DEGENERATING SLOPES WITH RESPECT TO HEEGAARD DISTANCE

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Abstract. Let $M = H_+ \cup_S H_-$ be a genus $g$ Heegaard splitting with Heegaard distance $n \geq \kappa + 2$: (1) Let $c_1, c_2$ be two slopes in the same component of $\partial_- H_-$, such that the natural Heegaard splitting $M' = H_+ \cup_S (H_- \cup_{c_1} 2 - \text{handle})$ has distance less than $n$, then the distance of $c_1$ and $c_2$ in the curve complex of $\partial_- H_-$ is at most $3M + 2$, where $\kappa$ and $M$ are constants due to Masur-Minsky. (2) Let $M^*$ be the manifold obtained by attaching a collection of handlebodies $\mathcal{H}$ to $\partial_- H_-$ along a map $f$ from $\partial\mathcal{H}$ to $\partial_- H_-$. If $f$ is a sufficiently large power of a generic pseudo-Anosov map, then the distance of the Heegaard splitting $M^* = H_+ \cup (H_- \cup_f \mathcal{H})$ is still $n$. The proofs rely essentially on Masur-Minsky’s theory of curve complex.

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1. Introduction

A Heegaard splitting of a compact orientable 3-manifold $M$ is a decomposition of it along an orientable embedded closed surface $S$ into two compression bodies $H_+$ and $H_-$. [21], and as a conscious extension of A. Casson and C. Gordon’s notion of strong irreducibility, J. Hempel [7] defined the Heegaard distance of a Heegaard splitting in terms of the curve complex $\mathcal{C}(S)$ of $S$, the Heegaard distance is the minimal distance between two curves $\alpha_+$ and $\alpha_-$ in $\mathcal{C}(S)$ which bound disks in $H_+$ and $H_-$ respectively. If a manifold has a Heegaard splitting with distance at least 3, then it is irreducible, atoroidal, $\partial$-irreducible, anannular and it is not a Seifert manifold due to Kobayashi and Hempel [7], so by Perelman’s proof of Geometrization conjecture of Thurston, the manifold is hyperbolic.

It is expected that high distance splitting has rigidity properties, Namazi [18] showed that if a 3-manifold $M$ has a Heegaard splitting with large distance, then the manifold has finite mapping class group, which is also predict by Geometrization conjecture. For other highly interesting construction of high distance Heegaard splittings, see [11] and [12], and see also [10] for related topics.

Thurston’s hyperbolic Dehn surgery theorem, see [24] and [20], says that for a noncompact finite volume complete hyperbolic 3-manifold, to each cusp, all but finitely many Dehn surgery resulting in a hyperbolic 3-manifold. It has been generalized to hyperbolic manifolds with totally geodesic boundaries by Scharlemann-Wu and Lackenby, see [9] and [22].

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Corollary 1.2. Minsky, see Proposition 4.6 of \[14\], we have: there are constants and Scharlemann-Wu \[22\], we have the following theorem about Heegaard distance:

\[ \text{Theorem 1.1.} \quad \text{There are constants } \kappa \text{ and } \mathfrak{M} \text{ depending on } g, \text{ such that if } M = H_+ \cup_S H_- \text{ is a genus } g \text{ Heegaard splitting of a bordered 3-manifold with Heegaard distance } n \geq \kappa + 2, \text{ and } c_1, c_2 \text{ are two degenerating slopes in the same component } F \text{ of } \partial_- H_-, \text{ then the distance of } c_1 \text{ and } c_2 \text{ in } \mathcal{C}(F) \text{ is bounded above by } 3\mathfrak{M} + 2. \]

The constants \( \kappa \) and \( \mathfrak{M} \) in the theorem are due to Masur-Minsky.

Since the curve complex has infinite diameter by Kobayashi-Luo or Masur-Minsky, see Proposition 4.6 of \[14\], we have:

Corollary 1.2. If \( M = H_+ \cup_S H_- \) is a genus \( g \) Heegaard splitting with Heegaard distance \( n \geq \kappa + 2 \) of a bordered 3-manifold, then there are infinite ways to attaching handlebodies to the boundary of \( M \), so that the resulting Heegaard splitting has distance \( n \).

