We characterize exotic C*-algebras of twisted, principal, \'{e}tale groupoids, together with the abelian subalgebra associated to the unit space, as precisely being the inclusions “A \( \subseteq \) B” of C*-algebras in which A is abelian, regular, and satisfies the extension property (pure states extend uniquely to B). When B is moreover nuclear, we deduce that the corresponding opaque ideal is trivial. As an application, we give a streamlined characterization of Kumjian’s C*-diagonals as the regular abelian subalgebras satisfying the extension property with vanishing opaque ideal.

### 1. Introduction.

Inspired by Feldman and Moore’s fundamental paper [9] on Cartan subalgebras of von Neumann algebras, Kumjian [11] introduced the notion of **diagonals** in the context of C*-algebras and proved that every inclusion

\[
A \subseteq B \quad (1.1)
\]

diag of an abelian C*-algebra A in another C*-algebra B, satisfying suitable hypotheses, is necessarily modeled by a twisted, principal, \'{e}tale groupoid.

The first main goal of the present paper is to prove a souped up version of Kumjian’s result, based on my recent work [7] with D. Pitts. The plan is to strip the hypotheses of [11: Theorem 3.1] to a bare minimum, while retaining its conclusion, except that the modeling will be done with a possibly exotic groupoid C*-algebra, rather than the reduced version adopted in [11].

Besides requiring the indispensable condition that (1.1) be a regular inclusion (Definition (2.2.iii) below), Kumjian assumes the existence of a faithful conditional expectation whose kernel is required to be spanned by the so called free normalizers, namely normalizers squaring to zero. In an important intermediate result [11: Proposition 1.4] Kumjian shows that C*-diagonals satisfy the extension property, meaning that each pure state of A admits a unique extension to a state on B (see Definition (2.1) below for more details).

We in turn adopt the extension property as the sole condition (besides regularity) in our main result, namely Theorem (3.2), where we prove that such inclusions are precisely the ones arising from exotic C*-algebras of twisted, principal, \'{e}tale groupoids.

The proof of our main result is achieved as a direct application of [7: Theorem 20.6], a result that characterizes exotic groupoid C*-algebras in full generality. In order to make use of it, all we need is to provide a canonical state [7: Definition 20.1] relative to every point of the spectrum of A. This is in turn obtained as a consequence of two well known results, namely that inclusions satisfying the extension property are maximal abelian [10: page 385], and automatically posses a conditional expectation [4: Theorem 2.2] (albeit not

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necessarily a faithful one). Since the proofs of these two results are very short, and often only done in the unital case, we include them for the convenience of the reader.

Our second main goal, actually the motivation for writing the present paper, is the study of the opaque ideal for regular inclusions. We first show that the regular inclusions with a trivial opaque ideal, and satisfying the extension property, are precisely Kumjian’s C*-diagonals. This result generalizes [6: Theorem 4.8] to the non-unital case. Furthermore, assuming the extension property, we give an affirmative answer to a question raised in [8: Question 4.2], by proving that the opaque ideal is trivial when the containing algebra is nuclear.

Last but not least, I’d like to thank mathoverflow user Darth Vader for bringing up Archbold’s paper in a comment to a question of mine [15], which triggered a long chain of ideas, eventually culminating with the present paper. Thanks are also due to David Pitts for several helpful suggestions.

2. Basic facts.

Throughout this section we shall fix a C*-algebra $B$, and we shall let

$$A \subseteq B$$

be a fixed closed *-subalgebra. In our main result, below, we will assume that $A$ is abelian, so the reader is welcome to assume this from now on, although in a few places commutativity of $A$ is not strictly required. When $A$ is explicitly assumed to be abelian, the spectrum of $A$ will be denoted by $X$, so that $A$ is *-isomorphic to $C_0(X)$ by Gelfand’s Theorem.

2.1. Definition. One says that $A$ has the extension property relative to $B$, provided:

(i) every pure state of $A$ admits a unique extension to a state on $B$, and
(ii) $A$ contains an approximate identity for $B$.

