Quantum Fusion of Strings (Flux Tubes) and Domain Walls

S. Bolognesi\textsuperscript{a}, M. Shifman\textsuperscript{a}, and M.B. Voloshin\textsuperscript{a,b}

\textsuperscript{a} William I. Fine Theoretical Physics Institute, University of Minnesota, 116 Church St. S.E., Minneapolis, MN 55455, USA

\textsuperscript{b} Institute of Theoretical and Experimental Physics, 117218, Moscow, Russia

Abstract

We consider formation of composite strings and domain walls as a result of fusion of two elementary objects (elementary strings in the first case and elementary walls in the second) located at a distance from each other. The tension of the composite object $T_2$ is assumed to be less than twice the tension of the elementary object $T_1$, so that bound states are possible. If in the initial state the distance $d$ between the fusing strings or walls is much larger than their thickness and satisfies the conditions $T_1 d^2 \gg 1$ (in the string case) and $T_1 d^3 \gg 1$ (in the wall case), the problem can be fully solved quasiclassically. The fusion probability is determined by the first, “under the barrier” stage of the process. We find the bounce configuration and its extremal action $S_B$. In the wall problem $e^{-S_B}$ gives the fusion probability per unit time per unit area. In the string case, due to a logarithmic infrared divergence, the problem is well formulated only for finite-length strings. The fusion probability per unit time can be found in the limit in which the string length is much larger than the distance between two merging strings.
1 Introduction

In this paper we will consider two problems of practical interest which arise in various settings, and can be solved purely quasiclassically. The formulation of the problems, and their solution, is very general. They refer to restructuring of solitonic objects (or branes) supported in various field theories. We were motivated by a specific problem that arose in [1], but here we will give a general discussion, and find a generic solution, so that our results can be used in all similar situations.

The first problem is about strings (flux tubes). Suppose we have two types of strings: “elementary” strings with tension $T_1$, and a composite string with tension $T_2$. We assume that the composite string is a bound state, i.e.

$$T_2 - 2T_1 < 0.$$  

(1)

By composite we mean that there is a conserved “charge” $Q$, and $Q = 1$ for the elementary string while $Q = 2$ for the composite one. The composite string can form as a result of a fusion of two elementary ones.

One can consider two parallel strings at a distance $d$ from each other. The parameter $d$ is assumed to be much larger than the string thickness. Quantum fluctuations of strings can result in a configuration with two elementary strings forming a composite one in the middle (see Fig. 1). A crucial characteristic of the process is a critical size $\ell_c$ of the merged segment. It is determined by the balance of two energies: the one gained due to the fact that $T_2 - 2T_1 < 0$ (the energy gain is $\ell_c (2T_1 - T_2)$), and the one lost due

![Figure 1: Two elementary strings merge into a composite one. Once the size $\ell$ of the merged segment reaches (in a quantum tunneling process) its critical value $\ell_c$, further expansion of its size becomes classical.](image)

to elongation of the strings. The first stage of dynamics producing a size $\ell_c$ merged segment occurs as a quantum tunneling, which can be described in the Euclidean space-time. Once the critical-size segment is attained, its further expansion proceeds as a purely classical process, with positive energy release and an accelerating expansion of the merged segment. The fusion probability is determined by the quantum tunneling stage.

The action corresponding to such fusion is large, provided

$$T_1 d^2 \gg 1.$$  

As was mentioned, we gain energy in the central domain because $T_2 < 2T_1$. We lose outside the central domain because of the string bending needed to match the asymptotic boundary conditions. An extremal (in fact, maximal) value of the action must exist. It is realized on a classical solution in Euclidean. This is a maximum with respect to the size variation $\ell$. This is the only instability, and is the usual one that gives the vacuum decay.

The problem is similar, in a sense, to that of metastable vacuum decay [2, 3] (for reviews see [4, 5]), but with an important difference (see Appendix A). In the false vacuum decay the energy balance is achieved between a bubble of a genuine vacuum (gain) versus the potential energy of its surface (loss). The Euclidean solution is provided by a bounce configuration. In the problem at hand, in which the three-string junction is assumed to carry no energy, the barrier is not due to the potential energy, but is rather associated with the kinetic energy term in the string Lagrangian. However, this is just a technical difference. A critical field configuration extremizing the Euclidean action still exists, and we will find it in the limit when the string thickness is negligible compared to the interstring distance $d$, see Fig. 1. This is an analog of the standard bounce [4].

In the formulation of the string fusion problem there are infrared subtleties related to the tails of the strings. We can define the problem in a

\begin{itemize}
  \item[\footnotesize1] In the present work it is assumed that the three-string junction mass is negligible.
  \item[\footnotesize2] This is due to the fact that at $\ell < \ell_c$ the energy gain is less than the energy loss; therefore the system under consideration tunnels under a barrier.
  \item[\footnotesize3] We will apply this term, bounce, to the extremal Euclidean string and wall configurations in the problems to be discussed below.
\end{itemize}
finite box, of length $L$, and compute, with exponential accuracy, the probability per unit of time of this fusion process $\Gamma(L)$. This is the content of Section 2.