Remark 1.1. In canonical Dehn surgery theory and handle addition theory, e.g. \[4\] and \[22\], two ”degenerating curves” are related by the intersection number. In the curve complex there is an up bound of distance in terms of the intersection number, see \[14\] or \[2\], but in general, there is no lower bound depending only on the intersection number, so our theorem is formulated by distance in the curve complex, and hence we can not obtain the bounded cardinality of degenerating curves in the Dehn surgery case but the bounded diameter. It is the disadvantage of our theorems, but we have the following:

Note that the curve complex of a torus is the well-known Farey graph, see \[14\], and there is a one-to-one correspondence between the slopes in a torus to the co-prime pairs of integers, so analogous to a theorem in \[24\], we have:

Theorem 1.3. If \( \partial M \) is a torus and \( M = H_+ \cup_S H_- \) is a Heegaard splitting with Heegaard distance \( n \geq \kappa + 2 \), then there is a cone in the up half plane, such that at most one co-prime lattice in the cone is a non-degenerating slope.

We also prove a result using pseudo-Anosov theory, that is, in the spirit of \[9\]:

Theorem 1.4. If \( M = H_+ \cup_S H_- \) is a genus \( g \) Heegaard splitting with Heegaard distance \( n \geq \kappa + 2 \), where \( H_+ \) is a handlebody and \( H_- \) is a compression body with \( \partial_- H_- = F_1 \cup F_2 \ldots \cup F_l \), \( H_j \) is a handlebody which has the same genus as \( F_1 \), fixed a homeomorphism \( \tau_j : F_j \to \partial H_j \), for generic pseudo-Anosov map \( f_j : \partial H_j \to \partial H_j \), there is \( n_j \) such that if \( H^* = H_- \cup_{j=1}^{l} \tau_j H_1 \cup_{j=1}^{m} \tau_j H_2 \ldots \cup_{j=1}^{m} \tau_j H_l \) with \( m_j \geq n_j \), then the resulting Heegaard splitting \( M^* = H_+ \cup_S H^* \) has Heegaard distance \( n \).

There are examples that after Dehn filling the Heegaard distance will degenerate drastically, for example, Minsky, Moriah and Schleimer \[17\] showed that there are
arbitrarily high Heegaard distance knots, and then do the trivial Dehn surgery, we get the distance zero Heegaard splittings of $S^3$. Our result shows that most handle additions and handlebody attachments do not degenerate the Heegaard distance if the initial distance is large, this is in the spirit of Thurston’s Hyperbolic Dehn surgery and its generalization, which show that most handle additions and handlebody attachments of a hyperbolic 3–manifold result in hyperbolic 3–manifolds.

The assumption that the distance is at least $\kappa + 2$ is essential in our proof, but in some sense, it is also necessary, see the following example:

**Example 1.5.** Let $T$ be a closed torus and $F$ be a torus with an open disk removed, then $F \times [0, 1]$ is a genus 2 handlebody, and attaching a 2–handle along $\partial F \times 0.5$ we get a distance 2 Heegaard splitting of $T^2 \times [0, 1]$, capping off any torus boundary of $T^2 \times [0, 1]$ by solid torus we get a distance zero Heegaard splitting of the solid torus.

In fact, firstly we ponder that if $M = H_+ \cup_S H_-$ is a unstabilized Heegaard splitting of an irreducible 3-manifold $M$ with boundary, is there any way to capping off the negative boundary of the compression body so that the resulting Heegaard splitting of the manifold is unstabilized, and then we consider the generalized problem on Heegaard distance.

In all of the above theorems, we assume that $H_+$ is a handlebody and $H_-$ is a compression body, the proof for the case that both $H_+$ and $H_-$ are compression bodies is easy from our proof of the above theorems. In fact, we just assume that the distance of $M = H_+ \cup_S H_-$ is $n \geq 3$ in this case, then the results similar to Theorem 1.1 and Theorem 1.4 can be proved in the same line.

