A general quasi-local flat space/boson star transition model and holography

Yan Peng

1 School of Mathematical Sciences, Qufu Normal University, Qufu, Shandong 273165, China

Abstract

We study a general flat space and boson star transition model on the asymptotically flat gravity background with Stückelberg mechanism and box boundary conditions. Similar to holographic theories, we disclose properties of phase transitions mostly from the condensation of a parameter related to behaviors of scalar fields around the box boundary. We find new types of first order quasi-local thermodynamical transitions in this general model. We mainly study effects of the scalar mass and Stückelberg mechanism on the critical phase transition points and the order of transitions. We also obtain solutions corresponding to higher energy states and examine stability of solutions with different energy states. Moreover, we mention that properties of transitions in this general asymptotically flat gravity system with box boundary conditions are strikingly similar to those in holographic insulator/superconductor transitions in the AdS gravity.

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* yanpengphy@163.com
I. INTRODUCTION

There is more and more evidence that scalar fields may exist in the universe. If the scalar field do indeed exist, such a boson cloud would gravitationally collapse into a new type of a compact object called boson stars. In 1968, Kaup constructed boson stars using a complex massive scalar field with gravitational interactions in a semiclassical manner. It was believed that these hypothetical boson stars could be an alternative explanation for parts of the dark matter. At present, there are lots of discussions on mini-, axidilaton, soliton, charged, oscillating and rotating boson stars according to the self-interaction potential of the scalar field and the spacetime symmetry, for references see. On the other side of AdS/CFT, the strongly interacting theories on the boundary are dual to the higher dimensional weakly coupled AdS gravity theories in the bulk. This correspondence has been used to construct holographic transition models to describe condensed matter physics where conventional calculational methods fail. Recently, Betti Hartmann and Jürg Riedel constructed a holographic transition model dual to glueball condensates on the background of asymptotically AdS boson star according to the AdS/CFT correspondence. Partly with the interest of holographic theories, there were also articles paying attentions to the similarity between transitions in quasi-local charged boson stars and those in the AdS gravity.

In the Einstein-Maxwell theory, it was shown that transitions of the gravity system in a box is similar to those of the AdS gravity. Lately, P. Basu, C. Krishnan and P.N.B. Subramanian further generalized such quasi-local gravity systems by adding a charged scalar field with mirror-like reflecting conditions on the box boundary. There are four possible phases as flat space, normal black hole, boson star and hairy black hole in this gravity system. It was also found that the overall phase structure of this Einstein-Maxwell-scalar system in a box is strikingly similar to that of holographic superconductors in AdS gravity. We also examined how the scalar mass and scalar charge could affect the critical phase transition points and the order of transitions and concluded that properties of flat space/boson star transitions in a box are qualitatively the same to those in holographic insulator/superconductor systems in AdS gravity. On the side of AdS gravity, it was found that the Stückelberg mechanism considering higher correction terms of the scalar field $\psi$ in the form $\psi^2 + \zeta \psi^4$ with $\zeta$ as the model parameter usually triggers the first order normal states/superconducting states phase transition. Except for holographic transitions in AdS gravity, we have also found in asymptotically flat background that this Stückelberg mechanism could trigger first order
flat space/boson star transitions in quasi-local gravity [39]. Lately, a new St"uckelberg mechanism with higher correction terms of the form $\psi^2 + \zeta \psi^6$ was discussed in the AdS spacetime [47, 48], which admits a novel type of first order transitions between superconducting states. The St"uckelberg mechanism provides a way to include higher correction terms of the scalar field, which is different from cases of boson stars usually introducing higher correction terms within the self-interacting potential of the scalar field, for a review see [4].

As a further step along this line, it is very interesting to extend this $\psi^2 + \zeta \psi^6$ St"uckelberg mechanism to the flat space/boson star system to establish a complete transition model and examine whether this new type of St"uckelberg mechanism could trigger a first order transition between boson star states. In addition, we also want to compare effects of the St"uckelberg mechanism in the quasi-local asymptotically flat gravity system and those of the AdS gravity.

We organize this paper as follows. In the next section, we will firstly construct a general flat space/boson star transition model containing a scalar field and a Maxwell field coupled on the background of the four dimensional asymptotically flat spacetime with box boundary conditions. Then we turn to study various types of phase transitions by choosing different values of the scalar mass and St"uckelberg mechanism parameters in section III. The summary of our main results will be presented in the end of this work.

