On the dynamics of contact Hamiltonian systems: I. Monotone systems

Liang Jin¹ and Jun Yan²

¹ Department of Mathematics, Nanjing University of Science and Technology, Nanjing 210094, People’s Republic of China
² School of Mathematical Sciences, Fudan University, Shanghai 200433, People’s Republic of China

E-mail: jl@njust.edu.cn and yanjun@fudan.edu.cn

Received 18 August 2020, revised 13 January 2021
Accepted for publication 19 February 2021
Published 12 May 2021

Abstract
This article is devoted to a description of the dynamics of the phase flow of monotone contact Hamiltonian systems. Particular attention is paid to locating the maximal attractor (or repeller), which could be seen as the union of compact invariant sets, and investigating its dynamical and topological properties. This is based on an analysis from the viewpoint of gradient-like systems.

Keywords: contact Hamiltonian system, maximal attractor (repeller), Hamilton–Jacobi equation, viscosity solution
Mathematics Subject Classification numbers: 35D40, 35F21, 37C70, 37J55.

1. Introduction and statement of the main results

Let $M$ be an $n$-dimensional connected, closed $C^\infty$ manifold. Equipped with the canonical two-form $\Omega = dx \wedge dp$, $(T^*M, \Omega)$ becomes a symplectic manifold. Hamiltonian systems are defined as the symplectic gradient vector fields of functions, called Hamiltonian, on $T^*M$. As the natural framework of classical and celestial mechanics, Hamiltonian systems received much attention and were extensively studied since the time of Newton. It is well known that Hamiltonian systems can only be used as the mathematical model of conservative dynamics. Therefore, to apply theoretical results to more physical systems, which may exchange energy with an environment, one needs more flexible alternatives for Hamiltonian systems.

Contact Hamiltonian systems become worth trying choices. As the direct analogies of Hamiltonian systems via characteristic theory for first order partial differential equations (PDE for short), such systems are determined by the standard contact form and a smooth function on the manifold of one-jets of functions on $M$ and have not been considered as much as their symplectic counterparts. However, in recent years, several applications of contact Hamiltonian systems have been found in equilibrium or irreversible thermodynamics, statistical physics

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and also, classical mechanics. We refer to [3, 4] for nice surveys of physical contents and applications of contact Hamiltonian systems.

An inspiring way of the mathematical study of contact Hamiltonian systems begins with a simple extension of Hamiltonian systems, the discounted systems. Their dynamics were investigated from different aspects including quasi-periodic motions [5, 6], Aubry–Mather sets in low dimensional models [10, 19–21], generalizations of Mather’s theory and weak KAM theory [22] and their applications to PDE problems [14, 17]. The variational theories for more general contact Hamiltonian systems with arbitrary degrees of freedom were explored in the series of works [26–29] and [7, 8] under Tonelli assumptions.

Among the above miscellaneous works, [22, 29] are of particular interests to us since much effort is paid to understand the global dynamics of monotone contact Hamiltonian systems. More precisely, in [29], the authors find a compact subset of the phase space containing all ω-limit sets of the phase flow; while in [22] the authors define and locates the maximal attractor, i.e., the union of all compact, invariant sets, which, at least to us, is a suitable and promising concept in the investigation of the dissipative feature of such systems. Inspired by these two works, the aim of this article is twofold, to study the maximal attractor and global dynamics of the system in more detail, and to present the results in an elementary way under more relaxed assumptions. We want to emphasis that our approach is based on an analysis from the viewpoint of gradient-like system (constructing Lyapunov functions of the phase flow) and is independent of the variational approach mentioned before.

Once and for all, we choose an auxiliary complete Riemannian metric on $M$ and, with slight abuse of notation, use $\| \cdot \|_v$ to denote the norm induced on the cotangent space $T^*_v M$. Using a canonical diffeomorphism, we identify $T^* M \times \mathbb{R}$ with the manifold of one-jets of functions on $M$. The natural contact structure on $T^* M \times \mathbb{R}$ is defined by $\xi = \ker \alpha$, where $\alpha = du - pdx$ denotes the standard one-form. As Lie did [18] in the 19th century, one would like to study the group of contact transformations, i.e., automorphisms of $T^* M \times \mathbb{R}$ preserving the contact structure. Among all contact transformations, there is an important subclass called infinitesimal ones, i.e., those transformations generated by vector fields on $T^* M \times \mathbb{R}$. Such a vector field must satisfies

$$\mathcal{L}_X \alpha = f \alpha, \quad i_X \alpha = -H,$$

(1)

where $\mathcal{L}_X$ denotes the Lie derivative along the vector field $X$ and $i_X$ denotes the inner product. The first equation, in which $f : T^* M \times \mathbb{R} \to \mathbb{R}$ is nowhere vanishing, gives the structure preserving property. Using Cartan’s formula,

$$f \alpha = \mathcal{L}_X \alpha = i_X (d\alpha) + d(i_X \alpha) = i_X \Omega - dH,$$

here the second equality uses the fact $d\alpha = \Omega$. In local coordinates, if we set $\dot{x} = i_{\dot{x}} \ dx$, $\dot{p} = i_{\dot{x}} \ dx$, $\dot{u} = i_{\dot{x}} \ du$, the above equation reads

$$\left( f + \frac{\partial H}{\partial u} \right) \ du + \left( \dot{p} + \frac{\partial H}{\partial x} - \dot{f} \cdot p \right) \ dx + \left( \frac{\partial H}{\partial p} - \dot{x} \right) \cdot p \equiv 0.$$

Thus $f(x, p, u) = -\frac{\partial H}{\partial u}(x, p, u)$ and using second equation of (1), one has

$$X_H : \begin{cases} \dot{x} = \frac{\partial H}{\partial p}(x, p, u), \\ \dot{p} = -\frac{\partial H}{\partial x}(x, p, u) - \frac{\partial H}{\partial u}(x, p, u) \cdot p, \\ \dot{u} = \frac{\partial H}{\partial p}(x, p, u) \cdot p - H(x, p, u). \end{cases}$$

(2)
On the other hand, by the wave-particle duality [2, chapters 1–2], (2) is the characteristic system for the Hamilton–Jacobi equation

\[ H(x, d_u u) = 0, \quad x \in M, \]

and reduces to Hamiltonian systems when \( H = H(x, p) \). Thus \( X_H \) could be seen as a natural generalization of Hamiltonian vector field via characteristic theory. Due to the above discussion, we call \( H \) a contact Hamiltonian and \( X_H \) the contact Hamiltonian vector field associated to \( H \). The phase flow generated by \( X_H \) is denoted by \( \Phi^t_H \), it consists of contact transformations.

The aim of this paper is to study the global dynamics of \( \Phi^t_H \) for contact Hamiltonian \( H \in C^2(T^*M \times \mathbb{R}, \mathbb{R}) \) of specific type, namely those satisfying

\begin{enumerate}[(H1)]  
  \item \( \partial^\mu H(x, p, u) \) is positive definite for every \((x, p, u) \in T^*M \times \mathbb{R}\),
  \item \( \lim_{\|\mu\|_1 \to +\infty} H(x, p, u) \to +\infty \) for every \((x, u) \in M \times \mathbb{R}\),
  \item \( \frac{\partial H}{\partial u}(x, p, u) \geq \lambda \) for some \( \lambda > 0 \) and every \((x, p, u) \in T^*M \times \mathbb{R}\),
\end{enumerate}

or alternatively,

\( (M+) \frac{\partial H}{\partial u}(x, p, u) \leq -\lambda \) for some \( \lambda > 0 \) and every \((x, p, u) \in T^*M \times \mathbb{R}\).

Physically, the orbit of such a flow generalizes the motion of particles in mechanical systems with friction or negative damping. Such systems occur in the study of self-excited oscillation theory, where \( \lambda \) is called the damping ratio and may change sign, see [30, introduction].

For any \( z \in T^*M \times \mathbb{R} \), let \( \alpha(z) \) (resp. \( \omega(z) \)) denotes the \( \alpha \)-limit set of \( z \) (resp. \( \omega \)-limit set of \( z \)) under \( \Phi^t_H \). It turns out that the global dynamics of \( \Phi^t_H \) is closely related to a compact, invariant set called maximal attractor (resp. repeller), which is formally

**Definition 1.1.** A global attractor (resp. repeller) for \( \Phi^t_H \) is a compact, invariant set \( K \subset T^*M \times \mathbb{R} \) such that for any \( z \in T^*M \times \mathbb{R}, \omega(z) \) (resp. \( \alpha(z) \)) \( \subset K \). Furthermore, a global attractor (resp. repeller) is said to be maximal if it is not properly contained in any other global attractor (resp. repeller).

**Remark 1.2.** Although there maybe more than one global attractors for \( \Phi^t_H \), the maximal attractor (resp. repeller) of \( \Phi^t_H \) is unique (if exists) and equals the union of compact \( \Phi^t_H \)-invariant sets. We shall denote it by \( K_H \).

The definition of global attractor (resp. repeller) is equivalent to that on [22, p 780]: for any neighbourhood \( \mathcal{O} \) of \( K \) and \( z \in T^*M \times \mathbb{R} \), there is \( T(z, \mathcal{O}) > 0 \) (resp. \( < 0 \)) such that \( \Phi^t_H(z) \in \mathcal{O} \) for \( t \geq T \) (resp. \( t \leq T \)). The notion of attractor (or repeller) is so important in the study of dynamical systems that much literature is devoted to giving a widely accepted definition, for example [12, 23, 24].

Since the phase space \( T^*M \times \mathbb{R} \) of \( \Phi^t_H \) is non-compact, the existence of maximal attractor or repeller is not a trivial fact. Using tools from Aubry–Mather theory and weak KAM theory, this fact was established in [22] for the case of discounted systems, i.e., \( H(x, p, u) = \lambda u + h(x, p), \lambda > 0 \), and in [29] for monotone contact Hamiltonian satisfying Tonelli assumptions.