The paper is organized as follows: We outline some fundamental results on the curve complex $\mathcal{C}(S)$ which we shall use in Section 2. In Section 3, based on some theorems by Masur-Minsky, we prove the Theorem 1.1 and theorem 1.3. In Section 4, build on some well-known facts and Lemma 3.1, we prove Theorem 1.4. In Section 5, we treat the case that both $H_+$ and $H_-$ are compression bodies with non-empty negative boundary.

### 2. Preliminaries on curve complex

For a compact surfaces $F$ of genus at least 1, Harvey [6] defined the curve complex, but we just use the 1–skeleton of it, we denote it also by $\mathcal{C}(F)$, whose vertices are one-to-one corresponding to the essential non-peripheral curves in $F$, and two vertices are connected by a length 1 arc when they are disjoint in the surface $F$ if $F$ is not homotopic to a torus or once-punctured torus, and two vertices are connected by a length 1 arc if they intersect in one point in the later cases. (Note that in fact in the torus and once-punctured torus case the definition is due to [14], which is different from [6]).

Let $T$ be a torus, fixed a longitude–meridian pair $\lambda - \mu$ of the slopes in $\mathcal{F} = \mathcal{C}(T)$, it is well-known then the slopes in $\mathcal{F} = \mathcal{C}(T)$ are determined by a pair of co-prime integers, so there is a one-to-one corresponding between the vertex of $\mathcal{C}(T)$ and $\hat{Q} = Q \cup \infty$, i.e, $\lambda$ corresponding to $\infty$ and $\mu$ corresponding to 0. $\mathcal{C}(T)$ is the well-known Faray graph, see [15].

Masur and Minsky [14] (See also Bowditch [2] and Hamenstadt [5] for other proofs) made the important progress by showing that the curve complex is hyperbolic in the sense of Gromov and Cannon.
Theorem 2.1. \( \mathcal{C}(F) \) is \( \delta \)-hyperbolic, where \( \delta \) depends only on the topology of \( F \).

Let \( H \) be a genus \( g \) handlebody, and \( S \) is its boundary, we denote by \( \mathcal{D}(H) \) the subset of \( \mathcal{C}(S) \) each of which bounds a disk in \( H \).

Recall that a subset \( A \) of a metric space \( X \) is quasi-convex if there is a constant \( \kappa \), such that \( \forall \ a, b \in A \), any geodesic \([a, b]\) is in the \( \kappa \)–neighborhood of \( A \). One key theorem by Masur and Minsky \cite{16} is:

**Theorem 2.2.** The disk set \( \mathcal{D}(H) \) is \( \kappa \)–quasi-convex in \( \mathcal{C}(S) \).

Let \( F \) be a compact surface, \( Y \) is an essential compact subsurfaces in \( F \), by essential subsurface we mean that \( \pi_1(Y) \to \pi_1(F) \) is injective, and \( Y \) has genus at least 1. (This definition is from \cite{15}, but we just concern with that \( Y \) has genus at least 1).

Masur and Minsky defined the subsurface projection \( \pi_Y \) from \( \mathcal{C}(F) \) to \( \mathcal{P}(\mathcal{C}(Y)) \), the power set of \( \mathcal{C}(Y) \): for each vertex \( v \) in \( \mathcal{C}(F) \), if \( v \cap Y = \emptyset \), then \( \pi(v) = \emptyset \), otherwise \( v \cap Y \) is a curve or a set of arcs in \( Y \), then do surgery with the boundary of \( Y \), we get a set of curves in \( Y \), which has diameter at most 2, and it is easy to show that that the subsurface projection map is 2–Lipschitz, see Lemma 2.3 of \cite{15}.

Another key theorem due to Masur and Minsky is the following \cite{15}:

**Theorem 2.3.** (Bounded Geodesic Image). Let \( Y \) be an essential subsurface of \( F \), and let \( \gamma \) be a geodesic segment, ray, or bi-infinite line in \( \mathcal{C}(F) \), such that \( \pi_Y(v) \neq \emptyset \) for every vertex \( v \) of \( \gamma \). There is a constant \( \mathcal{M} \) depending only on \( F \) so that \( \text{diam}_{\mathcal{C}(Y)}(\pi(\gamma)) \leq \mathcal{M} \).

