Integro-differential systems with variable exponents of nonlinearity

1 Introduction

Let \( n, N \in \mathbb{N} \) and \( T > 0 \) be fixed numbers, \( \Omega \subset \mathbb{R}^n \) be a bounded domain with the boundary \( \partial \Omega \), \( Q_{0,T} = \Omega \times (0,T) \). We seek a weak solution \( u = (u_1, \ldots, u_N) : Q_{0,T} \rightarrow \mathbb{R}^N \) of the problem

\[
\begin{aligned}
u_{k,t} + \alpha \Delta^2 u_k - \sum_{i=1}^{n} (a_{ik}(x,t)|u_{x_i}|^{p(x)-2}u_{x_i})_{x_i} + \Delta(b_{ik}(x,t)|u|^{p(x)-2}u_k) + (Nu)_k = \\
= \sum_{i,j=1}^{n} (f_{ijk}(x,t))_{x_i x_j} - \sum_{i=1}^{n} (f_{ik}(x,t))_{x_i} + f_{0k}(x,t), \quad (x,t) \in Q_{0,T}, \quad k = 1, N,
\end{aligned}
\]

\[
u|_{S_{0,T}} = 0, \quad \Delta u|_{S_{0,T}} = 0, \quad u|_{t=0} = u_0(x).
\]

Here \( \alpha > 0 \) is a number, \( \Delta := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \) is the Laplacian, \( \Delta^2 := \Delta(\Delta) \), \( |u| := (|u_1|^2 + \ldots + |u_N|^2)^{1/2} \), \( |u_{x_i}| := \left( \frac{\partial u}{\partial x_1}^2 + \ldots + \left| \frac{\partial u}{\partial x_i} \right|^2 \right)^{1/2}, i = 1, n \),

\[
(\mathcal{N}u)_k(x,t) := (Gu)_k(x,t) + (Bu)_k(x,t) + \phi_k((Eu)_k(x,t)), \quad (x,t) \in Q_{0,T},
\]

\[
(Gu)_k(x,t) := g_k(x,t)|u(x,t)|^{q(x)-2}u_k(x,t), \quad (x,t) \in Q_{0,T},
\]

\[
(Bu)_k(x,t) := -\beta_k(x,t)(u_k(x,t))^-, \quad (x,t) \in Q_{0,T},
\]

\[
(Eu)_k(x,t) := \int_{\Omega} \epsilon_k(x,t,y)(\tilde{u}_k(x+y,t) - \tilde{u}_k(x,t)) \, dy, \quad (x,t) \in Q_{0,T}.
\]

Equation (1) describes, for example, the long-scale evolution of the thin liquid films. The function \( u(x,t) \) is a height of the liquid films in the point \( x \) at the time \( t \), the fourth-order terms describe capillary force of the liquid.
surface tension, and the second-order terms describe the evaporation (condensation) process into the liquid (see [1–3] for more details). The investigation of the fourth-order degenerate parabolic equations of the thin liquid films was started in [4] by F. Bernis and A. Friedman (see also [2, 5–8], and the references given there). The Dirichlet problem for the Cahn–Hilliard equation (1) \( (N = 1, \alpha > 0, a_{i1} = g_1 = \epsilon_1 = f_{i,j1} = f_{i1} = f_{01} = 0, \) where \( i, j = 1, n) \) was considered in [5] where \( \gamma(x) = 2m, m \in \mathbb{N}, \) and \( b_1 < 0. \) The corresponding Neumann problem was studied in [6]. The Neumann problem for equation (1) \( (N = 1, \alpha > 0, a_{i1} > 0, p(x) \equiv \text{const} > 2, g_1 = \epsilon_1 = f_{i,j1} = f_{i1} = f_{01} = 0, \) where \( i, j = 1, n) \) was considered in [2] if \( \gamma(x) = 2 \) and \( b_1 > 0. \)

The initial-boundary value problems for the parabolic equations with variable exponents of the nonlinearity and without integral terms in equation were considered for instance in [9–15]. Integral terms (6) arise in many applications (see [16–18]). The second-order parabolic equations with variable exponents of the nonlinearity and integral term (6) were considered in [17, 19].

2 Notation and statement of theorem

Let \( \| \cdot \|_B \equiv \| \cdot : B \| \) be a norm of some Banach space \( B, B^N := B \times \ldots \times B \) \((N\) times) be the Cartesian product of the \( B, B^* \) be a dual space for \( B, \) and \( \langle \cdot, \cdot \rangle_B \) a scalar product between \( B^* \) and \( B. \) We use the notation \( X \subset Y \) if the Banach space \( X \) is continuously embedded into \( Y; \) the notation \( X \subset \subset Y \) means the continuous and dense embedding; the notation \( X \subset K Y \) means the compact embedding.

If \( w \in B, z = (z_1, \ldots, z_N) \in B^N, \) and \( v = (v_1, \ldots, v_N) \in B^N, \) then we set

\[
 \langle v, w \rangle := \left( \langle v_1, w_1 \rangle_B, \ldots, \langle v_N, w_N \rangle_B \right) \in \mathbb{R}^N, \quad \langle v, z \rangle := \sum_{k=1}^{N} \langle v_k, z_k \rangle_B \in \mathbb{R},
\]

and \( \| z : B^N \| := \| z_1 : B \| + \ldots + \| z_N : B \|. \)

Suppose that \( m, d \in \mathbb{N}, p \in [1, \infty], \) \( X \) is the Banach space, \( Q \) is a measurable set in \( \mathbb{R}^d, \) \( M(Q) \) is a set of all measurable functions \( v : Q \to \mathbb{R} \) (see [20, p. 120]), \( \text{Lip} (Q) \) is a set of all Lipschitz-continuous functions \( v : Q \to \mathbb{R} \) (see [21, p. 29]), \( C^{m}(Q) \) and \( C^{\infty}(Q) \) are determined from [22, p. 9], \( L^p(Q) \) is the Lebesgue space (see [22, p. 22, 24]), \( W^{m,p}(Q) \) and \( W^{m,p}_0(Q) \) are Sobolev spaces (see [22, p. 45]), \( H^m(Q) := W^{m,2}(Q), H^m_0(Q) := W^{m,2}_0(Q), C([0, T]; X) \) and \( C^m([0, T]; X) \) are determined from [23, p. 147], \( L^p(0, T; X) \) is determined from [23, p. 155], \( W^{m,p}(0, T; X) \) is determined from [24, p. 286], \( H^m(0, T; X) := W^{m,2}(0, T; X) \), and

\[
 B_+(Q) := \{ q \in L^\infty(Q) \mid \text{ess inf}_{y \in Q} q(y) > 0 \}.
\]

If \( q \in B_+(Q), \) then by definition, put

\[
 q_0 := \text{ess inf}_{y \in Q} q(y), \quad q^0 := \text{ess sup}_{y \in Q} q(y), \quad S_q(s) := \max\{s^{q_0}, s^{q^0}\}, \quad s \geq 0, \quad (8)
\]

\[
 q'(y) := \frac{q(y)}{q(y) - 1} \quad \text{for a.e. } y \in Q \left( \text{note that } \frac{1}{q(y)} + \frac{1}{q'(y)} = 1 \right) \quad \text{and } q' \in B_+(Q),
\]

\[
 \rho_q(v; Q) := \int_Q |v(y)|^{q'(y)} dy, \quad v \in M(Q),
\]

Assume that \( q \in B_+(Q), q_0 > 1, \) and \( m \in \mathbb{N}. \) The set

\[
 L^{q(y)}(Q) := \{ v \in M(Q) \mid \rho_q(v; Q) < +\infty \}
\]

is called a generalized Lebesgue space. It is well known that \( L^{q(y)}(Q) \) is a Banach space which is reflexive and separable (see [25, p. 599, 600, 604]) with respect to the Luxemburg norm

\[
 \| v : L^{q(y)}(Q) \| := \inf \{ \lambda > 0 \mid \rho_q(v/\lambda; Q) \leq 1 \}.
\]
The set $W^{m,q}(\Omega) := \{v \in L^q(\Omega) \mid D^\alpha v \in L^q(\Omega), \ |\alpha| \leq m \}$ is called a generalized Sobolev space. It is well known that $W^{m,q}(\Omega)$ is a Banach space which is reflexive and separable (see [25, p. 604]) with respect to the norm

$$\|v; W^{m,q}(\Omega)\| := \sum_{|\alpha| \leq m} ||D^\alpha v; L^q(\Omega)||.$$  

The closure of $C^\infty_0(\Omega)$ with respect to the norm (11) is called a generalized Sobolev space and is denoted by $W^{m,q}_0(\Omega)$.

The generalized Lebesgue space was first introduced in [26]. The properties of the generalized Lebesgue and Sobolev spaces were widely studied in [25, 27–30].

Let us define the set $\Upsilon(\Omega) \subset M(\Omega)$ as follows. For every $p \in \Upsilon(\Omega)$ there exist numbers $m \in \mathbb{N}$, $s_1, s_2, \ldots, s_m, s_m^* \in \mathbb{R}$, and open sets $\Omega_1, \ldots, \Omega_m \subset \Omega$ such that the following conditions hold:

1) $\Omega_1, \ldots, \Omega_m$ consist of the finite numbers of the components with the Lipschitz boundaries;
2) $\text{mes} \left( \Omega \setminus \bigcup_{j=1}^m \Omega_j \right) = 0$;
3) $1 = s_1 < s_2 < s_3 < \ldots < s_{m-1} < s_m < s_m^* = +\infty$;
4) for every $j \in \{1, \ldots, m\}$ the inequality $s_j \leq p(x) \leq s_j^*$ holds a.e. for $x \in \Omega_j$;
5) for every $k \in \{1, \ldots, m-1\}$ the inequality $s_k^* < R(s_k)$ holds, where

$$R(q) := \begin{cases} \frac{nq}{n-q} & \text{if } 1 \leq q < n, \\ \text{arbitrary } s > 1 \text{ if } n \leq q. \end{cases}$$

Note that $W^{1,q}_0(\Omega) \subset L^{p,q}(\Omega)$, where $q \in [1, +\infty)$ (see [23, p. 47]).

Suppose that $\Delta^0 v := v, \Delta^1 v := \Delta v, \Delta^r v := \Delta(\Delta^{r-1} v)$,

$$H^2(\Omega)_r := \{v \in H^2(\Omega) \mid v|_{\partial \Omega} = \Delta v|_{\partial \Omega} = \ldots = \Delta^{r-1} v|_{\partial \Omega} = 0, \ r \in \mathbb{N}.\}$$

By definition, put $Z := H^2(\Omega), X := W^{1,1}(\Omega), \mathcal{O} := L^q(\Omega), H := L^2(\Omega), \ V := Z \cap N \cap \mathcal{O} \cap H, \ U(0,T) := \{u : (0,T) \to V^N \mid D^\alpha u \in [L^2(Q,T)]^N, \ |\alpha| = 2, \ u|_{x_0}, \ldots, u|_{x_N} \in [L^p(Q,T)]^N, \ u \in [L^q(Q,T)]^N \cap [L^2(Q,T)]^N, \ u \in U(0,T)\}$

and

$$W(0,T) := \{w \in U(0,T) \mid w_t \in [U(0,T)]^N\}.$$ 

We will need the following assumptions:

(P): $p \in B_+(\Omega), p_0 > 1$, and one of the following alternatives holds:

(i) $p \in \Upsilon(\Omega); \quad$ (ii) $p^0 \leq R(p_0); \quad$ (iii) $p \in C(\bar{\Omega});$

(G): $\gamma > 0, \gamma_0 \leq 1$; $\epsilon_0 > 0$; $\gamma_0 > 0$;

(Z): $\alpha > 0, \gamma_0 \leq 2, s_0 := \min\{2, p_0, q_0\}, \gamma_0 := \max\{2, \alpha, \gamma, \gamma_0\}, r \in \mathbb{N}$, and

$$\gamma_0 \geq \frac{1}{2} \max\{2, 1 + \frac{n(p^0-2)}{2p^0}, \frac{n(q^0-2)}{2q^0}\}.$$

(A): $a_{ik} \in M(Q_0,T), a_{ik}(x,t) \leq a_0^{+} < +\infty$ for a.e. $(x,t) \in Q_{0,T}, \ where \ i = 1, n, k = 1, N;$(B): $b_0^{+} \in M(Q_0,T), b_0^{+} \leq b_0^{+} < +\infty$ for a.e. $(x,t) \in Q_0,T, \ where \ k = 1, N;$(C): $g_0 \in M(Q_0,T), g_0 < g_0 < g_0 < +\infty$ for a.e. $(x,t) \in Q_0,T, \ where \ k = 1, N;$(D): $\beta_0^{+} \in B_+(Q_0,T);$(E): $\phi_k \in \text{Lip}(\mathbb{R}), \ |\phi_k(x)| \leq |\phi^0(x)| \$ for every $x \in \mathbb{R}$, where $\phi^0 \in [0, +\infty), \ k = 1, N;$(F): $f_{ik} \in L^2(Q_0,T), f_{ik} \in L^p(Q_0,T), f_{ik} \in L^q(Q_0,T), \ where \ i, j = 1, n, k = 1, N;$(U): $u_0 \in H^N.$

Let us introduce the following notation. If $t \in (0,T)$ and if $k \in \{1, \ldots, N\}$, then we set

$$\langle (\Delta u)_k, w \rangle := \int_{\Omega} \alpha \Delta u_k(x) \Delta w(x) \, dx, \quad u \in Z^N, \quad w \in Z.$$
Likewise we define the operators
\[ G(t) \] and
\[ F(t) \]
where
\[ A(t) \] and
\[ V \]
are defined in (20), \((Nw)_1, \ldots, (Nw)_N\) are defined in (3). Clearly,
\[ F(t) \in [V^N]^*, \quad (N(t))(\mathcal{O}^N \cap H^N) \subset [\mathcal{O}^N \cap H^N]^*, \quad t \in [0, T]. \]

Likewise we define the operators \(G(t) : \mathcal{O}^N \to [\mathcal{O}^N]^*, \) \(B(t) : H^N \to H^N,\) and \(E(t) : H^N \to H^N,\) where \(t \in [0, T].\)

For the sake of convenience we have denoted \(\phi(Eu) = (\phi_1((Eu)_1), \ldots, \phi_N((Eu)_N))\) and \(\phi_k(Eu_k(t)) = \phi_k((Eu)_k(t)), k = 1, \ldots, N.\) By definition, put
\[ (u, v)_\Omega := \begin{cases} 
\int u(x)v(x) \, dx & \text{if } u, v : \Omega \to \mathbb{R}, \\
\int_{\partial \Omega} u(x)v(x) \, ds & \text{if } u, v : \Omega \to \mathbb{R}^N,
\end{cases} \]

**Definition 2.1.** A real-valued function \(u \in W(Q_{0,T}) \cap C([0, T], H^N)\) is called a weak solution of problem ((1), (2)) if \(u\) satisfies (2) and for every \(v \in U(Q_{0,T})\) we have
\[ (u, v)_{\Omega} + \int_0^T \left[ (K(t)u(t), w(t))_{V^N} + (N(t)u(t), v(t))_\Omega \right] dt = \int_0^T (F(t), v(t))_{V^N} dt. \]

**Theorem 2.2.** Suppose that conditions (P)-(U) and \(\partial \Omega \in C^{2r}\) are satisfied. Then problem ((1), (2)) has a weak solution.

