Bath assisted transport in a three-site spin chain: global vs local approach

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Within the standard weak-coupling limit, the reduced dynamics of open quantum spin chains with their two end spins coupled to two distinct heat baths at different temperatures are mainly derived using the so-called global and local approaches, in which, respectively, the spin self-interaction is and is not taken into account. In order to compare the differences between the two regimes, we examine the transport properties of an open three-site XX spin-chain. We analytically derive the exact steady state of the open chain in the global approach and its first order expansion with respect to the self-interaction in the local one. The steady state transport properties through the middle spin is then studied in order to compare the physical scenarios associated with the two regimes. By analytical and numerical means, we find that the local approach is in general not able to capture all aspects of the system transport properties and cannot be recovered from the global one in the limit of vanishing spin self-interaction. In particular, in the global approach, despite the local coupling of only the two ends of the chain to the baths, there emerge non-local effects in the form of spin sink and source terms which are not present in the local approach.

I. INTRODUCTION

Transport phenomena at the quantum scale have recently received an increasing attention, as they are of fundamental importance both in theory, for understanding the behaviour of driven many-body systems, and in applications, for the development of new quantum devices. The paradigmatic models for such studies are provided by linear chains of spins, coupled among themselves and further interacting with external baths through the spins at their ends; they allow modelling various instances of spin currents, possibly with controlled flux manipulation. Indeed, many results on the dynamics of specific realizations of such systems have been reported in the recent literature with direct applications to ultracold-atoms, light-harvesting complexes and quantum thermodynamics at large1, 2.

In presence of external baths, any quantum system needs to be treated as “open”, and its reduced dynamics, obtained by tracing over the baths degrees of freedom, becomes non unitary. In the so-called weak-coupling limit, in which the strength of the system-baths interaction is small, the system time evolution can be conveniently described in terms of a master equation in Gorini-Kossakowski-Sudarshan-Lindblad form, encoding effects of decoherence and dissipation20–29.

For a spin chain, i.e. for a system made of many interacting subsystems, the derivation of such master equation might be problematic. Indeed, due to the coupling among the spins, a so-called global master equation should emerge, that requires the diagonalization of the starting spin-chain Hamiltonian to be spelled out. The resulting dissipative dynamics is expected to favour environment induced excitation transfer between different sites (e.g. see 30–35). However, its explicit derivation could be quite difficult.

For these reasons, an alternative approach has been often followed for sufficiently small inter-spin couplings; this leads to a so-called local master equation; indeed, in its derivation the spin-spin interactions are neglected and only the local couplings of the spins at the two chain ends with the baths are taken into account (e.g. see 36–45). As a result, in this approach the decoherence and dissipative effects involve only the spins directly coupled to the external baths.

A stream of different investigations ensued with the purpose of comparing the virtues and weaknesses of the two point of views 38, 40, 41, 43, 44, 67, 78–91. The debate is still unsettled and both alternatives are regularly adopted in applications.

Aim of the present investigation is to contribute to the ongoing debate by analyzing a typical model of quantum transport: a spin-1/2 chain, with XX-type interaction, in presence of a constant transverse magnetic field, weakly coupled by means of its two end spins to two separate heat baths at different temperatures. In order to be able to obtain a completely analytic description of the chain reduced dynamics, we shall limit the discussion to a chain formed by just three sites and focus on the system transport properties corresponding to the rate of change in time of the average of the spin along the z direction at the middle site. The exact steady state of the corresponding master equation will be explicitly obtained in the global approach and up to the first order perturbation expansion with respect to the inter-spin interaction in the local approach, thus allowing for an analytic comparison of the asymptotic behaviours of the system transport properties in the two regimes.

The obtained results indicate that the local approach

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† The literature on the topic is vast; for instance, see [1]-[19]
is in general not able to capture all physical aspects of the transport phenomena emerging from the global treatment, that takes into account at all steps the self-interacting character of the spin system and is therefore, though harder to address analytically, yet more physically sound and complete. In particular, we will show that the global approach corresponds to a physical regime where, beside the currents, spin sink and source terms appear that are not present within the local approach. Moreover, we shall also see that the structure of the steady states in the two regimes makes the local one not recoverable from the global approach in the limit of vanishing inter-spin interaction.

The structure of the paper is as follows: in section II we shortly review the standard weak-coupling limit background for deriving master equations of Gorini-Kossakowski-Sudarshan-Lindblad type. In Sections III and IV we obtain the master equations in the global, respectively the local approach, we compute the stationary states and analyze the corresponding asymptotic transport properties. In section V we conclude by summarizing and discussing the results, while the more technical issues are presented in the Appendices.

II. OPEN XX SPIN CHAIN

As mentioned above, purpose of this work is the analytical study of the asymptotic transport properties of open quantum spin chains interacting with two thermal baths coupled to their end spins in the so-called global and local approaches; in order to achieve our goal, we restrict to the simplest setting of a three-site spin-1/2 chain, whereby the steady states of the open reduced dynamics can be analytically accessed in both regimes and the corresponding transport properties addressed by looking at the middle spin. In this section we shortly review the necessary techniques that will subsequently be applied to extract from the closed dynamics of the spin chain interacting with the thermal baths a fully physically consistent reduced Markovian master equation for the three spins of the chain alone.

The closed spin dynamics will be given by a nearest-neighbour XX-type inter-spin interaction in presence of a transverse constant magnetic field of strength $\Delta$, with Hamiltonian:

$$H_S = g \sum_{i=1}^{2} \left( \sigma_x^{(i)} \sigma_x^{(i+1)} + \sigma_y^{(i)} \sigma_y^{(i+1)} \right) + \Delta \sum_{i=1}^{3} \sigma_z^{(i)}, \quad (1)$$

where $\sigma_x^{(i)}, \sigma_y^{(i)}, \sigma_z^{(i)}$ are Pauli matrices attached to site $i$, and $g$ is the spin coupling constant; in absence of the inter-spin interaction, the magnetic field contribution plays the role of a ‘free’ Hamiltonian.

We then turn the spin chain into an open quantum system by coupling the two external spins to two independent Bosonic thermal baths (see Figure 1). We shall describe them by two sets of independent mode operators, $b_{\alpha}(\nu)$, $b_{\beta}^{\dagger}(\nu)$, labelled by the discrete index $\alpha = L, R$, distinguishing the two baths, and by the continuous variable $\nu$, obeying standard commutation relations, $[b_{\alpha}(\nu), b_{\beta}^{\dagger}(\nu')] = \delta_{\alpha, \beta} \delta(\nu - \nu')$. Despite their infinitely many degrees of freedom, for sake of simplicity we shall denote by $H_B^{(L)}$ and $H_B^{(R)}$ their free Hamiltonians and by

$$e^{itH_B} b_{\alpha}(\nu) e^{-itH_B} = e^{-i\nu t} b_{\alpha}(\nu) \quad (2)$$
$$e^{itH_B} b_{\alpha}^{\dagger}(\nu) e^{-itH_B} = e^{i\nu t} b_{\alpha}^{\dagger}(\nu) \quad (3)$$

their free dynamics with $H_B = H_B^{(L)} + H_B^{(R)}$.

The coupling of the baths to the spin chain, the $L$-bath to the first spin, the $R$-bath to the third one, is supposed to be weak and described by a typical system-environment Hamiltonian $H'$ of the form:

$$H' = \sum_{\alpha=L,R} \left( \sigma_+^{(\alpha)} B_{\alpha} + \sigma_-^{(\alpha)} B_{\alpha}^{\dagger} \right), \quad (4)$$

where

$$\sigma_{\pm}^{(L)} = \frac{1}{2} (\sigma_{x}^{(1)} \pm i \sigma_{y}^{(1)}) , \quad \sigma_{\pm}^{(R)} = \frac{1}{2} (\sigma_{x}^{(3)} \pm i \sigma_{y}^{(3)}) , \quad (5)$$

are spin variables of the first and the third site, while

$$B_{\alpha} = \int_0^{\infty} d\nu \ h_{\alpha}(\nu) h_{\alpha}^{\dagger}(\nu) , \quad [h_{\alpha}(\nu)]^{\dagger} = h_{\alpha}(\nu) , \quad (6)$$

are the corresponding bath operators, where $^{\dagger}$ means complex conjugation. Notice that the role of the real functions $h_{\alpha}(\nu)$ is that of smearing functions introducing an effective cutoff in the above $\nu$ integrals in order to make the bath operators $B_{\alpha}$ well-defined.

