Frobenius categories over a triangular matrix ring and comma categories

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ABSTRACT
We introduce the dual notions of $E(X, M, Y)$ and $M(X, M, Y)$, and investigate when they have enough injective objects or projective objects, when they are resolving or co-resolving, and when they are Frobenius categories. In the case that they are Frobenius categories, we establish a recollement of their stable categories. Finally some applications to comma categories are given.

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1. Introduction
Throughout this article, all rings are nonzero associative rings with identity and all modules are unitary and left unless otherwise stated. For a ring $R$, we write $R$-Mod for the category of all left $R$-modules. Given rings $A$, $B$ and an $(A, B)$-bimodule $M$, we can build an abelian category, denoted by $\text{Rep}(A, M, B)$, which contains $A$-Mod and $B$-Mod as full subcategories. The objects of $\text{Rep}(A, M, B)$ are all triples $\begin{bmatrix} X \\ Y \end{bmatrix}_\phi$ such that $X \in A - \text{Mod}$, $Y \in B - \text{Mod}$ and $\phi : M \otimes_B Y \rightarrow X$ is an $A$-homomorphism. A morphism from $\begin{bmatrix} X \\ Y \end{bmatrix}_\phi$ to $\begin{bmatrix} X_1 \\ Y_1 \end{bmatrix}_{\phi_1}$ is a pair $\begin{bmatrix} f \\ g \end{bmatrix}$ such that $f \in \text{Hom}_A(X, X')$, $g \in \text{Hom}_B(Y, Y')$ and the following diagram is commutative:

\[
\begin{array}{ccc}
M \otimes_B Y & \xrightarrow{\phi} & X \\
1_M \otimes g & \downarrow & \downarrow f \\
M \otimes_B Y' & \xrightarrow{\phi'} & X'.
\end{array}
\]
It is well-known that \( \text{Rep}(A, M, B) \) is equivalent to the module category \( \Lambda\text{-Mod} \), where

\[
\Lambda := \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}
\]

is a formal triangular matrix ring.

Formal triangular matrix rings play an important role in ring theory and the representation theory of algebra. They are often used to construct examples and counterexamples, which make the theory of rings and modules more abundant and interesting. So the properties of formal triangular matrix rings and modules over them have attracted many interests (see [1, 2, 7–9, 12, 13, 15]). These investigations usually focus to build some special subclasses or subcategories of \( \Lambda\text{-Mod} \) by using corresponding subclasses or subcategories of \( A\text{-Mod} \) and \( B\text{-Mod} \).

Projective objects and injective objects of \( \text{Rep}(A, M, B) \) are classified by Haghany and Varadarajan in [8] and [9], respectively.

Let \( U = \begin{bmatrix} X \\ Y \end{bmatrix} \in \text{Rep}(A, M, B) \).

**Theorem 1.1.** ([9, theorem 3.1]). \( U \) is a projective object of \( \text{Rep}(A, M, B) \) if and only if \( Y \) is a projective \( B\)-module, \( \phi \) is injective and \( \text{Coker}(\phi) \) is a projective \( A\)-module.

Denote by \( \phi \) the morphism \( \tau_{X,Y}(\phi) \), where \( \tau_{X,Y} \) is the natural isomorphism from \( \text{Hom}_A(M \otimes_B Y, X) \to \text{Hom}_B(Y, \text{Hom}_A(M, X)) \).

**Theorem 1.2.** ([8, proposition 5.1] and [1, p. 956]). \( U \) is an injective object of \( \text{Rep}(A, M, B) \) if and only if \( X \) is an injective \( A\)-module, \( \phi \) is surjective and \( \text{Ker}(\phi) \) is an injective \( B\)-module.

To simplify the notation, we introduce the category \( \text{Rep}_h(A, M, B) \). Its objects are all triples \( (X, Y) \), such that \( X \in A\text{-Mod}, Y \in B\text{-Mod} \) and \( \phi : Y \to \text{Hom}_A(M, X) \) is a \( B\)-homomorphism. A morphism from \( (X, Y) \) to \( (X_1, Y_1) \) is a pair \( (f, g) \) such that \( f \in \text{Hom}_A(X, X_1), g \in \text{Hom}_B(Y, Y_1) \) and the following diagram is commutative:

\[
\begin{array}{ccc}
Y & \xrightarrow{\phi} & \text{Hom}_A(M, X) \\
\downarrow{g} & & \downarrow{\text{Hom}_A(M, f)} \\
Y_1 & \xrightarrow{\phi_1} & \text{Hom}_A(M, X_1)
\end{array}
\]

It is easy to see that \( \text{Rep}_h(A, M, B) \) and \( \text{Rep}(A, M, B) \) are equivalent as abelian categories. Furthermore, using this notation, **Theorem 1.2** can be restated as follows:

**Theorem 1.3.** A triple \( (X, Y) \) is an injective object of \( \text{Rep}_h(A, M, B) \) if and only if \( X \) is an injective \( A\)-module, \( \text{Ker}(\phi) \) is an injective \( B\)-module and \( \phi \) is surjective.

Inspired by these results, we introduce and study the following notions.

**Definition 1.4.** Let \( \mathcal{X} \) (resp. \( \mathcal{Y} \)) be a full category of \( A\text{-Mod} \) (resp. \( B\text{-Mod} \)) closed under isomorphisms and containing zero module. We define \( \mathcal{M}(\mathcal{X}, \mathcal{M}, \mathcal{Y}) \) to be the full category of \( \text{Rep}(A, M, B) \) whose objects are triples \( \begin{bmatrix} X \\ Y \end{bmatrix} \in \text{Rep}(A, M, B) \) such that \( Y \in \mathcal{Y} \), \( \text{Coker}(\phi) \in \mathcal{X} \) and \( \phi \) is injective.

Dually, \( \mathcal{E}(\mathcal{X}, \mathcal{M}, \mathcal{Y}) \) is defined to be the full category of \( \text{Rep}_h(A, M, B) \) whose objects are triples \( \begin{bmatrix} X \\ Y \end{bmatrix} \in \text{Rep}_h(A, M, B) \) such that \( X \in \mathcal{X} \), \( \text{Ker}(\phi) \in \mathcal{Y} \) and \( \phi \) is surjective.

The notions of \( \mathcal{E}(\mathcal{X}, \mathcal{M}, \mathcal{Y}) \) and \( \mathcal{M}(\mathcal{X}, \mathcal{M}, \mathcal{Y}) \) are dual to each other and so are their properties. We write \( \mathcal{M}(A, M, \mathcal{Y}) \) and \( \mathcal{E}(A, M, \mathcal{Y}) \) for \( \mathcal{M}(\mathcal{X}, \mathcal{M}, \mathcal{Y}) \) and \( \mathcal{E}(\mathcal{X}, \mathcal{M}, \mathcal{Y}) \), respectively, if \( \mathcal{X} \) is
A – Mod. The categories \( \mathcal{M}(A, M, Y) \), \( \mathcal{M}(A, M, B) \) and etc., are similarly defined. Recall that \( \mathcal{M}(A, M, B) \) and \( \mathcal{E}(A, M, B) \) have been studied in [15] in the context of artinian algebras.

The article is arranged as follows. In Section 3, we study the properties of \( \mathcal{E}(X, M, Y) \). First, we classify projective objects of \( \mathcal{E}(X, M, Y) \) when it is an exact category and prove that \( \mathcal{E}(X, M, Y) \) has enough projective objects if and only if \( X \) and \( Y \) have enough projective objects under mild conditions. We discuss when \( \mathcal{E}(X, M, Y) \) is co-resolving and show there is a unique largest co-resolving subcategory of the form \( \mathcal{E}(X, M, Y) \). Based on these results, we characterize when \( \mathcal{E}(X, M, Y) \) is a Frobenius category. Finally, it is shown that the stable category of \( \mathcal{E}(X, M, Y) \) is a triangulated recollement of stable categories of \( X \) and \( Y \) when they are all Frobenius categories. In Section 4, we write down the dual results on \( \mathcal{M}(X, M, Y) \) without giving their proofs. In the last section, some of results in previous sections are extended to those of some special subcategories of comma categories.

2. Preliminaries

In this section, we recall some notions and results which we will use later.

