CHERN-SIMONS FORMS AND HIGHER CHARACTER MAPS OF LIE REPRESENTATIONS

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Abstract. This paper is a sequel to [4], where we study the derived representation scheme $DRep_g(a)$ parametrizing the representations of a Lie algebra $a$ in a reductive Lie algebra $g$. In [4], we constructed two canonical maps $Tr_g(a) : HC^*(a) \rightarrow H_*[DRep_g(a)]^G$ and $\Phi_g(a) : H_*[DRep_g(a)]^G \rightarrow H_*[DRep_h(a)]^W$ called the Drinfeld trace and the derived Harish-Chandra homomorphism, respectively. In this paper, we give an explicit formula for the Drinfeld trace in terms of Chern-Simons forms. As a consequence, we show that, if $a$ is an abelian Lie algebra, the composite map $\Phi_g(a) \circ Tr_g(a)$ is given by a canonical differential operator defined on differential forms on $A = \text{Sym}(a)$ and depending only on the Cartan data $(h, W, P)$, where $P \in \text{Sym}(h^*)^W$. We prove a combinatorial formula for this operator that plays an important role in the study of derived commuting schemes in [4].

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1. Introduction and Motivation

Let $g$ be a finite-dimensional reductive Lie algebra defined over a field $k$ of characteristic zero. For an arbitrary Lie algebra $a$, the set of all representations (i.e., Lie algebra homomorphisms) of $a$ in $g$ has a natural structure of an affine $k$-scheme called the representation scheme $\text{Rep}_g(a)$. In [4], we constructed a derived version of $\text{Rep}_g(a)$ by extending the representation functor $\text{Rep}_g$ to the category of differential graded (DG) Lie algebras and deriving it in the sense of non-abelian homological algebra [22]. The corresponding derived scheme $D\text{Rep}_g(a)$ is represented by a commutative DG algebra which (to simplify the notation) we also denote by $D\text{Rep}_g(a)$. The DG algebra $D\text{Rep}_g(a)$ is well defined up to homotopy; its homology $H_*[D\text{Rep}_g(a)]$ depends only on $a$ and $g$, with $H_0[D\text{Rep}_g(a)]$ being canonically isomorphic to the coordinate ring $k[\text{Rep}_g(a)]$ of $\text{Rep}_g(a)$. Following [4], we call $H_*[D\text{Rep}_g(a)]$ the representation homology of $a$ in $g$ and denote it by $H_*(a, g)$. The algebraic group $G$ associated with $g$ acts naturally on $\text{Rep}_g$ via the adjoint representation. This action extends, by functoriality, to representation homology, and we define $H_*(a, g)^G$ to be the $G$-invariant part of $H_*(a, g)$ (see Section 5.1 for a brief review of the above constructions).
In general, computing $H(a, g)$ and $H_*(a, g)^G$ is a difficult problem: an explicit presentation for these algebras is known only in a few nontrivial cases. It is therefore natural to look for some canonical maps relating representation homology to more accessible invariants. In [4], we defined two such maps:

\begin{align*}
(1.1) & \quad \text{Tr}_g(a) : HC^{(r)}(a) \rightarrow H_*(a, g)^G \\
(1.2) & \quad \Phi_g(a) : H_*(a, g)^G \rightarrow H_*(a, h)^W
\end{align*}

which we called the Drinfeld trace and the derived Harish Chandra homomorphism, respectively. The Drinfeld trace is defined on the $r$-th Hodge component of the cyclic homology of the universal enveloping algebra $Ua$ of the Lie algebra $a$ and depends on the choice of a $G$-invariant polynomial $P \in \text{Sym}^r(g^*)^G$ on the Lie algebra $g$: it should be thought of as a derived extension of the classical character maps for Lie representations. The Harish Chandra homomorphism $\Phi_g(a)$ is a graded algebra homomorphism that extends to representation homology the natural restriction map $k[\text{Rep}_g(a)]^G \rightarrow k[\text{Rep}_g(a)]^W$, where $h \subset g$ is a Cartan subalgebra of $g$ and $W$ is the associated Weyl group.

The maps (1.1) and (1.2) are particularly interesting when $a$ is a two-dimensional abelian Lie algebra. In this case, $D\text{Rep}_g(a)$ represents the derived commuting scheme of the Lie algebra $g$, a higher homological extension of the classical commuting scheme $\text{Rep}_g(a) = \{(x, y) \in g \times g : [x, y] = 0\}$ parametrizing the pairs of commuting elements in $g$. It turns out that if $a$ is graded, with generators having opposite parities, then $H_*(a, g)^G$ is a free graded commutative algebra generated by the (images of) Drinfeld traces (1.1) corresponding to free polynomial generators $\{P_1, \ldots, P_r\}$ of the invariant algebra $\text{Sym}(g^*)^G$. (As shown in [4], this result is equivalent to the strong Macdonald conjecture proved in [12].) On the other hand, when both generators of $a$ are even (e.g., have homological degree 0), it is conjectured in [4] (with some evidence provided) that the Harish Chandra homomorphism $\Phi_g(a)$ is actually an algebra isomorphism.

In the present paper, we study the maps (1.1) and (1.2) for arbitrary DG Lie algebras. We give a general formula for the Drinfeld trace in terms of Chern-Simons forms in a convolution DG algebra canonically attached to the pair $(a, g)$ (see Theorem 3.2). Our construction is inspired by an idea of Beilinson [1] who suggested that Chern-Simons classes of canonical $g$-torsors on convolution algebras should give additive analogues of Borel regulator maps. As a consequence, we show that the composite map

\begin{equation}
(1.3) \quad \Phi_g(a) \circ \text{Tr}_g(a) : HC^{(r)}(a) \rightarrow H_*(a, g)^G \rightarrow H_*(a, h)^W
\end{equation}

which we refer to as the reduced trace $\text{Tr}_g(a)$, depends only on the Cartan data $(h, W, P)$ (provided $\text{Sym}(g^*)^G$ is identified with $\text{Sym}(h^*)^W$ via the Chevalley isomorphism). If $a$ is an abelian Lie algebra (of any dimension), the cyclic homology groups $HC^{(r)}(a)$ can be expressed in terms of (algebraic) differential forms on the vector space $a$, and the reduced trace $\text{Tr}_g(a)$ is given by a canonical $W$-invariant differential operator defined on de Rham algebra of $\text{Sym}(a)$. We will give an explicit combinatorial formula for this operator, computing thus the higher character maps for all symmetric algebras (see Theorem 1.2 and Examples 4.4). We should mention that this formula plays a crucial role in [4], where it is used to verify the Harish Chandra quasi-isomorphism conjecture for the classical Lie algebras $gl_n$, $sl_n$, $so_{2n+1}$ and $sp_{2n}$ in the stable limit $n \rightarrow \infty$; however, it is given in [4] without a proof.

We now proceed with a summary of the contents of the paper. Section 2 contains preliminary material on de Rham complexes and cyclic homology of commutative algebras and cocommutative coalgebras. Although

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1. We will review the construction of these maps in Section 3 below.

2. In a sense, our Drinfeld trace map is Koszul dual to the ‘additive’ regulator map, the construction of which is outlined in [1], Section A6. We will give a detailed account on Beilinson’s construction in Appendix A.
most of this material is well known, we pay a special attention to identifications between various descriptions of cyclic homology. In particular, Theorem 2.5 provides an explicit formula for the isomorphism between the (reduced) cyclic homology of a symmetric algebra and its Koszul dual symmetric coalgebra. The proof of this theorem is somewhat technical and fairly long; we give it in Appendix B.

In Section 3 after brief recollections on derived representation schemes, we prove our main result (Theorem 3.2), expressing Drinfeld traces in terms of Chern-Simons forms. We also deduce two consequences – Lemma 3.1 and Lemma 3.2 – which are important for our trace computations. Lemma 3.1 implies that the reduced trace map \( \text{Tr}_h(a) \) depends only on \( h \) and the choice of a \( W \)-invariant polynomial on \( h \), while Lemma 3.2 essentially reduces the computation of \( \text{Tr}_h(a) \) to the rank one situation.

In Section 4, we compute the reduced trace maps for any finite-dimensional abelian Lie algebra \( a \) (or equivalently, for the symmetric algebra \( A = \mathcal{U}(a) = \text{Sym}(a) \)). The main result of this section (Theorem 4.2) gives an explicit formula for these traces in terms of a canonical differential operator acting on differential forms on \( A \).

In Section 5, we give another application of Theorem 3.2. This relies on several results from the literature. First, by a general formula from [3], the reduced trace maps for \( A = \text{Sym}(a) \) can be expressed in terms of symmetrized sums of Taylor components of an \( A_\infty \)-quasi-isomorphism inverting the minimal resolution of \( A \). Applying a homological perturbation formula due to Merkulov [20], we compute these components and transform the sums over \( S_n \) into sums over (equivalence classes of) binary trees. Comparing then the result with Theorem 3.2 gives an interesting combinatorial identity that expresses the sums of \( A_\infty \)-components over binary trees in terms of explicit Chern-Simons forms (see Corollary 5.1).

As mentioned above, our main formula for Drinfeld traces can be viewed as dual to Beilinson’s formula for ‘additive regulators’. However, in [1], Beilinson gives only a brief sketch of his construction with no explicit formulas. In [8], Feigin elaborates on ideas of [1] focusing on current Lie algebras \( \mathfrak{g}^M \) on smooth manifolds; he proposes an interesting conjecture on the structure of cohomology of \( \mathfrak{g}^M \) but still gives no explicit formula for Beilinson’s map. An explicit formula appears in Teleman’s paper [24], where it is given simply as part of a definition (cf. [24, (2.2)]), with no reference to [1]. We bridge this gap in Appendix A, where we provide a detailed account of [1, Section A.6] and show, in particular, that Teleman’s formula indeed arises from Beilinson’s construction. We also construct another natural map from Lie homology of current Lie algebras to cyclic homology of commutative algebras and compare it to Beilinson’s one. The relation between these two maps (cf. Theorem A.3) plays a key role in the proof of our main results in Section 3.

Finally, the second appendix – Appendix B – contains a detailed proof of Theorem 2.5. This result seems to be new and may be of independent interest.

**Notation.** Throughout this paper, \( k \) denotes a base field of characteristic zero. An unadorned tensor product \( \otimes \) stands for the tensor product over \( k \). Unless stated otherwise, all differential graded (DG) objects are equipped with differentials of degree \(-1\). For a graded vector space \( W \), \( \text{Sym}(W) \) denotes the graded symmetric algebra and \( \text{Sym}^c(W) \) the graded symmetric coalgebra of \( W \).

### 2. The de Rham complex and cyclic homology

In this section, we recall some basic facts on cyclic homology of commutative algebras needed for the present paper. We also record the dual statements for cocommutative coalgebras. The only (apparently) new result in this section is Theorem 2.5, the proof of which is given in the appendix.

#### 2.1. The de Rham algebra of a commutative DG algebra.

Let \( \text{DGCA}_k \) denote the category of commutative unital DG \( k \)-algebras with differential of degree \(-1\). Recall that, for \( A \in \text{DGCA}_k \), the DG module \( \Omega_A^1 \) of Kähler
differentials of $A$ is defined as the free DG $A$-module generated by the symbols $da$ (for $a \in A$), modulo the relations

$$d(\delta a) = \delta da \quad d(ab) = da.b + a.db.$$ 

Here, $\delta$ denotes the differentials intrinsic to $A$ and $\Omega^1_A$. For $a \in A$ homogeneous, $da$ has the same homological degree in $\Omega^1_A$ as $a$ has in $A$.

Consider the (homologically) graded algebra $\text{Sym}_A \Omega^1_A[1]$. Let $d$ denote the (unique) degree 1 derivation on $\text{Sym}_A \Omega^1_A[1]$ satisfying $d(a) = da$, $d(da) = 0$, $\forall a \in A$. Let $\delta$ denote the (unique) degree $-1$ derivation on $\text{Sym}_A \Omega^1_A[1]$ induced by the differential intrinsic to $A$. It is easy to verify that $d$ and $\delta$ are square 0 and (anti)commute. Hence, $(\text{Sym}_A \Omega^1_A[1], \delta, d)$ is an algebra object in the category of mixed complexes: we refer to it as the \textit{mixed algebra} of $A$. Note that $k \mapsto \text{Sym}_A \Omega^1_A[1]$ via the unit map $k \mapsto A$. We call the mixed complex $(\text{Sym}_A \Omega^1_A[1]/k, \delta, d)$ the \textit{mixed de Rham complex} of $A$ and denote it by $\text{DR}^*(A)$.

On the other hand, $A$ may be viewed as a \textit{cohomologically graded} algebra by inverting degrees $A^i = A_{-i}$. In this case, $d + \delta$ can also be viewed as a degree $+1$ differential on $\text{Sym}_A \Omega^1_A[1]$. We call the \textit{cochain algebra} $(\text{Sym}_A \Omega^1_A[1], d + \delta)$ the \textit{de Rham algebra} of $A$ and denote it by $\Omega^*_{\text{DR}}(A)$.

Note that $\text{Sym}_A^q((\Omega^1_A[1]) \cong \wedge^q A \Omega^1_A[k]$: explicitly, this isomorphism is given by $a_0 da_1 \ldots da_q \mapsto (-1)^{|a_0|+|a_2|+\ldots+(q-1)|a_{q-1}|}a_0 da_1 \ldots da_q$.

We refer to the complex $(\wedge^q A \Omega^1_A, \delta)$ as the \textit{complex of de Rham $q$-forms} of $A$ and denote it by $\Omega^q_{\text{DR}}$. Let $\Omega^q_A$ denote $\Omega^q_{\text{DR}}$ when $q > 0$ and $A/k$ when $q = 0$.

2.2. Cyclic homology of commutative DG algebras. Recall that if $(\mathcal{M}, \delta, d)$ is a mixed complex, its \textit{cyclic homology} is the homology of the total complex $CC(\mathcal{M})$ of the double complex defined by

$$CC^p_q(\mathcal{M}) = \begin{cases} \mathcal{M}_{q-p} & p \geq 0 \\ 0 & p < 0 \end{cases}$$

The horizontal differential $CC^p_q(\mathcal{M}) \to CC^{p-1}_q(\mathcal{M})$ is $d$ and the vertical differential $CC^p_q(\mathcal{M}) \to CC_{p,q-1}(\mathcal{M})$ is $\delta$.

The following theorem relates the cyclic and de Rham homologies of a commutative DG algebra.

**Theorem 2.1** ([13], Theorem 5.4.7). Assume that $A \in \text{DGCA}_k$ is smooth as a graded algebra. Then, $HC_*(A)$ is canonically isomorphic to the cyclic homology of the mixed de Rham complex $\text{DR}^*(A)$.

There is a natural direct sum decomposition

$$CC[\text{DR}^*(A)] = \bigoplus_{i \geq 0} CC^{(i)}[\text{DR}^*(A)],$$

where $CC^{(i)}[\text{DR}^*(A)] := \bigoplus_{n=1}^{2i} \Omega^2_{\text{DR}}[n]$ is the total complex of the double complex $C^{(i)}$, where

$$C^{(i)}_{p,q} = \begin{cases} [\Omega^1_{\text{DR}}]_{q-p} & p \geq 0 \\ 0 & p < 0 \end{cases}$$

The horizontal differential $C^{(i)}_{p,q} \to C^{(i)}_{p-1,q}$ is $d$ and the vertical differential $C^{(i)}_{p,q} \to C^{(i)}_{p,q-1}$ is $\delta$.

**Proposition 2.1** ([13], Proposition 5.4.9). The isomorphism of Theorem 2.1 is compatible with Hodge decomposition: in other words, it induces a canonical isomorphism

$$HC^{(i)}(A) \cong H_*(CC^{(i)}[\text{DR}^*(A)]).$$
Let $A = (\text{Sym}(W), \delta)$ where $W$ is a finite-dimensional graded $k$-vector space. Then, the de Rham algebra of $A$ is acyclic with respect to the de Rham differential. In other words, $\text{CC}^i(\text{DR}^*(A))$ is quasi-isomorphic to $\Omega^i_A/d\Omega^{i-1}_A$, with the quasi-isomorphism being induced by the projection

$$p : \text{CC}^i(\text{DR}^*(A)) = \text{Tot}(\text{CC}^i) \rightarrow \text{CC}^0 = \Omega^i_A/d\Omega^{i-1}_A.$$

As a consequence of this and Theorem 2.1, we obtain

**Theorem 2.2** ([16], Theorem 5.4.12). For $A = (\text{Sym}(W), \delta)$, there is a canonical isomorphism

$$\overline{HC}_n(A) \cong \bigoplus_{i \geq 0} H_{n-i}([\Omega^i_A/d\Omega^{i-1}_A, \delta]).$$

In other words, $\overline{HC}_*(A)$ is canonically isomorphic to $H_*[\text{DR}^*(A)/d\text{DR}^*(A), \delta]$. 

Recall that the reduced cyclic homology of $A$ is the homology of Connes’ (reduced) cyclic complex $\overline{\Omega}^A$. Since $A = (\text{Sym}(W), \delta)$ is commutative, there is a Hodge decomposition $\overline{\Omega}^A = \oplus_{i=1}^{\infty} \overline{\Omega}^A(i)$. The isomorphism in Theorem 2.2 is compatible with Hodge decomposition. To be precise, consider the antisymmetrization map

$$\varepsilon : \bar{A} \otimes \text{Sym}^i(\bar{A}[1]) \rightarrow \bar{A} \otimes \bar{A}[1]^{\otimes i}, \quad a_0 \otimes a_1 \wedge \ldots \wedge a_i \mapsto \sum_{\sigma \in S_i} (-1)^{f(\sigma, a_1, \ldots, a_i)} (a_0, a_{\sigma(1)}, \ldots, a_{\sigma(i)}),$$

where $(-1)^{f(\sigma, a_1, \ldots, a_i)}$ is obtained from the Koszul sign rule when permuting elements with degrees $|a_1| + 1, \ldots, |a_i| + 1$ via $\sigma$. By [16] Section 2.3.5, $\varepsilon$ induces a map of complexes

$$\varepsilon : \Omega^i_A/d\Omega^{i-1}_A \rightarrow \overline{\Omega}^A(i),$$

which is known to be a quasi-isomorphism when $A = (\text{Sym}(W), \delta)$. Its inverse is given by the composite map

$$I_{\text{HKR}} : \overline{\Omega}^A(i) \hookrightarrow \overline{\Omega}^A \rightarrow A \otimes \bar{A}[1]^{\otimes i}/\text{Im}(1-\tau) \rightarrow \Omega^i_A/d\Omega^{i-1}_A,$$

where the last arrow is defined by $(a_0, \ldots, a_i) \mapsto \frac{1}{i!} a_0 a_1 \ldots a_i$. The quasi-isomorphism 2.3 is induced by the classical Hochschild-Kostant-Rosenberg map on Hochschild chain complex of $A$; we therefore call $I_{\text{HKR}}$ the Hochschild-Kostant-Rosenberg map.

### 2.3. The de Rham coalgebra of a cocommutative DG coalgebra

We now formally dualize the constructions of Sections 2.1 and 2.2 replacing algebras by coalgebras. Let $C \in \text{DGCC}_k$. The notion of comodule $\Omega^1_C$ of Kähler codifferentials is dual to that of the module of Kähler differentials (cf. [21] Section 4). More precisely, $\Omega^1_C$ is the set of symmetric elements in the $C$-bicomodule $C \otimes C/\Delta(C)$. There is a universal (degree 0) coderivation $d : \Omega^1_C \rightarrow C, \omega \mapsto d\omega$. The tensor coalgebra $T^C_C(\Omega^1_C[-1])$ has a unique degree 1 coderivation $d$ such that $d(\omega) := d\omega$ for all $\omega \in \Omega^1_C$ and $d(\alpha) = 0$ for all $\alpha \in C$. It also has a degree $-1$ coderivation $\delta$ induced by the (co)differential on $C$. It is easy to check that $d$ and $\delta$ are square zero and (anti)commute. Further, both $d$ and $\delta$ restrict to (co)differentials on $\text{Sym}^C_C \Omega^1_C[-1]$, the largest cocommutative sub-coalgebra of $\Omega^1_C[-1]$. This makes $\text{Sym}^C_C \Omega^1_C[-1], d, \delta$ a coalgebra object in the category of mixed complexes. Via the counit of $C$, the mixed coalgebra $(\text{Sym}^C_C \Omega^1_C[-1], d, \delta)$ acquires a counit. We call the kernel of this counit the mixed de Rham complex of $C$ and denote it by $\text{DR}^*(C)$ and refer to $(\text{Sym}^C_C \Omega^1_C[-1], d, \delta)$ as the mixed coalgebra of $C$. On the other hand, one can view $d + \delta$ as a degree $-1$ (co)differential on $\text{Sym}^C_C \Omega^1_C[1]$; we refer to the resulting cocommutative DG coalgebra as the de Rham coalgebra of $C$ and denote it by $\text{DR}^*_C(C)$. The image of $\text{Sym}^C_C \Omega^1_C[-1] \cap [\Omega^1_C[-1]]^{\otimes Cq}$ in $[\Omega^1_C]^{\otimes Cq}[-q]$ will be denoted by $\Omega^q_C[-q]$. We refer to the complex $(\Omega^q_C, \delta)$ as
the complex of de Rham q forms of C. For $C \in \text{DGCC}_k$, let $\Omega^q_C := \Omega^q_{\text{DR}}$ for $q > 0$, with $\Omega^0_C$ being the kernel of the counit from $\Omega^0_{\text{DR}}$ to $k$.

