Dirac fields in a Bohm-Aharonov background and spectral boundary conditions

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We study the problem of a Dirac field in the background of an Aharonov-Bohm flux string. We exclude the origin by imposing spectral boundary conditions at a finite radius then shrinked to zero. Thus, we obtain a behaviour of the eigenfunctions which is compatible with the self-adjointness of the radial Hamiltonian and the invariance under integer translations of the reduced flux. After confining the theory to a finite region, we check the consistency with the index theorem, and discuss the vacuum fermionic number and Casimir energy.

1 Setting of the problem

We study the Dirac equation for a massless particle in four dimensional Minkowski space, in the presence of a flux tube located at the origin, i.e.,

\[(i \not{\partial} - \not{A}) \Psi = 0 \quad \not{H} = \nabla \wedge \not{A} = \frac{\kappa}{r} \delta(r) \hat{e}_z\]

(1)

where \(\kappa = \frac{\Phi}{2\pi}\) is the reduced flux.

As the gauge potential is \(z\)-independent, equation (1) can be decoupled into two uncoupled two-component equations by choosing:

\[
\gamma^0 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} -i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & i\sigma_1 \\ i\sigma_1 & 0 \end{pmatrix}
\]

(2)

In order to avoid singularities, we will consider that only \(r > r_0\) is accessible, and take the limit \(r_0 \to 0\), which is equivalent to having a punctured plane with the removed point corresponding to the string position.

By taking \(A_z = A_r = 0, A_\theta = \frac{\kappa}{r}\), for \(r > r_0\), the Hamiltonian can be seen to be block-diagonal, with its two-by-two blocks given by

\[
H_\pm = \begin{pmatrix} 0 & ie^{\mp i\theta} (\partial_r \pm B) \\ -ie^{\mp i\theta} (\partial_r \pm B) & 0 \end{pmatrix} \quad B = -\frac{i}{r} \partial_\theta - \frac{\kappa}{r}
\]

(3)

It should be noticed that these two “polarizations”, which we will label with \(s = \pm 1\), correspond to the two inequivalent choices for the gamma matrices in 2+1 dimensions. From now on, we will be working with \(s = 1\) (the case...
s = −1 can be studied in a similar way, and explicit reference will be made to it whenever necessary). In this case, we can write:

\[ H_s = \begin{pmatrix} 0 & L^1 \\ L & 0 \end{pmatrix} \], with \( L = -ie^{i\theta} (\partial_r + B) \) \( L^1 = ie^{-i\theta} (\partial_r + B) \) (4)

and its eigenfunctions:

\[ \Psi_E = \begin{pmatrix} \varphi_E (r, \theta) \\ \chi_E (r, \theta) \end{pmatrix} \] , satisfy:

\[ L \varphi_E = E \chi_E \]
\[ L^\dagger \chi_E = E \varphi_E \] (5)

Now, the two components in \( \Psi_E \) have different \( \theta \) dependence. In order to make this fact explicit, and to discuss boundary conditions at \( r = r_0 \), we introduce:

\[ \Psi_E = \frac{1}{\sqrt{r}} \begin{pmatrix} e^{-\frac{i\theta}{2}} \varphi_{1E} (r, \theta) \\ e^{\frac{i\theta}{2}} \chi_{1E} (r, \theta) \end{pmatrix} \]

so that

\[ L_1 \varphi_{1E} = iE \chi_{1E} \]
\[ L_1^\dagger \chi_{1E} = -iE \varphi_{1E} \] (6)

We expand \( \varphi_{1E} \) and \( \chi_{1E} \) in terms of eigenfunctions of B, which are of the form:

\[ c_n = e^{i(n+\frac{1}{2})\theta} \] , with \( \lambda_n (r) = \frac{n + \frac{1}{2} - \kappa}{r} \), \( n \in \mathbb{Z} \) (8)

once the condition has been imposed that \( \varphi_E \) and \( \chi_E \) in equation (5) are single-valued in \( \theta \).

