CLASS NUMBER AND REGULATOR COMPUTATION IN
PURELY CUBIC FUNCTION FIELDS OF UNIT RANK TWO

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ABSTRACT. We describe and give computational results of a procedure to compute the divisor class number and regulator of most purely cubic function fields of unit rank 2. Our implementation is an improvement to Pollard's Kangaroo method in infrastructures, using distribution results of class numbers as well as information on the congruence class of the divisor class number, and an adaptation that efficiently navigates these torus-shaped infrastructures. Moreover, this is the first time that an efficient “square-root” algorithm has been applied to the infrastructure of a global field of unit rank 2. With the exception of certain function fields defined by Picard curves, our examples are the largest known divisor class numbers and regulators ever computed for a function field of genus 3.

1. INTRODUCTION AND MOTIVATION

One of the more difficult problems in arithmetic geometry is the computation of the divisor class number of an algebraic curve over a finite field. In this paper, we give results on the application and optimization of a method of Scheidler and Stein [21, 22], combined with modifications to Pollard’s Kangaroo algorithm [15], to compute the exponent of the infrastructure of a purely cubic function field with complete splitting at infinity (i.e., unit rank 2) over a large base field. The regulator and the divisor class number are multiples of this exponent, and in many cases all three numbers are the same, whence our algorithm computes the regulator and the divisor class number in these cases as well. Our method greatly improves upon the method described in [12] to compute the regulator in this setting and is the first ever treatment of an efficient “square-root” algorithm in a two-dimensional infrastructure of a global field.

An algorithm due to Stein and Williams [30] uses techniques of Lenstra [13] and Schoof [24] to compute the divisor class number and regulator of a real quadratic function field in \( O(q^{(2g-1)/5} + \epsilon(g)) \) infrastructure operations, \( \epsilon \) where \(-1/4 \leq \epsilon(g) \leq 1/2\). This method was improved by Stein and Teske [27, 28, 29], who applied the Kangaroo algorithm to compute the 29-digit class number and regulator of a real quadratic function field of genus 3.

The algorithm of [30] was generalized to cubic and arbitrary function fields in [21, 22], respectively, and implemented in purely cubic function fields of unit rank 0 and 1 in [11]. In this paper, we provide an implementation and numerical examples for purely cubic function fields of unit rank 2. Our method is applied to compute divisor class numbers and regulators of up to 31 digits of function fields of genus 3. With the exception of the 55-digit class numbers computed by Bauer, Teske, and

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1Throughout this paper \([r]\) will denote the nearest integer to \(r \in \mathbb{R}\).
Weng in [4] [25] for cubic function function fields generated by Picard curves, our examples are the largest known class numbers and regulators ever computed for a function field of genus at least 3 over a large base field.

The remainder of this paper is organized as follows. We first give an overview of cubic function fields and their infrastructure. Then we outline our version of Pollard’s Kangaroo method for infrastructures, and explain how to recover the regulator and divisor class number in most cases. Finally, we discuss details of our implementation and provide numerical results.

2. Cubic Function Fields

For a general introduction to function fields, we direct the reader to [10, 31, 16]. Explicit details of purely cubic function fields and their arithmetic can be found in [20, 18, 19, 3, 21]. Let \( \mathbb{F}_q \) be a finite field and \( \mathbb{F}_q(x) \) the field of rational functions in \( x \) over \( \mathbb{F}_q \). Throughout this paper, we assume that \( \text{char}(\mathbb{F}_q) \geq 5 \). A cubic function field is a separable extension \( K/\mathbb{F}_q(x) \) of degree 3; we denote by \( g \) the genus of \( K \). A function field is purely cubic if it is of the form \( K = \mathbb{F}_q(x, y) \) where \( y^3 = F \) for some cube-free \( F \in \mathbb{F}_q[x] \).

2.1. Divisors and Ideals. Let \( \mathcal{D} \) denote the group of divisors of \( K \) defined over \( \mathbb{F}_q \), \( \mathcal{D}_0 \) the subgroup of divisors of degree 0 defined over \( \mathbb{F}_q \), and \( \mathcal{P} \) the subgroup of principal divisors defined over \( \mathbb{F}_q \). Then the (degree 0) divisor class group of \( K \) is the quotient group \( \mathcal{J} = \mathcal{D}_0/\mathcal{P} \) and its order \( h = |\mathcal{J}| \) is the (degree 0) divisor class number of \( K \). Let \( S \) be the set of places of \( K \) lying above the place at infinity of \( \mathbb{F}_q(x) \), \( \text{supp}(D) \) the support of \( D \in \mathcal{D} \), \( \mathcal{D}_0^S = \{ D \in \mathcal{D}_0 \mid \text{supp}(D) \subseteq S \} \), and \( \mathcal{P}^S = \mathcal{P} \cap \mathcal{D}_0^S \). Then the order \( R \) of the quotient group \( \mathcal{D}_0^S/\mathcal{P}^S \) is the \( (S-)\text{-regulator} \) of \( K \). Finally, let \( \mathcal{D}_S = \{ D \in \mathcal{D} \mid \text{supp}(D) \cap S = \emptyset \} \) and \( \mathcal{P}_S = \mathcal{P} \cap \mathcal{D}_S \). Then every \( D \in \mathcal{D} \) can be uniquely written in the form \( D = D_S + D^S \) with \( D_S \in \mathcal{D}_S \) and \( D^S \in \mathcal{D}^S \).

The maximal order of \( K/\mathbb{F}_q(x) \) is the integral closure of \( \mathbb{F}_q[x] \) in \( K \) and is denoted \( \mathfrak{O} \). Let \( \mathcal{I} \) denote the group of non-zero fractional ideals of \( \mathfrak{O} \) and \( \mathcal{H} \) the subgroup of non-zero principal fractional ideals. The ideal class group of \( K \) is the quotient group \( \text{Cl}(\mathfrak{O}) = \mathcal{I}/\mathcal{H} \), and its order \( h_\mathfrak{O} = |\text{Cl}(\mathfrak{O})| \) is called the ideal class number of \( K \). Let \( f \) be the greatest common divisor of the degrees of all the places in \( S \). By Schmidt [23] (see also Proposition 14.1 of [16]) there is an exact sequence

\[
(0) \rightarrow \mathcal{D}_0^S/\mathcal{P}^S \rightarrow \mathcal{J} \rightarrow \text{Cl}(\mathfrak{O}) \rightarrow \mathbb{Z}/f\mathbb{Z} \rightarrow (0),
\]

so that \( fh = h_\mathfrak{O}R \).

