Deconfinement, Screening and Abelian Projection at Finite Temperature

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Abstract

The behavior of static sources transforming according to different irreducible representations of the gauge group is studied in the context of finite temperature lattice gauge theory. We combine analytical and numerical approaches to extract information about confinement and screening both at low temperatures, and around the deconfinement phase transition. The idea that abelian projection in the Maximally Abelian Gauge reproduces the most important features of confinement and screening is tested at a quantitative level in these finite-$T$ theories. Our results show that while this abelian projection provides correct qualitative features, it fails at a detailed quantitative level.
1 Introduction

When one discusses the confinement properties of pure gauge theories, one normally restricts the attention to the static potential between two infinitely heavy sources transforming according to the fundamental representation of the gauge group. For the purposes of understanding the dynamical mechanisms of confinement and screening it is of interest to enlarge this framework by considering static sources transforming according to arbitrary irreducible representations of the gauge group. It is also of interest to study an extreme situation where the system is in equilibrium with a temperature $T$ so high that the confining properties are lost.

In this talk we shall describe some lattice gauge theory simulations that concern this combined situation: the behavior of static “quark” sources in different representations of the gauge group, with the system being in equilibrium with a heat bath of temperature $T$. We shall restrict our attention to the gauge groups $SU(2)$ and $SU(3)$. Some of the results for $SU(2)$ have been published already [1], but the bulk of the present contribution is based on previously unpublished material.

The static potential between two infinitely heavy sources is believed to depend crucially on the manner in which the chosen representation behaves under transformations restricted to the center $Z(\mathbb{N})$ of the gauge group. Representations that are insensitive to $Z(\mathbb{N})$ transformations should yield a screened potential, while those sensitive to these transformations should yield a confining potential. This is the standard picture of confinement and screening in non-Abelian gauge theories, dating back almost twenty years (see, e.g., ref. [2]). At this conference, a major theme was one explicit picture of confinement that a priori does not seem to refer to this $Z(\mathbb{N})$ symmetry at all, that of a “dual Meissner effect”. An explicit realization of this picture might well depend on a suitable gauge [3].

The so-called “Maximally Abelian Gauge” [4] has been argued (see, e.g., ref. [5]) to be superior in the sense that it should allow one to reconstruct, for example, the full $SU(\mathbb{N})$ string tension from Abelian observables – the Abelian projections. This is a very strong statement, reducing essentially all the dynamics of the confinement mechanism to an Abelian dual Meissner effect in that gauge. And the rôle played by the center symmetry is completely obscured.

As is well known, it is a global $Z(\mathbb{N})$ counterpart at finite-$T$ which makes the Polyakov line an order parameter for the confinement/deconfinement phase transition in a pure $SU(\mathbb{N})$ gauge theory. With very mild assumptions, this makes continuous deconfinement phase transitions in $(d + 1)$-dimensional $SU(\mathbb{N})$ gauge theories fall in the universality classes of globally $Z(\mathbb{N})$-symmetric spin systems in $d$ dimensions [6]. This promotes the global $Z(\mathbb{N})$ symmetry at finite $T$ to an observable: by measuring, e.g., critical exponents one can infer which global $Z(\mathbb{N})$ symmetry group is being spontaneously broken at the transition.

Under the global $Z(\mathbb{N})$ transformation, the Polyakov line in the fundamental representation transforms as $\langle Tr_F W(x) \rangle \rightarrow z Tr_F W(x)$. The Abelian projection of the fundamental-representation Polyakov line clearly transforms as well, and hence serves as an order parameter on equal footing with the full Polyakov line. But although both are order parameters for the confinement/deconfinement phase transition, there is of course no guarantee that they will agree even roughly in numerical values. We shall return to this question towards the end of this talk.

2 Finite-temperature Deconfinement: The Mean-Field Limit

The continuous finite-$T$ deconfinement phase transitions in pure gauge theories are particularly interesting from a theoretical point of view because they allow us to test universality ideas in a highly non-trivial setting. The universality predictions of Svetitsky and Yaffe [6] give exact statements about these transitions. It is not often we are in a position to test exact predictions for $SU(\mathbb{N})$ gauge theories, and it obviously merits a lot of attention.
The restriction to continuous phase transitions puts some limits on where the comparison can be carried out. But it is commonly accepted that the (3+1)-dimensional $SU(2)$ theory has a continuous transition, which hence should belong to the $Z(2)$ fixed point in 3 dimensions: the 3-d Ising fixed point. A lot of computational effort has gone into verifying this by treating the Polyakov line in the fundamental representation of $SU(2)$ as the analogue of the spin variable in the corresponding 3-dimensional Ising model. Obvious quantities to compare include the critical exponents, amplitude ratios etc. The agreement is impressive (see, e.g., ref. [7] for a recent study, and references therein). In (3+1)-dimensions this unfortunately exhausts our chances for testing the universality conjecture on $SU(N)$ theories, since the deconfinement phase transitions are believed to be 1st order for all higher values of $N$ in (3+1)-dimensions.

Since the deconfinement phase transition of pure $SU(2)$ gauge theory is interesting for theoretical purposes alone anyway, one might just as well test the universality conjecture in other dimensions. This gives much more freedom, since $SU(N)$ deconfinement phase transitions in (2+1)-dimensions appear to be continuous at least for $N = 2$ [8, 9] and $N = 3$ [10] (and probably for a few higher values of $N$). In (2+1) dimensions one has the additional bonus that conformal symmetry in the corresponding 2-dimensional $Z(N)$ spin theory provides many more detailed universality predictions [11]. Again using the Polyakov line in the fundamental representation of the gauge group, all results have been consistent with the universality hypothesis also in (2+1) dimensions.

