ON SZÁSZ-MIRAKYAN-JAIN OPERATORS PRESERVING EXPONENTIAL FUNCTIONS

G. C. Greubel
Newport News, VA, United States
jthomae@gmail.com

Abstract. In the present article we define the Jain type modification of the generalized Szász-Mirakjan operators that preserve constant and exponential mappings. Moments, recurrence formulas, and other identities are established for these operators. Approximation properties are also obtained with use of the Boham-Korovkin theorem.

Keywords. Szász-Mirakjan operators, Jain basis functions, Jain operators, Lambert W-function, Boham-Korovkin theorem.

2010 Mathematics Subject Classification: 33E20, 41A25, 41A36.

1. Introduction

In Approximation theory positive linear operators have been studied with the test functions \( \{1, x, x^2\} \) in order to determine the convergence of a function. Of interest are the Szász-Mirakjan operators, based on the Poisson distribution, which are useful in approximating functions on \([0, \infty)\) and are defined as, \([10], [12]\),

\[
S_n(f; x) = \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} e^{-nx} f \left( \frac{k}{n} \right).
\] (1.1)

In 1972, Jain \([9]\), used the Lagrange expansion formula

\[
\phi(z) = \phi(0) + \sum_{k=1}^{\infty} \frac{1}{k!} \left[ D^{k-1} \left( f^k(z) \phi'(z) \right) \right]_{z=0} \left( \frac{z}{f(z)} \right)^k
\] (1.2)

with \(\phi(z) = e^{\alpha z}\) and \(f(z) = e^{\beta z}\) to determined that

\[
1 = \alpha \sum_{k=0}^{\infty} \frac{1}{k!} (\alpha + \beta k)^{k-1} z^k e^{-(\alpha + \beta k)z}.
\] (1.3)

Jain established the basis functions

\[
L_{n,k}^{(\beta)}(x) = \frac{n x (nx + \beta k)^{k-1}}{k!} e^{-(nx + \beta k)}
\] (1.4)

with the normalization

\[
\sum_{k=0}^{\infty} L_{n,k}^{(\beta)}(x) = 1
\]
and considered the operators

$$B_n^\beta(f, x) = \sum_{k=0}^{\infty} L_{n,k}^{(\beta)}(x) f \left( \frac{k}{n} \right), \quad x \in [0, \infty).$$

(1.5)

In the reduction of $\beta = 0$ the Jain operators reduce to the Szász-Mirakjan operators.

Recently Acar, Aral, and Gonska [1] considered the Szász-Mirakjan operators which preserve the test functions $\{1, e^{ax}\}$ and established the operators

$$R_n^*(f; x) = e^{-n \gamma_n(x)} \sum_{k=0}^{\infty} \frac{(n \gamma_n(x))^k}{k!} f \left( \frac{k}{n} \right)$$

(1.6)

for functions $f \in C[0, \infty)$, $x \geq 0$, and $n \in \mathbb{N}$ with the reservation property

$$R_n^*(e^{2at}; x) = e^{2ax}.$$  

(1.7)

Here the Jain basis is used to extend the class of operators for the test functions $\{1, e^{-\lambda x}\}$ by defining Szász-Mirakyan-Jain operators which preserve the mapping of $e^{-\lambda x}$, for $\lambda, x > 0$. In the case of $\lambda = 0$ the Szász-Mirakyan-Jain operators are constant preserving operators. Moments, recurrence formulas, and other identities are established for these new operators. Approximation properties are also obtained with use of the Boham-Korovkin theorem. The Lambert W-function and related properties are used in the analysis of the properties obtained for the Szász-Mirakyan-Jain operators.

2. Szász-Mirakyan-Jain Operators

The Szász-Mirakyan-Jain operators, (SMJ), which are a generalization of the Szász-Mirakyan operators, are defined by

$$R_n^{(\beta)}(f; x) = n \alpha_n(x) \sum_{k=0}^{\infty} \frac{1}{k!} \left( n \alpha_n(x) + \beta k \right)^{k-1} e^{-(n \alpha_n(x) + \beta k)} f \left( \frac{k}{n} \right)$$

(2.1)

for $f \in C[0, \infty)$. It is required that these operators preserve the mapping of $e^{-\lambda x}$, as given by

$$R_n^{(\beta)}(e^{-\lambda x}; x) = e^{-\lambda x}$$

(2.2)

where $x \geq 0$ and $n \in \mathbb{N}$, and $\lambda \geq 0$. When $\beta = 0$ in (2.1) the operator reduces to that defined by Acar, Aral, and Gonska [1]. When $\beta = 0$ and $\alpha_n(x) = x$ the operator reduces to the well known Szász-Mirakyan operators given by (1.1). For $0 \leq \beta < 1$ and $\alpha_n(x) = x$ these operators reduce to the Szász-Mirakyan-Durrmeyer operators defined by Gupta and Greubel in [5].

Lemma 1. For $x \geq 0$, $\lambda \geq 0$, we have

$$\alpha_n(x) = \frac{-\lambda x}{n (z(\lambda/n, \beta) - 1)},$$

(2.3)

where $-\beta z(t, \beta) = W(-\beta e^{-\beta t})$ and $W(x)$ is the Lambert W-function.
Proof. Considering the mapping (2.2) it is required that
\[ e^{-\lambda x} = n \alpha_n(x) \sum_{k=0}^{\infty} \frac{1}{k!} (n \alpha_n(x) + \beta k)^{k-1} e^{-(n \alpha_n(x) + \beta k) \frac{e^{-\lambda k/n}}{k!}} \] (2.4)
Making use of (1.3) in the form
\[ e^{n \alpha_n(x) z} = n \alpha_n(x) \sum_{k=0}^{\infty} \frac{1}{k!} (n \alpha_n(x) + \beta k)^{k-1} e^{-(\beta z - \ln(z))k} \] (2.5)
and letting \( \beta z - \ln(z) = \beta + \frac{\lambda}{n} \) then
\[ e^{n \alpha_n(x) z} = n \alpha_n(x) \sum_{k=0}^{\infty} \frac{1}{k!} (n \alpha_n(x) + \beta k)^{k-1} e^{-(\beta + \lambda/n) k} \]
which provides
\[ e^{-\lambda x} = e^{n \alpha_n(x)(z-1)} \]
or
\[ \alpha_n(x) = -\frac{\lambda x}{n (z(\lambda/n, \beta) - 1)} \]
The value of \( z \) is determined by the equation \( \beta z - \ln(z) = \beta + \frac{\lambda}{n} \) which can be seen in the form
\[ z e^{-\beta z} = e^{-\beta - \lambda/n} \]
and has the solution
\[ z(\lambda/n, \beta) = -\frac{1}{\beta} W(-\beta e^{-\beta - \lambda/n}) \] (2.6)
where \( W(x) \) is the Lambert W-function. \( \Box \)

