Abstract. We study the optimal stopping problem of maximizing the variance of an unskilled linear diffusion. Especially, we reveal its close connection to game theory. Our main result shows that an optimal solution can often be found among stopping times that are mixtures of two hitting times. This and other revealed phenomena together with suggested solution methods could be helpful when facing more complex non-linear optimal stopping problems. The results are illustrated by a few examples.

1. Introduction

In classical optimal stopping problems one seeks a stopping time that optimizes the expectation of some process upon stopping, and optimal stopping with respect to higher moments has only recently been approached (see [5, 6, 13, 14]). In this paper we study the optimal stopping problem of finding a stopping time that maximizes the variance of a general unskilled linear diffusion $X_t$, i.e. we study the problem

$$
\sup_{\tau \in T} \text{Var}_x \{X_\tau\} = \sup_{\tau \in T} \left\{ \mathbb{E}_x \{X_\tau^2\} - \mathbb{E}_x \{X_\tau\}^2 \right\},
$$

where $T$ is a class of randomized stopping times generated from the filtration of $X$. We denote this the variance problem. The main difficulty in (1) is the fact that due to the non-linear term on the right hand side, the highly developed machinery for solving classical optimal stopping problems (e.g. [2, 12, 16]) is not readily usable.

The field of non-linear optimal stopping is new and thus present work deals with basic questions. The main importance on the results is thus the structure of solutions and the identification of tools for solution methods, rather than the specific solutions. However, the variance stopping problem is also interesting in its own right: Variance may be seen as a measure of risk and by maximizing the variance we get a tight upper bound for this risk.

As our main results, we have two observations to offer. The first observation is that for a general non-killed continuous diffusion the pure threshold rule cannot provide the value for all cases; There are cases where a randomized mixture of two exit times is an optimal stopping time and simple threshold rules offer purely weaker values. A somewhat similar result has been shown for some processes with jumps (see [6]). However, the importance of randomized stopping times for the jump processes studied in [6] can be narrowed down to mixtures of exit times with a single boundary which is either strict or vague. For general continuous diffusions the importance of randomized stopping times expands to

2010 Mathematics Subject Classification. 60G40; 60J60; 90C30; 91A05; 91A35.

Key words and phrases. Optimal stopping, Variance, Non-linear optimal stopping, Linear diffusion, Infinite zero-sum game.
include randomizations between exit times with strictly different exit boundaries and exit
times with both one and two boundaries. This reliance on randomized stopping times is
a remarkable difference from classical optimal stopping problems, which usually always
have an exit time solution whenever a solution exists. The second observation we make
is that the variance stopping problem is very closely related to game theory, and thus
well-known results from game theory are readily usable. To see the link, notice that
for a random variable $X_t$, we have $\sup_{\tau} \text{Var}_x(X_\tau) = \sup_{\tau} \inf_c E_x \{(X_\tau - c)^2\}$. By first
narrowing the class of stopping times within which the optimal stopping time is found,
one can utilize the theory of infinite convex two-player zero-sum games (see e.g. \cite{8, 18})
to solve the problem. In game theory context, one can find randomized optimal stopping
times more easily. For example, in \cite{7}, a gambling problem, introduced in \cite{4}, has been
studied employing the cumulative prospect theory from \cite{17}. It was shown in \cite{7} that the
optimal stopping strategy of a pre-committing gambler can be in this kind of a non-linear
setting, in fact, a randomized stopping time.

We also have some minor observations to offer. One observation is that similar to
previously studied non-linear optimal stopping problems (cf. \cite{6, 13, 14}) the variance
problem for general non-killed continuous diffusions has a strong dependence on the
initial value of the underlying process. Another observation is that we encounter quite a
strong transiency requirement in order to attain a non-trivial solution: if $X_t$ is recurrent
or it is ”not transient enough” then the problem is trivial. Lastly, we observe that if
the scale function satisfies a simple, typically satisfied monotonicity requirement, then
randomization is not needed, and in that case there exits a solution which is a pure exit
time with a single boundary given as the unique solution to a simple first order optimality
condition. All in all, our results indicate that although the variance problem is only a
small step away from the classical linear optimal stopping problems, there are some quite
substantial dissimilarities in the structure of the solutions.

The study on variance stopping began recently when Pedersen in \cite{13} proposed a
verification theorem for the variance stopping problem \cite{11} and used it for some classes
of continuous Itô-diffusions. The verification theorem states that in order to reach the
solution, it is sufficient to solve an embedded classical optimal stopping problem with
certain side conditions. The verification theorem has also been used successfully for
solving the variance problem for geometric Lévy processes in \cite{9}, and we apply it in this
paper as well.

Another non-linear optimal stopping problem which has been studied is the Mean-
variance problem given by

$$
\sup_{\tau} \left( E_x \{X_\tau\} - c \text{Var}_x \{X_\tau\} \right).
$$

This problem, together with related problems with side conditions, has been solved in
\cite{13} when the underlying process $X$ is a geometric Brownian motion, applying a Lagrange
multipliers method. The mean-variance problem has also been investigated in \cite{4} for
certain geometric Lévy processes utilizing an approach similar to \cite{13}. 
The structure of this paper is the following. In Section 2 we lay down our assumptions and divide the problem into different categories depending on whether the solution is trivial or not. In Section 3 we give our main results for non-trivial cases. The most of these results are proven in Sections 4 and 5 by utilizing the verification theorem from [13]. In Section 6 we prove the rest of the main results by applying game theoretical approach. Trivial and marginal cases are discussed in Section 7. In Section 8 we provide a step-by-step solution algorithm and illustrate our results with three examples. Lastly in Appendix A we give a proof for a technical auxiliary lemma.

2. Problem formulation

The mathematical formulation of the variance problem is the following. Let $X_t$ be a regular linear diffusion defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}, \mathbb{P})$, where $\mathcal{F}$ is the filtration generated from $X$. Let $X_t$ evolve on $I := (\alpha, \beta) \subseteq \mathbb{R}$. The boundaries can be natural, exit, entrance, absorbing or killing. We assume that the diffusion does not die inside the state space and that the scale function $S(x)$ and speed measure $m(x)$ are continuous. Now, in the variance problem we seek to identify the value function, $V$, and an optimal stopping time, $\tau^*$, such that

$$V(x) := \sup_{\tau \in \mathcal{T}} \text{Var}_x \{ X_\tau \} = \text{Var}_x \{ X_{\tau^*} \},$$

where the subscript $x$ refers to the initial point of the process $X$, and $\mathcal{T}$ is the class of randomized stopping times generated from $X$. We define the class of randomized stopping times in the following way. We assume that there exists a random variable $U$ uniformly distributed on $[0, 1]$ and independent of the process $X$. This may require that the probability space is expanded. We define an augmented filtration $\tilde{\mathcal{F}}$ as the filtration generated from both $U$ and the process $X$. That is, $\tilde{\mathcal{F}}_t = \sigma \{ U, (X_s)_{0 \leq s \leq t} \}$. Now $\mathcal{T}$ is defined as all stopping times generated from $\tilde{\mathcal{F}}$. We include stopping times which may take the value infinity, and thus it is common to rather refer to $\mathcal{T}$ as a set of Markov times (e.g. [15]).

Since we allow stopping times to take the value infinity we need to specify how we interpret $X_\infty$. Let $\zeta := \inf \{ t \geq 0 \mid X_t = \alpha \text{ or } \beta \}$ be the life time of the process $X_t$. We interpret $X_\infty := \lim_{t \to \zeta} X_t$, whenever the limit exists. In other cases, we let $X_\infty$ be undefined.

The definition of randomized stopping times we use is quite general. We find in this paper, that whenever an optimal stopping time exists we may find an optimal stopping time within the subclass which we denote by Bernoulli randomized stopping times. The Bernoulli randomized stopping times are the stopping times $\tau$ which may be written in the form: $\tau = \mathbb{1}_{(U < p)} \tau_1 + \mathbb{1}_{(U \geq p)} \tau_2$, where $p \in [0, 1]$ and where $\tau_1, \tau_2$ are stopping times with respect to the filtration $\mathcal{F}$.

We denote by $\tau_z := \inf \{ t \geq 0 \mid X_t = z \}$ the first hitting time to a state $z$ and by $\tau_{(z,y)} := \inf \{ t \geq 0 \mid X_t \notin (z, y) \}$ the first exit time from an open interval $(z, y)$. Lastly, $\tau_{(\alpha,z)} = \lim_{a \to \alpha} \tau_{(a,z)}$ and similarly $\tau_{(z,\beta)} = \lim_{b \to \beta} \tau_{(z,b)}$. 


2.1. Scale function and transiency of a diffusion. We denote by $S : \mathcal{I} \to \mathbb{R}$ the scale function of the diffusion (see e.g. II.4 in [3]). The minimal requirement for it is that it is increasing and continuous. Furthermore, we assume, without losing the generality, that either $S(\alpha) = 0$ or $S(\alpha) = -\infty$. If $S(\alpha) \in (-\infty, \infty)$, we can define $\tilde{S}(x) = S(x) - S(\alpha)$ to be a new scale function fulfilling our assumption. Here, and later, we understand $S(\alpha) = \lim_{a \downarrow \alpha} S(a)$.

It is known (e.g. II.6 in [3]) that the finiteness of a scale function at a boundary means that the corresponding end point is attractive, i.e. if $S(\alpha) = 0$, then $\lim_{t \to \infty} X_t = \alpha$ with positive probability. This is closely related to the transiency of the diffusion. Recall that a diffusion is recurrent, if $P_x(\tau_y < \infty) = 1$ for all $x, y \in \mathcal{I}$, and a diffusion which is not recurrent is said to be transient.

**Proposition 2.1** (Proposition 2.2 in [16]). Let $X_t$ be a regular diffusion on $\mathcal{I}$ with the boundary condition of killing for a regular end point. Then $X_t$ is transient if and only if $S(\alpha) > -\infty$ and/or $S(\beta) < \infty$, i.e. at least one of the end points is attractive.

Another important feature of the scale function is its relation to the hitting time distribution of a diffusion (see e.g. II.4 in [3]): for $\alpha < a < x < b < \beta$ we have

$$P_x(\tau_a < \tau_b) = \frac{S(b) - S(x)}{S(b) - S(a)}, \quad \text{and}$$

$$P_x(\tau_b < \tau_a) = \frac{S(x) - S(a)}{S(b) - S(a)}.$$  \hspace{1cm} (3)

Strictly speaking, (3) tells us the distributions under the condition that we hit either $a$ or $b$ for a given $a < x < b$, but not whether a diffusion eventually exits from an interval $(a, b)$ almost surely. That it actually exits almost surely from an arbitrary interval $(a, b)$ with compact closure in $\mathcal{I}$ has been proved e.g. in [3, Theorem 6.11] for Itô diffusions. For completeness, we include here the proof for a general diffusion.

**Lemma 2.2.** Let $X_t$ be a regular diffusion and let $x \in (z, y)$, where $\alpha < z < y < \beta$. Then $P_x(\min\{\tau_z, \tau_y\} < \infty) = 1$, i.e. $X_t$ exits from an open interval with a compact closure in finite time with probability 1.

**Proof.** If $X_t$ is recurrent, the claim is clear. Therefore, assume that $X_t$ is transient. By II.20 in [3] we can say that

$$P_x(\tau_y < \infty) = \frac{G_0(x, y)}{G_0(y, y)},$$

where $G_0$ is the Green function associated to a diffusion $X_t$ (e.g. II.11 in [3]). Furthermore, by II.11 in [3], we have

$$G_0(x, y) = \lim_{\substack{a \to \alpha \\, b \to \beta}} \frac{(S(x) - S(a)) (S(b) - S(y))}{(S(b) - S(a))}.$$  \hspace{1cm} (4)

Using this expression we can rewrite (4) as

$$P_x(\tau_y < \infty) = \lim_{a \to \alpha} \frac{S(x) - S(a)}{S(y) - S(a)},$$

where $\mathcal{I}$ is the state space and $\mathbb{R}$ is the real numbers. This completes the proof.
1. Assume that \( P(6) \in \mathcal{I} \). As the cases are analogous, we only prove the case (i). First notice that for every \( \alpha \), \( \beta \) \( P \) compact interval almost surely, then \( P_x(\tau_y < \infty) = 1 \). If \( S(\alpha) = -\infty \), we have \( P_x(\tau_y < \infty) = 1 \). If \( S(\beta) = \infty \), we have \( P_x(\tau_x < \infty) = 1 \). If both \( S(\alpha) \) and \( S(\beta) \) are finite. Letting \( z \to \alpha \) and \( y \to \beta \) we obtain (recalling that \( S(\alpha) = 0 \))

\[
\lim_{y \to \beta} P_x(\tau_y < \infty) = \frac{S(x)}{S(\beta)} \quad \text{and} \quad \lim_{z \to \alpha} P_x(\tau_x < \infty) = 1 - \frac{S(x)}{S(\beta)}.
\]

As these sum up to 1, we conclude that also in this case \( P_x(\min\{\tau_x, \tau_y\} < \infty) = 1 \). \( \square \)

A consequence of Lemma 2.2 is that for \( \alpha < a < b < \beta \) the stopping time \( \tau_{(a,b)} \) is finite a.s. and that for a transient diffusion \( P_x(\lim_{t \to \infty} X_t = \alpha \) or \( \lim_{t \to \infty} X_t = \beta \) = 1.

2. Assume now that \( P(2) \in \mathcal{I} \). Then \( V(x) = \infty. \)

Proof. As the cases are analogous, we only prove the case (i). First notice that for every \( x \in (a, b) \subset \mathcal{I} \),

\[
(5b) \quad P_x(\tau_x < \infty) = \lim_{b \to \beta} \frac{S(b) - S(x)}{S(b) - S(z)}.
\]

It follows from Proposition 2.1 that \( X \) is transient if and only if \( S(\alpha) > -\infty \) or \( S(\beta) < \infty \). If \( S(\alpha) = -\infty \), we have \( P_x(\tau_y < \infty) = 1 \). If \( S(\beta) = \infty \), we have \( P_x(\tau_x < \infty) = 1 \). If both \( S(\alpha) \) and \( S(\beta) \) are finite. Letting \( z \to \alpha \) and \( y \to \beta \) we obtain (recalling that \( S(\alpha) = 0 \))

\[
\lim_{y \to \beta} P_x(\tau_y < \infty) = \frac{S(x)}{S(\beta)} \quad \text{and} \quad \lim_{z \to \alpha} P_x(\tau_x < \infty) = 1 - \frac{S(x)}{S(\beta)}.
\]

As these sum up to 1, we conclude that also in this case \( P_x(\min\{\tau_x, \tau_y\} < \infty) = 1 \). \( \square \)

A consequence of Lemma 2.2 is that for \( \alpha < a < b < \beta \) the stopping time \( \tau_{(a,b)} \) is finite a.s. and that for a transient diffusion \( P_x(\lim_{t \to \infty} X_t = \alpha \) or \( \lim_{t \to \infty} X_t = \beta \) = 1.

