Two-dimensional position-dependent mass Lagrangians; Superintegrability and exact solvability

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Abstract: The two-dimensional extension of the one-dimensional PDM-Lagrangians and their nonlocal point transformation mappings into constant unit-mass exactly solvable Lagrangians is introduced. The conditions on the related two-dimensional Euler-Lagrange equations' invariance are reported. The mappings from superintegrable linear oscillators into sub-superintegrable nonlinear PDM-oscillators are exemplified by, (i) a sub-superintegrable Mathews-Lakshmanan type-I PDM-oscillator, for a PDM-particle moving in a harmonic oscillator potential, (ii) a sub-superintegrable Mathews-Lakshmanan type-II PDM-oscillator, for a PDM-particle moving in a constant potential, and (iii) a sub-superintegrable shifted Mathews-Lakshmanan type-III PDM-oscillator, for a PDM-particle moving in a shifted harmonic oscillator potential. Moreover, the superintegrable shifted linear oscillators and the isotonic oscillators are mapped into a sub-superintegrable PDM-nonlinear and a sub-superintegrable PDM-isotonic oscillators, respectively.

PACS numbers: 05.45.-a, 03.50.Kk, 03.65.-w

Keywords: Two-dimensional position-dependent mass Lagrangians, nonlocal point transformation, Euler-Lagrange equations invariance, superintegrability and sub-superintegrability.

I. INTRODUCTION

The mathematical challenge associated with the position-dependent mass (PDM) von Roos Hamiltonian [1], and the feasible applicability of the PDM settings in different fields of physics, has inspired a relatively intensive recent research attention on the quantum mechanical (see the sample of references [2–12]), classical mechanical and mathematical (see the sample of references [12–37]) domains in general. The position-dependent mass is, in principle, a position-dependent deformation in the standard constant mass settings that introduces its own PDM-byproducted reaction-type force $R_{PDM}(x, \dot{x}) = \frac{m'(x) \dot{x}^2}{2}$, and manifests deformation in the potential force field that may inspire nonlocal space-time point transformations. That is, if a PDM-particle is moving in a harmonic oscillator potential force field $V(x) = m(x) \omega^2 x^2/2$, for example, then one may use $q = x \sqrt{m(x)} \Rightarrow V(q) = \omega^2 q^2/2$ to retain the standard constant mass settings. In the process, some position-dependent deformation in time may be deemed vital (see \[22, 38\] for more details on this issue). For some comprehensive discussions on the quantum mechanical PDM related ordering ambiguity and on the classical-quantum mechanical correspondence, the reader may refer to (c.f., e.g., [12, 17]). However, on the issue of the classical mechanical equivalence between the Euler-Lagrange’s and Newton’s equations of motion one may refer to [22].

In a very recent study, Mustafa [38] has introduced a general nonlocal point transformation for one-dimensional PDM Lagrangians and provided their mappings into a constant "unit-mass" Lagrangians in the generalized coordinates. Therein, it has been shown that the applicability of such mappings not only results in the linearization of some nonlinear oscillators but also extends into the extraction of exact solutions of more complicated dynamical systems. Hereby, the exactly solvable Lagrangians (labeled as "reference/target-Lagrangians") are mapped along with their exact solutions into PDM-Lagrangians (labeled as "target/reference-Lagrangians"). It would be natural and interesting to extend/generalize Mustafa’s methodical proposal [38] to deal with Lagrangians in more than one-dimension. Therefore, the current methodical proposal is a parallel extension to [38] and deals with PDM Lagrangians in two-dimensions.

However, in handling such higher dimensional Lagrangians the notion/concept of superintegrability (c.f., e.g., [39–51] and related references cited therein) is unavoidable. A Lagrangian system is said to be superintegrable if it admits the Liouville-Arnold sense of integrability and introduces more constants of motion (also called integrals of motion) than the degrees of freedom the system is moving within (c.f., e.g., [45]). The set of the two-dimensional isotonic

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therefore, read nonlocal space-time transformation (see (10) below) the two-dimensional oscillator potentials

\[ V(x_1, x_2) = \frac{1}{2} \left( \omega_1^2 x_1^2 + \omega_2^2 x_2^2 + \frac{\beta_1}{x_1^2} + \frac{\beta_2}{x_2^2} \right), \]  

for example (a more general extension of the Smorodinsky-Winternitz isotonic oscillator potentials where \( \omega_1 = \omega_2 = \omega \)) is known to be superintegrable (c.f., e.g., [39, 42] [46, 47]). The details on their superintegrability (c.f., e.g., [41]) classification criteria lay far beyond the scope of our study here. They are considered as superintegrable potentials throughout the current methodical proposal, though we verify their superintegrability in brief to make the current methodical proposal self-contained. On the other hand, under some superintegrability of some shifted linear oscillators (a new superintegrable model to the best of our knowledge) and report their mapping into a sub-superintegrable PDM-Lagrangians. Hereby, we argue that, if the exactly solvable "sub-superintegrable PDM-Lagrangians" (as shall be so labeled hereinafter). In addition to our main objective to extend/generalize Mustafa’s methodical proposal [38] to deal with two-dimensional Lagrangians, we anticipate that the introduction of the terminology of "sub-superintegrability" would yet add a new flavour/concept to "superintegrability". The organization of our methodical proposal is in order.

The two-dimensional nonlocal space-time PDM transformation and the Euler-Lagrange equations’ invariance are discussed in section 2. Therein, one would observe that the obvious non-separability of the two-dimensional "PDM target-Lagrangians” \( L(x_1, x_2, \dot{x}_1, \dot{x}_2; t) \) is accompanied by the separability of the two-dimensional "reference-Lagrangians” \( L(q_1, q_2, \dot{q}_1, \dot{q}_2; \tau) \). Mandating in effect "sub-separability" of the two-dimensional "PDM target-Lagrangians” \( L(x_1, x_2, \dot{x}_1, \dot{x}_2; t) \), as a result of the two-dimensional nonlocal PDM transformations. In section 3, the mapping from the superintegrability of some linear oscillators into sub-superintegrability of nonlinear PDM-oscillators is introduced. To make the current methodical proposal self-contained, we discuss in short the superintegrability of the reference-Lagrangians (although similar discussions are available in the literature). Three two-dimensional sub-superintegrable Mathews-Lakshmanan type PDM-oscillators are used for clarification. They are, (i) a sub-superintegrable Mathews-Lakshmanan type-I PDM-oscillator for a PDM-particle moving in a harmonic oscillator potential, (ii) a sub-superintegrable Mathews-Lakshmanan type-II PDM-oscillator for a PDM-particle moving in a constant potential, (iii) a sub-superintegrable shifted Mathews-Lakshmanan type-III PDM-oscillator for a PDM-particle moving in a shifted harmonic oscillator potential. In the same section, we discuss (in short) the superintegrability of some shifted linear oscillators (a new superintegrable model to the best of our knowledge) and report their mapping into sub-superintegrable nonlinear PDM-Oscillators. We use, moreover, a superintegrable isotonic oscillator and map it into a sub-superintegrable PDM isotonic oscillator. Of course the mappings also include exact solvability of the "reference-Lagrangians” at hand as well. Our concluding remarks are given in section 4.

