Representations of Twisted Yangians
Associated with skew Young Diagrams

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To Professor I. M. Gelfand on his 90-th birthday

Abstract. Let $G_M$ be either the orthogonal group $O_M$ or the symplectic group $Sp_M$ over the complex field; in the latter case the non-negative integer $M$ has to be even. Classically, the irreducible polynomial representations of the group $G_M$ are labeled by partitions $\mu = (\mu_1, \mu_2, \ldots)$ such that $\mu'_1 + \mu'_2 \leq M$ in the case $G_M = O_M$, or $2\mu'_1 \leq M$ in the case $G_M = Sp_M$. Here $\mu' = (\mu'_1, \mu'_2, \ldots)$ is the partition conjugate to $\mu$. Let $W_\mu$ be the irreducible polynomial representation of the group $G_M$ corresponding to $\mu$.

Regard $G_N \times G_M$ as a subgroup of $G_{N+M}$. Then take any irreducible polynomial representation $W_\lambda$ of the group $G_{N+M}$. The vector space $W_\lambda(\mu) = \text{Hom}_{G_M}(W_\mu, W_\lambda)$ comes with a natural action of the group $G_N$. Put $n = \lambda_1 - \mu_1 + \lambda_2 - \mu_2 + \ldots$. In this article, for any standard Young tableau $\Omega$ of skew shape $\lambda/\mu$ we give a realization of $W_\lambda(\mu)$ as a subspace in the $n$-fold tensor product $\left(\mathbb{C}^N\right)^\otimes n$, compatible with the action of the group $G_N$. This subspace is determined as the image of a certain linear operator $F_\Omega(M)$ on $\left(\mathbb{C}^N\right)^\otimes n$, given by an explicit formula.

When $M = 0$ and $W_\lambda(\mu) = W_\lambda$ is an irreducible representation of the group $G_N$, we recover the classical realization of $W_\lambda$ as a subspace in the space of all traceless tensors in $\left(\mathbb{C}^N\right)^\otimes n$. Then the operator $F_\Omega(0)$ may be regarded as the analogue for $G_N$ of the Young symmetrizer, corresponding to the standard tableau $\Omega$ of shape $\lambda$. This symmetrizer is a certain linear operator on $\left(\mathbb{C}^N\right)^\otimes n$ with the image equivalent to the irreducible polynomial representation of the complex general linear group $GL_N$, corresponding to the partition $\lambda$. Even in the case $M = 0$, our formula for the operator $F_\Omega(M)$ is new.

Our results are applications of the representation theory of the twisted Yangian, corresponding to the subgroup $G_N$ of $GL_N$. This twisted Yangian is a certain one-sided coideal subalgebra of the Yangian corresponding to $GL_N$. In particular, $F_\Omega(M)$ is an intertwining operator between certain representations of the twisted Yangian in $\left(\mathbb{C}^N\right)^\otimes n$.

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0. Brief introduction

This article draws on the ideas beginning with the Schur–Weyl duality [W], passing through Gelfand–Zetlin bases [GZ1,GZ2] and continuing through Yangians [D], with the aim of realizing explicitly irreducible representations of the orthogonal or symplectic classical group $G_N$. This group corresponds to a symmetric or alternating, non-degenerate bilinear form $\langle \cdot , \cdot \rangle$ on the $N$-dimensional complex vector space $\mathbb{C}^N$. The irreducible representations of the group $G_N$ considered in this article are polynomial. By definition, they are subrepresentations of tensor powers of the defining representation acting on $\mathbb{C}^N$. This work provides new explicit realization of the representation of $G_N$ on the vector space (1.5). This vector space describes the multiplicities in the restriction of an irreducible representation of the group $G_{N+M}$ to the subgroup $G_M$. Results follow for the branching rules for restricting irreducible representations from $G_{N+M}$ to the subgroup $G_N \times G_M$, see [P].

Here is an overview of this article. Section 1 gives an exposition of the principal results, with detailed references given throughout. Section 2 recalls the classical realization [W] of any irreducible polynomial representation of the general linear group $GL_N$. This realization involves elements of the symmetric group rings, known as Young symmetrizers [Y1]. We also recall the approach to Young symmetrizers due to Cherednik [C2]. Following this approach, in Section 3 we construct analogues of the Young symmetrizers for the group $G_N$. This construction provides a realization of any irreducible polynomial representation of the group $G_N$, more explicit than in [W]. It is motivated by the representation theory of Yangians and of their twisted analogues [O2]. The main results concerning branching rules for the groups $GL_N$ and $G_N$ are stated as Theorems 1.6 and 1.8, respectively. Theorem 1.6 belongs to Cherednik [C3], its proof given in Section 4 is new. Theorem 1.8 is new, its proof is given in Section 5. It is hoped that this work will further motivate the interest of readers in Yangians, see [MN,MO,NO].

A word of explanation is necessary in regard to our scheme of referring to theorems, propositions, lemmas and corollaries. When referring to them, we indicate the subsections where they respectively appear. There will be no more than one of each of them in every subsection, so our scheme should cause no confusion. For example, Theorems 1.6 and 1.8 mentioned above are the theorems appearing in Subsections 1.6 and 1.8, respectively.

1. Main results

1.1. Let $\nu = (\nu_1, \nu_2, \ldots)$ be any partition of a non-negative integer $n$. The parts of $\nu$ are arranged in the non-increasing order: $\nu_1 \geq \nu_2 \geq \ldots \geq 0$. As usual, denote by $\nu' = (\nu'_1, \nu'_2, \ldots)$ the conjugate partition. In particular, $\nu'_1$ is the number of non-zero parts of $\nu$. Take any positive integer $N \geq \nu'_1$. Let
$V_\nu \subset (\mathbb{C}^N)^{\otimes n}$ be the irreducible polynomial representation of the complex general linear group $GL_N$ corresponding to the partition $\nu$. We will also regard representations of the group $GL_N$ as modules over the general linear Lie algebra $\mathfrak{gl}_N$. Then $V_\nu$ is an irreducible $\mathfrak{gl}_N$-module of the highest weight $(\nu_1, \ldots, \nu_N)$. Here we choose the Borel subalgebra in $\mathfrak{gl}_N$ consisting of the upper triangular matrices, and fix the basis of the diagonal matrix units $E_{11}, \ldots, E_{NN}$ in the corresponding Cartan subalgebra of $\mathfrak{gl}_N$.

Now let $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ be any two partitions. Take any non-negative integer $M$ such that $\lambda_i - \mu_i \leq N + M$ and $\lambda'_i - \mu'_i \leq M$ for each $i = 1, 2, \ldots$; see [M, Section I.5]. In this article, we will consider certain embeddings of $V_\lambda(\mu)$ into the $n$-fold tensor product $(\mathbb{C}^N)^{\otimes n}$ where

$$n = \lambda_1 - \mu_1 + \lambda_2 - \mu_2 + \ldots$$

These embeddings will be compatible with the action of the group $GL_N$.

Suppose that $\lambda_i \geq \mu_i$ for each $i = 1, 2, \ldots$. Consider the skew Young diagram

$$\lambda/\mu = \{(i, j) \in \mathbb{Z}^2 \mid i \geq 1, \lambda_i \geq j > \mu_i \}.$$ 

When $\mu = (0, 0, \ldots)$, this is the usual Young diagram of the partition $\lambda$. We will employ the standard graphic representation [M] of Young diagrams on the plane $\mathbb{R}^2$ with two matrix style coordinates. Here the first coordinate increases from top to bottom, while the second coordinate increases from left to right. The element $(i, j) \in \lambda/\mu$ is represented by the unit box with the bottom right corner at the point $(i, j) \in \mathbb{R}^2$.

The set $\lambda/\mu$ consists of $n$ elements. A standard tableau of shape $\lambda/\mu$ is any bijection $\Omega: \lambda/\mu \to \{1, \ldots, n\}$ such that $\Omega(i, j) < \Omega(i + 1, j)$ and $\Omega(i, j) < \Omega(i, j + 1)$ for all possible $i, j$. Graphically, the tableau $\Omega$ is represented by placing the numbers $\Omega(i, j)$ into the corresponding boxes of $\lambda/\mu$ on the plane $\mathbb{R}^2$. By filling the boxes with the numbers $1, \ldots, n$ by rows downwards, from left to right in every row, we get the row tableau $\Omega^r$ of shape $\lambda/\mu$. The column tableau $\Omega^c$ of shape $\lambda/\mu$ is also defined in the obvious way. Both $\Omega^r$ and $\Omega^c$ are standard tableaux. Below we represent $\Omega^r$ and $\Omega^c$ for the partitions $\lambda = (5, 3, 3, 3, 3, 3, 0, \ldots)$ and $\mu = (3, 3, 2, 0, 0, \ldots)$:
In Section 2, for every standard tableau $\Omega$ of shape $\lambda/\mu$ we will define a certain element $e_\Omega$ of the symmetric group ring $\mathbb{C}S_n$. If $\mu = (0, 0, \ldots)$, $e_\Omega$ is a diagonal matrix element of the irreducible representation of the group $S_n$ labeled by the partition $\lambda$; see (2.1). If $\Omega$ is the row or column tableau of shape $\lambda$, this matrix element is given explicitly by (2.3) and (2.4). When $\mu \neq (0, 0, \ldots)$, the element $e_\Omega \in \mathbb{C}S_n$ is defined by (2.12) and (2.13).

We denote by $E_\Omega$ the operator on the $n$-fold tensor product $(\mathbb{C}^N)^\otimes n$, corresponding to the element $e_\Omega \in \mathbb{C}S_n$ under the action of the group $S_n$ by permutations of tensor factors. The vector space $V_\lambda(\mu)$ will be realized as the image of the operator $E_\Omega$. Denote this image by $V_\Omega$. The subspace $V_\Omega$ in $(\mathbb{C}^N)^\otimes n$ is preserved by the natural action of the group $GL_N$. Thus $V_\Omega$ can be regarded as a representation of $GL_N$.

**Proposition.** The representations $V_\lambda(\mu)$ and $V_\Omega$ of $GL_N$ are equivalent.

We prove this proposition in Subsection 4.6. If $\mu = (0, 0, \ldots)$, then the representation $V_\Omega$ of $GL_N$ is equivalent to $V_\lambda$ by the classical duality theorem of Schur [W, Section IV.4]. Note that for any partitions $\lambda$ and $\mu$, the operator $E_\Omega$ on the space $(\mathbb{C}^N)^\otimes n$ does not depend on the integer $M$.

**1.2.** There is a description of the operator $E_\Omega$ on $(\mathbb{C}^N)^\otimes n$ of another kind. This description is obtained by the fusion procedure, due to Cherednik. For every $k = 1, \ldots, n$ put $c_k(\Omega) = j - i$ if $k = \Omega(i, j)$. The difference $c_k(\Omega)$ is the content of the box occupied by the number $k$ in the tableau $\Omega$. In the above example $n = 9$, and the sequences of contents $c_1(\Omega^r), \ldots, c_9(\Omega^r)$ and $c_1(\Omega^c), \ldots, c_9(\Omega^c)$ are respectively

$$3, 4, 0, -3, -2, -1, -4, -3, -2 \quad \text{and} \quad -3, -4, -2, -3, 0, -1, -2, 3, 4.$$

Introduce $n$ complex variables $t_1(\Omega), \ldots, t_n(\Omega)$ with the constraints

$$t_k(\Omega) = t_l(\Omega) \quad \text{if} \quad k \text{ and } l \text{ occur in the same row of } \Omega. \quad (1.2)$$

Alternatively to (1.2), as in [NT2, Section 2], we can impose the constraints

$$t_k(\Omega) = t_l(\Omega) \quad \text{if} \quad k \text{ and } l \text{ occur in the same column of } \Omega. \quad (1.3)$$

Thus the number of independent variables among $t_1(\Omega), \ldots, t_n(\Omega)$ equals the number of non-empty rows of the diagram $\lambda/\mu$ in the case (1.2), or the number of non-empty columns of $\lambda/\mu$ in the case (1.3).
Order lexicographically the set of all pairs \((k, l)\) with \(1 \leq k < l \leq n\). Take the ordered product over this set,

\[
\prod_{1 \leq k < l \leq n} \left( 1 - \frac{P_{kl}}{c_k(\Omega) - c_l(\Omega) + t_k(\Omega) - t_l(\Omega)} \right) \tag{1.4}
\]

where \(P_{kl}\) denotes the operator on the vector space \((\mathbb{C}^N)^{\otimes n}\) exchanging the \(k\)th and \(l\)th tensor factors. Consider (1.4) as a rational function of the constrained variables \(t_1(\Omega), \ldots, t_n(\Omega)\). The next result goes back to [C3].

**Theorem.** The product (1.4) is regular at \(t_1(\Omega) = \ldots = t_n(\Omega)\). The value of (1.4) at \(t_1(\Omega) = \ldots = t_n(\Omega)\) coincides with the operator \(E_\Omega\).

Most of the results of [C3] were given without proofs. In Section 2 of the present article we give all necessary details of the proof of this theorem.

1.3. The principal aim of this article is to give analogues of the operator \(E_\Omega\) for the classical complex Lie groups \(O_N\) and \(sp_N\). Let \(G_N\) be one of these two Lie groups. We will regard \(G_N\) as the subgroup in \(GL_N\), preserving a non-degenerate bilinear form \(\langle \ , \rangle\) on \(\mathbb{C}^N\), symmetric in the case \(G_N = O_N\), or alternating in the case \(G_N = sp_N\). In the latter case \(N\) has to be even. Throughout this article, we always assume that the integer \(N\) is positive.

The irreducible polynomial representations of \(G_N\) are labeled by the partitions \(\nu\) of \(n = 0, 1, 2, \ldots\) such that \(\nu_1' + \nu_2' \leq N\) if \(G_N = O_N\), and \(2\nu_1' \leq N\) if \(G_N = sp_N\); see [W, Sections V.7 and VI.3]. Denote by \(W_\nu\) the irreducible representation of \(G_N\) corresponding to \(\nu\). Take any two distinct numbers \(k, l \in \{1, \ldots, n\}\). By applying the bilinear form \(\langle \ , \rangle\) to a tensor \(w \in (\mathbb{C}^N)^{\otimes n}\) in the \(k\)th and \(l\)th tensor factors, we obtain a certain tensor \(\tilde{w} \in (\mathbb{C}^N)^{\otimes (n-2)}\). The tensor \(w\) is called *traceless* if \(\tilde{w} = 0\) for all distinct \(k\) and \(l\). Denote by \((\mathbb{C}^N)^{\otimes n}_0\) the subspace in \((\mathbb{C}^N)^{\otimes n}\) consisting of traceless tensors, this subspace is \(G_N\)-invariant. Choose any embedding of the irreducible representation \(V_\nu\) of the group \(GL_N\), into the space \((\mathbb{C}^N)^{\otimes n}\). Then \(W_\nu\) can be embedded into \((\mathbb{C}^N)^{\otimes n}\) as \(V_\nu \cap (\mathbb{C}^N)^{\otimes n}_0\).

Denote by \(g_N\) the Lie algebra of \(G_N\), so that \(g_N = so_N\) or \(g_N = sp_N\). We will regard \(g_N\) as a Lie subalgebra in \(gl_N\). We will also consider representations of the group \(G_N\) as \(g_N\)-modules. The \(g_N\)-module \(W_\nu\) is irreducible unless \(g_N = so_N\) and \(2\nu_1' = N\), in which case \(W_\nu\) is a direct sum of two irreducible \(so_N\)-modules; see Subsection 3.6. The \(so_N\)-module \(W_\nu\) may be reducible because the group \(O_N\) has two connected components.

Take a non-negative integer \(M\) and choose a non-degenerate bilinear form on the space \(\mathbb{C}^M\), symmetric in the case \(G_N = O_N\) or alternating in the case \(G_N = sp_N\). In the latter case the integer \(M\) has to be even. Consider the corresponding subgroup \(G_M \subset GL_M\). If \(M = 0\), the group \(G_M\) consists only of the unit element. Equip the direct sum \(\mathbb{C}^N \oplus \mathbb{C}^M = \)
\( \mathbb{C}^{N+M} \) with the bilinear form, which is the sum of the forms on the direct summands. We get an embedding of the direct product \( G_N \times G_M \) into \( G_{N+M} \) and \( G_M \), respectively. Here we assume that the partitions \( \lambda \) and \( \mu \) satisfy the conditions from [W, Sections V.7 and VI.3] for the groups \( G_{N+M} \) and \( G_M \), respectively. Thus for any given \( N \), we have \( W_\lambda(\mu) \neq \{0\} \) if and only if \( V_\lambda(\mu) \neq \{0\} \), see Subsection 1.1 above. Further, for any given \( N \) we have the inequality
\[
\dim W_\lambda(\mu) \leq \dim V_\lambda(\mu) .
\]

For every standard tableau \( \Omega \) of shape \( \lambda/\mu \), the results of the present article provide a distinguished embedding of the vector space \( W_\lambda(\mu) \) into \( V_\lambda(\mu) \), compatible with the action of the group \( G_N \subset GL_N \); see [KS].

1.4. As in the case of the general linear group \( GL_N \), suppose that \( \lambda_i \geq \mu_i \) for all \( i = 1, 2, \ldots \). Take the skew Young diagram \( \lambda/\mu \). For any standard tableau \( \Omega \) of shape \( \lambda/\mu \), we will now construct an embedding of \( W_\lambda(\mu) \) into the tensor product \( (\mathbb{C}^N)^\otimes_n \), where \( n \) is the number of elements in the set \( \lambda/\mu \). This embedding will be compatible with the action of the group \( G_N \).

Take any basis \( u_1, \ldots, u_N \) in the vector space \( \mathbb{C}^N \). Let \( v_1, \ldots, v_N \) be the dual basis in \( \mathbb{C}^N \), so that \( \langle u_i, v_j \rangle = \delta_{ij} \) for \( i, j = 1, \ldots, N \). The vector
\[
w(N) = \sum_{i=1}^{N} u_i \otimes v_i \in \mathbb{C}^N \otimes \mathbb{C}^N
\]
does not depend on the choice of the basis \( u_1, \ldots, u_N \) and is invariant under the action of the group \( G_N \) on \( \mathbb{C}^N \otimes \mathbb{C}^N \). Introduce the linear operator
\[
Q(N) : u \otimes v \mapsto \langle u, v \rangle \cdot w(N)
\]
in \( \mathbb{C}^N \otimes \mathbb{C}^N \); it commutes with the action of the group \( G_N \). The tableau \( \Omega \) defines the sequence of contents \( c_1(\Omega), \ldots, c_n(\Omega) \). In the case \( G_N = O_N \), we will use the variables \( t_1(\Omega), \ldots, t_n(\Omega) \) with the constraints (1.3). In the case \( G_N = Sp_N \) it is more convenient to use the variables \( t_1(\Omega), \ldots, t_n(\Omega) \) with the constraints (1.2). Take the ordered product over the pairs \((k,l)\)
\[
\prod_{1 \leq k < l \leq n} \left( 1 - \frac{Q_{kl}}{c_k(\Omega) + c_l(\Omega) + t_k(\Omega) + t_l(\Omega) + N + M} \right)
\]
where $Q_{kl}$ is the linear operator on $(\mathbb{C}^N)^\otimes n$, acting as $Q(N)$ in the $k$th and $l$th tensor factors, and acting as the identity in the remaining $n-2$ tensor factors. Here the pairs $(k,l)$ are ordered lexicographically, as in (1.4).

Throughout this article, we use the following convention. Whenever the double sign $\pm$ or $\mp$ appears, the upper sign corresponds to the case of a symmetric form $\langle \ ,\ \rangle$ while the lower sign corresponds to the case of an alternating form. Now multiply (1.9) by (1.4) on the right, and consider the result as a rational function of the constrained variables $t_1(\Omega), \ldots, t_n(\Omega)$.

**Theorem.**

a) If $G_N = O_N$ and the variables $t_1(\Omega), \ldots, t_n(\Omega)$ obey the constraints (1.3), then the ordered product of (1.9) and (1.4) is regular at $t_1(\Omega) = \ldots = t_n(\Omega) = -1/2$.

b) If $G_N = Sp_N$ and the variables $t_1(\Omega), \ldots, t_n(\Omega)$ obey (1.2), then the ordered product of (1.9) and (1.4) is regular at $t_1(\Omega) = \ldots = t_n(\Omega) = 1/2$.

c) The operator value $F_\Omega(M)$ of the ordered product of (1.9) and (1.4) at $t_1(\Omega) = \ldots = t_n(\Omega) = \mp 1/2$ is divisible on the left and on the right by $E_\Omega$.

Note that unlike $E_\Omega$, the operator $F_\Omega(M)$ on the vector space $(\mathbb{C}^N)^\otimes n$ may depend on the non-negative integer $M$. Observe that

$$c_k(\Omega) + c_l(\Omega) \geq 3 - 2\lambda'_1 \quad \text{if} \quad k \neq l.$$ 

If the partition $\lambda$ satisfies the condition $2\lambda'_1 \leq N + M$, every factor in (1.9) is regular at $t_1(\Omega) = \ldots = t_n(\Omega) = \mp 1/2$. Then by Theorem 1.2 we have

$$F_\Omega(M) = \prod_{1 \leq k < l \leq n} \left( 1 - \frac{Q_{kl}}{c_k(\Omega) + c_l(\Omega) + N + M} \right) \cdot E_\Omega.$$

The condition $2\lambda'_1 \leq N + M$ is satisfied when $G_N = Sp_N$, but may be not satisfied when $G_N = O_N$. The proof of Theorem 1.4 is given at the end of Subsection 3.4. This proof also provides an explicit formula for $F_\Omega(M)$ in the case $G_N = O_N$ for any $\lambda$.

When $G_N = O_N$ and $\Omega = \Omega^c$, this explicit formula for the operator $F_\Omega(M)$ on the space $(\mathbb{C}^N)^\otimes n$ is particularly simple. Namely, for $G_N = O_N$

$$F_{\Omega^c}(M) = \prod_{(k,l)} \left( 1 - \frac{Q_{kl}}{c_k(\Omega^c) + c_l(\Omega^c) + N + M} \right) \cdot E_{\Omega^c} \quad (1.10)$$

where the ordered product is taken over all pairs $(k,l)$ such that $k$ and $l$ appear in different columns of the tableau $\Omega^c$. For any such pair we have

$$c_k(\Omega^c) + c_l(\Omega^c) \geq 3 - \lambda'_1 - \lambda'_2 \geq 3 - N - M,$$
so that each of the denominators in (1.10) is non-zero. The operator $E_{Ω^r}$ on $(\mathbb{C}^N)^{⊗n}$ corresponds to the element $e_{Ω^r}$ of the symmetric group ring $\mathbb{C}S_n$.

If $M = 0$, the element $e_{Ω^r} \in \mathbb{C}S_n$ can be written explicitly by using (2.4).

When $G_N = Sp_N$, the definition of the operator $F_{Ω}(M)$ on $(\mathbb{C}^N)^{⊗n}$ can be simplified for the row tableau $Ω = Ω^r$. Namely, for $G_N = Sp_N$

$$F_{Ω^r}(M) = \prod_{(k,l)} \left( 1 - \frac{Q_{kl}}{c_k(Ω^r) + c_l(Ω^r) + N + M + 1} \right) \cdot E_{Ω^r}$$ (1.11)

where the ordered product is taken over all pairs $(k,l)$ such that $k$ and $l$ appear in different rows of the tableau $Ω^r$. The operator $E_{Ω^r}$ on $(\mathbb{C}^N)^{⊗n}$ corresponds to the element $e_{Ω^r}$ of the group ring $\mathbb{C}S_n$. If $M = 0$, the element $e_{Ω^r} \in \mathbb{C}S_n$ can be written explicitly by using (2.3). The simplified formulas (1.10) and (1.11) will be derived at the end of Subsection 3.4.

The vector space $W_{λ}(µ)$ will be realized as the image of the operator $F_{Ω}(M)$. Denote this image by $W_{Ω}(M)$. The subspace $W_{Ω}(M)$ in $(\mathbb{C}^N)^{⊗n}$ is preserved by the natural action of the group $G_N$ because the operator $F_{Ω}(M)$ commutes with this action by definition. Thus $W_{Ω}(M)$ can be regarded as a representation of the group $G_N$.

**Proposition.** Representations $W_{λ}(µ)$ and $W_{Ω}(M)$ of $G_N$ are equivalent.

The proof is given in Subsection 5.6. Due to Theorem 1.4, the image $W_{Ω}(M)$ of the operator $F_{Ω}(M)$ is contained in the subspace $V_{Ω} \subset (\mathbb{C}^N)^{⊗n}$. If $M = 0$, then we have the equality

$$W_{Ω}(0) = V_{Ω} \cap (\mathbb{C}^N)^{⊗n} \quad \text{for} \quad µ = (0,0,\ldots);$$ (1.12)

see Proposition 3.3. For general $M$ and $µ$, the image $W_{Ω}(M)$ of the operator $F_{Ω}(M)$ may differ from the intersection $V_{Ω} \cap (\mathbb{C}^N)^{⊗n}$. Still our proof of Proposition 1.4 is based on the equality (1.12). Even when $M = 0$ and $µ = (0,0,\ldots)$, our formulas for the operator $F_{Ω}(0)$ are new. The operator $F_{Ω}(0)$ can be regarded as an analogue of the Young symmetrizer [Y1] for the classical groups $O_N$ and $Sp_N$ instead of $GL_N$, see Subsection 3.3 below.

This provides a solution to a problem formulated by Weyl, see [W, p. 149].

If $µ \neq (0,0,\ldots)$, the operator $F_{Ω}(M)$ can also be defined via (3.26) and (3.35); see (2.12) and (2.13). Our definition of the operator $F_{Ω}(M)$ is motivated by the representation theory of Yangians [MNO], see below.

1.5. By definition, the vector space $V_{λ}(µ)$ is irreducible under the natural action of the subalgebra of $GL_M$-invariants in the universal enveloping algebra $U(\mathfrak{g}l_{N+M})$. We denote this subalgebra by $A_N(M)$; it coincides with the centralizer of the subalgebra $U(\mathfrak{g}l_M) \subset U(\mathfrak{g}l_{N+M})$. In Section 4 we describe the action of the algebra $A_N(M)$ on $V_{λ}(µ)$ explicitly, by using the Yangian $Y(\mathfrak{g}l_N)$ of the general linear Lie algebra $\mathfrak{g}l_N$. The Yangian $Y(\mathfrak{g}l_N)$ is
a deformation of the universal enveloping algebra of the polynomial current Lie algebra $\mathfrak{gl}_N[x]$ in the class of Hopf algebras, see [D] for instance.

The unital associative algebra $Y(\mathfrak{gl}_N)$ has a family of generators $T_{ij}^{(a)}$ where $a = 1, 2, \ldots$ and $i,j = 1, \ldots, N$. The defining relations for these generators can be written in terms of the formal power series

$$T_{ij}(x) = \delta_{ij} \cdot 1 + T^{(1)}_{ij} x^{-1} + T^{(2)}_{ij} x^{-2} + \ldots \in Y(\mathfrak{gl}_N)[[x^{-1}]]. \quad (1.13)$$

Here $x$ is the formal parameter. Let $y$ be another formal parameter. Then the defining relations in the associative algebra $Y(\mathfrak{gl}_N)$ can be written as

$$(x - y) \cdot [T_{ij}(x), T_{kl}(y)] = T_{kj}(x)T_{il}(y) - T_{kj}(y)T_{il}(x), \quad (1.14)$$

where $i,j,k,l = 1, \ldots, N$. The square brackets in (1.14) denote the usual commutator. If $N = 1$, the algebra $Y(\mathfrak{gl}_N)$ is commutative. Using the series (1.13), the coproduct $\Delta : Y(\mathfrak{gl}_N) \to Y(\mathfrak{gl}_N) \otimes Y(\mathfrak{gl}_N)$ is defined by

$$\Delta(T_{ij}(x)) = \sum_{k=1}^{N} T_{ik}(x) \otimes T_{kj}(x); \quad (1.15)$$

the tensor product at the right-hand side of the equality (1.15) is taken over the subalgebra $\mathbb{C}[[x^{-1}]] \subset Y(\mathfrak{gl}_N)[[x^{-1}]]$. The counit homomorphism $\varepsilon : Y(\mathfrak{gl}_N) \to \mathbb{C}$ is determined by the assignment $\varepsilon : T_{ij}(x) \mapsto \delta_{ij} \cdot 1$.