Remark 1. For the case \( \lambda \to 0 \) the resulting \( \alpha_n(x) \) is
\[ \lim_{\lambda \to 0} \alpha_n(x) = (1 - \beta) x. \]

Proof. For the case \( \lambda \to 0 \) the resulting \( z = z(\lambda/n, \beta) \) of (2.6) yields \( z(0, \beta) = 1 \). By considering
\[ \frac{\partial z}{\partial \lambda} = -\frac{1}{\beta} \frac{\partial}{\partial \lambda} W(-\beta e^{-\beta - \lambda/n}) = \frac{W(-\beta e^{-\beta - \lambda/n})}{n \beta (1 + W(-\beta e^{-\beta - \lambda/n}))} \]
and
\[ \lim_{\lambda \to 0} \frac{\partial z}{\partial \lambda} = -\frac{1}{n (1 - \beta)}. \]
Now, by use of L’Hospital’s rule,
\[ \lim_{\lambda \to 0} \alpha_n(x) = \frac{x}{n} \lim_{\lambda \to 0} \frac{\lambda}{z - 1} = \frac{x}{n} \lim_{\lambda \to 0} \frac{1}{\frac{\partial z}{\partial \lambda}} = (1 - \beta) x \]
as claimed. \( \Box \)
By taking the case of \( \lambda \to 0 \) the operators \( R_n^{(\beta)}(f; x) \) reduce from exponential preserving to constant preserving operators. In this case the operators \( R_n^{(\beta)}(f; x)|_{\lambda \to 0} \) are related to the Jain operators, \([1,5]\), by \( R_n^{(\beta)}(f; x) = B_n^\beta(f; (1 - \beta) x) \).

The SMJ operators are now completely defined by

\[
\begin{align*}
R_n^{(\beta)}(f; x) &= n \alpha_n(x) \sum_{k=0}^{\infty} \frac{1}{k!} (n \alpha_n(x) + \beta k)^{k-1} e^{-(n \alpha_n(x) + \beta k)} f \left( \frac{k}{n} \right), \\
\alpha_n(x) &= -\frac{\lambda x}{n(z(\lambda/n, \beta) - 1)}, \\
z(t, \beta) &= -\frac{1}{\beta} W(-\beta e^{-\beta-t})
\end{align*}
\]

and the requirement that \( R_n^{(\beta)}(e^{-\lambda t}; x) = e^{-\lambda x} \), for \( x \geq 0, \lambda \geq 0 \) and \( n \in \mathbb{N} \).

3. Moment Estimations

**Lemma 2.** The moments for the SMJ operators are given by:

\[
\begin{align*}
R_n^{(\beta)}(1; x) &= 1 \\
R_n^{(\beta)}(t; x) &= \frac{\alpha_n(x)}{1 - \beta} \\
R_n^{(\beta)}(t^2; x) &= \frac{\alpha_n^2(x)}{(1 - \beta)^2} + \frac{\alpha_n(x)}{n(1 - \beta)^3} \\
R_n^{(\beta)}(t^3; x) &= \frac{\alpha_n^3(x)}{(1 - \beta)^3} + \frac{3 \alpha_n^2(x)}{n^2(1 - \beta)^4} + (1 + 2 \beta) \frac{\alpha_n(x)}{n^2(1 - \beta)^5} \\
R_n^{(\beta)}(t^4; x) &= \frac{\alpha_n^4(x)}{(1 - \beta)^4} + \frac{6 \alpha_n^3(x)}{n(1 - \beta)^5} + (7 + 8 \beta) \frac{\alpha_n^2(x)}{n^2(1 - \beta)^6} + (1 + 8 \beta + 6 \beta^2) \frac{\alpha_n(x)}{n^3(1 - \beta)^7} \\
R_n^{(\beta)}(t^5; x) &= \frac{\alpha_n^5(x)}{(1 - \beta)^5} + \frac{10 \alpha_n^4(x)}{n(1 - \beta)^6} + 5(5 + 4 \beta) \frac{\alpha_n^3(x)}{n^2(1 - \beta)^7} \\
&\quad+ 15 (1 + 4 \beta + 2 \beta^2) \frac{\alpha_n^2(x)}{n^3(1 - \beta)^8} + \left( 1 + 22 \beta + 58 \beta^2 + 24 \beta^3 \right) \frac{\alpha_n(x)}{n^4(1 - \beta)^9}.
\end{align*}
\]

The proof follows directly from work of the author dealing with moment operators for the Jain basis, see \([4,5,6]\).

