INTRINSIC GEOMETRY AND ANALYSIS OF FINSLER STRUCTURES

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Abstract. In this short note, we prove that if $F$ is a weak upper semicontinuous admissible Finsler structure on a domain in $\mathbb{R}^n$, $n \geq 2$, then the intrinsic distance and differential structures coincide.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a domain and $F$ an admissible Finsler structure on $\Omega$ (the precise definition is given in Section 2 below). Associated to $F$, we have the following intrinsic distance defined by

$$\delta_F(x, y) = \sup_u \left\{ u(x) - u(y) : u \text{ is Lipschitz and } \| F(x, du(x)) \|_\infty \leq 1 \right\}.$$  \hspace{1cm} (1.1)

Above, $du(x)$ denotes the differential of the Lipschitz function $u$ at a point $x$. Recall that the well-known Rademacher’s theorem implies that $du(x)$ exists at almost every $x \in \Omega$ and thus the above definition makes sense. The ellipticity condition on $F$ implies that $\delta_F$ is locally comparable to the standard Euclidean distance. We define the point-wise Lipschitz constant of a Lipschitz function $u : \Omega \to \mathbb{R}$ by setting

$$\text{Lip}_{\delta_F} u(x) = \limsup_{y \to x} \frac{|u(y) - u(x)|}{\delta_F(x, y)}.$$  \hspace{1cm} (1.2)

Given a subset $K$ of $\mathbb{R}^n$, we set

$$\text{Lip}_{\delta_F}(u, K) = \sup_{x, y \in K, x \neq y} \frac{|u(x) - u(y)|}{\delta_F(x, y)}$$

and denote by $\text{Lip}_{\delta_F}(K)$ the collection of all functions $u : K \to \mathbb{R}$ with $\text{Lip}_{\delta_F}(u, K) < \infty$.

Sturm asked the following interesting question in [12]: is a diffusion process determined by the intrinsic distance? Mathematically, Sturm’s question can be formulated as follows: is it true that for all $u \in \text{Lip}_{\delta_F}(\Omega)$,

$$F(x, du(x)) = \text{Lip}_{\delta_F} u(x)$$

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almost everywhere with \( F(x, v) = \sqrt{\langle A(x)v, v \rangle} \)?

The answer to the question is yes when \( A \) is supposed to be continuous, as shown by Sturm in [12, Proposition 4]. He also pointed out that the answer to this question is not always positive [12, Theorem 2]:

for \( F(x, v) = \sqrt{\langle A(x)v, v \rangle} \), where \( A \) is a diffusion matrix, there exists \( \tilde{F}(x, v) = \sqrt{\langle \tilde{A}(x)v, v \rangle} \) such that \( \delta_F = \delta_{\tilde{F}} \) but

\[
F(x, v) < \tilde{F}(x, v)
\]

for all \( v \in \mathbb{R}^n \setminus \{0\} \); see also [11] for a different example.

The case \( F(x, v) = \sqrt{\langle A(x)v, v \rangle} \) gained deeper understanding in a recent paper [10], where the authors enhanced Sturm’s result by showing that if the diffusion matrix \( A \) is weak upper semicontinuous, then the differential and distance structures coincide. They also constructed an example, which shows that if \( A \) fails to be upper semicontinuous on a set of positive measure, then the differential and distance structure may fail to coincide.

The main purpose of this paper is to generalize the above result of [10] to more general Finsler structures. More precisely, we are going to prove the following result.

**Theorem 1.1.** Let \( n \geq 2 \) and \( F \) be an admissible Finsler structure on a domain \( \Omega \subset \mathbb{R}^n \). If \( F \) is weak upper semicontinuous on \( \Omega \), then the intrinsic distance and differential structure coincide. That is given a Lipschitz function \( u \) on \( \Omega \) (with respect to the Euclidean distance), for almost every \( x \in \Omega \), we have

\[
\text{Lip}_{\delta_F} u(x) = F(x, du(x)).
\]

The proof of [10, Theorem 2] relies heavily on the structure of \( F(x, v) = \sqrt{\langle A(x)v, v \rangle} \). It seems that there is little hope to adapt their proofs in the greater generality of this paper.