### 3 Auxiliary facts

#### 3.1 Properties of generalized Lebesgue and Sobolev spaces

The following Propositions are needed for the sequel.
Proposition 3.1 (see [31, p. 31]). If \( q \in \mathcal{B}_+(Q) \) and \( q_0 \geq 1 \), then for every \( \eta > 0 \) there exists a number \( Y_q(\eta) > 0 \) such that for every \( a, b \geq 0 \) and for a.e. \( y \in Q \) the generalized Young inequality

\[
abla \leq \eta a^{q(y)} + Y_q(\eta) b^{q(y)}
\]

holds. In addition, \( Y_q(\eta) \) depends on \( q_0, q \) and it is independent of \( y \), \( Y_2(\eta) = \frac{1}{4\eta}, \ Y_2(\frac{1}{2}) = \frac{1}{2} \), \( Y_q(+0) = +\infty \), and \( Y_q(+\infty) = 0 \).

Proposition 3.2. Assume that \( q \in \mathcal{B}_+(Q) \) and \( q_0 \geq 1 \). Then the following statements are satisfied:

(i) (see [25, p. 600]) if \( q(y) \geq r(y) \geq 1 \) for a.e. \( y \in Q \), then \( L^{r(y)}(Q) \subseteq L^{s(y)}(Q) \) and

\[
||v; L^{r(y)}(Q)|| \leq (1 + \text{mes}Q)||v; L^{s(y)}(Q)||, \quad v \in L^{q(y)}(Q);
\]

(ii) (see [30, p. 431]) for every \( u \in L^{q(y)}(Q) \) and \( v \in L^{q(y)}(Q) \) we get \( uv \in L^1(Q) \) and the following generalized Hölder inequality is true

\[
\int_Q |u(y)v(y)| \, dy \leq 2 ||u; L^{q(y)}(Q)|| \cdot ||v; L^{q(y)}(Q)||.
\]

Proposition 3.3 (see [32, p. 168]). Suppose that \( q \in \mathcal{B}_+(Q) \), \( q_0 \geq 1 \), \( S_q \) is defined by (8), and \( \rho_q \) is defined by (10). Then for every \( v \in \mathcal{M}(Q) \) the following statements are fulfilled:

(i) \( ||v; W^{1,p(x)}_0(\Omega)|| \leq S_{1/q}(\rho_q(v;Q)) \) if \( \rho_q(v;Q) < +\infty \);

(ii) \( \rho_q(v;Q) \leq S_q(||v; L^{r(x)}(Q)||) \) if \( ||v; L^{q(x)}(Q)|| < +\infty \).

Proposition 3.4. Suppose that \( q \in \mathcal{B}_+(Q) \) and \( q_0 \geq 1 \). Then the following statements hold:

(i) (see Theorem 3.10 [25, p. 610] and Theorem 2.7 [30, p. 443]) if either \( p \in \Upsilon(\Omega) \) or \( p \in C(\overline{\Omega}) \), then

\[
||v; W^{1,p(x)}_0(\Omega)|| = \sum_{i=1}^n ||v_{x_i}; L^{p(x)}(\Omega)||
\]

is a equivalent norm of \( W^{1,p(x)}_0(\Omega) \);

(ii) (see Lemma 5 [13, p. 48] and Theorem 3.1 [27, p. 76]) if \( u_{x_1}, \ldots, u_{x_n} \in L^{p(x)}(\Omega) \) and either \( p \in \Upsilon(\Omega) \) or \( p_0 \leq R(p_0) \) (see (12)), then \( u \in L^{p(x)}(\Omega) \) and the generalized Poincaré inequality

\[
||u; L^{p(x)}(\Omega)|| \leq C_1 \left( \sum_{i=1}^n ||u_{x_i}; L^{p(x)}(\Omega)|| + ||u; L^1(\Omega)|| \right),
\]

holds, where \( C_1 > 0 \) is independent of \( u \);

(iii) (see Lemma 2 [13, p. 46] and Theorem 3.2 [27, p. 77])

\[
L^{p_0}(0,T; L^{p(x)}(\Omega)) \supset L^{p(x)}(Q_{0,T}) \supset L^{p_0}(0,T; L^{p(x)}(\Omega)).
\]

3.2 Auxiliary functional spaces

Let \( \mathcal{L}(X,Y) \) be a space of bounded linear operators from \( X \) into \( Y \) (see [33, p. 32]), \( \langle \cdot, \cdot \rangle_H \) be the Cartesian product in the Hilbert space \( H_2^{2r}(\Omega) \), and \( H_2^{2r}(\Omega) \) is defined in (13), where \( r \in \mathbb{N} \). It is easy to verify that \( H_2^{2r}(\Omega) \) is the Hilbert space such that

\[
H_2^{2r}(\Omega) \supset H_2^{2r}(\Omega), \quad H_2^{2r}(\Omega) \supset L^2(\Omega) \supset [H_2^{2r}(\Omega)]^*.
\]

If \( \partial \Omega \subset C^1 \), then the following integration by parts formula is true

\[
\int_\Omega v \Delta u \, dx = \int_\Omega u \Delta v \, dx, \quad u, v \in H_2^{2r}(\Omega).
\]
Note that for every \( r \in \mathbb{N} \) the space \( H^2_{\Delta} (\Omega) \) is reflexive.

Let \( \{ w^j \}_{j \in \mathbb{N}} \) be a set of all eigenfunctions of the problem

\[
-\Delta w^j = \lambda_j w^j \quad \text{in} \quad \Omega, \quad w^j| \partial \Omega = 0, \quad j \in \mathbb{N}.
\]

(28)

Here \( \{ \lambda_j \}_{j \in \mathbb{N}} \subset \mathbb{R}_+ \) is the set of the corresponding eigenvalues. Suppose that \( \{ w^j \}_{j \in \mathbb{N}} \) is an orthonormal set in \( L^2(\Omega) \). It is easy to verify that solutions to problem (28) satisfy the equalities

\[
(-1)^r \Delta^r w = \lambda^r w, \quad w| \partial \Omega = \Delta^j w| \partial \Omega = \ldots = \Delta^{r-1} w| \partial \Omega = 0.
\]

(29)

The following propositions are needed for the sequel.

**Proposition 3.5** (see Theorem 8 [34, p. 230]). If \( \partial \Omega \subset C^{2r} \), then the set \( \{ w^j \}_{j \in \mathbb{N}} \) of all eigenfunction of the problem (28) is a basis for the space \( H^2_{\Delta} (\Omega) \).

**Proposition 3.6** (see Lemma 3 [34, p. 229]). If \( \partial \Omega \subset C^{2r} \), then there exists a constant \( C_2 > 0 \) such that for all \( v \in H^2_{\Delta} (\Omega) \) we obtain

\[
||v; H^2_{\Delta} (\Omega)|| \leq C_2 ||\Delta^r v; L^2 (\Omega)||.
\]

(30)

Define

\[
\mathcal{W}_r := [H^2_{\Delta} (\Omega)]^N, \quad \mathcal{W}_r^* := [\mathcal{W}_r]^*,
\]

(31)

where \( r \) is determined from condition (Z). We consider the space \( V^N \) (see (14) ) with respect to the norm

\[
||v; V^N|| := ||\Delta v; H^N|| + \sum_{i=1}^n ||v_{x_i}; [L^{p(x)} (\Omega)]^N|| + ||v; C^N|| + ||v; H^N||.
\]

Since \( r \) satisfies (Z) and (14) holds, it is easy to verify that

\[
\mathcal{W}_r \ominus V^N \ominus H^N \cong [H^N]^* \ominus [V^N]^* \ominus \mathcal{W}_r^*.
\]

(32)

The following Lemma is needed for the sequel.

**Lemma 3.7.** \( L^\infty (0, T; H^N) \cap C([0, T]; [V^N]^*) = C ([0, T]; H^N) \).

The proof is omitted (see for comparison Lemma 8.1 [35, p. 307]).

We consider the space \( U(Q_{0,T}) \) (see (15) ) with respect to the norm

\[
||u; U(Q_{0,T})|| := \sum_{i=1}^n ||u_{x_i,x_j}; [L^2(Q_{0,T})]^N|| + \sum_{i=1}^n ||u_{x_i}; [L^{p(x)} (Q_{0,T})]^N||
\]

\[
+ ||u; [L^{p(x)} (Q_{0,T})]^N|| + ||u; [L^2 (Q_{0,T})]^N||.
\]

(33)

It is easy to verify that the space \( U(Q_{0,T}) \) is reflexive. Taking into account the embedding of type (25) and inequality (30), we obtain

\[
L^{s_0} (0, T; V^N) \bar{\ominus} U(Q_{0,T}) \bar{\ominus} L^{s_0} (0, T; V^N),
\]

(33)

where \( s_0 \) and \( d_0 \) are determined from condition (Z). Whence,

\[
L^{s_0} (0, T; [V^N]^*) \bar{\ominus} [U(Q_{0,T})]^* \bar{\ominus} L^{s_0} (0, T; [V^N]^*).
\]

(34)

Similarly, using (32) we obtain

\[
L^{s_0} (0, T; \mathcal{W}_r) \bar{\ominus} U(Q_{0,T}) \bar{\ominus} [L^2 (Q_{0,T})]^N \bar{\ominus} [U(Q_{0,T})]^* \bar{\ominus} L^{s_0} (0, T; \mathcal{W}_r^*).
\]

(35)

Hence an arbitrary element of the spaces \( [U(Q_{0,T})]^* \) or \( U(Q_{0,T}) \) belongs to \( D^*(0, T; [V^N]^*) \). Therefore, we have distributional derivative of \( u \in U(Q_{0,T}) \subset D^*(0, T; [V^N]^*) \). Together with (34), we conclude that an arbitrary element \( w \in [U(Q_{0,T})]^* \) belongs to \( L^{s_0} (0, T; [V^N]^*) \). Thus, if \( u \in U(Q_{0,T}) \) belongs to \( L^{s_0} (0, T; V^N) \), then \( \{u, w \}_{U(Q_{0,T})} = f^T \{ u(t), v(t) \}_{V^N} dt \). In particular, this equality is true if \( u \in C ([0, T]; V^N) \).
Lemma 3.8. Suppose that conditions $(P)$ and $(Q)$ are satisfied, $u \in U(\mathcal{Q}_0, T)$, \( \{w^\mu\}_{\mu \in \mathbb{N}} \) is a basis for the space $V$. Then for every $\varepsilon > 0$ there exist a number $m \in \mathbb{N}$ and functions \( \{\varphi_{\mu k}\}_{\mu = 1, k = 1}^{m, N} \subset C^\infty([0, T]) \) such that \( ||u - \psi_m; U(\mathcal{Q}_0, T)|| < \varepsilon, \) where \( \psi_m = (\psi_{m1}, \ldots, \psi_{mN}) \) and \( \psi_{mk}(x, t) = \sum_{\mu=1}^m \varphi_{\mu k}(t)w^\mu(x), (x, t) \in \mathcal{Q}_0, k = 1, N. \)

The proof is omitted (see for comparison [36, p. 5] and [13, 27]).

### 3.3 Projection operator

Let $\mathcal{H}$ be the Hilbert space and $\mathcal{V}$ be the reflexive separable Banach space such that

\[
\mathcal{V} \supset \mathcal{H} \cong \mathcal{H}^* \supset \mathcal{V}^*.
\]  

(36)

Notice that if $g \in \mathcal{V}^*$ and $g \in \mathcal{H}$, then

\[
\langle g, v \rangle_{\mathcal{V}} = \langle g, v \rangle_{\mathcal{H}}, \quad v \in \mathcal{V}.
\]  

(37)

Suppose \( \{w^j\}_{j \in \mathbb{N}} \) is an orthonormal basis for the space $\mathcal{H}$, $m \in \mathbb{N}$ is a fixed number, $\mathcal{M}$ is a set of all linear combinations of the elements from $\{w^1, \ldots, w^m\}$, $\mathcal{M}^\perp$ is a orthogonal complements of $\mathcal{M}$ (see [37, p. 476]). Then (see [37, p. 526]) $\mathcal{M}$ is a closed subset of $\mathcal{H}$ and $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$.