There is a well-known isomorphism \( \Phi : \mathcal{D}_S \rightarrow \mathcal{I} \) given by \( D \mapsto \{ \alpha \in K^* \mid \text{div}(\alpha)_S \geq -D \} \) with inverse \( \mathcal{I} \mapsto -\sum_{p \in S} m_p \mathfrak{p} \), where \( \mathfrak{p} \) denotes any finite place of \( K \), \( m_p = \min\{ v_\mathfrak{p}(\alpha) \mid 0 \neq \alpha \in \mathcal{I} \} \), and \( v_\mathfrak{p} \) is the normalized discrete valuation corresponding to \( \mathfrak{p} \). Moreover, \( \Phi \) induces an isomorphism from \( \mathcal{D}_S/\mathcal{P}_S \) to \( \text{Cl}(\mathfrak{O}) \). If \( S \) contains an infinite place \( \infty_0 \) of degree 1, then \( \Phi \) can be extended to an isomorphism

\[
\Psi : \{ D \in \mathcal{D}_0 \mid v_\mathfrak{p}(D) = 0 \text{ for all } \mathfrak{p} \in S \setminus \{ \infty_0 \} \} \rightarrow \mathcal{I}
\]

by \( \Psi(D_S - \text{deg}(D_S)\infty_0) = \Phi(D_S) \), with the inverse given by \( \Psi^{-1}(f) = \Phi^{-1}(f) + \text{deg}(N_{K/\mathbb{F}_q}(f))\infty_0 \).
2.2. Units. By Proposition 14.1 and its Corollary 1 of [10], \( \mathcal{O}^* / \mathcal{F}_q^* \cong \mathcal{P}^S \) is a free abelian group of rank \( r = |S| - 1 \). We write \( S = \{ \infty_0, \ldots, \infty_r \} \) to denote the infinite places of \( K \), with \( v_i \) the normalized discrete valuation corresponding to \( \infty_i \), for \( 0 \leq i \leq r \). A set of generators of the free part of \( \mathcal{O}^* \) is called a system of fundamental units of \( \mathcal{O} \) and we write \( \{ \epsilon_1, \ldots, \epsilon_r \} \) for a given system of fundamental units.

We now restrict to the case \( r = 2 \), i.e., unit rank 2. Given any system of fundamental units \( \{ \epsilon_1, \epsilon_2 \} \), consider the \( 2 \times 2 \) matrix \( M = (v_i(\epsilon_j))_{1 \leq i, j \leq 2} \). If we transform \( M \) into Hermite Normal Form, then the resulting matrix entries correspond to valuations of another system of fundamental units, \( \{ \eta_1, \eta_2 \} \). This system is independent of the original system, and is unique up to constants in \( \mathbb{F}_q^* \). Furthermore, \( D_0^S = (\infty_1 - \infty_0, \infty_2 - \infty_0) \) and \( R = |D_0^S / \mathcal{P}^S| = \det(M) = v_1(\eta_1)v_2(\eta_2) \).

For the remainder of this paper, we assume that \( S \) contains an infinite place \( \infty_0 \) of degree 1, so that \( f = 1 \) and \( h = h_\infty R \). In this case, \( h_\infty \) is generally very small, so we operate in another set of ideals called the infrastructure of \( K \). To that end, we require the notion of a distinguished divisor and ideal.

2.3. Distinguished Divisors and Infrastructure. Let \( K \) be a cubic function field with an infinite place \( \infty_0 \) of degree 1 and maximal order \( \mathcal{O} \). A divisor \( D \) of \( K \) is said to be finitely effective if \( D_\infty \geq 0 \); that is, \( v_p(D) \geq 0 \) for all finite places \( p \) of \( K \). Following [3] [9] [11], a finitely effective divisor \( D \) is defined to be distinguished if

1. \( D \) is of the form \( D = D_\infty - \deg(D_\infty)\infty_0 \), and
2. \( E \) is any finitely effective divisor equivalent to \( D \) with \( \deg(E_\infty) \leq \deg(D_\infty) \) and \( E_\infty \geq D_\infty \), then \( D = E \).

A fractional ideal \( f \) of \( \mathcal{O} \) is said to be distinguished if \( \Psi^{-1}(f) \) is a distinguished divisor. Note that distinguished ideals are called reduced in [17] [20] [18] [12] [6].

A general treatment of infrastructures in function field extensions of arbitrary degree can be found in [6] [7]. The cubic scenario was first presented in [20] [18] [12], and we use a description based on [11] here.

By [11] Lemma 3.3.12 and Theorem 3.3.16, if \( K \) is a cubic function field with an infinite place of degree 1, then every divisor class contains at most one distinguished divisor. (In fact, almost all divisor classes contain a distinguished divisor; see [8].) This gives rise to the following definition. The (finite) set

\[ \mathcal{R} := \{ f \in \mathcal{H} \mid f \text{ is distinguished} \} \]

is the (principal) infrastructure of \( \mathcal{O} \) (or of \( K \)). While we use an ideal-theoretic definition of \( \mathcal{R} \) here, the isomorphism \( \Psi \) can be used to translate this into divisor-theoretic language. In particular, \( \mathcal{R} \) is in one-to-one correspondence with the set of distinguished representatives of the kernel of the map \( \mathcal{J} \to Cl(\mathcal{O}) \).