II. TWO-DIMENSIONAL NONLOCAL PDM-POINT TRANSFORMATIONS AND EULER-LAGRANGE’S INVARIANCE

The Lagrangian of a particle with a constant "unit mass" moving in the generalized coordinates \((q_1, q_2) \equiv (q_1(x_1), q_2(x_2))\), under the influence of a potential force field \(V(q_1, q_2)\), a deformed/re-scaled time \(\tau)\), is given by

\[ L(q_1, q_2, \dot{q}_1, \dot{q}_2; \tau) = \frac{1}{2} \left( \dot{\tilde{q}}_1^2 + \dot{\tilde{q}}_2^2 \right) - V(q_1, q_2) \quad \dot{\tilde{q}}_j = \frac{dq_j}{d\tau}; \quad j = 1, 2. \]  

The corresponding Euler-Lagrange equations

\[ \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0; \quad j = 1, 2, \]  

therefore, read

\[ \frac{d}{d\tau} \dot{q}_j + \frac{\partial}{\partial q_j} V(q_1, q_2) = 0; \quad j = 1, 2. \]
One should notice that for a "unit mass" particle moving in a free force field \( V(q_1, q_2) = 0 \implies dq_j (x_j) / d\tau = 0 \), hence the linear momenta \( \bar{q}_j (x_1) \) and \( \bar{q}_2 (x_2) \) are conserved quantities (in this particular case) and serve as fundamental integrals (i.e., constants of motion). However, for the set of potentials in the form of

\[
V(q_1, q_2) = V_1 (q_1) + V_2 (q_2) \neq 0; \quad V_j (q_j) \neq 0; \quad j = 1, 2
\]  

(which is the set of potential force fields of our interest in the current study) one may recast \( 5 \) as

\[
\frac{d}{d\tau} \bar{q}_j (x_j) + \frac{\partial}{\partial q_j} V_j (q_j) = 0 \implies \ddot{\bar{q}}_j (x_j) = \ddot{\bar{q}}_j (x) \frac{d}{d\tau} \bar{q}_j (x) + \bar{q}_j (x) \frac{\partial}{\partial q_j} V_j (q_j) = 0.
\]  

(7)

to obtain two integrals of motion \( I_1 = E_1 \) and \( I_2 = E_2 \) via

\[
\frac{d}{d\tau} \left[ \frac{1}{2} \dot{\bar{q}}_j^2 + V_j (q_j) \right] = 0 \implies \frac{dE_j}{d\tau} = \frac{dI_j}{d\tau} = 0; \quad j = 1, 2.
\]  

(8)

Yet, the conservation of the total energy \( E_{\text{tot}} \) is a natural and an immediate consequence of such settings. That is,

\[
\frac{d}{d\tau} [E_1 + E_2] = \frac{d}{d\tau} \left[ \frac{1}{2} (\dot{\bar{q}}_1^2 + \dot{\bar{q}}_2^2) + V_1 (q_1) + V_2 (q_2) \right] = 0 \implies \frac{dE_{\text{tot}}}{d\tau} = 0
\]  

(9)

The separability and/or integrability of the above system is obvious, therefore.

Next, let us introduce the nonlocal point transformation of the form

\[
q_j \equiv q_j (x_j) = \int \sqrt{g (\bar{x})} dx_j, \quad \tau = \int f (\bar{x}) dt \implies \frac{d\tau}{dt} = f (\bar{x}) \neq 0; \quad x_j = x_j (t), \quad \bar{x} = x_1, x_2.
\]  

(10)

Consequently, with an overhead dot to identify total time \( t \) derivative,

\[
\frac{dq_j}{d\tau} = \ddot{q}_j \equiv \dot{x}_j \sqrt{g (\bar{x})} \frac{f (\bar{x})}{\dot{f} (\bar{x})}, \quad \frac{d}{d\tau} \bar{q}_j (x_j) = \sqrt{g (\bar{x})} \left[ \dot{x}_j + \frac{1}{2} \frac{\dot{f} (\bar{x})}{f (\bar{x})} \right] \dot{x}_j.
\]  

(11)

Which when substituted in \( 7 \) would, in a straightforward manner, result in

\[
\left[ \dot{x}_1 \dot{\bar{x}}_1 + \dot{x}_2 \dot{\bar{x}}_2 + \left( \frac{1}{2} \frac{\dot{f} (\bar{x})}{f (\bar{x})} - \frac{\dot{f} (\bar{x})}{f (\bar{x})} \right) \dot{\bar{x}}_1^2 + \dot{\bar{x}}_2^2 \right] + \frac{f (\bar{x})^2}{g (\bar{x})} \frac{d}{dt} V (\bar{x}) = 0; \quad x_j = x_j (q_j),
\]  

(12)

where

\[
\frac{d}{dt} V (\bar{x}) = \left( \dot{x}_1 \frac{\partial}{\partial x_1} V_1 (x_1) + \dot{x}_2 \frac{\partial}{\partial x_2} V_2 (x_2) \right).
\]  

On the other hand, for a two-dimensional position-dependent mass particle moving in a force field \( V (x_1, x_2) = V_1 (x_1) + V_2 (x_2) \) the Lagrangian

\[
L (x_1, x_2, \dot{x}_1, \dot{x}_2; t) = \frac{1}{2} m (\bar{x}) \left( \dot{x}_1^2 + \dot{x}_2^2 \right) - [V_1 (x_1) + V_2 (x_2)];
\]  

