A FETI-DP PRECONDITIONER OF DISCONTINUOUS GALERKIN
METHOD FOR MULTISCALE PROBLEMS IN HIGH CONTRAST MEDIA

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ABSTRACT. In this paper we consider the second order elliptic partial differential equations
with highly varying (heterogeneous) coefficients on a two-dimensional region. The problems are
discretized by a composite finite element (FE) and discontinuous Galerkin (DG) Method. The
fine grids are in general nonmatching across the subdomain boundaries, and the subdomain
partitioning does not need to resolve the jumps in the coefficient. A FETI-DP preconditioner
is proposed and analyzed to solve the resulting linear system. Numerical results are presented
to support our theory.

1. Introduction

We consider the following problem: Find \( u^* \in H_0^1(\Omega) \) such that
\[
(1.1) \quad a(u^*, v) = (f, v) \quad \text{for all} \quad v \in H_0^1(\Omega),
\]
where
\[
a(u, v) := \int_\Omega \alpha(x) \nabla u \cdot \nabla v \, dx \quad \text{and} \quad (f, v) := \int_\Omega f v \, dx,
\]
where \( \Omega \subset \mathbb{R}^2 \) is a bounded polygonal domain. We assume that \( \alpha(x) \geq \alpha_0 > 0 \) and \( \alpha(x) \in L^\infty(\Omega) \) which may be discontinuous, while \( f(x) \in L^2(\Omega) \). The representative examples of the
problem (1.1) are subsurface flows in heterogeneous media [18, 19] where the heterogeneity varies
over a wide range of scales. The aim of this paper is to design and analyze a FETI-DP method
for solving such problems based on a composite FE/DG discretization.

Instead of using the full DG method over the whole domain, the composite FE/DG method
employs conforming FE methods inside the subdomains, while applies a DG discretization only
on the subdomain interfaces to deal with the nonmatching meshes across the interfaces; see [2, 5, 6, 7, 11]. The local bilinear forms of the discrete problem are composed of three symmetric
terms: the one associated with the energy, the one ensuring consistency and symmetry, and the
interior penalty term [25, 24] to handle the nonconforming FE spaces across the interfaces; see
cf. (2.6)-(2.9). Such discretization allows for nonmatching grids which provides greater flexibility
in the choice of mesh partitioning and memory storage. This may be useful particularly when
the coefficient varies roughly in one subdomain and mildly in the others.

FETI-DP methods, as well as FETI [15, 14] and BDDC [1, 20], have been well established as
a class of nonoverlapping domain decomposition methods for solving large-scale linear systems.
These methods have been used widely for standard continuous FE discretization, and verified
to be successful both theoretically and numerically; see [27] and references therein. FETI-DP
method was firstly introduced in [13] following by a theoretical analysis provided in [21]. In
FETI-DP algorithms, we need a relatively small number of continuity constraints across the

Key words and phrases. FETI-DP preconditioner, discontinuous Galerkin, multiscale problems.
The authors are thankful to M. Dryja for many fruitful discussions on the topics of this paper.

1
interface in each iteration step. The continuity of the solution across the subdomain interfaces is enforced by Lagrange multipliers, while the continuity at the subdomain vertices is enforced directly by assigning unique values. The methods were further improved in [12, 17, 27] to use the continuity constraints on the averages across the edges on subdomain interfaces. The FETI-DP methods have been developed more recently, and possess several advantages over the one-level FETI method; see cf. [27].

The FETI-DP method was firstly considered for composite FE/DG discretization in [7]. We will follow the same framework as described therein. In [7], the discontinuities of the coefficients are assumed to occur only across the subdomain interfaces. The main purpose of this paper is to extend the methodology to the case where the coefficients are allowed to have large jumps not only across but also along the subdomain interfaces and in the interior of the subdomains. We recall that such problems were investigated in the context of FETI methods in [22, 23].

In this paper, we will use the same DG bilinear form as in [11], construct our FETI-DP preconditioner as in [7], and define the components of the scaling matrix as proposed in [22]. For the theoretical aspect, we employ the cut off technique and the generalized discrete Sobolev type inequality, cf. [11], as well as the standard estimates of the edge and vertex functions, cf. [27]. It will be proved that the convergence of the FETI-DP method only weakly depends on the jump of coefficients, i.e., linearly depends on the contrast of the coefficients in the boundary layer. Here we define the boundary layer as the union of fine triangles that touch the subdomain boundaries. We also show that the condition number estimate of the proposed method is quadratic dependence on $H/h$ where $H$ is the subdomain diameter and $h$ is the fine mesh size. This quadratic dependence on $H/h$ can be relaxed to a weaker dependence of $H/h(1+\log H/h)^2$ under stronger assumptions on the coefficients in the interior of the subdomains.

The remaining part of this paper is organized as follows. In Section 2, we introduce the composite FE/DG formulation of problem (1.1). The FETI-DP method is presented in Section 3. The main results of the paper are given in Section 4 about the analysis of the condition number estimate. Numerical results are provided in Section 5 to confirm the theoretical analysis. In the last section we summarize our findings and discuss certain extensions.

Throughout this paper we denote a Sobolev space of order $k$ by the standard notation $H^k(\Omega)$ with norm given by $\| \cdot \|_{H^k(\Omega)}$ see e.g., [1] for exact definition. For $k = 0$ we use $L^2(\Omega)$ instead of $H^0(\Omega)$ and write the norm as $\| \cdot \|_{L^2(\Omega)}$. In addition, $A \simeq B$ stands for $C_1B \leq A \leq C_2B$ with positive constants $C_1$ and $C_2$ depending only on the shape regularity of the meshes.

2. DG Discretization

In this section we present the DG formulations of problem (1.1) that will be studied here.

Let the domain $\Omega = \bigcup_{i=1}^{N} \Omega_i$ and $\Omega_i$ be disjoint shape regular polygonal subdomains of diameters $H_i$. Denote the subdomain boundaries by $\partial \Omega_i$. For each $\Omega_i$, we introduce a shape regular triangulation $\mathcal{T}_h(\Omega_i)$ with the mesh size $h_i$. Note that the resulting triangulation of $\Omega$ is in general nonmatching across $\partial \Omega_i$.

We assume that the substructures $\{ \Omega_i \}_{i=1}^{N}$ form a geometrically conforming partition of $\Omega$, i.e., the intersection $\partial \Omega_i \cap \partial \Omega_j \ (i \neq j)$ is either empty, or a common vertex or edge of $\Omega_i$ and $\Omega_j$. Let us denote the common edge $E_{ij} = E_{ji} := \partial \Omega_i \cap \partial \Omega_j$. Although $E_{ij}$ and $E_{ji}$ are geometrically the same object, we will treat them separately since we consider different triangulations on $E_{ij} \subset \partial \Omega_i$ and on $E_{ji} \subset \partial \Omega_j$, with the mesh size of $h_i$ and $h_j$, respectively. In the text below, we use $E_{ijh}$ and $E_{ijh}$ to denote the set of nodal points of the triangulation on $E_{ij}$ and $E_{ji}$ with mesh sizes $h_i$ and $h_j$, respectively, and $E_{ijh}$ and $E_{jih}$ when the endpoints are included. Moreover, the two

inter
triangulations $\mathcal{T}_h(\Omega_i)$ and $\mathcal{T}_h(\Omega_j)$ can be merged to obtain a finer but the same mesh on $\bar{E}_{ij}$ and $E_{ji}$.

We also denote $E_{i\partial} := \partial\Omega_i \cap \partial\Omega$ when there is an intersection between $\partial\Omega_i$ and the global boundary $\partial\Omega$. Let us denote by $\mathcal{E}_i^0$ the set of indices to refer to the edges $E_{ij}$, i.e., $j$ of $\Omega_j$ which has a common edge $E_{ij}$ with $\Omega_i$, and by $\mathcal{E}_i^\partial$ the set of indices to refer to the edges $E_{i\partial}$. The set of indices of all edges of $\Omega_i$ is denoted by $\mathcal{E}_i := \mathcal{E}_i^0 \cup \mathcal{E}_i^\partial$.

For simplicity, we assume that the coefficient $\alpha(x) \geq \alpha_0 = 1$, which can be fulfilled by scaling \((1.1)\) with $1/\min_x \alpha(x)$. Without loss of generality again, we assume that $\alpha(x)$ is constant over each fine triangle. The analysis will depend on the coefficient in a boundary layer near subdomain boundaries. For each subdomain $\Omega_i$, we define the boundary layer $\Omega_i^b$ by
\[
\Omega_i^b := \bigcup \{ \bar{\tau} : \tau \in \Omega_i, \, \text{dist}(\tau, \partial\Omega_i) \leq h_i \},
\]
i.e., the union of fine triangles in $\mathcal{T}_h(\Omega_i)$ that touch the boundary $\partial\Omega_i$. Furthermore, we set
\[
(2.1) \quad \underline{\alpha}_i := \inf_{x \in \Omega_i^b} \alpha(x) \quad \text{and} \quad \overline{\alpha}_i := \sup_{x \in \Omega_i^b} \alpha(x).
\]
Let $\alpha_i(x)$ be $\alpha(x)$ restricted to $\Omega_i$. We define the harmonic averages along the edges $E_{ij}$ as follows:
\[
(2.2) \quad \alpha_{ij}(x) = \frac{2\alpha_i(x)\alpha_j(x)}{\alpha_i(x) + \alpha_j(x)} \quad \text{and} \quad h_{ij} = \frac{2h_i h_j}{h_i + h_j}.
\]
Note that the functions $\alpha_{ij}(x)$ and $h_{ij}$ are piecewise constant over the edge $E_{ij}$ on the mesh that is obtained by merging the partitions $\mathcal{T}_h(\Omega_i)$ and $\mathcal{T}_h(\Omega_j)$ along this common edge $E_{ij}$. It is easy to check that
\[
(2.3) \quad \min(\underline{\alpha}_i, \underline{\alpha}_j) \leq \alpha_{ij} \leq 2 \min(\underline{\alpha}_i, \underline{\alpha}_j) \quad \text{and} \quad \min(h_i, h_j) \leq h_{ij} \leq 2 \min(h_i, h_j).
\]

Let $V_h(\Omega_i)$ be the standard finite element space of continuous piecewise linear functions in $\Omega_i$. Define
\[
(2.4) \quad V_h(\Omega) = \prod_{i=1}^N V_h(\Omega_i) = V_h(\Omega_1) \times V_h(\Omega_2) \times \cdots \times V_h(\Omega_N),
\]
and represent functions $u \in V_h(\Omega)$ as $u = \{u_i\}_{i=1}^N$ with $u_i \in V_h(\Omega_i)$. We do not assume that functions in $V_h(\Omega_i)$ vanish on $\partial\Omega_i \cap \partial\Omega$.