2.2. Infinite values. We identify simple conditions under which the variance is infinite.

Proposition 2.3. Assume that one of the following holds.

(i) \( \beta = \infty \) and \( \lim_{b \to \infty} P_x(\tau_b < \infty) b^2 = \infty \).

(ii) \( \alpha = -\infty \) and \( \lim_{a \to -\infty} P_x(\tau_a < \infty) a^2 = \infty \).

Then \( V(x) = \infty \).

Proof. As the cases are analogous, we only prove the case (i). First notice that for every \( x \in (a, b) \subset \mathcal{I} \),

\[
(6) \quad \text{Var}_x \left\{ X_{\tau_{(a,b)}} \right\} = (a - b)^2 P_x(\tau_a < \tau_b) P_x(\tau_b < \tau_a).
\]

We split the proof in two according to whether \( P_x(\tau_a < \infty) = 1 \) for all \( b > x \), or not.

1. Assume that \( P_x(\tau_b < \infty) = 1 \) for all \( b > x \). Then we know that \( X_t \) hits \( n > x \) with probability 1 and from Lemma 2.2 we have that \( X_t \) exits from every interval \((a,n), \) \( x \in (a,n), \) with probability 1. It follows that we can choose a descending sequence \( a_n < x, \) in such a way that \( P_x(\tau_{a_n} < \tau_n) = P_x(\tau_{a_n} > \tau_n) = \frac{1}{2} \) for all \( n \in \mathbb{N}, n > x. \)

Substituting these into (6) gives

\[
\text{Var}_x \left\{ X_{\tau_{(a_n,n)}} \right\} = (a_n - n)^2 \frac{1}{4}.
\]

As we let \( n \to \infty, \) this tends to infinity.

2. Assume now that \( P_x(\tau_b < \infty) < 1 - \delta \) for some \( \delta \in (0,1) \) and all \( b > b^* \) for some \( b^* > x. \) If \( \alpha > -\infty \) then

\[
\text{Var}_x \left\{ X_{\tau_{(\alpha,b)}} \right\} = (b - \alpha)^2 P_x(\tau_b < \infty) (1 - P_x(\tau_b < \infty))
\]

\[
> P_x(\tau_b < \infty) (b - \alpha)^2 \delta \to \infty, \quad \text{as} \quad b \to \infty.
\]

If \( \alpha = \infty \) then take a descending sequence \( a_n \to -\infty. \) Since the process \( X \) exits every compact interval almost surely, then \( P(\tau_{a_n} < \tau_{b^*}) > \delta \) for every \( n, \) and \( P(\tau_{a_n} < \tau_{b^*}) \to \)
\(P(\tau_b = \infty)\) for \(n \to \infty\). Thus
\[
\text{Var}_x \left\{ X_{(a_n,b')} \right\} = (b^* - a_n)^2 P(\tau_{a_n} < \tau_{b^*})(1 - P(\tau_{a_n} < \tau_{b^*})) \\
> (b^* - a_n)^2 \delta(1 - P(\tau_{a_n} < \tau_{b^*})) \to \infty, \quad \text{as } n \to \infty. \quad \square
\]

Basically, the proposition says that if a diffusion is too likely to travel too far towards an unbounded end point the achievable variances are unbounded. Especially, as a corollary we see that attractive unbounded end point always leads to infinite values.

**Corollary 2.4.** Let \(X_t\) be transient, and assume that at least one unbounded boundary point is attractive. Then \(V(x) \equiv \infty\).

**Proof.** Assume \(\beta\) is an unbounded, attractive endpoint. That is, \(\beta = \infty\) and \(S(\alpha) = 0\). Then \(P(\lim_{t \to \infty} X_t = \beta) = \delta > 0\) and thus \(P_x(\tau_b < \infty) > \delta\) for all \(b > x\). Therefore, \(\lim_{b \to \infty} b^2 P_x(\tau_b < \infty) = \infty\), and the claim follows from Proposition 2.3. The case for the endpoint \(\alpha\) is analogous. \quad \square

### 2.3. Assumptions to get finite variance

As the recurrent case is quite simple to handle, we need to assume the diffusion to be transient. Moreover, the inspection of transient diffusions falls naturally into three parts: either exactly one of the end point is attractive or both are. That is, we assume that one of the following assumptions hold in order to get a finite, interesting problem.

**Assumption 2.5.**

Case (I) Let \(\alpha > -\infty\) and assume that \(\alpha\) is attractive and \(\beta\) is not (i.e. \(S(\alpha) = 0\) and \(S(\beta) = \infty\)) and that \(\lim_{b \to \beta} P_x(\tau_b < \infty) b^2 = 0\).

Case (II) Let \(\beta < \infty\) and assume that \(\beta\) is attractive and \(\alpha\) is not (i.e. \(S(\alpha) = -\infty\) and \(S(\beta) < \infty\)), and that \(\lim_{a \to \alpha} P_x(\tau_a < \infty) a^2 = 0\).

Case (III) Let \(-\infty < \alpha < \beta < \infty\) and assume that both end points are attractive (i.e. \(S(\alpha) = 0\) and \(S(\beta) < \infty\)).

Two cases not covered in the Proposition 2.3 or Assumption 2.5 are the ones where \(X_t\) is transient, and \(\beta = \infty\) with \(\lim_{b \to \infty} P_x(\tau_b < \infty) b^2 \in (0, \infty)\) and \(\alpha = -\infty\) with \(\lim_{a \to -\infty} P_x(\tau_a < \infty) a^2 \in (0, \infty)\). These special cases are discussed at Section 7.

It is a normal sight in non-discounted problems that one needs some transiency in order to get interesting results. However, we would like to stress that the squareness nature of the problem [1] forces quite a strong transiency requirement for a finite value: it is not enough that the process is transience, but it also needs to wander sufficiently rarely toward an unbounded end point.

Lastly, the following additional technical assumption will help to simplify the general result when facing randomized stopping times.

**Assumption 2.6.**

(I) Let the conditions of Case (I) from Assumption 2.5 hold. For each \(c \in \mathcal{I}\) let \(Z(c)\) denote the set of \(z\) maximizing the ratio \(\frac{z^2 - \alpha^2 - 2c(z-\alpha)}{S(z)}\). For each \(c \in \mathcal{I}\) for which \(Z(c)\) has more than one element we assume that
\[
\mathbb{E}_{Z(c)} \left\{ X_{\tau(a,z)} \right\} > c, \tag{7}
\]
where \( \hat{z}_l = \inf \{ Z(c) \} \) and \( \hat{z}_s = \sup \{ Z(c) \} \).

(II) Let the conditions of Case (II) from Assumption 2.5 hold. For each \( c \in \mathcal{I} \) let \( Y(c) \) denote the set of \( z \) maximizing the ratio \( \frac{y^2 - \beta^2 - 2c(y - \beta)}{S(\beta) - S(y)} \). For each \( c \in \mathcal{I} \) for which \( Y(c) \) has more than one element we assume that

\[
\mathbb{E}_{\hat{y}_l} \left\{ X_{\tau(\hat{y}_l, \beta)} \right\} > c,
\]

where \( \hat{y}_l = \inf \{ Y(c) \} \) and \( \hat{y}_s = \sup \{ Y(c) \} \).

Notice that \( Z(c) \) contains \( \inf \{ Z(c) \} \) and \( \sup \{ Z(c) \} \) since the ratio \( \frac{y^2 - \alpha^2 - 2c(y - \alpha)}{S(z)} \) is continuous in \( z \). Similar observation holds for \( Y(c) \).

This assumption is indeed quite a technical one. However, the sets \( Y(c) \) and \( Z(c) \) typically only contain one element, and when the sets contain more than one element the inequalities (7) and (8) are typically fulfilled. It seems that diffusions that do not satisfy Assumption 2.6 are very marginal ones, and one has to carefully construct a specific diffusion in order to find a counter example that does not satisfy the assumption above (cf. example in Subsection 8.4).

3. Results

3.1. Case (I): \( \alpha > -\infty \) is attractive while \( \beta \) is not. In this case, \( \mathbb{P}_x (\tau_b < \infty) = \frac{S(x)}{S(b)} \), and consequently the condition \( \lim_{b \to \beta} \mathbb{P}_x (\tau_b < \infty) b^2 = 0 \) can be written as \( \lim_{b \to \beta} \frac{b^2}{S(b)} = 0 \). Notice that \( \beta \) can be either bounded or unbounded. Theorem 3.1 provides the structure of an optimal stopping time for the variance problem.

**Theorem 3.1.** Let \( X_t \) be a regular diffusion on \( \mathcal{I} = (\alpha, \beta) \), where the boundaries can be natural, exit, entrance, killing or absorbing. We fix \( x \in \mathcal{I} \) and assume that the conditions of Case (I) in Assumption 2.3 hold.

(A) \( V(x) < \infty \), and there is an optimal stopping time which is a randomized stopping time given by \( \hat{\tau} := \xi(p^*) \tau_1 + (1 - \xi(p^*)) \tau_{(a,z)}^* \), where \( \xi(p^*) \) is a Bernoulli random variable with the parameter \( p^* \in [0, 1] \), and \( z_2^* > x \), and \( \tau_1 \) is of the form \( \tau_{(a,z_1)}^* \), \( \tau_{(a,b)}^* \), or \( \tau_1 \equiv 0 \), where \( x < z_1^* < z_2^* \), and \( \alpha < a^* < x < b^* < \beta \).

\( \tau_1 \) and \( \tau_{(a,z)}^* \) are given as solutions to \( \sup_x \mathbb{E}_x \{ (X_{\tau} - c^*)^2 \} \) that fulfill \( \mathbb{E}_x \{ X_{\tau_1} \} \geq c^* \) and \( \mathbb{E}_x \{ X_{\tau_{(a,z)}^*} \} \leq c^* \), where

\[
c^* = \inf \left\{ c \mid \exists z : z \text{ solves embedded quadratic problem } (12) \text{ and } c > \mathbb{E}_x \left\{ X_{\tau_{(a,z)}} \right\} \right\}
\]

and where \( p^* \in [0, 1] \) can be solved from \( \mathbb{E}_x \{ X_{\tau} \} = c^* \) Moreover, if Assumption 2.6 holds, then \( \tau_1 = \tau_{(a,z)}^* \) for some \( x \leq z_1^* < z_2^* \). (For solution algorithms and discussions, see Subsections 6.3 and 8.1.)

(B) If \( S \) is differentiable and \( \frac{S'(z)}{S(z)}(z - \alpha) \) is non-decreasing, then \( \tau_{(a,z^*)} \) is an optimal stopping time, where \( z^* = z^*(x) \) is the unique solution on \( (x, \beta) \) to

\[
\frac{S(z) - S(x)}{\frac{1}{2}S'(z) - S'(x)} = \frac{S'(z)}{S(z)}(z - \alpha).
\]

Furthermore, the value reads as \( V(x) = (z^* - \alpha)^2 \frac{S'(z)}{S(z)} \left( 1 - \frac{S(x)}{S(z)} \right) \).
The monotonicity condition in (B) may look quite peculiar, but it seems to be valid with the most well-known diffusions, such as geometric Brownian motion, Logistic diffusion, Brownian motion, Jacobi diffusion, etc. Moreover, this monotonicity condition simplifies considerably the solution.

The general case in part (A) is more complex to handle. In that case one can find an optimal stopping time which is one of the following: a hitting time of the form \(\tau_{(\alpha,z^*)}\), a randomization between two such times, a randomization between \(\tau_{(\alpha,z^*)}\) and 0 (stop immediately), or a randomization between \(\tau_{(\alpha,z^*)}\) and \(\tau_{(a^*,b^*)}\), where \(\alpha < a^* < x < b^* < \infty\). In literature, the optimality of the first three types have been reported before in variance stopping problems applying geometric Lévy processes (see [4]). However, the last type of optimal stopping time has not been reported before. We give an example of such in Subsection 8.4.

Although in the general case Theorem 3.1 offers no explicit solution, the proofs of Theorem 3.1(A) provide us an algorithm how to find the solution. The algorithm is presented in Subsection 8.1 (see also 6.3). It is based on the division of the state space into two regions: One where the randomized solution is optimal and the other where the usual hitting time policy is optimal.

We prove Theorem 3.1 in two different ways: One is along the lines of [6] and [13], and this is presented in Section 4 the other is via game theory, and this is presented in Section 6.

3.2. Case (II): \(\beta < \infty\) is attractive while \(\alpha\) is not. In this case, by (5b), \(\mathbb{P}_x(\tau_\alpha < \infty) = \frac{S(\beta) - S(x)}{S(\beta) - S(\alpha)}\) and consequently the condition \(\lim_{\alpha \to \infty} \mathbb{P}_x(\tau_\alpha < \infty) a^2 = 0\) can be written as \(\lim_{\alpha \to \infty} \frac{a^2}{S(\alpha)} = 0\). Notice that \(\alpha\) can be bounded or unbounded.

**Theorem 3.2.** Let \(X_t\) be a regular diffusion on \(I = (\alpha, \beta)\), where the boundaries can be natural, exit, entrance, killing or absorbing. We fix \(x \in I\) and assume that the conditions of Case (II) in Assumption 2.6 hold.

(A) \(V(x) < \infty\), and there is an optimal stopping time which is a randomized stopping time given by \(\hat{\tau} := \xi(p^*)\tau_{y_1^*,\beta} + (1 - \xi(p^*))\tau_2\), where \(\xi(p^*)\) is a Bernoulli random variable with a parameter \(p^* \in [0,1]\), and \(y_1^* < x\), and \(\tau_2\) is of the form \(\tau_{(y_2^*,\beta)}\), \(\tau_{(\alpha^*,b^*)}\), or \(\tau_2 \equiv 0\), where \(y_1^* < y_2^* < x\) and \(\alpha < a^* < x < b^* < \beta\).

\(\tau_{(y_1^*,\beta)}\) and \(\tau_2\) are given as solutions to \(\sup_x \mathbb{E}_x \left\{ (X_t - c^*)^2 \right\} \) that fulfil \(\mathbb{E}_x \left\{ X_{\hat{\tau}_{(y_1^*,\beta)}} \right\} \leq c^*\) and \(\mathbb{E}_x \{ X_{\tau_2} \} \geq c^*, \) where

\[ c^* = \sup \left\{ c \mid \exists y : \ y \text{ solves auxiliary problem } 12 \text{ and } c < \mathbb{E}_x \left\{ X_{\tau_{(y,\beta)}} \right\} \right\} \]

and where \(p^* \in [0,1]\) can be solved from \(c^* = \mathbb{E}_x \{ X_{\tau} \}\). Moreover, if Assumption 2.6 holds, then \(\tau_2 = \tau_{(y_2^*,\beta)}\). (For a solution algorithm, see Subsection 8.4.)