**Lemma 3.** Let, \( \phi = t - x \), then the central moments of the SMJ operators are:

\[
\begin{align*}
R_n^{(\beta)}(\phi^0; x) &= 1 \\
R_n^{(\beta)}(\phi^1; x) &= \frac{\alpha_n(x)}{1 - \beta} - x \\
R_n^{(\beta)}(\phi^2; x) &= \left( \frac{\alpha_n(x)}{1 - \beta} - x \right)^2 + \frac{\alpha_n(x)}{n(1 - \beta)^3} \\
R_n^{(\beta)}(\phi^3; x) &= \left( \frac{\alpha_n(x)}{1 - \beta} - x \right)^3 + \frac{3 \alpha_n(x)}{n(1 - \beta)^3} \left( \frac{\alpha_n(x)}{1 - \beta} - x \right) + (1 + 2 \beta) \frac{\alpha_n(x)}{n^2(1 - \beta)^5}
\end{align*}
\]
Proof. Utilizing the binomial expansion

\[ R_n^{(\beta)}(\phi^4; x) = \left( \frac{\alpha_n(x)}{1 - \beta} - x \right)^4 + \frac{6 \alpha_n(x)}{n (1 - \beta)^3} \left( \frac{\alpha_n(x)}{1 - \beta} - x \right)^2 \left( \frac{\alpha_n(x)}{1 - \beta} - x \right) + \frac{7 + 8 \beta}{n^2 (1 - \beta)^5} \alpha_n(x) \]

\[ \cdot \left( \frac{\alpha_n(x)}{1 - \beta} - x \right) + \frac{3 \alpha_n(x)}{n^3 (1 - \beta)^7} + \frac{3 \alpha_n(x)}{n^2 (1 - \beta)^5} \]

then

\[ R_n^{(\beta)}(\phi^5; x) = \left( \frac{\alpha_n(x)}{1 - \beta} - x \right)^5 + \frac{10 \alpha_n(x)}{n (1 - \beta)^3} \left( \frac{\alpha_n(x)}{1 - \beta} - x \right)^3 \left( \frac{\alpha_n(x)}{1 - \beta} - x \right) + \frac{5 \alpha_n(x)}{n^2 (1 - \beta)^5} \alpha_n(x) \]

\[ \cdot \left( \frac{\alpha_n(x)}{1 - \beta} - x \right) + \frac{\alpha_n(x)}{n^3 (1 - \beta)^7} \cdot \mu_1 + (5 + 4\beta) \left( \frac{\alpha_n(x)}{1 - \beta} - x \right) + 3 x \]

\[ \mu_2 = 3 (1 + 4\beta + 2\beta^2) \left( \frac{\alpha_n(x)}{1 - \beta} - x \right) + 2 (1 + 2\beta) x. \]

Proof. Utilizing the binomial expansion

\[ \phi^m = (t - x)^m = \sum_{k=0}^{m} (-1)^k \binom{m}{k} t^{m-k} x^k \]

then

\[ R_n^{(\beta)}(\phi^m; x) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} t^{m-k} x^k R_n^{(\beta)}(t^{m-k}; x). \]

With the use of (3.1) the first few values of \( m \) are:

\[ R_n^{(\beta)}(\phi^0; x) = R_n^{(\beta)}(t^0; x) = 1 \]

\[ R_n^{(\beta)}(\phi^1; x) = R_n^{(\beta)}(t; x) - x R_n^{(\beta)}(t^0; x) = \frac{\alpha_n(x)}{1 - \beta} - x \]

\[ R_n^{(\beta)}(\phi^2; x) = R_n^{(\beta)}(t^2; x) - 2x R_n^{(\beta)}(t; x) + x^2 R_n^{(\beta)}(t^0; x) \]

\[ = \frac{\alpha_n(x)}{(1 - \beta)^2} \frac{\alpha_n(x)}{n (1 - \beta)^3} - 2x \frac{\alpha_n(x)}{1 - \beta} + x^2 \]

\[ = \left( \frac{\alpha_n(x)}{1 - \beta} - x \right)^2 + \frac{\alpha_n(x)}{n (1 - \beta)^3} \]

The remainder of the central moments follow from (3.1) and (3.3). \( \square \)

Lemma 4. The central moments, given in Lemma 3, lead to the limits:

\[ \lim_{n \to \infty} n R_n^{(\beta)}(\phi; x) = \frac{\lambda x}{2! (1 - \beta)^2} \]

\[ \lim_{n \to \infty} n R_n^{(\beta)}(\phi^2; x) = \frac{x}{(1 - \beta)^2} \]
Proof. By setting $t = \lambda/n$ in (6.4) then

\[
\frac{(-\lambda)}{n (1 - \beta) (z(\lambda/n, \beta) - 1)} = 1 + \frac{v}{2!} + 2 \left(1 - 4\beta\right) \frac{v^2}{4!} + 6\beta^2 \frac{v^3}{4!} - (1 - 8\beta + 88\beta^2 + 144\beta^3) \frac{v^4}{6!} + 840\beta^2 (1 + 12\beta + 8\beta^2) \frac{v^5}{8!} + O(v^6),
\]

where $n (1 - \beta)^2 v = \lambda$. This expansion may be placed into the form

\[
\frac{\alpha_n(x)}{1 - \beta} - x = \frac{vx}{2!} \left(1 + (1 - 4\beta) \frac{v}{3!} + 12\beta^2 \frac{v^2}{4!} - O(v^3)\right).
\]

Multiplying by $n$ and taking the desired limit the resulting value is given by

\[
\lim_{n \to \infty} n R_n^{(\beta)} (\phi; x) = \frac{\lambda x}{2! (1 - \beta)^2}.
\]

It is evident that

\[
\left(\frac{\alpha_n(x)}{1 - \beta} - x\right)^2 = \left(\frac{vx}{2!}\right)^2 \left(1 + 2(1 - 4\beta) \frac{v}{3!} + 20 (1 - 8\beta + 52\beta^2) \frac{v^2}{6!} - O(v^3)\right)
\]

for which

\[
\left(\frac{\alpha_n(x)}{1 - \beta} - x\right)^2 + \frac{\alpha_n(x)}{n (1 - \beta)^3}
\]

\[
= \left(\frac{vx}{2!}\right)^2 \left(1 + 2(1 - 4\beta) \frac{v}{3!} + 20 (1 - 8\beta + 52\beta^2) \frac{v^2}{6!} - O(v^3)\right)
\]

\[
+ \frac{x}{n (1 - \beta)^2} \left(1 + \frac{v}{2!} + 2 (1 - 4\beta) \frac{v^2}{4!} - O(v^3)\right)
\]

Multiplying by $n$ and taking the limit yields

\[
\lim_{n \to \infty} n R_n^{(\beta)} (\phi^2; x) = \frac{x}{(1 - \beta)^2}.
\]

\[\square\]

