Estimates of linearization discs in $p$-adic dynamics with application to ergodicity*

Karl-Olof Lindahl
School of Mathematics and Systems Engineering
Växjö University, 351 95, Växjö, Sweden
Karl-Olof.Lindahl@vxu.se

Abstract

We give lower bounds for the size of linearization discs for power series over $\mathbb{C}_p$. For quadratic maps, and certain power series containing a 'sufficiently large' quadratic term, we find the exact linearization disc. For finite extensions of $\mathbb{Q}_p$, we give a sufficient condition on the multiplier under which the corresponding linearization disc is maximal (i.e. its radius coincides with that of the maximal disc in $\mathbb{C}_p$ on which $f$ is one-to-one). In particular, in unramified extensions of $\mathbb{Q}_p$, the linearization disc is maximal if the multiplier map has a maximal cycle on the unit sphere. Estimates of linearization discs in the remaining types of non-Archimedean fields of dimension one were obtained in [44, 46, 47].

Moreover, it is shown that, for any complete non-Archimedean field, transitivity is preserved under analytic conjugation. Using results by Ox-toby [52], we prove that transitivity, and hence minimality, is equivalent the unique ergodicity on compact subsets of a linearization disc. In particular, a power series $f$ over $\mathbb{Q}_p$ is minimal, hence uniquely ergodic, on all spheres inside a linearization disc about a fixed point if and only if the multiplier is maximal. We also note that in finite extensions of $\mathbb{Q}_p$, as well as in any other non-Archimedean field $K$ that is not isomorphic to $\mathbb{Q}_p$ for some prime $p$, a power series cannot be ergodic on an entire sphere, that is contained in a linearization disc, and centered about the corresponding fixed point.

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1 Introduction

In this paper we study iteration of power series \( f \) defined over \( \mathbb{C}_p \), the completion of the algebraic closure of the \( p \)-adic numbers \( \mathbb{Q}_p \). As in complex dynamics (i.e. iteration of complex-valued analytic functions, see e.g. [7, 20, 49]), the main features of the dynamics under \( f \in \mathbb{C}_p[[x]] \) is determined by the character of the periodic points of \( f \), i.e. the modulus of the multiplier at the periodic points. A periodic fixed point \( x_0 \) may be either attracting, indifferent or repelling depending on whether the multiplier \( \lambda = f'(x_0) \) is inside, on or outside the unit sphere. In this paper we consider non-resonant (i.e. \( \lambda \) not a root of unity) indifferent fixed points.

A power series over a complete valued field of the form

\[
f(x) = \lambda(x - x_0) + \text{(higher order terms)}
\]

is said to be linearizable at the fixed point \( x_0 \) if there exists a convergent power series solution \( g \) to the following form of the Schröder functional equation (SFE)

\[
g \circ f(x) = \lambda g(x), \quad \lambda = f'(x_0),
\]

which conjugates \( f \) to its linear part in some neighborhood of \( x_0 \). By the non-Archimedean Siegel theorem of Herman and Yoccoz [29], as in the complex field case [59], the condition

\[
|1 - \lambda^n| \geq Cn^{-\beta} \quad \text{for some real numbers } C, \beta > 0,
\]

on \( \lambda \) is sufficient for convergence also in the non-Archimedean field case. Their theorem applies to the multi-dimensional case. In dimension one, the condition [2] is always satisfied for non-resonant multipliers in fields of characteristic zero, i.e. the \( p \)-adic case studied in this paper, and the equal characteristic case of studied in [47]. This is not always true in fields of prime characteristic as shown in [44, 46].

As shown by Herman and Yoccoz, in the two-dimensional \( p \)-adic case there also exist examples where the Siegel condition is not satisfied and the corresponding conjugacy diverges. The multi-dimensional \( p \)-adic case has been taken further by Viegue in his thesis [63]. In this paper we only consider the one-dimensional non-resonant \( p \)-adic case so the conjugacy always converges.

The conjugacy function \( g \) is unique if we specify the image and derivative at \( x_0 \). It is customary to assume that \( g(x_0) = 0 \) and \( g'(x_0) = 1 \). By the local invertibility theorem, \( g \) has a local inverse \( g^{-1} \) at \( x_0 \). We will refer to the (indifferent) linearization disc of \( f \) about \( x_0 \), denoted by \( \Delta_f(x_0) \), as the largest disc \( U \subset \mathbb{C}_p \), with \( x_0 \in U \), such that [1] holds for all \( x \in U \), and \( g \) converges and is one-to-one on \( U \). The possibly larger disc, on which the the semi-conjugacy [1] holds, will be referred to as the semi-disc.

Note that, by definition, \( f \) must be one-to-one on the linearization disc \( \Delta_f(x_0) \). Moreover, the full conjugacy

\[
g \circ f \circ g^{-1}(x) = \lambda x,
\]

(3)

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is valid for all \( x \in g(\Delta f(x_0)) \). Let \( f^{\circ n} \) denote the \( n \)-fold composition of \( f \) with itself. On \( g(\Delta f(x_0)) \) we have \( g \circ f^{\circ n} \circ g^{-1}(x) = \lambda^n x \). Hence, there is a one-to-one correspondence between orbits under \( f \) and the multiplier map \( T_\lambda : x \mapsto \lambda x \), on \( \Delta f(x_0) \) and \( g(\Delta f(x_0)) \), respectively. In particular, since \( \lambda \) is not a root of unity, \( f \) can have no periodic points on the linearization disc, except the fixed point \( x_0 \). However, the semi-disc may contain other periodic points as well, as manifest in the papers \([4, 53]\). In fact, the semi-disc is contained in the \textit{quasi-periodicity domain} of \( f \), defined as the interior of the set of points on the projective line \( \mathbb{P}(\mathbb{C}_p) = \mathbb{C}_p \cup \{\infty\} \) that are recurrent by \( f \). In the case that \( f \) is a rational function, Rivera-Letelier \([55]\) gave several characterizations of the quasi-periodicity domain of \( f \) and described its local and global dynamics. In particular, he proved that analytic components of the domain of quasi-periodicity, which are \( p \)-adic analogues of Siegel discs and Herman rings in complex dynamics, are open affinoids (that is, they have simple geometry), and contains infinitely many indifferent periodic points.

Our aim in this paper is three-fold. First, we obtain lower (sometimes optimal) bounds for the size of linearization discs for \( f \in \mathbb{C}_p[[x]] \). These estimates extend results on quadratic polynomials over \( \mathbb{Q}_p \) by Ben-Menahem \([8]\), and Thiran, Verstegen, and Weyers \([62]\), and for certain polynomials with maximal multipliers over the \( p \)-adic integers \( \mathbb{Z}_p \) by Pettigrew, Roberts and Vivaldi \([53]\), as well as results on small divisors in \( \mathbb{C}_p \) by Khrennikov \([33]\).

Second, we prove that transitivity (the existence of a dense orbit) is preserved under analytic conjugation into linearization discs over an arbitrary complete non-Archimedean field. Using results by Oxtoby \([52]\), the transitivity of \( f \) on compact subsets of a linearization disc is proven to be equivalent to the ergodicity and unique ergodicity of \( f \).

Third, when the dynamics is defined over the \( p \)-adic numbers \( \mathbb{Q}_p \), we give necessary and sufficient conditions on the multiplier, that \( f \) is transitive, hence uniquely ergodic, on spheres inside the linearization disc. These results generalize results obtained by Bryk and Silva \([17]\), and by Gundlach, Khrennikov, and Lindahl \([28]\), for monomials \( f : x \mapsto cx^n \), and for 1-Lipschitz power series by Anashin \([1]\). On the other hand, we also prove that transitivity is not possible on a whole sphere in any proper extension of \( \mathbb{Q}_p \). Results on the transitivity and ergodic breakdown of the \( p \)-adic multiplier map \( x \mapsto \lambda x \) were obtained by Oselies and Zieschang \([51]\), and by Coelho and Parry \([21]\).

A classification of measure-preserving transformations of compact-open subsets of non-Archimedean local fields were obtained recently by Kingsbery, Levin, Preygel and Silva \([40]\). They show that if a \( C^1 \) transformation \( T \) is measure-preserving when restricted to a compact-open set \( X \) then \( X \) can be written as a disjoint union of invariant compact-open sets such that \( T \) restricted to each such set is either a local isometry or topologically and measurably conjugate to an ergodic Markov transformation. Concerning polynomials, the question of ergodicity is also answered, except in the case where the polynomial is 1-Lipschitz, as in the present paper (the power series \( f \) is certainly 1-Lipschitz on the entire linearization disc). Non-1-Lipschitz functions were also studied in \([2]\).

Let us also mention some related works on measure preserving transforma-
tions on the Berkovich space, which is a much larger space than the $p$-adics. The Berkovich space provides a bridge between non-Archimedean and complex dynamics. The works \[24, 25, 57\], construct a natural invariant measure for a wide class of rational functions, similar to existing constructions in complex dynamics.

Further results on the properties of the dynamics on $p$-adic linearization discs are provided in \[4, 53\]. Estimates for linearization discs in prime characteristic were obtained in \[44, 46\], and for fields of characteristic zero in the equal characteristic case \[47\]. See \[44\], for further comments on the non-Archimedean problem of linearization and its relation to the complex field case.

The construction of conjugacies in $p$-adic dynamics is related to standard and well-established techniques of local arithmetic geometry, see e.g. Lubin \[48\] and the construction of local canonical heights by Call and Silverman \[19\], and Hsia \[30\]. For indifferent, non-resonant, fixed points the conjugacy function is related to the ‘logarithm’ of the theory of one-parameter formal Lie-groups defined over the $p$-adics \[4, 48\]. As in \[48\], the Lie-logarithm is constructed as the limit

$$\lim_{n \to \infty} \frac{f^{op} - id_x}{p^n},$$

and is, up to a constant, the quotient between the conjugacy function $g$ and its derivative $g'$. The Lie-logarithm contains useful information about the dynamics of $f$. In particular, its roots are periodic points of $f$ \[48\]. See Li \[42, 43\] for various results on this matter, including the counting of periodic points of $p$-adic power series. Rivera-Letelier \[55\] proved if $f$ is a rational function, then the Lie-logarithm converges uniformly on the entire domain of quasi-periodicity.

For some additional references on non-Archimedean dynamics and its relationship, similarities, and differences with respect to the Archimedean theory of complex dynamics, see e.g. \[2, 3, 9–14, 22, 30, 31, 33–35, 37–39, 50, 56, 60, 61\]. Applications of $p$-adic numbers have been proposed in coding theory \[18\], round off errors \[16\], random number generation \[65\], and in biochemistry and physics \[5, 6, 32, 36, 54\].

## 2 Summary of results

Our most general result on the size of a linearization disc in $\mathbb{C}_p$ can be stated in the following way (see also Theorem 4.4 and Lemma 3.5).

**Theorem A** (Estimate of linearization discs in $\mathbb{C}_p$). Let $f \in \mathbb{C}_p[[x]]$ have an indifferent fixed point $x_0$, with multiplier $\lambda = f'(x_0)$, not a root of unity. Suppose that $f$ has the following expansion about $x_0$

$$f(x) = x_0 + \lambda (x - x_0) + \sum_{i \geq 2} a_i(x - x_0)^i, \quad \text{with } a = \sup_{i \geq 2} |a_i|^{1/(i-1)}.$$  \(5\)

Then, the linearization disc $\Delta_f(x_0)$, satisfies $D_{\sigma(\lambda, a)}(x_0) \subseteq \Delta_f(x_0) \subseteq D_{1/a}(x_0)$, where $\sigma(\lambda, a)$ is defined by \[44\]. Moreover, if the conjugacy function $g$ converges
on the closed disc $\mathbb{T}_{\sigma(\lambda,a)}(x_0)$, then $\Delta_f(x_0) \supseteq \mathbb{T}_{\sigma(\lambda,a)}(0)$. In particular, $f$ can have no periodic points in the punctured open disc $D_{\sigma(\lambda,a)}(x_0) \setminus \{x_0\}$.

The proof is based on estimates of the coefficients of the conjugacy function $g$. Applying a result of Benedetto [12] (Proposition 5.1 below), on these estimates we find a lower bound for the region of convergence of the inverse $g^{-1}$, and hence of the linearization disc.

Note that the estimate $\sigma = \sigma(\lambda,a)$ depends only on $\lambda$ and the real number $a$. To find the exact size of the linearization disc we do in general need more information about the coefficients of $f$. However, for a large class of quadratic polynomials, and certain power series containing a 'sufficiently large' quadratic term, we prove that

$$\tau = |1 - \lambda|^{-1/p} \sigma(\lambda,a) \quad \text{(6)}$$

is the exact radius of the linearization disc. More precisely, our main result can be stated in the following way (see also Theorem 5.1).

**Theorem B** (Linearization disc for quadratic maps). Let $p$ be an odd prime. Let

$$f(x) = x_0 + \lambda(x - x_0) + a(x - x_0)^2 \in \mathbb{C}_p[x - x_0],$$

with $\lambda$ not a root of unity. Suppose that $p^{-1} < |1 - \lambda| < 1$. Then, the linearization disc $\Delta_f(x_0)$ is equal to the disc $D_{\tau(\lambda,a)}(x_0)$, where the radius $\tau(\lambda,a) = |1 - \lambda|^{-1/p} \sigma(\lambda,a)$.

This result is extended in Theorem 5.2 to power series containing a 'sufficiently large' quadratic term. We also give sufficient conditions, there being a fixed point on the 'boundary' of the linearization disc, i.e. the sphere $S_\tau(x_0)$ about $x_0$ of radius $\tau$.

Note that $\tau(\lambda,a) < 1/a$. Hence, at least in this case, the linearization disc cannot contain the maximal disc $D_{1/a}(x_0)$ on which $f$ is one-to-one.

The relatively complicated expression for $\sigma$ stems from the presence of $p^s$th roots of unity in the punctured disc $D_{1}(1) \setminus \{1\}$, as described in Section 3.3. Some properties of $\sigma$ are discussed in Section 4.5. In particular, we prove the following result.

**Theorem C** (Asymptotic behavior of $\sigma$). Let $|\alpha - \lambda^m|$ be fixed. Then, the estimate $\sigma$ of the radius of the linearization disc goes to $1/a$ as $m$ or $s$ goes to infinity. If $s$ and $m$ are fixed, then $\sigma \to 0$ as $|\alpha - \lambda^m| \to 0$.

We now turn to the special case when the dynamics is restricted to $\mathbb{Q}_p$ and its finite extensions. In $\mathbb{Q}_p$, there are no $p^s$th roots of unity in $D_{1}(1) \setminus \{1\}$, and $\sigma$ takes a simpler form.

