On a universality class of anomalous diffusion

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We show that when transport is asymptotically dominated by ballistic excursions the moments of displacement exhibit a universal behaviour, and their values can be obtained after an appropriate rescaling of the mean square displacement in terms of the asymptotic behaviour of a minimal transport model that be call Fly-and-Die. Furthermore, we show that this universality extends to the displacement correlation function. We obtain the statistical properties defining a universality class to which the details and nature of the dynamics are irrelevant. We support this by studying several models exhibiting strong anomalous diffusion.

I. INTRODUCTION

When deviations from the asymptotic normal behaviour of fluctuations due to the break down of the central limit theorem occur, transport is generically termed as anomalous [1,2]. Anomalous transport is observed in a vast variety of systems and contexts, and an important fraction of current literature is devoted to the investigation of standard and anomalous diffusion, see for instance Refs. [1,2,11].

A transport process’s $x(t)$ is said to exhibit anomalous diffusion if the Mean Square Displacement (MSD) does not grow linearly in time. Such anomalous behaviour is found in a broad set of phenomena including molecules moving in a living cell [12], dynamics on cell membranes [13], solid-state disordered systems [14], telomeres inside the nucleus of mammalian cells [15], soil transport [16] heat transport in low-dimensional systems [17], among many others.

The statistical properties of transport are determined by the evolution of the probability density of the displacement $P(\Delta x, t)$. However, in most situations this density is unknown and transport is studied in terms of the asymptotic long-time behaviour of its moments

$$\langle \Delta x(t)^p \rangle = \langle (x(t) - x(0))^p \rangle \sim t^{\nu(p)} \, ,$$

where $p$ is the order of the moment and $\nu(p)$ an arbitrary function. In what follows we call the spectrum of the moments of the displacement as the function $\gamma_p = \nu(p)p/2$.

For scale-invariant transport the probability density accepts a scaling $P(\Delta x,t) = t^{-\nu} F(\Delta x/t^\nu)$ with $\nu(p) = \nu$ constant, meaning that all moments of the displacement are characterised by the single scale $\nu$. For processes with constant velocity $\nu$ takes values in $[0, 2]$ and a generalized diffusion coefficient can be defined through the variance of the displacement as

$$D_{\gamma} := \lim_{t \to \infty} \frac{\text{Var} [\Delta x(t)]}{2dt^\nu} \, .$$

Normal diffusion is obtained for $\nu = 1$ while anomalous diffusion (also termed weak anomalous diffusion) yields sub-diffusion for $0 < \nu < 1$ and super-diffusion for for $1 < \nu < 2$. For $\nu = 2$ the transport is ballistic.

Here we focus on strong anomalous diffusion for which the function $\nu(p)$ is not constant [18], meaning that transport results from a multi-scaling process. Most commonly, strong anomalous diffusion arises from a bi-scaling process, namely as a combination of normal or weak anomalous diffusion weighted by the bulk of the distribution (for $\Delta x/t^\nu \lesssim 1$) and rare but long-lasting ballistic excursions determining the tails of $P(\Delta x,t)$ for $\Delta x/t^\nu \gg 1$). This scaling leads to a piecewise linear function $\nu(p)$ and has been observed in a variety of systems such as polygonal billiards [19, 20], billiards with infinite horizon [21, 22], one-dimensional maps [23, 24], running sandpiles [25], stochastic models of inhomogeneous media [27, 28], the diffusion in laser-cooled atoms [29], the phase diffusion of a chaotic pendulum [30], and in experiments on the mobility of particles inside living cancer cells [31], among others. The more complex situation of a nonlinear function $p\gamma(p)$ has been observed in the motion of particles passively advected by dynamical membranes [32] and in the bulk-mediated diffusion on lipid bilayers [33].

In recent years a great deal of understanding on strong...
anomalous diffusion has been gained by means of generalisations of the central limit theorem in terms of non-normalisable densities [11, 34, 35].

Strong anomalous transport is therefore a generalisation of the more common weak anomalous transport and is believed to be generic for dynamics with fat-tailed waiting time distributions. Indeed, numerous investigations have been devoted to establish the relation between the properties of the dynamics and transport. For this deterministic maps have played a distinguished role. Roughly speaking chaos correspond to diffusion while non chaotic dynamics often exhibits anomalous diffusion due to the exponential decay of correlations in the former and slower than exponential in the later. [11, 2, 6, 20, 25]. Due to this, stochastic dynamics is often closer to that of chaotic dynamics [30, 37], but numerous questions remain open [3, 4, 27, 38, 41]. In particular, the asymptotic behaviour of correlation functions is not understood in general, although it is relevant e.g. to distinguish transport processes that are effectively different but have the same moments [2]. Numerous investigations have been devoted to this subject, see e.g. [12, 42].

In [25] a deterministic map named Slicer Map (SM) was recently introduced to shed light on these issues. It was shown that after an appropriate matching of parameters, the moments of the displacement scale with time like those of the Levy-Lorentz gas (LLg) previously introduced in [27], despite the very different nature of both dynamics. It was further proven [46], that this choice of parameters leads to the equality of the large time scalings of the position autocorrelation functions. This means that these two very different systems are indistinguishable on the level of the statistics of displacements, possibly suggesting a universality class.

