We study QCD axion or cosmological Goldstone bosons, such as ultra-light dark matter or quintessence, in a model with global symmetry highly protected by gauge symmetries. The global symmetry is accidental, obtained from an abelian quiver with scalar bifundamental fields. The Goldstone boson mass may receive explicit breaking contributions, but these are already much suppressed for a few quiver sites if the gauge charges of the scalars are appropriately chosen. The model can be obtained by latticizing an abelian 5d gauge theory on the linear dilaton background.
1 Introduction

New pseudo-Goldstone bosons (PGB’s) may play an important role in particle physics and cosmology, since they can solve the strong CP-problem [1–3], explain dark matter [4, 5] or inflation [6,7] and make dark energy dynamical [8–11]. Some of these models require that the PGB mass be very light, so there is a need to protect it from large explicit (e.g. gravitational) breaking of global symmetries [12–14]. Consequently, one often considers the symmetry from which the PGB’s originate as accidental, i.e. as unbroken by any gauge-invariant operator up to a given dimension. One often achieves this by interpreting these symmetries as remnants of extra-dimensional gauge symmetries (or, more precisely, by interpreting PGB’s as fifth components of vector fields which appear as scalars in 4d).

Inspired by the recent interest in 4-dimensional clockwork models [15–23], we discuss in this paper the properties and potential applications of the 4d quiver model obtained by latticizing a 5d (abelian) gauge theory in a linear dilaton background [24], with Dirichlet boundary conditions for the 4-d components of the gauge boson [18,20]. As a result of the 5d gauge invariance, the 4d field content is such that its most general renormalizable gauge-invariant lagrangian preserves an accidental global symmetry. Furthermore, the specific 5d background, or equivalently the specific 4d gauge charge assignment, ensures a strong protection of this accidental symmetry from explicit breaking terms, even when the discretization is crude (i.e. when the quiver has few sites).

The plan of the paper is as follows: in section 2 we recall the 4d model with a focus on the light scalar and examine its properties. In section 3, we discuss its potential identification with a QCD axion, while in section 4 we consider its applications to cosmological models that make use of a light particle (e.g. PGB dark matter and quintessence models). We present our conclusions in the last section. Some appendices cover additional material: appendix A contains the 5d deconstruction of an abelian vector field in a linear dilaton background whose low-energy limit matches that of our 4d picture, appendix B discusses the massive states of the model of section 2, appendices C.1 and C.2 describe realizations of the QCD axion discussed in section 3, appendix D displays a calculation of the axion-photons couplings of sections 3 and 4.1 and appendix E discusses the ranges of parameters of the model which allow the axion to be (a detectable kind of) dark matter.
2 Model

2.1 Gauge group and matter content

The (4d) setup we consider is an abelian quiver model with bifundamental scalar fields, first presented in [18] as the deconstruction [25, 26] of a 5d abelian gauge theory on an orbifolded linear dilaton background with Dirichlet boundary conditions for the 4d gauge field\(^1\) (see appendix A), and whose low-energy theory was derived in [20] as the 4d theory obtained after chiral symmetry breaking by some confining non-abelian gauge group. The precise matter content and charge assignment is given by the following quiver (where \(q\) and \(N\) are integers):

![Abelian quiver of the model](image)

with the following (most general renormalizable) lagrangian\(^2\):

\[
\mathcal{L} = -\frac{1}{4g_i^2} \sum_{i=1}^{N} F_{\mu\nu,i} F_{\mu
u}^{i} - \sum_{k=0}^{N} (|D_\mu \phi_k|^2 + m_k^2 |\phi_k|^2) - \sum_{k,l=0}^{N} \lambda_{kl} |\phi_k|^2 |\phi_l|^2 , \tag{2.1}
\]

where \(F_i\) is the field strength of the abelian vector field \(A_i\), with coupling constant \(g_i\), and with the covariant derivatives \(D_\mu \phi_k = (\partial_\mu - i(1 - \delta_{k,0}) A_\mu,k + iq(1 - \delta_{k,N}) A_{\mu,k+1}) \phi_k\). This lagrangian has a \(U(1)^{N+1}\) invariance, with a \(U(1)^N\) gauged subgroup.

This model is inspired by the so-called clockwork mechanism [15–23] and has been introduced in [18] as a possible realization of it, so we will comment on defining features of this mechanism if we recover them while we proceed, or discuss those which are different.

2.2 Spontaneous breaking and Goldstone mode

We are interested in obtaining Goldstone bosons, so we consider the spontaneous breaking of the full \(U(1)^{N+1}\) mentioned previously by choosing the parameters \(m_k^2\) and \(\lambda_{kl}\) of (2.1) so that all the scalar fields \(\phi_k\) get vev’s \(f_k\). The spectrum then consists after gauge fixing of...
$N$ massive vectors, $N+1$ massive real scalars (discussed in appendix B) and one Goldstone boson.

Since the vev’s $f_k$ break all the gauge symmetries, $N$ out of the $N+1$ phases of the $\phi_k$ are absorbed by the gauge vectors through the Higgs effect. The absorbed phase combinations depend on the charges and vev’s (we write $\phi_k = \frac{f_k + r_k}{\sqrt{2}} e^{i \theta_k}$):

$$L \supset -A_{\mu,i}(q f_{i-1} \partial_\mu \theta_{i-1} - f_i \partial_\mu \theta_i). \quad (2.2)$$

The last, uneaten phase $a$ remains in the spectrum after gauge fixing as a Goldstone boson associated to the accidental $U(1)_a$ global symmetry which is the ungauged factor of the $U(1)^{N+1}$ symmetry group of (2.1). The profile of this boson along the original phases is orthogonal to the $q f_{i-1} \theta_{i-1} - f_i \theta_i$ gauge Goldstone bosons profiles. If we canonically normalize the field and the vev’s are taken to be all equal, which will be assumed from now on\(^3\) (we then note $f_k = f$), it reads:

$$a = \frac{\theta_0 + q \theta_1 + ... q^N \theta_N}{\sqrt{1 + q^2 + ... + q^{2N}}}, \quad (2.3)$$

Eq. (2.3) displays the exponential localization discussed in clockwork models, and the charges of the original scalar fields under the global symmetry also match those which appear in those models. Indeed, $U(1)_a$ acts here as $\phi_k \rightarrow e^{iqk} \phi_k$.

### 2.3 Goldstone boson protection

The lagrangian (2.1), has an accidental exact $U(1)_a$ global symmetry at renormalizable level, hence the axion $a$ is massless. We expect however that global symmetries are broken by gravity effects [12–14], which forces us to include all higher order operators allowed by gauge invariance in the effective theory. For the quiver of Figure 1, these operators must be combinations of

$$|\phi_k|^2 \text{ and } \phi_0 \phi_1^q ... \phi_N^q. \quad (2.4)$$

Hence, operators that explicitly break the global symmetry must involve the second term and be of extremely high dimension as soon as $q$ and $N$ are both slightly bigger than one.

\(^3\)In the generic case, the axion profile is, up to a normalization factor,

$$a \sim \frac{\theta_0}{q^N f_0} + \frac{\theta_1}{q^{N-1} f_1} + ... + \frac{\theta_{N-1}}{q^{f_{N-1}}} + \frac{\theta_N}{f_N}. \quad (2.5)$$
We thus obtain in this setup a pseudo-Goldstone boson with a mass very well protected by
the gauge symmetry. The exponential dependence on $q$ and $N$ of the second operator of (2.4) can be used to make the boson mass “sufficiently” small with a reasonable number of
gauge groups, as we will emphasize later on. More specifically, if we use (2.3), we find:

$$\frac{\phi_0 \phi_1^q \ldots \phi_N^q}{M_c^{1+q+\ldots+q^N-4}} + \text{h.c.} \mid_{\text{axion terms}} = 2 \left( \frac{f}{\sqrt{2}M_c} \right)^{1+q+\ldots+q^N} M_c^4 \cos \left( \frac{a}{f_a} \right) - \frac{1}{2} m_a^2 a^2 , \tag{2.5}$$

where

$$f_a = \frac{f}{\sqrt{1 + q^2 + \ldots q^{2N}}} \tag{2.6}$$

and

$$m_a \sim \left( \frac{f}{\sqrt{2}M_c} \right)^{\frac{1}{2}(q+\ldots+q^N-1)} \sqrt{1 + q^2 + \ldots q^{2N}} M_c , \tag{2.7}$$

and $M_c$ is the cutoff of the theory, which we take close to the Planck mass $M_P$ when we
consider gravity-induced breaking effects (recall however that if a large number $N$ of particles
is present, the actual cutoff of the theory cannot be more than roughly $\frac{M_P}{\sqrt{N}} \ [29–32]$). Even
though $M_c$ may also be the scale of other breaking effects (such as the mass of heavy fermions
explicitly breaking $U(1)_a$ and running in loops, see Appendices A and C.1 for discussions on
this topic), we will for simplicity focus on gravitational scale breaking.

### 3 QCD axion

We dedicate this section to the study of the compatibility of the Peccei-Quinn (PQ) idea [1–3]
(see also [33] for a review) with the setup of section 24.

#### 3.1 Accidental Peccei-Quinn symmetry in the low-energy field

theory

We study in this section the low-energy effective field theory of the axion, assuming that every
other massive field has been integrated out. In order to identify $U(1)_a$ with a Peccei-Quinn
symmetry, we consider the following axionic coupling5:

$$i \log \left( \phi_0 \phi_1^q \ldots \phi_N^q \right) \text{Tr} \left( G^{\mu \nu} \tilde{G}_{\mu \nu} \right) + \text{h.c.} , \tag{3.1}$$

4 Accidental PQ symmetries have been studied in many different setups, see for example [34–44].

5 We will not pay attention to writing dimensionless quantities in the log’s since it does not affect the
discussion about axions which reside in the phases of the fields. Every expression in a log to appear in the
rest of the paper should then be thought of as a dimensionless one (e.g. a log(scalar) means a log(scalar
divided by a mass scale)).

6
where $G_{\mu\nu}$ is the gluon field strength and $\tilde{G}_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}G^{\rho\sigma}$ its dual. In this section, we will not discuss the origin of this coupling, which may arise from a string theory or from a UV-complete field theory (section 3.2 deals with the field theoretic case).

The operator in the log is, as we said in section 2.3, the first gauge-invariant term capable of coupling the axion of (2.3) to the gluons that we could have written (using a gauge-invariant term is necessary in order not to generate any $U(1)_i \times SU(3)^2$ gauge anomaly). This coupling has two major generic features: it involves all the quiver sites, and it implies a decrease in the decay constant of the axion compared to the scale of breaking $f$. Indeed, when we plug back the axion profile (2.3) in (3.1), we obtain:

$$i \log \left( \phi_0 \phi_1 \ldots \phi_N \right) \mathrm{Tr} \left( G^{\mu\nu} \tilde{G}_{\mu\nu} \right) + \text{h.c.} \supset -\frac{2\sqrt{1 + q^2 + \ldots q^N}}{f} a \mathrm{Tr} \left( G^{\mu\nu} \tilde{G}_{\mu\nu} \right),$$

(3.2)

where we recognize the effective axion decay constant of (2.6). This suggests that the present setup could describe intermediate scale axion decay constant obtained from high scale physics (such as string scale physics).\(^6\)

When non-perturbative effects of QCD turn on, (3.2) induces a potential for the axion:

$$\mathcal{L} \supset m_a^2 f_a^2 \cos \left( \frac{a}{f_a} - \theta_{\text{QCD}} \right).$$

(3.3)

We also include every gauge-invariant term to the potential, according to the discussion of section 2.3, and in particular generate a classical explicit breaking mass term (2.7) for the axion. In order to have $\left| \frac{a}{f_a} - \theta_{\text{QCD}} \right| < 10^{-10}$ at the minimum of the potential and solve the strong-CP problem, we must ensure [42–44] that:

$$\left[ m_{a,\text{QCD}} \sim \frac{m_{\pi} f_{\pi}}{f_a} \right] > 10^5 \left[ m_{a,\text{explicit}} \sim \left( \frac{f}{\sqrt{2} M_c} \right)^{\frac{1}{2}} (q^2 + \ldots q^N - 1) \frac{f}{f_a} M_c \right]$$

or equivalently

$$f \lesssim \left( 10^{-5} \sqrt{2} m_{\pi} f_{\pi} (\sqrt{2} M_c) \frac{1}{2} (q + \ldots + q^N - 3) \right)^{\frac{2}{q^2 + \ldots q^N + 1}}.$$

(3.4)

For example, when $q = 3, N = 2$ and $M_c = M_P$, it implies $f \lesssim 10^{12}$ GeV. If now $q = 3, N = 3$ and $M_c = M_P$, this becomes $f \lesssim 10^{16}$ GeV.\(^7\) This constraint is less and less demanding when

\(^6\)This feature, added to the fact that each site of the quiver contributes to the anomalous coupling, is qualitatively different from those of axion clockwork models, where the anomaly is generated at one site and the effective decay constant is bigger than the scale of new physics, often considered to be $\sim$ TeV.