The total Hamiltonian $H$ describing the complete system, the spin-chain together with the two external baths, can thus be written as

$$H = H_0 + \lambda H', \quad \text{where} \quad H_0 = H_S + H_B , \quad (7)$$

with $\lambda \ll 1$ a small dimensionless coupling constant. The Hamiltonian $H$ generates the evolution in time of the total density matrix $\rho_{\text{tot}}, \partial_t \rho_{\text{tot}}(t) = -i[H, \rho_{\text{tot}}(t)]$, starting at $t = 0$ from the initial total state $\rho_{\text{tot}}(0)$. We shall assume chain and baths to be initially prepared in an uncorrelated state, with the statistically independent thermal baths in their equilibrium Gibbs states, whence $\rho_\beta = \rho_\beta_L \otimes \rho_\beta_R$, characterized by temperatures $T_L \equiv 1/\beta_L$ and $T_R \equiv 1/\beta_R$, respectively. Namely,

$$\rho_\beta = \frac{e^{-\beta_L H_B^{(L)}}}{\text{Tr}(e^{-\beta_L H_B^{(L)}})} \otimes \frac{e^{-\beta_R H_B^{(R)}}}{\text{Tr}(e^{-\beta_R H_B^{(R)}})}, \quad (8)$$

FIG. 1. Three-spin chain in a two bath environment: the first spin is coupled to the left bath at temperature $T_L$, and the third spin is coupled to the right bath at temperature $T_R$. 

In this section we shortly review the standard weak-coupling limit. In section IV we obtain the master equations in the global approach, respectively the local approach, we compute the stationary states and analyze the corresponding transport properties addressed by looking at the middle spin. In this section we shortly review the necessary techniques that will subsequently be applied to extract from the closed dynamics of the spin chain interacting with the thermal baths a fully physically consistent reduced Markovian master equation for the three spins of the chain alone.
whence the thermal expectations

\[
\text{Tr}_B \left( \rho_B b^*_\alpha (\nu) b_{\alpha'} (\nu') \right) = \delta_{\alpha\alpha'} \delta(\nu - \nu') n_\alpha (\nu)
\]  

(9)

\[
\text{Tr}_B \left( \rho_B b_\alpha (\nu) b^*_\alpha (\nu') \right) = \delta_{\alpha\alpha'} \delta(\nu - \nu') \left( 1 + n_\alpha (\nu) \right)
\]  

(10)

with \( n_\alpha (\nu) = \frac{1}{e^{\omega_\alpha(\nu)} - 1} \). Finally, the spins will start in a generic initial state \( \rho (0) \), so that \( \rho_{\text{tot}} (0) = \rho (0) \otimes \rho_B \).

Being interested in studying the dynamics of the spin-system, one conveniently integrates over the unobserved bath degrees of freedom and concentrates on the analysis of the reduced time evolution, formally given by the partial trace \( \text{Tr}_B \), where the partial trace \( \text{Tr}_B \) is computed over the bath degrees of freedom. In the present situation, correlations in the baths can be assumed to decay much faster than the spin-system characteristic evolution time given by the inverse of its dominant energy scale; a physically consistent master equation for the reduced density matrix \( \rho(t) \) can then be obtained in the limit of vanishingly small coupling constant \( \lambda \).

In practice, the dynamics of the reduced system is obtained by suitably rescaling the time variable, \( t \to t/\lambda^2 \) and then taking the limit \( \lambda \to 0 \), following the mathematically precise procedure known as weak coupling limit [20]-[35]. The reduced density matrix \( \rho(t) \) is then found to obey the following evolution equation:

\[
\frac{\partial \rho(t)}{\partial t} = \mathcal{H}_S [\rho] + \mathcal{D}[\rho(t)] , \quad \mathcal{H}_S [\rho] \equiv -i \left[ \mathcal{H}_S , \rho \right] , \quad (11)
\]

where

\[
\mathcal{D}[\rho] = -\lim_{T \to \infty} \frac{1}{T} \int_0^T d\tau \mathcal{U}_S(-\tau) \mathcal{D}' \mathcal{U}_S(\tau) [\rho] , \quad (12)
\]

with unitary-time evolution given by

\[
\mathcal{U}_S(\tau) [\rho] = e^{-i \tau \mathcal{H}_S} \rho e^{i \tau \mathcal{H}_S} , \quad (13)
\]

and second order perturbative approximation

\[
\mathcal{D}'[\rho] = \lambda^2 \int_0^\infty \! dt \text{Tr}_B \left( e^{i \mathcal{H}_0 t} \mathcal{H}' e^{-i \mathcal{H}_0 t} , \left[ \mathcal{H}', \rho \otimes \rho_B \right] \right) , \quad (14)
\]

with \( \mathcal{H}_0 \) as defined in [7]. For the case at hand, the integrals in (12) and (14) can be explicitly computed and the master equation for \( \rho(t) \) cast in closed form:

\[
\frac{\partial \rho(t)}{\partial t} = \mathcal{H}[\rho(t)] + \mathcal{D}[\rho(t)] \equiv \mathcal{L}[\rho(t)] , \quad (15)
\]

\[
\mathcal{H}[\rho(t)] \equiv -i \left[ \mathcal{H}_{\text{eff}} , \rho(t) \right] . \quad (16)
\]

The Hamiltonian term \( \mathcal{H} = \mathcal{H}_S + \mathcal{H}_{LS} \) consists of two pieces: the system Hamiltonian generator \( \mathcal{H}_S \) corrected by another Hamiltonian generator \( \mathcal{H}_{LS}[\rho] = -i [\mathcal{H}_{LS}, \rho] \). Then, an effective Hamiltonian \( \mathcal{H}_{\text{eff}} = \mathcal{H}_S + \mathcal{H}_{LS} \) emerges that contains a bath induced Lamb-shift contribution \( \mathcal{H}_{LS} \) besides the starting system Hamiltonian. On the other hand, the dissipative part \( \mathcal{D} \) takes a standard Gorini-Kossakowski-Sudarshan-Lindblad form, whence the dynamical semigroup generated by (15) is composed by completely positive maps. Instead, let us remark that direct use of the standard second order perturbative approximation \( \mathcal{D}'[\rho] \), so popular in applications, often leads to physical inconsistencies resulting in a dynamics for \( \rho(t) \) that in general does not preserve the positivity of probabilities [33]. As we shall now discuss, the explicit expressions of \( \mathcal{H}_{LS} \) and \( \mathcal{D} \) depend on whether the global or local approach is adopted in the derivation, namely, on whether in the Hamiltonian \( \mathcal{H}_S \) in (13) one considers or not the inter-spin XX interaction terms.