The reason of this equivalence lies in the fact that the transport of SM and of LLg get an asymptotically relevant contribution from the ballistic flights, the flights that in a finite time \( t \) travel a distance \( vt \) with constant velocity \( v \) [11]. Such flights constitute rare events in both systems, because the probability of bouncing back tends to 1 as \( t \) grows. However, because the associated travelled distance is the largest possible, the contribution of ballistic flights to transport is sizeable. In particular, the ballistic flights of the SM give the smallest possible contribution to asymptotic transport regimes dominated by ballistic flights, since they contribute as much as the non-ballistic flights.

In this paper we show that this equivalence is much broader, characterising a universality class of transport asymptotically dominated by ballistic motion, irrespective of the details of the dynamics. Furthermore, we introduced a minimal deterministic map that we call Fly-and-Die (FnD), and propose it as representative of this class. We show that the asymptotic transport properties of any element in the class can be predicted from the FnD. For instance, the FnD model allows a simple calculation even of the \( n \)-point position correlation functions, that can be associated with any other element in the class, from mere knowledge of the scaling of the MSD.

This paper is organised as follows: In Section II we introduce the FnD dynamics and obtain analytical expressions for the transport representing the universality class. In Section III we summarise our previous results concerning a possible universal description of the moments of the displacement for the SM and the LLg. Section IV is devoted to the properties of billiard and billiard-like dynamics and refine the definition of the universality class. Section V summarises our conclusions.

II. THE FLY-AND-DIE DYNAMICS

In this section we introduce a minimal continuous-time model that we call Fly-and-Die (FnD), whose dynamics are, by construction, asymptotically dominated by ballistic excursions.

Consider the dynamics of a particle on the semi-infinite line \([0, \infty)\). Each trajectory in the FnD model is labeled by its initial condition \( x_0 \). Starting at \( x_0 \) the particle moves along the positive \( x \) axis with unit velocity up to a time \( t_c(x_0) \), and then it stops in position \( x_0 + t_c(x_0) \), namely

\[
x(x_0, t) = \begin{cases} 
  x_0 + t & \text{for } t \leq t_c(x_0) \\
  x_0 + t_c(x_0) & \text{for } t > t_c(x_0) 
\end{cases}
\]

Super-diffusive motion is expected to emerge when the distribution of the flight times \( t_c \) has a power-law tail. To be concrete, let us consider the case in which

\[
t_c(x_0) = b \left( \frac{1}{x_0} \right)^{1/\xi} 
\]

with \( \xi \) and \( b \) positive constants, and the initial condition \( x_0 \) uniformly distributed in the interval \([0, 1]\).

In simple words, an ensemble of particles with FnD dynamics consist of independent ballistic flights with a duration that is fully determined by the initial condition. In spite of the simplicity of these dynamics that allows us to easily obtain exact expressions for the statistics of the displacements and of the velocities, we shall show that its transport capture the properties of a whole class of dynamics asymptotically dominated by ballistic excursions.

Consider the probability distribution function \( P(> t) \) to perform a flight \( x(x_0, t) \) longer than \( t \). This probability amounts to the fraction of initial conditions \( x_0 \) such that \( t_c(x_0) > t \), and it is simply given by

\[
P(> t) = x_0(t) = \frac{b}{t^{\xi}}.
\]

Consequently, for \( p \neq \xi \) the \( p^{th} \) moment of the displacement \( \Delta x(t) \) for \( p \neq \xi \) takes the form

\[
\]
The trajectory is still flying at time $t_1$, hence $\Delta x(t_1) = t_1$ and $\Delta x(t_2) = t_2$.

$P(t_2) < x_0 < P(t_1)$: The trajectory was still flying at time $t_1$ but it died by the time $t_2$. Consequently, $\Delta x(t_1) = t_1$ and $\Delta x(t_2) = t_2(x_0)$.

$P(t_1) < x_0 < 1$: The trajectory died before $t_1$. Consequently, $\Delta x(t_1) = \Delta x(t_2) = t_c(x_0)$.

Splitting the integral and performing a calculation analogous to the derivation of Eq. (8) one finds

$$\phi(t_1, t_2) = \begin{cases} \frac{b t_1 \frac{b^2}{\eta - 1}}{\frac{2}{\eta} - \frac{2 - \eta}{2} \frac{b^2}{\eta - 1} - \frac{2 - \eta}{\eta} \frac{b^2}{\eta - 1}} & , \eta \neq 1 \\ b t_1 \ln \frac{b t_1}{t_1} + 2 b t_1 - b^2 & , \eta = 1 \end{cases}$$

For $t_1 = t_2$ this reduces to the mean-square displacement, Eq. (8).

In the literature one commonly normalizes the correlation function (9) by the second moment calculated at either of the two times, or by the geometric mean of those moments, $\langle |\Delta x(t_1)|^2 \rangle^{1/2} \langle |\Delta x(t_2)|^2 \rangle^{1/2}$. The normalised correlation is a function of $t_2/t_1$ in that case, and it may tend to 1 when $t_2/t_1$ tends to 1, because $\phi(t_1, t_2)$ may tend to $\langle |\Delta x(t_1)|^2 \rangle$ plus terms that become negligible in that limit.