Remark 2. Other limits may be determined by extending the work of Lemma 4, such as:

\[
\lim_{n \to \infty} R_n^{(\beta)} (\phi^m; x) = 0, \text{ for } m \geq 1
\]

\[
\lim_{n \to \infty} n R_n^{(\beta)} (\phi^m; x) = 0, \text{ for } m \geq 3
\]

\[
\lim_{n \to \infty} n^2 R_n^{(\beta)} (\phi^3; x) = \frac{2(1 + 2\beta) x + 3\lambda x^2}{2! (1 - \beta)^4}
\]

\[
(3.5)
\]

\[
\lim_{n \to \infty} n^2 R_n^{(\beta)} (\phi^4; x) = \frac{3 x^2}{(1 - \beta)^4}
\]

Lemma 5. Expansion on a general exponential weight is given by

\[
R_n^{(\beta)} (e^{-\mu t}; x) = e^{n \alpha_n(x) (z(\mu/n, \beta) - 1)},
\]
or
\[ R_n^{(\beta)}(e^{-\mu t}; x) = \text{Exp} \left[ -\lambda x \left( \frac{z(\mu/n, \beta) - 1}{z(\lambda/n, \beta) - 1} \right) \right] = \text{Exp} \left[ -\mu x \cdot \frac{\lambda}{\mu} \frac{z(\mu/n, \beta) - 1}{z(\lambda/n, \beta) - 1} \right] \] (3.6)

for \( \mu \geq 0 \) and has the expansion
\[ R_n^{(\beta)}(e^{-\mu t}; x) = e^{-\mu x} \left( 1 + \frac{\mu(\mu - \lambda)x}{2! n(1 - \beta)^2} + ((3\mu x - 4 - 8\beta)\mu - 3\mu x - 2 + 8\beta)\lambda \right) \frac{\mu(\mu - \lambda)x}{4! n^2(1 - \beta)^4} + \mathcal{O} \left( \frac{\mu(\mu - \lambda)x}{6! n^3(1 - \beta)^6} \right) \] (3.7)
where \(-\beta z(\mu/n, \beta) = W(-\beta e^{-\beta - \mu/n}), -\beta z(\lambda/n, \beta) = W(-\beta e^{-\beta - \lambda/n})\). In the limit as \( n \to \infty \) it is evident that
\[ \lim_{n \to \infty} R_n^{(\beta)}(e^{-\mu t}; x) = e^{-\mu x} \]
\[ \lim_{n \to \infty} n \left[ R_n^{(\beta)}(e^{-\mu t}; x) - e^{-\mu x} \right] = \frac{\mu(\mu - \lambda)x}{2!(1 - \beta)^2} e^{-\mu x}. \] (3.8)

**Proof.** It is fairly evident that
\[ R_n^{(\beta)}(e^{-\mu t}; x) = n\alpha_n(x) \sum_{k=0}^{\infty} \frac{1}{k!} (n\alpha_n(x) + \beta k)^{k-1} e^{-n\alpha_n(x) - (\beta + \mu/n)k} \]
which, by comparison to (2.5), leads to
\[ R_n^{(\beta)}(e^{-\mu t}; x) = e^{-n\alpha_n(x)(z(\mu/n, \beta) - 1)} = \text{Exp} \left[ -\lambda x \left( \frac{z(\mu/n, \beta) - 1}{z(\lambda/n, \beta) - 1} \right) \right]. \]
The expansion of (3.6), with use of (6.5), is given by
\[ R_n^{(\beta)}(e^{-\mu t}; x) = \sum_{k=0}^{\infty} \left( \frac{-\mu x}{k!} \right)^k \left( \frac{\lambda z(\mu/n, \beta) - 1}{\mu z(\lambda/n, \beta) - 1} \right)^k \]
\[ = \sum_{k=0}^{\infty} \left( \frac{-\mu x}{k!} \right)^k \left( 1 - \frac{k(\mu - \lambda)}{2!(1 - \beta)^2} + k((3k + 1 + 8\beta)\mu + (3k - 1 - 8\beta)\lambda) \frac{\mu - \lambda}{4!(1 - \beta)^4} + \mathcal{O} \left( \frac{\mu - \lambda}{6!(1 - \beta)^6} \right) \right) \]
\[ = e^{-\mu x} \left( 1 + \frac{\mu(\mu - \lambda)x}{2! n(1 - \beta)^2} + ((3\mu x - 4 - 8\beta)\mu - 3\mu x - 2 + 8\beta)\lambda \right) \frac{\mu(\mu - \lambda)x}{4! n^2(1 - \beta)^4} + \mathcal{O} \left( \frac{\mu(\mu - \lambda)x}{6! n^3(1 - \beta)^6} \right) \].

Taking the appropriate limits yields the desired results. \( \square \)

**Remark 3.** By use of Lemma 5 it may be stated that:
\[ \lim_{n \to \infty} n^2 R_n^{(\beta)}((e^{-t} - e^{-x})^4; x) = \frac{3x^2 e^{-4x}}{(1 - \beta)^4}. \] (3.9)
Proof. Since
\[ R_n^{(\beta)}((e^{-t} - e^{-x})^4; x) = R_n^{(\beta)}(e^{-4t}; x) - 4 e^{-x} R_n^{(\beta)}(e^{-3t}; x) + 6 e^{-2x} R_n^{(\beta)}(e^{-2t}; x) \]
\[ - 4 e^{-3x} R_n^{(\beta)}(e^{-t}; x) + e^{-4x} R_n^{(\beta)}(1; x) \]
then, by making use of \((3.7)\), it becomes evident that
\[ R_n^{(\beta)}((e^{-t} - e^{-x})^4; x) = 3 x^2 e^{-4x} n^2 \left( 1 - \beta \right) + O \left( \frac{1}{n^3 (1 - \beta)^6} \right). \]
Multiplying by \(n^2\) and taking the limit as \(n \to \infty\) yields the desired result. □

4. Analysis

**Theorem 1.** Given the sequence \(A_n : C^*[0, \infty) \to C^*[0, \infty)\) of positive linear operators which satisfies the conditions
\[ \lim_{n \to \infty} A_n(e^{-kt}; x) = e^{-kx}, \quad k = 0, 1, 2 \]
uniformly in \([0, \infty)\) then
\[ \lim_{n \to \infty} A_n(f; x) = f(x) \]
uniformly in \([0, \infty)\) for every \(f \in C^*[0, \infty)\).

