AVERAGE VALUES OF QUADRATIC HECKE CHARACTER SUMS

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Abstract. We study certain smoothed character sums involving \( \sum_{m,n} \left( \frac{m}{n} \right)_2 \), where \( \left( \frac{\cdot}{n} \right)_2 \) denotes the quadratic symbol in the Gaussian field. We extend some previously known results to obtain asymptotic formulas for the sums considered to larger ranges of \( m \) and \( n \).

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1. Introduction

An important theme in analytic number theory is the estimations of character sums. The classical Pólya-Vinogradov inequality (see for example [3, Chap. 23]) states that for any non-principal Dirichlet character \( \chi \) modulo \( q \) and \( M \in \mathbb{Z}, N \in \mathbb{N} \), we have

\[
\sum_{M < n \leq M+N} \chi(n) \ll q^{1/2} \log q.
\]

The above estimate has many interesting applications including the study of least quadratic non-residues, least primitive roots and primes in arithmetic progressions.

Another important topic studied in analytic number theory is the asymptotic behaviors of mean values. In view of the Pólya-Vinogradov inequality, we see that it is not possible to obtain an asymptotic formula for a single character sum. In order to obtain such formulas, we need to consider extra averagings.

In the case of quadratic Dirichlet character sums, we are led to consider the following sum:

\[
S(X, Y) = \sum_{m \leq X, (m, 1+i) = 1} \sum_{n \leq Y, (n, 1+i) = 1} \left( \frac{m}{n} \right)_2 \Phi \left( \frac{N(n)}{Y} \right) W \left( \frac{N(m)}{X} \right).
\]

Although it is relatively easy to obtain an asymptotic formula of \( S(X, Y) \) if \( Y = o(X/\log X) \) or \( X = o(Y/\log Y) \) with the help of the Pólya-Vinogradov inequality, the situation becomes much more subtle and delicate if \( X \) and \( Y \) are of comparable size. Using a Poisson summation formula developed in [9], J. B. Conrey, D. W. Farmer and K. Soundararajan [2] succeeded in obtaining an asymptotic formula of \( S(X, Y) \) for all \( X \) and \( Y \).

Thus one is motivated by the result of Conrey, Farmer and Soundararajan to study similar character sums. For example, let \( K = \mathbb{Q}(i) \) be the Gaussian field and \( \mathcal{O}_K = \mathbb{Z}[i] \) the ring of integers in \( K \). For every \( c \in \mathcal{O}_K \) such that \( (c, 1+i) = 1 \), let \( \left( \frac{\cdot}{c} \right)_2 \) be the quadratic residue symbols defined in Section 2.1. Suppose that \( \Phi(t), W(t) \) are two non-negative smooth functions compactly supported on \( \mathbb{R}^+ \), we define

\[
S_2(X, Y; \Phi, W) = \sum_{n \equiv 1 \mod (1+i)^3} \sum_{(m, 1+i) = 1} \left( \frac{(1+i)m}{n} \right)_2 \Phi \left( \frac{N(n)}{Y} \right) W \left( \frac{N(m)}{X} \right).
\]

A similar expression without the appearance of \( 1 + i \) in the quadratic symbol above has been studied previously by the authors in [5]. An asymptotic formula is obtained for such an expression, which is valid when \( Y = o(X) \) and \( X = o(Y^{2-\varepsilon}) \) for any \( \varepsilon > 0 \).

Our goal in this paper is to further continue our investigation on \( S_2(X, Y; \Phi, W) \), aiming at obtaining a valid asymptotic formula for all large \( Y \) going up to \( X \). The new input here is the method used by K. Soundararajan and M. P. Young in [10]. After using Poisson summation first to transform one of the sums in \( S_2 \) into a dual sum, we shall follow the approach in [10] to treat the resulting sums using multiple Dirichlet series. This allows us to better control
the emerging error terms.