III. GLOBAL APPROACH

In deriving the master equation (15) in the global approach, no additional approximations are made besides those relative to the weak coupling limit. Therefore, in order to compute the ergodic average in (12) one needs to explicitly find the spectrum and relative eigenvectors of the spin Hamiltonian \( \mathcal{H}_S \) in (1). The eight energy eigenvalues \( E_k \) and corresponding eigenvectors \( | E_k \rangle \) are collected in Appendix A. The spin operators \( \rho^{(\alpha)} \) can then be decomposed as

\[
A_\alpha (\omega) = \sum_{E_j - E_k = \omega} | E_k \rangle \langle E_k | \rho^{(\alpha)} | E_i \rangle \langle E_i | , \quad (17)
\]

where the sum \( \Sigma \) is over all energies eigenvalues \( E_k \) and \( E_i \) with a fixed energy difference \( \omega \). Under the working assumption that \( \Delta > \sqrt{2} g \) which avoids degeneracies, the allowed values of \( \omega \) are the following positive ones

\[
\omega_0 = 2 \Delta , \quad \omega_1 = 2(\Delta + \sqrt{2} g) , \quad \omega_2 = 2(\Delta - \sqrt{2} g) , \quad (18)
\]

and their negative counterparts \( -\omega_i , i = 0, 1, 2 \). Altogether, they are such that \( \sum_{\omega_i} A_\alpha (\omega) = \sigma^{(\alpha)}_\omega \) as implied by \( \sum_{k=1}^8 | E_k \rangle \langle E_k | = 1 \), while \( \mathcal{H}_S , A_\alpha (\omega) = -i \omega A_\alpha (\omega) \). Using the operators \( A_\alpha (\omega) \), the interaction Hamiltonian in (4) reads

\[
\mathcal{H}' = \sum_{\alpha=L,R} \sum_{\omega} \left( A_\alpha (\omega) B_\alpha \right. + \left. A_\alpha^\dagger (\omega) B_\alpha \right) . \quad (19)
\]

Inserting \( \mathcal{H}' \) into (14) and the latter expression into (12), environment correlation functions appear; due to the form (8) of the environment state \( \rho_B \), the only non vanishing correlations are the following ones

\[
G_\alpha (\pm t) \equiv \text{Tr}_B \left( \rho_B B_\alpha (\pm t) B_\alpha \right)
\]

(20)

\[
G_\alpha (\pm t) \equiv \text{Tr}_B \left( \rho_B B_\alpha \right) . \quad (21)
\]

\[
G_\alpha (\pm t) = \int_0^{+\infty} d\nu e^{\pm i \nu t} [h_\alpha (\nu)]^2 (1 + n_\alpha (\nu)) , \quad (20)
\]

\[
G_\alpha (\pm t) = \int_0^{+\infty} d\nu e^{\pm i \nu t} [h_\alpha (\nu)]^2 n_\alpha (\nu) , \quad (21)
\]
where \( B_\alpha(\pm t) = e^{\pm i t H_B} B_\alpha e^{\mp i t H_B} \). One then sees that, because of the ergodic average in (12), the environment affects the reduced dynamics of the spin chain via the “half Fourier” transforms

\[
\int_0^{+\infty} dt \ e^{\pm i t \omega} G_\alpha(t) , \quad \int_0^{+\infty} dt \ e^{\pm i t \omega} \tilde{G}_\alpha(t) .
\]  

Then, using that, in a distributional sense,

\[
\int_0^{+\infty} dt \ e^{\pm i t (\omega - \nu)} = \mp i P \frac{1}{\omega - \nu} + \pi \delta(\nu - \omega) ,
\]  

where \( P \) denotes the principal value, the dissipative term in the master equation (15) is collected from the action of the Dirac deltas when inserted in (22). It reads:

\[
\mathcal{D}[\rho] = \lambda^2 \sum_{\alpha=L,R} \sum_{\omega=\omega_0} \omega \ D_\omega^\alpha [\rho] ,
\]

with

\[
D_\omega^\alpha [\rho] = C_\omega^\alpha \left[ A_\alpha(\omega) \rho A_\alpha^\dagger(\omega) - \frac{1}{2} \left\{ A_\alpha(\omega) A_\alpha^\dagger(\omega), \rho \right\} \right]
\]

\[
+ \tilde{C}_\omega^\alpha \left[ A_\alpha^\dagger(\omega) \rho A_\alpha(\omega) - \frac{1}{2} \left\{ A_\alpha(\omega) A_\alpha^\dagger(\omega), \rho \right\} \right],
\]

where only the three positive values of \( \omega \) in (18) contribute because \( \nu \geq 0 \) in \( \delta(\nu - \omega) \); explicitly,

\[
C_\omega^\alpha = 2\pi |h_\alpha(\omega)|^2 (n_\alpha(\omega) + 1) , \quad \omega > 0
\]

\[
\tilde{C}_\omega^\alpha = 2\pi |h_\alpha(\omega)|^2 n_\alpha(\omega) , \quad \omega > 0
\]

On the other hand, from the action of the principal value in (23) when inserted in (22), one gets the Lamb-shift correction \( H_{LS} \) to the Hamiltonian contribution \( H \) in (16). It amounts to \((-i)\) the commutator with the following Hamiltonian:

\[
H_{LS} = \lambda^2 \sum_{\alpha=L,R} \sum_{\omega} \left[ S_\omega^\alpha(\omega) A_\alpha^\dagger(\omega) A_\alpha(\omega)
\right.
\]

\[
+ \tilde{S}_\omega^\alpha(\omega) A_\alpha(\omega) A_\alpha^\dagger(\omega) \right] ,
\]

where the sum runs over all positive and negative \( \omega \)'s and the coefficients \( S_\omega^\alpha(\omega) \) and \( \tilde{S}_\omega^\alpha(\omega) \) read

\[
S_\omega^\alpha = P \int_0^{+\infty} d\nu \ |h_\alpha(\nu)|^2 \frac{1 + n_\alpha(\nu)}{\omega - \nu} ,
\]

\[
\tilde{S}_\omega^\alpha = P \int_0^{+\infty} d\nu \ |h_\alpha(\nu)|^2 \frac{n_\alpha(\nu)}{\nu - \omega} .
\]

Notice that, using the eigenprojections of \( H_S \) and the structure of the operators \( A_\alpha(\omega) \), one retrieves a diagonal expression for the Lamb-shift Hamiltonian:

\[
H_{LS} = \sum_{k=1}^{8} \eta_k |E_k\rangle \langle E_k| , \quad \eta_k \in \mathbb{R},
\]

which thus commutes with the system Hamiltonian \( H_S \).

Finally, the operators \( A_\alpha(\omega) \) appearing in (25) and (26), the so-called Lindblad operators, are explicitly given by:

\[
A_L(\omega_0) = \frac{1}{2} \left( \sigma_z^{(1)} - \sigma_z^{(2)} \sigma_z^{(3)} \right) ,
\]

\[
A_L(\omega_1) = \frac{1}{4} \left( \sigma_z^{(1)} - \sqrt{2} \sigma_z^{(2)} \sigma_z^{(3)} + \sigma_z^{(2)} \sigma_z^{(3)} \right) ,
\]

\[
A_L(\omega_2) = \frac{1}{4} \left( \sigma_z^{(1)} + \sqrt{2} \sigma_z^{(2)} \sigma_z^{(3)} + \sigma_z^{(2)} \sigma_z^{(3)} \right) ,
\]

\[
A_R(\omega_0) = \frac{1}{2} \left( 1 - \sigma_z^{(2)} \sigma_z^{(3)} \right) ,
\]

\[
A_R(\omega_1) = \frac{1}{4} \left( 3 - \sqrt{2} \sigma_z^{(2)} \sigma_z^{(3)} + \sigma_z^{(2)} \sigma_z^{(3)} \right) ,
\]

\[
A_R(\omega_2) = \frac{1}{4} \left( 3 + \sqrt{2} \sigma_z^{(2)} \sigma_z^{(3)} + \sigma_z^{(2)} \sigma_z^{(3)} \right) ,
\]

for \( \omega_i > 0 \), while the expressions for negative \(-\omega_i \) are obtained form \( A_\alpha(-\omega_i) = A_\alpha^\dagger(\omega_i) \), \( i = 0, 1, 2 \).

Notice that the operators \( A_\alpha(\omega) \) are non local, as they couple different spin sites: as we will see, they induce bath driven excitation transfer among different sites.

