PINNING WITH A VARIABLE MAGNETIC FIELD OF THE TWO DIMENSIONAL GINZBURG-LANDAU MODEL

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Abstract. We study the Ginzburg-Landau energy of a superconductor with a variable magnetic field and a pinning term in a bounded smooth two dimensional domain \( \Omega \). Supposing that the Ginzburg-Landau parameter and the intensity of the magnetic field are large and of the same order, we determine an accurate asymptotic formula for the minimizing energy. This asymptotic formula displays the influence of the pinning term. Also, we discuss the existence of non-trivial solutions and prove some asymptotics of the third critical field.

1. Introduction

We consider a bounded, open and simply connected set \( \Omega \subset \mathbb{R}^2 \) with smooth boundary. We suppose that \( \Omega \) models an inhomogeneous superconducting sample submitted to an applied external magnetic field. The energy of the sample is given by the so called pinned Ginzburg-Landau functional,

\[
E_{\kappa,H,a,B_0}(\psi,A) = \int_\Omega \left( |(\nabla - i\kappa H A)\psi|^2 + \frac{\kappa^2}{2}(a(x,\kappa) - |\psi|^2)^2 \right) \, dx + \kappa^2 H^2 \int_\Omega |\text{curl}\, A - B_0|^2 \, dx.
\]

Here \( \kappa \) and \( H \) are two positive parameters such that \( \kappa \) describes the properties of the material, and \( H \) measures the variation of the intensity of the applied magnetic field. The modulus \( |\psi|^2 \) of the wave function (order parameter) \( \psi \in H^1(\Omega; \mathbb{C}) \) measures the density of the superconducting electron Cooper pairs. The magnetic potential \( A \) belongs to \( H^1_{\text{div}}(\Omega) \) where

\[
H^1_{\text{div}}(\Omega) = \{ A = (A_1, A_2) \in H^1(\Omega)^2 : \text{div}\, A = 0 \text{ in } \Omega, A \cdot \nu = 0 \text{ on } \partial\Omega \},
\]

with \( \nu \) being the unit interior normal vector of \( \partial\Omega \). The function \( \kappa H \text{curl}\, A \) gives the induced magnetic field.

When \( \psi \equiv 0 \) and \( (\psi,A) \) is a minimizer or a critical point of the functional, we call this pair normal. In our case it is easy to see normal minimizers (if any) are necessarily in the form \( (0, A) \) with \( A \in H^1_{\text{div}}(\Omega) \) such that \( \text{curl}\, A = B_0 \). This solution is unique and denoted by \( F \). A natural question will be to determine under which condition this normal solution is a minimizer.

The function \( B_0 \in C^\infty(\overline{\Omega}) \) is the intensity of the external magnetic field which is variable in our problem. Let

\[
\Gamma = \{ x \in \overline{\Omega} : B_0(x) = 0 \}.
\]

We assume that either \( \Gamma \) is empty or that \( B_0 \) satisfies :

\[
\begin{cases}
|B_0| + |\nabla B_0| > 0 & \text{in } \overline{\Omega} \\
\nabla B_0 \times \vec{n} \neq 0 & \text{on } \Gamma \cap \partial\Omega.
\end{cases}
\]

The assumption in (1.4) implies that for any open set \( \omega \) relatively compact in \( \Omega \), \( \Gamma \cap \omega \) is either empty, or consists of a union of smooth curves.

The energy \( E_{\kappa,H,a,B_0} \) considered here is slightly different from the classical Ginzburg-Landau energy in the sense that there is a varying term denoted by \( a(x,\kappa) \) penalizing the variations of the order parameter \( \psi \) and called the pinning term. This term arises also naturally in the
microscopic derivation of the Ginzburg-Landau theory from BCS theory (see [17]) without any a priori assumption on the sign of $a$.

In this paper, we will assume that the pining term $a$ satisfies:

**Assumption 1.1.** The function $a(x, \kappa)$ is real, defined on $\overline{\Omega} \times [\kappa_0, +\infty)$, and satisfies for some $\kappa_0 > 0$ the following assumptions:

\begin{align}
(A_1) \quad & \forall \kappa \geq \kappa_0, a(\cdot, \kappa) \in C^1(\overline{\Omega}). \\
(A_2) \quad & \sup_{x \in \overline{\Omega}, \kappa \geq \kappa_0} |a(x, \kappa)| < +\infty. \\
(A_3) \quad & \sup_{x \in \overline{\Omega}, \kappa \geq \kappa_0} |\nabla_x a(x, \kappa)| < +\infty. \\
(A_4) \quad & \text{There exists a positive constant } C_1, \text{ such that,} \\
\quad & \forall \kappa \geq \kappa_0, \quad \mathcal{L}(\partial\{a(x, \kappa) > 0\}) \leq C_1 \kappa^2, \tag{1.8}
\end{align}

where $\mathcal{L}$ is the "length" of $\partial\{a(x, \kappa) > 0\}$ in $\Omega$ in a sense that will be explained in (3.1).

Let us introduce for later use,

\[ L(\kappa) = \sup_x |\nabla_x a(x, \kappa)|, \tag{1.9} \]

\[ \bar{a} = \sup_{x \in \overline{\Omega}, \kappa \geq \kappa_0} a(x, \kappa) \tag{1.10} \]

and

\[ \underline{a} = \inf_{x \in \overline{\Omega}, \kappa \geq \kappa_0} a(x, \kappa). \tag{1.11} \]

The assumption in $(A_3)$ gives a uniform control for any $\kappa$ of the oscillation of $a(\cdot, \kappa)$ which will be made precise later by an assumption on $L(\kappa)$. Notice that the normal state $(0, F)$ is a critical point of the functional in (1.1). It is standard, starting from a minimizing sequence, to prove the existence of minimizers in $H^1(\Omega; \mathbb{C}) \times H^1_{\text{div}}(\Omega)$ of the functional $\mathcal{E}_{\kappa, H, a, B_0}$. A minimizer $(\psi, A)$ of (1.1) is a weak solution of the Ginzburg-Landau equations,

\[
\begin{cases}
-\nabla - i\kappa H A)^2 \psi = \kappa^2 (a(x, \kappa) - |\psi|^2) \psi \quad & \text{in } \Omega \quad (a) \\
-\nabla \cdot \text{curl}(A - F) = \frac{1}{\kappa H} \text{Im}(\bar{\psi} (\nabla - i\kappa H A) \psi) \quad & \text{in } \Omega \quad (b) \\
\nu \cdot (\nabla - i\kappa H A) \psi = 0 \quad & \text{on } \partial \Omega \quad (c) \\
\text{curl} A = \text{curl} F \quad & \text{on } \partial \Omega \quad (d) \tag{1.12}
\end{cases}
\]

Here, $\text{curl} A = \partial_{x_1} A_2 - \partial_{x_2} A_1$ and $\nabla \cdot \text{curl} A = (\partial_{x_2} (\text{curl} A), -\partial_{x_1} (\text{curl} A))$.

Let us introduce the magnetic Schrödinger operator in an open set $\tilde{\Omega}$ in $\mathbb{R}^2$:

\[ P_{A,V}^{\tilde{\Omega}} = -(\nabla - i A)^2 + V(x), \tag{1.13} \]

where $A \in H^1_{\text{div}}(\tilde{\Omega})$ and $V$ is a continuous function bounded from below.

The form domain of $P_{A,V}^{\tilde{\Omega}}$ is

\[ \mathcal{V}(\tilde{\Omega}) = \{ u \in L^2(\tilde{\Omega}), \quad (\nabla - i A) u \in L^2(\tilde{\Omega}), \quad (V + C)^{1/2} u \in L^2(\tilde{\Omega}) \}, \]

and its operator domain is given by

\[ D(P_{A,V}^{\tilde{\Omega}}) := \{ u \in \mathcal{V}(\tilde{\Omega}), \quad P_{A,V}^{\tilde{\Omega}} u \in L^2(\tilde{\Omega}), \quad \nu \cdot (\nabla - i A) u = 0 \text{ on } \partial \tilde{\Omega} \}. \]
Then, \( a,c \) reads
\[
P_{A,V}^\Omega \psi = -\kappa^2 |\psi|^2 \psi,
\]
with \( A = \kappa H A, \psi \in D(P_{A,V}^\Omega) \) and \( V = -\kappa^2 a \).

There are many papers on the Ginzburg-Landau functional with a pinning term, most of them study the influence of the pinning term on the location of vortices, i.e. the zeros of the minimizing order parameter. For the functional without a magnetic field (i.e. \( B_0 = 0 \) in (1.1)), the influence of the pinning term is studied in [28] and more recently in [32] and the references therein. The pinning term (i.e. the function \( a \)) in [28] is a step function independent of \( \kappa \); more complicated \( \kappa \)-dependent periodic step functions are considered in [32]. The magnetic version of the functional in [28] is studied in [25, 26].

In [2], Aftalion, Sandier and Serfaty considered a smooth and \( \kappa \) dependent pinning term \( a \) satisfying:
\[
(\text{H}_1) \quad L(\kappa) \ll \kappa H.
\]
\[
(\text{H}_2) \quad \text{There exist a continuous function } a(x), \text{ a positive constant } a_0 \text{ and, for all } \kappa \geq 0, \text{ there exist two functions } \sigma(\kappa) = o \left( \frac{\ln |\ln \kappa|}{\kappa} \right) \text{ and } \beta(x,\kappa) \geq 0 \text{ such that,}
\]
\[
\min_{B(x,\sigma(\kappa))} \beta(x,\kappa) = 0, \quad a(x,\kappa) = a(x) + \beta(x,\kappa), \quad \text{and } 0 < a_0 \leq a(x) \leq 1.
\]
The study contains the case when \( a(x,\kappa) = a(x) (\beta = 0) \) but also cases with a \( \kappa \)-control of the \( x \)-oscillation of \( \beta(\cdot,\kappa) \) which could increase with \( \kappa \). In the scales of this paper, the results in [2] are valid when the parameter \( H \) is of order \( \frac{|\ln \kappa|}{\kappa} \) as \( \kappa \to +\infty \).

Extending the discussion, the functional in (1.1) is close to models of Bose-Einstein condensates (see e.g. [1, 3]).

In this paper, we will analyze how the pinning term appears in the asymptotics of the energy in the presence of a strong external variable magnetic field (see Theorem 1.2 below). Also, we discuss the influence of the pinning on the asymptotic expression of the third critical field \( H_{C_3} \) (see Theorems 1.6 and 1.7).

We focus on the regime of large values of \( \kappa, \kappa \to +\infty \) and we study the ground state energy defined as follows,
\[
E_g(\kappa,H,a,B_0) = \inf \left\{ \mathcal{E}_{\kappa,H,a,B_0}(\psi,A) : (\psi,A) \in H^1(\Omega;\mathbb{C}) \times H^1_{\text{div}}(\Omega) \right\}.
\]

More precisely, we give an asymptotic estimate which is valid in the simultaneous limit \( \kappa \to +\infty \) and \( H(\kappa) \to +\infty \) with the constraint that \( \frac{H(\kappa)}{\kappa} \) remains asymptotically of uniform size, that is satisfying
\[
\lambda_{\min} \leq \frac{H(\kappa)}{\kappa} \leq \lambda_{\max} \quad (\kappa \geq \kappa_0),
\]
where \( \lambda_{\min}, \lambda_{\max} \) are positive constants such that \( \lambda_{\min} < \lambda_{\max} \).

The behavior of \( E_g(\kappa,H,a,B_0) \) involves a function \( \hat{f} : [0,+\infty) \to [0,\frac{1}{2}] \) introduced in [5, Theorem 2.1]. The function \( \hat{f} \) is increasing, continuous and \( \hat{f}(b) = \frac{1}{2}, \) for all \( b \geq 1 \).

**Theorem 1.2.** Suppose that Assumption 1.1 and (1.15) hold, and
\[
L(\kappa) = O(\kappa^{\frac{1}{2}}) \quad \text{as } \kappa \to +\infty.
\]
The ground state energy in \([1.14]\) satisfies
\[
\mathcal{E}_\kappa(K, H, a, B_0) = \kappa^2 \int_{\{a(x, \kappa) > 0\}} a(x, \kappa)^2 \hat{f} \left( \frac{H |B_0(x)|}{\kappa a(x, \kappa)} \right) \, dx \\
+ \frac{\kappa^2}{2} \int_{\{a(x, \kappa) \leq 0\}} a(x, \kappa)^2 \, dx + o(\kappa^2) , \quad \text{as } \kappa \to +\infty . \tag{1.17}
\]

When \(\Omega \cap \{a(x, \kappa) > 0\} = \emptyset\), we obtain directly from \([1.14]\)
\[
\mathcal{E}_{\kappa,H,a,B_0}(\psi,A) \geq \frac{\kappa^2}{2} \int_{\Omega} a(x, \kappa)^2 \, dx = \mathcal{E}_{\kappa,H,a,B_0}(0,F) .
\]

Hence the minimizer of \(\mathcal{E}_{\kappa,H,a,B_0}\) is the normal state. In physical terms, this case corresponds to the case when we are above the critical temperature.

We will describe later cases when the remainder term in \((1.17)\) is indeed small compared with the leading order term (see Section \([6]\)).

The assumptions in Theorem \([1.2]\) contain the case when the function \(a\) is constant and equals 1, which was proved in \([4]\) under Assumption \((1.15)\).

Along the proof of Theorem \([1.2]\) we obtain an estimate of the ‘magnetic energy’ as follows:

\textbf{Corollary 1.3.} Under the assumptions of Theorem \([1.2]\) we have
\[
(kH)^2 \int_{\Omega} |\nabla A - B_0|^2 \, dx = o(\kappa^2) , \quad \text{as } \kappa \to +\infty . \tag{1.18}
\]

If \(D\) is a domain in \(\Omega\), we introduce the local energy in \(D\) of \((\psi,A) \in H^1(\Omega; \mathbb{C}) \times H^1_{\text{div}}(\Omega)\) by:
\[
\mathcal{E}_0(\psi,A; a,D) = \int_D |\nabla - i\kappa H A| \psi|^2 \, dx + \frac{\kappa^2}{2} \int_D (a(x, \kappa) - |\psi|^2)^2 \, dx . \tag{1.19}
\]

The next theorem gives an estimate of the local energy \(\mathcal{E}_0(\psi,A; a,D)\).

\textbf{Theorem 1.4.} Under the assumptions of Theorem \([1.2]\) if \((\psi,A)\) is a minimizer of \((1.1)\) and \(D\) is regular set such that \(\overline{D} \subset \Omega\), then
\[
\mathcal{E}_0(\psi,A; a,D) = \kappa^2 \int_{D \cap \{a(x, \kappa) > 0\}} a(x, \kappa)^2 \hat{f} \left( \frac{H |B_0(x)|}{\kappa a(x, \kappa)} \right) \, dx \\
+ \frac{\kappa^2}{2} \int_{D \cap \{a(x, \kappa) \leq 0\}} a(x, \kappa)^2 \, dx + o(\kappa^2) , \quad \text{as } \kappa \to +\infty . \tag{1.20}
\]

\textbf{Theorem 1.4} will be useful in the proof of the next theorem which gives the asymptotic behavior of the order parameter \(\psi\), when \((\psi,A)\) is a global minimizer.

\textbf{Theorem 1.5.} Under the assumptions of Theorem \([1.2]\) if \((\psi,A)\) is a minimizer of \((1.1)\) and \(D\) is a regular set such that \(\overline{D} \subset \Omega\), then
\[
\int_D |\psi(x)|^4 \, dx = -\int_{D \cap \{a(x, \kappa) > 0\}} a(x, \kappa)^2 \left\{ 2\hat{f} \left( \frac{H |B_0(x)|}{\kappa a(x, \kappa)} \right) - 1 \right\} \, dx + o(1) , \quad \text{as } \kappa \to +\infty . \tag{1.21}
\]

Formula \((1.21)\) indicates that \(\psi\) is asymptotically localized in the region where \(a > 0\). When \(a(x, \kappa) = 1\), Theorem \([1.5]\) was proved in \([4]\).

The techniques that we are going to use here are inspired from those of \([4]\) and \([5]\) (where the case \(a = 1\) was treated). At a technical level, our proof is slightly different than the proofs in \([4] \text{[14]} \text{[30]}\) since we do not use the uniform elliptic estimates. These important estimates are frequently used in the papers about the Ginzburg-Landau functional (see \([13]\)) with a constant pinning term. They appeared first in \([30]\) and were then extended to the full regime in \([12]\).
We introduce the following critical fields (cf. e.g. [11, 30])

\[ N_{\text{cp}}(\kappa) = \{ H > 0 : \mathcal{E}_{\kappa,H,a,B_0} \text{ has a non-normal critical point} \}. \]  

Notice that the above set is bounded (see Theorem 8.5). We also introduce the two sets:

\[ N(\kappa) = \{ H > 0 : \mathcal{E}_{\kappa,H,a,B_0} \text{ has a non-normal minimizer} \}. \]
\[ N^\text{loc}(\kappa) = \{ H > 0 : \mu_1(\kappa,H) < 0 \}. \]

Here, \( \mu_1(\kappa,H) \) is the ground state energy of the semi-bounded quadratic form

\[ Q^\Omega_{\kappa,H,F,-\kappa^2a}(\phi) = \int_\Omega (|\nabla - i\kappa HF\phi|^2 - \kappa^2 a(x,\kappa)|\phi|^2) \, dx, \]  

i.e.

\[ \mu_1(\kappa,H) = \inf_{\phi \in H^1(\Omega)} \left( \frac{Q^\Omega_{\kappa,H,F,-\kappa^2a}(\phi)}{\|\phi\|_{L^2(\Omega)}^2} \right). \]

Note that \( \mu_1(\kappa,H) \) is the lowest eigenvalue of \( P^\Omega_{\kappa,H,F,-\kappa^2a} \). Here, we refer to [9, 27, 33, 34] for previous contributions.

We introduce the following critical fields (cf. e.g. [11, 30]).

\[ \overline{H}_{C_3}^\text{cp}(\kappa) = \sup N_{\text{cp}}(\kappa), \quad \overline{H}_{C_3}^\text{pp}(\kappa) = \inf (\mathbb{R}_+ \setminus N_{\text{cp}}(\kappa)), \]  

\[ \overline{H}_{C_3}(\kappa) = \sup N(\kappa), \quad \overline{H}_{C_3}^\text{loc}(\kappa) = \inf (\mathbb{R}_+ \setminus N(\kappa)), \]
\[ \overline{H}_{C_3}^\text{loc}(\kappa) = \sup N^\text{loc}(\kappa), \quad \overline{H}_{C_3}^\text{loc}(\kappa) = \inf (\mathbb{R}_+ \setminus N^\text{loc}(\kappa)). \]

Below \( H_{C_3} \), normal states will lose their stability and above \( \overline{H}_{C_3} \), the normal state is (up to a gauge transformation) the only critical point of the functional in (1.1).

Our aim is to determine the asymptotics of all the critical fields as \( \kappa \to +\infty \). This involves spectral quantities related to three models depending on \( \Gamma \) being empty or not.

Let us introduce

\[ \Theta_0 = \inf_{\xi \in \mathbb{R}} \mu(\xi), \]

where \( \mu \) is the lowest eigen value of the operator

\[ H_{\kappa \xi} : = -\frac{d^2}{dt^2} + (t + \xi)^2 \quad \text{in} \quad L^2(\mathbb{R}_+), \]

subject to the Neumann boundary condition \( u'(0) = 0 \).

**Theorem 1.6.** Suppose that \( \Gamma = \{ x \in \Omega : B_0(x) = 0 \} = \emptyset \) and that \( a \in C^1(\Omega) \) satisfies \( \{ a > 0 \} \neq \emptyset \). Then, as \( \kappa \to +\infty \), all the six critical fields satisfy an asymptotic expansion in the form:

\[ H_{C_3}(\kappa) = \max \left( \sup_{x \in \Omega} \frac{a(x)}{|B_0(x)|}, \sup_{x \in \partial \Omega} \frac{a(x)}{|B_0(x)|} \right) \kappa + \mathcal{O}(\kappa^2). \]  

We introduce

\[ \lambda_0 = \inf_{\tau \in \mathbb{R}} \lambda(\tau), \]

where

\[ \lambda(\tau) = \int_\Omega (|\nabla - i\kappa HF\phi|^2 - \kappa^2 a(\phi)^2) \, dx. \]
Theorem 1.7. Suppose that \( \Gamma = \{ x : B_0(x) = 0 \} \neq \emptyset \), that (1.4) holds and that \( a \in C^1(\Omega) \) satisfies \( \{ a > 0 \} \neq \emptyset \). As \( \kappa \to +\infty \), the six critical fields in (1.27), (1.29) satisfy the asymptotic expansion:

\[
H_{C_3}(\kappa) = \max \left( \sup_{x \in \Gamma \setminus \emptyset} \frac{a(x)^3}{\lambda^2_0 \nabla B_0(x)}, \sup_{x \in \partial \Omega} \frac{a(x)^3}{\lambda(\mathbb{R}^2_+, \theta(x))^2 |\nabla B_0(x)|} \right) \kappa^2 + \mathcal{O} \left( \kappa^{\frac{7}{2}} \right).
\]

Here \( \theta(x) \) denotes the angle between \( \nabla B_0(x) \) and the inward normal vector \(-\nu(x)\).

Organization of the paper. The rest of the paper is split into twelve sections. Section 2 analyzes the model problem with a constant magnetic field and a constant pinning term. Section 3 establishes an upper bound on the ground state energy. Section 4 contains useful estimates on minimizers. The estimates in Section 4 are used in Section 5 to establish a lower bound of the ground state energy and to finish the proof of Theorem 1.2, Corollary 1.3 and Theorem 1.4. In Section 6 we discuss the conclusion in Theorem 1.2 by providing various examples of pinning terms obeying Assumption 1.1. Section 7 is devoted to the proof of Theorem 1.5. Section 8 generalizes a theorem of Giorgi-Phillips concerning the breakdown of superconductivity under a large applied magnetic field. Sections 9 and 10 are devoted to the proof of Theorem 1.6. The proof of Theorem 1.7 is the purpose of Sections 11 and 12.

Notation. Throughout the paper, we use the following notation:

- If \( b_1(\kappa) \) and \( b_2(\kappa) \) are two positive functions on \([\kappa_0, +\infty)\), we write \( b_1(\kappa) \ll b_2(\kappa) \) if \( b_1(\kappa)/b_2(\kappa) \to 0 \) as \( \kappa \to +\infty \).
- If \( b_1(\kappa) \) and \( b_2(\kappa) \) are two functions with \( b_2(\kappa) \neq 0 \), we write \( b_1(\kappa) \sim b_2(\kappa) \) if \( b_1(\kappa)/b_2(\kappa) \to 1 \) as \( \kappa \to +\infty \).
- If \( b_1(\kappa) \) and \( b_2(\kappa) \) are two positive functions, we write \( b_1(\kappa) \approx b_2(\kappa) \) if there exist positive constants \( c_1, c_2 \) and \( \kappa_0 \) such that \( c_1 b_2(\kappa) \leq b_1(\kappa) \leq c_2 b_2(\kappa) \) for all \( \kappa \geq \kappa_0 \).
- Let \( a_+(\bar{x}_0, \kappa) = [a(\bar{x}_0, \kappa)]_+ \) and \( a_-(\bar{x}_0, \kappa) = [a(\bar{x}_0, \kappa)]_- \) where, for any \( x \in \mathbb{R} \), \( [x]_+ = \max(x, 0) \) and \( [x]_- = \max(-x, 0) \).
- Given \( R > 0 \) and \( x = (x_1, x_2) \in \mathbb{R}^2 \), \( Q_R(x) = (-R/2 + x_1, R/2 + x_1) \times (-R/2 + x_2, R/2 + x_2) \) denotes the square of side length \( R \) centered at \( x = (x_1, x_2) \) and we write \( Q_R = Q_R(0) \).
2.1. **A useful function.** Consider $R > 0$, $b > 0$, $\zeta \in \{-1, +1\}$ and $\alpha \in \mathbb{R}$. We define the following Ginzburg-Landau energy with constant magnetic field on $H^1(Q_R)$ by

$$ u \mapsto F_{b,Q_R}^{\zeta,\alpha}(u) = \int_{Q_R} \left( |b(\nabla - i\zeta A_0)u| + \frac{1}{2} (\alpha - |u|^2)^2 \right) \, dx, \quad (2.1) $$

where

$$ A_0(x) = \frac{1}{2}(-x_2, x_1), \quad \forall x = (x_1, x_2) \in \mathbb{R}^2. \quad (2.2) $$

We have two cases according to the sign of $\alpha$:

- **Case 1.** $\alpha > 0$:

  We notice that

  $$ F_{b,Q_R}^{\zeta,\alpha}(u) = \alpha^2 F_{b,Q_R}^{\zeta,1}(\tilde{u}), \quad (2.3) $$

  where

  $$ \tilde{b} = \frac{b}{\alpha} \quad \text{and} \quad \tilde{u} = \frac{u}{\sqrt{\alpha}}. \quad (2.4) $$

  We introduce the two ground state energies

  $$ e_N(b, R, \alpha) = \inf \left\{ F_{b,Q_R}^{\zeta,1}(u) : u \in H^1(Q_R; \mathbb{C}) \right\}, \quad (2.5) $$

  $$ e_D(b, R, \alpha) = \inf \left\{ F_{b,Q_R}^{\zeta,1}(u) : u \in H^1_0(Q_R; \mathbb{C}) \right\}. \quad (2.6) $$

  As $F_{b,Q_R}^{\zeta,1}(u) = F_{b,Q_R}^{\zeta,1}(\tilde{u})$, it is immediate that,

  $$ \inf F_{b,Q_R}^{\zeta,1}(u) = \inf F_{b,Q_R}^{\zeta,1}(\tilde{u}). \quad (2.7) $$

  Using (2.5) and (2.6), we get from (2.3)

  $$ e_N(b, R, \alpha) = \alpha^2 e_N \left( \frac{b}{\alpha}, R, 1 \right) = \alpha^2 e_N \left( \frac{b}{\alpha}, R \right), \quad (2.8) $$

  and

  $$ e_D(b, R, \alpha) = \alpha^2 e_D \left( \frac{b}{\alpha}, R, 1 \right) = \alpha^2 e_D \left( \frac{b}{\alpha}, R \right). \quad (2.9) $$

  As a consequence of (2.3) and (2.4), $\tilde{u}$ is a minimizer of $F_{b,Q_R}^{\zeta,1}$ if and only if $u$ is a minimizer of $F_{b,Q_R}^{\zeta,\alpha}$. In particular any minimizer of $F_{b,Q_R}^{\zeta,\alpha}$ satisfies

  $$ |u| \leq \sqrt{\alpha}. \quad (2.10) $$

  Recall from [14, Theorem 2.1] that,

  $$ \hat{f}(b) = \lim_{R \to \infty} \frac{e_D(b, R)}{R^2}. \quad (2.11) $$

  The next proposition was proved in [5, Lemma 2.2, Proposition 2.4] in the case $\alpha = 1$. Its present form can be deduced immediately from (2.8).

