A TRANSFER MORPHISM FOR ALGEBRAIC KR-THEORY

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Abstract. In this paper, we construct a transfer morphism for algebraic KR-theory of schemes with involution (aka. Hermitian K-theory of schemes with involution), and we prove a version of dévissage theorem for algebraic KR-theory. We also apply this dévissage theorem to compute algebraic KR-theory of \( \mathbb{P}^1 \) with the involution \( \tau \) given by \([X : Y] \mapsto [Y : X]\), and to prove the \( \mathbb{A}^1 \)-invariance of algebraic KR-theory of schemes with involution.

1. Introduction

In 1970s, researchers found that in order to understand quadratic forms, it was helpful to study Witt groups of function fields of algebraic varieties, which has led to substantial progress in quadratic form theory (see the books [Sch85], [Lam05], [KS80] and [Knu91]). However, Witt groups of many function fields are very difficult to understand. Instead of computing Witt groups of function fields of algebraic varieties, Knebusch [Kne77] proposed to study Witt groups of algebraic varieties themselves, which would certainly reveal some information about Witt groups of their function fields. In the introduction of loc. cit., Knebusch suggested we develop a version of Witt groups of schemes with involution, which not only enlarges the theory of trivial involution, but also provides new insights into topology. For instance, L-theory of the laurent polynomial ring \( k[T, T^{-1}] \) with the non-trivial involution \( T \mapsto T^{-1} \) was found very useful in understanding geometric manifolds [Ran98], and Witt groups can be identified with L-theory if two is invertible.

More generally, Witt groups fit into the framework of Hermitian K-theory, [Ba73], [Kar80] and [Sch17]. More precisely, the negative homotopy groups of the Hermitian K-theory spectra are Witt groups, cf. [Sch17, Proposition 6.3]. It is also worth mentioning that Hermitian K-theory has been successfully applied to solve several problems in the classification of vector bundles and the theory of Euler classes (cf. [AF14a], [AF14b] and [FS09]). In light of this, I decide to develop the current paper under the framework of Hermitian K-theory of dg categories of Schlichting [Sch17].

The simplest example of Witt groups of schemes with involution is \( W(\text{Spec}(\mathbb{C}), \sigma) \) where \( \sigma \) is the complex conjugation (here we consider \( \text{Spec}(\mathbb{C}) \) over \( \text{Spec}(\mathbb{Z}[\frac{1}{2}]) \)). In fact, we have \( W(\text{Spec}(\mathbb{C}), \sigma) \cong \mathbb{Z} \) by ranks and signatures of a Hermitian form over \( \mathbb{C} \) (cf. [Knu91 I.10.5]). This computation is different from \( W(\text{Spec}(\mathbb{C})) \cong \mathbb{Z}/2\mathbb{Z} \) in which every rank two quadratic form is hyperbolic.

It is also known that topological Hermitian K-theory of spaces with involution can be identified with Aityah’s KR-theory ([At66]). In the mean time, Hermitian K-theory of schemes with involution is a natural lifting of the topological Hermitian K-theory to the algebraic world. Therefore, we choose the name Algebraic KR-theory for Hermitian K-theory of schemes with involution to emphasize its analogy in topology.

In this paper, we define a transfer morphism for the algebraic KR-theory which generalizes Gille’s transfer morphism on Witt groups with trivial involution [Gil03]. More precisely, we prove in Theorem 3.1 the following:

**Theorem 1.1.** Let \((X, \sigma_X)\) and \((Z, \sigma_Z)\) be schemes with involution and with \( \frac{1}{2} \in \mathcal{O}_X \). Let \((I, \sigma_I)\) be a dualizing complex with involution on \((X, \sigma_X)\) (cf. Definition 2.17). If \( \pi : Z \to X \) is a finite morphism
of schemes with involution, then there is a map of spectra (called the transfer morphism)
\[ T_{Z/X} : GW[i](Z, \sigma_Z, (\pi^*I^*, \sigma_{\pi^*I})) \to GW[i](X, \sigma_X, (I^*, \sigma_I)). \]

In light of the work of Balmer-Walter \cite{BW02} and Gillet \cite{Gil07}, we also prove a version of the dévissage theorem for algebraic K-ring-theory. The following result is proved in Theorem 5.1.

**Theorem 1.2.** Let \((X, \sigma_X)\) and \((Z, \sigma_Z)\) be schemes with involution and with \(\frac{1}{2} \in \mathcal{O}_X\). Let \((I, \sigma_I)\) be a dualizing complex with involution on \((X, \sigma_X)\) (cf. Definition 2.13). If \(\pi : Z \to X\) is a closed immersion which is invariant under involutions, then there is an equivalence of spectra
\[ D_{Z/X} : GW[i](Z, \sigma_Z, (\pi^*I^*, \sigma_{\pi^*I})) \to GW[i](X, \sigma_X, (I^*, \sigma_I)). \]

If \(X\) and \(Z\) are both regular, we can identify the coherent algebraic K-ring-theory with the vector bundle algebraic K-theory, and we have a more precise formulation of the dévissage theorem as follows (cf. Theorem 6.11).

**Theorem 1.3** (Dévissage). Let \((X, \sigma_X)\) be a regular scheme with involution. Let \((\mathcal{L}, \sigma_{\mathcal{L}})\) be a dualizing coefficient with \(\mathcal{L}\) a locally free \(\mathcal{O}_X\)-module of rank one. If \(Z\) is a regular closed subscheme of \(X\) of codimension \(d\) which is invariant under \(\sigma\), then there is a stable equivalence of spectra
\[ D_{Z/X, \mathcal{L}} : GW[i-d](Z, \sigma_Z, (\omega_{X/Z} \otimes_{\mathcal{O}_X} \mathcal{L}, \sigma_{\omega_{X/Z} \otimes \sigma_{\mathcal{L}}})) \to GW[i](X, \sigma_X, (\mathcal{L}, \sigma_{\mathcal{L}})). \]

It turns out that if the involution is non-trivial, the canonical double dual identification has a certain sign varying according to the data of dualizing coefficients (cf. Definition 2.8). This sign is not important if the involution is trivial, but it is crucial if the involution is non-trivial. To illustrate, we use the dévissage theorem to compute the following (cf. Theorem 7.1).

**Theorem 1.4.** Let \(S\) be a regular scheme with \(\frac{1}{2} \in \mathcal{O}_S\). Then, we have a stable equivalence of spectra
\[ GW[i](\mathbb{P}^1_S, \tau_{\mathbb{P}^1_S}^\varepsilon) \cong GW[i](S) \oplus GW[i+1](S) \]
where \(\tau_{\mathbb{P}^1_S}^\varepsilon : \mathbb{P}^1_S \to \mathbb{P}^1_S : [x : y] \mapsto [y : x]\). In particular, we have the following isomorphism on Witt groups
\[ W^i(\mathbb{P}^1_S, \tau_{\mathbb{P}^1_S}^\varepsilon) \cong W^i(S) \oplus W^{i+1}(S) \]

It is well-known that \(W^i(\mathbb{P}^1_S) \cong W^i(S) \oplus W^{i-1}(S)\) if the involution on \(\mathbb{P}^1_S\) is trivial (cf. \cite{Ne09} and \cite{Wal03}). By our dévissage theorem, the involution \(\sigma : \mathbb{P}^1_S \to \mathbb{P}^1_S : [x : y] \mapsto [y : x]\) induces a sign \((-1)\) on the double dual identification, and we have
\[ W^i(\mathbb{P}^1_S, \sigma) \cong W^i(S) \oplus W^{i-1}(S, -\text{can}) \]
but \(W^{i-1}(S, -\text{can}) \cong W^{i+1}(S)\).

We can also use the dévissage to prove the following \(C_2\)-equivariant \(\mathbb{A}^1\)-invariance (cf. Theorem 7.5).

**Theorem 1.5.** Let \(S\) be a regular scheme with involution \(\sigma_S\) and with \(\frac{1}{2} \in \mathcal{O}_S\). Let \(\sigma_{\mathbb{A}^1_S} : \mathbb{A}^1_S \to \mathbb{A}^1_S\) be the involution on \(\mathbb{A}^1_S\) with the indeterminate fixed by \(\sigma_{\mathbb{A}^1_S}\) and such that the following diagram commutes.
\[ \begin{array}{ccc}
\mathbb{A}^1_S & \xrightarrow{\sigma_{\mathbb{A}^1_S}} & \mathbb{A}^1_S \\
\downarrow p & & \downarrow p \\
S & \xrightarrow{\sigma_S} & S
\end{array} \]

The pullback
\[ p^* : GW[i](S, \sigma_S) \to GW[i](\mathbb{A}^1_S, \sigma_{\mathbb{A}^1_S}) \]
is an isomorphism.
Let Heller, Krishina and Østrær [HKO14]. More precisely, we write down the following result.

C representability of Hermitian K-theory in the Nisnevich excessive [Sch17, Theorem 9.6] adapted to schemes with involution, we obtain the B which can be considered as a C

An Definition 2.1. i.e. regard N be covered by invariant affine opens.

Notations and prerequisites

2.1. Notations on rings. Let R be a ring. It is convenient to keep the following notations on the ring R in this paper

- R-Mod is the category of (left) R-modules
- M(R) is the category of finitely generated R-modules
- M_f(R) is the category of finite length R-modules
- M_J(R) is the category of finitely generated R-modules supported in an ideal J of R. Recall that Supp(M) = \{p ∈ Spec(R)|M_p \neq 0\} and V(J) = \{p ∈ Spec(R)|p ⊃ J\}. More precisely, M_J(R) is the full subcategory of M(R) consisting of those modules M such that Supp(M) ⊂ V(J).
- D^b(\textbf{R}-Mod) is the derived category of bounded complexes of R-modules with coherent cohomology.
- D^b_{f}(R) is the full subcategory of D^b_c(R-Mod) of complexes, whose cohomology modules are finite length R-modules.
- D^b(J) is the full triangulated subcategory of D^b_c(R-Mod) of complexes whose cohomology modules are annihilated by some power of the ideal J.

Definition 2.1. An involution σ on a ring R is a morphism σ : R → R of rings such that σ^2 = id_R.

Definition 2.2. A duality coefficient (I, i) on the category R-Mod consists of an R-module I equipped with an R-module isomorphism i : I → I^{\text{op}} such that i^2 = id.

2.1.1. Opposite modules and duality. Let (R, τ) be a ring with involution. Let M be a left R-module. Define M^{\text{op}} (or simply M^{op}) to be the same as M as a set. For an element in M^{op}, we use the symbol m^{op} to denote the element m coming from the set structure of M. Consider M^{op} as a right R-module via m^{op}a = (am)^{op}.