We refer to [11, subsection 2.1] for the definition of an exact category. Let \( C \) be a full subcategory of an abelian category which is closed under extensions. Then, \( C \) carries a canonical exact structure. An object \( P \in C \) is projective if every short exact sequence \( 0 \rightarrow X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow 0 \) in \( C \) induces a short exact sequence \( 0 \rightarrow \text{Hom}_C(P, X_1) \rightarrow \text{Hom}_C(P, X_2) \rightarrow \text{Hom}_C(P, X_3) \rightarrow 0 \). The exact category \( C \) has enough projective objects if every \( X \in C \) induces a short exact sequence \( 0 \rightarrow K \rightarrow P \rightarrow X \rightarrow 0 \) in \( C \) such that \( P \) is projective. Injective objects of \( C \) and that \( C \) has enough injective objects are defined dually. We write \( I(C) \) and \( P(C) \) for the classes of injective objects and projective objects of \( C \), respectively. An exact category is a Frobenius category if it has enough projective objects and injective objects, and if its projective and injective objects coincide, i.e. \( I(C) = P(C) \).

The notion of a Frobenius category is very important because its stable category is a triangulated category and many triangulated categories including derived category and homotopy category of an abelian category are algebraic. Recall that a triangulated category is algebraic if it is the stable category of a Frobenius category. Let \( C \) be a Frobenius category. By definition, the stable category \( \text{St}C \) has the same objects as \( C \) and for any \( X, Y \in C \),

\[
\text{Hom}_{\text{St}C}(X, Y) := \text{Hom}_C(X, Y)/\mathcal{P}(X, Y).
\]

Here, \( \mathcal{P}(X, Y) \) stands for the set of morphisms from \( X \) to \( Y \) that factor through a projective object. For each \( X \in B \) we fix a short exact sequence \( 0 \rightarrow X \rightarrow I(X) \rightarrow TX \rightarrow 0 \) such that \( I(X) \) is an injective object of \( C \). Let \( \xi : 0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0 \) be a short exact sequence in \( C \). Then, it induces a commutative diagram:

\[
\begin{array}{ccc}
0 & \rightarrow & X \\
& \downarrow & \downarrow \text{f} \downarrow & \downarrow \text{g} \\
& \rightarrow & Y \\
& \downarrow & \downarrow \text{h} \\
0 & \rightarrow & I(X) \\
& \rightarrow & TX \\
& \rightarrow & 0.
\end{array}
\]

Figure 1. A standard triangle.

The sequence \( (f, g, h) \) is called a standard triangle induced by \( \xi \). The image in \( \text{St}C \) of such a sequence is an exact triangle of \( \text{St}C \) and every exact triangle of \( \text{St}C \) arises in this way, see [11, proposition 3.3.2].
Lemma 2.1. Let $\mathcal{C}$ and $\mathcal{D}$ be Frobenius categories. If $F : \mathcal{C} \to \mathcal{D}$ is an additive functor satisfying (1) $F$ is exact, (2) $F(I(\mathcal{C})) \subseteq I(\mathcal{D})$, then, $F : \text{StC} \to \text{StD}$ is a triangulated functor.

Proof. Note that $F$ turns Figure 1 in $\mathcal{C}$ into the following commutative diagram in $\mathcal{D}$

\[
\begin{array}{ccccccc}
0 & \longrightarrow & F(X) & \xrightarrow{F(f)} & F(Y) & \xrightarrow{F(g)} & F(Z) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & F(X) & \longrightarrow & F(I(X)) & \longrightarrow & F(TX) & \longrightarrow & 0,
\end{array}
\]

whose rows are exact. This implies $\tilde{F}$ maps an exact triangle in $\text{StC}$ into an exact triangle in $\text{StD}$.

The tools of pushout and pullback are indispensable. We record the following results for the later use.

Lemma 2.2. (see e.g. [10, proposition 5.2]). Let

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{a} & & \downarrow{b} \\
C & \xrightarrow{g} & D
\end{array}
\]

be a commutative diagram in an abelian category.

1. If it is a pullback diagram, then,
   i. $\text{Kera} \cong \text{Kerb}$, $\text{Kerf} \cong \text{Kerg}$;
   ii. both $\tilde{g} : \text{Coker} a \to \text{Coker} b$ and $\tilde{b} : \text{Coker} f \to \text{Coker} g$ are injective;
   iii. $(b, g)$ is the pushout of $(a, f)$ if and only if either $\tilde{g}$ or $\tilde{b}$ is an isomorphism. Dually,

2. If it is a pushout diagram, then,
   i. $\text{Coker} a \cong \text{Coker} b$, $\text{Coker} f \cong \text{Coker} g$;
   ii. both $\tilde{f} : \text{Kera} \to \text{Kerb}$ and $\tilde{a} : \text{Kerf} \to \text{Kerg}$ are surjective.
   iii. $(a, f)$ is the pullback of $(b, g)$ if and only if either $\tilde{f}$ or $\tilde{a}$ is an isomorphism.

Lemma 2.3. (see e.g. [3, proposition 2.12]). If we have one of the following commutative diagram in an abelian category with exact rows:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & E & \longrightarrow & A & \xrightarrow{f} & B & \longrightarrow & 0 \\
\downarrow & & \downarrow{a} & & \downarrow{b} & & \downarrow & \\
0 & \longrightarrow & E & \longrightarrow & C & \xrightarrow{g} & D & \longrightarrow & 0,
\end{array}
\]

and

\[
\begin{array}{ccccccc}
0 & \longrightarrow & A & \xrightarrow{f} & B & \longrightarrow & E & \longrightarrow & 0 \\
\downarrow{a} & & \downarrow{b} & & \downarrow & & \downarrow & \\
0 & \longrightarrow & C & \xrightarrow{g} & D & \longrightarrow & E & \longrightarrow & 0,
\end{array}
\]
then, the following

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{a} & & \downarrow{b} \\
C & \xrightarrow{g} & D
\end{array}
\]

is a pushout-pullback diagram. (This means \((a, f)\) is a pullback \((g, b)\), and \((g, b)\) is a pushout \((a, f)\) at the same time)

3. The properties of \(E(\mathcal{X}, M, \mathcal{Y})\)

In this section, \(A\) and \(B\) are rings and \(M\) is an \((A, B)\)-bimodule. Let \(\mathcal{X}\) and \(\mathcal{Y}\) be subcategories of \(A\)-Mod and \(B\)-Mod, respectively, closed under isomorphisms and containing the zero module. We will study properties of \(E(\mathcal{X}, M, \mathcal{Y})\) (see Definition 1.4 for what it means). First of all, we need to define some useful functors.

1. \(p : \mathcal{X} \to E(\mathcal{X}, M, \mathcal{Y})\), \(X \mapsto (X, \text{Hom}_A(M, X))_1\);
2. \(q : \mathcal{Y} \to E(A, M, \mathcal{Y})\), \(Y \mapsto (0, Y)_0\);
3. \(P : E(\mathcal{X}, M, \mathcal{Y}) \to \mathcal{X}\), \((X, Y)_\varphi \mapsto X\);
4. \(Q : E(\mathcal{X}, M, \mathcal{Y}) \to \mathcal{Y}\), \((X, Y)_\varphi \mapsto \text{Ker}(\varphi)\);
5. \(Q : E(\mathcal{X}, M, \mathcal{Y}) \to B\text{-Mod}\), \((X, Y)_\varphi \mapsto Y\).

The actions to morphisms of these functors are defined in a natural way. For examples, if \(f \in \text{Mor}\mathcal{X}\), then, \(p(f) = (f, \text{Hom}(M, f))\); if \((f, g) \in \text{Mor}E(\mathcal{X}, M, \mathcal{Y})\), then, \(P(f, g) = f, Q(f, g) = g\) and \(Q(f, g) = \bar{g}\), where \(\bar{g}\) is given by the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \to & \text{Ker}(\varphi) & \to & Y & \xrightarrow{\varphi} & \text{Hom}_A(M, X) & \to & 0 \\
& & \downarrow{\bar{g}} & & \downarrow{g} & & \downarrow{\text{Hom}_A(M, f)} & & \\
0 & \to & \text{Ker}(\varphi_1) & \to & Y_1 & \xrightarrow{\varphi_1} & \text{Hom}_A(M, X_1) & \to & 0.
\end{array}
\]

Note that \(Q(X, Y)_\varphi\) may not belong to \(\mathcal{Y}\) even if \((X, Y)_\varphi \in E(\mathcal{X}, M, \mathcal{Y})\). If the categories involved are exact, then, the functors \(q, Q, P\) and \(Q\) are exact. Conversely, \(p\) is faithful and left exact, but not always exact.

