STRATIFIED CRITICAL POINTS ON THE REAL MILNOR FIBRE AND INTEGRAL-GEOMETRIC FORMULAS

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Dedicated to professor David Trotman on his 60th birthday

Abstract. Let \((X, 0) \subset (\mathbb{R}^n, 0)\) be the germ of a closed subanalytic set and let \(f\) and \(g : (X, 0) \to (\mathbb{R}, 0)\) be two subanalytic functions. Under some conditions, we relate the critical points of \(g\) on the real Milnor fibre \(X \cap f^{-1}(\delta) \cap B_\epsilon, 0 < |\delta| \ll \epsilon \ll 1\), to the topology of this fibre and other related subanalytic sets. As an application, when \(g\) is a generic linear function, we obtain an “asymptotic” Gauss-Bonnet formula for the real Milnor fibre of \(f\). From this Gauss-Bonnet formula, we deduce “infinitesimal” linear kinematic formulas.

1. Introduction

Let \(F = (f_1, \ldots, f_k) : (\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0), 2 \leq k \leq n\), be a complete intersection with isolated singularity. The Lê-Greuel formula [21, 22] states that

\[
\mu(F') + \mu(F) = \dim \mathcal{O}_{\mathbb{C}^n, 0} / I,
\]

where \(F' : (\mathbb{C}^n, 0) \to (\mathbb{C}^{k-1}, 0)\) is the map with components \(f_1, \ldots, f_{k-1}\), \(I\) is the ideal generated by \(f_1, \ldots, f_{k-1}\) and the \((k \times k)\)-minors \(\frac{\partial(f_1, \ldots, f_k)}{\partial(x_1, \ldots, x_k)}\) and \(\mu(F)\) (resp. \(\mu(F')\)) is the Milnor number of \(F\) (resp \(F'\)). Hence the Lê-Greuel formula gives an algebraic characterization of a topological data, namely the sum of two Milnor numbers. However, since the right-hand side of the above equality is equal to the number of critical points of \(f_k\), counted with multiplicity, on the Milnor fibre of \(F'\), the Lê-Greuel formula can be also viewed as a topological characterization of this number of critical points.

Many works have been devoted to the search of a real version of the Lê-Greuel formula. Let us recall them briefly. We consider an analytic map-germ \(F = (f_1, \ldots, f_k) : (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0), 2 \leq k \leq n\), and we denote by \(F'\) the map-germ \((f_1, \ldots, f_{k-1}) : (\mathbb{R}^n, 0) \to (\mathbb{R}^{k-1}, 0)\). Some authors investigated the following difference:

\[
D_{\delta, \delta'} = \chi(F'^{-1}(\delta) \cap \{f_k \geq \delta'\} \cap B_\epsilon) - \chi(F'^{-1}(\delta) \cap \{f_k \leq \delta'\} \cap B_\epsilon),
\]

where \((\delta, \delta')\) is a regular value of \(F\) such that \(0 \leq |\delta'| \ll |\delta| \ll \epsilon\).

In [12], we proved that

\[
D_{\delta, \delta'} \equiv \dim \mathcal{O}_{\mathbb{R}^n, 0} / I \mod 2,
\]
where $\mathcal{O}_{\mathbb{R}^n,0}$ is the ring of analytic function-germs at the origin and $I$ is the ideal generated by $f_1, \ldots, f_{k-1}$ and all the $k \times k$ minors $\frac{\partial (f_k, f_1, \ldots, f_{k-1})}{\partial (x_{i_1}, \ldots, x_{i_k})}$. This is only a mod 2 relation and we may ask if it is possible to get a more precise relation.

When $k = n$ and $f_k = x_1^2 + \cdots + x_n^2$, according to Aoki et al. ([1], [3]), $D_{\delta,0} = \chi (F_t^{-1}(\delta) \cap B_\epsilon) = 2\deg H$ and $2\deg H$ is the number of semi-branches of $F_{t-1}(0)$, where

$$H = \frac{\partial (f_n, f_1, \ldots, f_{n-1})}{\partial (x_1, \ldots, x_n)} (f_1, \ldots, f_{n-1}).$$

They proved a similar formula in the case $f_k = x_n$ in [2] and Szafraniec generalized all these results to any $f_k$ in [23].

When $k = 2$ and $f_2 = x_1$, Fukui [18] stated that $D_{\delta,0} = -\text{sign}(-\delta)^n \deg H$, where $H = (f_1, \frac{\partial f_1}{\partial x_2}, \ldots, \frac{\partial f_1}{\partial x_n})$. Several generalizations of Fukui’s formula are given in [19], [11], [20] and [13].

In all these papers, the general idea is to count algebraically the critical points of a Morse perturbation of $f_k$ on $F_{t-1}(\delta) \cap B_\epsilon$ and to express this sum in two ways: as a difference of Euler characteristics and as a topological degree. Using the Eisenbud-Levine formula [16], this latter degree can be expressed as a signature of a quadratic form and so, we obtain an algebraic expression for $D_{\delta,\delta'}$.

In this paper, we give a real and stratified version of the Lê-Greuel formula. We restrict ourselves to the topological aspect and relate a sum of indices of critical points on a real Milnor fibre to some Euler characteristics (this is also the point of view adopted in [7]). More precisely, we consider a germ of a closed subanalytic set $(X, 0) \subset (\mathbb{R}^n, 0)$ and a subanalytic function $f : (X, 0) \to (\mathbb{R}, 0)$. We assume that $X$ is contained in an open set $U$ of $\mathbb{R}^n$ and that $f$ is the restriction to $X$ of a $C^2$-subanalytic function $F : U \to \mathbb{R}$. We denote by $X^f$ the set $X \cap f^{-1}(0)$ and we equip $X$ with a Thom stratification adapted to $X^f$. If $0 < |\delta| \ll \epsilon \ll 1$ then the real Milnor fibre of $f$ is defined by

$$M_f^\delta = f^{-1}(\delta) \cap X \cap B_\epsilon.$$

We consider another subanalytic function $g : (X, 0) \to (\mathbb{R}, 0)$ and we assume that it is the restriction to $X$ of a $C^2$-subanalytic function $G : U \to \mathbb{R}$. We denote by $X^g$ the intersection $X \cap g^{-1}(0)$. Under two conditions on $g$, we study the topological behaviour of $g_{|M_f^\epsilon}$.

We recall that if $Z \subset \mathbb{R}^n$ is a closed subanalytic set, equipped with a Whitney stratification and $p \in Z$ is an isolated critical point of a subanalytic function $\phi : Z \to \mathbb{R}$, restriction to $Z$ of a $C^2$-subanalytic function $\Phi$, then the index of $\phi$ at $p$ is defined as follows:

$$\text{ind} (\phi, Z, p) = 1 - \chi (Z \cap \{ \phi = \phi(p) - \eta \} \cap B_\epsilon (p)),$$
where $0 < \eta \ll \epsilon \ll 1$ and $B_\epsilon(p)$ is the closed ball of radius $\epsilon$ centered at $p$. Let $p_1^\delta, \ldots, p_r^\delta$ be the critical points of $g$ on $X \cap f^{-1}(\delta) \cap \tilde{B}_\epsilon$, where $\tilde{B}_\epsilon$ denotes the open ball of radius $\epsilon$. We set

$$I(\delta, \epsilon, g) = \sum_{i=1}^r \text{ind}(g, X \cap f^{-1}(\delta), p_i^\delta),$$

$$I(\delta, \epsilon, -g) = \sum_{i=1}^r \text{ind}(-g, X \cap f^{-1}(\delta), p_i^\delta).$$

Our main theorem (Theorem 3.10) is the following:

$$I(\delta, \epsilon, g) + I(\delta, \epsilon, -g) = 2\chi(M_{\delta, \epsilon}^f) - \chi(X \cap f^{-1}(\delta) \cap S_\epsilon) - \chi(X^g \cap f^{-1}(\delta) \cap S_\epsilon).$$

As a corollary (Corollary 3.11), when $f : (X, 0) \to (\mathbb{R}, 0)$ has an isolated stratified critical point at 0, we obtain that

$$I(\delta, \epsilon, g) + I(\delta, \epsilon, -g) = 2\chi(M_{\delta, \epsilon}^f) - \chi(\text{Lk}(X^f)) - \chi(\text{Lk}(X^f \cap X^g)),$$

where $\text{Lk}(-)$ denotes the link at the origin.

Then we apply these results when $g$ is a generic linear form to get an asymptotic Gauss-Bonnet formula for $M_{\delta, \epsilon}^f$ (Theorem 4.5). In the last section, we use this asymptotic Gauss-Bonnet formula to prove infinitesimal linear kinematic formulas for closed subanalytic germs (Theorem 5.5), that generalize the Cauchy-Crofton formula for the density due to Comte [8].

The paper is organized as follows. In Section 2, we prove several lemmas about critical points on the link of a subanalytic set. Section 3 contains real stratified versions of the Lé-Greuel formula. In Section 4, we establish the asymptotic Gauss-Bonnet formula and in Section 5, the infinitesimal linear kinematic formulas.

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2. LEMMAS ON CRITICAL POINTS ON THE LINK OF A STRATUM

In this section, we study the behaviour of the critical points of a $C^2$-subanalytic function on the link of stratum that contains 0 in its closure, for a generic choice of the distance function to the origin.

Let $Y \subset \mathbb{R}^n$ be a $C^2$-subanalytic set such that 0 belongs to its closure $\overline{Y}$. Let $\theta : \mathbb{R}^n \to \mathbb{R}$ be a $C^2$-subanalytic function such that $\theta(0) = 0$. We will first study the behaviour of the critical points of $\theta|_Y : Y \to \mathbb{R}$ in
the neighborhood of 0, and then the behaviour of the critical points of the restriction of \( \theta \) to the link of 0 in \( Y \).

**Lemma 2.1.** The critical points of \( \theta|_Y \) lie in \( \{ \theta = 0 \} \) in a neighborhood of 0.

**Proof.** By the Curve Selection Lemma, we can assume that there is a \( C^1 \)-subanalytic curve \( \gamma : [0, \nu] \to Y \) such that \( \gamma(0) = 0 \) and \( \gamma(t) \) is a critical point of \( \theta|_Y \) for \( t \in ]0, \nu[ \). Therefore, we have

\[
(\theta \circ \gamma)'(t) = \langle \nabla \theta|_Y(\gamma(t)), \gamma'(t) \rangle = 0,
\]

since \( \gamma'(t) \) is tangent to \( Y \) at \( \gamma(t) \). This implies that \( \theta \circ \gamma(t) = \theta \circ \gamma(0) = 0 \). \( \square \)

Let \( \rho : \mathbb{R}^n \to \mathbb{R} \) be another \( C^2 \)-subanalytic function such that \( \rho^{-1}(a) \) intersects \( Y \) transversally. Then the set \( Y \cap \{ \rho \leq a \} \) is a manifold with boundary. Let \( p \) be a critical point of \( \theta|_{Y \cap \{ \rho \leq a \}} \) which lies in \( Y \cap \{ \rho = a \} \) and which is not a critical point of \( \theta|_Y \). This implies that

\[
\nabla \theta|_Y(p) = \lambda(p) \nabla \rho|_Y(p),
\]

with \( \lambda(p) \neq 0 \).