**Theorem D** (linearization discs in $\mathbb{Q}_p$ for odd primes). Let $p$ be an odd prime, and let $f \in \mathbb{Q}_p[[x - x_0]]$ be of the form (2). Let $\Delta_f(x_0, \mathbb{Q}_p) = \Delta_f(x_0) \cap \mathbb{Q}_p$ be the corresponding linearization disc in $\mathbb{Q}_p$. Then, $\Delta_f(x_0, \mathbb{Q}_p) \supseteq D_{\sigma_1}(x_0, \mathbb{Q}_p)$, where

$$\sigma_1 = a^{-1/p^s} |1 - \lambda^m|^{1/p^s}.$$
and $m \geq 1$ is the smallest integer such that $|1 - \lambda^m| < 1$. Furthermore, if $|1 - \lambda^m| = p^{-1}$ and $m = p - 1$, then $\Delta_f(x_0, Q_p)$ is either the open or closed disc of radius $1/a$ about $x_0$. In particular, if either $\max_{i \geq 2} |a_i|^{1/(i-1)}$ is attained (as for polynomials) or $f$ diverges on $S_{1/a}(x_0, Q_p)$, then $\Delta_f(x_0, Q_p) = D_{1/a}(x_0, Q_p)$.

Note that the condition $|1 - \lambda^m| = p^{-1}$ and $m = p - 1$, imply that $\lambda$ has a maximal cycle modulo $p^2$ in the sense that it is a generator of the group of units $(\mathbb{Z}/p^2\mathbb{Z})^\times$. In this case $\lambda$ is said to be maximal.

**Theorem E** (linearization discs in $Q_2$). Let $f \in Q_2[[x - x_0]]$ be of the form (5). Then, the following two statements hold:

1. If $|1 - \lambda| < 1/2$, then the linearization disc $\Delta_f(x_0, Q_2)$ contains the open disc of radius $\sigma_1 = |1 - \lambda|/2a$ about $x_0$.

2. If $|1 - \lambda| = 1/2$, then the linearization disc $\Delta_f(x_0, Q_2)$ contains the open disc of radius $\sigma_3 = \sqrt{1 + |1 - \lambda|}/a$ about $x_0$.

**Theorem F** (Maximal linearization discs in extensions of $Q_p$). Let $K$ be a finite extension of $Q_p$ of degree $n$, with ramification index $e$, residue field $k$ of degree $[k : \mathbb{F}_p] = n/e$, and uniformizer $\pi$. Let $f \in K[[x]]$ be a power series of the form (6) and $\alpha$ a root of unity such that there is no closer root of unity to $\lambda^{p^{n/e} - 1}$ than $\alpha$.

Suppose that $\lambda$ has a maximal cycle modulo $\pi^2$ and

$$\log_p e \leq (p^{n/e} - 3)p/(p - 1) - \nu\left(\frac{\alpha - \lambda^{p^{n/e} - 1}}{1 - \lambda^{p^{n/e} - 1}}\right) + \log_p(p - 1),$$

where $\nu$ is the valuation. Then, the linearization disc $\Delta_f(x_0, K) = \Delta_f(x_0) \cap K$ is maximal in the sense that $\Delta_f(x_0, K)$ is either the open or closed disc of radius $1/a$. In particular, if either $\max_{i \geq 2} |a_i|^{1/(i-1)}$ is attained (as for polynomials) or $f$ diverges on $S_{1/a}(x_0)$, then $\Delta_f(x_0, K) = D_{1/a}(x_0, K)$.

Note that if the ramification index $e$ is not divisible by $p - 1$, then $\alpha = 1$ so that the $\nu$-term vanishes in this case. Also note that the linearization disc may be maximal even if $\lambda$ does not have a maximal cycle modulo $\pi^2$, see Theorem 4.3.

In the final section of this paper we note some facts concerning transitivity, minimality and ergodicity on linearization discs. In particular, we show that transitivity is preserved under analytic conjugation into a linearization disc. More precisely.

**Theorem G** (Transitivity and conjugation in non-Archimedean fields). Let $K$ be a complete non-Archimedean field. Suppose that the power series $f(x) = x_0 + \lambda(x - x_0) + O((x - x_0)^2) \in K[[x - x_0]]$ is analytically conjugate to $T_\lambda$, on the linearization disc $\Delta_f(x_0)$ in $K$, via a conjugacy function $g$, with $g(x_0) = 0$ and $|g'(x_0)| = 1$. Suppose also that the subset $X \subseteq \Delta_f(x_0)$ is invariant under $f$. Then, the following statements hold:
1) \( f \) is transitive on \( X \) if and only if \( T_\lambda \) is transitive on \( g(X) \).

2) If \( X \) is compact and \( f \) is transitive on \( X \), then \( f \) is minimal on \( X \). Moreover, \( f(X) = X \) and \( g(X) = T_\lambda(g(X)) \).

Moreover, if \( X \) is compact, the following are equivalent

- \( f \) is uniquely ergodic.
- \( f \) is ergodic for any \( f \)-invariant measure \( \mu \) on the Borel sigma-algebra \( B(X) \) that is positive on non-empty open sets.

In fact, the minimality of \( f \) is equivalent to its unique ergodicity.

**Theorem H** (Unique ergodicity in non-Archimedean fields). Let \( K, f, \Delta_f(x_0), \) and \( g \) be as in Theorem \( \mathbb{G} \). Suppose that the subset \( X \subset \Delta_f(x_0) \) is non-empty, compact and invariant under \( f \). The following statements are equivalent:

1. \( T_\lambda : g(X) \to g(X) \) is minimal.
2. \( f : X \to X \) is minimal.
3. \( f : X \to X \) is uniquely ergodic.
4. \( f \) is ergodic for any \( f \)-invariant measure \( \mu \) on the Borel sigma-algebra \( B(X) \) that is positive on non-empty open sets.

The unique invariant measure \( \mu \) is the normalized Haar measure \( \mu \) for which the measure of a disc is equal to the radius of the disc.

Note that the conjugacy function \( g \) maps spheres in the linearization disc into spheres about the origin. In \( \mathbb{Q}_p \), the multiplier map \( T_\lambda : x \mapsto \lambda x \) is minimal on each sphere \( S \) about the origin if and only if \( \lambda \) is a generator of \( (\mathbb{Z}/p^2\mathbb{Z})^* \). Moreover, if \( \lambda \) is a generator of \( (\mathbb{Z}/p^2\mathbb{Z})^* \), then as a consequence of Theorem \( \mathbb{D} \) the linearization disc \( \Delta_f(x_0, \mathbb{Q}_p) \) includes the the open disc \( D_{1/a}(0) \).

**Theorem I** (Ergodic spheres in \( \mathbb{Q}_p \)). Let \( p \) be an odd prime, and let the series \( f \in \mathbb{Q}_p[[x - x_0]] \) be of the form \( \mathbb{F} \). Let \( S \subset \mathbb{Q}_p \) be a non-empty sphere of radius \( r < 1/a \) about \( x_0 \), i.e. \( r \) is an integer power of \( p \). Then, the following statements are equivalent:

1. \( \lambda \) is a generator of \( (\mathbb{Z}/p^2\mathbb{Z})^* \).
2. \( f : S \to S \) is minimal.
3. \( f : S \to S \) is uniquely ergodic.
4. \( f \) is ergodic for any \( f \)-invariant measure \( \mu \) on the Borel sigma-algebra \( B(S) \) that is positive on non-empty open sets.

By Theorem \( \mathbb{D} \) the estimate of the radius \( 1/a \) is maximal in the sense that there exist examples of such \( f \), which either diverges on the sphere \( S_{1/a}(x_0) \) or satisfy \( f(x) = x_0 \) for at least one \( x \in S_{1/a}(x_0) \). We have, however, not been able to rule out the possibility that in some cases we may allow \( r = 1/a \), see Lemma \( \mathbb{B} \).

Also note that if \( \lambda \in S_1(0) \) is not a generator of \( (\mathbb{Z}/p^2\mathbb{Z})^* \), then \( T_\lambda \) and hence \( f(x) = \lambda x + O(x^2) \) may still be minimal on some subset of a sphere. A complete classification of the ergodic breakdown of \( \mathbb{Q}_p \) with respect to \( T_\lambda \) is given in [51].
We also note (lemma 3.9) that in a finite proper extension of \( \mathbb{Q}_p \), a power series cannot be ergodic on an entire sphere, that is contained in a linearization disc, and centered about the corresponding fixed point. In fact, if \( K \) is a non-Archimedean field, then ergodicity on a linearization sphere is only possible if \( K \) is isomorphic to a field of \( p \)-adic numbers. For transitivity to occur, \( K \) must be locally compact. Therefore, \( K \) is either a \( p \)-adic field or a field of prime characteristics. Let \( K \) be a locally compact field of prime characteristic, with uniformizer \( \pi \). If \( x \in K \) and \( x \equiv 1 \mod \pi \), then \( x^{p^n} \equiv 1 \mod \pi^{p^n} \). As a consequence, \( T_\lambda \) cannot be transitive on a sphere in \( K \), see Lemma 3.10.

**Theorem J** (Ergodic non-Archimedean linearization spheres). Let \( K \) be a complete non-Archimedean field and let \( f \) be holomorphic on a disc \( U \) in \( K \). Suppose that \( f \) has a linearization disc \( \Delta \subset U \) and \( S \subset \Delta \) is a sphere about the corresponding fixed point \( x_0 \in K \). Then \( f : S \to S \) is ergodic if and only if \( K \) is isomorphic to \( \mathbb{Q}_p \) and the multiplier is a generator of the group of units \((\mathbb{Z}/p^2\mathbb{Z})^*\). Furthermore, if \( K = \mathbb{Q}_p \) and \( \lambda \) is a generator of the group of units \((\mathbb{Z}/p^2\mathbb{Z})^*\), then the radius of \( \Delta \) is \( 1/a \) (considered as a disc in \( \mathbb{Q}_p \)).

### 3 Preliminaries

Throughout this paper \( K \) is a non-Archimedean field, complete with respect to a nontrivial absolute value \( | \cdot | \). That is, \( | \cdot | \) is a multiplicative function from \( K \) to the nonnegative real numbers with \( |x| = 0 \) precisely when \( x = 0 \), satisfying the following strong or ultrametric triangle inequality:

\[
|x + y| \leq \max(|x|, |y|), \quad \text{for all } x, y \in K,
\]

and nontrivial in the sense that it is not identically 1 on \( K^* \), the set of all nonzero elements in \( K \). One useful consequence of ultrametricity is that for any \( x, y \in K \) with \( |x| \neq |y| \), the inequality (7) becomes an equality. In other words, if \( x, y \in K \) with \( |x| < |y| \), then \( |x + y| = |y| \).

In this context it is standard to denote by \( \mathcal{O} \), the ring of integers of \( K \), given by \( \mathcal{O} = \{x \in K : |x| \leq 1 \} \), by \( \mathcal{M} \) the unique maximal ideal of \( \mathcal{O} \), given by \( \mathcal{M} = \{x \in K : |x| < 1 \} \), and by \( k \) the corresponding residue field

\[
k = \mathcal{O}/\mathcal{M}.
\]

Note that if \( K \) has positive characteristic \( p \), then also \( \text{char} k = p \); but if char \( K = 0 \), then \( k \) could have characteristic 0 or \( p \). Note also that if \( x, y \in \mathcal{O} \) reduce to residue classes \( \overline{x}, \overline{y} \in k \), then \( |x - y| = 1 \) if \( \overline{x} \neq \overline{y} \), and it is strictly less than 1 otherwise.

In this paper we mainly consider the case when \( K \) is either a \( p \)-adic field, i.e. a finite extension of a field of \( p \)-adic numbers \( \mathbb{Q}_p \), or a field of complex \( p \)-adic numbers \( \mathbb{C}_p \). Recall that the \( p \)-adic numbers are constructed in the following way. For any prime \( p \), there is a unique absolute value on \( \mathbb{Q} \) such that \( |p| = 1/p \). The field \( \mathbb{Q}_p \) of \( p \)-adic rationals is defined to be the corresponding completion of \( \mathbb{Q} \); \( \mathbb{C}_p \) is then the completion of an algebraic closure of \( \mathbb{Q}_p \). Let us also remark
that the residue field of $\mathbb{Q}_p$ is the field $\mathbb{F}_p$, of $p$ elements, whereas the the residue field of $\mathbb{C}_p$ is the algebraic closure of $\mathbb{F}_p$.

Given $K$ with absolute value $|\cdot|$ we define the value group as the image

$$|K^*| = \{|x| : x \in K^*\}. \quad (8)$$

Note that, since $|\cdot|$ is multiplicative, $|K^*|$ is a multiplicative subgroup of the positive real numbers. We will also consider the full image $|K| = |K^*| \cup \{0\}$.

The absolute value $|\cdot|$ is said to be discrete if the value group is cyclic, that is if there is a uniformizer $\pi \in K$ such that $|K^*| = \{|\pi|^n : n \in \mathbb{Z}\}$. Note that if $K = \mathbb{Q}_p$, then $p$ is a uniformizer of $K$, and the value group consists of all integer powers of $p$. If $K = \mathbb{C}_p$, then $|K^*|$ consists of all rational powers of $p$.

In particular, the absolute value on $\mathbb{C}_p$ is not discrete.

Recall that $K$ is locally compact (w.r.t. $|\cdot|$) if and only if (i) $|\cdot|$ is discrete, and (ii) the residue field $k$ is finite. If $K$ is a $p$-adic field, then $K$ is locally compact and each integer $x \in \mathcal{O}$ has a unique representation as a Taylor series in $\pi$ of the form

$$x = \sum_{i=0}^{\infty} x_i \pi^i, \quad x_i \in \mathcal{R}, \quad (9)$$

where $\mathcal{R}$ is a complete system of representatives of the residue field $k$.

Given a prime $p$, a $p$-adic number $x$ can be expressed in base $p$ as

$$x = \sum_{k=0}^{\infty} x_k p^k, \quad x_k \in \{0, ..., p-1\},$$

for some integer $\nu$ such that $x_{\nu} \neq 0$ and $x_k = 0$ for all $k < \nu$. The absolute value of $x$ is given by $|x| = p^{-\nu}$. If $x$ is an integer, its $p$-adic expansion contains no negative powers of $p$ and hence $|x| \leq 1$.

For future reference, let us note the following lemma.

**Lemma 3.1.** Given a rational number $x$, denote by $\lfloor x \rfloor$ the integer part of $x$. Let $n \geq 1$ be an integer and let $S_n$ be the sum of the coefficients in the $p$-adic expansion of $n$. Then,

$$\nu(n!) = \frac{n - S_n}{p - 1} \leq \frac{n - 1}{p - 1}, \quad (10)$$

with equality if $n$ is a power of $p$. Consequently, for all integers $a \geq 1$,

$$\frac{\nu([\frac{n}{a}]!)}{n} \to \frac{1}{a(p - 1)}, \quad (11)$$

as $n$ goes to infinity.

For a proof of (10), the reader can consult [58, Lemma 25.5].

Furthermore, to each finite extension $K$ of $\mathbb{Q}_p$ of degree $n$, there is an associated residue class degree $f = [k : \mathbb{F}_p]$, and a ramification index $e$ such that

$$|K^*| = \{p^{l/e} : l \in \mathbb{Z}\}. \quad (12)$$
For example, by adjoining $\sqrt{p}$ to $\mathbb{Q}_p$ we get a ramified extension with ramification index $e = 2$. The degree of the extension $n = [K : \mathbb{Q}_p]$, the residue class degree $f$, and the ramification index $e$ satisfy the relation

$$n = e \cdot f.$$ 

A finite extension of degree $n$ is called unramified, if $e = 1$ (or equivalently, $f = n$), and ramified, if $e > 1$ (or equivalently, $f < n$).