Let \( F' \) be a compact surface with one boundary, and \( F \) be the surface obtained by capping off the boundary by a disk, then there is a natural projection map \( P : \mathcal{C}(F') \to \mathcal{C}(F) \) by amalgamating the curves which are identified up to the disk: if \( c_1, c_2 \) and the boundary of \( F' \) co-bounded a 3–punctured sphere, we define \( P(c_1) = P(c_2) \) in \( \mathcal{C}(F) \). Note that \( P \) is a distance decreasing map.

### 3. The proof of the Theorem 1.1 and Theorem 1.3

In this section, we prove Theorem 1.1, Corollary 1.2 and Theorem 1.3.

Denoted by \( F_j, j = 1, 2, 3, \ldots, k \) the components of \( \partial_- H_- \), then there is a set of disks \( \mathcal{E} = \{E_1, E_2, E_3 \ldots E_k\} \) in \( H_- \) which divides \( H_- \) into a handlebody(or a 3-ball)and copies of \( F_j \times I \), we denote the components of \( \partial_+ H_- \setminus \partial \mathcal{E} \) which corresponding to \( F_j \) by \( F'_j \), then each \( F'_j \) is a genus larger or equal to 1 surface with a disk removed, we also assume that \( \partial E_j = \partial F'_j \), see Figure 1. Fixed a homeomorphism \( \eta_j \) between \( F'_j \cup E_j \) and \( F_j \), we have the canonical isometry from \( \mathcal{C}(F'_j \cup E_j) \) to \( \mathcal{C}(F_j) \), we also denote it by \( \eta_j \), and note that for two curves in \( \mathcal{C}(F'_j) \), the distance of them under the composition \( \eta_j P_j : \mathcal{C}(F'_j) \to \mathcal{C}(F'_j \cup E_j) \to \mathcal{C}(F_j) \) does not depend on the homeomorphism \( \eta_j \), so we simply denote \( \eta_j P_j \) also by \( P_j \).

Our crucial observation is the follow:

**Lemma 3.1.** \( \pi : \mathcal{D}_+ \to \mathcal{C}(F'_j) \) is bounded with constant \( \mathcal{M} \) depending only on \( g(S) \).

**Proof:** \( \forall \ a \in \mathcal{D}_+ \), since \( E_j \) is an essential disk in \( H_- \), we have \( d(\alpha, \partial E_j) \geq n \geq \kappa + 2 \). For any pair \( x, y \in \mathcal{D}_+ \), let \([x, y]\) be a geodesic in \( \mathcal{C}(S) \), then it is in the
κ–neighborhood of \( \mathscr{D}_+ \) by Theorem 2.2, so any vertex \( v \) of \([x, y]\) has distance at most \( \kappa \) from a curve, say \( \alpha \), in \( \mathscr{D}_+ \), we have that \( d(v, \partial F_j^i) \geq 2 \). Note that distance at least 2 means that there intersect essentially in the Heegaard surface \( S \), then we can use Theorem 2.3, which conclude that \( d_{\mathcal{E}(F_i^j)}(\pi(x), \pi(y)) \leq \mathfrak{M} \), where \( \mathfrak{M} \) is a constant depending only on \( S \).

\[ \text{Figure 1} \]

The proof of Theorem 1.1:

We assume that two degenerating slopes \( c_1 \) and \( c_2 \) are in the same boundary component \( F = F_j \) of \( \partial M \), we also denote \( E_j \) by \( E \) and \( F_j^i \) by \( F' \) for simplicity. We perform 2–handle addition along \( c_i \), and let \( \mathscr{D}_i \) be the disk set of \( H_i = H_\alpha \cup c_i \). 2–handle. We assume that \( a_i \in \mathscr{D}_+ \) and \( b_i \in \mathscr{D}_i \) realize the Heegaard distance of \( M_i = H_+ \cup H_i \), so \( d_{\mathcal{E}(S)}(a_i, b_i) \leq n - 1 \).

