On the F-expanding of Homoclinic class

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Abstract

We establish a closing property for thin trapped homoclinic classes. Taking advantage of this property, we proved that if the homoclinic class \( H(p) \) admits a dominated splitting \( T_{H(p)}M = E \oplus F \), where \( E \) is thin trapped (see Definition 1.3) and all periodic points homoclinically related to \( p \) are uniformly \( F \)-expanding at the period (see Definition 1.4), then \( F \) is expanded (see Definition 1.3).

1 Introduction

Let \( f \) be a diffeomorphism on a compact manifold \( M \) with metric \( d \). Given two hyperbolic periodic points \( p \) and \( q \) of \( f \), denoted by \( W^s(p) \), \( W^u(q) \) the stable manifolds of \( p \) and \( q \), denoted by \( W^u(p) \), \( W^u(q) \) the unstable manifolds of \( p \) and \( q \). We call that they are homoclinically related if \( W^s(\text{orb}(p)) \cap W^u(\text{orb}(q)) \neq \emptyset \) and \( W^s(\text{orb}(q)) \cap W^u(\text{orb}(p)) \neq \emptyset \). And we denote this relation by \( p \sim q \). The homoclinic class of a hyperbolic periodic point \( p \) is defined as \( H(p) \equiv \{ q : q \in P(f), q \sim p \} \), where \( P(f) \) denotes the set of all hyperbolic periodic points of \( f \).

In general, the hyperbolicity of the periodic points contained in a compact invariant set \( \Lambda \) is not enough to get the hyperbolicity of \( \Lambda \). For example, Kupka-Smale Theorem [9, P91] affirms that every periodic orbit of \( C^r \)-generic \((r \geq 1)\) diffeomorphisms is hyperbolic and the stable and unstable manifolds of the periodic orbits are pairwise transverse. It means that the homoclinic class is usually not hyperbolic although it contains many hyperbolic invariant sets: any set of periodic orbits homoclinically related to \( p \). Bonatti, Gan and Yang [2, Main Theorem] gave a sufficient criterion for the hyperbolicity of a homoclinic class. They also raised a question: Can we obtain the hyperbolicity of an invariant compact set by using the hyperbolicity of the periodic orbits in the set?

Our main work shows that one can get some “hyperbolicity” of \( H(p) \) under the topological conditions (see Definition 1.3) on the periodic orbits homoclinically related to \( p \). Now, we will introduce precisely the notions for these conditions.

**Definition 1.1.** The invariant splitting \( T_\Lambda M = E \oplus F \) over a compact \( f \)-invariant set \( \Lambda \) is a dominated splitting if there are two constants \( C > 0, \lambda \in (0, 1) \) such that

\[
\| Df^n|_{E_x} \| \cdot \| Df^{-n}|_{F_{f^n(x)}} \| \leq C\lambda^n, \text{ for any } x \in \Lambda \text{ and every } n \in \mathbb{N}.
\]

**Definition 1.2.** A periodic point \( p \) is hyperbolic if there are constants \( C > 0, \lambda \in (0, 1) \) and an invariant splitting \( T_p M = E_p \oplus F_p \), such that for every \( n \in \mathbb{N} \), one has that

\[
\| Df^n|_{E_p} \| \leq C\lambda^n, \quad \| Df^{-n}|_{F_p} \| \leq C\lambda^n.
\]
For the dominated splitting $T_\Lambda M = E \oplus F$, a plaque family tangent to the bundle $E$ is a family of continuous maps $W$ from the linear bundle $E$ to $M$ satisfying:

1. for each $x \in \Lambda$, the map $W_x : E_x \to M$ is a $C^1$-embedding that satisfies $W_x(0) = x$ and whose image is tangent to $E_x$ at $x$;
2. $(W_x)_{x \in \Lambda}$ is a continuous family of $C^1$-embeddings.

Let $W(x)$ be the image of embedding $W_x$. A plaque family $W$ is locally invariant if there is $\delta > 0$ such that for every $x \in \Lambda$, one has that $f \circ W_x(B(0, \delta)) \subseteq W(f(x))$, where $B(0, \delta) \subseteq E_x$ is the ball whose radius is $\delta$. Plaque Family Theorem [7, Theorem 5.5] shows that there always exists a locally invariant plaque family tangent to $E$. A plaque family is called trapped if for every $x \in \Lambda$, one has that $f(W(x)) \subseteq W(f(x))$.

**Definition 1.3.** For the dominated splitting $T_\Lambda M = E \oplus F$, $E$ is thin trapped if for any neighborhood $U$ of the section $0$ in $E$, there is

1. a continuous family $\{\varphi_x\}_{x \in \Lambda}$ of $C^1$-diffeomorphisms of the spaces $\{E_x\}_{x \in \Lambda}$ supported in $U$;
2. a constant $\delta > 0$ such that $f(W_x \circ \varphi_x(B(0, \delta))) \subseteq W_{f(x)} \circ \varphi_{f(x)}(B(0, \delta))$ for any $x \in \Lambda$.

$F$ is expanded on $\Lambda$, if there are $C > 0$ and $\lambda \in (0, 1)$ such that

$$\|Df^{-n}|_{F_x}\| \leq C\lambda^n, \text{ for any } x \in \Lambda \text{ and } n \in \mathbb{N}^+$$

**Definition 1.4.** Let $\Lambda = \{q_n\}_{n \in \mathbb{N}}$ be a sequence of hyperbolic periodic points of diffeomorphism $f$, for the dominated splitting $T_\Lambda M = E \oplus F$, $f$ is uniformly $F$-expanding at the period on $\Lambda$ if there are two constants $C > 0$, $\lambda \in (0, 1)$ such that for any $n \in \mathbb{N}$, one has that

$$\prod_{j=1}^{\pi(q_n)} \|Df^{-1}|_{F_{j(q_n)}}\| \leq C\lambda^{\pi(q_n)},$$

where $\pi(q_n)$ is the period of the periodic point $q_n$. We also call that $F$ is uniformly $\lambda$-expanding at the period on $\Lambda$.

Now, we introduce our main result and the sketch of our proof for the main result.

**Main Theorem.** Let $f$ be a diffeomorphism on a compact Riemannian manifold, $p$ be a hyperbolic periodic point. Assume that the homoclinic class $H(p)$ admits a dominated splitting $T_{H(p)} M = E \oplus F$ and $E$ is thin trapped with $\dim E = \text{ind} (p)$. If $f$ is uniformly $F$-expanding at the period on all periodic points homoclinically related to $p$, then $F$ is expanded on $H(p)$.

This Main Theorem also gives a criterion for getting “weak periodic point” which means that they have a Lyapunov exponent arbitrarily close to zero in a given homoclinic class.

**Theorem A.** Let $p$ be a hyperbolic periodic point of a diffeomorphism $f$ on a compact Riemannian manifold. If the homoclinic class $H(p)$ satisfying:

- the homoclinic class $H(p)$ admits a partially hyperbolic splitting $T_{H(p)} M = E^s \oplus E^c \oplus E^u$ with $E^s$ is uniformly contracted, $E^u$ is uniformly expanding and $\dim E^c = 1$;
- $\dim E^s = \text{ind} (p)$ and $H(p)$ is not hyperbolic.
then for every \( \varepsilon > 0 \), one can find a periodic point \( q \in H(p) \) homoclinically related to \( p \) such that
\[
\frac{1}{\pi(q)} \log(\|Df^{\pi(q)}|_{E^q}\|) \leq \varepsilon.
\]

The Main Theorem gives a criterion for getting some hyperbolicity of a homoclinic class from the same property of the periodic points in this homoclinic class. We should consider the question: how to establish the relations between non-periodic points in the compact set and periodic points? The Anosov Closing Lemma [8, P269, Theorem 6.4.15] implies that for any point in a hyperbolic set whose orbit nearly returns to the point, there is a periodic orbit closely shadowing this nearly-returning orbit. Gan [5, Theorem 1.1] showed that any quasi-hyperbolic pseudoorbit with recurrence can be shadowed by periodic orbit. But in our assumptions, the homoclinic class is not hyperbolic set and we can not get the quasi-hyperbolic pseudoorbit with respect to the dominated splitting on the homoclinic class. Even though the recurrent orbit (non-periodic) can be shadowed by periodic orbit, we also need to consider that: how to expand the property of periodic orbit to other orbit (non-periodic orbit).

