Planetary and light motions from Newtonian theory: an amusing exercise

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Abstract

We attempt to see how closely we can formally obtain the planetary and light path equations of general relativity by employing certain operations on the familiar Newtonian equation. This paper is intended neither as an alternative to nor as a tool for grasping Einstein’s general relativity. Though the exercise can be understood by readers at large, it is especially recommended to teachers of relativity for an appreciative understanding of its peculiarity as well as its pedagogical value in the teaching of differential equations.

Everyone knows Newton’s theory of gravity and some know Einstein’s theory of general relativity (GR). Undoubtedly, GR is one of the most beautiful self-consistent modern creations in the realm of theoretical physics. It has been wonderfully tested against various astronomical observations to date including those in the solar system. However, at a popular level, a naive question is often asked as to whether the GR effects could have been interpreted using a more mundane theory than the abstract theory of GR in which gravity—which is as real a force as any other—has been ‘geometrized’. For instance, some ask the question: what is the difference between the bending of light rays in GR and that occurring in a refractive optical medium? The answer lies in the well-known fact that the propagation of light rays in a gravity field in the manner of GR can be exactly rephrased as propagation in an equivalent optical refractive medium with appropriate constitutive equations [1]. The refractive index can be employed in a new set of optical–mechanical equations so that a single equation covers motions of both massive and massless particles in a spherically symmetric field [2–4]. An approach of this kind provides a useful and interesting window to look at familiar observed GR results but, by no means, implies a replacement of GR.
The whole point of the above paragraph is that one inevitably needs to know the metric solutions of GR in advance. Only after knowing them, can one derive appropriate refractive indices and the method of optical–mechanical analogy in terms of these indices then exactly reproduces the GR geodesic equations. That is to say, we might employ different working methods but the physical content remains essentially that of GR. (There have been attempts to set aside GR altogether and propose alternative physics by introducing a variable test mass [5], or even assuming variable speeds of light in flat space [6]. These ideas have their own values and we are not going to discuss them here.)

The object of the present paper is somewhat different: we are not going to suggest any working method of the kind described above, but present an interesting calculation. (However, it must not be weighed against the grand edifice of GR.) Using PPN-like approximations on the Newtonian theory, we shall formally obtain planetary and light path equations. They resemble the path equations of GR only fortuitously and this is the amusing part. Apart from this, the contents could be instructive in exemplifying the role of numerically smaller terms in the differential equations.

To begin with, one recalls an earlier discussion of Møller [7] that has shown that the bending of light rays is due partly to the geometrical curvature of space and partly to the variation of light speed in a Newtonian potential. In fact, the ratio is exactly 50:50. The GR null trajectory equations can be integrated, once assuming a Euclidean space with a variable light speed and again a curved space with a constant light speed. This analysis and arguments clearly elucidate the complementary roles of curved space and Newtonian theory in the best possible manner. This complementarity motivates us to examine how far, if at all, we are able to introduce curvature effects in the path equations of the Newtonian theory. That is, we try to obtain, from the familiar Newtonian theory itself, the form of the known GR path equations of motion without geometrizing gravity. (It is known that the gravitational redshift is a prediction of GR, but it is also known that it can be predicted from the equivalence principle without using GR equations [8]. Hence we shall not address this result here.)

Let us start from the usual Kepler problem of a massive test particle moving around a spherical gravitating mass $M$ under the Newtonian inverse square law. Let $T$ and $V$ denote the kinetic and potential energies, respectively. Then $T + V = \text{constant} = \frac{h_0^2}{c_0^2}$ (say) implies in relativistic units

$$\frac{1}{2}[r^2 + r^2 \dot{\phi}^2] - mc_0^2 r^{-1} = \frac{E_0}{2},$$

where $m = GMc_0^{-2}$ and a dot denotes differentiation with respect to the Newtonian time $t$; $c_0$ is the speed of light in vacuum. The central nature of the force implies constancy of the angular momentum (the Lagrangian is independent of $\phi$) such that

$$r^2 \ddot{\phi} = h_0.$$  

With $u = \frac{1}{r}$, we can rewrite equation (1) as

$$h_0^2 \left[ u^2 + \left( \frac{du}{d\phi} \right)^2 \right] - 2mc_0^2 = E_0,$$

where the constant $E_0$ has the dimension of $c_0^2$. For bound material orbits $E_0 < 0$. Customarily, by differentiating again with respect to $\phi$, one finds a second-order differential equation that yields a Keplerian ellipse given by