There have, however, been some indications that the full picture may be less simple. Suppose we ask for the critical dynamics of Polyakov lines in other representations of the gauge group. If we associate the Polyakov line in the fundamental representation with the $Z(N)$ spin variable, what are in spin-model language the analogues of the Polyakov lines in higher representations? The behavior under $Z(N)$ transformations is supposed to give the answer. Thus, for representations transforming trivially under $Z(N)$ there should be no critical dynamics at all, since the associated Polyakov lines are not order parameters of the phase transition. These lines should, for continuous transitions, go smoothly from non-vanishing values in the confined to perhaps somewhat higher values in the deconfined phase. Representations that do transform under $Z(N)$, on the other hand, are equally good order parameters. From that point of view there should be no compelling reason for singling out the fundamental representation as the analogue of the spin operator. The only way to avoid a contradiction would seem to be that all these representations should display the same critical behavior near $T_c$ as the fundamental representations.

To what extent are these general arguments supported by Monte Carlo results? A clear change in behavior of the adjoint Polyakov line at the deconfinement “phase transition” (there is of course no genuine transition in a finite volume) as measured on small lattices [12] was already one indication that there could be difficulties with numerical investigations of this problem. With the same level of statistics (and the same lattice sizes) that were used routinely to confirm the universality arguments based on the fundamental Polyakov line, a surprisingly different behaviour was found for the higher representations of $SU(2)$ lattice gauge theory in (3+1) dimensions [13, 14]. (For the continuous deconfinement transitions of $SU(2)$ and $SU(3)$ lattice gauge theories in (2+1) dimensions, see ref. [8, 10]). These numerical simulations indicated that sources of higher representations that were sensitive to $Z(N)$ would correspond to different magnetization exponents, one exponent for each irreducible representation. But these results could all be criticized [13, 8] on the grounds that they also seemed to indicate critical behavior for Polyakov lines that simply are not order parameters for the transition, those of transforming trivially under $Z(N)$. Indeed, in a Monte Carlo study of (3+1)-dimensional $SU(2)$ lattice gauge by Kiskis [15] the expected behavior (adjoint Polyakov line non-vanishing across the transition, the isospin 3/2 Polyakov line behaving like the fundamental) was

\[2\] Of course, there are universal effects that can be studied around 1st order transitions as well, such as finite-volume scaling.
Finally extracted very close to the finite-volume “critical point” $T_c$. In hindsight, the difficulty with higher representation sources was perhaps to be expected. It had been observed earlier that also representations transforming trivially under the center group could feel a linearly rising potential (with a slope different from the string tension of the fundamental representation) at intermediate distances [16]. At a certain range of distance scales all representations appear to carry with them their own dynamics. In the limit of an infinite number of colours, factorization is sufficient to show that all irreducible representations are confined by a linearly rising potential if the fundamental representation is, with string tensions that depend on the representations [17]. Essentially, the intermediate distance region in which a non-zero string tension exists for all representations grows with $N$, the number of colours, reaching infinity as $N \to \infty$. So for the relatively small lattices sizes available, one is almost always probing the intermediate distance regime, where all representations feel a linearly rising potential – a “confining string”.

Conventional wisdom has it that the Gaussian fixed point with mean-field exponents is the relevant one for all dimensionalities $d$ between a certain specified “upper critical dimension” $d_u$ and $d = \infty$. We shall not try to contradict the validity of this statement, but we shall show that although it is an exceptional case, it does not exclude the possibility of finding new and unexpected critical behavior around this Gaussian fixed point.

It is important to realize that all of these issues can be addressed even in the strong-coupling region of the lattice theory. In fact, in this regime the universality arguments are even strengthened (since the effective Polyakov-line interactions can be shown explicitly to be short-ranged [18, 19]), and the question of the critical behavior of higher-representation sources near the phase transition point is as meaningful in the strong-coupling regime as near the continuum limit. The advantage of going to the strong-coupling regime is of course that the question here can be studied in a much simplified setting which still captures all the essentials. The leading-order effective Polyakov-line action reads [19]:

$$S_{\text{eff}}[W] = \frac{1}{2} J \sum_{x,j} \left\{ Tr_F W(x) Tr_F W^\dagger(x + j) + Tr_F W^\dagger(x) Tr_F W(x + j) \right\} .$$

(1)

Here the sum on $j$ runs over nearest neighbours. The effective coupling $J$ is related to the gauge coupling $g$ and $N_\tau$, the number of time-like links in the compactified temporal direction. To lowest order, for $SU(2)$, it is

$$J(g, N_\tau) = \left( \frac{I_2(4/g^2)}{I_1(4/g^2)} \right)^{N_\tau},$$

(2)

with $I_n$ indicating the $n$th order modified Bessel function. For $SU(2)$ we will use a notation in which $Tr_n W$ means the trace taken in the representation of isospin $n/2$. $Tr_1 W$ is thus $Tr_F W$, and $Tr_2 W$ is the trace in the adjoint representation, etc. Higher orders in the expansion (1) (and corrections to the effective coupling (2)) can be computed in a systematic expansion [20], but we will not need these corrections for the present purpose. The effective action (1) becomes asymptotically exact in the strong coupling limit.

