A LIE GROUP AS A 4-DIMENSIONAL QUASI-KÄHLER MANIFOLD WITH NORDEN METRIC

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Abstract. A 4-parametric family of 4-dimensional quasi-Kähler manifolds with Norden metric is constructed on a Lie group. This family is characterized geometrically. The condition for such a 4-manifold to be isotropic Kähler is given.

INTRODUCTION

It is a fundamental fact that on an almost complex manifold with Hermitian metric (almost Hermitian manifold), the action of the almost complex structure on the tangent space at each point of the manifold is isometry. There is another kind of metric, called a Norden metric or a $B$-metric on an almost complex manifold, such that action of the almost complex structure is anti-isometry with respect to the metric. Such a manifold is called an almost complex manifold with Norden metric [2] or with $B$-metric [3]. See also [1] for generalized $B$-manifolds. It is known [2] that these manifolds are classified into eight classes.

The purpose of the present paper is to exhibit, by construction, almost complex structures with Norden metric on Lie groups as 4-manifolds, which are of certain classes, called quasi-Kähler manifold with Norden metrics. It is proved that the constructed 4-manifold is isotropic Kähler (a notion introduced by García-Río and Matsushita [4]) if and only if it is scalar flat.

1. ALMOST COMPLEX MANIFOLDS WITH NORDEN METRIC

1.1. Preliminaries. Let $(M, J, g)$ be a $2n$-dimensional almost complex manifold with Norden metric, i.e. $J$ is an almost complex structure and $g$ is a metric on $M$ such that

\[ J^2X = -X, \quad g(JX, JY) = -g(X, Y) \]

for all differentiable vector fields $X, Y$ on $M$, i.e. $X, Y \in \mathfrak{X}(M)$.

The associated metric $\tilde{g}$ of $g$ on $M$ given by $\tilde{g}(X, Y) = g(X, JY)$ for all $X, Y \in \mathfrak{X}(M)$ is a Norden metric, too. Both metrics are necessarily of signature $(n, n)$. The manifold $(M, J, \tilde{g})$ is an almost complex manifold with Norden metric, too.

Further, $X, Y, Z, U$ ($x, y, z, u$, respectively) will stand for arbitrary differentiable vector fields on $M$ (vectors in $T_pM$, $p \in M$, respectively).

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The Levi-Civita connection of $g$ is denoted by $\nabla$. The tensor field $F$ of type $(0,3)$ on $M$ is defined by

$$F(X, Y, Z) = g((\nabla_X J)Y, Z).$$

It has the following symmetries

$$F(X, Y, Z) = F(X, Z, Y) = F(X, JY, JZ).$$

Further, let $\{e_i\}$ ($i = 1, 2, \ldots, 2n$) be an arbitrary basis of $T_pM$ at a point $p$ of $M$. The components of the inverse matrix of $g$ are denoted by $g^{ij}$ with respect to the basis $\{e_i\}$.

The Lie form $\theta$ associated with $F$ is defined by

$$\theta(z) = g^{ij}F(e_i, e_j, z).$$

A classification of the considered manifolds with respect to $F$ is given in [2]. Eight classes of almost complex manifolds with Norden metric are characterized there according to the properties of $F$. The three basic classes are given as follows

$$W_1 : F(x, y, z) = \frac{1}{4n} \{g(x, y)\theta(z) + g(x, z)\theta(y)$$
$$ + g(x, Jy)\theta(Jz) + g(x, Jz)\theta(Jy)\};$$
$$W_2 : \mathfrak{S}_{x,y,z} F(x, y, Jz) = 0, \quad \theta = 0;$$
$$W_3 : \mathfrak{S}_{x,y,z} F(x, y, z) = 0,$$

where $\mathfrak{S}$ is the cyclic sum by three arguments.

The special class $W_0$ of the Kähler manifolds with Norden metric belonging to any other class is determined by the condition $F = 0$.