(13)

in the Cartesian coordinates, would yield two Euler-Lagrange equations

\[
m (\bar{x}) \ddot{x}_j + \dot{m} (\bar{x}) \ddot{x}_j - \frac{1}{2} \frac{\partial m (\bar{x})}{\partial x_j} \left( \dot{x}_1^2 + \dot{x}_2^2 \right) + \frac{\partial}{\partial x_j} V_j (x_j) = 0; \quad j = 1, 2.
\]  

(14)

The non-separability of this system is obviously manifested by the position-dependent mass term. However, when multiplied, from the left, by \( \dot{x}_j \) it reads

\[
m (\bar{x}) \dot{x}_j \ddot{x}_j + \dot{m} (\bar{x}) \dot{x}_j \dot{\bar{x}}_j - \frac{1}{2} \left( \dot{x}_j \frac{\partial m (\bar{x})}{\partial x_j} \right) \left( \dot{x}_1^2 + \dot{x}_2^2 \right) + \left( \dot{x}_j \frac{\partial}{\partial x_j} V_j (x_j) = 0; \quad j = 1, 2.
\]  

(15)

Consequently, the addition of the two equations of \( 15 \) yields

\[
\dot{x}_1 \ddot{x}_1 + \dot{x}_2 \ddot{x}_2 + \frac{1}{2} \frac{\dot{m} (\bar{x})}{m (\bar{x})} \left( \dot{x}_1^2 + \dot{x}_2^2 \right) + \frac{1}{m (\bar{x})} \frac{d}{dt} V (\bar{x}) = 0,
\]  

(16)
and hence
\[
\frac{d}{dt} \left[ \frac{1}{2} m(\bar{x}) \left( \dot{x}_1^2 + \dot{x}_2^2 \right) + V_1(x_1) + V_2(x_2) \right] = 0 \implies \frac{dE_{tot}}{dt} = 0.
\] (17)

Hereafter, we emphasize that the conservation of the total energy \(E_{tot}\) in (17) can never be considered as an immediate consequence of the sum of two fundamental integrals of motion \(E_{x_1}\) and \(E_{x_2}\). The time evolutions of \(E_{x,i}\)'s do not satisfy (15). i.e., one may easily show that
\[
\frac{d}{dt} E_{x,i} = \frac{d}{dt} \left[ \frac{1}{2} m(\bar{x}) \dot{x}_i^2 + V_j(x_j) \right] = m(\bar{x}) \ddot{x}_i \dot{x}_j + \frac{1}{2} m(\bar{x}) \dot{x}_j^2 + \ddot{x}_j \frac{\partial}{\partial x_j} V_j(x_j) \neq 0
\] (18)

compared to (15). The third term in (15) is missing in (18). Moreover, the comparison between (12) and (16) obviously suggests that the Euler-Lagrange equations of motion (12) and (16) are identical if and only if \(g(\bar{x})\) and \(g(\bar{x})\) satisfy the conditions
\[
g(\bar{x}) = \frac{m(\bar{x}) f(\bar{x})^2}{2} \iff 1 \frac{\dot{g}(\bar{x})}{g(\bar{x})} - \frac{1}{f(\bar{x})} \frac{m(\bar{x})}{2} \iff q_j = \int \sqrt{m(\bar{x}) f(\bar{x})} \, dx_j
\] (19)

As such, it is clear that the functional nature/structure of the position-dependent mass \(m(\bar{x})\) determines the nature/structure of the nonlocal transformation functions \(g(\bar{x})\) and \(f(\bar{x})\). For the sake of simplicity, however, we shall work with \(m(\bar{x}) \equiv m(r) \equiv g(\bar{x}) \equiv g(r)\) and \(f(\bar{x}) \equiv f(r)\). Hence, \(m(r), g(r), f(r)\) are well behaved functions of explicit dependence on \(r = \sqrt{x_1^2 + x_2^2}\), if not otherwise mentioned.

At this point, one may safely conclude that under such invertible (i.e., the Jacobian determinant \(\text{det} (\partial x / \partial q_i) \neq 0\) nonlocal transformation, (10) and (19)), the two-dimensional Euler-Lagrange equations, (12) and (16), remain invariant.

That is, the Lagrangian \(L(q_1, q_2, \bar{q}_1, \bar{q}_2; \tau)\), in (3), non-locally transforms (via (10) and (19)) into the PDM Lagrangian \(L(x_1, x_2, \dot{x}_1, \dot{x}_2; t)\), in (13), and leaves in the process the corresponding two-dimensional Euler-Lagrange equations of motions, (12) and (16), invariant. The mapping
\[
L(q_1, q_2, \bar{q}_1, \bar{q}_2; \tau) \iff \left\{ \begin{array}{l}
q_j = \int \sqrt{m(r)} f(r) \, dx_j; \ j = 1, 2 \\
\tau = \int f(r) \, dt \\
\bar{q}_j = \dot{x}_j \sqrt{m(r)}; \ j = 1, 2 \\
\end{array} \right. \iff L(x_1, x_2, \dot{x}_1, \dot{x}_2; t),
\] (20)

between the "unit mass" Lagrangian \(L(q_1, q_2, \bar{q}_1, \bar{q}_2; \tau)\) and PDM Lagrangian \(L(x_1, x_2, \dot{x}_1, \dot{x}_2; t)\) is clear, therefore.

Nevertheless, the obvious non-separability of the two-dimensional Euler-Lagrange equations associated with the PDM Lagrangian \(L(x_1, x_2, \dot{x}_1, \dot{x}_2; t)\) (hence, \(L(x_1, x_2, \dot{x}_1, \dot{x}_2; t)\) is non-separable) is accompanied by the separability of the two-dimensional Euler-Lagrange equations associated with unit mass Lagrangian \(L(q_1, q_2, \bar{q}_1, \bar{q}_2; \tau)\) (hence, \(L(q_1, q_2, \bar{q}_1, \bar{q}_2; \tau)\) is separable). This should, in turn, mandate the notion of "sub-separability" of the two-dimensional Lagrangian \(L(x_1, x_2, \dot{x}_1, \dot{x}_2; t)\) as a result of the two-dimensional PDM nonlocal transformations in (20). Likewise, if the two-dimensional unit mass Lagrangian \(L(q_1, q_2, \bar{q}_1, \bar{q}_2; \tau)\) admits superseparability and/or superintegrability then the two-dimensional PDM Lagrangian \(L(x_1, x_2, \dot{x}_1, \dot{x}_2; t)\) may very well be labeled as sub-superseparable and/or sub-superintegrable. The latter forms the focal point of our study here and shall be clarified in the forthcoming experimental examples.