**Proposition 2.1.** For all $M > 0$, there exist universal constants $C_M$ and $R_M$ such that $\forall R \geq R_M$, $\forall b > 0$, $\forall \alpha > 0$ such that $0 < \frac{b}{\alpha} \leq M$, we have

$$ e_N(b, R, \alpha) \geq e_D(b, R, \alpha) - C_M \alpha^2 R \left( \frac{b}{\alpha} \right)^{\frac{1}{2}} \quad (2.12) $$

$$ \alpha^2 \hat{f} \left( \frac{b}{\alpha} \right) \leq \frac{e_D(b, R, \alpha)}{R^2} \leq \alpha^2 \hat{f} \left( \frac{b}{\alpha} \right) + C_M \frac{\alpha^2 \sqrt{b}}{R}. \quad (2.13) $$
Case 2. α ≤ 0:
When α ≤ 0, we write α = −α0, α0 ≥ 0 and (2.1) becomes
\[
F_{b, Q_R}^{ζ, α}(u) = \int_{Q_R} \left( b|(∇ - iζA_0)u|^2 + \frac{1}{2} (α_0 + |u|^2)^2 \right) dx.
\]
(2.14)
It is clear that,
\[
F_{b, Q_R}^{ζ, α}(u) \geq \frac{1}{2} α_0^2 R^2 \quad \text{and} \quad F_{b, Q_R}^{ζ, α}(0) = \frac{1}{2} α_0^2 R^2.
\]
As a consequence, we have
\[
\frac{1}{2} α_0^2 R^2 \leq e_D(b, R, α) \leq F_{b, Q_R}^{ζ, α}(0) = \frac{1}{2} α_0^2 R^2.
\]
When α = 0, it is easy to show that
\[
F_{b, Q_R}^{ζ, α}(u) = 0.
\]
Notice that the only minimizer of \(F_{b, Q_R}^{ζ, α}\) is \(u = 0\). Thus, for any \(α ≤ 0\), we obtain
\[
\frac{e_D(b, R, α)}{R^2} = \frac{1}{2} α^2.
\]
(2.15)

3. Upper bound of the energy

The aim of this section is to give an upper bound of the ground state energy \(E_ε(κ, H, a, B_0)\) introduced in (1.14) under Assumption (1.15). For this we cover \(Ω\) by (the closure of) disjoint open squares \(Q_ε(γ)\), whose centers \(γ\) belong to a square lattice \(Γ_ε = ℓZ × ℓZ\).

We will get an upper bound by matching together approximate minimizers, in each square \(Q_ε(γ)\) contained in \(Ω\), obtained by freezing the pinning term and the magnetic field at a suitable point \(γ\). The size \(ℓ\) of the square will be chosen as a function of \(κ\). We start with estimates in a given square \(Q_ε(x_0)\) and will take later \(x_0 = γ\).

About Assumption (A4).

We first explain what was meant in Assumption (A4). By \(L(∂{a > 0}) \leq C_1 κ^{\frac{1}{2}}\) we mean the existence of \(C_2 > 0\) and \(κ_0\) such that:
\[
∀κ ≥ κ_0, ∀ℓ ≤ C_2 κ^{-\frac{1}{4}}, \text{card } \{γ ∈ Γ_ε \cap Ω \text{ with } Q_ε(γ) \cap ∂{a > 0} \cap Ω ≠ ∅\} \leq C_1 κ^{\frac{1}{2}} ℓ^{-1}.
\]
(3.1)

Using Assumption (1.9), for any \(x_0 \in Q_ε(x_0)\) and \(κ ≥ κ_0\), we observe that,
\[
|a(x, κ) - a(x_0, κ)| ≤ \left( \sup_x |∇_x a(x, κ)| \right) |x - x_0| ≤ \frac{ℓ}{\sqrt{2}} L(κ), \quad ∀x ∈ Q_ε(x_0).
\]
(3.2)

**Definition 3.1 (ρ-admissible).** Let \(ρ ∈ (0, 1)\). We say that triple \((ℓ, x_0, x_0)\) is \(ρ\)-admissible if \(Q_ε(x_0) \subset \{ |B_0| > ρ \} \cap Ω\) and \(x_0 \in Q_ε(x_0)\). In this case, we also say that the pair \((ℓ, x_0)\) is \(ρ\)-admissible and the corresponding square \(Q_ε(x_0)\) is \(ρ\) admissible.

We recall from [5] Section 3 the definition of the test function,
\[
\tilde{w}_{ℓ, x_0, x_0}(x) = \begin{cases} e^{iκHx_0, x_0} \tilde{u}_R \left( \frac{R}{ℓ}(x - x_0) \right) & \text{if } x ∈ Q_ε(x_0) \subset \{ |B_0| > ρ \} \cap Ω, \\ e^{iκHx_0, x_0} \tilde{u}_R \left( \frac{R}{ℓ}(x - x_0) \right) & \text{if } x ∈ Q_ε(x_0) \subset \{ |B_0| < −ρ \} \cap Ω, \end{cases}
\]
(3.3)
where \(\tilde{u}_R ∈ H_1^0(Ω)\) is a minimizer of \(F_{b, Q_R}^{ζ, α−1}\) satisfying by (2.10) \(∥\tilde{u}_R∥ ≤ 1\) and \(φ_{x_0, x_0}\) is the function introduced in [3] Lemma A.3 that satisfies
\[
|F(x) - B_0(x_0)A_0(x - x_0) - ∇φ_{x_0, x_0}| ≤ C ℓ^2, \quad ∀x ∈ Q_ε(x_0).
\]
(3.4)
Here $B_0 = \text{curl} F$ and $A_0$ is the magnetic potential introduced in (2.2).

Let us introduce the function:

$$w_{\ell, x_0, \tilde{x}_0}(x) = \sqrt{a_+(\tilde{x}_0, \kappa)} \tilde{w}_{\ell, x_0, \tilde{x}_0}(x), \quad \forall x \in Q_\ell(\tilde{x}_0).$$

(3.5)

Using the bound $|\tilde{w}_{\ell, x_0, \tilde{x}_0}| \leq 1$, which is immediately deduced from the bound of $|\tilde{u}|$, we get from (3.5),

$$|w_{\ell, x_0, \tilde{x}_0}|^2 \leq a_+(\tilde{x}_0, \kappa).$$

(3.6)

**Proposition 3.2.** Under Assumptions (1.4), (1.7), there exist positive constants $C$ and $\kappa_0$ such that if $\kappa \geq \kappa_0$, $\ell \in (0, 1)$, $\delta \in (0, 1)$, $\rho > 0$, $\ell^2 \kappa \rho > 1$ and $(\ell, x_0, \tilde{x}_0)$ is a $p$-admissible triple, then,

$$\frac{1}{|Q_\ell(x_0)|} E_0(w_{\ell, x_0, \tilde{x}_0}, F; a, Q_\ell(x_0)) \leq (1 + \delta)\kappa^2 \left[ a_+(\tilde{x}_0, \kappa)^2 \frac{1}{\kappa} \left( \frac{H |B_0(\tilde{x}_0)|}{a_+(\tilde{x}_0, \kappa)} + \frac{1}{2} a_-(\tilde{x}_0, \kappa)^2 \right) \right] + C \left( \frac{1}{\kappa \ell} + \delta^{-1} \ell^2 L(\kappa)^2 + \delta^{-1} \kappa^2 \ell^4 \right) \kappa^2.$$

(3.7)

Proof.

Let

$$R = \ell \sqrt{\kappa H |B_0(\tilde{x}_0)|} \quad \text{and} \quad b = \frac{H |B_0(\tilde{x}_0)|}{\kappa}.$$

First we estimate $\frac{\ell^2}{2} \int_{Q_\ell(x_0)} (a(x, \kappa) - |w_{\ell, x_0, \tilde{x}_0}|^2) \, dx$ from above. Using (3.2), we get the existence of a constant $C > 0$ such that for any $\delta \in (0, 1)$ and any $\kappa \geq \kappa_0$,

$$\frac{\kappa^2}{2} \int_{Q_\ell(x_0)} \left( a(x, \kappa) - |w_{\ell, x_0, \tilde{x}_0}|^2 \right) \, dx \leq (1 + \delta)\kappa^2 \left[ a_+(\tilde{x}_0, \kappa)^2 \frac{1}{\kappa} \left( \frac{H |B_0(\tilde{x}_0)|}{a_+(\tilde{x}_0, \kappa)} + \frac{1}{2} a_-(\tilde{x}_0, \kappa)^2 \right) \right] + C \left( \frac{1}{\kappa \ell} + \delta^{-1} \ell^2 L(\kappa)^2 + \delta^{-1} \kappa^2 \ell^4 \right) \kappa^2 \quad \text{(3.8)}$$

First we estimate $\frac{\ell^2}{2} \int_{Q_\ell(x_0)} |(\nabla - i \kappa H F) w_{\ell, x_0, \tilde{x}_0}|^2 \, dx$ from above. The estimate of $\int_{Q_\ell(x_0)} |(\nabla - i \kappa H F) w_{\ell, x_0, \tilde{x}_0}|^2 \, dx$ from above is the same as in [5] Proposition 3.1]. We have

$$\int_{Q_\ell(x_0)} |(\nabla - i \kappa H F) w_{\ell, x_0, \tilde{x}_0}|^2 \, dx \leq (1 + \delta) \int_{Q_\ell(x_0)} \left| (\nabla - i \kappa H (B_0(\tilde{x}_0) A_0(x - x_0) + \nabla \varphi_{x_0, \tilde{x}_0})) w_{\ell, x_0, \tilde{x}_0} \right|^2 \, dx + C \delta^{-1} \kappa^2 \ell^4 \kappa^2 |w_{\ell, x_0, \tilde{x}_0}|^2.$$

(3.9)

From (1.10), by collecting (3.9), (3.10) and (3.6), we find that,

$$E_0(w_{\ell, x_0, \tilde{x}_0}, F; a, Q_\ell(x_0)) \leq (1 + \delta) E_0(w_{\ell, x_0, \tilde{x}_0}, B_0(\tilde{x}_0) A_0(x - x_0) + \nabla \varphi_{x_0, \tilde{x}_0}; a(\tilde{x}_0, \kappa), Q_\ell(x_0)) + C \delta^{-1} \ell^4 L(\kappa)^2 + \kappa^4 \ell^6 a_+(\tilde{x}_0, \kappa)).$$

(3.11)

As we did in [5], we use the change of variable $y = \frac{R}{\ell}(x - x_0)$ and obtain

$$E_0(w_{\ell, x_0, \tilde{x}_0}, B_0(\tilde{x}_0) A_0(x - x_0) + \nabla \varphi_{x_0, \tilde{x}_0}; a(\tilde{x}_0, \kappa), Q_\ell(x_0))$$

$$= \int_{Q_R} \left[ a_+(\tilde{x}_0, \kappa) \left| \left( \frac{R}{\ell} \nabla - i \frac{R}{\ell} \zeta_\ell A_0(y) \right) \tilde{u}_R(y) \right|^2 + \frac{\kappa^2}{2} \left( a(\tilde{x}_0, \kappa) - a_+(\tilde{x}_0, \kappa) |\tilde{u}_R(y)|^2 \right)^2 \right] \frac{\ell^2}{R^2} dy.$$
Here, we denote by $\zeta$ the sign of $B_0(x_0)$. We distinguish between two cases:

**Case 1:** When $a(\bar{x}_0, \kappa) > 0$, we get

$$E_0(w_{\ell, x_0, \bar{x}_0}, B_0(\bar{x}_0)A_0(x - x_0) + \nabla \varphi_{x_0, \bar{x}_0}; a(\bar{x}_0, \kappa), Q_\ell(x_0)) = \frac{a(\bar{x}_0, \kappa)^2}{b} f_{a(\bar{x}_0, \kappa), Q_\ell}^C + f_{b/a(\bar{x}_0, \kappa), Q_\ell}^C (\tilde{u}_R).$$

From (2.7) and (2.8), we obtain,

$$E_0(w_{\ell, x_0, \bar{x}_0}, B_0(\bar{x}_0)A_0(x - x_0) + \nabla \varphi_{x_0, \bar{x}_0}; a(\bar{x}_0, \kappa), Q_\ell(x_0)) = \frac{1}{b} e_D(b, R, a(\bar{x}_0, \kappa)).$$

(3.12)

As a consequence of the upper bound in (2.13), the ground state energy $e_D(b, R, a(\bar{x}_0, \kappa))$ in (3.12) is bounded for all $b > 0$ and $R \geq 1$ by:

$$e_D(b, R, a(\bar{x}_0, \kappa)) \leq \frac{a(\bar{x}_0, \kappa)^2}{b} f \left( \frac{b}{a(\bar{x}_0, \kappa)} \right) + C_M a(\bar{x}_0, \kappa) \frac{3}{4} R \frac{\sqrt{b}}{R}.$$  

(3.13)

With the choice of $R$ in (3.8), we have effectively $R \geq 1$ which follows from the assumption $R \geq \ell \sqrt{\kappa H} > 1$.

We get from (3.12) and (3.13) the estimate

$$E_0(w_{\ell, x_0, \bar{x}_0}, \zeta | B_0(\bar{x}_0)|A_0(x - x_0) + \nabla \varphi_{x_0, \bar{x}_0}; a(\bar{x}_0, \kappa), Q_\ell(x_0)) \leq a(\bar{x}_0, \kappa)^2 \frac{R^2}{b} f \left( \frac{b}{a(\bar{x}_0, \kappa)} \right) + C_M a(\bar{x}_0, \kappa) \frac{3}{4} R \frac{\sqrt{b}}{R},$$

(3.14)

with $(b, R)$ defined in (3.8).

By collecting the estimates in (3.11)-(3.14) we get,

$$E_0(w_{\ell, x_0, \bar{x}_0}, F; a(\bar{x}_0, \kappa), Q_\ell(x_0)) \leq (1 + \delta) a(\bar{x}_0, \kappa)^2 \frac{R^2}{b} f \left( \frac{b}{a(\bar{x}_0, \kappa)} \right) + C_M a(\bar{x}_0, \kappa) \frac{3}{4} R \frac{\sqrt{b}}{R} + C \delta^{-1} \kappa^2 \ell^4 L(\kappa)^2 + \kappa^4 \ell^4 \bar{a}.$$  

(3.15)

Here, we have used the fact that $a(\bar{x}_0, \kappa) \leq \sup_{x \in \Pi, k \geq k_0} a(x, \kappa) = \bar{a}$.

**Case 2:** When $a(\bar{x}_0, \kappa) \leq 0$, we have,

$$E_0(w_{\ell, x_0, \bar{x}_0}, F; a(\bar{x}_0, \kappa), Q_\ell(x_0)) = \frac{\kappa^2}{2} \int_{Q_\ell(x_0)} a(x, \kappa)^2 dx.$$

From (3.2), we get the existence of a constant $C > 0$ such that for any $\delta \in (0, 1)$,

$$E_0(w_{\ell, x_0, \bar{x}_0}, F; a(\bar{x}_0, \kappa), Q_\ell(x_0)) \leq (1 + \delta) \frac{\kappa^2}{2} a(\bar{x}_0, \kappa)^2 \ell^2 + C \delta^{-1} \kappa^2 \ell^4 L(\kappa)^2.$$  

(3.16)

The results of cases 1-2, we obtain,

$$E_0(w_{\ell, x_0, \bar{x}_0}, F; a(\bar{x}_0, \kappa), Q_\ell(x_0)) \leq (1 + \delta) \kappa^2 \left[ a_+(\bar{x}_0, \kappa)^2 \frac{H}{\kappa} \left( 1 + \frac{1}{2} a_-(\bar{x}_0, \kappa)^2 \right) \right] \ell^2 + C \left( \frac{\kappa}{\ell} \pi^2 + \delta^{-1} \kappa^2 \ell^2 L(\kappa)^2 + \delta^{-1} \kappa^4 \ell^4 \bar{a} \right) \ell^2,$$

(3.17)

which finishes the proof of Proposition 3.2.

**Application 3.3.**

We select $\ell$, $\rho$, $\delta$ and the constraint on $L(\kappa)$ as follows:

$$\hat{\ell} = \kappa^{-\frac{2}{7}} \ell, \quad \rho = \kappa^{-\frac{17}{24}} \rho, \quad L(\kappa) \leq C \kappa^\frac{1}{2}.$$

(3.18)
and

\[ \delta = \kappa^{-\frac{1}{12}} \quad (3.19) \]

Under Assumption \( \{1.15\} \), this choice permits to verify the assumptions in Proposition 3.2 and to obtain error terms of order \( o(\kappa^2) \). We have indeed as \( \kappa \to \infty \)

\[ \frac{\kappa}{\ell} = \kappa^{\frac{19}{12}} \ll \kappa^2, \]

\[ \delta^{-1} \kappa^2 \ell^2 L(\kappa)^2 \leq \kappa^{\frac{23}{12}} \ll \kappa^2, \]

\[ \delta^{-1} \kappa^4 \ell^4 = \kappa^{\frac{31}{12}} \ll \kappa^2, \]

\[ \ell^2 \kappa H \rho = \kappa^{\frac{1}{12}} \gg 1. \]

**Theorem 3.4.** Under Assumptions \( \{1.4\}, \{1.8\} \), if \( \{1.15\} \) holds and \( L(\kappa) \leq C \kappa^{\frac{1}{2}} \), then, the ground state energy \( E_g(\kappa, H, a, B_0) \) in \( \{1.14\} \) satisfies

\[ E_g(\kappa, H, a, B_0) \leq \kappa^2 \int_{\{a(x, \kappa) > 0\}} a(x, \kappa)^2 \hat{f} \left( \frac{H |B_0(x)|}{\kappa a(x, \kappa)} \right) dx + \frac{\kappa^2}{2} \int_{\{a(x, \kappa) \leq 0\}} a(x, \kappa)^2 dx + o(\kappa^2), \quad \text{as} \ \kappa \to \infty. \quad (3.20) \]

*Proof.* Let \( \ell \in (0, 1) \), \( \delta \) and \( \rho \) be chosen as in \( (3.18) \) and \( (3.19) \). We consider the lattice \( \Gamma_\ell := \ell \mathbb{Z} \times \ell \mathbb{Z} \) and write, for \( \gamma \in \Gamma_\ell \), \( Q_{\gamma, \ell} = Q_{\ell}(\gamma) \). In the next decomposition we keep the \( \rho \)-admissible boxes \( Q_{\ell}(\gamma) \) in \( \Omega \) which in addition are either contained in \( \{a > 0\} \) or in \( \{a \leq 0\} \). Hence we introduce

\[ I_{\ell, \rho}^+ = \{ \gamma; \overline{Q_{\gamma, \ell}} \subset \Omega \cap \{|B_0| > \rho; a > 0\} \}, \quad I_{\ell, \rho}^- = \{ \gamma; \overline{Q_{\gamma, \ell}} \subset \Omega \cap \{|B_0| > \rho; a \leq 0\} \}, \quad (3.21) \]

and

\[ N^+ = \text{card} I_{\ell, \rho}^+, \quad N^- = \text{card} I_{\ell, \rho}^-. \quad (3.22) \]

Under Assumption \( \{1.8\} \), we have,

\[ N^+ + N^- = |\Omega| \ell^{-2} + \mathcal{O}(\kappa^{\frac{1}{2}} \ell^{-1} + \ell^{-1} + \rho \ell^{-2}), \quad \text{as} \ \kappa \to +\infty. \quad (3.23) \]

In \( (3.23) \), \( \kappa^{\frac{1}{2}} \ell^{-1} \) appears when treating the boundary of the set \( \{a(x, \kappa) > 0\} \) (using Assumption \( \{A_1\} \) as explained in \( (3.1) \), \( \ell^{-1} \) appears in the treatment of the boundary and \( \rho \ell^{-2} \) appears when treating the neighborhood of \( \Gamma \).

In each \( \rho \)-admissible \( Q_{\ell}(\gamma) \), we consider some \( \tilde{\gamma} \) (to be chosen later) such that \( (\ell, \gamma, \tilde{\gamma}) \) be a \( \rho \)-admissible triple. We consider \( w_{\ell, \gamma, \tilde{\gamma}} \) and extend it by 0 outside of \( Q_{\gamma, \ell} \), keeping the same notation for this extension. Then we define

\[ s(x) = \sum_{\gamma \in I_{\ell, \rho}^+ \cup I_{\ell, \rho}^-} w_{\ell, \gamma, \tilde{\gamma}}(x). \quad (3.24) \]

We compute the Ginzburg-Landau energy of the test configuration \( (s, \mathbf{F}) \) in \( \Omega \). Since \( \text{curl} \mathbf{F} = B_0 \), we get,

\[ E_{\kappa, H, a, B_0}(s, \mathbf{F}, \Omega) = \sum_{\gamma \in I_{\ell, \rho}^+ \cup I_{\ell, \rho}^-} E_0(w_{\ell, \gamma, \tilde{\gamma}}, \mathbf{F}; a(\tilde{\gamma}, \kappa), Q_{\gamma, \ell}). \quad (3.25) \]

Notice that for any \( \tilde{\gamma} \in Q_{\gamma, \ell} \), \( a(\tilde{\gamma}, \kappa) \) satisfies \( (3.2) \) with \( x = \gamma \) and \( \tilde{x}_0 = \tilde{\gamma} \), and \( B_0(\tilde{\gamma}) \) satisfies \( (3.4) \). We recall that \( \hat{f} \) is a continuous, non-decreasing function (see \[ \{5\} \) Theorem 2.1]) and that \( B_0 \) and \( a(\cdot, \kappa) \) are in \( C^1 \). Then, in each box \( Q_{\gamma, \ell} \), we select \( \tilde{\gamma} \in Q_{\gamma, \ell} \) such that

\[ |a(\tilde{\gamma}, \kappa)|^2 \hat{f} \left( \frac{H B_0(\tilde{\gamma})}{\kappa a(\tilde{\gamma}, \kappa)} \right) = \inf_{\gamma \in Q_{\gamma, \ell}} |a(\gamma, \kappa)|^2 \hat{f} \left( \frac{H B_0(\tilde{\gamma})}{\kappa a(\tilde{\gamma}, \kappa)} \right) \quad (\text{if} \ \gamma \in I_{\ell, \rho}^+). \]
Using Proposition 3.2 and noticing that \(|Q_{\gamma,\ell}| = \ell^2\), we get the existence of \(C > 0\) such that, for any \(\delta \in (0,1)\)

\[
\sum_{\gamma \in \tilde{I}_{\ell,\rho}} \mathcal{E}_0(w_{\ell,\gamma}; F; a(\gamma, \kappa), Q_{\gamma,\ell}) \leq \kappa^2(1 + \delta) \sum_{\gamma \in \tilde{I}_{\ell,\rho}} \inf_{\gamma \in Q_{\gamma,\ell}} |a(\gamma, \kappa)|^2 \hat{f} \left( \frac{H B_0(\gamma)}{\kappa a(\gamma, \kappa)} \right) \ell^2 + \kappa^2(1 + \delta) \sum_{\gamma \in Q_{\gamma,\ell}} \frac{|a(\gamma, \kappa)|^2}{2} \ell^2 + C \sum_{\gamma \in \tilde{I}_{\ell,\rho}} \left( \frac{\kappa^2}{\ell} + \delta^{-1} \kappa^2 \ell^2 (\kappa^2 + \delta^{-1} \kappa^4 \ell^4) \right) \ell^2. \tag{3.26}
\]

We recognize the lower Riemann sum of the function \(x \mapsto |a(x, \kappa)|^2 \hat{f} \left( \frac{H B_0(x)}{\kappa a(x, \kappa)} \right) \) in \((\cup_{\gamma \in \tilde{I}_{\ell,\rho}} Q_{\gamma,\ell})\) and the function \(x \mapsto |a(x, \kappa)|^2 \) in \((\cup_{\gamma \in \tilde{I}_{\ell,\rho}} Q_{\gamma,\ell})\). Notice that \(\{\cup_{\gamma \in \tilde{I}_{\ell,\rho}} Q_{\gamma,\ell}\} \subset \Omega\). Thanks to Application 3.3 using (3.23) and the non negativity of \(\hat{f}\), we get by collecting (3.25)-(3.26) that,

\[
\mathcal{E}_{n,H,a,B_0}(s, F, \Omega) \leq \kappa^2 \int_{\{a(x, \kappa) > 0\}} a(x, \kappa)^2 \hat{f} \left( \frac{H B_0(x)}{\kappa a(x, \kappa)} \right) \, dx + \frac{\kappa^2}{2} \int_{\{a(x, \kappa) \leq 0\}} a(x, \kappa)^2 \, dx + C \kappa^2. \tag{3.27}
\]

Since \((\psi, A)\) is a minimizer of the functional \(\mathcal{E}_{n,H,a,B_0}\) in (1.1), we get

\[
E_g(\kappa, H, a, B_0) \leq \mathcal{E}_{n,H,a,B_0}(s, F, \Omega). \tag{4.1}
\]

This finishes the proof of Theorem 3.4. 

\[ \square \]

4. A PRIORI ESTIMATES OF MINIMIZERS

The aim of this section is to give a priori estimates for the solutions of the Ginzburg-Landau equations (1.12). In the case when \(a(x, \kappa) = 1\) the starting point is an \(L^\infty\) estimate of \(\psi\). This estimate can be easily extended in the general case considered in this paper when (1.12) and (1.12) hold. Let us introduce:

\[
\overline{\rho}(\kappa) = \sup_{x \in \overline{\Omega}} a(x, \kappa). \tag{4.1}
\]

**Proposition 4.1.** Let \(\kappa > 0\); if \((\psi, A)\) is a critical point (see (1.12)), then,

\[
|\psi(x)|^2 \leq \max \{\overline{\rho}(\kappa), 0\}, \quad \forall x \in \overline{\Omega}. \tag{4.2}
\]

**Proof.** We distinguish between two cases:

**Case 1:** \(\overline{\rho}(\kappa) \leq 0\).

Multiplying the equation for \(\psi\) in (1.12) by \(\overline{\psi}\) and integrating over \(\Omega\), we get

\[
\int_{\Omega} |(\nabla - i\kappa H A)\psi|^2 \, dx = \kappa^2 \int_{\Omega} (a(x, \kappa) - |\psi|^2)|\psi|^2 \, dx. \tag{4.3}
\]

Since \((a(x, \kappa) - |\psi|^2) \leq -|\psi|^2\), we obtain that \(|\psi|^2 = 0\) almost everywhere.