Henceforth, we will restrict to the case \( r = 2 \). We consider the lattice \( \Lambda := \langle (v_1(\eta_1), 0), (v_1(\eta_2), v_2(\eta_2)) \rangle \subseteq \mathbb{Z}^2 \). If \( f \in \mathcal{R} \), then there is a function \( \alpha \in K^* \) such that \( f^\ast = (\alpha^{-1})^\ast \). The coset \( (v_1(\alpha), v_2(\alpha)) + \Lambda \) is uniquely determined by \( f \). We define the distance of \( f \) to be \( \delta(f) := (\delta_1(f), \delta_2(f)) + \Lambda := (v_1(\alpha), v_2(\alpha)) + \Lambda \). Since \( \delta : \mathcal{R} \to \mathbb{Z}^2 / \Lambda \) is injective, \( \mathcal{R} \) can be thought of as a subset of \( \mathbb{Z}^2 / \Lambda \). In other words, \( \mathcal{R} \) is structured as discrete points on the surface of a torus.

In practice, we do not know \( \Lambda \), and finding \( \delta(f) \) given only \( f \) is computationally infeasible. We therefore define the (extended principal) infrastructure as

\[ \mathcal{R} := \{ (f, v) \in \mathcal{R} \times \mathbb{Z}^2 \mid \delta(f) = v + \Lambda \} \]
For \( a = (f, v) \in \mathcal{R} \), call \( \delta(a) := (\delta_1(a), \delta_2(a)) := v \) the \textit{distance} and \( \text{id}(a) := f \) the \textit{ideal part} of \( a \). Finally, we will call \( v_1(\eta_1) = \exp(D_{\mathbb{R}}^S/P^S) \) the \textit{exponent} of \( \mathcal{R} \) and denote it \( \exp(\mathcal{R}) \). Since \( R = v_1(\eta_1)v_2(\eta_2) \), we have \( \exp(\mathcal{R}) \mid R \).

### 2.4. Infrastructure Arithmetic

Infrastructures have two main operations: the \textit{baby step} and \textit{giant step} operations. Roughly speaking, a baby step maps an infrastructure element to another element close to it, in terms of distance, while a giant step reduces the product of two distinguished ideals. We will also describe a third operation called the \textit{below} operation, which finds an infrastructure element of (or close to and just below) a given distance. Moreover, these operations can be computed efficiently; for full details and proofs of this arithmetic in purely cubic function fields, we refer the reader to \([20, 18, 12, 3, 11]\).

In unit rank 2 infrastructures, there are three types of baby steps as follows. Let \( f \) be a distinguished ideal of \( \mathcal{O} \) and denote it \( \exp(\mathcal{O}) \) and \( \text{id}(f) \), then the \textit{baby step} \( f \mapsto v(f) \), and we write \( v(f) = (v_1(\phi), v_2(\phi)) \). Finally, we will call \( v(f) \) the \textit{fractional ideal}.

### Theorem 1 (Theorem 3.4 of [12])

Let \( \mathcal{O} \) be the maximal order of a purely cubic function field \( K \) of unit rank 2 and \( f \) a distinguished fractional ideal of \( \mathcal{O} \). For any \( i \in \{0, 1, 2\} \), there exists an element \( \phi = \phi_i(f) \in \mathcal{H}_i(f) \), unique up to a factor in \( \mathbb{F}_q^* \), such that \( \phi \geq_i \alpha \) for all \( \alpha \in \mathcal{H}_i(f) \). Furthermore, \( \langle \phi^{-1} \rangle f \) is also a distinguished fractional ideal.

Let \( i \in \{0, 1, 2\} \). If \( a = (f, v) \in \mathcal{R} \), \( \phi = \phi_i(f) \), and \( g = \langle \phi^{-1} \rangle f \in \tilde{\mathcal{R}} \), then the operation \( a \mapsto b := (g, v + (v_1(\phi), v_2(\phi))) \) is called a \textit{baby step (in the i-direction)}, and we write \( b_{\text{bs}}(a) = b \). See also Figure 1 on how baby steps behave with high probability.

The giant step operation is analogous to multiplication. If \( a_1 = (f_1, v_1) \), \( a_2 = (f_2, v_2) \in \mathcal{R} \), then \( f_1f_2 \) is generally not distinguished. However, by [11] Theorem 5.3.17, there is a function \( \psi \in K^* \) such that \( v_i(\psi) \geq 0 \), for each \( i = 0, 1, 2 \), and \( v_0(\psi) + v_1(\psi) + v_2(\psi) \leq 2q \), yielding \( a_1 * a_2 := (\langle \psi^{-1} \rangle f_1f_2, v_1 + v_2 + (v_1(\psi), v_2(\psi))) \in \mathcal{R} \). Thus, \( \delta(a_1 * a_2) = \delta(a_1) + \delta(a_2) + (v_1(\psi), v_2(\psi)) \), so that \( \delta(a_1 * a_2) \geq \delta(a_1) + \delta(a_2) \). We call \( * \) the \textit{giant step} operation. Under \( * \), \( \mathcal{R} \) is an abelian group-like structure, failing only associativity, and by [8], existence of inverses for very few elements.

A third required operation is the computation of the infrastructure element \textit{below} any ordered pair \((a, b)\) of integers \(a, b \in \mathbb{N}\). This is the unique element \( B(a, b) := a \in \mathcal{R} \) such that \( \delta(a) = (a - i, b) \) with \( i \geq 0 \) minimal. From [8], it follows that \( \delta(B(a, b)) = (a, b) \) with probability \( 1 - O(1/q) \).

Navigating \( \mathcal{R} \) is not as straightforward as the cyclic infrastructures of fields of unit rank 1. This is due to the existence of “hidden” elements and “holes”. An element in \( \mathcal{R} \) is \textit{hidden} if it cannot be reached via baby steps. A \textit{hole} \( d \in \mathbb{Z}^2 \) is an element that does not lie in the image of the distance map \( \delta \), i.e., there exists no element \( a \in \mathcal{R} \) with \( \delta(a) = d \). By [8], the probability of encountering a hidden
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Figure 1. The Typical Baby Step Behaviors According to Proposition 1 (1)

Figure 2. Algorithm 1 – The Most Common Scenario

Invoke Algorithm 1 here.

