Finite-time Identification of Stable Linear Systems
Optimality of the Least-Squares Estimator

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Abstract—We present a new finite-time analysis of the estimation error of the Ordinary Least Squares (OLS) estimator for stable linear time-invariant systems. We characterize the number of observed samples (the length of the observed trajectory) sufficient for the OLS estimator to be \((\varepsilon, \delta)\)-PAC, i.e., to yield an estimation error less than \(\varepsilon\) with probability at least \(1 - \delta\). We show that this number matches existing sample complexity lower bounds [1], [2] up to universal multiplicative factors (independent of \((\varepsilon, \delta)\) and of the system). This paper hence establishes the optimality of the OLS estimator for stable systems, a result conjectured in [1]. Our analysis of the performance of the OLS estimator is simpler, sharper, and easier to interpret than existing analyses. It relies on new concentration results for the covariates matrix.

I. INTRODUCTION

We investigate the canonical problem of identifying Linear Time Invariant (LTI) systems of the form:

\[
  x_0 = 0 \quad \text{and} \quad \forall t \geq 0, \quad x_{t+1} = A x_t + \eta_{t+1}, \quad (1)
\]

where \(x_t \in \mathbb{R}^d\) denotes the state at time \(t\), \(A \in \mathbb{R}^{d \times d}\) is initially unknown but stable (i.e., its spectral radius satisfies \(\rho(A) < 1\)), and \((\eta_t)_{t \geq 1}\) are i.i.d. sub-gaussian zero-mean noise vectors with covariance matrix \(I_d\) (for simplicity). The objective is to estimate the matrix \(A\) from an observed trajectory of the system. Most work on this topic is concerned with the convergence properties of some specific estimation methods (e.g., ordinary least squares (OLS), ML estimator) [3], [4], [5]. Recently however, there have been intense research efforts towards understanding the finite-time behavior of classical estimation methods [6], [7], [1], [8], [9], [10]. The results therein aim at deriving bounds on the sample complexity of existing estimation methods (mostly the OLS), namely at finding upper bounds on the number of observations \(T\) sufficient to identify \(A\) with prescribed levels \((\varepsilon, \delta)\) of accuracy and confidence\(^1\):

\[
  \mathbb{P}(|A_t - A| \leq \varepsilon) \geq 1 - \delta \quad \text{for any } t \geq \tau \quad \text{if } A_t \text{ denotes the estimator of } A \text{ after } t \text{ observations.}
\]

Lower bounds of the sample complexity (valid for any estimator) have been also derived in [1], [2]. Sample complexity upper bounds appearing in the aforementioned papers are often hard to interpret and to compare. The main difficulty behind deriving such bounds stems from the fact that the data used to estimate \(A\) is correlated (we observe a single trajectory of the system). In turn, existing analyses rely on involved concentration results for random matrices [11], [12] and self-normalized processes [13]. The most technical and often tedious part of these analyses deals with deriving concentration results for the covariates matrix \(X\) defined as \(X^\top = (x_1, \ldots, x_t)\), and the tightness of these results directly impacts that of the sample complexity upper bound (refer to §II for more details).

In this paper, we present a novel analysis of the error \(|A_t - A|\) of the OLS estimator. In this analysis, we derive tight concentration results for the entire spectrum of the covariates matrix \(X\). To this aim, we first show that the spectrum of \(X\) can be controlled when \(\|X M^\top X M - I_d\|\) is upper bounded, where \(M = \left(\sum_{s=0}^{t-1} \Gamma_s(A)\right)^{-1/2}\) and \(\Gamma_s(A) = \sum_{k=0}^{s} A^k(A^k)^\top\) is the finite-time controllability gramian of the system. We then derive a concentration inequality for \(\|X M^\top X M - I_d\|\) by (i) expressing this quantity as the supremum of a chaos process [14]; and (ii) applying Hanson-Wright inequality [15] and an \(\varepsilon\)-net argument to upper bound this supremum.

Our main result is simple and easy to interpret. For any \(\varepsilon > 0\) and \(\delta \in (0, 1)\), we establish that the OLS estimator is \((\varepsilon, \delta)\)-PAC after \(t\) observations, i.e.,

\[
  \mathbb{P}(|A_t - A| \leq \varepsilon) \geq 1 - \delta, \quad \text{provided that:}
\]

\[
  \lambda_{\min} \left(\sum_{s=0}^{t-1} \Gamma_s(A)\right) \geq c \max \left\{ \frac{1}{\varepsilon^2}, J(A)^2 \right\} \left(\log(1/\delta) + d\right), \tag{2}
\]

where \(\lambda_{\min}(W)\) denotes the smallest eigenvalue of \(W\), \(c\) is a universal constant, and 
\(J(A) = \sum_{s>0} \|A^s\|\) depends on \(A\) only and is finite when \(\rho(A) < 1\) (\(J(A) \leq 1/(1 - \|A\|)\) if \(\|A\| < 1\)).

