HAUSDORFF DIMENSION OF CERTAIN SETS ARISING IN ENGEL CONTINUED FRACTIONS

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Abstract. In the present paper, we are concerned with the Hausdorff dimension of certain sets arising in Engel continued fractions. In particular, the Hausdorff dimension of sets
\( \{ x \in [0,1) : b_n(x) \geq \phi(n) \text{ i.m. } n \in \mathbb{N} \} \) and \( \{ x \in [0,1) : b_n(x) \geq \phi(n), \forall n \geq 1 \} \)
are completely determined, where i.m. means infinitely many, \( \{b_n(x)\}_{n \geq 1} \) is the sequence of partial quotients of the Engel continued fraction expansion of \( x \) and \( \phi \) is a positive function defined on natural numbers.

1. Introduction. Given a real number, there are various ways to represent it as an expansion of digits or partial quotients, such as continued fractions (see Khintchine [17]) and series expansions (see Galambos [9] and Schweiger [23]). One of the most well-known representation of real numbers is regular continued fractions (RCFs). The regular continued fraction expansion of a real number can be induced by the RCF-map (or Gauss transformation) \( T : [0,1) \to [0,1) \) given by
\[
T(0) := 0 \text{ and } T(x) = \frac{1}{x - \lfloor \frac{1}{x} \rfloor}, \forall x > 0,
\]
where \( \lfloor x \rfloor \) denotes the greatest integer not exceeding \( x \). Indeed, putting \( a_1(x) = \lfloor \frac{1}{x} \rfloor \) and \( a_{n+1}(x) = a_1(T^n(x)) \) for any \( n \geq 1 \), every real number \( x \in [0,1) \) can be written uniquely as
\[
x = \frac{1}{a_1(x) + \frac{1}{a_2(x) + \ldots + \frac{1}{a_n(x) + \ldots}}}, \tag{1}
\]
The form (1) is said to be the regular continued fraction (RCF) expansion of \( x \) and \( a_n(x), n \in \mathbb{N} \) are called the partial quotients of the RCF expansion of \( x \). If there exists some \( n \in \mathbb{N} \) such that \( T^k(x) = 0 \) for all \( k \geq n \), we say that the RCF expansion of \( x \) is finite and denote (1) by \( [a_1(x), a_2(x), \ldots, a_n(x)] \). Otherwise, it is said to

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be *infinite* and denote (1) by \([a_1(x), a_2(x), \ldots, a_n(x), \ldots]\). It is known that a real number has an infinite RCF expansion if and only if it is irrational. That is to say, there is a one-to-one correspondence between irrational numbers and the sequences of partial quotients. So it will help us to understand irrational numbers better by studying some properties of the corresponding partial quotients. A well-known result on partial quotients is the Borel-Bernstein theorem (see [17, Theorem 30]), which states that for almost all \(x \in [0, 1]\) in the sense of Lebesgue measure, \(a_n(x) \geq \phi(n)\) holds for infinitely many \(n \in \mathbb{N}\) or just for finitely many \(n \in \mathbb{N}\) according as the series \(\sum_{n \geq 1} 1/\phi(n)\) diverges or converges, where \(\phi\) is a positive function defined on natural numbers. As a consequence of this result, many sets consisting of all real numbers whose partial quotients are subject to some kind of restrictions have null Lebesgue measure. In fractal geometry, Hausdorff dimension provides a very useful tool to measure such sets of Lebesgue measure zero and it is attained much attention in studying the exceptional sets arising in regular continued fractions. It is worth pointing out that the first published work in this region is due to Jarník [15], in which he investigated the set of real numbers whose partial quotients are bounded. In 1941, Good [10] gave a quite overall study of sets with some restrictions on partial quotients, including the set \(\{x \in [0, 1] : a_n(x) \to \infty \text{ as } n \to \infty\}\). For any positive function \(\phi\) defined on natural numbers, he also attempted to investigate the set
\[
E(\phi) = \{x \in [0, 1] : a_n(x) \geq \phi(n) \text{ i.m. } n \in \mathbb{N}\}
\]
but did not give the exact value of its Hausdorff dimension, where i.m. means infinitely many. Furthermore, many authors tried to perfect Good’s work on the Hausdorff dimension of \(E(\phi)\), for instance, the Hausdorff dimension of the set \(\{x \in [0, 1] : a_n(x) \in B, \forall n \geq 1 \text{ and } a_n(x) \to \infty \text{ as } n \to \infty\}\) (\(B \subseteq \mathbb{N}\) is infinite) are derived by combining the results of Hirst [13] and Wang and Wu [24], see also Cusick [2]. Later, Feng et al. [8] and Luczak [20] considered the Hausdorff dimension of the set \(E(\phi)\) in the case of \(\phi(n) = a^b n^c\) with \(a, b > 1\). At last, in 2008, Wang and Wu [25] completely solved the problem on the Hausdorff dimension of \(E(\phi)\). Over the last twenty years, with the fast developing of dynamical systems, there is a close connection between representation of real numbers and dynamical systems. Many expansions of real numbers can be generated by some infinite iterated function system (iIFS, see [11, 21, 22]). In particular, regular continued fractions can be generated by the iIFS \(f_n : [0, 1] \to [0, 1]\) defined by
\[
f_n(x) = \frac{1}{x + n}, \quad x \in [0, 1].
\]
The problem on the Hausdorff dimension of \(E(\phi)\) has been well improved in the context of iIFSs (see [1, 16, 26]). It should be pointed out that regular continued fractions is a 2-decaying iIFS system in the context of Jordan and Rams [16]. See Cao et al. [1], Liao and Rams [19], Zhang and Cao [26] for more general results from regular continued fractions to \(d\)-decaying iIFS systems \((d > 1)\).

In the present paper, we are interested in a variation of the regular continued fraction expansion, namely Engel continued fractions (ECFs). We emphasize that ECFs can not be generated by some infinite iterated function system and hence that the general results on the Hausdorff dimension of \(E(\phi)\) in the case of [1] and [16] can not be applied to ECFs, which is the main motivation of this paper. In 2002, Hartono et al. [12] first introduced this new continued fraction algorithm with non-decreasing partial quotients. Let \(T_E : [0, 1] \to [0, 1]\) be the ECF-map given
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Figure 1. RCF-map and ECF-map

This means that the ECF-map $T_E(x)$ is in fact equal to the RCF-map $T(x)$ normalized by $[1/x]$. That is to say, we can obtain the ECF-map by shrinking each branch of the RCF-map according to a certain ratio (as shown in the above figure). The main common thing of these two maps is that they are both piecewise maps with infinitely many nonlinear branches. The difference of them is that all branches of the RCF-map are full, while, for the ECF-map, these branches are not full except for the first one on the right, which implies that partial quotients are non-decreasing (see below statement). Besides, it is well known that RCF-map is ergodic and has finite invariant measure equivalent to Lebesgue measure; however, Hartono et al. [12] showed that $T_E$ has no finite invariant measure equivalent to Lebesgue measure, but that it has infinitely many $\sigma$-finite, infinite invariant measures. Similar to regular continued fractions, every real number $x \in [0,1)$ can be represented in the following form

$$x = \frac{1}{b_1(x) + \frac{b_1(x)}{b_2(x) + \ddots + \frac{b_{n-1}(x)}{b_n(x) + \ddots}}} \tag{2}$$

where $b_1(x) = [1/x] \in \mathbb{N}$ and $b_{n+1}(x) = b_1(T_E^n(x))$ with $b_{n+1}(x) \geq b_n(x)$ for all $n \in \mathbb{N}$. The form (2) is said to be the ECF expansion of $x$ denoted by $[b_1(x), b_2(x), \ldots, b_n(x), \ldots]$ and $b_n(x), n \in \mathbb{N}$ are called the partial quotients of the ECF expansion of $x$. The algorithm of ECFs producing non-decreasing partial quotients, has very different behaviors with respect to RCFs. For instance, Kraaikamp and Wu [18] proved a strong law of large numbers for $\log b_n(x)$, i.e.,
\[ \lim_{n \to \infty} \frac{1}{n} \log b_n(x) = 1 \]  

