A New Regularization Method in 3-Dimensional Momentum Space

Liang-gang Liu$^{1,2}$, Xiang-qian Luo$^{1,2}$, Wei Chen$^2$
$^1$CCAST (World Laboratory) P. O. Box 8730, Beijing, 100080
$^2$Department of Physics, Zhongshan University
Guangzhou 510275, P. R. China

March 28, 2022

Abstract

We propose a new method to calculate the 4-dimensional divergent integrals. By calculating the one loop integral as an example, the regularization of the integrals in 3-dimension momentum space are given in details. We find that the new method gives the same results as the traditional dimensional regularization method gives, but the new method has the advantage that it gives the real and the imaginary part separately.

PACS number(s): 11.10.Gh
1 Introduction

It is well known that the dimensional regularization method is a powerful and elegant tool to calculate and regularize the divergent integrals in 4-dimensional energy-momentum space\(^{(1,2)}\). This method is widely used both in particle physics as well as in nuclear physics\(^{(3,4)}\) in calculating the loops integration and renormalization. But in some cases, such as of the zero-point fluctuation energy \(\int_{-\infty}^{\infty} dp \sqrt{\not{p}^2 + m^2}\), where \(m\) is the mass of nucleon or mesons, cannot be calculated by using the dimensional regularization formulae directly. In this case, a cutoff momentum is introduced to truncate the upper limit of the integration to make the divergence controllable\(^{(5)}\). In our studies, we found this kind of 3-dimensional divergent integrals can be regularized by a new method \(^{(6)}\). In this paper, we will demonstrate that this method is also applicable to calculate other divergent integrals in 4-dimensional energy-momentum space.

In the next section, we will show briefly the traditional dimensional regularization of the integrals \(F_1\) and \(F_2\). In sect. 3, we give the details of the calculation of \(F_1\), \(F_2\) by our new method. A summary is given in the last section.

2 4-Dimensional regularization of \(F_1\) and \(F_2\)

In the one loop level of vacuum fluctuation, the polarization insertion to the vertex function or propagators can be expressed in term of two functions \(^{(7)}\):

\[
F_1 = \frac{i}{(2\pi)^4} \int_{-\infty}^{\infty} d^4q \frac{1}{q^2 - m^2 + i\epsilon},
\]

\[
F_2(k^2, m_1^2, m_2^2) = \frac{i}{(2\pi)^4} \int_{-\infty}^{\infty} d^4q \frac{1}{(q^2 - m_1^2 + i\epsilon)(q - k)^2 - m_2^2 + i\epsilon)},
\]

Here \(k, q\) are four energy-momentum.\(^{[1]}\) The subscript \(\infty\) indicates the integration is in the whole 4-dimensional space. By use of the dimensional regularization technique\(^{(1,2,3)}\), \(F_1\) and \(F_2\) are readily calculated, the results are follows:

\[
F_1 = -\frac{m^2}{16\pi^2} (\Gamma(\epsilon) - \ln m^2 + 1 + \Theta(\epsilon)),
\]

\[
F_2(k^2, m_1^2, m_2^2) = -\frac{1}{16\pi^2} (\Gamma(\epsilon) - \int_0^1 dx \ln M_k^2(x) + \Theta(\epsilon)),
\]

\[
M_k^2(x) = k^2 x^2 - (k^2 + m_1^2 - m_2^2)x + m_1^2,
\]

\(^{[1]}\)We follow the convention of J. D. Bjorken AND S. D. Drell, Relativistic Quantum Fields, McGraw-Hill, New York, (1965).
where $\epsilon \to 0^+$, $\Gamma(\epsilon)$ is gamma function, $\Gamma(\epsilon) \sim \frac{1}{\epsilon}$ so $F_1, F_2$ are divergent. Notice that $M_k^2(x)$ is not positive-definite, provided that $k^2 > (m_1 + m_2)^2$, which is the threshold that the incoming particle can decay into particles $m_1, m_2$. Thus a imaginary part of $F_2$ appears,

$$\text{Im} F_2(k^2, m_1^2, m_2^2) = -\frac{1}{16\pi} \frac{\sqrt{\Delta}}{k^2} \theta(k^2 > (m_1 + m_2)^2), \quad (6)$$

$$\Delta = |k^2 - (m_1 + m_2)^2||k^2 - (m_1 - m_2)^2|. \quad (7)$$

## 3 Regularization in 3-dimensional space

We propose in the following a new regularization method in 3-dimensional space. Taking $F_1$ and $F_2$ as examples to demonstrate that the results are the same as above.

### 3.1 Regularization of $F_1$

The denominate in eq. (1) can be expressed as $(q_0 - E_q + i\epsilon)(q_0 + E_q - i\epsilon)$, here $E_q = \sqrt{q^2 + m^2}$. By use of Cauchy theorem, the integration of $q_0$ in the complex $q_0$ plane gives:

$$F_1 = \frac{1}{2(2\pi)^3} \int_\infty dE_q.$$  \quad (8)

Now we assume the 3-dimensional integration is performed in n-dimension with the limit $n \to 3$, so $dq \to d^n q = q^{n-1} dq d\Omega_n$ with $\int_{\infty} d\Omega_n = 4\pi$ in the limit $n \to 3$, here $q \equiv |q|$. Then eq. (8) can be reduced as follows:

$$F_1 = \frac{1}{8\pi^2} (m^2)^{\frac{n-3}{2}} \int_0^\infty dt t^{\frac{n}{2} - 1} (1 + t)^{-\frac{3}{2}}$$  \quad (9)

$$= \frac{1}{8\pi^2} (m^2)^{\frac{n-3}{2}} \frac{\Gamma(\frac{n}{2}) \Gamma(\frac{1}{2} - \frac{n}{2})}{\Gamma(\frac{1}{2})}.$$

Expand it in a Laurent series about $\epsilon = \frac{3-n}{2}$, it gives a result which is the same as eq. (3).