Define an unique orthogonal projection $P_m : \mathcal{H} \to \mathcal{M}$ by the rule (see [37, p. 527])

\[
P_{mh} := \sum_{j=1}^m (h, w^j)_{\mathcal{H}} w^j, \quad h \in \mathcal{H}.
\]  

(38)

This is a linear self-adjoint continuous operator (see Theorem 7.3.6 [37, p. 515]) such that

\[
\|P_{mh}\|_{\mathcal{H}} \leq \|h\|_{\mathcal{H}}, \quad h \in \mathcal{H}.
\]  

(39)

If $\{w^j\}_{j \in \mathbb{N}} \subset \mathcal{V}$, then let us define an operator $\hat{P}_m : \mathcal{V} \to \mathcal{V}$ (not necessarily self-adjoint) by the rule

\[
\hat{P}_m v := P_m v \quad \text{for every} \quad v \in \mathcal{V}.
\]  

(40)

We shall find a conjugate operator $\hat{P}_m^* : \mathcal{V}^* \to \mathcal{V}^*$. Take elements $v \in \mathcal{V}$, $z \in \mathcal{V}^*$. Then

\[
\langle z, P_m v \rangle_{\mathcal{V}} = \left( \sum_{j=1}^m (v, w^j)_{\mathcal{H}} w^j \right)_z \mathcal{V} = \sum_{j=1}^m (v, w^j)_{\mathcal{H}} \langle z, w^j \rangle_{\mathcal{V}} = \left( v, \sum_{j=1}^m (z, w^j)_{\mathcal{V}} w^j \right)_{\mathcal{H}}.
\]

Since $v, w^1, \ldots, w^m \in \mathcal{V}$, (37) yields that

\[
\left( v, \sum_{j=1}^m (z, w^j)_{\mathcal{V}} w^j \right)_{\mathcal{H}} = \left( \sum_{j=1}^m (z, w^j)_{\mathcal{V}} w^j, v \right)_{\mathcal{H}} = \left( \sum_{j=1}^m (z, w^j)_{\mathcal{V}} w^j, v \right)_{\mathcal{V}}.
\]

Thus,

\[
\langle z, P_m v \rangle_{\mathcal{V}} = \langle \hat{P}_m^* z, v \rangle_{\mathcal{V}}, \quad \text{where}
\]

\[
\hat{P}_m^* z = \sum_{j=1}^m (z, w^j)_{\mathcal{V}} w^j, \quad z \in \mathcal{V}^*.
\]  

(41)

In addition, (41) implies that $\hat{P}_m^*(\mathcal{V}^*) \subset \mathcal{V}$.

Lemma 3.9. Assume that $\{w^j\}_{j \in \mathbb{N}}$ is an orthonormal basis for the space $\mathcal{H}$ such that $\{w^j\}_{j \in \mathbb{N}} \subset \mathcal{V}$, $\psi_1^m, \ldots, \psi_m^m \in \mathbb{R}$ are some numbers, and $F \in \mathcal{V}^*$. Then $z^m = \sum_{s=1}^m \psi_s w^s \in \mathcal{V}$ satisfies

\[
\begin{align*}
\langle z^m, w^1 \rangle_{\mathcal{V}} &= \langle F, w^1 \rangle_{\mathcal{V}}, \\
\vdots \\
\langle z^m, w^m \rangle_{\mathcal{V}} &= \langle F, w^m \rangle_{\mathcal{V}}.
\end{align*}
\]  

(42)
iff the following equality holds

\[ z^m = \bar{P}_m^* F \quad \text{in} \quad \mathcal{V}^*. \]  

(43)

**Proof.** Clearly, (43) implies (42). We shall prove that (42) implies (43). Take \( v \in \mathcal{V} \). There exist numbers \( \alpha_1^m, \ldots, \alpha_m^m \in \mathbb{R} \) such that \( P_m v = \bar{P}_m v = \sum_{\mu=1}^m \alpha_{\mu}^m w^\mu \). Multiplying both sides of \( \mu \)-th equality of (42) by \( \alpha_{\mu}^m \) and summing the obtained equalities, we get \( \langle z^m, \bar{P}_m v \rangle_{\mathcal{V}} = \langle F, \bar{P}_m v \rangle_{\mathcal{V}} \). Hence, \( \langle \bar{P}_m^* z^m, v \rangle_{\mathcal{V}} = \langle \bar{P}_m^* F, v \rangle_{\mathcal{V}} \) for every \( v \in \mathcal{V} \). Thus,

\[ \bar{P}_m^* z^m = \bar{P}_m^* F \quad \text{in} \quad \mathcal{V}^*. \]

(44)

Taking into account (37), the inclusions \( z^m, w^1, \ldots, w^m \in \mathcal{V} \), and the orthonormality condition for \( \{w^j\}_{j \in \mathbb{N}} \subset \mathcal{H} \), from (41) we obtain

\[ \bar{P}_m^* z^m = \sum_{j=1}^m \langle z^m, w^j \rangle_{\mathcal{V}} w^j = \sum_{j=1}^m \left( \sum_{s=1}^m \psi_s^m w^s, w^j \right) \mathcal{H} w^j = \sum_{s=1}^m \psi_s^m w^s = z^m. \]

Therefore, (42) yields (43).

In the sequel, we only consider the case \( \mathcal{H} = L^2(\Omega), \mathcal{V} = H^2_\Delta(\Omega) \) (see (13)), and \( \{w^j\}_{j \in \mathbb{N}} \) is determined from problem (28). Then (38) implies that (see (21))

\[ (P_m u)(x) = \sum_{j=1}^m (u, w^j)_{\Omega} w^j(x), \quad x \in \Omega, \quad u : \Omega \to \mathbb{R}. \]

(45)

This operator \( P_m : L^2(\Omega) \to L^2(\Omega) \) is a linear self-adjoint continuous projection operator such that \( \|P_m\|_{L^2(\Omega) \to L^2(\Omega)} = 1 \).

To prove that \( \bar{P}_m \) belongs to \( \mathcal{L}(H^2_\Delta(\Omega), H^2_\Delta(\Omega)) \), we take \( v \in H^2_\Delta(\Omega) \). Then \( \Delta^\gamma \bar{P}_m v \in L^2(\Omega) \) and Corollary 6.2.10 [38, p. 171] implies that there exists a function \( h \in L^2(\Omega) \) such that \( \|h\|_{L^2(\Omega)} = 1 \) and \( (h, \Delta^\gamma \bar{P}_m v)_{L^2(\Omega)} = \|\Delta^\gamma \bar{P}_m v\|_{L^2(\Omega)} \).

By (45), (40), (29), and (27) we obtain

\[ \|\bar{P}_m v\|_{H^2_\Delta(\Omega)} \leq \|\Delta^\gamma \bar{P}_m v\|_{L^2(\Omega)} = \left( \int \left( \frac{1}{2} \sum_{j=1}^m (v, w^j)_{\Omega} \Delta^\gamma w^j \right)^2 \right)^{1/2} \]

(46)

Suppose now that \( f \in L^s(0, T; \mathcal{H}), s > 1 \). If \( P_m : \mathcal{H} \to \mathfrak{M} \) is determined from (38), then \( P_m f(t) \in \mathcal{H} \) for every \( t \in [0, T] \),

\[ P_m f(t) = \sum_{j=1}^m (f(t), w^j)_{\mathcal{H}} w^j, \]

(47)

and from (39) we get

\[ \int_0^T \|P_m f(t)\|^2_{\mathcal{H}} dt \leq \int_0^T \|f(t)\|^2_{\mathcal{H}} dt, \]

(48)

Finally assume that \( \bar{P}_m : \mathcal{V} \to \mathcal{V} \) is determined from (40), \( \mathcal{H} = L^2(\Omega) \), and \( \mathcal{V} = H^2_\Delta(\Omega) \). Taking into account (46) and (48), we have that

\[ \|\bar{P}_m u; L^s(0, T; H^2_\Delta(\Omega))\| \leq \|u; L^s(0, T; H^2_\Delta(\Omega))\|, \quad u \in L^s(0, T; H^2_\Delta(\Omega)), \quad s \geq 1. \]

(49)

Clearly, we can prove (38)-(49) if we replace \( L^2(\Omega), H^2_\Delta(\Omega) \) by \( [L^2(\Omega)]^N, [H^2_\Delta(\Omega)]^N \) respectively.
3.4 Differentiability of the nonlinear expressions

Take a function $\sigma \in \mathcal{M}(\Omega)$ and by definition, put

$$
\psi_{\sigma(x)}(s) := \begin{cases} 
\sigma(x) & \text{if } s > 0, \\
0 & \text{if } s \leq 0,
\end{cases} \quad x \in \Omega.
$$

Similarly to Theorem A.1 [39, p. 47], we obtain that if $v \in W^{1,p}(0, T; L^p(\Omega))$ ($1 \leq p \leq \infty$), then $v^+ := \max\{u, 0\} \in W^{1,p}(0, T; L^p(\Omega))$ and $(u^+)_{tt} = \nabla^2 u$ almost everywhere in $Q_{0,T}$, where

$$
\nabla^2 u := \begin{cases} 
1 & \text{if } s > 0, \\
0 & \text{if } s \leq 0.
\end{cases}
$$

The function $v^- := \max\{-u, 0\}$ has a similar property.

The following Propositions are needed for the sequel.

**Proposition 3.10.** (see Theorem 2 [24, p. 286]). If $X$ is a Banach space and $1 \leq p \leq \infty$, then $W^{1,p}(0, T; X)$ and $W^{1,p}(0, T; X)$ and the following integration by parts formula holds:

$$
\int_s^T u_t(t) \, dt = u(\tau) - u(s), \quad 0 \leq s < \tau \leq T, \quad u \in W^{1,p}(0, T; X).
$$

**Proposition 3.11.** (the Aubin theorem, see [40] and [41, p. 393]). If $s, h > 1$ are fixed numbers, $\mathcal{W}, \mathbb{L}, \mathbb{B}$ are the Banach spaces, and $\mathcal{W} \subseteq \mathbb{L} \subseteq \mathbb{B}$, then

$$
\{u \in L^s(0, T; \mathcal{W}) \mid u_t \in L^h(0, T; \mathbb{B})\} \subseteq L^s(0, T; \mathbb{L}) \cap C([0, T]; \mathbb{B}).
$$

**Lemma 3.12.** Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded $C^{0,1}$-domain. Then the integration by parts formula

$$
\int_{Q_{s,T}} w_t z \, dx \, dt = \int_{Q_{s,T}} w z \, dx \bigg|_{t=s}^{t=T} - \int_{Q_{s,T}} w_z \, dx \, dt, \quad 0 \leq s < \tau \leq T,
$$

holds if one of the following alternatives hold:

(i) $w \in L^{q(x)}(Q_{0,T})$, where $q \in B_+(\Omega)$ and $q_0 > 1$, $w_t \in L^1(Q_{0,T})$, $z \in L^\infty(Q_{0,T})$, $z_t \in L^{q(x)}(Q_{0,T})$;

(ii) $w, w_t \in L^1(Q_{0,T})$, $z, z_t \in L^\infty(Q_{0,T})$.

**Proof.** (i). Take $W := \{w \in L^{q(x)}(Q_{0,T}) \mid w_t \in L^1(Q_{0,T})\}$. If $\varphi \in C^1([0, T])$ and $z \in Z$, then $\varphi z \in W^{1,1}(0, T; L^{\frac{q_0}{q_0-1}}(\Omega))$. Using (52) with $u = \varphi(t)z(x, t)$, we get

$$
\int_s^T \varphi(t)z(x, t) \, dt = \varphi(t)z(x, t) - \varphi(s)z(x, s) - \int_s^T \varphi(t)z_t(x, t) \, dt, \quad x \in \Omega.
$$

Take a function $v \in C^1(\overline{\Omega})$. By (54), we obtain that

$$
\int_{Q_{s,T}} \varphi_t v z \, dx \, dt = \int_{Q_{s,T}} \varphi v z \, dx \bigg|_{t=s}^{t=T} - \int_{Q_{s,T}} \varphi v z_t \, dx \, dt.
$$

Clearly, $C^1([0, T]; C^1(\overline{\Omega})) \cap W \cap W^{1,1}(0, T; L^1(\Omega))$. Then the set

$$
\left\{ \sum_{i=1}^m \varphi_i(t)v_i(x) \mid m \in \mathbb{N}, \varphi_1, \ldots, \varphi_m \in C^1([0, T]), v_1, \ldots, v_m \in C^1(\overline{\Omega}) \right\}
$$

is dense in $W$ and (55) yields (53).

We shall omit the proof of (ii) because it is analogous to the previous one. $\square$
Lemma 3.13. Suppose that $\sigma \in B_+(\Omega)$. Let $p, q \in B_+(\Omega)$, $p_0, q_0 > 1$, $p(y) \geq \sigma(y)$ and $q(y) \leq \frac{p(y)}{\sigma(y)}$ for a.e. $y \in \Omega$, and $\psi_{\sigma(y)}$ is determined from (50) if we replace $\sigma(x)$ by $\sigma(y)$. Then for every $u \in L^{p(y)}(\Omega)$ we have that

$$\rho_\sigma(\psi_{\sigma(y)}(u); Q) \leq \rho_\sigma(u; Q),$$

where $C_3 > 0$ is independent of $u$.