**Definition 2.2.** We say that \( p \in Y \cap \{ \rho = a \} \) is an outwards-pointing (resp. inwards-pointing) critical point of \( \theta|_{Y \cap \{ \rho \leq a \}} \) if \( \lambda(p) > 0 \) (resp. \( \lambda(p) < 0 \)).

Now let us assume that \( \rho : \mathbb{R}^n \to \mathbb{R} \) is a distance function to the origin which means that \( \rho \geq 0 \) and \( \rho^{-1}(0) = \{ 0 \} \) in a neighborhood of 0. By Lemma 2.1 we know that for \( \epsilon > 0 \) small enough, the level \( \rho^{-1}(\epsilon) \) intersects \( Y \) transversally. Let \( p^\epsilon \) be a critical point of \( \theta|_{Y \cap \rho^{-1}(\epsilon)} \) such that \( \theta(p^\epsilon) \neq 0 \). This means that there exists \( \lambda(p^\epsilon) \) such that

\[
\nabla \theta|_Y(p^\epsilon) = \lambda(p^\epsilon) \nabla \rho|_Y(p^\epsilon).
\]

Note that \( \lambda(p^\epsilon) \neq 0 \) because \( \nabla \theta|_Y(p^\epsilon) \neq 0 \) for \( \theta(p^\epsilon) \neq 0 \).

**Lemma 2.3.** The point \( p^\epsilon \) is an outwards-pointing (resp. inwards-pointing) for \( \theta|_{Y \cap \{ \rho \leq \epsilon \}} \) if and only if \( \theta(p^\epsilon) > 0 \) (resp. \( \theta(p^\epsilon) < 0 \)).

**Proof.** Let us assume that \( \lambda(p^\epsilon) > 0 \). By the Curve Selection Lemma, there exists a \( C^1 \)-subanalytic curve \( \gamma : [0, \nu] \to Y \) passing through \( p^\epsilon \) such that \( \gamma(0) = 0 \) and for \( t \neq 0 \), \( \gamma(t) \) is a critical point of \( \theta|_{Y \cap \{ \rho = \rho(\gamma(t)) \}} \) with \( \lambda(\gamma(t)) > 0 \). Therefore we have

\[
(\theta \circ \gamma)'(t) = \langle \nabla \theta|_Y(\gamma(t)), \gamma'(t) \rangle = \lambda(\gamma(t)) \langle \nabla \rho|_Y(\gamma(t)), \gamma'(t) \rangle.
\]

But \( (\rho \circ \gamma)' > 0 \) for otherwise \( (\rho \circ \gamma)' \leq 0 \) and \( \rho \circ \gamma \) would be decreasing. Since \( \rho(\gamma(t)) \) tends to 0 as \( t \) tends to 0, this would imply that \( \rho \circ \gamma(t) \leq 0 \), which is impossible. We can conclude that \( (\theta \circ \gamma)' > 0 \) and that \( \theta \circ \gamma \) is strictly increasing. Since \( \theta \circ \gamma(t) \) tends to 0 as \( t \) tends to 0, we see that \( \theta \circ \gamma(t) > 0 \) for \( t > 0 \). Similarly if \( \lambda(p^\epsilon) < 0 \) then \( \theta(p^\epsilon) < 0 \). \( \square \)
Now we will study these critical points for a generic choice of the distance function. We denote by \( \text{Sym}(\mathbb{R}^n) \) the set of symmetric \( n \times n \)-matrices with real entries, by \( \text{Sym}^+(\mathbb{R}^n) \) the open dense subset of such matrices with non-zero determinant and by \( \text{Sym}^{+,*}(\mathbb{R}^n) \) the open subset of these invertible matrices that are positive definite or negative definite. Note that these sets are semi-algebraic. For each \( A \in \text{Sym}^{+,*}(\mathbb{R}^n) \), we denote by \( \rho_A \) the following quadratic form:

\[
\rho_A(x) = \langle Ax, x \rangle.
\]

We denote by \( \Gamma_{\theta,A}^Y \) the following subanalytic polar set:

\[
\Gamma_{\theta,A}^Y = \left\{ x \in Y \mid \text{rank} \left[ \nabla_{\theta Y}(x), \nabla_{\rho_A|Y}(x) \right] < 2 \right\},
\]

and by \( \Sigma_Y^\theta \) the set of critical points of \( \theta|_Y \). Note that \( \Sigma_Y^\theta \subset \{ \theta = 0 \} \) by Lemma 2.1.

**Lemma 2.4.** For almost all \( A \) in \( \text{Sym}^{+,*}(\mathbb{R}^n) \), \( \Gamma_{\theta,A}^Y \setminus (\Sigma_Y^\theta \cup \{ 0 \}) \) is a \( C^1 \)-subanalytic curve (possible empty) in a neighborhood of 0.

**Proof.** We can assume that \( \dim Y > 1 \). Let

\[
Z = \left\{ (x, A) \in \mathbb{R}^n \times \text{Sym}^{+,*}(\mathbb{R}^n) \mid x \in Y \setminus (\Sigma_Y^\theta \cup \{ 0 \}) \right\}
\]

and \( \text{rank} \left[ \nabla_{\theta Y}(x), \nabla_{\rho_A|Y}(x) \right] < 2 \} \). Let \((y, B)\) be a point in \( Z \). We can suppose that around \( y \), \( Y \) is defined by the vanishing of \( k \) subanalytic functions \( f_1, \ldots, f_k \) of class \( C^2 \). Hence in a neighborhood of \((y, B)\), \( Z \) is defined be the vanishing of \( f_1, \ldots, f_k \) and the minors

\[
\frac{\partial(f_1, \ldots, f_k, \theta, \rho_A)}{\partial(x_{i_1}, \ldots, x_{i_k+2})}.
\]

Furthermore, since \( y \) does not belong to \( \Sigma_Y^\theta \), we can assume that

\[
\frac{\partial(f_1, \ldots, f_k, \theta)}{\partial(x_1, \ldots, x_k, x_{k+1})} \neq 0,
\]

in a neighborhood of \( y \). Therefore \( Z \) is locally defined by \( f_1 = \cdots = f_k = 0 \) and

\[
\frac{\partial(f_1, \ldots, f_k, \theta, \rho_A)}{\partial(x_1, \ldots, x_{k+1}, x_{k+2})} = \cdots = \frac{\partial(f_1, \ldots, f_k, \theta, \rho_A)}{\partial(x_1, \ldots, x_{k+2}, x_n)} = 0.
\]

Let us write \( M = \frac{\partial(f_1, \ldots, f_k, \theta)}{\partial(x_1, \ldots, x_{k+1}, x_{k+1})} \) and for \( i \in \{ k+2, \ldots, n \} \), \( m_i = \frac{\partial(f_1, \ldots, f_k, \theta, \rho_A)}{\partial(x_1, \ldots, x_{k+2}, x_i)} \). If \( A = [a_{ij}] \) then

\[
\rho_A(x) = \sum_{i=1}^{n} a_{ii} x_i^2 + 2 \sum_{i \neq j} a_{ij} x_i x_j,
\]

and so \( \frac{\partial \rho_A}{\partial x_i} = 2 \sum_{j=1}^{n} a_{ij} x_j \). For \( i \in \{ k+1, \ldots, n \} \) and \( j \in \{ 1, \ldots, n \} \), we have

\[
\frac{\partial m_i}{\partial a_{ij}} = 2 x_j M.
\]
Since $y \neq 0$, one of the $x_j$’s does not vanish in the neighborhood of $y$ and we can conclude that the rank of
\[ [\nabla f_1(x), \ldots, \nabla f_k(x), \nabla m_{k+2}(x, A), \ldots, \nabla m_n(x, A)] \]
is $n - 1$ and that $Z$ is a $C^1$-subanalytic manifold of dimension $\frac{n(n+1)}{2} + 1$.
Now let us consider the projection $\pi_2 : Z \to \text{Sym}^{+,*}(\mathbb{R}^n)$, $(x, A) \mapsto A$. Bertini-Sard’s theorem implies that the set $D_{\pi_2}$ of critical values of $\pi_2$ is a subanalytic set of dimension strictly less than $\frac{n(n+1)}{2}$. Hence, for all $A \notin D_{\pi_2}$, $\pi_2^{-1}(A)$ is a $C^1$-subanalytic curve (possibly empty). But this set is exactly $\Gamma_{\theta, A}^Y \setminus (\Sigma^Y_\theta \cup \{0\})$.

Let $R \subset Y$ be a subanalytic set of dimension strictly less than $\dim Y$. We will need the following lemma.

**Lemma 2.5.** For almost all $A$ in $\text{Sym}^{+,*}(\mathbb{R}^n)$, $\Gamma_{\theta, A}^Y \setminus (\Sigma^Y_\theta \cup \{0\}) \cap R$ is a subanalytic set of dimension at most 0 in a neighborhood of 0.

**Proof.** Let us put $l = \dim Y$. Since $R$ admits a locally finite subanalytic stratification, we can assume that $R$ is a $C^2$-subanalytic manifold of dimension $d$ with $d < l$. Let $W$ be the following subanalytic set:

\[ W = \left\{ (x, A) \in \mathbb{R}^n \times \text{Sym}^{+,*}(\mathbb{R}^n) \mid x \in R \setminus (\Sigma^Y_\theta \cup \{0\}) \right\} \]

and rank $[\nabla \theta_{|Y}(x), \nabla \rho_A|_{Y}(x)] < 2$.

Using the same method as in the previous lemma, we can prove that $W$ is a $C^1$-subanalytic manifold of dimension $\frac{n(n+1)}{2} + d - l$ and conclude, remarking that $d - l \leq -1$. \qed

Now we introduce a new $C^2$-subanalytic function $\beta : \mathbb{R}^n \to \mathbb{R}$ such that $\beta(0) = 0$. We denote by $\Gamma_{\theta, \beta, A}^Y$ the following subanalytic polar set:

\[ \Gamma_{\theta, \beta, A}^Y = \left\{ x \in Y \mid \text{rank} \left[ \nabla \theta_{|Y}(x), \nabla \beta_{|Y}(x), \nabla \rho_A|_{Y}(x) \right] < 3 \right\}, \]

and by $\Gamma_{\theta, \beta}^Y$ the following subanalytic polar set:

\[ \Gamma_{\theta, \beta}^Y = \left\{ x \in Y \mid \text{rank} \left[ \nabla \theta_{|Y}(x), \nabla \beta_{|Y}(x) \right] < 2 \right\}. \]

**Lemma 2.6.** For almost all $A$ in $\text{Sym}^{+,*}(\mathbb{R}^n)$, $\Gamma_{\theta, \beta, A}^Y \setminus (\Gamma_{\theta, \beta}^Y \cup \{0\})$ is a $C^1$-subanalytic set of dimension at most 2 (possibly empty) in a neighborhood of 0.