Given a map f : A → B of rings, a module M over A, and a module N over B. Regarding S as an A-module, we can form the tensor product

f^*M := M ⊗_A B

which can be considered as a B-module. We also have the restriction of scalars

f_s N := N_A,

i.e. regard N as an A-module via a · n := f(a) · n.
Apply $\sigma : R \to R$ to the map $f : A \to B$ with $R = A = B$. It will be convenient to write $f : A \to B$ instead of $\sigma : R \to R$ for the next lemma, although they mean the same thing.

**Lemma 2.5.** Let $M$ be an $A$-module.

1. We have $\sigma_M : M := M_B = M^{op}$.
2. We have a natural $A$-module isomorphism $\sigma^* M := M \otimes_A B \cong M^{op}$.

**Proof.** (1) is clear. For (2), we define a map $\eta_M : M \otimes_A B \to M^{op} : m \otimes b \mapsto m \bar{b}$. The inverse is given by $M^{op} \to M \otimes_A B : m \mapsto m \otimes 1$. 

**Remark 2.4.** One can form two categories with duality. Define a functor $\#^\sigma_I : (R\text{-Mod})^{op} \to R\text{-Mod} : M \mapsto \text{Hom}_R(\sigma^* M, I)$ (resp. $\#^\sigma_I : (R\text{-Mod})^{op} \to R\text{-Mod} : M \mapsto \text{Hom}_R(\sigma_*, M, I)$). The double dual identification is the $R$-module homomorphism $\text{can}^\sigma_{I,M} : M \to M^{\#^\sigma_I \#^\sigma_I}$ (resp. $\text{can}^\sigma_{I,M} : M \to M^{\#^\sigma_I \#^\sigma_I}$) given by $\text{can}^\sigma_{I,M}(x)(f \otimes b) = i(f(x \otimes b))$ (resp. $\text{can}^\sigma_{I,M}(x)(f^{op}) = i(f(x^{op}))$). The triplet $(R\text{-Mod}, \#^\sigma_I, \text{can}^\sigma_{I,M})$ (resp. $(R\text{-Mod}, \#^\sigma_I, \text{can}^\sigma_{I,M})$) is a category with duality. In this paper, we sometimes write $(R\text{-Mod}, \#^\sigma_I)$ (resp. $(R\text{-Mod}, \#^\sigma_I)$) for simplicity. The natural isomorphism $\eta_M : \sigma^* M \to M^{op}$ in the proof of Lemma 2.3 induces an isomorphism of categories with duality $(\text{id}, \eta) : (R\text{-Mod}, \#^\sigma_I) \to (R\text{-Mod}, \#^\sigma_I)$.

**Lemma 2.5.** Let $R, S$ be rings with involution. Let $f : R \to S$ be a homomorphism of rings with involution. Let $M, I$ be $R$-modules with $R$-module homomorphisms $\sigma_M : M \to M^{op}$ and $\sigma_I : I \to I^{op}$. Let $N$ be an $S$-module with an $S$-module homomorphism $\sigma_N : N \to N^{op}s$. Then

1. The map $\sigma_{M,I}^R : \text{Hom}_R(M, I) \to \text{Hom}_R(M, I)^{op} : f \mapsto \sigma_I f \sigma_M$ is a well-defined $R$-module homomorphism.
2. The set $\text{Hom}_R(N, I)$ can be considered as an $S$-module by the action $s \cdot f : n \mapsto f(s \cdot n)$. Also, $f : \text{Hom}_R(N, I) = \text{Hom}_R(f, N, I)$. Moreover, the map $\sigma_{N,I}^S : \text{Hom}_R(N, I) \to \text{Hom}_R(N, I)^{op} : f \mapsto \sigma_I f \sigma_N$ is a well-defined $S$-module homomorphism and $f : \sigma_{N,I}^S = \sigma_{N,I}^S$.
3. If $N$ and $I$ are both involutions, then $\sigma_{N,I}^S$ is an involution.

2.2. **Notations on schemes.** Let $X$ be a scheme. We continue to fix the following notations about the scheme $X$ in this paper.

- $\mathcal{O}_X\text{-Mod}$ is the category of $\mathcal{O}_X$-modules.
- $\mathcal{Q}(X)$ is the category of quasi-coherent $\mathcal{O}_X$-modules.
- $\mathcal{V}(X)$ is the category of finite rank locally free coherent modules over $X$.
- $\mathcal{M}(X)$ is the category of finitely generated $\mathcal{O}_X$-modules supported in a closed scheme $Z$ of $X$. Recall that $\text{Supp}(\mathcal{F}) = \{ x \in X | \mathcal{F}_x \neq 0 \}$. More precisely, $\mathcal{M}(X)$ is the full subcategory of $\mathcal{M}(X)$ consisting of those modules $\mathcal{F}$ such that $\text{Supp}(\mathcal{F}) \subset Z$.
- $\mathcal{D}_c(\mathcal{Q}(X))$ is the derived category of bounded complexes of quasi-coherent sheaves on $X$ with coherent cohomology.
- $\mathcal{D}_c^b(\mathcal{Q}(X))$ is the full triangulated subcategory of $\mathcal{D}_c(\mathcal{Q}(X))$ of complexes $\mathcal{F}^\bullet$ such that $\{ x \in X | \mathcal{H}^i(\mathcal{F}^\bullet)_x \neq 0 \text{ for some } i \in \mathbb{Z} \} \subset Z$.
- $\mathcal{D}(X)$ is the derived category $\mathcal{D}(\mathcal{V}(X))$.
- $\mathcal{D}_c(\mathcal{V}(X))$ is the derived category of bounded complexes of quasi-coherent sheaves on $X$ with coherent cohomology.
- $\mathcal{D}_c^b(\mathcal{V}(X))$ is the full triangulated category of $\mathcal{D}_c(\mathcal{V}(X))$ of complexes $\mathcal{F}^\bullet$ such that $\{ x \in X | \mathcal{H}^i(\mathcal{F}^\bullet)_x \neq 0 \text{ for some } i \in \mathbb{Z} \} \subset Z$.
- $\mathcal{Q}(X)$ is the dg category of bounded complexes of quasi-coherent $\mathcal{O}_X$-modules with coherent cohomology.
- $\mathcal{Q}(X)$ is the full dg subcategory of $\mathcal{Q}(X)$ of complexes $\mathcal{F}^\bullet$ such that $\{ x \in X | \mathcal{H}^i(\mathcal{F}^\bullet)_x \neq 0 \text{ for some } i \in \mathbb{Z} \} \subset Z$.
- $\mathcal{V}(X)$ is the dg category $\mathcal{V}(\mathcal{V}(X))$ of bounded complexes of locally free coherent $\mathcal{O}_X$-modules.
- $\text{Ch}^b_s(X)$ is the full dg subcategory $\text{Ch}^b(X)$ of those complexes $\mathcal{F}^\bullet$ such that $\{x \in X | H^i(\mathcal{F}^\bullet)x \neq 0 \text{ for some } i \in \mathbb{Z} \} \subseteq \mathbb{Z}$.

**Definition 2.6.** An **involution** $\sigma_X$ on a ringed space $X$ is a morphism $\sigma_X : X \to X$ of ringed spaces such that $\sigma^2 = \text{id}_X$.

**Remark 2.7.** If $X = \text{Spec}(R)$, to give a scheme with involution $(X, \sigma)$ is the same as giving a ring with involution $(R, \sigma)$.

**Definition 2.8.** A **duality coefficient** $(I, i)$ for the category $\mathcal{O}_X$-$\text{Mod}$ of $\mathcal{O}_X$-modules on $X$ consists of an $\mathcal{O}_X$-module $I$ equipped with an $\mathcal{O}_X$-module isomorphism $i : I \to \sigma_*I$ such that $\sigma_*i \circ i = \text{id}$. An isomorphism $(I, i) \to (I', i')$ of duality coefficients is an isomorphism of $\mathcal{O}_X$-modules $\alpha : I \to I'$ such that the following diagram commutes

$$
\begin{array}{c}
I & \xrightarrow{\alpha} & I' \\
\downarrow i & & \downarrow i' \\
\sigma_*I & \xrightarrow{\sigma_*\alpha} & \sigma'_*I'
\end{array}
$$

2.2.1. **Dualities on schemes with involution.** Let $(X, \sigma)$ be a scheme with involution. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. By the adjointness, we have a well-known natural isomorphism of groups

$$
\text{Hom}_{\mathcal{O}_X}(\sigma^*\sigma^*\mathcal{F}, \mathcal{F}) \cong \text{Hom}_{\mathcal{O}_X}(\sigma^*\mathcal{F}, \sigma_*\mathcal{F})
$$

Note that $\mathcal{F} \cong \sigma^*\sigma^*\mathcal{F}$. Then, we have a map

$$
\eta_{\mathcal{F}} : \sigma^*\mathcal{F} \to \sigma_*\mathcal{F}
$$

in $\text{Hom}_{\mathcal{O}_X}(\sigma^*\mathcal{F}, \sigma_*\mathcal{F})$ corresponding to $\text{id} : \mathcal{F} \to \mathcal{F}$ in $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$. The map $\eta_{\mathcal{F}}$ is an isomorphism because it is the map $\eta_M : M \otimes_A S \to M^{op} : m \otimes s \mapsto m\hat{s}$ affine locally.

**Remark 2.9.** By replacing $R$ with $\mathcal{O}_X$, $\sigma_*$ with $\sigma^*$, and $\sigma^*$ with $\sigma_*$, we have obtained categories with duality $(\mathcal{O}_X, \#_I^\sigma)$ and $(\mathcal{O}_X, \#_I^\sigma)$.

**Lemma 2.10.** The map $\eta_{\mathcal{F}} : \sigma^*\mathcal{F} \to \sigma_*\mathcal{F}$ induces an isomorphism of categories with duality $(\mathcal{O}_X, \#_I^\sigma) \to (\mathcal{O}_X, \#_I^\sigma)$.

**Proof.** This can be checked affine locally, which is Remark 2.4. □

**Remark 2.11.** This result tells us that we can work on either duality $\#_I^\sigma$ or $\#_I^\sigma$. Thus, we use them interchangeably in this paper. For computations, it would be easier to use the duality $\#_I^\sigma$. For proving the devissage theorem, to use $\#_I^\sigma$ would be easier.