For more information on $p$-adic numbers and their field extensions the reader can consult [27].

### 3.1 Non-Archimedean discs

Let $K$ be a complete non-Archimedean field. Given an element $x \in K$ and real number $r > 0$ we denote by $D_r(x)$ the open disc of radius $r$ about $x$, by $\overline{D}_r(x)$ the closed disc, and by $S_r(x)$ the sphere of radius $r$ about $x$. To omit confusion, we sometimes write $D_r(x, K)$ rather than $D_r(x)$ to emphasize that the disc is considered as a disc in $K$.

If $r \in |K^*|$ (that is if $r$ is actually the absolute value of some nonzero element of $K$), we say that $D_r(x)$ and $\overline{D}_r(x)$ are rational. Note that $S_r(x)$ is non-empty if and only if $\overline{D}_r(x)$ is rational. If $r \notin |K^*|$, then we will call $D_r(x)$ an irrational disc. In particular, if $a \in K \subset \mathbb{C}_p$ and $r = |a|^s$ for some rational number $s \in \mathbb{Q}$, then $D_r(x)$ and $\overline{D}_r(x)$ are rational considered as discs in the algebraic closure $\mathbb{C}_p$. However, they may be irrational considered as discs in $K$. Note that all discs are both open and closed as topological sets, because of ultrametricity. However, as we will see in Section 3.2 below, power series distinguish between rational open, rational closed, and irrational discs.

### 3.2 Non-Archimedean power series

Let $K$ be a complete non-Archimedean field with absolute value $| \cdot |$. Let $f$ be a power series over $K$ of the form

$$f(x) = \sum_{i=0}^{\infty} a_i (x - \alpha)^i, \quad a_i \in K.$$ 

Then, $f$ converges on the open disc $D_{R_f}(\alpha)$ of radius

$$R_f = \frac{1}{\limsup |a_i|^{1/i}},$$

and diverges outside the closed disc $\overline{D}_{R_f}(\alpha)$ in $K$. The power series $f$ converges on the sphere $S_{R_f}(\alpha)$ if and only if

$$\lim_{i \to \infty} |a_i| R_f^i = 0.$$
Definition 3.1. Let $U \subset K$ be a disc, let $\alpha \in U$ and let $f : U \rightarrow K$. We say that $f$ is holomorphic on $U$ if we can write $f$ as a power series

$$f(x) = \sum_{i=0}^{\infty} a_i (x - \alpha)^i \in K[[x - \alpha]]$$

which converges for all $x \in U$.

Holomorphicity is well-defined since, contrary to the complex field case, it does not matter which $\alpha \in U$ we choose in the definition of holomorphicity, see e.g. [58].

The basic mapping properties of non-Archimedean power series on discs are given by the following generalization by Benedetto [12], of the Weierstrass Preparation Theorem [15, 26, 41].

Proposition 3.1 (Lemma 2.2 [12]). Let $K$ be algebraically closed. Let $f(x) = \sum_{i=0}^{\infty} a_i (x - \alpha)^i$ be a nonzero power series over $K$ which converges on a rational closed disc $U = \overline{D}_R(\alpha)$, and let $0 < r \leq R$. Let $V = \overline{D}_r(\alpha)$ and $V' = D_r(\alpha)$. Then

$$s = \max\{|a_i|r^i : i \geq 0\},$$

$$d = \max\{i \geq 0 : |a_i|r^i = s\},$$

$$d' = \min\{i \geq 0 : |a_i|r^i = s\}$$

are all attained and finite. Furthermore,

a. $s \geq |f'(x_0)| \cdot r$.

b. if $0 \in f(V)$, then $f$ maps $V$ onto $\overline{D}_s(0)$ exactly $d$-to-1 (counting multiplicity).

c. if $0 \in f(V')$, then $f$ maps $V'$ onto $D_s(0)$ exactly $d'$-to-1 (counting multiplicity).

We will consider the case $a_0 = 0$ in more detail. For our purpose, it is then often more convenient to state Proposition 3.1 in the following way.

Proposition 3.2. Let $K$ be algebraically closed and let $h(x) = \sum_{i=1}^{\infty} c_i (x - \alpha)^i$ be a power series over $K$.

1. Suppose that $h$ converges on the rational closed disc $\overline{D}_R(\alpha)$. Let $0 < r \leq R$ and suppose that

$$|c_i|r^i \leq |c_1|r \quad \text{for all } i \geq 2. \quad \text{(14)}$$

Then, $h$ maps the open disc $D_r(\alpha)$ one-to-one onto $D_{|c_1|r}(0)$. Furthermore, if

$$d = \max\{i \geq 1 : |c_i|r^i = |c_1|r\},$$

then $h$ maps the closed disc $\overline{D}_r(\alpha)$ onto $\overline{D}_{|c_1|r}(0)$ exactly $d$-to-1 (counting multiplicity).
2. Suppose that \( h \) converges on the rational open disc \( D_R(\alpha) \) (but not necessarily on the sphere \( S_R(0) \)). Let \( 0 < r \leq R \) and suppose that

\[
|c_i|r^i \leq |c_1|r \quad \text{for all } i \geq 2.
\]

Then, \( h \) maps \( D_r(\alpha) \) one-to-one onto \( D_{|c_1|r}(0) \).

As a consequence of Proposition 3.1, \( f \) satisfies the following Lipschitz condition.

**Proposition 3.3** (Lemma 2.7 [12]). Let \( f \) be a non-constant power series defined on a disc \( U \subset K \) of radius \( r > 0 \), and suppose that \( f(U) \) is a disc of radius \( s > 0 \). Then for any \( x, y \in U \),

\[
|f(x) - f(y)| \leq \frac{s}{r}|x - y|.
\]

Also note the following non-Archimedean analogue of the Complex Koebe 1/4-Theorem.

**Proposition 3.4** (Non-Archimedean Koebe 1-Theorem, Lemma 3.5 [12]). Let \( K \) be algebraically closed. Let \( f \) be a power series over \( K \) which is convergent and one-to-one on a disc \( U \subset K \), with \( 0 \in U \). Suppose that \( f(0) = 0 \) and \( f'(0) = 1 \). Then \( f(U) = U \).

If \( f \) and \( U \) satisfy the condition of the Koebe theorem, then by the Lipschitz condition in Proposition 3.3, \( f : U \rightarrow U \) is not only bijective but also isometric. We have the following lemma.

**Lemma 3.2.** Let \( K \) be algebraically closed. Let \( f \) be a power series over \( K \), which converges and is one-to-one on a disc \( U \subset K \). Suppose that there is an element \( x_0 \in U \) such that \( f(x_0) = x_0 \) and \( |f'(x_0)| = 1 \). Then \( f : U \rightarrow U \) is bijective and isometric.

**Proof.** First, assume that \( U \) is rational closed. Consider the function \( h(x) = f(x) - x_0 \). By definition, \( h \) is also one-to-one on \( U \). Moreover, \( h(x_0) = 0 \) and \( |h'(x_0)| = 1 \). Thus, in view of Proposition 3.1, \( h(U) \) is a rational closed disc and the radius of \( h(U) \) is the same as that of \( U \). It follows that \( f(U) \) is rational closed and that the radius of \( f(U) \) is the same as that of \( U \). Because both \( U \) and \( f(U) \) contain \( x_0 \), we have \( f(U) = U \). The remaining case is when \( U \) is open. Write \( U \) as the union \( \cup U_i \) of rational closed discs containing \( x_0 \). Then \( f(U) = \cup f(U_i) = \cup U_i = U \).

Next, we show that \( f : U \rightarrow U \) is isometric. As the radius of \( h(U) \) is the same as that of \( U \), we have by Proposition 3.3 that \( |h(x) - h(y)| \leq |x - y| \). On the other hand, since \( h : U \rightarrow h(U) \) is bijective, we have

\[
|x - y| = |h^{-1} \circ h(x) - h^{-1} \circ h(y)| \leq |h(x) - h(y)|.
\]

Consequently, \( |h(x) - h(y)| = |x - y| \) so that \( h : U \rightarrow h(U) \), and hence \( f : U \rightarrow U \), is isometric. \( \Box \)
In fact, a power series \( f \) over a complete non-Archimedean field \( K \), is always one-to-one (and hence isometric) on some non-empty disc about an indifferent fixed point \( x_0 \in K \). This is a consequence of the local invertibility theorem \([58]\). The maximal such disc is given by the following lemma.

**Lemma 3.3.** Let \( K \) be algebraically closed. Let \( f \in K[[x]] \) be convergent on some non-empty disc about \( x_0 \in K \). Suppose that \( f(x_0) = x_0 \) and \( |f'(x_0)| = 1 \), and write

\[
 f(x) = x_0 + \lambda(x - x_0) + \sum_{i \geq 2} a_i(x - x_0)^i, \quad a = \sup_{i \geq 2} |a_i|^{1/(i-1)}.
\]

Let \( M \) be the largest disc, with \( x_0 \in M \), such that \( f : M \to M \) is bijective (and hence isometric). Then \( M = D_{1/a}(x_0) \) if either \( \max_{i \geq 2} |a_i|^{1/(i-1)} \) is attained (as for polynomials) or \( f \) diverges on \( S_{1/a}(x_0) \). Otherwise, \( M = \overline{D}_{1/a}(x_0) \).

**Proof.** Because \( f \) is convergent, we must have

\[
 a = \sup_{i \geq 2} |a_i|^{1/(i-1)} < \infty.
\]

Moreover, \( f \) is certainly convergent on the open disc \( D_{1/a}(x_0) \). As in the proof of Lemma 3.2 it is sufficient to consider the mapping properties of the map \( h(x) = f(x) - x_0 \).

First, in view of Proposition 3.2 \( h : D_{1/a}(x_0) \to D_{1/a}(0) \) is one-to-one, since by definition

\[
 |a_i|(1/a)^i \leq 1/a = |a_1|(1/a).
\]

Second, if \( h \) converges on the closed disc \( \overline{D}_{1/a}(x_0) \) and \( \max_{i \geq 2} |a_i|^{1/(i-1)} \) is attained for some \( i \geq 2 \), then

\[
 d = \max \{ i \geq 1 : |a_i|(1/a)^i = |a_1|(1/a) \} \geq 2.
\]

By Proposition 3.2 \( h \) is not one-to-one on \( \overline{D}_{1/a}(x_0) \).

Third, if \( h \) converges on the closed disc \( \overline{D}_{1/a}(x_0) \) and \( \max_{i \geq 2} |a_i|^{1/(i-1)} \) is never attained. Then,

\[
 |a_i|(1/a)^i < 1/a = |a_1|(1/a),
\]

for all \( i \geq 2 \), so that \( d = 1 \). In other words, \( h : \overline{D}_{1/a}(x_0) \to \overline{D}_{1/a}(0) \) is one-to-one in this case. However, \( h \) cannot be one-to-one on any (rational) disc strictly containing \( \overline{D}_{1/a}(x_0) \); if \( r < a = \sup_{i \geq 2} |a_i|^{1/(i-1)} \), then \( |a_N|^{1/(N-1)} \geq r \) and hence

\[
 |a_N|(1/r)^N \geq 1/r = |a_1|(1/r),
\]

for some \( N \geq 2 \). This completes the proof. \( \square \)

It follows from the proof above that if \( f \) converges on the sphere \( S_{1/a}(x_0) \) but fails to be one-to-one there, then there is a point \( x \in S_{1/a}(x_0) \) such that \( f(x) = x_0 = f(x_0) \). This is always the case when \( f \) is a polynomial.
That $f$ may diverge on $S_{1/a}(x_0)$ follows since, for example, the power series $f(x) = \lambda x + \sum_{i=2}^{\infty} (a_2 i^{i-1} x^i)$ converges if and only if $|x| < 1/|a_2| = 1/a$

Furthermore, for every $x \in M$, $|f(x) - x_0| = |x - x_0|$ and hence all spheres in $M$ are invariant under $f$.

**Remark 3.1.** Recall that the discs $D_{1/a}(0)$ and $\mathcal{D}_{1/a}(0)$ are rational if and only if $a = \sup_{i \geq 2} |a_i|^{1/(i-1)} \in |K|$. If the maximum $a = \max_{i \geq 2} |a_i|^{1/(i-1)}$ exists, and $K$ is algebraically closed, then $a \in |K|$. This is always the case if $f$ is a polynomial. If $f$ is not a polynomial and the maximum fails to exist we may have $\sup_{i \geq 2} |a_i|^{1/(i-1)} \notin |K|$. Let $K = \mathbb{C}_p$. Let $\beta$ be an irrational number and let $p_n/q_n$ be the $n$-th convergent of the continued fraction expansion of $\beta$. Let the sequence $\{a_i \in \mathbb{Q}_p\}_{i \geq 2}$ satisfy

$$|a_i| = \begin{cases} p^n, & \text{if } i - 1 = q_n \text{ and } p_n/q_n < \beta, \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$\sup_{i \geq 2} |a_i|^{1/(i-1)} = p^\beta \notin |K| = \{p^r : r \in \mathbb{Q}\} \cup \{0\}.$$ 

For more information on non-Archimedean power series the reader can consult [58]. From a dynamical point of view, the paper [12] contains many useful results on non-Archimedean analogues of complex analytic mapping theorems relevant for dynamics.

### 3.3 Linearization discs

The results above have some important implications for linearization discs. We use the following definition of a linearization disc. Let $K$ be a complete non-Archimedean field. Suppose that $f \in K[[x]]$ has an indifferent fixed point $x_0 \in K$, with multiplier $\lambda = f'(x_0)$, not a root of unity. By [29], there is a unique formal power series solution $g$, with $g(x_0) = 0$ and $g'(x_0) = 1$, to the following form of the Schröder functional equation

$$g \circ f(x) = \lambda g(x).$$

If the formal solution $g$ converges on some non-empty disc about $x_0$, then the corresponding linearization disc of $f$ about $x_0$, denoted by $\Delta_f(x_0)$, is defined as the largest disc $U \subset K$, with $x_0 \in U$, such that the Schröder functional equation holds for all $x \in U$, and $g$ converges and is one-to-one on $U$. We will often refer to $g$ as the conjugacy function.

This notion of a linearization disc is well-defined since by the proof of Lemma 3.3 there always exist a largest disc on which $g$ is one-to-one (provided that $g$ is convergent). Also note that by the non-Archimedean Siegel theorem [29] and the fact that $\mathbb{C}_p$ is of characteristic zero, the formal solution $g$ always converges if the state space $K = \mathbb{C}_p$.

As one might expect from previous results, both $f$ and the conjugacy $g$ turn out to be one-to-one and isometric on a non-Archimedean linearization disc.
Lemma 3.4. Let $K$ be algebraically closed. Suppose that $f \in K[[x]]$ has a linearization disc $\Delta_f(x_0)$ about $x_0 \in K$. Let $g$, with $g(x_0) = 0$ and $g'(x_0) = 1$, be the corresponding conjugacy function. Then, both $g : \Delta_f(x_0) \to g(\Delta_f(x_0))$ and $f : \Delta_f(x_0) \to \Delta_f(x_0)$ are bijective and isometric. In particular, if $x_0 = 0$, then $g(\Delta_f(x_0)) = \Delta_f(x_0)$. Furthermore, $\Delta_f(x_0) \subseteq M \subseteq D_{1/a}(x_0)$, where $M$ and $a$ are defined as in Lemma 3.3.

