AN ALTERNATIVE PROOF OF HILL’S CRITERION OF FREENESS FOR ABELIAN GROUPS

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Abstract. In this note, we provide a different proof of Hill’s criterion of freeness for abelian groups. Our proof hinges on the construction of suitable $G(\aleph_0)$-families of subgroups of the links in Hill’s theorem and, ultimately, on the construction of such a family of pure subgroups of the group itself.

Resumen. En este trabajo, se proporciona una nueva demostración del criterio de Hill para grupos abelianos libres. La demostración se basa en la construcción de una $G(\aleph_0)$-familia de subgrupos en los eslabones del teorema de Hill y, prioritariamente, en la construcción de una familia tal de subgrupos puros.

1. Introduction

In 1934, Lev Pontryagin proved that a countable, torsion-free abelian group is free if and only if every finite rank, pure subgroup is free [3]. Equivalently, every properly ascending chain of subgroups of the same finite rank is finite. From the proof of this criterion, it follows that a torsion-free abelian group $G$ is free if there exists an ascending chain

$$0 = G_0 < G_1 < \cdots < G_n < \cdots \quad (n < \omega),$$

consisting of pure subgroups of $G$ whose union is equal to $G$, such that every $G_n$ is free and countable. Here, a subgroup $H$ of the abelian group $G$ is pure if solubility in $G$ of every equation of the form $nx = h \in H$, with $n \in \mathbb{Z}$, implies its solubility in $H$. Also, we say that $G$ is torsion-free if $n = 0$ or $g = 0$, whenever $n \in \mathbb{Z}$ and $g \in G$ satisfy $ng = 0$.

Later, in 1970, Hill established that, in order for an abelian group $G$ to be free, it is sufficient to prove that it is the union of a countable ascending chain $\{G_n\}$ consisting of free, pure subgroups $G_n$. In other words, he proved the following theorem, establishing thus that the countability condition on the cardinality of the links of the chain was superfluous.

Theorem 1.1 (Hill’s criterion of freeness). A torsion-free abelian group $G$ is free if there exists a countable ascending chain

$$0 = G_0 < G_1 < \cdots < G_n < \cdots \quad (n < \omega)$$

of subgroups of $G$, such that:

(a) every $G_n$ is free,

(b) every $G_n$ is a pure subgroup of $G$, and

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Palabras y frases clave. Grupo abeliano, libertad, criterio de Hill, pureza, $G(\aleph_0)$-familia.
In this note, we give a proof of Hill’s criterion different from the one provided in [1]. Our proof hinges on the construction of suitable classes of subgroups of the groups \( G_n \) and, ultimately, on the construction of such a family consisting of pure subgroups of \( G \). Section 3 of this work contains the proof of Theorem 1.1, while Section 2 presents some preliminary results.

2. Preparatory Lemmas

The following is a general result which will be used in the proof of Theorem 1.1. We refer to [2] for definitions of the set-theoretical concepts.

Lemma 2.1. An abelian group \( G \) is free if there exists a continuous, well-ordered, ascending chain

\[
0 = A_0 < A_1 < \cdots < A_\gamma < A_{\gamma+1} < \cdots \quad (\gamma < \tau)
\]

of subgroups of \( G \), such that:

(a) every factor group \( A_{\gamma+1}/A_\gamma \) is free, and
(b) \( G = \bigcup_{\gamma<\tau} A_\gamma \).

Proof. The conclusion follows from the fact that \( G \) is isomorphic to the direct sum of the factor groups \( A_{\gamma+1}/A_\gamma \), for \( \gamma < \tau \). \( \Box \)

Recall that a \( G(\aleph_0) \)-family of an abelian group \( G \) is a collection \( \mathcal{B} \) of subgroups of \( G \), which satisfies the following properties:

(i) 0 and \( G \) belong to \( \mathcal{B} \),
(ii) \( \mathcal{B} \) is closed under unions of ascending chains, and
(iii) for every \( A_0 \in \mathcal{B} \) and every countable set \( H \subseteq G \), there exists \( A \in \mathcal{B} \) which contains both \( A_0 \) and \( H \), such that \( A/A_0 \) is countable.

Clearly, every abelian group has a \( G(\aleph_0) \)-family, namely, the collection of all its subgroups.

