MATHEMATICAL MODELLING OF FRACTIONAL ORDER CIRCUITS

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Abstract. In this work a classical derivation of fractional order circuits models is presented. Generalized constitutive equations in terms of fractional Riemann-Liouville derivatives are introduced in the Maxwell's equations. Next the Kirchhoff voltage law is applied in a RCL circuit configuration. A fractional differential equation model is obtained with Caputo derivatives. Thus standard initial conditions apply.

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1. Introduction

In the las decades there has been a great interest on Fractional Calculus and its applications. Models involving fractional derivatives and operators have been found to better describe some real phenomena. A historical review of applications is presented in Machado et al [6].

In this work we are interested on applications to linear systems, in particular to fractional circuits models. Research on the topic is very active and lively. Form the perspective of Systems Theory, an inviting introduction to Fractional Systems is presented in Duarte [3]. See also the work of Radwan & Salama [2]. Focusing
on the applications to fractional order circuits is the work of Elwakil [5]. Therein the motivation for fractional order circuits from Biochemistry and Medicine is described. A more extensive survey on models from Biology and Biomedicine for fitting impedance data is presented in Freeborn [4]. With regards to the Cole Model, the Laplace transform model of a fractional order circuit, it is pointed out that: "while this model is effective at representing experimentally collected bioimpedance data, it does not provide an explanation of the underlying mechanisms".

Our modest aim is to provide some insight on this issue. Our purpose is to show that the fractional order circuit model can be obtained from a classical modelling approach. We consider the RCL circuit configuration and apply the Maxwell’s equations as customary. The novelty is on introducing generalised constitutive equations relating the electric flux density, the magnetic field intensity, and the electric current density with the electric and magnetic fields. These laws are in terms of fractional time derivatives in the sense of Riemann-Liouville. In particular, a generalised Ohm’s law is introduced which is consistent with the empirical Curie’s law. This law is the point of departure of our work, it is fully discussed in Westerlund & Ekstam [10].

The fractional order circuit model to be obtained it is, as expected, a fractional ordinary differential equation in the Riemann-Liouville sense. The underlying mechanisms of physical phenomena where this model applies, are partially explained by the generalised constitutive laws to be introduced.

It is well known that initial conditions need not be well defined in the Riemann-Liouville case. We show that it is straightforward to obtain an equivalent model with derivatives in the Caputo sense. Consequently, initial conditions can be prescribed as usual.

The outline is as follows.

In Section 2 we list some basic definitions and properties of Fractional Calculus, these shall be used freely throughout. In Section 3 we review Curie’s law and recall the Maxwell’s equations in continuous media. From Curie’s law we make a case for a generalised Ohm’s law in lossy media. The classical modelling of RCL circuits is carried out in Section 4. Then, generalized laws for the electric flux density and magnetic field intensity are introduced leading to fractional versions of circuit componentes. In Section 5 we apply the Kirchhoff’s law of voltages to obtain the desired fractional order circuit model.

2. Preliminaries on Fractional Calculus

For later reference let us recall some basic properties of Fractional Calculus, see Diethelm [2].

Let $s > 0$ and $f : \mathbb{R} \to \mathbb{R}$. The Riemann-Liouville fractional integral of order $s$ centered at 0, $J^s f$ is given by:
Hereafter, let $0 < \alpha \leq 1$. The Riemann-Liouville derivative $D^\alpha f(x)$ of order $\alpha$ centered at 0 is defined by

\begin{equation}
D^\alpha f = DJ^{1-\alpha} f.
\end{equation}

An alternative definition is given by Caputo, $\hat{D}^\alpha f$, namely

\begin{equation}
\hat{D}^\alpha f = J^{1-\alpha} Df.
\end{equation}

Let $\mathbb{R}^+ = \{t \in \mathbb{R} : t > 0\}$ and $\mathbb{R}_0^+ = \{t \in \mathbb{R} : t \geq 0\}$. Let us denote by $C^0(\mathbb{R}^+)$ and $C^0(\mathbb{R}_0^+)$ spaces of continuous functions Also let $L^1_{loc}(\mathbb{R}^+)$ be the space of locally integrable functions.

