Superconformal Symmetry in Three-dimensions

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Abstract

Three-dimensional $\mathcal{N}$-extended superconformal symmetry is studied within the superspace formalism. A superconformal Killing equation is derived and its solutions are classified in terms of supertranslations, dilations, Lorentz transformations, $R$-symmetry transformations and special superconformal transformations. Superconformal group is then identified with a supermatrix group, $\text{OSp}(\mathcal{N}|2, \mathbb{R})$, as expected from the analysis on simple Lie superalgebras. In general, due to the invariance under supertranslations and special superconformal transformations, superconformally invariant $n$-point functions reduce to one unspecified $(n-2)$-point function which must transform homogeneously under the remaining rigid transformations, i.e. dilations, Lorentz transformations and $R$-symmetry transformations. After constructing building blocks for superconformal correlators, we are able to identify all the superconformal invariants and obtain the general form of $n$-point functions. Superconformally covariant differential operators are also discussed.

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1 Introduction and Summary

Based on the classification of simple Lie superalgebras \([1]\), Nahm analyzed all possible superconformal algebras \([2]\). According to Ref. \([2]\), not all spacetime dimensions allow the corresponding supersymmetry algebra to be extended to a superconformal algebra contrary to the ordinary conformal symmetry. The standard supersymmetry algebra admits an extension to a superconformal algebra only if \(d \leq 6\). Namely the highest dimension admitting superconformal algebra is six, and in \(d = 3, 4, 5, 6\) dimensions the bosonic part of the superconformal algebra has the form

\[
\mathcal{L}_C \oplus \mathcal{L}_R, \tag{1.1}
\]

where \(\mathcal{L}_C\) is the Lie algebra of the conformal group and \(\mathcal{L}_R\) is a \(R\)-symmetry algebra acting on the superspace Grassmann variables. Explicitly for Minkowskian spacetime

\[
d = 3; \quad o(2, 3) \oplus o(N),
\]

\[
d = 4; \quad \begin{cases} o(2, 4) \oplus u(N), & N \neq 4 \\
o(2, 4) \oplus su(4) \end{cases},
\] \(d = 5; \quad o(2, 5) \oplus su(2),
\]

\[
d = 6; \quad o(2, 6) \oplus sp(N),
\]

where \(N\) or the number appearing in \(R\)-symmetry part is related to the number of supercharges.

On six-dimensional Minkowskian spacetime it is possible to define Weyl spinors of opposite chiralities and so the general six-dimensional supersymmetry may be denoted by two numbers, \((N, \bar{N})\), where \(N\) and \(\bar{N}\) are the numbers of chiral and anti-chiral supercharges. The \(R\)-symmetry group is then \(Sp(N) \times Sp(\bar{N})\). The analysis of Nahm shows that to admit a superconformal algebra either \(N\) or \(\bar{N}\) should be zero. Although both \((1, 1)\) and \((2, 0)\) supersymmetry give rise \(N = 4\) four-dimensional supersymmetry after dimensional reduction, only \((2, 0)\) supersymmetry theories can be superconformal \([3]\). On five-dimensional Minkowskian spacetime Nahm’s analysis seems to imply a certain restriction on the number of supercharges as the corresponding \(R\)-symmetry algebra is to be \(su(2)\).
The above analysis is essentially based on the classification of simple Lie superalgebras and identification of the bosonic part with the usual spacetime conformal symmetry rather than Poincaré symmetry, since the former forms a simple group, while the latter does not. This approach does not rely on any definition of superconformal transformations on superspace.

The present paper deals with superconformal symmetry in three-dimensions and lies in the same framework as our sequent work on superconformal symmetry in other dimensions, \( d = 4, 6 \) \[4–6\]. Namely we analyze superconformal symmetry directly in terms of coordinate transformations on superspace. We first define the superconformal group on superspace and derive the superconformal Killing equation. Its general solutions are identified in terms of supertranslations, dilations, Lorentz transformations, \( R \)-symmetry transformations and special superconformal transformations. Based on the explicit form of the solutions the superconformal group is independently identified to agree with Nahm’s analysis and some representations are obtained.

Specifically, in Ref. \([4]\) we identified four-dimensional \( \mathcal{N} \neq 4 \) extended superconformal group with a supermatrix group, \( SU(2, 2|\mathcal{N}) \), having dimensions \((15 + \mathcal{N}^2|8\mathcal{N})\), while for \( \mathcal{N} = 4 \) case we pointed out that an equivalence relation must be imposed on the supermatrix group and so the four-dimensional \( \mathcal{N} = 4 \) superconformal group is isomorphic to a quotient group of the supermatrix group. In fact \( \mathcal{N} = 4 \) superconformal group is a semi-direct product of U(1) and a simple Lie supergroup containing SU(4). The U(1) factor can be removed by imposing tracelessness condition on the supermatrix group so that the dimension reduces from \((31|32)\) to \((30|32)\) and the \( R \)-symmetry group shrinks from U(4) to Nahm’s result, SU(4)\[4\]. In Ref. \([6]\) by solving the superconformal Killing equation we show that six-dimensional \((\mathcal{N}, 0)\) superconformal group is identified with a supermatrix group, \( OSp(2, 6|\mathcal{N}) \), having dimensions \((28 + \mathcal{N}(2\mathcal{N} + 1)|16\mathcal{N})\), while for \((\mathcal{N}, \tilde{\mathcal{N}}), \mathcal{N}, \tilde{\mathcal{N}} > 0\) supersymmetry, we verified that although dilations may be introduced, there exist no special superconformal transformations as expected from Nahm’s result.

The main advantage of our formalism is that it enables us to write general expression for two-point, three-point and \( n \)-point correlation functions of quasi-primary superfields which transform simply under superconformal transformations. In Refs. \([4–6]\) we explicitly constructed building blocks for superconformal correlators in four- and six-dimensions, and

\[\text{Similarly if five-dimensional superconformal group is not simple, this will be a way out from the puzzling restriction on the number of supercharges in five-dimensional superconformal theories, as the corresponding } R \text{-symmetry group can be bigger than Nahm’s result, SU(2). However this is at the level of speculation at present.}\]
proved that these building blocks actually generate the general form of correlation functions. In general, due to the invariance under supertranslations and special superconformal transformations, \( n \)-point functions reduce to one unspecified \((n - 2)\)-point function which must transform homogeneously under the rigid transformations only - dilations, Lorentz transformations and \( R \)-symmetry transformations [4]. This feature of superconformally invariant correlation functions is universal for any spacetime dimension if there exists a well defined superinversion in the corresponding dimension, since superinversion plays a crucial role in its proof. For non-supersymmetric case, contrary to superinversion, the inversion map is defined of the same form irrespective of the spacetime dimension and hence \( n \)-point functions reduce to one unspecified \((n - 2)\)-point function in any dimension which transform homogeneously under dilations and Lorentz transformations.

The formalism is powerful for applications whenever there exist off-shell superfield formulations for superconformal theories, and such formulations are known in four-dimensions for \( \mathcal{N} = 1, 2, 3 \) [7–11] and in three-dimensions for \( \mathcal{N} = 1, 2, 3, 4 \) [12–17]. In fact within the formalism Osborn elaborated the analysis of \( \mathcal{N} = 1 \) superconformal symmetry for four-dimensional quantum field theories [20], and recently Kuzenko and Theisen determine the general structure of two- and three- point functions of the supercurrent and the flavour current of \( \mathcal{N} = 2 \) superconformal field theories [21]. A common result contained in Refs. [21] is that the three-point functions of the conserved supercurrents in both \( \mathcal{N} = 1 \) and \( \mathcal{N} = 2 \) superconformal theories allow two linearly independent structures and hence there exist two numerical coefficients which can be calculated in specific perturbation theories using supergraph techniques.

The contents of the present paper are as follows. In section 2 we review supersymmetry in three-dimensions. In particular, we verify that supersymmetry algebra with \( N \) Dirac supercharges is equivalent to \( 2N \)-extended Majorana supersymmetry algebra, so that in the present paper we consider \( \mathcal{N} \)-extended Majorana superconformal symmetry with an arbitrary natural number, \( \mathcal{N} \).

In section 3, we first define the three-dimensional \( \mathcal{N} \)-extended superconformal group in terms of coordinate transformations on superspace as a generalization of the definition of ordinary conformal transformations. We then derive a superconformal Killing equation, which is a necessary and sufficient condition for a supercoordinate transformation to be superconformal. The general solutions are identified in terms of supertranslations, dilations, Lorentz transformations, \( R \)-symmetry transformations and special superconformal transformations, where \( R \)-symmetry is given by \( O(\mathcal{N}) \) as in eq.(1.2). We also present a definition of superinversion in three-dimensions through which supertranslations and special
superconformal transformations are dual to each other. The three-dimensional $\mathcal{N}$-extended superconformal group is then identified with a supermatrix group, $\text{OSp}(\mathcal{N}|2,\mathbb{R})$, having dimensions $(10 + \frac{1}{2}\mathcal{N}(\mathcal{N} - 1)|4\mathcal{N})$ as expected from the analysis on simple Lie superalgebras $[2,22]$.

In section 4, we obtain an explicit formula for the finite non-linear superconformal transformations of the supercoordinates, $z$, parameterizing superspace and discuss several representations of the superconformal group. We also construct matrix or vector valued functions depending on two or three points in superspace which transform covariantly under superconformal transformations. For two points, $z_1$ and $z_2$, we find a matrix, $I(z_1, z_2)$, which transforms covariantly like a product of two tensors at $z_1$ and $z_2$. For three points, $z_1, z_2, z_3$, we find ‘tangent’ vectors, $Z_i$, which transform homogeneously at $z_i$, $i = 1, 2, 3$. These variables serve as building blocks of obtaining two-point, three-point and general $n$-point correlation functions later.

In section 5, we discuss the superconformal invariance of correlation functions for quasi-primary superfields and exhibit general forms of two-point, three-point and $n$-point functions. Explicit formulae for two-point functions of superfields in various cases are given. We also identify all the superconformal invariants.

In section 6, superconformally covariant differential operators are discussed. The conditions for superfields, which are formed by the action of spinor derivatives on quasi-primary superfields, to remain quasi-primary are obtained. In general, the action of differential operator on quasi-primary fields generates an anomalous term under superconformal transformations. However, with a suitable choice of scale dimension, we show that the anomalous term may be cancelled. We regard this analysis as a necessary step to write superconformally invariant actions on superspace, as the kinetic terms in such theories may consist of superfields formed by the action of spinor derivatives on quasi-primary superfields.

In the appendix, the explicit form of superconformal algebra and a method of solving the superconformal Killing equation are exhibited.
2 Preliminary

2.1 Gamma Matrices

With the three-dimensional Minkowskian metric, $\eta^{\mu\nu} = \text{diag}(+1, -1, -1)$, the $2 \times 2$ gamma matrices, $\gamma^\mu$, $\mu = 0, 1, 2$, satisfy

$$\gamma^\mu \gamma^\nu = \eta^{\mu\nu} + i \epsilon^{\mu\nu\rho} \gamma_\rho.$$  \hfill (2.1)

The hermiticity condition is

$$\gamma^0 \gamma^\mu \gamma^0 = \gamma^{\mu \dagger}.$$  \hfill (2.2)

Charge conjugation matrix, $\epsilon$, satisfies

$$\epsilon \gamma^\mu \epsilon^{-1} = -\gamma^{\mu t},$$  \hfill (2.3)

where $\gamma^\mu$ forms a basis for $2 \times 2$ traceless matrices with the completeness relation

$$\gamma^{\mu \alpha \beta} \gamma^\mu \gamma^\delta = 2 \delta^\alpha_\beta \delta^\gamma_\delta - \delta^\alpha_\gamma \delta^\beta_\delta.$$  \hfill (2.4)

2.2 Three-dimensional Superspace

The three-dimensional supersymmetry algebra has the standard form with $P_\mu = (H, -P)$

$$\{Q^\alpha_i, \bar{Q}_{j\beta}\} = 2 \delta^i_j \gamma^{\alpha \beta} P_\mu,$$  \hfill (2.5)

$$[P_\mu, P_\nu] = [P_\mu, Q^\alpha_i] = [P_\mu, \bar{Q}_{i\alpha}] = \{Q^\alpha_i, Q^{j\beta}\} = \{Q_{i\alpha}, \bar{Q}_{j\beta}\} = 0,$$

where $1 \leq \alpha \leq 2$, $1 \leq i \leq N$ and $Q^i, \bar{Q}_{ij}$ satisfy

$$\bar{Q}_{i} = Q^{i \dagger} \gamma^0.$$  \hfill (2.6)

Now we define for $1 \leq a \leq 2N$, $1 \leq i, j \leq N$

$$Q^a = \left( \frac{1}{\sqrt{2}}(Q^i + \epsilon^{-1} \bar{Q}_{i}^\dagger) \right),$$

$$i \frac{1}{\sqrt{2}}(Q^i - \epsilon^{-1} \bar{Q}_{j}^\dagger),$$  \hfill (2.7)

\[To emphasize the anti-symmetric property of the 2 \times 2 charge conjugation matrix in three-dimensions we adopt the symbol, $\epsilon$, instead of the conventional one, $C$.\]
\[ Q_a \equiv Q^{at} \gamma^0 = \left( \frac{1}{\sqrt{2}} (\bar{Q}_i - Q^{it} \epsilon), \quad -i \frac{1}{\sqrt{2}} (\bar{Q}_j + Q^{jt} \epsilon) \right). \] (2.8)

\[ Q^a, \bar{Q}_b \text{ satisfy the Majorana condition} \]

\[ \bar{Q}_a = Q^{at} \gamma^0 = -Q^{at} \epsilon, \]

\[ Q^a = \epsilon^{-1} \bar{Q}^t_a. \] (2.9)

With this notation we note that the three-dimensional \( N \)-extended supersymmetry algebra (2.3) is equivalent to the \( 2N \)-extended Majorana supersymmetry algebra

\[ \{Q^{a\alpha}, \bar{Q}_{b\beta}\} = 2 \delta^{a\alpha} \bar{Q}_{b\beta} P_\mu, \]

\[ [P_\mu, P_\nu] = [P_\mu, Q^{a\alpha}] = 0. \] (2.10)

This can be generalized by replacing \( 2N \) with an arbitrary natural number, \( N \), and hence \( N \)-extended Majorana supersymmetry algebra.