Proof. By the conjugacy relation $g \circ f(x) = \lambda g(x)$ and the fact that the map $g : \Delta_f(x_0) \to g(\Delta_f(x_0))$ is one-to-one, $f : \Delta_f(x_0) \to \Delta_f(x_0)$ is also one-to-one and hence bijective and isometric by Lemma 3.2.

Recall that $g(x_0) = 0$ and $g'(x_0) = 1$. That $g : \Delta_f(x_0) \to g(\Delta_f(x_0))$ is bijective and isometric then follows by same arguments as those applied to $h$, in the proof of Lemma 3.2.

As a consequence, the radius of a linearization disc $\Delta_f(x_0)$ is equal to that of $g(\Delta_f(x_0))$. In particular, the radius of a linearization disc is independent of the location of the fixed point $x_0$. Therefore, we shall, without loss of generality, henceforth assume that $x_0 = 0$.

The forthcoming sections are very much devoted to estimates of the maximal disc on which $g$ is one-to-one. Before dealing with this more delicate problem, note the following remark.

Remark 3.2. All the results in this and the previous section, except for Proposition 3.1, hold also in the case that $K$ is not algebraically closed, with the modification that the mappings are are one-to-one but not necessarily surjective. However, with certain restrictions on the multiplier $\lambda$, e.g. $f : \Delta_f(x_0) \cap \mathbb{Q}_p \to \Delta_f(x_0) \cap \mathbb{Q}_p$ may also be surjective, see Corollary 6.1. In fact, as stated in Theorem 6.1 if $f$ is transitive on a compact subset $X$ of a linearization disc, then $f(X) = X$.

3.4 The formal solution

As noted in the previous section, we may, without loss of generality, assume that $f$ has its fixed point at the origin, and that $f \in F_{\lambda,a}$, as defined below. Let $\lambda \in \mathbb{C}_p$ be such that

$$|\lambda| = 1, \quad \text{but } \lambda^n \neq 1, \quad \forall n \geq 1,$$

and let $a$ be a real number. We shall associate with the pair $(\lambda, a)$ a family $\mathcal{F}_{\lambda,a}$ of power series defined by

$$\mathcal{F}_{\lambda,a} := \left\{ \lambda x + \sum_{i \geq 2} a_i x^i \in \mathbb{C}_p[[x]] : a = \sup_{i \geq 2} |a_i|^{1/(i-1)} \right\}.$$  

It follows that each $f \in \mathcal{F}_{\lambda,a}$ is convergent on $D_{1/a}(0)$, and by Lemma 3.3 $f : D_{1/a}(0) \to D_{1/a}(0)$ is bijective and isometric.
As \( \mathbb{C}_p \) is of characteristic zero, we may, by the non-Archimedean Siegel theorem [29], associate with \( f \) a unique convergent power series solution \( g \) to the Scröder functional equation, of the form

\[
g(x) = x + \sum_{k \geq 2} b_k x^k,
\]

and a corresponding linearization disc about the origin

\[
\Delta_f := \Delta_f(0).
\]

Recall that by Lemma 3.4, since \( x_0 = 0 \), the linearization disc \( \Delta_f \) is the largest disc \( U \subset \mathbb{C}_p \) about the origin such that the full conjugacy \( g \circ f \circ g^{-1}(x) = \lambda x \) holds for all \( x \in U \).
Given \( f \in \mathcal{F}_{\lambda,a} \), Lemma 3.4 yields the following concerning \( \Delta_f \).

**Lemma 3.5.** Let \( f \in \mathcal{F}_{\lambda,a} \). Then \( f \) has a linearization disc \( \Delta_f \) about the origin in \( \mathbb{C}_p \). Let \( g \), with \( g(0) = 0 \) and \( g'(0) = 1 \), be the corresponding conjugacy function. Then, the following two statements hold:

1) Both \( g : \Delta_f \to \Delta_f \) and \( f : \Delta_f \to \Delta_f \) are bijective and isometric.

2) \( \Delta_f \subseteq \overline{D}_{1/a}(0) \). If \( a = \max_{i \geq 2} |a_i|^{1/(i-1)} \) or \( f \) diverges on the sphere \( S_{1/a}(0) \), then \( \Delta_f \subseteq D_{1/a}(0) \).

Our results on lower bounds for linearization discs are based on the following lemma.

**Lemma 3.6.** Let \( f \in \mathcal{F}_{\lambda,a} \). Then, the coefficients of the conjugacy function \( g \) satisfy

\[
|b_k| \leq \left( \prod_{n=1}^{k-1} |1 - \lambda^n| \right)^{-1} a^{k-1},
\]

for all \( k \geq 2 \).

**Proof.** The coefficients of the conjugacy \( g \) must satisfy the recurrence relation

\[
b_k = \frac{1}{\lambda(1-\lambda^{k-1})} \sum_{l=1}^{k-1} b_l \left( \sum_{\alpha_1! \cdots \alpha_k!} \frac{l!}{\alpha_1! \cdots \alpha_k!} a_1^{\alpha_1} \cdots a_k^{\alpha_k} \right)
\]

where \( \alpha_1, \alpha_2, \ldots, \alpha_k \) are nonnegative integer solutions of

\[
\begin{cases}
\alpha_1 + \ldots + \alpha_k = l, \\
\alpha_1 + 2\alpha_2 + \ldots + k\alpha_k = k,
\end{cases}
\]

\[
1 \leq l \leq k - 1.
\]

Note that the factorial factors \( l! / \alpha_1! \cdots \alpha_k! \) are always integers and thus of modulus less than or equal to 1. Also recall that \( |a_i| \leq a^{i-1} \). It follows that

\[
|b_k| \leq \left( \prod_{n=1}^{k-1} |1 - \lambda^n| \right)^{-1} a^\alpha,
\]

for some integer \( \alpha \). In view of equation (19) we have

\[
\sum_{i=2}^{k} (i-1)\alpha_i = k - l.
\]

Consequently, since \( |a_i| \leq a^{i-1} \), we obtain

\[
\prod_{i=2}^{k} |a_i|^{\alpha_i} \leq \prod_{i=2}^{k} a^{(i-1)\alpha_i} = a^{k-l}.
\]
Now we use induction over $k$. By definition $b_1 = 1$ and, according to the recursion formula (18), $|b_2| \leq |1 - \lambda|^{-1}|a_2| \leq |1 - \lambda|^{-1}|a|$. Suppose that

$$|b_l| \leq \left( \prod_{n=1}^{l-1} |1 - \lambda^n| \right)^{-1} a^{l-1}$$

for all $l < k$. Then

$$|b_k| \leq \left( \prod_{n=1}^{k-1} |1 - \lambda^n| \right)^{-1} a^{k-1} \max \left\{ \prod_{i=2}^{k} |a_i|^{\alpha_i} \right\},$$

and the lemma follows by the estimate (20).

In the following sections we show how to calculate the distance $|1 - \lambda^n|$ for an arbitrary integer $n \geq 1$. Applying Proposition 3.2 to the estimate in the above lemma we can then estimate the disc on which the conjugacy function $g$ is one-to-one.

3.5 Geometry of the unit sphere and the roots of unity

Let $\Gamma$ be the group of all roots of unity in $\mathbb{C}_p$. It has the important subgroup $\Gamma_u$ ($u$ for unramified), given by

$$\Gamma_u = \{ \xi \in \mathbb{C}_p : \xi^m = 1 \text{ for some } m \text{ not divisible by } p \}. \quad (21)$$

**Proposition 3.5** ($\Gamma_u$ and the unit sphere). The unit sphere $S_1(0)$ in $\mathbb{C}_p$ decomposes into the disjoint union

$$S_1(0) = \cup_{\xi \in \Gamma_u} D_1(\xi).$$

In particular, $\Gamma_u \cap D_1(1) = \{1\}$ and consequently $|1 - \xi| = 1$ for all $\xi \neq 1$. To each $\lambda \in S_1(0)$ there is a unique $\xi \in \Gamma_u$ and $h \in D_1(1)$ such that $\lambda = \xi h$.

**Proof.** See [58, p. 103].

Let us note that $\Gamma_u$ is isomorphic to the multiplicative subgroup $\overline{\mathbb{F}_p} \setminus \{0\}$ of the residue field in $\mathbb{C}_p$.

Another important subgroup of $\Gamma$ is $\Gamma_r$ ($r$ for ramified), given by

$$\Gamma_r = \{ \zeta \in \mathbb{C}_p : \zeta^p = 1 \text{ for some integer } s \geq 0 \}. \quad (22)$$

By elementary group theory $\Gamma_u \cap \Gamma_r = \{1\}$ and $\Gamma = \Gamma_u \times \Gamma_r$, see e.g. the paper [58, p. 103]). Most importantly, the $p^s$th roots of unity in $\Gamma_r$ are located on spheres about the point $x = 1$ of radius $R(t)$, where

$$R(t) := \begin{cases} 0, & \text{if } t = 0, \\ p^{-\frac{t}{p^s-1}} - 1, & \text{if } t \geq 1. \end{cases} \quad (23)$$

This fact is fundamental for our estimates of linearization discs.
Proposition 3.6 (The geometry of $\Gamma_r$). If $\zeta \in \Gamma_r$ is a primitive $p^s$th root of unity for some $s \geq 0$, then

$$|1 - \zeta| = R(s).$$

Moreover, if $\zeta_1, \zeta_2 \in S_{R(s)}(1)$ and $\zeta_1 \neq \zeta_2$, then

$$|\zeta_1 - \zeta_2| = R(s).$$

Furthermore, for each $s \geq 1$, there are $p^s - p^{s-1}$ different roots of unity on the sphere $S_{R(s)}(1)$.

Proof. See for example [23].

As a consequence we have the following lemma.

Lemma 3.7. Let $\lambda \in D_1(1)$ be not a root of unity. Then, there exist $\alpha \in \Gamma_r$ such that $|\alpha - \lambda| \leq |\gamma - \lambda|$, for all $\gamma \in \Gamma$. Furthermore, if $|1 - \lambda| \neq R(s)$ for every $s \geq 0$. Then, $\alpha = 1$ is the only root of unity with this property.

3.6 Transitivity of the multiplier map

In this section, $K$ is a finite extension of $\mathbb{Q}_p$ of degree $[K : \mathbb{Q}_p] = e \cdot f$, with ramification index $e$, residue field degree $f = [k : \mathbb{F}_p]$, and uniformizer $\pi$.

Let $\lambda \in \mathbb{C}_p$ be an element on the unit sphere $S_1(0)$. We are concerned with calculating the distance

$$|1 - \lambda^n|$$

for each integer $n \geq 1$. In view of Proposition 3.5 there is an integer $m$, not divisible by $p$, such that $\lambda = \xi h$ for some $m$th root of unity $\xi \in \Gamma_u$ and $h \in D_1(1)$.

In other words, the following integer exists

$$m = m(\lambda) := \min\{n \in \mathbb{Z} : n \geq 1, |1 - \lambda^n| < 1\}.$$ (24)

Also note that, since the residue field is of characteristic $p > 0$, $m$ is not divisible by $p$. In fact, if $\lambda$ belongs to a finite algebraic extension $K$ of $\mathbb{Q}_p$, then

$$1 \leq m \leq p^f - 1,$$

where $p^f$ is the number of elements in the residue field $k$ of $K$. In particular, if $\lambda \in \mathbb{Q}_p$ we have $1 \leq m \leq p - 1$.

Lemma 3.8. Let $K$ be a finite extension of $\mathbb{Q}_p$, with uniformizer $\pi$ and ramification index $e$. Let $x \in K$, and $\delta = \min\{1 + e, p\}$. If $x \equiv 1 \mod \pi$, then we have $x^p \equiv 1 \mod \pi^\delta$. Moreover, if $x \not\equiv 1 \mod \pi^2$, then $x^l \not\equiv 1 \mod \pi^2$, $1 \leq l \leq p - 1$. 

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Proof. First, suppose $x \in 1 + O(\pi)$. Then
\[
x^p \in (1 + O(\pi))^p = 1 + pO(\pi) + \sum_{k=2}^{p-1} \binom{p}{k} O(\pi^k) + O(\pi^p) = 1 + O(\pi^\delta),
\]
where the last equality follows from the fact that $|p| = |\pi|^e$.

Second, suppose $x \in 1 + a_1 \pi + O(\pi^2)$, $a_1 \neq 0$, and $1 \leq l \leq p - 1$. Then
\[
x^l \in (1 + a_1 \pi + O(\pi^2))^l = \sum_{k=0}^{l} \binom{l}{k} (a_1 \pi + O(\pi^2))^k = 1 + la_1 \pi + O(\pi^2).
\]

Remark 3.3. Note that [53] only consider the case $\lambda \in \mathbb{Q}_p$ for odd primes, whereas we also consider extensions of $\mathbb{Q}_p$, including the case $p = 2$. Therefore, a maximal $\lambda$ does not always give a dense orbit as explained below.

It follows from Lemma 3.8 that if $\lambda$ is maximal, then the multiplication map $T_\lambda : \lambda \mapsto \lambda x$ has cycle length $p^f - 1$ modulo $\pi$, and length $p(p^f - 1)$ modulo $\pi^2$. As a consequence, $T_\lambda$ cannot act as a permutation modulo $\pi^2$ if $e \geq 2$. In particular, $T_\lambda$ cannot have a dense orbit on the unit sphere in $K$ in this case. We have the following Lemma.

Lemma 3.9. Let $K$ be a proper extension of $\mathbb{Q}_p$ and let $\lambda \in K$, with $|\lambda| = 1$. Then the map $T_\lambda : x \mapsto \lambda x$, cannot be transitive on any sphere about the origin in $K$. Furthermore, if $\lambda \in \mathbb{Q}_p$, then $T_\lambda$ is transitive on a sphere about the origin (in fact all spheres inside the unit disc) in $\mathbb{Q}_p$ if and only if $\lambda$ is maximal and $p$ is odd.

Proof. Let $[K : \mathbb{Q}_p] = e \cdot f \geq 2$. First, suppose that $f > 1$. By Lemma 3.8 $\lambda^{p(p^f-1)} \equiv 1 \mod \pi^2$. Consequently, $T_\lambda$ cannot act as a permutation on all $p^f(p^f - 1)$ elements in the group of units modulo $\pi^2$.

Second, suppose $f = 1$ but $e > 1$. By Lemma 3.8 $\lambda^{p(p-1)} \equiv 1 \mod \pi^3$. Consequently, $T_\lambda$ cannot act as a permutation on all $p^3(p-1)$ units modulo $\pi^3$.

The last statement of the lemma follows from the fact that a integer $\lambda$ is a generator of the group of units modulo $p^k$ for every integer $k \geq 2$ if and only if $\lambda$ is a generator modulo $p^2$, which happen if and only if $\lambda$ is maximal and $p$ is odd.