**Proof.** We can assume that $\dim Y > 2$. Let

\[ Z = \left\{ (x, A) \in \mathbb{R}^n \times \text{Sym}^{+,*}(\mathbb{R}^n) \mid x \in Y, \text{rank} \left[ \nabla \theta_{|Y}(x), \nabla \beta_{|Y}(x) \right] = 2 \right\} \]

and rank $[\nabla \theta_{|Y}(x), \nabla \beta_{|Y}(x), \nabla \rho_A|_{Y}(x)] < 3$.

Let $(y, B)$ be a point in $Z$. We can suppose that around $y$, $Y$ is defined by the vanishing of $k$ subanalytic functions $f_1, \ldots, f_k$ of class $C^2$. Hence in a
neighborhood of \((y, B)\), \(Z\) is defined by the vanishing of \(f_1, \ldots, f_k\) and the minors

\[
\frac{\partial(f_1, \ldots, f_k, \theta, \beta, \rho_A)}{\partial(x_{i_1}, \ldots, x_{i_{k+3}})}.
\]

Since \(y\) does not belong to \(\Gamma^{Y}_{\theta, \beta}\), we can assume that

\[
\frac{\partial(f_1, \ldots, f_k, \theta, \beta)}{\partial(x_1, \ldots, x_k, x_{k+1}, x_{k+2})} \neq 0,
\]
in a neighborhood of \(y\). Therefore \(Z\) is locally defined by \(f_1, \ldots, f_k = 0\) and

\[
\frac{\partial(f_1, \ldots, f_k, \theta, \beta, \rho_A)}{\partial(x_1, \ldots, x_{k+2}, x_{k+3})} = \cdots = \frac{\partial(f_1, \ldots, f_k, \theta, \beta, \rho_A)}{\partial(x_1, \ldots, x_{k+2}, x_n)} = 0.
\]

It is clear that we can apply the same method as Lemma 2.4 to get the result. \(\square\)

3. Lê-Greuel Type Formula

In this section, we prove the Lê-Greuel type formula announced in the introduction.

Let \((X, 0) \subset (\mathbb{R}^n, 0)\) be the germ of a closed subanalytic set and let \(f : (X, 0) \to (\mathbb{R}, 0)\) be a subanalytic function. We assume that \(X\) is contained in an open set \(U\) of \(\mathbb{R}^n\) and that \(f\) is the restriction to \(X\) of a \(C^2\)-subanalytic function \(F : U \to \mathbb{R}\). We denote by \(X^f\) the set \(X \cap f^{-1}(0)\) and by [11], we can equip \(X\) with a Thom stratification \(V = \{V_\alpha\}_{\alpha \in A}\) adapted to \(X^f\). This means that \(\{V_\alpha \in V \mid V_\alpha \not\subseteq X^f\}\) is a Whitney stratification of \(X \setminus X^f\) and that for any pair of strata \((V_\alpha, V_\beta)\) with \(V_\alpha \not\subseteq X^f\) and \(V_\beta \subset X^f\), the Thom condition is satisfied.

Let us denote by \(\Sigma_V f\) the critical locus of \(f\). It is the union of the critical loci of \(f\) restricted to each stratum, i.e. \(\Sigma_V f = \cup_\alpha \Sigma(f_{|V_\alpha})\), where \(\Sigma(f_{|V_\alpha})\) is the critical set of \(f_{|V_\alpha} : V_\alpha \to \mathbb{R}\). Since \(\Sigma_V f \subset f^{-1}(0)\) (see Lemma 2.1), the fibre \(f^{-1}(\delta)\) intersects the strata \(V_\alpha\)’s, \(V_\alpha \not\subseteq X^f\), transversally if \(\delta\) is sufficiently small. Hence it is Whitney stratified with the induced stratification \(\{f^{-1}(\delta) \cap V_\alpha \mid V_\alpha \not\subseteq X^f\}\).

By Lemma 2.1 we know that if \(\epsilon > 0\) is sufficiently small then the sphere \(S_\epsilon\) intersects \(X^f\) transversally. By the Thom condition, this implies that there exists \(\delta(\epsilon) > 0\) such that for each \(\delta\) with \(0 < |\delta| \leq \delta(\epsilon)\), the sphere \(S_\epsilon\) intersects the fibre \(f^{-1}(\delta)\) transversally as well. Hence the set \(f^{-1}(\delta) \cap B_\epsilon\) is a Whitney stratified set equipped with the following stratification:

\[
\{f^{-1}(\delta) \cap V_\alpha \cap \tilde{B}_\epsilon, f^{-1}(\delta) \cap V_\alpha \cap S_\epsilon \mid V_\alpha \not\subseteq X^f\}.
\]

Definition 3.1. We call the set \(f^{-1}(\delta) \cap X \cap B_\epsilon\), where \(0 < |\delta| \ll \epsilon \ll 1\), a real Milnor fibre of \(f\).

We will use the following notation: \(M^1_f = f^{-1}(\delta) \cap X \cap B_\epsilon\).

Now we consider another subanalytic function \(g : (X, 0) \to (\mathbb{R}, 0)\) and we assume that it is the restriction to \(X\) of a \(C^2\)-subanalytic function \(G : U \to \mathbb{R}\).
We denote by $X^g$ the intersection $X \cap g^{-1}(0)$. Under some restrictions on $g$, we will study the topological behaviour of $g|_{M^\delta, \epsilon_f}$.

First we assume that $g$ satisfies the following Condition (A):

- Condition (A): $g : (X, 0) \to (\mathbb{R}, 0)$ has an isolated stratified critical point at $0$.

This means that for each strata $V_{\alpha}$ of $V$, $g : V_{\alpha} \setminus \{0\} \to \mathbb{R}$ is a submersion in a neighborhood of the origin.

In order to give the second assumption on $g$, we need to introduce some polar sets. Let $V_{\alpha}$ be a stratum of $V$ not contained in $X_f$. Let $\Gamma_{V_{\alpha} f, g}$ be the following set:

$$\Gamma_{V_{\alpha} f, g} = \{ x \in V_{\alpha} | \text{rank}[\nabla f_{|V_{\alpha}}(x), \nabla g_{|V_{\alpha}}(x)] < 2 \} ,$$

and let $\Gamma_{f, g}$ be the union $\bigcup \Gamma_{V_{\alpha} f, g}$ where $V_{\alpha} \not\subseteq X_f$. We call $\Gamma_{f, g}$ the relative polar set of $f$ and $g$ with respect to the stratification $V$. We will assume that $g$ satisfies the following Condition (B):

- Condition (B): the relative polar set $\Gamma_{f, g}$ is a 1-dimensional $C^1$-subanalytic set (possibly empty) in a neighborhood of the origin.

Note that Condition (B) implies that $\Gamma_{f, g} \cap X_f \subset \{0\}$ in a neighborhood of the origin because the frontiers of the $\Gamma_{V_{\alpha} f, g}$’s are 0-dimensional.

From Condition (A) and Condition (B), we can deduce the following result.

Lemma 3.2. We have $\Gamma_{f, g} \cap X^g \subset \{0\}$ in a neighborhood of the origin.

Proof. If it is not the case then there is a $C^1$-subanalytic curve $\gamma : [0, \nu[ \to \Gamma_{f, g} \cap X^g$ such that $\gamma(0) = 0$ and $\gamma([0, \nu[) \subset X^g \setminus \{0\}$. We can also assume that $\gamma([0, \nu[)$ is contained in a stratum $V$. For $t \in ]0, \nu[$, we have

$$0 = (g \circ \gamma)'(t) = \langle \nabla g_{|V} (\gamma(t)), \gamma'(t) \rangle .$$

Since $\gamma(t)$ belongs to $\Gamma_{f, g}$ and $\nabla g_{|V} (\gamma(t))$ does not vanish for $g : (X, 0) \to (\mathbb{R}, 0)$ has an isolated stratified critical point at $0$, we can conclude that $\langle \nabla f_{|V}(\gamma(t)), \gamma'(t) \rangle = 0$ and that $(f \circ \gamma)'(t) = 0$ for all $t \in ]0, \nu[$. Therefore $f \circ \gamma \equiv 0$ because $f(0) = 0$ and $\gamma([0, \nu[)$ is included in $X_f$. This is impossible by the above remark. \hfill \Box

Let $B_1, \ldots, B_l$ be the connected components of $\Gamma_{f, g}$, i.e. $\Gamma_{f, g} = \bigcup_{i=1}^l B_i$. Each $B_i$ is a $C^1$-subanalytic curve along which $f$ is strictly increasing or decreasing and the intersection points of the $B_i$’s with the fibre $M_f^{\delta, \epsilon}$ are exactly the critical points (in the stratified sense) of $g$ on $X \cap f^{-1}(\delta) \cap \bar{B}_i$.

Let us write

$$M_f^{\delta, \epsilon} \cap \bigcup_{i=1}^l B_i = \{ p_1^{\delta, \epsilon}, \ldots, p_r^{\delta, \epsilon} \} .$$

Note that $r \leq l$.

Let us recall now the definition of the index of an isolated stratified critical point.
**Definition 3.3.** Let \( Z \subset \mathbb{R}^n \) be a closed subanalytic set, equipped with a Whitney stratification. Let \( p \in Z \) be an isolated critical point of a subanalytic function \( \phi : Z \to \mathbb{R} \), which is the restriction to \( Z \) of a \( C^2 \)-subanalytic function \( \Phi \). We define the index of \( \phi \) at \( p \) as follows:

\[
\text{ind}(\phi, Z, p) = 1 - \chi(Z \cap \{ \phi = \phi(p) - \eta \} \cap B_\varepsilon(p)),
\]

where \( 0 < \eta \ll \varepsilon \ll 1 \) and \( B_\varepsilon(p) \) is the closed ball of radius \( \varepsilon \) centered at \( p \).

Our aim is to give a topological interpretation to the following sum:

\[
\sum_{i=1}^{r} \text{ind}(g, X \cap f^{-1}(\delta), p^\delta_\varepsilon) + \text{ind}(-g, X \cap f^{-1}(\delta), p^\delta_\varepsilon).
\]

For this, we will apply stratified Morse theory to \( g|_{M^\delta_\varepsilon} \). Note that the points \( p^\delta_\varepsilon \)'s are not the only critical points of \( g|_{M^\delta_\varepsilon} \) and other critical points can occur on the “boundary” \( M^\delta_\varepsilon \cap S_\varepsilon \).

