THE FUNDAMENTAL INEQUALITY FOR COCOMPACT FUCHSIAN GROUPS

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ABSTRACT. We prove that the hitting measure is singular with respect to Lebesgue measure for any random walk on a cocompact Fuchsian group generated by translations joining opposite sides of a symmetric hyperbolic polygon.

Let $G < SL_2(\mathbb{R})$ be a Fuchsian group, and $\mu$ be a finitely supported, generating probability measure on $G$. We consider the random walk

$$w_n := g_1 g_2 \cdots g_n$$

where each $(g_i)$ is i.i.d. with distribution $\mu$. Let us fix a base point $o \in \mathbb{H}^2$. Then the hitting measure $\nu$ of the random walk on $S^1 = \partial \mathbb{D}$ is

$$\nu(A) := \mathbb{P} \left( \lim_{n \to \infty} w_n o \in A \right)$$

for any Borel set $A \subseteq \partial \mathbb{D}$. As the boundary circle $\partial \mathbb{D}$ also carries the Lebesgue measure, it is natural to compare the two. In this note we prove the following.

**Theorem 1.** Let $P$ be a symmetric hyperbolic polygon in the Poincaré disk $\mathbb{D}$, with $2m$ sides, which is the standard fundamental domain for a cocompact Fuchsian group $G$. Let $T := \{t_1, t_2, \ldots, t_{2m}\}$ be the hyperbolic translations which identify opposite sides of $P$. Then, for any symmetric measure $\mu$ supported on the set $T$, the hitting measure on $S^1 = \partial \mathbb{D}$ is singular with respect to Lebesgue measure.

**Figure 1.** A symmetric hyperbolic octagon. Sides of the same color are identified by the Fuchsian group $G$. 

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The theorem addresses a conjecture of Kaimanovich-Le Prince [KP11]. In the 70’s, Furstenberg [Fu71] proved that for any discrete subgroup of $SL_2(\mathbb{R})$ there exists a measure $\mu$ such that the hitting measure (which is also the unique harmonic measure) of the corresponding random walk is absolutely continuous with respect to Lebesgue measure. However, such measures are inherently infinitely supported, as they arise from discretization of Brownian motion (see also [LS84]).

For finitely supported measures, the question of singularity of hitting measure has been asked several times. In particular, in [KP11] it was conjectured that every finitely supported measure on $SL_d(\mathbb{R})$ yields a singular hitting measure. For non-discrete subgroups of $SL_2(\mathbb{R})$, there exist finitely supported measures which are absolutely continuous at infinity ([Bo12], [BPS12]).

In the non-cocompact, discrete, case, singularity has been proven in many contexts and with different proofs ([GL90], [BHM11], [DKN09], [KP11], [GMT15], [DG18], [RT19]), which exploit in various ways the fact that the cusp subgroup is highly distorted in $G$. However, the cocompact, discrete, case is way harder, as the hyperbolic metric and the word metric are quasi-isometric to each other.

Recently, singularity of hitting measure for cocompact Fuchsian groups whose fundamental domain is a regular polygon has been proven by [Ko20] and [CLP17], except for a finite number of cases with few sides (in particular, the current paper is the first one to deal with the case of a hyperbolic octagon, even the regular one).

If one replaces the random walk with Brownian motion, then, on a negatively curved surface, the hitting measure is absolutely continuous if and only if the curvature is constant [Le95].

This problem is closely related to the following “numerical characteristics” of random walks. Recall that the entropy of $\mu$ is given by

$$h := \lim_{n \to \infty} \frac{-\sum_{g \in G} \mu^n(g) \log \mu^n(g)}{n}$$

and the drift is

$$\ell := \frac{d_{\mathbb{H}}(o, w_n o)}{n},$$

where $d_{\mathbb{H}}$ denotes the hyperbolic metric and the limit exists almost surely. The drift also equals the classical Lyapunov exponent for random matrix products [FK60]. Finally, the volume growth of $G$ is

$$v := \limsup_{n \to \infty} \frac{1}{n} \log \# \{ g \in G : d_{\mathbb{H}}(o, go) \leq n \}.$$

The inequality

$$h \leq \ell v \tag{1}$$

has been established by Guivarc’h [Gu80] and is called the fundamental inequality by Vershik [Ve00]. It is an old question (e.g. [Ve00, Question A]) to characterize under which conditions the inequality (1) is strict. Moreover, for discrete, cocompact actions, (1) is strict if and only if the hitting measure is singular with respect to Lebesgue measure ([BHM11], [GMM18]; see Theorem 4).