The antipode $S$ on $Y(\mathfrak{gl}_N)$ can be defined by using the element

$$T(x) = \sum_{i,j=1}^{N} E_{ij} \otimes T_{ij}(x) \in \text{End}(\mathbb{C}^N) \otimes Y(\mathfrak{gl}_N)[[x^{-1}]], \quad (1.16)$$

where the matrix units $E_{ij}$ are regarded as basis elements of the algebra $\text{End}(\mathbb{C}^N)$. The formal power series (1.16) in $x^{-1}$ is invertible, because its leading term is 1. The anti-automorphism $S$ is defined by the assignment $\text{id} \otimes S : T(x) \mapsto T(x)^{-1}$.

We also use the involutive automorphism $\xi_N$ of the algebra $Y(\mathfrak{gl}_N)$ defined by the assignment $\text{id} \otimes \xi_N : T(x) \mapsto T(-x)^{-1}. \quad (1.17)$

For references and more details on the definition of the Yangian $Y(\mathfrak{gl}_N)$ see [MNO, Section 1]. Some of these details are also given in Section 4 below.

The defining relations (1.14) show that for any $z \in \mathbb{C}$, the assignment

$$\tau_z : T_{ij}(x) \mapsto T_{ij}(x - z) \quad \text{for all} \quad i,j = 1, \ldots, N \quad (1.18)$$

defines an automorphism $\tau_z$ of the algebra $Y(\mathfrak{gl}_N)$. Here the formal power series $T_{ij}(x - z)$ in $(x - z)^{-1}$ should be re-expanded in $x^{-1}$. Regard the
matrix units $E_{ij} \in \mathfrak{gl}_N$ as generators of the universal enveloping algebra $U(\mathfrak{gl}_N)$. The relations (1.14) show that the assignment
\[
\alpha_N : T_{ij}(x) \mapsto \delta_{ij} \cdot 1 - E_{ji} x^{-1}
\] defines a homomorphism $\alpha_N : Y(\mathfrak{gl}_N) \to U(\mathfrak{gl}_N)$ of associative algebras. By definition, the homomorphism $\alpha_N$ is surjective.

By pulling the standard action of the algebra $U(\mathfrak{gl}_N)$ on the space $\mathbb{C}^N$ back through the composition of the homomorphisms $\alpha_N \circ \tau_z : Y(\mathfrak{gl}_N) \to U(\mathfrak{gl}_N)$, we obtain a module over the algebra $Y(\mathfrak{gl}_N)$, called an evaluation module. To indicate the dependence on the parameter $z$, let us denote this $Y(\mathfrak{gl}_N)$-module by $V_\lambda(z)$. The operator $E_\Omega$ on the vector space $(\mathbb{C}^N)^\otimes n$ admits the following interpretation in terms of the tensor product of evaluation modules over the Hopf algebra $Y(\mathfrak{gl}_N)$. Let $P_0$ be the linear operator on $(\mathbb{C}^N)^\otimes n$ reversing the order of the tensor factors. This operator corresponds to the element of the maximal length in the symmetric group $S_n$.

Proposition. The operator $E_\Omega P_0$ is an intertwiner of the $Y(\mathfrak{gl}_N)$-modules
\[
V(c_1(\Omega)) \otimes \ldots \otimes V(c_n(\Omega)) \to V(c_1(\Omega)) \otimes \ldots \otimes V(c_n(\Omega)).
\]

By Proposition 1.5, the image $V_\lambda$ of the operator $E_\Omega$ is a submodule in the tensor product of evaluation $Y(\mathfrak{gl}_N)$-modules $V(c_1(\Omega))\otimes\ldots\otimes V(c_n(\Omega))$. This interpretation of $E_\Omega$ and $V_\lambda$ is due to Cherednik [C3]. We obtain Proposition 1.5 as a particular case of Proposition 4.2.

If $M = 0$ and $\mu = (0,0,\ldots)$, the image of the operator $E_\Omega$ on $(\mathbb{C}^N)^\otimes n$ is equivalent to $V_\lambda$ as a representation of the group $GL_N$. Proposition 1.5 then turns $V_\lambda$ into $Y(\mathfrak{gl}_N)$-module. The resulting $Y(\mathfrak{gl}_N)$-module can also be obtained from the irreducible $\mathfrak{gl}_N$-module $V_\lambda$ by pulling back through the homomorphism $\alpha_N$; this is a particular case of Theorem 1.6 below.

1.6. Olshanski [O1] has defined a homomorphism from the algebra $Y(\mathfrak{gl}_N)$ to the subalgebra $A_N(M)$ of $GL_M$-invariants in $U(\mathfrak{gl}_{N+M})$, for each non-negative integer $M$. Along with the centre of the algebra $U(\mathfrak{gl}_{N+M})$, the image of this homomorphism generates the algebra $A_N(M)$. We will use the following version of this homomorphism, to be denoted by $\alpha_{NM}$.

Let the indices $i,j$ range over the set $\{1,\ldots,N+M\}$. Consider the basis of the matrix units $E_{ij}$ in the Lie algebra $\mathfrak{gl}_{N+M}$. We assume that the subalgebras $\mathfrak{gl}_N$ and $\mathfrak{gl}_M$ in $\mathfrak{gl}_{N+M}$ are spanned by elements $E_{ij}$ where
\[
1 \leq i,j \leq N \quad \text{and} \quad N + 1 \leq i,j \leq N + M,
\]
respectively. The subalgebra in the Yangian $Y(\mathfrak{gl}_{N+M})$ generated by $T_{ij}^{(a)}$ where $1 \leq i,j \leq N$, by definition coincides with the Yangian $Y(\mathfrak{gl}_N)$. Let us
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denote by $\varphi_M$ this natural embedding $Y(\mathfrak{gl}_N) \to Y(\mathfrak{gl}_{N+M})$. Consider also the involutive automorphism $\xi_{N+M}$ of the algebra $Y(\mathfrak{gl}_{N+M})$, see (1.17). The image of the homomorphism

$$\alpha_{N+M} \circ \xi_{N+M} \circ \varphi_M : Y(\mathfrak{gl}_N) \to U(\mathfrak{gl}_{N+M})$$

belongs to the subalgebra $A_N(M) \subset U(\mathfrak{gl}_{N+M})$. Moreover, this image along with the centre of the algebra $U(\mathfrak{gl}_{N+M})$, generates the subalgebra $A_N(M)$. For the detailed proofs of these two assertions, see [MO, Section 2]. In the present article, we use the homomorphism $Y(\mathfrak{gl}_N) \to U(\mathfrak{gl}_{N+M})$

$$\alpha_{N+M} = \alpha_{N+M} \circ \xi_{N+M} \circ \varphi_M \circ \xi_N.$$  \hspace{1cm} (1.22)

Note that when $M = 0$, the homomorphism (1.22) coincides with $\alpha_N$.

Take any formal power series $g(x) \in \mathbb{C}[[x^{-1}]]$ with the leading term 1. The assignment

$$T_{ij}(x) \mapsto g(x) \cdot T_{ij}(x)$$ \hspace{1cm} (1.23)

defines an automorphism of the algebra $Y(\mathfrak{gl}_N)$, see (1.13) and (1.14). Put

$$g_\mu(x) = \prod_{k \geq 1} \frac{(x - \mu_k + k)(x + k - 1)}{(x - \mu_k + k - 1)(x + k)}.$$ \hspace{1cm} (1.24)

In the product (1.24) over $k$, only finitely many factors differ from 1. So $g_\mu(x)$ is a rational function of $x$. We have $g_\mu(\infty) = 1$; therefore $g_\mu(x)$ expands as a power series in $x^{-1}$ with the leading term 1.

As in Subsection 1.1, suppose that $\lambda'_1 \leq N + M$ and $\mu'_1 \leq M$. The space $V_\mu(\mathfrak{gl}_N)$ comes with a natural action of the algebra $A_N(M)$. Regard $V_\mu(\mathfrak{gl}_N)$ as a module over the algebra $Y(\mathfrak{gl}_N)$, by using the composition of the homomorphism $\alpha_{N+M} : Y(\mathfrak{gl}_N) \to A_N(M)$ with the automorphism of $Y(\mathfrak{gl}_N)$ defined by (1.23), where $g(x) = g_\mu(x)$. By Proposition 1.5, the image $V_\Omega$ of the operator $E_\Omega$ can also be regarded as an $Y(\mathfrak{gl}_N)$-module.

**Theorem.** The $Y(\mathfrak{gl}_N)$-modules $V_\lambda(\mu)$ and $V_\Omega$ are equivalent.

This result goes back to [C3, Theorem 2.6]. The proof of Theorem 1.6 is given in Subsections 4.4 to 4.6 of the present article. The algebra $A_N(M)$ acts on $V_\lambda(\mu)$ irreducibly; the central elements of $U(\mathfrak{gl}_{N+M})$ act on $V_\lambda(\mu)$ as scalar operators. So Theorem 1.6 has a corollary, see [NT2, Section 4].

**Corollary.** The $Y(\mathfrak{gl}_N)$-module $V_\Omega$ is irreducible.

The Yangian $Y(\mathfrak{gl}_N)$ contains the universal enveloping algebra $U(\mathfrak{gl}_N)$ as a subalgebra. The embedding $U(\mathfrak{gl}_N) \to Y(\mathfrak{gl}_N)$ can be defined by the assignment

$$E_{ij} \mapsto -T^{(1)}_{ji}.$$ \hspace{1cm} (1.25)
This embedding provides an action of \( U(\mathfrak{g} N) \) on the \( Y(\mathfrak{g} N) \)-module \( V_\lambda(\mu) \). On the other hand, the vector space \( V_\lambda(\mu) \) comes with a natural action of \( U(\mathfrak{g} N) \) as a subalgebra of \( U(\mathfrak{g} N + M) \). This natural action of \( U(\mathfrak{g} N) \) on \( V_\lambda(\mu) \) coincides with its action as a subalgebra in \( Y(\mathfrak{g} N) \), see Subsection 4.3.

The subspace \( V_\Omega \subset \left( \mathbb{C}^N \right)^{\otimes n} \) is preserved by the standard action of the Lie algebra \( \mathfrak{g} N \) on \( \left( \mathbb{C}^N \right)^{\otimes n} \), because this subspace is the image of the operator \( E_\Omega \) corresponding to an element of the symmetric group ring \( \mathbb{C} S_n \). Hence \( U(\mathfrak{g} N) \) acts naturally on the vector space \( V_\Omega \) as well. This natural action of \( U(\mathfrak{g} N) \) on \( V_\Omega \) coincides with its action as a subalgebra in \( Y(\mathfrak{g} N) \), see again Subsection 4.3.

Note that the natural action of the Lie algebra \( \mathfrak{g} N \) on the vector space \( V_\lambda(\mu) \) may be reducible. Using Theorem 1.6 and its Corollary 1.6, we can identify the vector space \( V_\lambda(\mu) \) with the subspace \( V_\Omega \) in \( \left( \mathbb{C}^N \right)^{\otimes n} \) uniquely, up to multiplication in \( V_\Omega \) by a non-zero complex number. Theorem 1.6 can be regarded as sharpening of Proposition 1.1. Moreover, we will obtain Proposition 1.1 in the course of the proof of Theorem 1.6. In the proof of Theorem 1.6, we will use Proposition 2.4.

1.7. Let us now consider the universal enveloping algebra \( U(\mathfrak{g} N + M) \). Denote by \( B_N(M) \) the subalgebra of \( G_M \)-invariants in \( U(\mathfrak{g} N + M) \). Then \( B_N(M) \) contains the subalgebra \( U(\mathfrak{g} N) \subset U(\mathfrak{g} N + M) \). In the case \( \mathfrak{g} N = \mathfrak{sp} N \), \( B_N(M) \) coincides with the centralizer of the subalgebra \( U(\mathfrak{sp} M) \subset U(\mathfrak{sp} N + M) \). In the case \( \mathfrak{g} N = \mathfrak{so} N \), \( B_N(M) \) is contained in the centralizer of the subalgebra \( U(\mathfrak{so} M) \subset U(\mathfrak{so} N + M) \), but may not coincide with the centralizer.

In the case \( G_N = \mathfrak{sp} N \), the vector space \( W_\lambda(\mu) \) is irreducible under the action of the algebra \( B_N(M) \). In the case \( G_N = O_N \), the \( B_N(M) \)-module \( W_\lambda(\mu) \) is either irreducible, or splits as a direct sum of two irreducible \( B_N(M) \)-modules. It is irreducible if \( W_\lambda \) is irreducible as a \( \mathfrak{so} N + M \)-module, that is, if \( 2\lambda' \neq N + M \). But the condition \( 2\lambda' \neq N + M \) is not necessary for the irreducibility of the \( B_N(M) \)-module \( W_\lambda(\mu) \) in the case \( G_N = O_N \); see [P, Proposition 10.1]. In any case, \( W_\lambda(\mu) \) is irreducible under the joint action of the algebra \( B_N(M) \) and the subgroup \( G_N \subset G_{N+M} \).

In Section 5 we explicitly describe the action of \( B_N(M) \) in \( W_\lambda(\mu) \), by using the twisted Yangian \( Y(\mathfrak{g} N, \sigma) \). Here \( \sigma \) is the involutive automorphism of the Lie algebra \( \mathfrak{g} N \), such that \( -\sigma \) is the operator conjugation with respect to the bilinear form \( \langle \cdot, \cdot \rangle \) on \( \mathbb{C}^N \). Then \( \mathfrak{g} N \subset \mathfrak{g} N \) is the subalgebra of \( \sigma \)-fixed points. The associative algebra \( Y(\mathfrak{g} N, \sigma) \) is a deformation of the universal enveloping algebra of the twisted polynomial current Lie algebra

\[ \{ A(x) \in \mathfrak{g} N [x] : \sigma(A(x)) = A(-x) \} \].

The deformation \( Y(\mathfrak{g} N, \sigma) \) is not a Hopf algebra, but a coideal subalgebra in the Hopf algebra \( Y(\mathfrak{g} N) \). The definition of the twisted Yangian \( Y(\mathfrak{g} N, \sigma) \) was motivated by the work of Sklyanin [S] on quantum integrable systems.
with boundary conditions. This definition was given by Olshanski [O2] with an assistance from the author of the present article, see [MO, pp. 273–274].

As in Subsection 1.5, let the indices $i, j$ range over the set $\{1, \ldots, N\}$. The subalgebra $Y(\mathfrak{gl}_N, \sigma)$ of the associative algebra $Y(\mathfrak{gl}_N)$ is defined in terms of the generating series (1.13) as follows. Let $\tilde{T}(x)$ be the element of the algebra $\text{End}(\mathbb{C}^N) \otimes Y(\mathfrak{gl}_N)[[x^{-1}]]$, obtained by applying to $T(x)$ the conjugation with respect to $\langle , \rangle$ in the first tensor factor, and by changing $x$ to $-x$. Then consider the element

$$
\tilde{T}(x) T(x) \in \text{End}(\mathbb{C}^N) \otimes Y(\mathfrak{gl}_N)[[x^{-1}]].
$$

(1.26)

The subalgebra $Y(\mathfrak{gl}_N, \sigma)$ in $Y(\mathfrak{gl}_N)$ is generated by the coefficients of all the formal power series from $Y(\mathfrak{gl}_N)[[x^{-1}]]$, appearing in the expansion of the element (1.26) relative to the basis of matrix units $E_{ij}$ in $\text{End}(\mathbb{C}^N)$.

To give the defining relations for these generators of $Y(\mathfrak{gl}_N, \sigma)$, let us introduce the extended twisted Yangian $X(\mathfrak{gl}_N, \sigma)$. The unital associative algebra $X(\mathfrak{gl}_N, \sigma)$ has a family of generators $S_{ij}^{(a)}$ where $a = 1, 2, \ldots$. Put

$$
S_{ij}(x) = \delta_{ij} \cdot 1 + S_{ij}^{(1)} x^{-1} + S_{ij}^{(2)} x^{-2} + \ldots \in X(\mathfrak{gl}_N, \sigma)[[x^{-1}]].
$$

(1.27)

The defining relations for these generators are given in Subsection 5.1, using

$$
S(x) = \sum_{i,j=1}^N E_{ij} \otimes S_{ij}(x) \in \text{End}(\mathbb{C}^N) \otimes X(\mathfrak{gl}_N, \sigma)[[x^{-1}]].
$$

(1.28)

One can define a homomorphism $\pi_N : X(\mathfrak{gl}_N, \sigma) \to Y(\mathfrak{gl}_N, \sigma)$ by assigning

$$
id \otimes \pi_N : S(x) \mapsto \tilde{T}(x) T(x).
$$

(1.29)

By definition, the homomorphism $\pi_N$ is surjective. Further, the algebra $X(\mathfrak{gl}_N, \sigma)$ has a distinguished family of central elements $D^{(1)}, D^{(2)}, \ldots$. These elements of $X(\mathfrak{gl}_N, \sigma)$ are defined in Subsection 5.1, using the series

$$
D(x) = 1 + D^{(1)} x^{-1} + D^{(2)} x^{-2} + \ldots \in X(\mathfrak{gl}_N, \sigma)[[x^{-1}]].
$$

(1.30)

By [MNO, Theorem 6.4] the kernel of the homomorphism $\pi_N$ coincides with the (two-sided) ideal generated by the central elements $D^{(1)}, D^{(2)}, \ldots$.

Thus the algebra $Y(\mathfrak{gl}_N, \sigma)$ is defined by the generators $S_{ij}^{(a)}$ satisfying the relation $D(x) = 1$ and the reflection equation (5.1). This terminology has been used by physicists; see [MNO, Section 3] for the references, and for more details on the definition of the algebra $Y(\mathfrak{gl}_N, \sigma)$. In the present article we need the algebra $X(\mathfrak{gl}_N, \sigma)$, which is determined by (5.1) alone, because
this algebra admits an analogue of the automorphism $\xi_N$ of $Y(\mathfrak{gl}_N)$. By [MNO, Proposition 6.5] the assignment
\[
\text{id} \otimes \eta_N : S(x) \mapsto S(-x - \frac{N}{2})^{-1}
\] (1.31)
defines an involutive automorphism $\eta_N$ of the algebra $X(\mathfrak{gl}_N, \sigma)$. However, $\eta_N$ does not determine an automorphism of the algebra $Y(\mathfrak{gl}_N, \sigma)$, because the map $\eta_N$ does not preserve the ideal of $X(\mathfrak{gl}_N, \sigma)$ generated by the central elements $D^{(1)}, D^{(2)}, \ldots$; see [MNO, Subsection 6.6].

Note that when $z \neq 0$, the automorphism $\tau_z$ of the algebra $Y(\mathfrak{gl}_N)$ does not preserve the subalgebra $Y(\mathfrak{gl}_N, \sigma) \subset Y(\mathfrak{gl}_N)$; see (1.18). There is no analogue of the automorphism $\tau_z$ for the algebra $X(\mathfrak{gl}_N, \sigma)$.

Let us now regard $E_{ij}$ as basis vectors of the Lie algebra $\mathfrak{gl}_N$. The defining relations (5.1) of the algebra $X(\mathfrak{gl}_N, \sigma)$ imply that the assignment
\[
\beta_N : S_{ij}(x) \mapsto \delta_{ij} \cdot 1 - \frac{E_{ji} + \sigma(E_{ji})}{x \pm \frac{1}{2}}
\] (1.32)
defines a homomorphism of associative algebras $\beta_N : X(\mathfrak{gl}_N, \sigma) \to U(\mathfrak{gl}_N)$; see [MNO, Proposition 3.11]. According to our general convention, here the upper sign in $\pm$ corresponds to the case $G_N = O_N$ while the lower sign corresponds to $G_N = S\mathbb{P}_N$. The homomorphism $\beta_N$ is surjective. Moreover, $\beta_N$ factors through $\pi_N$. Note that the homomorphism $Y(\mathfrak{gl}_N, \sigma) \to U(\mathfrak{gl}_N)$ corresponding to $\beta_N$ cannot be obtained from $\alpha_N : Y(\mathfrak{gl}_N) \to U(\mathfrak{gl}_N)$ by restriction to the subalgebra $Y(\mathfrak{gl}_N, \sigma)$, because the image of $Y(\mathfrak{gl}_N, \sigma)$ relative to $\alpha_N$ is not contained in the subalgebra $U(\mathfrak{gl}_N) \subset U(\mathfrak{gl}_N)$. The link between the homomorphisms $\alpha_N$ and $\beta_N$ was given by [N3, Proposition 2.4], see also [MN, Lemma 3.8] and Lemma 5.4 of the present article.

The formulas (1.15), (1.16) and the definition (1.26) imply that for any choice of the symmetric or alternating non-degenerate form $\langle \ , \ \rangle$ on $\mathbb{C}^N$, the subalgebra $Y(\mathfrak{gl}_N, \sigma)$ in $Y(\mathfrak{gl}_N)$ is also a right coideal:
\[
\Delta(Y(\mathfrak{gl}_N, \sigma)) \subset Y(\mathfrak{gl}_N, \sigma) \otimes Y(\mathfrak{gl}_N).
\] (1.33)

Although this fact is underlying for our results, it is not directly used in the present article. Using (1.33), for any $Y(\mathfrak{gl}_N, \sigma)$-module $W$ and any $Y(\mathfrak{gl}_N)$-module $V$, one turns the vector space $W \otimes V$ into a $Y(\mathfrak{gl}_N, \sigma)$-module again. In our case $W$ is going to be the trivial $Y(\mathfrak{gl}_N, \sigma)$-module $\mathbb{C}$ defined via the restriction to $Y(\mathfrak{gl}_N, \sigma)$ of the counit homomorphism $\varepsilon : Y(\mathfrak{gl}_N) \to \mathbb{C}$.

Let us extend $\sigma$ to an automorphism of the associative algebra $U(\mathfrak{gl}_N)$. For any $z \in \mathbb{C}$, define the twisted evaluation module $\tilde{V}(z)$ over the algebra $Y(\mathfrak{gl}_N)$ by pulling the standard action of the algebra $U(\mathfrak{gl}_N)$ on the vector space $\mathbb{C}^N$ back through the composition of homomorphisms
\[
\sigma \circ \alpha_N \circ \tau_{-z} : Y(\mathfrak{gl}_N) \to U(\mathfrak{gl}_N),
\] (1.34)
see (1.20). It follows from the definition (1.26) that the evaluation module \( V(z) \) and the twisted evaluation module \( \tilde{V}(z) \) over \( Y(\mathfrak{gl}_N) \) have the same restriction to \( Y(\mathfrak{gl}_N, \sigma) \subset Y(\mathfrak{gl}_N) \); see Subsection 5.2 for the explanation. The linear operator \( F_\Omega(M) \) on \((\mathbb{C}^N)^\otimes n\) has the following interpretation, in terms of the restrictions to \( Y(\mathfrak{gl}_N, \sigma) \) of tensor products of evaluation modules over the Hopf algebra \( Y(\mathfrak{gl}_N) \); see Proposition 1.5. Put

\[
d_k(\Omega) = c_k(\Omega) + \frac{M}{2} + \frac{1}{2} \quad \text{for each } k = 1, \ldots, n.
\]

**Proposition.** The operator \( F_\Omega(M) \) is an intertwiner of \( Y(\mathfrak{gl}_N, \sigma) \)-modules

\[
\tilde{V}(d_1(\Omega)) \otimes \ldots \otimes \tilde{V}(d_n(\Omega)) \rightarrow V(d_1(\Omega)) \otimes \ldots \otimes V(d_n(\Omega)).
\]

Therefore the image \( W_\Omega(M) \) of the operator \( F_\Omega(M) \) is a submodule in the restriction of the tensor product of evaluation \( Y(\mathfrak{gl}_N) \)-modules

\[
V(d_1(\Omega)) \otimes \ldots \otimes V(d_n(\Omega))
\]

to \( Y(\mathfrak{gl}_N, \sigma) \subset Y(\mathfrak{gl}_N) \). This interpretation is the source of the definition of \( F_\Omega(M) \) and \( W_\Omega(M) \). Proof of Proposition 1.7 is given in Subsection 5.3.

If \( \mu = (0,0,\ldots) \) and \( M = 0 \), the image of the operator \( F_\Omega(M) \) on \((\mathbb{C}^N)^\otimes n\) is equivalent to \( W_\lambda \) as a representation of the group \( G_N \); see (1.12). Proposition 1.7 turns \( W_\lambda \) into a \( Y(\mathfrak{gl}_N, \sigma) \)-module. This \( Y(\mathfrak{gl}_N, \sigma) \)-module can also be obtained from the \( g_N \)-module \( W_\lambda \) by pulling back through the homomorphism \( \beta_N \); this fact is a particular case of Theorem 1.8 below.

1.8. Extending the results of [O1] from \( \mathfrak{gl}_N \) to other classical Lie algebras, \( \mathfrak{so}_N \) and \( \mathfrak{sp}_N \), Olshanski [O2] defined a homomorphism from \( Y(\mathfrak{gl}_N, \sigma) \) to the subalgebra \( B_N(M) \) of \( G_M \)-invariants in \( U(\mathfrak{g}_N+M) \), for each non-negative integer \( M \). Along with the subalgebra of \( G_N+M \)-invariants in \( U(\mathfrak{g}_N+M) \), the image of this homomorphism generates the algebra \( B_N(M) \). We will use the following version of this homomorphism for the algebra \( X(\mathfrak{gl}_N, \sigma) \), to be denoted by \( \beta_{N,M} \).

Let the indices \( i,j \) range over \( \{1, \ldots, N+M\} \). In Subsection 1.6 we chose the basis of the matrix units \( E_{ij} \) in the Lie algebra \( \mathfrak{gl}_{N+M} \) so that the subalgebras \( \mathfrak{gl}_N \) and \( \mathfrak{gl}_M \) in \( \mathfrak{gl}_{N+M} \) are spanned by elements \( E_{ij} \), where the indices \( i,j \) satisfy (1.21). Now assume that \( \mathfrak{g}_N \subset \mathfrak{gl}_N \) and \( \mathfrak{g}_M \subset \mathfrak{gl}_M \).

Consider the extended twisted Yangian \( X(\mathfrak{gl}_{N+M}, \sigma) \), where \(-\sigma\) is the conjugation with respect to the form \( \langle , \rangle \) on \( \mathbb{C}^{N+M} \). By definition, the subalgebra in \( X(\mathfrak{gl}_{N+M}, \sigma) \) generated by those \( S^{(a)}_{ij} \) where \( 1 \leq i,j \leq N \), coincides with \( X(\mathfrak{g}_N, \sigma) \). Let us denote by \( \psi_M \) this natural embedding of the algebra \( X(\mathfrak{g}_N, \sigma) \) into \( X(\mathfrak{gl}_{N+M}, \sigma) \).