\( P_\mu, Q^{a\alpha}, 1 \leq a \leq N \) generate a supergroup, \( G_T \), with parameters, \( z^M = (x^\mu, \theta^{a\alpha}) \), which are coordinates on superspace. The general element of \( G_T \) is written in terms of these coordinates as

\[ g(z) = e^{i(\bar{Q}_a \theta^a)}. \] (2.11)

Corresponding to eq. (2.9) \( \theta^a \) also satisfies the Majorana condition

\[ \bar{\theta}_a = \theta^{at} \gamma^0 = -\theta^{at} \epsilon, \] (2.12)

so that

\[ \bar{Q}_a \theta^a = \bar{\theta}_a Q^a, \quad g(z) \dagger = g(z)^{-1} = g(-z). \] (2.13)

The Baker-Campbell-Hausdorff formula with the supersymmetry algebra (2.10) gives

\[ g(z_1)g(z_2) = g(z_3), \] (2.14)

where

\[ x^\mu_3 = x^\mu_1 + x^\mu_2 + i \bar{\theta}_{1a} \gamma^\mu \theta^a_2, \quad \theta^a_3 = \theta^a_1 + \theta^a_2 \] (2.15)

Letting \( z_1 \to -z_2 \) we may get the supertranslation invariant one forms, \( e^M = (e^\mu, d\theta^{a\alpha}) \), where

\[ e^\mu(z) = dx^\mu - i \bar{\theta}_a \gamma^\mu d\theta^a. \] (2.16)
The exterior derivative, \( d \), on superspace is defined as

\[
d \equiv d z^M \frac{\partial}{\partial z^M} = e^M D_M = e^\mu \partial_\mu - d \theta^{\alpha \alpha} D_{\alpha \alpha},
\]

(2.17)

where \( D_M = (\partial_\mu, -D_{\alpha \alpha}) \) are covariant derivatives

\[
\partial_\mu = \frac{\partial}{\partial x^\mu}, \quad D_{\alpha \alpha} = -\frac{\partial}{\partial \theta^{\alpha \alpha}} + i (\bar{\theta}_a \gamma^\mu)_\alpha \frac{\partial}{\partial x^\mu}.
\]

(2.18)

We also define

\[
\bar{D}^{\alpha \alpha} = \epsilon^{-1 \alpha \beta} D_{\alpha \beta} = \frac{\partial}{\partial \bar{\theta}^{\alpha \alpha}} - i (\gamma^\mu \theta^a)_\alpha \frac{\partial}{\partial x^\mu},
\]

(2.19)

satisfying the anti-commutator relations

\[
\{ \bar{D}^{\alpha \alpha}, D_{\beta \beta} \} = 2i \delta^a \gamma^{\alpha \beta} \partial_\mu.
\]

(2.20)

Under an arbitrary superspace coordinate transformation, \( z \rightarrow z' \), \( e^M \) and \( D_M \) transform as

\[
e^M(z') = e^N(z) R_M^{-1} \cdot N_M(z), \quad D'_M = R^{-1}_M N(z) D_N,
\]

(2.21)

so that the exterior derivative is left invariant

\[
e^M(z) D_M = e^M(z') D'_M,
\]

(2.22)

where \( R_M^{-1} N_M(z) \) is a \((3 + 2N) \times (3 + 2N)\) supermatrix of the form

\[
R_M^{-1} N_M(z) = \begin{pmatrix}
R^\mu_\nu(z) & \partial_\mu \theta^{h \beta} \\
- B^{\alpha \alpha}_a(z) & -D_{\alpha \alpha} \theta^{h \beta}
\end{pmatrix},
\]

(2.23)

with

\[
R^\mu_\nu(z) = \frac{\partial x^\nu}{\partial x^\mu} - i \bar{\theta}^a \gamma^\nu \frac{\partial \theta^a}{\partial x^\mu},
\]

(2.24)

\[
B^{\mu \alpha}_a(z) = D_{\alpha \alpha} e^\mu + i \bar{\theta}^a \gamma^\mu D_{\alpha \alpha} \theta^b.
\]

(2.25)

For Majorana spinors it is useful to note from eqs.(2.2,2.3,3.3a)

\[
\xi_\alpha \rho^a = \bar{\rho}_a \xi^a,
\]

(2.26a)
\[
\rho^a \bar{\varepsilon}_a + \varepsilon^a \bar{\rho}_a + \bar{\rho}_a \varepsilon^a 1 = 0, \tag{2.26b}
\]
\[
\bar{\rho}_a \gamma^{\mu_1} \gamma^{\mu_2} \cdots \gamma^{\mu_n} \varepsilon^a = (-1)^n \bar{\varepsilon}_a \gamma^{\mu_n} \cdots \gamma^{\mu_2} \gamma^{\mu_1} \rho^a, \tag{2.26c}
\]
\[
(\bar{\rho}_a \gamma^{\mu_1} \gamma^{\mu_2} \cdots \gamma^{\mu_n} \varepsilon^a)^* = \bar{\varepsilon}_a \gamma^{\mu_n} \cdots \gamma^{\mu_2} \gamma^{\mu_1} \rho^a. \tag{2.26d}
\]

In particular
\[
\theta^a \bar{\theta}_a = -\frac{1}{2} \bar{\theta}_a \theta^a 1. \tag{2.27}
\]

3 Superconformal Symmetry in Three-dimensions

In this section we first define the three-dimensional superconformal group on superspace and then discuss its superconformal Killing equation along with the solutions.

3.1 Superconformal Group & Killing Equation

The superconformal group is defined here as a group of superspace coordinate transformations, \( z \xrightarrow{g} z' \), that preserve the infinitesimal supersymmetric interval length, \( e^2 = \eta_{\mu\nu} e^\mu e^\nu \), up to a local scale factor, so that
\[
e^2(z) \rightarrow e^2(z') = \Omega^2(z; g) e^2(z), \tag{3.1}
\]
where \( \Omega(z; g) \) is a local scale factor.

This requires \( B_{\alpha a}^\mu (z) = 0 \)
\[
D_{\alpha a} x^\mu + i \bar{\theta}_b \gamma^\mu D_{\alpha a} \theta^b = 0, \tag{3.2}
\]
and
\[
e^\mu(z') = e^\nu(z) R^\nu_{\mu}(z; g), \tag{3.3}
\]
\[
R^\lambda_{\nu}(z; g) R^\nu_{\rho}(z; g) \eta_{\lambda\rho} = \Omega^2(z; g) \eta_{\mu\nu}, \quad \det R(z; g) = \Omega^3(z; g). \tag{3.4}
\]

Hence \( \mathcal{R}_M^N \) in eq.(2.23) is of the form
\[
\mathcal{R}_M^N(z; g) = \begin{pmatrix} R^\nu_{\mu}(z; g) & \partial_\mu \theta^{\rho b} \\ 0 & -D_{\alpha a} \theta^{\rho b} \end{pmatrix}. \tag{3.5}
\]

\[\text{More explicit form of } \mathcal{R}_M^N \text{ is obtained later in eq.(4.40).}\]
Infinitesimally $z' \simeq z + \delta z$, eq. (3.2) gives
\[ D_{a\alpha} h^\mu = 2i(\bar{\lambda}_a \gamma^\mu)_\alpha, \] (3.6)
or equivalently
\[ D^{a\alpha} h^\mu = -2i(\gamma^\mu \lambda^a)_\alpha, \] (3.7)
where we define
\[ \lambda^a = \delta \theta^a, \quad \bar{\lambda}_a = \delta \bar{\theta}_a, \] (3.8)
\[ h^\mu = \delta x^\mu - i\bar{\theta}^a \gamma^\mu \delta \theta^a. \]

Infinitesimally from eq. (2.24) $R_{\mu \nu}$ is of the form
\[ R_{\mu \nu} \simeq \delta_{\nu}^\mu + \partial_{\mu} h_{\nu}, \] (3.9)
so that the condition (3.4) reduces to the ordinary conformal Killing equation
\[ \partial_{\mu} h_{\nu} + \partial_{\nu} h_{\mu} \propto \eta_{\mu \nu}. \] (3.10)

We note that eq. (3.10) follows from eq. (3.6). Using the anti-commutator relation for $D_{a\alpha}$ (2.20) we get from eqs. (3.6, 3.7, A.1a)
\[ \delta_{a\alpha} \partial_{\nu} h_{\mu} = \frac{1}{2} \left( D^{a\alpha} (\bar{\lambda}_b \gamma^\mu \gamma^\nu)_\alpha - (\gamma^\nu \gamma^\mu D_{ba} \lambda^a)_\alpha \right), \] (3.11)
and hence
\[ \delta_{a}^b \left( \partial_{\mu} h_{\nu} + \partial_{\nu} h_{\mu} \right) = (D^{a\alpha} \bar{\lambda}_{ba} - D_{ba} \lambda^{a\alpha}) \eta_{\mu \nu}, \] (3.12)
which implies eq. (3.10). Thus eq. (3.6) or eq. (3.7) is a necessary and sufficient condition for a supercoordinate transformation to be superconformal.

From eqs. (3.6, 3.7) $\lambda^{a\alpha}, \bar{\lambda}_{a\alpha}$ are given by
\[ \lambda^{a\alpha} = i \frac{1}{6} \bar{D}^{a\beta} h^\alpha_{\beta}, \quad \bar{\lambda}_{a\alpha} = -i \frac{1}{6} D_{a\beta} h^\beta_{\alpha}, \] (3.13)
where
\[ h^\alpha_{\beta} = h^{\mu \gamma}_{\alpha \beta}. \] (3.14)

Substituting these expressions back into eqs. (3.6, 3.7) gives using eqs. (2.1, 2.4)
\[ D_{a\alpha} h^\mu = -i \frac{1}{2} \epsilon_{\nu \lambda} D_{a\beta} h^{\nu \gamma}_{\lambda \alpha}, \] (3.15)
\[ \bar{D}^{a\alpha} h^\mu = i \frac{1}{2} \epsilon_{\nu \lambda} \gamma_{\beta} D_{a\beta} h^{\nu \lambda}_{\alpha}. \]
or equivalently

\[ D_{\alpha\beta} h^\gamma = \frac{2}{3} \delta_\alpha^\beta D_{\alpha\delta} h^\gamma - \frac{1}{3} \delta^\beta_\gamma D_{\alpha\delta} h^\alpha, \]

\[ \bar{D}^{\alpha\beta} h^\gamma = \frac{2}{3} \delta_\alpha^\beta D^{\alpha\delta} h^\gamma - \frac{1}{3} \delta^\beta_\gamma D^{\alpha\delta} h^\alpha. \]

Eq.(3.15) or eq.(3.16) may therefore be regarded as the fundamental superconformal Killing equation and its solutions give the generators of extended superconformal transformations in three-dimensions. The general solution is

\[ h^\mu(z) = 2x \cdot b x^\mu - (x^2 - \frac{1}{4}(\bar{\theta}_a \theta^a)^2) b^\mu + \epsilon_{\mu\nu\lambda} x^\nu b^\lambda \bar{\theta}_a \theta^a \]

\[-2 \bar{\rho}_a x_{-} \gamma^\mu \theta^a + w^\mu_{\nu} x^\nu + \frac{1}{4} \epsilon_{\mu\nu\lambda} w^\nu \lambda \bar{\theta}_a \theta^a + \lambda x^\mu\]

\[ + it^a \bar{\theta}_b \gamma^\mu \theta^a + 2i \varepsilon^a \gamma^\mu \theta^a + a^\mu, \]

where \( a^\mu, b^\mu, \lambda, w_{\mu\nu} = -w_{\nu\mu} \) are real, \( \varepsilon^a, \rho^a \) satisfy the Majorana condition (2.9) and \( t \in \text{so}(N) \) satisfying

\[ t^\dagger = t = -t. \]

We also set

\[ x = x^\mu \gamma, \quad x_{\pm} = x \pm \frac{i}{2} \bar{\theta}_a \theta^a 1. \]

Eq.(3.17) gives

\[ \lambda^a = x_{+} b_{-} \gamma^a - i x_{+} \rho^a + 2(\bar{\rho}_b \theta^b) \theta^a + (w + \frac{1}{2} \lambda) \theta^a - \theta^b t^a + \varepsilon^a \]

satisfying the Majorana condition

\[ \bar{\lambda}_a = \lambda^a \gamma^0 = -\lambda^a \varepsilon, \]

where we put

\[ w = \frac{1}{4} w_{\mu
u} \gamma^\mu \gamma^\nu. \]

For later use it is worth to note

\[ \gamma^0 w \gamma^0 = -w^\dagger, \quad \epsilon w \epsilon^{-1} = -w^t. \]

\[ ^4 \text{A method of obtaining the solution (3.17) is demonstrated in Appendix E.} \]
3.2 Extended Superconformal Transformations

In summary, the generators of superconformal transformations in three-dimensions acting on the three-dimensional superspace, $\mathbb{R}^{3|2N}$, with coordinates, $z^M = (x^\mu, \theta^{\alpha})$, can be classified as

1. Supertranslations, $a, \varepsilon$

$$\delta x^\mu = a^\mu - i\bar{\theta}_a \gamma^\mu \varepsilon^a, \quad \delta \theta^a = \varepsilon^a.$$  \hspace{1cm} (3.24)

   This is consistent with eq.(2.15).

2. Dilations, $\lambda$

$$\delta x^\mu = \lambda x^\mu, \quad \delta \theta^a = \frac{i}{2} \lambda \theta^a.$$  \hspace{1cm} (3.25)

3. Lorentz transformations, $w$

$$\delta x^\mu = w^\mu \nu x^\nu, \quad \delta \theta^a = w \theta^a.$$  \hspace{1cm} (3.26)

4. $R$-symmetry transformations, $t$

$$\delta x^\mu = 0, \quad \delta \theta^a = -\theta^b t_{b}{}^a.$$  \hspace{1cm} (3.27)

   where $t \in \mathfrak{so}(N)$ of dimension $\frac{1}{2}N(N-1)$.

5. Special superconformal transformations, $b, \rho$

$$\delta x^\mu = 2x \cdot b x^\mu - (x^2 + \frac{1}{4}(\bar{\theta}_a \theta^a)^2) b^\mu - \bar{\rho}_a x \gamma^\mu \theta^a,$$

$$\delta \theta^a = x \cdot b \gamma \theta^a - i x \cdot \rho^a + 2(\bar{\rho}_b \theta^b) \theta^a.$$  \hspace{1cm} (3.28)

As we consider infinitesimal transformations we obtain $\text{SO}(N)$ as $R$-symmetry group. However finitely $R$-symmetry group can be extended to $\text{O}(N)$ which leaves the supertranslation invariant one form (2.16) invariant manifestly.
3.3 Superinversion

In three-dimensions we define superinversion, \( z^M \xrightarrow{i_s} z'^M = (x'^\mu, \theta'^a) \in \mathbb{R}^{3|2N} \), by

\[
x'_\pm = -x^-_\pm, \quad \theta'^a = ix^-_a \theta^a.
\] (3.29)

As a consistency check we note from \( x_+ x_- = (x^2 + \frac{1}{4} (\bar{\theta}_a \theta^a)^2) 1 \)

\[
\bar{\theta}_a = \theta'^a \gamma^0 = -\theta'^a \epsilon, \quad x'_+ - x'_- = i\bar{\theta}_a \theta^a 1.
\] (3.30)

It is easy to verify that superinversion is idempotent

\[
i^2 = 1.
\] (3.31)

Using

\[
e(z) = e^\mu(z) \gamma_\mu = dx_+ + 2i \theta^a \bar{\theta}_a,
\] (3.32)

we get under superinversion

\[
e(z') = x'^-1 e(z)x^-1.
\] (3.33)

and hence

\[
e^2(z') = \Omega^2(z; i_s)e^2(z), \quad \Omega(z; i_s) = \frac{1}{x^2 + \frac{1}{4} (\theta_a \theta^a)^2}.
\] (3.34)

Eq.(3.33) can be rewritten as

\[
e^\mu(z') = e^\mu(z) R^\mu(z; i_s), \quad R^\mu(z; i_s) = \frac{1}{2} \text{tr}(\gamma_\nu x'^-1 \gamma^\mu x'^-1).
\] (3.35)

Explicitly

\[
R^\mu(z; i_s) = \frac{1}{(x^2 + \frac{1}{4} (\theta_a \theta^a)^2)^2} \left( 2x_\nu x^\mu - (x^2 - \frac{1}{4} (\theta_a \theta^a)^2) \delta^\mu_\nu - \epsilon_\nu^\mu \lambda x_\lambda \theta_a \theta^a \right).
\] (3.36)

Note that

\[
\gamma^\nu R^\mu(z; i_s) = x'^-1 \gamma^\nu x'^-1, \quad R^\mu(z; i_s) \gamma_\mu = x'^-1 \gamma_\nu x'^-1.
\] (3.37)

If we consider a transformation, \( z \xrightarrow{g \circ i_s} z' \), where \( g \) is a three-dimensional superconformal transformation, then we get

\[
h^\mu(z) = 2x_\nu x^\nu - (x^2 - \frac{1}{4} (\bar{\theta}_a \theta^a)^2) \delta^\nu_\mu + e^\nu_\lambda x^\nu a^\lambda \bar{\theta}_a \theta^a
\]

\[
-2\bar{\epsilon}_a \gamma^\mu x_\gamma \theta^a + w^\mu_\nu x^\nu + \frac{1}{4} \epsilon^\mu_\nu \lambda x^\nu \theta_\lambda \bar{\theta}_a \theta^a - \lambda x^\mu
\]

\[
+ i t_a \gamma^\mu \theta^a + 2i \bar{\theta}_a \gamma^\mu \theta^a + b^\mu.
\] (3.38)
Hence, under superinversion, the superconformal transformations are related by

\[
\mathcal{K} \equiv \begin{pmatrix}
    a^\mu \\
    b^\mu \\
    \varepsilon^a \\
    \rho^a \\
    \lambda \\
    w^\mu_{\nu} \\
    t^a_b
\end{pmatrix} \rightarrow \begin{pmatrix}
    b^\mu \\
    a^\mu \\
    \varepsilon^a \\
    \rho^a \\
    -\lambda \\
    w^\mu_{\nu} \\
    t^a_b
\end{pmatrix}.
\]

(3.39)

In particular, special superconformal transformations (3.28) can be obtained by

\[
z \xrightarrow{i_s \circ (b, \rho) \circ i_s} z',
\]

(3.40)

where \((b, \rho)\) is a supertranslation.