**Lemma 2.12.** Let $\alpha : (I, i) \to (I', i')$ be an isomorphism of duality coefficients. Then, the identity functor induces a duality preserving equivalence of categories with duality

$$(\text{id}, \alpha_* : (\mathcal{O}_X, \#_I^\sigma) \to (\mathcal{O}_X, \#_{I'}^\sigma))$$

**Proof.** The duality compatibility isomorphism is defined by

$$
\alpha_* : \text{Hom}_{\mathcal{O}_X}(\sigma_*M, I) \to \text{Hom}_{\mathcal{O}_X}(\sigma_*M, I') : f \mapsto \alpha \circ f
$$

which is natural in $M$. Now it is straightforward to check the axioms. □

2.3. **Dualizing complexes with involution.** Recall the following definition of a dualizing complex from [Gil07b, Definition 1.7] or [Ha66].

**Definition 2.13.** Let $X$ be a scheme. A **dualizing complex** $I^\bullet$ on $X$ is a complex

$$
I^\bullet := (\cdots \to 0 \to I^m \xrightarrow{d^m} I^{m+1} \to \cdots \to I^{n-1} \xrightarrow{d^{n-1}} I^n \to 0 \to \cdots) \in \mathcal{D}^b_c(\mathcal{Q}(X))
$$

of injective modules such that the natural morphism of complexes

$$
c_{I, M} : M^\bullet \to \text{Hom}_{\mathcal{O}_X}(\text{Hom}_{\mathcal{O}_X}(M^\bullet, I^\bullet), I^\bullet) : m \mapsto (c_{I, M}(m) : f \mapsto (-1)^{|f||m|}f(m))
$$
is an isomorphism for any $M^\bullet \in D^b_c(Q(X))$. A dualizing complex is called minimal if $H^r$ is an essential extension of $\ker(d^r)$ for all $r \in \mathbb{Z}$.

**Remark 2.14.** Any dualizing complex is quasi-isomorphic to a minimal dualizing complex [Gil07b, Proposition 1.9].

**Definition 2.15.** Let $(X, \sigma)$ be a noetherian scheme with involution and $\frac{1}{2}$. A dualizing complex with involution on $(X, \sigma)$ is a pair $(I^\bullet, \sigma_I)$ consisting of a dualizing complex $I^\bullet$ and an isomorphism of complexes of $\mathcal{O}_X$-modules $\sigma_I : I^\bullet \to \sigma_* (I^\bullet)$ such that $\sigma_*(\sigma_I) \circ \sigma_I = \text{id}_I$.

**Remark 2.16.** Note that $(I^p, \sigma_{I^p})$ is a duality coefficient for the category $\mathcal{O}_X$-Mod.

**Lemma 2.17.** Let $(X, \sigma)$ be a Gorenstein scheme of finite Krull dimension with involution and $\frac{1}{2}$. Let $(L, \sigma_L)$ be a dualizing coefficient with $L$ a locally free $\mathcal{O}_X$-module of rank one. Then, there exists a unique (up to chain homotopy equivalence) dualizing complex $(J^\bullet, \sigma_J)$ with involution quasi-isomorphic to $(\mathcal{L}, \sigma_{\mathcal{L}})$.

**Proof.** Let $X^{(p)} = \{x \in X | \dim \mathcal{O}_{X,x} = p\}$. Recall [Ha66, p241] that we have a Cousin complex

$$0 \to \mathcal{O}_X \xrightarrow{d^{-1}} I^0 \xrightarrow{d^0} I^1 \to \cdots \to I^n \to 0 \to \cdots$$

with $I^0 = \bigoplus_{x \in X^{(p)}} i_+ (\mathcal{O}_{X,x})$ and $I^p = \bigoplus_{x \in X^{(p)}} i_+ (\text{coker}(d^{p-2}))(x)$, where the map $i_x : \text{Spec}(\mathcal{O}_{X,x}) \to X$ is the canonical map. If $X$ is Gorenstein, the Cousin complex provides a minimal dualizing complex $I^\bullet$ on $X$ quasi-isomorphic to $\mathcal{O}_X$. This can be checked locally on closed points by [Ha66, Corollary V.2.3 p259]. For the case of Gorenstein local rings, see [Sha69, Theorem 5.4].

We define a map $\sigma_I : I^p \to \sigma_* I^p$ as follows. We consider the following commutative diagram

$$\begin{array}{ccc}
\text{Spec}(\mathcal{O}_{X,x}) & \xrightarrow{\sigma_{\mathcal{O}_{X,x}}} & \text{Spec}(\mathcal{O}_{X,\sigma(x)}) \\
\downarrow i_x & & \downarrow i_{\sigma(x)} \\
X & \xrightarrow{\sigma_X} & X
\end{array}$$

Assume that $M$ is an $\mathcal{O}_X$-module together with a map $M \to \sigma_* M$. If $x = \sigma(x)$, the map $M \to \sigma_* M$ induces a canonical map $i_x, i_x^* M \to \sigma_* i_x, i_x^* M$ of $\mathcal{O}_X$-modules by using the functoriality isomorphism $\sigma_* i_x, i_x^* M \cong i_{\sigma(x)}^* \sigma_* i_x, i_x^* M \cong i_{\sigma(x)}^* i_{\sigma(x)}^* M$. If $x \neq \sigma(x)$, the map $M \to \sigma_* M$ still induces a canonical map

$$i_x, i_x^* M \oplus i_{\sigma(x)}^* i_{\sigma(x)}^* M \to \sigma_* i_{\sigma(x)}^* i_{\sigma(x)}^* M \oplus \sigma_* i_x, i_x^* M$$

by the functoriality isomorphism above. The map $\sigma_I : I^p \to \sigma_* I^p$ is obtained by applying this situation inductively (Note that $x \in X^{(p)}$ if and only if $\sigma(x) \in X^{(p)}$). It is clear that $\sigma_*(\sigma_I) \circ \sigma_I = \text{id}_I$ by our construction.

If $\mathcal{L}$ is a line bundle on $X$, then $J^\bullet := \mathcal{L} \otimes I^\bullet$ is a dualizing complex, cf. [Ha66, Proof (1) of Theorem V.3.1, p. 266]. Moreover, the complex $J^\bullet$ is quasi-isomorphic to $\mathcal{L}$. Note that we have a canonical map $\sigma_* (\mathcal{L} \otimes I^\bullet) \to \sigma_* \mathcal{L} \otimes \sigma_* I^\bullet$ which is an isomorphism in this case. These observations reveal a canonical involution $\sigma_J : J^\bullet \to \sigma_* J^\bullet$. \qed

**2.4. Setup for algebraic $KR$-theory.** Let $(X, \sigma)$ be a scheme with involution. In this paper, we use the framework of Schlichting [Sch17]. Recall that $\text{Qcoh}^b(X)$ is a closed symmetric monoidal category under tensor product complexes $M^\bullet \otimes_{\mathcal{O}_X} N^\bullet$ given by

$$(M^\bullet \otimes_{\mathcal{O}_X} N^\bullet)^n := \bigoplus_{i+j=n} M^i \otimes_{\mathcal{O}_X} N^j, \quad d(m \otimes n) = dm \otimes n + (-1)^{|m||n|} m \otimes dn$$

and internal homomorphism complexes $\mathcal{H}om_{\mathcal{O}_X}(M^\bullet, N^\bullet)$ given by

$$\mathcal{H}om_{\mathcal{O}_X}(M^\bullet, N^\bullet)^n := \bigoplus_{j-i=n} \mathcal{H}om_{\mathcal{O}_X}(M^i, N^j), \quad df = d \circ f - (-1)^{|f|} f \circ d.$$
Let \((I^*, \sigma_I)\) be a dualizing complex with involution. Then, we define the duality functor
\[
\#_{\sigma}^*: (\text{Qcoh}^b(X))^{\text{op}} \to \text{Qcoh}^b(X) : E^* \mapsto \text{Hom}_{O_X}(\sigma_* E^*, I^*),
\]
and the double dual identification \(\text{can}_{\sigma, E}^* : E \to E^* \#_{\sigma}^* \#_{\sigma}^*\) given by the composition
\[
E \xrightarrow{\text{can}_{\sigma, E}} \text{Hom}_{O_X}(\text{Hom}_{O_X}(E, I), I) \to \text{Hom}_{O_X}(\text{Hom}_{O_X}(E, \sigma_* I), I) \to \text{Hom}_{O_X}(\sigma_* \text{Hom}_{O_X}(\sigma_* E, I), I)
\]
where the first map \(\text{can}_{\sigma, E}\) is defined by \(\text{can}_{\sigma, E}(x)(f) = (-1)^{|x||f|} f(x)\), the second map is induced by \(\sigma_I : I \to \sigma_* I\), and the third map is induced by \(\sigma_* \text{Hom}_{O_X}(M, N) = \text{Hom}_{O_X}(\sigma_* M, \sigma_* N)\).

**Lemma 2.18.** The quadruple
\[
(\text{Qcoh}^b(X), \text{quis}, \#_{\sigma}^* , \text{can}_{\sigma}^*)
\]
is a dg category with weak equivalence and duality.

**Proof.** We have the duality preserving map defined globally. It is enough to check
\[
(\text{Qcoh}^b(X), \text{quis}, \#_{\sigma}^* , \text{can}_{\sigma}^*)
\]
is a dg category with weak equivalence and duality when \(X = \text{Spec}(A)\). The affine case has been done by [Knu91] Chapter I.2 where \(A\) is more generally allowed to be non-commutative.