Since the Heegaard distance degenerates, we have that \( b_1 \cap F' \neq \emptyset \), we assume that \( b_1 \) bounds disk \( B_i \) in \( H_1 \), and \( B_1 - \mathcal{E} \) is a set of disks, then there is at least one component of \( B_1 - \mathcal{E} \) which is essential in \( F \times I \cup c_1 \cup \text{2-handle} = H \), we denote the boundary of it by \( c \). In other words, \( c \in \pi(b_1) \). See Figure 2.

Note that \( H \) is a punctured solid torus or a compression body according to where the genus of \( F \) is 1 or large. If \( H \) is a punctured solid torus or \( g(F) \geq 2 \) and \( c_1 \) is separating in \( F \), then there is only one essential disk in \( H \), its boundary and \( c_1 \) co-bounded an annulus in \( F \times I \), in this case, we have \( d_{\mathcal{E}(F)}(P(c), c_1) = 0 \). If \( g(F) \geq 2 \) and \( c_1 \) is non-separating in \( F \), there is just one non-separating disk in \( H \), and a set of separating disk in \( H \), the boundary of each separating disk, say \( \alpha' \), co-bounded with \( \alpha \subseteq F \) an annulus in \( F \times I \), and \( \alpha \) is disjoint from \( c_1 \), since \( \alpha \) is the boundary of a neighborhood of \( c_1 \cup d \), where \( d \) is a curve which intersects \( c_1 \) with just one point, and we have \( d_{\mathcal{E}(F)}(P(c), c_1) \leq 1 \).

Let \( c_1' \) be the curve in \( F' \) which co-bounded an annulus in \( F \times I \) with \( c_1 \), note that \( d_{\mathcal{E}(S)}(a_1, c_1') \geq n - 1 \), suppose otherwise, since \( d_{\mathcal{E}(S)}(\mathscr{D}_-, c_1') \leq 1 \), and \( d_{\mathcal{E}(S)}(a_1, \mathscr{D}_-) \leq n - 1 \), a contradiction to the assumption the initial Heegaard splitting has distance \( n \). Let \([a_1, b_1]\) be a geodesic in \( \mathcal{E}(S) \), we claim that each vertex of \([a_1, b_1]\) intersect \( F' \) essentially: otherwise, suppose that \( v \) is a vertex which is disjoint from \( F' \), then \( d(v, c_1') = 1 \), and we must have that \( d(v, b_1) = d(v, c_1) \) since \( a_1 \) and \( b_1 \) realize the distance of \( M^1 = H_+ \cup (H_\alpha \cup c_1 \cup \text{2-handle}) \). Then we have \( d(a_1, v) \leq n - 1 - 1 = n - 2 \), and \( v \cap \partial E = \emptyset \), so \( d(a_1, \partial E) \leq n - 1 \), which is a contradiction to the assumption that the initial Heegaard splitting has distance \( n \).
From the above claim, and Theorem 2.3, we have $d_{\mathcal{C}(F')}(\pi(a_1), \pi(b_1)) \leq \mathfrak{M}$. We also claim that for each vertex $v$ in $[a_1, b_1]$, $P\pi(v)$ is not empty in $\mathcal{C}(F)$: this is due to the assumption that $F'$ has just one boundary.

Then since the natural projection map $P$ is distance decreasing, and each of the projection is not empty, we have $d_{\mathcal{C}(F)}(P\pi(a_1), P\pi(b_1)) \leq \mathfrak{M} + 1$. Together with $d_{\mathcal{C}(F)}(c_1, P\pi(b_1)) \leq 1$, we have $d_{\mathcal{C}(F)}(c_1, \pi(a_1)) \leq 2\mathfrak{M} + 1$. Similarly, we also have $d_{\mathcal{C}(F)}(P\pi(a_2), P\pi(b_2)) \leq 2\mathfrak{M}$.