Recall that if $(\mathcal{M}, \delta, d)$ is a mixed complex, its negative cyclic homology is the homology of the complex $\text{CC}^-(\mathcal{M})$, where $\text{CC}^-(\mathcal{M})$ is the total complex of the double complex $C(\mathcal{M})$, where

$$C_{p,q}(\mathcal{M}) = \begin{cases} 
\mathcal{M}_{q-p}, & p \leq 0 \\
0, & p > 0
\end{cases}$$

The horizontal differential $C_{p,q}(\mathcal{M}) \to C_{p-1,q}(\mathcal{M})$ is $d$ and the vertical differential $C_{p,q}(\mathcal{M}) \to C_{p,q-1}(\mathcal{M})$ is $\delta$.

The next theorem is dual to Theorem 2.1

**Theorem 2.3.** Let $C := (\text{Sym}^c(W), \delta)$ where $W$ is a graded $k$-vector space of finite (total) dimension. Then there is a canonical isomorphism

$$\overline{HC}_*(C) \cong H_*([\text{CC}^-]^{\bullet}[\text{DR}^\bullet(C)]).$$

There is a natural direct sum decomposition

$$\text{CC}^-[\text{DR}^\bullet(C)] = \bigoplus_{i \geq 0} \text{CC}^{-,(i)}[\text{DR}^\bullet(C)],$$

where $\text{CC}^{-,(i)}[\text{DR}^\bullet(C)] := \bigoplus_{n \geq i} \Omega^2_i - n[-n]$ is the total complex of the double complex $C^{(i)}$, where

$$C^{(i)}_{p,q} = \begin{cases} 
[\Omega^p_{\text{DR}}]_{q+i}, & p \geq 0 \\
0, & p < 0
\end{cases}$$

The horizontal differential $C^{(i)}_{p,q} \to C^{(i)}_{p-1,q}$ is $d$ and the vertical differential $C^{(i)}_{p,q} \to C^{(i)}_{p,q-1}$ is $\delta$. Dual to Proposition 2.1 we have

**Proposition 2.2.** The isomorphism in Theorem 2.3 is compatible with Hodge decomposition: in other words, it induces an isomorphism

$$\overline{HC}_i^*(C) \cong H_i([\text{CC}^-]^{\bullet}[\text{DR}^\bullet(C)]), \quad \forall i \geq 0.$$

Note that when $C = (\text{Sym}^c(W), \delta)$ where $W$ is a finite-dimensional graded vector space, $\text{CC}^{-,(i)}(\text{DR}^\bullet(C))$ is quasi-isomorphic to $\text{Ker}(d : \Omega_C^i \to \Omega_{C}^{i-1})[-i]$. This quasi-isomorphism is induced by the natural inclusion

$$\iota : \text{Ker}(d : \Omega_C^i \to \Omega_C^{i-1})[-i] \hookrightarrow \text{CC}^{-,(i)}(\text{DR}^\bullet(C)).$$

We thus obtain the following statement (which is dual to Theorem 2.2).

**Theorem 2.4.** There is a canonical isomorphism

$$\overline{HC}_n(C) \cong \bigoplus_{q \geq 0} H_{n+q}([\text{Ker}(d : \Omega_C^q \to \Omega_C^{q-1}), \delta]).$$

In other words, $\overline{HC}_*(C)$ is canonically isomorphic to $H_*[\text{Ker}(d : \text{DR}^\bullet(C) \to \text{DR}^\bullet(C)), \delta]$.

Dually to Theorem 2.2 the isomorphism in Theorem 2.3 respects Hodge decomposition. Further, $\overline{HC}_*(C)$ is the homology of Connes’ reduced complex $\overline{C}^\lambda(C)$ of $C$. Dually to 2.1 Lemma 1.2, $\overline{C}^\lambda(C) = (\Omega(C))$, where $\Omega(C)$ is the cobar construction of $C$ and $R_k := R/(k + [R, R])$ for any $R \in \text{DGCA}_k$. Dually to Theorem 2.2 the isomorphism in Theorem 2.3 is by the map

$$\varepsilon : \overline{C}^\lambda(C) \cong \bigoplus_i \text{Ker}(d : \Omega^i(C) \to \Omega^{i-1}(C))[-i]$$
obtained by taking (graded) linear dual of $\mathbb{R}^n$ applied to $E := \text{Hom}_k(C, k)$. The inverse map is given by the (graded) dual of the Hochschild-Kostant-Rosenberg map. We refer to this last map as the co-HKR map and denote it by $I_{\text{HKR}}$.

2.4. The mixed Hopf algebra of a vector space. In this subsection, we let $A := \text{Sym}(W)$ and $C := \text{Sym}^c(W[1])$ where $W$ is a finite-dimensional graded vector space (with trivial differential). Note that $C$ is Koszul dual to $A$ in the sense of $\mathbb{R}^{\mathbb{R}}$ and where $d$ increases homological degree by 1. From this, one sees that $d$ isomorphism, the universal coderivation $\theta : \Omega^1_C \rightarrow C$ becomes the map

$$C \otimes_k W[1] \rightarrow C, \quad v \otimes \alpha \mapsto v \cdot \alpha,$$

where $\cdot$ denotes the product in $\text{Sym}(W[1])$.

Let $\mathcal{H}(W) := \text{Sym}(W \oplus W[1])$, equipped with the graded Hopf algebra structure of the symmetric algebra of the graded vector space $W \oplus W[1]$.

**Proposition 2.3.** The de Rham differential $d$ on $\mathcal{H}(W)$ makes $(\mathcal{H}(W), 0, d)$ a Hopf algebra object in the category of mixed complexes. As an algebra, $(\mathcal{H}(W), 0, d)$ is the mixed algebra of $A$. As a coalgebra, $(\mathcal{H}(W), 0, d)$ is the mixed coalgebra of $C$.

**Proof.** Indeed, as a graded coalgebra, $\mathcal{H}(W) = \text{Sym}^c(W) \otimes \text{Sym}^c(W[1])$. Further, by Lemma 2.1 for any $v_1, \ldots, v_q \in W$ and $\alpha \in \text{Sym}^c(W[1])$, we have

$$d_C(v_1 \ldots v_q \otimes \alpha) = \sum (-1)^{|v_i|(|v_i|+\ldots+|v_q|)}(-1)^{|v_1|+\ldots+|v_i|+\ldots+|v_q|}v_1 \ldots \hat{v}_i \ldots v_q \otimes s(v_i)\alpha,$$

where $d_C$ is the de Rham coderivation on the mixed coalgebra of $C$ and where $s : W \rightarrow W[1]$ is the operator increasing homological degree by 1. From this, one sees that $d_C$ is equal to the de Rham differential $d$. It follows that the de Rham differential is a differential of degree 1 on the graded Hopf algebra $\mathcal{H}(W)$ and that $(\mathcal{H}(W), 0, d)$ viewed as a DG coalgebra is equal to the mixed coalgebra of $C$. This verifies the first and third assertions in the desired proposition. The second assertion is obvious. \hfill $\square$

**Remark.** By Proposition 2.3, the identity map is an isomorphism of mixed complexes between the mixed algebra of $A$ and the mixed coalgebra of $C$. However, under this isomorphism, the $p$-forms on $A$ having coefficients of polynomial degree $q$ are identified with $q$-forms on $C$ having coefficients of polynomial degree $p$.

Let $d_A$ denote the differential $d$ on the mixed de Rham complex $\text{DR}^*(A)$ of $A$ and let $d_C$ denote the differential $d$ on $\text{DR}^*(C)$. Combining the isomorphisms of Theorem 2.2 and Theorem 2.4 with the identifications of Proposition 2.3, we have

**Proposition 2.4.** There is a natural isomorphism $\mathcal{H}C^*(A) \cong \mathcal{H}C^+_{n+1}(C)$ given by the composite map

$$\mathcal{H}C_n(A) \cong [\text{Coker}(d_A)]_n \xrightarrow{d_A} [\text{Ker}(d_A)]_{n+1} = [\text{Ker}(d_C)]_{n+1} \cong \mathcal{H}C^+_{n+1}(C).$$

On the other hand, for any associative algebra $A \in \mathfrak{A}$ and any cofibrant resolution $R \rightarrow A$ in $\mathfrak{DGA}_k$, there is a quasi-isomorphism of complexes

$$T : \mathcal{C}^\wedge(A) \rightarrow R_k$$

This quasi-isomorphism is constructed in [3] Section 4.3, where it is denoted $s^\wedge(\theta)$.
determined by a twisting cochain \( f : B(A) \to R[1] \) whose components \( f_n : A^{\otimes n} \to R_{n-1} \) are the Taylor components of an \( A_\infty \)-inverse to the quasi-isomorphism \( R \sim A \). Explicitly, (2.5) is induced on \( n \) chains by the map (cf. [3] Theorem 4.2)

\[
T_n : A^{\otimes n+1} \to R_g, \quad (a_0, \ldots, a_n) \mapsto \sum_{p \in \mathbb{Z}_{n+1}} (-1)^{nk}[f_n(a_p, a_{1+p}, \ldots, a_{n+p})],
\]

where \([f] \) is the image of \( f \in R \) in \( R_g \).

Now, if \( A = \text{Sym}(W) \) and \( C = \text{Sym}^c(W[1]) \), we take \( R = \Omega(C) \) to be the cobar construction of \( C \), so that \( R_g = \overline{\Omega}(C)[-1] \). In this case, we have the following result which refines Proposition 2.4.

**Theorem 2.5.** The isomorphism \( \overline{\PiC}_* (A) \sim \overline{\PiC}_* (C) \) induced by \( T \) is given by

\[
\overline{\PiC}_* (A) \xrightarrow{\text{HKR}} \text{Coker}(d_A)_* \xrightarrow{d} \text{Ker}(d_C)_{*+1} \xrightarrow{\text{HKR}} \overline{\PiC}_* (C),
\]

where \( I_{\text{HKR}} \) and \( I_{\text{HKR}}^c \) are the Hochschild-Kostant-Rosenberg maps defined in Section 2.2 and Section 2.3, respectively.

A detailed proof of Theorem 2.5 will be given in Appendix D.

3. Chern-Simons forms and Drinfeld traces

3.1. Derived representation schemes of Lie algebras. We review our basic construction of derived representation schemes of Lie algebras and associated character maps. For details and proofs, we refer the reader to [2], Sections 6 and 7.

3.1.1. Lie representation functor. Let \( g \) be a finite-dimensional Lie algebra over \( k \). Given an (arbitrary) Lie algebra \( a \in \text{LieAlg}_k \), we are interested in classifying the representations of \( a \) in \( g \). The corresponding moduli scheme \( \text{Rep}_g(a) \) is defined by its functor of points:

\[
\text{Rep}_g(a) : \text{CommAlg}_k \to \text{Sets}, \quad B \mapsto \text{Hom}_{\text{Lie}}(a, g(B))
\]

that assigns to a commutative \( k \)-algebra \( B \) the set of families of representations of \( a \) in \( g \) parametrized by the \( k \)-scheme Spec\( (B) \). The functor \( \text{Rep}_g(a) \) is represented by a commutative algebra \( a_g \), which has the following canonical presentation (cf. [3] Proposition 6.1):

\[
a_g = \frac{\text{Sym}_k(a \otimes g^*)}{\langle (x \otimes \xi_1) \cdot (y \otimes \xi_2) - (y \otimes \xi_1) \cdot (x \otimes \xi_2) - [x \cdot y] \otimes \xi^* \rangle},
\]

where \( g^* \) is the vector space dual to \( g \) and \( \xi^* \mapsto \xi^*_1 \wedge \xi^*_2 \) is the linear map \( g^* \to \wedge^2 g^* \) dual to the Lie bracket on \( g \). The universal representation \( \varrho_g : a \to g(a_g) \) is given by the natural map

\[
a \to a \otimes g^* \otimes g \hookrightarrow \text{Sym}_k(a \otimes g^*) \otimes g \to a_g \otimes g = g(a_g), \quad x \mapsto \sum_i [x \otimes \xi_i^*] \otimes \xi_i,
\]

where \( \{\xi_i\} \) and \( \{\xi_i^*\} \) are dual bases in \( g \) and \( g^* \). The algebra \( a_g \) has a canonical augmentation \( \varepsilon : a_g \to k \) induced by the zero map \( a \otimes g^* \to 0 \). Thus the assignment \( a \mapsto a_g \) defines a functor with values in the category of augmented commutative algebras:

\[
(\varepsilon)_g : \text{LieAlg}_k \to \text{CommAlg}_{k/k}.
\]

We call \( (\varepsilon)_g \) the representation functor in \( g \). Geometrically, one can think of \( (a_g, \varepsilon) \) as a coordinate ring \( k[\text{Rep}_g(a)] \) of the based affine scheme \( \text{Rep}_g(a) \), with the basepoint corresponding to the trivial representation.
Next, observe that, for any $a \in \mathfrak{Lie Alg}_k$, the Lie algebra $\mathfrak{g}$ acts naturally on $a_\mathfrak{g}$ by derivations: this action is induced by the coadjoint action $\text{ad}_\mathfrak{g}^\ast$ of $\mathfrak{g}$ and is functorial in $a$. We write $(-)^{\text{ad}_\mathfrak{g}} : \mathfrak{Lie Alg}_k \to \text{Comm Alg}_{k/k}$ for the subfunctor of $(-)_g$ defined by taking the $\text{ad}_\mathfrak{g}^\ast$-invariants:

$$a_\mathfrak{g}^{\text{ad}_\mathfrak{g}} := \{ x \in a_\mathfrak{g} : \text{ad}_\mathfrak{g}^\ast(x) = 0, \forall \xi \in \mathfrak{g} \}.$$  

If $\mathfrak{g}$ is a reductive Lie algebra and $G$ is the associated algebraic group, the algebra $a_\mathfrak{g}^{\text{ad}_\mathfrak{g}}$ represents the affine quotient scheme $\text{Rep}_\mathfrak{g}(a)/G$ parametrizing the closed orbits of $G$ in $\text{Rep}_\mathfrak{g}(a)$.

3.1.2. Derived functors. In general, the scheme $\text{Rep}_\mathfrak{g}(a)$ is quite singular. One way to ‘resolve singularities’ is to replace $\text{Rep}_\mathfrak{g}(a)$ by a smooth DG scheme $\text{DRep}_\mathfrak{g}(a)$ having $\text{Rep}_\mathfrak{g}(a)$ as its ‘0-th homology’. This approach to resolution of singularities is advocated in [6], where it is applied to a number of classical moduli problems in algebraic geometry. Our construction of $\text{DRep}_\mathfrak{g}(a)$ is more algebraic and inspired by [3]. The idea is to work with the representation functor (3.2) (instead of the representation scheme $\text{Rep}_\mathfrak{g}(a)$) and define $\text{DRep}_\mathfrak{g}(a)$ in terms of the derived functor of (3.3) using the ‘abstract’ homotopy theory in the category of DG Lie algebras.

In more detail, the functor (3.2) can be extended to the category of DG Lie algebras by formula (3.1):

$$( - )_g : \mathfrak{DGLA}_k \to \mathfrak{DGCA}_{k/k}, \quad a \mapsto a_\mathfrak{g}.$$  

It is shown in [4] that, for a fixed DG Lie algebra $a \in \mathfrak{DGLA}_k$, the corresponding commutative DG algebra $a_\mathfrak{g}$ represents an affine DG scheme parametrizing the DG Lie representations of $a$ in $\mathfrak{g}$. The homotopy theories in $\mathfrak{DGLA}_k$ and $\mathfrak{DGCA}_{k/k}$ are determined by the classes of weak equivalences which in both cases are taken to be the quasi-isomorphisms. The corresponding homotopy categories $\text{Ho}(\mathfrak{DGLA}_k)$ and $\text{Ho}(\mathfrak{DGCA}_{k/k})$ are thus defined by formally inverting all morphisms in $\mathfrak{DGLA}_k$ and $\mathfrak{DGCA}_{k/k}$ inducing isomorphisms on homology. Now, the key point is that, although the functor (3.3) is not homotopy invariant (it does not preserve quasi-isomorphisms and hence does not descend to $\text{Ho}(\mathfrak{DGLA}_k)$), it is a left Quillen functor and hence has a well-behaved left derived functor (see [4, Theorem 6.4])

$$L(-)_g : \text{Ho}(\mathfrak{DGLA}_k) \to \text{Ho}(\mathfrak{DGCA}_{k/k}).$$  

The derived functor $L(-)_g$ provides, in a precise sense, the ‘best possible’ approximation to $(-)_g$ at the level of homotopy categories. For a fixed DG Lie algebra $a \in \mathfrak{DGLA}_k$, we then define $\text{DRep}_\mathfrak{g}(a) := L(a)_g$.

In a similar fashion, we construct a derived functor of the invariant functor (3.3):

$$L(-)^{\text{ad}_\mathfrak{g}}_g : \text{Ho}(\mathfrak{DGLA}_k) \to \text{Ho}(\mathfrak{DGCA}_{k/k}),$$  

and define $\text{DRep}_\mathfrak{g}(a)^{\text{ad}_\mathfrak{g}} := L(a)^{\text{ad}_\mathfrak{g}}_g$. Notice that we use the notation $\text{DRep}_\mathfrak{g}(a)$ and $\text{DRep}_\mathfrak{g}(a)^{\text{ad}_\mathfrak{g}}$ for the commutative DG algebras in $\text{Ho}(\mathfrak{DGCA}_{k/k})$ representing the eponymous derived schemes (rather than for the derived schemes themselves).

As usual, derived functors are computed by applying the original functors to appropriate resolutions. In our case, to compute $L(-)_g$ and $L(-)^{\text{ad}_\mathfrak{g}}_g$ we use the cofibrant resolutions in $\mathfrak{DGLA}_k$, which, in practice, are given by semi-free DG Lie algebras (cf. Section 3.1.2). Thus, we have isomorphisms

$$\text{DRep}_\mathfrak{g}(a) \cong (Qa)_g, \quad \text{DRep}_\mathfrak{g}(a)^{\text{ad}_\mathfrak{g}} \cong (Qa)^{\text{ad}_\mathfrak{g}},$$  

where $Qa$ is any cofibrant resolution of $a$ in $\mathfrak{DGLA}_k$.

Finally, for any DG Lie algebra $a$, we define the representation homology of $a$ in $\mathfrak{g}$ by

$$H_\ast(a, \mathfrak{g}) := H_\ast(\text{DRep}_\mathfrak{g}(a)).$$  

This is a graded commutative $k$-algebra, which depends on $a$ and $\mathfrak{g}$ (but not on the choice of resolution of $a$). If $a \in \mathfrak{Lie Alg}_k$ is an ordinary Lie algebra, then there is an isomorphism of commutative algebras $H_0(a, \mathfrak{g}) \cong a_\mathfrak{g}$.
which justifies the name ‘derived representation scheme’ for \( \text{DRep}_g(a) \). In addition, if \( g \) is a reductive Lie algebra, then

\[
H_\bullet[\text{DRep}_g(a)]_{\text{adg}} \cong H_\bullet(a;g)_{\text{adg}}.
\]

Thus, the homology of \( \text{DRep}_g(a)_{\text{adg}} \) can be viewed as the invariant part of the representation homology of \( a \).