**Definition 2.19.** We define the coherent algebraic \(KR\)-theory spectra of \((X, \sigma_X)\) with respect to \((I, \sigma_I)\) as
\[
GW_i^{[i]}(X, \sigma_X, (I, \sigma_I)) := GW_i^{[i]}(\text{Qcoh}^b(X), \text{quis}, \#_{\sigma}^* , \text{can}_{\sigma}^*)
\]
for every \(i \in \mathbb{Z}\) where \(GW_i^{[i]}(\text{Qcoh}^b(X), \text{quis}, \#_{\sigma}^* , \text{can}_{\sigma}^*)\) is the Grothendieck-Witt spectrum of the dg category with weak equivalences and duality \((\text{Qcoh}^b(X), \text{quis}, \#_{\sigma}^* , \text{can}_{\sigma}^*)\). The Witt group of \((X, \sigma_X)\) with respect to \((I, \sigma_I)\) is defined as
\[
W^i(X, \sigma_X, (I, \sigma_I)) := W^i(D^b_C(\mathcal{Q}(X)), \#_{\sigma}^* , \text{can}_{\sigma}^*)
\]
where \(W^i(D^b_C(\mathcal{Q}(X)), \#_{\sigma}^* , \text{can}_{\sigma}^*)\) is the Witt group of the triangulated category \((D^b_C(\mathcal{Q}(X)), \#_{\sigma}^* , \text{can}_{\sigma}^*)\) with duality. Let \(Z\) be an invariant closed subscheme of \(X\). We can also define
\[
GW_Z^{[i]}(X, \sigma_X, (I, \sigma_I)) := GW_i^{[i]}(\text{Qcoh}^b(Z), \text{quis}, \#_{\sigma}^* , \text{can}_{\sigma}^*)
\]
and
\[
W_Z^i(X, \sigma_X, (I, \sigma_I)) := W^i(D^b_C(Z), \#_{\sigma}^* , \text{can}_{\sigma}^*)
\]

Let \((\mathcal{L}, \sigma_{\mathcal{L}})\) be a dualizing coefficient with \(\mathcal{L}\) a locally free \(O_X\)-module of rank one. Then, we have a dg category with duality
\[
(\text{Ch}^b(X), \text{quis}, \#_{\sigma}^\mathcal{L} , \text{can}_{\sigma}^\mathcal{L})
\]
where
\[
\#_{\sigma}^\mathcal{L} : (\text{Ch}^b(X))^{\text{op}} \to \text{Ch}^b(X) : E^* \mapsto \text{Hom}_{O_X}(\sigma_* E^*, \mathcal{L}),
\]
and the double dual identification \(\text{can}_{\sigma, \mathcal{L}}^* : E \to E^* \#_{\sigma}^\mathcal{L} \#_{\sigma}^\mathcal{L}\) given by the composition
\[
E \xrightarrow{\text{can}_{\sigma, \mathcal{L}}} \text{Hom}_{O_X}(\text{Hom}_{O_X}(E, \mathcal{L}), \mathcal{L}) \to \text{Hom}_{O_X}(\text{Hom}_{O_X}(E, \sigma_* \mathcal{L}), \mathcal{L}) \to \text{Hom}_{O_X}(\sigma_* \text{Hom}_{O_X}(\sigma_* E, \mathcal{L}), \mathcal{L})
\]
where the first map \(\text{can}_{\sigma, \mathcal{L}}\) is defined by \(\text{can}_{\sigma, \mathcal{L}}(x)(f) = (-1)^{|x||f|} f(x)\), the second map is induced by \(\sigma_{\mathcal{L}} : \mathcal{L} \to \sigma_* \mathcal{L}\), and the third map is induced by \(\sigma_* \text{Hom}_{O_X}(M, N) = \text{Hom}_{O_X}(\sigma_* M, \sigma_* N)\).

**Definition 2.20.** We define the algebraic \(KR\)-theory spectra of \((X, \sigma_X)\) with respect to \((\mathcal{L}, \sigma_{\mathcal{L}})\) as
\[
GW_i^{[i]}(X, \sigma_X, (\mathcal{L}, \sigma_{\mathcal{L}})) := GW_i^{[i]}(\text{Ch}^b(X), \text{quis}, \#_{\sigma}^\mathcal{L} , \text{can}_{\sigma}^\mathcal{L})
\]
for every \(i \in \mathbb{Z}\). The Witt group of \((X, \sigma_X)\) with respect to \((\mathcal{L}, \sigma_{\mathcal{L}})\) is defined as
\[
W^i(X, \sigma_X, (\mathcal{L}, \sigma_{\mathcal{L}})) := W^i(D^b_C(X), \#_{\sigma}^\mathcal{L} , \text{can}_{\sigma}^\mathcal{L})
\]
Let \(Z\) be an invariant closed subscheme of \(X\). We can also define
\[
GW_Z^{[i]}(X, \sigma_X, (\mathcal{L}, \sigma_{\mathcal{L}})) := GW_i^{[i]}(\text{Ch}^b_Z(X), \text{quis}, \#_{\sigma}^\mathcal{L} , \text{can}_{\sigma}^\mathcal{L})
\]
and
\[ W'_2(X, \sigma_X, (\mathcal{L}, \sigma_\mathcal{L})) := W^i(D^b_Z(X), \#_{\sigma_\mathcal{L}}) \]

2.5. Dualizing complex via a finite morphism. Let \((Z, \sigma_Z)\) be a scheme with involution, and let \(\pi : Z \to X\) be a finite morphism. Assume that \(\pi\) is compatible involutions on \(Z\) and \(X\), i.e., the following diagram commutes

\[
\begin{array}{ccc}
Z & \xrightarrow{\pi} & X \\
\sigma_Z & \downarrow & \sigma_X \\
Z & \xrightarrow{\pi} & X 
\end{array}
\]

Following Hartshorne [Ha66], we write
\[ \pi^b I^\bullet := \bar{\pi}^* \text{Hom}_{O_X}(\pi_* O_Z, I^\bullet) \]
where \(\bar{\pi} : (Z, \sigma_Z) \to (X, \pi_* O_Z) =: \bar{X}\) is the canonical flat map of ringed spaces. If \(I^\bullet\) is a dualizing complex, then \(\pi^b I^\bullet\) is a dualizing complex [Ill7a Section 2.4]. Note that there is a canonical involution \(\sigma_{\bar{X}} : \bar{X} \to \bar{X}\) defined by \(\sigma_X : X \to X\) on the topological spaces and \(\pi_* \sigma_Z : \pi_* O_Z \to \sigma_X \pi_* \sigma_Z O_Z = \pi_* \sigma Z \pi_* O_Z\) on sheaves of rings. Then, we can define an involution \(\sigma_{\pi^b I^\bullet} : \pi^b I^\bullet \to \pi Z^\pi I^\bullet\) on \(\pi^b I^\bullet\) by the following composition
\[
\begin{align*}
\pi^b I^\bullet &= \bar{\pi}^* \text{Hom}_{O_X}(\pi_* O_Z, I^\bullet) \\
&\xrightarrow{\pi_* |_{\sigma_Z}} \bar{\pi}^* \text{Hom}_{O_X}(\pi_* \sigma_Z O_Z, I^\bullet) \\
&\xrightarrow{1 - \sigma_{\pi^b I^\bullet}} \bar{\pi}^* \text{Hom}_{O_X}(\pi_* O_Z, I^\bullet) \\
&= \sigma_{\pi^b I^\bullet} \pi^b I^\bullet
\end{align*}
\]
It follows that we obtain a dualizing complex with involution \((\pi^b I^\bullet, \sigma_{\pi^b I^\bullet})\) on \(Z\). Thus, we have a well-defined spectrum \(GW^{[i]}(Z, \sigma_Z, (\pi^b I^\bullet, \sigma_{\pi^b I^\bullet}))\) for every \(i \in \mathbb{Z}\).

Note that if \(Z = \text{Spec}(B), X = \text{Spec}(A)\) are affine, then \(\text{Hom}_{A}(B, I)\) has a structure of \(B\)-module by defining the action \(b \cdot f\) as \(b \cdot f : m \mapsto f(bm)\) for \(b, m \in B\). The involution \(\sigma_{\pi^b I^\bullet} : \pi^b I^\bullet \to \pi Z^\pi I^\bullet\) is precisely the map of \(B\)-modules
\[ \sigma_{B, I}^\pi : \text{Hom}_{A}(B, I) \to \text{Hom}_{A}(B, I)^\text{op} : f \mapsto \sigma f \sigma B \]

3. The transfer morphism

**Theorem 3.1.** Let \((X, \sigma_X)\) and \((Z, \sigma_Z)\) be schemes with involution and with \(\frac{1}{2} \in O_X\). If \(\pi : Z \to X\) is a finite morphism of schemes with involution, then there is a map of spectra
\[ T_{Z/X} : GW^{[i]}(Z, \sigma_Z, (\pi^b I^\bullet, \sigma_{\pi^b I^\bullet})) \to GW^{[i]}(X, \sigma_X, (I^\bullet, \sigma_I)). \]

**Definition 3.2.** The map \(T_{Z/X} : GW^{[i]}(Z, \sigma_Z, (\pi^b I^\bullet, \sigma_{\pi^b I^\bullet})) \to GW^{[i]}(X, \sigma_X, (I^\bullet, \sigma_I))\) in the above theorem is called the transfer morphism.

**Proof of Theorem 3.1**. The transfer map \(T_{Z/X} : GW^{[i]}(Z, \sigma_Z, (\pi^b I^\bullet, \sigma_{\pi^b I^\bullet})) \to GW^{[i]}(X, \sigma_X, (I^\bullet, \sigma_I))\) is induced by the duality preserving functor in Lemma 3.3 below. \[\square\]

**Lemma 3.3.** There is a duality preserving functor
\[ \pi_* : (\text{Qcoh}^b(Z), \#_{\sigma_Z} I^\bullet) \to (\text{Qcoh}^b(X), \#_{\sigma_X} I^\bullet) \]
with the duality compatibility natural transformation
\[ \eta : \pi_* \circ \#_{\pi^b I^\bullet} \to \#_{I^\bullet} \circ \pi_* \]
given by the following composition of isomorphisms

\[ \begin{align*}
\pi_*[\sigma \pi^! I]_{O_Z} &\xrightarrow{\eta} [\pi_* \sigma Z \pi^! I]_{O_X} \\
\xrightarrow{|p,1|} &\xrightarrow{|[1,\xi]|} [\sigma X \pi_* \pi^! I]_{O_X}
\end{align*} \]

for \( G \in \text{Qcoh}(Z) \) where \( \xi : \pi_* \pi^! I \to I \) is the evaluation at one.\(^1\)

**Proof.** We can check the conditions of the duality preserving functor affine locally, which is checked in Lemma 3.4 below.

**Lemma 3.4.** Let \( \pi : R \to S \) be a finite morphism of rings with involution. There is a duality preserving functor

\[ \pi_* : (\text{Ch}^b(S\text{-Mod}), \#_{\pi^!}) \to (\text{Ch}^b(R\text{-Mod}), \#_{\pi^!}) \]

with the duality compatibility natural transformation

\[ \eta : \pi_* \circ \#_{\pi^!} \to \#_{\pi^!} \circ \pi_* \]

induced by the isomorphism of \( R \)-modules

\[ \text{Hom}_S(M^{op}, \text{Hom}_R(S,I)) \to \text{Hom}_R(M^{op}, I) : f \mapsto (\xi \circ f) : m^{op} \mapsto f(m^{op})(1_S) \]

for \( M \in S\text{-Mod} \).

