Decay of the Cosmic Vacuum Energy

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Energy-momentum conservation suggests that a vacuum in thermal equilibrium with a bath of radiation during inflation should gradually diminish the vacuum energy. We find that coupling to a bath of black-body radiation at temperature $T = H/2\pi$ requires the Hubble rate, $H$, to evolve as in the “intermediate inflation” scenario, with $H \propto t^{-1/3}$, rather than as a constant. Such behaviour does not conflict with observations when the vacuum energy is described by a slowly-rolling scalar field, but will change the asymptotic states of the universe. We find that this scenario introduces a curvature singularity at early times. The scale factor takes a finite non-zero value at this singularity, while the energy densities in radiation and the vacuum diverge to positive and negative infinity, respectively. This shows that inflation is possible even when the energy density of the vacuum is large and negative. Furthermore, the introduction of an additional non-interacting perfect fluid into the space-time reveals that radiation can dominate over dust at late times, in contrast to what occurs in the standard cosmological model.

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I. INTRODUCTION

Inflationary cosmology is based on the hypothesis of a period of accelerated expansion in the very early history of the universe. This surge in the expansion solves the horizon problem, and is generically expected to drive the observable curvature of space and any expansion or curvature anisotropies to unobservably small values today. In addition, inflationary cosmology also provides a natural mechanism for creating the seeds of structure formation from tiny quantum mechanical fluctuations. These fluctuations are a manifestation of the thermal nature of quantum fields in curved spaces. But the existence of a thermal space also implies the existence of a bath of radiation. In this paper we will consider the gravitational consequences of this radiation on the expansion of the universe and on the observables that emerge from a Planckian description of radiation during inflation.

The energy density of radiation with a black-body spectrum at the Gibbons-Hawking temperature, $T = H/2\pi$, is given in Planck units as

$$\rho_r = 4\sigma T^4 = \frac{g_* H^4}{480\pi^2}, \quad (1)$$

where $\sigma = g_\ast \pi^2/120$ is the Stefan-Boltzmann constant, $H$ is the Hubble expansion rate, and $g_\ast$ is the effective number of relativistic degrees of freedom.

In a Friedmann-Lemaître-Robertson-Walker (FLRW) geometry, the energy-conservation equation for this radiation fluid is given by

$$\dot{\rho}_r + 3H \rho_r = Q, \quad (2)$$

where $Q = Q(t)$ is an energy exchange term that is required in order for Eq. (1) to be satisfied at all times, and overdots denote derivatives with respect to the co-moving proper time, $t$. The $Q$ term parameterizes the energy flow into the radiation field, and thermalization is assumed to be instantaneous.

Energy-momentum conservation now requires that $T^\mu_\nu = 0$, where $T^\mu_\nu$ is the total energy-momentum tensor of all matter fields in the space-time. If we are considering a space-time that contains effectively just radiation ($r$) and vacuum ($v$) energy, with $T^\mu_\nu = T^r_\mu_\nu + T^v_\mu_\nu$, then we must therefore also have

$$\dot{\rho}_v = -Q, \quad (3)$$

where $\rho_r = -\rho_v$ is the energy density of the vacuum, and $p_v$ is its pressure. This equation shows that the vacuum energy density must be decaying, and is the cosmological counterpart of the requirement that radiating black holes must reduce in mass, in order for the total energy-momentum in the space-time to be conserved [2].

The equations above, together with the Friedmann equation,

$$H^2 = \frac{8\pi}{3} (\rho_r + \rho_v),$$

can be combined to show that the energy-exchange term, $Q$, is given by

$$Q = \frac{g_*}{120\pi^2} H^3 (\dot{H} + H^2).$$

This in turn implies

$$\dot{H} + \frac{g_* H^4}{90\pi} = 0, \quad (4)$$

which, using $H = \dot{a}/a$, can be integrated to find the form
of the expansion scale factor to be:

\[ a(t) \propto \exp \left\{ \frac{3}{2} \left( \frac{30\pi}{g_s} \right)^\frac{2}{3} (t - t_0)^\frac{2}{3} \right\}, \quad (5) \]

where \( t_0 = \text{constant} \). This type of expansion is of a type known as “intermediate inflation” [3], which generally has \( a(t) \propto \exp\{\lambda t^{2/3}\} \), where \( A > 0 \) and \( 0 < n < 1 \) are constants. Intermediate inflation has been studied by several authors [3, 2, 8], and is known to arise in rainbow gravity theories [9].

The particular form of intermediate inflation with \( a(t) \propto \exp\{\lambda t^{2/3}\} \) is special. When generated from a minimally coupled scalar field in a suitably chosen potential, it is the only form of intermediate inflation (other than perfect de Sitter) that gives an exact Harrison-Zeldovich spectrum of first-order density perturbations. Unlike standard slow-roll scenarios, however, it is also known that this type of intermediate inflation can produce large amounts of gravitational radiation [3-6].