### III. FROM SUPERINTEGRABILITY TO SUB-SUPERINTEGRABILITY; TWO-DIMENSIONAL PDM-OSCILLATORS

Although the superintegrability of some of the reference superintegrable oscillators, we use here, is verified in the literature, we recycle them in such a way that serves our methodological proposal and keeps it self-contained. The superintegrable linear oscillators, the superintegrable shifted-oscillators (is a new superintegrable oscillator model, to the best of our knowledge), and the superintegrable isotonic oscillators are used here as illustrative examples. They are mapped along with their exact solutions into sub-superintegrable PDM-oscillators.
A. Superintegrable linear oscillators into sub-superintegrable nonlinear PDM-oscillators

Consider a unit mass particle moving under the influence of the two-dimensional oscillators potential

\[ V(q_1, q_2) = \frac{1}{2} \left( \omega_1^2 q_1^2 + \omega_2^2 q_2^2 \right) \]  

in the generalized coordinates. Then, the corresponding two-dimensional Lagrangian

\[ L(q_1, q_2, \dot{q}_1, \dot{q}_2; \tau) = \frac{1}{2} (\dot{q}_1^2 + \dot{q}_2^2) - \frac{1}{2} (\omega_1^2 q_1^2 + \omega_2^2 q_2^2) \]  

leads to Euler-Lagrange equations

\[ \frac{d}{d\tau} \ddot{q}_j + \omega_j^2 q_j = 0 \implies q_j(\tau) = A_j \cos(\omega_j \tau + \varphi), \ j = 1, 2, \]  

subjected to the boundary conditions \( q_j(0) = A_j, \dot{q}_j(0) = 0 \), say. It obviously admits separability and is known to satisfy the superintegrability conditions (c.f. e.g., [39–42]) via the use of a complex factorization technique (c.f., e.g., [43, 44]). That is, if we introduce the two complex functions

\[ Q_j = \ddot{q}_j + i\omega_j q_j = i \omega_j Q_j; \ j = 1, 2, \]  

then the functions

\[ Q_{jk} = Q_j^{*k} (Q_k^*)^{-j}; \ j, k = 1, 2, \]  

represent complex constants of motion with vanishing deformed/rescaled-time evolution \( \tau \),

\[ \frac{d}{d\tau} Q_{jk} = i (\omega_j \omega_k - \omega_k \omega_j) Q_{jk} = 0. \]  

Moreover, one can, in a straightforward manner, verify that

\[ Q_{jj} = \ddot{q}_j^2 + \omega_j^2 q_j^2 \implies 2E_1 = I_1 = Q_{11}, 2E_2 = I_2 = Q_{22}. \]  

where \( I_1 \) and \( I_2 \) are two fundamental integrals of motion. Yet, for the isotropic oscillator \( \omega_1 = \omega_2 = \omega_0 \), for example,

\[ Q_{12} = (\ddot{q}_1 \ddot{q}_2 + \omega_0^2 q_1 q_2) + i\omega_0 (q_1 \ddot{q}_2 - q_2 \ddot{q}_1) = I_3 + iI_4 \]  

identifies two more integrals of motion

\[ I_3 = \text{Re} Q_{12} = \ddot{q}_1 q_2 + \omega_0^2 q_1 q_2; \quad I_4 = \text{Im} Q_{12} = \omega_0 (q_1 \ddot{q}_2 - q_2 \ddot{q}_1). \]  

which are, in the general case \( \omega_1 \neq \omega_2 \), polynomials in the momenta. Therefore, our two-dimensional Lagrangian (22) admits superintegrability in the generalized coordinates \((q_1, q_2)\) and in the deformed/rescaled time \( \tau \). The details on such superintegrability are far beyond the scope of our the current methodical proposal, though can be traced through the comprehensive article of Ranada [42] and related references cited therein.

1. Sub-superintegrable Mathews-Lakshmanan type-I PDM-oscillators

Let us now consider a PDM particle \( m(r) \) moving in the harmonic oscillator force field

\[ V(x_1, x_2) = \frac{1}{2} m(r) \omega^2 (x_1^2 + x_2^2), \]  

with the corresponding PDM Lagrangian

\[ L(x_1, x_2, \dot{x}_1, \dot{x}_2; t) = \frac{1}{2} m(r) \left[ \dot{x}_1^2 + \dot{x}_2^2 - \omega^2 (x_1^2 + x_2^2) \right]. \]
This Lagrangian indulges one and only one integral offered by the total energy $E_{tot}$ as in (17), and the corresponding Euler-Lagrange equations are non-separable. However, with the substitution

$$q_j = x_j \sqrt{m(r)}; \ j = 1, 2,$$

one can non-locally transform $L(x_1, x_2, \dot{x}_1, \dot{x}_2; t)$ of (30) into $L(q_1, q_2, \dot{q}_1, \dot{q}_2; \tau)$ of (22). Moreover, if this substitution is used along with the transformation in (20) we get

$$q_j = x_j \sqrt{m(r)} \implies \frac{dq_j}{dx_j} = \frac{1}{\sqrt{m(r)}} \left[1 + \frac{m'(r)}{2m(r)} \left(\frac{x_j^2}{r}\right)\right],$$

and

$$q_j = \int \sqrt{m(r)} f(r) \, dx_j \implies \frac{dq_1}{dx_1} + \frac{dq_2}{dx_2} = 2\sqrt{m(r)} f(r).$$

hence

$$f(r) = 1 + \frac{m'(r)}{4m(r)} r.$$

At this point, one should be aware that we are interested in $m(r)$ and $f(r)$ that are only explicit function in $r = \sqrt{x_1^2 + x_2^2}$. Obviously, moreover, the choice of $f(r)$ would determine the position-dependent mass function $m(r)$ (of course the other way around works as well). This is clarified in the following assumption. Let us assume that

$$f(r) = m(r) - \frac{1}{4m(r)} \frac{m'(r)}{r}$$

to obtain a position-dependent mass of the form

$$f(r) = m(r) - \frac{1}{4m(r)} \frac{m'(r)}{r} = 1 + \frac{1}{4m(r)} m(r) \implies m(r) = \frac{1}{1 \pm \beta r^2}; \ \beta \geq 0.\ \ \ \ (35)$$