2 Preliminaries

In this section, we introduce some important tools and some basic facts but useful for the proof of the Main Theorem.

2.1 Hyperbolic time

Let \( f \) be a diffeomorphism on a compact manifold \( M \). For every \( x \in M \) and \( n \in \mathbb{N}^+ \), \((x, n)\) denotes the segment of orbit \((x, n) \triangleq \{x, f(x), \ldots, f^{n-1}(x)\}\). We consider a compact invariant set \( \Lambda \) which has a dominated splitting \( T_{\Lambda}M = E \oplus F \). By [1, P289, Appendix B], one can fix an admissible compact neighborhood of \( \Lambda \) and denotes by \( M(f, U) = \bigcap_{i \in \mathbb{Z}} f^i U \) the maximal invariant set in \( U \). The dominated splitting \( E \oplus F \) extends in a unique way on \( M(f, U) \).

**Definition 2.1.1.** For any \( \lambda \in (0, 1) \) and \( x \in M(f, U) \), an orbit segment \((x, n)\) is an uniform \( \lambda \)-string, if \( \prod_{j=k+1}^n \|Df^{-1}|_{F_{f^j(x)}}\| \leq \lambda^{n-k}, \) for \( k = 0, 1, \ldots, n-1 \). And \( n \in \mathbb{N}^+ \) is called \( \lambda \)-hyperbolic time of \( x \).

Denoted by \( HT(x, \lambda) \) the set of all \( \lambda \)-hyperbolic times of \( x \) and the \( n \)-th \( \lambda \)-hyperbolic time of \( x \) is denoted by \( HT_n(x, \lambda) \). For a periodic point \( x \), denoted by \( \Gamma_1(x, \lambda) \) the largest \( \lambda \)-hyperbolic time in the period \( \pi(x) \) and the smallest \( \lambda \)-hyperbolic time larger than the period \( \pi(x) \) is denoted by \( \Gamma_2(x, \lambda) \). The following lemma, given by Pliss ([11]), gives us a tool to prove that a point satisfying the assumption that \( \liminf_{m \to \infty} \frac{1}{m} \sum_{i=1}^m \log \|Df^{-1}|_{F_{f^i(x)}}\| \leq \log \lambda \), for same \( \lambda \in (0, 1) \), has many (positive density at infinity) hyperbolic times.

**Lemma 2.1.2** (Pliss Lemma [11]). Given constants \( A \) and \( C_2 < C_1 < 0 \) with \( A \geq |C_2| \), there is \( \theta \in (0, 1) \) such that if any real numbers \( \{a_j\}_{j=1}^N \) satisfy the conditions
\[
(1) \ |a_j| \leq A, \text{ for } j = 1, 2, \ldots, N;
(2) \ \sum_{j=1}^N a_j \leq NC_2.
\]
then there is an integer \( l \geq \theta N \) and a sequence numbers \( 1 \leq n_1 < n_2 < \cdots < n_l \leq N \) such that
\[
\sum_{j=n+1}^{n_l} a_j \leq (n_i - n)C_1, \quad \forall 0 \leq n < n_i, \ i = 1, 2, \cdots, l.
\]

**Lemma 2.1.3.** Given \( \mu < \mu_1 < \mu_2 < 0 \) and \( A > |\mu_1| \), for any real number sequence \( \{a_i\}_{i=1}^{\infty} \) with \( |a_i| \leq A \), if there is a number \( m \in \mathbb{N}^+ \) such that \( \sum_{i=1}^{m} a_i = m \mu \) and \( a_{i+m} = a_i \) for every \( i \in \mathbb{N} \), then there is an integer \( N \) for every \( a \) such that \( N(x, n) \), if \( x, n \) be the set of points such that the orbit segment \( (x, n) \) is contained in the \( \varepsilon \)-neighborhood of \( \Lambda(r) \). Then, \( d(\Lambda_0, \Lambda(r)) \leq \varepsilon \), where \( \Lambda(r) \) is the set of all \( r \)-obstruction orbit segment.

**Lemma 2.1.6.** Assume that the homoclinic class \( H(p) \) of hyperbolic periodic point \( p \) admits a dominated splitting \( T_{H(p)}M = E \bigoplus F \) and \( F \) is uniformly \( \lambda \)-expanding at the period on the
set of periodic points homoclinically related to $p$. If there is a $r$-obstruction point $b \in H(p)$ with $r \in (\lambda, 1)$, then there exists a sequence periodic points $\{q_n : n \in \mathbb{N}^+\} \subset H(p)$ homoclinically related to $p$, such that $\lim_{n \to \infty} q_n = b$ and $HT_1(q_n, \mu) \to \infty$ as $n \to \infty$, where $\lambda < \mu < r$. Moreover, $\Gamma_2(q_n, \mu) - \Gamma_1(q_n, \mu) \to \infty$ as $n \to \infty$.

**Proof.** Since $b \in H(p)$, by the definition of $H(p) \triangleq \{q : q \in P(f), q \sim \rho\}$, there is a sequence of periodic points $\{q_n : n \in \mathbb{N}^+\} \subset H(p)$ homoclinically related to $p$, such that $\lim_{n \to \infty} q_n = b$. As $F$ is uniformly $\lambda$-expanding at the period on the set of periodic points homoclinically related to $p$, there are two constants $C > 0$, $\lambda \in (0, 1)$ such that $\prod_{j=1}^{n} || Df^{-1}_{f^{j}(q_n)} || \leq C \lambda^{q_n}$, for every $n \in \mathbb{N}^+$. Let $a_i = \log || Df^{-1}_{f^{i}(q_n)} ||$, one has that $\lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m} a_i \leq \log \lambda$. By the Lemma 2.1.3, one gets that $q_n$ has infinitely many $\mu$-hyperbolic times. Now, we prove that $HT_1(q_n, r_2) \to \infty$ as $n \to \infty$ by contradiction. Otherwise, we suppose that there exists constant $C_1 > 0$ such that $HT_1(q_n, \mu) \leq C_1$, for all $n$. Then there is a subsequence $\{q_{n_k}\}$ of $\{q_n\}$, such that $HT_1(q_{n_k}, \mu)$ is constant, denoting by $L$. Hence for all $q_{n_k}$, one has that $\prod_{i=1}^{L} || Df^{-1}_{f^{i}(q_{n_k})} || \leq \mu^L$. By the continuous of $Df^{-1}$, one gets that

$$\prod_{i=1}^{L} || Df^{-1}_{f^{i}(b)} || = \lim_{k \to \infty} \prod_{i=1}^{L} || Df^{-1}_{f^{i}(q_{n_k})} || \leq \mu^L \leq r^L.$$ 

This means that $b$ is not a $r$-obstruction point which contradicts our condition. Since $\Gamma_2(q_n, \mu) - \Gamma_1(q_n, \mu) \geq \Gamma_2(q_n, \mu) - \pi(q_n) \geq HT_1(q_n, \mu)$, one has that $\Gamma_2(q_n, \mu) - \Gamma_1(q_n, \mu) \to \infty$ as $n \to \infty$. 