The result of the above computation immensely simplifies if one calculates the exponent in a logarithmic approximation. In this approximation it turns out possible (in the limit $2T_1 - T_2 \ll T_1$) to generalize the analysis of the parallel string fusion to cover the case of nonparallel strings. This problem will be addressed in Sect. 3.

The third problem we will deal with, in Section 4, is similar in nature, but it refers to parallel domain walls, rather than strings. Adding an extra dimension to the solitonic objects to be fused has a crucial effect. The infrared problem we had to deal with in the case of the string fusion now disappears, even in the infinite volume. Then, we can readily calculate the fusion probability per unit time and per unit area, with the exponential accuracy. This is done in Sect. 4. In Sect. 5 we deal with the wall merger at strong binding. An instructive example of the domain wall fusion in super-Yang–Mills theory is considered in Sect. 6.

It is not difficult to generalize our analysis to branes with arbitrary $p$ spatial dimensions, usually called $p$-branes, in a space-time with $D + 1$ dimensions. We outline this, and summarize our results in Sect. 7.

In summary, our solution of the string/wall fusion problem is general and independent of dynamics of the underlying microscopic theory provided the following assumptions are met: (i) $\delta/d \rightarrow 0$ where $\delta$ is the string or wall thickness; (ii) $T_1 d^2$ (in the string case) or $T_1 d^3$ (in the wall case) $\gg 1$; (iii) the three-string (or three-wall) junction contribution to the extremal action is negligible. The latter condition is met in many instances of practical interest. In the string case we must also assume that $L \gg d$. At weak binding the constraint on $d$ softens; it is sufficient to require $T_1 d^3 \sqrt{\frac{T_1}{2T_1 - T_2}} \gg 1$.

## 2 Parallel Strings

To compute the decay probability, it is convenient to Wick-rotate the time direction. Then in the Euclidean space-time we have a problem of two static
2-branes, which can fuse due to quantum fluctuations. The Euclidean string action is the string tension multiplied by the area of the branes.

We want to find a bounce solution, which corresponds to two surfaces at asymptotic distance \( d \), and an interior “bubble” in which they overlap to form a bound state (Fig. 2). The tunneling rate is then determined by the difference \( S_B \) between the (Euclidean) action on the bounce configuration and that on the trivial configuration, with two flat world sheets for each string, one at \( z = d/2 \) and another at \( z = -d/2 \),

\[
\Gamma = C \exp(-S_B),
\]

where \( C \) is a pre-exponential factor. Getting this factor requires calculation of the path integrals over fluctuations around the bounce solution, as well as around the trivial flat world-sheet configurations. This issue will not be addressed here. In what follows we will discuss only the exponential factor determined by classical solutions to the Euclidean equations of motion.

One can see, however, that for infinite strings a bounce solution does not exist. This is due to an infrared peculiarity of two-dimensional surfaces. Whenever we pull such a surface in the perpendicular direction, it never becomes flat asymptotically. Its asymptotic behavior, from the solution to

\[\text{Fig. 2: World surface for two elementary strings forming a composite one. The Euclidean time is denoted by } \tau.\]
the Laplace equation, is always logarithmic. Thus, we have no chance to recover the required boundary condition, that at \( x, \tau \to \infty \) the \( z \) coordinate of the surface tends to \( \pm d/2 \).

This infrared behavior can be regularized provided we assume that \( z = \pm d/2 \) is achieved at some finite distance in the \( \{x, \tau\} \) plane. The most physically transparent regularization of the Euclidean version of the problem consists of a strip (infinite in the \( \tau \) direction), with the boundary conditions \( z = \pm d/2 \) implemented at its edges (Fig. 3). We parametrize the coordinates as \( x, \tau \) and \( z \), where \( x = \mp L/2 \) present two edges of the strip, \( \tau \) corresponds to the Euclidean time, and \( z = \pm d/2 \) are the vertical locations of the two parallel strings, so that the boundary condition for the bounce configuration sought for is as follows: at \( x = \pm L/2 \) the value of \( z \) is fixed at \( +d/2 \) for one string and at \( -d/2 \) for the other. We use \( x \) and \( \tau \) to parametrize the brane, and \( z = f(x, \tau) \) determines the height of the branes. In fact, it is sufficient to consider only the upper side of the picture since it is symmetric under reflection \( z \to -z \). This will be referred to as a “strip” boundary condition.

Below we will find that \( S_B \) depends on \( L \) only through \( \ln L \). Aiming at logarithmic accuracy (i.e. keeping \( \ln L \) and omitting nonlogarithmic constants assuming \( \ln L \) to be large), we can replace the strip boundary conditions by much simpler ones, to be referred to as “round” boundary conditions (Fig. 3). The round boundary condition is convenient for two reasons. First, it will help us to calibrate our solution. Second, the results obtained with the round boundary condition are useful in extending the problem to the case of non-parallel strings. The problem with the round boundary conditions is that, by itself, it has no Minkowski physical interpretation. In the Euclidean space, instead, it is just a problem of fusion due to thermal fluctuations, with the position of the 2-branes fixed at the circle.