**Proof.** By assumption, we have the commutative diagram

\[ \begin{array}{ccc}
R & \xrightarrow{\pi^!} & R \\
\downarrow & & \downarrow \\
S & \xrightarrow{\sigma} & S
\end{array} \]

(3.2)

It is evident that \( \eta_M \) is an isomorphism. The inverse is given by

\[ \text{Hom}_R(M^{op}, I) \to \text{Hom}_S(M^{op}, \text{Hom}_R(S,I)) : g \mapsto g' : (m^{op})(s) = g((m \cdot \sigma_S(s))^{op}) \]

We need to show that the diagram is commutative

\[ \begin{array}{ccc}
\pi_* M & \xrightarrow{} & \pi_* (M^{\#_{\pi^!} \#_{\pi^!}}) \\
\downarrow & & \downarrow \\
(\pi_* M)^{\#_{\pi^!} \#_{\pi^!}} & \xrightarrow{} & (\pi_* M^{\#_{\pi^!} \#_{\pi^!}})^{\#_{\pi^!}}
\end{array} \]

Elements in this diagram act as follows.

\[ \begin{aligned}
m &\mapsto \text{can}^{\pi^!}(m) \\
\eta_{M^{\#_{\pi^!}}} &\mapsto \eta^{\pi^!}(\text{can}^{\pi^!}(m)) \\
\text{can}^I(m) &\mapsto \eta^I(\text{can}^I(m))
\end{aligned} \]

To prove the commutativity, we need to show that, for any \( f \in M^{\#_{\pi^!}} \),

\[ \eta_{M^{\#_{\pi^!}}} (\text{can}^{\pi^!}(m))(f) = \eta^I (\text{can}^I(m))(f). \]

It is straightforward to check that they both equal \( \sigma_I(f(m)(1_S)) \).

---

1 This map is called trace in [Ha66] and [Gil07a], but on affine schemes this map is in fact the evaluation at one.
4. The local case and finite length modules

Let $(R, m, k)$ be a local ring with an involution $\sigma_R$. Suppose that $I^\bullet$ is a dualizing complex of $R$, $\sigma_I$ is an involution of $I^\bullet$ and $m$ is invariant under the involution of $R$. In the previous section, we have defined a transfer morphism

$$T_{m/R}: W^i(k, \sigma_k, (\pi^2 I^\bullet, \sigma_{\pi^1})) \to W^i(R, \sigma_R, (I^\bullet, \sigma_I))$$

which factors through

$$D_{m/R}: W^i(k, \sigma_k, (\pi^2 I^\bullet, \sigma_{\pi^1})) \to W^i_m(R, \sigma_R, (I^\bullet, \sigma_I)).$$

Moreover, the map $D_{m/R}$ factors as

$$W^i(k, \sigma_k, (\pi^2 I^\bullet, \sigma_{\pi^1})) \to W^i(M_{fl}(R), \sigma_R, (I^\bullet, \sigma_I)) \to W^i_m(R, \sigma_R, (I^\bullet, \sigma_I))$$

where the first map is induced by the functor $\pi_* : \mathcal{D}^b(M(k)) \to \mathcal{D}^b(M_{fl}(R))$ and the second map is induced by the inclusion $\mathcal{D}^b(M_{fl}(R)) \to \mathcal{D}^b_f(R) = D^b_m(R)$.

**Theorem 4.1.** Let $(R, m, k)$ be a local ring with an involution $\sigma_R$. Suppose that $I^\bullet$ is a dualizing complex of $R$, $\sigma_I$ is an involution of $I^\bullet$ and $m$ is invariant under the involution of $R$. Then, there is an isomorphism of groups

$$D_{m/R}: W^i(k, \sigma_k, (\pi^2 I^\bullet, \sigma_{\pi^1})) \to W^i_m(R, \sigma_R, (I^\bullet, \sigma_I)).$$

**Lemma 4.2.** The natural map $W^i(M_{fl}(R), \sigma_R, (I^\bullet, \sigma_I)) \to W^i_m(R, \sigma_R, (I^\bullet, \sigma_I))$ is an isomorphism of groups.

**Proof.** The inclusion

$$\mathcal{D}^b(M_{fl}(R)) \to \mathcal{D}^b_f(R) = D^b_m(R)$$

is an equivalence of category, by [Ke99, Section 1.15] \(\square\)

**Lemma 4.3.** The map $\pi_* : W^i(k, \sigma_k, (\pi^2 I^\bullet, \sigma_{\pi^1})) \to W^i(M_{fl}(R), \sigma_R, (I^\bullet, \sigma_I))$ is an isomorphism.

**Proof.** Let $I^\bullet := (\cdots \to 0 \to I^n \to I^{n+1} \to \cdots \to I^1 \to 0 \to \cdots)$ be the dualizing complex on $R$. Let $E := I^n$ and let $\pi^2 E$ be the $k$-module $\text{Hom}_R(k, E)$. Let

$$\sigma_{\pi^2 E} : \text{Hom}_R(k, E) \to \text{Hom}_R(k, E)^{op} : f \mapsto \sigma_{\pi^2 E}(f)$$

be the involution, where $\sigma_{\pi^2 E}(f)(a) = \sigma_E f \sigma_k(a)$.

Consider the following diagram

$$\begin{array}{ccc}
W^{i+n}(k, \sigma_k, (\pi^2 E, \sigma_{\pi^2 E})) & \xrightarrow{\cong} & W^{i+n}(M_{fl}(R), \sigma_R, (E, \sigma_E)) \\
\xrightarrow{v_1} & & \xrightarrow{v_2} \\
W^i(k, \sigma_k, (\pi^2 I^\bullet, \sigma_{\pi^1})) & \xrightarrow{\pi_*} & W^i(M_{fl}(R), \sigma_R, (I^\bullet, \sigma_I))
\end{array}$$

The map $h$ is the transfer morphism, and it is an isomorphism by [QSS99] for $i + n$ even. If $i + n$ is odd, $W^{i+n}(k, \sigma_k, (\pi^2 E, \sigma_{\pi^2 E})) = W^{i+n}(M_{fl}(R), \sigma_R, (E, \sigma_E)) = 0$ by [BW02, Proposition 5.2]. The map $v_1$ is induced by the identity

$$(\text{id}, \theta) : (\mathcal{D}^b k, \#^2 \pi^2 I^\bullet) \to (\mathcal{D}^b k, \#^2 \pi^2 E[n])$$

where $\theta : \#^2 \pi^2 I^\bullet \to \#^2 \pi^2 E[n]$ is the natural transformation given by the isomorphism

$$\theta : \text{Hom}_k(M^{op}, \text{Hom}_R(k, I^\bullet)) \to \text{Hom}_k(M^{op}, \text{Hom}_R(k, E[n]))$$

of complexes induced by the projection $I^\bullet \to E[n]$. \(\theta_M\) is an isomorphism, because as an $R$-module

$$\text{Hom}_k(M^{op}, \text{Hom}_R(k, I^\bullet)) \cong \text{Hom}_R(M^{op}, I^\bullet) = 0$$

if $j \neq n$ by [GL02, Lemma 3.3 (4)] and the isomorphism $\eta$ in Lemma 4.3. The isomorphism $v_2$ is defined analogously to the map $v_1$. \(\square\)
Lemma 4.4. Let \((R, m, k)\) be a local ring with involution and \(\frac{1}{2} \in \mathcal{O}_X\). Assume that \(I^\bullet\) is a dualizing complex of \(R\), \(\sigma_I\) an involution of \(I^\bullet\) and \(m\) is invariant under the involution of \(R\). Let \(J\) be an invariant ideal of \(R\).

\[
\begin{array}{c}
R \\
\pi
\end{array} \xrightarrow{p} k
\]

We have a commutative diagram

\[
\begin{array}{ccc}
W^i(k, \sigma_k, (q^p I^\bullet, \sigma_{q^p I})) & \xrightarrow{q_*} & W^i_{m/J}(R/J, \sigma_{R/J}, (p^I I^\bullet, \sigma_{p^I I})) \\
\cong & & \cong \\
W^i(k, \sigma_k, (\pi^I I^\bullet, \sigma_{\pi^I I})) & \xrightarrow{\pi_*} & W^i_m(R, \sigma_R, (I^\bullet, \sigma_I))
\end{array}
\]

Therefore, the map \(p_*\) is an isomorphism

Proof. Define a map \(\gamma : q^p I^\bullet \to \pi^I I\)

\[
\gamma : \text{Hom}_{R/J}(k, \text{Hom}_R(R/J, I)) \to \text{Hom}_R(k, I)
\]

by sending \(f : k \to \text{Hom}_R(R/J, I)\) to \(\gamma(f) : a \mapsto f(a)(1)\). This map is an isomorphism, cf. [Ha66 Proposition 6.2 p166]. The commutativity of Diagram (4.1) follows immediately from the following commutative diagram of triangulated categories with duality

\[
\begin{array}{ccc}
(D^b k, \#q^p I) & \xrightarrow{q_*} & (D^b_{m/J}(R/J), \#p^I I) \\
\downarrow & & \downarrow p_* \\
(D^b k, \#\pi^I I) & \xrightarrow{\pi_*} & (D^b_R, \#I)
\end{array}
\]

where the left vertical arrow is induced by the isomorphism \(\gamma : q^p I^\bullet \to \pi^I I\). □

5. The dévissage theorem for a closed immersion

Let \(Z \hookrightarrow X\) be a closed immersion now. Since \(\pi_* \mathcal{G} \in \text{Qcoh}_{c,Z}(X)\) for all \(\mathcal{G} \in \text{Qcoh}(X)\), we have a map

\[
D_{Z/X} : \text{GW}^{[i]}(Z, \sigma_Z, (\pi^I I^\bullet, \sigma_{\pi^I I})) \to \text{GW}^{[i]}_Z(X, \sigma_X, (I^\bullet, \sigma_I)).
\]

In fact, the transfer morphism \(T_{Z/X}\) is the composition

\[
\text{GW}^{[i]}(Z, \sigma_Z, (\pi^I I^\bullet, \sigma_{\pi^I I})) \xrightarrow{D_{Z/X}} \text{GW}^{[i]}_Z(X, \sigma_X, (I^\bullet, \sigma_I)) \to \text{GW}^{[i]}(X, \sigma_X, (I^\bullet, \sigma_I))
\]

where the second map is the extension of support.