## 2.2 Exponents properties in Homoclinic class

In this section, let $p$ be a hyperbolic periodic point. The homoclinic class $H(p)$ admits a dominated splitting $T_{H(p)} M = E \oplus F$ with $\dim(E) = \text{ind}(p)$. We assume that the bundle $E$ is thin trapped and $f$ is uniformly $F$-expanding at the period on the set of periodic points homoclinically related to $p$. First, we introduce some properties of $C^1$ diffeomorphisms.

**Lemma 2.2.1.** Let $f$ be a $C^1$ diffeomorphism on a compact manifold $M$. For any $x \in M$, if there is $C = C(x) > 0$ and $\mu_1$, $\lambda_1 \in (0, 1)$ such that $C \mu_1^n \leq \prod_{i=0}^{n-1} || Df_{f^i(x)} || \leq C \lambda_1^n$, for every $n \in \mathbb{N}^+$, then for any $\mu_2 < \mu_1 < \lambda_1 < \lambda_2$, there is $C_0 = C_0(x)$ and $r = r(\mu_1, \mu_2, \lambda_1, \lambda_2)$ such that

$$C_0 \mu_2^n \leq \prod_{i=0}^{n-1} || Df_{f^i(y)} || \leq C_0 \lambda_2^n, \ \forall \ y \in B(x, r), \ \forall \ n \in \mathbb{N}^+.$$ 

**Proof.** Since $f$ is $C^1$ diffeomorphism, for $0 < \mu_2 < \mu_1 < \lambda_1 < \lambda_2 < 1$, there is $r_1 > 0$ such that

$$\frac{\mu_2}{\mu_1} \leq \frac{|| Df_x ||}{|| Df_z ||} \leq \frac{\lambda_2}{\lambda_1}, \ \text{for any} \ d(x, y) \leq r_1.$$ 

For any $y \in B(x, r_1)$, by the Mean Value Theorem, there exists $\xi \in B(x, r_1)$ such that $d(f(x), f(y)) = || Df_\xi || \cdot d(x, y)$. Since $\xi \in B(x, r_1)$, one has that

$$d(f(x), f(y)) \leq \frac{\lambda_2}{\lambda_1} || Df_x || \cdot d(x, y) \leq C \frac{\lambda_2}{\lambda_1} r_1 = C_2 x.$$ 

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Taking \( r = \min\{r_1, r_1/C\} \), \( C_0 = \max\{C, 1/C\} \), we claim that \( d(f^j(x), f^j(y)) \leq r_1 \), for every \( y \in B(x, r) \) and every \( j \in \mathbb{N}^+ \). Next, we prove the claim. We may assume that \( d(f^j(x), f^j(y)) \leq r_1 \) for any \( y \in B(x, r) \) and every \( j = 1, 2, \cdots, m \). Then, for any \( y \in B(x, r) \), by the Mean Value Theorem, there is \( \eta \in B(x, r) \) such that

\[
d(f^{m+1}(x), f^{m+1}(y)) = \| Df_{f^{m+1}}^{m+1} \| d(x, y) \leq \| Df_{f^m \eta}^m \| \| Df_{f^{m-1} \eta}^{m-1} \| \cdots \| Df_{f \eta} \| d(x, y)
\]

\[
\leq \frac{\lambda_2}{\lambda_1} \| Df_{f \eta} \| \frac{\lambda_2}{\lambda_1} \| Df_{f^{m-1} \eta} \| \cdots \frac{\lambda_2}{\lambda_1} \| Df_{f_{f^m \eta}} \| \| Df_{f^{m+1} \eta} \| d(x, y)
\]

\[
\leq \frac{\lambda_2^{m+1}}{\lambda_1^{m+1}} \prod_{i=0}^m \| Df_{f^i \eta} \| r \leq \frac{\lambda_2^{m+1}}{\lambda_1^{m+1}} \lambda_1^{m+1} r = C \lambda_2^{m+1} r < r_1
\]

Therefore, \( f^{m+1}(B(x, r)) \subset B(f^{m+1}(x), r_1) \). This proves our claim. Therefore, for \( y \in B(x, r) \) and \( n \in \mathbb{N}^+ \), one has that

\[
\frac{\mu_2^n}{\mu_1^n} \leq \prod_{i=0}^{n-1} \| Df_{f^i \eta} \| \leq \frac{\lambda_2}{\lambda_1} \frac{\lambda_2^{n+1}}{\lambda_1^{n+1}}.
\]

Consequently, one has that \( C_0 \mu_2^n \leq \prod_{i=0}^{n-1} \| Df_{f^i \eta} \| \leq C_0 \lambda_2^n \), for \( y \in B(x, r) \) and \( n \in \mathbb{N}^+ \).

Proposition 2.2.2. Assume that the homoclinic class \( H(p) \) admits a dominated splitting \( T_{H(p)} \mathbb{M} = E \oplus \mathbb{F} \) with \( \dim E = \text{ind}(p) \). If \( f \) is uniformly \( F \)-expanding at the period on all periodic points homoclinically related to \( p \), then there exists constant \( N \in \mathbb{N}^+ \) such that a local plaque family tangent to the bundle \( F \) are the unstable manifolds of all hyperbolic periodic points with period larger than \( N \).

Proof. Since \( f \) is uniformly \( F \)-expanding, by the Definition 1.4, there are two constants \( C > 0, \lambda \in (0, 1) \) such that

\[
\prod_{j=1}^{\pi(x)} \| Df^{-1} \big|_{F_{f^j(x)}} \| \leq C \lambda^{\pi(x)} \quad \text{for any hyperbolic periodic point } x \in H(p).
\]

Let \( N \) be the smallest constant which satisfies that \( C \lambda^N < 1 \), define

\[
A = \{ x \in H(p) : x \text{ is periodic point with period larger than } N \}.
\]

For any \( x \in A \) and \( \varepsilon > 0 \), since \( T_{H(p)} \mathbb{M} = E \oplus \mathbb{F} \) is a dominated splitting, by Plaque Family Theorem [7, Theorem 5.5], there always exists a locally invariant plaque family tangent to the bundle \( E \) and \( F \). Denote by \( W^F_{p}(x) \) the plaque family tangent to the bundle \( F \) at point \( x \). Since \( x \) is a hyperbolic periodic point, we may assume that \( T_{x} \mathbb{M} = E^{s} \oplus F^{s} \) is the hyperbolic splitting. By the Definition 1.2, there are constants \( C_1 > 0, \lambda_1 \in (0, 1) \) such that for every \( n \in \mathbb{N} \), one has that

\[
\| Df^n \|_{E^{s}} \leq C_1 \lambda_1^n, \quad \| Df^{-n} \|_{F^{u}} \leq C_1 \lambda_1^n.
\]

Since \( \dim E = \text{ind}(p) \), we only need to prove that \( F \subseteq F^{u} \). If \( F \not\subseteq F^{u} \), then there is a vector \( v \in F \) such that \( v \not\in F^{u} \). Thus, one has a decomposition \( v = v^s \oplus v^u \), where \( 0 \neq v^s \in E^s \) and \( v^u \in F^u \). Therefore, for every \( m \in \mathbb{N}^+ \), one has that

\[
C_1^{-1} \lambda_1^{-m \pi(x)} \| v^s \| \leq \| Df^{-m \pi(x)} v \| \leq ( \prod_{j=1}^{m \pi(x)} \| Df^{-1} \big|_{F_{f^j(x)}} \|) \| v \| \leq (C \lambda^{\pi(x)})^m \| v \|.
\]

For \( m \) large enough, we have that \( C_1 \lambda_1^{-m \pi(x)} \| v^s \| > 1 \) and \( (C \lambda^{\pi(x)})^m \| v \| < 1 \). This means that our assumption \( F \not\subseteq F^{u} \) is fault. Consequently, \( F \subseteq F^{u} \).
Next, we introduce that in the homoclinic class \( H(p) \), there is a dense set such that every point in this set has stable manifolds of uniformly size. Given \( \varepsilon > 0 \), a sequence points \( \{x_0, \ldots, x_m\} \) is called a periodic \( \varepsilon \)-orbit or periodic pseudoorbit, if \( x_m = x_0 \) and \( d(f(x_i), x_{i+1}) < \varepsilon \), for \( i = 0, \ldots, m − 1 \).

