GLOBAL STABILITY OF V-SHAPED TRAVELING FRONTS IN COMBUSTION AND DEGENERATE MONOSTABLE EQUATIONS

ZHEN-HUI BU AND ZHI-CHENG WANG
School of Mathematics and Statistics, Lanzhou University
Lanzhou, Gansu 730000, China

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Abstract. This paper is concerned with the global stability of V-shaped traveling fronts in reaction-diffusion equations with combustion and degenerate monostable nonlinearity. The existence of such curved fronts has been recently proved by [39]. In this paper, by constructing new subsolutions, we show the asymptotic stability of V-shaped traveling fronts.

1. Introduction and main results. Parabolic differential equations can be used to model a host of natural processes such as biological invasions, combustion, chemical reactions, population dynamics, the spreading of diseases and others. An important class of solutions modeling the propagation of reaction is traveling wave front. In the past years, there had been many works devoted to the study of it, see [1, 6, 8, 9, 10, 11, 19, 22, 24, 36] and many other works. These literatures mainly focused on planar traveling wave solutions in one-dimensional or high-dimensional spaces. In recent years, there were many studies on nonplanar traveling wave solutions such as V-shaped waves in two-dimensional spaces, pyramidal traveling waves in three-dimensional spaces and conical traveling waves in high-dimensional spaces. The investigation on these multi-dimensional traveling waves has important applications to multi-dimensional chemical waves, multi-dimensional curved flames and nerve transmission phenomena.

For the following equation

\[ u_t = \Delta u + f(u) \quad \text{in} \quad \mathbb{R} \times \mathbb{R}^N \]  

(1)

with bistable nonlinearity, the existence and stability results of nonplanar traveling wave solutions were established by many authors in the dimensional \( N \geq 2 \), see [12, 15, 16, 20, 26, 27, 31, 32, 33, 34, 35]. For the Fisher–KPP nonlinearity, Hamel and Nadirashvili [17] and Huang [18] proved that the equation (1) exists nonplanar traveling fronts and these fronts are stable with \( N \geq 2 \). For the equation (1) with combustion nonlinearity, Bonnet and Hamel [2] established the existence of two-dimensional V-shaped traveling fronts with \( N = 2 \). Furthermore, Hamel and Monneau [13], investigated the questions of the existence, of the uniqueness and of the qualitative properties with conical shape in \( \mathbb{R}^N \) (\( N \geq 2 \)) and Hamel et al. [14] established the asymptotic stability of V-shaped traveling wave fronts.
in $\mathbb{R}^2$. Very recently, by constructing new supersolutions, the authors of this paper [5, 39] established the existence and stability of V-shaped traveling waves in two-dimensional spaces and pyramidal traveling fronts in three-dimensional spaces for the equation (1) with combustion and degenerate Fisher-KPP nonlinearity. For more results about nonplanar traveling fronts of reaction-diffusion systems and periodic reaction-diffusion equations, we refer to [4, 7, 25, 30, 43, 37, 38, 39, 41, 42].

In this paper, we investigate the following equation

$$
\frac{\partial u}{\partial t} = \Delta u + f(u) \quad \text{in } [0, +\infty) \times \mathbb{R}^2, \quad (2)
$$

$$
 u \mid_{t=0} = u_0 \quad \text{in } \mathbb{R}^2. \quad (3)
$$

Here a given initial function $u_0$ is continuous and bounded. The Laplacian $\Delta$ denotes $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. The nonlinear terms $f(u)$ are assumed to satisfy the following condition:

(A1) $f(u)$ is of class $C^{1+\theta} ([-\varepsilon, 1 + \varepsilon], \mathbb{R})$ for some constants $\theta \in (0, 1]$ and $\varepsilon \in (0, 1)$ and such that

$$
f(0) = f(1) = 0, \quad f'(0) = 0, \quad f'(1) < 0, \quad f(u) \geq 0 \quad \text{for } u \in (0, 1).
$$

In the one-dimensional space, a traveling wave front of (2) connecting the equilibria $0$ and $1$. the minimal wave speed of one-dimensional traveling wave fronts of (4) connecting equilibria $0$ and $1$.

Under the conditions (A1) and (A2) hold when the nonlinear term $f$ is of combustion type. In this case, the solutions $(c_*, \phi)$ of (4) are unique and in the sense that $c_*$ is unique and $\phi$ is also unique up to a translation in $\varrho$, see [1]. If the function $f(u) = u^p(1 - u)$ ($p > 1$), then it follows from [21] and the references therein that there exists a positive number $c_*$ such that the equation (4) has solutions $(k, \phi)$ if and only if $k \geq c_*$. Moreover, the traveling wave front $\phi(\varrho)$ with minimal speed $c_*$ satisfies

$$
\phi(\varrho) \sim A e^{c_* \varrho} \quad \text{as } \varrho \to -\infty \text{ with } A > 0.
$$

Thus, in this case, the assumptions (A1) and (A2) also hold and the number $c_*$ is just the minimal wave speed of one-dimensional traveling wave fronts of (4) connecting equilibria 0 and 1.

Under the conditions (A1) and (A2), we can obtain that there exist some positive constants $\overline{K}_1, \overline{K}_2, \overline{K}_3, \overline{K}_4, \overline{K}_5$ and $\Lambda$ such that

$$
| f'(u_1) - f'(u_2) | \leq \overline{K}_1 | u_1 - u_2 |^\theta, \quad \forall \ u_1, u_2 \in [-\varepsilon, 1 + \varepsilon], \quad (5)
$$

$$
\overline{K}_2 e^{c_* \varrho} \leq | \phi(\varrho) |, | \phi'(\varrho) |, | \phi''(\varrho) | \leq \overline{K}_3 e^{c_* \varrho}, \quad \forall \ \varrho > 0, \quad (6)
$$

$$
\overline{K}_4 e^{-\Lambda \varrho} \leq | \phi(\varrho) - 1 |, | \phi'(\varrho) |, | \phi''(\varrho) | \leq \overline{K}_5 e^{-\Lambda \varrho}, \quad \forall \ \varrho > 0.
$$
Furthermore, we have
\[ \lim_{\varrho \to -\infty} \left| \frac{\varphi''(\varrho)}{\varphi(\varrho)} \right| = c_*^2. \]

Without loss of generality, we use the moving coordinate of speed \( c \) toward the -y direction. Put \( z = y + ct \) and \( v(t, x, y) = v(t, x, z) \), we have
\[ v_t - v_{xx} - v_{zz} + cv_z - f(v) = 0 \quad \text{in } [0, +\infty) \times \mathbb{R}^2, \quad (7) \]
\[ v_{|t=0} = v_0 \quad \text{in } \mathbb{R}^2. \quad (8) \]

We write the solution as \( v(t, x, z; v_0) \). If \( V \) is a traveling wave with speed \( c \), it satisfies
\[ \mathcal{L}[V] := V_{xx} + V_{zz} - cv_z + f(V) = 0 \quad \text{in } \mathbb{R}^2. \quad (9) \]

Throughout this paper, we always assume that \( c > c_* \). Let
\[ m_* = \frac{\sqrt{c^2 - c_*^2}}{c_*} \quad \text{and} \quad K(\lambda) := \lambda^2 - \frac{c_*}{c} \lambda \quad \text{for any } \lambda \in \mathbb{R}. \]
It is clear that the equation \( K(\lambda) = 0 \) has two roots 0 and \( \frac{c_*}{c} \). In addition, we can easily obtain that \( K(c_*\beta) < 0 \) if \( \beta \in (0, \frac{c_*}{c}) \).

Note that \( \phi \left( \frac{c_*}{c}(z \pm m_*x) \right) \) satisfy (9). They are so called planar traveling fronts. Since the maximum of subsolutions is also a subsolution, it turns out that
\[ V^-(x, z) := \max \left\{ \phi \left( \frac{c_*}{c}(z - m_*x) \right), \phi \left( \frac{c_*}{c}(z + m_*x) \right) \right\} = \phi \left( \frac{c_*}{c}(z + m_*|x|) \right) \]
is a subsolution of (9). \( V^-(x, z) \) is strictly monotone increasing in \( z \). In particular, the solution \( v(t, x, z; V^-) \) monotonically converges to a traveling curved front \( V(x, z) \) (Figure 1), namely
\[ \lim_{t \to +\infty} v(t, x, z; V^-) = V(x, z), \quad \forall (x, z) \in \mathbb{R}^2. \]

The following theorem has been proved in [39].

**Theorem 1.1.** ([39]) Assume that (A1) and (A2) hold. For each \( c > c_* \), there exists \( V(x, z) \) satisfying (9) and
\[ \lim_{R \to \infty} \sup_{x^2 + z^2 > R^2} \frac{|V(x, z) - V^-(x, z)|}{(V^-(x, z))^\beta} = 0, \quad \forall \beta \in (0, 1). \quad (10) \]

Moreover, one has
\[ V^-(x, z) < V(x, z) \quad \text{for } (x, z) \in \mathbb{R}^2, \]
\[ V_z(x, z) > 0 \quad \text{for } (x, z) \in \mathbb{R}^2. \]
\[ \inf_{\delta \leq V(x, z) \leq 1 - \delta} V_z(x, z) > 0 \quad \text{for all } \delta \in (0, 1/2). \quad (11) \]

For any \( \beta \in (0, 1) \), if \( v_0(x, z) \in C(\mathbb{R}^2, [0, 1]) \) satisfies
\[ \lim_{R \to \infty} \sup_{x^2 + z^2 > R^2} \frac{|v_0(x, z) - V^-(x, z)|}{(V^-(x, z))^\beta} = 0 \quad (12) \]
and
\[ V^-(x, z) \leq v_0(x, z) \quad \text{for } (x, z) \in \mathbb{R}^2, \quad (13) \]
then the solution \(v(t, x, z; v_0)\) of (7)-(8) satisfies

\[
\lim_{t \to \infty} \left\| \frac{v(t, \cdot, \cdot; v_0) - V(\cdot, \cdot)}{(V^{-}(\cdot, \cdot))^\beta} \right\|_{L^\infty(\mathbb{R}^2)} = 0.
\]

Figure 1. The profiles of the traveling curved front \(V\) and the thick solid curve shows the level line \(V = 0.5\).

Here we would like to give some comments on the traveling curved front \(V(x, z)\) in Theorem 1.1. We note that (10) implies that the curved front \(V(x, z)\) converges asymptotically to two planar traveling fronts \(\phi \left( \frac{c}{z} (z + m_\alpha |x|) \right)\) along the two half-lines \(z + m_\alpha |x| = \text{const}\.\) and the level sets of the solution \(V\) have two asymptotic lines \(z + m_\alpha |x| = \text{const}\..\) See Figure 1 and Figure 2 for the level sets of the traveling curved front \(V\). In fact, two asymptotic lines \(z + m_\alpha |x| = \text{const}\.\) are the level set of \(V^{-}(x, z) = \phi \left( \frac{c}{z} (z + m_\alpha |x|) \right)\). The level sets of \(V\) look like V-shaped curves in the \(x-z\) plane, so the curved front \(V(x, z)\) is also called a V-shaped front. We denote the angle between two asymptotic lines \(z + m_\alpha |x| = \text{const}\.\) by \(2\alpha \in (0, \pi)\), which is also the angle of the V-shaped front \(V(x, z)\). Then we have \(\alpha = \arccot m_\alpha \in (0, \frac{\pi}{2})\), \(c = c_\alpha / \sin \alpha\) and \(\phi \left( \frac{c}{z} (z + m_\alpha |x|) \right) = \phi (z \sin \alpha + |x| \cos \alpha)\). For the combustion nonlinearity, it is well known that (1) admits a unique planar wave front \((c_\alpha, \phi)\) satisfying (4), in the sense that \(c_\alpha\) is unique and \(\phi\) is also unique up to a translation. But for V-shaped fronts of (1) in \(\mathbb{R}^2\), the admissible wave speed is no longer unique, but a half continuum \((c_\alpha, +\infty)\). Following the formula \(c = c_\alpha / \sin \alpha\), we know that the angle \(\alpha\) is the smaller as the speed \(c\) is larger. The physical meaning is that the curvature (of the flame, eg.) increase with the speed (of the fuel flow,
Theorem 1.3. In [14], Hamel et al. proved the following result. The space $V$ and of a solution $v(t, x, z)$, see [2]. Mathematically, the traveling curved front $V(x, z)$ evolves from two oblique planar waves $\phi(z \sin \alpha + |x| \cos \alpha)$ and describes their interaction. Thus, the propagation speed $c$ of $V(x, z)$ depends on the angle $2\alpha$ between two oblique planar waves $\phi(z \sin \alpha + |x| \cos \alpha)$.

Theorem 1.1 shows that the solution $v(t, x, z; v_0)$ converges to the V-shaped traveling curved front when the initial value $v_0(x, z) \in C(\mathbb{R}^2, [0, 1])$ is larger than the construction subsolution $V^-(x, z)$ and satisfies (12). However, by the continuity of the initial function, we can also expect that the solution of equation (7)-(8) converges to $V(x, z)$ under the assumption (12) even if an initial value is less than $V^-(x, z)$. In this paper, our aim is to prove that the theorem holds true if the initial value only satisfies (12) without the assumption (13).

Theorem 1.2. Assume that the conditions (A1) and (A2) hold. For any $\beta \in (0, \frac{1}{2})$, if the initial value $v_0(x, z) \in C(\mathbb{R}^2, [0, 1])$ satisfies (12), then the solution $v(t, x, z; v_0)$ of (7)-(8) satisfies

$$\lim_{t \to \infty} \left\| \frac{v(t, \cdot, \cdot; v_0) - V(\cdot, \cdot)}{(V^-(\cdot, \cdot))^\beta} \right\|_{L^\infty(\mathbb{R}^2)} = 0.$$ 

Remark 1. The stability of the traveling curved front $V$ implies that for each $c > c_\ast$, the solution $(c, V)$ of (9) and (10) is unique. Namely, if $V(x, z)$ is another V-shaped traveling front of (9) and (10) with speed $c$, then one has $V(x, z) = V(x, z)$ for all $(x, z) \in \mathbb{R}^2$.

In fact, for the reaction-diffusion equations with combustion nonlinearity, Hamel et al. [14] had proved the stability of the V-shaped traveling fronts in weighted spaces. In order to make a comparison with our result, we state the stability result in [14] specifically as follows.

Choose $\tilde{\alpha} \in (0, \pi/2)$. Denote by $UC(\mathbb{R}^2)$ the space of all bounded uniformly continuous functions from $\mathbb{R}^2$ to $\mathbb{R}$. We fix a $C^\infty$ function $h : \mathbb{R} \to \mathbb{R}$ such that $h(x) = |x|$ for $|x|$ large enough. For $\omega_1 > 0$, we set

$$q(x, y) = e^{-\omega_1(h(x) \sin \tilde{\alpha} - y \cos \tilde{\alpha})}$$

and

$$G_{\omega_1} = \left\{ u \in UC(\mathbb{R}^2), \lim_{R \to \infty} \sup_{\mathbb{R}^2} |u(x, z)| = 0, u/q \in L^\infty(\mathbb{R}^2) \right\}.$$ 

The space $G_{\omega_1}$ is a Banach space with the norm

$$\|u\|_{G_{\omega_1}} = \|u\|_{L^\infty(\mathbb{R}^2)} + \|u/q\|_{L^\infty(\mathbb{R}^2)}.$$ 

In [14], Hamel et al. proved the following result.

Theorem 1.3. [14] Let $f$ satisfy (A1) and $f \equiv 0$ on $[0, \theta_1]$ for some constant $\theta_1 \in (0, 1)$. Let $v(t, x, z; v_0)$ be a solution of the Cauchy problem (7)-(8) with initial value $v_0(x, z) \in UC(\mathbb{R}^2)$ such that $0 \leq v_0(x, z) \leq 1$. Assume the existence of $\omega_1, C_0 > 0$ and of a solution $V(x, z)$ of (9) such that $|v_0(x, z) - V(x, z)| \leq C_0 e^{-\omega_1 \sqrt{x^2 + z^2}}$ in $\mathbb{R}^2$. Also assume that there exists $\tilde{a}, \tilde{b} \in \mathbb{R}^2$ such that $v_0(x, z) \leq V(x + \tilde{a}, z + \tilde{b})$ in $\mathbb{R}^2$. Then there are four constants $\bar{T} \geq 0$, $K \geq 0$, $\omega_1 > 0$ and $\omega_2 > 0$ such that

$$\|v(t, x, z; v_0) - V(x, z)\|_{G_{\omega_1}} \leq Ke^{-\omega_2 t}, \quad \forall \ t > \bar{T}.$$
Theorem 1.3 establishes the asymptotic stability of V-shaped traveling fronts of (9) if the nonlinearity $f$ is only of combustion type and the initial value is less than the translation of the $V$-shaped traveling fronts. In contrast to Theorem 1.3, by constructing new subsolutions and using the comparison principle, which are motivated by Ninomiya and Taniguchi [26], in Theorem 1.2 we prove the asymptotic stability of V-shaped traveling fronts of (9) when the nonlinear terms $f$ are of combustion and degenerate monostable type, and the initial value only satisfies (12) which is weaker than the assumptions in Theorem 1.3.