For the representation of the asymptotics on doubly logarithmic scale this is not convenient. Therefore we prefer to normalize the correlation function by the time difference $h = t_2 - t_1$. For $\eta \neq 1$ this yields

$$C_h(t_1, t_2) = \langle \eta/b \phi(t_1, t_2) + (2 - \eta) \frac{b^2}{\eta - 1} \frac{b^2}{\eta - 1} \rangle$$

$$= \frac{\eta}{\eta - 1} \frac{t_1}{h} \left( \frac{t_1}{h} + 1 \right)^{\eta - 1} - \frac{2 - \eta}{\eta - 1} \left( \frac{t_1}{h} \right)^{\eta}$$

$$= \frac{\eta}{\eta - 1} \left( \frac{t_1}{h} \right)^{\eta} \left[ \left( 1 + \frac{h}{t_1} \right)^{\eta - 1} - \frac{2 - \eta}{\eta} \right]$$

which gives rise to the asymptotic scaling

$$C_h(t_1, t_2) \sim \begin{cases} \frac{2}{t_1} \left( \frac{t_1}{h} \right)^{\eta} & \text{for } t_1 \gg h \\ \frac{2}{t_1} \frac{t_1}{h} & \text{for } t_1 \ll h, \eta \neq 1 \\ \frac{2}{t_1} \frac{t_1}{h} & \text{for } t_1 \ll h, \eta = 1 \end{cases}$$

Equation (10a) is shown in Fig. 2 for $\xi = 0.5$, $b = 1$ and $h = 1$.

In the following sections we show that these results characterise a whole class of different dynamics.

III. MAPPING THE MOMENTS OF THE DISPLACEMENT

In this section we explore other dynamics and show how their global properties of transport can be expressed in terms of the results found in the previous section for
the Fly-and-Die model. To exhibit the generality of those results, we consider a deterministic dynamical model called the Slicer Map (SM) \[23\], and a continuous time stochastic dynamics called the Lévy-Lorentz gas (LLg) \[40\].

### A. The Slicer Map

The SM is a one-parameter deterministic exactly solvable map \( S_n : [0, 1] \times \mathbb{Z} \to [0, 1] \times \mathbb{Z} \) defined by \[23, 40\]

\[
S_\alpha(x, m) := \begin{cases} 
(x, m-1) & \text{if } 0 \leq x \leq \ell_m \text{ or } \frac{1}{2} < x \leq 1 - \ell_m, \\
(x, m+1) & \text{if } \ell_m < x \leq \frac{1}{2} \text{ or } 1 - \ell_m < x \leq 1,
\end{cases}
\]

where the so-called “slicer” \( \ell_m \) is given by

\[
\ell_m := \frac{1}{(|m| + 2^{1/\alpha})^\alpha} \quad \text{with } \alpha \in \mathbb{R}^+.
\]

For \( 1/2 < x < 1 \) each iteration of the map increases the values of \( m \) by one, until \( x > \ell_m \). Subsequently, the trajectory enters a stable period-two cycle, oscillating back and forth between the two neighbouring sites \( m \) and \( m - 1 \). Similarly, for \( 0 < x < 1/2 \) each iteration of the map decreases the values of \( m \) by one, until \( x < -\ell_m \), and then the trajectory enters a stable period-two cycle.

The SM was inspired by the dynamics of polygonal billiards. In it the distance between two trajectories does not change in time, as long as they are mapped by the same branch of the map which, for each \( m \in \mathbb{Z} \), are defined by \( \ell_m \). The distance between two points \( x_1 \) and \( x_2 \) jumps discontinuously when they reach a cell \( m \in \{x_1, x_2\} \). Thus, the dynamics is reminiscent of polygonal billiard dynamics \[23, 13, 20\], where initial conditions are only separated when they are reflected by different sides of the polygon. The corners act as slicers of the bundle of initial conditions (see section sec:pol for more details). The analogy between the two systems also includes the fact the SM has vanishing Lyapunov exponent and it preserves the phase space volume.

Upon varying the parameter \( \alpha \) describing the position of the slicers, the SM exhibits all transport regimes, sub-diffusion (\( 2 \geq \alpha > 1 \), super-diffusion (\( 1 > \alpha > 0 \), and diffusion (\( \alpha = 1 \)).

The moments of the displacement of a particle under the SM where obtained in \[25\]. To compare with moments of the FnD model, here we redefine the SM on a symmetric unit interval simply by shifting the origin by \(-1/2\). Consider an ensemble of initial conditions with \( m = 0 \) and \( x \) uniformly distributed in the right half of the unit interval \([0, 1/2]\). In the limit \( n \gg 2^{1/\alpha} \) and for \( p \neq \alpha \) one obtains

\[
\langle (x_n - x_0)^p \rangle \sim 2 \int_0^{\ell_n} dx n^p + 2 \int_{\ell_n}^{1/2} dx \left(x^{-1/\alpha} - 2^{-1/\alpha}\right)^p
\]

\[
\sim 2 n^p \ell_n + \frac{2}{1 - p/\alpha} \left(2^{-1+p/\alpha} - \ell_n^{-1-p/\alpha}\right) + O(1)
\]

\[
\sim \frac{2p}{p - \alpha} n^{p-\alpha} + O(1)
\]

Instead, for \( p = \alpha \) one obtains

\[
\langle (x_n - x_0)^\alpha \rangle \sim 2 \ln \frac{n^{\alpha}}{2}.
\]

Therefore, the moments of displacement are

\[
\langle (x_n - x_0)^p \rangle \sim \begin{cases} 
\text{const} & \text{for } p < \alpha, \\
2 \ln \frac{n^\alpha}{2} & \text{for } p = \alpha, \\
\frac{2p}{p - \alpha} n^{p-\alpha} & \text{for } p > \alpha
\end{cases}
\]

The spectrum of the moments of displacement is shown in Fig. 1 (blue curve) for \( \alpha = 0.1 \).