In [GMM18] it is proven that, replacing $d_{\mathbb{H}}$ above with a word metric on $G$, the inequality is strict unless the group $G$ is virtually free. Observe that on cocompact
Fuchsian groups any word metric is quasi-isometric to the hyperbolic metric; however, being quasi-isometric is not strong enough a condition to guarantee that the hitting measure is in the same class.

Note that for a cocompact Fuchsian group it is well-known that $v = 1$ (see e.g. [PR94]). Thus, Theorem 1 implies:

**Corollary 2.** Under the hypotheses of Theorem 1, the inequality $h < \ell$ is strict.

The approach of this paper is based on the fact that cocompact forces at least some of the generators to have long enough translation lengths (this is related to the collar lemma: two intersecting closed geodesics cannot be both short at the same time; also, the quotient Riemann surface has a definite positive area). Indeed, in Theorem 7 we prove a criterion for singularity in terms of the translation lengths of the generators, and then we show (Theorem 8) that our generating set $S$ satisfies the inequality:

$$\sum_{g \in S} \frac{1}{1 + e^{\ell(g)}} < 1$$

where $\ell(g)$ denotes the translation length of $g$ in the hyperbolic metric. Interestingly, our geometric inequality has exactly the same form as the one in [CS92], [ACCS96].

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1. Preliminary results

Let $\mu$ be a probability measure on a countable group $G$. We assume that $\mu$ is generating, i.e. the semigroup generated by the support of $\mu$ equals $G$. We define the step space as $(G^\mathbb{N}, \mu^\mathbb{N})$, and the map $\pi : G^\mathbb{N} \to G^\mathbb{N}$ as $\pi((g_n)_{n \in \mathbb{N}}) := (w_n)_{n \in \mathbb{N}}$, with for any $n$

$$w_n := g_1 g_2 \ldots g_n.$$  

The target space of $\pi$ is denoted by $\Omega$ and called the path space; as a set, it equals $G^\mathbb{N}$, and is equipped with the measure $\mathbb{P}_\mu := \pi_*(\mu^\mathbb{N})$.

Then, we define the first-passage function $F_\mu(x, y)$ as

$$F_\mu(x, y) := \mathbb{P}_\mu(\exists n : w_n x = y)$$

for any $x, y \in G$, and the Green metric $d_\mu$ on $G$, introduced in [BB07], as

$$d_\mu(x, y) := -\log F_\mu(x, y).$$

The following fact is well-known.

**Lemma 3.** Let $p : G \to H$ be a group homomorphism, let $\mu$ be a probability measure on $G$, and let $\mu := p_* \mu$. Then, for any $x, y \in G$,

$$d_\mu(p(x), p(y)) \leq d_\mu(x, y).$$
Proof. Since \( p \) induces a map from paths in \( G \) to paths in \( H \), we have \( \pi^n(p(g)) \geq \mu^n(g) \) for any \( g \in G \), any \( n \geq 0 \). Hence

\[
P_\pi(p(x), p(y)) \geq P_\mu(x, y)
\]

for any \( x, y \in G \), from which the claim follows. \( \square \)

We shall use the following criterion, which relates the absolute continuity of the hitting measure to the fundamental inequality. Recall that a group action is geometric if it is isometric, properly discontinuous, and cocompact.

**Theorem 4.** ([BHM11, Corollary 1.4, Theorem 1.5], [Ta19]) Let \( \Gamma \) be a non-elementary hyperbolic group acting geometrically on \( \mathbb{H}^2 \), endowed with the geometric distance \( d = d_{\mathbb{H}} \) induced from the action. Consider a generating probability measure \( \mu \) on \( \Gamma \) with finite support. Let us also assume that \( \mu \) is symmetric. Then the following conditions are equivalent:

1. The equality \( h = \ell v \) holds.
2. The Hausdorff dimension of the hitting measure \( \nu \) on \( S^1 \) is equal to \( 1 \).
3. The measure \( \nu \) is equivalent to the Lebesgue measure on \( S^1 \).
4. For any \( o \in \mathbb{H}^2 \), there exists a constant \( C > 0 \) such that for any \( g \in \Gamma \) we have

\[
|d_\mu(e, g) - d_{\mathbb{H}}(o, go)| \leq C.
\]

For each \( g \in G \), let \( \ell(g) \) denote its translation length, namely

\[
\ell(g) := \lim_{n \to \infty} \frac{d_{\mathbb{H}}(o, g^na)}{n}.
\]

Equivalently, \( \ell(g) \) is the length of the corresponding closed geodesic on the quotient surface. The mechanism to utilize Theorem 4 is through the following lemma, similar to the one from [Ko20].