The discrete problem obtained by the DG method is of the form: Find $u_h^* = \{u_{h,i}^*\}_{i=1}^N \in V_h(\Omega)$ with $u_{h,i}^* \in V_h(\Omega_i)$ such that
\[
(2.5) \quad a_h(u_{h,i}^*, v) = (f, v) \quad \text{for all} \quad v = \{v_i\}_{i=1}^N \in V_h(\Omega),
\]
where
\[
a_h(u, v) := \sum_{i=1}^N a_i(u, v) \quad \text{and} \quad (f, v) := \sum_{i=1}^N \int_{\Omega_i} f v_i \, dx.
\]
Here each local bilinear form $a_i(\cdot, \cdot)$ is given as the sum of three symmetric bilinear forms:
\[
(2.6) \quad a_i(u, v) := a_i(u, v) + s_i(u, v) + p_i(u, v),
\]
where
\[
(2.7) \quad a_i(u, v) := \int_{\Omega_i} \alpha_i(x) \nabla u_i \cdot \nabla v_i \, dx,
\]
\begin{align}
(2.8) \quad s_i(u,v) & := \sum_{j \in \mathcal{E}_i} \frac{1}{l_{ij}} \int_{E_{ij}} \alpha_{ij}(x) \left( \frac{\partial u_i}{\partial n_j}(v_j - v_i) + \frac{\partial v_i}{\partial n_j}(u_j - u_i) \right) ds, \\
\text{and} \quad
(2.9) \quad p_i(u,v) & := \sum_{j \in \mathcal{E}_i} \frac{1}{l_{ij}} \delta \int_{E_{ij}} \alpha_{ij}(x)(u_j - u_i)(v_j - v_i) ds.
\end{align}

Here $\frac{\partial}{\partial n}$ denotes the outward normal derivative on $\partial \Omega_i$, and $\delta$ is a positive penalty parameter. When $j \in \mathcal{E}_i^0$, we set $l_{ij} = 2$, and let $\alpha_{ij}$ and $h_{ij}$ be defined in (2.2). When $j \in \mathcal{E}_i^0$, we set $l_{ij} = 1$, $u_\partial = 0$, $v_\partial = 0$, and define $\alpha_{i\partial} = \alpha_i$ and $h_{i\partial} = h_i$.

We introduce the bilinear form
\begin{equation}
(2.10) \quad d_h(u,v) := \sum_{i=1}^N d_i(u,v)
\end{equation}
with
\begin{equation}
(2.11) \quad d_i(u,v) := a_i(u,v) + p_i(u,v).
\end{equation}

It is easy to check that $d_h(\cdot, \cdot)$ is symmetric and positive definite, which can induce a broken $H^1$ norm in $V_h(\Omega)$ by
\[ ||u||^2_h := d_h(u,u) = \sum_{i=1}^N \left( ||\alpha^{1/2} \nabla u_i||^2_L^2(\Omega_i) + \sum_{j \in \mathcal{E}_i} \frac{1}{l_{ij}} \frac{\delta}{h_{ij}} ||\alpha^{1/2}(u_i - u_j)||^2_L^2(E_{ij}) \right) \]
for any $u = \{u_i\}_{i=1}^N \in V_h(\Omega)$.

The next lemma characterizes the equivalence between the bilinear forms $a_h(\cdot, \cdot)$ and $d_h(\cdot, \cdot)$. This equivalence implies the existence and uniqueness of the solution to the discrete problem (2.5), and also allows us to use the bilinear form $d_h(\cdot, \cdot)$ instead of $a_h(\cdot, \cdot)$ for preconditioning.

**Lemma 2.1.** There exists $\delta_0 > 0$ such that for $\delta \geq \delta_0$ and for all $u \in V_h(\Omega)$, we have
\begin{equation}
(2.12) \quad \gamma_0 d_i(u,u) \leq a_i(u,u) \leq \gamma_1 d_i(u,u) \quad \text{for all} \quad i = 1, \cdots, N,
\end{equation}
and
\begin{equation}
(2.13) \quad \gamma_0 d_h(u,u) \leq a_h(u,u) \leq \gamma_1 d_h(u,u),
\end{equation}
where $\gamma_0$ and $\gamma_1$ are positive constants independent of $h_i$, $H_i$, $\alpha_i(x)$, and $u$. For the proof we refer to Lemma 2.1 of [11].

### 3. FETI-DP Preconditioner for the Schur Complement Systems

In this section, we will give the formulation of our FETI-DP method using the framework introduced in [27] [7].

#### 3.1. Schur Complement Systems and Discrete Harmonic Extensions

Firstly, we borrow the notations from [7]. Let
\[ \Omega'_i := \overline{\Omega_i} \cup \{ \cup_{j \in \mathcal{E}_i} \overline{E_{ji}} \}, \]
i.e., the union of $\overline{\Omega_i}$ and the $\overline{E_{ji}} \subset \partial \Omega_j$ with $j \in \mathcal{E}_i^0$, and let
\[ \Gamma_i := \partial \Omega_i \setminus \partial \Omega, \quad \Gamma'_i := \Gamma_i \cup \{ \cup_{j \in \mathcal{E}_i} \overline{E_{ji}} \}, \quad \text{and} \quad I_i := \Omega'_i \setminus \Gamma'_i. \]
Then we set
\begin{equation}
\Gamma := \bigcup_{i=1}^{N} \Gamma_i, \quad \Gamma' := \prod_{i=1}^{N} \Gamma_i', \quad \text{and} \quad I := \prod_{i=1}^{N} I_i.
\end{equation}

We introduce $W_i(\Omega'_i)$ as the FE space of functions defined on the nodal values of $\Omega'_i$. That is,
\begin{equation}
W_i(\Omega'_i) = W_i(\Omega_i) \times \prod_{j \in \mathcal{E}} W_i(E_{ji}),
\end{equation}
where $W_i(E_{ji})$ is the trace of the space $V_h(\Omega_j)$ on $E_{ji} \subset \partial \Omega_j$ with $j \in \mathcal{E}$. In the following, we use the same notation to denote both FE functions and their vector representations. The local bilinear form $a_i(\cdot, \cdot)$ in (2.6) is defined over $W_i(\Omega'_i) \times W_i(\Omega'_i)$, and the associated stiffness matrix is given by
\begin{equation}
\langle A'_i u_i, v_i \rangle = a'_i(u_i, v_i) \quad \text{for all} \quad u_i, v_i \in W_i(\Omega'_i),
\end{equation}
where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product associated to the vectors with nodal values. We will decompose $u_i \in W_i(\Omega'_i)$ as $u_i = (u_{i,I}, u_{i,I'})$, where $u_{i,I}$ represents values of $u_i$ at interior nodal points on $I_i$ and $u_{i,I'}$ at the nodal points on $\Gamma'_i$. Note that for subdomains $\Omega_i$ which intersect $\partial \Omega$ by edges, the nodal values of $W_i(\Omega'_i)$ on $\partial \Omega_i \setminus \Gamma'_i$ are treated as unknowns and belong to $I_i$. Hence, we can rewrite
\begin{equation}
W_i(\Omega'_i) = W_i(I_i) \times W_i(\Gamma'_i),
\end{equation}
and the matrix $A'_i$ as
\begin{equation}
A'_i = \begin{pmatrix}
A'_{i,I,I} & A'_{i,I,I'} \\
A'_{i,I',I} & A'_{i,I',I'}
\end{pmatrix},
\end{equation}
where the block rows and columns correspond to the nodal points of $I_i$ and $\Gamma'_i$, respectively.

The Schur Complement $S'_i$ of $A'_i$, with respect to the nodal points of $\Gamma'_i$, takes the form
\begin{equation}
S'_i := A'_{i,I',I} - A'_{i,I',I'} (A'_{i,I,I'})^{-1} A'_{i,I,I'}.
\end{equation}
Note that $S'_i$ satisfies the energy minimizing property
\begin{equation}
\langle S'_i u_i, v_i \rangle = \min a'_i(w_i, w_i)
\end{equation}
subject to the condition that $w_i = (w_{i,I}, w_{i,I'}) \in W_i(\Omega'_i)$ and $w_{i,I'} = u_{i,I'}$ on $\Gamma'_i$. The bilinear form $a'_i(\cdot, \cdot)$ is symmetric and nonnegative with respect to $W_i(\Omega'_i)$, see Lemma 2.1. The minimizing function of (3.7) is called the discrete harmonic extension in the sense of $a'_i(\cdot, \cdot)$, denoted by $H'_i u_{i,I'}$, and satisfies
\begin{equation}
a'_i(H'_i u_{i,I'}, v_i) = 0 \quad \text{for all} \quad v_i \in \tilde{W}_i(\Omega'_i)
\end{equation}
with $H'_i u_{i,I'} = u_{i,I'}$ on $\Gamma'_i$. Here $\tilde{W}_i(\Omega'_i)$ is the subspace of $W_i(\Omega'_i)$ of functions which vanish on $\Gamma'_i$. We also introduce $H_i u_{i,I'} \in W_i(\Omega'_i)$, the standard discrete harmonic extension in the sense of $a_i(\cdot, \cdot)$, which is defined by
\begin{equation}
a_i(H_i u_{i,I'}, v_i) = 0 \quad \text{for all} \quad v_i \in \tilde{W}_i(\Omega'_i)
\end{equation}
with $H_i u_{i,I'} = u_{i,I'}$ on $\Gamma'_i$.

Note that the extensions, $H_i$ and $H'_i$, differ from each other in the sense that $H_i u_{i,I'}$ at the interior nodes $I_i$ depends only on the nodal values of $u_{i,I'}$ on $\Gamma_i$ while $H'_i u_{i,I'}$ depends on the nodal values of $u_{i,I'}$ on $\Gamma'_i$. The next lemma shows the equivalence between $H_i$ and $H'_i$ in the energy form induced by $d_i(\cdot, \cdot)$. This equivalence will allow us to take advantages of all the
discrete Sobolev results known for $\mathcal{H}_i$ discrete harmonic extensions. The fundamental idea of the proof comes from [6], and we still include the proof here for completeness.

