We summarize results concerning the Bernstein property of differential equations.

In this short overview we will look at entire solutions of partial differential equations of second order. We say a solution to be entire if it is defined over the entire plane ($\mathbb{R}^2$) or over the entire space ($\mathbb{R}^n$).

As we will see, some differential equations possess only linear functions as entire solutions, i.e., in these cases the linearity of an entire solution follows from its mere existence, without any boundedness conditions. If a partial differential equation has only affine linear functions as entire solutions, we say that it has the Bernstein property, according to a celebrated result of S. N. Bernstein [B], which states that every $C^2$-solution of the minimal surface equation
\[
(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0
\]
over the entire plane $\mathbb{R}^2$ is necessarily affine linear.

We start with the following operator introduced in [ZT]:
\[
L_{\gamma, \varepsilon}[u] := (2\varepsilon + (\gamma + 1)u_x^2 + (\gamma - 1)u_y^2)\ u_{xx} + 4u_x u_y u_{xy} + (2\varepsilon + (\gamma - 1)u_x^2 + (\gamma + 1)u_y^2)\ u_{yy}
\]
with $\gamma, \varepsilon \in \mathbb{R}$ and consider the equation
\[
L_{\gamma, \varepsilon}[u] = 0 \quad \text{over } \mathbb{R}^2.
\]
Without loss of generality we can choose $\varepsilon \in \{-1; 0; 1\}$, for we can obtain entire $C^2$-solutions of $L_{\gamma, \varepsilon}[u] = 0$ (with $\varepsilon \neq 0$) via an appropriate scaling of the solutions of $L_{\gamma, \pm 1}[u] = 0$ and vice versa.

Our equation $L_{\gamma, \varepsilon}[u] = 0$ is elliptic if $\varepsilon \gamma > 0$ and $|\gamma| \geq 1$. We start with this case and consider other cases later.

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*Peter Lewintan*, University of Duisburg-Essen, Germany.
First we choose $\gamma = \varepsilon = -1$, so that $L_{-1,-1}[u] = 0$ corresponds to the familiar minimal surface equation over $\mathbb{R}^2$. As we have already mentioned, it has the Bernstein property. The extension of this result to higher dimensions is well-known:

The Bernstein theorem was extended to $n \leq 7$, i.e., each entire $C^2$-solution of the minimal surface equation

$$\text{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) = 0 \quad \text{over } \mathbb{R}^n$$

has to be affine linear, cf. [dG] for $n = 3$, [Al] for $n = 4$ and [Sm] for $n \leq 7$.

Surprisingly, the Bernstein theorem fails for dimensions $n \geq 8$ as there exist entire non-linear solutions to the minimal surface equation, cf. [BdG]. However, under suitable growth conditions on the solution $u$ or its gradient $Du$, one can prove Bernstein-type theorems in every dimension $n \in \mathbb{N}$, cf. e.g. [BdGM], [Mo], [CNS], [Ni], [EH].

Let us now turn to higher codimension $k > 1$, so that instead of the minimal surface equation, we consider a system of partial differential equations, the so-called minimal surface system.

Already in the simplest case of dimension $n = 2$ and codimension $k = 2$ we obtain several entire non-linear solutions $f \in C^2(\mathbb{R}^2, \mathbb{R}^2)$ of the corresponding system

$$(1 + |f_y|^2)f_{xx} - 2f_x \cdot f_y f_{xy} + (1 + |f_x|^2)f_{yy} = 0,$$

for every holomorphic function $f : \mathbb{C} \to \mathbb{C}$, regarded as a map $\mathbb{R}^2 \to \mathbb{R}^2$, solves it. If $k > 1$, the boundedness of the gradient is a sufficient condition for a Bernstein-type theorem only when $n \leq 3$, cf. [CO] and [Fin]. An analogous result for $n \geq 4$ is wrong, as follows from the Lawson-Osserman cone [LO]. Under sufficiently strong assumptions, one can still achieve the linearity of solutions, cf. e.g. [HJW], [JX], [Wg], [JXY].