Eq. (5) is a significant departure from the usual exponential expansion, and occurs even though the energy density of radiation may be small. This can be attributed to the dual requirements of an almost constant density of radiation, as well as the exponential dilution of that radiation with inflationary expansion. Therefore, the vacuum energy must constantly replace the quickly dissipating radiation, and even though the amount of radiation required at any given time may be small, it must effectively be replenished at every moment of time.

II. ENERGY EXCHANGE DURING INFLATION

It is natural to consider the effects of the interaction introduced in Section I on the observables that result from thermal fluctuations during inflation. To do this, we model the vacuum energy as a scalar field with a self-interaction potential, \( V(\phi) \), such that

\[ \rho_v = \frac{1}{2} \dot{\phi}^2 + V(\phi) \quad \text{and} \quad p_v = \frac{1}{2} \phi^2 - V(\phi). \quad (6) \]

In terms of these new variables, the Friedmann and conservation equations can be manipulated into the form

\[ \dot{H} - \frac{g_s H^4}{180\pi} + 3H^2 - 8\pi V = 0 \]

\[ \dot{\phi}^2 + 2V - \frac{3H^2}{4\pi} + \frac{g_s H^4}{240\pi^2} = 0. \quad (8) \]

This simple system of equations represents a 2-dimensional dynamical system, the solutions of which can be found only after the form of the potential \( V(\phi) \) is specified. In the absence of the terms involving \( g_s \), it can be seen that these equations reduce to the usual ones for a scalar field-filled Friedmann model.

In order to proceed further, we define the slow-roll parameters by [10]

\[ \epsilon_H \equiv \frac{3\dot{\phi}^2}{2V + \dot{\phi}^2} = -\frac{\dot{H} + g_s H^4}{H^2(1 - 2g_s H^2/180\pi)} \]

\[ \eta_H \equiv -\frac{\ddot{\phi}}{H \dot{\phi}} = \frac{\dot{H} + g_s H^2 H}{2H(1 - 2g_s H^2/180\pi)} \quad (10) \]

As usual, the spectral index of the primordial curvature perturbations, \( n_s \), will be given by

\[ n_s - 1 = \frac{d}{d\ln k} \ln P_R, \]

where the right-hand side is evaluated at horizon crossing. Using \( P_R = H^4/(2\pi)^2 \dot{\phi}^2 \) [10], this gives

\[ n_s - 1 = \frac{(4\dot{H} - H \ddot{H})}{H^2(1 - 2g_s H^2/180\pi)} \]

\[ = -4\epsilon_H \left( 1 - \frac{g_s H^2}{180\pi} \right) + 2\eta_H - \frac{2g_s H^2}{45\pi}, \quad (13) \]

where we have used Eqs. (9) and (10) to write this expression in terms of \( \epsilon_H \) and \( \eta_H \). Likewise, we can define the spectral index of tensor perturbations to be given by

\[ n_T - 1 = \frac{d}{d\ln k} \ln P_{\text{grav}}. \]

Using \( P_{\text{grav}} = 4H^2/\pi \) [10], this gives

\[ n_T - 1 = \frac{2\dot{H}}{H^2} \]

\[ = -2\epsilon_H \left( 1 - \frac{g_s H^2}{180\pi} \right) - \frac{g_s H^2}{45\pi}. \quad (16) \]

Both Eqs. (13) and (16) can be seen to reduce to the usual expressions, when \( g_s \) vanishes. It is also the case that the consistency relation between \( n_s \) and \( n_T \) is unchanged by the relaxation due to radiation creation. That is, \( n_s - 1 = 2n_T + 2\eta_H \).

Further observables of interest are given by the amplitude of scalar and tensor fluctuations, which are given respectively by

\[ A_s \equiv \frac{H^4}{(2\pi)^2 \dot{\phi}^2} \quad \text{and} \quad A_T \equiv \frac{16H^2}{\pi}. \quad (17) \]

The tensor-to-scalar ratio, \( r \), is then defined by

\[ r \equiv \frac{A_T}{A_s} = \frac{16\dot{H}}{H^2} - \frac{8g_s H^2}{45\pi} \]

\[ = 16\epsilon_H \left( 1 - \frac{g_s H^2}{180\pi} \right). \quad (19) \]