2.1 String fusion with round boundary conditions

In this section we will require \( z(R) = \pm d/2 \) where

\[
R \gg d
\]

is assumed (for the definition of \( R \) see Fig. 3). In the leading logarithmic (in \( R \)) approximation the problem of merging of two parallel strings can be
solved for arbitrary relation between the tensions $T_2$ and $T_1$ as long as the merger is possible, i.e. $T_2 < 2T_1$. We do not have to require $2T_1 - T_2 \ll T_1$. Thus, in this section we lift this constraint.

Thus, we replaced the strip space-time boundary for the world sheets by a disk of a large radius $R \sim L/2$, so that a bounce centered at the origin (i.e. $x = 0$ and $\tau = 0$) is $O(2)$ axially symmetric and is described by a function $z(r)$ where

$$r = \sqrt{x^2 + \tau^2}. \quad (5)$$

The slice of the solution $z(r)$ passing through the $x = \tau = 0$ line is shown in Fig. III (where $\ell_c = 2r_c$ and $L$ must be replaced by $2R$). The contribution of each string’s world sheet to the action for such a centrally symmetric configuration is given by the integral

$$S = 2\pi T \int r \, dr \sqrt{1 + z'^2} \quad (6)$$

with an appropriate tension $T$. Here $z' = dz/dr$, and the integrand represents the area of the circular element of the surface.
The central part of the bounce configuration, a disk of radius \( r_c \) located at \( z = 0 \), is filled by the string with tension \( T_2 \). The profile \( z(r) \) for each of the strings with tension \( T_1 \) is determined by the equations of motion, which extremize the surface area of two world sheets. The difference \( S_B \) of the action on the bounce and that on the trivial configuration can be thus written as

\[
S_B = \pi \left( T_2 - 2T_1 \right) r_c^2 + 4\pi T_1 \int_{r_c}^R r \, dr \left( \sqrt{1 + z'^2} - 1 \right),
\]

where \( z(r) \) stands for the vertical profile of one of the two world sheets (for definiteness we consider the upper one) and the contribution of the other simply doubles the coefficient in front of the integral in Eq. (7).

We find that the simplest way to analyze the solution for \( z(r) \) is using an “integral of motion”, which follows from the symmetry under \( z \) translation, which we call \( r_0 \):

\[
\frac{r \, z'}{\sqrt{1 + z'^2}} = r_0.
\]  

The left-hand side is independent on \( z \), which, in fact, tells us that the vertical component of the capillarity force acting on any horizontal section of the film is constant. The relation between the constant \( r_0 \) and the radius \( r_c \) of the bounce is found from the condition of equilibrium of the boundary of the disk, where the string \( T_2 \) bifurcates into two strings \( T_1 \). This condition is that the net horizontal force at the boundary vanishes,

\[
\frac{2T_1}{\sqrt{1 + z'^2}} \bigg|_{r=r_c} = T_2.
\]

After setting \( r = r_c \) in Eq. (8) and eliminating \( z' \big|_{r=r_c} \) from Eqs. (8) and (9), one readily finds the following relation:

\[
r_0 = r_c \sqrt{1 - \frac{T_2^2}{4T_1^2}}.
\]  

The solution to the equation of motion (8) satisfying the boundary condition
\[ z(R) = d/2 \] at \( R \gg r_c \) has the form\(^5\)

\[ z = r_0 \ln \frac{r + \sqrt{r^2 - r_0^2}}{2R} + \frac{d}{2}. \] (11)

The parameter \( r_c \) can be determined from the condition \( z(r_c) = 0 \) in terms of \( d \) and \( R \). To this end we substitute Eq. (10) in

\[ \ln \frac{2R}{r_c + \sqrt{r_c^2 - r_0^2}} = \frac{d}{2r_0}, \] (12)

which can be solved numerically. Figure 4 presents \( r_c/d \) as a function of \( R/d \) at a representative value of \( T_2/(2T_1) = 0.95 \). (A matching with Eq. (19) below starts emerging at the right edge of the plot.)

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\(^5\)To be more exact, in Eq. (10) \( z(R) = d/2 \) up to terms \( O\left(\frac{r_0^2}{R^2} \frac{d}{r_0}\right) \). As we will see shortly, roughly speaking, \( r_0 \sim d/\left[2 \ln(R/d)\right] \). Hence, up to logarithms, the relative error is \( O\left(\frac{d}{R^2}\right) \) and is negligible due to condition (11).
rather than \( r_c \), namely,
\[
S_B = \pi T_1 r_0 (d - r_0) .
\] (13)

Note that the boundary condition at \( r = r_c \) written in terms of \( r_0 \) reads
\[
\begin{align*}
    r_0 \ln \left( \frac{r_0}{2R} \sqrt{\frac{2T_1 + T_2}{2T_1 - T_2}} \right) &= -\frac{d}{2} .
\end{align*}
\] (14)

Equations (13) and (14) provide the solution for the exponential factor in the probability of the merger of two strings for arbitrary ratio of the tensions \( T_2/(2T_1) \). In particular at \( T_2 = 0 \) the problem is equivalent to that of spontaneous reconnection of two parallel strings [6]. From (10) we learn that in this limit
\[
r_0 = r_c ,
\] (15)
and the bounce configuration is described by a configuration discussed in [7]. In the present paper we will focus on the opposite limit in which the binding of the strings is parametrically small, i.e.
\[
2T_1 - T_2 \ll T_1 .
\] (16)

It is convenient to introduce a dimensionless small parameter \( \delta \) for the binding,
\[
\delta = \sqrt{1 - \frac{T_2^2}{4T_1^2}} .
\] (17)

One can readily verify that in the limit \( \delta \ll 1 \) the gradient of the deviation of the string profile from a flat string is small, and the equations for the profile of the string world sheet in the bounce configuration can be linearized. This allows one to consider the fusion problem in a more physical strip geometry, i.e. with the strip boundary conditions. This can be done both for parallel strings and slightly nonparallel ones.