It is useful to write the effective Polyakov-line action (1) for $SU(2)$ in terms of a new variable $\Phi(x) \equiv \frac{1}{2} Tr_1 W(x)$. The partition function then takes the following form:

$$Z = \int_{-1}^{1} [d\Phi] \exp \left[ 4J \sum_{x,j} \Phi(x) \Phi(x + j) + \sum_x \tilde{V}[\Phi^2] \right] ,$$

(3)

with a local potential $\tilde{V}[\Phi^2] = \frac{1}{2} \ln [1 - \Phi(x)^2]$.
There are two simple limiting cases in which the effective Polyakov-line action (1) can be solved exactly. One is the large-$N$ limit \( [2] \), where the deconfinement phase transition turns out to be of first order (in agreement with large-$N$ reduction arguments based directly on the full gauge theory \( [22] \)), and where universality arguments hence cannot be addressed. The other exactly solvable case is the mean-field limit in which \( d \to \infty \), with \( d \) being the number of spatial dimensions \( [13] \). In this limit one finds a genuine second-order critical point for the gauge group \( SU(2) \) at a critical coupling \( J_c \to 0 \) as \( d \to \infty \). For \( J > J_c \) all higher-representation expectation values \( \langle Tr_n W \rangle \) are non-zero \( [13] \):

\[
\langle Tr_n W \rangle = (n + 1) \frac{I_{n+1}(2a)}{I_1(2a)},
\]

with \( a = 2dJ \langle Tr_1 W \rangle \) being the self-consistent mean-field solution for the fundamental representation. Surprisingly, they all display non-trivial critical behavior close to \( J_c \):

\[
\langle Tr_n W \rangle \sim (J - J_c)^{\beta_n},
\]

where \( \beta_n = n/2 \). For the fundamental representation this just corresponds to the mean-field Ising magnetization exponent \( \beta_1 = \beta = 1/2 \), in complete agreement with the universality arguments.

For the higher representations this new critical behavior is highly unexpected. In this unphysical, but exactly solvable, limit all standard screening arguments appear to break down, and we are seeing new behaviour which is not predicted by universality \( [13] \).

It is normally assumed that the relevant \( Z(2) \) spin system universality class to which the \( SU(2) \) finite-$T$ phase transition should belong (if continuous) would display “classical” mean field exponents all the way down from \( d = \infty \) to the upper critical dimension \( d_u \) (in this case with \( d_u = 4 \), the critical behaviour being modified by logarithmic corrections just at \( d = d_u \)). At a first glance this might seem to indicate that the non-trivial behaviour (5) for all representations should remain valid for all \( d > 4 \) (\( d = 4 \) just being the limiting case), with non-trivial critical scaling even for representations of integer isospin, and with new critical exponents for all isospin half-integer representations as well.

The first conclusion simply cannot be correct (and we will show explicitly below why the argument is invalid), because at strong coupling one can compute, for example, the adjoint Polyakov line and see that to first non-trivial order in \( 1/g^2 \) it is non-zero. What about the odd-$n$ representations? Could it be that they display new non-trivial critical behaviour of the kind (5) even at \( d = 4 \)? Getting accurate Monte Carlo results close to \( T_c \) in the full \( SU(2) \) gauge theory is extremely difficult and time-consuming. To circumvent this problem, a Monte Carlo simulation has instead been performed on the effective Polyakov-line action for \( SU(2) \) in \( d = 4 \) spatial dimensions. From a conceptual point of view this is no restriction at all, since – as we have emphasized before – all these questions are as important in the context the strong-coupling effective action (1) as in the full gauge theory.

The critical coupling of the model was determined using the fourth order cumulant \( [22] \):

\[
U_1 = 1 - \langle m^4 \rangle / (3 \langle m^2 \rangle^2),
\]

where \( m \) is the single lattice average over \( \Phi \), i.e. the Polyakov line in the fundamental representation. In these runs the lattice sizes were \( L = 4, 6, 8, 12 \) and 16. The resulting curves are plotted in Fig. 1. The crossings of the cumulant provide estimates of the critical coupling \( J_c \). The error vanishes like

\[^{3}\text{Still, the large-N solution does display a number of interesting features such as the simultaneous deconfinement of all higher representations at the transition temperature } T_c, \text{ independently of whether these transformations transform trivially under the center symmetry (in this case } U(1) \text{) or not. All representations of the Polyakov line are hence equally good order parameters in this special case, and all display a discontinuous jump at the phase transition.}\]

\[^{4}\text{One also finds separate exponents } \delta_n = 3/n \text{ for the behaviour of } \langle Tr_n W \rangle \sim h^{1/b_n} \text{ at } J = J_c \text{ in a small magnetic field coupled to } Tr_1 W.\]
The crossings of the Binder cumulant for $L = 12$ and $L = 16$ lattices were taken as the best estimate of the critical coupling, giving $4J_c = 0.5507(2)$.

Next, consider the fourth-order cumulant $U_n$ for higher representations. Fig. 2 shows the results for $n = 2$. With increasing lattice sizes the cumulant converges toward $2/3$ for couplings both below and above the critical point. The value $U = 2/3$ signals a finite expectation value of the observable: the Polyakov line in the adjoint representation is not an order parameter. The fourth-order cumulant for the $n = 3$ representation takes values close to $2/3$ in the broken phase and values close to zero in the high temperature phase; it behaves as an order parameter. But the curves do not display crossings close to the critical coupling predicted from the cumulant for the fundamental representation. The curve for $L = 16$ comes close to that of the fundamental representation, so we might expect that for still larger lattices the cumulant for the $n = 3$ representation converges towards the fundamental ones, and that the crossings can then be observed at the critical coupling $J_c$. The data for the higher representation were so affected by errors that no reliable results for the cumulants could be extracted.

