Research Article

A Posteriori Error Estimates for Hughes Stabilized SUPG Technique and Adaptive Refinement for a Convection-Diffusion Problem

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The motive of the present work is to propose an adaptive numerical technique for singularly perturbed convection-diffusion problem in two dimensions. It has been observed that for small singular perturbation parameter, the problem under consideration displays sharp interior or boundary layers in the solution which cannot be captured by standard numerical techniques. In the present work, Hughes stabilization strategy along with the streamline upwind/Petrov-Galerkin (SUPG) method has been proposed to capture these boundary layers. Reliable a posteriori error estimates in energy norm on anisotropic meshes have been developed for the proposed scheme. But these estimates prove to be dependent on the singular perturbation parameter. Therefore, to overcome the difficulty of oscillations in the solution, an efficient adaptive mesh refinement algorithm has been proposed. Numerical experiments have been performed to test the efficiency of the proposed algorithm.

1. Introduction

Singularly perturbed problems occur frequently in various branches of applied science and engineering, e.g., fluid dynamics, aerodynamics, oceanography, quantum mechanics, chemical reactor theory, reaction-diffusion processes, and radiating flows. In general, it has been observed that singularly perturbed problems exhibit singularities as the singular perturbation parameter $\varepsilon \to 0$. Therefore, it becomes essential to implement some robust numerical technique to capture these singularities. In literature, there exist various numerical techniques to handle these singularities. Adaptive mesh refinement techniques are one of such techniques. Very few researchers have proposed adaptive refinement strategies for singularly perturbed convection-diffusion problems. Generally, adaptive refinement techniques are based on two types of error estimates, namely, a priori error estimates and a posteriori error estimates. Nicaise [1] developed a posteriori residual error estimates for convection-diffusion-reaction problems using some cell-centered finite volume methods. Based on a posteriori error estimates, the author proposed an adaptive algorithm. John [2, 3] did numerical study of various a posteriori error estimates and indicators for convection-diffusion problems. On the basis of error estimates, the author proposed numerical solution of singularly perturbed convection-dominated problems on adaptive refined grid. Repin and Nicaise [4] derived a posteriori error estimates for linear convection-diffusion-reaction problems using functional arguments. Verfurth [5] derived a posteriori error estimates for convection-dominated stationary convection-diffusion equation using locally refined isotropic meshes. Zhao et al. [6] proposed adaptive numerical technique for convection-diffusion equations based on semir robust residual a posteriori error estimates for lower order nonconforming finite element approximations of streamline diffusion method.

From literature, we know that the classical finite element methods [7] fail to provide satisfactory results for small values of singular perturbation parameter, i.e., when $\varepsilon \to 0$. It has also been observed that streamline upwind/Petrov-Galerkin (SUPG) method provides good approximate solution in the region where there is no sharp change in the solution but fails
in the small subregions of sharp boundary layers. It has been observed that occurrence of these nonphysical oscillations in the region of sharp boundary layers in the discrete solution of SUPG method is based on the fact that this scheme is not monotonicity preserving. To overcome this difficulty, in the present work, we have proposed Hughes stabilization strategy [8] along with the SUPG method. It involves suitable addition of one more term which is multiple of a function in the direction where spurious oscillations were seen in approximate SUPG solution. This additional term is added on the left-hand side of SUPG discretization of convection-diffusion problem. The a posteriori error estimates have been derived for the proposed scheme. Based on these estimates, an anisotropic mesh refinement strategy has been proposed for singularly perturbed problems.

The outline of the paper is as follows.

In Section 2, the continuous problem under consideration and its streamline upwind Petrov-Galerkin finite element approximation have been presented. In Section 3, some auxiliary tools which are required for deriving reliable error bounds have been presented. In Section 4, we have discussed residual-based a posteriori error estimates and derived error bounds on anisotropic meshes. An adaptive refinement algorithm based on derived a posteriori error estimates has been proposed in Section 5. Section 6 deals with some numerical experiments which have been performed to analyze the robustness and efficiency of the proposed adaptive refinement strategy. In the last section, conclusion has been presented.