**Lemma 5.** Suppose that the hitting measure is absolutely continuous. Then for any \( g \in G \) we have

\[
\ell(g) \leq d_\mu(e, g).
\]

Proof. If not, then \( \ell(g) > d_\mu(e, g) \geq 0 \), hence \( g \) is loxodromic. Let us pick some \( o \in \mathbb{H}^2 \) which lies on the axis of \( g \), so that \( d_{\mathbb{H}}(o, g^k o) = \ell(g^k) = k\ell(g) \) for any \( k \). Moreover, by the triangle inequality for the Green metric one has \( d_\mu(e, g^k) \leq k d_\mu(e, g) \), hence

\[
d_{\mathbb{H}}(o, g^k o) - d_\mu(e, g^k) \geq k\ell(g) - k d_\mu(e, g) = k(\ell(g) - d_\mu(e, g))
\]

thus, since \( \ell(g) - d_\mu(e, g) > 0 \),

\[
\sup_{k \in \mathbb{N}} |d_{\mathbb{H}}(o, g^k o) - d_\mu(e, g^k)| = +\infty,
\]

which contradicts Theorem 4. \( \square \)

Let \( F \) be a free group, freely generated by a finite set \( S \). Recall the (hyperbolic) boundary \( \partial F \) of \( F \) is the set of infinite, reduced words in the alphabet \( S \cup S^{-1} \). Given a finite, reduced word \( g \), we denote as \( C(g) \subseteq \partial F \) the cylinder determined by \( g \), namely the set of infinite, reduced words which start with \( g \).
Lemma 6. Consider a random walk on the free group

\[ F_m = \langle s_1^{\pm 1}, \ldots, s_m^{\pm 1} \rangle, \]

defined by a symmetric probability measure \( \mu \) on the generators. If we denote \( x_i := F_\mu(e, s_i) = F_\mu(e, s_i^{-1}) \), and the hitting measure on the boundary of \( F_m \) by \( \nu \), then

\[ \nu(C(s_i)) = \frac{x_i}{1 + x_i}. \]

A similar lemma is stated in [La18, Exercise 5.14].

Proof. For any infinite word \( w = s_{j_1} s_{j_2} s_{j_3} \ldots \) there exist two possibilities:

1. There exists a subword \( s_{j_1} \ldots s_{j_k} \) such that it equals \( s_i \) in \( F_m \)
2. No subword \( s_{j_1} \ldots s_{j_k} \) equals \( s_i \), so it belongs to the set of paths which
never hit \( s_i \).

In the first case we denote this subword by \( w_1 \), and we consider \( w^{-1}_1 \) and we
apply the same procedure, but replacing \( s_i \) with \( s_i^{-1} \) at each subsequent step. This
procedure yields the equality

\[ \nu(C(s_i)) = P(e \to s_i \not\to e) + P(e \to s_i \to e \to s_i \not\to e) + \cdots = \sum_{n=0}^{\infty} F_\mu(e, s_i)^{2n+1}(1 - F_\mu(e, s_i)) = \sum_{n=1}^{\infty} (-1)^{n+1} F_\mu(e, s_i)^{n} = \frac{x_i}{1 + x_i}. \]

\[ \square \]

2. A criterion for singularity

Theorem 7. Let \( \mu \) be a symmetric, finitely supported measure on a cocompact Fuchsian group, and let \( S \) be the support of \( \mu \). Suppose that

\[ \sum_{g \in S} \frac{1}{1 + e^{\ell(g)}} < 1. \]

Then the hitting measure \( \nu \) on \( \partial D \) is singular with respect to Lebesgue measure.

Proof. Let \( F \) be a free group of rank \( m \), with generators \( \langle h_i \rangle_{i=1}^m \), and let \( \tilde{\mu} \) be a
measure on \( F \) with \( \tilde{\mu}(h_i) = \mu(g_i) \). Moreover, let us denote

\[ x_i := F_\tilde{\mu}(e, h_i) = F_\tilde{\mu}(\exists n : w_n = h_i). \]

Then we have

\[ \sum_{i=1}^{m} \frac{x_i}{1 + x_i} = \frac{1}{2}. \]