3.1.3. Representation homology vs Lie (co)homology. Next, we explain how representation homology is related to Lie cohomology. A key to this relation is a fundamental theorem of Quillen \cite{23} that establishes a duality (an example of Koszul duality) between the category of DG Lie algebras and the category \( \text{dgCc}_k/k \) of cocommutative co-augmented DG coalgebras. Quillen’s duality is given by a pair of adjoint functors

\[
\Omega_{\text{comm}}: \text{dgCc}_k/k \rightleftarrows \text{DGLA}_k : B_{\text{Lie}}
\]

inducing an equivalence of the corresponding homotopy categories \( \text{Ho}(\text{dgCc}_k/k) \) and \( \text{Ho}(\text{DGLA}_k) \). The functor \( B_{\text{Lie}} \) is defined by the classical Chevalley-Eilenberg complex of a DG Lie algebra \( B_{\text{Lie}}(a) := C(a;k) \); it is a Lie analogue of the bar construction for associative algebras. The functor \( \Omega_{\text{comm}} \) is defined by taking the free Lie algebra of the graded vector space \( \tilde{C}[-1] \), where \( \tilde{C} \) is the cokernel of the co-augmentation map of a coalgebra \( C \in \text{dgCc}_k/k \): this is a Lie analogue of the classical cobar construction of Adams.

A DG coalgebra \( C \in \text{dgCc}_k/k \) is said to be Koszul dual to a Lie algebra \( a \) if there is a quasi-isomorphism \( \Omega_{\text{comm}}(C) \cong a \). Such quasi-isomorphisms are a common source of cofibrant resolutions in \( \text{DGLA}_k \). In fact, we can choose a cofibrant resolution of \( a \) in a functorial way by taking \( C = B_{\text{Lie}}(a) \), but it is often convenient to work with ‘smaller’ coalgebras. The cocommutativity of \( C \) ensures that \( g^*(C) := g^* \otimes C \) has a natural structure of a DG Lie coalgebra in the same way as \( g(B) = g \otimes B \) has a natural structure of a DG Lie algebra for a commutative algebra \( B \). We let \( C^c(g^*(\tilde{C});k) \) denote the Chevalley-Eilenberg complex of the Lie coalgebra \( g^*(C) \), which is defined by a dual construction to the classical Chevalley-Eilenberg complex of the Lie algebra \( g(B) \). We also define a relative Chevalley-Eilenberg complex \( C^c(g^*(C),g^*;k) \) for the pair of Lie coalgebras \( g^*(C) \rightarrow g^* \), which is dual to the classical relative Chevalley-Eilenberg complex for the pair of Lie algebras \( g \rightarrow g^* \). Note that \( C^c(g^*(\tilde{C});k) \) and \( C^c(g^*(C),g^*;k) \) are naturally commutative DG algebras in the same way as the classical Chevalley-Eilenberg complexes \( C(g(B);k) \) and \( C(g(B),g;k) \) are naturally cocommutative DG coalgebras.

The next theorem is one of the main observations of \cite{4} (see op. cit., Theorem 6.5).

**Theorem 3.1.** Let \( C \in \text{dgCc}_k/k \) be Koszul dual to \( a \in \text{DGLA}_k \). Then there are isomorphisms in \( \text{Ho}(\text{dgCa}_k/k) \):

\[
\text{DRep}_g(a) \cong C^c(\tilde{g}^*(\tilde{C});k), \quad \text{DRep}_g(a)_{\text{adg}} \cong C^c(\tilde{g}^*(C),\tilde{g}^*;k).
\]

As a consequence of Theorem 3.1 we have

\[
H_\bullet(a,g) \cong H_\bullet(\tilde{g}^*(\tilde{C});k), \quad H_\bullet(a,g)_{\text{adg}} \cong H_\bullet(\tilde{g}^*(C),\tilde{g}^*;k).
\]

These isomorphisms can be interpreted by saying that representation homology is Koszul dual to the classical Lie homology of current Lie algebras.

3.1.4. Drinfeld trace maps and cyclic homology. Our next goal is to describe certain natural maps with values in representation homology. These maps can be viewed as (derived) characters of finite-dimensional Lie representations. From now on, we assume that \( g \) is a reductive Lie algebra over \( k \). We let \( I(g) := \text{Sym}^r(g)^{\text{adg}} \) denote the space of invariant polynomials on \( g \), and write \( I^r(g) \subset I(g) \) for the subspace of homogeneous polynomials of degree \( r \).

For a Lie algebra \( a \) and a fixed integer \( r \geq 1 \), we define \( \lambda^{(r)}(a) := \text{Sym}^r(a)/[a,\text{Sym}^r(a)] \), which is the space of coinvariants of the adjoint representation of \( a \) in \( \text{Sym}^r(a) \). Note that the vector space \( \lambda^{(r)}(a) \) comes with a
natural map $a \times a \times \ldots \times a \to \lambda^{(r)}(a)$, which is the universal symmetric ad-invariant form on $a$ of degree $r$. The assignment $a \mapsto \lambda^{(r)}(a)$ defines a functor on the category of Lie algebras that naturally extends to the category of DG Lie algebras. Theorem 7.1 of [13] implies that $\lambda^{(r)}$ has a left derived functor

$$L\lambda^{(r)} : \text{Ho}(\text{DGLA}) \to \text{Ho}(\text{Com}_k), \quad a \mapsto \lambda^{(r)}(Qa),$$

whose homology we denote by $HC^{(r)}_\bullet(a)$. If $a$ is an ordinary Lie algebra, $HC^{(1)}_\bullet(a)$ is isomorphic to the classical (Chevalley-Eilenberg) homology $H_\bullet(a, k)$ of $a$, while $HC^{(2)}_\bullet(a)$ is the Lie analogue of cyclic homology introduced by Getzler and Kapranov in [13].

In general, for any DG Lie algebra $a \in \text{DGLA}_k$, we constructed in [13] a natural isomorphism

$$(3.4) \quad \text{H}_\bullet(C(Ua)) \cong \bigoplus_{r=1}^\infty HC^{(r)}_\bullet(a),$$

which can be viewed as a Koszul dual of Hodge decomposition of the (reduced) cyclic homology of the universal enveloping algebra of $a$. This isomorphism justifies the notation for the homology groups $HC^{(r)}_\bullet(a)$.

Now, observe that, for any commutative algebra $B$, there is a natural symmetric invariant $r$-linear form $a(B) \times a(B) \times \ldots \times a(B) \to \lambda^{(r)}(a) \otimes B$ on the current Lie algebra $a(B)$. Hence, by the universal property of $\lambda^{(r)}$, we have a canonical map

$$(3.5) \quad \lambda^{(r)}[a(B)] \to \lambda^{(r)}(a) \otimes B.$$  

Applying $\lambda^{(r)}$ to the universal representation $g_\theta : a \to g(a_\theta)$ and composing with $\lambda^{(r)}$, we define

$$(3.6) \quad \lambda^{(r)}(a) \longrightarrow \lambda^{(r)}[g(a_\theta)] \longrightarrow \lambda^{(r)}(g) \otimes a_\theta.$$  

On the other hand, for the Lie algebra $g$, we have a canonical (nondegenerate) pairing

$$(3.7) \quad I^r(g) \times \lambda^{(r)}(g) \to k$$

induced by the linear pairing between $g^*$ and $g$. Replacing the Lie algebra $a$ in $(3.6)$ by its cofibrant resolution $L \sim a$ and using $(3.7)$, we define the morphism of complexes

$$(3.8) \quad I^r(g) \otimes \lambda^{(r)}(L) \xrightarrow{(3.6)} I^r(g) \otimes \lambda^{(r)}(g) \otimes L_\theta \xrightarrow{(3.7)} L_\theta.$$  

For a fixed polynomial $P \in I^r(g)$, this morphism induces a map on homology $\text{Tr}_g(a) : HC^{(r)}_\bullet(a) \to H_\bullet(a, g)$, which we call the Drinfeld trace associated to $P$. (We warn the reader that $\text{Tr}_g(a)$ does depend on the choice of $P$ but we suppress this in our notation.) It is easy to check that the image of $(3.8)$ is contained in the invariant subalgebra $L_\theta^{ad g}$ of $L_\theta$, hence the Drinfeld trace is actually a map

$$(3.9) \quad \text{Tr}_g(a) : HC^{(r)}_\bullet(a) \to H_\bullet(a, g)^{ad g}.$$  

We will give an explicit formula for $(3.9)$ in Section 3.3 below.
3.2. Chern-Simons forms. In this section, all DG algebras will be cohomologically graded. Let \( \mathcal{A} \) be a commutative DG algebra, and let \( \mathfrak{g} \) be a finite-dimensional Lie algebra. Recall that a \( \mathfrak{g} \)-valued connection on \( \mathcal{A} \) is an element \( \theta \in \mathcal{A}^1 \otimes \mathfrak{g} \); its curvature is given by \( \Omega := d\theta + \frac{1}{2}[\theta, \theta] \) in \( \mathcal{A}^2 \otimes \mathfrak{g} \), and it is easy to verify that \( d\Omega = [\Omega, \theta] \), which is usually called the Bianchi identity.

Now, fix \( P \in \mathcal{I}^{r+1}(\mathfrak{g}) \), a homogeneous ad-invariant polynomial of degree \( r + 1 \). Given \( \alpha \in \mathcal{A} \otimes \text{Sym}^{r+1}(\mathfrak{g}) \), regard \( P \) as a linear map \( \text{Sym}^{r+1}(\mathfrak{g}) \rightarrow k \) and define \( P(\alpha) \) by applying to \( \alpha \) the evaluation map \( \text{Id}_\mathcal{A} \otimes \text{ev}_P : \mathcal{A} \otimes \text{Sym}^{r+1}(\mathfrak{g}) \rightarrow \mathcal{A} \). Thus, \( P(\alpha) \) is an element of \( \mathcal{A} \) having the same cohomological degree as \( \alpha \). A simple calculation, using the Bianchi identity, shows that \( dP(\Omega^{r+1}) = 0 \) for any \( \theta \in \mathcal{A}^1 \otimes \mathfrak{g} \). Thus \( P(\Omega^{r+1}) \) is a cocycle in \( \mathcal{A} \) of degree \( 2r + 2 \). In fact, this cocycle is always exact, and among all coboundaries representing \( P(\Omega^{r+1}) \), one can specify a natural element \( TP(\theta) \in \mathcal{A}^{2r+1} \) called the Chern-Simons form [7]. Explicitly, this form is defined by the formula (cf. [7] (3.1))

\[
TP(\theta) := (r + 1) \int_0^1 P(\theta, \Omega_t^t) dt,
\]

where \( \Omega_t = r\Omega + \frac{1}{2}(t^2 - t)[\theta, \theta] \) is the curvature for the family of connections \( \theta_t := t\theta \), \( t \in [0, 1] \), contracting \( \theta \) to 0. We refer to \( TP(\theta) \) as the Chern-Simons form of \( P \) and \( \theta \). A classical calculation (see [7] Prop. 3.2) gives

**Proposition 3.1.** \( dTP(\theta) = P(\Omega^{r+1}) \).

We remark that the Chern-Simons form can be also defined directly, without integration, by the following formula (cf. [7] (3.5))

\[
(3.10) \quad TP(\theta) = \sum_{i=0}^r A_i \Psi_{i,P},
\]

where \( A_i := \frac{(-1)^i(r+1)!i!}{2^i(r-i)!i!(r+1+i)!} \) and \( \Psi_{i,P} := P(\theta[\theta, \theta]^i(\Omega^{r+i-1})) \).

An example of a commutative DG algebra with \( \mathfrak{g} \)-valued connection is the Weil algebra \( W(\mathfrak{g}) \) of \( \mathfrak{g} \). Recall (cf. [19] Section 6.9) that \( W(\mathfrak{g}) := \text{Sym}(\mathfrak{g}[1] \oplus \mathfrak{g}[2]) \cong \wedge(\mathfrak{g}^*) \otimes \text{Sym}(\mathfrak{g}^*) \), where the generators \( \mathfrak{g}^* \) of \( \wedge(\mathfrak{g}^*) \) are in cohomological degree 1 and the generators \( \mathfrak{g}^* \) of \( \text{Sym}(\mathfrak{g}^*) \) are in cohomological degree 2. The differential is given by the identity map on the generators of cohomological degree 1 and vanishes on the generators of cohomological degree 2. It is therefore, easy to see that \( W(\mathfrak{g}) \) is acyclic, i.e., quasi-isomorphic to \( k \). The identity map \( \mathfrak{g}^* \rightarrow \mathfrak{g}^* = W(\mathfrak{g}) \) gives a \( \mathfrak{g} \)-valued connection \( \theta_W \) on \( W(\mathfrak{g}) \). There is a second isomorphism \( W(\mathfrak{g}) \cong \wedge(\mathfrak{g}^*) \otimes \text{Sym}(\mathfrak{g}^*) \) such that \( \theta_W \) is still the identity from \( \mathfrak{g}^* \) to \( \mathfrak{g}^* \) viewed as the space of generators of \( \mathfrak{g}^* \) and the curvature \( \Omega_{\mathcal{W}} \) of \( \theta_W \) is the identity map from \( \mathfrak{g}^* \) to the space of generators of \( \text{Sym}(\mathfrak{g}^*) \). Thus, under this second isomorphism, the element \( P(\Omega_W^{r+1}) \) is identified with the degree \( 2r+2 \) element \( P \in \text{Sym}(\mathfrak{g}^*) \subset \wedge(\mathfrak{g}^*) \otimes \text{Sym}(\mathfrak{g}^*) \) for any \( P \in \mathcal{I}^{r+1}(\mathfrak{g}) \). It follows from Proposition 3.1 that \( P = dTP(\theta_W) \).

The Weil algebra is the **universal** commutative DG algebra with a \( \mathfrak{g} \)-valued connection. Indeed, a connection \( \theta \) on a commutative DG algebra \( \mathcal{A} \) can be viewed as a map \( \mathfrak{f} \) vector spaces \( \theta : \mathfrak{g}^* \rightarrow \mathcal{A}^1 \). This defines a (characteristic) homomorphism \( c : W(\mathfrak{g}) \rightarrow \mathcal{A} \) by \( c(\mu) = \theta(\mu), c(\mu) = d_\mathcal{A}(\theta(\mu)) \) for all \( \mu \in \mathfrak{g}^* \). Here each \( \mu \in \mathfrak{g}^* \) is viewed as a degree 1 element of \( \wedge(\mathfrak{g}^*) \) and \( \bar{\mu} := d_{\mathcal{W}}(\mu) \), \( \mu \) viewed as a degree 2 element of \( \text{Sym}(\mathfrak{g}^*) \). Clearly, \( \theta = c(\theta_W), \Omega = c(\Omega_W), \) etc. From this, it follows that for any \( P \in \mathcal{I}^{r+1}(\mathfrak{g}) \),

\[
P(\Omega^{r+1}) = c(P), \quad TP(\theta) = c(TP(\theta_W)).
\]

Thus, \( TP(\theta_W) \) is the universal Chern-Simons form.
3.3. **Main theorem.** Let $C := \langle \text{Sym}^r(W), \delta \rangle$ be a semi-cofree, cocommutative, conilpotent coaugmented DG coalgebra cogenerated by a finite-dimensional graded vector space $W$. Assume further that the corestriction of $\delta$ to $W$ vanishes on $\text{Sym}^r(W)$ for $r \gg 0$. Motivated by Beilinson’s construction (see Appendix A), we consider the convolution (DG) algebra

$$A := \text{Hom}(\text{DR}_\bullet(C), \mathcal{C}^c(g^*\langle \hat{C} \rangle; k)).$$

Note that taking bigraded linear duals gives an isomorphism of convolution algebras

$$A \cong A_E := \text{Hom}(\mathcal{C}(g(\hat{E}); k), \text{DR}_\bullet(E)), \tag{3.11}$$

where $E := \langle \text{Sym}(W^*), \delta^* \rangle$. Equip $A$ with a cohomological grading by inverting all homological degrees.

The DG algebra $A$ has a natural $g$-valued connection $\theta \in A^1 \otimes g$ given by

$$\theta(c) = \sum_{\alpha} \xi^*_\alpha (s^{-1}c) \otimes \xi_\alpha, \tag{3.12}$$

where $\{\xi_\alpha\}$ is a basis of $g$ and $\{\xi^*_\alpha\}$ is the dual basis of $g^*$ and $\xi^*_\alpha (\hat{c}) := \xi^*_\alpha \otimes \hat{c}$ ($\hat{c}$ being the image of $c$ in $\hat{C}$). Similarly, the curvature $\Omega \in A^2 \otimes g$ of the connection $\theta$ vanishes on $\Omega^{\bullet, \bullet}_E$ for all $j \neq 1$ and satisfies

$$\Omega(\omega) = \sum_{\alpha} \xi^*_\alpha (s^{-1}d\omega) \otimes \xi_\alpha, \tag{3.13}$$

for $\omega \in \Omega^{\bullet, \bullet}_E$. Further, $A$ is a commutative DG algebra, making $A \otimes g$ a DG Lie algebra. Hence, $[\theta, \theta] \in A^2 \otimes g$. Thus, for any $P \in I^{r+1}(g)$, the Chern-Simons form $TP(\theta)$ associated with $\theta$ arises as an element of $A^{3r+1}$.

Recall that the map $s^{2r}$ increasing homological degree by $2r$ gives a map of graded vector spaces from $\mathcal{C}^{-(r)}[\text{DR}_\bullet(C)]$ to $\text{DR}_\bullet(C)$. Dually, $s^{2r}$ gives a map of graded vector spaces from $\text{DR}_\bullet(E)$ to $\mathcal{C}^{(r)}[\text{DR}_\bullet(E)]$. Let $\theta_E$ denote the image of the connection $\theta$ of $A$ under (3.11). It is known (see Theorem A.1) that the (shifted) Chern-Simons form $s^{2r}TP(\theta_E)$ gives a map of complexes

$$s^{2r}TP(\theta_E) : \mathcal{C}(g(\hat{E}); k) \rightarrow \mathcal{C}^{(r)}[\text{DR}_\bullet(E)][1].$$

Taking (bigraded) linear duals, we see that the (shifted) Chern-Simons form $TP(\theta) s^{2r}$ gives a map of complexes

$$TP(\theta) s^{2r} : \mathcal{C}^{-(r)}[\text{DR}_\bullet(C)][-1] \rightarrow \mathcal{C}^c(g^*(\hat{C}); k). \tag{3.14}$$

By Theorem 2.3 the inclusion

$$\iota : \text{Ker}(d : \Omega^r_C \rightarrow \Omega^{r-1}_C)[-r] \rightarrow \mathcal{C}^{-(r)}[\text{DR}_\bullet(C)]$$

is a quasi-isomorphism. Let $\varepsilon : \mathcal{C}^\lambda[\bullet](C) \rightarrow \text{Ker}(d : \Omega^r_C \rightarrow \Omega^{r-1}_C)[-r]$ be as in (2.4). By Theorem 3.4

$$\text{DR}_\bullet(g)(a) \cong \mathcal{C}^c(g^*(\hat{C}); k), \tag{3.15}$$

where $a$ is the DG Lie algebra Koszul dual to $C$. On the other hand, recall from [4] Prop. 7.4 that

$$\text{H}^{\bullet+1}(C) \cong \mathcal{H}_\bullet (\text{Rep}_g(a)). \tag{3.16}$$

[4] Prop. 7.4 further implies that the isomorphism (3.16) respects the Hodge decomposition (3.4) to give an isomorphism

$$\text{H}^{\bullet+1}(C) \cong \mathcal{H}^{(r)}_\bullet(\mathcal{C}^c(g^*(\hat{C}); k)) \cong \mathcal{H}^{(r+1)}_\bullet(\mathcal{C}^c(g^*(\hat{C}); k)). \tag{3.17}$$

5 Equipping $W$ and $W^*$ with weight 1 makes the coalgebras $\text{DR}_\bullet(C), \mathcal{C}(g(\hat{E}); k)$ and the algebras $\mathcal{C}^c(g^*(\hat{C}); k), \text{DR}_\bullet(E)$ bigraded. While taking bigraded duals, we stick to the convention that homological degrees are inverted while weights are preserved. Thus, $\text{DR}_\bullet(C)$ is the bigraded dual of $\text{DR}_\bullet(E)$, etc. (even though the differentials are not necessarily weight-preserving).

6 The isomorphism (3.11) transforms $\theta$ into the connection $A_E$. 

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With these identifications, $TP(\theta)s^{2r}$ can be interpreted as a map from $HC^{(r+1)}_\ast(a)$ to $H_\ast(a, g)$.