**Case 2:** \(\overline{\rho}(\kappa) > 0\).

We will show that \(\psi \in C^0(\overline{\Omega})\). In fact, \((\psi, A)\) satisfies (1.12), \(\psi \in L^p(\Omega)\) for all \(2 \leq p < +\infty\) and \(A \in H^{1}_{\text{div}}(\Omega) \hookrightarrow L^p(\Omega)\). Thus, \(\psi \in W^{2,q}(\Omega)\) for all \(q < 2\). As a consequence of the continuous Sobolev embedding of \(W^{j+m,q}(\Omega)\) into \(C^j(\overline{\Omega})\) for any \(q \geq \frac{2}{m}\), we obtain that \(\psi \in C^0(\overline{\Omega})\). Define for any \(\kappa > 0\) the following open set:

\[
\Omega_+ = \left\{ x \in \Omega : |\psi(x)| > \sqrt{\overline{\rho}(\kappa)} \right\}. \tag{4.4}
\]
and the following functions on $\Omega_+$

\[
\phi = \frac{\psi}{|\psi|}, \quad \hat{\psi} = \left[|\psi| - \sqrt{\pi(\kappa)}\right]_+ \phi.
\]

It is clear that

\[
\nabla \left[|\psi| - \sqrt{\pi(\kappa)}\right]_+ = 1_{\Omega_+} \nabla \left(\left[|\psi| - \sqrt{\pi(\kappa)}\right]\right) = 1_{\Omega_+} \nabla |\psi|.
\]

Notice that $\psi \in H^1(\Omega)$, so applying Proposition 3.1.2, we get the property that \(\nabla \left[|\psi| - \sqrt{\pi(\kappa)}\right]_+ \in L^2(\Omega)\), which implies that \(\left[|\psi| - \sqrt{\pi(\kappa)}\right]_+ \in H^1(\Omega)\).

We introduce an increasing cut-off function $\chi \in C^\infty(\mathbb{R})$ such that,

\[
\chi(t) = \begin{cases} 0 & \text{for } t \leq \frac{1}{4} \sqrt{\pi(\kappa)} \\ 1 & \text{for } t \geq \frac{3}{4} \sqrt{\pi(\kappa)} \end{cases},
\]

and define

\[
\hat{\phi} = \chi(|\psi|)\frac{\psi}{|\psi|}.
\]

Since $\chi(|\psi|)\frac{\psi}{|\psi|}$ is smooth with bounded derivatives and $\psi \in H^1(\Omega)$, the chain rule gives that $\hat{\phi} \in H^1(\Omega)$. Furthermore,

\[
(\nabla - i\kappa H A)\hat{\phi} = 1_{\Omega_+} \hat{\phi} \nabla |\psi| + \left[|\psi| - \sqrt{\pi(\kappa)}\right]_+ (\nabla - i\kappa HA)\hat{\phi}.
\]

Using (4.5) and (4.6), we get

\[
1_{\Omega_+}(\nabla - i\kappa H A)\psi = 1_{\Omega_+}(\nabla - i\kappa H A)(|\psi| \hat{\phi}) = 1_{\Omega_+} \{\hat{\phi} \nabla |\psi| + |\psi|(\nabla - i\kappa HA)\hat{\phi}\}.
\]

We have on $\Omega_+$ that $|\phi| = |\hat{\phi}| = 1$. Therefore

\[
\phi \nabla \hat{\phi} + \hat{\phi} \nabla \phi = \phi \nabla \phi + \hat{\phi} \nabla \phi
\]

\[
= |\phi|^2
\]

\[
= 0.
\]

So, $\text{Re}(1_{\Omega_+} \phi \nabla \hat{\phi}) = 0$. This implies by using (4.7) and (4.8) that

\[
\text{Re} \left\{ (\nabla - i\kappa H A)\hat{\phi} \cdot (\nabla - i\kappa H A)\psi \right\} = 1_{\Omega_+} \left( |\nabla |\psi||^2 + \left(|\psi| - \sqrt{\pi(\kappa)}\right)|\psi||(\nabla - i\kappa HA)\hat{\phi}|^2 \right).
\]

Multiplying (1.12) by $\overline{\psi}$ and using (1.12), it results from an integration by parts over $\Omega$ that

\[
0 = \text{Re} \left\{ \int_{\Omega} (\nabla - i\kappa H A)\overline{\psi}(\nabla - i\kappa H A)\psi + \overline{\psi}(|\psi|^2 - a)\psi \, dx \right\}
\]

\[
\geq \text{Re} \left\{ \int_{\Omega} (\nabla - i\kappa H A)\overline{\psi}(\nabla - i\kappa H A)\psi + \overline{\psi} \left(|\psi|^2 - \pi(\kappa)\right) \psi \, dx \right\}
\]

\[
\geq \int_{\Omega_+} |\nabla |\psi||^2 + (|\psi| - \pi(\kappa)) |\psi||(\nabla - i\kappa HA)\hat{\phi}|^2
\]

\[
+ \left(|\psi| + \sqrt{\pi(\kappa)}\right) \left(|\psi| - \sqrt{\pi(\kappa)}\right)^2 |\psi| \, dx.
\]

Since the integrand is non-negative in $\Omega_+$, we easily conclude that $\Omega_+$ has measure zero, and consequently, we get that $|\psi| \in L^\infty(\Omega)$.

Since $\Omega_+$ has measure zero and $\psi \in C^0(\overline{\Omega})$, we get

\[
|\psi(x)|^2 \leq \pi(\kappa), \quad \forall x \in \overline{\Omega}.
\]

\[\square\]
Corollary 4.2. Let \( \kappa > 0 \); if \((\psi, A) \in H^1(\Omega; \mathbb{C}) \times H^1_{\text{div}}(\Omega)\) is a critical point, we have,
\[
|\psi(x)|^2 \leq \max \{\pi, 0\}, \quad \forall x \in \overline{\Omega},
\]
where \(\overline{\Omega} = \sup_{x} \overline{\Omega}(\kappa)\) was introduced in (1.10).

The following estimates play an essential role in controlling the errors resulting from various approximations (see Section 5). These estimates are simpler than the delicate elliptic estimates in [12] and [30].

Proposition 4.3. Suppose that (1.15) holds. Let \(\beta \in (0, 1)\). There exist positive constants \(\kappa_0\) and \(C\) such that, if \(\kappa \geq \kappa_0\) and \((\psi, A)\) is a minimizer of (1.1), then
\[
\|\text{curl}(A - F)\|_{L^2(\Omega)} \leq \frac{C}{H}. \tag{4.10}
\]
\[
\|A - F\|_{H^2(\Omega)} \leq \frac{C}{H}, \tag{4.11}
\]
\[
\|A - F\|_{C^{0,\beta}(\overline{\Omega})} \leq \frac{C}{H}. \tag{4.12}
\]

Here we recall that \(F\) is the magnetic potential defined by
\[
\text{curl} F = B_0, \quad F \in H^1_{\text{div}}(\Omega). \tag{4.13}
\]

Proof. Under Assumption (1.15), Theorem 3.4 yields
\[
\|\text{curl}(A - F)\|_{L^2(\Omega)} \leq \frac{1}{\kappa H} E_g(\kappa, H, a, B_0)^{\frac{1}{2}}
\]
\[
\leq \frac{1}{\kappa H} \left( \kappa^2 \int_{\{a(x, \kappa) > 0\}} a(x, \kappa)^2 f \left( \frac{H |B_0(x)|}{\kappa a(x, \kappa)} \right) \, dx + \kappa^2 \frac{1}{2} \int_{\{a(x, \kappa) \leq 0\}} a(x, \kappa)^2 \, dx \right)^{\frac{1}{2}}. \tag{4.14}
\]

Using (1.6) and the bound \(f(b) \leq \frac{1}{2}\), we get,
\[
\|\text{curl}(A - F)\|_{L^2(\Omega)} \leq \frac{C}{H}. \tag{4.15}
\]

As in [5], Proposition 4.1], we prove that
\[
\|A - F\|_{H^2(\Omega)} \leq \frac{C}{H}. \tag{4.16}
\]

Now, the estimate in \(C^{0,\beta}\)-norm is a consequence of the continuous Sobolev embedding of \(H^2(\Omega)\) in \(C^{0,\beta}(\overline{\Omega})\). \(\square\)

5. Lower bounds for the global and local energies

In this section, we suppose that \(\mathcal{D}\) is an open set with smooth boundary such that \(\overline{\mathcal{D}} \subset \Omega\) (or \(\mathcal{D} = \Omega\)). We will give a lower bound of the ground state energy \(E_g(\kappa, H, a, B_0)\) introduced in (1.14).

Proposition 5.1. Under Assumptions (1.4)-(1.7), there exist for all \(\beta \in (0, 1)\) positive constants \(C\) and \(\kappa_0\) such that if \(\kappa \geq \kappa_0\), \(\ell \in (0, \frac{1}{2})\), \(\delta \in (0, 1)\), \(\rho > 0\), \(\ell^2 \kappa H \rho > 1\), \((\psi, A)\) is a minimizer of (1.1), \(h \in C^1(\Omega)\), \(|h|_\infty \leq 1\) and \((\ell, x_0, \tilde{x}_0)\) is a \(\rho\)-admissible triple, then,
\[
\frac{1}{|Q_\ell(x_0)|} E_0(h\psi, A; a, Q_\ell(x_0)) \geq (1 - \delta) \kappa^2 \left\{ a_+(\tilde{x}_0, \kappa)^2 f \left( \frac{H |B_0(\tilde{x}_0)|}{\kappa a_+(\tilde{x}_0, \kappa)} \right) + \frac{1}{2} a_-(\tilde{x}_0, \kappa)^2 \right\}
\]
\[
- C \kappa^2 \left( \delta^{-1} \ell^2 L(\kappa)^2 + \delta^{-1} \kappa^2 \ell^4 + \delta^{-1} \ell^2 \beta + (\kappa \ell)^{-1} + \ell L(\kappa) \right), \tag{5.1}
\]
where \(L(\kappa)\) is introduced in (1.9).
Proof. We distinguish between two cases according to the sign of \(a(x_0, \kappa)\).

We begin with the case when \(a(x_0, \kappa) \leq 0\). We have,

\[
\mathcal{E}_0(h\psi, A; a, Q_\ell(x_0)) = \int_{Q_\ell(x_0)} |(\nabla - i\kappa H A)h\psi|^2\, dx + \frac{\kappa^2}{2} \int_{Q_\ell(x_0)} (a(x, \kappa) - |h\psi|^2)^2\, dx
\]

\[
\geq \frac{\kappa^2}{2} \int_{Q_\ell(x_0)} a(x, \kappa)^2\, dx - \kappa^2 \int_{Q_\ell(x_0)} a(x, \kappa)|h\psi|^2\, dx.
\]

Using (3.2), (4.9) and the assumptions on \(h\), the simple decomposition \(a(x, \kappa) = a(x_0, \kappa) + (a(x, \kappa) - a(x_0, \kappa))\) yields for any \(\delta \in (0, 1)\)

\[
\frac{\kappa^2}{2} \int_{Q_\ell(x_0)} a(x, \kappa)^2\, dx \geq (1 - \delta) \frac{\kappa^2}{2} \int_{Q_\ell(x_0)} a(x_0, \kappa)^2\, dx
\]

\[
+ (1 - \delta^{-1}) \frac{\kappa^2}{2} \int_{Q_\ell(x_0)} (a(x, \kappa) - a(x_0, \kappa))^2\, dx
\]

\[
\geq (1 - \delta) \frac{\kappa^2}{2} a(x_0, \kappa)^2 |Q_\ell(x_0)| - C^2 \delta^{-1} \kappa^2 \ell^2 L(\kappa)^2 |Q_\ell(x_0)|,
\]  

(5.2)

and

\[
-\kappa^2 \int_{Q_\ell(x_0)} a(x, \kappa)|h\psi|^2\, dx \geq -\kappa^2 \int_{Q_\ell(x_0)} a(x_0, \kappa)|h\psi|^2\, dx - C \ell L(\kappa) \kappa^2 |Q_\ell(x_0)|
\]

\[
\geq -C \ell L(\kappa) \kappa^2 |Q_\ell(x_0)|.
\]  

(5.3)

Collecting (5.2) and (5.3), we get,

\[
\frac{1}{|Q_\ell(x_0)|} \mathcal{E}_0(h\psi, A; a, Q_\ell(x_0)) \geq (1 - \delta) \frac{\kappa^2}{2} a(x_0, \kappa)^2 - C^2 \delta^{-1} \kappa^2 \ell^2 L(\kappa)^2 - C' \ell L(\kappa) \kappa^2.
\]

(5.4)

Now, we treat the case when \(a(x_0, \kappa) > 0\). Let \(\phi_{x_0}(x) = (A(x_0) - F(x_0)) \cdot x\), where \(F\) is the magnetic potential introduced in (4.13). Using the estimate of \(\|A - F\|_{C^0, \beta(\Omega)}\) given in Proposition 4.3, we get for any \(\beta \in (0, 1)\) the existence of a constant \(C\) such that for all \(x \in Q_\ell(x_0)\),

\[
|A(x) - \nabla \phi_{x_0} - F(x)| \leq C \frac{\ell \beta}{H}.
\]

(5.5)

Let \(\bar{x}_0 \in Q_\ell(x_0)\) and \(\varphi = \varphi_{x_0, \bar{x}_0} + \phi_{x_0}\) with \(\varphi_{x_0, \bar{x}_0}\) satisfying (3.4). We define the function in \(Q_\ell(x_0)\),

\[
u(x) = e^{-i\kappa H \varphi} h\psi(x).
\]

(5.6)

Similarly to (3.9), we have, for any \(\delta \in (0, 1)\),

\[
\frac{\kappa^2}{2} \int_{Q_\ell(x_0)} (a(x, \kappa) - |h\psi|^2)^2\, dx \geq (1 - \delta) \frac{\kappa^2}{2} \int_{Q_\ell(x_0)} (a(x_0, \kappa) - |h\psi|^2)^2\, dx - C^4 \delta^{-1} \kappa^2 \ell^4 L(\kappa)^2.
\]

(5.7)

Using the same techniques as in [4] Lemma 4.1, we get, for any \(\beta \in (0, 1)\),

\[
\int_{Q_\ell(x_0)} |(\nabla - i\kappa H A)h\psi|^2\, dx \geq (1 - \delta) \int_{Q_\ell(x_0)} |(\nabla - i\kappa H (\zeta_0 B_0(x_0) + \nabla \varphi)) h\psi|^2\, dx
\]

\[
- C^4 \delta^{-1} (\kappa H)^2 \left( \ell^4 + \frac{\ell^2 \beta}{H^2} \right) \int_{Q_\ell(x_0)} |h\psi|^2\, dx.
\]

(5.8)
Thus, by collecting (5.7) and (5.8), using (1.7), 4.9 and \( \| h \|_{L^\infty(\Omega)} \leq 1 \), we get
\[
\mathcal{E}_0(h\psi, A; a(\vec{x}_0, \kappa), Q_{\ell}(x_0)) \geq (1 - \delta)\mathcal{E}_0(e^{-\imath \kappa H} \psi(x), \zeta_\ell B_0(\vec{x}_0)|A_0(x-x_0);a(\vec{x}_0, \kappa), Q_{\ell}(x_0)) \\
- C\delta^{-1}\kappa^2 \ell^2 L(\kappa)^2 - C_1\delta^{-1} \kappa^2 H^2 \left( \ell^4 + \frac{\ell^2 \beta}{H^2} \right) \ell^2.
\]
(5.9)

Let \( R \) and \( b \) be as in (3.8). Let us introduce the function \( v_{\ell,x_0,\vec{x}_0} \) in \( Q_R \) as follows:
\[
v_{\ell,x_0,\vec{x}_0}(x) = \begin{cases} 
  u \left( \frac{\ell}{R} x + x_0 \right) & \text{if } x \in Q_R \cap \{ B_0 > \rho \} \cap \Omega \\
  \pi \left( \frac{\ell}{R} x + x_0 \right) & \text{if } x \in Q_R \cap \{ B_0 < -\rho \} \cap \Omega,
\end{cases}
\]
where \( u \) is defined in (5.6).

Similarly to (3.12), we use the change of variable \( y = \frac{\ell}{R}(x - x_0) \) and get
\[
\mathcal{E}_0(e^{-\imath \kappa H} \psi(x), \zeta_\ell \kappa H|B_0(\vec{x}_0)|A_0(x-x_0);a(\vec{x}_0, \kappa), Q_{\ell}(x_0)) = \frac{1}{b} \mathcal{F}_{b,Q_R}^{+1,a(\vec{x}_0, \kappa)}(v_{\ell,x_0,\vec{x}_0}),
\]
(5.11)

where \( \mathcal{F}_{b,Q_R}^{+1,a(\vec{x}_0, \kappa)} \) is introduced in (2.1).

Since \( v_{\ell,x_0,\vec{x}_0} \in H^1(Q_R) \) then, using (2.12) and (2.13), we get
\[
\frac{1}{b} \mathcal{F}_{b,Q_R}^{+1,a(\vec{x}_0, \kappa)}(v_{\ell,x_0,\vec{x}_0}) \geq \frac{1}{b} c_N(b, R, a(\vec{x}_0, \kappa)) \\
\geq \frac{1}{b} c_D(b, R, a(\vec{x}_0, \kappa)) - C_M a(\vec{x}_0, \kappa) \frac{R}{\sqrt{b}} \\
\geq a(\vec{x}_0, \kappa)^2 \frac{R^2}{b} \mathcal{F} \left( \frac{b}{a(\vec{x}_0, \kappa)} \right) - \tilde{C}_M \frac{R}{\sqrt{b}}.
\]
(5.12)

Inserting (5.12) into (5.11), we get
\[
\mathcal{E}_0(e^{-\imath \kappa H} \psi(x), \zeta_\ell \kappa H|B_0(\vec{x}_0)|A_0(x-x_0);a(\vec{x}_0, \kappa), Q_{\ell}(x_0)) \geq a(\vec{x}_0, \kappa)^2 \frac{R^2}{b} \mathcal{F} \left( \frac{b}{a(\vec{x}_0, \kappa)} \right) \\
- \tilde{C}_M \frac{R}{\sqrt{b}}.
\]
(5.13)

Having in mind (3.8) and (5.13), we get from (5.9),
\[
\frac{1}{Q_{\ell}(x_0)} \mathcal{E}_0(h\psi, A; a(\vec{x}_0, \kappa), Q_{\ell}(x_0)) \geq (1 - \delta)\kappa^2 a(\vec{x}_0, \kappa)^2 \mathcal{F} \left( \frac{H B_0(\vec{x}_0)}{\kappa a(\vec{x}_0, \kappa)} \right) \\
- C\delta^{-1}\kappa^2 \ell^2 L(\kappa)^2 - C_1\delta^{-1} \kappa^2 H^2 \left( \ell^4 + \frac{\ell^2 \beta}{H^2} \right) - C_2 \frac{\kappa}{\ell}.
\]
(5.14)

The estimates in (5.4) and (5.14) achieve the proof of Proposition 5.1.

\[ \square \]

**Application 5.2.** We keep the same choice of \( \ell, \rho, L(\kappa) \) and \( \delta \) as in (3.18), (3.19) and choose:
\[
\beta = \frac{3}{4}.
\]
(5.15)

This choice and Assumption 1.15 permit to have the assumptions in Proposition 5.1 satisfied and make the error terms in its statement of order \( o(\kappa^2) \). We have as \( \kappa \to \infty \),
\[
\delta^{-1}\kappa^4 \ell^4 = \kappa^{\frac{21}{14}} \ll \kappa^2,
\delta^{-1}\kappa^2 \ell^2 \beta = \kappa^{\frac{9}{14}} \ll \kappa^2,
\delta^{-1}\kappa^2 \ell^2 L(\kappa)^2 = \kappa^{\frac{13}{14}} \ll \kappa^2,
\kappa^2 = \kappa^{\frac{10}{11}} \ll \kappa^2,
\ell L(\kappa) \kappa^2 = \kappa^{\frac{23}{22}} \ll \kappa^2,
\]
\[ \ell^2 \kappa \hbar \rho = \kappa^3 \gg 1. \]

The next theorem presents a lower bound of the local energy in a relatively compact smooth domain \( D \) in \( \Omega \). We deduce the lower bound of the global energy by replacing \( D \) by \( \Omega \).

**Theorem 5.3.**

Under Assumptions \((1.4)-(1.8)\), if \((1.15)\) holds, \( L(\kappa) \leq C \kappa^{\frac{1}{2}} \) with \( C > 0 \), \( h \in C^1(\overline{\Omega}) \), \( \|h\|_{\infty} \leq 1 \), \((\psi, A)\) is a minimizer of \((1.1)\) and \( D \) an open set in \( \Omega \), then as \( \kappa \to +\infty \),

\[ \mathcal{E}(h\psi, A; a, B_0, D) \geq \mathcal{E}_0(h\psi, A; a, D) \geq \kappa^2 \int_{D \cap \{a(x) > 0\}} a(x, \kappa)^2 \hat{f} \left( \frac{H \, |B_0(x)|}{\kappa \, a(x, \kappa)} \right) \, dx \]

\[ + \frac{\kappa^2}{2} \int_{D \cap \{a(x) \leq 0\}} a(x, \kappa)^2 \, dx + o(\kappa^2). \quad (5.16) \]

**Proof.** The proof is similar to the one in Theorem 3.4 and we keep the same notation. Let

\[ D_{\ell,\rho}^+ = \left( \bigcup_{\gamma \in \mathcal{I}_{\ell,\rho}} Q_{\gamma,\ell} \right) \quad \text{and} \quad D_{\ell,\rho}^- = \left( \bigcup_{\gamma \in \mathcal{I}_{\ell,\rho}} \overline{Q_{\gamma,\ell}} \right), \]

where \( \gamma \in \mathcal{I}_{\ell,\rho}^+ \) and \( \gamma \in \mathcal{I}_{\ell,\rho}^- \) are introduced in (3.21).

Thanks to Proposition 5.1, we can easily prove the existence of positive constant \( C \) such that for any \( \delta \in (0, 1) \) and \( \beta \in (0, 1) \),

\[ \mathcal{E}_0(h\psi, A; a, D) \geq \kappa^2 (1 - \delta) \left\{ \int_{D_{\ell,\rho}^+ \cap \{a(x) > 0\}} a(x, \kappa)^2 \hat{f} \left( \frac{H \, |B_0(x)|}{\kappa \, a(x, \kappa)} \right) \, dx \right\} \]

\[ + \frac{1}{2} \int_{D_{\ell,\rho}^- \cap \{a(x) \leq 0\}} a(x, \kappa)^2 \, dx \quad - C \, r(\kappa, \ell, \delta, \rho, L(\kappa), \beta), \]

where

\[ r(\kappa, \ell, \delta, \rho, L(\kappa), \beta) = \kappa^2 \ell + \kappa^2 \rho + \frac{\kappa}{\ell} + \delta^{-1} \kappa^2 \ell^2 L(\kappa)^2 + \delta^{-1} \kappa^4 \ell^4 + \delta^{-1} \kappa^2 \ell^2 \beta + \ell \, L(\kappa) \kappa^2. \quad (5.17) \]

Notice that using the regularity of \( \partial D \), \((1.4)\) and \((1.8)\) (see (3.1)), we get the existence of constants \( C_1 \) and \( C_2 \) such that,

\[ \forall \ell \leq C_2 \kappa^{-\frac{1}{2}}, \quad \forall \rho \in (0, 1), \quad |D \setminus D_{\ell,\rho}^+| + |D \setminus D_{\ell,\rho}^-| \leq C_1 (\kappa^2 \ell + \rho). \quad (5.18) \]

This implies by using \((1.7)\) and the upper bound \( \hat{f} \leq \frac{1}{2} \),

\[ \int_{D_{\ell,\rho}^+ \cap \{a(x) > 0\}} a(x, \kappa)^2 \hat{f} \left( \frac{H \, |B_0(x)|}{\kappa \, a(x, \kappa)} \right) \, dx \geq \int_{D_{\ell,\rho}^+ \cap \{a(x) > 0\}} a(x, \kappa)^2 \hat{f} \left( \frac{H \, |B_0(x)|}{\kappa \, a(x, \kappa)} \right) \, dx \]

\[ - \frac{1}{2} \pi |D \setminus D_{\ell,\rho}| \quad (5.19) \]

and

\[ \frac{1}{2} \int_{D_{\ell,\rho}^- \cap \{a(x) \leq 0\}} a(x, \kappa)^2 \, dx \geq \frac{1}{2} \int_{D_{\ell,\rho}^- \cap \{a(x) \leq 0\}} a(x, \kappa)^2 \, dx - \frac{1}{2} \pi |D \setminus D_{\ell,\rho}^-|, \quad (5.20) \]

where \( \pi \) is introduced in \((1.10)\).

Collecting \((5.19)\) and \((5.20)\), using Assumptions \((1.6)\) and \((5.18)\), we find that,

\[ \mathcal{E}_0(h\psi, A; a, D) \geq \kappa^2 (1 - \delta) \left\{ \int_{D \cap \{a(x) > 0\}} a(x, \kappa)^2 \hat{f} \left( \frac{H \, |B_0(x)|}{\kappa \, a(x, \kappa)} \right) \, dx \right\} \]

\[ + \frac{1}{2} \int_{D \cap \{a(x) \leq 0\}} a(x, \kappa)^2 \, dx \quad - C \, \hat{r}(\kappa, \ell, \delta, \rho, L(\kappa), \beta), \quad (5.21) \]
where \( \hat{r}(\kappa, \ell, \delta, \rho, L(\kappa), \beta) \) satisfies (5.17).

Under Assumption (1.15), the choice of the parameters \( \rho, \ell, L(\kappa) \) in (3.18), \( \delta \) in (3.19) and \( \beta \) in (5.15), implies that all error terms are of lower order compared to \( \kappa^2 \).

As a consequence of (1.15), the inequality (5.21) becomes as \( \kappa \to +\infty \)

\[
\mathcal{E}_0(h\psi, A; a, D) \geq \kappa^2 \left\{ \int_{D \cap \{a(x,\kappa) > 0\}} a(x,\kappa)^2 \hat{f} \left( \frac{H |B_0(x)|}{\kappa a(x,\kappa)} \right) \, dx + \frac{1}{2} \int_{D \cap \{a(x,\kappa) \leq 0\}} a(x,\kappa)^2 \, dx \right\} + o(\kappa^2). \tag{5.22}
\]

Moreover, we know that

\[
\mathcal{E}(h\psi, A; a, B_0, D) \geq \mathcal{E}_0(h\psi, A; a, D).
\]

This achieves the proof of Theorem 5.3.

\( \square \)

As we now show, Theorem 5.3 permits to achieve the proof of two statements presented in the introduction:

**Proof of Corollary 1.3.**

If \((\psi, A)\) is a minimizer of (1.1), we have

\[
E_0(\kappa, H) = \mathcal{E}_0(\psi, A; a, \Omega) + (\kappa H)^2 \int_{\Omega} |\text{curl} (A - F)|^2 \, dx, \tag{5.23}
\]

where \( \mathcal{E}_0(\psi, A; a, \Omega) \) is defined in (1.19).