One can therefore derive the following relation between \( n_s \) and \( r \):
Recent observations imply that the amplitude of curvature perturbations generated during inflation is given by 

$$n_s - 1 = -r/4 - 2g_sH^2/45\pi + 2\eta_H. \tag{20}$$

while the scalar spectral index is inferred to be

$$n_s - 1 = -0.0365 \pm 0.0094, \tag{22}$$

and the tensor-to-scalar ratio is constrained by 

$$r \lesssim 0.2. \tag{23}$$

Together, these constraints imply

$$\epsilon_H \left(1 - \frac{g_sH^2}{180\pi}\right) \lesssim 0.01, \tag{24}$$

$$\eta_H - \frac{g_sH^2}{45\pi} \lesssim 0.01, \tag{25}$$

and

$$|H| \lesssim 10^{-5}. \tag{26}$$

For \(g_s \ll 10^{10}\) therefore, we have that observational constraints on \(\epsilon_H\) and \(\eta_H\) are unchanged, and that the expressions for the spectral indices of scalar and tensor fluctuations, as well as their amplitudes, are effectively given by the usual expressions in terms of the slow-roll parameters. The level of non-gaussianity is expected to be that for standard slow-roll inflationary models, and \(f_{NL} \sim O(\epsilon_H\text{ and } \eta_H) \sim 10^{-2}\), which is unobservably small \(^{11}\).

### III. ENERGY EXCHANGE IN THE PRESENCE OF A NON-INTERACTING PERFECT FLUID

If we also include in the universe a non-interacting perfect fluid, with equation of state

$$p = (\gamma - 1)\rho,$$

then the Friedmann equation becomes

$$H^2 = \frac{8\pi}{3} (\rho_r + \rho_v + \rho). \tag{27}$$

The energy conservation equations for \(\rho_r\) and \(\rho_v\) can again be written as in Eqs. [2] and [3]. The energy density in the non-interacting field, which we take to be separately conserved, is given by

$$\dot{\rho} + 3\gamma H \rho = 0. \tag{28}$$

It is straightforward to see that if the radiation is again taken to be specified by a black body, so that Eq. [1] is applicable, then the energy-exchange term must again be given by \(Q = g_sH^3(\dot{H} + H^2)/120\pi^2\). In the presence of the additional non-interacting fluid, however, we no longer have Eq. [1]. Instead, we find that the energy density in the non-interacting fluid and in the vacuum can be written, respectively, as

$$\rho = -\frac{\dot{H}}{4\pi\gamma} - \frac{g_sH^4}{360\pi\gamma^2}, \tag{29}$$

and

$$\rho_v = \frac{3H^2}{8\pi} + \frac{\dot{H}}{4\pi\gamma} + \frac{(4 - 3\gamma)g_sH^4}{1440\pi\gamma^2}. \tag{30}$$

This allows us to construct the following differential equation for \(H\):

$$\ddot{H} + \left(3\gamma + \frac{2g_sH^2}{45\pi}\right)H\dot{H} + \frac{g_s\gamma H^5}{30\pi} = 0. \tag{31}$$

Defining the new variables \(X\) and \(\omega\) via

$$X = -\frac{\dot{H}}{H} - \frac{g_sH^4}{90\pi\gamma} \quad \text{and} \quad \omega^2 = \left(\frac{g_s}{90\pi\gamma}\right)H^4, \tag{32}$$

we can rewrite Eq. [31] as

$$\sqrt{\frac{405\pi\gamma}{2g_s}}X \frac{d\omega}{dX} = \omega^2 + X. \tag{33}$$

The solution to this equation is

$$\omega^2 = X \left(\frac{Y_1 \left(\frac{8g_sX}{405\pi\gamma}\right)}{Y_0 \left(\frac{8g_sX}{405\pi\gamma}\right)} + c_1 J_1 \left(\frac{8g_sX}{405\pi\gamma}\right)\right)^2, \tag{34}$$

where \(J_i(x)\) and \(Y_i(x)\) are Bessel functions, and \(c_1\) is a constant of integration. Using the definitions for \(X\) and \(\omega\), this equation can be used to find the following relation between the energy density in the non-interacting fluid and the Hubble rate:

$$H^4 = \frac{360\pi^2\gamma\rho}{g_s} \frac{Y_1 \left(\frac{32g_s\rho}{405\pi\gamma}\right)}{Y_0 \left(\frac{32g_s\rho}{405\pi\gamma}\right)} + c_1 J_1 \left(\frac{32g_s\rho}{405\pi\gamma}\right)^2. \tag{35}$$

This equation can be used, together with Eq. [28], to solve for \(a(t)\). It can also be used to determine the relative energy density of any two fluids at any given \(a(t)\), or equivalently, redshift.

As an example of this, the values of \(\Omega_r = 8\pi\rho_r/3H^2\) for radiation, the vacuum energy, and non-interacting dust (the case \(\gamma = 1\)) are displayed in Fig. 1. To construct this plot we have taken \(g_s = 2\) and \(c_1 = 10^{-4}\). The value of \(a(t)\) has also been rescaled so that the initial singularity occurs at \(a(t) = 0\).