2. Continuous Problem

Consider the following convection-diffusion equation in two dimensions:

\[-\nabla \cdot \epsilon \nabla u + a \cdot \nabla u + bu = f \text{ in } \Omega,\]

\[u = 0 \text{ on } \partial \Omega_D,\]

\[\frac{\partial u}{\partial n} = g \text{ on } \partial \Omega_N,\]

where \( \Omega \subset \mathbb{R}^2 \) is a bounded domain with Lipschitz-continuous boundary \( \partial \Omega = \partial \Omega_D \cup \partial \Omega_N \), \( \epsilon \) is a positive perturbation parameter, and \( a, b, f \) are sufficiently smooth. Here, \( \partial \Omega_D \) and \( \partial \Omega_N \) denote the Dirichlet and Neumann boundaries of the domain, respectively.

\( W^{1,\infty}(\Omega) \) and \( L^\infty(\Omega) \) represent the usual Sobolev and Lebesgue spaces, respectively. The notation \((\_\_\_)\) has been used for inner product \((\_\_,\_\_)_\Omega\).

Throughout the paper, we assume that \( -(1/2)\nabla a + b \geq b_0 \) for any open bounded subset \( K \subset \bar{\Omega} \), let \( H^1(K) \) be the standard Sobolev space. Further, we define

\[ V_0 = \{ v \in H^1(\Omega), \quad v = 0 \text{ on } \partial \Omega_D \}. \]

Let

\[ \| v \|_K^2 = \epsilon \| \nabla v \|_K^2 + b_0 \| v \|_K^2 \]

be energy norm on bounded subset \( K \subset \Omega \). The weak formulation of equations (1a), (1b) and (1c) is given by the following.

Find \( u \in H^1(\Omega) \) such that

\[ B(u, v) = (F, v) \]

where

\[ B(u, v) = \epsilon (\nabla u, \nabla v) + (a \cdot \nabla u, v) + (bu, v), \]

\[ (F, v) = (f, v) + (g, v)_{\partial \Omega_N} \quad \forall v \in V_0. \]

The existence and uniqueness of the solution of the above weak formulation (4) are guaranteed using Lax Milgram Lemma together with condition (1c). Let \( \Gamma_0 \) be the admissible and shape-regular triangulation of domain \( \Omega \) consisting of triangles. Let \( L \) be any two-dimensional element with edge \( E \). Let \( n_{LE} = (n_x, n_y) \) be unit outward normal vector to \( L \) along \( E \) (see Figure 1). Fixing one of the two normal vectors, let \( n_E \) be the normal vector for each edge \( E \).

It has been observed that the solution of singularly perturbed problem displays boundary layers if the Peclet number, as discussed below, is large. Define local mesh Peclet number as

\[ \text{Pe}_K = \frac{\| a \|_{\infty,K} h^K_{\min}}{2\epsilon}, \]

where \( h^K_{\min} \) is the minimal length of element \( K \) as defined in the next section.

Let \( V_h = \{ v_h \in H^1(\Omega); v_h|_K \in P_1(K) \} \), where \( P_1(K) \) is the space of all linear polynomials over the element \( K \) and \( V_0 = \{ v_h \in V_h: v_h|_{\partial \Omega_D} = 0 \} \). Next, we discuss the SUPG method along with the Hughes stabilization technique for approximating the solution of problem (1a), (1b) and (1c). The SUPG method [9] for problem (1a), (1b) and (1c) is defined as follows.

Find \( u_h \in V_h \) such that

\[ B_p(u_h, v_h) = (F, v_h)_{\partial \Omega_N} \]

where \( B_p(u_h, v_h) = B(u_h, v_h) + (R_h(u_h), \sigma_a \cdot \nabla v_h), \quad R_h(u) = -\epsilon \Delta_h u + a \cdot \nabla u + bu - f, \quad \sigma \) is nonnegative stabilization parameter, and \( B(u, v) \) and \( (F, v) \) are defined in (5).
finite element discretization of convection-diffusion equation where
\[
a_h = \begin{cases} 
\frac{(a \cdot \nabla u_h) \nabla u_h}{|\nabla u_h|^2}, & \text{if } |\nabla u_h| \neq 0, \\
0, & \text{if } |\nabla u_h| = 0,
\end{cases}
\]
and \( \sigma \) is a nonnegative stabilization parameter. This additional term increases the robustness of SUPG method in the boundary layer region by controlling oscillations. Using Hughes stabilization technique to SUPG finite element method, equation (1a), (1b) and (1c) is discretized as follows.