Many variations of the above notion appear in the literature. Quite often [10,3,5] it is required that $B$ be unital and that $A$ contain the unit of $B$, which obviously implies point (ii) above. In [4], on the other hand, point (ii) is replaced by the equivalent requirement that no pure state of $B$ annihilates $A$ (the equivalence follows from the Krein-Milman Theorem and [1: Lemma 2.32]).

A nice conceptual equivalent formulation is as follows: denoting by $\mathcal{P}(A)$ the set of all pure states on $A$, and by $\mathcal{P}_0(A) = \mathcal{P}(A) \cup \{0\}$, and similarly for $B$, then the extension property may be characterized by saying that every member of $\mathcal{P}_0(A)$ admits a unique extension to a member of $\mathcal{P}_0(B)$. The catch is that the zero linear functional on $A$ admits a unique extension to a member of $\mathcal{P}_0(B)$ iff no pure state of $B$ restricts to zero on $A$.

The following notion was introduced by Kumjian in [11]:

2.2. Definition.

(i) An element $b \in B$ is said to be a normalizer for $A$, if

$$b^*ab \subseteq A, \quad \text{and} \quad bab^* \subseteq A.$$
(ii) The set of all normalizers is denoted by $N(A, B)$.

(iii) One says that $A$ is a regular subalgebra of $B$ when (2.1.ii) holds, and $N(A, B)$ spans a dense subspace of $B$.

2.3. Remarks.

(a) Observe that, assuming (2.1.ii), one has that $n^*n \in A$ for every $n$ in $N(A, B)$. This is because if $\{u_i\}_i$ is an approximate unit for $B$ contained in $A$, then

$$n^*n = \lim_{i \to \infty} n^*u_in \in A.$$  

(b) On the other hand, if we assume that $N(A, B)$ spans $B$, and that $n^*n$ lie in $A$ for every normalizer $n$, then (2.1.ii) follows. Indeed, taking any normalizer $n$, we have

$$n = \lim_{k \to \infty} n(n^*n)^{1/k} \in \overline{BA},$$

from where it follows that $B = \overline{BA}$, and from this it is easily seen that any approximate unit for $A$ is also an approximate unit for $B$.

(c) This said, it would perhaps be slightly more elegant to strengthen the definition of normalizers by adding the requirement that $n^*n$, and perhaps also $nn^*$, lie in $A$. Should this be done, the above definition of a regular subalgebra could be streamlined by requiring only that $N(A, B)$ span $B$, as (2.1.ii) would be automatic.

(d) We should also remark that D. Pitts has recently shown [13] that, when $A$ is a maximal abelian subalgebra of $B$, then the fact that $N(A, B)$ span $B$ (with the standard definition of normalizers) alone implies (2.1.ii). The reader should however be warned that, unlike (2.2.iii), Pitts defines regularity without condition (2.1.ii).

In order to prove our main result we need two auxiliary facts. The first one refers to the existence of a conditional expectation from $B$ to $A$, a fact that has been proved by Anderson [3: Theorem 3.4] for $A$ unital and abelian, and generalized by Archbold [4: Theorem 2.2] for some situations where $A$ does not need to be commutative. In case $A$ is abelian and regular the proof is much shorter, so it is perhaps worth spelling it out.

2.4. Lemma. [3,4] Assume that $A$ is abelian, regular, and satisfies the extension property. For every $x$ in $X$ (the spectrum of $A$), let $\varphi_x$ be the pure state defined by

$$\varphi_x(a) = a(x), \quad \forall a \in A,$$

and let $\psi_x$ be the unique state on $B$ extending $\varphi_x$. Moreover, for each $b$ in $B$, consider the scalar-valued function $E(b)$ defined on $X$ by

$$E(b)|_x = \psi_x(b), \quad \forall x \in X.$$ 

Then

(i) $E(b)$ is continuous on $X$,

(ii) $E(b)$ belongs to $A$ (that is, in case $X$ is not compact, $E(b)$ vanishes at $\infty$),

(iii) The ensuing map $E : B \to A$ is the unique conditional expectation from $B$ to $A$. 

