Non-perturbative renormalization in kaon decays.

A. Donini\textsuperscript{a}, V. Giménez\textsuperscript{b}, G. Martinelli\textsuperscript{a}, G.C. Rossi\textsuperscript{c}, M. Talevi\textsuperscript{a}\textsuperscript{*}, M. Testa\textsuperscript{a} and A. Vladikas\textsuperscript{c}

\textsuperscript{a}Dip. di Fisica, Università di Roma “La Sapienza” and I.N.F.N., Sezione di Roma I, Piazzale Aldo Moro 2, I-00185 Roma, Italy.

\textsuperscript{b}Dep. de Física Teórica and IFIC, Universitat de València, E-46100 Burjassot (Valencia), Spain.

\textsuperscript{c}Dip. di Fisica, Università di Roma “Tor Vergata” and I.N.F.N., Sezione di Roma II, Via della Ricerca Scientifica 1, I-00133 Roma, Italy.

We discuss the application of the MPSTV non-perturbative method \cite{note} to the operators relevant to kaon decays. This enables us to reappraise the long-standing question of the \( \Delta I = 1/2 \) rule, which involves power-divergent subtractions that cannot be evaluated in perturbation theory. We also study the mixing with dimension-six operators and discuss its implications to the chiral behaviour of the \( B_K \) parameter.

1. Introduction

A qualitative theoretical understanding of the \( \Delta I = 1/2 \) rule in \( K \to \pi\pi \) decays has proven to be a formidable task since the calculation of hadronic matrix elements in the low-energy non-perturbative regime is needed. In the continuum, with an active charm quark and the GIM mechanism at work, the operator basis given by \( O_{\pm}^{\pm LL} = \frac{1}{2}[(\bar{s}d)_{L}(\bar{u}u)_{L} \pm (\bar{s}u)_{L}(\bar{d}u)_{L}] - (u \to c) \).

In the framework of lattice QCD with Wilson-like fermions, the renormalization strategy is complicated by chiral symmetry breaking. In fact, the Wilson term induces the mixing of \( O_{\pm}^{\pm LL} \) with lower-dimensional operators, with power-divergent coefficients, which need to be subtracted non-perturbatively. Non-perturbative (NP) renormalization has witnessed a great progress recently \cite{note,note2}. In this talk, we want to outline the strategy for the renormalization of the lattice operators \( O_{\pm}^{\pm LL} \) using the NP method (NPM) of \cite{note}. In the following, we shall concentrate only on the octet component of \( O_{\pm}^{\pm} \), which we will denote with \( O_{0}^{\pm} \). In sec. 2 we also reconsider the mixing with dimension-six operators \cite{note3} and its implications on the chiral behaviour of \( B_K \), a problem also addressed in \cite{note} using the Ward Identities (WI). We stress that the NPM of \cite{note} and the WI are completely equivalent, provided the renormalization scale is large enough, since the renormalized operators with the good chiral properties are unique. Anyway, the NPM of \cite{note} must be used to calculate the overall constant, which the WI cannot determine.

2. Renormalization strategy

On the lattice, the general prescription to restore the chiral symmetry broken by the Wilson term is to subtract all the operators of dimension equal or less than \( O_{0}^{\pm} \). The operators which need to be subtracted are dictated by the symmetries of the action: charge conjugation (C), parity (P) and \( s \leftrightarrow d \) flavour switching symmetry (S) \cite{note}.

There are different ways of calculating the \( K \to \pi\pi \) matrix elements, which correspond to different renormalization structures \cite{note,note2}. We considering here a general structure of the form

\[ \hat{O}^{\pm} = Z^{\pm} \left[ O_{0}^{\pm} + \sum_{i=1}^{4} Z_{i}^{\pm} O_{i}^{\pm} + Z_{5}^{\pm} O_{5} + Z_{3}^{\pm} O_{3} \right] \] (2)