(B) If \(S\) is differentiable and \(\frac{S'(y)}{S(\beta) - S(y)}(\beta - y)\) is non-increasing, then \(\tau_{(y^*,\beta)}\) is an optimal stopping time, where \(y^*\) is the unique solution on \((\alpha,x)\) to

\[
\frac{S(x) - S(y)}{S(x) - \frac{1}{2}S(y) - \frac{1}{2}S(\beta)} = \frac{S'(y)}{S(\beta) - S(y)}(\beta - y).
\]


Furthermore, the value reads as \( V(x) = (\beta - y^*)^2 \frac{S(\beta) - S(x)}{S(\beta) - S(y^*)} \left( \frac{S(x) - S(y^*)}{S(\beta) - S(y^*)} \right) \).

3.3. Case (III): \(-\infty < \alpha < \beta < \infty\), both end points attractive. As the state space is now finite, the value is always bounded with \( \frac{1}{4} (\beta - \alpha)^2 \) and hence is finite.

Furthermore, the diffusion hits one or the other end point almost surely as we let \( t \to \infty \), with \( \mathbb{P}_x (\lim_{t \to \infty} X_t = \beta) = \frac{S(x)}{S(\beta)} = 1 - \mathbb{P}_x (\lim_{t \to \infty} X_t = \alpha) \).

In this case, the solution reads as follows.

**Theorem 3.3.** Let \( X_t \) be a regular diffusion on \( \mathcal{I} = (\alpha, \beta) \), where the boundaries can be natural, exit, entrance, killing or absorbing. We fix \( x \in \mathcal{I} \) and assume that the conditions of Case (III) in Assumption 2.5 holds. Then \( V(x) < \infty \). Furthermore:

(I) Assume that \( \mathbb{E}_x \left\{ X_{\tau(\alpha,\beta)} \right\} \leq \frac{1}{2} (\beta + \alpha) \). Then the statements (A) and (B) of Theorem 3.1 hold true when requiring that \( z^* \) in (3.1) is either a unique solution to (B), or, if the root does not exist, \( z^* = \beta \).

(II) Assume that \( \mathbb{E}_x \left\{ X_{\tau(\alpha,\beta)} \right\} > \frac{1}{2} (\beta + \alpha) \). Then the statements (A) and (B) of Theorem 3.1 hold when requiring that \( y^* \) in (3.1) is either a unique solution to (B), or, if the root does not exist, \( y^* = \alpha \).

In Case (III) both end points are attractive. Consequently, the solution depends on the initial point even greater than before, as the optimal policy type — whether to use a stopping time \( \tau(\alpha,x) \) or \( \tau(y,\beta) \) — depends on which boundary is closer. We can use \( \mathbb{E}_x \left\{ X_{\tau(\alpha,\beta)} \right\} \) to measure this "closeness".

4. First proof of Theorem 3.1 (Case (I)) — Optimal stopping approach

Here we prove Theorem 3.1 (Case (I)) by utilizing the verification theorem from [13].

In our proof of Theorem 3.1 we assume that \( \alpha = 0 \). This simplifies arguments and causes no loss of generality. To see that this indeed is the case, let \( X_t \) be a regular linear diffusion defined on a state space \( \mathcal{I} = (\alpha, \beta) \), with \( \alpha \in (-\infty, \infty) \), and let \( S(x) \) be the scale function and \( m(x) \) the speed measure associated to \( X_t \). We assume, like earlier, that \( S(\alpha) = 0 \). We face the optimal stopping problem

\[
V(x) = \sup_\tau \text{Var}_x \{ X_\tau \}, \quad X_0 = x,
\]

If \( \alpha \neq 0 \), we define an auxiliary process \( Y_t \) on a state space \( (0, \beta - \alpha) \) by defining a scale function \( \hat{S} \), a speed measure \( \hat{m} \), and a starting point \( Y_0 \) through

\[
\hat{S}(y) := S(y + \alpha), \quad \hat{m}(y) := m(y + \alpha), \quad Y_0 = y := x - \alpha.
\]

Then \( Y \) is a well defined diffusion on \( (0, \beta - \alpha) \) inheriting its boundary behaviour from \( X_t \), and we can now study the optimal variance stopping problem

\[
\hat{V}(y) = \sup_\tau \text{Var}_y \{ Y_\tau \}, \quad Y_0 = y.
\]

The desired solution concerning the diffusion \( X_t \) can be retrieved easily from this by inverting the transformations in (11).
Therefore, in this section we assume that \( \alpha = 0 \), and that the conditions of Case (I) in Assumption 2.5 hold.

4.1. Preliminaries and solution to an embedded quadratic problem. To find the solution, we utilize the following verification theorem.

**Proposition 4.1** (Theorem 2.1 in [13]). Assume that a constant \( c^* = c^*(x) \) is such that the value function for an auxiliary problem

\[
V^{c^*}(x) = \sup_{\tau} \mathbb{E}_x \left\{ (X_\tau - c^*)^2 \right\}
\]

is finite and the optimal stopping time \( \tau^{c^*} \) that produces the value \( V^{c^*}(x) \) satisfies the condition \( c^* = \mathbb{E}_x \left\{ X_{\tau^{c^*}} \right\} \). Then \( \tau^{c^*} \) is also an optimal stopping time for the variance stopping problem (1).

In order to use the verification theorem, we need to solve the auxiliary embedded quadratic problem

\[
(12) \quad V^c(x) = \sup_{\tau} \mathbb{E}_x \left\{ (X_\tau - c)^2 \right\},
\]

for all \( c \in \mathcal{I} \). This is done in the following lemma.

**Lemma 4.2.** Let \( c \in \mathcal{I} \) and let \( z_c \) be the greatest point that maximizes \( \frac{z^2 - 2cz}{S(z)} \). For \( x \leq z_c \), \( \tau_{(0,z_c)} \) is an optimal stopping time to (12) and the value reads as

\[
V^c(x) = \frac{z_c^2 - 2cz_c}{S(z_c)} S(x) + c^2.
\]

In addition, for \( x > z_c \), the hitting time \( \tau_{D_c} = \inf\{t \geq 0 \mid X_t \in D_c\} \) is an optimal stopping time, where \( D_c = \{x \mid V^c(x) = (x - c)^2\} \) is the stopping set associated to the embedded problem with parameter \( c \).

**Proof.** 1. Assume first that \( c < \frac{1}{2} \beta \). For an arbitrary \( z \geq x \) we have

\[
\mathbb{E}_x \left\{ (X_{\tau_{(0,z)}} - c)^2 \right\} = \frac{z^2 - 2cz}{S(z)} S(x) + c^2.
\]

As we have assumed that \( \lim_{z \to \beta} \frac{z^2 - 2cz}{S(z)} = 0 \), the maximizer of the ratio \( \frac{z^2 - 2cz}{S(z)} \) must be smaller than \( \beta \). On the other hand, the ratio \( \frac{z^2 - 2cz}{S(z)} \) is non-positive for all \( z \leq 2c \) and positive for all \( z > 2c \). Therefore, we see that there must be at least one point that maximizes \( \frac{z^2 - 2cz}{S(z)} \), and it is between \( (2c, \beta) \). Let \( z_c \) be the greatest of such points (which exists since \( \frac{z^2 - 2cz}{S(z)} \) is \( z \)-continuous). Recalling that \( X_\infty = 0 \) a.s., we have for
all stopping times } \tau \}
\begin{align*}
\mathbb{E}_x \{(X_\tau - c)^2\} &= \mathbb{E}_x \left\{ \frac{X_\tau^2 - 2cX_\tau}{S(X_\tau)} S(X_\tau) \mid X_\tau \neq 0 \right\} \mathbb{P}(X_\tau \neq 0) \\
&= \mathbb{E}_x \left\{ \frac{X_\tau^2 - 2cX_\tau}{S(X_\tau)} \right\} + c^2
\end{align*}

where the first inequality follows by the maximality of } z_c \text{ and the second one follows from the fact that } S(X_t) \text{ is a positive local martingale and hence supermartingale. Notice that } S(\beta) = \infty \text{ do not bring any additional concerns as } \mathbb{E}_x (X_\tau = \beta) = 0. \text{ As the value } \frac{z_c^2 - 2cz_c}{S(z_c)}S(x) + c^2 \text{ is attained by } \tau_{(0,z_c)} \text{ for all } x < z_c, \text{ it is an optimal stopping time for all } x < z_c.

2. Assume now that } c \geq \frac{1}{\sqrt{2}} \beta. \text{ As } (0,2c) \text{ belongs to a continuation region, we see at once that if } c \geq \frac{1}{\sqrt{2}} \beta, \text{ the optimal stopping time is } \tau_{(0)}.

3. Finally, let us prove the optimality of } \tau_{D_c} \text{ for every } x > z_c.

By items above it is known that } (0,2c) \text{ is in the continuation region and that } z_c \in D_c. \text{ Moreover, for all stopping times } \tau \text{ and sequences } b_n \text{ such that } b_n \to \infty \text{ as } n \to \infty \text{ we have, by Case (I) of Assumption 2.5, that } \lim_{n \to \infty} \mathbb{E}_x \left\{ (X_{\tau \wedge \tau_n} - c)^2 \right\} = \mathbb{E}_x \left\{ (X_\tau - c)^2 \right\}. \text{ It follows that near the boundaries } 0 \text{ and } \infty \text{ we cannot have any unpleasant behaviour and consequently by } [11, \text{ Theorem 6.3(III)}] \text{ } \tau_{D_c} \text{ is an optimal stopping time.} \quad \square

Notice that we solved the auxiliary problem explicitly only for } x \in (0,z_c). \text{ In most cases this is enough: the auxiliary embedded quadratic problem is usually needed with only such a } c \text{ that } x < z_c. \text{ There are, however, a few exceptions in which cases we apply the fact that } \tau_{D_c} \text{ is an optimal stopping time. Furthermore, as the diffusion } X_t \text{ is continuous, we know, in fact, that } \tau_{D_c} = \tau_{(a,b)} \text{ for some } 0 \leq a \leq x \leq b \leq \beta, \text{ where either } a > 0 \text{ or } b < \beta.

In the proof of our main theorem, Theorem 3.1, we need the following five, more or less technical, results.

**Lemma 4.3.** Let Assumptions 2.2 Case (I) hold. And let } z_c \text{ denote the greatest maximizer of } \frac{z_c^2 - 2cz_c}{S(z_c)}. \text{ Then each of the following is true. }

1. } F(a,b,c) = \mathbb{E}_x \left\{ (X_{\tau_{(a,b)}} - c)^2 \right\} - \sup_\tau \mathbb{E}_x \left\{ (X_\tau - c)^2 \right\} \text{ is continuous.}

2. There exist } a_L, b_L, \text{ and } c_L < \frac{1}{\sqrt{2}} \beta \text{ such that } F(a_L,b_L,c_L) = 0 \text{ and } \mathbb{E}_x \left\{ X_{\tau_{(a_L,b_L)}} \right\} \geq c_L.

3. With } c_L \text{ given from part (2) there exist } c_U > c_L \text{ such that } F(0,z_{c_U},c_U) = 0 \text{ and } \mathbb{E}_x \left\{ X_{\tau_{(0,z_{c_U})}} \right\} < c_U.

4. } G(a,b) = \mathbb{E}_x \left\{ X_{\tau_{(a,b)}} \right\} \text{ is continuous.}

5. if } c_1 \leq c_2 \text{ and } c_1, c_2 \in [0,\frac{1}{\sqrt{2}} \beta) \text{ then } \sup \{ z_c \mid c \in [c_1,c_2] \} < \beta.
Proof of Theorem 3.1. The main idea of the proof is to utilize the verification technique provided by Proposition 4.1. That is, we need to search for a combination of a stopping time $\tau^*$ and a parameter $c$ such that $\tau^*$ is optimal for the embedded problem (12) and $c = \mathbb{E}_x \{ X_{\tau^*} \}$. If these two requirements are fulfilled, then we know from Proposition 4.1 that $\tau^*$ is also optimal for the variance problem. In the following, let $z_c$ be the greatest point that maximizes $z^2 - 2cz S(z)$. Graphically, we search for an intersection between the two curves $(z_c, c)$ and $(z, \mathbb{E}_x \{ X_{\tau(0,z)} \})$ (cf. Figure 1). If the two curves intersect in a point $(z^*, c^*)$, then $\tau^* = \tau(0,z^*)$ is optimal for the variance problem. If the two curves do not intersect, then we instead find a randomized solution.

![Figure 1. Illustration of the existence proof. For a fixed $x$ (here $x_1 < x_2$) the curves $(z_c, c)$ and $(z, \mathbb{E}_x \{ X_{\tau(0,z)} \})$ intersects. If the curve $(z_c, c)$ is discontinuous at the meeting point (here at $c^*$), the obtained optimal stopping rule is a randomized one.](image)

Define $c^* := \inf \{ c \in (c_L, \frac{1}{2} \beta) : z_c > x > \mathbb{E}_x \{ X_{\tau(0,z_c)} \} < c \}$. By Lemma 4.3(3) $c^*$ is well defined.

1. We first prove that there exists a hitting time $\tau_{(0,z^+)}$ which is optimal for $\sup_{\tau} \mathbb{E}_x [(X_\tau - c^*)^2]$ and satisfies $\mathbb{E}_x [X_{\tau_{(0,z^+)}}] \leq c^*$.

From the definition of $c^*$ it follows that there is a decreasing sequence $c_n \downarrow c^*$ such that for all $n$ we have $c_n < \frac{1}{2} \beta$ and $F(0, z_{c_n}, c_n) = 0$ and $c_n \geq \mathbb{E}_x [X_{\tau_{(0,z_{c_n})}}]$. As $c_n \in [c^*, c_1]$ and $c_1 < \frac{1}{2} \beta$ it follows from Lemma 4.3(5) that there is a $K \in (x, \beta)$ such that $\sup_n \{ z_{c_n} \} < K$.