holds for almost all \( x \in [0, 1) \) in the sense of Lebesgue measure. Moreover, they also showed that the set of real numbers in which such a strong law of large numbers does not hold, has full Hausdorff dimension. Furthermore, Fan et al. [4] established a central limit theorem for \( \log b_n(x) \). Following this line of research, Fang et al. [7] considered the large and moderate deviation principles for ECF expansions (see also [5, 6]). For the Hausdorff dimension of some sets in ECFs, Zhong and Tang [27] considered a Hirst’s problem in the context of ECFs and their results indicate that there is a difference between RCFs and ECFs in this problem. Recently, Hu et al. [14] studied the efficiency of approximating real numbers by their convergents of ECFs. In particular, they estimated the Hausdorff dimension of the set of points whose ECF-convergents are the best approximations infinitely often and obtained the Hausdorff dimensions of the related sets defined by some growth rates of partial quotients in ECF expansions (i.e., the Luczak’s problem in the context of ECFs). We are interested in the Hausdorff dimension of \( E(\phi) \) in the context of Engel continued fractions to significantly extend the results of Zhong and Tang [27] and Hu et al. [14]. More precisely, we would like to completely give the exact Hausdorff dimension of the sets 

\[ F(\phi) = \{ x \in [0, 1) : b_n(x) \geq \phi(n) \text{ i.m. } n \in \mathbb{N} \} \]

and 

\[ \tilde{F}(\phi) = \{ x \in [0, 1) : b_n(x) \geq \phi(n), \forall n \geq 1 \} , \]

where \( \phi \) is a positive function defined on natural numbers. We also study the Hausdorff dimension of certain sets defined by the growth rate of partial quotients in ECFs and other related sets.

The rest of this paper is organized as follows. In Section 2, we introduce some definitions and notations of Engel continued fractions. Section 3 is devoted to estimating the Hausdorff dimension of 

\[ \left\{ x \in [0, 1) : na^{b^n} \leq b_n(x) \leq (n + 1)a^{b^n}, \forall n \geq 1 \right\} \]

and 

\[ \{ x \in [0, 1) : b_n(x) \geq a^{b^n} \text{ i.m. } n \} \]  

with \( a, b > 1 \), which play an important role in studying the Hausdorff dimension of \( F(\phi), \tilde{F}(\phi) \) and other related sets. The exact formulas on the Hausdorff dimension of certain sets related to the growth rate of partial quotients are presented in Section 4. In particular, we completely determine the Hausdorff dimension of \( F(\phi), \tilde{F}(\phi) \) and other related sets. In Section 5, we consider the Hausdorff dimension of the set related to the ratio between two consecutive partial quotients, which shares a dichotomy law according to Borel-Bernstein type theorem. Throughout the paper, we use \( \cdot \) to denote the diameter of a subset of \([0,1)\), \( \mathcal{H}^s \) the \( s \)-dimensional Hausdorff measure and \( \dim_H \) the Hausdorff dimension.

2. Preliminaries. In this section, we recall some definitions and several arithmetic properties of Engel continued fractions. We first give an elementary arithmetic property of the Engel continued fraction expansion in representing real numbers, see Hartono et al. [12] (see also Fan et al. [4]).

Proposition 1. ([12, Theorem 2.1]) Let \( x \in [0, 1) \) be a real number. Then \( x \) has a finite ECF expansion (i.e. \( T_E^n(x) = 0 \) for some \( n \geq 1 \)) if and only if \( x \) is rational.
Definition 2.1. An $n$-block $(b_1, \cdots, b_n)$ is said to be admissible for ECF expansions if there exists $x \in [0,1)$ such that $b_j(x) = b_j$ for all $1 \leq j \leq n$. An infinite sequence $(b_1, \cdots, b_n, \cdots)$ is called an admissible sequence if $(b_1, \cdots, b_n)$ is admissible for any $n \geq 1$.

The following proposition, due to Fan et al. [4], gives a characterization of all admissible sequences occurring in the ECF expansion.

Proposition 2. ([4, Proposition 2.2]) A sequence of positive integers $(b_1, \cdots, b_n, \cdots)$ is admissible for ECF expansions if and only if for all $n \geq 1$,

$$b_{n+1} \geq b_n.$$ 

Definition 2.2. Let $(b_1, \cdots, b_n) \in \mathbb{N}^n$ be admissible. We call

$$I(b_1, \cdots, b_n) = \{x \in [0,1) : b_1(x) = b_1, \cdots, b_n(x) = b_n\}$$

the cylinder of order $n$ of the ECF expansion.

In other words, $I(b_1, \cdots, b_n)$ is the set of points beginning with $(b_1, \cdots, b_n)$ in their ECF expansions. The following result gives the structure and the length of cylinders, see Hartono et al. [12] (see also Fan et al. [4]).

Proposition 3. ([12, Section 2]) Let $(b_1, \cdots, b_n) \in \mathbb{N}^n$ be admissible. Then the cylinder $I(b_1, \cdots, b_n)$ is an interval with two endpoints

$$[b_1, \cdots, b_{n-1}, b_n] \quad \text{and} \quad [b_1, \cdots, b_{n-1}, b_n + 1]$$

and hence that its length satisfies

$$|I(b_1, \cdots, b_n)| = \frac{\prod_{i=1}^{n-1} b_i}{Q_n(Q_n + Q_{n-1})},$$

where the quantity $Q_n$ satisfies the recursive formula $Q_n = b_nQ_{n-1} + b_{n-1}Q_{n-2}$ under the conventions $Q_{-1} = 0$ and $Q_0 = 1$.

By the recursive formula of $Q_n$, we obtain that

$$b_1b_2 \cdots b_n \leq Q_n \leq 2^n b_1b_2 \cdots b_n,$$  \hspace{1cm} (5)

which is very useful in estimating the length of the cylinder in the next section.

We use the notation $\lambda$ to denote the Lebesgue measure on $[0,1)$ and treat partial quotients $\{b_n\}_{n \geq 1}$ as the random variables defined on probability space $([0,1), \mathcal{B}([0,1)), \lambda)$, where $\mathcal{B}([0,1))$ means the Borel $\sigma$-algebra on $[0,1)$. We know that $\{b_n\}_{n \geq 1}$ does not form a homogeneous Markov chain (see Remark 5 of [7]) but has the following property, which is important in the metric theory of Engel continued fractions (see [4, 7, 18]). We also emphasize that such a property is not true for regular continued fractions.

Proposition 4. ([7, Proposition 3.4]) Let $\{b_n\}_{n \geq 1}$ be the sequence of partial quotients of the ECF expansion. Then for any $k \geq j \geq 1$, we have

$$\lambda\{x \in [0,1) : b_1(x) = j\} = \frac{1}{j(j+1)}$$

and the conditional probabilities

$$\frac{j}{k(k+2)} \leq \lambda\{x \in [0,1) : b_{n+1}(x) = k \mid b_n(x) = j\} \leq \frac{j+1}{k(k+1)}$$ \hspace{1cm} for all $n \geq 1$. 