Then the corresponding two-dimensional PDM Lagrangian of (30) reads a two-dimensional Mathews-Lakshmanan type-I oscillator

$$L(x_1, x_2, \dot{x}_1, \dot{x}_2; t) = \frac{1}{2} \left[\frac{x_1^2 + x_2^2}{1 \pm \beta (x_1^2 + x_2^2)} - \omega^2 \left(x_1^2 + x_2^2\right)\right].\ \ \ \ (36)$$

A Lagrangian of this type is neither separable nor superintegrable. Nevertheless, our two-dimensional PDM Lagrangian $L(x_1, x_2, \dot{x}_1, \dot{x}_2; t)$ (36) nonlocally transforms into a superintegrable two-dimensional Lagrangian $L(q_1, q_2, \dot{q}_1, \dot{q}_2; \tau)$ (22). Hence the two-dimensional PDM Lagrangian $L(x_1, x_2, \dot{x}_1, \dot{x}_2; t)$ of (36) is a sub-superintegrable PDM Lagrangian and the corresponding Euler-Lagrange equations (16) admit exact solutions

$$x_j(t) = A_j \cos(\Omega t + \varphi); \ \Omega^2 = \frac{\omega^2}{1 \pm \beta (A_1^2 + A_2^2)}.$$

Likewise, the reversed process is equally valid. That is, the relation

$$\begin{align*}
\text{Superintegrale} \left\{ \begin{array}{l}
L(q_1, q_2, \dot{q}_1, \dot{q}_2; \tau)
\end{array} \right\}
\iff
\text{Nonlocal transformation}
\left\{ \begin{array}{l}
q_j = x_j \sqrt{m(r)}; \ j = 1, 2,
\dot{q}_j = \dot{x}_j \sqrt{m(r)}
\end{array} \right\}
\iff
\text{Sub-superintegrable}
\left\{ \begin{array}{l}
L(x_1, x_2, \dot{x}_1, \dot{x}_2; t)
of (36) 
\end{array} \right\}
\end{align*}$$

provides the exact mapping from the superintegrability and exact solvability of (22) into the sub-superintegrability and exact solvability of the PDM Mathews-Lakshmanan type-I Lagrangian (36).
Consider a PDM particle \( m(r) \) moving in a constant potential force field of the form
\[
V(x_1, x_2) = \frac{1}{2} m(r) \omega^2 (\xi_1^2 + \xi_2^2)
\]
with the corresponding Lagrangian
\[
L(x_1, x_2, \dot{x}_1, \dot{x}_2; t) = \frac{1}{2} m(r) \left[ \dot{x}_1^2 + \dot{x}_2^2 - \omega^2 (\xi_1^2 + \xi_2^2) \right].
\]
where \( \xi_1, \xi_2 \in \mathbb{R} \) are constants. This Lagrangian has the total energy \( E_{tot} \) of (17) as the only integral of motion and the corresponding Euler-Lagrange equations are non-separable. However, the substitution of \( q_j = \xi_j \sqrt{m(r)} \) would nonlocally transform it into the superintegrable Lagrangian \( L(q_1, q_2, \dot{q}_1, \dot{q}_2; \tau) \) of (22) for \( \omega_1 = \omega_2 = \omega \). Under such settings,
\[
q_j = \int \sqrt{m(r)} f(r) \, dx_j = \xi_j \sqrt{m(r)} \implies f(r) = \frac{\xi_j m'(r)}{2 m(r)} \left( \frac{x_j}{r} \right),
\]
and for \( \xi_1 = \xi_2 = \xi / \sqrt{2} \), this would imply that
\[
2 f(r)^2 = \left( \frac{m'(r)}{2 m(r)} \right)^2 \left( \frac{\xi^2 x_1^2 + \xi^2 x_2^2}{r^2} \right) \implies f(r) = \frac{\xi^2 m'(r)}{4 m(r)}.
\]
Consequently, the corresponding two-dimensional PDM Euler-Lagrange equation (17), for
\[
m(r) = \frac{1}{1 \pm \beta r^2},
\]
reads
\[
\frac{d}{d\tau} \left[ \frac{x_1^2 + x_2^2 + \omega^2 \xi^2}{2(1 \pm \beta r^2)} \right] = 0,
\]
and admits solutions of the forms
\[
x_j(t) = A_j \cos(\Omega t + \varphi); \quad \Omega^2 = \frac{\pm \omega^2 \beta \xi^2}{1 \pm \beta (A_1^2 + A_2^2)}; \quad \beta \geq 0; \quad j = 1, 2
\]
Under such settings, one can show that for \( \beta = \mp 1/\xi^2 \) the Lagrangian of (36) and the Lagrangian of (40) indulge the very same dynamical properties as documented in the corresponding Euler-Lagrange equation (16). Hence, the Lagrangian at hand here is a Mathews-Lakshmanan type-II PDM-oscillators Lagrangian.

Obviously, moreover, our non-separable and non-superintegrable Lagrangian (40) non-locally transforms into a separable and superintegrable Lagrangian (22). Our PDM Lagrangian (40) is a \textit{sub-superintegrable} Mathews-Lakshmanan type-II PDM-oscillators Lagrangian, therefore. In short, the relation
\[
\begin{align*}
\begin{array}{c}
\text{Superintegrable} \\
L(q_1, q_2, \dot{q}_1, \dot{q}_2; \tau)
\end{array}
\end{align*}
\iff
\begin{align*}
\begin{array}{c}
\text{Nonlocal transformation} \\
q_j = \xi_j \sqrt{m(r)}; \quad j = 1, 2
\end{array}
\end{align*}
\iff
\begin{align*}
\begin{array}{c}
\text{Sub-superintegrable} \\
L(x_1, x_2, \dot{x}_1, \dot{x}_2; t)
\end{array}
\end{align*}
\iff
\begin{align*}
\begin{array}{c}
\text{Nonlocal transformation} \\
f(r) = \frac{\xi m'(r)}{4 m(r)}; \quad \xi_j = \frac{\xi}{\sqrt{2}}
\end{array}
\end{align*}
\iff
\begin{align*}
\begin{array}{c}
\text{Nonlocal transformation} \\
m(r) = 1/(1 \pm \beta r^2); \quad \beta = \mp 1/\xi^2
\end{array}
\end{align*}
\iff
\begin{align*}
\begin{array}{c}
\text{Nonlocal transformation} \\
\Omega^2 = \frac{\omega^2}{1 \pm \beta (A_1^2 + A_2^2)}
\end{array}
\end{align*}
\]

represents the sought after mapping from/to \textit{superintegrable} Lagrangian \( L(q_1, q_2, \dot{q}_1, \dot{q}_2; \tau) \) of (22) to/from the \textit{sub-superintegrable} Lagrangian \( L(x_1, x_2, \dot{x}_1, \dot{x}_2; t) \) of (39).
3. Sub-superintegrable Mathews-Lakshmanan type-III PDM shifted-oscillators