From Eq. (14) one can obtain an asymptotic expression for the probability distribution function of the displacement \( P(x, t) \), where we refer to the displacement as \( x \) to ease notation, and extend the discrete time to a continuous time \( t \). We restrict ourselves to the super-diffusive regime, namely \( 0 < \alpha < 1 \).

By symmetry all the odd moments are zero. The moment generating function is defined as

\[
\tilde{P}(k, t) = \int_{-\infty}^{\infty} e^{itx} P(x, t) dx.
\]

Inspecting the moments of displacement of Eq. (14) we can reconstruct the generating function in Fourier space as

\[
\tilde{P}(k, t) = 1 + \frac{1}{t^\alpha} \sum_{j=1}^{\infty} 4j \frac{t^{2j}}{2 - \alpha} \frac{(-1)^j k^{2j}}{(2j)!},
\]

where the moment’s order \( p = 2j \). This sum converges to give

\[
\tilde{P}(k, t) = 1 - \frac{2}{2 - \alpha} k t^{1-\alpha} \sin(kt).
\]

Inverting (17) we obtain that in the asymptotic large time limit

\[
\lim_{t \to \infty} P(x, t) = 2\pi \delta(x) + \frac{\sqrt{2\pi} t^{1-\alpha}}{2 - \alpha} \delta'(x + t) - \delta'(x - t),
\]

where \( \delta(x) \) is the Dirac’s delta function and \( \delta' \) denotes its derivative with respect to its argument, yielding the probability measure moving away from the origin.

The displacement autocorrelation function of the SM \( \phi(m, n) = \langle (x_m - x_0) (x_n - x_0) \rangle \) was obtained in \[40\]. In the limit of large time \( m \) (or \( n \)) one obtains
\[
\phi(m, n) \sim 2 \int_0^{\ell_m} dx \, n \, m + 2 \int_0^{\ell_m} dx \, m \left( x^{-1/\alpha} - 2^{1/\alpha} \right) + 2 \int_{\ell_m}^{1/2} dx \left( x^{-1/\alpha} - 2^{1/\alpha} \right)^2 \\
\sim \frac{2}{1 - \alpha} m \, n^{1-\alpha} - \frac{2 \alpha}{(1 - \alpha)(2 - \alpha)} m^{2-\alpha} + \frac{2}{(\alpha - 1)(\alpha - 2)} 2^{2/\alpha}. \tag{19}
\]

For \( m = n \), the expression reduces to the mean square displacement at finite time \( n \)
\[
\phi(n, n) = \langle (x_n - x_0)^2 \rangle \sim \frac{4 \beta}{r_0} \langle r \rangle ^{-\beta - 1}, \quad r \in [r_0, \infty), \tag{20}
\]
where \( \beta > 0 \) and \( r_0 \) is the minimum distance between scatterers.

The spectrum of the moments of the displacement were derived in \[23\] for a nonequilibrium initial condition, namely for particles starting at a scatterer position. The resulting spectrum is
\[
\langle |r(t)|^p \rangle \sim \begin{cases} 
\frac{t^{\eta_p}}{r_0} & \text{for } \beta < 1, \; p < \beta, \\
\frac{t^{\eta_p}}{\beta^{p-\beta}} & \text{for } \beta < 1, \; p > \beta, \\
t^{\eta_p} & \text{for } \beta > 1, \; p < 2\beta - 1, \\
t^{\eta_p + p - \beta} & \text{for } \beta > 1, \; p > 2\beta - 1.
\end{cases} \tag{21}
\]

The behaviour of Eq. \[21\] is shown in Fig. \[1\] (red curve) for \( \beta = 1.4 \).

In particular, for the mean-square displacement, \( p = 2 \), one obtains \( \langle r^2(t) \rangle \sim t^{\eta} \) with
\[
\eta = \begin{cases} 
2 - \frac{\beta^2}{(1 + \beta)} & \text{for } \beta < 1, \\
\frac{5}{2} \beta & \text{for } 1 \leq \beta < 3/2, \\
1 & \text{for } 3/2 \leq \beta.
\end{cases} \tag{22}
\]

Unlike the FnD and the SM, the LLg model only exhibits super-diffusive transport for nonequilibrium initial conditions for \( 0 < \beta < 3/2 \), and diffusive transport for \( \beta \geq 3/2 \).

Compared to the previous models, the Levy-Lorentz gas is much more complicated, and accessibility of exact analytical expressions is limited. As a matter of fact, there does not exist analytical expressions for the autocorrelation functions of the LLg. Extensive numerical simulations have been previously reported in \[40\] for the autocorrelation function of the displacement \( \langle \Delta x(t_1)\Delta x(t_2) \rangle \) for i) \( t_2 \) at fixed \( t_1 \); ii) \( t_1 \) and \( t_2 \) for a fixed time difference \( h = t_2 - t_1 \); iii) \( t_1 \) with \( t_2 = t_1 + t_1^q \) for different fixed values of \( q \).