In [1], the authors have shown that an estimator can be \((\varepsilon, \delta)\)-PAC after \(t\) observations uniformly over the set \(\mathcal{L} = \{A : \exists O \in \mathbb{R}^{d \times d}, A = \rho O \text{ and } O^\top O = I_d\}\) of scaled-orthogonal matrices for some fixed \(\rho > 0\) only if the following necessary condition holds: for \(A \in \mathcal{L}\),

\[
  \lambda_{\min} \left(\sum_{s=0}^{t-1} \Gamma_s(A)\right) \geq c \frac{1}{\varepsilon^2} \left(\log(1/\delta) + d\right), \tag{3}
\]

for some universal constant \(c > 0\). Hence our sufficient condition (2) cannot be improved when the accuracy level \(\varepsilon\) is sufficiently low, i.e., when \(\varepsilon = O(J(A)^{-1})\). This result was actually conjectured in [1]. More recently in [2], we also proved that for any arbitrary \(A\), a necessary condition for the existence of a \((\varepsilon, \delta)\)-PAC estimator after \(t\) observations.
is $\lambda_{\min}(\sum_{s=0}^{t-1} \Gamma_s(A)) \geq c_1 \varepsilon^2 \log(1/\delta)$, but we believe that this sample complexity lower bound can be improved to (3) (for arbitrary matrix $A$, not only for orthogonal matrices). Anyway, when $(\varepsilon, \delta)$ approach 0, and more precisely when $\varepsilon = O(\mathcal{J}(A)^{-1})$ and $\delta = O(\varepsilon^{-d})$, the necessary condition derived in [2] and the sufficient condition (2) are identical. They would be also identical for any $\delta > 0$, if we manage to show that the sample complexity lower bound (3) holds for any matrix.

Another way of presenting our results is to consider the rate at which the estimation error decays with the number of observations $t$. We obtain: with probability at least $1 - \delta$,

$$\|A_t - A\| \leq c \sqrt{\frac{\log(1/\delta) + d}{\lambda_{\min}(\sum_{s=0}^{t-1} \Gamma_s(A))}}. \tag{4}$$

One can readily check that $\lambda_{\min}(\sum_{s=0}^{t-1} \Gamma_s(A)) \geq t$. Hence, $\|A_t - A\|$ decays as $\sqrt{\log(1/\delta)}$, which is the best possible decay rate. In §II, we compare our result to those of existing analyses of the OLS estimator. Our result is tighter than state-of-the-art results, and it is derived using much simpler arguments.

**Notations.** Throughout the paper, $\|W\|$ denotes the operator norm of the matrix $W$, and the Euclidian norm of a vector $w$ is denoted either by $\|w\|$ or $\|w\|_2$. The unit sphere in $\mathbb{R}^d$ is denoted by $S^{d-1}$. The singular values of any real matrix $W \in \mathbb{R}^{m \times d}$ with $m \geq d$ are denoted by $s_k(W)$ for $k = 1, \ldots, d$, arranged in a non-increasing order. For any square matrix $W$, $W^\dagger$ represents its Moore-Penrose pseudoinverse. The vectorization operation vec(·) acts on matrices by stacking its columns into one long column vector. Next, we recall the definition of sub-gaussian random variables and vectors. The $\psi_2$-norm of a r.v. $X \in \mathbb{R}$ is defined as $\|X\|_{\psi_2} = \inf\{K > 0 : \mathbb{E}[\exp(X^2/K^2)] \leq 2\}$. $X$ is said sub-gaussian if $\|X\|_{\psi_2} < \infty$. Now a random vector $X \in \mathbb{R}^d$ is sub-gaussian if its one-dimensional marginals $X^\top x$ are sub-gaussian r.v.’s for all $x \in \mathbb{R}^d$. Its $\psi_2$-norm is then defined as $\|X\|_{\psi_2} = \sup_{x \in S^{d-1}} \|X^\top x\|_{\psi_2}$. Note that a Gaussian vector with covariance matrix equal to the identity is sub-gaussian, and its $\psi_2$-norm is 1. A vector $X \in \mathbb{R}^d$ is isotropic if $\mathbb{E}[(X^\top x)^2] = \|x\|^2$, for all $x \in \mathbb{R}^d$ or equivalently if $\mathbb{E}XX^\top = I_d$.