$$u = \frac{1}{p}(1 + e\cos\phi),$$

where $e$ is the eccentricity, and $p = \frac{h_0^2}{2m}$ is the semi-latus rectum.
Let us redefine the radial variable \( u \rightarrow u' \) through the equations
\[
  u' = u \Phi(u),
\]
\[
  \Phi(u) = \left(1 + \frac{mu}{2}\right)^{-2},
\]
\[
  u' = \frac{1}{r}.
\]
(Aside: these transformations are not unfamiliar to those conversant with GR.) After some straightforward algebra, we get
\[
du' = \Phi(u) \Omega(u) du,
\]
where
\[
  \Omega(u) = \left(1 + \frac{mu}{2}\right)^{-1} \left(1 - \frac{mu}{2}\right),
\]
\[
  \Omega(u') = \left(1 - 2mu'\right)^{1/2},
\]
\[
  \Phi(u') = \frac{1}{4} \left[1 + (1 - 2mu')^{1/2}\right]^2.
\]
Note that \( \Phi(u) \) of equation (6) is numerically the same as \( \Phi(u') \) of equation (11). The same applies between \( \Omega(u) \) of equation (9) and \( \Omega(u') \) of equation (10). The following expansions can also be directly verified:
\[
  2mu = 2mu' + 2m^2u'^2 + 5m^3u'^3 + \cdots = 2mu' + O(m^2u'^2).
\]
This implies that, to first order, \( r \simeq r' \). Also,
\[
  \Phi^2(u')\Omega^2(u') = 1 - 4mu' + O(m^2u'^2).
\]
Let us now express equation (3) in terms of the new variable \( u' \). Multiplying both sides of equation (3) by \( \Phi^2 \Omega^2 \) and using equations (5)–(13), we get
\[
  h_0^2 \left[ \frac{1}{\Omega^2} u'^2 + \left(\frac{du'}{d\phi}\right)^2 \right] = c_0^2 \left[ E_0c_0^{-2} + 2mu' + O(m^2u'^2) \right] \Phi^2 \Omega^2.
\]
Simplifying further using equations (10) and (13), we have
\[
  h_0^2 \left[ u'^2 + \left(\frac{du'}{d\phi}\right)^2 - 2mu'^3 \right] = c_0^2 \left[ E_0c_0^{-2} + 2mu'(1 - 2E_0c_0^{-2}) + O(m^2u'^2) \right].
\]
Apply this equation to a practical situation, the solar system. At the site of Mercury, the planet nearest to the Sun, \( mu \simeq mu' \simeq 2.5 \times 10^{-8} \). Let us ignore the terms \( O(m^2u'^2) \) in comparison to the \( mu' \) term. Then equation (15) reduces to
\[
  h_0^2 \left[ u'^2 + \left(\frac{du'}{d\phi}\right)^2 - 2mu'^3 \right] = E_0 + 2mu'c_0^2(1 - 2E_0c_0^{-2}).
\]
Differentiating with respect to \( \phi \), we get
\[
  u' + \frac{d^2u'}{d\phi^2} = \frac{1}{p'^2} + 3mu'^2,
\]
where
\[
  \frac{1}{p'} = \frac{mc_0^2}{h_0^2}, \quad h' = \frac{h_0}{(1 - 2E_0c_0^{-2})^{3/2}}
\]
is a rescaled constant.
The final equation (17) seems suggestive with the usual perturbation term $3mu^2$ appearing: it is exactly of the same form as the GR path equation! One notes that the constant $h'$ involves the test particle energy $E_0$ similar to what one finds in the GR treatment. To see this, compare with equation (17) the corresponding GR equation given by (take henceforth $G = 1$)

$$u + \frac{d^2u}{d\phi^2} = \frac{1}{p} + 3mu^2,$$  

(19)

in which $p$ is given by $p = \frac{v_I}{m u_0^2}$, where $u_0$ is the test particle rest mass, $J = \frac{\alpha u_0^2}{c^2}$ is the constant angular momentum rescaled by the energy at infinity $U_0 = \sqrt{\frac{\hbar^2}{E_0 c^2} - 1}$ and the constant $U_3 = r^2 \frac{d\phi}{d\lambda}$, $\lambda$ being the affine parameter [9]. As usual, considering low velocity, we can take $U_0 = m_0 c^2$ and identifying the asymptotic value of $J$ as $h_0$, we have

$$p \simeq \frac{h_0^2}{M}.$$  

(20)

With this value of $p$, the GR perturbation term $3mu^2$ then gives the well-known perihelion advance of the Keplerian ellipse.