The next step is to study the behaviour of these “boundary” critical points for a generic choice of the distance function to the origin. Let \( \rho : \mathbb{R}^n \to \mathbb{R} \) be a \( C^2 \)-subanalytic function which is a distance function to the origin. We denote by \( \tilde{S}_\varepsilon \) the level \( \rho^{-1}(\varepsilon) \) and by \( \tilde{B}_\varepsilon \) the set \( \{ \rho \leq \varepsilon \} \). We will focus on the critical points of \( g|_{X \cap \tilde{S}_\varepsilon} \) and \( g|_{X \cap f^{-1}(\delta) \cap \tilde{S}_\varepsilon} \), with \( 0 < |\delta| \ll \varepsilon \ll 1 \).

For each stratum \( V \) of \( X^f \), let

\[
\Gamma^V_{g, \rho} = \{ x \in V \mid \text{rank}[\nabla g|_V(x), \nabla \rho|_V(x)] < 2 \},
\]

and let \( \Gamma^X_{g, \rho} \cap X^g \subset \{ 0 \} \) in a neighborhood of the origin.

**Lemma 3.4.** We have \( \Gamma^X_{g, \rho} \cap X^g \subset \{ 0 \} \) in a neighborhood of the origin.

**Proof.** Same proof as Lemma 5.2. \( \square \)

Therefore if \( \varepsilon > 0 \) is small enough, \( g|_{\tilde{S}_\varepsilon \cap X^f} \) has a finite number of critical points. They do not lie in the level \( \{ g = 0 \} \) so by Lemma 2.3 they are outwards-pointing for \( g|_{X^f \cap \tilde{B}_\varepsilon} \) if they lie in \( \{ g > 0 \} \) and inwards-pointing if they lie in \( \{ g < 0 \} \).

Let us study now the critical points of \( g|_{X \cap f^{-1}(\delta) \cap \tilde{S}_\varepsilon} \). We will need the following lemma.

**Lemma 3.5.** For every \( \varepsilon > 0 \) sufficiently small, there exists \( \delta(\varepsilon) > 0 \) such that for \( 0 < |\delta| \leq \delta(\varepsilon) \), the points \( p^\delta_\varepsilon \) lie in \( \tilde{B}_{\varepsilon/4} \).

**Proof.** Let

\[
W = \{(x, r, y) \in U \times \mathbb{R} \times \mathbb{R} \mid \rho(x) = r, y = f(x) \text{ and } x \in \Gamma_{f, g}\}.
\]
Then $W$ is a subanalytic set of $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ and since it is a graph over $\Gamma_{f,g}$, its dimension is less or equal to 1. Let

$$\pi : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$$

$$\begin{array}{ccc}
(x, r, y) & \mapsto & (r, y),
\end{array}$$

be the projection on the last two factors. Then $\pi|_W : W \to \pi(W)$ is proper and $\pi(W)$ is a closed subanalytic set in a neighborhood of the origin.

Let us write $Y_1 = \mathbb{R} \times \{0\}$ and let $Y_2$ be the closure of $\pi(W) \setminus Y_1$. Since $Y_2$ is a curve for $W$ is a curve, 0 is isolated in $Y_1 \cap Y_2$. By Łojasiewicz's inequality, there exists a constant $C > 0$ and an integer $N > 0$ such that $|y| \geq C r^N$ for $(r, y)$ in $Y_2$ sufficiently close to the origin. So if $x \in \Gamma_{f,g}$ then $|f(x)| \geq C \rho(x)^N$ if $\rho(x)$ is small enough.

Let us fix $\epsilon > 0$ small. If $0 < |\delta| \leq \frac{1}{4}(\frac{\epsilon}{1})^N$ and $x \in f^{-1}(\delta) \cap \Gamma_{f,g}$ then $\rho(x) \leq \frac{\epsilon}{4}$.

For each stratum $V \notin X^f$, let

$$\Gamma_{f,g,\rho}^V = \{ x \in V \mid \text{rank}[\nabla f|_V(x), \nabla g|_V(x), \nabla \rho|_V(x)] < 3 \},$$

and let $\Gamma_{f,g,\rho} = \bigcup_{V \notin X^f} \Gamma_{f,g,\rho}^V$. By Lemma 2.6, we can assume that $\Gamma_{f,g,\rho} \setminus \Gamma_{f,g}$ is a $C^1$-subanalytic manifold of dimension 2. Let us choose $\epsilon > 0$ small enough so that $\tilde{S}_\epsilon$ intersects $\Gamma_{f,g,\rho} \setminus \Gamma_{f,g}$ transversally. Therefore $(\Gamma_{f,g,\rho} \setminus \Gamma_{f,g}) \cap \tilde{S}_\epsilon$ is subanalytic curve. By Lemma 3.3, we can find $\delta(\epsilon) > 0$ such that $f^{-1}([\delta(\epsilon), -\delta(\epsilon)]) \cap \tilde{S}_\epsilon \cap \Gamma_{f,g,\rho}$ is empty and so

$$f^{-1}([-\delta(\epsilon), \delta(\epsilon)]) \cap (\Gamma_{f,g,\rho} \setminus \Gamma_{f,g}) \cap \tilde{S}_\epsilon = f^{-1}([-\delta(\epsilon), \delta(\epsilon)]) \cap \Gamma_{f,g,\rho} \cap \tilde{S}_\epsilon.$$

Let $C_1, \ldots, C_t$ be the connected components of $f^{-1}([-\delta(\epsilon), \delta(\epsilon)]) \cap \Gamma_{f,g,\rho} \cap \tilde{S}_\epsilon$ whose closure intersects $X^f \cap \tilde{S}_\epsilon$. Note that by Thom’s $(a.f)$-condition, for each $i \in \{1, \ldots, t\}$, $\overline{C_i} \cap X^f$ is subset of $\Gamma_{f,g,\rho}^V$. Let $z_i$ be a point in $\overline{C_i} \cap X^f$.

Since $C_i \cap X^f = \emptyset$, there exists $0 < \delta_i(\epsilon) \leq \delta(\epsilon)$ such that the fibre $f^{-1}(\delta_i)$, $0 < |\delta| \leq \delta_i(\epsilon)$, intersects $C_i$ transversally in a neighborhood of $z_i$.

Let us choose $\delta$ such that $0 < |\delta| \leq \text{Min}\{\delta_i(\epsilon) \mid i = 1, \ldots, t\}$. Then the fibre $f^{-1}(\delta)$ intersect the $C_i$’s transversally and $f^{-1}(\delta) \cap (\cup_i C_i)$ is exactly the set of critical points of $g|_{f^{-1}(\delta) \cap X \cap \tilde{S}_\epsilon}$. We have proved:

**Lemma 3.6.** For $0 < |\delta| \ll \epsilon \ll 1$, $g|_{f^{-1}(\delta) \cap X \cap \tilde{S}_\epsilon}$ has a finite number of critical points, which are exactly the points in $\Gamma_{f,g,\rho} \cap \tilde{S}_\epsilon \cap f^{-1}(\delta)$.

Let $\{s_{1,\epsilon}^\delta, \ldots, s_{u}^\delta\}$ be the set of critical points of $g|_{f^{-1}(\delta) \cap X \cap \tilde{S}_\epsilon}$.

**Lemma 3.7.** For $i \in \{1, \ldots, u\}$, $g(s_{i}^\delta) \neq 0$ and $s_{i}^\delta$ is outwards-pointing (resp. inwards-pointing) if and only if $g(s_{i}^\delta) > 0$ (resp. $g(s_{i}^\delta) < 0$).

**Proof.** Note that $s_{i}^\delta$ is necessarily outwards-pointing or inwards-pointing because $s_{i}^\delta \notin \Gamma_{f,g}$.
Assume that for each \( \delta > 0 \) small enough, there exists a point \( s_i^{\delta, \epsilon} \) such that \( g(s_i^{\delta, \epsilon}) = 0 \). Then we can construct a sequence of points \((\sigma_n)_{n \in \mathbb{N}}\) such that \( g(\sigma_n) = 0 \) and \( \sigma_n \) is a critical point of \( g|_{f^{-1}(\frac{1}{n}) \cap X \cap \tilde{S}_\epsilon} \). We can also assume that the points \( \sigma_n \)’s belong to the same stratum \( S \) and that they tend to \( \sigma \in V \) where \( V \subseteq X^f \) and \( V \subset \partial \tilde{S} \). Therefore we have a decomposition:

\[
\nabla g|_S(\sigma_n) = \lambda_n \nabla f|_S(\sigma_n) + \mu_n \nabla \rho|_S(\sigma_n).
\]

Now by Whitney’s condition (a), \( T_{\sigma_n}S \) tends to a linear space \( T \) such that \( T \sigma V \subset T \). So \( \nabla g|_S(\sigma_n) \) tends to a vector in \( T \) whose orthogonal projection on \( T \sigma V \) is exactly \( \nabla g|_V(\sigma) \). Similarly \( \nabla \rho|_S(\sigma_n) \) tends to a vector in \( T \) whose orthogonal projection on \( T \sigma V \) is exactly \( \nabla \rho|_V(\sigma) \). By Thom’s condition, \( \nabla f|_S(\sigma_n) \) tends to a vector in \( T \) which is orthogonal to \( T \sigma V \), so we see that \( \nabla g|_V(\sigma) \) and \( \nabla \rho|_V(\sigma) \) are colinear which means that \( \sigma \) is a critical point of \( g|_{X^f \cap \tilde{S}_\epsilon} \). But since \( g(\sigma_n) = 0 \), we find that \( g(\sigma) = 0 \), which is impossible by Lemma 3.3. This proves the first assertion.