The proof of this theorem 1 can be found in [2, 3, 8] and has, in essence, been demonstrated by \((3.7)\) for \(\mu \geq 0\). An estimate of the rate of convergence for the SMJ operators will require the use of the modulus of continuity
\[ \omega(F, \delta) = \operatorname{Sup}_{x,t>0} |F(t) - F(x)| \]
and can be seen as, for the case where \(F(e^{-t}) = f(t)\),
\[ \omega^*(f; \delta) = \operatorname{Sup}_{x,t>0, |e^{-t} - e^{-x}| \leq \delta} |f(t) - f(x)| \]
and is well defined for \(\delta \geq 0\) and all functions \(f \in C^*[0, \infty)\). In the present case the modulus of continuity has the property
\[ |f(t) - f(x)| \leq \left( 1 + \frac{(e^{-x} - e^{-t})^2}{\delta^2} \right) \omega^*(f; \delta), \quad \delta > 0. \tag{4.1} \]
Further properties and use of the modulus of continuity can be found in [3, 8]. The following theorem can also be found in the later.

**Theorem 2.** If a sequence of positive linear operators \(A_n : C^*[0, \infty) \to C^*[0, \infty)\) satisfy the equalities:
\[ \|A_n(1; x) - 1\|_{[0, \infty)} = a_n \]
\[ \|A_n(e^{-t}; x) - e^{-x}\|_{[0, \infty)} = b_n \]
\[ \|A_n(e^{-2t}; x) - e^{-2x}\|_{[0, \infty)} = c_n, \]
where $a_n, b_n$ and $c_n$ are bounded and finite, in the limit $n \to \infty$, then
\[
\|A_n(f; x) - f(x)\|_{0,\infty} \leq a_n |f(x)| + (2 + a_n) \omega^*(f, \sqrt{a_n + 2b_n + c_n}),
\]
for every function $f \in C^*(0,\infty)$, and satisfies
\[
\|A_n(f; x) - f(x)\|_{0,\infty} \leq 2 \omega^*(f, \sqrt{2b_n + c_n})
\]
for constant preserving operators.

Proof. Since
\[
A_n((e^{-t} - e^{-x})^2; x) = [A_n(e^{-2t}; x) - e^{-2x}] - 2 e^{-x} [A_n(e^{-t}; x) - e^{-x}] + e^{-2x} [A_n(1; x) - 1]
\]
then, by use of (4.1),
\[
A_n(|f(t) - f(x)|; x) \leq \left( A_n(1; x) + \frac{1}{\delta^2} A_n((e^{-t} - e^{-x})^2; x) \right) \omega^*(f, \delta)
\]
\[
\leq \left( 1 + a_n + \frac{a_n + 2b_n + c_n}{\delta^2} \right) \omega^*(f, \delta).
\]
By choosing $\delta = \sqrt{a_n + 2b_n + c_n}$ then
\[
A_n(|f(t) - f(x)|; x) \leq (2 + a_n) \omega^*(f, \sqrt{a_n + 2b_n + c_n}).
\]
Now, making use of
\[
|A_n(f; x) - f(x)| \leq |f| |A_n(1; x) - 1| + A_n(|f(t) - f(x)|; x)
\]
leads to the uniform estimation of convergence in the form
\[
\|A_n(f; x) - f(x)\|_{0,\infty} \leq a_n |f(x)| + (2 + a_n) \omega^*(f, \sqrt{a_n + 2b_n + c_n}).
\]
For constant preserving operators the property $\|A_n(1; x) - 1\|_{0,\infty} = a_n = 0$ holds and leads to
\[
\|A_n(f; x) - f(x)\|_{0,\infty} \leq 2 \omega^*(f, \sqrt{2b_n + c_n}).
\]

Remark 4. The SMJ operators satisfy
\[
\|P_n^{(\beta)}(f; x) - f(x)\|_{0,\infty} \leq 2 \omega^*(f, \sqrt{2b_n + c_n}).
\]

Proof. By using Lemma 2 it is evident that $P_n^{(\beta)}(1; x) = 1$ and yields $a_n = 0$. By using (3.7), of Lemma 5, it is seen that
\[
P_n^{(\beta)}(e^{-\mu t}; x) - e^{-\mu x} = e^{-\mu x} \left( \frac{\mu(\mu - \lambda)x}{2! n (1 - \beta)^2} - \frac{\Lambda(x, \mu, \lambda) \mu(\mu - \lambda)x}{4! n^2 (1 - \beta)^4} + O \left( \frac{1}{n^3 (1 - \beta)^6} \right) \right),
\]
where $\Lambda(x, \mu, \lambda) = (3\mu x - 4 - 8\beta) \mu - (3\mu x - 2 + 8\beta) \lambda$, and provides
\[
\|P_n^{(\beta)}(e^{-\mu t}; x) - e^{-\mu x}\| = \left\| \frac{\mu(\mu - \lambda)x e^{-\mu x}}{2! n(1 - \beta)^2} \left( 1 + \frac{2 \Lambda(x, \mu, \lambda)}{4! n (1 - \beta)^2} + O \left( \frac{1}{n^2 (1 - \beta)^4} \right) \right) \right\|
\]
which, for \( \mu \in \{1, 2\} \), the remaining limiting values, \( b_n \) and \( c_n \) can be seen to be bounded and finite. It is also evident that in the limiting case, \( n \to \infty \), \( b_n \) and \( c_n \) tend to zero. By the resulting statements of Theorem 2 it is determined that

\[
\| R_n^{(\beta)}(f; x) - f(x) \|_{[0, \infty)} \leq 2 \omega^*(f, \sqrt{2b_n + c_n}).
\]

as claimed. \( \square \)

For the SMJ operators a quantitative Voronovskaya-type theorem can be defined in the following way.