In our case, the parallel of $p$ from equation (17) is

$$p' := \left(\frac{mc_0^2}{\hbar^2}\right)^{-1} = \frac{h_0^2}{M(1 - 2E_0 c^{-2})}.$$  

(21)

Its asymptotic value can be computed using equation (1). For near circular orbits, the kinetic and potential energies are roughly of the same order of magnitude such that the velocity is $v^2 \sim \frac{M}{r} = mc_0^2$. Then, from equation (1), and noting that $u \simeq u'$ asymptotically, we can write $E_0 = \alpha mu_0^2$ where $-1 < \alpha < 1$. Then the denominator becomes $M(1 - 2\alpha mu')$. The term $2\alpha mu' \simeq 10^{-8}$ can be easily ignored compared to unity and we are left with

$$p' \simeq \frac{h_0^2}{M}.$$  

(22)

just as in equation (20). So we can replace $p'$ in equation (17) by its asymptotic value $p$ given either by equation (20) or (22).

For the motion of light, the situation is different: the dimensionless quantity $E_0 c^{-2}$ must be fixed to the value $\frac{1}{2}$ so that $p' \to \infty$. Recall that only a nonzero value for light ($E_0 \neq 0$) in the Newtonian theory is consistent with the zero value in GR\(^5\). (The zero rest mass of photons is a special relativistic or GR concept but is not a Newtonian concept.) Consequently, we have the equation of the light ray trajectory exactly as in GR:

$$u' + \frac{d^2u'}{d\phi^2} = 3mu'^2.$$  

(23)

Thus equations (17) and (23), respectively, seem to provide the same GR results as far as the weak field tests for the perihelion advance and the bending of light are concerned. To examine the situation more closely, recall what steps were involved. The first step is the radial rescaling $u \to u'$ which has no physical import. The second step is that, in arriving at

\(^5\) See, for instance, the treatise by Weinberg [10]. If we start with the usual GR geodesic equations, then, in the low velocity, weak field limit, they reduce to $r^2 \dot{\psi} \simeq h_0$ and $\frac{1}{2}(v^2 + \frac{\hbar^2}{m^2}) - \frac{\mu}{r} \simeq \frac{1}{2} \dot{E}$. For photons, $\dot{E} = 0$ so that for the total energy, we are left with a nonzero value $\frac{1}{2}$. If we start with the Newtonian equations, we instead get for light motion the value $\frac{1}{4}$ from equation (1) because $\dot{E}_0 = \frac{1}{2}$. The discrepant factor of 2 is actually a contribution from GR but it makes no difference to us as we have essentially started from the Newtonian theory. It is the nonzero value on the right-hand side of equation (1) for light that is consistent with the zero value in GR.
equation (16), we have ignored terms like $O(m^2 u^2)$ on numerical grounds. Note that it is only equation (15) *per se*, and *not* equation (17), that inverts exactly to the original equation (3) in the $(u, \varphi)$ coordinates describing the inverse square law. As we see, equation (17) produces an additional $3mu^2$ term! Strictly speaking, equation (17) is approximate to the extent we ignored the smaller terms compared to unity (of the order of $10^{-16}$ and less!) in arriving at it. Treating this equation (17) as an *exact* equation means that we are retaining the cubic additional term as the only perturbation while disregarding the remaining smaller perturbations. This is the only nontrivial step we have adopted in the above computation.

If we had retained the smaller terms in equation (15), then it could tell the original situation: the exact Newtonian orbits. It is our nontrivial, but numerically justified, omission of the smaller terms that has brought forth equations similar to those in GR. Thus, the exact solution of equation (15) is still a Keplerian ellipse but its expression does not look as familiar as in equation (4). Instead, in the primed coordinates, it looks like

$$u' = u \Phi(u) = \frac{\Phi(u)}{p} (1 + e \cos \varphi), \quad (24)$$

where $u$ is given by equation (4). Expressions might differ in look depending on the choice of coordinates, but the orbital shapes do not change.