The following are well known facts about derivatives and integrals in the sense of Riemann-Liouville:

- $J^r J^s = J^{r+s}$, $r, s > 0$. (semigroup property)
- For $\lambda > -1$

\begin{equation*}
D^\alpha t^\lambda = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda - \alpha + 1)} t^{\lambda-\alpha},
\end{equation*}

\text{giving in particular } D^\alpha t^{\alpha-1} = 0.
- $D^\alpha J^\alpha f = f$ for all $f \in C^0(\mathbb{R}^+) \cap L^1_{loc}(\mathbb{R}^+)$.  
- If $u \in C^0(\mathbb{R}^+) \cap L^1_{loc}(\mathbb{R}^+)$, then the fractional differential equation

\begin{equation*}
D^\alpha u = 0
\end{equation*}

has $u = ct^{\alpha-1}$, $c \in \mathbb{R}$, as unique solutions.
- If $f, D^\alpha f \in C^0(\mathbb{R}^+) \cap L^1_{loc}(\mathbb{R}^+)$ then

\begin{equation*}
J^\alpha D^\alpha f(t) = f(t) + ct^{\alpha-1}
\end{equation*}

for some $c \in \mathbb{R}$. When $f \in C^0(\mathbb{R}_0^+)$, $c = 0$.

Our aim is to stress the ideas for the derivation of a fractional circuit model. Thus, we shall assume that the functions under consideration are sufficiently regular. It will become apparent that, with some technicalities, the arguments that follow are valid for functions with weaker properties.
For instance, consider the space $C^0_r(\mathbb{R}^+)$, $r \geq 0$ of functions $f \in C^0(\mathbb{R}^+)$ such that $t^r f \in C^0(\mathbb{R}^+_0)$. Let $r < 1 - s$ and $f \in C^0_r(\mathbb{R}^+_0)$ with $D^s f \in C^0(\mathbb{R}^+) \cap L_{\text{loc}}(\mathbb{R}^+)$. Then $J^s D^s f = f$.

Finally, we list some properties for the Caputo derivative:

- $\dot{D} f = D f$
- If $f(0) = 0$ then $\dot{D} f = D f$ and $\dot{D} \dot{D} f = \dot{D} D f = \dot{D}^{1+\alpha} f$

3. Generalized constitutive equations in Electromagnetism

3.1. Curie’s law. Consider a dielectric material and apply at $t = 0$ a constant dc voltage $v_0$. The current produced is

\begin{equation}
\dot{i}(t) = \frac{v_0}{ht^\alpha}, \quad 0 < \alpha < 1, \quad t > 0.
\end{equation}

The constant $h$ is related to the capacitance of the capacitor and the kind of dielectric. Whereas the constant $\alpha$ is associated to the losses of the capacitor. The losses decrease with $\alpha$ approaching 1.

Equation (3) is known as Curie’s law, see Curie [1].

Let $v(t) = v_0 H(t)$, where $H(t)$ is the Heaviside function. In terms of the Riemann-Liouville derivative, Curie’s law reads

\begin{equation}
\dot{i}(t) = CD^{\alpha} v(t), \quad C = \frac{\Gamma(1 - \alpha)}{h}.
\end{equation}

Based on exhaustive experimental evidence, Westerlund & Ekstam [10] contend that all dielectrics and insulators follow Curie’s Law. We may consider this phenomenon as a particular case of transport in complex media. Consequently, models involving fractional derivatives seem appropriate. More on this below.

Remark. As noted above, the definition of fractional derivative is not unique. Besides the fractional derivatives of Riemann-Liouville and Caputo there are other definitions, one commonly used is that of Grünwald-Letinkov. A case for the Caputo derivative is that for fractional differential equations, initial conditions are straightforward. In particular, the Caputo derivative of constant functions is zero. The latter precludes the use of Caputo derivative for describing Curie’s law in terms of fractional derivatives. Consequently, our choice of the Riemann-Liouville derivative is necessary.
3.2. Generalized constitutive equations. Let $f : \mathbb{R}^3 \times \mathbb{R}^+ \to \mathbb{R}^3$. Thus $f(\cdot, t)$ is a vector field. Let us denote by $J^{1-\alpha}_t f$ the Riemann-Liouville integral of $f$ applied component wise. Similarly, the Riemann-Liouville and Caputo fractional derivatives are denoted as $D^\alpha_t f$ and $\dot{D}^\alpha_t f$, respectively. As before when dealing with fractional time derivatives, we consider $f(x, t) \equiv f(x)H(t)$, for $f$ independent of $t$.