Consider also the involutive automorphism \( \eta_{N+M} \) of \( X(\mathfrak{gl}_{N+M}, \sigma) \), see (1.31). The image of the homomorphism

\[
\beta_{N+M} \circ \eta_{N+M} \circ \psi_M : X(\mathfrak{gl}_N, \sigma) \rightarrow U(\mathfrak{g}_{N+M})
\]

\[ (1.37) \]
belongs to the subalgebra $B_N(M) \subset U(g_{N+M})$. This image along with the subalgebra of $G_{N+M}$-invariants in $U(g_{N+M})$, generates the subalgebra $B_N(M)$. The proofs of these two assertions are contained in [MO, Section 4]. In the present article, we use the homomorphism $X(g_N, \sigma) \to U(g_{N+M})$

$$\beta_{NM} = \beta_{N+M} \circ \eta_{N+M} \circ \psi_M \circ \eta_N. \quad (1.38)$$

This is an analogue of the homomorphism (1.22). Note that when $M = 0$, the homomorphism (1.38) coincides with $\beta_N$.

For any formal power series $g(x) \in \mathbb{C}[\llbracket x^{-1} \rrbracket]$ with the leading term 1, the assignment $S_{ij}(x) \mapsto g(x) \cdot S_{ij}(x)$ (1.39) defines an automorphism of the algebra $X(g_N, \sigma)$, this follows from (1.27) and (5.1). Note that (1.39) determines an automorphism of the quotient $Y(g_N, \sigma)$ of $X(g_N, \sigma)$ if and only if $g(x) = g(-x)$; see Subsection 5.1.

As in Subsection 1.3, suppose that the partitions $\lambda$ and $\mu$ satisfy the conditions from [W] for the groups $G_{N+M}$ and $G_M$, respectively. Consider the vector space $W_\lambda(\mu)$ which comes with a natural action of the algebra $B_N(M)$. Regard $W_\lambda(\mu)$ as a module over the algebra $X(g_N, \sigma)$, using the composition of the homomorphism $\beta_{NM} : X(g_N, \sigma) \to U(g_{N+M})$ with the automorphism of the algebra $X(g_N, \sigma)$, defined by (1.39) where

$$g(x) = g_\mu(x - \frac{M}{2} \pm \frac{1}{2}); \quad (1.40)$$

see (1.24). By Proposition 1.7, the image $W_\Omega(M)$ of the operator $F_\Omega(M)$ of the operator $F_\Omega(M)$ can also be regarded as an $Y(g_N, \sigma)$-module.

**Theorem.** The action of the algebra $X(g_N, \sigma)$ on $W_\lambda(\mu)$ factors through the homomorphism $\pi_N : X(g_N, \sigma) \to Y(g_N, \sigma)$. The $Y(g_N, \sigma)$-modules $W_\lambda(\mu)$ and $W_\Omega(M)$ are equivalent.

Together with the explicit construction of the subspace $W_\Omega(M)$ in $(\mathbb{C}^N)^\otimes n$, this analogue of Theorem 1.6 is the main result of the present article. The proof of Theorem 1.8 is given in Subsections 5.4 to 5.6.

1.9. The twisted Yangian $Y(g_N, \sigma)$ contains the enveloping algebra $U(g_N)$ as a subalgebra. The embedding $U(g_N) \to Y(g_N, \sigma)$ can be defined by

$$E_{ij} + \sigma(E_{ij}) \mapsto -\pi_N(S_{ji}^{(1)}), \quad (1.41)$$

see [MNO, Proposition 3.12]. This embedding yields an action of $U(g_N)$ on the $Y(g_N, \sigma)$-module $W_\lambda(\mu)$, see the first statement of Theorem 1.8. On the other hand, the vector space $W_\lambda(\mu)$ comes with a natural action of the subgroup $G_N \subset G_{N+M}$. The corresponding action of $U(g_N)$ on $W_\lambda(\mu)$ coincides with its action as a subalgebra in $Y(g_N, \sigma)$, see Subsection 5.7.
The subspace $W_{\Omega}(M) \subset (\mathbb{C}^N)^{\otimes n}$ is preserved by the standard action of the group $G_N$ on $(\mathbb{C}^N)^{\otimes n}$. So $U(\mathfrak{g}_N)$ acts naturally on the vector space $W_{\Omega}(M)$ as well. This natural action of $U(\mathfrak{g}_N)$ on $W_{\Omega}(M)$ coincides with its action as a subalgebra in $Y(\mathfrak{gl}_N, \sigma)$, see again Subsection 5.7. Theorem 1.8 thus agrees with Proposition 1.4.

The $G_M$-invariant elements of $U(\mathfrak{g}_{N+M})$ act on the space $W_{\lambda}(\mu)$ as scalar operators. In the case $G_N = Sp_N$, the $Y(\mathfrak{gl}_N, \sigma)$-module $W_{\lambda}(\mu)$ is therefore irreducible. In the case $G_N = O_N$, $W_{\lambda}(\mu)$ is irreducible under the joint action of the algebra $Y(\mathfrak{gl}_N, \sigma)$ and of the group $G_N$. In our proof of Theorem 1.8 we construct a surjective linear operator $W_{\lambda}(\mu) \to \Omega(\sigma)$, which intertwines the actions of both $Y(\mathfrak{gl}_N, \sigma)$ and $G_N$. Hence our proof of Theorem 1.8 has a corollary, cf. Corollary 1.6.

**Corollary.** a) If $G_N = Sp_N$, the $Y(\mathfrak{gl}_N, \sigma)$-module $W_{\Omega}(M)$ is irreducible.

b) If $G_N = O_N$, $W_{\Omega}(M)$ is irreducible under action of $Y(\mathfrak{gl}_N, \sigma)$ and $O_N$.

In the case when $G_N = O_N$, the $Y(\mathfrak{gl}_N, \sigma)$-module $W_{\Omega}(M)$ is either irreducible, or splits as a direct sum of two irreducible submodules; see the beginning of Subsection 1.7 for more details.

Using Corollary 1.9, we can identify $W_{\lambda}(\mu)$ with the subspace $W_{\Omega}(M)$ in $(\mathbb{C}^N)^{\otimes n}$ uniquely, up to multiplication in $W_{\Omega}(M)$ by a non-zero complex number. This identification is compatible with the action of $G_N$. Thus we sharpen Proposition 1.4. Moreover, we will obtain Proposition 1.4 in the course of the proof of Theorem 1.8. In this proof, we will use Proposition 3.5.

By Theorem 1.4, $W_{\Omega}(M)$ is a vector subspace in the image $V_{\Omega}$ of the operator $E_{\Omega}$. As explained in Subsection 1.6, we can identify the vector space $V_{\lambda}(\mu)$ with $V_{\Omega}$ uniquely, up to multiplication in $V_{\Omega}$ by a non-zero complex number. Using the identification of $W_{\lambda}(\mu)$ with $W_{\Omega}(M)$ as above, we obtain a distinguished embedding of the vector space $W_{\lambda}(\mu)$ into $V_{\lambda}(\mu)$; see the inequality (1.6). This embedding is compatible with the action of the subgroup $G_N \subset GL_N$, and depends on the choice of standard tableau $\Omega$ of shape $\lambda/\mu$. This result supports a thesis of Cherednik [C3], that Yangians are “hidden symmetries” of the classical representation theory.

The subspace $V_{\Omega}$ in $(\mathbb{C}^N)^{\otimes n}$ can also be regarded as a submodule in the tensor product of evaluation $Y(\mathfrak{gl}_N)$-modules (1.36); this follows by setting $z = \frac{\lambda}{\mu} = \frac{1}{2}$ in Proposition 4.2. Denote this $Y(\mathfrak{gl}_N)$-submodule by $V_{\Omega}(M)$. By Proposition 1.7, then $W_{\Omega}(M)$ is a submodule in the restriction of $V_{\Omega}(M)$ to the subalgebra $Y(\mathfrak{gl}_N, \sigma) \subset Y(\mathfrak{gl}_N)$.

Let us now consider the case when $N = 2$ and $G_N = Sp_2$. Then the equality of dimensions in (1.6) is attained, see [P, Proposition 10.3]. In this case, the restriction of the $Y(\mathfrak{gl}_2)$-module $V_{\Omega}(M)$ to the subalgebra $Y(\mathfrak{gl}_2, \sigma) \subset Y(\mathfrak{gl}_2)$ is irreducible, and coincides with $W_{\Omega}(M)$. So $W_{\Omega}(M)$ is an irreducible $Y(\mathfrak{gl}_2)$-module in this case. By using Theorem 1.8 along with
[NT1, Corollary 2.13], one then derives [M1, Theorem 5.2] which describes $W_\lambda(\mu)$ as $\mathfrak{gl}_2$-module, and which has been pivotal for this work of Molev. In another special case when $N = 2$ and $G_N = O_2$, our results on the vector space $W_\lambda(\mu)$ are different from those implied by [M2, Theorem 3.2] and [M3, Theorem 2.3]: unlike Molev, in this case we work with the non-connected Lie group $O_{2+M}$ rather than with its Lie algebra $\mathfrak{so}_{2+M}$. In any case, the methods of this article are different from those of [M1,M2,M3].

2. Young symmetrizers

2.1. We begin this section with recalling several classical facts [Y1,Y2] about the irreducible representations of the symmetric group $S_l$ over the complex field $\mathbb{C}$. These representations are labeled by partitions of $l$. We will identify partitions with their Young diagrams. Denote by $U_\lambda$ the irreducible representation of $S_l$ corresponding to the partition $\lambda$. We will also regard representations of the group $S_l$ as modules over the group ring $\mathbb{C}S_l$. Fix the chain $S_1 \subset S_2 \subset \ldots \subset S_l$ of subgroups with the standard embeddings.

There is a decomposition of the space $U_\lambda$ into a direct sum of one-dimensional subspaces, labeled by the standard tableaux of shape $\lambda$. The one-dimensional subspace $U_\Lambda \subset U_\lambda$ corresponding to a standard tableau $\Lambda$ is defined as follows. For any $m \in \{1, \ldots, l-1\}$ take the tableau obtained from $\Lambda$ by removing the numbers $m+1, \ldots, l$. Let the Young diagram $\mu$ be the shape of the resulting tableau. Then the subspace $U_\Lambda$ is contained in an irreducible $\mathbb{C}S_m$-submodule of $U_\lambda$ corresponding to $\mu$. Any basis of $U_\lambda$ formed by vectors $u_\Lambda \in U_\Lambda$ is called a Young basis. Fix an $S_l$-invariant inner product $(\ , \ )$ on $U_\lambda$. All the subspaces $U_\Lambda \subset U_\lambda$ are then pairwise orthogonal. We choose the vectors $u_\Lambda \in U_\Lambda$ so that $(u_\Lambda, u_\Lambda) = 1$.

For any standard tableau $A$, denote by $S_A$ (respectively by $S'_A$) the subgroup in $S_l$ preserving the collections of numbers appearing in every row (every column) of the tableau $A$. Then introduce the elements of $\mathbb{C}S_l$

$$p_A = \sum_{s \in S_A} s \quad \text{and} \quad q_A = \sum_{s \in S'_A} s \cdot \text{sgn} s.$$  

The product $p_A q_A \in \mathbb{C}S_l$ is the Young symmetrizer corresponding to $A$. By [Y1] the normalized product $p_A q_A \cdot \dim U_\lambda / l!$ is a minimal idempotent in the group ring $\mathbb{C}S_l$. The left ideal in $\mathbb{C}S_l$ generated by the element $p_A q_A$ is equivalent to the representation $U_\lambda$, under the action of the group $S_l$ on this ideal via left multiplication.

For any standard tableau $A$ consider the diagonal matrix element of the representation $U_\lambda$ corresponding to the vector $u_A$,

$$e_A = \sum_{s \in S_l} (u_A, s \cdot u_A) s \in \mathbb{C}S_l.$$  

(2.1)
We have the equality
\[ e_A^2 = e_A \cdot l! / \dim U_\lambda. \] (2.2)

There is an explicit formula for the element \( e_A \) of the group ring \( \mathbb{C}S_l \). This formula is particularly simple when \( A \) is the row tableau \( A^r \), or the column tableau \( A^c \). Using the lemma from [W, Section IV.2], one can obtain

**Proposition.**

a) There are equalities of elements of \( \mathbb{C}S_l \)
\[ e_{A^r} = p_{A^r} q_{A^r} p_{A^r} / \lambda_1! \lambda_2! \ldots, \] (2.3)
\[ e_{A^c} = q_{A^c} p_{A^c} q_{A^c} / \lambda'_1! \lambda'_2! \ldots. \] (2.4)

b) There exist invertible elements \( p, q \in \mathbb{C}S_l \) such that
\[ p_{A^r} q_{A^r} p_{A^r} = p_{A^r} q_{A^r} p \quad \text{and} \quad q_{A^c} p_{A^c} q_{A^c} = q_{A^c} p_{A^c} q. \]

For any \( k = 1, \ldots, l-1 \) let \( s_k \in S_l \) be the transposition of \( k \) and \( k+1 \). An expression for the matrix element \( e_A \) with arbitrary \( A \) can be obtained from either of (2.3) and (2.4) by using the formulas [Y2] for the action of the generators \( s_1, \ldots, s_{l-1} \) of the group \( S_l \) on the vectors of the Young basis. Fix any standard tableau \( \Lambda \). For every \( k = 1, \ldots, l \) let \( c_k = c_k(\Lambda) \) be the content of the box occupied by \( k \) in \( \Lambda \). Consider the tableau \( s_k \Lambda \) obtained from \( \Lambda \) by exchanging the numbers \( k \) and \( k+1 \). The resulting tableau may be non-standard. This happens exactly when \( k \) and \( k+1 \) stand (next to each other) in the same row or column of \( \Lambda \). But \( c_k \neq c_{k+1} \) always; put \( h = (c_{k+1} - c_k)^{-1}. \)

So far the vector \( u_\Lambda \) has been determined up to a multiplier \( z \in \mathbb{C} \) with \( |z| = 1 \). Due to [Y2] all the vectors of the Young basis can be further normalized so that for any standard tableau \( \Lambda \) and \( k = 1, \ldots, l-1 \)
\[ s_k \cdot u_\Lambda = \begin{cases} hu_\Lambda + \sqrt{1-h^2} u_{s_k \Lambda} & \text{if } s_k \Lambda \text{ is standard;} \\ hu_\Lambda & \text{otherwise.} \end{cases} \] (2.5)

This normalization determines all the vectors of the Young basis up to a common multiplier \( z \in \mathbb{C} \) with \( |z| = 1 \). If the tableau \( s_k \Lambda \) is standard, then
\[ \sqrt{1-h^2} u_{s_k \Lambda} = (s_k - h) u_\Lambda \] (2.6)
due to (2.5). Then by the definition (2.1) we have the relation
\[ (1-h^2) e_{s_k \Lambda} = (s_k - h) e_A(s_k - h). \] (2.7)

For any standard tableau \( \Lambda \) there is a sequence of transpositions \( s_{k_1}, \ldots, s_{k_b} \) such that \( \Lambda = s_{k_b} \ldots s_{k_1} A^r \) and the tableaux \( s_{k_a} \ldots s_{k_1} A^r \) are standard for all \( a = 1, \ldots, b \); see for instance [N1, Section 2]. Using (2.7) repeatedly, one can derive from (2.3) an explicit formula for the element \( e_A \in S_l \). Another explicit formula for \( e_A \) can be derived in a similar way from (2.4).
2.2. There is an expression for the element $e_A \in \mathbb{C}S_l$ of another kind. This expression is obtained by the so-called fusion procedure [C2]. For every two distinct indices $i, j \in \{1, \ldots, l\}$ introduce the rational function of $x, y \in \mathbb{C}$

$$f_{ij}(x, y) = 1 - \frac{(ij)}{x - y},$$

valued in $\mathbb{C}S_l$; here $(ij) \in S_l$ is the transposition of $i$ and $j$. As direct calculation shows, these rational functions satisfy the relations

$$f_{ij}(x, y) f_{ik}(x, z) f_{jk}(y, z) = f_{jk}(y, z) f_{ik}(x, z) f_{ij}(x, y)$$

(2.9)

for pairwise distinct indices $i, j, k$. For pairwise distinct $i, j, m, n$ we also have

$$f_{ij}(x, y) f_{mn}(z, w) = f_{mn}(z, w) f_{ij}(x, y).$$

(2.10)

Now take $l$ complex variables $t_1, \ldots, t_l$. Order lexicographically the set of all pairs $(i, j)$ with $1 \leq i < j \leq l$. The ordered product over this set,

$$\prod_{1 \leq i < j \leq l} f_{ij}(c_i + t_i, c_j + t_j)$$

(2.11)

is a rational function of $t_1, \ldots, t_l$ with values in $\mathbb{C}S_l$. This function depends only on the differences $t_i - t_j$. Denote by $T_A$ the set of tuples $(t_1, \ldots, t_l)$ such that $t_i = t_j$ whenever the numbers $i$ and $j$ appear in the same row of the standard tableau $\Lambda$. Alternatively, we can choose $T_A$ to be the set of all tuples $(t_1, \ldots, t_l)$ such that $t_i = t_j$ whenever the numbers $i$ and $j$ appear in the same column of $\Lambda$. The following proposition goes back to [C2].

**Proposition.** Restriction to $T_A$ of the rational function (2.11) is regular at $t_1 = \ldots = t_l$. The value of this restriction at $t_1 = \ldots = t_l$ coincides with the element $e_A \in \mathbb{C}S_l$.

In its present form, Proposition 2.2 has been proved in [N2, Section 2]. The proof actually provides an explicit formula for the element $e_A \in \mathbb{C}S_l$, different from those obtained by using (2.3) or (2.4).

2.3. We need a generalization of Proposition 2.2 to standard tableaux of skew shapes [C2]. Take any $m \in \{0, \ldots, l - 1\}$. Let $\mathcal{T}$ be standard tableau obtained from $\Lambda$ by removing the boxes with the numbers $m + 1, \ldots, l$. Let $\mu$ be the shape of the tableau $\mathcal{T}$. Define a standard tableau $\Omega$ of the skew shape $\lambda/\mu$ by setting $\Omega(i, j) = \Lambda(i, j) - m$ for all $(i, j) \in \lambda/\mu$. Every standard tableau $\Omega$ of shape $\lambda/\mu$ is obtained from a certain $\Lambda$ in this way.

Put $n = l - m$. Denote by $\iota_m$ the embedding of the symmetric group $S_n$ into $S_l$ as a subgroup preserving the subset $\{m + 1, \ldots, l\}$; we extend
the map \( \iota_m \) to \( \mathbb{C} S_n \) by linearity. Denote by \( S_{mn} \) the subgroup \( S_m \times \iota_m(S_n) \) in \( S_l \). Introduce the linear map

\[
\theta_m : \mathbb{C} S_l \to \mathbb{C} S_{mn} : s \mapsto \begin{cases} 
s & \text{if } s \in S_{mn}; \\
0 & \text{otherwise}. \end{cases} \tag{2.12}
\]

By definition, the element \( e_A \in \mathbb{C} S_l \) is divisible on the left and on the right by \( e_T \in \mathbb{C} S_n \). Hence there exists an element \( e_{\Omega} \in \mathbb{C} S_n \) such that

\[
\theta_m(e_A) = e_T \cdot \iota_m(e_{\Omega}). \tag{2.13}
\]

The element \( e_{\Omega} \in \mathbb{C} S_n \) does not depend on the choice of standard tableau \( T \) of the shape \( \mu \), because the boxes with the numbers \( 1, \ldots, m \) have in \( A \) and \( T \) the same contents; see (2.7). The generalization of Proposition 2.2 from \( A \) to \( \Omega \) is based on the following simple observation.

**Lemma.** The image under the map \( \theta_m \) of the product (2.11) equals

\[
\prod_{i \leq j \leq m} f_{ij}(c_i + t_i, c_j + t_j) \cdot \prod_{m < i < j \leq l} f_{ij}(c_i + t_i, c_j + t_j). \tag{2.14}
\]

**Proof.** The pairs \((i, j)\) in the product (2.11) are ordered lexicographically:

\[
(1,2), \ldots,(1,l),(2,3), \ldots,(2,l), \ldots, \ldots, (l-1,l). \tag{2.15}
\]

Now expand the product (2.11) as a sum of the products of transpositions \((i_1,j_1) \ldots (i_d,j_d) = s\) with coefficients from the field \( \mathbb{C}(t_1, \ldots, t_l) \); the sum is taken over subsequences \((i_1,j_1), \ldots, (i_d,j_d)\) in the sequence (2.15). Let \((i_b,j_b)\) be the first pair in the subsequence such that \( i_b \leq m < j_b \), we suppose the pair exists. Let \((i_c,j_c)\) be the last pair in the subsequence such that \( i_c = i_b \). Note that \( j_c \geq j_b > m \), while for any pair \((i_a,j_a)\) with \( a < b \) we have \( i_a < j_a \leq m \). Hence \( s(i_b) = j_c > m \) for \( i_b \leq m \), and \( \theta_m(s) = 0 \).

Now consider the ordered product on the right-hand side of (2.14),

\[
\prod_{m < i < j \leq l} f_{ij}(c_i + t_i, c_j + t_j). \tag{2.16}
\]

**Corollary.** Restriction of (2.16) to \( T_A \) is regular at \( t_{m+1} = \ldots = t_l \). The value of this restriction at \( t_{m+1} = \ldots = t_l \) coincides with \( \iota_m(e_{\Omega}) \in \mathbb{C} S_l \).

**Proof.** Consider the restriction of the rational function (2.14) to \( T_A \). By Proposition 2.2 and Lemma 2.3, this restriction is regular at \( t_1 = \ldots = t_l \); the value of this restriction at \( t_1 = \ldots = t_l \) is \( \theta_m(e_A) \). By Proposition 2.2 applied to the tableau \( T \) instead of to \( A \), the value at \( t_1 = \ldots = t_m \) of the restriction of the product over \( 1 \leq i < j \leq m \) in (2.14) is \( e_T \). Therefore the value at \( t_{m+1} = \ldots = t_l \) of restriction to \( T_A \) of the product (2.16) is the element \( \iota_m(e_{\Omega}) \), determined by the relation (2.13). \( \square \)
The proof of Proposition 2.2 from [N2, Section 2] implies the following.

**Proposition.** The element $e_A$ is divisible on the left and right by $\iota_m(e_\Omega)$.

2.4. For any two standard tableaux $\Lambda$ and $\Lambda'$ of the same shape $\lambda$, consider the matrix element of the representation $U_\lambda$ corresponding to the pair of vectors $u_A$ and $u_{A'}$ of the Young basis,

$$e_{AA'} = \sum_{s \in S_l} (u_A, s \cdot u_{A'}) s \in \mathbb{C}S_l.$$  

If $\Lambda = \Lambda'$, then we have the equality $e_{AA'} = e_A$ by the definition (2.1). Now take any $m \in \{0, \ldots, l-1\}$. Put $n = l - m$ as in Subsection 2.3. Consider the element $\theta_m(e_{AA'}) \in S_{mn}$, see the definition (2.12). Let $\Upsilon$ and $\Upsilon'$ be the standard tableaux obtained by removing the boxes with the numbers $m + 1, \ldots, l$ from the tableaux $\Lambda$ and $\Lambda'$, respectively.

**Lemma.** We have $\theta_m(e_{AA'}) = 0$ unless $\Upsilon$ and $\Upsilon'$ are of the same shape.

**Proof.** By definition, the element $e_{AA'} \in \mathbb{C}S_l$ is divisible by $e_\Upsilon \in \mathbb{C}S_m$ on the left, and by $e_{\Upsilon'} \in \mathbb{C}S_m$ on the right. The image $\theta_m(e_{AA'}) \in \mathbb{C}S_{mn}$ inherits these two divisibility properties. If the tableaux $\Upsilon$ and $\Upsilon'$ are not of the same shape, the irreducible representations $U_\Upsilon$ and $U_{\Upsilon'}$ of $S_m$ are not equivalent, and $e_\Upsilon s e_{\Upsilon'} = 0$ for any $s \in S_m$. □

Consider the permutational action of the symmetric group $S_n$ on the tensor product $(\mathbb{C}^N)^\otimes n$. We denote by $E_\Omega$ the linear operator on $(\mathbb{C}^N)^\otimes n$ corresponding of the element $e_\Omega \in \mathbb{C}S_n$. In the notation of Subsection 1.2, we have $c_k(\Omega) = c_{m+k}$ for $k = 1, \ldots, n$. Moreover, when $(t_1, \ldots, t_l) \in T_A$ we can put $t_k(\Omega) = t_{m+k}$ for $k = 1, \ldots, n$. Then we obtain Theorem 1.2 as a reformulation of Corollary 2.3, see the definition (2.8).

Consider the image $V_\Omega$ of the operator $E_\Omega$. Recall that the standard tableau $\Omega$ is of shape $\lambda/\mu$. Take any non-negative integer $M$ such that $\lambda'_1 \leq N + M$ and $\mu'_1 \leq M$. Consider the vector space $V_\lambda(\mu)$ defined by (1.1).