### 3.4 Superconformal Algebra

The generator of infinitesimal superconformal transformations, \(\mathcal{L}\), is given by

\[
\mathcal{L} = h^\mu \partial_\mu - \lambda^{\alpha \alpha} D_{a a}.
\]

(3.41)

If we write the commutator of two generators, \(\mathcal{L}_1, \mathcal{L}_2\), as

\[
[\mathcal{L}_2, \mathcal{L}_1] = \mathcal{L}_3 = h_3^\mu \partial_\mu - \lambda_3^{\alpha \alpha} D_{a a},
\]

(3.42)

then \(h_3^\mu, \lambda_3^{\alpha \alpha}\) are given by

\[
h_3^\mu = h_2^\nu \partial_\nu h_1^\mu - h_1^\nu \partial_\nu h_2^\mu + 2i \bar{\lambda}_1 \gamma^\mu \lambda_2^a, \\
\lambda_3^a = \mathcal{L}_2 \lambda_1^a - \mathcal{L}_1 \lambda_2^a.
\]

(3.43)
and $h_3^\mu$, $\lambda_3^a$ satisfy eq.(3.7) verifying the closure of the Lie algebra. Explicitly with eqs.(3.17, 3.20) we get

$$
a_3^\mu = w_1^\mu a_2^\nu + \lambda_1 a_2^\mu + i \varepsilon_{1a} \gamma^\mu \varepsilon_2^a - (1 \leftrightarrow 2),$$

$$
\varepsilon_3^a = w_1 \varepsilon_2^a + \frac{1}{2} \lambda_1 \varepsilon_2^a - i a_2 \cdot \rho_1^a \varepsilon_2^a - (1 \leftrightarrow 2),$$

$$
\lambda_3 = 2 a_2 \cdot b_1 - 2 \tilde{\rho}_1 \varepsilon_2^a - (1 \leftrightarrow 2),$$

$$
w_3^{\mu\nu} = w_1^\mu w_2^\nu + 2(a_2^\mu b_1^\nu - a_2^\nu b_1^\mu) + 2 \tilde{\rho}_1 \gamma^{[\mu} \gamma_{\nu]} \varepsilon_2^a - (1 \leftrightarrow 2),$$

$$
b_3^\mu = w_1^\mu b_2^\nu - \lambda_1 b_2^\mu + i \tilde{\rho}_1 \gamma^\mu \rho_2^a - (1 \leftrightarrow 2),$$

$$
\rho_3^a = w_1 \rho_2^a - \frac{1}{2} \lambda_1 \rho_2^a - i b_2 \cdot \gamma_1 \varepsilon_2^a - \rho_2 \varepsilon_1 t_1^a - (1 \leftrightarrow 2),$$

$$
t_{3a}^b = (t_1 t_2)_a^b + 2(\bar{\rho}_2 \varepsilon_1^b - \varepsilon_1 \rho_2^b) - (1 \leftrightarrow 2).$$

From eq.(3.44) we can read off the explicit forms of three-dimensional superconformal algebra as exhibited in Appendix C.

If we define a $(4 + 2N) \times (4 + 2N)$ supermatrix, $M$, as

$$
M = \begin{pmatrix}
  w + \frac{1}{2} \lambda & i a \cdot \gamma & \sqrt{2} \varepsilon^b \\
i b \cdot \gamma & w - \frac{1}{2} \lambda & \sqrt{2} \rho^b \\
-\sqrt{2} \tilde{\rho}_a & -\sqrt{2} \varepsilon_a & t_a^b
\end{pmatrix},
$$

then the relation above (3.44) agrees with the matrix commutator

$$
[M_1, M_2] = M_3.
$$

In general, $M$ can be defined as a $(4, 2N)$ supermatrix subject to

$$
BMB^{-1} = -M^\dagger, \quad B = \begin{pmatrix}
  0 & \gamma^0 & 0 \\
\gamma^0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix},
$$

$$
CMC^{-1} = -M^t, \quad C = \begin{pmatrix}
  0 & \epsilon & 0 \\
\epsilon & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}.
$$
Supermatrix of the form (3.45) is the general solution of these two equations.

The 4 × 4 matrix appearing in $M$,

$$
\begin{pmatrix}
    w + \frac{1}{2}\lambda & ia \cdot \gamma \\
    ib \cdot \gamma & w - \frac{1}{2}\lambda
\end{pmatrix},
$$

(3.48)
corresponds to a generator of $SO(2,3) \cong Sp(2,\mathbb{R})$ as demonstrated in Appendix D. Thus, the $\mathcal{N}$-extended Majorana superconformal group in three-dimensions may be identified with the supermatrix group generated by supermatrices of the form $M$ (3.45), which is $OSp(\mathcal{N}|2,\mathbb{R}) \equiv G_S$ having dimensions $(10 + \frac{1}{2}\mathcal{N}(\mathcal{N} - 1)|4\mathcal{N})$. 
4 Coset Realization of Transformations

In this section, we first obtain an explicit formula for the finite non-linear superconformal transformations of the supercoordinates and discuss several representations of the superconformal group. We then construct matrix or vector valued functions depending on two or three points in superspace which transform covariantly under superconformal transformations. These variables serve as building blocks of obtaining two-point, three-point and general \( n \)-point correlation functions later.

4.1 Superspace as a Coset

To obtain an explicit formula for the finite non-linear superconformal transformations, we first identify the superspace, \( \mathbb{R}^{3|2N} \), as a coset, \( G_S/G_0 \), where \( G_0 \subset G_S \) is the subgroup generated by matrices, \( M_0 \), of the form (3.45) with \( a^\mu = 0 \), \( \varepsilon^a = 0 \) and depending on parameters \( b^\mu, \rho^a, \lambda, w_{\mu\nu}, t_{a\ b} \). The group of supertranslations, \( G_T \), parameterized by coordinates, \( z^M \in \mathbb{R}^{3|2N} \), has been defined by general elements as in eq.(2.11) with the group property given by eqs.(2.14, 2.15). Now we may represent it by supermatrices

\[
G_T(z) = \exp \begin{pmatrix} 0 & ix & \sqrt{2}\theta^b \\ 0 & 0 & 0 \\ -\sqrt{2}\bar{\theta}_a & 0 \end{pmatrix} = \begin{pmatrix} 1 & ix_\epsilon & \sqrt{2}\theta^b \\ 0 & 1 & 0 \\ 0 & -\sqrt{2}\bar{\theta}_a & \delta^b_a \end{pmatrix}. \tag{4.1}
\]

Note \( G_T(z)^{-1} = G_T(-z) \).

In general an element of \( G_S \) can be uniquely decomposed as \( G_TG_0^{-1} \). Thus for any element \( G(g) \in G_S \) we may define a superconformal transformation, \( z \xrightarrow{g} z' \), and an associated element \( G_0(z; g) \in G_0 \) by

\[
G(g)^{-1}G_T(z)G_0(z; g) = G_T(z'). \tag{4.2}
\]

If \( G(g) \in G_T \) then clearly \( G_0(z; g) = 1 \). Infinitesimally eq.(4.2) becomes

\[
\delta G_T(z) = MG_T(z) - G_T(z)\hat{M}_0(z), \tag{4.3}
\]

where \( M \) is given by eq.(3.45) and \( \hat{M}_0(z) \), the generator of \( G_0 \), has the form

\[
\hat{M}_0(z) = \begin{pmatrix} \hat{w}(z) + \frac{1}{2}\hat{\lambda}(z) & 0 & 0 \\ ib\cdot\gamma & \hat{w}(z) - \frac{1}{2}\hat{\lambda}(z) & \sqrt{2}\hat{\rho}^b(z) \\ -\sqrt{2}\hat{\rho}_a(z) & 0 & \hat{t}_{a\ b}(z) \end{pmatrix}. \tag{4.4}
\]

\[\text{The subscript, } T, \text{ denotes supertranslations.}\]
The components depending on \( z \) are given by

\[
\hat{w}(z) + \frac{1}{2} \hat{\lambda}(z) = w + \frac{1}{2} \lambda + x_+ b \cdot \gamma + 2 \theta^a \bar{\rho}_a, \\
\hat{w}(z) - \frac{1}{2} \hat{\lambda}(z) = w - \frac{1}{2} \lambda - b \cdot \gamma x_- - 2 \rho^a \tilde{\theta}_a, \\
\hat{\lambda}(z) = \lambda + 2x \cdot b - 2 \tilde{\theta}_a \rho^a, \\
\hat{\rho}^a(z) = \rho^a + ib \cdot \gamma \theta^a, \\
\hat{\theta}_a(z) = \tilde{\rho}_a - i \theta_a b \cdot \gamma = (\hat{\rho}^a(z))^\dagger \gamma^b = -\hat{\rho}^a(z)^\dagger \epsilon, \\
\hat{t}_{ab}(z) = t_{ab} + 2i \tilde{\theta}_a b \cdot \gamma \theta^b + 2 \tilde{\theta}_a \rho^b - 2 \rho_a \theta^b. \\
\]

(4.5)

\( \hat{w}(z) \) can be also written as \( \hat{w}(z) = \frac{1}{4} \hat{w}_{\mu\nu}(z) \gamma^\mu \gamma^\nu \) with

\[
\hat{w}_{\mu\nu}(z) = w_{\mu\nu} + 2(x_\mu b_\nu - x_\nu b_\mu) + \epsilon_{\mu\nu\lambda}(b^\lambda \tilde{\theta}_a \theta^a + 2i \bar{\rho}_a \gamma^\lambda \theta^a). \\
\]

(4.6)

Writing \( \delta G_T(z) = \mathcal{L}G_T(z) \) we may verify that \( \mathcal{L} \) is identical to eq. (3.41).

The definitions (4.5) can be summarized by

\[
D_{ab} \lambda^{b\beta}(z) = -\frac{1}{2} \delta_a^b \delta_\alpha^\beta \hat{\lambda}(z) - \delta_a^b \hat{w}_\alpha^\beta(z) + \delta_\alpha^\beta \hat{t}_{ab}(z), \\
\hat{\partial}_\nu h_\mu(z) = \hat{w}_{\mu\nu}(z) + \eta_{\mu\nu} \hat{\lambda}(z), \\
\]

(4.7a)

(4.7b)

and they give

\[
[D_{aa}, \mathcal{L}] = -D_{aa} \lambda^{b\beta} D_{b\beta} = \left( \frac{1}{2} \delta_a^b \delta_\alpha^\beta \hat{\lambda}(z) + \delta_a^b \hat{w}_\alpha^\beta(z) - \delta_\alpha^\beta \hat{t}_{ab}(z) \right) D_{b\beta}. \\
\]

(4.8)

For later use we note

\[
D_{aa} \hat{w}_{\mu\nu}(z) = 2(\hat{\rho}_a(z) \gamma_{[\mu} \gamma_{\nu]}), \\
D_{aa} \hat{\lambda}(z) = 2\hat{\rho}_{aa}(z), \\
D_{aa} \hat{t}_{ab}(z) = 2(\delta_{ab} \delta_{cd} - \delta_a^c \delta_b^d) \hat{\rho}_{da}. \\
\]

(4.9)
The above analysis can be simplified by reducing $G_0(z; g)$. To achieve this we let

$$Z_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and then

$$M_0 Z_0 = Z_0 H_0, \quad H_0 = \begin{pmatrix} w - \frac{1}{2} \lambda & \sqrt{2} \rho^b \\ 0 & t^b_a \end{pmatrix}.$$  \hspace{1cm} (4.11)

Now if we define

$$Z(z) \equiv G_T(z) Z_0 = \begin{pmatrix} i x_+ & \sqrt{2} \theta^b \\ 1 & 0 \\ -\sqrt{2} \bar{\theta}_a & \delta^b_a \end{pmatrix},$$

then $Z(z)$ transforms under infinitesimal superconformal transformations as

$$\delta Z(z) = L Z(z) = M Z(z) - Z(z) H(z),$$

where $H(z)$ is given by

$$\hat{M}_0(z) Z_0 = Z_0 H(z), \quad H(z) = \begin{pmatrix} \hat{w}(z) - \frac{1}{2} \hat{\lambda}(z) & \sqrt{2} \hat{\rho}^b(z) \\ 0 & \hat{t}^b_a(z) \end{pmatrix}.$$  \hspace{1cm} (4.14)

From eqs. (3.42, 3.46) considering

$$[\mathcal{L}_2, \mathcal{L}_1] Z(z) = \mathcal{L}_3 Z(z),$$

we get

$$H_3(z) = \mathcal{L}_2 H_1(z) - \mathcal{L}_1 H_2(z) + [H_1(z), H_2(z)],$$

which gives separate equations for $\hat{w}, \hat{\lambda}, \hat{\rho}$ and $\hat{t}^b_a$, thus $\hat{\lambda}_3 = \mathcal{L}_2 \hat{\lambda}_1 - \mathcal{L}_1 \hat{\lambda}_2$, etc.