Finally, we introduce notations specifically related to our linear system. Let $X$ and $E$ be the matrices defined by $X^\top = (x_1, \ldots, x_t)$ and $E^\top = (\eta_2, \ldots, \eta_{t+1})$. We further define the noise vector $\xi$ by $\xi^\top = (\eta_2, \ldots, \eta_{t+1})$.

**II. Related work**

The finite-time analysis of the OLS estimator has received a lot of attention recently, see [16], [1], [8] and references therein.

In [16], the authors prove that the OLS estimator is $(\varepsilon, \delta)$-PAC if the observed trajectory is longer than $\frac{c_1}{\varepsilon^2} \log(1/\delta)$. In this upper bound, the constant $c_1(A)$ depends on $A$ in a complicated manner. The bound does not exhibit the right scaling in $\log(1/\delta)$ and $\varepsilon$. It also has a worse scaling with the dimension $d$ than our bound.

The main result in [1] (Theorem 2.1) states that the OLS estimator $(\varepsilon, \delta)$-PAC after $t$ observations under the following condition:

$$t \geq \frac{1}{c_2 \lambda_{\min}(\Gamma_k(A))} (d \log(d/\delta) + \log \det(\Gamma_t(A)\Gamma_k(A)^{-1})), \tag{5}$$

for some $k$ satisfying

$$\frac{t}{k} \geq c'(d \log(d/\delta) + \log \det(\Gamma_t(A)\Gamma_k(A)^{-1})).$$

This result is difficult to interpret, and choosing $k$ to optimize the bound seems involved. The authors of [1] present a simplified result in Corollary 2.2, removing the dependence in $k$. However, the result requires that $t \geq T_0$, and we can show that $T_0$ actually depends on $A$ (the authors do not express this dependence). Corollary A.2 presented in the appendix of [1] is more explicit, and states that OLS is $(\varepsilon, \delta)$-PAC if

$$t \geq c(d \log (d \text{cond}(S)/t) + \sum b_i^2 \log t),$$

where $b_1, b_2, \ldots$ are the block sizes of the Jordan decomposition of $A$, and $S$ is the diagonalizing matrix in this decomposition. Note that the term $\sum b_i^2$ may be in the worst case of the order $d^2$. Compared to our sample complexity upper bound (2), the above bound has a worse dependence in the dimension $d$. It has the advantage not to have the term $\mathcal{J}(A)$, but this advantage disappears when $\varepsilon \sim c'(\mathcal{J}(A))^{-1}$. The analysis of [1] relies on the decomposition of the estimation error $\|A_t - A\| \leq \|X\|\|U^\top E\|$, where $U$ is obtained from the singular value decomposition $X = USV^\top$. Deriving an upper bound of $\|X\|$ is then the most involved step of the analysis. To this aim, the authors adapt the so-called small ball method [17] (which typically requires independence), and introduce the Block Martingale Small Ball condition (BMSB) (which indeed requires the introduction of the term $k$ in the result).

The authors of [8] use the same decomposition of $A_t - A$ as ours. This decomposition is $\|A_t - A\| \leq \|E^\top X ((X^\top X)^{-1}) 2 \|\|((X^\top X)^{-1}) 2 \|$. The first term corresponds to a self-normalized process, and can be analyzed using the related theory [13]. The analysis of the second term again requires to control the singular values of $X$. In turn, the analysis of $s_d(X)$ presented in [8] is involved, and unfortunately leads to bounds that are not precise in the system parameters $A$ and the dimension $d$. Overall, the upper bound of the sample complexity proposed in [8] is $O(C(d) \frac{d}{\varepsilon^2} \log(1/\delta))$. The dependence in $d$ is not explicit, and that in $A$ is unknown and hidden in constants.

**III. Main result**

Consider the linear system (1), and assume that the noise vectors are i.i.d. isotropic vectors with independent coordinates of $\psi_2$-norm upper bounded by $K$. We observe a single trajectory of the system $(x_1, \ldots, x_{t+1})$, and builds from these observations the OLS estimator: $A_i = \cdots$
arg min $\sum_{A \in \mathbb{R}^{d \times d}} \| x_{s+1} - Ax_s \|^2$. $A_t$ enjoys the following closed form expression:

$$A_t = \left( \sum_{s=0}^{t} x_{s+1}x_s^\top \right) \left( \sum_{s=0}^{t} x_sx_s^\top \right)^\dagger.$$  

(5)

The estimation error is also explicit:

$$A_t - A = \left( \sum_{s=1}^{t} \eta_{s+1}x_s^\top \right) \left( \sum_{s=1}^{t} x_sx_s^\top \right)^\dagger,$$  