Indeed, if \( \tilde{\nu} \) is the hitting measure on \( \partial F \), by Lemma 6 the measure of the cylinder \( C(h_i) \) starting with \( h_i \) is

\[ \tilde{\nu}(C(h_i)) = \frac{x_i}{1 + x_i}, \]

from which (4) follows. If the hitting measure \( \nu \) on \( S^1 = \partial D \) is absolutely continuous, then by Lemma 5 and Lemma 3 we get

\[ \ell(g_i) \leq d_\mu(e, g_i) \leq d_\tilde{\mu}(e, h_i) = -\log x_i \]

for any \( i \). Hence

\[ \frac{x_i}{1 + x_i} \leq \frac{1}{1 + e^{\ell(g_i)}}. \]
and, by summing over all \(i\)'s,
\[
1 = 2 \sum_{i=1}^{m} \frac{x_i}{1 + x_i} \leq 2 \sum_{i=1}^{m} \frac{1}{1 + e^{\ell(g_i)}} < 1,
\]
which contradicts (3). Hence, \(\nu\) is singular with respect to Lebesgue measure. \(\Box\)

3. PARAMETERIZATION OF THE SPACE OF POLYGONS

Following \[Ga79\], a hyperelliptic polygon is a polygon \(P\) in the hyperbolic plane, with \(2m\) sides \((L_i)_{i=1}^{2m}\) which are geodesic for the hyperbolic metric, and such that:

(1) The sum of all angles of \(P\) is \(2\pi\);
(2) Opposite sides have equal length.

Given a hyperelliptic polygon \(P\), one considers for each pair \((L_i, L_{i+m})\) of opposite sides a hyperbolic translation \(g_i\) such that \(g_i(L_i) = L_{i+m}\). By \[Ga79\], the set \((g_i)_{i=1}^{m}\) generates a cocompact, hyperelliptic Fuchsian group, and moreover, every hyperelliptic Fuchsian group has a fundamental domain which is a hyperelliptic polygon \(P\) constructed as above.

Theorem 8. Let \(P\) be a hyperelliptic polygon, and let \(G\) be its associated cocompact Fuchsian group. Let \(S := \{g_1, \ldots, g_{2m}\}\) be the set of hyperbolic translations identifying opposite sides of \(P\). Then we have
\[
\sum_{g \in S} \frac{1}{1 + e^{\ell(g)}} < 1.
\]

Figure 2. Angles at the center and at the vertices of a symmetric hyperbolic octagon.

Remarks.

Note that if one replaces (1) above by the sum of the angles of \(P\) being equal to \(\frac{2\pi}{k}\), with \(k > 1\) an integer, Theorem 8 (hence also Theorem 1) still holds, with the same proof. By Poincaré’s theorem, this still yields a cocompact (though not torsion-free) Fuchsian group.
The inequality (5) has the same form as the main inequality in [ACCS96] and [CS92] for free Kleinian groups. It is also reminiscent of McShane’s identity [McS98], where one obtains the equality by taking the infinite sum over all group elements of a punctured torus group. Our inequality, however, does not follow from either of them; in fact, is is in a way stronger than these, as a cocompact surface group can be deformed to a finite covolume group and then to a Schottky (hence free) group by increasing the translation lengths of the generators. It is interesting to point out that the above inequalities have an interpretation in terms of hitting measures of stochastic processes (see also [LT18]). Here, we go along the opposite route: we prove the geometric inequality (5) and then we use it to conclude properties about the hitting measure.

Finally, there are generating sets of \( G \) for which (5) fails. Indeed, the mechanism behind the inequality is that, since all curves corresponding to \( (g_i)_{i=1}^m \) intersect each other, by the collar lemma, at most one of them can be short. In general, on a surface of genus \( g \) one can choose a configuration of \( 3g-3 \) short curves, and construct a Dirichlet domain for which the corresponding side pairing does not satisfy (5).

**Proof.** A way to parameterize the space of all symmetric hyperbolic polygons is to note that, by [Bu10, Example 2.2.7],

\[
\cos(\gamma_i) = -\cosh(a_i) \cosh(a_{i+1}) \cos(\alpha_i) + \sinh(a_i) \sinh(a_{i+1})
\]

with \( i = 1, \ldots, m \), where \( (a_i) \) are the distances between the base point and the \( i \)th side, \( (\alpha_i) \) are the angles at the origin and \( (\gamma_i) \) are the angles at the vertices. Since \( \ell(g_i) \geq 2a_i \), it is enough to show

\[
\sum_{i=1}^m \frac{1}{1 + e^{2a_i}} < \frac{1}{2}
\]

under the constraints \( \sum_{i=1}^m a_i = \pi \) and \( \sum_{i=1}^m \gamma_i = \pi \).

The fundamental geometric idea in our approach to Theorem 8 is that two intersecting curves cannot be both short, as a consequence of the collar lemma [Bu78]. For instance, we get:

**Lemma 9.** Suppose that there exists \( a_i \) such that \( \sinh(a_i) \leq \frac{2(m-1)}{m(m-2)} \). Then the hitting measure is singular.