We will construct those subsolutions in Section 2 and give the proof of Theorem 1.2 in Section 3. Throughout this paper, we always assume that the conditions (A1) and (A2) hold. For the sake of convenience, in this paper we always let the function $V(x,z)$ be the V-shaped traveling front established in Theorem 1.1. In addition, we always denote

$$\lambda_1 := \sup_{e \in \mathbb{R}} \left| \frac{\phi'(q)}{\phi(q)} \right|, \quad \lambda_2 := \sup_{e \in \mathbb{R}} \left| \frac{\phi''(q)}{\phi(q)} \right|, \quad \lambda_3 := \sup_{u \in [-\epsilon, 1+\epsilon]} |f'(u)|,$$

$$B_1 := \sup_{e \in \mathbb{R}} |\phi'(q)|, \quad B_2 := \sup_{e \in \mathbb{R}} |\phi''(q)|, \quad B_3 := \sup_{e \in \mathbb{R}} |\phi\phi''(q)|, \quad B_4 := \sup_{e \in \mathbb{R}} |\phi^2\phi''(q)|$$

and fix $\epsilon_1 \in (0, \epsilon)$ such that

$$\frac{3}{2} f'(1) \leq f'(u) \leq \frac{1}{2} f'(1), \quad \forall u \in (1 - 2\epsilon_1, 1 + 2\epsilon_1).$$

2. Construction of subsolutions. Consider

$$W_t = \frac{W_{xx}}{1 + W_x^2} + c_4 \sqrt{1 + W_x^2}, \quad x \in \mathbb{R}, \quad t > 0. \quad (14)$$

For any $c > c_4$, there exists a unique solution $\varphi(x; c)$ of (14) with asymptotic lines $y = m_\ast |x|$ satisfying

$$c = \frac{\varphi_{xx}}{1 + \varphi_x^2} + c_4 \sqrt{1 + \varphi_x^2}, \quad x \in \mathbb{R}.$$  

Lemma 2.1. (Brazhnik [3], Ninomiya and Taniguchi [27, 28]) There exist positive constants $\gamma$, $k_i$ $(i = 1, 2, 3)$ and $\omega_\pm$ such that

$$\max \{ |\varphi''(x)|, \quad |\varphi'''(x)| \} \leq k_1 \text{sech}(\gamma x),$$

$$k_2 \text{sech}(\gamma x) \leq \frac{c}{\sqrt{1 + \varphi'(x)^2}} - c_4 \leq k_3 \text{sech}(\gamma x),$$

$$m_\ast |x| \leq \varphi(x), \quad (15)$$

$$\omega_- \leq \hat{\omega}(x) \leq \omega_+ \quad (16)$$

for any $x \in \mathbb{R}$, where

$$\hat{\omega}(x) = \frac{c(\varphi(x) - m_\ast |x|)}{c - c_4 \sqrt{1 + \varphi'(x)^2}}.$$  

By Lemma 2.1, it is easy to obtain that there exists a constant $a > 0$ such that

$$\varphi'(x)^2 \leq m_\ast^2 \quad \text{and} \quad m_\ast |x| \leq \varphi(x) \leq m_\ast |x| + a \quad \text{for all} \quad x \in \mathbb{R}.$$  

Define

$$\psi(\zeta) := \begin{cases} -\frac{1}{m_\ast \gamma} \ln (1 + \exp(-\gamma \zeta)), & \zeta \leq 0, \\ \frac{\zeta}{2m_\ast} + \frac{1}{4m_\ast \gamma} \text{sech}(\gamma \zeta) - \frac{\ln 2}{m_\ast \gamma} - \frac{1}{4m_\ast \gamma}, & \zeta > 0. \end{cases}$$
In this case, inequality (21) had been proved by [26]. We need only to prove the case that \( \zeta > \zeta_0 \) hold. Now we give a simpler proof of the inequality (21). In fact, when

\[
\frac{\zeta}{m_*} - \frac{\ln \zeta}{m_*} \leq \psi(\zeta) \leq \frac{\zeta}{m_*}, \quad \zeta \leq 0, \\
\frac{\zeta}{2m_*} - \frac{1 + 4 \ln 2}{4 m_*} \leq \psi(\zeta) \leq \frac{\zeta}{2m_*}, \quad \zeta > 0,
\]

and

\[
1 + \frac{1}{16 m_*^2} \leq 1 + \psi'(\zeta) \leq 1 + \frac{1}{m_*^2} = \frac{c^2}{c^2 - c_*^2}, \quad \forall \zeta \in \mathbb{R}.
\]

In addition, there exist some constants \( K_i > 0 \) \( (i = 1, 2) \) so that \( \psi(\zeta) \) satisfies

\[
\max \left\{ \left| \frac{\psi'(\zeta)}{m_*} \right|, \left| \frac{1}{m_*} \right| \right\} \leq K_1 \text{sech}(\gamma \zeta) \quad \text{for} \quad \zeta \leq 0,
\]

\[
\left| \frac{\psi'(\zeta)}{2m_*} \right| \leq K_1 \text{sech}(\gamma \zeta) \quad \text{for} \quad \zeta > 0,
\]

\[
\max \left\{ |\psi''(\zeta)|, |\psi'''(\zeta)| \right\} \leq K_1 \text{sech}(\gamma \zeta) \quad \text{for} \quad \zeta \in \mathbb{R},
\]

\[
c_* - \frac{c \psi'(\zeta)}{\sqrt{1 + \psi'(\zeta)^2}} \geq K_2 \min \{ 1, \exp(\gamma \zeta) \} \quad \text{for} \quad \zeta \in \mathbb{R}.
\]

By the definition of the function \( \psi \), we can easily get that the inequalities (18)-(20) hold. Now we give a simpler proof of the inequality (21). In fact, when \( \zeta \leq 0 \), the inequality (21) had been proved by [26]. We need only to prove the case that \( \zeta > 0 \). In this case, there is

\[
c_* - \frac{c \psi'(\zeta)}{\sqrt{1 + \psi'(\zeta)^2}} \geq c_* - \frac{c}{\sqrt{1 + \frac{1}{4 m_*^2}}} = c_* - \frac{c}{\sqrt{1 + 4 m_*^2}} > 0.
\]

That is, the function \( c_* - \frac{c \psi'(\zeta)}{\sqrt{1 + \psi'(\zeta)^2}} \) has a strict positive lower bound if \( \zeta > 0 \). Thus the inequality (21) holds for all \( \zeta \in \mathbb{R} \).

In the following we give at first the definition of mild super- and subsolutions of (7) and establish a comparison principle.

**Definition 2.2.** ([40, Definition 2.1]) A continuous function \( w : [0, q) \to UC(\mathbb{R}^2) \), \( q > 0 \), is called a mild subsolution (mild supersolution) of (7) on \([0, q)\) if

\[
w(t, x, z) \leq (\geq) T(t - s)w(s, x, z) + \int_s^t T(t - r)f(w(r, x, z))dr
\]

for all \( 0 \leq s < t < q \), where

\[
T(t)w(0, x, z) = \frac{1}{4\pi t} \int_{\mathbb{R}^2} \exp \left( -\frac{(x - x')^2 + (z - ct - z')^2}{4t} \right) w(0, x', z')dx'dz'.
\]

The following lemma follows from [23], see also [40].

**Lemma 2.3.** Assume that (A1) hold. Then for any pair of mild supersolution \( w^+(t, x, z) \) and mild subsolution \( w^-(t, x, z) \) of (7) on \([0, +\infty)\) with \( -\epsilon \leq w^+(t, x, z), w^-(t, x, z) \leq 1 + \epsilon \) for \((t, x, z) \in [0, +\infty) \times \mathbb{R}^2\), and \( w^+(0, x, z) \geq w^-(0, x, z) \) for \((x, z) \in \mathbb{R}^2\), we have \( w^+(t, x, z) \geq w^-(t, x, z) \) for \((t, x, z) \in [0, +\infty) \times \mathbb{R}^2\).

In the remainder of this paper, we always let \( b := \frac{c}{8k_1} \). Now, we construct a mild subsolution to (9) as follows.
Theorem 2.4. For any $\beta \in \left(0, \frac{1}{2}\right)$, there exist a positive constant $\varepsilon_0(\beta)$ and a positive constant $\alpha_0(\beta, \varepsilon)$ such that, for any $0 < \varepsilon < \varepsilon_0(\beta)$, $0 < \alpha \leq \alpha_0(\beta, \varepsilon)$ and $l \in \mathbb{R}$, the function
\[
V(x, z; \beta, \varepsilon, \alpha, l) = \phi \left( \frac{\psi(\alpha z) + \alpha|x| + \alpha l}{\alpha \sqrt{1 + \psi'(\alpha z)^2}} \right) - \varepsilon \operatorname{sech}(\gamma \alpha z) \phi^\beta \left( \frac{c \omega|\alpha z|}{b m_*} + l + \frac{\psi(\alpha z)}{\alpha} \right)
\]
is a mild subsolution of (9) in $\mathbb{R}^2$. Moreover, there is
\[
\lim_{R \to \infty} \sup_{\mathbb{R}^2} \left| \frac{V(x, z; \beta, \varepsilon, \alpha, 0) - V^- (x, z)}{\phi^\beta \left( \frac{c \omega|\alpha z|}{b m_*} \right)} \right| \leq \left(1 + \exp \left( \frac{\lambda_2 a^{\beta}}{bm_*} \right) \right)^{\varepsilon}.
\]
Proof. Fix $\beta \in \left(0, \frac{1}{2}\right)$. In the following we always assume $0 < \alpha < \varepsilon < \varepsilon_1$. Set
\[
\xi = \alpha z, \quad \sigma(\xi) = \varepsilon \operatorname{sech}(\gamma \xi), \\
\mu(x, z) = \frac{\psi(\alpha z) + \alpha|x| + \alpha l}{\alpha \sqrt{1 + \psi'(\alpha z)^2}}, \quad \eta(x, z) = \frac{c \omega|\alpha z|}{b m_*} \left( \phi(b x) + l + \frac{\psi(\alpha z)}{\alpha} \right).
\]
Then we have $V(x, z; \beta, \varepsilon, \alpha, l) = \phi(\mu) - \sigma(\xi) \phi^\beta(\eta)$. The chain rule gives
\[
\eta_x = \frac{c \omega}{c} \phi'(b x), \quad \eta_{xx} = \frac{c \omega}{c} b \phi''(b x), \quad \eta_z = \frac{c \omega}{c} m_* \phi'(\xi), \quad \eta_{zz} = \frac{c \omega}{c} m_* \alpha \psi''(\xi),
\]
\[
\mu_x = \begin{cases} 
\frac{1}{\sqrt{1 + \psi'(\xi)^2}}, & x < 0, \\
\frac{1}{\sqrt{1 + \psi'(\xi)^2}}, & x > 0
\end{cases}
\]
\[
\mu_x|_0^- = -\frac{1}{\sqrt{1 + \psi'(\xi)^2}}, \quad \mu_x|_0^+ = \frac{1}{\sqrt{1 + \psi'(\xi)^2}},
\]
\[
\mu_z = \frac{\psi'(\xi)}{\sqrt{1 + \psi'(\xi)^2}} - \frac{\alpha \psi'(\xi) \psi''(\xi)}{1 + \psi'(\xi)^2},
\]
\[
\mu_{zz} = \frac{\alpha \psi''(\xi) (1 - \psi'(\xi)^2)}{(1 + \psi'(\xi)^2)^2} + \frac{\alpha^2 g(\xi)}{(1 + \psi'(\xi)^2)^2},
\]
where
\[
g(\xi) = 3 \psi'(\xi)^2 \psi''(\xi)^2 - (1 + \psi'(\xi)^2)(\psi''(\xi)^2 + \psi'(\xi) \psi'''(\xi)).
\]
Thus using $\phi''(\mu) - c \psi'(\mu) + f(\phi(\mu)) = 0$, for $x \leq 0$, by direct calculation we obtain
\[
\mathcal{L}[V] := V_{xx} + V_{zz} - c V_x + f(V)
\]
\[
= \phi'(\mu) \mu_x - \sigma(\xi) \beta \phi^{\beta-1}(\eta) \phi'(\eta) \eta_x x
\]
\[
+ \phi'(\mu) \mu_z - \alpha \sigma'(\xi) \phi^\beta(\eta) - \sigma(\xi) \beta \phi^{\beta-1}(\eta) \phi'(\eta) \eta_z z
\]
\[- c \phi'(\mu) \mu_z - \alpha \sigma'(\xi) \phi^\beta(\xi) - \sigma(\xi) \beta \phi^{\beta-1}(\eta) \phi'(\eta) \eta_z x
\]
\[+ f(\phi(\mu) - \sigma(\xi) \phi^\beta(\eta))
\]
\[
= \phi''(\mu) \mu_z^2 - \sigma(\xi) \beta \phi^{\beta-1}(\eta) \phi'(\eta) \eta_z^2
\]
\[- \sigma(\xi) \beta \phi^{\beta-1}(\eta) \phi'(\eta) \eta_{xx} + \phi''(\mu) \mu_z + \phi'(\mu) \mu_z - \alpha^2 \sigma''(\xi) \phi^\beta(\eta)
\]
\[- 2 \alpha \beta \sigma'(\xi) \phi^{\beta-1}(\eta) \phi'(\eta) \eta_z - \sigma(\xi) \beta (\beta - 1) \phi^{\beta-2}(\eta) \phi'(\eta) \eta_z^2
\]

is a mild subsolution of (9) in $\mathbb{R}^2$, we divide our proof into three steps.

**Step 1.** There exist a positive constant $\alpha_1(\beta)$ and a large constant $X$ such that for $0 < \alpha < \min\{\alpha_1(\beta), \varepsilon\}$ and $(x, z) \in (-\infty, 0) \times \mathbb{R}$ with $\mu(x, z) < -X$, the inequality

$$L[V] > 0$$

holds.

Since $\lim_{\varepsilon \to -\infty} \frac{\phi'(\varepsilon)}{\phi(\varepsilon)} = c_*$, then there exists a constant $X_1 > 0$ large enough such that

$$\frac{1}{2} c_* < \frac{\phi'(\varepsilon)}{\phi(\varepsilon)} < \frac{3}{2} c_*$$

and

$$\beta^2 \left( \frac{\phi'(\varepsilon)}{\phi(\varepsilon)} \right)^2 - \frac{c_*}{8} \phi'(\varepsilon) < \frac{1}{2} K(c_* \beta)$$

(23)

for $\varepsilon < -X_1$. It follows from $\lim_{\varepsilon \to -\infty} \frac{\phi'(\varepsilon)}{\phi(\varepsilon)} = c_*$, $\lim_{\varepsilon \to -\infty} \frac{\phi''(\varepsilon)}{\phi(\varepsilon)} = c_*^2$, (16) and the definition of $\psi(\xi)$ that there holds

$$c_*^2 m_*^2 \left( \frac{-\phi'(\varepsilon)}{\phi(\varepsilon)} + \phi''(\varepsilon) \right) \left( \frac{\phi'(\varepsilon)}{m_*} \right)^2 \psi(\xi) \to 0$$
uniformly for $x \in \mathbb{R}$ and $\xi \in \mathbb{R}$ as $\varrho \to -\infty$. Thus, there exists $X_2 > 0$ large enough such that

$$
\left| \frac{c^2 m^2}{c^2} \left( -\left( \frac{\phi'(\varrho)}{\phi(\varrho)} \right)^2 + \phi''(\varrho) \right) \left( \frac{\phi'(bx)}{m} \right)^2 + \psi'(\xi)^2 \right| < -\frac{1}{16} K(c, \beta)
$$

(24)

for any $\varrho < -X_2$, $\xi \in \mathbb{R}$ and $x \leq 0$. By (5), there exists $X_3 > 0$ large enough such that

$$
\left| f'(\phi(\varrho) - \tau \sigma(\xi)\phi^\beta(\hat{\varrho})) \right| \leq K_1 \left| \phi(\varrho) - \tau \sigma(\xi)\phi^\beta(\hat{\varrho}) \right|^\vartheta < -\frac{1}{16} K(c, \beta)
$$

(25)

for any $\varrho, \hat{\varrho} < -X_3$ and $\xi \in \mathbb{R}$. By the definition of $\sigma(\xi)$, we can put

$$
A_1 := \sup_{\xi \in \mathbb{R}} \frac{|\sigma'(\xi)|}{\sigma(\xi)} \text{ and } A_2 := \sup_{\xi \in \mathbb{R}} \frac{|\sigma''(\xi)|}{\sigma(\xi)}.
$$

Take

$$
\alpha_1(\beta) = \min \left\{ \frac{-1}{16} K(c, \beta) \mid A_2 c + 2\lambda_1 A_1 c + \alpha c m_1 K_1 + c^2 A_1, \beta \right\}
$$