**Lemma 3.1.** For any $u_{i,G} \in W_i(\Gamma'_i)$, there exists a constant $C > 0$ independent of $h_i, H_i, \alpha_i(x)$ and $u_{i,G}$, such that

$$
(3.10) \quad d_i(\mathcal{H}_i u_{i,G}, \mathcal{H}_i u_{i,G}) \leq d_i(\mathcal{H}'_i u_{i,G}, \mathcal{H}'_i u_{i,G}) \leq C d_i(\mathcal{H}_i u_{i,G}, \mathcal{H}_i u_{i,G}).
$$

**Proof.** Here and below, for simplicity of presentation, we omit the subscript $G$ and denote $u_{i,G}$ by $u_i$ if there is no confusion.

The left-hand inequality of (3.10) follows from the energy minimizing property of the discrete harmonic extension $\mathcal{H}_i$ in the sense of $a_i(\cdot, \cdot)$, and the fact that $\mathcal{H}_i u_i = \mathcal{H}'_i u_i = u_i$ on $\Gamma'_i$. Here we remain to prove the right-hand inequality.

It is easy to verify that $\mathcal{H}_i \mathcal{H}'_i u_i = \mathcal{H}_i u_i$ since the extensions keep the boundary values. Note that we can represent $\mathcal{H}'_i u_i \in W_i(\Omega'_i)$ as

$$
(3.11) \quad \mathcal{H}'_i u_i = \mathcal{H}_i \mathcal{H}'_i u_i + \mathcal{P}_i \mathcal{H}'_i u_i,
$$

where $\mathcal{P}_i \mathcal{H}'_i u_i$ is the projection of $\mathcal{H}'_i u_i$ into $W_i(\Omega'_i)$ in the sense of $a_i(\cdot, \cdot)$, i.e., $\mathcal{P}_i \mathcal{H}'_i u_i \in W_i(\Omega'_i)$ and satisfies

$$
\begin{align*}
\forall \mathcal{H}'_i u_i, \mathcal{P}_i \mathcal{H}'_i u_i &\in W_i(\Omega'_i), \\
\forall v_i &\in \hat{W}_i(\Omega'_i),
\end{align*}
$$

Choosing $v_i = \mathcal{P}_i \mathcal{H}'_i u_i$, by Cauchy-Schwarz inequality, we obtain

$$
(3.12) \quad a_i(\mathcal{P}_i \mathcal{H}'_i u_i, \mathcal{P}_i \mathcal{H}'_i u_i) \leq a_i(\mathcal{H}'_i u_i, \mathcal{H}'_i u_i).
$$

Hence,

$$
(3.13) \quad d_i(\mathcal{H}'_i u_i, \mathcal{H}'_i u_i) = d_i(\mathcal{H}'_i u_i, \mathcal{H}_i \mathcal{H}'_i u_i) + d_i(\mathcal{H}'_i u_i, \mathcal{P}_i \mathcal{H}'_i u_i) = d_i(\mathcal{H}'_i u_i, \mathcal{H}_i u_i) + d_i(\mathcal{H}'_i u_i, \mathcal{P}_i \mathcal{H}'_i u_i).
$$

Since the bilinear form $d_i(\cdot, \cdot)$ is symmetric and nonnegative, by Cauchy-Schwarz inequality again, we have

$$
(3.14) \quad d_i(\mathcal{H}'_i u_i, \mathcal{H}_i u_i) \leq \varepsilon d_i(\mathcal{H}'_i u_i, \mathcal{H}'_i u_i) + \frac{1}{4\varepsilon} d_i(\mathcal{H}_i u_i, \mathcal{H}_i u_i)
$$

with arbitrary $\varepsilon > 0$.

Since $\mathcal{P}_i \mathcal{H}'_i u_i \in \hat{W}_i(\Omega'_i)$, using the formulations (2.9) and (3.8), we get

$$
\begin{align*}
d_i(\mathcal{H}'_i u_i, \mathcal{P}_i \mathcal{H}'_i u_i) &= a_i(\mathcal{H}'_i u_i, \mathcal{P}_i \mathcal{H}'_i u_i),
\end{align*}
$$

and

$$
0 = a_i(\mathcal{H}'_i u_i, \mathcal{P}_i \mathcal{H}'_i u_i) = a_i(\mathcal{H}'_i u_i, \mathcal{P}_i \mathcal{H}'_i u_i) + s_i(\mathcal{H}'_i u_i, \mathcal{P}_i \mathcal{H}'_i u_i),
$$

which together imply that

$$
\begin{align*}
d_i(\mathcal{H}_i u_i, \mathcal{P}_i \mathcal{H}'_i u_i) &= - s_i(\mathcal{H}'_i u_i, \mathcal{P}_i \mathcal{H}'_i u_i).
\end{align*}
$$

We proceed the same lines of Lemma 2.1 in [11], and finally obtain

$$
(3.15) \quad d_i(\mathcal{H}_i u_i, \mathcal{P}_i \mathcal{H}'_i u_i) \leq C \left( 2a_i(\mathcal{P}_i \mathcal{H}'_i u_i, \mathcal{P}_i \mathcal{H}'_i u_i) + \frac{1}{2\varepsilon} p_i(\mathcal{H}_i u_i, \mathcal{H}_i u_i) \right) \leq C \left( 2a_i(\mathcal{H}'_i u_i, \mathcal{H}'_i u_i) + \frac{1}{2\varepsilon} p_i(\mathcal{H}_i u_i, \mathcal{H}_i u_i) \right) \leq C \left( 2d_i(\mathcal{H}_i u_i, \mathcal{H}_i u_i) + \frac{1}{2\varepsilon} d_i(\mathcal{H}_i u_i, \mathcal{H}_i u_i) \right).
$$
where we have used \[3.12\].

Combining \[3.14\] and \[3.15\], we have
\[
d_i(\mathcal{H}_i u_i, \mathcal{H}_i u_i) \leq C \left( \epsilon d_i(\mathcal{H}_i u_i, \mathcal{H}_i u_i) + \frac{1}{4\epsilon} d_i(\mathcal{H}_i u_i, \mathcal{H}_i u_i) \right).
\]

The right-hand side of \[3.16\] follows by choosing a sufficiently small $\epsilon$.

Lemma \[2.1\] and Lemma \[3.1\] together directly give the following corollary.

**Corollary 3.2.** For any $u_i, u_i \in W_i(\Gamma_i)$, there exist positive constants $C_0$ and $C_1$ independent of $h_i, H_i, \alpha_i(x)$ and $u_i, u_i$, such that
\[
C_0 d_i(\mathcal{H}_i u_i, \mathcal{H}_i u_i) \leq d_i(\mathcal{H}_i u_i, \mathcal{H}_i u_i) \leq C_1 d_i(\mathcal{H}_i u_i, \mathcal{H}_i u_i).
\]

Let us introduce the product spaces
\[
W(\Omega') := \prod_{i=1}^{N} W_i(\Omega'_i) \quad \text{and} \quad W(\Gamma') := \prod_{i=1}^{N} W_i(\Gamma'_i).
\]

That is, a function $u \in W(\Omega')$ means that $u = \{u_i\}_{i=1}^{N}$ with $u_i \in W_i(\Omega'_i)$, and a function $u_{i'} \in W(\Gamma')$ means that $u_{i'} = \{u_i\}_{i=1}^{N}$ with $u_i \in W_i(\Gamma'_i)$; see \[3.2\] and \[3.4\] for the definitions of $W_i(\Omega'_i)$ and $W_i(\Gamma'_i)$, and also \[3.1\] for notation. We also define
\[
S' := \text{diag}\{S'_1, \ldots, S'_N\},
\]
where $S'_i$ is given in \[3.6\].

### 3.2. FEIT-DP Problem

Secondly, we formulate \[2.5\] as a constrained minimization problem.

With a similar decomposition as \[3.2\], we can partition $W_i(\Gamma'_i)$ as
\[
W_i(\Gamma'_i) = W_i(\Gamma_i) \times \prod_{j \in \mathcal{E}_i^0} W_i(\overline{E}_{ij}),
\]
where $W_i(\Gamma_i)$ is the trace of the space $V_h(\Omega_i)$ on $\Gamma_i$. A function $u_i \in W_i(\Gamma'_i)$ can be written as
\[
u_i = \{(u_i)_i, \{(u_i)_j\}_{j \in \mathcal{E}_i^0}\},
\]
where $(u_i)_i$ is $u_i$ restricted to $\overline{E}_{ij}$ and $(u_i)_j$ is $u_i$ restricted to $\overline{E}_{ij}$ for all $j \in \mathcal{E}_i^0$.

We consider $\widehat{W}(\Gamma')$ as the subspace of $W(\Gamma')$ which contains the continuous functions on $\Gamma$. A function $u = \{u_i\}_{i=1}^{N} \in W(\Gamma')$ is defined to be continuous on $\Gamma$ in the sense that for all $1 \leq i \leq N$ we have
\[
\begin{align*}
(u_i)_i(x) = (u_j)_i(x) & \quad \text{for all } x \in \overline{E}_{ij} \quad \text{for all } j \in \mathcal{E}_i^0, \\
(u_i)_j(x) = (u_j)_j(x) & \quad \text{for all } x \in \overline{E}_{ij} \quad \text{for all } j \in \mathcal{E}_i^0.
\end{align*}
\]
We say that $u = \{u_i\}_{i=1}^{N} \in W(\Omega')$, where $u_i = \{u_i, u_i\}$ with $u_i, u_i \in W_i(I_i)$ and $u_i, u_i \in W_i(\Gamma'_i)$, is continuous on $\Gamma$ if \{\{u_i, u_i\}\}_{i=1}^{N} \in W(\Gamma')$ satisfies the continuity condition \[3.21\]. The subspace of $W(\Omega')$ of functions which are continuous on $\Gamma$ is denoted by $\widehat{W}(\Omega')$; c.f., Definition 3.3 in [7]. Note that there is a one-to-one correspondence between vectors in $V_h(\Omega)$ and $\widehat{W}(\Omega')$.