**Theorem 2.2.3.** ([8], Anosov Closing Lemma) If \( \Lambda \) is a hyperbolic set of diffeomorphism \( f \), then there is an open neighborhood \( U \) of \( \Lambda \) and two constants \( C > 0, \varepsilon_0 > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0) \) and any periodic \( \varepsilon \)-orbit \( \{x_0, \ldots, x_m\} \subset U \), there exist a periodic point \( y \in U \) satisfying \( f^m(y) = y \) and \( d(f^k(y), x_k) < C\varepsilon \), for \( k = 0, 1, \ldots, m − 1 \).

**Theorem 2.2.4.** Let \( p \) be a hyperbolic periodic point of diffeomorphism \( f \). Assume that the homoclinic class \( H(p) \) has a dominated splitting \( T_{H(p)}M = E \oplus_x F \) with \( \dim(E) = \text{ind}(p) \). If \( E \) is thin trapped and \( f \) is uniform \( F \)-expanding at the period on the set of periodic points which are homoclinically related to \( p \), then for any \( \varepsilon > 0 \), there is a \( \varepsilon \)-dense set \( \mathcal{P} \subseteq H(p) \) of hyperbolic periodic points homoclinically related to the orbit of \( p \), such that every point \( q \in \mathcal{P} \) has stable manifolds of uniformly size.

**Proof.** We may assume that \( p \) is a hyperbolic fix point (if not, consider \( g = f^{\varepsilon(p)} \)). Since \( \dim(E) = \text{ind}(p) \), for the dominated splitting \( T_pM = E \oplus_x F \), by [6, Theorem 1], taking suitable Riemann norm, there exists \( \lambda_1 \in (0, 1) \) such that \( \|Df|_{E_x}\| \leq \lambda_1 \) and \( \|Df^{-1}|_{F_p}\| \leq \lambda_1 \). Hereafter, we fixed the numbers \( 0 < \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < 1 \). Since \( f \) is \( C^1 \), for \( \lambda_2 \in (\lambda_1, 1) \), there is \( r > 0 \) such that for any \( x, y \in H(p) \) with \( d(x, y) < r \), one has that

\[
\frac{\|Df|_{E_x}\|}{\|Df|_{E_y}\|} \leq \frac{\lambda_2}{\lambda_1} \quad \text{and} \quad \frac{\|Df^{-1}|_{F_p}\|}{\|Df^{-1}|_{F_p}\|} \leq \frac{\lambda_2}{\lambda_1}.
\]

By the definition of \( H(p) \), for any \( \varepsilon < r \), one can take a \( \frac{\varepsilon}{2} \)-dense set

\[
B = \{x \in H(p) : x \in W^s(p) \cap W^u(p)\} \text{ in } H(p).
\]

**Claim.** For every \( x \in B \), \( \Lambda_x \triangleq \text{Orb}(p) \cup \text{Orb}(x) \) is a hyperbolic set.

**Proof.** Since \( x \in W^s(p) \cap W^u(p) \), there exists \( n_0 \in \mathbb{N} \) such that for any \( n \geq n_0 \), one has that \( d(f^n(x), p) < r \) and \( d(f^{-n}(x), p) < r \). Therefore,

\[
\frac{\|Df|_{E_{m(x)}\}}{\|Df|_{E_x}\|} \leq \frac{\lambda_2}{\lambda_1}, \quad \frac{\|Df^{-1}|_{F_{-n(x)}\}}{\|Df^{-1}|_{F_p}\|} \leq \frac{\lambda_2}{\lambda_1}, \quad \text{for any } n \geq n_0.
\]

Let \( C_1 = \max_{y \in H(p)}\left\{\frac{\|Df|_{E_x}\|}{\lambda_2}, \frac{\|Df^{-1}|_{F_p}\|}{\lambda_2}\right\} \), taking \( C = \max\{1, C_1\} \), for any \( y \in \Lambda_x \) and any \( n \in \mathbb{N} \), one has that

\[
\|Df^n|_{E_x}\| \leq C\lambda_2^n, \quad \|Df^{-n}|_{F_p}\| \leq C\lambda_2^n.
\]

Hence, \( \Lambda_x = \text{Orb}(p) \cup \text{Orb}(x) \) is a hyperbolic set.

Fixed \( x \in B \), by the Theorem 2.2.3, for the hyperbolic set \( \Lambda_x \), there exist an open neighborhood \( U \supset \Lambda_x \) and \( C_2, \varepsilon_0 > 0 \), such that for any \( \varepsilon_1 \in (0, \varepsilon_0) \) and any periodic \( \varepsilon_1 \)-orbit \( \{x_0, \ldots, x_m\} \subset U \), there is a periodic point \( y \in U \) satisfying \( f^m(y) = y \) and \( d(f^k(y), x_k) < C_2\varepsilon_1 \), for \( k = 0, 1, \ldots, m − 1 \). Taking \( \varepsilon_2 = \min(\varepsilon/2C_2, \varepsilon/2, r) \), since \( x \in W^s(p) \cap W^u(p) \), there exists \( m_0 \in \mathbb{N} \) such that for any \( m \geq m_0 \), one has that \( d(f^m(x), p) < \varepsilon_2/2 \) and \( d(f^{-m}(x), p) < \varepsilon_2/2 \).
Let $K = \sup_{y \in H(p)} \{\|Df_y\|, 2\} < +\infty$, for $\lambda_2 < \lambda_3 < 1$, taking an integer $m_1 \geq C/2 \log(\lambda_2/\lambda_3)$, where $C = \log \lambda_3 + (2m_0 - 2) \log \lambda_2 - (2m_0 - 1) \log K$. One can get a $\varepsilon_2$-closed orbit segment: \[ \{f^{-m_1}(x), f^{-m_1+1}(x), \ldots, f^{-m_0}(x), f^{-m_0+1}(x), \ldots, f^{-1}(x), x, f(x), \ldots, f^{m_0}(x), \ldots, f^{m_1}(x)\}. \]

By the Theorem 2.2.3, the closed orbit segment can be $\varepsilon/2$-shadowed by a periodic point $q_x$ with $\pi(q_x) = 2m_1 + 1$. Therefore,

$$
\prod_{i=0}^{\pi(q_x)-1} \|Df|_{E_{f^i(q_x)}}\| \leq K^{2m_0-1} \cdot \lambda_{2}^{2m_1-2m_0+2} \leq \lambda_3^{\pi(q_x)}.
$$

Hence, one has that

$$
\prod_{i=0}^{n\pi(q_x)-1} \|Df|_{E_{f^i(q_x)}}\| \leq \lambda_3^{n\pi(q_x)}, \text{ for any } n \in \mathbb{N}.
$$

Let $\mathcal{P} = \{q_x : x \in B\}$, then $\mathcal{P} \subseteq H(p)$ is a set of hyperbolic periodic points homoclinically related to the orbit of $p$. For $\lambda_3 < \lambda_4$, any $q \in \mathcal{P}$ and any $n \in \mathbb{N}$, by the Lemma 2.1.2, there are $\theta = \theta(\lambda_3, \lambda_4) \in (0, 1)$ and positive integers $n_1 < n_2 < \cdots < n_l \leq n$ with $l \geq \theta n \pi(q)$, such that

$$
\prod_{i=k}^{n_j-1} \|Df|_{E_{f^i(q)}}\| \leq \lambda_4^{n_j-k}, \text{ for } \forall \; k \in \{0, 1, \ldots, n_j-1\}, \; j = 1, 2, \ldots, l.
$$