**Proof.** From the collar lemma [Bu78] we have

\[
\sinh(a_i) \sinh(a_j) \geq 1
\]

for all \( i \neq j \). Recall that

\[
\frac{2}{1 + e^{2a}} = 1 - \tanh(a)
\]

hence, if we set \( s := \sinh(a_1) \), we obtain for \( i \neq 1 \) that \( \sinh(a_i) \geq \frac{1}{s} \) thus

\[
\tanh(a_i) = \frac{\sinh(a_i)}{\sqrt{1 + \sinh(a_i)^2}} \geq \frac{1}{\sqrt{1 + s^2}}
\]

hence

\[
\sum_{i=1}^m \tanh(a_i) \geq \frac{s}{\sqrt{1 + s^2}} + \frac{m-1}{\sqrt{1 + s^2}} > m - 1
\]

if and only if \( s < \frac{2(m-1)}{m(m-2)} \). \(\square\)
To actually prove Theorem 8, however, we need an improvement on the previous estimate. Let us rewrite equation (6) above as

$$\cos(\alpha_i) = \tanh(a_i) \tanh(a_{i+1}) - \frac{\cos(\gamma_i)}{\cosh(a_i) \cosh(a_{i+1})}$$

and, recalling that

$$\tanh^2(x) + \frac{1}{\cosh^2(x)} = 1$$

we obtain, by setting \(z_i = \tanh(a_i)\),

$$(7) \quad \cos(\alpha_i) = z_i z_{i+1} - \cos(\gamma_i) \sqrt{1 - z_i^2} \sqrt{1 - z_{i+1}^2}$$

with \(0 \leq z_i \leq 1\). Finally, we want to show

$$m \sum_{i=1}^{m} \frac{1}{1 + e^{2a_i}} = \sum_{i=1}^{m} \frac{1 - z_i}{2} < \frac{1}{2},$$

which is equivalent to

$$\sum_{i=1}^{m} z_i > m - 1.$$  

Now, let us first assume that \(\gamma_i \leq \pi/2\) for all \(1 \leq i \leq m\). Then (7) yields

$$\cos(\alpha_i) \leq z_i z_{i+1}$$

hence the constraint becomes

$$(9) \quad \sum_{i=1}^{m} \arccos(z_i z_{i+1}) \leq \pi.$$  

Note that \(z_1 \to 0\) implies \(\cos(\alpha_1) \leq z_1 z_2 \to 0\) thus \(\alpha_1 \to \pi/2\) and \(\cos(\alpha_i) \leq z_i z_{i+1} \to 0\) thus \(\alpha_i \to \pi/2\), hence also \(\alpha_2, \alpha_3, \ldots, \alpha_{m-1} \to 0\), which implies \(z_2, z_3, \ldots, z_m \to 1\).

4. AN OPTIMIZATION PROBLEM

By the above discussion, we reduced the proof of Theorem 8 (at least in the case all angles of \(P\) are acute) to the following optimization problem.

**Theorem 10.** Let \(m \geq 3\) and \(0 \leq x_i \leq 1\) with \(\sum_{i=1}^{m} x_i = m - 1\). Then

$$\sum_{i=1}^{m} \arccos(x_i x_{i+1}) \geq \pi.$$  

Moreover, equality holds if and only if there exists an index \(i\) such that \(x_i = 0\) and \(x_j = 1\) for all \(j \neq i\).

In the statement of Theorem 10 and elsewhere from now on, all indices \(i\) are meant modulo \(m\). The next is the main technical lemma.

**Lemma 11.** Let \(m \geq 3\) and \(0 \leq x_i \leq 1\) with \(\sum_{i=1}^{m} x_i = 1\). Then

$$\sum_{i=1}^{m} \sqrt{x_i + x_{i+1} - x_i x_{i+1}} \geq \sqrt{4 + 3 \sum_{i=1}^{m} x_i x_{i+1}}.$$
Figure 3. The graph of $f(x) := \sum_{i=1}^{3} \arccos((1 - x_{i})(1 - x_{i+1}))$ subject to the constraint $\sum_{i=1}^{3} x_{i} = 1$, compared with the constant function at height $\pi$. The lack of convexity (or concavity) of $f$ makes the proof of Theorem 10 trickier.