Then it follows from (18)-(20) that

$$
\alpha^2 \frac{\sigma''(\xi)}{\sigma(\xi)} + 2\alpha \frac{c_x m_x}{c} \frac{\phi'(\varrho)}{\phi(\varrho)} \frac{\sigma'(\xi)}{\sigma(\xi)} \psi'(\xi) + \alpha \beta \frac{c_x m_x}{c} \frac{\phi'(\varrho)}{\phi(\varrho)} \psi''(\xi) - c \frac{\sigma'(\xi)}{\sigma(\xi)}
$$

$$
\leq \alpha A_2 + 2\alpha \lambda_1 A_1 \frac{c_x}{c} + \alpha c m_1 K_1 + \alpha c A_1 < -\frac{1}{16} K(c, \beta)
$$

(26)

for any $0 < \alpha < \alpha_1$, $\xi \in \mathbb{R}$ and $\varrho \in \mathbb{R}$. By (18)-(20), for $\alpha \in (0, \varepsilon)$, there exist positive constants $A_3, A_4, A_5$ and $A_6$ such that

$$
\left| \mu_z^2 + \frac{1}{1 + \psi'(\xi)} - 1 \right|
$$

$$
= \alpha^2 \frac{\psi'(\xi)^2 \psi''(\xi)}{(1 + \psi'(\xi))^2} \mu^2 - 2\alpha \psi'(\xi)^2 \psi''(\xi)
$$

$$
\leq \alpha A_3 \sigma(\xi) \mu^2 + A_3 \sigma(\xi)|\mu|
$$

$$
= \sigma(\xi)(\alpha A_3 \mu^2 + A_4 |\mu|)
$$

(27)

and

$$
\left| \mu_{zz} + \frac{c c \alpha \psi'(\xi) \psi''(\xi)}{1 + \psi'(\xi)} \right| \leq A_5 \sigma(\xi) + A_6 \sigma(\xi)|\mu| = \sigma(\xi)(A_5 + A_6 |\mu|)
$$

(28)

for any $(x, z) \in (-\infty, 0) \times \mathbb{R}$. By (6), there is $X_4 > 0$ large enough such that

$$
A_3 \frac{|\phi''(\varrho)|}{\phi^3(\varrho)} \varrho^2 + A_4 \frac{|\phi''(\varrho)|}{\phi^3(\varrho)} |\varrho| < -\frac{1}{16} K(c, \beta), \quad \varrho < -X_4,
$$

(29)

$$
A_5 \frac{|\phi''(\varrho)|}{\phi^3(\varrho)} + A_6 \frac{|\phi''(\varrho)|}{\phi^3(\varrho)} |\varrho| < -\frac{1}{16} K(c, \beta), \quad \varrho < -X_4.
$$

(30)

Since $b = \frac{c}{8K_1}$, we obtain

$$
b \beta c_x \frac{\phi'(\varrho)}{\phi(\varrho)} \frac{\phi''(bx)}{m} - c_x m_x \beta \frac{\phi'(\varrho)}{\phi(\varrho)} \psi'(\xi)
$$
\[
\begin{align*}
&= c_\ast \beta \phi'(\eta) \left( \frac{b}{c} \varphi''(bx) - m_\ast \psi'(\xi) \right) \\
&\leq c_\ast \beta \phi'(\eta) \left( \frac{b \eta}{c} \frac{\xi}{\eta} - \frac{1}{4} \right) = -\frac{1}{8} c_\ast \beta \phi'(\eta). 
\end{align*}
\]

Since \( 1 < 1 + \psi'(\xi) \leq 1 + \frac{1}{m_\ast^2} = \frac{\xi^2}{c^2 - \psi^2} \) and \( m_\ast|x| \leq \varphi(x) \leq m_\ast|x| + a \), then we have \( \frac{c_\ast m_\ast^2}{c} (1 + \psi'(\xi)^2) \leq 1 \) for any \( \xi \in \mathbb{R} \) and

\[
\frac{c}{c_\ast m_\ast} \eta(x, z) - \frac{a}{b m_\ast} \eta(x, z) < \mu(x, z) < \eta(x, z) \quad \text{for} \quad (x, z) \in (-\infty, 0) \times \mathbb{R} \quad \text{with} \quad \mu(x, z) < 0.
\]

Choose \( X = \max \left\{ \frac{a}{c_\ast m_\ast} X_1, \frac{a}{b m_\ast} X_2, \frac{c}{c_\ast m_\ast} X_3, \frac{a}{b m_\ast} X_4 \right\} \). Then for \( 0 < \alpha < \min \{ \alpha_1(\beta), \varepsilon \} \) and \( (x, z) \in (-\infty, 0) \times \mathbb{R} \) with \( \mu(x, z) < -X \), we obtain

\[
\mathcal{L}[\mathcal{V}] = \left( \mu_x^2 + \frac{1}{1 + \psi'(\xi)^2} - 1 \right) \phi''(\mu) + \left( \mu_{xx} + \frac{c \alpha \psi'(\xi) \psi''(\xi)}{1 + \psi'(\xi)^2} \phi'(\mu) \right)
\]

\[
+ \left( c_\ast - \frac{c \psi'(\xi)}{\sqrt{1 + \psi'(\xi)^2}} \right) \psi'(\mu)
\]

\[
- \sigma(\xi) \phi^\beta(\eta) \left\{ \alpha^2 \frac{\sigma''(\xi)}{\sigma(\xi)} + 2 \alpha \beta \frac{c m_\ast \phi(\eta)}{c} \frac{\sigma'(\xi) \psi'(\xi)}{\sigma(\xi)} \right. \\
+ b \beta \frac{c m_\ast \phi'(\eta)}{c} \psi''(bx) + \alpha \beta \frac{c m_\ast \phi'(\eta)}{c} \psi''(\xi) - \alpha \sigma'(\xi) \right. \\
+ \beta \frac{c_\ast m_\ast^2}{c^2} \left( \frac{\phi'(\eta)}{\phi(\eta)} \right)^2 \left( \frac{\phi''(\eta)}{\phi(\eta)} \right)^2 \left( \frac{\psi'(bx)}{m_\ast} \right)^2 + \psi'(\xi)^2 \right.
\]

\[
- \beta \left( \frac{\phi'(\eta)}{\phi(\eta)} \right)^2 - c_\ast m_\ast \beta \phi'(\eta) - \psi'(\xi) \right) \right.
\]

\[
+ f(\phi(\mu) - \sigma(\xi) \phi^\beta(\eta)) - f(\phi(\mu))
\]

\[
\geq - \sigma(\xi) \phi^\beta(\eta) \left( \alpha A_3 \mu^2(x, z) + A_4 |\mu(x, z)| \right) \frac{|\phi''(\mu)|}{\phi^\beta(\eta)}
\]

\[
- \sigma(\xi) \phi^\beta(\eta) (A_5 + A_6 |\mu(x, z)|) \frac{|\phi'(\mu)|}{\phi^\beta(\eta)}
\]

\[
- \sigma(\xi) \phi^\beta(\eta) \left\{ \alpha^2 \frac{\sigma''(\xi)}{\sigma(\xi)} + 2 \alpha \beta \frac{c m_\ast \phi(\eta)}{c} \frac{\sigma'(\xi) \psi'(\xi)}{\sigma(\xi)} \right. \\
+ \alpha \beta \frac{c m_\ast \phi'(\eta)}{c} \psi''(\xi) - \alpha \sigma'(\xi) \right. \\
+ \beta \frac{c_\ast m_\ast^2}{c^2} \left( \frac{\phi'(\eta)}{\phi(\eta)} \right)^2 \left( \frac{\phi''(\eta)}{\phi(\eta)} \right)^2 \left( \frac{\psi'(bx)}{m_\ast} \right)^2 + \psi'(\xi)^2 \right.
\]

\[
- \beta \left( \frac{\phi'(\eta)}{\phi(\eta)} \right)^2 - c_\ast m_\ast \beta \phi'(\eta) - \psi'(\xi) \right) \right.
\]

\[
+ f(\phi(\mu) - \sigma(\xi) \phi^\beta(\eta)) - f(\phi(\mu))
\]
\[ -\sigma(\xi)\phi^\beta(\eta)f'(\phi(\mu) - \tau\sigma(\xi)\phi^\beta(\eta)) \geq \sigma(\xi)\phi^\beta(\eta) \left( aA_3\mu^2(x, z) + A_1\mu(x, z) \right) \frac{|\phi''(\mu)|}{\phi^\beta(\mu)} \]

\[ -\sigma(\xi)\phi^\beta(\eta) \left( A_5 + A_6\mu(x, z) \right) \frac{|\phi'(\mu)|}{\phi^\beta(\mu)} \]

\[ -\sigma(\xi)\phi^\beta(\eta) \left\{ \alpha^2\frac{\sigma''(\xi)}{\sigma(\xi)} + 2\alpha\frac{c_\alpha m_*}{c} \phi(\eta) \frac{\phi'(\xi)}{\phi(\xi)} \frac{\phi'(\xi)}{\phi(\xi)} \right\} \]

\[ + \alpha\beta \frac{c_\alpha m_*}{c} \phi(\eta) \frac{\phi''(\xi)}{\phi(\xi)} \psi''(\xi) - c_\alpha \frac{\phi'(\xi)}{\phi(\xi)} \]

\[ + \beta \frac{c_\alpha m_*}{c} \left( \phi(\eta) \right)^2 \frac{c_\alpha m_*}{c} \frac{\phi''(\xi)}{\phi(\xi)} \frac{\phi(\eta)}{\phi(\xi)} \left( \frac{\phi'(\xi)}{m_*} \right)^2 + \psi''(\xi) \]

\[ \geq \sigma(\xi)\phi^\beta(\eta) \left\{ \frac{1}{16} K(c_\alpha \beta) + \frac{1}{16} K(c_\alpha \beta) + \frac{1}{16} K(c_\alpha \beta) \right\} \]

\[ + \frac{1}{16} K(c_\alpha \beta) - \frac{1}{2} K(c_\alpha \beta) + \frac{1}{16} K(c_\alpha \beta) \]

\[ > 0, \]

where the first inequality is obtained by (27), (28) and \( \varphi'(x)^2 \leq m_*^2 \), the second inequality is obtained by (31) and \( \frac{c_\alpha m_*}{c} (1 + \varphi'(\xi)^2) \leq 1 \), and the third inequality is obtained by (23), (24), (25), (26), (29) and (30).

**Step 2.** There exist a positive constant \( \alpha_2(\beta) \) and a large constant \( X' > 0 \) such that for \( \alpha \in (0, \alpha_2(\beta)) \), \( \varepsilon \in (0, \varepsilon_1) \) and \( (x, z) \in (-\infty, 0] \times \mathbb{R} \) with \( \mu(x, z) > X' \), the inequality

\[ \mathcal{L}[\mathcal{V}] > 0 \]

holds.

Since \( \phi(\varrho) \to 1 \), \( \phi'(\varrho) \to 0 \) and \( \phi''(\varrho) \to 0 \) as \( \varrho \to +\infty \), there is \( X'_1 > 0 \) large enough such that

\[ 1 - \varepsilon_1 < \phi(\varrho), \quad \left| \frac{\varphi'(\varrho)}{\varphi(\varrho)} \right| < \frac{|f'(1)|}{2c_*}, \quad \left| \frac{\varphi''(\varrho)}{\varphi(\varrho)} \right| < \frac{|f'(1)|}{16} \quad \text{for any } \varrho > X'_1. \]  

(32)

Take \( X'_2 > 0 \) large enough such that

\[ (A_3\varrho^2 + A_4) \left| \frac{\varphi'(\varrho)}{\varphi(\varrho)} \right| < \frac{|f'(1)|}{16}, \quad (A_5\varrho + A_6) \left| \frac{\varphi'(\varrho)}{\varphi(\varrho)} \right| < \frac{|f'(1)|}{16} \]

(33)

for any \( \varrho > X'_2 \) and \( \varrho > X'_2 \). Let

\[ \alpha_2(\beta) = \min \left\{ \frac{1}{16} |f'(1)|, \frac{1}{A_2c + 2\lambda_1 A_1 c_* + \alpha c_* m_* \lambda_1 K_1 + c^2 A_1}, \beta \right\} \]

Then

\[ \alpha A_2 + 2\alpha \lambda_1 A_1 \frac{c_*}{c} + \alpha \lambda_1 K_1 \frac{c_* m_*}{c} + \alpha c A_1 < \frac{1}{16} |f'(1)| \]  

(34)
for any $\alpha \in (0, \alpha_2(\beta))$. Since $1 < 1 + \psi'(\xi)^2 \leq 1 + \frac{1}{m_*^2} = \frac{c^2}{c \varepsilon^2}$ and $m_*|x| \leq \varphi(x) \leq m_*|x| + a$, then we have $\frac{c^2}{c \varepsilon^2} \left(1 + \psi'(\xi)^2\right) \leq 1$ for any $\xi \in \mathbb{R}$ and
$$\frac{c}{c \varepsilon m_*} \eta(x, z) > \mu(x, z) > \eta(x, z) - \frac{c \varepsilon a}{cb} \quad \text{for} \quad (x, z) \in (-\infty, 0) \times \mathbb{R} \quad \text{with} \quad \mu(x, z) > 0.$$ Choose $X' = \max \left\{ \frac{c}{c \varepsilon m_*} X'_1, \frac{c}{c \varepsilon m_*} X'_2, X'_1, X'_2 \right\}$. Then for $\alpha \in (0, \alpha_2(\beta))$, $\varepsilon \in (0, \varepsilon_1)$ and $(x, z) \in (-\infty, 0] \times \mathbb{R}$ with $\mu(x, z) > X'$, one has

$$\mathcal{L}[V] \geq \left( \mu_z^2 + \frac{1}{1 + \psi'(\xi)^2} - 1 \right) \phi''(\mu) + \left( \mu_{zz} + \frac{c \varepsilon \psi'(\xi) \psi''(\xi)}{1 + \psi'(\xi)^2} - \mu \right) \phi'(\mu)$$

$$- \sigma(\xi) \phi' = \alpha \phi(\xi) \left[ \alpha^2 A_2 + 2 \alpha \lambda_1 A_1 \frac{c}{c} + \frac{c \phi'(\eta)}{8 \phi(\eta)} + \alpha \lambda_1 K_1 \frac{c m_*}{c} + \alpha A_1 \right]$$

$$+ \frac{c^2 m_*^2}{c^2} \left( \frac{\phi'(\eta)}{\phi(\eta)} \left( 1 + \psi'(\xi)^2 \right) + f'(\phi(\mu) - \tau \sigma(\xi) \phi'(\eta) \right)$$

$$\geq \sigma(\xi) \phi'(\eta) \left[ - \frac{1}{16} |f'(1)| - \frac{1}{16} |f'(1)| - \frac{1}{16} |f'(1)| \right]$$

$$\geq 0,$$

where the first inequality is obtained by (17), (21), $\beta < 1$ and $\phi' > 0$, the second inequality is obtained by (27), (28), $b = \frac{c}{c \varepsilon^2}$ and $\psi'(x)^2 \leq m_*^2$, and the third inequality is obtained by (32), (33) and (34).

**Step 3.** There exists a positive constant $\varepsilon_0^\beta$ such that for any $\varepsilon \in (0, \varepsilon_0^\beta)$ and $(x, z) \in (-\infty, 0] \times \mathbb{R}$ with $\mu(x, z) \in [-X, X']$, the inequality

$$\mathcal{L}[V] > 0$$

holds.

For $(x, z) \in (-\infty, 0] \times \mathbb{R}$ with $\mu(x, z) \in [-X, X']$, it follows from (21), (27) and (28) that

$$\mathcal{L}[V] \geq \left( \mu_z^2 + \frac{1}{1 + \psi'(\xi)^2} - 1 \right) \phi''(\mu) + \left( \mu_{zz} + \frac{c \varepsilon \psi'(\xi) \psi''(\xi)}{1 + \psi'(\xi)^2} - \mu \right) \phi'(\mu)$$

$$+ \left( c_* - \frac{c \varepsilon \psi'(\xi) \psi''(\xi)}{1 + \psi'(\xi)^2} \right) \phi'(\mu)$$
for any \( \varepsilon > 0 \), mild subsolution of (9) in \( \varepsilon < \varepsilon_0 \) for all \( x, z \) in \( \mathbb{R}^2 \) for any \( \varepsilon \in (0, \varepsilon_0(\beta)) \), \( \alpha \in (0, \alpha_0(\beta, \varepsilon)) \) and \( l \in \mathbb{R} \). Since

\[
\mathcal{V}(x, z; \beta, \varepsilon, \alpha, l) = \mathcal{V}(x, z; \beta, \varepsilon, \alpha, l)
\]

for all \( (x, z) \in \mathbb{R}^2 \) for any \( \varepsilon \in (0, \varepsilon_0(\beta)) \), \( \alpha \in (0, \alpha_0(\beta, \varepsilon)) \) and \( l \in \mathbb{R} \), we have

\[
\mathcal{L}[\mathcal{V}] \geq 0, \quad \forall \ (x, z) \in (0, +\infty) \times \mathbb{R}
\]

Similar to the proof of Lemma 3.2 of [40], we can obtain

\[
\mathcal{V}(x, z) \leq (T(t-s)\mathcal{V})(x, z) + \int_s^t (T(t-r)f(\mathcal{V}))(x, z)dr
\]

for all \( 0 \leq s < t < +\infty \) and \( (x, z) \in \mathbb{R}^2 \), which implies that \( \mathcal{V}(x, z; \beta, \varepsilon, \alpha, l) \) is a mild subsolution of (9) in \( \mathbb{R}^2 \) if \( \varepsilon \in (0, \varepsilon_0(\beta)) \), \( \alpha \in (0, \alpha_0(\beta, \varepsilon)) \) and \( l \in \mathbb{R} \).