**Theorem 3.** Let \( f, f', f'' \in C^*[0, \infty) \) then

\[
\left| n[R_n^{(\beta)}(f; x) - f(x)] - \frac{\lambda x}{2!(1-\beta)^2} f'(x) - \frac{x}{n(1-\beta)^2} f''(x) \right| \\
\leq |\mu_n(x, \beta)| |f'(x)| + |\nu_n(x, \beta)| |f''(x)| \\
+ 2(2\nu_n(x, \beta) + \frac{x}{(1-\beta)^2} + \zeta_n(x, \beta)) \omega^*(f''; \frac{1}{\sqrt{n}})
\]

where

\[
\mu_n(x, \beta) = n R_n^{(\beta)}(\phi; x) - \frac{\lambda x}{2!(1-\beta)^2},
\]

\[
\nu_n(x, \beta) = \frac{1}{2!} \left(n R_n^{(\beta)}(\phi^2; x) - \frac{x}{(1-\beta)^2}\right)
\]

\[
\zeta_n(x, \beta) = n^2 \sqrt{R_n^{(\beta)}((e^{-x} - e^{-t})^4; x)} \sqrt{R_n^{(\beta)}(\phi^4; x)}.
\]

**Proof.** The Taylor expansion for the function \( f(x) \) is seen by

\[
f(t) = f(x) + f'(x)(t-x) + \frac{f''(x)}{2!}(t-x)^2 + \theta(t, x)(t-x)^2
\]

where \( 2\theta(t, x) = f''(\eta) - f''(x) \) for \( x \leq \eta \leq t \). Applying the SMJ operator to the Taylor expansion it is determined that

\[
|R_n^{(\beta)}(f(t); x) - f(x) R_n^{(\beta)}(1; x) - f'(x) R_n^{(\beta)}(\phi; x) - \frac{f''(x)}{2!} R_n^{(\beta)}(\phi^2; x)| \\
\leq |R_n^{(\beta)}(\theta(t, x) \phi^2; x)|.
\]

Using the results of lemma 4 and 5 this can be seen by

\[
\left| n[R_n^{(\beta)}(f; x) - f(x)] - \frac{\lambda x}{2!(1-\beta)^2} f'(x) - \frac{x}{n(1-\beta)^2} f''(x) \right| \\
\leq \left| n R_n^{(\beta)}(\phi; x) - \frac{\lambda x}{2!(1-\beta)^2} \right| |f'(x)| + \frac{1}{2!} \left| n R_n^{(\beta)}(\phi^2; x) - \frac{x}{(1-\beta)^2} \right| |f''(x)| \\
+ |n R_n^{(\beta)}(\theta(t, x) \phi^2; x)|
\]
or
\[
\left| n \left( R_n^{(\beta)}(f; x) - f(x) \right) - \frac{\lambda x}{2! (1 - \beta)^2} f'(x) - \frac{x}{2! (1 - \beta)^2} f''(x) \right|
\leq |\mu_n(x, \beta)| |f'(x)| + |\nu_n(x, \beta)| |f''(x)| + |n R_n(\theta(t, x) \phi^2; x)|
\]
where
\[
\mu_n(x, \beta) = n R_n^{(\beta)}(\phi; x) - \frac{\lambda x}{2! (1 - \beta)^2}
\]
\[
\nu_n(x, \beta) = \frac{1}{2!} \left( n R_n^{(\beta)}(\phi^2; x) - \frac{x}{(1 - \beta)^2} \right).
\]

By using (3.8) it is given that
\[
|\theta(t, x)| \leq \left( 1 + \frac{(e^{-t} - e^{-x})^2}{\delta^2} \right) \omega^*(f''; \delta)
\]
which becomes, when \(|e^{-t} - e^{-x}| \leq \delta\) is taken into consideration, \(|\theta(t, x)| \leq 2 \omega^*(f''; \delta)\). If \(|e^{-t} - e^{-x}| > \delta\) then \(|\theta(t, x)| \leq (2/\delta^2) (e^{-t} - e^{-x})^2 \omega^*(f''; \delta)\). Therefore, it can be concluded that
\[
|\theta(t, x)| \leq 2 \left( 1 + \frac{(e^{-t} - e^{-x})^2}{\delta^2} \right) \omega^*(f''; \delta).
\]
The term \(n R_n^{(\beta)}(\theta(t, x) \phi^2; x)\) becomes
\[
n R_n^{(\beta)}(\theta(t, x) \phi^2; x) \leq 2 n \left( R_n^{(\beta)}(\phi^2; x) + \frac{1}{\delta^2} R_n^{(\beta)}((e^{-t} - e^{-x})^2 \phi^2; x) \right) \omega^*(f''; \delta)
\]
which, by applying the Cauchy-Swarz inequality, becomes
\[
n R_n^{(\beta)}(\theta(t, x) \phi^2; x) \leq 2 n \left( R_n^{(\beta)}(\phi^2; x) + \frac{1}{\delta^2} \zeta_n(x, \beta) \right) \omega^*(f''; \delta),
\]
where
\[
\zeta_n(x, \beta) = n^2 \sqrt{R_n^{(\beta)}((e^{-x} - e^{-t})^4; x)} \sqrt{R_n^{(\beta)}(\phi^4; x)}.
\]
Now, by choosing \(\delta = 1/\sqrt{n}\), the desired result is obtained. \(\square\)

**Remark 5.** By use of Lemma 4 it is clear that \(\mu_n(x, \beta) \to 0\) and \(\nu_n(x, \beta) \to 0\) as \(n \to \infty\). Using (3.3) and (3.9) the limit of \(\zeta_n(x, \beta)\) becomes
\[
\lim_{n \to \infty} \zeta_n(x, \beta) = 3 x^2 e^{-2x} \frac{1}{(1 - \beta)^4}
\]
and yields
\[
\lim_{n \to \infty} \left( 2 \nu_n(x, \beta) + \frac{x}{(1 - \beta)^2} + \zeta_n(x, \beta) \right) = \frac{x}{(1 - \beta)^2} + \frac{3 x^2 e^{-2x}}{(1 - \beta)^4}.
\]
Corollary 1. Let \( f, f', f'' \in C^*[0, \infty) \) then the inequality
\[
\lim_{n \to \infty} n |R^{(\beta)}_n(f; x) - f(x)| = \frac{\lambda x}{2!(1-\beta)^2} f'(x) + \frac{x}{(1-\beta)^2} f''(x)
\]
holds for all \( x \in [0, \infty) \).