### A. Spin transport properties

To study the transport properties of the system, we shall concentrate on the rate of change in time of the average of \( \sigma_z^{(2)} \), that is on the quantity

\[
\frac{d}{dt} \text{Tr}[\sigma_z^{(2)}(t)] = \frac{d}{dt} \text{Tr}[\sigma_z^{(2)}(t)\rho(0)] ,
\]

where in the second equality the time-evolution has been conveniently transferred to the spin operator. In fact, the system dynamics can be equivalently formulated in terms of evolving spin observables \( \mathcal{O}(t) \) for any fixed initial state \( \rho(0) \): the spin observables obey the so-called “dual” master equation, obtained from (15) through the identity \( \langle \mathcal{O} \rangle \equiv \text{Tr}[\mathcal{O}(t)\rho(t)] = \text{Tr}[\mathcal{O}(t)\rho(0)] \), valid for any initial state \( \rho(0) \), so that, in general:

\[
\frac{\partial \mathcal{O}(t)}{\partial t} = i[H_{eff}, \mathcal{O}] + \tilde{D}[\mathcal{O}] \equiv \tilde{L}[\rho(t)] ,
\]

with

\[
\tilde{D}[\mathcal{O}] = \lambda^2 \sum_{\alpha=L,R} \sum_{i=0}^{2} \tilde{D}_\omega^\alpha [\mathcal{O}] ,
\]

\[
\tilde{D}_\omega^\alpha [\mathcal{O}] = C_\omega^\alpha \left[ A_\alpha^\dagger(\omega) \mathcal{O} A_\alpha(\omega) - \frac{1}{2} \left\{ A_\alpha^\dagger(\omega) A_\alpha(\omega), \mathcal{O} \right\} \right]
\]

\[
+ \tilde{C}_\omega^\alpha \left[ A_\alpha^\dagger(\omega) \mathcal{O} A_\alpha(\omega) - \frac{1}{2} \left\{ A_\alpha(\omega) A_\alpha^\dagger(\omega), \mathcal{O} \right\} \right] .
\]
The Hamiltonian contribution to the rate of change in time of the average of \( \sigma_z \), namely the one obtained from the first piece in the r.h.s. of (34), can be expressed in terms of the following operator spin currents:
\[
J^{(i,i+1)} = 4i \left( \sigma^-_i \sigma^+_{i+1} - \sigma^+_i \sigma^-_{i+1} \right), \quad i = 1, 2 ,
\]
as
\[
i \left[ H_{\text{eff}}, \sigma^{(2)}_z \right] = (1 + \kappa) \left( J^{(1,2)} - J^{(2,3)} \right),
\]
where the Lamb shift contribution is characterized by a constant
\[
\kappa = \frac{i}{8\sqrt{2}} \sum_{\alpha=L,R} \sum_{\omega=\pm\omega_1,\pm\omega_2} \left( S^{(\alpha)}_\omega - S^{(\alpha)}_\omega \right).
\]
Notice that the operator differences in (38) contribute to the continuity equation as current divergence terms.

Furthermore, it turns out that \( \omega_0 \) is not contributing to the \( \omega \) sum. An analogous behaviour holds for the dissipative contribution, as \( \tilde{D}_\alpha^{(L,R)} \left[ \sigma^{(2)}_z \right] \equiv 0 \), while for the remaining two values one has (the plus sign refers to \( \omega_1 \), the minus sign to \( \omega_2 \)):
\[
\tilde{D}_\omega^{(\alpha)} \left[ \sigma^{(2)}_z \right] = -\frac{\pi |h_\alpha(\omega)|^2}{2} \left\{ 1 + \left(1 + 2 n_\alpha(\omega) \right) \right. \\
\times \left[ \sigma^\alpha_\omega \pm \frac{1}{\sqrt{2}} \left( Q^{(1,2)} - Q^{(2,3)} \right) \right] \right\},
\]
with
\[
Q^{(i,i+1)} = \sigma^-_i \sigma^+_{i+1} + \sigma^+_i \sigma^-_{i+1}, \quad i = 1, 2 .
\]
The rate of change in time of the average of \( \sigma_z^{(2)} \) in (33) gives finally rise to the following continuity equation:
\[
\frac{d}{dt} \text{Tr}[\sigma_z^{(2)} \rho(t)] = (1 + \kappa) \text{Tr} \left[ \left( J^{(1,2)} - J^{(2,3)} \right) \rho(t) \right] + \text{Tr} \left[ \left( Q_L + Q_R \right) \rho(t) \right],
\]
with
\[
Q_\alpha = \lambda^2 \sum_{\omega=\pm\omega_1,\pm\omega_2} \tilde{D}_\omega^{(\alpha)} \left[ \sigma^{(2)}_z \right], \quad \alpha = L, R .
\]
One thus sees that, besides the current divergence contributions, the continuity equation contains also extra terms that are due to the presence of the two heat baths; these terms cannot be cast as current differences and are interpretable as source or sink contributions. As we shall see below, their role is crucial to fully understand the transport properties of the system.

### B. Steady state

Although the master equation (15), or equivalently (34), does not allow for a simple analytic solution, it admits a unique steady state, that we will explicitly compute, so that the asymptotic expression of the rate of change in time of the average of \( \sigma_z^{(2)} \) in (43) can be accessed analytically and studied numerically.

The uniqueness of the steady state can be easily established by recalling that this is the case for all master equations for which the commutant of \( i.e. \) the operators commuting with) the set of the corresponding Lindblad operators turns out to be the identity \([92]-[97]\). In the present case, it is convenient to work in the system energy eigenbasis. A generic system operator can then be written as \( X = \sum_{k=1}^{8} x_{k\ell} |E_k\rangle \langle E_{k'}| \), so that the Lindblad operators listed in (32) in the spin ‘computational basis’ can be re-expressed in the energy eigenbasis as reported in Appendix B. By explicit computation, one then shows that the only matrix \( X \) commuting with all the elements in (32) is a multiple of the identity since the entries \( x_{k\ell} \) become then of the form \( x_{k\ell} = \xi \delta_{k\ell} \), with a same \( \xi \in \mathbb{C} \). To obtain the explicit form of the steady state \( \rho_\infty \), one has to impose the vanishing of the r.h.s. of the master equation (15).

\[
\mathcal{L}[\rho_\infty] \equiv \mathcal{H}[\rho_\infty] + \mathcal{D}[\rho_\infty] = 0 .
\]

Using the expressions of the operators \( A_\alpha(\omega) \) in terms of the matrix units \( |E_j\rangle \langle E_{j'}| \) constructed by means of the eigenvectors of \( H_S \) as given in (B1) of Appendix B, one finds that \( \mathcal{D} \) maps the \( H_S \) eigenprojections into linear combinations of themselves. Therefore, asking that \( \mathcal{D}[\rho_\infty] = 0 \) on
\[
\rho_\infty = \sum_{k=1}^{8} \mu_k |E_k\rangle \langle E_k| , \quad \mu_k \in \mathbb{R} ,
\]
namely on a matrix diagonal with respect to the \( H_S \) eigenbasis, amounts to solving a system consisting of 8 linear equations in the real unknowns \( \mu_k \). Furthermore, due to the diagonal form (31) of the Lamb-shift Hamiltonian, it also turns out that \( \mathcal{H}[\rho_\infty] = 0 \). Then, if the coefficients \( \mu_k \) determined by \( \mathcal{D}[\rho_\infty] = 0 \) are positive, by the uniqueness of the stationary state, \( \rho_\infty \) as in (46) solves (45) (see Appendix B for details).

With the explicit steady state at disposal, one can now study the fate of the various contributions to the rate of change in time of the average of \( \sigma_z^{(2)} \) in (43) for asymptotically long times. First of all, as the spin-currents \( J^{(i,i+1)} \) in (47) have zero expectations with respect to the energy eigenstates \( |E_k\rangle \), they vanish in the steady state:

\[
\text{Tr} \left[ J^{(1,2)} \rho_\infty \right] = \left[ J^{(2,3)} \rho_\infty \right] = 0 .
\]

Instead, for the sink/source terms, one finds:
\[
\text{Tr} \left[ Q_L \rho_\infty \right] = \lambda^2 \frac{\pi}{4} \sum_{\omega=\pm\omega_1,\pm\omega_2} \left[ h_\omega(\omega) \right]^2 \\
\times \sum_{\omega=\pm\omega_1,\pm\omega_2} \frac{n_L(\omega) \left[ s(\omega) - \tau(\omega) \right] - \tau(\omega)}{s(\omega) + \tau(\omega)} ,
\]
with
In presence of external baths, any quantum system can be described by a master equation of the form
\[ \frac{d}{dt} \rho = \mathcal{L}[\rho] , \]
where the definition in [3] have been used and \( n_\alpha = 1/(e^{\beta_\alpha} - 1) \) are now the only two relevant Boson distribution functions. For simplicity, we have reabsorbed in a redifinition of the coupling constant \( \lambda \) the irrelevant factor \( 2\pi |H(\Delta)|^2 \), with \( h_L = h_R = h \), coming from the Fourier transforms of the thermal correlation functions (compare with [26] and [27]). Similarly, the Lamb-shift contributions to the Hamiltonian piece can be reabsorbed in a redifinition of the constant magnetic field strength \( \Delta \), so that in practice \( H_{\text{eff}} = H_S \), but now with \( q \neq 0 \). As a result, the local approach yields the following master equation for the spin density matrix:
\[ \frac{d\rho(t)}{dt} = -i[H_S, \rho(t)] + \mathcal{D}_L[\rho(t)] + \mathcal{D}_R[\rho(t)] = \mathcal{L}[\rho(t)] . \]