To see an example where Theorem 1.1 applies more generally than [10, Theorem 2], we may choose suitable weighted \( L^p \)-norm with \( 1 \leq p < \infty \). For instance, consider \( F(x, v) = \left( \sum_{i=1}^n w(x) |v_i|^p \right)^{1/p} \), where the weight function \( w \) is upper semicontinuous and satisfies the ellipticity condition \( 0 < c \leq w(x) \leq C < \infty \) for all \( x \in \mathbb{R}^n \).

Theorem 1.1 can be regarded as an improved version of [8, Proposition 2.4] from \( L^\infty \)-norm to pointwise equality.

Our proof of Theorem 1.1 completely differs from that used in [10] and it is simpler than [10], even in their setting. The crucial observation is Proposition 3.1 below, a special case of a result due to De Cecco and Palmieri [9], which states that the intrinsic distance \( \delta_F \) (infinitesimally) coincides with \( d^*_x \), where \( d^*_x \) is the distance induced by the Finsler structure \( F \). The weak upper semicontinuity is crucial for our proof, since it implies that the “metric density” of a curve with respect to the metric length coincides with its “differential density”; see Section 4 below for
the precise meaning. Our approach is more geometric and was influenced a lot by the recent studies in Finsler geometry [6, 7, 2, 4]. Some of the ideas from this paper were successfully used in our companion paper [9] on certain $L^\infty$-variational problems associated to measurable Finsler structures. It is known (e.g. [1, 11]) that the intrinsic distance and differential structures coincide even for abstract Dirichlet forms on metric measure spaces. It would be interesting to know that whether a version of Theorem 1.1 holds in the abstract setting as there.

This paper is organized as follows. Section 2 contains all the preliminaries related to Finsler structures. Section 3 and Section 4 contain an overview of the necessary background that are needed for our proof of Theorem 1.1. In Section 5, we prove Theorem 1.1. The appendix contains a separate proof of Proposition 3.1 under the weak upper semicontinuity assumption.

2. Preliminaries on Finsler structures

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a domain, i.e., an open connected set.

**Definition 2.1** (Finsler structures). We say that a function $F : \Omega \times \mathbb{R}^n \to [0, \infty)$ is a Finsler structure on $\Omega$ if

- $F(\cdot, v)$ is Borel measurable for all $v \in \mathbb{R}^n$, $F(x, \cdot)$ is continuous for a.e. $x \in \Omega$;
- $F(x, v) > 0$ for a.e. $x$ if $v \neq 0$;
- $F(x, \lambda v) = |\lambda|F(x, v)$ for a.e. $x \in \Omega$ and for all $\lambda \in \mathbb{R}$ and $v \in \mathbb{R}^n$.

**Definition 2.2** (Admissible Finsler structures). A Finsler structure $F$ is said to be admissible if

- $F(x, \cdot)$ is convex for a.e. $x \in \Omega$;
- $F$ is locally equivalent to the Euclidean norm or elliptic, i.e., there exists a continuous function $\lambda : \Omega \to [1, \infty)$ such that
  \[
  \frac{1}{\lambda(x)}|v| \leq F(x, v) \leq \lambda(x)|v|
  \]
  for a.e. $x \in \Omega$ and for all $v \in \mathbb{R}^n$.

It is straightforward to verify that the standard $L^p$-norm ($1 \leq p < \infty$), i.e., $F(x, v) = (\sum_{i=1}^n v_i^p)^{1/p}$, is an admissible Finsler structure on $\mathbb{R}^n$. From the geometric point of view, there are many other interesting examples and we refer the interested readers to [2] for the details.

Recall that a function $u : \Omega \to \mathbb{R}$ is said to be upper semicontinuous at $x \in \Omega$ if
\[
u(x) \geq \limsup_{y \to x} u(y).
\]
Following [10], we say that $u$ is weak upper semicontinuous in $\Omega$ if $u$ is upper semicontinuous at almost every $x \in \Omega$. Let $F$ be an admissible
Finsler structure on $\Omega$. We say that $F$ is weak upper semicontinuous on $\Omega$ if for each $v \in \mathbb{R}^n$, the function $F(\cdot, v)$ is weak upper semicontinuous on $\Omega$.