Find \( u_h \in V^h \) such that
\[
B_{\rho_K}(u_h, v_h) = \langle F, v_h \rangle \forall v_h \in V^h_0,
\]
where
\[
B_{\rho_K}(u_h, v_h) = B(u_h, v_h) + \langle R_h(u_h), \rho a \cdot \nabla v_h \rangle + \langle R_h(\sigma a_h \cdot \nabla v_h) \rangle
\]
and \( \rho_K \) be the stabilization parameter over each element \( K \). Ross et al. [7] showed that the approximate solution \( u_h \) obtained using SUPG finite element discretization exists and is unique provided stabilization parameter \( \rho_K \) is small and satisfies
\[
0 \leq \rho_K \leq \frac{1}{2} \min \left\{ b_0 \| b \|_{\infty, K}, \left( \frac{h_{\text{min}}^K}{\left( h_{\text{min}}^K \right)^2} e^{-\frac{1}{2} \nu^{-2}} \right) \right\},
\]
where \( h_{\text{min}}^K \) is the minimal length of element \( K \) and the constant \( \nu \) satisfies the inequality
\[
\| \nabla \nabla v_h \|_K \leq \nu \left( h_{\text{min}}^K \right)^{-1} \| \nabla v_h \|_K \forall v_h \in V^h_0.
\]

From inequality (12), it can easily be observed that \( \nu = 0 \) for piecewise linear functions in \( V^h_0 \). Therefore, the above bounds reduce to \( 0 \leq \rho_K \leq b_0 \| b \|_{\infty, K}^{-2} \). In order to simplify the calculations, we introduce the notation \( \epsilon \leq d \) which means that there exists a positive constant \( A \) independent of \( c, d, T_h \), and \( \epsilon \) such that \( \epsilon \leq Ad \). Further, we assume that
\[
\rho_K \leq h_{\text{min}}^K \| a \|_{\infty, K}^{-1} \forall K \in \Gamma_h.
\]

Also, for any mesh function \( v_h \in V^h_0 \), using (12) and scaling arguments, we can get
\[
\| \nabla v_h \|_K \leq \left( h_{\text{min}}^K \right)^{-1} \| v_h \|_K.
\]

Using energy norm def. (3),
\[
\| v_h \|_K \leq b_0^{1/2} \| v_h \|_K.
\]

Thus, we have
\[
\| \nabla v_h \|_K \leq \left( h_{\text{min}}^K \right)^{-1} b_0^{-1/2} \| v_h \|_K.
\]

Again, from energy norm, we have
\[
\| \nabla v_h \|_K \leq \epsilon^{-1/2} \| v_h \|_K.
\]

Using (16) and (17), we get
\[
\| \nabla v_h \|_K \leq \min \left\{ \left( h_{\text{min}}^K \right)^{-1} b_0^{-1/2}, \epsilon^{-1/2} \right\} \| v_h \|_K.
\]

3. Some Important Notations and Tools

Since singularly perturbed convection-diffusion problems exhibit sharp boundary layers when Peclet number becomes large or the singular perturbation parameter becomes smaller, in such situations, elements with large aspect ratio (anisotropic meshes) are recommended. In this section, we will discuss some important results on anisotropic meshes.

3.1. Notations. Consider an arbitrary triangle \( K \in \Gamma_h \) with \( Q_0Q_1 \) as the longest edge (see Figure 2). Denote two orthogonal vectors \( q_{i,k} \) with length \( h_{i,k} = |q_{i,k}|, i = 1, 2, \) where \( q_{i,k} \) is taken along the longest edge \( Q_iQ_j \). From Figure 2, it can be verified that \( h_{1,k} \geq h_{2,k} \). Define \( h_{\text{min}}^K = h_{2,k} \). These \( q_{i,k} \)'s correspond to two anisotropic directions. Further, define an orthogonal matrix \( C_K = (q_{1,k}, q_{2,k}) \in \mathbb{R}^{2 \times 2} \). Let \( a_K \) be the scaling factor defined as
\[
a_K = \min \left\{ b_0^{-1/2}, \epsilon^{-1/2} h_{\text{min}}^K \right\}.
\]

