypergeometric series (cf. [7, Corollary 2.33]):
\[
\sum_{k=0}^{\infty} \frac{(x)_{k}^3 (1-x)_{k}^3 k(1+3k)(3+9k+7k^2) + x(1-x)(1+6k+6k^2 + x - x^2)}{64^k}
\]

\[
= \frac{\sin(\pi x)}{\pi},
\] (2.10)

where we have replaced \(\sin(\pi x)/\pi x\) by \(\sin(\pi x)/\pi\) for correction. When \(0 < x < 1\), it is obvious that the series on the left-hand side is uniformly convergent. Employing the operator \(\mathcal{D}_x\) on both sides of (2.10) and taking advantage of (2.8), there holds
\[
3 \sum_{k=0}^{\infty} \frac{(x)_{k}^3 (1-x)_{k}^3 k(1+3k)(3+9k+7k^2) + x(1-x)(1+6k+6k^2 + x - x^2)}{64^k}
\]
\[
\times \frac{1}{(x - 1 + i)(-x + i)} \sum_{i=1}^{k} \frac{1}{(x - \frac{1}{2} + n + i)(\frac{1}{2} - x + n + i)}
\]
\[
+ \sum_{k=0}^{\infty} \frac{(x)_{k}^3 (1-x)_{k}^3 1 + 6k + 6k^2 + 2x(1-x)}{64^k} = \frac{\cos(\pi x)}{1 - 2x}.
\]

9
The $x \to \frac{1}{2}$ case of the upper identity provides

$$
\sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k^3}{k!^3 6^{4k}} \left\{ (42k + 5) \left[ 4H_{2k}^{(2)} - H_k^{(2)} \right] + \frac{8}{1 + 2k} \right\} = \frac{8\pi}{3} \tag{2.11}
$$

Hence we deduce (1.10) from the linear combination of (2.9) and (2.11).

\[ \square \]

3 Proof of Theorems 1.3-1.6

Firstly, we are ready to prove Theorem 1.3.

**Proof of Theorem 1.3.** For achieving the purpose, we require two identities for basic hypergeometric series (cf. [6 Equation (5.1d)] and [8 Theorem 17]):

$$
\sum_{k=0}^{n} \frac{1 - aq^{3k-1}}{1 - aq^{-1}} \frac{(aq^{3k-1}, aq^{2n}, a/q; q^2)_k}{(aq^{2k}/b, aq^{2k}/c, bc/q; q^2)_k} \frac{(b/q, c/q, aq^2/bc; q)_k}{q^k} = \frac{(aq, b, c, aq^3/bc; q^2)_n}{(aq^2/b, aq^2/c, bc/q; q^2)_n}, \tag{3.1}
$$

$$
\sum_{k=0}^{\infty} \frac{1 - aq^{3k}}{1 - aq^{-1}} \frac{(aq^{3k}, aq^{2n}, a/q; q^2)_k}{(aq^2/b, aq^2/c, bc/q; q^2)_k} \frac{(b/q, c/q, aq^2/bc; q)_k}{q^k} = \sum_{k=0}^{\infty} \frac{1 - aq^{2k}}{1 - a} \frac{(b, c, d, e; q)_k}{(aq, b, aq/c, aq/d, aq/e; q)_k} \left( \frac{a^2 q}{bcde} \right)^k, \tag{3.2}
$$

where

$$
A_k(a, b, c, d, e; q) = \frac{(1 - q^{2k} a/c)(1 - q^{1+3k} a^2/bcd)}{(1 - a)(1 - q^{1+2k}a^2/bcd)} + q^{2k} a \left( 1 - q^{k} e \right) \left( 1 - a \right) + \frac{q^{1+4k} a^2}{ce} \frac{(1 - q^{1-k} a/bc)(1 - q^{1+k} a/bc)(1 - q^{1+k} a/2bde)(1 - q^{2+3k} a^2/bde)}{(1 - q^{1+2k}a/b)(1 - q^{1+2k}a/c)(1 - q^{1+2k}a/d)(1 - q^{1+2k}a/2bcde)(1 - q^{2+2k}a^2/bcd)} \times \frac{(1 - q^{1-k} a/2bde)(1 - q^{1+4k} a^2/bcd)}{(1 - q^{1+2k}a^2/bcd)(1 - q^{2+2k}a^2/bcd)}. \tag{3.3}
$$

Notice that the $(a, b, c) = (0, q^2, q^2)$ case of (3.1) is

$$
\sum_{k=0}^{n} \frac{(q; q)_k (q^{2n}; q^2)_k}{(q^3; q^2)_k (q^{2n}; q)_k} q^k = \frac{(q^2; q^2)_n}{(q, q^3; q^2)_n}.
$$
Letting \( n \to \infty \) in the above identity, we obtain (1.11).