The model was also simulated for various $J > J_c$ on lattices of sizes up to $L = 16$, the aim being an approximate determination of the critical exponents $\beta_n$ directly from the Monte Carlo measurements of the different representations. A $J$-dependent “effective” exponent $\beta_n^{eff}$ can be defined by

$$\beta_n^{eff} = (J - J_c) \frac{d \langle Tr_n W \rangle/dJ}{\langle Tr_n W \rangle}. \quad (7)$$

The derivative of $\langle Tr_n W \rangle$ with respect to $J$ can then be computed from the relation

$$\frac{d}{dJ} \langle Tr_n W \rangle = \langle (Tr_n W) \cdot \tilde{S} \rangle - \langle Tr_n W \rangle \langle \tilde{S} \rangle. \quad (8)$$

The final results for this are presented in fig. 4 (for the odd representations) and fig. 5 (for the even ones). Only values which were consistent on the two largest lattice sizes were considered. It turned out that for the coupling close to $J_c$ even the $L = 16$ lattice was not sufficient to give a stable result for the $n = 5$ representation. The curves plotted in these two figures are improved mean-field predictions. They will be explained in detail below.

Figs. 4 and 5 demonstrate fairly convincingly that the odd representations converge toward an effective $\beta_n^{eff} = 1/2$ independent of $n$, while the even representations converge toward $\beta_n^{eff} = 0$ (as expected if these representations remain finite at $J_c$). But the plots also reveal an interesting phenomenon for larger values of $(J - J_c)/J_c$: the effective $J$-dependent exponents $\beta_n^{eff}$ quickly reach a regime of couplings where they are essentially equally spaced, growing linearly with $n$. Although they never actually reach the mean-field prediction (5), they get quite close, and they certainly obey the rule $\beta_n^{eff} \sim n \cdot \beta_1^{eff}$ to surprisingly high accuracy. This is just as for the original observations in the full $(3 + 1)$-dimensional $SU(2)$ gauge theory. It appears that this approximate linear relation between the $\beta_n$’s, when measured not too close to the critical point, can be viewed as the “remnant” of the $d = \infty$ solution. It is then only very close to the critical point the behaviour changes, and the single critical exponent $\beta$ emerges for the odd representations, while the even representations run smoothly across the transition point. We can estimate this narrow window in the original gauge coupling $4/g^2$ by using the relation (2). In the case of $N_f = 2$ the transition occurs at $4/g_c^2 = 1.6424(4)$. In order to obtain $\beta^{eff} < 0.625$ (i.e. 25% above the correct value $\beta = 0.5$) for the $n = 3$ representation we would have to take $4/g^2 < 1.66$.

While these results may have clarified the situation in the $d = 4$ theory, we are still left with the surprising $d = \infty$ results where mean field theory is believed to be exact. How can they be explained? Consider the representation of the effective Polyakov-line action given in eq. (3). This is a $Z(2)$-invariant effective scalar field theory in $d$ dimensions, as expected on general grounds. But it is a very particular effective scalar theory, one that embodies the underlying $SU(2)$ structure (in
the restrictions on the integration interval of $\Phi(x)$, and in the very special form of the local potential $V[\Phi^2]$, which reflects the Haar measure for $SU(2)$.

Since we at this point wish to focus on the $d = \infty$ results, we can restrict ourselves to “classical” mean-field considerations. It is instructive to generalize the partition function above to an arbitrary local potential $V[\Phi^2]$ and relax the limitation on the integrations over $\Phi(x)$ to be in the interval $[-1, 1]$. The $d = \infty$ solution is then found by considering the single-site partition function

$$Z_{SS} = \int_{-\infty}^{\infty} [d\Phi] \exp \left[ v\Phi + V[\Phi^2] \right] ,$$

where $v = 4dJ\langle \Phi \rangle$ will be determined by the self-consistency solution. Clearly, for $n$ being any non-negative integer, $\langle \Phi^{2n+1} \rangle = 0$ unless the global $Z(2)$ symmetry is spontaneously broken. Call the critical coupling at which this occurs $J_c$. If the phase transition is continuous, $\langle \Phi \rangle$ will be small just above $J_c$, and it is meaningful to expand in $v$ (no matter how large $d$ is taken, once fixed). The result is, for the expectation values of the first two non-trivial mean-field moments of $\Phi$:

$$\langle \Phi^2 \rangle = \langle \Phi^2 \rangle_0 + \frac{1}{2} \left[ \langle \Phi^4 \rangle - \left( \langle \Phi^2 \rangle_0 \right)^2 \right] v^2 + \ldots$$
$$\langle \Phi^3 \rangle = \langle \Phi^4 \rangle_0 v + \frac{1}{2} \left[ \frac{1}{3} \langle \Phi^6 \rangle_0 - \langle \Phi^2 \rangle_0 \langle \Phi^4 \rangle_0 \right] v^3 + \ldots ,$$

where the subscript “0” indicates the (constant) expectation value in the unbroken phase $J < J_c$. Higher moments can be worked out analogously, by expanding both the partition function $Z_{SS}$ and the unweighted averages in powers of $v$. Using the recursion relation $\chi_{n+1} = \chi_n \chi_1 - \chi_{n-1}$ for $SU(2)$ characters, we find the general $d = \infty$ predictions:

$$\langle Tr_2 W \rangle = \left[ 4\langle \Phi^2 \rangle - 1 \right] + 2 \left[ \langle \Phi^4 \rangle - \left( \langle \Phi^2 \rangle_0 \right)^2 \right] v^2 + \ldots$$
$$= A_2 + B_2 v^2 + \ldots$$
$$\langle Tr_3 W \rangle = \left[ 8\langle \Phi^4 \rangle_0 - 4\langle \Phi^2 \rangle_0 \right] v + 4 \left[ \frac{1}{3} \langle \Phi^6 \rangle_0 - \langle \Phi^2 \rangle_0 \langle \Phi^4 \rangle_0 + \frac{1}{2} \langle \Phi^2 \rangle_0 \right] v^3 + \ldots$$
$$= A_3 v + B_3 v^3 + \ldots ,$$