### C. Equivalence of the statistics of the displacement

Recently in \[25\] it was discovered that restricted to the super-diffusive regime, namely for \( 0 < \alpha < 1 \), the spectrum of moments of the displacement of the SM does coincide with that of the LLg once the respective parameters \( \alpha \) of Eq. \[14\] and \( \beta \) of Eq. \[21\] are rescaled so that the MSD of both models coincide, namely depending on the value of \( \beta \) this is achieved if \[25\]
\[
\alpha = \begin{cases} 
\frac{\beta^2}{1 + \beta} & \text{for } 0 < \beta \leq 1, \\
\beta - \frac{\beta}{2} & \text{for } 0 < \beta \leq \frac{3}{2}, \\
1 & \text{for } \frac{3}{2} < \beta.
\end{cases} \tag{23}
\]

Moreover, recalling Eq. \[7\] we see that this extends to the whole spectrum of the FnD simply by taking \( \xi = \alpha \) and identifying \( b = 2 \), except for the constant value for \( p < \alpha \), which requires a more detailed analysis \[23\]. For \( p = 1 \) and \( p = 2 \) one thus determines the offset values
\[
\langle (x_n - x_0)^{2\alpha} \rangle \sim \frac{2^{1/\alpha}}{\alpha - 1} \quad \text{for } \alpha > 1, \tag{24a}
\]
\[
\langle (x_n - x_0)^{2\alpha} \rangle \sim \frac{2^{2/\alpha}}{(\alpha - 1)(\alpha - 2)} \quad \text{for } \alpha > 2. \tag{24b}
\]

It should however be noted that for the SM a single linear relation between \( \xi \) and \( \alpha \) suffices while three different relations are needed to connect \( \xi \) (or \( \alpha \)) to \( \beta \) in the LLg case. This indicates that the equivalence is not complete, although no discrepancies are revealed by the moments for given values of \( \alpha \), \( \beta \) and \( \xi \). The equivalence of the moments of displacement becomes evident in Fig. \[1\] where
we expect that for the spectra of the three models are compared. Rescaling the parameters of each model such that the MSD (indicated by the open squares) coincide with that of FnD (green curve), the scaling in time of all the moments with $p \geq 2$ are described by Eq. (7). Note, however, that lower moments than $p = 2$ do not coincide as these are determined not by the contribution of the rare ballistic excursions but by the typical events described by the bulk of the distribution $P(\Delta x, t)$ (cf. Eq. (25)).

The equivalence of the statistics of the particle displacement, in spite of the quite different dynamics and transport of these models, suggests a certain universality. As discussed above, these models share the fact that in the large time limit, transport is dominated by ballistic excursions. So far, the spectrum of moments of the displacement of the models that we have studied exhibit strong anomalous diffusion. The bilinear shape of the spectra (see Fig. 1) originates from the existence of two different time scales ruling the behaviour of the probability distribution function of the displacement $P(\Delta x, t)$, one dominating the dynamics around the center of the distribution of the displacement and another dominating the dynamics at the tails. This bi-scaling has been intensively studied and generalized to the existence of a non-normalizable probability density called infinite covariant density [34, 47, 48]. This has been proven to be the case for the Fly-and-Die dynamics. We can generally assume that the center of the distribution $P(\Delta x, t)$ is given by a Lévy stable distribution with some parameter $\nu$ [35]. Then the center of the distribution is determined by the lowest moments, and from [4] we expect that for $p < \nu$

$$P_{\text{bulk}}(\Delta x) \sim \Theta x^{-(1-\nu)},$$

where $\Theta$ is a time independent constant, corresponding to the value of $\langle (\Delta x)^p \rangle$ for $p < \xi$. Note that in general the exponent $\nu$ will be related to the model parameter $\xi$.

Furthermore, from the asymptotic result [45] for higher moments $p \gg 1$, we expect at the tails of the distribution will obey the scaling

$$P_{\text{tail}}(\Delta x, t) \sim \frac{1}{t^\beta} f \left( \frac{\Delta x}{t^\mu} \right),$$

for some parameters $\beta$ and $\mu$, some function $f(x)$ and $x/t^\mu \gg 1$. Since the FnD dynamics is dominated in the infinite time limit by ballistic trajectories, we expect $\mu = 1$.

Expressions (25), (26) need to match at intermediate times scales [35]. This means that when $\Delta x/t^\mu \ll 1$ the time dependence on the scaling must vanish since $\Theta$ is time independent. This is the case when

$$\mu(1-\nu) - \beta = 0,$$

from where we determine $\beta = 1 - \nu$. Therefore, the tail of the distribution function must satisfy

$$P_{\text{tail}}(\Delta x, t) \sim t^{-(1+\nu)} f \left( \frac{\Delta x}{t} \right).$$

From Eq. (28) the higher moments of the $p > \nu$ are obtain as

$$\langle |x(t)|^p \rangle = t^{-(1+\nu)} \int x^p f \left( \frac{x}{t} \right) dx = t^{-(1+\nu)} \int \left( \frac{x}{t} \right)^p f \left( \frac{x}{t} \right) d \left( \frac{x}{t} \right) = t^{p-\nu} \int (\tilde{x})^p f(\tilde{x}) d(\tilde{x}) \propto t^{p-\nu},$$

where we have denoted the displacement $\Delta x \rightarrow x$ to ease notation. Comparing this with Eq. (7), suggest that the bulk of the distribution is ruled by the scaling exponent of the FnD dynamics, namely $\nu = \xi$.