**Theorem 3.2.** For any invariant polynomial $P \in I^{r+1}(g)$, the following diagram commutes:

\[
\begin{array}{ccc}
HC^{(r+1)}_\ast(a) & \xrightarrow{\circ \epsilon} & HC^{(r)}_\ast(C) \\
\downarrow{\text{Tr}_g(a)} & & \downarrow{\text{θ}} \\
H_\ast(a, g) & \xrightarrow{\text{Thm. 3.1}} & H_\ast(g^\ast(\tilde{C}); k)
\end{array}
\]

Thus, the map $\text{Tr}_g(a)$ is given by the formula

\[
\text{Tr}_g(a) = \frac{1}{(r + 1)!}TP(\theta)s^{2r},
\]

where $\theta$ is defined in (3.12).

**Proof.** For brevity, we denote the maps induced on homologies by a given map of complexes by the same symbol as that of the original map of complexes. Let $\mathcal{L} := \Omega_{\text{Com}}(C)$. Note that $\tilde{C}[-1] \to \mathcal{L}$ as graded $k$-vector spaces. Recall from Section 3.1.4 (see [4, Sec. 7] for more details) that the Drinfeld trace was constructed as a map of complexes

\[
\text{Tr}_g(L) : \lambda^{(r+1)}(\mathcal{L}) \to \mathcal{C}^c(g^\ast(\tilde{C})[1]).
\]

Dually, one obtains a map of complexes

\[
(3.18) \quad \mathcal{C}(g(\bar{E}); k) \to \text{Sym}^{r+1}(\mathcal{C}(\bar{E})) \cap B(E)^{\hat{}} .
\]

where $B(E)^{\hat{}}$ denotes the cocommutator subspace of the coalgebra $B(E)$. The isomorphism

\[
(3.19) \quad C^{\lambda,(r)}(C)[-1] \cong \lambda^{(r+1)}(\mathcal{L})
\]

inducing (3.17) on homologies is obtained by taking $k$-linear duals on the isomorphism

\[
\text{Sym}^{r+1}(\mathcal{C}(\bar{E})) \cap B(E)^{\hat{}} \cong C^{\lambda,(r)}(E)[1],
\]

whose inverse is explicitly given by the map $N$ which acts on $E[1]^{\otimes n}$ by $1 + \tau + \ldots + \tau^{n-1}$ where $\tau$ is the $n$-cycle $(0, 1, \ldots, n - 1)$. Let $\varphi_P$ denote the composite map

\[
\mathcal{C}(g(\bar{E}); k) \xrightarrow{(3.13)} \text{Sym}^{r+1}(\mathcal{C}(\bar{E})) \cap B(E)^{\hat{}} \xrightarrow{\text{Tr}_g(L)} C^{\lambda,(r)}(E)[1].
\]

The composite map

\[
C^{\lambda,(r)}(C)[-1] \xrightarrow{(3.10)} \lambda^{(r+1)}(\mathcal{L}) \xrightarrow{\varphi_P} C^{\lambda,(r)}(g^\ast(\tilde{C}); k)
\]

is thus equal to the map obtained by applying $\varphi_P$ to $E$ and taking (bigraded) linear duals.

It is known (see Theorem [A.3]) that the diagram

\[
\begin{array}{ccc}
\mathcal{C}(g(\bar{E}); k) & \xrightarrow{TP(\theta)s^{2r}} & \mathcal{C}^{\lambda,(r)}(E)[1] \\
\downarrow{\varphi_P} & & \downarrow{\text{ε}} \\
\text{CC}^{(r)}[\text{DR}(E)][1] & \xrightarrow{\text{θ}} & C^{\lambda,(r)}(E)[1]
\end{array}
\]

commutes on homologies, where $p$ is as in (2.21) and $\varepsilon$ is as in (2.22). The desired result follows immediately from the above fact by taking bigraded linear duals.

\[\square\]

It is easy to verify that (for the dual statement, see Proposition [A.5].
Proposition 3.2. \( TP(\theta)s^{2r} \circ \ell = P(\theta.\Omega^r)s^{2r} \).

As a consequence of Theorem \[3.2\] and Proposition \[3.2\] we have

Corollary 3.1. For any \( P \in I^{r+1}(g) \), the Drinfeld trace map \( \text{Tr}_g(a) : \text{HC}^{(r+1)}_*(a) \to H_*(a,g)^{adg} \) is given by

\[
(3.20) \quad \text{Tr}_g(a) = \frac{1}{(r+1)!} P(\theta.\Omega^r)s^{2r},
\]

where \( \theta \) is defined in \[3.12\].

We now specialize to the case \( g = gl_V \) and use formula \[3.20\] to express the derived character maps for associative algebras.

3.4. The case of \( gl_V \). Recall that, for any \( k \)-algebra \( A \) and finite-dimensional vector space \( V \), there is a derived scheme \( D\text{Rep}_V(A) \), analogous to \( D\text{Rep}_g(a) \), that parametrizes the representations of \( A \) in \( V \) as an associative algebra (see \[3\]). There exist also natural maps

\[
(3.21) \quad \text{Tr}_V(A) : \text{HC}_*(A) \to H_*(A,V)
\]

relating the (reduced) cyclic homology of \( A \) to its representation homology \( H_*(A,V) := H_*[D\text{Rep}_V(A)] \). As shown in \[4\], the representation homology \( H_*(A,V) \) and the character maps \[3.21\] can be expressed in Koszul dual terms of Lie coalgebras. To be precise, by \[4\, \text{Theorem 3.1}\], there is an isomorphism of algebras

\[
(3.22) \quad H_*(A,V) \cong H_*(gl_V(C); k),
\]

where \( gl_V(C) := \text{End}(V)^* \otimes C \) is a DG Lie coalgebra defined over a (co-associative) coalgebra \( C \in \text{DGC}_{k/k} \) Koszul dual to the algebra \( A \). On the other hand, since the cobar construction \( R := \Omega(C) \) provides a cofibrant resolution of \( A \) in \( \text{DGA}_{k/k} \), by a theorem of Feigin-Tsygan, we can identify \( \text{HC}(A) \cong H_*[R_q] \), where \( R_q := R/[R,R] \). With these identifications, the trace map \( \text{Tr}_V(A) \) becomes

\[
\text{Tr}_V(A) : H_*[R_q] \to H_*(gl_V(C); k).
\]

Now, let \( A = \mathcal{U}a \) be the universal enveloping algebra of a Lie algebra \( a \). In this case, the Koszul dual coalgebra \( C \) can be chosen to be cocommutative, with \( C := \Omega_{\text{conn}}(C) \), giving a cofibrant resolution of the Lie algebra \( a \) in \( \text{DGLA}_{k/k} \). As a result, \[3.22\] becomes an isomorphism (cf. \[4\, \text{Prop. 6.3}\]):

\[
H_*(A,V) \cong H_*(a, gl_V). \tag*{(3.22)}
\]

Identifying \( gl_V = \text{End}(V) \), we define polynomials \( \text{Tr} \in I^{q+1}(gl_V) \) by \( \text{Tr}(X, \ldots, X) := \text{Tr}_V(X^{q+1}) \), where \( \text{Tr}_V : \text{End}(V) \to k \) is the usual matrix trace on \( V \). With this choice of invariant polynomials, the direct sum of the Drinfeld traces gives a map \( \oplus_{q \geq 1} \text{HC}^{(q)}_*(a) \to H_*(a, gl_V) \), which upon isomorphism \[3.3\], coincides with the trace map \[3.21\]. By Corollary \[3.1\] we now conclude

\[
(3.23) \quad \text{Tr}_V(A) = \sum_{q=0}^{\infty} \frac{1}{(q+1)!} \text{Tr}(\theta.\Omega^q)s^{2q}.
\]

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3.5. **Reduced trace maps.** Unfortunately, despite its simple appearance, the trace formula (3.24) is still inaccessible to computations. The problem is that we need to know an explicit presentation for representation homology which is available only in a few nontrivial cases. One way to get around this problem is to restrict the Drinfeld traces to 'diagonal representations' using with an appropriate (derived) version of the classical Harish-Chandra homomorphism. The derived Harish-Chandra homomorphism plays a crucial role in \[ \mathbb{H} \] and we briefly recall its construction here.

We keep the assumption that \( g \) is a finite-dimensional reductive Lie algebra. Let \( h \) be a Cartan subalgebra of \( g \), and let \( W \) be the corresponding Weyl group. For any Lie algebra \( a \), the representation scheme \( \text{Rep}_a(a) \) is naturally a (closed) subscheme of \( \text{Rep}_g(a) \). In fact, the inclusion \( h \hookrightarrow g \) induces a morphism of schemes \( \text{Rep}_h(a)/W \to \text{Rep}_g(a)/G \), where \( \text{Rep}_h(a)/W \) is the quotient of \( \text{Rep}_h(a) \) relative to the natural action of \( W \) on \( h \). This yields a morphism of functors \( (-)^\text{ad} \to (-)_W^\text{ad} \) which extends to a morphism of the derived functors: \( \mathbb{L}(-)^\text{ad} \to \mathbb{L}(-)_W^\text{ad} \). As a result, for any DG Lie algebra \( a \), we get a canonical map of commutative DG algebras

\[
(3.24) \quad \Phi_a(a) : \text{DRep}_a(a)^{\text{ad}} \to \text{DRep}_a(W)^{\text{ad}}.
\]

which we called the *derived Harish-Chandra homomorphism* (cf. \[ \mathbb{H} \] Section 7). Explicitly, using Theorem 3.1, we realize \( \Phi_a(a) \) as a morphism of complexes \( C^*(\mathfrak{g}(C), \mathfrak{g}^*; k) \to C^*(\mathfrak{h}(C), \mathfrak{h}^*; k)_W^\text{ad} \) corresponding to the projection \( \mathfrak{g}^* \to \mathfrak{h}^* \). At the homology level, this induces a map \( H_\bullet(a, g)^{\text{ad}} \to H_\bullet(a, h)_W^\text{ad} \).

Now, let \( I(h)_W^\text{ad} := \text{Sym}(h^*)_W^\text{ad} \) denote the space of \( W \)-invariant polynomials on \( h \). Recall that \( \mathfrak{g}^* \to \mathfrak{h}^* \) extends to an isomorphism of algebras \( I(g) \xrightarrow{\sim} I(h)_W^\text{ad} \) which is called the Chevalley isomorphism for \( g \).

**Lemma 3.1.** For every integer \( r \geq 1 \), the following diagram commutes

\[
\begin{array}{ccc}
I^r(g) \otimes \text{HC}^r_\bullet(a) & \xrightarrow{\sim} & I^r(h)_W^\text{ad} \otimes \text{HC}^r_\bullet(a) \\
\downarrow \qquad \text{Tr}_a(a) & & \downarrow \text{Tr}_a(a) \\
H_\bullet(a, g)^{\text{ad}} & \xrightarrow{\Phi_a(a)} & H_\bullet(a, h)_W^\text{ad}
\end{array}
\]

**Proof.** Let \( P \in I^r(g) \) and let \( P_W \in I^r(h)_W^\text{ad} \) denote the image of \( P \) under the Chevalley isomorphism. Similarly, let \( \theta_h \) denote the connection (3.12) on the convolution algebra \( A_h := \text{Hom}(\text{DR}_h(C), C^*(\mathfrak{h}(C); k)) \) and let \( \Omega_h \) denote the curvature of \( \theta_h \). The inclusion \( h \hookrightarrow g \) induces an inclusion \( A_h \otimes \text{Sym}(h) \hookrightarrow A_g \otimes \text{Sym}(g) \) of commutative DG algebras. Clearly, for any element \( \alpha \in A_h \otimes \text{Sym}(h) \), \( P_W(\alpha) = P(\alpha) \), where \( \alpha \) on the right hand side is viewed as an element of \( A_g \otimes \text{Sym}(g) \). By (3.24),

\[
\text{Tr}_h(a) = \frac{1}{r!} P_W(\theta_h^{r-1} \Omega_h^{r-1}), \quad \text{Tr}_g(a) = \frac{1}{r!} P(\theta \Omega^{r-1}).
\]

It therefore suffices to verify that \( \Phi_g(\theta \Omega^{r-1}) = \theta_h \Omega_h^{r-1} \) as elements of \( A_h \otimes \text{Sym}(g) \). This follows from the fact that \( \Phi_g \) is a DG algebra homomorphism and \( \Phi_g(\theta) = \theta_h \), which is easy to verify by direct calculation. \( \square \)

**Lemma 3.1** shows that, modulo the Chevalley isomorphism, the composite map

\[
(3.25) \quad \text{HC}^r_\bullet(a) \xrightarrow{\text{Tr}_a(a)} H_\bullet(a, g)^{\text{ad}} \xrightarrow{\Phi_a(a)} H_\bullet(a, h)_W^\text{ad}
\]
equals \( \text{Tr}_a(a) \), which depends only on \( h \), \( W \) and the choice of an invariant polynomial \( P \in I(h)_W^\text{ad} \) but not on the Lie algebra \( g \). We call \( \text{Tr}_a(a) \) the *reduced trace map*. This map is more accessible than the Drinfeld trace, since \( H_\bullet(a, h)_W^\text{ad} \) is easy to compute in many cases.
In fact, the computation of $\text{Tr}_h(a)$ reduces to the rank one case. To be precise, let $h = k$ be a one-dimensional Lie algebra with a preferred basis. Denote by $\text{Tr}(a)$ the Drinfeld trace map for a corresponding to the canonical element in $\text{Sym}(h^*)$ of degree $r$. Then, for an arbitrary $h$, the map $\text{Tr}_h(a)$ factors through $\text{Tr}(a)$. To see this, choose a Koszul dual coalgebra $C \in \text{DGCC}_{k/k}$ for $a$, and let $R := \Omega(C)$. Then, for a given $h$, choose a linear basis $\{\xi_{\alpha}\} \subset h$ and define a DG algebra homomorphism

$$\vartheta_h : R_{ab} \rightarrow C^c(h^*(\bar{C}); k) \otimes \text{Sym}(h)$$

by sending the canonical generators $s^{-1}c$ of $R_{ab}$ to the elements

$$\vartheta_h(s^{-1}c) = \sum_{\alpha} \xi_{\alpha}^*(s^{-1}c) \otimes \xi_{\alpha},$$

where $\{\xi_{\alpha}^*\} \subset h^*$ is the dual basis to $\{\xi_{\alpha}\}$. Note that the map $\vartheta_h$ thus defined is independent on the choice of basis $\{\xi_{\alpha}\}$. Now, for any $P \in I(h)^W$, the evaluation at $P$ on the second factor gives a map $C^c(h^*(\bar{C}); k) \otimes \text{Sym}(h) \rightarrow C^c(h^*(\bar{C}); k)^W$. Write $P(\vartheta_h)$ for the composition of this map with $\vartheta_h$:

$$P(\vartheta_h) : R_{ab} \rightarrow C^c(h^*(\bar{C}); k)^W.$$  

On homology, this induces a map $H_*(a, k) \rightarrow H_*(a, h)^W$.

Lemma 3.2. For any $P \in I(h)^W$, the trace map \((3.25)\) factors as

$$\text{Tr}_h(a) = P(\vartheta_h) \circ \text{Tr}(a).$$

Proof. Let $\theta_0$ denote $\vartheta_h$ for $h := k$ and let $\Omega_0$ denote the curvature of $\theta_0$. By \((3.20)\),

$$\text{Tr}(a) = \frac{1}{r!}[\theta_0, \Omega_0^{-1}], \quad \text{Tr}_h(a) = \frac{1}{r!}P(\theta_h, \Omega_h^{-1}).$$

The desired result now follows from the observation that $\theta_h = \vartheta_h(\theta_0)$. \hfill $\square$

Example 3.1. Let $h := h_n$ be the subalgebra of diagonal matrices in $gl_n$ and let $W = S_n$ be the symmetric group acting on $h_n$ by permuting the diagonal entries. Let $P_q \in I_q(h_n)^{S_n}$ be the symmetric polynomial given by the $(q + 1)$-th power sum. Then $C(h^*(\bar{C}); k)^W = [R_{ab}^{S_n}]^{S_n}$, where $S_n$ acts on $R_{ab}^{S_n}$ by permuting the factors. Writing $S^n[R_{ab}] := [R_{ab}^{S_n}]^{S_n}$, we see that $\sum_{q=0}^{\infty} P_q(\vartheta_h) : R_{ab} \rightarrow S^n[R_{ab}]$ is precisely the symmetrization map

$$\text{Sym} : R_{ab} \rightarrow S^n[R_{ab}], \quad r \mapsto \sum_{i=1}^n (1, \ldots, r, \ldots, 1),$$

where $r$ in the $i$-th factor is in the $i$-th summand. Now, let $A := Ua$. With the above choice of invariant polynomials, the direct sum of the reduced Drinfeld traces $\text{Tr}_{h_n}(a)$ gives a map $\oplus_{q \geq 1} H^q_c(a) \rightarrow S^n[R_{ab}]$, which upon isomorphism \((3.4)\), coincides with the trace map

$$\text{Tr}_{h_n}(A) : H^\bullet_c(a) \rightarrow S^n[H_*(R_{ab})]$$

constructed in \[4\] Section 4. We therefore, obtain a special case (for universal enveloping algebras) of \[4\] Prop. 4.2 as a consequence of Lemma 3.2. We remind the reader that, for $n = 1$, the reduced trace map

$$\text{Tr}_V(a) : H^\bullet_c(a) \rightarrow H_*(R_{ab})$$

coincides with the trace $\text{Tr}_V$ in \((3.21)\) for $V = k$.

Thus, thanks to Lemma 3.2 computing the trace map $\text{Tr}_h(a)$ for any $h$ and any invariant polynomial $P \in I(h)^W$ reduces to computing the map $\text{Tr}(a) = \text{Tr}_h(a)$ for $h$ being one-dimensional. In the next section, we will given an explicit formula for $\text{Tr}_V(a)$ for an arbitrary abelian Lie algebra $a$ in terms of differential forms on $\text{Sym}(a)$.  

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4. Traces of symmetric algebras

4.1. Symmetric algebras. In this section, \( A := \text{Sym}(W) \) will denote the symmetric algebra of a vector space \( W \) of finite dimension \( N \). We may think of \( A \) as the universal enveloping algebra of the abelian Lie algebra \( a = W \) with trivial bracket, so that the results of Section 3.4 will apply. We will write \( \overline{\text{Tr}}(a) \) for \( a = W \) as \( \overline{\text{Tr}}(A) \) or simply as \( \overline{\text{Tr}} \) when there is no danger of confusion.

Recall that \( A \) has a minimal cofibrant resolution \( R = \Omega(C) \) given by the cobar construction of the Koszul dual coalgebra \( C = \text{Sym}^c(W[1]) \). The algebra \( R \) is the tensor algebra generated by the vector space \( \text{Sym}(W[1])[-1] \), whose elements of degree \( k - 1 \) we denote by

\[
\lambda(v_1, v_2, \ldots, v_k) := s^{-1}(dv_1 \ldots dv_k) \in \text{Sym}^k(W[1])[-1].
\]

Here, \( dv \) denotes \( v \) viewed as an element of \( W[1] \). With this notation, the differential on \( R \) satisfies

\[
\begin{align*}
\delta \lambda(v_1, v_2) &= -[v_1, v_2], \\
\delta \lambda(v_1, v_2, v_3) &= -[v_1, \lambda(v_2, v_3)] - [v_2, \lambda(v_3, v_1)] - [v_3, \lambda(v_1, v_2)].
\end{align*}
\]

In general, one can verify without much difficulty the following formula.

**Lemma 4.1.** The differential \( \delta \) on the minimal resolution \( R \) of \( A = \text{Sym}(W) \) is defined by

\[
\delta \lambda(v_1, \ldots, v_n) = \sum_{p+q=n} \sum_{1 \leq p \leq q} (-1)^p (-1)^q \left[ \lambda(v_{\sigma(1)}, \ldots, v_{\sigma(p)}), \lambda(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)}) \right],
\]

where \( \text{Sh}(p, q) \) denotes the set of \( (p, q) \)-shuffles. Hence, \( \delta R \subset [R, R] \).

Let \( R_{ab} \) denote the abelianization of \( R \). By Lemma 4.1, \( R_{ab} \) is the graded symmetric algebra of \( \text{Sym}(W[1])[-1] \) equipped with zero differential. Explicitly (omitting the shifts), we can write

\[
R_{ab} = \text{Sym}(W) \otimes \text{Sym}(\Lambda^2 W \oplus \Lambda^3 W \oplus \ldots \oplus \Lambda^N W)
\]

with understanding that the elements of \( \Lambda^k W \) have (homological) degree \( k - 1 \).

