Ruin probabilities under general investments and heavy-tailed claims

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Abstract. In this paper we study the asymptotic decay of finite time ruin probabilities for an insurance company that faces heavy-tailed claims, uses predictable investment strategies and makes investments in risky assets whose prices evolve according to quite general semimartingales. We show that the ruin problem corresponds to determining hitting probabilities for the solution to a randomly perturbed stochastic integral equation. We derive a large deviation result for the hitting probabilities that holds uniformly over a family of semimartingales and show that this result gives the asymptotic decay of finite time ruin probabilities under arbitrary investment strategies, including optimal investment strategies.

1. Introduction

Consider the following model for the evolution of the risk reserve of an insurance company. The cumulative premiums minus claims up to time $t$ are modeled by a Lévy process, denoted $\varepsilon Y_t$, whose downward jumps are assumed to have a heavy-tailed (regularly varying) distribution. The insurance company has the opportunity to deposit its capital to a bank account giving instantaneous interest rate $r_t$ and to invest its capital by taking positions in $n$ risky assets with spot prices $S^k_t$, $k = 1, \ldots, n$. We assume that the spot prices form strictly positive semimartingales and that the interest rates form a càdlàg adapted process. We let $\pi^0_t$ denote the fraction of the risk reserve deposited to the bank account and let, for $k = 1, \ldots, n$, $\pi^k_t$ denote the fraction invested in the $k$th risky asset at time $t$. It is assumed that $\pi_t = (\pi^0_t, \ldots, \pi^n_t)$ is a càdlàg predictable process. By construction $\pi^0_t + \cdots + \pi^n_t = 1$.

With this notation the evolution of the risk reserve $X^\varepsilon_t$ over time is given by the stochastic integral equation

$$X^\varepsilon_t = x + \int_{0^+}^t \pi^0_s X^\varepsilon_{s^-} - r_s ds + \sum_{k=1}^n \int_{0^+}^t \pi^k_s X^\varepsilon_{s^-} \frac{dS^k_s}{S^k_{s^-}} + \varepsilon Y_t, \quad t \geq 0,$$

where $x > 0$ denotes the initial capital. In this paper the ruin probability over a finite time interval, which without loss of generality is taken to be $[0,1]$, is studied. Since this probability cannot be computed without assuming a particular (simple)
parametric model, we will rely on asymptotic approximations. In this paper the asymptotic decay of the ruin probability $P(\inf_{t \in [0,1]} X_t^\varepsilon < 0)$ is determined, as $\varepsilon \to 0$. The investment strategies are allowed to depend on $\varepsilon$; natural examples would be strategies that are functions of the reserve- and premium-minus-claims processes. Moreover, the asymptotic decay of the ruin probability under optimal investment strategies is obtained (see Theorem 1 and Corollary 1).

The formulation of the ruin problem can be restated in terms of hitting probabilities for the solution to the stochastic integral equation

$$X_t^\varepsilon = x + \int_{0^+}^t X_s^\varepsilon dZ_s + \varepsilon Y_t, \quad t \in [0, 1],$$

where $Z$ is a semimartingal. In particular, the stochastic integral equations (1) and (2) coincide on $[0, 1]$ if

$$Z_t = \int_{0^+}^t \pi_s^0 r_s ds + \sum_{k=1}^n \int_{0^+}^t \pi_s^k dS^k_s.$$

If the quadratic covariation process $[Z, Y] = 0$ a.s. it follows from Itô’s formula (see Lemma 1) that the solution $X_t^\varepsilon$ to (2) is given by

$$X_t^\varepsilon = \mathcal{E}(Z)_t \left( x + \varepsilon \int_{0^+}^t \frac{dY_s}{\mathcal{E}(Z)_s} \right), \quad t \in [0, 1],$$

and $X_t^0 = x \mathcal{E}(Z)_t$, where $\mathcal{E}(Z)$ denotes the Doléans-Dade exponential (11, p. 84)

$$\mathcal{E}(Z)_t = e^{Z_t - \frac{1}{2} [Z, Z]^c_t} \prod_{s \in (0, t]} (1 + \Delta Z_s) e^{-\Delta Z_s}.$$

Here $[Z, Z]^c$ is the continuous part of the quadratic variation process and $\Delta Z_t = Z_t - Z_{t-}$. Note that if $Z$ has jumps bounded below by $-1$ and $\inf_{t \in (0,1]} \Delta Z_t > -1$, then $\mathcal{E}(Z)_t$ is strictly positive and it follows that $\inf_{t \in [0,1]} X_t^0 > 0$. However, the process $\varepsilon Y$ may cause $X_t^\varepsilon$ to be negative but as $\varepsilon \to 0$ such events become more and more rare. Using a functional large deviation result for stochastic integrals driven by regularly varying Lévy processes the asymptotic decay of the hitting probability $P(\inf_{t \in [0,1]} X_t^\varepsilon < 0)$ as $\varepsilon \to 0$, is obtained (under a natural moment condition on $\mathcal{E}(Z)$). This immediately gives the asymptotic decay of the ruin probability.

Letting $\varepsilon \to 0$ in the ruin problem means that we are studying the decay of the ruin probability when the premiums-minus-claims process becomes (arbitrary) small compared to the risk reserve. Alternatively, one can keep $\varepsilon$ fixed and let the initial capital $x \to \infty$. This is the more popular approach in the risk theory literature. From (1) we see that

$$P \left( \inf_{t \in [0,1]} X_t^\varepsilon < 0 \right) = P \left( \inf_{t \in [0,1]} \left\{ x + \varepsilon \int_{0^+}^t \frac{dY_s}{\mathcal{E}(Z)_s} \right\} < 0 \right) = P \left( \inf_{t \in [0,1]} \int_{0^+}^t \frac{dY_s}{\mathcal{E}(Z)_s} < -\frac{x}{\varepsilon} \right)$$

and hence the asymptotic analysis in the two cases is identical.
Of particular interest is the asymptotic decay of the ruin probability under an optimal investment strategy; i.e. a strategy that minimizes the ruin probability. We prove a large deviation result for hitting probabilities for $X^\varepsilon$ in (4) with $Z$ as in (3) which holds uniformly over a family $\Pi$ of investment strategies $\pi$:

$$\lim_{\varepsilon \to 0} \inf_{\pi \in \Pi} P(\inf_{t \in [0,1]} X^\varepsilon_{t; Z} < 0) = x^{-\alpha} \inf_{\pi \in \Pi} \int_0^1 E\mathcal{E}(Z)^{-\alpha}_t dt,$$

where $\nu$ is the Lévy measure of $Y_1$. Roughly speaking our result says that, for small $\varepsilon$, the optimal strategy (which may depend on $\varepsilon$) does not yield much smaller ruin probability than, what we call, an asymptotically optimal strategy. That is, a strategy that minimizes the integral on the right-hand side in (5). This is relevant, because finding asymptotically optimal strategies is much easier than finding optimal strategies. In some cases an asymptotically optimal strategy can be explicitly calculated (see Proposition 6 below).

In the special case where the asset price follows a geometric Brownian motion and the premiums-minus-claims process is a compound Poisson process, the optimal investment strategy, for the infinite time horizon ruin problem with interest rate $r = 0$, is characterized in [6]. There the authors use stochastic control theory to characterize the optimal strategy as a solution to a partial differential equation. In the case of heavy-tailed claim sizes, the asymptotic value (as the initial capital $x \to \infty$) of the optimal fraction invested in the risky asset is determined in [5] and [13]. It coincides with the asymptotically optimal strategy (in the finite time horizon case) determined in Example 2 below. When the asset price follows an exponential Lévy process the asymptotic decay of the ruin probability for constant investments $\pi$ was recently studied in [9]; also in the case of an infinite time horizon.

The paper is organized as follows. Section 2 is devoted to the asymptotic decay, as $\varepsilon \to 0$, of hitting probabilities for the solution $X^\varepsilon$ in (2). This result is applied to finite time horizon ruin problems in Section 3, where we also consider asymptotically optimal strategies. All the proofs and some auxiliary results are given in Section 4.

Throughout the paper we refer to [11] for definitions and notation. We assume that all the random elements considered are defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,1]}, P)$ satisfying the usual hypotheses, see p. 3 in [11].

2. Hitting probabilities for the solution to a stochastic integral equation

In this section we investigate hitting probabilities for the solution to a stochastic integral equation that is perturbed by small but heavy-tailed random noise. The main result is Theorem 1 that gives a large deviation result for hitting probabilities which holds uniformly over a family of semimartingales.

