HARMONIC APPROXIMATION BY FINITE SUMS OF MODULI

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ABSTRACT. Let $h(B_d)$ denote the space of real-valued harmonic functions on the unit ball $B_d$ of $\mathbb{R}^d$, $d \geq 2$. Given a radial weight $w$ on $B_d$, consider the following problem: construct a finite family $\{f_1, f_2, \ldots, f_J\}$ in $h(B_d)$ such that the sum $|f_1| + |f_2| + \cdots + |f_J|$ is equivalent to $w$. We solve the problem for weights $w$ with a doubling property. Moreover, if $d$ is even, then we characterize those $w$ for which the problem has a solution.

1. Introduction

1.1. Weight functions and radial weights. By definition, $w : [0,1) \to (0, +\infty)$ is a weight function if $w$ is a non-decreasing, continuous, unbounded function. Let $B$ denote the unit ball of a real or complex Euclidean space. We extend $w$ to a radial weight on $B$ setting $w(z) = w(|z|)$, $z \in B$.

For functions $u, v : B \to (0, +\infty)$, we write $u \asymp v$ and we say that $u$ and $v$ are equivalent if

$$C_1 u(z) \leq v(z) \leq C_2 u(z), \quad z \in B,$$

for some constants $C_1, C_2 > 0$. The definition of equivalent functions on $[0,1)$ is analogous.

1.2. Main result. Let $h(B_d)$ denote the space of real-valued harmonic functions on the unit ball $B_d$ of $\mathbb{R}^d$, $d \geq 2$. Harmonic approximation mentioned in the title of the present paper refers to the following notion:

Definition 1.1. A weight function $w : [0,1) \to (0, +\infty)$ is called harmonically approximable if there exists a finite family $\{f_1, f_2, \ldots, f_J\} \subset h(B_d)$ such that

$$|f_1| + |f_2| + \cdots + |f_J| \asymp w,$$

where $w$ is extended to a radial weight on $B_d$. Clearly, property (1.1) does not change if $w$ is replaced by an equivalent weight function.

To study harmonic approximation, we use doubling and log-convex weight functions. A weight function $w$ is called doubling if there exists a constant $A > 1$ such that

$$w(1-s/2) \leq A w(1-s), \quad 0 < s \leq 1.$$

A weight function $w$ is called log-convex if $\log w(r)$ is a convex function of $\log r$, $0 < r < 1$. The doubling condition (1.2) restricts the growth of $w$; a log-convex weight function may grow arbitrarily rapidly. Also, it is known that every doubling weight function is equivalent to a log-convex one.

Theorem 1.1. Let $d \geq 2$ and let $w$ be a weight function on $[0,1)$.

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(i) If \( w \) is harmonically approximable, then \( w \) is equivalent to a log-convex weight function.

(ii) If \( d \) is even, then \( w \) is harmonically approximable if and only if \( w \) is equivalent to a log-convex weight function.

(iii) If \( d \) is odd and \( w \) is doubling, then \( w \) is harmonically approximable.

1.3. Comments.

1.3.1. Holomorphically approximable weights. Let \( H(\mathbb{B}_m) \) denote the space of holomorphic functions on the unit ball \( \mathbb{B}_m \) of \( \mathbb{C}^m \), \( m \geq 1 \).

Definition 1.2. A weight function \( w \) is called holomorphically approximable if there exists a finite family \( \{f_1, f_2, \ldots, f_J\} \subset H(\mathbb{B}_m) \) such that property (1.1) holds in \( \mathbb{B}_m \).

A characterization of holomorphically approximable weight functions is given by the following result.

Theorem 1.2 ([2, Theorems 1.2 and 1.3]). Let \( m \geq 1 \). A weight function \( w \) is holomorphically approximable if and only if \( w \) is equivalent to a log-convex weight function.

In particular, Theorem 1.1(ii) follows from Theorem 1.1(i) and Theorem 1.2.

