Performance of multifractal detrended fluctuation analysis on short time series

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(Dated: November 12, 2013)

PACS numbers: 05.45.Df,05.45.Tp,89.65.Gh

I. INTRODUCTION

There are many processes of interest in nature and in society which exhibit a fractal or multifractal behavior \cite{Bunde2004,Borner2004}. One source of information from these processes are time series obtained from records of measurements or observations. These time series may be affected from experimental or observational non-stationary uncertainties which have to be disentangled from the potential intrinsic fluctuations and correlations of the studied system. This is a very complex task and many methods to achieve this goal have been proposed \cite{Peng1994}. One method which has proved to be quite useful to detect reliably long-range correlations in data with trends is the detrended fluctuation analysis (DFA) introduced by Peng et al. \cite{Peng1994}.

Later, this method has been generalized to the analysis of multifractal time series (MFDFA) by Kantelhardt et al. \cite{Kantelhardt2002} and has been extended to multidimensional series \cite{Havlin2001} and to investigate the power-law correlations between simultaneously-recorded time series \cite{Havlin2002}. MFDFA has been compared favorably to other methods \cite{Havlin2002} and applied to a wide range of fields. Just to name a few cases – currently \cite{Kantelhardt2002} has been cited hundreds of times – MFDFA has been used to study series from geophysics \cite{Bunde2004,Borner2004}, physiology \cite{Peng1994,Borner2004}, financial markets \cite{Havlin2002,Havlin2001}, and, of particular interest here, to study the exchange rate of different currencies \cite{Havlin2002}. MFDFA works very well for time series with some \(2^{16}\) elements or more, but nevertheless it is important to evaluate the performance of this, and any other method (see e.g. \cite{Bunde1994}) on shorter time series, mainly for two reasons: first, there are many records of interest which are short and second, there are processes for which long records are available, but where it is expected that the multifractal behavior changes with time and the study of short fragments of these long series could yield important insight on those cases.

In this work the performance of MFDFA is studied as a function of the decreasing length of the series. The evaluation is performed using computer simulated data sets with known fractal and multifractal behavior. The results are applied to the analysis of the daily exchange rate between the US Dollar and the Euro. This time series is relatively short, around 3500 entries, given that the Euro currency debuted at the beginning of 1999. Furthermore in its short life the Euro has gone through a dubitative start, followed by a strong couple of years and since around 2008 it has been immersed in a crisis which has threatened its existence. This turbulent history makes it interesting to ask if its dynamics have changed with time.

The paper is organized as follows: in the next section the MFDFA method is briefly described and the notation used in the rest of the paper is introduced. Section II presents the analysis of the mono- and multifractal synthetic data. Section IV discusses the application of the results to a time series from finance, namely the daily exchange rate between the US Dollar and the Euro. Finally, the conclusions of this work are presented in Section V.

II. MULTIFRACTAL DETRENDED FUNCTIONAL ANALYSIS

The MFDFA method introduced in \cite{Kantelhardt2002} will be described briefly here. The input to the method is a time series \(x(i)\) of finite length \(N\). It is assumed that the time series has a compact support; i.e., that only a negligible fraction of the elements \(x(i)\) are zero. The algorithm has 5 steps:

1. Compute the profile \(Y(j)\), where \(j = 1,\ldots,N\):

\[
Y(j) = \sum_{i=0}^{j} [x(i) - < x >].
\] \hspace{1cm} (1)

2. Divide the new series \(Y(j)\) in \(N_s\) non-overlapping contiguous segments of size \(s\) starting from the beginning of the series and then repeat starting from the end to obtain \(2N_s\) segments.
3. Calculate, for all segments $\nu$ and all sizes $s$, the local polynomial trend of order $m$, $P^m_{\nu s}$, via a least-square fit and compute the variance:

$$F^2(\nu, s) = \frac{1}{s} \sum_{i=1}^{s} \{ Y[(\nu - 1)s + i] - P^m_{\nu s}(i) \}^2. \quad (2)$$

4. Average over all segments of a given size $s$ to obtain the $q$-order fluctuations:

$$F_q(s) = \left\{ \frac{1}{2N_s} \sum^{2N_s}_{\nu=1} [F^2(\nu, s)]^{q/2} \right\}^{1/q}, \quad (3)$$

or, for $q = 0$,

$$F_0(s) = \exp \left\{ \frac{1}{4N_s} \sum^{2N_s}_{\nu=1} \ln [F^2(\nu, s)] \right\}. \quad (4)$$