Note that the de Rham algebra of \( A \) can be identified as \( \Omega^*_A = \text{Sym}(W \oplus sW) = \text{Sym}(W) \otimes \Lambda(W) \), and for each \( k \geq 1 \), there is a canonical (injective) map

\[
s^{-1} : \Omega^k_A \to R_{ab}, \quad a dv_1 \ldots dv_k \mapsto a \lambda(v_1, \ldots, v_k).
\]

This map shifts homological degree by \(-1 \), whence its notation.

Next, we recall that there is a canonical isomorphism (cf. Theorem 2.2)

\[
\overline{\text{HC}}_*(A) \cong \Omega^*_A/d\Omega^*_{A}^{-1}.
\]

On the other hand, regarding \( A = \text{Sym}(W) \) as the universal enveloping algebra of the abelian Lie algebra \( a = W \), we have the (dual) Hodge decomposition \((3.4)\) of \( \overline{\text{HC}}_*(A) \). Under the isomorphism \((3.3)\), the image of the direct summand \( \overline{\text{HC}}^{(r)}_*(a) \) in \((3.4)\) is precisely \( \text{Sym}^r(W) \otimes \Lambda^*(W)/d[\text{Sym}^{r+1}(W) \otimes \Lambda^{*+1}(W)] \). Thus, for the abelian Lie algebra \( a = W \), we have an isomorphism

\[
\overline{\text{HC}}^{(r)}_*(a) \cong \text{Sym}^r(W) \otimes \Lambda^*(W)/d[\text{Sym}^{r+1}(W) \otimes \Lambda^{*+1}(W)].
\]

Upon the isomorphism \((3.4)\), the direct sum of the reduced Drinfeld traces \( \overline{\text{Tr}}(a) : \overline{\text{HC}}^{(r)}_*(a) \to R_{ab} \) becomes the reduced trace \( \overline{\text{Tr}}(A) : \overline{\text{HC}}_*(A) \to R_{ab} \) in \((3.20)\). Let \( \varepsilon \) be as in \((2.2)\) and let \( T \) be the quasi-isomorphism
in \((4.10)\). By Theorem 4.1, the isomorphisms \(T \circ \varepsilon, \varepsilon^{-1} \circ d : \Omega^*_{A}/d\Omega^*_{A} \rightarrow H_\bullet(R_b)\) coincide. It follows that \(\text{Tr}(A)\)

is identified with the composite map

\[
\Omega^*_{A}/d\Omega^*_{A} \xrightarrow{\varepsilon^{-1} \circ d} H_\bullet(R_b) \xrightarrow{\text{Tr}(A)} R_{ab},
\]

Let \(\omega \in \Omega^p_{A}\) be a form whose polynomial coefficients are homogeneous of degree \(q + 1\). The homological degree of \(d\omega\) in \(\text{DR}_\bullet(C)\) is \(p + 1\). On the other hand, \(d\omega\) can also be viewed as an element of \(\text{DR}_\bullet(C)\), where its homological degree is \(2q + p + 1\). We may therefore, suppress \(s^j\) from the notation when we apply \((3.23)\) to \([d\omega] \in H_{p+1}[\text{DR}_\bullet(C)]\) by reinterpreting \(d\omega\) as an element of \(\text{DR}_\bullet(C)\). With these conventions, \((3.23)\) immediately implies:

**Theorem 4.1.** The reduced trace map \(\text{Tr}(A) : \Omega^*_{A}/d\Omega^*_{A} \rightarrow R_{ab}\) is given by

\[
\text{Tr}(A)(\omega) = \sum_{q=0}^{\infty} \frac{1}{(q + 1)!} [\theta\Omega^q](d\omega).
\]

We now illustrate Theorem 4.1 for some concrete examples.

4.2. **Traces in low homological degrees.** Let \(x_1, \ldots, x_N\) be a basis for \(W\) and let \(dx_1, \ldots, dx_N\) denote the corresponding basis elements in \(W[1]\) (i.e., \(dx_i := sx_i\)). In this case, the de Rham coalgebra of \(C\) is \(\text{Sym}^\circ(W[1]) \otimes \text{Sym}^\circ(W[2])\) equipped with the de Rham differential of \(A\), with the difference between \(\text{DR}_\bullet(C)\) and \(\text{DR}_\bullet(A)\) being the interpretation of the generators of \(A\) as degree 2 cogenerators of \(\text{DR}_\bullet(C)\) rather than degree 0 generators of \(A\). Let \(f(x_1, \ldots, x_N)\) be a homogenous polynomial of degree \(r\) in \(x_1, \ldots, x_N\). For notational brevity, the element \(f(x_1, \ldots, x_N)dx_{i_1} \ldots dx_{i_p}\) of \(\text{DR}_\bullet(A)\) (which is of cohomological degree \(p\) and is in \(\Omega^p_{A}\)) shall continue to be denoted by \(f(x_1, \ldots, x_N)dx_{i_1} \ldots dx_{i_p}\) when viewed as an element of \(\Omega^\bullet_{C} \subset \text{DR}_\bullet(C)\) (where its homological degree is \(2r + p\)). Let \(R := \Omega(C)\). Recall that we denote the element \(s^{-1}(dv_1 \ldots dv_p) \in R_{ab}\) by \(\lambda(v_1, \ldots, v_p)\) for \(v_1, \ldots, v_p \in W\). In what follows, let \(g := g_1 = k\). Choose the element 1 as the basis as well as the dual basis of \(g\). With these choices, \((3.12)\) becomes

\[
\theta(f(x_1, \ldots, x_N)dx_{i_1} \ldots dx_{i_p}) = f(0, \ldots, 0)\lambda(x_{i_1}, \ldots, x_{i_p}).
\]

Similarly, \((3.13)\) becomes

\[
\Omega(f(x_1, \ldots, x_N)dx_{i_1} \ldots dx_{i_p}) = \begin{cases} 
\lambda(f, x_{i_1}, \ldots, x_{i_p}) & \text{if } f \in W \\
0 & \text{if } f \notin W
\end{cases}
\]

4.2.1. **Homological degree 0.** Let \(f(x_1, \ldots, x_N)\) have the form \(u_1 \ldots u_r du_{r+1}\), where \(u_1, \ldots, u_r \in W\). By \((4.8)\), we need to evaluate \([\theta\Omega^q](u_1 \ldots u_r du_{r+1})\). Note that

\[
\Delta^{q+1}(u_1 \ldots u_r du_{r+1}) = \sum \pm u_{S_1} du_{T_1} \otimes \cdots \otimes u_{S_{q+1}} du_{T_{q+1}},
\]

where the summation above runs over \(S_1 \subset \ldots \subset S_{q+1} = \{1, \ldots, r\}\) and \(T_1 \subset \ldots \subset T_{q+1} = \{r + 1\}\). Hence,

\[
[\theta\Omega^q](u_1 \ldots u_r du_{r+1}) = \sum \pm \theta(u_{S_1} du_{T_1}) \Omega(u_{S_2} du_{T_2}) \cdots \Omega(u_{S_{q+1}} du_{T_{q+1}}).
\]

It follows from \((4.9)\) and \((4.10)\) that the only summands contributing to the R.H.S of \((4.11)\) are those for which \(S_1 = \emptyset, |S_2| = \ldots = |S_{q+1}| = 1\) and \(T_1 \neq \emptyset\). Hence, the R.H.S of \((4.11)\) is nonzero only when \(q = r\), in which case it equals

\[
r!u_1 \ldots u_{r+1} = r!\varepsilon(u_1 \ldots u_r du_{r+1}),
\]
where $\iota_\epsilon$ denotes contraction with the Euler vector field $\epsilon := \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$. Therefore, for $f$ homogenous in $A$ of degree $r + 1$, $r \geq 0$,

$$
\text{Tr}(A)(f) = \frac{1}{(r+1)!} [\theta, \Omega^r](df) = \frac{1}{r+1} \iota_\epsilon(df) = f. \tag{4.12}
$$

4.2.2. Homological degree 1. Let $\omega = \sum_{i=1}^N f_i dx_i$ where the coefficients $f_i$ are homogenous of degree $r + 1$. Note that the summands of $d\omega$ are of the form $u_1 \ldots u_r du_{r+1} du_{r+2}$, where $u_1, \ldots, u_{r+2} \in W$. By (4.8), we need to evaluate $[\theta, \Omega^q](u_1 \ldots u_r du_{r+1} du_{r+2})$. Note that

$$
\Delta^{q+1}(u_1 \ldots u_r du_{r+1} du_{r+2}) = \sum \pm u_{s_1} du_{t_1} \otimes \ldots \otimes u_{s_{q+1}} du_{t_{q+1}},
$$

where the summation above runs over $S_1 \sqcup \ldots \sqcup S_{q+1} = \{1, \ldots, r\}$ and $T_1 \sqcup \ldots \sqcup T_{q+1} = \{r+1, r+2\}$. Hence,

$$
[\theta, \Omega^q](u_1 \ldots u_r du_{r+1} du_{r+2}) = \sum \pm \theta(u_{s_1} du_{t_1}) \Omega(u_{s_2} du_{t_2}) \ldots \Omega(u_{s_{q+1}} du_{t_{q+1}}). \tag{4.13}
$$

It follows from (4.9) and (4.10) that the only summands contributing to the R.H.S of (4.13) are those for which $S_1 = \emptyset$, $|S_2| = \ldots = |S_{q+1}| = 1$ and $T_1 \neq \emptyset$. Hence, the R.H.S of (4.13) is nonzero only when $q = r$. In this case, the summands for which $T_1 = \{r+1, r+2\}$ contribute

$$
r! \lambda(u_{r+1}, u_{r+2}) u_1 \ldots u_r =: r! s^{-1}(u_1 \ldots u_r du_{r+1} du_{r+2})
$$
to $[\theta, \Omega^r](u_1 \ldots u_r du_{r+1} du_{r+2})$. The summands for which $|T_1| = 1$ together add up to

$$
r! \left( \sum_{p=1}^r u_1 \ldots u_{r+1} \lambda(u_p, u_{r+2}) - u_1 \ldots u_{r+1} \lambda(u_p, u_{r+1}) \right)
= r! s^{-1}(d\iota_\epsilon - 2)(u_1 \ldots u_r du_{r+1} du_{r+2}).
$$

Hence,

$$
\text{Tr}(A)(\omega) = \frac{1}{(r+1)!} [\theta, \Omega^r](d\omega) = \frac{1}{(r+1)!} [r! s^{-1} d\omega + r! s^{-1}(d\iota_\epsilon - 2)(d\omega)]
= \frac{1}{(r+1)!} [r! s^{-1} d\omega + r! s^{-1}(d\iota_\epsilon + \iota_\epsilon d - 2)(d\omega)]
= \frac{1}{(r+1)!} [r! s^{-1} d\omega + r! s^{-1} r(d\omega)]
= s^{-1} d\omega.
$$

4.2.3. Homological degree 2. Let $\omega = \sum_{i<j} f_{ij} dx_i dx_j$ be a two-form whose coefficients $f_{ij}$ are homogenous polynomials of degree $r + 1$. Note that the summands of $d\omega$ are of the form $u_1 \ldots u_r du_{r+1} du_{r+2} du_{r+3}$, where $u_1, \ldots, u_{r+3} \in W$. By (4.8), we need to evaluate $[\theta, \Omega^q](u_1 \ldots u_r du_{r+1} du_{r+2} du_{r+3})$. Note that

$$
\Delta^{q+1}(u_1 \ldots u_r du_{r+1} du_{r+2} du_{r+3}) = \sum \pm u_{s_1} du_{t_1} \otimes \ldots \otimes u_{s_{q+1}} du_{t_{q+1}},
$$

where the summation above runs over $S_1 \sqcup \ldots \sqcup S_{q+1} = \{1, \ldots, r\}$ and $T_1 \sqcup \ldots \sqcup T_{q+1} = \{r+1, r+2, r+3\}$. Hence,

$$
[\theta, \Omega^q](u_1 \ldots u_r du_{r+1} du_{r+2} du_{r+3}) = \sum \pm \theta(u_{s_1} du_{t_1}) \Omega(u_{s_2} du_{t_2}) \ldots \Omega(u_{s_{q+1}} du_{t_{q+1}}). \tag{4.14}
$$
It follows from (4.9) and (4.10) that the only summands contributing to the R.H.S of (4.14) are those for which $S_1 = \emptyset$, $|S_2| = \ldots = |S_{q+1}| = 1$ and $T_1 \neq \emptyset$. Hence, the R.H.S of (4.14) is nonzero only when $q = r$. The summands for which $T_1 = \{r+1, r+2, r+3\}$ contribute

$$r!s^{-1}(u_1 \ldots u_r du_{r+1} du_{r+2} du_{r+3})$$

to the R.H.S of (4.14). Similarly, the summands for which $|T_1| = 1$ and one of $|T_2|, \ldots, |T_{r+1}|$ is 2 add up to

$$r!s^{-1} \sum_{p=1}^{r} u_1 \ldots \hat{u}_p \ldots u_r (u_{r+1} \lambda(u_p, u_{r+2}, u_{r+3}) - u_{r+2} \lambda(u_p, u_{r+1}, u_{r+3}) + u_{r+3} \lambda(u_p, u_{r+1}, u_{r+2}))$$

$$= r!s^{-1}(dt_e - 3)(u_1 \ldots u_r du_{r+1} \ldots du_{r+3}).$$

Similarly, the summands for which $|T_1| = 2$ add up to $-2r! \cdot D^{(2,2)}(u_1 \ldots u_r du_{r+1} \ldots du_{r+3})$, where

$$D^{(2,2)}(u_1 \ldots u_r du_{r+1} \ldots du_{r+3}) := \sum_{1 \leq k \neq l \leq r} u_1 \ldots \hat{u}_k \ldots \hat{u}_l \ldots u_r [u_{r+3} \lambda(u_k, u_{r+1}) \lambda(u_l, u_{r+2}) + u_{r+2} \lambda(u_k, u_{r+1}) \lambda(u_l, u_{r+3}) + u_{r+1} \lambda(u_k, u_{r+2}) \lambda(u_l, u_{r+3})]$$

Finally the summands for which $|T_1| = 1$ and two of $|T_2|, \ldots, |T_{r+1}|$ are 1 add up to $r! \hat{D}^{(2,2)}(u_1 \ldots u_r du_{r+1} \ldots du_{r+3})$ where

$$\hat{D}^{(2,2)}(u_1 \ldots u_r du_{r+1} \ldots du_{r+3}) := \sum_{1 \leq k \neq l \leq r} u_1 \ldots \hat{u}_k \ldots \hat{u}_l \ldots u_r [u_{r+3} \lambda(u_k, u_{r+1}) \lambda(u_l, u_{r+2}) + u_{r+2} \lambda(u_k, u_{r+1}) \lambda(u_l, u_{r+3}) + u_{r+1} \lambda(u_k, u_{r+2}) \lambda(u_l, u_{r+3})]$$

A direct computation shows that

$$(4.15) \hat{D}^{(2,2)} \circ d = -(r - 1)D^{(2,2)} \circ d$$

on 2-forms with polynomial coefficients that are homogenous of degree $r + 1$. Hence,

$$\mathbf{Tr}(A)(\omega) = \frac{1}{(r + 1)!}[\theta, \Omega^r](d\omega)$$

$$= \frac{1}{(r + 1)!}[r!s^{-1}d\omega + r!s^{-1}(dt_e - 3)(d\omega) - 2r!D^{(2,2)}(d\omega) + r!\hat{D}^{(2,2)}(d\omega)]$$

$$= \frac{1}{(r + 1)!}[r!s^{-1}d\omega + r!s^{-1}(dt_e + r_d - 3)(d\omega) - 2r!D^{(2,2)}(d\omega) - (r - 1)r!D^{(2,2)}(d\omega)]$$

$$= \frac{1}{(r + 1)!}[r!s^{-1}d\omega + r!s^{-1}r(d\omega) - (r + 1)r!D^{(2,2)}(d\omega)]$$

$$= s^{-1}d\omega - D^{(2,2)}(d\omega).$$

4.3. Traces as differential operators.

4.3.1. We aim to provide a formula for reduced traces of $A = \text{Sym}(W)$ in arbitrary homological degree. To this end, for each $1 \leq p \leq k \leq \dim(W)$, let us define a linear map

$$(4.16) W \otimes \Lambda^k(W) \to \Lambda^p(W) \otimes \Lambda^{k+1-p}(W)$$

by

$$u \otimes v_1 \wedge \ldots \wedge v_k \mapsto \frac{1}{p!} \sum_{j_1 < \ldots < j_{p-1}} (-1)^{\sum j_i - \binom{p-1}{p-2}} (u \wedge v_{j_1} \wedge \ldots \wedge v_{j_{p-1}}) \otimes (v_1 \wedge \ldots \wedge \hat{v}_{j_1} \wedge \ldots \wedge \hat{v}_{j_{p-1}} \wedge \ldots \wedge v_k)$$
with convention that this is the identity map if \( p = 1 \). By duality, \([14,16]\) gives a canonical map
\[
(4.17) \quad \Delta^{(p,k+1-p)}_{k} : \Lambda^{k}(W) \to W^{*} \otimes \Lambda^{p}(W) \otimes \Lambda^{k+1-p}(W).
\]
Now, for any multi-index \((i_1, \ldots, i_m) \in \mathbb{N}^m\) such that \(i_1 + \ldots + i_m = k + m - 1\), we can construct
\[
(4.18) \quad \Delta^{(i_1,\ldots,i_m)}_{k} : \Lambda^{k}(W) \to \text{Sym}^{m-1}(W^{*}) \otimes \Lambda^{i_1}(W) \otimes \ldots \otimes \Lambda^{i_m}(W)
\]
by iterating \([1.17]\):
\[
\Lambda^{k}(W) \to W^{*} \otimes \Lambda^{k+1-i_m}(W) \otimes \Lambda^{i_m}(W) \to \ldots \to (W^{*})^{\otimes m-1} \otimes \Lambda^{i_1}(W) \otimes \ldots \otimes \Lambda^{i_m}(W)
\]
and then projecting \((W^{*})^{\otimes m-1} \to \text{Sym}^{m-1}(W^{*})\).

Finally, interpreting the elements \(\text{Sym}(W^{*})\) as constant coefficient differential operators on \(\text{Sym}(W)\), we define the following differential operator on forms
\[
(4.19) \quad D^{(i_1,\ldots,i_m)}_{k} : \text{Sym}(W) \otimes \Lambda^{k}W \xrightarrow{1 \otimes \Delta^{(i_1,\ldots,i_m)}} \text{Sym}(W) \otimes \text{Sym}^{m-1}(W^{*}) \otimes \Lambda^{i_1}(W) \otimes \ldots \otimes \Lambda^{i_m}(W) \xrightarrow{\text{act} \otimes s^{-1} \otimes \text{act}} R_{ab}
\]
where \(\text{act} : \text{Sym}^{m-1}(W^{*}) \otimes \text{Sym}(W) \to \text{Sym}(W) \mapsto R_{ab}\) is the action map\(^7\) and \(s^{-1}\) is the embedding defined by \(s^{-1}(v_1 \wedge \ldots \wedge v_k) := \lambda(v_1, \ldots, v_k)\).

For example, the first order differential operator \(D^{(p,k+1-p)}_{k} : \Omega^{k}(A) \to R_{ab}\) is explicitly given by
\[
(u_1 \ldots u_n) dv_1 \ldots dv_k \mapsto \frac{1}{p!} \sum_{i=1}^{n} (u_1 \ldots \hat{u}_i \ldots u_n) \sum_{j_1 < \ldots < j_{p-1}} \pm \lambda(u_i, v_{j_1}, \ldots, v_{j_{p-1}}) \lambda(v_1, \ldots, \hat{v}_{j_1}, \ldots, \hat{v}_{j_{p-1}}, \ldots, v_k),
\]
where the sign \(\pm\) is given by \((-1)^{\sum_{j} \binom{j_{p-1}}{2}}\) for \(j_1 < \ldots < j_{p-1}\). In particular,
\[
D^{(2,2)}[(u_1 \ldots u_n)dv_1 dv_2 dv_3] = \frac{1}{2} \sum_{i=1}^{n} (u_1 \ldots \hat{u}_i \ldots u_n) [\lambda(u_i, v_1) \lambda(v_2, v_3) - \lambda(u_i, v_2) \lambda(v_1, v_3) + \lambda(u_i, v_3) \lambda(v_1, v_2)]
\]

4.3.2. The trace formula. In general, we have

**Theorem 4.2.** Let \(\omega \in \text{Sym}^{r+1}(W) \otimes \Lambda^{l}(W) \subset \Omega^{l}_{A}\) be a l-form with homogeneous polynomial coefficients of degree \(r+1\). Then,
\[
\text{Tr}(A)(\omega) = s^{-1}(d\omega) + \sum_{i_1 + \ldots + i_m = l+m} c^{(i_1,\ldots,i_m)} D^{(i_1,\ldots,i_m)}(d\omega),
\]
where the sum runs over all tuples \((i_1, \ldots, i_m)\) such that \(i_1, \ldots, i_{m-1} \geq 2\) and \(i_m \geq 1\) adding up to \(l + m\) and where \((r+1)c^{(i_1,\ldots,i_m)}\) depends only on \(i_1, \ldots, i_m\).