II. EQUATIONS OF MOTION AND BOUNDARY CONDITIONS

We begin with introducing the flat space/boson star transition model containing a scalar field and a Maxwell field coupled on the background of the four dimensional asymptotically flat spacetime confined in a box. In this paper, we choose a fixed radial coordinate $r = r_b$ as the time-like box boundary. And the corresponding general Lagrange density in St"uckelberg form reads [43–48]:

$$\mathcal{L} = R - \left[ \frac{1}{4} F^{MN} F_{MN} - \partial \psi^2 - G(\psi)(\partial \theta - qA_\mu)^2 - m^2 \psi^2 \right],$$  

where $m$ is the mass of the scalar field $\psi(r)$ and $A_M$ stands for the ordinary Maxwell field. $q$ is the charge of the scalar field representing the coupling between the scalar field and the Maxwell field. We establish a St"uckelberg transition model in asymptotically flat gravity by considering a simple form $G(\psi) = \psi^2 + q^5 \zeta \psi^6$ with $\zeta$ as the model parameter, which has been discussed in holographic superconductor models in AdS gravity [47, 48]. Different from the self-interacting terms of the scalar field usually studied in boson star background [4], our higher correction terms of the scalar field here are coupled. Using the gauge symmetry $A_\mu \rightarrow A_\mu + \partial \alpha$, $\theta \rightarrow \theta + \alpha$, we can set $\theta = 0$ without loss of generality.
We choose matter fields with only radial dependence as
\[ A = \phi(r)dt, \quad \psi = \psi(r). \] (2)

Considering the matter fields’ backreaction on the metric, we take the deformed four dimensional boson star solution as
\[ ds^2 = -g(r)h(r)dt^2 + \frac{dr^2}{g(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \] (3)

Here, we impose \( g(\infty) = 1 \) and \( h(\infty) = 1 \) to recover the flat spacetime at the infinity.

With above assumptions, we obtain equations of motion as
\[
\frac{1}{2} \psi'(r)^2 + \frac{g'(r)}{rg(r)} + \frac{q^2G(\psi)\phi(r)^2}{2g(r)^2h(r)} + \frac{\phi'(r)^2}{g(r)h(r)} - \frac{1}{r^2g(r)} + \frac{1}{r^2} + \frac{m^2}{2g} \psi^2 = 0, \] (4)

\[
h'(r) - rh(r)\psi'(r)^2 - \frac{q^2G(\psi)\phi(r)^2}{g(r)^2} = 0, \] (5)

\[
\phi'' + \frac{2\phi'(r)}{r} - \frac{h'(r)\phi'(r)}{2h(r)} - \frac{q^2G(\psi)\phi(r)}{2g(r)} = 0, \] (6)

\[
\psi'' + \frac{g'(r)\psi'(r)}{g(r)} + \frac{h'(r)\psi'(r)}{2h(r)} + \frac{2\psi'(r)}{r} + \frac{q^2G'(\psi)\phi(r)^2}{2g(r)^2h(r)} - \frac{m^2}{g} \psi = 0, \] (7)

Where \( G'(\psi) = \frac{dG(\psi)}{d\psi} = 2\psi + 6q^2\zeta\psi^6 \). In order to study properties of transitions in detail, we search for numerical solutions satisfying boundary conditions since these equations are nonlinear and coupled. Here, we integrate the equations from \( r = 0 \) to box boundary \( r = r_b \) with shooting methods. Around \( r = 0 \), the solutions can be putted in the form
\[
\psi(r) = a + br^2 + \cdots, \]
\[
\phi(r) = aa + bbr^2 + \cdots, \]
\[
g(r) = 1 + Ar^2 + \cdots, \]
\[
h(r) = AA + BBr^2 + \cdots, \] (8)

with the dots representing higher order terms. Putting these expansions into equations of motion and considering leading terms, we could use three independent parameters \( a, aa \) and \( AA \) to describe the solutions. With the rescaling \( r \to \alpha r \), we can also set the box boundary as \( r_b = 1 \). Around \( r_b = 1 \), the matter fields behave as
\[
\psi \to \psi_1 + \psi_2(1 - r) + \cdots, \] (9)
\[
\phi \to \phi_1 + \phi_2(1 - r) + \cdots, \] (10)
where $\mu = \phi(1) = \phi_1$ is the chemical potential. With the symmetry $h \rightarrow \beta^2 h$, $\phi \rightarrow \phi$, $t \rightarrow \frac{t}{\beta}$, we can make a transformation of the solutions to set $g_{tt}(1) = -1$ \[38\]. Since we impose mirror boundary conditions for the scalar fields or $\psi(r_b) = 0$, we can take $\psi_1 = 0$ and try to use the other parameter $\psi_2$ to disclose properties of transitions in this quasi-local ensemble similar to approaches in holographic superconductor theories on the background of AdS gravity. Our box boundary condition here is independent of the mass of scalar fields, which is different from that in holographic superconductor theories where asymptotic behaviors of scalar fields at infinity boundary usually depend on the scalar mass.