Consider the stochastic integral equation

$$X^\varepsilon_t = x + \int_{0}^{t} X^\varepsilon_{s-} dZ_s + \varepsilon Y_t, \quad t \in [0,1],$$

where $X^\varepsilon_t$ is the solution to the stochastic integral equation perturbed by small but heavy-tailed random noise $Z$ and $Y_t$ is the premiums-minus-claims process. The limiting behavior of the ruin probability $P(\inf_{t \in [0,1]} X^\varepsilon_{t; Z} < 0)$ as $\varepsilon \to 0$ is determined by the large deviation principle:

$$\lim_{\varepsilon \to 0} \inf_{\pi \in \Pi} P(\inf_{t \in [0,1]} X^\varepsilon_{t; Z} < 0) = x^{-\alpha} \inf_{\pi \in \Pi} \int_0^1 E\mathcal{E}(Z)^{-\alpha}_t dt,$$

where $\nu$ is the Lévy measure of $Y_1$. Roughly speaking our result says that, for small $\varepsilon$, the optimal strategy (which may depend on $\varepsilon$) does not yield much smaller ruin probability than, what we call, an asymptotically optimal strategy. That is, a strategy that minimizes the integral on the right-hand side in (5). This is relevant, because finding asymptotically optimal strategies is much easier than finding optimal strategies. In some cases an asymptotically optimal strategy can be explicitly calculated (see Proposition 6 below).

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$$\lim_{\varepsilon \to 0} \inf_{\pi \in \Pi} P(\inf_{t \in [0,1]} X^\varepsilon_{t; Z} < 0) = x^{-\alpha} \inf_{\pi \in \Pi} \int_0^1 E\mathcal{E}(Z)^{-\alpha}_t dt,$$
where $Y$ is a Lévy process and $Z$ is a semimartingal. If the quadratic covariation process $[Z,Y] = 0$ a.s. it follows from Itô’s formula (see Lemma 1 below) that the solution $X^\varepsilon$ to (6) is given by

$$
X^\varepsilon_t = \mathcal{E}(Z)_t \left( x + \varepsilon \int_{0+}^t \frac{dY_s}{\mathcal{E}(Z)_{s-}} \right), \quad t \in [0,1],
$$

and $X^\varepsilon_0 = x\mathcal{E}(Z)_t$. Suppose $Z$ has jumps bounded below by $-1$, i.e. $\inf_{t \in [0,1]} \Delta Z_t > -1$. Then $\mathcal{E}(Z)_t$ is strictly positive and it follows that $\inf_{t \in [0,1]} X^\varepsilon_t > 0$. However, for $\varepsilon > 0$ the process $Y$ may cause $X^\varepsilon_t$ to take negative values and as $\varepsilon \to 0$ this event becomes more and more rare. We are concerned with the asymptotic decay of the probability that $\inf_{t \in [0,1]} X^\varepsilon_t < 0$. Using the explicit solution (7) it follows that

$$
\left\{ \inf_{t \in [0,1]} X^\varepsilon_t < 0 \right\} = \left\{ \inf_{t \in [0,1]} \int_{0+}^t \frac{dY_s}{\mathcal{E}(Z)_{s-}} < -\frac{x}{\varepsilon} \right\}.
$$

Hence, it is sufficient to consider hitting probabilities for the stochastic integral on the right hand side.

Suppose, for now, that the Lévy measure $\nu$ of $Y_1$ is regularly varying. That is, there is an $\alpha > 0$ and a $p \in [0,1]$ such that, for all $\lambda > 0$,

$$
\lim_{u \to \infty} \frac{\nu(-\infty,-\lambda u)}{\nu(-\infty,-u) \cup (u,\infty)} = p \lambda^{-\alpha}, \quad \lim_{u \to 0} \frac{\nu(\lambda u,\infty)}{\nu(-\infty,-u) \cup (u,\infty)} = (1-p) \lambda^{-\alpha}.
$$

Using (5) together with a functional large deviation result in (7) for stochastic integral processes driven by regularly Lévy processes, the asymptotic decay of the hitting probability can be obtained. A modification of Example 3.2 in (7) is the following.

**PROPOSITION 1.** Let $Y$ be a Lévy process and suppose that the Lévy measure $\nu$ of $Y_1$ satisfies (2) with $p > 0$. Let $Z$ be a semimartingale such that $\inf_{t \in [0,1]} \Delta Z_t > -1$ a.s., $[Z,Y] = 0$ a.s., and for some $\delta > 0$

$$
E \sup_{t \in [0,1]} \mathcal{E}(Z)^{-\alpha-\delta} < \infty.
$$

Then the solution $X^\varepsilon$ to (6) satisfies

$$
\lim_{\varepsilon \to 0} \frac{P(\inf_{t \in [0,1]} X^\varepsilon_t < 0)}{\nu(-\infty,-\varepsilon^{-1})} = x^{-\alpha} \int_0^1 E \mathcal{E}(Z)^{-\alpha} dt.
$$

Note that the moment condition (10) only concerns the behavior of $\mathcal{E}(Z)$ near 0. This conditions implies that the probability that the unperturbed system $X^0$ is close to 0 is sufficiently small. If $Z$ is a Lévy process satisfying $\inf_{t \in [0,1]} \Delta Z_t > -1$ a.s., then whether (11) holds or not depends only on the decay of the Lévy measure of $Z_1$ near $-1$. In this case the following is a more easily checked sufficient condition.

**PROPOSITION 2.** Let $Z$ be a Lévy process and let $\eta$ be the Lévy measure of $Z_1$. If $\eta(-\infty,-1] = 0$ and $\int_{-1}^{\alpha} (1+z)^{-\alpha-\delta} \eta(dz) < \infty$ for some $\alpha \in (0,1)$, then (10) holds.
Since this is a special case of Proposition 4 below we omit the proof.

**Remark 1.** Note that for the Lévy process \( Z \) in Proposition 2, \( \eta(\mathbb{R})/(\mathbb{R})(\exp\{\eta(\mathbb{R})/\lambda\}) = 0 \) implies that \( \inf_{t \in (0,1]} \Delta Z_t > -1 \) a.s.

The moment condition on the Lévy measure \( \eta \) is such that the distribution of the jumps of the Lévy process \( Z \) can be regularly varying at \(-1\) as long as the index of regular variation is strictly less than \(-\alpha\). That is, the risky asset may, for instance, have heavy-tailed negative returns as long as the tail is not too heavy compared to that of the Lévy measure \( \nu \).

If \( Z \) is a Lévy process, then the constant \( \int_0^1 E\mathbb{E}(Z)^{-\alpha} dt \) appearing in Proposition 1 can be explicitly computed.

**Proposition 3.** Let \( Z \) be a Lévy process on the form \( Z_t = rt + \sigma B_t + J_t \), where \( r \in \mathbb{R}, \sigma \geq 0, B \) is a standard Brownian motion and \( J \) is a compound Poisson process independent of \( B \). If \( J \neq 0 \) and the Lévy measure \( \eta \) of \( J_1 \) satisfies \( \eta(\mathbb{R})(\exp\{\eta(\mathbb{R})/\lambda\}) < \infty \) for some \( a \in (0,1) \), then

\[
\int_0^1 E\mathbb{E}(Z)^{-\alpha} dt = C(\alpha, r, \sigma) \frac{1}{\lambda} (e^{\lambda} - 1)
\]

with \( \lambda = \exp\{\eta(\mathbb{R})/\exp\{\eta(\mathbb{R})^{-1} \int (1 + x)^{-\alpha} \eta(dx) \} - 1\} \).

The proof is given in Section 4.

**Remark 2.** The expectation \( E\mathbb{E}(Z)^{-\alpha} \) can be computed explicitly also in the case when the Lévy process \( J \) is not necessarily a compound Poisson process by combining Theorem 25.17 in [12] and Lemma 2.7 in [8]. However, this results in a more complicated expression.

Proposition 1 is sufficient for determining the asymptotic decay of finite time ruin probabilities in quite general models. Not surprisingly, the result can be shown to hold without any assumption about the decay of the right tail of the Lévy measure \( \nu \). Indeed, it is only the negative jumps of \( \varepsilon Y \) that can cause the process \( X^\varepsilon \) to take negative values. What is more important is that the result is very robust to changes in the semimartingale \( Z \). Next we explore this robustness in detail.