Indeed, given a log-convex weight function extended to a radial weight on \( B_{2m} \) and \( \mathbb{B}_m \), it suffices to identify \( B_{2m} \) and \( \mathbb{B}_m \), and to take the real and imaginary parts of holomorphic functions provided by Theorem 1.2.

To the best of the author’s knowledge, Theorem 1.1(iii) is new for any weight function \( w \).

1.3.2. Optimal \( J \) in (1.1). We always have \( J \geq 2 \) in (1.1) by the mean value property for harmonic functions. The corresponding optimal number \( J = J(d) \) is probably of independent interest.

1.3.3. \( L^2 \)-means. For \( f \in h(B_d) \), put

\[
M_2^2(f, r) = \int_{\partial B_d} |f(ry)|^2 \, d\sigma_d(y), \quad 0 \leq r < 1,
\]

where \( \sigma_d \) is the normalized Lebesgue measure on the sphere \( \partial B_d \). The proof of Theorem 1.1(i) uses the following modification of property (1.1) in terms of \( L^2 \)-means:

\[
\text{there exists } J \in \mathbb{N} \text{ and } f_1, \ldots, f_J \in h(B_d) \text{ such that}
M_2^2(f_1, r) + \cdots + M_2^2(f_J, r) \preceq w^2(r), \quad 0 \leq r < 1.
\]

If the above property holds, then \( w \) is called harmonically \( L^2 \)-approximable. We use an analogous definition for \( H(\mathbb{B}_m) \).

As mentioned above, we have \( J \geq 2 \) in (1.1) for any weight function \( w \). However, if (1.1) is replaced by (1.3), then we need just one function \( f \in h(B_d) \); see Proposition 1.3 below. The situation is similar for \( H(\mathbb{B}_m) \): one has \( J \geq 2 \) in the holomorphic analog of (1.1) by the maximum principle, but the approximation problem for \( L^2 \)-means is solvable by one function \( f \in H(\mathbb{B}_m) \).
Proposition 1.3. Let $w$ be a weight function on $[0,1)$. Then the following properties are equivalent:

(1.4) $w$ is equivalent to a log-convex function;

(1.5) $w$ is harmonically $L^2$-approximable;

(1.6) there exists $f \in h(B_d)$ such that $M_2(f, r) \asymp w(r), \quad 0 \leq r < 1$;

(1.7) $w$ is holomorphically $L^2$-approximable;

(1.8) there exists $f \in H(\mathbb{B}_m)$ such that $M_2(f, r) \asymp w(r), \quad 0 \leq r < 1$.

1.3.4. Growth spaces. One may consider property (1.1) as a reverse estimate in the growth space $h_w(B_d)$, $d \geq 2$. By definition, $h_w(B_d)$ consists of those $f \in h(B_d)$ for which

\begin{equation}
|f(x)| \leq Cw(x), \quad x \in B_d.
\end{equation}

Property (1.1) guarantees that estimate (1.9) is, in a sense, reversible for a finite family of test functions in $h_w(B_d)$. Such test functions are known to be useful in the studies of various concrete operators on the growth spaces (see, for example, [1, 6] and references therein).

1.4. Organization of the paper. The results related to the log-convexity are collected in Section 2: we prove Theorem 1.1(i) and Proposition 1.3. Theorem 1.1(iii), the main technical result of the present paper, is obtained in Section 3.

2. Log-convex weight functions

2.1. Equivalence to a log-convex weight function is necessary. The following lemma is standard (see, for example, [5]).

Lemma 2.1. Let $w$ be a weight function. If $w$ is harmonically $L^2$-approximable, then $w$ is equivalent to a log-convex weight function.