III. PROPERTIES OF SCALAR CONDENSATION

In this part, we firstly study properties of transitions through the free energy of the gravity system. We show the free energy as a function of the chemical potential in Fig. 1 with $q = 100$, $m^2 = -2$ and various $\zeta$. It can be seen from (a), (b) and (c) in Fig. 1 that, for the small model parameters ($\zeta = 0$, $0.004$ or $0.006$), $F$ decreases smoothly near the critical chemical potential $\mu_c = 0.02805$ indicating the second order phase transitions from flat space state into boson star state. What’s more, in (c) the plot with $\zeta = 0.006$, besides the second order phase transition at the critical chemical potential $\mu_c = 0.02805$, the free energy develops a “swallow tail” at $\tilde{\mu}_c = 0.02925$, a typical signal for a first order phase transition to occur, indicating that there is a new phase transition between the boson star phases. (d) of Fig. 1 shows that as $\zeta$ increases to $0.01$, $\Delta F$ develops a discontinuity in the first derivative of the free energy with respect to the chemical potential at the critical points $\mu_c = 0.02793$, which implies that strong Stuckelberg mechanism triggers the first order flat space/boson star phase transition.

Inspired by holographic theories, we also try to disclose properties of transitions from condensation behaviors of the parameter $\psi_2$ in Fig. 2. We can see that the parameter $\psi_2$ increases monotonically as a function of the chemical potential in (a) with $\zeta = 0$ and (b) with $\zeta = 0.004$ around the phase transition point $\mu = 0.02805$ corresponding to the second order phase transition around $\mu_c = 0.02805$ in (a) and (b) of Fig. 1. However, in (c) as we choose the parameter $\zeta = 0.006$, the curve starts from $\psi_2 = 0$ at the critical chemical potential $\mu = 0.02805$ signaling a second order phase transition, and a jump of the parameter appears at $\mu = 0.02925$ corresponding to the first order transition around $\tilde{\mu}_c = 0.02925$ in (c) of Fig.1. As the model parameter $\zeta$ grows up, it can be seen in (d) of Fig. 2 that the parameter $\psi_2$ starts from a finite value 0.4451 at $\mu = 0.02793$ and the jump of $\psi_2$ corresponds to the first order flat space/boson star transition around $\mu_c = 0.02793$ in (d) of Fig. 1. We conclude that the jump of the slope of the parameter $\psi_2$ with respect to the chemical
FIG. 1: (Color online) The free energy as a function of $\mu$ with $q = 100$, $m^2 = -2$ and various $\zeta$: (a) the case $\zeta = 0$, (b) the case $\zeta = 0.004$, (c) the case $\zeta = 0.006$, (d) the case $\zeta = 0.01$. The blue dashed line in each panel corresponds to the free energy of the flat space and red solid curves are with the boson star phases.

potential is with the second order phase transition and the jump of the parameter $\psi_2$ implies a first order phase transition, which are qualitatively the same to those in holographic superconductor theories.

FIG. 2: (Color online) The condensation behaviors of $\psi_2$ as a function of $\mu$. We take $q = 100$, $m^2 = -2$ and various $\zeta$ as: (a) the case $\zeta = 0$, (b) the case $\zeta = 0.004$, (c) the case $\zeta = 0.006$, (d) the case $\zeta = 0.01$. The red solid line in each panel corresponds to the stable boson star phase and the red dashed line in each panel corresponds to the unstable boson star phase.