### 2.2 Linearizing the problem in the weak binding limit

In the linearized approximation (valid if \( |\nabla z| \ll 1 \)) the classical equation of motion for the string profile is
\[
\Delta z = 0 .
\] (18)
If $\delta \ll 1$ the above condition is met. It is not difficult to solve Eq. (18) with the round boundary condition. Alternatively, one can expand the full nonlinear solution in the disk geometry. One finds in the leading logarithmic in $R$ approximation at $\delta \ll 1$

$$r_0 = r_c \delta \quad \text{and} \quad r_0 \approx \frac{d}{2 \ln(R\delta/d)}.$$  \hspace{1cm} (19)

Then the action $S_B$ can be readily derived from Eq. (13), namely,

$$S_B \approx \pi T_1 \frac{d^2}{2 \ln(R\delta/d)} \left\{1 + O\left(\frac{1}{\ln(R/d)}\right)\right\}. \hspace{1cm} (20)$$

Please, remember that the exponent determining the decay rate is $e^{-S_B}$. The condition (2) justifies the quasiclassical approximation.

It is interesting to note that the bounce action is mainly determined by the tension $T$ and the distance $d$, rather than by the binding parameter $\delta$. The formula (20) also tells us that the bounce action becomes small, and the semiclassical treatment becomes inapplicable, for exponentially long strings. However, it is clear from the overall proportionality of the fusion rate to the string length that for such long strings the probability of fusion becomes of order one. It can also be readily verified that introducing of a small mass $\mu$ for the three-string junction, neglected throughout this paper, does not change the infrared dependence of the bounce action. Indeed, the $\mu$-induced contribution to the action is

$$\Delta S_B = 2\pi \mu r_c \approx \frac{\pi \mu d}{\delta \ln(R\delta/d)}, \hspace{1cm} (21)$$

which has the same logarithmic behavior at large $R$ as $S_B$ in Eq. (20). Thus, the condition under which the junction mass can be neglected does not depend on the string length and reduces to

$$\mu \ll d \delta T_1. \hspace{1cm} (22)$$

### 2.3 Strip boundary condition in linear approximation

In the linear approximation in which the equation of motion for $z(x, \tau)$ reduces to the two-dimensional Laplace equation (18), the solution can be
constructed as a real (or imaginary) part of a holomorphic function of the complex variable

\[ w = x + i\tau. \]  

(23)

Using this construction and the analogy with two-dimensional electrostatics (the so-called image charges method, see Appendix B), one can readily find the solution for the strip \(-L/2 \leq x \leq L/2\) with the boundary conditions \(z(\pm L/2) = d/2\). For the bounce centered at \(x = l\) and \(\tau = 0\) the solution has the form

\[ z(x, \tau) = r_0 \operatorname{Re} \left[ \ln \left( \frac{\sin \frac{\pi(x-l)}{2L}}{\cos \frac{\pi(x+l)}{2L}} \right) \right] + \frac{d}{2}, \]  

(24)

with the constant \(r_0\) being determined by the condition of equilibrium of the bifurcation boundary, corresponding to \(z = 0\), similarly to Eq. (10). Clearly, in the strip geometry the \(O(2)\) symmetry is lost and the world sheet boundary for the string \(T_2\) is no longer a disk. However for large \(L\) and for the bounce center not too close to the strip edge, the exact solution (24) can be approximated by a logarithmic one,

\[ z \approx r_0 \operatorname{Re} \left[ \ln \left( \frac{w-l}{L} \right) \right] + \text{const}. \]  

(25)

It corresponds to an approximately circular bifurcation boundary with the radius \(r_c\) related to the parameter \(r_0\) as in Eq. (19). The applicability conditions for this approximation are that \(\ell\) is not parametrically close to \(L/2\) and also \(L \gg r_c\). The bounce action \(S_B\) on such configuration in the logarithmic in \(L\) approximation coincides with that in Eq. (20) with \(R\) being replaced by \(L\),

\[ S_B \approx \pi T_1 \frac{d^2}{2 \ln(L\delta/d)} \left\{ 1 + O\left( \frac{1}{\ln(L/d)} \right) \right\}, \]

\[ \delta \approx \sqrt{\frac{2T_1 - T_2}{T_1}}. \]  

(26)