The \((a, b, c) = (q^3, q^2, q^2)\) case of (3.1) reads

\[
\sum_{k=0}^{n} \frac{1 - q^{3k+2} (q^3 q^2 q^2)^k (q^{2n+3} q^{2n} q^2)^k}{1 - q^2} \frac{(q^3, q^2, q^2)_k}{(q^{2n+3}, q^{2n}, q^2)_k} q^k = \frac{(q^2; q^4 n^3 (q^2)^{2})_n}{(q; q)_n (q^3; q^2)_n}.
\]

Letting \( n \to \infty \) in the upper identity, we get (1.12).

Performing the replacements \((a, b, c, d, e) \to (x, a, b, xq/c, xq/d)\) in (3.2) and then letting \( x \to 0 \), we find

\[
\sum_{k=0}^{\infty} \frac{(a, b, c/a, c/b, d/a, d/b; q)_k}{(c, d; 1)_{1+2k} (cd/abq; q)_{2+2k}} q^{6(k)} \frac{1}{(ab)} B_k(a, b, c, d; q)
\]

\[
= \sum_{k=0}^{\infty} \frac{(a, b; q)_k}{(c, d; q)_k} \left( \frac{cd}{ab} \right)^k q^{6k},
\]

where

\[
B_k(a, b, c, d; q) = (1 - q^{2k} a)(1 - q^{2k} b)(1 - q^{2k} cd/ab)(1 - q^{3k-1} cd/b) - \frac{q^{3k-1} cd}{a} (1 - q^k a) (1 - q^k c/b) (1 - q^k d/b) (1 - q^{3k} cd/a).
\]

When \( d = q \), the series on the right-hand side of (3.3) can be evaluated by the \(q\)-Gauss summation formula (cf. [10, Appendix II. 8]):

\[
\varphi_1 \left[ \begin{array}{c} a, b \\ c \\ q, \frac{c}{ab} \end{array} \right] = (c/a, c/b; q)_\infty \left( \frac{c/a, c/b; q}{c, c/ab; q} \right)_\infty.
\]

So we have

\[
\sum_{k=0}^{\infty} \frac{(a, b, q/a, q/b, c/a, c/b; q)_k}{(q; q)_{1+2k} (c/a, c/b; q)_2} q^{3k^2-k} \left( \frac{c^2}{ab} \right)^k C_k(a, b, c; q)
\]

\[
= \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty},
\]

where

\[
C_k(a, b, c; q) = (1 - q^{2k} c)(1 - q^{1+2k} c')(1 - q^{1+2k} c/ab)(1 - q^{3k} c/b) - \frac{q^{3k} c}{a} (1 - q^k a) (1 - q^k c/b) (1 - q^{1+k} /b) (1 - q^{1+3k} c/a).
\]

Letting \((a, b, c, q) \to (q, q, q^2, q^2)\) in (3.4), we prove (1.13).

Secondly, we start to prove Theorem 1.4.
Proof of Theorem 1.4. Apply the operator $\mathcal{D}_b$ on both sides of the $c \rightarrow q^4/b$ case of (3.1) to discover

$$
\sum_{k=0}^{n} \frac{1 - aq^{3k-1}}{1 - aq^{-1}} (q^{-2n}, aq^{2n}, a/q; q^2)_k \frac{(a/q^2, b/q, q^3/b; q)_k (q, aq^{2n}, q^{-2n}; q)_k}{(q, aq^{2n}, q^{-2n}; q)_k} q^k D_k(a, b) \n$$

$$
= \frac{(aq, a/q, b, q^4/b; q^2)_n}{(q, q^3, aq^2/b, ab/q^2; q^2)_n} E_n(a, b),
$$

where

$$
D_k(a, b) = \sum_{i=1}^{k} \frac{q^{i-2}}{1 - bq^{i-2}} - \sum_{i=1}^{k} \frac{q^{i+2}/b^2}{1 - q^{i+2}/b} + \sum_{i=1}^{k} \frac{aq^{2i}/b^2}{1 - aq^{2i}/b} - \sum_{i=1}^{k} \frac{aq^{2i-4}}{1 - abq^{2i-4}},
$$

$$
E_n(a, b) = \sum_{j=1}^{n} \frac{q^{2j-2}}{1 - bq^{2j-2}} - \sum_{j=1}^{n} \frac{q^{2j+2}/b^2}{1 - q^{2j+2}/b} + \sum_{j=1}^{n} \frac{aq^{2j}/b^2}{1 - aq^{2j}/b} - \sum_{j=1}^{n} \frac{aq^{2j-4}}{1 - abq^{2j-4}}.
$$