The majority of our computations will take place in the subset $R_0 := \{ a \in R \mid \delta_2(a) = 0 \}$ of $R$. As such, if we encounter an element $b \notin R_0$, either via a baby step or a giant step, then we must find an element in $R_0$ close to $b$. Algorithm 1 finds such an element $a \in R_0$ with overwhelming probability and is based on Theorem 1. The idea of the algorithm is to first find an element of non-negative 2-distance. If the element has positive 2-distance at this point, then we expect that a step in the 0-direction followed by a series of steps in the 2-direction produces an element $a \in R_0$ with $\delta_1(a) \geq \delta_1(b)$ and small $\delta_2(a) \geq 0$. If at this point, $\delta_2(a) \neq 0$, we repeat the process until an element $a \in R_0$ is found. Since the number of holes is very small by [8], one iteration almost always suffices.

Figure 2 illustrates the most common scenario in which Algorithm 1 is used, namely when we encounter a hole in $R$ when taking a baby step in the 0-direction. This baby step generally results in an element with a 2-distance of 1.

3. The Kangaroo Method in $R$

If we are given integers $E, U \in \mathbb{N}$ such that the divisor class number $h \in (E - U, E + U)$, then the Kangaroo method may be optimized to compute a multiple

Proposition 1. Let $K/F_q(x)$ be a cubic function field of genus $g$.

1. If $a \in R$, then with probability $1 - O(1/q)$, we have

$$\delta(bs_i(a)) - \delta(a) = \begin{cases} (1, 0) & \text{if } i = 0, \\ (-1, 1) & \text{if } i = 1, \\ (0, -1) & \text{if } i = 2. \end{cases}$$

2. If $a_1 = (f, v_1), a_2 = (f, v_2) \in R$ and $a_1 \ast a_2 = (\langle \psi^{-1} \rangle f_1 f_2, v_3)$, so that $\delta(a_1 \ast a_2) = \delta(a_1) + \delta(a_2) + (v_1(\psi), v_2(\psi))$, then with probability $1 - O(1/q)$, we have

$$\hat{\psi}(g) := v_1(\psi) = v_2(\psi) = \begin{cases} \lfloor g/3 \rfloor & \text{if } g \equiv 2 \pmod{3}, \\ (g + 1)/3 & \text{if } g \equiv 2 \pmod{3}. \end{cases}$$

The proof follows from [11] using [8]. The first statement is visualized in Figure 1.
Later, we will show how to determine the regulator unit rank 2. After our description, we will optimize its running time in Theorem 2.

Important improvements that apply in particular to operating in infrastructures of parallelized Kangaroo method of van Oorschot and Wiener [34, 29] and explain Kangaroo method is preferred for larger computations because it requires very little storage and can be parallelized efficiently. Specifically, we will describe the parallelized Kangaroo method of van Oorschot and Wiener [34, 29] and explain important improvements that apply in particular to operating in infrastructures of unit rank 2. After our description, we will optimize its running time in Theorem 2. Later, we will show how to determine the regulator $R$ and the divisor class number $h$ from $\exp(R)$ in many cases.

There are two key elements to adapting the Kangaroo algorithm to infrastructures of unit rank 2 function fields. Firstly, the units of $\mathcal{O}$ correspond to elements $(\mathcal{O}, v) \in \mathcal{R}$, where $v \in \Lambda$. Secondly, there exists a unit $\epsilon \in \mathcal{O}^*$ such that $(v_1(\epsilon), v_2(\epsilon)) = (h, 0)$; $\epsilon = \eta_1 i$, for some $i \in \mathbb{N}$. Therefore, we restrict our search to elements $a$ with $\delta_2(a) = 0$, i.e., we operate in $\mathcal{R}_0 \subseteq \mathcal{R}$. In Figure 3 we illustrate the Kangaroo method in our setting. The $v_1$ and $v_2$ axes are labeled to give a reference for distance. The black dots correspond to units, with $\eta_1$, $\eta_2$, and $\epsilon$ labeled. The infrastructure $\mathcal{R}$ is the gray parallelogram on the left, with $\mathcal{R}_0$ the thick line at its base. Copies of $\mathcal{R}$ tile $\mathcal{R}$ in the $v_1v_2$-plane. A sample interval $(E - U, E + U)$ is shown in the top figure, highlighted in gray, containing the unit $\epsilon$ at $(h, 0)$. This interval is then expanded in the bottom figure to show how the Kangaroo method proceeds. Infrastructure elements (kangaroos) are initialized at $(0, 0)$ and $(h, 0)$, which then jump, via baby steps and giant steps, along the $v_1$-axis until their paths merge. The jumps are represented by the arcs. Once these paths merge, a multiple of $\exp(\mathcal{R})$ can be determined, and from that we can determine $\exp(\mathcal{R})$ itself. This often makes it possible to determine $R$ and $h$ as well; in fact, in many cases $\exp(\mathcal{R}) = R = h$ (see the discussion in Section 4). However, Figure 3 illustrates the most general situation.

We now describe in detail a modification of the parallelized Kangaroo method using notation similar to that of [27, 29] for hyperelliptic function fields. Let $m$ be the (even) number of available processors. The algorithm uses two herds of kangaroos: a herd $\{T_1, \ldots, T_{m/2}\}$ of tame kangaroos and a herd $\{W_1, \ldots, W_{m/2}\}$ of wild kangaroos. A kangaroo is a sequence of elements in $\mathcal{R}$, and we write $T_j = \{t_{A,j}\}_{A \in \mathbb{N}_0}$ and $W_k = \{w_{B,k}\}_{B \in \mathbb{N}_0}$, for $1 \leq j, k \leq m/2$. Each tame and wild kangaroo is initialized via $t_{0,j} = B(E + (j-1)\nu, 0) \in \mathcal{R}_0$ and $w_{0,k} = B((k-1)\nu, 0) \in \mathcal{R}_0$, respectively, for some small $\nu \in \mathbb{Z}$. From these initial positions, the kangaroos make jumps (i.e., baby and giant steps) in $\mathcal{R}_0$ until a collision between a tame and a wild kangaroo occurs. That is, the kangaroos jump until a tame and a

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**Algorithm 1 (red₀) Finding an Element in $\mathcal{R}_0$**

**Input:** An element $b \in \mathcal{R}$ such that $\delta_2(b) \neq 0$.

**Output:** An element $\text{red}_0(b) := a \in \mathcal{R}_0$ close to $b$ such that $\delta_2(a) = 0$.

1: while $\delta_2(b) < 0$ do $a := b$, $b := bs_1(a)$
2: while $\delta_2(b) > 0$ do /* Now $\delta_2(b) \geq 0$. */
3: $a := b$, $b := bs_0(a)$
4: while $\delta_2(b) > 0$ do $a := b$, $b := bs_2(a)$
5: if $\delta_2(b) < 0$ then $b := a$
6: return $a := b$
wild kangaroo have the same ideal part. In this case, if $id(t_{A,i}) = id(w_{B,i'})$, for some $A, B \in \mathbb{N}$ and $1 \leq i, i' \leq m/2$, then $\delta_1(t_{A,i}) \equiv \delta_1(w_{B,i'}) \pmod{\exp(R)}$, so $h_0 := \delta_1(t_{A,i}) - \delta_1(w_{B,i'})$ is a multiple of $\exp(R)$.

