Mathematical Proof Construction: Students’ Ability in Higher Education

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Abstract. This study aimed to categorizing and describing students’ ability in constructing a mathematics proof. This study used descriptive-qualitative approach which involved 46 mathematics undergraduate students. The results showed that the students’ ability in mathematical proof constructing consist 2 phases, that is, visual proof and symbolic proof. Students were not using the concepts and definitions in order to constructing mathematics proof. Consequently, reinforcement of concepts will improve students’ ability to constructing an argument or explanation in mathematical proof.

1. Introduction
Proof should not be a new thing for mathematics students in higher education. It is the fundamental of mathematics understanding and is essential to develop, build, and communicate in mathematics [1, 2]. The aim of a mathematical proof can be stated as proving the trueness or falseness of an argument for every case and condition [3]. There are three points to define mathematics proof, namely precise arguments, getting things right, and testing intuition product [4, 5]. Construction of mathematical proof is a mathematical task in which students are provided with some information (assumptions, axioms, definitions) and asked to apply inference rules (using previous facts, applying theorems) until the expected conclusions are reached [6]. Process of proof involves cognitive aspect. It is also an activity that led to the discovery of a mathematical fact, build a conjecture, constructing a reason or explanation for something, including exploration, generalization, reasoning, argument, and validation.

A research result showed that that the majority of prospective teachers in Germany can’t construct completely formal proof for secondary school mathematics subject [7]. Others study reveal that undergraduate mathematics majors had insufficient understandings of continuous functions for determining the validity of a given statement and producing proofs and counterexamples [8]. The other study indicated that pre-service middle school mathematics teachers are successful in evaluating discussions in the cases regarding proof by contradiction [9]. Furthermore, this study showed that nearly half of the students explain their correct answers by referring to Proof by contradiction [10]. It means that proving is an essential activity in mathematics but there are serious difficulties encountered by mathematics undergraduates regarding their ability in constructing proof.

This study aimed to categorizing and describing mathematics undergraduate students' ability in constructing a mathematics proof. Based on the perspective of cognitive development, there are four
stages to categorizing mathematical proof, that is, (i) enactive proof, (ii) visual proof, (iii) symbolic proof, and (iv) formal proof [11]. Enactive proof is a proof involving use concrete objects to show the truth of the statement. Visual proof means validating the mathematical statement involving graphic or image. Symbolic proof means validating the mathematical statement involving the manipulation of symbols of algebra. The formal proof means validating the mathematical statement using the definition of the concept and an axiomatic deductive system. Mathematics undergraduates will learn advanced mathematics without difficulty when they categorized at formal proof.

2. Research Method
Forty-six mathematics undergraduate students at university in East Java, Indonesia participated in this study. They were selected by purposive sampling, that is, second year students who had taken set theory course in their previous semester. Each participant completed test of mathematics proof by direct proof. The instrument comprised mathematical statement that’s were designed to require an understanding of subset and represent basic types of proof. This study utilized descriptive research design using the survey technique. The data collected were coded, summarized and were treated according to the research purposes. Data of students’ ability in constructing proof were analyzed using theoretical framework of David Tall [11]. Table 1 shows phases and Coding for validating a statement.

Table 1. Phases and Coding for Verification Statement Aspect

| Phases          | Description                              | Coding |
|-----------------|------------------------------------------|--------|
| Enactive proof  | involving concrete objects               | En     |
| Visual proof    | involving charts, diagrams, or pictures  | Vis    |
| Symbolic Proof  | involving the manipulation of algebra symbols | Sym   |
| Formal Proof    | involving definitions, concepts and axiomatic deductive system | For |

3. Result and Discussion
Table 2 present the quantitative data of the undergraduate students’ ability about proving the following statement: Let A, B, and C are set. Prove that if \( A \subseteq B \), then \( A - C \subseteq B - C \). Students can use direct proof method to verify correctness of statement. The direct way to proceed is to assume that \( A \subseteq B \) is true and show (deduce) that \( A - C \subseteq B - C \) is also true. Axiomatic deductive system is needed to constructing proof statement. Students should deriving definitions of subset, that is, \( A \) is a subset of \( B \) if and only if every element of \( A \) is an element of \( B \), and write \( A \subseteq B \iff (\forall x)(x \in A \implies x \in B) \). Furthermore, the definitions about set operation, that is, differences two set defines by \( A - B = \{x : x \in A \text{ and } x \notin B\} \), is also can help student to constructing proof correctly [12]. Interestingly, the result shows that most of students prefer draw diagrams to construct the proof of statement and the others students use manipulation of algebra symbols. Students do not come up with any definitions, concepts, and axiomatic deductive system when constructing proof.

Table 2. Undergraduates Proof Ability Categories

| Phases          | Numbers of Student | Proof | | |
|-----------------|--------------------|-------|---|---|
| Enactive proof  | 0                  | 0     | 0| |
| Visual proof    | 36                 | 19.6  | 58.7| |
| Symbolic Proof  | 10                 | 0     | 21.7| |
| Formal Proof    | 0                  | 0     | 0| |
Table 2 shows that students’ ability in constructing mathematical proof categorized in two phases, that is, visual proof and symbolic proof. Visualization is the process of realizing the mental representation. In this case, visualization can be appears by a Venn diagram. Symbolic proof means constructing the truth of statement mathematics proof using algebraic symbol manipulation. Unfortunately, the students made mistake of proof in these two phases. More than a half students who drawn a Venn diagram are not quite right. Others result shows that symbolic proof that shown by students were incorrect. All of students who solving proof problems used a specific example are choosing element of each set but they fault to use definition and properties of set correctly. Inaccurate used of Venn diagram and algebraic symbol manipulation indicate that there are misconceptions in understanding the concept of subset and the operation of two sets.