where \( O_{0}^{\pm} \) are the bare operators, \( O_{i}^{\pm}, i = 1, \ldots, 4 \) are dimension-six operators of wrong chirality (cf. sec. 3), \( O_{5} \) is a dimension-five of the
form \( \bar{\sigma} \Sigma_{\mu\nu} F_{\mu\nu} d \) (\( \Sigma_{\mu\nu} = \sigma_{\mu\nu} \) or \( \bar{\sigma} \sigma_{\mu\nu} \)) and \( O_3 \) is a dimension-three operator of the form \( \bar{\sigma} \Gamma d \) (\( \Gamma = 1 \) or \( \gamma_5 \)). According to the NPM, the mixing \( Z \)'s are determined by finding a set of projectors on the tree-level amputated Green functions (GF), with off-shell quark and gluon external states, the choice of which depends on the nature of the operators at hand. For the \( \Delta I = 1/2 \) operators we identify the following set of external states: \( q\bar{q}, q\bar{q}g, q\bar{q}g\bar{q} \), with the momenta given below in eq. (3). For each choice of external states, i.e. for each different set of GF, we need different type of projectors. Let us denote with \( \mathbf{P}_3 \) the projector on the \( q\bar{q} \) GF of the operator \( O_3 \), with \( \mathbf{P}_5 \) the projector on the \( q\bar{q}g \) GF of the operator \( O_5 \), and with \( \mathbf{P}_j \), \( j = 1, \ldots, 4 \) the set of mutually orthogonal projectors on the operators \( O_i \), \( i = 1, \ldots, 4 \) \( \mathbb{C}^4 \). Applying the projectors to the corresponding NP GF of the renormalized operators \( \hat{O}^{\pm} \), with an appropriate choice of the external states, we require that the renormalized operators be proportional to the bare operators, \( \hat{O}^{\pm}(\mu) \propto O_0^{\pm}(\mu) \) (up to terms of \( O(\alpha) \)), i.e. we impose the following renormalization conditions (trace over colour and spin is understood in the projection operation):

\[
\begin{align*}
\mathbf{P}_3\langle q(p)| \hat{O}^{\pm}|q(p)\rangle &= 0 \\
\mathbf{P}_5\langle q(p-k)| g(k)| \hat{O}^{\pm}|q(p)\rangle &= 0 \\
\mathbf{P}_j\langle q(p)| \hat{O}^{\pm}|q(p)\rangle &= 0, \quad j = 1, \ldots, 4
\end{align*}
\]  

(3)

where \( p \) and \( k \) denote the momentum of the external quark and gluon legs. The system of equations (3) completely determines in a NP way the renormalization constants, as we have six conditions (non-homogeneous due to the matrix elements of \( O_0^{\pm} \), cf. eq. (3)) in six unknown mixing constants, \( Z_i^{\pm} \), \( i = 1, \ldots, 4, Z_5^{\pm}, Z_3^{\pm} \).

Unfortunately, since solving eq. (3) involves delicate cancellations between large contributions, it may very likely result in a very noisy determination, even with large statistics. An equivalent strategy we can adopt is:

1. We introduce an intermediate subtraction for the dimension-five and -six operators

\[
\hat{O}_i \equiv O_i^{\pm} + C_3^{(\pm,i)} O_3, \quad i = 0, \ldots, 4, \\
\hat{O}_5 \equiv O_5 + C_3^{(5)} O_3, 
\]  

(4)

and determine the power-divergent mixing constants \( C_3^{(\pm,i)} \) and \( C_3^{(5)} \) by imposing

\[
\begin{align*}
\mathbf{P}_3\langle q(p)| \hat{O}_i^{\pm}|q(p)\rangle &= 0, \quad i = 0, \ldots, 4, \\
\mathbf{P}_5\langle q(p)| \hat{O}_5|q(p)\rangle &= 0.
\end{align*}
\]  

(5)

2. The finite mixing constants \( Z_i^{\pm} \) and \( Z_5^{\pm} \), which in principle can be calculated in perturbation theory (PT), are then determined from the system

\[
\begin{align*}
\mathbf{P}_3\langle q(p-k)| g(k)| \hat{O}_i^{\pm}|q(p)\rangle &= \mathbf{P}_5\langle q(p-k)| g(k)| \hat{O}_5|q(p)\rangle \\
\mathbf{P}_j\langle q(p)| \hat{O}_i^{\pm}|q(p)\rangle &= \mathbf{P}_j\langle q(p)| \hat{O}_5|q(p)\rangle, \quad j = 1, \ldots, 4.
\end{align*}
\]  

(6)