It turns out that the existence of maximal attractor (resp. repeller) does not depend on the variational properties of the system (2). To locate it, we find two Lyapunov functions for \( \Phi^t_H \) so that the intersection of their sub-level sets is forward (resp. backward) invariant, and by the assumptions on \( H \), is compact. Among these two functions, the second one relies on the viscosity solution of the Hamilton–Jacobi equation mentioned before, and is of particular interest.
Besides another proof of the existence result, our construction offers some information on the topological structure of the maximal attractor (resp. repeller), which are summarized into

**Theorem A.** Assume $H \in C^2(T^*M \times \mathbb{R}, \mathbb{R})$ satisfies (H1), (H2) and (M−) (resp. (M+)).

then

1. $\Phi^t_H$ is forward (resp. backward) complete, i.e., $\Phi^t_H$ is well-defined for all $t \in [0, +\infty)$ (resp. $t \in (-\infty, 0]$).
2. The maximal attractor (resp. repeller) $K_H$ for $\Phi^t_H$ exists and $\alpha(z) \neq \emptyset$ (resp. $\omega(z) \neq \emptyset$) if and only if $z \in K_H$. For every neighbourhood $\mathcal{O}$ of $K_H$ and every compact set $S \subset T^*M \times \mathbb{R}$, there is $T(S, \mathcal{O}) > 0$ (resp. $T(S, \mathcal{O}) < 0$) such that $\Phi^t_H(S) \subset \mathcal{O}$ for all $t \geq T$ (resp. $t \leq T$).
3. There exists a decreasing basis of neighbourhoods $\{\mathcal{O}_t\}_{t \geq 0}$ of $K_H$ such that every $\mathcal{O}_t$ is homotopic equivalent to $M$. In particular, $K_H$ is connected. Here, the notion decreasing means for any $0 \leq l \leq t$, $\mathcal{O}_l \subset \mathcal{O}_t$.

**Remark 1.3.** (A2) obviously implies definition 1.1. The statement (A2) means that we may choose $T(z, \mathcal{O})$ to be uniform for all $z$ in a compact set $S$ of $T^*M \times \mathbb{R}$.

To study the dynamics of $\Phi^t_H$ on $K_H$, we shall focus on monotone contact Hamiltonians satisfying the additional assumptions:

1. $\frac{\partial H}{\partial p}(x, 0, u) = 0$ for every $(x, u) \in M \times \mathbb{R}$,
2. and, if we set $\mathcal{F}_H = \{(x_0, 0, u_0) \in T^*M \times \mathbb{R} : \partial_x H(x_0, 0, u_0) = 0, H(x_0, 0, u_0) = 0\},$
3. $x_0$ is a nondegenerate critical point of the function $x \mapsto H(x, 0, u_0)$ for every $(x_0, 0, u_0) \in \mathcal{F}_H$.

We further explain the assumptions (H3) and (H4) by giving the following

**Example 1.4.** There is a natural class of contact Hamiltonians satisfying (H3), namely ones satisfying the symmetry assumption:

$$H(x, p, u) = H(x, -p, u) \quad \text{for every } (x, p, u) \in T^*M \times \mathbb{R}.$$ 

In particular, the Hamiltonian $H : T^*M \times \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$H(x, p, u) = K(x, p) + V(x) + \lambda u,$$

is a monotone contact Hamiltonian satisfying (H3), where $K(x, p) = \frac{1}{2}||p||^2$ denotes the kinetic energy of moving particles and $V: M \rightarrow \mathbb{R}$ is a $C^2$ potential. In addition, if $V$ is a Morse function, then (H4) is satisfied.

**Remark 1.5.** (H3) means that the convex function $p \mapsto H(x, p, u)$ attains its minimum at $0 \in T_x^*M$ for every $(x, u) \in M \times \mathbb{R}$. According to (H3), $\mathcal{F}_H$ consists of all equilibria of $X_H$.

**Remark 1.6.** The monotone contact Hamiltonians satisfying (H4) form an open and dense subset of the space of all monotone contact Hamiltonians, with respect to the strong $C^2$-topology posed on $C^2(T^*M \times \mathbb{R}, \mathbb{R})$. For a more detailed discussion, we refer to the subsection A.2 of the appendix.
To describe the dynamics on $K_H$, we need an elementary class of invariant sets other than equilibria. For two distinct equilibria $z_0, z_1 \in F_H$, we use $\Sigma(z_0, z_1)$ to denote the set of all $z \in T^*M \times \mathbb{R}$ satisfying
\[
\lim_{t \to -\infty} \Phi^t_H(z) = z_0, \quad \lim_{t \to +\infty} \Phi^t_H(z) = z_1.
\]
Let
\[
\Sigma_H = \bigcup_{z_0, z_1 \in F_H} \Sigma(z_0, z_1),
\]
then $\Sigma_H$ is clearly $\Phi^t_H$-invariant.

With additional assumptions (H3) and (H4), the coordinate function $u$ serves as a Lyapunov function on the maximal attractor. From this observation, we conclude that the structure of the maximal attractor is simple, consisting of equilibria and heteroclinic orbits between them, and the dynamics behave in an invertible way. We also give a discussion of this phenomenon at the end of section 4.1 via the example (3). We formulate the above outline into our second main result:

**Theorem B.** Assume $H \in C^2(T^*M \times \mathbb{R}, \mathbb{R})$ satisfies (H1)–(H4) and (M−) (resp. (M+) ), then

(B1) The maximal attractor (resp. repeller) $K_H = F_H \cup \Sigma_H$. In particular, $K_H$ is path-connected.

(B2) For any $z \in T^*M \times \mathbb{R}$, both $\alpha(z), \omega(z)$ consist of at most one equilibrium in $F_H$.

(B3) For two equilibria $z_0 = (x_0, 0, u_0), z_1 = (x_1, 0, u_1) \in F_H$, assume $\Sigma(z_0, z_1) \neq \emptyset$, then $u_0 < u_1$.

The remaining of this article is organized as follows. In section 2, we analysis system (2) from the viewpoint of gradient-like systems, i.e., deriving two Lyapunov functions that are crucial for the proof of theorem A. Section 3 is devoted to the proof of theorem A. In section 4, we prove theorem B and apply it to the discounted systems. The appendix contains preliminaries which may be helpful for understanding the main body of this paper.

**2. Analysis from the viewpoint of gradient-like systems**

An important feature of the monotone contact Hamiltonian systems is that the phase flow possesses various Lyapunov functions on corresponding domains of the phase space. In this section, we shall give a detailed analysis of $\Phi^t_H$ in this direction, then use these results to derive some forward or backward invariant sets.

**2.1. Settings and first Lyapunov function**

Let $TM$ and $T^*M$ denote the tangent and cotangent bundle of $M$. A point of $TM$ will be denoted by $(x, \dot{x})$, where $x \in M$ and $\dot{x} \in T_xM$, and a point of $T^*M$ by $(x, p)$, where $p \in T^*_xM$ is a linear form on $T_xM$. The canonical pairing between tangent and cotangent bundles are denoted by $\langle \cdot, \cdot \rangle$. We shall use either $z$ or the local coordinates $(x, p, u)$ to denote a point of $T^*M \times \mathbb{R}$. In this and later sections, we always assume the contact Hamiltonian $H \in C^2(T^*M \times \mathbb{R}, \mathbb{R})$ satisfies (H1) and (H2).

Since $H \in C^2(T^*M \times \mathbb{R}, \mathbb{R})$, (2) shows that $X_H$ is a $C^1$ vector field on $T^*M \times \mathbb{R}$. Thus the fundamental theorems for ordinary differential equations states that, for every $z = (x, p, u)$
$\in T^*M \times \mathbb{R}$, there is a unique integral curve of $X_H$ passing through $z$, with the maximum existence interval $(a(z), b(z))$, where $a(z) < 0 < b(z)$. We use either $\Phi^t_H(z)$ or $z(t) = (x(t), p(t), u(t)), \quad t \in (a(z), b(z))$

to denote the integral curve through $z$. Notice that $z(t)$ is $C^1$ with respect to $t$, by (2), one can compute as

$$H(z(t)) = \langle dH, X_H \rangle(z(t)) = \left[ \frac{\partial H}{\partial x} \cdot \dot{x} + \frac{\partial H}{\partial p} \cdot \dot{p} + \frac{\partial H}{\partial u} \cdot \dot{u} \right](z(t))$$

$$= \left[ \frac{\partial H}{\partial x} \frac{\partial H}{\partial p} \left( \frac{\partial H}{\partial x} + \frac{\partial H}{\partial u} \cdot p \right) + \frac{\partial H}{\partial u} \left( \frac{\partial H}{\partial p} \cdot p - H \right) \right](z(t))$$

$$= -\frac{\partial H}{\partial u}(z(t)) \cdot H(z(t)),$$

which states that

$$H(z(t)) = e^{-\int_0^t \frac{\partial H}{\partial u}(z(s)) \, ds} \cdot H(z), \quad t \in (a(z), b(z)). \quad (4)$$

Combining (4) and (M_4), one has the well-known

**Proposition 2.1. (First Lyapunov function).** For any $z \in T^*M \times \mathbb{R}$, $H: T^*M \times \mathbb{R} \rightarrow \mathbb{R}$ never changes its sign along the integral curve $\Phi^t_H(z)$. Moreover, assume $H$ satisfies (M_-) (resp. (M_+)), then

$$|H(z(t))| \leq e^{-\lambda t} \cdot |H(z)|, \quad t \in [0, b(z))$$

(resp. $|H(z(t))| \leq e^{\lambda t} \cdot |H(z)|, \quad t \in (a(z), 0]$).

As an immediate consequence, we have

**Corollary 2.2.** Assume $H: T^*M \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies (M_-) (resp. (M_+)), then for $\delta > 0$, $H^{-1}(0)$ and $H^{-1}((-\infty, \delta))$ is forward (resp. backward) invariant under $\Phi^t_H$.