3.6.1 Non-transitivity in characteristic $p$

To motivate Theorem 3.4, we also show that the multiplier map can never be transitive on a whole sphere for fields of prime characteristics.
Proposition 3.7. Let $K$ be a locally compact field of prime characteristic, with uniformizer $\pi$. If $x \in K$ and $x \equiv 1 \mod \pi$, then $x^{p^n} \equiv 1 \mod \pi^{p^n}$ for all integers $n \geq 0$.

Proof. Suppose $x \in 1 + O(\pi)$. Then

$$x^p \in (1 + O(\pi))^{p^n} = 1 + p^n O(\pi) + \sum_{i=2}^{p-1} \left( \frac{p^n}{i} \right) O(\pi^i) + O(\pi^{p^n}) = 1 + O(\pi^{p^n}).$$

\[ \square \]

Lemma 3.10. Let $K$ be a locally compact field of prime characteristic and let $\lambda \in K$, with $|\lambda| = 1$. Then, the map $T_\lambda: x \mapsto \lambda x$, cannot be transitive on any sphere about the origin in $K$.

Proof. By local compactness of $K$, there is a uniformizer $\pi \in K$, and hence $T_\lambda$ is transitive on the unit sphere if and only if it is transitive on the group of units modulo $\pi^n$ for every integer $n \geq 2$. In particular, $T_\lambda$ has to be transitive modulo $\pi^{p^2}$ which is impossible by the following arguments. As $K$ is locally compact, the residue field $k$ is finite. Let $c$ be the cardinality of $k$. By definition $c \geq p$, and hence there are $(c-1)p^{2-1} > p^2$ units modulo $\pi^{p^2}$. On the other hand, by Proposition 3.7 $\lambda^{p^2} \equiv 1 \mod \pi^{p^2}$. Consequently, $T_\lambda$ cannot be transitive modulo $\pi^{p^2}$ and hence not on the unit sphere or any other sphere about the origin in $K$.

\[ \square \]

3.7 Arithmetic of the multiplier

Lemma 3.11. Let $\lambda \in \mathbb{C}_p$ be an element on the unit sphere, but not a root of unity. Let $m$ be defined by (24), and let $s$ be the integer for which $R(s) \leq |1 - \lambda^m| < R(s + 1)$. Then, the following three statements hold:

1. If $m$ does not divide $n$, then $|1 - \lambda^n| = 1$.

2. If $m$ is a divisor of $n$ and $0 \leq s \leq \nu(n)$ we have

$$|1 - \lambda^n| = \begin{cases} n|p^s|1 - \lambda^m|p^{s'}|, & \text{if } R(s) < |1 - \lambda^m| < R(s + 1), \\ n|p^s|1 - \lambda^m|p^{s'-1}|\alpha - \lambda^m|, & \text{if } |1 - \lambda^m| = R(s). \end{cases}$$

Here $\alpha \in \Gamma_\nu$ is chosen so that $|\alpha - \lambda^m| \leq |\gamma - \lambda^m|$, for all $\gamma \in \Gamma$.

3. If $m$ is a divisor of $n$ and $s > \nu(n)$ so that $|1 - \lambda^m| > R(\nu(n))$, then

$$|1 - \lambda^n| = |1 - \lambda^m|^p^{\nu(n)}.$$

Remark 3.4. In the second statement of the lemma, $|n|p^s \leq 1$ since we assume that $s \leq \nu(n)$ in this case.
Remark 3.5. By Lemma 3.7 we may have $| \alpha - \lambda^m | < | 1 - \lambda^m |$ only if $\lambda^m$ belong to the same sphere about 1 as $\alpha \in \Gamma_r$. In all other cases we may choose $\alpha = 1$. In particular, we always have $| \alpha - \lambda^m | \leq | 1 - \lambda^m |$.

Remark 3.6. If $\lambda \in \mathbb{C}_2$ and $| 1 - \lambda^m | = 2^{-1/(2-1)} = 2^{-1}$, then the ‘closest’ root of unity $\alpha = -1 = \sum_{k=0}^{\infty} 2^k$.

Proof. It is enough to consider the case $m = 1$. This proof is based on the factorization of the polynomial $\lambda^n - 1$ from which we find
\[
| \lambda^n - 1 | = \prod_{\theta^n=1} | \lambda - \theta |. \tag{25}
\]

As noted in Section 3.5 the roots of unity in $\mathbb{C}_p$ is the direct product $\Gamma = \Gamma_u \times \Gamma_r$. This representation enables us to write $| \lambda^n - 1 |$ in the form
\[
| \lambda^n - 1 | = | \lambda - 1 | \prod_{\xi^n=1} | \lambda - \xi | \prod_{\zeta^n=1} | \zeta | \prod_{(\xi \xi)^n=1} | \zeta - \xi \zeta |, \tag{26}
\]
where $\xi \in \Gamma_r \setminus \{1\}$ and $\zeta \in \Gamma_u \setminus \{1\}$. Recall that we assume that $\lambda \in D_1(1)$. In view of Proposition 3.3, $\xi, (\xi \xi) \notin D_1(1)$ and consequently $| \lambda - \xi | = | \lambda - \xi \zeta | = 1$.

Moreover, for $n = ap^{\nu(n)}$, we have that $\zeta^n = 1$ if and only if $\zeta^{p^{\nu(n)}} = 1$. It follows that (26) can be reduced to
\[
| \lambda^n - 1 | = | \lambda - 1 | \prod_{\zeta^{p^{\nu(n)}}=1} | \lambda - \zeta |. \tag{27}
\]

If $p$ does not divide $n$ so that $\nu(n) = 0$, then $| 1 - \lambda^n | = | 1 - \lambda |$ as required.

In the remaining cases $\nu(n) \geq 1$ and we have to take the factors $| \lambda - \zeta |$ into account. Note that $| \lambda - \zeta | = | (\lambda - 1) + (1 - \zeta) |$ and by ultrametricity
\[
| \lambda - \zeta | = \max \{| \lambda - 1 |, | 1 - \zeta | \}, \tag{28}
\]
if $| \lambda - 1 | \neq | 1 - \zeta |$. Thus we can compute (27) by counting the number of roots $\xi$ that are closer and farther to 1 compared to $\lambda$, respectively.

Recall the following facts from Proposition 3.6. If $\zeta \in \Gamma_r \setminus \{1\}$ is a primitive $p^s$th root of unity for some $s \geq 1$, then $| 1 - \zeta | = p^{-r_s}$ where $r_s = 1/(p^s - p^{s-1})$. The sphere $S_{p^{-r_s}}(1)$ contains $p^s - p^{s-1} = 1/r_s$ roots of unity. Note that these spheres have radii ordered as $p^{-r_1} < p^{-r_2} < \ldots < 1$.

Now we consider the case $| 1 - \lambda | < p^{-r_1}$. In this case $\lambda$ is closer to 1 than any of the roots $\zeta$ and therefore, in view of (28), $| \lambda - \zeta | = | 1 - \zeta |$ for every $\zeta \in \Gamma_r \setminus \{1\}$. From (27) we thus have that
\[
| \lambda^n - 1 | = | \lambda - 1 | (p^{-r_1})^{1/r_1} \ldots (p^{-r_{\nu(n)}})^{1/r_{\nu(n)}} = | \lambda - 1 | p^{-\nu(n)}
\]
as required.

Now we consider the case $p^{-r_s} < | \lambda - 1 | < p^{-r_{s+1}}$, $1 \leq s \leq \nu(n)$. We have in view of (28) that $| 1 - \zeta | = | 1 - \lambda |$ for all $\zeta$ such that $| 1 - \zeta | < | \lambda - 1 |$. This
is the case for $p^s$ roots. All other roots are further from 1 than $\lambda$ is. For these roots $|\lambda - \zeta| = |1 - \zeta|$. Hence, the right-hand side of (27) becomes

$$|\lambda - 1| p^s (p^{-r_{s+1}})^{1/r_{s+1}} \cdots (p^{-r_{\nu(n)}})^{1/r_{\nu(n)}} = |1 - \lambda| p^s p^{-\nu(n)-s}$$

(29)

as required.

Now we consider the case $|\lambda - 1| = p^{-r_s}$ for some $1 \leq s \leq \nu(n)$. Let $\alpha \in S_{p^{-r_s}}(1)$ be such that $|\alpha - \lambda| \leq |\zeta - \lambda|$ for all $\zeta \in \Gamma_r$. Note that $|\zeta - \lambda| \leq p^{-r_s}$ for all $\zeta \in S_{p^{-r_s}}(1)$ and that $\alpha$ is unique if and only if $|\lambda - \alpha| < p^{-r_s}$. By Proposition 3.6 $|\lambda - \zeta| = p^{-r_s}$ for all $\zeta \neq \alpha$ on the sphere $S_{p^{-r_s}}(1)$. For the right-hand side of (27) we obtain

$$|\lambda - 1| p^{s-1}|\lambda - \alpha| (p^{-r_{s+1}})^{1/r_{s+1}} \cdots (p^{-r_{\nu(n)}})^{1/r_{\nu(n)}},$$

Consequently,

$$|\lambda^n - 1| = |\lambda - 1| p^{s-1}|\lambda - \alpha| p^{-\nu(n)-s}$$

(30)

as required.

Finally, we consider $|\lambda - 1| > p^{-r_{\nu(n)}}$. In this case $|\lambda - 1| > |1 - \zeta|$ for all $\zeta$ that are $p^{\nu(n)}$th roots of unity. Consequently, $|\lambda - \zeta| = |\lambda - 1|$ and we obtain

$$|\lambda^n - 1| = |\lambda - 1| p^{\nu(n)},$$

as proposed in the lemma. This completes the proof. $\square$

4 Estimates of linearization discs

We will estimate the size of the linearization disc for a power series $f \in \mathcal{F}_{\lambda,a}$. The estimates are divided into three different cases according to the three sections below.

4.1 Case I

In this section we assume that

$$R(0) < |1 - \lambda^m| < R(1).$$

(31)

In what follows $\sigma_1$ will be the real number defined by

$$\sigma_1 := a^{-1} p^{-\frac{1}{m(p-1)}|1 - \lambda^m|^{1/p}}.$$

(32)

Lemma 4.1. Suppose $\lambda$ is such that $R(0) < |1 - \lambda^m| < R(1)$. Then,

$$\left(\prod_{n=1}^{k-1} |1 - \lambda^n| \right)^{-1} a^{k-1} \leq p^{-\frac{1}{m(p-1)}} \sigma_1^{-(k-1)},$$

(33)

with equality if $(k-1)/m$ is an integer power of $p$. 23
Proof. By Lemma 3.11

\[ |1 - \lambda^n| = \begin{cases} 1, & \text{if } m \nmid n, \\ |n||1 - \lambda^m|, & \text{if } m \mid n, \end{cases} \tag{34} \]

in this case. Let \( N = \lfloor k - 1/m \rfloor \) denote the integer part of \( k - 1/m \). Then, by (34)

\[ \prod_{n=1}^{k-1} |1 - \lambda^n| = |N!||1 - \lambda^m|^N. \tag{35} \]

By Lemma 3.1

\[ |N!| \geq p^{-\frac{k-1}{p-1}} \geq p^{-\frac{k-1}{(p-1)(p-1)}}, \]

where each inequality become an equality if \((k - 1)/m\) is an integer power of \( p \). It follows by (35) that

\[ \left( \prod_{n=1}^{k-1} |1 - \lambda^n| \right)^{-1} a^{k-1} \leq p^{-\frac{1}{p-1}} \sigma_1^{-(k-1)}, \]

with equality if \((k - 1)/m\) is an integer power of \( p \).

We will prove the following theorem.

**Theorem 4.1.** Let \( f \in \mathcal{F}_{\lambda,a} \) and suppose \( \lambda \) is such that \( R(0) < |1 - \lambda^m| < R(1) \). Then, the linearization disc \( \Delta_f \supseteq D_{\sigma_1}(0) \). Moreover, if the conjugacy function \( g \) converges on the closed disc \( \overline{D}_{\sigma_1}(0) \), then \( \Delta_f \supseteq \overline{D}_{\sigma_1}(0) \).

**Proof.** In view of Lemma 3.1 and Lemma 4.1 we have

\[ \left( \limsup |b_k|^{1/k} \right)^{-1} \geq \sigma_1. \]

This implies that \( g \) converges on the open disc of radius \( \sigma_1 \). Moreover, by Lemma 4.1

\[ |b_k| \sigma_1^k \leq p^{-\frac{1}{p-1}} \sigma_1 < \sigma_1 = |b_1| \sigma_1. \]

It follows by Proposition 3.2 that \( g : D_{\sigma_1}(0) \to D_{\sigma_1}(0) \) is a bijection. The strict inequality \( |b_k| \sigma_1^k < |b_1| \sigma_1 \) implies that, if \( g \) converges on the closed disc \( \overline{D}_{\sigma_1}(0) \), then \( g : \overline{D}_{\sigma_1}(0) \to \overline{D}_{\sigma_1}(0) \) is bijective. Recall that by Lemma 3.3

\( f : D_{1/a}(0) \to D_{1/a}(0) \) is a bijection. Moreover, \( 1/a > \sigma_1 \). Consequently, the linearization disc \( \Delta_f \) includes the disc \( D_{\sigma_1}(0) \) or \( \overline{D}_{\sigma_1}(0) \), depending on whether the conjugacy function \( g \) converges on the closed disc \( \overline{D}_{\sigma_1}(0) \) or not.

The theorem has some important consequences for linearization discs in \( \mathbb{Q}_p \).

In fact, as we will see below, a linearization disc in \( \mathbb{Q}_p \) may coincide with the maximal disc on which \( f \) is one-to-one and even with the region of convergence of \( f \). Recall that the value group \( |\mathbb{Q}_p^*| \) contains only integer powers of \( p \). This implies that if \( \lambda \in \mathbb{Q}_p \), then \( |1 - \lambda^m| \leq p^{-1} < p^{-1/(p-1)} \) if \( p > 2 \). Consequently, Theorem 1.1 applies if \( \lambda \in \mathbb{Q}_p \) for some prime \( p > 2 \). In particular, if \( f \) is a power series over \( \mathbb{Q}_p \), we have the following corollary.
Corollary 4.1 (linearization discs in \( \mathbb{Q}_p \)). Let \( f \in \mathcal{F}_{\lambda,a} \cap \mathbb{Q}_p[[x]] \) for some odd prime \( p \). Let \( \Delta_f(0, \mathbb{Q}_p) = \Delta_f \cap \mathbb{Q}_p \) be the corresponding linearization disc, about the origin, in \( \mathbb{Q}_p \). Then, \( \Delta_f(0, \mathbb{Q}_p) \supseteq D_{\sigma_1}(0, \mathbb{Q}_p) \). Moreover, if the conjugacy function \( g \) converges on the closed disc \( \overline{D}_{\sigma_1}(0) \), then the linearization disc \( \Delta_f(0, \mathbb{Q}_p) \supseteq \overline{D}_{\sigma_1}(0, \mathbb{Q}_p) \). Furthermore, if \( \lambda \) is maximal, then, the linearization disc \( \Delta_f(0, \mathbb{Q}_p) \) is maximal in the sense that \( \Delta_f(0, \mathbb{Q}_p) \) is either the open or closed disc of radius \( 1/a \). In particular, if either \( \max_{i \geq 2} |a_i|^{1/(i-1)} \) is attained (as for polynomials) or \( f \) diverges on \( S_{1/a}(0, \mathbb{Q}_p) \), then \( \Delta_f(0, \mathbb{Q}_p) = D_{1/a}(0, \mathbb{Q}_p) \).