3. **Auxiliary results.** Let \( a, b > 1 \). The main purpose of this section is to determine the Hausdorff dimension of
\[
E(a, b) = \left\{ x \in [0, 1) : na^b \leq b_n(x) \leq (n + 1)a^b, \forall n \geq 1 \right\}
\]
and
\[
F(a, b) = \left\{ x \in [0, 1) : b_n(x) \geq a^b \text{ i.m. } n \in \mathbb{N} \right\},
\]
which play an important role in the next section. Since \( E(a, b) \subseteq F(a, b) \), we first give a lower bound of \( \dim_H E(a, b) \) and an upper bound of \( \dim_H F(a, b) \). For the upper bound of \( \dim_H F(a, b) \), we point out that Zhong and Tang [27] in fact gave an estimation by using the ideas of Luczak [20] in the case of regular continued fractions (see also Hu et al. [14]).

**Lemma 3.1** ([27]). Let \( a, b > 1 \). Then \( \dim_H F(a, b) \leq 1/b \).

Now it remains to calculate the lower bound of \( \dim_H E(a, b) \). To do this, we need the following lemma, which serve as an important tool to estimate the lower bound of the Hausdorff dimension of a fractal set (see Example 4.6 in [3]).

**Lemma 3.2** ([3]). Let \([0, 1] = E_0 \supset E_1 \supset \cdots \) be a decreasing sequence of sets and 
\( E = \bigcap_{n \geq 1} E_n \). We assume that each \( E_n \) is a union of a finite number of disjoint closed intervals (called basic intervals of order \( n \)) and each basic interval in \( E_{n-1} \) contains \( m_n \) intervals of \( E_n \) which are separated by gaps of lengths at least \( \varepsilon_n \). If \( m_n \geq 2 \) and \( \varepsilon_{n-1} > \varepsilon_n > 0 \), then
\[
\dim_H E \geq \liminf_{n \to \infty} \frac{\log(m_1m_2 \cdots m_{n-1})}{-\log(m_n\varepsilon_n)}.
\]

By constructing such a subset of \( E(a, b) \) and using the result of Lemma 3.2, we determine the lower bound of \( \dim_H E(a, b) \).

**Lemma 3.3.** Let \( a, b > 1 \). Then \( \dim_H E(a, b) \geq 1/b \).

**Proof.** Let \( a, b > 1 \). For any \( n \geq 1 \), we define
\[
\mathcal{D}_n = \left\{ (\sigma_1, \cdots, \sigma_n) \in \mathbb{N}^n : k a^b \leq \sigma_k \leq (k + 1)a^b, \forall 1 \leq k \leq n \right\}
\]
with the convention \( \mathcal{D}_0 = \emptyset \). It should be noted that \( \mathcal{D}_1 \) is not empty since \( \sigma_1 \) can at least take value \((a^b) + 1\). Hence that \( \mathcal{D}_n \) is not empty for all \( n \geq 1 \). For any \((\sigma_1, \cdots, \sigma_n) \in \mathcal{D}_n \) and \( 1 \leq k \leq n-1 \), we know \( \sigma_{k+1} \geq (k+1)a^{b+1} > (k+1)a^b \geq \sigma_k \) and \( \sigma_1 \geq 1 \). Hence \((\sigma_1, \cdots, \sigma_n)\) is admissible for ECF expansions by Proposition 2. This is to say, all elements in \( \mathcal{D}_n \) are admissible for ECF expansions. For any \((\sigma_1, \cdots, \sigma_n) \in \mathcal{D}_n \), put
\[
J(\sigma_1, \cdots, \sigma_n) = \bigcup_{\sigma_{n+1}} \text{cl}(I(\sigma_1, \cdots, \sigma_n, \sigma_{n+1})),
\]
where the union is taken over all \( \sigma_{n+1} \) satisfying \((\sigma_1, \cdots, \sigma_n, \sigma_{n+1}) \in \mathcal{D}_{n+1} \) and \( \text{cl}(A) \) denotes the closure of a set \( A \). Note that \( J(\sigma_1, \cdots, \sigma_n) \) is a closed interval in \([0, 1]\) and we call it the basic interval of order \( n \). For all \( n \geq 1 \), denote
\[
E_n = \bigcup_{(\sigma_1, \cdots, \sigma_n) \in \mathcal{D}_n} J(\sigma_1, \cdots, \sigma_n),
\]
with the convention \( E_0 = [0, 1] \) and
\[
E := \bigcap_{n \geq 0} E_n.
\]
Then $E$ is a subset of $E(a,b)$. Next we will estimate the Hausdorff dimension of $E$. To do this, we should first calculate the number of basic intervals of order $n$ contained in the basic interval of order $(n-1)$ and the length of the gap between two of them. By the definitions of $D_n$ and $J(\sigma_1, \cdots, \sigma_n)$, we know such a number is

$$m_n := [(n+1)a^{b^n}] - [na^{b^n}].$$

Now it remains to estimate the length of the gap. For any two different blocks $(\sigma_1, \cdots, \sigma_n)$ and $(\sigma'_1, \cdots, \sigma'_n)$ in $D_n$, we have that one of the following two intervals

$$\bigcup_{\sigma_n \leq \sigma_{n+1} < (n+1)a^{b^n+1}} I(\sigma_1, \cdots, \sigma_n, \sigma_{n+1}) \quad (7)$$

and

$$\bigcup_{\sigma_n' \leq \sigma_{n+1}' < (n+1)a^{b^n+1}} I(\sigma'_1, \cdots, \sigma'_n, \sigma'_{n+1}) \quad (8)$$

is the gap between $J(\sigma_1, \cdots, \sigma_n)$ and $J(\sigma'_1, \cdots, \sigma'_n)$. We only calculate the length of the interval in (7) since the calculation for the interval in (8) is similar. Let $\eta = a^{b(\beta-1)} > 1$. Then we have that $a^{b(n+1)} \geq \eta b^n$ holds for all $n \geq 1$. If $\sigma_n \leq (n+1)a^{b^n}$, then it follows that $(n+1)a^{b^n+1} \geq \eta \sigma_n$ for all $n \geq 1$. By (4) and (5), we deduce that

$$|I(\sigma_1, \cdots, \sigma_n, \sigma_{n+1})| \geq \frac{\sigma_1 \cdots \sigma_n}{2Q_{n+1}} \geq \frac{1}{2^{2n+1}\sigma_1 \cdots \sigma_n \sigma_{n+1}^2},$$

which implies that

$$\sum_{\sigma_n \leq \sigma_{n+1} < (n+1)a^{b^n+1}} |I(\sigma_1, \cdots, \sigma_n, \sigma_{n+1})| \geq \sum_{\sigma_n \leq \sigma_{n+1} \leq \eta \sigma_n} |I(\sigma_1, \cdots, \sigma_n, \sigma_{n+1})| \geq \frac{1}{2^{2n+1}\sigma_1 \cdots \sigma_n} \sum_{\sigma_n \leq \sigma_{n+1} \leq \eta \sigma_n} \frac{1}{\sigma_n^{n+1}} \geq \left(1 - \frac{1}{\eta}\right) \frac{1}{2^{2n+1}\sigma_1 \cdots \sigma_n^2}.$$

Note that $ka^k \leq \sigma_k \leq (k+1)a^k$ for all $1 \leq k \leq n$, we conclude that

$$\sum_{\sigma_n \leq \sigma_{n+1} < (n+1)a^{b^n+1}} |I(\sigma_1, \cdots, \sigma_n, \sigma_{n+1})| \geq \left(1 - \frac{1}{\eta}\right) \frac{1}{2^{2n+1}(n+2)a^{b^n+1} + b^{n+1}} := \varepsilon_n.$$