Now we prove (22) with \( l = 0 \). In this case, the function \( \mu(x, z) = \frac{\psi(\alpha z) + \alpha |x|}{\alpha \sqrt{1 + \psi(\alpha z)^2}} \).

Since the function \( \phi(\varrho + \hat{\varrho})e^{-\lambda_1 \hat{\varrho}} \) is decreasing in \( \hat{\varrho} \in \mathbb{R} \), we have

\[
\frac{\phi^\beta \left( \frac{c}{m} \frac{x}{\varrho} + \frac{\psi(\alpha z)}{\alpha} \right)}{\phi^\beta \left( \frac{c}{m} \frac{x}{\varrho} + \frac{\psi(\alpha z)}{\alpha} \right)} \leq \frac{\phi^\beta \left( \frac{c}{m} \frac{x}{\varrho} + \frac{\psi(\alpha z)}{\alpha} \right)}{\phi^\beta \left( \frac{c}{m} \frac{x}{\varrho} + \frac{\psi(\alpha z)}{\alpha} \right)} \leq \exp \left( \frac{\lambda_1 \alpha \beta}{b m_*} \right).
\]
Thus,

$$\frac{\varepsilon \operatorname{sech}(\gamma \alpha z) \phi^\beta \left( \frac{c_m \gamma z}{c} \left( \frac{z (\phi (x, z) + \psi (x, z))}{\operatorname{ln} 2 (\alpha)} \right) \right)}{\phi^\beta \left( \frac{c_m \gamma z}{c} (|x| + \psi (x, z) ) \right)} \leq \exp \left( \frac{\lambda_1 a \beta}{b m_\ast} \right) \varepsilon \quad \text{for } (x, z) \in \mathbb{R}^2.$$ 

Therefore, in order to prove (22) with \( l = 0 \), it is sufficient to show that

$$\lim_{R \to \infty} \sup_{|x^2 + z^2 > R^2} \frac{|\phi (\mu (x, z)) - V^{-} (x, z)|}{\phi^\beta \left( \frac{c_m \gamma z}{c} (|x| + \psi (x, z) \alpha) \right)} = 0. \quad (35)$$

Let

$$\zeta (x, z) := \frac{c_m \gamma z}{c} (|x| + \psi (x, z) \alpha).$$

Then (17) implies

$$\frac{c_m \gamma z}{c} (z + m_\ast |x|) - \frac{c_m \gamma z}{c} \ln 2 \leq \zeta (x, z) \leq \frac{c_m \gamma z}{c} (z + m_\ast |x|), \quad z \leq 0, \quad (36)$$

and

$$\frac{c_m \gamma z}{c} (z + m_\ast |x|) - \frac{c_m \gamma z}{c} \ln 2 \leq \zeta (x, z) \leq \frac{c_m \gamma z}{c} (z + m_\ast |x|) \leq \frac{c_m \gamma z}{c} (z + m_\ast |x|), \quad z > 0. \quad (37)$$

First, we consider the case that \( z + m_\ast |x| \to +\infty \). It follows from (36), (37) and \( \mu (x, z) = \frac{1}{\sqrt{1 + \psi (x, z)^2}} c_m \gamma z \zeta (x, z) \) that we have

$$\lim_{R \to \infty} \sup_{z + m_\ast |x| > R} \frac{|\phi (\mu (x, z)) - V^{-} (x, z)|}{\phi^\beta \left( \zeta (x, z) \right)} \leq \lim_{R \to \infty} \sup_{z + m_\ast |x| > R} \frac{|\phi (\mu (x, z)) - 1|}{\phi^\beta \left( \zeta (x, z) \right)} + \lim_{R \to \infty} \sup_{z + m_\ast |x| > R} \frac{|V^{-} (x, z) - 1|}{\phi^\beta \left( \zeta (x, z) \right)} = 0.$$

Next, we consider the case that \( z + m_\ast |x| \to -\infty \). It is obvious that \( z \to -\infty \) if \( z + m_\ast |x| \to -\infty \). In addition, there is \( \frac{c_m \gamma z}{c} \zeta (x, z) \leq \mu (x, z) \leq \zeta (x, z) \) if \( \zeta (x, z) < 0 \). Therefore, when \( z + m_\ast |x| < 0 \), we have \( \zeta (x, z) < 0 \) and \( \mu (x, z) < 0 \). Then we obtain

$$\frac{|\phi (\mu (x, z)) - V^{-} (x, z)|}{\phi^\beta \left( \zeta (x, z) \right)} \leq \frac{|\phi' (\theta \mu (x, z) + (1 - \theta) \zeta (z + m_\ast |x|)) (\mu (x, z) - \zeta (z + m_\ast |x|))|}{\phi^\beta \left( \zeta (x, z) \right)} \leq \frac{K_3 e^{\gamma \zeta (\theta \mu (x, z) + (1 - \theta) \zeta (z + m_\ast |x|))}}{K_2 e^{\beta c_m \zeta}} \left( \mu (x, z) - \frac{c_m \gamma z}{c} (z + m_\ast |x|) \right) \leq \frac{K_3 e^{\gamma \zeta (\theta \mu (x, z) + (1 - \theta) \zeta (z + m_\ast |x|))}}{K_2 e^{\beta c_m \zeta}} \left( \frac{c_m \gamma z}{c} \zeta (|x| + |x|) \right) \leq \frac{K_3 e^{\gamma \zeta (\theta \mu (x, z) + (1 - \theta) \zeta (z + m_\ast |x|))}}{K_2 e^{\beta c_m \zeta}} \left( \frac{c_m \gamma z}{c} (z + m_\ast |x|) \right) \leq \frac{K_3 e^{\gamma \zeta (\theta \mu (x, z) + (1 - \theta) \zeta (z + m_\ast |x|))}}{K_2 e^{\beta c_m \zeta}} \left( \frac{c_m \gamma z}{c} (z + m_\ast |x|) \right).
Consequently,
\[
\lim_{R \to +\infty} \sup_{z + m_* |x| < -R} \frac{|\phi(\mu(x, z)) - V^-(x, z)|}{\phi^\beta(\zeta(x, z))} = 0.
\]

Finally, we consider the case that \( z + m_* |x| \) is bounded, but \( x^2 + z^2 \to +\infty \). Without loss of generality, let \( |z + m_* |x|| \leq R_0 \), where \( R_0 \) is sufficiently large positive constant. Since \( x^2 + z^2 \to +\infty \) and \( |z + m_* |x|| \leq R_0 \), there must be \( z \to -\infty \) and \( |x| \to +\infty \). From the definition of \( \psi \), it follows that
\[
\lim_{R \to +\infty} \sup_{|z + m_* |x|| \leq R_0, x^2 + z^2 > R^2} \left| \mu(x, z) - \frac{c_*}{c} (z + m_* |x|) \right| = 0.
\]

Due to \( |z + m_* |x|| \leq R_0 \) and (36), there is
\[
\zeta(x, z) \geq -\frac{c_*}{c} R_0 - \frac{c_* \ln 2}{c \alpha \gamma} \quad \text{for} \ x \in \mathbb{R} \text{ and } z < 0.
\]

Therefore
\[
\lim_{R \to +\infty} \sup_{|z + m_* |x|| \leq R_0, x^2 + z^2 > R^2} \frac{|\phi(\mu(x, z)) - V^-(x, z)|}{\phi^\beta(\zeta(x, z))} \leq \lim_{R \to +\infty} \sup_{|z + m_* |x|| \leq R_0, x^2 + z^2 > R^2} \frac{\sup_{\phi \in \mathbb{R}} \phi'(\phi)}{\phi^\beta(\zeta(x, z))} \left| \mu(x, z) - \frac{c_*}{c} (z + m_* |x|) \right| = 0.
\]

By the above arguments, we have proved (22). This completes the proof. \( \square \)

In the following, we always let \( \kappa := \min \left\{ -\frac{1}{16} K(c_* \beta), \frac{1}{16} |f'(1)|, 1 \right\} \). We construct a mild subsolution of equation (7) on \([0, +\infty) \times \mathbb{R}^2\) as follows.

**Lemma 2.5.** For any \( \beta \in (0, \frac{1}{5}) \), there exist positive constants \( \bar{\rho}(\beta), \bar{\varepsilon}_0(\rho, \beta), \bar{\alpha}_0(\beta) \) and \( \bar{\delta}_0(\rho, \beta) \) such that for \( \rho \geq \bar{\rho}(\beta), \delta \in (0, \bar{\delta}_0(\rho, \beta)), \varepsilon \in (0, \min\{\varepsilon_0(\beta, \varepsilon, \bar{\alpha}_0(\beta))\}) \) and \( \alpha \in (0, \min\{\alpha_0(\beta, \varepsilon, \bar{\alpha}_0(\beta))\}) \), the function
\[
W(t, x, z; \beta, \rho, \varepsilon, \alpha, \delta)
:= \phi \left( \psi(az) + \alpha|z| - \alpha \rho \delta (1 - e^{-\kappa t}) \right)
\]
\[
- \varepsilon \text{sech}(\gamma az) \phi^\beta \left( \frac{c_* m_*}{c} \left( \frac{\varphi(bx)}{bm_*} - \rho \delta (1 - e^{-\kappa t}) + \frac{\psi(az)}{\alpha} \right) \right)
\]
\[
- \delta e^{-\kappa t} \phi^\beta \left( \frac{c_* m_*}{c} \left( \frac{\varphi(bx)}{bm_*} - \rho \delta (1 - e^{-\kappa t}) + \frac{\psi(az)}{\alpha} \right) \right)
\]
is a mild subsolution of (7) on \((t, x, z) \in [0, +\infty) \times \mathbb{R}^2\).

**Proof.** Fix \( \beta \in (0, \frac{1}{5}) \). Let
\[
\omega(t, x, z) := \frac{c_* m_*}{c} \left( \frac{\varphi(bx)}{bm_*} - \rho \delta (1 - e^{-\kappa t}) + \frac{\psi(az)}{\alpha} \right)
\]
\[
\hat{\omega}(t, x, z) := \frac{\psi(az) - \alpha x - \alpha \rho \delta (1 - e^{-\kappa t})}{\alpha \sqrt{1 + \psi''(az)^2}}.
\]
For \( x \leq 0 \), let

\[
W(t, x, z; \beta, \rho, \varepsilon, \alpha, \delta) = \phi \left( \frac{\psi(\alpha z) - \alpha x - \alpha \rho \delta(1 - e^{-\kappa t})}{\alpha \sqrt{1 + \psi'(\alpha z)^2}} \right) - \varepsilon \text{sech}(\gamma \alpha z) \phi' \left( \frac{\phi'(bx)}{b_m} - \rho \delta(1 - e^{-\kappa t}) + \frac{\psi(\alpha z)}{\alpha} \right),
\]

and hence

\[
W(t, x, z; \beta, \rho, \varepsilon, \alpha, \delta) = W(t, x, z; \beta, \rho, \varepsilon, \alpha, \delta) - \delta e^{-\kappa t} \phi^2(\omega(t, x, z)),
\]

\[
W_t = -\frac{\kappa \rho \delta e^{-\kappa t}}{\sqrt{1 + \psi'(\alpha z)^2}} \phi'(\omega) + \kappa \rho \delta \varepsilon e^{-\kappa t} \text{sech}(\gamma \alpha z) \frac{c_m}{c} \phi^{\beta-1}(\omega) \phi'(\omega).
\]

It follows from Theorem 2.4 that, for any \((t, x, z) \in [0, +\infty) \times (-\infty, 0) \times \mathbb{R}\) with \(\varepsilon \in (0, \varepsilon_0(\beta))\) and \(\alpha \in (0, \alpha_0(\beta, \varepsilon))\), there is

\[
-W_{xx} - W_{zz} + cW_z - f(W) \leq 0.
\]

By direct calculation and using (39), we have

\[
\mathcal{N}[W] := W_t - W_{xx} - W_{zz} + cW_z - f(W) = W_x + \delta \kappa e^{-\kappa t} \phi^\beta(\omega) + \frac{c_m}{c} \rho \delta^2 \kappa \beta e^{-2\kappa t} \phi^{\beta-1}(\omega) \phi'(\omega)
\]

\[
- W_{xx} + \left( \frac{c_m}{c} \beta e^{-\kappa t} \phi^{\beta-1}(\omega) \phi'(\omega) \phi'(bx) \right)
\]

\[
- W_{zz} + \left( \frac{c_m}{c} \beta e^{-\kappa t} \phi^{\beta-1}(\omega) \phi'(\omega) \phi'(ax) \right)
\]

\[
+ cW_z - c_m \beta e^{-\kappa t} \phi^{\beta-1}(\omega) \phi'(\omega) \phi'(ax) - f(W)
\]

\[
= W_x + \frac{c_m^2}{c^2} \delta \beta e^{-\kappa t} \phi^{\beta-1}(\omega) \phi'(\omega) \phi'(bx)
\]

\[
- W_{xx} + \frac{c_m^2}{c^2} \delta \beta (\beta - 1) e^{-\kappa t} \phi^{\beta-2}(\omega) \phi'(\omega)^2 \left( \phi'(bx) \right)^2
\]

\[
+ \frac{c_m^2}{c^2} \delta \beta e^{-\kappa t} \phi^{\beta-1}(\omega) \phi''(\omega) \phi'(bx)
\]

\[
- W_{zz} + \frac{c_m^2}{c^2} \delta \beta (\beta - 1) e^{-\kappa t} \phi^{\beta-2}(\omega) \phi'(\omega)^2 \phi'(ax)^2
\]

\[
+ \frac{c_m^2}{c^2} \delta \beta e^{-\kappa t} \phi^{\beta-1}(\omega) \phi'(\omega) \phi'(ax)
\]

\[
+ \frac{c_m^2}{c^2} \delta \beta e^{-\kappa t} \phi^{\beta-1}(\omega) \phi''(\omega) \phi'(ax)^2 + cW_z
\]

\[
- c_m \delta \beta e^{-\kappa t} \phi^{\beta-1}(\omega) \phi'(\omega) \phi'(ax) - f(W) - f(W) + f(W)
\]

\[
\leq W_x + \delta \kappa e^{-\kappa t} \phi^\beta(\omega) + \frac{c_m}{c} \rho \delta^2 \kappa \beta e^{-2\kappa t} \phi^{\beta-1}(\omega) \phi'(\omega)
\]

\[
+ \frac{c_m^2}{c^2} \delta \beta (\beta - 1) e^{-\kappa t} \phi^{\beta-2}(\omega) \phi'(\omega)^2 \left( \frac{\phi'(bx)}{m}\right)^2 + \psi'(ax)^2
\]
\[ 0 \leq \phi(x) \leq m_*|x| + a, \text{ then we have } \frac{c^2 m_*^2}{c} (1 + \psi'(\xi)^2) \leq 1 \text{ for any } \xi \in \mathbb{R} \]

for any \((t, x, z) \in [0, +\infty) \times (-\infty, 0] \times \mathbb{R}\) with \(\omega(t, x, z) < 0\). Thus, by the definition of \(W(t, x, z; \beta, \rho, \varepsilon, \alpha, \delta)\) and (5), there exists \(X_3 > 0\) large enough such that

\[
\left| f'(W(t, x, z; \beta, \rho, \varepsilon, \alpha, \delta) - \tau \delta e^{-\kappa t} \phi^\prime(\omega)) \right| \\
\leq K_1 |W(t, x, z; \beta, \rho, \varepsilon, \alpha, \delta) - \tau \delta e^{-\kappa t} \phi^\prime(\omega)|^\theta < -\frac{1}{16} K(c_* \beta) \tag{40}
\]

for any \((t, x, z) \in [0, +\infty) \times (-\infty, 0] \times \mathbb{R}\) with \(\omega(t, x, z) < -X_3\).

Since \(\phi(q) \rightarrow 1, \phi'(q) \rightarrow 0\) and \(\phi''(q) \rightarrow 0\) as \(q \rightarrow +\infty\), there is \(X'_1 > 0\) large enough such that

\[
\left| \frac{\phi'(q)}{\phi(q)} \right| < \frac{|f'(1)|}{2c_*}, \quad \left| \frac{\phi''(q)}{\phi(q)} \right| < \frac{|f'(1)|}{16} \text{ for } q > X'_1. \tag{41}\]

Since \(1 < 1 + \psi'(\xi)^2 \leq 1 + \frac{1}{m_*^2} = \frac{c^2}{c^2 + \varepsilon^2}\) and \(m_*|x| \leq \phi(x) \leq m_*|x| + a, \text{ then we have } \frac{c^2 m_*^2}{c} (1 + \psi'(\xi)^2) \leq 1 \text{ for any } \xi \in \mathbb{R} \]

for any \((t, x, z) \in [0, +\infty) \times (-\infty, 0] \times \mathbb{R}\) with \(\omega(t, x, z) > 0\). Thus, by the definition of \(W(t, x, z; \beta, \rho, \varepsilon, \alpha, \delta)\), there exists \(X'_2 > 0\) large enough such that

\[
W(t, x, z; \beta, \rho, \varepsilon, \alpha, \delta) > 1 - \varepsilon_1 \tag{42}\]

for any \((t, x, z) \in [0, +\infty) \times (-\infty, 0] \times \mathbb{R}\) with \(\omega(t, x, z) > X'_2 + \frac{c_* a}{c}\).