1.2. Curvature properties. Let $R$ be the curvature tensor field of $\nabla$ defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z.$$

The corresponding tensor field of type $(0,4)$ is determined as follows

$$R(X, Y, Z, U) = g(R(X, Y)Z, U).$$

The Ricci tensor $\rho$ and the scalar curvature $\tau$ are defined as usually by

$$\rho(y, z) = g^{ij}R(e_i, y, z, e_j), \quad \tau = g^{ij}\rho(e_i, e_j).$$

It is well-known that the Weyl tensor $W$ on a $2n$-dimensional pseudo-Riemannian manifold ($n \geq 2$) is given by

$$W = R - \frac{1}{2n-2} \psi_1(\rho) - \frac{\tau}{2n-1} \pi_1,$$

where

$$\psi_1(\rho)(x, y, z, u) = g(y, z)\rho(x, u) - g(x, z)\rho(y, u)$$
$$+ \rho(y, z)g(x, u) - \rho(x, z)g(y, u);$$
$$\pi_1 = \frac{1}{2} \psi_1(g) = g(y, z)g(x, u) - g(x, z)g(y, u).$$

Moreover, the Weyl tensor $W$ is zero if and only if the manifold is conformally flat.
Let $\alpha = \{x, y\}$ be a non-degenerate 2-plane (i.e. $\pi_1(x, y, y, x) \neq 0$) spanned by vectors $x, y \in T_p M, p \in M$. Then, it is known, the sectional curvature of $\alpha$ is defined by the following equation

$$k(\alpha) = k(x, y) = \frac{R(x, y, y, x)}{\pi_1(x, y, y, x)}.$$  

The basic sectional curvatures in $T_p M$ with an almost complex structure and a Norden metric $g$ are

- **holomorphic sectional curvatures** if $J\alpha = \alpha$;
- **totally real sectional curvatures** if $J\alpha \perp \alpha$ with respect to $g$.

In [6], a **holomorphic bisectional curvature** $h(x, y)$ for a pair of holomorphic 2-planes $\alpha_1 = \{x, Jx\}$ and $\alpha_2 = \{y, Jy\}$ is defined by

$$h(x, y) = -\frac{R(x, Jx, y, Jy)}{\sqrt{\pi_1(x, Jx, x, Jx)\pi_1(y, Jy, y, Jy)}}$$

where $x, y$ do not lie along the totally isotropic directions, i.e. the both of the couples $(g(x, x), g(x, Jx))$ and $(g(y, y), g(y, Jy))$ are different from the couple $(0, 0)$. The holomorphic bisectional curvature is invariant with respect to the basis of the 2-planes $\alpha_1$ and $\alpha_2$. In particular, if $\alpha_1 = \alpha_2$, then the holomorphic bisectional curvature coincides with the holomorphic sectional curvature of the 2-plane $\alpha_1 = \alpha_2$.

### 1.3. Isotropic Kähler manifolds

The square norm $||\nabla J||^2$ of $\nabla J$ is defined in [4] by

$$||\nabla J||^2 = g^{ij}g^{kl}g\left((\nabla e_i) J e_k, (\nabla e_j) J e_l\right).$$

Having in mind the definition (1.2) of the tensor $F$ and the properties (1.3), we obtain the following equation for the square norm of $\nabla J$

$$||\nabla J||^2 = g^{ij}g^{kl}g^{pq}F_{ikp}F_{jlq},$$

where $F_{ikp} = F(e_i, e_k, e_p)$.

**Definition 1.1** ([5]). An almost complex manifold with Norden metric satisfying the condition $||\nabla J||^2 = 0$ is called an **isotropic Kähler manifold with Norden metric**.

**Remark 1.1.** It is clear, if a manifold belongs to the class $W_0$, then it is isotropic Kählerian but the inverse statement is not always true.

### 1.4. Quasi-Kähler manifolds with Norden metric

The only class of the three basic classes, where the almost complex structure is not integrable, is the class $W_3$ – the class of the **quasi-Kähler manifolds with Norden metric**.