As a conjugate of $Z(z)$ we define $\bar{Z}(z)$ by

$$\bar{Z}(z) = \begin{pmatrix} \gamma^0 & 0 \\ 0 & 1 \end{pmatrix} Z(z)^\dagger B = \begin{pmatrix} \epsilon^{-1} & 0 \\ 0 & 1 \end{pmatrix} Z(z)^\dagger C = \begin{pmatrix} 1 & -ix_+ & -\sqrt{2} \theta^b \\ 0 & \sqrt{2} \bar{\theta}_a & \delta^b_a \end{pmatrix}.$$ \hspace{1cm} (4.17)

This satisfies

$$\bar{Z}(z) = \bar{Z}(0) G_T(z)^{-1},$$

and corresponding to eq. (4.13) we have

$$\delta \bar{Z}(z) = L \bar{Z}(z) = \bar{H}(z) \bar{Z}(z) - \bar{Z}(z) M,$$

where

$$\bar{H}(z) = \begin{pmatrix} \hat{w}(z) + \frac{1}{2} \hat{\lambda}(z) & 0 \\ -\sqrt{2} \hat{\rho}_a(z) & \hat{t}^b_a(z) \end{pmatrix}.$$  \hspace{1cm} (4.20)
4.2 Finite Transformations

Finite superconformal transformations can be obtained by exponentiation of infinitesimal transformations. To obtain a superconformal transformation, \( z \rightarrow z' \), we therefore solve the differential equation

\[
\frac{d}{dt} z^M_t = \mathcal{L}^M(z_t), \quad z_0 = z, \quad z_1 = z',
\]

where, with \( \mathcal{L} \) given in eq.(3.41), \( \mathcal{L}^M(z) \) is defined by

\[
\mathcal{L} = \mathcal{L}^M(z) \partial_M.
\]

From eq.(4.13) we get

\[
\frac{d}{dt} Z(z_t) = M Z(z_t) - Z(z_t) H(z_t),
\]

which integrates to

\[
Z(z_t) = e^{tM} Z(z) K(z, t),
\]

where \( K(z, t) \) satisfies

\[
\frac{d}{dt} K(z, t) = -K(z, t) H(z_t), \quad K(z, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Hence for \( t = 1 \) with \( K(z, 1) \equiv K(z; g) \), \( z \rightarrow z' \), eq.(4.24) becomes

\[
Z(z') = G(g)^{-1} Z(z) K(z; g), \quad G(g)^{-1} = e^M.
\]

\( G_0(z; g) \) in eq.(4.2) is related to \( K(z; g) \) from eq.(4.24) by

\[
G_0(z; g) Z_0 = Z_0 K(z; g).
\]

In general \( K(z; g) \) is of the form

\[
K(z; g) = \begin{pmatrix} \Omega(z; g) L(z; g) & \sqrt{2} \Sigma^b(z; g) \\ 0 & U^b(z; g) \end{pmatrix},
\]

where \( \Omega(z; g) \) is identical to the local scale factor in eq.(3.1), \( U(z; g) \in SO(\mathcal{N}) \)

\[
U^{-1} = U^\dagger = U^t, \quad \det U = 1,
\]
and \( L(z; g) \) satisfies
\[
\det L(z; g) = 1, \tag{4.30a}
\]
\[
L^{-1}(z; g) = \epsilon^{-1}L(z; g)^t \epsilon = \gamma^0 L(z; g)^t \gamma^0. \tag{4.30b}
\]

From eq.\( \text{(4.26)} \) \( \bar{Z}(z) \) transforms as
\[
\bar{Z}(z') = \bar{K}(z; g) \bar{Z}(z) G(g), \tag{4.31}
\]
where
\[
\bar{K}(z; g) = \begin{pmatrix} \gamma^0 & 0 \\ 0 & 1 \end{pmatrix} K(z)^t \begin{pmatrix} \gamma^0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \epsilon^{-1} & 0 \\ 0 & 1 \end{pmatrix} K(z)^t \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.32}
\]
\[
= \begin{pmatrix} \Omega(z; g) \frac{1}{2} L^{-1}(z; g) & 0 \\ \sqrt{2} \Sigma_a(z; g) & U_{ab}(z; g) \end{pmatrix}.
\]

If we define for superinversion, \( z \rightarrow z' \), \( \text{(3.29)} \)
\[
G(i_s)^{-1} = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad K(z; i_s) = \begin{pmatrix} -i(\epsilon x_+)^{-1} \sqrt{2}i x_+^{-1} \theta^b \\ -i x_+^t \delta^b_a \delta^a_b \\ 0 \end{pmatrix}, \tag{4.33}
\]
with
\[
V_a^b(z) = \delta_a^b + 2i \tilde{\theta}_a x_+^{-1} \theta^b, \tag{4.34}
\]
an analogous formula to eq.\( \text{(4.26)} \) can be obtained for superinversion
\[
G(i_s)^{-1} Z(z) K(z; i_s) = \begin{pmatrix} 1 & 0 \\ -i x_+^t \sqrt{2} \tilde{\theta}_a^q \\ \sqrt{2} \theta^q_a \delta^b_a \end{pmatrix} = \bar{Z}(z')^t. \tag{4.35}
\]

Similarly we have
\[
\bar{K}(z; i_s) \bar{Z}(z) G(i_s) = Z(z')^t, \tag{4.36}
\]
where
\[
\bar{K}(z; i_s) = \begin{pmatrix} i \epsilon x_+^{-1} & 0 \\ -\sqrt{2} i \tilde{\theta}_a x_+^{-1} \sqrt{2} \theta^b_a \end{pmatrix}. \tag{4.37}
\]
Note that
\[ V^{-1}(z) = V^\dagger = V(z)^t = V(-z) = 1 - 2i\bar{\theta}x_-^{-1}\theta, \quad (4.38a) \]
\[ \theta^a V_a^b(z) = x_- x_+^{-1}\theta^b, \quad V_a^b(z)\bar{\theta}_b = \bar{\theta}_a x_+^{-1}x_-, \quad (4.38b) \]
\[ R_{\mu\nu}(z; g)\gamma_\nu = \Omega(z; g)L^{-1}(z; g)\gamma_\mu L(z; g), \quad (4.38c) \]
\[ \gamma^\nu R_{\nu\mu}(z; g) = \Omega(z; g)L(z; g)\gamma^\mu L^{-1}(z; g). \quad (4.38d) \]
where \( R_{\mu\nu}(z; g) \) is identical to the definition (3.3). We may normalize \( R_{\mu\nu}(z; g) \) as
\[ \hat{R}_{\mu\nu}(z; g) = \Omega(z; g)\frac{1}{2}\text{tr}(\gamma_\mu L(z; g)\gamma^\nu L^{-1}(z; g)) \in SO(1, 2). \quad (4.39) \]

### 4.3 Representations

Based on the results in the previous subsection, it is easy to show that the matrix, \( \mathcal{R}_M(z; g) \), given in eq. (3.3) is of the form
\[ \mathcal{R}_M(z; g) = \begin{pmatrix} \Omega(z; g)\hat{R}_{\mu\nu}(z; g) & i\Omega(z; g)\frac{1}{2}(L^{-1}(z; g)\gamma_\mu\Sigma^b(z; g))\beta \\ 0 & \Omega(z; g)\frac{1}{2}L^{-1}\beta_{\alpha}(z; g)U_{a}^b(z; g) \end{pmatrix}. \quad (4.40) \]

Since \( \mathcal{R}_M(z; g) \) is a representation of the three-dimensional superconformal group, each of the following also forms a representation of the group, though it is not a faithful representation
\[ \Omega(z; g) \in D, \quad \hat{R}(z; g) \in SO(1, 2), \quad (4.41) \]
\[ L(z; g), \quad U(z; g) \in O(\mathcal{N}), \]
where \( D \) is the one dimensional group of dilations.

Under the successive superconformal transformations, \( g'' : z \xrightarrow{g} z' \xrightarrow{g'} z'' \), they satisfy
\[ L(z; g)L(z'; g') = L(z; g''), \quad \text{and so on.} \quad (4.42) \]

### 4.4 Functions of Two Points

In this subsection, we construct matrix valued functions depending on two points, \( z_1 \) and \( z_2 \), in superspace which transform covariantly like a product of two tensors at \( z_1 \) and \( z_2 \)
under superconformal transformations.

If $F(z)$ is defined for $z \in \mathbb{R}^{3|2N}$ by

$$F(z) = \bar{Z}(0)G_T(z)Z(0) = \begin{pmatrix} i x_- & \sqrt{2} \theta^b \\ -\sqrt{2} \bar{\theta}_a & \delta^b_a \end{pmatrix}, \quad (4.43)$$

then $F(z)$ satisfies

$$F(-z) = \begin{pmatrix} \gamma^0 & 0 \\ 0 & 1 \end{pmatrix} F(z) \begin{pmatrix} \gamma^0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \epsilon^{-1} & 0 \\ 0 & 1 \end{pmatrix} F(z) \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}, \quad (4.44)$$

and the superdeterminant of $F(z)$ is given by

$$\text{sdet} F(z) = - \text{det} x_+ = x^2 + \frac{1}{4} (\bar{\theta}_a \theta^a)^2. \quad (4.45)$$

We also note

$$\begin{pmatrix} 1 \\ -i \sqrt{2} \theta_a x^{-1}_- \end{pmatrix} F(z) \begin{pmatrix} 1 \\ i \sqrt{2} x^{-1}_- \theta^b \end{pmatrix} = \begin{pmatrix} i x_- & 0 \\ 0 & V^b_a(-z) \end{pmatrix}, \quad (4.46)$$

where $V^b_a(-z)$ is identical to eq.(4.38a) and from eqs.(4.45, 4.46) it is evident that

$$\text{det} V(z) = 1. \quad (4.47)$$

Hence, with eq.(4.38a), $V(z) \in \text{SO}(N)$.

Now with the supersymmetric interval for $\mathbb{R}^{3|2N}$ defined by

$$G_T(z_2)^{-1} G_T(z_1) = G_T(z_{12}), \quad z_{12}^M = (x^\mu_{12}, \theta^a_{12}, \bar{\theta}_{12a}) = -z_2^M, \quad (4.48)$$

we may write

$$\bar{Z}(z_2)Z(z_1) = F(z_{12}) = \begin{pmatrix} i x_{12}^- & \sqrt{2} \theta^b_{12} \\ -\sqrt{2} \bar{\theta}_{12a} & \delta^b_a \end{pmatrix}, \quad (4.49)$$

and

$$\text{sdet} F(z_{12}) = x_{12}^2 + \frac{1}{4} (\bar{\theta}_{12a} \theta^a_{12})^2, \quad \text{det} V(z_{12}) = 1, \quad (4.50)$$
From eq. (4.53b) we get

\[ x_{12-} = x_{1-} - x_{2+} - 2i\theta_2^b \bar{\theta}_{1a} = x_{12} - i\frac{1}{2} \bar{\theta}_{12a} \theta_{12}^a \, 1, \]

\[ x_{12+} = x_{1+} - x_{2-} + 2i\theta_1^a \bar{\theta}_{2a} = x_{12} + i\frac{1}{2} \bar{\theta}_{12a} \theta_{12}^a \, 1. \]  

(4.51)

From eqs. (4.26, 4.31) \( F(z_{12}) \) transforms as

\[ F(z'_{12}) = \bar{K}(z_2; g)F(z_{12})K(z_1; g). \]  

(4.52)

Explicitly with eqs. (4.28, 4.32) we get the transformation rules for \( x'_{12\pm} \) and \( \theta_{12}^a \)

\[ x'_{12-} = \Omega(z_1; g) \frac{1}{2} \Omega(z_2; g) \frac{1}{2} L^{-1}(z_2; g) x_{12-} L(z_1; g), \]

(4.53a)

\[ x'_{12+} = \Omega(z_1; g) \frac{1}{2} \Omega(z_2; g) \frac{1}{2} L^{-1}(z_1; g) x_{12+} L(z_2; g), \]

\[ \theta_{12}^a = \Omega(z_1; g) \frac{1}{2} L^{-1}(z_1; g) (\theta_{12}^b U^a_b(z_2; g) + i x_{12-} \Sigma^a_2), \]

(4.53b)

\[ \theta_{21}^a = \Omega(z_2; g) \frac{1}{2} L^{-1}(z_2; g) (\theta_{21}^b U^a_b(z_1; g) - i x_{12-} \Sigma^a_1). \]

In particular

\[ x_{12}^2 + \frac{1}{4} (\bar{\theta}_{12a} \theta_{12}^a)^2 = \Omega(z_1; g) \Omega(z_2; g) (x_{12}^2 + \frac{1}{4} (\bar{\theta}_{12a} \theta_{12}^a)^2). \]  

(4.54)

From eqs. (4.38d, 4.53a) \( \text{tr}(\gamma^\mu x_{12-} \gamma^\nu x_{12+}) \) transforms covariantly as

\[ \text{tr}(\gamma^\mu x'_{12-} \gamma^\nu x'_{12+}) = \text{tr}(\gamma^\lambda x_{12-} \gamma^\rho x_{12+})R^\mu_\lambda(z_2; g)R^\nu_\rho(z_1; g). \]  

(4.55)

From eq. (4.53b) we get

\[ \begin{pmatrix} 1 & -i\sqrt{2}x_{12-}^\dagger \theta_{12}^c \\ 0 & \delta_a^c \end{pmatrix} K(z_1; g) \begin{pmatrix} 1 & i\sqrt{2}x_{12+}^\dagger \theta_{12}^b \\ 0 & \delta_a^b \end{pmatrix} \]

\[ = \begin{pmatrix} \Omega(z_1; g) \frac{1}{2} L(z_1; g) & 0 \\ 0 & U(z_1; g) \end{pmatrix}, \]  

(4.56a)

\[ \begin{pmatrix} 1 & 0 \\ -i\sqrt{2}\bar{\theta}_{12a} x_{12-}^\dagger & \delta_a^c \end{pmatrix} \bar{K}(z_2; g) \begin{pmatrix} 1 & 0 \\ i\sqrt{2}\bar{\theta}_{12b} x_{12+}^\dagger & \delta_a^b \end{pmatrix} \]

\[ = \begin{pmatrix} \Omega(z_2; g) \frac{1}{2} L^{-1}(z_2; g) & 0 \\ 0 & U^{-1}(z_2; g) \end{pmatrix}. \]  

(4.56b)
Using this and eq. (4.46) we can rederive eq. (4.53a) and obtain

\[ V(z_{12}') = U^{-1}(z_1; g)V(z_{12})U(z_2; g), \]

\[ V(z_{21}') = U^{-1}(z_2; g)V(z_{21})U(z_1; g). \]  

(4.57)

### 4.5 Functions of Three Points

In this subsection, for three points, \( z_1, z_2, z_3 \) in superspace, we construct ‘tangent’ vectors, \( Z_i \), which transform homogeneously at \( z_i, \ i = 1, 2, 3. \)

With \( z_{21} \overset{L}{\rightarrow} (z_{21})', \ z_{31} \overset{L}{\rightarrow} (z_{31})' \), we define \( Z_i^M = (X_i^a, \Theta_i^a) \in \mathbb{R}^{3|2N} \) by

\[ G_T((z_{31})')^{-1}G_T((z_{21})') = G_T(Z_1). \]  

(4.58)

Explicit expressions for \( Z_i^M \) can be obtained by calculating

\[ \tilde{Z}((z_{31})')Z((z_{21})') = F(Z_1) = \begin{pmatrix} iX_{1-} & \sqrt{2}X_{1-}^a \\ -\sqrt{2}\Theta_{1a} & \delta_a^b \end{pmatrix}. \]  

(4.59)

We get

\[ X_{1-} = x_{31+}^{-1}x_{23-}^{-1}x_{21-}^{-1}, \]

\[ \Theta_1^a = i(x_{21+}^{-1}\theta_2^a - x_{31+}^{-1}\theta_3^a), \quad \bar{\Theta}_{1a} = -i(\bar{\theta}_{21a}x_{21-}^{-1} - \bar{\theta}_{31a}x_{31-}^{-1}). \]  

(4.60)

Using

\[ x_{23-} = x_{21-} - x_{31+} - 2i\theta_{31}^{a}\bar{\theta}_{21a}, \]  

one can assure

\[ X_{1+} - X_{1-} = -2i\Theta_1^a\bar{\Theta}_{1a} = i\bar{\Theta}_{1a}\Theta_1^a, \]  

\[ (4.61) \]

(4.62)

From eq. (4.53a) under superconformal transformations, \( z \overset{g}{\rightarrow} z' \), \( X_1^a, \Theta_1^a, \bar{\Theta}_{1a} \) transform as

\[ X_1^a = \Omega(z_1; g)^{-1}L^{-1}(z_1; g)X_{1+}L(z_1; g), \]  

(4.63a)

\[ \Theta_1^a = \Omega(z_1; g)^{-\frac{1}{2}}L^{-1}(z_1; g)\Theta_{1+}U_a(z_1; g), \]  

(4.63b)

\[ \bar{\Theta}_{1a} = \Omega(z_1; g)^{-\frac{1}{2}}U_{1-}^{-1}\theta_{1b}(z_1; g)\bar{\Theta}_{1b}L(z_1; g), \]  

(4.63c)
so that
\[ X'_\mu = \Omega(z_1; g)^{-1} X'^{\nu} \hat{R}_{\nu}^{\mu}(z_1; g). \] (4.64)

Thus \( \mathcal{Z}_1 \) transforms homogeneously at \( z_1 \), as ‘tangent’ vectors do.