To summarize, we constructed a subset of $E(a,b)$: $E = \bigcap_{n \geq 0} E_n$, where $E_0 = [0, 1]$ and $E_n$ is defined in (6) which is a union of a finite number of disjoint basic intervals of order $n$. Moreover, each basic intervals of order $(n-1)$ contains $m_n$ basic intervals of order $n$ and the length of the gap between two of them is at least $\varepsilon_n$. Note that $m_n \geq 2$ and $\varepsilon_n > \varepsilon_{n+1} > 0$ for all $n \geq 1$, by Lemma 3.2, we obtain that

$$\dim_H E(a,b) \geq \dim_H E \geq \liminf_{n \to \infty} \frac{\log(m_1m_2 \cdots m_n)}{-\log(m_{n+1} \varepsilon_{n+1})} = \liminf_{n \to \infty} \frac{b + \cdots + b^n}{b + \cdots + b^n + b^{n+1}} = \frac{1}{b}.$$
For $a, b > 1$, let
\[ \tilde{F}(a, b) = \left\{ x \in [0, 1) : b_n(x) \geq a b^n, \forall n \geq 1 \right\}. \]

Note that $E(a, b) \subseteq \tilde{F}(a, b) \subseteq F(a, b)$, in view of Lemmas 3.1 and 3.3, we have

**Proposition 5.** Let $a, b > 1$. Then $\dim_H E(a, b) = \dim_H \tilde{F}(a, b) = \dim_H F(a, b) = 1/b$.

Being similar to the proof of Lemma 3.3, we can obtain the following proposition, which shows that the set of $x$'s such that $b_n(x)$ tends to infinity with exponential rates as $n$ goes to infinity has full Hausdorff dimension.

**Proposition 6.** For any $\alpha > 0$, then
\[ \dim_H \left\{ x \in [0, 1) : n e^{\alpha n} \leq b_n(x) \leq (n + 1) e^{\alpha n}, \forall n \geq 1 \right\} = 1. \]

**Proof.** The proof is very similar to the proof of Lemma 3.3, so we just list the key steps. Let $\alpha > 0$. The symbolic space $\tilde{D}_n$ can be defined as
\[ \tilde{D}_n = \left\{ (\sigma_1, \cdots, \sigma_n) \in \mathbb{N}^n : ke^{\alpha k} \leq \sigma_k \leq (k + 1)e^{\alpha k}, \forall 1 \leq k \leq n \right\}. \]

It is not difficult to show that all elements in $\tilde{D}_n$ are admissible for ECF expansions. For any $(\sigma_1, \cdots, \sigma_n) \in \tilde{D}_n$, the basic interval of order $n$ is defined as
\[ \tilde{J}(\sigma_1, \cdots, \sigma_n) = \bigcup_{\sigma_{n+1}} \text{cl}(I(\sigma_1, \cdots, \sigma_n, \sigma_{n+1})). \]

Let
\[ \tilde{E}_n = \bigcup_{(\sigma_1, \cdots, \sigma_n) \in \tilde{D}_n} \tilde{J}(\sigma_1, \cdots, \sigma_n) \]
with the convention $\tilde{E}_0 = [0, 1]$ and $\tilde{E} := \bigcap_{n \geq 0} \tilde{E}_n$. Then $\tilde{E}$ is the desired subset. Using the similar arguments on the calculations of $m_n$ and $\varepsilon_n$ in the proof of Lemma 3.3, we know that $\tilde{J}(\sigma_1, \cdots, \sigma_{n-1})$ contains
\[ \tilde{m}_n = \lfloor (n + 1)e^{\alpha n} \rfloor - \lfloor ne^{\alpha n} \rfloor \]
basic intervals of order $n$ in $\tilde{E}_n$ and the lengths of gaps between two of them are at least
\[ \tilde{\varepsilon}_n = \left(1 - \frac{1}{e^{\alpha}}\right) \frac{1}{2^{2n+1}(n+2)!e^{\alpha(1+2+\cdots+n-1)+2n}}. \]

Therefore, we have
\[ \dim_H \left\{ x \in [0, 1) : ne^{\alpha n} \leq b_n(x) \leq (n + 1)e^{\alpha n}, \forall n \geq 1 \right\} \geq \dim_H \tilde{E} \geq \liminf_{k \to \infty} \frac{\alpha(1+2+\cdots+n)}{\alpha(1+2+\cdots+n+(n+1))} = 1. \]

As a consequence of Proposition 6, we have the following two corollaries.

**Corollary 1 ([18]).** For any $\alpha > 0$, then
\[ \dim_H \left\{ x \in [0, 1) : \lim_{n \to \infty} \frac{\log b_n(x)}{n} = \alpha \right\} = 1. \]
Corollary 2. For any $\alpha > 0$, then
\[ \dim_H \{ x \in [0,1) : b_n(x) \geq e^{\alpha n} \text{ i.m. } n \in \mathbb{N} \} = 1 \]
and
\[ \dim_H \{ x \in [0,1) : b_n(x) \geq e^{\alpha n}, \forall n \geq 1 \} = 1. \]

4. Growth rate of partial quotients. In this section, we are concerned with the Hausdorff dimension of certain sets related to the growth rate of partial quotients. We give a complete answer on the Hausdorff dimension of $F(\phi)$ and $\tilde{F}(\phi)$ in the first part. Then the exact Hausdorff dimension of other related sets are presented.

4.1. The Hausdorff dimension of $F(\phi)$ and $\tilde{F}(\phi)$.

Theorem 4.1. Let $\phi$ be a positive function defined on natural numbers and
\[ F(\phi) = \{ x \in [0,1) : b_n(x) \geq \phi(n) \text{ i.m. } n \in \mathbb{N} \}. \]
Suppose that $\alpha = \liminf_{n \to \infty} \frac{\log \phi(n)}{n}$.

1. If $0 \leq \alpha < 1$, then $F(\phi)$ has full Lebesgue measure.
2. If $1 \leq \alpha < \infty$, then $\dim_H F(\phi) = 1$.
3a. If $\alpha = \infty$, let $\beta = \liminf_{n \to \infty} \frac{\log \log \phi(n)}{n}$.
3b. If $1 < \beta < \infty$, then $\dim_H F(\phi) = \beta^{-1}$.
3c. If $\beta = \infty$, then $\dim_H F(\phi) = 0$.

Remark 1. Let $\log \beta = \liminf_{n \to \infty} \frac{\log \log \phi(n)}{n}$. If we only focus on the Hausdorff dimension of $F(\phi)$, then the results in Theorem 4.1 can be shortly written as
\[ \dim_H F(\phi) = \begin{cases} 1, & \text{if } \beta = 1; \\ \beta^{-1}, & \text{if } 1 < \beta < \infty; \\ 0, & \beta = \infty, \end{cases} \]
since a set with full Lebesgue measure must have full Hausdorff dimension.

Proof. (1) $0 \leq \alpha < 1$. Note that
\[ \lim_{n \to \infty} \frac{\log b_n(x)}{n} = 1 \]
holds for almost all $x \in [0,1)$ in the sense of Lebesgue measure, so for such a $x \in [0,1)$ and $\varepsilon > 0$, there exists $N := N(x, \varepsilon) > 0$ such that
\[ e^{(1-\varepsilon)n} \leq b_n(x) \leq e^{(1+\varepsilon)n} \tag{9} \]
for all $n \geq N$. Since $\alpha = \liminf_{n \to \infty} \frac{\log \phi(n)}{n}$, for any $0 < \varepsilon < (1 - \alpha)/2$, $\frac{\log \phi(n)}{n} \leq \alpha + \varepsilon$, i.e., $\phi(n) \leq e^{n(\alpha + \varepsilon)}$ holds for infinitely many $n \in \mathbb{N}$. Let
\[ \mathbb{L} = \{ n : n \geq N \text{ and } \phi(n) \leq e^{n(\alpha + \varepsilon)} \}. \]
Then $\mathbb{L}$ is an infinite subset of $\mathbb{N}$. Combine with $0 < \varepsilon < (1 - \alpha)/2$ and (9), we have
\[ b_n(x) \geq e^{(1-\varepsilon)n} > e^{n(\alpha + \varepsilon)} \geq \phi(n) \]
holds for $n \in \mathbb{L}$. That is,
\[ \left\{ x \in [0,1) : \lim_{n \to \infty} \frac{\log b_n(x)}{n} = 1 \right\} \subset \left\{ x \in [0,1) : b_n(x) \geq \phi(n), \forall n \in \mathbb{L} \right\} \subset F(\phi), \]
which implies that $F(\phi)$ has full Lebesgue measure.