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Proof. The continuity of \( E(b) \) follows immediately from [7: Proposition 8.4], once we realize that the set \( F_b \) mentioned there coincides with \( X \), and that \( \varepsilon_b = E(b) \).

It is also clear that \( E(b) \) is bounded with \( \| E(b) \|_\infty \leq \| b \| \), hence \( E \) defines a contractive, positive map

\[
E : B \rightarrow C^b(X),
\]

where \( C^b(X) \) stands for the C*-algebra of all bounded, continuous, complex functions on \( X \). If \( b \in B \) and \( a \in A \), then

\[
E(ab) = aE(b) = E(ba),
\]

by [7: Lemma 7.3]. Therefore, letting \( \{ u_i \} \) be an approximate unit for \( B \) contained in \( A \), we have that

\[
E(b) = \lim_{i \to \infty} E(u_i b) = \lim_{i \to \infty} u_i E(b) \in C_0(X) = A,
\]

proving (ii).

Regarding (iii), it is now clear that \( E \) is a conditional expectation from \( B \) to \( A \). To show that \( E \) is unique, suppose that \( F \) is another one. Then, for every \( x \) in \( X \), we may define a state \( \rho_x \) on \( B \) by

\[
\rho_x : b \in B \mapsto F(b)|_x \in \mathbb{C}.
\]

Evidently \( \rho_x \) extends \( \varphi_x \), so \( \rho_x = \psi_x \) by uniqueness. Therefore, for every \( b \) in \( B \), and every \( x \) in \( X \), we have that

\[
F(b)|_x = \rho_x(b) = \psi_x(b) = E(b)|_x,
\]

proving that \( F = E \). \( \square \)

The second auxiliary tool needed to prove our main result is the fact that the extension property implies maximal commutativity. In the case of unital algebras, this fact is implicitly mentioned in the introduction to [10], incidentally the same paper where the famous Kadison-Singer problem was first posed. This is also stated in [5: Corollary 2.7], where [10] is referenced for a proof. In what follows we give a proof of this result for the sake of completeness, because the available references apparently only deal with the unital case, and also because the proof is really very short.

2.5. Lemma. [10] If \( A \) is abelian and satisfies the extension property relative to \( B \), then \( A \) is maximal abelian.

Proof. If \( b \in B \) commutes with \( A \), we must prove that \( b \in A \). By decomposing \( b \) in its real and imaginary parts, we may assume that \( b \) is self-adjoint. The closed *-algebra \( C \) of \( B \) generated by \( A \cup \{ b \} \) is therefore abelian, so we may write \( C = C_0(\Omega) \), for some locally compact space \( \Omega \), while the inclusion

\[
C_0(X) = A \hookrightarrow C = C_0(\Omega)
\]

is necessarily implemented by a proper, surjective map \( h : \Omega \rightarrow X \) (notice that (2.1.ii) is crucial for such a map to exist). Observe that \( h \) must be one-to-one because, if \( s \) and \( t \) are distinct points in \( \Omega \) such that \( h(s) = h(t) = x \), then both \( \varphi_s \) and \( \varphi_t \) extend \( \varphi_x \) from \( A \) to \( C \). Further state extensions of \( \varphi_s \) and \( \varphi_t \) to \( B \) would then violate the extension property. This proves that \( h \) is bijective. Since \( \Omega \) is locally compact and \( h \) is proper, it follows that \( h \) is a homeomorphism, so \( A = C \), and hence \( b \in A \). \( \square \)
3. Exotic C*-algebras of twisted, principal, étale groupoids.

Recall that, given any twisted étale groupoid \((\mathcal{G}, \mathcal{L})\), the full and reduced C*-algebras of \((\mathcal{G}, \mathcal{L})\), denoted

\[ C^*(\mathcal{G}, \mathcal{L}) \quad \text{and} \quad C^*_{\text{red}}(\mathcal{G}, \mathcal{L}), \]

are respectively defined to be the completions of \(C_c(\mathcal{G}, \mathcal{L})\) under the maximal and reduced norms, namely

\[ \|\cdot\|_{\text{max}} \quad \text{and} \quad \|\cdot\|_{\text{red}}. \]

If we instead complete \(C_c(\mathcal{G}, \mathcal{L})\) under some C*-norm \(\mu\) satisfying

\[ \|f\|_{\text{max}} \geq \mu(f) \geq \|f\|_{\text{red}}, \quad \forall f \in C_c(\mathcal{G}, \mathcal{L}), \]

the resulting object is often denoted \(C^*_\mu(\mathcal{G}, \mathcal{L})\), and called an exotic\(^1\) groupoid C*-algebra.