By Lemma 3.1, $d(P\pi(a_1), P\pi(a_2)) \leq 2\mathfrak{M}$. Then we have $d_{\mathcal{C}(S)}(c_1, c_2) \leq 3\mathfrak{M} + 2$. □

**The proof of corollary 1.2:** This is the easy corollary of Theorem 1.4, but it also can be obtained from Theorem 1.1.

First if some of $F_j$ is a genus at least 2 surface, then $\mathcal{C}(F_j)$ is a diameter infinite graph and each separating curve $c$ is distance 1 with a non-separating curve. So there are infinitely many ways to choose the separating curve $c_j$ and then do $2-$handle addition along $c_j$ we obtain a distance $n$ Heegaard splitting of a manifold $M^*$ with $\partial M^*$ is a set of tori, then for each torus in $\partial M^*$, the curve complex is the Farey graph, which is also a diameter infinite graph, we have infinite ways to perform Dehn filling and obtain distance $n$ Heegaard splitting. $2-$handle additions and Dehn fillings succeeded in the same boundary is the process of handlebody attachment, so we perform handlebody attachments to the manifold with the Heegaard distances do not degenerate.

Since the distance of $M = H_+ \cup_S H_-$ is $n \geq \kappa + 2 \geq 3$, so $M$ is hyperbolic with totally geodesic boundary or with toroidal cusps by Hempel’s theorem on Heegaard distance and Thurston’s Hyperbolicity theorem on Haken manifolds, and $M^*$ is a hyperbolic 3-manifolds with toroidal cusps. By Thurston’s Dehn surgery theorem, all but finitely many Dehn surgery on $M^*$ resulting hyperbolic 3–manifolds with volume converge to the volume of $M^*$. So the manifolds construct above have infinitely many different volumes, and the handlebody attachments are different.

so the non-degenerating slopes are different even up to homeomorphism of $M$. This means that the handlebody attachments about are actually infinitely many. □

**The proof of Theorem 1.3:**
Since vertexes in \( \mathcal{F} = \mathcal{C}(T) \) are determined by a pair of co-prime integers, the pair of co-prime integers are subset of lattices in \( \mathbb{C} \), the plane, but the co-prime pairs \( (a, b) \) and \( (-a, -b) \) correspond to the same slope in \( \mathcal{F} = \mathcal{C}(T) \), so we just take lattices in the up half space \( \mathbb{H} \). We also denote by \( b/a \) the lattice \( (a, b) \) in \( \mathbb{H} \).

Let \( L_\alpha = \{(x, y)|y = \alpha x\} \) be a ray in \( \mathbb{H} \), and for \( \alpha > 0 \), the \( \theta \)-neighborhood of \( L_\alpha \) denoted by \( L_\alpha(\theta) \) is the set \( L_\alpha(\theta) = \{(x, y)|(\alpha - \theta)x > y > (\alpha + \theta)x\} \), which is a cone.

We have the following:

**Lemma 3.2.** Let \( \mathcal{B}_n \subset \mathcal{F} \) be the \( n \)-neighbourhood of \( 1/0 \), then there is a cone \( L_\alpha(\theta) \) such that \( \mathcal{B}_n \) intersect with \( L_\alpha(\theta) \) by at most one point, say 0/1.

**Proof:** We first claim that for each \( n \), there is a set of cones \( L_{\alpha_n}^n(\theta^n_j) \) which intersect only on \( 0/1 = 0 \) and three lines \( X_\pm = \{(x, y)|x = \pm 1\} \) and \( Y = \{(x, y)|y = 1\} \) with \( \mathcal{B}_n \subseteq \cup \cup_{j=1}^n L_{\alpha_n}^n(\theta^n_j) \cup X_\pm \cup Y \cup \{1/0, 0/1\} \).

We prove the claim by induction on \( n \). If \( n = 1 \), note that \( d(1/0, k/1) = 1 \) in \( \mathcal{F} \) since the intersection number of two slope \( b/a \) and \( y/x \) is \( \{bx - ay\} \), and so the \( 1 \)-neighborhood of \( 1/0 = \infty \) lies in two vertical lines \( X_\pm \) except itself and 0/1.