**Proposition 3.1.** If both \(\mathcal{X}\) and \(\mathcal{Y}\) are closed under extensions, then, \(E(\mathcal{X}, M, \mathcal{Y})\) is closed under extensions;

**Proof.** Let

\[
0 \to (X_1, Y_1)_{\varphi_1} \xrightarrow{(f_1, g_1)} (X_2, Y_2)_{\varphi_2} \xrightarrow{(f_2, g_2)} (X_3, Y_3)_{\varphi_3} \to 0
\]

be a short exact sequence in \(\text{Rep}_b(A, M, B)\) such that \((X_i, Y_i)_{\varphi_i} \in E(\mathcal{X}, M, \mathcal{Y})\) for \(i = 1, 2\). Since the sequence \(0 \to X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \to 0\) is exact, it follows that \(X_2 \in \mathcal{X}\). In addition, we have the following commutative diagram:
where both rows are exact. By the Snake Lemma we obtain the exact sequence:

\[ 0 \rightarrow \text{Ker}(\phi_1) \rightarrow \text{Ker}(\phi_2) \rightarrow \text{Ker}(\phi_3) \rightarrow \text{Coker}(\phi_1) \rightarrow \text{Coker}(\phi_2) \rightarrow \text{Coker}(\phi_3). \]

Note that \( \text{Coker}(\phi_1) = \text{Coker}(\phi_3) = 0 \) and \( \text{Ker}(\phi_i) \in X \) for \( i = 1, 2 \), it follows that \( \text{Coker}(\phi_2) = 0 \) and \( \text{Ker}(\phi_2) \in X \).

If \( \mathcal{E}(X, M, Y) \) is closed under extensions, then, so is \( Y \), since \( q \) is an exact functor; but \( X \) may not be closed under extensions in general. If \( p \) is an exact functor, then, \( X \) is also closed under extensions.

Next, we aim to classify all projective objects of \( \mathcal{E}(X, M, Y) \) when it is an exact category. We begin with a lemma, which deals with this question under the hypothesis as weak as possible. To this end, we introduce condition (*)..

(*) Any short exact sequence \( 0 \rightarrow X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \rightarrow 0 \) in \( X \) induces a short exact sequence \( 0 \rightarrow (X_1, Y_1)_{\phi_1} \xrightarrow{(f_1, g_1)} (X_2, Y_2)_{\phi_2} \xrightarrow{(f_2, g_2)} (X_3, Y_3)_{\phi_3} \rightarrow 0 \) in \( \mathcal{E}(X, M, Y) \).

In the case when \( p \) is exact, this condition holds automatically.

**Lemma 3.2.** Suppose that both \( X \) and \( Y \) are closed under extensions and let \( (P, Q)_{\phi} \in \mathcal{E}(X, M, Y) \).

If the following conditions holds:

i. \( P \) is a projective object in \( X \);

ii. \( Q \) is \( Y \)-projective, (this means \( \text{Hom}_A(Q, \cdot) \) takes a short exact sequence in \( Y \) to a short exact sequence in \( B\text{-Mod} \), but \( Y \) may not belong to \( Y \));

iii. the map \( \phi \) satisfies the following property: given any short exact sequence in \( B\text{-Mod} \):

\[ 0 \rightarrow \text{Ker}(\phi) \rightarrow Y \xrightarrow{\phi} \text{Hom}_A(M, X) \rightarrow 0 \]

with \( \text{Ker}(\phi) \in Y \) and \( X \in X \), and any \( A \)-homomorphism \( f : P \rightarrow X \), there exists a \( B \)-homomorphism: \( g : Q \rightarrow Y \) such that \( \text{Hom}(M, f) \circ \phi = \phi \circ g \), that is, the following diagram is commutative:

\[ \begin{array}{ccc}
Q & \xrightarrow{\phi} & \text{Hom}_A(M, P) \\
\downarrow{\bar{g}} & & \downarrow{\text{Hom}(M, f)} \\
0 & \xrightarrow{\text{Ker}(\phi)} & Y \xrightarrow{\phi} \text{Hom}_A(M, X) \rightarrow 0.
\end{array} \]

then, \( (P, Q)_{\phi} \) is a projective object of \( \mathcal{E}(X, M, Y) \).

The converse is also true if condition (*) holds.

**Proof.** First, we assume that \( (P, Q)_{\phi} \) satisfies the three conditions mentioned above, and we proceed to prove \( (P, Q)_{\phi} \) is a projective object. For this, we let

\[ 0 \rightarrow (X_1, Y_1)_{\phi_1} \xrightarrow{(f_1, g_1)} (X_2, Y_2)_{\phi_2} \xrightarrow{(f_2, g_2)} (X_3, Y_3)_{\phi_3} \rightarrow 0 \]
be a short exact sequence in $\mathcal{E}(X, M, Y)$ and $(\alpha, \beta) : (P, Q) \rightarrow (X_3, Y_3)_{\phi_3}$, a morphism in $\text{Rep}_h(A, M, B)$. Since $P$ is a projective object of $X$, there is an $A$-morphism $\rho : P \rightarrow X_2$ such that $\alpha = f_2 \rho$. Denote $\text{Hom}(M, h)$ by $h^*$ for any $A$-homomorphism $h$. By (iii), there exists a $B$-homomorphism $t$ such that $\rho^* \phi = \phi_2 t$. Considering the following commutative diagram, where all rows and columns are exact.

The exactness of the upmost row follows from the Snake Lemma. Note that

$$\phi_3 \beta = \alpha^* \phi = f_2^* \rho^* \phi = f_2^* \phi_2 t = \phi_3 g_2 t,$$

it follows that $\phi_3 (\beta - g_2 t) = 0$, and so there exists a $B$-homomorphism $j : Q \rightarrow \text{Ker}(\phi)$ such that $\beta - g_2 t = i_3 j$ by the universal property of the Kernel. By (ii), we have $j = g_2 k$ for some $B$-homomorphism $k : Q \rightarrow \text{Ker}(\phi_2)$.

Now, set $s = t + i_2 k$. Then,

$$\phi_2 s = \phi_2 t + \phi_2 i_2 k = \phi_2 t + \rho^* \phi$$

and

$$g_2 s = g_2 t + g_2 i_2 k = g_2 t + i_3 g_2 k$$

$$= g_2 t + i_3 j = g_2 t + \beta - g_2 t$$

$$= \beta.$$

Hence, $(\rho, s) : (P, Q) \rightarrow (X_2, Y_2)_{\phi_2}$ is a morphism in $\text{Rep}_h(A, M, B)$ and it lifts the morphism $(\alpha, \beta)$, that is, $(\alpha, \beta) = (f_2, g_2)(\rho, s)$. Consequently, $(P, Q)_{\phi}$ is a projective object of $\mathcal{E}(X, M, Y)$, as desired.

Conversely, assume that $(P, Q)_{\phi}$ is a projective object in $\mathcal{E}(X, M, Y)$. First, we prove that (iii) holds. Let

$$0 \rightarrow \text{Ker}(\phi) \rightarrow Y^* \text{Hom}_A(M, X) \rightarrow 0$$

be a short exact sequence in $B$-Mod with $\text{Ker}(\phi) \in \mathcal{Y}$ and $X \in \mathcal{X}$, and $f : P \rightarrow X$ an $A$-homomorphism. Then, we have the following exact sequences in $\mathcal{E}(X, M, Y)$:
0 \to (0, \text{Ker}(\phi))_0 \to (X, Y)_\phi \xrightarrow{(1, \phi)} (X, \text{Hom}_A(M, X))_1 \to 0.

For the morphism \((f, \text{Hom}_A(M, f)) : (P, Q)_\phi \to (X, \text{Hom}_A(M, X))_1\), there exists a morphism, say \((j, k) : (P, Q)_\phi \to (Y, \text{Hom}_A(M, X))_\phi\) such that \((f, \text{Hom}(M, f)) = (1, \phi)(j, g)\). Particularly, we have \(\phi g = \text{Hom}_A(M, f)\). This proves (iii).