where $A_2$, $B_2$, $A_3$ and $B_3$ are (non-universal) constants. This shows the behaviour expected from universality arguments. The adjoint Polyakov line will remain non-vanishing across the phase transition at $J_c$ (and is hence not an order parameter), and the isospin-3/2 representation scales near $J_c$ as $v$, i.e., as the fundamental representation. But if we take the particular potential $\tilde{V}[\Phi^2]$ of eq. (3), and restrict the integration over $\Phi$ to the interval $[-1, 1]$, then devious cancellations occur. One finds $\langle \Phi^2 \rangle_0 = 1/4$ and $\langle \Phi^4 \rangle_0 = 1/8$, leading to

$$\langle Tr_2 W \rangle = \frac{1}{8} v^2 + \ldots = 2d^2 J^2 \langle \Phi \rangle^2 + \ldots$$
$$\langle Tr_3 W \rangle = \left[ \frac{4}{3} \langle \Phi^6 \rangle_0 + \frac{3}{8} \right] v^3 + \ldots .$$

It is thus suddenly the non-leading terms in the general expansion of the Polyakov lines that become important, due to the amplitudes of the leading terms vanishing in this limit. The cancellations required for this phenomenon are actually simple consequences of the orthogonality relations for $SU(2)$ characters, as follows if one performs the mean field calculation directly in $SU(2)$ language. They occur similarly for all higher representations, leading, of course, eventually to the general $d = \infty$ solution (5).
We are now in a better position to understand the $d = \infty$ results. As shown above, the appearance of new exponents for each of the odd-$n$ representations in the limit $d = \infty$ is due to very delicate cancellations that make the amplitudes of the leading terms in the expansion close to the critical point vanish. Although the same mechanism is responsible for the fact that also even-$n$ representations display non-trivial critical behaviour in the $d = \infty$ theory, that phenomenon is of course far more difficult to understand from the point of view of physics. The even-$n$ Polyakov-line representations simply ought not to be order parameters for the deconfinement transition, even in the $d = \infty$ limit, since such sources should be screened both above and below the critical point. The resolution of this apparent paradox lies in the fact that the critical coupling $J_c$ actually vanishes (like $1/d$) when $d \to \infty$, as follows directly from the mean-field solution (4). This behaviour is not an artifact of the mean-field solution; it can be checked to hold as well in the exact solution of the $N = \infty$ theory [21]. In terms of the gauge coupling $g$ this entails, for fixed $N\tau$, $g \to \infty$. Although this makes the strong-coupling effective Lagrangian analysis more and more accurate, it also pushes the confinement/deconfinement phase transition right to the extreme limit $g = \infty$ where all sources are “confined” ($\langle Tr_n W \rangle = 0$ for all $n$ at $g = \infty$ in the full gauge theory simply as a consequence of the orthogonality property of the group characters). It is for this simple reason that the mean-field solution, correctly, predicts critical behaviour for all representations of $SU(2)$.

3 Improved Mean Field Theory

The limit $d = \infty$ of finite-temperature gauge theories is thus in many respects highly singular. This, together with the Monte Carlo data presented above for the $d = 4$ $SU(2)$ theory, indicates that the usual assumption of $d = \infty$ exponents being valid down to the upper critical dimension $d_u$ simply fails in this case. Can we understand the singular nature of the $d = \infty$ limit in an analytical way? As explained above, there are actually no reasons to doubt that mean field theory predicts the $d = \infty$ behaviour correctly. The only resolution would then be that any finite dimensionality $d$ should correspond to radically different behaviour close to the critical point, i.e., that $1/d$-corrections discontinuously should alter the critical indices. To see whether this is the case, we have considered a slightly improved mean-field solution of the same effective Polyakov-line action (1). (This improvement appears to be equivalent to what is known as the Bethe-approximation, see, e.g., ref. [24]).

The Bethe-approximation can be seen as the lowest order improvement in a whole class of mean-field improvements. Let us consider a lattice field theory defined by the action

$$ S = -\beta \sum_{<xy>} \phi_x \phi_y + \sum_x V(\phi_x) $$

where we assume for simplicity of the argument that $\phi$ is a real variable. In standard mean-field approximation one replaces the neighbours of a spin by an external field.

$$ Z_M = \int d\phi \exp(\beta 2dH\phi - V(\phi)) $$

The value of the external field is fixed by the self-consistency condition $H = \langle \phi \rangle(H)$.

The idea to improve the mean-field approximation is to consider a finite lattice rather than a single site. Again the missing neighbours of the spins at the boundary are replaced by an external field, which just acts on those spins at the boundary.

$$ Z_{1M} = \int [d\phi] \exp(+\beta \sum_{<xy>} \phi_x \phi_y + \sum_x \beta n_x H \phi_x - \sum_x V(\phi_x)) $$
where $n_x$ is the number of missing neighbours at the site $x$. Having a larger lattice there is some ambiguity in the choice of the self-consistency condition. Assuming that the best approximation of the true physics is obtained in center of the lattice we require $\langle \phi_c(H) \rangle = \langle \phi_{mc}(H) \rangle$, where $c$ denotes a site at the center of the lattice and $nc$ next to the center. It turns out that the critical exponents obtained from this type of improved mean-field are in general identical to those of standard mean-field. However non-universal quantities like off-critical expectation values and the critical temperature converge systematically towards the true values as the size of the lattice increases. This behaviour can be nicely illustrated at the example of the two dimensional Ising model. One obtains from square lattices of size $L = 2, 3, 4$ and $5$ the inverse critical temperatures $\beta_c = 0.285745...$, $0.380902...$, $0.388503...$ and $0.400785...$ respectively. These numbers have to be compared with the exact solution $\beta_c = 0.440686...$ and the standard mean-field result $\beta_c = 0.25$. For a discussion of even more general improved mean-field methods see ref. [25] and references therein.