Before stating the result, we note that, as pointed out in [3], one can regard \((\frac{\chi}{n})_4\) as a Hecke character \(\chi_m \pmod{16m}\) of trivial infinite type when \((m, 1 + i) = 1\), thus justifying our view of \(S_2\) as Hecke character sums. Moreover, the factor \((\frac{\chi}{n})_4\) is inserted in the expression of \(S_2\) for a purely technical reason. One can obtain a similar asymptotic formula for \(S_2\) without this factor.

Our main result is as follows.

**Theorem 1.1.** Let \(\Phi\) and \(W\) be two non-negative, smooth, compactly supported functions on \(\mathbb{R}^+\). Let \(S_2(X, Y; \Phi, W)\) be defined in (4.2). For large \(X\) and \(Y\) with \(X \geq Y\), \(\theta = \frac{131}{416}\), we have for any \(\varepsilon > 0\),

\[
S_2(X, Y; \Phi, W) = C(\Phi, W)XY^{1/2} + O \left( \frac{X}{X} \right),
\]

where \(C(\Phi, W)\) is given in (3.11).

We remark here that the result of Conrey, Farmer and Soundararajan on \(S(X, Y)\) defined in (1.1) is that

\[
S(X, Y) = \frac{2}{\pi^2} C \left( \frac{Y}{X} \right) X^{3/2} + O \left( \left( Y \right) \log XY \right).
\]

Moreover, it was shown in [2] that

\[
C(\alpha) = \sqrt{\alpha} + O \left( \alpha^{3/2} \right), \quad \alpha \to 0.
\]

It follows that when \(Y < X^{1-\varepsilon}\), we can recast \(S(X, Y)\) as

\[
S(X, Y) = \frac{2}{\pi^2} XY^{1/2} + O \left( \frac{Y^{3/2}}{Y} \right).
\]

The error term given in Theorem (1.1) is essentially of the same order of magnitude as that in the above expression. Moreover, the asymptotic formula given in (1.3) is now valid for all \(Y \leq X^{1-\varepsilon}\).

2. Preliminaries

2.1. Quadratic and quartic symbols. The symbol \((\frac{\cdot}{4})_4\) is the quartic residue symbol in the ring \(\mathbb{Z}[i]\). For a prime \(\pi \in \mathbb{Z}[i]\) with \(N(\pi) \neq 2\), the quartic symbol is defined for \(a \in \mathbb{Z}[i]\), \((a, \pi) = 1\) by \((\frac{a}{\pi})_4 = \epsilon(\pi^{-1})^{-1/4} \pmod{\pi}\), with \((\frac{\cdot}{4})_4 \in \{\pm 1, \pm i\}\). When \(\pi a\), we define \((\frac{\cdot}{4})_4 \equiv 0\). Then the quartic character can be extended to any composite \(n\) with \((N(n), 2) = 1\) multiplicatively. We extend the definition of \((\frac{\cdot}{4})_4\) to \(n = 1\) by setting \((\frac{\cdot}{4})_4 = 1\). We further define \((\frac{\cdot}{i}) = (\frac{\cdot}{4})_4^2\) to be the quadratic residue symbol for these \(n\)’s.

Note that in \(\mathbb{Z}[i]\), every ideal coprime to 2 has a unique generator congruent to 1 modulo \((1 + i)^3\). Such a generator is called primary.

2.2. Gauss sums. For any \(n, r \in \mathbb{Z}[i]\), \(n \equiv 1 \pmod{(1 + i)^3}\), the quadratic Gauss sum \(g_2(r, n)\) is defined by

\[
g_2(r, n) = \sum_{x \mod n} \left( \frac{x}{n} \right) \tilde{e}_i \left( \frac{rx}{n} \right), \quad \text{where} \quad \tilde{e}_i(z) = \exp \left( 2\pi i \left( \frac{z}{2i} \right) \right).
\]

The following property of \(g_2(r, n)\) can be easily derived from its definition.

\[
g_2(rs, n) = \left( \frac{\cdot}{n} \right)_2 g_2(r, n), \quad (s, n) = 1, \quad n \text{ primary}.
\]

We write \(N(n)\) for the norm of any \(n \in \mathbb{Z}[i]\) and \(\varphi(n)\) for the number of elements in the reduced residue class of \(\mathcal{O}_K/\langle n \rangle\). The next lemma allows us to evaluate \(g_2(r, n)\) for \(n \equiv 1 \pmod{(1 + i)^3}\) explicitly.