(6)

which we can re-write as (using the notation introduced in the introduction)

$$A_t - A = E^\top X(XX^\top)^\dagger.$$  

(7)

Before stating our main result, we define a quantity that will arise naturally in our analysis. We define the truncated block Toeplitz matrix $\Gamma$ as

$$\Gamma = \begin{bmatrix} I_d & \cdots & 0 \\ A & \cdots & 0 \\ A^2 & \cdots & A \\ \vdots & \ddots & \vdots \\ A^{t-2} & \cdots & A^2 \\ A^{t-1} & \cdots & A \\ \end{bmatrix}. \quad (8)$$

When $\rho(A) < 1$, we can upper bound $\| \Gamma \|$ by a quantity $J(A)$, independent of $t$, as shown in Lemma 5 in [18]:

$$\| \Gamma \| \leq J(A) = \sum_{s \geq 0} \| A^s \|. \quad (9)$$

Our main result is the following Theorem.

**Theorem 1 (OLS performance):** Let $A_t$ denote the OLS estimator. For any $0 < \delta < 1$, and any $\varepsilon > 0$, we have:

$$\mathbb{P}(\|A_t - A\| > \varepsilon) < \delta,$$

as long as the following condition holds

$$\lambda_{\min} \left( \sum_{s=0}^{t-1} \Gamma_s(A) \right) \geq c \max \left\{ \frac{1}{\varepsilon^2}, \| \Gamma \|^2 \right\} \left( \log \left( \frac{1}{\delta} \right) + d \right),$$  

(10)

where $c = c'K^4$ and $c' > 0$ is a universal constant.

IV. SPECTRUM OF THE COVARIATES MATRIX

In this section, we analyze the spectrum of the covariates matrix $X$. Such an analysis is at the core of the proof of Theorem 1. Indeed, relating $s_d(X)$ to the spectrum of the matrix $M = \left( \sum_{s=0}^{t-1} \Gamma_s(A) \right)^{-\frac{1}{2}}$ is close to what Theorem 1 states. We are actually able to control the entire spectrum of $X$ with high probability, as stated in Theorem 2 below. We believe that this result is of independent interest; but it is not directly used in the proof of Theorem 1. The latter relies on Lemma 2, which is also the main ingredient of the proof of Theorem 2.

**Theorem 2:** Let $\varepsilon > 0$. Let $M = \left( \sum_{s=0}^{t-1} \Gamma_s(A) \right)^{-\frac{1}{2}}$. Then:

$$\frac{1}{\| M \|} (1 - K^2 \varepsilon) \leq s_d(X) \leq \cdots \leq s_1(X) \leq (1 + K^2 \varepsilon) \frac{1}{s_d(M)}$$

holds with probability at least

$$1 - 2 \exp \left( -c_1 \varepsilon^2 \frac{1}{\| M \|^2\| \Gamma \|^2 + c_2 d} \right),$$

for some universal constants $c_1, c_2 > 0$.

Theorem 2 is a direct consequence of the following two lemmas. Lemma 1 is a corollary of [19, Ch. 4, Lemma 4.5.6]), and is proved in [18]. Informally, in this lemma, we may think of $X$ as a tall random matrix and $M^{-1}$ as a deterministic normalizing matrix. If the latter is chosen appropriately, then one would hope that $XM$ is an approximate isometry in the sense of Lemma 1, which in turn will provide a two-sided bound on the spectrum of $X$.

**Lemma 1 (Approximate isometries):** Let $X \in \mathbb{R}^{t \times d}$, and let $M \in \mathbb{R}^{d \times d}$ be a full rank matrix. Let $\varepsilon > 0$ and assume that

$$||(XM)^\top XM - I_d|| \leq \max(\varepsilon, \varepsilon^2).$$  

(11)

Then the following holds

$$\frac{1}{s_1(M)} (1 - \varepsilon) \leq s_d(X) \leq \cdots \leq s_1(X) \leq (1 + \varepsilon) \frac{1}{s_d(M)}.$$

Lemma 2 is the main ingredient of the proof of Theorem 2, but also of that of Theorem 1. The lemma is established by expressing $||(XM)^\top XM - I_d||$ as the supremum of a chaos process, which we can control combining Hanson-Wright inequality and a classical $\varepsilon$-net argument. In the next two subsections, we formally introduce chaos processes and provide related concentration inequalities; we also present the $\varepsilon$-net argument used to complete the proof of Lemma 2.