Our first main result is a complete characterization of exotic C*-algebras of twisted, principal, étale groupoids.

3.2. Theorem. Given a C*-algebra \(B\) and a closed *-subalgebra \(A \subseteq B\), the following are equivalent:

(a) \(A\) is abelian, regular, and satisfies the extension property,

(b) \(B\) is isomorphic to an exotic C*-algebra for a twisted, principal, étale groupoid \((\mathcal{G}, \mathcal{L})\), via an isomorphism carrying \(A\) onto \(C_0(\mathcal{G}(0))\).

Proof. Assuming (a) we will prove (b) by means of a direct application of [7: Theorem 20.6]. For this we let \(N = N(A, B)\), and we will prove that the state \(\psi_x\) (introduced in the proof of (2.4)) is an \(N\)-canonical state [7: Definition 20.1] relative to \(x\), for every \(x\) in \(X\). In other words, we must show that \(\psi_x\) vanishes on any element \(n \in N\), except possibly at those for which \(x\) is trivial relative to \(n\) [7: Definition 18.1], meaning that there exists \(v\) in \(A\) such that \(v(x) = 1\), and \(nv \in A\).

Seen from the contra-positive point of view, our task is to show that, for every \(x\) in \(X\), and every \(n\) in \(N\), one has that

\[ \psi_x(n) \neq 0 \quad \Rightarrow \quad x \text{ is trivial relative to } n. \]

Given that \(\psi_x(n) \neq 0\), we deduce from (2.4.i) that there exists a neighborhood \(V\) of \(x\) such that \(\psi_y(n) \neq 0\), for every \(y\) in \(V\). The extension property precisely says that every point of \(X\) is free in the sense of [7: Definition 9.1], so the isotropy algebra \(B(y)\) is isomorphic to \(\mathbb{C}\) by [7: Proposition 9.3]. Moreover, identifying \(B(y)\) with \(\mathbb{C}\), we have that

\[ E_y(n) = \psi_y(n), \]

by [7: Proposition 9.4], where \(E_y : B \rightarrow B(y)\) is the corresponding localizing projection [7: Definition 3.6] (for the pair of ideals \((J_y, J_y)\)). From \(\psi_y(n) \neq 0\), and employing [7:

\(^1\) Some authors use the word exotic only in case \(\mu\) is distinct from either the full or reduced norms. Nevertheless, as should be clear from the above, we shall not make that distinction.
Proposition 7.5.iii], we deduce that \( n \) lies in \( N_{\psi,\eta} \), that is, \( \psi \) belongs to the domain of \( \beta_n \) (see [7: Proposition 4.4.ii]), and \( \beta_n(y) = y \). In other words, \( \beta_n \) coincides with the identity map on \( V \).

Choose any \( v \) in \( A \) such that \( v(x) = 1 \), and \( v = 0 \) outside \( V \). Setting \( m := nv \), it follows from [7: Proposition 4.4.iii] that \( x \) lies in the domain of \( \beta_m \), and that \( \beta_m \) is the identity on its domain. We may then invoke [7: Proposition 4.7] to conclude that \( m \) lies in the commutant \( A' \), hence also in \( A \) by (2.5). This shows that \( nv \in A \), so we see that \( x \) is trivial relative to \( n \), as desired. An application of [7: Theorem 20.6] then takes care of everything, except for the fact that \( G \) is principal.

To see that \( G \) is principal, let \( B_1 = C^*_\text{red}(G, \mathcal{L}) \). Since \( \mu \) dominates the reduced norm, \( B_1 \) is a quotient of \( B \), and as \( A \) satisfies the extension property relative to \( B \), the same is evidently true with respect to \( B_1 \). By [7: Corollary 14.14] (adopting the second alternative in [7: 14.8]), we then deduce that every isotropy group \( G(x) \) is trivial, so \( G \) is principal.