Proof. Set $\Delta_{i} := x_{i} + x_{i+1} - x_{i}x_{i+1}$. Note that

$$\Delta_{i} \geq \max\{x_{i}, x_{i+1}\}$$

hence

$$\sqrt{\Delta_{i} \Delta_{i+1}} \geq x_{i+1}. \tag{10}$$

Moreover, since $m \geq 2$, we have $x_{i+1} + x_{i+2} \leq \sum_{i=1}^{m} x_{i} = 1$, hence if we multiply by $(x_{i+1} + x_{i+2})$, we obtain

$$\Delta_{i} = x_{i} + x_{i+1} - x_{i}x_{i+1}$$
$$\geq (x_{i} + x_{i+1})(x_{i+1} + x_{i+2}) - x_{i}x_{i+1}$$
$$\geq x_{i+1}^{2} + x_{i+1}x_{i+2}.$$

Similary, we obtain

$$\Delta_{i+2} = x_{i+2} + x_{i+3} - x_{i+2}x_{i+3}$$
$$\geq (x_{i+2} + x_{i+3})(x_{i+1} + x_{i+2}) - x_{i+2}x_{i+3}$$
$$\geq x_{i+2}^{2} + x_{i+1}x_{i+2}.$$

Thus, Cauchy-Schwarz yields

$$\sqrt{\Delta_{i} \Delta_{i+2}} \geq \sqrt{x_{i+1}^{2} + x_{i+1}x_{i+2}} \sqrt{x_{i+2}^{2} + x_{i+1}x_{i+2}} \geq 2x_{i+1}x_{i+2}. \tag{11}$$

By squaring both sides, our desired inequality is equivalent to

$$\sum_{i=1}^{m} \Delta_{i} + 2 \sum_{1 \leq i < j \leq m} \sqrt{\Delta_{i} \Delta_{j}} \geq 4 + 3 \sum_{i=1}^{m} x_{i}x_{i+1},$$
thus, using $\sum_{i=1}^{m} \Delta_i = 2 - \sum_{i=1}^{m} x_i x_{i+1}$, it is enough to prove

$$
\sum_{1 \leq i < j \leq m} \sqrt{\Delta_i \Delta_j} \geq 1 + 2 \sum_{i=1}^{m} x_i x_{i+1}.
$$

Now, note that

$$
\sum_{1 \leq i < j \leq m} \sqrt{\Delta_i \Delta_j} = \sum_{i=1}^{m} \sqrt{\Delta_i \Delta_{i+1}} + M
$$

with

(13) $M = 0$ \quad \text{if } m = 3

(14) $M = \sum_{i=1}^{2} \sqrt{\Delta_i \Delta_{i+2}}$ \quad \text{if } m = 4

(15) $M \geq \sum_{i=1}^{m} \sqrt{\Delta_i \Delta_{i+2}}$ \quad \text{if } m \geq 5.

Thus, for $m \geq 5$ we have, using (15), (10) and (11),

$$
\sum_{1 \leq i < j \leq m} \sqrt{\Delta_i \Delta_j} \geq \sum_{i=1}^{m} \sqrt{\Delta_i \Delta_{i+1}} + \sum_{i=1}^{m} \sqrt{\Delta_i \Delta_{i+2}}
\quad \geq \sum_{i=1}^{m} x_{i+1} + 2 \sum_{i=1}^{m} x_{i+1} x_{i+2}
\quad \geq 1 + 2 \sum_{i=1}^{m} x_{i+1} x_{i+2}
$$

which yields (12), hence completes our proof. The cases $m = 3$ and $m = 4$ need to be dealt with separately. If $m = 3$, we obtain, by multiplying by $\sum_{i=1}^{3} x_i = 1$,

$$
\Delta_i = x_{i}^{2} + x_{i+1}^{2} + \sum_{i=1}^{3} x_i x_{i+1}
$$

so by Cauchy-Schwarz we get

$$
\sqrt{\Delta_i \Delta_{i+1}} \geq x_{i+1}^{2} + x_{i+1} x_{i+2} + \sum_{i=1}^{3} x_i x_{i+1}
$$

hence

$$
\sum_{i=1}^{3} \sqrt{\Delta_i \Delta_{i+1}} \geq \sum_{i=1}^{3} x_{i}^{2} + 4 \sum_{i=1}^{3} x_i x_{i+1}
\quad = \left( \sum_{i=1}^{3} x_i \right)^{2} + 2 \sum_{i=1}^{3} x_i x_{i+1}
\quad = 1 + 2 \sum_{i=1}^{3} x_i x_{i+1}
$$
which yields (12), as desired. Finally, if \( m = 4 \), then we note
\[
\sum_{1 \leq i < j \leq 4} \sqrt{\Delta_i \Delta_j} = \sum_{i=1}^{4} \sqrt{\Delta_i \Delta_{i+1}} + \sum_{i=1}^{2} \sqrt{\Delta_i \Delta_{i+2}}
\]
and, again by Cauchy-Schwarz,
\[
\sqrt{\Delta_1 \Delta_3} \geq \sqrt{x_1^2 + x_2^2 + x_1 x_3 + 2 x_1 x_3^2 + x_2 x_3^2} \geq 2 x_1 x_3 + 2 x_2 x_3
\]
and similarly
\[
\sqrt{\Delta_2 \Delta_4} \geq 2 x_2 x_4
\]
thus, using (10),
\[
\sum_{1 \leq i < j \leq 4} \sqrt{\Delta_i \Delta_j} \geq 4 \sum_{i=1}^{4} x_i + 2 \sum_{i=1}^{4} x_i x_{i+1} = 1 + 2 \sum_{i=1}^{4} x_i x_{i+1}
\]
which is again (12). This completes the proof.