**Lemma 2:** Let $M = \left( \sum_{s=0}^{t-1} \Gamma_s(A) \right)^{-\frac{1}{2}}$. Then:

$$||(XM)^\top XM - I_d|| \leq \max(\varepsilon, \varepsilon^2) K^2$$

holds with probability at most

$$2 \exp \left( -c_1 \varepsilon^2 \frac{1}{\| M \|^2\| \Gamma \|^2 + c_2 d} \right)$$

for some positive absolute constants $c_1, c_2$.

A. Chaos Processes and Hanson-Wright Inequality

In this section, we introduce chaos processes, and provide Hanson-Wright inequality, an instrumental concentration result related to these processes. A chaos in probability theory refers to a quadratic form $\xi^\top W\xi$, where $W$ is a deterministic matrix in $\mathbb{R}^{d \times d}$ and $\xi$ is a random vector with independent coordinates. If the vector $\xi$ is isotropic, we simply have $\mathbb{E} \xi^\top W\xi = \text{tr} W$. The process $(\xi^\top W\xi)_{W \in W}$ indexed by a set $W$ of matrices is referred to as a chaos process.

In our analysis, we use the following concentration result on chaos, due to Hanson-Wright [20], [21], see [15] (Theorems 2.1).

$\xi^\top W\xi = 2A mapping T : X \to Y$ is said to be an isometry if $d_X(x, y) = d_Y(Tx, Ty)$ for all $x, y \in X$. $\xi^\top W\xi$
Theorem 3 (Concentration of anisotropic random vectors):
Let \( B \in \mathbb{R}^{m \times d} \) and \( \xi \in \mathbb{R}^d \) be a random vector with zero-mean, unit-variance, sub-gaussian independent coordinates. Then for all \( \varepsilon > 0 \),
\[
\mathbb{P} \left( \| B \|_F^2 - \| B \|_F^2 > \varepsilon \| B \|_F^2 \right) \leq 2 \exp \left( -c \min \left( \frac{\varepsilon^2}{K^4}, \frac{\varepsilon}{K^2} \right) \frac{\| B \|_F^2}{K^2} \right),
\]
where \( c \) is an absolute positive constant and \( K = \| \xi \|_{\psi_2} \).

B. The \( \varepsilon \)-net argument
The epsilon net argument is a simple, yet powerful tool in the non-asymptotic theory of random matrices and high dimensional probability [22], [19], [23]. Here, we provide two instances of this argument.

Lemma 3: Let \( W \) be an \( m \times d \) random matrix, and \( \varepsilon \in (0, 1) \). Furthermore, let \( N \) be an \( \varepsilon \)-net of \( S^{d-1} \) with minimal cardinality. Then for all \( \rho > 0 \), we have
\[
\mathbb{P} (\| W \| > \rho) \leq \left( \frac{2}{\varepsilon} + 1 \right) \max_{x \in N} \mathbb{P} (\| W x \|_2 > (1 - \varepsilon) \rho).
\]

Lemma 4: Let \( W \) be an \( d \times d \) symmetric random matrix, and \( \varepsilon \in (0, 1/2) \). Furthermore, let \( N' \) be an \( \varepsilon \)-net of \( S^{d-1} \) with minimal cardinality. Then for all \( \rho > 0 \), we have
\[
\mathbb{P} (\| W \| > \rho) \leq \left( \frac{2}{\varepsilon} + 1 \right) \max_{x \in N'} \mathbb{P} (\| x^T W x \| > (1 - 2\varepsilon) \rho) \mathbb{P} (\| W \| \leq \rho).
\]

Lemma 3 and Lemma 4 exploit variational forms of the operator norm, namely \( \| W \| = \sup_{x \in S^{d-1}} \| W x \|_2 \) and, when \( W \) is symmetric, \( \| W \| = \sup_{x \in S^{d-1}} \| x^T W x \| \). The proof of these standard lemmas can be found for example in [19, Chapter 4].