**Lemma 2.3.** [8 Lemma 2.4]

(i) For \(m, n \text{ primary} \) and \((m, n) = 1\), we have

\[
g_2(k, mn) = g_2(k, m)g_2(k, n).
\]
(ii) Let \( \varpi \) be a primary prime in \( \mathbb{Z}[i] \). Suppose \( \varpi^h \) is the largest power of \( \varpi \) dividing \( k \). (If \( k = 0 \) then set \( h = \infty \).)

Then for \( l \geq 1 \),

\[
g_2(k, \varpi^l) = \begin{cases} 
0 & \text{if } l \leq h \text{ is odd}, \\
\varphi(\varpi^l) = \#(\mathbb{Z}[i]/(\varpi^l))^* & \text{if } l \leq h \text{ is even}, \\
-N(\varpi)^{l-1} & \text{if } l = h + 1 \text{ is even}, \\
(k\varpi^{-h})N(\varpi)^{-1/2} & \text{if } l = h + 1 \text{ is odd}, \\
0 & \text{if } l \geq h + 2.
\end{cases}
\]

As an immediate consequence of the above lemma, we see that for any \( k, n \in \mathbb{Z}[i] \) with \( n \) primary,

\[
g_2(k, n) \ll N(n). \tag{2.3}
\]

2.4. Poisson summation. The proof of Theorems 1.1 requires the following Poisson summation formula (see [6] for details):

**Lemma 2.5.** [6, Corollary 2.12] Let \( n \in \mathbb{Z}[i], n \equiv 1 \pmod{(1+i)^3} \) and \( (\cdot \mid n)_2 \) be the quadratic residue symbol \( \pmod{n} \).

For any Schwartz class function \( W \), we have

\[
\sum_{m \in \mathbb{Z}[i]} \frac{(m)}{n} W\left(\frac{N(m)}{X}\right) = \frac{X}{2N(n)} \left(\frac{1+i}{n}\right) \sum_{k \in \mathbb{Z}[i]} (-1)^N(k) g_2(k, n) \hat{W}_i\left(\sqrt{\frac{N(k)X}{2N(n)}}\right).
\]

where \( \hat{W}_i(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(N(x + yi)) \bar{e}_i(-t(x + yi)) \, dx \, dy \), \( t \geq 0 \).

2.6. Evaluation of \( \hat{W}_i(t) \). We will require some simple estimates on \( \hat{W}_i(t) \) and its derivatives. First note that for any \( t \geq 0 \), \( \hat{W}_i(t) \in \mathbb{R} \) since

\[
\hat{W}_i(t) = \int_{\mathbb{R}^2} \cos(2\pi ty) W(x^2 + y^2) \, dx \, dy.
\]

Changing to polar coordinates, we get

\[
\hat{W}_i(t) = 4 \int_0^{\pi/2} \int_0^\infty \cos(2\pi tr \sin \theta) W(r^2) \, r \, dr \, d\theta = 2 \int_0^{\pi/2} \int_0^\infty \cos(2\pi tr^{1/2} \sin \theta) W(r) \, r \, dr \, d\theta.
\]

Now apply Mellin inversion yields

\[
\hat{W}_i(t) = 2 \int_0^{\pi/2} \int_0^\infty \cos(2\pi tr^{1/2} \sin \theta) \left(\frac{1}{2\pi r} \right)^{1/2} \int_{-1/2}^{1/2} \hat{W} \left(1 + \frac{s}{2}\right) r^{-s/2} ds \, dr \, d\theta.
\]