**Lemma 12.** For \( 0 \leq x \leq 1 \) we have the inequalities:

1. \[
\frac{2}{\pi} \arccos(1 - x) \geq \frac{2}{3} \sqrt{x + \frac{1}{3} x^2}
\]
   with equality if and only if \( x = 0 \) or \( x = 1 \);
2. \[
\frac{2}{3} \sqrt{4 + 3x} + \frac{2 - x}{3} \geq 2
\]
   with equality if and only if \( x = 0 \).

**Proof.** For the first inequality, let \( f(x) := \frac{2}{\pi} \arccos(1 - x^2) - \frac{2}{3} x - \frac{1}{3} x^2 \). One checks that \( f(0) = f(1) = 0 \) and \( f\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2} - \frac{\sqrt{2}}{3} > 0 \); moreover, \( f'(x) \) has a unique zero in \([0, 1]\). Hence, \( f(x) \geq 0 \) for all \( 0 \leq x \leq 1 \), which implies (1).

To prove (2), let \( g(x) := \frac{2}{\pi} \sqrt{4 + 3x} + \frac{2 - x}{3} \). Then one checks \( g(0) = 2 \) and \( g'(x) = \frac{1}{\sqrt{4 + 3x}} - \frac{1}{3} > 0 \) for \( 0 \leq x \leq 1 \), which implies \( g(x) \geq 2 \) for all \( 0 \leq x \leq 1 \). \( \square \)

**Proof of Theorem 10.** By replacing \( x_i \) by \( 1 - x_i \) and setting \( f(x) := \frac{2}{\pi} \arccos(1 - x) \), our claim is equivalent to
\[
\sum_{i=1}^{m} f(x_i + x_{i+1} - x_i x_{i+1}) \geq 2
\]
under the constraint \( \sum_{i=1}^{m} x_i = 1 \), with \( m \geq 3 \) and \( 0 \leq x_i \leq 1 \).

Let us set \( \Delta_i := x_i + x_{i+1} - x_i x_{i+1} \) and \( \sigma := \sum_{i=1}^{m} x_i x_{i+1} \). Observe that \( 2\sigma \leq (\sum_{i=1}^{m} x_i)^2 = 1 \). Then we have by Lemma 12
\[
\sum_{i=1}^{m} f(\Delta_i) \geq \frac{2}{3} \sum_{i=1}^{m} \sqrt{\Delta_i} + \frac{1}{3} \sum_{i=1}^{m} \Delta_i
\]
and using Lemma 11 and the fact \( \sum_{i=1}^{m} \Delta_i = 2 - \sigma \), we obtain
\[
\geq \frac{2}{3} \sqrt{4 + 3\sigma} + \frac{1}{3} (2 - \sigma) \geq 2
\]
where in the last step we apply Lemma 12 (2). This completes the proof of the inequality. By Lemma 12 (1), equality implies that \( \Delta_i = 0, 1 \) for every \( i \), which in
Figure 4. The hyperbolic pentagon of Lemma 13.

5. The obtuse angle case

The proof in the previous section works as long as all angles $\gamma_i$ are less than $\pi/2$. If one of them is obtuse (note that only one of them may be so, since the sum satisfies $\sum_{i=1}^{m} \gamma_i = \pi$), we have a geometric argument to reduce ourselves to that case.

Lemma 13. Let $ABCDE$ be a hyperbolic pentagon, with right angles $\hat{B}$ and $\hat{E}$, and suppose that $\hat{C} < \pi/2$ and $\hat{C} + \hat{D} < \pi$. Let $P$ be the midpoint of $CD$, and let $\hat{F}$ be the foot of the orthogonal projection of $P$ to $BC$. Let $\hat{G}$ be the intersection of the lines $FP$ and $ED$. Then the angle $\delta = \angle DGF$ satisfies $\delta < \pi/2$.