It has been found in holographic superconductor models that the metric solutions can be used to study
properties of transitions in AdS gravity. Here, we want to examine whether metric solutions can be used to detect properties of transitions in flat space/boson star system on the background of asymptotically flat quasi-local gravity. In Fig. 3, fixing the coordinate \( r = 1 \) (on the box boundary), we depict the metric \( g(1) \) as a function of the chemical potential. In (a), (b) and (c) of Fig. 3, we find \( g(1) \) has a discontinuous slope with respect to \( \mu \) at the critical chemical potential \( \mu = 0.02805 \), which corresponds to the second order phase transition points \( \mu_c = 0.02805 \) in (a), (b) and (c) of Fig. 1. Further, around the first order phase transition points \( \tilde{\mu}_c = 0.02925 \) in (c) and \( \mu_c = 0.02793 \) in (d) of Fig. 1, we find jump of \( g(1) \). Then we concluded that the discontinuity in the first derivative of \( g(1) \) corresponds to the second order phase transition and the jump of \( g(1) \) corresponds to the first order phase transition, which are in qualitatively accordance with cases in holographic transition models [48].

According to the above discussions, we can define two parameters \( \bar{\zeta} \) and \( \tilde{\zeta} \) as two threshold values for each fixed scalar mass. When \( \zeta \leq \tilde{\zeta} \), there is only the second order flat space/boson star transition at the critical chemical potential \( \mu_c \). And for \( \tilde{\zeta} < \zeta < \bar{\zeta} \), the system experiences a second order flat space/boson star transition at \( \mu_c \) and an additional first order transition between the boson star phases at \( \tilde{\mu}_c \). In the case of \( \zeta \geq \bar{\zeta} \), a first order flat space/boson star transition appears at the critical chemical potential \( \mu_c \). With various mass \( m^2 \), we show \( \tilde{\zeta} \) and \( \bar{\zeta} \) as a function of \( m^2 \) with \( q = 100 \) in the left panel of Fig. 4. It can be easily seen from the curves that \( \tilde{\zeta} \) and \( \bar{\zeta} \) decrease very slowly and are almost a constant. In addition, we arrive at a
relation that \( \tilde{\zeta} \approx 2\zeta \), which also holds in holographic superconductor transitions \[48\]. We further plot the first order critical transition points \( \mu_c \) and \( \tilde{\mu}_c \) as a function of the scalar mass in the right panel of Fig. 4. From the picture, we conclude that more negative scalar mass corresponds to a smaller critical chemical potential or smaller mass makes the first order phase transitions easier to occur.

![Graph showing critical parameters and phase transitions](image)

**FIG. 4**: (Color online) The left panel corresponds to effects of the scalar mass on the critical parameter \( \zeta \) (red) and \( \tilde{\zeta} \) (blue) with \( q = 100 \). The red line in the right panel show the effects of the scalar mass on the first order critical flat space/boson star phase transition points \( \mu_c \) with \( q = 100 \) and \( \zeta = 0.01 \). And the blue line in the right panel represents the critical chemical potential \( \tilde{\mu}_c \) of the first order transition between boson star states as a function of the scalar mass with \( q = 100 \) and \( \zeta = 0.006 \).

**A. The stability of various solutions**

For each fixed value \( \psi(0) \), we find discrete values of \( \phi(0) \) satisfying the box boundary conditions \( \psi_2 = 0 \). As we choose various \( \phi(0) \), we obtained different families of solutions. The solutions can be labeled by the number of times that \( \psi(r) \) vanishes. We show scalar fields corresponding to the first and second states in Fig. 5 with \( q = 100, m^2 = -2, \zeta = 0 \) and \( \psi(0) = 0.1 \). The scalar field of the first state in red solid line starts from \( \psi(0) = 0.1 \) at \( r = 0 \) and decreases monotonically to zero as approaching the box boundary and the higher states of the green curves correspond to scalar fields with oscillations.

![Graph showing scalar fields](image)

**FIG. 5**: (Color online) The behaviors of the scalar fields \( \psi(r) \) with \( q = 100, m^2 = -2, \zeta = 0 \) and \( \psi(0) = 0.1 \). The curves correspond to the solutions: the 1st state \( \phi(0) = 0.02805 \) (red) and the 2nd state \( \phi(0) = 0.06121 \) (green).
In order to study the stability of states, we show the free energy of systems with different states in cases of $q = 100$, $m^2 = -2$ and $\zeta = 0$ in Fig. 6. It can be easily seen from the picture that the bottom red line has the lowest free energy and thus is the stable phase, whereas the higher energy state in green solid line with larger free energy is unstable. In this paper, we only take the first two states as an example. With analysis of more higher states, we can directly prove that the solution in (a) of Fig. 1, Fig. 2 and Fig. 3 corresponding to the first state in red line is stable.