Throughout the rest of this paper we weaken the assumption (9) and only assume that the Lévy measure \( \nu \) of \( Y_1 \) has a regularly varying left tail. That is, for some \( \alpha > 0 \),

\[
\lim_{u \to \infty} \frac{\nu(\mathbb{R})}{\nu(\mathbb{R})} = \lambda^{-\alpha}, \quad \lambda > 0.
\]

In particular, the right tail of \( \nu \) is allowed to decay arbitrarily slowly. Proposition 1 can be extended to hold uniformly over a family of semimartingales in the sense of the following theorem.
Theorem 1. Let $Y$ be a Lévy process and suppose that the Lévy measure $\nu$ of $Y_1$ satisfies (11). Let $\Gamma$ be any non-empty family of semimartingales $Z$ such that $[Z,Y] = 0$ a.s. and $\inf_{t \in [0,1]} \Delta Z_t > -1$ a.s. for every $Z \in \Gamma$, and such that

$$\sup E \sup_{\nu \in [0,1]} E(Z_t^{-\alpha - \delta}) < \infty \text{ for some } \delta > 0.$$  

Then the solutions $X^\varepsilon = X^\varepsilon_{\varepsilon Z}$, for $Z \in \Gamma$, to (10) satisfy

$$\lim \sup_{\varepsilon \to 0} E(\varepsilon Z_t^{-\alpha}) = \lim_{\varepsilon \to 0} \inf_{Z \in \Gamma} E(Z_t^{-\alpha}) = 0$$

and

$$\lim \inf_{\varepsilon \to 0} E(Z_t^{-\alpha}) = \inf_{Z \in \Gamma} E(Z_t^{-\alpha}) = 0.$$  

In particular, if there exists $Z^\ast \in \Gamma$ such that

$$\int_0^1 E(Z_t^{-\alpha}) dt = \inf_{Z \in \Gamma} \int_0^1 E(Z_t^{-\alpha}) dt,$$

then

$$\lim \inf_{\varepsilon \to 0} E(\varepsilon Z_t^{-\alpha}) = \lim_{\varepsilon \to 0} \inf_{Z \in \Gamma} E(Z_t^{-\alpha}) = \inf_{Z \in \Gamma} E(Z_t^{-\alpha}).$$

The proof is given in Section 4.

3. Asymptotic decay of finite time ruin probabilities

Consider an insurance company whose cumulative premiums minus claims are modeled by a Lévy process $\varepsilon Y$. The Lévy measure $\nu$ of $Y_1$ is assumed to satisfy (11). That is, the left tail of $\nu$ is regularly varying. Suppose that the insurance company has the opportunity to use a dynamic investment strategy. Assume that there are $n$ risky assets whose spot prices $S^k_t$ form strictly positive semimartingales and a bank account that gives non-negative instantaneous interest rate $\kappa$, where $r = \{r_t \}_{t \in [0,1]}$ is a càdlàg adapted stochastic process. Let $\pi = \{ (\pi^0_t, \ldots, \pi^n_t) \}_{t \in [0,1]}$ be a càdlàg predictable stochastic process, where $\pi^k_t$ denotes the fraction of the risk reserve invested in the $k$th risky asset and $\pi^0_t = 1 - \pi^1_t - \cdots - \pi^n_t$ is the fraction invested in the bank account, at time $t$. With this notation the evolution of the risk reserve follows the stochastic integral equation

$$X^\varepsilon_{\pi} = x + \int_0^t \pi^0_s X^\varepsilon_{\pi} r_s ds + \sum_{k=1}^n \int_0^t \pi^k_s X^\varepsilon_{\pi} dS^k_s + \varepsilon Y_t, \quad t \in [0,1].$$

Since $S^k_t$ is a strictly positive semimartingale, $S^k_t = E(U^k_t)$, where $U^k_t$ is the semimartingale given by $U^k_t = S^k_t + \int_0^t (S^k_{s-})^{-1} dS^k_s$ (see Lemma 2.2 in [8]). Hence, $X^\varepsilon_{\pi}$ is of the form (15) where the semimartingale $Z^\pi$ is given by

$$Z^\pi_t = \int_0^t \pi^0_s r_s ds + \sum_{k=1}^n \int_0^t \pi^k_s dS^k_s, \quad t \in [0,1].$$
Note also that if $[S^k, Y] = 0$ for all $k$, then $[Z^\pi, Y] = 0$.

An investment strategy $\pi$ will be called optimal if it minimizes the ruin probability within a reasonably large class of strategies. That is, $\pi^*(\varepsilon)$ is optimal if

$$P\left( \inf_{t \in [0,1]} X_i^{\varepsilon, \pi^*(\varepsilon)} < 0 \right) \leq P\left( \inf_{t \in [0,1]} X_i^{\varepsilon, \pi} < 0 \right),$$

for every strategy $\pi$ in the class. It is generally difficult to find optimal strategies, even in relatively simple models, and it typically involves solving a partial differential equation. An easier problem is to look for, what we will call, an asymptotically optimal strategy. That is, a strategy $\pi_{as}$ that minimizes $\int_0^1 E\left( (Z^\pi_t)^{-\alpha} \right) dt$. Using Theorem 1 we find that, for small $\varepsilon$, the ruin probability for the optimal strategy $\pi^*(\varepsilon)$ is not much smaller than for an asymptotically optimal investment strategy $\pi_{as}$. More precisely, the asymptotic decay under an optimal strategy is the same as under an asymptotically optimal strategy. We summarize the findings of this section in the following corollary to Theorem 1.

**Corollary 1.** Let $Y$ be a Lévy process and suppose that the Lévy measure $\nu$ of $Y_t$ satisfies (11). Let $X^{\varepsilon, \pi}$ be the solution to (13), where each strictly positive semimartingale $S^k$ satisfies $[S^k, Y] = 0$ a.s. and $\pi$ belongs to a non-empty family $\Pi$ of càglàd predictable processes. Suppose that $\Gamma = \{Z^\pi; \pi \in \Pi\}$, where $Z^\pi$ is given by (14), satisfies the conditions of Theorem 1. Then

$$\limsup_{\varepsilon \to 0} \sup_{\pi \in \Pi} \left| \frac{P(\inf_{t \in [0,1]} X_i^{\varepsilon, \pi} < 0)}{\nu(-\infty, -\varepsilon^{-1})} \right| - x^{-\alpha} \int_0^1 E\left( (Z^\pi_t)^{-\alpha} \right) dt = 0$$

and

$$\liminf_{\varepsilon \to 0} \sup_{\pi \in \Pi} \left| \frac{P(\inf_{t \in [0,1]} X_i^{\varepsilon, \pi} < 0)}{\nu(-\infty, -\varepsilon^{-1})} \right| = \inf_{\pi \in \Pi} \lim_{\varepsilon \to 0} \frac{P(\inf_{t \in [0,1]} X_i^{\varepsilon, \pi} < 0)}{\nu(-\infty, -\varepsilon^{-1})} = \inf_{\pi \in \Pi} x^{-\alpha} \int_0^1 E\left( (Z^\pi_t)^{-\alpha} \right) dt.$$

**Remark 3.** The conditions of Theorem 1 that $\Gamma$ needs to satisfy have natural interpretations. First, it is assumed that $\inf_{t \in [0,1]} \Delta Z^\pi_t > -1$ a.s. for all $\pi \in \Pi$. This means that the company cannot be ruined simply by investing in the risky assets. At least one insurance claim is necessary for the risk reserve to become negative. The second condition is that

$$\sup_{\pi \in \Pi} \sup_{t \in [0,1]} E\left( (Z^\pi_t)^{-\alpha - \delta} \right) < \infty \quad \text{for some } \delta > 0.$$

This condition says that it is sufficiently unlikely that the company is near ruin due to unsuccessful investments only.

In the setting of (the first statement of) Corollary 1 it would be natural to consider strategies $\pi$ for which $\pi_t$ is some function of the reserve process, the interest rate and asset prices, and the premiums-minus-claims process up to (but not including) time $t$. In this case we might take

$$\pi_t = \pi_t^\varepsilon = f(X_{t-}, r_{t-}, S_{t-}^1, \ldots, S_{t-}^n, \varepsilon Y_{t-})$$
for some function $f$ and set $\Pi = \{ \pi^\varepsilon; \varepsilon \geq 0 \}$. In a given application one would choose a suitable small $\varepsilon > 0$ and use the approximation
\[
P\left( \inf_{t \in [0,1]} X_t^{\pi^\varepsilon} < 0 \right) \approx \nu(-\infty, -\varepsilon^{-1}) x^{-\alpha} \int_0^1 EE(Z_t^{\pi})^{-\alpha} dt
\]
to estimate the ruin probability.

We will now present two specific models or sets of assumptions for which the conditions of Theorem 1 hold and hence the conclusions of Corollary 1 hold. We note that whether the moment condition (12) holds depends both on the model for the risky assets $S^k$ and the set of investment strategies $\Pi$.

Consider first the case where the dynamics for the risky assets are given by
\[
S_k = S_0 + \int_{0^+}^t \mu_s - S_s - dB_s + \int_{0^+}^t \sigma_s - S_s - dB_s, \quad t \in [0,1],
\]
where $\mu$ and $\sigma$ are càdlàg adapted processes with $\inf_{t \in [0,1]} \sigma_t > 0$ a.s., and $B$ is a Brownian motion.