**Proof.** Assume $H(z) \in [e_-, e_+]$ for some $e_- \leq 0 \leq e_+$. Then using proposition 2.1, we have

$$e_- \leq e^{-\lambda t} e_- \leq H(z(t)) \leq e^{-\lambda t} e_+ \leq e_+, \quad t \in [0, b(z)),$$

(resp. $e_- \leq e^{\lambda t} e_- \leq H(z(t)) \leq e^{\lambda t} e_+ \leq e_+, \quad t \in (a(z), 0]$),

which implies that $z(t) \in H^{-1}([e_-, e_+])$. This completes the proof. \qed

2.2. HJ equation and second Lyapunov function

As is mentioned in the introduction, (2) is the characteristic system of an associated Hamilton–Jacobi PDE, so that the dynamics of $\Phi^t_H$ is closely related to the solution of this equation.

Assume $H: T^*M \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies (M_-) (resp. (M_+)), we consider the Hamilton–Jacobi equation

$$H(x, d_x u, u) = 0, \quad x \in M \quad (HJ_-)$$

(resp. $H(x, -d_x u, -u) = 0, \quad x \in M$). \quad (HJ_+)
Notice that if \( H(x, p, u) \) satisfies (H1), (H2) and (M_+), then \( H(x, -p, -u) \) satisfies (H1), (H2) and (M_-). From the viscosity solution theory of Hamilton–Jacobi equations, we know that (H2) (resp. (H_+)) admits a unique (see theorem A.5, appendix) solution \( u_- \) (resp. \( u_+ \)) \( \in C(M; \mathbb{R}) \), which is in general not \( C^1 \) and should be understood in the viscosity sense. By the theory of viscosity solutions [16], \( u_\pm \) is Lipschitz continuous on \( M \). Rademacher theorem implies \( du_\pm \) exists for almost every point on \( M \). Thus for every \( x \in M \), the set
\[
D^*u_\pm(x) = \{ p \in T^*_xM : \exists \{ x_k \} \subseteq M \setminus \{ x \}, x = \lim_{k \to \infty} x_k, p = \lim_{k \to \infty} du_\pm(x_k) \}
\]
is non-empty and compact. A more detailed analysis [9, theorem 3.2.1, proposition 5.3.1, theorem 5.3.6] shows that

**Proposition 2.3.** \( u_- \) (resp. \( u_+ \)) : \( M \to \mathbb{R} \) is locally semiconcave (resp. semiconvex) and for every \( x \in M \), \( p \in D^*u_\pm(x) \),
\[
H(x, p, u_\pm(x)) = 0. \]

In particular, for every \( x \in T_xM \), the directional derivative
\[
\partial u_\pm(x, \dot{x}) = \lim_{h \to 0^+} \frac{u_\pm(x + h\dot{x}) - u_\pm(x)}{h}
\]
exists and
\[
\partial u_-(x, \dot{x}) = \min_{p \in D^*u_-(x)} \langle p, \dot{x} \rangle,
\partial u_+(x, \dot{x}) = \max_{p \in D^*u_+(x)} \langle p, \dot{x} \rangle.
\]

**Remark 2.4.** The function \( u_- \) (resp. \( u_+ \)) coincides with the backward (resp. forward) weak KAM solution for (H1) defined in [29].

The next lemma is well-known in convex analysis, it is used to derive the second Lyapunov function \( \Phi_H \). The proof is omitted, interested reader may refer to [25, theorem 23.5].

**Lemma 2.5.** Assume \( H : T^*M \times \mathbb{R} \to \mathbb{R} \) is a \( C^1 \) function satisfying (H1), then
\[
\langle p, \dot{x} \rangle - H(x, p, u) = \sup_{p' \in T^*_xM} \{ \langle p', \dot{x} \rangle - H(x, p', u) \}
\]
if and only if \( \dot{x} = \frac{\partial u}{\partial p}(x, p, u) \).

**Remark 2.6.** The coercive assumption (H2) cannot ensure that: for every \( (x, \dot{x}) \in TM \),
\[
\sup_{p' \in T^*_xM} \{ \langle p', \dot{x} \rangle - H(x, p', u) \}
\]
is attained by some \( p \in T^*_xM \). As is well known in convex analysis [9, appendix, theorem A.2.3], the conclusion is ensured by the superlinearity assumption on \( H \), i.e.,
\[
\lim_{\|p\|_x \to +\infty} \frac{H(x, p, u)}{\|p\|_x} = +\infty.
\]
For \( z = (x, p, u) \), assume \( H \) satisfies (M_-) (resp. (M_+)), let us define
\[
F(z) = u_-(x) - u \quad \text{resp.} \quad F(z) = u - u_+(x),
\]
(7)
then the following theorem shows that, besides \(|H|, F\) serves as another Lyapunov function on \(F^{-1}([0, +\infty))\).

**Theorem 2.7 (Second Lyapunov function).** For any \(z = (x, p, u) \in F^{-1}([0, +\infty))\),
\[
\mathcal{L}_{x_H}F(z) \leq -\lambda F(z) \quad \text{(resp. } \mathcal{L}_{-x_H}F(z) \leq -\lambda F(z)).
\] (8)

The left hand of (8) vanishes if and only if \(u = u_{-t}(x), p \in D' u_{-t}(x)\) (resp. \(u = u_{+t}(x), p \in D' u_{+t}(x))\) and for any \(p' \in D' u_{-t}(x)\) (resp. \(p' \in D' u_{+t}(x)\)),
\[
\langle p - p', \frac{\partial H}{\partial p}(x, p, u) \rangle \leq 0.
\]

**Proof.** Notice that the existence of direction derivative follows from proposition 2.3. In this proof, we set \(\dot{x} = \langle \text{d}x, X_H \rangle\), \(\dot{u} = \langle \text{d}u, X_H \rangle\).

Assume \(H\) satisfies (M.) and \(F(z) \geq 0\). For any \(p' \in D' u_{-t}(x)\), we use (2) to compute
\[
\mathcal{L}_{x_H}F(z) = \partial u_{-t}(x, \dot{x}) - \dot{u}
\leq \langle p', \dot{x} \rangle = \langle p', \dot{x} \rangle - \langle [p, \dot{x}] - H(x, p, u) \rangle
\leq \langle p', \dot{x} \rangle - \langle [p, \dot{x}] - H(x, p', u) \rangle
= H(x, p', u) - H(x, p', u - (x))
\leq \lambda(u - u_{-t}(x)) = -\lambda F(z),
\]
The third equality follows from proposition 2.3. The first inequality follows from (6), the second one uses lemma 2.5 and the third one is due to (M.).

Assume \(H\) satisfies (M.) and \(F(z) \geq 0\). We use (2) to compute
\[
\mathcal{L}_{-x_H}F(z) = -\dot{u} - \partial u_{+t}(x, -\dot{x})
= -\dot{u} + \langle p, \dot{x} \rangle = \langle p, \dot{x} \rangle - \langle [p, \dot{x}] - H(x, p, u) \rangle
\leq \langle p, \dot{x} \rangle - \langle [p, \dot{x}] - H(x, p, u) \rangle
= H(x, p, u) - H(x, p, u + (x))
\leq \lambda(u - u_{+t}(x)) = -\lambda F(z),
\]
where \(\tilde{p} \in D' u_{+t}(x)\) satisfies \(\partial u_{+t}(x, -\dot{x}) = \langle \tilde{p}, -\dot{x} \rangle\). The fourth equality follows from proposition 2.3. The first inequality uses lemma 2.5, the second one is due to (M.).

Note that by the inequality (8), if the left hand of (8) vanishes, then \(F(z) = 0\) and \(u = u_{-t}(x)\) (resp. \(u = u_{+t}(x)\)). In this case, \(\mathcal{L}_{x_H}F(z) = -\lambda F(z)\) (resp. \(\mathcal{L}_{-x_H}F(z) = -\lambda F(z)\)). Thus every inequality in the above computation occurs to be an equality. In particular, let \(\tilde{p} \in D' u_{-t}(x)\) (resp. \(\tilde{p} \in D' u_{+t}(x)\)) satisfies
\[
\langle p, \dot{x} \rangle = \partial u_{-t}(x, \dot{x}) \quad \text{(resp. } \langle p, \dot{x} \rangle = \partial u_{+t}(x, -\dot{x})\).
\]
The fourth equality reads as
\[
\langle p, \dot{x} \rangle - H(x, p, u) = \langle p, \dot{x} \rangle - H(x, \tilde{p}, u),
\]
since \(\dot{x} = \frac{\partial H}{\partial \dot{p}}(x, p, u)\), lemma 2.5 gives \(p = \tilde{p}\). By equation (6), we have for any \(p' \in D' u_{-t}(x)\) (resp. \(p' \in D' u_{+t}(x)\)),
\[
\langle p, \dot{x} \rangle = \langle p, \dot{x} \rangle \leq \langle p', \dot{x} \rangle, \quad \text{(resp. } \langle p, -\dot{x} \rangle = \langle p, -\dot{x} \rangle \geq \langle p', -\dot{x} \rangle\).
or equivalently
\[ \langle p - p', \frac{\partial H}{\partial p}(x, p, u) \rangle \leq 0. \]

The above theorem leads us to define
\[ U = \{ z \in T^*M \times \mathbb{R} : F(z) \leq 0 \} \]
\[ = \{ (x, p, u) \in T^*M \times \mathbb{R} : u \geq u_-(x) \} \]
(resp. \( = \{ (x, p, u) \in T^*M \times \mathbb{R} : u \leq u_+(x) \} \),
and for \( \delta > 0 \),
\[ U_\delta = \{ z \in T^*M \times \mathbb{R} : F(z) < \delta \} \]
\[ = \{ (x, p, u) \in T^*M \times \mathbb{R} : u > u_-(x) - \delta \} \]
(resp. \( = \{ (x, p, u) \in T^*M \times \mathbb{R} : u < u_+(x) + \delta \} \).

By equation (8), \( F \) is monotone decreasing along forward (resp. backward) \( \Phi_t^H \)-orbit segments in \( F^{-1}([0, +\infty)) \), then following corollary is a direct consequence of this fact.