Proof. Assume that

\begin{equation}
w^2(r) \asymp M_2^2(f_1, r) + \cdots + M_2^2(f_J, r), \quad 0 \leq r < 1,
\end{equation}

for a family $\{f_1, \ldots, f_J\} \subset h(B_d)$. Put $M_2^2(r) = M_2^2(f_1, r) + \cdots + M_2^2(f_J, r)$. We claim that $M_2(r)$ is log-convex. Indeed, for $j = 1, \ldots, J$, we have

\begin{equation}
f_j(x) = \sum_{k=0}^{\infty} P_{j,k}(x), \quad x \in B_d,
\end{equation}

where $P_{j,k}$ is a harmonic homogeneous polynomial of degree $k$, and the series converges uniformly on compact subsets of $B_d$. For $k_1 \neq k_2$, $P_{k_1}$ and $P_{k_2}$ are orthogonal in $L^2(\partial B_d)$, hence,

\begin{equation}
M_2^2(f_j, r) = \sum_{k=0}^{\infty} \|P_{j,k}\|^2_{L^2(\partial B_d)} r^{2k}, \quad 0 \leq r < 1.
\end{equation}

So, $M_2^2(r) = \sum_{k=0}^{\infty} a_k^2 r^{2k}$ for certain $a_k \in \mathbb{R}$. Therefore, Hadamard’s three circles theorem or direct computations guarantee that $M_2^2(r)$ and $M_2(r)$ are log-convex, as required. \hfill \Box

Applying Lemma 2.1, we obtain Theorem 1.1(i) and several implications in Proposition 1.3.
Proof of Theorem 1.1. We are given a family \(\{f_1, \ldots, f_J\} \subset h(B_d)\) such that
\[
w(x) \asymp |f_1(x)| + \cdots + |f_J(x)|, \quad x \in B_d,
\]
or, equivalently,
\[
w^2(r) \asymp |f_1(ry)|^2 + \cdots + |f_J(ry)|^2, \quad 0 \leq r < 1, \ y \in \partial B_d.
\]
Integrating over the sphere \(\partial B_d\) with respect to Lebesgue measure \(\sigma_d\), we obtain
\[
w^2(r) \asymp M_2^2(f_1, r) + \cdots + M_2^2(f_J, r), \quad 0 \leq r < 1.
\]
So, by Lemma 2.1, \(w\) is equivalent to a log-convex weight function. \(\square\)

Clearly, the above argument also guarantees that every holomorphically approximable weight function is equivalent to a log-convex one; see [2] for a different proof.

Proof of Proposition 1.3. The implications \((1.6) \Rightarrow (1.4)\) and \((1.8) \Rightarrow (1.7)\) are trivial. By Lemma 2.1, \((1.5)\) implies \((1.4)\), and \((1.7)\) implies \((1.4)\). So, to finish the proof of Proposition 1.3, it suffices to show that \((1.4)\) implies \((1.6)\) and \((1.8)\). \(\square\)

2.2. Approximation by integral means. In this section, we show that \((1.4)\) implies \((1.6)\). The proof of the implication \((1.4) \Rightarrow (1.8)\) is analogous.

Lemma 2.2. Let \(w\) be a log-convex weight function on \([0, 1)\). Then there exists a sequence \(\{a_k\}_{k=0}^{\infty} \subset \mathbb{R}\) such that
\[
\sum_{k=0}^{\infty} a_k^2 r^{2k} \asymp w^2(r), \quad 0 \leq r < 1,
\]
where the series converges uniformly on compact subsets of \([0, 1)\).