Let us recall Maxwell’s Equations in (macroscopic) continuous media,

\begin{align*}
\nabla \cdot \mathcal{D} &= \rho \\
\nabla \times \mathcal{E} &= -\frac{\partial \mathcal{B}}{\partial t} \\
\nabla \cdot \mathcal{B} &= 0 \\
\nabla \times \mathcal{H} &= \frac{\partial \mathcal{D}}{\partial t} + \mathcal{J}
\end{align*}

The physical quantities are the electric field $\mathcal{E}$, the magnetic field $\mathcal{B}$, the electric flux density $\mathcal{D}$, the magnetic field intensity $\mathcal{H}$, electric current density $\mathcal{J}$, and electric charge density $\rho$.

To have a complete system of equations, constitutive relations are necessary for a specific medium. These are relations for $\mathcal{D}$, $\mathcal{H}$ and $\mathcal{J}$ in terms of $\mathcal{E}$, $\mathcal{B}$. For complex material media, e.g. lossy media, these relations need not be simple. They may depend on past history, may be nonlinear, etc.

Consider Ohm’s law

$$\mathcal{J} = \sigma \mathcal{E}, \quad \sigma \geq 0,$$

with $\sigma$ the positive constant of conductivity of the medium considered. This law is valid in isotropic and homogeneous media, without memory.

Curie’s law \([3]\) indicates that there is loss of energy in the process under study, and the fractional differential law \([4]\) models a medium with memory.

Consequently, for lossy media in Electromagnetism we propose the generalised Ohm’s law

\begin{equation}
\mathcal{J}(x, t) = \sigma_\alpha D^{1-\alpha}_t \mathcal{E}(x, t), \quad 0 < \alpha \leq 1.
\end{equation}

When $\alpha = 1$ the classical law is recovered.
Remark. Law (9) is in analogy to a generalized Darcy law in Fluid Mechanics. Indeed, the continuity equation for electric charges, derived from equations (5) and (8) reads

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot J = 0. \]  

If \( \rho \) is fluid density, Equation (10) is the continuity equation in Fluid Mechanics. For highly heterogeneous media, anomalous behaviour has been observed, such as non-Darcian flow. Generalized laws have been proposed to better describe this behaviour. If \( u(x, t) \) is the superficial velocity and \( P(x, t) \) is the pressure, the generalized Darcy law

\[ u(x, t) = -\frac{k}{\mu} \Gamma(\alpha) D_t^{1-\alpha} \nabla P(x, t), \quad 0 < \alpha \leq 1 \]  

has proven useful. See Moreles et al. [8] and references within.

4. Fractional Circuit Elements

Here we follow the classical derivation of the RCL circuit components, while introducing generalized constitutive relations for \( D \) and \( B \) analogue to (9).

4.1. Fractional Resistor. Consider a lossy material in the form of a cylinder of length \( l \) and cross section \( S \) of area \( A \). Assume a constant electric field in the direction of the axis of the cylinder \( z \). The voltage \( v(t) \) between the ends of the cylinder is

\[ v(t) = \int_0^l \mathcal{E} \cdot zdz \]

(12)

whereas the total current \( i(t) \) is

\[ i(t) = \int_S \mathcal{J} \cdot zdS \]

(13)

Assume (9) is valid. We have

\[ i(t) = A \sigma_\alpha D_t^{1-\alpha} \mathcal{E} \cdot z. \]

Hence

\[ i(t) = \frac{A \sigma_\alpha}{l} D_t^{1-\alpha} v(t) \]

Let \( R_\alpha = \frac{l}{A \sigma_\alpha} \). It follows that the fractional resistor satisfies the Curie’s law
\( i(t) = \frac{1}{R_\alpha} D_t^{1-\alpha} v(t) \)

Notice that if \( \alpha = 1 \), we have the classical law
\[
v = R i.
\]

4.2. **Fractional Capacitor.** Consider two parallel plates confining a lossy material. Both plates have charge of equal magnitude but opposite sign. If distance between the plates is small compared with their size, there is no charge in the region between the plates. Charge will reside mostly in the inner surfaces of the plates. The electric field is normal to the plates away from the edges, say in the \( x \) direction, and zero in the interior of the plates.