A sufficient condition for the moment condition (12) in Theorem 1 to hold is given next.
Proposition 5. Take $\alpha > 0$. Suppose that the evolution of the risk reserve follows (13), where $S$ is given by (15) and $\pi$ belongs to a family $\Pi$ of càdlàg predictable processes for which for all $p > 0$ and some $\gamma > \alpha$

\[
E \exp \left\{ (2\gamma^2 + \gamma) \int_0^1 \pi_t^2 \sigma_t^2 dt \right\} < \infty,
\]

\[
E \sup_{\pi \in \Pi} \exp \left\{ p \int_0^t [r_s + \pi_s(\mu_s - r_s)] ds \right\} < \infty.
\]

If $Z^\pi$ is given by (14), then there exists a $\delta > 0$ such that

\[
\sup_{\pi \in \Pi} E \sup_{t \in [0,1]} E(Z^\pi)^{1-\alpha-\delta} < \infty.
\]

Example 1. Let the dynamics of $V_t = \sigma_t^2$ be given by the Cox-Ingersoll-Ross (CIR) model

\[
V_t = V_0 + \kappa \int_0^t (\theta - V_s) ds + \delta \int_0^t \sqrt{V_s} dW_s,
\]

where $\kappa, \theta, \delta$ are positive constants and $W$ is standard Brownian motion. Corollaries 3.2 and 3.3 in [1] give necessary and sufficient conditions for the integrated squared volatility process to have finite exponential moments:

(16) \[
E \exp \left\{ u \int_0^t V_s ds \right\} < \infty, \quad u > 0.
\]

If $u \leq \kappa^2/(2\delta^2)$, then (16) holds for all $t > 0$. If $u > \kappa^2/(2\delta^2)$, then (16) holds for all $t < t^*$, where $t^* = 2\gamma^{-1}(\pi + \arctan(-\gamma/\kappa))$ with $\gamma = \sqrt{2\delta^2 u - \kappa^2}$.

When the risky asset is modeled by (15) it is possible to find the asymptotically optimal strategy explicitly.

Proposition 6. Take $\alpha > 0$. Suppose that the evolution of the risk reserve follows (13), where $S$ is given by (15) and $\pi$ belongs to the family $\Pi$ of càdlàg predictable processes for which

(17) \[
E \exp \left\{ \frac{\alpha^2}{2} \int_0^1 \pi_t^2 \sigma_t^2 dt \right\} < \infty.
\]

If $Z^\pi$ is given by (14) and if $\pi^*$ is given by $\pi^*_t = \frac{\mu - r_t}{(1+\alpha)\sigma_t}$ and satisfies (17), then

\[
\int_0^1 E\mathbb{E}(Z^{\pi^*})^{-\alpha} dt \leq \int_0^1 E\mathbb{E}(Z^\pi)^{-\alpha} dt
\]

for every $\pi \in \Pi$.

Remark 4. Note that the asymptotically optimal investment strategy looks just like the solution to the Merton problem (see e.g. [4] p. 169) with HARA utility. This comes from the fact that here minimizing $\int_0^1 E\mathbb{E}(Z^{\pi})^{-\alpha} dt$ is equivalent to minimizing $E\mathbb{E}(Z^{\pi})^{-\alpha}$ which is very similar to maximizing $E\mathbb{E}(Z^{\pi})^{\alpha}$ as is done in the Merton problem.
Example 2. Suppose \( r \) and \( \mu \) and \( \sigma \) are constants, i.e. the spot price process \( S \) of the risky asset is a geometric Brownian motion. Then, the asymptotically optimal strategy \( \pi^* \) is given by \( \pi^*_t = \frac{\mu - r}{(1 + \alpha)\sigma^2} \) and the asymptotic decay of the finite time ruin probability is

\[
\lim_{\varepsilon \to 0} \frac{P(\inf_{t \in [0, 1]} X_t^{\pi^*} < 0)}{\nu(-\infty, -\varepsilon^{-1})} = x^{-\alpha} \int_0^1 E \mathcal{E}(Z^{\pi^*})_{-\alpha} \, dt
\]

\[
= x^{-\alpha} \int_0^1 E \exp \left\{ -\alpha(1 - \pi^*)rt - \alpha \pi^* \mu t + \alpha \frac{(\pi^*)^2 \sigma^2 t}{2} - \alpha \pi^* \sigma B_t \right\} dt
\]

\[
= x^{-\alpha} \int_0^1 \exp \left\{ \left( -\alpha r - \frac{\alpha(\mu - r)^2}{2(1 + \alpha)\sigma^2} \right) t \right\} dt
\]

\[
= x^{-\alpha} \frac{1 - \exp \left\{ \left( -\alpha r - \frac{\alpha(\mu - r)^2}{2(1 + \alpha)\sigma^2} \right) \right\}}{\alpha r + \frac{\alpha(\mu - r)^2}{2(1 + \alpha)\sigma^2}}.
\]

This may be compared to the strategy \( \pi = 0 \) with no investment in the risky asset. Proposition 3 yields

\[
\lim_{\varepsilon \to 0} \frac{P(\inf_{t \in [0, 1]} X_t^0 < 0)}{\nu(-\infty, -\varepsilon^{-1})} = x^{-\alpha} \int_0^1 E \mathcal{E}(Z^0)_{-\alpha} \, dt = x^{-\alpha} \frac{1 - e^{-\alpha r}}{\alpha r}.
\]

Note that the reduction of the asymptotic decay of the ruin probability using the asymptotically optimal strategy compared to no investment depend crucially on the (Sharpe) ratio \( \gamma = (\mu - r) / \sigma \). If the constant

\[
R = \lim_{\varepsilon \to 0} \frac{P(\inf_{t \in [0, 1]} X_t^{\pi^*} < 0)}{P(\inf_{t \in [0, 1]} X_t^0 < 0)} = \frac{1 - e^{-\alpha r} e^{-\alpha \gamma^2 / 2(1 + \alpha)}}{1 - e^{-\alpha r}} \frac{\alpha r}{\alpha r + \alpha \gamma^2 / 2(1 + \alpha)}
\]

is studied for reasonable parameter choices, \((r, \alpha) = (0.05, 2)\) say, then one finds that it is necessary to have the opportunity to invest in a very attractive risky asset, \( \gamma > 1 \) say, to have any significant reduction of the ruin probability.

As mentioned in the introduction, Example 2 above is closely related to the studies in \([6, 5, 13]\) of the infinite horizon case with \( r = 0 \). Translating the results to our notation the authors obtain the following limit as \( \varepsilon \to 0 \) of the optimal strategy \( \pi^*(\varepsilon) \):

\[
\lim_{\varepsilon \to 0} \pi^*(\varepsilon) = \frac{\mu}{(1 + \alpha)\sigma^2}.
\]

This coincides with the asymptotically optimal strategy calculated above.

4. Proofs and auxiliary results

Lemma 1. The stochastic integral equation (6) has a unique solution which is given by (7).
Proof of Lemma 11. First some notation. Let $A_t = Z_t - \frac{1}{2}[Z, Z]_t$, $B_t = \prod_{s \in (0,t)} (1 + \Delta Z_s) e^{-\Delta Z_s}$, $C_t = x + \varepsilon \int_{0+}^{t} dY_s$, and $X_t^\varepsilon = e^{A_t} B_t C_t$. Then $[A, B]_t^\varepsilon = [B, C]_t^\varepsilon = 0$, and $[A, A]_t^\varepsilon = [Z, Z]_t^\varepsilon$. By Itô’s formula (see [11] Theorem 33)

$$X_t^\varepsilon - x = \int_{0+}^{t} X_s^\varepsilon dA_s + \int_{0+}^{t} e^{A_s} C_s dB_s + \int_{0+}^{t} e^{A_s} B_s dC_s$$

$$+ \frac{1}{2} \int_{0+}^{t} X_s^\varepsilon d[A, A]_s + \int_{0+}^{t} e^{A_s} B_s d[A, C]_s$$

$$+ \sum_{s \in (0,t]} (X_s^\varepsilon - X_s^\varepsilon - X_s^\varepsilon \Delta A_s - e^{A_s} C_s \Delta B_s - e^{A_s} B_s \Delta C_s)$$

$$= \int_{0+}^{t} X_s^\varepsilon dZ_s + \sum_{s \in (0,t]} e^{A_s} C_s \Delta B_s + \varepsilon \int_{0+}^{t} dY_s + \int_{0+}^{t} e^{A_s} B_s d[A, C]_s$$

$$+ \sum_{s \in (0,t]} (X_s^\varepsilon - X_s^\varepsilon - X_s^\varepsilon \Delta A_s - e^{A_s} C_s \Delta B_s - e^{A_s} B_s \Delta C_s)$$

$$= \int_{0+}^{t} X_s^\varepsilon dZ_s + \varepsilon Y_t + \int_{0+}^{t} e^{A_s} B_s d[A, C]_s$$

$$+ \sum_{s \in (0,t]} (e^{A_s} B_s - (1 + \Delta Z_s)(C_s - C_s-) - e^{A_s} B_s \Delta C_s)$$

$$= \int_{0+}^{t} X_s^\varepsilon dZ_s + \varepsilon Y_t + \int_{0+}^{t} e^{A_s} B_s d[A, C]_s$$

$$+ \sum_{s \in (0,t]} (e^{A_s} B_s - (1 + \Delta Z_s) \Delta C_s - e^{A_s} B_s \Delta C_s)$$