For the rest of this section, we will assume the hypotheses of Theorem 1.1. Under these circumstances, we fix a basis \( X_n \) of \( G_n \) for every \( n < \omega \), and let \( \mathcal{B}_n \) be the family of all subgroups of \( G_n \) generated by subsets of \( X_n \). Clearly, every member of \( G_n \) is a direct summand of \( G_n \) and, thus, a pure subgroup of \( G \).

Lemma 2.2. The collection \( \mathcal{B}_n' = \{ A \in \mathcal{B}_n : A + G_i \text{ is pure in } G, \text{ for every } i < \omega \} \) is a \( G(\aleph_0) \)-family of pure subgroups of \( G_n \), for every \( n < \omega \).

Proof. All we need to check is that the countability condition is satisfied, since the other conditions of a \( G(\aleph_0) \)-family are obvious. So, let \( A_0 \in \mathcal{B}_n' \), and let \( H_0 \) be a countable subset of \( G_n \). Moreover, let \( m < \omega \), and assume that we have already constructed a chain

\[
A_0 < A_1 < \cdots < A_m
\]

of groups in \( \mathcal{B}_n \), such that:

1. \( H_0 \) is contained in \( A_1 \),
2. for every \( j < m \), the group \( A_{j+1}/A_j \) is countable, and
3. for every \( j < m \) and every \( i < \omega \), \( (A_{j+1} + G_i)/(A_0 + G_i) \) contains the purification of \( (A_j + G_i)/(A_0 + G_i) \) in \( G/(A_0 + G_i) \).
To find the next member of (4), for every $i < \omega$, let $V_i \subseteq G_n$ be a complete set of representatives of the purification of $(A_m + G_i)/(A_0 + G_i)$ in $G/(A_0 + G_i)$. The sets $V_i$ are clearly countable, so that $H_{m+1} = H_0 \cup \bigcup_{i < \omega} V_i$ is likewise countable. Therefore, there exists $A_{m+1} \in B_n$ containing both $A_m$ and $H_{m+1}$, such that $A_{m+1}/A_m$ is countable. Inductively, we construct a chain

$$A_0 < A_1 < \cdots < A_m < \ldots \quad (m < \omega)$$

of groups in $B_n$, satisfying properties 1, 2 and 3 above, for every $m < \omega$.

Evidently, the union $A$ of the links of (5) is a member of $B_n$, $A/A_0$ is countable, and our construction guarantees that $(A + G_i)/(A_0 + G_i)$ is pure in $G/(A_0 + G_i)$. Thus, $A + G_i$ is pure in $G$ and, consequently, $A$ belongs to $B_n'$. \hfill \Box

Lemma 2.3. The collection $B = \{ A \subseteq G : A \cap G_n \in B_n', \text{ for every } n < \omega \}$ is a $G(N_0)$-family of pure subgroups of $G$.

Proof. Again, only the countability condition merits attention; so, let $A_0 \in B$, and let $H \subseteq G$ be countable. For every $k < \omega$, let $A_k^n = A_0 \cap G_k$. Moreover, let $n < \omega$, and assume that we have already constructed a finite ascending chain

$$A_0 < A_1 < \cdots < A_n$$

of subgroups of $G$, such that all factor groups $A_m/A_0$ are countable, for every $m \leq n$. Furthermore, suppose that each link $A_m$ in (6) may be expressed as the union of a countable ascending chain

$$0 = A_0^m < A_1^m < \cdots < A_k^m < \ldots \quad (k < \omega)$$

of subgroups of $G$, such that:

(a) $A_k^n \in B_k'$, for every $k < \omega$ and every $m \leq n$,

(b) $A_k^n$ is countable over $A_0 \cap G_k$, for every $k < \omega$ and every $m \leq n$, and

(c) $A_k^n \subseteq A_m \cap G_k < A_k^{n+1}$, for every $k < \omega$ and $m + 1 \leq n$.

For every $k < \omega$, the group $(A_n \cap G_k)/(A_0 \cap G_k)$ is countable, so we may fix a countable set of representatives $Y_k$ of $A_n \cap G_k$ modulo $A_0 \cap G_k$. Moreover, there exists $B_k \in B_k'$ containing both $A_0 \cap G_k$ and $Y_k$, such that $B_k$ is countable over $A_0 \cap G_k$. Thus, any set of representatives $H_k$ of $B_k$ modulo $A_0 \cap G_k$ is countable.