3 Nonparallel strings

When the number of the space-time dimensions is \(3 + 1\) (or more), it is possible to have nonintersecting and nonparallel strings. Then geometry of
the problem can be characterized by two parameters: the minimal distance \( d \), and an angle \( \alpha \), which we will assume to be small (Fig. 5). We chose the spatial axes in the following way, the \( z \) axis runs along the common perpendicular to the strings, and the origin of the coordinates is placed in the middle of the segment of this perpendicular connecting the strings. The \( x \) axis runs along the bisector of the angle between the projection of the strings on the \( \{x, y\} \) plane, so that the strings are parametrized as \((\alpha \ll 1)\)

\[
\begin{align*}
  z &= d/2, \quad y = \tan \frac{\alpha x}{2}, \\
  z &= -d/2, \quad y = -\tan \frac{\alpha x}{2}.
\end{align*}
\] (27)

We will refer to this configuration as “twisted strings.” In choosing the infrared regularization we aim at logarithmic accuracy of the bounce action, so that it is sufficient to consider the axially symmetric geometry of the world sheet with a large radius \( R \), i.e. the round boundary conditions (Fig. 3). Namely, the following constraints will be imposed:

\[
 z(R) = \pm \frac{d}{2}
\] (28)

Figure 5: Geometry for nonparallel strings.
for two strings under consideration, while the boundary conditions for $y$ are

$$y(x, \tau) \bigg|_{r=R} = \pm \frac{\alpha}{2} x .$$

(29)

In the linearized approximation the equations of motion for the orthogonal deviations of the string $z(x, \tau)$ and $y(x, \tau)$ are independent from one another. If $\alpha$ is small, “twisting” in the variable $y$ can be considered as a perturbation over the solution $z(x, \tau)$ for parallel strings (Sect. [2]). It is clear from the symmetry of the problem that $y$ must identically vanish in the central part of the bounce, i.e. at $r \leq r_c$ one has $y = 0$. Then between the circles at $r = r_c$ and $r = R$ the function $y$ is harmonic and changes from $y(r_c) = 0$ to the values prescribed by the boundary conditions (29). Invoking the well-known central harmonics for the two-dimensional Laplace operator, we find the explicit form of the profile $y(x, \tau)$ for the upper and the lower strings,

$$y = \pm \frac{\alpha}{2} \frac{x}{1 - r_c^2/R^2} \left(1 - \frac{r_c^2}{r^2}\right).$$

(30)

Thus, the twist in the $y$ direction results in an additional contribution to the (linearized) bounce action which takes the form

$$\delta_y S_B = \frac{7\pi \alpha^2}{16} T_1 r_c^2 .$$

(31)

Using the relations (19) for $r_c$ one finds that the fusion probability for twisted strings $\Gamma(\alpha)$ is reduced in comparison with the parallel strings,

$$\Gamma(\alpha) = \Gamma(0) \exp \left[-\frac{7\pi \alpha^2}{64 \delta^2} T_1 \frac{d^2}{\ln^2(R\delta/d)}\right].$$

(32)

Here $\Gamma(0)$ is the merger probability for the parallel strings. It can be noted that, although the contribution to the exponential factor associated with the twist is of a higher order in $1/\ln(Rd^{-1})$ than that in $\Gamma(0)$, it has a nontrivial singular dependence on $\delta$. The latter dependence implies that the string merger at weak binding takes place only if the angle between them is also small, i.e. at $\alpha < \alpha_{\text{max}}$ where

$$\alpha_{\text{max}} \sim \delta \frac{\ln(R\delta/d)}{\sqrt{T_1 d}} .$$

(33)
Fusion of parallel domain walls

We will analyze the domain wall fusion at $2T_1 - T_2 \ll T_1$, when the linearized approximation is applicable. In contradistinction with the string problem, in the domain wall problem we do not need any infrared regularization, since the solution of the three-dimensional Laplace equation falls off as $1/r$ rather than logarithmically. The coordinates that parametrize the Euclidean 3-brane are $x$, $y$, $\tau$, while the walls are given by the height functions $z = f(x, y, \tau)$. The boundary conditions are set at infinite $r = (x^2 + y^2 + \tau^2)^{1/2}$,

$$z(r) \to \pm \frac{d}{2} \quad \text{at} \quad r \to \infty. \quad (34)$$

The solutions of the linearized equation $\Delta z = 0$ for the top and bottom walls are

$$z_1(x, y, \tau) = -\frac{A}{r} + \frac{d}{2},$$

$$z_2(x, y, \tau) = \frac{A}{r} - \frac{d}{2}, \quad (35)$$

where the bounce is assumed to be centered at the origin. The two $T_1$ branes meet at $r = r_c$ where

$$r_c = \frac{2A}{d}. \quad (36)$$

It is obvious that $r_c$ is the radius of the word volume of the composite wall (i.e. the world volume radius of the $T_2$ brane configuration) at the moment it leaves Euclidean and enters the Minkowski space ($\tau = 0$).