Using (1.17) and (5.22) (with \( D = \Omega \)), then under Assumption (1.15) as \( \kappa \to +\infty \)

\[
\mathcal{E}_0(\psi, A; a, \Omega) = \kappa^2 \int_{\{a(x,\kappa) > 0\}} a(x,\kappa)^2 \hat{f} \left( \frac{H |B_0(x)|}{\kappa a(x,\kappa)} \right) \, dx + \frac{\kappa^2}{2} \int_{\{a(x,\kappa) \leq 0\}} a(x,\kappa)^2 \, dx + o(\kappa^2). \tag{5.24}
\]

Putting (5.24) and (1.17) into (5.23), we finish the proof of Corollary 1.3.

\( \square \)

**Proof of Theorem 1.4.**

Noticing that (5.22) is valid when \( h = 1 \) and \( D \) replaced by \( \overline{D}/ := \Omega \setminus D \) for any open domain \( D \subset \Omega \) with smooth boundary, then we get:

\[
\mathcal{E}_0(\psi, A; a, \overline{D}/) \geq \kappa^2 \left\{ \int_{\overline{D}/ \cap \{a(x,\kappa) > 0\}} a(x,\kappa)^2 \hat{f} \left( \frac{H |B_0(x)|}{\kappa a(x,\kappa)} \right) \, dx + \frac{1}{2} \int_{\overline{D}/ \cap \{a(x,\kappa) \leq 0\}} a(x,\kappa)^2 \, dx \right\} + o(\kappa^2). \tag{5.25}
\]

We can decompose \( \mathcal{E}_0(\psi, A; a, D) \) as follow:

\[
\mathcal{E}_0(\psi, A; a, D) = \mathcal{E}_0(\psi, A; a, \Omega) - \mathcal{E}_0(\psi, A; a, \overline{D}/).
\]

Using (5.24) and (5.25), we get

\[
\mathcal{E}_0(\psi, A; a, D) \leq \kappa^2 \left\{ \int_{D \cap \{a(x,\kappa) > 0\}} a(x,\kappa)^2 \hat{f} \left( \frac{H |B_0(x)|}{\kappa a(x,\kappa)} \right) \, dx + \frac{1}{2} \int_{D \cap \{a(x,\kappa) \leq 0\}} a(x,\kappa)^2 \, dx \right\} + o(\kappa^2). \tag{5.26}
\]

\( \square \)
6. STUDY OF EXAMPLES

In this section, we will describe situations where the remainder term in \( \text{(1.17)} \) is indeed small as \( \kappa \to +\infty \) compared with the leading order term

\[
E^L_{\kappa}(\kappa, H, a, B_0) := \kappa^2 \left( \int_{\{\kappa(x, \kappa) > 0\}} a(x, \kappa)^2 \frac{\hat{f}}{\kappa} \left( \sigma \frac{|B_0(x)|}{a(x)} \right) dx + \frac{1}{2} \int_{\{\kappa(x, \kappa) \leq 0\}} a(x, \kappa)^2 dx \right),
\]

where,

\[
\sigma = \frac{H}{\kappa}.
\]

Note that \( 0 < \lambda_{\min} \leq \sigma \leq \lambda_{\max} \), so that \( \sigma \) will be considered as an independent parameter in \([\lambda_{\min}, \lambda_{\max}]\).

We will also explore, case by case how one can verify Assumption \((A_4)\) as formulated precisely in \( (3.1) \).

6.1. The case of a \( \kappa \)-independent pinning.

**Proposition 6.1.** Suppose \( (1.4) \) and \( (1.15) \) hold. Let \( a(x, \kappa) = a(x) \) where \( a(x) \in C^1(\Omega) \) is a function independent of \( \kappa \) and satisfies,

\[
\begin{cases}
\{ x \in \Omega : a(x) > 0 \} \neq \emptyset, \\
or \\
\{ x \in \Omega : a(x) < 0 \} \neq \emptyset.
\end{cases}
\]

There exist positive constants \( C \) and \( \kappa_0 \) such that,

\[
\forall \kappa \geq \kappa_0, \quad E^L_{\kappa}(\kappa, H, a, B_0) \geq C \kappa^2.
\]

**Proof.** Since \( a(x, \kappa) = a(x) \), the energy \( E^L_{\kappa} \) becomes:

\[
E^L_{\kappa}(\kappa, H, a, B_0) := \kappa^2 \left( \int_{\{a(x) > 0\}} a(x)^2 \frac{\hat{f}}{\kappa} \left( \sigma \frac{|B_0(x)|}{a(x)} \right) dx + \frac{1}{2} \int_{\{a(x) \leq 0\}} a(x)^2 dx \right).
\]

Each term being positive, it is clear that the leading term is positive if \( \{x \in \Omega : a(x) < 0\} \neq \emptyset \). If \( \{x \in \Omega : a(x) < 0\} = \emptyset \) and \( \{x \in \Omega : a(x) > 0\} \neq \emptyset \), there exist \( \rho_0 > 0 \), \( a_0 > 0 \) and a disk \( D(x_0, r_0) \) such that

\[
D(x_0, r_0) \subset \{a(x) > a_0\} \cap \{|B_0| > \rho_0\}.
\]

Using the monotonicity of \( \hat{f} \) and the bound of \( a(x) \) in \( (1.6) \), we may write

\[
\int_{\{a(x) > 0\}} a(x)^2 \frac{\hat{f}}{\kappa} \left( \frac{H |B_0(x)|}{a(x)} \right) dx \geq \int_{D(x_0, r_0)} a(x)^2 \frac{\hat{f}}{\kappa} \left( \frac{|B_0(x)|}{a(x)} \right) dx
\]

\[
\geq \pi r_0^2 a_0^2 \hat{f} \left( \frac{\rho_0}{a} \right) \sigma,
\]

where \( \sigma \) is introduced in \( (1.10) \).

In particular, when \( (1.15) \) is satisfied, there exists \( \kappa_0 > 0 \) such that

\[
\forall \kappa \geq \kappa_0, \quad \int_{\{a(x) > 0\}} a(x)^2 \frac{\hat{f}}{\kappa} \left( \frac{H |B_0(x)|}{a(x)} \right) dx \geq \pi r_0^2 a_0^2 \hat{f} \left( \frac{\rho_0}{a} \lambda_{\min} \right).
\]

\( \Box \)
Proposition 6.2 (Verification of \((A_4)\)). Suppose that the function \(a\) satisfies (see Fig.1),
\[
\begin{align*}
\{ & |a| + |\nabla a| > 0 \quad \text{in } \overline{\Omega}, \\
& \nabla a \times \vec{n} \neq 0 \quad \text{on } \tilde{\Gamma} \cap \partial \Omega,
\end{align*}
\] (6.6)
where \(\tilde{\Gamma}\) defined as follows:
\[
\tilde{\Gamma} = \{ x \in \overline{\Omega} : a(x) = 0 \}. \tag{6.7}
\]
Then Assumption \((A_4)\) is satisfied.

Proof. From \((6.6)\), we observe that,
\[
\text{card } \{ \gamma \in \Gamma_\ell \cap \Omega \text{ with } Q_\ell(\gamma) \cap \partial \{ a > 0 \} \neq \emptyset \} = \text{card } \{ \gamma \in \Gamma_\ell \cap \Omega \text{ with } Q_\ell(\gamma) \cap \tilde{\Gamma} \neq \emptyset \}.
\]
Let \(\epsilon \in (0, 1)\), we introduce the domain
\[
D_\epsilon = \{ x \in \Omega : \text{dist}(x, \tilde{\Gamma}) \leq \epsilon \}.
\]

Now we give a rough upper bound for the area of \(D_\epsilon\).

By assumption \(\tilde{\Gamma}\) consists of a finite number of connected curves, which are either closed in \(\Omega\) or join two points of \(\partial \Omega\). Let us consider the first case, we denote by \(\tilde{\Gamma}^{(1)}\) such a curve. We can parametrize this curve using the standard tubular coordinates \((s, t)\), where \(s\) measures the arc-length in \(\tilde{\Gamma}^{(1)}\) and \(t\) measures the distance to \(\tilde{\Gamma}^{(1)}\) (see [13] Appendix F) for the detailed construction of these coordinates).

In the neighborhood of \(\tilde{\Gamma}^{(1)}\), we choose one point \(\gamma_0\) on \(\tilde{\Gamma}^{(1)}\) corresponding to \((0, 0)\). Let \(N \in \mathbb{N}\) and \(L\) the length of \(\tilde{\Gamma}^{(1)}\). We consider for \(i = 0, ..., N, s_i = \frac{i}{N} L\) (modulo \(L\mathbb{Z}\)) and \(\gamma_i = (s_i, 0)\). Notice that, there exists a positive constant \(C\) such that,
\[
|\text{dist}(\gamma_i, \gamma_{i+1})| = (1 + \epsilon_i)|s_i - s_{i+1}|, \quad \left( -\frac{C}{N} \leq \epsilon_i < 0 \right).
\]

Thus,
\[
\left| \left\{ x \in \Omega : \text{dist} \left( x, \tilde{\Gamma}^{(1)} \right) \leq \frac{L}{N} \right\} \right| \leq \sum_i \left| Q_{\frac{L}{N}}((s_i, 0)) \right|.
\]

Coming back to our problem, we select \(N = \left\lceil \frac{L}{\epsilon} \right\rceil\) and we note that
\[
\frac{L}{N + 1} \leq \epsilon \leq \frac{L}{N},
\]
which implies that,
\[ |D\epsilon| \leq \frac{\mathcal{L}^2}{N} \left( 1 + O \left( \frac{1}{N} \right) \right) \]
\[ \leq \mathcal{L} \epsilon \left( 1 + O \left( \frac{1}{N} \right) \right) = \epsilon \mathcal{L}(1 + O(\epsilon)) . \]

Hence we have shown that,
\[ \limsup_{\epsilon \to 0} \frac{|D\epsilon|}{\epsilon} \leq \mathcal{L} . \]

In a similar fashion, we prove that
\[ \liminf_{\epsilon \to 0} \frac{|D\epsilon|}{\epsilon} \geq \mathcal{L} . \]
and, as a consequence, we end up with the following conclusion:
\[ \lim_{\epsilon \to 0} \frac{|D\epsilon|}{\epsilon} = \mathcal{L} . \] (6.8)

Coming back to Assumption (A4), we now observe that all the \( Q_\ell(\gamma) \) touching \( \tilde{\Gamma} \) are inside \( D_{\sqrt{2}} \), hence we get, by comparison of the area
\[ \ell^2 \text{card} \{ \gamma \in \Gamma_\epsilon \cap \Omega \text{ with } Q_\ell(\gamma) \cap \tilde{\Gamma} \neq \emptyset \} \leq C\ell , \]
and consequently, there exist positive constants \( C_1, C_2 \) and \( \kappa_0 \) such that
\[ \forall \kappa \geq \kappa_0 , \forall \ell \leq C_2\kappa^{-\frac{1}{2}} , \text{card} \{ \gamma \in \Gamma_\epsilon \cap \Omega \text{ with } Q_\ell(\gamma) \cap \partial\{a > 0\} \neq \emptyset\} \leq C_1 \ell^{-1} , \]
which is a stronger form of (A4).

6.2. The case with a \( \kappa \)-dependent oscillation.

6.2.1. Preliminaries. We start with two lemmas which are standard in homogenization theory (see [3] Section 16-17)

Lemma 6.3. Let \( D \subset \mathbb{R}^2 \) be a bounded open set and \( \varphi \) be a \( \Gamma_{T_1, T_2} \)-periodic continuous function in \( \mathbb{R}^2 \) with \( \Gamma_{T_1, T_2} = T_1\mathbb{Z} \times T_2\mathbb{Z} \). There exists a positive constant \( M_0 \) such that if \( M \geq M_0 \), then,
\[ \int_D \varphi(Mx) \, dx = \frac{|D|}{T_1 T_2} \int_0^{T_1} \int_0^{T_2} \varphi(t_1, t_2) dt_1 dt_2 + O(M^{-1}) . \]

Lemma 6.4. Let \( D \subset \mathbb{R}^2 \) be a bounded open set and \( \phi : \mathbb{R}^2 \times \overline{D} \rightarrow \mathbb{R}^2 \) be a continuous function satisfying:
\[ \phi(t + T, x) = \phi(t, x) , \quad \forall T \in T_1\mathbb{Z} \times T_2\mathbb{Z} , \] (6.9)
and uniformly Lipschitz, i.e. with the property that there exist constants \( C > 0 \) and \( \epsilon_0 \), such that,
\[ |\phi(t, x) - \phi(t, \tilde{x})| \leq C|x - \tilde{x}| , \quad \forall t \in \mathbb{R}^2 , \forall x, \tilde{x} \in \overline{D}, \text{ s.t. } |x - \tilde{x}| < \epsilon_0 . \] (6.10)
There exists a positive constant \( M_0 \) such that if \( M \geq M_0 \), then,
\[ \int_D \phi(Mx, x) \, dx = \int_D \overline{\phi}(x) \, dx + O(M^{-1}) , \]
where,
\[ \overline{\phi}(x) = \frac{1}{T_1 T_2} \int_0^{T_1} \int_0^{T_2} \phi((t_1, t_2), x) dt_1 dt_2 . \] (6.11)
6.2.2. First example:

Proposition 6.5. Suppose that (1.4) and (1.15) hold. Let \( a(x, \kappa) = \alpha(\kappa^{\frac{1}{2}} x) \) where \( \alpha(\cdot) \in C^1(\Omega) \) is a \( \Gamma_{T_1,T_2} \)-periodic function. Then the leading order term \( E^L_\kappa \) defined in (6.1) satisfies,

\[
E^L_\kappa(\kappa, H, a, B_0) = \kappa^2 \int_\Omega \phi_+(x) \, dx + \kappa^2 |\Omega| \phi_- + o(\kappa^2), \quad \text{as} \; \kappa \to +\infty.
\]

Here,

\[
\phi_+(x) = \frac{1}{T_1 T_2} \int_0^{T_1} \int_0^{T_2} \alpha_+(t_1, t_2)^2 \left( \frac{|B_0(x)|}{\alpha_+(t_1, t_2)} \right) dt_1 dt_2,
\]

and

\[
\phi_- = \frac{1}{T_1 T_2} \int_0^{T_1} \int_0^{T_2} \alpha_-(t_1, t_2)^2 dt_1 dt_2.
\]

Proof. We first estimate the second term in (6.1). We apply Lemma 6.3 with \( D = \Omega, M = \kappa^{\frac{1}{2}} \) and \( \varphi = \alpha^2 \), we obtain,

\[
\int_\Omega a_-(x, \kappa)^2 \, dx = \frac{|\Omega|}{T_1 T_2} \int_0^{T_1} \int_0^{T_2} \alpha_-(t_1, t_2)^2 dt_1 dt_2 + O(\kappa^{-\frac{3}{2}}),
\]

and consequently,

\[
\kappa^2 \int_{\{a(x) \leq 0\}} a(x, \kappa)^2 \, dx = \kappa^2 \frac{|\Omega|}{T_1 T_2} \int_0^{T_1} \int_0^{T_2} \alpha_-(t_1, t_2)^2 dt_1 dt_2 + O(\kappa^3).
\]

Now, we estimate the first term in (6.1). We first prove that \( \hat{f} \) is a Lipschitz function in \([b_0, 1]\) with \( b_0 \in (0, 1) \). We consider this restriction because when \( b \to 0_+ \) (see [18, Theorem 2.1]), \( \hat{f} \) satisfies,

\[
\hat{f}(b) = \frac{b}{2} \ln \frac{1}{b}(1 + o(1)), \quad (6.12)
\]

and \( \hat{f} \) is not a Lipschitz function at 0. We recall the definition of \( \hat{f} \)

\[
\hat{f}(b) = \lim_{R \to +\infty} \frac{e_D(b, R)}{R^2} \quad (\forall b \in [0,1]),
\]

where

\[
e_D(b, R) = \inf_{u \in b_0, Q_R} \int_{Q_R} \left( b |(\nabla - iA_0)u|^2 + \frac{1}{2} (1 - |u|^2)^2 \right) \, dx.
\]

From the definition, we can conclude that \( \hat{f} \) is concave and hence locally Lipschitz in \((0, +\infty)\) (see [18, Theorem 2.35]). For completion we write below a proof making explicit the Lipschitz constant. For \( b' > 0 \), let \( u_{b', R} \in H^1_0(Q_R) \) be a minimizer of \( F_{b', Q_R}^{1,1} \). Then for all \( b \in (0, 1) \), we have,

\[
e_D(b, R) \leq F_{b_0, Q_R}^{1,1}(u_{b', R}) \leq e_D(b', R) + \| (\nabla - iA_0)u_{b', R} \|_{L^2(Q_R)}^2 |b - b'|.
\]

Now, we estimate \( \| (\nabla - iA_0)u_{b', R} \|_{L^2(Q_R)}^2 \) from above. Coming back to the definition, we get the existence of a positive constant \( C \), such that for any \( b \in [b_0, 1] \) and for any \( b' \in [b_0, 1] \),

\[
\| (\nabla - iA_0)u_{b', R} \|_{L^2(Q_R)}^2 \leq \frac{e_D(b', R)}{b'}.
\]

This implies that,

\[
e_D(b, R) \leq e_D(b', R) + \frac{e_D(b', R)}{b'} |b - b'|.
\]
Dividing by $R^2$ and taking the limit as $R \to +\infty$, we obtain
\[
\hat{f}(b) \leq \hat{f}(b') + \frac{|\hat{f}(b')|}{|b'-b|} |b - b'|.
\]
Using the asymptotic behavior of $\hat{f}$ in (6.12) as $b' \to 0_+$, we finally obtain the existence of $C$ such that
\[
\hat{f}(b) \leq \hat{f}(b') + C \left( \log \frac{1}{|b_0|} \right) |b - b'|, \, \forall b, b' \text{ with } 1 > b > b_0 \text{ and } 1 > b' > b_0.
\]
Exchanging $b$ and $b'$, we have proved the

**Lemma 6.6.** $\hat{f}$ is locally Lipschitz in $(0, +\infty)$. More precisely, there exists $C$ such that for any $b_0 > 0$,
\[
|\hat{f}(b) - \hat{f}(b')| \leq C \left( \log \frac{1}{b_0} \right) |b - b'|, \, \forall b, b' \text{ with } 1 > b > b_0 \text{ and } 1 > b' > b_0. \tag{6.13}
\]
In addition, we have
\[
|\hat{f}(b) - \hat{f}(b')| \leq 2 |b - b'|, \, \forall b, b' \text{ with } b > \frac{1}{2} \text{ and } b' > \frac{1}{2}. \tag{6.14}
\]

To continue, we consider
\[
\mathbb{R}^2 \times \Omega_\rho \ni (t, x) \mapsto \phi(t, x) = \alpha_+(t)^2 \hat{f} \left( \frac{\sigma |B_0(x)|}{\alpha_+(t)} \right),
\]
where, $\Omega_\rho := \Omega \cap \{|B_0| > \rho\}$. The periodicity condition in (6.9) is clear. Let us verify the Lipschitz property. Let
\[
b_0 = \frac{\lambda_{\min}}{\alpha_0} \rho,
\]
where, $\lambda_{\min}$ is introduced in (1.15) and $\alpha_0 = \sup \alpha_+ (t)$.

Let $\epsilon > 0$, $\mathcal{I}_+ = \{t \in \mathbb{R} : \alpha_+(t) \geq \epsilon\}$ and $\mathcal{I}_- = \{t \in \mathbb{R} : \alpha_+(t) \leq \epsilon\}$, we distinguish between two cases:

**Case 1:** ($\alpha_+(t) \geq \epsilon$). We observe that for $(x, t) \in \Omega_\rho \times \mathcal{I}_+$, we have
\[
b_0 \leq |\frac{B_0(x)}{\alpha_+(t)}| \leq \frac{\sigma |B_0(x)|}{\epsilon}.
\]
Thus, for any $t \in \mathcal{I}_+$ and for any $x, x' \in \Omega_\rho$, we get
\[
\left| \alpha_+(t)^2 \hat{f} \left( \frac{\sigma |B_0(x)|}{\alpha_+(t)} \right) - \alpha_+(t)^2 \hat{f} \left( \frac{\sigma |B_0(x')|}{\alpha_+(t)} \right) \right| = \alpha_+(t)^2 |\hat{f}(b) - \hat{f}(b')| \leq C \left( \log \frac{1}{\rho} \right) \left| |B_0(x)| - |B_0(x')| \right|. \tag{6.15}
\]
Therefore, using also the Lipschitz property for $x \mapsto |B_0(x)|$, we get that $\Omega_\rho \ni x \mapsto \phi(t, x)$ is uniformly Lipschitz for $t \in \mathcal{I}_+$.

**Case 2:** ($\alpha_+(t) \leq \epsilon$). We observe that for $(x, t) \in \Omega_\rho \times \mathcal{I}_-$,
\[
\frac{\sigma |B_0(x)|}{\alpha_+(t)} \geq \frac{\sigma |B_0(x)|}{\epsilon}.
\]
We note that $\hat{f}(b) = \frac{1}{2}, \forall b \geq 1$ (see [14] Theorem 2.1]). For this reason we choose
\[
\epsilon = \frac{\lambda_{\min}}{2} \rho,
\]
which implies that for \((x,t) \in \Omega_\rho \times \mathcal{I}_-\),
\[
\sigma \frac{|B_0(x)|}{\alpha_+(t)} \geq 2 \quad \text{and} \quad \hat{f}\left(\sigma \frac{|B_0(x)|}{\alpha_+(t)}\right) = \frac{1}{2}.
\]
Thus, for any \(t \in \mathcal{I}_-\) and for any \(x,x' \in \overline{\Omega}_\rho\), we get
\[
\left|\alpha_+(t)^2 \hat{f}\left(\sigma \frac{|B_0(x)|}{\alpha_+(t)}\right) - \alpha_+(t)^2 \hat{f}\left(\sigma \frac{|B_0(x')|}{\alpha_+(t)}\right)\right| = \left|\frac{\alpha_+(t)^2 - \alpha_+(t)^2}{2}\right| = 0.
\]
Hence we get that \(\Omega_\rho \ni x \mapsto \phi(t,x)\) is uniformly Lipschitz for \(t \in \mathcal{I}_-\).

Now, we apply Lemma 6.4 with \(D = \Omega_\rho\) and \(M = \kappa_\frac{1}{2}\) and we obtain,
\[
\int_{\Omega_\rho} a_+(x,\kappa)^2 \hat{f}\left(\sigma \frac{|B_0(x)|}{a_+(x,\kappa)}\right) dx = \int_{\Omega_\rho} \overline{\phi}(x) dx + \mathcal{O}(\rho^{-1/2}),
\]
where \(\overline{\phi}\) is introduced in (6.11).

Coming back to the integral over \(\Omega\), we get, for any \(\rho \in (0,\rho_0)\) and for any \(\kappa \geq \kappa_0\) with \(\rho_0\) small enough and \(\kappa_0\) large enough,
\[
\int_{\Omega} a_+(x,\kappa)^2 \hat{f}\left(\sigma \frac{|B_0(x)|}{a_+(x,\kappa)}\right) dx = \int_{\Omega} \overline{\phi}(x) dx + \mathcal{O}(\rho) + \mathcal{O}(\rho^{-1/2}).
\]
Here, we have used the fact that \(\overline{\phi}\) is a bounded function in \(\Omega\). Let us show that the remainder term \(s(\kappa)\) in the right hand side in (6.18) is \(o(1)\). The remainder term has the form \(s_1(\kappa) + s_2(\kappa)\) with \(s_1(\kappa) = \mathcal{O}(\rho)\) and \(s_2(\kappa) = \mathcal{O}(\rho^{-1/2})\). Let us show that it is \(o(1)\). Given \(\varepsilon > 0\), there exists \(\rho_\varepsilon > 0\) such that \(|s_1(\kappa)| \leq \frac{\varepsilon}{2}\), for all \(\kappa \geq \kappa_0\). Then, \(\rho = \rho_\varepsilon\) being chosen, we can find \(\kappa_\varepsilon \geq \kappa_0\) such that, for any \(\kappa \geq \kappa_\varepsilon\), \(|s_2(\kappa)| \leq \frac{\varepsilon}{2}\).

\[
\text{Figure 2. Schematic representation of a domain with a } \kappa\text{-dependent oscillation pinning and with vanishing magnetic field along } \Gamma.
\]

\[
\text{Proposition 6.7 (Verification of (A4)). Suppose that the function } \alpha \text{ defined in Proposition 6.5 satisfies}
\]
\[
|\alpha| + |\nabla \alpha| > 0 \quad \text{in } \mathbb{R}^2.
\]

Then Assumption (A4) is satisfied.
Proof. Using (6.19), a change of variable $y = \kappa^{\frac{1}{2}} x$ and $\gamma' = \kappa^{\frac{1}{2}} \gamma$ yields,

$$\text{card} \{ \gamma \in \Gamma \cap \Omega \text{ with } Q_{\ell}(\gamma) \cap \partial \{ x \in \Omega : a(x, \kappa) > 0 \} \neq \emptyset \} = \text{card} \{ \gamma' \in \Gamma_{\kappa^{\frac{1}{2}} \ell} \cap \kappa \frac{1}{2} \Omega \text{ with } Q_{\kappa^{\frac{1}{2}} \ell}(\gamma') \cap \hat{\Gamma} \neq \emptyset \},$$

where,

$$\hat{\Gamma} = \{ y \in \mathbb{R}^2 \mid \alpha(y) = 0 \}.$$

Let $\epsilon \in (0, 1)$, we introduce the domain

$$\hat{D}_{\epsilon,M} = \{ y \in M \cdot \Omega : \text{dist}(y, \hat{\Gamma}) \leq \epsilon \}.$$

Thanks to (6.8) and the periodicity assumption, we get the existence of positive constants $C$, $M_0$ and $\epsilon_0$ such that, for any $\epsilon \in (0, \epsilon_0)$, $M \geq M_0$

$$|\hat{D}_{\epsilon,M}| \leq C M \epsilon.$$

In the sequel, we choose $M = \kappa^{\frac{1}{2}}$ and $\epsilon = M \sqrt{2} \ell$. We note that, there exist constants $c > 0$ and $\kappa_0 > 0$ such that,

$$\forall \kappa \geq \kappa_0, \forall \ell \leq c \kappa^{-\frac{1}{2}}, \quad 0 < \epsilon \leq \epsilon_0.$$

We now observe that all the $Q_{\kappa^{\frac{1}{2}} \ell}(\gamma)$ touching $\hat{\Gamma}$ are inside $\hat{D}_{\kappa^{\frac{1}{2}} \sqrt{2} \ell, \kappa^{\frac{1}{2}}}$, hence we get, by comparison of the areas

$$\kappa \ell^2 \text{card} \{ \gamma' \in \Gamma_{\kappa^{\frac{1}{2}} \ell} \cap \kappa \frac{1}{2} \Omega \text{ with } Q_{\kappa^{\frac{1}{2}} \ell}(\gamma') \cap \hat{\Gamma}_{\kappa} \neq \emptyset \} \leq C \sqrt{2} \kappa \ell.$$

There exist positive constants $C_1$ and $C_2$, such that,

$$\forall \kappa \geq \kappa_0, \forall \ell \leq C_2 \kappa^{-\frac{1}{2}}, \text{ card } \{ \gamma \in \Gamma_{\ell} \cap \Omega \text{ with } Q_{\ell}(\gamma) \cap \partial \{ x \in \Omega : a(x, \kappa) > 0 \} \neq \emptyset \} \leq C_1 \ell^{-1}.$$

\[ \square \]

6.2.3. Second example. This example was considered by Aftalion, Sandier and Serfaty (see (H2)).

**Proposition 6.8.** Suppose that (1.4) and (1.15) hold. Let $a(x, \kappa) = a(x) + \beta(x, \kappa)$, where $\beta(x, \kappa)$ is a nonnegative function and $\{ a > 0 \} \cap \Omega \neq \emptyset$, (see Fig. 3). There exist positive constants $\tau_1$ and $\kappa_0$ such that,

$$\forall \kappa \geq \kappa_0, \quad E^L_{a}(\kappa, H, a, B_0) \geq \tau_1 \kappa^2.$$

**Proof.** We can write,

$$\kappa^2 \int_{\{ a(x, \kappa) > 0 \}} a(x, \kappa)^2 \left( \frac{H |B_0(x)|}{\kappa a(x, \kappa)} \right) dx \geq \kappa^2 \int_{\{ a(x) > 0 \}} a(x)^2 \left( \frac{H |B_0(x)|}{\kappa a(x)} \right) dx \geq \kappa^2 \int_{\{ a(x) > 0 \}} a(x)^2 \left( \frac{H |B_0(x)|}{\kappa a} \right) dx. \quad (6.20)$$

Here we have used that $\hat{f}$ is increasing, the nonnegativity of $\beta$ to get $a(x, \kappa) \geq a(x)$, Assumption (A2) to estimate $\hat{f}$ from below, and $\{ a(x) > 0 \} \subset \{ a(x, \kappa) > 0 \}$.