Theorem 5.1. Let \((X, \sigma_X)\) and \((Z, \sigma_Z)\) be schemes with involution and with \(\frac{1}{2} \in \mathcal{O}_X\). If \(\pi : Z \to X\) is a closed immersion which is invariant under involutions, then there is an equivalence of spectra

\[
D_{Z/X} : \text{GW}^{[i]}(Z, \sigma_Z, (\pi^I I^\bullet, \sigma_{\pi^I I})) \to \text{GW}^{[i]}_Z(X, \sigma_X, (I^\bullet, \sigma_I)).
\]

Proof. The result follows by Theorem 5.2 and the Karoubi induction. □

Theorem 5.2. Let \((X, \sigma_X)\) and \((Z, \sigma_Z)\) be schemes with involution and with \(\frac{1}{2} \in \mathcal{O}_X\). If \(\pi : Z \to X\) is a closed immersion which is invariant under involutions, then there is an isomorphism of groups

\[
D_{Z/X} : W^i(Z, \sigma_Z, (\pi^I I^\bullet, \sigma_{\pi^I I})) \to W^i_Z(X, \sigma_X, (I^\bullet, \sigma_I)).
\]
Proof. We borrow the idea of [BW02], [Gil02], [Gil07a] to use the filtration of derived category filtered by codimension of points to reduce the problem to the local case. All we need to check is that this approach is compatible with the duality given by a non-trivial involution.

Let $X^p_T := \{ x \in X | m \leq \mu_T(x) \leq p - 1 \}$ and $X^{(p)} := \{ x \in X | \mu_T(x) = p \}$. We define

$$\mathcal{D}^p_{Z,X} := \left\{ M^* \in \mathcal{D}^{b,c,Z}_c(\mathbb{Q}(X) | (M^*)_x \text{ is acyclic for all } x \in X^p_T \right\}$$

considered as a full subcategory of $\mathcal{D}^{b,c,Z}_c(\mathbb{Q}(X))$. Note that the duality functor $\#^i$ maps $\mathcal{D}^p_{Z,X}$ into itself. Therefore, $(\mathcal{D}^{p}_{Z,X}, \#^i)$ is a triangulated category with duality. The subcategories $\mathcal{D}^{p}_{Z,X}$ provide a finite filtration

$$\mathcal{D}^{b}_{c,Z}(\mathbb{Q}(X)) = \mathcal{D}^{m}_{Z,X} \supset \mathcal{D}^{m+1}_{Z,X} \supset \cdots \supset \mathcal{D}^{p}_{Z,X} \supset \cdots \supset \mathcal{D}^{0}_{Z,X} \supset (0)$$

which induces exact sequences of triangulated categories with duality

$$\mathcal{D}^{p+1}_{Z,X} \longrightarrow \mathcal{D}^{p}_{Z,X} \longrightarrow \mathcal{D}^{p}_{Z,X}/\mathcal{D}^{p+1}_{Z,X}$$

On the other hand, we define

$$\mathcal{D}^{p}_{Z} := \left\{ M^* \in \mathcal{D}^{b}_{c,Z}(\mathbb{Q}(Z) | (M^*)_z \text{ is acyclic for all } z \in Z^p_Z \right\}$$

as a full subcategory of $\mathcal{D}^{b}_{c,Z}(\mathbb{Q}(Z))$. By the same reason above, we have a finite filtration

$$\mathcal{D}^{p}_{c,Z}(\mathbb{Q}(Z)) = \mathcal{D}^{m}_{Z} \supset \mathcal{D}^{m+1}_{Z} \supset \cdots \supset \mathcal{D}^{p}_{Z} \supset \cdots \supset \mathcal{D}^{0}_{Z} \supset (0)$$

which induces exact sequences of triangulated categories with duality

$$\mathcal{D}^{p+1}_{Z} \longrightarrow \mathcal{D}^{p}_{Z} \longrightarrow \mathcal{D}^{p}_{Z}/\mathcal{D}^{p+1}_{Z}$$

Since $\mu_T(z) = \mu_{\pi_{(T)}(z)}$ for all $z \in Z$, we have $\pi_*(\mathcal{D}^{p}_{Z}) \subset \mathcal{D}^{p}_{Z,X}$. It follows that we obtain a map of exact sequences of triangulated categories with duality

$$\xymatrix{ (\mathcal{D}^{p+1}_{Z}, \#^i) \ar[r]_{\pi_*} \ar[d]_{\pi_*} & (\mathcal{D}^{p}_{Z}, \#^i) \ar[r]_{\pi_*} \ar[d]_{\pi_*} & (\mathcal{D}^{p}_{Z,X}, \#^i) \ar[d]_{\pi_*} \ar[r]_{\pi_*} & (\mathcal{D}^{p}_{Z,X}/\mathcal{D}^{p+1}_{Z,X}, \#^i) \ar[d]_{\pi_*} \ar[r]_{\pi_*} & \cdots \ar[r] \cdots \\
(\mathcal{D}^{p+1}_{Z,X}, \#^i) \ar[r]_{} & (\mathcal{D}^{p}_{Z,X}, \#^i) \ar[r]_{} & (\mathcal{D}^{p}_{Z,X}/\mathcal{D}^{p+1}_{Z,X}, \#^i) \ar[r]_{} & \cdots \ar[r] \cdots \cdots \\
\cdots \ar[r]_{} & W_i(\mathcal{D}^{p}_{Z,X}/\mathcal{D}^{p+1}_{Z,X}, \#^i) \ar[r]_{} & W_i(\mathcal{D}^{p}_{Z,X}/\mathcal{D}^{p+1}_{Z,X}, \#^i) \ar[r]_{} & W_i(\mathcal{D}^{p+1}_{Z,X}/\mathcal{D}^{p+1}_{Z,X}, \#^i) \ar[r]_{} & \cdots \ar[r] \cdots \cdots }$$

(5.1)

which induces a map of long exact sequences of groups

$$\xymatrix{ \cdots \ar[r]_{} & W_i(\mathcal{D}^{p}_{Z,X}, \#^i) \ar[r]_{} & W_i(\mathcal{D}^{p}_{Z,X}/\mathcal{D}^{p+1}_{Z,X}, \#^i) \ar[r]_{} & W_i(\mathcal{D}^{p+1}_{Z,X}/\mathcal{D}^{p+1}_{Z,X}, \#^i) \ar[r]_{} & \cdots }$$

\begin{lemma} \label{lem:localization}
The localization functors

$$\text{loc} : \mathcal{D}^{p}_{Z,X}/\mathcal{D}^{p+1}_{Z,X} \rightarrow \coprod_{x \in Z \cap X^{(p)}} \mathcal{D}^{b}_{m_{X,x}(\mathcal{O}_{X,x})}$$

and

$$\text{loc} : \mathcal{D}^{p}_{Z}/\mathcal{D}^{p+1}_{Z} \rightarrow \coprod_{z \in Z^{(p)}} \mathcal{D}^{b}_{m_{Z,z}(\mathcal{O}_{Z,z})}$$

induce equivalences of categories.
\end{lemma}

Proof. This result is well-known in the literature, see [Gille, Theorem 5.2] for instance. \qed
Note that the morphism \( \pi_* : W^i(D^b_{Z/X} / D^{p+1}_{Z/X}, \#_{\pi^*}) \to W^i(D^b_{Z/X} / D^{p+1}_{Z/X}, \#_{\pi^*}) \) is an isomorphism. This can be concluded by the commutative diagram

\[
\begin{array}{ccc}
W^i(D^b_{Z/X} / D^{p+1}_{Z/X}, \#_{\pi^*}) & \xrightarrow{\pi_*} & W^i(D^b_{Z/X} / D^{p+1}_{Z/X}, \#_{\pi^*}) \\
\downarrow & & \downarrow \cong \\
W^i(D^b_{Z/X} / D^{p+1}_{Z/X}, \#_{\pi^*}) & \cong & \oplus_{\omega \in \mathbb{Z}/p} W^i_{m_{X,z}}(O_{X,z}, \sigma_{O_{X,z}}(I_{Z,z}, \omega_{Z,z}))
\end{array}
\]

where the right morphism is an isomorphism by Lemma 4.4.

\[ \beta \]

6. A GEOMETRICAL COMPUTATION OF THE DÉVISSAGE FOR REGULAR SCHEMES

Let \((X, \sigma_X)\) be a regular scheme with involution. Assume \((Z, \sigma_Z)\) is a regular scheme with involution and assume further that \(Z\) is regularly embedded in \(X\) of codimension \(d\). We have a normal bundle \(N := N_{X/Z}\) of \(X\) on \(Z\) which is locally free of rank \(d\). If \(Z\) is invariant under \(\sigma_X\), then we can define an involution \(\sigma_N : N \to \sigma_N\) induced from \(\sigma_X : X \to X\) by the invariant ideal sheaf of \(Z\) in \(X\). The involution \(\sigma_N\) induces an involution \(\det(\sigma_N) : \det(N) \to \det(N)\) by taking determinants. Recall that the canonical sheaf is defined by

\[ \omega_{X/Z} := \mathcal{H}om_{\mathcal{O}_Z}(\det(N), \mathcal{O}_Z) \]

Now, \(\det(\sigma_N)\) induces an involution

\[ \sigma_{\omega_{X/Z}} : \omega_{X/Z} \to \sigma_N \omega_{X/Z} : f \mapsto \sigma_N \circ (\sigma_N f) \circ \det(\sigma_N) \]

**Theorem 6.1** (Dévissage). Let \((X, \sigma_X)\) be a regular scheme with involution. If \(Z\) is a regular closed subscheme of \(X\) of codimension \(d\) which is invariant under \(\sigma\), then there is a stable equivalence of spectra

\[ D_{Z/X, \mathcal{L}} : GW^{i-d}(Z, \sigma_Z, (\omega_{X/Z} \otimes_{\mathcal{O}_X} \mathcal{L}, \sigma_{\omega_{X/Z}} \otimes \sigma_{\mathcal{L}})) \to GW^{i-d}(X, \sigma_X, (\mathcal{L}, \sigma_{\mathcal{L}})) \]

By the Karoubi induction, it is enough to prove the following result on Witt groups.

**Theorem 6.2.** Let \((X, \sigma_X)\) be a regular scheme with involution and with \(\frac{1}{d} \in \mathcal{O}_X\). If \(Z\) is a smooth closed subscheme of \(X\) of codimension \(d\) which is invariant under \(\sigma\), then there is an isomorphism of groups

\[ W^{i-d}(Z, \sigma_Z, (\omega_{X/Z} \otimes_{\mathcal{O}_X} \mathcal{L}, \sigma_{\omega_{X/Z}} \otimes \sigma_{\mathcal{L}})) \to W^i(Z, \sigma_Z, (\mathcal{L}, \sigma_{\mathcal{L}})) \]

**Proof.** Let \((\mathcal{L}, \sigma_{\mathcal{L}}) \to (\mathcal{I}^*, \sigma_I)\) be a minimal injective resolution compatible with the involution \(\sigma_X\) where \(\mathcal{I}^* = I^0 \to I^1 \to \cdots \to I^d\), which exists by Lemma 2.17. We have the dévissage isomorphism on the level of coherent Witt groups

\[ D_{Z/X} : W^i(Z, \sigma_Z, (\pi^d \mathcal{I}^*, \sigma_{\pi^d})) \to W^i(Z, \sigma_Z, (\mathcal{I}^*, \sigma_I)) \]

We want to construct a quasi-isomorphism \(\beta : \omega_{X/Z} \otimes_{\mathcal{O}_X} \mathcal{L}[-d] \to \pi^d \mathcal{I}^*\) which is compatible with involutions, then we can use a variant of Lemma 2.17 to conclude the result.