Proof. Clearly, $\frac{p(y)}{\sigma(y)} \geq 1$ for a.e. $y \in \Omega$, $\frac{\rho_{\sigma(y)}(u)}{\rho_{\sigma(y)}(u)} = |u|^{p(y)} \leq |u|^{p(y)} \in L^1(\Omega)$. Then by [42, p. 297], we obtain $\psi_{\sigma(y)}(u) \in L^\frac{p(y)}{\sigma(y)}(\Omega)$. Moreover, (56) and

$$\|\psi_{\sigma(y)}(u); L^q(\Omega)\| \leq C_4\|\psi_{\sigma(y)}(u); L^\frac{p(y)}{\sigma(y)}(\Omega)\| \leq C_4\rho_{\sigma(y)}(u; Q)$$

hold. This inequality and (56) imply (57). 

Lemma 3.14. Suppose that $p \in B_+(\Omega)$, $p_0 > 1$, $\theta \in M(\Omega \times \mathbb{R})$, for a.e. $x \in \Omega$ the function $\mathbb{R} \ni \xi \mapsto \theta(x, \xi) \in \mathbb{R}$ is continuously differentiable, and there exists a number $M > 0$ such that

$$|\theta(x, \xi) - \theta(x, \eta)| \leq M|\xi - \eta|, \quad |\theta_x(x, \xi)| \leq M$$

for a.e. $x \in \Omega$ and for every $\xi, \eta, \xi, \zeta \in \mathbb{R}$. If $u, u_t \in L^p(x)(\mathbb{Q}_{0,T})$, then $\theta(x, u), (\theta(x, u))_{t, t} \in L^p(x)(\mathbb{Q}_{0,T})$ and

$$\left(\theta(x, u)\right)_t = \theta_x(x, u) u_t.$$ 

Proof. Since $u_t \in L^p(x)(\mathbb{Q}_{0,T})$, there exists a sequence $\{u^m\}_{m \in \mathbb{N}} \subset C^1(\mathbb{Q}_{0,T})$ such that $u^m \rightharpoonup u$ and $u^m \rightharpoonup u_t$ strongly in $L^p(x)(\mathbb{Q}_{0,T})$ and almost everywhere in $\mathbb{Q}_{0,T}$. Clearly,

$$\left(\theta(x, u^m(x, t))\right)_t = \lim_{h \to 0} \frac{\theta(x, u^m(x, t + h)) - \theta(x, u^m(x, t))}{h} u^m(x, t + h) - u^m(x, t) = \theta_x(x, u^m(x, t)) u^m_t(x, t),$$

where $\mathbb{N} \ni \xi \mapsto \theta(x, \xi) \in \mathbb{R}$ is determined from (50) if we replace $\sigma(x)$ by $\sigma(y)$ and for every $\xi, \eta, \xi, \zeta \in \mathbb{R}$.

Corollary 3.15. Suppose that $-\infty < a < b < +\infty$ and one of the following alternatives holds: (i) $I = [a, b]$; (ii) $I = [a, +\infty)$; (iii) $I = (-\infty, b]$. Assume also that $p \in B_+(\Omega)$, $p_0 > 1$, $\theta \in M(\Omega \times I)$, a.e. for $x \in \Omega$ the function $\mathbb{R} \ni \xi \mapsto \theta(x, \xi) \in \mathbb{R}$ is continuously differentiable, and there exists a number $M > 0$ such that a.e. or $x \in \Omega$ and for every $\xi, \eta, \xi, \zeta \in I$, (58) holds. If $u, u_t \in L^p(x)(\mathbb{Q}_{0,T})$ and $u(x, t) \in I$ a.e. for $(x, t) \in \mathbb{Q}_{0,T}$, then $\theta(x, u), (\theta(x, u))_{t} \in L^p(x)(\mathbb{Q}_{0,T})$ and (59) holds.
Proof. For the sake of convenience, only the case $I = (-\infty, b]$ is considered (see for comparison [44, p. 98]). Let us extend $\theta$ outside $I$ as follows

$$
\Theta(x, \xi) := \begin{cases} 
\theta(x, \xi) & \text{if } \xi \leq b, \\
\theta(x, b)\xi + \theta(x, b) - \theta(x, b)b & \text{if } \xi > b,
\end{cases} \quad x \in \Omega.
$$

Then $\Theta$ satisfies the conditions of Lemma 3.14 and $\Theta(x, u(x, t)) = \theta(x, u(x, t))$ for a.e. $(x, t) \in Q_{0,T}$. This completes the proof.

Lemma 3.16. Suppose that $p \in \mathcal{B}_+(\Omega), p_0 > 1$, $\theta \in \mathcal{M}(\Omega \times \mathbb{R})$, for a.e. $x \in \Omega$ the function $\mathbb{R} \ni \xi \mapsto \theta(x, \xi) \in \mathbb{R}$ is continuous and the function $\mathbb{R} \setminus \{\xi_1, \ldots, \xi_N\} \ni \xi \mapsto \theta(x, \xi) \in \mathbb{R}$ is differentiable, and (58) holds for a.e. $x \in \Omega$, where $\xi, \eta \in \mathbb{R}, \xi \in \mathbb{R} \setminus \{\xi_1, \ldots, \xi_N\}$. If $u, u_t \in L^{p(x)}(Q_{0,T})$, then $\theta(x, u), (\theta(x, u)_t)_t \in L^{p(x)}(Q_{0,T})$ and (59) holds.

Proof. For the sake of convenience, only the case $N = 1$ and $\xi_1 = 0$ is considered (see for comparison [44, p. 100]). It is easy to verify that

$$
\theta(x, u) := \theta(x, u^+) + \theta(x, -u^-) - \theta(x, 0).
$$

(60)

Since $u, u_t \in L^{p(x)}(Q_{0,T}) \subset L^{p_0}(Q_{0,T})$, we have that $(u^\pm)_t \in L^{p_0}(Q_{0,T})$ and $(u^\pm)_t = \pm \check{\theta}(u)u_t$, where $\check{\theta}$ is determined from (51). Then by Corollary 3.15, we obtain the formulas of type (59) for every term in (60). Therefore, (59) holds. By (58) and (59), we get $(\theta(x, u))_t \in L^{p(x)}(Q_{0,T})$.

Lemma 3.17. Suppose that $\beta \in \mathcal{B}_+(\Omega), \psi_{\beta(x)}$ is determined from (50) if we replace $\sigma$ by $\beta$, and

$$
\chi_k(s) := \begin{cases} 
1 & \text{if } s > \frac{1}{k}, \\
0 & \text{if } s \leq \frac{1}{k},
\end{cases} \quad k \in \mathbb{N}.
$$

(61)

If $u \in C^1(Q_{0,T})$ and $v, v_t \in L^1(Q_{0,T})$, then

$$
\lim_{k \rightarrow +\infty} \int_{Q_{0,T}} \chi_k(u) \beta(x) \psi_{\beta(x)-1}(u)u_t \, v \, dx \, dt = \int_{\Omega} \psi_{\beta(x)}(u) \, v \, dx \bigg|_{t=0}^{t=T} - \int_{Q_{0,T}} \psi_{\beta(x)}(u) \, v_t \, dx \, dt.
$$

(62)

Proof. By definition, set

$$
\psi_{\beta(x), k}(s) := \begin{cases} 
k^{\beta(x)} & \text{if } s \geq k, \\
 s^{\beta(x)} & \text{if } \frac{1}{k} < s < k, \\
 1 & \text{if } s \leq \frac{1}{k},
\end{cases}
\quad \check{\xi}_{\beta(x), k}(s) := \begin{cases} 
\beta(x) s^{\beta(x)-1} & \text{if } \frac{1}{k} < s < k, \\
0 & \text{if } s \leq \frac{1}{k} \text{ and } s \geq k,
\end{cases}
$$

$k \in \mathbb{N}, k \geq 2, x \in \Omega$. Clearly, $\psi_{\beta(x), k}(s) \rightarrow \psi_{\beta(x)}(s)$, where $s \in \mathbb{R}, x \in \Omega$. In addition, for $k \in \mathbb{N}, k \geq 2$ and $x \in \Omega$ the function $s \mapsto \psi_{\beta(x), k}(s)$ has the Lipschitz property in $\mathbb{R}$ and it is not differentiable only in the point $s = \frac{1}{k}$ and $s = k$. Moreover, $\frac{\partial}{\partial s} \psi_{\beta(x), k}(s) = \check{\xi}_{\beta(x), k}(s)$ if $s \neq \frac{1}{k}$ and $s \neq k$. Whence, by Lemma 3.16, we obtain

$$
(\psi_{\beta(x), k}(u))_t = \check{\xi}_{\beta(x), k}(u)u_t \quad \text{almost everywhere in } Q_{0,T}.
$$

(63)

Thus, $\psi_{\beta(x), k}(u), (\psi_{\beta(x), k}(u))_t \in L^{\infty}(Q_{0,T})$. Using case (ii) of Lemma 3.12 with $z = \psi_{\beta(x), k}(u)$ and $w = v$, we get (53), i.e.

$$
\int_{Q_{0,T}} (\psi_{\beta(x), k}(u))_t \, v \, dx \, dt = \int_{\Omega} \psi_{\beta(x), k}(u) \, v \, dx \bigg|_{t=0}^{t=T} - \int_{Q_{0,T}} \psi_{\beta(x), k}(u) \, v_t \, dx \, dt.
$$

(64)
Let $M := \max_{(x,t) \in \Omega(0,T)} |u(x,t)|$, $k_0 \in \mathbb{N}$, $k_0 \geq \max\{2, M\}$. Since $|u| \leq M \leq k_0 \leq k$, from (63) we have
\[
\left(\psi_{\beta(x,k)}(u)\right) = \tilde{\psi}_{\beta(x,k)}(u) = \chi_k(u) \beta(x) \psi_{\beta(x)-1}(u)u_t,
\]
where $k \geq k_0$. By $|\psi_{\beta(x,k)}(u(x,t))| \leq M^2(x) \forall (x,t) \in \overline{Q_{0,T}}$ and Lebesgue's Dominated Convergence Theorem (see [33, p. 90]), we obtain
\[
\lim_{k \to +\infty} \int_{\omega} \psi_{\beta(x,k)}(u) v \, dx = \int_{\omega} \psi_{\beta(x)}(u) v \, dx \quad \text{if} \quad t = 0 \quad \text{and} \quad t = T,
\]
\[
\lim_{k \to +\infty} \int_{Q_{0,T}} \psi_{\beta(x,k)}(u) v_t \, dx dt = \int_{Q_{0,T}} \psi_{\beta(x)}(u) v_t \, dx dt.
\]
Therefore, (62) follows from (64).

\[ \square \]

**Theorem 3.18.** Suppose that $\sigma \in B_+\left(\Omega, \sigma_0 > 1, \right.$ and the function $\psi_{\sigma(x)}$ is determined from (50). Then the following statements are satisfied:
1) if $u \in C^1(\overline{Q_{0,T}})$, then $\psi_{\sigma(x)}(u)_t \left(\psi_{\sigma(x)}(u)\right)_t \in L^\infty(\overline{Q_{0,T}})$ and
\[
\left(\psi_{\sigma(x)}(u)\right)_t = \sigma(x) \psi_{\sigma(x)-1}(u) u_t; \quad (65)
\]
2) if $u, u_t \in L^p(\overline{Q_{0,T}})$, where $p \in L^\infty(\Omega)$ and $p(x) \geq \sigma(x)$ for a.e. $x \in \Omega$, then
\[
\psi_{\sigma(x)}(u), \left(\psi_{\sigma(x)}(u)\right)_t \in L^p(\overline{Q_{0,T}}), \text{ equality (65) is true, and the estimate}
\]
\[
\rho_p/\sigma\left(\psi_{\sigma(x)}(u)\right)_t \leq C_2 S_1/\sigma \left(\psi_{\sigma(x)}(u)\right)_t \leq C_2 S_1/\sigma \left(\rho_p(u_t; Q_{0,T})\right);
\]
holds, where $C_2 > 0$ is independent of $u$.

**Proof.** First let us prove Case 1. Take a function $u \in C^1(\overline{Q_{0,T}})$. If $v, v_t \in C(\overline{Q_{0,T}})$, $\chi_k$ is determined from (61), and $k \in \mathbb{N}$, then $|\chi_k(u) \sigma(x) \psi_{\sigma(x)-1}(u) u_t| \leq C_0$, where $C_0 > 0$ is independent of $k, x, t$. Hence, Lebesgue's Dominated Convergence Theorem (see [33, p. 90]) yields that
\[
\lim_{k \to +\infty} \int_{Q_{0,T}} \chi_k(u) \sigma(x) \psi_{\sigma(x)-1}(u) u_t v \, dx dt = \int_{Q_{0,T}} \sigma(x) \psi_{\sigma(x)-1}(u) u_t v \, dx dt.
\]
Using (62) with $\beta = \sigma > 1$, we obtain
\[
\int_{Q_{0,T}} \sigma(x) \psi_{\sigma(x)-1}(u) u_t v \, dx dt = \int_{Q_{0,T}} \psi_{\sigma(x)}(u) v \, dx \bigg|_{t=T}^{t=0} - \int_{Q_{0,T}} \psi_{\sigma(x)}(u) v_t \, dx dt. \quad (67)
\]
Taking in (67) the function $v \in C^\infty_c(\overline{Q_{0,T}})$, we get
\[
\int_{Q_{0,T}} \sigma(x) \psi_{\sigma(x)-1}(u) u_t v \, dx dt = - \int_{Q_{0,T}} \psi_{\sigma(x)}(u) v_t \, dx dt
\]
(notice that $\sigma \psi_{\sigma(x)-1}(u) u_t \in L^\infty(\overline{Q_{0,T}})$ because $\sigma_0 > 1$). Therefore, (65) holds.