To make the jumps, define a set of small \(^{(relative \ to \ U)}\) random positive integers \(\{s_1, \ldots, s_{64}\}\), the jump set \(J = \{a_1, \ldots, a_{64}\}\), where \(a_i = B(s_i - \psi(g), -\psi(g))\), for \(1 \leq i \leq 64\), and a hash function \(w : R \to \{1, \ldots, 64\}\). Also, for a real number \(\tau \geq 1\), let \(S_\tau \subseteq R_0\) such that approximately every \(\tau\)-th element of \(R_0\) belongs to \(S_\tau\). Each kangaroo jumps through \(R_0\) via an iteration of a giant step and possibly one or more baby steps. Initially, each tame and wild kangaroo will take baby steps, if necessary, until it is in \(S_\tau\). Then each kangaroo \(t_i\) takes the giant step \(t_{i+1} := t_i \ast a_w(t_i)\), for \(l \geq 0\), followed by baby steps in the 0-direction, correcting via Algorithm 1 if necessary, until an element in \(S_\tau\) is found.

If there is a collision between two kangaroos of the same herd, then we must re-initialize one of the two kangaroos. If \(t_i\) is one of two kangaroos in a collision, then choose a small \(c \in \mathbb{N}\), set \(t_{i+1} := t_i \ast B(c, -\psi(g))\), and take baby steps until an element in \(S_\tau\) is found. Then \(t\) continues jumping on its new path as usual. The other kangaroo may continue without interruption.

Using the idea of van Oorschot and Wiener \cite{34}, we will only store distinguished points to reduce the storage requirement. In order to avoid confusion in terminology, such points will be called (kangaroo) traps instead. Let \(\theta \in \mathbb{N}\) be a sufficiently large power of 2 and define another hash function \(z : R_0 \to \{0, \ldots, \theta - 1\}\). Set a trap, that is, store a kangaroo \(t\), if \(z(t) = 0\). Since kangaroos travel along the same path following a collision, any collision will eventually land in a trap.

Finally, if there exist \(a, b \in \mathbb{N}_0\), such that \(b > 1\) and \(h \equiv a \pmod b\), then we make adjustments to take advantage of this information. First, change the estimate \(E\) to \(E - (E \pmod b) + a\), so that \(E \equiv a \pmod b\) for the revised value of \(E\). Next, choose \(\nu\) and the \(s_i\) such that \(b \mid \nu\) and \(b \mid s_i\), for each \(1 \leq i \leq 64\). Finally, restrict \(S_\tau\) to elements \(a \in R_0\) such that \(\delta_1(a) \equiv a \pmod b\) and require that approximately every \(7r\)-th element of \(R_0\) lies in \(S_\tau\). The remaining initializations and procedures are the same as before.

In Algorithm 2 we formalize the procedures described above. The following result is a generalization of and an improvement upon similar ideas in \cite{27, 29} and establishes optimal choices for \(\tau\) and the average jump distance \(\beta = Mean(s_i)\) to minimize the expected heuristic running time of the Kangaroo method. The proof
Algorithm 2 Computing \( h_0 \) via the Kangaroo Algorithm

**Input:** A purely cubic function field \( K/F_q(x) \) of unit rank 2; \( a, b \in \mathbb{N}_0 \) such that \( h \equiv a \pmod{b} \) (or \( b = 1 \) and \( a = 0 \) if no non-trivial \( b \) is known); and an even integer \( m \), the number of processors.

**Output:** A multiple \( h_0 \) of \( \exp(R) \).

1. Compute the genus \( g \), choose \( \rho \) from Table 1 and choose \( \hat{\alpha} := \hat{\alpha}(g) \) from Table 1
2. Set \( \beta := [(m/2)\sqrt{(2\rho - 1)\alpha U}] - \rho + 1 \), \( \nu := b[2\beta/(bm)] \), \( \theta := 2^{\lfloor \log(\beta)/2 \rfloor} \), \( j := k := 0 \).
3. Choose random integers \( g + 1 + \hat{\psi}(g) \leq s_i \leq 2\beta \), with \( 1 \leq i \leq 64 \), such that \( \text{Mean}\{s_i\} = \beta \) and \( b \mid s_i \).
4. Compute the jump set \( J := \{a_1, \ldots, a_{64}\} \), where \( a_i := B(s_i - \hat{\psi}(g), -\hat{\psi}(g)) \).
5. While \( \delta_1(a_i) \neq s_i - \hat{\psi}(g) \) for any \( i \) do
   6. Replace \( s_i := s_i + b \) and recompute \( a_i := B(s_i - \hat{\psi}(g), -\hat{\psi}(g)) \) for inclusion in \( J \).
7. Define hash functions \( w : R \rightarrow \{1, \ldots, 64\} \) and \( z : R \rightarrow \{0, \ldots, \theta - 1\} \).
8. For \( i = 1 \) to \( \lfloor m/2 \rfloor \) do
   9. Initialize the tame kangaroos, \( T_i : t_{0,i} := B(E + (i - 1)\nu, 0) \).
10. Initialize the wild kangaroos, \( W_i : w_{0,i} := B((i - 1)\nu, 0) \).
11. While \( t_{0,i} \notin S_r \) do \( t_{0,i} := \text{red}_0(bs_0(t_{0,i})) \).
12. While \( w_{0,i} \notin S_r \) do \( w_{0,i} := \text{red}_0(bs_0(w_{0,i})) \).
13. While a collision between a tame and a wild kangaroo has not been found do
   14. For \( i = 1 \) to \( \lfloor m/2 \rfloor \) do
15. If \( z(t_{j,i}) = 0 \) or \( z(w_{k,i}) = 0 \) then store the respective element(s).
16. Compute \( t_{j+1,i} := \text{red}_0(t_{j,i} \cdot a_{w(t_{j,i})}) \) and \( w_{k+1,i} := \text{red}_0(w_{k,i} \cdot a_{w(w_{k,i})}) \).
17. While \( t_{j+1,i} \notin S_r \) do \( t_{j+1,i} := \text{red}_0(bs_0(t_{j+1,i})) \).
18. While \( w_{k+1,i} \notin S_r \) do \( w_{k+1,i} := \text{red}_0(bs_0(w_{k+1,i})) \).
19. Increment \( j := j + 1 \) and \( k := k + 1 \).
20. If \( t_{A,i} = w_{B,i} \) then return \( h_0 := \delta_1(t_{A,i}) - \delta_1(w_{B,i}) \).