Relying on standard numerical integration for a continuous field $\phi$ only the smallest lattices are managable. In our study we considered a lattice consisting out of a central site (C) and its $2d$ neighbours (O). Due to factorization this problem is not harder to solve than a two-site problem.

In order to get a rough idea about the improvement that can be obtained we computed the "star"-approximation for the 3D Ising model on a simple cubic lattice. From standard mean-field one gets $\beta_c = 1/6 = 0.16...$. The improvement gives $\beta_c = 0.202732...$, which has to be compared with the MC-estimate $\beta_c = 0.221652(3)$ [26].

The partition function of this system is given by

$$Z = \int_{-1}^{+1} d\Phi_C \sqrt{1 - \Phi_C^2} \prod_{O} \int_{-1}^{+1} d\Phi_O \sqrt{1 - \Phi_O^2} exp(4J(\Phi_C + W)\Phi_O)$$

in the case of the effective Polyakov-line action. The integration over the $\phi_O$ fields leads to

$$Z \propto \int_{-1}^{+1} d\Phi_C \sqrt{1 - \Phi_C^2} \left[ I_1(4J(\Phi_C + W)) \right]^{2d} \tag{16}$$

The remaining one-dimensional integration we have performed numerically. In order to fix the external field $W$ we require, as above, that the magnetization of the fundamental representation is equal for the central site (C) and its neighbours (O). Expectation values are evaluated on the central site. We have solved the self-consistency equations for $d = 4, 8, 16$ and $32$. The value found for the critical coupling $J_c = 0.5352...$ in $4-d$ deviates from the Monte Carlo value by only $2.8\%$, while standard mean field theory is off by $9.2\%$. But a more striking consequence of the improvement is seen in the behaviour of the even- $n$ representations, which are now non-vanishing for all values of the coupling $J \neq 0$. The numerical values at or below $J_c$, however, decrease rapidly with $d$. For the adjoint representation it is reduced by a factor of approximately $2$ at $J_c$ when one doubles the dimension, while for the $n = 4$ representation it drops by almost a factor of $4$. In this fashion the present solution matches the usual mean-field results in the limit $d = \infty$.

With the improved mean field theory we can finally make a much more accurate comparison with our $d = 4$ Monte Carlo results. In figs. 4 and 5 we have thus plotted (as smooth curves) the corresponding predictions for the $J$-dependent effective exponents $\beta^{eff}_{\phi}$ defined as in eq. (8). Qualitatively the behaviour of the Monte Carlo data is quite well reproduced.

Clearly, as $d \to \infty$ the window in which the conventional results are reproduced shrinks to zero. In fact, one can easily estimate from the improved mean-field solution (16) that this window decreases in size as $1/d$, eventually disappearing at $d = \infty$. In the more conventional language, this is the point at which the amplitudes of the leading terms in the expansion for the Polyakov lines vanish.
4 New Results for SU(3)

The effective Polyakov-line action (1) for the finite temperature SU(3) gauge theory can be written in a similar form as (3). Any \( U \in SU(3) \) is unitary equivalent to the diagonal matrix
\[
\text{diag}(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3})
\]
with \( \phi_1 + \phi_2 + \phi_3 = 0 \) mod \( 2\pi \). The Haar measure takes the form
\[
\int \text{d}U \cdots = \frac{1}{6} \int_0^{2\pi} \frac{d\phi_1}{2\pi} \int_0^{2\pi} \frac{d\phi_2}{2\pi} \prod_{1 \leq k \leq l \leq 3} |e^{i\phi_k} - e^{i\phi_l}|^2 \cdots \tag{18}
\]
and the trace of the \( n^{th} \) power of a SU(3) matrix is given by
\[
s_n = \text{Tr}U^n = \sum_{j=1}^3 \exp(in\phi_j). \tag{19}
\]
The characters of the irreducible representations can be written as finite polynomials in the \( s_n \) and their complex conjugate. In the following we shall consider
\[
\begin{align*}
\chi_1(U) &= 1 \quad \tag{20} \\
\chi_3(U) &= s_1 \quad \tag{21} \\
\chi_6(U) &= \frac{1}{2}(s_1^2 + s_2) \quad \tag{22} \\
\chi_8(U) &= |s_1|^2 - 1 \quad \tag{23} \\
\chi_{10}(U) &= \frac{1}{6}(s_1^2 + 3s_1s_2 + 2s_3) \quad \tag{24}
\end{align*}
\]
where the index of \( \chi \) gives the dimension of the representation. Since the SU(3) finite temperature gauge theory is expected to undergo a first order phase transition, we have to compute the free energy to verify this assumption and to obtain the critical temperature. In order to extract a result from the star-lattice, we try to compensate for boundary effects in the optimal way. In addition to the star lattice we consider a lattice consisting of just two sites. The star lattice has \( 2d \) internal links and \( 2d(2d - 1) \) external links. The two site lattice has 1 internal link and \( 2(2d - 1) \) external links. The best approximation we expect from a linear combination where the external links are cancelled. Therefore we chose
\[
J_{IM}(H) = -\ln Z_{\text{star}}(H) + d\ln Z_2(H) \tag{25}
\]
as approximation for the free energy.