$$= \int_{0+}^{t} X_s^\varepsilon dZ_s + \varepsilon Y_t + \int_{0+}^{t} e^{A_s} B_s d[A, C]_s + \sum_{s \in (0,t]} e^{A_s} B_s \Delta Z_s \Delta C_s$$

$$= \int_{0+}^{t} X_s^\varepsilon dZ_s + \varepsilon Y_t + \int_{0+}^{t} e^{A_s} B_s d[A, C]_s$$

$$= \int_{0+}^{t} X_s^\varepsilon dZ_s + \varepsilon Y_t + \varepsilon [Z, Y]_t$$

$$= \int_{0+}^{t} X_s^\varepsilon dZ_s + \varepsilon Y_t.$$
Proof of Proposition 3. First consider the case $J = 0$. The constant $C(\alpha, r, \sigma)$ is computed as follows
\[
\int_0^1 E^\mathbb{P}(Z_t^{-\alpha}) dt = \int_0^1 E\exp\{-\alpha((r - \sigma^2/2)t + \sigma B_t)\} dt
\]
\[
= \int_0^1 \exp\{-\alpha(r - \sigma^2/2)t\}E\exp\{-\alpha\sigma B_t\} dt
\]
\[
= \int_0^1 \exp\{\sigma^2(\alpha^2 + \alpha)/2 - \alpha r\} t dt
\]
\[
= C(\alpha, r, \sigma).
\]
Now consider the case $J \neq 0$. Note that the Dolean-Dade exponential of a sum of two independent processes is the product of the two Dolean-Dade exponentials. To complete the proof we just repeat the computations at the end of the proof of Proposition 2. This gives
\[
\mathcal{E}(J_t^{-\alpha}) = \exp\{t\eta(R)(\exp\{\eta(R)^{-1} \int (1 + x)^{-\alpha} \eta(dx)\} - 1)\}.
\]

□

Proof of Theorem 1. From (8) it follows that, provided that the limit exists,
\[
\lim_{\varepsilon \to 0} \inf_{Z \in \Gamma} \mathbb{P}(\inf_{t \in [0,1]} \int_0^t A_s dY_s < -x/\varepsilon) = \int_0^1 \mathbb{E}[A_t^\alpha] dt.
\]
Applying Theorem 2 below completes the proof.

□

Theorem 2. Let $Y$ be a Lévy process such that the Lévy measure $\nu$ of $Y_1$ satisfies (11) for some $\alpha > 0$. Let $A$ be a family of caglād predictable strictly positive processes satisfying $\sup_{A \in A} \mathbb{E} \sup_{t \in [0,1]} |A_t|^{|\alpha + \varepsilon|} < \infty$ for some $\varepsilon > 0$. Then

(i) $\lim_{x \to \infty} \inf_{A \in A} \frac{P(\inf_{t \in [0,1]} \int_0^t A_s dY_s < -x)}{\nu(-\infty, -x)} = \inf_{A \in A} \int_0^1 EA_t^\alpha dt$,

(ii) $\lim_{x \to \infty} \sup_{A \in A} \left| \frac{P(\inf_{t \in [0,1]} \int_0^t A_s dY_s < -x)}{\nu(-\infty, -x)} - \int_0^1 EA_t^\alpha dt \right| = 0.$
Proof. We use the notation \((A \cdot Y)\) for the stochastic integral process given by \((A \cdot Y)_t = \int_0^t A_s dY_s\). We first show that (ii) implies (i):

\[
\limsup_{x \to \infty} \inf_{A \in A} \frac{P(\inf_{t \in [0,1]} (A \cdot Y)_t < -x)}{\nu(-\infty, -x)} \\
\leq \limsup_{x \to \infty} \inf_{A \in A} \left( \frac{P(\inf_{t \in [0,1]} (A \cdot Y)_t < -x)}{\nu(-\infty, -x)} - \int_0^1 E A_t^\alpha dt + \int_0^1 E A_t^\alpha dt \right) \\
\leq \limsup_{x \to \infty} \sup_{A \in A} \left| \frac{P(\inf_{t \in [0,1]} (A \cdot Y)_t < -x)}{\nu(-\infty, -x)} - \int_0^1 E A_t^\alpha dt \right| + \inf_{A \in A} \int_0^1 E A_t^\alpha dt \\
= \inf_{A \in A} \int_0^1 E A_t^\alpha dt,
\]

Hence, (ii) implies (i).

It remains to show (ii). We decompose \(Y\) (the Lévy-Itô decomposition) into a sum \(Y = \tilde{Y} + J\) of independent Lévy processes, where \(\tilde{Y}\) has jumps whose norms are bounded by 1 and \(J = Y - \tilde{Y}\) is a compound Poisson process with representation \(J_t = \sum_{k=1}^{N_t} Z_k\). Moreover, we can decompose \(J\) into a sum \(J = J_x + (J - J_x)\) of independent compound Poisson processes, where \(J_x\) consists of the jumps \(\Delta Y_t\) of \(Y\) with \(|\Delta Y_t| > x^\beta\) for some \(\beta \in (1/2, 1)\). Let \(M_x = \#\{t \in [0,1] : |\Delta Y_t| > x^\beta\}\) and let \(\tau_{x,1}, \ldots, \tau_{x,M_x}\) be the time points of these jumps. Then \(J_x = \{J_x(t)\}_{t \in [0,1]}\) is given by \(J_x = \sum_{k=1}^{M_x} Z_{x,k} I_{[\tau_{x,k},1]}\), where \(Z_{x,k} = \Delta Y_{\tau_{x,k}}\). Note that, for any \(\delta > 0\),

\[
P\left(\inf_{t \in [0,1]} (A \cdot Y)_t < -x\right) = P\left(\inf_{t \in [0,1]} (A \cdot Y)_t < -x, \sup_{t \in [0,1]} |(A \cdot \tilde{Y})_t| > \delta x\right) \\
+ P\left(\inf_{t \in [0,1]} (A \cdot Y)_t < -x, \sup_{t \in [0,1]} |(A \cdot \tilde{Y})_t| \leq \delta x\right) \\
\leq P\left(\sup_{t \in [0,1]} |(A \cdot \tilde{Y})_t| > \delta x\right) \\
+ P\left(\inf_{t \in [0,1]} (A \cdot J)_t < -(1 - \delta)x\right)
\]
and
\[ P(\inf_{t \in [0,1]} (A \cdot Y)_t < -x) = P(\inf_{t \in [0,1]} (A \cdot Y)_t < -x, \sup_{t \in [0,1]} |(A \cdot \tilde{Y})_t| > \delta x) \]
\[ + P(\inf_{t \in [0,1]} (A \cdot Y)_t < -x, \sup_{t \in [0,1]} |(A \cdot \tilde{Y})_t| \leq \delta x) \]
\[ \geq P(\inf_{t \in [0,1]} (A \cdot J)_t < -(1+\delta)x). \]

Hence, in order to prove (ii) it is sufficient to prove that
\[ \lim_{x \to \infty} \sup_{A \in A} \left| \frac{P(\inf_{t \in [0,1]} (A \cdot J)_t < -x)}{\nu(-\infty, -x)} - \int_{0}^{1} EA_{A} \right| = 0 \]
and that
\[ \lim_{x \to \infty} \sup_{A \in A} \frac{P(\inf_{t \in [0,1]} (A \cdot \tilde{Y})_t < -x)}{\nu(-\infty, -x)} = 0. \]

Similarly, in order to prove [18] it is sufficient to prove that
\[ \lim_{x \to \infty} \sup_{A \in A} \left| \frac{P(\inf_{t \in [0,1]} (A \cdot J_x)_t < -x)}{\nu(-\infty, -x)} - \int_{0}^{1} EA_{A} \right| = 0 \]
and that
\[ \lim_{x \to \infty} \sup_{A \in A} \frac{P(\inf_{t \in [0,1]} (A \cdot (J - J_x))_t < -x)}{\nu(-\infty, -x)} = 0. \]

However, [19] follows from Lemma 5.5 in [7] (Lemma 5.5 in [7] is proved without the supremum over $A$ but the proof holds also for the present stronger statement). We now show [21]. Decompose $J - J_x$ into the sum $J - J_x = (J - J_x)^+ + (J - J_x)^-$, where
\[ (J - J_x)^+ = \sum_{k=1}^{N_x} Z_k I_{[1,x]}(Z_k), \quad (J - J_x)^- = \sum_{k=1}^{N_x} Z_k I_{[-x,-1]}(Z_k). \]

Note that
\[ \sup_{A \in A} \frac{P(\inf_{t \in [0,1]} (A \cdot (J - J_x))_t < -x)}{\nu(-\infty, -x)} \leq \sup_{A \in A} \frac{P(\inf_{t \in [0,1]} (A \cdot (J - J_x)^-)_t < -x)}{\nu(-\infty, -x)} \]
\[ = \sup_{A \in A} \frac{P(\sup_{t \in [0,1]} (A \cdot (J - J_x)^-)_t > x)}{\nu(-\infty, -x)} \]
and that [22] $\to 0$ as $x \to \infty$ by Lemma 5.3 and Remark 5.1 in [7] (Lemma 5.3 in [7] is proved without the supremum over $A$ but the proof holds also for the present stronger statement). Hence, we have shown [21].