**Proposition.** If $V_\lambda(\mu) \neq \{0\}$, then $V_\Omega \neq \{0\}$.

**Proof.** Take the operator $E_A$ on the vector space $(\mathbb{C}^{N+M})^\otimes l$, corresponding to the element $e_A \in \mathbb{C}S_l$. Realize the irreducible representation $V_\lambda$ of the group $GL_{N+M}$ from (1.1) as the image $V_A$ of $E_A$. Split $(\mathbb{C}^{N+M})^\otimes l$ into the direct sum of subspaces, obtained from the subspaces $(\mathbb{C}^M)^\otimes k \otimes (\mathbb{C}^N)^\otimes (l-k)$ by some permutations of the $l$ tensor factors; here $k = 0, \ldots, l$. Each of these subspaces is preserved by the action of the subgroup $GL_M \subset GL_{N+M}$. The vector space $V_A$ is the sum of the images of all these subspaces with respect to $E_A$, but only the images with $k = m$ may contribute to

$$V_\lambda(\mu) = \text{Hom}_{GL_M}(V_\mu, V_A).$$
Further, consider the projections of $V_A$ onto the direct summands of $(\mathbb{C}^{N+M}) \otimes I$ obtained from the subspaces $(\mathbb{C}^M)^{\otimes k} \otimes (\mathbb{C}^N)^{\otimes (l-k)}$ by some permutations of the $l$ tensor factors. Again, irreducible representations of $GL_M$ equivalent to $V_\mu$ may occur only in the projections with $k = m$. Let us denote by $I_m$ the projector onto the direct summand\

$$(\mathbb{C}^M)^{\otimes m} \otimes (\mathbb{C}^N)^{\otimes n} \subset (\mathbb{C}^{N+M}) \otimes I.$$ (2.17)\

Let $P_s$ be the operator on $(\mathbb{C}^{N+M}) \otimes I$ corresponding to permutation $s \in S_l$. Now suppose that $V_\lambda(\mu) \neq \{0\}$. By the above argument, then there exist permutations $s'$ and $s''$ in $S_l$ such that\

$$\text{Hom}_{GL}(V_\mu, I_m P_{s'} E_A P_{s''} \cdot (\mathbb{C}^M)^{\otimes m} \otimes (\mathbb{C}^N)^{\otimes n}) \neq \{0\}. \quad (2.18)$$\

The product $P_{s'} E_A P_{s''}$ in (2.18) can be written as a linear combination of the operators $E_{\lambda', \lambda''}$ on $(\mathbb{C}^{N+M}) \otimes I$, corresponding to the matrix elements $e_{\lambda', \lambda''} \in C S_l$. In this linear combination the coefficients are taken from $C$, while $\lambda'$ and $\lambda''$ range over all standard tableaux of shape $\lambda$. By (2.18), there exists at least one pair of tableaux $\lambda'$ and $\lambda''$ such that\

$$\text{Hom}_{GL}(V_\mu, I_m E_{\lambda', \lambda''} \cdot (\mathbb{C}^M)^{\otimes m} \otimes (\mathbb{C}^N)^{\otimes n}) \neq \{0\}. \quad (2.19)$$\

Restriction of the operator $I_m E_{\lambda', \lambda''}$ to the subspace (2.17) coincides with the restriction of the operator on $(\mathbb{C}^{N+M}) \otimes I$, corresponding to the element $\theta_m(e_{\lambda', \lambda''}) \in C S_m$. Consider the tableaux $T'$ and $T''$, obtained by removing the boxes with the numbers $m + 1, \ldots, l$ from the tableaux $\lambda'$ and $\lambda''$, respectively. Due to Lemma 2.4, the inequality (2.19) implies that $T'$ and $T''$ are of the same shape. But then $e_{\lambda', \lambda''} = e_{\lambda''} e$ for some invertible element $e \in C S_m$, see (2.6). Then\

$$E_{\lambda', \lambda''} \cdot (\mathbb{C}^M)^{\otimes m} \otimes (\mathbb{C}^N)^{\otimes n} = E_{\lambda''} \cdot (\mathbb{C}^M)^{\otimes m} \otimes (\mathbb{C}^N)^{\otimes n}.$$\

By applying the relation (2.13) to the tableau $\lambda'$ instead of $\lambda$, the inequality (2.19) now implies that the tableau $T'$ is of shape $\mu$. Moreover, the left-hand side of (2.19) then equals $V_{\Omega'}$ for some standard tableau $\Omega'$ of skew shape $\lambda/\mu$. By (2.7), the inequality $V_{\Omega'} \neq \{0\}$ implies that $V_{\Omega'} \neq \{0\}$. \hfill $\Box$

The space $V_\lambda(\mu)$ comes with an action of the subgroup $GL_N \subset GL_L$. The subspace $V_\Omega \subset (\mathbb{C}^N)^{\otimes n}$ is preserved by the action of the group $GL_N$. Let us now consider $V_\lambda(\mu)$ and $V_\Omega$ as representations of the group $GL_N$. In Subsection 4.6 we will prove that these representations are equivalent, as stated in Proposition 1.1. Note that the operator $E_{\Omega}$ on $(\mathbb{C}^N)^{\otimes n}$ does not depend on $M$. It is well known that the dimension of the vector space $V_\lambda(\mu)$ is the same for all integers $M$ such that $\lambda'_1 \leq N + M$ and $\mu'_1 \leq M$; see for instance [M, Section I.5].
2.5. In this subsection, we collect a few results which we need for the proof of Theorem 1.6. Let us keep fixed a standard tableau \( \Lambda \) of non-skew shape \( \lambda \). Here \( \lambda \) is a partition of \( l \). For every \( k = 1, \ldots, l \) we denote \( c_k = c_k(\Lambda) \).

Consider the rational functions (2.8) with pairwise distinct indices \( i, j \in \{1, \ldots, l+1\} \); these functions take values in the group ring \( \mathbb{C}S_{l+1} \). Take the element \( e_{\Lambda} \in \mathbb{C}S_l \) defined by (2.1). Consider the image of \( e_{\Lambda} \) under the embedding \( \iota_1 : \mathbb{C}S_l \to \mathbb{C}S_{l+1} \), see the beginning of Subsection 2.3. For the proof of the following proposition, see [N2, Section 2].

**Proposition.** We have equality of rational functions in \( x \), valued in \( \mathbb{C}S_{l+1} \)

\[
f_{12}(x,c_1) \cdots f_{1,l+1}(x,c_l) \cdot \iota_1(e_{\Lambda}) = \left( 1 - \sum_{k=1}^{l} \frac{(1k+1)}{x} \right) \cdot \iota_1(e_{\Lambda}).
\]

Now for each \( m = 0, \ldots, l-1 \) define a linear map \( \gamma_m : \mathbb{C}S_{l+1} \to \mathbb{C}S_{l+1} \) as follows. By definition, for any group element \( s \in S_{l+1} \)

\[
\gamma_m(s) = \begin{cases} s & \text{if } s(1) \neq 2, \ldots, m+1; \\ 0 & \text{otherwise}. \end{cases} \tag{2.20}
\]

**Lemma.** For any \( z_1, \ldots, z_l \in \mathbb{C} \) we have equality of rational functions in \( x \)

\[
\gamma_m(f_{1,l+1}(x,z_1) \cdots f_{12}(x,z_1)) = f_{1,l+1}(x,z_1) \cdots f_{1,m+2}(x,z_{m+1}).
\]

**Proof.** Let us expand the product \( f_{1,l+1}(x,z_1) \cdots f_{12}(x,z_1) \) as a sum of the products of transpositions \((1i_a)\cdots(1i_1)\) with coefficients from \( \mathbb{C}(x) \); here the sum is taken over all subsequences \( i_1, \ldots, i_a \) in the sequence \( 2, \ldots, l+1 \). By the definition (2.20) we have \( \gamma_m(1) = 1 \), while for \( a \geq 1 \)

\[
\gamma_m((1i_a)\cdots(1i_1)) = \begin{cases} (1i_a)\cdots(1i_1) & \text{if } i_1 > m+1; \\ 0 & \text{otherwise}. \quad \Box \end{cases}
\]

We will need a reformulation of this lemma. For each \( m = 0, \ldots, l-1 \) define a linear map \( \gamma'_m : \mathbb{C}S_{l+1} \to \mathbb{C}S_{l+1} \) as follows: for \( s \in S_{l+1} \),

\[
\gamma'_m(s) = \begin{cases} s & \text{if } s^{-1}(1) \neq 2, \ldots, m+1; \\ 0 & \text{otherwise}. \tag{2.21} \end{cases}
\]

**Corollary.** For \( z_1, \ldots, z_l \in \mathbb{C} \) we have equality of rational functions in \( x \)

\[
\gamma'_m(f_{12}(x,z_1) \cdots f_{1,l+1}(x,z_1)) = f_{1,m+2}(x,z_{m+1}) \cdots f_{1,l+1}(x,z_l).
\]

This result is derived from Lemma 2.5 by using the anti-automorphism of the group ring \( \mathbb{C}S_{l+1} \), such that \( s \mapsto s^{-1} \) for every group element \( s \).
3. Traceless tensors

3.1. For any positive integer $L$ choose a non-degenerate bilinear form $\langle \,, \rangle$ on the vector space $\mathbb{C}^L$, symmetric or alternating. In the latter case $L$ has to be even. For any positive integer $l$ consider the commutant of the image of the classical group $G_L = O_L$ or $G_L = Sp_L$ in the operator algebra $\text{End}((\mathbb{C}^L) \otimes l)$. This commutant is called the Brauer centralizer algebra, see [W, Section V.2]. We will denote this algebra by $C_l(L)$.

Consider the linear operator $Q(L)$ on $\mathbb{C}^L \otimes \mathbb{C}^L$, see (1.8). This operator commutes with the action of the group $G_L$ in $\mathbb{C}^L \otimes \mathbb{C}^L$. For any distinct indices $i,j \in \{1, \ldots, l\}$ let $Q_{ij}$ be the operator on $(\mathbb{C}^L) \otimes l$ acting as $Q(L)$ in the $i$th and $j$th tensor factors, and acting as the identity in the remaining $l-2$ tensor factors. Note that $Q_{ij} = Q_{ji}$. Here we also have

$$Q_{ij}^2 = L \cdot Q_{ij} \quad \text{for any distinct} \quad i,j \in \{1, \ldots, l\}. \quad (3.1)$$

Consider the image of the symmetric group ring $\mathbb{C}S_l$ in $\text{End}((\mathbb{C}^L) \otimes l)$. The algebra $C_l(L)$ is generated by this image and all the operators $Q_{ij}$ on $(\mathbb{C}^L) \otimes l$ with $i < j$. Let $P_{ij}$ be the operator on $(\mathbb{C}^L) \otimes l$ corresponding to transposition $(ij) \in S_l$. We have the equalities

$$Q_{ij} (1 \mp P_{ij}) = 0 \quad \text{for any distinct} \quad i,j \in \{1, \ldots, l\}. \quad (3.2)$$

According to our general convention, the upper sign in $\mp$ corresponds to the case $G_L = O_L$, while the lower sign corresponds to the case $G_L = Sp_L$.

For any two distinct indices $i,j \in \{1, \ldots, l\}$ consider the rational function of $x,y \in \mathbb{C}$

$$R_{ij}(x,y) = 1 - \frac{P_{ij}}{x - y} \quad (3.3)$$

which takes values in $\text{End}((\mathbb{C}^L) \otimes l)$; this is the Yang rational R-matrix. The function (3.3) corresponds to (2.8) under the action of the symmetric group $S_l$ on $(\mathbb{C}^L) \otimes l$. Due to (2.9) we have the relation

$$R_{ij}(x,y) R_{ik}(x,z) R_{jk}(y,z) R_{ik}(x,z) R_{ij}(x,y) = R_{jk}(y,z) R_{ik}(x,z) R_{ij}(x,y) \quad (3.4)$$

for any pairwise distinct indices $i,j,k$; it is called the Yang–Baxter relation. Note that

$$R_{ij}(x,y) R_{ji}(y,x) = 1 - \frac{1}{(x - y)^2}. \quad (3.5)$$

The operator $Q(L)$ on $\mathbb{C}^L \otimes \mathbb{C}^L$ can be obtained from the permutation operator on $\mathbb{C}^L \otimes \mathbb{C}^L$ by conjugation in any one of the two tensor factors relative to the bilinear form $\langle \,, \rangle$. For any $i \neq j$ introduce the functions

$$\tilde{R}_{ij}(x,y) = 1 + \frac{Q_{ij}}{x + y} \quad \text{and} \quad \tilde{R}_{ij}(x,y) = 1 - \frac{Q_{ij}}{x + y + L}; \quad (3.6)$$
then we have
\[ \tilde{R}_{ij}(x,y) \tilde{R}_{ij}(x,y) = 1. \] (3.7)

As \( Q_{ij} = Q_{ji} \), we also have the relations
\[ \tilde{R}_{ij}(x,y) = \tilde{R}_{ji}(y,x) \quad \text{and} \quad \tilde{R}_{ij}(x,y) = \tilde{R}_{ji}(y,x). \] (3.8)

By applying to the relation (3.4) the conjugation relative to the form \( \langle \ , \rangle \) in the \( i \)th tensor factor, and by changing \( x \) to \( -x \), we get the relation
\[ \tilde{R}_{ik}(x,z) \tilde{R}_{ij}(x,y) R_{jk}(y,z) = R_{jk}(y,z) \tilde{R}_{ik}(x,z) \tilde{R}_{ij}(x,y). \] (3.9)

By using (3.7), we obtain from (3.9) the relation
\[ R_{ij}(x,y) R_{ik}(x,z) R_{jk}(y,z) = R_{jk}(y,z) R_{ik}(x,z) \tilde{R}_{ij}(x,y). \] (3.10)

Using (3.7) and (3.8), we also obtain from (3.9) the relation
\[ \tilde{R}_{ij}(x,y) R_{ik}(x,z) R_{jk}(y,z) = R_{jk}(y,z) R_{ik}(x,z) \tilde{R}_{ij}(x,y). \] (3.11)

The relations (3.4),(3.5) and (3.7) to (3.11) are equalities of functions that take their values in the Brauer centralizer algebra \( C_l(L) \).

3.2. The irreducible modules over the algebra \( C_l(L) \), or equivalently, the irreducible representations of the group \( G_L \) appearing in the tensor product \( (C^L)^{\otimes l} \), are labeled by the partitions of \( l, l-2, \ldots \) satisfying the conditions from [W] for \( G_L \) as described in Subsection 1.3. In the present article, we consider only the irreducible \( C_l(L) \)-modules which are labeled by partitions of \( l \). The corresponding irreducible representations of the group \( G_L \) appear in the subspace
\[ (C^L)^{\otimes l}_0 \subset (C^L)^{\otimes l} \] (3.12)
of traceless tensors. The images of the groups \( S_l \) and \( G_L \) in \( \text{End}((C^L)^{\otimes l}_0) \) span the commutants of each other, see [W, Sections V.7 and VI.3] again.

By definition, all the operators \( Q_{ij} \) on \( (C^L)^{\otimes l}_0 \) vanish on the subspace (3.12). Denote by \( I_l(L) \) the two-sided ideal in \( C_l(L) \) generated by all the operators \( Q_{ij} \). The quotient algebra \( C_l(L)/I_l(L) \) can be identified with the image of the group ring \( \mathbb{C}S_l \) in the operator algebra \( \text{End}((C^L)^{\otimes l}_0) \). The image of the element \( P_{ij} \in C_l(L) \) in the quotient algebra is then identified with the operator on the subspace (3.12) corresponding to \( (ij) \in S_l \).

Take any partition \( \lambda \) of \( l \) satisfying the above mentioned conditions. The irreducible representation \( W_\lambda \subset (C^L)^{\otimes l}_0 \) of the group \( G_L \) corresponds to the irreducible representation \( U_\lambda \) of the group \( S_l \). We will regard \( U_\lambda \) as a \( C_l(L) \)-module using the canonical homomorphism
\[ C_l(L) \rightarrow C_l(L)/I_l(L). \] (3.13)
Then $U_{\Lambda}$ is the irreducible $Cl(L)$-module corresponding to the partition $\lambda$.

Fix any standard tableau $\Lambda$ of shape $\lambda$. As in Section 2, let $c_k = c_k(\Lambda)$ be the content of the box occupied by $k$ in $\Lambda$. Consider the product

$$\prod_{1 \leq i < j \leq l} R_{ij}(c_i + t_i, c_j + t_j) \cdot \prod_{1 \leq i < j \leq l} R_{ij}(c_i + t_i, c_j + t_j)$$

as a rational function of the complex variables $t_1, \ldots, t_l$ which takes values in the algebra $Cl(L)$. Here the pairs $(i, j)$ with $1 \leq i < j \leq l$ are ordered lexicographically, as usual. Denote by $T_{\Lambda}$ the set of tuples $(t_1, \ldots, t_l)$ such that $t_i = t_j$ whenever the numbers $i$ and $j$ appear in $\Lambda$ the same column for $GL = O_L$, or in the same row for $GL = Sp_L$. Denote by $E_{\Lambda}$ the linear operator on $(Cl(L))^\otimes l$ corresponding to the element $e_{\Lambda} \in Cl(L)$, see (2.1).

**Proposition.** Restriction to $T_{\Lambda}$ of the rational function (3.14) is regular at $t_1 = \ldots = t_l = \mp \frac{1}{2}$. The value $F_{\Lambda}$ of this restriction at $t_1 = \ldots = t_l = \mp \frac{1}{2}$ is divisible on the left and on the right by the operator $E_{\Lambda}$ on $(Cl(L))^\otimes l$.

**Proof.** Denote by $\omega$ the involutive anti-automorphism of the algebra $Cl(L)$, such that all the elements $P_{ij}$ and $Q_{ij}$ are $\omega$-invariant. The element $E_{\Lambda}$ of $Cl(L)$ corresponds to (2.1). Therefore $E_{\Lambda}$ is also $\omega$-invariant. Applying $\omega$ to the product (3.14) just reverses the ordering of the factors. The initial ordering can then be restored by using the relations (3.4) and (3.10). So any value of the function (3.14) is $\omega$-invariant. Thanks to this observation, it suffices to prove the divisibility of $F_{\Lambda}$ by $E_{\Lambda}$ only on the right.

Take $l$ complex variables $x_1, \ldots, x_l$. Choose any $k \in \{1, \ldots, l - 1\}$ and write $x_1', \ldots, x_l'$ for the sequence of variables obtained by exchanging the terms $x_k$ and $x_{k+1}$ in the sequence $x_1, \ldots, x_l$. Using (3.4), we obtain

$$P_{k,k+1} R_{k+1,k} (x_{k+1}, x_k) \cdot \prod_{1 \leq i < j \leq l} R_{ij}(x_i, x_j)$$

$$= \prod_{1 \leq i < j \leq l} R_{ij}(x'_i, x'_j) \cdot R_{k,k+1} (x_k, x_{k+1}) P_{k,k+1}. \quad (3.15)$$

Using (3.10) along with the second relation in (3.8), we obtain the equality

$$P_{k,k+1} R_{k+1,k} (x_{k+1}, x_k) \cdot \prod_{1 \leq i < j \leq l} \bar{R}_{ij}(x_i, x_j)$$

$$= \prod_{1 \leq i < j \leq l} \bar{R}_{ij}(x'_i, x'_j) \cdot R_{k+1,k} (x_{k+1}, x_k) P_{k,k+1}. \quad (3.16)$$
Now suppose that the index \( k \in \{1, \ldots, l - 1\} \) is chosen so that the tableau \( s_k \Lambda \) is standard. Then \( |c_k - c_{k+1}| > 1 \). Set \( x_i = c_i + t_i \) for each \( i = 1, \ldots, l \) in the two equalities (3.15) and (3.16). Then these two equalities show that when multiplying (3.14) on the left by
\[
P_{k,k+1}R_{k+1,k}(c_{k+1} + t_{k+1}, c_k + t_k),
\]
and dividing on the right by
\[
R_{k,k+1}(c_k + t_k, c_{k+1} + t_{k+1})P_{k,k+1},
\]
we get the analogue of (3.14) for the standard tableau \( s_k \Lambda \) instead of \( \Lambda \). As \( |c_k - c_{k+1}| > 1 \), the functions (3.17) are regular at \( t_k = t_{k+1} \). Moreover, the values of the functions (3.17) at \( t_k = t_{k+1} \) are invertible. By (2.7) and (3.5),
\[
P_{k,k+1}R_{k+1,k}(c_k + \frac{1}{2}, c_{k+1} + \frac{1}{2})E_{\Lambda} = E_{s_k \Lambda}R_{k,k+1}(c_k + \frac{1}{2}, c_{k+1} + \frac{1}{2})P_{k,k+1}.
\]
Hence it suffices to prove Proposition 3.2 for the tableau \( s_k \Lambda \) instead of \( \Lambda \).

First consider the case \( G_L = O_L \). Here we select the upper sign in \( \mp \). There is a sequence of transpositions \( s_{k_1}, \ldots, s_{k_b} \) such that \( s_{k_b} \ldots s_{k_1} \Lambda = \Lambda_c \) and the tableaux \( s_{k_a} \ldots s_{k_b} \Lambda^r \) are standard for all \( a = 1, \ldots, b \); see Subsection 2.1. Therefore it suffices to prove Proposition 3.2 only for \( \Lambda = \Lambda_c \). Then the set \( T_{\Lambda^r} \) consists of all tuples \((t_1, \ldots, t_l)\) such that \( t_i = t_j \) whenever \( i \) and \( j \) appear in the same column of \( \Lambda_c \).

Suppose that the numbers \( k \) and \( k+1 \) appear in the same column of \( \Lambda_c \). Using the relations (3.4) we can demonstrate that in (3.14), the second ordered product over \( 1 \leq i < j \leq l \) is divisible on the left by the function \( R_{k,k+1}(t_k + c_k, t_{k+1} + c_{k+1}) \). Restriction of this function to \( T_{\Lambda^r} \) equals \( 1 - P_{k,k+1} \). Now reorder the pairs \((i,j)\) in the first ordered product over \( 1 \leq i < j \leq l \) in (3.14) as follows. For every \( a = 1, \ldots, \lambda_1 \) first come all the pairs \((i,j)\) where both \( i \) and \( j \) occur in the \( a \)th column of the tableau \( \Lambda^r \). These pairs are ordered between themselves lexicographically. After them come all the pairs \((i,j)\) where \( i \) occurs in the \( a \)th column, while \( j \) occurs in the \( b \)th column, for all \( b > a \). These pairs are again ordered lexicographically. After these come the pairs \((i,j)\) where \( i \) occurs in the \( b \)th column, for some \( b > a \). This reordering involves only transpositions of commuting factors, so the value of the product (3.14) will not change.

Now take any \( a \in \{1, \ldots, \lambda_1\} \). Let \( c, c+1, \ldots, d \) be the consecutive numbers appearing in the \( a \)th column of \( \Lambda^r \). In the restriction to \( T_{\Lambda^r} \) of the reordered product (3.14), the product of all the factors succeeding
\[
\prod_{c \leq i < j \leq d} R_{ij}(c_i + t_i, c_j + t_j),
\]

\[ (3.19) \]
is divisible on the left by $1 - P_{k,k+1}$ whenever $c \leqslant k < d$. This follows from the relations (3.10). Moreover, then the product of the succeeding factors is divisible on the left by $1 - P_{ij}$ for $c \leqslant i < j \leqslant d$. Using the relations (3.2) with the upper sign in $\mp$, we now eliminate all the products (3.19) from the restriction of (3.14) to $T_{A^c}$, consecutively for $a = 1, \ldots, \lambda_1$. The restriction to $T_{A^c}$ of the product of the remaining factors in (3.14) is regular at $t_1 = \ldots = t_l = -\frac{1}{2}$. Thus we get the first statement of Proposition 3.2. Moreover, we get the explicit formula

$$F_{A^c} = \prod_{(i,j)} \left( 1 - \frac{Q_{ij}}{c_i(A^c) + c_j(A^c) + L - 1} \right) \cdot E_{A^c} \quad \text{for } G_L = O_L \quad (3.20)$$

where the ordered product is taken over all pairs $(i,j)$ such that $i$ and $j$ appear in different columns of $A^c$. For any such pair we have

$$c_i(A^c) + c_j(A^c) \geqslant 3 - \lambda'_1 - \lambda'_2 \geqslant 3 - L,$$

so that each of the denominators in (3.20) is non-zero. The operator $E_{A^c}$ on $(\mathbb{C}^L)^{\otimes l}$ corresponds to the element $e_{A^c} \in \mathbb{C}S_l$, and we used Proposition 2.2. The operator $F_{A^c}$ given by (3.20) is evidently divisible by $E_{A^c}$ on the right.

The proof of Proposition 3.2 in the case $G_L = Sp_L$ is much simpler. As we observed in Subsection 1.4, in this case every factor $R_{ij}(c_i + t_i, c_j + t_j)$ in the product (3.14) is regular at $t_1 = \ldots = t_l = \frac{1}{2}$, for any choice of the tableau $A$. Using Proposition 2.2, we get from (3.14) the explicit formula

$$F_A = \prod_{1 \leqslant i < j \leqslant l} \left( 1 - \frac{Q_{ij}}{c_i + c_j + L + 1} \right) \cdot E_A \quad \text{for } G_L = Sp_L. \quad (3.21)$$

Thus $F_A$ is divisible by $E_A$ on the right. Thanks to the observation made in the beginning of the proof, $F_A$ is also divisible by $E_A$ on the left. \qquad \square

Observe that in the case $G_L = O_L$, for any tableau $A$ and any two distinct numbers $i, j \in \{1, \ldots, l\}$ we have $c_i + c_j \geqslant 3 - L$ unless both $i$ and $j$ appear in the first column of $A$. Therefore the factor $R_{ij}(c_i + t_i, c_j + t_j)$ in the product (3.14) may have a pole at $t_1 = \ldots = t_l = -\frac{1}{2}$, only if both $i$ and $j$ appear in the first column of $A$. However, this observation does not facilitate significantly the proof of Proposition 3.2. Our proof also has a

**Corollary.** If the tableau $s_kA$ is standard for some $k \in \{1, \ldots, l-1\}$, then

$$P_{k,k+1}R_{k+1,k}(c_{k+1}, c_k) F_A = F_{s_kA} R_{k,k+1}(c_k, c_{k+1}) P_{k,k+1}. $$
In the case $G_L = O_L$, our proof of Proposition 3.2 also provides the formula (3.20) for the operator $F_{Λc}$. In the case $G_L = SpL$, there is a simplified formula for the operator $F_{Λr}$, similar to (3.20). We have

$$F_{Λr} = \prod_{(i,j)} \left( 1 - \frac{Q_{ij}}{c_i(A^r) + c_j(A^r) + L + 1} \right) \cdot E_{Λc} \quad \text{for} \quad G_L = SpL \quad (3.22)$$

where the ordered product is taken over all pairs $(i,j)$ such that $i$ and $j$ appear in different rows of $A^r$. The proof of the formula (3.22) is similar to that of (3.20). Here we work with the rows of the tableau $A^r$. But here from the very beginning of the proof we can set $t_1 = \ldots = t_l = \frac{1}{2}$ in the first product over $1 \leq i < j \leq l$ in (3.14) where $Λ = A^r$, at the same time replacing the second product in (3.14) by $E_{Λr}$. If $k$ and $k+1$ appear in the same row of $Λ$, due to (2.5) the operator $E_{Λr}$ is divisible on the left by

$$1 + P_{k,k+1} = R_{k,k+1}(c_k + \frac{1}{2}, c_{k+1} + \frac{1}{2}).$$

We omit other details of the proof of the formula (3.22).

3.3. For any standard tableau $Λ$, denote by $V_{Λ}$ and $W_{Λ}$ the images in the space $(\mathbb{C} L)^{\otimes l}$ of the operators $E_{Λ}$ and $F_{Λ}$, respectively. We have $W_{Λ} \subset V_{Λ}$ by the second statement of Proposition 3.2. Consider the subspace (3.12) of the traceless tensors. The next proposition is pivotal for the present article.

**Proposition.** We have the equality of vector spaces $W_{Λ} = V_{Λ} \cap (\mathbb{C} L)^{\otimes l}_0$.

**Proof.** For any distinct indices $i, j \in \{1, \ldots, l\}$ the operator $P_{ij}$ preserves the subspace (3.12) while the operator $Q_{ij}$ vanishes on this subspace. So the action of (3.14) on this subspace coincides with the action of the second product over $1 \leq i < j \leq l$ in (3.14). Hence by the definition of the operator $F_{Λ}$ we get the equality $F_{Λ} \cdot w = E_{Λ} \cdot w$ for any vector $w \in (\mathbb{C} L)^{\otimes l}_0$. Therefore

$$W_{Λ} \supset E_{Λ} \cdot (\mathbb{C} L)^{\otimes l}_0 = V_{Λ} \cap (\mathbb{C} L)^{\otimes l}_0.$$

We already noted that $W_{Λ} \subset V_{Λ}$. It remains to prove that $W_{Λ} \subset (\mathbb{C} L)^{\otimes l}_0$. Equivalently, we have to prove that $Q_{ij} F_{Λ} = 0$ for any distinct $i$ and $j$.

Firstly let us prove the equality $Q_{12} F_{Λ} = 0$ for every $Λ$. Consider the case $G_L = O_L$. In every standard tableau $Λ$, the number 1 always occupies the upper left corner. The number 2 appears in $Λ$ either next to the right of 1, or next down from 1. In the latter case, the operator $E_{Λ}$ is divisible on the left by $1 - P_{12}$. By Proposition 3.2, the operator $F_{Λ}$ is then also divisible on the left by $1 - P_{12}$. But $Q_{12}(1 - P_{12}) = 0$ for $G_L = O_L$ by (3.2).