To prove the second one, we use the same method. Assume that for each \( \delta > 0 \) small enough, there exists a point \( s_i^{\delta, \epsilon} \) such that \( g(s_i^{\delta, \epsilon}) > 0 \) and \( s_i^{\delta, \epsilon} \) is an inwards-pointing critical point for \( g|_{X^f \cap f^{-1}(\delta) \cap \tilde{S}_\epsilon} \). Then we can construct a sequence of points \((\tau_n)_{n \in \mathbb{N}}\) such that \( g(\tau_n) > 0 \) and \( \tau_n \) is an inwards-pointing critical point for \( g|_{f^{-1}(\frac{1}{n}) \cap X \cap \tilde{S}_\epsilon} \). We can also assume that the points \( \tau_n \)’s belong to the same stratum \( S \) and that they tend to \( \tau \in V \) where \( V \subseteq X^f \) and \( V \subset \partial \tilde{S} \). Therefore, we have a decomposition:

\[
\nabla g|_S(\tau_n) = \lambda_n \nabla f|_S(\tau_n) + \mu_n \nabla \rho|_S(\tau_n),
\]

with \( \mu_n < 0 \). Using the same arguments as above, we find that \( \nabla g|_V(\tau) = \mu \nabla \rho|_S(\tau) \) with \( \mu \leq 0 \) and \( g(\tau) \geq 0 \). This contradicts the remark after Lemma 3.3. Of course, this proof works for \( \delta < 0 \). \( \square \)

Let \( \Gamma_{g, \rho} \) be the following polar set:

\[
\Gamma_{g, \rho} = \{ x \in U \mid \text{rank} \{ \nabla g(x), \nabla \rho(x) \} < 2 \}.
\]

By Lemma 2.5 and Lemma 2.11 we can assume that \( \Gamma_{g, \rho} \setminus \{ g = 0 \} \) does not intersect \( X^f \setminus \{ 0 \} \) in a neighborhood of 0 and so \( \Gamma_{g, \rho} \setminus \{ g = 0 \} \) does not intersect \( X^f \cap \tilde{S}_\epsilon \) for \( \epsilon > 0 \) sufficiently small. Since the critical points of \( g|_{X^f \cap \tilde{S}_\epsilon} \) lie outside \( \{ g = 0 \} \), they do not belong to \( \Gamma_{g, \rho} \setminus \tilde{S}_\epsilon \) and so the critical points of \( g|_{f^{-1}(\delta) \cap X \cap \tilde{S}_\epsilon} \) do not neither if \( \delta \) is sufficiently small. Hence at each critical point of \( g|_{f^{-1}(\delta) \cap X \cap \tilde{S}_\epsilon} \), \( g|_{\tilde{S}_\epsilon} \) is a submersion. We are in position to apply Theorem 3.1 and Lemma 2.1 in [15]. For \( 0 < |\delta| \ll \epsilon \ll 1 \), we set

\[
I(\delta, \epsilon, g) = \sum_{i=1}^{r} \text{ind}(g, X \cap f^{-1}(\delta), p_i^{\delta, \epsilon}),
\]

\[
I(\delta, \epsilon, -g) = \sum_{i=1}^{r} \text{ind}(-g, X \cap f^{-1}(\delta), p_i^{\delta, \epsilon}).
\]
Theorem 3.8. We have
\[ I(\delta, \epsilon, g) + I(\delta, \epsilon, -g) = 2\chi(X \cap f^{-1}(\delta) \cap \tilde{B}_{\epsilon}) - \chi(X \cap f^{-1}(\delta) \cap \tilde{S}_{\epsilon}) - \chi(X^g \cap f^{-1}(\delta) \cap \tilde{S}_{\epsilon}). \]

Proof. Let us denote by \( \{a^+_j\}_{j=1}^{\alpha^+} \) (resp. \( \{a^-_j\}_{j=1}^{\alpha^-} \)) the outwards-pointing (resp. inwards-pointing) critical points of \( g : X \cap f^{-1}(\delta) \cap \tilde{S}_{\epsilon} \to \mathbb{R} \). Applying Morse theory type theorem ([15], Theorem 3.1) and using Lemma 2.1 in [15], we can write
\[ I(\delta, \epsilon, g) + \sum_{j=1}^{\alpha^-} \text{ind}(g, X \cap f^{-1}(\delta) \cap \tilde{S}_{\epsilon}, a^-_j) = \chi(X \cap f^{-1}(\delta) \cap \tilde{B}_{\epsilon}) \] (1),
\[ I(\delta, \epsilon, -g) + \sum_{j=1}^{\alpha^+} \text{ind}(-g, X \cap f^{-1}(\delta) \cap \tilde{S}_{\epsilon}, -a^+_j) = \chi(X \cap f^{-1}(\delta) \cap \tilde{B}_{\epsilon}) \] (2).

Let us evaluate
\[ \sum_{j=1}^{\alpha^-} \text{ind}(g, X \cap f^{-1}(\delta) \cap \tilde{S}_{\epsilon}, a^-_j) + \sum_{j=1}^{\alpha^+} \text{ind}(-g, X \cap f^{-1}(\delta) \cap \tilde{S}_{\epsilon}, a^+_j). \]

Since the outwards-pointing critical points of \( g|_{X \cap f^{-1}(\delta) \cap \tilde{S}_{\epsilon}} \) lie in \( \{g > 0\} \) and the inwards-pointing critical points of \( g|_{X \cap f^{-1}(\delta) \cap \tilde{S}_{\epsilon}} \) lie in \( \{g < 0\} \), we have
\[ \chi(X \cap f^{-1}(\delta) \cap \tilde{S}_{\epsilon} \cap \{g \geq 0\}) - \chi(X \cap f^{-1}(\delta) \cap \tilde{S}_{\epsilon} \cap \{g = 0\}) = \sum_{j=1}^{\alpha^+} \text{ind}(g, X \cap f^{-1}(\delta) \cap \tilde{S}_{\epsilon}, a^+_j) \] (3),

and
\[ \chi(X \cap f^{-1}(\delta) \cap \tilde{S}_{\epsilon} \cap \{g \leq 0\}) - \chi(X \cap f^{-1}(\delta) \cap \tilde{S}_{\epsilon} \cap \{g = 0\}) = \sum_{j=1}^{\alpha^-} \text{ind}(-g, X \cap f^{-1}(\delta) \cap \tilde{S}_{\epsilon}, a^-_j) \] (4).

Therefore making (3) + (4) and using the Mayer-Vietoris sequence, we find
\[ \chi(X \cap f^{-1}(\delta) \cap \tilde{S}_{\epsilon}) - \chi(X \cap f^{-1}(\delta) \cap \tilde{S}_{\epsilon} \cap \{g = 0\}) = \sum_{j=1}^{\alpha^+} \text{ind}(g, X \cap f^{-1}(\delta) \cap \tilde{S}_{\epsilon}, a^+_j) + \sum_{j=1}^{\alpha^-} \text{ind}(-g, X \cap f^{-1}(\delta) \cap \tilde{S}_{\epsilon}, a^-_j) \] (5).

Moreover we have
\[ \chi(X \cap f^{-1}(\delta) \cap \tilde{S}_{\epsilon}) = \sum_{j=1}^{\alpha^+} \text{ind}(g, X \cap f^{-1}(\delta) \cap \tilde{S}_{\epsilon}, a^+_j) \]
There are non-zero and do not point in opposite direction in \(\Omega\)

Let us assume now that \((X, 0)\) is equipped with a Whitney stratification \(W = \bigcup_{\alpha \in \Delta} W_\alpha\) and \(f : (X, 0) \to (\mathbb{R}, 0)\) has an isolated critical point at 0. In this situation, our results apply taking for \(V\) the following stratification:

\[
\{ W_\alpha \setminus f^{-1}(0), W_\alpha \cap f^{-1}(0) \setminus \{0\}, \{0\} \mid W_\alpha \in W \}. 
\]

**Corollary 3.9.** If \(f : (X, 0) \to (\mathbb{R}, 0)\) has an isolated stratified critical point at 0, then

\[
I(\delta, \epsilon, g) + I(\delta, \epsilon, -g) = 2\chi(X \cap f^{-1}(\delta) \cap \tilde{S}_\epsilon) - \chi(X \cap X^g \cap \tilde{S}_\epsilon) - \chi(X^f \cap X^g \cap \tilde{S}_\epsilon).
\]

**Proof.** For each stratum \(W\) of \(X\), let

\[
\Gamma^W_{f, \rho} = \{ x \in W \mid \text{rank}[\nabla f|_W(x), \nabla \rho|_W(x)] < 2 \},
\]

and let \(\Gamma^W_{f, \rho} = \bigcup_W \Gamma^W_{f, \rho}\). By Lemma 3.4 applied to \(X\) and \(f\) instead of \(X^f\) and \(g\), \(\Gamma^W_{f, \rho} \cap \{f = 0\} \subset \{0\}\) in a neighborhood of the origin and so 0 is a regular value of \(f : X \cap \tilde{S}_\epsilon \to \mathbb{R}\) for \(\epsilon\) sufficiently small. By Thom-Mather’s second isotopy lemma, \(X \cap f^{-1}(0) \cap \tilde{S}_\epsilon\) is homeomorphic to \(X \cap f^{-1}(\delta) \cap \tilde{S}_\epsilon\) for \(\delta\) sufficiently small.

Now let \(p\) be a stratified critical point of \(f : X^g \to \mathbb{R}\). By Lemma 2.1 we know that \(p\) belongs to \(f^{-1}(0) \cap X^g\) and so \(p\) is also a critical point of \(g : X^f \to \mathbb{R}\). Hence \(p = 0\) by Condition (A), and \(f : X^g \to \mathbb{R}\) has an isolated stratified critical point at 0. As above, we conclude that \(X^f \cap X^g \cap \tilde{S}_\epsilon\) is homeomorphic to \(X^g \cap f^{-1}(\delta) \cap \tilde{S}_\epsilon\).  

Let \(\omega(x) = \sqrt{x_1^2 + \cdots + x_n^2}\) be the euclidian distance to the origin. As explained by Durfee in [10], Lemma 1.8 and Lemma 3.6, there is a neighborhood \(\Omega\) of 0 in \(\mathbb{R}^n\) such that for every stratum \(V\) of \(X^f\), \(\nabla \omega|_V\) and \(\nabla \rho|_V\) are non-zero and do not point in opposite direction in \(\Omega \setminus \{0\}\). Applying
Durfee’s argument ([10], Proposition 1.7 and Proposition 3.5), we see that $X^f \cap \tilde{S}_\epsilon$ is homeomorphic to $X^f \cap S_{\epsilon'}$ for $\epsilon, \epsilon' > 0$ sufficiently small. Similarly $X^f \cap X^g \cap \tilde{S}_\epsilon$ and $X^f \cap X^g \cap S_{\epsilon'}$ are homeomorphic. Now let us compare $X \cap f^{-1}(\delta) \cap B_\epsilon$ and $X \cap f^{-1}(\delta) \cap B_{\epsilon'}$. Let us choose $\epsilon'$ and $\epsilon$ such that

$$X \cap f^{-1}(\delta) \cap B_{\epsilon'} \subset X \cap f^{-1}(\delta) \cap \tilde{B}_\epsilon \subset \Omega.$$

If $\delta$ is sufficiently small then, for every stratum $V \not\subset X^f$, $\nabla \omega_{|V \cap f^{-1}(\delta)}$ and $\nabla \rho_{|V \cap f^{-1}(\delta)}$ are non-zero and do not point in opposite direction in $\tilde{B}_\epsilon \setminus \tilde{B}_{\epsilon'}$. Otherwise, by Thom’s $(a_f)$-condition, we would find a point $p$ in $X^f \cap (B_\epsilon \setminus \tilde{B}_{\epsilon'})$ such that either $\nabla \omega_{|S}(p)$ or $\nabla \rho_{|S}(p)$ vanish or $\nabla \omega_{|S}(p)$ and $\nabla \rho_{|S}(p)$ point in opposite direction, where $S$ is the stratum of $X^f$ that contains $p$. This is impossible if we are sufficiently close to the origin. Now, applying the same arguments as Durfee [10], Proposition 1.7 and Proposition 3.5, we see that $X \cap f^{-1}(\delta) \cap \tilde{B}_\epsilon$ is homeomorphic to $X \cap f^{-1}(\delta) \cap B_{\epsilon'}$ and that $X \cap f^{-1}(\delta) \cap \tilde{S}_\epsilon$ is homeomorphic to $X \cap f^{-1}(\delta) \cap S_{\epsilon'}$.