In order to construct the next link in (6), assume that the groups in the ascending chain $0 = A_0^{n+1} < A_1^{n+1} < \cdots < A_k^{n+1}$ have been built as needed, for some $k < \omega$, and let $Z_k \subseteq G_k$ be a set of representatives of $A_k^{n+1}$ modulo $A_0 \cap G_k$. Then, there exists $A_k^{n+1} \in B_k'$ which contains $A_0 \cap G_{k+1}$ and the countable set $Z_k \cup H_{k+1} \cup (H \cap G_{k+1})$, such that $A_k^{n+1}$ is countable over $A_0 \cap G_{k+1}$.

Clearly, the group $A = \bigcup_{n < \omega} A_n$ contains both $A_0$ and $H$, and is countable over $A_0$. Moreover, our construction guarantees that $A \cap G_k \in B_k$, for every $k < \omega$. We conclude that $A \in B$. \hfill \Box

Before we prove our next result, it is important to notice that $A + G_n$ is a pure subgroup of $G$, for every $A \in B$ and every $n < \omega$. Indeed, that $(A + G_n) \cap G_{n+1}$ is pure in $G$ follows from the fact that $A \cap G_{n+1} \in B_{n+1}'$. Next, assume that $(A + G_n) \cap G_k$ is pure in $G$, for some $k > n$. It is easy to check that

$$\frac{(A + G_k) \cap G_{k+1}}{(A + G_n) \cap G_{k+1}} \cong \frac{G_k}{(A + G_n) \cap G_k},$$

whence it follows that $(A + G_n) \cap G_{k+1}$ is pure in $G$. The claim is readily established after noticing that $A + G_n = \bigcup_{k < \omega} (A + G_n) \cap G_k$. 

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Lemma 2.4. For every $A \in \mathcal{B}$, finite rank, pure subgroups of $G/A$ are free.

Proof. Let $A \in \mathcal{B}$, and let $D$ be a pure subgroup of $G$ containing $A$, such that $D/A$ is of finite rank. If $S = \{d_1, \ldots, d_n\}$ is a complete set of representatives of a maximal independent system of $D$ modulo $A$, then there exists $k < \omega$ such that $S \subseteq G_k$. Then $A + (D \cap G_k) = D \cap (A + G_k)$ is a pure subgroup of $G$ containing $S$, which lies between $A$ and $D$. Therefore, $D = A + (D \cap G_k)$. The fact that $A \cap G_k \in \mathcal{B}_k$ implies that $A \cap G_k$ is a summand of $G_k$. Therefore, there exists a finite rank, free group $B$, such that $D \cap G_k = (A \cap G_k) \oplus B$. Notice that

$$D = A + (D \cap G_k) = A + ((A \cap G_k) \oplus B) = A \oplus B,$$

which implies that $D/A$ is free. □

3. Proof of the main result

Proof of Theorem 1.1. Let $\alpha$ be any nonzero ordinal, and let

$$0 = A_0 < A_1 < \cdots < A_\gamma < A_{\gamma+1} \cdots \quad (\gamma < \alpha)$$

be an ascending chain of subgroups in $\mathcal{B}$, such that all factor groups $A_{\gamma+1}/A_\gamma$ are free. If $\alpha$ is a limit ordinal, then we let $A_\alpha = \bigcup_{\gamma < \alpha} A_\gamma$. Otherwise, there exists an ordinal $\beta$ such that $\alpha = \beta + 1$. In this case, if there exists $x \in G \setminus A_\beta$, we let $A_{\beta+1} \in \mathcal{B}$ contain both $x$ and $A_\beta$, such that $A_{\beta+1}/A_\beta$ be countable. Lemma 2.4 implies now that finite rank, pure subgroups of $A_{\beta+1}/A_\beta$ are free. Consequently, $A_{\beta+1}/A_\beta$ is free by Pontryagin’s criterion.

Using transfinite induction, we construct a continuous, well-ordered, ascending chain of subgroups of $G$ satisfying properties (a) and (b) of Lemma 2.1. We conclude that $G$ is free. □

References

[1] P. Hill. On the freeness of abelian groups: a generalization of Pontryagin’s theorem. Bullet. Amer. Math. Soc., 76(5):1118–1120, 1970.
[2] T. Jech. Set Theory. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 3th edition, 2003.
[3] L. Pontryagin. The theory of topological commutative groups. Annals of Math., 35(2):361–388, 1934.