Let us remark that the definitional condition from (1.5) implies the vanishing of the Lie form $\theta$ for the class $W_3$. 
The curvature tensor $R$ satisfies the following identity

$$R(X, JZ, Y, JU) + R(X, JY, U, JZ) + R(X, JY, Z, JU) + R(X, JZ, U, JY) + R(JX, Z, JY, U) + R(JX, Y, JU, Z) + R(JX, Y, JZ, U) + R(JX, Z, JU, Y) + R(JX, U, JY, Z) + R(JX, U, JZ, Y) = -\bar{S}_{X,Y,Z} g\left(\nabla_X JY + \nabla_Y JX, \nabla_Z JU + \nabla_U JZ\right).$$

As it is known, a manifold with Levi-Civita connection $\nabla$ is locally symmetric if and only if its curvature tensor is parallel, i.e. $\nabla R = 0$. According to (1.9), (1.10), $\nabla g = 0$ and (1.15), we establish the truthfulness of the following

**Theorem 1.1.** Every $2n$-dimensional $W_3$-manifold with vanishing Weyl tensor and constant scalar curvature is locally symmetric.

Let us remark that a necessary condition for a $W_3$-manifold to be isotropic Kählerian is given by the following

**Theorem 1.2** ([5]). Let $(M, J, g)$, dim $M \geq 4$, be a $W_3$-manifold and $R(x, Jx, y, Jy) = 0$ for all $x, y \in T_p M$. Then $(M, J, g)$ is an isotropic Kähler $W_3$-manifold.

2. The Lie Group as a 4-Dimensional $W_3$-Manifold

Let $V$ be a 4-dimensional vector space and consider the structure of the Lie algebra defined by the brackets $[E_i, E_j] = C^k_{ij} E_k$, where $\{E_1, E_2, E_3, E_4\}$ is a basis of $V$ and $C^k_{ij} \in \mathbb{R}$. Then the Jacobi identity for $C^k_{ij}$ holds.

$$C^k_{ij} C^l_{ks} + C^k_{js} C^l_{ki} + C^k_{ri} C^l_{kj} = 0 \quad (2.1)$$

Let $G$ be the associated connected Lie group and $\{X_1, X_2, X_3, X_4\}$ is a global basis of left invariant vector fields induced by the basis of $V$. Then we define an almost complex structure by the conditions

$$JX_1 = X_3, \quad JX_2 = X_4, \quad JX_3 = -X_1, \quad JX_4 = -X_2. \quad (2.2)$$

Let us consider the left invariant metric defined by the the following way

$$g(X_1, X_1) = g(X_2, X_2) = -g(X_3, X_3) = -g(X_4, X_4) = 1, \quad g(X_i, X_j) = 0 \quad \text{for} \quad i \neq j. \quad (2.3)$$

The introduced metric is a Norden metric because of (2.2).

In this way, the induced 4-dimensional manifold $(G, J, g)$ is an almost complex manifold with Norden metric. We then determine the structural constants $C^k_{ij}$ such that the manifold $(G, J, g)$ is of class $W_3$.

Let $\nabla$ be the Levi-Civita connection of $g$. Then the following well-known condition is valid

$$2g(\nabla_X Y, Z) = X g(Y, Z) + Y g(X, Z) - Z g(X, Y) + g([X, Y], Z) + g([Z, X], Y) + g([Z, Y], X). \quad (2.4)$$
Having in mind (2.2), (2.3) and (2.4) we get immediately the following equation for the tensor $F$ defined by (1.2):

$$F(X_i, X_j, X_k) = g \left( [X_i, JX_j] + [X_j, JX_i], X_k \right) + g \left( [X_j, JX_k] + [X_k, JX_j], X_i \right) + g \left( [X_k, JX_i] + [X_i, JX_k], X_j \right).$$

(2.5)

The condition for $(G, J, g)$ to be of $\mathcal{W}_3$-manifold is the vanishing of (2.5). Having in mind (2.2), (2.3) and the skew-symmetry of the Lie brackets, we determine at the first stage the constants $C^k_{ij}$ from the vanishing of the right side of (2.5) and hence we obtain the commutators of the basis vector fields as follows:

$$[X_1, X_3] = \lambda_2 X_2 + \lambda_4 X_4,$$

$$[X_2, X_4] = \lambda_1 X_1 + \lambda_3 X_3,$$

$$[X_2, X_3] = \lambda_5 X_1 + \lambda_6 X_2 + \lambda_7 X_3 + \lambda_8 X_4,$$

$$[X_3, X_4] = \lambda_9 X_1 + \lambda_{10} X_2 + \lambda_{11} X_3 + \lambda_{12} X_4,$$