Let $G(a, b)$ be defined from Lemma 4.3(4) and define

$$A(c) := \{ z \mid F(0, z, c) = 0 \text{ and } c \geq \mathbb{E}_x \{ X_{\tau(0,z)} \} \}.$$ 

By Lemma 4.3(1) and Lemma 4.3(2) $F$ and $G$ are continuous, and thus $A(c)$ is closed. Let $z_n = \inf A(c_n) \vee x$. Then $z_n \in A(c_n)$: If $\inf A(c_n) > x$, the claim follows from the fact that $A(c_n)$ is closed. If $x > \inf A(c_n)$, then there exists $\bar{z} < x$ for which $F(0, \bar{z}, c_n)$
and \( c_n \geq E_x \left\{ X_{\tau(0,x)} \right\} = x \), but now as effectively \( \tau(0,x) \) gives the same values as \( \tau(0,\bar{z}) \), we have \( F(0, \bar{z}, c_n) = F(0, 0, c_n) \) and \( E_x \left\{ X_{\tau(0,\bar{z})} \right\} = x \leq c_n \), meaning that \( x \in A(c_n) \).

It follows that \( z_n \in A(c_n) \).

Furthermore, \( z_n \in [x, K] \) for all \( n \in \mathbb{N} \) and thus \( (z_n) \) has a convergent subsequence \( (z_{n_k}) \). We denote its limit by \( \bar{z}^+ \). Now \( (0, z_{n_k}, c_{n_k}) \in F^{-1}\{\{0\}\} \), which is closed because of Lemma 4.3.4. Thus \( (0, z_{n_k}, c_{n_k}) \to (0, \bar{z}^+, c^*) \) for which, by continuity of \( F \) and \( G \), \( F(0, \bar{z}^+, c^*) = 0 \) and \( E_x \left\{ X_{\tau(0,\bar{z}^+)} \right\} \leq c^* \). That is, \( \tau(0,\bar{z}^+) \) is optimal for \( \sup_{x} E_x \{ (X_x - c^*)^2 \} \) and \( c^* \geq E_x \left\{ X_{\tau(0,\bar{z}^+)} \right\} \).

2. Next we show that there exists a hitting time \( \tau^- \) such that \( \tau^- \) is optimal for \( \sup_x E_x[(X_{\tau^-} - c^*)^2] \) and \( E_x[X_{\tau^-}] \geq c^* \). Denote by \( z_{c^*^-} := \lim \sup_{c^*} z_c \). It follows from Lemma 4.3.1 that \( z_{c^*^-} < \beta \). We split the proof in two cases.

(i) Assume \( z_{c^*^-} > x \). From the definition of \( z_{c^*^-} \) it follows that there is an increasing sequence \( c_n \uparrow c^* \) such that \( z_{c_n} \geq x \) and \( z_{c_n} \to z_{c^*^-} \) when \( n \to \infty \). From the definition of \( c^* \) it follows that for all \( n \) we have \( E_x \left\{ X_{\tau(0, z_{c_n})} \right\} \geq c_n \). From Lemma 4.3.1 we get that \( E_x \left\{ X_{\tau(0, z_{c_{c_n}})} \right\} \) converges to \( E_x \left\{ X_{\tau(0, z_{c^*^-})} \right\} \) when \( n \to \infty \), and to. By Lemma 4.3.1 we can deduce that \( F(0, z_{c_n}, c_n) \) converges to \( F(0, z_{c^*^-}, c^*) \). Since \( F(0, z_{c_n}, c_n) = 0 \) for all \( n \), also \( F(0, z_{c^*^-}, c^*) = 0 \). Thus, \( \tau_{z^-} = \tau(0, z_{c^*^-}) \) is optimal for \( \sup_x E_x \{ (X_x - c^*)^2 \} \) and \( E_x \left\{ X_{\tau(0, z_{c^*^-})} \right\} \geq c^* \).

(ii) Assume \( z_{c^*^-} \leq x \). Now, there must exist a sequence \( c_n \uparrow c^* \) such that for every \( n \) it holds that \( c_n < \frac{1}{2} z_{c_n} < x \) and that \( D_{c_n} \) defined by Lemma 4.2 the stopping time \( \tau_{D_{c_n}} \) is optimal for \( \sup_{x} E_x \left\{ (X_x - c_n)^2 \right\} \). Recall that \( \tau_{D_{c_n}} = \tau_{(a_n, b_n)} \) for some \( a_n \leq x \leq b_n \). As \( z_{c_n} \in D_{c_n} \) and \( z \leq x \), we have \( a_n \in \{ z_c, x \} \). Hence, there exists a convergent subsequence of \( a_n \), and thus, we may assume that the sequence \( a_n \) has been chosen in such a way that \( z_{c_n} \to z_{c^*^-} \) and \( a_n \) is convergent. Let \( a^* \) be the limit of the sequence \( a_n \). Note that for \( a < b \)

\[
E_x \left\{ (X_{\tau(a, b)} - c)^2 \right\} = P(\tau_a < \tau_b)(a - c)^2 + P(\tau_b < \tau_a)(b - c)^2
\]

\[
= \frac{S(b) - S(x)}{S(b) - S(a)} (a - c)^2 + \frac{S(x) - S(a)}{S(b) - S(a)} (b - c)^2
\]

\[
= \frac{S(b) - S(x)}{S(b) - S(a)} (a - c)^2 + \frac{S(x) - S(a)}{S(b) - S(a)} \frac{S(b)}{S(b) - S(a)} \left( \frac{b^2 \cdot 2c - b^2}{S(b)} \right).
\]

Since \( \frac{b^2}{S(b)} \to 0 \) as \( b \) goes to \( \beta \), the expectation above converges to \( (a - c)^2 \) when \( b \) goes to \( \beta \), and since this value is lower than \( (x - c)^2 \), we must have \( b_n < \beta \) for every \( n \). Now, we split the proof in further two cases depending on whether \( a^* = x \) or not.

Assume that \( a^* = x \). Then

\[
\sup_{x} E_x \left\{ (X_x - c_n)^2 \right\} = P(\tau_{a_n} < \tau_{b_n})(a_n - c_n)^2 + P(\tau_{b_n} < \tau_{a_n})(b_n - c_n)^2
\]
3. Combining all together, we have that both $\tau^E$ is optimal for the variance problem. If $\tau = 0$ is optimal for $\sup_x \mathbb{E}_x [(X_\tau - c)^2]$. Besides, since $c_n < x$ for every $n$, then $c^* < x$ and $c^* \leq \mathbb{E}_x [X_0]$, and thus we can choose $\tau^- = 0$.

Assume now that $a^* < x$. To show that $b_n$ is bounded from above, we fix $\delta > 0$. For every $\varepsilon > 0$ there is a bound $b_\varepsilon$ such that for every $c \in (c^* - \delta, c^*)$, $a \in [a^*, x)$ and $b > b_\varepsilon$ the value

$$
(S(x) - S(a)) \frac{S(b)}{S(b) - S(a)} \left(\frac{b^2}{S(b)} + \frac{c^2}{S(b)} - 2c \frac{1}{b} \frac{b^2}{S(b)}\right) < \varepsilon.
$$

Choose $\varepsilon_2 = (x - a^*)/2$. Then, from a certain step $a_n < x - \varepsilon_2$ and $c_n \in (c^* - \delta, c^*)$ on, we get for every $b > b_\varepsilon$ that

$$
\mathbb{E}_x [(X_{\tau(a_n,b)} - c_n)^2] \leq (a_n - c_n)^2 + \varepsilon \leq (x - \varepsilon_2 - c_n)^2 + \varepsilon
$$

$$
= (x - c_n)^2 + \varepsilon_2^2 - 2\varepsilon_2 (x - c_n) + \varepsilon
$$

$$
\leq (x - c_n)^2 + \varepsilon_2^2 - 2\varepsilon_2 (x - c^*) + \varepsilon.
$$

Here $\varepsilon_2^2 - 2\varepsilon_2 (x - c^*) < 0$, and thus, if we take $\varepsilon_1 < -\varepsilon_2^2 - 2\varepsilon_2 (x - c^*)$, then $\mathbb{E}_x [(X_{\tau(a_n,b)} - c_n)^2] < (x - c_n)^2$ for every $b > b_\varepsilon$. In other words, it is more advantageous to stop right away than to have $b_n > b_\varepsilon$. Consequently, for $c_n \in (c^* - \delta, c^*)$ we must have $b_n \leq b_\varepsilon$. Thereby, from a certain step onward, the sequence $b_n$ is bounded from above, and particularly $b_n$ has a convergent subsequence. We denote the limit of the subsequence $b^*$. It follows that there is a sequence $c_n \uparrow c^*$ such that for every $c_n$ there is a hitting time $\tau_{(a_n,b_n)}$ that is optimal for $\sup_x \mathbb{E}_x [(X_\tau - c_n)^2]$ and $a_n \rightarrow a^*$ and $b_n \rightarrow b^*$. Thus, $F(a_n, b_n, c_n) = 0$ and from Lemma 13.10 it follows that also $F(a^*, b^*, c^*) = 0$. Since $a^* \geq z_{c^* -}$, we have $G(a^*, b^*) \geq a^* \geq z_{c^* -}$, and since $z_{c_n} > c_n$ for all $n$, we also have $G(a^*, b^*) \geq c^*$. That is, $\tau^- = \tau_{(a^*, b^*)}$ is optimal for $\sup_x \mathbb{E}_x [(X_\tau - c^*)^2]$ and $c^* \leq \mathbb{E}_x [X_{\tau_{(a^*, b^*)}}]$, and so we can choose $\tau^- = \tau_{(a^*, b^*)}$.

3. Combining all together, we have that both $\tau_{(0,z^*+)}$ and $\tau^-$ solves $\mathbb{E}_x \{(X_\tau - c^*)^2\}$. Let

$$
p^* = \frac{\mathbb{E}_x \{X_{\tau^*} - c^*\}}{\mathbb{E}_x \{X_{\tau^-} \} - \mathbb{E}_x \{X_{\tau_{(0,z^*)}} \}},
$$

and let $\xi(p^*)$ be a Bernoulli random variable with $P(\xi(p^*) = 1) = 1 - P(\xi(p^*) = 0) = p^*$. If $\mathbb{E}_x \{X_{\tau_{(0,z^*)}} \} = c^*$, then it follows from Proposition 4.1 that $\tau_{(0,z^*)}$ is optimal for the variance problem. If $\mathbb{E}_x \{X_{\tau^-} \} = c^*$, then it follows from Proposition 1.1 that $\tau^-$ is optimal for the variance problem. If $\mathbb{E}_x \{X_{\tau_{(0,z^*)}} \} < c^*$, then define $\tau^* = \xi(p^*)\tau_{(0,z^*)} + (1 - \xi(p^*))\tau^-$. Then

$$
\mathbb{E}_x \{X_{\tau^*} \} = p^* \mathbb{E}_x \{X_{\tau_{(0,z^*)}} \} + (1 - p^*) \mathbb{E}_x \{X_{\tau^-} \} = c^*,
$$

and

$$
\mathbb{E}_x \{(X_{\tau^*} - c^*)^2\} = \mathbb{E}_x \{(X_{\tau_{(0,z^*)}} - c^*)^2\} p^* + \mathbb{E}_x \{(X_{\tau^-} - c^*)^2\} (1-p^*) = \mathbb{E}_x \{(X_{\tau^-} - c^*)^2\},
$$

and
and thus \( \tau^* \) is optimal \( \sup_x E_x \{ (X_{\tau} - c')^2 \} \). Therefore, by Proposition 1.1 \( \tau^* \) is optimal for the variance problem. \( \square \)

The differentiability of \( S \) has a role in the next proof when we for a large class of processes show a simpler way to reach the solution.

**Proof of Theorem 5.2**. Assume that \( S \) is differentiable, that \( \frac{S'(x)}{S(x)} \) is non-decreasing and that Case (I) of Assumption 2.5 hold.

From the existence proof, we found that genuine randomization can occur only if \( c \mapsto z \) has a discontinuity (cf. Figure 1). It follows that, if for all \( c \in I \) there exists only one \( z_c \) for which \( \tau_{(0,z_c)} \) maximizes the embedded quadratic problem, then naturally the randomization is no longer needed and \( \tau_{(0,z^+)} \) is an optimal stopping time to the variance stopping problem.

To show this uniqueness, let \( c \in I \).

From Lemma 4.2 we know that for \( x < z_c \) the optimal stopping time to the embedded quadratic problem \( (12) \) is \( \tau_{(0,z_c)} \), where \( z_c \) is a point that maximizes \( \frac{z^2 - 2cz}{S(z)} \). It is clear from Lemma 4.2 that there is at least one such maximizer. We show that under the assumed monotonicity condition, it is the only one.

By straight derivation, the first order optimality condition can be written as

\[
\frac{d}{dz} \frac{z^2 - 2cz}{S(z)} = 0 \iff \frac{z - c}{\frac{1}{2}z - c} = \frac{S'(z)}{S(z)} z.
\]

As was noticed in the proof of Lemma 4.2, the maximum point of the ratio \( \frac{z^2 - 2cz}{S(z)} \) is on \((2c, \beta)\). Moreover, it can be easily checked that since \( c > 0 \), the ratio \( \frac{z - c}{\frac{1}{2}z - c} \) is \( z \)-decreasing and positive for \( z > 2c \). As we assumed the positive mapping \( \frac{S'(z)}{S(z)} z \) to be non-decreasing, we see that for any \( c \in I \) there is at most one \( z_c > 2c \) satisfying the first order optimality condition \( (13) \). Consequently, the optimal stopping time \( \tau_{(0,z_c)} \) is unique.

All in all, we can conclude that \( \tau_{(0,z^*)} \) is an optimal stopping time to the variance stopping problem \( (11) \) for some \( z^* \).

To find this \( z^* \), we notice that as \( \mathbb{P}_x (\tau_x < \infty) = \frac{S(x)}{S(z)} \), we can calculate for \( z > x \):

\[
v(x,z) := \text{Var}_x \{ X_{\tau_{(0,z)}} \} = z^2 \frac{S(x)}{S(z)} \left( 1 - \frac{S(x)}{S(z)} \right).
\]

As \( \lim_{z \to \beta} v(x,z) = 0 \) (by Case (I) of Assumption 2.5), there exists at least one \( z^* < \beta \) that maximizes \( v(x,z) \) and it must satisfy the optimality condition \( \frac{\partial v(x,z)}{\partial z} = 0 \). (Notice that trivially \( \text{Var}_x \{ X_{\tau_{(0,z)}} \} = 0 \) for \( z < x \).)