(2) $1 \leq \alpha < \infty$. In this case, being similar to the case (1), for any $\varepsilon > 0$, $\phi(n) \leq e^{n(\alpha + \varepsilon)}$ holds for infinitely many $n \in \mathbb{N}$. Let $$\mathbb{L} = \left\{ n : \phi(n) \leq e^{n(\alpha + \varepsilon)} \right\}.$$ Then $\mathbb{L}$ is an infinite subset of $\mathbb{N}$. Therefore, $$\{ x \in [0, 1) : b_n(x) \geq e^{n(\alpha + \varepsilon)} \forall n \geq 1 \} \subseteq \{ x \in [0, 1) : b_n(x) \geq \phi(n), \forall n \in \mathbb{L} \} \subseteq F(\phi).$$ It follows from Corollary 2 that $F(\phi)$ has full Hausdorff dimension.

(3) $\alpha = \infty$. In this case, let $$\log \beta = \liminf_{n \to \infty} \frac{\log \phi(n)}{n}.$$ (3a) $\beta = 1$. For any $\varepsilon > 0$, $\frac{\log \phi(n)}{n} \leq \log(1 + \varepsilon)$, i.e., $\phi(n) \leq e^{(1+\varepsilon)n}$ holds for infinitely many $n \in \mathbb{N}$. Let $$\mathbb{L} = \left\{ n : \phi(n) \leq e^{(1+\varepsilon)n} \right\}.$$ Then $\mathbb{L}$ is an infinite subset of $\mathbb{N}$ and hence $$\{ x \in [0, 1) : b_n(x) \geq e^{(1+\varepsilon)n} \forall n \geq 1 \} \subseteq \{ x \in [0, 1) : b_n(x) \geq \phi(n), \forall n \in \mathbb{L} \} \subseteq F(\phi).$$ It follows from Proposition 5 that $$\dim_H F(\phi) \geq \dim_H \left\{ x \in [0, 1) : b_n(x) \geq e^{(1+\varepsilon)n} \forall n \geq 1 \right\} = \frac{1}{1+\varepsilon}.$$ Letting $\varepsilon \to 0^+$, we have $\dim_H F(\phi) = 1$.

(3b) $1 < \beta < \infty$. For any $0 < \varepsilon < \beta - 1$, there exists $N := N(\varepsilon) > 0$ such that for all $n \geq N$, we have $\phi(n) \geq e^{(\beta-\varepsilon)n}$. At the same time, $\phi(n) \leq e^{(\beta+\varepsilon)n}$ for infinitely many $n \in \mathbb{N}$. Let $$\mathbb{L} = \left\{ n : \phi(n) \leq e^{(\beta+\varepsilon)n} \right\}.$$ Then $\mathbb{L}$ is an infinite subset of $\mathbb{N}$. So, $$F(\phi) \subseteq \left\{ x \in [0, 1) : b_n(x) \geq e^{(\beta-\varepsilon)n} \text{ i.m. } n \in \mathbb{N} \right\}$$ and $$\left\{ x \in [0, 1) : b_n(x) \geq e^{(\beta+\varepsilon)n} \forall n \geq 1 \right\} \subseteq \{ x \in [0, 1) : b_n(x) \geq \phi(n), \forall n \in \mathbb{L} \} \subseteq F(\phi).$$ In view of Proposition 5, we deduce that $$\frac{1}{\beta + \varepsilon} \leq \dim_H F(\phi) \leq \frac{1}{\beta - \varepsilon}.$$ Letting $\varepsilon \to 0^+$, we have $\dim_H F(\phi) = \beta^{-1}$.

(3c) $\beta = \infty$. In this case, for any $B > 1$, there exists $N := N(B) > 0$ such that for all $n \geq N$, we have $\phi(n) \geq e^{Bn}$. Then, $$F(\phi) \subseteq \left\{ x \in [0, 1) : b_n(x) \geq e^{Bn} \text{ i.m. } n \in \mathbb{N} \right\},$$ which implies that $\dim_H F(\phi) \leq B^{-1}$ by Proposition 5. Letting $B \to \infty$, we know $\dim_H F(\phi) = 0$. \hfill \Box

Similarly, we have the following result.
Theorem 4.2. Let $\phi$ be a positive function defined on natural numbers and
\[ \widetilde{F}(\phi) = \{ x \in [0, 1) : b_n(x) \geq \phi(n), \forall n \geq 1 \}. \]

Suppose that $\log \beta = \limsup_{n \to \infty} \frac{\log \phi(n)}{n}$.

(1) If $\beta = 1$, then $\dim \tilde{H} \phi = 1$.

(2) If $1 < \beta < \infty$, then $\dim \tilde{H} \phi = \beta^{-1}$.

(3) If $\beta = \infty$, then $\dim \tilde{H} \phi = 0$.

To prove Theorem 4.2, we need the following lemma.

Lemma 4.3. Let $N$ be a positive integer and $\phi$ be a positive function defined on natural numbers. Denote
\[ \overline{F}_N(\phi) = \{ x \in [0, 1) : b_n(x) \geq \phi(n), \forall n > N \}. \]

Then $\dim \tilde{H} \overline{F}_N(\phi) = \dim \tilde{H} \phi$.

Proof. Given integers $B_1 \leq B_2 \leq \cdots \leq B_N$ with $\phi(k) \leq B_k$ for all $1 \leq k \leq N$, we define $f_{B_1, \ldots, B_N} : \overline{F}_N(\phi) \to \overline{F}(\phi)$ as $f_{B_1, \ldots, B_N}(x) = [B_1, \ldots, B_N + x]$. In view of the countable stability of Hausdorff dimension and invariant property under bi-Lipschitz map (see [3]), it suffices to show that $f_{B_1, \ldots, B_N}$ is a bi-Lipschitz map. In fact, for any $x, y \in \overline{F}_N(\phi)$, we have
\[ f_{B_1, \ldots, B_N}(x) = \frac{P_N + xB_NP_{N-1}}{Q_N + xB_NQ_{N-1}} \quad \text{and} \quad f_{B_1, \ldots, B_N}(y) = \frac{P_N + yB_NP_{N-1}}{Q_N + yB_NQ_{N-1}}, \]

where $P_N$ and $Q_N$ satisfy the recursive formula:
\[ P_k = B_kP_{k-1} + B_{k-1}P_{k-2} \quad \text{and} \quad Q_k = B_kQ_{k-1} + B_{k-1}Q_{k-2}, \forall k \geq 2, \]

with the convention $P_0 = 0, P_1 = 1$ and $Q_0 = 1, Q_1 = B_1$. Hence that
\[ P_NQ_{N-1} - P_{N-1}Q_N = (-1)^{N-1}B_1 \cdots B_{N-1} \]

and
\[ |f_{B_1, \ldots, B_N}(x) - f_{B_1, \ldots, B_N}(y)| = \left| \frac{P_N + xB_NP_{N-1}}{Q_N + xB_NQ_{N-1}} - \frac{P_N + yB_NP_{N-1}}{Q_N + yB_NQ_{N-1}} \right| = \frac{B_1 \cdots B_N}{(Q_N + xB_NQ_{N-1})(Q_N + yB_NQ_{N-1})} \cdot |x - y|, \]

Therefore,
\[ \frac{B_1 \cdots B_N}{(Q_N + B_NQ_{N-1})^2} \leq \frac{|f_{B_1, \ldots, B_N}(x) - f_{B_1, \ldots, B_N}(y)|}{|x - y|} \leq \frac{B_1 \cdots B_N}{Q_N^2}. \]

That is, $f_{B_1, \ldots, B_N}$ is a bi-Lipschitz map.