\(^7\)These values show the compatibility of our setup with astrophysical ($f_a \gtrsim 10^9$ GeV) and cosmological ($f_a \lesssim 10^{11}$ GeV) bounds on the axion decay constant (the upper bound can be relaxed, if the PQ symmetry is assumed to be broken during inflation, as soon as one allows for tuning in the cosmic initial conditions for the axion), see [45].
q or N grow, and it shows that it is easy to protect the PQ solution in our setup with a few quiver sites, even when the axion scale is high (e.g. string like).

Axion couplings to photons, which are the subject of most axion searches, are also part of this low-energy discussion. They can be derived when we consider the axionic generalizations of (3.1):

\[
\mathcal{L} \supset \frac{iC}{32\pi^2} \log\left(\phi_0\phi_1^q...\phi_N^q\right) G^{a,\mu\nu} \tilde{G}_\mu^a + \frac{i\mathcal{E}}{32\pi^2} \log\left(\phi_0\phi_1^q...\phi_N^q\right) F^{\mu\nu} \tilde{F}_{\mu\nu}
\]

\[
\to -\sqrt{1 + q^2 + ... q^{2N}}(\mathcal{E} - \frac{2C}{3} 4 + m_u/m_d) a F^{\mu\nu} \tilde{F}_{\mu\nu}
\]

\[
\approx -\sqrt{1 + q^2 + ... q^{2N}}(\mathcal{E} - 1.92C) a F^{\mu\nu} \tilde{F}_{\mu\nu}
\]

where \( F \) is the photon field strength, \( \tilde{F} \) its dual, \( m_{u,d} \) are quark masses, \( C \) and \( \mathcal{E} \) anomalous constants (which we will specify when we deal with precise models in what follows), the arrow indicates that we took into account the mixing between the axion and the mesons which arises from (3.2) and (3.3) [46] and we used \( m_u \approx 0.6m_d \) in the last line. These couplings feature the dependence on the decreased effective decay constant (2.6) we already encountered in (3.2).

### 3.2 Axionic couplings from heavy fermion loops

We now discuss the UV origin of (3.1) in terms of loops of heavy fermions coupling to the axion.

Let us first recall how axionic couplings are generated via quarks loops. (Global) anomalies with respect to \( SU(3)_c \) are mediated by colored fermions\(^8\) with some charge under the (global) symmetry, which run in loops between gluons and scalars, whose phase contains part of the axion mode. The schematic procedure\(^9\) is:

\[
\mathcal{L} = -|\partial \sigma|^2 - \overline{Q} \gamma^\mu (\partial_\mu - iA_\mu^a T^a) Q - (y\sigma L Q_R + h.c.) \text{ where } \sigma = \frac{f}{\sqrt{2}} e^{i\frac{\pi}{2}}
\]

\[
\to -\frac{1}{2} (\partial \sigma)^2 - \overline{Q} \gamma^\mu (\partial_\mu + \frac{yf}{\sqrt{2}} - iA_\mu^a T^a) Q + \frac{y}{\sqrt{2}} a \overline{Q} \gamma_5 Q
\]

\[
\text{triangle loop} \longrightarrow -\frac{a}{32\pi^2 f} G \overline{G} = \frac{i}{32\pi^2} \log(\sigma_{\text{axion terms}}) G \overline{G}
\]

\(^8\)In our case, heavy fermions must obtain their mass from Yukawa couplings since Dirac or Majorana mass terms require vector-like representations.

\(^9\)An example of the triangle loop calculation, including the numerical coefficients, is presented in appendix D.
where $\sigma$ is a scalar field, $Q_{L,R}$ are left and right handed colored fermions, $T^a$ are the generators of $SU(3)_c$, $A$ is the gluon field of field strength $G$ and $y$ is a Yukawa coupling.

We then see how to generate (3.1) from fermions loops, starting from the following lagrangian\(^\text{10}\):

$$\mathcal{L} \supset - y_0 \phi_0 Q_{L,0} Q_{R,0} - \phi_1 Q_{L,1} Q_{R,1} - \phi_2 Q_{L,2} Q_{R,2} + \ldots + h.c. $$

\[ \text{triangle loops} \xrightarrow{i \over 32\pi^2} i \over 32\pi^2 \left( \log(\phi_0) + q \log(\phi_1) + \ldots q^N \log(\phi_N) \right) \overline{G} \overline{G} = \overline{G} \overline{G}. \]

This procedure is actually the minimal one (with dimension four Yukawa couplings) that generates an $SU(3)_c^2 \times U(1)_a$ anomaly without generating gauge anomalies (or said differently, that generates (3.1)). It requires adding colored fermions at each site, in accordance with the fact that (3.1) involves all quiver links. In terms of couplings defined in (3.5), it has $\mathcal{C} = 1$ and $\mathcal{E} = 0$. Note that the lagrangian (3.7) (and (3.9) below) respects the global symmetry $U(1)_a$. Therefore, it cannot generate the scalar potential (2.5) by quantum corrections. However, since we had to nonetheless include (2.5) as a gravity correction, we should also consider all gauge-invariant non-renormalizable fermionic operators in addition to (3.7). Those operators could classically break $U(1)_a$ and generate both the mass of the axion and its couplings to the gluons. In appendix C.1, we present a model with such fermionic operators.

The number of additional fermions grows exponentially with $N$: for instance, in order to use (2.6) to bring a Planck scale $f$ down to an intermediate scale $f_a = 10^{10-11}$ GeV, we need $\sim q^N \gtrsim 10^{7-8}$ additional fermions (which would however be close to the Planck mass and would thus not spoil gauge coupling unification, or perturbativity far below the Planck mass). Alternatively, if we start with $f$ already at intermediate scale, the strong CP-problem is for instance solved when $f \sim 10^{11}$ GeV, $q = 3$ and $N = 2$. This is enough to ensure the gauge protection according to the discussion following (3.4), with $1 + 3 + 3^2 = 13$ additional Dirac fermions in the 3 of $SU(3)_c$. The new fermions spoil asymptotic freedom but keep perturbativity of the strong interactions below the Planck mass. In this specific example, we get a (detectable) coupling to photons from (3.5):

$$\mathcal{L} \supset (1.7 \times 10^{13} \text{ GeV})^{-1} a F^{\mu\nu} \tilde{F}_{\mu\nu}. \quad (3.8)$$

If one wants to circumvent the conclusions of (3.7), one can also assign gauge charges to

\(^{10}\text{This procedure, as well as (3.9), is uniquely determined by the fermionic gauge charges, see Appendix C.1.}\)
the fermions so that their lowest gauge-invariant mass terms are of higher dimension. One example of this type is

$$L \supset -y_0 \phi_0 \bar{Q}_{L,0} Q_{R,0} - \frac{1}{M_c^{q+...+qN-1}} \phi_1 \phi_2^q ... \phi_N^{qN-1} \bar{Q}_{L}^{qN-2-q} Y_{ij} Q_R^{j} + h.c. \quad (3.9)$$

where $M_c$ is the cutoff of the theory. The action (3.9) couples the axion to the gluons via a number of additional fermions independent on $N$, but the high dimension of the second coupling in the first line of (3.9) lowers the mass of the $Q_i$ fermions. Since these fermions are colored and unobserved at the LHC, we must impose $m_{Q_i} \gtrsim$ a few TeV, which gives, if one takes as an example $f = \sqrt{2} M_c$ and $M_c = M_P$,

$$\left( \frac{f}{\sqrt{2} M_c} \right)^{1+q+...+q^{N-1}} M_c \gtrsim \text{TeV} \Rightarrow \frac{q^N}{q-1} \lesssim 15. \quad (3.10)$$

The bound is even more stringent as soon as we decrease $f$ in order to satisfy (3.4). It imposes in particular that we cannot reduce the decay constant of the axion using (2.6) and (3.9) from the Planck scale down to the intermediate scale of invisible axion models. One can interpolate between (3.7) and (3.9), but then there will either be limitations on $q$ and $N$ due to the high dimension of the mass terms or a number of fermions that grows with $N$ (or both).

This discussion goes beyond the particular case of the quiver of Figure 1 and concerns every theory with a protected PQ symmetry: the higher the quality of an accidental PQ symmetry, the higher the number of fermions required to make it anomalous with axionic couplings generated by fermion loops.$^{11}$

Indeed, any axionic coupling term in such a theory free of gauge anomalies must be of the form:

$$i \log(O) G \tilde{G} + h.c. \quad (3.11)$$

where $O$ is by construction gauge-invariant and not invariant under the anomalous global symmetry. If it arises from loops of heavy fermions, it is through the scheme discussed above:

$$L \supset -\sum_i (O_i \bar{\psi}_{i,L} \psi_{i,R}) \overset{\text{triangles}}{\longrightarrow} \frac{i}{32\pi^2} \log \left( \prod_i O_i \right) G \tilde{G} \quad (3.12)$$

$^{11}$This applies in particular if we enlarge the scalar content of the theory depicted in Figure 1 to additional scalar fields while keeping the quiver as the main source of protection, as in appendix C.2.
(where we assumed that we removed from the sum every pair of vector-like fermions), and \( O = \prod_i O_i \). However, the very notion of accidental axion symmetry means that \( O \) is an operator of high dimension, so the targeted quality of the axion global symmetry imposes a lower bound on \( \dim(O) = \sum_i \dim(O_i) \), while the definition we adopt for “heavy” fermions (in our case, unobserved at the LHC) puts an upper bound on \( \dim(O_i) \) for each \( i \). The two limits together imply a lower bound on the number of heavy fermions.

4 Axion-like particles

In this section, we study the case of axion-like particles (ALP’s), which generically refers to pseudo-Goldstone bosons not designed to solve the strong-CP problem, and whose interactions are consequently less constrained than those of the QCD axion.

Since section 3.2 showed us that making our accidental \( U(1)_a \) anomalous with respect to some external gauge group is demanding, especially when \( N \) grows, we will focus on those ALP’s which do not have any anomalous couplings. We will see that, in such models, gravitational contributions are sufficient to make the ALP a good dark matter candidate. Furthermore, there exist operators which make the dark matter ALP detectable, if for example some standard model particles are charged under the quiver gauge symmetry. This only requires limited additions to the particle content of Figure 1, even when the number of quiver sites is large. However, while the \( U(1)_a \) protection by the quiver is strong enough to generate quintessence-like mass scales, the need for trans-Planckian field values of usual axion quintessence models is still present in our setup, and is exacerbated by the reduction of the axion effective decay constant.