Globally, we have a map \(\beta : \omega_{X/Z} \otimes_{\mathcal{O}_X} \mathcal{L}[-d] \to \pi^d \mathcal{I}^*\) of \(\mathcal{O}_Z\)-modules which is a quasi-isomorphism. This map is defined by the fundamental local isomorphism

\[ H^i \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{I}^*) = \text{Ext}_{\mathcal{O}_X}^{i+1}(\mathcal{O}_Z, \mathcal{L}) = \begin{cases} 0 & \text{if } i \neq d \\ \omega_{X/Z} \otimes_{\mathcal{O}_X} \mathcal{L} & \text{if } i = d \end{cases} \]

(see [Ha66], p 179).

We need to check that \(\beta\) is compatible with the involution. This can be checked affine locally. So, we assume \(X = \text{Spec}(R)\) is affine and \(Z = \text{Spec}(R/J)\) with \(J\) defined by a regular sequence \((x_1, \cdots, x_d)\) of length \(d\). Let \(E = \oplus_{1 \leq i \leq d} Re_i\) be a free \(R\)-module with basis \(\{e_1, \cdots, e_d\}\). We also assume that \((\mathcal{L}, \sigma_{\mathcal{L}})\) is a rank one projective \(R\)-module with an involution \(\sigma_{\mathcal{L}}\), and \(L \to I^*\) is an injective resolution with involution \(\sigma_L\) compatible with \(\sigma_{\mathcal{L}}\).
Let \( s : E \to R : (y_i e_i)_{1 \leq i \leq d} \mapsto \sum_{i=1}^{d} y_i x_i \). There is a koszul resolution of \( R/J \)

\[
\cdots \to 0 \to \bigwedge^d E \to \bigwedge^{d-1} E \to \cdots \to \bigwedge^0 E \to R/J
\]

with differential

\[
d^i : \bigwedge^i E \to \bigwedge^{i-1} E : \alpha_1 \wedge \cdots \wedge \alpha_i \mapsto \sum_{t=1}^{i} (-1)^{t+1} s(\alpha_t) \alpha_1 \wedge \cdots \wedge \hat{\alpha}_t \wedge \cdots \wedge \alpha_i
\]

Since \( (x_1, \cdots, x_d) \) is a regular sequence, the koszul complex provides a minimal projective resolution of \( R/J \). Set \( P^{-i} = P_i = \bigwedge^i E \).

Since \( J \) is invariant under the involution \( \sigma_R \), we see that \( (\sigma_R(x_1), \cdots, \sigma_R(x_d)) \) is also a regular sequence and generates the ideal \( J \). It follows that we can write \( \sigma(x_i) = \sum_j a_{ij} x_j \) for some \( a_{ij} \in R \).

Write \( A = (a_{ij})_{1 \leq i, j \leq d} \) for this \( d \times d \) matrix with coefficients in \( R \). Note that \( J/J^2 \) can be viewed as a free \( R/J \)-module with basis \( ( \bar{x}_1, \cdots, \bar{x}_d ) \) by [Liu06, Section 6.3.1, Lemma 3.6 p 228]. Let \( \bar{A} = (\bar{a}_{ij})_{1 \leq i, j \leq d} \) be the matrix of coefficients consisting of residue classes in \( R/J \). Observe that \( \bar{A} \) is an involutionary matrix in \( M_d(R/J) \) (though \( A \) need not to be).

Define a homomorphism of free \( R \)-modules \( \sigma_E : E \to E^{\text{op}} \) by

\[
\bigoplus_{i=1}^{d} R e_i \xrightarrow{A} \bigoplus_{i=1}^{d} R e_i \xrightarrow{\sum_{i=1}^{d} \sigma_R} \bigoplus_{i=1}^{d} (R)^{\text{op}} e_i = (\bigoplus_{i=1}^{d} (R)e_i)^{\text{op}}
\]

We check that the diagram

\[
\begin{array}{ccc}
E & \xrightarrow{s} & R \\
\downarrow \sigma_E & & \downarrow \sigma_R \\
E^{\text{op}} & \xrightarrow{s^{\text{op}}} & R^{\text{op}}
\end{array}
\]

of \( R \)-modules commutes. The homomorphism \( \sigma_E \) induces homomorphisms of \( R \)-modules

\[
\bigwedge^i \sigma_E : \bigwedge^i E \to (\bigwedge^i E)^{\text{op}}
\]

on exterior powers \( \bigwedge^i E \).

We review the fundamental local isomorphism

\[
\text{Ext}_R^i(R/J, L) \to \text{Hom}_R(\det(J/J^2), L)
\]
of the \( R/J \)-modules (see \cite[Chapter III.7 p 176]{Ha66}). To illustrate this isomorphism, we draw the following diagram of \( R \)-modules

\[
\begin{array}{cccc}
\Hom_R(\det(J/J^2), L/JL) & \\
\beta & \\
\Hom_R(P^0, L) \rightarrow \cdots \rightarrow \Hom_R(P^d, L) & \\
\downarrow & \\
\Hom_R(R/J, I^0) \rightarrow \Hom_R(P^0, I^0) \rightarrow \cdots \rightarrow \Hom_R(P^d, I^0) & \\
\downarrow & \\
\Hom_R(R/J, I^1) \rightarrow \Hom_R(P^0, I^1) \rightarrow \cdots \rightarrow \Hom_R(P^d, I^1) & \\
\downarrow & \\
\Hom_R(R/J, I^2) \rightarrow \Hom_R(P^0, I^2) \rightarrow \cdots \rightarrow \Hom_R(P^d, I^2) & \\
\downarrow & \\
\cdots & \\
\downarrow & \\
\Hom_R(R/J, I^n) \rightarrow \Hom_R(P^0, I^n) \rightarrow \cdots \rightarrow \Hom_R(P^d, I^n) & \\
\end{array}
\]

Note that by this diagram, we have zigzags

\[
\Hom_R(R/J, I^\bullet) \rightarrow \Tot(\Hom_R(P^\bullet, I^\bullet)) \leftarrow \Hom_R(P^\bullet, L)
\]

of quasi-isomorphisms of \( R \)-modules. Now, we deduce

\[
\Ext^i_R(R/J, L) := H^i(\Hom_R(R/J, I^\bullet)) \cong H^i\Hom_R(P^\bullet, L)
\]

and we define a map of \( R \)-modules

\[
\tilde{\beta} : \Hom_R(\det E, L) \rightarrow \Hom_R(\det(J/J^2), L/JL) : f \mapsto \tilde{\beta}(f)
\]

where \( \tilde{\beta}(f)(\tilde{x}_1 \wedge \cdots \wedge \tilde{x}_d) := f(x_1 \wedge \cdots \wedge x_d) \). One checks that the composition

\[
\Hom_R(P^{d-1}, L) \rightarrow \Hom_R(P^d, L) \overset{\beta}{\rightarrow} \Hom_R(\det(J/J^2), L/JL)
\]

is zero. Thus, we get a map of \( R \)-modules

\[
H^d\Hom(P^\bullet, L) \rightarrow \Hom_R(\det(J/J^2), L/JL)
\]

which can be considered as a map of \( R/J \)-modules, and it is an isomorphism.

Now, we investigate involutions. Using notations in Lemma 2.5, we consider the maps of complexes of \( R/J \)-modules \( \sigma_R^{R/J}/R/J \) and \( \sigma_{\det(J/J^2), L/JL}^{R/J} \) on the \( R/J \)-modules \( \Hom_R(R/J, I^\bullet) \) and \( \Hom_R(\det(J/J^2), L/JL) \), and the maps of complexes of \( R \)-modules \( \sigma_R^{P,I} \) and \( \sigma_{P,L}^{R} \) on the \( R \)-modules \( \Hom_R(P^\bullet, I^\bullet) \) and \( \Hom_R(P^\bullet, L) \).

Note that \( \sigma_R^{R/J}/R/J \) and \( \sigma_{\det(J/J^2), L/JL}^{R/J} \) are involutions, but \( \sigma_R^{P,I} \) and \( \sigma_{P,L}^{R} \) are not. We form the following commutative diagrams

\[
\begin{array}{cccc}
\Hom_R(R/J, I^\bullet) & \rightarrow & \Tot(\Hom_R(P^\bullet, I^\bullet)) & \leftarrow \Hom_R(P^\bullet, L) & \rightarrow & \Hom_R(\det(J/J^2), L/JL)[-d] \\
\downarrow & & & & \downarrow \\
\Hom_R(R/J, I^\bullet)^{op} & \rightarrow & \Tot(\Hom_R(P^\bullet, I^\bullet))^{op} & \leftarrow \Hom_R(P^\bullet, L)^{op} & \rightarrow & \Hom_R(\det(J/J^2), L/JL)^{op}[-d]
\end{array}
\]
of $R$-modules. Since horizontal maps are all quasi isomorphisms and they induce maps of $R/J$-modules on homologies, we get the desired commutative diagram

$$H^d\text{Hom}_R(R/J, I^\bullet) \xrightarrow{\cong} \text{Hom}_R(\det(J/J^2), L/JL)$$

$$\downarrow \hspace{1cm} \downarrow$$

$$H^d\text{Hom}_R(R/J, I^\bullet)^{\text{op}} \xrightarrow{\cong} \text{Hom}_R(\det(J/J^2), L/JL)^{\text{op}}$$

of $R/J$-modules. □

7. SOME COMPUTATIONS

Let $S$ be a scheme with $\frac{1}{2}$ (not assumed to be regular). Let $\mathbb{P}^1_S$ be the projective line $\text{Proj}(\mathcal{O}_S[X, Y])$.

7.1. $\mathbb{P}^1$ WITH SWITCHING INVOLUTION. Let $\tau_{\mathbb{P}^1_S} : \mathbb{P}^1_S \to \mathbb{P}^1_S : [X : Y] \mapsto [Y : X]$ be the involution of switching coordinates of $\mathbb{P}^1_S$.