Moreover, since $\nu = \nu(\xi)$ is exclusively determined by the scaling of the center of the distribution and, given that we have not used further details of the dynamics to determine (29), we may conclude that for any transport process whose asymptotic behaviour is dominated by ballistic excursions $x(t) \sim t$, the scaling of higher moments of the particle's displacement $p > \nu$ is universal given by

$$\langle |x_t - x_0|^p \rangle \sim t^{p-\nu}.$$
Therefore, holding FnD as the reference model for which the MSD scaling is \( \gamma_2 = 2 - \xi \), if the spectrum of moments of the displacement \( \gamma_p \) for any other model is given by Eq. (30), then its spectrum is described by FnD for any moment \( p \geq 2 \) after the simple rescaling \( \gamma_p \to \gamma_p + \nu - \xi \). Since lower moments are not determined by the ballistic time scale but by the bulk of the distribution, these are model dependent (see Fig. 1). More importantly, this universality extends to the autocorrelation function of the displacement.

Indeed, the displacement autocorrelation of the SM Eq. (19) exhibits the same scaling as Eq. (10a) after the identifying \( \eta = 2 - \alpha \) as for the moments, and \( h = n - m \).

Rearranging Eq. (19) accordingly to these identifications yields

\[
\frac{2}{\eta} \phi(m, n) - \frac{1}{(n-1)} 2^{-1+2/\alpha} \frac{h^n}{n!} = \frac{\eta}{\eta - 1} \left( \frac{m}{h} \right)^n \left[ \left(1 + \frac{h}{m} \right)^{\eta - 1} - \frac{2 - \eta}{\eta} \right]. \tag{31}
\]

This expression agrees with Eq. (10a) up to the constant offset added to \( \phi(m, n) \).

Fig. 2 shows the autocorrelation function of the displacement for the FnD (Eq. 10a) for \( \xi = 0.5 \), the SM (Eq. 19) for \( \alpha = 0.1 \), and the LLg numerically obtained for \( \beta = 0.9 \).

To plot the autocorrelation function for the Slicer Map in Eq. (19), we have extended discrete time to continuous time and in addition identified \( m \to t_2 \), \( n \to t_1 \), \( h = m - n \). Taking \( \alpha = 2 - \eta \) we obtain

\[
\frac{2}{\eta} \phi(t_2, t_1) h^n = \frac{\eta}{\eta - 1} \left( \frac{m}{h} \right)^n \left[ \left(1 + \frac{h}{m} \right)^{\eta - 1} + \left( \frac{h}{m} \right)^{\eta} \right] - \frac{2 - \eta}{\eta} (1 + \frac{h}{m})^{\eta}, \tag{32}
\]

which is the expression shown in Fig. 2 for \( b = 2 \).

Equation (10a) suggests a data collapse when plotting \( C_b(t_1, t_2) \) as function of \( t_1/h \). For large \( t_1/h \), \( \phi \) asymptotically scales with \( t_2 \) like the mean-square displacement, corresponding to \( h \to 0 \), does. The asymptotics for small \( t_1/h \) corresponds to \( t_2 \gg t_1 \), and if \( t_1 \) is fixed, the correlation function decays with \( h \) like \( 1/h \).

This is shown in the inset Fig. 2. After rescaling the autocorrelation functions of the three models perfectly collapse for times \( t_1/h \gtrsim 1 \). As for the moments of the displacement the short time behaviour \( t_1/h < 1 \) is determined by the typical events described by the bulk of the distribution of the displacements and thus, it is system dependent.

Since the properties of transport are fully determined by the behaviour of the moments of the displacement and time autocorrelations, this observation reinforces the definition of a universality class for those dynamics asymptotically dominated by ballistic excursions and the Fly-and-Die dynamics as representative of this class.

In the next section we further refine the definition of the universality class by considering billiard dynamics.

\[\text{FIG. 2. (Colour online). Displacement autocorrelation function } C_b(t_1, t_2) \text{ for the FnD of Eq. (10a) with } \xi = 0.5, b = 1 \text{ and } h = 1 \text{ (green curve), the SM of Eq. (32) with } \alpha = 0.1, b = 2 \text{ and } h = 1 \text{ (blue curve), and the LLg with } \beta = 0.9 \text{ (red plus symbols). Inset: Autocorrelation } C_b(t_1, t_2) \text{ with the appropriate identification of the parameters as explained in the text.}\]

IV. A UNIVERSALITY CLASS

A class of dynamics characterised by rare ballistic excursions is that of billiards with infinite horizon, namely billiards for which the mean free path is infinite [21]. A paradigmatic example is the Lorentz gas for which short excursions transport is neither by Gaussian nor by Lévy statistics [50]. Another particular example is that of polygonal billiards that, without infinite horizon, the asymptotic transport is still dominated by ballistic excursions due to the existence of perfect “ballistic” paths [22]. In this section we explore how the analytic results of FnD do universally describe these systems and use the results to refine the definition of a universality class.

A. Lorentz gas

We consider first a paradigmatic billiard model, the periodic Lorentz gas consisting of an array of periodically placed circular scatterers on a triangular lattice, of radius \( R \) on the plane (in the original billiard the scatterers had random positions). The separation between nearest-neighbour scatterers is set to \( \Delta = \frac{\pi}{3} \cos \frac{\pi}{3} \) so that the horizon is infinite for \( R < 1 \). The dynamics consists of the free flights of a pointwise particle of unit velocity undergoing specular collisions with the scatterers (see Inset of Fig. 3).