We represent triangles by \( K \) or \( K' \) or \( K_i \) and its edges by \( E \). Further define its height over edge \( E \) as
\[
h_{E,K} = 2 \cdot \frac{|K|}{|E|},
\]
where \( |K| \) represents the area of triangle \( K \). Let \( w_{E} \) be the bounded domain formed by using two triangles having common edge \( E \). Further, define \( w_{E} \) to be the domain consisting of triangle \( K \) and its edge neighboring triangles.
Let
\[
P_{w_k} = \max_{K} \frac{\nabla w_k \cdot \nabla P_{e_k}}{\varepsilon_{K}}\tag{21}
\]
be mesh Peclet number on the domain \(w_k\) where \(P_{e_k}\) is defined in (7). For an interior edge \(E = K_1 \cap K_2\), define parameters \(h_E = (h_{E,K_1} + h_{E,K_2})/2\), \(h_{E,\min} = (h_{K_1,\min} + h_{K_2,\min})/2\), and \(\alpha_E = (\alpha_{K_1} + \alpha_{K_2})/2\).

For boundary edge \(E \in \partial K \cap \partial \Omega\), we define \(h_E = h_{E,K}\), \(h_{E,\min} = h_{K,\min}^E\), and \(\alpha_E = \alpha_K\).

Since the mesh considered is assumed to be shape-regular and admissible, along with these requirements, we take
\[
h_{i,K}^E \sim h_{i,K}^E \forall K, K' \text{ with } K \cap K' \neq \emptyset, i = 1, 2\tag{22}
\]
and the number of triangles with node \(y_j\) is bounded uniformly.

### 3.2. Interpolation

In order to obtain reliable error upper bounds, we define a suitable matching function [11, 12] to measure the alignment of anisotropic mesh \(\Gamma_h\) and anisotropic function.

**Definition 1** (matching function). Let \(u \in H^1(\Omega)\) and \(\Gamma_h \in F\) be the triangulation of \(\Omega\). We define \(M_1 : H^1(\Omega) \times F \rightarrow \mathbb{R}\) by
\[
M_1(u, \Gamma_h) := \frac{\left( \sum_{K \in \Gamma_h} \left( \frac{h_{K,\min}^E}{h_{E,\min}} \right)^{-2} \cdot \left\| \nabla u \right\|_{K}^2 \right)^{1/2}}{\left\| \nabla u \right\|}
\]
where \(C_K \in \mathbb{R}^{2 \times 2}\) as defined earlier.

We can easily verify that \(M_1(u, \Gamma_h)\) is for isotropic meshes. Similarly, it can easily be observed that \(M_1(u, \Gamma_h)\) is for anisotropic meshes suitably aligned with anisotropic function \(u\). Therefore, \(M_1(u, \Gamma_h) \approx C\) for anisotropic meshes.

To propose reliable error estimates in energy norm, we will use Clément interpolation operator \(R_C\) [13] for \(u \in H^1(\Omega)\) as standard Lagrange interpolation cannot be defined for these functions.

**Lemma 2.** Let \(u \in H^1_0(\Omega)\) and \(\alpha_K\) be the scaling factor defined by (19). Then, the Clément interpolation operator \(R_C : H^1_0(\Omega) \rightarrow V^h_0\) satisfies
\[
\sum_{K \in \Gamma_h} \alpha_K^{-2} \cdot \left\| u - R_K u \right\|_{K}^2 \leq C^2 \left\| u \right\|^2
\]
\[
e^{1/2} \sum_{E \in \partial \Omega \cap \partial \Omega_h} \alpha_E^{-1} \cdot \left\| u - R_E u \right\|_{E}^2 \leq C^2 \left\| u \right\|^2.
\]

**Proof.** The proof is discussed in [14].

### 4. Residual Error Estimates

In this section, firstly we discuss exact and approximate residuals. Further, we will develop reliable error upper bounds for Hughes stabilized SUPG finite element solution on anisotropic meshes. It is shown that the error bounds obtained depend on anisotropic interpolation.

#### 4.1. Exact Residuals

We define exact element residual \(R_K\) and exact edge residual \(R_E\) as
\[
R_K = f - (\varepsilon \Delta u + a \cdot \nabla u + b \nabla u) \quad \text{on} \quad K, \\
R_E(x) = \begin{cases} 
\varepsilon \cdot \lim_{i \rightarrow \Delta \Omega_K} \left[ \partial \nabla u \right] - \partial \nabla u(x + s) - \partial \nabla u(x - s) & \text{if} \quad E \subset \Omega \setminus \partial \Omega, \\
g - \varepsilon \partial \nabla u & \text{if} \quad E \subset \partial \Omega, \\
0 & \text{if} \quad E \subset \partial \Omega_{h},
\end{cases}
\]
where \(n_{\perp E} \subset \Omega \setminus \partial \Omega\) is the unitary normal vector and \(n_{\perp E} \subset \Omega \setminus \partial \Omega_{h}\) is the outer unitary normal vector.