Eq. (4.63a) can be summarized as
\[
F(\mathcal{Z}'_1) = \begin{pmatrix} \Omega(z_1; g)^{-\frac{1}{2}} L^{-1}(z_1 g) & 0 \\ 0 & U^{-1}(z_1; g) \end{pmatrix} F(\mathcal{Z}_1) \begin{pmatrix} \Omega(z_1; g)^{-\frac{1}{2}} L(z_1 g) & 0 \\ 0 & U(z_1; g) \end{pmatrix}.
\] (4.65)

Direct calculation using eq. (4.38b) shows that
\[ V(\mathcal{Z}_1) = V(z_12) V(z_23) V(z_31). \] (4.66)

Similarly for \( R_{\mu}^{\nu}(z; i_s) \) given in eq. (3.33) we obtain from eqs. (3.37, 4.60)
\[
R(\mathcal{Z}_1; i_s) = \left( x_{12}^2 + \frac{1}{4} (\theta_{12a} \theta_{12}^a)^2 \right)^2 \left( x_{31}^2 + \frac{1}{4} (\theta_{31a} \theta_{31}^a)^2 \right)^2 \left( x_{23}^2 + \frac{1}{4} (\theta_{23a} \theta_{23}^a)^2 \right)^2 \left( R(z_{12}; i_s) R(z_{23}; i_s) R(z_{31}; i_s) \right). \] (4.67)

From eqs. (4.55, 4.57) \( V_{ab}^{\mu}(\mathcal{Z}_1) \), \( R_{\mu}^{\nu}(\mathcal{Z}_1; i_s) \) transform homogeneously at \( z_1 \) under superconformal transformation, \( z \rightarrow g \rightarrow z' \),
\[
V(\mathcal{Z}_1') = U^{-1}(z_1; g) V(\mathcal{Z}_1) U(z_1; g),
\] (4.68a)
\[
R(\mathcal{Z}_1'; i_s) = \Omega(z_1; g)^2 R^{-1}(z_1; g) R(\mathcal{Z}_1; i_s) R(z_1; g).
\] (4.68b)

It is useful to note
\[
\det X_{1\pm} = -X_1^2 - \frac{1}{4}(\tilde{\Theta}_a \Theta_1^a)^2 = -\frac{x_{23}^2 + \frac{1}{4} (\theta_{23a} \theta_{23}^a)^2}{\left( x_{12}^2 + \frac{1}{4} (\theta_{12a} \theta_{12}^a)^2 \right) \left( x_{31}^2 + \frac{1}{4} (\theta_{31a} \theta_{31}^a)^2 \right)}. \] (4.69)

By taking cyclic permutations of \( z_1, z_2, z_3 \) in eq. (4.60) we may define \( \mathcal{Z}_2, \mathcal{Z}_3 \). We find \( \mathcal{Z}_2, \mathcal{Z}_3 \) are related to \( \mathcal{Z}_1 \) as
\[
\tilde{X}_{2-} = -x_{21} + x_{1+} x_{12+}, \quad \tilde{\Theta} = i x_{21} + \Theta_1^b V_b^a(z_{12}), \] (4.70a)
\[
\tilde{X}_{3-} = x_{31}^{-1} \tilde{X}_{1+} x_{13+}^{-1}, \quad \Theta_3^b = i x_{31}^{-1} \tilde{\Theta}_1^b V_b^a(z_{13}), \] (4.70b)

where \( \tilde{Z} = (\tilde{X}, \tilde{\Theta}) \) is defined by superinversion, \( \mathcal{Z} \rightarrow \tilde{\mathcal{Z}} \).
5 Superconformal Invariance of Correlation Functions

In this section we discuss the superconformal invariance of correlation functions for quasi-primary superfields and exhibit general forms of two-point, three-point and $n$-point functions without proof, as the proof is essentially identical to those in our earlier work [4, 5].

5.1 Quasi-primary Superfields

We first assume that there exist quasi-primary superfields, $Ψ^I(z)$, which under the superconformal transformation, $z \to z'$, transform as

$$Ψ^I \to Ψ'^I, \quad Ψ'^I(z') = Ψ^I(z)D_J^I(z; g).$$

(5.1)

$D(z; g)$ obeys the group property so that under the successive superconformal transformations, $g'' : z \to z' \to z''$, it satisfies

$$D(z; g)D(z'; g') = D(z; g''),$$

(5.2)

and hence

$$D(z; g)^{-1} = D(z'; g^{-1}).$$

(5.3)

We choose here $D(z; g)$ to be a representation of $SO(1, 2) \times O(N) \times D$, which is a subgroup of the stability group at $z = 0$, and so we decompose the spin index, $I$, of superfields into $SO(1, 2)$ index, $ρ$, and $O(N)$ index, $r$, as $Ψ^I \equiv Ψ_ρ^r$. Now $D_J^I(z; g)$ is factorized as

$$D_J^I(z; g) = D_ρ^σ(L(z; g))D_r^s(U(z; g))Ω(z; g)^{-η},$$

(5.4)

where $D_ρ^σ(L)$, $D_r^s(U)$ are representations of $SO(1, 2)$ and $O(N)$ respectively, while $η$ is the scale dimension of $Ψ_ρ^r$.

Infinitesimally

$$δΨ_ρ^r(z) = -(L + ηǎλ(z))Ψ_ρ^r(z) - Ψ_σ^r(z)(s^α_β)^ρ_σ \tilde{w}_α^β(z) - Ψ_ρ^s(z)\frac{1}{2}(s_{ab})_s^r\tilde{t}^{ab}(z),$$

(5.5)

where $\tilde{t}^{ab}(z) = δ^{ac}\tilde{t}_c^b(z)$, and $s^α_β$, $s_{ab}$ satisfy

$$[s^α_β, s^γ_δ] = δ^α_δ s^γ_β - δ^γ_β s^α_δ,$$

(5.6)

$$[s_{ab}, s_{cd}] = -η_{ac}s_{bd} + η_{ad}s_{bc} + η_{bc}s_{ad} - η_{bd}s_{ac}.$$
$s_{ab}$ is the generator of $O(N)$, while $s^{\alpha \beta}$ is connected to the generator of $SO(1, 2)$, $s_{\mu \nu}$, through

$$s_{\mu \nu} \equiv \frac{1}{2} s^{\alpha \beta} (\gamma_{[\mu} \gamma_{\nu]}) \epsilon_{\alpha \beta}, \quad s^{\alpha \beta} = -\frac{1}{2} s_{\mu \nu} (\gamma_{[\mu} \epsilon_{\nu]}^{\alpha \beta}),$$

$$[s_{\mu \nu}, s_{\lambda \rho}] = -\eta_{\mu \lambda} s_{\nu \rho} + \eta_{\mu \rho} s_{\nu \lambda} + \eta_{\nu \lambda} s_{\mu \rho} - \eta_{\nu \rho} s_{\mu \lambda}, \quad s^{\alpha \beta} \tilde{w}^{\beta \alpha}(z) = \frac{1}{2} s_{\mu \nu} \tilde{w}^{\mu \nu}(z).$$

From eqs. (4.15, 4.16) using eq. (5.6) we have

$$\delta_3 \Psi^r = [\delta_2, \delta_1] \Psi^r. \quad (5.8)$$

It is useful to consider the conjugate superfield of $\Psi^r$, $\bar{\Psi}^r(z)$, which transforms as

$$\bar{\Psi}^{\rho r}(z') = \Omega(z; g)^{-\eta} D^\rho_{\sigma}(L^{-1}(z; g)) D_r^s(U^{-1}(z; g)) \bar{\Psi}^s(z). \quad (5.9)$$

Superconformal invariance for a general $n$-point function requires

$$\langle \Psi^{I_1}_1(z_1) \Psi^{I_2}_2(z_2) \cdots \Psi^{I_n}_n(z_n) \rangle = \langle \Psi^I_1(z_1) \Psi^I_2(z_2) \cdots \Psi^I_n(z_n) \rangle. \quad (5.10)$$

### 5.2 Two-point Correlation Functions

The solution for the two-point function of the quasi-primary superfields, $\Psi^r$, $\bar{\Psi}^r$, has the general form

$$\langle \bar{\Psi}^r(z_1) \Psi^s(z_2) \rangle = C_{\Psi} \frac{I^\rho_{\sigma}(\tilde{x}_{12+}) I^s_r(V(z_{12}))}{\eta} \left( y_{12}^2 + \frac{1}{4} (\bar{\theta}_{12} \theta_{12}^a)^2 \right)^{\frac{1}{2}}, \quad (5.11)$$

where we put

$$\tilde{x}_{12+} = \frac{x_{12+}}{\left( y_{12}^2 + \frac{1}{4} (\bar{\theta}_{12} \theta_{12}^a)^2 \right)^{\frac{1}{2}}},$$

and $I^\rho_{\sigma}(\tilde{x}_{12+})$, $I^s_r(V(z_{12}))$ are tensors transforming covariantly according to the appropriate representations of $SO(1, 2)$, $O(N)$ which are formed by decomposition of tensor products of $\tilde{x}_{12+}$, $V(z_{12})$. Under superconformal transformations, $I^\rho_{\sigma}(\tilde{x}_{12+})$ and $I^s_r(V(z_{12}))$ satisfy from eqs. (4.33, 4.34)

$$D(L^{-1}(z_1; g)) I(\tilde{x}_{12+}) D(L(z_2; g)) = I(\tilde{x}'_{12+}), \quad (5.13a)$$

$$D(U^{-1}(z_1; g)) I(V(z_{12})) D(U(z_2; g)) = I(V(z'_{12})). \quad (5.13b)$$

27
As examples, we first consider real scalar, spinorial and gauge superfields, $S(z), \phi_\alpha(z), \bar{\phi}_\alpha(z), \zeta^a(z), \bar{\zeta}^a(z)$. They satisfy

\begin{equation}
S(z) = S(z)^* ,
\end{equation}

\begin{equation}
\bar{\phi}_\alpha(z) = \epsilon^{-1\alpha\beta} \phi_\beta(z) = (\gamma^0 \phi(z)^\dagger)^\alpha ,
\end{equation}

\begin{equation}
\zeta^a(z) = \zeta^a(z)^* = \zeta_a(z) ,
\end{equation}

and transform as

\begin{equation}
S'(z') = \Omega(z; g)^{-\eta} S(z) ,
\end{equation}

\begin{equation}
\phi'_\alpha(z') = \Omega(z; g)^{-\eta} \phi_\beta(z) L^\beta_\alpha(z; g) ,
\end{equation}

\begin{equation}
\bar{\phi}'_\alpha(z') = \Omega(z; g)^{-\eta} L^{-1\alpha\beta}(z; g) \bar{\phi}_\beta(z) ,
\end{equation}

\begin{equation}
\zeta'^a(z') = \Omega(z; g)^{-\eta} U^{-1 \beta}_a(z; g) \zeta_\beta(z) ,
\end{equation}

\begin{equation}
\zeta'_a(z') = \Omega(z; g)^{-\eta} U^{-1 \beta}_a(z; g) \zeta_\beta(z) .
\end{equation}

The two-point functions of them are

\begin{equation}
\langle S(z_1)S(z_2) \rangle = C_S \frac{1}{(x_{12}^2 + \frac{1}{4}(\bar{\theta}_{12a} \theta_{12}^a)^2)^{\eta}} ,
\end{equation}

\begin{equation}
\langle \bar{\phi}_\alpha(z_1)\phi_\beta(z_2) \rangle = iC_\phi \frac{(x_{12}^2)^\alpha_\beta}{(x_{12}^2 + \frac{1}{4}(\bar{\theta}_{12a} \theta_{12}^a)^2)^{\eta + \frac{1}{2}}} ,
\end{equation}

\begin{equation}
\langle \zeta_a(z_1)\zeta^b(z_2) \rangle = C_\zeta \frac{V_{ab}(z_{12})}{(x_{12}^2 + \frac{1}{4}(\bar{\theta}_{12a} \theta_{12}^a)^2)^{\eta}} ,
\end{equation}

\begin{equation}
\langle \zeta'^a(z_1)\zeta'^b(z_2) \rangle = C_\zeta' \frac{V_{ab}(z_{12})}{(x_{12}^2 + \frac{1}{4}(\bar{\theta}_{12a} \theta_{12}^a)^2)^{\eta}} .
\end{equation}

Note that to have non-vanishing two-point correlation functions, the scale dimensions, $\eta$, of the two fields must be equal.