### A. Spin transport properties

In the local approach, the transport properties of the spin chain are also addressed by looking at the rate of change in time of the average of \( \sigma_\alpha^{(2)} \) by means of the definition given in [33]. Recalling [35], one shows that the Hamiltonian contribution to (33) can be recast again in terms of the difference of the two spin-currents \( J^{(1,2)} \) and \( J^{(2,3)} \) defined in [37]. However, no bath contributions can now arise, as the dissipative pieces in [53] do not involve the middle spin. Therefore, in the local approach, the continuity equation reads
\[ \frac{d}{dt} \text{Tr}[\sigma_\alpha^{(2)} \rho(t)] = \text{Tr} \left[ (J^{(1,2)} - J^{(2,3)}) \rho(t) \right] , \]
with no bath-induced sink/source terms.

### B. Steady state

Although the steady states of boundary-driven XX spin-chains have been studied before in terms of matrix
The eight operators $\mathcal{F}$ are diagonal, while the remaining six $\mathcal{E}$ are hermitian, off-diagonal. Clearly, the steady state $\rho_\infty$ must be a normalized, linear combination of these operators, and the condition $\mathcal{L}[\rho_\infty] = 0$, yielding a system of 14 linear equations in the unknown coefficients, will fix it completely. However, the expression of these coefficients turns out to be rather cumbersome and a compact, explicit version for them hard to find. It thus proves more convenient to seek a perturbative expression for the stationary state.

As in the local approach the coupling $g$ between the spins is considered to be small, $g \ll \Delta, \lambda$, we treat the spin-interaction as a perturbation and rewrite the dynamical generator $\mathcal{L}$ in (54) as

$$\mathcal{L} = \mathcal{L}_0 + g \mathcal{L}_1 \,,$$

where

$$\mathcal{L}_0[\rho] = -i \left[ \Delta \sum_{i=1}^{3} \sigma_z^{(i)} , \rho \right] + \mathcal{D}_L[\rho] + \mathcal{D}_R[\rho] \,,$$

$$\mathcal{L}_1[\rho] = -i \left[ \sum_{i=1}^{2} \left( \sigma_x^{(i)} \sigma_x^{(i+1)} + \sigma_y^{(i)} \sigma_y^{(i+1)} \right) , \rho \right] \,.$$

Then, expressing the steady state $\rho_\infty$ as a power series expansion:

$$\rho_\infty = \sum_{n=0}^{\infty} g^n \rho^{(n)} \,,$$

the steady state condition $\mathcal{L}[\rho_\infty] = 0$ reduces to:

$$\mathcal{L}_0[\rho^{(0)}] + \sum_{n=1}^{\infty} g^n \left( \mathcal{L}_0[\rho^{(n)}] + \mathcal{L}_1[\rho^{(n-1)}] \right) = 0 \,,$$

leading to the following recursive relations that must be satisfied for all $n$:

$$\mathcal{L}_0[\rho^{(0)}] = 0 \,, \quad \mathcal{L}_0[\rho^{(n+1)}] = -\mathcal{L}_1[\rho^{(n)}] \,.$$ 

Therefore, once a stationary state $\rho^{(0)}$ of $\mathcal{L}_0$ is chosen, its first order perturbation is obtained as

$$\rho^{(1)} = -\mathcal{L}_0^{-1} \circ \mathcal{L}_1[\rho^{(0)}] \,,$$

through the inversion of $\mathcal{L}_0$, and similarly for the higher order terms:

$$\rho^{(n+1)} = -\mathcal{L}_0^{-1} \circ \mathcal{L}_1[\rho^{(n)}] \,.$$ 

Though $\mathcal{L}_0$ is in general not invertible, $\mathcal{L}_0^{-1}$ can be defined on a subspace that does not contain elements of the kernel of $\mathcal{L}_0$ (see Appendix C for further details).

In the case at hand, the state $\rho^{(0)}$ such that $\mathcal{L}_0[\rho^{(0)}] = 0$ is of the form:

$$\rho^{(0)} = \rho_L \otimes \rho_r \otimes \rho_R \,,$$

where, using the ‘computational basis’ of spins as in Appendix A,

$$\rho_\alpha = \frac{1}{1 + 2 n_\alpha} \begin{pmatrix} n_\alpha & 0 \\ 0 & 1 + n_\alpha \end{pmatrix} \,, \quad \alpha = L, R \,,$$

are thermal states, while $\rho_r$ is an arbitrary diagonal density matrix:

$$\rho_r = \begin{pmatrix} r & 0 \\ 0 & 1 - r \end{pmatrix} \,, \quad 0 \leq r \leq 1 \,.$$ 

The $\mathcal{L}_0$-stationary state is not unique; indeed, the commutant of the Lindblad operators appearing in (58) is not the identity, rather the linear span generated by the two operators:

$$P_{\pm} = 1 \otimes \frac{1 \pm \sigma_z}{2} \otimes 1 \,.$$ 

The action of the perturbation $\mathcal{L}_1$ on $\rho^{(0)}$ can now be straightforwardly obtained; in the tensor product basis and with the ordering used in Appendix A, one finds (only the non-vanishing entries are explicitly shown):

$$\mathcal{L}_1[\rho^{(0)}] = i \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -a & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & c & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & d & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -c & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & d & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \,.$$ 

---

2 See [62] and references therein.
with
\[ a = 2r_L(r - r_R) \], \]
\[ b = 2r_R(r_L - r) \], \]
\[ c = 2(r_L + r_R)(1 - r_R) \], \]
\[ d = 2(r - r_R)(1 - r_L) \] \hspace{1cm} (70)

and
\[ r_\alpha = \frac{n_{\alpha}}{1+2n_{\alpha}} \], \hspace{0.5cm} \alpha = L, R \] \hspace{1cm} (71)

One can check that the just obtained \( \mathcal{L}_1[\rho^{(0)}] \) is not in the kernel of \( \mathcal{L}_0 \), so that \( \rho^{(1)} \) in (65) can be safely computed and found to be of the same form as the matrix in (62), but with the four constants replaced by
\[ a' = \frac{4(r_R - r)2n_L + r_L(1 + 2n_R)}{1 + 2n_R(3 + 4n_L + 2n_R)} \], \]
\[ b' = \frac{4(r - r_L)2n_R + r_L(1 + 2n_L)}{1 + 2n_R(3 + 2n_L + 4n_R)} \], \]
\[ c' = \frac{4(r - r_L)3 + 2n_L + 2n_R - r_R(1 + 2n_L)}{1 + 2n_L(3 + 2n_L + 4n_R)} \], \]
\[ d' = \frac{4(r_R - r)3 + 2n_R + 2n_L - r_L(1 + 2n_R)}{1 + 2n_L(3 + 4n_L + 2n_R)} \] \hspace{1cm} (72)

The corresponding expression for \( \rho^{(1)} \) should give the first-order correction in the expansion of the steady state \( \rho_\infty \); however, while we know \( \rho_\infty \) to be unique, both \( \rho^{(0)} \) and \( \rho^{(1)} \) still have \( r \) as a free parameter. This situation is common in perturbation theory [95]-[101]; in order to fix \( r \), one needs to examine the next perturbative order using (62), and apply \( \mathcal{L}_1 \) to \( \rho^{(1)} \). By requiring \( \mathcal{L}_1[\rho^{(1)}] \) not to belong to the kernel of \( \mathcal{L}_0 \), so that the second order perturbative contribution \( \rho^{(2)} \) can be determined, fixes uniquely the parameter \( r \):
\[ r = \frac{r_L(1 + 2n_L) + r_R(1 + 2n_R)}{2(1 + n_L + n_R)} \] \hspace{1cm} (73)

Notice that \( 0 \leq r \leq 1 \), since \( 0 \leq r_{LR, L} \leq 1 \).