Here and after, for any function \( f \), its Mellin transform, \( \hat{f} \), is defined to be

\[
\hat{f}(s) = \int_0^\infty f(t) t^{s} \frac{dt}{t}.
\]
We make some changes of variables (first \( r^{1/2} \to r \), then \( 2\pi tr \sin \theta \to r \) and \( s \to -s \)) to get that
\[
\hat{W}(t) = \frac{4}{2\pi i} \int_{(1/2)} \hat{W}(1 - \frac{s}{2}) \int_0^{\pi/2} \int_0^\infty \cos(r) \left( \frac{r}{2\pi t \sin \theta} \right)^s \frac{dr}{r} d\theta ds
\]
(2.4)
\[
= \frac{4}{2\pi i} \int_{(1/2)} \hat{W}(1 - \frac{s}{2})(2\pi t)^{-s} \left( \int_0^{\pi/2} (\sin \theta)^{-s} d\theta \int_0^\infty \cos(r)r^s \frac{dr}{r} \right) ds
\]
\[
= \frac{\pi}{2\pi i} \int_{(1/2)} \hat{W}(1 - \frac{s}{2})(\pi t)^{-s} \frac{\Gamma(s/2)}{\Gamma((1-s)/2)} ds,
\]
where the last line above follows from the relation (see [4, Section 2.4]) that
\[
\int_0^{\pi/2} (\sin \theta)^{-s} d\theta \int_0^\infty \cos(r)r^s \frac{dr}{r} = \frac{\pi}{2} 2^{s-1} \frac{\Gamma(s/2)}{\Gamma((1-s)/2)}.
\]

2.7. Analytical behavior of a Dirichlet series. Let \( g_2(k, n) \) be defined as in (2.1). We now fix a generator for every prime ideal \((\varpi) \subset \mathcal{O}_K\) by taking \( \varpi \) to be primary if \((\varpi, 1+i) = 1\) and \( 1+i \) for the ideal \((1+i)\) (noting that \((1+i)\) is the only prime ideal in \( \mathcal{O}_K \) that lies above the integral ideal \((2)\) \(\subset \mathbb{Z}\)). We also fix 1 as the generator for the ring \( \mathbb{Z}[i] \) itself and extend the choice of the generator for any ideal of \( \mathcal{O}_K \) multiplicatively. We denote the set of such generators by \( G \).

We say any element \( k \in \mathcal{O}_K \) is square-free if the ideal \((k)\) is not divisible by the square of any prime ideal. Now, for any square-free \( k_1 \in \mathcal{O}_K \), we define
\[
J_{k_1}(v, w) = \sum_{n \equiv 1 \pmod{1+i}} \sum_{k_2} \frac{1}{N(n)^w N(k_2)^{2u}} \frac{g_2(k_1 k_2^2, n)}{N(n)},
\]
where we use the convention throughout that all sums over \( k_2 \) are restricted to \( k_2 \in G \).

It follows from (2.3) that \( g_2(k_1 k_2^2, n) \ll N(n) \), so that \( J_{k_1}(v, w) \) converges absolutely if \( \Re(w) \) and \( \Re(v) \) are both strictly greater than 1. For any Hecke character \( \chi \), we write \( L(s, \chi) \) for the associated L-function. The next lemma describes certain analytical behavior of \( J_{k_1}(v, w) \).

Lemma 2.8. The function \( J_{k_1}(v, w) \) defined above may be written as
\[
L \left( \frac{1}{2} + w, \chi_{k_1} \right) J_{2, k_1}(v, w),
\]
where \( J_{2, k_1}(v, w) \) is a function uniformly bounded by \( N(k_1)^c \) for any \( \varepsilon > 0 \) in the region \( \Re(v) \geq 1/2 + \varepsilon \), and \( \Re(w) \geq \varepsilon \).

Proof. It follows from Lemma 2.3 that the summands on the right-hand side of (2.5) are jointly multiplicative in terms of \( n \) and \( k_2 \), so that \( J_{k_1}(v, w) \) can be expressed as an Euler product over all primary primes \( \varpi \) with each factor at \( \varpi \) being
\[
J_{\varpi, k_1}(v, w) := \sum_{k_2 \geq 0, n \geq 0} \frac{1}{N(\varpi)^{nw+2k_2v}} \frac{g_2(k_1 \varpi^{2k_2}, \varpi^n)}{N(\varpi)^n}.
\]
Applying (2.3) again, we have \( J_{\varpi, k_1}(v, w) \ll 1 \) uniformly for all \( \varpi \).