Returning to the original Bernstein theorem, we can extend it to further differential equations. Such classes of elliptic differential equations (over $\mathbb{R}^2$), entire $C^2$-solutions of which are necessarily affine linear, were given e.g. in [Be], [F], [Je], [Si1], [Si2]. The minimal surface equation is included in all these classes. Equations of "minimal surface type" possess the Bernstein property for $n \leq 7$, cf. [Si1], and we need additional growth conditions in other dimensions, cf. [Win].

For an elaborated account of the minimal surface case we refer to the monograph [DHT].

Now we choose $\gamma = \varepsilon = 1$, so that $L_{1,1}[u] = 0$ corresponds to the "wrong minimal surface equation"

$$(1 + u_x^2)u_{xx} + 2u_x u_y u_{xy} + (1 + u_y^2)u_{yy} = 0.$$

In [Si2] Simon posed the question whether this equation has the Bernstein property. We can answer in two different ways:

- By the separation ansatz $u(x,y) = g(x) + h(y)$ we construct entire non-linear $C^2$-solutions of this equation explicitly. In a similar manner we can determine further (not necessarily elliptic) differential equations without the Bernstein property, cf. [Lew].

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1This cone is an example for a non-analytic Lipschitz solution in higher codimensions.
We use an explicit criterion of J. C. C. Nitsche and J. A. Nitsche [NN], which ensures the existence of entire non-linear $C^2$-solutions of certain elliptic differential equations including the wrong minimal surface equation.

The Nitsche criterion states:

The **Euler-Lagrange** equation arising from the regular variational integral

$$\int_{\mathbb{R}^2} F(|Du|^2) dx$$

has entire non-linear $C^2$-solutions if the integral

$$\int_1^\infty \frac{1 + w\lambda(w)}{2 + w\lambda(w)} \cdot \frac{dw}{w} \quad \text{with} \quad \lambda(w) := 2 \frac{F''(w)}{F'(w)}$$

diverges.

With this criterion we can treat all the other combinations of $\gamma$ and $\varepsilon$ in the elliptic case, more precisely:

In the elliptic case ($\varepsilon \gamma > 0$ and $|\gamma| \geq 1$) we obtain our equation $L_{\gamma, \varepsilon}[u] = 0$ as Euler-Lagrange equation of the functional

$$\mathcal{F}_{\gamma, \varepsilon}(u) := \int_{\mathbb{R}^2} F_{\gamma, \varepsilon}(|Du|^2) dx$$

by setting $w := |Du|^2$ and

$$F_{\gamma, \varepsilon}(w) := \begin{cases} (2|\varepsilon| + |\gamma - 1|w)^{\frac{\gamma}{\gamma - 1}}, & \text{for } (\gamma > 1, \varepsilon > 0) \text{ or } (\gamma \leq -1, \varepsilon < 0), \\ e^{\frac{\gamma}{|\varepsilon|}}, & \text{for } (\gamma = 1, \varepsilon > 0). \end{cases}$$

In all these cases we gain

$$\lambda_{\gamma, \varepsilon}(w) = 2 \frac{F''_{\gamma, \varepsilon}(w)}{F'_{\gamma, \varepsilon}(w)} = \frac{2}{2\varepsilon + (\gamma - 1)w}.$$ 

The integral

$$\int_1^\infty \frac{1 + w\lambda_{\gamma, \varepsilon}(w)}{2 + w\lambda_{\gamma, \varepsilon}(w)} \cdot \frac{dw}{w} = \frac{1}{2} \int_1^\infty \left( \frac{1}{2\varepsilon + \gamma w} + \frac{1}{w} \right) dw$$

diverges for all admissible $\gamma \neq -1$. By the Nitsche criterion, in all these cases there exist entire non-linear $C^2$-solutions of the corresponding Euler-Lagrange equation, i.e. for $\gamma \geq 1, \varepsilon > 0$ and $\gamma < -1, \varepsilon < 0$ resp. the equation $L_{\gamma, \varepsilon}[u] = 0$ does not have the Bernstein property.