Next, since any short sequence in \(B, \text{Mod}: 0 \to Y_1 \to Y_2 \to Y_3 \to 0\) induces a short sequence \(0 \to (0, Y_1)_0 \to (0, Y_2)_0 \to (0, Y_3)_0 \to 0\), condition (ii) follows.

Finally, we prove (i) under the additional condition (*). Let \(0 \to X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \to 0\) be a short exact sequence in \(\mathcal{X}\) and let \(\alpha : P \to X_3\) be a \(\mathcal{A}\)-homomorphism. Then, by condition (*), there exists a short exact sequence of the form:

\[
0 \to (X_1, Y_1)_\phi \xrightarrow{(f_1, g_1)} (X_2, Y_2)_\phi \xrightarrow{(f_2, g_2)} (X_3, Y_3)_\phi \to 0
\]

in \(\mathcal{E}(\mathcal{X}, M, \mathcal{Y})\). Using (iii), which have been proved before, there exists a morphism \((\alpha, \beta) : (P, Q)_\phi \to (X_3, Y_3)_\phi\) in \(\mathcal{E}(\mathcal{X}, M, \mathcal{Y})\). It follows that there exists a morphism \((\alpha', \beta') : (P, Q)_\phi \to (X_2, Y_2)_\phi\) such that \((\alpha, \beta) = (\alpha', \beta')(f_2, g_2)\) and this particularly implies \(\alpha = \alpha f_2\). Hence, \(P\) is a projective object of \(\mathcal{X}\), proving (i).

Hereafter we need the additional hypothesizes that \(\text{Hom}_A(M, X) \in \mathcal{Y}\) for \(X \in \mathcal{X}\) and \(\text{Ext}^1_A(M, X) = 0\) for any \(X \in \mathcal{X}\). For short, we will denote them by \(\text{Hom}_A(M, \mathcal{X}) \subseteq \mathcal{Y}\) and \(\text{Ext}^1_A(M, \mathcal{X}) = 0\), respectively. \(\text{Lemma 3.2}\) can be simplified a lot under these hypothesizes.

**Corollary 3.3.** Suppose that \(\text{Ext}^1_A(M, \mathcal{X}) = 0\) and that \(\mathcal{E}(\mathcal{X}, M, \mathcal{Y})\) is closed under extensions. If either \(\mathcal{Y}\) has enough injective objects or \(\text{Hom}_A(M, \mathcal{X}) \subseteq \mathcal{Y}\), then, for any \((P, Q)_\phi \in \mathcal{E}(\mathcal{X}, M, \mathcal{Y})\), \((P, Q)_\phi\) is a projective object of \(\mathcal{E}(\mathcal{X}, M, \mathcal{Y})\) if and only if \(P\) is a projective object of \(\mathcal{X}\) and \(Q\) is \(\mathcal{Y}\)-projective.

**Proof.** Since \(\text{Ext}^1_A(M, \mathcal{X}) = 0\), the functor \(\text{p}\) is exact, and particularly, condition (*) holds. Thus, if \((P, Q)_\phi\) is a projective object of \(\mathcal{E}(\mathcal{X}, M, \mathcal{Y})\), then, \(P\) is a projective object of \(\mathcal{X}\) and \(Q\) is \(\mathcal{Y}\)-projective by \(\text{Lemma 3.2}\). Conversely, we assume that \(P\) is a projective object of \(\mathcal{X}\) and \(Q\) is \(\mathcal{Y}\)-projective. If \(\mathcal{Y}\) has enough injective objects, then, \(\text{Ext}^1_A(Q, Y) = 0\) for \(Y \in \mathcal{Y}\) by the Long Exact Sequence Theorem, and so condition (iii) in \(\text{Lemma 3.2}\) is satisfied (by noting that \(\text{Ext}^1_A(Q, \text{Ker}(\phi)) = 0\)) if \(\text{Hom}_A(M, \mathcal{X}) \subseteq \mathcal{Y}\), then, the short exact sequence \(0 \to \text{Ker}(\phi) \to Y \xleftarrow{\phi} \text{Hom}_A(M, X) \to 0\) in condition (iii) lies in \(\mathcal{Y}\), and thus, condition (iii) is also satisfied. Consequently, \((P, Q)_\phi\) is a projective object of \(\mathcal{E}(\mathcal{X}, M, \mathcal{Y})\) by \(\text{Lemma 3.2}\).

We now consider when \(\mathcal{E}(\mathcal{X}, M, \mathcal{Y})\) has enough projective objects.

**Theorem 3.4.** Suppose that \(\mathcal{E}(\mathcal{X}, M, \mathcal{Y})\) is closed under extensions and assume further that \(\text{Ext}^1_A(M, \mathcal{X}) = 0\) and \(\text{Hom}_A(M, \mathcal{X}) \subseteq \mathcal{Y}\). Then, \(\mathcal{E}(\mathcal{X}, M, \mathcal{Y})\) has enough projective objects if and only if both \(\mathcal{X}\) and \(\mathcal{Y}\) have enough projective objects.

**Proof.** The “only if” part follows since \(\mathcal{P}\) and \(\mathcal{Q}\) are exact functors. Conversely, assume that both \(\mathcal{X}\) and \(\mathcal{Y}\) have enough projective objects. Let \((X, Y)_f \in \mathcal{E}(\mathcal{X}, M, \mathcal{Y})\). Then, there exists a short exact sequence in \(\mathcal{X}\):

\[
0 \to K \to P \xrightarrow{\phi} X \to 0,\]

where \(P\) is a projective object of \(\mathcal{X}\). From this, we obtain the following commutative diagram:
where \((\beta, g)\) is the pullback of \((f, \text{Hom}(M, z))\). By Lemma 2.2, we see that \(\ker(g) \cong \ker(f), \ker(\beta) \cong \text{Hom}_A(M, K)\), and that both rows and both columns are short exact sequences. Since \(Y \in \mathcal{Y}\) and \(\ker(\beta) \cong \text{Hom}_A(M, K) \in \mathcal{Y}\), it follows that \(T \in \mathcal{Y}\) and so we have the following exact sequence in \(\mathcal{Y}\):

\[
0 \to N \to Q \xrightarrow{e} T \to 0,
\]

where \(Q\) is a projective object of \(\mathcal{Y}\). Considering the following commutative diagram:

\[
\begin{array}{ccccccccc}
0 & \to & \ker(\beta e) & \to & Q & \xrightarrow{\beta e} & Y & \to & 0 \\
\downarrow{\bar{e}} & & \downarrow{e} & & \downarrow{e} & & \downarrow{e} & & \\
0 & \to & \ker(\beta) & \to & T & \xrightarrow{\beta} & Y & \to & 0.
\end{array}
\]

In view of Lemma 2.3, the down-left square is a pushout–pullback diagram. It follows that the left column is also a short exact sequence and so \(\ker(\beta e) \in \mathcal{Y}\). Similarly, we get the following commutative diagram

\[
\begin{array}{ccccccccc}
\ker(e') & \xrightarrow{\cong} & N & \downarrow{e'} & \downarrow{e'} & \downarrow{e'} & \downarrow{e'} & & \\
0 & \to & \ker(ge) & \to & Q & \xrightarrow{ge} & \text{Hom}_A(M, P) & \to & 0 \\
\downarrow{e} & & \downarrow{e} & & \downarrow{e} & & \downarrow{e} & & \\
0 & \to & \ker(g) & \to & T & \xrightarrow{g} & \text{Hom}_A(M, P) & \to & 0
\end{array}
\]

and so \(\ker(ge) \in \mathcal{Y}\). Finally, putting these facts together, we obtain the following commutative diagram:
and it induces the following short exact sequence in $\mathcal{E}(\mathcal{X}, M, \mathcal{Y})$:

$$0 \rightarrow (K, \text{Ker}(\beta e)) \rightarrow (P, Q) \rightarrow (X, Y) \rightarrow 0.$$ 

By Corollary 3.3, we have $(P, Q)_{ge}$ is a projective object of $\mathcal{E}(\mathcal{X}, M, \mathcal{Y})$, and thus, our proof is complete. □

Let $\mathcal{C}$ be an abelian category which has enough injective objects. A full subcategory $\mathcal{D}$ of $\mathcal{C}$ is a co-resolving subcategory of $\mathcal{C}$ or just co-resolving if $\mathcal{I}(\mathcal{C}) \subseteq \mathcal{D}$ and $\mathcal{D}$ is closed under direct summands, extensions and the cokernels of injective homomorphisms. We turn to consider when $\mathcal{E}(\mathcal{X}, M, \mathcal{Y})$ is a co-resolving subcategory of $\text{Rep}_h(A, M, B)$.