![Figure 1](image1.png)

**Figure 1.** One of Venn Diagram by Student

Figure 1 shows inaccurate Venn diagram used by students in the process of proof. Venn diagram is one of visual image which is important to understand mathematics concept and completing mathematics assignments [13]. The use of diagram is almost always helps students to successfully complete the task when unable to work symbolically [14]. Further, visual representation can impact positively and negatively [13]. There are two issues related to visual proof through the use of Venn diagrams, namely (1) the students have difficulty in formal proof, and (2) proving through informal proof are easier.

![Figure 2](image2.png)

**Figure 2.** One of manipulate algebra symbol by student

Figure 2 shows that at the phases of symbolic proof, students count operations of difference two
sets as operations that apply to the difference between two integers. As a result, the procedure has been done can’t be instructions to generate conclusions. One of the causes difficulty in constructing mathematical proof such as lack of conceptual understanding and can’t express definition of concept [15]. Students use the shadow concept to create a specific example that serves as an example of a generic to find properties. Specific or generic examples can be used to look for patterns or similarities in the nature of a definition of the concept of manipulation without following the rules of logic [16].

The results showed that the students have not yet reached a formal proof. Students have difficulty to proof not only because of the lack of knowledge about the content material [17]. Sometimes, students know the definition and can explain it in an informal but can’t use the definition to construct proof. Further, explained that the source of the difficulty of proof due to three aspects, (1) the understanding of the concept (definitions, pictures, and usability), (2) lack of knowledge of the logic and methods of verification, and (3) the limitations of language and notation. Students focus more on procedure than content. In addition, the students realized that they memorize proof than the understand what is proof and how to write it.

4. Conclusion
Proof has an important role in mathematics. Proof used to eliminate uncertainty about mathematical propositions and verify the correctness of mathematical statements. Based on developmental psychology perspective, students’ ability to construct mathematical proof in formal proof phases has not been reached. Lectures should provide a generic ‘bridge’ to smooth the transition to formal proof. Besides, lecturer must encourage students to use aspects of mathematics such as concept and definition to solve proving problems. Thus, undergraduate students’ difficulties in constructing mathematical proofs can be reduced.

References
[1] Ball D L and Bass H 2003 Making Mathematics Reasonable in School A Research Companion to Principles and Standards for School Mathematics ed J Kilpatrick, W G Martin, and D Schifter (Reston, VA: National Council of Teachers of Mathematics) p 27–44.
[2] Carpenter T P, Franke M L, and Levi L 2003 Thinking Mathematically: Integrating Arithmetic & Algebra in Elementary School (Portsmouth, NH: Heinemann)
[3] Baki A 2008 Mathematics Education from Theory to Practice. (Ankara: Harf Educational Publications)
[4] Wilder R L 1994 The Nature of Mathematical Proof J. The American Mathematical Monthly 51 309–323.
[5] Artemov S N 2001. Explicit Provability and Constructive Semantics. The Bulletin of Symbolic Logic. 7 1-36
[6] Weber K 2002 Beyond Proving and Explaining: Proofs that Justify the Use of Definitions and Axiomatic Structures and Proofs that Illustrate Technique J. For the learning of Mathematics 22 14–17
[7] Schwarz, Bjorn and Kaiser, Gabriele 2009 Professional Competence of Future Mathematics Teachers on Argumentation and Proof and How Evaluate It Proceedings of the ICMI study 19 Conference: Proof and Proving in Mathematics Education
[8] Ko Y, Knuth E 2009 Undergraduate Mathematics Majors’ Writing Performance Producing Proofs And Counterexamples About Continuous Functions J. Mathematical Behavior 28 68-77
[9] Demiray E, Bostan M I 2016 Pre-service Middle School Mathematics Teachers’ Evaluations of Discussions: The Case of proof by Contradiction J. Mathematics Education Research 29 1-23
[10] Weber K 2001 Student Difficulty in Constructing Proofs: The Need for Strategic Knowledge J. Educational Studies in Mathematics 48 101-119
[11] Tall D 1995 Cognitive Development, Representations, and Proof Paper presented on Conference on Justifying and Proving in School Mathematics (London: Institute of Education
[12] Smith D, Eggen M, and Andre R S 2011 A Transition to Advanced Mathematics 7th Edition (Boston; Richard Stratton Publisher)
[13] Alcock L and Simpson A 2004 Convergence of sequences and Series: Interactions Between Visual Reasoning and the Learner’s Beliefs about Their Own Role *Educational Studies in Mathematics* **57** 1–32
[14] Gibson D 1998 Students’ Use Diagrams to Develop Proofs in an Introductory Analysis Course
[15] Hawro J 2007 University Students’ Difficulties with Formal Proving and Attempts to Overcome Them *Proceedings of the Fifth Congress of The European Society for Research in Mathematics Education* ed Pitta-Pantazi D (Cyprus: Larcana) p 2290–2299
[16] Alcock L and Inglis M 2008 Doctoral Students’ Use of Examples in Evaluating and Proving Conjectures *Educational Studies in Mathematics* **69** p 111–129
[17] Moore R C 1994 Making the Transition to Formal Proof *Journal of Educational Studies in Mathematics* **27** 249 - 266