Consequently, in the region between the plates equation \((5)\) becomes
\[
\nabla \cdot \mathbf{D} = 0.
\]

We propose the constitutive law
\[
\mathbf{D} = \epsilon_\beta D_t^{1-\beta} \mathcal{E}, \quad 0 < \beta \leq 1.
\]

Since \( \mathcal{E} = -\nabla \phi \), the potential \( \phi(x, y, z) \equiv \phi(x) \) solves the equation
\[
D_x D_t^{1-\beta} D_x \phi = 0
\]

We have
\[
D_x \left( \frac{1}{\Gamma(\beta)} t^{\beta-1} D_x \phi \right) = 0,
\]
and for \( t > 0 \)
\[
D_x^2 \phi = 0.
\]

Let the plates be at \( x = a \) and \( x = b \). As customary, suppose that the potentials are constant on each plate and are \( \phi(a) = v_a \) and \( \phi(b) = v_b \). Then
\[
\phi(x) = (v_b - v_a)x + bv_a - av_b \quad \frac{b}{b-a},
\]
and the electric field
\[
\mathcal{E} = \frac{v_{ab}}{b-a},
\]
where \( v_{ab} = v_a - v_b \).
Now choose $\delta$ such that $\delta << |b - a|$ and construct the cylindrical surface $S_C$, with axis in the $x$-direction. The plane top and bottom, of area $A$, are at $x = a + \frac{\delta}{2}$ and $x = a - \frac{\delta}{2}$.

From the integral form of equation (5) in the region confined by $S_C$ we obtain

(18) $$qA = \int_{S_C} \mathbf{D} \cdot n \, ds,$$

where $q$ is the constant surface density of charge, positive on $x = a$.

Let $v(t) = v_{ab} H(t)$ and $Q(t)$ the total charge on the plate at $x = a$, namely $Q(t) = qAH(t)$. From (15) and (17) we have

$$Q = \frac{\epsilon \beta A}{b - a} D_{1-\beta}^{-1} v$$

We are led to a generalized governing equation of a capacitor

(19) $$Q(t) = C_{\beta} D_{1-\beta} t v(t)$$

with

$$C_{\beta} = \frac{\epsilon \beta A}{b - a}.$$

4.3. Fractional Inductor. For later reference let us recall Ampere's law and Faraday's law of induction. Namely

(20) $$i(t) = \oint_C \mathbf{H} \cdot t \, dl$$

(21) $$\oint_C \mathbf{E} \cdot t \, dl = -\frac{d}{dt} \int_S \mathbf{B} \cdot n \, ds$$

The corresponding generalized constitutive relation between the magnetic field intensity $\mathbf{H}$ and the magnetic field $\mathbf{B}$ is

(22) $$\mu_\gamma \mathbf{H} = D_{1-\gamma} t^{1-\gamma} \mathbf{B}, \quad 0 < \gamma \leq 1.$$

For the inductor it is considered a toroidal frame of rectangular cross section. The inner and outer radii of the frame are $r_1$ and $r_2$ respectively, and the height is $h$. A coil consisting of $n$ turns of wire is tightly wound on the frame. There is a current of magnitude $i(t), t \geq 0$ in the conducting wire.

Set cylindrical coordinates such that the $z$-axis is the axis of symmetry and the frame is located between $r = r_1$ and $r = r_2$. Let $C_r$ be a circular path within the
toroidal frame of radius \( r \) with \( r_1 < r < r_2 \). As customary, we assume axis symmetry so that the magnetic field \( B \equiv B(r, z, t) \) only depends on \( r, z, \) and \( t \).

By Ampere's law applied to the surface of the disk bounded by \( C_r \), we have

\[
n_i(t) = \int_{C_r} \mathbf{H} \cdot \mathbf{t} \, dl,
\]

where \( \mathbf{t} \) is the unit tangent to \( C_r \).