5. For signals with fractal properties there is a range of sizes, $s_{\text{min}} < s < s_{\text{max}}$, at a given order $q$ for which

$$F_q(s) \sim s^{h(q)}. \quad (5)$$

The $h(q)$ are called generalized Hurst exponents and are the output of the MFDFA algorithm. Note that $h(q)$ is related to the singularity spectrum $f(\alpha)$, where $\alpha$ is called the Hölder exponent, through the following relations:

$$\alpha = h(q) + qh'(q), \quad (6)$$

and

$$f(\alpha) = q[\alpha - h(q)] + 1, \quad (7)$$

where $h'(q)$ denotes the derivative of $h$ with respect to $q$.

III. PERFORMANCE ON SYNTHETIC DATA

In this section synthetic signals are used to evaluate the performance of the MFDFA method as a function of length. The goal of this section is to get an insight on what is the shortest length of series from each model that can be reliably analyzed; what is the magnitude of the precision that can be expected for such a length and in which range of $q$ is the analysis valid. Both monofractal and multifractal models are studied and compared to the corresponding analytic predictions.

Note that these studies yield only estimations of possible shortest lengths and precision of the analysis and not definite predictions, because real time series are much more complex than their synthetic counterparts. On the other hand, these studies show where it is necessary to be specially careful when assigning a mono- or multifractal behavior to a real time series or when assessing the amount of multifractality present in a real time series.

Note that in all the synthetic cases studied here the signals were detrended with a polynomial of order two in the third step of the MFDFA method explained in section III.

A. Computer generated time series

Series of length $2^k$ with $k = 20, 18, 16, 14, 12$ and $10$ were generated for each one of the synthetic models that were analyzed. In each case the number of independent realizations were $10, 40$ and $100$ for $k = 20, 18$ and $16–10$ respectively.

Monofractal models are interesting to evaluate the performance of a method because they have the simplest functional form for the generalized Hurst exponents: a constant. Three different models were studied. The case
of white noise characterized by a Hurst exponent $H = 0.5$ and the absence of long range correlations; and the cases $H = 0.75$ and $H = 0.25$ which present long-range correlations and anti-correlations respectively.

Multifractal models exhibit richer behavior in the generalized Hurst exponent presenting thus a harder challenge to the MFDFA method. Three types of stochastic binomial cascades were used as multifractal models: log-Poisson, log-Gamma and log-Normal. All three have been shown to have different and interesting multifractal behavior as discussed for example in [28].

B. Monofractal signals

The key assumption of the MFDFA method is that $F_q(s) \sim s^{h(q)}$ for some range in $s$, so that $h(q)$ can be extracted, in that $s$ range, by a fit to a line in a log-log scale. This has been shown to be the case for long monofractal series in a wide range of $q$; e.g., [10, 15]. Here the main interest is the dependance on the decreasing length of the series. In particular it is important to determine if the range in $s$ depends on the length of the series. To obtain a statistically stable answer to this question, the average $F_q(s)$, denoted by $\langle F_q(s) \rangle$, of all independent realizations of a given model was used.

The behavior for the case of white noise, $H = 0.5$, is depicted in Figure 1. The upper panel in Figure 1 shows $\langle F_q(s) \rangle$ for three representative values of $q$ and the six lengths studied in this work. There are two important observations to be made from this panel: (i) there is a range of box sizes $s$ where $F_q(s)$ exhibits a power law behavior and, (ii) the fact that the symbols for the different lengths can not be distinguish by eye means that this power law behavior does not depend on the length of the series, at least for the lengths considered here.

Remember that the fifth point of the MFDFA algorithm requires the range of sizes where the power law behavior is valid in order to extract from this range the generalized Hurst exponent. Looking again at the upper panel in Figure 1 it is observed that, as expected, the choice of $s_{\text{max}}$ depends on the length of the series, while the choice of $s_{\text{min}}$ depends on $q$.

To make easier to visualize this last point, the lower panel in Figure 1 shows the local difference $\Delta(\langle F_q(s) \rangle)$ defined as

$$\Delta(\langle F_q(s) \rangle) = \frac{\ln(\langle F_q(s_{i+1}) \rangle) - \ln(\langle F_q(s_i) \rangle)}{\ln(s_{i+1}) - \ln(s_i)} \quad (8)$$

where $i$ runs over all sizes. $\Delta(\langle F_q(s) \rangle)$ is shown for different values of $q$ at a fix length $L = 10^{20}$. It is clear that $\Delta(\langle F_q(s) \rangle)$ is constant over a large range of sizes $s$. At larger values of $s$ there are strong fluctuations due to the small number of boxes that can be formed at large sizes. At lower values of $s$ the behavior of $\Delta(\langle F_q(s) \rangle)$ depends on $q$. Note that a qualitatively similar behavior is found for the monofractal models defined by $H = 0.25$ and $H = 0.75$.