Let us now use a position-dependent mass with a different functional structure moving in a shifted-oscillator force field and described by the Lagrangian

\[ L(x_1, x_2, \dot{x}_1, \dot{x}_2; t) = \frac{1}{2} m(r_s) \left\{ \dot{x}_1^2 + \dot{x}_2^2 - \omega^2 \left[ (x_1 + \gamma_1)^2 + (x_2 + \gamma_2)^2 \right] \right\}. \]  

(46)

where \( r_s = \sqrt{(x_1 + \gamma_1)^2 + (x_2 + \gamma_2)^2} \) is introduced for convenience. The form of the shifted oscillator potential is clear here. At this point, we may recollect that the functional nature/structure of the position-dependent mass \( m(\bar{x}) \) determines the nature/structure of \( g(q) \) and \( f(\bar{x}) \) in our nonlocal point transformation as suggested by equation (19). Therefore, \( f(r) \rightarrow f(r_s) \) and \( g(r) \rightarrow g(r_s) \) for our Lagrangian (46) at hand. Moreover, it is a straightforward manner, and in parallel with (32)-(35), one may show that the substitution of

\[ q_j = (x_j + \xi_j) \sqrt{m(r_s)} \implies f(r_s) = 1 + \frac{1}{4} \frac{m'(r_s)}{m(r_s)} r_s \]

would consequently, for the choice

\[ f(r_s) = m(r_s) - \frac{1}{4} \frac{m'(r_s)}{m(r_s)} r_s = 1 + \frac{1}{4} \frac{m'(r_s)}{m(r_s)} r_s, \]

yield

\[ m(r_s) = \frac{1}{1 + \beta r_s^2} = \frac{1}{1 + \beta \left( (x_1 + \gamma_1)^2 + (x_2 + \gamma_2)^2 \right)}. \]

(49)

Then the corresponding two-dimensional PDM Lagrangian of (46) reads a two-dimensional Mathews-Lakshmanan type-III PDM shifted-oscillators Lagrangian

\[ L(x_1, x_2, \dot{x}_1, \dot{x}_2; t) = \frac{1}{2} \frac{\omega^2}{1 + \beta (A_1^2 + A_2^2)} \cdot \]

(50)

Our two-dimensional PDM Lagrangian \( L(x_1, x_2, \dot{x}_1, \dot{x}_2; t) \) (50) nonlocaly transforms into a superintegarble two-dimensional Lagrangian \( L(q_1, q_2, \dot{q}_1, \dot{q}_2; \tau) \) of (22). Hence our PDM-Lagrangian \( L(x_1, x_2, \dot{x}_1, \dot{x}_2; t) \) (50) is a sub-superintegrable Lagrangian and the corresponding Euler-Lagrange equation (16) admits exact solutions of the form

\[ x_j(t) = A_j \cos(\Omega t + \varphi) - \gamma_j; \]

\[ \Omega^2 = \frac{\omega^2}{1 + \beta (A_1^2 + A_2^2)}; \]

(51)

The process is summed up as

\[
\begin{align*}
\text{Superintegrable} & \quad \left\{ \begin{array}{l}
\text{Lagrangian:} \\
q_j = x_j \sqrt{m(r_s)}; \quad j = 1, 2 \\
\dot{q}_j = \dot{x}_j \sqrt{m(r_s)} \\
f(r_s) = m(r_s) - \frac{1}{4} \frac{m'(r_s)}{m(r_s)} r_s \\
m(r_s) = 1/ \left( 1 + \beta r_s^2 \right)
\end{array} \right\} \\
\text{of (22) with} \\
q_j(\tau) = A_j \cos(\omega \tau + \varphi)
\end{align*}
\]

\[ \iff \]

\[
\begin{align*}
\text{Sub-superintegrable} & \quad \left\{ \begin{array}{l}
\text{Lagrangian:} \\
q_j = A_j \cos(\Omega t + \varphi) - \gamma_j \\
\Omega^2 = \frac{\omega^2}{1 + \beta (A_1^2 + A_2^2)}
\end{array} \right\},
\end{align*}
\]

(52)

to represent the mapping from the superintegrability harmonic oscillators (22) into sub-superintegrability of the above Mathews-Lakshmanan type-III PDM shifted-oscillators (50).

B. Superintegrable shifted-linear oscillators into sub-superintegrable nonlinear PDM-oscillators

Consider a unit mass particle moving in the two-dimensional shifted-oscillators potential

\[ V(q_1, q_2) = \frac{1}{2} \left[ \alpha_1^2 (q_1 + \eta_1)^2 + \alpha_2^2 (q_2 + \eta_2)^2 \right] \implies V_j(q_j) = \frac{1}{2} \alpha_j^2 (q_j + \eta_j)^2; \quad j = 1, 2,
\]

(53)
in the generalized coordinates, with the constant shifts $\eta_1$, $\eta_2 \in \mathbb{R}$. Then, the corresponding two-dimensional Lagrangian

$$L(q_1, q_2, \dot{q}_1, \dot{q}_2; \tau) = \frac{1}{2} (\dot{q}_1^2 + \dot{q}_2^2) - \frac{1}{2} \left[ \alpha_1^2 (q_1 + \eta_1)^2 + \alpha_2^2 (q_2 + \eta_2)^2 \right],$$

(54)
yields the Euler-Lagrange equations

$$\frac{d}{d\tau} \dot{q}_j + \alpha_j^2 (q_j + \eta_j) = 0 \implies q_j (\tau) = A_j \cos (\alpha_j \tau + \varphi) - \eta_j; \ j = 1, 2,$$

(55)
with the initial conditions that $q_j (0) = A_j - \eta_j$, $\dot{q}_j (0) = 0$, say. The separability of this Lagrangian is obvious.