The total Euclidean action is the sum of two contributions

$$S = (T_2 - 2T_1) \frac{4\pi}{3} r_c^3 + 2T_1 \int_{r_c}^{\infty} 4\pi r^2 dr \frac{z'^2}{2}$$

$$= - (2T_1 - T_2) \frac{4\pi}{3} r_c^3 + T_1 \pi r_c d^2, \quad (37)$$

where the first one comes from the composite wall in the middle, while the second from two tails of elementary walls. The action (37) is regularized: we subtracted the contribution of two parallel undistorted walls (Fig. (6a)). In deriving this action we used Eq. (36).
Please, note that the signs are opposite. The first term is negative, and dominant at large $r_c$. The second is positive and dominant at small $r_c$. The bounce solution which is at the tip of hill can be obtained by extremizing Eq. (37) with respect to $r_c$,

$$
 r_c = \frac{d}{2} \sqrt{\frac{T_1}{2T_1 - T_2}}, \\
 S_B = \frac{\pi}{3} T_1 d^3 \sqrt{\frac{T_1}{2T_1 - T_2}}.
$$

The probability of the wall fusion per unit time and unit area is proportional to $e^{-S_B}$.

Now we can check that the linearization approximation is valid. The necessary condition is $|z'| \ll 1$ which is equivalent to $A/r_c^2 \ll 1$. Equations (36) and (38) imply

$$
\frac{A}{r_c^2} \sim \frac{d}{r_c} \sim \sqrt{\frac{2T_1 - T_2}{T_1}}.
$$

This condition is met at weak binding, i.e.

$$
\frac{2T_1 - T_2}{T_1} \ll 1.
$$

Note that it does not depend on the inter wall distance $d$. However, the distance must be much larger than the wall thickness. It must be large
enough to ensure $S_B \gg 1$. In particular, the choice $T_1 d^3 \gg 1$ does the job. Another condition that was assumed in the consideration above is that the tension of the three-wall junction (closed circles on Fig. 6b) is negligible, so that the junction contribution to the action can be ignored.

5 Parallel walls at strong binding

If Eq. (40) is not satisfied, the wall binding is strong, and the bounce action must be treated beyond the linear approximation,

$$S_B = \frac{4\pi}{3} (T_2 - 2T_1) r_c^3 + 2T_1 \int_{r_c}^{\infty} 4\pi r^2 dr \left( \sqrt{1 + z'^2} - 1 \right). \quad (41)$$

It is not difficult to see that there exists an “integral of motion” analogous to (8),

$$\frac{r^2 z'}{\sqrt{1 + z'^2}} = r_0^2. \quad (42)$$

The solution for the wall bounce must be such that the left-hand side of (42) is $r$ independent. We use the notation $r_0^2$ for this constant. Then the classical equation of motion reduces to

$$\frac{dz}{dr} = \pm \frac{r_0^2}{\sqrt{r^4 - r_0^4}}, \quad (43)$$

implying the following relation between the parameters $r_0$, $r_c$ and $d$:

$$\frac{r_0^2}{r_c} \int_{r_c}^{\infty} \frac{dr}{\sqrt{r^4 - r_0^4}} = \frac{d}{2}. \quad (44)$$

Another relation between $r_0$ and $r_c$ is the equilibrium condition coinciding with Eq. (9). Considering the expression (42) for the integral of motion in the wall case, one readily finds from this condition that

$$r_0^4 = r_c^4 \left( 1 - \frac{T_2^2}{4T_1^2} \right). \quad (45)$$
Invoking Eqs. (42) and (43), the bounce action in Eq. (41) can be transformed to

\[ S_B = - (2T_1 - T_2) \frac{4\pi}{3} r_c^3 + 8\pi T_1 \int_{r_c}^{\infty} dr \left( \frac{r^4}{\sqrt{r^4 - r_0^4}} - r^2 \right) \]

\[ \equiv \frac{4\pi}{3} T_2 r_c^3 + \frac{8\pi}{3} T_1 \int_{r_c}^{\infty} dr \frac{r_0^4}{r^4 - r_0^4} \]

\[ - \frac{8\pi}{3} T_1 \left[ r_c^3 - \int_{r_c}^{\infty} dr \left( \frac{3r^4 - r_0^4}{\sqrt{r^4 - r_0^4}} - 3r^2 \right) \right] . \]  

The second term in the first line is an elliptic integral which we will replace by its value mandated by the condition (44). The expression in the square brackets in the second line reduces to a combination of elementary functions. Indeed,

\[ r_c^3 - \int_{r_c}^{\infty} dr \left( \frac{3r^4 - r_0^4}{\sqrt{r^4 - r_0^4}} - 3r^2 \right) = r_c \sqrt{r_c^4 - r_0^4} . \]  

This can be directly verified by differentiating both sides in Eq. (47) over \( r_c \).