$$[X_4, X_1] = (\lambda_2 + \lambda_3) X_1 + (\lambda_1 + \lambda_6) X_2 + (\lambda_4 + \lambda_7) X_3 + (\lambda_3 + \lambda_8) X_4,$$

$$[X_2, X_1] = (\lambda_9 + \lambda_4) X_1 + (\lambda_{10} - \lambda_3) X_2 + (\lambda_{11} - \lambda_2) X_3 + (\lambda_{12} + \lambda_1) X_4,$$

where the coefficients $\lambda_i \in \mathbb{R}$ ($i = 1, 2, \ldots, 12$) must satisfy the Jacobi condition (2.1).

Conversely, let us consider an almost complex manifold with Norden metric $(G, J, g)$ defined by the above manner, where the conditions (2.6) hold. We check immediately that the right side of (2.5) vanishes. Then the left side of (2.5) is zero and according (1.5) we obtain that $(G, J, g)$ belongs to the class $\mathcal{W}_3$.

Now we put the condition the Norden metric $g$ to be a Killing metric of the Lie group $G$ with the corresponding Lie algebra $\mathfrak{g}$, i.e.

$$g(\text{ad}X(Y), Z) = -g(Y, \text{ad}X(Z)),$$

(2.7)

where $X, Y, Z \in \mathfrak{g}$ and $\text{ad}X(Y) = [X, Y]$. It is equivalent to the condition the metric $g$ to be an invariant metric, i.e.

$$g([X, Y], Z) + g([X, Z], Y) = 0.$$

(2.8)

Then, according to (2.8), the equations (2.6) take the following form:

$$[X_1, X_3] = \lambda_2 X_2 + \lambda_4 X_4,$$

$$[X_2, X_4] = \lambda_1 X_1 + \lambda_3 X_3,$$

$$[X_2, X_3] = -\lambda_2 X_1 - \lambda_3 X_4,$$

$$[X_3, X_4] = -\lambda_4 X_1 + \lambda_3 X_2,$$

$$[X_4, X_1] = \lambda_1 X_2 + \lambda_4 X_3,$$

$$[X_2, X_1] = -\lambda_2 X_3 + \lambda_1 X_4,$$

(2.9)

By direct verification we prove that the commutators from (2.9) satisfy the Jacobi identity. The Lie groups $G$ thus obtained are of a family which is characterized by four parameters $\lambda_i$ ($i = 1, \ldots, 4$).

Therefore, for the manifold $(G, J, g)$ constructed above, we establish the truthfulness of the following
Theorem 2.1. Let \((G, J, g)\) be a 4-dimensional almost complex manifold with Norden metric, where \(G\) is a connected Lie group with corresponding Lie algebra \(\mathfrak{g}\) determined by the global basis of left invariant vector fields \(\{X_1, X_2, X_3, X_4\}\); \(J\) is an almost complex structure defined by (2.2) and \(g\) is an invariant Norden metric determined by (2.3) and (2.8). Then \((G, J, g)\) is a quasi-Kähler manifold with Norden metric if and only if \(G\) belongs to the 4-parametric family of Lie groups determined by the conditions (2.9).

3. Geometric characteristics of the constructed manifold

Let \((G, J, g)\) be the 4-dimensional quasi-Kähler manifold with Norden metric introduced in the previous section.

From the Levi-Civita connection (2.4) and the condition (2.8) we have

\[
\nabla_{X_i} X_j = \frac{1}{2} [X_i, X_j] \quad (i, j = 1, 2, 3, 4).
\]

3.1. The components of the tensor \(F\). Then by direct calculations, having in mind (1.2), (2.2), (2.3), (2.9) and (3.1), we obtain the nonzero components of the tensor \(F\) as follows

\[
-F_{122} = -F_{144} = 2F_{212} = 2F_{221} = 2F_{234} = 2F_{243} = 2F_{414} = -2F_{423} = -2F_{432} = 2F_{441} = \lambda_1,
\]

\[
2F_{112} = 2F_{121} = 2F_{134} = 2F_{143} = -2F_{211} = -2F_{233} = -2F_{314} = 2F_{323} = 2F_{332} = -2F_{341} = \lambda_2,
\]

\[
2F_{214} = -2F_{223} = -2F_{232} = 2F_{241} = F_{322}
\]

\[
F_{344} = -2F_{412} = -2F_{421} = -2F_{434} = -2F_{443} = \lambda_3,
\]

\[
-2F_{114} = 2F_{123} = 2F_{132} = -2F_{141} = -2F_{312} = -2F_{321} = -2F_{334} = -2F_{343} = F_{411} = F_{433} = \lambda_4,
\]

where \(F_{ijk} = F(X_i, X_j, X_k)\).