Next we define the nodal points associated with the corner variables by
\[
V_i' := V_i \cup \{\cup_{j \in \mathcal{E}_i^0} \partial E_{ij}\} \quad \text{where} \quad V_i := \{\cup_{j \in \mathcal{E}_i^0} \partial E_{ij}\}.
\]
Lemma 3.3. Let \( \tilde{W}(\Omega') \subset W(\Omega') \) and \( \tilde{W}(\Gamma') \subset W(\Gamma') \) as the space of functions that are continuous on all the \( \mathcal{V}_i' \). A function \( u = \{u_i\}_{i=1}^N \in W(\Gamma') \) is defined to be continuous at the corners \( \mathcal{V}_i' \) in the sense that for all \( 1 \leq i \leq N \) we have

\[
\begin{cases}
(u_i)_i(x) = (u_j)_j(x) & \text{at } x \in \partial E_{ij} \text{ for all } j \in \mathcal{E}_i^0, \\
(u_i)_i(x) = (u_j)_j(x) & \text{at } x \in \partial E_{ji} \text{ for all } j \in \mathcal{E}_i^0.
\end{cases}
\] (3.23)

We say that \( u = \{u_i\}_{i=1}^N \in \tilde{W}(\Omega') \), where \( u_i = (u_{i,I}, u_{i,\Gamma}) \) with \( u_{i,I} \in W_i(I_i) \) and \( u_{i,\Gamma} \in W_i(\Gamma_i') \), is continuous on \( \mathcal{V}_i' \) if \( \{u_i\}_{i=1}^N \in W(\Gamma') \) satisfies the continuity condition (3.23). The subspace of \( W(\Omega') \) of functions which are continuous on \( \mathcal{V}_i' \) is denoted by \( \tilde{W}(\Omega') \); c.f., Definition 4.1 in [7].

We now consider the subspace \( \tilde{W}(\Omega') \subset \tilde{W}(\Gamma') \subset W(\Gamma') \).

We can represent \( u \in \tilde{W}(\Omega') \) as \( u = (u_I, u_{II}, u_{\Delta}) \), where the subscript \( I \) refers to the interior degrees of freedom at nodal points \( I \); see (3.1), the \( \Pi \) refers to the primal(\( \Pi \)) variables at the corners \( \mathcal{V}_i' \) for all \( 1 \leq i \leq N \), and the \( \Delta \) refers to the dual(\( \Delta \)) variables at the remaining nodal points on \( \Gamma_i'(\mathcal{V}_i') \) for all \( 1 \leq i \leq N \). Similarly, a vector \( u \in \tilde{W}(\Gamma') \) can be uniquely decomposed as \( u = (u_{II}, u_{\Delta}) \). Therefore, we can represent \( \tilde{W}(\Gamma') = \tilde{W}_I(\Gamma') \times W_\Delta(\Gamma') \), where \( \tilde{W}_I(\Gamma') \) and \( W_\Delta(\Gamma') \) refer to the \( \Pi \)- and \( \Delta \)-degrees of freedom of \( \tilde{W}(\Gamma') \), respectively.

Let \( \tilde{A} \) be the stiffness matrix obtained by restricting the block diagonal matrix \( A' \) from \( W(\Omega') \) to \( \tilde{W}(\Omega') \), where \( A' = \text{diag}\{A'_{I_1}, \ldots, A'_{N}\} \). Note that the matrix \( \tilde{A} \) is no longer block diagonal since there are couplings between primal(\( \Pi \)) variables. Using the decomposition \( u = (u_I, u_{II}, u_{\Delta}) \in \tilde{W}(\Omega') \), we can partition \( \tilde{A} \) as

\[
\tilde{A} = \begin{pmatrix} A'_{II} & A'_{I\Pi} & A'_{I\Delta} \\ A'_{I\Pi} & \tilde{A}_{\Pi\Pi} & A'_{I\Pi} \\ A'_{I\Delta} & A'_{I\Pi} & A'_{\Delta\Delta} \end{pmatrix}.
\] (3.24)

Note that the only coupling across subdomains are through the \( \Pi \) variables where the matrix \( \tilde{A} \) is subassembled.

Once the variables of \( I \) and \( \Pi \) sets are eliminated, the Schur complement matrix associated with the \( \Delta \)-variables is obtained of the form

\[
\tilde{S} := A'_{\Delta\Delta} - (A'_{\Delta I} A'_{I\Pi}) \left( \begin{pmatrix} A'_{II} & A'_{I\Pi} \\ A'_{I\Pi} & \tilde{A}_{\Pi\Pi} \end{pmatrix} \right)^{-1} \begin{pmatrix} A'_{I\Delta} \\ A'_{I\Pi} \end{pmatrix}.
\] (3.25)

Note that \( \tilde{S} \) is defined on the vector space \( W_\Delta(\Gamma') \).

Lemma 3.3. Let \( \tilde{A} \) and \( \tilde{S} \) be defined in (3.24) and (3.25). For any \( u_\Delta \in W_\Delta(\Gamma') \), it holds

\[
\langle \tilde{S} u_\Delta, u_\Delta \rangle = \min \langle \tilde{A} w, w \rangle,
\]

where the minimum is taken over \( w = (w_I, w_{II}, w_\Delta) \in \tilde{W}(\Omega') \) with \( w_\Delta = u_\Delta \).

The proof of the above lemma can be found in Lemma 6.22 of [27] and Lemma 4.2 of [21].

Next we introduce some notations to define the jump matrix \( B_\Delta \). The vector space \( W_\Delta(\Gamma') \) can be further decomposed as

\[
W_\Delta(\Gamma') := \prod_{i=1}^N W_{i,\Delta}(\Gamma_i'),
\] (3.26)
where the local space $W_{i,\Delta}(\Gamma'_i)$ includes functions associated with variables at the nodal points of $\Gamma'_i \setminus \Gamma'_i$. Hence, a vector $u_{\Delta} \in W_{\Delta}(\Gamma')$ can be represented as $u_{\Delta} = \{u_{i,\Delta}\}_{i=1}^N$ with $u_{i,\Delta} \in W_{i,\Delta}(\Gamma'_i)$. Moreover, the vector $u_{i,\Delta} \in W_{i,\Delta}(\Gamma'_i)$ can be partitioned as 

$$u_{i,\Delta} = \{(u_{i,\Delta})_i, \{(u_{i,\Delta})_j\}_{j \in \mathcal{E}_i}\}$$

with $(u_{i,\Delta})_i = u_{i,\Delta}|_{\Gamma'_i \setminus \Gamma_i}$ and $(u_{i,\Delta})_j = u_{i,\Delta}|_{E_{ij}}$. In order to measure the jump of $u_{\Delta} \in W_{\Delta}(\Gamma')$ across the $\Delta-$nodes, we introduce the space

$$\tilde{W}_{\Delta}(\Gamma) := \prod_{i=1}^N V_i(\Gamma_i \setminus \Gamma_i),$$

where $V_i(\Gamma_i \setminus \Gamma_i)$ is the restriction of $V_i(\Omega_i)$ to $\Gamma_i \setminus \Gamma_i$. The jumping matrix $B_{\Delta} : W_{\Delta}(\Gamma') \rightarrow \tilde{W}_{\Delta}(\Gamma)$ is constructed as follows: let $u_{\Delta} = \{u_{i,\Delta}\}_{i=1}^N \in W_{\Delta}(\Gamma')$ and let $v := B_{\Delta}u_{\Delta}$ where $v = \{v_i\}_{i=1}^N \in \tilde{W}_{\Delta}(\Gamma)$ satisfies

$$(3.27) \quad v_i = (u_{i,\Delta})_i - (u_{j,\Delta})_i \text{ on } E_{ij} \text{ for all } j \in \mathcal{E}_i.$$

The jumping matrix $B_{\Delta}$ can be written as

$$(3.28) \quad B_{\Delta} = (B_{\Delta}^{(1)}, B_{\Delta}^{(2)}, \ldots, B_{\Delta}^{(N)}),$$

where the rectangular matrix $B_{\Delta}^{(i)}$ consists of columns of $B_{\Delta}$ attributed to the $i-$th components of the product space $W_{\Delta}(\Gamma')$. The entries of $B_{\Delta}^{(i)}$ consist of values of $\{0, 1, -1\}$. It is easy to see that $\text{Range}(B_{\Delta}) = \tilde{W}_{\Delta}(\Gamma)$, and $B_{\Delta}$ has full rank. In addition, if $u = (u_{\Omega}, u_{\Delta}) \in \tilde{W}(\Gamma')$ and $B_{\Delta}u_{\Delta} = 0$ then $u \in \tilde{W}(\Gamma')$.

We can reformulate the discrete problem (2.5) on the space of $W_{\Delta}(\Gamma')$, as a minimization problem with constraints given by the continuity requirement: Find $u_{\Delta}^* \in W_{\Delta}(\Gamma')$ such that

$$(3.29) \quad J(u_{\Delta}^*) = \min J(v_{\Delta}),$$

where the minimum is taken over $v_{\Delta} \in W_{\Delta}(\Gamma')$ with constraints $B_{\Delta}v_{\Delta} = 0$. The objective function

$$(3.30) \quad J(v_{\Delta}) := \frac{1}{2}\langle \tilde{S}v_{\Delta}, v_{\Delta} \rangle - \langle \tilde{g}_{\Delta}, v_{\Delta} \rangle,$$

where $\tilde{S}$ is defined in (3.29) and

$$\tilde{g}_{\Delta} := f_{\Delta} - (A'_{\Delta I} A'_{\Delta II}) \left( \begin{array}{cc} A'_{II} & A'_{III} \\ A''_{II} & A''_{III} \end{array} \right)^{-1} \left( \begin{array}{c} f_I \\ f_{II} \end{array} \right).$$

Here $f = \{f_i\}_{i=1}^N \in V_h(\Omega)$, where $f_i$ is the load vector associated with the subdomain $\Omega_i$, and $f$ can be represented as $f = (f_I, f_{II}, f_{\Gamma_I})$. The forcing term $f_{\Delta} \in W_{\Delta}(\Gamma')$ is defined by $f_{\Delta} = \{f_{i,\Delta}\}_{i=1}^N$, where the entries $f_{i,\Delta}$ are defined as $f_{\Omega_i} f_{v_{i,\Delta}} dx$ when $v_{i,\Delta}$ are the canonical basis functions of $W_{i,\Delta}(\Gamma'_i)$.