Proof. By [2] Theorem 1.2], there exist \(f_1, f_2 \in H(B_1)\) such that \(|f_1(z)| + |f_2(z)| \asymp w(|z|), z \in B_1\), hence,
\[
|f_1(r\zeta)|^2 + |f_2(r\zeta)|^2 \asymp w^2(r), \quad 0 \leq r < 1, \ z \in \partial B_1.
\]
Integrating the above equivalence with respect to Lebesgue measure on the unit circle \(\partial B_1\), we obtain \((2.1)\) with \(a_k^2 = |\hat{f}_1(k)|^2 + |\hat{f}_2(k)|^2\). \(\square\)

Proof of Proposition 1.3. Let \(\mathcal{H}_k = \mathcal{H}_k(d)\) denote the space of harmonic homogeneous polynomials of degree \(k\) in \(d\) real variables. The same symbol is used for the restriction of \(\mathcal{H}_k\) to the sphere \(\partial B_d\). Let \(Z_k(\cdot, \cdot)\) denote the reproducing kernel for \(\mathcal{H}_k \subset L^2(\partial B_d)\). Fix a point \(x \in \partial B_d\). So, we have \(Z_k(\cdot) = Z_k(x, \cdot) \in \mathcal{H}_k\). Set \(Y_k = (\dim \mathcal{H}_k)^{-\frac{1}{2}} Z_k\). Then
\[
\begin{align*}
\|Y_k\|_{L^\infty(\partial B_d)} &= \sqrt{\dim \mathcal{H}_k}; \\
\|Y_k\|_{L^2(\partial B_d)} &= 1.
\end{align*}
\]
See, for example, [3] for the above properties of zonal harmonics \(Z_k\).

Given a log-convex weight function \(w\), let the sequence \(\{a_k\}_{k=0}^{\infty}\) be that provided by Lemma 2.2. Put
\[
f(x) = \sum_{k=0}^{\infty} a_k Y_k(x), \quad x \in B_d.
\]
Property (2.2) and explicit formulas for \( \dim \mathcal{H}_k \) guarantee that the above series converges uniformly on compact subsets of \( B_d \). Hence, \( f \in h(B_d) \). Since (2.4) is an orthonormal series, we have

\[
M^2_2(f, r) = \sum_{k=0}^{\infty} a_k^2 r^{2k} \approx w^2(r), \quad 0 \leq r < 1,
\]

by Lemma 2.2. So, the proof of the proposition is finished. \( \square \)

3. Doubling weight functions

In the present section, we prove Theorem 1.1(iii). Both in [1] and [2], the required holomorphic functions in the complex ball \( B_m, m \geq 2 \), are constructed as appropriate lacunary series of Aleksandrov–Ryll–Wojtaszczyk polynomials. As far as the author is concerned, the existence of analogous homogeneous harmonic polynomials in \( \mathbb{R}^{2n+1}, n = 1, 2, \ldots \), remains an open problem. So, in the proof of Theorem 1.1(iii), we use series of special harmonic functions that are not polynomials.

3.1. Building blocks.

Lemma 3.1 (see [4, Lemma 3]). Let \( d \geq 2 \) and let \( p \in \mathbb{N} \). Then there exist constants \( \alpha = \alpha(d) \in \mathbb{N}, Q = Q(d) \in \mathbb{N} \) and \( C = C(p, d) > 0 \) with the following property: for every \( n \in \mathbb{Z}_+ \), there exist functions \( u_{q,n} \in h(B_d) \), \( q = 1, \ldots, Q \), such that

1. \( |u_{q,n}(x)| \leq 1, \quad x \in B_d; \)
2. \( \max_{1 \leq q \leq Q} |u_{q,n}(x)| \geq \frac{1}{4}, \quad 0 < 1 - |x| < 2^{-\alpha-n}; \)
3. \( |u_{q,n}(x)| \leq C(p, d)2^{-np}(1 - |x|)^{-p}, \quad x \in B_d. \)

Proof. Let \( n \in \mathbb{Z}_+ \). By [4, Lemma 3], there exists a function \( v_1 \in h(B_d) \) with properties (3.1), (3.3), and such that

\[
|v_1(ry)| \geq \frac{1}{4}, \quad 0 < 1 - r < 2^{-\alpha-n} \quad \text{and} \quad y \in E_n,
\]

where

\[
E_n = \left\{ y \in \partial B_d : y = (t \cos \varphi, t \sin \varphi, y_3, \ldots, y_d), \quad t \geq \frac{3}{4}, \varphi \in \Phi_n \right\},
\]

\[
\Phi_n = \bigcup_{j=0}^{2^n-1} \left\{ \left( 2j + \frac{1}{4} \right) 2^{-n}, \left( 2j + \frac{3}{4} \right) 2^{-n} \right\}.
\]