**Theorem 3.5.** The following statements are equivalent:

1. $\mathcal{E}(\mathcal{X}, M, \mathcal{Y})$ is a co-resolving subcategory of $\text{Rep}_h(A, M, B)$;
2. Both $\mathcal{X}$ and $\mathcal{Y}$ are co-resolving and $\text{Ext}^1_A(M, \mathcal{X}) = 0$.

**Proof.** Assume that $\mathcal{E}(\mathcal{X}, M, \mathcal{Y})$ is a co-resolving subcategory of $\text{Rep}_h(A, M, B)$. Let $X \in \mathcal{X}$. Then, we have a short exact sequence

$$0 \rightarrow X \rightarrow E \rightarrow N \rightarrow 0,$$

where $E$ is an injective $A$-module. This gives rise to an injective homomorphism

$$(x, \text{Hom}(M, x)) : (X, \text{Hom}_A(M, X)) \rightarrow (E, \text{Hom}_A(M, E)).$$

Write $\text{Coker}(x, \text{Hom}(M, x))$ as $(N', Y)_f$. Then, $(N', Y)_f \in \mathcal{E}(\mathcal{X}, M, \mathcal{Y})$ and the following sequence

$$0 \rightarrow (X, \text{Hom}_A(M, X)) \rightarrow (E, \text{Hom}_A(M, E)) \rightarrow (N', Y)_f \rightarrow 0$$

is exact in $\text{Rep}_h(A, M, B)$. From this it follows that $N' \cong N$ and

$$0 \rightarrow \text{Hom}_A(M, X) \rightarrow \text{Hom}_A(M, E) \rightarrow Y \rightarrow 0$$

is a short exact sequence in $B$-Mod. In addition, we have the following commutative diagram:
In view of the Snake Lemma, it follows that \( \text{Ker}(f) = 0 \) and so \( f \) is an isomorphism. Hence, the second row can be completed into a short exact sequence in \( B\text{-Mod} \) and this implies that \( \text{Ext}^1_A(M, X) = 0 \) by the Long Exact Sequence Theorem.

Since \( \text{Ext}^1_A(M, X) = 0 \), \( \mathfrak{p} \) is an exact functor and so both \( \mathcal{X} \) and \( \mathcal{Y} \) is closed under extensions. It is clear that both \( \mathcal{X} \) and \( \mathcal{Y} \) is closed under direct summands. Finally, the exactness of \( \mathfrak{p} \) and \( \mathfrak{q} \) implies both \( \mathcal{X} \) and \( \mathcal{Y} \) is closed under the cokernels of injective homomorphisms and Theorem 1.3 implies that \( \mathcal{X} \) and \( \mathcal{Y} \) contains all the injective \( A \)-modules and injective \( B \)-modules, respectively.

Conversely, assume that both \( \mathcal{X} \) and \( \mathcal{Y} \) are co-resolving and \( \text{Ext}^1_A(M, X) = 0 \). It is clear that \( E(X, M, Y) \) contains all injective objects of \( \text{Rep}_h(A, M, B) \) by Theorem 1.3 and is closed under direct summands and extensions. Let

\[
0 \to (X, Y) \xrightarrow{(x, y)} (X', Y') \xrightarrow{f'} (U, V) \xrightarrow{g} 0
\]

be a short exact sequence in \( \text{Rep}_h(A, M, B) \) such that its first and second nonzero terms belong to \( E(X, M, Y) \). Then, \( U \in \mathcal{X} \). In addition, we have the following commutative diagram:

\[
\begin{array}{ccc}
0 & \to & Y \\
& \downarrow f & \downarrow \beta \\
0 & \to & \text{Hom}_A(M, X) \to \text{Hom}_A(M, X') \to \text{Hom}_A(M, U) \to 0.
\end{array}
\]

Note that both rows are short exact sequences. In view of the Snake Lemma, we have \( g \) is surjective, and the following sequence

\[
0 \to \text{Ker}(f) \to \text{Ker}(f') \to \text{Ker}(g) \to 0
\]

is exact. Hence, \( \text{Ker}(g) \in \mathcal{Y} \) and this implies \( (U, V) \in E(X, M, Y) \), completing the proof.

In view of Theorem 3.5, we call \( \mathcal{X} \) to be \( M^+\)-co-resolving provided that \( \mathcal{X} \) is a co-resolving subcategory of \( A\text{-Mod} \) such that \( \text{Ext}^i_A(M, X) = 0 \) for any \( X \in \mathcal{X} \). The following result shows that there are a unique largest \( M^+\)-co-resolving subcategory in \( A\text{-Mod} \) and a unique largest co-resolving subcategory in \( \text{Rep}_h(A, M, B) \) of the form \( E(X, M, Y) \).

**Proposition 3.6.** Set \( \mathbb{X} := \{X \in A \text{-Mod} | \text{Ext}^i_A(M, X) = 0, \forall i \geq 1\} \).

1. \( \mathbb{X} \) is an \( M^+\)-co-resolving subcategory.
2. Any \( M^+\)-co-resolving subcategory of \( A\text{-Mod} \) is included in \( \mathbb{X} \).
3. \( E(\mathbb{X}, M, B) \) is the largest co-resolving subcategory of \( \text{Rep}_h(A, M, B) \) of the form \( E(X, M, Y) \).
4. \( E(A, M, B) \) is a co-resolving subcategory of \( \text{Rep}_h(A, M, B) \) if and only if \( M \) is a projective \( A \)-module.

**Proof.**

1. That \( \mathbb{X} \) contains all injective \( A \)-module follows directly from the definition of the functor \( \text{Ext}^i_A(M, -) \). Since \( \text{Ext}^i_A(M, -) \) commutes with products and particularly, finite direct sums,
$\mathcal{X}$ is closed under direct summands. Finally, that $\mathcal{X}$ is closed under extensions and the co-kernels of injective homomorphisms follows from the Long Exact Sequence Theorem.

2. Let $\mathcal{X}$ be an $M\mathcal{P}$-co-resolving subcategory of $A$-Mod. Take $X \in \mathcal{X}$ and assume that

$$0 \to X \to E_0 \xrightarrow{f_0} E_1 \xrightarrow{f_1} E_2 \xrightarrow{f_2} E_3 \to \cdots$$

is an injective resolution of $X$. Since $\mathcal{X}$ is co-resolving, it follows that $X_i := \text{Im}(f_i) \in \mathcal{X}$ for $i = 0, 1, 2, \ldots$. Hence,

$$\text{Ext}^1_A(M, X) \cong \text{Ext}^{-1}_A(M, X_0) \cong \cdots \cong \text{Ext}^1_A(M, X_{i-2}) = 0$$

for all $i \geq 1$, and so $\mathcal{X} \subseteq \mathcal{X}$, as desired.

3. It is clear from (2).

4. By (3), $\mathcal{E}(A, M, B)$ is co-resolving if and only if $\mathcal{X} = A - \text{Mod}$. The latter means $\text{Ext}^1_A(M, X) = 0$ for any $X \in A - \text{Mod}$ and for $i \geq 1$, which is equivalent to saying that $M$ is a projective $A$-module.

Theorem 3.7. Suppose $\text{Hom}_A(M, \mathcal{X}) \subseteq \mathcal{Y}$ and that $\mathcal{E}(\mathcal{X}, M, \mathcal{Y})$ is a co-resolving subcategory of $\text{Rep}_A(A, M, B)$. Then, the following statements are equivalent:

1. $\mathcal{E}(\mathcal{X}, M, \mathcal{Y})$ is a Frobenius category;
2. Both $\mathcal{X}$ and $\mathcal{Y}$ are Frobenius category and $\text{Hom}_A(M, \mathcal{I}(\mathcal{X})) \subseteq \mathcal{I}(\mathcal{Y})$;
3. Both $\mathcal{X}$ and $\mathcal{Y}$ are Frobenius category and $\text{Hom}_A(M, \mathcal{P}(\mathcal{X})) \subseteq \mathcal{P}(\mathcal{Y})$.