C. Proof of Lemma 2

Step 1 (\( \| (X M)^T X M - I_d \| \) as the supremum of a chaos process)
Note that we can write \( \text{vec}(X^T) = \Gamma \xi \). It follows from the fact that \( \xi \) is isotropic that
\[
\mathbb{E} (X^T X) = \mathbb{E} \sum_{s=1}^t x_s x_s^T = \sum_{s=1}^{t-1} \sum_{k=0}^{s-1} A^k (A^k)^T = \sum_{s=0}^{t-1} \Gamma_s(A).
\]
Noting that \( \mathbb{E} X X^T \) is a symmetric positive definite matrix, we may define the inverse of its positive definite symmetric square root matrix \( M = \left( \sum_{s=0}^{t-1} \Gamma_s(A) \right)^{-1/2} \). We think of the inverse of this matrix as a normalization of \( X \). Now, we have:
\[
\| (X M)^T X M - I_d \| = \sup_{u \in S^{d-1}} \| u^T ((X M)^T X M - I_d) u \| \\
= \sup_{u \in S^{d-1}} \| u^T (X M)^T X M u - \mathbb{E} u^T (X M)^T X M u \| \\
= \sup_{u \in S^{d-1}} \| ||(X M)^T X M - \mathbb{E}|| ||(X M)^T X M - \mathbb{E}|| \|_F \|_2 \\
= \sup_{u \in S^{d-1}} \| ||\sigma_{M u}^T \Gamma \xi ||^2 - ||\sigma_{M u}^T \Gamma \xi ||^2 \|_F \\
= \sup_{u \in S^{d-1}} \| ||\sigma_{M u}^T \Gamma \xi ||^2 - ||\sigma_{M u}^T \Gamma \xi ||^2 \|_F, \quad (13)
\]
where in the first equality, we used the variational form of the operator norm for symmetric matrices, where in fourth equality, the \( t d \times t \) matrix \( \sigma_{M u} \) is defined as
\[
\sigma_{M u} = \begin{bmatrix} M u & O \\ O & M u \end{bmatrix},
\]
and the last equality follows from the fact that \( \xi \) is isotropic. In fact, by definition of \( M \), we have \( ||\sigma_{M u}^T \Gamma ||_F^2 = 1 \) for all \( u \in S^{d-1} \). We have proved that \( ||(X M)^T X M - I_d||_F \) is the supremum of a chaos process \( \langle \xi^T W \xi \rangle_W \), where the parametrizing matrix \( W \) is
\[
W = \Gamma^T \sigma_{M u} \sigma_{M u}^T \Gamma.
\]