Rotating the set \( E_n \), we obtain \( v_1, v_2, v_3, v_4 \in h(B_d) \) such that

\[
|v_1(ry)| + \cdots + |v_4(ry)| \geq \frac{1}{4}, \quad 0 < 1 - r < 2^{-\alpha-n} \quad \text{and} \quad y \in E_0,
\]

where \( E_0 = \{ y \in \partial B_d : y^2_1 + y^2_2 \geq \frac{9}{16} \} \). Now, observe that there exists a finite family of rotations \( \{ T_k \}_{k=1}^{K} \) such that

\[
\bigcup_{k=1}^{K} T_k E_0 = \partial B_d.
\]
So, the family $v_m T_k^{-1} \in h(B_d)$, $m = 1, 2, 3, 4, k = 1, \ldots, K$, has the required properties for the number $n \in \mathbb{Z}_+$ under consideration. □

3.2. Construction. We are given a doubling weight function $w : [0, 1) \to (0, +\infty)$. Without loss of generality, assume that $w(0) = 1$. We use the auxiliary function
$$\Phi(x) = w \left( 1 - \frac{1}{x} \right), \quad x \geq 1.$$ 
So, we have $\Phi(1) = 1$ and $w(t) = \Phi \left( \frac{1}{1+t} \right), 0 \leq t < 1$. The doubling condition (1.2) rewrites as
(3.4) $\Phi(2x) \leq A \Phi(x), \quad x \geq 1.$
Without loss of generality, we assume that $A \geq 2$.
For $k = 0, 1, \ldots$, set
(3.5) $n_k = \max \{ j \in \mathbb{Z}_+ : \Phi(2^j) \leq A^k \}.$
Since $\Phi(1) = 1$, $n_0$ is correctly defined. Also, we have $\Phi(2^{n_k}) \leq A\Phi(2^{n_k}) \leq A^{k+1}$ by (3.4) and (3.5); hence, $n_{k+1} > n_k$ and $n_{k+\ell} - n_k \geq \ell$ for $\ell \in \mathbb{N}$. In what follows, we often use these properties without explicit reference.

Let the functions $u_{q,n}$ be those provided by Lemma 3.1. For $J \in \mathbb{N}$, consider the series
$$F_{q,j}(x) = \sum_{k=0}^{\infty} A^{k+j} u_{q,n_{j+k}}, \quad x \in B_d, \quad q = 1, 2, \ldots, Q, \quad j = 0, 1, \ldots, J - 1.$$ 
Estimates obtained below guarantee that the above series uniformly converges on compact subsets of $B_d$; in particular, $F_{q,j}$ is a harmonic function. The exact value of the constant $J$ will be selected later.

3.3. Basic estimates. Observe that it suffices to obtain the two-sided estimate
(3.6) $\sum_{q=1}^{Q} \sum_{j=0}^{J-1} |F_{q,j}(x)| \preceq \Phi \left( \frac{1}{1 - |x|} \right)$
for $2^{-\alpha-n_{j+m+1}} \leq 1 - |x| \leq 2^{-\alpha-n_{j+m}}, j = 0, 1, \ldots, J - 1, m = 0, 1, \ldots$.
Indeed, (3.6) guarantees that
$$1 + \sum_{q=1}^{Q} \sum_{j=0}^{J-1} |F_{q,j}(x)| \preceq \Phi \left( \frac{1}{1 - |x|} \right)$$
for $1 > |x| \geq 1 - 2^{-\alpha-n_0}$, hence, for all $x \in B_d$.