#### 4.2. Approximate Residuals

Let \(Q\) be the approximation operator used to approximate the element residual and the face residual, i.e.,
\[
r_K = Q(R_K) \in P^0(K) \quad \forall \ K \in \Gamma_h, \\
r_E = Q(R_E) \in P^0(E) \quad \forall \ E,
\]
where we have denoted (approximate) element residual by \(r_K\) and the (approximate) face residual by \(r_E\). Since the finite element solution \(u_h\) is linear, thus
\[
r_E = R_E \quad \forall E \subset \Omega \setminus \partial \Omega_N.
\]

#### 4.3. Residual Error Estimator

Residual error estimator \(\eta_K\) and the approximation term \(\zeta_K\) over triangle \(K\) are given by
\[
\eta_K^2 = \alpha_K^2 \cdot \left\| r_K \right\|_{K} + e^{-1/2} \cdot \alpha_K \cdot \sum_{E \in \partial K \cap \partial \Omega_h} \left\| r_E \right\|_{E}^2
\]
\[
\zeta_K^2 = \alpha_K^2 \cdot \left\| R_K - R_E \right\|_{h} + e^{-1/2} \cdot \alpha_K \cdot \sum_{E \in \partial K \cap \partial \Omega_h} \left\| R_E - R_E \right\|_{E}^2
\]
where \(\alpha_K\) is the scaling factor defined earlier and \(\alpha_K^2 = 3\alpha_K^2\).

Further, we define global error estimators as
Next, we derive reliable upper error bounds on anisotropic meshes.

**Theorem 1** (residual error estimation). Let \( v \in H_0^1(\Omega) \) be the exact solution and \( v_h \in V_h^0 \) be the finite element solution obtained by the proposed scheme. Then, the error in energy norm is bounded above globally by

\[
\| v - v_h \| \leq C \left[ \sum_{K \in T_h} 3\alpha_K^2 (\| r_K - R_h \|^2 + \| R_h \|^2_K) + \sum_{E \in \partial \Omega, \partial \Omega_0} \epsilon^{-1/2} \alpha_E (\| r_E - R_h \|^2 + \| R_h \|^2_E) \right]^{1/2}.
\]

Using Cauchy Schwarz inequality, we get

\[
\sum_{K \in T_h} (R_K, u - R_C u)_K \leq \left( \sum_{K \in T_h} \alpha_K^2 \| R_K \|^2_K \right)^{1/2} \left( \sum_{K \in T_h} \alpha_K^{-2} \| u - R_C u \|^2_K \right)^{1/2},
\]

\[
\sum_{E \in \partial \Omega, \partial \Omega_0} (R_E, u - R_C u)_E \leq \left( \sum_{E \in \partial \Omega, \partial \Omega_0} \epsilon^{-1/2} \alpha_E \| R_E \|^2_E \right)^{1/2} \left( \sum_{E \in \partial \Omega, \partial \Omega_0} \epsilon^{-1/2} \alpha_E^{-1} \| u - R_C u \|^2_E \right)^{1/2}.
\]

Further, using Lemma 2, we get

\[
\sum_{K \in T_h} (R_K, u - R_C u)_K \leq \left( \sum_{K \in T_h} \alpha_K^2 \| R_K \|^2_K \right)^{1/2} C \| u \|,
\]

\[
\sum_{E \in \partial \Omega, \partial \Omega_0} (R_E, u - R_C u)_E \leq \left( \sum_{E \in \partial \Omega, \partial \Omega_0} \epsilon^{-1/2} \alpha_E \| R_E \|^2_E \right)^{1/2} C \| u \|.
\]

Therefore, the term \( B(v - v_h, u - R_C u) \) is bounded above by

\[
B(v - v_h, u - R_C u) \leq \left( \sum_{K \in T_h} \alpha_K^2 \| R_K \|^2_K + \sum_{E \in \partial \Omega, \partial \Omega_0} \epsilon^{-1/2} \alpha_E \| R_E \|^2_E \right)^{1/2} C \| u \|.
\]