For \( \varpi \nmid k_1 \), we can replace \( g_2(k_1 \varpi^{2k_2}, \varpi^n) \) in the definition of \( J_{\varpi, k_1}(v, w) \) by an explicit evaluation of it using Lemma 2.3. Keeping only the non-zero terms, this yields an alternative expression for \( J_{\varpi, k_1}(v, w) \), which we denote by \( J_{\varpi, k_1}^{\text{gen}}(v, w) \). One checks that we have
\[
J_{\varpi, k_1}^{\text{gen}}(v, w) := \sum_{k_2 \geq 0} \left( \sum_{n \equiv 0 \pmod{2}} \frac{1}{N(\varpi)^{nw+2k_2v}} \frac{\varphi(\varpi^n)}{N(\varpi)^n} + \sum_{n=2k_2+1} \frac{1}{N(\varpi)^{nw+2k_2v}} \frac{\chi_{k_1}(\varpi)}{N(\varpi)^{1/2}} \right).
\]
We now extend the above definition of \( J_{\varpi,k_1}(v,w) \) to all other \( \varpi \) (including \( \varpi = 1 + i \)).

In the region \( \Re(v) \geq 1/2 + \varepsilon, \Re(w) \geq \varepsilon \), it follows from the definition of \( J_{\varpi,k_1}(v,w) \) that the contribution from terms \( k_2 \geq 1 \) is \( \ll 1/N(\varpi)^{1+2\varepsilon} \). The contribution of the term \( k_2 = 0 \) is \( 1 + \chi_{k_1}(\varpi)N(\varpi)^{-1/2-w} \).

We now define

\[
J_{2,k_1}(v,w) = \left( L\left( \frac{1}{2} + w, \chi_{ik_1} \right) \right)^{-1} J_{k_1}(v,w) = J_{2,k_1}^{\text{gen}}(v,w) J_{2,k_1}^{\text{non-gen}}(v,w),
\]

where

\[
J_{2,k_1}^{\text{gen}}(v,w) = \prod_{\varpi} \left( 1 - \frac{\chi_{ik_1}(\varpi)}{N(\varpi)^{1/2+w}} \right) J_{\varpi,k_1}(v,w), \quad J_{2,k_1}^{\text{non-gen}}(v,w) = \prod_{\varpi \mid (1+i)k_1} J_{\varpi,k_1}(v,w)^{-1} J_{\varpi,k_1}(v,w),
\]

and we define \( J_{\varpi,k_1}(v,w) = 1 \) when \( \varpi = 1 + i \).

Our arguments above imply that \( J_{2,k_1}^{\text{gen}}(v,w) \) is uniformly bounded by 1 in the region \( \Re(v) \geq 1/2 + \varepsilon \), and \( \Re(w) \geq \varepsilon \). As one checks easily that \( J_{2,k_1}^{\text{non-gen}}(v,w) \) is uniformly bounded by \( N(k_1)^{\varepsilon} \) for any \( \varepsilon > 0 \) in the same region, the assertions of the lemma now follow.

\[\square\]

2.9. \textbf{A mean value estimate.} For \( k \in \mathcal{O}_K \), recall that \( \chi_k \) denote the quadratic symbol \( (\frac{k}{\varpi}) \). We shall use \( \sum^* \) to denote a sum over square-free elements in \( \mathcal{O}_K \) throughout the paper. In the proof of Theorem 1.1 we need the following mean value estimate for quadratic Hecke \( L \)-functions.

\textbf{Lemma 2.10.} \textit{For any complex number} \( \sigma + it \) \textit{with} \( \sigma \geq 1/2 \), \textit{we have}

\[
(2.6) \quad \sum^*_{N(k) \leq X} |L(\sigma + it, \chi_{ik_1})|^2 \ll_\varepsilon (X(1 + |t|))^{1+\varepsilon},
\]

\textit{and that}

\[
(2.7) \quad \sum^*_{N(k) \leq X} |L(\sigma + it, \chi_{ik_1})| \ll_\varepsilon (X(1 + |t|)^{1/2})^{1+\varepsilon}.
\]