4.1 ALP’s potentials and dark sector candidates

Since most ALP’s are used in cosmology (see [45] for a review), let us first discuss the cosmological relevance of our setup. In the non-anomalous setup that we chose to consider in this section, we think of any ALP potential as generated by some classical explicit breaking in a UV theory. The lowest-dimensional gauge-invariant potential of this type for the particle \( a \) of (2.3) is (2.5), which very weakly breaks \( U(1)_a \) and entitle us to call \( a \) a pseudo-Goldstone boson, as we discussed in section 2.3. It is a typical periodic potential, consistent with the ALP’s origin as a periodic phase degree of freedom, and such potentials are very useful in
cosmology: the smallness of the masses and the specific potential they provide make ALP’s good dark matter or dynamical dark energy candidates via the misalignment mechanism. The relic density can be calculated once we are given the initial value $a_{\text{init}}$ of the ALP field after inflation and its mass $m_a$ (given in (2.7) for our setup):

$$\Omega_a \approx \begin{cases} 
\text{Dark matter}: & 2 \times 10^2 \left( \frac{m_a}{10^{-22}\text{eV}} \right)^{1/2} \left( \frac{a_{\text{init}}}{M_P} \right)^2 \\
\text{Dark energy}: & 8 \times 10^{-2} \left( \frac{m_a}{10^{-33}\text{eV}} \right)^2 \left( \frac{a_{\text{init}}}{M_P} \right)^2 
\end{cases} \quad (4.1)$$

$a_{\text{init}}$ is given by $a_{\text{init}} = \epsilon_{\text{init}} f_a$, where $f_a$ (which defines the periodicity of the ALP potential) is given in (2.6) and $\epsilon_{\text{init}}$ depends on one’s taste for tuning (the spontaneous breaking of $U(1)_a$ must happen before inflation, see the discussion of appendix E). In order for these formulas to be valid, i.e. for the ALP to behave like CDM before radiation-matter equality or like dark energy today, we supplement (4.1) with:

$$m_a \begin{cases} 
\text{DM}: & \gtrsim 10^{-28} \text{ eV} \\
\text{DE}: & \lesssim 10^{-33} \text{ eV} 
\end{cases} \quad (4.2)$$

(where the bound for DM can be pushed up to $m_a \gtrsim 10^{-22} \text{ eV}$ when non-linear cosmological observables are taken into account).

In our setup, obtaining masses as low as those which appear in (4.2) without tuning is easy (for instance, (2.7) equals $\sim 10^{-33} \text{ eV}$ when $f = 0.13M_P, q = 3, N = 4$). However, we can see from the comparison of (4.1) and (4.2) that axion quintessence demands initial values which are higher than the Planck mass. This can be achieved with some tuning on $\epsilon_{\text{init}}$ or when the effective decay constant of the axion is increased compared to the mass scales of the model (as in clockwork models which, however, have no mass protection mechanism built in). Since our effective decay constant (2.6) is reduced, the latter is not an option while the former is not enough to reach the correct energy density (if we insist on keeping $f$ below the Planck mass): indeed if we impose $m_a \lesssim 10^{-33} \text{ eV}$, we can only obtain $\Omega_a \lesssim 0.05$ and would need at least 13 of such ALPs to reach the observed dark energy density.

In contrast, natural dark matter candidates do arise in our model. In Figure 2, we scan the parameters $f$ and $M_c$ for some values of $q$ and $N$ (see appendix E for a more complete treatment) which satisfy the condition (4.1) for $\Omega_{\text{DM}} = 0.3$ and (4.2), allowing for $\epsilon_{\text{init}}$ to range from 0.1 to $\pi - 0.1$, and allowing a constant multiplying the potential (2.5) ranging from 0.033 to 30.

We see in Figure 2 that we obtain suitable DM candidates, and that the dependence on $q$ and $N$ of the mass (2.7) allows us to reach very low ALP’s masses. These small
masses, combined with the high scale $f$ of their associated new physics, are hard to realize in a pure field theoretical framework and are usually thought of as coming from a string axiverse \[47, 48\]. Our setup then provides an economical, in the sense of a low number of gauge groups, realization of such values. For instance, the smallest masses discussed in the literature for ultra light dark matter, $m_a \sim 10^{-21} - 10^{-22}$ eV, require $M_c \approx M_P$ and are obtained for $f \approx 0.2M_P$, $q = 3$ and $N = 4$ (for the choices of $q$ and $N$ displayed). This example shows that, even though we scan different values of $M_c$ in Figure 2, a gravitational origin ($M_c = M_P$) for (2.5) is sufficient to reproduce the cosmological relic density of dark matter.

In order to conclude that such ALP’s are to play a role in the cosmic evolution, we must check that their lifetime can be comparable to or bigger than the age of the universe. In generic models, there is a decay channel of an ALP into two photons, usually coming from a $U(1)_P \times U(1)_e$ anomaly. Even though there is no anomaly in the models of this section, non-anomalous, CP-even and gauge invariant operators that enable this decay exist. For instance, they can arise if we couple one of the quiver sites of Figure 1 to an anomaly-free set of electrically charged fermions displayed in Table 1, while we keep the standard model particles uncharged under the quiver gauge group.
Table 1: Anomaly-free set of fermions coupling the ALP to the photon field (the three first columns indicate the gauge charges of the fields whereas the last one gives the PQ charges induced by (4.3), as functions of $q_1$ and $q_2$ which are arbitrary)

|     | $U(1)_i$ | $U(1)_{i+1}$ | $U(1)_{em}$ | $U(1)_{a}$ |
|-----|----------|--------------|-------------|------------|
| $\psi_{L,1}$ | -1    | 0            | $ne$        | $q_1$     |
| $\psi_{R,1}$ | 0       | $-q$         | $ne$        | $q_1 + q^i$ |
| $\psi_{L,2}$ | 1       | 0            | $-ne$       | $q_2$     |
| $\psi_{R,2}$ | 0       | $q$          | $-ne$       | $q_2 - q^i$ |

With such charges, one can write Yukawa couplings $y_{1,2}$ to $\phi_i$:

$$L \supset -y_1 \phi_i \bar{\psi}_{R,1} \psi_{L,1} - y_2 \phi_i \bar{\psi}_{L,2} \psi_{R,2} + h.c. . \quad (4.3)$$

The effective operators describing the decay $a \to \gamma \gamma$ then are (see Appendix D for the computation):

$$L \supset \frac{n^2 e^2 q^i}{192\pi^2 \sqrt{1 + q^2 + ... q^{2N}}} f \left( \frac{1}{m_1^2} - \frac{1}{m_2^2} \right) (-\Box aF \tilde{F} + 2 \partial_\mu aF_{\nu \eta} \partial^\nu \tilde{F}^{\mu \eta}) , \quad (4.4)$$

where $m_1 = \frac{m f}{\sqrt{2}}, m_2 = \frac{m f}{\sqrt{2}}$ and $F$ is the photon field strength. Notice that, contrary to anomalous couplings such as (3.1), non-anomalous interactions are site-localized, and exhibit clockwork-like effects due to the profile (2.3). This feature will be present in all the operators discussed in this section. We can also see that the non-anomalous nature of the ALP-photons coupling makes this interaction of derivative type and of higher dimension than usual anomalous $aF \tilde{F}$ terms, so this decay does not make our ALP’s unstable over the cosmic history. Indeed these couplings give a decay rate:

$$\Gamma_{a \to \gamma \gamma} = \frac{q^2 n^4 \alpha^2 m_a^7}{1024 \pi^3 (1 + q^2 + ... q^{2N} f^2 \left( \frac{1}{m_1^2} - \frac{1}{m_2^2} \right)^2} , \quad (4.5)$$

where $\alpha$ is the fine structure constant. Hence, we conclude that this decay channel is harmless with respect to the cosmic evolution of our ALP’s. Indeed, as guessed above, the non-anomalous nature of the ALP-photons coupling forces the $m_a$ factor to appear in the decay rate (4.5) at a higher power than in the case of usual $aF \tilde{F}$-induced decays and ensures a long ALP lifetime. The clockwork-like dependence of (4.5) only tends to weaken the ALP couplings to photons when matter is coupled to the first quiver sites.

\[12\] For ultra-light dark matter with $m \sim 10^{-21}$ eV, if we choose $f = m_1 = 2m_2 = 0.3M_P$ and $n = 1$, the decay rate is $\sim 10^{-300} \text{s}^{-1}$. 

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The example of Table 1 is a realization of the more general gauge-invariant non-anomalous operators, coupling the axion to the photon field, which we can write within the effective field theory:

\[
\frac{1}{\Lambda^4} D^\mu \phi_i \phi_i^* F^{\mu \nu} \tilde{F}_{\mu \nu} \frac{i q^i f}{2 \sqrt{1 + ... q^2 N \Lambda^4}} (\Box a F \tilde{F}, \partial_\mu a \partial^n F^{\mu \nu} \tilde{F}_{\eta \nu} \text{ and } \partial_\mu a \partial^n \tilde{F}^{\mu \nu} F_{\eta \nu}),
\]

(4.6)

where \(D\) is the covariant derivative, there is no summation over the index \(i\) and \(\Lambda\) the scale at which this operator is generated. For instance, in the example of Table 1 \(\Lambda\) is equal to the mass of the \(\psi\) fermions. Since (4.6) preserves \(U(1)_a\), \(\Lambda\) does not have to be equal to \(M_c\) which was the scale of classical explicit breaking, even though there could also be \(U(1)_a\) preserving interactions at scale \(M_c\) (for instance there could be gravitational contributions of the form (4.6) where \(\Lambda = M_P\)). Thus, in a minimal, agnostic approach, we should consider effective theory operators such as (4.6), supplemented by the potential (2.5) where three independent scales are used: the scale \(f\), and \(f_a\) which follows, which are the scales of spontaneous breaking of \(U(1)_a\), are given by the quiver and the renormalizable scalar potential in (2.1). The scale \(M_c\), at which \(U(1)_a\) is explicitly broken, must verify \(M_c > f\) for the effective lagrangian to be valid and \(M_c \lesssim M_P\) since gravity anyway breaks \(U(1)_a\). Finally, \(\Lambda\) is a scale of additional physics which generates couplings of the quiver fields to other sectors of the theory, like the SM. It must respect \(\Lambda \gtrsim f\), since the new physics can lie at (almost) scale \(f\), like in the example of Table 1, but should not be at a lower scale than the effective theory one.

From (4.6) we can calculate the decay rate of an axion into two photons:

\[
\Gamma_{a \rightarrow \gamma \gamma} \sim \frac{q^2 m_a^2 f^2}{16 \pi (1 + q^2 + ... q^{2N}) \Lambda^8}.
\]

(4.7)

This result generalizes (4.5) and of course does not spoil the conclusions made with \(\Lambda \sim f\) since the dependence on \(m_a\), responsible of the low value of \(\Gamma_{a \rightarrow \gamma \gamma}\), has not changed and \(\Lambda \gtrsim f\) can only weaken the decay rate.