However, the verification of the superintegrability of such a Lagrangian follows (step-by-step) from the complex factorization recipe (c.f., e.g., [12, 14]) by introducing the complex functions

$$Q_j = \dot{q}_j + i \alpha_j (q_j + \eta_j) \implies \frac{d}{d\tau} Q_j = i \alpha_j Q_j; \ j = 1, 2.$$  

(56)

Which would, in turn, suggest that

$$Q_{jk} = Q_j^\alpha (Q_k^\alpha)^\eta_j \implies \frac{d}{d\tau} Q_{jk} = i (\alpha_j \alpha_k - \alpha_k \alpha_j) Q_{jk} = 0; \ j, k = 1, 2.$$  

(57)
That is, the complex function $Q_{jk}$ represent complex constants of motion with vanishing deformed/rescaled-time evolution $\tau$. Yet, it is an easy task to show that the two fundamental integrals $I_1$ and $I_2$ are given through the relation

$$Q_{j2} = \dot{q}_j^2 + \alpha_j^2 (q_j + \eta_j)^2 \implies 2E_1 = I_1 = Q_{11}, \ 2E_2 = I_2 = Q_{22},$$

(58)
Whereas, for $\alpha_1 = \alpha_2 = \alpha_o$, for example,

$$Q_{12} = [\dot{q}_1 \dot{q}_2 + \alpha_o^2 (q_1 + \eta_1) (q_2 + \eta_2)] + i \alpha_o ((q_1 + \eta_1) \dot{q}_2 - (q_2 + \eta_2) \dot{q}_1),$$

(59)
which identifies two real integrals of motion $I_3$ and $I_4$ such that

$$I_3 = \text{Re} \ Q_{12} = \dot{q}_1 \dot{q}_2 + \alpha_o^2 (q_1 + \eta_1) (q_2 + \eta_2); \quad I_4 = \text{Im} \ Q_{12} = \alpha_o ((q_1 + \eta_1) \dot{q}_2 - (q_2 + \eta_2) \dot{q}_1).$$

(60)
Therefore, our two-dimensional Lagrangian (54) admits superintegrability in the generalized coordinates $(q_1, q_2)$ and in the deformed/rescaled time $\tau$.

Next, under the nonlocal transformation (20), along with the substitutions

$$q_j = x_j \sqrt{m(r)} - \eta_j \implies \frac{dq_j}{dx_j} = \sqrt{m(r)} \left[ 1 + \frac{m'(r)}{2m(r)} \left( \frac{x_j^2}{r} \right) \right] \implies f(r) = 1 + \frac{1}{1 \pm \beta r^2},$$

(61)
This result looks very much the same as that of $f(r)$ used in (33). Thus it would, again, with the assumption that

$$f(r) = m(r) - \frac{1}{4} \frac{m'(r)}{m(r)} r = 1 + \frac{1}{4} \frac{m'(r)}{m(r)} r \implies m(r) = \frac{1}{1 \pm \beta r^2},$$

lead to the sub-superintegrable Mathews-Lakshmanan type-I PDM-oscillator (36). The corresponding Euler-Lagrange equation (16) admits exact solutions as those in (37). Then, the sub-superintegrability of our Mathews-Lakshmanan type-I PDM-Lagrangian (36) turns out to be a consequence of the superintegrability of the linear oscillators (22) and/or the superintegrability of the shifted-oscillators (54). Likewise, the sub-superintegrable and exact solvable Mathews-Lakshmanan type-I PDM-Lagrangian (36) may very well be nonlocally transformed into two superintegrable Lagrangians, a superintegrable linear oscillator (22) and a superintegrable shifted-oscillator (54). That is, the relation

$$\begin{align*}
\text{Superintegrable} & \quad L(q_1, q_2, \dot{q}_1, \dot{q}_2; \tau) \quad \text{of (54), with} \\
q_j (\tau) = A_j \cos (\alpha_j \tau + \varphi) - \eta_j
\end{align*}$$

$$\begin{align*}
\text{Nonlocal transformation} & \quad q_j = x_j \sqrt{m(r)} - \eta_j; \ j = 1, 2, \\
\dot{q}_j = \dot{x}_j \sqrt{m(r)} \\
f(r) = m(r) - \frac{1}{4} \frac{m'(r)}{m(r)} r \\
m(r) = 1/ (1 \pm \beta r^2)
\end{align*}$$

$$\begin{align*}
\text{Sub-superintegrable} & \quad L(x_1, x_2, \dot{x}_1, \dot{x}_2; t) \\
of (36), with \\
x_j (t) = A_j \cos (\Omega t + \varphi) \\
\Omega^2 = \frac{\omega^2}{1 + \beta (A_1^2 + A_2^2)}
\end{align*}$$

(62)
would describe the mapping from the superintegrable Lagrangians (54) into the sub-superintegrable ones of (36).
C. Superintegrable Isotonic Oscillator into sub-superintegrable PDM-deformed Isotonic oscillator

A "unit mass" particle moving in a two-dimensional isotonic oscillator potential field

\[ V(q_1, q_2) = \frac{1}{2} \left( \omega_1^2 q_1^2 + \omega_2^2 q_2^2 + \omega_1^2 q_1^2 + \omega_2^2 q_2^2 \right) \implies V_j(q_j) = \frac{1}{2} \left( \omega_j^2 q_j^2 + \frac{\beta_j}{q_j^2} \right), \]

where \( \omega_j = n_j \omega_0; \ j = 1, 2 \), is described by the Lagrangian

\[ L(q_1, q_2, \dot{q}_1, \dot{q}_2; \tau) = \frac{1}{2} \left( \dot{q}_1^2 + \dot{q}_2^2 \right) - \frac{1}{2} \left( \omega_1^2 q_1^2 + \omega_2^2 q_2^2 + \frac{\beta_1}{q_1^2} + \frac{\beta_2}{q_2^2} \right), \]

which is known to be the superintegrable Smorodinsky-Winternitz type Lagrangian (c.f., e.g., [29, 38, 40, 42, 47]). Its superintegrability can be verified through the two complex substitutions

\[ Q_j = \left( \dot{q}_j^2 - \omega_j^2 q_j^2 + \frac{\beta_j}{q_j^2} \right) + 2\omega_j q_j \dot{q}_j \implies \frac{d}{d\tau} Q_j = 2\omega_j Q_j, \ j = 1, 2, \]

that satisfy (25) with \( Q_{jk} \) representing complex constants of motion and leads to more than two integrals of motion that manifest superintegrability. Moreover, the corresponding Euler-Lagrange equations of which read two Ernako-Pinney’s like equations

\[ \frac{d}{d\tau} \dot{q}_j = -\omega_j^2 q_j + \frac{\beta_j}{q_j^2}, \]

with the corresponding exact solutions

\[ q_j = \frac{A_j}{\omega_j} \sin(\omega_j \tau + \delta_j) \implies \beta_j = -A_j^2; \ j = 1, 2. \]