Proceeding like in (6.4), there exist $\tau_1 > 0$ and $\kappa_0 > 0$ such that,

$$\forall \kappa \geq \kappa_0, \quad \kappa^2 \int_{\{ a(x, \kappa) > 0 \}} a(x, \kappa)^2 \left( \frac{H |B_0(x)|}{\kappa a(x, \kappa)} \right) dx \geq \tau_1 \kappa^2. \quad (6.21)$$

\[ \square \]
6.2.4. Third example: This example is similar to the previous example, but here we suppose that
\[ \beta(x, \kappa) = \alpha(\kappa^{1/2}x), \]
where \( \alpha(\cdot) \) is a \( \Gamma_{T_1,T_2} \)-periodic positive function in \( \mathbb{R}^2 \).

**Proposition 6.9.** Suppose that (1.4) and (1.15) hold. Let \( a(x, \kappa) = a(x) + \alpha(\kappa^{1/2}x) \), where \( \alpha(\cdot) \) is a \( \Gamma_{T_1,T_2} \)-periodic positive bounded function in \( \mathbb{R}^2 \), \( a(\cdot) \in C^1(\Omega) \) and \( \{a < 0\} \cap \Omega = \emptyset \). Then the leading order term \( E^L_k \) defined in (6.1) satisfies,
\[ E^L_k(\kappa, H, a, B_0) = \kappa^2 \int_{\Omega} \bar{\phi}(x) \, dx + o(\kappa^2), \quad \text{as } \kappa \to +\infty. \]
Here,
\[ \bar{\phi}(x) = \frac{1}{T_1 T_2} \int_0^{T_1} \int_0^{T_2} (a(x) + \alpha(t_1, t_2))^2 \hat{f} \left( \frac{\sigma \left| B_0(x) \right|}{a(x) + \alpha(t_1, t_2)} \right) \, dt_1 dt_2. \]
The proof of Proposition 6.9 is similar to that of Proposition 6.5.

6.3. Upper bound of the main term.

It is easy to show that \( E^L_k \) is less than \( C\kappa^2 \) for some \( C > 0 \). Indeed, using the bound of \( a \) in (1.6) and the bound \( \hat{f}(b) \leq \frac{1}{2} \), we have,
\[ \kappa^2 \int_{\{a(x, \kappa) > 0\}} a(x, \kappa)^2 \hat{f} \left( \frac{H \left| B_0(x) \right|}{\kappa a(x, \kappa)} \right) \, dx \leq C\kappa^2, \]
and
\[ \kappa^2 \int_{\{a(x, \kappa) \leq 0\}} a(x, \kappa)^2 \, dx \leq C\kappa^2. \]

7. Proof of Theorem 1.5

The technique that will be used in this proof has been introduced by Helffer-Kachmar in [21] for the case \( a(x, \kappa) = 1 \). The proof is decomposed into three steps:

**Step 1: Case \( D = \Omega \).**

Let \((\psi, A)\) be a solution of (1.12). Thanks to (4.3), we have,
\[
\int_\Omega |(\nabla - i\kappa H A)\psi|^2 \, dx = \kappa^2 \int_\Omega (a(x, \kappa) - |\psi|^2)|\psi|^2 \, dx
\]
\[
= \frac{\kappa^2}{2} \int_\Omega (a(x, \kappa)^2 - |\psi(x)|^4) \, dx - \frac{\kappa^2}{2} \int_\Omega (a(x, \kappa) - |\psi|^2)^2 \, dx.
\]

Having in mind the definition of \(E_0(\psi, A; a, \Omega)\), we get,
\[
\frac{\kappa^2}{2} \int_\Omega (a(x, \kappa)^2 - |\psi(x)|^4) \, dx = E_0(\psi, A; a, \Omega).
\]

Using (5.24), we get that as \(\kappa \to +\infty\)
\[
\frac{\kappa^2}{2} \int_\Omega (a(x, \kappa)^2 - |\psi(x)|^4) \, dx = \kappa^2 \int_\{a(x, \kappa)^2 \leq 0\} a(x, \kappa)^2 \left( \frac{H |B_0(x)|}{\kappa a(x, \kappa)} \right) \, dx
\]
\[
+ \frac{\kappa^2}{2} \int_\{a(x, \kappa) > 0\} a(x, \kappa)^2 \, dx + o(\kappa^2).
\]

Notice that
\[
\int_\Omega a(x, \kappa)^2 \, dx = \int_\{a(x, \kappa) \leq 0\} a(x, \kappa)^2 \, dx + \int_\{a(x, \kappa) > 0\} a(x, \kappa)^2 \, dx.
\]

Therefore, dividing (7.2) by \(\kappa^2\), we get
\[
\int_\Omega |\psi(x)|^4 \, dx = - \int_\{a(x, \kappa) > 0\} a(x, \kappa)^2 \left\{ 2 \frac{H |B_0(x)|}{\kappa a(x, \kappa)} \right\} \, dx + o(1).
\]

**Step 2: Upper bound.**

Let \(D \subset \Omega\) be a regular domain and, for \(\ell \in (0, 1)\),
\[
D_\ell = \{ x \in D : \text{dist}(x, \partial D) \geq \ell \}.
\]

We introduce a cut-off function \(\chi_\ell \in C_0^\infty(\mathbb{R}^2)\) such that
\[
0 \leq \chi_\ell \leq 1 \text{ in } \mathbb{R}^2, \quad \text{supp} \chi_\ell \subset D, \quad \chi_\ell = 1 \text{ in } D_\ell \quad \text{and} \quad |\nabla \chi_\ell| \leq \frac{C}{\ell} \text{ in } \mathbb{R}^2,
\]
where \(C\) is a positive constant. We multiply both sides of (1.12) by \(\chi_\ell^2 \psi\). It results from an integration by parts that
\[
\int_D \left( \frac{H}{\kappa a(x, \kappa)} \right) \, dx = \frac{O(\ell^{-1})}{\kappa^2}.
\]

Here, we have used the fact that \(|\nabla \chi_\ell|^2 = O(\ell^{-2})\), \(|D_\ell| = O(\ell)\) and the bound of \(\psi\) in (4.9).

We notice that \(\chi_\ell^4 \leq \chi_\ell^2 \leq 1\). We add to both sides the term \(\frac{\kappa^2}{2} \int_D a^2 \, dx\) to obtain,
\[
\int_D \left( \frac{|\nabla - i\kappa H A|}{\kappa} \right) \chi_\ell^2 |\psi|^4 + \frac{\kappa^2}{2} a^2 - \frac{\kappa^2}{2} |\chi_\ell^4 |\psi|^4 + \frac{\kappa^2}{2} |\chi_\ell^2 |\psi|^4 \right) \, dx \leq C \ell^{-1} + \frac{\kappa^2}{2} \int_D a^2 \, dx.
\]

This implies that
\[
E_0(\chi_\ell \psi, A; a, D) \leq \frac{\kappa^2}{2} \int_D (a^2 - \chi_\ell^4 |\psi|^4) \, dx + C \ell^{-1}.
\]

Using (7.5), we get
\[
\int_D |\psi|^4 \, dx \leq \int_D \chi_\ell^4 |\psi|^4 \, dx + \int_D (1 - \chi_\ell^4) |\psi|^4 \, dx
\]
\[
\leq \int_D \chi_\ell^4 |\psi|^4 \, dx + C' \ell,
\]
and consequently,
\[ E_0(\chi_{\ell}, A; a, D) \leq \frac{\kappa^2}{2} \int_D (a^2 - |\psi|^4) \, dx + C(\ell^{-1} + \ell). \] (7.8)

Using (5.22) with \( h = \chi_{\ell} \) and taking the choice of \( \ell \) defined in (3.18), we get, as \( \kappa \to +\infty \),
\[ \frac{\kappa^2}{2} \int_D (a^2 - |\psi|^4) \, dx \geq \kappa^2 \int_{D \cap \{ a(x, \kappa) > 0 \}} a(x, \kappa)^2 \left( \frac{\kappa |B_0(x)|}{\kappa a(x, \kappa)} \right) \, dx + \frac{\kappa^2}{2} \int_{D \cap \{ a(x, \kappa) \leq 0 \}} a(x, \kappa)^2 \, dx \]
\[ + o(\kappa^2). \] (7.9)

Notice that,
\[ \int_D a(x, \kappa)^2 \, dx = \int_{D \cap \{ a(x, \kappa) \leq 0 \}} a(x, \kappa)^2 \, dx + \int_{D \cap \{ a(x, \kappa) > 0 \}} a(x, \kappa)^2 \, dx. \]

Therefore,
\[ -\frac{\kappa^2}{2} \int_D |\psi|^4 \, dx \geq \kappa^2 \int_{D \cap \{ a(x, \kappa) > 0 \}} a(x, \kappa)^2 \left( \frac{\kappa |B_0(x)|}{\kappa a(x, \kappa)} \right) \, dx - \frac{\kappa^2}{2} \int_{D \cap \{ a(x, \kappa) > 0 \}} a(x, \kappa)^2 \, dx \]
\[ + o(\kappa^2). \] (7.10)

Dividing both sides by \( -\frac{\kappa^2}{2} \), we obtain, as \( \kappa \to +\infty \),
\[ \int_D |\psi|^4 \, dx \leq -\int_{D \cap \{ a(x, \kappa) > 0 \}} a(x, \kappa)^2 \left\{ 2 \frac{\kappa |B_0(x)|}{\kappa a(x, \kappa)} - 1 \right\} \, dx + o(1). \] (7.11)

Remark 7.1. We can replace \( D \) by \( D^c \) such that the estimate in (7.11) is still true. That is:
\[ \int_{D^c} |\psi|^4 \, dx \leq -\int_{D^c \cap \{ a(x, \kappa) > 0 \}} a(x, \kappa)^2 \left\{ 2 \frac{\kappa |B_0(x)|}{\kappa a(x, \kappa)} - 1 \right\} \, dx + o(1). \] (7.12)

**Step 3: Lower bound.**

We can decompose \( \int_D |\psi|^4 \, dx \) as follows:
\[ \int_D |\psi|^4 \, dx = \int_{\Omega} |\psi|^4 \, dx - \int_D |\psi|^4 \, dx \]

Thanks to Remark 7.1 using the asymptotics in (7.3) obtained in Step 1 when \( D = \Omega \) and the upper bound in Step 2, we get
\[ \int_D |\psi|^4 \, dx \leq -\int_{D \cap \{ a(x, \kappa) > 0 \}} a(x, \kappa)^2 \left\{ 2 \frac{\kappa |B_0(x)|}{\kappa a(x, \kappa)} - 1 \right\} \, dx + o(1). \] (7.13)

**8. Extension of the Giorgi-Phillips Theorem**

In this section we extend a result of Giorgi-Phillips [19], in the two cases when the external magnetic field \( B_0 \) is variable (i.e. \( \Gamma \neq \emptyset \)) and when the external magnetic field \( B_0 \) is constant (i.e. \( \Gamma = \emptyset \)), with a pinning term. We recall that the normal solution \((0, F)\) is a trivial solution of the Ginzburg-Landau system (1.12). We will show that this solution is a global minimizer, when \( \kappa \) and \( H \) are sufficiently large. We first establish a priori estimates for a critical point \((\psi, A)\) of the G-L-functional.
8.1. Estimates of $A$ and of $\| (\nabla - \mathrm{i} \kappa H F ) \psi \|$.  

We need the following estimates on $A$ and on $\| (\nabla - \mathrm{i} \kappa H F ) \psi \|$ which are independent of the assumption of $\Gamma$.

**Theorem 8.1.** There exist positive constants $C_1$, $C_2$ and $C_3$ such that, if (1.16) hold, $\kappa > 0$, $H > 0$ and $(\psi, A)$ is a solution of (1.12), then,

\[
\| (\nabla - \mathrm{i} \kappa H A ) \psi \|_{L^2(\Omega)} \leq C_1 \kappa \| \psi \|_{L^2(\Omega)}, 
\]

\[
\| \text{curl}(A - F) \|_{L^2(\Omega)} \leq C_2 \frac{H}{\kappa} \| \psi \|_{L^2(\Omega)} \| \psi \|_{L^4(\Omega)}, 
\]

\[
\| (\nabla - \mathrm{i} \kappa H F ) \psi \|_{L^2(\Omega)} \leq C_3 \kappa \| \psi \|_{L^2(\Omega)}.
\]

**Proof.** We first prove (8.1). In the case when $a \leq 0$ with $a$ introduced in (1.10), we get using (4.9) that $\psi = 0$ and hence (8.1) is proved.

In the case when $a > 0$, thanks to (4.9), we have,

\[
0 \leq (a - |\psi|^2) \leq a.
\]

(8.4)

We recall that if $(\psi, A)$ is a solution of (1.12) then, (see (4.3))

\[
\int_{\Omega} |(\nabla - \mathrm{i} \kappa H A ) \psi|^2 \, dx = \kappa^2 \int_{\Omega} (a(x, \kappa) - |\psi|^2) |\psi|^2 \, dx.
\]

Using (1.6) and (8.4), we obtain (8.1).

Now, we prove (8.2). We obtain from the equation in (1.12) the following estimate (see [13, Equation (11.9b)]):

\[
\kappa H \int_{\Omega} | \text{curl}(A - F) |^2 \, dx \leq \| (\nabla - \mathrm{i} \kappa H A ) \psi \|_{L^2(\Omega)} \| (A - F) \psi \|_{L^2(\Omega)}.
\]

Using (8.1) and applying Hölder’s inequality, we get

\[
\kappa H \int_{\Omega} | \text{curl}(A - F) |^2 \, dx \leq C \kappa \| \psi \|_{L^2(\Omega)} \| \psi \|_{L^4(\Omega)} \| A - F \|_{L^4(\Omega)}.
\]

We get by regularity of the curl-div system (see [13, A.7]),

\[
\| A - F \|_{H^1(\Omega)} \leq C \| \text{curl}(A - F) \|_{L^2(\Omega)},
\]

(8.5)

where $C$ is a positive constant.

By the Sobolev embedding theorem, we get,

\[
\| A - F \|_{L^4(\Omega)} \leq C_{\text{Sob}} \| A - F \|_{H^1(\Omega)} \leq \tilde{C} \| \text{curl}(A - F) \|_{L^2(\Omega)}.
\]

(8.6)

Consequently,

\[
\kappa H \int_{\Omega} | \text{curl}(A - F) |^2 \, dx \leq \kappa \| \psi \|_{L^2(\Omega)} \| \psi \|_{L^4(\Omega)} \| \text{curl}(A - F) \|_{L^2(\Omega)},
\]

which leads to (8.2).

Finally, we prove (8.3). Using (8.2) and (8.6), Hölder’s inequality gives,

\[
\| (A - F) \psi \|_{L^2(\Omega)}^2 \leq \| A - F \|_{L^4(\Omega)}^2 \| \psi \|_{L^4(\Omega)}^2 \leq \frac{C'}{H^2} \| \psi \|_{L^4(\Omega)}^4 \| \psi \|_{L^2(\Omega)}^2,
\]

(8.7)
Using (8.1), (8.7) and the bound of $\psi$ above, Young’s inequality gives,

$$
\| (\nabla - i\kappa H) \psi \|_{L^2(\Omega)}^2 \leq 2 \| (\nabla - i\kappa H) \psi \|_{L^2(\Omega)}^2 + 2 (\kappa H)^2 \| (A - F) \psi \|_{L^2(\Omega)}^2
\leq 2 C'' \kappa^2 \| \psi \|_{L^2(\Omega)}^2.
$$  \hspace{1cm} (8.8)

\[\Box\]

8.2. The case $\Gamma = \emptyset$.

For $\xi \in \mathbb{R}$, we consider the Neumann realization $h^{N,\xi}$ in $L^2(\mathbb{R}^+)$ associated with the operator $-\frac{d^2}{dt^2} + (t + \xi)^2$, i.e.

$$
h^{N,\xi} := -\frac{d^2}{dt^2} + (t + \xi)^2, \quad \mathcal{D}(h^{N,\xi}) = \{ u \in B^2(\mathbb{R}^+) : u'(0) = 0 \},
$$  \hspace{1cm} (8.9)

where,

$$
B^2(\mathbb{R}^+) = \{ u \in L^2(\mathbb{R}^+) : t^p u^{(q)} \in L^2(\mathbb{R}^+), \forall p, q \in \mathbb{N} \text{ s.t. } p + q \leq 2 \}.
$$

M. Dauge and B. Helffer [10] (see also Fournais-Helffer [13, Proposition 4.2.2]) have proved that the lowest eigenvalue $\mu$ of $h^{N,\xi}$ admits a minimum $\Theta_0$, which is attained at a unique point $\xi_0 < 0$, and satisfies:

$$
\Theta_0 = \inf_\xi \mu(\xi) = \mu(\xi_0) < 1.
$$  \hspace{1cm} (8.10)

Moreover

$$
\Theta_0 = \xi_0^2.
$$  \hspace{1cm} (8.11)

We introduce the notation:

$$
\inf_{x \in \Omega} |B_0(x)| = b_0 \quad \text{and} \quad \inf_{x \in \partial \Omega} |B_0(x)| = b'_0.
$$  \hspace{1cm} (8.12)

We denote by $\mu^N(\mathcal{B}F; \Omega)$ the lowest eigenvalue of the Schrödinger operator $P_{\mathcal{B}F,0}^\Omega$ (see [13]) with Neumann condition in $L^2(\Omega)$:

$$
\mu^N(\mathcal{B}F; \Omega) = \inf_{\psi \in H^1(\Omega)} \frac{(P_{\mathcal{B}F,0}^\Omega \psi, \psi)}{\| \psi \|_{L^2(\Omega)}^2}.
$$  \hspace{1cm} (8.13)

In [13], it is proved that

**Theorem 8.2.** Suppose that $\Omega \subset \mathbb{R}^2$ is an open bounded set with smooth boundary and $\Gamma = \emptyset$. Then,

$$
\lim_{B \to +\infty} \frac{\mu^N(\mathcal{B}F; \Omega)}{B} = \min(b_0, \Theta_0 b'_0).
$$  \hspace{1cm} (8.14)

In the next theorem, we give a simple proof of the result which says that $(0, F)$ is the unique minimizer of the functional when $H$ is sufficiently large and when the magnetic field $B_0$ is constant with pinning term.

**Theorem 8.3.** Let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded, simply-connected open set and $\Gamma = \emptyset$. Then, there exist positive constants $C$ and $\kappa_0$, such that, if

$$
H \geq C\kappa, \quad \kappa \geq \kappa_0,
$$

then $(0, F)$ is the unique solution to (1.12).
Proof. We assume that we have a non normal critical point \((\psi, A)\) for \(E_{\kappa,H,a,B_0}\). This means that \((\psi, A) \in H^1(\Omega) \times H^1_{\text{div}}(\Omega)\) is a solution of (1.12) and
\[
\int_{\Omega} |\psi|^2 \, dx > 0. \tag{8.15}
\]
Therefore, we get from (4.9) that,
\[
|\psi(x)|^2 \leq a \quad \forall x \in \Omega,
\]
where \(a\) is introduced in (1.10).

Let \(B = \kappa H\). (8.16)

Theorem 8.1 tells us that,
\[
\| (\nabla - iBF)\psi \|_{L^2(\Omega)}^2 \leq C\kappa^2 \| \psi \|_{L^2(\Omega)}^2.
\]

Since \(\psi\) satisfies (8.15), this implies by assumption that the lowest Neumann eigenvalue \(\mu^N_{B_0}(\Omega)\) of \(P_{B_0,0}^\Omega\) in \(L^2(\Omega)\) satisfies,
\[
\mu^N_{B_0}(\Omega) \leq C\kappa^2. \tag{8.17}
\]

Thanks to Theorem 8.2, we get the existence of a constant \(C > 0\), such that, if \(H \geq C\kappa\), then \((0,F)\) is the unique solution to (1.12). \(\Box\)

8.3. The case \(\Gamma \neq \emptyset\).

We recall the definition of \(\lambda_0\) in (1.31), the definition of \(\Gamma\) in (1.3) and for any \(\theta \in (0,\pi)\) we recall that \(\lambda(\mathbb{R}^2_+,\theta)\) is the bottom of the spectrum of the operator \(P_{A_{\text{app}},\theta}^{\mathbb{R}^2_+}\) with
\[
A_{\text{app},\theta} = - \left( \frac{x_2^2}{2} \cos \theta, \frac{x_1^2}{2} \sin \theta \right).
\]

Define
\[
\alpha_1(B_0) = \min \left\{ \lambda_0^2, \min_{x \in \Gamma \cap \Omega} |\nabla B_0(x)|, \min_{x \in \Gamma \cap \partial \Omega} \lambda(\mathbb{R}^2_+,\theta(x))^2 |\nabla B_0(x)| \right\}. \tag{8.18}
\]

In [34], it is proved that

**Theorem 8.4.** Suppose that (1.4) holds and \(\Gamma \neq \emptyset\). Then
\[
\lim_{B \to +\infty} \frac{\mu^N_{B_0}(\Omega)}{B^2} = \alpha_1(B_0)^{\frac{1}{2}}. \tag{8.19}
\]

In the next theorem, we give a simple proof of the result which says that \((0,F)\) is the unique minimizer of the functional when \(H\) is sufficiently large and when \(B_0\) is variable. This result was obtained in [19] for the case with constant magnetic field and with a constant pinning term.

**Theorem 8.5.** Let \(\Omega \subset \mathbb{R}^2\) be a smooth, bounded, simply-connected open set, the pinning term \(a\) satisfying (1.6), and the magnetic field \(B_0\) satisfying (1.4). Then, there exist positive constants \(C\) and \(\kappa_0\), such that, if
\[
H \geq C\kappa^2, \quad \kappa \geq \kappa_0.
\]
Then \((0,F)\) is the unique solution to (1.12).

**Proof.** Similarly to the proof of Theorem 8.3 we assume that we have a non normal critical point \((\psi, A)\) for \(E_{\kappa,H,a,B_0}\).

Therefore, we get from (8.3) that,
\[
\mu^N_{B_0}(\Omega) \leq C\kappa^2 \quad (B = \kappa H).
\]

Thanks to Theorem 8.4 we get a contradiction, if \(H \geq C\kappa^2\) and \(C\) is sufficiently large. \(\Box\)
9. Asymptotics of $\mu_1(\kappa, H)$: The Case with Non Vanishing Magnetic Field

The aim of this section is to give an estimate for the lowest eigenvalue $\mu_1(\kappa, H)$ of the operator $P^\Omega_{\kappa H, -\kappa^2 a}$ (see (1.26)) in the case when $\Gamma = \emptyset$ with a $\kappa$-independent pinning (i.e. $a(x, \kappa) = a(x)$).

Recall that the set $\Gamma$ is introduced in (1.3).

9.1. Lower bound.

Without loss of generality we suppose that $B_0 > 0$ in $\overline{\Omega}$. Our results will be formulated by introducing:

$$\Lambda_1(B_0, a, \sigma) = \min \left\{ \inf_{x \in \Omega} \{ \sigma B_0(x) - a(x) \}, \inf_{x \in \partial \Omega} \{ \Theta_0 \sigma B_0(x) - a(x) \} \right\},$$

(9.1)

where $\sigma$ is a positive constant.

In the case when the pinning term is constant (i.e. $a(x) = a_0$), (9.1) becomes as follows:

$$\Lambda_1(B_0, a, \sigma) = \sigma \min \left\{ \inf_{x \in \Omega} \{ B_0(x) \}, \Theta_0 \inf_{x \in \partial \Omega} \{ B_0(x) \} \right\} - a_0.$$ 

This case was treated by Pan and Kwek [29].