Proof of Theorem 4.2. (1) $\beta = 1$. In this case, for any $\varepsilon > 0$, there exists $N := N(\varepsilon) > 0$ such that for all $n > N$, we have $\phi(n) \leq e^{(1+\epsilon)n}$. Hence,
\[ \overline{F}_N(\phi) \supseteq \{ x \in [0, 1) : b_n(x) \geq e^{(1+\epsilon)n}, \forall n \geq 1 \}. \]

It follows from Proposition 5 and Lemma 4.3 that $\dim \tilde{H} \phi \geq 1/(1+\varepsilon)$. Let $\varepsilon \to 0^+$, we have $\dim \tilde{H} \phi = 1$.

(2) $1 < \beta < \infty$. Since $\log \beta = \limsup_{n \to \infty} \frac{\log \phi(n)}{n}$, for any $0 < \varepsilon < \beta - 1$, we have...
\( \phi(n) \geq e^{(\beta - \varepsilon)n} \) holds for infinitely many \( n \in \mathbb{N} \). Also there exists \( N := N(\varepsilon) > 0 \) such that for all \( n > N \), we have \( \phi(n) \leq e^{(\beta + \varepsilon)n} \). So,

\[
\tilde{F}(\phi) \subseteq \left\{ x \in [0, 1) : b_n(x) \geq e^{(\beta - \varepsilon)n} \ \text{i.m.} \ n \in \mathbb{N} \right\}
\]

and

\[
\tilde{F}_N(\phi) \supseteq \left\{ x \in [0, 1) : b_n(x) \geq e^{(\beta + \varepsilon)n}, \forall n \geq 1 \right\}.
\]

Combining these with Lemma 3.1, Proposition 5 and Lemma 4.3, we conclude that

\[
\frac{1}{\beta + \varepsilon} \leq \dim_H \tilde{F}(\phi) \leq \frac{1}{\beta - \varepsilon}.
\]

Let \( \varepsilon \to 0^+ \), we have \( \dim_H F(\phi) = \beta^{-1} \).

(3) \( \beta = \infty \). For any \( B > 1 \), we have \( \limsup_{n \to \infty} \frac{\log \phi(n)}{n} > \log B \). So, \( \phi(n) \geq e^{B^n} \) holds for infinitely many \( n \in \mathbb{N} \). Then

\[
\tilde{F}(\phi) \subseteq \left\{ x \in [0, 1) : b_n(x) \geq e^{B^n} \ \text{i.m.} \ n \in \mathbb{N} \right\},
\]

which implies that \( \dim_H \tilde{F}(\phi) \leq B^{-1} \) by Proposition 5. Let \( B \to \infty \), we have \( \dim_H \tilde{F}(\phi) = 0 \). \( \square \)

4.2. Other related sets. As we have mentioned, Kraaikamp and Wu [18] proved a strong law of large numbers for \( \log b_n \), i.e.,

\[
\lim_{n \to \infty} \frac{1}{n} \log b_n(x) = 1
\]

holds for almost all \( x \in [0, 1) \) in the sense of Lebesgue measure. Moreover, they also showed that for any \( \alpha > 0 \),

\[
\dim_H \left\{ x \in [0, 1) : \lim_{n \to \infty} \frac{\log b_n(x)}{n} = \alpha \right\} = 1.
\]

We further investigate this topic by considering a more fast growth speed of \( \log b_n \), which indicates that the corresponding Hausdorff dimension will decay with an inverse function rate when the growth speed of \( \log b_n \) is exponential.

**Theorem 4.4.** Let \( b > 1 \). For any \( \alpha > 0 \), we have

\[
\dim_H \left\{ x \in [0, 1) : \lim_{n \to \infty} \frac{\log b_n(x)}{b^n} = \alpha \right\} = \frac{1}{b}.
\]

**Proof.** On the one hand, for any \( \alpha > 0 \), it is clear that \( E(a, b) \) is a subset of the desired set, which implies that

\[
\dim_H \left\{ x \in [0, 1) : \lim_{n \to \infty} \frac{\log b_n(x)}{b^n} = \alpha \right\} \geq \frac{1}{b}
\]

by Lemma 5. On the other hand, let \( x \in [0, 1) \) satisfy

\[
\lim_{n \to \infty} \frac{\log b_n(x)}{b^n} = \alpha.
\]

For any \( 0 < \varepsilon < \alpha \), there exists \( N := N(x, \varepsilon) > 0 \) such that for all \( n \geq N \), we have \( b_n(x) \geq e^{(\alpha - \varepsilon)b^n} \). So,

\[
\left\{ x \in [0, 1) : \lim_{n \to \infty} \frac{\log b_n(x)}{b^n} = \alpha \right\} \subseteq \left\{ x \in [0, 1) : b_n(x) \geq e^{(\alpha - \varepsilon)b^n} \ \text{i.m.} \ n \in \mathbb{N} \right\}.
\]
Applying Lemma 3.1 to \( a = e^{\alpha - \varepsilon} > 1 \), we deduce that
\[
\dim_H \left\{ x \in [0, 1) : \lim_{n \to \infty} \frac{\log b_n(x)}{b^n} = \alpha \right\} \leq \frac{1}{b}.
\]

The proof is completed. \( \square \)

By (3), we have
\[
\lim_{n \to \infty} \frac{\log b_{n+1}(x)}{\log b_n(x)} = 1
\]
holds for almost all \( x \in [0, 1) \) in the sense of Lebesgue measure. The following result will study the Hausdorff dimension of the set of points in which such a limit attains any other values.

**Theorem 4.5.** For any \( b \geq 1 \), we have
\[
\dim_H \left\{ x \in [0, 1) : \lim_{n \to \infty} \frac{\log b_{n+1}(x)}{\log b_n(x)} = b \right\} = \frac{1}{b}.
\]

**Proof.** If \( b = 1 \), since \((\log b_n(x))/n \to 1 \) as \( n \to \infty \) holds for almost all \( x \in [0, 1) \), we know
\[
\left\{ x \in [0, 1) : \lim_{n \to \infty} \frac{\log b_{n+1}(x)}{\log b_n(x)} = 1 \right\}
\]
has full Lebesgue measure and it of course has full Hausdorff dimension. Now let \( b > 1 \). Note that
\[
\left\{ x \in [0, 1) : \lim_{n \to \infty} \frac{\log b_n(x)}{b^n} = 1 \right\} \subseteq \left\{ x \in [0, 1) : \lim_{n \to \infty} \frac{\log b_{n+1}(x)}{\log b_n(x)} = b \right\},
\]
it follows from Theorem 4.4 that
\[
\dim_H \left\{ x \in [0, 1) : \lim_{n \to \infty} \frac{\log b_{n+1}(x)}{\log b_n(x)} = b \right\} \geq \frac{1}{b}.
\]

According to the definition of Hausdorff dimension, the inverse inequality can be obtained by considering a natural covering system. For any \( x \in [0, 1) \) satisfying
\[
\lim_{n \to \infty} \frac{\log b_{n+1}(x)}{\log b_n(x)} = b
\]
and \( 0 < \varepsilon < b - 1 \), there exists \( N := N(x, \varepsilon) > 0 \) such that for all \( n \geq N \), we have \( b_n^{b-\varepsilon}(x) \leq b_{n+1}(x) \leq b_n^{b+\varepsilon}(x) \) and hence
\[
\left\{ x \in [0, 1) : \lim_{n \to \infty} \frac{\log b_{n+1}(x)}{\log b_n(x)} = b \right\} \subseteq \bigcup_{N \geq 1} B(b, N),
\]
where
\[
B(b, N) = \left\{ x \in [0, 1) : b_n^{b-\varepsilon}(x) \leq b_{n+1}(x) \leq b_n^{b+\varepsilon}(x), \forall n \geq N \right\}.
\]
This implies that
\[
\dim_H \left\{ x \in [0, 1) : \lim_{n \to \infty} \frac{\log b_{n+1}(x)}{\log b_n(x)} = b \right\} \leq \sup_{N \geq 1} \left\{ \dim_H B(b, N) \right\}
\]
by the countable stability of the Hausdorff dimension. Inspired by Lemma 4.3, we know \( \dim_H B(b, 1) = \dim_H B(b, 1) \) for all \( N \geq 1 \).