**Corollary 2.8.** Assume \( H : T^*M \times \mathbb{R} \to \mathbb{R} \) satisfies \((M_-) \) (resp. \( (M_+) \)), \( U \) and \( U_\delta, \delta > 0 \) are forward (resp. backward) invariant under \( \Phi_t^H \).

Now we would like to compare some constructions in [22] with ours. In [22], the authors treated the contact Hamiltonian of discounted type, i.e., \( H(x, p, u) = \lambda u + h(x, p), \lambda > 0 \). In this case, the third equation in (2) is independent of first two, so (2) could be defined on \( T^*M \), forgetting the \( u \)-component. They constructed a function \( F_\lambda : T^*M \to \mathbb{R} \),

\[ F_\lambda(x, p) = \lambda u_-(x) + h(x, p), \quad (9) \]

playing similar role as \( F \) defined here. Let \( z = (x, p, u) \), \( F_\lambda \) can be seen as a function on \( T^*M \times \mathbb{R} \) by

\[ F_\lambda(x, p) = \lambda F(z) + H(z). \]

With this relation and the formula for \( H \) in hand, it is not difficult to adapt the above proof to show [22, section 5, lemma 1]. Conversely, theorem 2.7 is a partial (because the assumptions \((M_\pm) \) are inequalities) extension of [22, section 5, lemma 1] in non-differentiable case.

3. Proof of theorem A

For a discounted Hamiltonian \( H(x, p, u) = \lambda u + h(x, p) \), it is found [22, main theorem (i)] and [29, theorem 1.6] that all orbits of (2) are attracted by some compact, \( \Phi_t^H \)-invariant set \( K_H \). In this section, we shall show that the orbits of monotone contact Hamiltonian systems possess similar property, some topological information of \( K_H \) is also discussed.

The three subsections correspond to three conclusions in theorem A. In the following, \( U_0 \geq 0 \) denotes the \( C^0 \)-norm of \( u_- \) (resp. \( u_+ \)) \in \( C(M, \mathbb{R}) \) associated to the contact Hamiltonian \( H \) satisfying \((M_-) \) (resp. \((M_+)\)).
3.1. Forward or backward completeness

To show the forward (resp. backward) completeness of the flow, we need to build compact forward (resp. backward) invariant sets. This is due to the following simple corollary of (H2).

The proof, using a restatement of the assumption (see appendix A.1, (H2′)), is direct, so we omit it.

**Lemma 3.1.** Assume \( H : T^*M \times \mathbb{R} \to \mathbb{R} \) satisfies (M−) (resp. (M+)), then for every \( e, U \in \mathbb{R} \),

\[
\mathcal{Y}(e, U) := \{ (x, p, u) \in T^*M \times \mathbb{R} : H(x, p, u) \leq e, u \geq U \} \\
(\text{resp.} := \{ (x, p, u) \in T^*M \times \mathbb{R} : H(x, p, u) \leq e, u \leq U \})
\]

is compact.

We define

\[
\mathcal{Y} = H^{-1}(0) \cap \mathcal{U},
\]

and for \( \delta > 0 \),

\[
\mathcal{Y}_\delta = H^{-1}((-\infty, \delta)) \cap \mathcal{U}_\delta. \tag{10}
\]

Denote by \( \overline{\mathcal{Y}_\delta} \) the closure of \( \mathcal{Y}_\delta \), it follows directly from corollary 2.2, 2.8 and lemma 3.1 that

**Theorem 3.2.** Assume \( H : T^*M \times \mathbb{R} \to \mathbb{R} \) satisfies (M−) (resp. (M+)), \( \mathcal{Y} \) and \( \overline{\mathcal{Y}_\delta} \) are compact and forward (resp. backward) invariant under \( \Phi_t^\prime \).

An important corollary of the above proposition is

**Proof of (A1).** Assume \( H : T^*M \times \mathbb{R} \to \mathbb{R} \) satisfies (M−) (resp. (M+)), by theorem 2.7 and corollary 2.8, for every \( z \in T^*M \times \mathbb{R} \), we have the following dichotomy:

- there is \( T \in [0, b(z)) \) (resp. \( T \in (a(z), 0] \)) such that \( \Phi_t^\prime(z) \in \mathcal{U} \) for \( t \in [T, b(z)) \) (resp. \( t \in (a(z), T] \));
- \( \{ \Phi_t^\prime(z) \}_{t \in [0, b(z))} \subseteq \mathcal{U}^c \), then it follows that

\[
0 \leq F(\Phi_t^\prime(z)) \leq e^{-\lambda t}F(z) \leq F(z), \quad t \in [0, b(z)) \tag{11}
\]

(resp. \( 0 \leq F(\Phi_t^\prime(z)) \leq e^{M t}F(z) \leq F(z), \quad t \in (a(z), 0] \)).

Set \( e_z = |H(z)|, U_z = -U_0 - F(z) \) (resp. \( U_z = U_0 + F(z) \)), combining proposition 2.1 and the above, for every \( z \in T^*M \times \mathbb{R} \),

\[
\Phi_t^\prime(z) \in \mathcal{Y}(e_z, U_z), \quad t \in [0, b(z)) \ (\text{resp.} \ t \in (a(z), 0]).
\]

Now we use lemmas 3.1 and A.1 to see that \( b(z) = \infty \) (resp. \( a(z) = -\infty \)) and the conclusion follows.

Together with (5) and (11), we obtain that for every \( z \in T^*M \times \mathbb{R} \),

\[
\omega(z) \ (\text{resp.} \ a(z)) \in H^{-1}(0) \cap \mathcal{U} = \mathcal{Y} \tag{12}
\]

**Remark 3.3.** It is equivalent to say, if \( H \) satisfies (M−) (resp. (M+)), then \( \Phi_t^\prime : T^*M \times \mathbb{R} \to T^*M \times \mathbb{R} \) is well-defined for any \( t \geq 0 \) (resp. \( t \leq 0 \)).
3.2. Existence and location

As the second step, we prove the existence of maximal attractor (resp. repeller) by locating the maximal compact, $\Phi^\ast_{H_t}$-invariant set. This fact serves as a stronger version of (12).

**Theorem 3.4.** Any compact $\Phi^\ast_{H_t}$-invariant set is contained in $\mathcal{Y}$.

**Proof.** Assume there is $z = (x, p, u) \in T^\ast M \times \mathbb{R}$ and a compact set $S \subset T^\ast M \times \mathbb{R}$ such that $\Phi^\ast_{H_t}(z) = z(t) = (x(t), p(t), u(t)) \in S$ for all $t \in (a(z), b(z))$. We shall prove $z \in \mathcal{Y}$.

By corollary A.2, it is easy to see that $(a(z), b(z)) = \mathbb{R}$.

There is $e(S) > 0$ such that $|H(z')| \leq e(S)$ for any $z' \in S$. It follows from proposition 2.1 that for any $T > 0$, $|H(z)| \leq e^{-\lambda T} |H(z(-T))| \leq e^{-\lambda T} e(S)$, we send $T$ goes to infinity to find $z \in H^{-1}(0)$.

There is $U(S) > 0$ such that $|F(z')| \leq U(S)$ for any $z' \in S$. Similarly, one uses theorem 2.7 to deduce that for any $T > 0$ $F(z) \leq e^{-\lambda T} F(z(-T)) \leq e^{-\lambda T} U(S)$, Sending $T$ goes to infinity to find $F(z) \leq 0$ and $z \in U \cap H^{-1}(0) = \mathcal{Y}$.

As the consequences of (A1) and theorem 3.4 implies

**Theorem 3.5.** The maximal global attractor $K_H$ for $\Phi^\ast_{H}$ exists and

$$K_H \subset \mathcal{Y}.$$ 

Moreover, $\alpha(z) \neq \emptyset$ (resp. $\omega(z) \neq \emptyset$) if and only if $z \in K_H$.

**Proof.** Let $K_H$ be the closure of the union of all compact $\Phi^\ast_{H_t}$-invariant sets. By theorem 3.4, $K_H$ is a closed subset of $\mathcal{Y}$, thus is compact. It is easy to verify, by continuity of $\Phi^\ast_{H_t}$, that for all $z \in K_H$, $(a(z), b(z)) = \mathbb{R}$ and $K_H$ itself is $\Phi^\ast_{H_t}$-invariant.

Assume $H$ satisfies (M.) (resp. (M.)), for every $z \in T^\ast M \times \mathbb{R}$, by (12), $\omega(z)$ (resp. $\alpha(z)$) is a compact $\Phi^\ast_{H_t}$-invariant set in $\mathcal{Y}$, thus is a subset of $K_H$. So any neighbourhood $\mathcal{O}$ of $K_H$ is also a neighbourhood of $\omega(z)$ (resp. $\alpha(z)$) and, by the definition of $\omega$-limit set (resp. $\alpha$-limit set), there is $T(z, \mathcal{O}) > 0$ (resp. $T(z, \mathcal{O}) < 0$) such that $\Phi^\ast_{H_t}(z) \in \mathcal{O}$ for all $t \geq T$ (resp. $t \leq T$). Thus by definition 1.1, $K_H$ is a global attractor (resp. repeller) of $\Phi^\ast_{H_t}$ and its maximality is implied by the definition.

Assume $H$ satisfies (M.) (resp. (M.)) and $\alpha(z) \neq \emptyset$ (resp. $\omega(z) \neq \emptyset$). From lemma 5.1, we conclude that $(a(z), b(z)) = \mathbb{R}$. Using equation (4), it is clear that $z \in H^{-1}(0)$. Thus, by theorem 2.7,

$$\lim_{t \to -\infty} F(z(t)) \quad \text{(resp. } \lim_{t \to +\infty} F(z(t))) = \sup_{r \in \mathbb{R}} F(z(t)) \text{ exists.}$$

Now lemma 3.1 ensures that $\operatorname{Im}(z(t))$ is a compact $\Phi^\ast_{H_t}$-invariant set and is contained in $K_H$. This completes the proof. 