Note that $A$ and $\tilde{S}$ are both symmetric and positive definite; see also Lemma 3.3. By introducing a set of Lagrange multipliers $\lambda \in \tilde{W}_{\Delta}(\Gamma)$, to enforce the continuity constraints, we obtain the following saddle point formulation of (3.29): Find $u_{\Delta} \in W_{\Delta}(\Gamma')$ and $\lambda^* \in \tilde{W}_{\Delta}(\Gamma)$ such that

$$(3.31) \quad \begin{cases} \tilde{S}u_{\Delta} + B_{\Delta}^T \lambda^* \equiv \tilde{g}_{\Delta} \\ B_{\Delta}u_{\Delta}^* = 0. \end{cases}$$

This reduces to

$$(3.32) \quad F\lambda^* = d,$$
where
\begin{equation}
F := B_\Delta \tilde{S}^{-1} B^T_\Delta \quad \text{and} \quad d := B_\Delta \tilde{S}^{-1} \tilde{g}_\Delta.
\end{equation}
Once $\lambda^*$ is computed, we can back solve and obtain
\begin{equation}
u^*_\Delta = \tilde{S}^{-1}(\tilde{g}_\Delta - B^T_\Delta \lambda^*).
\end{equation}

3.3. FEIT-DP Preconditioner. We will now define a preconditioner for $F$ in (3.33).

Let us introduce the diagonal scaling matrix $D_\Delta(i)$, which maps $W_{i,\Delta}(\Gamma'_i)$ into itself, for all $1 \leq i \leq N$. Each of the diagonal entries of $D_\Delta(i)$ corresponds to one $\Delta$–node, and it is given by the weighted counting function $\Delta$
\begin{equation}
\delta_j^i(x) := \frac{\alpha_j}{\alpha_j + \beta_i} \quad \text{for all} \; x \in \{E_{ijh} \cup E_{jih}\} \quad \text{for all} \; j \in E_i^0,
\end{equation}
where $\alpha_i$ is defined in (2.1). Note that one edge is shared by two subdomains. The union of all these functions $\delta_j^i(x)$ provides a partition of unity on all $\Delta$–nodes.

We also define
\begin{equation}B_{D,\Delta} := \left(B^{(1)}_{\Delta}, \ldots, B^{(N)}_{\Delta}\right).
\end{equation}
An important role will be played by the operator
\begin{equation}
\Delta := B^T_{D,\Delta} B_\Delta,
\end{equation}
which maps $W_\Delta(\Gamma')$ into itself. It is easy to check that for $w_\Delta = \{w_{i,\Delta}\}_{i=1}^N \in W_\Delta(\Gamma')$ and $v_\Delta := \Delta w_\Delta$, we have
\begin{align}
\delta_j^i(x)(w_{i,\Delta})_i(x) = \delta_j^i(x)[(w_{i,\Delta})_j(x) - (w_\Delta)_i(x)] \quad \text{for all} \; x \in E_{ijh},
\end{align}

\begin{align}
\delta_j^i(x)(w_{i,\Delta})_j(x) = \delta_j^i(x)[(w_{i,\Delta})_j(x) - (w_\Delta)_j(x)] \quad \text{for all} \; x \in E_{jih},
\end{align}
where $\delta_j^i(x)$ is defined in (3.35). Hence, $\Delta$ preserves jumps in the sense that
\begin{equation}
B_\Delta \Delta = B_\Delta,
\end{equation}
which implies that $\Delta$ is a projection with $\Delta = \Delta$.

Define
\begin{equation}
S'_\Delta := \mathrm{diag}\{S'_{1,\Delta}, \ldots, S'_{N,\Delta}\},
\end{equation}
where $S'_{i,\Delta}$ is the local Schur complement $S'_{i}$, see (3.6), restricted to $W_{i,\Delta}(\Gamma'_i)$ from $W_{i}(\Gamma'_i)$, i.e., $S'_{i,\Delta}$ is obtained from $S'_i$ by deleting rows and columns associated with the variables at nodal points of $V'_i \subset \Gamma'_i$.

The FETI-DP method is the standard preconditioned conjugate gradient algorithm for solving the preconditioned system
\begin{equation}
M^{-1}F \lambda = M^{-1}d
\end{equation}
with the preconditioner
\begin{equation}
M^{-1} := B_{D,\Delta} S'_\Delta B^T_{D,\Delta} = \sum_{i=1}^N B^{(i)}_{\Delta} D^{(i)}_{\Delta} S'_{i,\Delta} D^{(i)}_{\Delta}(B^{(i)}_{\Delta})^T.
\end{equation}
Note that $M^{-1}$ is a block diagonal matrix and each block is invertible since $S'_{i,\Delta}$ and $D^{(i)}_{\Delta}$ are invertible, and $B^{(i)}_{\Delta}$ has full rank.
4. Condition Number Estimate for FETI-DP Preconditioner

The main result of our paper is included in the following theorem, which gives an estimate of the condition number for the preconditioned FETI-DP operator $M^{-1}F$.

**Theorem 4.1.** For any $\lambda \in \tilde{W}_\Delta(\Gamma)$, there exists a positive constant $C$ independent of $h_i, H_i, \alpha(x)$ and $\lambda$ such that

\[
\langle M\lambda, \lambda \rangle \leq \langle F\lambda, \lambda \rangle \leq C\beta \langle M\lambda, \lambda \rangle,
\]

where

\[
\beta = \left(\frac{H}{h}\right)^2 \max_{i=1}^{N} \frac{\bar{\alpha}_i}{\alpha_i},
\]

with $H/h = \max_{i=1}^{N} H_i/h_i$. If for any $1 \leq i \leq N$ the coefficient $\alpha(x)$ in the subdomain $\Omega_i$ satisfies

\[
\alpha(x) \geq \alpha_i \quad \text{for all} \quad x \in \Omega_i,
\]

then we have

\[
\beta = \frac{H}{h} (1 + \log \frac{H}{h})^2 \max_{i=1}^{N} \frac{\bar{\alpha}_i}{\alpha_i}.
\]

**Proof.** By the general abstract theory for FETI-DP method, see [21] and Theorem 6.35 of [27], the proof of the lower and upper bound in (4.1) follows by checking Lemma 4.2 and Lemma 4.3 as below, respectively.

For clarity, we will use the following norms for $w = (w_\Pi, w_\Delta) \in \tilde{W}(\Gamma')$ with $w_\Delta \in W_\Delta(\Gamma')$:

\[
\|w\|_{S'}^2 := \langle S'w, w \rangle, \quad \|w_\Delta\|_{\bar{S}}^2 := \langle \bar{S}w_\Delta, w_\Delta \rangle,
\]

and

\[
\|w_\Delta\|_{S_\Delta}^2 := \langle S_\Delta w_\Delta, w_\Delta \rangle = \langle S' \begin{pmatrix} 0 \\ w_\Delta \end{pmatrix}, \begin{pmatrix} 0 \\ w_\Delta \end{pmatrix} \rangle,
\]

where $S'$, $\bar{S}$ and $S_\Delta$ are defined in (3.18), (3.25) and (3.41), respectively.

**Lemma 4.2.** For any $\mu \in \tilde{W}_\Delta(\Gamma)$ there exists a $w_\Delta \in W_\Delta(\Gamma')$ such that

\[
\mu = B_\Delta w_\Delta
\]

with

\[
P_\Delta w_\Delta = w_\Delta
\]

and

\[
\|w_\Delta\|_{\bar{S}} \leq \|P_\Delta w_\Delta\|_{S_\Delta}.
\]

**Proof.** For any $\mu \in \tilde{W}_\Delta(\Gamma)$, there exists an element $v_\Delta \in W_\Delta(\Gamma')$ such that

\[
\mu = B_\Delta v_\Delta,
\]

since $B_\Delta$ has full rank.

Note that $P_\Delta$ is a projection which maps $W_\Delta(\Gamma')$ to itself. By choosing

\[
w_\Delta = P_\Delta v_\Delta \in W_\Delta(\Gamma'),
\]

we can easily obtain

\[
P_\Delta w_\Delta = P_\Delta^2 v_\Delta = P_\Delta v_\Delta = w_\Delta.
\]
and

\[ B_\Delta w_\Delta = B_\Delta P_\Delta v_\Delta = B_\Delta v_\Delta = \mu, \]

where we have used (3.40).

It follows from Lemma 3.3 that

\[ \| w_\Delta \|^2_\mathcal{S} = \min \langle \tilde{A} v, v \rangle \leq \min \langle \tilde{A} \hat{v}, \hat{v} \rangle = \| w_\Delta \|^2_\mathcal{S} = \| P_\Delta w_\Delta \|^2_\mathcal{S}, \]

where the first minimum is taken over \( v \) and \( \hat{v} \).

Proof. For any \( w_\Delta \in W_\Delta(\Gamma') \) it holds that

\[ \| P_\Delta w_\Delta \|^2_\mathcal{S} \leq C\beta \| w_\Delta \|^2_\mathcal{S}, \]

where \( \beta \) is defined in (4.2) or/and (4.4), and \( C \) is a positive constant independent of \( h_i, H_i, \alpha(x) \) and \( w_\Delta \).

Let \( \hat{w} = \{ \hat{w}_i \}_{i=1}^N \) with \( \hat{w}_i \in W_r(\Gamma'_i) \) be defined by

\[ (\hat{w}_i)_i(x) = I_{E_{ij}}(w_i)_i(x) \text{ for all } x \in \bar{E}_{ijh} \text{ for all } j \in \mathcal{E}_i^0, \]

and

\[ (\hat{w}_i)_j(x) = I_{E_{ij}}(w_i)_j(x) \text{ for all } x \in \bar{E}_{jih} \text{ for all } j \in \mathcal{E}_i^0. \]

Note that \( \hat{w} \in \tilde{W}(\Gamma') \); see (3.21). Therefore, representing \( \hat{w} = (\hat{w}_\Pi, \hat{w}_\Delta) \), we have \( B_\Delta \hat{w}_\Delta = 0 \).