is similar to the analogous result in cubic function fields of unit rank 1. We therefore omit the proof and refer the reader to [11].

**Theorem 2.** Let \( K/F_q(x) \) be a purely cubic function field of unit rank 2 such that \( h \equiv a \pmod{b} \) for some \( a, b \in \mathbb{N} \). Then the expected heuristic running time, over all cubic function fields over \( F_q(x) \) of genus \( g \), to compute a multiple \( h_0 \) of \( \exp(R) \) via Algorithm 2 is minimized by choosing \( \tau = \rho/b \) and an average jump distance of \( \beta = [(m/2)\sqrt{(2\rho - 1)\alpha U}] - \rho + 1 \). Here, \( m \) is the (even) number of processors, \( \rho = T_G/T_B \), \( T_G \) and \( T_B \) are the respective times required to compute a giant step and a baby step in \( R \), and \( \alpha = \alpha(q, g) < 1/2 \) is the mean value of \( |h - E|/U \) over all cubic function fields over \( F_q(x) \) of genus \( g \). With these choices, the expected heuristic running time is \( 4\sqrt{\alpha U/(2\rho - 1)} + \theta m + O(1)(2 - 1/\rho)T_G \), as \( q \to \infty \), where traps are set on average every \( \theta \) iterations.

Following the recommendations given in [33, 29], we make choices for the remaining variables. First, we choose the \( s_i \) randomly such that \( g + 1 + \hat{\psi}(g) \leq s_i \leq 2\beta \), for \( 1 \leq i \leq 64 \). (The lower bound is an application of Theorem 5.3.10 of [11].)
COMPUTATIONS IN CUBIC FUNCTION FIELDS OF UNIT RANK TWO

Table 1. Giant Step to Baby Step Ratio \( \rho = \frac{T_G}{T_B} \) and Estimate \( \hat{\alpha}(g) \) of Mean \( \frac{|h - E|}{U} \)

| \( g \) | \( \deg(G) \) | \( \deg(H) \) | \( \rho \) | \( \deg(G) \) | \( \deg(H) \) | \( \rho \) | \( g \) | \( q \) | \( \lambda \) | \( \alpha(g) \) |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 2   | 2   | 2   | 3.04839 |   |   |   | 3   | 100003 | 1   | 0.27187490 |
| 3   | 4   | 1   | 3.02410 | 3   | 3   | 4.42018 | 4   | 10009  | 1   | 0.19186318 |
| 5   | 5   | 2   | 5.61416 | 5   | 997  | 6.38660 | 5   | 997    | 2   | 0.19190607 |
| 6   | 7   | 1   | 5.96440 | 4   | 4   | 6.38660 | 6   | 463    | 2   | 0.15975657 |
| 7   | 9   | 0   | 7.87264 | 6   | 3   | 8.21655 | 7   | 97     | 2   | 0.12602172 |

Table 1 lists values of \( \rho \) for various unit rank 2 situations of genera \( 2 \leq g \leq 7 \).

In each case, we computed the ratios using \( 10^6 \) baby steps and \( 10^6 \) giant steps in a function field \( \mathbb{F}_q(x,y) \) with \( q = 10^8 + 39 \) and \( y^3 = GH^2 \), where \( G \) and \( H \) were random, monic, co-prime, irreducible polynomials with \( \deg(G) \geq \deg(H) \).

In the next section, we briefly review the method of [21] implemented here to compute the divisor class number of a cubic function field.

4. Computing \( h \) and \( R \) – the Idea

Algorithm 3 lists the three main phases of the method of Scheidler and Stein [21, 22] to compute a multiple \( h_0 \) of \( \exp(R) \). If \( \exp(R) \) is large enough, then these three steps determine the divisor class number \( h \) of a cubic function field. Step 4 determines \( \exp(R) \) and in certain cases, Step 5 computes the regulator \( R \) and ideal class number \( h_0 \) of \( \mathcal{O} \).

If \( \exp(R) \leq 2U \), then there may be more than one multiple of \( \exp(R) \) in the interval \( (E - U, E + U) \), in which case \( h \) cannot be determined. Nonetheless, \( h \) is limited to a smaller subset, since \( \exp(R) \mid h \). We know that \( R/\exp(R) \) is a divisor of \( d := \gcd(\exp(R), h/\exp(R)) \); if \( d = 1 \), then \( R = \exp(R) \).

By results of Achter and Pries [1, 2], the class numbers of purely cubic function fields of genus \( g \) over a finite field \( \mathbb{F}_q \) behave like random integers in the Hasse-Weil interval \( [\sqrt{q} - 1)^2g, (\sqrt{q} + 1)^2g] \) with respect to divisibility. Therefore, the class numbers are very often square-free, whence the divisor class group \( \mathcal{J} \) is cyclic. In that case, \( \exp(R) = R \), and if one assumes that \( \mathcal{D}_0^* / \mathcal{P}^S \) behaves like a random subgroup of \( \mathcal{J} \), then \( R \) is large. Therefore, Algorithm 3 can determine \( R \) and \( h \) in many cases.