In conclusion, the steady state of the master equation (54), up to the first order in the coupling \( g \), is given by
\[ \rho_\infty = \rho^{(0)} + g \rho^{(1)} \] \hspace{1cm} (74)

where \( \rho^{(0)} \) and \( \rho^{(1)} \) are as in (65) and (69), (72), with the parameter \( r \) as in (73).

A first interesting conclusion that can be drawn from the previous results is that the local regime does not emerge from the global one by letting \( g = 0 \). Indeed, in general, already the order zero expansion with respect to \( g \) of the stationary state \( \rho_\infty \) in (40) derived in the global approach differs from the order zero term \( \rho^{(0)} \) in the local approach. In order to appreciate this fact, consider the coefficients given in (B6) in Appendix B: they depend on \( g \) through the Hamiltonian \( H_S \) eigenvalues. By setting \( g = 0 \), from (18) one finds \( \omega_1 = \omega_2 = \omega_3 = 2\Delta \), whence in the expressions (49) and (60) \( n_{L,R}(\omega) = n_{L,R} \) and \( h_{L,R}(\omega) = h_{L,R} \) for all \( \omega_{1,2,3} = 2\Delta \). By further imposing \( h_L = h_R =: h \), as already done before to get (53), one finds \( \tau = x h^2 \) and \( s = h^2(2 + x) \) with \( x = n_L + n_R \).

Then, the factor \( h \) disappears from the coefficients of the vector (B7) and, using it to compute the stationary state in (B2) of Appendix B one obtains:
\[ \rho_\infty = \rho_x \otimes \rho_x \otimes \rho_x \] \hspace{0.5cm} (75)

Then the states \( \rho_\infty \) and \( \rho^{(0)} \) in (65) can coincide only for equal left and right temperatures.

The expectation that the two regimes correspond to different physical scenarios is strikingly confirmed when one analyzes the asymptotic behaviour of the spin currents \( J^{(1,2)} \) and \( J^{(2,3)} \) that enter the expression of the rate of change in time of the average of \( \sigma_x^{(2)} \) in (55). Using (74), one finds that only the first order term \( \rho^{(1)} \) in the perturbative expansion contributes to the asymptotic average of the currents:
\[ \text{Tr}[J^{(1,2)} \rho_\infty] = 8g(b' + c') \]
\[ \text{Tr}[J^{(2,3)} \rho_\infty] = 8g(a' + d') \] \hspace{1cm} (76)

These mean values are in general nonzero and vanish only when the two bath temperatures \( T_L \) and \( T_R \) are equal, as in this case \( n_L = n_R =: n \) and \( r_L = r_R =: r = \frac{1}{2n} \).

However, the total rate of change in time of the average of \( \sigma_x^{(2)} \) in (55) is always zero, as it should be in a steady state; indeed, the two expressions in (76) are equal, as the condition \( a' + d' = b' + c' \) is precisely the one that fixes the parameter \( r \) to assume the value in (73).

V. DISCUSSION

Stimulated by the ongoing debate on the different available approaches that can be adopted for analyzing the transport properties of open quantum systems, we have studied the dynamics of a three-site spin-1/2 chain, with \( XX \)-type interaction in presence of a constant magnetic field, weakly coupled at the two ends to two separate heat baths, at different temperatures. The merit of such a simplification is that it allows for an analytical determination of the asymptotic states and corresponding transport properties of the spin chain. Master equations generating the reduced spin dynamics have been derived in the weak coupling (Markovian) limit using both the global approach, where the full \( XX \) Hamiltonian is always taken into account, and the local approach, where instead the spin-spin interactions are neglected. Both types of master equations admit unique stationary states, whose forms have been explicitly derived in the global regime and up to its first order perturbation expansion with respect to the inter-spin interaction in the local regime, thus allowing a complete analytic treatment of the system asymptotic transport properties.
In particular, we have focused on the behaviour of the rate of change in time of the average of the middle spin $z$ component $\sigma_z^{(2)}$, the corresponding continuity equation allows defining spin currents involving the first two sites, $J^{(1,2)}$, whose damped (due to the baths) oscillatory (due to the Hamiltonian) behaviour is depicted in Figure 2 in both approaches. For the last two spins the behaviour of the spin current $J^{(2,3)}$ is similar. Notice that in contrast with the global case, the local master equation supports non-vanishing asymptotic values for the two spin currents, becoming zero only when the bath temperatures are equal; these values are nevertheless equal so that the asymptotic global rate of change of the average of $\sigma_z^{(2)}$ is zero, as it should be in a stationary state.

However, the most striking difference in the description of the system transport properties comes from the presence of additional bath induced pieces in the global approach continuity equation, as it also contains source and sink contributions; their behavior is shown in Figure 4, where it is also visually clear that their long-time values become equal in modulus but opposite in sign, assuring again vanishing rate of change of the average of $\sigma_z^{(2)}$ in the steady state. The origin of these source/sink contributions can be traced to the $XX$-type self-coupling among the spins: this interaction is fully taken into account by the global master equation, that indeed allows bath assisted excitation transfer between different sites. A further indication of the radical differences between the two regimes is the fact that an expansion of the steady state in the global regime with respect to the inter-spin coupling does not lead to the first order approximation of the steady state in the local regime. These results indicate that the use of the local approach to model quantum transport phenomena might not be in general appropriate, as it appears unable to capture all their physical aspects. The global treatment, by consistently considering at all steps the self-interacting character of the spin system, appears more physically motivated and suitable; in particular, it accounts for non-local transport effects, not of the current divergence type, which are only possible due to the inter-spin interaction as they disappear in the local approach.\footnote{The physical advantages of the global approach, at least for long times, seem to be confirmed by a recent result\cite{21} which shows that the dynamics of an open spin chain with Markovian approximations can be better described by a suitable convex interpolation of the dynamics in the local and global regimes with the global one prevailing asymptotically in time.}

**Appendix A**

In this Appendix we collect the explicit expressions of the eight eigenvalues and eigenvectors of the system Hamiltonian $H_S$ in \Footnote{For the energy levels one gets: $E_{1,2} = \pm 3\Delta$, $E_{3,4} = \pm \Delta$, $E_{5,6} = \pm (\Delta + 2g\sqrt{2})$, $E_{7,8} = \pm (\Delta - 2g\sqrt{2})$. (A1)}
The corresponding eigenstates, written in the ‘computational basis’ of tensor product spin states, \( |jk\ell\rangle \equiv |j\rangle \otimes |k\rangle \otimes |\ell\rangle \) with the convention \( \sigma_{z}|i\rangle = (-1)^i|i\rangle \), \( i = 0, 1 \), explicitly read:

\[
\begin{align*}
|E_1\rangle &= |000\rangle \quad (A2) \\
|E_2\rangle &= |111\rangle \quad (A3) \\
|E_3\rangle &= \frac{1}{\sqrt{2}} (|001\rangle - |100\rangle) \quad (A4) \\
|E_4\rangle &= \frac{1}{\sqrt{2}} (|011\rangle - |110\rangle) \quad (A5) \\
|E_5\rangle &= \frac{1}{2} (|001\rangle + \sqrt{2}|010\rangle + |100\rangle) \quad (A6) \\
|E_6\rangle &= \frac{1}{2} (|011\rangle - \sqrt{2}|101\rangle + |110\rangle) \quad (A7) \\
|E_7\rangle &= \frac{1}{2} (|011\rangle + \sqrt{2}|101\rangle + |110\rangle) \quad (A8) \\
|E_8\rangle &= \frac{1}{2} (|001\rangle - \sqrt{2}|010\rangle + |100\rangle) \quad (A9)
\end{align*}
\]

In addition, when writing system states as \( 8 \times 8 \) density matrices, we shall use the natural tensor product ordering, namely: \( |000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle \).