Thus, we have

\[ \varphi_{1E} (r, \theta) = \sum_{n=-\infty}^{\infty} f_n (r) e^{i(n+\frac{1}{2})\theta} \]
\[ \chi_{1E} (r, \theta) = \sum_{n=-\infty}^{\infty} g_n (r) e^{i(n+\frac{1}{2})\theta} \] (9)

which leads, for noninteger \( \kappa \) (\( \kappa = k + \alpha \), with \( k \) the integer part of \( \kappa \) and \( \alpha \) its fractional part) to

\[ \Psi_E (r, \theta) = \sum_{n=-\infty}^{\infty} e^{in\theta} \left( \begin{pmatrix} A_n J_{n-\kappa} (|E|r) + B_n J_{\kappa-n} (|E|r) \\ A_n J_{n+1-\kappa} (|E|r) - B_n J_{\kappa-n-1} (|E|r) \end{pmatrix} e^{i\theta} \right) \] (10)

(Of course, for integer \( \kappa \), a linear combination of Bessel and Neumann functions must be taken). Finally, for \( s = -1 \), the upper and lower components of \( \Psi_E \) interchange, and \( E \rightarrow -E \).
2 Boundary conditions at the origin

As is well known, the radial Dirac Hamiltonian in the background of an Aharonov-Bohm gauge field requires a self-adjoint extension for the critical subspace \( n = k \). In fact, imposing regularity of both components of the Dirac field at the origin is too strong a requirement, except for integer flux. Rather, one has to apply the theory of Von Neumann deficiency indices, which leads to a one parameter family of allowed boundary conditions, characterized by

\[
i \lim_{r \to 0} (Mr)^{\nu + 1} g_n(r) \sin \left( \frac{\pi}{4} + \frac{\Theta}{2} \right) = \lim_{r \to 0} (Mr)^{-\nu} f_n(r) \cos \left( \frac{\pi}{4} + \frac{\Theta}{2} \right) \tag{11}\]

with \( \nu \) varying between \(-1\) and \(0\) (\( \nu = -\alpha \) for \( s = 1 \); \( \nu = \alpha - 1 \) for \( s = -1 \)). Here, \( \Theta \) parametrizes the admissible self-adjoint extensions, and \( M \), a mass parameter, is introduced for dimensional reasons. Which of these boundary conditions to impose depends on the physical situation under study.

One possibility is to take a finite flux tube, ask for continuity of both components of the Dirac field at finite radius and then let this radius go to zero. Thus, one of the possible self-adjoint extensions is obtained, which corresponds to \( \Theta = \frac{\pi}{2} \text{sgn} (\kappa) \). As pointed out, for instance, in this kind of procedure leads to a boundary condition that breaks the invariance under \( \kappa \to \kappa + n \) (\( n \in \mathbb{Z} \)). Now, this is a large gauge symmetry, which of course is singular when considering the whole plane, but is not so when the origin is removed or, equivalently, the plane has the topology of a cylinder.

To preserve the aforementioned symmetry we propose, instead, to exclude the origin, by imposing spectral boundary conditions of the Atiyah-Patodi-Singer(APS) type as defined in at a finite radius \( r_0 \) and then letting \( r_0 \to 0 \).

We consider the development in eq.(9) and, for \( s = 1 \), impose at \( r = r_0 \) :

\[
f_n (r_0) = 0, \text{ for } \lambda_n (r_0) \leq 0 \quad g_n (r_0) = 0, \text{ for } \lambda_n (r_0) > 0 \tag{12}\]

As is well known, imposing this kind of boundary condition is equivalent to removing the boundary, by attaching a semi-infinite tube at its position and then extending the Dirac equation by a constant extension of the gauge field, while asking that zero modes be square integrable (except for \( \lambda_n = 0 \), where a constant zero mode remains, with a nonzero lower component).

After using the dominant behaviour of Bessel functions for small arguments, and taking the zero radius limit, we have the following result for the eigenfunctions in eq.(10):

\[
\Psi_E (r, \theta) = \sum_{n=-\infty}^{k} B_n \left( \frac{J_{k+\alpha-n} (|E|r) e^{in\theta}}{-|E| J_{k+\alpha-n-1} (|E|r) e^{i(n+1)\theta}} \right) + \]


\[ + \sum_{n=k+1}^{\infty} A_n \left( \frac{J_{n-k-\alpha}(|E|r) e^{in\theta}}{i\frac{|E|}{E}} J_{n+1-k-\alpha}(|E|r) e^{i(n+1)\theta} \right) \quad \alpha \geq \frac{1}{2} \quad (13) \]

\[ \Psi_E(r, \theta) = \sum_{n=-\infty}^{k-1} B_n \left( \frac{J_{k+n}(|E|r) e^{in\theta}}{i\frac{|E|}{E}} J_{k+n-1}(|E|r) e^{i(n+1)\theta} \right) \]

\[ + \sum_{n=k}^{\infty} A_n \left( \frac{J_{n-k}(|E|r) e^{in\theta}}{i\frac{|E|}{E}} J_{n+1-k}(|E|r) e^{i(n+1)\theta} \right) \quad \alpha < \frac{1}{2} \quad (14) \]