Suppose that 2 appears in $Λ$ next to the right of 1. Then for any $j \neq 1$

$$c_1 + c_j \geq 1 - \lambda_1' \geq 1 + \lambda_2' - L \geq 2 - L.$$
So none of the factors $\bar{R}_{1j}(c_1 + t_1, c_j + t_j)$ in the product (3.14) has a pole at $t_1 = \ldots = t_l = -\frac{1}{2}$. Some of the factors $\bar{R}_{ij}(c_i + t_i, c_j + t_j)$ with $i \neq 1$ may have poles at $t_1 = \ldots = t_l = -\frac{1}{2}$. Then consider the standard tableau of shape $\lambda$, which is obtained by placing the numbers 1 and 2 in the first row as in $A$, and by filling the remaining boxes of the Young diagram $\lambda$ by columns. Let us denote this new tableau by $A'$. Arguing as in the proof of Proposition 3.2, we show that the equality $Q_{12} F_A = 0$ is sufficient to prove for $A = A'$. This argument involves multiplying the product (3.14) on the left by (3.17) and dividing on the right by (3.18), for some $k \in \{3, \ldots, l-1\}$ such that the tableau $s_k A$ is standard. Further, arguing as in the proof of Proposition 3.2, we show that $F_{A'}$ is divisible on the left by

$$\bar{R}_{12}(c_1 - \frac{1}{2}, c_2 - \frac{1}{2}) = 1 - Q_{12}/L.$$  

The equality $Q_{12} F_{A'} = 0$ now follows from (3.1).

In the case $G_L = Sp_L$, the proof of the equality $Q_{12} F_A = 0$ for any standard tableau $A$ is much simpler. In this case for any $A$, every factor $\bar{R}_{ij}(c_i + t_i, c_j + t_j)$ in the product (3.14) is regular at $t_1 = \ldots = t_l = \frac{1}{2}$. If the numbers 1 and 2 appear in the first row of $A$, then the operator $E_A$ is divisible on the left by $1 + P_{12}$. The operator $F_A$ is then also divisible on the left by $1 + P_{12}$. But for $G_L = Sp_L$ we have $Q_{12}(1 + P_{12}) = 0$. If 1 and 2 appear in the first column of $A$, then the operator $F_A$ is divisible on the left by

$$\bar{R}_{12}(c_1 + \frac{1}{2}, c_2 + \frac{1}{2}) = 1 - Q_{12}/L.$$  

The equality $Q_{12} F_A = 0$ now follows from (3.1).

Let us now prove the equality $Q_{ij} F_A = 0$ for any pair of distinct indices $i$ and $j$. If for some $k \in \{1, \ldots, l-1\}$ the tableau $s_k A$ is standard, then by Corollary 3.2

$$P_{k,k+1} F_A = \frac{F_A}{c_{k+1} - c_k} + F_{s_k A} R_{k,k+1}(c_k, c_{k+1}) P_{k,k+1}.$$  

If the tableau $s_k A$ is not standard, then $P_{k,k+1} F_A$ equals $F_A$ or $-F_A$; this equality follows from the second statement of Proposition 3.2 by (2.5).

For any permutation $s \in S_l$, let $P_s$ be the corresponding operator on the space $(C^L)^{\otimes l}$. For some elements $R_{A'}(s) \in C_l(L)$ that may also depend on $A$, we have

$$P_s F_A = \sum_{A'} F_{A'} R_{A'}(s),$$  

where $A'$ ranges over all standard tableaux of shape $\lambda$. Let us now choose the permutation $s$ so that $s(i) = 1$ and $s(j) = 2$. Then we get

$$Q_{ij} F_A = P_s^{-1} Q_{12} P_s F_A = \sum_{A'} P_s^{-1} Q_{12} F_{A'} R_{A'}(s) = 0. \quad \square$$
By definition, the associative algebra $C_l(L)$ is semisimple. Denote by $C_\lambda(L)$ the simple ideal of $C_l(L)$ corresponding to the irreducible $C_l(L)$-module $U_\lambda$. Proposition 3.3 gives the following characterization of the $F_A$.

**Corollary.** The operator $F_A \in C_l(L)$ is the unique element of the simple ideal $C_\lambda(L)$, with the image under (3.13) corresponding to $e_A \in CS_l$.

**Proof.** By definition, we have $F_A \in E_A + I_l(L)$; see Proposition 2.2. Hence the image of $F_A$ under the homomorphism (3.13) corresponds to the element $e_A \in CS_l$. By Proposition 3.3, we also have $F_A \in C_\lambda(L)$, see the beginning of Subsection 3.2. But any element of the ideal $C_\lambda(L) \subset C_l(L)$ is uniquely determined by the image of this element under (3.13). \qed

Note that due to (2.2), the above characterization of the operator $F_A$ implies the equality

$$F_A^2 = F_A \cdot l! / \dim U_\lambda. \quad (3.23)$$

### 3.4

We will now extend the results of Subsection 3.2 to standard tableaux of skew shapes. Take any $m \in \{0, \ldots, l-1\}$. As in Subsection 2.3, denote by $\Upsilon$ the standard tableau obtained from $\Lambda$ by removing the boxes with the numbers $m+1, \ldots, l$. Let $\mu$ be the shape of the tableau $\Upsilon$. Define a standard tableau $\Omega$ of the skew shape $\lambda/\mu$ by setting $\Omega(i,j) = \Lambda(i,j) - m$ for all $(i,j) \in \lambda/\mu$. Put $n = l - m$, the number of elements in the set $\lambda/\mu$.

Take any $M \in \{0, \ldots, L-1\}$ and put $N = L - M$. Then choose the decomposition $C^L = C^N \oplus C^M$ so that the subspaces $C^N$ and $C^M$ in $C^L$ are orthogonal relative to the form $\langle \ , \ \rangle$ on $C^L$. In the case $G_L = Sp_L$ both $N$ and $M$ have to be even. If $M > 0$, the restriction of the form $\langle \ , \ \rangle$ from $C^L$ to $C^M$ is non-degenerate. Consider the corresponding subspace of traceless tensors

$$(C^M)^{\otimes m}_0 \subset (C^M)^{\otimes m}. \quad (3.24)$$

We will assume that the irreducible representation $V_\mu$ of the group $G_M$ appears in the subspace (3.24), so that the partition $\mu$ of $m$ satisfies the conditions from [W] for $G_M$ as described in Subsection 1.3. In particular, if $M = 1$, then $G_M = O_1$ and we have to take $m \in \{0,1\}$.

Recall that in Subsection 2.4, we denoted by $I_m$ the projector to the direct summand (2.17). Now, there is a unique $G_M$-invariant projector

$$H_m : (C^M)^{\otimes m} \to (C^M)^{\otimes m}_0.$$

Let us denote by $J_m$ the composition of the operators $I_m$ and $H_m \otimes 1$. This composition is a projector to the subspace

$$(C^M)^{\otimes m}_0 \otimes (C^N)^{\otimes n} \subset (C^L)^{\otimes l}. \quad (3.25)$$
Regard $J_m$ as an operator on the vector space $(\mathbb{C}^L)^{\otimes l}$. Then, for any linear operator $A$ on $(\mathbb{C}^L)^{\otimes l}$, define an operator on the same space,

$$A^\vee = J_m A J_m.$$  

We may also regard $A^\vee$ as an operator on the subspace (3.25). The projector $J_m$ commutes with the action of subgroup $G_N \times G_M \subset G_L$ on $(\mathbb{C}^L)^{\otimes l}$. So for $A \in \mathbb{C}^L(\mathbb{L})$,

$$A^\vee \in (C_m(M)/I_m(M)) \otimes C_n(N).$$

Here the quotient algebra $C_m(M)/I_m(M)$ is identified with the image of the symmetric group ring $\mathbb{C}S_m$ in $\text{End}((\mathbb{C}^M)^{\otimes m})$. We get a linear map

$$\Theta_m : \mathbb{C}^L(\mathbb{L}) \rightarrow (C_m(M)/I_m(M)) \otimes C_n(N) : A \mapsto A^\vee. \quad (3.26)$$

As the map $\Theta_m$ is linear, for any distinct $i,j \in \{1, \ldots, l\}$ we have

$$R^\vee_{ij}(x,y) = 1 - P^\vee_{ij} \quad \text{and} \quad \overline{R}^\vee_{ij}(x,y) = 1 - \frac{Q^\vee_{ij}}{x+y+L}. \quad (3.27)$$

If $i \leq m < j$ or $i > m \geq j$, then we have $P^\vee_{ij} = 0$ and $Q^\vee_{ij} = 0$. If $i,j \leq m$ then $Q^\vee_{ij} = 0$ also, while the operator $P^\vee_{ij}$ on (3.25) acts in the tensor factor $(\mathbb{C}^M)^{\otimes m}$ as the permutation corresponding to $(ij) \in S_m$, and it also acts as the identity in $(\mathbb{C}^N)^{\otimes n}$. If $i,j > m$ then each of the operators $P^\vee_{ij}$ and $Q^\vee_{ij}$ acts as the identity on $(\mathbb{C}^M)^{\otimes m}$. Then the action of $P^\vee_{ij}$ on $(\mathbb{C}^N)^{\otimes n}$ corresponds to the transposition $(i-m,j-m) \in S_n$. Then $Q^\vee_{ij}$ acts as $Q(N)$ in the $(i-m)$-th and $(j-m)$-th tensor factors of $(\mathbb{C}^N)^{\otimes n}$, and acts as the identity in the remaining $n-2$ tensor factors of $(\mathbb{C}^N)^{\otimes n}$. Our extension of the results of Subsection 3.2 to standard tableaux of skew shapes is based on the next observation, see Lemma 2.3.

**Proposition.** The image of the product (3.14) under the map $\Theta_m$ equals

$$\prod_{1 \leq i < j \leq m} R^\vee_{ij}(c_i + t_i, c_j + t_j) \quad (3.27)$$

$$\times \prod_{m < i < j \leq l} \overline{R}_i^\vee(c_i + t_i, c_j + t_j) \cdot \prod_{m < i < j \leq l} R^\vee_{ij}(c_i + t_i, c_j + t_j). \quad (3.28)$$

**Proof.** When multiplying the product (3.14) on the left by the operator $J_m$, the factors $\overline{R}_i^\vee(c_i + t_i, c_j + t_j)$ with $i < j \leq m$ and $i \leq m < j$ cancel. Indeed, since $(\mathbb{C}^M)^{\otimes m}_0 \subset (\mathbb{C}^L)^{\otimes m}_{-}$, we have $J_m Q_{ij} = 0$ for $i < j \leq m$. Since the subspaces $\mathbb{C}^M$ and $\mathbb{C}^N$ are orthogonal with respect to the form $\langle \cdot, \cdot \rangle$ on
\[ \mathbb{C}^L, \] we also have \( J_m Q_{ij} = 0 \) for \( i \leq m < j \). Thus by multiplying (3.14) by \( J_m \) on the left and right, we obtain the product

\[
J_m \cdot \prod_{m < i < j \leq l} \overrightarrow{R}_{ij}(c_i + t_i, c_j + t_j) \cdot \prod_{1 \leq i < j \leq l} R_{ij}(c_i + t_i, c_j + t_j) \cdot J_m. \tag{3.29}
\]

Expand the product of all factors \( R_{ij}(c_i + t_i, c_j + t_j) \) in (3.29) as a sum of the operators \( P_s \) on \( (\mathbb{C}^L)^{\otimes l} \) corresponding to permutations \( s \in S_l \), with the coefficients from \( \mathbb{C}(t_1, \ldots, t_l) \). The operators \( \overrightarrow{R}_{ij}(c_i + t_i, c_j + t_j) \) with \( m < i < j \) commute with the operator on \( (\mathbb{C}^L)^{\otimes l} \), which acts as the projector to the subspace \( \mathbb{C}^M \subset \mathbb{C}^L \) along \( \mathbb{C}^N \) in each of the first \( m \) tensor factors, and which acts trivially in the last \( n \) tensor factors of \( (\mathbb{C}^L)^{\otimes l} \). Hence the summand in (3.29) corresponding to \( P_s \) vanishes, unless the permutation \( s \in S_l \) preserves the subset \( \{1, \ldots, m\} \subset \{1, \ldots, l\} \). By Lemma 2.3, the product (3.29) then equals

\[
J_m \cdot \prod_{m < i < j \leq l} \overrightarrow{R}_{ij}(c_i + t_i, c_j + t_j) \times \prod_{1 \leq i < j \leq m} R_{ij}(c_i + t_i, c_j + t_j) \cdot \prod_{m < i < j \leq l} R_{ij}(c_i + t_i, c_j + t_j) \cdot J_m = J_m \cdot \prod_{m < i < j \leq l} \overrightarrow{R}_{ij}(c_i + t_i, c_j + t_j) \cdot J_m \tag{3.30}
\]

\[
\times \prod_{1 \leq i < j \leq m} R_{ij}^\vee(c_i + t_i, c_j + t_j) \cdot \prod_{m < i < j \leq l} R_{ij}^\vee(c_i + t_i, c_j + t_j).
\]

Now expand the expression displayed in the line (3.30), as a sum over all subsequences

\[
(i_1, j_1), \ldots, (i_d, j_d) \tag{3.31}
\]

in the sequence of lexicographically ordered pairs \((i, j)\) with \( m < i < j \leq l \). The summand of (3.30) corresponding to the subsequence (3.31) is equal, up to a coefficient from \( \mathbb{C}(t_1, \ldots, t_l) \), to the product \( J_m Q_{i_1 j_1} \cdots Q_{i_d j_d} J_m \). We will prove that for some permutation \( s \) of the set \( \{i_1, j_1\} \cup \ldots \cup \{i_d, j_d\} \), and for some pairwise distinct elements \( i_1, j_1, \ldots, i_c, j_c \) of this set, we have

\[
Q_{i_1 j_1} \cdots Q_{i_d j_d} = P_s Q_{i_1 j_1} \cdots Q_{i_c j_c}. \tag{3.32}
\]

Since

\[
J_m P_s Q_{i_1 j_1} \cdots Q_{i_c j_c} J_m = P_s^\vee Q_{i_1 j_1}^\vee \cdots Q_{i_c j_c}^\vee,
\]

we obtain the product

\[
J_m \cdot \prod_{m < i < j \leq l} \overrightarrow{R}_{ij}(c_i + t_i, c_j + t_j) \cdot \prod_{1 \leq i < j \leq m} R_{ij}(c_i + t_i, c_j + t_j) \cdot J_m = J_m \cdot \prod_{m < i < j \leq l} \overrightarrow{R}_{ij}(c_i + t_i, c_j + t_j) \cdot J_m \tag{3.30}
\]

\[
\times \prod_{1 \leq i < j \leq m} R_{ij}^\vee(c_i + t_i, c_j + t_j) \cdot \prod_{m < i < j \leq l} R_{ij}^\vee(c_i + t_i, c_j + t_j).
\]
we will then also have the equality
\[ J_m Q_{i_1j_1} \cdots Q_{i_cj_c} J_m = Q_{i_1j_1}^\vee \cdots Q_{i_cj_c}^\vee. \]

The last equality shows that the expression (3.30) equals the first product over \( m < i < j \leq l \) in (3.28). Hence Proposition 3.4 will follow from (3.32).

Since \( Q_{ij} = Q_{ji} \) for any \( i \neq j \), we can assume that \( i_1 < j_1, \ldots, i_c < j_c \) in (3.32). We will prove the equality (3.32) under this assumption. We will also show that the indices \( i_1, j_1, \ldots, i_c, j_c \) in (3.32) can be so chosen that
\[ c \leq d, \quad \text{and} \quad (i_b, j_b) \preceq (i_d, j_d) \quad \text{for all} \quad b = 1, \ldots, c. \tag{3.33} \]

Here \( \preceq \) indicates the lexicographical ordering. The operators \( Q_{i_bj_b} \) in (3.32) pairwise commute, so their ordering is irrelevant. We proceed by induction on \( d = 0, 1, 2, \ldots \). If \( d = 0 \), there is nothing to prove. Now suppose that for some subsequence (3.31), the equality (3.32) is true along with (3.33).

Suppose that \( i < j \) and \( (i_d, j_d) \preceq (i, j) \). By the induction assumption, we have
\[ Q_{i_1j_1} \cdots Q_{i_dj_d} Q_{ij} = P_s Q_{i_1j_1} \cdots Q_{i_cj_c} Q_{ij}. \]

This equality makes the induction step, unless \( i \) or \( j \) coincides with one of the indices \( i_1, j_1, \ldots, i_c, j_c \). Suppose there is such a coincidence. Let \( b \) be the maximal number such that \( \{i_b, j_b\} \cap \{i, j\} \neq \emptyset \). Here \( i_b \neq j \) because \( (i_b, j_b) \preceq (i, j) \). Then there are three possibilities:
\[ i = i_b < j; \quad i_b < j_b = i < j; \quad i_b < i < j = j_b. \tag{3.34} \]

Consider, for instance, the first of the possibilities (3.34). Here we have
\[ Q_{i_bj_b} Q_{ij} = Q_{i_bj_b} Q_{i_bj} = P_{j_bj} Q_{i_bj} \cdot \]

For each \( a = 1, \ldots, c \) we have \( i_a \neq j \). By our choice of \( b \), we have \( j_a \neq j \) for \( a = b, \ldots, c \). If \( j_a \neq j \) also for \( a = 1, \ldots, b - 1 \), then we have the equality
\[ P_s Q_{i_1j_1} \cdots Q_{i_bj_b} Q_{ij} = P_s P_{j_bj} Q_{i_1j_1} \cdots Q_{i_b-1j_b-1} Q_{i_b+1j_b+1} \cdots Q_{i_cj_c} Q_{i_bj}, \]

which makes the induction step. If \( j_b = j \) for some number \( a < b \), then the number \( a \) is unique, and we have the equalities
\[ P_s Q_{i_1j_1} \cdots Q_{i_bj_b} Q_{ij} = P_s Q_{i_1j_1} \cdots Q_{i_{a-1}j_{a-1}} P_{j_bj_a} Q_{i_b+1j_b+1} \cdots Q_{i_cj_c} Q_{i\bar{b}\bar{a}} = \]
\[ P_s P_{j_bj_a} Q_{i_1j_1} \cdots Q_{i_{a-1}j_{a-1}} Q_{i\bar{a}j_b} Q_{i_{a+1}j_{a+1}} \cdots Q_{i_b-1j_b-1} Q_{i_b+1j_b+1} \cdots Q_{i_cj_c} Q_{i\bar{b}\bar{a}}. \]

Thus we again make the induction step, because \( (i_a, \bar{a}) \preceq (i, j) = (i_b, \bar{b}) \) implies \( i_a < i_b \), so that here we have \( i_a < j_b \) and \( (i_a, j_b) \preceq (i_b, \bar{b}) \).

The second and the third possibilities in (3.34) are treated similarly. We omit the details of this treatment. □
Consider the product (3.28) as a rational function of the variables \( t_1, \ldots, t_l \). This function may actually depend only on \( t_{m+1}, \ldots, t_l \). The values of this function are linear operators on the subspace (3.25), which act as the identity in the tensor factor \((\mathbb{C}^M)^\otimes_0^m\) of (3.25). Regard \( E_\Omega\) as the operator acting on the tensor factor \((\mathbb{C}^N)^\otimes_n^m\) of (3.25).

**Corollary.** Restriction of (3.28) to \( T_A \) is regular at \( t_{m+1} = \ldots = t_l = \mp \frac{1}{2} \). The value of this restriction at \( t_{m+1} = \ldots = t_l = \mp \frac{1}{2} \) is divisible on the left and right by the operator id \( \otimes E_\Omega \).

**Proof.** Consider the restriction of the rational function (3.28) to \( T_A \). Due to Propositions 3.2 and 3.4, this restriction is regular at \( t_1 = \ldots = t_l \); the value of this restriction at \( t_1 = \ldots = t_l \) is \( \Theta_m(F_A) \). By Proposition 2.2 applied to the tableau \( Y \) instead of to \( A \), the value at \( t_1 = \ldots = t_m \) of the restriction of the function (3.27) to \( T_A \) equals the operator \( (E_T \mid (\mathbb{C}^M)^\otimes^m_0) \otimes \text{id} \); here the first tensor factor is the restriction of the operator \( E_T \) in \((\mathbb{C}^M)^\otimes^m_1\) to the subspace (3.24). Hence the restriction to \( T_A \) of the product (3.28) is regular at \( t_{m+1} = \ldots = t_l \). The second statement of Corollary 3.4 now follows from Proposition 2.3. \( \square \)

In the notation of Subsection 1.2, \( c_k(\Omega) = c_{m+k} \) for each \( k = 1, \ldots, n \). When \((t_1, \ldots, t_l) \in T_A\), we can put \( t_k(\Omega) = t_{m+k} \) for \( k = 1, \ldots, n \). Then we get Theorem 1.4 as a reformulation of Corollary 3.4. Moreover, the value at \( t_{m+1} = \ldots = t_l = \mp \frac{1}{2} \) of the restriction of (3.28) to \( T_A \) is \( \text{id} \otimes F_\Omega(M) \).

If \( M = 0 \), then \( \mu = (0,0, \ldots) \) and \( \Omega = \Lambda \), so that \( F_\Omega(M) = F_\Lambda \). For arbitrary \( M \) and \( \Omega \), our proof of Corollary 3.4 shows that the operator \( F_\Omega(M) \) on \((\mathbb{C}^N)^\otimes n \), as defined by Theorem 1.4, satisfies the relation

\[
\Theta_m(F_A) = (E_T \mid (\mathbb{C}^M)^\otimes^m_0) \otimes F_\Omega(M).
\]

This relation, along with Proposition 3.2, can be regarded as an alternative definition of the operator \( F_\Omega(M) \). Using the relation (3.35) along with the proof of Proposition 3.4, we also obtain the simplified formulas (1.10) and (1.11) from the formulas (3.20) and (3.22), respectively.

**3.5.** The image of the operator \( F_\Omega(M) \) acting on the vector space \((\mathbb{C}^N)^\otimes n\) is denoted by \( W_\Omega(M) \). If \( M = 0 \), then \( \mu = (0,0, \ldots) \) and \( \Omega = \Lambda \). In this case \( W_\Omega(0) = W_\Lambda \), so that we have the equality (1.12) by Proposition 3.3. For arbitrary \( M \geq 0 \), we will need the following analogue of Proposition 2.4.

**Proposition.** If \( W_\lambda(\mu) \neq \{0\} \), then \( W_\Omega(M) \neq \{0\} \).

**Proof.** Let us realize the irreducible representation \( W_\lambda \) of the group \( G_{N+M} \) from (1.5) as the image \( W_\Lambda \subset (\mathbb{C}^{N+M})^\otimes t \) of the operator \( F_\Lambda \). Define the vectors \( w(M) \in (\mathbb{C}^M)^\otimes 2 \) and \( w(N+M) \in (\mathbb{C}^{N+M})^\otimes 2 \) as in (1.7). Note that

\[
w(N+M) = w(N) + w(M).
\]
The vector space \((\mathbb{C}^{N+M})^\otimes l\) is the sum of its subspaces, obtained by some permutations of the \(l\) tensor factors from the subspaces of the form
\[
(\mathbb{C}^M)^\otimes k \otimes (\mathbb{C}^w(M))^\otimes d \otimes (\mathbb{C}^N)^\otimes (l-k-2d); \tag{3.37}
\]
here \(k\) and \(d\) are non-negative integers such that \(k + 2d \leq l\). This sum may be not direct, but every summand is preserved by the action of the subgroup \(G_M \subset G_{N+M}\); see [W, Section V.6]. For instance, consider the subspace (3.37) itself. The image of this subspace under the operator \(F_A\) coincides with the image of the subspace
\[
(\mathbb{C}^M)^\otimes k \otimes (\mathbb{C}^w(N))^\otimes d \otimes (\mathbb{C}^N)^\otimes (l-k-2d). \tag{3.38}
\]
This coincidence follows from (3.36) and from the equalities \(F_A Q_{ij} = 0\) for
\[
(i,j) = (k + 1, k + 2), \ldots, (k + 2d - 1, k + 2d); \tag{3.39}
\]
here \(Q_{ij}\) are operators on \((\mathbb{C}^{N+M})^\otimes l\). These operator equalities are implied by Proposition 3.3; see also the beginning of the proof of Proposition 3.2. But the subspace (3.38) is contained in \((\mathbb{C}^M)^\otimes k \otimes (\mathbb{C}^N)^\otimes (l-k)\). Note that
\[
\text{Hom}_{G_M}(\mathbb{W}_\mu, F_A \cdot (\mathbb{C}^M)^\otimes k \otimes (\mathbb{C}^N)^\otimes (l-k)) \neq \{0\} \Rightarrow k = m.
\]

Further, define a projector \(J^{(d)}_k\) from the vector space \((\mathbb{C}^{N+M})^\otimes l\) to the subspace (3.37) as follows. Let \(H^{(d)}_k\) be the linear operator on the tensor product \((\mathbb{C}^M)^\otimes (k+2d)\), acting as the (unique) \(G_M\)-invariant projector
\[
H_k : (\mathbb{C}^M)^\otimes k \rightarrow (\mathbb{C}^M)^\otimes k
\]
in the first \(k\) tensor factors of \((\mathbb{C}^M)^\otimes (k+2d)\), and acting as the operator \((Q(M)/M)^\otimes d\) in the last \(2d\) tensor factors of \((\mathbb{C}^M)^\otimes (k+2d)\). Then \(J^{(d)}_k\) is the composition of the projector \(I_{k+2d}\) to the direct summand
\[
(\mathbb{C}^M)^\otimes (k+2d) \otimes (\mathbb{C}^N)^\otimes (l-k-2d) \subset (\mathbb{C}^{N+M})^\otimes l,
\]
with the operator \(H^{(d)}_k \otimes 1\). Note that the projector \(J^{(d)}_k\) is \(G_M\)-equivariant.