Let $Q^\Omega_{B \Phi, -\frac{B}{\sigma} a}$ be the quadratic form of $P^\Omega_{B \Phi, -\frac{B}{\sigma} a}$, i.e.

$$Q^\Omega_{B \Phi, -\frac{B}{\sigma} a}(\psi) = \int \Omega \left( |(\nabla - iB\Phi)\psi|^2 - \frac{B}{\sigma} a(x)|\psi|^2 \right) dx.$$ 

(9.2)

**Proposition 9.1.** Let $\Omega \subset \mathbb{R}^2$ be an open bounded set with smooth boundary, $I$ a closed interval in $(0, +\infty)$ and $\Gamma = \emptyset$. There exist positive constant $C$ and $B_0$ such that if $\sigma \in I$, $B \geq B_0$, $\psi \in H^1(\Omega) \setminus \{0\}$ and $a \in C^1(\overline{\Omega})$, then,

$$\frac{Q^\Omega_{B \Phi, -\frac{B}{\sigma} a}(\psi)}{||\psi||^2_{L^2(\Omega)}} \geq \frac{B}{\sigma} \Lambda_1(B_0, a, \sigma) - C B^3.$$ 

(9.3)

**Proof.** The proof is a consequence of the following inequality that we take from [13] Prop. 9.2.1,

$$\forall \psi \in H^1(\Omega), \quad \int \Omega |(\nabla - iB\Phi)|\psi|^2 dx \geq \int \Omega (U(x) - CB^{3/4})|\psi|^2 dx,$$

where

$$U(x) = \begin{cases} B B_0(x) & \text{if dist}(x, \partial \Omega) \geq B^{-3/8}, \\ \Theta_0 B_0(x) & \text{if dist}(x, \partial \Omega) < B^{-3/8}, \end{cases}$$

(9.4)

$B \geq B_0$, $\Theta_0$ and $C$ are two constants independent of $B$.

Clearly, there exist two constants $C' > 0$ and $B_0 > 0$ such that, for all $\sigma \in I$, we have,

$$U(x) - \frac{B}{\sigma} a(x) \geq \frac{B}{\sigma} \Lambda_1(B_0, a, \sigma) - C' B^{3/4}.$$ 

□

Coming back to our initial parameters $\kappa$ and $H$, we obtain:

**Theorem 9.2.** Let $\Omega \subset \mathbb{R}^2$ be an open bounded set with smooth boundary and $\Gamma = \emptyset$. Suppose that (1.15) holds and $a \in C^1(\overline{\Omega})$, then,

$$\mu_1(\kappa, H) \geq \kappa^2 \Lambda_1 \left( B_0, a, \frac{H}{\kappa} \right) + \mathcal{O}(\kappa^2), \quad \text{as } \kappa \to +\infty.$$ 

Here, $\Lambda_1$ is introduced in (9.1).
Proof. We apply Proposition \([9.1]\) with
\[
B = \kappa H , \quad \sigma = \frac{H}{\kappa} \quad \text{and} \quad I = [\lambda_{\min}, \lambda_{\max}] .
\]
Let us verify that the conditions of the proposition are satisfied for this choice. It is trivial that \(\sigma \in I\). Now, as \(\kappa \to +\infty\), we have,
\[
B = \sigma \kappa^2 \to +\infty.
\]
This implies that, as \(\kappa \to +\infty\),
\[
\mu_1(\kappa, H) \geq \kappa^2 \Lambda_1 \left( B_0, a, \frac{H}{\kappa} \right) + O(\kappa^{\frac{3}{2}}).\]
This finishes the proof of the theorem. \(\Square\)

9.2. Upper bound.

**Proposition 9.3** (Upper bound in the bulk). Suppose that \(\Omega \subset \mathbb{R}^2\) is an open bounded set with smooth boundary \(\partial \Omega\), \(\lambda_{\text{max}} > 0\) and \(\Gamma = \varnothing\). For any \(x_0 \in \Omega\), there exist positive constants \(C\) and \(B_0\) such that, if \(\sigma \in (0, \lambda_{\text{max}}]\), \(B \geq B_0\) and \(a \in C^1(\overline{\Omega})\), then,
\[
\mu_{B,\sigma} \leq \frac{B}{\sigma} \left\{ \sigma B_0(x_0) - a(x_0) \right\} + C B^\frac{3}{2}.\tag{9.5}
\]
Here,
\[
\mu_{B,\sigma} = \inf_{\psi \in H^1(\Omega) \setminus \{0\}} \frac{Q^\Omega_{BF,-\frac{\sigma}{\mu} a}(\psi)}{\|\psi\|_{L^2(\Omega)}^2},\tag{9.6}
\]
where \(Q^\Omega_{BF,-\frac{\sigma}{\mu} a}\) is introduced in \([9.2]\).

**Proof.** Thanks to \([9.2]\), we have,
\[
Q^\Omega_{BF,-\frac{\sigma}{\mu} a}(u) = \int_\Omega |(\nabla - i BF)u(x)|^2 dx - \frac{B}{\sigma} \int_\Omega a(x)|u(x)|^2 dx .
\]
The upper bound of the first term in the right hand side above is based on the construction of Gaussian quasi-mode (see \([13, \text{Subsection 2.4.2}]\) for the case with constant pinning) centered at \(x_0 \in \Omega\),
\[
\varphi_1(x) = e^{i BF \phi_0} \chi \left( B^{\frac{1}{2}} (x - x_0) \right) \left( \sqrt{B} B_0(x_0) (x - x_0) \right).
\]
Here, \(\chi\) is a cut-off function equal to 1 in a neighborhood of 0 such that \(\text{supp} \chi \subset D(0,1)\), the function \(\phi_0\) satisfies \([3.4]\) and the function \(u\) defined as follows:
\[
u(x) = \frac{\pi^{-1}}{\sqrt{2}} e^{-\frac{|x|^2}{2}}.
\]
We note that \(\text{supp} \varphi_1 \subset \Omega\) for \(B\) large enough. As in \([13, (2.35)]\), we get the existence of a positive constant \(B_0\) such that, for any \(B \geq B_0\),
\[
\int_\Omega |(\nabla - i BF)\varphi_1(x)|^2 dx \leq B B_0(x_0) + O(B^\frac{3}{2}).\tag{9.7}
\]
To derive the upper bound for the second term, we use Taylor’s formula for the function \(a\) near \(x_0\),
\[
|a(x) - a(x_0)| \leq C B^{-\frac{3}{2}} , \quad \left(x \in D \left(x_0, B^{-\frac{1}{2}} \right)\right).	ag{9.8}
\]
Using \([9.8]\), since \(\text{supp} \varphi_1 \subset D \left(x_0, B^{-\frac{1}{2}} \right)\), we get,
\[
- \int_\Omega a(x)|\varphi_1(x)|^2 dx \leq -a(x_0) \int_\Omega |\varphi_1(x)|^2 dx + C B^{-\frac{1}{2}} \int_\Omega |\varphi_1(x)|^2 dx ,\tag{9.9}
\]
We recall the definition of $\Theta$. Here, we notice that, if the infimum of $\sigma B_0(x) - a(x)$ was attained on $\partial \Omega$, (i.e. there exists $x_0 \in \partial \Omega$ such that $\inf_{x \in \Omega} \{ \sigma B_0(x) - a(x) \} = \sigma B_0(x_0) - a(x_0)$), we would have,

$$
\sigma B_0(x_0) - a(x_0) < \Theta \sigma B_0(x_0) - a(x_0),
$$

which is impossible, since $\Theta < 1$. Hence, we can choose $x_0 \in \Omega$, such that,

$$
\sigma B_0(x_0) - a(x_0) = \inf_{x \in \Omega} \{ \sigma B_0(x) - a(x) \},
$$

and we apply Proposition 9.3 with

$$
B = \kappa H \quad \text{and} \quad \sigma = \frac{H}{\kappa}.
$$

Thus, we get the existence of a positive constant $\kappa_0$ such that, if,

$$
\kappa \geq \kappa_0 \quad \text{and} \quad \kappa_0 \kappa^{-1} < H < \lambda_{\max} \kappa, \quad (9.11)
$$

then,

$$
\mu_1(\kappa, H) \leq \kappa^2 \inf_{x \in \Omega} \left\{ \frac{H}{\kappa} B_0(x) - a(x) \right\} + O(\kappa), \quad \text{as } \kappa \to +\infty. \quad (9.12)
$$

**Proposition 9.5** (Upper bound near the boundary). Suppose that $\Omega \subset \mathbb{R}^2$ is an open bounded set with a smooth boundary, $\lambda_{\max} > 0$ and $\Gamma = \emptyset$. For any $x_0 \in \partial \Omega$ and for any $\sigma \in (0, \lambda_{\max})$, we have,

$$
\mu_{B, \sigma} \leq \frac{B}{\sigma} \left( \sigma \Theta B_0(x_0) - a(x_0) \right) + O(B^\frac{1}{2}), \quad \text{as } B \to +\infty. \quad (9.13)
$$

Here, $\Theta_0$ is introduced in (8.10).

**Proof.** We recall the definition of $\mu_{B, \sigma}$ as follows:

$$
\mu_{B, \sigma} = \inf_{u \in H^1(\Omega) \setminus \{0\}} \left( \frac{\int_{\Omega} |(\nabla - iB\nabla)u(x)|^2 dx}{\int_{\Omega} |u(x)|^2 dx} - \frac{\mathcal{B}}{\sigma} \int_{\Omega} a(x)|u(x)|^2 dx \right).
$$

The first term in the right hand side is studied by Helffer-Morame (see [23, Theorem 9.1] with $h = B^{-1}$ and $\mu_{B, \sigma} = \frac{\mu(\lambda_1(h))}{\kappa}$) or Fournais-Helffer (see [13, Section 9.2.1]). They proved for any $x_0 \in \partial \Omega$ the existence of $B_0$ such that for $B \geq B_0$ one can construct a trial function $\tilde{u}$ such that,

$$
\frac{\int_{\Omega} |(\nabla - iB\nabla)\tilde{u}(x)|^2 dx}{\int_{\Omega} |\tilde{u}(x)|^2 dx} \leq \mathcal{B}_0 B_0(x_0) + O(B^\frac{1}{2}), \quad \text{as } B \to +\infty.
$$

The estimates of the second term in the right hand side are just as in (9.10) and this achieves the proof of the proposition. \qed

**Remark 9.6.** $\partial \Omega$ being compact, we can choose $x_0 \in \partial \Omega$, such that,

$$
\sigma \Theta B_0(x_0) - a(x_0) = \inf_{x \in \partial \Omega} \{ \sigma \Theta B_0(x) - a(x) \},
$$

and we apply Proposition 9.3 with

$$
B = \kappa H \quad \text{and} \quad \sigma = \frac{H}{\kappa},
$$
which implies under Assumption [9.11] that,
\[
\mu_1(\kappa, H) \leq \kappa^2 \inf_{x \in \partial \Omega} \left\{ \frac{H}{\kappa} \Theta_0 B_0(x) - a(x) \right\} + O(\kappa), \quad \text{as } \kappa \to +\infty. \tag{9.14}
\]

Remarks [9.4] and [9.9] lead to the conclusion in:

**Theorem 9.7.** Let \( \Omega \subset \mathbb{R}^2 \) is an open bounded set with a smooth boundary and \( \Gamma = \emptyset \). Suppose that [9.11] hold and \( a \in C^1(\bar{\Omega}) \), we have
\[
\mu_1(\kappa, H) \leq \kappa^2 \Lambda_1 \left( B_0, a, \frac{H}{\kappa} \right) + O(\kappa), \quad \text{as } \kappa \to +\infty.
\]
Here, \( \Lambda_1 \) is introduced in [9.1].

Notice that the conclusion in Theorem 9.7 is valid under the assumption \( \kappa H \geq B_0 \) with \( B_0 > 0 \) sufficiently large. Lemma [9.8] below takes care of the regime where \( \kappa H = O(1) \).

**Lemma 9.8.** Let \( C_{\max} > 0 \). Suppose that \( \{a > 0\} \neq \emptyset \). There exists a constant \( \kappa_0 > 0 \) such that, if
\[
\kappa \geq \kappa_0 \quad \text{and} \quad 0 \leq H \leq C_{\max} \kappa^{-1},
\]
then
\[
\mu_1(\kappa, H) < 0.
\]

**Remark 9.9.** The conclusion in Lemma 9.8 is valid in both cases where \( \Gamma = \emptyset \) and \( \Gamma \neq \emptyset \).

**Proof of Lemma 9.8.**

Let \( \ell > 0 \). Choose \( x_0 \in \Omega \) such that \( a(x_0) > 0 \). We introduce a cut-off function \( \chi_{\ell} \in C^\infty_c(\mathbb{R}^2) \) satisfying:
\[
0 \leq \chi_{\ell} \leq 1 \text{ in } \mathbb{R}^2, \quad \supp \chi_{\ell} \subset B(x_0, \ell), \quad \chi_{\ell} = 1 \text{ in } B(x_0, \ell/2) \quad \text{and} \quad |\nabla \chi_{\ell}| \leq \frac{C}{\ell}. \tag{9.15}
\]

The min-max principle yields,
\[
\mu^{(1)}(\kappa, H)\|\chi_{\ell}\|_{L^2(\Omega)}^2 \leq \int_{\Omega} |(\nabla - i\kappa H F)\chi_{\ell}|^2 dx - \kappa^2 \int_{\Omega} a(x)|\chi_{\ell}(x)|^2 dx.
\]

Using the assumptions on \( \chi_{\ell} \) and the fact that \( F \in C^\infty(\bar{\Omega}) \), a trivial estimate is,
\[
\int_{\Omega} |(\nabla - i\kappa H F)\chi_{\ell}|^2 dx = \int_{B(x_0, \ell)} |\nabla \chi_{\ell}(x)|^2 dx + \kappa^2 H^2 \int_{B(x_0, \ell)} |F \chi_{\ell}(x)|^2 dx
\]
\[
\leq C (1 + (\kappa H \ell)^2). \tag{9.16}
\]

We write by Taylor’s formula applied to the function \( a \) near \( x_0 \),
\[
- \kappa^2 \int_{\Omega} a(x)|\chi_{\ell}(x)|^2 dx \leq -a(x_0) \kappa^2 \ell^2 + C \kappa^2 \ell^3. \tag{9.17}
\]

Collecting [9.16] and [9.17], we obtain,
\[
\mu^{(1)}(\kappa, H)\|\chi_{\ell}\|_{L^2(\Omega)}^2 \leq -a(x_0) \kappa^2 \ell^2 + C(\kappa^2 \ell^3 + 1 + (\kappa H \ell)^2).
\]

We select \( \ell = \kappa^{-\frac{1}{2}} \) and note that \( \kappa H < C_{\max} \). We find that,
\[
\mu^{(1)}(\kappa, H)\|\chi_{\ell}\|_{L^2(\Omega)}^2 \leq -a(x_0) \kappa + C \left( \kappa^{\frac{1}{2}} + 1 + C_{\max}^2 \kappa^{-1} \right).
\]

Since \( \chi_{\ell} \neq 0 \) and \( a(x_0) > 0 \), we deduce that, for \( \kappa \) sufficiently large,
\[
\mu^{(1)}(\kappa, H) < 0.
\]
\[\square\]
10. Proof of Theorem 1.6

10.1. Analysis of $H_{c_{3,0}}$ and $H_{C_{3}}^{\text{loc}}$.

In this subsection we give a lower bound of the critical field $H_{c_{3,0}}^{\text{loc}}$ (see (1.29)) and we give an upper bound of the critical field $H_{C_{3}}^{\text{loc}}$ in the case when the magnetic field $B_0$ is constant with a pinning term.

**Proposition 10.1.** Suppose that $\{a > 0\} \neq \emptyset$ and $\Gamma = \emptyset$. There exist constants $C > 0$ and $\kappa_0 \geq 0$ such that if

$$\kappa \geq \kappa_0, \quad H \leq \kappa \max \left( \sup_{x \in \Omega} \frac{a(x)}{B_0(x)}, \sup_{x \in \partial \Omega} \frac{a(x)}{\Theta_0 B_0(x)} \right) - C \kappa^{\frac{1}{2}}, \quad (10.1)$$

then,

$$\mu_1(\kappa, H) < 0.$$  

Moreover,

$$\kappa \max \left( \sup_{x \in \Omega} \frac{a(x)}{B_0(x)}, \sup_{x \in \partial \Omega} \frac{a(x)}{\Theta_0 B_0(x)} \right) - C \kappa^{\frac{1}{2}} \leq H_{C_{3}}^{\text{loc}}.$$  

**Proof.** To apply the previous results, we take $\lambda_{\text{max}} = \max \left( \sup_{x \in \Omega} \frac{a(x)}{B_0(x)}, \sup_{x \in \partial \Omega} \frac{a(x)}{\Theta_0 B_0(x)} \right) + 1$.

We have two cases:

**Case 1.** If

$$\sup_{x \in \Omega} \frac{a(x)}{B_0(x)} > \sup_{x \in \partial \Omega} \frac{a(x)}{\Theta_0 B_0(x)},$$

then, there exists $x_0 \in \Omega$ (the supremum of $\frac{a(x)}{B_0(x)}$ can not be attained on the boundary, since $\frac{a(x)}{\Theta_0 B_0(x)} > \frac{a(x)}{B_0(x)}$), such that,

$$\sup_{x \in \Omega} \frac{a(x)}{B_0(x)} = \frac{a(x_0)}{B_0(x_0)}.$$  

If (10.1) is satisfied for some $C > 0$, then,

$$\frac{H}{\kappa} \leq \frac{a(x_0)}{B_0(x_0)} - C \kappa^{-\frac{1}{2}},$$

that we can write in the form,

$$\kappa^2 \left( \frac{H}{\kappa} B_0(x_0) - a(x_0) \right) \leq -CM \kappa^{\frac{3}{2}},$$

where $M > 0$ is a constant independent of $C$.

Suppose that $\kappa H \geq B_0$ where $B_0$ is selected sufficiently large such that we can apply Remark 9.4 (Thanks to Lemma 9.8 $\mu_1(\kappa, H) < 0$ when $\kappa H < B_0$).

Remark 9.4 tells us that there exist positive constants $C_1$ and $\kappa_0$ such that, for $\kappa \geq \kappa_0$,

$$\mu_1(\kappa, H) \leq \kappa^2 \inf_{x \in \Omega} \left( \frac{H}{\kappa} B_0(x) - a(x) \right) + C_1 \kappa$$

$$\leq \kappa^2 \left( \frac{H}{\kappa} B_0(x_0) - a(x_0) \right) + C_1 \kappa^{\frac{3}{2}} \quad (10.2)$$

$$\leq (C_1 - CM) \kappa^{\frac{3}{2}}. \quad (10.3)$$

By choosing $C$ such that $CM > C_1$, we get,

$$\mu_1(\kappa, H) < 0.$$
Case 2. Here, we suppose that
\[
\sup_{x \in \partial \Omega} \frac{a(x)}{\Theta_0 B_0(x)} \geq \sup_{x \in \Omega} \frac{a(x)}{B_0(x)}.
\]

By compactness, there exists \( x_0' \in \partial \Omega \), such that,
\[
\sup_{x \in \partial \Omega} \frac{a(x)}{\Theta_0 B_0(x)} = \frac{a(x_0')}{\Theta_0 B_0(x_0')}.
\]

If \((10.1)\) is satisfied for some \( C > 0 \), then,
\[
\kappa^2 \left( \frac{H}{\kappa} \Theta_0 B_0(x_0') - a(x_0') \right) \leq -CM' \kappa^2.
\]

Thanks to Remark 9.6, we get the existence of positive constants \( C_2 \) and \( \kappa_0 \) such that, for \( \kappa \geq \kappa_0 \),
\[
\mu_1(\kappa, \kappa, H) \leq \kappa^2 \inf_{x \in \partial \Omega} \left( \frac{H}{\kappa} \Theta_0 B_0(x) - a(x) \right) + C_2 \kappa
\]
\[
\leq \kappa^2 \left( \frac{H}{\kappa} \Theta_0 B_0(x_0') - a(x_0') \right) + C_2 \kappa^2
\]
\[
\leq (C_2 - CM') \kappa^2.
\]

By choosing \( C \) such that \( CM' > C_2 \), we get,
\[
\mu_1(\kappa, H) < 0.
\]

This finishes the proof of the proposition.

**Proposition 10.2.** Suppose that \( \{a > 0\} \neq \emptyset \), \( \lambda_{\max} > 0 \) and \( \Gamma = \emptyset \). There exist constants \( C > 0 \) and \( \kappa_0 > 0 \) such that if \( \kappa \geq \kappa_0 \), \( \lambda_{\max} \kappa \geq H > \kappa \max \left( \sup_{x \in \Omega} \frac{a(x)}{B_0(x)}, \sup_{x \in \partial \Omega} \frac{a(x)}{\Theta_0 B_0(x)} \right) + C \kappa \frac{1}{2} \),
\[
(10.6)
\]
then,
\[
\mu_1(\kappa, H) > 0.
\]

Moreover,
\[
\overline{H}_{C_3}^{\text{loc}} \leq \kappa \max \left( \sup_{x \in \Omega} \frac{a(x)}{B_0(x)}, \sup_{x \in \partial \Omega} \frac{a(x)}{\Theta_0 B_0(x)} \right) + C \kappa \frac{1}{2}.
\]

**Proof.** To apply the previous results, we take
\[
\lambda_{\min} = \frac{1}{2} \max \left( \sup_{x \in \Omega} \frac{a(x)}{B_0(x)}, \sup_{x \in \partial \Omega} \frac{a(x)}{\Theta_0 B_0(x)} \right).
\]

If \((10.6)\) holds for some \( C > 0 \), then, for any \( x \in \Omega \), we have,
\[
\frac{H}{\kappa} B_0(x) - a(x) \geq C \kappa^{-\frac{1}{2}},
\]
\[
(10.7)
\]
and, for any \( x' \in \partial \Omega \), we have,
\[
\frac{H}{\kappa} \Theta_0 B_0(x') - a(x') \geq C \kappa^{-\frac{1}{2}}.
\]
\[
(10.8)
\]
Having in mind the definition of \( \Lambda_1 \) in \((9.1)\), the estimates in \((10.7)\) and in \((10.8)\) give us that for \( \kappa \) large enough,
\[
\Lambda_1 \left( B_0, a, \frac{H}{\kappa} \right) \geq C \kappa^{-\frac{1}{2}}.
\]
Thanks to Theorem 9.2, we get the existence of positive constants $C'$ and $\kappa_0$ such that, for $\kappa \geq \kappa_0$,

$$
\mu_1(\kappa, H) \geq \kappa^2 \Lambda_1 \left(B_0, a, \frac{H}{\kappa}\right) - C' \kappa^\frac{3}{2} \\
\geq (C - C') \kappa^\frac{3}{2}.
$$

(10.9)

To finish this proof, we choose $C > C'$.

As a consequence, we have proved Theorem 1.6 for $H^\loc_{C^3}$ and $\overline{H}^\loc_{C^3}$.

10.2. Analysis of $H^\loc_{C^3}$ and $\overline{H}^\loc_{C^3}$.

In this subsection we give a lower bound of the critical field $H^\loc_{C^3}$ (see (1.27)) and we give an upper bound of the critical field $\overline{H}^\loc_{C^3}$ in the case when the magnetic field $B_0$ is constant with a pining term. We start with a proposition which measures the effect of the localization at the boundary when $H$ is sufficiently large.

**Proposition 10.3.** Suppose that $\Gamma = \emptyset$ and (10.6) holds. There exists a positive constant $C$, such that if $(\psi, A)$ is a solution of (1.12), then the following estimate holds:

$$
\|\psi\|^2_{L^2(\Omega)} \leq C \kappa^{-\frac{3}{2}} \|\psi\|^2_{L^4(\Omega)}.
$$

(10.10)

**Proof.**

The techniques that will be used in this proof are similar with the ones in [14, Lemma 2.6]. If $H$ satisfies (10.6) for some $C > 0$, then, for any $x \in \Omega$, we have,

$$
\kappa H B_0(x) - \kappa^2 a(x) \geq C \kappa^\frac{3}{2}.
$$

(10.11)

First, we let $\chi \in C^\infty(\mathbb{R})$ be a standard cut-off function such that

$$
\chi = 1 \text{ in } [1, \infty] \quad \text{and} \quad \chi = 0 \text{ in } ]- \infty, 1/2[.
$$

(10.12)

Next, we define $\lambda = \kappa^{-\frac{3}{2}}$, and $\chi_\lambda$ as follows:

$$
\chi_\lambda(x) = \chi \left( \frac{\text{dist}(x, \partial\Omega)}{\lambda} \right), \quad \forall x \in \Omega.
$$

(10.13)

Referring to (7.6), we have

$$
\int_\Omega \left| (\nabla - i\kappa H A) \chi_\lambda \psi \right|^2 \, dx = \kappa^2 \int_\Omega |\nabla \chi_\lambda|^2 |\psi|^2 \, dx.
$$

(10.14)

We estimate $\int_\Omega \left| (\nabla - i\kappa H A) \chi_\lambda \psi \right|^2 \, dx$ from below. As in [21, Proposition 6.2], we can prove that,

$$
\int_\Omega \left| (\nabla - i\kappa H A) \chi_\lambda \psi \right|^2 \, dx \geq \kappa H \int_\Omega \text{curl} F \left| \chi_\lambda \psi \right|^2 \, dx - \kappa H \|\text{curl}(A - F)\|_{L^2(\Omega)} \|\chi_\lambda \psi\|^2_{L^4(\Omega)}.
$$

Noticing that $\text{curl} F = B_0(x)$ and $\|\text{curl}(A - F)\|_{L^2(\Omega)} \leq \frac{c}{H} \|\psi\|_{L^2(\Omega)}$, we have,

$$
\int_\Omega \left| (\nabla - i\kappa H A) \chi_\lambda \psi \right|^2 \, dx \geq \kappa H \int_\Omega B_0(x) |\chi_\lambda \psi|^2 \, dx - \kappa C \|\psi\|^2_{L^2(\Omega)} \|\chi_\lambda \psi\|^2_{L^4(\Omega)}.
$$

Implementing a Cauchy-Schwarz inequality, we get

$$
\int_\Omega \left| (\nabla - i\kappa H A) \chi_\lambda \psi \right|^2 \, dx \geq \kappa H \int_\Omega B_0(x) |\chi_\lambda \psi|^2 \, dx - c^2 \|\psi\|^2_{L^2(\Omega)} - \kappa^2 \|\chi_\lambda \psi\|^4_{L^4(\Omega)}.
$$

(10.15)

Inserting (10.15) into (10.14), we obtain,

$$
\int_\Omega \left( \kappa H B_0(x) - \kappa^2 a(x) \right) |\chi_\lambda \psi|^2 \, dx \leq c^2 \int_\Omega |\psi|^2 \, dx + \int_\Omega |\nabla \chi_\lambda|^2 |\psi|^2 \, dx - \kappa^2 \int_\Omega \left( \chi_\lambda^2 - \chi_{\lambda^4} \right) |\psi|^4 \, dx.
$$
Theorem 10.4. Suppose that \( \Gamma = \emptyset \) and \( \{a > 0\} \neq \emptyset \). There exists \( C > 0 \) and \( \kappa_0 \) such that, if \( H \) satisfies
\[
H > \kappa \max \left( \sup_{x \in \Omega} \frac{a(x)}{B_0(x)}, \sup_{x \in \Omega} \frac{a(x)}{\Theta_0 B_0(x)} \right) + C \kappa^\frac{1}{2},
\]
then \( (0, F) \) is the unique solution to (1.12).
Moreover,
\[
\mathcal{H}_{C_3}^{cp} \leq \kappa \max \left( \sup_{x \in \Omega} \frac{a(x)}{B_0(x)}, \sup_{x \in \Omega} \frac{a(x)}{\Theta_0 B_0(x)} \right) + C \kappa^\frac{1}{2}.
\]