For details on how to compute \( E \) and \( U \) in Step 1, along with the complete analysis of the running time of Algorithm 3 see [21, 22]. Further implementation details may be found in [11]. Here we merely state that by [21, 22], for \( g \geq 3 \), the complexity of Step 1 of Algorithm 3 is \( O\left(q^{[(2g - 1)/5]} + \varepsilon(g)\right) \) giant steps, as \( q \to \infty \).
Algorithm 3 Computing \( h \) and/or \( R \) – the Idea

1. Determine \( E, U \in \mathbb{N} \) such that \( h \in (E - U, E + U) \).
2. Determine extra information about \( h \) such as congruences or the distribution of \( h \) in the interval \( (E - U, E + U) \).
3. Compute a multiple \( h_0 \) of \( \exp(R) \) via Algorithm 2.
4. Compute \( R^* := \exp(R) \) via Algorithm 4.
5. If \( R^* > 2U \), let \( h \) be the unique multiple of \( R^* \) in \( (E - U, E + U) \). If \( \gcd(R^*, h/R^*) = 1 \), then \( R = R^* \) and \( h_0 = h/R^* \).

where \(-1/4 \leq \varepsilon(g) \leq 1/2\). If \( g \leq 2 \), then there is no asymptotic improvement in using the bounds described in [21, 22] versus the Hasse-Weil bounds.

Next, we discuss some practical issues arising in our implementation of Algorithm 3. We omit details on Step 3 since they were already given in Section 3.

5. Implementation Details

5.1. Implementation Details for Phase 2. For Phase 2 of Algorithm 3, we use extra information about \( h \) to effectively reduce the size of the interval, \( (E - U, E + U) \), determined in Phase 1. The method to compute \( E \) and \( U \) uses a truncated Euler product representation of the zeta function of the function field, and we consider finite places (i.e., monic irreducible polynomials) up to a degree bound \( \lambda \).

It has been shown in both the quadratic and cubic function field cases that \( h \) is not uniformly distributed in this interval, and tends to be close to the approximation \( E \) [28, 11].

Let \( \alpha(q, g) = \text{Mean}(|h - E|/U) \), where the mean is taken over all cubic function fields of genus \( g \) over \( \mathbb{F}_q(x) \). In Theorem 2, we described how to apply \( \alpha(q, g) \) to minimize the expected running time of Algorithm 2. For a fixed genus \( g \), we assume that the limit \( \alpha(g) = \lim_{q \to \infty} \alpha(q, g) \) exists, as is the case for hyperelliptic function fields [28]. However, \( \alpha(q, g) \) and \( \alpha(g) \) are very difficult to compute precisely, so instead we applied approximations \( \hat{\alpha}(g) \) of \( \alpha(g) \) for \( 3 \leq g \leq 7 \). Table 1 (Table 6.5 of [11]) lists these approximations for selected values of \( g \), based on a sampling of 10000 cubic function fields of genus \( g \) over a fixed field \( \mathbb{F}_q \). However, these averages may be applied to cubic function fields over any finite field. In Table 1, \( \lambda \) is the degree bound used to compute the estimate \( E \).

A second component of Phase 2 of Algorithm 3 finds information about \( h \) modulo small primes. In [4], Bauer, Teske, and Weng consider purely cubic function fields defined by Picard curves. In this case, they proved the following result about \( h \) modulo powers of 3.

Proposition 2 (Lemma 2.2 of [4]). Let \( K = \mathbb{F}_q(x, y) \) be the function field of a Picard curve \( C : y^3 = F(x) \), where \( q \equiv 1 \pmod{3} \). If \( F \) has \( k \) distinct irreducible factors over \( \mathbb{F}_q[x] \), then \( 3^{k-1} \mid h \). If \( F \) is irreducible, then \( h \equiv 1 \pmod{3} \).

The genus 3 curves we used in our computations are birationally equivalent to Picard curves, so we applied this proposition to these curves.

5.2. Implementation Details for Phase 4. Algorithm 4 outlines the procedure for Step 4 of Algorithm 3. This step will determine \( \exp(R) \) given a multiple \( h_0 \) of \( \exp(R) \). Here, we adapt Algorithm 4.4 of [30] to the case of cubic function fields of unit rank 2, using the fact that \( \exp(R) \) is the smallest factor \( R^* \) of \( h_0 \) such that...
Algorithm 4 Computing \( \exp(\mathcal{R}) \): Step 4 of Algorithm 3

**Input:** A multiple \( h_0 \) of \( \exp(\mathcal{R}) \) and a lower bound \( l \) of \( \exp(\mathcal{R}) \).

**Output:** The exponent \( \exp(\mathcal{R}) \) of the infrastructure \( \mathcal{R} \).

1. Set \( h^* := 1 \).
2. Factor \( h_0 = \prod_{i=1}^{k} p_i^{e_i} \).
3. for \( i = 1 \) to \( k \) do
   4. if \( p_i < h_0/l \) then
      5. Find \( 1 \leq c_i \leq a_i \) minimal such that \( \text{id}(B(h_0/p_i^{c_i}, 0)) \neq \mathcal{O} \).
   6. Set \( h^* := p_i^{c_i-1}h^* \).
5. return \( \exp(\mathcal{R}) = R^* := h_0/h^* \).

\( \text{id}(B(R^*, 0)) = \mathcal{O} \). Recall that \( B(a, 0) \), for \( a \in \mathbb{N} \), may be impossible to determine because of a hidden element having distance \( (a, 0) \). Nevertheless, the probability of this occurring is negligible, so that we can assume that Algorithm 4 produces the correct output.

We briefly comment on the running time of Algorithm 4 relative to the running time of Algorithm 3 especially in light of the factorization in Step 2. First, current heuristic methods to factor the integer \( h_0 \) require a subexponential number of bit operations in \( \log(h_0) \). Furthermore, Step 3 only requires a polynomial number (in \( g \) and \( \log(q) \)) of infrastructure operations. Therefore, determining \( \exp(\mathcal{R}) \) from \( h_0 \) will not dominate the overall running time of Algorithm 3. The class numbers that we found required only a few seconds to factor. In fact, we simply used a basic implementation of Pollard’s Rho method for factoring [14].