3.2. The square norm of the Nijenhuis tensor. Let \(N\) be the Nijenhuis tensor of the almost complex structure \(J\) on \(G\), i.e.

\[
N(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY], \quad X, Y \in \mathfrak{g}.
\]

Having in mind (2.9), we obtain the nonzero components \(N_{ij} = N(X_i, X_j)\) \((i, j = 1, 2, 3, 4)\) as follows

\[
N_{12} = -N_{34} = 2(\lambda_4 X_1 - \lambda_3 X_2 + \lambda_2 X_3 - \lambda_1 X_4),
\]

\[
N_{14} = -N_{23} = 2(\lambda_2 X_1 - \lambda_1 X_2 - \lambda_4 X_3 + \lambda_3 X_4).
\]

Therefore its square norm \(\|N\|^2 = g^{ik} g^{ks} g(N_{ij}, N_{ks})\) has the form

\[
\|N\|^2 = -32 (\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2),
\]

where \(g\) is the Norden metric on \(G\).
where the inverse matrix of $g$ has the form

$$
(g^{ij}) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
\end{pmatrix}
$$

**Proposition 3.1.** The manifold $(G, J, g)$ is isotropic Kählerian if and only if its Nijenhuis tensor is isotropic.

### 3.3. The square norm of $\nabla J$

According to (2.3) and (3.6), from (1.13) we obtain the square norm of $\nabla J$ as

$$
||\nabla J||^2 = 4 \left( \lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2 \right).
$$

The last equation in accordance with Definition 1.1 implies the following

**Proposition 3.2.** The manifold $(G, J, g)$ is isotropic Kählerian if and only if the condition $\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2 = 0$ holds.

**Remark 3.1.** The last theorem means that the set of vectors with the coordinates $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ at an arbitrary point $p \in G$ describes the isotropic cone in $T_p G$.

### 3.4. The components of $R$

Let $R$ be the curvature tensor of type $(0,4)$ determined by (1.7) and (1.9) on $(G, J, g)$. We denote its components by $R_{ijks} = R(X_i, X_j, X_k, X_s)$ ($i, j, k, s = 1, 2, 3, 4$). Using (3.1) and the Jacobi identity we receive

$$
R_{ijks} = -\frac{1}{4}g \left( [[X_i, X_j], X_k], X_s \right).
$$

From the last formula and (2.9) we get the nonzero components of $R$ as follows

$$
R_{1221} = -\frac{1}{4} \left( \lambda_1^2 + \lambda_2^2 \right), \quad R_{1331} = \frac{1}{4} \left( \lambda_1^2 - \lambda_2^2 \right), \\
R_{1441} = -\frac{1}{4} \left( \lambda_1^2 - \lambda_2^2 \right), \quad R_{2332} = \frac{1}{4} \left( \lambda_2^2 - \lambda_3^2 \right), \\
R_{2442} = \frac{1}{4} \left( \lambda_2^2 - \lambda_3^2 \right), \quad R_{3443} = \frac{1}{4} \left( \lambda_3^2 + \lambda_4^2 \right), \\
R_{1341} = R_{2342} = -\frac{1}{4} \lambda_1 \lambda_2, \quad R_{2132} = -R_{4134} = \frac{1}{4} \lambda_1 \lambda_3, \\
R_{1231} = -R_{4234} = \frac{1}{4} \lambda_1 \lambda_4, \quad R_{2142} = -R_{3143} = \frac{1}{4} \lambda_2 \lambda_3, \\
R_{1241} = -R_{3243} = \frac{1}{4} \lambda_2 \lambda_4, \quad R_{3123} = R_{4124} = \frac{1}{4} \lambda_3 \lambda_4.
$$