FIG. 2. Mean generalized Hurst exponent for (a) $H = 0.25$, (b) $H = 0.50$ and (c) $H = 0.75$ and for different lengths of the series compared to the theoretical expectation represented by the solid line.
To simplify the analysis a value of $s_{\text{min}}$ was chosen so that the dependence on $q$ was avoided. The actual values used on the fits were: $s_{\text{min}} = 50$, 40 and 30 for $H = 0.25$, 0.50 and 0.75 respectively. The values of $s_{\text{max}}$ used to extract $h(q)$ were $s_{\text{max}} = 10^4$ for lengths $2^k$ with $k = 16$, 18 and 20; $s_{\text{max}} = 3000$ for $k = 14$; $s_{\text{max}} = 800$ for $k = 12$ and $s_{\text{max}} = 200$ for $k = 10$. These values were used for all three Hurst parameters.

Within the stipulated $s$ regions each realization of a series presented the power like behavior of equation (5) and for each realization the values of the generalized Hurst exponents were extracted by a linear least squares fit. These values were averaged over all realizations of a given value of $H$ and a given length. The average, denoted $\langle h(q) \rangle$, was compared to the theoretical expectations. The results for all three cases are shown in Figure 2. The general trend is the same for the three cases: (i) the values of $h(q)$ are under/over predicted at large/small values of $q$ and the best agreement are for values of $q$ close to zero, (ii) the concordance between theory and simulation is best for long series and deteriorates as the length of the series decreases.

The first trend mentioned implies that in case large ranges in $q$ are studied, the results would mimic a multifractal behavior specially if the amount of multifractality is estimated from the difference in the values of $h$ at large negative and positive values of $q$ as sometimes is done.

If the analysis is restricted to a more central region in $q$, say $|q| < 5$, then MFDFA yields good results for all lengths. In this range of $q$ the agreement of theory and simulation for $H = 0.25$ is between 2% at $q = 5$ and 8% at $q = -5$ for series of $2^{12}$ elements and it goes up to 10% at $q = -5$ for the shortest series while is between 1% and 3% for the longest series. For the case $H = 0.5$ the worst agreement happens for the shortest series and it is just 5%, but in general the results are within 3% of the theoretical expectations. The situation is even slightly better for $H = 0.75$ with most comparisons between $h(q)$ as predicted by theory and found with the simulations below 2% and the worst cases, for the shortest series at $q = 5$ and $q = -5$ only 6% away of the predictions.

The previous discussion referred to the average $h(q)$. The fluctuation between the different values of $h(q)$ corresponding to each realization have been evaluated with the standard deviation. For lengths of $2^{14}$ or longer the standard deviation is well below 5% over all values of $q$. For the shorter series and $H = 0.25$ the fluctuations for $-5 < q < 5$ reach 10% and 20% for lengths $2^{12}$ and $2^{10}$ respectively. The situation is better for $H = 0.5$ and $H = 0.75$ where the corresponding fluctuations are 8% and 15%; and 6% and 12% respectively.

C. Multifractal signals

Following [28], stochastic binomial cascades were used as models of multifractal behavior. The cascades are built as follows. Consider a unit of some property, commonly named mass, in the interval $[0,1]$. Next split the interval in two halves and assign a random fraction of the mass to each of the new intervals. Repeat the procedure for each half. After $k$ steps the mass is distributed in $2^k$ intervals of size $2^{-k}$ yielding a series of length $2^k$. The assignment of the random fraction to each half is not arbitrary, the density function providing the random fractions has to conserve the mass in the average. The random fractions are independent, identically distributed random variables drawn from a specific distribution. Mandelbrot has shown (see [28] and references therein) that this process produces signals with multifractal properties.

Analytical results for the multifractal properties of stochastic binomial cascades are available for a number of distributions [28]. Here the cases of Log-Normal, Log-Gamma and Log-Poisson cascades are used to evaluate the performance of the MFDFA method on short time series. Long time series for three of the five models analyzed here have been studied in [15] and those results agree with the findings below.