Using the standard Galerkin orthogonality condition and standard scaling results, the above equation reduces to
\[ B(v - v_h, R_c u) = \sum_{K} \rho_K (R_K, a \cdot \nabla R_c u) - \sum_{K} \sigma_K (R_K, a_h \cdot \nabla R_c u) \]
\[ \leq \sum_{K \in \mathcal{T}_h} \rho_K \| R_K \|_K \| a \|_{\infty,K} \| \nabla R_c u \|_K \]
\[ + \sum_{K \in \mathcal{T}_h} \sigma_K \| R_K \|_K \| a_h \|_{\infty,K} \| \nabla R_c u \|_K \]
\[ \leq \sum_{K \in \mathcal{T}_h} \rho_K \| R_K \|_K \| a \|_{\infty,K} \left( h_K^{0.5} \right)^{-1} \alpha_K \| \nabla R_c u \|_K \]
\[ + \sum_{K \in \mathcal{T}_h} \sigma_K \| R_K \|_K \| a_h \|_{\infty,K} \left( h_K^{0.5} \right)^{-1} \alpha_K \| \nabla R_c u \|_K. \]
\[ (40) \]

We know that for Clément operator \[11\]
\[ \| R_c u \| \leq C \| u \| \forall u \in H^1_0(\Omega). \]
\[ (41) \]

Thus, we have
\[ B(v - v_h, R_c u) \leq \left( \sum_{K \in \mathcal{T}_h} \rho_K \| R_K \|_K^2 \| a \|_{\infty,K}^2 \left( h_K^{0.5} \right)^{-2} \alpha_K^2 \right)^{1/2} \]
\[ \cdot C \| u \| + \left( \sum_{K \in \mathcal{T}_h} \sigma_K \| R_K \|_K^2 \| a_h \|_{\infty,K}^2 \left( h_K^{0.5} \right)^{-2} \alpha_K^2 \right)^{1/2} C \| u \|. \]
\[ (42) \]

It may be noted that the effect of nonlinear term in the \( L^\infty \) norm will be bounded by that of the term \( \| a \|_{\infty,K} \) as shown below, i.e.,
\[ a_h = (a \cdot \nabla u_h) / |\nabla u_h|^2, |\nabla u_h| \neq 0, \]
\[ a_h \leq \frac{\| a \| \| \nabla u_h \| \| \nabla u_h \|}{|\nabla u_h|^2}, \text{ Using Cauchy - Schwarz inequality}, \]
\[ \| a_h \| \leq \frac{\| a \| \| \nabla u_h \| \| \nabla u_h \|}{|\nabla u_h|^2}, \leq \| a \|. \]
\[ (43) \]

Since \( a \), the convection coefficient, is assumed to be smooth in the domain under consideration, it is bounded above. Hence, the nonlinear term \( \| a_h \|_{\infty,K} \) is taken as bounded above by some constant and is absorbed in the constant term.

We know that
\[ \rho_K \leq h_K^{0.5} \| a \|_{\infty,K} \forall K \in \Gamma_h, \]
\[ \sigma_K \leq h_K^{0.5} \| a_h \|_{\infty,K} \forall K \in \Gamma_h. \]
\[ (44) \]

Therefore,
\[ B(v - v_h, R_c u) \leq \left( \sum_{K \in \mathcal{T}_h} \alpha_K^2 \| R_K \|_K^2 + \sum_{E \in \partial \Omega_0} \alpha_E \| R_E \|_E^2 \right)^{1/2} \cdot C \| u \| \cdot \{ M_1(u, \Gamma_h) = C \}. \]
\[ (45) \]

Since
\[ \| v - v_h \| \leq \| v - v_h \|, \]
\[ (46) \]

using equation (45) and equation (37) in equation (32), we get
\[ \| v - v_h \| \leq \left( \sum_{K \in \mathcal{T}_h} 3 \alpha_K^2 \| R_K \|_K^2 + \sum_{E \in \partial \Omega_0} \alpha_E \| R_E \|_E^2 \right)^{1/2} C. \]
\[ (47) \]