### 3.2 Regularization of $F_2(k^2, m_1^2, m_2^2)$

$F_2(k^2, m_1^2, m_2^2)$ can be rewritten as follows:

$$F_2(k^2, m_1^2, m_2^2) = \frac{i}{(2\pi)^4} \int_0^1 dx \int d^n q dq_0 \frac{1}{[q_0^2 - q^2 - M_k^2(x)]^2}. \quad (10)$$
Since $M_k^2(x)$ can be both positive and negative, we should locate the poles in the denominator of eq. (11). The analysis shows that

\[ M_k^2(x) > 0 \text{ for } k^2 < (m_1 + m_2)^2, \]

or \[ k^2 > (m_1 + m_2)^2 \text{ and } 0 < x < x_1, \]

or \[ k^2 > (m_1 + m_2)^2 \text{ and } x_2 < x < 1, \]

$M_k^2(x) < 0$ for \[ k^2 > (m_1 + m_2)^2 \text{ and } x_1 < x < x_2, \]

where \[ x_{1,2} = \frac{1}{2k^2}(k^2 + m_1^2 - m_2^2 ± \sqrt{\Delta}). \] (12)

Then eq. (11) can be written as follows:

\[
F_2(k^2, m_1^2, m_2^2) = \frac{i}{(2\pi)^4}\{\theta[k^2 < (m_1 + m_2)^2]\int_0^1 dx \int_\infty^\infty d^n q \, dq_0 \frac{1}{(q_0^2 - E_q(x)^2 + i\varepsilon)^2}
\]

\[
+ \theta[k^2 > (m_1 + m_2)^2]\left(\int_0^{x_1} dx + \int_1^{x_2} dx\right) \int_\infty^\infty d^n q \, dq_0 \frac{1}{(q_0^2 - E_q(x)^2 + i\varepsilon)^2}
\]

\[
+ \theta[k^2 > (m_1 + m_2)^2]\int_{x_1}^{x_2} dx \int_\infty^\infty d^n q \, dq_0 \frac{1}{(q_0^2 - q^2(x) + |M_k^2(x)|)^2},
\]

\[
\text{here } E_q(x)^2 = q^2 + M_k^2(x). \]

It can be verified that

\[
\int_\infty^\infty d^n q \, dq_0 \frac{1}{(q_0^2 - E_q(x)^2 + i\varepsilon)^2} = \int_\infty^\infty d^n q \, dq_0 \frac{1}{(q_0^2 - q^2(x) + |M_k^2(x)|)^2}
\]

\[
= i\pi^2(\Gamma(\varepsilon) - \ln M_k^2(x) + O(\varepsilon)).
\] (14)

Substitute eq. (14) into eq. (13) we obtain

\[
F_2(k^2, m_1^2, m_2^2) = -\frac{1}{16\pi^2}(\Gamma(\varepsilon) - \int_0^1 dx \ln|M_k^2(x)|) + O(\varepsilon)
\]

\[
+ \frac{i}{(2\pi)^4} \cdot (-i\pi^2) \cdot (x_2 - x_1) \cdot i\pi \theta[k^2 > (m_1 + m_2)^2]
\]

\[
= -\frac{1}{16\pi^2}(\Gamma(\varepsilon) - \int_0^1 dx \ln|M_k^2(x)|)
\]

\[
- \frac{i}{16\pi} \frac{\sqrt{\Delta}}{k^2} \theta[k^2 > (m_1 + m_2)^2] + O(\varepsilon).
\]

We can see that it is the same as eq. (4) and eq. (6), therefore, demonstrate that 3-dimensional regularization gives the same results as traditional 4-dimensional regularization method. This method can be extended to calculate other divergent integrals, such as zero-point fluctuation energy mentioned in the Introduction. The calculation gives

\[ -\frac{1}{(2\pi)^3} \int_\infty d^q \sqrt{q^2 + m^2} \rightarrow -\frac{1}{(2\pi)^3} \int_\infty d^n q \sqrt{q^2 + m^2} \]

\[ = -\frac{1}{32\pi^2} m^4(\Gamma(\varepsilon) - \ln m^2 + \frac{3}{2}) + O(\varepsilon), \] (16)
which is again the same as that given in ref. (8).

4 Summary

We have demonstrated by calculating integrals $F_1$ and $F_2$ that the new 3-dimensional regularization method gives the same results as those the traditional 4-dimensional regularization method gives. We also show that the new method can be used to calculate the zero-point fluctuation energy of relativistic particles. In fact, this method can be generalized further to evaluate the quantum effect in two or one dimensional systems, it has a wide application perspective in physics.

References

(1) G. ‘t Hooft and M. Veltman, Nucl. Phys., B44 189 (1972).

(2) P. Ramond, Field Theory: A Modern Primer, Addison-Wesley, Redwood City, (1981).

(3) B. D. Serot, J. D. Walecka, Recent Progress in Quantum Hadrodynamics, to be published in the International Journal of Modern Physics E.

(4) L. G. Liu, W. Bentz and A. Arima, Ann. Phys., 194, 387 (1989).

(5) M. Nakano, et al, Phys. Rev., C55, 1 (1997).

(6) L. G. Liu, Quantum Effects in Nuclear Matter, Ph. D. thesis, University of Tokyo, (1989).

(7) J. A. Mignaco, E. Remiddi, IL Nuovo Cimento, 1A, 376 (1971).

(8) S. A. Chin, Ann. Phys., 108, 301 (1977).