If we now let $\varepsilon$ tend to zero with $|\gamma| > 1$, the integrand converges to

$$F_{\gamma, 0}(w) = (|\gamma - 1|w)^{\frac{\gamma}{\gamma - 1}}.$$
By the substitution $p = \frac{2\gamma}{\gamma - 1}$ (for $\gamma \neq 1$) we obtain

$$F_{\gamma,0}(|Du|^2) = c(p) \frac{1}{p} |Du|^p.$$ 

Thus, we can associate our functional $F_{\gamma,0}$ with the functional

$$F_p(u) := \frac{1}{p} \int_{\mathbb{R}^2} |Du|^p \, dx \quad \text{where} \quad p = \frac{2\gamma}{\gamma - 1}.$$ 

The minimizers of the latter functional are the so-called $p$-harmonic functions, the solutions of

$$\Delta_p u := \text{div}(|Du|^{p-2}Du) = 0 \quad \text{over} \quad \mathbb{R}^2.$$ 

We can regard solutions of $L_{\gamma,0}[u] = 0$ as solutions of $\Delta_p u = 0$. It is common to introduce the $p$-harmonic functions in the weak sense and not as solutions of $L_{\gamma,0}[u] = 0$. As far as the author is aware, it is not clear whether entire $C^2$-solutions of $L_{\gamma,0}[u] = 0$ with $|\gamma| > 1$ are necessarily affine linear. However, under suitable growth conditions, cf. [KSZ], each entire $p$-harmonic function is affine linear. This statement also holds true in higher dimensions.

For $\gamma = -1$ we get $p = 1$ and the equation $L_{-1,0}[u] = 0$, i.e.

$$u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy} = 0,$$

which corresponds to

$$\Delta_1 u := \text{div} \left( \frac{Du}{|Du|} \right) = 0.$$ 

The equation $L_{-1,0}[u] = 0$ does not have the Bernstein property, for there are solutions of the form $u(x, y) = g(x)$, with an arbitrary $g \in C^2(\mathbb{R}, \mathbb{R})$, or $u(x, y) = e^x + y$.

In a more interesting case $\gamma$ tends to 1 and $p$ tends to $+\infty$. Indeed, the equation $L_{1,0}[u] = 0$ corresponds to the equation of the so-called $\infty$-harmonic function over $\mathbb{R}^2$

$$u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0.$$ 

The latter has the Bernstein property, cf. [Ar]. One can extend this result to higher dimensions if additional regularity be assumed, more precisely:

Each entire $C^4$-solution of

$$\Delta_{\infty} u := \sum_{j,k=1}^n u_{x_j} u_{x_k} u_{x_jx_k} = 0 \quad \text{over} \quad \mathbb{R}^n$$

is necessarily affine linear, cf. [Yu]. It is not clear whether $C^2$-regularity suffices here. If so, the equation $\Delta_{\infty} u = 0$ would have the Bernstein property in all dimensions, in contrast to the minimal surface equation.

Concerning all the other combinations of $\varepsilon$ and $\gamma$, we state the following:
For \( \varepsilon = \gamma = 0 \) we have entire non-linear \( C^2 \)-solutions: \( u(x, y) = x^2 + y^2 \) solves \( L_{0,0}[u] = 0 \).

Also, for \( \varepsilon > 0 \) and \( \gamma = -1 \) our equation does not have the Bernstein property, for \( L_{-1,1}[u] = 0 \) admits solutions of the form \( u(x, y) = x + h(y) \), with an arbitrary \( h \in C^2(\mathbb{R}, \mathbb{R}) \). However, \( L_{-1,1}[u] = 0 \) corresponds to the maximal surface equation

\[
(1 - u_y^2)u_{xx} + 2u_xu_yu_{xy} + (1 - u_x^2)u_{yy} = 0,
\]

and under constraint \( |Du|_{C^0} < 1 \) its entire solutions are affine linear. This is also valid in higher dimensions, cf. \([Ca]\) and \([CY]\).

For \( \varepsilon < 0 \) and \( \gamma = 1 \) our equation \( L_{\gamma, \varepsilon}[u] = 0 \) arises in the study of isentropic irrotational steady plane flows, cf. \([CF]\).

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