By straight derivation we get the first order necessary optimality condition

\[
\frac{\partial v(x,z)}{\partial z} = 0 \iff \frac{S(z) - S(x)}{\frac{1}{2}S(z) - S(x)} = \frac{S'(z)}{S(z)} z.
\]

It is easily seen that \( \frac{S(z) - S(x)}{\frac{1}{2}S(z) - S(x)} \) is negative for all \( z < S^{-1}(2S(x)) \) and \( z \)-decreasing and positive for all \( z > S^{-1}(2S(x)) \). As \( \frac{S'(z)}{S(z)} z \) is positive and assumed to be non-decreasing, we see that there is at most one solution to \( (14) \). Consequently, the unique solution on
\[(S^{-1}(2S(x)) \land \beta) \text{ to } \text{(14)}, \text{ is the optimal stopping boundary } z^*, \text{ the stopping time } \tau_{(0,z^*)} \text{ is an optimal stopping time to the problem } \text{(1)} \text{ and the value reads as } v(x,z^*). \]

5. Proofs of Cases (II) and (III)

5.1. Proof of Theorem 3.2 — Case (II). The proof of this theorem can be returned to Case (I) in the following way. Let \( S(x) \) be the scale function and \( m(x) \) the speed measure associated to \( X_t \). If \( \beta \neq 0 \), we can define an auxiliary process \( Y_t \) on a state space \((\alpha - \beta, 0)\) by defining the scale function \( \hat{S} \), speed measure \( \hat{m} \), and starting point \( Y_0 \) through

\[
\hat{S}(y) := S(y + \beta), \quad \hat{m}(y) := m(y + \beta), \quad Y_0 = y := x - \beta. \tag{15}
\]

Then \( Y \) is well defined diffusion on \((\alpha - \beta, 0)\) inheriting its boundary behaviour from \( X_t \).

After this we can define another auxiliary process \( Z_t \) on a state space \((0, \beta - \alpha)\) by defining the scale function \( \check{S} \), speed measure \( \check{m} \), and starting point \( Z_0 \) through

\[
\check{S}(z) := -\hat{S}(-z) + \hat{S}(0), \quad \check{m}(z) := m(-z), \quad Z_0 = z := -y. \tag{16}
\]

Then \( Z \) is well defined diffusion on \((0, \beta - \alpha)\), with lower end point inheriting its behaviour from the upper end point of \( Y \) and vice versa, and its scale function vanishing at the lower boundary: \( \check{S}(0) = 0 \). It follows that the optimal variance stopping problem

\[
\hat{V}(z) = \sup_{\tau} \text{Var}_z \{ Z_{\tau} \}, \quad Z_0 = z
\]

can be solved utilizing Theorem 3.1. Consequently, the desired result concerning the diffusion \( X_t \) on \((\alpha, \beta)\) can be retrieved from this by inverting the transformations in \text{(15)} and \text{(16)}.

5.2. Proof of Theorem 3.3 — Case (III). We need to consider only the case \( \alpha = 0 \). If \( \alpha \neq 0 \), we can make the same transformation we did in Case (I) to retrieve the case \( \alpha = 0 \).

The main difference to the other cases is the fact that the type of the optimal solution depends on the location of the starting point \( x \). This phenomenon arises when solving the embedded quadratic problem \text{(12)}: The type of the solution is different depending on whether \( c \leq \frac{1}{2} \beta \) or not, as we prove in the following lemma.

**Lemma 5.1.** Let \( z_c \) be the greatest point on \([0, \beta]\) that maximizes \( \frac{x^2 - 2zx}{S(z)} \), and \( y_c \) the smallest point on \([0, \beta]\) that maximizes \( \frac{y^2 - 2cy - \beta^2 + 2c\beta}{S(\beta) - S(y)} \).

(A) Assume that \( c \leq \frac{1}{2} \beta \). Then \( y_c = \beta \) and \( z_c \in (2c, \beta) \) and, for all \( x < z_c \), the optimal stopping time to the embedded quadratic problem \text{(12)} is \( \tau_{(0,z_c)} \) and the value reads as

\[
V^c(x) = \frac{z_c^2 - 2cz_c}{S(z_c)} S(x) + c^2.
\]

(B) Assume that \( c > \frac{1}{2} \beta \). Then \( y_c \in [0, 2c - \beta) \) and \( z_c = 0 \) and, for all \( x > y_c \), the optimal stopping time to the embedded quadratic problem \text{(12)} is \( \tau_{(y_c, \beta)} \) and the value

reads as
\[ V^c(x) = \frac{y^2 - 2cy_c - \beta^2 + 2c\beta}{S(\beta) - S(y_c)}S(x) + (\beta - c)^2. \]

**Proof.** (A) It is easily seen that when \( c \leq \frac{1}{2}\beta \), then \( \frac{y^2 - 2cy - \beta^2 + 2c\beta}{S(\beta) - S(y)} \) is positive on \((2c, \beta]\) and negative elsewhere in the state space. On the other hand \( \frac{y^2 - 2cy - \beta^2 + 2c\beta}{S(\beta) - S(y)} \) is negative on \([0, \beta)\). Therefore \( y_c = \beta \) (the value with a stopping rule \( \tau_{(y_c, \beta)} \) is maximized when \( y = \beta \) and \( z_c \in (2c, \beta] \). The proof for the solution for all \( x < z_c \) is analogous to Lemma 4.12.

(B) It is easily seen that when \( c > \frac{1}{2}\beta \), then \( \frac{y^2 - 2cy - \beta^2 + 2c\beta}{S(\beta) - S(y)} \) is negative on \((0, \beta)\). On the other hand \( \frac{y^2 - 2cy - \beta^2 + 2c\beta}{S(\beta) - S(y)} \) is positive on \([0, 2c - \beta)\) and negative elsewhere in the state space. Therefore \( y_c \in [0, 2c - \beta) \) and \( z_c = 0 \).

Let \( x > y_c \). Then for all stopping times \( \tau \) we have
\[
\mathbb{E}_x \{ (X_\tau - c)^2 \} = \mathbb{E}_x \left\{ \frac{X^2_\tau - 2cX_\tau - \beta^2 + 2c\beta}{S(\beta) - S(X_\tau)}(S(\beta) - S(X_\tau)) \right\} + (\beta - c)^2 \leq \frac{y^2 - 2cy - \beta^2 + 2c\beta}{S(\beta) - S(y)} \mathbb{E}_x \{ S(\beta) - S(X_\tau) \} + (\beta - c)^2 = \frac{y^2 - 2cy - \beta^2 + 2c\beta}{S(\beta) - S(y)}(S(\beta) - S(x)) + (\beta - c)^2,
\]
where the first inequality follows by the maximality of \( y_c \) and the second one follows from the fact that \( S(X_t) \) is a bounded local martingale on \( \mathcal{I} \) and hence martingale.

As this value is attained with \( \tau_{(y_c, \beta)} \) we know that it must be the optimal stopping time. \( \square \)

The rest of the proof is analogous to Case (I) and Case (II). One only needs to do separately the cases \( \mathbb{E}_x \{ X_{\tau_{(0, \beta)}} \} \leq \frac{1}{2}\beta \) and \( \mathbb{E}_x \{ X_{\tau_{(0, \beta)}} \} > \frac{1}{2}\beta \).

6. Second proof — A game theoretic approach

Here we illustrate the close relation between the variance problem and a classical game theoretic problem by using game theoretic tools to provide the proof the main theorem, Theorem 6.1(A), under Assumption 2.6. In this way we complete the proof of Theorem 3.1.

6.1. Preliminaries — Short introduction to game theory. The variance problem can be written in the form
\[
\sup_{\tau} \text{Var}_x \{ X_\tau \} = \sup_{\tau} \mathbb{E}_x \{ (X_\tau - \mathbb{E}_x \{ X_\tau \})^2 \} = \inf_{c} \sup_{\tau} \mathbb{E}_x \{ (X_\tau - c)^2 \} =: \sup_{\tau} \inf_{c} A(\tau, c).
\]
Here, \( A(\tau, c) \) is convex with respect to \( c \), and consequently we can interpret the problem as an infinite, convex two-player zero-sum game.

**Definition 6.1.** (A) We say that a game has a **value** \( V \), if
\[
\sup_{\tau} \inf_{c} A(\tau, c) = \inf_{c} \sup_{\tau} A(\tau, c).
\]
If the value exists, we can write \( V = \inf_c \sup_{\tau} A(\tau, c) = \sup_{\tau} \inf_c A(\tau, c) \).

(B) A \textbf{pure strategy} is a strategy that uses a single stopping time \( \tau \) or a level \( c \in I \).

(C) A \textbf{mixed strategy} is a strategy that mixes pure strategies using some known probability distribution.

We only need mixed strategies of the form \( \hat{\tau} := \xi(p)\tau_1 + (1 - \xi(p))\tau_2 \), where \( P(\xi(p) = 1) = 1 - P(\xi(p) = 0) = p \).

We have the following known result concerning the value and optimal strategies regarding infinite convex games on compact regions (Theorem 4.3.1 in [8], see also Sections 2.11-2.13 in [18]).

**Proposition 6.2.** Assume that a payoff function \( B(x, y) \) of a game is continuous on both variables, that \( B''_{yy}(x, y) > 0 \) and that the strategies \( x \) and \( y \) take values from compact sets \( X \) and \( Y \).

Then

\[
\sup_{x \in X} \inf_{y \in Y} B(x, y) = \inf_{y \in Y} \sup_{x \in X} B(x, y).
\]

Furthermore, the sup-player has an optimal mixed strategy involving at most 2 pure strategies and the optimal strategy of the inf-player is a pure strategy.

6.2. \textbf{Solving the game.} We assume that Case (I) in Assumption 2.5 holds and that \( \alpha = 0 \). Let \( x \in \mathbb{R}_+ \) be fixed. In addition, to complete the proof of Theorem 3.1 and also to ease the subsequent deductions to more clearly illustrate game theoretic approach, we assume that Assumption 2.6 holds.

In order to use the previous Proposition 6.2 in our setting, we have to restrict the admissible strategy sets of the players. This is done in the following. Recall that \( T \) is the set of all \( \mathcal{F} \)-stopping times and \( z_c \) is the greatest point that maximizes \( z_c^2 - 2z_c S(z) \).

**Lemma 6.3.** (A) Let \( \hat{c}_x = \inf\{c > 0 \mid z_c > x\} \). There exists \( M_x < \infty \) such that

\[
\sup_{\tau \in T} \inf_{c \in \mathbb{R}_+} A(\tau, c) = \sup_{\tau \in T} \inf_{c \in [\hat{c}_x, M_x]} A(\tau, c).
\]

(B) There exists \( N_x \) such that

\[
\inf_{c \in \mathbb{R}_+} \sup_{\tau \in T} A(\tau, c) \leq \inf_{c \in [\hat{c}_x, M_x]} \sup_{x \in [x, N_x]} A(\tau(0, x), c).
\]

**Proof.** (A) If the inf-player chooses \( c = 2x \), then the sup-player faces an optimal stopping problem

\[
\sup_{\tau} \mathbb{E}_x \{(X_\tau - 2x)^2\},
\]

which clearly has a value by Lemma 4.2. Let us denote this value by \( \hat{M}_x \).

On the other hand, if the inf-player chooses \( c > 2x + \sqrt{M_x} =: M_x \), we see at once that by stopping immediately, the sup-player receives \( (x - c)^2 > (x + \sqrt{M_x})^2 > \hat{M}_x \). I.e. choosing too large \( c \) is a non-rational act for the inf-player. It follows that for each \( x \), we can narrow down the possible strategies for the inf-player to be in a compact set \([0, M_x]\).
Let us next show that we must have \( c > \hat{c}_x \). There are two possibilities: Either \( z_{\hat{c}_x} = x \), or \( z_{\hat{c}_x} > x \), where \( z_{\hat{c}_x} \) is the greatest \( z_c \) that maximizes the ratio \( \frac{z^2 - 2cz}{S(z)} \). These two cases are considered separably:

1. Let \( z_{\hat{c}_x} = x \). Then
\[
\sup_{\tau} A(\tau, \hat{c}) = (x - \hat{c}) < (x - c)^2 \quad \text{for all } c < \hat{c}.
\]
Consequently, it is non-rational for the inf-player to choose \( c < \hat{c} \).

2. Assume that \( z_{\hat{c}_x} > x \), and let \( z_{\hat{c}_x} \) be the smallest \( z_c \) that maximizes \( \frac{z^2 - 2cz}{S(z)} \). We know that \( z_{\hat{c}_x} < x \). By Assumption 2.6[I], we have
\[
\mathbb{E}_x \left\{ X_{\tau(0, z_{\hat{c}_x}^+)} \right\} > \mathbb{E}_{z_{\hat{c}_x}^-} \left\{ X_{\tau(0, z_{\hat{c}_x}^+)} \right\} > \hat{c}_x.
\]
Based on this, we get
\[
\frac{d}{dc} A(\tau(0, z_{\hat{c}_x}^+), c) = 2 \left( c - \mathbb{E}_x \left\{ X_{\tau(0, z_{\hat{c}_x}^+)} \right\} \right) < 0, \quad \text{for all } c < \hat{c}_x.
\]
Consequently, we can argue the following: Let \( c < \hat{c} \). Then
\[
\sup_{\tau} A(\tau, c) \geq A(\tau(0, z_{\hat{c}_x}^+), c) > A(\tau(0, z_{\hat{c}_x}^+), \hat{c}_x) = \sup_{\tau} A(\tau, \hat{c}_x),
\]
where the last equality follows from the facts that \( x \leq z_{\hat{c}_x} \) and that \( z_{\hat{c}_x} \) maximizes \( \frac{z^2 - 2cz}{S(z)} \) (cf. Lemma 1.2). Consequently, it is non-rational for the inf-player to choose \( c < \hat{c}_x \).

(B) Let \( M_x \) be as it was in item (A) above. Choose \( N_x := \sup \{ z_c \mid c \in [\hat{c}_x, M_x] \} \). We can now deduce the following
\[
\inf_{c \in \mathbb{R}_+} \sup_{\tau \in \mathcal{T}} A(\tau, c) \leq \inf_{c \in [\hat{c}_x, M_x]} \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left\{ \frac{X^2 - 2cX}{S(X)} S(X_{\tau}) \right\} + c^2 \leq \inf_{c \in [\hat{c}_x, M_x]} \sup_{\tau \in \mathcal{T}} \frac{z^2 - 2cz}{S(z)} S(x) + c^2
\]
\[
= \inf_{c \in [\hat{c}_x, M_x]} \sup_{z \in [x, N_x]} \frac{z^2 - 2cz}{S(z)} S(x) + c^2 = \inf_{c \in [\hat{c}_x, M_x]} \sup_{z \in [x, N_x]} A(\tau(0, z), c).
\]
Here the second inequality follows from the fact that \( z_c \) is the maximizer of \( \frac{z^2 - 2cz}{S(z)} \), and the last equality from the fact that for all \( c \in [\hat{c}_x, M_x] \) the maximizer \( z_c \in [x, N_x] \) and hence we can apply Lemma 1.2 in the last equality.

We saw in the proof that Assumption 2.6[I] basically guarantee that for an inf-player it would not be beneficial to play a \( c < \hat{c}_x \).

Now we can state our main result in the game theoretic framework.