\textit{Proof.} The estimate (2.6) follows from [4 Corollary 1.4] and (2.7) follows from (2.6) by Cauchy’s inequality. \(\square\)

3. \textbf{Proof of Theorem 1.1}

Applying the Poisson summation formula, Lemma 2.3,

\[
S_2(X,Y; \Phi, W) = \frac{X}{2} \sum_{k \in \mathbb{Z}[i]} (-1)^{N(k)} \sum_{n \equiv 1 \bmod (1+i)^3} g_2(k,n) \Phi \left( \frac{N(n)}{Y} \right) \tilde{W}_i \left( \sqrt{\frac{N(k)X}{2N(n)}} \right)
\]

\[
= \frac{X \tilde{W}_i(0)}{2} \sum_{n \equiv 1 \bmod (1+i)^3} g_2(0,n) \Phi \left( \frac{N(n)}{Y} \right) + \frac{X}{2} \sum_{k \in \mathbb{Z}[i], k \neq 0} (-1)^{N(k)} \sum_{n \equiv 1 \bmod (1+i)^3} g_2(k,n) \Phi \left( \frac{N(n)}{Y} \right) \tilde{W}_i \left( \sqrt{\frac{N(k)X}{2N(n)}} \right)
\]

\[:= M_0 + M', \quad \text{say}.\]

The term \( M_0 \) can be treated exactly as in [5 Section 3.1] and the result there gives

\[
(3.1) \quad M_0 = \pi X Y^{1/2} 24 \zeta_K(2) \tilde{W}_i(0) \Phi \left( \frac{1}{2} \right) + O \left( X Y^{\theta/2} \right),
\]

where \( \zeta_K(s) \) is the Dedekind zeta function of \( K \) and \( \theta = 131/416 \) arises from the currently best known error term of the Gauss circle problem (see [7]).
3.1. The Term $M'$. Now suppose $k \neq 0$. Applying (2.30) gives

$$M' = \frac{\pi}{2} \sum_{k \in \mathbb{Z}[i]} \sum_{n \equiv 1 \pmod{1+i^3}} (-1)^{N(k)} g_2(k, n) \Phi \left( \frac{N(n)}{Y} \right) \frac{1}{2\pi i} \int_{(1/2)} \hat{W} \left( 1 - \frac{s}{2} \right) \left( \pi \sqrt{\frac{N(k)X}{2N(n)}} \right)^{-s} \frac{\Gamma(s/2)}{\Gamma((1-s)/2)} ds.$$  

We take the Mellin inversion of $\Phi$ to further recast $M'$ as

$$M' = \frac{\pi}{2} \sum_{k \in \mathbb{Z}[i]} \sum_{n \equiv 1 \pmod{1+i^3}} (-1)^{N(k)} g_2(k, n) \frac{\pi \Phi(u)}{\Gamma((1-s)/2)} \hat{W} \left( 1 - \frac{s}{2} \right) \left( \pi \sqrt{\frac{N(k)X}{2N(n)}} \right)^{-s} \frac{\Gamma(s/2)}{\Gamma((1-s)/2)} duds,$$

where the last equality above follows from the observation that the integral over $s$ may be taken over any vertical lines with real part between 0 and 2.

Let $f(k) = g_2(k, n)/N(k)^s$ and we write $k = k_1 k_2^2$ with $k_1$ square-free and $k_2 \in G$. Recall here that $G$ is the set of generators of all ideals in $\mathcal{O}_K$ defined in Section 2.27. We break the sum over $k_1$ into two sums, depending on whether $k_1$ and $1+i$ are relatively prime or not, getting

$$\sum_{k \in \mathbb{Z}[i]} (-1)^{N(k)} f(k) = \sum_{k_1} f(k_1 k_2^2) + \sum_{k_1} (-1)^{N(k_2)} f(k_1 k_2^2).$$

Note that

$$\sum_{k_2} (-1)^{N(k_2)} f(k_1 k_2^2) = \sum_{1+i|k_2} f(k_1 k_2^2) - \sum_{(k_2, 1+i) = 1} f(k_1 k_2^2) = 2 \sum_{1+i|k_2} f(k_1 k_2^2) - \sum_{k_2} f(k_1 k_2^2),$$

It follows that we have

$$\sum_{k \in \mathbb{Z}[i]} (-1)^{N(k)} f(k) = \sum_{k_1} f(k_1 k_2^2) + \sum_{k_1} (-1)^{N(k_2)} f(k_1 k_2^2) \left( 2 \sum_{k_2} f(k_1 k_2^2) - \sum_{k_2} f(k_1 k_2^2) \right),$$

where the last equality above follows by a relabelling of $k_1$.