![FIG. 6: (Color online) The picture is for the behaviors of free energy as a function of the chemical potential with $q = 100$, $m^2 = -2$ and $\zeta = 0$. The curves from bottom to top correspond to the first (red) and the second (green) energy states. The blue dashed line shows the free energy of the pure asymptotically flat space.](image)

**IV. CONCLUSIONS**

We investigated a general flat space/boson star model with St"uckelberg mechanism on the background of four dimensional asymptotically flat quasi-local gravity. We mentioned that the St"uckelberg mechanism in this work provided a way of including higher correction terms of the scalar field different from the usual approach of including higher corrections within the self-interacting potential. We mainly disclosed properties of phase transitions through a parameter $\psi_2$ related to asymptotic behaviors of scalar fields on the boundary similar to approaches in holographic superconductor theories. For each fixed values of the scalar mass, we found two critical parameters $\tilde{\zeta}$ and $\bar{\zeta}$, which are useful in determining the order of transitions. We further plotted the critical parameters $\tilde{\zeta}$ and $\bar{\zeta}$ as a function of the scalar mass and found that the parameters decrease very slowly with respect to the scalar mass. In particular, we arrived at a relation $\bar{\zeta} \approx 2\tilde{\zeta}$, which also holds in the holographic theories. Moreover, we found that the more negative scalar mass makes the two types of first order phase transitions easier to happen. We also managed to disclose properties of phase transitions by analyzing the metric solutions. We founded that the jump of slop of the metric solutions with respect to the chemical potential corresponds to a second order transition and the jump of the metric solution corresponds to a first
order phase transitions. We also studied stability of various solutions of different energy states and proved that the first state is stable. As a summary, we concluded that St"uckelberg mechanism brings richer physics in the flat space/boson star system and properties of phases transitions in this general model is qualitatively the same to holographic insulator/superconductor transitions with St"uckelberg mechanism on the AdS gravity background.

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[1] Kaup D J 1968 Klein-Gordon geon Phys. Rev. 172 1331
[2] Schunck F E 1998 A scalar field matter model for dark halos of galaxies and gravitational redshift Preprint astroph/ 9802258.
[3] Mielke E W and Schunck F E 2002 Non-topological scalar soliton as dark matter halo Phys. Rev. D 66 023503.
[4] F. E. Schunck and E. W. Mielke, General relativistic boson stars, Class. Quant. Grav. 20 (2003).
[5] Friedberg R, Lee T D and Pang Y 1987 Mini-soliton stars Phys. Rev. D 35 3640.
[6] Ruffini R and Bonazzola S 1969 Systems of self-gravitating particles in general relativity and the concept of an equation of state Phys. Rev. 187 1767.
[7] Derrick G H 1964 Comments on nonlinear wave equations as models for elementary particles J. Math. Phys. 5 1252.
[8] Torres D F 2002 Accretion disc onto a static non-baryonic compact object Nucl. Phys. B 626 377.
[9] Lee T D 1987 Soliton stars and the critical masses of black holes Phys. Rev. D 35 3637.
[10] Colpi M, Shapiro S L and Wasserman I 1986 Boson stars: Gravitational equilibria of self-interacting scalar fields Phys. Rev. Lett. 57 2485.
[11] Capozziello S, Lambiase G and Torres D F 2000 Čerenkov radiation and scalar stars Class. Quantum Grav. 17 3171.
[12] Jetzer P and van der Bij J J 1989 Charged boson stars Phys. Lett. B 227 341.
[13] Mielke E W and Scherzer R 1981 GeonCtype solutions of the nonlinear Heisenberg-Klein-Gordon equation Phys. Rev. D 24 2111.
[14] Lee T D and Pang Y 1992 Non-topological solitons Phys. Rep. 221 251.
[15] Liddle A R and Madsen M S 1992 The structure and formation of boson stars Int. J. Mod. Phys. D 1 101.
[16] J.M. Maldacena, The large-N limit of superconformal field theories and super-gravity, Adv. Theor. Math. Phys. 2 (1998) 231.
[17] S.S. Gubser, I.R. Klebanov, A.M. Polyakov, Gauge theory correlators from non-critical string theory, Phys. Lett.B 428 (1998) 105.
[18] E. Witten, Anti-deSitter space and holography, Adv. Theor. Math. Phys. 2 (1998) 253.
[19] S.A. Hartnoll, Lectures on holographic methods for condensed matter physics, Class. Quantum Gravity 26 (2009) 224002.
[20] C.P. Herzog, Lectures on holographic superfluidity and superconductivity, J.Phys. A 42 (2009) 343001.
[21] G.T. Horowitz, Introduction to holographic superconductors, Lect. Notes Phys. 828 (2011) 313.
[22] T. Nishioka, S. Ryu, T. Takayanagi, Holographic superconductor/insulator transition at zero temperature, J.High Energy Phys. 03 (2010) 131.
[23] G.T. Horowitz, B. Way, Complete phase diagrams for a holographic superconductor/insulator system, J.High Energy Phys. 11 (2010) 011.
[24] R. Gregory, S. Kanno, J. Soda, Holographic superconductors with higher curvature corrections, J.High Energy Phys. 10 (2009) 010.
[25] L. Barclay, R. Gregory, S. Kanno, P. Sutcliffe, GaussCBonnet holographic superconductors, J.High Energy Phys. 12 (2010) 029.
[26] Q. Pan, B. Wang, E. Papantonopoulos, J. Oliviera, A. Pavan, Holographic superconductors with various condensates in EinsteinGaussCBonnet gravity, Phys. Rev.D 81 (2010) 106007.
[27] Hua Bi Zeng, Yu Tian, Zhe Yong Fan, Chiang-Mei Chen, Nonlinear transport in a two dimensional holographic superconductor, Phys. Rev.D 93 (2016) 121901.
[28] Ya-Peng Hu, Huai-Fan Li, Hua-Bi Zeng, Hai-Qing Zhang, Holographic Josephson junction from massive gravity, Phys. Rev. D 93 (2016) 104009.