In [22, section 5, proposition 9], using (9), the authors showed that

$$\pi(K_H) \subseteq F_{-1}^{-1}((-\infty, 0]).$$

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where \( \pi : T^*M \times \mathbb{R} \to T^*M \) is the projection forgetting \( u \). This coincides with our result since \( \pi(Y) = F_{\lambda}^{-1}((-\infty, 0]) \) in the setting of [22] and

**Remark 3.6.** Since \( H \) satisfies \( (M_\land) \), \( Y \subset H^{-1}(0) \), \( \pi|_Y \) is a homeomorphism onto its image.

### 3.3. More information on \( K_H \)

The aim of this section is to prove the remaining conclusion of \( (A2) \) and \( (A3) \), thus complete the proof of theorem A.

Assume \( H \) satisfies \( (M^-) \) (resp. \( (M^+) \)), let us define

\[
\hat{K}_H = \bigcap_{t \geq 0} \Phi^t_H(Y_\delta) \text{ (resp. } \hat{K}_H = \bigcap_{t \leq 0} \Phi^t_H(Y_\delta) \text{)},
\]

(13) then \( \hat{K}_H \) is compact and by theorem 3.2, for every \( T \geq \tau > 0 \) (resp. \( T \leq \tau < 0 \)),

\[
\hat{K}_H \subseteq \bigcap_{t \in [0,T]} \Phi^t_H(Y_\delta) = \Phi^T_H(Y_\delta) \subseteq \Phi^\tau_H(Y_\delta)
\]

(14)

This directly implies that

**Theorem 3.7.** \( \hat{K}_H \) is the maximal compact \( \Phi^t_H \)-invariant set. In particular,

\[ K_H = \hat{K}_H. \]

**Proof.** We shall only consider the case that \( H \) satisfies \( (M^-) \). For the other case, we only need to replace all \( \geq \) by \( \leq \) in the following argument.

\( \hat{K}_H \) is \( \Phi^t_H \)-invariant: for every \( \tau \geq 0 \), by theorem 3.2,

\[
\Phi^\tau_H(\hat{K}_H) = \bigcap_{t \geq 0} \Phi^{t+\tau}_H(Y_\delta) = \bigcap_{t \geq 0} \Phi^t_H(\Phi^\tau_H(Y_\delta)) \subseteq \bigcap_{t \geq 0} \Phi^t_H(Y_\delta) = \hat{K}_H,
\]

where the first equality holds since \( \Phi^\tau_H \) is injective. By the definition of \( \hat{K}_H \),

\[
\hat{K}_H \subseteq \bigcap_{t \geq \tau} \Phi^t_H(Y_\delta) = \bigcap_{t \geq 0} \Phi^{t+\tau}_H(Y_\delta) = \Phi^\tau_H\left( \bigcap_{t \geq 0} \Phi^t_H(Y_\delta) \right) = \Phi^\tau_H(\hat{K}_H).
\]

Therefore we obtain

\[
\Phi^\tau_H(\hat{K}_H) = \hat{K}_H, \quad \text{for } \tau \geq 0.
\]

(15)

Since \( \hat{K}_H \) is compact, lemma 5.1 and (15) show that for every \( z \in \hat{K}_H \),

\[(a(z), b(z)) = \mathbb{R} \quad \text{and} \quad \Phi^\tau_H(\hat{K}_H) = \hat{K}_H, \quad \text{for } \tau \in \mathbb{R}.
\]

\( \hat{K}_H \) is maximal: assume \( S \) is a compact \( \Phi^t_H \)-invariant set. From theorem 3.4 and definition of \( \overline{Y_\delta} \),

\[ S \subseteq Y \subseteq \overline{Y_\delta}. \]
The above relation and $\Phi_H^t$-invariance of $\mathcal{K}$ give
\[
\mathcal{S} = \bigcap_{i \geq 0} \Phi_H^i(\mathcal{S}) \subseteq \bigcap_{i \geq 0} \Phi_H^i(\mathcal{Y}_\delta) = \hat{\mathcal{K}}_H.
\]
Combining the above discussions, theorem 3.5 and the compactness of $\hat{\mathcal{K}}_H$, we have $\mathcal{K}_H = \hat{\mathcal{K}}_H$. \hfill $\square$

**Remark 3.8.** Assume $H$ satisfies (M−) (resp. (M+)), by repeating the proof above, it is clear that
\[
\mathcal{K}_H = \bigcap_{i \geq 0} \Phi_H^i(\mathcal{S}) \quad \text{(resp. } \mathcal{K}_H = \bigcap_{i \leq 0} \Phi_H^i(\mathcal{S}),)
\]
$\mathcal{S}$ is any compact, forward (resp. backward) invariant set containing $\mathcal{Y}$.

The following lemma shows that $\Phi_H^T(\mathcal{Y}_\delta)$ is a good approximation of $\mathcal{K}_H$ (in the sense of topology) when $T > 0$ is large enough.

**Lemma 3.9.** For every open neighbourhood $\mathcal{O}$ of $\mathcal{K}_H$, there is $T(\mathcal{Y}_\delta, \mathcal{O}) > 0$ such that $\Phi_H^T(\mathcal{Y}_\delta) \subseteq \mathcal{O}$.

**Proof.** We assume, contrary to the conclusion, that there exist $T_n > 0$ and $z_n \in \Phi_H^{T_n}(\mathcal{Y}_\delta)$ with
\[
\lim_{n \to \infty} T_n = \infty, \quad z_n \notin \mathcal{O}, \quad (16)
\]
Since $z_n \in \mathcal{Y}_\delta \setminus \mathcal{O}$, which is clearly compact, then
\[
z_n \to z^* \in \mathcal{Y}_\delta \setminus \mathcal{O}. \quad (17)
\]
Now by (16), for every $T > 0$, there is $N \in \mathbb{N}$ such that if $n \geq N$, then $T_n \geq T$ and $z_n \in \Phi_H^{T_n}(\mathcal{Y}_\delta) \subseteq \Phi_H^{T_n}(\mathcal{Y}_\delta)$. It follows from compactness of $\Phi_H^T(\mathcal{Y}_\delta)$ that $z^* \in \Phi_H^T(\mathcal{Y}_\delta)$ for every $T > 0$, then
\[
z^* \in \bigcap_{i \geq 0} \Phi_H^i(\mathcal{Y}_\delta) = \mathcal{K}_H,
\]
which contradicts (17). \hfill $\square$

Now we are ready to complete the

**Proof of (A2).** The existence of maximal attractor (resp. repeller) is settled by theorem 3.5.

We turn to the second part of the conclusion.

Fixing $\delta > 0$, for any compact set $\mathcal{S} \subset T^* M \times \mathbb{R}$, we define
\[
T_1(\mathcal{S}) := \max \left\{ -\frac{1}{\lambda} \ln \left( \frac{\delta}{\max_{z \in \mathcal{S}} |H(z)|} \right), -\frac{1}{\lambda} \ln \left( \frac{\delta}{\max_{z \in \mathcal{S}} F(z)} \right) \right\},
\]
notice that $T_1 \leq 0$ if and only if $S \subseteq \mathcal{Y}_\delta$. By proposition 2.1 and 2.7,
\[
\Phi_H^t(\mathcal{S}) \subset \mathcal{Y}_\delta, \quad \text{for } t > T_1. \quad (18)
\]
We use lemma 3.9 to obtain $T_2(\mathcal{O}, \mathcal{Y}_\delta) > 0$ such that
\[
\Phi_H^t(\mathcal{Y}_\delta) \subset \mathcal{O}, \quad \text{for } t > T_2. \quad (19)
\]
Lemma 3.10. \( Y_\delta \) is homotopic equivalent to \( M \). In particular, \( Y_\delta \) is path-connected.

**Proof.** For every \((x, u) \in M \times \mathbb{R} \), by (H1) and (H2), there is a \( C^1 \) map \((x, u) \mapsto P_s(x, u) \in T_s^* M\) satisfying
\[
\frac{\partial H}{\partial p}(x, P_s(x, u), u) = 0.
\] (20)

Assume \( H \) satisfies (M−) (resp. (M+)), set
\[
\begin{align*}
\Sigma_− &:= \{(x, P_s(x, u_−(x)), u_−(x)) : x \in M\} \subset T^* M \times \mathbb{R}, \\
\Sigma_+ &:= \{(x, P_s(x, u_+(x)), u_+(x)) : x \in M\} \subset T^* M \times \mathbb{R}.
\end{align*}
\]
then it is clear that
\begin{itemize}
  \item \( H|_{\Sigma_\pm} \leq 0 \), thus \( \Sigma_\pm \subset Y_\delta \).
  \item \( \Sigma_\pm \) is homeomorphic to \( M \).
\end{itemize}

Now for \( z = (x, p, u), t \in [0, 1] \), define
\[
U_\pm(x, t) = (1 - t)u + tu_\pm(x)
\]
and continuous maps
\[
\begin{align*}
G_1(z, t) &= (x, (1 - t)p + tP_s(x, u), u), \\
G_{2, \pm}(z, t) &= (x, P_s(x, U_\pm(x, t)), U_\pm(x, t)), \quad (z, t) \in Y_\delta \times [0, 1].
\end{align*}
\]

**Claim.** \( G_1, G_{2, \pm} \) maps \( Y_\delta \times [0, 1] \) into \( Y_\delta \).

**Proof of the claim.** Observe that fixing any \((x, u), \) by (H1) and (20),
\begin{itemize}
  \item[(a)] the convex function \( H(x, \cdot, u) \) attains its minima at \( P_s(x, u) \),
  \item[(b)] for \( z = (x, p, u) \) with any \( p \in T_s^* M \), as a function in \( t \in [0, 1] \),
\end{itemize}
\[
H \circ G_1(z, t) = H(x, (1 - t)p + tP_s(x, u), u)
\]
is monotone decreasing.

Thus by (b), for any \( z \in Y_\delta \),
\[
H \circ G_1(z, t) \leq H \circ G_1(z, 0) = H(x, p, u) \leq \delta
\]
and \( u \) is unchanged under \( G_1 \), we have \( \text{Im}(G_1) \subset Y_\delta \).