**Appendix B**

In this Appendix, we provide additional information on the determination of the stationary state of the dynamics obtained in the global approach.

The proof of the uniqueness of the stationary state requires re-expressing the Lindblad operators listed in (32) in the basis of the energy eigenstates given in the previous Appendix. One easily finds:

\[
\begin{align*}
A_L(\omega_0) &= -\frac{1}{\sqrt{2}} \left( |E_3\rangle \langle E_1| + |E_2\rangle \langle E_4| \right) \\
&\quad + |E_7\rangle \langle E_5| - |E_6\rangle \langle E_8| \\
A_L(\omega_1) &= \frac{1}{2} \left(|E_5\rangle \langle E_3| - |E_4\rangle \langle E_5| + |E_2\rangle \langle E_7| \right) \\
A_L(\omega_2) &= \frac{1}{2} \left(|E_5\rangle \langle E_3| + |E_7\rangle \langle E_3| + |E_2\rangle \langle E_5| + |E_4\rangle \langle E_8| \right) \\
A_R(\omega_0) &= \frac{1}{\sqrt{2}} \left( |E_3\rangle \langle E_1| - |E_2\rangle \langle E_4| \right) \\
&\quad + |E_7\rangle \langle E_5| - |E_6\rangle \langle E_8| \\
A_R(\omega_1) &= \frac{1}{2} \left(|E_5\rangle \langle E_3| + |E_7\rangle \langle E_3| + |E_2\rangle \langle E_5| + |E_4\rangle \langle E_8| \right) \\
A_R(\omega_2) &= \frac{1}{2} \left(|E_5\rangle \langle E_3| + |E_7\rangle \langle E_3| + |E_2\rangle \langle E_5| + |E_4\rangle \langle E_8| \right)
\end{align*}
\]

(B1)

On the other hand, for the explicit derivation of the stationary state, a diagonal ansatz in the spin energy basis suffices:

\[
\rho_\infty = \sum_{k=1}^{8} \mu_k |E_k\rangle \langle E_k|, \quad (B2)
\]

The eight unknown constants \( \mu_k \) are determined by imposing \( \rho_\infty \) to be in the kernel of the dissipator \( D \) in (24). Inserting the expression \( \rho_\infty \) into the stationary condition \( D[\rho_\infty] = 0 \) leads to a set of linear equations that can be represented as

\[
\mathcal{M} \cdot \vec{\mu} = 0, \quad (B3)
\]

where \( \vec{\mu} \) is a 8-dimensional vector with components \( \mu_k \), while \( \mathcal{M} \) is an \( 8 \times 8 \) matrix with entries:

\[
\mathcal{M}_{k,\ell} = \sum_{\alpha=L,R} \sum_\omega \text{Tr} \left[D^{(\omega)}_\alpha |E_k\rangle \langle E_\ell| |E_k\rangle \langle E_\ell| \right]. \quad (B4)
\]

It can be explicitly expressed in terms of the six quantities reported in (19)-(50). Setting \( \tau_i := \tau(\omega_i) \) and \( s_i := s(\omega_i), i = 0, 1, 2 \), it reads (only the nonvanishing entries are explicitly shown):

\[
\mathcal{M} = \begin{pmatrix}
1 & \tau_0 & \tau_1/2 & s_1/2 & s_4/2 & s_1/2 & s_0 & s_1/2 \\
\tau_0 & 1 & \tau_1/2 & s_2/2 & s_1/2 & s_2/2 & s_3/2 & s_0 \\
\tau_1/2 & s_2/2 & 1 & s_0 & s_3/2 & s_1/2 & s_2/2 & s_0 \\
\tau_0 & s_0 & s_1/2 & 1 & s_0 & s_3/2 & s_1/2 & s_0 \\
\tau_1/2 & s_3/2 & s_0 & s_0 & 1 & s_0 & s_3/2 & s_0 \\
s_0 & s_3/2 & s_0 & s_0 & s_0 & 1 & s_0 & s_1 \\
s_1/2 & s_2/2 & s_0 & s_0 & s_0 & s_0 & 1 & s_0 \\
s_1/2 & s_2/2 & s_0 & s_0 & s_0 & s_0 & s_0 & 1
\end{pmatrix}
\]

(B5)

with the diagonal terms given by

\[
\begin{align*}
m_1 &= -(s_0 + \frac{s_1 + s_2}{2}), \quad m_2 = -(\tau_0 + \frac{\tau_1 + \tau_2}{2}), \\
m_3 &= -(\tau_0 + \frac{s_1 + s_2}{2}), \quad m_4 = -(s_0 + \frac{s_3 + s_2}{2}), \\
m_5 &= -(s_0 + \frac{s_1 + s_2}{2}), \quad m_6 = -(\tau_0 + \frac{s_1 + s_2}{2}), \\
m_7 &= -(\tau_0 + \frac{s_1 + s_2}{2}), \quad m_8 = -(s_0 + \frac{s_3 + s_2}{2}).
\end{align*}
\]

(B6)

Together with normalization, \( \text{Tr}[\rho_\infty] = \sum_k \mu_k = 1 \), equation (B3) uniquely fixes the components of \( \vec{\mu} \),

\[
\vec{\mu} = \frac{1}{(s_0 + \tau_0)(s_1 + \tau_1)(s_2 + \tau_2)} \begin{pmatrix}
\tau_0 \tau_1 \tau_2 \\
\tau_0 \tau_1 \tau_2 \\
\tau_0 \tau_1 \tau_2 \\
\tau_0 \tau_1 \tau_2 \\
\tau_0 \tau_1 \tau_2 \\
\tau_0 \tau_1 \tau_2 \\
\tau_0 \tau_1 \tau_2 \\
\tau_0 \tau_1 \tau_2
\end{pmatrix}
\]

(B7)

and hence the expression of the steady state. Notice indeed that the quantities \( \tau_i := \tau(\omega_i) \) and \( s_i := s(\omega_i) \) in (19)-(50) are all positive, whence \( \rho_\infty \) is a positive and normalized \( 8 \times 8 \) matrix.
Appendix C

In this Appendix we shall discuss some general questions regarding the determination in perturbation theory of the steady state of a quantum dynamical semigroup, i.e. the dynamics generated by a master equation in Groini-Kossakowski-Sudarshan-Lindblad form.

1. General setting

Let $\gamma_t = e^{tL}$ the one-parameter semigroup generated by a master equation as in (15),

$$\frac{\partial \rho(t)}{\partial t} = L[\rho(t)] ,$$

acting on the state space $S_d$ of the system, that we assume to be $d$-dimensional, and denote by $\gamma_t$ the corresponding ‘dual’ semigroup acting on the system observables (see the discussion leading to equation (14)), $\gamma_t : M_d \rightarrow M_d$, where $M_d$ is the set of $d \times d$ complex matrices. Further, define $G$ to be the linear map from $S_d$ into itself constructed through the time-average

$$G : S_d \ni \rho \mapsto G[\rho] = \lim_{T \to +\infty} \frac{1}{T} \int_0^T dt \gamma_t[\rho] \in S_d .$$