**Proof.** By \([4.38]\),
\[
\text{Tr}(A)(\omega) = \sum_{q=0}^{\infty} \frac{1}{(q+1)!} [\theta \Omega^{q}](d\omega).
\]
Note that \(d\omega\) is a \(k\)-linear combination of summands of the form \(u_1 \ldots u_r du_{r+1} \ldots du_{r+l+1}\), where \(u_1, \ldots, u_{r+l+1} \in W\). Further observe that
\[
\Delta^{q+1}(u_1 \ldots u_r du_{r+1} \ldots du_{r+l+1}) = \sum \pm u_{S_1} du_{T_1} \otimes \ldots \otimes u_{S_{q+1}} du_{T_{q+1}},
\]

\(^7\)View \(\text{Sym}(W) \otimes \Lambda^{k}(W)\) as the module of sections of the bundle \(\Omega^{k}_{X}\), where \(X := \text{Spec}[\text{Sym}(W)]\). Similarly view \(R_{ab}\) as the module of sections of the corresponding (homologically graded) vector bundle \(\mathcal{F}\) on \(X\). By construction, \(D^{(i_1,\ldots,i_m)}_{k}\) can indeed be viewed as a global section of \(D^{<m}_{X} \otimes \text{Hom}(\Omega^{k}_{X}, \mathcal{F})\), where \(D^{<m}_{X}\) is the sheaf of differential operators of order less than \(m\) on \(X\).
where the summation above runs over $S_1 \sqcup \ldots \sqcup S_{q+1} = \{1, \ldots, r\}$ and $T_1 \sqcup \ldots \sqcup T_{q+1} = \{r+1, \ldots, r+l+1\}$. Hence,

$$[\theta, \Omega'](u_1 \ldots u_r du_{r+1} \ldots du_{r+l+1}) = \sum \pm \theta(u_{S_1} du_{T_1}) \Omega(u_{S_2} du_{T_2}) \ldots \Omega(u_{S_{q+1}} du_{T_{q+1}}).$$

It follows from (4.3) and (4.10) that the only summands contributing to the R.H.S of (4.20) are those for which $S_1 = \emptyset, |S_2| = \ldots = |S_{q+1}| = 1$ and $T_1 \neq \emptyset$. Hence, the R.H.S of (4.20) is nonzero only when $q = r$. Given any tuple $(i_1, \ldots, i_m)$ with $i_1, \ldots, i_{m-1} \geq 2$ and $i_m \geq 1$ such that $i_1 + \ldots + i_m = l + m$, the summands on the R.H.S of (4.20) with $|T_1| = i_m$ and $m - 1$ among $|T_2|, \ldots, |T_{r+1}|$ being equal to $i_1, \ldots, i_{m-1}$ contribute

$$r! \hat{D}^{(i_1, \ldots, i_m)}(u_1 \ldots u_r du_{r+1} \ldots du_{r+l+1}),$$

where

$$\hat{D}^{(i_1, \ldots, i_m)} = \hat{c}^{(i_1, \ldots, i_m)} D^{(i_1, \ldots, i_m)},$$

where the constant $\hat{c}^{(i_1, \ldots, i_m)}$ depends only on $(i_1, \ldots, i_m)$. The summands on the R.H.S for which $|T_1| = l + 1$ contribute $r! s^{-1}(u_1 \ldots u_r du_{r+1} \ldots du_{r+l+1})$. Similarly,

$$\hat{D}^{(i_1,1)}(u_1 \ldots u_r du_{r+1} \ldots du_{r+l+1}) = \sum_{i=1}^{r} \sum_{j=1}^{l+1} (-1)^{j-1} u_1 \ldots u_r u_{r+j} s^{-1}(du_{r+1} \ldots du_{r+j} \ldots du_{r+l+1})$$

$$= s^{-1}(d \epsilon - l - 1)(u_1 \ldots u_r du_{r+1} \ldots du_{r+l+1}).$$

It follows that

$$\frac{1}{(r+1)!} [\theta, \Omega'](\eta) = \frac{1}{r+1} s^{-1}(\eta) + \frac{1}{r+1} s^{-1}(d \epsilon - l - 1)(\eta) + \sum_{i_1 + \ldots + i_m = l + m} \frac{1}{r+1} \hat{D}^{(i_1, \ldots, i_m)}(\eta)$$

$$= \frac{1}{r+1} s^{-1}(\eta) + \frac{1}{r+1} s^{-1}(d \epsilon - l - 1)(\eta) + \sum_{i_1 + \ldots + i_m = l + m} \frac{1}{r+1} \hat{c}^{(i_1, \ldots, i_m)} D^{(i_1, \ldots, i_m)}(\eta),$$

for any $\eta \in \text{Sym}^r(W) \otimes \Lambda^{l+1}(W) \subset \Omega^{l+1}$. Hence, for $\omega \in \text{Sym}^{r+1}(W) \otimes \Lambda^l(W)$,

$$\frac{1}{(r+1)!} [\theta, \Omega'](d \omega) = \frac{1}{r+1} s^{-1}(d \omega) + \frac{1}{r+1} s^{-1}(d \epsilon - l - 1)(d \omega) + \sum_{i_1 + \ldots + i_m = l + m} \frac{1}{r+1} \hat{c}^{(i_1, \ldots, i_m)} D^{(i_1, \ldots, i_m)}(d \omega)$$

$$= s^{-1}(d \omega) + \sum_{i_1 + \ldots + i_m = l + m} \frac{1}{r+1} \hat{c}^{(i_1, \ldots, i_m)} D^{(i_1, \ldots, i_m)}(d \omega).$$

This proves the desired result. \square

**Remark.** Comparing the formula in Theorem 4.2 with the computation in Section 4.2.3 when $l = 2$, we see that $c^{(2,2)} = \frac{1}{r+1}$ and $c^{(2,2,1)} \neq 0$. However, since $D^{(2,2,1)}(d \omega) = -(r+1) D^{(2,2)}(d \omega)$, the summands $\frac{1}{r+1} D^{(2,2)}(d \omega)$ and $\frac{1}{r+1} D^{(2,2,1)}(d \omega)$ of $\Theta(A)(\omega)$ add up to $- D^{(2,2)}(d \omega)$. 

4.4. **Examples.** We illustrate the formulas of Section 4.2 for polynomial algebras in two and three variables.
4.4.1. Polynomials of two variables. Let \( \dim(W) = 2 \). Choose a basis in \( W \) and identify \( A = k[x, y] \). Then \( R = k(x, y, t) \) with \( \deg x = \deg y = 0 \) and \( \deg t = 1 \). The differential on \( R \) is defined by \( \delta t = [x, y] \), so that \( t = -\lambda(x, y) \), cf. (2.4). Section 4.2.2 says that \( \text{Tr}(A) : \Omega_A^1 \to R_{ab} \) is given by

\[
\text{Tr}(A)[P dx + Q dy] = s^{-1}[(Q_x - P_y)dxdy] = (Q_x - P_y)\lambda(x, y) = (P_y - Q_x)t.
\]

This formula can also be obtained directly from the explicit formulas of [3, Ex. 4.1].

4.4.2. Polynomials of three variables. Let \( A = k[x, y, z] \). Using the notation of [2, Ex. 6.3.2], we write the minimal resolution of \( A \) in the form \( R = k(x, y, z, \xi, \theta, t) \), where \( \deg x = \deg y = \deg z = 0 \), \( \deg \xi = \deg \theta = \deg \lambda = 1 \) and \( \deg t = 2 \). The differential on \( R \) is defined by

\[
\delta \xi = [y, z], \quad \delta \theta = [z, x], \quad \delta \lambda = [x, y]; \quad \delta t = [x, \xi] + [y, \theta] + [z, \lambda]
\]

Comparing with (1.2) we see that

\[
\xi = \lambda(z, y), \quad \theta = \lambda(x, z), \quad \lambda = \lambda(y, x), \quad t = \lambda(x, y, z).
\]

By Section 4.2.2 \( \text{Tr}(A) : \Omega_A^1 \to R_{ab} \) is given by

\[
\text{Tr}(A)[P dx + Q dy + R dz] = s^{-1}[(Q_x - P_y)dxdy + (R_y - Q_z)dydz + (P_z - R_x)dzdx] = (Q_x - P_y)\lambda(x, y) + (R_y - Q_z)\lambda(y, z) + (P_z - R_x)\lambda(z, x) = (P_y - Q_x)\lambda + (Q_z - R_y)\xi + (R_x - P_z)\theta.
\]

Next, to compute \( \text{Tr}(A)_2 \) we take \( \omega \in \Omega^2(A) \) in the form

\[
\omega = P dx dy + Q dy dz + R dz dx.
\]

The trace formula in Section 4.2.3 implies (after a tedious but straightforward calculation) that

\[
\text{Tr}[\omega] = (P_z + Q_x + R_y)t + (P_z + Q_x + R_y)x \theta \lambda + (P_z + Q_x + R_y)z \lambda \xi + (P_z + Q_x + R_y)z \xi \theta.
\]

5. Reduced traces: a combinatorial description

In this section, we will give another formula for reduced trace maps of symmetric algebras in terms of binary trees. Throughout, we will keep the notation and assumptions of the previous section.

Our starting point is Theorem 4.2 of [3] that gives a general formula for the derived character maps in terms of Taylor components \( f_{k+1} : A^\otimes(k+1) \to R \) of an \( A_\infty \)-quasi-isomorphism \( f : A \to R \) inverting a DG algebra resolution of \( A \). We apply this formula to the minimal resolution \( R := \Omega(C) \) of the symmetric algebra \( A = \text{Sym}(W) \); as a consequence, we get the following

Proposition 5.1. The map \( \text{Tr}(A) : \Omega_A^*/d\Omega_A^{*−1} \to R_{ab} \) is given by the formula

\[
(5.1) \quad \text{Tr}(A)[a_0 da_1 \ldots da_k] = \sum_{\sigma \in S_{k+1}} (-1)^\sigma \tilde{f}_{k+1}(a_{\sigma(0)}, a_{\sigma(1)}, \ldots, a_{\sigma(k)}) ,
\]

where \( \tilde{f}_{k+1} : A^\otimes(k+1) \to R \to R_{ab} \) are the components of the \( A_\infty \)-morphism \( A \xrightarrow{T} R \to R_{ab} \).

Proof. \( \text{Tr}(A) \) is induced by the composition \( \tilde{T} : \Omega_A^*/d\Omega_A^{*−1} \xrightarrow{T} \Omega_A^*/d\Omega_A^{*−1} \xrightarrow{\text{can}} R_{ab} \). Here, \( T \) is as in (2.5). By [3 (2.41)], \( \tilde{T} \) is given by

\[
\sum_{\tau \in \mathcal{E}_{k+1}} (-1)^\tau \sum_{\sigma \in S_k} (-1)^\sigma \tilde{f}_{k+1}(a_{\tau\sigma(0)}, a_{\tau\sigma(1)}, \ldots, a_{\tau\sigma(k)}) ,
\]
where $\sigma$ ranges over the subgroup of permutations of $\{0,1,\ldots,k\}$ preserving 0 and $\tau$ ranges over the cyclic subgroup $\mathbb{Z}_{k+1}$ of $\mathbb{S}_{k+1}$ generated by $i \mapsto i + 1$. Since $\mathbb{S}_{k+1}$ is the product of its subgroups $\mathbb{S}_k$ and $\mathbb{Z}_{k+1}$, the above sum equals the right-hand side of (5.1). \hfill $\Box$

5.1. Merkulov’s construction. Let $\pi : R \xrightarrow{\sim} A$ be a fixed semi-free resolution. Choose a linear section $f_1$ of $\pi$ and identify $A$ with its image in $R$ under $f_1$. Since $R$ is quasi-isomorphic to $A$, the complex $R_\bullet$ is acyclic in all degrees $\geq 1$. For each $i \geq 0$, we fix a decomposition of $R_i$ such that $R_0 = A \oplus B_0$ and $R_i = B_i \oplus L_i$ for $i \geq 1$. Here $B_i = d_{i+1}(R_{i+1}) \subseteq R_i$, and $L_i = s_i(R_i/B_i) \subseteq R_i$, where $s_i : R_i/B_i \hookrightarrow R_i$ is a section of the canonical projection $p_i : R_i \twoheadrightarrow R_i/B_i$.

Next, we pick a homotopy $h : R \to R[1]$ between the maps $Id_R$ and $f_1 \circ \pi$ satisfying $h_1|_{L_i} = 0$, $h_0|_{A} = 0$ and $h_1|_{B_i} : B_i \xrightarrow{\sim} L_{i+1}$. One can construct the components $h_i : R_i \to R_{i+1}$ of $h$ inductively:

Since $d_0 = 0$, the equation for $h_0$ simply is

$$(5.2) \quad d_1h_0 = Id_R - f_1\pi.$$ 

For $n \geq 1$, we have $\pi|_{R_n} \equiv 0$. Hence $h_n$ is defined by

$$(5.3) \quad d_{n+1}h_n = Id_R - h_{n-1}d_n.$$ 

Now, given $h : R \to R[1]$, for $i \geq 1$ we define the operations $\mu_i : R^{\otimes i} \to R$ by

- There is no $\mu_1$, but we formally set $h\mu_1 := -Id_R$;
- $\mu_2 : R \otimes R \to R$ is the multiplication map $\mu_2(a_1 \otimes a_2) = a_1a_2$;
- For $i \geq 2$, $\mu_i$ is a map of degree $i - 2$ defined by

$$(5.4) \quad \mu_i := \sum_{s+t=i \atop s,t \geq 1} (-1)^{s+1} \mu_2(h\mu_s \otimes h\mu_t).$$

Finally, for $k \geq 1$, we define $f_{k+1} : A^{\otimes (k+1)} \to R$ by

$$(5.5) \quad f_{k+1} := -h_{k-1} \circ \mu_{k+1} \circ f_1^{\otimes (k+1)}.$$ 

The following observation is due to Merkulov [20] (see also [18] Prop. 2.3, Lemma 2.5). 

Theorem 5.1. The maps (5.5) define an $A_{\infty}$-quasi-isomorphism $f : A \to R$ inverse to $\pi$.

Remark. In general, if $R$ is any DG algebra, the above construction also yields a (minimal) $A_{\infty}$ structure on $H_\bullet(R)$. The corresponding higher multiplications are defined by $m_k := \pi \circ \mu_k \circ f_1^{\otimes (k+1)}$, $k \geq 2$. In the case when $H_\bullet(R) = A$ is an ordinary algebra, the operations $m_3, m_4, \ldots$ are trivial for degree reasons, while $m_2$ coincides with the induced multiplication on $A$, since $m_2(a_1, a_2) = \pi(f_1(a_1)f_1(a_2)) = \pi(f_1(a_1))\pi(f_1(a_2)) = a_1a_2$.

5.2. Traces and binary trees. Substituting (5.4) into formula (5.1) we get

$$(5.6) \quad \text{T}(a_0a_1 \ldots a_k) = \sum_{\sigma \in \mathbb{S}_{k+1}} (-1)^{1+\sigma} h_{k-1} \mu_{k+1} (f_1(a_{\sigma(0)}), \ldots, f_1(a_{\sigma(k)}))$$

Merkulov’s construction provides us with the recursive formula (5.4) for $\mu_{k+1}$, and hence for $f_{k+1}$, in terms of operations $\mu_i$ with $i < k + 1$ and homotopy $h$.

By $\Sigma_k$ we denote the set of rooted planar binary trees with $k + 1$ leaves. Operation $f_T$ for a tree $T \in \Sigma_k$ is defined in the following way. First of all, we will label all the leaves and internal vertices in the following way. Every leaf we will label by 0. After that, if a vertex $v$ has left son with label $l$ and right son with label $r$, then we label $v$ by $l + r + 1$. After that, we insert $f_1$ into each leaf, and if a vertex $v$ was labeled by some number
l, we insert $h_{l-1}µ_2$ into $v$. In the very last vertex (the one that is adjacent with the root) we insert $-h_{k-1}µ_2$. Then moving along the tree down to the root we can read off the map $f_{k+1}$. For example, $f_T$ for the trees

$$T_1 = \quad \quad \quad T_2 = \quad \quad \quad$$

will be just $f_{T_1}(a_0, a_1, a_2) = -h_1µ_2(f_1(a_0) ⊗ h_0µ_2(f_1(a_1) ⊗ f_1(a_2))) = -h_1(\tilde{a}_0 · h_0(\tilde{a}_1 · \tilde{a}_2))$ and $f_{T_2}(a_0, a_1, a_2) = -h_3(h_0(\tilde{a}_0 · \tilde{a}_1) · \tilde{a}_2)$.

There is an algorithm how to determine the sign $(-1)^T$ that corresponds to a tree $T$. First we label leaves by ‘+1’. After that, for any vertex $v$ that has left son labeled by a sign $l$ and right son labeled by $r$, we label $v$ by $l · r · (-1)^{s+1}$, where $s$ is the number of leaves to the left from $v$. Then the sign of the tree $(-1)^T$ is by definition the sign of the last vertex (the one that is adjacent with the root). For example, for the trees $T_1$ and $T_2$ above we will have $(-1)^{T_1} = 1$ and $(-1)^{T_2} = -1$.

The construction defined above (almost) coincides with the construction given in [15 Sect. 6.4]. The only difference is that we expanded all higher multiplications $µ_i$ with $i > 2$.

**Lemma 5.1 (cf. [15]).**

(5.7) \[ f_{k+1} = \sum_{T ∈ S_k} (-1)^T f_T \]

**Proof.** The proof can be obtained by easy induction on the number of vertices. □

Now if we apply the result of the lemma 5.1 to the formula (5.6) we will get the following expression for the reduced trace:

(5.8) \[ \overline{\text{Tr}}(a_0da_1 \ldots da_k) = \sum_{σ ∈ S_{k+1}} (-1)^σ \sum_{T ∈ S_k} (-1)^T \overline{f}_T (\tilde{a}_σ(0), \ldots, \tilde{a}_σ(k)). \]

Two labeled planar rooted binary trees $(σ, T)$ and $(σ', T')$ are equivalent if there exists a rooted tree isomorphism $φ : T → T'$ that preserves the labels on leaves (labeling is given by a choice of $σ ∈ S_{k+1}$ which we think of as labels on $k + 1$ leaves). Let’s denote the set of equivalence classes of pairs $(σ, T)$ by $⟨S⟩_k$.

For any tree $T$ define $[f]_T$ to be a map, obtained from $f_T$ by replacing any $µ_2(a ⊗ b)$ appearing in $f_T$ by $[a, b]$. For example, if $f_T(a_0, a_1, a_2) = -h_3(h_0(\tilde{a}_0 · \tilde{a}_1) · \tilde{a}_2)$, then $[f]_T = -h_3[h_0(\tilde{a}_0, \tilde{a}_1), \tilde{a}_2]$.

**Lemma 5.2.**

(5.9) \[ \overline{\text{Tr}}(a_0da_1 \ldots da_k) = \sum_{[σ, T] ∈ ⟨S⟩_k} (-1)^{σ_0} · (-1)^{T_0} [f]_{T_0}(\tilde{a}_{σ_0(0)}, \ldots, \tilde{a}_{σ_0(k)}). \]

Here $(σ_0, T_0)$ is a representative of the class $[σ, T]$.

**Proof.** Straightforward induction on the number of vertices. □

As an example, let us consider the case $k = 3$. There are 5 elements in $S_3$:

$$T_1 = \quad \quad \quad T_2 = \quad \quad \quad T_3 = \quad \quad \quad T_4 = \quad \quad \quad T_5 = \quad \quad \quad$$

Their signs are going to be $(-1)^{T_1} = -1$, $(-1)^{T_2} = +1$, $(-1)^{T_3} = -1$, $(-1)^{T_4} = +1$ and $(-1)^{T_5} = -1$. 