For the map \( G_{2, \pm} \), first notice that
\[
U_-(x, t) > u_-(x) - \delta \quad \text{(resp. } U_+(x, t) < u_+(x) + \delta),
\]
thus \( \text{Im}(G_{2, \pm}) \subset \mathcal{U}_\delta \). Since \( H \) is monotone in \( u \), for any \((x, p) \in T^* M\),
\begin{align*}
\text{either } &H(x, p, U_+(x, t)) \leq H(x, p, u), & (21) \\
\text{or } &H(x, p, U_-(x, t)) \leq H(x, p, u_+(x)). & (22)
\end{align*}
Thus we obtain, for any $z \in \mathcal{Y}_\delta$,

either $H \circ G_{2, \pm}(z, t) = H(x, P_y(x, U_{\pm}(x, t)), U_{\pm}(x, t))$

$sle H(x, p, U_{\pm}(x, t)) \leq H(x, p, u) \leq \delta$.

where the first inequality uses (a) and the second inequality use (22);

or $H \circ G_{2, \pm}(z, t)$

$sle H(x, P(x, u_{\pm}(x)), U_{\pm}(x, t))$

$sle H(x, P(x, u_{\pm}(x)), u_{\pm}(x)) \leq 0,$

where the first inequality uses (b) with $u = U_{\pm}(x, t)$ and the second inequality use (23). Hence, $\text{Im}(G_{2, \pm}) \subset H^{-1}((\infty, \delta])$. This completes the proof of claim.

Besides, it is easy to see that $G_1(\cdot, 1) = G_2(\cdot, 0)$, thus we construct $G_{\pm} : \mathcal{Y}_{\delta} \times [0, 1] \to \mathcal{Y}_{\delta}$ by the usual concatenation

$G_{\pm}(z, t) = \begin{cases} G_1(z, 2t), & t \in \left[0, \frac{1}{2}\right]; \\
G_{2, \pm}(z, 2t - 1) \quad \text{(resp. } G_{2, +}(z, 2t - 1)), & t \in \left[\frac{1}{2}, 1\right]. 
\end{cases}$

Then we have $G_-$ (resp. $G_+$) is continuous and

- $G(z, 1) \in \Sigma_-$ (resp. $G(z, 1) \in \Sigma_+$) for any $z \in \mathcal{Y}_{\delta}$,
- $G(\cdot, 1) = \text{id}_{\Sigma_-}$ (resp. $G(\cdot, 1) = \text{id}_{\Sigma_+}$) for all $t \in [0, 1]$,

thus $G_-$ (resp. $G_+$) is a strong deformation retraction from $\mathcal{Y}_{\delta}$ to $\Sigma_-$ (resp. $\Sigma_+$). This finishes the proof. \(\square\)

**Remark 3.11.** By the same proof except some obvious adaption, lemma 3.10 holds with $\mathcal{Y}_{\delta}$ replaced by $\mathcal{Y}$.

We use lemmas 3.9 and 3.10 to give

**Proof of (A3).** Assume $H$ satisfies (M-) (resp. (M+)). Fix $\delta > 0$, we define for $t \geq 0$,

$O_t = \Phi_H^t(\mathcal{Y}_\delta) \quad \text{(resp. } O_t = \Phi_H^{2t}(\mathcal{Y}_\delta)).$

Since $\Phi_H^t, t \geq 0$ (resp. $t \leq 0$) is a diffeomorphism, we have $O_t$ is homotopically equivalent to $M$. Now theorem 3.2 and lemma 3.9 shows that $\{O_t\}_{t \geq 0}$ is a decreasing basis of neighbourhoods of $\mathcal{K}_H$.

To show $\mathcal{K}_H$ is connected. We argue by contradiction: assume there are two disjoint compact sets $\mathcal{S}, \mathcal{S}'$ such that $\mathcal{K}_H = \mathcal{S} \amalg \mathcal{S}'$, where $\amalg$ denotes the disjoint union. Choosing open neighbourhoods $\mathcal{S} \subset O, \mathcal{S}' \subset O'$ such that

$O \cap O' = \emptyset,$ \hspace{1cm} (23)

and $O \amalg O'$ is an open neighbourhood of $\mathcal{K}_H$.

By lemma 3.9, there is $T > 0$ such that $\Phi_H^T(\mathcal{Y}_\delta) \subseteq O \amalg O'$. By lemma 3.10, $\Phi_H^T(\mathcal{Y}_\delta)$ is path-connected, thus $z \in \mathcal{S}, z' \in \mathcal{S}'$ is connected by a path in $\Phi_H^T(\mathcal{Y}_\delta) \subseteq O \amalg O'$. This contradicts (23). \(\square\)
4. Proof of theorem B

In this section, we shall assume that $H : T^*M \times \mathbb{R} \to \mathbb{R}$ satisfies the additional assumptions (H3) and (H4) and then present a proof of theorem B. The crucial tool is a last Lyapunov function on $\mathcal{Y}$, guaranteed by the strict convexity of $H$.

4.1. Third Lyapunov function

We show that the coordinate function $u : T^*M \times \mathbb{R} \to \mathbb{R}$ serves as a Lyapunov function on $\mathcal{Y}$.

**Theorem 4.1 (Third Lyapunov function).** $u : T^*M \times \mathbb{R} \to \mathbb{R}$ is monotone increasing along $\Phi_t^H$-orbits in $H^{-1}(0)$. Moreover, $z_0 \in H^{-1}(0)$ is non-wandering under $\Phi_t^H$ if and only if $z_0 \in \mathcal{F}_H$.

**Proof.** Notice that the contact Hamiltonian $H$ satisfies (H1), thus $H(x, \cdot, u)$ is strictly convex on $T^*M$. Then for any $z = (x, p, u) \in T^*M \times \mathbb{R}$,

$$\frac{\partial H}{\partial p}(z) \cdot p \geq 0 \quad \text{and} \quad \frac{\partial H}{\partial p}(z) \cdot p = 0 \quad \text{if and only if} \quad p = 0.$$

For $z = (x, p, u) \in H^{-1}(0)$,

$$\dot{u} = \frac{\partial H}{\partial p}(z) \cdot p - H(z) = \frac{\partial H}{\partial p}(z) \cdot p \geq 0,$$

which verifies the first conclusion. From the above discussions, $\dot{u} = 0$ implies that

$$p = 0, \quad \dot{x} = \frac{\partial H}{\partial p}(x, 0, u) = 0. \quad (25)$$

Assume $z_0 = (x_0, p_0, u_0) \in H^{-1}(0)$ is non-wandering under $\Phi_t^H$, then along the $\Phi_t^H$-orbits initiating from $z_0$, $\dot{u} \equiv 0$. Now (25) implies that $p \equiv 0, \dot{x} \equiv 0$ and $z_0 \in \mathcal{F}_H$. □

Physically, $u : \mathcal{Y} \to \mathbb{R}$ plays the role of entropy in systems which realize the transfer between mechanical energy and internal energy. This may be seen from the mechanical system with frictions, i.e., $H(x, p, u) = \lambda u + \frac{1}{2} \|p\|^2 + V(x)$, where the Newton’s equation associated to (2) reads

$$\ddot{x} = -\nabla V(x) - \lambda \dot{x}, \quad (26)$$

where the term $-\lambda \dot{x}$ gives the friction. Thus the work done by friction along a $\Phi_t^H$-orbit $z : [a, b] \to \mathcal{Y}$,

$$W(z) = \int_a^b (\lambda \dot{x}, p) \, dt = \lambda \int_a^b \frac{\partial H}{\partial p} \cdot p \, dt = \lambda \int_a^b \dot{u} \, dt = \lambda [u(b) - u(a)] \geq 0.$$

Since on $\mathcal{Y}$, the contact Hamiltonian $H \equiv 0$, then the mechanical energy $\frac{1}{2} \|p\|^2 + V(x)$ decrease with the quantity $\lambda [u(b) - u(a)]$, and this part of energy converts to heat. Thus the variation of $u$ is proportional to the increase of internal energy.

4.2. Dynamics on $\mathcal{F}_H$

At the beginning, we assume $H$ satisfies (H3) and show something more general. Recall that

$$\mathcal{F}_H = \{(x_0, 0, u_0) \in T^*M \times \mathbb{R} : \partial_x H(x_0, 0, u_0) = 0, H(x_0, 0, u_0) = 0\},$$

thus by (2), $\mathcal{F}_H$ is the set of equilibria for $\Phi_t^H$. 3329
For two connected, compact, disjoint subsets $F_0, F_1$ of $F_H$, we use $\Sigma(F_0, F_1)$ to denote all $z \in \Gamma M \times \mathbb{R}$ satisfying

$$\alpha(z) = F_0, \omega(z) = F_1,$$

and $\Sigma_H$ to denote the union of all $\Sigma(F_0, F_1)$.

The compactness of $K_H$ follows from theorem 3.5. Let $z \in K_H$, $K_H$ is invariant under $\Phi_H^t$, hence

$$\alpha(z) \subseteq K_H, \omega(z) \subseteq K_H.$$

Elementary knowledge from dynamical system shows that $\alpha(z), \omega(z)$ are closed, connected and non-wandering with respect to $\Phi_H$. Thus, using theorem 4.1, $\alpha(z)$ and $\omega(z)$ are compact, connected subsets of $F_H$ and we conclude that

$$u|_{\alpha(z)} \equiv u_0, u|_{\omega(z)} \equiv u_1.$$

Assume further that $z \in K_H \setminus F_H$ and $z(t) = (x(t), p(t), u(t))$ is the $\Phi_H^t$-orbit through $z$. By (24), for any $t_0 < t_1$,

$$u(t_1) - u(t_0) = \int_{t_0}^{t_1} \dot{u} \, dt = \int_{t_0}^{t_1} \frac{\partial H}{\partial p}(z(t)) \cdot p(t) \, dt > 0$$

since $p(t)$ is not identically 0 on $[t_0, t_1]$. Thus $u$ is strictly increasing along $z(t)$ and we conclude that

$$u_0 < u_1, \quad \alpha(z) \cap \omega(z) = \emptyset. \quad (27)$$

Since the choice of $z \in K_H$ is arbitrary, the above discussion leads to

$$K_H = F_H \cup \Sigma_H. \quad (28)$$

**Proof of theorem B.** From (A2), if $H$ satisfies (M+) (resp. (M-)), then $\alpha(z) \neq \emptyset$ (resp. $\omega(z) \neq \emptyset$) if and only if $z \in K_H$. Assume $H$ satisfies (H4), then $F_H$ is a finite set. The connectedness of $F_0, F_1$ implies that both of them are singleton, so we assume $F_0 = \{z_0 = (x_0, 0, u_0)\}, F_1 = \{z_1 = (x_1, 0, u_1)\}$. This proves (B2). It follows from the definition of $\Sigma(F_0, F_1)$ that

$$\lim_{t \to -\infty} z(t) = z_0, \quad \lim_{t \to \infty} z(t) = z_1, \quad (29)$$

and the structure of $K_H$ follows from (28).