**Theorem 6.4.** Let \( x > 0 \) be fixed and assume that Assumptions 2.7[I] and 2.6[I] hold. Then there exist \( c^* \) and \( \tau^* \) such that the value is attained and
\[
V(x) = A(\tau^*, c^*) = \sup_{z \in [0, N_x]} \inf_{c \in [\hat{c}_x, M_x]} A(\tau(0, z), c) = \inf_{c \in [\hat{c}_x, M_x]} \sup_{z \in [0, N_x]} A(\tau(0, z), c).
\]
Moreover, \( c^* \) is a pure strategy and \( \tau^* \) potentially a mixed strategy, i.e. for some \( z_1 \leq z_2 \) we can write \( \tau^* = \xi(p)\tau(0, z_1) + (1 - \xi(p))\tau(0, z_2) \), where \( \xi(p) \) is a Bernoulli random variable with a parameter \( p \in [0, 1] \).
Proof. It is clear that using the compact strategy sets \([x, N_x]\) and \([\hat{c}_x, M_x]\) with payoff function \(A(\tau_{(0,z)}, c)\), Proposition 6.2 is readily usable. What is left to do, is to show that we can restrict to study these compact strategy sets. Applying Lemma 6.3 we can make the following deduction

\[
\inf_{c \in \mathbb{R}^+} \sup_{\tau \in T} A(\tau, c) \leq \inf_{c \in [\hat{c}_x, M_x]} \sup_{z \in [x, N_x]} A(\tau_{(0,z)}, c) = \sup_{\tau \in T} \inf_{c \in [\hat{c}_x, M_x]} A(\tau_{(0,z)}, c) \\
\leq \sup_{\tau \in T} \inf_{c \in \mathbb{R}^+} A(\tau, c) = \inf_{c \in \mathbb{R}^+} \sup_{\tau \in T} A(\tau, c) \\
\leq \inf_{c \in \mathbb{R}^+} \sup_{\tau \in T} A(\tau, c),
\]

and hence there must be equality between all entities, and the claim follows from Proposition 6.2.

The previous theorem illustrates the fact that in the variance stopping problem one can utilize known game theoretic results. This may be valuable information when working with more complex non-linear optimal stopping problem.

6.3. Finding the solution. The game theoretic proof for the existence of a value (Proposition 6.2) also offers a way to identify the solution, and this is presented in this subsection (cf. Sections 2.11-2.12 in [18] and [8]).

Before writing the procedure, we need to define the following concept of essential strategies. Let \(c^*\) be an optimal strategy for the inf-player. We call a pure strategy \(\tau_{z_i}\) essential, if \(V(x) = A(\tau_{z_i}, c^*)\). Notice that if \(\tau_{z_i}\) is essential, it is not necessarily optimal as we may have \(\inf_c A(\tau_{z_i}, c) < A(\tau_{z_i}, c^*)\).

Lemma 6.5. Let \(x \in \mathcal{I} = (0, \beta)\) and assume that Assumptions 2.5 Case (I) and 2.6 are satisfied. Then utilizing the following procedure one finds the optimal strategies:

1. Optimal strategy for the inf-player is \(c^*\) and it is the threshold that solves

\[
\inf_c \frac{z_c^2 - 2cz_c}{S(z_c)}S(x) + c^2,
\]

where \(z_c\) is as in Lemmas 4.2 and 6.2. Furthermore, \(V(x) = \inf_c \frac{z_c^2 - 2cz_c}{S(z_c)}S(x) + c^2\).

2. Let \((\tau_{z_i})_{i \in \mathbb{I}}\) be a set of essential strategies for the sup-player.

3. If \(A_c'(\tau_{z_i}, c^*) = 0\) for some essential strategy \(\tau_{z_i}\), \(i \in \mathbb{I}\), then \(\tau^* = \tau_{z_i}\) is the optimal strategy for the sup-player and it is a pure strategy.

4. If such a strategy does not exist, then there exists \(\tau_{z_1}\) and \(\tau_{z_2}\) for which

\[
A_c'(\tau_{z_1}, c^*) < 0 \\
A_c'(\tau_{z_2}, c^*) > 0,
\]

and the optimal strategy for the sup-player is a mixed strategy \(\tau^* = \xi(p^*)\tau_{z_1} + (1 - \xi(p^*))\tau_{z_2}\) with \(\xi(p^*)\) being a Bernoulli random variable with a parameter \(p^*\). Here \(p^*\) can be solved from

\[
pA_c'(\tau_{z_1}, c^*) + (1 - p)A_c'(\tau_{z_2}, c^*) = 0.
\]
Proof. The first step follows from Theorem 2.13.2 and Corollaries 2.11.3, 2.11.4 from [18], while the other steps follow from Sections 2.11-2.12 in [18]. □

The preceding lemma points out the possibilities that the game theory can offer when solving non-linear optimal stopping problems: proofs are very short as one can lean on known results. Also, worth noticing is that none of the proofs above use differentiability of $S$ anywhere. The presented algorithm above is fine, but unfortunately it does not yet tell us the whole picture as one would need to apply the lemma infinite times, one for each starting point $x \in I$, in order to reach the solution. The final algorithm to solve the variance stopping problem for all $x \in I$ is presented in Subsection S.1 below.

7. Special cases

For the sake of completeness let us study here briefly the special cases which are not yet covered.

7.1. Recurrent case. In the recurrent case we have $-S(\alpha) = S(\beta) = \infty$.

Lemma 7.1. Let $X_t$ be recurrent.

(A) Assume that $\alpha = -\infty$ or $\beta = \infty$. Then $V(x) \equiv \infty$.

(B) Assume that $-\infty < \alpha < \beta < \infty$. Then the optimal stopping time is $\tau_{(\alpha,\beta)}$ and the value reads $V(x) = \frac{1}{4}(\beta - \alpha)^2$.

Proof. (A) Let $\beta = \infty$. As $X_t$ is recurrent, we have $P_x(\tau_b < \infty) = 1$ for all $b$, and hence the claim follows straightly from Proposition 2.3. The case $\alpha = -\infty$ is analogous.

(B) First note that an arbitrary random variable $Y$ on an interval $[\alpha, \beta]$ has the highest possible variance if $P(Y = \alpha) = P(Y = \beta) = \frac{1}{2}$. In this case $\text{Var}(Y) = \frac{1}{4}(\alpha - \beta)^2$. Since $X_t$ takes values on $(\alpha, \beta)$ we must have $V(x) \leq \frac{1}{4}(\alpha - \beta)^2$. Let us next show that also the reversed inequality holds.

As $X$ is recurrent we can choose sequences $a_n$ and $b_n$ in such a way that $a_n \to \alpha$ and $b_n \to \beta$ as $n \to \infty$, and that $P_x(X_{\tau(a_n,b_n)} = a_n) = P_x(X_{\tau(a_n,b_n)} = b_n) = \frac{1}{2}$ for all $n \in \mathbb{N}$. To show that these sequences exist, let $x$ be, for simplicity, such that $S(x) = 0$. Choose $a_n < x$ to be any decreasing sequence for which $\lim_{n \to \infty} a_n = \alpha$ and choose $b_n$ to satisfy $S(b_n) = -S(a_n)$. Then the sequences $a_n$ and $b_n$ satisfy required properties since

$$P_x(\tau_{b_n} < \tau_{a_n}) = \frac{-S(a_n)}{S(b_n) - S(a_n)} = \frac{1}{\frac{S(b_n)}{S(a_n)} + 1} = \frac{1}{2} \text{ for all } n \in \mathbb{N}.$$ 

Thus $\text{Var}_x(X_{\tau(a_n,b_n)}) = \frac{1}{4}(a_n - b_n)^2 \to \frac{1}{4}(\alpha - \beta)^2$ as $n \to \infty$. Since we have $V(x) \geq \text{Var}_x(X_{\tau(a_n,b_n)})$ for all $n \in \mathbb{N}$, we must also have $V(x) \geq \frac{1}{4}(\alpha - \beta)^2$ proving the claim. □

The result states intuitively clear fact of how, in recurrent case, we should use the whole span of the state space. Notice that the optimal stopping time $\tau_{(\alpha,\beta)}$ is infinite almost surely. However, as was presented in the previous proof, for every $\varepsilon > 0$ there exists $a$ and $b$ such that $\tau_{(a,b)}$ is an almost surely finite $\varepsilon$-stopping time.
7.2. Transient case with \( \lim_{b \to \infty} \mathbb{P}_x (\tau_b < \infty) b^2 \in (0, \infty) \). Let us consider briefly Case (I) of Assumption 2.5 with the condition that \( \beta = \infty \) and \( \lim_{b \to \infty} \mathbb{P}_x (\tau_b < \infty) b^2 = \lim_{b \to \infty} \frac{b^2}{S(b)} \in (0, \infty) \).

It can be shown, mimicking the proof of Lemma 4.2, that also in this special case the optimal stopping time to an embedded quadratic problem

\[
V^c(x) = \sup_\tau \mathbb{E}_x \{(X_\tau - c)^2\}
\]

is \( \tau_{(\alpha,z)} \) for all \( c \in \mathcal{I} \) and \( x < z_c \). However, the main difference is that now we may have \( z_c = \infty \), and \( \tau_{(\alpha,\infty)} \) is unattainable in finite time almost surely. Nevertheless, we can write \( z_c = \arg\max \left\{ \frac{z^2 - 2xz}{S(z)} \right\} \), and the value \( V^c(x) = CS(x) + c^2 < \infty \), where \( C = \sup_z \left\{ \frac{z^2 - 2xz}{S(z)} \right\} \). Unfortunately, general existence proofs of Theorem 3.1 cannot be utilized straightforwardly as they require \( \varepsilon \)-optimality strategies do exist, we can modify our Theorem 6.4 to work also in this case. Moreover, the proof when \( \frac{S'(z)}{S(z)} \) is non-decreasing, Theorem 3.1 can be quite straightforwardly modified to work also in this case. Summarizing, the following result holds.

**Lemma 7.2.** Let \( X_t \) be a regular diffusion on \( \mathcal{I} = (\alpha, \infty) \), where the boundaries can be natural, exit, entrance, killing or absorbing. Let Assumption 2.5(I) hold. We fix \( x \in \mathcal{I} \) and assume that \( \lim_{b \to \infty} \mathbb{P}_x (\tau_b < \infty) b^2 = \lim_{b \to \infty} \frac{b^2}{S(b)} \in (0, \infty) \).

(A) If Assumption 2.6(I) holds, then the value exists and there exists \( c^* > 0 \) such that \( V(x) = \sup_\tau \mathbb{E}_x \{(X_\tau - c^*)^2\} \). Moreover, there exists an \( \varepsilon \)-optimal stopping time of the form \( \tau_{(0,z)} \), or possible randomization between two such times.

(B) If \( S \) is differentiable and \( \frac{S'(z)}{S(z)}(z - \alpha) \) is non-decreasing, then the value exists and the optimal stopping time to the optimal variance stopping problem \( \Pi \) is \( \tau_{(\alpha,z^*)} \), where \( z^* \) is either the unique solution to

\[
\frac{S(z) - S(x)}{S(z) - S(x)} = \frac{S'(z)}{S(z)}(z - \alpha),
\]

or, if the root does not exist, \( z^* = \infty \). Furthermore, the value reads as \( V(x) = \left( z^* - \alpha \right)^2 \frac{S(x)}{S(z^*)} \left( 1 - \frac{S(z)}{S(z^*)} \right) \).

In item (B) whenever \( z^* = \infty \), the value \( V(x) \) is understood as a limit \( V(x) = \lim_{z \to \infty} \left( z^* - \alpha \right)^2 \frac{S(x)}{S(z^*)} \left( 1 - \frac{S(z)}{S(z^*)} \right) \).

The only case not yet covered is the transient case \( \lim_{b \to \infty} \mathbb{P}_x (\tau_b < \infty) b^2 \in (0, \infty) \) where Assumption 2.6 does not hold. It seems that in that case the obstacles in the proof of Theorem 6.4 could be also avoided by applying \( \varepsilon \)-optimal stopping times and taking into account that \( z_c \) might be \( \infty \). If done rigorously, this is quite a laborious and lengthy case. Hence we shall omit the inspection of this very marginal case.

8. Examples

8.1. Solution algorithm. Before proceeding to our examples, let us introduce a solution algorithm how to find the solution for all \( x \in \mathcal{I} \). The algorithm is written for Case (I):
We assume that $\alpha = 0$, $\beta > 0$, where $\alpha$ is attractive while $\beta$ is not and $\frac{\beta^2}{\gamma^2} \to 0$ as $b \to \beta$. In this case, if $S$ is differentiable and $\frac{S'(x)}{S(x)} x$ is non-decreasing, the solution is easy to find applying Theorem 3.1(B). So, we assume now that the above mentioned mapping is not non-decreasing.

Under these assumptions the solution is potentially a randomized solution, and there is no explicit way to tell what is the optimal stopping time. However, we can construct an algorithm based on the fact that the solution exists and is either a stopping time $\tau_{(0,z)}$ for some $z$ or a randomization between stopping times $\tau_{(0,z)}$ and $\tau_{(a,b)}$, where $0 \leq a \leq x \leq b \leq z$. In the algorithm we separate these two cases.

Step 1. (i) For each $c \in I$, solve the embedded quadratic problem

$$V^c(x) = \sup_z E_x \left\{ (X_{\tau_{(0,x)}}, - c)^2 \right\}.$$  

A threshold $z_c$ that maximizes the ratio $\frac{z^2}{S(x)}$ is the maximizer for this embedded quadratic problem (cf. Lemma 3.12). Let $C = \{ c \in I \mid \exists z_c \text{ maximizing } (18) \}$ be the set of all $c$, for which there exists more than one maximizer for (18). Typically, the set $C$ is finite.

(ii) Take $c \in C$. Let $Z_c = \left\{ z \mid z = \arg \max \left\{ \frac{z^2}{S(x)} \right\} \right\}$ be the set of maximizers of (18) for $c$, and denote by $\underline{c}_c := \inf\{ Z_c \}$ and $\overline{c}_c := \sup\{ Z_c \}$ the smallest and greatest of such points. Check whether assumption 2.6[I] holds or not, i.e. is the condition $E_x \left\{ X_{\tau_{(0,c)}} \right\} > c$ met.

Step 2a. Assumption 2.6[I] holds.