For a real vector superfield, $J^\mu(z)$, where the representation of SO(1,2) is given by $\hat{R}_\mu^\nu(z; g)$, we have

\begin{equation}
\langle J^\mu(z_1)J^\nu(z_2) \rangle = C_V \frac{J^{\mu\nu}(z_{12})}{(x_{12}^2 + \frac{1}{4}(\bar{\theta}_{12a} \theta_{12}^a)^2)^{\eta}} ,
\end{equation}

28
\[ I^{\mu\nu}(z) = \hat{R}^{\mu\nu}(z; i_s) = \frac{1}{2} \text{tr} (\gamma^\mu \hat{x}_+ \gamma^\nu \hat{x}_-) . \]  

(5.21)

From eq. (3.37) we note

\[ I^{\mu\nu}(z) = I^{\nu\mu}(-z) , \quad I^{\mu\nu}(z) I_{\lambda\nu}(z) = \delta^{\mu\lambda} . \]  

(5.22)

If we define

\[ J^\alpha_\beta(z) = J^\mu(z) (\gamma_\mu)^\alpha_\beta , \]  

(5.23)

d then from eqs. (2.4, 5.20) and

\[ D_{\alpha\omega}(z_1)(x_{12}^+ + \frac{1}{4}(\bar{\theta}^a \theta^a_{12})^2) = 2i(\bar{\theta}^a \gamma_\mu x_{12}^-)_\alpha , \]  

(5.24)

we get

\[ D_{\alpha\omega}(z_1) \langle J^\alpha_\beta(z_1) J^\nu(z_2) \rangle = 2i C_V (2 - \eta) \left( \frac{(\bar{\theta}^a \gamma_\mu x_{12}^-)_{\beta'}}{(x_{12}^2 + \frac{1}{4}(\bar{\theta}^a \theta^a_{12})^2)^{\eta+1}} \right) . \]  

(5.25)

Hence \( \langle J^\alpha_\beta(z_1) J^\nu(z_2) \rangle \) is conserved if \( \eta = 2 \)

\[ D_{\alpha\omega}(z_1) \langle J^\alpha_\beta(z_1) J^\nu(z_2) \rangle = 0 \quad \text{if} \quad \eta = 2 . \]  

(5.26)

The anti-commutator relation for \( D_{\alpha\omega} \) (2.20) implies also

\[ \frac{\partial}{\partial x_{1\alpha}^\mu} \langle J^\mu(z_1) J^\nu(z_2) \rangle = 0 \quad \text{if} \quad \eta = 2 . \]  

(5.27)

This agrees with the non-supersymmetric general result that two-point correlation function of vector field in \( d \)-dimensional conformal theory is conserved if the scale dimension is \( d - 1 \) .

### 5.3 Three-point Correlation Functions

The solution for the three-point correlation function of the quasi-primary superfields, \( \Psi_r^r \), has the general form

\[ \langle \Psi_r^r(z_1) \Psi_s^s(z_2) \Psi_t^t(z_3) \rangle = \frac{H_{\rho\sigma\tau\nu}(Z_1) I^{\rho\sigma}(\hat{x}_{12}^+ ) I^{\tau\nu}(\hat{x}_{13}^+ ) I_{\sigma s}(V(z_{12})) I_{\tau t}(V(z_{13}))}{\left( x_{12}^2 + \frac{1}{4}(\bar{\theta}^a \theta^a_{12})^2 \right)^{\eta_2} \left( x_{13}^2 + \frac{1}{4}(\bar{\theta}^a \theta^a_{13})^2 \right)^{\eta_3}} . \]  

(5.28)
where $Z_{1}^{M} = (X_{1}^{\mu}, \Theta_{1}^{a}) \in \mathbb{R}^{3|2N}$ is given by eq.(4.58).

Superconformal invariance (5.10) is now equivalent to

$$
H_{\rho_1}^{r_1 \sigma_1} s_1 \tau_1^{t_1} (Z) D^{\rho_1} (L) D^{\sigma_1} (L) D^{r_1} (L) = H_{\rho_1}^{r_1 \sigma_1} s_1 \tau_1^{t_1} (Z'),
$$

(5.29a)

$$
Z'^{M} = (X^{\nu}, L^{-1} \Theta^{a}),
$$

$$
H_{\rho}^{r_1 \sigma_1} s_1 \tau_1^{t_1} (Z) D_{\rho}^{r_1} (U) D_{\sigma_1}^{s_1} (U) D_{\tau_1}^{t_1} (U) = H_{\rho}^{r_1 \sigma_1} s_1 \tau_1^{t_1} (Z''),
$$

(5.29b)

$$
Z''^{M} = (X^{\mu}, \Theta^{b} U^{a}_b),
$$

$$
H_{\rho}^{r_1 \sigma_1} s_1 \tau_1^{t_1} (Z) = \lambda^{\eta_1 + \eta_2 - \eta_3} H_{\rho}^{r_1 \sigma_1} s_1 \tau_1^{t_1} (Z'''),
$$

(5.29c)

$$
Z'''^{M} = (\lambda X^{\mu}, \lambda^{1/2} \Theta^{a}),
$$

where $U \in O(N)$, $\lambda \in \mathbb{R}$ and $2 \times 2$ matrix, $L$, satisfies

$$
L^{-1} = \gamma_{0}^{0} L_{0}^{0} = \epsilon^{-1} L_{1}^{1}, \quad \det L = 1,
$$

(5.30)

$$
\hat{R}_{\nu}(L) = \frac{1}{2} \text{tr}(\gamma_{\nu} L_{0}^{0} L^{1}).
$$

In general there are a finite number of linearly independent solutions of eq.(5.29a), and this number can be considerably reduced by taking into account the symmetry properties, superfield conservations and the superfield constraints [5, 20, 21].

5.4 $n$-point Correlation Functions - in general

The solution for $n$-point correlation functions of the quasi-primary superfields, $\Psi_{\rho}^{r}$, has the general form

$$
\langle \Psi_{1\rho_1}^{r_1} (z_1) \cdots \Psi_{n\rho_n}^{r_n} (z_n) \rangle
$$

$$
= H_{\rho_1}^{r_1 \rho_2} \cdots \rho_n^{r_n} (Z_{1(1)}, \cdots, Z_{1(n-2)}) \prod_{k=2}^{n} \frac{I_{p}^{k} (\tilde{x}_{1k}^{+}) I_{r}^{k} (V(z_{1k}))}{\left( x_{1k}^{2} + \frac{1}{4}(\tilde{\theta}_{1ka}^{a})^{2} \right)^{\eta_{k}}},
$$

(5.31)

where, in a similar fashion to eq.(4.58), with $z_{k1} \to \tilde{z}_{k1}$, $k \geq 2$, $Z_{1(1)}, \cdots, Z_{1(n-2)}$ are given by

$$
G_{T}(\tilde{z}_{m1})^{-1} G_{T}(\tilde{z}_{j1}) = G_{T}(Z_{1(j-1)}), \quad j = 2, 3, \cdots, n - 1.
$$

(5.32)
We note that all of them are ‘tangent’ vectors at $z_1$.

Superconformal invariance (5.10) is equivalent to

\[
H_{\rho_1^r \cdots \rho_n^r} (Z_1, \ldots, Z_{n-2}) \prod_{k=1}^n D^\rho_k (L) = H_{\rho_1^r \cdots \rho_n^r} (Z'_1, \ldots, Z'_{n-2}),
\]

(5.33a)

\[
Z'^M_{(j)} = (X^\nu_{(j)}, L^{\nu \Theta^a_{(j)})},
\]

\[
H_{\rho_1^r' \cdots \rho_n^r'} (Z_1, \ldots, Z_{n-2}) \prod_{k=1}^n D^\rho_k (U) = H_{\rho_1^r \cdots \rho_n^r} (Z''_1, \ldots, Z''_{n-2}),
\]

(5.33b)

\[
Z''^M_{(j)} = (X^\mu_{(j)}, \Theta^b_{(j)} U^a_{(j)}),
\]

\[
H_{\rho_1^r \cdots \rho_n^r} (Z_1, \ldots, Z_{n-2}) = \lambda^{-\eta_1 + \eta_2 + \cdots + \eta_n} H_{\rho_1^r \cdots \rho_n^r} (Z''_1, \ldots, Z''_{n-2}),
\]

(5.33c)

Thus $n$-point functions reduce to one unspecified $(n-2)$-point function which must transform homogeneously under the rigid transformations, $SO(1,2) \times O(N) \times D$.

From

\[
X_{1(j-1)+} = x^{-1}_{j1+} x_{jn} x_{n1-}^{-1}, \quad X_{1(j-1)-} = x^{-1}_{n1+} x_{jn} x_{j1-}^{-1},
\]

(5.34)

we get

\[
X_{(l,m)+} \equiv X_{(l-1)+} - X_{(m-1)-} + 2i \Theta^a_{(l-1)} \bar{\Theta} (m-1)a = x^{-1}_{l1+} x_{lm} x_{m1-}^{-1},
\]

(5.35)

and hence

\[
\det X_{(l,m)\pm} = -\frac{x_{lm}^2 + \frac{1}{4}(\bar{\Theta}_{lm} \Theta_{lm})^2}{\left( x_{1l}^2 + \frac{1}{4}(\bar{\Theta}_{1l} \Theta_{1l})^2 \right) \left( x_{1m}^2 + \frac{1}{4}(\bar{\Theta}_{1m} \Theta_{1m})^2 \right)}. \tag{5.36}
\]

Now if we define

\[
\Delta_{lm} = -\frac{1}{2(n-1)(n-2)} \sum_{i=1}^n \eta_k + \frac{1}{2(n-2)} (\eta_l + \eta_m), \tag{5.37}
\]

then using the following identity which holds for any matrix, $S_{lm}$, and number, $\lambda$,

\[
\left( \prod_{l \neq m} (S_{lm})^{\Delta_{lm}} \right) \left( \prod_{k=2}^n (S_{1k} S_{k1})^{-\frac{1}{2} \eta_k} \right) = \lambda^{-\frac{1}{2}(-\eta_1 + \eta_2 + \cdots + \eta_n)} \prod_{2 \leq l \neq m} \left( \frac{\lambda S_{lm}}{S_{1l} S_{lm}} \right)^{\Delta_{lm}}, \tag{5.38}
\]

31
we can rewrite the \(n\)-point correlation functions \((5.31)\) as

\[
\langle \Psi_{1\rho_1} r_1(z_1) \cdots \Psi_{n\rho_n} r_n(z_n) \rangle = K_{\rho_1 r_1} \cdots \rho_n r_n (Z_{1(1)}, \cdots, Z_{1(n-2)}) \prod_{k=2}^{n} I_{\rho_k}^{r_k} (\hat{x}_{1k} + I_{r_k}^{r_k} (V(z_k))) ,
\]

(5.39)

where

\[
K_{\rho_1 r_1} \cdots \rho_n r_n (Z_{1(1)}, \cdots, Z_{1(n-2)}) = H_{\rho_1 r_1} \cdots \rho_n r_n (Z_{1(1)}, \cdots, Z_{1(n-2)}) \prod_{2 \leq l \neq m} (- \det X_{(l,m)})^{\Delta_{lm}} .
\]

(5.40)

Note the difference in eq.(5.31) and eq.(5.39), namely the denominator in the latter is written in a democratic fashion.

Superconformal invariance \((5.33a)\) is equivalent to

\[
K_{\rho_1 r_1} \cdots \rho_n r_n (Z_{1(1)}, \cdots, Z_{1(n-2)}) = H_{\rho_1 r_1} \cdots \rho_n r_n (Z'_{1(1)}, \cdots, Z'_{1(n-2)}) \prod_{2 \leq l \neq m} (- \det X_{(l,m)})^{\Delta_{lm}} ,
\]

(5.41)

In particular, \(K\) is invariant under dilations contrary to \(H\).

5.5 Superconformal Invariants

In the case of correlation functions of quasi-primary scalar superfields, eqs.(5.10,5.39,5.41) imply that \(K(Z_{1(1)}, \cdots, Z_{1(n-2)})\) is a function of the superconformal invariants and furthermore that all of the superconformal invariants can be generated by contracting the indices of \(Z_{1(j)}^\mu = (X_{1(j)}^\mu, \Theta_{1(j)}^a)\) to make them \(\text{SO}(1,2) \times \text{O}(N) \times \text{D}\) invariant according to the recipe by Weyl [25]. To do so we first normalize \(Z_{1(j)}^\mu\) as

\[
\hat{Z}_{1(j)}^\mu = (\hat{X}_{1(j)}^\mu, \hat{\Theta}_{1(j)}^a) ,
\]

\[
\hat{X}_{1(j)}^\mu = \frac{X_{1(j)}^\mu}{(X_{1(1)}^2)^{\frac{1}{4}}} , \quad \hat{\Theta}_{1(j)}^a = \frac{\Theta_{1(j)}^a}{(X_{1(1)}^2)^{\frac{1}{4}}} .
\]

(5.42)
By virtue of eqs. (A.3a, A.7a) all the $SO(1,2) \times O(N) \times D$ invariants or three-dimensional superconformal invariants are

$$\hat{X}_{1(j)} \cdot \hat{X}_{1(k)} \cdot \hat{\theta}_{1(j)} \cdot \hat{\theta}_{1(k)} \cdot \hat{\theta}_{1(m)} \cdot \hat{\theta}_{1(n)}$$

In particular, from eq. (5.36) we note that they produce cross ratio type invariants depending on four points, $z_r, z_s, z_t, z_u$, the non-supersymmetric of which are well known - see e.g. Ref. [26]

$$x_{rs} + \frac{1}{4}(\bar{\theta}_{rsa} \theta_{sa})^2 x_{tu} + \frac{1}{4}(\bar{\theta}_{tua} \theta_{ta})^2 x_{su} + \frac{1}{4}(\bar{\theta}_{rsa} \theta_{sa})^2 x_{tu} + \frac{1}{4}(\bar{\theta}_{tua} \theta_{ta})^2 x_{su} + \frac{1}{4}(\bar{\theta}_{rsa} \theta_{sa})^2 x_{tu} + \frac{1}{4}(\bar{\theta}_{tua} \theta_{ta})^2 x_{su}.$$ (5.44)

If we restrict the $R$-symmetry group to be $SO(\mathcal{N})$ instead of $O(\mathcal{N})$ then the followings are also superconformal invariants in the case of even $\mathcal{N}$, according to Weyl [25]

$$\epsilon^{a_1 b_1 \cdots a_{\mathcal{N}/2} b_{\mathcal{N}/2}} \prod_{j=1}^{\mathcal{N}} T_{ja} b_j, \quad T_{ja} b_j = \hat{\theta}_{1(j_1)} a_1 \hat{\theta}_{1(j_2)} a_2$$ or $\hat{\theta}_{1(j_1)} a_1 \hat{\theta}_{1(j_2)} a_2$$ (5.45)

which we may call pseudo-invariants.

6 Superconformally Covariant Operators

In general acting on a quasi-primary superfield, $\Psi_\rho^r(z)$, with the spinor derivative, $D_{aa}$, does not lead to a quasi-primary field. For a superfield, $\Psi_\rho^r$, from eqs. (4.8, 4.9, 5.5) we have

$$D_{aa} \delta \Psi_\rho^r = -(\mathcal{L} + (\eta + \frac{1}{2}) \lambda) D_{aa} \Psi_\rho^r$$

$$- D_{a\beta} \Psi_\rho^r \hat{w}^\beta_\alpha - D_{aa} \Psi_\sigma^r (s^\beta \gamma \hat{w}^\gamma_\beta)^\sigma_\rho$$

$$- D_{a\alpha} \Psi_\rho^r \hat{w}^\alpha_\beta - D_{aa} \Psi_\rho^r a_{\alpha} \frac{1}{2}(s_{bc} \hat{w}^\beta_{bc})_s^r$$

$$+ 2 \hat{\rho}_{\rho^r} (\Psi Y_{b^3 a^3})_\rho^r,$$

where $Y_{b^3 a^3}$ is given by

$$Y_{b^3 a^3} = 2 \hat{\rho}_{\rho^r} (\Psi Y_{b^3 a^3})_\rho^r.$$

For conformally covariant differential operators in non-supersymmetric theories, see e.g. [27, 28].
To ensure that $D_{aa} \Psi^\rho_r$ is quasi-primary it is necessary that the terms proportional to $\hat{\rho}$ vanish and this can be achieved by restricting $D_{aa} \Psi^\rho_r$ to an irreducible representation of $SO(1, 2) \times O(\mathcal{N})$ and choosing a particular value of $\eta$ so that $\Psi Y = 0$. The change of the scale dimension, $\eta \rightarrow \eta + \frac{1}{2}$, in eq.(6.1) is also apparent from eq.(2.21)

$$D_{aa} = \Omega(z; g)^\frac{1}{2} L^{-1\beta}_\alpha(z; g) U_a^b(z; g) D'_{b\beta}.$$ (6.3)

As an illustration we consider tensorial fields, $\Psi^{a_1 \ldots a_m \alpha_1 \ldots \alpha_n}$, which transform as

$$\delta \Psi^{a_1 \ldots a_m \alpha_1 \ldots \alpha_n} = - (L + \eta \hat{\lambda}) \Psi^{a_1 \ldots a_m \alpha_1 \ldots \alpha_n}$$

$$- \sum_{p=1}^m \Psi^{a_1 \ldots b \ldots a_m \alpha_1 \ldots \alpha_n} \hat{t}_a^p - \sum_{q=1}^n \Psi^{a_1 \ldots a_m \alpha_1 \ldots \beta \ldots \alpha_n} \hat{w}_a^\beta \alpha_q.$$ (6.4)

Note that spinorial indices and gauge indices, $\alpha, a$ may be raised or lowered by $\epsilon^{-1\alpha\beta}, \epsilon_{\alpha\beta}, \delta^{ab}, \delta_{ab}$.