Using the definition of \( P_\Delta \), we have

\[ P_\Delta w_\Delta = B_{D,\Delta}^T B_{D,\Delta} w_\Delta = B_{D,\Delta}^T B_{D,\Delta} (w_\Delta - \hat{w}_\Delta) = P_\Delta (w_\Delta - \hat{w}_\Delta). \]

Define \( v \in \tilde{W}(\Gamma') \) to be equal to \( P_\Delta (w_\Delta - \hat{w}_\Delta) \) at the \( \Delta \)-nodes, and equal to zero at the \( \Pi \)-nodes. Let us represent \( v = \{ v_i \}_{i=1}^N \) with \( v_i \in W_r(\Gamma'_i) \) and

\[ v_i = \{(v_i)_i, \{(v_i)_j\}_{j \in \mathcal{E}_i^0}\}, \]

where \( (v_i)_i \in W_r(\Gamma_i) \); see (3.19) and (3.20). Using (3.38) and (3.39), it is easy to check that

\[ (v_i)_i = \delta^I_i(x)[(w_i - \hat{w}_i)_i - (w_j - \hat{w}_j)_i], \]

and

\[ (v_i)_j = \delta^I_j(x)[(w_i - \hat{w}_i)_j - (w_j - \hat{w}_j)_j]. \]
We denote by $V_h(\partial \Omega_i)$ the space of continuous and piecewise linear functions on the local boundaries $\partial \Omega_i$. It is obvious that $(v_i)_i \in V_h(\partial \Omega_i)$. By the definitions of $S'_\Delta$ and $S'$, (4.11), (4.10), and (4.5), we have

$$ \|P_{\Delta}w_{\Delta}\|_{S'_\Delta}^2 = \|v\|_{S'}^2 = \sum_{i=1}^N \|v_i\|_{S'_i}^2, $$

where

$$ \|v_i\|_{S'_i}^2 = \langle S'_i v_i, v_i \rangle = a'_i(H_i'v_i, H_i'v_i) $$

with the discrete harmonic extension $H_i'$ defined in (4.8).

With (4.7), to prove (4.6), we need to show that

$$ \sum_{i=1}^N a'_i(H_i'v_i, H_i'v_i) \leq C \beta \sum_{i=1}^N a'_i(H_i'w_i, H_i'w_i). $$

By Corollary 3.2 it remains to prove

$$ \sum_{i=1}^N d_i(H_i v_i, H_i v_i) \leq C \beta \sum_{i=1}^N d_i(H_i w_i, H_i w_i), $$

with

$$ d_i(H_i v_i, H_i v_i) = \|\alpha_i^{1/2} \nabla (H_i v_i)\|_{L^2(\Omega_i)}^2 + \sum_{j \in E_i} \sum_{\delta E_{ij}} \frac{1}{h_{ij}} \|\alpha_i^{1/2} [(v_i)_i - (v_i)_j]\|_{L^2(E_{ij})}^2 $$

(4.10)

$$ \leq I_1 + I_2. $$

First we consider the term $I_2$ of (4.10). For $j \in E_i^0$, the proof is trivial due to the specific choices of parameters. For $j \in E_i^1$, it follows from (4.8) and (4.9) that

$$ \|(v_i)_i - (v_i)_j\|_{L^2(\Omega_i)}^2 = (\delta_j^i(x))^2 \|(w_i - \hat{w}_j)_i - (w_i - \hat{w}_j)_j\|_{L^2(\Omega_i)}^2 $$

$$ \leq \|(w_i)_i - (w_i)_j\|_{L^2(\Omega_i)}^2 + \|(w_j)_i - (w_j)_j\|_{L^2(\Omega_i)}^2, $$

since $\hat{w} \in \hat{W}(\Gamma')$, and $\delta_j^i \in (0, 1)$. Here $e$ is a fine edge on the mesh that is obtained by merging $T_h(\Omega_i)$ and $T_h(\Omega_j)$ along $E_{ij}$. We recall that $\alpha_{ij}$ is constant on each $e \subset E_{ij}$ and denoted by $\alpha_{ij}^e$.

By summing up, we finally get

$$ I_2 \leq C \sum_{j \in E_i} \sum_{\delta E_{ij}} \sum_{e \subset E_{ij}} \alpha_i^e \|(w_i)_i - (w_i)_j\|_{L^2(\Omega_i)}^2 + $$

$$ + C \sum_{j \in E_i} \sum_{\delta E_{ij}} \sum_{e \subset E_{ij}} \alpha_i^e \|(w_j)_i - (w_j)_j\|_{L^2(\Omega_i)}^2 $$

(4.11)

$$ = C \sum_{j \in E_i} \frac{1}{h_{ij}} \|\alpha_i^{1/2} [(w_i)_i - (w_i)_j]\|_{L^2(E_{ij})}^2 $$

$$ + C \sum_{j \in E_i} \frac{1}{h_{ij}} \|\alpha_i^{1/2} [(w_j)_i - (w_j)_j]\|_{L^2(E_{ij})}^2 $$

$$ \leq C \{d_i(H_i w_i, H_i w_i) + \sum_{j \in E_i} d_j(H_j w_j, H_j w_j)\}. $$

Now we estimate the first term $I_1$ of (4.10). Here we introduce two semi-norms defined on $\Gamma_i$ as follows: for any $u_i \in W_i(\Gamma_i)$,

$$ |u_i|_{B_i}^2 := \min\{\|\alpha_i^{1/2} \nabla \hat{u}_i\|_{L^2(\Omega_i)}^2 : \hat{u}_i \in V_h(\Omega_i) \text{ and } \hat{u}_i|_{\partial \Omega_i} = u_i\}, $$

(4.12)
and
\begin{equation}
|u_i|^2_{H^{1/2}(\partial \Omega_i)} := \min\{\|\nabla \tilde{u}_i\|_{L^2(\Omega_i)}^2 : \tilde{u}_i \in V_h(\Omega_i) \text{ and } \tilde{u}_i|_{\partial \Omega_i} = u_i\}.
\end{equation}

Denote by \( \tilde{H} \): \( W_i(\Gamma_i) \rightarrow V_h(\Omega_i) \) as the discrete harmonic extension in the sense of \( a_i(\cdot, \cdot) \). Hence the function \( \tilde{H}_i u_i \) is the minimizing function of \( (4.12) \).

Note that \( H_i v_i \) at the interior nodes depends only on the nodal values of \( v_i \) on \( \Gamma_i \), i.e., \( H_i(v_i)_i = H_i v_i \) in the interior of subdomains \( \Omega_i \). This implies that
\begin{equation}
I_i = \|\alpha_i^{1/2} \nabla \tilde{H}_i(v_i)_i\|_{L^2(\Omega_i)}^2
= \|(v_i)_i\|^2_{H_i}
\leq C_{\alpha_i} \left( \left\|\left((v_i)_i\right)^2_{H^{1/2}(\partial \Omega_i)} + \frac{1}{H_i} \|(v_i)_i\|_{L^2(\Omega_i)}^2 \right\| \right),
\end{equation}
where we have used the second inequality of Lemma 4.1 in \[22\].

We can write \((v_i)_i\) as
\begin{equation}
(v_i)_i = \sum_{j \in E_i} I^h(\theta_{E_{ij}}(v_i)_i) = \sum_{j \in E_i} I^h(\theta_{E_{ij}} \delta_j(x)(w_i - \tilde{w}_i)_i - (w_j - \tilde{w}_j)_i),
\end{equation}
where \( I^h \) is the usual Lagrange interpolation operator, and for \( j \in E_i \) the finite element cut-off function \( \theta_{E_{ij}}(x) \) equals to 1 for all \( x \in E_{ijh} \) and vanishes on all the other nodes; see Definition 4.2 of \[22\].

By \( (3.35) \), we know that
\begin{equation}
\theta_i \left( \delta_j(x) \right)^2 \leq C_{\alpha_i} \min(\alpha_i, \alpha_j).
\end{equation}
Putting \( (4.15) \) into \( (4.14) \), we obtain
\begin{equation}
I_i \leq C \min(\alpha_i, \alpha_j) \sum_{j \in E_i} \left\{ |\psi_{ij}|_{H^{1/2}(\partial \Omega_i)}^2 + \frac{1}{H_i} |\psi_{ij}|_{L^2(\Omega_i)}^2 \right\}
\leq C \min(\alpha_i, \alpha_j) \sum_{j \in E_i} \left\{ |a_{ij}|_{H^{1/2}(\partial \Omega_i)}^2 + |b_{ij}|_{H^{1/2}(\partial \Omega_i)}^2 + \frac{1}{H_i} |a_{ij}|_{L^2(\Omega_i)}^2 + \frac{1}{H_i} |b_{ij}|_{L^2(\Omega_i)}^2 \right\},
\end{equation}
where
\begin{align*}
\psi_{ij} &= I^h(\theta_{E_{ij}}(w_i - \tilde{w}_i)_i - (w_j - \tilde{w}_j)_i))
= I^h(\theta_{E_{ij}}[(w_i - \tilde{w}_i)_i - (w_j - \tilde{w}_j)_i]) + I^h(\theta_{E_{ij}}[(w_j - \tilde{w}_j)_j - (w_j - \tilde{w}_j)_j])
:= a_{ij} + b_{ij}.
\end{align*}
As stated in \[22\] that \( |I^h(\theta_{E_{ij}}(w_j - \tilde{w}_j)_j)|_{H^{1/2}(\partial \Omega_i)} \simeq |I^h(\theta_{E_{ij}}(w_j - \tilde{w}_j)_j)|_{H^{1/2}(\partial \Omega_i)} \), since the discrete harmonic extensions from \( E_{ij} \) to \( \Omega_i \) and \( \Omega_j \) are equivalent in the corresponding \( H^1 \)–seminorms. Here we refer to Lemma 4.19 of \[27\] with the two dimensional case, and have
\begin{equation}
|a_{ij}|_{H^{1/2}(\partial \Omega_i)}^2 = |I^h(\theta_{E_{ij}}[(w_i - \tilde{w}_i)_i - (w_j - \tilde{w}_j)_j])|^2_{H^{1/2}(\partial \Omega_i)}
\leq C \left( |I^h(\theta_{E_{ij}}[(w_i - \tilde{w}_i)_i])|^2_{H^{1/2}(\partial \Omega_i)} + |I^h(\theta_{E_{ij}}[(w_j - \tilde{w}_j)_j])|^2_{H^{1/2}(\partial \Omega_i)} \right)
\leq C \left( \frac{1}{H_i} \|(w_i - \tilde{w}_i)_i\|^2_{L^2(E_{ij})} + \frac{1}{H_j} \|(w_j - \tilde{w}_j)_j\|^2_{L^2(E_{ij})} \right).
\end{equation}
Let $\hat{w}_i$ be the convex combination of the values of $w_i$ at the end points of $E_{ij}$, we can employ the generalized discrete Sobolev inequality, c.f. Lemma 4.5 of [22], and obtain

$$
\min(\alpha_i, \alpha_j) \sum_{j \in \mathcal{E}_i} |a_{ij}|^{2}_{H^{1/2}(\partial \Omega_i)} 
$$

$$
\leq C \sum_{j \in \mathcal{E}_i} \sum_{k=i,j} \bar{\alpha}_k \left( \frac{H_k}{h_k} \right)^2 |\mathcal{H}_k(w_k)|^2_{H^1(\Omega_k)} 
$$

$$
\leq C \sum_{j \in \mathcal{E}_i} \sum_{k=i,j} \bar{\alpha}_k \left( \frac{H_k}{h_k} \right)^2 |(w_k)|_{H^2_k} 
$$

$$
\leq C \left( \frac{H}{h} \right)^2 \sum_{j \in \mathcal{E}_i} \sum_{k=i,j} \bar{\alpha}_k d_k(\mathcal{H}_k w_k, \mathcal{H}_k w_k). 
$$