(i) For the chosen $c \in C$, define a randomized stopping time $\hat{\tau}_c(p) := \xi(p)\tau_{(0,\underline{c}_c)} + (1 - \xi(p))\tau_{(a,\overline{c}_c)}$ with $\xi(p)$ being a Bernoulli random variable with a parameter $p$, and define $\underline{c}_c$ and $\overline{c}_c$ to be the smaller and greater, respectively, solutions to the equations

$$E_x \left\{ X_{\hat{\tau}_c(1)} \right\} = c \iff \underline{c}_c = S^{-1}\left( \frac{c}{\underline{c}_c} S(\underline{c}_c) \right),$$

$$E_x \left\{ X_{\hat{\tau}_c(0)} \right\} = c \iff \overline{c}_c = S^{-1}\left( \frac{c}{\overline{c}_c} S(\overline{c}_c) \right).$$

Then for all $x \in (\underline{c}_c, \overline{c}_c)$ there exists a unique $p^*(x) \in (0,1)$ satisfying the condition $E_x \left\{ X_{\tau_{(p^*(x))}} \right\} = c$.

(ii) Repeat the step (i) for all $c \in C$ which satisfies Assumption 2.6[I].

(iii) Define $J_A := \bigcup_{c \in C} (\underline{c}_c, \overline{c}_c)$ to be the set of points $x$ for which Assumption 2.6[I] is satisfied.

Step 2b. Assumption 2.6[I] does not hold.

(i) For the chosen $c \in C$, $\overline{c}_c$ and $\underline{c}_c$ are again the solutions to

$$E_x \left\{ X_{\tau_{(0,\underline{c}_c)}} \right\} = c \iff \underline{c}_c = S^{-1}\left( \frac{c}{\underline{c}_c} S(\underline{c}_c) \right),$$

$$E_x \left\{ X_{\tau_{(0,\overline{c}_c)}} \right\} = c \iff \overline{c}_c = S^{-1}\left( \frac{c}{\overline{c}_c} S(\overline{c}_c) \right).$$
(ii) For $x \in (\underline{x}_c, \overline{x}_c]$, we can use a randomized stopping time $\hat{\tau}_c(p) = \xi(p)\tau(0, \underline{x}_c) + (1 - \xi(p))\tau(0, \overline{x}_c)$. However, for $x \in (\underline{x}_c, \overline{x}_c)$ we need to define a randomized stopping time $\hat{\tau}_c(p) := \xi(p)\tau_{D_c} + (1 - \xi(p))\tau_{(0, \overline{x}_c)}$. Here $\tau_{D_c}$ is an optimal stopping time, where $D_c$ is the stopping set for an embedded problem with a parameter $c$.

(iii) Repeat the steps (i) – (ii) for all $c \in C$, which does not satisfy Assumption 2.6(I).

(iv) Define $\mathcal{J}_A := \bigcup_{c \in C}(\underline{x}_c, \overline{x}_c)$ to be the set of points $x$ for which Assumption 2.6(I) is not satisfied.

Step 3. The following is an optimal stopping time:

\[
\begin{align*}
\tau_{(0, x^*(x))}, & \quad x \in \mathcal{I} \setminus (\mathcal{J}_A \cup \mathcal{J}_A) \\
\hat{\tau}_c(x) = \xi(p^*_x)\tau(0, \underline{x}_c) + (1 - \xi(p^*_x))\tau(0, \overline{x}_c), & \quad x \in \mathcal{J}_A \\
\hat{\tau}_c(x) = \xi(p^*_x)\tau_{D_c} + (1 - \xi(p^*_x))\tau(0, \overline{x}_c), & \quad x \in \mathcal{J}_A & \mathcal{J}_A \\
\hat{\tau}_c(x) = \xi(p^*_x)\tau_{D_c} + (1 - \xi(p^*_x))\tau(0, \overline{x}_c), & \quad x \in \mathcal{J}_A & \mathcal{J}_A.
\end{align*}
\]

First, when $x \in \mathcal{I} \setminus (\mathcal{J}_A \cup \mathcal{J}_A)$, then $z^*(x)$ is a maximizer of $(z - \alpha)^2 \frac{S(z)}{S(x)} \left(1 - \frac{S(z)}{S(x)}\right)$ and hence, if $S$ is differentiable, a solution (not necessarily unique, as there may be local extreme points!) to the first order optimality condition

\[
\frac{S(z) - S(x)}{2S(z) - S(x)} = \frac{S'(z)}{S(z)} z^*.
\]

Second, when $x \in \mathcal{J}_A$, then $x \in (\underline{x}_c, \overline{x}_c)$ for some $c$. The points $\underline{x}_c$ and $\overline{x}_c$ are maximizers associated with this $c$ and $p^*_x$ can be solved from

\[
\text{E}_x \left\{ X_{\hat{\tau}_c(p^*_x)} \right\} = c \quad \Leftrightarrow \quad p^*_x = \frac{c - \overline{x}_c}{S(\overline{x}_c) - S(\underline{x}_c)}.
\]

Lastly, when $x \in \mathcal{J}_A$, then $x \in (\underline{x}_c, \overline{x}_c)$ for some $c$. One then needs to find $D_c$ associated with this $c$ and $p^*_c$ can be solved from

\[
\text{E}_x \left\{ X_{\xi(p^*_c)\tau_{D_c} + (1 - \xi(p^*_c))\tau(0, \overline{x}_c)} \right\} = c.
\]

Step 4. The value reads as

\[
V_c(x) = \inf \left\{ V^c(x) : (z^*(x) - \alpha)^2 \frac{S(z^*)}{S(x)} \left(1 - \frac{S(z^*)}{S(x)}\right), \quad x \in \mathcal{I} \setminus (\mathcal{J}_A \cup \mathcal{J}_A) \right\}, \quad x \in (\underline{x}_c, \overline{x}_c) \subset \mathcal{J}_A \cup \mathcal{J}_A.
\]

Here $z^*(x)$ is as in Step 3 and $c \in C$, $\underline{x}_c$, and $\overline{x}_c$ are the constants associated with the interval $(\underline{x}_c, \overline{x}_c)$.

In the algorithm we first identify the regions in which the solution is a randomized stopping time solution, after which we know that in everywhere else, a familiar threshold stopping time is an optimal one.
There are three observations to make. First, in Step 3 when \( x \in \mathcal{I} \setminus (\mathcal{J}_A \cup \mathcal{J}_A) \), the optimal stopping threshold \( z^*(x) \) is a solution to the first order optimality condition, but now as \( \frac{S(z)}{S(x)} \) is not non-decreasing, it is not necessarily a unique solution. Therefore, one needs to check which solution is the maximizer. Second observation is that the value for \( x \in (\bar{x}, \tau_c) \), given in Step 4, can be written with a constant stopping boundary \( \tau_c \) (or equivalently with \( \bar{x} \)). The reason for this is that the value of the variance stopping problem equals to the value of the embedded quadratic problem, and for \( x \in (\bar{x}, \tau_c) \) the corresponding constant \( c \) is unaltered.

Lastly, if Assumption 2.6(I) does not hold, we see that Step 2 differs quite remarkably, as we do not know how \( D_c \) looks like in general case. However, for \( x \in ([\bar{x}], \bar{x}] \) we are in the safe waters and we can randomize between \( \tau_{(0, \bar{x})} \) and \( \tau_{(0, x)} \). This follows from the fact that as \( x < \bar{x} \), we can apply Lemma 4.2 to conclude that \( \tau_{(0, x)} \) is an optimal stopping time.

### 8.2. Geometric Brownian motion

Let us first illustrate our results with geometric Brownian motion (which is also one of the examples considered in [13]).

Now the state space is \( \mathcal{I} = (0, \infty) \) and diffusion is a solution to the stochastic differential equation

\[
dX_t = \mu X_t dt + \sigma X_t dW_t, \quad X_0 = x,
\]

where \( \mu \in \mathbb{R} \) and \( \sigma \in \mathbb{R}_+ \) are given coefficients. The scale function is given by

\[
S(x) = \begin{cases} 
\frac{1-2\mu}{1-2\sigma^2}, & \mu \neq \frac{1}{2}\sigma^2 \\
\log(x), & \mu = \frac{1}{2}\sigma^2.
\end{cases}
\]

We have two trivial cases:

1. Assume that \( \mu > \frac{1}{2}\sigma^2 \). Then \( S(\infty) = 0 \) and so \( \infty \) is attractive (by Proposition 2.1) and \( V(x) = \infty \) by Corollary 2.4.

2. Assume that \( \mu = \frac{1}{2}\sigma^2 \). Then \( -S(0) = \infty = S(\infty) \), and gBm is recurrent leading to a value \( V(x) = \infty \) (by Lemma 7.1).

The third case is the most interesting one:

3. Assume that \( \mu < \frac{1}{2}\sigma^2 \). Then \( S(0) = 0 \) and \( S(\infty) = \infty \), so that \( 0 \) is attractive while \( \infty \) is not. Furthermore \( \mathbb{P}_x(\tau_b < \infty) = \frac{S(x)}{S(b)} \), so that

\[
\lim_{b \to \infty} \mathbb{P}_x(\tau_b < \infty) b^2 = \lim_{b \to \infty} \frac{b^2}{S(b)} S(x) = \begin{cases} 
\infty, & \mu > -\frac{1}{2}\sigma^2 \\
x^2 \in (0, \infty), & \mu = -\frac{1}{2}\sigma^2 \\
0, & \mu < -\frac{1}{2}\sigma^2.
\end{cases}
\]

Hence we have yet another three cases with the first one being trivial:

i.) Assume further that \( \mu > -\frac{1}{2}\sigma^2 \). Then \( \lim_{b \to \infty} \mathbb{P}_x(\tau_b < \infty) b^2 = \infty \) and consequently \( V(x) = \infty \) by Proposition 2.3.

ii.) Assume further that \( \mu < -\frac{1}{2}\sigma^2 \). Then \( \lim_{b \to \infty} \mathbb{P}_x(\tau_b < \infty) b^2 = 0 \) and consequently all the conditions of Case (I) in Assumption 2.5 is satisfied. In addition,
as \( \frac{S'(z)}{S(z)} z = 1 - \frac{2}{\sigma^2} \) is a constant and hence non-decreasing, we apply Theorem 3.1[13]: For a fixed \( x > 0 \), the optimal stopping time is \( \tau_{(0,z^*(x))} \), where \( z^*(x) \) is a unique solution to a first order optimality condition

\[
\frac{S(z^*) - S(x)}{S(z^*) - S(x)} = \frac{S'(z^*)}{S(z^*)} z^* \iff z^* = \left( \frac{2\mu}{\mu + \frac{1}{2}\sigma^2} \right) \frac{z^2 - 2\mu}{x}.
\]

Moreover, the value reads as

\[
V(x) = z^*(x)^2 \frac{S(x)}{S(z^*(x))} \left( 1 - \frac{S(x)}{S(z^*(x))} \right) = x^2 \left( \left( \frac{2\mu}{\mu + \frac{1}{2}\sigma^2} \right) \frac{z^2 - 2\mu}{\sigma^2} - \left( \frac{2\mu}{\mu + \frac{1}{2}\sigma^2} \right) \frac{z^2}{\sigma^2} \right).
\]

We notice that the optimal stopping time and value are identical to what was obtained in Theorem 3.2 in [13].

iii. Assume further that \( \mu = -\frac{1}{2}\sigma^2 \), so that \( S(x) = \frac{1}{2}x^2 \) and \( \lim_{b \to \infty} \mathbb{P}_x(\tau_b < \infty) b^2 = x^2 \in (0, \infty) \). Now we can apply Lemma 7.2[13] to conclude the result in this case. We see that the first order optimality condition

\[
\frac{S(z) - S(x)}{S(z) - S(x)} = \frac{S'(z)}{S(z)} z \iff x^2 = 0
\]

does not have a solution for any \( x \in (0, \infty) \). Consequently, by Lemma 7.2[13], the optimal ”stopping time” is \( \tau_{(0,\infty)} \), and the value reads as

\[
V(x) = \lim_{z \to \infty} z^2 x^2 \left( 1 - \frac{x^2}{z^2} \right) = x^2.
\]

Especially we see that this value is finite, but at the same time it is not attainable almost surely. However, by choosing \( Z > x \) to be a large number, we get with a stopping time \( \tau_{(0,Z)} \)

\[
\text{Var}_x \left\{ X_{\tau_{(0,Z)}} \right\} = x^2 - \frac{x^4}{Z^2},
\]

which we can get as close to \( V(x) \) as we like. Here \( \tau_{(0,Z)} \) is a finite stopping time with probability \( \mathbb{P}_x(\tau_Z < \infty) = \frac{x^2}{Z^2} \).

We would like to mention that this case (\( \mu = -\frac{1}{2}\sigma^2 \)) was not considered in [13].

8.3. Jacobi diffusion. Next we illustrate our results on a finite state space when both boundaries are attractive. To that end, let \( \mathcal{I} = (0, 1) \) and consider a Jacobi diffusion \( X_t \) (see e.g. Chapter 2 in [10] for a basic characteristics), which is a solution to a SDE

\[
dX_t = (a - bX_t)dt + \sigma \sqrt{X_t(1 - X_t)}dW_t.
\]

Here \( W_t \) is a standard Brownian motion. Moreover, we assume that \( a, b, \sigma \in \mathbb{R}_+ \) are such that \( 0 < \frac{b}{a} < 1 \), so that the mean-reverting level lies in the interval \( (0, 1) \). Furthermore, for illustrative purposes, we assume that \( 2b - 2a < \sigma^2 \) and \( 2a < \sigma^2 \) so that we can write
down the scale function explicitly as

\[ S(x) = \text{Beta}(x, -B, -A), \]

where Beta is the incomplete beta function, \( B := \frac{2a}{\sigma^2} - 1 \in (-1, 0) \), and \( A := \frac{2b - 2a}{\sigma^2} - 1 \in (-1, 0) \). Now \( S(0) = 0 \) and \( S(1) < \infty \) so that both end points are attractive and we have Case (III) of Assumption 2.5 to consider and the solution can be read from Theorem 3.3. Especially, as the state space is finite, the value of a variance stopping problem (1) is always finite.

Notice that in [13] the Jacobi diffusion was also examined but in a case where only the lower boundary 0 was an attractive point instead of both end points. This affects greatly to the outcome as in our case, following Theorem 3.3, the solution depends on which boundary is closer, and the closeness is measured by inspecting whether

\[ E_x\{X_{\tau(0,1)}\} = S(x) \]

is over or under \( \frac{1}{2} \). It can be proved that the monotonicities of \( S'(x) \) and \( \frac{S'(y)}{S(1) - S(y)}(1 - y) \) are satisfied so that the solution is of the type:

\[ \tau_{(0,z^*)}, \quad \text{if } S(x) \leq \frac{1}{2} S(1); \]

\[ \tau_{(y^*,1)}, \quad \text{if } S(x) > \frac{1}{2} S(1). \]

Notice that in [13], where only 0 was an attractive point, the optimal stopping time was always of the type \( \tau_{(0,z^*)} \).