Proof. We first observe that it results from Giorgi-Phillips like Theorem 8.3 that it remains only to prove the theorem under the stronger Assumption (10.6). Suppose now that \( (\psi, A) \) is a solution of (1.12) with \( \psi \neq 0 \), we observe that
\[
0 < \kappa^2 \|\psi\|_{L^2(\Omega)}^2 = -\int_\Omega \left( |(\nabla - i \kappa H A)\psi|^2 - \kappa^2 a(x)|\psi|^2 \right) dx := T.
\]
We can write,
\[
-\frac{T}{\kappa} \geq (1 - \sqrt{\frac{\kappa}{\kappa - 1}}) \int_\Omega |(\nabla - i \kappa H F)\psi|^2 dx - \kappa \int_\Omega a(x)|\psi|^2 dx - \frac{(\kappa H)^2}{\sqrt{\kappa - 1}} \int_\Omega |(A - F)\psi|^2 dx
\]
\[
\geq \mu_1(\kappa, H) \|\psi\|^2_{L^2(\Omega)} - \sqrt{\frac{\kappa}{\kappa - 1}} \kappa \|\psi\|^2_{L^2(\Omega)} - \frac{(\kappa H)^2}{\sqrt{\kappa - 1}} \|\psi\|^2_{L^2(\Omega)}.
\]
We refer to (8.3) and (8.7), we have,
\[
-\frac{T}{\kappa} \geq \mu_1(\kappa, H) \|\psi\|^2_{L^2(\Omega)} - C \sqrt{\frac{\kappa}{\kappa - 1}} \kappa \|\psi\|^2_{L^2(\Omega)}.
\]
Thanks to Proposition 10.3 using (10.17), we get,
\[
\|\psi\|^2_{L^2(\Omega)} \leq C \kappa^{- \frac{11}{8}} \sqrt{T}.
\]
As a consequence of (10.20), (10.19) becomes,
\[-\mathcal{T} \geq \mu_1(\kappa, H) \|\psi\|_{L^2(\Omega)}^2 - C' \kappa^{-\frac{5}{8}} \mathcal{T}.\]  
(10.21)
Having in mind that \(\psi \neq 0\) and \(\mathcal{T} > 0\) (see (10.17)), we deduce for \(\kappa\) sufficiently large \(\mu_1(\kappa, H) < 0\), which is in contradiction with Proposition 10.2. Therefore, we conclude that \(\psi = 0\), which is what we needed to prove. \(\square\)

**Proposition 10.5.** Suppose that \(\Gamma = \emptyset\) and \(\{a > 0\} \neq \emptyset\). There exists \(C > 0\) and \(\kappa_0\) such that, if \(H\) satisfies
\[H \leq \kappa \max \left( \sup_{x \in \Omega} \frac{a(x)}{B_0(x)}, \sup_{x \in \Omega} \frac{a(x)}{\Theta_0 B_0(x)} \right) - C \kappa^\frac{1}{2},\]  
(10.22)
then there exists a solution \((\psi, A)\) of (1.12) with \(\|\psi\|_{L^2(\Omega)} \neq 0\).
Moreover,
\[\kappa \max \left( \sup_{x \in \Omega} \frac{a(x)}{B_0(x)}, \sup_{x \in \Omega} \frac{a(x)}{\Theta_0 B_0(x)} \right) - C \kappa^\frac{1}{2} \leq H_{CP}^{eq}.
\]

**Proof.** We use \((t\psi, F)\), with \(t\) sufficiently small and \(\psi\), an eigenfunction associated with \(\mu_1(\kappa, H)\), as a test configuration for the functional (1.1), i.e.
\[\int_{\Omega} \left( \|\nabla - i\kappa HF\|_{L^2(\Omega)}^2 - \kappa^2 a(x)\|\psi\|_{L^2(\Omega)}^2 \right) dx = \mu_1(\kappa, H) \|\psi\|_{L^2(\Omega)}^2.
\]
Proposition 10 tells us that there exists a constant \(C\) such that, under Assumption (10.22), \(\mu_1(\kappa, H) < 0\).
Therefore,
\[C_1(\kappa, H) := \int_{\Omega} \left( \|\nabla - i\kappa HF\|_{L^2(\Omega)}^2 - \kappa^2 a(x)\|\psi\|_{L^2(\Omega)}^2 \right) dx < 0.
\]
We can write,
\[E_{\kappa,H,a,B_0}(t\psi, F) = t^2 \int_{\Omega} \left( \|\nabla - i\kappa HF\|_{L^2(\Omega)}^2 - \kappa^2 a(x)\|\psi\|_{L^2(\Omega)}^2 \right) dx + t^4 \frac{\kappa^2}{2} \int_{\Omega} \|\psi\|_{L^2(\Omega)}^4 dx + \frac{\kappa^2}{2} \int_{\Omega} a(x) dx
\]
\[= t^2 \left( C_1(\kappa, H) + \frac{\kappa^2}{2} \int_{\Omega} \|\psi\|_{L^2(\Omega)}^4 dx \right) + E_{\kappa,H,a,B_0}(0, F).
\]
We choose \(t\) such that,
\[C_1(\kappa, H) + \frac{\kappa^2}{2} \int_{\Omega} \|\psi\|_{L^2(\Omega)}^4 dx < 0.
\]
Thus, we get
\[E_{\kappa,H,a,B_0}(t\psi, F) < E_{\kappa,H,a,B_0}(0, F).
\]
Hence a minimizer, which is a solution of (1.12), will be non-trivial. \(\square\)

**10.3. End of the proof of Theorem 1.6.** First, we will prove the following inclusion,
\[N^{loc}(\kappa) \subset N(\kappa).
\]
We see that if \(H \notin N(\kappa)\), then \((0, F)\) is a local minimizer of \(E_{\kappa,H,a,B_0}\). Thus, the Hessian of the functional \(E_{\kappa,H,a,B_0}\) at the normal state \((0, F)\) should be positive.
For every \((\tilde{\phi}, \tilde{A}) \in H^1(\Omega) \times H^1_{\text{div}}(\Omega)\) we have,
\[E_{\kappa,H,a,B_0}(t\tilde{\phi}, F + t\tilde{A}) = t^2 \left[ Q_{\kappa HF, -\kappa^2 a}(\tilde{\phi}) + (\kappa H)^2 \int_{\Omega} |\text{curl } \tilde{A}|^2 dx \right] + O(t^3).
\]
This implies that the Hessian of the functional \(E_{\kappa,H,a,B_0}\) at the normal state \((0, F)\) can be written as follows:
\[\text{Hess}_{(0,F)}(\tilde{\phi}, \tilde{A}) = Q^\Omega_{\kappa HF, -\kappa^2 a}(\tilde{\phi}) + (\kappa H)^2 \int_{\Omega} |\text{curl } \tilde{A}|^2 dx.
\]
Since \( \text{Hess}_{(0,F)}[\tilde{\phi}, \tilde{A}] \geq 0 \), we get that \( \mu_1 \kappa H \geq 0 \), and consequently \( H \notin N^{\text{loc}}(\kappa) \). Hence we obtain the above inclusion.

On the other hand, if \( (\psi, A) \) is a minimizer of the functional in (1.1) with \( \psi \neq 0 \), then \( (\psi, A) \) is a solution of (1.12), and we have the following inclusion,

\[
\mathcal{N}(\kappa) \subset N^{\text{cp}}(\kappa),
\]

and consequently,

\[
\mathcal{N}^{\text{loc}}(\kappa) \subset \mathcal{N}(\kappa) \subset N^{\text{cp}}(\kappa).
\]  \hspace{1cm} (10.23)

Having in mind the definition of all the critical fields in (1.27), (1.28) and (1.29), we deduce that,

\[
\mathcal{N}(\kappa) \subset \mathcal{N}^{\text{cp}}(\kappa).
\]  \hspace{1cm} (10.24)

Using (10.23), we observe that,

\[
\mathbb{R}^+ \setminus \mathcal{N}^{\text{cp}}(\kappa) \subset \mathbb{R}^+ \setminus \mathcal{N}(\kappa) \subset \mathbb{R}^+ \setminus \mathcal{N}^{\text{loc}}(\kappa).
\]

From the definition of all the critical fields, we conclude that,

\[
H^{\text{loc}}_{C_3}(\kappa) \leq H^{\text{cp}}_{C_3}(\kappa) \leq H^{\text{cp}}_{C_3}(\kappa).
\]  \hspace{1cm} (10.25)

We note that \( H^{\text{loc}}_{C_3}(\kappa) \leq H^{\text{loc}}_{C_3}(\kappa) \) and \( H^{\text{cp}}_{C_3}(\kappa) \leq H^{\text{cp}}_{C_3}(\kappa) \). Therefore, all the critical fields are contained in the interval \([H^{\text{loc}}_{C_3}(\kappa), H^{\text{cp}}_{C_3}(\kappa)]\).

As a consequence, we have proved Theorem 1.6 for the six critical fields.

Remark 10.6. As in [13], it would be interesting to show that all the critical fields coincide when \( \kappa \) is large enough.

11. Asymptotics of \( \mu_1(\kappa, H) \): The Case with Vanishing Magnetic Field

In this section we give an estimate for the lowest eigenvalue \( \mu_1(\kappa, H) \) of the operator \( P_{\kappa H F, -\kappa^2 a}^\Omega \) (see (1.26)) in the case when \( \Gamma = \emptyset \) with a \( \kappa \)-independent pinning, i.e. \( a(\kappa, x) = a(x) \). The results in this section are valid under the assumption \( \Gamma \neq \emptyset \), where the set \( \Gamma \) is introduced in (1.3). Let

\[
\mathcal{B} = \kappa H \quad \text{and} \quad \hat{\sigma} = \frac{H}{\kappa^2} \cdot \hspace{1cm} (11.1)
\]

We observe that,

\[
P_{\kappa H F, -\kappa^2 a}^\Omega = P_{\mathcal{B}F, -(\hat{\sigma})^2 a}^\Omega.
\]

We will give an estimate for the lowest eigenvalue \( \mu_{\mathcal{B}, \hat{\sigma}} \) of \( P_{\mathcal{B}F, -(\hat{\sigma})^2 a}^\Omega \). After a change of notation, we deduce an estimate for \( \mu_1(\kappa, H) \).
11.1. **Lower bound.** In the absence of a pinning term, that is when \( a = 1 \), Pan and Kwek [31] gave the lower bound for the lowest eigenvalue \( \mu(BF) \) of \( P_{BF,0}^D \) when \( B \to +\infty \). In this subsection, we determine a lower bound for \( \mu_1 \) when \( \kappa \to +\infty \) and the pinning term is present. We first recall the definition of \( \lambda_0 \) in (1.31), the definition of \( \Gamma \) in (1.3) and for any \( \theta \in (0, \pi) \) we recall that \( \lambda(\mathbb{R}^2_+, \theta) \) is the bottom of the spectrum of the operator \( P_{A_{app,\theta},0}^+ \), with
\[
A_{app,\theta} = -\left( \frac{x^2}{2} \cos \theta, \frac{x^2}{2} \sin \theta \right).
\]

We then define for any \( \tilde{\sigma} > 0 \),
\[
\hat{\Lambda}_1 (B_0, a, \tilde{\sigma}) = \min \left\{ \inf_{x \in \Gamma \cap \Omega} \left\{ \lambda_0 \left( \tilde{\sigma} |\nabla B_0(x)| \right)^2 - a(x) \right\}, \inf_{x \in \Gamma \cap \partial \Omega} \left\{ \lambda(\mathbb{R}^2_+, \theta(x)) \left( \tilde{\sigma} |\nabla B_0(x)| \right)^2 - a(x) \right\} \right\}.
\] (11.2)

Here, for \( x \in \Gamma \cap \partial \Omega \), \( \theta(x) \) denotes the angle between \( \nabla B_0(x) \) and the inward normal vector \( -\nu(x) \).

We start with a proposition that states a lower bound of \( \mu_1(\kappa, H) \) in the case when \( \Gamma \neq \emptyset \).

**Proposition 11.1.** Let \( I \) be a closed interval in \((0, \infty)\). There exist two positive constants \( B_0 > 0 \) and \( C > 0 \) such that if \( B \geq B_0 \), \( \tilde{\sigma} \in I \), \( \psi \in H^1(\Omega) \setminus \{0\} \) and \( a \in C^1(\overline{\Omega}) \), then,
\[
\frac{Q_{\Omega}^{B_{BF}, -\left( \frac{\kappa}{\sigma} \right)^\frac{2}{3} a} (\psi)}{\|\psi\|_{L^2(\Omega)}^2} \geq \left( \frac{B}{\tilde{\sigma}} \right)^\frac{3}{2} \left( \hat{\Lambda}_1 (B_0, a, \tilde{\sigma}) - CB^{-\frac{1}{29}} \right). \tag{11.3}
\]

**Proof.** Let \( \ell = B^{-7/29} \). We define the following sets,
\[
D_1 = \{ x \in \Omega : \text{dist}(x, \Gamma) < 2 \ell \}, \quad D_2 = \{ x \in \Omega : \text{dist}(x, \Gamma) > \ell \}.
\]

Let \( h_j \) be a partition of unity satisfying
\[
\sum_{j=1}^2 h_j^2 = 1, \quad \sum_{j=1}^2 |\nabla h_j|^2 \leq C \ell^{-2} = CB^{14/29} \quad \text{and} \quad \text{supp } h_j \subset D_j \quad (j \in \{1, 2\}).
\]

There holds the following decomposition,
\[
Q_{\Omega}^{B_{BF}, -\left( \frac{\kappa}{\sigma} \right)^\frac{2}{3} a} (\psi) = Q_{\Omega}^{D_1, B_{BF}, -\left( \frac{\kappa}{\sigma} \right)^\frac{2}{3} a} (h_1 \psi) + Q_{\Omega}^{D_2, B_{BF}, -\left( \frac{\kappa}{\sigma} \right)^\frac{2}{3} a} (h_2 \psi) - \sum_{j=1}^2 \int_{\Omega} |\nabla h_j|^2 |\psi|^2 \, dx
\]
\[
\geq Q_{\Omega}^{D_1, B_{BF}, -\left( \frac{\kappa}{\sigma} \right)^\frac{2}{3} a} (h_1 \psi) + Q_{\Omega}^{D_2, B_{BF}, -\left( \frac{\kappa}{\sigma} \right)^\frac{2}{3} a} (h_2 \psi) - CB^{14/29} \int_{\Omega} |\psi|^2 \, dx. \tag{11.4}
\]

We cover the curve \( \Gamma \) by a family of disks
\[
D(\omega_j, \ell) \subset \{ x \in \mathbb{R}^2 : \text{dist}(x, \Gamma) \leq 2\ell \} \quad \text{and} \quad D_1 \subset \bigcup_j D(\omega_j, \ell) \quad (\omega_j \in \Gamma).
\]

Consider a partition of unity satisfying
\[
\sum_{j} \chi_j^2 = 1, \quad \sum_{j} |\nabla \chi_j|^2 \leq C \ell^{-2} \quad \text{and} \quad \text{supp } \chi_j \subset D(\omega_j, \ell).
\]

Moreover, we can add the property that:
\[
\text{either supp } \chi_j \cap \Gamma \cap \partial \Omega = \emptyset, \quad \text{either } \omega_j \in \Gamma \cap \partial \Omega.
\]
We may write,
\[
Q_{\frac{\mathcal{BF}}{2}}^{D_1}(h_1 \psi) = \sum_{\text{int}} Q_{\frac{\mathcal{BF}}{2}}^{D_1} \left( \chi_jh_1 \psi \right) + \sum_{\text{bnd}} Q_{\frac{\mathcal{BF}}{2}}^{D_1} \left( \chi_jh_1 \psi \right) - \sum_j \int_{D_1} |\nabla \chi_j|^2 |h_1 \psi|^2 \, dx .
\]
(11.5)

where ‘int’ is in reference to the $j$’s such that $\omega_j \in \Gamma \cap \Omega$ and ‘bnd’ is in reference to the $j$’s such that $\omega_j \in \Gamma \cap \partial \Omega$.

For the last term on the right side of (11.5), we get using the assumption on $\chi_j$:
\[
\int_{D_1} |\nabla \chi_j|^2 |h_1 \psi|^2 \, dx \leq C \ell^{-2} \int_{D_1} |h_1 \psi|^2 \, dx = C \mathcal{B}^{29/14} \int_{D_1} |h_1 \psi|^2 \, dx .
\]
(11.6)

We have to find a lower bound for $Q_{\frac{\mathcal{BF}}{2}}^{D_1}(h_1 \psi)$ for each $j$ such that $\omega_j \in \Gamma \cap \Omega$ and for each $j$ such that $\omega_j \in \Gamma \cap \partial \Omega$. Thanks to [33], we have,
\[
\int_{\Omega} |(\nabla - i\mathcal{BF})\chi_jh_1 \psi|^2 \, dx \geq \mathcal{B}^2 \int_{\Omega} \left( \lambda_0 |\nabla B_0(\omega_j)| \right)^2 \chi_jh_1 \psi^2 \, dx.
\]

Using Taylor’s formula, we can write in every disk $D(w_j, \ell)$,
\[
|a(x) - a(w_j)| \leq C\ell = C\mathcal{B}^{-7/29} \leq C\mathcal{B}^{-1/18} .
\]
(11.7)

In that way, we get,
\[
\sum_{\text{int}} Q_{\frac{\mathcal{BF}}{2}}^{D_1} \left( \chi_jh_1 \psi \right) \\
\geq \sum_{\text{int}} \left( \frac{B}{\sigma} \right)^2 \left( \lambda_0 \left( \tilde{\sigma} |\nabla B_0(\omega_j)| \right)^2 - a(\omega_j) - C\mathcal{B}^{-1/18} \right) \int |\chi_jh_1 \psi|^2 \, dx \\
\geq \left( \frac{B}{\sigma} \right)^2 \left( \inf_{x \in \Gamma \cap \Omega} \left\{ \lambda_0 \left( \tilde{\sigma} |\nabla B_0(x)| \right)^2 - a(x) \right\} - C\mathcal{B}^{-1/18} \right) \sum_{\text{int}} \int |\chi_jh_1 \psi|^2 \, dx .
\]
(11.8)

In a similar fashion, the analysis in [33] and (11.7) yields,
\[
\sum_{\text{bnd}} Q_{\frac{\mathcal{BF}}{2}}^{D_1} \left( \chi_jh_1 \psi \right) \\
\geq \sum_{\text{bnd}} \left( \frac{B}{\sigma} \right)^2 \left( \lambda(\mathbb{R}_+, \theta(\omega_j)) \left( \tilde{\sigma} |\nabla B_0(\omega_j)| \right)^2 - a(\omega_j) - C\mathcal{B}^{-1/18} \right) \int |\chi_jh_1 \psi|^2 \, dx \\
\geq \left( \frac{B}{\sigma} \right)^2 \left( \inf_{x \in \Gamma \cap \Omega} \left\{ \lambda(\mathbb{R}_+, \theta(x)) \left( \tilde{\sigma} |\nabla B_0(x)| \right)^2 - a(x) \right\} - C\mathcal{B}^{-1/18} \right) \sum_{\text{bnd}} \int |\chi_jh_1 \psi|^2 \, dx .
\]
(11.9)

We insert (11.8), (11.9) and (11.6) into (11.5) to obtain,
\[
Q_{\frac{\mathcal{BF}}{2}}^{D_1}(h_1 \psi) \geq \left( \frac{B}{\sigma} \right)^2 \left( \tilde{\Lambda}(B_0, a, \tilde{\sigma}) - C\mathcal{B}^{-1/18} \right) \int |h_1 \psi|^2 \, dx .
\]
(11.10)

Now, we will bound $\int_{\Omega} |(\nabla - i\mathcal{BF})h_2 \psi|^2 \, dx$ from below. Let $\ell_1 < \ell$, we cover $D_2$ by a family of disks
\[
D(\omega_j', \ell_1) \subset \{ x \in \mathbb{R}^2 : \text{dist}(x, \Gamma) \geq \ell_1 \} \quad (\omega_j' \in \overline{\Omega}) .
\]

Consider a partition of unity satisfying
\[
\sum_j \chi_j^2 = 1 , \quad \sum_j |\nabla \chi_j|^2 \leq C\ell_1^{-2} \quad \text{and} \quad \text{supp} \chi_j \subset D(\omega_j', \ell_1) .
\]
There holds the decomposition formula,
\[
\int_{\Omega} |(\nabla - iB\mathbf{F})h_2\psi|^2 \, dx = \sum_j \int_{\Omega} |(\nabla - iB\mathbf{F})\chi_j h_2\psi|^2 \, dx - \sum_j \int_{\Omega} |\nabla \chi_j|^2 |h_2\psi|^2 \, dx \\
\geq \sum_j \int_{\Omega} |(\nabla - iB\mathbf{F})\chi_j h_2\psi|^2 \, dx - C\ell_1^{-2} \int_{\Omega} |h_2\psi|^2 \, dx,
\] (11.11)

We observe that there exists a gauge function \( \varphi_j \) satisfying (see \[4, Equation (A.3)]),
\[
|F(x) - (B_0(\omega'_j)A_0(x - \omega'_j) + \nabla \varphi_j)| \leq C\ell_1^2 \text{ in } D(\omega'_j, \ell'_1).
\]

Using Cauchy-Schwarz inequality, we may write,
\[
\int_{\Omega} |(\nabla - iB\mathbf{F})\chi_j h_2\psi|^2 \, dx \geq \frac{1}{2} \int_{\Omega} |(\nabla - iB B_0(\omega'_j)A_0(x - \omega'_j))e^{-iB\varphi_j} \chi_j h_2\psi|^2 \, dx \\
- C B^2 \ell_1^4 \int_{\Omega} |\chi_j h_2\psi|^2 \, dx.
\]

We are reduced to the analysis of the Neumann realization of the Schrödinger operator with a constant magnetic field equal to \( B B_0(\omega'_j) \) in our case.

Notice that by the assumption on \( h_2 \), there exist constants \( M > 0 \) and \( B_0 > 0 \) such that, for all \( j \), \( |B_0(\omega'_j)| \geq M \ell \) in the support of \( h_2 \). Thus,
\[
\forall j, \quad B|B_0(\omega'_j)| \geq M B \ell = MB_{22/29} \gg 1.
\]

Moreover, the magnetic potentials \( A_0(x) \) and \( A_0(x - \omega'_j) \) are gauge equivalent since
\[
A_0(x - \omega'_j) = A_0(x) - A_0(\omega'_j) = A_0(x) - \nabla (A_0(\omega'_j) - x).
\]

Thanks to Theorem 8.2, there exists a constant \( B_0 \) such that, for any \( B \geq B_0 \), we write by the min-max principle,
\[
\sum_j \int_{\Omega} |(\nabla - iB\mathbf{F})\chi_j h_2\psi|^2 \, dx \geq \frac{\Theta_0 B |B_0(\omega'_j)|}{2} \sum_{int} \int_{\Omega} |\chi_j h_2\psi|^2 \, dx - C B^2 \ell_1^4 \sum_{int} \int_{\Omega} |\chi_j h_2\psi|^2 \, dx \\
\geq \left( \frac{\Theta_0}{2} B \ell - C B^2 \ell_1^4 \right) \sum_j \int_{\Omega} |\chi_j h_2\psi|^2 \, dx \\
= \left( \frac{\Theta_0}{2} B \ell - C B^2 \ell_1^4 \right) \int_{\Omega} |h_2\psi|^2 \, dx.
\] (11.12)

Putting (11.12) into (11.11), we obtain
\[
Q_{Bz_{\mathbf{F}}, -\left( \frac{a}{2} \right)^2}^D (h_2\psi) = \int_{\Omega} |(\nabla - iB\mathbf{F})h_2\psi|^2 \, dx - \left( \frac{B}{\sigma} \right)^{2/3} \int_{\Omega} a(x) |h_2\psi|^2 \, dx \\
\geq \left( \frac{\Theta_0}{2} B \ell - C B^2 \ell_1^4 - C \ell_1^{-2} \right) \int_{\Omega} |h_2\psi|^2 \, dx - \left( \frac{B}{\sigma} \right)^{2/3} \int_{\Omega} a(x) |h_2\psi|^2 \, dx.
\] (11.13)

We choose \( \ell_1 = B^{-\rho} \) and \( \frac{9}{22} < \rho < \frac{11}{22} \). We observe that,
\[
B^2 \ell_1^4 = B^{-4\rho} \ll B_{22/29} = B \ell, \quad \ell_1^{-2} = B^{2\rho} \ll B \ell, \quad B^{2/3} \ll B_{22/29} = B \ell.
\]

In this way, we infer from (11.13), that there exists a constant \( c > 0 \) such that, for \( B \) sufficiently large,
\[
Q_{Bz_{\mathbf{F}}, -\left( \frac{a}{2} \right)^2}^D (h_2\psi) \geq c B_{22/29} \int_{\Omega} |h_2\psi|^2 \, dx \geq \left( \frac{B}{\sigma} \right)^{\frac{2}{3}} \Lambda_1 (B_0, a, \sigma) \int_{\Omega} |h_2\psi|^2 \, dx.
\] (11.14)
Collecting (11.4), (11.10) and (11.14), we finish the proof of Proposition 11.1. □

Theorem 11.2 is valid under the assumption that,

$$0 < \hat{\lambda}_{\min} < \hat{\lambda}_{\max} < \infty$$

are constants independent of $\kappa$ and $H$.

Theorem 11.2. Let $\Omega \subset \mathbb{R}^2$ is an open bounded set with a smooth boundary and $\Gamma \neq \emptyset$. Suppose that (11.15) hold and $a \in C^1(\overline{\Omega})$, we have

$$\mu_1(\kappa, H) \geq \kappa^2 \hat{\Lambda}_1 \left( B_0, a, \frac{H}{\kappa^2} \right) + \mathcal{O}(\kappa^{\frac{11}{6}}),$$

as $\kappa \to +\infty$.

Here, $\hat{\Lambda}_1$ is introduced in (11.2).

Proof. We apply Proposition 11.1 with $B = \kappa H$, $\hat{\sigma} = \frac{H}{\kappa^2}$ and $I = [\hat{\lambda}_{\min}, \hat{\lambda}_{\max}]$.