**Theorem 3.10.** We have

$$I(\delta, \epsilon, g) + I(\delta, \epsilon, -g) = 2\chi(M_f^{\delta, \epsilon}) - \chi(X \cap f^{-1}(\delta) \cap S_\epsilon) - \chi(X^g \cap f^{-1}(\delta) \cap S_\epsilon).$$

□

**Corollary 3.11.** If $f : (X, 0) \to (\mathbb{R}, 0)$ has an isolated stratified critical point at 0, then

$$I(\delta, \epsilon, g) + I(\delta, \epsilon, -g) = 2\chi(M_f^{\delta, \epsilon}) - \chi(Lk(X^f)) - \chi(Lk(X^f \cap X^g)).$$

□

Let us remark if $\dim X = 2$ then in Theorem 3.10 and in Corollary 3.11 the last term of the right-hand side of the equality vanishes. If $\dim X = 1$ then in Theorem 3.10 and in Corollary 3.11 the last two terms of the right-hand side of the equality vanish.

4. **An infinitesimal Gauss-Bonnet formula**

In this section, we apply the results of the previous section to the case of linear forms and we establish a Gauss-Bonnet type formula for the real Milnor fibre.

We will first show that generic linear forms satisfy Condition (A) and Condition (B). For $v \in S^{n-1}$, let us denote by $v^*$ the function $v^*(x) = \langle v, x \rangle$.

**Lemma 4.1.** There exists a subanalytic set $\Sigma_1 \subset S^{n-1}$ of positive codimension such that if $v \notin \Sigma_1$, $\{v = 0\}$ intersects $X \setminus \{0\}$ transversally (in the stratified sense) in a neighborhood of the origin.

**Proof.** It is a particular case of Lemma 3.8 in [14]. □

**Corollary 4.2.** If $v \notin \Sigma_1$ then $v^*_X : (X, 0) \to (\mathbb{R}, 0)$ has an isolated stratified point at 0.
Proof. By Lemma 2.1, we know that the stratified critical points of \( v^*_X \) lie in \( \{ v^* = 0 \} \). But since \( \{ v^* = 0 \} \) intersects \( X \setminus \{ 0 \} \) transversally, the only possible critical point of \( v^*_X : (X, 0) \to (\mathbb{R}, 0) \) is the origin. \( \square \)

Lemma 4.3. There exists a subanalytic set \( \Sigma_2 \subset S^{n-1} \) of positive codimension such that if \( v \notin \Sigma_2 \), then \( \Gamma_{f, v^*} \) is a \( C^1 \)-subanalytic curve (possibly empty) in a neighborhood of 0.

Proof. Let \( V \) be stratum of dimension \( e \) such that \( V \not\subset X^f \). We can assume that \( e \geq 2 \). Let
\[
M_V = \left\{ (x, y) \in V \times \mathbb{R}^n \mid \text{rank}[\nabla f|_V(x), \nabla y^*_V(x)] < 2 \right\}.
\]
It is a subanalytic manifold of class \( C^1 \) and of dimension \( n + 1 \). To see this, let us pick a point \((x, y)\) in \( M_V \). In a neighborhood of \( x \), \( V \) is defined by the vanishing of \( k = n - e \) \( C^2 \)-subanalytic functions \( f_1, \ldots, f_k \). Since \( V \) is not included in \( X^f \), \( f : V \to \mathbb{R} \) is a submersion and we can assume that in a neighborhood of \( x \), the following \((k + 1) \times (k + 1)\)-minor:
\[
\frac{\partial(f_1, \ldots, f_k, f)}{\partial(x_1, \ldots, x_k, x_{k+1})},
\]
does not vanish. Therefore, in a neighborhood of \((x, y)\), \( M_V \) is defined by the vanishing of the following \((k + 2) \times (k + 2)\)-minors:
\[
\frac{\partial(f_1, \ldots, f_k, f, y^*)}{\partial(x_1, \ldots, x_k, x_{k+1}, x_{k+2})}, \ldots, \frac{\partial(f_1, \ldots, f_k, f, y^*)}{\partial(x_1, \ldots, x_k, x_{k+1}, x_n)}.
\]
A simple computation of determinants shows that the gradient vectors of these minors are linearly independent. As in previous lemmas, we show that \( \Sigma_{f, v^*} \) is one-dimensional considering the projection
\[
\pi_2 : M_V \to \mathbb{R}^n \quad (x, y) \mapsto y.
\]
Since \( \Gamma_{f, v^*} = \bigcup_{V \not\subset X^f} \Gamma_{f, v^*} \), we get the result. \( \square \)

Let \( \Sigma = \Sigma_1 \cup \Sigma_2 \), it is a subanalytic subset of \( S^{n-1} \) of positive codimension and if \( v \notin \Sigma \) then \( v^* \) satisfies Conditions (A) and (B). In particular, \( v|_{f^{-1}(\delta) \cap X \cap B_\epsilon} \) has a finite number of critical points \( p^{\delta, \epsilon}_{\delta, \epsilon} \). We recall that
\[
I(\delta, \epsilon, v^*) = \sum_{i=1}^{r_v} \text{ind}(v^*, X \cap f^{-1}(\delta), p^{\delta, \epsilon}_{\delta, \epsilon}),
\]
\[
I(\delta, \epsilon, -v^*) = \sum_{i=1}^{r_v} \text{ind}(-v^*, X \cap f^{-1}(\delta), p^{\delta, \epsilon}_{\delta, \epsilon}).
\]
In this situation, Theorem 3.10 and Corollary 3.11 become
Corollary 4.4. If \( v \notin \Sigma \) then
\[
I(\delta, \epsilon, v^*) + I(\delta, \epsilon, -v^*) = 2\chi(M_{f}^{\delta, \epsilon}) - \chi(X \cap f^{-1}(\delta) \cap S_\epsilon) - \chi(X v^* \cap f^{-1}(\delta) \cap S_\epsilon).
\]
Furthermore, if \( f : (X, 0) \to (\mathbb{R}, 0) \) has an isolated stratified critical point at 0, then
\[
I(\delta, \epsilon, v^*) + I(\delta, \epsilon, -v^*) = 2\chi(M_{f}^{\delta, \epsilon}) - \chi(\text{Lk}(X f)) - \chi(\text{Lk}(X f \cap X v^*)�)
\]

As an application, we give a Gauss-Bonnet formula for the Milnor fibre \( M_{f}^{\delta, \epsilon} \). Let \( \Lambda_0(X \cap f^{-1}(\delta), -) \) be the Gauss-Bonnet measure on \( X \cap f^{-1}(\delta) \) defined by
\[
\Lambda_0(X \cap f^{-1}(\delta), U') = \frac{1}{s_{n-1}} \int_{S^{n-1}} \sum_{x \in U'} \text{ind}(v^*, X \cap f^{-1}(\delta), x) dx,
\]
where \( U' \) is a Borel set of \( X \cap f^{-1}(\delta) \) (see [6], page 299) and \( s_{n-1} \) is the volume of the unit sphere \( S^{n-1} \). Note that if \( x \) is not a critical point of \( v^*|_{X \cap f^{-1}(\delta)} \) then \( \text{ind}(v^*, X \cap f^{-1}(\delta), x) = 0 \). We are going to evaluate
\[
\lim_{\epsilon \to 0} \lim_{\delta \to 0} \Lambda_0(X \cap f^{-1}(\delta), M_{f}^{\delta, \epsilon}).
\]

Theorem 4.5. We have
\[
\lim_{\epsilon \to 0} \lim_{\delta \to 0} \Lambda_0(X \cap f^{-1}(\delta), M_{f}^{\delta, \epsilon}) = \chi(M_{f}^{\delta, \epsilon}) - \frac{1}{2} \chi(X \cap f^{-1}(\delta) \cap S_\epsilon)
\]
\[
- \frac{1}{2s_{n-1}} \int_{S^{n-1}} \chi(X \cap f^{-1}(\delta) \cap \{v^* = 0\} \cap S_\epsilon) dv.
\]
Furthermore, if \( f : (X, 0) \to (\mathbb{R}, 0) \) has an isolated stratified critical point at 0, then
\[
\lim_{\epsilon \to 0} \lim_{\delta \to 0} \Lambda_0(X \cap f^{-1}(\delta), M_{f}^{\delta, \epsilon}) = \chi(M_{f}^{\delta, \epsilon}) - \frac{1}{2} \chi(\text{Lk}(X f))
\]
\[
- \frac{1}{2s_{n-1}} \int_{S^{n-1}} \chi(\text{Lk}(X f \cap X v^*)) dv.
\]

Proof. By definition, we have
\[
\Lambda_0(X \cap f^{-1}(\delta), M_{f}^{\delta, \epsilon}) = \frac{1}{s_{n-1}} \int_{S^{n-1}} \sum_{x \in M_{f}^{\delta, \epsilon}} \text{ind}(v^*, X \cap f^{-1}(\delta), x) dx.
\]
It is not difficult to see that
\[
\Lambda_0(X \cap f^{-1}(\delta), M_{f}^{\delta, \epsilon}) = \\
\frac{1}{2s_{n-1}} \int_{S^{n-1}} \left[ \sum_{x \in M_{f}^{\delta, \epsilon}} \text{ind}(v^*, X \cap f^{-1}(\delta), x) + \text{ind}(-v^*, X \cap f^{-1}(\delta), x) \right] dv.
\]
Note that if \( v \notin \Sigma \) then
\[
\sum_{x \in M^{k,\epsilon}_f} \text{ind}(v^*, X \cap f^{-1}(\delta), x) + \text{ind}(-v^*, X \cap f^{-1}(\delta), x)
\]
is equal to \( I(\delta, \epsilon, v^*) + I(\delta, \epsilon, -v^*) \) and is uniformly bounded by Hardt’s theorem. By Lebesgue’s theorem, we obtain
\[
\lim_{\epsilon \to 0} \lim_{\delta \to 0} \Lambda_0(X \cap f^{-1}(\delta), M^{k,\epsilon}_f) = \frac{1}{2s_{n-1}} \int_{S^{n-1}} \lim_{\epsilon \to 0} \lim_{\delta \to 0} [I(\delta, \epsilon, v^*) + I(\delta, \epsilon, -v^*)] dv.
\]
We just have to apply the previous corollary to conclude. \( \square \)

5. **INFINITESIMAL LINEAR KINEMATIC FORMULAS**

In this section, we apply the results of the previous section to the case of a linear function in order to obtain “infinitesimal” linear kinematic formulas for closed subanalytic germs.