Note that if $(n, 1+i) = 1$, $g_2(k, n) = g_2(2k, n)$ by (2.22). It follows that we have $f(2k_1 k_2^2) = 4^{-s} f(k_1 k_2^2)$ so that

$$\sum_{k \in \mathbb{Z}[i]} (-1)^{N(k)} f(k) = (2^{1-2s} - 1) \sum_{k_1} f(k_1 k_2^2) + \sum_{k_2} f(k_1 k_2^2) + \sum_{k_2} f(k_1 k_2^2).$$

We apply the above expression to recast $M'$ as

$$M' = \frac{\pi}{2} X \left( \sum_{k_1} \mathcal{M}_1(s, u, k_1) + \sum_{k_1} \mathcal{M}_2(s, u, k_1) \right) ,$$
where

\[
\mathcal{M}_1(s, u, k_1) = \left( \frac{1}{2\pi i} \right)^2 \int_{(3/2)} \int W \left( 1 - \frac{s}{2} \right) (X N(k_1))^{-s/2} \left( 2^{1-2s} - 1 \right) \left( \frac{\pi}{2} \right)^{-s} \frac{\Gamma(s/2)}{\Gamma(1-s/2)} \Phi(u) Y^u J_{k_1} \left( \frac{s}{2}, u - \frac{s}{2} \right) duds
\]

(3.3)

\[
= \left( \frac{1}{2\pi i} \right)^2 \int_{(3/2)} \int \widehat{W} \left( 1 - \frac{s}{2} \right) (X N(k_1))^{-s/2} \left( 2^{1-2s} - 1 \right) \left( \frac{\pi}{2} \right)^{-s} \frac{\Gamma(s/2)}{\Gamma(1-s/2)} \Phi(u + \frac{s}{2}) Y^{u+s/2} J_{k_1} \left( \frac{s}{2}, u \right) duds.
\]

and \(J_{k_1}(v, w)\) is defined in (2.5). The formula for \(\mathcal{M}_2(s, u, k_1, l)\) is identical to (3.3) except the factor \(2^{1-2s} - 1\) is removed.

To evaluate \(\mathcal{M}_1(s, u, k_1)\) and \(\mathcal{M}_2(s, u, k_1)\), we apply Lemma 2.8 to replace \(J_{k_1}(s, u)\) by \(L(1/2 + u, \chi_{ik_1})J_{2,k_1}(s, u)\) and shift the line of integration over \(u\) to the line \(\Re(u) = \varepsilon\). We encounter a pole of the Dirichlet \(L\)-functions at \(u = 1/2\) for \(k_1 = \pm i^3\) only. We denote the sum of the possible residues by \(R\) and the remaining integrals by \(I\).

To treat \(R\), we move the line of integration over \(s\) to the line \(\Re(s) = 2 + \varepsilon\). We observe that as \(\Phi\) and \(W\) are compactly supported on \(\mathbb{R}^+\), integrating by parts implies that for any integers \(E_i \geq 0, 1 \leq i \leq 2\),

\[
\left| \widehat{W}(s) \Phi(u) \right| \ll \frac{1}{|as|(1+|s|)E_1(1+|u|)^E_2}.
\]

(3.4)

We further deduce from Stirling’s formula (see [8, (5.112)]) that

\[
\frac{\Gamma(s/2)}{\Gamma((1-s)/2)} \ll |s|^{\Re(s)-1/2}.
\]

(3.5)