The \( 1 \)-neighborhood of \( 0/1 \) lies in the horizontal line \( Y \) except itself and 0/0. For a fixed \( k \neq 0 \), if \( d(k/1, b/a) = 1 \), then \( b = ka \pm 1 \), so \( b/a = k \pm 1/a \) which is in the \( \epsilon_k \)-neighborhood of \( y = kx \) for fixed \( \epsilon_k > 0 \) small enough and \( a \) sufficiently large. So there are only finitely points in the \( 1 \)-neighborhood of \( k/1 \) which is not in \( L_k(\epsilon_k) \), we call them exceptional points, and for each exceptional point \( v/w \), we take a small cone \( L_{v/w}(\epsilon_{v/w}). \)

We first choose \( \epsilon_1 \) small enough and then \( \epsilon_2 \) so that \( L_1(\epsilon_1) \cap L_2(\epsilon_2) = \{0/1\} \), and then we choose \( \epsilon_3, \epsilon_4, \ldots \). Then we treat the finite exceptional point \( v/w \) for the \( 1 \)-neighborhood of \( 1/1 \) one-by-one, choose the \( \epsilon_{v/w} \) small enough such the cone intersects other cones only on \( \{0/1\} \), then we treat the finite exceptional point \( p/q \) for the \( 1 \)-neighborhood of \( 2/1 \) in the same way, and then the exceptional point for \( 3/1 \).

We return these cones by \( L_{\alpha_n}^n(\theta^n_j), j = 1, 2, 3 \ldots \), and which intersect only on \( \{0/1 = 0\} \) such that \( \mathcal{B}_n \subseteq \cup \cup_{j=1}^n L_{\alpha_n}^n(\theta^n_j) \cup X_\pm \cup Y \cup \{1/0, 0/1\} \).

Now assume by induction that the claim is true for \( n = k - 1 \), then for each \( b/a \) which is distance \( k - 1 \) with 1/0, which is in a cone constructed above, say in \( L_{\alpha_1}^k(\theta_1^k) \), choose \( \epsilon \) small enough such that \( L_{b/a}(\epsilon) \subset L_{\alpha_1}^k(\theta_1^k) \), from the same line above, all but finitely \( 1 \)-neighborhood of \( b/a \) lies in \( L_{b/a}(\epsilon) \). For each exceptional point \( v/w \) to \( b/a \), if it is also contained by one of the cone constructed above, do nothing; otherwise, we choose a small cone contains it.

Perform the above process for each point which is distance \( k - 1 \) with 1/0, we get a set of cones such that \( \mathcal{B}_k \) are embraced by these cones together with \( X_\pm \cup Y \cup \{1/0, 0/1\} \). By induction, the claim follows.

From the claim, we then choose a small cone \( L_\alpha(\beta) \) which is disjoint from \( L_{\alpha_n}^n(\theta^n_j), j = 1, 2, 3 \ldots \) but not \( 0/1 \), this ends the proof of the Lemma. \( \square \)

Now from Lemma 3.2 and Theorem 1.1, Theorem 1.3 follows. \( \square \)

4. The proof of the Theorem 1.4

In this section, we prove Theorem 1.4, first a few facts:
Let $f_j : F_j \to F_j$ be a pseudo-Anosov map, recall that for each $f_j$, there are two fixed points in $P\mathcal{MF}(F_j)$, the projective measured foliation space of $F_j$, say $l_+$ and $l_-$, which are attractor and repeller respectively, see [3]. For the fixed handlebody $H_j$, there is the limit set of the mapping class group of $H_j$ acts on $P\mathcal{MF}(F_j)$, say $\Lambda_j$, which is the closure of the disk set, see [12]. Since $\Lambda_j$ has measure zero in $P\mathcal{MF}(F_j)$ by Kerchhoff [8], we say $f_j$ is generic pseudo-Anosov if $l_+ \cap \Lambda_j = \emptyset$, see [12]. Note that generic pseudo-Anosov $f_j$ can not be extended to a homeomorphism of the handlebody $H_j$ and $f_j$ acts isometrically on $\mathcal{C}(F_j)$.