Now we will estimate the Hausdorff dimension of the set \( B(b, 1) \). To do this, some symbols are needed. For any \( n \geq 1 \), define
\[
\mathcal{D}_n = \{ (\sigma_1, \cdots, \sigma_n) \in \mathbb{N}^n : \sigma_1 \in \mathbb{N}, \sigma_k^{\alpha - \varepsilon} \leq \sigma_{k+1} \leq \sigma_k^{\alpha + \varepsilon}, \forall 1 \leq k \leq n - 1 \}.
\]
Here we should notice that all elements in $D_n$ are admissible for ECF expansions. In fact, for any $(\sigma_1, \ldots, \sigma_n) \in D_n$, we have that $\sigma_k \geq \sigma_{k-1}^{b-\varepsilon} > \sigma_{k-1}$ by $0 < \varepsilon < b-1$. So the element of $D_n$ can be used to define cylinders. For any $(\sigma_1, \ldots, \sigma_n) \in D_n$, we define

$$J(\sigma_1, \ldots, \sigma_n) = \bigcup_{\sigma_n^{b-\varepsilon} \leq \sigma_{n+1} \leq \sigma_n^{b+\varepsilon}} \text{cl}(I(\sigma_1, \ldots, \sigma_n, \sigma_{n+1})), $$

where $I(\sigma_1, \ldots, \sigma_n, \sigma_{n+1})$ is the cylinder whose definition is given in Section 2. Thus, we obtain that

$$B(b, 1) = \bigcap_{n \geq 1} \bigcup_{(\sigma_1, \ldots, \sigma_n) \in D_n} J(\sigma_1, \ldots, \sigma_n).$$

Note that

$$\frac{|I(\sigma_1, \ldots, \sigma_n, \sigma_{n+1})|}{|I(\sigma_1, \ldots, \sigma_n)|} \leq \frac{\sigma_n + 1}{\sigma_{n+1}(\sigma_{n+1} + 1)},$$

so we derive that

$$|J(\sigma_1, \ldots, \sigma_n)| = \sum_{\sigma_n^{b-\varepsilon} \leq \sigma_{n+1} \leq \sigma_n^{b+\varepsilon}} |I(\sigma_1, \ldots, \sigma_n, \sigma_{n+1})| = |I(\sigma_1, \ldots, \sigma_n)| \sum_{\sigma_n^{b-\varepsilon} \leq \sigma_{n+1} \leq \sigma_n^{b+\varepsilon}} \frac{|I(\sigma_1, \ldots, \sigma_n, \sigma_{n+1})|}{|I(\sigma_1, \ldots, \sigma_n)|} \leq |I(\sigma_1, \ldots, \sigma_n)| \sum_{\sigma_n^{b-\varepsilon} \leq \sigma_{n+1} \leq \sigma_n^{b+\varepsilon}} \frac{\sigma_n + 1}{\sigma_{n+1}(\sigma_{n+1} + 1)}$$

$$\leq |I(\sigma_1, \ldots, \sigma_n)| \cdot (\sigma_n + 1) \cdot \left(\frac{1}{\sigma_n^{b-\varepsilon}} - \frac{1}{\sigma_n^{b+\varepsilon} + 1}\right) := \Phi(n, b, \varepsilon).$$

(10)

Now let $0 < \varepsilon < \min\{1/2, b-1\}$. Then we have $(1 + b - \varepsilon)(b - \varepsilon) - (b + \varepsilon) = (b - \varepsilon)^2 - 2\varepsilon > 0$. For any $s > \frac{b+\varepsilon}{(1+b-\varepsilon)(b-\varepsilon)-(b+\varepsilon)}$, by (10) and (11), we deduce that

$$\sum_{(\sigma_1, \ldots, \sigma_n) \in D_n} |J(\sigma_1, \ldots, \sigma_n)|^s \leq \sum_{(\sigma_1, \ldots, \sigma_n) \in D_n} \left(\Phi(n, b, \varepsilon)\right)^s \leq \left(\sum_{(\sigma_1, \ldots, \sigma_{n-1}) \in D_{n-1}} \left(\Phi(n-1, b, \varepsilon)\right)^s\right) \cdot \Upsilon(n-1, b, \varepsilon, s),$$

(11)

where $\Upsilon(n-1, b, \varepsilon, s)$ is defined as

$$\Upsilon(n-1, b, \varepsilon, s) = \sum_{\sigma_n^{b-\varepsilon} \leq \sigma_{n+1} \leq \sigma_n^{b+\varepsilon}} \left(\frac{1}{\sigma_n + 1}\right)^s \cdot \left(\frac{1}{\sigma_n^{b-\varepsilon} + 1} - \frac{1}{\sigma_n^{b+\varepsilon} + 1}\right)^s.$$
for all $n \geq 1$ and
\[
\left( \frac{\sigma_n^{2\varepsilon}}{\sigma_n-1} \right)^{-s} \leq \left( \frac{\sigma_n^{2\varepsilon} - \sigma_n^{b+\varepsilon}}{\sigma_n-1+1} \right)^{-s} \leq 1
\]
holds for sufficiently large $n$ since $(\sigma_n^{2\varepsilon} - \sigma_n^{b+\varepsilon})/(\sigma_n-1+1) \to 0$ as $n \to \infty$. Observe that $\sigma_n^{b+\varepsilon} \leq \sigma_n \leq \sigma_n^{b+\varepsilon}$ and $s > 1/(1+b-\varepsilon)(b+\varepsilon)$, we have
\[
\ldots
\]
for sufficiently large $n$. Combing these with (12), we conclude that
\[
\sum_{(\sigma_1, \ldots, \sigma_n) \in D_n} |J(\sigma_1, \ldots, \sigma_n)|^s \leq \sum_{(\sigma_1, \ldots, \sigma_n) \in D_n} (\Phi(n, b, \varepsilon))^s
\]
holds for sufficiently large $n$. By the definition of $s$-dimensional Hausdorff measure, we finally obtain that
\[
\mathcal{H}^s(B(b, 1)) \leq \liminf_{n \to \infty} \sum_{(\sigma_1, \ldots, \sigma_n) \in D_n} |J(\sigma_1, \ldots, \sigma_n)|^s < \infty,
\]
which implies that $\dim H B(b, 1) \leq 1/(b+\varepsilon)(b+\varepsilon)$ and hence that
\[
\dim H \left\{ x \in [0, 1] : \lim_{n \to \infty} \frac{\log b_{n+1}(x)}{\log b_n(x)} = b \right\} \leq \frac{b+\varepsilon}{(1+b-\varepsilon)(b+\varepsilon)}.
\]
Letting $\varepsilon \to 0^+$, the proof is completed. \hfill \Box

5. Ratio between two consecutive partial quotients. This section is devoted to dealing with the Hausdorff dimension of certain sets related to the ratio between two consecutive partial quotients. Such a ratio can be viewed as an indirect way to consider the growth rate of partial quotients. For any $x \in [0, 1)$ and $n \geq 1$, let
\[
R_n(x) = \frac{b_{n+1}(x)}{b_n(x)}.
\]
Let $\phi$ be a positive function defined on natural numbers and $R(\phi) = \{ x \in (0, 1) : R_n(x) \geq \phi(n) \ i.m. \ n \}$. For the size of $R(\phi)$, Fan et al. [4] gave a zero-one law for the Lebesgue measure of $R(\phi)$.