Since $\sigma_0 > 1$, from (50) we have $\psi_{\sigma(x)} \in L^\infty(\overline{Q_{0,T}})$ and from (65) we have $\left(\psi_{\sigma(x)}(u)\right)_t \in L^\infty(\overline{Q_{0,T}})$.

Now let us prove Case 2. Suppose $u \in U$, where $U := \{u \in L^p(\overline{Q_{0,T}}) \mid u_t \in L^p(\overline{Q_{0,T}})\}$. Clearly, $C^\infty([0,T]; C^1(\Omega)) \supset W^{1,\rho_p}(0,T; L^p(\Omega)) \supset U \supset W^{1,\rho_p}(0,T; L^p(\Omega))$. Then there exists a sequence $\{u^m\}_{m \in \mathbb{N}} \subset C^\infty(\overline{Q_{0,T}})$ such that $u^m \to u$ and $u^m_t \to u_t$ strongly in $L^p(\overline{Q_{0,T}})$, $u^m \to u$ in $C([0,T]; L^p(\Omega))$.

Assume that $v, v_t \in C(\overline{Q_{0,T}})$. By (67), for every $m \in \mathbb{N}$ we obtain
\[
\int_{Q_{0,T}} \sigma(x) \psi_{\sigma(x)-1}(u^m) u^m_t v \, dx dt = \int_{Q_{0,T}} \psi_{\sigma(x)}(u^m) v \, dx \bigg|_{t=T}^{t=0} - \int_{Q_{0,T}} \psi_{\sigma(x)}(u^m) v_t \, dx dt. \quad (68)
\]
Since $1 < \sigma(x) \leq p(x)$, we get $\frac{\rho(x)}{\sigma(x)} > 1$ for a.e. $x \in \Omega$. Therefore,

$$
\psi_{\sigma(x)}^{-1}(u^m_{m \to \infty}) \to \psi_{\sigma(x)}^{-1}(u) \quad \text{strongly in } L^{\frac{\rho(x)}{\sigma(x)-1}}(Q_{0,T}).
$$

Clearly, $[L^{\frac{\rho(x)}{\sigma(x)-1}}(Q_{0,T})]^* \cong L^{\frac{\rho(x)}{\rho(x)-(\sigma(x)-1)}}(Q_{0,T})$. Since $p(x) \geq (\sigma(x) - 1) + 1$, we have that $p(x) \geq \frac{\rho(x)}{\rho(x)-(\sigma(x)-1)}$ for a.e. $x \in \Omega$. Therefore,

$$
u_{m \to \infty}^m \to \nu \quad \text{strongly in } L^{\frac{\rho(x)}{\rho(x)-(\sigma(x)-1)}}(Q_{0,T}).
$$

By Lemma 5.2 [23, p. 19], we obtain

$$
\int_{Q_{0,T}} \sigma(x)\psi_{\sigma(x)}^{-1}(u^m_{m \to \infty}) v \, dx \, dt \to \int_{Q_{0,T}} \sigma(x)\psi_{\sigma(x)}^{-1}(u) v \, dx \, dt
$$

and $\sigma_{\sigma(x)}^{-1}(u)\nu_{m \to \infty} \in L^1(Q_{0,T})$. It is easy to verify that

$$
\int_{Q_{0,T}} \psi_{\sigma(x)}(u^m_{m \to \infty}) v \, dx \, dt \to \int_{Q_{0,T}} \psi_{\sigma(x)}(u) v \, dx \, dt
$$

and $\psi_{\sigma(x)}(u)\nu_{m \to \infty} \in L^1(Q_{0,T})$. It is easy to verify that

By (65) and generalized Young’s inequality, we obtain

$$
\left| \int_{Q_{0,T}} \psi_{\sigma(x)}(u) \, dx \, dt \right| \leq C_{1/\sigma} \left| \int_{Q_{0,T}} |u|^{\frac{\rho(x)}{\sigma(x)}} \, dx \, dt \right| \left| \int_{Q_{0,T}} |\nu|^{\frac{\rho(x)}{\sigma(x)}} \, dx \, dt \right|.
$$

This implies (66) and completes the proof of Theorem 3.18. \qed

Note that the case $\sigma(x) \equiv \in [0, 1]$ is considered in [45].

**Theorem 3.19.** Suppose that $r \in B_+(\Omega)$. Then the following statements are satisfied:

1) If $r_0 > 1$, then the equality

$$
|u|^{r(x)} = (r(x)|u|^{r(x)-2} u)_{t} = (r(x)|u|^{r(x)-2} u)_{t} + (r(x)|u|^{r(x)-2} u)_{t}
$$

is true if one of the following alternatives holds:

(i) $u \in C^1(Q_{0,T})$ (here we have $|u|^{r(x)}, (|u|^{r(x)})_{t} \in L^\infty(Q_{0,T})$);

(ii) $u, \nu_{m \to \infty} \in L^{r(x)}(Q_{0,T})$ and $r(x) \geq r(x)$ for a.e. $x \in \Omega$ (here we have $|u|^{r(x)}, (|u|^{r(x)})_{t} \in L^{r(x)}(Q_{0,T})$).

2) If $r_0 \geq 2$, then the equality

$$
|u|^{r(x)-2} u_{t} = (r(x)-1)|u|^{r(x)-2} u_{t}
$$

is true if one of the following alternatives hold:

(i) $u \in C^1(Q_{0,T})$ (here we have $|u|^{r(x)-2} \nu_{m \to \infty}^{r(x)-2} \nu_{m \to \infty} \in L^\infty(Q_{0,T})$);

(ii) $u, \nu_{m \to \infty} \in L^{r(x)}(Q_{0,T})$ and $r(x) \geq r(x)-1$ for a.e. $x \in \Omega$ (here $|u|^{r(x)-2} \nu_{m \to \infty}^{r(x)-2} \nu_{m \to \infty} \in L^{r(x)-1}(Q_{0,T})$).

**Proof.** Suppose that $\psi_{r(x)-2}$ is determined from (50) if we replace $\sigma$ by $r-2$. Then the proof follows from Theorem 3.18 since

$$
|x|^{r(x)} = \psi_{r(x)}(x) + \psi_{r(x)}(-x), \quad |x|^{r(x)-2} = \psi_{r(x)-1}(x) - \psi_{r(x)-1}(-x), \quad x \in \Omega, \quad s \in \mathbb{R}.
$$

\qed
3.5 Cauchy’s problem for system of ordinary differential equations

Take $Q = (0, T) \times \mathbb{R}^\ell$, where $\ell \in \mathbb{N}$. In this section, we seek a weak solution $\varphi : [0, T] \rightarrow \mathbb{R}^\ell$ of the problem
\[
\varphi'(t) + L(t, \varphi(t)) = M(t), \quad t \in [0, T], \quad \varphi(0) = \varphi^0, \quad (74)
\]
where $M : [0, T] \rightarrow \mathbb{R}^\ell$, $L : Q \rightarrow \mathbb{R}^\ell$ are some functions (for the sake of convenience we have assumed that $L(t, 0) = 0$ for every $t \in [0, T]$) and $\varphi^0 = (\varphi^0_1, \ldots, \varphi^0_\ell) \in \mathbb{R}^\ell$.

The following Definitions are needed for the sequel.

**Definition 3.20.** A real-valued function $\varphi \in W^{1,1}_0(0, T; \mathbb{R}^\ell)$ is called a weak solution of problem (74) if it satisfies the initial value condition and satisfies the equation almost everywhere.

**Definition 3.21.** We shall say that a function $L : Q \rightarrow \mathbb{R}^\ell$ satisfies the Carathéodory condition if for every $\xi \in \mathbb{R}^\ell$ the function $(0, T) \ni t \mapsto L(t, \xi) \in \mathbb{R}^\ell$ is measurable and if for a.e. $t \in (0, T)$ the function $R^\ell \ni \xi \mapsto L(t, \xi) \in \mathbb{R}^\ell$ is continuous.

**Definition 3.22** (see [46, p. 241]). We shall say that a function $L : Q \rightarrow \mathbb{R}^\ell$ satisfies the $L^p$-Carathéodory condition if $L$ satisfies the Carathéodory condition and for every $R > 0$ there exists a function $h_R \in L^p(0, T)$ such that
\[
|L(t, \xi)| \leq h_R(t) \quad (75)
\]
a.e. for $t \in (0, T)$ and for every $\xi \in \overline{D}_R := \{ y \in \mathbb{R}^\ell \mid |y| \leq R \}$.

**Proposition 3.23** (Gronwall-Bellman’s Lemma [47, p. 25]). Suppose that $A, B \in L^1(0, T)$ and $y \in C([0, T])$ are nonnegative functions. If for every $\tau \in [0, T]$ we have
\[
y(\tau) \leq C + \int_0^\tau A(t)y(t) + B(t) \, dt, \quad (76)
\]
where $C$ is a nonnegative number, then the following inequality is true
\[
y(\tau) \leq \left( C + \int_0^\tau B(t) e^{\frac{-\tau}{\rho} \int_0^t A(s) \, ds} \, dt \right) e^{\int_0^\tau A(t) \, dt}, \quad \tau \in [0, T]. \quad (77)
\]
We will need the following Theorem.

**Theorem 3.24** (Carathéodory-LaSalle’s Theorem). Suppose that $p \geq 2$, function $L : Q \rightarrow \mathbb{R}^\ell$ satisfies $L^p$-Carathéodory condition, $M \in L^p(0, T; \mathbb{R}^\ell)$, and $\varphi^0 \in \mathbb{R}^\ell$. If there exists a nonnegative functions $\alpha, \beta \in L^1(0, T)$ such that for every $\xi \in \mathbb{R}^\ell$ and for a.e. $\tau \in [0, T]$ the inequality
\[
(L(t, \xi), \xi)_{\mathbb{R}^\ell} \geq -\alpha(t)|\xi|^2 - \beta(t) \quad (78)
\]
holds, then problem (74) has a global weak solution $\varphi \in W^{1,p}(0, T; \mathbb{R}^\ell)$.

**Proof.** We modify the method employed in the proof of Theorem 3 [48, p. 240]. According to the Carathéodory Theorem [49, p. 17], we have a local weak solution $\varphi \in W^{1,p}(0, b; \mathbb{R}^\ell)$ ($b \in (0, T]$) to the Cauchy problem (74) such that for every $\tau \in [0, b]$ the equality
\[
\varphi(\tau) = \varphi^0 + \int_0^\tau M(t) \, dt - \int_0^\tau L(t, \varphi(t)) \, dt \quad (79)
\]
holds. If $b = T$, then Theorem 3.24 is proved. If $b < T$, then we take $\varphi^1 := \varphi(b)$ and consider the equation from (74) with new initial value condition $\varphi(b) = \varphi^1$. Using the Carathéodory Theorem and (79), we extend solution
to problem (74) into \([b, b_1]\), where \(b_1 \leq T\) etc. Thus, similarly to [50, p. 22-24], we have one of the following possibility:
1) solution to problem (74) can be extended into \([0, T]\);
2) there exists a weak solution to problem (74) which is defined on right maximal interval of existence \([0, \tilde{b}]\), where \(\tilde{b} \leq T\).

We shall prove that Case 2 is impossible. Assume the converse. Then for every \(\tau \in (0, \tilde{b})\) this local weak solution \(\varphi\) belongs to \(W^{1,p}(0, \tau; \mathbb{R}^\ell)\). Define

\[
R := \left\{\left(\phi(0)^2 + \int_0^T [2\beta(t) + |M(t)|^2] \, dt\right)^{1/2}\right\},
\]

where \(\alpha\) and \(\beta\) are determined from (78). Since \(L\) satisfies the \(L^p\)-Carathéodory condition and \(R\) is determined from (80), there exists a function \(h_R \in L^p(0, T)\) such that for a.e. \(t \in (0, T)\) and for every \(\xi \in \overline{D_R} := \{y \in \mathbb{R}^\ell \mid |y| \leq R\}\) inequality (75) holds.