With the same argument as [4.17], we get

$$
\sum_{j \in \mathcal{E}_i} |b_{ij}|^2_{H^{1/2}(\partial \Omega_i)} \leq C \sum_{j \in \mathcal{E}_i} \frac{1}{h_j} \|(w_j - \hat{w}_j) - (w_j - \hat{w}_j)_i\|^2_{L^2(E_{ij})}. 
$$

Let $Q_i$ be the $L_2$ projection on $V_h(E_{ij})$, the restriction of $V_h(\partial \Omega_i)$ to $E_{ij}$ with $h_i$—triangulation on $E_{ij}$. Using the inverse inequality, and the $L_2$ stability of the $L_2$ projection we have

$$
\|Q_i[(w_j)_j - (w_j)]\|^2_{L^2(E_{ij})} + \|Q_i[(w_j)_j - \hat{w}_j]\|^2_{L^2(E_{ij})} + \|Q_i[w_j - \hat{w}_j]\|^2_{L^2(E_{ij})} + \|Q_i[(w_j)_j - (w_j)]\|^2_{L^2(E_{ij})} 
$$

$$
\leq C(\|Q_i[(w_j)_j - (w_j)]\|^2_{L^2(E_{ij})} + \min(\alpha_i, \alpha_j) \sum_{j \in \mathcal{E}_i} |a_{ij}|^{2}_{H^{1/2}(\partial \Omega_i)} + \|Q_i[w_j - \hat{w}_j]\|^2_{L^2(E_{ij})} + \|Q_i[(w_j)_j - \hat{w}_j]\|^2_{L^2(E_{ij})}) 
$$

$$
\leq C(\|Q_i[(w_j)_j - (w_j)]\|^2_{L^2(E_{ij})} + \min(\alpha_i, \alpha_j) \sum_{j \in \mathcal{E}_i} |a_{ij}|^{2}_{H^{1/2}(\partial \Omega_i)} + \|Q_i[w_j - \hat{w}_j]\|^2_{L^2(E_{ij})} + \|Q_i[(w_j)_j - \hat{w}_j]\|^2_{L^2(E_{ij})}) 
$$

since $(\hat{w}_j)_i$ and $(\hat{w}_j)_j$ are linear on $E_{ij}$ and $E_{ji}$, respectively. Let $(\hat{w}_j)_i$ be the average of $(w_j)_j$ on $E_{ji}$. By (4.42) in [4] we obtain

$$
\max_{\partial E_{ji}}((w_j)_i - (w_j)_j)^2 
$$

$$
\leq C \left( \frac{1}{h_i} \|(w_j)_i - (w_j)_j\|^2_{L^2(E_{ji})} + \max_{\partial E_{ji}}((Q_i(w_j - \hat{w}_j)_j)^2 + \max_{\partial E_{ji}}((w_j - \hat{w}_j)_j)^2 \right) 
$$

$$
\leq C \left( \frac{1}{h_i} \|(w_j)_i - (w_j)_j\|^2_{L^2(E_{ji})} + \frac{H^2}{h_j} |\mathcal{H}_j(w_j)_j|_{H^1(\Omega_j^h)} \right), 
$$

where we have used (4.10) in [22], and the $H^{1/2}$ stability of the $L^2$ projection. Substituting (4.21) into (4.20), we have

$$
\|Q_i[(w_j) - (w_j)]\|^2_{L^2(E_{ij})} + \|Q_i[(w_j) - \hat{w}_j]\|^2_{L^2(E_{ij})} + \|Q_i[w_j - \hat{w}_j]\|^2_{L^2(E_{ij})} + \|Q_i[(w_j) - (w_j)]\|^2_{L^2(E_{ij})} 
$$

$$
\leq C \left( \frac{H^2}{h_j} |\mathcal{H}_j(w_j)_j|_{H^1(\Omega_j^h)} \right), 
$$

(4.22)
where we have used (2.2). Putting the above inequality into (4.19), we obtain

$$\min(\alpha_i, \alpha_j) \sum_{j \in E_i} \frac{1}{h_i} \|q_{ij}\|^2_{L^2(\partial Q_i)} \leq C \sum_{j \in E_i} \left( \frac{H_j}{h_j} \max_{e \in E_i} \frac{\alpha_{ij}}{\alpha_j} \right) \sum_{e \in E_i} \alpha_{ij}^2 \| (w_j)_i - (w_j)_j \|_{L^2(e)}^2 + \left( \frac{H_j}{h_j} \right)^2 \frac{\alpha_{ij}}{\alpha_j} \| (w_j)_j \|^2_{L^2(e)} \right)$$

(4.23)

$$\leq C \sum_{j \in E_i} \left( \frac{H_j}{h_j} \max_{e \in E_i} \frac{\alpha_{ij}}{\alpha_j} \right) \frac{1}{h_j} \alpha_{ij}^{1/2} \| (w_j)_i - (w_j)_j \|_{L^2(e)}^2 + \left( \frac{H_j}{h_j} \right)^2 \frac{\alpha_{ij}}{\alpha_j} \| (w_j)_j \|^2_{L^2(e)} \right)$$

$$\leq C \left( \frac{H_j}{h_j} \right)^2 \max_{i=1}^{N} \frac{\alpha_i}{\alpha_j} \sum_{j \in E_i} d_j (H_j w_j, H_j w_j),$$

where we used the fact that for all $e \in E_i$

$$\min(\alpha_i, \alpha_j) \leq \max(\frac{\alpha_i}{\alpha_j}, \alpha_j) \min(\alpha_i, \alpha_j) \leq \max(\frac{\alpha_i}{\alpha_j}, \alpha_j) \alpha_{ij}. $$

Using the $L^2$ continuity of the nodal interpolation operator $I^h$, and proceeding the same lines of (4.18), we have

$$\min(\alpha_i, \alpha_j) \sum_{j \in E_i} \frac{1}{h_i} \|q_{ij}\|^2_{L^2(\partial Q_i)} \leq C \min(\alpha_i, \alpha_j) \sum_{j \in E_i} \frac{1}{h_i} \| (w_i - \hat{w}_i)_i \|_{L^2(E_i)}^2 + \| (w_j - \hat{w}_j)_j \|_{L^2(E_j)}^2$$

(4.24)

$$\leq C \left( \frac{H_j}{h_j} \right)^2 \sum_{j \in E_i} \sum_{k=1}^N \frac{\alpha_k}{\alpha_j} d_k (H_k w_k, H_k w_k),$$

and

$$\min(\alpha_i, \alpha_j) \sum_{j \in E_i} \frac{1}{h_i} \|b_{ij}\|^2_{L^2(\partial Q_i)} \leq C \min(\alpha_i, \alpha_j) \sum_{j \in E_i} \frac{1}{h_i} \| (w_j - \hat{w}_j)_j \|_{L^2(E_j)}^2$$

(4.25)

$$\leq C \left( \frac{H_j}{h_j} \right)^2 \sum_{j \in E_i} \frac{\alpha_j}{\alpha_j} d_j (H_j w_j, H_j w_j),$$

since $H_i \simeq H_j$. Combining the inequalities (4.18), (4.23), (4.24) and (4.25), we have

$$I_1 \leq C \left( \frac{H_j}{h_j} \right)^2 \max_{i=1}^{N} \frac{\alpha_i}{\alpha_j} \sum_{j \in E_i} \sum_{k=1}^N d_k (H_k w_k, H_k w_k).$$

(4.26)

Substituting (4.26) and (4.11) into (4.10), we get

$$d_i (H iv_i, H iv_i) \leq C \left( \frac{H_j}{h_j} \right)^2 \max_{i=1}^{N} \frac{\alpha_i}{\alpha_j} \sum_{j \in E_i} \sum_{k=1}^N \alpha_k d_k (H_k w_k, H_k w_k).$$

(4.27)