Using triangle inequalities,
\[ \| R_K \|_K^2 \leq \| r_K - r_K \|_K^2 + \| r_K \|_K^2, \]
\[ \| R_E \|_E^2 \leq \| r_E - r_E \|_E^2 + \| r_E \|_E^2, \]
\[ (48) \]

we get
\[ \| v - v_h \| \leq C \left[ \sum_{K \in \mathcal{T}_h} 3 \alpha_K^2 \left( \| r_K - r_K \|_K^2 + \| r_K \|_K^2 \right) + \sum_{E \in \partial \Omega_0} \alpha_E \left( \| r_E - r_E \|_E^2 + \| r_E \|_E^2 \right) \right]^{1/2} C. \]
\[ (49) \]

5. Adaptive Refinement Algorithm

In this section, we propose an adaptive refinement strategy based on the a posteriori error estimates obtained in the last section. We propose the following adaptive refinement algorithm:

1. Discretize the domain using triangular elements. Triangulation is being carried out using red refinement
2. Solve the problem using the proposed scheme described in Section 2
3. Over each element \( K \), the residual error estimates \( \eta_K \) have been calculated as defined in Section 4
4. Mark the elements \( \{ K'_e \}_{e=1}^M \) satisfying \( \eta_{K'_e} > C \max_{L'} \eta_{L'}, \) where \( C \) is a user chosen constant from \( (0, 1) \), for refinement
5. Refine these marked elements using green refinement procedure
Refine all those elements having hanging nodes also to avoid the discontinuity in the solution.

Solve the problem again on the new adapted mesh.

Repeat the process of grid refinement until the solution has been obtained up to a given desired accuracy.

6. Numerical Results

In this section, numerical experiments have been carried out in order to test the efficiency and robustness of the proposed adaptive refinement technique based on the derived error estimates.

Example 4. Consider the following singularly perturbed convection-diffusion problem:

\[-\nabla \cdot \varepsilon \nabla u + 2u_x + 3u_y + u = f \text{ in } \Omega = (0, 1)^2,\]

\[u = 0 \text{ on } \partial \Omega.\]  \hspace{1cm} (50)

The right-hand side function \(f\) is so chosen to satisfy the exact solution

\[u = \sin (x) \left(1 - e^{-2(1-x)/\varepsilon}\right) y^2 \left(1 - e^{-3(1-y)/\varepsilon}\right).\]  \hspace{1cm} (51)

The solution of the above problem exhibits exponential boundary layers along the lines \(x = 1\) and \(y = 1\). For adaptive refinement, an anisotropic triangular mesh has been taken into consideration. In Figures 3(a) and 3(b), we present a portion of adaptive triangular mesh for \(\varepsilon = 2^{-3}\) with different degrees of freedom. Figures 4(a) and 4(b) present adaptive refined meshes for \(\varepsilon = 2^{-5}\) with different degrees of freedoms.

In Figures 5 and 6, the numerical solution obtained using the proposed refinement algorithm for different values of the singular perturbation parameter \(\varepsilon = 2^{-3}\) and \(\varepsilon = 2^{-5}\) has been plotted. It can be easily seen that even very sharp boundary layers have been efficiently captured using the proposed refinement algorithm. From the solutions, it can also be observed that the problem is very sensitive to the singular perturbation parameter, i.e., even for \(\varepsilon = 2^{-5}\), very sharp boundary layers appear in the solution. In Figure 7, energy norm errors for \(\varepsilon = 2^{-5}\) have been presented. The behavior of effectivity index \(\psi = \|v - v_h\|/\text{ErrorEstimator}\) which is used to measure reliability of the estimator is shown in Figure 8.

7. Conclusion

In the presented work, an adaptive numerical technique has been proposed for singularly perturbed convection-diffusion problems in two dimensions. The singularly perturbed problem under consideration has been solved using Hughes stabilization technique under SUPG finite element framework. Anisotropic meshes have been considered for the domain discretization. Reliable a posteriori error estimates have been developed in energy norm on anisotropic meshes. Based on these a posteriori error estimates, an
adaptive mesh refinement strategy has been proposed. It has been depicted through numerical experiments that the proposed adaptive refinement strategy is very much efficient in capturing sharp boundary layers as the singular perturbation parameter $\varepsilon$ approaches to 0.

Data Availability

There is no data were used to support this study

Conflicts of Interest

The authors declare that they have no competing interest.

Authors’ Contributions

All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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