As a result, the bounce action takes the form

\[ S_B = \frac{4\pi}{3} \left( T_2 r_c^3 - 2 T_1 r_c \sqrt{r_c^4 - r_0^4} + T_1 r_0^2 d \right) . \]  

After substituting in the latter expression the relation (45) one readily finds that the first two terms in parentheses cancel, and one is left with a simple formula for the bounce action in terms of \( r_0 \) and \( d \),

\[ S_B = \frac{4\pi}{3} T_1 r_0^2 d . \]  

In order to express the parameter \( r_0 \) in terms of the separation distance \( d \) and the wall tensions \( T_1 \) and \( T_2 \) one needs to solve the transcendental equation (44) which results in an elliptic function. In the limit of weak coupling one recovers the results of the previous section, while in the extreme strong coupling limit, \( T_2 \to 0 \), one has \( r_c = r_0 \) and we find

\[ r_0 = \frac{\Gamma \left( \frac{3}{4} \right)}{2 \sqrt{\pi} \Gamma \left( \frac{5}{4} \right)} d = 0.381 \ldots d , \]  

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so that the bounce action can be written as

$$S_B \bigg|_{T_2=0} = \frac{1}{3} \left[ \frac{\Gamma \left( \frac{3}{4} \right)}{\Gamma \left( \frac{5}{4} \right)} \right]^2 T_1 d^3 = 0.609 \ldots T_1 d^3. \quad (50)$$

Thus, an estimate $S_B = \text{const.} \pi T_1 d^3 \delta^{-1}$ works in the entire range of variation of $T_2$ in the problem of the domain wall fusion: from $T_2 = 2T_1$ (where $\delta \to 0$) down to $T_2 = 0$.

6 An instructive example

In this section we will consider a particular example in which weakly bound domain walls naturally appear. Supersymmetric Yang–Mills theory supports [8] critical domain walls whose tensions are exactly known [8, 9]. Namely, the tension of the $k$-wall is given by the formula

$$T_k = N^2 \Lambda^3 \sin \left( \frac{\pi k}{N} \right) \quad (51)$$

for SU($N$) gauge group. Here $\Lambda$ is a dynamical scale parameter. The tension of the composite $k$ walls is less than $k$ times the elementary wall tension (i.e. the $k = 1$ wall).

Hence, at large $N$ we have

$$2T_1 - T_2 = \frac{\pi^3}{N} \Lambda^3, \quad T_1 = \pi N \Lambda^3. \quad (52)$$

Invoking Eq. (58) we conclude that the probability of two parallel wall fusion (per unit time per unit area) is proportional to

$$\Gamma \sim \exp \left[ -\frac{\pi}{3} N^2 (\Lambda d)^3 \right]. \quad (53)$$

7 Conclusions

In this paper we have considered a generic problem of restructuring (fusion) of solitonic objects, due to binding energy. We paid particular attention to
formulating, and solving, the problem in a generic way, without any reference to particular underlying theories, or mechanisms responsible for the soliton existence and binding (i.e. no reference to microscopic physics). In this way our results can be applied in every instance in which our assumptions (summarized at the very end of Introduction) are met. Although we have focused on strings and walls in $3 + 1$ dimensions, our results can be easily generalized to higher-dimensional branes and/or higher space-time dimensions. For example, if we consider two $p$ branes in $(p + 2)$-dimensional space-time, the fusion probability per unite volume and unit time is

$$\Gamma_p \sim \exp \left\{-C_p 2^p (p-1)^p \left(\frac{T_1}{2T_1 - T_2}\right)^{(p-1)/2} (T_1 d^{p+1})\right\}, \quad (54)$$

where $C_\ell$ is the volume coefficient,

$$C_\ell = \frac{\pi^{\ell/2}}{\Gamma \left(\frac{\ell}{2} + 1\right)}. \quad (55)$$

The problem for strings in $3 + 1$ dimensions has a peculiar infrared behavior, and requires a regularization. For domain walls, or higher-dimensional branes, no infrared regularization is needed.

The study and computation of the fluctuations around the bounce solution is left for future work. This problem is essential in order to compute the coefficient $\mathcal{C}$ in Eq. (3). In some circumstances, it is just a numerical coefficient, which have little impact on the physical behavior. In other cases it can be of crucial importance. Assume we want to study the problem of infinite, parallel strings. We thus take the problem considered in Sects. 2.1 and 2.2 and send the cutoff $R \to \infty$. In this case $r_c$ and $r_0$ both go to zero, and the bounce action formally vanishes. The center of the bounce is located at the center of the circle of radius $R$. Translation of the center becomes approximately a flat direction, as $R \to \infty$. This is a usual divergence that is absorbed in the volume dependence, so that in fact we compute the decay probability per unit time and unit length. But there is also another (nearly) flat direction – the one due to a rescaling $(x, \tau) \to \lambda(x, \tau)$. This is peculiar to the case of strings. In the weak binding limit, the right-hand side of (7) becomes (nearly) scale invariant. This extra (approximate) zero mode must be properly treated in calculating the factor $\mathcal{C}$. 

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Is worth to mention a particular example of binding energy between strings. It arises in type IIB string theory where \((p, q)\) strings are bound states of \(p\) F1-strings and \(q\) D1-strings. Networks of \((p, q)\)-strings have been studied recently, since they naturally arise at the end of some string theory inflation scenarios [11]. Tunneling effects may be relevant for the evolution of these strings networks.

Finally, we would like to mention that work on the D-brane fusion was carried out (e.g. [10]) in string theory, in a setting specific to string theory.