Note that $G$ projects onto the stationary manifold of $\gamma_t$:

$$G \circ \gamma_t = \gamma_t \circ G = G^2 \; , \; \gamma_t[\rho] = \rho \iff G[\rho] = \rho ,$$

or equivalently in terms of the generator:

$$G \circ L = L \circ G = 0 \; , \; L[\rho] = 0 \iff G[\rho] = \rho .$$

Let us first discuss the conditions for the inverse $L^{-1}$ of the generator $L$ to exist. Clearly, it can be well defined only on a subspace that does not contain elements of the kernel of $L$. To this purpose, consider the operator $F := \text{id} - G$. Notice that

$$\mathcal{L} \circ F = \mathcal{F} \circ \mathcal{L} ,$$

so that the range of $\mathcal{F}$, $\text{Ran}(\mathcal{F})$, is mapped into itself by the generator $\mathcal{L}$. Extending $\mathcal{G}$ by linearity on the whole of $M_d$, one obtains that, if $\mathcal{G}[X] = 0$ for $X \in M_d$, then automatically $X \in \text{Ran}(\mathcal{F})$ as

$$\mathcal{G}[X] = 0 \implies X = X - \mathcal{G}[X] = \mathcal{F}[X] .$$

Thus $\text{Ker}(\mathcal{G}) \subseteq \text{Ran}(\mathcal{F})$. Moreover,

$$\text{Ran}(\mathcal{F}) \cap \text{Ker}(\mathcal{L}) = 0 .$$

Indeed, if $X = Y - \mathcal{G}[Y]$ and $\mathcal{L}[X] = 0$, from (C4) it follows that

$$\mathcal{L}[X] = 0 = \mathcal{L}[Y] - \mathcal{L} \circ \mathcal{G}[Y] \implies \mathcal{L}[Y] = 0 \implies \mathcal{G}[Y] = Y \implies X = Y - Y = 0 .$$

The inverse $\mathcal{L}^{-1}$ can then be defined as the map from $\text{Ran}(\mathcal{F})$ into $\text{Ran}(\mathcal{F})$ such that

$$\mathcal{L} \circ \mathcal{L}^{-1} = \mathcal{L}^{-1} \circ \mathcal{L} .$$

As inverse of $\mathcal{L}$ on $\text{Ran}(\mathcal{F})$, $\mathcal{L}^{-1}$ satisfies

$$\mathcal{L}^{-1} \circ \mathcal{G} = \mathcal{G} \circ \mathcal{L}^{-1} ,$$

whence $\mathcal{L}^{-1}(\text{Ran}(\mathcal{F})) \subseteq \text{Ran}(\mathcal{F})$ and $\text{Tr} \circ \mathcal{L}^{-1} = 0$; indeed, the trace-preserving property of $\gamma_t$, and thus of $\mathcal{G}$, entails $\text{Tr}[X] = 0$ for all $X \in \text{Ran}(\mathcal{F})$.

2. Perturbative expansion

Suppose now that the generator $\mathcal{L}$ of $\gamma_t$ has the form as in (57),

$$\mathcal{L}_g = \mathcal{L}_0 + g \mathcal{L}_1 ,$$

where $g$ is a small perturbative parameter. We seek a perturbative expansion of the stationary states $\mathcal{L}[\rho_\infty] = 0$, in the form

$$\rho_\infty = \sum_{n=0}^{\infty} g^n \rho^{(n)} ;$$

leading to the following recursive relations:

$$\mathcal{L}_0[\rho^{(0)}] = 0 \; , \; \mathcal{L}_0[\rho^{(n+1)}] = -\mathcal{L}_1[\rho^{(n)}] .$$

Therefore, once a stationary state $\rho^{(0)}$ of $\mathcal{L}_0$ is chosen, its first order perturbation can be obtained by the inversion of $\mathcal{L}_0$:

$$\rho^{(1)} = -\mathcal{L}_0^{-1} \circ \mathcal{L}_1[\rho^{(0)}] .$$

From the previous subsection, we know that this can be done by ensuring that $\mathcal{L}_1[\rho^{(0)}] \in \text{Ran}(\mathcal{F})$. If there are more than one stationary state $\rho^{(0)}$ of $\mathcal{L}_0$, then, according to (C6), this property can be enforced by adjusting $\rho^{(0)}$ so that

$$\mathcal{G}_0 \circ \mathcal{L}_1[\rho^{(0)}] = 0 ,$$

where $\mathcal{G}_0$ is the average map associated with the semigroup generated by $\mathcal{L}_0$. This implies $\mathcal{L}_1[\rho^{(0)}] \in \text{Ran}(\mathcal{F}_0)$ and thus $\notin \text{Ker}(\mathcal{F}_0)$, where $\mathcal{F}_0 = \text{id} - \mathcal{G}_0$. This same argument can be applied at all orders since all of them ask for the inversion of $\mathcal{L}_0$.

3. Application to the XX spin-chain

The application of the previous general considerations to the specific case discussed in the main text is straightforward. Following the definitions and conventions of Section IV, one first realizes that the average map $\mathcal{G}_0$ with respect to $\gamma_t^{(0)} = e^{t\mathcal{L}_0}$ can be cast in the form:

$$\mathcal{G}_0[\rho] = \lambda_+ (\rho) \rho_+ + \lambda_- (\rho) \rho_- ,$$

where the $\lambda_\pm(\rho)$ are given by

$$\lambda_\pm(\rho) = \text{Tr} \left[ \frac{\mathcal{G}_0[\rho]}{\rho} \right] \lambda_\pm .$$

The application of the previous general considerations to the specific case discussed in the main text is straightforward.
where $\lambda_+ (\rho) \geq 0$, $\lambda_- (\rho) + \lambda_- (\rho) = 1$ and
\[
\rho_\pm = \rho_L \otimes \frac{1 \mp \sigma_z}{2} \otimes \rho_R .
\] (C17)
Now, the operators $P_\pm$ in (68) are such that:
\[
P_\pm \rho_\pm = \rho_\pm P_\pm = \rho_\pm , \quad P_\pm \rho_\mp = 0 .
\] (C18)
Moreover, they are left invariant by the dual semigroup (see (34) and the discussion preceding it); indeed, the dual of the dissipative part and of the Hamiltonian contribution of the corresponding generator $\tilde{\mathcal{L}}_0$ are such that
\[
\tilde{\mathcal{D}}_{L,R}[1] = 0 \, , \quad \left[ \Delta \sum_{i=1}^{3} \sigma_3^{(i)} , P_\pm \right] = 0 ,
\] (C19)
so that, $\tilde{\mathcal{L}}_0[P_\pm] = 0$, whence $\tilde{\mathcal{G}}_0[P_\pm] = P_\pm$ under the dual $\tilde{\mathcal{G}}_0$ of the average map $\mathcal{G}_0$. Then, using (C16)
\[
\text{Tr} \left[ P_\pm \mathcal{G}_0[\rho] \right] = \text{Tr} \left[ \mathcal{G}_0[P_\pm \rho] \right] = \text{Tr} \left[ P_\pm \rho \right] = \lambda_+ (\rho) \text{Tr} \left[ P_\pm \rho_+ \right] + \lambda_- (\rho) \text{Tr} \left[ P_\pm \rho_- \right] ,
\] (C20)
yields $\lambda_\pm (\rho) = \text{Tr} \left[ P_\pm \rho \right]$, and thus
\[
\mathcal{G}_0[\rho] = \rho_L \otimes \left( \frac{\text{Tr} \left[ \rho P_+ \right]}{\text{Tr} \left[ \rho P_- \right]} \begin{pmatrix} 0 \\ \text{Tr} \left[ \rho P_- \right] \end{pmatrix} \right) \otimes \rho_R .
\] (C21)
In order to obtain the $n$-th order perturbation,
\[
\rho^{(n)} = - \mathcal{L}_0^{-1} \circ \mathcal{L}_1[\rho^{(n-1)}] ,
\] (C22)
one now needs to invert $\mathcal{L}_0$. According to the general construction developed above, in order to do that we first proceed to ensure that
\[
\mathcal{G} \circ \mathcal{L}_1[\rho^{(n-1)}] = \rho_L \otimes \left( \begin{pmatrix} \Gamma_+^{(n-1)} & 0 \\ 0 & \Gamma_-^{(n-1)} \end{pmatrix} \right) \otimes \rho_R = 0 ,
\] (C23)
where
\[
\Gamma_\pm^{(n-1)} = \text{Tr} \left[ \mathcal{L}_1[\rho^{(n-1)}] P_\pm \right] .
\] (C24)
This request together with $P_+ + P_- = 1$ and the trace-preserving character of $\mathcal{L}_1$ implies that one may need to adjust $\rho^{(n-1)}$ so that one of the following two equivalent conditions holds true:
\[
\Gamma_+^{(n-1)} = - \Gamma_-^{(n-1)} = 0 .
\] (C25)
In the case $n = 2$, this is precisely the condition fixing uniquely the value of the parameter $r$ given in (73).

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