The value of the external field \( H \) is then fixed by minimizing the free energy with respect to \( H \). The occurrence of two minima means that the phase transition is of first order. We find for \( d=3 \)
\( J_c \approx 0.1343 \) and \( J_c \approx 0.1369 \) for mean-field and improved mean-field respectively. These results have to be compared with the Monte Carlo estimate \( J_c = 0.13722 \) obtained by K.Rummukainen [27].

In fig. 6 we give the expectation value of the Polyakov line for various representations. We also give the solution for the second minimum of the free energy. These results are relevant for a supercooling of the system. In contrast to the lowest-order mean field solution [13], the expectation value of the Polyakov line in the adjoint representation does not vanish in the low-temperature phase.

5 Abelian Projections

The abelian projection theory of quark confinement, put forward by ’t Hooft in ref. [3], is a proposed identification of the physical degrees of freedom that are the most relevant to the infrared dynamics.
The idea is based on a gauge choice, which reduces the underlying $SU(N)$ gauge symmetry to an abelian $U(1)^{N-1}$ symmetry generated by the Cartan subalgebra of the gauge group. The non-abelian theory, with such a gauge choice, resembles a generalization of compact QED, in which (abelian) charged quarks and gluons interact by an exchange of “photons,” which are the (abelian) neutral gluons corresponding to generators of the Cartan subalgebra. The abelian gauge field also contains monopoles, which may be identified from certain degeneracies in the gauge-fixing condition. At this stage, the identification of abelian neutral gauge fields as the "photon" fields of an abelian gauge theory, with remaining gauge fields labeled as abelian charged vector bosons, is simply kinematics.

The hypothesis of the abelian projection theory is that it is the photon/monopole fields, i.e. the abelian gauge fields, that are the crucial degrees of freedom with respect to quark confinement. Confinement, in this picture, is due to condensation of monopoles associated with the $U(1)^{N-1}$ gauge fields, as in compact $QED_3$ \[28\]. Now, if the crucial degrees of freedom are those of an abelian gauge theory, it is reasonable to suppose that the vacuum fluctuations of these abelian gauge fields would dominate the vacuum fluctuations of the theory at large scales. Suppose, for example, that the gauge group is $SU(2)$, and the gauge choice is the maximal abelian gauge, which (on the lattice) maximizes the quantity

$$Q = \sum_{x,\mu} \text{Tr}[U_{\mu}(x)\sigma^3U_{\mu}^\dagger(x)\sigma^3]$$ \hspace{1cm} (26)

There is then a remaining $U(1)$ symmetry, and the corresponding "abelian" gauge field is $A_3^\mu(x)$. If it is the vacuum fluctuations in $A_3^\mu$ which dominate at large scales, then it might be a reasonable approximation, e.g. for purposes of calculating large Wilson loops or perhaps also Polyakov lines, to ignore the contribution from the other color components, i.e.

$$\langle W_j(C) \rangle = \frac{1}{2j+1} \langle \text{Tr} \exp[i \int dx^\mu A_\mu^a T^j_a] \rangle$$

$$\approx \frac{1}{2j+1} \langle \text{Tr} \exp[i \int dx^\mu A_\mu^3 T^j_3] \rangle$$ \hspace{1cm} (27)

where $T^j_a$ are the $SU(2)$ generators in the $j$-representation, and we normalize $W_j$ to a maximum value of 1. This approximation was originally applied by Polyakov, to calculate Wilson loops in the D=3 dimensional Georgi-Glashow model in the Higgs phase \[28\]. In the Georgi-Glashow model, the $U(1)$ symmetry is singled out by a unitary gauge choice. In the context of the abelian projection theory, the approximation is known as "abelian dominance." \[5\]

The validity of abelian dominance is rather important to the abelian projection theory, because if this approximation turns out to be wrong, if in fact it is not true that fluctuations in $A_3^\mu(x)$ dominate the vacuum at large scales, then it is unclear in what respect the monopole/photon degrees of freedom of the abelian projection theory are the most crucial for quark confinement.

On the lattice, abelian dominance is tested by calculating, via lattice Monte Carlo, Wilson loops obtained from "abelian-projected" link configurations in maximal abelian gauge. In $SU(2)$ gauge theory this means setting the off-diagonal components of the gauge-fixed $U_{\mu}(x)$ matrices to zero, and then rescaling each link by a constant to restore unitarity. Abelian dominance seems to work quite well, in numerical simulations, for extracting string tensions from Wilson loops in the fundamental representation \[5\]. Very recently, however, it has been shown in ref. \[29\] that, for $SU(2)$ gauge theory in $D = 3$ dimensions, abelian dominance fails entirely when applied to Wilson loops in higher group representations. This failure is connected with the “Casimir scaling” of interquark forces. It is well known that the force between heavy quarks in any group representation, in a distance interval from onset of confinement to the onset of color screening, is proportional to the quadratic Casimir of the representation. This fact has been seen both in $D = 3$ and $D = 4$ dimensions, for $SU(2)$ and $SU(3)$ gauge groups \[16, 30\]. Moreover, the interval between the onset of confinement and the onset
of screening is quite large in the scaling region; it is not even clear that color screening has been seen yet in $D = 3$ dimensions (c.f. Poulis and Trottier in [30]). For $SU(2)$, the numerical results in this interval, in accord with Casimir scaling, are that the $j = \frac{3}{2}$ tension is about 5 times the $j = \frac{1}{2}$ tension, and the $j = 1$ tension is about $\frac{2}{7}$ the $j = \frac{1}{2}$ string tension [16, 30]. In contrast to this Casimir scaling of interquark forces, it was found in ref. [29] that string tensions extracted from loops built from abelian projected configurations are roughly equal for the $j = 1/2$ and $j = 3/2$ representations, and are consistent with zero for $j = 1$. For the $j > \frac{1}{2}$ representations, abelian dominance is not even approximately true.