Employing the operator $\mathcal{D}_b$ on both sides of the last equation, it is easy to show that

$$
\sum_{k=0}^{n} \frac{1 - aq^{3k-1}}{1 - aq^{-1}} (q^{-2n}, aq^{2n}, a/q; q^2)_k \frac{(a/q^2, b/q, q^3/b; q)_k (q, aq^{2n}, q^{-2n}; q)_k}{(q, aq^{2n}, q^{-2n}; q)_k} q^k \left\{ D_k(a, b)^2 - F_k(a, b) \right\}
$$

$$
= \frac{(aq, a/q, b, q^4/b; q^2)_n}{(q, q^3, aq^2/b, ab/q^2; q^2)_n} \left\{ E_n(a, b)^2 - G_n(a, b) \right\}, \quad (3.5)
$$

where

$$
F_k(a, b) = \sum_{i=1}^{k} \frac{q^{2i-4}}{(1 - bq^{i-2})^2} - \sum_{i=1}^{k} \frac{(q^{i+2}/b - 2)q^{i+2}/b^3}{(1 - q^{i+2}/b)^2} + \sum_{i=1}^{k} \frac{(aq^{2i}/b - 2)aq^{2i}/b^3}{(1 - aq^{2i}/b)^2} - \sum_{i=1}^{k} \frac{a^2q^{4i-8}}{(1 - abq^{2i-4})^2},
$$

$$
G_n(a, b) = \sum_{j=1}^{n} \frac{q^{4j-4}}{(1 - bq^{2j-2})^2} - \sum_{j=1}^{n} \frac{(q^{2j+2}/b - 2)q^{2j+2}/b^3}{(1 - q^{2j+2}/b)^2} + \sum_{j=1}^{n} \frac{(aq^{2j}/b - 2)aq^{2j}/b^3}{(1 - aq^{2j}/b)^2} - \sum_{j=1}^{n} \frac{a^2q^{4j-8}}{(1 - abq^{2j-4})^2}.
$$

The $(a, b) = (0, q^2)$ case of (3.5) produces

$$
\sum_{k=0}^{n} q^k \frac{(q; q)_k (q^{-2n}; q^2)_k}{(q^3; q^2)_k (q^{-2n}; q)_k} \sum_{i=1}^{k} \frac{q^i}{[i]^2} = \frac{(q^2; q^2)_n}{(q, q^3; q^2)_n} \sum_{j=1}^{n} \frac{q^{2j}}{[2j]^2}.
$$

Letting $n \rightarrow \infty$ in this identity, we catch hold of (1.14). \qed
Thirdly, we shall prove Theorem 1.6.

Proof of Theorem 1.5. The \((a, b) = (q^3, q^2)\) case of (3.5) provides

\[
\sum_{k=0}^{n} q^k \frac{1 - q^{3k+2}}{1 - q^2} \frac{(q; q)_k^2 (q^2; q^2)_k (q^{2n+3}; q^{-2n}; q^2)_k}{(q^3; q^3)_k^3} \left\{ \sum_{i=1}^{k} \frac{q^i}{[i]^2} - \sum_{i=1}^{k+1} \frac{q^{2i-1}}{[2i - 1]^2} \right\} = \frac{(q^2; q^2)_n^3 (q^4; q^2)_n}{(q; q)_n (q^3; q^2)_n^3}
\]

(3.6)

The \((a, b, c) = (q^3, q^2, q^2)\) case of (3.1) can be expressed as

\[
\sum_{k=0}^{n} q^k \frac{1 - q^{3k+2}}{1 - q^2} \frac{(q; q)_k^2 (q^2; q^2)_k (q^{2n+3}; q^{-2n}; q^2)_k}{(q^3; q^3)_k^3} \left\{ \sum_{i=1}^{k} \frac{q^i}{[i]^2} - \sum_{i=1}^{k+1} \frac{q^{2i-1}}{[2i - 1]^2} \right\} = \frac{(q^2; q^2)_n^3 (q^4; q^2)_n}{(q; q)_n (q^3; q^2)_n^3}
\]

(3.7)

According to the linear combination of (3.6) and (3.7), we have

\[
\sum_{k=0}^{n} q^k \frac{1 - q^{3k+2}}{1 - q^2} \frac{(q; q)_k^2 (q^2; q^2)_k (q^{2n+3}; q^{-2n}; q^2)_k}{(q^3; q^3)_k^3} \left\{ \sum_{i=1}^{k} \frac{q^i}{[i]^2} - \sum_{i=1}^{k+1} \frac{q^{2i-1}}{[2i - 1]^2} \right\} = \frac{(q^2; q^2)_n^3 (q^4; q^2)_n}{(q; q)_n (q^3; q^2)_n^3}
\]

(3.8)

Letting \(n \to \infty\) in this identity, we find (1.15).