Let us verify that the conditions of the proposition are satisfied for this choice. Thanks to (11.15), $\hat{\sigma} \in I$. Now, as $\kappa \to +\infty$, we have,

$$B = \hat{\sigma} \kappa^3 \to +\infty.$$

This implies that, as $\kappa \to +\infty$,

$$\mu_1(\kappa, H) \geq \kappa^2 \hat{\Lambda}_1 \left( B_0, a, \frac{H}{\kappa^2} \right) + \mathcal{O}(\kappa^{\frac{11}{6}}).$$

This finish the proof of the theorem. □

11.2. Upper bound.

The next theorem is a generalization of the results in [34] and [33] valid when the pinning term $a(\kappa, x) = a(x)$ is independent of $\kappa$ and non-constant.

We denote by $\mu_{B, \hat{\sigma}}$ the lowest eigenvalue of the operator $P^{\Omega}_{B, \hat{\sigma}} \cdot \left( \frac{\nabla}{\hat{\sigma}} \right)^2 a$, i.e.

$$\mu_{B, \hat{\sigma}} = \inf_{\psi \in H^1(\Omega) \setminus \{0\}} \frac{Q^{\Omega}_{B, \hat{\sigma}} \left( \frac{\nabla}{\hat{\sigma}} \right)^2 a (\psi)}{\|\psi\|^2_{L^2(\Omega)}}.$$

Proposition 11.3. Suppose that $\Gamma \neq \emptyset$ and $\hat{\lambda}_{\max} > 0$. There exist positive constants $C$ and $B_0$ such that, for $\hat{\sigma} \in (0, \hat{\lambda}_{\max}]$, $a \in C^1(\overline{\Omega})$ and $B \geq B_0$, we have,

$$\mu_{B, \hat{\sigma}} \leq \left( \frac{B}{\sigma} \right)^\frac{3}{2} \left( \hat{\Lambda}(B_0, a, \hat{\sigma}) - CB^{-\frac{1}{18}} \right).$$

Proof. Let $x_0 \in \Gamma$. In [34] [33], a quasi-mode $u(B, x_0; x)$ is constructed such that, supp $u(B, x_0; \cdot) \subset \overline{\Omega} \cap B(0, B^{-1/18})$ and,

$$\forall B \geq B_0, \quad \int_{\Omega} |(\nabla - iBF)u(B, x_0; x)|^2 dx \leq B^\frac{3}{2} \left( \hat{\Lambda}(x_0) + CB^{-1/18} \right),$$
where $B_0$ and $C$ are constants independent of the point $x_0$ and the parameter $B$, and

$$
\Lambda(x_0) = \begin{cases} 
\lambda_0 |\nabla B_0(x_0)|^{\frac{2}{3}} & \text{if } x_0 \in \Gamma \cap \Omega, \\
\lambda(\mathbb{R}^2, \theta(x_0)) |\nabla B_0(x_0)|^{\frac{2}{3}} & \text{if } x_0 \in \Gamma \cap \partial \Omega.
\end{cases}
$$

Using the smoothness of the function $a(\cdot)$, we get in the support of $u(B, x_0; \cdot)$,

$$|a(x) - a(x_0)| \leq CB^{-1/18}.$$

Thus, we deduce that,

$$
\frac{Q^\Omega_{BF, -(\frac{B}{\sigma})}^\frac{3}{2}a(u(B, x_0; \cdot))}{\|u(B, x_0; \cdot)\|^2_{L^2(\Omega)}} \leq \left( \frac{B}{\sigma} \right)^{\frac{3}{2}} \left( \hat{\sigma}^{\frac{3}{2}}\Lambda(x_0) - a(x_0) + CB^{-1/18} \right).
$$

Thanks to the min-max principle, we deduce that,

$$
\mu_{B, \sigma} \leq \left( \frac{B}{\sigma} \right)^{\frac{3}{2}} \left( \hat{\sigma}^{\frac{3}{2}}\Lambda(x_0) - a(x_0) + CB^{-1/18} \right).
$$

Since this is true for all $x_0 \in \Gamma$, we deduce that,

$$
\mu_{B, \sigma} \leq \left( \frac{B}{\sigma} \right)^{\frac{3}{2}} \left( \hat{\Lambda}_1(B_0, a, \hat{\sigma}) + CB^{-1/18} \right),
$$

where $\hat{\Lambda}_1(B_0, a, \hat{\sigma})$ is introduced in (11.2).

Proposition [11.3] permits to obtain:

**Theorem 11.4.** Let $\hat{\lambda}_{\max} > 0$. Suppose that $\Gamma \neq \emptyset$ and $a \in C^1(\Omega)$. There exist two constants $C_1 > 0$ and $\kappa_0 > 0$ such that, if,

$$
\kappa \geq \kappa_0, \quad \text{and} \quad \kappa_0\kappa^{-1} < H < \hat{\lambda}_{\max}\kappa^2
$$

then

$$
\mu_1(\kappa, H) \leq \kappa^2 \hat{\Lambda}_1 \left( B_0, a, \frac{H}{\kappa^2} \right) + C_1 \kappa^{\frac{11}{6}}, \quad \text{as } \kappa \to +\infty.
$$

**Proof.** To apply the results of Proposition [11.3] we take $B = \kappa H$ and $\hat{\sigma} = \frac{H}{\kappa^2}$. We see for $\kappa$ sufficiently large that $\hat{\sigma} \in (0, \hat{\lambda}_{\max})$ and $B$ large.

Theorem [11.4] is valid when $\kappa H \geq \kappa_0$ and $\kappa_0$ is sufficiently large.

12. **Proof of Theorem 1.7**

12.1. **Analysis of $H^1_{C_3}$ and $H^1_{C_3}$.**

In this subsection we will prove Theorem 1.7 for $H^1_{C_3}$ and $H^1_{C_3}$. We first recall some useful results from [34] about the relation between the eigenvalues $\lambda_0$ and $\lambda(\mathbb{R}^2, \theta)$, introduced in (1.31) and in (1.33).

**Theorem 12.1.**

(i) $\lambda(\mathbb{R}^2, 0) = \lambda_0$.

(ii) If $0 < \theta < \pi$, then $\lambda(\mathbb{R}^2, \theta) < \lambda_0$.

The next proposition gives the region where $\mu_1(\kappa, H) < 0$ that allows us to obtain an information about $H^1_{C_3}$ (see (1.29)) in the case when the magnetic field $B_0$ is constant with a pining term.
Proposition 12.2. Suppose that \( \{a > 0\} \neq \emptyset \) and \( \Gamma \neq \emptyset \). There exist constants \( C > 0 \) and \( \kappa_0 \geq 0 \) such that if
\[
\kappa \geq \kappa_0, \quad H \leq \max \left( \frac{\sup_{x \in \Gamma \cap \Omega} \frac{a(x)^{\frac{3}{2}}}{\lambda_0^2 |\nabla B_0(x)|}}{\sup_{x \in \Gamma \cap \Omega} \frac{a(x)^{\frac{3}{2}}}{\lambda(\mathbb{R}^2_+, \theta(x))^{\frac{3}{2}} |\nabla B_0(x)|} } \right) \kappa^2 - C \kappa^{\frac{11}{2}}, \tag{12.1}
\]
then,
\[
\mu_1(\kappa, H) < 0.
\]
Moreover,
\[
\max \left( \frac{\sup_{x \in \Gamma \cap \Omega} \frac{a(x)^{\frac{3}{2}}}{\lambda_0^2 |\nabla B_0(x)|}}{\sup_{x \in \Gamma \cap \Omega} \frac{a(x)^{\frac{3}{2}}}{\lambda(\mathbb{R}^2_+, \theta(x))^{\frac{3}{2}} |\nabla B_0(x)|} } \right) \kappa^2 - C \kappa^{\frac{11}{2}} \leq H^{\text{loc}}_{\Gamma_3}.
\]

Proof. We have two cases:

Case 1. Here, we suppose that,
\[
\sup_{x \in \Gamma \cap \Omega} \frac{a(x)^{\frac{3}{2}}}{\lambda_0^2 |\nabla B_0(x)|} > \sup_{x \in \Gamma \cap \Omega} \frac{a(x)^{\frac{3}{2}}}{\lambda(\mathbb{R}^2_+, \theta(x))^{\frac{3}{2}} |\nabla B_0(x)|}.
\]
Thanks to the assumption in (1.4), we have, for all \( x \in \Gamma \cap \partial \Omega \), \( 0 < \theta(x) < \pi \). Theorem 12.1 then tells us that,
\[
\forall x \in \Gamma \cap \partial \Omega, \quad \frac{a(x)^{\frac{3}{2}}}{\lambda(\mathbb{R}^2_+, \theta(x))^{\frac{3}{2}} |\nabla B_0(x)|} > \frac{a(x)^{\frac{3}{2}}}{\lambda_0^2 |\nabla B_0(x)|}.
\]
Thus, there exists \( x_0 \in \Omega \cap \Gamma \) such that (the supremum of \( \frac{a(x)^{\frac{3}{2}}}{\lambda_0^2 |\nabla B_0(x)|} \) in \( \Gamma \cap \Omega \) can not be attained on the boundary),
\[
\sup_{x \in \Gamma \cap \Omega} \frac{a(x)^{\frac{3}{2}}}{\lambda_0^2 |\nabla B_0(x)|} = \frac{a(x_0)^{\frac{3}{2}}}{\lambda_0^2 |\nabla B_0(x_0)|}.
\]
If (12.1) is satisfied for some \( C > 0 \), then,
\[
\frac{H}{\kappa^2} \leq \frac{a(x_0)^{\frac{3}{2}}}{\lambda_0^2 |\nabla B_0(x_0)|} - C \kappa^{-\frac{1}{6}},
\]
that we can write in the form,
\[
\kappa^2 \left( \lambda_0 \left( \frac{H}{\kappa^2} |\nabla B_0(x_0)| \right)^{\frac{3}{2}} - a(x_0) \right) \leq -C M \kappa^{\frac{11}{2}}, \tag{12.2}
\]
where \( M > 0 \) is a constant independent of \( C \).

Suppose that \( \kappa H \geq B_0 \) where \( B_0 \) is selected sufficiently large such that we can apply Theorem 11.4 (Thanks to Lemma 9.8 \( \mu_1(\kappa, H) < 0 \) when \( \kappa H < B_0 \)).

By Theorem 11.4 there exist positive constants \( C_1 \) and \( \kappa_0 \) such that, for \( \kappa \geq \kappa_0 \),
\[
\mu_1(\kappa, H) \leq \kappa^2 \inf_{x \in \Gamma \cap \Omega} \left( \lambda_0 \left( \frac{H}{\kappa^2} |\nabla B_0(x)| \right)^{\frac{3}{2}} - a(x) \right) + C_1 \kappa^{\frac{11}{2}}
\]
\[
\leq \kappa^2 \left( \lambda_0 \left( \frac{H}{\kappa^2} |\nabla B_0(x_0)| \right)^{\frac{3}{2}} - a(x_0) \right) + C_1 \kappa^{\frac{11}{2}}
\]
\[
\leq (C_1 - C M) \kappa^{\frac{11}{2}}. \tag{12.3}
\]
By choosing $C$ such that $CM > C_1$, we get,

$$\mu_1(\kappa, H) < 0.$$ 

**Case 2.** Here, we suppose that

$$\sup_{x \in \Gamma \cap \Omega} \frac{a(x)^{\frac{3}{2}}}{\lambda \left(\mathbb{R}^2_{+}, \theta(x) \right)^{\frac{3}{2}} |\nabla B_0(x)|} \geq \sup_{x \in \Gamma \cap \partial \Omega} \frac{\lambda \left(\mathbb{R}^2_{+}, \theta(x) \right)^{\frac{3}{2}} |\nabla B_0(x)|}{\lambda \left(\mathbb{R}^2_{+}, \theta(x) \right)^{\frac{3}{2}} |\nabla B_0(x)|},$$

The assumption in (12.1) and the upper bound in Theorem 11.4 give us, for all $\kappa \geq \kappa_0$, $\kappa H \geq B_0$ and $B_0$ a sufficiently large constant,

$$\mu_1(\kappa, H) \leq (C_1 - C \tilde{M}) \kappa \frac{11}{6},$$

where $\tilde{M} > 0$ is a constant independent of $C$. By choosing $C$ such that $C \tilde{M} > C_1$, we get,

$$\mu_1(\kappa, H) < 0.$$ 

This finishes the proof of the proposition. 

The next proposition gives us a lower bound of $H^{loc}_{C_3}$ (see (1.2)). This is obtained by localizing the region where $\mu_1(\kappa, H) > 0$ holds.

**Proposition 12.3.** Suppose that $\{a > 0\} \neq \emptyset$, $\tilde{\lambda}_{\text{max}} > 0$ and $\Gamma = \emptyset$. There exist constants $C > 0$ and $\kappa_0 > 0$ such that if

$$\kappa \geq \kappa_0, \quad \tilde{\lambda}_{\text{max}} \kappa \geq H$$

then,

$$\mu_1(\kappa, H) > 0.$$ 

Moreover,

$$H^{loc}_{C_3} \leq \max \left( \sup_{x \in \Gamma \cap \Omega} \frac{a(x)^{\frac{3}{2}}}{\lambda \left(\mathbb{R}^2_{+}, \theta(x) \right)^{\frac{3}{2}} |\nabla B_0(x)|}, \sup_{x \in \Gamma \cap \partial \Omega} \frac{a(x)^{\frac{3}{2}}}{\lambda \left(\mathbb{R}^2_{+}, \theta(x) \right)^{\frac{3}{2}} |\nabla B_0(x)|} \right) \kappa^2 + C \kappa \frac{11}{6}.$$ 

**Proof.** Having in mind the definition of $\tilde{\lambda}_1$ in (11.2), under the assumption in (12.4), we get for $\kappa$ large enough,

$$\tilde{\lambda}_1 \left( B_0, a, \frac{H}{\kappa^2} \right) \geq C M \kappa^{-\frac{1}{2}},$$

where $M > 0$ is a constant independent of the constant $C$.

Thanks to Theorem 11.2 we get the existence of positive constants $C'$ and $\kappa_0$ such that, for $\kappa \geq \kappa_0$,

$$\mu_1(\kappa, H) \geq (C M - C') \kappa \frac{11}{6}.$$ 

To finish the proof, we choose $C$ sufficiently large such that $CM > C'$. 

□
12.2. Analysis of $H^{cp}_{C_\lambda}$ and $\mathcal{R}^{cp}_{C_\lambda}$.

Proposition 12.4 below is an adaptation of an analogous result obtained in [21] for the functional in (1.1) with a constant pinning term. Proposition 12.4 is valid when $\Gamma \neq \emptyset$. Proposition 12.4 says that, if $(\psi, A)$ is a critical point of the functional in (1.1) and $H$ is of order $\kappa^2$, then $|\psi|$ is concentrated near the set $\Gamma$.

**Proposition 12.4.** Let $\varepsilon > 0$. There exist two positive constants $C$ and $\kappa_0$ such that, if

$$\kappa \geq \kappa_0, \quad H \geq \varepsilon \kappa^2,$$

and $(\psi, A)$ is a solution of (1.12), then

$$\|\psi\|_{L^2(\Omega)}^2 \leq C \kappa^{-\frac{1}{2}} \|\psi\|_{L^4(\Omega)}^2.$$

**Proof.** Let $\lambda = \kappa^{-\frac{1}{2}}$ and $\Omega_\lambda = \{ x \in \Omega : \text{dist}(x, \partial \Omega) > \lambda \} \& \text{dist}(x, \Gamma) > \lambda \}$. We introduce a function $h \in C^\infty_c(\Omega)$ satisfying

$$0 \leq h \leq 1 \text{ in } \Omega, \quad h = 1 \text{ in } \Omega_\lambda, \quad \text{supp } h \subset \Omega_{\lambda/2},$$

and

$$|\nabla h| \leq \frac{C}{\lambda} \text{ in } \Omega,$$

where $C$ is a positive constant.

Using (8.2), we can prove that (see the detailed proof in [21] Eq. (6.6)) when $a$ is constant,

$$\kappa H \int_{\Omega} |B_0(x)| |\psi|^2 \, dx - c \kappa \|\psi\|_{L^2(\Omega)} \|h\psi\|_{L^4(\Omega)}^2 \leq \int_{\Omega} |(\nabla - i\kappa HA)\psi|^2 \, dx.$$

Now, the Cauchy-Schwarz inequality yields,

$$c \kappa \|\psi\|_{L^2(\Omega)} \|h\psi\|_{L^4(\Omega)}^2 \leq c^2 \|\psi\|_{L^2(\Omega)}^2 + \kappa^2 \|h\psi\|_{L^4(\Omega)}^4,$$

which implies that

$$\int_{\Omega} (\kappa H |B_0(x)| - \kappa^2 a(x)) |\psi|^2 \, dx \leq \int_{\Omega} |(\nabla - i\kappa HA)\psi|^2 \, dx - \kappa^2 \int_{\Omega} a(x) |\psi|^2 \, dx$$

$$+ c^2 \|\psi\|_{L^2(\Omega)}^2 + \kappa^2 \|h\psi\|_{L^4(\Omega)}^4.$$

We may use a localization formula as the one in (10.14) (but with $\chi_\kappa = h$) to write,

$$\int_{\Omega} (\kappa H |B_0(x)| - \kappa^2 a(x)) |\psi|^2 \, dx \leq c^2 \int_{\Omega} |\psi|^2 \, dx + \int_{\Omega} |\nabla h|^2 |\psi|^2 \, dx + \kappa^2 \int_{\Omega} (h^4 - h^2) |\psi|^4 \, dx$$

$$\leq c^2 \int_{\Omega} |\psi|^2 \, dx + \int_{\Omega} |\nabla h|^2 |\psi|^2 \, dx.$$

Here, we have used the fact that $h^4 \leq h^2$ since $0 \leq h \leq 1$.

By assumption (1.4), $\nabla B_0$ does not vanish on $\Gamma$, hence

$$|B_0(x)| \geq \frac{1}{M} \kappa^{-\frac{1}{2}} \text{ in } \{ \text{dist}(x, \Gamma) \geq \lambda \},$$

(12.8)

for some constant $M > 0$.

Thus, by using (1.10) and (12.6), we get,

$$\left(\frac{\varepsilon}{M} \kappa^{-\frac{1}{2}} - \kappa^2 \sigma\right) \int_{\Omega} |\psi|^2 \, dx \leq c^2 \int_{\Omega} |\psi|^2 \, dx + \int_{\Omega} |\nabla h|^2 |\psi|^2 \, dx.$$
Writing \( \int_{\Omega} |\psi|^2 \, dx = \int_{\Omega} |h\psi|^2 \, dx + \int_{\Omega} (1 - h^2) |\psi|^2 \, dx \) and using the assumption on \( h \), we have,
\[
\left( \frac{\varepsilon}{M} \kappa^2 - \kappa^2 \alpha - \varepsilon^2 \right) \int_{\Omega} |h\psi(x)|^2 \, dx \leq (c^2 + C \kappa) \int_{\Omega \setminus \Lambda} |\psi|^2 \, dx.
\]

For \( \kappa \) large enough, \( \frac{\varepsilon}{M} \kappa^2 - \kappa^2 \alpha - \varepsilon^2 \geq \frac{\varepsilon}{2M} \kappa^2 \) and
\[
\int_{\Omega} |h\psi(x)|^2 \, dx \leq 2 \frac{M}{\varepsilon} C \kappa^{-\frac{3}{2}} \int_{\Omega \setminus \Lambda} |\psi|^2 \, dx.
\]

Thanks to the assumption on the support of \( h \), we get further,
\[
\int_{\Omega} |\psi(x)|^2 \, dx \leq \left( 2 \frac{M}{\varepsilon} C \kappa^{-\frac{3}{2}} + 1 \right) \int_{\Omega \setminus \Lambda} |\psi|^2 \, dx.
\]

Recall that \( \lambda = \kappa^{-\frac{1}{2}} \). The Cauchy Schwarz inequality yields,
\[
\int_{\Omega \setminus \Lambda} |\psi(x)|^2 \, dx \leq |\Omega \setminus \Lambda|^{1/2} \left( \int_{\Omega \setminus \Lambda} |\psi|^4 \, dx \right)^{1/2} \leq C \kappa^{-\frac{1}{2}} \left( \int_{\Omega} |\psi|^4 \, dx \right)^{1/2}.
\]

This finishes the proof of the proposition. \( \square \)

Now, we can give an upper bound of the critical field \( \mathcal{H}_{C_3}^{p} \) in the case when \( \Gamma \neq \emptyset \) and with a pining term.

**Theorem 12.5.** Suppose that \( \Gamma \neq \emptyset \) and \( \{ a > 0 \} \neq \emptyset \). There exists \( C > 0 \) and \( \kappa_0 \) such that, if \( H \) satisfies

\[
H > \max \left( \sup_{x \in \Gamma \cap \Omega} \frac{a(x)^{\frac{3}{2}}}{\lambda_0^3 |\nabla B_0(x)|}, \sup_{x \in \Gamma \cap \Omega} \frac{a(x)^{\frac{3}{2}}}{\lambda |\nabla B_0(x)|} \right) \kappa^2 + C \kappa^{\frac{11}{4}}, \tag{12.9}
\]

then \((0, F)\) is the unique solution to \((1.12)\).

Moreover,
\[
\mathcal{H}_{C_3}^{p} \leq \max \left( \sup_{x \in \Gamma \cap \Omega} \frac{a(x)^{\frac{3}{2}}}{\lambda_0^3 |\nabla B_0(x)|}, \sup_{x \in \Gamma \cap \Omega} \frac{a(x)^{\frac{3}{2}}}{\lambda |\nabla B_0(x)|} \right) \kappa^2 + C \kappa^{\frac{11}{4}}.
\]

**Proof.** In light of the result in Theorem 8.5, we may assume the extra condition that \( H \leq \lambda_{\text{max}} \kappa^2 \) for a sufficiently large constant \( \lambda_{\text{max}} \).

We take the constant \( C \) in \((12.9)\) as in Proposition 12.3. In that way, under the assumption in \((12.9)\), we have

\[
\mu_1(\kappa, H) < 0. \tag{12.10}
\]

Suppose now that \((\psi, A)\) is a solution of \((1.12)\) with \( \psi \neq 0 \). Similarly, as in the proof of Theorem 10.4 we have,

\[
- \nabla \cdot (\kappa H^{\frac{1}{2}} |\nabla \psi|^{\frac{1}{2}}) \psi \geq \mu_1(\kappa, H) \|\psi\|^2_{L^2(\Omega)} - C \sqrt{T} \kappa \|\psi\|^2_{L^2(\Omega)}, \tag{12.11}
\]

where \( T = \kappa^2 \|\psi\|^4_{L^4(\Omega)} \) is introduced in \((10.17)\).

To apply the result of Proposition 12.4, we take

\[
\varepsilon = \frac{1}{2} \max \left( \sup_{x \in \Gamma \cap \Omega} \frac{a(x)^{\frac{3}{2}}}{\lambda_0^3 |\nabla B_0(x)|}, \sup_{x \in \Gamma \cap \Omega} \frac{a(x)^{\frac{3}{2}}}{\lambda |\nabla B_0(x)|} \right),
\]

and get,

\[
\|\psi\|^2_{L^2(\Omega)} \leq C \kappa^{-\frac{1}{2}} \|\psi\|^2_{L^4(\Omega)} = C \kappa^{-\frac{3}{2}} \sqrt{T}. \tag{12.12}
\]
Putting (12.12) into (12.11), we obtain,

\[ -\tau \geq \mu_1(\kappa, H) \Vert \psi \Vert_{L^2(\Omega)}^2 - C^\kappa \tau^{\frac{1}{4}}. \]

We conclude that, for \( \kappa \geq \kappa_0 \) and \( \kappa_0 \) a sufficiently large constant, \( \mu_1(\kappa, H) < 0 \), which is in contradiction with (12.10). Therefore, we conclude that \( \psi = 0 \).

Following the argument given in Proposition 10.5, we get:

**Proposition 12.6.** Suppose that \( \Gamma \neq \emptyset \) and \( \{ a > 0 \} \neq \emptyset \). There exists \( C > 0 \) and \( \kappa_0 \) such that, if \( \kappa \geq \kappa_0 \) and \( H \) satisfies

\[ H \leq \max \left( \sup_{x \in \Gamma \cap \Omega} \frac{a(x)^{\frac{2}{3}}}{\lambda_0^3 |\nabla B_0(x)|}, \sup_{x \in \Gamma \cap \Omega} \frac{a(x)^{\frac{1}{2}}}{\lambda(\mathbb{R}_+^2, \theta(x))^2 |\nabla B_0(x)|} \right) \kappa^2 - C \kappa^{\frac{11}{3}}, \quad (12.13) \]

then there exists a solution \((\psi, A)\) of (1.12) with \( \Vert \psi \Vert_{L^2(\Omega)} \neq 0 \).

Moreover,

\[ \max \left( \sup_{x \in \Gamma \cap \Omega} \frac{a(x)^{\frac{1}{2}}}{\lambda_0^3 |\nabla B_0(x)|}, \sup_{x \in \Gamma \cap \Omega} \frac{a(x)^{\frac{1}{2}}}{\lambda(\mathbb{R}_+^2, \theta(x))^2 |\nabla B_0(x)|} \right) \kappa^2 - C \kappa^{\frac{11}{3}} \leq H_{C_3}^\text{loc}. \]

**End of the proof of Theorem 1.7.** All the critical fields are contained in the interval \([H_{C_3}^\text{loc}, \overline{H}_{C_3}^\text{loc}]\) (the proof of this statement is exactly as the one given for (10.24) and (10.25)). By Proposition 12.2 and Theorem 12.5 we get the existence of positive constants \( C \) and \( \kappa_0 \), such that for \( \kappa \geq \kappa_0 \),

\[ \max \left( \sup_{x \in \Gamma \cap \Omega} \frac{a(x)^{\frac{1}{2}}}{\lambda_0^3 |\nabla B_0(x)|}, \sup_{x \in \Gamma \cap \Omega} \frac{a(x)^{\frac{1}{2}}}{\lambda(\mathbb{R}_+^2, \theta(x))^2 |\nabla B_0(x)|} \right) \kappa^2 - C \kappa^{\frac{11}{3}} \leq H_{C_3}^\text{loc} \leq \overline{H}_{C_3}^\text{loc}, \]

\[ \leq \max \left( \sup_{x \in \Gamma \cap \Omega} \frac{a(x)^{\frac{2}{3}}}{\lambda_0^3 |\nabla B_0(x)|}, \sup_{x \in \Gamma \cap \Omega} \frac{a(x)^{\frac{3}{2}}}{\lambda(\mathbb{R}_+^2, \theta(x))^2 |\nabla B_0(x)|} \right) \kappa^2 + C \kappa^{\frac{11}{3}}. \quad (12.14) \]

As a consequence, we have proved that the asymptotics in Theorem 1.7 is valid for the six critical fields in (1.27), (1.28) and (1.29).

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