Let \(X := \max\left\{X_1, X_2, X_3, X'_1, X'_2 + \frac{c_* a}{c}\right\}\), where \(X_1\) and \(X_2\) be defined as Theorem 2.4. Set

\[
\bar{\beta} := \frac{\min_{X_1 \leq \phi' \leq X_2} X \phi'(q)}{2c_*} \quad \text{and} \quad \beta_1 := \frac{\bar{\beta} c_* m_*}{2c},
\]

Take \(\bar{\rho}(\beta) > 0\) large enough such that

\[
-\frac{p \bar{\beta}_1}{2} + \kappa + \lambda_2 + \frac{c_*}{8} \lambda_1 + m_* K_1 \lambda_1 + \lambda_3 < 0, \quad \forall \rho \geq \bar{\rho}(\beta). \tag{43}\]

Let \(\xi(\rho, \beta) := \min\left\{\frac{\bar{\beta}}{16 \rho \lambda_1 m_*}, \frac{|f'(1)|}{16 \rho \lambda_1 c_* m_*}, 1\right\}\) for \(\rho \geq \bar{\rho}(\beta)\). Then for any \(0 < \varepsilon < \xi(\rho, \beta)\), there is

\[
-\frac{\phi'(\omega)}{\sqrt{1 + \psi'(\alpha z)^2}} + \frac{c_* m_*}{c} \operatorname{sech}(\gamma \alpha z) \phi^\beta(\omega) \frac{\phi'(\omega)}{\phi(\omega)} - \frac{c_* m_*}{c} \beta + \varepsilon \lambda_1 \frac{c_* m_*}{c} \leq -\beta_1 \tag{44}\]
for any \((t, x, z) \in [0, +\infty) \times (-\infty, 0] \times \mathbb{R}\) with \(-X \leq \omega(t, x, z) \leq X\). Let
\[
\tilde{\delta}_0(\rho, \beta) := \min \left\{ \varepsilon_1, \frac{\beta_1 c}{2\lambda_1 c_m}, \frac{K(c_1\beta)c}{16\rho c_m\lambda_1}, \frac{|f'(1)| c}{16\rho c_m\lambda_1} \right\}
\]
and
\[
\tilde{\alpha}_0(\beta) := \min \left\{ \frac{K(c_1\beta)c}{16\lambda_1 c_m K_1}, \frac{|f'(1)| c}{16\lambda_1 c_m K_1} \right\}.
\]
Let \(\rho \geq \bar{\rho}(\beta)\). For \((t, x, z) \in [0, +\infty) \times (-\infty, 0] \times \mathbb{R}\) with \(\omega(t, x, z) < -X\), we have
\[
\mathcal{N}[W] := W_t - W_{xx} - W_{zz} + c W_z - f(W) 
\]
\[
\leq \delta e^{-\mu t} \phi^2(\omega) \left[ -\frac{\kappa \rho}{\sqrt{1 + \psi'(\alpha z)^2}} \frac{\phi'(\omega)}{\phi(\omega)} + \kappa \rho \beta \text{sech}(\gamma \alpha z) \frac{c_m}{c} \frac{\phi'(\omega)}{\phi(\omega)} \right] 
\]
\[
+ \kappa + \frac{c_m}{c} \rho \beta \frac{\phi'(\omega)}{\phi(\omega)} \psi''(\alpha z) + f'(W - \tau \delta e^{-\mu t} \phi^2(\omega)) \right] 
\]
\[
\leq \delta e^{-\mu t} \phi^2(\omega) \left[ \kappa \rho \beta \text{sech}(\gamma \alpha z) \frac{c_m}{c} \frac{\phi'(\omega)}{\phi(\omega)} + \frac{c_m}{c} \rho \frac{\phi'(\omega)}{\phi(\omega)} \psi''(\alpha z) \right] 
\]
\[
+ \frac{c_m^2}{c^2} \phi'(\omega) \psi''(\alpha z) - \beta \frac{\phi'(\omega)}{\phi(\omega)} \psi''(\alpha z) 
\]
\[
\leq \delta e^{-\mu t} \phi^2(\omega) \left[ \frac{1}{16} K(c_1\beta) + \frac{1}{2} K(c_1\beta) - \frac{1}{16} K(c_1\beta) \right] < 0,
\]
where the second inequality is obtained by \(\phi' > 0, \psi'(x)^2 \leq m_2^2\) and (31), and the third inequality is obtained by applying (23), (24), (40), \(\frac{c_m^2}{c^2} (1 + \psi'(\alpha z)^2) \leq 1\) for any \(z \in \mathbb{R}\) with \(0 < \varepsilon < \min\{\varepsilon_0(\beta), \bar{\varepsilon}_0(\rho, \beta)\}\), \(0 < \delta < \delta_0(\rho, \beta)\) and \(0 < \alpha < \min\{\alpha_0(\beta, \varepsilon), \bar{\alpha}_0(\beta)\}\).

For \((t, x, z) \in [0, +\infty) \times (-\infty, 0] \times \mathbb{R}\) with \(\omega(t, x, z) > X\), we have
\[
\mathcal{N}[W] := W_t - W_{xx} - W_{zz} + c W_z - f(W)
\]
\[
\begin{align*}
&\leq W_t + \delta \kappa e^{-\kappa t} \phi^\beta(\omega) + \frac{c_* m_*}{c} \rho \delta^2 \kappa \beta e^{-2\kappa t} \phi^\beta - 1(\omega) \phi'(\omega) \\
&+ \frac{c_* m_*}{c^2} \delta\beta e^{-\kappa t} \phi^\beta - 1(\omega) \phi''(\omega) \left( \left( \frac{\phi'(bx)}{m_*} \right)^2 + \psi'(\alpha z)^2 \right) \\
&+ \frac{c_* m_*}{c} \beta e^{-\kappa t} \phi^\beta(\omega) \psi'(\alpha z) \\
&+ \frac{c_* m_*}{c} \delta \beta e^{-\kappa t} \phi^\beta(\omega) \phi''(bx) - \delta e^{-\kappa t} \phi^\beta(\omega) f'(W - \tau_\delta e^{-\kappa t} \phi^\beta(\omega)) \\
= &\delta e^{-\kappa t} \phi^\beta(\omega) \left[ - \frac{\kappa \rho}{\sqrt{1 + \psi'(\alpha z)^2}} \phi'(\omega) + \kappa \rho \delta \varepsilon \text{sech} (\gamma \alpha z) \frac{c_* m_*}{c} \phi'(\omega) + \kappa \\
&+ \frac{c_* m_*}{c} \beta \phi'(\omega) + \frac{c_* m_*}{c} \frac{\phi'(bx)}{m_*} \phi''(\omega) \left( \left( \frac{\phi'(bx)}{m_*} \right)^2 + \psi'(\alpha z)^2 \right) \\
&+ \frac{c_* m_*}{c} \beta \phi'(\omega) \psi'(\alpha z) + \frac{c_* m_*}{c} \phi'(bx) + f'(W - \tau e^{-\kappa t} \phi^\beta(\omega)) \right] \\
\leq &\delta e^{-\kappa t} \phi^\beta(\omega) \left[ \frac{c_* m_*}{c} \phi'(\omega) + \kappa + \frac{c_* m_*}{c} \phi'(bx) + f'(W - \tau e^{-\kappa t} \phi^\beta(\omega)) \right] \\
< &0,
\end{align*}
\]

where the second inequality is obtained by \( \phi' > 0 \) and \( \phi'(x)^2 \leq m_*^2 \), and the third inequality is obtained by applying \((41), (42)\) and \( b = \frac{1}{8k^2} \) with \( 0 < \varepsilon < \min\{\varepsilon_0(\beta), \varepsilon_0(p, \beta)\} \), \( 0 < \delta < \delta_0(p, \beta) \) and \( 0 < \alpha < \min\{\alpha_0(\beta), \varepsilon_0(\beta), \varepsilon_0(p, \beta)\} \).

For \( (t, x, z) \in [0, \infty) \times (-\infty, 0) \times \mathbb{R} \) with \( |\omega(t, x, z)| \leq X \), applying \((38), (43)\) and \((44)\) with \( 0 < \varepsilon < \min\{\varepsilon_0(\beta), \varepsilon_0(p, \beta)\} \), \( 0 < \delta < \delta_0(p, \beta) \) and \( 0 < \alpha < \min\{\alpha_0(\beta), \varepsilon_0(\beta), \varepsilon_0(p, \beta)\} \), we obtain

\[
N[W] := W_t - W_{xx} - W_{zz} + cW_z - f(W)
\]

\[
\leq - \frac{\kappa \rho \delta e^{-\kappa t}}{\sqrt{1 + \psi'(\alpha z)^2}} \phi'(\omega) + \kappa \rho \delta \varepsilon \text{sech} (\gamma \alpha z) \frac{c_* m_*}{c} \phi^\beta - 1(\omega) \phi'(\omega) \\
+ \delta \kappa e^{-\kappa t} \phi^\beta(\omega) + \frac{c_* m_*}{c} \rho \delta^2 \kappa \beta e^{-2\kappa t} \phi^\beta - 1(\omega) \phi'(\omega) \\
+ \frac{c_* m_*}{c^2} \delta \beta (\beta - 1) e^{-\kappa t} \phi^\beta - 2(\omega) \phi'(\omega)^2 \left( \left( \frac{\phi'(bx)}{m_*} \right)^2 + \psi'(\alpha z)^2 \right) \\
+ \frac{c_* m_*}{c^2} \delta \beta e^{-\kappa t} \phi^\beta - 1(\omega) \phi''(\omega) \left( \left( \frac{\phi'(bx)}{m_*} \right)^2 + \psi'(\alpha z)^2 \right) \\
+ \frac{c_* m_*}{c} \delta \beta e^{-\kappa t} \phi^\beta - 1(\omega) \phi''(bx) + \frac{c_* m_*}{c} \beta e^{-\kappa t} \phi^\beta(\omega) \phi''(\omega) + \frac{c_* m_*}{c} \delta \beta e^{-\kappa t} \phi^\beta - 1(\omega) \phi^\beta(\omega) \psi'(\alpha z) \\
- c_* m_* \delta \beta e^{-\kappa t} \phi^\beta - 1(\omega) \phi'(\omega) \psi'(\alpha z) + \delta e^{-\kappa t} \phi^\beta(\omega) f'(W - \tau \delta e^{-\kappa t} \phi^\beta(\omega))
\]
\[
\begin{align*}
\leq & -\frac{\kappa \rho \delta e^{-\kappa t}}{\sqrt{1 + \psi'(\alpha z)^2}} \phi'(\omega) + \kappa \rho \beta \varepsilon \delta e^{-\kappa t} \text{sech}(\gamma \alpha z) \frac{c_s m_s}{c} \phi^{\beta - 1}(\omega) \phi'(\omega) \\
& + \delta \kappa e^{-\kappa t} \phi'(\omega) + \frac{c_s m_s}{c} \rho \delta \kappa e^{-2\kappa t} \phi^{\beta - 1}(\omega) \phi'(\omega) \\
& + \frac{c_s^2 m_s^2}{c^2} \delta \beta e^{-\kappa t} \phi^{\beta - 1}(\omega) \phi''(\omega) \left( \frac{\phi'(bx)}{m_s} \right)^2 + \psi'(\alpha z)^2 \\
& + \frac{c_s}{c} \delta \beta \kappa e^{-\kappa t} \phi^{\beta - 1}(\omega) \phi''(bx) + \frac{c_s m_s}{c} \delta \beta \kappa e^{-\kappa t} \phi^{\beta - 1}(\omega) \phi'(\omega) \phi''(\alpha z) \\
& + \delta e^{-\kappa t} \phi^{\beta}(\omega) f'(W - \tau \delta e^{-\kappa t} \phi^{\beta}(\omega)) \\
& \leq \delta e^{-\kappa t} \left[ \rho k \left( -\frac{\phi'(\omega)}{\sqrt{1 + \psi'(\alpha z)^2}} + \beta \varepsilon \frac{c_s m_s}{c} \text{sech}(\gamma \alpha z) \frac{\phi'(\omega)}{\phi(\omega)} \right) + \kappa \\
& + \frac{c_s m_s}{c} \rho \delta \kappa \beta \sup_{\omega \in \mathbb{R}} \frac{\phi'(\omega)}{\phi(\omega)} + \sup_{\omega \in \mathbb{R}} \frac{\phi''(\omega)}{\phi(\omega)} + \frac{c_s b}{c} \sup_{x \in \mathbb{R}} |\phi''(bx)| \sup_{\omega \in \mathbb{R}} \frac{\phi'(\omega)}{\phi(\omega)} \\
& + m_s \sup_{\omega \in \mathbb{R}} |\phi''(\alpha z)| \sup_{\omega \in \mathbb{R}} \frac{\phi'(\omega)}{\phi(\omega)} + \sup_{u \in [-\epsilon, 1 + \epsilon]} |f'(u)| \right] \\
& \leq \delta e^{-\kappa t} \left( -\rho k \beta_1 + \kappa + \frac{c_s m_s}{c} \rho \delta \kappa \beta \lambda_1 + \lambda_2 + \frac{c_s}{8} \lambda_1 + m_s K_1 \lambda_1 + \lambda_3 \right) \\
& \leq \delta e^{-\kappa t} \left( -\rho k \beta_1 + \kappa + \lambda_2 + \frac{c_s}{8} \lambda_1 + m_s K_1 \lambda_1 + \lambda_3 \right) < 0.
\end{align*}
\]

Thus by the above arguments, we have \( \mathcal{N}[W] \leq 0 \) in \([0, +\infty) \times (-\infty, 0) \times \mathbb{R} \) if 
\[
\rho \geq \bar{\rho}(\beta), \quad \varepsilon \in (0, \min\{\varepsilon_0(\beta), \bar{\varepsilon}_0(\rho, \beta)\}), \\
\delta \in (0, \bar{\delta}_0(\rho, \beta)) \quad \text{and} \quad \alpha \in (0, \min\{\alpha_0(\varepsilon, \beta), \bar{\alpha}_0(\beta)\}).
\]

Since
\[
W(t, -x, z; \beta, \rho, \varepsilon, \alpha, \delta) = W(t, x, z; \beta, \rho, \varepsilon, \alpha, \delta) \quad \text{for} \ (t, x, z) \in [0, +\infty) \times \mathbb{R}^2,
\]
we also have 
\[
\mathcal{N}[W] \leq 0 \quad \text{for} \ (t, x, z) \in [0, +\infty) \times (0, +\infty) \times \mathbb{R}.
\]
Similar to the proof of Lemma 3.2 of [40], we have
\[
W(t, x, z) \leq (T(t - s)W(s))(x, z) + \int_s^t (T(t - r) f(W(r)))(x, z) dr
\]
for all \( 0 \leq s < t < +\infty \) and \((x, z) \in \mathbb{R}^2\). Then for 
\[
\rho \geq \bar{\rho}(\beta), \quad \varepsilon \in (0, \min\{\varepsilon_0(\beta), \bar{\varepsilon}_0(\rho, \beta)\}), \\
\delta \in (0, \bar{\delta}_0(\rho, \beta)) \quad \text{and} \quad \alpha \in (0, \min\{\alpha_0(\varepsilon, \beta), \bar{\alpha}_0(\beta)\}),
\]
the function \( W(t, x, z; \beta, \rho, \varepsilon, \alpha, \delta) \) is a mild subsolution of equation (7) on \([0, +\infty) \times \mathbb{R}^2\). This completes the proof. \( \square \)

**Lemma 2.6.** For any \( \beta \in (0, \frac{1}{8}) \) and \( d \in \{m_s, 1\} \), there exist positive constants \( \bar{\rho}(\beta, d) \) and \( \bar{\delta}_0(\rho, \beta, d) \) such that for any \( \rho \geq \bar{\rho}(\beta, d), \delta \in (0, \bar{\delta}_0(\rho, \beta, d)) \) and \( \theta \in \mathbb{R}, \)
the function

\[ U(t, x, z; \beta, \rho, \delta, \theta, d) := V \left( x, z + \theta - d\rho \frac{1}{b} \left( 1 - e^{-\kappa t} \right) \right) - \delta e^{-\kappa t}\phi^{(d)} \left( \frac{c_s}{c} \left( z + \theta - d\rho \left( 1 - e^{-\kappa t} \right) + \frac{x}{b} \right) \right) \]

is a subsolution of (7) on \((t, x, z) \in [0, +\infty) \times \mathbb{R}^2\).