Similarly a function $u : \Omega \to \mathbb{R}$ is said to be lower semicontinuous at $x \in \Omega$ if

$$u(x) \leq \liminf_{y \to x} u(y),$$

and $u$ is weak lower semicontinuous in $\Omega$ if $u$ is lower semicontinuous at almost every $x \in \Omega$. Let $F$ be an admissible Finsler structure on $\Omega$. We say that $F$ is weak lower semicontinuous on $\Omega$ if for each $v \in \mathbb{R}^n$, the function $F(\cdot, v)$ is weak lower semicontinuous on $\Omega$.

Let $F$ be an admissible Finsler structure for $\Omega$. We introduce the dual of $F$: $\Omega \times \mathbb{R}^n \to [0, \infty)$ in the standard way.

**Definition 2.3 (Dual Finsler structures).** The dual $F^*$ of an admissible Finsler structure $F : \Omega \times \mathbb{R}^n \to [0, \infty)$ is defined as

$$F^*(x, w) = \sup_{v \in \mathbb{R}^n} \{ \langle v, w \rangle : F(x, v) \leq 1 \} = \max_{v \neq 0} \{ \langle w, \frac{v}{F(x, v)} \rangle \},$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^n$.

The following proposition follows immediately from Definition 2.3; see for instance [8, Section 1.2] or [3, Section 2] for more information.

**Proposition 2.4 (Basic properties of a dual Finsler structure).** Let $F$ be an admissible Finsler structure on $\Omega$. Then the dual function $F^*$ satisfies the following properties

- $F^*(\cdot, v)$ is Borel measurable and $F^*(x, \cdot)$ is Lipschitz;
- $F^*(x, \cdot)$ is locally equivalent to the Euclidean norm, i.e.
  $$\frac{1}{\lambda(x)}|v| \leq F^*(x, v) \leq \lambda(x)|v|.$$

3. **Comparison of intrinsic distances**

Let $(\Omega, F(\cdot, \cdot), d^F_c, \delta_F)$ be a Finsler manifold with an admissible Finsler structure $F$. For an admissible Finsler structure $F$ on $\Omega$, we may associate a distance in the standard way by setting

$$d^*_c(x, y) := \sup_N \inf_{\gamma \in \Gamma_N} \left\{ \int_0^1 F^*(\gamma(t), \gamma'(t))dt \right\},$$
where the supremum is taken over all subsets $N$ of $\Omega$ such that $|N| = 0$ and $\Gamma^x_N(\Omega)$ denotes the set of all Lipschitz curves in $\Omega$ with end points $x$ and $y$ transversal to $N$, i.e. such that $\mathcal{H}^1(N \cap \gamma) = 0$. For an admissible Finsler metric $F$, $d^*_c$ is indeed an intrinsic distance; for the definition of an intrinsic distance and this fact, see [6, 7]. Above, we use $|E|$ to denote the $n$-dimensional Lebesgue measure of a set $E \subset \mathbb{R}^n$ and $\mathcal{H}^1$ the one-dimensional Hausdorff measure.

The following fundamental result, which relates $\delta_F$ and $d^*_c$, was a special case of [6, Theorem 3.7].

**Proposition 3.1.** Let $F$ be an admissible Finsler structure on $\Omega$. Then for almost every $x \in \Omega$, it holds

$$\lim_{y \to x} \frac{\delta_F(x, y)}{d^*_c(x, y)} = 1.$$  

(3.1)

Since we have assumed the weak upper semicontinuity on our admissible Finsler structure in our main result Theorem 1.1, we give a separate proof of Proposition 3.1 under this extra assumption in the appendix.

### 4. Comparison of metric derivatives

For any distance $d$ on $\Omega$ and each Lipschitz (with respect to $d$) curve $\gamma : [a, b] \to \Omega$, the length of $\gamma$ with respect to $d$ is denoted by $L_d(\gamma)$, i.e.,

$$L_d(\gamma) := \sup \left\{ \sum_{i=1}^{k} d(\gamma(t_i), \gamma(t_{i+1})) \right\},$$

where the supremum is taken over all partitions $\{[t_i, t_{i+1}]\}$ of $[a, b]$.