Finally, we begin to prove Theorem 1.6.

Proof of Theorem 1.6. Setting \((a, b, c, d, e) = (q^{1/2}, q^{1/2}, x, q/x, q^{-n})\) in (3.2) and calculating the series on the right-hand side by the \(q\)-analogue of Dougall’s \(sF_4\) summation formula (cf. [10, Appendix II. 21]):

\[
\phi_5^{(6)} \left[ \begin{array}{c}
q^{a \frac{3}{2}} - qa \frac{3}{2}, aq, b, c, q^{-n} \\
a \frac{3}{2}, -a \frac{3}{2}, aq/b, aq/c, aq^{n+1}, q, -aq^{n+1} / bc
\end{array} \right] = \frac{(aq, aq/bc; q)_n}{(aq/b, aq/c; q)_n},
\]

there is

\[
\sum_{k=0}^{n} \frac{(q^{a \frac{3}{2}}; q)_k^2 (q^{a \frac{3}{2} + n}; q)_k (x; q)_k^2 (q/x; q)_k^2 (xq^{\frac{a + 1}{2}} + q^{\frac{a + n}{2}}/x; q)_k}{(q, q^{a \frac{3}{2} + n}, q^{a \frac{3}{2} + n}; q)_2 k (xq^{\frac{a + 1}{2}}, q^{\frac{a + n}{2}} / x; q)_2 k} \times (-1)^k q^{\frac{a}{2} + k} \cdot \frac{U_k(x, n; q)}{(xq^{\frac{a}{2}}, q^{\frac{a}{2}} / x; q)_n},
\]

where

\[
U_k(x, n; q) = \frac{1 - q^{\frac{a}{2} + 3k}}{1 - q^{\frac{a}{2}}} + \frac{q^{\frac{a}{2} + 2k + n}(1 - q^{k-n})(1 - xq^k)(1 - q^{1+k}/x)(1 - xq^{\frac{a}{2} + 3k+n})}{(1 - q^2)(1 + q^{k+n})(1 - q^{\frac{a}{2} + 2k + n})(1 - q^{\frac{a}{2} + 2k})}
\]
\[ \frac{q^{2+4k+n}(1 - q^{1+k+n})(1 - q^{k-n})(1 - xq^k)^2(1 - q^{1+k}/x)(1 - xq^{1+k+n}/x)}{x(1 - q^{1/2})(1 + q^{1/2+k})(1 - q^{1/2+2k+n})(1 - q^{1/2+2k})^2(1 - xq^{1/2+2k})(1 - q^{1/2+2k}/x)} \]

Via the operator \( D_x \) and the last equation, it is clear that

\[ \sum_{k=0}^{n} \left( \frac{q^{1+n}; q^{2+n}; q}{q^{2+n}; q} \right) (x; q)_k (q^2/x; q)_k (xq^{2+n}; q^{3+n}/x; q)_k 
\times (-1)^k q^{5k^2+1k+n} U_k (x, n; q) V_k (x, n; q) 
+ \sum_{k=0}^{n} \left( \frac{q^{3/2}; q^{3/2}; q}{q^{3/2}; q} \right) (x; q)_k (q^2/x; q)_k (xq^{3/2+n}; q^{3+n}/x; q)_k 
\times (-1)^k q^{5k^2+1k+n} D_x U_k (x, n; q) 
= \frac{(q^{1/2}; q^{3/2}; q)_n}{(xq^{1/2}; q^{3/2}/x; q)_n} \left\{ \sum_{j=1}^{n} \frac{q^{-j}}{1 - xq^{-j}} - \sum_{j=1}^{n} \frac{q^{j+1/2}/x^2}{1 - xq^{j+1/2}/x} \right\}, \tag{3.8} \]

where

\[ V_k (x, n; q) = 2 \sum_{i=1}^{k} \frac{q^i/x^2}{1 - q^i/x} - 2 \sum_{i=1}^{k} \frac{q^{-i-1}}{1 - xq^{-i-1}} + \sum_{i=1}^{2k} \frac{q^{-i}}{1 - xq^{-i}} 
- \sum_{i=1}^{2k} \frac{q^{i+1/2}/x^2}{1 - xq^{i+1/2}/x} + \sum_{i=1}^{k} \frac{q^{i+1/2+n}/x^2}{1 - xq^{i+1/2+n}/x} - \sum_{i=1}^{k} \frac{q^{i+1/2+n}}{1 - xq^{i+1/2+n}}. \]