Proof. If \( \lambda \) is maximal, then \( |1 - \lambda^m| = p^{-1} \) and \( m = p - 1 \). Consequently, \( D_{\sigma_1}(0) = \overline{D}_{\sigma_1}(0) = D_{1/a}(0) \), considered as discs in \( \mathbb{Q}_p \).

Moreover, a power series \( f \in \mathcal{F}_{\lambda,a} \cap \mathbb{Q}_p \) may diverge on \( S_{1/a}(0) \). For example, the power series \( f(x) = \lambda x + \sum_{i=2}^{\infty} (a_2)^{i-1} x^i \) converges if and only if \( |x| < 1/|a_2| = 1/a \).

To see that \( f \) may have a zero on the sphere \( S_{1/a}(0) \), consider the following example. Let \( f(x) = \lambda x + a_2 x^2 \). Then \( a = |a_2| \). But \( x = -\lambda/a_2 \in \mathbb{Q}_p \) is a zero of \( f \) located on the sphere \( S_{1/a}(0) \) in \( \mathbb{Q}_p \).

Remark 4.1. If \( p \) is an odd prime and \( f(x) = \lambda x + O(x^2) \) is a power series over \( \mathbb{Z}_p \), with multiplier \( \lambda \) such that \( |1 - \lambda^m| = p^{-1} \), and \( m = p - 1 \). Then, the linearization disc in \( \mathbb{Q}_p \) includes the open unit disc \( D_1(0) \). This result was also obtained in [53, Proposition 2.2 ].

Let \( K \) be an unramified field extension of \( \mathbb{Q}_p \). Then the value group \( |K^*| \) contains only integer powers of \( p \). Hence, if \( \lambda \in K \), then Theorem 4.1 applies and we have the following corollary.

Corollary 4.2. Corollary 4.1 holds for any unramified extension \( K \) of \( \mathbb{Q}_p \).

The case \( p = 2 \), will be treated in case III below.

4.2 Case II

In this section we assume that \( s \geq 1 \) and

\[
R(s) < |1 - \lambda^m| < R(s + 1).
\]

In what follows \( \sigma_2 \) will be the real number defined by

\[
\sigma_2 := a^{-1} p^{-\frac{1}{m}} |1 - \lambda^m|^{-\frac{1}{m}} (1 + \frac{1}{s(m+1)}).
\]

Lemma 4.2. Suppose \( \lambda \) satisfies (36) for some \( s \geq 1 \). Then,

\[
\left( \prod_{n=1}^{k-1} |1 - \lambda^n| \right)^{-1} a^{k-1} \leq p^{-\frac{1}{m}} \sigma_2^{-(k-1)},
\]

with equality if \((k-1)/mp^{s+1}\) is an integer power of \( p \).
Proof. By lemma 3.11

\[ |1 - \lambda^n| = \begin{cases} 
1, & \text{if } m \nmid n, \\
|n|p^s|1 - \lambda^m|p^r, & \text{if } mp^{s+1} | n, \\
|1 - \lambda^m|p^{r(n)}, & \text{if } m | n \text{ but } mp^{s+1} \nmid n
\end{cases} \]  

(39)
in this case. Throughout this proof \( N \) will be the integer

\[ N = \left\lfloor \frac{k - 1}{mp^{s+1}} \right\rfloor. \]  

(40)

Note that

\[ \prod_{mp^{s+1} | n}^{k-1} |n| = |mp^{s+1} \cdot 2mp^{s+1} \ldots | = |N||mp^{s+1}|N, \]

and since \( p \nmid m \),

\[ \prod_{mp^{s+1} | n}^{k-1} |n| = |N|p^{-(s+1)N}. \]  

(41)

Moreover,

\[ \prod_{mp^{s+1} | n}^{k-1} p^s|1 - \lambda^m|p^r = p^sN|1 - \lambda^m|p^rN, \]  

(42)

and

\[ \prod_{mp^{s+1} | n}^{k-1} |1 - \lambda^m|p^{r(n)} = |1 - \lambda^m|\sum_{j=0}^{s} \left( \left\lfloor \frac{k - 1}{mp^j} \right\rfloor - \left\lfloor \frac{k - 1}{mp^{j+1}} \right\rfloor \right) p^j. \]  

(43)

Combining the three products (41), (42), and (43) we obtain

\[ \prod_{n=1}^{k-1} |1 - \lambda^n| = |N|p^{-N}|1 - \lambda^m|\Sigma_1, \]  

(44)

where

\[ \Sigma_1 = p^sN + \sum_{j=0}^{s} \left( \left\lfloor \frac{k - 1}{mp^j} \right\rfloor - \left\lfloor \frac{k - 1}{mp^{j+1}} \right\rfloor \right) p^j. \]

Simplifying, we obtain

\[ \Sigma_1 = \left\lfloor \frac{k - 1}{m} \right\rfloor + \sum_{j=1}^{s} \left\lfloor \frac{k - 1}{mp^j} \right\rfloor (p^j - p^{j-1}). \]

Consequently,

\[ \Sigma_1 \leq \frac{k - 1}{m} \left( 1 + sp - \frac{1}{p} \right), \]
with equality if \((k - 1)/mp^s + 1\) is an integer power of \(p\). By Lemma 3.1
\[
|N| p^{-N} \geq p^{-\frac{k-1}{mp^s} - N} = p^{-\frac{k-1}{mp^s} + 1} \geq p^{-\frac{k-1}{mp^s} + \frac{1}{mp^s}},
\]
where each inequality become an equality if \((k - 1)/mp^s + 1\) is an integer power of \(p\). Applying these estimates to the identity \(\text{(44)}\) we obtain the inequality \(\text{(38)}\) as required.

By similar arguments as those applied in the proof of Theorem 4.1, we obtain the following result.

**Theorem 4.2.** Let \(f \in \mathcal{F}_{\lambda, a}\) and suppose \(\lambda\) satisfies
\[
R(s) < |1 - \lambda| < R(s + 1),
\]
for some integer \(s \geq 1\). Then, \(\Delta_f \supseteq D_{\sigma_2}(0)\). Moreover, if the conjugacy function \(g\) converges on the closed disc \(\overline{D}_{\sigma_2}(0)\), then \(\Delta_f \supseteq \overline{D}_{\sigma_2}(0)\).

### 4.3 Case III

In this section it will be assumed that \(s \geq 1\) and
\[
|1 - \lambda^m| = R(s). \tag{45}
\]
In what follows \(\sigma_3\) will be the real number defined by
\[
\sigma_3 := \sigma_2 \cdot \left(\frac{|\alpha - \lambda^m|}{|1 - \lambda^m|}\right)^{1/mp^s}, \tag{46}
\]
where \(\alpha \in \Gamma_r\) is chosen such that \(|\alpha - \lambda^m| \leq |\gamma - \lambda^m|\), for all \(\gamma \in \Gamma\).

**Lemma 4.3.** Suppose \(\lambda\) satisfies \(\text{(43)}\) for some \(s \geq 1\). Then
\[
\left(\prod_{n=1}^{k-1} |1 - \lambda^n|\right)^{-1} a^{k-1} \leq p^{-\frac{k-1}{mp^s} + \sigma_3^{-1}(k-1)}, \tag{47}
\]
with equality if \((k - 1)/mp^s + 1\) is an integer power of \(p\).

**Proof.** By Lemma 3.1
\[
|1 - \lambda^n| = \begin{cases} 
1, & \text{if } m \nmid n, \\
|n|p^s|1 - \lambda^m|p^{s-1}|\alpha - \lambda^m|, & \text{if } mp^s \mid n, \\
|1 - \lambda^m|p^{s(n)}, & \text{if } m \mid n \text{ but } mp^s \nmid n
\end{cases}, \tag{48}
\]
in this case. Throughout this proof we let \(M\) be the integer
\[
M = \left\lfloor \frac{k - 1}{mp^s} \right\rfloor.
\]

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Note that
\[ k - 1 \prod_{mp^r|n} |n| = |M!|p^{-sM}, \tag{49} \]
\[ k - 1 \prod_{mp^r|n} p^s|1 - \lambda^m|p^s = p^sM|1 - \lambda^m|(p^s - 1)^M, \tag{50} \]
\[ k - 1 \prod_{mp^r|n} |\alpha - \lambda^m| = |\alpha - \lambda^m|^M, \tag{51} \]
and
\[ k - 1 \prod_{mp^r|n} |1 - \lambda^m|p^{\sigma(n)} = |1 - \lambda^m|\Sigma_{j=0}^{s-1}\left(\left\{\frac{k - 1}{mp^j}\right\} - \left\{\frac{k - 1}{mp^{j+1}}\right\}\right)p^j. \tag{52} \]

Combining the three products (49), (50), (51), and (50) we obtain
\[ \prod_{n=1}^{k - 1}|1 - \lambda^n| = |M!||1 - \lambda^m|\Sigma_{2} \left(\frac{|\alpha - \lambda^m|}{|1 - \lambda^m|}\right)^M, \tag{53} \]
where
\[ \Sigma_{2} = p^sM + \sum_{j=0}^{s-1}\left(\left\{\frac{k - 1}{mp^j}\right\} - \left\{\frac{k - 1}{mp^{j+1}}\right\}\right)p^j. \]

Simplifying, we obtain
\[ \Sigma_{2} = \Sigma_{1} = \left\lfloor\frac{k - 1}{m}\right\rfloor + \sum_{j=1}^{s}\left\lfloor\frac{k - 1}{mp^j}\right\rfloor(p^j - p^{j-1}). \]

As in the proof of Lemma 4.2 we have
\[ \Sigma_{1} \leq \frac{k - 1}{m} \left(1 + sp - 1\right), \]
with equality if \((k - 1)/mp^s\) is an integer power of \(p\). By Lemma 3.1
\[ |M!| \geq p^{-M - 1} \geq p^{-\frac{k - 1}{mp^s} + \frac{1}{p - 1}}, \]
where each inequality become an equality if \((k - 1)/mp^s\) is an integer power of \(p\). Furthermore, by definition
\[ |\alpha - \lambda^m| \leq |1 - \lambda^m| = R(s). \]

Applying these estimates to the identity (53) we obtain the inequality (47) as required.

\[ \square \]

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By similar arguments as those applied in the proof of Theorem 4.1, we obtain the following result.

**Theorem 4.3.** Let $f \in \mathcal{F}_{\lambda,a}$ and suppose 

$$|1 - \lambda^m| = R(s),$$

for some integer $s \geq 1$. Then, $\Delta_f \supseteq D_{\sigma_3}(0)$. Moreover, if the conjugacy function $g$ converges on the closed disc $\overline{D}_{\sigma_3}(0)$, then $\Delta_f \supseteq \overline{D}_{\sigma_3}(0)$.

**Corollary 4.3.** Let $K$ be an unramified extension of $\mathbb{Q}_2$. Let $f$ be of the form

$$f(x) = \lambda(x - x_0) + \sum a_i(x - x_0)^i \in K[[x - x_0]], a = \sup |a_i|^{1/(i-1)}.$$

Then, the following two statements hold:

1. If $|1 - \lambda| < 1/2$, then the linearization disc $\Delta_f(x_0, K)$ contains the open disc of radius $\sigma_1 = |1 - \lambda|/2a$ about $x_0$.

2. If $|1 - \lambda| = 1/2$, then the linearization disc the linearization disc $\Delta_f(x_0, K)$ contains the open disc of radius $\sigma_3 = \sqrt{|1 + \lambda|/a}$ about $x_0$.

**Proof.** As $p = 2$ we must have $m = 1$ and since $K$ is unramified we must have $s \leq 1$. The first statement is then a direct consequence of Theorem 4.1. As to the second statement, $|1 - \lambda| = 1/2$. Theorem 4.3 applies with $m = 1$, $s = 1$ and $\alpha = -1$. Hence, $\sigma_3 = \sqrt{|1 + \lambda|/2a}$, and since $K$ is unramified we may as well exclude the factor $2^{3/2}$.

**Remark 4.2.** Note that $\sqrt{|1 + \lambda|} \leq 1/2$, so even if $\lambda$ is maximal (as in the second statement of Corollary 4.3 above), it seems that the radius of the linearization disc in $\mathbb{Q}_2$ may not be the maximal, $1/a$ as obtained for $\mathbb{Q}_p$ with $p$ odd in Corollary 4.1.

### 4.4 Statement of the general estimate

Suppose $\lambda^m$ belongs to the annulus

$$\{z \in \mathbb{C}_p : R(s) < |1 - z| < R(s + 1)\}.$$ 

Then, in view of Lemma 3.7, $\alpha = 1$ is the closest root of unity to $\lambda^m$. It follows that $\sigma_3 = \sigma_2$ in this case. Consequently, the estimate $\sigma_3$ of the radius of the linearization disc, holds for all $\lambda$ such that $R(s) \leq |1 - \lambda^m| < R(s + 1)$ for some $s \geq 1$. Furthermore, if we put $s = 0$ and $\alpha = 1$, then $\sigma_3 = \sigma_1$.

Hence, $\sigma_3$ may serve as a general bound if we include the case $s = 0$. Recall that, by definition, $R(0) = 0$. Our estimates can thus be summarized according to the following theorem.
Theorem 4.4. Let \( f \in \mathcal{F}_{\lambda,a} \). Suppose \( \lambda \) is not a root of unity, and
\[
R(s) \leq |1 - \lambda^m| < R(s + 1),
\]
for some \( s \geq 0 \). Then, the linearization disc \( \Delta_f \supseteq D_\sigma(0) \) where
\[
\sigma = \sigma(\lambda, a) := a^{-1} R(s + 1) \frac{|1 - \lambda^m|}{|1 - \lambda^m|} \left( \frac{|\alpha - \lambda^m|}{|1 - \lambda^m|} \right)^{1/(p-1)}.
\]
Moreover, if the conjugacy function converges on the closed disc \( \overline{D}_\sigma(0) \), then \( \Delta_f \supseteq \overline{D}_\sigma(0) \).

4.5 Asymptotic behavior of the estimate of the radius of the linearization disc

In this section we consider the following question. What happens to the estimate
\[
\sigma = a^{-1} p^{-\frac{1}{m(p-1)p^r}} |1 - \lambda^m| \frac{1}{p} \left( \frac{|\alpha - \lambda^m|}{|1 - \lambda^m|} \right)^{1/(p-1)},
\]
of the radius of the linearization disc, as \( m \) or \( s \) goes to infinity? We will show that in each of these two cases \( \sigma \) approach \( 1/a \). As stated in Lemma 3.5, for \( f \in \mathcal{F}_{\lambda,a} \), the value \( 1/a \) is the maximal radius of a linearization disc. On the other hand, if \( s \) and \( m \) are fixed, then \( \sigma \to 0 \) as \( |\alpha - \lambda^m| \to 0 \).

The result is based on the following Lemma.