One might think that though equation (17) looks different from equation (15), it still represents a Keplerian ellipse in the $(u', \varphi)$ coordinates. This is not the case since equation (17) is now nonlinear. We can find its solution by standard procedures starting with the zeroth-order solution $u'_0 = \frac{1}{p}(1 + e \cos \varphi)$, which is the solution of $u' + \frac{d^2 u'}{d \varphi^2} = \frac{1}{p}$. Equation (17) then gives the observed perihelion advance as $\frac{\pi \mathcal{M}}{p}$. (Note that if one starts with the same $u'_0$ in equation (15) or its second derivative form, one would eventually end up with equation (24) as the final solution.) Likewise, the exact equation for a straight line is

$$u' = \frac{1}{R} \Phi(u) \cos \varphi, \quad (25)$$

where $R$ is the distance from the origin. To zeroth order, $u'_0 = \frac{\cos \varphi}{p}$ is a solution of $u' + \frac{d^2 u'}{d \varphi^2} = 0$. By usual methods again with equation (23), one finds a total observed bending of light rays $\Delta \varphi \simeq \frac{2M}{R}$.

The procedure leading to equation (17) has some similarity with that in GR. In the curved spacetime of GR, one needs to consider the coordinate independent proper length $l$ instead of the radial coordinate $r$. Thus, in the Schwarzschild metric, $l$ is given by

$$l = \int \frac{dr}{\sqrt{1 - \frac{2m}{r}}} = \sqrt{r(-2m + r) + 2m \sqrt{2m - r} \arctan \sqrt{r/(2m - r)}} \sqrt{r(1 - \frac{2m}{r})}. \quad (26)$$

In terms of $(l, \varphi)$ coordinates, the GR equation (19) cannot maintain its form or assume another exact closed form due to the fact that $r$ cannot be expressed in terms of $l$ in a closed form. However, in the weak field region, $r \simeq l$, and we can maintain the form of equation (19) as it is, while ignoring higher order terms in $l$. In the present calculation, the background is Euclidean and so we can express $l$, using equation (8), as $l = \int dr = \int \Phi(r') \Omega^{-1}(r') dr'$. In our calculation, we have ignored higher order terms in $u'$ in the weak field region so that $r \simeq r'$ and we ended up with equation (17).

Can we physically interpret our nontrivial step as a modification of the Newtonian force law? In this context, it is to be noted that, historically, Newton himself attempted to modify his force law to explain some phenomenon (for details, see [5]). One might also recall other...
efforts, for instance, Sommerfeld’s calculation [11] for the precession of an electron in a Coulomb potential due to a proton ($Z = 1$):

$$\frac{d}{dt} \left( m_0 \frac{\vec{v}}{\sqrt{1 - v^2}} \right) = \frac{Ze^2}{r^2} \hat{r},$$

(27)

where $\hat{r}$ is a unit vector in the radial direction and $e$ is the electronic charge. However, it produces only $(1/6)^{th}$ of the observed perihelion advance of planets if the Coulomb potential on the right is replaced by the Newtonian potential. One could try the above special relativistic equation with another kind of force law on the right [12],

$$\frac{d}{dt} \left( m_0 \frac{\vec{v}}{\sqrt{1 - v^2}} \right) = \frac{Mm_0}{r^2(1 - v^2)^2} \hat{r},$$

(28)

where $v^2 = \dot{r}^2 + r^2 \dot{\phi}^2$ does produce the observed perihelion advance, but the difficulty is that its first integral does not produce the conserved relativistic energy. This is understandable because the potential is velocity dependent. Coming back to our calculation, one might say that equation (17) (which is the same as equation (19)) corresponds to a potential $V(r) = -\frac{M}{r} - \frac{M}{r^3}$ but then the last term leads to a dimensional mismatch (see [5]). Because of this, our procedure cannot be interpreted as a modification of the Newtonian force law. Also, there was absolutely no use of the concept of geometric curvature in the calculation; it was completely Euclidean.

Thus, we conclude that the similarity between equations (17) and (19) is only a fortuitous though amusing coincidence; it is just a mirage resulting from the choice of coordinates. There is absolutely no reason to prefer $(u', \phi)$ coordinates over others and, in this case, the formal coincidence will be lost. Nonetheless, the procedure illustrates something of pedagogical importance in the treatment of differential equations: one should be watchful with smaller terms! Their removal can *nonlinearize* a given linear equation (like going from equation (15) to (17)) and, conversely, their restoration can *linearize* a known nonlinear equation (like returning from equation (17) to (15)).

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