Taking into account (see (78)) the following inequalities

\[
(L(t, \varphi(t)), \varphi(t))_{\mathbb{R}^\ell} \geq -\alpha(t)|\varphi(t)|^2 - \beta(t), \quad (M(t, \varphi(t)))_{\mathbb{R}^\ell} \leq |M(t)| \cdot |\varphi(t)| \leq \frac{1}{2}|M(t)|^2 + \frac{1}{2}|\varphi(t)|^2,
\]

from (74) we get

\[
(\varphi'(t), \varphi(t))_{\mathbb{R}^\ell} - \alpha(t)|\varphi(t)|^2 - \beta(t) \leq \frac{1}{2}|M(t)|^2 + \frac{1}{2}|\varphi(t)|^2, \quad t \in [0, \tilde{b}).
\]

Hence,

\[
\int_0^\tau (\varphi'(t), \varphi(t))_{\mathbb{R}^\ell} \, dt \leq \int_0^\tau \left[(\alpha(t) + \frac{1}{2})|\varphi(t)|^2 + \beta(t) + \frac{1}{2}|M(t)|^2\right] \, dt, \quad \tau \in (0, \tilde{b}).
\]

Since \(\varphi \in W^{1,p}(0, \tau; \mathbb{R}^\ell)\) and \(p \geq 2\), we obtain

\[
|\varphi|^2 \in W^{1,2}(0, \tau), \quad (\varphi(t))^2 = 2(\varphi'(t), \varphi(t))_{\mathbb{R}^\ell}, \quad t \in (0, \tau).
\]

Hence Proposition 3.10 implies that

\[
\int_0^\tau (\varphi'(t), \varphi(t))_{\mathbb{R}^\ell} \, dt = \frac{1}{2}|\varphi(t)|^2 - \frac{1}{2}|\varphi(0)|^2.
\]

Whence (81) has a form (76), where \(C = |\varphi(0)|^2\),

\[
y(t) = |\varphi(t)|^2, \quad A(t) = 2\alpha(t) + 1, \quad B(t) = 2\beta(t) + |M(t)|^2, \quad t \in (0, \tau).
\]

Therefore, from (77) we get

\[
y(t) \leq \left(C + \int_0^t B(s) \, ds\right) e^{\int_0^t A(s) \, ds} \leq \left(C + \int_0^t B(t) \, dt\right) e^{\int_0^t A(t) \, dt} \leq R^2,
\]

where \(R\) is determined from (80). Thus \(|\varphi(t)| \leq R, \tau \in (0, \frac{1}{2})\), i.e. the point \(\varphi(t)\) belongs to \(D_R\), where \(t \in (0, \tilde{b})\).

By (75), we have that \(|L(t, \varphi(t))| \leq h_R(t), \tau \in (0, \tilde{b})\). Therefore, (79) yields that

\[
|\varphi(t_2) - \varphi(t_1)| = \left|\int_{t_1}^{t_2} L(t, \varphi(t)) \, dt\right| \leq \left|\int_{t_1}^{t_2} h_R(t) \, dt\right| \xrightarrow{t_1, t_2 \to \tilde{b} - 0} 0.
\]

Finally we have an existence of the finite limit \(\lim_{t \to \tilde{b} - 0} \varphi(t)\). Then solution to problem (74) can be extended to \([0, \tilde{b}]\) by the rule \(\varphi(\tilde{b}) := \lim_{t \to \tilde{b} - 0} \varphi(t) < \infty\). This contradiction completes the proof Theorem 3.24.

If \(L\) is slowly continuous with respect to \(\varphi\), then Theorem 3.24 follows from Theorem 3 [48, p. 240]. If \(M \equiv 0\) and \(L\) is continuous, then Theorem 3.24 coincides with Lemma 4 [51, p. 67].
3.6 Some integral expressions

The following lemmas will be needed in the sequel.

**Lemma 3.25** (see for comparison Lemma 2.3 [31, p. 26]). *Suppose that condition (Q) is satisfied, \( g \in L^\infty(Q_0,T) \), \( z \in L^{q(x)}(\Omega), m \in \mathbb{N}, \xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m, w^1, \ldots, w^m \in L^{q(x)}(\Omega) \), and \( w(x,\xi) = \sum_{i=1}^m \xi_i w^i(x) \). Then the function

\[
I(t,\xi) := \int_\Omega g(x,t) |w(x,\xi)|^{q(x)-2} w(x,\xi) z(x) \, dx, \quad t \in (0,T), \quad \xi \in \mathbb{R}^m,
\]

satisfies the \( L^\infty \)-Carathéodory condition.

**Proof.** Step 1. The Fubini Theorem [33, p. 91] yields that \( I(\cdot,\xi) \in L^1(0,T) \). Then the function \([0,T] \ni t \mapsto I(t,\xi) \in \mathbb{R} \) is measurable.

Step 2. We prove that the function \( \mathbb{R} \ni \xi \mapsto I(t,\xi_1,\ldots,\xi_m) \in \mathbb{R} \) is continuous at the point \( \xi^0 \in \mathbb{R} \). Take \( \xi = (\xi_1, \xi_2, \ldots, \xi_m), \xi^0 = (\xi^0_1, \xi^0_2, \ldots, \xi^0_m) \), where \( |\xi - \xi^0| \leq 1 \).

By Theorem 2.1 [52, p. 2], we get

\[
|I(t,\xi) - I(t,\xi^0)| \leq C_{10} |\eta_1| |\eta_2|^{q(x)-2} |\eta_2|^{q(x)-1-\beta(x)} |\eta_1 - \eta^0_2|^{\beta(x)},
\]

where \( 0 < \beta(x) \leq \min\{1, q(x)-1\} \), \( \eta_1, \eta_2 \in \mathbb{R} \), \( C_{10} > 0 \) is independent of \( \eta_1, \eta_2, x \). Hence,

\[
I(t,\xi) - I(t,\xi^0) = \int_\Omega g \left( |w(x,\xi)|^{q(x)-2} w(x,\xi) - |w(x,\xi^0)|^{q(x)-2} w(x,\xi^0) \right) z \, dx
\]

\[
\leq C_{11} \int_\Omega \left( |w(x,\xi)| + |w(x,\xi^0)| \right)^{q(x)-1-\beta(x)} |w(x,\xi) - w(x,\xi^0)|^{\beta(x)} |z| \, dx = C_{11}(I_1 + I_2),
\]

where

\[
I_1 = \int_{\Omega_1} h(x,\xi,\xi^0) \, dx, \quad I_2 = \int_{\Omega_2} h(x,\xi,\xi^0) \, dx,
\]

\( \Omega_1 = \{x \in \Omega \mid q(x) \leq 2\}, \Omega_2 = \{x \in \Omega \mid q(x) > 2\} \), and

\[
h(x,\xi,\xi^0) = \left( |w(x,\xi)| + |w(x,\xi^0)| \right)^{q(x)-1-\beta(x)} |w(x,\xi) - w(x,\xi^0)|^{\beta(x)} |z(x)|, \quad x \in \Omega.
\]

By taking \( \beta(x) = q(x)-1 \), where \( x \in \Omega_1 \), we obtain

\[
I_1 = \int_{\Omega_1} |w(x,\xi) - w(x,\xi^0)|^{q(x)-1} |z(x)| \, dx = \int_{\Omega_1} |\xi_1 - \xi_1^0|^{q(x)-1} |w^1(x)|^{q(x)-1} |z(x)| \, dx
\]

\[
\leq |\xi_1 - \xi_1^0|^{q(x)-1} \int_{\Omega_1} |w^1(x)|^{q(x)-1} |z(x)| \, dx = C_{12} |\xi_1 - \xi_1^0|^{q(x)-1} \rightarrow 0, \quad \xi_1 \rightarrow \xi_1^0.
\]

By taking \( \beta(x) = 1 \), where \( x \in \Omega_2 \), we obtain

\[
I_2 = \int_{\Omega_2} \left( |w(x,\xi)| + |w(x,\xi^0)| \right)^{q(x)-2} |w(x,\xi) - w(x,\xi^0)| \cdot |z(x)| \, dx
\]

\[
= |\xi_1 - \xi_1^0| \int_{\Omega_2} \left( |w(x,\xi)| + |w(x,\xi^0)| \right)^{q(x)-2} |w^1(x)| \cdot |z(x)| \, dx \leq C_{13} |\xi_1 - \xi_1^0| \rightarrow 0, \quad \xi_1 \rightarrow \xi_1^0.
\]

Therefore, by (84), we obtain that \( |I(t,\xi) - I(t,\xi^0)| \rightarrow 0 \). Continuing in the same way, we see that \( I \) is continuous with respect to \( \xi_2, \ldots, \xi_m \).

Step 3. Taking into account the results of Step 1 and Step 2, we obtain that the function \( I \) satisfies the Carathéodory condition. Since \( g \in L^\infty(Q_0,T) \), the \( L^\infty \)-Carathéodory condition holds. \( \square \)
Lemma 3.26. Suppose that condition (E) is satisfied,

$$
(Eu)(x,t) := \int_\Omega \epsilon(x,t,y) \left( \overline{u}(x+y,t) - \overline{u}(x,t) \right) \, dy, \quad (x,t) \in Q_{0,T},
$$

(85)

where $u \in L^1(Q_{0,T})$, $\overline{u}$ is the zero extension of $u$ from $Q_{0,T}$ into $(\mathbb{R}^n \setminus \Omega) \times (0,T)$. Then for every $s > 1$ the operator $E : L^s(Q_{0,T}) \to L^s(Q_{0,T})$ is linear bounded continuous and

$$
||Eu; L^s(Q_{0,T})|| \leq C_{14}||u; L^s(Q_{0,T})||, \quad u \in L^s(Q_{0,T}), \quad \tau \in (0,T),
$$

(86)

where $C_{14} > 0$ is independent of $u$ and $\tau$.

The proof is trivial.

Lemma 3.27. Suppose that $\phi \in \text{Lip}(\mathbb{R})$, $\epsilon \in L^\infty(Q_{0,T} \times \Omega)$, $z \in L^2(\Omega)$, $m \in \mathbb{N}$, $\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m$, $w^1, \ldots, w^m \in L^2(\Omega)$, $w(x, \xi) = \sum_{l=1}^m \xi_l w^l(x)$, $x \in \Omega$, and the operator $E$ is determined from (85). Then the function

$$
J(t, \xi) := \int_\Omega \phi((Ew(\cdot, \xi))(x,t)) z(x) \, dx, \quad t \in (0,T), \quad \xi \in \mathbb{R}^m,
$$

(87)

satisfies the $L^\infty$-Carathéodory condition.

Proof. Step 1. Lemma 3.26 implies that $Ew \in L^2(Q_{0,T})$ if $\xi \in \mathbb{R}^m$. Hence $\phi(Ew) \in L^2(Q_{0,T}) \subset L^1(Q_{0,T})$. The Fubini Theorem [33, p. 91] yields that $J(t, \xi) \in L^1(0,T)$. Then the function $[0,T] \ni t \mapsto J(t, \xi) \in \mathbb{R}$ is measurable.

Step 2. Take a point $t \in (0,T)$. We prove that the function $\mathbb{R} \ni \xi \mapsto J(t, \xi_1, \ldots, \xi_m) \in \mathbb{R}$ is continuous at the point $\xi^0 \in \mathbb{R}$. Take $\xi = (\xi^1, \xi^2, \ldots, \xi^m), \xi^0 = (\xi^1_0, \xi^2_0, \ldots, \xi^m_0)$. Then

$$
|J(t, \xi) - J(t, \xi^0)| \leq C_{15} \int_\Omega \left| \phi((Ew(\cdot, \xi))(x,t)) - \phi((Ew(\cdot, \xi^0))(x,t)) \right| \cdot |z(x)| \, dx
$$

$$
\leq C_{15} \int_\Omega \left| (Ew(\cdot, \xi))(x,t) - (Ew(\cdot, \xi^0))(x,t) \right| \cdot |z(x)| \, dx
$$

$$
= C_{15} \int_\Omega \left| \epsilon(x,t,y) \left( w(x+y, \xi) - w(x, \xi) - (w(x+y, \xi^0) - w(x, \xi^0)) \right) \right| \, dy \cdot |z(x)| \, dx
$$

$$
\leq C_{16} |\xi_1 - \xi^0_1| \int_\Omega \int_\Omega \left| w^1(x+y) + w^1(x) \right| \cdot |z(x)| \, dx \, dy = C_{17} |\xi_1 - \xi^0_1| \rightarrow 0.
$$

Continuing in the same way, we see that $J$ is continuous with respect to $\xi_2, \ldots, \xi_m$.

Step 3. Taking into account the results of Step 1 and Step 2, we obtain that the function $J$ satisfies the Carathéodory condition. Since $\epsilon \in L^\infty(Q_{0,T} \times \Omega)$, the $L^\infty$-Carathéodory condition holds. $\square$

Clearly, the operator $\Lambda(t) : Z^N \to [Z^N]^*$ (see (16)) is linear, bounded, continuous and monotone. Similarly as in Theorem 3.4 [53, p. 545], we prove that $A(t) : X^N \to [X^N]^*$ (see (17) ) is bounded, semiconcious and monotone if $p \in B_+(\Omega), p_0 > 1$, and condition (A) is satisfied. The operator $G(t) : \mathcal{O}^N \to [\mathcal{O}^N]^*$ (see (4)) is bounded, semicontinuous and monotone. Similarly to (86), we get the estimate

$$
||(Ew(t); L^s(\Omega))^N|| \leq C_{18} ||w; L^s(\Omega)^N||, \quad w \in [L^s(\Omega)]^N, \quad t \in [0,T],
$$

(88)

where $s > 1$ and $C_{18} > 0$ is independent of $w$ and $t$. Using condition $(\Phi)$, we get that the operator $[L^s(\Omega)]^N \ni u \mapsto \phi(Eu) \in [L^s(\Omega)]^N$ is bounded and continuous.
Lemma 3.28. Suppose that conditions (Γ), (B), and (Z) are satisfied, the operator Ψ is determined from (18). Then Ψ(t) : Z^N → [Z^N]^m is bounded and semicontinuous. Moreover,

\[ |\Psi(t)u, v| \leq C_{19}S_{1/\gamma'}(S_\gamma(||u; H^N||)||v; Z^N||) ||v; Z^N||, \quad u, v \in Z^N, \quad t \in (0, T), \] (89)

where \( S_{1/\gamma'} \) and \( S_\gamma \) are defined by (8), \( C_{19} > 0 \) is independent of \( u, v \) and \( t \).