For $\Psi^{a_1 \ldots a_m \alpha_1 \ldots \alpha_n}$ we have

$$(\Psi^{\beta}_{b \alpha a})^{a_1 \ldots a_m \alpha_1 \ldots \alpha_n} = - (\eta + \frac{1}{2} n) \delta_{ba} \delta_\alpha^\beta \Psi^{a_1 \ldots a_m \alpha_1 \ldots \alpha_n}$$

$$+ \delta^\beta_\alpha \sum_{p=1}^m (\delta_{aa_p} \Psi^{a_1 \ldots b \ldots a_m \alpha_1 \ldots \alpha_n} - \delta_{ba_p} \Psi^{a_1 \ldots a \ldots a_m \alpha_1 \ldots \alpha_n})$$

$$+ \delta_{ba} \sum_{q=1}^n \delta_\alpha^\beta \alpha_q \Psi^{a_1 \ldots \alpha_a \ldots a_m \alpha_1 \ldots \alpha_n}.$$ (6.5)

In particular, eq.(6.3) shows that the following are quasi-primary

$$D_{[b(\beta \Psi^{a_1 \ldots a_m | a_1 \ldots a_n})]}$$ if $\eta = m + \frac{1}{2} n,$ (6.6a)

$$D_{[b|\beta \Psi^{a_1 \ldots a_m}]^\beta}$$ if $\eta = m - \frac{3}{2}.$ (6.6b)

where $( ), [ ]$ denote the usual symmetrization, anti-symmetrization of the indices respectively and obviously eq.(6.6a) is nontrivial if $1 \leq m + 1 \leq \mathcal{N}$. Note that due to the term containing $\delta_{aa_p}$ in eq.(6.5) one should anti-symmetrize the gauge indices.
Now we consider the case where more than one spinor derivative, $D_{\alpha\alpha}$, act on a quasi-primary superfield. In this case, it is useful to note

\[ D_{\lbrack \alpha \rbrack D_{\beta \rbrack}^\alpha} = 0, \]  

(6.7)

and

\[ D_{\alpha \alpha} \hat{\partial}_{\beta\beta} = -i \delta_{ab} (\epsilon b \cdot \gamma)_{\alpha\beta}. \]  

(6.8)

From eq. (6.5) one can derive

\[
D_{b_1(\beta_1} \cdots D_{b_l\beta_l} \delta \Psi_{a_1 \cdots a_m]\alpha_1 \cdots \alpha_n) \\
n = 2l(-\eta + m + \frac{1}{2} n + \frac{3}{4} (l-1)) \hat{\rho}_{[b_1(\beta_1 D_{b_2\beta_2} \cdots D_{b_l\beta_l} \Psi_{a_1 \cdots a_m]}\alpha_1 \cdots \alpha_n) + \text{homogeneous terms}. \\
\]

(6.9)

Hence the following is quasi-primary

\[
D_{b_1(\beta_1} \cdots D_{b_l\beta_l} \Psi_{a_1 \cdots a_m}\alpha_1 \cdots \alpha_n) \quad \text{if} \quad \eta = m + \frac{1}{2} n + \frac{3}{4} (l-1). \\
\]

(6.10)

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Appendix

A Useful Equations

Some useful identities relevant to the present paper are

\[ \frac{1}{2} \text{tr}(\gamma^\mu \gamma^\nu) = \eta^\mu\nu, \]  
\[ (A.1a) \]

\[ \gamma^\mu \gamma^\nu \gamma^\rho = \eta^\mu\nu \gamma^\rho - \eta^\mu\rho \gamma^\nu + \eta^\nu\rho \gamma^\mu + i \varepsilon^\mu\nu\rho, \]  
\[ (A.1b) \]

\[ -\frac{i}{2} \varepsilon^\mu\nu\rho \gamma_\nu \gamma_\rho = \gamma^\mu. \]  
\[ (A.1c) \]

\[ \varepsilon^{\mu\nu\kappa} \varepsilon_{\lambda\rho\kappa} = \delta^\mu_\lambda \delta^\nu_\rho - \delta^\mu_\rho \delta^\nu_\lambda, \]  
\[ (A.2a) \]

\[ \varepsilon^{\mu\nu\kappa} \varepsilon_{\lambda\nu\kappa} = 2 \delta^\mu_\lambda, \]  
\[ (A.2b) \]

\[ \varepsilon^{\mu\nu\kappa} \varepsilon_{\mu\nu\kappa} = \varepsilon^{\mu\nu\kappa}. \]  
\[ (A.2c) \]

\[ \rho \bar{\epsilon} = -\frac{1}{2} (\bar{\epsilon} \gamma^\mu \rho \gamma_\mu + \bar{\epsilon} \rho \, 1), \]  
\[ (A.3a) \]

\[ \epsilon^{-1} \bar{\epsilon} \rho \, \epsilon = -\frac{1}{2} (\bar{\epsilon} \gamma^\mu \rho \gamma_\mu - \bar{\epsilon} \rho \, 1). \]  
\[ (A.3b) \]

For Majorana spinors

\[ D_{aa} \theta^b \beta = -\delta_a^b \delta_\alpha^\beta, \quad D_{aa} \bar{\theta}_b \beta = \delta_{ab} \epsilon_\alpha^\beta, \]  
\[ (A.4a) \]

\[ D^{aa} \theta^b \beta = -\delta^b \epsilon^{-1} \delta^\alpha_\beta, \quad D^{aa} \bar{\theta}_b \beta = \delta_a^b \delta_\alpha^\beta, \]  
\[ (A.4b) \]

\[ D_{aa} \bar{\theta}_b \theta^b = 2 \bar{\theta}_{aa}, \]  
\[ (A.4c) \]

\[ D_{aa} x^+_\gamma = 2 i \delta^\beta_\alpha \bar{\theta}_{a \gamma}, \]  
\[ (A.4d) \]

\[ D_{aa} x^-_\gamma = 2 i (\delta^\beta_\alpha \bar{\theta}_{a \gamma} - \delta^\beta_{\alpha \gamma} \bar{\theta}_{aa}). \]  
\[ (A.4e) \]
\[
\gamma^0 x_\pm \gamma^0 = x^\dagger_\pm, \quad \text{(A.5a)}
\]
\[
\epsilon x_\pm \epsilon^{-1} = -x^\dagger_\pm, \quad \text{(A.5b)}
\]
\[
det x_+ = det x_- = -x^2 - \frac{1}{4}(\bar{\theta}_a \theta^a)^2. \quad \text{(A.5c)}
\]
\[
\delta_\alpha^\delta \delta_\beta^\gamma - \delta_\alpha^\gamma \delta_\beta^\delta = \epsilon_{\alpha\beta} \epsilon^{-1}\gamma^\delta. \quad \text{(A.6)}
\]
\[
\bar{\theta} \gamma^\mu \theta \gamma^\mu \psi' = -2\bar{\theta} \psi' \bar{\theta} \psi - \bar{\theta} \theta' \bar{\psi} \psi', \quad \text{(A.7a)}
\]
\[
\epsilon_{\mu_1 \cdots \mu_d} \epsilon_{\nu_1 \cdots \nu_d} = \sum_{p=1}^d \text{sign}(p) \delta_{\mu_1 \nu_p} \cdots \delta_{\mu_d \nu_p} \quad : \text{permutations}, \quad \text{(A.7b)}
\]
\[
\epsilon_{\mu_1 \cdots \mu_d} x^{\mu_1}_{(1)} \cdots x^{\mu_d}_{(d)} = \pm \sqrt{\epsilon^{1 \cdots d} x^{(1)} \cdots x^{(d)} \cdot \bar{x}^{(d)} \cdot \bar{x}^{(1)}}. \quad \text{(A.7c)}
\]

**B Solution of Superconformal Killing Equation**

From the well known solution of the ordinary conformal Killing equation (3.10) [26], we may write the general solution of the superconformal Killing equation (3.15) as

\[
h^\mu(z) = 2x \cdot b(\theta) x^\mu - (x^2 - \frac{1}{4}(\bar{\theta}_a \theta^a)^2) b^\mu(\theta) + \epsilon_{\nu\lambda} x^\nu b^\lambda(\theta) \bar{\theta}_a \theta^a + w^\mu(\theta) x^\nu + \frac{1}{4} \epsilon_{\nu\lambda} w^\nu(\theta) \bar{\theta}_a \theta^a + \lambda(\theta) x^\mu + a^\mu(\theta). \quad \text{(B.1)}
\]

Substituting this expression into eq. (3.15) leads three independent equations corresponding to the second, first and zeroth order in \(x\). Considering the quadratic terms or the coefficients of \(x_\rho x_\lambda\), we get

\[
\eta^{\rho\lambda} D_{aa} b^\lambda(\theta) + \eta^{\mu\lambda} D_{aa} b^\rho(\theta) - \eta^\rho\lambda D_{aa} b^\mu(\theta)
\]
\[
= -i \frac{1}{2} \epsilon_\mu \gamma^\lambda \gamma^\rho \alpha (\eta^{\nu\rho} D_{aa} b^\lambda(\theta) + \eta^{\nu\lambda} D_{aa} b^\rho(\theta) - \eta^\nu\lambda D_{aa} b^\rho(\theta)). \quad \text{(B.2)}
\]
Contracting this with $\eta_{\rho\lambda}$ gives
\[ D_{aa} b^\mu(\theta) = -i\frac{1}{2}\epsilon^\mu_{\nu\kappa} \gamma^\kappa_{\alpha} D_{a\beta} b^\nu(\theta), \]  
while contraction with $\eta_{\mu\lambda}$ leads
\[ D_{aa} b^\rho(\theta) = i\frac{1}{3}\epsilon^\rho_{\nu\kappa} \gamma^\kappa_{\alpha} D_{a\beta} b^\nu(\theta). \]  
Thus
\[ D_{aa} b^\mu(\theta) = 0, \]  
$b^\mu(\theta)$ is constant. Straightforward calculation shows that $2x \cdot b x^\mu - (x^2 - \frac{1}{4}(\bar{\theta}_a \theta^a)^2) b^\mu + \epsilon^\mu_{\nu\lambda} x^\nu b^\lambda \bar{\theta}_a \theta^a$ is a solution of the superconformal Killing equation (3.15).

Now the linear in $x$ terms become
\[ D_{aa}(\epsilon^{\mu\nu\kappa} v^\kappa(\theta) + \eta^{\mu\nu} \lambda(\theta)) = i\frac{1}{2} D_{a\beta}(\eta^{\mu\nu} v(\theta) \cdot \gamma - v^\mu(\theta) \gamma^\nu - \epsilon^{\mu\nu\kappa} \lambda(\theta) \gamma^\kappa)_{\alpha}, \]  
where $v^\kappa(\theta)$ is the dual form of $w^{\mu\nu}(\theta)$
\[ v^\kappa = \frac{1}{2} \epsilon^\kappa_{\mu\nu} w^{\mu\nu}, \quad w^{\mu\nu} = \epsilon^{\mu\nu\kappa} v^\kappa. \]  
Contracting eq.(B.6) with $\eta_{\mu\nu}$ gives
\[ D_{aa} \lambda(\theta) = i\frac{1}{3} D_{a\beta} v^\beta_{\alpha}(\theta), \quad v^\beta_{\alpha}(\theta) = v^\mu(\theta) \gamma^\nu_{\mu\alpha}. \]  
Substituting this back into eq.(B.6) leads
\[ 0 = 5\epsilon^{\mu\nu\rho} D_{aa} v^\rho(\theta) - i\eta^{\mu\nu} D_{a\beta} v^\beta_{\alpha}(\theta) + 4i D_{a\beta} v^\mu(\theta) \gamma^\nu_{\mu\alpha} - i D_{a\beta} v^\nu(\theta) \gamma^\mu_{\nu\alpha}. \]  
Contraction with $\epsilon^\kappa_{\mu\nu}$ shows that $v^\kappa(\theta)$ satisfies the superconformal Killing equation (3.15)
\[ D_{aa} v^\kappa(\theta) = -i\frac{1}{2} \epsilon^\kappa_{\mu\nu} D_{a\beta} v^\mu(\theta) \gamma^\nu_{\mu\alpha}. \]  
Eqs.(B.8,B.10) are actually equivalent to eq.(4.4), since from eq.(B.10) successively
\[ D_{aa} v^\kappa(\theta) \gamma^\rho_{\alpha\beta} = i\frac{1}{2} \epsilon^{\kappa\rho\mu} D_{a\beta} v^\mu_{\alpha}(\theta) + \frac{1}{2} \eta^{\kappa\rho} D_{aa} v^\alpha_{\beta}(\theta) - \frac{1}{2} D_{aa} v^\rho(\theta) \gamma^\nu_{\rho\alpha} \gamma^\kappa_{\nu\beta}, \]
\[ D_{aa} v^\nu(\theta) \gamma^\rho_{\mu\beta} = \frac{1}{3} \eta^{\rho\kappa} D_{aa} v^\kappa_{\beta}(\theta), \]
\[ D_{aa} v^\kappa(\theta) \gamma^\rho_{\beta\alpha} = i\epsilon^\kappa_{\rho\mu} D_{a\beta} v^\mu_{\alpha}(\theta), \]
\[ D_{aa} v^\kappa(\theta) \gamma^\rho_{\alpha\beta} = \frac{1}{3} \eta^{\kappa\rho} D_{aa} v^\alpha_{\beta}(\theta), \]
and the last expression makes eq. (B.9) hold.