In all multifractal cases studied here, a region where equation (5) was fulfilled could be identified. The same procedure outlined above was used to obtain the values of $s_{\text{min}}$ and $s_{\text{max}}$. In the case of the multifractal models the value of $s_{\text{min}}$ did not depend on $q$, nor in the length of the series, and a value $s_{\text{min}} = 40$ was used in all cases. For $s_{\text{max}}$ the same values as for the monofractal signals were used; namely $s_{\text{max}} = 10^4$ for lengths $2^k$ with $k = 16$, 18 and 20; $s_{\text{max}} = 3000$ for $k = 14$; $s_{\text{max}} = 800$ for $k = 12$ and $s_{\text{max}} = 200$ for $k = 10$.

Log-Poisson cascade. This is a random discrete model which depends on only one parameter which represents the mean and variance of a Poisson distribution. The series studied here were generated with the value 1.4.
which ensures that the mass is conserved in the average for binomial cascades.

The results of MF DFA for this model are shown in Figure 4. Series of all lengths under investigation reproduced the theoretical prediction with the same precision of better than half a percent for \(-0.5 < q < 2.0\). For larger values of \(q\) the simulation over-estimates the prediction; at \(q = 4\) the difference between prediction and simulation is 3% independent of the length of the series.

For smaller values of \(q\) the method is not able to yield the predicted shape of the generalized Hurst exponent and seems to saturate to a value depending on the length of the series. At \(q = -1\) the agreement is still of the order of 1% for the shortest series and better for the longest one, but at \(q = -2\) the difference between prediction and simulation is already up to 20% for the longest series and 30% for the shortest one.

Note that in this case the estimation of multifractality as the difference of the generalized Hurst exponent at a large negative \(q\) and a large positive \(q\) would yield a smaller multifractality than expected from theory. This behavior is the opposite to the behavior shown by monofractal series which tend to yield a bigger multifractality than what is present in the model.

For this multifractal model, the fluctuations among independent realizations, as quantified by the standard deviation, have a similar behavior for both sets of parameters. For the parameter set \([1, \ln(2)]\) at \(q = 0\) they grow from 1% to 12% for lengths from \(2^{20}\) to \(2^{10}\). For \(q = 5\) they are around 8-10% except for \(2^{10}\) which reaches 25%. For \(q \rightarrow -1\) the fluctuations are bigger: around 15% for all lengths except the shortest where the standard deviation grows to 30%. For the second parameter set, \([2, 1/0.6]\), the qualitative behavior is similar but quantitatively the fluctuations are substantially smaller, being about half of the fluctuations for the parameter set \([1, \ln(2)]\).

**Log-Normal cascade.** This model is characterized by two parameters which correspond to the mean and standard deviation of a Normal distribution. Two different sets of parameters were used to evaluate the performance of MF DFA: \([1, 0.5]\) and \([1.2, 0.75]\). The interpretation of these parameters for \(h(q)\) is straightforward: in this case \(h(q)\) is a straight line, the value of \(h(q = 0)\) corresponds to the mean of the Normal distribution while

![FIG. 4. Mean generalized Hurst exponent for Log-Gamma binomial cascades with parameters (a) \([1, \ln(2)]\) and (b) \([2, 1/0.6]\) for different lengths of the series compared to the theoretical expectation represented by the solid line.](image-url)
FIG. 5. Mean generalized Hurst exponent for Log-Normal binomial cascades with parameters (a) [1.1, 0.5] and (b) [1.2, 0.75] for different lengths of the series compared to the theoretical expectation represented by the solid line.

The slope is directly related to its variance.

The results are shown in Figure 5. As in the previous multifractal models there is no strong dependence with the length of the series down to the shortest series that were studied. Simulation and theory agree in the middle region of $q$ and the range in $q$ where the agreement is good depends on the value of the parameters. At large values of $q$ the simulation yield values larger than expected from the theory and at small values of $q$ the theory is above the simulations. So, also in these cases the amount of multifractality could be underestimated.

For the first set of parameters, [1.1, 0.5] the agreement between theory and simulation in the range from $q = -2$ to $q = 3$ is 2% or better. The range in $q$ with a similar agreement between theory and simulation is reduced to the range from $q = -1$ to $q = 2$ for the second set of parameters, [1.2, 0.75].

For this multifractal model, the fluctuations among independent realizations, as quantified by the standard deviation, have a similar behavior for both sets of parameters. For $|q| = -3$ they grow from 2-3% to 10% for lengths decreasing from $2^{20}$ to $2^{12}$ and are smaller for $q = 0$. For the shortest length the fluctuations reach 20% at large $|q|$ and around 12% at $q = 0$.