Step 2 (Uniform bound on the chaos) Let \( \rho > 0 \), and \( u \in S^{d-1} \). Again, recalling that \( \xi \) is a zero-mean, subgaussian random vector with independent coordinates, from Henson-Wright inequality (see Theorem 3), we deduce that:
\[
||\sigma_{M u}^T \Gamma \xi ||^2 - ||\sigma_{M u}^T \Gamma ||^2 \geq \rho ||\sigma_{M u}^T \Gamma ||^2,
\]
holds with probability at most
\[
2 \exp \left( -c \min \left( \frac{\rho^2}{K^4}, \frac{\rho}{K^2} \right) \frac{\| \sigma_{M u}^T \Gamma \|_F^2}{\| \sigma_{M u} \|_F^2} \right)
\]
for some positive universal constant \( c \). Noting that \( ||\sigma_{M u}^T \Gamma ||_F^2 = 1 \), and \( ||\sigma_{M u}^T \Gamma ||^2 \leq ||M||^2 ||\Gamma||^2 \), an upper bound on the above probability is
\[
2 \exp \left( -c \min \left( \frac{\rho^2}{K^4}, \frac{\rho}{K^2} \right) \frac{1}{||M||^2 ||\Gamma||^2} \right).
\]
Step 3 (\( \varepsilon \)-net argument) Recalling the equalities (12) and (13) in step 1, as a consequence of step 2, we have: for all \( \varepsilon \in (0, 1/2) \), for all \( u \in S^{d-1} \), the following
\[
\mathbb{P} (\| u^T ((X M)^T X M - I_d) u \| > (1 - 2\varepsilon) \rho) \leq 2 \cdot \exp \left( -c \min \left( \frac{(1 - 2\varepsilon)^2 \rho^2}{K^4}, \frac{(1 - 2\varepsilon) \rho}{K^2} \right) \frac{1}{||M||^2 ||\Gamma||^2} \right).
\]
Now choosing \( \varepsilon = \frac{1}{4} \) and applying Lemma 4, we obtain that
\[
|| (X M)^T X M - I_d || > \rho
\]
holds with probability at most
\[
2 \cdot d \exp \left( -c \min \left( \frac{\rho^2}{K^4}, \frac{\rho}{K^2} \right) \frac{1}{4||M||^2 ||\Gamma||^2} \right),
\]
where we used \( \min \left( \frac{\rho^2}{4K^4}, \frac{\rho}{2K^2} \right) \geq \frac{1}{2} \min \left( \frac{\rho^2}{K^4}, \frac{\rho}{K^2} \right) \). By choosing \( \rho = \max(\varepsilon, \varepsilon^2) K^2 \), which is equivalent to \( \varepsilon^2 = \min(\frac{\rho^2}{K^4}, \frac{\rho}{K^2}) \), we obtain that
\[
|| (X M)^T X M - I_d || > \max(\varepsilon, \varepsilon^2) K^2
\]
holds with probability at most
\[
2 \exp \left( -c_1 \varepsilon^2 \frac{1}{||M||^2 ||\Gamma||^2} + c_2 d \right),
\]
for some positive absolute constants \( c_1, c_2 \). This completes the proof of Lemma 2.
V. PROOF OF THEOREM 1
We first decompose the estimation error as:
\[ |A_t - A| \leq \| E^\top X ((X^\top X)^\frac{1}{2}) \| (\| (X^\top X)^\frac{1}{2} \| > \epsilon) \]
We introduce the matrix \( M = \left( \sum_{s=1}^{t-1} \Gamma_s(A) \right)^{-\frac{1}{2}} \), and the events \( E_1 \) and \( E_2 \) defined as:
\[ E_1 = \left\{ \| E^\top X ((X^\top X)^\frac{1}{2}) \| (\| (X^\top X)^\frac{1}{2} \| > \epsilon) \right\} \]
\[ E_2 = \left\{ \| (XM)^\top XM - I_d \| \leq \frac{1}{2} \right\} \]
Observe that:
\[ P(|A_t - A| > \epsilon) \leq P(E_1 \cap E_2) + P(\overline{E_2}) \]
A. Upper bound of \( P(\overline{E_2}) \)
We use Lemma 2. Let \( \rho = \max(\epsilon^2, \epsilon)K^2 \), which is equivalent to writing \( \epsilon^2 = \min(\frac{\epsilon^2}{K^2}, \frac{\rho^2}{K^2}) \). Then choose \( \rho = \frac{1}{2} \). With this choice, Lemma 2 implies that
\[ \| (XM)^\top XM - I_d \| \leq \frac{1}{2} \]
holds with probability at most
\[ 2 \exp \left(-c_1 \min \left( \frac{1}{2K^2}, \frac{1}{4K^2} \right) \| M \|^2 \frac{\| I \|^2}{2} + c_2d \right) \]
Since \( \xi \) is sub-gaussian and isotropic, we have \( K \geq 1 \). We conclude that
\[ P(\overline{E_2}) \leq \frac{\delta}{\tau} \text{ when } \frac{1}{\| M \|^2} \geq \frac{16K^4\| I \|^2}{\epsilon} \left( \log \left( \frac{4}{\delta} \right) + d \log(9) \right) \] (14)
B. Upper bound of \( P(E_1 \cap E_2) \)
We derive an upper bound on the above probability using a similar technique as in [8] (see also [24] where similar decompositions are used for the analysis of autoregressive processes). We first use the event \( E_2 \) to simplify the condition \( \{ \| E^\top X ((X^\top X)^\frac{1}{2}) \| (\| (X^\top X)^\frac{1}{2} \| > \epsilon) \} \) until we get a quantity that can be analyzed using concentration results on self-normalized processes.
When the event \( E_2 \) occurs, we have:
\[ \frac{3}{2} I_d \geq (XM)^\top XM \geq \frac{1}{2} I_d \]
or equivalently
\[ \frac{3}{2} \sum_{s=0}^{t-1} \Gamma_s(A) = \frac{3}{2} M^{-2} \geq X^\top X \geq \frac{1}{2} M^{-2} = \frac{1}{2} \sum_{s=0}^{t-1} \Gamma_s(A). \]
Define \( S = \frac{1}{2} M^{-2} \) and \( \beta = \sqrt{s_d(S)} \). Note that when event \( E_2 \) occurs,
\[ s_d(X) \geq \beta = \sqrt{\frac{1}{2} \lambda_{\min} \left( \sum_{s=0}^{t-1} \Gamma_s(A) \right)} . \]
Thus \( \| X^\top \| \leq \frac{1}{\beta} \) and we obtain:
\[ E_1 \cap E_2 \subseteq \{ \| E^\top X (X^\top X)^{-\frac{1}{2}} \| > \epsilon \beta \} \cap E_2. \]
Next, again when \( E_2 \) occurs, we have \( 2X^\top X \geq X^\top X + S \), and thus \( 2(X^\top X + S)^{-1} \geq (X^\top X)^{-1} \). We deduce that:
\[ \| E^\top X (X^\top X)^{-\frac{1}{2}} \| \leq \sqrt{2} \| E^\top X (X^\top X + S)^{-\frac{1}{2}} \|. \]
Hence,
\[ \{ \| E^\top X (X^\top X)^{-\frac{1}{2}} \| > \epsilon \} \cap E_2 \]
\[ \subseteq \{ \sqrt{2} \| E^\top X (X^\top X + S)^{-\frac{1}{2}} \| > \epsilon \beta \} \cap E_2. \]
Furthermore, note that under \( E_2 \), we have \( 3S \geq X^\top X \geq S \), which implies that for all \( \delta \in (0, 1) \),
\[ \epsilon \beta \geq \frac{\log \left( \frac{2 \cdot 5^d (\text{det}(S + S)S^{-1})^{\frac{1}{2}}}{\delta} \right)}{\log \left( \frac{2 \cdot 5^d (\text{det}(S + S)S^{-1})^{\frac{1}{2}}}{\delta} \right)} \]
Now consider the condition
\[ \epsilon \geq \frac{4\sqrt{cK}}{\beta} \sqrt{\log \left( \frac{2 \cdot 5^d (\text{det}(S + S)S^{-1})^{\frac{1}{2}}}{\delta} \right)} \]
\[ = \frac{4\sqrt{cK}}{\beta} \sqrt{\log \left( \frac{2}{\delta} \right) + d \log(10)} \]
Since \( \beta^2 = \frac{2}{\| M \|^2} \), the above condition is equivalent to
\[ \frac{1}{\| M \|^2} \geq \frac{16cK^2}{\epsilon^2} \left( \log \left( \frac{2}{\delta} \right) + d \log(10) \right) \] (15)
We deduce that, under the above condition,
\[ \{ \sqrt{2} \| E^\top X (X^\top X + S)^{-\frac{1}{2}} \| > \epsilon \beta \} \cap E_2 \]
\[ \subseteq \{ \| E^\top X (X^\top X + S)^{-\frac{1}{2}} \| > \epsilon \} \cap E_2 \]
We are now ready to apply Corollary 1 in [18] for self-normalized processes, and conclude that the event
\[ \{ \| E^\top X (X^\top X + S)^{-\frac{1}{2}} \| \}
\[ > \frac{4\sqrt{cK}}{\beta} \sqrt{\log \left( \frac{2 \cdot 5^d (\text{det}(S + S)S^{-1})^{\frac{1}{2}}}{\delta} \right)} \]
occurs with probability at most \( \delta/2 \). This implies that:
\[ P \left( \sqrt{2} \| E^\top X (X^\top X + S)^{-\frac{1}{2}} \| > \epsilon \beta \} \cap E_2 \right) \leq \frac{\delta}{2} \]
as long as condition (15) holds. Hence \( P(E_1 \cap E_2) \leq \frac{\delta}{2} \) when (15) holds.
C. Concluding steps