Since $K_H$ is compact, for any two equilibria $z', z'' \in \Sigma(z', z'')$ is closed. Let $z_0 \in F_H$ and $P_0$ the path-component of $K_H$ containing $z_0$. By (28) and (29), $P_0$ consists of finitely many equilibria and heteroclinic orbits between them and is therefore a union of finitely many closed sets. Thus $P_0$ is a closed subset of $K_H$. Since any two path-components are disjoint, by (A3), there is only one path-component and $K_H$ is path-connected. This proves (B1).

Finally, notice that (B3) is a direct consequence of (27).

□
4.3. Applications to discounted systems

To apply our results, we consider the discounted Hamiltonian

$$H(x, p, u) = \lambda u + h(x, p), \quad \lambda > 0,$$

(30)

where $h : T^*M \to \mathbb{R}$ satisfies (H1)–(H4) (these assumptions are independent of $u$). Then $X_H$ (or system (2)) could be reduced to the vector field (or discounted system)

$$X_{h, \lambda} : \begin{cases}
\dot{x} = \frac{\partial h}{\partial p}(x, p), \\
\dot{p} = -\frac{\partial h}{\partial x}(x, p) - \lambda p.
\end{cases}$$

(31)

defined on $T^*M$. Denote the phase flow of $X_{h, \lambda}$ by $\phi_{h, \lambda}^t : T^*M \to T^*M$. Using (A1), $\phi_{h, \lambda}^t$ is forward complete. Let $\Omega = d\alpha$, $X_{h, \lambda}$ is also called conformally symplectic since, by (31), $\mathcal{L}_{X_{h, \lambda}} \Omega = -\lambda \Omega$. This leads to

$$\phi_{h, \lambda}^t \omega = e^{-\lambda t} \omega \quad \text{for all } t \geq 0.$$  

(32)

For any $(x, p) \in T^*M$, set $(x(t), p(t)) := \phi_{h, \lambda}^t(x, p)$ and

$$u(t) = e^{-\lambda t} \left[ h(x, p) + \int_0^t e^{\lambda s} \left( \frac{\partial h}{\partial p} \cdot p - \dot{h} \right) ((x(s), p(s))ds \right],$$

then $z(t) = (x(t), p(t), u(t))$ satisfies (2) with the Hamiltonian (30). Thus by (4), we have for $t \in \mathbb{R}$,

$$\lambda u(t) + h(x(t), p(t)) = 0.$$  

(33)

Thus we obtain a converse version of remark 3.6.

**Lemma 4.2.** Assume for $(x, p) \in T^*M$, there is a compact subset $S \subset T^*M$ such that $\{(x(t), p(t)) : t \in \mathbb{R}\} \subset S$, then for $u(t)$ defined above,

$$(x(t), p(t), u(t)) \in K_H, \quad \text{for all } t \in \mathbb{R}.$$  

**Proof.** It follows directly from (33) that $u(t), t \in \mathbb{R}$ is bounded, thus the closure of $\{(x(t), p(t), u(t)) : t \in \mathbb{R}\}$ is a compact, $\phi_{h, \lambda}^t$-invariant set. \hfill $\square$

Denote the set of equilibria of $\phi_{h, \lambda}^t$ by

$$\mathcal{F}_{h, \lambda} = \{(x_0, 0) \in T^*M : \partial_x h(x_0, 0) = 0\},$$

for $(x_0, 0), (x_1, 0) \in \mathcal{F}_{h, \lambda}$ the set of all $(x, p) \in T^*M$ satisfying

$$\lim_{t \to +\infty} \phi_{h, \lambda}^t(x, p) = (x_0, 0), \quad \lim_{t \to +\infty} \phi_{h, \lambda}^t(x, p) = (x_1, 0),$$

by $\Sigma(x_0, x_1)$ and

$$\Sigma_{h, \lambda} = \cup_{(x_0, 0), (x_1, 0) \in \mathcal{F}_{h, \lambda}} \Sigma(x_0, x_1).$$

Similarly to definition 1.1, one can define the maximal attractor $\mathcal{K}_{h, \lambda}$ for $\phi_{h, \lambda}^t$, it is also a maximal compact, $\phi_{h, \lambda}^t$-invariant set. Now remark 3.6 and the above lemma help us translate theorem B into:

**Theorem 4.3.** Assume $h \in C^2(T^*M, \mathbb{R})$ satisfies (H1)–(H4), then
(a) The maximal attractor $\mathcal{K}_{h,\lambda} = \mathcal{F}_{h,\lambda} \cup \Sigma_{h,\lambda}$. In particular, $\mathcal{K}_{h,\lambda}$ is path-connected.

(b) For any $(x, p) \in T^*M$, both $\alpha(x, p, \omega(x, p))$ consist of at most one equilibrium in $\mathcal{F}_{h,\lambda}$.

(c) For two distinct equilibria $(x_0, 0), (x_1, 0) \in \mathcal{F}_{h,\lambda}$, assume $\Sigma(x_0, x_1) \neq \emptyset$, then $h(x_0, 0) > h(x_1, 0)$.

**Remark 4.4.** By (32), $\mathcal{K}_{h,\lambda}$ is of measure zero with respect to the Lebesgue measure $\Omega^x$ on $T^*M$. One could verify the above theorem on the dissipative pendulum $H(x, p, u) = \lambda u + \frac{1}{2} p^2 + \cos(x)$, $(x, p, u) \in T^*T \times \mathbb{R}$, see also [22, section 8] for more discussions and pictures. Our theorem shows that the dynamical behaviour of dissipative pendulum is typical.

**Acknowledgments**

L. Jin is supported in part by the National Natural Science Foundation of China (Grant Nos. 11901293, 11971232) and Start-up Foundation of Nanjing University of Science and Technology (No. AE89991/114). J. Yan is supported in part by the National Natural Science Foundation of China (Grant Nos. 11790273 and 11631106). The authors would like to thank the referees for their careful reading and for sharing their insights and helpful comments on the article under review, which help us to improve the article substantially. They also want to thank Prof. K. Wang, Prof. L. Wang and Dr. K. Zhao for helpful discussions on this topic.

**Appendix. Preliminaries**

This section serves as a supplementary explanation of several terms arising in the context. In particular, readers who are not familiar with the notion of viscosity solution may find more information about them.

**A.1. Vector field $X_H$ and its phase flow**

Let $X_H$ be the contact Hamiltonian vector field, $X_H$ is $C^1$ by the local expression (2). Recall that the local existence theorem [1, p 276, corollary] implies: for every $z \in T^*M \times \mathbb{R}$, there exists a neighbourhood $\mathcal{O}$ of $z, a_c < 0 < b_c$ and a map

$$\Phi_H : C([a_c, b_c] \times \mathcal{O}, T^*M \times \mathbb{R}) \ni (t, z') \mapsto \Phi_H(t; z')$$

called the phase flow, generated by $X_H$ satisfying for every $t \in [a_c, b_c]$.

- $\Phi_H(t; \cdot) : \mathcal{O} \rightarrow T^*M \times \mathbb{R}$ is a diffeomorphism onto its image.
- $\frac{d}{dt} \Phi_H(t; z') = X_H(\Phi_H(t; z'))$ and $\Phi_H(0; z') = z'$.

For every $z \in T^*M \times \mathbb{R}$, let $(a(z), b(z))$ be the maximum existence interval of the integral curve through $z$, the extension theorem [1, p 102, corollary 9] states that $\Phi_H$ is well-defined on some neighbourhood of $\text{Im}(z|_{[a, b]}), [a, b] \subset (a(z), b(z))$.

In the context of this article, we use the brief notation $\Phi_H^z(\cdot)$ to replace $\Phi_H(t; \cdot)$. By the above discussion, $\Phi_H^z(z), z \in (a(z), b(z))$ coincides with the unique integral curve $z(t) = (x(t), p(t), u(t))$ of $X_H$ through $z$ and

**Proposition A.1.** Assume $b(z) < \infty$ (resp. $a(z) > -\infty$), then

$$\lim_{t \rightarrow b(z)^-} |u(t)| + \|p(t)\|_{X(t)} = \infty \quad \text{(resp.} \lim_{t \rightarrow a(z)^+} |u(t)| + \|p(t)\|_{X(t)} = \infty).$$
Proof. It is enough to consider the case \( b(z) < \infty \) and we argue by contradiction: assume there is \( M > 0 \) and \( t_n < b(z) \), \( n \geq 1 \) such that

\[
\lim_{n \to \infty} t_n = b(z) \quad \text{and} \quad \limsup_{n \to \infty} |u(t_n)| + \|p(t_n)\|_{x(t_n)} \leq M.
\]

Thus, by passing to a subsequence, \( z_n = (x(t_n), p(t_n), u(t_n)) \) converges to some \( z_0 \) and by the definition of \( b(z) \),

\[
\lim_{n \to \infty} b(z_n) = \lim_{n \to \infty} [b(z) - t_n] = 0.
\]  

(34)

Applying the definition of local phase flow at \( z_0 \), for \( n \) large enough, \( \Phi^n(z_0) \) is well-defined on \([a_{z_0}, b_{z_0}]\). This leads to the conclusion \( b(z_0) > b_{z_0} > 0 \). This contradicts (34).