Notice that our procedure leads precisely to a self adjoint extension satisfying the condition of minimal irregularity (the radial functions diverge at \( r \to 0 \) at most as \( r^{-p} \), with \( p \leq \frac{1}{2} \)). It corresponds to the values of the parameter \( \Theta \):

\[ \Theta = \begin{cases} \frac{\pi}{2} & \text{for } \alpha \geq \frac{1}{2} \\ -\frac{\pi}{2} & \text{for } \alpha < \frac{1}{2} \end{cases} \quad (15) \]

As shown in, \( \Theta = \pm \frac{\pi}{2} \) are the only two possible values of the parameter which correspond to having a Dirac delta magnetic field at the origin. Moreover, this extension is compatible with periodicity in \( \kappa \). In fact, the dependence on \( k \) can be reduced to an overall phase factor in the eigenfunctions.

As regards charge conjugation (\( \Psi_E \to \sigma_3 \Psi_E^* \); \( \kappa \rightarrow -\kappa \)), it is respected by the eigenfunctions, except for \( \alpha = \frac{1}{2} \). This is due to the already commented presence of a constant zero mode on the cylinder. However, for the representation \( s = -1 \) of \( 2 \times 2 \) Dirac matrices, APS boundary conditions must be reversed, for \( \lambda \neq 0 \), as compared to (12), since the operator \( B \) changes into \( -B \). For \( \lambda = 0 \) the lower component will be taken to be zero at \( r_0 \) which, as we will show later, allows for charge conjugation to be a symmetry of the whole model. In this case, the resulting extension corresponds to

\[ \Theta = \begin{cases} \frac{\pi}{2} & \text{for } \alpha \geq \frac{1}{2} \\ -\frac{\pi}{2} & \text{for } \alpha < \frac{1}{2} \end{cases} \quad (16) \]

It is worth pointing that, for integer \( \kappa = k \), our procedure leads (both for \( s = \pm 1 \)) to the requirement of regularity of both components at the origin.

### 3 The theory in a bounded region

From now on, we will confine the Dirac fields inside a bounded region, by introducing a boundary at \( r = R \), and imposing there boundary conditions of the APS type, complementary to the ones considered at \( r = r_0 \). For \( s = 1 \)

\[ f_n(R) = 0, \text{ for } \lambda_n(R) > 0 \quad g_n(R) = 0, \text{ for } \lambda_n(R) \leq 0 \quad (17) \]
We start by studying the zero modes of our theory, which are of the form

\[ \Psi_0 (r, \theta) = \left( e^{-i \frac{\theta}{2}} \sum_{n=-\infty}^{\infty} A_n r^n e^{i(n + \frac{1}{2}) \theta}, e^{i \frac{\theta}{2}} \sum_{n=-\infty}^{\infty} B_n r^n e^{i(n + \frac{1}{2}) \theta} \right) \]  

(18)

It is easy to see that no zero mode remains after imposing the boundary conditions at both the internal and external radius, even without taking \( r_0 \rightarrow 0 \). This is in agreement with the APS index theorem. In fact, according to such theorem

\[ n_+ - n_- = A + b (r_0) + b (R) \]  

(19)

where \( n_+ (n_-) \) is the number of chirality positive (negative) zero energy solutions, \( A \) is the anomaly, or bulk contribution, and \( b \) are the surface contributions coming from both boundaries.

\[ b (R) = \frac{1}{2} (h_R - \eta (R)) \quad b (r_0) = \frac{1}{2} (\eta (r_0) - h_{r_0}) \]  

(20)

with \( \eta (r) \) the spectral asymmetry of the boundary operator \( B \) and \( h_r \) the dimension of its kernel.