### 4.2 Detection via NMR

The non-anomalous couplings to photons of (4.6), too weak to destabilize the cosmic history of our ALP’s, are also too weak to be probed by current ALP’s searches, which rely on a
dimension 5 anomalous $aF\bar{F}$ coupling. Non-anomalous dimension 5 generic couplings of a Goldstone pseudoscalar $a$ to a detector’s matter can be parametrized [46] as follows:

$$
\frac{g_{aee}}{f_a} \partial_\mu a \bar{e} \gamma^\mu \gamma_5 e \text{ and } \frac{g_{aNN}}{f_a} \partial_\mu a \bar{N} \gamma^\mu \gamma_5 N,
$$

(4.8)

where $g$’s are dimensionless coupling constants, $f_a$ is again the axion decay constant, and $N$ and $e$ are respectively the nucleon and electron fields. In our setup, they can be generated in field theory if we charge the first family of the standard model under $U(1)_{i,i+1}$, according to Table 2\(^{13}\), in a way which gives them $U(1)_a$ charges. At lowest order, the most general

| Fields | $SU(3)$ | $SU(2)$ | $U(1)_Y$ | $U(1)_i$ | $U(1)_{i+1}$ | $U(1)_a$ |
|--------|---------|---------|---------|---------|------------|---------|
| $Q_L$  | 3       | 2       | $\frac{1}{6}$ | 0       | 0          | $qq$    |
| $u_R$  | 3       | 1       | $\frac{2}{3}$ | $-q$    | 1          | $q_Q + q_H + q^+$ |
| $d_R$  | 3       | 1       | $-\frac{1}{3}$ | $q$     | $-1$       | $q_Q - q_H - q^+$ |
| $L_L$  | 1       | 2       | $-\frac{1}{2}$ | 0       | 0          | $q_L$   |
| $e_R$  | 1       | 1       | $-1$ | $q$      | $-1$       | $q_L - q_H - q^+$ |
| $H$    | 1       | 2       | $-\frac{1}{2}$ | 0       | 0          | $q_H$   |

Table 2: SM charges that produce an ALP-spin coupling (the first columns indicate the gauge charges of the fields whereas the last one gives the PQ charges induced by (4.9), as functions of $q_Q, q_H$ and $q_L$ which are arbitrary)

lagrangian is the SM lagrangian where only the first family Yukawa terms have been modified:

$$
\mathcal{L} \supset -\frac{1}{M_c} \left( \bar{u}_R H \phi_i Y_u Q_L + \bar{d}_R (H \phi_i)^* Y_d Q_L + \bar{e}_R (H \phi_i)^* Y_e L_L \right) + h.c.
$$

$$
\supset - \frac{v f}{2M_c} \left( \bar{u} [e^{-i \frac{q_{a}}{\sqrt{1+...q_{2N}}} \gamma_5} Y_u] u + \bar{d} [e^{-i \frac{q_{a}}{\sqrt{1+...q_{2N}}} \gamma_5} Y_d] d + \bar{e} [e^{-i \frac{q_{a}}{\sqrt{1+...q_{2N}}} \gamma_5} Y_e] e \right),
$$

(4.9)

where $v$ is the Higgs vev and where we assumed that these higher order Yukawa couplings come from the same physics which generated (2.5), even though the fact that the precise scale $M_c$ divides these operators is of no importance for what follows. One can make the\(^{13}\)All anomalies involving at least one standard model factor are canceled with these charges. One must however add additional fermions only charged under $U(1)_{i,i+1}$ to cancel the $U(1)_{i,i+1} \times U(1)_{i,i+1} \times U(1)_{i,i+1}$ and $U(1)_{i,i+1}$-gravity anomalies. See appendix C.1 for explicit examples on how this can be done.
fermion masses in (4.9) real with an appropriate chiral redefinition of the fermions, and obtain from their kinetic terms the expected couplings:

\[
\mathcal{L} \supset -\frac{iq^i a}{2\sqrt{1 + ... + q^{2N} f}} (\bar{u}\gamma_5\gamma^\mu u + \bar{d}\gamma_5\gamma^\mu d + \bar{e}\gamma_5\gamma^\mu e),
\]

(4.10)

where no anomalous term appeared since the \(U(1)_a\) symmetry is anomaly-free and where we note that, similar to what was observed previously for axion-photons couplings, the ALP-spin coupling of (4.10) is site-dependent due to the clockwork profile (2.3).

If the mass (2.7) of the ALP is such that it constitutes part of the dark matter, these couplings may soon be tested via Nuclear Magnetic Resonance\(^{14}\) (NMR) by the CASPEr-Wind experiment [49]. As an illustration, in Figure 3 we assume that the coupling (4.10) is located at site \(i = 0\) of the quiver and restrict ourselves to the \((q, N)\) values displayed in Figure 2 and to the gravitational breaking of the axionic symmetry (i.e. to the case where \(M_c = M_P\), see appendix E for a more general study). We then see that CASPEr-Wind can detect some of the ALP’s discussed in this paper (one example is for \(f \lesssim 5 \times 10^{15}\) GeV, \(q = 2\) and \(N = 4\)). Thus the present model, while invisible to experiments based on axion-photons couplings, can be probed and constrained by NMR-based ALP’s searches. Note however that, in order for (4.9) to be consistent with the observed values of the fermion masses, there should not be a too strong hierarchy between \(f\) and \(M_c\). Notice also that possible FCNC effects induced by such Yukawa couplings (see for ex. [50]) are completely unobservable due to the high values of \(f\) and \(M_c\).

Like we did in (4.6), we can generalize such couplings in the gauge-invariant effective theory:

\[
\frac{1}{\Lambda^2} D_\mu \phi_i \phi_i^* \bar{e} \gamma^\mu \gamma_5 e \quad \text{and} \quad \frac{1}{\Lambda^2} D_\mu \phi_i \phi_i^* \bar{N} \gamma^\mu \gamma_5 N
\]

\[
\text{terms linear in } a \frac{iq^i f}{2\sqrt{1 + ... q^{2N} \Lambda^2}} (\partial_\mu a \bar{e} \gamma^\mu \gamma_5 e \quad \text{and} \quad \partial_\mu a \bar{N} \gamma^\mu \gamma_5 N),
\]

(4.11)

Once again, the scale \(\Lambda\) is a priori undetermined since (4.11) does not break \(U(1)_a\), for instance \(\Lambda = M_P\) if (4.11) is of gravitational origin. However, taking \(\Lambda = M_P\) does not allow to detect our DM candidates, contrary to the case of (4.10) where it is equal to \(f\).

\(^{14}\)Bounds already exist on axion-mediated spin-dependent forces between particles, but they do not constrain models with high or intermediate scale axion decay constants.
Figure 3: Sensitivity of CASPER-Wind to the ALP’s
(colored regions indicate axions suitable to saturate the DM relic density, blue curves set the limit of the upper left part of the plot where the sensitivity of CASPER-Wind allows for a DM detection. Both axes are log-scale)

5 Conclusion

A generic problem for QCD axion models is to control non-anomalous contributions to the axion potential coming from classical explicit breaking of the global Peccei-Quinn symmetry e.g by gravitational interactions. Such contributions would shift the axion field in the minimum to unacceptably large for solving the strong CP problem values. Similarly, such contributions would jeopardize the possibility of having ultralight axion-like particles as candidates for dark matter or dark energy, generating too large masses for them. In the latter case, one often invokes new anomalous gauge interactions as a source of properly adjusted explicit breaking and ignores non-anomalous gravitational contributions.

In this paper we have investigated the scenario of the QCD axion or ultralight axion-like particles obtained as pseudo-Goldstone modes of an accidental global symmetry coming from an abelian gauge theory quiver with scalar bifundamental fields. The underlying gauge symmetries protect very efficiently the global symmetry against the contribution from non-anomalous explicit breaking effects to the axion potential, and in particular to its mass. Such contributions are a function of the gauge charge assignment for the scalars and the number
of the quiver sites. Already with a few quiver sites they are sufficiently small for the pseudo-Goldstone boson to play the role of the QCD axion or a cosmologically relevant axion-like particle. We have mainly focused on gravity-breaking effects, which account for the fact that quantum gravity does not respect global symmetries, but other sources of non-anomalous explicit breaking effects can be considered as well.

In our model, a genuine QCD axion obtained from a renormalizable theory requires an additional fermionic content, with the number of heavy colored fermions growing exponentially with the number $N$ of gauge groups, but its mass protection against gravitational effects is already sufficient for $N = 2$. Stable pseudo-Goldstone dark matter models can also be easily built with small number $N$ of gauge groups. A very interesting aspect of them is that gravitational contributions, allowed but controlled by the gauge symmetry protection, can be used in a constructive way, to induce a just right axion potential without referring to any new strongly-interacting sector and its chiral anomalies. Such a dark matter axion-like particle can be coupled to the standard model with a small number of extra particles, if any, that does not depend on $N$. Such models can be tested via Nuclear Magnetic Resonance experiments, which record the matter spin precession due to the oscillation of the dark matter field. Pseudo-Goldstone quintessence models of dynamical dark energy can also be obtained in such a setup, but their construction faces usual challenges, such as a trans-Planckian axion decay constant, in order to recover the observed energy density.

Whereas a large number of sites $N$ is not needed for mass protection of the PGB, it could be an useful option to use in order to decrease the axion decay constant from a large (Planck or string) value to an intermediate scale, since qualitatively $f_{\text{eff}} \sim f/q^N$. Such a high $N$ also connects the 4d model to the deconstruction of a five-dimensional abelian vector model on a linear dilaton background, with Dirichlet boundary conditions for the 4d components of the gauge field, which shares the same low-energy limit as the 4d theory. This gives some intuition to understand some features of the 4d model. However, the main differences between our axion models compared to similar models based on deconstruction of flat extra dimensions [51, 52] are twofold. On the one side, the PGB mass comes from operators of very high dimension even for few sites $N = 2 - 4$. On the other side, the generation of these operators, breaking explicitly the global symmetry, from field theory loops does not have a five dimensional interpretation.
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A 5d deconstruction on a linear dilaton background

A.1 Abelian gauge field

We recall in this appendix the link between the 5-dimensional deconstructed theory of an abelian gauge field on a linear dilaton background and the low-energy modes of the 4d model defined in eq. (2.1).

We start by considering a 5d manifold which is the product of the 4d Minkowski space with an interval of length $L$, with a 5d theory living on it:

$$S = \int d^5z \sqrt{-g} L(z^M) = \int d^4x \int_0^L dy \sqrt{-g} L(x^\mu, y) , \quad (A.1)$$

where we split the 5d coordinates $z^M$ into 4d Minkowski coordinates $x^\mu$, and the position along the interval $y$.

We discretize the fifth dimension interval down to a regularly-spaced lattice of $N+2$ sites. Defining $\Delta_4 = \frac{L}{N+1}, y_i = i\Delta_4$ (where $i$ runs from 0 to $N+1$), this amounts to replacing:

$$\int_0^L dy f(x, y) \rightarrow \sum_{i=0}^N \Delta_4 f(x, y_i) , \quad (A.2)$$

$$\partial_y f(x, y_i) \rightarrow \frac{f(x^\mu, y_{i+1}) - f(x, y_i)}{\Delta_4} .$$

We choose to denote $f(x, y_i) = f_i(x)$ in what follows. We do not wish to study the dynamics of the background and restrict to the following static metrics:

$$ds^2 = g_{MN}dz^Mdz^N = e^{-2a(y)}(\eta_{\mu\nu}dx^\mu dx^\nu + e^{2b(y)}dy^2) . \quad (A.3)$$