By the summation of the above inequality for all $1 \leq i \leq N$ and noting that the number of edges of each subdomain can be bounded independently of $N$, we finally obtain (4.6) with $\beta$ satisfying (4.2).
Next we consider the special case when the coefficient \( \alpha(x) \) in the subdomains \( \Omega_i \) satisfies (4.3) for all \( 1 \leq i \leq N 
\)
\[
I_1 \leq C \min(\tau_i, \tau_j) \{ |c_{ij}|^2_{H^{1/2}(\partial \Omega_i)} + |d_{ij}|^2_{H^{1/2}(\partial \Omega_i)} + \frac{1}{h_i} \|c_{ij}\|_{L^2(\Omega_i)}^2 + \frac{1}{h_i} \|d_{ij}\|_{L^2(\Omega_i)}^2 \},
\]
(4.28)
where
\[
c_{ij} = (w_i - \hat{w}_i)_j - (w_j - \hat{w}_j)_j \quad \text{and} \quad d_{ij} = (w_j - \hat{w}_j)_j - (w_j - \hat{w}_j)_i.
\]
It is well-known that; c.f. [27],
\[
\min(\tau_i, \tau_j) |c_{ij}|^2_{H^{1/2}(\partial \Omega_i)} \leq \min(\tau_i, \tau_j) \left( |(w_i - \hat{w}_i)_j|^2_{H^{1/2}(\partial \Omega_i)} + |(w_j - \hat{w}_j)_j|^2_{H^{1/2}(\partial \Omega_i)} \right) \leq \min(\tau_i, \tau_j) \sum_{j \in E_i} \left( |(w_i - \hat{w}_i)_j|^2_{H^{1/2}(E_{ij})} + |(w_j - \hat{w}_j)_j|^2_{H^{1/2}(E_{ij})} \right)
\]
(4.29)
\[
\leq C \sum_{j \in E_i} \sum_{k=i,j} \tau_k (1 + \log \frac{h_k}{h_i})^2 \| \nabla \hat{H}_k(w_k) |_{\Omega_k}^2 \|
\]
\[
= C \sum_{j \in E_i} \sum_{k=i,j} \tau_k (1 + \log \frac{h_k}{h_i})^2 \| \nabla \hat{H}_k w_k |_{\Omega_k}^2 \|
\]
\[
\leq C \sum_{j \in E_i} \sum_{k=i,j} \tau_k (1 + \log \frac{h_k}{h_i})^2 d_k(\hat{H}_k w_k, \hat{H}_k w_k),
\]
\[
\text{since } \alpha(x) \geq \alpha_k \text{ for all } x \in \Omega_k.
\]
Using (4.44) in [7], we have
\[
\min(\tau_i, \tau_j) |d_{ij}|^2_{H^{1/2}(\partial \Omega_i)} = \min(\tau_i, \tau_j) \sum_{j \in E_i} \left( |(w_j - \hat{w}_j)_j|^2_{H^{1/2}(\partial \Omega_i)} \right) \leq C (1 + \log \frac{h}{h_i})^2 \sum_{j \in E_i} \left( |(w_j - \hat{w}_j)_j|^2_{L^2(\Omega_i)} \right) \leq C (1 + \log \frac{h}{h_i})^2 \sum_{j \in E_i} \left( |(w_j - \hat{w}_j)_j|^2_{L^2(\Omega_i)} \right)
\]
(4.30)
\[
\text{where we have used (2.3), and the fact that } \delta \text{ is practically chosen such that } \delta = O(1).
\]
Proceeding with the same lines of (4.29), we can obtain
\[
\min(\tau_i, \tau_j) \frac{1}{h_i} \|c_{ij}\|_{L^2(\Delta)}^2 \leq \min(\tau_i, \tau_j) \frac{1}{h_i} \sum_{j \in E_i} \left( |(w_i - \hat{w}_i)_j|^2_{L^2(\Omega_i)} + |(w_j - \hat{w}_j)_j|^2_{L^2(\Omega_i)} \right) \leq \min(\tau_i, \tau_j) \frac{H}{h_i} \sum_{j \in E_i} \left( |(w_i - \hat{w}_i)_j|^2_{H^{1/2}(\Delta)} + |(w_j - \hat{w}_j)_j|^2_{H^{1/2}(\Omega_i)} \right) \leq \min(\tau_i, \tau_j) \frac{H}{h_i} \sum_{j \in E_i} \sum_{k=i,j} \tau_k d_k(\hat{H}_k w_k, \hat{H}_k w_k),
\]
(4.31)
\[
\text{since } (w_i - \hat{w}_i)_j = 0 \text{ at the end points of } E_{ij}.
\]
Using the inverse inequality, and the $L^2$ stability of the $L^2$ projection we have

$$
\|(w_j - \hat{w}_j)_j - (w_j - \hat{w}_j)_i\|_{L^2(E_{ij})}^2 \\
\leq C\{\|Q_i((w_j)_j - (w_j)_i)\|_{L^2(E_{ij})}^2 + \|Q_i((w_j - \hat{w}_j)_j)\|_{L^2(E_{ij})}^2 + \\
\|((w_j - \hat{w}_j)_j)\|_{L^2(E_{ij})}^2 + \|((\hat{w}_j)_i - (w_j)_j)\|_{L^2(E_{ij})}^2 \}
$$

(4.32)

$$
\leq C\{\|((w_j)_j - (w_j)_i)\|_{L^2(E_{ij})}^2 + H_i((w_j - \hat{w}_j)_j)_{H^1/2(E_{ij})}^2 + H_i\max((w_j)_i - (w_j)_j)^2 \}
$$

$$
\leq C\{\|((w_j)_j - (w_j)_i)\|_{L^2(E_{ij})}^2 + H_i(1 + \log \frac{H_j}{h_j})^2\|\nabla H_j w_j\|_{L^2(\Omega_j)}^2 + \\
+ \frac{H_i}{h_i}\|(w_j)_j - (w_j)_i\|_{L^2(E_{ij})}^2 + H_i(1 + \log \frac{H_j}{h_j})\|\nabla H_j w_j\|_{L^2(\Omega_j)}^2 \}
$$

where we have used (4.43) in \[7\]. Hence,

$$
\min(\alpha_i, \alpha_j) \frac{1}{h_i} \|((w_j - \hat{w}_j)_j - (w_j - \hat{w}_j)_i)\|_{L^2(E_{ij})}^2 \\
\leq C\{\frac{H_i}{h_i} \max(\frac{\alpha_i}{\alpha_i}, \frac{\alpha_j}{\alpha_j}) \frac{1}{h_i} \|\alpha_i^{1/2}(w_j)_j - (w_j)_i\|_{L^2(E_{ij})}^2 + \\
\frac{H_i}{h_i}(1 + \log \frac{H_j}{h_j})^2\frac{\alpha_j}{\alpha_j} \|\nabla H_j w_j\|_{L^2(\Omega_j)}^2 \}
$$

This immediately gives

$$
\min(\alpha_i, \alpha_j) \frac{1}{h_i} \|d_{ij}\|_{L^2(\partial \Omega_i)}^2 \leq C\frac{H_i}{h_i}(1 + \log \frac{H_j}{h_j})^2 \max_{i=1}^N \frac{\alpha_i}{\alpha_j} \sum_{j \in E, j \neq i} d_j(H_j w_j, H_j w_j).
$$

With the same arguments as in (4.26) and (4.27), we finally obtain (4.6) with $\beta$ satisfying (4.4).

\[\square\]

5. Numerical Experiments

Let the domain $\Omega$ be a unit square $(0, 1)^2$. For the experiments, we partition the domain $\Omega$ into $4 \times 4$ square subdomains. The distribution of coefficients in each example is presented by figures. We use the proposed FETI-DP method for the discontinuous Galerkin formulation (Section 3) of the problem, and iterate with the preconditioned conjugate gradient (PCG) method. The iteration in each test stops whenever the $L^2$ norm of the residual is reduced by a factor of $10^{-6}$. The penalty parameter $\delta$ is chosen to be 5 in all the experiments.

Example 5.1. In our first example, c.f. left picture of Fig. 1, the coefficient denotes a 'binary' medium with $\alpha(x) = \hat{\alpha}$ on a square shaped inclusion (shaded region) lying inside one subdomain $\Omega_i$ at a distance of $h$ from both the horizontal and the vertical edges of $\partial \Omega_i$, and $\alpha(x) = 1$ in the rest of the domain. We study the behavior of the preconditioner as $h$ and $\hat{\alpha}$ varies, respectively.

It follows from Tab. 1 that the condition numbers are independent of the values of $\hat{\alpha}$ since the coefficient contrast in the boundary layer is exactly equal to 1. This is consistent with our theoretical results.

Adopting different fine mesh sizes $h$, we obtain the log-log plot of the condition numbers in terms of $H/h$ for $\hat{\alpha} = 10^0$. The left plot of Fig. 2 shows a dependence worse than linear growth, which is expected to become harder as $h$ goes finer. This confirms the estimate of (4.2) that contains a logarithmic factor besides the linear dependence.
**Example 5.2.** The distribution of coefficient is shown in the right picture of Fig. 1, with inclusions in two neighbouring subdomains with coefficient values both larger and smaller than in the boundary layers.

Similar as the above example, we investigate the dependence of the condition numbers on the mesh ratio $H/h$. The right plot of Fig. 2 tells us the robustness of the quadratic dependence in the estimate of (4.4).

![Subdomain partition and coefficient distribution.](image1)

![Subdomain partition and coefficient distribution.](image2)

**Fig. 1.** Subdomain partition and coefficient distribution. Left: Example 5.1; Right: Example 5.2.

**Tab. 1.** Example 5.1: PCG iterations and condition numbers (in parentheses).

| $H = 32h$ | $H = 64h$ | $H = 128h$ | $H = 256h$ |
|-----------|-----------|------------|------------|
| $\hat{\alpha} = 10^2$ | 13(8.568) | 18(17.39) | 22(31.91) | 27(55.93) |
| $\hat{\alpha} = 10^4$ | 13(9.470) | 17(20.30) | 22(42.39) | 29(89.93) |
| $\hat{\alpha} = 10^6$ | 13(9.481) | 19(20.34) | 22(42.58) | 29(90.72) |

**Tab. 2.** Example 5.2: PCG iterations and condition numbers (in parentheses).

| $H = 32h$ | $H = 64h$ | $H = 128h$ | $H = 256h$ |
|-----------|-----------|------------|------------|
| 19(26.51) | 25(92.54) | 36(346.3) | 57(1333)   |

**Example 5.3.** We employ this example to investigate the dependence of our method on the coefficient variation in the boundary layers. The distribution of the coefficient is depicted in Fig. 3. The coefficient $\alpha(x) = \hat{\alpha}$ in the edge islands (shaded region), and $\alpha(x) = 1$ else where.

The numerical results reported in Tab. 3 confirm our theoretical results in Theorem 4.1, i.e., a linear dependence of the condition number on the coefficient variation in the boundary layers. It is worth further investigation to provide techniques to remove this dependence. In [22], the authors used a pointwise weight to define the scaling matrix and finally made the performance of the method completely independent of the coefficient contrast for some special cases. However, there was no theoretical support to explain this robustness and this technique is not valid for the present example either.
Fig. 2. Log-log plot of condition numbers vs. $H/h$. Left: Example 5.1 with $\hat{\alpha} = 10^6$, the slope of least square is 1.1; Right: Example 5.2, the slope of least square is 1.9.

Fig. 3. Example 5.3: subdomain partition and coefficient distribution. The length of each inclusion is $H/8$ and the height is $H/2$.

Tab. 3. Example 5.3: PCG iterations and condition numbers (in parentheses).

| $\hat{\alpha}$ | $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ |
|---------------|--------|--------|--------|--------|--------|
| $H = 64h$     | 44(64.63) | 66(6.37e+2) | 91(6.36e+3) | 121(6.36e+4) | 145(6.36e+5) |

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