With infinite horizon the probability to observe a trajectory with length between \( l \) and \( l + dl \) decays as \( l^{-3} \), meaning that the variance of the of the trajectory’s length diverges. In a seminal work Bleher showed that for the infinite horizon Lorentz gas the displacement scaled by
\( \sqrt{\ln(t)} \) exhibits Gaussian statistics \( \xi_1 \). Since then the asymptotic scaling of the MSD has generated a great deal of research to understand those logarithmic corrections to linear scaling of the MSD \( \xi_2 \). Only recently the behaviour of the MSD, resulting from the combination of the bulk Gaussian fluctuations and the statistics of the rare events has been thoroughly explained \( \xi_3 \).

At infinite horizon \( (R < 1) \), the Lorentz gas exhibits strong anomalous diffusion \( \xi_4 \), having a spectrum of the moments of displacement

\[
\gamma_p = \begin{cases} 
\frac{p}{2} & , \quad p \leq 2 \\
\frac{p}{p - 1} & , \quad p \geq 2
\end{cases}
\]

(33)

Instead, when horizon is finite \( (R \geq 1) \), transport is diffusive (beyond logarithmic corrections), and the moments of displacement behave in time with \( \gamma_p = p/2 \) for any order \( p \). This is shown in Fig. 3 resulting from numerical evolution of \( 10^6 \) trajectories during a total time of \( 10^6 \). Initial positions \( r(0) \) and vector velocities were chosen randomly. After verifying that all the moments scale asymptotically as a power law of time Fig. 3 shows the spectrum \( \gamma_p \) for \( R = 0.1 \) (solid squares) and \( R = 1.1 \) (open squares).

Note that for the Lorentz gas the MSD belongs to both families of scaling: Gaussian fluctuations for small deviations \( p < 2 \), and ballistic large deviations \( p > 2 \). Therefore, the FnD model with \( \xi = 1 \) describes the spectrum of moments of the infinite horizon Lorentz gas for \( p > 2 \). We argue that this result can be extended to any billiard with infinite horizon, as long as the MSD belongs to the asymptotic ballistic family. For billiards with finite horizon but asymptotically dominated by ballistic rare excursions the situation is slightly more restricted, as discussed in the next section \( \xi_5 \).

Furthermore, we have numerically computed the autocorrelation function for the displacement at arbitrary times \( t_1 \neq t_2 \)

\[
C(t_1, t_2) = \langle \Delta r(t_1) \Delta r(t_2) \rangle ,
\]

(34)

where \( \langle \cdot \rangle \) refers to average over the ensemble of trajectories. As in the previous sections we define \( h = t_2 - t_1 \). The results are shown in Fig. 3 as a function of \( t_1/h \) for infinite horizon with \( R = 0.1 \) (blue cross symbols). The scaling of the autocorrelation function for \( t_1/h \gg 1 \) inherits the same scaling as MSD, namely \( C(t_1, t_2) \sim t_1/h \). On the other hand, for \( t_1/h \ll 1 \) the scaling of the autocorrelation function will generally depend on the details of the short time scale. For the infinite horizon Lorentz gas we obtain \( C(t_1, t_2) \sim \sqrt{t_1/h} \). In Fig. 3 we also show the autocorrelation function of FnD for \( \xi = 0.5 \). Matching the MSD \( \langle |\Delta r(t)|^p \rangle = 1 \) for the infinite horizon Lorentz gas with that of FnD \( \langle |\Delta r(t)|^p \rangle = 1.5 \) implies a scaling of the autocorrelation function of \( C(t_1, t_2) \sqrt{t_1/h} \). The rescaled autocorrelation is shown in the inset of Fig. 3. Collapse with the autocorrelation of FnD for \( t_1/h \gg 1 \) is accomplished by multiplication of \( C(t_1, t_2) \) by an appropriate constant factor.

Given that transport at large times is determined by the MSD and all the moments of the displacement of higher order and by the autocorrelation function, we conclude that the transport of the infinite horizon Lorentz gas is fully determined by FnD. Therefore, it belongs to the universality class of the FnD of asymptotic transport dominated by ballistic excursions.

We finish this section by noting that with finite horizon the scaling of the autocorrelation function is the same as that with infinite horizon. This is indeed the case because for the Lorentz gas, the MSD is the same in both cases. Nevertheless, with finite horizon the spectrum of moments of the Lorentz gas is not described by the corresponding spectrum of FnD of Eq. (17) after matching of the MSD. Therefore the finite horizon Lorentz gas does not belong to the universality class of FnD.

B. Polygonal channel

To study further evidence for the existence of a universality class represented by the FnD model, in this section we investigate a drastically different billiards, generically named polygonal channels. These billiards have been widely studied in the past due to their peculiar transport behaviour \( \xi_6 \). This billiard corresponds to the dynamics of a pointlike particle undergoing specular reflections with upper and lower zigzagging boundaries. The boundaries are repetitions of an elementary cell whose geometry is fully specified by four parameters as shown in the inset of Fig. 4 (here we follow the notation that has been used in \( \xi_7 \)).