IV. APPLICATION TO THE EXCHANGE RATE BETWEEN THE US DOLLAR AND THE EURO

As mentioned before there is a twofold interest in the study of short time series: (i) many series of interest are short and (ii) the dynamics of longer series may change with time requiring the analysis of shorter pieces of the long series to get an insight into this process. Both aspects are relevant in the case of exchange rates because some important currencies are either relatively new or its exchange to other currency have been subject to new policies in the near past. The former is the case for the Euro which was born at the beginning of 1999. The
multifractality of the US Dollar to Euro exchange rate has been studied in [25] with a different emphasis than here. Other exchange rates involving asian currencies have been studied in [24, 26]. In particular [26] separates the already short series in two ranges in order to study the effect of the Asian currency crisis on the fractal behavior of different Asian exchange rates.

Other more general works which do not only analyze exchange rates, but also other financial records to study the statistics of return intervals between events above a certain threshold in the context of multifractal models are presented in [29–32].

A. The time series

The daily exchange rate between the US Dollar and the Euro is analyzed with the MFDFA method. The data, shown in the upper panel of Figure 6, has been obtained from the web page of the Board of Governors of the Federal Reserve System (www.federalreserve.gov).

It contains 3420 data entries from January 4th, 1999 to August 3rd, 2012. There are approximately 250 entries each year. The analysis has been carried out not in the daily exchange rate \( r_i \) but on the logarithmic differences of the rate in consecutive days \( R_i = \ln(r_{i+1}) - \ln(r_i) \). This variable, shown in the lower panel of Figure 6, has been chosen in order to be able to compare directly with the results from [24, 25].

FIG. 7. MF DFA of the exchange rate between the US Dollar and the Euro. (a) \( q \)-order fluctuations for \( q \) 2, 0 and -2 (from top to bottom). (b) The generalized Hurst exponents obtained from an application of the MFDFA algorithm.

FIG. 8. Generalized Hurst exponents (open circles) for periods of four years of exchange rates between the US Dollar and the Euro. The first period starts with the first data available in 1999 (top panel) and the last period starts with the first available data in 2009. The solid line is the result of the MFDFA method on the complete time series from 1999 to 2012.
TABLE I. Values of the mean and standard variation for $h(q)$ in each period shown in Figure 8 as well as the corresponding values of $\Delta_{\text{max}} \equiv \max\{h(q)\} - \langle h(q) \rangle / \langle h(q) \rangle$.

| Period       | $\langle h(q) \rangle$ | $\sigma(h(q))$ | $\Delta_{\text{max}}$ |
|--------------|------------------------|----------------|---------------------|
| 1999–2002    | 0.522                  | 0.007          | 0.039               |
| 2000–2003    | 0.572                  | 0.017          | 0.057               |
| 2001–2004    | 0.531                  | 0.011          | 0.034               |
| 2002–2005    | 0.548                  | 0.012          | 0.045               |
| 2003–2006    | 0.551                  | 0.026          | 0.092               |
| 2004–2007    | 0.519                  | 0.069          | 0.244               |
| 2005–2008    | 0.566                  | 0.045          | 0.156               |
| 2006–2009    | 0.560                  | 0.052          | 0.176               |
| 2007–2010    | 0.584                  | 0.043          | 0.144               |
| 2008–2011    | 0.522                  | 0.020          | 0.100               |
| 2009–2012    | 0.507                  | 0.013          | 0.055               |

B. Analysis of the full exchange rate time series

The full time series has been analyzed using the MFDFA method. It has been found that equation 15 is fulfilled for all the range in box sizes $s$, starting from $s = 10$, for $q$ values within -5 and 5. An example for three $q$ values is shown in the upper panel of Figure 7. The fit to extract $h(q)$ has been performed from $s_{\text{min}} = 20$ to the maximum available box size $s$.

The result of the MF DFA is presented in the lower panel of Figure 7. The observed behavior is similar to that observed in the two lower panels of Figure 2 for the case of monofractal signals with Hurst exponent $H = 0.5$ and $H = 0.75$. The region of small $|q|$ has a linear behavior with a very small slope while at higher/lower values of $q$, the generalized Hurst exponent $h(q)$ increases/decreases with a higher slope. The difference between $h(q)$ at $q = 0$ and $q = \pm 5$ is less than 10% while the difference w.r.t $q = \pm 2$ is less than 2%.