3.4. Lower estimate in (3.6). In fact, we are going to prove the following somewhat stronger property:
(3.7) $\sum_{q=1}^{Q} |F_{j,q}(x)| \geq C \Phi \left( \frac{1}{1 - |x|} \right), \quad j = 0, 1, \ldots, J - 1,$
for a universal constant $C > 0$ for $x \in B_d$ such that $2^{-\alpha-n_{j+m+1}} \leq 1 - |x| \leq 2^{-\alpha-n_{j+m}}, m = 0, 1, \ldots$. So, fix an $m \in \{0, 1, \ldots\}$ and assume, without loss of generality, that $j = 0$. Now, consider a point $x$. 
Since \(1 - |x| \leq 2^{-\alpha - n J_m}\), property (3.2) guarantees that \(|u_{q,n J_m}(x)| \geq \frac{1}{4}\) for some \(q = q(x) \in \{1, 2, \ldots, Q\}\). Fix such a \(q\) and consider the series
\[
F_q(x) = F_{q,0}(x) = \sum_{k=0}^{\infty} A^{J_k} u_{q,n J_k}(x).
\]

For a sufficiently large \(J\), we will show that
\[
|F_q(x)| \geq C \Phi \left( \frac{1}{1 - |x|} \right),
\]
where \(C > 0\) is a universal constant.

Represent the series \(F_q(x)\) as the sum of the following three functions:

\[
(3.8) \quad \sum_{k=0}^{m-1} A^{J_k} u_{q,n J_m}(x) + \sum_{k=m+1}^{\infty} A^{J_k} u_{q,n J_m}(x) := f_1(x) + f_2(x) + f_3(x).
\]

First, by (3.1),
\[
|f_1(x)| \leq \sum_{k=0}^{m-1} A^{J_k} \leq A^{J(m-1)+1}.
\]

Second, by the definition of \(q\),
\[
|f_2(x)| \geq \frac{A^{J_m}}{4}.
\]

Finally, given \(p \in \mathbb{N}\), property (3.3) guarantees that
\[
|f_3(x)| \leq C(p, d) \sum_{k=m+1}^{\infty} A^{J_k} 2^{-p n J_k} (1 - |x|)^{-p} 
\leq C(p, d) \sum_{k=m+1}^{\infty} A^{J_k} 2^{-p(n J_k - n J_m + 1)} 2^{p(\alpha + n J_m + 1)}.
\]

Now, fix \(p \in \mathbb{N}\) such that \(A < 2 \cdot 2^p\). In particular, the constants \(C(p, d)\) and \(2^p\) are also fixed. We claim that the series under consideration converges. Indeed, we have \(n J_k - n J_m + 1 \geq J(k - m) - 1\), thus,
\[
|f_3(x)| \leq A^{J_m} C(p, d) 2^{p(\alpha+1)} \sum_{k=m+1}^{\infty} A^{J(k-m)} 2^{-p(n J_k - n J_m + 1)}
\leq A^{J_m} C(p, d) 2^{p(\alpha+1)} \sum_{s=1}^{\infty} \left( \frac{A}{2^p} \right)^J_s
\leq A^{J_m} C(p, d) 2^{p(\alpha+1)} \sum_{s=1}^{\infty} \left( \frac{1}{2} \right)^J_s.
\]

Since \(p\) is fixed, we may select so large \(J\) that
\[
C(p, d) 2^{p(\alpha+1)} \sum_{s=1}^{\infty} \left( \frac{1}{2} \right)^J_s < \frac{1}{16}.
\]
Therefore, we have
\[
|f_3(x)| \leq \frac{A^{J_m}}{16}.
\]
In sum, we obtain

\begin{equation}
|F_q(x)| \geq \frac{A^{J_m}}{4} - A^{J(m-1)+1} - \frac{A^{J_m}}{16} \geq \frac{A^{J_m}}{8}
\end{equation}

for a sufficiently large \( J \). Fix such a \( J \in \mathbb{N} \). Note that the choice of appropriate \( J \) does not depend on \( m \).