In a similar way, by using the operator \((Q(N)/N)^\otimes d\) instead of the operator \((Q(M)/M)^\otimes d\), define a projector \(J^{(d)}_k\) from \((\mathbb{C}^{N+M})^\otimes l\) to the subspace (3.38). In the notation of Subsection 3.4, we have
\[
J^{(0)}_k = J^{(0)}_k = J_k.
\]

Let us apply the projector \(J^{(d)}_k\) to the subspace \(W_A \subset (\mathbb{C}^{N+M})^\otimes l\). If an irreducible representation of \(G_M\) equivalent to \(W_A\) occurs in the projection of \(W_A\) to (3.37), it also occurs in the projection of \(W_A\) to (3.38). This follows
from (3.36) and the equalities $Q_{ij}F_A = 0$ for the pairs of indices $i$ and $j$ given by (3.39). Either occurrence implies that $k = m$. But again we have

$$J^{(d)}_m \cdot F_A \subset J^{(0)}_m \cdot F_A = J_m \cdot F_A.$$ 

Now suppose that $W_{\lambda}(\mu) \neq \{0\}$. By the above argument, there exist permutations $s'$ and $s''$ in $S_l$ such that

$$\text{Hom}_{G_M}(W_{\mu} \cdot J_m P_{s'} F_A P_{s''} \cdot (\mathbb{C}^M)^{\otimes m} \otimes (\mathbb{C}^N)^{\otimes n}) \neq \{0\} \quad (3.40)$$

Let $\lambda'$ and $\lambda''$ range over the set of all standard tableaux of shape $\lambda$. Denote by $F_{\lambda'A'''}$ the unique element of the simple ideal $C_\lambda(L) \subset C_l(L)$, with the image under (3.13) corresponding to the matrix element $e_{\lambda'A'''} \in \mathbb{C}S_l$. If $\lambda = \lambda' = \lambda''$, then $F_A = F_{\lambda'A'''}$ by Corollary 3.3. The product $P_{s'} F_A P_{s''}$ in (3.40) can be written as a linear combination of the operators $F_{\lambda'A'''}$ on $(\mathbb{C}^{N+M})^{\otimes l}$, with the coefficients from $\mathbb{C}$. Due to (3.40), there exists at least one pair of tableaux $\lambda'$ and $\lambda''$ such that

$$\text{Hom}_{G_M}(W_{\mu} \cdot J_m F_{\lambda'A'''} \cdot (\mathbb{C}^M)^{\otimes m} \otimes (\mathbb{C}^N)^{\otimes n}) \neq \{0\} \quad (3.41)$$

According to Subsection 3.4, the restriction of the operator $J_m F_{\lambda'A'''}$ to the subspace (3.25) is denoted by $\Theta_m(F_{\lambda'A'''})$. Consider the tableaux $\tau'$ and $\tau''$, obtained by removing the boxes with the numbers $m+1, \ldots, l$ from the tableaux $\lambda'$ and $\lambda''$, respectively. The operator $\Theta_m(F_{\lambda'A'''})$ on the subspace (3.25) is divisible on the left and right, respectively by

$$(E_{\tau'} | (\mathbb{C}^M)^{\otimes m}) \otimes \text{id} \quad \text{and} \quad (E_{\tau''} | (\mathbb{C}^M)^{\otimes m}) \otimes \text{id}.$$ 

These divisibility properties follow from Proposition 3.2; see the proof of Lemma 2.4. Now the inequality (3.41) implies that the tableaux $\tau'$ and $\tau''$ are of the same shape. But then $F_{\lambda'A'''} = F_{\lambda'} E$ for the operator $E$ on the subspace (3.25) corresponding to some invertible element $e \in \mathbb{C}S_{mn}$; see the end of the proof of Proposition 2.4. Then

$$F_{\lambda'A'''} \cdot (\mathbb{C}^M)^{\otimes m} \otimes (\mathbb{C}^N)^{\otimes n} = F_{\lambda'} \cdot (\mathbb{C}^M)^{\otimes m} \otimes (\mathbb{C}^N)^{\otimes n}.$$ 

By applying the relation (3.35) to the tableau $\lambda'$ instead of $A$, the inequality (3.41) now implies that the tableau $\tau'$ is of shape $\mu$. Moreover, the left-hand side of (3.41) equals $W_{\Omega'}(M)$ for some standard tableau $\Omega'$. The inequality $W_{\Omega'}(M) \neq \{0\}$ implies that $W_{\Omega}(M) \neq \{0\}$.

The space $W_{\lambda}(\mu)$ comes with an action of the subgroup $G_N \subset G_L$. The subspace $W_{\Omega}(M) \subset (\mathbb{C}^N)^{\otimes n}$ is preserved by the action of the group $G_N$ because $F_{\Omega}(M) \in C_n(N)$. Let us consider $W_{\lambda}(\mu)$ and $W_{\Omega}(M)$ as representations of the group $G_N$. In Subsection 5.6 we will prove that these representations are equivalent, as stated in Proposition 1.4.
3.6. Let $\mathfrak{g}_L$ be the Lie algebra of $G_L$, so that $\mathfrak{g}_L = \mathfrak{so}_L$ or $\mathfrak{g}_L = \mathfrak{sp}_L$. We regard $\mathfrak{g}_L$ as a Lie subalgebra in $\mathfrak{gl}_L$. Let us now consider representations of the group $G_L$ as $\mathfrak{g}_L$-modules. If $G_L = Sp_L$, the $\mathfrak{sp}_L$-modules $W_\lambda$ for different partitions $\lambda$ of $l = 0, 1, 2, \ldots$ with $2\lambda'_1 \leq L$ are irreducible and pairwise non-equivalent. One obtains all irreducible finite-dimensional $\mathfrak{sp}_L$-modules in this way. If we choose the Borel subalgebra, and then fix the basis in the corresponding Cartan subalgebra of $\mathfrak{sp}_L$ as in [KT, Subsection 1.1], then $W_\lambda$ is the irreducible $\mathfrak{sp}_L$-module of highest weight $(\lambda_1, \ldots, \lambda_{L/2})$. However, in the present article we do not use the highest weight theory of $\mathfrak{sp}_L$-modules.

If $G_L = O_L$, we have $\lambda'_1 + \lambda'_2 \leq L$. Therefore, by changing the length of the first column of the Young diagram of the partition $\lambda$ to $L - \lambda'_1$, we obtain the Young diagram of a certain partition. Denote this partition by $\lambda^*$. The two representations $W_\lambda$ and $W_{\lambda^*}$ are called associated, and are equivalent as $\mathfrak{so}_L$-modules. The $\mathfrak{so}_L$-module $W_\lambda$ is irreducible unless $\lambda = \lambda^*$, that is, unless $2\lambda'_1 = L$. In the last case $W_\lambda$ splits into a direct sum of two irreducible $\mathfrak{so}_L$-modules. The irreducible $\mathfrak{so}_L$-modules corresponding to different partitions $\lambda$ are pairwise non-equivalent, except for the pairs of modules $W_\lambda$ and $W_{\lambda^*}$ with $\lambda \neq \lambda^*$. In this way one obtains all irreducible finite-dimensional non-spinor $\mathfrak{so}_L$-modules, see [W, Section V.9]. We can choose the Borel subalgebra, and fix the basis in the corresponding Cartan subalgebra of $\mathfrak{so}_L$ as in [KT, Subsection 1.1]. If $2\lambda'_1 = L$, then $W_\lambda$ splits into a direct sum of two irreducible $\mathfrak{so}_L$-modules of highest weights

$$(\lambda_1, \ldots, \lambda_{L/2-1}, \lambda_{L/2}) \quad \text{and} \quad (\lambda_1, \ldots, \lambda_{L/2-1}, -\lambda_{L/2}).$$

If $2\lambda'_1 < L$, then $W_\lambda$ is the irreducible $\mathfrak{so}_L$-module of the highest weight $(\lambda_1, \ldots, \lambda_{L/2})$. But again, the highest weight theory of $\mathfrak{so}_L$-modules is not used in the present article.

Observe that when $G_L = O_L$, the first column length of at least one of the two partitions $\lambda$ and $\lambda^*$ does not exceed $L/2$. Therefore, when regarding $W_\lambda$ as a $\mathfrak{so}_L$-module, we can assume that $2\lambda'_1 < L$. Under this assumption, for any two distinct indices $i, j \in \{1, \ldots, l\}$ we have

$$c_i + c_j \geq 3 - 2\lambda'_1 \geq 3 - L.$$ 

Then every factor $R_{ij}(c_i + t_i, c_j + t_j)$ in the product (3.14) is regular at $t_1 = \ldots = t_l = -\frac{1}{2}$, for any choice of the standard tableau $\Lambda$ of shape $\lambda$.

By Proposition 2.2, we then obtain from (3.14) the explicit formula

$$F_\Lambda = \prod_{1 \leq i < j \leq l} \left( 1 - \frac{Q_{ij}}{c_i + c_j + L - 1} \right) \cdot E_\Lambda \quad \text{for} \quad \mathfrak{g}_L = \mathfrak{so}_L,$$

which is similar to the explicit formula (3.21).
4. Yangian representations

4.1. Using the formal power series (1.13) in \( x^{-1} \), introduce the elements of the algebra \( \text{End}(\mathbb{C}^N) \otimes Y(\mathfrak{gl}_N) [[x^{-1}]] \)

\[ T_1(x) = \sum_{i,j=1}^{N} E_{ij} \otimes 1 \otimes T_{ij}(x) \quad \text{and} \quad T_2(x) = \sum_{i,j=1}^{N} 1 \otimes E_{ij} \otimes T_{ij}(x). \]

The defining relations (1.14) of the algebra \( Y(\mathfrak{gl}_N) \) are equivalent to

\[ R_{12}(x,y) T_1(x) T_2(y) = T_2(y) T_1(x) R_{12}(x,y). \] (4.1)

After multiplying both sides of the equality (4.1) by \( x - y \), it becomes an equality of formal Laurent series in \( x^{-1} \) and \( y^{-1} \) with the coefficients from the algebra \( \text{End}(\mathbb{C}^N) \otimes Y(\mathfrak{gl}_N) \). In (4.1), we identify the element \( R_{12}(x,y) \) of \( \text{End}(\mathbb{C}^N) \otimes Y(\mathfrak{gl}_N) \) as defined by (3.3), with the element

\[ R_{12}(x,y) \otimes 1 \in (\text{End}(\mathbb{C}^N)) \otimes Y(\mathfrak{gl}_N). \]

Using the defining relations of \( Y(\mathfrak{gl}_N) \) in the form (4.1), together with (3.5), one shows that the assignment (1.17) defines an automorphism of the algebra \( Y(\mathfrak{gl}_N) \). Evidently, this automorphism is involutive.

For any \( z \in \mathbb{C} \) consider the evaluation \( Y(\mathfrak{gl}_N) \)-module \( V(z) \), see (1.20). Let us denote by \( \rho_z \) the corresponding homomorphism \( Y(\mathfrak{gl}_N) \rightarrow \text{End}(\mathbb{C}^N) \). Then \( T(x) \mapsto R_{12}(x,z) \) under the homomorphism

\[ \text{id} \otimes \rho_z : \text{End}(\mathbb{C}^N) \otimes Y(\mathfrak{gl}_N) \rightarrow (\text{End}(\mathbb{C}^N))^\otimes_2; \] (4.2)

see the definitions (1.16),(1.18),(1.19) and (3.3). More generally, consider the tensor product of evaluation \( Y(\mathfrak{gl}_N) \)-modules \( V(z_1) \otimes \ldots \otimes V(z_n) \), for any \( z_1, \ldots, z_n \in \mathbb{C} \). The corresponding homomorphism

\[ \rho_{z_1 \ldots z_n} : Y(\mathfrak{gl}_N) \rightarrow (\text{End}(\mathbb{C}^N))^\otimes_n \] (4.3)

is the composition of the map \( \rho_{z_1} \otimes \ldots \otimes \rho_{z_n} \) with \( n \)-fold comultiplication map \( Y(\mathfrak{gl}_N) \rightarrow Y(\mathfrak{gl}_N))^\otimes_n \). By definition (1.15), then

\[ T(x) \mapsto R_{12}(x,z_1) \ldots R_{1,n+1}(x,z_n) \] (4.4)

under the homomorphism

\[ \text{id} \otimes \rho_{z_1 \ldots z_n} : \text{End}(\mathbb{C}^N) \otimes Y(\mathfrak{gl}_N) \rightarrow (\text{End}(\mathbb{C}^N))^\otimes(n+1). \] (4.5)

Now consider the standard action of the algebra \( U(\mathfrak{gl}_N) \) on \( (\mathbb{C}^N)^\otimes_n \). Denote by \( \varpi_n \) the homomorphism \( U(\mathfrak{gl}_N) \rightarrow \text{End}((\mathbb{C}^N)^\otimes_n) \). Using the
matrix units $E_{ij} \in \text{End}(\mathbb{C}^N)$, we can write the operator of transposition of the tensor factors in $(\mathbb{C}^N)^{\otimes 2}$ as the sum

$$
\sum_{i,j=1}^{N} E_{ij} \otimes E_{ji}.
$$

So by the definition (1.19) of the homomorphism $\alpha_N : \mathcal{Y}(\mathfrak{gl}_N) \to \mathcal{U}(\mathfrak{gl}_N)$,

$$
T(x) \mapsto 1 - \sum_{k=1}^{n} \frac{P_{1,k+1}}{x},
$$

under the homomorphism

$$
\text{id} \otimes (\varpi_n \circ \alpha_N) : \text{End}(\mathbb{C}^N) \otimes \mathcal{Y}(\mathfrak{gl}_N) \to (\text{End}(\mathbb{C}^N))^{\otimes (n+1)}. \tag{4.7}
$$

4.2. Now take any standard tableau $\Omega$ with $n$ boxes. Consider the operator $E_\Omega$ on the vector space $(\mathbb{C}^N)^{\otimes n}$, as defined in Subsection 1.1. We denote by $P_0$ the linear operator on $(\mathbb{C}^N)^{\otimes n}$ reversing the order of the tensor factors. Setting $z = 0$ in the following proposition, we obtain Proposition 1.5.

**Proposition.** Put $z_k = c_k(\Omega) + z$ for $k = 1, \ldots, n$. Then the operator $E_\Omega P_0$ is an intertwiner of the $\mathcal{Y}(\mathfrak{gl}_N)$-modules

$$
V(z_n) \otimes \ldots \otimes V(z_1) \longrightarrow V(z_1) \otimes \ldots \otimes V(z_n). \tag{4.8}
$$

**Proof.** The action of $\mathcal{Y}(\mathfrak{gl}_N)$ on the module at the right-hand side of (4.8) can be described by the assignment (4.4). The action of $\mathcal{Y}(\mathfrak{gl}_N)$ on the module at the left-hand side of (4.8) can be described by the assignment

$$
T(x) \mapsto R_{12}(x,z_n) \ldots R_{1,n+1}(x,z_1). \tag{4.9}
$$

Take $n$ complex variables $x_1, \ldots, x_n$. Using (3.4) repeatedly, we obtain the equality of rational functions in $x, x_1, \ldots, x_n$

$$
R_{12}(x,x_1) \ldots R_{1,n+1}(x,x_n) \cdot \prod_{1 \leq i < j \leq n} R_{i+1,j+1}(x_i,x_j) \cdot (1 \otimes P_0)
$$

$$
= \prod_{1 \leq i < j \leq n} R_{i+1,j+1}(x_i,x_j) \cdot (1 \otimes P_0) \cdot R_{12}(x,x_n) \ldots R_{1,n+1}(x,x_1),
$$

with values in the tensor product $(\text{End}(\mathbb{C}^N))^{\otimes (n+1)}$. Using the constrained variables $t_1(\Omega), \ldots, t_n(\Omega)$ from Subsection 1.2, put

$$
x_k = c_k(\Omega) + t_k(\Omega) \quad \text{for each} \quad k = 1, \ldots, n.
$$
Then at $t_1(\Omega) = \ldots = t_n(\Omega) = z$ we have $x_k = z_k$ for $k = 1, \ldots, n$. By Theorem 1.2, the above equality of rational functions in $x, x_1, \ldots, x_n$ yields

$$R_{12}(x, z_1) \ldots R_{1, n+1}(x, z_n) \cdot (1 \otimes E^\Omega P_0) = (1 \otimes E^\Omega P_0) \cdot R_{12}(x, z_1) \ldots R_{1, n+1}(x, z_1).$$

(4.10)

In view of (4.4) and (4.9), the equality (4.10) proves Proposition 4.2. \qed

Due to Proposition 4.2, for any $z \in \mathbb{C}$ the subspace $V_\Omega$ in $(\mathbb{C}^N)^\otimes_n$ can be regarded as a submodule in the tensor product of evaluation $Y(\mathfrak{gl}_N)$-modules

$$V(c_1(\Omega) + z) \otimes \ldots \otimes V(c_n(\Omega) + z).$$

This fact will be also used in Section 5, for the particular value $z = \frac{M}{2} + \frac{1}{2}$.

4.3. Consider the embedding $U(\mathfrak{gl}_N) \to Y(\mathfrak{gl}_N)$ as defined by (1.25). By the definition (1.19), the homomorphism $\alpha_N : Y(\mathfrak{gl}_N) \to U(\mathfrak{gl}_N)$ is identical on the subalgebra $U(\mathfrak{gl}_N) \subset Y(\mathfrak{gl}_N)$. The automorphism $\xi_N$ of $Y(\mathfrak{gl}_N)$ is also identical on this subalgebra, see the definition (1.17). It follows that the restriction of $\alpha_{NM}$ to this subalgebra coincides with the natural embedding $U(\mathfrak{gl}_N) \to U(\mathfrak{gl}_{N+M})$. For the particular choice (1.24) of the series $x(x)$, the automorphism (1.23) is identical on the subalgebra $U(\mathfrak{gl}_N) \subset Y(\mathfrak{gl}_N)$, because the coefficient of $g_\mu(x)$ at $x^{-1}$ is 0. So the action of the subalgebra $U(\mathfrak{gl}_N) \subset Y(\mathfrak{gl}_N)$ on the $Y(\mathfrak{gl}_N)$-module $V_\lambda(\mu)$ coincides with its natural action, corresponding to the natural action of the group $GL_N$ on $V_\lambda(\mu)$.

Furthermore, the action of the subalgebra $U(\mathfrak{gl}_N) \subset Y(\mathfrak{gl}_N)$ on any evaluation module $V(z)$ over $Y(\mathfrak{gl}_N)$ coincides with the natural action of $U(\mathfrak{gl}_N)$ on the vector space $\mathbb{C}^N$ of $V(z)$. The assignment (1.25) determines a Hopf algebra embedding, because by (1.15) we have

$$\Delta(T_{ij}^{(1)}) = T_{ij}^{(1)} \otimes 1 + 1 \otimes T_{ij}^{(1)}.$$ 

Therefore the action of the subalgebra $U(\mathfrak{gl}_N) \subset Y(\mathfrak{gl}_N)$ on the tensor product of evaluation $Y(\mathfrak{gl}_N)$-modules $V(c_1(\Omega)) \otimes \ldots \otimes V(c_n(\Omega))$ coincides with the natural action of $U(\mathfrak{gl}_N)$ on the vector space $(\mathbb{C}^N)^\otimes_n$. The $Y(\mathfrak{gl}_N)$-module $V_\Omega$ is defined as a submodule of this tensor product of evaluation modules. Hence the action of $U(\mathfrak{gl}_N) \subset Y(\mathfrak{gl}_N)$ on this submodule coincides with the natural action of $U(\mathfrak{gl}_N)$ on the subspace $V_\Omega \subset (\mathbb{C}^N)^\otimes_n$.

4.4. In the remainder of this section we prove Theorem 1.6. Fix a standard tableau $\Lambda$ of non-skew shape $\lambda$, such that the tableau $\Omega$ is obtained from $\Lambda$ by removing the boxes with numbers $1, \ldots, m$. Here $\lambda$ is a partition of $l$, and $m = l - n$. As in Sections 2 and 3, write $c_k = c_k(\Lambda)$ for $k = 1, \ldots, l$. The standard tableau of non-skew shape $\mu$, obtained by removing the boxes with the numbers $m + 1, \ldots, l$ from the tableau $\Lambda$, is denoted by $\Upsilon$. 

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Put \( L = N + M \). Take the vector space \( \mathbb{C}^L = \mathbb{C}^N \oplus \mathbb{C}^M \). In the present subsection the rational functions \( R_{12}(x,y), \ldots, R_{1,t+1}(x,y) \) of the variables \( x \) and \( y \) will take values in the algebra \( (\text{End}(\mathbb{C}^L))^\otimes (t+1) \).

We always denote by \( E_A \) the operator on the vector space \( (\mathbb{C}^L)^\otimes l \), corresponding to \( e_A \in \mathbb{C}S_l \). By Proposition 2.5, we have the equality

\[
R_{12}(x,c_1) \ldots R_{1,t+1}(x,c_t) \cdot (1 \otimes E_A)
= \left(1 - \sum_{k=1}^t \frac{P_{1,k+1}}{x}\right) \cdot (1 \otimes E_A),
\]

of rational functions in \( x \), with values in the algebra \( (\text{End}(\mathbb{C}^L))^\otimes (t+1) \). The operator \( P_{1,k+1} \) on \( (\mathbb{C}^L)^\otimes (t+1) \) corresponds to \( (1 k + 1) \in S_{t+1} \), see (3.3). The equality (4.11) is the starting point for our proof of Theorem 1.6.

First consider the case, when \( M = 0 \) and \( \mu = (0,0,\ldots) \). In this case \( N = L, n = l \) and the standard tableau \( Q = \lambda \) has a non-skew shape. In this case, the left-hand side of the equality (4.11) describes the action of the Yangian \( Y(\mathfrak{gl}_L) \) on the submodule \( V_R = V_A \) of the tensor product \( V(c_1) \otimes \ldots \otimes V(c_l) \) of evaluation \( Y(\mathfrak{gl}_L) \)-modules; see (4.4). The right-hand side of (4.11) describes the action of \( Y(\mathfrak{gl}_L) \) on the module \( V_\lambda(\mu) = V_\lambda \), defined in Subsection 1.6; see (4.6). Indeed, here \( g_\mu(x) = 1 \) and \( \alpha_{NM} = \alpha_N \). The image of the operator \( E_Q = E_A \), as a \( \mathfrak{gl}_L \)-submodule in \( (\mathbb{C}^L)^\otimes l \), is equivalent to the \( \mathfrak{gl}_L \)-module \( V_\lambda \). Thus the equality (4.11) shows that the \( Y(\mathfrak{gl}_L) \)-modules \( V_A \) and \( V_\lambda \) are equivalent.

Let us now prove Theorem 1.6 for \( M \geq 1 \). As in Subsection 2.4, split the vector space

\[
(\mathbb{C}^L)^\otimes l = (\mathbb{C}^N \oplus \mathbb{C}^M)^\otimes l
\]

into the direct sum of subspaces, obtained from \( (\mathbb{C}^M)^\otimes k \otimes (\mathbb{C}^N)^\otimes (l-k) \) by some permutations of the \( l \) tensor factors. Here \( k = 0, \ldots, l \). Consider the projector onto the direct summand (2.17),

\[
I_m : (\mathbb{C}^L)^\otimes l \rightarrow (\mathbb{C}^M)^\otimes m \otimes (\mathbb{C}^N)^\otimes n.
\]

Note that the operator (4.12) is \( GL_N \times GL_M \)-equivariant; here we use the embedding \( GL_N \times GL_M \rightarrow GL_L \) chosen in the beginning of Subsection 1.6.

Consider the \( Y(\mathfrak{gl}_L) \)-module, obtained by pulling the tensor product of the evaluation \( Y(\mathfrak{gl}_L) \)-modules corresponding to \( c_1, \ldots, c_l \) back through the automorphism \( \xi_L \) of \( Y(\mathfrak{gl}_L) \); see (1.17). The action of \( Y(\mathfrak{gl}_L) \) on this module is described by the assignment

\[
\sum_{i,j=1}^L E_{ij} \otimes T_{ij}(x) \rightarrow R_{1,l+1}(-x,c_l)^{-1} \ldots R_{12}(-x,c_1)^{-1}
= f(x) \cdot R_{1,l+1}(x,-c_l) \ldots R_{12}(x,-c_1),
\]

(4.13)
where

\[ f(x) = \prod_{k=1}^{l} \frac{(x + c_k)^2}{(x + c_k)^2 - 1}; \]  \hspace{1cm} (4.15)

see (3.5) and (4.4). Denote by \( V_I \) the restriction of this \( Y(\mathfrak{gl}) \)-module to the subalgebra \( Y(\mathfrak{gl}_N) \subset Y(\mathfrak{gl}_L) \); here we use the natural embedding \( \varphi_M : Y(\mathfrak{gl}_N) \rightarrow Y(\mathfrak{gl}_L) \). That is, \( \varphi_M : T_{ij}(x) \mapsto T_{ij}(x) \) for \( 1 \leq i, j \leq N \) by definition. Further, denote by \( V'_I \) the \( Y(\mathfrak{gl}_N) \)-module obtained by pulling the \( Y(\mathfrak{gl}_N) \)-module \( V_I \) back through the automorphism \( \xi_N \) of \( Y(\mathfrak{gl}_N) \). Note that the vector space of the \( Y(\mathfrak{gl}_N) \)-modules \( V_I \) and \( V'_I \) is \((\mathbb{C}^L)^{\otimes l} \).

Take the vector space \((\mathbb{C}^M)^{\otimes m} \otimes (\mathbb{C}^N)^{\otimes n} \). For \( k = m + 1, \ldots, l \) put

\[ R_{1,k+1}^{\wedge}(x,y) = 1 - \frac{P_{1,k+1}^{\wedge}}{x-y} \]  \hspace{1cm} (4.16)

where \( P_{1,k+1}^{\wedge} \) denotes the operator on \( \mathbb{C}^N \otimes (\mathbb{C}^M)^{\otimes m} \otimes (\mathbb{C}^N)^{\otimes n} \), acting as transposition in the first and \((k+1)\)-th tensor factors, and acting as the identity in the remaining \( l-1 \) tensor factors. One can define an action of the algebra \( Y(\mathfrak{gl}_N) \) on the vector space \((\mathbb{C}^M)^{\otimes m} \otimes (\mathbb{C}^N)^{\otimes n} \) by the assignment

\[ \sum_{i,j=1}^{N} E_{ij} \otimes T_{ij}(x) \mapsto f(x) \cdot R_{1,l+1}^{\wedge}(x,-c_l) \ldots R_{1,m+2}^{\wedge}(x,-c_{m+1}). \]

Denote by \( V_{mn} \) the \( Y(\mathfrak{gl}_N) \)-module defined by the above displayed assignment. Further, denote by \( V_{mn}^* \) the \( Y(\mathfrak{gl}_N) \)-module obtained by pulling the \( Y(\mathfrak{gl}_N) \)-module \( V_{mn} \) back through the automorphism \( \xi_N \) of \( Y(\mathfrak{gl}_N) \). The action of the algebra \( Y(\mathfrak{gl}_N) \) on \( V_{mn}^* \) is then described by the assignment

\[ \sum_{i,j=1}^{N} E_{ij} \otimes T_{ij}(x) \mapsto h(x) \cdot R_{1,m+2}^{\wedge}(x,c_{m+1}) \ldots R_{1,l+1}^{\wedge}(x,c_l), \]

where

\[ h(x) = f(-x)^{-1} \cdot \prod_{k=m+1}^{l} \frac{(x - c_k)^2}{(x - c_k)^2 - 1} = \prod_{k=1}^{m} \frac{(x - c_k)^2 - 1}{(x - c_k)^2}. \]  \hspace{1cm} (4.17)

Note that the \( Y(\mathfrak{gl}_N) \)-module \( V_{mn}^* \) can also be obtained as follows. Take the \( Y(\mathfrak{gl}_N) \)-module, obtained by pulling the tensor product of evaluation \( Y(\mathfrak{gl}_N) \)-modules with the parameters \( c_{m+1} = c_1(\Omega), \ldots, c_l = c_n(\Omega) \) back through the automorphism \( T_{ij}(x) \mapsto h(x)T_{ij}(x) \) of the algebra \( Y(\mathfrak{gl}_N) \). The vector space of this \( Y(\mathfrak{gl}_N) \)-module is \((\mathbb{C}^N)^{\otimes n} \). Then by regarding the tensor product \((\mathbb{C}^M)^{\otimes m} \otimes (\mathbb{C}^N)^{\otimes n} \) as \( Y(\mathfrak{gl}_N) \)-module where every element of the Hopf algebra \( Y(\mathfrak{gl}_N) \) acts on \((\mathbb{C}^M)^{\otimes m} \) via the counit homomorphism \( \varepsilon : Y(\mathfrak{gl}_N) \rightarrow \mathbb{C} \), we obtain the \( Y(\mathfrak{gl}_N) \)-module \( V_{mn}^* \).
Proposition. The projection \((4.12)\) is an intertwining operator of \(\mathcal{Y}(\mathfrak{gl}_N)\)-modules \(V_l \to V_{mn}\).

Proof. This result follows by comparing \((4.13),(4.14)\) with the definition of the \(\mathcal{Y}(\mathfrak{gl}_N)\)-module \(V_{mn}\), and by using Lemma 2.5 in the case 
\[z_1 = -c_1, \ldots, z_l = -c_l. \quad (4.18)\]

Corollary. The projection \((4.12)\) is an intertwining operator of \(\mathcal{Y}(\mathfrak{gl}_N)\)-modules \(V_l^* \to V_{mn}^*\).