When \( \Delta y_h + \Delta y_l < H \) the channel has an infinite horizon, meaning that a trajectory can move for an infinitely large horizontal distance before encountering a boundary. As for the Lorentz gas, in this case the probability density
we show the spectrum of moments of dis-
5
4
2
3
6
7
1
0 2 4 6 8
FIG. 4. (Colour online). Spectrum of the moments of
disp.
Gaussian diffusive fluctuations and the bal-
horizontal (∆ = 1, ∆ = 0.77, ∆ = 0.45 and: H = 1.27 (blue solid cir-
circles), H = 1.17 (green open circles) and H = 1.07 (red open squares).
The first set has horizontal while the last
two have infinite horizon. The dashed line corresponds to γ = p/2 while the dotted curves are
γ = p − ν with (from top to bottom) ν = 1, 1.5 and 2. Inset: Schematic geometry
of the polygonal channel.
of the trajectory’s length l scales as ∼ t−3. With finite
horizon (Δy + Δy > H), these infinitely long trajecto-
does not exist. Nevertheless, a dense family of special
trajectories exist, for which the transport is asympto-
tically dominated by ballistic excursions [19, 20, 22]. The
dynamics of the polygonal channel differ substantially
from the dynamics in a Lorentz gas. First, the dynamics
in the polygonal channel is integrable, with “non ex-
ponential” separation of trajectories produced by the
channel corners. Secondly, the ballistic excursions appear not
as the result of infinite horizon but due to the very poor
mixing of the collision angles [19, 20]. These properties
make the dynamics of polygonal billiard much richer.
In Fig. 4 we show the spectrum of moments of dis-
placement γ for three different channels with ∆x = 1,
∆y = 0.77, ∆y = 0.45 and different widths H (pre-
viously considered in [19]). For H = 1.27 (blue solid
circles), the polygonal channel has infinite horizon and
as for the Lorentz gas, the MSD lies at the transition
between the Gaussian diffusive fluctuations and the
ballistic large fluctuations. As we argued in the previous
section, when the horizon is infinite, the spectrum γ is
described by Eq. (34).
More interesting is the behaviour of the polygonal
channel with finite horizon. We show in Fig. 5 two ex-
amples with H = 1.17 (green open circles) and H = 1.07
(red open squares). Strictly speaking in these two chan-
nels no ballistic trajectories exist. However, due to the
poor mixing properties of the collision angle, there are
special trajectories that move in the same direction for
arbitrary long distances before reversing their motion,
thus mimicking ballistic trajectories [20]. This is clearly
evidenced in Fig. 5 where for all three polygonal channels
the behaviour of the moments of large order is given by
p − ν, with ν = 1 for infinite horizon (H = 1.27), and
ν = 1.5 (for H = 1.17), ν = 2 (for H = 1.07) for finite
horizon.
In distinction with billiards with infinite horizon, with
finite horizon the polygonal channels has and MSD that
does not necessarily belong to the ballistic family of tra-
jectories. For H = 1.17 the ballistic scaling sets in at
p = 3 and for H = 1.07 at p = 4. Therefore, matching the
MSD with that of FnD is not enough in these cases to
describe their spectrum of moments for higher order
p > 2.
As in the previous sections we have numerically com-
puted the autocorrelation function of the displacement at
arbitrary times, Eq. (34). We show the results in Fig. 6
for the polygonal channel for infinite horizon H = 1.27
(red dashed curve) and finite horizon H = 1.07 (black
dot-dashed curve). Since MSD has the same value for all
channels, and it also coincide with the MSD of Lorentz
gas, the same rescaling collapses all autocorrelations with
that of FnD (inset of Fig. 5).
Overall we can conclude that with infinite horizon
the polygonal channel belongs to the universality class of
 FnD so that its asymptotic transport is described by
FnD, while with finite horizon it does not. Also one
could include the polygonal channel with finite horizon
to the same universality class by extending the definition
of the class to match the lowest moment of displacement
belonging to both families of trajectories with FnD.
As a final remark we mention that other polygonal bi-
lliards like those with parallel boundaries (∆y = ∆y),
does not exhibit strong anomalous diffusion. While the
spectrum of moments obtained is described by γ = p − ν
for all p, the autocorrelation function of the displacement
does not satisfies the scaling C0(t1, t2)/h vs t1/h sug-
gested by FnD. Such models are therefore not elements
of the FnD class.

V. CONCLUSIONS

The moments of the displacement and the position
auto-correlation functions of many systems that show
super-diffusive transport are dominated by ballistic tra-
jectories. Here we argue that the moments of order 2
and larger, as well as the autocorrelation function of the
the displacements characterise a wide class of such sys-
tems, that have totally different microscopic dynamics
but enjoy the same transport properties. Elements in the
class, whose MSD scaling exponent η matches asymptoti-
cally, scale in the same way, as in Eq. (24) and are hardly
distinguishable as far as the statistics of positions are
concerned. Other observables, like the statistics of veloc-
ities, are needed to distinguish the different systems in
the class.
We have introduced the Fly and Die map as the sim-
plest representative of this universality class. For this
of the FnD dynamics, which allowed us to establish a scaling relation for the correlation function \( \phi(t_1, t_2) \) that only depends on the ratio \( t_1/h \). Given a certain value of \( \eta \), this provides a collapse on a single line of all our data.

Therefore we conclude that a transport model belongs to the universality class of FnD if:

1. The dynamics are asymptotically dominated by ballistic rare events. This means that the spectrum of higher moments will grow as \( t^{p+\nu} \) with \( \nu \in \mathbb{R} \).

2. The MSD belongs to the ballistic family but scales slower than \( t^2 \). However, MSD must be described by the small fluctuations, i.e., by the bulk of the distribution (\( \nu < 2 \) in Eq. (26)).

3. The autocorrelation function of the displacement admits the scaling \( C_b(t_1, t_2)/h \) vs \( t_1/h \).

The excellent agreement between the numerical data and the FnD predictions establishes a new way to analyse correlations in anomalous transport and it suggests that the FnD map can be taken as prototypical of the transport processes dominated by ballistic flights.

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