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There are 15 equivalence classes in \((\Sigma)_3\). There are 3 classes \([[(\sigma,T)]]\) with \(T = T_5\) and \(\sigma \in \Sigma_1 := \{(0123), (0213), (0312)\}\). There are 12 classes \([[(\sigma,T)]]\) with \(T = T_4\) and \(\sigma \in \Sigma_2 = \{((\sigma(0), \sigma(1), \sigma(2), \sigma(3)) \in \Sigma_1 | \sigma(2) < \sigma(3)\}\). So for \((T_3, \Sigma_3)\), we will have the following explicit formula

\[
\mathcal{Tr}(a_0 a_1 a_2 a_3) = \sum_{\sigma \in \Sigma_1} (-1)^\sigma \tilde{h}_2 \left[ h_0 [\tilde{a}_{\sigma(0)}, \tilde{a}_{\sigma(1)}], h_0 [\tilde{a}_{\sigma(2)}, \tilde{a}_{\sigma(3)}] \right]
+ \sum_{\sigma \in \Sigma_2} (-1)^{1+\sigma} \tilde{h}_2 \left[ \tilde{a}_{\sigma(0)}, h_1 [\tilde{a}_{\sigma(1)}, h_0 [\tilde{a}_{\sigma(2)}, \tilde{a}_{\sigma(3)}]] \right]
\]

More explicitly,

\[
\mathcal{Tr}(a_0 a_1 a_2 a_3) = \tilde{h}_2 \left[ h_0 [\tilde{a}_0, \tilde{a}_1], h_0 [\tilde{a}_2, \tilde{a}_3] \right] - \tilde{h}_2 \left[ h_0 [\tilde{a}_0, \tilde{a}_2], h_0 [\tilde{a}_1, \tilde{a}_3] \right] + \tilde{h}_2 \left[ h_0 [\tilde{a}_0, \tilde{a}_3], h_0 [\tilde{a}_1, \tilde{a}_2] \right]
- \tilde{h}_2 \left[ \tilde{a}_0, h_1 [\tilde{a}_1, h_0 [\tilde{a}_2, \tilde{a}_3]] \right] + \tilde{h}_2 \left[ \tilde{a}_0, h_1 [\tilde{a}_2, h_0 [\tilde{a}_1, \tilde{a}_3]] \right] - \tilde{h}_2 \left[ \tilde{a}_0, h_1 [\tilde{a}_3, h_0 [\tilde{a}_1, \tilde{a}_2]] \right]
+ \tilde{h}_2 \left[ \tilde{a}_1, h_1 [\tilde{a}_0, h_0 [\tilde{a}_2, \tilde{a}_3]] \right] - \tilde{h}_2 \left[ \tilde{a}_2, h_1 [\tilde{a}_0, h_0 [\tilde{a}_1, \tilde{a}_3]] \right] + \tilde{h}_2 \left[ \tilde{a}_3, h_1 [\tilde{a}_0, h_0 [\tilde{a}_1, \tilde{a}_2]] \right]
- \tilde{h}_2 \left[ \tilde{a}_1, h_1 [\tilde{a}_2, h_0 [\tilde{a}_0, \tilde{a}_3]] \right] + \tilde{h}_2 \left[ \tilde{a}_1, h_1 [\tilde{a}_3, h_0 [\tilde{a}_0, \tilde{a}_2]] \right] - \tilde{h}_2 \left[ \tilde{a}_2, h_1 [\tilde{a}_3, h_0 [\tilde{a}_0, \tilde{a}_1]] \right]
+ \tilde{h}_2 \left[ \tilde{a}_2, h_1 [\tilde{a}_1, h_0 [\tilde{a}_0, \tilde{a}_3]] \right] - \tilde{h}_2 \left[ \tilde{a}_3, h_1 [\tilde{a}_1, h_0 [\tilde{a}_0, \tilde{a}_2]] \right] + \tilde{h}_2 \left[ \tilde{a}_3, h_1 [\tilde{a}_2, h_0 [\tilde{a}_0, \tilde{a}_1]] \right]
\]

By Theorem 4.1 and Lemma 5.2 we have

**Corollary 5.1.**

\[
\sum_{[(\sigma,T)] \in (\Sigma)_k} (-1)^{\sigma_0} \cdot (-1)^{T_0} \left[ \tilde{h}_2 \left[ \tilde{a}_{\sigma_0(0)}, \ldots, \tilde{a}_{\sigma_0(k)} \right] \right] = \sum_{q=0}^{\infty} \frac{1}{(q+1)!} [\theta, \Omega^q](da_0 da_1 \ldots da_k).
\]

**APPENDIX A. CHERN-SIMONS FORMS, LIE AND CYCLIC HOMOLOGY**

We now give a detailed exposition of the construction of an additive analog of the Borel regulator map as outlined in [1, Sec. A.6]. We then compare this map with a related construction dual to the Drinfeld traces (see Theorem A.3 below).

**A.1. The convolution algebra.** Let \(A\) be a commutative DG algebra and let \(g\) be a finite-dimensional Lie algebra. Invert degrees to turn \(A\) into a cohomologically graded DG algebra. Then, the Chevalley-Eilenberg complex \(C(g(A);k)\) is a cocommutative (cohomologically graded) DG coalgebra. As a result, one has the commutative DG convolution algebra

\[
\mathcal{A} := \text{Hom}(C(g(A);k), \text{DR}_\bullet(A)).
\]

Note that as cohomologically graded algebras, \(\mathcal{A} \cong \bigoplus_{i,j} A^{i,j} \) where

\[
A^{i,j} := \text{Hom}(\wedge^i g(A), \Omega_A^j)[-i-j].
\]

Equip \(\mathcal{A}\) with the connection \(\theta \in (A^{0,1})^1 \otimes g \subset A^1 \otimes g\) given by the formula

\[
\theta(a \otimes \xi) = a \otimes \xi \in A \otimes g, \quad \forall a \in A, \xi \in g.
\]

The following proposition follows from a straightforward computation.

**Proposition A.1.** The curvature \(\Omega\) of \(\theta\) lies in the summand \(A^{1,1} \otimes g\) of \(A^2 \otimes g\). Explicitly,

\[
\Omega \in \text{Hom}_C(g(A)[1], \Omega_A^1[-1]), \quad \xi \otimes a \mapsto da \otimes \xi.
\]
Similarly, the element $[\theta, \theta] \in A^{0,2} \otimes g$ is given by

$$[\theta, \theta][(\xi_1 \otimes a_1) \wedge (\xi_2 \otimes a_2)] = -2(-1)^{|a_1|} a_1 a_2 \otimes [\xi_1, \xi_2].$$

**Proposition A.2.** The element $\Omega^n \in A^{n,n} \otimes g^\otimes n = \text{Hom}_C(\wedge^n g(A)[n], \Omega^n_A[-n]) \otimes g^\otimes n$ is given by

$$(\xi_1 \otimes a_1) \wedge \ldots \wedge (\xi_n \otimes a_n) \mapsto \sum_{\sigma \in S_n} (-1)^{f(\sigma,|a_1|,\ldots,|a_n|)} (\xi_{\sigma(1)} \otimes a_{\sigma(1)}) \otimes \ldots \otimes (\xi_{\sigma(n)} \otimes a_{\sigma(n)}).$$

**Proof.** By Proposition A.1, the summand of $\Delta(n)[(\xi_1 \otimes a_1) \wedge \ldots \wedge (\xi_n \otimes a_n)]$ contributing to $\Omega^\otimes n \circ \Delta(n)[(\xi_1 \otimes a_1) \wedge \ldots \wedge (\xi_n \otimes a_n)]$ is given by

$$\sum_{\sigma \in S_n} (-1)^{f(\sigma,|a_1|,\ldots,|a_n|)} (\xi_{\sigma(1)} \otimes a_{\sigma(1)}) \otimes \ldots \otimes (\xi_{\sigma(n)} \otimes a_{\sigma(n)}).$$

Here, $(-1)^{f(\sigma,|a_1|,\ldots,|a_n|)}$ is the sign obtained after applying $\sigma$ to a product of elements of degrees $|a_1| + 1, \ldots, |a_n| + 1$ in a commutative graded algebra. By a second use of Proposition A.1,

$$\Omega^n[(\xi_1 \otimes a_1) \wedge \ldots \wedge (\xi_n \otimes a_n)] = \sum_{\sigma} (-1)^{f(\sigma,|a_1|,\ldots,|a_n|)} da_{\sigma(1)} \ldots da_{\sigma(n)} \otimes \xi_{\sigma(1)} \otimes \ldots \otimes \xi_{\sigma(n)}$$

$$= \sum_{\sigma} da_1 \ldots da_n \otimes \xi_{\sigma(1)} \otimes \ldots \otimes \xi_{\sigma(n)}.$$

This finishes the proof of the proposition. □

**Proposition A.3.** For any $P \in I^{r+1}(g)$, we have

$$P(\theta, [\theta, \theta]^n) \wedge \Omega^{2r-n}[(\xi_0 \otimes a_0) \wedge \ldots \wedge (\xi_n \otimes a_n)] = \sum_{\sigma \in S_{n+1}} \pm A_{\sigma, P} a_{\sigma(0)} \ldots a_{\sigma(2n-2r)} da_{\sigma(2n-2r+1)} \ldots da_{\sigma(n)}$$

where

$$A_{\sigma, P} = c_{n,r} P(\xi_{\sigma(0)}, \xi_{\sigma(1)}, \xi_{\sigma(2)}), \ldots, [\xi_{\sigma(2n-2n-2r)}, \xi_{\sigma(2n-2r+1)}, \ldots, \xi_{\sigma(n)}].$$

Here $c_{n,r}$ is a nonzero constant depending only on $n$ and $r$, with $c_{n,n} = 1$; the sign $\pm$ in the sum is obtained by applying $\sigma$ to a product of elements of degrees $|a_0| + 1, \ldots, |a_n| + 1$ in a commutative graded algebra.

**Remark.** The formula of Proposition A.3 appears in [24] as an ad hoc definition (see op.cit., (2.2)). Proposition A.3 thus explains the origin of this formula and clarifies the computations of [24].

**Proof.** Let $\beta_i := \xi_i \otimes a_i$ for brevity. Indeed, the component of $\Delta(n+1)[(\xi_0 \otimes a_0) \wedge \ldots \wedge (\xi_n \otimes a_n)]$ in $g(A) \otimes \wedge^2 g(A)^{2n-r} \otimes g(A)^{2r-n}$ is given by

$$C' \sum_{\sigma \in S_{n+1}} \pm \beta_{\sigma(0)} \otimes \beta_{\sigma(1)} \wedge \beta_{\sigma(2)} \otimes \ldots \otimes \beta_{\sigma(2n-2r-1)} \wedge \beta_{\sigma(2n-2r)} \in g(A)^{2n-r} \otimes g(A)^{2r-n}\beta_{\sigma(2n-2r+1)} \otimes \ldots \otimes \beta_{\sigma(n)}$$

where $C'$ is a positive constant depending only on $n$ and $r$. By Proposition A.1, the desired proposition follows (with $C = C'(-2)^{n-r}$). □

**A.2. Lie and cyclic homology.** Let $P \in I^{r+1}(g)$. Let $A$, $\theta$ be as in Section A.1. Recall from (3.10) that $TP(\theta) = \sum_{n=0}^{2r} A_{n-r} \Psi_{n-r,P}$ where $\Psi_{n-r,P} \in \text{Hom}(\text{Sym}^{n+1}(g(A)[1]), \Omega_A^{2n-r}[n-2r+1])$. Also recall that in Section A.1 the original homological grading in $A$ was inverted to give a cohomological grading for $C(g(A); k)$ and allow $\text{DR}_*(A)$ and $A$ to have their natural cohomological gradings. Invert homological degrees once again to restore the original homological grading of $A$, and thereby $C(g(A); k)$. This inverts the natural cohomological
gradings of $\text{DR}_\bullet(A)$ and $\mathcal{A}$, giving them a homological grading. Let $s$ denote the operator increasing homological degree by 1. Then, $s^{2r}TP(\theta) \in \oplus_{n=0}^{2r} \text{Hom}(\text{Sym}^{n+1}(g(A)[1]), \Omega^{2r-n}_A[n])$. Recall that

$$\text{CC}^{(r)}(\text{DR}^\bullet(A)) := \oplus_{n=0}^{2r} \Omega^{2r-n}_A[n], d + \delta,$$

where $d : \Omega^{2r-n-1}[n-1] \rightarrow \Omega^{2r-n}[n]$ is viewed as a differential with homological degree $-1$ that vanishes when $n = r$. It follows that $s^{2r}TP(\theta)$ gives a map of degree $-1$ from $\mathcal{C}(g(A); k)$ to $\text{CC}^{(r)}(\text{DR}^\bullet(A))$.

**Theorem A.1.** Let $A$ be a smooth commutative DG algebra. Then $s^{2r}TP(\theta)$ induces a map on homologies $H_{\bullet+1}(g(A); k) \rightarrow \overline{\text{HC}}^{(r)}_\bullet(A)$.

**Proof.** By Proposition 3.1, $d(TP(\theta)) = P(\Omega^{r+1})$. Hence,

$$\sum_{n=r}^{2r} A_{n-r}[(d + \delta)\Psi_{n-r, p} + \Psi_{n-r, p}(d + \delta)] = P(\Omega^{r+1}).$$

Comparing the components of both sides in $\text{Hom}(\text{Sym}^{r+1}(g(A)[1]), \Omega^{r+1}_A[-r-1])$, we see that

(A.2) $d\Psi_{0, p} = P(\Omega^{r+1})$

(A.3) $\delta\Psi_{0, p} + \sum_{n=r+1}^{2r} A_{n-r}[(d + \delta)\Psi_{n-r, p} + \Psi_{n-r, p}(d + \delta)] = 0$

We now note that the left hand side of (A.3) is exactly $ds^{2r}TP(\theta)$, provided $s^{2r}TP(\theta)$ is viewed as a degree $-1$ element of $\text{Hom}(\mathcal{C}(g(A); k), \text{CC}^{(r)}(\text{DR}^\bullet(A)))$: indeed, the differential of $\text{CC}^{(r)}(\text{DR}^\bullet(A))$ restricted to the graded subspace $\Omega^{r}_A[r]$ of $\text{CC}^{(r)}(\text{DR}^\bullet(A))$ is exactly $\delta$. Hence, $s^{2r}TP(\theta)$ gives a map of complexes from $\mathcal{C}(g(A); k)[1]$ to $\text{CC}^{(r)}(\text{DR}^\bullet(A))$. The induced map on homologies gives a map of graded vector spaces from $H_{\bullet+1}(g(A); k)$ to $\overline{\text{HC}}^{(r)}_\bullet(A)$ as desired. □

Note that if $A$ is a smooth augmented commutative DGA, then the connection $\theta$ restricts to a connection on the convolution DGA $\text{Hom}(\mathcal{C}(g(\bar{A}); k), \text{DR}_\bullet(A))$. It follows that Theorem A.1 can be modified for smooth augmented commutative DG algebras, giving

**Theorem A.2.** Let $A$ be a smooth augmented commutative DG algebra. Then $s^{2r}TP(\theta)$ induces a map on homologies $H_{\bullet+1}(g(\bar{A}); k) \rightarrow \overline{\text{HC}}^{(r)}_\bullet(A)$.

Next, we compare $s^{2r}TP(\theta)$ with another construction of a map from $H_{\bullet+1}(g(\bar{A}); k)$ to $\overline{\text{HC}}^{(r)}_\bullet(A)$ that we give below.

A.3. From Lie to cyclic homology: the second construction. Let $A$ be an augmented commutative DG algebra. There is a more direct construction of a map from the Lie homology of $g(\bar{A})$ to the (shifted) reduced cyclic homology of $A$, which works even if $A$ is not smooth. This construction is dual the construction of the Drinfeld traces recalled from [4] in Section 3.1.3. We begin with the map

$$g(\bar{A})[1] \cong \bar{A}[1] \otimes g, \; \xi \otimes a \mapsto a \otimes \xi.$$

Precomposing this with the natural projection $\text{Sym}^c(g(\bar{A})[1]) \rightarrow g(\bar{A})[1]$, we obtain map of graded vector spaces

$$\text{Sym}^c(g(\bar{A})[1]) \rightarrow \bar{A}[1] \otimes g,$$

which is equivalent to a map of graded vector spaces

(A.4) $\text{Sym}^c(g(\bar{A})[1]) \otimes g^* \rightarrow \bar{A}[1].$
Note that the Lie coalgebra structure on \( g^* \) together with the cocommutative (conilpotent) DG coalgebra structure on \( \text{Sym}^c(g(\bar{A}[1])) \) makes \( \text{Sym}^c(g(\bar{A}[1])) \otimes g^* \) a (conilpotent) graded Lie coalgebra. It follows that \( \text{Sym}^c(g(\bar{A}[1])) \otimes g^* \to \mathcal{L}^c(\bar{A}[1]), \) where \( \mathcal{L}^c(W) \) denotes the cofree Lie coalgebra cogenerated by a graded vector space \( W \). Equip \( \text{Sym}^c(g(\bar{A}[1])) \) with the (co)differential in \( C(g(\bar{A}); k) \) and equip \( \mathcal{L}^c(\bar{A}[1]) \) with differential in \( B_{\text{con}}(A) \) (see [2] Section 6.2.1). Then we have

**Lemma A.1.** The map (A.5) is a map of (conilpotent) DG Lie coalgebras.

The map (A.5) therefore induces a map of cocommutative DG coalgebras

\[
C(g(\bar{A}); k) \otimes \text{Sym}^c(g^*) \to \text{Sym}^c(\mathcal{L}^c(\bar{A}[1])).
\]

There is an isomorphism of complexes \( \text{Sym}^*: T^c(\bar{A}[1]) \cong \text{Sym}^c(\mathcal{L}^c(\bar{A}[1])) \) dual to the symmetrization map (where \( T^c(\bar{A}[1]) \) is equipped with the bar differential). We therefore, obtain a map of complexes

\[
\varphi(-, -) : C(g(\bar{A}); k) \otimes \text{Sym}^c(g^*) \to B(A).
\]

Let \( B(A)^\bullet \) denote the cocommutator subcomplex of \( B(A) \). Let \( \text{Sym}^{r+1}(\mathcal{L}^c(\bar{A}[1])) \) continue to denote the image of \( \text{Sym}^{r+1}(\mathcal{L}^c(\bar{A}[1])) \) in \( B(A) \) under the inverse of the isomorphism \( \text{Sym}^* \). Then, for a polynomial \( P \in I^{r+1}(g) \),

**Proposition A.4.** \( \varphi(-, P) \) gives a map of complexes

\[
\varphi_P : C(g(\bar{A}); k) \to \text{Sym}^{r+1}(\mathcal{L}^c(\bar{A}[1])) \cap B(A)^\bullet \cong C_{{\lambda}^{(r)}(A)}[1].
\]

Let \( p : C_{{\lambda}^{(r)}(A)}[1] \to \Omega_{\lambda}^\bullet/d\Omega_{\lambda}^r[1] \) be as in (2.1). Let \( \varepsilon : \Omega_{\lambda}^\bullet/d\Omega_{\lambda}^r[1] \to C_{{\lambda}^{(r)}(A)}[1] \) be as in (2.2) and let \( \pi_r : C_{{\lambda}^{(r)}(A)} \to \Omega_{\lambda}^\bullet/d\Omega_{\lambda}^r[1] \) be as in (2.3). For notational brevity, use the same symbol to denote a map of complexes \( X \to Y \) and the induced map \( X[i] \to Y[i] \) for any \( i \). Then,

**Theorem A.3.** Let \( A = (\text{Sym}(W), \delta) \) be augmented over \( k \). Then, the following diagram commutes in \( \mathcal{D}(k) \):

\[
\begin{array}{ccc}
C(g(\bar{A}); k) & \xrightarrow{\varphi_P} & B(A)^\bullet \cong C_{{\lambda}^{(r)}(A)}[1] \\
\xrightarrow{\text{def}.} s^{2r}TP(\theta) & & \\
CC_{{\lambda}^{(r)}(A)}[1] & \xrightarrow{\varepsilon \circ p} & C_{{\lambda}^{(r)}(A)}[1]
\end{array}
\]

**Proof.** We begin the proof with the following proposition.

**Proposition A.5.**

\[
p \circ s^{2r}TP(\theta) = s^{2r}P(\theta, \Omega^r)
\]

**Proof.** By (3.10), \( TP(\theta) = \sum_{n=1}^{2r} A_n \cdot P(\theta, \theta^n \Omega^{2r-n}) \). Note that when \( s^rTP(\theta) \) is interpreted as a map of graded vector spaces from \( C(g(\bar{A}); k) \) to \( C_{{\lambda}^{(r)}(A)} \), the image of \( P(\theta, \theta^n \Omega^{2r-n}) \) lies in \( \Omega^{2r-n}[n] \). It follows that \( p \circ P(\theta, \theta^n \Omega^{2r-n}) = 0 \) for \( n > r \). Since \( A_0 = 1 \), the desired proposition follows. \( \square \)
By Proposition A.5 it suffices to verify that in \( D(k) \),
\[
\varphi_P = \frac{1}{(r+1)!} \varepsilon \circ s^{2r} P(\theta \Omega^r). \tag{A.8}
\]
Further, since \( I_{HKR} : \mathcal{C}^{\wedge}(r)(A)[1] \to \Omega^r_A/d\Omega^{r-1}_A[1+r] \) is a quasi-isomorphism inverting \( \varepsilon \), it suffices to verify that
\[
I_{HKR} \circ \varphi = \frac{1}{(r+1)!} s^{2r} P(\theta \Omega^r). \tag{A.9}
\]
Note that both sides of (A.9) are honest maps of complexes. By Proposition A.2 \( s^{2r} P(\theta \Omega^r) \) vanishes on all chains in \( \mathcal{C}(g(\bar{A}); k) \) that are not in \( \text{Sym}^{r+1}(g(\bar{A})[1]) \). Similarly, \( \varphi_P \) vanishes on chains in \( \text{Sym}^n(g(\bar{A})[1]) \) for any \( n < r + 1 \). \( \varphi_P \) maps chains that are in \( \text{Sym}^n(g(\bar{A})[1]) \), \( n > r + 1 \) to chains in \( \text{Sym}^{r+1}(\mathcal{L}^c(\bar{A}[1])) \cap B(A)^{\natural} \) that are a linear combination of summands with at least one factor in \( \mathcal{L}^{c \geq 2}(\bar{A}[1]) \). Hence, \( I_{HKR} \circ \varphi_P \) vanishes on chains that are not in \( \text{Sym}^{r+1}(g(\bar{A})[1]) \).