Introducing

\[ \tau_1 = 16K^4 \| \Gamma \|_2^2 c_2 \left( \log \left( \frac{4}{\delta} \right) + d \log(9) \right), \]

\[ \tau_2 = 16c_2 K^2 \left( \log \left( \frac{2}{\delta} \right) + d \log(10) \right), \]

we may re-write Condition (14) as \( \frac{1}{\| M \|^2} \geq \tau_1 \) and Condition (15) as \( \frac{1}{\| M \|^2} \geq \tau_2 \). Thus, we have

\[ P \left( \| A_1 - A_2 \| > \varepsilon \right) \leq P \left( \mathcal{E}_1 \cap \mathcal{E}_2 \right) + P \left( \mathcal{E}_2^c \right) \leq \frac{\delta}{2} + \frac{\delta}{2} \]

provided that \( \frac{1}{\| M \|^2} \geq \max(\tau_1, \tau_2) \). Finally observe that:

\[
\max(\tau_1, \tau_2) \leq 16K^2 \max \left( \frac{c_1}{c_2}, \frac{K^2}{c_2} \right) \| \Gamma \|_2^2 \left( \log \left( \frac{4}{\delta} \right) + d \log(10) \right) \\
\leq C \max \left( \frac{1}{\varepsilon^2}, \| \Gamma \|_2^2 \right) \left( \log \left( \frac{4}{\delta} \right) + d \log(10) \right)
\]

where \( C = 16K^2 \max \left( \frac{c_1}{c_2}, \frac{K^2}{c_2} \right) \). We conclude that

\[ P \left( \| A_1 - A_2 \| > \varepsilon \right) \leq \delta \]

holds as long as

\[
\frac{1}{\| M \|^2} \geq C \max \left( \frac{1}{\varepsilon^2}, \| \Gamma \|_2^2 \right) \left( \log \left( \frac{4}{\delta} \right) + d \log(10) \right).
\]

This completes the proof of Theorem 1.

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