Since $q$ is a periodic point, if $n \to \infty$, then there is a point $q'$ which is an iteration of $q$, such that

$$
\prod_{i=0}^{m-1} \|Df|_{E_{f^i(q')}}\| \leq \lambda_4^m, \text{ for any } m \in \mathbb{N}.
$$

It means that $q'$ has stable manifolds of $\delta$-size, where $\delta$ is only relate to $\lambda_4$. Since $E$ is thin trapped, for every $y \in \{q', \ldots, f^{\pi(q')-1}(q')\}$ and $\delta > 0$, one has that $f^i(W^s_\delta(y)) \subseteq W^s_\delta(f^i y) = W^s_\delta(q')$, for some $i \in \{0, 1, \ldots, \pi(q') - 1\}$. Since $W^s_\delta(q') = W^s_\delta(q')$, the point $y$ also has stable manifolds of $\delta$-size. Therefore, every point $x \in \mathcal{P}$ has stable manifolds of uniformly size.

For any $b \in H(p)$, by the choice of the set $B$, there is $x \in B$ such that $d(b, x) < \frac{\varepsilon}{2}$. For this point $x$, there is a $q_x \in \mathcal{P}$ such that $d(x, q_x) < \frac{\varepsilon}{2}$. Then, $d(b, q_x) < \varepsilon$. Therefore, $\mathcal{P}$ is a $\varepsilon$-dense subset of $H(p)$.

\[ \square \]

3 The closing property of thin trapped in $H(p)$

It is well-known that pseudoorbits near a hyperbolic set can be shadowed by a real orbit. This is called Pseudo-Orbit Tracing Property. This property plays an important role in the study of stability of dynamical systems (see \cite{12}, \cite{4}, \cite{13} and \cite{10} ). Gan \cite[Theorem 1.1]{5} showed that quasi-hyperbolic pseudoorbits can be shadowed by a real orbit. In this section, we introduce the Pseudo-Orbit Tracing Property in $H(p)$. Before heading to the main block, we clarify some notations and identify some constants.

**Definition 3.1.** Assume that the homoclinic class $H(p)$ admits the dominated splitting $T_{H(p)}M = E \bigoplus F$, for $x \in H(p)$, $n \in \mathbb{N}^+$, an orbit arc $(x, n) \triangleq \{x, f(x), \ldots, f^{n-1}(x)\}$ is called $\lambda$-thin trapped, if $E$ is thin trapped and

$$
\prod_{j=1}^{k} \|Df^{-1}|_{E_{f^{n-j}(x)}}\| \leq \lambda^k, \; \forall \; k = 1, \ldots, n - 1.
$$
Proof. Under our assumptions, one can get the conclusions from the Proposition 3.4, uniformly size. 

Definition 3.3. For \( \delta > 0 \) and \( N \in \mathbb{N}^+ \), a sequence of points \( \{x_1, x_2, \cdots, x_N\} \) is called \( \delta \)-shadowed by a periodic point \( x \), if \( N \) is the period of \( x \) and \( d(f^n(x), x_n) < \delta \), for \( 1 \leq n \leq N \).

From the next proposition, we can see that the thin trapped homoclinic classes \( H(p) \), which admit the dominated splitting \( T_{H(p)}M = E \bigoplus F \) with \( \dim(E) = \text{ind}(p) \), has local product structures.

Proposition 3.4. ( [3, Lemma 3.4]) Given \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for any \( x, y \in H(p) \) with \( d(x, y) < \delta \), \( W_{cs}^s(x) \) and \( W_{cu}^s(y) \) are transversally intersect at a single point belonging to \( H(p) \), where \( W_{cs}^s(x) \subset W^s(x) \) is centered at \( x \) with length \( 2\epsilon \), \( * = cs \) or \( cu \).

Hereafter, we assume that the homoclinic classes \( H(p) \) admits the dominated splitting \( T_{H(p)}M = E \bigoplus F \) with \( \dim(E) = \text{ind}(p) \). For this dominated splitting, \( E \) is thin trapped and \( f \) is uniformly \( F \)-expanding at the period on all periodic points homoclinically related to \( p \). One can find dense hyperbolic periodic points which has stable and unstable manifolds of uniformly size.

Lemma 3.5. Given \( \delta, \epsilon > 0 \), one can find a \( \delta \)-dense set \( P \subseteq H(p) \) of hyperbolic periodic points homoclinically related to the orbit of \( p \), such that for any \( x \in P \), \( W_{cs}^s(x) \) and \( W_{cu}^s(x) \) are the stable and unstable manifolds of \( x \), respectively.

Proof. Under our assumptions, one can get the conclusions from the Proposition 2.2.2 and Proposition 2.2.4.

From the Theorem 2.2.4, one can see that \( P \) is a hyperbolic set. By the Definition 1.2 and Definition 3.4, for the dominated splitting \( T_{H(p)}M = E \bigoplus F \), there are \( C > 0 \), \( \lambda \in (0, 1) \) such that for any \( x \in P \), one has that

\[
\| Df^n|_{E_x} \| \leq C \lambda^n, \quad \forall \ n \in \mathbb{N}^+ \text{ and } \prod_{j=1}^{\pi(x)} \| Df^{-1}|_{F_{f^{j}(x)}} \| \leq C \lambda^{\pi(x)}.
\]

Hereafter, taking \( n_0 \) be the smallest positive integer such that \( C \lambda^{n_0} \leq \frac{1}{2} \), we give the definition of \( (n_0, \varepsilon) \)-closed pseudoorbit for the constant \( \varepsilon \).

Definition 3.6. Assume that the homoclinic class \( H(p) \) admits the dominated splitting \( T_{H(p)}M = E \bigoplus F \). Given \( \lambda \in (0, 1) \) and \( \varepsilon > 0 \), a \( (\lambda, \varepsilon) \)-thin trapped closed pseudoorbit \( \{(x_i, n_i)\}_{i=0}^m \) is called \((n_0, \varepsilon)\)-closed pseudoorbit, if \( n_i \geq n_0 \), for \( i = 0, 1, \cdots, m \).

Theorem 3.7. Given \( \varepsilon > 0 \), there is a constant \( \delta_0 > 0 \) such that for any \( \delta \in (0, \delta_0) \), if a finite number of orbit segment \( \{(x_i, n_i)\}_{i=0}^m \subset P \) is a \((n_0, \delta)\)-closed pseudoorbit, then the \((n_0, \delta)\)-closed pseudoorbit can be \( \varepsilon \)-shadowed by a periodic point.

Proof. For the given \( \varepsilon > 0 \), by the Proposition 3.4, there exists \( \delta_1 \) such that for any \( x, y \in H(p) \) with \( d(x, y) < \delta_1 \), one has that

\[
W_{cs}^s(x) \cap W_{cu}^{cs}(y) \neq \emptyset \quad \text{and} \quad W_{cu}^{cu}(x) \cap W_{cs}^{cu}(y) \neq \emptyset.
\]
Applying the Proposition 3.4 again, for the constant \( \alpha = \min\{\varepsilon, \delta_1\} \), one obtains a constant \( \delta_2 \) such that
\[
W^{cs}_\alpha(x) \cap W^{eu}_\alpha(y) \neq \emptyset \quad \text{and} \quad W^{cu}_\alpha(x) \cap W^{cs}_\alpha(y) \neq \emptyset \quad \text{for any} \quad x, y \in H(p) \quad \text{with} \quad d(x, y) < \delta_2.
\]
Due to the Lemma 3.5, for this \( \delta_2 \), there is a \( \delta_2 \)-dense subset \( \mathcal{P} \subseteq H(p) \) of hyperbolic periodic points homoclinically related to the orbit of \( p \), such that for any \( x \in \mathcal{P} \), \( W^{cu}_\alpha(x) \) are the stable manifolds of \( x \), denote by \( W^{s}_\alpha(x) \) and \( W^{cu}_\alpha(x) \) are the unstable manifolds of \( x \), denote by \( W^{u}_\alpha(x) \).