Substituting \( B \) from (22) we obtain

\[
\mu_\gamma n_i(t) = D_1^{1-\gamma} \int_{C_r} B \cdot \mathbf{t} \, dl.
\]

After application of the Riemann-Liouville integral of order \( 1 - \gamma \) we are led to

\[
\mu_\gamma n J_1^{1-\gamma} i(t) = \int_{C_r} B \cdot \mathbf{t} \, dl.
\]

If \( B \) is the magnitude of the magnetic field in the direction of \( \mathbf{t} \), then

\[
\int_{C_r} B \cdot \mathbf{t} \, dl = \int_0^{2\pi} B(r, z, t) dl = 2\pi r B(r, z, t).
\]

So

\[
B(r, z, t) = \frac{\mu_\gamma n}{2\pi r} J_1^{1-\gamma} i(t).
\]

Let us apply Faraday’s law to one such surface \( S \). Then the normal to this surface is \( \mathbf{n} = \mathbf{t} \). So if \( v_S(t) \) is the induced voltage

\[
v_S(t) = -\frac{d}{dt} \int_S B \cdot \mathbf{n} \, dS
\]

\[
= -\frac{d}{dt} \int_0^h \int_{r_1}^{r_2} B(r, z, t) dr dz
\]

\[
= -\frac{\mu_\gamma n}{2\pi} \int_0^h \int_{r_1}^{r_2} \frac{1}{r} dr dz \frac{d}{dt} J_1^{1-\gamma} i(t)
\]

\[
= -\frac{\mu_\gamma n h}{2\pi} \log(r_2/r_1) D_1^{\gamma} i(t).
\]

Let \( v \) be the total voltage dropped. Since there are \( n \) such coils, we have

(23) \[ v(t) = L_\gamma D_1^{\gamma} i(t) \]

with \( L_\gamma = \frac{\mu_\gamma n}{2\pi} \log(r_2/r_1) \).

Equation (23) is a generalization of the governing equation of an inductor.
5. Fractional RCL Circuit

Table 1 summarizes the equations obtained in the previous section. In the second column the fractional generalizations are shown, while the third column presents the limit behavior of the fractional models. As expected the classical governing equations, representing ideal components of a circuit, are obtained.

| Component | Fractional Equation | Classical Equation |
|-----------|---------------------|--------------------|
| Resistor  | $i(t) = \frac{1}{R_\alpha} D_t^{1-\alpha} v(t)$ | $i(t) = \frac{1}{R} v(t)$ |
| Capacitor | $Q(t) = C_\beta D_t^{1-\beta} v(t)$ | $Q(t) = C v(t)$ |
| Inductor  | $v(t) = L_\gamma D_t^{1-\gamma} i(t)$ | $v(t) = L D_t[i(t)]$ |

**Table 1.** Fractional and classical governing equations of circuit components

For $v$ regular enough we have for the fractional resistor

$$v(t) = R_\alpha J_t^{1-\alpha} i(t),$$

whereas for the fractional capacitor

$$v(t) = \frac{1}{C_\beta} J_t^{1-\beta} Q(t).$$

Consider a resistance, a capacitor, an inductor in series with a power source $e(t)$, as shown in Figure 1. The first three components have fractional governing equations.

![Fractional RCL circuit](image)

**Figure 1.** Fractional RCL circuit

By Kirchhoff’s law of voltages we have:

$$e(t) = R_\alpha J_t^{1-\alpha} i(t) + \frac{1}{C_\beta} J_t^{1-\beta} Q(t) + L_\gamma D_t^{1-\gamma} i(t),$$

but from the continuity equation (10) it is known that $i(t) = D_t Q(t)$. We are led to the Fractional RCL circuit equation

$$L_\gamma D_t^{\gamma} D_t Q(t) + R_\alpha J_t^{1-\alpha} D_t Q(t) + \frac{1}{C_\beta} J_t^{1-\beta} Q(t) = e(t).$$
In terms of Caputo derivatives

\[ L_γ \hat{D}_t^{γ} Q(t) + R_α \hat{D}_t^{α} Q(t) + \frac{1}{C_β} J_t^{1-β} Q(t) = e(t). \]

This is a fractional integro-differential equation of order \(1 + γ\) in \(t\), with derivatives in the sense of Caputo. Thus, \(Q(0)\) and \(D_t Q(0)\) are the natural initial conditions.

If \(Q(t)\) is regular enough, then \(J_t^{1-β} Q(0) = 0\) and the Riemann-Liouville and Caputo derivatives coincide. Applying the Caputo derivative of order \(1 - β\) in \(25\) we obtain

\[ L_γ \hat{D}_t^{1-β} D_t \hat{D}_t^{γ} Q(t) + R_α \hat{D}_t^{1-β} \hat{D}_t^{α} Q(t) + \frac{1}{C_β} Q(t) = \hat{D}_t^{1-β} e(t). \]

This is the form commonly found in applications. See for instance Magin [7].

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