We note that intermediate inflation, of the type given in Eq. [5], is approached at both late and early times...
(when the energy density in dust becomes negligible). However, at intervening times the energy density in the non-interacting dust dominates over the radiation and vacuum energy, and leads to \( a(t) \sim t^{2/3} \) for a finite period of time. We note that while the vacuum energy dominates at late times, it is the energy density in radiation that dominates at early times. In this latter limit the vacuum energy becomes negative, and diverges as the initial singularity is approached. Nonetheless, the evolution of the universe approaches intermediate inflationary evolution \( (3) \) as the initial singularity is approached, demonstrating that inflation can occur even when the energy density in the vacuum is large and negative.

We can also use Eq. \((33)\) to determine the leading order behaviour of \( \rho \) at late times, as \( H \to 0 \). This is given by

\[
\lim_{H \to 0} \rho = \left( \frac{405 \gamma}{8 g_s} \right) \exp \left\{ -2 \gamma_E - c_1 \pi - \frac{135 \pi \gamma}{g_s H^2} \right\}, \quad (36)
\]

where \( \gamma_E \) denotes Euler’s constant. For \( \gamma > 0 \), this can be seen to decay with \( H \) must faster than the energy density in radiation, as prescribed by Eq. \((1)\), showing that at late times the energy density in radiation generically dominates over matter, in contrast to the usual case in cosmological models with non-interacting radiation and dust \( [14] \). In Fig. 2 we plot the energy density in dust as a fraction of the energy density in radiation, again for \( g_s = 2 \) and \( c_1 = -10^4 \). It can be seen that there is a transient period when the dust dominates over the radiation, but that the radiation dominates over the dust at both late and early times. The opposite result is true for \( \gamma \lesssim 0 \), in which case the non-interacting fluid dominates over radiation at late times, while being sub-dominant at early times. If we had introduced an effective perfect ‘fluid’ with \( \gamma = 2/3 \), to mimic the presence of negative spatial curvature in the Friedmann equation, then the same general evolution occurs and the curvature ‘fluid’ does not dominate at late times, which shows that flatness is approached as in standard inflation.

Interestingly, it can also be seen that as \( H \to \infty \) we have \( \rho \to \text{constant} \) (for \( \gamma > 0 \)). This shows that at early times, when the energy densities in the vacuum and radiation dominate, the energy density in dust is not divergent, but rather approaches a finite non-zero value. This is quite different to the normal behaviour that occurs in the absence of energy exchange between radiation and the vacuum, and shows that in the present case a finite density of matter at the beginning of inflation is not necessarily washed away to vanishingly small values at late times. Similar behaviour (displaying \( \rho \to 0 \) as \( H \to \infty \)) has been found when inviscid perfect fluids are added to a bulk viscous fluid in cosmological models \( [15] \).

IV. DISCUSSION

In this paper, we have studied the cosmological consequences of the vacuum being in thermal equilibrium with a bath of black-body radiation. In this situation, energy is exchanged between the vacuum and the radiation. In the absence of other matter fields, the assumption of a vacuum equation of state \( p_v = -\rho_v \), and a temperature \( T = H/2\pi \) for the vacuum, results in intermediate inflation with \( H \propto t^{-1/3} \), and the introduction of an initial curvature singularity.

We have studied the observational consequences of this energy exchange when the vacuum is treated as a slowly-rolling minimally-coupled scalar field with a self-interaction potential. We find that observational constraints on the amplitude of scalar and tensor perturbations, and the spectral index of primordial curvature
perturbations, result in expressions that are very close to the usual ones, written in terms of the slow-roll parameters. There are therefore no strong observational constraints to distinguish this scenario from a standard slow-roll inflation.

We then proceeded to calculate the evolution of the universe when it contains a non-interacting barotropic perfect fluid, in addition to the interacting radiation and vacuum energy. We found that, as long as the non-interacting fluid has an equation of state $p > -\rho$, it is dominated at both late and early times by the radiation. We also find that it is possible for the non-interacting fluid to dominate for a finite period at intermediate times, and that this is possible even if the fluid has finite energy density at the initial singularity.

While the generic end-state of these models is intermediate inflation driven by the vacuum energy, we find that the generic initial state is intermediate inflation in which the radiation diverges to $+\infty$ and the vacuum energy diverges to $-\infty$. The occurrence of early universe inflation in the presence of a large negative vacuum energy is an intriguing consequence of this scenario, and would appear to be consistent with the picture of the negative Planck-sized vacuum energy that is generically expected to result from the lowest-order supergravity terms in string and M-theories [16].

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