Let \( \delta_0 = \frac{\delta_2 - \alpha/2}{2} \), given \( \delta \in (0, \delta_0) \), if a finite number of orbit segment \( \{(x_i, n_i)\}_{i=0}^m \subseteq \mathcal{P} \) is a \((n_0, \delta)\)-closed pseudoorbit, then we can construct a sequence of points which are the intersection points of some stable manifolds and \( W^{cu}_\alpha \) plaques.

Since \( \{(x_i, n_i)\}_{i=0}^m \subseteq \mathcal{P} \) is a \((n_0, \delta)\)-closed pseudoorbit, by the Proposition 3.4, one has that
\[
z_1 \in W^{nu}_\alpha(f^{n_1}x_1) \cap W^{s}_\alpha(x_2) \neq \emptyset.
\]
For the hyperbolic set \( \mathcal{P} \), by the Definition 1.2 and Definition 1.4, one has that \( ||Df^n|_{E_x}|| \leq C\lambda^n \), for any \( n \in \mathbb{N}^+ \). Therefore, one has that
\[
d(f^{n_2}(x_2), f^{n_2}(z_1)) \leq C\lambda^{n_2}d(x_2, z_1) \leq C\lambda^{n_0}d(x_2, z_1) \leq \frac{d(x_2, z_1)}{2} \leq \frac{\alpha}{2}.
\]
Consequently, one has that \( d(f^{n_2}(z_1), x_3) \leq d(f^{n_2}(x_2), f^{n_2}(z_1)) + d(f^{n_2}(x_2), x_3) \leq \delta_2 \). By the Proposition 3.4, one has that
\[
z_2 \in W^{cu}_\alpha(f^{n_2}z_1) \cap W^{s}_\alpha(x_3) \neq \emptyset.
\]
Similarly,
\[
z_i \in W^{cu}_\alpha(f^{n_i}z_{i-1}) \cap W^{s}_\alpha(x_{i+1}) \neq \emptyset, \quad \text{for} \quad i = 3, 4, \ldots, m - 1 \quad \text{and} \quad z_m \in W^{cu}_\alpha(f^{nm}z_{m-1}) \cap W^{s}_\alpha(x_1) \neq \emptyset.
\]
Since the compactness of the set \( H(p) \), the limits of \( \lim_{k \to +\infty} f^{k(\sum_{i=1}^m n_i)} (z_m) \) exists and the limits point \( z = \lim_{k \to +\infty} f^{k(\sum_{i=1}^m n_i)} (z_m) \) also belongs to the set \( H(p) \). Due to
\[
f^{\sum_{i=1}^m n_i}(z) = f^{\sum_{i=1}^m n_i} \left( \lim_{k \to +\infty} f^{k(\sum_{i=1}^m n_i)} (z_m) \right) = \lim_{k \to +\infty} f^{(k+1)(\sum_{i=1}^m n_i)} (z_m) = z,
\]
the point \( z \) is a periodic point. From our construction of the sequence of points which are the intersection points of some stable manifolds and \( W^{cu}_\alpha \) plaques and the properties of the \((n_0, \delta)\)-closed pseudoorbit, one has that the \((n_0, \delta)\)-closed pseudoorbit can be \( \varepsilon \)-shadowed by the periodic point \( z \).

\[\Box\]

4 Proof of the Main Theorem

We prove the Main Theorem by contradiction under the assumptions that

- \( p \) is a hyperbolic periodic point;
- \( H(p) \) admits a dominated splitting \( T_{H(p)}M = E \bigoplus_{\gamma} F \) with \( \dim(E) = \text{ind}(p) \);
- \( E \) is thin trapped and \( f \) is uniformly \( F \)-expanding at the period on the set of periodic points homoclinically related to \( p \);
- \( F \) is not uniformly expanding on \( H(p) \).
Building closed pseudoorbit.

**Lemma 4.1.** For every \( r \in (0, 1] \), there exists a \( r \)-obstruction point \( b \) in \( H(p) \).

**Proof.** (By contradiction.) If the lemma is wrong, then there is \( r \in (0, 1] \) such that there is no \( r \)-obstruction point in \( H(p) \) by the Definition 2.1.4. Therefore, for any \( x \in H(p) \), there exists \( n = n(x) > 0 \) such that \( \prod_{i=1}^{n} \|Df^{-1}|F(x)|\| < r^n \). Let \( U(x) \) be the neighborhood of \( x \) such that for any \( y \in U(x) \), one has that \( \prod_{i=1}^{n(x)} \|Df^{-1}|F(x)|\| < r^{n(x)} \). Since \( H(p) \) is a compact set, there exist finite points \( x_1, \cdots, x_m \) such that \( H(p) \subseteq \bigcup_{i=1}^{m} U(x_i) \). Let

\[
N = \max_{1 \leq i \leq m} \{n(x_i)\}, \quad C = \max_{x \in H(p)} \left\{ \frac{\|Df^{-1}|F(x)|\|}{r}, \cdots, \frac{\prod_{i=1}^{N} \|Df^{-1}|F(x)|\|}{r^N} \right\}.
\]

For any \( x \in H(p) \) and \( n \in \mathbb{N}^+ \), by splitting every orbit segment \((x, f^nx)\) in segments of the form \((f^nx, f^{n+1}x)\), one has that \( \prod_{i=1}^{n} \|Df^{-1}|F(x)|\| \leq Cr^n \). Hence \( F \) is uniformly expanding. This contradicts our hypothesis that \( F \) is not uniformly expanding on \( H(p) \). \( \square \)

Now we construct a periodic closed pseudoorbit \( P \) as follows. Hereafter, we fixed a sequence numbers \( 0 < \lambda < r_4 < r_3 < r_2 < r_1 \leq 1 \) and \( \varepsilon > 0 \). According to the Lemma 3.5, one can find a \( \varepsilon/2 \)-dense set \( \mathcal{P} \subseteq H(p) \) of hyperbolic periodic points homoclinically related to the orbit of \( p \), such that every point belonging to \( \mathcal{P} \) has stable and unstable manifolds of uniformly size. By the Lemma 4.1, there is a \( r_1 \)-obstruction point \( b_1 \in H(p) \). Therefore, by the Lemma 2.1.6, \( HT_1(q_0, \mu_2) \sim \infty \) as \( n \to \infty \). This means that there is a sequence periodic points in \( H(p) \) and the period of those periodic points tends to infinity. Consequently, we may assume that the period of every periodic point in the \( \varepsilon/2 \)-dense set \( \mathcal{P} \subseteq H(p) \) is large enough.