The case $a = ky, b = 0$ describes the so-called linear dilaton background in the conformally flat frame, whereas $a = b = ky$ is the Randall-Sundrum metric.
We will study a five-dimensional abelian theory of lagrangian
\[ \sqrt{-g} \mathcal{L} = \sqrt{-\hat{g}} \left( -\frac{1}{4} g^{MN} g^{PQ} F_{MN} F_{PQ} \right) = -\frac{e^{-5a+b}}{4} \left( e^{4a} F_{\mu\nu} F_{\mu\nu} + 2e^{4a-2b} F_{\mu}^4 F_{\mu}^4 \right) \] (A.4)
where the 4d indices are contracted using the Minkowski metric. We impose 5d Dirichlet boundary conditions for \( A_{\mu} \) and Neumann conditions for \( A_{4} \):
\[ A_{\mu}(x, y = 0, L) = 0 \]
\[ \partial_4 (e^{-a-b} A_{4})(x, y = 0, L) = 0 \] (A.5)
Deconstruction now yields:
\[
\int_{0}^{L} dy \sqrt{-g} \mathcal{L} = \sum_{i=0}^{N} \Delta_i \left( -\frac{e^{-a_i+b_i}}{4} F_{i,\mu\nu} + \frac{e^{-a_i-b_i}}{2} \left( \partial_{\mu} A_{i,4} - \frac{A_{i+1,\mu} - A_{i,\mu}}{\Delta_4} \right)^2 \right) 
\]
(A.6)
where we defined \( A_{i,\mu} = e^{-a_i+b_i} \sqrt{\Delta_4} A_{i,\mu} \) (with \( F \) its associated field strength) and \( A'_{i,\mu} = e^{-a_i-b_i} \sqrt{\Delta_4} A_{i,\mu} \). Dropping the primes and using the boundary conditions, we finally obtain
\[
\mathcal{L} = -\frac{1}{4} \sum_{i=1}^{N} F_{i,\mu\nu}^2 - \frac{1}{2} \sum_{i=1}^{N-1} \left( \partial_{\mu} A_{i,4} - \frac{e^{a_{i+1}-a_i-b_i-1-b_i}}{\Delta_4} A_{i+1,\mu} - e^{-b_i} A_{i,\mu} \right)^2 
- \frac{1}{2} \left( \partial_{\mu} A_{0,4} - \frac{e^{b_0} A_{1,\mu}}{\Delta_4} \right)^2 
- \frac{1}{2} \left( \partial_{\mu} A_{N,4} + \frac{e^{-b_N} A_{N,\mu}}{\Delta_4} \right)^2 
\]
(A.7)
Specializing to the linear dilaton background for which \( a = ky, b = 0 \), the lattice action now becomes
\[
\mathcal{L} = -\frac{1}{4} \sum_{i=1}^{N} F_{i,\mu\nu}^2 - \frac{1}{2} \sum_{i=1}^{N-1} \left( \partial_{\mu} A_{i,4} - \frac{e^{b_i} A_{i+1,\mu} - A_{i,\mu}}{\Delta_4} \right)^2 
- \frac{1}{2} \left( \partial_{\mu} A_{0,4} - \frac{e^{b_0} A_{1,\mu}}{\Delta_4} \right)^2 
- \frac{1}{2} \left( \partial_{\mu} A_{N,4} + \frac{A_{N,\mu}}{\Delta_4} \right)^2 
\]
(A.8)
where we made the replacement \( k \rightarrow \frac{k}{\Delta_4} \). Defining \( q = e^{b_i}, f = \frac{1}{\Delta_4}, \phi_i = \frac{f}{\sqrt{2}} e^{\frac{i A_{i,4}}{\Delta_4}}, D_{\mu} \phi_i = (\partial_{\mu} - i(1 - \delta_{i,0}) A_{\mu,i} + i q(1 - \delta_{i,N}) A_{\mu,i+1}) \phi_i \), we can rewrite (A.8) as
\[
\mathcal{L} = -\frac{1}{4} \sum_{i=1}^{N} F_{\mu\nu,i} F_{\mu\nu}^i - \sum_{k=0}^{N} |D_{\mu} \phi_k|^2 
\]
(A.9)
thus establishing the link between the low-energy limit of the 4d model of section 2 and the deconstruction on a linear dilaton background of an 5d abelian vector mode with boundary conditions (A.5).
Finally, the Wilson line $e^{i \int dy \, A_4(x, y)}$ gets mapped to the $U(1)_a$-violating potential of (2.4):

$$e^{i \int dy \, A_4(x, y)} = e^{i \sum_{i=0}^{N} q_i A'_i(x)} = \frac{\phi_0 \phi'_1 ... \phi'_N}{(\sqrt{2})^{1+q+...q_N}}.$$  \hspace{1cm} (A.10)

### A.2 Charged bulk fermion

Deconstructed fermions might be useful in order to get insights on how $U(1)_a$ can be made anomalous or classically broken [51,52]. However, as we will see below, this procedure is not applicable in our setup. Indeed, let us consider the action of a bulk fermion charged under the abelian symmetry of the previous section:

$$\sqrt{-g} \mathcal{L} = \sqrt{-g} \left( -\frac{1}{2} \overline{\Psi}[\gamma^\mu (\partial_\mu - i e_4 A_\mu) + m] \Psi + \text{h.c.} \right)$$

$$= e^{-5a+b} \left( -\frac{1}{2} \overline{\Psi}[e^a \gamma^\mu (\partial_\mu - i e_4 A_\mu) + m] \Psi - e^{a-b} \overline{\Psi} \gamma^4 (\partial_4 - i e_4 A_4) \Psi + \text{h.c.} \right)$$

$$= e^{-4a+b} \left( -\frac{1}{2} \overline{\Psi} [\gamma^\mu (\partial_\mu - i e_4 A_\mu) + e^{-a} m] \Psi - e^{-b} \overline{\Psi} \gamma^4 (\partial_4 - i e_4 A_4) \Psi + \text{h.c.} \right),$$  \hspace{1cm} (A.11)

where we did not include the spin connection of the metric (A.3), calculable from the vielbein $e^M_A = \delta^M_A (e^{a-b} \delta^M_5)$, since it cancels out in the action, and $\gamma^4$ can be taken equal to the 4d $\gamma_5$. Deconstructing, using the normalized bosonic fields and defining $\Psi'_i = \sqrt{\Delta_4} e^{-2a_i + b_i} \Psi_i$ we get:

$$\int dy \, \sqrt{-g} \mathcal{L} \rightarrow \sum_{i=0}^{N} \Delta_4 e^{-4a_i + b_i} \left( -\frac{1}{2} \overline{\Psi_i}[\gamma^\mu (\partial_\mu - i e_4 A_{i,\mu}) + e^{-a_i} m] \Psi_i 

- \frac{e^{-b_i}}{2} \overline{\Psi_i} \gamma_5 (\Psi_{i+1} - \Psi_i - i e_4 A_{i,4} \Psi_{i+1}) \right) + \text{h.c.} \right)$$

$$= \sum_{i=0}^{N} \left( -\frac{1}{2} \overline{\Psi_i}[\gamma^\mu (\partial_\mu - i e_4 A'_{i,\mu}) + e^{-a_i} m] \Psi'_i 

- \frac{1}{2} \overline{\Psi'_i} \gamma_5 \left( \frac{e^{2a_i+1-2a_i-b_i+1} - e^{-b_i} \Psi'_{i+1}}{\Delta_4} - i \frac{e_4}{\sqrt{\Delta_4}} e^{2a_i+1-2a_i-b_i+1} A'_{i,4} \Psi'_{i+1} \right) \right) + \text{h.c.} \right).$$  \hspace{1cm} (A.12)

We now restrict the discussion to the linear dilaton background, with the vector boundary conditions of the previous section, to supplement with boundary conditions for the fermion. If we choose $\Psi_{0,L} = \Psi_{N+1,R} = 0$, the deconstructed lagrangian becomes (where we defined
$e = \frac{\epsilon_k}{\sqrt{\Delta}}$, and dropped the primes):

$$\mathcal{L}_{4d} = - \frac{1}{2} \sum_{i=1}^{N} \overline{\Psi}_i [\gamma^\mu (\partial_\mu - ie^{k/2} e A_{i,\mu}) + e^{-k} m] \Psi_i - \frac{1}{2} \overline{\Psi}_{0,R} \gamma^\mu \partial_\mu \Psi_{0,R} - \frac{1}{2} \overline{\Psi}_{N+1,L} \gamma^\mu \partial_\mu \Psi_{N+1,L}$$

$$- \frac{1}{2} \sum_{i=1}^{N-1} \overline{\Psi}_i \gamma_5 \left( \frac{e^{2k}}{\Delta_4} - ie^{k(N/2+2)} e A_{i,4} \right) \Psi_{i+1} - \frac{1}{2} \overline{\Psi}_{0,R} \left( \frac{e^{2k}}{\Delta_4} - ie^{k} e A_{0,4} \right) \Psi_{1,L}$$

$$- \frac{1}{2} \overline{\Psi}_{N,R} \left( \frac{e^{2k}}{\Delta_4} - ie^{k(N/2+2)} e A_{0,4} \right) \Psi_{N+1,L} + h.c. .$$

(A.13)

However, we cannot UV complete this lagrangian as we did in (A.9) since its $k$-dependence prevents from recognizing the low-energy expansion of the $\phi_i$'s. Only when the background is flat ($k = 0 \iff q = 1$) one can follow such a procedure (when $e = 1$):

$$\mathcal{L}_{4d \text{ UV, flat}} = - \frac{1}{2} \sum_{i=1}^{N} \overline{\Psi}_i [\gamma^\mu (\partial_\mu + ie A_{i,\mu}) + m] \Psi_i - \frac{1}{2} \overline{\Psi}_{0,R} \gamma^\mu \partial_\mu \Psi_{0,R} - \frac{1}{2} \overline{\Psi}_{N+1,L} \gamma^\mu \partial_\mu \Psi_{N+1,L}$$

$$- \frac{1}{\sqrt{2}} \sum_{i=1}^{N-1} \overline{\Psi}_i \gamma_5 \phi_i \Psi_{i+1} - \frac{1}{\sqrt{2}} \overline{\Psi}_{0,R} \phi_0 \Psi_{1,L} - \frac{1}{\sqrt{2}} \overline{\Psi}_{N,R} \phi_N \Psi_{N+1,L} + h.c. .$$

(A.14)

This lagrangian respects $U(1)_a$ but makes it anomalous at the loop level. If one now includes an allowed mass term $-\frac{m}{2} \overline{\Psi}_{0,R} \Psi_{N+1,L} + h.c.$, $U(1)_a$ is classically broken by non-local effects, which can then generate the potential $\phi_0 \phi_1 ... \phi_N$ from fermionic loops [51, 52]. When $q \neq 1$, none of this can be implemented. This reminds us that in section 3.2 we needed $\sim q^N$ fermions to make $U(1)_a$ anomalous at the loop level, while deconstruction only provides us with $\sim N$ fermions.

Nevertheless, in order to make $U(1)_a$ anomalous like in section 3.2, or to classically break it like in appendix C.1, one can consider purely four dimensional setups.
B Massive vectors of the 4d model

The model of eq. (2.1) contains massive modes in addition to the Goldstone boson $a$. The vector bosons mass matrix is:

$$M_{\text{vect}}^2 = 2 \begin{pmatrix} q^2 |\phi_0|^2 + |\phi_1|^2 & -q |\phi_1|^2 & 0 & \ldots & 0 & 0 \\ -q |\phi_1|^2 & q^2 |\phi_1|^2 + |\phi_2|^2 & -q |\phi_2|^2 & 0 & \ldots & 0 \\ 0 & -q |\phi_2|^2 & \ldots & \ldots & \ldots & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & 0 \end{pmatrix}$$

= $f^2 \begin{pmatrix} 1 + q^2 & -q & 0 & \ldots & 0 & 0 \\ -q & 1 + q^2 & -q & 0 & \ldots & 0 \\ 0 & -q & \ldots & \ldots & \ldots & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & -q \\ 0 & \ldots & \ldots & 0 & -q & 1 + q^2 \end{pmatrix}$ \hspace{1cm} (B.1)

after gauge symmetry breaking, with eigenvalues and eigenvectors:

$$\{m_j^2 = f^2(1 + q^2 - 2q \cos\left(\frac{j\pi}{N+1}\right)) \text{ and } A'_j = (\sin\left(\frac{jk\pi}{N+1}\right)), k = 1...N, j = 1...N \} \hspace{1cm} (B.2)$$

All vectors are massive since all gauge symmetries are broken. We recognize in (B.2) the specific (band-like) massive spectrum of clockwork models.

The masses of the Higgs-like $r_k$ scalar fields depend on the choices of parameters in (2.1) and do not necessarily lie in a band.