Conversely, assuming (b), it is immediate that \( A \) is abelian and regular, so it suffices to prove the extension property. Considering the natural epimorphism

\[
\pi : C^*(G, \mathcal{L}) \to C^*_\mu(G, \mathcal{L}),
\]

arising from the first inequality in (3.1), observe that \( \pi \) restricts to the identity map on the respective copies of \( C_0(G^{(0)}) \). Therefore, for each \( x \in X = G^{(0)} \), there are at least as many extensions of \( \varphi_x \) to \( C^*(G, \mathcal{L}) \), as there are to \( C^*_\mu(G, \mathcal{L}) \). This said, it is enough to prove the extension property under the assumption that \( B = C^*(G, \mathcal{L}) \).

Given any \( x \) in \( X \), we have by [7: Theorem 14.12.ii] that the isotropy algebra \( B(x) \) is isomorphic to the (twisted) group \( C^* \)-algebra of the isotropy group \( G(x) \). However this group is trivial due to the fact that \( G \) is principal, whence \( B(x) = \mathbb{C} \).

Using [7: Proposition 9.3] we then conclude that every \( x \) in \( X \) is free, meaning that \( \varphi_x \) admits a unique extension to a state on \( B \), according to [7: Definition 9.1]. \( \square \)

We should point out that one could attempt to prove (3.2) using a different strategy, based on a result by Donsig and Pitts [6: Theorem 4.8], where it is shown that the left kernel of the conditional expectation \( E \) given by (2.4), namely

\[
L = \{ b \in B : E(b^*b) = 0 \},
\]

is a two-sided ideal in \( B \), trivially intersecting \( A \). The quotient \( B/L \) therefore contains a faithful, regular copy of \( A \). Donsig and Pitts go on to prove that \( A \) is then a \( C^* \)-diagonal in \( B/L \), so Kumjian’s Theorem may be applied to model \( B/L \) as the reduced \( C^* \)-algebra of a twisted, principal, étale groupoid \( (G, \mathcal{L}) \).

One would then need to show that the \( C^* \)-algebra of \( (G, \mathcal{L}) \), namely the twisted Weyl groupoid for \( B/L \), admits a representation in \( B \) (as opposed to \( B/L \)), which would then lead to a sequence of \( * \)-homomorphisms

\[
C^*(G, \mathcal{L}) \to B \to B/L \simeq C^*_\text{red}(G, \mathcal{L}),
\]

from where it would follow that \( B \) is an exotic version of \( C^*(G, \mathcal{L}) \), as desired. We don’t presently have a clear strategy for providing such a representation, so we feel that the proof given above might be the best way to proceed.
Under the hypothesis of (3.2), and assuming that $A$ and $B$ are unital, Pitts shows that there are maximal and minimal C*-norms on the *-subalgebra of $B$ linearly spanned by $N(A, B)$ [12: Corollary 7.5]. The completions of this subalgebra under such norms is likely to coincide with the full and reduced groupoid C*-algebras, respectively, so this would perhaps lead to yet another avenue for proving (3.2).

Recalling that a groupoid C*-algebra is nuclear iff the groupoid is amenable [2], we have the following consequence:

**3.3. Corollary.** Under the conditions of (3.2), if $B$ is moreover nuclear, then the groupoid $G$ mentioned there is amenable, and hence

$$B \simeq C^*(G, \mathcal{L}) \simeq C^*_\mu(G, \mathcal{L}) \simeq C^*_{\text{red}}(G, \mathcal{L}).$$

*Proof.* Since the norm $\mu$ given by (3.2) dominates the reduced norm, we get a surjective *-homomorphism

$$B \simeq C^*_\mu(G, \mathcal{L}) \to C^*_{\text{red}}(G, \mathcal{L}),$$

which implies that $C^*_{\text{red}}(G, \mathcal{L})$ is a quotient of $B$, and hence a nuclear algebra. This in turn implies that $G$ is amenable. \qed

4. **The opaque ideal.**

The original motivation for pursuing the ideas that led to this paper was the study of the opaque ideal of regular inclusions. In order to describe it, let us assume throughout this section that:

- $B$ is a C*-algebra, and that
- $A \subseteq B$

is an abelian, regular, sub-C*-algebra. As usual we will denote the spectrum of $A$ by $X$.