**Theorem 7.1.** Let $S$ be a regular scheme with $\frac{1}{2}$. Then, we have a stable equivalence of spectra

$$GW^{[i]}(\mathbb{P}^1_S, \tau_{\mathbb{P}^1_S}) \cong GW^{[i]}(S) \oplus GW^{[i+1]}(S)$$

In particular, we have the following isomorphism on Witt groups

$$W^{[i]}(\mathbb{P}^1_S, \tau_{\mathbb{P}^1_S}) \cong W^{[i]}(S) \oplus W^{[i+1]}(S)$$

**Lemma 7.2.** Let $\sigma_{\mathbb{A}^1_S}^{-} : \mathbb{A}^1_S \to \mathbb{A}^1_S$ be the involution given by $t \mapsto -t$. Then, the pullback

$$p^* : GW^{[i]}(S) \to GW^{[i]}(\mathbb{A}^1_S, \sigma_{\mathbb{A}^1_S}^{-})$$

is an isomorphism.

**Proof.** Let $\text{pt} : S \to \mathbb{A}^1_S$ be the rational point corresponding to 0. Then, there is a commutative diagram

$$\begin{array}{ccc}
S & \xrightarrow{\text{pt}} & \mathbb{A}^1_S \\
\downarrow & & \downarrow \\
S & \xrightarrow{\sigma_{\mathbb{A}^1_S}^{-}} & \mathbb{A}^1_S
\end{array}$$

which induces a commutative diagram

$$\begin{array}{ccc}
GW^{[i]}(S) & \xrightarrow{p^*} & GW^{[i]}(\mathbb{A}^1_S, \sigma_{\mathbb{A}^1_S}^{-}) \\
\downarrow & & \downarrow \\
GW^{[i]}(S) & \xrightarrow{\text{pt}^*} & GW^{[i]}(\mathbb{A}^1_S, \sigma_{\mathbb{A}^1_S}^{-})
\end{array}$$

This implies the pullback $p^* : GW^{[i]}(S) \to GW^{[i]}(\mathbb{A}^1_S, \sigma_{\mathbb{A}^1_S}^{-})$ is split injective. Now, consider the composition

$$GW^{[i]}(\mathbb{A}^1_S, \sigma_{\mathbb{A}^1_S}^{-}) \to GW^{[i]}(S) \xrightarrow{p^*} GW^{[i]}(\mathbb{A}^1_S, \sigma_{\mathbb{A}^1_S}^{-})$$

We want to show that it is an isomorphism. We use the following $C_2$-equivariant homotopy

$$\begin{array}{ccc}
\mathbb{A}^1 \times \mathbb{A}^1 & \to & \mathbb{A}^1 \\
\downarrow & & \downarrow \\
\mathbb{A}^1 \times \mathbb{A}^1 & \to & \mathbb{A}^1
\end{array}$$

This diagram tells us that

$$(\mathbb{A}^1_S, \sigma_{\mathbb{A}^1_S}^{-}) \xrightarrow{p} (S, \text{id}) \xrightarrow{\text{pt}} (\mathbb{A}^1_S, \sigma_{\mathbb{A}^1_S}^{-})$$

is homotopic to $\text{id} : (\mathbb{A}^1_S, \sigma_{\mathbb{A}^1_S}^{-}) \to (\mathbb{A}^1_S, \sigma_{\mathbb{A}^1_S}^{-})$. The result follows. □
Let $\mathbb{P}^0_{[1:1]} \to \mathbb{P}^1_S$ be the closed point cut out by the homogeneous ideal $(X - Y)$. Then, $\mathbb{P}^0_{[1:1]}$ is invariant under the involution $\tau_{\mathbb{P}^1_S}$ on $\mathbb{P}^1_S$.

**Lemma 7.3.** Let $S$ be a regular scheme with $\frac{1}{2}$. The dévissage theorem provides an isomorphism

$$D_{p^i/p^j} : GW^{[i]}(\mathbb{P}^1_S, \tau_{\mathbb{P}^1_S}) \xrightarrow{\cong} GW^{[i+1]}(S)$$

**Proof.** Let $I = (X - Y)$ be the homogeneous ideal defining $\mathbb{P}^0_{[1:1]}$ in $\mathbb{P}^1_S$. Then, the involution $\tau_{\mathbb{P}^1_S} : \mathbb{P}^1_S \to \mathbb{P}^1_S$ induces an involution on $I = (X - Y)$ which sends $X - Y$ to $Y - X$. By Theorem 6.1 we conclude that

$$GW^{[i]}(\mathbb{P}^1_S, \tau_{\mathbb{P}^1_S}) \xrightarrow{\cong} GW^{[i-1]}(S, -\text{can})$$

Also, $GW^{[i-1]}(S, -\text{can}) \cong GW^{[i+1]}(S)$. \hfill $\square$

**Proof of Theorem 7.2.** By [Sch17], we have the following homotopy fibration sequence

$$GW^{[i]}(\mathbb{P}^1_S, \tau_{\mathbb{P}^1_S}) \to GW^{[i]}(\mathbb{P}^1_S, \tau_{\mathbb{P}^1_S}) \to GW^{[i]}(\mathbb{P}^1_S - \mathbb{P}^0_{[1:1]}, \tau_{\mathbb{P}^1_S - \mathbb{P}^0_{[1:1]}})$$

Note that there is an invariant isomorphism

$$(A^1_S, \sigma^+_{A^1_S}) \to (\mathbb{P}^1_S - \mathbb{P}^0_{[1:1]}, \tau_{\mathbb{P}^1_S - \mathbb{P}^0_{[1:1]}})$$

by sending $t$ to $[t + \frac{1}{2} : t - \frac{1}{2}]$. Therefore, we get a homotopy fibration sequence

$$GW^{[i]}(\mathbb{P}^1_S, \tau_{\mathbb{P}^1_S}) \to GW^{[i]}(\mathbb{P}^1_S, \tau_{\mathbb{P}^1_S}) \to GW^{[i]}(A^1_S, \sigma^+_{A^1_S}) \xrightarrow{\cong} GW^{[i]}(S)$$

By Lemma 7.2 we have the isomorphism $p^* : GW^{[i]}(S) = GW^{[i]}(A^1_S, \sigma^+_{A^1_S})$ which provides a splitting for the above homotopy fibration sequence. Combining with Lemma 7.3 we conclude that

$$GW^{[i]}(\mathbb{P}^1_S, \tau_{\mathbb{P}^1_S}) \cong GW^{[i]}(\mathbb{P}^1_S, \tau_{\mathbb{P}^1_S}) \oplus GW^{[i]}(S) \cong GW^{[i+1]}(S) \oplus GW^{[i]}(S)$$

The result follows. \hfill $\square$

7.2. $C_2$-equivariant $A^1$-invariance. Consider the involution $\sigma_{\mathbb{P}^1_S} : \mathbb{P}^1_S \to \mathbb{P}^1_S$ given by the graded morphism $\mathcal{O}_S[X, Y] \to \mathcal{O}_S[X, Y]$ of graded sheaves of $\mathcal{O}_S$-algebras, such that $\mu \mapsto \mathcal{O}_S(\mu)$ if $\mu \in \mathcal{O}_S$, and $X \mapsto X, Y \mapsto Y$. There is an element

$$\beta := \begin{pmatrix} \mathcal{O}_{\mathbb{P}^1}(-1) & X \\ Y & Y \end{pmatrix} \in GW^{[1]}(\mathbb{P}^1_S, \sigma_{\mathbb{P}^1_S})$$

**Proposition 7.4.** The map

$$(q^* \circ \beta \cup q^*(-)) : GW^{[i]}(S, \sigma_S) \oplus GW^{[i-1]}(S, \sigma_S) \to GW^{[i]}(\mathbb{P}^1_S, \sigma_{\mathbb{P}^1_S})$$

of spectra is an equivalence.

**Proof.** The proof of [Sch17] Theorem 9.10] can be applied without modification. \hfill $\square$
Theorem 7.5. Let $S$ be a regular scheme with involution $\sigma_S$ and with $\frac{1}{2}$. Let $\sigma_{A_1^S} : A_1^S \rightarrow A_1^S$ be the involution on $A_1^S$ with the indeterminant fixed by $\sigma_{A_1^S}$ and such that the following diagram commutes

$$
\begin{array}{ccc}
A_1^S & \xrightarrow{\sigma_{A_1^S}} & A_1^S \\
p & & p \\
S & \xrightarrow{\sigma_S} & S
\end{array}
$$

Then, the pullback

$$p^* : GW[i](S, \sigma_S) \rightarrow GW[i](A_1^S, \sigma_{A_1^S})$$

is an isomorphism.

Proof. Let $pt : S \rightarrow \mathbb{P}_S^1$ be the rational point with $X = 0$ and $Y = 1$. It follows that there is a commutative diagram

$$
\begin{array}{ccc}
S & \xrightarrow{\sigma_S} & S \\
pt & & pti \\
\mathbb{P}_S^1 & \xrightarrow{\sigma_{\mathbb{P}_S^1}} & \mathbb{P}_S^1
\end{array}
$$

By Schlichting [Sch17], we have the following localization sequence

$$GW_{pt}^i(\mathbb{P}_S^1, \sigma_{\mathbb{P}_S^1}) \rightarrow GW^i(\mathbb{P}_S^1, \sigma_{\mathbb{P}_S^1}) \rightarrow GW^i(A_1^S, \sigma_{A_1^S})$$

We write out the following commutative diagram

$$
\begin{array}{ccc}
GW^i(S, \sigma_S) & \xrightarrow{D_{pt/\mathbb{P}_S^1}} & GW^i(S, \sigma_S) \oplus GW^i(S, \sigma_S) \\
(0, 1) & & (1, 0) \\
GW^i(\mathbb{P}_S^1, \sigma_{\mathbb{P}_S^1}) & \xrightarrow{(q^* \cup q^*(-))} & GW^i(\mathbb{P}_S^1, \sigma_{\mathbb{P}_S^1}) \\
p^* & & p^*
\end{array}
$$

(7.1)

The main reason for the commutativity of the left square is that we have the locally free resolution

$$0 \rightarrow O_{\mathbb{P}_S^1}(-1) \xrightarrow{X} O_{\mathbb{P}_S^1} \rightarrow O_{pt} \rightarrow 0$$

By dévissage, we conclude $D_{pt/\mathbb{P}_S^1}$ is an equivalence. By Proposition 7.4, we know that the middle map $(q^* \cup q^*(-))$ in Diagram (7.1) is an equivalence. It follows that the right arrow in Diagram (7.1)

$$p^* : GW[i](S, \sigma_S) \rightarrow GW[i](A_1^S, \sigma_{A_1^S})$$

is an equivalence. \qed

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