In our case, the boundary contributions cancel. As regards the volume part, it also vanishes for the gauge field configuration under study, and we have \( n_+ - n_- = 0 \), which is consistent with the absence of zero modes. For \( s = -1 \) both boundary contributions interchange, and identical conclusions hold regarding the index.

The nonzero energy spectrum can be determined by imposing \((s = 1)\) the boundary conditions \((17)\) at \( r = R \) on the eigenfunctions in eqs. \((13)\) and \((14)\). Thus, one gets:

\[ E_{n,l} = \begin{cases} 
  \pm \frac{j_{n+\alpha,l}}{R}, & n \geq 1, \alpha \geq \frac{1}{2} \\
  \pm \frac{j_{n+\alpha,l}}{R}, & n \geq -1, \alpha \geq \frac{1}{2} 
\end{cases} \quad E_{n,l} = \begin{cases} 
  \pm \frac{j_{n-\alpha,l}}{R}, & n \geq 0, \alpha < \frac{1}{2} \\
  \pm \frac{j_{n-\alpha,l}}{R}, & n \geq 0, \alpha < \frac{1}{2} 
\end{cases} \]  

(21)

where \( j_{\nu,l} \) is the \( l \)-th positive root of \( J_{\nu} \). The same spectrum results for \( s = -1 \).

For both \( s \) values, the energy spectrum is symmetric with respect to zero. This fact, together with the absence of zero modes results in a null vacuum expectation value for the fermionic charge:

\[ \langle N \rangle_+ = -\frac{1}{2} (n_+ - n_-) = 0 \]  

(22)

For the same reasons \( \langle N \rangle_- = 0 \), so the total fermionic number of the theory is null.
It is interesting to note that the contribution to the fermionic number coming from \( r_0 \) coincides, for each \( s \) value, with the result presented for the whole punctured plane in [4] (for details, see [10]), except for \( \alpha = \frac{1}{2} \). This last fact is associated with charge conjugation non invariance in each subspace. However, the sum of both contributions cancels for all \( \alpha \).

We go now to the evaluation of the Casimir energy which, in the framework of the \( \zeta \) regularization, is given by

\[
E_C = -2\mu (\mu R)^2 \left. \zeta_\nu(z) \right|_{z=-1}
\]  

(23)

where the parameter \( \mu \) was introduced for dimensional reasons. Here, \( \zeta_\nu(z) \) is the so-called partial zeta function, defined as in reference [4].

For the problem at hand,

\[
\sum_\nu \zeta_\nu = \begin{cases} 
\sum_{n=-\infty}^{\infty} \zeta|n-\alpha| + \zeta_{\alpha-1} & \text{for } \alpha \geq \frac{1}{2} \\
\sum_{n=-\infty}^{\infty} \zeta|n-\alpha| + \zeta_{-\alpha} & \text{for } \alpha < \frac{1}{2}
\end{cases}
\]  

(24)

Notice that, as a consequence of the invariance properties of the imposed boundary conditions, the Casimir energy is both periodic in \( \kappa \) and invariant under \( \alpha \to 1 - \alpha \), as well as continuous at integer values of \( \kappa \).

For any value of \( \kappa \), it is the sum of the energy corresponding to a scalar field in the presence of a flux string and subject to Dirichlet boundary conditions, plus a partial zeta coming, for fractionary \( \kappa \), from the presence of an eigenfunction which is singular at the origin or, for integer \( \kappa \), from the duplication of \( J_0 \). Both contributions can be studied following the methods employed in [11]. So, here we won’t go into the details of such calculation.

The scalar field contribution presents a pole at \( z = -1 \), with an \( \alpha \)-independent residue. However, the pole in the partial zeta appearing in [13], [24] has an \( \alpha \)-dependent residue. Due to this fact, and the consequent need to introduce \( \alpha \)-dependent counterterms, an absolute meaning cannot be assigned to the finite part of the Casimir energy.

As a final remark, it is worth stressing that this approach to the problem of self-adjointness at the origin is, to our knowledge, the first proposal of a physical application of APS boundary conditions in this context.

Contrary to the treatment of the origin, APS boundary conditions were imposed at the exterior boundary for merely formal reasons (the existence of an index theorem for this case). The vacuum energy under local (bag-like) external boundary conditions is at present under study.
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