Proof. Similar to [54, p. 159], we use the generalized Hölder inequality, Proposition 3.3 with \( q = \gamma \), and notation (7). We get the estimate

\[ |\Psi(t)u, v| = \left( \int_{\Omega} \sum_{k=1}^{N} b_k(x, t)|u|^{\gamma(x)-2}v_k \, dx \right) \leq b_0 \left( \int_{\Omega} |u|^{\gamma(x)-1} |\Delta v| \, dx \right) \]

\[ \leq 2b_0 ||u|^{\gamma(x)-1}; L^{\gamma'(x)}(\Omega)|| \cdot ||\Delta v; L^{\gamma(x)}(\Omega)|| \leq 2b_0 S_{1/\gamma'} \left( \int_{\Omega} |u|^{(\gamma(x)-1)\gamma'(x)} \, dx \right) \]

\[ \cdot ||\Delta v; L^{\gamma(x)}(\Omega)|| \leq C_{20}S_{1/\gamma'}(S_\gamma)||u; [L^{\gamma(x)}(\Omega)]^N|| \cdot ||\Delta v; [L^{\gamma(x)}(\Omega)]^N||. \]

Since \( \gamma^0 \leq 2 \), we obtain that (89) holds and the operator Ψ is bounded. We omit the proof that Ψ is semicontinuous (it is similar to the proof of Lemma 3.25).

Let us consider the Banach space \( \mathcal{V} \) such that \( \mathcal{V} \supset Z^N \). Let us define the family of operators \( \Psi_\gamma(t) : \mathcal{V} \to \mathcal{V}^m \) by the rule

\[ \langle \Psi_\gamma(t)u, v \rangle := \langle \Psi(t)u, v \rangle, \quad u, v \in \mathcal{V}, \quad t \in [0, T]. \]

By (89), we obtain

\[ |\Psi_\gamma(t)u, v| \leq C_{21}S_{1/\gamma'}(S_\gamma)||u; [\mathcal{V}]|| ||v; \mathcal{V}||, \quad u, v \in \mathcal{V}, \quad t \in (0, T), \] (90)

where \( C_{21} > 0 \) is independent of \( u, v \) and \( t \). Then \( \Psi_\gamma : \mathcal{V} \to \mathcal{V}^m \) is bounded. We will replace this space \( \mathcal{V} \) by \( \mathcal{V}^N \) and \( \mathcal{W}_0 \). For the sake of convenience we have replaced \( \Psi_{\gamma, N} \) and \( \Psi_{\gamma, 0} \) by Ψ and we have replaced \( \langle \cdot, \cdot \rangle_{V, N} \) and \( \langle \cdot, \cdot \rangle_{V, 0} \) by \( \langle \cdot, \cdot \rangle \). The same notation we need for \( \Lambda(t), A(t), \) and \( \mathcal{K}(t), t \in (0, T) \). According to the above remarks, we have that the operator \( \mathcal{K}(t) \) (see (19)) is bounded and semicontinuous from \( \mathcal{V}^N \) into \( \mathcal{V}^N \) and is bounded from \( \mathcal{W}_0 \) into \( \mathcal{W}_0^m \).

Lemma 3.29. Suppose that conditions (Γ), (Q), (A)-(E), (7), and (21) hold. Assume also that \( \alpha > 0, p \in B_+(\Omega), p_0 > 1, \{w^j\}_{j \in \mathbb{N}} \subset \mathcal{V}, m \in \mathbb{N}, L = (L_{11}, L_{21}, \ldots, L_{m1}, \ldots, L_{1N}, L_{2N}, \ldots, L_{mN}), \)

\[ L_{\mu k}(t, \xi) = (g_k(t)(z)_{\mu k}(w^\mu)(\xi)_{\mu k} + (N(t)z)_{\mu k}(w^\mu)(\xi)_{\mu k}) \Omega, \quad k = 1, N, \quad \mu = 1, m, \quad t \in (0, T), \]

\[ \xi = (\xi_{11}, \xi_{21}, \ldots, \xi_{m1}, \ldots, \xi_{1N}, \xi_{2N}, \ldots, \xi_{mN}), \quad z = (z_1, \ldots, z_N), \]

and

\[ z_k(x) = \sum_{\ell=1}^{m} \xi_{k\ell} w^\ell(x), \quad x \in \Omega, \quad k = 1, N. \]

Then

\[ (L(t, \xi), \xi)_{m} \geq \int_{\Omega} \left[ \frac{\alpha}{2} |\Delta z|^2 + a_0 \sum_{i=1}^{n} |z_i|^{p(x)} + g_0 |z|^q(x) - C_{22} |z|^2 \right] \, dx - C_{23}, \quad t \in (0, T), \] (92)

where \( C_{22}, C_{23} \) > 0 are independent of \( z, \xi \) and \( t \).

Proof. Clearly,

\[ (L(t, \xi), \xi)_{m} = \langle \mathcal{K}(t)z, z \rangle + \langle N(t)z, z \rangle \Omega = \sum_{k=1}^{N} \int_{\Omega} \left[ a|\Delta z_k|^2 + \sum_{i=1}^{n} a_{i k}(t) |z_i|^{p(x)-2} |z_k, x_i|^2 \right] \]

\[ + \left( \int_{\Omega} \left[ \frac{\alpha}{2} |\Delta z|^2 + a_0 \sum_{i=1}^{n} |z_i|^{p(x)} + g_0 |z|^q(x) - C_{22} |z|^2 \right] \, dx \right) - C_{23}, \quad t \in (0, T). \]
Taking into account (A), (G), and (BB), we obtain
\[ \sum_{k=1}^{N} \left| b_{k} |z|^{q(x)-2} z_{k} \Delta z_{k} + g_{k}(t)|z|^{q(x)-2}|z_{k}|^{2} + \beta_{k}(t)|(z_{k})^{-2} \right| dx + (\phi(Ez(t)), z)_{\Omega}. \] (93)

Using the generalized Young inequality, we get
\[ \sum_{k=1}^{N} \left| b_{k} |z|^{q(x)-2} z_{k} \Delta z_{k} \right| = b^{0}|z|^{q(x)-1}| \Delta z | \leq C_{24}(x_{1})|z|^{q(x)} + x_{1}| \Delta z |^{q(x)} \]
\[ \leq x_{1}| \Delta z |^{2} + C_{25}(x_{1})(1 + |z|^{2}), \] (95)

where \( x_{1} > 0, C_{25}(x_{1}) > 0 \) is independent of \( x, t, k \) and \( m \).

Taking into account condition (\( \Phi \)), Cauchy-Bunyakowski-Schwarz’s inequality, and (88), we obtain
\[ \left| (\phi(Ez(t)), z)_{\Omega} \right| \leq \phi^{0} \int_{\Omega} |Ez(t)||z| \, dx \leq C_{26}||Ez(t)||L^{2}(\Omega)^{N}|| \cdot ||z||L^{2}(\Omega)^{N}|| \]
\[ \leq C_{18}||z||L^{2}(\Omega)^{N}|| \cdot ||z||L^{2}(\Omega)^{N}|| \leq C_{27} \int_{\Omega} |z|^{2} \, dx, \] (96)

where \( C_{27} > 0 \) is independent of \( z, t \) and \( m \).

Using (94)-(96) and choosing \( x_{1} = \frac{2}{\alpha} \) we can show that (93) yields (92). \( \square \)

4 Proof of main Theorem

The solution will be constructed via Faedo-Galerkin’s method.

**Step 1.** Let \( \{w_{j}\}_{j \in \mathbb{N}} \) be a set of all eigenfunctions of the problem (28) which are an orthonormal in \( L^{2}(\Omega), \)
\[ M_{m}^{N} := \left\{ x \mapsto \left( \sum_{\mu=1}^{m} \alpha_{\mu k}^{m} w_{\mu}(x), \ldots, \sum_{\mu=1}^{m} \alpha_{\mu N}^{m} w_{\mu}(x) \right) \mid \alpha_{\mu k}^{m} \in \mathbb{R}, \ k = 1, N, \ \mu = 1, m \right\}, \ m \in \mathbb{N}, \]
\( r \) is determined from condition (Z), \( W_{r} \) and \( W_{r}^{\ast} \) are defined by (31), and \( V \) is defined by (14). Taking into account Proposition 3.5 and (32), we obtain that \( M_{m}^{N} := \bigcup_{m \in \mathbb{N}} M_{m}^{N} \) is dense in \( W_{r} \) and \( V^{N}. \)

Take \( m \in \mathbb{N} \) and \( u_{m}^{0} := (u_{01}^{m}, \ldots, u_{m}^{N}) \), where
\[ u_{k}^{m}(t, x) := \sum_{\mu=1}^{m} \varphi_{\mu k}^{m}(t) w_{\mu}(x), \ (x, t) \in Q_{0, T}, \ k = 1, N, \]
\( \varphi^{m} := (\varphi_{11}^{m}, \varphi_{21}^{m}, \ldots, \varphi_{1N}^{m}, \varphi_{2N}^{m}, \ldots, \varphi_{mN}^{m}) \) is a solution to the problem
\[ \langle u_{k}^{m}(t), w_{\mu} \rangle + \langle \mathcal{K}(t)u_{m}^{m}(t), w_{\mu} \rangle + \langle N(t)u_{m}^{m}(t), w_{\mu} \rangle_{\Omega} = \langle F(t), w_{\mu} \rangle, \ t \in (0, T). \] (97)
\[ \varphi_{\mu k}^{m}(0) = \beta_{\mu k}^{m}, \ k = 1, N, \ \mu = 1, m \] (88)
(see (3), (19), and (20) for definition of the elements of \( N, \mathcal{K}, \) and \( F \)), the functions \( u_{m}^{0} := (u_{10}^{m}, \ldots, u_{m0}^{N}) \) satisfies the condition
\[ u_{m}^{0} \xrightarrow{m \to \infty} u_{0} \ \text{strongly in} \ H^{N}. \]
and \( u^{m}_{0k}(x) := \sum_{\mu=1}^{m} \beta_{\mu k}^{m} w^{\mu}(x), x \in \Omega, k = 1, N \). Clearly,

\[
u^{m}(0) = \left( \sum_{\mu=1}^{m} \phi_{\mu 1}^{m}(0) w^{\mu}(x), \ldots, \sum_{\mu=1}^{m} \phi_{\mu N}^{m}(0) w^{\mu}(x) \right) = u^{m}_{0}.
\]

The problem \((97), (98)\) coincides with \((74)\) if \( \ell = mN \),

\[
\psi^{0} = (\beta_{11}^{m}, \beta_{21}^{m}, \ldots, \beta_{m1}^{m}, \beta_{1N}^{m}, \beta_{2N}^{m}, \ldots, \beta_{mN}^{m}).
\]

\[ M = (M_{11}, M_{21}, \ldots, M_{m1}, M_{1N}, M_{2N}, \ldots, M_{mN}), \quad M_{\mu k}(t) = (F_{k}(t), w^{\mu}), \]

\[ L = (L_{11}, L_{21}, \ldots, L_{m1}, L_{1N}, L_{2N}, \ldots, L_{mN}). \]

\[ L_{\mu k}(t, \psi^{m}) = \left( (K(t)u^{m}(t))_{k}, w^{\mu} \right) + \left( (N(t)u^{m}(t))_{k}, w^{\mu} \right), \quad k = 1, N, \quad \mu = 1, m, \quad t \in (0, T). \]

By (F), we have \( M \in L^{2}(0, T; \mathbb{R}^{mN}) \). Taking into account the lemmas such as Lemmas 3.27 and 3.25, we see that \( L \) satisfies the \( L^{\infty} \)-Carathéodory condition. From (92) we obtain

\[
(L(t, \psi^{m}), \psi^{m})_{\mathbb{R}^{mN}} \geq -C_{28} \int_{\Omega} |u^{m}|^{2} dx - C_{29}
\]

\[
\geq -C_{30}(m) \int_{\Omega} \sum_{k=1}^{N} \sum_{\mu=1}^{m} |\phi_{\mu k}^{m}|^{2} |w^{\mu}(x)|^{2} dx - C_{29} = -C_{31}(m) |\psi^{m}|^{2} - C_{29},
\]

where \( C_{29}, C_{31} > 0 \) are independent of \( t, \psi^{m} \). Then Carathéodory-LaSalle’s Theorem 3.24 implies that there exists a solution \( \psi^{m} \in H^{1}(0, T; \mathbb{R}^{mN}) \) to problem \((97), (98)\). If we combine the condition \( \partial \Omega \in C^{2, r} \) with Proposition 3.5 and embedding (26), we get \( \{w^{f}_{j}\}_{j \in \mathbb{N}} \subset W_{r} \subset [H^{2r}(\Omega)]^{N} \). Thus,

\[
u^{m} \in H^{1}(0, T; W_{r}) \subset H^{1}(0, T; [H^{2r}(\Omega)]^{N}) \subset [H^{1}(Q_{0, T})]^{N}.
\]

**Step 2.** Multiplying both sides of the corresponding equality \((97)\) by \( \phi_{\mu k}^{m}(t) \), summing the obtained equalities, and integrating in \( t \in (0, r) \subset (0, T) \), we get

\[
\int_{Q_{0, r}} (u^{m}_{t}, u^{m}) \, dx \, dt + \int_{0}^{r} (L(t, \psi^{m}(t)), \psi^{m}(t))_{\mathbb{R}^{mN}} \, dt
\]

\[
= \int_{Q_{0, r}} \left[ \sum_{i, j=1}^{n} (f_{i j}, u_{x_{i}x_{j}}^{m}) + \sum_{i=1}^{n} (f_{i}, u_{x_{i}}^{m}) + (f_{0}, u^{m}) \right] \, dx \, dt, \quad r \in (0, T). \]