There is ample motivation, then, to critically examine the hypothesis of abelian dominance in the case of Polyakov lines in various representations. Again, the abelian-projected link variables in $SU(2)$ gauge theory are constructed by truncating the full link variable

$$U = a_0 I + i \sum_{k=1}^{3} a_k \sigma^k$$

(28)

to the diagonal component, followed by a rescaling, i.e.

$$U \rightarrow U' = \frac{a_0 I + i a_3 \sigma^3}{\sqrt{a_0^2 + a_3^2}}$$

(29)

The abelian links $U'$ are then used in the computation of the Polyakov lines.

We work in maximal abelian gauge. In this gauge there is nothing special about the time direction; string tensions have been extracted from loops in all space-time orientations. For spatially asymmetric gauges, there is a danger that abelian dominance (even for fundamental representations) may only work for loops oriented in certain directions, as found by one of us (J.G.) and Iwasaki in ref. [32].

Fig. 7 shows the Polyakov lines in the fundamental representation, computed using the full and abelian-projected configurations. The lattice size was $12^3 \times 4$, and shown are raw data before averaging, after 500 iterations. The errors are insignificant in this context, since we are not interested in a very detailed comparison of numbers close to $T_c$. We have indicated the full Polyakov lines by $W$, and the abelian-projected Polyakov lines by $V$, both normalized to a maximum value of 1. The results are similar to those found in ref. [31]. There is clearly a steep rise in Polyakov line values at the deconfinement transition, for both the full and abelian-projected configurations. There is, just as clearly, a quantitative disagreement in the full and abelian-projected results. The disagreement becomes much more pronounced for Polyakov lines at $j = 1$, shown in Fig. 8; in this case it is not even clear that we have qualitative agreement! Note in particular the fact that the value of the adjoint line formed from abelian-projected configurations tends to a non-zero constant ($\frac{1}{3}$) at $\beta = 0$. Finally, the $j = 3/2$ results are shown in Fig. 8; again there is considerable quantitative disagreement between the values of the full and abelian-projected lines.

The origin of the asymptotic value of $\frac{1}{3}$, obtained as $\beta \rightarrow 0$ for the $j = 1$ representation Polyakov line, is not hard to understand. Computing adjoint lines with abelian-projected configurations, we have

$$\langle W_1(C) \rangle_{ab\text{-proj.}} = \frac{1}{3} \langle \text{Tr} \exp [i \int dx^\mu A_\mu^3 T_3^1] \rangle$$

$$= \frac{1}{3} \sum_{m=-1}^{1} \langle \langle m \rangle \exp [i \int dx^\mu A_\mu^3 T_3^1 |m\rangle \}$$

$$= \frac{1}{3} \sum_{m=-1}^{1} \langle \exp [im \int dx^\mu A_\mu^3] \rangle$$

12
\[ \frac{1}{3} + \frac{1}{3} \langle \exp[i \oint dx^\mu A_\mu^3] + \text{c.c.} \rangle \to \frac{1}{3} \quad \text{as} \quad \beta \to 0 \quad (30) \]

The adjoint representation of a Polyakov line built entirely out of abelian-projected fields has an abelian neutral component, corresponding to \( m = 0 \) in the above sum, which remains unaffected by the vacuum fluctuations (no matter how large) of the abelian gauge field. The behavior of adjoint Polyakov lines constructed from the full, unprojected configurations is, of course, very different. The value of adjoint Polyakov lines in the unprojected case is related to the energy of a gluon bound to the adjoint source. This energy becomes infinite, in lattice units, as \( \beta \to 0 \), which is why the corresponding adjoint Polyakov line vanishes in that limit. The fact that the abelian-projected adjoint loop is finite (\( \frac{1}{3} \)) in the same limit indicates that the abelian dominance approximation is insensitive to the actual dynamics of color screening. It also indicates that for the full, unprojected Polyakov lines, the vacuum fluctuations of the "charged boson" fields, i.e. \( A_1^\mu \) and \( A_2^\mu \), must play an essential role.

Evidently, the abelian projection approximation (eq. (29)) in the maximal abelian gauge is not a very good approximation for Polyakov lines, with the quantitative disagreement between full and abelian projected values being especially severe for the higher \( (j > \frac{1}{2}) \) representations. This disagreement is in line with the results for Creutz ratios found in ref. [29] (certain other criticisms of the abelian projection theory can be found in ref. [32]) and in general suggests a failure of the abelian dominance approximation in maximal abelian gauge. A failure of abelian dominance, in turn, would imply that an effective abelian theory at the confinement scale, invoking only the monopoles and photons identified by abelian projection, is inadequate to describe the actual non-perturbative behavior of non-abelian gauge theory.

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Fig. 2
Fig. 3
Fig. 5
$J_c = 0.1369...$
\[ j = \frac{1}{2} \]

Diagram showing the relationship between \( \frac{4}{g^2} \) and \( \text{Tr} V \) and \( \text{Tr} W \).
Tr V and Tr W

j = 3/2

"TrV" ◊
"TrW" +