To solve eq. (B.10) we first note from

\[ D_{a\alpha} v^\beta \gamma(\theta) = \frac{2}{3} \delta_{\alpha}^\beta D_{a\delta} v^\delta \gamma(\theta) - \frac{1}{3} \delta_{\gamma}^\delta D_{a\delta} v^\delta \alpha(\theta) , \]  

(B.12)

that

\[ D_{b\beta} D_{a\alpha} v^\gamma \delta(\theta) = -\frac{2}{3} \delta_{\gamma}^\gamma D_{a\omega} D_{b\beta} v^\omega \delta(\theta) + \frac{1}{3} \delta_{\delta}^\delta D_{a\omega} D_{b\beta} v^\omega \alpha(\theta) \]

\[ = \frac{4}{9} \delta_{\alpha}^\gamma D_{b\omega} D_{a\beta} v^\omega \delta(\theta) - \frac{2}{9} \delta_{\gamma}^\gamma D_{b\omega} D_{a\beta} v^\omega \beta(\theta) - \frac{1}{9} \delta_{\alpha}^\beta D_{b\omega} D_{a\beta} v^\omega \gamma(\theta) \]

\[ + \frac{1}{9} \delta_{\gamma}^\gamma D_{b\omega} D_{a\alpha} v^\omega \beta(\theta) . \]  

(B.13)

Contraction with \( \delta_{\gamma}^\beta \) gives

\[ D_{b\gamma} D_{a\alpha} v^\beta \delta(\theta) = -D_{b\gamma} D_{a\delta} v^\beta \alpha(\theta) , \]  

(B.14)

so that eq. (B.13) becomes

\[ D_{b\beta} D_{a\alpha} v^\gamma \delta(\theta) = \frac{2}{3} \delta_{\alpha}^\gamma D_{b\omega} D_{a\beta} v^\omega \delta(\theta) - \frac{1}{3} \delta_{\gamma}^\delta D_{b\omega} D_{a\beta} v^\omega \alpha(\theta) , \]  

(B.15)

which is in fact equivalent to eq. (B.13).

From eq. (B.15) and \( D_{b\beta} D_{a\alpha} v^\gamma \delta(\theta) = -D_{a\alpha} D_{b\beta} v^\gamma \delta(\theta) \) we get

\[ 2 \delta_{\alpha}^\gamma D_{b\omega} D_{a\beta} v^\omega \delta(\theta) + 2 \delta_{\beta}^\gamma D_{a\omega} D_{b\alpha} v^\omega \delta(\theta) = \delta_{\delta}^\gamma (D_{a\omega} D_{b\alpha} v^\omega \beta(\theta) + D_{b\omega} D_{a\beta} v^\omega \alpha(\theta)) . \]  

(B.16)

Contracting with \( \delta_{\gamma}^\alpha \) gives

\[ 3 D_{b\omega} D_{a\beta} v^\omega \delta(\theta) = -2 D_{a\omega} D_{b\beta} v^\omega \delta(\theta) + D_{a\omega} D_{b\delta} v^\omega \beta(\theta) . \]  

(B.17)

Hence from eq. (B.14) we can put

\[ D_{b\omega} D_{a\alpha} v^\omega \beta = \Gamma_{ab}(\theta) \epsilon_{\alpha \beta} , \]

(B.18)

\[ \Gamma_{ab}(\theta) = \epsilon^{\gamma}_{\beta \delta} \]  

\[ \frac{1}{2} D_{b\beta} D_{a\alpha} (\epsilon(\theta) \epsilon^{-1})^{\beta \alpha} = -\Gamma_{ba}(\theta) , \]

so that eq. (B.15) becomes with eq. (A.6)

\[ D_{b\beta} D_{a\alpha} v^\gamma \delta(\theta) = \frac{1}{3} (2 \delta_{\alpha}^\gamma \epsilon_{\beta \delta} - \epsilon_{\beta \alpha} \delta_{\gamma}^\delta) \Gamma_{ab}(\theta) . \]  

(B.19)
Thus
\[ D_{cv} \Gamma_{ab}(\theta) = \frac{1}{2} D_{b\beta} D_{a\alpha} D_{cv}(v(\theta) \epsilon^{-1})^{\beta\alpha} \]
\[ = \frac{1}{2} D_{b\gamma} \Gamma_{ca}(\theta) \]
\[ = 0. \]  
(B.20)

Therefore \( \Gamma_{ab}(\theta) \) is independent of \( \theta \) and \( v(\theta) \) is at most quadratic in \( \theta \).

From eq. (B.19) we get
\[ D_{b\beta} D_{a\gamma} v^\gamma_\delta(\theta) = \epsilon_{\beta\gamma} \Gamma_{ab}. \]  
(B.21)

Integrating this gives
\[ D_{a\gamma} v^\gamma_\alpha(\theta) = 6i(t_a \tilde{b}_\alpha + \tilde{\rho}_{aa}), \]  
(B.22)

where \( 6it^a = \Gamma_{ab} \) so that
\[ t^i = -t, \]  
(B.23)

and the spinor, \( \tilde{\rho}_{aa} \), appears as a constant of integration.

Now eq. (B.12) becomes with eq. (2.4)
\[ D_{a\alpha} v^\beta_\gamma(\theta) = 2i(t_a \tilde{b}_\beta \gamma^\mu + \tilde{\rho}_a \gamma^\mu)_{\alpha \gamma} \]  
(B.24)

Integrating this gives
\[ v^\mu(\theta) = it_a \tilde{b}_\beta \gamma^\mu \theta^\alpha + 2i\tilde{\rho}_a \gamma^\mu \theta^\alpha + v^\mu. \]  
(B.25)

Eq. (B.8) becomes
\[ D_{aa} \lambda(\theta) = -2t_a \tilde{b}_\beta \lambda_{ba} + 2\tilde{\rho}_{aa}, \]  
(B.26)

so that
\[ D_{b\beta} D_{a\alpha} \lambda(\theta) = -i\frac{1}{3} \epsilon_{\alpha\beta} \Gamma_{ab}. \]  
(B.27)

However from \( D_{b\beta} D_{a\alpha} \lambda(\theta) + D_{aa} D_{b\beta} \lambda(\theta) = 0 \) we note \( \Gamma_{ab} = 0 \). Hence
\[ w^{\mu\nu}(\theta) = \tilde{\rho}_a (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) \theta^\alpha + w^{\mu\nu}, \]  
(B.28a)

\[ \lambda(\theta) = -2\tilde{\rho}_a \theta^\alpha + \lambda. \]  
(B.28b)

With these expressions straightforward calculation shows that \( w^{\mu\nu}(\theta) x^\nu + \frac{1}{4} \epsilon^{\mu\nu\lambda\rho} w^{\nu\lambda}(\theta) \tilde{b}_\alpha \theta^\alpha + \lambda(\theta) x^\mu \) is a solution of the superconformal Killing equation (3.15).

The remaining terms are
\[ D_{aa} a^\mu(\theta) = -\frac{i}{2} \epsilon^{\mu\lambda\rho} D_{a\beta} a^\lambda(\theta) \gamma^\rho \beta_{\alpha}, \]  
(B.29)
the general solution of which we already obtained. From eq. (B.25)
\[ a^\mu(\theta) = it_a b^\mu \bar{\theta} \gamma^\mu \theta^a + 2i \bar{\epsilon} a^\mu \theta^a + a^\mu. \] (B.30)
For \( a^\mu(\theta) \) to be real \( t \) must be anti-hermitian and hence with eq. (B.23) \( t \in o(N) \).

All together, we obtain the general solution of the superconformal Killing equation (3.17).

## C Basis for Superconformal Algebra

We write the superconformal generators in general as
\[ \mathcal{K} \cdot \mathcal{P} = a^\mu P_\mu + \bar{\epsilon}_a Q^a + \lambda D + \frac{1}{2} w^{\mu \nu} M_{\mu \nu} + b^\mu K_\mu + \bar{\rho}_a S^a + \frac{1}{2} t^{ab} A_{ab}, \] (C.1)
for
\[ \mathcal{K} = (a^\mu, b^\mu, \bar{\epsilon}_a, \rho_a, \lambda, w^{\mu \nu}, t_a^b), \] (C.2a)
\[ \mathcal{P} = (P_\mu, K_\mu, Q^a, S^a, D, M_{\mu \nu}, A_{ab}^b), \] (C.2b)
where we put \( t^{ab} = \delta^{ac} t_c^b \) and the \( R \)-symmetry generators, \( A_{ab} = A_a \delta_{cb} \), satisfy the \( o(N) \) condition, \( A^\dagger = A^\dagger = -A \).

The superconformal algebra can now be obtained by imposing
\[ [\mathcal{K}_1 \cdot \mathcal{P}, \mathcal{K}_2 \cdot \mathcal{P}] = -i \mathcal{K}_3 \cdot \mathcal{P}, \] (C.3)
where \( \mathcal{K}_3 \) is given by eq. (3.44). From this expression, we can read off the following superconformal algebra.

- **Poincaré algebra**
  \[ [P_\mu, P_\nu] = 0, \]
  \[ [M_{\mu \nu}, P_\lambda] = i(\eta_{\mu \lambda} P_\nu - \eta_{\nu \lambda} P_\mu), \] (C.4)
  \[ [M_{\mu \nu}, M_{\lambda \rho}] = i(\eta_{\mu \lambda} M_{\nu \rho} - \eta_{\mu \rho} M_{\nu \lambda} - \eta_{\nu \lambda} M_{\mu \rho} + \eta_{\nu \rho} M_{\mu \lambda}). \]

- **Supersymmetry algebra**
  \[ \{Q^{a \alpha}, \bar{Q}_{b \beta}\} = 2 \delta^{a \beta}_{b \gamma} \gamma^{\mu \alpha} \gamma_\mu P_\mu, \]
  \[ [M_{\mu \nu}, Q^a] = i \frac{1}{2} \gamma_{[\mu} \gamma_{\nu]} Q^a, \] (C.5)
  \[ [P_\mu, Q^{a \alpha}] = 0. \]
• Special superconformal algebra

\[ [K_\mu, K_\nu] = 0, \quad [M_{\mu\nu}, K_\lambda] = i(\eta_{\mu\lambda} K_\nu - \eta_{\nu\lambda} K_\mu), \]

\[ \{S^{a\alpha}, \bar{S}_{b\beta}\} = 2\delta^a_b \gamma^{\mu\alpha\beta} K_\mu, \]

\[ [M_{\mu\nu}, S^a] = i\frac{1}{2} \gamma_{[\mu} \gamma_{\nu]} S^a, \]

\[ [K_\mu, S^{a\alpha}] = 0. \]  \hspace{1cm} \text{(C.6)}

• Cross terms between \((P, Q)\) and \((K, S)\)

\[ [P_\mu, K_\nu] = 2i(M_{\mu\nu} + \eta_{\mu\nu} D), \]

\[ [P_\mu, S^a] = -\gamma_\mu Q^a, \]

\[ [K_\mu, Q^a] = -\gamma_\mu S^a, \]  \hspace{1cm} \text{(C.7)}

\[ \{Q^{a\alpha}, \bar{S}_{b\beta}\} = -i\delta^{a}_b (2\delta^{\alpha}_b D + (\gamma^{[\mu} \gamma^{\nu]})^{\alpha}_b M_{\mu\nu}) + 2i\delta^{a}_b A^{a}_b. \]

• Dilations

\[ [D, P_\mu] = -i P_\mu, \quad [D, K_\mu] = i K_\mu, \]

\[ [D, Q^a] = -i\frac{1}{2} Q^a, \quad [D, S^a] = i\frac{1}{2} S^a, \]  \hspace{1cm} \text{(C.8)}

\[ [D, D] = [D, M_{\mu\nu}] = [D, A^{a}_b] = 0. \]

• R-symmetry, \(o(\mathcal{N})\)

\[ [A_{ab}, A_{cd}] = i(\delta_{ac} A_{bd} - \delta_{ad} A_{bc} - \delta_{bc} A_{ad} + \delta_{bd} A_{ac}), \]

\[ [A_{ab}, Q^c] = i(\delta^c_a \delta_{bd} - \delta^c_b \delta_{ad}) Q^d, \]

\[ [A_{ab}, S^c] = i(\delta^c_a \delta_{bd} - \delta^c_b \delta_{ad}) S^d, \]  \hspace{1cm} \text{(C.9)}

\[ [A_a^b, P_\mu] = [A_a^b, K_\mu] = [A_a^b, M_{\mu\nu}] = 0. \]
D Realization of $\text{SO}(2,3) \cong \text{Sp}(2,\mathbb{R})$ structure in $M$

We exhibit explicitly the relation of the three-dimensional conformal group to $\text{SO}(2,3) \cong \text{Sp}(2,\mathbb{R})$ by introducing five-dimensional gamma matrices, $\Gamma^A$, $A = 0, 1, \cdots, 4$

\[
\Gamma^\mu = \begin{pmatrix} \gamma^\mu & 0 \\ 0 & -\gamma^\mu \end{pmatrix}, \quad \Gamma^3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \Sigma^4 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \tag{D.1}\]

They satisfy with $G^{AB} = \text{diag}(+1, -1, -1, -1, +1)$

\[
\Gamma^A \Gamma^B + \Gamma^B \Gamma^A = 2G^{AB}, \tag{D.2}\]

and

\[
\begin{pmatrix} 0 & \gamma^0 \\ \gamma^0 & 0 \end{pmatrix} \Gamma^A \begin{pmatrix} 0 & \gamma^0 \\ \gamma^0 & 0 \end{pmatrix} = -\Gamma^A, \quad \begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \end{pmatrix} \Gamma^A \begin{pmatrix} 0 & \epsilon^{-1} \\ \epsilon^{-1} & 0 \end{pmatrix} = \Gamma^A. \tag{D.3}\]

For the supermatrix, $M$, given in eq. (3.45), we may now express the $4 \times 4$ part in terms of $\Gamma^{AB} \equiv \frac{1}{4} \left[ \Gamma^A, \Gamma^B \right]$ as

\[
m \equiv \begin{pmatrix} w + \frac{1}{2} \lambda & i a \cdot \gamma \\ ib \cdot \gamma & w - \frac{1}{2} \lambda \end{pmatrix} = \frac{1}{2} w_{AB} \Gamma^{AB}, \tag{D.4}\]

where $w_{34}, w_{\mu3}, w_{\mu4}$ are given by

\[
w_{34} = \lambda, \quad w_{\mu3} = a_\mu - b_\mu, \quad w_{\mu4} = a_\mu + b_\mu. \tag{D.5}\]

$\Gamma^{AB}$ generates the Lie algebra of $\text{SO}(2,3)$

\[
[\Gamma^{AB}, \Gamma^{CD}] = -G^{AC} \Gamma^{BD} + G^{AD} \Gamma^{BC} + G^{BC} \Gamma^{AD} - G^{BD} \Gamma^{AC}. \tag{D.6}\]

In general, $m$ can be defined as a $4 \times 4$ matrix subject to two conditions

\[
b m + m^\dagger b = 0, \quad b = \begin{pmatrix} 0 & \gamma^0 \\ \gamma^0 & 0 \end{pmatrix}, \tag{D.7a}\]

\[
c m + m^\dagger c = 0, \quad c = \begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \end{pmatrix}, \tag{D.7b}\]

To show $\text{SO}(2,3) \cong \text{Sp}(2,\mathbb{R})$ we take, without loss of generality, $\gamma^0 = i \epsilon$ and $\epsilon$ to be real. Now if we define

\[
\tilde{m} = p m p^{-1}, \quad p = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix}. \tag{D.8}\]
then from
\[ p^{-1} = p^\dagger = p^\dagger, \quad pcp^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = j, \] (D.9)
we note that eq. (D.7a) is equivalent to the sp(2, \mathbb{R}) condition
\[ \tilde{m}^* = \tilde{m}, \quad j\tilde{m} + \tilde{m}^*j = 0. \] (D.10)

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