### 3.5. The components of $\rho$ and the value of $\tau$

Having in mind (1.8) and (3.6) we obtain the components $\rho_{ij} = \rho(X_i, X_j)$ ($i, j = 1, 2, 3, 4$) of the Ricci tensor and the scalar curvature $\tau$

$$
\rho_{11} = -\frac{1}{2} \left( \lambda_1^2 + \lambda_2^2 - \lambda_3^2 \right), \quad \rho_{22} = -\frac{1}{2} \left( \lambda_1^2 + \lambda_2^2 - \lambda_4^2 \right), \\
\rho_{33} = \frac{1}{2} \left( \lambda_2^2 - \lambda_3^2 - \lambda_4^2 \right), \quad \rho_{44} = \frac{1}{2} \left( \lambda_1^2 + \lambda_3^2 - \lambda_4^2 \right), \\
\rho_{12} = -\frac{1}{2} \lambda_3 \lambda_4, \quad \rho_{13} = \frac{1}{2} \lambda_1 \lambda_3, \quad \rho_{14} = \frac{1}{2} \lambda_2 \lambda_3, \\
\rho_{23} = \frac{1}{2} \lambda_1 \lambda_4, \quad \rho_{24} = \frac{1}{2} \lambda_2 \lambda_4, \quad \rho_{34} = -\frac{1}{2} \lambda_1 \lambda_2.
$$

$$
\tau = -\frac{3}{2} \left( \lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2 \right),
$$

i.e. the scalar curvature on $(G, J, g)$ is constant.

The last equation and Proposition 3.2 imply immediately
Proposition 3.3. The manifold \((G, J, g)\) is isotropic Kählerian if and only if it is scalar flat.

3.6. The Weyl tensor. Now, let us consider the Weyl tensor \(W\) on \((G, J, g)\) defined by (1.9) and (1.10). Taking into account (2.3), (3.9), (3.10) and (3.11), we establish the truthfulness of the following

Theorem 3.4. The manifold \((G, J, g)\) has vanishing Weyl tensor.

Theorem 3.5. The manifold \((G, J, g)\) is locally symmetric.

3.7. The sectional curvatures and the holomorphic bisectional curvature. Let us consider the characteristic 2-planes \(\alpha_{ij}\) spanned by the basis vectors \(\{X_i, X_j\}\) at an arbitrary point of the manifold:

- holomorphic 2-planes - \(\alpha_{13}, \alpha_{24}\);
- totally real 2-planes - \(\alpha_{12}, \alpha_{14}, \alpha_{23}, \alpha_{34}\).

Then, using (1.10), (1.11), (2.3) and (3.9), we obtain the corresponding sectional curvatures

\[
\left\{
\begin{array}{ll}
k(\alpha_{13}) &= -\frac{1}{4} (\lambda_2^2 - \lambda_3^2), \\
k(\alpha_{24}) &= -\frac{1}{4} (\lambda_2^2 - \lambda_3^2), \\
k(\alpha_{12}) &= -\frac{1}{4} (\lambda_1^2 + \lambda_2^2), \\
k(\alpha_{14}) &= -\frac{1}{4} (\lambda_1^2 + \lambda_2^2), \\
k(\alpha_{23}) &= -\frac{1}{4} (\lambda_2^2 - \lambda_3^2), \\
k(\alpha_{14}) &= -\frac{1}{4} (\lambda_2^2 - \lambda_3^2).
\end{array}
\right.
\]

Taking into account (1.12), (2.3) and (3.9), we obtain that the holomorphic bisectional curvature of the unique pair of basis holomorphic 2-planes \(\{\alpha_{13}, \alpha_{24}\}\) vanishes, i.e.

\[(3.13)\quad h(X_1, X_2) = 0.\]

3.8. The isotropic-Kählerian property. Having in mind the Propositions 3.1–3.3 in the previous subsections, we give the following

Theorem 3.6. The following conditions are equivalent for the manifold \((G, J, g)\):

(i) \((G, J, g)\) is isotropic Kähler manifold;
(ii) the Nijenhuis tensor is isotropic;
(iii) the condition \(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2 = 0\) holds;
(iv) the scalar curvature vanishes.

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