**Step 1.** By the Lemma 4.1, there is a \( r_1 \)-obstruction point \( b_1 \in H(p) \). Therefore, one can find a sequence \( \{q_n : q_n \sim p \} \) such that \( \lim_{n \to \infty} q_n = b_1 \). For \( \varepsilon/2 > 0 \), \( r_2 < \lambda < r_1 \leq 1 \), by the Lemma 2.1.5, there exists \( N_1 = N_1(\varepsilon, \lambda_1) > 0 \) such that if \( (x, f^{N_1}(x)) \) is a \( \lambda_1 \)-obstruction segment, then \( d(x, \Lambda(\lambda_1)) < \varepsilon/2 \). Taking a \( x_1 \in \mathcal{P} \) with \( \Gamma_2(x_1, \lambda_1) - \Gamma_1(x_1, \lambda_1) - 1 \geq N_1 \), we get uniform \( \lambda_1 \)-strings \((x_1, f^{HT_1(x_1, \lambda_1)}(x_1))\), \((f^{HT_1(x_1, \lambda_1)}(x_1), f^{\Gamma_1(x_1, \lambda_1)}(x_1))\) and \( \lambda_1 \)-obstruction segment \((f^{\Gamma_1(x_1, \lambda_1)}(x_1), f^{2\Gamma_1(x_1, \lambda_1)}(x_1))\). Then, there exists a \( \lambda_1 \)-obstruction point \( b_2 \in H(p) \) such that \( d(b_2, f^{\Gamma_1(x_1, \lambda_1)}(x_1)) < \varepsilon/2 \).

**Step 2.** For the \( \lambda_1 \)-obstruction point \( b_2 \in H(p) \), we also use \( \{q_n \} \) denote the sequence \( \{q_n : q_n \sim p \} \) such that \( \lim_{n \to \infty} q_n = b_2 \). For \( \varepsilon/2 > 0 \), \( r_2 < \lambda_2 < \lambda_1 < r_1 \), by the Lemma 2.1.5, there exists \( N_2 = N_2(\lambda_2, \varepsilon) > 0 \) such that if \( (x, f^{N_2}(x)) \) is a \( \lambda_2 \)-obstruction segment, then \( d(x, \Lambda(\lambda_2)) < \varepsilon/2 \). By the Lemma 2.1.6, \( HT_1(q_0, \mu_2) \sim \infty \) as \( n \to \infty \). Let \( B \) be the subset of \( \mathcal{P} \) such that for every \( q_n \in B \), one has that

\[
\prod_{i=1}^{N} \|Df^{-1}|F(x)|\| \leq Cr^n.
\]

where \( m = \inf_{x \in H(p)} \|Df^{-1}|F(x)|\| \). Taking \( x_2 \in B \) with \( \Gamma_2(x_2, \mu_2) - \Gamma_1(x_2, \mu_2) - 1 \geq N_2 \), we get uniform \( \lambda_2 \)-strings \((x_2, f^{HT_1(x_2, \lambda_2)}(x_2))\), \((f^{HT_1(x_2, \lambda_2)}(x_2), f^{\Gamma_1(x_2, \lambda_2)}(x_2))\) and \( \lambda_2 \)-obstruction segment \((f^{\Gamma_1(x_2, \lambda_2)}(x_2), f^{2\Gamma_1(x_2, \lambda_2)}(x_2))\). Therefore, there exists a \( \lambda_2 \)-obstruction point \( b_3 \in H(p) \) such that \( d(b_3, f^{\Gamma_1(x_2, \lambda_2)}(x_2)) < \varepsilon/2 \).
Step 3. For $\varepsilon > 0$, taking $\lambda_j$ with $r_4 < r_3 < r_2 < \cdots < \lambda_j < \lambda_{j-1} < \cdots < \lambda_2 < \lambda_1 < r_1$, where $j = 1, 2, \cdots$, by repeating the Step 2, one can get a sequence point $\{x_n\}$ satisfying the following properties:

- $(x_j, f^{HT_1(x_j, \lambda_j)}(x_j))$, $(f^{HT_1(x_j, \lambda_j)}(x_j), f^{\Gamma_1(x_j, \lambda_j)}(x_j))$ are uniform $\lambda_j$-strings;
- $(f^{\Gamma_1(x_j, \lambda_j)}(x_j), f^{\Gamma_2(x_j, \lambda_j)}(x_j))$ are $\lambda_j$-obstruction segment;
- $d(x_j, f^{\Gamma_1(x_j-1, \lambda_j-1)}(x_j-1)) \leq \varepsilon$;
- $r_3^{HT_1(x_j, \lambda_j)-1} \cdot m \cdot m^{\Gamma_1(x_j-1, \lambda_j-1)-HT_1(x_j-1, \lambda_j-1)} \geq r_4^{HT_1(x_j, \lambda_j)+\Gamma_1(x_j-1, \lambda_j-1)-HT_1(x_j-1, \lambda_j-1)}$;

From the above step, we construct a pseudoorbit which is not closed. As $H(p)$ is compact set, there exist $m_0 \in \mathbb{N}$ and $k > 0$, such that

$$d(f^{HT_1(x_{m_0}, \lambda_{m_0})}(x_{m_0}), f^{HT_1(x_{m_0+k}, \lambda_{m_0+k})}(x_{m_0+k})) < \varepsilon.$$ 

Let $K_j = HT_1(x_{m_0+j}, \lambda_{m_0+j})$, $j = 1, 2, \cdots, k$. Let $y_{m_0+j} = f^{HT_1(x_{m_0+j}, \lambda_{m_0+j})}(x_{m_0+j})$, $j = 0, \cdots, k-1$ and $L_j = \Gamma_1(x_{m_0+j}, \lambda_{m_0+j}) - HT_1(x_{m_0+j}, \lambda_{m_0+j})$, $j = 0, 1, \cdots, k-1$. Therefore, we get the closed pseudoorbit $P$ which is the union of uniform $\lambda_1$-strings as

$$(y_{m_0}, f^{L_0}(y_{m_0})), (x_{m_0+1}, f^{K_1}(x_{m_0+1})), (y_{m_0+1}, f^{L_1}(y_{m_0+1})), (x_{m_0+2}, f^{K_2}(x_{m_0+2})), \cdots, (y_{m_0+k-1}, f^{L_{k-1}}(y_{m_0+k-1})), (x_{m_0+k}, f^{K_k}(x_{m_0+k})).$$

where $y_{m_0+j} = f^{K_j}(x_{m_0+j})$ and $d(f^{L_j}(y_{m_0+j}), x_{m_0+j+1}) < \varepsilon$, $j = 0, 1, \cdots, k - 1$.

Estimation about periodic orbit. From the construction of closed pseudoorbit, by the Theorem 3.7, for $\delta > 0$, there exist $\eta_0 = \eta_0(\lambda, \delta) > 0$, such that for any $\eta \in (0, \eta_0]$, there exists a periodic point $\delta$-shadows $(\lambda, \eta)$-thin trapped closed pseudoorbit. In fact, we also has the following properties as the Lemma 4.2 for the periodic point.

**Lemma 4.2.** For the fixed $r_4 < r_3 < r_1$, there is a $\delta_0 > 0$ such that all those periodic points which $\delta_0$-shadows $(\lambda, \eta)$-thin trapped closed pseudoorbit have stable and unstable manifolds of uniformly size.