We start recalling known facts on the geometry of subanalytic sets. We need some notations:

- for \( k \in \{0, \ldots, n\} \), \( G^k_n \) is the Grassmann manifold of \( k \)-dimensional linear subspaces in \( \mathbb{R}^n \) and \( g^k_n \) is its volume,
- for \( k \in \mathbb{N} \), \( b_k \) is the volume of the \( k \)-dimensional unit ball and \( s_k \) is the volume of the \( k \)-dimensional unit sphere.

In [17], Fu developed integral geometry for compact subanalytic sets. Using the technology of the normal cycle, he associated with every compact subanalytic set \( X \subset \mathbb{R}^n \) a sequence of curvature measures
\[
\Lambda_0(X, -), \ldots, \Lambda_n(X, -),
\]
called the Lipschitz-Killing measures. He proved several integral geometry formulas, among them a Gauss-Bonnet formula and a kinematic formula. Later another description of the measures using stratified Morse theory was given by Broecker and Kuppe [6] (see also [5]). The reader can refer to [14], Section 2, for a rather complete presentation of these two approaches and for the definition of the Lipschitz-Killing measures.

Let us give some comments on these Lipschitz-Killing curvatures. If \( \dim X = d \) then
\[
\Lambda_{d+1}(X, U') = \cdots = \Lambda_n(X, U') = 0,
\]
for any Borel set \( U' \) of \( X \) and \( \Lambda_d(X, U') = \mathcal{L}_d(U') \), where \( \mathcal{L}_d \) is the \( d \)-dimensional Lebesgue measure in \( \mathbb{R}^n \). Furthermore if \( X \) is smooth then for any Borel set \( U' \) of \( X \) and for \( k \in \{0, \ldots, d\} \), \( \Lambda_k(X, U') \) is related to the classical Lipschitz-Killing-Weil curvature \( K_{d-k} \) through the following equality:
\[
\Lambda_k(X, U') = \frac{1}{s_{n-d-k-1}} \int_{U'} K_{d-k}(x) dx.
\]
In [14], Section 5, we studied the asymptotic behaviour of the Lipschitz-Killing measures in the neighborhood of a point of $X$. Namely we proved the following theorem ([14], Theorem 5.1).

**Theorem 5.1.** Let $X \subset \mathbb{R}^n$ be a closed subanalytic set such that $0 \in X$. We have:

$$\lim_{\epsilon \to 0} \Lambda_0(X, X \cap B_\epsilon) = 1 - \frac{1}{2} \chi(\text{Lk}(X)) - \frac{1}{2g_n^{n-1}} \int_{G_n^{n-1}} \chi(\text{Lk}(X \cap H))dH.$$ 

Furthermore for $k \in \{1, \ldots, n-2\}$, we have:

$$\lim_{\epsilon \to 0} \frac{\Lambda_k(X, X \cap B_\epsilon)}{b_k \epsilon^k} = -\frac{1}{2g_n^{n-k-1}} \int_{G_n^{n-k-1}} \chi(\text{Lk}(X \cap H))dH$$

$$+ \frac{1}{2g_n^{n-k+1}} \int_{G_n^{n-k+1}} \chi(\text{Lk}(X \cap L))dL,$$

and:

$$\lim_{\epsilon \to 0} \frac{\Lambda_{n-1}(X, X \cap B_\epsilon)}{b_{n-1} \epsilon^{n-1}} = \frac{1}{2g_n^n} \int_{G_n^n} \chi(\text{Lk}(X \cap H))dH,$$

$$\lim_{\epsilon \to 0} \frac{\Lambda_n(X, X \cap B_\epsilon)}{b_n \epsilon^n} = \frac{1}{2g_n^n} \int_{G_n^n} \chi(\text{Lk}(X \cap H))dH.$$ 

In the sequel, we will use these equalities and Theorem 4.5 to establish linear kinematic types formulas for the quantities $\lim_{\epsilon \to 0} \Lambda_k(X, X \cap B_\epsilon)$, $k = 1, \ldots, n$. Let us start with some lemmas. We work with a closed subanalytic set $X$ such that $0 \in X$, equipped with a Whitney stratification $\{W_\alpha\}_{\alpha \in A}$.

**Lemma 5.2.** Let $f$ be a $C^2$-subanalytic function such that $f|_X : X \to \mathbb{R}$ has an isolated stratified critical point at 0. Then for $0 < \delta \ll \epsilon \ll 1$, we have

$$\chi(M_f^{\delta, \epsilon}) + \chi(M_f^{-\delta, \epsilon}) = \chi(\text{Lk}(X)) + \chi(\text{Lk}(X^f)).$$

**Proof.** With the same techniques and arguments as the ones we used in order to establish Corollary 4.11, we can prove that

$$\text{ind}(f, X, 0) + \text{ind}(-f, X, 0) = 2\chi(X \cap B_\epsilon) - \chi(\text{Lk}(X)) - \chi(\text{Lk}(X^f)).$$

We conclude thanks to the following equalities

$$\text{ind}(f, X, 0) = 1 - \chi(M_f^{-\delta, \epsilon}), \quad \text{ind}(-f, X, 0) = 1 - \chi(M_f^{\delta, \epsilon}),$$

and

$$\chi(X \cap B_\epsilon) = 1.$$ 

**Corollary 5.3.** There exist a subanalytic set $\Sigma_1 \subset S^{n-1}$ of positive codimension such that if $v \notin \Sigma$ then for $0 < \delta \ll \epsilon \ll 1$,

$$\chi(M_v^{\delta, \epsilon}) + \chi(M_v^{-\delta, \epsilon}) = \chi(\text{Lk}(X)) + \chi(\text{Lk}(X \cap \{v^* = 0\})).$$

**Proof.** Apply Corollary 4.2 and Lemma 5.2.
Lemma 5.4. Let $S \subset \mathbb{R}^n$ be $C^2$-subanalytic manifold. Let $H \in G_{n-k}^n$, $k \in \{1, \ldots, n\}$ and let $G_{H^⊥}^1$ be the Grassmann manifold of lines in the orthogonal complement $H^⊥$ of $H$. There exists a subanalytic set $\Sigma'_H \subset G_{H^⊥}^1$ of positive codimension such that if $\nu \notin \Sigma'_H$ then $H \oplus \nu$ intersects $S \setminus \{0\}$ transversally.

Proof. Assume that $S$ has dimension $e$ and that $H$ is given by the equations $x_1 = \ldots = x_k = 0$ so that $H^⊥ = \mathbb{R}^k$ with coordinate system $(x_1, \ldots, x_k)$. Let $W$ be defined by

$$W = \left\{(x, v_1, \ldots, v_{k-1}) \in \mathbb{R}^n \times (\mathbb{R}^k)^{k-1} \mid x \in S \setminus \{0\} \quad \text{and} \quad \langle x, v_1 \rangle = \ldots = \langle x, v_{k-1} \rangle = 0\right\},$$

where $v_i \in \mathbb{R}^k \times \{0\} \subset \mathbb{R}^n$. Let us show that $W$ is a $C^2$-subanalytic manifold of dimension $e + (k-1)^2$. Let $(y, w)$ be a point in $W$. We can assume that around $y$, $S$ is defined by the vanishing of $n - e$ $C^2$-subanalytic functions $f_1, \ldots, f_{n-e}$. Hence in a neighborhood of $(y, w)$, $W$ is defined by the equations:

$$f_1(x) = \ldots = f_{n-e}(x) = 0 \quad \text{and} \quad \langle x, v_1 \rangle = \ldots = \langle x, v_{k-1} \rangle = 0.$$

Because $y \neq 0$, we see that the gradient vectors of this $n - e + k - 1$ functions are linearly independent at $(y, w)$. This enables us to conclude that $W$ is a $C^2$-subanalytic manifold of dimension $e + (k-1)^2$. Let $\pi_2$ be the following projection:

$$\pi_2 : W \to (\mathbb{R}^n)^{n-k}, \ (x, v_1, \ldots, v_{n-k}) \mapsto (v_1, \ldots, v_{n-k}).$$

Bertini-Sard’s theorem implies that the set of critical values of $\pi_2$ is a subanalytic set of positive codimension. If $(v_1, \ldots, v_{k-1})$ lies outside this subanalytic set then the $(n-k+1)$-plane $\{x \in \mathbb{R}^n \mid \langle x, v_1 \rangle = \ldots = \langle x, v_{k-1} \rangle = 0\}$ contains $H$ and intersects $S \setminus \{0\}$ transversally.

Now we can present our infinitesimal linear kinematic formulas. Let $H \in G_{n-k}^n$, $k \in \{1, \ldots, n\}$, and let $S_{H^⊥}^{k-1}$ be the unit sphere of the orthogonal complement of $H$. Let $v$ be an element in $S_{H^⊥}^{k-1}$. For $\delta > 0$, we denote by $H_{v, \delta}$ the $(n-k)$-dimensional affine space $H + \delta v$ and we set

$$\beta_0(H, v) = \lim_{\epsilon \to 0} \lim_{\delta \to 0} \Lambda_0(H_{\delta, v} \cap X, H_{\delta, v} \cap X \cap B_\epsilon).$$