Combining (3.4), (3.5) together with the estimation that \(J_{2,\pm i^3}(s, u) \ll 1\) by Lemma 2.8 we estimate the remaining integral over \(s\) on the line \(\Re(s) = 2 + \varepsilon\) to arrive at

\[
R \ll Y^{1/2} \left( \frac{Y}{X} \right)^{1+\varepsilon}.
\]

(3.6)

Now, to estimate the contribution of \(I\) to (3.2), we move the line of integration over \(s\) to \(\Re(s) = c = 4 + 2\varepsilon\). We find by Lemma 2.8 that for any \(\varepsilon > 0\),

\[
J_{k_1}(s, u) \ll (N(k_1))^\varepsilon \left| L \left( \frac{1}{2} + u, \chi_{ik_1} \right) \right|.
\]

Using (3.4) with \(E_1 = E_2 = 1\) and (3.5), together with the above bound, we find that

\[
I \ll Y^\varepsilon \left( \frac{Y}{X} \right)^{c/2} \int_{(c)} \int N(k_1)^{c/2-\varepsilon} \left| L \left( \frac{1}{2} + u, \chi_{ik_1} \right) \right| \frac{|s|^{\Re(s)-1/2} du ds}{|1-s/2|(1+|1-s/2|)u+s/2(1+|u+s/2|)}.
\]

(3.7)

Now Lemma 2.10 and partial summation give the bound

\[
\sum_{N(k_1)}^* \frac{1}{N(k_1)^{c/2-\varepsilon}} \left| L \left( \frac{1}{2} + u, \chi_{ik_1} \right) \right| \ll (1 + |\Im(u)|)^{1/2+\varepsilon} \ll \left( (1 + |u + s|)^{1/2+\varepsilon} + |s|^{1/2+\varepsilon} \right).
\]

(3.8)

Inserting (3.8) in (3.7) renders the bound

\[
I \ll Y^\varepsilon \left( \frac{Y}{X} \right)^{c/2} \ll Y^\varepsilon \left( \frac{Y}{X} \right)^{2+\varepsilon}.
\]

(3.9)

As \(X \geq Y\), we deduce from (3.2), (3.6) and (3.9) that

\[
M' \ll XY^{1/2} \left( \frac{Y}{X} \right)^{1+\varepsilon}.
\]

(3.10)
3.2. Conclusion. Combining (3.1) and (3.10) gives that if $X \geq Y$, then

$$S_2(X, Y; \Phi, W) = \pi Y^{1/2} \tilde{W}_i(0) \Phi \left( \frac{1}{2} \right) + O \left( XY^{\theta/2} + XY^{1/2} \left( \frac{Y}{X} \right)^{1+\varepsilon} \right).$$

The assertion of Theorem 1.1 now follows from the above expression upon setting

$$(3.11) \quad C(\Phi, W) = \frac{\pi}{24 \zeta(2) \tilde{W}_i(0) \Phi \left( \frac{1}{2} \right)}.$$

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References

[1] V. Blomer, L. Goldmakher, and B. Louvel, $L$-functions with n-th order twists, Int. Math. Res. Not. IMRN 2014 (2014), no. 7, 1925–1955.
[2] J. B. Conrey, D. W. Farmer, and K. Soundararajan, Transition mean values of real characters, J. Number Theory 82 (2000), no. 1, 109–120.
[3] H. Davenport, Multiplicative Number Theory, Third edition, Graduate Texts in Mathematics, vol. 74, Springer-Verlag, Berlin, etc., 2000.
[4] P. Gao, Moments and non-vanishing of central values of quadratic Hecke $L$-functions in the Gaussian field (Preprint). arXiv:2002.11899.
[5] P. Gao and L. Zhao, Mean values of some Hecke characters, Acta Arith. 187 (2019), no. 2, 125–141.
[6] M. N. Huxley, Integer points, exponential sums and the Riemann zeta function, Number theory for the millennium, II (Urbana, IL, 2000), 2002, pp. 275–290.
[7] K. Soundararajan and M. P. Young, The second moment of quadratic twists of modular $L$-functions, J. Eur. Math. Soc. (JEMS) 12 (2010), no. 5, 1097–1116.