[29] Yunqi Liu, Yungui Gong, Bin Wang, Non-equilibrium condensation process in holographic superconductor with nonlinear electrodynamics, J. High Energy Phys. 02 (2016) 116.

[30] Xiao-Mei Kuang, Eleftherios Papantonopoulos, Building a holographic super-conductor with a scalar field coupled kinematically to Einstein tensor, J. High Energy Phys. 08 (2016) 161.

[31] F. Aprile, J.G. Russo, Models of holographic superconductivity, Phys. Rev. D 81 (2010) 026009.

[32] A. Salvio, Holographic superfluids and superconductors in dilaton gravity, J. High Energy Phys. 09 (2012) 134.

[33] R. Cai, H. Zhang, Holographic superconductors with Horava-Lifshitz black holes, Phys. Rev. D 81 (2010) 066003.

[34] J. Jing, Q. Pan, S. Chen, Holographic superconductors with Power-C Maxwell field, J. High Energy Phys. 11 (2011) 045.

[35] G.T. Horowitz, M.M. Roberts, Holographic superconductors with various condensates, Phys. Rev. D 78 (2008) 126008.

[36] J.P. Gauntlett, J. Sonner, T. Wiseman, Holographic superconductivity in M-Theory, Phys. Rev. Lett. 103 (2009) 151601.

[37] Betti Hartmann, and Jürg Läpple, Glueball condensates as holographic duals of supersymmetric Q-balls and boson stars, Phys. Rev. D 86, 104008 (2012).

[38] Pallab Basu, Chethan Krishnan, P.N. Bala Subramanian, Hairy black holes in a box JHEP 11(2016)041.

[39] Yan Peng, Studies of a general flat space/boson star transition model in a box through a language similar to holographic superconductors, JHEP 07(2017)092.

[40] Robert M. Wald, The Thermodynamics of Black Holes, book: Living Rev. Rel 420016.

[41] P. Hut, Charged black holes and phase transitions, Mon. Not. R. astr. Soc. (1977) 180, 379.

[42] G. W. Gibbons and M. J. Perry, Black Holes in Thermal Equilibrium, Phys. Rev. Lett. 36, 985.

[43] S. Franco, A. Garcia-Garcia and D. Rodriguez-Gomez, A general class of holographic superconductors, JHEP 04(2010)092.

[44] S. Franco, A.M. Garcia-Garcia and D. Rodriguez-Gomez, A holographic approach to phase transitions, Phys. Rev. D 81 (2010) 041901.

[45] Q. Pan and B. Wang, General holographic superconductor models with Gauss-Bonnet corrections, Phys. Lett. B 693 (2011) 383.

[46] Y. Peng, Q. Pan and B. Wang, Various types of phase transitions in the AdS soliton background, Phys. Lett. B 699 (2011) 383.

[47] R.-G. Cai, S. He, L. Li and L.-F. Li, Entanglement entropy and Wilson loop in Stuckelberg holographic insulator/superconductor model, JHEP 10 (2012) 107.

[48] Yan Peng and Yunqi Liu, A general holographic metal/superconductor phase transition model, JHEP 02(2015)082.