Lemma 4.4. For all \( s \geq 0 \)
\[
\sigma \geq a^{-1} p^{-\frac{1}{m(p-1)p^r}} |1 - \lambda^m| \frac{1}{p} \left( \frac{|\alpha - \lambda^m|}{|1 - \lambda^m|} \right)^{1/(p-1)},
\]
In particular,
\[
\sigma \geq a^{-1} p^{-\frac{1}{m(p-1)}} |\alpha - \lambda^m| \frac{1}{p}.
\]
Proof. Recall that for \( s = 0 \), we have that \( |\alpha - \lambda^m| = |1 - \lambda| \). This completes the case \( s = 0 \).

For \( s \geq 1 \), we have \( |1 - \lambda^m| \geq p^{-1/(p^{r-1}(p-1))} \), and by the definition of \( \sigma \)
\[
\sigma \geq a^{-1} p^{-\frac{1}{m(p-1)p^r}} |1 - \lambda^m| \frac{1}{p} \left( \frac{|\alpha - \lambda^m|}{|1 - \lambda^m|} \right)^{1/(p-1)}.
\]
Recall from Remark 3.5 that we always have \( |\alpha - \lambda^m| \leq |1 - \lambda^m| \). Hence, we also obtain the following bound for \( \sigma \)
\[
\sigma \geq a^{-1} p^{-\frac{1}{m(p-1)}} |\alpha - \lambda^m| \frac{1}{p},
\]
as required.
Note that if $|1 - \lambda^m| \neq R(s)$ for all $s \geq 0$, then $\alpha = 1$. Hence, by increasing $s$, we push $\lambda^m$ further away from the closest root of unity, and make $\sigma$ closer to its maximum value $1/a$. Loosely speaking, according to the the Lemma, the farther $\lambda^m$ is from the ‘closest’ root of unity, the closer $\sigma$ is to its maximum value $1/a$. On the other hand, if $s$ and $m$ are fixed, then $\sigma \to 0$ as $|\alpha - \lambda^m| \to 0$. In particular, we have the following theorem.

**Theorem 4.5.** Let $|\alpha - \lambda^m|$ be fixed. Then, the estimate $\sigma$ of the radius of the linearization disc goes to $1/a$ as $m$ or $s$ goes to infinity. If $s$ and $m$ are fixed, then $\sigma \to 0$ as $|\alpha - \lambda^m| \to 0$.

### 4.6 Maximal linearization discs in finite extensions of $\mathbb{Q}_p$

Let $K$ be a finite extension of $\mathbb{Q}_p$, and let $f \in F_{\lambda,a} \cap K[[x]]$. The disc

$$\Delta_f(0, K) = \Delta_f \cap K$$

will be referred to as the corresponding linearization disc in $K$. We say that $\Delta_f(0, K)$ is maximal if it contains the open disc $D_{1/a}(0, K) = D_{1/a}(0) \cap K$. 

**Theorem 4.6.** Let $K$ be a finite extension of $\mathbb{Q}_p$, with ramification index $e$. Let $f \in F_{\lambda,a} \cap K[[x]]$ and let $s$ be the integer for which $R(s) < |1 - \lambda^m| \leq R(s + 1)$. Let $\epsilon$ be the integer satisfying $\nu(1 - \lambda^m) = \epsilon/e$. Suppose that

$$s < \left(\frac{m}{\epsilon - 2}\right) \frac{p}{p - 1} - \nu \left(\frac{\alpha - \lambda^m}{1 - \lambda^m}\right).$$

(55)

Then, the linearization disc $\Delta_f(x_0, K)$ is maximal. In particular, if either $\max_{i \geq 2} |a_i|^{1/(i-1)}$ is attained (as for polynomials) or $f$ diverges on $S_{1/a}(x_0, K)$, then $\Delta_f(x_0, K) = D_{1/a}(x_0, K)$.

**Proof.** We first consider the case $R(s) < |1 - \lambda^m| < R(s + 1)$. Consequently, $e^s - 1(p - 1) < \epsilon < e^s(p - 1)$. In view of (54) with $\alpha = 1$,

$$\nu(a\sigma) = \frac{1}{m(p - 1)p^s} + \frac{\epsilon(1 + s(p - 1)/p)}{em} = \frac{e + e^s(p - 1)(1 + s(p - 1)/p)}{em(p - 1)p^s},$$

and since $\epsilon < e^s(p - 1)$ we have

$$\nu(a\sigma) < \frac{1}{e} \cdot \frac{e(2 + s(p - 1)/p)}{m}.$$

It follows that

$$\nu(a\sigma) < 1/e$$

if $s < p(m/e - 2)/(p - 1)$ as required.

We now consider the case $|1 - \lambda^m| = R(s + 1)$. Hence, $e = e^s(p - 1)$. In view of (54),

$$\nu(a\sigma) = \frac{1}{m(p - 1)p^{s+1}} + \frac{\epsilon(1 + (s + 1)(p - 1)/p)}{em} + \frac{\nu}{mp^{s+1}}.$$

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Hence,
\[
\nu(a\sigma) = \frac{e + ep^{s+1}(p-1)(1 + (s + 1)(p - 1)/p) + ev(p - 1)}{em(p - 1)p^{s+1}},
\]
and since \(e = ep^s(p - 1)\) we have
\[
\nu(a\sigma) = \frac{1}{e} \cdot \frac{ep(1/p + 1 + (s + 1)(p - 1)/p) + (p - 1)v}{mp},
\]
or equivalently,
\[
\nu(a\sigma) = \frac{1}{e} \cdot \frac{\epsilon(2 + s(p - 1)/p) + v(p - 1)/p}{m}.
\]
It follows that
\[
\nu(a\sigma) < 1/e
\]
if \(s < p(m/e - 2)/(p - 1) - \nu\) as required.

One might ask whether the condition (55) is really necessary or just a consequence of lack of precision in our estimates of the radius of the linearization disc. However, in section 5 (Corollary 5.1) we show that there are examples where the condition (55) is not satisfied and the corresponding linearization disc is strictly contained in the disc \(D_{1/\alpha}(0)\).

**Corollary 4.4.** Let \(K\) be a finite extension of \(\mathbb{Q}_p\) of degree \(n\), with ramification index \(e\), and residue field \(k\) of degree \(\left[ k : \mathbb{F}_p \right] = n/e\). Let \(\lambda\) be maximal, \(f \in \mathcal{F}_{\lambda, a} \cap K[[x]]\), and let \(s\) be the integer for which \(R(s) < |1 - \lambda p^{\gamma/e - 1}| \leq R(s + 1)\). Suppose that
\[
s < (p^{n/e} - 3)\frac{p}{p - 1} - \nu \left( \frac{\alpha - \lambda p^{\gamma/e - 1}}{1 - \lambda p^{\gamma/e - 1}} \right).
\]
Then, the linearization disc \(\Delta_f(0, K)\) is maximal. In particular, if either the maximum \(\max_{i \geq 2} |a_i|^{1/(i-1)}\) is attained (as for polynomials) or \(f\) diverges on \(S_{1/\alpha}(0, K)\), then \(\Delta_f(0, K) = D_{1/\alpha}(0, K)\).

**Proof.** Recall that since \(\lambda\) is maximal we have \(m = p^{n/e} - 1\) and moreover \(|1 - \lambda p^{\gamma/e - 1}| = p^{-1/e}\). The corollary then follows from theorem 4.6 with \(\epsilon = 1\) and \(m = p^{n/e} - 1\).

**Remark 4.3.** For example, if \(K\) is an unramified extension, then \(e = 1\) and \(s = 0\). Furthermore, if \(\lambda\) is maximal and \(p\) is odd, then \(\alpha = 1\). Hence, if \(p > 3\), the condition (55) holds and the linearization disc is maximal. On the other hand, if \(p = 3\), then condition (55) is not satisfied. However, by Corollary 4.2 we know that the linearization disc is maximal also for \(p = 3\). This shows that the condition (55) is not necessary in this case.
5 The quadratic case

To see the results at work we provide examples, where we can find the exact size of the linearization disc. We begin with quadratic polynomials, and then show how we can extend this result to power series containing a ‘sufficiently large’ quadratic term. We also give sufficient conditions, on the multiplier \( \lambda \), that there is a fixed point on the ‘boundary’ of a linearization disc for quadratic polynomials.

Given a quadratic polynomial \( f \) of the form \( f(x) = \lambda x + a_2 x^2 \in \mathbb{C}_p \), the radius of the corresponding linearization disc can be estimated by \( \sigma \), defined by (54). If \( \lambda \) is located inside the annulus \( \{ z : p^{-1} < |1 - z| < 1 \} \), we can actually find the exact size of the linearization disc.

**Theorem 5.1.** Let \( p \) be an odd prime and let \( \lambda \in \mathbb{C}_p \), not a root of unity, belong to the annulus \( \{ z : p^{-1} < |1 - z| < 1 \} \). Let \( f \) be a quadratic polynomial of the form \( f(x) = \lambda x + a_2 x^2 \in \mathbb{C}_p \). Then, the coefficients of the conjugacy \( g \) satisfy

\[
|b_k| = \frac{|a_2|^{k-1}|1 - \lambda|^k}{\prod_{n=1}^{k-1} |1 - \lambda^n|}. \tag{57}
\]

Moreover, the linearization disc about the origin, \( \Delta_f = D_\tau(0) \), where the radius \( \tau = |1 - \lambda|^{-1/p} \sigma \).

**Proof.** We first prove that the coefficients of the conjugacy, defined by the recursive formula (18), satisfy the identity (57).

Recall that one consequence of ultrametricity is that for any \( x, y \in \mathbb{K} \) with \( |x| \neq |y| \), the inequality (7) becomes an equality. In other words, if \( x, y \in \mathbb{K} \) with \( |x| < |y| \), then \( |x + y| = |y| \). The idea of the proof is to find a dominating term in the right hand side of (18) which is strictly greater than all the others. Then, the absolute value of the coefficient \( b_k \), of the conjugacy \( g \), is equal to the absolute value of the dominating term. The proof is similar to that performed in [44, p 760–761], for fields of prime characteristic.

Since the term \( l!/(\alpha_1!\alpha_2!) \) is always an integer

\[
\frac{l!}{\alpha_1!\alpha_2!} \leq 1,
\]

with equality if and only if \( l!/(\alpha_1!\alpha_2!) \) is not divisible by \( p \). We will show that most of the time the \( b_{k-1} \)-term is the greatest. In fact,

\[
\frac{(k-1)!}{\alpha_1!\alpha_2!} = k - 1. \tag{58}
\]

As \( k \) runs from 1, \ldots, \( p \), the number \( k-1 \) will never be divisible by \( p \). Recall that we assume in this proof that \( |1 - \lambda| < 1 \) (so that \( m = 1 \)). Hence, \( |1 - \lambda^n| < 1 \), for all integers \( n \geq 1 \). Therefore, the \( b_{k-1} \)-term will be strictly greater than all the other terms in the right hand side of (18), and thus by the ultrametric
triangle inequality (7), we have

\[ |b_k| = \frac{|b_{k-1}| |k-1| |a_2|}{|1 - \lambda^{k-1}|} = \frac{|a_2|^{k-1}}{|1 - \lambda^{k-1}| |1 - \lambda^{k-2}||1 - \lambda^{k-3}| \cdots |1 - \lambda|}. \]  

(59)

But if \( k = p + 1 \), then \( |k - 1| = p - 1 \) so that for \( l = k - 1 \) in (18), we obtain

\[ |b_{k-1}| \frac{(k-1)!}{\alpha_1! \alpha_2!} |a_2| = \frac{p^{-1} |a_2|^p}{|1 - \lambda^{p-1}| |1 - \lambda^{p-2}| \cdots |1 - \lambda|}. \]  

(60)

Then, the \( b_{k-2} \)-term will dominate. In fact,

\[ \frac{(k-2)!}{\alpha_1! \alpha_2!} = \frac{(k-2)(k-3)}{2} = 1, \text{ if } p|k - 1, \ p > 2. \]  

(61)

As a consequence, \( l = k - 2 \) gives

\[ |b_{p-1}| \frac{(k-2)!}{\alpha_1! \alpha_2!} |a_2|^2 = \frac{|a_2|^p}{|\lambda^{p-2} - 1||\lambda^{p-3} - 1| \cdots |1 - \lambda|}. \]  

(62)

Note that, since \( m = 1 \) in our case, we have \( |1 - \lambda^{p-1}| = |1 - \lambda|. \) Moreover, by assumption, \( |1 - \lambda| > p^{-1} \). Hence, the \( b_{k-2} \)-term (62) is strictly greater than the \( b_{k-1} \)-term (60), and all \( b_l \)-terms for which \( l < k - 2 \). Consequently,

\[ |b_{p+1}| = \frac{|b_{p-1}| |a_2|^2}{|\lambda^{p-1} - 1|} = \frac{|a_2|^p}{|\lambda^{p-1}||\lambda^{p-2} - 1||\lambda^{p-3} - 1| \cdots |1 - \lambda|}. \]  

(63)

Note the lack of the factor \(|\lambda^{p-1} - 1|\). Now, since according to Lemma 3.11

\[ |1 - \lambda^p| < |1 - \lambda| = |1 - \lambda^{p-1}|, \]

we have that

\[ |1 - \lambda^p| \prod_{j=1}^{p-2} |1 - \lambda^j| < |1 - \lambda^{p-1}| \prod_{j=1}^{p-2} |1 - \lambda^j|. \]

Therefore we have for \( k = p + 1 \) that the \( b_{k-1} \)-term is again strictly greater than all the others in the right hand side of (18) so that

\[ |b_{p+2}| = \frac{|b_{p+1}| |a_2|}{|\lambda^{p+1} - 1|} = \frac{|a_2|^{p+1}}{|1 - \lambda^{p+1}||1 - \lambda^p||1 - \lambda^{p-2}||1 - \lambda^{p-3}| \cdots |1 - \lambda|}. \]

The \( b_{k-1} \)-term will dominate until \( p \) divides \( k - 1 \) again, i.e. for \( k = 2p + 1 \) (which means that we “loose” the factor \(|\lambda^{2p-1} - 1|\)). Repeated application of these arguments yields that

\[ |b_k| = \frac{|a_2|^{k-1} \prod_{i=1, p \leq k-1} |1 - \lambda^i|^{-1}}{\prod_{n=1}^{k-1} |1 - \lambda^n|}. \]  

(64)
Note that $|1 - \lambda| = |1 - \lambda|$, since $m = 1$ in this case. Hence, we obtain (57) as required.

It remains to prove that that the corresponding linearization disc is the open disc $D_{\tau}(0)$. Recall that the estimates for the $b_k$'s in the previous sections where based on the estimate (17). Moreover,

$$\left(1 - \lambda\left|\frac{1}{p-1}\right|\right)^{1/k} \to |1 - \lambda|^{1/k}, \quad k \to \infty.$$ 

This suggests that $g$ converges on an open disc of radius

$$\tau = |1 - \lambda|^{-\frac{1}{p-1}} \sigma.$$ 

In fact, $g$ diverges on the sphere of radius $\tau$; let $I \geq s + 1$ be an integer, then by Lemma 4.1, 4.2, and 4.3 and (57) we have

$$|b_{p^I+1}|^{1/p^I+1} = p^{-\frac{1}{p(I+1)}} |1 - \lambda|^{-\frac{1}{p-1}} \sigma = p^{-\frac{1}{p-1}} \tau,$$

which does not approach zero as $I$ goes to infinity. Furthermore, in a similar way, applying Lemma 4.1, 4.2, and 4.3 to the identity (57) we obtain

$$|b_k|^{1/k} \leq p^{-\frac{1}{p-1}} \tau.$$ 

Consequently, $g$ is one-to-one on $D_{\tau}(0)$. It follows that the linearization disc of the quadratic polynomial $f$ is the disc $\Delta_f = D_{\tau}(0)$. 