C Realizations of benchmark QCD axion models

We discuss the compatibility of usual benchmark invisible QCD axion models, namely KSVZ [53, 54] and DFSZ [55, 56] models, with our setup. In these models, the $U(1)_{PQ}$ anomaly with respect to QCD is respectively carried by additional heavy colored particles or by the standard model quarks, and the PQ symmetry arises from the introduction of a SM singlet scalar field (as well as an extra Higgs doublet for the DFSZ model). The phase shift symmetry of this singlet is not gauge protected in their original realization, consequently so we replace it by the accidental symmetry of our quiver model. We will also discuss, in the case of the KSVZ model, how the additional fermions can break $U(1)_a$ and generate (2.5) as a quantum correction.
C.1 KSVZ model: anomaly mediated by additional particles

The original KSVZ model was already (anonymously and briefly) introduced in (3.6), where \( \sigma \) is a SM gauge singlet, and some quiver versions of it were already described in (3.7) and (3.9). There, the needed couplings were ad hoc, in contrast with the fact that we talked about an accidental Peccei-Quinn symmetry. However, we can choose the fermions charges so that the procedure of (3.7) (respectively (3.9)) is automatically implied by the most general renormalizable gauge-invariant lagrangian (respectively the lowest-order gauge-invariant lagrangian which renders all the additional fermions massive), given the gauge charges of the different fields involved. This is for instance achieved if the fermions charges are those displayed in Table 3 (respectively Table 4).

\[
\begin{array}{cccccc}
Q_{L,0} & U(1)_1 & U(1)_2 & U(1)_3 & \ldots & U(1)_N & SU(3)_c & U(1)_a \\
-q & 0 & 0 & \ldots & 0 & 3 & q_R + 1 \\
Q_{L,1}^{i=1 \ldots q} & 1 & -q & 0 & \ldots & 0 & 3 & q_R + q \\
Q_{L,2}^{i=1 \ldots q^2} & 0 & 1 & -q & \ldots & 0 & 3 & q_R + q^2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
Q_{L,N}^{i=1 \ldots q^N} & 0 & 0 & 0 & \ldots & 1 & 3 & q_R + q^N \\
Q_{R}^{i=1 \ldots (1+q+\ldots+q^N)} & 0 & 0 & 0 & \ldots & 0 & 3 & q_R \\
\end{array}
\]

Table 3: Colored fermions charged under the quiver gauge group of Figure 1, canceling \( SU(3)_c^2 - U(1)_a \) anomalies and leading to a QCD axion \( (U(1)_a \) charges are those imposed by (C.1), functions of \( q_R \) which is arbitrary)

\[
\begin{array}{cccccc}
Q_{L,0} & U(1)_1 & U(1)_2 & \ldots & U(1)_N & SU(3)_c & U(1)_a \\
-q & 0 & \ldots & 0 & 3 & q_R + 1 \\
Q_{L,1}^{i=1 \ldots q} & 1 & 0 & \ldots & 0 & 3 & q_R + q + q^3 + \ldots + q^{q-1} \\
Q_{R}^{i=1 \ldots (1+q)} & 0 & 0 & \ldots & 0 & 3 & q_R \\
\end{array}
\]

Table 4: Colored fermions with mass terms from higher dimensional operators \( (q_R \) is arbitrary)

For example, the most general renormalizable lagrangian associated with Table 3 is, with such charges:

\[
\mathcal{L} \supset -\phi_0 Q_{L,0} Y_0 Y_0 Q^i_R - \phi_1 Q_{L,1}^{i=1 \ldots q} Y_{1,ij} Q^i_R + \ldots + h.c. \ , \quad (C.1)
\]

and it defines the \( U(1)_a \) charges of the fermion bilinears which make \( U(1)_a \) accidentally conserved, which in turn determine the \( U(1)_a \times SU(3)^2 \) anomaly and justify the procedure
Along with these colored fermions, one must also add fermions only charged under the quiver gauge group to cancel the $U(1)_i \times U(1)_j \times U(1)_k$ anomalies. A way of achieving this for (3.7) is presented in Table 5.

| $\psi_{R,0}$    | $U(1)_1$ | $U(1)_2$ | $U(1)_3$ | ... | $U(1)_N$ | $SU(3)_c$ | $U(1)_a$ |
|------------------|-----------|-----------|-----------|------|-----------|-----------|-----------|
| $\psi^{+1...q}$  | $-q$      | 0         | 0         | ...  | 0         | 1         | $q_L+1$   |
| $\psi^{+1...q}$  | 1         | $-q$      | 0         | ...  | 0         | 1         | $q_L+q$   |
| $\psi^{+1...q}$  | 0         | 1         | $-q$      | ...  | 0         | 1         | $q_L+q^2$ |
| $\psi^{+1...q}$  | ...       | ...       | ...       | ...  | ...       | ...       | ...       |
| $\psi^{+1...q}$  | 0         | 0         | 0         | ...  | 1         | 1         | $q_L+q^N$ |

Table 5: SM-singlet fermions charged under the quiver gauge group of Figure 1, canceling cubic quiver anomalies of Table 3 ($q_L$ is arbitrary)

One can check at the level of these fermionic contents that the models are gauge-anomaly-free, and at the level of their most general renormalizable lagrangian that they preserve an anomalous $U(1)_a$ global symmetry.

Still, we only considered renormalizable lagrangian, so we could ask whether Planck-suppressed fermionic terms will be generated along with (2.5), whether such terms explicitly break $U(1)_a$ and whether they can induce quantum corrections to the axion mass. In the cases discussed above, we can supplement (C.1) by:

$$\mathcal{L} \supset - \frac{\phi_1^{q_1}...\phi_N^{q_N}}{M_p^{q_1+...+q_N-1}} Q_{L,0} Y_i Q_i + \frac{\phi_0^{q_0} \phi_1^{q_1}...\phi_N^{q_N}}{M_p^{q_1+...+q_N-1}} Q_{L,1} Y_{i,j} Q_{i} + \ldots + h.c. , \quad (C.2)$$

which now explicitly breaks $U(1)_a$ and induces loop corrections to $m_a^2$. However, such corrections are proportional to the factor $\frac{Y}{M_p^{q_1+...+q_N-1}}$ since $U(1)_a$ is perturbatively preserved when those terms are equal to zero. Hence, by comparing with (2.7) where $m_a^2 \sim \frac{1}{M_p^{q_1+...+q_N-1}}$, we conclude that (2.7) gives the leading contribution to the axion mass$^{15}$.

However, this conclusion depends on the choice of gauge charges. For instance, if one chooses $q = 3, N = 2$ and the gauge charges of Table 6, one can write the following lagrangian:

$^{15}$When one takes into account the $\psi$ fields of Table 5, one could also write gauge-invariant Majorana mass terms for the $\psi_L$'s, but these do not break $U(1)_a$.  

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Table 6: Colored fermions giving the major contribution to the axion mass (\(U(1)_a\) charges are those imposed by (C.3), functions of \(q_0\) which is arbitrary))

| \(Q_{\ell,0}\) | \(U(1)_1\) | \(U(1)_2\) | \(SU(3)_c\) | \(U(1)_a\) |
|----------------|-----------|-----------|--------------|------------|
| \(Q_{R,0}\)   | 3         | 0         | 3            | \(q_0 - 1\) |
| \(Q^{i=1...3}_{\ell,1}\) | 1         | 0         | 3            | \(q_0 + q + q^2\) |
| \(Q^{i=1...3}_{R,1}\)   | 0         | 3         | 3            | \(q_0 + q^2\) |
| \(Q^{i=1...9}_{\ell,2}\) | 0         | 1         | 3            | \(q_0 + q^2\) |
| \(Q^{i=1...9}_{R,2}\)   | 0         | 0         | 3            | \(q_0\) |

(C.3)

\[
\mathcal{L} \supset - \frac{Q_{L,0}}{M_P} M Q_{R,2} - \phi_0 Q_{L,0} Y_{00} Q_{R,0} - \phi_1 Q_{L,1} Y_{11} Q_{R,1} - \phi_2 Q_{L,2} Y_{22} Q_{R,2} - \frac{\phi^2}{M_P} Q_{L,2} Y_{21} Q_{R,1} - \frac{\phi_1^2 \phi_2^2}{M_P} Y_{10} Q_{R,0} + h.c.
\]

(where we omitted flavour indices and some gauge invariant terms which do not break \(U(1)_a\) and thus have no impact on the discussion). The first line of (C.3) fix the \(U(1)_a\) charges displayed in Table 6, whereas the last one breaks this charge assignment since it has a global charge \(-1 - q^2 - q^4\). However, as soon as one of the \(M,Y^{(')}\) is zero, \(U(1)_a\) is conserved. Consequently, (C.3) induces a loop correction to \(m_a^2\) proportional to \(\frac{1}{M_P}\) whereas the square of (2.7) is proportional to \(\frac{1}{M_P^2}\). Thus, in this case, gravitational corrections to the fermion lagrangian induce a mass for the axion which competes with the pure scalar breaking of (2.5).

C.2 DFSZ model: anomaly mediated by standard model quarks

We focus now on the DFSZ model, since, contrary to the KSVZ model, the original model has the important feature that the anomaly is only carried by the standard model quarks. It makes uses of two Higgs doublets \(H_{1,2}\), an extra singlet scalar \(\sigma\) and can be summarized as follows:

\[
\mathcal{L} \supset - \overline{u}_R H_1 Y_u Q_L - \overline{d}_R H_2 Y_d Q_L - \overline{e}_R H_2 Y_e L_L - \lambda H_1 H_2 \sigma^2
\]

\[
\frac{u,d \text{ triangles}}{32\pi^2} \frac{i}{32\pi^2} \log(H_1 H_2) G \tilde{G} - \lambda H_1 H_2 \sigma^2.
\]

(C.4)

The first line of (C.4) is invariant under a global \(U(1)\) which acts on the scalars as \(\sigma \rightarrow e^{i\alpha} \sigma, H_{1,2} \rightarrow e^{-i\alpha} H_{1,2}\). The symmetry is spontaneously broken and, according to the second line of (C.4), anomalous with respect to QCD.

In order to adapt this construction to the case of our quiver, it is important to disentangle two features of (C.4): the \(\log(H_1 H_2)\) operator originates from the Yukawa terms of the SM
quarks which run into loops\textsuperscript{16}, whereas the $H_1H_2\sigma^2$ term (and the rest of the tree-level lagrangian) defines which symmetry is respected\textsuperscript{17}. Thus, if we want to apply this logic to $U(1)_a$, we must identify gauge charges of $H_1$ and $H_2$ which will preserve the accidental $U(1)_a$, and identify a gauge-invariant operator $O$, charged under $U(1)_a$, which will induce an axionic coupling $\log(O)\tilde{G}\tilde{G}$ to the gluons. We can immediately understand from section 3.2 that $O$ must be of high dimension, so it must be generated by more colored particles than standard model quarks alone. It would thus be more precise to talk about a mixed DFSZ-KSVZ model, where the anomaly is mediated by both standard model quarks and additional fermions. In particular, we loose the pleasant economical quark content of the original DFSZ model, since one needs a growing number of additional particles as in the KSVZ case.

As an (unoptimized) example of this procedure, we choose the matter content and gauge charges of Table 7 in addition to that of Figure 1\textsuperscript{18}. With these charges, one has the following most general renormalizable interaction terms:

$$
\mathcal{L} \supset -u_R H_1 Y_u Q_L - d_R H_2 Y_d Q_L - \phi_0 Q_{R,\text{EW}} Y_{\text{EW},i} Q^i_{L,\text{EW}} - \phi_0^q Q_{R,0} Y_{0,i} Q^i_L - \phi_1^q Q_{R,1} Y_{1,i} Q^i_L - \phi_2^q Q_{R,2} Y_{2,i} Q^i_L - ... - \lambda H_1 H_2 \phi_0^2 + h.c.
$$

\text{triangles} \rightarrow - \frac{i}{32\pi^2} \log \left( (H_1 H_2)^3 \phi_0^5 \phi_1^q \phi_2^q \phi_5^q \phi_N^q \right) \tilde{G}\tilde{G} - (\lambda H_1 H_2 \phi_0^2 + h.c.),

where we identify $O = (H_1 H_2)^3 \phi_0^5 \phi_1^q \phi_2^q \phi_5^q \phi_N^q$. $U(1)_a$ charges are assigned to $H_{1,2}$ so that $H_1 H_2 \phi_0^2$ is invariant, and $\log(O)\tilde{G}\tilde{G}$ makes $U(1)_a$ anomalous. The axion effective decay constant displays the same asymptotic dependence than (2.6): $f_a \sim \frac{f}{q^2}$.