Dividing both sides of (3.8) by \( 1 - q/x^2 \) and then letting \( (x, q, n) \to (q, q^2, \infty) \), there holds

\[ \sum_{k=0}^{\infty} q^{6k^2} \frac{(q; q^2)_k^6}{(q^2; q^2)_k^{3/2}} \left\{ \sum_{i=1}^{2k} \frac{q^{2i}}{(2i)^2} - 2 \sum_{i=1}^{2k} \frac{q^{2i-1}}{(2i-1)^2} \right\} 
\times \frac{1 + 2q^{1+2k} - q^{1+6k}(2 + 2q^2 + q^{1+2k} + q^{3+2k} - 3q^{3+6k})}{(1 - q)(1 - q^{2+4k})(1 + q^{1+2k})^2} 
+ \sum_{k=0}^{\infty} q^{6k+8k^2} \frac{(q; q^2)_k^6}{(q^2; q^2)_k^{3/2}} \frac{(1 - q)(1 + q^{1+2k} + q^{2+4k})^2}{(1 - q^{2+4k})(1 + q^{1+2k})^4} 
= \frac{(q, q^2; q^2)^{\infty}}{(q^2; q^2)^{\infty}} \sum_{j=1}^{\infty} \frac{q^{2j}}{[2j]^2}. \tag{3.9} \]

The \( (a, b, c) = (x, q/x, q) \) case of (3.4) can be stated as

\[ \sum_{k=0}^{\infty} \frac{(x, q/x; q)_k^3}{(q^2; q^2)_k^{3/2}} q^{3k^2} \left\{ (1 - q^{1+2k})^3(1 - xq^{3k}) - \frac{q^{1+3k}}{x}(1 - xq^k)^3(1 - q^{2+3k}/x) \right\} 
= (1 - q) \frac{(x, q/x; q)^{\infty}}{(q^2; q^2)^{\infty}}. \tag{3.10} \]
When $0 < x < 1$, it is obvious that the series on the left-hand side of (3.10) is uniformly convergent. Through the operator $D_x$ and (3.10), it is not difficult to see that

$$
3 \sum_{k=0}^{\infty} \frac{(x, q/x; q^2)_k^3}{(q^2; q^2)_k^3} q^{3k^2} \left\{ (1 - q^{1+2k})^3(1 - xq^k)^3 - \frac{q^{1+3k}}{x}(1 - xq^k)^3(1 - q^{2+3k}/x) \right\}
\times \left\{ \sum_{i=1}^{k} \frac{q^i/x^2}{1 - q^i/x} - \sum_{i=1}^{k} \frac{q^i-1}{1 - xq^{i-1}} \right\}
+ \sum_{k=0}^{\infty} \frac{(x, q/x; q^2)_k^3}{(q^2; q^2)_k^3} q^{3k^2+3k} (q - x^2)(x + 3xq^{2+4k} - 2x^2q^{1+3k} - 2q^{2+3k})
\frac{1}{x^3}
= (1 - q)^3 \frac{(x, q/x; q^2)}{(q^2; q^2)_\infty^2} \left\{ \sum_{j=1}^{\infty} \frac{q^j/x^2}{1 - q^j/x} - \sum_{j=1}^{\infty} \frac{q^j-1}{1 - xq^{j-1}} \right\}. \tag{3.11}
$$

Dividing both sides of (3.11) by $1 - q/x^2$ and then letting $(x, q) \to (q, q^2)$, we can verify that

$$
3 \sum_{k=0}^{\infty} q^{6k^2} \frac{(q; q^2)_k^6}{(q^4; q^2)_k^3} \frac{(1 - q^{2+4k})(1 - q^{1+6k}) - q^{1+6k}(1 - q^{1+2k})^3(1 - q^{3+6k})}{(1 - q)^4}
\times \sum_{j=1}^{k} \frac{q^{2i-1}}{[2i - 1]^2} + \sum_{k=0}^{\infty} q^{6k^2+6k} \frac{(q; q^2)_k^6}{(q^4; q^2)_k^3} q - 4q^{4+6k} + 3q^{5+8k}
\frac{1}{(1 - q)^2}
= (1 + q)^2 \frac{(q^3; q^2)^2}{(q^2, q^4; q^2)_\infty} \sum_{j=1}^{\infty} \frac{q^{2j-1}}{[2j - 1]^2}. \tag{3.12}
$$

By means of the linear combination of (3.9) and (3.12) multiplied, respectively, by $(-64)$ and $3$, we are led to (1.16). \qed

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