Then we set

$$\beta_0(H) = \frac{1}{s_{k-1}} \int_{S_{H^⊥}^{k-1}} \beta_0(H, v) dv.$$

Theorem 5.5. For $k \in \{1, \ldots, n\}$, we have

$$\lim_{\epsilon \to 0} \frac{\Lambda_k(X, X \cap B_\epsilon)}{b_k \epsilon^k} = \frac{1}{s_k^{n-k}} \int_{G_{n-k}^n} \beta_0(H) dH.$$
Proof. We treat first the case \( k \in \{1, \ldots, n-2\} \). By Theorem 5.1, we know that
\[
\lim_{\epsilon \to 0} \frac{\Lambda_k(X, X \cap B_\epsilon)}{b_k \epsilon^k} = -\frac{1}{2g_{n-k-1}} \int_{G_{n-k-1}} \chi(Lk(X \cap H))dH + \frac{1}{2g_{n-k+1}} \int_{G_{n-k+1}} \chi(Lk(X \cap L))dL.
\]
By Lemma 3.8 in [14], we know that generically \( H \) intersects \( X \setminus \{0\} \) transversally in a neighborhood of the origin. Let us fix \( H \) that satisfies this generic property. For any \( v \in S^{n-k}_{H^\perp} \), let \( v \) be the line generated by \( v \) and let \( L_v \) be the \((n-k+1)\)-plane defined by \( L_v = H \oplus v \). By Lemma 5.4 we know that for \( v \) generic in \( S^{n-k}_{H^\perp} \), \( L_v \) intersects \( X \setminus \{0\} \) transversally in a neighborhood of the origin. Therefore, \( v^*_X \cap L_v \) has an isolated singular point at 0 and we can apply Theorem 5.5. We have
\[
\lim_{\epsilon \to 0} \lim_{\delta \to 0} \Lambda_0(X \cap L_v \cap \{v^* = \delta\}, X \cap L_v \cap \{v^* = \delta\} \cap B_\epsilon) = \chi(X \cap L_v \cap \{v^* = \delta\} \cap B_\epsilon) - \frac{1}{2} \chi(Lk(X \cap L_v \cap \{v^* = 0\}))
\]
\[
- \frac{1}{2s_{n-k}} \int_{S_{n-k}^{n-k}} \chi(Lk(X \cap L_v \cap \{v^* = 0\} \cap \{w^* = 0\}))dw,
\]
where \( S_{n-k}^{n-k} \) is the unit sphere of \( L_v \). Let us remark that \( L_v \cap \{v^* = \delta\} \) is exactly \( H_{v,\delta} \) and that \( L_v \cap \{v^* = 0\} \) is \( H \). We can also apply Lemma 5.2 to \( v^*_X \cap L_v \) to obtain the following relation:
\[
\beta_0(H, v) + \beta_0(H, -v) = \chi(Lk(X \cap L_v))
\]
\[
- \frac{1}{s_{n-k}} \int_{S_{n-k}^{n-k}} \chi(Lk(X \cap H \cap \{w^* = 0\}))dw.
\]
Since \( \beta(H) \) is equal to
\[
\frac{1}{2s_{n-k}} \int_{S_{n-k}^{n-k}} \left[ \beta_0(H, v) + \beta_0(H, -v) \right] dv,
\]
we find that
\[
\beta(H) = \frac{1}{2s_{n-k}} \int_{S_{n-k}^{n-k}} \chi(Lk(X \cap L_v))dv
\]
\[
- \frac{1}{2s_{n-k}} \int_{S_{n-k}^{n-k}} \int_{S_{n-k}^{n-k}} \chi(Lk(X \cap H \cap \{w^* = 0\}))dwdv.
\]
Replacing spheres with Grassman manifolds in this equality, we obtain
\[
\beta(H) = \frac{1}{2g_{n-k}} \int_{G_{n-k}^{n-k}} \chi(Lk(X \cap H \oplus \nu))d\nu - \frac{1}{2g_{n-k}^1g_{n-k+1}} \int_{G_{n-k}^{n-k}} \int_{G_{n-k}^{n-k}} \chi(Lk(X \cap H \cap K))dKd\nu.
\]
and proceeding as above, we find

\[ \frac{1}{g_n^{n-k}} \int_{G_n^{n-k}} \beta(H) dH = \frac{1}{2g_k^{n-k} g_n^{n-k}} \int_{L^{n-k}} \int_{G_n^{n-k}} \chi(\text{Lk}(X \cap H \oplus \nu)) d\nu dH - \]

\[ \frac{1}{2g_n^{n-k} g_k^{n-k}} \int_{G_n^{n-k}} \int_{G_n^{n-k}} \chi(\text{Lk}(X \cap H \cap K)) dK dH. \]

Let us compute

\[ \mathcal{I} = \frac{1}{2g_n^{n-k} g_k^{n-k}} \int_{G_n^{n-k}} \int_{G_n^{n-k}} \chi(\text{Lk}(X \cap H \oplus \nu)) d\nu dH. \]

Let \( \mathcal{H} \) be the flag variety of pairs \((L, H), L \in G_n^{n-k+1} \text{ and } H \in G_n^{n-k} \). This variety is a bundle over \( G_n^{n-k} \), each fibre being a \( G_k^1 \). Hence we have

\[ \int_{G_n^{n-k}} \int_{G_n^{n-k}} \chi(\text{Lk}(X \cap H \oplus \nu)) d\nu dH = \int_{G_n^{n-k+1}} \int_{G_n^{n-k}} \chi(\text{Lk}(X \cap L)) dH dL = \]

\[ g_k^{n-k} \int_{G_n^{n-k+1}} \chi(\text{Lk}(X \cap L)) dL. \]

Finally, we get that

\[ \mathcal{I} = \frac{g_k^{n-k}}{2g_n^{n-k} g_k^{n-k}} \int_{G_n^{n-k+1}} \chi(\text{Lk}(X \cap L)) dL = \]

\[ \frac{1}{2g_n^{n-k+1}} \int_{G_n^{n-k+1}} \chi(\text{Lk}(X \cap L)) dL. \]

Let us compute now

\[ \mathcal{J} = \frac{1}{2g_n^{n-k} g_k^{n-k}} \int_{G_n^{n-k+1}} \int_{G_n^{n-k+1}} \int_{G_n^{n-k}} \chi(\text{Lk}(X \cap H \cap K)) dK d\nu dH. \]

First, as we have just done above, we can write

\[ \mathcal{J} = \frac{1}{2g_n^{n-k} g_k^{n-k}} \int_{G_n^{n-k+1}} \int_{G_n^{n-k+1}} \int_{G_n^{n-k}} \chi(\text{Lk}(X \cap H \cap K)) dK d\nu dH = \]

\[ \frac{1}{g_n^{n-k+1}} \int_{G_n^{n-k+1}} \chi(\text{Lk}(X \cap J)) dJ, \]

and so

\[ \mathcal{J} = \frac{1}{2g_n^{n-k} g_k^{n-k}} \int_{G_n^{n-k+1}} \int_{G_n^{n-k+1}} \int_{G_n^{n-k}} \chi(\text{Lk}(X \cap J)) dJ dH dL. \]

Then we remark (see [13], Corollary 3.11 for a similar argument) that

\[ \frac{1}{g_n^{n-k+1}} \int_{G_n^{n-k+1}} \chi(\text{Lk}(X \cap H \cap K)) dK = \frac{1}{g_n^{n-k+1}} \int_{G_n^{n-k+1}} \chi(\text{Lk}(X \cap J)) dJ, \]

and proceeding as above, we find

\[ \int_{G_n^{n-k}} \int_{G_n^{n-k+1}} \chi(\text{Lk}(X \cap J)) dJ dH = g_2^{2} \int_{G_n^{n-k+1}} \chi(\text{Lk}(X \cap J)) dJ, \]
so
\[ J = \frac{g_1^2}{2gn^{n-k}g_kn_k} \int_{G_{n-k}^{n-k}} \chi(\text{Lk}(X \cap J))dJ. \]

To finish the computation, we consider the flag variety of pairs \((L, J)\), \(L \in G_{n-k}^{n-k+1}\) and \(J \in G_{L}^{n-k-1}\). It is a bundle over \(G_{n-k}^{n-k-1}\), each fibre being a \(G_{k+1}^2\). Hence we have
\[ J = \frac{g_1^2g_k^2}{2gn^{n-k}g_kn_k} \int_{G_{n-k}^{n-k}} \chi(\text{Lk}(X \cap J))dJ \]
\[ = \frac{1}{2gn^{n-k}g_kn_k} \int_{G_{n-k}^{n-k}} \chi(\text{Lk}(X \cap J))dJ. \]

This ends the proof for the case \(k \in \{1, \ldots, n-2\}\). For \(k = n-1\) or \(n\), the proof is the same. We just have to remark that in these cases
\[ \beta_0(H, v) + \beta_0(H, -v) = \chi(\text{Lk}(X \cap L_v)), \]
and if \(k = n-1\), \(\dim L_v = 2\) and if \(k = n\), \(\dim L_v = 1\). \(\square\)

Let us end with some remarks on the limits \(\lim_{\epsilon \to 0} \frac{\Lambda_k(X, X \cap B_{\epsilon})}{b_k \epsilon^k}\). We already know that if \(\dim X = d\) then \(\lim_{\epsilon \to 0} \frac{\Lambda_k(X, X \cap B_{\epsilon})}{b_k \epsilon^k} = 0\) for \(k \geq d + 1\). This is also the case if \(l < d_0\), where \(d_0\) is the dimension of the stratum that contains 0. To see this let us first relate the limits \(\lim_{\epsilon \to 0} \frac{\Lambda_k(X, X \cap B_{\epsilon})}{b_k \epsilon^k}\) to the polar invariants defined by Comte and Merle in [9]. They can be defined as follows. Let \(H \in G_{n-k}^{n-k}\), \(k \in \{1, \ldots, n\}\), and let \(v\) be an element in \(S_{k-1}^H\). For \(\delta > 0\), we set
\[ \lambda_0(H, v) = \lim_{\epsilon \to 0} \lim_{\delta \to 0} \chi(H_{\delta, \epsilon} \cap X \cap B_{\epsilon}), \]
and then
\[ \sigma_k(X, 0) = \frac{1}{s_{k-1}} \int_{S_{H-1}^{k-1}} \lambda_0(H, v)dv. \]

Moreover, we put \(\sigma_0(X, 0) = 1\).

**Theorem 5.6.** For \(k \in \{0, \ldots, n-1\}\), we have
\[ \lim_{\epsilon \to 0} \frac{\Lambda_k(X, X \cap B_{\epsilon})}{b_k \epsilon^k} = \sigma_k(X, 0) - \sigma_{k+1}(X, 0). \]

Furthermore, we have
\[ \lim_{\epsilon \to 0} \frac{\Lambda_n(X, X \cap B_{\epsilon})}{b_n \epsilon^n} = \sigma_n(X, 0). \]
Proof. It is the same proof as Theorem 5.5. For example if $k \in \{0, \ldots, n-1\}$, we just have to remark that
\[ \lambda_0(H, v) + \lambda_0(H, -v) = \chi(\text{Lk}(X \cap Lv)) + \chi(\text{Lk}(X \cap H)), \]
by Lemma 5.2, which implies that
\[ \sigma_k(X, 0) = \frac{1}{2g^n-k+1} \int_{G^{n-k+1}} \chi(\text{Lk}(X \cap L))dL + \frac{1}{2g^n-k} \int_{G^{n-k}} \chi(\text{Lk}(X \cap H))dH. \]
\[ \square \]

It is explained in [9] that $\sigma_k(X, 0) = 1$ if $0 \leq k \leq d_0$, so if $k < d_0$ then
\[ \lim_{\epsilon \to 0} \frac{\Lambda_k(X, X \cap B_{\epsilon})}{b_k} = 0. \]

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