Proof. We first observe the following fact: if $\mathcal{C}$ is a co-resolving subcategory of $R$-Mod, where $R$ is a ring, then, the injective objects of $\mathcal{C}$ are the same as those of $R$-Mod. To see this, let $U$ be an injective object of $\mathcal{C}$. Then, there is a short exact sequence in $\mathcal{C} : 0 \to U \to V \to N \to 0$ such that $V$ is an injective object of $R$-Mod. It is not hard to see that this sequence splits and so $U$ is a direct summand of $V$. From this, it follows that $U$ is an injective object of $R$-Mod. The converse is trivial.

(2) $\iff$ (3) It is clear from the definition of a Frobenius category.

In the rest part of this proof, we write $\mathcal{E}$ for $\mathcal{E}(\mathcal{X}, M, \mathcal{Y})$.

(1) $\implies$ (2) If $Y \in \mathcal{I}(\mathcal{Y})$, then, $(0, Y)_0 \in \mathcal{I}(\mathcal{E})$ by Theorem 1.3. Since $\mathcal{I}(\mathcal{E}) = \mathcal{P}(\mathcal{E})$, it follows that $(0, Y)_0 \in \mathcal{P}(\mathcal{E})$ and this implies $Y \in \mathcal{P}(\mathcal{Y})$ and so $\mathcal{I}(\mathcal{Y}) \subseteq \mathcal{P}(\mathcal{Y})$ by Corollary 3.3. Similarly, we have $\mathcal{I}(\mathcal{Y}) \supseteq \mathcal{P}(\mathcal{Y})$.

Let $X \in \mathcal{I}(\mathcal{X})$. Then, $(X, \text{Hom}_A(M, X))_1 \in \mathcal{I}(\mathcal{E})$ by Theorem 1.3. It follows that $(X, \text{Hom}_A(M, X))_0 \in \mathcal{P}(\mathcal{E})$, and so $X \in \mathcal{P}(\mathcal{X})$ and $\text{Hom}_A(M, X) \subseteq \mathcal{P}(\mathcal{Y})$. Hence, $\text{Hom}_A(M, \mathcal{I}(\mathcal{X})) \subseteq \mathcal{I}(\mathcal{Y})$ and $\mathcal{I}(\mathcal{X}) \subseteq \mathcal{P}(\mathcal{Y})$. Conversely, we take $X \in \mathcal{P}(\mathcal{X})$. Since $\mathcal{Y}$ has enough projective objects by Theorem 3.4, there is a short exact sequence $0 \to K \to P \xrightarrow{\phi} \text{Hom}_A(M, X) \to 0$ in $\mathcal{Y}$ such that $P \in \mathcal{P}(\mathcal{Y})$. It follows that $(X, P)_0 \in \mathcal{P}(\mathcal{E}) = \mathcal{I}(\mathcal{E})$ and so $X \in \mathcal{P}(\mathcal{X})$. These facts together with Theorem 3.4 imply both $\mathcal{X}$ and $\mathcal{Y}$ are Frobenius categories.

(2) $\implies$ (1) If $(X, Y)_0 \in \mathcal{P}(\mathcal{E})$, then, $X \in \mathcal{P}(\mathcal{X}) = \mathcal{P}(\mathcal{X})$ and $Y \in \mathcal{P}(\mathcal{Y}) = \mathcal{P}(\mathcal{Y})$ by Corollary 3.3. Since $\text{Hom}_A(M, X) \subseteq \mathcal{P}(\mathcal{Y})$, the short exact sequence $0 \to \text{Ker}(\phi) \to Y \xrightarrow{\phi} \text{Hom}_A(M, X) \to 0$ splits and it follows that $\ker(\phi) \subseteq \mathcal{I}(\mathcal{Y})$. Hence, $(X, Y)_0 \in \mathcal{I}(\mathcal{E})$.

Conversely, if $(X, Y)_0 \in \mathcal{I}(\mathcal{E})$, then, $\ker(\phi) \subseteq \mathcal{I}(\mathcal{Y})$ and $X \in \mathcal{I}(\mathcal{X})$. From this it follows that $0 \to \ker(\phi) \to Y \xrightarrow{\phi} \text{Hom}_A(M, X) \to 0$ splits, and thus, $Y = \ker(\phi) \oplus \text{Hom}_A(M, X) \subseteq \mathcal{I}(\mathcal{Y}) = \mathcal{P}(\mathcal{Y})$. Consequently $(X, Y)_0 \in \mathcal{P}(\mathcal{E})$ and so $\mathcal{I}(\mathcal{E}) = \mathcal{P}(\mathcal{E})$. Now, this fact together with Theorem 3.4 implies $\mathcal{E}$ is a Frobenius category.

In the final part of this section, we always assume $\mathcal{E}(\mathcal{X}, M, \mathcal{Y})$ is a co-resolving subcategory of $\text{Rep}_A(A, M, B)$ and that $\text{Hom}_A(M, \mathcal{X}) \subseteq \mathcal{Y}$. Under these assumptions, if $(X, Y)_0 \in \mathcal{E}(\mathcal{X}, M, \mathcal{Y})$, then...
then, $Y \in \mathcal{Y}$. Hence, we now regard $Q$ as a functor from $\mathcal{E}(\mathcal{X}, M, \mathcal{Y})$ to $\mathcal{Y}$. We establish the following recollement of triangulated categories when $\mathcal{E}(\mathcal{X}, M, \mathcal{Y})$ is a Frobenius category in a similar way as in [15, theorem 1.3].

![Recollement Diagram](image)

**Figure 2.** A recollement diagram.

**Lemma 3.8.** $\langle Q, q \rangle$, $\langle q, Q \rangle$ and $\langle P, p \rangle$ are adjoint pairs.

**Proof.** Let $Y_1 \in \mathcal{Y}$ and $(X, Y)_\varphi \in \mathcal{E}(\mathcal{X}, M, \mathcal{Y})$. Then,

\[
\text{Hom}((X, Y)_\varphi, q(Y_1)) \cong \text{Hom}((X, Y)_\varphi, (0, Y_1)_0) \\
\cong \text{Hom}(Y, Y_1) \cong \text{Hom}(Q(X, Y)_\varphi, Y_1)
\]

and

\[
\text{Hom}(q(Y_1), (X, Y)_\varphi) \cong \text{Hom}((0, Y_1)_0, (X, Y)_\varphi) \\
= \{g \in \text{Hom}(Y_1, Y) | \varphi g = 0\} \cong \text{Hom}(Y_1, \text{Ker}(\varphi)) \\
= \text{Hom}(Y_1, Q(X, Y)_\varphi)
\]

This shows that $\langle Q, q \rangle$ and $\langle q, Q \rangle$ are adjoint pairs. Similarly, $\langle P, p \rangle$ is an adjoint pair.

**Lemma 3.9.** If $\mathcal{E}(\mathcal{X}, M, \mathcal{Y})$ is a Frobenius category, then, there is a triangulated equivalence:

\[
\text{St} \mathcal{E}(\mathcal{X}, M, \mathcal{Y})/\text{q}(\text{St} \mathcal{Y}) \cong \text{St} \mathcal{X}.
\]

**Proof.** Denote $\mathcal{E}(\mathcal{X}, M, \mathcal{Y})$ by $\mathcal{E}$. We claim that $(X, Y)_\varphi$ and $(0, Y)_0$ are isomorphic in $\text{St} \mathcal{E}$ for any $(X, Y)_\varphi \in \mathcal{E}$ with $X \in \mathcal{T}(\mathcal{X})$. Let $0 \to \text{Ker}(\varphi) \to K \to N \to 0$ be a short exact sequence in $\mathcal{Y}$ with $K \in \mathcal{T}(\mathcal{Y})$. Then, we obtain the following commutative diagram:

\[
\begin{array}{ccc}
0 & \to & \text{Ker}(\varphi) & \to & Y \\
& & & & \uparrow \varphi \\
0 & \to & K & \to & Q \\
& & & & \downarrow \varphi_1 \\
& & & & \text{Hom}_A(M, X) \\
\end{array}
\]

where the left square is a pushout. This gives rise to a short exact sequence in $\mathcal{E}$:

\[
0 \to (X, Y)_\varphi \to (X, Q)_\varphi \to (0, N)_0 \to 0.
\]

The fact that $(X, Q)_\varphi \in \mathcal{T}(\mathcal{E})$ together with the short exact sequence $0 \to (0, Y)_0 \to (0, Q)_0 \to (0, N)_0 \to 0$ implies $(X, Y)_\varphi \cong (0, Y)_0$ in $\text{St} \mathcal{E}$, as claimed.