Applying the above proposition, we have the well-known extension theorem

**Corollary A.2.** For \( z \in T^*M \times \mathbb{R} \), if there is a compact subset \( K \subset T^*M \times \mathbb{R} \) such that

\[
z(t) \in K, \quad \text{for all } t \in (a(z), b(z)),
\]

then \((a(z), b(z)) = \mathbb{R} \).

A.2. Genericity of (H4)

This sub-appendix is devoted to a discussion on the condition (H4). Once and for all, we equip \( C^2(T^*M \times \mathbb{R}, \mathbb{R}) \) and \( C^2(M, \mathbb{R}) \) with strong (or Whitney) \( C^2 \)-topology [15, p 35]. Note that the set of monotone contact Hamiltonian is open in \( C^2(T^*M \times \mathbb{R}, \mathbb{R}) \) with respect to this topology. For \( u \in C^2(M, \mathbb{R}) \), we define \( \|u\|_{C^2} \) to be the supreme of the absolute value of \( u \) and all of its derivatives up to two on \( M \).

Notice that for a monotone contact Hamiltonian \( H \), the partial derivative \( \frac{\partial H}{\partial u} \) never vanishes. Then for each \( x \in M \), there exists a unique \( u_H(x) \) such that

\[
H(x, 0, u_H(x)) = 0, \quad \text{for all } x \in M.
\]  

(35)

By the implicit function theorem, \( u_H \in C^2(M, \mathbb{R}) \) and the map \( \text{H} \leftrightarrow u_H \) is continuous with respect to the \( C^2 \)-topology on the corresponding function spaces. It is not difficult to show that

**Proposition A.3.** A monotone contact Hamiltonian \( H \) satisfies (H4) if and only if \( u_H \) is a Morse function on \( M \).

Proof. Taking differential with respect to \( x \) on the both sides of the equation (35), we obtain

\[
\partial_x H(x, 0, u_H(x)) + \partial_u H(x, 0, u_H(x)) \, du_H(x) = 0.
\]  

(36)

For any \((x_0, 0, u_0) \in \mathcal{F}_H\),

\[
H(x_0, 0, u_0) = 0, \quad \partial_x H(x_0, 0, u_0) = 0,
\]

the definition of \( u_H \) gives \( u_0 = u_H(x_0) \) and

\[
\partial_u H(x_0, 0, u_H(x_0)) = 0.
\]

Now the equation (36) gives \( du_H(x_0) = 0 \) since \( \frac{du_H}{dx} \) never vanishes. Thus \((x_0, 0, u_0) \in \mathcal{F}_H \) if and only if \( x_0 \) is critical point of \( u_H \). Combining this fact, we take \( x \)-differential again on both sides of (36) to obtain that, for any \((x_0, 0, u_0) \in \mathcal{F}_H\),

\[
\partial_{xx} H(x_0, 0, u_H(x_0)) + \partial_h H(x_0, 0, u_H(x_0)) \, d^2 u_H(x_0) = 0,
\]  

(37)
where $d^2u_H$ denotes the Hessian of $u_H$.

If $H$ satisfies (H4), then for any $(x_0, 0, u_0) \in F_H, \partial_{x_0} H(x_0, 0, u_H(x_0))$ is non-degenerate as a quadratic form. By (37) and the fact that $\frac{\partial u_H}{\partial x_0}$ never vanishes, this is equivalent to that the Hessian $d^2u_H(x_0)$ is non-degenerate for any critical point $x_0$ of $u_H$. Thus $u_H$ is a Morse function. □

We claim that the set of monotone contact Hamiltonian satisfying (H4) is an open and dense subset of the set of monotone contact Hamiltonian with respect to the strong $C^2$-topology.

**Proof of the openness:** by the above proposition, if $H$ satisfies (H4), $u_H$ is a Morse function. Since the map $H \mapsto u_H$ is continuous, the for a $H^t C^2$-close to $H$, it is a monotone contact Hamiltonian and the associated function $u_{H^t}$ is $C^2$-close to $u_H$. It is known that the set of Morse functions is open with respect to the strong $C^2$-topology on $C^2(M, \mathbb{R})$, thus if $u_{H^t}$ is close enough to $H$, then $u_{H^t}$ must be a Morse function.

**Proof of the denseness:** since the set of Morse functions is dense with respect to the $C^2$-topology on $C^2(M, \mathbb{R})$, for any $\epsilon > 0$ and $u_H \in C^2(M, \mathbb{R})$, there is a Morse function $u' \in C(M, \mathbb{R})$ such that $\|u' - u_H\|_{C^2} < \epsilon$, then

$$
\|H(x, 0, u'(x))\|_{C^2} = \|H(x, 0, u_H(x)) - H(x, 0, u'(x))\|_{C^2} < C \epsilon,
$$

where

$$
C := \sup \left\{ \frac{\partial H}{\partial u}(x, 0, u) \mid (x, u) \in M \times \mathbb{R}, |u| \leq \|u_H\|_{C^2} + \epsilon \right\}.
$$

Let $\eta$ be a $C^\infty$ bump function such that

$$
\eta = \begin{cases}
1, & \{(x, p, u) \in T^*M \times \mathbb{R} \mid ||p||_1, \leq 1, |u| \leq \|u_H\|_{C^0} + 1\}; \\
0, & \{(x, p, u) \in T^*M \times \mathbb{R} \mid ||p||_1, \geq 2, |u| \geq 2\|u_H\|_{C^0} + 2\}.
\end{cases}
$$

We construct

$$
H'(x, p, u) := H(x, p, u) - \eta(x, p, u) H(x, 0, u'(x)),
$$

Then $H'$ is a monotone contact Hamiltonian with $u_{H'} = u'$ and is close to $H$ in the strong $C^2$-topology.

### A.3. Lipschitz estimate of viscosity solutions

Let $H : T^*M \times \mathbb{R} \to \mathbb{R}$ be a contact Hamiltonian satisfying (H1), (H2) and (M−) (resp. (M+)). Since $M$ is compact, the assumption (H2) and (M−) (resp. (M+)) implies

(H2') for every $e, U \in \mathbb{R}$, there is $P(e, U) > 0$ such that if $||p||_1 > P, u \geq U$ (resp. $u \leq U$), then $H(x, p, u) > e$.

It is well-known that (H−) does not admit $C^1$ solutions in general. The following definition is originally due to Crandall and Lions [13] and is used extensively in the study of Hamilton–Jacobi equations.

**Definition A.4.** Let $u : M \to \mathbb{R}$ be a continuous function.

We call $u$ a viscosity sub-solution (resp. super-solution) of (H−) if for every $x \in M$, $\phi \in C^1(M, \mathbb{R})$ such that $u - \phi$ attains a local maximum (resp. minimum) at $x$,

$$
H(x, \phi(x), u(x)) \leq 0 \quad \text{(resp. } H(x, \phi(x), u(x)) \geq 0\text{)}.
$$
We call $u$ a viscosity solution of $(\mathcal{H}J_-)$ if it is both a viscosity sub- and super-solution of $(\mathcal{H}J_-)$.

The following property is crucial in deducing the uniqueness and the Lipschitz estimate of viscosity solution of $(\mathcal{H}J_-)$. The proof is standard and we refer to [11, theorem 3.2].

**Theorem A.5 (Comparison principle).** Assume $H : T^*M \times \mathbb{R} \to \mathbb{R}$ satisfies $(\mathcal{M}^-)$ and $u, v \in C(M, \mathbb{R})$ are respectively viscosity sub- and super-solutions of $(\mathcal{H}J_-)$. Then $u \leq v$ on $M$.

**Remark A.6.** Assume $H : T^*M \times \mathbb{R} \to \mathbb{R}$ satisfies $(\mathcal{M}^+)$, then $\tilde{H}(x, p, u) := H(x, -p, -u)$ satisfies $(\mathcal{M}^-)$ and the equation $(\mathcal{H}J_-)$ for $\tilde{H}$ is just $(\mathcal{H}J_+)$. Thus theorem A.5 also applies to $(\mathcal{H}J_+)$. To give an estimate of $u_\pm$, the idea is to find constant sub- and super-solutions of $(\mathcal{H}J_\pm)$ and apply the comparison principle. Assume $H$ satisfies $(\mathcal{M}^-)$ (resp. $(\mathcal{M}^+)$), we define constants $U_0 = \min_{x \in M} uH(x), \quad U = \max_{x \in M} uH(x),$ it follows that $H(x, 0, U) \leq H(x, 0, uH(x)) = 0 \leq H(x, 0, U) \quad \text{(resp. } H(x, 0, -(-U)) \leq H(x, 0, uH(x)) = 0 \leq H(x, 0, -(-U))).$

Thus
- $u \equiv U$ (resp. $u \equiv -U$) is a sub-solution of $(\mathcal{H}J_-)$ (resp. $(\mathcal{H}J_+)$),
- $u \equiv U$ (resp. $u \equiv -U$) is a super-solution of $(\mathcal{H}J_-)$ (resp. $(\mathcal{H}J_+)$).

Note that $u_\pm$ is By comparison principle for$(\mathcal{H}J_-)$ (resp. $(\mathcal{H}J_+)$),
\[ U_0 \leq u_-(x) \leq U \quad \text{(resp. } U_0 \leq u_+(x) \leq U). \tag{38} \]
Combining definition A.4 and proposition 2.3, for almost every $x \in M$,
\[ H(x, du_-(x), U) \leq 0 \quad \text{(resp. } H(x, du_+(x), U) \leq 0). \]
Thus by $(\mathcal{H}_2')$, for almost every $x \in M$,
\[ \|du_-(x)\|_x \leq P(0, U) \quad \text{(resp. } \|du_+(x)\|_x \leq P(0, U)). \tag{39} \]
Notice that (38) and (39) give the desired estimate.

**ORCID iDs**

Liang Jin [https://orcid.org/0000-0002-9986-2414](https://orcid.org/0000-0002-9986-2414)

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