By (102), similar to Case 1.ii of Theorem 3.19 with \( p(x) = r(x) = 2 \), we obtain

\[
|u^{m}|^{2} \in W^{1, 2}(0, T; L^{1}(\Omega)). \quad (|u^{m}|^{2})_{t} = 2(u^{m}_{t}, u^{m}).
\]

Then, the integration by parts formula and (99) yield that

\[
\int_{Q_{0, r}} (u^{m}_{t}, u^{m}) \, dx \, dt = \frac{1}{2} \int_{\Omega} |u^{m}(x, r)|^{2} \, dx - \frac{1}{2} \int_{\Omega} |u^{m}_{0}(x)|^{2} \, dx.
\]

By (92), we get

\[
\int_{0}^{r} (L(t, \psi^{m}(t)), \psi^{m}(t))_{\mathbb{R}^{mN}} \, dt \geq \int_{Q_{0, r}} \left[ \frac{\alpha}{2} |\Delta u^{m}|^{2} + a_{0} \sum_{i=1}^{n} |u_{x_{i}}^{m}|^{2 p(x)} + g_{0} |u^{m}| \bar{p}(x) - C_{32} |u^{m}|^{2} \right] \, dx - C_{33}.
\]
where $C_{32}, C_{33} > 0$ are independent of $m$ and $\tau$. In addition, Young’s inequality, the condition $\partial \Omega \in C^2$, and estimate (30) yield that

$$\left| \int_{Q_{0,\tau}} \sum_{i,j=1}^n (f_{ij}, u^m_{x_i x_j}) \, dx \right| \leq \int_{Q_{0,\tau}} \sum_{i,j=1}^n \left[ x_1 |u^m_{x_i x_j}|^2 + \frac{1}{4x_1} |f_{ij}|^2 \right] \, dx \, dt$$

$$\leq \int_{Q_{0,\tau}} \left[ x_1 C_{34} |\Delta u^m|^2 + \frac{1}{4x_1} \sum_{i,j=1}^n |f_{ij}|^2 \right] \, dx \, dt,$$

where $x_1 > 0$, the constant $C_{34} > 0$ is independent of $m$ and $x_1$. By (23), we get

$$\left| \sum_{i=1}^n (f_{ij}, u^m_{x_i}) + (f_0, u^m) \right| \leq \left[ x_2 \sum_{i=1}^n |u_{x_i}^{\rho(x)}| + Y_{\rho(x)} \sum_{i=1}^n |f_{ij}|^2 + x_3 |u^m|^q(x) + Y_q(x_3) |f_0|^q(x) \right].$$

According to the above remarks, from (103) we have the following inequality

$$\frac{1}{2} \int_{Q_{0,\tau}} |u^m|^2 \, dx + \int_{Q_{0,\tau}} \left[ \left( \frac{\partial}{\partial t} - x_1 C_{34} \right) |\Delta u^m|^2 + (a_0 - x_2) \sum_{i=1}^n |u_{x_i}^{\rho(x)}| + (g_0 - x_3) |u^m|^q(x) \right] \, dx \, dt$$

$$\leq \frac{1}{2} \int_{Q_{0,\tau}} |u^m_0|^2 \, dx + C_{35}(x_1, x_2, x_3) \left( 1 + \int_{Q_{0,\tau}} \sum_{i,j=1}^n |f_{ij}|^2 + \sum_{i=1}^n |f_{ij}|^2 + |f_0|^q(x) \right) \, dx \, dt$$

$$+ \int_{Q_{0,\tau}} |u^m|^2 \, dx \, dt, \quad \tau \in (0, T], \quad (104)$$

where $C_{35} > 0$ is independent of $m$ and $\tau$.

Let $y(t) := \int_{Q_{0,\tau}} |u^m(x, t)|^2 \, dx$, $t \in [0, T]$. Choosing $x_1, x_2, x_3 > 0$ sufficiently small, from (104) we can obtain that $y(\tau) \leq C_{36} + C_{37} \int_0^\tau y(t) \, dt$, $\tau \in (0, T)$. Then the Gronwall-Bellman Lemma yields that

$$\int_{\Omega} |u^m(x, \tau)|^2 \, dx \leq C_{38}, \quad \tau \in (0, T], \quad (105)$$

and so

$$\int_{Q_{0,\tau}} |u^m|^2 \, dx \, dt \leq C_{38} T, \quad \tau \in (0, T]. \quad (106)$$

Using (104), (106), and choosing $x_1, x_2, x_3 > 0$ sufficiently small, we get

$$\int_{Q_{0,\tau}} \left[ |\Delta u^m|^2 + \sum_{i=1}^n |u_{x_i}^{\rho(x)}|^2 + |u^m|^q(x) \right] \, dx \, dt \leq C_{39}, \quad \tau \in (0, T]. \quad (107)$$

Here $C_{38}, C_{39} > 0$ are independent of $m$ and $\tau$.

By (105)-(107), we have that there exists a sequence $\{u^{m_j}\} \subset \{u^m\}$ such that

$$u^{m_j} \rightharpoonup u \quad \text{weakly in} \quad L^\infty(0, T; H^N) \quad \text{and weakly in} \quad U(Q_{0,T}). \quad (108)$$

Step 3. We define the element $F \in [U(Q_{0,T})]^*$ and the operator $A : U(Q_{0,T}) \to [U(Q_{0,T})]^*$ by the rules

$$\langle F, v \rangle_{U(Q_{0,T})} := \int_0^T \langle F(t), v(t) \rangle \, dt, \quad v \in U(Q_{0,T}). \quad (109)$$

$$\langle Av, v \rangle_{U(Q_{0,T})} := \int_0^T \left[ \langle K(t) u(t), v(t) \rangle + \langle N(t) u(t), v(t) \rangle \right] \, dt, \quad u, v \in U(Q_{0,T}). \quad (110)$$
Using (24), (86), (106) and (107), we get
\[
(Au^m, v)_{U(Q_0, T)} = \int_{Q_0, T} \sum_{k=1}^{N} \left[ \alpha \Delta u_k^m \Delta v_k + \sum_{i=1}^{n} a_{ik} |u_k^m|^p(x) - u_k^m v_k + b_k |u_k^m|^q(x) - 2u_k^m \Delta v_k \right] dx dt \leq C_{40} \int_{Q_0, T} \left[ \Delta u^m \cdot \Delta v \right] dx dt
\]
\[
+ \sum_{i=1}^{n} |u_k^m|^p(x) - 1 |v| + |u_k^m|^q(x) - 1 |v| + |u_k^m| \cdot |v| + |Eu^m| \cdot |v| \right] dx dt
\]
\[
\leq C_{40} \left( \sum \left| \Delta u^m \right| L^2(Q_0, T) \right) \cdot \sum |v| + L^2(Q_0, T) \left| \Delta v \right| + \sum_{i=1}^{n} \left| \Delta u^m \right| L^2(Q_0, T) \left| \Delta v \right| + \sum_{i=1}^{n} \left| \Delta u^m \right| L^2(Q_0, T) \left| \Delta v \right|
\]
\[
\times \left| \Delta u^m \right| \left| \Delta v \right| + 2 \left| \Delta u^m \right| L^2(Q_0, T) \left| \Delta v \right| + \left| \Delta v \right| L^2(Q_0, T) \left| \Delta u^m \right| + \left| \Delta v \right| L^2(Q_0, T) \left| \Delta u^m \right|
\]
\[
\leq C_{41} \left| v \right| U(Q_0, T) \left| v \right|
\]
where $C_{41} > 0$ is independent of $m, v$. Then
\[
\|Au^m; [U(Q_0, T)]^*\| \leq C_{41}
\]
and so
\[
Au^m \xrightarrow{j \to \infty} \chi \text{ weakly in } [U(Q_0, T)].
\]

**Step 3.** Suppose that the numbers $r$ and $s^0$ are determined from condition (Z), the spaces $\mathcal{W}_r$ and $\mathcal{W}_r^*$ are defined by (31), $P_m : H^N \to H^N$ is the projection operator from (45) (see also (21)), $\hat{P}_m$ is defined by (40), where $\mathcal{H} = H^N$ and $\mathcal{V} = \mathcal{W}_r$. Similarly to [54, p. 77] and [55, p. 62-63], using Lemma 3.9, notation (109) and (110), we rewrite (97) as
\[
u^m_r = \hat{P}_m^* (\mathcal{F} - Au^m).
\]

By (49), we get
\[
\left\| \hat{P}_m f \right\| L^{s^0}(0, T; \mathcal{W}_r^\pm) \leq \left\| f \right\| L^{s^0}(0, T; \mathcal{W}_r^\pm), \quad f \in L^{s^0}(0, T; \mathcal{W}_r).
\]

Since $\left\| D^* \right\|_{\mathcal{L}(A^*, A)} = \left\| D \right\|_{\mathcal{L}(A, B)}$ for every $D \in \mathcal{L}(A, B)$ (see [42, p. 231]), using (114), we have
\[
\left\| \hat{P}_m^* h \right\| L^{s^0}(0, T; \mathcal{W}_r^\pm) \leq \left\| h \right\| L^{s^0}(0, T; \mathcal{W}_r^\pm), \quad h \in L^{s^0}(0, T; \mathcal{W}_r^\pm).
\]

Taking into account (115), (35), and (109), we obtain
\[
\left\| \hat{P}_m^* \mathcal{F} \right\| L^{s^0}(0, T; \mathcal{W}_r^\pm) \leq \left\| \mathcal{F} \right\| L^{s^0}(0, T; \mathcal{W}_r^\pm) \leq C_{42} \left\| \mathcal{F} \right\| L^{s^0}(0, T; \mathcal{W}_r^\pm) \leq C_{43}.
\]

By (115), (110), (111), and (35), we get
\[
\left\| \hat{P}_m^* Au^m \right\| L^{s^0}(0, T; \mathcal{W}_r^\pm) \leq \left\| Au^m \right\| L^{s^0}(0, T; \mathcal{W}_r^\pm) \leq C_{44} \left\| Au^m \right\| L^{s^0}(0, T; \mathcal{W}_r^\pm) \leq C_{44}.
\]

Using (113), (116), and (117) (see for comparison [54, 55]), we obtain
\[
\left\| u^m_r \right\| L^{s^0}(0, T; \mathcal{W}_r^\pm) \leq C_{46}.
\]

Here $C_{43}, \ldots, C_{46} > 0$ are independent of $m$. Therefore,
\[
u^m_r \xrightarrow{j \to \infty} u_t \text{ weakly in } L^{s^0}(0, T; \mathcal{W}_r^\pm).
Step 4. Suppose the numbers \( r \) and \( s_0 \) are determined from condition (Z). Then (32) implies that \( V^N \subseteq H^N \subseteq \mathcal{W}^s_{r} \). By (33), (106), and (107), we get

\[
\|u^m; L^{s_0}(0, T; V^N)\| \leq C_{47}\|u^m; U(Q_{0, T})\| \leq C_{48}.
\]

where \( C_{48} > 0 \) is independent of \( m \).

Taking into account (120), (118), the Aubin theorem (see Proposition 3.11), and Lemma 1.18 [23, p. 39], we obtain

\[
\lim_{j \to \infty} u^m_j \to u \quad \text{strongly in} \quad L^2(0, T; H^N) \quad \text{and in} \quad C([0, T]; \mathcal{W}^s_{r}).
\]

(121)

\[
\lim_{j \to \infty} u^m_j \to u \quad \text{almost everywhere in} \quad Q_{0, T}.
\]

(122)

Clearly, \( V^N \subseteq [H^1_0(\Omega)]^N \subseteq \mathcal{W}^s_{r} \). Then (120), (118), and the Aubin theorem yield that

\[
\lim_{j \to \infty} u^m_j \to u \quad \text{strongly in} \quad L^2(0, T; [H^1_0(\Omega)]^N).
\]

(123)

Hence for every \( i \in \{1, \ldots, n\} \) we have

\[
\int_{Q_{0, T}} |u^m_j - u_{x_i}|^2 \, dx \, dt \leq \|u^m_j - u; L^2(0, T; [H^1_0(\Omega)]^N)\|^2 \quad \to 0.
\]

Thus \( u^m_j \to u_{x_i} \) strongly in \([L^2(Q_{0, T})]^N\) and so Lemma 1.18 [23, p. 39] implies that

\[
\lim_{j \to \infty} u^m_j \to u_{x_i} \quad \text{almost everywhere in} \quad Q_{0, T}, \quad i = 1, \ldots, n.
\]

(124)

By (122) and (124), we obtain the equality \( \chi = Au \).

Step 5. Using (97) and (102), we obtain

\[
- \int_0^T (u^{m_j}(t), w) \varphi'(t) \, dt + (Au^{m_j}, w \varphi)_{U(Q_{0, T})} = (F, w \varphi)_{U(Q_{0, T})},
\]

(125)

where \( \varphi \in C^\infty((0, T), \omega \in \mathcal{M}_{k}^N, k \in \mathbb{N}, k \leq m_j, j \in \mathbb{N}. \) Letting \( j \to +\infty \) and using Lemma 3.8, we get the equality \( u_t + Au = F \). Whence, \( u_t = F - Au \in [U(Q_{0, T})]^s, u \in W(Q_{0, T}), \) and (22) holds. Moreover, we obtain the inclusion \( u_t \in L^2(0, T; [V^N]^s) \) because (34) is true. Hence, \( u \in C([0, T]; [V^N]^s) \). By (108), we have that \( u \in L^\infty(0, T; H^N) \). Thus, Lemma 3.7 yields that \( u \in C([0, T]; H^N) \) and so \( u \) is a weak solution to initial-boundary value problem (1), (2).  

\[\square\]

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