Referring to the evaluation states $\varphi_x$ introduced in the proof of (2.4), let $J_x$ be the kernel of $\varphi_x$, so that $J_x$ is an ideal in $A$ (although quite likely not an ideal in $B$). The **opaque ideal** for the inclusion “$A \subseteq B$” is defined in [7: Definition 11.4] as being

$$\Delta = \bigcap_{x \in X} J_x B.$$

See [7: Proposition 11.2.ii] for a proof that the opaque ideal is indeed a closed two-sided ideal in $B$.

It is not very often that one comes across an example of an inclusion with a nonzero opaque ideal, and in fact in well behaved cases, such as when $B$ is abelian [8: Section 4], or when $B$ is a reduced groupoid C*-algebra [7: Proposition 15.3.i], the opaque ideal is zero. For an example of a nonzero $\Delta$ the reader is referred to [8: Proposition 4.1]. The fact that this example employs an action of the free group causes one to suspect that the lack of amenability of the free group is the main culprit behind this phenomena. In fact [8 : Question 4.2] may be thought of as an attempt to rule out bad behavior of non-amenable groups via the requirement of nuclearity.
Let us now give a characterization of the opaque ideal in terms of extended pure states. Recalling that \( \mathcal{P}(A) \) denotes the set of all pure states on \( A \), let us define the set of extended pure states on \( B \) by

\[
\mathcal{E}\mathcal{P}(B) = \{ \psi \in \mathcal{S}(B) : \psi|_A \in \mathcal{P}(A) \},
\]

where \( \mathcal{S}(B) \) is the state space of \( B \).

In plain words \( \mathcal{E}\mathcal{P}(B) \) consists of all states on \( B \) obtained as the extension of some pure state on \( A \). Notice that, although every pure state on \( A \) admits a pure extension to \( B \), some might also admit a mixed (i.e. non-pure) extension. In other words, \( \mathcal{E}\mathcal{P}(B) \) might very well contain mixed states. An exception is of course the situation in which \( A \) has the extension property relative to \( B \), in which case \( \mathcal{E}\mathcal{P}(B) \) is nothing but the set formed by the \( \psi_x \), all of which are necessarily pure.

4.1. Proposition.

(i) If \( x \in X \), and if \( \psi \) is a state on \( B \) extending \( \varphi_x \), then \( \psi \) vanishes on \( J_xB \).

(ii) Every \( \psi \) in \( \mathcal{E}\mathcal{P}(B) \) vanishes on \( \Delta \).

(iii) \( \Delta = \{ b \in B : \psi(b^*b) = 0, \text{ for all } \psi \in \mathcal{E}\mathcal{P}(B) \} \).

(iv) In case \( A \) has the extension property relative to \( B \), we moreover have that

\[
\Delta = \{ b \in B : E(b^*b) = 0 \},
\]

where \( E \) is the unique conditional expectation given by (2.4).

Proof. If \( \psi \) extends \( \varphi_x \) then, given \( a \in J_x \) and \( b \in B \), we have by Cauchy-Schwartz that

\[
|\psi(ab)|^2 \leq \psi(aa^*)\psi(b^*b) = \langle a^*a, x \rangle \psi(bb^*) = 0,
\]

proving (i), while (ii) follows immediately from (i).