Proof. Let $F'$ be the symmetric point to $F$ with respect to $P$. Then $CFP$ and $DPF'$ are equal triangles. Hence $E\hat{D}F' = E\hat{D}P + P\hat{D}F' = E\hat{D}C + B\hat{C}D < \pi$, hence $F'$ lies on the segment $PG$. Moreover, $DF'P = \angle CF'P = \pi/2$, hence $\delta = \angle DG\leq < \pi/2$. □

Let us use the notation

$$\varphi(x_1, x_2, \ldots, x_m) := \sum_{i=1}^{m} \frac{1}{1 + e^{2x_i}}.$$ 

Proposition 14. Let $P$ be a hyperelliptic hyperbolic polygon with $2m$ sides and center $o$, and let $\ell_1, \ldots, \ell_m$ be the distances between $o$ and the midpoints of the sides. Then there exists a symmetric hyperbolic $2m$-gon $P'$ whose angles are all acute and such that

$$\varphi(\ell_1, \ell_2, \ldots, \ell_m) \leq \varphi(d'_1, d'_2, \ldots, d'_m)$$

where $d'_i$ is the distance between $o$ and the $i$th side of $P'$. 

\[\]
Proof. Let us denote as $d_i$ the distance between $o$ and the $i$th side of $P$. Note that by definition $d_i \leq \ell_i$ for all $i$.

If the polygon $P$ only has acute angles, we take $P = P'$ and note that by definition $d'_i = d_i \leq \ell_i$, which yields the claim.

Suppose now that the hyperbolic polygon $P$ has one obtuse angle, and let $\ell_1$ correspond to one of the two sides which are adjacent to the obtuse angle. Consistently with this choice, let us denote as $s_1, s_2, \ldots, s_{2m}$ the sides of $P$.

Let us now consider the hyperbolic pentagon delimited by $s_{2m}, s_1, s_2$, and the orthogonal projections from $o$ to $s_2$ and $s_{2m}$. Let us call this pentagon $ABCDE$, where $o = A$, the side $s_1$ is denoted $\overrightarrow{DC}$, the orthogonal projection from $o$ to $s_2$ is $B$, and the orthogonal projection from $o$ to $s_{2m}$ is $E$.

Using Lemma 13 let us replace $P$ by a new polygon $P'$ obtained substituting the pentagon $ABCDE$ by the pentagon $ABFGE$, which satisfies $\tilde{F} = \pi/2$ and $\tilde{G} \leq \pi/2$. If we denote by $d'_i$ the distance between $o = A$ and $FG$, then we have

$$d'_i = d(A, \overrightarrow{FG}) \leq d(A, P) = \ell_i.$$  

On the other hand, note that for $i = 2, \ldots, m$ the distance between $o$ and the $i$th side is the same for $P$ and $P'$. That is, $d_i = d'_i$ for $i = 2, \ldots, m$. Hence,

$$\varphi(\ell_1, \ell_2, \ldots, \ell_m) \leq \varphi(\ell_1, d_2, \ldots, d_m) \leq \varphi(d'_1, d_2, \ldots, d'_m) = \varphi(d'_1, d'_2, \ldots, d'_m)$$

which proves the claim.

This deals with polygons with one obtuse angle, completing the proof of Theorem 8. Let us see the details.

Proof of Theorem 8 Let us first suppose that $\gamma_i \leq \pi/2$ for all $i$. We know by \([\text{0}]\) that $\sum_{i=1}^m \arccos(z_i z_{i+1}) \leq \pi$ with $0 < z_i < 1$. Then we need to show that $\sum_{i=1}^m z_i > m - 1$. Suppose not, then there exists $z_i$ with $\sum_{i=1}^m z_i \leq m - 1$. Then there exists $(z'_i)_{i=1}^m$ with $0 \leq z_i \leq z'_i \leq 1$ for all $i$, so that $\sum z'_i = m - 1$. Then we have, by Theorem 10 $\pi \leq \sum_{i=1}^m \arccos(z'_i z'_{i+1}) \leq \sum_{i=1}^m \arccos(z_i z_{i+1}) \leq \pi$, hence $\sum_{i=1}^m \arccos(z'_i z'_{i+1}) = \pi$, which by the second part of Theorem 10 implies $z'_i = 0$ for some $i$, hence also $z_i = 0$, which is a contradiction. If there is an $i$ for which $\gamma_i > \pi/2$, then we reduce to the previous case by applying Proposition 14.

Proof of Theorem 7 Theorem 8 shows that the criterion of Theorem 7 holds, proving the singularity of hitting measure.

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