5. Further Considerations

Having established several results for the Szász-Mirakyan-Jain operators further considerations can be considered. One such consideration could be an application of a theorem found in a recent work of Gupta and Tachev, [7]. In order to do so the following results are required.

Lemma 6. Let \( z_\mu = z(\mu/n, \beta) \), \( \phi = t-x \), and \( f = \text{Exp}[n \alpha_n(x) (z_\mu-1)] \). The exponentially weighted moments are then given by:
\[
R^{(\beta)}_n(e^{-\mu x} \phi^0; x) = f
\]
\[
R^{(\beta)}_n(e^{-\mu x} \phi^1; x) = \left[ \frac{\alpha_n(x) z_\mu}{1 - \beta z_\mu} - x \right] f
\]
\[
R^{(\beta)}_n(e^{-\mu x} \phi^2; x) = \left[ \left( \frac{\alpha_n(x) z_\mu}{1 - \beta z_\mu} - x \right)^2 + \frac{\alpha_n(x) z_\mu}{n(1-\beta z_\mu)^2} \right] f
\]
\[
R^{(\beta)}_n(e^{-\mu x} \phi^3; x) = \left[ \left( \frac{\alpha_n(x) z_\mu}{1 - \beta z_\mu} - x \right)^3 + \frac{3 \alpha_n(x) z_\mu}{n(1-\beta z_\mu)^3} \left( \frac{\alpha_n(x) z_\mu}{1 - \beta z_\mu} - x \right) 
+ (1 + 2 \beta z_\mu) \frac{\alpha_n(x) z_\mu}{n^2(1-\beta z_\mu)^5} \right] f
\]
\[
R^{(\beta)}_n(e^{-\mu x} \phi^4; x) = \left[ \left( \frac{\alpha_n(x) z_\mu}{1 - \beta z_\mu} - x \right)^4 + \frac{6 \alpha_n(x) z_\mu}{n(1-\beta z_\mu)^3} \left( \frac{\alpha_n(x) z_\mu}{1 - \beta z_\mu} - x \right)^2 
+ (7 + 8 \beta z_\mu) \frac{\alpha_n(x) z_\mu}{n^2(1-\beta z_\mu)^5} \cdot \left( \frac{\alpha_n(x) z_\mu}{1 - \beta z_\mu} - x \right) + (1 + 8 \beta z_\mu + 6 \beta^2 z_\mu) \right] f
\]
\[
\frac{\alpha_n(x) z_\mu}{n^3(1 - \beta z_\mu)} + \frac{3 \alpha_n(x) z_\mu}{n^2(1-\beta z_\mu)^5} \right] f
\]
(5.1)

Proof. By using (2.4) then
\[
R^{(\beta)}_n(e^{-\mu x} \phi^m; x) = n \alpha_n \sum_{k=0}^{\infty} \frac{1}{k!} (n \alpha_n + \beta k)^{k-1} e^{-(n \alpha_n + \beta k)} e^{-\mu k/n} \left( \frac{k}{n} - x \right)^m
\]
\[
= (-1)^m \left( \frac{d}{d\mu} + x \right)^m e^{n \alpha_n(x) (z_\mu-1)}.
\]
For the case \( m = 1 \) it is given that

\[
R_n^{(β)}(e^{-μt} φ; x) = -\left( \frac{d}{dμ} + x \right) e^{α_n(x)(z_μ - 1)} = \left[ \frac{α_n(x) z_μ}{1 - β z_μ} - x \right] e^{α_n(x)(z_μ - 1)}.
\]

The remainder of the moments follow.

**Remark 6.** The ratio of \( R_n^{(β)}(e^{-μt} φ^4; x) \) and \( R_n^{(β)}(e^{-μt} φ^2; x) \) as \( n \to ∞ \) is

\[
\lim_{n \to ∞} \frac{R_n^{(β)}(e^{-μt} φ^4; x)}{R_n^{(β)}(e^{-μt} φ^2; x)} = 0,
\]

with order of convergence \( O(n^{-2}) \).

**Proof.** Consider the expansion of

\[
\frac{α_n(x) z_μ}{1 - β z_μ} = \frac{1 - β}{1 - β z_μ} \cdot \frac{α_n(x)}{1 - β}
\]

by making use of the expansion used in the proof of Lemma 4, (6.3), and by

\[
\frac{1 - β}{1 - β z_μ} = 1 - \frac{β \mu}{n(1 - β)^2} + \frac{3 β^2 \mu^2}{2! \cdot n^2(1 - β)^4} - \frac{(β + 14 β^2) \mu^3}{3! \cdot n^3(1 - β)^6} + O \left( \frac{μ^4}{n^4(1 - β)^8} \right)
\]

then

\[
\frac{α_n(x) z_μ}{1 - β z_μ} - x = \frac{x}{2 n(1 - β)^2} \left( (λ - 2μ) + \frac{σ(λ, μ)}{3! \cdot n(1 - β)^2} + O \left( \frac{1}{n^2(1 - β)^4} \right) \right).
\]

where \( σ(λ, μ) = (1 - 4β)λ - 6λμ + 6(1 - 2β + 3β^2)μ^2 \). By squaring this result and taking the limit it is determined that

\[
\lim_{n \to ∞} \frac{R_n^{(β)}(e^{-μt} φ^4; x)}{R_n^{(β)}(e^{-μt} φ^2; x)} = \lim_{n \to ∞} \frac{(λ - 2μ)^2 x^2}{4 n^2(1 - β)^4} \left( 1 + O \left( \frac{1}{n} \right) \right) \to 0.
\]