Regarding (iii), notice that the inclusion “\( \subseteq \)” follows from (ii). In order to prove the reverse inclusion, let \( b \) be an element belonging to the set in the right-hand-side of (iii). To prove that \( b \) lies in \( \Delta \), and making use of [7: Proposition 11.2], it suffices to prove that \( E_x(b^*b) = 0 \), for every \( x \) in \( X \). Arguing by contradiction, suppose that this is not so for some \( x \). In this case one can find a state \( \rho \) on the isotropy algebra \( B(x) \), such that \( \rho(E_x(b^*b)) \neq 0 \). By [7: Proposition 3.20] the composition

\[
\psi : B \xrightarrow{E_x} B(x) \xrightarrow{\rho} \mathbb{C}
\]

is a state on \( B \) vanishing on \( J_x \), and hence extending \( \varphi_x \), so that \( \psi \in \mathcal{E}\mathcal{P}(B) \). We then have that

\[
0 \neq \rho(E_x(b^*b)) = \psi(b^*b),
\]

a contradiction, hence proving (iii). The last point follows easily from (iii), the fact that

\[
\mathcal{E}\mathcal{P}(B) = \{ \psi_x : x \in X \},
\]

and the definition of \( E \) in (2.4). \qed
In [12: Definition 2.4], Pitts introduces the set \( \text{Mod}(A, B) \) formed by the \( A \)-modular states, which, in our case, coincides with \( \mathcal{EP}(B) \). Furthermore, in [6: Theorem 4.8], Donsig and Pitts prove that, when \( A \) and \( B \) are unital, and \( A \) has the extension property relative to \( B \), then the subspace of \( B \) consisting of the elements annihilated by all modular states coincides with the left kernel of the conditional expectation \( E \), hence the opaque ideal by (4.1.iii). They also show that the quotient of \( B \) by said ideal is a C*-diagonal, thus obtaining a characterization of C*-diagonals in terms of the opaque ideal.

We spell out the details of this characterization below, for the sake of completeness, and also to allow for the non-unital case as well. See [13: Proposition 8] for similar characterizations.

4.2. Proposition. [6: Theorem 4.8] Let \( B \) be a C*-algebra and let \( A \subseteq B \) be a regular, abelian sub-C*-algebra. Then the following are equivalent:

(a) \( A \) is a C*-diagonal in \( B \) in the sense of [11: Definition 1.3],

(b) \( A \) satisfies the extension property relative to \( B \), and the opaque ideal vanishes.

Proof. Assuming (a) we have that \( A \) satisfies the extension property relative to \( B \) by [11: Proposition 1.4]. Also, the hypotheses guarantee the existence of a faithful conditional expectation, so the expectation \( E \) of (2.4), being the unique such, must be faithful. Therefore the opaque ideal vanishes by (4.1.iv).

Conversely, by (3.2) we may assume that \( B = C^*_\mu(\mathcal{G}, \mathcal{L}) \), and \( A = C_0(\mathcal{G}(0)) \), where \((\mathcal{G}, \mathcal{L})\) is a twisted, principal, étale groupoid. The composition

\[
B = C^*_\mu(\mathcal{G}, \mathcal{L}) \xrightarrow{\pi} C^*_\text{red}(\mathcal{G}, \mathcal{L}) \xrightarrow{F} C_0(\mathcal{G}(0)) = A,
\]

where \( \pi \) is the natural map, and \( F \) is the standard conditional expectation on the reduced algebra, is clearly a conditional expectation, hence coincides with \( E \) by (2.4). Since \( F \) is well known to be faithful, it follows that

\[
\ker(\pi) = \{ b \in B : F(\pi(b^*b)) = 0 \} = \{ b \in B : E(b^*b) = 0 \} \overset{(4.1.iv)}{=} \Delta.
\]

The assumption that \( \Delta = \{ 0 \} \) then tells us that \( B \) coincides with \( C^*_\text{red}(\mathcal{G}, \mathcal{L}) \), so the conclusion follows from [11: Theorem 2.9]. \( \square \)

To conclude, let us give an affirmative answer to [8: Question 4.2] in the special case that \( A \) satisfies the extension property.

4.3. Theorem. Let \( B \) be a C*-algebra and let \( A \subseteq B \) be a regular, abelian sub-C*-algebra. If \( B \) is nuclear, and \( A \) has the extension property relative to \( B \), then \( \Delta = \{ 0 \} \).

Proof. By (3.3), we have that \( B \simeq C^*_\text{red}(\mathcal{G}, \mathcal{L}) \), so the conclusion follows from [7: Proposition 15.3.1]. \( \square \)

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