Proof. Fix \(\beta \in (0, \frac{1}{2})\). Let

\[ \omega(t, x, z) := \frac{c_s}{c} \left( z + \theta - d\rho \left( 1 - e^{-\kappa t} \right) + \frac{x}{b} \right) . \]

The strategy of the proof is to find a positive constant \(X\) and show the inequality

\[ U_t - U_{xx} - U_{zz} + cU_z - f(U) \leq 0 \]

for any \((t, x, z) \in [0, +\infty) \times \mathbb{R}^2\) by considering three cases \(\omega(t, x, z) < -X, \omega(t, x, z) > X\) and \(|\omega(t, x, z)| \leq X\), respectively.

By direct calculation, we have

\[ \mathcal{N}[U] := U_t - U_{xx} - U_{zz} + cU_z - f(U) \]

\[ = -d\rho \delta e^{-\kappa t}V_z + \delta ke^{-\kappa t}\phi^{(d)}(\omega) + \frac{c_s}{c} d\rho \delta^2 \kappa \beta e^{-2\kappa t}\phi^{(d-1)}(\omega)\phi'(\omega) \]

\[ - V_{xx} + \left( \frac{c_s}{c} \delta \beta e^{-\kappa t}\phi^{(d-1)}(\omega)\phi'(\omega) \phi'(bx) \right) \]

\[ - V_{zz} + \left( \frac{c_s}{c} \delta \beta e^{-\kappa t}\phi^{(d-1)}(\omega)\phi'(\omega) \right) \]

\[ + cV_z - c_s \delta \beta e^{-\kappa t}\phi^{(d-1)}(\omega)\phi'(\omega) \]

\[ - f \left( V \left( x, z + \theta - d\rho \left( 1 - e^{-\kappa t} \right) \right) - \delta e^{-\kappa t}\phi^{(d)}(\omega) \right) \]

\[ = -d\rho \delta e^{-\kappa t}V_z + \delta ke^{-\kappa t}\phi^{(d)}(\omega) + \frac{c_s}{c} d\rho \delta^2 \kappa \beta e^{-2\kappa t}\phi^{(d-1)}(\omega)\phi'(\omega) \]

\[ - V_{xx} + \left( \frac{c_s}{c} \delta \beta (\beta - 1)e^{-\kappa t}\phi^{(d-2)}(\omega)\phi'(\omega)^2 \phi'(bx)^2 \right) \]

\[ + \frac{c_s}{c} \delta \beta e^{-\kappa t}\phi^{(d-1)}(\omega)\phi'(\omega) \phi'(bx)^2 \]

\[ + \frac{c_s}{c} \delta \beta e^{-\kappa t}\phi^{(d-1)}(\omega)\phi'(\omega) \phi'(bx)^2 \]

\[ - V_{zz} + \left( \frac{c_s}{c} \delta \beta (\beta - 1)e^{-\kappa t}\phi^{(d-2)}(\omega)\phi'(\omega)^2 \right) \]

\[ + \frac{c_s}{c} \delta \beta e^{-\kappa t}\phi^{(d-1)}(\omega)\phi'(\omega) \phi'(bx)^2 \]

\[ - f \left( V \left( x, z + \theta - d\rho \left( 1 - e^{-\kappa t} \right) \right) - \delta e^{-\kappa t}\phi^{(d)}(\omega) \right) \]

\[ = -d\rho \delta e^{-\kappa t}V_z + \delta ke^{-\kappa t}\phi^{(d)}(\omega) + \frac{c_s}{c} d\rho \delta^2 \kappa \beta e^{-2\kappa t}\phi^{(d-1)}(\omega)\phi'(\omega) \]

\[ + \frac{c_s}{c} \delta \beta (\beta - 1)e^{-\kappa t}\phi^{(d-2)}(\omega)\phi'(\omega)^2 \phi'(bx)^2 \]

\[ + \frac{c_s}{c} \delta \beta e^{-\kappa t}\phi^{(d-1)}(\omega)\phi'(\omega) \phi'(bx)^2 \]
+ \frac{c}{c^2} \delta \beta e^{-\kappa t} \phi^{b-1}(\omega) \phi'(\omega) \phi''(bx) + \frac{c^2}{c^2} \delta \beta (\beta - 1) e^{-\kappa t} \phi^{b-2}(\omega) \phi'(\omega)^2
+ \frac{c}{c^2} \delta \beta e^{-\kappa t} \phi^{b-1}(\omega) \phi''(bx) - c_1 \delta \beta e^{-\kappa t} \phi^{b-1}(\omega) \phi'(\omega)
+ \delta e^{-\kappa t} \phi'(\omega) f'(V (x, z + \theta - d\rho \delta (1 - e^{-\kappa t})) - \tau \delta e^{-\kappa t} \phi'(\omega))

where \tau(t, x, z) \in (0, 1).

By (10) and (16), we obtain

\[ b \beta \frac{c \phi'(\omega)}{\phi(\omega)} \phi''(bx) - c_1 \beta \frac{\phi'(\omega)}{\phi(\omega)} =\]

\[ = c_1 \beta \frac{\phi'(\omega)}{\phi(\omega)} \left( \frac{b}{c} \phi''(bx) - 1 \right) \]

\[ \leq c_1 \beta \frac{\phi'(\omega)}{\phi(\omega)} \left( \frac{b}{c} - 1 \right) = \frac{7}{8} c_1 \beta \frac{\phi'(\omega)}{\phi(\omega)}. \] (45)

Since \( \lim_{\varrho \to -\infty} \frac{\phi'(\varrho)}{\phi(\varrho)} = c_1 \), then there exists \( X_1 > 0 \) large enough such that

\[ \frac{1}{2} c_1 < \frac{\phi'(\varrho)}{\phi(\varrho)} < \frac{3}{2} c_1 \text{ and } \beta^2 \left( \frac{\phi'(\varrho)}{\phi(\varrho)} \right)^2 - \frac{7}{8} c_1 \beta \phi'(\varrho) < \frac{1}{2} K(\beta c_1) \] (46)

for any \( \varrho < -X_1 \). It follows that \( \lim_{\varrho \to -\infty} \frac{\phi'(\varrho)}{\phi(\varrho)} = c_1 \), \( \lim_{\varrho \to -\infty} \frac{\phi''(\varrho)}{\phi(\varrho)} = c_1^2 \) and (16) that there holds

\[ \frac{c_1^2}{c^2} \left( - \left( \frac{\phi'(\varrho)}{\phi(\varrho)} \right)^2 + \frac{\phi''(\varrho)}{\phi(\varrho)} \right) (1 + \phi'(bx)^2) \to 0 \]

uniformly in \( x \in \mathbb{R} \) as \( \varrho \to -\infty \). Thus, there exists \( X_2 > 0 \) large enough such that

\[ \left| \frac{c_1^2}{c^2} \left( - \left( \frac{\phi'(\varrho)}{\phi(\varrho)} \right)^2 + \frac{\phi''(\varrho)}{\phi(\varrho)} \right) (1 + \phi'(bx)^2) \right| < -\frac{1}{16} K(\beta c_1), \quad \forall \varrho < -X_2, \ x \in \mathbb{R}. \] (47)

By (5), (10) and \( m_* |x| \leq \frac{\phi(bx)}{b} \leq m_* |x| + \frac{\varphi}{b} \), there exists \( X_3 > 0 \) large enough such that

\[ |f'(V (x, z + \theta - d\rho \delta (1 - e^{-\kappa t})) - \tau \delta e^{-\kappa t} \phi'(\omega))| \]

\[ \leq K_1 |V (x, z + \theta - d\rho \delta (1 - e^{-\kappa t})) - \tau \delta e^{-\kappa t} \phi'(\omega)| \theta < -\frac{1}{16} K(\beta c_1) \] (48)

for any \( (x, z) \in \mathbb{R}^2 \) and \( \omega < -X_3 \).

Since \( \phi(\varrho) \to 1, \phi'(\varrho) \to 0 \) and \( \phi''(\varrho) \to 0 \) as \( \varrho \to +\infty \), there is \( X'_1 > 0 \) large enough such that

\[ \left| \frac{\phi'(\varrho)}{\phi(\varrho)} \right| < \frac{|f'(1)|}{2c_1}, \quad \left| \frac{\phi''(\varrho)}{\phi(\varrho)} \right| < \frac{|f'(1)|}{16}, \quad \forall \varrho > X'_1. \] (49)

By (10) and \( m_* |x| \leq \frac{\phi(bx)}{b} \leq m_* |x| + \frac{\varphi}{b} \), there exists \( X'_2 > 0 \) large enough such that

\[ V (x, z + \theta - d\rho \delta (1 - e^{-\kappa t})) > 1 - \varepsilon_1 \quad \text{for any} \ (x, z) \in \mathbb{R}^2 \ \text{with} \ \omega > X'_2. \] (50)
Let $X := \max\{X_1, X_2, X_3, X'_1, X'_2\}$ and $\hat{M} := X + \frac{c_1}{2}$. By (11), there exists a positive constant $\beta_2$ such that
\[
\min_{-\hat{M} \leq \omega \leq \hat{M}} V_z(x, z + \theta - d\rho \delta (1 - e^{-\kappa t})) \geq \beta_2.
\]
Take $\tilde{\rho}(\beta, d) > 0$ large enough such that
\[
-\frac{\rho \delta \kappa \beta_2}{2} + \kappa + \lambda_2 + \frac{c_s}{8} \lambda_1 + \lambda_3 < 0, \quad \forall \rho \geq \tilde{\rho}(\beta, d).
\]
Let $\tilde{\delta}_0(\rho, \beta, d) := \min\{\epsilon_1, \frac{c\beta_2}{2\lambda_1 c_s}, \frac{K(\beta c_s)}{16c_\beta \rho d\Lambda_1}, \frac{|f'(1)|c}{16c_{\beta} \rho d\Lambda_1}\}$ for $\rho \geq \tilde{\rho}(\beta, d)$.

Note that $\frac{c^2}{2} \left(1 + \varphi'(bx)^2\right) \leq 1$ for any $x \in \mathbb{R}$. Let $\rho \geq \tilde{\rho}(\beta, d)$. For $(t, x, z) \in [0, +\infty) \times \mathbb{R}^2$ with $\omega(t, x, z) < -X$, applying $V_z > 0$, (45), (46), (47) and (48) with $0 < \delta < \tilde{\delta}_0(\rho, \beta, d)$, we have
\[
\mathcal{N}[U] := U_t - U_{xx} - U_{zz} + cU_z - f(U)
\leq \delta e^{-\kappa t} \phi^2(\omega) \left[ \kappa + \frac{c_s}{c} d\rho \delta \kappa \beta e^{-\kappa t} \frac{\phi'(\omega)}{\phi(\omega)} 
\right.
\]
\[
+ \frac{c^2}{2} \beta \left( \frac{\phi'(\omega)}{\phi(\omega)} \right)^2 + \frac{c^2}{2} \beta^2 \left( \frac{\phi'(\omega)}{\phi(\omega)} \right)^2 + \frac{c^2}{2} \beta^2 \left( \frac{\phi'(\omega)}{\phi(\omega)} \right)^2 + \frac{c^2}{2} \beta^2 \left( \frac{\phi'(\omega)}{\phi(\omega)} \right)^2
\]
\[
\leq \delta e^{-\kappa t} \phi^2(\omega) \left[ \kappa + \frac{c_s}{c} d\rho \delta \kappa \beta e^{-\kappa t} \frac{\phi'(\omega)}{\phi(\omega)} 
\right.
\]
\[
+ \frac{c^2}{c^2} \beta \left( \frac{\phi'(\omega)}{\phi(\omega)} \right)^2 + \frac{c^2}{c^2} \beta^2 \left( \frac{\phi'(\omega)}{\phi(\omega)} \right)^2 + \frac{c^2}{c^2} \beta^2 \left( \frac{\phi'(\omega)}{\phi(\omega)} \right)^2
\]
\[
+ \frac{c^2}{c^2} \beta^2 \left( \frac{\phi'(\omega)}{\phi(\omega)} \right)^2 - \frac{7}{8} c_s \beta \frac{\phi'(\omega)}{\phi(\omega)}
\]
\[
+ f'(V(x, z + \theta - d\rho \delta (1 - e^{-\kappa t})) - \tau \delta e^{-\kappa t} \phi^2(\omega))
\]
\[
\leq \delta e^{-\kappa t} \phi^2(\omega) \left[ \kappa - \frac{1}{16} K(\beta c_s) - \frac{1}{16} K(\beta c_s) + \frac{1}{2} K(\beta c_s) - \frac{1}{16} K(\beta c_s) \right]
\]
\[
< 0.
\]
For $(t, x, z) \in [0, +\infty) \times \mathbb{R}^2$ with $\omega(t, x, z) > X$, applying $V_z > 0$, (49) and (50) with $0 < \delta < \tilde{\delta}_0(\rho, \beta, d)$, one has
\[
\mathcal{N}[U] := U_t - U_{xx} - U_{zz} + cU_z - f(U)
\leq -d\rho \kappa e^{-\kappa t} V_z + \delta \kappa e^{-\kappa t} \phi^2(\omega) + \frac{c_s}{c} d\rho \delta^2 \kappa \beta e^{-2\kappa t} \phi^2(\omega) \phi'(\omega)
\]
\[
+ \frac{c^2}{c^2} \delta \beta(\beta - 1) e^{-\kappa t} \phi^2(\omega) \phi'(\omega)^2 (1 + \varphi'(bx)^2)
\]
By the above arguments, we can obtain that
\[ \mathcal{N}[U] := U_t - U_{xx} - U_{xz} + cU_x - f(U) \leq 0 \quad \text{in} \quad [0, +\infty) \times \mathbb{R}^2 \]
if \( \rho \geq \hat{\rho} (\beta, d) \), \( 0 < \delta < \hat{\delta}_0 (\rho, \beta, d) \) and \( \theta \in \mathbb{R} \). That is, \( U(t, x, z; \beta, \rho, \delta, \theta, d) \) is a subsolution of (7) on \( (t, x, z) \in [0, +\infty) \times \mathbb{R}^2 \). This completes the proof. \( \square \)
3. **Proof of Theorem 1.2.** In this section, we prove Theorem 1.2. Firstly, we list two lemmas which have been proved in [39].

**Lemma 3.1.** ([39]) For any \( \nu \in (0, 1) \), if the initial value \( v_0(x, z) \in C \left( \mathbb{R}^2, [0, 1 + \epsilon] \right) \) satisfies

\[
\lim_{R \to +\infty} \sup_{x^2 + z^2 > R^2} \frac{|v_0(x, z) - V^-(x, z)|}{(V^-(x, z))^\nu} = 0,
\]

then for any \( T > 0 \), we have

\[
\lim_{R \to +\infty} \sup_{x^2 + z^2 > R^2} \frac{|v(T, x, z; v_0) - V^-(x, z)|}{(V^-(x, z))^\nu} = 0.
\]

**Lemma 3.2.** ([39]) For any \( \nu \in (0, 1) \), on has

\[
\lim_{R \to +\infty} \sup_{z + m, |x| < R} \frac{V_\nu(x, z)}{(V^-(x, z))^\nu} = 0.
\] (52)

Now we begin to prove Theorem 1.2. For simplicity, throughout this section we denote \( V(x, z; \beta, \varepsilon, \alpha, 0) \) by \( V(x, z) \). Take

\[\beta \in \left(0, \frac{1}{8}\right)\] and \( \rho \geq \max \{\tilde{\rho}(\beta), \bar{\rho}(\beta, \varepsilon), \bar{\rho}(\beta, 1)\} \),

and let

\[\delta_0 := \min \{\delta_0(\beta, \beta), \tilde{\delta}_0(\beta, \beta, m_*), \bar{\delta}_0(\beta, \beta, 1)\},\]

\[\varepsilon_0 := \min \{\varepsilon_0(\beta, \varepsilon), \bar{\varepsilon}_0(\beta, \varepsilon)\}, \quad \alpha_0 := \min \{\alpha_0(\beta, \varepsilon), \bar{\alpha}_0(\beta)\}.
\]

For any \( \delta \in (0, \delta_0), \varepsilon \in (0, \varepsilon_0) \) and \( \alpha \in (0, \alpha_0) \) and some positive constant \( M \) which will be specified later, define

\[V_1(x, z) := \phi \left( \frac{\psi(\alpha z) + \alpha|x| - \alpha \rho \delta}{\alpha \sqrt{1 + \psi'(\alpha z)^2}} - \varepsilon \text{sech}(\gamma \alpha z) \rho^\beta \left( \frac{c_* m_*}{c} \left( \frac{\varphi(b x)}{bm_*} - \rho \delta + \psi(\alpha z) \right) \right) \right),\]

\[V^\delta(x, z) := \max \{V_1(x, z), V(x, z - M - m_* \rho \delta)\}.
\]

**Lemma 3.3.** There is

\[
\lim_{R \to +\infty} \sup_{x^2 + z^2 > R^2} \left| V^\delta(x, z) - V(x, z - m_* \rho \delta) \right| = 0.
\] (53)

**Proof.** The proof is similar to the Lemma 4.10 of Wang [37]. We omit it here. \(\square\)

**Lemma 3.4.** The limit of \( v(t, x, z; V^\delta) \) exists in \( L^\infty(\mathbb{R}^2) \) as \( t \to \infty \) and the limit function

\[V^\delta_t(x, z) := \lim_{t \to \infty} v(t, x, z; V^\delta)\]

satisfies

\[\mathcal{L}[V^\delta_t] = 0, \quad V^\delta_t(x, z) \leq V^\delta(x, z) \text{ and } V^\delta_t(x, z) \geq V(x, z - m_* \rho \delta) \text{ on } \mathbb{R}^2.
\]

**Proof.** By the comparison principle, we have \( 0 \leq v(t, x, z; V^\delta) \leq 1 \). Since the function \( V^\delta(x, z) \) is a mild subsolution of (9), \( v(t, x, z; V^\delta) \) is increasing in \( t \) and the limiting function \( V^\delta_t(x, z) \) exists in \( L^\infty(\mathbb{R}^2) \) with

\[\mathcal{L}[V^\delta_t] = 0 \text{ and } V^\delta_t(x, z) \leq V^\delta(x, z) \leq 1.
\]
For the detail, see Sattinger [29].