It remains to prove [20]. Let
\[ M_x^- = \# \{ t \in (0,1]; \Delta Y_t < -x^\beta \}, \quad (J_x^-)_t = \sum_{k=1}^{N_x} Z_k I_{(-\infty,-x^\beta]}(Z_k). \]
Note that
\[
P(\tau_x, Z_1^x < -x, M_x = 1) \leq P(\inf_{t \in [0,1]} (A \cdot J_x)_t < -x)
\]
\[
\leq P(\inf_{t \in [0,1]} (A \cdot J_x^-)_t < -x, M_x^- = 1)
\]
\[
+ P(\inf_{t \in [0,1]} (A \cdot J_x^-)_t < -x, M_x^- \geq 2)
\]
\[
= P(\tau_x, Z_1^x < -x, M_x = 1)
\]
\[
+ P(\inf_{t \in [0,1]} (A \cdot J_x^-)_t < -x, M_x^- \geq 2)
\]
\[
\leq P(\tau_x, Z_1^x < -x, M_x = 1) + P(M_x^- \geq 2)
\]
and that \(\lim_{x \to \infty} P(M_x^- \geq 2) / \nu(\infty, -x) = 0\) by Lemma 5.4 in [7]. Applying Lemma 2 below shows (20) and hence completes the proof. □

**Lemma 2.** With the notation above it holds that
\[
\lim_{x \to \infty} \sup_{A \in \mathcal{A}} \left| \frac{P(\tau_x, Z_1^x < -x, M_x = 1)}{\nu(\infty, -x)} - \int_0^1 EA^\alpha_t dt \right| = 0.
\]

**Proof.** Let \(\xi\) be the Poisson random measure with intensity measure \(\text{Leb} \times \nu\), where \(\text{Leb}\) is Lebesgue measure on \([0,1]\), that determines the jumps of \(Y\) and note that
\[
P(\tau_x, Z_1^x < -x, M_x = 1)
\]
\[
= \int P(\xi([0,1] \times (-\infty, -\max(x/y, x^\beta)]) = 1) dP(\tau_x \leq y)
\]
\[
\leq \int \nu(-\infty, -x/y) e^{-\nu(-\infty, -x/y)} dP(\tau_x \leq y)
\]
\[
\leq \int \nu(-\infty, -x/y) dP(\tau_x \leq y).
\]

Set
\[
\Delta(x, A) = \frac{P(\tau_x, Z_1^x < -x, M_x = 1)}{\nu(\infty, -x)} - \int_0^1 EA^\alpha_t dt
\]
\[
= \frac{P(\tau_x, Z_1^x < -x, M_x = 1)}{\nu(\infty, -x)} - \underbrace{EA^\alpha_{\tau_x}}_{\Delta_1(x, A)} + \underbrace{EA^\alpha_{\tau_x,1} - \int_0^1 EA^\alpha_t dt}_{\Delta_2(x, A)}
\]
We need to show two things:

\(\text{(A): } \lim_{x \to \infty} \sup_{A \in \mathcal{A}} |\Delta_1(x, A)| = 0\) and \(\text{(B): } \lim_{x \to \infty} \sup_{A \in \mathcal{A}} |\Delta_2(x, A)| = 0\).

(A): This is essentially a uniform version of what is often called Breiman’s result (see Lemma 2.2 in [10]). Take an arbitrary \(C > 0\) and note that for \(x\) sufficiently
large

\[
\Delta_1(x, A) = \int_{[0,C]} \left( \frac{\nu(-\infty, -x/y)e^{-\nu(-\infty, -x/y)}}{\nu(-\infty, -x)} - y^\alpha \right) dP(A_{x,1} \leq y) \\
- EA_{x,1}^0 I_{(C,\infty)}(A_{x,1}) \\
+ \int_{(C,\infty)} \frac{P(A_{x,1} \leq y)}{\nu(-\infty, -x)} dP(A_{x,1} \leq y) \\
= \Delta_{11}(x, A) - \Delta_{12}(A) + \Delta_{13}(x, A).
\]

Since \(\sup_{A\in\mathcal{A}} EA_{x,1}^0 < \infty\), \(\lim_{C\to\infty} \sup_{A\in\mathcal{A}} \Delta_{12}(A) = 0\). The uniform convergence theorem for regularly varying functions (Theorem 1.5.2 in \[3\]) implies that for every \(C > 0\)

\[
\sup_{A\in\mathcal{A}} |\Delta_{11}(x, A)| \leq \sup_{A\in\mathcal{A}} \int_{[0,C]} \left| \frac{\nu(-\infty, -x/y)e^{-\nu(-\infty, -x/y)}}{\nu(-\infty, -x)} - y^\alpha \right| dP(A_{x,1} \leq y) \\
\leq \sup_{y\in[0,C]} \left| \frac{\nu(-\infty, -x/y)e^{-\nu(-\infty, -x/y)}}{\nu(-\infty, -x)} - y^\alpha \right| \to 0.
\]

The Potter bounds (Theorem 1.5.6 in \[3\]) says that for any \(A > 1\) and \(\delta \in (0, \varepsilon)\) there exists \(x_0 = x_0(A, \delta)\) such that

\[
\frac{\nu(-\infty, -x/y)}{\nu(-\infty, -x)} \leq Ay^{\alpha+\delta}
\]

whenever \(x, x/y \geq x_0\) and \(y > C > 1\). Hence,

\[
\sup_{A\in\mathcal{A}} |\Delta_{13}(x, A)| \leq \int_{(C,\infty/x_0)} y^{\alpha+\delta} dP(A_{x,1} \leq y) \\
\leq A \sup_{A\in\mathcal{A}} \int_{(C,\infty/x_0)} y^{\alpha+\delta} dP(A_{x,1} \leq y) + A \sup_{A\in\mathcal{A}} P(A_{x,1} \leq -x/x_0) \\
\to 0
\]

by first letting \(n \to \infty\) and then \(C \to \infty\), since \(\sup_{A\in\mathcal{A}} EA_{x,1}^{\alpha+\varepsilon} < \infty\).

(B): We now show that \(\lim_{x \to \infty} \sup_{A\in\mathcal{A}} |\Delta_2(x, A)| = 0\). Let \(A_x = (-\infty, -x\beta) \cup (x\beta, \infty)\) and note that for all \(t \in [0, 1]\),

\[
P(\tau_{x,1} \leq t) = P(\xi([0,t] \times A_x) \geq 1 \mid \xi([0,1] \times A_x) \geq 1) = \frac{1 - \exp\{-t\nu(A_x)\}}{1 - \exp\{-\nu(A_x)\}},
\]

\[
\frac{d}{dt} P(\tau_{x,1} \leq t) = \frac{\nu(A_x) \exp\{-t\nu(A_x)\}}{1 - \exp\{-\nu(A_x)\}}.
\]
Since \( \tau_{x,1} \) and \( A_{\tau_{x,1}} \) are independent it holds that
\[
\sup_{A \in A} |E A^n_{\tau_{x,1}} - \int_0^1 E A^n_t dt| = \sup_{A \in A} \left| \int_0^1 E A^n_t \left( \frac{\nu(A_x) \exp\{-t \nu(A_x)\}}{1 - \exp\{-\nu(A_x)\}} - 1 \right) dt \right|
\leq \sup_{A \in A} E \sup_{t \in [0, 1]} A^n_t \int_0^1 \left| \frac{\nu(A_x) \exp\{-t \nu(A_x)\}}{1 - \exp\{-\nu(A_x)\}} - 1 \right| dt
\to 0
\]
as \( x \to \infty \) by the bounded convergence theorem. The proof is complete.

**Proof of Proposition [4]** We first prove the claim in the case \( n = 1 \). Then we show that the claim in the case of a general \( n \) follows from the one-dimensional case by applying Hölder’s inequality.