We end this subsection with the next lemma. Consider the rational functions \(g_\mu(x)\) and \(h(x)\), defined by \((1.24)\) and \((4.17)\), respectively.

Lemma. We have the equality \(g_\mu(x) h(x) = 1\).

Proof. Consider the product at the right-hand side of the equalities \((4.17)\). This product is symmetric in \(c_1, \ldots, c_m\) and therefore does not depend on the choice of a standard tableau \(\Upsilon\) of shape \(\mu\). We choose \(\Upsilon\) to be the row tableau of shape \(\mu\). For any index \(i \geq 1\), take the boxes in the \(i\)th row of the Young diagram \(\mu\) in their natural order, from the leftmost to the rightmost box. The contents of these boxes form the sequence \(1 - i, \ldots, \mu_i - i\) which is increasing by 1; this sequence is empty if \(\mu_i = 0\). Therefore
\[
h(x) = \prod_{k=1}^m \left( \frac{x - c_k + 1}{x - c_k} \cdot \frac{x - c_k - 1}{x - c_k} \right) 
= \prod_{i \geq 1} \left( \frac{x + i}{x - \mu_i + i} \cdot \frac{x - \mu_i + i - 1}{x + i - 1} \right) = g_\mu(x)^{-1}. \quad \square
\]

4.5. Let us continue our proof of Theorem 1.6. Consider the image of the subspace \((2.17)\) under the operator \(E_\Lambda\) on \((\mathbb{C}L)^{\otimes l}\). Note that this image is contained in the subspace \(V_\Lambda \subset (\mathbb{C}L)^{\otimes l}\). Consider the \(\mathcal{Y}(\mathfrak{gl}_N)\)-module \(V_l\) defined in Subsection 4.4; the vector space of this module is \((\mathbb{C}L)^{\otimes l}\).

Proposition. The image of the subspace \((2.17)\) under the operator \(E_\Lambda\) is an \(\mathcal{Y}(\mathfrak{gl}_N)\)-submodule of \(V_l\).

Proof. The action of the coefficients of the series \(T_{ij}(x)\) with \(1 \leq i, j \leq N\) on the \(\mathcal{Y}(\mathfrak{gl}_N)\)-module \(V_l\) is described by the assignment \((4.13)\). Here we use the natural embedding \(\varphi_M : \mathcal{Y}(\mathfrak{gl}_N) \to \mathcal{Y}(\mathfrak{gl}_L)\). Consider the product \((4.14)\) in the algebra \((\text{End}(\mathbb{C}L))^{\otimes (l+1)}(x)\). We have a relation in this algebra,
\[
R_{1,t+1}(x, -c_1) \ldots R_{12}(x, -c_1) \cdot (1 \otimes E_\Lambda) = (1 \otimes E_\Lambda) \times R_{12}(x, -c_1) \ldots R_{1,t+1}(x, -c_1); \quad (4.19)
\]
see (4.10). Expand the product in the second line of the display (4.19), as
\[ R_{12}(x,-c_1) \ldots R_{1,l+1}(x,-c_l) = \sum_{i,j=1}^{L} E_{ij} \otimes A_{ij}(x) \]
for certain rational functions \( A_{ij}(x) \in (\text{End}(\mathbb{C}^L)) \otimes I(x) \). Then consider the restrictions of the operator values of the functions \( A_{ij}(x) \) with \( 1 \leq i, j \leq N \) to the subspace (2.17). Using Corollary 2.5 in the case (4.18), we prove that
\[ \sum_{i,j=1}^{N} E_{ij} \otimes (A_{ij}(x) | (\mathbb{C}^M) \otimes (\mathbb{C}^N) \otimes n) \]
\[ = R_{1,m+2}^{\wedge}(x,-c_{m+1}) \ldots R_{1,l+1}^{\wedge}(x,-c_l) . \]
In particular, the operator values of the functions \( A_{ij}(x) \) with \( 1 \leq i, j \leq N \) preserve the subspace (2.17). Now Proposition 4.5 follows from (4.19). \( \square \)

The \( Y(\mathfrak{g}_N) \)-module \( V_l^* \) is obtained from \( V_l \) by pulling back through an automorphism of \( Y(\mathfrak{g}_N) \). Therefore Proposition 4.5 has a corollary.

**Corollary.** The image of the subspace (2.17) under the operator \( E_A \) is an \( Y(\mathfrak{g}_N) \)-submodule of \( V_l^* \).

4.6. In this subsection we complete the proof of Theorem 1.6. Consider the image of the subspace (2.17) under the linear operator
\[ I_m E_A : (\mathbb{C}^L) \otimes I \to (\mathbb{C}^M) \otimes m \otimes (\mathbb{C}^N) \otimes n . \]
This image coincides with the vector subspace
\[ V_T \otimes V_\Omega \subset (\mathbb{C}^M) \otimes m \otimes (\mathbb{C}^N) \otimes n . \] (4.20)
Indeed,
\[ I_m E_A | (\mathbb{C}^M) \otimes m \otimes (\mathbb{C}^N) \otimes n = E_T \otimes E_\Omega ; \]
see (2.12) and (2.13). It now follows from Corollaries 4.4 and 4.5 that the vector subspace (4.20) is a submodule in the \( Y(\mathfrak{g}_N) \)-module \( V_{mn}^* \). Let us denote this submodule of \( V_{mn}^* \) by \( V \). Note that \( V \) is a subquotient of the \( Y(\mathfrak{g}_N) \)-module \( V_l^* \) by definition.

The description of the \( Y(\mathfrak{g}_N) \)-module \( V_{mn}^* \) given after (4.17) yields the following description of the \( Y(\mathfrak{g}_N) \)-module \( V \). Take the \( Y(\mathfrak{g}_N) \)-module \( V_\Omega \) as defined in Subsection 1.5. Pull \( V_\Omega \) back through the automorphism \( T_i(x) \mapsto h(x) T_i(x) \) of \( Y(\mathfrak{g}_N) \). Extend the resulting action of \( Y(\mathfrak{g}_N) \) on the vector space \( V_\Omega \) to the vector space \( V_T \otimes V_\Omega \) so that every element of \( Y(\mathfrak{g}_N) \) acts on \( V_T \) as the identity. Then we obtain the \( Y(\mathfrak{g}_N) \)-module \( V \).
The subspace $V_{\mathcal{Y}} \subset (\mathbb{C}^M)^{\otimes m}$ is equivalent to $V_{\mu}$ as a representation of the group $GL_M$. Consider the subspace $V_{\lambda} \subset (\mathbb{C}^L)^{\otimes l}$ as a representation of $GL_L$, equivalent to $V_{\lambda}$. Then regard $V_{\lambda}$ as $Y(\mathfrak{gl}_N)$-module by pulling back through the homomorphism $\alpha_{NM} : Y(\mathfrak{gl}_N) \to U(\mathfrak{gl}_L)$; see definition (1.22).

**Proposition.** The $Y(\mathfrak{gl}_N)$-module $V$ is a subquotient of $Y(\mathfrak{gl}_N)$-module $V_{\lambda}$.

**Proof.** By the definition (1.22), we have

$$\alpha_{NM} = \alpha_L \circ \xi_L \circ \varphi_M \circ \xi_N.$$ 

Consider $V_{\lambda}$ as a submodule in the tensor product of the evaluation $Y(\mathfrak{gl}_L)$-modules with the parameters $c_1, \ldots, c_L$. We have already shown that the action of $Y(\mathfrak{gl}_L)$ on this submodule factors through the homomorphism $\alpha_L : Y(\mathfrak{gl}_L) \to U(\mathfrak{gl}_L)$. So the $Y(\mathfrak{gl}_N)$-module $V_{\lambda}$ as defined above can also be obtained by pulling the action of $Y(\mathfrak{gl}_L)$ on $V_{\lambda}$ back through the injective homomorphism $\xi_L \circ \varphi_M \circ \xi_N : Y(\mathfrak{gl}_N) \to Y(\mathfrak{gl}_L)$.

Thus $V_{\lambda}$ is a submodule in the $Y(\mathfrak{gl}_N)$-module $V_l^*$. But by definition, $V$ is a quotient of a certain $Y(\mathfrak{gl}_N)$-submodule of $V_l^*$. The latter submodule of $V_l^*$ is contained in $V_{\lambda}$. $\square$

Consider the restriction of the representation $V_{\lambda}$ of the group $GL_L$ to the subgroup $GL_M$. Realize the vector space (1.1) as

$$\text{Hom}_{GL_M}(V_{\mathcal{Y}}, V_{\lambda}).$$

(4.21)

Since the image of the homomorphism $\alpha_{NM}$ is contained in the subalgebra of $GL_M$-invariants $A_N(M) \subset U(\mathfrak{gl}_L)$, the action of the algebra $Y(\mathfrak{gl}_N)$ on $V_{\lambda}$ induces an action of $Y(\mathfrak{gl}_N)$ on $V_{\lambda}$ that factors through the homomorphism $\alpha_L : Y(\mathfrak{gl}_L) \to U(\mathfrak{gl}_L)$. This action of $Y(\mathfrak{gl}_N)$ on (4.21) is irreducible, see [MO, Section 2].

The operator (4.12) is $GL_N \times GL_M$-equivariant, and the vector space $V_{\mathcal{Y}} \otimes V_{\mathcal{Z}}$ of the $Y(\mathfrak{gl}_N)$-module $V$ has a natural action of the groups $GL_N$ and $GL_M$. The action of $GL_M$ on $V$ commutes with the action of the algebra $Y(\mathfrak{gl}_N)$. By Proposition 4.6, the $Y(\mathfrak{gl}_N)$-module

$$\text{Hom}_{GL_M}(V_{\mathcal{Y}}, V)$$

(4.22)

is a subquotient of (4.21). Since the $Y(\mathfrak{gl}_N)$-module (4.21) is irreducible, it must be equivalent to the $Y(\mathfrak{gl}_N)$-module (4.22); see Proposition 2.4.

The $Y(\mathfrak{gl}_N)$-module (4.22) can also be obtained by pulling the $Y(\mathfrak{gl}_N)$-module $V_{\mathcal{Z}}_{\mu}$ as defined in Subsection 1.5, back through the automorphism (1.23) of $Y(\mathfrak{gl}_N)$, where $g(x) = g_{\mu}(x)^{-1}$. Here we use Lemma 4.4. The proof of Theorem 1.6 is now complete.
Note that (4.22) is also a subquotient of (4.21) as a representation of the group $GL_N$. Thus we obtain Proposition 1.1 together with Theorem 1.6. Proposition 1.1 could be proved independently of Theorem 1.6. We chose the present proofs, because they have analogues for the classical groups $O_N$ and $Sp_N$ instead of $GL_N$. These analogues will be given in the next section.

5. Twisted Yangians

5.1. Using the formal power series (1.27) in $x^{-1}$, introduce the elements of the algebra \( \text{End}(\mathbb{C}^N)^{\otimes 2} \otimes X(gl_N, \sigma) \) \([x^{-1}]\)

\[
S_1(x) = \sum_{i,j=1}^N E_{ij} \otimes 1 \otimes S_{ij}(x) \quad \text{and} \quad S_2(x) = \sum_{i,j=1}^N 1 \otimes E_{ij} \otimes S_{ij}(x).
\]

The defining relations of the algebra \( X(gl_N, \sigma) \) can be written as

\[
R_{12}(x,y) S_1(x) \tilde{R}_{12}(x,y) S_2(y) = S_2(y) \tilde{R}_{12}(x,y) S_1(x) R_{12}(x,y). \quad (5.1)
\]

After multiplying both sides of the equality (5.1) by \( x^2 - y^2 \), it becomes an equality of formal Laurent series in \( x^{-1} \) and \( y^{-1} \) with the coefficients from the algebra \( \text{End}(\mathbb{C}^N)^{\otimes 2} \otimes X(gl_N, \sigma) \). In (5.1), we identify the elements \( R_{12}(x,y) \) and \( \tilde{R}_{12}(x,y) \) of \( \text{End}(\mathbb{C}^N)^{\otimes 2}(x,y) \) as defined by (3.3) and (3.6), respectively with the elements \( R_{12}(x,y) \otimes 1 \) and \( \tilde{R}_{12}(x,y) \otimes 1 \) of \( \text{End}(\mathbb{C}^N)^{\otimes 2}(x,y) \otimes X(gl_N, \sigma) \).

Using (3.5) and (3.7) where \( L = N \), one derives from the defining relations (5.1) that the assignment (1.31) defines an automorphism of the algebra \( X(gl_N, \sigma) \). For details of this argument see [MNO, Subsection 6.5].

By dividing each side of the equality (5.1) by \( S_2(y) \) on the left and right, and then setting \( y = -x \), we obtain the equality

\[
Q_{12} S_1(x) R_{12}(x,-x) S_2(-x)^{-1} = S_2(-x)^{-1} R_{12}(x,-x) S_1(x) Q_{12}.
\]

As in (5.1), here we identify the element \( Q_{12} \in \text{End}(\mathbb{C}^N)^{\otimes 2} \) with

\[
Q_{12} \otimes 1 \in \text{End}(\mathbb{C}^N)^{\otimes 2} \otimes X(gl_N, \sigma).
\]

Since the image of the operator \( Q_{12} = Q(N) \) in \( (\mathbb{C}^N)^{\otimes 2} \) is one-dimensional, the last displayed equality implies the existence of a formal power series \( D(x) \) in \( x^{-1} \) with coefficients in \( X(gl_N, \sigma) \) and leading term 1, such that

\[
Q_{12} S_1(x) R_{12}(x,-x) S_2(-x)^{-1} = \left(1 \pm \frac{1}{2x}\right) D(x) Q_{12}. \quad (5.2)
\]
By using (3.5) when $y = -x$, one derives from (5.2) that $D(x) D(-x) = 1$.

By [MNO, Theorem 6.3], all coefficients of the series $D(x)$ belong to the centre of the algebra $X(\mathfrak{gl}_N, \sigma)$. By [MNO, Theorem 6.4], the kernel of the surjective homomorphism $\pi_N : X(\mathfrak{gl}_N, \sigma) \rightarrow Y(\mathfrak{gl}_N, \sigma)$ is generated by the coefficients of the series $1 - D(x)$. For any series $g(x) \in \mathbb{C}[[x^{-1}]]$ with the leading term 1, the definition (5.2) of the series $D(x)$ shows that the assignment (1.39) determines an automorphism of the quotient algebra $Y(\mathfrak{gl}_N, \sigma)$ of $X(\mathfrak{gl}_N, \sigma)$, if and only if $g(x) = g(-x)$.

5.2. For any $z \in \mathbb{C}$, consider the restriction of the evaluation $Y(\mathfrak{gl}_N)$-module $V(z)$ to the subalgebra $Y(\mathfrak{gl}_N, \sigma) \subset Y(\mathfrak{gl}_N)$. By definition, this subalgebra is generated by the coefficients of all the formal power series from $Y(\mathfrak{gl}_N)[[x^{-1}]]$, appearing in the expansion of the element (1.26) relative to the basis of matrix units $E_{ij}$ in $\text{End}(\mathbb{C}^N)$. Under the homomorphism (4.2) corresponding to the $Y(\mathfrak{gl}_N)$-module $V(z)$, we have $\tilde{T}(x) \mapsto R_{12}(x, z)$; see (3.6) and Subsection 4.1. Therefore, under the homomorphism (4.2)

$$\tilde{T}(x) T(x) \mapsto \tilde{R}_{12}(x, z) R_{12}(x, z). \quad (5.3)$$

Now consider the twisted evaluation $Y(\mathfrak{gl}_N)$-module $\tilde{V}(z)$, see (1.34). Let us denote by $\tilde{\rho}_z$ the corresponding homomorphism $Y(\mathfrak{gl}_N) \rightarrow \text{End}(\mathbb{C}^N)$. Then $T(x) \mapsto \tilde{R}_{12}(x, z)$ under the homomorphism

$$\text{id} \otimes \tilde{\rho}_z : \text{End}(\mathbb{C}^N) \otimes Y(\mathfrak{gl}_N) \rightarrow (\text{End}(\mathbb{C}^N))^\otimes 2.$$ 

Therefore under this homomorphism

$$\tilde{T}(x) T(x) \mapsto R_{12}(x, z) \tilde{R}_{12}(x, z). \quad (5.4)$$

The obvious equality of the right-hand sides of (5.3) and (5.4) explains why the restrictions of the $Y(\mathfrak{gl}_N)$-modules $V(z)$ and $\tilde{V}(z)$ to the subalgebra $Y(\mathfrak{gl}_N, \sigma) \subset Y(\mathfrak{gl}_N)$ coincide.

Consider the restriction of the tensor product of evaluation $Y(\mathfrak{gl}_N)$-modules $V(z_1) \otimes \ldots \otimes V(z_n)$ to the subalgebra $Y(\mathfrak{gl}_N, \sigma) \subset Y(\mathfrak{gl}_N)$, for any $z_1, \ldots, z_n \in \mathbb{C}$. The action of $Y(\mathfrak{gl}_N)$ on this tensor product defines the homomorphism (4.3). Then

$$\tilde{T}(x) T(x) \mapsto \tilde{R}_{1,n+1}(x, z_n) \ldots \tilde{R}_{12}(x, z_1) R_{12}(x, z_1) \ldots R_{1,n+1}(x, z_n) \quad (5.5)$$

under the homomorphism (4.5), see the formula (4.4).

Furthermore, consider the restriction of the tensor product of twisted evaluation $Y(\mathfrak{gl}_N)$-modules $\tilde{V}(z_1) \otimes \ldots \otimes \tilde{V}(z_n)$ to $Y(\mathfrak{gl}_N, \sigma) \subset Y(\mathfrak{gl}_N)$. The action of $Y(\mathfrak{gl}_N)$ on this tensor product defines a homomorphism

$$\tilde{\rho}_{z_1 \ldots z_n} : Y(\mathfrak{gl}_N) \rightarrow (\text{End}(\mathbb{C}^N))^\otimes n.$$
This homomorphism is the composition of the map \( \tilde{\rho}_1 \otimes \ldots \otimes \tilde{\rho}_n \) with the \( n \)-fold comultiplication map \( Y(\mathfrak{gl}_N) \to Y(\mathfrak{gl}_N)^{\otimes n} \). By the definition (1.15),

\[
T(x) \mapsto \tilde{R}_{12}(x,z_1) \ldots \tilde{R}_{1,n+1}(x,z_n)
\]

under the homomorphism

\[
\text{id} \otimes \tilde{\rho}_1 \ldots \tilde{\rho}_n : \text{End}(\mathbb{C}^N) \otimes Y(\mathfrak{gl}_N) \to (\text{End}(\mathbb{C}^N))^{\otimes (n+1)}. \tag{5.6}
\]

Therefore, under (5.6)

\[
\tilde{T}(x)T(x) \mapsto R_{1,n+1}(x,z_n) \ldots R_{12}(x,z_1) \tilde{R}_{12}(x,z_1) \ldots \tilde{R}_{1,n+1}(x,z_n). \tag{5.7}
\]

Note that when \( n > 1 \), the images of the product \( \tilde{T}(x)T(x) \) under the homomorphisms (4.5) and (5.6) may be non-equal; see Subsection 5.3.

Let us now consider the extended twisted Yangian \( X(\mathfrak{gl}_N,\sigma) \), and the homomorphism \( \beta_N : X(\mathfrak{gl}_N,\sigma) \to U(\mathfrak{gl}_N) \) defined by the assignment (1.32). Consider the standard action of the algebra \( U(\mathfrak{g}_N) \) on the tensor product \( (\mathbb{C}^N)^{\otimes n} \). The corresponding homomorphism \( U(\mathfrak{g}_N) \to \text{End}((\mathbb{C}^N)^{\otimes n}) \) is just the restriction of the homomorphism \( \varpi_n : U(\mathfrak{gl}_N) \to \text{End}((\mathbb{C}^N)^{\otimes n}) \) to the subalgebra \( U(\mathfrak{g}_N) \subset U(\mathfrak{gl}_N) \), see the end of Subsection 4.1. Using the matrix units \( E_{ij} \in \text{End}(\mathbb{C}^N) \), we can write the operator \( -Q(N) \) on \( (\mathbb{C}^N)^{\otimes 2} \) as

\[
\sum_{i,j=1}^N E_{ij} \otimes \sigma(E_{ji}) = \sum_{i,j=1}^N \sigma(E_{ij}) \otimes E_{ji}. \tag{5.8}
\]

Hence

\[
S(x) \mapsto 1 - (x \pm \frac{1}{2})^{-1} \sum_{k=1}^n (P_{1,k+1} - Q_{1,k+1}) \tag{5.9}
\]

under the homomorphism

\[
\text{id} \otimes (\varpi_n \circ \beta_N) : \text{End}(\mathbb{C}^N) \otimes X(\mathfrak{gl}_N,\sigma) \to (\text{End}(\mathbb{C}^N))^{\otimes (n+1)}. \tag{5.10}
\]

Note that under the homomorphism (4.7),

\[
\tilde{T}(x)T(x) \mapsto \left( 1 + x^{-1} \sum_{k=1}^n Q_{1,k+1} \right) \left( 1 - x^{-1} \sum_{k=1}^n P_{1,k+1} \right); \tag{5.10}
\]

see (4.6). The elements of the algebra \( \text{End}(\mathbb{C}^N)^{\otimes (n+1)}(x) \) at the right-hand sides of the assignments (5.9) and (5.10) are different. This difference was explained by [N3, Proposition 2.4], see also Lemma 5.4 below.

5.3. Now take any standard tableau \( \Omega \) of shape \( \lambda/\mu \). We assume that the partitions \( \lambda \) of \( l \) and \( \mu \) of \( m \) satisfy the conditions from [W] for the groups
Proof of Proposition 1.7. The action of the algebra \( Y(\mathfrak{g}_l, \sigma) \) on the tensor products \( V(d_1(\Omega)) \otimes \ldots \otimes V(d_n(\Omega)) \) and \( \tilde{V}(d_1(\Omega)) \otimes \ldots \otimes \tilde{V}(d_n(\Omega)) \) is explicitly described by the formulas (5.5) and (5.7), respectively, where

\[
z_1 = d_1(\Omega), \ldots, z_n = d_n(\Omega).
\]

As in Subsection 4.2, take \( n \) complex variables \( x_1, \ldots, x_n \). Using (3.4), (3.9) and (3.11) for \( L = N \) repeatedly, we obtain the equality of rational functions in the variables \( x, x_1, \ldots, x_n \)

\[
\tilde{R}_{1,n+1}(x, x_n) \cdots \tilde{R}_{12}(x, x_1) \ R_{12}(x, x_1) \cdots R_{1,n+1}(x, x_n)
\]

\[
\times \prod_{1 \leq i < j \leq n} \tilde{R}_{i+1,j+1}(x_i, x_j) \cdot \prod_{1 \leq i < j \leq n} R_{i+1,j+1}(x_i, x_j)
\]

\[
= \prod_{1 \leq i < j \leq n} \tilde{R}_{i+1,j+1}(x_i, x_j) \cdot \prod_{1 \leq i < j \leq n} R_{i+1,j+1}(x_i, x_j)
\]

\[
\times R_{1,n+1}(x, x_n) \cdots R_{12}(x, x_1) \tilde{R}_{12}(x, x_1) \ldots \tilde{R}_{1,n+1}(x, x_n)
\]

with values in the tensor product \( (\text{End}(\mathbb{C}^N))^{\otimes (n+1)} \). Using the variables \( t_1(\Omega), \ldots, t_n(\Omega) \) constrained as in Theorem 1.4, put

\[
x_k = c_k(\Omega) + t_k(\Omega) + \frac{M}{2} \quad \text{for each} \quad k = 1, \ldots, n.
\]

At \( t_1(\Omega) = \ldots = t_n(\Omega) = \mp \frac{1}{2} \), the above equality of rational functions in \( x, x_1, \ldots, x_n \) then yields the equality of rational functions in \( x \),

\[
\tilde{R}_{1,n+1}(x, d_n(\Omega)) \cdots \tilde{R}_{12}(x, d_1(\Omega)) \ R_{12}(x, d_1(\Omega)) \cdots R_{1,n+1}(x, d_n(\Omega))
\]

\[
\times F_\Omega(M) = F_\Omega(M) \times
\]

\[
R_{1,n+1}(x, d_n(\Omega)) \cdots R_{12}(x, d_1(\Omega)) \tilde{R}_{12}(x, d_1(\Omega)) \ldots \tilde{R}_{1,n+1}(x, d_n(\Omega)).
\]

In view of (5.5),(5.7) and (5.11), this equality proves Proposition 1.7. \( \square \)

5.4. In this and the next two subsections, we prove Theorem 1.8. We use the same method as in the proof of Theorem 1.6. As in Subsection 4.4, fix a standard tableau \( \Lambda \) of shape \( \lambda \), such that the tableau \( \Omega \) is obtained from \( \Lambda \) by removing the boxes with the numbers \( 1, \ldots, m \). Here \( \lambda \) is a partition of \( l \), and \( m = l - n \). For \( k = 1, \ldots, l \) write \( d_k = c_k(\Lambda) \mp \frac{1}{2} \). The
standard tableau of shape $\mu$, obtained by removing the boxes with numbers $m+1, \ldots, l$ from the tableau $\Lambda$, is denoted by $\Upsilon$.

Put $L = N + M$. Consider the vector space $\mathbb{C}^L = \mathbb{C}^N \oplus \mathbb{C}^M$. In the present subsection, the rational functions

$$R_{12}(x, y), \ldots, R_{1,l+1}(x, y) \quad \text{and} \quad \tilde{R}_{12}(x, y), \ldots, \tilde{R}_{1,l+1}(x, y)$$

will take values in the algebra $(\text{End}(\mathbb{C}^L))^\otimes (l+1)$. Take the linear operator $F_{\Lambda}$ on the vector space $(\mathbb{C}^L)^\otimes l$, this operator is defined by Proposition 3.2.

**Lemma.** We have the equality in $(\text{End}(\mathbb{C}^L))^\otimes (l+1)$

$$\tilde{R}_{1,l+1}(x, d) \cdots \tilde{R}_{12}(x, d) \cdot (1 \otimes F_{\Lambda}) = \left(1 - \sum_{k=1}^{l} \frac{P_{1,k+1} - Q_{1,k+1}}{x \pm \frac{1}{2}}\right) \cdot (1 \otimes F_{\Lambda}). \quad (5.12)$$

**Proof.** Due to Proposition 3.2, the operator $F_{\Lambda}$ is divisible on the left by the operator $E_{\Lambda}$. Using Proposition 2.5 twice, we can rewrite the product at the left hand side of the desired equality (5.12) as

$$\left(1 + \sum_{k=1}^{l} \frac{Q_{1,k+1}}{x \mp \frac{1}{2}}\right) \left(1 - \sum_{k=1}^{l} \frac{P_{1,k+1}}{x \pm \frac{1}{2}}\right) \cdot (1 \otimes F_{\Lambda}); \quad (5.13)$$

see (4.11). Due to Proposition 3.3, for any distinct indices $i, j \in \{1, \ldots, l\}$

$$Q_{1,i+1} P_{1,j+1} \cdot (1 \otimes F_{\Lambda}) = P_{1,j+1} Q_{j+1,i+1} \cdot (1 \otimes F_{\Lambda}) = 0.$$

Therefore, by using the relations

$$Q_{1,k+1} P_{1,k+1} = \pm Q_{1,k+1} \quad \text{for each} \quad k = 1, \ldots, l$$

we can rewrite the product (5.13) as at the right-hand side of (5.12). $\Box$

The equality (5.12) is the starting point for our proof of Theorem 1.8. First, consider the case when $M = 0$ and $\mu = (0, 0, \ldots)$. In this case $N = L$, $n = l$ and the standard tableau $\Omega = \Lambda$ has a non-skew shape. In this case, the left-hand side of (5.12) describes the action of the twisted Yangian $Y(\mathfrak{g}_L, \sigma)$ on its module $W_\Omega(0) = W_\Lambda(0)$; see (5.5) and the end of Subsection 1.7.