With Lemma 6 and Remark 5 use could be made of Theorem 5 of Gupta and Tachev, [7], which can be stated as

**Theorem 4.** Let \( E \) be a subspace of \( C[0, ∞) \) which contains the polynomials and suppose \( L_n : E \to C[0, ∞) \) is a sequence of linear positive operators preserving linear functions. Suppose that for each constant \( μ > 0 \), and fixed \( x \in [0, ∞) \), the operators \( L_n \) satisfy

\[
L_n(e^{-μt} (t - x)^2; x) ≤ Q(μ, x) R_n^{(β)}(e^{-μt} (t - x)^2; x).
\]

Additionally, if \( f \in C^2[0, ∞) \cap E \) and \( f^n \in Lip(α, μ) \), for \( 0 < α ≤ 1 \), then, for \( x ∈ [0, ∞) \),

\[
\left| L_n(f; x) - f(x) - \frac{f''(x)}{2} μ_n^{R_n^{(β)}} \right| ≤ \left\{ e^{-μx} + \frac{Q(μ, x)}{2} + \sqrt{\frac{Q(2μ, x)}{4}} \right\} \frac{μ_n^{R_n^{(β)}}}{μ_{n, 2}^{R_n^{(β)}}} \cdot \omega_1 \left( f^n, \frac{μ_{n, 4}^{R_n^{(β)}}}{μ_{n, 2}^{R_n^{(β)}}}; μ \right)
\]

where \( μ_{n, 2}^{R_n^{(β)}} = R_n^{(β)}(e^{-μt} (t - x)^2; x) \).
6. Appendix

Expansion of the function $f(ae^t)$ in powers of $t$ is given by

$$f(ae^t) = \sum_{k=0}^{\infty} \left[ D_k^f(ae^t) \right]_{t=0} \frac{tk}{k!} = f(a) + \sum_{k=1}^{\infty} p_k(a) \frac{tk}{k!},$$

(6.1)

where

$$p_n(a) = \left[ D_n^f(ae^t) \right]_{t=0} = \sum_{r=1}^{n} S(n, n-r) a^r f^{(r)}(a),$$

(6.2)

with $S(n, m)$ being the Stirling numbers of the second kind. Applying this expansion to the Lambert W-function the formula $W(xe^x) = x$ and the $n^{th}$-derivative coefficients, Oeis A042977, [11, 13] are required to obtain

$$-\frac{1}{\beta} W(-\beta e^{-\beta t}) = 1 + (1 - \beta) \sum_{n=1}^{\infty} \frac{B_{n-1}(\beta) u^n}{n!},$$

(6.3)

where $(1 - \beta)^2 u = t$ and $B_n(x)$ are the Eulerian polynomials of the second kind. Let $z(t)$ be the left-hand side of (6.3), $-\beta z(t) = W(-\beta e^{-\beta t})$, to obtain

$$t \frac{1}{(1 - \beta)(z(t) - 1)} = 1 - \frac{u}{2!} + 2 \left(1 - 4\beta\right) \frac{u^2}{4!} - 6\beta^2 \frac{u^3}{4!} - (1 - 8\beta + 88\beta^2 + 144\beta^3) \frac{u^4}{6!} - 840\beta^2 (1 + 12\beta + 8\beta^2) \frac{u^5}{8!} + O(u^6).$$

(6.4)

The ratio of $z(x) - 1$ to $z(t) - 1$ is given by

$$\frac{t \frac{z(x) - 1}{x - z(t) - 1}}{x} = 1 + \frac{(x-t)}{2!(1-\beta)^2} + \delta_1 \frac{(x-t)}{4!(1-\beta)^4} + \delta_2 \frac{(x-t)}{4!(1-\beta)^6} + O \left( \frac{(x-t)}{8!(1-\beta)^8} \right),$$

(6.5)

where

$$\delta_1 = 4(1 + 2\beta)x - 2(1 - 4\beta)t$$

$$\delta_2 = (1 + 8\beta + 6\beta^2)x^2 - (1 - 4\beta - 6\beta^2)xt + 6\beta^2 t^2$$

References

[1] Acar, T., Aral, A., Gonska, H., On Szász-Mirakyan Operators Preserving $e^{2ax}$, $a > 0$, Mediterr. J. Math., December 2016
[2] Altomare, F. & Campiti, M., Korovkin-type Approximation Theory and its Applications, De Gruyter Series Studies in Mathematics, Vol. 17, Walter de Gruyter & Co., Berlin, New York, 1994
[3] Boyanov, B. D., Veselinov, V. M., A note on the approximation of functions in an infinite interval by linear positive operators, Bull. Math. Soc. Sci. Math. Roum 14(62), 1970, 9-13. (no. 1)
[4] Greubel, G. C., A note on Jain basis functions, arXiv:1612.09385 [math.CA], 2016
[5] Gupta, V., and Greubel, G. C., Moment Estimations of new Szász-Mirakyan-Durrmeyer operators, Appl. Math. Comp., 271, 2015, 540-547
[6] Gupta, V., and Greubel, G. C., A Note on modified Phillips operators, arXiv:1604.08847 [math.CA], 2016
[7] Gupta, V. and Tachev, G., *On approximation properties of Phillips operators preserving exponential functions*, Mediterr. J. Math. (2017), 14:177

[8] Holhos, A., *The rate of approximation of functions in an infinite interval by positive linear operators*, Studia Univ. "Babeș-Bolyai", Mathematica 55(2), 2010, 133-142

[9] Jain, G. C., *Approximation of functions by a new class of linear operators*, J. Aust. Math. Soc. 13 (3), 1972, 271-276

[10] Mirakyan, G. M., *Approximation of continuous functions with the aid of polynomials* (Russian), Dokl. Akad. Nauk. SSSR, 31, 1941, 201-205

[11] Online Encyclopedia of Integer Sequences, sequence A042977, http://oeis.org/A042977

[12] Szász, O., *Generalization of S. Bernsteins polynomials to the infinite interval*, J. Res. Nat. Bur. Stand., 45, 1950, 239-245

[13] Weisstein, Eric W. "Lambert W-Function." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/LambertW-Function.html