Let \( \alpha \) and \( \delta \) be as in Proposition [2]. According to the Lévy-Itô decomposition we can decompose \( U \) into the sum of three independent Lévy processes: \( U = F + G + H \), where \( F \) is a Gaussian process with drift, \( G \) has zero mean and jumps satisfying \(|\Delta G_t| < \varepsilon\) for some small \( \varepsilon \), and \( H \) is a compound Poisson process. Set \( \pi := \pi^1 \) so that \( \pi^0 = 1 - \pi \). Then
\[
Z^n_t = \int_0^t (1 - \pi_s) r_s - ds + \int_0^t \pi_s dF^k_s + \int_0^t \pi_s dG^k_s + \int_0^t \pi_s dH^k_s
\]
\[
=: \int_0^t (1 - \pi_s) r_s - ds + F^n_t + G^n_t + H^n_t.
\]
We note that \( E(Z^n)_t = e^{\int_0^t (1 - \pi_s) r_s - ds} E(F^n)_t E(G^n)_t E(H^n)_t \) and hence that
\[
E \sup_{t \in [0, 1]} E(Z^n)^{-(\alpha + \delta/2)} \leq E \left( \sup_{t \in [0, 1]} e^{-(\alpha + \delta/2) \int_0^t (1 - \pi_s) r_s - ds} \sup_{t \in [0, 1]} E(F^n)^{-(\alpha + \delta/2)} \right)
\cdot \left( \sup_{t \in [0, 1]} E(G^n)^{-(\alpha + \delta/2)} \right)
\cdot \left( \sup_{t \in [0, 1]} E(H^n)^{-(\alpha + \delta/2)} \right).
\]

We note that \( e^{-(\alpha + \delta/2) \int_0^t (1 - \pi_s) r_s - ds} \leq 1 \) for all \( t \) since \( \pi_s \in [0, 1] \) and \( r_s \geq 0 \) for all \( s \). Using Hölder’s inequality with \( 1 < p < (\alpha + \delta)/(\alpha + \delta/2) \) and \( 1/p + 1/q = 1 \) the above expression is less than or equal to
\[
\left( E \sup_{t \in [0, 1]} E(F^n)^{-q(\alpha + \delta/2)} \right)^{1/q} \left( E \sup_{t \in [0, 1]} E(G^n)^{-q(\alpha + \delta/2)} \right)^{1/q} \left( E \sup_{t \in [0, 1]} E(H^n)^{-p(\alpha + \delta/2)} \right)^{1/p}.
\]

Using the Cauchy-Schwartz inequality and the fact that \( \pi_t \in [0, 1] \) an upper bound for the above expression is
\[
\left( E \sup_{t \in [0, 1]} E(F^n)^{-2q(\alpha + \delta/2)} \right)^{1/2q} \left( E \sup_{t \in [0, 1]} E(G^n)^{-2q(\alpha + \delta/2)} \right)^{1/2q} \left( E \sup_{t \in [0, 1]} E(H^n)^{-p(\alpha + \delta/2)} \right)^{1/p}.
\]
The proof is complete when we have shown that each of these three factors exists finitely. We start with the first factor \( I \) and show that, for any \( \beta > 0 \),
\[
E \sup_{t \in [0, 1]} E(F^n)^{-\beta} < \infty.
\]
Write $F_t = at + \sigma B_t$ where $a \in \mathbb{R}$, $\sigma > 0$ and $B$ is a Brownian motion. Then $\mathcal{E}(F^\tau)$ is given by $\mathcal{E}(F^\tau)_t = \exp\{ \int_0^t (a\pi_s - \sigma^2 \pi_s^2/2)ds + \int_0^t \sigma \pi_s dB_s \}$ and we have

$$E \sup_{t \in [0,1]} \mathcal{E}(F^\tau)_t^{-\beta} = E \sup_{t \in [0,1]} \exp\{-\beta \int_0^t \pi_s - dB_s \} \exp\{-\beta \int_0^t (a\pi_s - \sigma^2 \pi_s^2/2)ds \} \leq \exp\{t\beta(\sigma^2/2 - \min\{a,0\})\} E \sup_{t \in [0,1]} \exp\{-\beta \int_0^t \pi_s dB_s \}.$$

Set $M_t^I := -\sigma \int_0^t \pi_s dB_s$ and note that for any $\lambda > 0$, $\lambda M^I$ is a continuous martingale and hence (see [11], Theorem 39, p. 138)

$$E \exp\{\beta M_t^I \} \leq E \exp\{4\beta^2 [M^I, M^I]_t \} \leq E \exp\{4\beta^2 \sigma^2 t \} < \infty.$$

Then Lemma 3 below completes the proof of part I.

Next we consider II and show that, for any $\beta > 0$,

$$E \sup_{t \in [0,1]} \mathcal{E}(G^\tau)_t^{-\beta} < \infty.$$

Denote by $\xi$ the Poisson random measure associated with the jumps of $G$ such that

$$G_t = \int_0^t \int_{\{|x| < \varepsilon\}} x(\xi(ds, dx) - ds\eta(dx)) =: \int_0^t \int_{\{|x| < \varepsilon\}} x\tilde{\xi}(ds, dx).$$

Then, by Itô’s formula (see also [2], p. 248)

$$\mathcal{E}(G^\tau)_t = \exp \left\{ \int_0^t \int_{\{|x| < \varepsilon\}} \log(1 + \pi_s x)^2 \tilde{\xi}(ds, dx) \right\} + \int_0^t \int_{\{|x| < \varepsilon\}} \left( \log(1 + \pi_s x) - \pi_s x \right) ds\eta(dx)$$

which gives

$$\mathcal{E}(G^\tau)_t^{-\beta} = \exp \left\{ -\beta \int_0^t \int_{\{|x| < \varepsilon\}} \log(1 + \pi_s x)\tilde{\xi}(ds, dx) \right\} - \beta \int_0^t \int_{\{|x| < \varepsilon\}} \left( \log(1 + \pi_s x) - \pi_s x \right) ds\eta(dx)$$

Set $M_t^{II} := -\int_0^t \int_{\{|x| < \varepsilon\}} \log(1 + \pi_s x)\tilde{\xi}(ds, dx)$ and note that (see e.g. [2], p. 209) that $M^{II}$ is a local martingale. For $|y| < \varepsilon$ and a constant $k = k(\varepsilon) > 0$ it holds that $|\log(1 + y) - y| \leq ky^2$. Hence, since $|\pi_t| \leq 1$,

$$E \sup_{t \in [0,1]} \mathcal{E}(G^\tau)_t^{-\beta} \leq E \sup_{t \in [0,1]} e^{\beta M_t^{II}} \exp \left\{ \beta t \int_{\{|x| < \varepsilon\}} kx^2 \eta(dx) \right\} \leq KE \sup_{t \in [0,1]} e^{\beta M_t^{II}}.$$

Moreover, the quadratic variation of $M^{II}$ is given by (see [2], p. 230)

$$[M^{II}, M^{II}]_t = \int_0^t \int_{\{|x| < \varepsilon\}} (\log(1 + \pi_s x))^2 \tilde{\xi}(ds, dx)$$
and hence (see e.g. Lemma 4.2.2, p. 197, in [2])

\[ E(M_t^{II})^2 = E[M_1^{II}, M_1^{II}]_t = \int_0^t \int_{\{|x|<\varepsilon\}} E(\log(1 + \pi_s x))^2 \nu(dx)ds. \]

This quantity is finite because \(|\log(1 + y)| \leq |y| + ky^2\) for \(|y| < \varepsilon\) so it follows in particular that \(M_t^{II}\) is a (square-integrable) martingale. By Lemma [3] below it is sufficient to show \(Ee^{\beta M_t^{II}} < \infty\).

We introduce

\[ A_t = \frac{1}{2} \int_0^t \int_{\{|x|<\varepsilon\}} -(1 + \pi_s x)^{2\beta} + 1 + 2\beta \log(1 + \pi_s x) \eta(dx)ds. \]

By a Taylor expansion we get, for \(|y| < \varepsilon\) and a constant \(k = k(\varepsilon) > 0\),

\[ |(1 + y)^{2\beta} + 1 + 2\beta \log(1 + y)| \leq ky^2. \]

This implies that \(|A_t| < Ct\) a.s. for each \(t\) and some constant \(C > 0\). It follows by Cauchy-Schwartz inequality that

\[ E\exp\{\beta M_t^{II}\} \leq \left( E\exp\{2\beta M_t^{II} - 2A_t\} \right)^{1/2} \left( E\exp\{2A_t\} \right)^{1/2}. \]

We have constructed \(A_t\) in such a way that \(\exp\{2\beta M_t^{II} - 2A_t\}\) is a nonnegative local martingale starting at 1; this follows from Corollary 5.2.2, p. 253, in [2]. Hence \(\exp\{2\beta M_t^{II} - 2A_t\}\) is also a supermartingale and its expectation is bounded by 1. Since \(A_t\) is bounded we finally arrive at \(E\exp\{\beta M_t^{II}\} < \infty\). This completes the proof of part II.