Now, observe that \( \Phi \) is an increasing function, hence, by (3.4) and (3.5),

\begin{equation}
\Phi \left( \frac{1}{1 - |x|} \right) \leq \Phi \left( 2^{\alpha+n_{J_{m+1}}} \right) \leq A^\alpha \Phi \left( 2^n \right) \leq A^{\alpha+1+J_m}
\end{equation}

for \( 2^{-\alpha-n_{J_{m+1}}} < 1 - |x| \).

Finally, by (3.9) and (3.10), we obtain

\[
\sum_{q=1}^{Q} \sum_{j=0}^{J_{m+1}} |F_{q,j}(x)| \geq |F_q(x)| \geq \frac{A^{J_m}}{8} \geq \frac{A^{\alpha-1}}{8} \Phi \left( \frac{1}{1 - |x|} \right)
\]

for \( 2^{-\alpha-n_{J_{m+1}}} < 1 - |x| \) \( \leq 2^{-\alpha-n_{J_m}} \). In other words, the required lower estimate (3.7) holds for \( j = 0 \) and \( m = 0, 1, \ldots \). The parameter \( J \) is fixed, so, the above argument remains the same for arbitrary \( j \in \{0, 1, \ldots, J\} \). Therefore, the proof of the lower estimate in (3.6) is finished.

3.5. **Upper estimate in (3.6)**. We assume that the constants \( J \) and \( p \) are fixed according to the restrictions of section 3.4 in particular, \( A < 2 \cdot 2^p \).

Given \( m \in \{0, 1, \ldots\} \) and \( j \in \{0, 1, \ldots, J-1\} \), suppose that

\[
2^{-\alpha-n_{J_{m+j+1}}} \leq 1 - |x| \leq 2^{-\alpha-n_{J_{m+j}}}
\]

Using (3.1), (3.3) and the inequality \( n_k - n_{J_{m+j+1}} \geq k - Jm - j - 1 \), we obtain

\[
\sum_{q=1}^{Q} \sum_{j=0}^{J_{m+1}} |F_{q,j}(x)|
\]

\[
\leq Q \left( \sum_{k=0}^{J_{m+1}} A^k + C(p, d) \sum_{k=J_{m+1}}^{\infty} A^k 2^{-p(n-k-1)}(1 - |x|)^{-p} \right)
\]

\begin{equation}
\leq Q \left( A^{J_{m+1}} + C(p, d)2^{p\alpha} \sum_{k=J_{m+1}}^{\infty} A^k 2^{-p(n_k-n_{J_{m+j+1}})} \right)
\end{equation}

\[
\leq Q \left( A^{J_{m+1}} + C(p, d)2^{p\alpha} A^{J_{m+j+1}} \sum_{s=0}^{\infty} \left( \frac{1}{2} \right)^s \right)
\]

\[
\leq C(p, d, Q(d), \alpha(d)) A^{J_{m+j+1}}.
\]

Also, for \( 1 - |x| \leq 2^{-\alpha-n_{J_{m+j}}} \), we have

\begin{equation}
\Phi \left( \frac{1}{1 - |x|} \right) \geq \Phi \left( 2^{\alpha+n_{J_{m+j}}} \right) \geq \Phi \left( 2^{J_{m+j}} \right) \geq A^{J_{m+j}}
\end{equation}

by the definition of \( n_{J_{m+j}} \).

By (3.11) and (3.12),

\[
\sum_{q=1}^{Q} \sum_{j=0}^{J_{m+1}} |F_{q,j}(x)| \leq C(p, d, A) \Phi \left( \frac{1}{1 - |x|} \right)
\]
for all $x \in B_d$ under consideration. In other words, the upper estimate in (3.6) holds. The proof of Theorem 1.1(iii) is finished.

**References**

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