**Theorem 5.1** ([4]). The set $R(\phi)$ has null or full Lebesgue measure according as the series $\sum_{n \geq 1} 1/\phi(n)$ converges or diverges.
We will give a complete description of $R(\phi)$ from the viewpoint of fractal dimensions. To do this, let
\[
R(a,b) = \left\{ x \in [0,1) : R_n(x) \geq a^{b^n} \text{ i.m. } n \in \mathbb{N} \right\}
\]
and
\[
\tilde{R}(a,b) = \left\{ x \in [0,1) : R_n(x) \geq a^{b^n}, \forall n \geq 1 \right\}
\]
for $a, b > 1$. We first give the exact Hausdorff dimensions of $R(a,b)$ and $\tilde{R}(a,b)$.

**Lemma 5.2.** Let $a, b > 1$. Then $\dim_H R(a,b) = \dim_H \tilde{R}(a,b) = 1/b$.

**Proof.** Let $a, b > 1$. On the one hand, we know
\[
R(a,b) \subseteq \left\{ x \in [0,1) : b_{n+1}(x) \geq a^{b^n} \text{ i.m. } n \in \mathbb{N} \right\}
\]
by the definition of $R_n(x)$ and hence that
\[
R(a,b) \subseteq \left\{ x \in [0,1) : b_{n+1}(x) \geq c^{b^{n+1}} \text{ i.m. } n \in \mathbb{N} \right\}
\]
with $c = a^{b^{-1}} > 1$, which implies that $\dim_H R(a,b) \leq 1/b$ by Proposition 5. On the other hand, for $a, b > 1$, let $c > 0$ satisfy the equation $c^{b^{-1}} = a$, then $c > 1$. We claim that
\[
E(c,b) \subseteq \tilde{R}(a,b).
\]
This is because that if $nc^{b^n} \leq b_n(x) \leq (n+1)c^{b^n}$ holds for some $x \in [0,1)$ and for all $n \geq 1$, we have
\[
R_n(x) = \frac{b_{n+1}(x)}{b_n(x)} \geq \frac{(n+1)c^{b^{n+1}}}{(n+1)c^{b^n}} = c^{(b-1)b^n} = a^{b^n}.
\]
Thus, it follows from Proposition 5 that $\dim_H \tilde{R}(a,b) \geq 1/b$. Since $\tilde{R}(a,b) \subseteq R(a,b)$, we conclude that $\dim_H R(a,b) = \dim_H \tilde{R}(a,b) = 1/b$. \qed

The proofs of the following two theorems are very similar to the proof Theorems 4.1 and 4.2, and are left to the reader.

**Theorem 5.3.** Suppose that $\log \beta = \lim \inf_{n \to \infty} \frac{\log \log \phi(n)}{n}$.
(1) If $\beta = 1$, then $\dim_H R(\phi) = 1$.
(2) If $1 < \beta < \infty$, then $\dim_H R(\phi) = \beta^{-1}$.
(3) If $\beta = \infty$, then $\dim_H R(\phi) = 0$.

**Theorem 5.4.** Let $\phi$ be a positive function defined on natural numbers and
\[
\tilde{R}(\phi) = \left\{ x \in [0,1) : R_n(x) \geq \phi(n), \forall n \geq 1 \right\}.
\]
Suppose that $\log \beta = \lim \sup_{n \to \infty} \frac{\log \log \phi(n)}{n}$.
(1) If $\beta = 1$, then $\dim_H \tilde{R}(\phi) = 1$.
(2) If $1 < \beta < \infty$, then $\dim_H \tilde{R}(\phi) = \beta^{-1}$.
(3) If $\beta = \infty$, then $\dim_H \tilde{R}(\phi) = 0$.

For a further investigation on the ratio $R_n$, we consider
\[
L_n(x) = \max\{R_1(x), \ldots, R_n(x)\}
\]
for $x \in [0,1)$ and $n \geq 1$. In [4], Fan et al. proved an iterated logarithm type theorem for $L_n$. We emphasize that the 0-1 law is also true for $L_n$ as a consequence of Theorem 5.1.
Proposition 7. Let \( \phi \) be a non-decreasing and positive function defined on natural numbers. The set
\[
L(\phi) := \{ x \in [0, 1) : L_n(x) \geq \phi(n) \text{ i.m. } n \in \mathbb{N} \}
\]
has Lebesgue measure zero or one according as the series \( \sum_{n \geq 1} 1/\phi(n) \) converges or diverges.

Proof. Let \( \phi \) be a non-decreasing and positive function defined on natural numbers. If \( \sup_{n \geq 1} \phi(n) < \infty \), then the series \( \sum_{n \geq 1} 1/\phi(n) \) diverges and hence that for almost all \( x \in [0, 1) \), \( R_n(x) \geq \phi(n) \) holds for infinitely many \( n \in \mathbb{N} \) by Theorem 5.1. Note that \( L_n(x) \geq R_n(x) \), so \( L_n(x) \geq \phi(n) \) holds infinitely often for almost all \( x \in [0, 1) \). Now let \( \phi(n) \) tend to \( \infty \) as \( n \) goes to \( \infty \). In this case, we claim that \( L(\phi) = R(\phi) \).

From the proof of Proposition 7, we can see that if \( \sup_{n \geq 1} \phi(n) < \infty \), then \( R_n(x) \geq \phi(n) \) and \( L_n(x) \geq \phi(n) \) both hold infinitely often for almost all \( x \in [0, 1) \). Of course, \( R(\phi) \) and \( L(\phi) \) have full Hausdorff dimension. If \( \phi(n) \) tends to \( \infty \) as \( n \) goes to \( \infty \), then \( R(\phi) = L(\phi) \). Therefore, we always have that \( R(\phi) \) and \( L(\phi) \) share the same Hausdorff dimension when \( \phi \) is a non-decreasing and positive function defined on natural numbers. However, the following result shows that the assumption of non-decreasing property on function \( \phi \) can be relaxed.

Theorem 5.5. Let \( \phi \) be a positive function defined on natural numbers. Suppose that \( \log \beta = \lim_{n \to \infty} \inf \frac{\log \phi(n)}{n} \).

(1) If \( \beta = 1 \), then \( \dim_H L(\phi) = 1 \).
(2) If \( 1 < \beta < \infty \), then \( \dim_H L(\phi) = \beta^{-1} \).
(3) If \( \beta = \infty \), then \( \dim_H L(\phi) = 0 \).

Theorem 5.6. Let \( \phi \) be a positive function defined on natural numbers and
\[
\widetilde{L}(\phi) = \{ x \in [0, 1) : L_n(x) \geq \phi(n), \forall n \geq 1 \}.
\]
Suppose that \( \log \beta = \lim_{n \to \infty} \sup \frac{\log \phi(n)}{n} \).

(1) If \( \beta = 1 \), then \( \dim_H \widetilde{L}(\phi) = 1 \).
(2) If \( 1 < \beta < \infty \), then \( \dim_H \widetilde{L}(\phi) = \beta^{-1} \).
(3) If \( \beta = \infty \), then \( \dim_H \widetilde{L}(\phi) = 0 \).

The proofs of Theorems 5.5 and 5.6 can be finished by observing that for any \( a, b > 1 \),
\[
\dim_H \{ x \in [0, 1) : L_n(x) \geq a^n \text{ i.m. } n \in \mathbb{N} \} = 1/b
\]
and
\[
\dim_H \{ x \in [0, 1) : L_n(x) \geq a^n, \forall n \geq 1 \} = 1/b.
\]
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