Finally we show that \(E \sup_{t \in [0,1]} \mathcal{E}(H^\pi_t)^{-\alpha-\delta} < \infty\). First we note that if \(H^\pi\) consists of only the negative jumps of \(H\), then

\[ E \sup_{t \in [0,1]} \mathcal{E}(H^\pi_t)^{-\alpha-\delta} \leq E \sup_{t \in [0,1]} \mathcal{E}(H^{-\pi}_t)^{-\alpha-\delta}. \]

We may write \(H_t^{-\pi} = \sum_{k=1}^{N_t} \pi_{\tau_k} Z_k\), where \(\{N_t\}\) is a Poisson process with intensity \(\eta(-1, -\varepsilon)\) and arrival sequence \(\tau_1, \tau_2, \ldots\), independent of the sequence (of jump sizes) \(\{Z_k\}\) with probability distribution \(\eta(\cdot \cap (-1, -\varepsilon))/\eta(-1, -\varepsilon)\). Then \(\mathcal{E}(H^{-\pi})_t = \prod_{k=1}^{N_t} (1 + \pi_{\tau_k} Z_k)\). Hence,

\[ E \sup_{t \in [0,1]} \mathcal{E}(H^\pi_t)^{-\alpha-\delta} \leq E \sup_{t \in [0,1]} \mathcal{E}(H^{-\pi}_t)^{-\alpha-\delta}\]

\[ = E \left( \prod_{k=1}^{N_t} (1 + \pi_{\tau_k} Z_k) \right)^{-\alpha-\delta}\]

\[ \leq E \left( \prod_{k=1}^{N_t} (1 + Z_k) \right)^{-\alpha-\delta}\]

\[ = E e^{-(\alpha+\delta) \sum_{k=1}^{N_t} \log(1 + Z_k)}\]

\[ = \exp\{\eta(-1, -\varepsilon)(\exp\{M(-\alpha - \delta)\} - 1)\},\]
where $M$ is the moment generating function of $\log(1 + Z_1)$. Since

$$M(-\alpha - \delta) = E(1 + Z_1)^{-\alpha - \delta} = \eta(-1, -\varepsilon)^{-1} \int_{-1}^{-\varepsilon} (1 + z)^{-\alpha - \delta} \eta(dz) < \infty,$$

the claim, for the case $n = 1$, follows.

For a general $n$ we may, with the similar notation as above, write

$$\mathcal{E}(Z^n)_t = e^{f^n_t + \pi^n_0 r^n_0} \prod_{k=1}^n \mathcal{E}(E^n_{\pi,k}) \mathcal{E}(G^n_{\pi,k}) \mathcal{E}(H^n_{\pi,k}).$$

We know from the proof for the case $n = 1$ that only the factors $\mathcal{E}(H^n_{\pi,k})_t$ may cause problems with existence of moments. Using Hölder’s inequality and following the arguments above we find that

$$E \sup_{t \in [0,1]} \mathcal{E}(H^n_{\pi,k})_t^{-\alpha - \delta} < \infty \quad \text{for each } k,$$

which follows from the assumptions on the Lévy measures $\eta^k$, is sufficient to ensure that $E \sup_{t \in [0,1]} \mathcal{E}(Z^n)_t^{-\alpha - \delta/n} < \infty$. This completes the proof.

**Lemma 3.** Let $M$ be a martingale and set $M^*_t = \sup_{s \in [0,t]} |M_s|$. Then, for $\lambda > 0$, $P(M^*_t \geq x) \leq e^{-\lambda x} Ee^{\lambda |M_t|}$. Moreover, if $Ee^{\lambda M_t} < \infty$ for all $\lambda > 0$, then $Ee^{\lambda M^*_t} < \infty$ for all $\lambda > 0$.

**Proof.** Take $\lambda > 0$. Without loss of generality we may assume that $Ee^{\lambda |M_t|} < \infty$. Note that since $x \mapsto e^{\lambda |x|}$ is convex, $e^{\lambda |M_t|}$ is a submartingale. Let $\tau = \min\{t, \inf\{s > 0 : |M_s| > x\}\}$. Then,

$$Ee^{\lambda |M_t|} \geq Ee^{\lambda |M_t|} = Ee^{\lambda |M_t^*| I_{\{M_t^* \geq x\}}} + Ee^{\lambda |M_t^*| I_{\{M_t^* < x\}}} \geq e^{\lambda x} P(M^*_t \geq x) + Ee^{\lambda |M_t|} I_{\{M_t^* < x\}}.$$

Hence, $P(M^*_t \geq x) \leq e^{-\lambda x} Ee^{\lambda |M_t|} I_{\{M_t^* \geq x\}} \leq e^{-\lambda x} Ee^{\lambda |M_t|}$. For the last statement, take $\xi > \lambda > 0$. Then

$$Ee^{\lambda M^*_t} = \int_0^\infty P(e^{\lambda M^*_t} > x) dx$$

$$= 1 + \lambda \int_0^\infty e^{\lambda x} P(M^*_t > x) dx \leq 1 + \lambda Ee^{\xi |M_t|} \int_0^\infty e^{(\lambda - \xi)x} dx < \infty.$$

**Proof of Proposition**. Set $f_s = r_{s-} + \pi_s(\mu_{s-} - r_{s-})$ and $g_s = \pi_s \sigma_{s-}$ and take $\beta \in (\alpha, \gamma)$. Then

$$\mathcal{E}(Z^n)_t^{-\beta} = \exp\left\{ -\beta \left( \int_0^t f_s ds + \int_0^t g_s dB_s - \frac{1}{2} \int_0^t g_s^2 ds \right) \right\}.$$
Hölder’s inequality gives, with $1/p + 1/q = 1$ and $q$ small so that $q\beta < \gamma$,

$$E \sup_{t \in [0,1]} E(Z_t)^{-\beta} \leq \left( E \sup_{t \in [0,1]} \exp \left\{ -p\beta \int_{0}^{t} f_s ds \right\} \right)^{1/p} \left( E \sup_{t \in [0,1]} \exp \left\{ -q\beta \int_{0}^{t} g_s dB_s + \frac{q\beta}{2} \int_{0}^{t} g^2_s ds \right\} \right)^{1/q}.$$  

Take $r \leq 2\gamma$ and note that $-r \int_{0}^{t} g_s dB_s$ is a continuous local martingale and that

$$E \exp \left\{ \frac{r^2}{2} \int_{0}^{1} g^2_s ds \right\} < \infty.$$  

It follows from Theorem 41 on page 140 in [11] that $M = \{M_t\}$ given by

$$M_t = \exp \left\{ -r \int_{0}^{t} g_s dB_s - \frac{r^2}{2} \int_{0}^{t} g^2_s ds \right\}$$  

is a nonnegative martingale. Hence, $K = \{K_t\}$ is a submartingale so Theorem 20 on page 11 in [11] gives

$$E \sup_{t \in [0,1]} K_t \leq \left( E \sup_{t \in [0,1]} \exp \left\{ -q\beta \int_{0}^{1} g_s dB_s + \frac{q\beta}{2} \int_{0}^{1} g^2_s ds \right\} \right)^{1/q}.$$  

To show that the expectation in (23) is finite we set $r := q\beta < \gamma$ and note that

$$\exp \left\{ -r \int_{0}^{1} g_s dB_s + \frac{r^2}{2} \int_{0}^{1} g^2_s ds \right\} = \left( \exp \left\{ -2r \int_{0}^{1} g_s dB_s - 2r^2 \int_{0}^{1} g^2_s ds \right\} \right)^{1/2} \left( \exp \left\{ (2r^2 + r) \int_{0}^{1} g^2_s ds \right\} \right)^{1/2}. $$

Hence, with $\gamma = q\beta$ for $q$ sufficiently small, the Cauchy-Schwarz inequality yields that the expectation in (23) is finite. \(\square\)

**Proof of Proposition 6.** The process $M$ given by $M_t = \alpha \int_{0}^{t} \pi_s \sigma_s dB_s$ is a continuous local martingale if (17) holds. The Novikov condition (17) and Theorem 41, p. 140, in [11] guarantee that $\mathcal{E}(M)$ given by

$$\mathcal{E}(M)_t = \exp \left\{ \alpha \int_{0+}^{t} \pi_s \sigma_s dB_s - \frac{\alpha^2}{2} \int_{0+}^{t} \pi^2_s \sigma^2_s ds \right\}$$

is a uniformly integrable martingale. Hence, for every $\pi \in \Pi$, the measure $Q_\pi$ given by

$$E \left( \frac{dQ_\pi}{dP} \middle| \mathcal{F}_t \right) = \mathcal{E}(M)_t$$
is a probability measure (equivalent to $P$). Therefore we may write

$$E(E(Z^{\pi})^{-\alpha}) = E_{Q,\pi} \exp \left\{ \alpha \int_0^t \left( -(1 - \pi_s)r_s - \pi_s \mu_s + \frac{1 + \alpha}{2} \pi_s^2 \sigma_s^2 \right) ds \right\}. $$

Hence, minimizing $E(E(Z^{\pi})^{-\alpha})$ with respect to $\pi$ is equivalent to minimizing the integrand on the right-hand side above. Since $\pi \mapsto -(1 - \pi)r - \pi \mu + \frac{1 + \alpha}{2} \pi^2 \sigma^2$ has a unique minimum at $\pi^* = \frac{\mu - r}{(1 + \alpha)\sigma}$ the claim follows. □

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