Yet, under the nonlocal transformation setting in (38) our superintegrable Lagrangian \( L(q_1, q_2, \dot{q}_1, \dot{q}_2; \tau) \) in (64) nonlocaly transforms into a sub-superintegrable Smorodinsky-Winternitz like PDM-oscillators Lagrangian

\[ L(x_1, x_2, \dot{x}_1, \dot{x}_2; t) = \frac{1}{2} \left\{ x_1^2 + x_2^2 - \frac{\omega_1^2 x_1^2 + \omega_2^2 x_2^2}{1 \pm \lambda (x_1^2 + x_2^2)} - \left[ 1 \pm \lambda (x_1^2 + x_2^2) \right] \left( \frac{\beta_1}{x_1^2} + \frac{\beta_2}{x_2^2} \right) \right\}. \]

The Euler-Lagrange equation (16) for which admits exact solutions of the form

\[ x_j = \sqrt{\frac{A_j}{\Omega}} \sin(\Omega \tau + \delta_j); \ j = 1, 2, \]

where

\[ \Omega^2 = \begin{cases} \omega^2 + \lambda^2 (A_1 + A_2)^2 & ; \ \omega_1 = \omega_2 = \omega, \beta_1 \neq \beta_2 \\ \frac{A_1 \omega_1^2 + A_2 \omega_2^2 + \lambda^2 (A_1 + A_2)^2}{A_1 + A_2} & ; \ \omega_1 \neq \omega_2, \beta_1 \neq \beta_2 \\ (\omega_1^2 + \omega_2^2) / (16 A^2 \lambda^2) & ; \ \omega_1 \neq \omega_2, \beta_1 = \beta_2 \equiv A_1 = A_2 = A \end{cases}. \]

IV. CONCLUDING REMARKS

In this article, and in parallel with our recent methodical proposal in [38], we have introduced the two-dimensional extension of the one-dimensional PDM-Lagrangians and their nonlocal transformation mappings’ recipes into constant unit-mass exactly solvable Lagrangians. Hereby, the two-dimensional nonlocal point transformations (10) and the related Euler-Lagrange equations invariance conditions (19) are reported. However, dealing with Lagrangians in more
than one-dimension renders superintegrability to be unavoidably in the process. We have, therefore, asserted that, if a set of "superintegrable reference/target-Lagrangians" $L(q_1, q_2, \tilde{q}_1, \tilde{q}_2; \tau)$ is mapped into a set of "PDM target/reference-Lagrangians" $L(x_1, x_2, \dot{x}_1, \dot{x}_2; t)$, then the set of "PDM target/reference-Lagrangians" $L(x_1, x_2, \dot{x}_1, \dot{x}_2; t)$ is a set of "sub-superintegrable PDM-Lagrangians" (as shall be so labelled hereinafter). Two sets of illustrative examples are used. The first set is devoted to the mappings from superintegrable linear oscillators into sub-superintegrable nonlinear PDM-oscillators. Where, three two-dimensional sub-superintegrable Mathews-Lakshmanan type PDM-oscillators are used: (i) a sub-superintegrable Mathews-Lakshmanan type-I PDM-oscillator for a PDM-particle moving in a harmonic oscillator potential, (ii) a sub-superintegrable Mathews-Lakshmanan type-II PDM-oscillator for a PDM-particle moving in a constant potential, and (iii) a sub-superintegrable shifted Mathews-Lakshmanan type-III PDM-oscillator for a PDM-particle moving in a shifted harmonic oscillator potential. The second set, nevertheless, includes some superintegrable shifted linear oscillators (new to the best of our knowledge) and isotonic oscillators that are mapped into sub-superintegrable PDM-nonlinear and sub-superintegrable PDM-isotonic oscillators, respectively. The mappings included exact solvability as well. Our observations are in order.

Whilst the two-dimensional PDM Mathews-Lakshmanan type-I and type-III, and the PDM shifted nonlinear oscillators share the same total energy

$$E_{tot} = \frac{1}{2} \omega^2 \frac{(A_1^2 + A_2^2)}{1 \pm \beta (A_1^2 + A_2^2)},$$

(71)

the two-dimensional PDM Mathews-Lakshmanan type-II oscillators admit total energy

$$E_{tot} = \frac{\omega^2 \xi^2}{1 - \xi^2 (A_1^2 + A_2^2)},$$

(72)

and the two-dimensional PDM-deformed isotonic oscillators indulge total energy

$$E_{tot} = \frac{1}{2} \left\{ \frac{\omega_1^2 A_1 + \omega_2^2 A_2}{\Omega \pm \lambda (A_1 + A_2)} - [\Omega \pm \lambda (A_1 + A_2)] (A_1 + A_2) \right\}.$$

(73)

Yet, the two-dimensional PDM Mathews-Lakshmanan type-I and type-III inherit the dynamical properties and trajectories of each other. On the other hand, the sub-superintegrability of the linear oscillator (22) and/or the shifted-oscillators (54) may very well be attributed to the superintegrability of the linear oscillator (22) and/or the shifted-oscillators (54).

Finally, the generalization of the current methodical proposal into more than two-dimensional recipes looks eminent and feasible. This is very obviously documented in the description of our equations (6) to (12), where $j = 1, 2$ is used. Strictly speaking, for a three-dimensional case $j = 1, 2 \rightarrow j = 1, 2, 3$ and $\bar{x} = x_1, x_2 \rightarrow \bar{x} = x_1, x_2, x_3$ in (10) and so on so forth. It would be interesting to study and explore the consequences of such generalization.
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