By Lemma 2.1, the functor $\mathcal{P} : \mathcal{E}(\mathcal{X}, M, \mathcal{Y}) \to \mathcal{X}$ induces a triangulated functor $\mathcal{P} : \text{St} \mathcal{E}(\mathcal{X}, M, \mathcal{Y}) \to \text{St} \mathcal{X}$, whose Kernel is $\tilde{\text{q}}(\text{St} \mathcal{Y})$ by the claim above. \(\Box\)

For a subcategory $\mathcal{S}$ of an additive category $\mathcal{T}$, we set $\mathcal{S}^\perp := \{X \in \mathcal{T} | \text{Hom}(S, X) = 0, \forall S \in \mathcal{S}\}$ and $\perp \mathcal{S} := \{X \in \mathcal{T} | \text{Hom}(X, S) = 0, \forall S \in \mathcal{S}\}$. The following result is known essentially, see [4, 5].
Lemma 3.10. Let $S$ be a thick triangulated subcategory of a triangulated category $T$. Then, the following are equivalent:

1. The quotient functor $Q : T \to T/S$ has a right (resp. left) adjoint;
2. $(S, S^\perp)$ (resp. $(S^\perp, S)$) is a torsion pair of $T$;
3. The embedding functor $i : S \to T$ has a (resp. left) right adjoint.

The proof of Lemma 3.10 is given in the Appendix. We also need the following easy result, see e.g. [14].

Lemma 3.11. Let $(F, G)$ be an adjoint pair, where $F : C \to D$ and $G : D \to C$ are additive functors. If $\mathcal{X}$ and $\mathcal{Y}$ are additive subcategories of $C$ and $D$, respectively, such that $F\mathcal{X} \subseteq \mathcal{Y}$ and $G\mathcal{Y} \subseteq \mathcal{X}$, then, $(F, G)$ with $\tilde{F} : \mathcal{C}/\mathcal{X} \to \mathcal{D}/\mathcal{Y}$ and $\tilde{G} : \mathcal{D}/\mathcal{Y} \to \mathcal{C}/\mathcal{X}$ is also an adjoint pair.

Combining the facts above we obtain:

Theorem 3.12. With the same assumptions as well as one of the equivalent statements of Theorem 3.7, the diagram in (Figure 2) is a recollement of triangulated categories.

Proof. In view of Lemmas 2.1 and 3.11, the functors $\tilde{Q}, \tilde{q}, \bar{Q}$ and $\bar{p}$ are all triangulated functors and $(\tilde{Q}, \tilde{q}), (\bar{q}, \bar{Q}), (\bar{p}, \bar{P})$ are adjoint pairs. From these, we obtain the left side of (Figure 2), and it can be completed into (Figure 2) by Lemmas 3.9 and 3.10.

4. The properties of $\mathcal{M}(\mathcal{X}, M, \mathcal{Y})$

The theory of $\mathcal{M}(\mathcal{X}, M, \mathcal{Y})$ is perfectly dual to that of $\mathcal{E}(\mathcal{X}, M, \mathcal{Y})$, whether on their results or on their proofs. Hence, we will only present the results on $\mathcal{M}(\mathcal{X}, M, \mathcal{Y})$, but omitting all the proofs in this section.

Lemma 4.1. If both $\mathcal{X}$ and $\mathcal{Y}$ are closed under extensions, then, $\mathcal{M}(\mathcal{X}, M, \mathcal{Y})$ is closed under extensions.

Proposition 4.2. Suppose that $\mathcal{M}(\mathcal{X}, M, \mathcal{Y})$ is closed under extensions and that $\text{Tor}_1^B(M, \mathcal{Y}) = 0$. If either $\mathcal{X}$ has enough projective objects or $M \otimes_B \mathcal{Y} \subseteq \mathcal{X}$, then, for any $[U/V]_\phi \in \mathcal{M}(\mathcal{X}, M, \mathcal{Y})$, $[U/V]_\phi$ is an injective object of $\mathcal{M}(\mathcal{X}, M, \mathcal{Y})$ if and only if $U$ is an injective object of $\mathcal{X}$ and $V$ is $\mathcal{Y}$-injective.

Proposition 4.3. Suppose that $\mathcal{M}(\mathcal{X}, M, \mathcal{Y})$ is closed under extensions and assume further that $\text{Tor}_1^B(M, \mathcal{Y}) = 0$ and $M \otimes_B \mathcal{Y} \subseteq \mathcal{X}$. Then, $\mathcal{M}(\mathcal{X}, M, \mathcal{Y})$ has enough injective objects if and only if both $\mathcal{X}$ and $\mathcal{Y}$ have enough injective objects.

Let $C$ be an abelian category which has enough projective objects. A full subcategory $\mathcal{D}$ of $C$ is a resolving subcategory of $C$ or just resolving if $P(C) \subseteq \mathcal{D}$ and $\mathcal{D}$ is closed under direct summands, extensions and the kernels of surjective homomorphisms.

Theorem 4.4. The following statements are equivalent:

1. $\mathcal{M}(\mathcal{X}, M, \mathcal{Y})$ is a resolving subcategory of $\text{Rep}(A, M, B)$;
2. Both $\mathcal{X}$ and $\mathcal{Y}$ are resolving and $\text{Tor}_1^B(M, \mathcal{Y}) = 0$.

A full subcategory $\mathcal{Y}$ of $B-\text{Mod}$ is called $M^\perp$-resolving if it is resolving and $\text{Tor}_1^B(M, Y) = 0$ for any $Y \in \mathcal{Y}$.
Proposition 4.5. Set $\mathcal{Y} := \{ Y \in B - \text{Mod} | \text{Tor}_i^B(M, Y) = 0, \forall i \geq 1 \}$. Then,

1. $\mathcal{Y}$ is an $M^\perp$-resolving subcategory;
2. Any $M^\perp$-resolving subcategory of $B$-Mod is included in $\mathcal{Y}$;
3. $\mathcal{M}(A, M, \mathcal{Y})$ is the largest resolving subcategory of $\text{Rep}(A, M, B)$ of the form $\mathcal{M}(\mathcal{X}, M, \mathcal{Y})$;
4. $\mathcal{M}(A, M, B)$ is a resolving subcategory of $\text{Rep}(A, M, B)$ if and only if $M$ is a flat $B$-module.

Theorem 4.6. Suppose that $\mathcal{M}(\mathcal{X}, M, \mathcal{Y})$ is a resolving subcategory of $\text{Rep}(A, M, B)$ and that $M \otimes_B \mathcal{Y} \subseteq \mathcal{X}$. Then, the following statements are equivalent:

1. $\mathcal{M}(\mathcal{X}, M, \mathcal{Y})$ is a Frobenius category;
2. Both $\mathcal{X}$ and $\mathcal{Y}$ are Frobenius categories and $M \otimes_B \mathcal{I}(\mathcal{Y}) \subseteq \mathcal{I}(\mathcal{X})$;
3. Both $\mathcal{X}$ and $\mathcal{Y}$ are Frobenius categories and $M \otimes_B \mathcal{P}(\mathcal{Y}) \subseteq \mathcal{P}(\mathcal{X})$.

Theorem 4.7. Under the same assumptions together with one of the equivalent statements of Theorem 4.6, we have the following recollement of triangulated categories.

Here, the functors $\overline{P}_M$, $\overline{P}_M$, $\overline{P}_M$, $\overline{Q}_M$ are induced by the following ones, respectively:

- $\overline{P}_M : \mathcal{M}(\mathcal{X}, M, \mathcal{Y}) \to \mathcal{X}$ such that $X \mapsto \text{Coker}(\varphi)$;
- $\overline{P}_M : \mathcal{X} \to \mathcal{M}(\mathcal{X}, M, \mathcal{Y})$, $X \mapsto \begin{bmatrix} X \\ 0 \end{bmatrix}$;
- $\overline{P}_M : \mathcal{M}(\mathcal{X}, M, \mathcal{Y}) \to \mathcal{X}$ such that $X \mapsto X$;
- $\overline{Q}_M : \mathcal{M}(\mathcal{X}, M, \mathcal{Y}) \to \mathcal{Y}$ such that $Y \mapsto Y$;
- $\overline{Q}_M : \mathcal{Y} \to \mathcal{M}(\mathcal{X}, M, \mathcal{Y})$, $Y \mapsto \begin{bmatrix} M \otimes_B Y \\ Y \end{bmatrix}$.