It is worth noticing that a $\mu^2 H_1 H_2$ or $\mu H_1 H_2 \sigma$ term was not included in (C.4) in order to maintain a global symmetry, whereas we now cannot write something else than (C.5) that would respect gauge symmetries, which was the original goal when we introduced the quiver. The first allowed $U(1)_a$-violating operator is again $\phi_0^q \phi_1^q \psi N^q$ and the discussion around eq. (3.4) applies.

\textsuperscript{16}Actually, since three quark families run in the loops, the correct operator is $\log((H_1 H_2)^3)$.

\textsuperscript{17}and thus which combination of the phases of the scalars is a genuine massless Goldstone boson. If $\sigma$ is assumed to get an intermediate scale vev, this boson is mostly located on the phase of $\sigma$ and evades the astrophysical constraints on an electroweak scale axion.

\textsuperscript{18}All anomalies involving a standard model factor are canceled. The cubic, as well as the mixed abelian-gravitational anomalies of the quiver gauge group can be canceled by adding heavy SM-singlet fermions with charges identical to those of the additional fermions in Table 7, with SM representations turned into multiplicities, in the spirit of Table 5.
### D Couplings of the axion to gauge vectors

We compute the axion-photon-photon coupling for the model of Figure 1 and Table 1. However, the calculation performed here is very general and can also be seen as a derivation of (3.6).

One considers first a theory with a gauge group (which we keep unspecified until the end, where we will identify it with QCD or electromagnetism) of generators $T^a$, coupling constant $g$ and vector $A_\mu^a$ (with field strength $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + ...$), a complex scalar field $\sigma$ and two chiral fermions $\psi_{L,R}$ in the fundamental representation of the gauge group, $\mathbf{1}$.
with a Yukawa coupling to the scalar:
\[
\mathcal{L} = -\frac{1}{4g^2} F_{\mu\nu}^2 - \bar{\psi}_L \gamma^\mu D_\mu \psi_L - \bar{\psi}_R \gamma^\mu D_\mu \psi_R - |\partial \sigma|^2 - V(|\sigma|^2) - (y \sigma \bar{\psi}_L \psi_R + h.c.) ,
\]  
(D.1)

where \( D_\mu = \partial_\mu + i A_\mu^a T_a \). This lagrangian has a \( U(1) \) global symmetry under which \( \sigma \to e^{i\alpha} \sigma \) and \( \bar{\psi}_L \psi_R \to e^{-i\alpha} \bar{\psi}_L \psi_R \). The transformation of the fermion bilinear makes this global symmetry anomalous.

We choose \( V(|\sigma|^2) \) so that \( \sigma \) gets a vev \( f \). We then work out the axion dynamics by parametrizing \( \sigma = \frac{f}{\sqrt{2}} e^{i\varphi} \):
\[
\mathcal{L} \supset -\frac{1}{4g^2} F_{\mu\nu}^2 - \bar{\psi} \gamma^\mu (D_\mu + \frac{yf}{\sqrt{2}}) \psi - \frac{1}{2} (\partial \varphi)^2 + i \frac{y}{\sqrt{2}} a \bar{\psi} \gamma^3 \psi ,
\]  
(D.2)

where we only kept the linear terms in \( a \) and merged the two chiral fermions in a Dirac fermion.

One gets a coupling between the axion \( a \) and the gauge boson \( A \) at one loop via the two following diagrams:

Figure 4: Feynman diagrams leading to the axion-vector-vector couplings
The effective coupling is \( c^{\mu,ab} a A_{\mu}^a A_{\nu}^b \), here in momentum space with \( M_\psi = \frac{y_f}{\sqrt{2}} \):

\[
c^{\mu,ab} = - \int d^4 k \, \text{Tr} \left( \frac{i}{\sqrt{2}} \gamma_5 \frac{-i - i\frac{k}{M_\psi} + M_\psi}{(2\pi)^4} \frac{i}{k^2 + M_\psi^2 - i\epsilon} \frac{-i - i(k+p) + M_\psi}{(2\pi)^4} \frac{-i}{(k+p)^2 + M_\psi^2 - i\epsilon} \right) \\
\times (-i(2\pi)^4)(-i\gamma^\nu T^b) \frac{-i}{(2\pi)^4} \frac{-i(k+p+q) + M_\psi}{(k+p+q)^2 + M_\psi^2 - i\epsilon} \\
+ \frac{i}{\sqrt{2}} \gamma_5 \frac{-i - i\frac{k}{M_\psi} + M_\psi}{(2\pi)^4} \frac{i}{k^2 + M_\psi^2 - i\epsilon} \frac{i}{(2\pi)^4} \frac{-i - i(k+q) + M_\psi}{(k+q)^2 + M_\psi^2 - i\epsilon} \\
\times (-i(2\pi)^4)(-i\gamma^\nu T^a) \frac{-i}{(2\pi)^4} \frac{-i(k+p+q) + M_\psi}{(k+p+q)^2 + M_\psi^2 - i\epsilon}
\]

\[
= \int d^4 k \frac{1}{(2\pi)^4} \frac{y}{\sqrt{2}} \left[ \text{Tr} \left( \gamma_5(-i\gamma^\mu + M_\psi)\gamma^\nu(-i(k+p) + M_\psi)\gamma^\nu(-i(k+p+q) + M_\psi) \right) \right] \\
\times \text{Tr}(T^a T^b). \quad (D.3)
\]

Since the first non-zero trace including \( \gamma_5 \) is \( \text{Tr}(\gamma_5 \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) = 4i\epsilon_{\mu\nu\rho\sigma} \) and that traces of an odd number of gamma matrices are zero, we have:

\[
\text{Tr} \left( \gamma_5(-i\gamma^\mu + M_\psi)\gamma^\nu(-i(k+p) + M_\psi)\gamma^\nu(-i(k+p+q) + M_\psi) \right) \\
= 4iM_\psi \epsilon_{\mu\nu\rho\sigma} p_{\rho} q_{\sigma}. \quad (D.4)
\]

We also use the Feynman trick followed by a Wick rotation to calculate:

\[
\int d^4 k \frac{1}{(k^2 + M_\psi^2 - i\epsilon)((k+p)^2 + M_\psi^2 - i\epsilon)((k+q)^2 + M_\psi^2 - i\epsilon)} \\
= 2i\pi^2 \Gamma(1) \int_0^1 dx \int_0^{1-x} dy \frac{1}{M_\psi^2 + (x+y)(1-x-y)q^2 + x(1-x)p^2 + 2x(1-x-y)pq}. \quad (D.5)
\]

We can expand this result at first order in \( \frac{p}{M_\psi}, \frac{q}{M_\psi} \):

\[
c^{\mu,ab} = -\frac{4\pi^2}{(2\pi)^4} f \text{Tr}(T^a T^b) \epsilon_{\mu\nu\rho\sigma} p_{\rho} q_{\sigma} \\
\times \left[ \int_0^1 dx \int_0^{1-x} dy \left( 1 - (x+y)(1-x-y) \frac{q^2}{M_\psi^2} - x(1-x) \frac{p^2}{M_\psi^2} - 2x(1-x-y) \frac{pq}{M_\psi^2} \right) \\
+ \int_0^1 dx \int_0^{1-x} dy \left( 1 - (x+y)(1-x-y) \frac{p^2}{M_\psi^2} - x(1-x) \frac{q^2}{M_\psi^2} - 2x(1-x-y) \frac{pq}{M_\psi^2} \right) \right] \\
= -\frac{1}{4\pi^2} f \delta^{ab} \epsilon_{\mu\nu\rho\sigma} p_{\rho} q_{\sigma} \left( \frac{1}{2} - \frac{q^2}{12M_\psi^2} - \frac{p^2}{12M_\psi^2} - \frac{pq}{12M_\psi^2} \right). \quad (D.6)
\]
which, with the identification $pA(p) \rightarrow -i\partial A(x)$, gives finally the one-loop coupling between the axion and the vector bosons:

$$\mathcal{L} \supset -\frac{\epsilon_{\mu\nu\rho\sigma}}{32\pi^2 f} a F^a_{\mu\nu} F^a_{\rho\sigma} + \frac{\epsilon_{\mu\nu\rho\sigma}}{192\pi^2 M_{\psi}^2 f} (-\Box a F^a_{\mu\nu} F^a_{\rho\sigma} + 2 \partial_\mu a \partial^\eta F^a_{\rho\sigma} F^a_{\eta\varsigma}).$$  \hspace{1cm} (D.7)

The first term of (D.7) is the usual axionic coupling to gauge fields, while the other terms match similar calculations already performed in the literature (see for example [57]).

If one now adds to the theory (D.1) another set of fermions coupled in the following way:

$$\mathcal{L} \supset -\bar{\psi}'_L \gamma^\mu D_\mu \psi'_L - \bar{\psi}'_R \gamma^\mu D_\mu \psi'_R - (y' \sigma^\ast \bar{\psi}'_L \psi'_R + h.c.)$$

there is no anomaly anymore, but there remains non-anomalous couplings to the gauge fields (where we defined $M_{\psi}' = y' f$):

$$\mathcal{L} \supset \frac{\epsilon_{\mu\nu\rho\sigma}}{192\pi^2 f} \left( \frac{1}{M_{\psi}^2} - \frac{1}{M_{\psi}^2} \right) (-\Box a F^a_{\mu\nu} F^a_{\rho\sigma} + 2 \partial_\mu a \partial^\eta F^a_{\rho\sigma} F^a_{\eta\varsigma}).$$  \hspace{1cm} (D.9)

Specializing to electromagnetism, normalizing the photon field $A_\mu \rightarrow e A_\mu$ and choosing

$$\sigma = \phi_i = \frac{f}{\sqrt{2}} e^{i \frac{\phi_i}{\sqrt{1+q^2+...+q^N}}}$$

one obtains (4.4).

## E Scan of the parameters which allow for (detectable) ALP DM

We extend in this appendix the analysis performed in section 4 to more values of $q$ and $N$, since the DM examples in Figure 2 have been arbitrarily chosen. Figure 5 displays all DM candidates in our setup when $q \leq 6$ and $N \leq 5$, with tuning restrictions identical to those used in Figure 2. As mentioned in section 4, those results were obtained assuming that $U(1)_a$ was broken above the inflation scale. Indeed, the (compact) axion value range suggested by (2.3) is $[-(1+q+...q^N)\pi f, (1+q+...q^N)\pi f]$ whereas the potential (2.5) has periodicity $a \rightarrow a + \frac{2\pi f}{\sqrt{1+q^2+...+q^N}}$. Thus, (2.5) has $1+q+...q^N$ minima over the possible range of values for $a$, and the present setup would suffer from a severe domain wall problem if $U(1)_a$ was unbroken during inflation. Furthermore, we can see from Figures 2 and 5 that most of our ALP DM candidates require $f$ to be high (whereas the inflation scale, given by the Hubble rate during inflation, verifies $H_{\text{inflation}} \lesssim 10^{14}$ GeV). Consequently, we only focus on the broken case.
Figure 5: Range of parameters for a DM ALP
(axions suitable to saturate the DM relic density are found in colored regions, all axes are log-scale)

(which may suffer from isocurvature fluctuations issues, which are however negligible when $f$ is close to $M_P$).