To prove that $V_\delta^*(x,z) \geq V(x,z - m_* \rho \delta)$ for $(x,z) \in \mathbb{R}^2$, we use an argument by contradiction. Define

$$\varpi := \inf \left\{ \lambda \mid V(x,z - \lambda) \leq V_\delta^*(x,z) \text{ for all } (x,z) \in \mathbb{R}^2 \right\}.$$ 

Assume $\varpi > m_* \rho \delta$. By (53), we have $V(\cdot - \varpi) \neq V_\delta^*(\cdot, \cdot)$. Thus the strong maximum principle implies

$$V(x,z - \varpi) < V_\delta^*(x,z) \quad \text{for all } (x,z) \in \mathbb{R}^2.$$ 

(54)

It follows from Lemma 3.2 and $\lim_{R \to \infty} \sup_{z + m_* |x| > R} V(x,z) = 0$ that there exists a constant $\mathcal{R} > 0$ large enough such that

$$\sup_{z + m_* |x| \leq -\mathcal{R} + \varpi} \frac{V_x(x,z)}{(V^-(x,z))^\beta} < \frac{1}{2\rho},$$ 

(55)

$$\sup_{z + m_* |x| \geq \mathcal{R} - 2\rho} V_z(x,z) < \frac{1}{4\rho},$$ 

(56)

$$\sup_{z + m_* |x| \geq \mathcal{R} - 2\rho} (V^-(x,z))^\beta > \frac{1}{2}.$$ 

(57)

By (54), $V_\delta^*(x,z) \leq V_\delta^*(x,z)$ and $\lim_{x \to \pm \infty} \sup_{|z + m_* |x| | \leq \mathcal{R}} \left| V_\delta^*(x,z) - V(x,z - m_* \rho \delta) \right| = 0$, there exists a constant $0 < \sigma < \min \left\{ \frac{\delta_0}{2}, \frac{\varpi - m_* \rho \delta}{2\rho} \right\}$ small enough such that

$$V(x,z - \varpi + 2\rho \sigma) \leq V_\delta^*(x,z) \quad \text{for } (x,z) \in \mathbb{R}^2 \text{ with } |z + m_* |x| | \leq \mathcal{R}.$$ 

On the other hand, for $(x,z) \in \mathbb{R}^2$ with $z + m_* |x| \leq -\mathcal{R}$, it follows from (55) that

$$\frac{V(x,z - \varpi) - V(x,z - \varpi + 2\rho \sigma)}{\phi^\beta \left( \frac{c_*}{c} (z - \varpi + 2\rho \sigma + \varphi(bx)/b) \right)} \geq \frac{V(x,z - \varpi) - V(x,z - \varpi + 2\rho \sigma)}{(V^-(x,z - \varpi + 2\rho \sigma))^\beta}$$

$$= \frac{-2\rho \sigma \int_0^1 V_z(x,z - \varpi + 2\tau \rho \sigma)d\tau}{(V^-(x,z - \varpi + 2\rho \sigma))^\beta}$$

$$\geq -2\rho \sigma \int_0^1 V_z(x,z - \varpi + 2\tau \rho \sigma)d\tau$$

$$\geq -\sigma,$$

which yields

$$V(x,z - \varpi) - V(x,z - \varpi + 2\rho \sigma) \geq -\sigma \phi^\beta \left( \frac{c_*}{c} (z - \varpi + 2\rho \sigma + \varphi(bx)/b) \right).$$

For $(x,z) \in \mathbb{R}^2$ with $z + m_* |x| \geq \mathcal{R}$, it follows from (56) and (57) that

$$V(x,z - \varpi) - V(x,z - \varpi + 2\rho \sigma)$$

$$= -2\rho \sigma \int_0^1 V_z(x,z - \varpi + 2\tau \rho \sigma)d\tau$$

$$\geq -\frac{1}{2} \sigma \geq -\sigma \phi^\beta \left( \frac{c_*}{c} \left( z - \varpi + 2\rho \sigma + \frac{\varphi(bx)}{b} \right) \right).$$

Finally, we obtain

$$V(x,z - \varpi + 2\rho \sigma) - \sigma \phi^\beta \left( \frac{c_*}{c} \left( z - \varpi + 2\rho \sigma + \frac{\varphi(bx)}{b} \right) \right) \leq V_\delta^*(x,z), \forall (x,z) \in \mathbb{R}^2.$$
By Lemma 2.6 and the comparison principle, one gets
\[ U(t, x, z; \beta, \rho, \sigma, -\varpi + 2 \rho \sigma, 1) \leq V^0_x(x, z), \quad \forall (x, z) \in \mathbb{R}^2, \quad t > 0. \]
Letting \( t \to \infty \), we have \( V(x, z - (\varpi - \rho \sigma)) \leq V^0_x(x, z) \) for all \((x, z) \in \mathbb{R}^2\). This contradicts the definition of \( \varpi \). This completes the proof of the lemma. \( \square \)

Let
\[ W_1(t, x, z; \beta, \rho, \varepsilon, \alpha, \delta, -M, m_*) \]
\[ := \max \{ W(t, x, z; \beta, \rho, \varepsilon, \alpha, \delta), U(t, x, z; \beta, \rho, \delta, -M, m_*) \} . \]

Obviously,
\[ W_1(0, x, z; \beta, \rho, \varepsilon, \alpha, \delta, -M, m_*) \]
\[ = \max \left\{ V(x, z) - \delta \phi^\beta \left( \frac{c_* m_*}{c} \left( \frac{\varphi(bx)}{b} + \frac{\psi(\alpha z)}{\alpha} \right) \right), \right. \]
\[ \left. V(x, z - M) - \delta \phi^\beta \left( \frac{c_*}{c} \left( z - M + \frac{\varphi(bx)}{b} \right) \right) \right\} . \]

**Lemma 3.5.** For any initial function \( v_0(x, z) \in C(\mathbb{R}^2; [0, 1]) \) satisfying (12) and any constants \( 0 < \delta < \delta_0, 0 < \varepsilon < \varepsilon_0 \) and \( T > 0 \), there exist constants \( M > 0 \) and \( \delta_1 > 0 \) such that for \( 0 < \alpha < \delta_1 \) and \((x, z) \in \mathbb{R}^2\),
\[ W_1(0, x, z; \beta, \rho, \varepsilon, \alpha, \delta, -M, m_*) \leq v(T, x, z; v_0) \]
holds.

**Proof.** For simplicity, we denote \( v(t, x, z; v_0) \) by \( v(t, x, z) \). Since the initial value \( 0 \leq v_0(x, z) \leq 1 \) satisfies (12), then the strong parabolic maximum principle implies \( 0 < v(t, x, z) < 1 \) for all \((t, x, z) \in (0, +\infty) \times \mathbb{R}^2\).

We divide our proof into the following two steps.

**Step 1.** For any \( \delta \in (0, \delta_0) \) and \( T > 0 \), we prove that there exists a constant \( M > 0 \) such that
\[ V(x, z - M) - \delta \phi^\beta \left( \frac{c_*}{c} \left( z - M + \frac{\varphi(bx)}{b} \right) \right) \leq v(T, x, z) \quad \text{for} \quad (x, z) \in \mathbb{R}^2. \]

By (10) and Lemma 3.1, we obtain that there exists a constant \( \mathcal{R} > 0 \) large enough such that
\[ V(x, z) - V^-(x, z) \leq \delta \left( V^-(x, z) \right)^{\beta + \frac{1}{2}} \quad \text{and} \quad V(x, z) - \delta \left( V^-(x, z) \right)^\beta \leq v(T, x, z) \]
for all \((x, z) \in \mathbb{R}^2 \) with \( x^2 + z^2 > \mathcal{R}^2 \). Then there exists a sufficiently large positive constant \( R_1 \) such that
\[ V(x, z) - V^-(x, z) \leq \delta \left( V^-(x, z) \right)^{\beta + \frac{1}{2}} \quad \text{and} \quad \left( V^- (x, z) \right)^{1-\beta} + \delta \left( V^- (x, z) \right)^{\frac{1}{2}} - \delta < 0 \]
for all \((x, z) \in \mathbb{R}^2 \) with \( z + m_* |x| < -R_1. \) Thus, we have
\[ V(x, z) - \delta \left( V^-(x, z) \right)^\beta \]
\[ < V^- (x, z) + \delta \left( V^- (x, z) \right)^{\beta + \frac{1}{2}} - \delta \left( V^- (x, z) \right)^\beta \]
\[ = \left( V^- (x, z) \right)^\beta \left[ (V^- (x, z))^{1-\beta} + \delta (V^- (x, z))^{\frac{1}{2}} - \delta \right] \]
\[ < 0 < v(T, x, z). \]
For \((x, z) \in \mathbb{R}^2\) with \(z + m_*|x| \geq -R_1\), we have

\[ V(x, z) - \delta \left(V^-(x, z)\right)^\beta \leq 1 - \delta \phi^\beta \left(\frac{c_*}{c}R_1\right). \]

It follows from Lemma 3.1 that we get

\[
\lim_{R \to +\infty} \sup_{z + m_*|x| \geq -R_1} |v(T, x, z + R) - V^-(x, z + R)| = 0.
\]

Since

\[
\lim_{R \to +\infty} \sup_{z + m_*|x| \geq -R_1} |V^-(x, z + R) - 1| = 0,
\]

then there exists a constant \(M > 0\) such that

\[
1 - \delta \phi^\beta \left(\frac{c_*}{c}R_1\right) \leq v(T, x, z + M) \quad \text{for} \quad (x, z) \in \mathbb{R}^2 \quad \text{with} \quad z + m_*|x| \geq -R_1.
\]

Combining the arguments above and (15), we have

\[
v(T, x, z + M) \geq V(x, z) - \delta \left(V^-(x, z)\right)^\beta \geq V(x, z) - \delta \phi^\beta \left(\frac{c_*}{c} \left(z + \frac{\varphi(bx)}{b}\right)\right)
\]

for all \((x, z) \in \mathbb{R}^2\). Thus

\[
v(T, x, z) \geq V(x, z - M) - \delta \phi^\beta \left(\frac{c_*}{c} \left(z - M + \frac{\varphi(bx)}{b}\right)\right) \quad \text{for} \quad (x, z) \in \mathbb{R}^2.
\]

This completes the proof of Step 1.

**Step 2.** For any \(\delta \in (0, \delta_0), \varepsilon \in (0, \hat{\varepsilon}_0)\) and \(T > 0\), we show that there exists a constant \(\hat{\alpha}_1 > 0\) such that, for \(\alpha \in (0, \hat{\alpha}_1),

\[
V(x, z) - \delta \phi^\beta \left(\frac{c_*}{c} \left(z + \frac{\varphi(bx)}{b}\right)\right) \leq v(T, x, z) \quad \forall (x, z) \in \mathbb{R}^2.
\]

By \(V(x, z) \leq \phi \left(\frac{|x| + \psi(\alpha z)/\alpha}{\sqrt{1 + \psi'(\alpha z)^2}}\right)\), we know that it is sufficient to prove

\[
\phi \left(\frac{|x| + \psi(\alpha z)/\alpha}{\sqrt{1 + \psi'(\alpha z)^2}}\right) - \delta \phi^\beta \left(\frac{c_*}{c} \left|x + \frac{\psi(\alpha z)}{\alpha}\right|\right) \leq v(T, x, z) \quad \text{for} \quad (x, z) \in \mathbb{R}^2.
\]

Since \(\lim_{\rho \to -\infty} \phi^{1-\beta}(\rho) = 0\), then there exists a constant \(R_2 > 0\) sufficiently large such that

\[
\phi^{1-\beta}(\rho) < \delta \quad \text{for} \quad \rho < \max \left\{\frac{c_*}{c}R_2, \frac{c_*m_*}{c}R_2\right\}.
\]
Firstly, for any \((x, z) \in \mathbb{R}^2\) with \(z + m_*|x| < -R_2\), we have

\[
\phi \left( \frac{|x| + \psi(az)/\alpha}{\sqrt{1 + \psi'(az)^2}} \right) - \delta \phi^\beta \left( \frac{c_* m_*}{c} \left( \frac{|x| + \psi(az)}{\alpha} \right) \right) \\
\leq \phi \left( \frac{c_* m_*}{c} \left( \frac{|x| + \psi(az)}{\alpha} \right) \right) - \delta \phi^\beta \left( \frac{c_* m_*}{c} \left( \frac{|x| + \psi(az)}{\alpha} \right) \right) \\
= \phi^\beta \left( \frac{c_* m_*}{c} \left( \frac{|x| + \psi(az)}{\alpha} \right) \right) \left( \phi^{1-\beta} \left( \frac{c_* m_*}{c} \left( z + m_*|x| \right) \right) - \delta \right) \\
< 0 < v(T, x, z)
\]

from \((17)\) and \((59)\).

Secondly, according to Lemma 3.1, there exists a constant \(R_3 > 0\) sufficiently large such that

\[
\max \{ \phi(0), 1 - \delta \phi^\beta(0) \} < v(T, x, z) \quad \text{for} \quad (x, z) \in \mathbb{R}^2 \quad \text{with} \quad z + m_*|x| > R_3.
\]

Since there hold

\[
\phi \left( \frac{|x| + \psi(az)/\alpha}{\sqrt{1 + \psi'(az)^2}} \right) - \delta \phi^\beta \left( \frac{c_* m_*}{c} \left( \frac{|x| + \psi(az)}{\alpha} \right) \right) \leq \phi(0)
\]

for \(|x + \psi(az)/\alpha| \leq 0\), and

\[
\phi \left( \frac{|x| + \psi(az)/\alpha}{\sqrt{1 + \psi'(az)^2}} \right) - \delta \phi^\beta \left( \frac{c_* m_*}{c} \left( \frac{|x| + \psi(az)}{\alpha} \right) \right) \leq 1 - \delta \phi^\beta(0)
\]

for \(|x + \psi(az)/\alpha| > 0\), we have

\[
\phi \left( \frac{|x| + \psi(az)/\alpha}{\sqrt{1 + \psi'(az)^2}} \right) - \delta \phi^\beta \left( \frac{c_* m_*}{c} \left( \frac{|x| + \psi(az)}{\alpha} \right) \right) < v(T, x, z) \quad (61)
\]

for \((x, z) \in \mathbb{R}^2\) with \(z + m_*|x| > R_3\).