On chains that are in \( \text{Sym}^{r+1}(g(\bar{A})[1]) \), Proposition A.3 gives
\[
P(\theta \Omega^r)((\xi_0 \otimes a_0) \wedge \ldots \wedge (\xi_r \otimes a_r)) = \sum_{\sigma \in S_{r+1}} \pm a_{\sigma(0)} da_{\sigma(1)} \ldots da_{\sigma(r)} P(\xi_{\sigma(0)}, \ldots, \xi_{\sigma(r)}).
\]

On the other hand, by (A.6),
\[
[(\xi_0 \otimes a_0) \wedge \ldots \wedge (\xi_r \otimes a_r)] \otimes P \mapsto a_0 \wedge \ldots \wedge a_r P(\xi_0, \ldots, \xi_i).
\]
Thus,
\[
\varphi(-, P) : (\xi_0 \otimes a_0) \wedge \ldots \wedge (\xi_r \otimes a_r) \mapsto \sum_{\sigma \in S_{r+1}} \pm a_{\sigma(0)} \otimes \ldots \otimes a_{\sigma(r)} P(\xi_{\sigma(0)}, \ldots, \xi_{\sigma(r)}).
\]
Recall from [21] Section 1.3 that the identification between \( \mathcal{C}(A)[1] \cong B(A)^{\natural} \) is given by the operator \( N \) which acts on \( A[1] \otimes n \) by \( 1 + \tau + \ldots + \tau^{n-1} \) where \( \tau \) denotes the \( n \)-cycle \( (0, 1, \ldots, n-1) \). It follows that
\[
N^{-1} \left( \sum_{\sigma \in S_{r+1}} \pm a_{\sigma(0)} \otimes \ldots \otimes a_{\sigma(r)} P(\xi_{\sigma(0)}, \ldots, \xi_{\sigma(r)}) \right) = \frac{1}{r+1} \sum_{\sigma \in S_{r+1}} \pm a_{\sigma(0)} \otimes \ldots \otimes a_{\sigma(r)} P(\xi_{\sigma(0)}, \ldots, \xi_{\sigma(r)}).
\]
Hence,
\[
\varphi_P((\xi_0 \otimes a_0) \wedge \ldots \wedge (\xi_r \otimes a_r)) = \frac{1}{r+1} \sum_{\sigma \in S_{r+1}} \pm a_{\sigma(0)} \otimes \ldots \otimes a_{\sigma(r)} P(\xi_{\sigma(0)}, \ldots, \xi_{\sigma(r)}).
\]
Therefore,
\[
I_{HKR} \circ \varphi_P((\xi_0 \otimes a_0) \wedge \ldots \wedge (\xi_r \otimes a_r)) = \frac{1}{(r+1)!} \sum_{\sigma \in S_{r+1}} \pm a_{\sigma(0)} da_{\sigma(1)} \ldots da_{\sigma(r)} P(\xi_{\sigma(0)}, \ldots, \xi_{\sigma(r)}).
\]
This proves the desired theorem. \( \square \)

**Appendix B. The HKR and co-HKR maps**

In this Appendix, we present a concise proof of Theorem 2.5. As explained in Section B.3, the method used for this proof can also be used to give a direct formula for traces of symmetric algebras from which Theorem 1.2 can be derived.

For the rest of this proof, the sign \( \pm \) in front of each summand is the sign obtained after applying \( \sigma \) to a product of elements of degrees \(|a_0| + 1, \ldots, |a_n| + 1\) in a commutative graded algebra.
B.1. Recollections and notation. Let \( W \) be a finite-dimensional \( k \)-vector space. As usual, let \( A = \text{Sym}(W) \) and let \( C = \text{Sym}^c(W[1]) \) be the coalgebra Koszul dual to \( A \). Fixing a basis \( \{x_1, \ldots, x_N\} \) of \( W \), we identify \( A \) with the polynomial algebra \( A = k[x_1, \ldots, x_N] \). Let \( dx_j := sx_i \), so that \( C \) is identified with the polynomial coalgebra \( C \cong k[dx_1, \ldots, dx_N] \), equipped with the shuffle coproduct. Hence,
\[
\text{DR}^\bullet(A) \cong \text{DR}^\bullet(C) \cong k[x_1, \ldots, x_N, dx_1, \ldots, dx_N]/k.
\]
The de Rham differential then is defined in the obvious manner by setting \( d(x_j) = dx_j \). Furthermore, we denote by \( R = \Omega(C) \) the minimal quasi free resolution of \( A \) as before. Concretely, \( R \) is free as a graded algebra generated by symbols
\[
x_I := s^{-1}(\prod_{i \in I} dx_i)
\]
for \( I \subset \{1, \ldots, N\} \), where the product is taken in lexicographic order to fix the sign. It will be convenient for us to extend this notation also to multisets, i.e., sets with multiplicities, by declaring that \( x_I = 0 \) if the multiset \( I \) contains any symbols with multiplicity greater than one.

Furthermore, given arbitrary elements \( u_1, \ldots, u_M \in W \) we will use similar notation and write
\[
u_I := s^{-1}(\prod_{i \in I} du_i)
\]
and
\[du_I := \prod_{i \in I} du_i\]
where in both cases the product is taken in the lexicographic order.

B.2. Remark on the map \( T \). Let \( A_1 \) and \( A_2 \) be dg (or \( A_\infty \)) algebras. Recall from [14, Sec. 4] that an \( A_\infty \)-morphism \( f : A_1 \to A_2 \) is equivalent to a twisting cochain \( f : B(A_1) \to A_2 \), where \( B(A_1) \) denotes the bar construction of \( A_1 \). A twisting cochain \( f \) induces a map between cyclic bar complexes
\[
\phi_f : C^\lambda(A_1) \to C^\lambda(A_2)
\]
given by the formula (using cyclic indexing, i.e., \(-1 \equiv n \) etc.)
\[
(a_0, \ldots, a_n) \mapsto \sum_{r=1}^{n} \sum_{j_1+\cdots+j_r=n+1} \sum_{i=1}^{j_1} \pm(f_{j_1}(a_{1-i}, \ldots, a_{j_1-i}), f_{j_2}(a_{j_1-i+1}, \ldots, a_{j_1+j_2-i}), \ldots, f_r(\ldots, a_{-i})).
\]
This formula is compatible with composition of \( A_\infty \) morphisms, making \( C^\lambda(\cdot) \) a functor from the category of dg algebras with \( A_\infty \) morphisms to the category of chain complexes. If we require in addition that the \( A_\infty \) morphisms be unital, i.e., \( f_1(1_{A_1}) = 1_{A_2} \) and \( f_n(\ldots, 1_{A_1}, \ldots) = 0 \) for \( n > 1 \), then \( \phi_f \) descends to a map
\[
\phi_f : C^\lambda(A_1) \to C^\lambda(A_2)
\]
between reduced cyclic complexes.

In particular, let \( A_1 = A \) and let \( A_2 = R \), and let \( \theta : B(A) \to R \) be the twisting cochain corresponding to a (unital) \( A_\infty \) right inverse to the canonical projection \( R \to A \). The composite map
\[
C^\lambda(A) \xrightarrow{\phi_f} C^\lambda(R) \xrightarrow{R}\n\]
is precisely the map \( T \) from [23]. Functoriality of \( C^\lambda(\cdot) \) immediately implies the following lemma.

Lemma B.1. The map \( T : C^\lambda(A) \to R\) induces the same map in homology as the zigzag of quasi-isomorphisms
\[
C^\lambda(A) \leftarrow C^\lambda(R) \to R\).
\]

The Lemma provides us with an alternative definition of \( T \) without \( A_\infty \) morphisms, and hence with another equivalent formulation of Theorem [25].
B.3. Proof of Theorem 2.5. We have the following diagram of quasi-isomorphisms:

\[
\begin{array}{c}
\text{DR}^\bullet(A)/d\text{DR}^\bullet(A) \xrightarrow{\text{HKR}} \overline{C}^\Lambda \xrightarrow{\text{B.1}} \overline{C}^\Lambda(R) \xrightarrow{\text{B.2}} R_\text{dR} = \overline{C}^\Lambda(C)[1] \xrightarrow{\text{ker}(dc)[1]}
\end{array}
\]

Here, \(\varepsilon\) is the quasi-isomorphism inverting the co-HKR map (see (2.3)). Using Lemma B.1 the statement of Theorem 2.5 is equivalent to the assertion that the isomorphism in homology induced by (B.1) above is just the de Rham differential. To check this, let us start with the element of \(\text{DR}^\bullet(A)/d\text{DR}^\bullet(A)\) represented by \(u_1u_2\cdots u_n du_{n+1}\cdots du_{n+p}\) for elements \(u_1, \ldots, u_{n+p} \in W\).

**Lemma B.2.** A representative in \(\overline{C}^\Lambda(R)\) of the homology class corresponding to \(\alpha := u_1u_2\cdots u_n du_{n+1}\cdots du_{n+p}\) is given by

\[
\beta = \frac{1}{n!} \sum_{\sigma \in S_n} \sum_{m=0}^{p} \sum_{f} (-1)^f u_{\sigma(1)} \cdots u_{\sigma(n)} \otimes u_{f^{-1}(1)} \otimes \cdots \otimes u_{f^{-1}(m)}
\]

where the sum is over maps \(f : \{n+1, \ldots, n+p\} \to \{1, \ldots, n, \bar{1}, \ldots, \bar{m}\}\) such that each \(\bar{j}\) is hit at least once. The sign depends only on \(f\), and is determined by the permutation of the factors \(du_j\) (which are counted as odd) appearing in a term in the above formula.

**Proof.** Let us first verify that the stated element indeed maps to \(\alpha\) under the composition of the two left-most maps in (B.1). Indeed, the only elements that survive the projection \(R \to A\) are those \(u_f\)'s with \(|f| = 1\). In other words, only the top summand \(m = p\) contributes, and for \(m = p\) the only allowed maps \(f\) are permutations of \(\{\bar{1}, \ldots, \bar{m}\}\). There are \(p!\) such maps, canceling the prefactor in the definition of the Hochschild-Kostant-Rosenberg morphism. The symmetrization \(\frac{1}{n!} \sum_{\sigma \in S_n}\) is inessential and can be omitted before or after mapping via the HKR morphism. Hence we exactly recover \(\alpha\) as desired.

Next we have to show that \(\beta\) is a cocycle. We claim that, in fact, each \(\sigma\)-summand in (B.2) is separately closed. This is verified by a straightforward, but tedious computation. We sketch this computation for the term where \(\sigma\) is the identity. Abbreviating the first tensor factor as \(X\) to save space, the Hochschild differential of this term is

\[
\sum_{m=0}^{p} \sum_{f} (-1)^f \delta \left( X \otimes u_{f^{-1}(1)} \otimes \cdots \otimes u_{f^{-1}(m)} \right) =
\]

\[
\sum_{m=0}^{p} \sum_{f} (-1)^f \left( \pm X u_{f^{-1}(1)} \otimes u_{f^{-1}(2)} \otimes \cdots \otimes u_{f^{-1}(m)} \pm X \otimes u_{f^{-1}(1)} u_{f^{-1}(2)} \otimes \cdots \otimes u_{f^{-1}(m)} \pm \cdots \right).
\]

Note that the terms \(u_{f^{-1}(1)} u_{f^{-1}(2)}\) appearing in the above summands for some \(m\) reproduce \(\delta_R u_{f^{-1}(i)}\) in the summand for \(m - 1\), where \(\delta_R\) is the differential on \(R\). It can also be seen without difficulty that the sign before the above summands with \(u_{f^{-1}(i)} u_{f^{-1}(2)}\) coming from \(\delta_R\) is precisely \((-1)^{f+|X|+|u_{f^{-1}(i)}|+2}\) while the sign on the summands with \(u_{f^{-1}(1)} u_{f^{-1}(2)}\) coming from the Hochschild differential \(\delta\) is \((-1)^{f+|X|+|u_{f^{-1}(1)}|+1}\). Thus, the above summands with \(u_{f^{-1}(i)} u_{f^{-1}(2)}\) coming from the Hochschild differential cancel out similar summands from \(\delta_R\) applied to \(f\) corresponding to \(m - 1\). A similar statement holds, of course, if we replace 1 and 2 by \(i\) and \(j\). Furthermore, by a similar argument, the terms \(X \otimes u_{f^{-1}(1)}\) and \(u_{f^{-1}(m)} X\) appearing in some summand \(m\) cancel out the terms yielding \(\delta_R X\) for one lower \(m\). This shows that (B.2) is indeed closed under the total differential. \(\square\)
Under the map $\overline{C}^1(R) \to R_z$ the element $\beta$ above is sent to

(B.3) \[
\frac{1}{n!} \sum_{\sigma \in S_n} \sum_f (\!\!\!\! u_{\sigma(1)} \!\!\!\! \beta^{-1}(1) \!\!\!\! u_{\sigma(2)} \!\!\!\! \beta^{-1}(2) \cdots \!\!\!\! u_{\sigma(n)} \!\!\!\! \beta^{-1}(n)).
\]

Here the sum is over maps $f : \{n + 1, \ldots, n + p\} \to \{1, \ldots, n\}$. The corresponding element in $\overline{C}^1(C)[1]$ is obtained by replacing $u_f$’s by the corresponding $du_f$’s, and putting tensor signs between factors. This gives

(B.4) \[
\frac{1}{n!} \sum_{\sigma \in S_n} \sum_f (-1)^f du_{\sigma(1)} \otimes du_{\sigma(2)} \otimes \cdots \otimes du_{\sigma(n)}.
\]

We need to compute the image of the above element under the map $\varepsilon$ (see (B.4)). The map dual to $\varepsilon$ has been explicitly described in (2.4). In particular, all terms that have more than one non-linear tensor factor are sent to 0 under $\varepsilon$. Hence, in the above sum over $f$ we retain only maps $f : \{n + 1, \ldots, n + p\} \to \{1, \ldots, n\}$ that have a single element in the image. We will assume that $p > 0$, leaving the simpler case of $p = 0$ to the reader. Then the summands of (B.4) contributing nontrivially add up to

\[
\frac{1}{n!} \sum_{\sigma \in S_n} \sum_{i=1}^n du_{\sigma(1)} \otimes \cdots \otimes du_{\sigma(i)} du_{\{n+1 \ldots , n+p\}} \otimes \cdots \otimes du_{\{n\}} = \frac{1}{n!} \sum_{\sigma \in S_n} du_{\sigma(1)} du_{\{n+1 \ldots , n+p\}} \otimes du_{\{2\} \otimes \cdots \otimes du_{\{n\}}.}
\]

One can now easily verify that $\varepsilon(\text{B.4}) = \sum_{i=1}^n du_i du_{n+1} \cdots du_{n+p} u_1 \cdots u_n = d\alpha \in \ker d$. This proves Theorem 2.5. \hfill \Box

B.4. Simplified trace formula. Note that we can get a formula for the trace map by just mapping the element \text{B.3} of $R_z$ to the abelianization $R_{ab} \cong k[x_I | I \subset \{1, \ldots, N\}, I \neq \emptyset]$.

Theorem B.1. The trace map $\overline{T}(A) : DR^*(A) / dDR^*(A) \to R_{ab}$ satisfies the formula

$\overline{T}(A)(u_1 u_2 \cdots u_n du_{n+1} \cdots du_{n+p}) = \sum_f (-1)^f u_{\{1\} \otimes \beta^{-1}(1)} u_{\{2\} \otimes \beta^{-1}(2)} \cdots u_{\{n\} \otimes \beta^{-1}(n)}.$

where the sum is over maps $f : \{n + 1, \ldots, n + p\} \to \{1, \ldots, n\}$.

B.5. Another proof of Theorem 4.2. For $\omega \in \Sym^n(W) \otimes \Lambda^p(W)$, let $F(\omega) := \frac{1}{(n+1)!} [\theta, \Omega^n](\omega)$. Explicitly, \[(n+1)!F(\omega) \text{ is given by the right hand side of (4.20) (for } q = n). \text{ We can directly verify that for } \omega \in \Sym^n(W) \otimes \Lambda^p(W), \text{ the right hand side of the formula in Theorem B.1 (applied to } \omega \text{) coincides with } F(\omega). \text{ This gives us another route to the proof of Theorem 4.2. Indeed,}

\[F(u_1 u_2 \cdots u_n du_{n+1} \cdots du_{n+p}) = \frac{1}{n+1} \sum_f \pm u_{f^{-1}(0)} u_{\{1\} \otimes \beta^{-1}(1)} u_{\{2\} \otimes \beta^{-1}(2)} \cdots u_{\{n\} \otimes \beta^{-1}(n)},\]

and where the sum is over all maps $f : \{n + 1, \ldots, n + p\} \to \{0, 1, \ldots, n\}$. Computing $F(\omega)$ for $\omega = u_1 u_2 \cdots u_n du_{n+1} \cdots du_{n+p}$ we obtain

\[F(\omega) = F \left( \sum_{i=1}^n u_1 \cdots \hat{u_i} \cdots u_n du_{i+1} \cdots du_{n+p} \right) = \frac{1}{n} \sum_{i=1}^n \sum_f \pm u_{f^{-1}(0)} u_{\{1\} \otimes \beta^{-1}(1)} \cdots \hat{u_i} \cdots u_{\{n\} \otimes \beta^{-1}(n)} \cdots \hat{u_n} \cdots u_{n+p}.\]
where the second sum in the second line is over maps $f : \{i, n + 1, \ldots, n + p\} \to \{0, 1, \ldots, \hat{i}, \ldots, n\}$. We will decompose this sum according to $j := f(i)$. In particular, we split off the $j = 0$-piece. This yields

$$F(d\omega) = \frac{1}{n} \sum_{i=1}^{n} \sum_{f} \pm u_{\{i\} \cup f^{-1}(0)} u_{\{1\} \cup f^{-1}(1)} \cdots \hat{u}_{i} \cdots u_{\{n\} \cup f^{-1}(n)}$$

$$+ \frac{1}{n} \sum_{i,j=1 \atop i \neq j}^{n} \sum_{f} \pm u_{f^{-1}(0)} u_{\{1\} \cup f^{-1}(1)} \cdots \hat{u}_{i} \cdots u_{\{j\} \cup f^{-1}(j)} \cdots u_{\{n\} \cup f^{-1}(n)}$$

where the sums are now over maps $f : \{n + 1, \ldots, n + p\} \to \{0, 1, \ldots, \hat{i}, \ldots, n\}$. Note that the second line is zero by antisymmetry of the summand in $i$ and $j$. The first line remains and can be re-written as

$$F(d\omega) = \sum_{f} \pm u_{\{1\} \cup f^{-1}(1)} u_{\{2\} \cup f^{-1}(2)} \cdots u_{\{n\} \cup f^{-1}(n)}$$

which agrees with the formula of Theorem B.1. \qed

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