3. Mixing with dimension-six operators

In ref. \([7]\) the NP mixing with equal dimensional operators of the \( \Delta S = 2 \) operator, which has the same Lorentz structure of \( O_0^+ \), has been studied, with the mixing limited to a basis of three operators, denoted here with \( O_i^+ \), \( i = 1, 2, 3 \), found by an explicit 1-loop calculation \([11]\). A more careful analysis based on CPS \([8]\) has shown that beyond 1-loop, the operators \( O_0^+ \) are allowed to mix only with \( O_i^+ \), \( i = 1, 2, 3 \), present in 1-loop PT, but also with

\[
O_4^+ = \frac{1}{16N_c} [(SS + PP - \frac{1}{3} TT) \pm 2 \leftrightarrow 4],
\]

(7)

which is not present at the 1-loop level, where \( \Gamma \bar{\Gamma} = (\bar{\psi}_1 \Gamma \psi_2)(\bar{\psi}_3 \Gamma \psi_4) \) and \( N_c = 3 \).

The operators \( O_i^+ \), \( i = 0, \ldots, 4 \) are Fierz eigenstates in Dirac-colour space with eigenvalue \( \pm 1 \), respectively \([8]\). We stress that Fierz transformations are not a symmetry of the action, but only of the operators, thus they cannot determine the operator mixing. Nevertheless, since Fierz transformations have been used in the perturbative calculations \([11]\), we have also used them to classify and reorganize the NP mixing.

We have extended the method for the renormalization of the \( \Delta S = 2 \) operator presented in ref. \([7]\) including the fourth operator \( O_4 \), and have determined the mixing constants \( Z_i \), \( i = 1, \ldots, 4 \), and the overall constant \( Z^{\Delta S=2} \). We have performed the calculation using the same parameters of ref. \([8]\), on a \( 16^3 \times 32 \) lattice, at \( \beta = 6.0 \), with a hopping parameter \( \kappa = 0.1425 \) for the SW-Clover quark propagator, in the lattice Landau gauge,
Figure 1. NP mixing $Z$'s as a function of $\mu^2 a^2$. The solid (dashed) line is from SPT (BPT).

but with higher statistics, i.e. on an ensemble of 200 configurations. We have calculated the $Z$'s at different renormalization scales $\mu^2 a^2$ (cf. fig. 1). We note that $Z_2$ and $Z_4$ are very well defined and almost scale-independent in a large “window” of $\mu^2 a^2$, whereas $Z_1$ and $Z_3$ present a smaller window, i.e. a more pronounced scale-dependence. Moreover, $Z_4$ which is absent in 1-loop PT, is not negligible. We stress that the large fluctuations at small $\mu^2 a^2$ do not spoil the validity of the NPM, since in that region the perturbative matching to a continuum scheme is not reliable, as for any NP lattice method.

Using the NP $Z$'s of fig. 1, the chiral behaviour of the $B_K$ parameter studied in refs. [7,9] can be revisited. Since the mixing with $O_4$ starts at $\mathcal{O}(\alpha_0)$ we do not expect drastic changes in the chiral behaviour. If the chiral behaviour were sensibly different, we would not trust the matching to the continuum which has an uneludable perturbative uncertainty. In fig. 2 the result using the NP $Z$'s at $\mu^2 a^2 = 0.96$ is shown. We will concentrate here on the novelties introduced by the NP mixing with the complete basis ($N = 4$) with respect to the basis found in 1-loop PT ($N = 3$) [8], referring to [8] for a detailed analysis. Using the usual parametrization $\langle O^{\Delta S=2} \rangle = \alpha + \beta m_K^2 + \gamma (p\cdot q) + ...$, we have found that the use of the complete set of operators leaves the values of $\beta$ and $\gamma$ almost unchanged compared to the $N = 3$ case, whereas the value of $\alpha$ is slightly lowered, passing from $\alpha^{N=3} = 0.017(13)$ to $\alpha^{N=4} = -0.004(15)$. We note that the two values of $\alpha$ are compatible with each other and with zero. Within the current statistical limitations forced by the thinning approximation, we can only state that the value of $B_K$, proportional to $\gamma$, is unaltered, while the correct chiral behaviour of the continuum, signaled by the vanishing of $\alpha$, is recovered. These results are independent of the scale $\mu$ used, albeit not too small. We argue that this stability is due to the extremely clean determination of $Z_2$.

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