Thirdly, for \((x, z) \in \mathbb{R} \times (0, +\infty)\) with \(-R_2 \leq z + m_*|x| \leq R_3\), it follows from \((17)\) that

\[
|x + \psi(az)/\alpha| \leq \frac{1}{m_*} \left( z + m_*|x| - \frac{\ln 2}{\gamma \alpha} \right).
\]

Thus for any \(0 < \alpha < \min \left\{ \frac{\ln 2}{R_2 \gamma}, \frac{\ln 2}{R_3 + \frac{\ln 2}{\phi^{-1}(\delta^{\gamma})}} \right\} := \tilde{\alpha}\), we have

\[
|x + \psi(az)/\alpha| < 0 \quad \text{and} \quad \phi^{1-\beta} \left( \frac{c_*}{c} \left( R_3 - \frac{\ln 2}{\gamma \alpha} \right) \right) < \delta. \quad (62)
\]

It follows from Lemma 3.1 that

\[
\lim_{R \to \infty} \sup_{-R_2 \leq z + m_*|x| \leq R_3, z < -R} \frac{|v(T, x, z) - V^-(x, z)|}{(V^-(x, z))^\beta} = 0,
\]
then there exists a constant $\tilde{R}_1 > 0$ large enough such that for $(x, z) \in \mathbb{R} \times (-\infty, -\tilde{R}_1)$ with $-R_2 \leq z + m_* |x| \leq R_3$,

$$v(T, x, z) \geq V^-(x, z) - \frac{\delta}{2} \phi^\beta \left( -\frac{c_* m_*}{c} R_2 \right).$$ (63)

By the definition of $\psi$, for $-R_2 \leq z + m_* |x| \leq R_3$, we obtain

$$\lim_{\xi \to -\infty} \left| \phi \left( \frac{z + m_* |x|}{m_* \sqrt{1 + \psi'(\xi)^2}} \right) - V^-(x, z) \right| = 0,$$

then there exists a constant $\tilde{R}_2 > 0$ large enough such that for $(x, z) \in \mathbb{R} \times (-\infty, -\tilde{R}_1)$ with $-R_2 \leq z + m_* |x| \leq R_3$ and $\xi < -\tilde{R}_2$,

$$\phi \left( \frac{z + m_* |x|}{m_* \sqrt{1 + \psi'(\xi)^2}} \right) \leq V^-(x, z) + \frac{\delta}{2} \phi^\beta \left( -\frac{c_* m_*}{c} R_2 \right).$$ (64)

Let $\tilde{R}_3 := \max \{ \tilde{R}_1, \tilde{R}_2 \}$. Since for $(x, z) \in \mathbb{R} \times [-\tilde{R}_3, 0]$ with $-R_2 \leq z + m_* |x| \leq R_3$,

$$|x| + \frac{\psi(\alpha z)}{\alpha} = \frac{\psi(\alpha z)}{\alpha} - \frac{\alpha z}{m_*} + \frac{z}{m_*} + |x| \leq \max_{\xi \in [-R_*, 0]} \left( \frac{\psi(\xi) - \xi}{\alpha} \right) + \frac{R_3}{m_*},$$

then by (17), there exists a constant $\tilde{\alpha}_2 > 0$ such that for $\alpha \in (0, \tilde{\alpha}_2)$,

$$|x| + \frac{\psi(\alpha z)}{\alpha} \leq \max_{\xi \in [-R_*, 0]} \left( \frac{\psi(\xi) - \xi}{\alpha} \right) + \frac{R_3}{m_*} < 0,$$ (65)

and

$$\phi^{1-\beta} \left( \frac{c_* m_*}{c} \left( |x| + \frac{\psi(\alpha z)}{\alpha} \right) \right) \leq \phi^{1-\beta} \left( \frac{\max_{\xi \in [-R_*, 0]} \left( \frac{\psi(\xi) - \xi}{\alpha} \right)}{\alpha} + \frac{R_3}{m_*} \right) < \delta.$$ (66)

Let $\tilde{\alpha}_1 := \min \{ \tilde{\alpha}_1, \tilde{\alpha}_2 \}$ and $\alpha \in (0, \tilde{\alpha}_1)$. Therefore, for $(x, z) \in \mathbb{R} \times (0, +\infty)$ with $-R_2 \leq z + m_* |x| \leq R_3$, by (62), we have

$$\phi \left( \frac{|x| + \psi(\alpha z)}{\sqrt{1 + \psi'(\alpha z)^2}} \right) - \delta \phi^\beta \left( \frac{c_* m_*}{c} \left( |x| + \frac{\psi(\alpha z)}{\alpha} \right) \right)$$

$$\leq \phi \left( \frac{c_* m_*}{c} \left( |x| + \frac{\psi(\alpha z)}{\alpha} \right) \right) - \delta \phi^\beta \left( \frac{c_* m_*}{c} \left( |x| + \frac{\psi(\alpha z)}{\alpha} \right) \right)$$

$$= \phi^\beta \left( \frac{c_* m_*}{c} \left( |x| + \frac{\psi(\alpha z)}{\alpha} \right) \right) \left( \phi^{1-\beta} \left( \frac{c_* m_*}{c} \left( |x| + \frac{\psi(\alpha z)}{\alpha} \right) \right) - \delta \right)$$
\[ \leq \phi^\beta \left( \frac{c_m}{c} \left( \frac{|x| + \psi(\alpha z)}{\alpha} \right) \right) \left( \phi^{1-\beta} \left( \frac{c_m}{c} \left( z + m_*|x| - \frac{\ln 2}{\gamma \alpha} \right) \right) - \delta \right) \]
\[ \leq \phi^\beta \left( \frac{c_m}{c} \left( |x| + \frac{\psi(\alpha z)}{\alpha} \right) \right) \left( \phi^{1-\beta} \left( \frac{c_m}{c} \left( R_3 - \frac{\ln 2}{\gamma \alpha} \right) \right) - \delta \right) \]
\[ < 0 < v(T, x, z). \] (67)

Finally, we show that for \((x, z) \in \mathbb{R} \times (-\infty, 0]\) with \(-R_2 \leq z + m_*|x| \leq R_3\), the inequality (58) holds. For this purpose, we divide the proof into the following four cases.

**Case 1.** For \((x, z) \in \mathbb{R} \times (-\infty, 0]\) with \(-R_2 \leq z + m_*|x| \leq R_3\) and \(|x| + \frac{\psi(\alpha z)}{\alpha} < -R_2\), by (59), we have

\[ \phi \left( \frac{|x| + \psi(\alpha z)/\alpha}{\sqrt{1 + \psi'(\alpha z)^2}} \right) - \delta \phi^\beta \left( \frac{c_m}{c} \left( |x| + \frac{\psi(\alpha z)}{\alpha} \right) \right) \]
\[ \leq \phi \left( \frac{c_m}{c} \left( |x| + \frac{\psi(\alpha z)}{\alpha} \right) \right) - \delta \phi^\beta \left( \frac{c_m}{c} \left( |x| + \frac{\psi(\alpha z)}{\alpha} \right) \right) \]
\[ = \phi^\beta \left( \frac{c_m}{c} \left( |x| + \frac{\psi(\alpha z)}{\alpha} \right) \right) \left( \phi^{1-\beta} \left( \frac{c_m}{c} \left( |x| + \frac{\psi(\alpha z)}{\alpha} \right) \right) - \delta \right) \]
\[ < 0 < v(T, x, z). \] (68)

**Case 2.** For \((x, z) \in \mathbb{R} \times (-\infty, -\tilde{R}_1]\) with \(-R_2 \leq z + m_*|x| \leq R_3\), \(-R_2 \leq |x| + \frac{\psi(\alpha z)}{\alpha}\) and \(\alpha z < -\tilde{R}_2\), by (63) and (64), we get

\[ \phi \left( \frac{|x| + \psi(\alpha z)/\alpha}{\sqrt{1 + \psi'(\alpha z)^2}} \right) - \delta \phi^\beta \left( \frac{c_m}{c} \left( |x| + \frac{\psi(\alpha z)}{\alpha} \right) \right) \]
\[ \leq \phi \left( \frac{z + m_*|x|}{m_*\sqrt{1 + \psi'(\alpha z)^2}} \right) - \delta \phi^\beta \left( - \frac{c_m}{c} R_2 \right) \]
\[ \leq V^-(x, z) - \frac{\delta}{2} \phi^\beta \left( - \frac{c_m}{c} R_2 \right) \leq v(T, x, z). \] (69)

**Case 3.** For \((x, z) \in \mathbb{R} \times (-\infty, -\tilde{R}_1]\) with \(-R_2 \leq z + m_*|x| \leq R_3\), \(-R_2 \leq |x| + \frac{\psi(\alpha z)}{\alpha}\) and \(-\tilde{R}_2 \leq \alpha z \leq 0\), by (65) and (66), we have

\[ \phi \left( \frac{|x| + \psi(\alpha z)/\alpha}{\sqrt{1 + \psi'(\alpha z)^2}} \right) - \delta \phi^\beta \left( \frac{c_m}{c} \left( |x| + \frac{\psi(\alpha z)}{\alpha} \right) \right) \]
\[ \leq \phi \left( \frac{c_m}{c} \left( |x| + \frac{\psi(\alpha z)}{\alpha} \right) \right) - \delta \phi^\beta \left( \frac{c_m}{c} \left( |x| + \frac{\psi(\alpha z)}{\alpha} \right) \right) \]
\[ = \phi^\beta \left( \frac{c_m}{c} \left( |x| + \frac{\psi(\alpha z)}{\alpha} \right) \right) \left( \phi^{1-\beta} \left( \frac{c_m}{c} \left( |x| + \frac{\psi(\alpha z)}{\alpha} \right) \right) - \delta \right) \]
\[ = \phi^\beta \left( \frac{c_m}{c} \left( |x| + \frac{\psi(\alpha z)}{\alpha} \right) \right) \left( \phi^{1-\beta} \left( \frac{c_m}{c} \left( \psi(\alpha z) - \frac{\alpha z}{m_*} + z + |x| \right) \right) - \delta \right) \]
\[ \leq \phi^\beta \left( \frac{c_m}{c} \left( |x| + \frac{\psi(\alpha z)}{\alpha} \right) \right) \left( \phi^{1-\beta} \left( \frac{c_m}{c} \left( R_3 - m_* \right) \right) - \delta \right) \]
\[
\leq \phi^\beta \left( \frac{c_\ast m_\ast}{c} \left( |x| + \frac{\psi(\alpha z)}{\alpha} \right) \right) \left( \phi^{1-\beta} \left( \frac{c_\ast m_\ast}{c} \left( \Xi + R_3 \right) \right) - \delta \right) < 0 \leq v(T, x, z),
\]
where \( \Xi = \max_{-R_3 \leq \xi \leq 0} \left( \psi(\xi) - \frac{\xi}{m_\ast} \right) / \alpha. \)

**Case 4.** For \((x, z) \in \mathbb{R} \times [-R_1, 0] \) with \(-R_2 \leq z + m_\ast |x| \leq R_3 \) and \(-R_2 \leq |x| + \frac{\psi(\alpha z)}{\alpha}, \)
by (65) and (66), we have
\[
\phi \left( \frac{|x| + \psi(\alpha z) / \alpha}{\sqrt{1 + \psi'(\alpha z)^2}} \right) - \delta \phi^\beta \left( \frac{c_\ast m_\ast}{c} \left( |x| + \frac{\psi(\alpha z)}{\alpha} \right) \right)
\leq \phi \left( \frac{c_\ast m_\ast}{c} \left( |x| + \frac{\psi(\alpha z)}{\alpha} \right) \right) - \delta \phi^\beta \left( \frac{c_\ast m_\ast}{c} \left( |x| + \frac{\psi(\alpha z)}{\alpha} \right) \right)
= \phi^\beta \left( \frac{c_\ast m_\ast}{c} \left( |x| + \frac{\psi(\alpha z)}{\alpha} \right) \right) \left( \phi^{1-\beta} \left( \frac{c_\ast m_\ast}{c} \left( |x| + \frac{\psi(\alpha z)}{\alpha} \right) \right) - \delta \right) < 0 < v(T, x, z).
\]

Therefore, for \((x, z) \in \mathbb{R} \times (-\infty, 0] \) with \(-R_2 \leq z + m_\ast |x| \leq R_3 \), by (68), (69), (70) and (71) we have
\[
\phi \left( \frac{|x| + \psi(\alpha z) / \alpha}{\sqrt{1 + \psi'(\alpha z)^2}} \right) - \delta \phi^\beta \left( \frac{c_\ast m_\ast}{c} \left( |x| + \frac{\psi(\alpha z)}{\alpha} \right) \right) \leq v(T, x, z).
\]
Combining (60), (61), (67) and (72) yields
\[
V(x, z) - \delta \phi^\beta \left( \frac{c_\ast m_\ast}{c} \left( \frac{\varphi(bx)}{b m_\ast} + \frac{\psi(\alpha z)}{\alpha} \right) \right) \leq v(T, x, z) \text{ for } (x, z) \in \mathbb{R}^2.
\]
This completes the proof of Step 2.

By the above arguments of Step 1 and Step 2, we have
\[
W_1(0, x; z; \beta, \rho, \varepsilon, \alpha, \delta, -M, m_\ast) \leq v(T, x; z; v_0) \text{ for } (x, z) \in \mathbb{R}^2.
\]
The proof of the lemma is completed. \(\square\)

**Proof of Theorem 1.2.** For simplicity, we denote \(v(t, x, z; v_0)\) by \(v(t, x, z)\). Note that
\[
v_0^\pm (x, z) := \min \{v_0(x, z), V^-(x, z)\}
\leq v_0(x, z)
\leq \max \{v_0(x, z), V^-(x, z)\} := v_0^\mp (x, z).
\]
By the comparison principle, we have
\[
v(t, x, z; v_0^-) \leq v(t, x, z; v_0) \leq v(t, x, z; v_0^+) \text{ for } (t, x, z) \in [0, +\infty) \times \mathbb{R}^2.
\]
Assume that \(v_0(x, z) \in C(\mathbb{R}^2, [0, 1])\) satisfies (12) for \(\beta \in (0, \frac{1}{8})\). It is clear that the function \(v_0^\pm (x, z)\) also satisfy
\[
\lim_{R \to +\infty} \sup_{x^2 + z^2 > R^2} \left| \frac{v_0^\pm (x, z) - V^-(x, z)}{(V^-(x, z))^\beta} \right| = 0.
\]
It follows from Theorem 1.1 that
\[
\lim_{t \to +\infty} \sup_{(x, z) \in \mathbb{R}^2} \left| \frac{v(t, x, z; v_0^+) - V(x, z)}{(V^-(x, z))^\beta} \right| = 0.
\]
Therefore, to complete the proof of Theorem 1.2, it is sufficient to show
\[
\lim_{t \to +\infty} \sup_{(x,z) \in \mathbb{R}^2} \frac{|v(t,x,z;v_0^m) - V(x,z)|}{(V^-(x,z))^\beta} = 0.
\]

Without loss of generality, we assume that \(v_0(x,z) \in C(\mathbb{R}^2,[0,1])\) satisfies \(v_0(x,z) \leq V^-(x,z)\) for \((x,z) \in \mathbb{R}^2\) and (12) for any \(\beta \in (0,\frac{1}{2})\). Fix \(T > 0\) and \(\varepsilon \in (0,\varepsilon_0)\). For any \(\varsigma > 0\), let \(\delta \in \min \left\{ \frac{1}{2}\delta_0, \frac{\varsigma}{2M \rho \delta_1} \right\}\), where
\[
D_1 = \sup_{(x,z) \in \mathbb{R}^2} \frac{V_0(x,z)}{(V^-(x,z))^\beta} \quad \text{and} \quad D_2 = \sup_{(x,z) \in \mathbb{R}^2, \theta \in [0,1]} \frac{(V^-(x,z + \theta))^\beta}{(V^-(x,z))^\beta}.
\]

Let \(M > 0\) and \(\alpha_1 > 0\) be determined by Lemma 3.5. Then for \(\alpha \in (0, \min\{\alpha_0, \alpha_1\})\), Lemma 3.5 immediately shows that
\[
W_1(t,x,z;\beta,\rho,\varepsilon,\alpha,\delta,-M,m_\ast) \leq v(t + T, x, z) \leq v(t + T, x, z; V^-) \leq V(x,z)
\]
for all \((t,x,z) \in [0, +\infty) \times \mathbb{R}^2\). Applying the comparison principle, we have
\[
v(t',x,z;W_1^\delta) \leq v(t' + T, x, z) \leq v(t' + T, x, z; V^-) \leq V(x,z)
\]
for \((t,x,z) \in [0, +\infty) \times \mathbb{R}^2\) and \(t' \geq 0\), where
\[
W_1^\delta(x,z) = W_1(t, x, z; \beta, \rho, \varepsilon, \alpha, \delta,-M,m_\ast).
\]
By letting \(t \to \infty\), we obtain
\[
v(t',x,z;V^\delta) \leq \liminf_{t \to \infty} v(t' + T, x, z) \leq \limsup_{t \to \infty} v(t' + T, x, z) \leq V(x,z)
\]
for \((t',x,z) \in [0, +\infty) \times \mathbb{R}^2\). Thus letting \(t' \to \infty\) again, by Lemma 3.4, we obtain
\[
V(x,z - m_\ast \rho \delta) \leq V^\delta(x,z) \leq \liminf_{t \to \infty} v(t, x, z) \leq \limsup_{t \to \infty} v(t, x, z) \leq V(x,z)
\]
for \((x,z) \in \mathbb{R}^2\). Since
\[
V(x,z) - V(x,z - m_\ast \rho \delta)
= m_\ast \rho \delta \int_0^1 V_\tau(x,z - \tau m_\ast \rho \delta)d\tau
= m_\ast \rho \delta \int_0^1 \frac{V_\tau(x,z - \tau m_\ast \rho \delta)}{(V^- (x,z - \tau m_\ast \rho \delta))^\beta} (V^- (x,z - \tau m_\ast \rho \delta))^{\beta} d\tau
\leq \varsigma (V^- (x,z))^\beta,
\]
then we have
\[
V(x,z) \leq V(x,z - m_\ast \rho \delta) + \varsigma (V^- (x,z))^\beta, \quad \forall (x,z) \in \mathbb{R}^2.
\]
Thus we get
\[
V(x,z) - \varsigma (V^- (x,z))^\beta \leq \liminf_{t \to \infty} v(t, x, z) \leq \limsup_{t \to \infty} v(t, x, z) \leq V(x,z)
\]
for \((x,z) \in \mathbb{R}^2\). By the arbitrariness of \(\varsigma > 0\), we obtain
\[
\lim_{t \to \infty} \left\| \frac{v(t,\cdot,z) - V(\cdot,z)}{(V^- (\cdot,z))^\beta} \right\|_{L^\infty(\mathbb{R}^2)} = 0.
\]
This completes the proof of Theorem 1.2. \(\square\)
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E-mail address: buzhh14@lzu.edu.cn
E-mail address: wangzhch@lzu.edu.cn