Given a curve $\gamma$, the metric derivative of $\gamma$ at $t$ is defined to be

$$|\gamma'(t)|_d := \limsup_{s \to 0} \frac{d(\gamma(t+s), \gamma(t))}{s}.$$  

If $\gamma : [a, b] \to \Omega$ is Lipschitz with respect to $d$, then its length can be computed by integrating the metric derivative, i.e.

$$L_d(\gamma) = \int_a^b |\gamma'(t)|_d dt.$$  

In other words, for a Lipschitz curve, the metric derivative is the metric density of its length.

For any intrinsic distance $d$, which is locally bi-Lipschitz equivalent to the Euclidean distance, we may associate a Finsler structure $\Delta_d$ in the following manner. For each $x \in \Omega$ and for every direction $v$, we define

$$\Delta_d(x, v) := \limsup_{t \to 0^+} \frac{d(x, x + tv)}{t}.$$  

(4.1)
It can be proved that for every Lipschitz curve $\gamma : [a, b] \to \Omega$, we have
\[
\mathcal{L}_d(\gamma) = \int_a^b \Delta_d(\gamma(t), \gamma'(t)) dt.
\]
In particular, $\Delta_d(\gamma(t), \gamma'(t)) = |\gamma'(t)|_d$ for a.e. $t \in [a, b]$.

**Remark 4.1.** For any admissible Finsler structure $F$, one always has
\[
\Delta_d^*(x, v) \leq F^*(x, v) \quad \text{for a.e. } x \in \Omega \text{ and all } v \in \mathbb{R}^n;
\]
see [8, Proposition 1.6]. However, the equality does not necessary hold; See [7, Example 5.1] for a counter-example.

In addition, for an admissible Finsler structure $F$, the dual Finsler structure $F^*$ always induces a lower semicontinuous length structure; see [4, Section 2.4.2]. Moreover, if the Finsler metric $F$ is weak upper semicontinuous on $\Omega$, then the following stronger result holds.

**Proposition 4.2** (Proposition 2.9, [3]). If the Finsler structure $F$ is weak upper semicontinuous on $\Omega$, then for a.e. $x \in \Omega$ and all $v \in \mathbb{R}^n$, it holds
\[
\Delta_d^*(x, v) = F^*(x, v).
\]

5. **Coincidence of distance structure and differential structure**

In this section, we are ready to prove our main result Theorem 1.1.

**Proposition 5.1.** For each $u \in \text{Lip}_{\delta_F} (\Omega)$, $F(x, du(x)) \leq \text{Lip}_{\delta_F} u(x)$ for a.e. $x \in \Omega$.

**Proof.** Since both sides are positively 1-homogeneous with respect to $u$, we only need to show that for a.e. $x \in \Omega$, if $\text{Lip}_{\delta_F} u(x) = 1$, then $F(x, du(x)) \leq 1$.

Note that by Proposition 3.1, for a.e. $x \in \Omega$, $\text{Lip}_{\delta_F} u(x) = \text{Lip}_{d_*} u(x)$.

Fix such an $x$. For each $v \in \mathbb{R}^n$, we have
\[
du(x)v = \lim_{t \to 0} \frac{u(x + tv) - u(x)}{t} \\
\leq \limsup_{t \to 0} \frac{d_*^2(x, x + tv)}{t} \cdot \limsup_{t \to 0} \frac{u(x + tv) - u(x)}{d_*^2(x, x + tv)} \\
\leq \Delta_d^*(x, v) \text{Lip}_{d_*} u(x) \leq F^*(x, v),
\]
where in the last inequality, we have used the inequality (4.2).

Therefore,
\[
F(x, du(x)) = F^{**}(x, du(x)) = \max_{v \neq 0} \left\{ du(x) \left( \frac{v}{F^*(x, v)} \right) \right\} \leq 1
\]
as desired. This completes our proof. \qed
Theorem 5.2. Let $F$ be an admissible Finsler structure on $\Omega$. If $F$ is weak upper semicontinuous on $\Omega$, then for any Lipschitz function $u$ in $(\Omega, \delta_F)$, $F(\cdot, du(\cdot))$ is an upper gradient of $u$. In particular, this implies that

$$\text{Lip}_{\delta_F} u(x) \leq F(x, du(x))$$

for a.e. $x \in \Omega$.