The theorem implies, in particular, that $f$ can have no periodic points (except the fixed point at the origin) in the disc $D_{\tau}(0)$. However, there may be periodic points on the boundary. We will give sufficient conditions that there is a fixed point on the boundary $S_{\tau}(0)$. Note that $f$ has a fixed point $\hat{x} = (1 - \lambda)/a_2$. Solving the equation

$$\tau = |1 - \lambda|/|a_2|,$$

yields that $\hat{x}$ is located on $S_{\tau}(0)$ if $s = 1$ and $\alpha = 1$ and $|1 - \lambda| = p^{-1/2(p-1)}$, or if $s \geq 2$ and $|1 - \lambda| = p^{-t(s)}$, where $t(s) = \lfloor(s-1)p^{s-1}(p-1)/p^{s-1} \rfloor$. Furthermore, $\hat{x}$ cannot be located on $S_{\tau}(0)$ if $s = 0$; the only solution to (66) for $s = 0$ is $|1 - \lambda| = p^{-p/(p-1)}$, and hence, $\lambda$ does not belong to the annulus \( \{ z : p^{-1} < |1 - z| < 1 \} \).

As in [44, Corollary 2.1] the previous result on quadratic polynomials works also for power series containing a dominating quadratic term in the following sense.

**Theorem 5.2.** Let $p$ be an odd prime and let $\lambda \in \mathbb{C}_p$, not a root of unity, belong to the annulus \( \{ z : p^{-1} < |1 - z| < 1 \} \). Let $f \in \mathbb{C}_p[[x]]$ be a power series of the form

$$f(x) = \lambda x + a_2 x^2 + \sum_{i \geq 3} a_i x^i,$$
where
\[ |1 - \lambda^{1/p}|a_2| > 1, \quad |1 - \lambda^{1/p}|a_2| > |a_i|, \quad i \geq 3. \] (67)

Then, the coefficients of the conjugacy \( g \) satisfy (67). Moreover, the linearization disc about the origin, \( \Delta_f = D_\tau(0) \), where \( \tau = |1 - \lambda|^{-1/p} \sigma \).

Proof. By the condition (67), the same terms as in the proof of Theorem 5.1 will be strictly larger than all the others in (18). The reason for the factor \( |1 - \lambda|^{1/p} \), is the lack of the factor \( |1 - \lambda^{p-1}| \) in (63).

Corollary 5.1. Let \( f \) be a power series satisfying the conditions of Theorem 5.2, then the radius of the linearization disc
\[ \tau < p^{-1/p-1} a^{-1}. \]

In particular, the linearization disc cannot contain the disc \( D_{1/a}(0) \cap K \) for any algebraic extension \( K \) of \( \mathbb{Q}_p \).

6 Minimality and ergodicity

6.1 Minimality and conjugation in non-Archimedean fields

In this section we consider power series defined over an arbitrary complete non-Archimedean field \( K \), rather than just over \( \mathbb{C}_p \). The notion of transitivity and minimality on subsets of \( K \) are defined as follows. Let \( X \) be a subset of \( K \) and let \( f \in K[[x]] \) be a power series which converges on \( X \). Suppose that \( X \) is invariant under \( f \), i.e. \( f(X) \subseteq X \). The map \( f : X \to X \) is said to be transitive if there is an element \( x \in X \), such that its forward orbit \( \{f^n(x)\}_{n=0}^\infty \) is dense in \( X \). We say that \( f : X \to X \) is minimal on \( X \) if for every \( x \in X \), its forward orbit \( \{f^n(x)\}_{n=0}^\infty \) is dense in \( X \).

We will look for dense orbits near indifferent non-resonant fixed points of \( f \). By Lemma 3.4, the dynamics on a linearization disc \( \Delta_f(x_0, K) \) is located on invariant spheres, about the fixed point \( x_0 \). As in previous sections we will assume (without loss of generality) that \( f \) has an indifferent fixed point at the origin. Given \( \lambda \in K \), let \( T_\lambda : K \to K \) be the multiplication map, \( x \mapsto \lambda x \). We will prove that transitivity of \( f(x) = \lambda x + O(x^2) \in K[[x]] \), on some subset \( X \), of the corresponding linearization disc \( \Delta_f \) about the origin, is equivalent to transitivity of \( T_\lambda \) on \( g(X) \). Moreover, transitivity and minimality are equivalent if \( X \) is compact.

Theorem 6.1 (Transitivity is preserved under analytic conjugation). Let \( f(x) = \lambda x + O(x^2) \in K[[x]] \) be analytically conjugate to \( T_\lambda \), on a linearization disc \( \Delta_f \) about the origin, via a conjugacy function \( g \), such that \( g(0) = 0 \) and \( g'(0) = 1 \). Suppose that the subset \( X \subseteq \Delta_f \) is invariant under \( f \). Then, the following statements hold:

1) \( f \) is transitive on \( X \) if and only if \( T_\lambda \) is transitive on \( g(X) \).
2) If $X$ is compact and $f$ is transitive on $X$, then $f$ is minimal on $X$. Moreover, $f(X) = X$ and $g(X) = T_\lambda(g(X))$.

Proof. Let $f$ be analytically conjugate to $T_\lambda$, with conjugacy function $g$. Suppose that $X \subseteq \Delta_f$ is invariant under $f$. By the conjugacy relation we must have $g(f(X)) = T_\lambda(g(X))$. It follows that $T_\lambda(g(X)) \subseteq g(X)$. In other words, $g(X)$ is invariant under $T_\lambda$.

Now, suppose that $T_\lambda$ is transitive on $g(X)$. Then, there is an $x \in X$ such that the orbit \(\{T_\lambda^n(x)\}\) is dense in $g(X)$. Recall that by Lemma 3.4, $g : X \to g(X)$ is bijective and isometric. Hence, given $\epsilon > 0$ and $y \in X$, there is an integer $n \geq 1$ such that in view of the conjugacy relation

\[
\epsilon > |T_\lambda^n \circ g(x) - g(y)| = |g^{-1} \circ T_\lambda^n \circ g(x) - g^{-1} \circ g(y)| = |f_\lambda^n(x) - y|.
\]

It follows that the orbit \(\{f_\lambda^n(x)\}\) is dense in $X$. Accordingly, $f$ is transitive on $X$. Likewise, transitivity of $f$ implies transitivity of $T_\lambda$.

Now we consider the second statement of the theorem. Suppose that the subset $X \subset \Delta_f$ is compact, and that $f : X \to X$ is transitive so that $T_\lambda$ is transitive on $g(X)$. In view of the conjugacy relation we have $g(f_\lambda^n(x)) = T_\lambda^n(g(x))$. Hence, transitivity of $T_\lambda$ implies that $g(X)$ is dense in $T_\lambda(g(X))$. Continuity of $g$ and $T_\lambda$, and compactness of $X$, gives $g(X) = T_\lambda(g(X))$. Consequently, $f(X) = g^{-1} \circ T_\lambda \circ g(X) = X$. Hence, $f : X \to X$ is not only one-to-one and isometric but also surjective.

It is well known that a transitive bijective isometry is minimal, see e.g. [64]. Accordingly, minimality and transitivity are equivalent on compact subsets of non-Archimedean linearization discs. This completes the proof. \(\square\)

**Remark 6.1.** By Lemma 3.4, $f : \Delta_f \to \Delta_f$ is also an isometry. Therefore, if $f$ is minimal on some subset $X \subseteq \Delta_f$ and $X \neq \{0\}$, then $X \subseteq S$ for some sphere $S \subset \Delta_f$.

### 6.2 Minimality in $\mathbb{Q}_p$

It follows from the previous section that transitivity, and hence minimality, on spheres about an indifferent fixed point can be characterized in terms of the multiplier map. By Lemma 3.9, the multiplier map $T_\lambda$ is transitive on a sphere about the origin in $\mathbb{Q}_p$ if and only if $\lambda$ is maximal and $p$ is odd. Moreover, in view of Theorem 6.1, minimality and transitivity of $T_\lambda$ coincide, as proven earlier by various authors [17, 21, 28, 51] in the $p$-adic setting.

Recall that, if $\lambda$ is maximal, then the corresponding linearization disc is maximal as stated in Corollary 4.1.

**Theorem 6.2.** Let $f \in F_{\lambda,a} \cap \mathbb{Q}_p[[x]]$ for some prime $p$. Then, $f$ is minimal on each sphere $S \subset \mathbb{Q}_p$ of radius $r < 1/a$ about the origin, if and only if $\lambda$ is maximal and $p$ is odd. Moreover, if $\lambda$ is maximal and $p$ is odd, then $g(S) = f(S) = S$.  

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Proof. Suppose that \( f \) is minimal on a rational sphere \( S \subset \Delta_f(0, \mathbb{Q}_p) \) about the origin. In view of Theorem 6.1 \( T_\lambda \) is minimal on \( g(S) \). Hence, by the conjugacy relation, \( f^{o_n}(g^{-1}(x)) = g^{-1}(\lambda^n(x)) \), the image \( g^{-1}(S) \) is dense in \( S \). By continuity of \( g^{-1} \) and compactness of \( S \), \( g^{-1}(S) = S \). Accordingly, \( g(S) = S \). It follows that \( T_\lambda \) is minimal on \( S \). Consequently, \( \lambda \) is maximal.

On the other hand, suppose that \( \lambda \) is maximal. Then, \( T_\lambda \) is minimal on each sphere \( S_r(0) \subset \mathbb{Q}_p \). In view of Corollary 4.1 the semi-conjugacy relation \( g(f^{o_n}(x)) = \lambda^n g(x) \), holds for all \( x \in \Delta_f(0, \mathbb{Q}_p) \supseteq D_{1/a}(0) \). By similar arguments as above, we conclude that \( g(S_r(0)) = S_r(0) \) if \( r < 1/a \). It follows that \( T_\lambda \) is minimal on \( g(S_r(0)) \) for \( r < 1/a \). Consequently, \( f \in \mathcal{F}_{\lambda,a} \cap \mathbb{Q}_p[[x]] \) is minimal on \( S_r(0) \) for each \( r < 1/a \).

\[ \square \]

As shown in Section 3.2 \( f : \Delta_f \rightarrow \Delta_f \) is not only one-to-one, but also surjective, in the algebraic closure \( \mathbb{C}_p \). By Theorem 6.2 \( f \) may also be surjective on \( \Delta_f(\mathbb{Q}_p) = \Delta_f \cap \mathbb{Q}_p \).

**Corollary 6.1.** Let \( f \in \mathcal{F}_{\lambda,a} \cap \mathbb{Q}_p[[x]] \) for some odd prime \( p \). Suppose that \( \lambda \) is maximal, then \( f : \Delta_f(0, \mathbb{Q}_p) \rightarrow \Delta_f(0, \mathbb{Q}_p) \) is bijective.

### 6.3 Unique ergodicity

Let \( K \) be a complete non-Archimedean field. In the following \( X \) will be a compact subset of \( K \). For example, if \( K = \mathbb{C}_p \), \( X \) could be any disc or any sphere in a finite extension of \( \mathbb{Q}_p \).

A continuous map \( T : X \rightarrow X \) is uniquely ergodic if there exists only one probability measure \( \mu \), defined on the Borel \( \sigma \)-algebra \( \mathcal{B}(X) \) of \( X \), such that \( T \) is measure-preserving (i.e. \( \mu(A) = \mu(T^{-1}(A)) \) for all \( A \in \mathcal{B}(X) \)) with respect to \( \mu \). As \( \mu \) is unique it follows that \( T \) must be ergodic with respect to \( \mu \). Recall that \( T \) is ergodic if for any \( A \in \mathcal{B} \), whenever \( T^{-1}(A) = A \), then \( \mu(A)\mu(A^c) = 0 \).

(Here \( A^c \) denotes the complement of \( A \).)

As shown by Oxtoby [52], a bijective isometry of a compact metric space is uniquely ergodic, hence ergodic, if and only if it is minimal. See Bryk and Silva [17] for a shorter proof in the \( p \)-adic setting.

In view of Lemma 3.3 and Theorem 6.1 \( f(x) \in \lambda x + O(x^2) \in K[[x]] \) is certainly bijective and isometric on compact invariant subsets of a linearization disc \( \Delta_f \) about the origin in \( K \). Consequently, transitivity, minimality, ergodicity and unique ergodicity are all equivalent and preserved under analytical conjugation on compact subsets of \( \Delta_f \).

**Theorem 6.3.** Let \( K \) be a complete non-Archimedean field. Let the power series \( f(x) = \lambda x + O(x^2) \in K[[x]] \) be analytically conjugate to \( T_\lambda \), on a linearization disc \( \Delta_f \) about the origin, via a conjugacy function \( g \), such that \( g(0) = 0 \) and \( g'(0) = 1 \). Suppose that the subset \( X \subset \Delta_f \) is non-empty, compact and invariant under \( f \). The following statements are equivalent:

1. \( T_\lambda : g(X) \rightarrow g(X) \) is minimal.
2. $f : X \rightarrow X$ is minimal.

3. $f : X \rightarrow X$ is uniquely ergodic.

4. $f$ is ergodic for any $f$-invariant measure $\mu$ on $B(X)$ that is positive on non-empty open sets.

Now, we return to the $p$-adic case. Note that a rational sphere $S \subset \mathbb{C}_p$ is not compact since $\mathbb{C}_p$ is not locally compact. (Recall that a sphere $S \subset K$ is rational if and only if it is non-empty, i.e. the radius is a number in the value group $|K^*|$.) However, in $\mathbb{Q}_p$ (or any finite extension of $\mathbb{Q}_p$), every rational sphere is compact. By Theorem 6.2 we have the following result.

**Theorem 6.4.** Let $f \in \mathcal{F}_{\lambda,a} \cap \mathbb{Q}_p[[x]]$ for some odd prime $p$. Let $S \subset \mathbb{Q}_p$ be a rational sphere of radius $r < 1/a$ about the origin. Then, the following statements are equivalent:

1. $\lambda$ is maximal.
2. $f : S \rightarrow S$ is minimal.
3. $f : S \rightarrow S$ is uniquely ergodic.
4. $f$ is ergodic for any $f$-invariant measure $\mu$ on $B(S)$ that is positive on non-empty open sets.

As proven in [17, 21, 51], in this case the unique invariant measure $\mu$ is the normalized Haar measure for which the measure of a disc is equal to the radius of the disc.

Note that the estimate of the radius $1/a$ is maximal in the sense that there exist examples of such $f$ which diverges or have a zero on the sphere $S_{1/a}(0)$, see Lemma 3.3 and Corollary 4.1.

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