**Proof.** For the fixed $r_4 < r_3 < r_1$, by the Theorem 2.2.1, there is a constant $\delta_0$ such that for any $x \in B(q_n, \delta_0)$, one has that

$$\prod_{i=1}^{\Gamma_1(q_n, \lambda_n)} \|Df^{-1}|_{F(f^ix)}\| \geq r_4^{\Gamma_1(q_n, \lambda_n)} \quad \text{and} \quad \prod_{i=1}^{\Gamma_1(q_n, \lambda_n)} \|Df^{-1}|_{F(f^ix)}\| \leq r_1^{\Gamma_1(q_n, \lambda_n)}.$$ 

By the Theorem 3.7, for $\delta_0 > 0$, there exist $\eta_0 = \eta_0(\lambda, \delta_0) > 0$, such that for any $\eta \in (0, \eta_0]$, there exists a periodic point $\delta_0$-shadows $(\lambda, \eta)$-thin trapped closed pseudoorbit. In the construction of closed pseudoorbit, taking $\varepsilon = \eta_0/4$, one can construct $(\lambda, \eta_0)$-thin trapped closed pseudoorbit. Therefore, there exists this periodic point $y$ $\delta_0$-shadows $(\lambda, \eta_0)$-thin trapped closed pseudoorbit. This means $y \in B(q_n, \delta_0)$ for any $q_n$ in $(\lambda, \eta_0)$-thin trapped closed pseudoorbit. Therefore,

$$\prod_{i=1}^{\pi(y)} \|Df^{-1}|_{F(f^{i}(y))}\| \geq r_4^{\pi(y)} \quad \text{and} \quad \prod_{i=1}^{\pi(y)} \|Df^{-1}|_{F(f^i(y))}\| \leq r_1^{\pi(y)}.$$ 

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Since $T_{H(p)} M = E \bigoplus F$ is a dominated splitting, by the Definition 1.1, one can take an suitable norm such that $\|Df|_{E(x)}\| \cdot \|Df^{-1}|_{F(f(x))}\| \leq \lambda_0$, where $0 < \lambda_0 < \lambda < 1$. Hence

$$
\prod_{i=0}^{\pi(y)-1} \|Df|_{E(f^i(y))}\| \leq \frac{\lambda_0^{\pi(y)}}{\prod_{i=1}^{\pi(y)} \|Df^{-1}|_{F(f^i(y))}\|} \leq (\frac{\lambda_0}{\lambda})^{\pi(y)}.
$$

Taking $\lambda = \max\{\lambda_0/\lambda, r_1\}$, one has that

$$
\prod_{i=0}^{\pi(y)-1} \|Df|_{E(f^i(y))}\| \leq \lambda^{\pi(y)} \quad \text{and} \quad \prod_{i=1}^{\pi(y)} \|Df^{-1}|_{F(f^i(y))}\| \leq \lambda^{\pi(y)}.
$$

This means that those periodic points have stable and unstable manifolds of uniformly size.

According to the Theorem 2.2.4, by the choice of the set $\mathcal{P}$ of hyperbolic periodic points homoclinically related to the orbit of $p$, the hyperbolic periodic points belonging to $\mathcal{P}$ have stable and unstable manifold of uniformly size. By the Lemma 4.2, those periodic points which $\delta_1$-shadows $(\lambda, \eta)$-thin trapped closed pseudoorbit have stable and unstable manifolds of uniformly size. Therefore, there is a $\delta_1 > 0$ such that if periodic point $\delta_1$-shadows $(\lambda, \eta)$-thin trapped closed pseudoorbit, then periodic point is homoclinically related to the hyperbolic periodic points that given in the construction of the closed pseudoorbit. Therefore, for $\delta = \min\{\delta_0, \delta_1\}$, where $\delta_0$ is given in the Lemma 4.2, by the Theorem 3.7, the closed pseudoorbit $P$ can be $\delta$-shadowed by a periodic point $\tilde{p}$. And one also has that $W^s_\epsilon(\tilde{p}) \cap W^u_\epsilon(p) \neq \emptyset$ and $W^u_\epsilon(p) \cap W^s_\epsilon(\tilde{p}) \neq \emptyset$. This means that $\tilde{p}$ is homoclinically related to $p$. By the Lemma 4.2, one has the estimation $\prod_{i=1}^{\pi(\tilde{p})} \|Df^{-1}|_{F(f^i(\tilde{p}))}\| \geq r_1^{\pi(\tilde{p})}$. This contradicts that $f$ is uniformly $F$-expanding at the period on $H(p)$. Consequently, the assumption that $F$ is not uniformly expanding on $H(p)$, is invalid. Therefore, $F$ is uniformly expanding on $H(p)$. Here, we finish the proof of the Main Theorem.

Now, we give the proof of the Theorem A under the Main Theorem.

Proof. (The proof of Theorem A.) In the assumption of the Theorem A, for the dominated splitting $T_{H(p)} = E \bigoplus F$, we may assume that the splitting $F$ splits in $F = E^c \bigoplus E^u$. Therefore, the Main Theorem shows that $f$ is not uniformly $F$-expanding at the period on all periodic points homoclinically related to $p$.

For every $\varepsilon > 0$, taking $r < 1$ such that $\log(r^{-1}) < \varepsilon$. Since $f$ is not uniformly $F$-expanding at the period on all periodic points homoclinically related to $p$, there is a periodic point $q$ homoclinically related to $p$ such that $\prod_{i=1}^{\pi(q)} \|Df^{-1}|_{F(f^i(q))}\| \geq r_1^{\pi(q)}$. Since $\dim E^c = 1$, one has

$$
\|Df^{\pi(q)}|_{E^c(q)}\|^{-1} = \prod_{i=1}^{\pi(q)} \|Df^{-1}|_{E^c(f^i(q))}\| \geq \prod_{i=1}^{\pi(q)} \|Df^{-1}|_{F(f^i(q))}\| \geq r_1^{\pi(q)}.
$$

Therefore, $\frac{1}{\pi(q)} \log(\|Df^{\pi(q)}|_{E^c(q)}\|) \leq \varepsilon$.

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REFERENCES

References

[1] Bonatti, Christian and Díaz, Lorenzo J. and Viana, Marcelo, *Dynamics beyond uniform hyperbolicity*, Encyclopaedia of Mathematical Sciences, vol. 102, Springer-Verlag, Berlin, 2005, A global geometric and probabilistic perspective, Mathematical Physics, III.

[2] Bonatti, Christian and Gan, Shaobo and Yang, Dawei, *On the hyperbolicity of homoclinic classes*, Discrete Contin. Dyn. Syst., 25, (2009), no. 4, 1143–1162.

[3] Crovisier, Sylvain and Pujals, Enrique R., *Essential hyperbolicity and homoclinic bifurcations: a dichotomy phenomenon/mechanism for diffeomorphisms*, Invent. Math., 201, (2015), no. 2, 385–517.

[4] Gan, Shaobo, *The star systems $X^*$ and a proof of the $C^1$ $\Omega$-stability conjecture for flows*, J. Differential Equations, 163, (2000), no. 1, 1–17.

[5] Gan, Shaobo, *A generalized shadowing lemma*, Discrete Contin. Dyn. Syst., 8, (2002), no. 3, 627–632.

[6] Gourmelon, Nikolaz, *Adapted metrics for dominated splittings*, Ergodic Theory Dynam. Systems, 27, (2007), no. 6, 1839–1849.

[7] Hirsch, M. W. and Pugh, C. C. and Shub, M., *Invariant manifolds*, Lecture Notes in Mathematics, Vol. 583, Springer-Verlag, Berlin-New York, 1977, ii+149.

[8] Katok, Anatole and Hasselblatt, Boris, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications, Vol. 54, Cambridge University Press, Cambridge, 1995, With a supplementary chapter by Katok and Leonardo Mendoza, xviii+802.

[9] Palis, Jr., Jacob and de Melo, Welington, *Geometric theory of dynamical systems*, An introduction, Translated from the Portuguese by A. K. Manning, Springer-Verlag, New York-Berlin, 1982, xii+198.

[10] Pilyugin, Sergei Yu., *Shadowing in dynamical systems*, Lecture Notes in Mathematics, Vol. 1706, Springer-Verlag, Berlin, 1999, xviii+271.

[11] Pliss, V. A., *On a conjecture of Smale*, Differencialnye Uravnenija, 8, (1972), 268–282.

[12] Wen Lan, *On the $C^1$ stability conjecture for flows*, J. Differential Equation, 129, (1996), no. 2, 334–357.

[13] Wen Lan, *On the preperiodic set*, Discrete Contin. Dynam. Systems, 6, (2000), no. 1, 237–241.

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