We also allow in Figure 6 (which, as Figure 3, compares the sensitivity of the CASPER-Wind experiment with the predictions of our model) for more values of $q$ and $N$, but also for $M_c < M_P$. The upper panel of Figure 6 couples the standard model with the first site of the quiver while the lower panel couples it to the last site of the quiver (which, as visible in the plot, increases the coupling and thus the detectability of the setup). We see from Figure 6 that CASPER-Wind experiments are more sensitive to high scale (e.g. gravitational) values of $M_c$. 
Figure 6: Sensitivity of CASPER-Wind to the ALP’s
(colored regions indicate axions suitable to saturate the DM relic density; detection
happens in the upper left part of the plot, blue lines are identical to those of Figure 3, all
axes are log-scale)
References

[1] R. D. Peccei and H. R. Quinn, “CP Conservation in the Presence of Instantons,” *Phys. Rev. Lett.* **38** (1977) 1440–1443.

[2] S. Weinberg, “A New Light Boson?,” *Phys. Rev. Lett.* **40** (1978) 223–226.

[3] F. Wilczek, “Problem of Strong p and t Invariance in the Presence of Instantons,” *Phys. Rev. Lett.* **40** (1978) 279–282.

[4] J. Preskill, M. B. Wise, and F. Wilczek, “Cosmology of the Invisible Axion,” *Phys. Lett.* **120B** (1983) 127–132.

[5] L. F. Abbott and P. Sikivie, “A Cosmological Bound on the Invisible Axion,” *Phys. Lett.* **120B** (1983) 133–136.

[6] K. Freese, J. A. Frieman, and A. V. Olinto, “Natural inflation with pseudo - Nambu-Goldstone bosons,” *Phys. Rev. Lett.* **65** (1990) 3233–3236.

[7] J. E. Kim, H. P. Nilles, and M. Peloso, “Completing natural inflation,” *JCAP* **0501** (2005) 005, arXiv:hep-ph/0409138 [hep-ph].

[8] B. Ratra and P. J. E. Peebles, “Cosmological Consequences of a Rolling Homogeneous Scalar Field,” *Phys. Rev.* **D37** (1988) 3406.

[9] J. A. Frieman, C. T. Hill, A. Stebbins, and I. Waga, “Cosmology with ultralight pseudo nambu-goldstone bosons,” *Phys. Rev. Lett.* **75** (Sep, 1995) 2077–2080. https://link.aps.org/doi/10.1103/PhysRevLett.75.2077.

[10] J. E. Kim and H. P. Nilles, “A Quintessential axion,” *Phys. Lett.* **B553** (2003) 1–6, arXiv:hep-ph/0210402 [hep-ph].

[11] J. E. Kim, “Modeling the small dark energy scale with a quintessential pseudoscalar boson,” *J. Korean Phys. Soc.* **64** (2014) 795–805, arXiv:1311.4545 [hep-ph].

[12] S. W. Hawking, “Quantum Coherence Down the Wormhole,” *Phys. Lett.* **B195** (1987) 337.

[13] S. B. Giddings and A. Strominger, “Loss of Incoherence and Determination of Coupling Constants in Quantum Gravity,” *Nucl. Phys.* **B307** (1988) 854–866.
[14] T. Banks and N. Seiberg, “Symmetries and Strings in Field Theory and Gravity,” 
*Phys. Rev.* D83 (2011) 084019, arXiv:1011.5120 [hep-th].

[15] K. Choi and S. H. Im, “Realizing the relaxion from multiple axions and its UV 
completion with high scale supersymmetry,” *JHEP* 01 (2016) 149, arXiv:1511.00132 
[hep-ph].

[16] D. E. Kaplan and R. Rattazzi, “Large field excursions and approximate discrete 
symmetries from a clockwork axion,” *Phys. Rev.* D93 no. 8, (2016) 085007, 
arXiv:1511.01827 [hep-ph].

[17] G. F. Giudice and M. McCullough, “A Clockwork Theory,” *JHEP* 02 (2017) 036, 
arXiv:1610.07962 [hep-ph].

[18] A. Ahmed and B. M. Dillon, “Clockwork Goldstone Bosons,” *Phys. Rev.* D96 no. 11, 
(2017) 115031, arXiv:1612.04011 [hep-ph].

[19] N. Craig, I. Garcia Garcia, and D. Sutherland, “Disassembling the Clockwork 
Mechanism,” *JHEP* 10 (2017) 018, arXiv:1704.07831 [hep-ph].

[20] R. Coy, M. Frigerio, and M. Ibe, “Dynamical Clockwork Axions,” *JHEP* 10 (2017) 
002, arXiv:1706.04529 [hep-ph].

[21] G. F. Giudice and M. McCullough, “Comment on ”Disassembling the Clockwork 
Mechanism”,,” arXiv:1705.10162 [hep-ph].

[22] K. Choi, S. H. Im, and C. S. Shin, “General Continuum Clockwork,” 
arXiv:1711.06228 [hep-ph].

[23] G. F. Giudice, Y. Kats, M. McCullough, R. Torre, and A. Urbano, “Clockwork / 
Linear Dilaton: Structure and Phenomenology,” arXiv:1711.08437 [hep-ph].

[24] I. Antoniadis, S. Dimopoulos, and A. Giveon, “Little string theory at a TeV,” *JHEP* 
05 (2001) 055, arXiv:hep-th/0103033 [hep-th].

[25] N. Arkani-Hamed, A. G. Cohen, and H. Georgi, “(De)constructing dimensions,” *Phys. 
Rev. Lett.* 86 (2001) 4757–4761, arXiv:hep-th/0104005 [hep-th].
[26] C. T. Hill, S. Pokorski, and J. Wang, “Gauge invariant effective Lagrangian for Kaluza-Klein modes,” Phys. Rev. D64 (2001) 105005, arXiv:hep-th/0104035 [hep-th].

[27] J. de Blas, A. Falkowski, M. Perez-Victoria, and S. Pokorski, “Tools for deconstructing gauge theories in AdS(5),” JHEP 08 (2006) 061, arXiv:hep-th/0605150 [hep-th].

[28] S. Weinberg, The Quantum theory of fields. Vol. 1: Foundations. Cambridge University Press, 2005.

[29] G. Veneziano, “Large N bounds on, and compositeness limit of, gauge and gravitational interactions,” JHEP 06 (2002) 051, arXiv:hep-th/0110129 [hep-th].

[30] N. Arkani-Hamed, S. Dimopoulos, and S. Kachru, “Predictive landscapes and new physics at a TeV,” arXiv:hep-th/0501082 [hep-th].

[31] G. Dvali, “Black Holes and Large N Species Solution to the Hierarchy Problem,” Fortsch. Phys. 58 (2010) 528–536, arXiv:0706.2050 [hep-th].

[32] G. Dvali and M. Redi, “Black Hole Bound on the Number of Species and Quantum Gravity at LHC,” Phys. Rev. D77 (2008) 045027, arXiv:0710.4344 [hep-th].

[33] R. D. Peccei, “The Strong CP problem and axions,” Lect. Notes Phys. 741 (2008) 3–17, arXiv:hep-ph/0607268 [hep-ph]. [\,3(2006)].

[34] J. E. Kim, “Invisible Axion and Neutrino Oscillation in SU(11),” Phys. Rev. D24 (1981) 3007.

[35] H. M. Georgi, L. J. Hall, and M. B. Wise, “Grand Unified Models With an Automatic Peccei-Quinn Symmetry,” Nucl. Phys. B192 (1981) 409–416.

[36] S. Dimopoulos, P. H. Frampton, H. Georgi, and M. B. Wise, “AUTOMATIC INVISIBLE AXION WITHOUT DOMAIN WALLS,” Phys. Lett. 117B (1982) 185.

[37] K. Kang, I.-G. Koh, and S. Ouvry, “The Strong CP Problem and Axion Invisibility,” Phys. Lett. 119B (1982) 361.

[38] A. G. Dias, V. Pleitez, and M. D. Tonasse, “Naturally light invisible axion in models with large local discrete symmetries,” Phys. Rev. D67 (2003) 095008, arXiv:hep-ph/0211107 [hep-ph].
[39] K. Harigaya, M. Ibe, K. Schmitz, and T. T. Yanagida, “Peccei-Quinn symmetry from a gauged discrete R symmetry,” Phys. Rev. D88 no. 7, (2013) 075022, arXiv:1308.1227 [hep-ph].

[40] M. Redi and R. Sato, “Composite Accidental Axions,” JHEP 05 (2016) 104, arXiv:1602.05427 [hep-ph].

[41] H. Fukuda, M. Ibe, M. Suzuki, and T. T. Yanagida, “A "gauged" U(1) Peccei-Quinn symmetry,” Phys. Lett. B771 (2017) 327–331, arXiv:1703.01112 [hep-ph].

[42] S. M. Barr and D. Seckel, “Planck-scale corrections to axion models,” Phys. Rev. D 46 (Jul, 1992) 539–549. https://link.aps.org/doi/10.1103/PhysRevD.46.539.

[43] R. Holman, S. D. H. Hsu, T. W. Kephart, E. W. Kolb, R. Watkins, and L. M. Widrow, “Solutions to the strong CP problem in a world with gravity,” Phys. Lett. B282 (1992) 132–136, arXiv:hep-ph/9203206 [hep-ph].

[44] M. Kamionkowski and J. March-Russell, “Planck scale physics and the Peccei-Quinn mechanism,” Phys. Lett. B282 (1992) 137–141, arXiv:hep-th/9202003 [hep-th].

[45] D. J. E. Marsh, “Axion Cosmology,” Phys. Rept. 643 (2016) 1–79, arXiv:1510.07633 [astro-ph.CO].

[46] M. Srednicki, “Axion Couplings to Matter. 1. CP Conserving Parts,” Nucl. Phys. B260 (1985) 689–700.

[47] P. Svrcek and E. Witten, “Axions In String Theory,” JHEP 06 (2006) 051, arXiv:hep-th/0605206 [hep-th].

[48] A. Arvanitaki, S. Dimopoulos, S. Dubovsky, N. Kaloper, and J. March-Russell, “String Axiverse,” Phys. Rev. D81 (2010) 123530, arXiv:0905.4720 [hep-th].

[49] P. W. Graham and S. Rajendran, “New Observables for Direct Detection of Axion Dark Matter,” Phys. Rev. D88 (2013) 035023, arXiv:1306.6088 [hep-ph].

[50] G. Marques-Tavares and M. Teo, “Light axions with large hadronic couplings,” arXiv:1803.07575 [hep-ph].
[51] C. T. Hill and A. K. Leibovich, “Deconstructing 5-D QED,” *Phys. Rev.* D66 (2002) 016006, arXiv:hep-ph/0205057 [hep-ph].

[52] C. T. Hill and A. K. Leibovich, “Natural theories of ultralow mass PNGB’s: Axions and quintessence,” *Phys. Rev.* D66 (2002) 075010, arXiv:hep-ph/0205237 [hep-ph].

[53] J. E. Kim, “Weak Interaction Singlet and Strong CP Invariance,” *Phys. Rev. Lett.* 43 (1979) 103.

[54] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, “Can Confinement Ensure Natural CP Invariance of Strong Interactions?,” *Nucl. Phys.* B166 (1980) 493–506.

[55] M. Dine, W. Fischler, and M. Srednicki, “A Simple Solution to the Strong CP Problem with a Harmless Axion,” *Phys. Lett.* 104B (1981) 199–202.

[56] A. R. Zhitnitsky, “On Possible Suppression of the Axion Hadron Interactions. (In Russian),” *Sov. J. Nucl. Phys.* 31 (1980) 260. [Yad. Fiz.31,497(1980)].

[57] E. Dudas, L. Heurtier, Y. Mambrini, and B. Zaldivar, “Extra U(1), effective operators, anomalies and dark matter,” *JHEP* 11 (2013) 083, arXiv:1307.0005 [hep-ph].