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\section*{Appendix A: Connection with field theory}

In the case of weak binding, and parallel branes, the problem can be recast in a simple field-theoretic formulation. Let us discuss it for the specific case of parallel strings. The “true” action is the Nambu-Goto one, with a tension that depends upon the distance between the two strings. For small perpendicular fluctuations \(z\), i.e. at weak binding, we can use \(x\) to parametrize the space coordinate on the world sheets of the strings, and the following action ensues:

\[
S = -T(z) \int dx d\tau \sqrt{1 - \partial_\mu z \partial^\mu z} \\
= \int dx d\tau \ T(z) \left(-1 + \frac{1}{2} \partial_\mu z \partial^\mu z + \ldots\right), \quad (A.1)
\]

where \(z\) is the distance between two strings, and \(T(z)\) is the “combined tension” as a function of the distance. If we define a scalar field \(\phi\),

\[
\phi = z \sqrt{T(z)}, \quad (A.2)
\]
assuming that the $z$ dependence of $T$ is adiabatically slow, we get the action in the form

$$S = \int dx d\tau \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right),$$

(A.3)

with the canonically normalized kinetic term and

$$V(\phi) = T(z(\phi)).$$

(A.4)

The shape of the effective potential $V(\phi)$ is very similar to $T(z)$, although the change of variables from $z$ to $\phi$ changes the functional dependence in the vicinity of the minimum at $z = 0$, Fig. 7. This effective potential is of a “false plateau” type. It is flat almost everywhere, apart from a small domain near zero, where it drops off by $2T_1 - T_2$.

Details of the potential shape near the minimum depend on microscopic physics that is responsible for the binding energy. The results presented in this paper do depend on these details. What was crucial was the fact that the effective potential is essentially constant and flat everywhere, and drops to zero at a very short distance (the string/wall thickness) from the origin.

![Figure 7: Potential as a function of the distance $z$.](image)

The flat plateaux means that we have a classical moduli space of vacua, which, in turn, corresponds to the fact that there are no long-range forces between the two strings (or walls). The tunneling is similar to the conventional false vacuum decay studied for metastable potentials [4]. A key technical difference is that a barrier is present only in the form of kinetic energy.

We can add the following regulator to the potential:

$$V_\epsilon = \epsilon z^2 (z - d)^2, \quad \epsilon \to 0.$$

(A.5)
The total potential $T(z) + V_\epsilon(z)$ maintains $z = 0$ as a true vacuum. The plateau is lifted, and $z = d$ is a metastable vacuum. In this formulation, the decay probability is determined by the conventional bounce of the type relevant to the false vacuum decay. At $\epsilon \to \infty$ the thin wall approximation is valid. On the other hand, at $\epsilon \to 0$ (the case we are interested in) we find ourselves completely outside of the thin wall approximation, albeit, as we see in the bulk of the paper, a bounce-like solution exists and can be explicitly found.

The bounce becomes exceedingly shallower as we decrease $\epsilon$. The number of space-time dimensions is crucial here. In $(1 + 1)$-dimensional theory, the solution asymptotically is logarithmic, and the boundary condition $z_1 - z_2 = d$ can only be imposed at a finite distance. In three or more dimensions the fall-off is power-like, and the boundary condition can be imposed at infinity.

**Appendix B: Holomorphic potentials**

A good way to directly derive Eq. (24) in Sect. 2.3 is provided by analogy with two-dimensional electrostatics through the well-known in electrostatics image charges method. To this end we use the trick of putting auxiliary “mirror” charges as shown in Fig. 8. We extend the strip to the entire complex $w = x + i\tau$ plane, put the original unit charge at $w = l$, and then add a series of ‘−1’ mirror charges at $w_k^- = (2k+1) L - l$, and a series of ‘+1’ mirror charges at $w_k^+ = 2k L + l$, with $k$ being any integer. (In fact $k = 0$ in the latter case corresponds to the original charge whose potential is being calculated with the strip boundary conditions.) In this way we certainly satisfy, due to the symmetry of the system of mirror charges, the boundary conditions that $z$ must vanish at $\text{Re } w = \pm L/2$. We then can find the solution for the harmonic function $z(x, \tau)$ in terms of the real part of a holomorphic potential $\Phi(w)$ produced by the constructed system of charges: $z = C_1 \text{Re}\Phi(w) + C_2$ with $C_1$ and $C_2$ being arbitrary constants. The holomorphic potential from
the considered system of charges is given by

$$\Phi(w) = \sum_{k \in \mathbb{Z}} \ln(w - w_k^+) - \sum_{k \in \mathbb{Z}} \ln(w - w_k^-)$$

$$= \ln \prod_{k \in \mathbb{Z}} (w - w_k^+) - \ln \prod_{k \in \mathbb{Z}} (w - w_k^-). \quad (B.1)$$

Using the Euler’s formula representing the sin function as a product, one can explicitly find the products in Eq. (B.1) up to inessential (although infinite) multiplicative constant:

$$\prod_{k \in \mathbb{Z}} (w - w_k^+) = \text{const} \cdot \sin \frac{\pi (w - l)}{2L},$$

$$\prod_{k \in \mathbb{Z}} (w - w_k^-) = \text{const} \cdot \cos \frac{\pi (w + l)}{2L}. \quad (B.2)$$
From here the harmonic function \((24)\) ensues. It is certainly easy to check, \textit{a posteriori}, that the function in Eq.\((24)\) satisfies the necessary boundary conditions in the discussed problem.

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