Proof. First, note that our assumption on $F$ implies that $F$ satisfies the following uniform upper semicontinuity property, for a.e. $x \in \Omega$,

$$\forall \varepsilon > 0, \exists \delta > 0 : F(y, v) \leq (1 + \varepsilon) F(x, v) \quad \text{for all} y \in B(x, \delta), v \in \mathbb{R}^n. \quad (5.1)$$

By homogeneity of $F$ (with respect to $v$), it suffices to prove (5.1) for all $v \in S$ (the unit sphere). Suppose by contradiction, that (5.1) fails. Then there exist some $x \in \Omega$ and some $\varepsilon_0 > 0$ such that for each $k \in \mathbb{N}$, there exist some $y_k \in B(x, \frac{1}{k})$ and $v_k \in S$ so that

$$F(y_k, v_k) > (1 + \varepsilon_0) F(x, v_k). \quad (5.2)$$

By compactness of $S$, we may assume (up to another subsequence if necessary) $v_k \to v \in S$ as $k \to \infty$. Then

$$F(x, v) = \limsup_{k \to \infty} F(x, v_k) \geq \limsup_{k \to \infty} \limsup_{y \to x} F(y, v_k)$$

$$\geq \limsup_{k \to \infty} F(y_k, v_k) \geq \limsup_{k \to \infty} (1 + \varepsilon_0) F(x, v_k)$$

$$= (1 + \varepsilon_0) F(x, v),$$

which is a contradiction.

Secondly, by Rademacher’s Theorem, it suffices to prove Theorem 5.2 when $u(x) = \langle v, x \rangle$ is linear. We may additionally assume that $v \neq 0$. By the fundamental theorem of calculus and the definition of $F^*$, we have

$$|u(x) - u(y)| = |\langle v, y - x \rangle| = \left| \int_0^1 \frac{d}{dt} u(\gamma(t)) dt \right|$$

$$= \left| \int_0^1 \langle v, \gamma'(t) \rangle dt \right| \leq (1 + \varepsilon) F(x, v) \int_0^1 F^*(\gamma(t), \gamma'(t)) dt$$

whenever $x, y$ and $\gamma(t)$ belongs to the “$\delta$-neighborhood of $x$” where (5.1) holds; it follows that

$$\frac{|\langle v, y - x \rangle|}{d^*_\delta(x, y)} \leq (1 + \varepsilon) F(x, v),$$

whenever $|x - y| < \delta$. Letting $y \to x$ and $\varepsilon \to 0$ concludes our proof. □
Appendix: Proof of Proposition 3.1 when $F$ is weak upper semicontinuous

Proof. The inequality $\delta_F(x, y) \leq d_c^*(x, y)$ follows directly from definitions. Indeed, for each Lipschitz function $u$ with $\|F(\cdot, du(\cdot))\|_{L^\infty(\Omega)} \leq 1$, each $x, y \in \Omega$, for each Lipschitz curve $\gamma$ joining $x$ and $y$ that is transversal to the zero measure set $N = \{x \in \Omega : F(x, du(x)) > 1\}$,

$$
\begin{aligned}
&u(x) - u(y) = \int_0^1 du(\gamma(t)) (\gamma'(t))dt \\
&\leq \int_0^1 F^*(\gamma(t), \gamma'(t))dt = L_{d_c^*} (\gamma),
\end{aligned}
$$

where $L_{d_c^*}$ denotes the length of the curve $\gamma$ with respect to the metric $d_c^*$. Taking infimum over all admissible curves on the right-hand side and then supermum over all admissible functions over the left-hand side, we obtain via Proposition 4.2 that

$$
\delta_F(x, y) \leq d_c^*(x, y).
$$

In particular,

$$
\limsup_{y \to x} \frac{\delta_F(x, y)}{d_c^*(x, y)} \leq 1.
$$

We are left to prove that

$$
\liminf_{y \to x} \frac{\delta_F(x, y)}{d_c^*(x, y)} \geq 1. \tag{5.3}
$$

We divide the proof of this equation into two steps.