The proof of Theorem 1.4: By Lemma 3.1, the projection of $\mathcal{D}_+$ into $\mathcal{C}(F_j)$ has finite diameter. Let $\mathcal{D}_j$ be the set of disks in $H_j$, by Theorem 1.1 of [1], we have that there are two constants $a$ and $b$, such that in the curve complex of $\mathcal{C}(F_j)$, we have $n/a - b \leq d_{\mathcal{C}(F_j)}(\mathcal{D}_j, f_j^n(\mathcal{D}_j)) \leq na + b$, so we have that the distance of $\mathcal{D}_j$ and $f_j^n(\mathcal{D}_j)$ is large for $n$ sufficiently large, and then we have $d_{\mathcal{C}(F_j)}(P\pi(\mathcal{D}_+), f_j^n(\mathcal{D}_j))$ is large.

If the Heegaard distance of $M^* = H_+ \cup_S H^*$ is less then $n$, we assume that $a$ bounds a disk $\mathcal{D}_+$ in $H_+$ and $a^*$ bounds a disk $D^*$ which realize the Heegaard distance, then $D^* - \mathcal{E}$ is a set of disks, and at least one, say $D_1$, is essential in $H_j$. As in the proof of Theorem 1.1, we have $d(P\pi(\mathcal{D}_+), \partial D_j)$ is less then $\mathfrak{M} + 1$, which is a contradiction to the that $d(P\pi(\mathcal{D}_+), f_j(\mathcal{D}_j))$ large. □

5. Generalization for $H_+$ is a compression body with non-empty negative boundary

In this section, we generalize the main theorems to the case that both $H_+$ and $H_-$ are compression bodies with non-empty negative boundary. In fact, in this case the assumption on the initial distance $n$ can be weaken to $n \geq 3$:

Lemma 5.1. If $M = H_+ \cup_S H_-$ is a Heegaard splitting with distance $n \geq 3$, where $H_+$ and $H_-$ are compression bodies with non-empty negative boundary. Let $\mathcal{D}_+$ be the disk set of $H_+$, then for any component $F$ of $\partial_-H_-$, $P\pi(\mathcal{D}_+)$ has diameter at most $\mathfrak{M}$.

Proof: Since $\partial_-H_+$ is non-empty, there is a curve $c$ in $\mathcal{C}(S)$, which is disjoint from $\mathcal{D}_+$. So $\forall x, y \in \mathcal{D}_+$, we have $d_{\mathcal{C}(S)}(x, y) = 1$ or 2. Note that $x \cap F' \neq \emptyset$ by the assumption that $n$ is at least 3, where $F'$ is the compact surface with one boundary corresponding to $F$ as in the Section 4, and if $c \cap F' = \emptyset$, so $d_{\mathcal{C}(S)}(c, \partial F') = 1$, then with $d_{\mathcal{C}(S)}(c, x) = 1$ we have $d_{\mathcal{C}(S)}(x, \partial F') = 2$, a contradiction to $n \geq 3$. Then for the length 1 or 2 geodesic $[x, y]$, we can use Theorem 2.3, the lemma follows. □

Now, with Lemma 5.1, similarly to the proofs in Section 3 and Section 4, we have:

Theorem 5.2. If $M = H_+ \cup_S H_-$ is a genus $g$ Heegaard splitting of a bordered 3–manifold with Heegaard distance $n \geq 3$, where $H_+$ and $H_-$ are compression bodies with non-empty negative boundary. If $c_1$, $c_2$ are two degenerating slopes in the same component $F$ of $\partial_-H_-$, then the distance of $c_1$ and $c_2$ in $\mathcal{C}(F)$ is bounded above by $3\mathfrak{M} + 2$.

Theorem 5.3. If $M = H_+ \cup_S H_-$ is a genus $g$ Heegaard splitting with Heegaard distance $n \geq 3$, where $H_+$ and $H_-$ are compression bodies with non-empty negative boundary with $\partial_-H_- = F_1 \sqcup F_2 \ldots \sqcup F_l$, $H_j$ is handlebody which has the same genus
as \( F_j \), fixed a homeomorphism \( \tau_j : F_j \to \partial H_j \), for generic pseudo-Anosov map \( f_j : \partial H_j \to \partial H_j \), there is \( n_j \) such that if \( H^* = H_1 \cup \cdots \cup H_m \cup \cdots \), \( m_j \geq n_j \), then the resulting Heegaard splitting \( M^* = H_+ \cup H^* \) has Heegaard distance \( n \).

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