6. Computational Results

In this section, we tested the practical effectiveness of the Kangaroo algorithm to compute the divisor class number and extracted the ideal class number and regulator of six purely cubic function fields of unit rank 2: five of genus 3 and one of genus 4. We remark that this is the first time that Algorithm 3 has been implemented for cubic function fields of unit rank 2.

The genus 3 curves that we used for the examples in this section were each of the form \( C_i : y^3 = G_i(x)x^2 \), where

- \( G_1(x) = x^4 + 858028x^3 + 786068x^2 + 69746x + 675670 \),
- \( G_2(x) = x^4 + 9655935x^3 + 8633555x^2 + 1319425x + 1437614 \),
- \( G_3(x) = x^4 + 63268943x^3 + 53257730x^2 + 59385220x + 16188628 \),
- \( G_4(x) = x^4 + 834364201x^3 + 8363484x^2 + 953863416x + 850202733 \),
- \( G_5(x) = x^4 + 9994854268x^3 + 7631258748x^2 + 7469686108x + 292775976 \),

and the genus 4 curve was of the form \( C_6 : y^3 = G_6(x) \), where

\( G_6(x) = x^6 + 4207x^5 + 3340x^4 + 9858x^3 + 7507x^2 + 36x + 1019 \).

Each \( G_i \) is irreducible over the field \( \mathbb{F}_q \) used, and \( q \equiv 1 \pmod{3} \) is prime.

In Table 2, we list the ideal class number \( h_\mathcal{O} \), the regulator \( R \), and the ratio \( |h - E|/U \) for these six examples. For the genus 3 examples, we used \( \rho = 3.02410 \) and \( \tau = \rho/3 = 1.00803 \), and for the genus 4 example, we used \( \rho = \tau = 4.03846 \).

Based on the last column, we see that the estimate \( E \) was better than average except for the computations with curves \( C_3 \) and \( C_4 \). The \( C_4 \) through \( C_6 \) examples
were computed via a parallelized approach, using up to 64 processors. The largest divisor class number we computed had 31 decimal digits.

Data from the Kangaroo computations is given in Table 3. Here, “BS Jumps” and “GS Jumps” refer to the respective number of baby steps and giant steps computed using the Kangaroo method in each example, \( \lg \theta \) is the base 2 logarithm of the value of \( \theta \) used for setting traps, “Traps” is the total number of traps that were set, \( m \) is the number of processors (or kangaroos) that were used, “Coll.” is the number of useless collisions in the given example, and “Time” refers to the total time taken by the computation in minutes, hours, and days. For timing and technical considerations, we implemented our algorithms in C++ using NTL, written by Shoup [26], compiled using \texttt{g++}, and run on IBM cluster nodes with Intel Pentium 4 Xeon 2.4 GHz processors and 2 GB of RAM running Redhat Enterprise Linux 3.

| Curve | \( q \) | \( g \) | \( h_\mathcal{O} \) | \( R \) | \( |h − E|/U \) |
|---|---|---|---|---|---|
| \( C_1 \) | 1000003 | 3 | 1 | 1002847489604613721 | 0.1498574 |
| \( C_2 \) | 10000141 | 3 | 1 | 100039743576015862929 | 0.1263140 |
| \( C_3 \) | 100000039 | 3 | 1 | 100009485874807321192993 | 0.3799612 |
| \( C_4 \) | 1000000009 | 3 | 1 | 1000036037504733195527721763 | 0.3814163 |
| \( C_5 \) | 10000200031 | 3 | 1 | 1000028959108091361595659615907 | 0.2262216 |
| \( C_6 \) | 10009 | 4 | 1 | 10081785007075827 | 0.1218925 |

Table 2. Regulators and Ideal Class Numbers

| Curve | \( q \) | \( g \) | BS Jumps | GS Jumps | \( \lg \theta \) | Traps | \( m \) | Coll. | Time |
|---|---|---|---|---|---|---|---|---|---|
| \( C_1 \) | 1000003 | 3 | 18825 | 2353928 | 10 | 2337 | 2 | – | 98.5 \( m \) |
| \( C_2 \) | 10000141 | 3 | 72149 | 9040560 | 12 | 2198 | 2 | – | 5.67 \( h \) |
| \( C_3 \) | 100000039 | 3 | 1537984 | 192895918 | 14 | 11768 | 2 | – | 6.84 \( d \) |
| \( C_4 \) | 1000000009 | 3 | 12942441 | 1624509536 | 18 | 6147 | 40 | 0 | 37.4 \( d \) |
| \( C_5 \) | 10000200031 | 3 | 404518765 | 50755701157 | 20 | 46235 | 64 | 7 | 1543 \( d \) |
| \( C_6 \) | 10009 | 4 | 1814587 | 596886 | 12 | 109 | 6 | 2 | 80.3 \( m \) |

Table 3. Regulator Computation Data

7. Conclusions and Future Work

Using current implementations of the arithmetic in the infrastructure of a purely cubic function field of unit rank 2, divisor class numbers and regulators up to 31 digits were computed using the method of Scheidler and Stein [21] and the Kangaroo algorithm as a subroutine. The largest example among these class numbers and regulators was the largest ever computed for a function field of genus at least 3, with the exception of function fields defined by a Picard curve. This was also the first time that a “square-root” algorithm was efficiently applied to the infrastructure \( \mathcal{R} \) of a global field of unit rank 2. Moreover, we made improvements to the Kangaroo method in \( \mathcal{R} \) by showing how to take advantage of information on the congruence class of the divisor class number and how to use the ratio \( \rho = T_G/T_B \) more effectively.
A procedure to determine the regulator given the divisor class number and infrastructure exponent using methods of Buchmann, Jacobson, and Teske is work in progress. In addition, efficient ideal and infrastructure arithmetic needs to be developed for arbitrary (i.e., not necessarily purely) cubic function fields as well as for characteristic 2 and 3 in order to apply this method to such function fields. Finally, it is unknown if we can take advantage of the torus structure of $R$ to compute $R$ using more efficient techniques.

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