Step 1: assume that $F(\cdot, v)$ is continuous.

Fix $x \in \Omega$ and $\varepsilon > 0$. Since $F(\cdot, v)$ and $F^*(\cdot, v)$ are continuous in $B(x, \delta)$, we may assume that for all $z \in B(x, \delta)$,

$$
(1 - \varepsilon) F(z, v) \leq F(x, v) \leq (1 + \varepsilon) F(z, v)
$$

and

$$
(1 - \varepsilon) F^*(z, v) \leq F^*(x, v) \leq (1 + \varepsilon) F^*(z, v).
$$

Note that the issue is local, we are now restricting ourself to the ball $B(x, \delta)$.

Consider the curve $\gamma(t) = x + t(y - x)$, we have

$$
d_c^*(x, y) \leq L_{d_c^*} (\gamma) = \int_0^1 F^*(\gamma(t), \gamma'(t))dt \leq (1 + \varepsilon) F^*(x, y - x).
$$

By the definition of a dual Finsler structure, we know that there exists some $\tilde{v} \neq 0$ such that $F^*(x, y - x) = \langle y - x, \frac{\tilde{v}}{F(x, \tilde{v})} \rangle$. Set

$$
v := \frac{\tilde{v}}{(1 + \varepsilon) F(x, \tilde{v})}.
$$
Then $F(x, v) = \frac{1}{1 + \varepsilon}$ and $\langle v, y - x \rangle = \frac{1}{1 + \varepsilon} F^*(x, y - x)$. Note that for all $z \in B(x, \delta)$, $F(z, v) \leq (1 + \varepsilon) F(x, v) \leq 1$ and so the function $u(z) := \langle v, z \rangle$ is an admissible function for $\delta_F(x, y)$. This means that

$$\delta_F(x, y) \geq u(y) - u(x) = 1/(1 + \varepsilon) F^*(x, y - x) \geq \frac{1}{(1 + \varepsilon)^2} d^*_c(x, y).$$

It is clear that (5.3) follows from the above inequality by letting $\varepsilon \to 0$.

**Step 2:** Assume that $F(\cdot, v)$ is weak upper semicontinuous.

In this case, $F^*$ is weak lower semicontinuous, it is a well-known fact that there exists a sequence of admissible Finsler norms $F^*_n(\cdot, v)$, which is continuous in the first variable, such that

$$F_n(x, v) \leq F^*_n(x, v) \leq \cdots \to F^*(x, v);$$

and $d^{*n} \to d^*_c$ as $n \to \infty$, where $d^{*n}$ is the distance induced by the Finsler structure $F_n$; see for instance [5, Section 4]. Let $F_n = F^*_{nn}$ denote the dual of $F^*_n$, then it is easy to check from our definition that

$$F_n(x, v) \geq F_{n+1}(x, v) \geq \cdots \to F(x, v).$$

It follows that

$$\frac{\delta_F(x, y)}{d^*_c(x, y)} = \lim_{n \to \infty} \frac{\delta_{F_n}(x, y)}{d^{*n}_c(x, y)},$$

where $\delta_{F_n}$ is the intrinsic distance induced by $F_n$ similar as $\delta_F$. Given $\varepsilon > 0$, there exists $N_0$ such that for all $n \geq N_0$,

$$\frac{\delta_F(x, y)}{d^*_c(x, y)} \geq (1 - \varepsilon) \frac{\delta_{F_n}(x, y)}{d^{*n}_c(x, y)}.$$

On the other hand, by step 1,

$$\liminf_{y \to x} \frac{\delta_{F_n}(x, y)}{d^{*n}_c(x, y)} \geq 1.$$

We thus obtain

$$\liminf_{y \to x} \frac{\delta_{F_n}(x, y)}{d^{*n}_c(x, y)} \geq \liminf_{y \to x} (1 - \varepsilon) \frac{\delta_{F_n}(x, y)}{d^{*n}_c(x, y)} \geq 1 - \varepsilon.$$

The claim follows by letting $\varepsilon \to 0$.

\[ \Box \]

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