To illustrate numerically this example on Jacobi diffusion, let us choose \( a = 0.02 \), \( b = 0.038 \) and \( \sigma = 0.26 \). Then the the mean-reverting level \( \frac{a}{b} \approx 0.53 \in (0,1) \). With these choices \( A \approx -0.47 \) and \( B \approx -0.41 \), and the state \( S^{-1}(\frac{S(1)}{2}) \approx 0.43 \). Below this state, the optimal stopping time is \( \tau_{(0,z^*(x))} \) and above it is \( \tau_{(y^*(x),1)} \). The optimal stopping boundaries \( z^*(x) \) and \( y^*(x) \) are illustrated in Figure 2.

8.4. Randomized solution. The mapping \( S'(x) \) is non-decreasing with the usual diffusions, and consequently the solution is a "pure strategy" stopping time with the most familiar diffusions.

In order to illustrate the randomized stopping time concept, we construct a specific diffusion: Let a state space be \( \mathcal{I} = \mathbb{R}_+ \) and define the scale function by

\[ S(x) := \begin{cases} \frac{x^2 - \frac{1}{2} x}{4x - 6}, & x < 2 \\ \frac{x^2 - \frac{1}{2} x}{10x - 22}, & x \in [2, 2.1) \\ \frac{x^2 - \frac{1}{2} x}{11x + 0.8}, & x \in [2.1, 12) \\ \frac{x^2 - \frac{1}{2} x}{2x + 0.8}, & x \geq 12. \end{cases} \]

One can easily check that such an \( S(x) \) is increasing, continuous, \( S(0) = 0 \), and \( S(\infty) = \infty \), so that 0 is attractive and \( \infty \) is not. Moreover, we can straightforwardly check that \( \lim_{b \to \infty} \frac{b^2}{S(0)} = 0 \) and conclude that the conditions of Case (I) in Assumption 2.5 are satisfied. Notice that \( S \) is not continuously differentiable over the points 2, 2.1, and 12, but the general proof does not require differentiability so that we can now apply Theorem...
The optimal stopping time is \( \tau^* = \tau(z^*, y^*) \), if we interpret \( z^*(x) = 1 \) for all \( x \geq S^{-1}(\frac{1}{2}) \) and \( y^*(x) = 0 \) for all \( x \leq S^{-1}(\frac{1}{2}) \).

3.1(A). Observe that also the monotonicity condition of \( S(x) \) is not met, as it is strictly decreasing on \( (2, 1) \).

We now follow the algorithm from Subsection 8.1

Step 1. (i) For \( c > 0 \), we solve the auxiliary embedded quadratic problem

\[
V^c(x) = \sup_\tau E_x \left\{ (X_\tau - c)^2 \right\}.
\]

It can be shown that in this particular example, for a given \( c \in \mathbb{R}_+ \setminus \{\bar{c}\} \), \( \bar{c} = \frac{3}{4} \), there exists a unique state \( z_c \) maximizing a ratio \( \frac{z^2 - 2cz}{S(z)} \) (cf. Lemma
Moreover, \( z_c \equiv 2 \) for all \( c < \bar{c} \) and \( z_c \equiv 12 \) for all \( c \in (\bar{c}, 5.54) \), cf. Figure 3. Now \( C = \{ \bar{c} \} \).

(ii) For a \( \bar{c} \) there exist two states \( 2 = \bar{z} < \bar{z} = 12 \) both maximizing the ratio \( \frac{z^2 - 2z_c}{S(z)} \). Now \( \mathbb{E}_x \left\{ X_{\tau(0, z)} \right\} \approx 0.095 < 0.75 = \bar{c} \) so that Assumption 2.6[H] does not hold. Thus we need to continue to Step 2b.

Step 2b. (i) Now we can solve \( \bar{x} \) and \( \bar{z} \):

\[
\mathbb{E}_x \left\{ X_{\tau(0, \bar{z})} \right\} = \frac{S(x)}{S(2)} \bar{c} = \bar{c} \quad \implies \quad \bar{x} = S^{-1}(0.1875) = 0.75
\]

\[
\mathbb{E}_{\bar{z}} \left\{ X_{\tau(0, \bar{z})} \right\} = \frac{S(\bar{z})}{S(12)} 12 = \bar{c} \quad \implies \quad \bar{z} = S^{-1}(3.9375) \approx 2.958.
\]

(ii) For \( x \in (\bar{z}, \bar{z}) = (0.75, 2] \), we can use a randomized stopping time \( \hat{\tau}(p) = \xi(p)\tau(0, \bar{z}) + (1 - \xi(p))\tau(0, \bar{z}) \) to produce an optimal stopping time. However, for \( x \in (\bar{z}, \bar{z}) \approx (0.75, 2.958) \), we need a randomized stopping time \( \hat{\tau}(p) = \xi(p)\tau_{D_x} + (1 - \xi(p))\tau(0, \bar{z}) \) so that we need to solve \( D_x \) for the embedded problem with a parameter \( \bar{c} \).

Applying standard optimal stopping arguments (e.g. from [16] or [15]) we can conclude that for \( x \in (\bar{z}, \bar{z}) \), \( \tau_{D_x} = \tau_{(\bar{c}, \bar{c})} = \tau_{(2, 12)} \) is an optimal stopping time for \( V^\bar{c}(x) \).

(iv) Now \( J_A = (\bar{z}, \bar{z}) \approx (0.75, 2.958) \).

Step 2. An optimal stopping time is

\[
\begin{align*}
\tau(0, z_*(x)), & \quad x \in (0, \bar{z}) \quad ((0, 0.75]) \\
\xi(p^*_x)\tau(0, \bar{z}) + (1 - \xi(p^*_x))\tau(0, \bar{z}), & \quad x \in (\bar{z}, \bar{z}) \quad (= (0.75, 2]) \\
\xi(p^*_x)\tau(2, 12) + (1 - \xi(p^*_x))\tau(0, \bar{z}), & \quad x \in (\bar{z}, \bar{z}) \quad (\approx (2, 2.958)) \\
\tau(0, z^*(x)), & \quad x \in [\bar{z}, \infty) \quad (\approx [2.958, \infty)).
\end{align*}
\]

Here \( z_*(x) \) is the smallest and \( z^*(x) \) the greatest solution to the first order optimality condition

\[
\frac{S(z) - S(x)}{\frac{1}{2}(S(z) - S(x))} = \frac{S'(z)}{S(z)} \bar{z}.
\]

Moreover, for \( x \in (0.75, 2] \), \( p^*_x \) is the unique solution to

\[
\mathbb{E}_x \left\{ X_{\hat{\tau}(p)} \right\} = \bar{c} \iff p^*_x \approx \frac{0.7875}{x} - 0.05.
\]

For \( x \in (2, 2.958) \), \( p^*_x \) is the unique solution to

\[
\mathbb{E}_x \left\{ X_{\hat{\tau}(p)} \right\} = \bar{c} \iff p^*_x \approx \frac{6.25}{63 - S(x)} - \frac{369.14}{63 - S(x)}.
\]

Step 3. The value reads as

\[
V(x) = \begin{cases} 
  z_*(x) \left( \frac{S(x)}{S(z_*(x))} \left( 1 - \frac{S(x)}{S(z_*(x))} \right) \right), & x \in (0, \bar{z}) \\
  0.5625 + 2S(x), & x \in (\bar{z}, \bar{z}) \\
  z^*(x) \left( \frac{S(x)}{S(z^*(x))} \left( 1 - \frac{S(x)}{S(z^*(x))} \right) \right), & x \in [\bar{z}, \infty),
\end{cases}
\]
where $z_*(x)$ and $z^*(x)$ are as in Step 2 above.

In this example we saw how an optimal stopping time can be a mixture between two different types of stopping times, namely $\tau_z$ and $\tau_{(a,b)}$. There are also examples where we randomize between $\tau_z$ and 0. In this way we see how the variance stopping problem can offer surprising solutions despite its simple formulation.

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Appendix A. Omitted proofs

A.1. Proof of Lemma 4.3
(1) For \(a \leq x < b\):

\[
E_x \left\{ (X_{\tau(a,b)} - c)^2 \right\} = \frac{S(b) - S(x)}{S(b) - S(a)} (a - c)^2 + \frac{S(x) - S(a)}{S(b) - S(a)} (b - c)^2.
\]

This is continuous on \([0, x] \times (x, \beta) \times [0, \infty)\).

\[
E_x \left\{ (X_{\tau(a,b)} - c)^2 \right\} = (b - c)^2 + \frac{S(b) - S(x)}{S(b) - S(a)} (a^2 - b^2 + 2c(b - a))
\]

to therefore if \((a_n, b_n, c_n) \in [0, x] \times (x, \beta) \times [0, \infty)\) for all \(n\) and \(a_n \to a \in [0, x)\) and \(b_n \to x\) and \(c_n \to c \in [0, \infty)\) then

\[
E_x \left\{ (X_{\tau(a_n,b_n)} - c_n)^2 \right\} \to (x - c)^2 + 0 \cdot (a^2 - x^2 + 2c(x - a)) = (x - c)^2.
\]

If instead \(a_n \to x\) then

\[
E_x \left\{ (X_{\tau(a_n,b_n)} - c_n)^2 \right\} \to (x - c)^2 + 1 \cdot 0 = (x - c)^2.
\]

Thereby \((a, b, c) \mapsto E_x \left\{ (X_{\tau(a,b)} - c)^2 \right\}\) is continuous on \((a, b, c) \in [0, x] \times [x, \beta) \times [0, \infty)\).

In addition, recall that for every \(c\) there exists \((a, b) \in [0, x] \times [x, \beta)\) such that \(\tau_{(a,b)}\) is optimal. Thus sup \(E_x \left\{ (X_{\tau(a)} - c)^2 \right\} = \sup_{0 \leq a \leq b} E_x \left\{ (X_{\tau(a,b)} - c)^2 \right\}\), and this is continuous as \((a, b, c) \mapsto E_x \left\{ (X_{\tau(a,b)} - c)^2 \right\}\).

(2) We consider two different cases: when there exists a \(c < \frac{1}{2} \beta\) such that \(z_c < x\), and when there is no such \(c\).

(i) Assume first that there exists \(\hat{c} < \frac{1}{2} \beta\) for which \(z_{\hat{c}} < x\). It is known that for every \(c\) the embedded quadratic problem has a hitting time solution \(\tau_{D_c}\), where

\[D_c := \{y \in [0, \beta) : (y - c)^2 \geq \sup_x E_y \left\{ (X_{\tau} - c)^2 \right\} \}\]

and from \(\cdots \ z_c \in D_c\.

Hence, \(E_x \left\{ X_{\tau_{D_c}} \right\} \geq z_c > \hat{c}\). The first inequality follows from \(z \leq x\) and \(z \in D_c\). The second inequality follows from the fact that \(z_c > 2c\) for every \(c < \frac{1}{2} \beta\).

Given the starting value \(x\), \(\tau_{D_c} = \tau_{(a,b)}\) for some \(a \leq x < b\), and thus we can choose \((a_L, b_L, c_L) = (a, b, \hat{c})\).

(ii) Now assume that for all \(c \in (0, \frac{1}{2} \beta)\) we have \(z_c > x\). Let \(y = \sup \{z_c : c \in (0, x)\}\).

Then \(y > x\), and it follows from Lemma 4.3.5 that \(y < \beta\). Now, for all \(c \in (0, x)\) we have

\[
E_x \left\{ X_{\tau(0, z_c)} \right\} = \frac{S(x)}{S(z_c)} z_c \leq \frac{S(x)}{S(y)} x \geq \frac{S(x)}{S(y)} x =: \delta > 0,
\]

where the coefficient \(\frac{S(x)}{S(y)} < 1\) so that \(\delta \in (0, x)\).

It follows that for all \(c \in (0, \delta)\) we have \(E_x \left\{ X_{\tau(0, z_c)} \right\} > \delta > c\). Hence, the claim is proved by choosing \(c_L = \delta \wedge \frac{1}{4} \beta\) and \((a_L, b_L, c_L) = (0, z_{c_L}, c_L)\).

(3) Let \(c \to \frac{1}{2} \beta\). Then, because \(z_c > 2c\), we have \(z_c \to \beta\). Moreover, since \(\lim_{z \to \beta} \frac{z^2}{S(z)} = 0\) we have \(E_x \left\{ X_{\tau(0, z_c)} \right\} = \frac{S(x)}{S(z_c)} z_c \to 0\) as \(z_c \to \beta\). Therefore, if we choose \(c_H\)
high enough then \( c_H > c_L \) and \( E_x \{ X_{t(0,z_{c_H})} \} < c_H \) and \( z_{c_H} > x \) and thereby \( F(0,z_{c_H},c_H) = 0 \).

\[(4) \text{ for } a \leq x < b:\]
\[
E_x \{ X_{\tau(a,b)} \} = \frac{S(b) - S(a)}{S(b) - S(a)} a + \frac{S(x) - S(a)}{S(b) - S(a)} b = a + \frac{S(x) - S(a)}{S(b) - S(a)} (b - a),
\]
this is continuous on \([0,x] \times (x,\beta)\). If \((a_n,b_n) \in [0,x] \times (x,\beta)\) for all \(n\) and \(a_n \to a \in [0,x]\) and \(b_n \to x\) then
\[
E_x \{ X_{\tau(a_n,b_n)} \} \to a + 1 \cdot (x - a) = x.
\]
If instead \(a_n \to x\), then
\[
E_x \{ X_{\tau(a_n,b_n)} \} = x + 0 = x.
\]
Thereby \((a,b) \mapsto E_x \{ X_{\tau(a,b)} \}\) is continuous on \([0,x] \times [x,\beta]\).

\[(5) \text{ If } c_1 = c_2 = 0 \text{ the result is obvious. Assume in the following } c_2 > 0. \text{ On one hand let } x = \frac{1}{2}(\beta + 2c_2) \text{ and } \varepsilon = \frac{2^2 - 2c_2 x}{S(x)} > 0. \text{ Then } x \in [0,\beta) \text{ and } \frac{2^2 - 2c_2 x}{S(x)} > \varepsilon \text{ for all } c \in [c_1,c_2]. \]
On the other hand recall that \(\lim_{z \to \beta} \frac{S^2(z)}{S(z)} = 0\) and therefore also \(\lim_{z \to \beta} \frac{S(z)}{S(z)} = 0\). Thus, since the set \([c_1,c_2]\) is bounded, then when \(z\) goes to \(\beta\), we have \(\frac{2^2 - 2c_2 x}{S(z)}\) converges uniformly to 0 over \(c \in [c_1,c_2]\). Specifically, there exists a \(Z < \beta\) such that for \(z > Z\) and \(c \in [c_1,c_2]\) then \(\frac{2^2 - 2c_2 x}{S(z)} < \varepsilon\). If we combine these two observations it follows that \(z_c \leq Z < \beta\) for \(c \in [c_1,c_2]\).