The right-hand side of the equality (5.12) describes the action of the extended twisted Yangian $X(\mathfrak{g}_L, \sigma)$ on its module $W_\lambda(\mu) = W_\lambda$, defined in Subsection 1.8; see (5.9). Indeed, here we have $g_\mu(x) = 1$ and $\beta_{NM} = \beta_N$. The image of the operator $F_\Omega(0) = F_{\Lambda}$, as a $\mathfrak{g}_L$-submodule in $(\mathbb{C}^L)^\otimes l$, is equivalent to the $\mathfrak{g}_L$-module $W_\lambda$. Thus the equality (5.12) shows that in the case $M = 0$ the action of the algebra $X(\mathfrak{g}_L, \sigma)$ on $W_\lambda$ factors through
the homomorphism $\pi_L : X(\mathfrak{gl}_L, \sigma) \to Y(\mathfrak{gl}_L, \sigma)$; see (1.29). The equality (5.12) also shows that $W_A$ and $W_\lambda$ are equivalent as $Y(\mathfrak{gl}_L, \sigma)$-modules.

Now suppose that $M \geq 1$. In Subsection 3.4, we introduced a projector

$$J_m : (\mathbb{C}^L)^\otimes l \to (\mathbb{C}^M)^\otimes m \otimes (\mathbb{C}^N)^\otimes n. \quad (5.14)$$

Note that the linear operator (5.14) is $G_M \times G_N$-equivariant by definition.

Take the tensor product of the evaluation $Y(\mathfrak{gl}_L)$-modules with the parameters $d_1, \ldots, d_l$. Consider the restriction of this tensor product to the subalgebra $Y(\mathfrak{gl}_L, \sigma) \subset Y(\mathfrak{gl}_L)$. Pull this restriction back through the homomorphism $\pi_L : \eta_L|_{X(\mathfrak{gl}_L)} : X(\mathfrak{gl}_L, \sigma) \to Y(\mathfrak{gl}_L, \sigma)$, see (1.29) and (1.31). The action of $X(\mathfrak{gl}_L, \sigma)$ on this module is described by the assignment

$$\sum_{i,j=1}^L E_{ij} \otimes S_{ij}(x) \mapsto R_{1,1+1}(-x + \frac{L}{2}, d_1) \cdots R_{12}(-x + \frac{L}{2}, d_1) \cdots R_{1,1+1}(-x + \frac{L}{2}, d_1) \cdots R_{12}(-x + \frac{L}{2}, d_1)$$

$$= f(x + \frac{L}{2} \mp \frac{1}{2}) \cdot R_{1,1+1}(x + \frac{L}{2}, -d_1) \cdots R_{12}(x + \frac{L}{2}, -d_1) \cdots R_{1,1+1}(x + \frac{L}{2}, -d_1) \cdots R_{12}(x + \frac{L}{2}, -d_1), \quad (5.15)$$

$$= \sum_{i,j=1}^L E_{ij} \otimes S_{ij}(x) \mapsto R_{1,1+1}(x + \frac{L}{2}, -d_1) \cdots R_{12}(x + \frac{L}{2}, -d_1) \cdots R_{1,1+1}(x + \frac{L}{2}, -d_1) \cdots R_{12}(x + \frac{L}{2}, -d_1), \quad (5.16)$$

see (5.5). Here the function $f(x) \in \mathbb{C}(x)$ is defined by (4.15); in the notation of Subsection 4.4 we have $d_k = c_k \mp \frac{1}{2}$ for every $k = 1, \ldots, l$. Denote by $W$ the restriction of this $X(\mathfrak{gl}_L, \sigma)$-module to the subalgebra $X(\mathfrak{gl}_N, \sigma)$ of $X(\mathfrak{gl}_L, \sigma)$; here we use the natural embedding $\psi_M : X(\mathfrak{gl}_N, \sigma) \to X(\mathfrak{gl}_L, \sigma)$. That is, $\psi_M : S_{ij}(x) \mapsto S_{ij}(x)$ for $1 \leq i, j \leq N$.

Further, denote by $W^*_i$ the $X(\mathfrak{gl}_N, \sigma)$-module obtained by pulling the $X(\mathfrak{gl}_L, \sigma)$-module $W$ back through the automorphism $\eta_N$ of $X(\mathfrak{gl}_N, \sigma)$. Note that the vector space of the $Y(\mathfrak{gl}_N)$-modules $W_i$ and $W^*_i$ is $(\mathbb{C}^L)^\otimes l$.

Take the vector space $(\mathbb{C}^M)^\otimes m \otimes (\mathbb{C}^N)^\otimes n$. For $k = m + 1, \ldots, l$ we will keep using the notation (4.16). But from now on we will regard $P^\wedge_{1,k+1}$ as an operator on the subspace

$$(\mathbb{C}^N \otimes (\mathbb{C}^M)^\otimes 0 \otimes (\mathbb{C}^N)^\otimes n) \subset (\mathbb{C}^N \otimes (\mathbb{C}^M)^\otimes m \otimes (\mathbb{C}^N)^\otimes n). \quad (5.17)$$

For $k = m + 1, \ldots, l$ consider the operator on $\mathbb{C}^N \otimes (\mathbb{C}^M)^\otimes m \otimes (\mathbb{C}^N)^\otimes n$, acting as $Q(N)$ in the first and $(k + 1)$-th tensor factors, and acting as the identity in the remaining $l - 1$ tensor factors. The restriction of this operator to the subspace (5.17) will be denoted by $Q^\wedge_{1,k+1}$. Using this notation, put

$$\widehat{R}_{1,k+1}^\wedge(x, y) = 1 + \frac{Q^\wedge_{1,k+1}(x)}{x + y}.$$
One can define an action of $X(gl_N, \sigma)$ on the space $(C^M)_0^m \otimes (C^N)^{\otimes n}$ by the assignment
\[
\sum_{i,j=1}^N E_{ij} \otimes S_{ij}(x) \mapsto f(x + \frac{L_2}{2} \pm \frac{1}{2}) \cdot R_{1,l+1}^\wedge(x + \frac{L_2}{2}, -d_l) \cdots R_{1,m+2}^\wedge(x + \frac{L_2}{2}, -d_{m+1})
\[
\times \tilde{R}_{1,m+2}^\wedge(x - \frac{L_2}{2}, -d_{m+1}) \cdots \tilde{R}_{1,l+1}^\wedge(x - \frac{L_2}{2}, -d_l);
\]
(5.18)

see below for the proof of this assertion. Denote by $W_{mn}$ the $X(gl_N, \sigma)$-module defined by the assignment (5.18). Denote by $W_{mn}^*$ the $X(gl_N, \sigma)$-module obtained by pulling the $X(gl_N, \sigma)$-module $W_{mn}$ back through the automorphism $\eta_N$ of $X(gl_N, \sigma)$. Using the function $h(x) \in \mathbb{C}(x)$ defined by (4.17), the action of $X(gl_N, \sigma)$ on $W_{mn}^*$ is described by
\[
\sum_{i,j=1}^N E_{ij} \otimes S_{ij}(x) \mapsto
\[
\sum_{i,j=1}^N E_{ij} \otimes S_{ij}(x) \mapsto h(x - \frac{M}{2} \pm \frac{1}{2}) \cdot R_{1,l+1}^\wedge(x, d_l + \frac{M}{2}) \cdots R_{1,m+2}^\wedge(x, d_{m+1} + \frac{M}{2})
\[
\times R_{1,m+2}(x, d_{m+1} + \frac{M}{2}) \cdots R_{1,l+1}(x, d_l + \frac{M}{2}).
\]

Note that the $X(gl_N, \sigma)$-module $W_{mn}^*$ can also be obtained as follows. Take the tensor product of the evaluation $Y(gl_N)$-modules with parameters $d_{m+1} + \frac{M}{2} = d_1(\Omega), \ldots, d_l + \frac{M}{2} = d_n(\Omega)$.

Consider the restriction of this tensor product to $Y(gl_N, \sigma) \subset Y(gl_N)$. Then regard this restriction as $X(gl_N, \sigma)$-module by using the homomorphism $\pi_N : X(gl_N, \sigma) \to Y(gl_N, \sigma)$. Pull this $X(gl_N, \sigma)$-module back through the automorphism $S_{ij}(x) \mapsto h(x - \frac{M}{2} \pm \frac{1}{2}) \cdot S_{ij}(x)$ (5.19)

of the algebra $X(gl_N, \sigma)$. The vector space of the resulting $X(gl_N, \sigma)$-module is $(C^N)^{\otimes n}$. Then by regarding $(C^M)_0^m \otimes (C^N)^{\otimes n}$ as $X(gl_N, \sigma)$-module where every element of $X(gl_N, \sigma)$ acts on $(C^M)_0^m$ trivially, we obtain the $X(gl_N, \sigma)$-module $W_{mn}^*$.

**Proposition.** The projector (5.14) is an intertwiner of $X(gl_N, \sigma)$-modules $W_l \to W_{mn}$.
Proof. Expand the product at the right-hand-side of the equality (5.16) as the sum
\[ \sum_{i,j=1}^{L} E_{ij} \otimes B_{ij}(x) \]
for certain rational functions $B_{ij}(x) \in (\text{End}(\mathbb{C}^L))^\otimes l(x)$. It suffices to show that the sum
\[ \sum_{i,j=1}^{N} E_{ij} \otimes (J_m B_{ij}(x)) \quad (5.20) \]
is equal to the product at the right-hand-side of the assignment (5.18), multiplied by $J_m$ on the right. To show this, let us expand the right-hand-side of the equality (5.16) as the sum of the products
\[ P_{1i_a} \ldots P_{1i_1} Q_{1j_1} \ldots Q_{1j_b} \quad (5.21) \]
with coefficients from $\mathbb{C}(x)$; here the sum is taken over all subsequences $i_1, \ldots, i_a$ and $j_1, \ldots, j_b$ in the sequence $2, \ldots, l+1$. Consider four cases.

1) Suppose that $j_1 \leq m+1$. Also suppose that $a = 0$ or $i_1 > m+1$. In this case, the product (5.21) is divisible on the left by $Q_{1j_1}$ or by $Q_{i_1j_1}$. The product (5.21) does not contribute to the sum (5.20) in this case, because the subspaces $C^M$ and $C^N$ in $\mathbb{C}^L$ are orthogonal.

2) Suppose that $i_1 \leq m+1$ and $b = 0$. In this case, the product (5.21) does not contribute to the sum (5.20); see the proof of Lemma 2.5.

3) Suppose that $i_1 \leq m+1$ and $b > 0$. Then the product (5.21) is divisible on the left by $Q_{i_1k}$ for some index $k \in \{1, \ldots, l+1\}$. If $k = 1$ or $k > m+1$, then (5.21) does not contribute to the sum (5.20), because the subspaces $C^M$ and $C^N$ in $\mathbb{C}^L$ are orthogonal. If $1 < k \leq m+1$, then (5.21) does not contribute to (5.20), because the subspace $(C^M)^\otimes m \subset (\mathbb{C}^L)^\otimes m$ consists of traceless tensors with respect to the bilinear form $\langle \; , \; \rangle$ on $\mathbb{C}^L$.

4) It remains to consider the case, when both $i_1, \ldots, i_a$ and $j_1, \ldots, j_b$ are subsequences in the sequence $m+2, \ldots, l+1$. Suppose this is the case. Write (5.21) as the sum
\[ \sum_{i,j=1}^{L} E_{ij} \otimes B_{ij} \]
for certain elements $B_{ij} \in (\text{End}(\mathbb{C}^L))^\otimes l$. If $b > 0$, the product (5.21) can also be written as
\[ P_{1i_a} \ldots P_{1i_1} P_{j_{b-1}j_b} \ldots P_{j_1j_2} Q_{1j_b} \cdot \]
Using this observation and Lemma 2.5, we prove that in our remaining case
\[ \sum_{i,j=1}^{N} E_{ij} \otimes J_m B_{ij} = P_{1i_a} \ldots P_{1i_1} Q_{1j_1} \ldots Q_{1j_b} J_m . \]
Corollary. The projector (5.14) is an intertwiner of $X(\mathfrak{gl}_N, \sigma)$-modules $W^*_m \to W^*_{mn}$.

5.5. Let us continue our proof of Theorem 1.8. Consider the image of the subspace (3.25) under the operator $F_{\Lambda}$ on $(\mathbb{C}^L)^{\otimes l}$. Note that this image is contained in the subspace $W_A \subset (\mathbb{C}^L)^{\otimes l}$. Consider the $X(\mathfrak{gl}_N, \sigma)$-module $W_{l}$ defined in Subsection 5.4; the vector space of this module is $(\mathbb{C}^L)^{\otimes l}$.

Proposition. The image of the subspace (3.25) under the operator $F_{\Lambda}$ is an $X(\mathfrak{gl}_N, \sigma)$-submodule of $W_l$.

Proof. The action of the coefficients of the series $S_{ij}(x)$ with $1 \leq i, j \leq N$ on the $X(\mathfrak{gl}_N, \sigma)$-module $W_l$ is described by the assignment (5.15). Here we use the natural embedding $\psi_M : X(\mathfrak{gl}_N, \sigma) \to X(\mathfrak{gl}_L, \sigma)$. Consider the product in the algebra $(\text{End}(\mathbb{C}^L)^{\otimes (l+1)}(x),$ displayed at the right-hand side of the equality (5.16). We have a relation in this algebra,

$$R_{1,l+1}(x + \frac{L}{2}, -d_l) \ldots R_{12}(x + \frac{L}{2}, -d_1) \times \tilde{R}_{12}(x - \frac{L}{2}, -d_l) \ldots \tilde{R}_{1,l+1}(x - \frac{L}{2}, -d_1) \cdot (1 \otimes F_{\Lambda}) = (1 \otimes F_{\Lambda}) \times \tilde{R}_{12}(x - \frac{L}{2}, -d_l) \ldots \tilde{R}_{1,l+1}(x + \frac{L}{2}, -d_1) ... R_{12}(x + \frac{L}{2}, -d_l) \ldots R_{1,l+1}(x + \frac{L}{2}, -d_1); \quad (5.22)$$

see Subsection 5.3. Consider the $2l$ factors displayed in the last two lines of the relation (5.22). Expand the product of these factors as the sum

$$\sum_{i,j=1}^L E_{ij} \otimes C_{ij}(x)$$

where $C_{ij}(x) \in (\text{End}(\mathbb{C}^L)^{\otimes l}(x)$. Consider the restrictions of the operator values of the functions $C_{ij}(x)$ with $1 \leq i, j \leq N$ to the subspace (3.25). By an argument similar to the one used in the proof of Proposition 5.4,

$$\sum_{i,j=1}^N E_{ij} \otimes (C_{ij}(x) | (\mathbb{C}^M)^{\otimes m} \otimes (\mathbb{C}^N)^{\otimes n}) = \tilde{R}_{1,l+1}(x - \frac{L}{2}, -d_l) \ldots \tilde{R}_{1,m+2}(x - \frac{L}{2}, -d_1) \times R_{1,m+2}(x + \frac{L}{2}, -d_l) \ldots R_{1,l+1}(x + \frac{L}{2}, -d_1).$$

In particular, the operator values of the functions $C_{ij}(x)$ with $1 \leq i, j \leq N$ preserve the subspace (3.25). Now Proposition 5.5 follows from (5.22). \qed
The X(\(\mathfrak{gl}_N, \sigma\))-module \(W^*_l\) has been obtained from \(W_l\) by pulling back through an automorphism of X(\(\mathfrak{gl}_N, \sigma\)). So Proposition 5.5 has a corollary.

**Corollary.** The image of the subspace (3.25) under the operator \(F_A\) is an X(\(\mathfrak{gl}_N, \sigma\))-submodule of \(W^*_l\).

**5.6.** In this subsection we will complete the proof of Theorem 1.8. Consider the image of the subspace (3.25) under the linear operator

\[
J_m F_A : (C^L)^{\otimes l} \to (C^M)^{\otimes m} \otimes (C^N)^{\otimes n}.
\]

Due to Proposition 3.3, this image coincides with the vector subspace

\[
W_T \otimes W\Omega (M) \subset (C^M)^{\otimes m} \otimes (C^N)^{\otimes n}.
\]  

(5.23)

Indeed,

\[
J_m F_A | (C^M)^{\otimes m} \otimes (C^N)^{\otimes n} = (E_T | (C^M)^{\otimes m}) \otimes F\Omega (M);
\]

see (3.26) and (3.35). It now follows from Corollaries 5.4 and 5.5 that the vector subspace (5.23) is a submodule in the X(\(\mathfrak{gl}_N, \sigma\))-module \(W^*_{mn}\). Let us denote this submodule of \(W^*_{mn}\) by \(W^*_l\). Note that \(W^*_l\) is a subquotient of the X(\(\mathfrak{gl}_N, \sigma\))-module \(W^*_l\) by definition.

The description of the X(\(\mathfrak{gl}_N, \sigma\))-module \(W^*_{mn}\) given just before stating Proposition 5.4, yields the following description of the X(\(\mathfrak{gl}_N, \sigma\))-module \(W_T\). Take the Y(\(\mathfrak{gl}_N, \sigma\))-module \(W\Omega (M)\) as defined in Subsection 1.7. Then regard \(W\Omega (M)\) as X(\(\mathfrak{gl}_N, \sigma\))-module by using the homomorphism \(\pi_N\). Pull the X(\(\mathfrak{gl}_N, \sigma\))-module \(W\Omega (M)\) back through the automorphism (5.19) of the algebra X(\(\mathfrak{gl}_N, \sigma\)). Now extend the resulting action of X(\(\mathfrak{gl}_N, \sigma\)) on the vector space \(W\Omega (M)\), to the vector space \(W_T \otimes W\Omega (M)\) so that every element of X(\(\mathfrak{gl}_N, \sigma\)) acts on \(W_T\) as the identity. Then we obtain the X(\(\mathfrak{gl}_N, \sigma\))-module \(W_T\). Note that the subspace \(W_T\) of the vector space (3.24) is equivalent to \(W^*_l\) as a representation of \(G_M\); see Proposition 3.3.

Consider the vector subspace \(W_A \subset (C^L)^{\otimes l}\) as a representation of \(G_L\), equivalent to \(W^*_l\). Then regard \(W_A\) as X(\(\mathfrak{gl}_N, \sigma\))-module by pulling back through the homomorphism \(\beta_{NM} : X(\mathfrak{gl}_N, \sigma) \to U(\mathfrak{g}_L)\), see (1.37).

**Proposition.** The X(\(\mathfrak{gl}_N, \sigma\))-module \(W\) is a subquotient of the X(\(\mathfrak{gl}_N, \sigma\))-module \(W_A\).

**Proof.** By (1.37), we have \(\beta_{NM} = \beta_L \circ \eta_L \circ \psi_M \circ \eta_N\). Consider \(W_A\) as a submodule in the restriction of the tensor product of the evaluation Y(\(\mathfrak{gl}_L\))-modules with the parameters \(d_1, \ldots, d_l\) to Y(\(\mathfrak{gl}_L, \sigma\)) \(\subset Y(\mathfrak{g}_L)\). Then regard \(W_A\) as a X(\(\mathfrak{gl}_L, \sigma\))-module, using the homomorphism \(\pi_L\). We have already shown that the resulting action of X(\(\mathfrak{gl}_L, \sigma\)) in \(W_A\) factors through the homomorphism \(\beta_L : X(\mathfrak{gl}_L, \sigma) \to U(\mathfrak{g}_L)\). Hence the X(\(\mathfrak{gl}_N, \sigma\))-module \(W_A\).
as defined above can also be obtained by pulling the just determined action of \( \mathfrak{gl}_L, \sigma \) on \( W_A \), back through the injective homomorphism

\[
\eta_L \circ \psi_M \circ \eta_N : X(\mathfrak{gl}_N, \sigma) \to X(\mathfrak{gl}_L, \sigma).
\]

Thus \( W_A \) is a submodule in the \( X(\mathfrak{gl}_N, \sigma) \)-module \( W_\ast \). But by definition, \( W \) is a quotient of a certain \( X(\mathfrak{gl}_N, \sigma) \)-submodule of \( W_\ast \). The latter submodule of \( W_\ast \) is contained in \( W_A \). \( \Box \)

We can now complete our proof of Theorem 1.8. Let us consider the restriction of the representation \( W_\Lambda \) of the group \( G_L \) to the subgroup \( G_M \). Realize the vector space (1.5) as

\[
\text{Hom}_{G_M}(W_T, W_A).
\]

Since the image of the homomorphism \( \beta_{ NM} \) is contained in the subalgebra of \( G_M \)-invariants \( B_N(M) \subset \text{U}(\mathfrak{gl}_L) \), the action of the algebra \( X(\mathfrak{gl}_N, \sigma) \) on \( W_A \) induces an action of \( X(\mathfrak{gl}_N, \sigma) \) on (5.24). If \( G_L = \text{Sp}_L \), then this action of \( X(\mathfrak{gl}_N, \sigma) \) on (5.24) is irreducible, see [MO, Section 4]. If \( G_L = O_L \), then (5.24) is irreducible under the joint action of the algebra \( X(\mathfrak{gl}_N, \sigma) \) and the subgroup \( G_N \subset G_L \).

The operator (5.14) is \( G_M \times G_N \)-equivariant, and the vector space \( W_T \otimes W_\Omega(M) \) of the \( X(\mathfrak{gl}_N, \sigma) \)-module \( W \) comes with the natural action of the group \( G_M \times G_N \). The action of \( G_M \) on \( W \) commutes with the action of the algebra \( X(\mathfrak{gl}_N, \sigma) \). By Proposition 5.6, the \( X(\mathfrak{gl}_N, \sigma) \)-module

\[
\text{Hom}_{G_M}(W_T, W),
\]

is a subquotient of the \( X(\mathfrak{gl}_N, \sigma) \)-module (5.25). It is also a subquotient of (5.24) as a representation of the group \( G_N \). Since (5.24) is irreducible under the joint action of \( X(\mathfrak{gl}_N, \sigma) \) and \( G_N \), this action must be equivalent to the joint action of \( X(\mathfrak{gl}_N, \sigma) \) and \( G_N \) on (5.25); see Proposition 3.5.

The \( X(\mathfrak{gl}_N, \sigma) \)-module (5.25) can also be obtained in the following way. Take the \( Y(\mathfrak{gl}_N, \sigma) \)-module \( W_\Omega(M) \) as defined in Subsection 1.7. Regard \( W_\Omega(M) \) as \( X(\mathfrak{gl}_N, \sigma) \)-module by using the homomorphism \( \pi_N \). Finally, pull this \( X(\mathfrak{gl}_N, \sigma) \)-module back through the automorphism (1.24) of \( X(\mathfrak{gl}_N, \sigma) \), where

\[
g(x) = g_\mu(x - \frac{M}{2} \pm \frac{1}{2})^{-1}.
\]

Here we use Lemma 4.4. This explicit description of the \( X(\mathfrak{gl}_N, \sigma) \)-module (5.25) completes the proof of Theorem 1.8. We also obtain Proposition 1.4.

5.7. In this subsection, we prove analogues of the results of Subsection 4.3 for the twisted Yangian \( Y(\mathfrak{gl}_N, \sigma) \). Due to (1.31) and (1.32), for the elements \( S_{ij}^{(1)} \in X(\mathfrak{gl}_N, \sigma) \) we have

\[
\eta_N(S_{ij}^{(1)}) = S_{ij}^{(1)} \quad \text{and} \quad \beta_N(S_{ij}^{(1)}) = -E_{ji} - \sigma(E_{ji});
\]

here the matrix unit \( E_{ji} \) is regarded as a generator of the algebra \( \text{U}(\mathfrak{gl}_N) \).
Therefore for any non-negative integer $M$, by the definition (1.38) of the homomorphism $\beta_{NM} : X(\mathfrak{gl}_N, \sigma) \to U(\mathfrak{g}_{N+M})$ we obtain the equality

$$\beta_{NM}(S^{(1)}_{ij}) = -E_{ji} - \sigma(E_{ji});$$  \hspace{1cm} (5.26)

at the right-hand side of this equality we have an element of the subalgebra $U(\mathfrak{g}_N) \subset U(\mathfrak{g}_{N+M})$. This equality shows that the element $S^{(1)}_{ij} \in X(\mathfrak{gl}_N, \sigma)$ acts as $-E_{ji} - \sigma(E_{ji})$ on the $X(\mathfrak{gl}_N, \sigma)$-module $W_\lambda(\mu)$. Here we use the fact that the coefficient of the series (1.40) at $x^{-1}$ is zero; see also (1.24).

Consider the surjective homomorphism $\pi : X(\mathfrak{gl}_N, \sigma) \to Y(\mathfrak{gl}_N, \sigma)$, and the embedding $U(\mathfrak{g}_N) \to Y(\mathfrak{gl}_N, \sigma)$ defined by (1.41). Using the first statement of Theorem 1.8, we can regard $W_\lambda(\mu)$ as $Y(\mathfrak{gl}_N, \sigma)$-module. The right-hand side of the equality (5.26), as an element of the subalgebra $U(\mathfrak{g}_N) \subset Y(\mathfrak{gl}_N, \sigma)$, coincides with $\pi(S^{(1)}_{ij})$; see (1.41). Independently of Theorem 1.8, this coincidence shows that the action of the elements $S^{(1)}_{ij}$ on the $X(\mathfrak{gl}_N, \sigma)$-module $W_\lambda(\mu)$ factors through the homomorphism $\pi$. Moreover, this coincidence shows that the natural action of the algebra $U(\mathfrak{g}_N)$ on $W_\lambda(\mu)$ coincides with its action as a subalgebra in $Y(\mathfrak{gl}_N, \sigma)$.

Let us now consider the $Y(\mathfrak{gl}_N, \sigma)$-module $W_\Omega(M)$. It is a submodule in the restriction of the tensor product (1.36) of evaluation $Y(\mathfrak{gl}_N)$-modules to the subalgebra $Y(\mathfrak{gl}_N, \sigma) \subset Y(\mathfrak{gl}_N)$, see also (1.35). Observe that the embedding $U(\mathfrak{g}_N) \to Y(\mathfrak{gl}_N, \sigma)$ as defined by (1.41) can also be obtained by restricting the embedding $U(\mathfrak{g}_N) \to Y(\mathfrak{g}_N) \subset Y(\mathfrak{gl}_N)$; see (1.25) and (1.29). Here we use the equality (5.8) in $\mathfrak{g}_N \otimes \mathfrak{g}_N$. But the action of the subalgebra $U(\mathfrak{g}_N) \subset Y(\mathfrak{g}_N)$ on the $Y(\mathfrak{gl}_N)$-module (1.36) coincides with the natural action of $U(\mathfrak{g}_N)$ on the vector space $(\mathbb{C}^N)^{\otimes n}$; see Subsection 4.4. Therefore the natural action of $U(\mathfrak{g}_N)$ on $W_\Omega(M)$ coincides with its action as a subalgebra in $Y(\mathfrak{gl}_N, \sigma)$.

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