5. Applications to comma categories

Assume that $\mathcal{A}$ and $\mathcal{B}$ are abelian categories and let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ be additive functors. The comma category $(\mathcal{B}, F, A)$ is defined as follows:

- $\text{obj}(\mathcal{B}, F, A)$ : triples $(B, FA)$, where $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $f : B \to FA \in \text{Mor}\mathcal{B}$;
- $\text{Mor}(\mathcal{B}, F, A)$: pairs $(x, \beta) : (B, FA)_f \to (B', FA')_{f'}$, where $x \in \text{Hom}_\mathcal{A}(A, A')$ and $\beta \in \text{Hom}_\mathcal{B}(B, B')$ and the following diagram is commutative:
Dually, the comma category \((GB,A)\) is defined as follows:

\[
\text{obj}(GB,A) : \text{triples } (GB,A)_{f}, \text{ where } A \in A, B \in B \text{ and } f : GB \to A \in \text{Mor}A;
\]

\[
\text{Mor}(GB,A) : \text{pairs } (\alpha, \beta) : (GB,A)_{f} \to (GB',A')_{f'}, \text{ where } \alpha \in \text{Hom}_{A}(A,A') \text{ and } \beta \in \text{Hom}_{B}(B,B') \text{ and the following diagram is commutative:}
\]

\[
\begin{align*}
GB & \xrightarrow{f} A \\
\downarrow \alpha & \quad \downarrow \beta \\
GB' & \xrightarrow{f'} A'.
\end{align*}
\]

The following result is well-known, see e.g. [6].

**Lemma 5.1.** (1) If \(F\) is left exact, then, \((B,F,A)\) is an abelian category.

If \(G\) is right exact, then, \((GB,A)\) is an abelian category.

In general it is difficult to identify projective objects or injective objects of a comma category. In this section, we point out that some proofs in previous sections can be transplanted into the case of comma categories and this allow us to classify projective objects of \(E(Y,F,X)\) and injective objects of \(M(G,Y,X)\) when \(F\) and \(G\) are exact functors, where \(E(Y,F,X)\) and \(M(G,Y,X)\) are exact subcategories of comma categories which are defined below.

**Definition 5.2.** Let \(X\) (resp. \(Y\)) be full subcategories of \(A\) (resp. \(B\)) closed under isomorphisms and containing 0. By definition, \(E(Y,F,X)\) is a full subcategories of \((B,F,A)\) whose objects are \((Y,FX)_{f}\) such that \(f\) is surjective, \(X \in X\) and \(\text{Ker}(f) \in Y\); and \(M(G,Y,X)\) is a full subcategories of \((GB,A)\) whose objects are \((GY,X)_{f}\) such that \(f\) is injective, \(Y \in Y\) and \(\text{Coker}(f) \in X\).

We now present the results of this section.

**Proposition 5.3.** Suppose that \(F\) is left exact and both \(X\) and \(Y\) are closed under extensions.

1. \(E(Y,F,X)\) is closed under extensions;
2. If assume further that \(F\) is exact and \(FX \subseteq Y\), then,
   i. a triple \((Y,FX)_{f}\) is a projective object of \(E(Y,F,X)\) if and only if \(X\) is a projective object of \(X\) and \(Y\) is a projective object of \(Y\);
   ii. \(E(Y,F,X)\) has enough projective objects if and only if both \(X\) and \(Y\) have enough projective objects.

**Proof.** The proofs of (1), 2(i) and 2(ii) are essentially the same as those of Proposition 3.1, Corollary 4.2 and Theorem 3.4, respectively. \(\square\)

Dually we have the following result.

**Proposition 5.4.** Suppose that \(G\) is right exact and both \(X\) and \(Y\) are closed under extensions.

1. \(M(G,Y,X)\) is closed under extensions;
2. If assume further that \(G\) is exact and \(GY \subseteq X\), then,


i. a triple $(GY, X)_f$ is a projective object of $\mathcal{M}(GY, X)$ if and only if $X$ is an injective object of $X$ and $Y$ is an injective object of $Y$;

ii. $\mathcal{M}(GY, X)$ has enough injective objects if and only if both $X$ and $Y$ have enough injective objects.

**Appendix**

In this appendix, we give a Proof of Lemma 3.10. Some preparations are needed.

**Lemma A1.** Let $S$ be a thick triangulated subcategory of a triangulated category $T$ and $Q : T \to T/S$ the quotient functor. If $Q$ has a right (resp. left) adjoint $R$ (resp. $L$), then, $R$ (resp. $L$) is fully faithful.

**Proof.** This follows from [11, lemmas 1.2.6, 3.2.1 and proposition 3.2.2].

**Lemma A2.** Let $Q : T \to T/S$ be the quotient functor.

1. If $T \in T$ and $X \in S^\perp$, then, $\text{Hom}_T(T, X) \cong \text{Hom}_{T/S}(Q(T), Q(X))$.  
2. If $T \in T$ and $X \in S^\perp$, then, $\text{Hom}_T(X, T) \cong \text{Hom}_{T/S}(Q(X), Q(T))$.

**Proof.** This follows from [11, lemmas 1.2.6, 3.2.1 and proposition 3.2.2].

We now prove the right case of Lemma 3.10. (The proof of the left case is dual, and thus, omitted.)

**Proof.** (1) $\Rightarrow$ (2) Let $R$ be the right adjoint to $Q$ and $\eta : 1_T \to RQ$ the unit of $(Q, R)$. Then, for any $T \in T$, the morphism $T^{\eta}_{RQ(T)}$ fits into an exact triangle of $T$:

$$S \to T^{\eta}_{RQ(T)} \to S[1].$$

This gives rise to an exact triangle $Q(S) \to Q(T) \to Q(T)[1]$, and this implies $Q(T)[1] = S[1]$. Hence, $(S, S^\perp)$ is a torsion pair.

(2) $\Rightarrow$ (1) Let $Q' : S^\perp \to T/S$ be the restriction functor of $Q$. Then, $Q'$ is fully faithful by Lemma A2. Since $(S, S^\perp)$ is a torsion pair, there is for any $T \in T$ an exact triangle $S \to T \to Z \to Y[1]$ such that $S \in S$ and $Z \in S^\perp$. Note that $Q(T) \cong Q(Z)$, it follows that $Q'$ is dense (i.e. there is for any $X \in T/S$ an object $Z \in S^\perp$ such that $X \cong Q'(Z)$), and so it is an equivalence.

Now, set $R = jF$, where $j : S^\perp \to T$ is the embedding functor and $F$ is the quasi-inverse to $Q'$. Then, for any $T \in T$ and $X \in T/S$, we have

$$\text{Hom}_T(T, R(X)) = \text{Hom}_T(T, F(X))$$

$$\cong \text{Hom}_{T/S}(Q(T), QF(X)) = \text{Hom}_{T/S}(Q(T), X).$$

Here, the second isomorphism follows from Lemma A2. Consequently, $R$ is the right adjoint to $Q$.

(2) $\Rightarrow$ (3). For any $T \in T$, let $u(T) \to v(T) \to u(T)[1]$ be an exact triangle such that $u(T) \in S$ and $v(T) \in S^\perp$.

Given $S, T \in T$. Note that $\text{Hom}(u(T), \phi_S) : \text{Hom}_T(u(T), u(S)) \to \text{Hom}_T(u(T), S)$ is an isomorphism, there exists for any morphism $f : T \to S$ a unique morphism $u(f) : u(T) \to u(S)$ such that the following diagram is commutative:

\[
\begin{array}{ccc}
  u(T) & \xrightarrow{\phi_T} & T \\
  \downarrow{u(f)} & & \downarrow{f} \\
  u(S) & \xrightarrow{\phi_S} & S \\
\end{array}
\]

This implies that $u(T)$ is uniquely determined up to isomorphism by $T$ and that $u$ is a functor from $T$ to $S$.

By an easy check, $u$ is the right adjoint to $i$. 

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