Based on this behavior and the similitudes with the monofractal synthetic series, it is tempting to advance the hypothesis that the series exhibits a monofractal behavior with long-range correlations and a Hurst exponent around 0.54; i.e., close to white noise. On the other hand there are many studies with longer financial time series, including exchange rates (e.g., 28, 31) which point to a multifractal behavior, and the lower panel of Figure 7 does not show a constant $h(q)$ so one could also argue the multifractal scenario. In this case, and for this short time series, the multifractality if present would be weak.

C. Analysis of shorter sections of the time series

From the analysis of monofractal synthetic signals with $H = 0.5$ and $H = 0.75$, it was concluded that series of lengths as short as $2^{10}$ could be analyzed using MFDFA with a precision of some 5% at the largest values of $|q| < 5$, and even better precision for $-3 \leq q \leq 3$.

For the case of the exchange rates a length of $2^{10}$ corresponds to four years and a few days of data. So the MFDFA has been applied to segments of $2^{10}$ data points, where each segment started with the first available data in each year from 1999 to 2009. The last period starting in 2009 and ending in August 3rd, 2012 has 902 data points.

It has been found that it was possible to apply the MFDFA method to these shorter time series and that in each period a behavior as expected from equation 15 was found. The generalized Hurst exponents found for each period are shown in Figure 8.

To quantify somehow the behavior in each period three quantities have been computed: the mean value of each generalized Hurst exponent over all the $q$, the corresponding standard deviation and the relative maximum difference between $h(q)$ and the mean. (In each set $\{h(q)\}$ there were 51 different values of $q$ going from $q = -5$ to $q = 5$ in steps of 0.2.) The results are shown in Table I.

The first few periods present a behavior consistent with a monofractal signal with a slightly different Hurst exponent. Given the uncertainties expected from the application of MFDFA to short series is difficult to decide if the differences between these first few periods are due to the shortness of the time series – which is the likely explanation – or if they reflect some deeper dynamics – which would be very interesting.

But for the period 2004–2007, and in a lesser fashion for those periods surrounding it, the size of the standard deviation of $h(q)$ for the period or the biggest relative difference between the values of $h(q)$ and the corresponding mean are simply too big to be explained with the expectations from monofractal signals. Indeed, for this case the behavior is closer to a Log-Normal expectation.

Note that consecutive periods have a 75% overlap so their behavior is correlated. This means that the behavior observed in the period 2004–2007 could have started some time earlier and lasted for a few years. The last period shows a return to a monofractal compatible behavior, quite close to a white noise; i.e. without long-range correlations.

It is interesting to notice that around 2003 the Euro started to show signals of being a strong currency and also to notice that the current crisis with the Euro currency started around 2008. The results found here seem to indicate that the stronger phase of the Euro shows a multifractal behavior while the weaker phases are closer to a monofractal behavior with a Hurst exponent just above 0.5.

A full analysis of these tantalizing indications is outside the scope of this analysis. For the purposes of this article it is important to notice that the analysis of short time series can be successfully carried out using the MFDFA technique and that it seems that changes in the fractal behavior of exchange rate time series can be observed.
V. CONCLUSIONS

In conclusion, the performance of the MFDFA method has been studied for several mono- and multifractal models as a function of decreasing length. For all models, and all lengths, a region in \( q \) has been found where the agreement of the simulation and the theoretical predictions for the generalized Hurst exponent is of few percent. Outside these regions not only the agreement is worst, but also the results could lead to a wrong assignment of a multifractal behavior for a monofractal signal or a reduced multifractality for a multifractal signal.

The results found in this study have been applied to the daily exchange rate between the USD and the Euro. It has been found that the result of the analysis of the series spanning the 12 years of existence of the Euro is compatible both, with a monofractal behavior close to white noise and with a weak multifractal behavior. Furthermore the analysis of 4 year periods seems to indicate that sometime before 2004 the dynamics of the exchange rate changed from either (a) a mono- to a multifractal behavior and that after some years the dynamics have changed back to a monofractal behavior or (b) the multifractal behavior changed from weak to strong to weak in the mentioned periods.

These results show that with due care, the analysis of short time series is possible with MFDFA and that the analysis of short periods of longer time series could help to discover a change of dynamics in the system under study.

ACKNOWLEDGMENTS

This work has been partially supported by Conacyt Mexico, and by the project LK11209 from the MSMT of the Czech Republic.

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