Certain extensions of vertex operator algebras of affine type

Haisheng Li
Department of Mathematical Sciences, Rutgers University, Camden, NJ 08102

Abstract We generalize Feigin and Miwa’s construction of extended vertex operator (super)algebras $A_k(sl(2))$ for other types of simple Lie algebras. For all the constructed extended vertex operator (super)algebras, irreducible modules are classified, complete reducibility of every module is proved and fusion rules are determined modulo the fusion rules for vertex operator algebras of affine type.

1 Introduction

In the development of vertex operator algebra theory, one of the most important problems is to construct new solvable vertex operator (super)algebras in the sense that irreducible modules and fusion rules can be completely determined and that intertwining operators can be explicitly constructed. To a certain extent, such algebras give rise to solvable physical models. One of many ways to get such vertex operator (super)algebras is to consider certain extensions of some well known algebras. For example ([MS], [Li5]), the vertex operator algebra $L(k,0)$ associated to the affine Lie algebra $\hat{sl}_2$ with a positive even integral level $k$ can be extended to a vertex operator (super)algebra $L(k,0)+L(k,k)$. When $k$ is odd, $L(k,0)+L(k,k)$ does not have an extended vertex operator (super)algebra structure because of the failure of the locality. (For a certain class of vertex operator algebras, e.g., vertex operator algebras associated to positive-definite even lattices, as proved in [DL], the sum of a copy of each irreducible module does have a nice structure, called an abelian intertwining algebra. See [DL], [FFR], [M] and [Hua2] for notions of various generalized structures.)

In [FM], Feigin and Miwa constructed a family of extended vertex operator (super)algebras $A_k$ from the vertex operator algebras $L(k,0)$ associated to the affine Lie algebra $\hat{sl}_2$ with an arbitrary positive integral level $k$, and they classified all irreducible modules and determined all fusion rules. In addition they obtained very interesting results on the monomial basis for irreducible modules. This paper was mainly motivated by [FM] and [DLM2]. As the main results of this paper we generalize their results except the monomial basis result to affine Lie algebras of other types by using a different approach.

The algebras $A_k$ were defined in [FM] by a set of mutually local vertex operators (or fields). On the other hand, in terms of vertex operator algebra language, $A_k$ are extensions (by an infinite sum of irreducible modules) of vertex operator algebra $L(k,0)\otimes$...
$M(1,0)$, where $M(1, 0)$ is the vertex operator algebra associated to an infinite-dimensional Heisenberg Lie algebra of rank one, or a single free bosonic field. The essential building block of $A_k$ is the irreducible $L(k, 0) \otimes M(1,0)$-module $L(k, k) \otimes M(1, \alpha)$ for some $\alpha \in \mathbb{C}$.

The $L(k, 0)$-module $L(k, k)$ has been known to be a simple current ([FG], [GW], [SY]) in the sense that the left multiplication of the equivalence class $[L(k, k)]$ in the Verlinde algebra gives rise to a permutation on the standard basis. It was known to physicists (cf. [MS], [SY]) that a simple current with integer weights can be included to generate an extended vertex operator algebra. In [Li5], as an exercise by using an explicit construction of simple currents given in [Li4] we studied the extension of a certain vertex operator algebra by a self-dual (or order 2) simple current where $L(k, 0) + L(k, k)$ is a special case. For such extended algebras, all their irreducible modules were classified and the complete reducibility of every module was proved. A little bit latter, the results of [Li5] were greatly extended in [DLM2].

The construction of simple currents given in [Li4] was based on a result of [Li2]. Let $V$ be a vertex operator algebra and let $h$ be a weight one primary vector in $V$ such that the component operators $h(m)$ of the vertex operator $Y(h, z)$ satisfy the Heisenberg algebra relation and such that $h(0)$ is semisimple on $V$ with only rational eigenvalues. Clearly, $\sigma_h := e^{2\pi i h(0)}$ is an automorphism of $V$. Define

$$\Delta(h, z) = z^{h(0)} \exp \left( \sum_{n=1}^{\infty} \frac{h(n)}{-n} (-z)^{-n} \right),$$

an element of $(\text{End} V)\{z\}$. It was proved in [Li2] that for any $V$-module $W$,

$$(W^{(h)}, Y_h(\cdot, z)) := (W, Y(\Delta(h, z), \cdot, z))$$

is a $\sigma_h$-twisted $V$-module. In particular, this gives an (untwisted) $V$-module if $h(0)$ acting on $V$ has only integral eigenvalues. It was furthermore proved in [Li4] that if $I(\cdot, z)$ is an intertwining operator of type $\left( \frac{w_1}{w_1 w_2} \right)$ (in the sense of [FHL]), then $I(\Delta(h, z), \cdot, z)$ is an intertwining operator of type $\left( \frac{w_1^{(h)}}{w_1 w_2^{(h)}} \right)$. By using this result and the invertibility of $\Delta(h, z)$, it was proved that $V^{(h)}$ is a simple current. As a matter of fact, for certain well known vertex operator algebras almost all simple currents can be constructed in this way. For example, when $V$ is the vertex operator algebra $V_L$ associated with a positive-definite even lattice $L$, it was proved that all irreducible $V_L$-modules can be constructed this way. In this case, this construction is intimately related to the construction of twisted modules by using shifted vertex operators in [Le]. When $V = L(\ell, 0)$ associated to an affine Lie algebra $\hat{g}$ with $g \neq E_8$ and with a positive integral level $\ell$, all the simple currents can also be constructed this way. The merit of this construction of a simple current is on the canonicalness of the vector space and the vertex operator map (in terms of the algebra $V$ and the element $h$). With this construction, certain intertwining operators can be
constructed canonically. The canonical construction of intertwining operators of certain types is the basis of [Li5] and [DLM2].

The essential results of [DLM2] can be described as follows: Let $V$ be a vertex operator algebra and $H$ be a subspace of $V_{(1)}$ such that the components $h(n)$ of vertex operators $Y(h, z)$ for $h \in H$, $n \in \mathbb{Z}$ satisfy the Heisenberg algebra relation. Let $L$ be a subgroup of $H$ such that for every $\alpha \in L$, $\alpha(0)$ acts semisimply on $V$ with only integral eigenvalues. Consider the space $V[L] = \mathbb{C}[L] \otimes V$, where for $\alpha \in L$, $e^{\alpha} \otimes V$ is identified with $V^{(\alpha)}$ equipped with the $V$-module structure $Y_\alpha$. Extend the $V$-module structure on $V[L]$ in a certain canonical way to a vertex operator map $Y$ on $V[L]$. Then it was proved that $V[L]$ equipped with the defined vertex operator map $Y$ is a generalized vertex algebra in the sense of [DL]. It was proved that $V[L]$ is a vertex operator (super)algebra when $L$ satisfies certain conditions. When $V = M_h(1, 0)$ with $h = \mathbb{C} \otimes \mathbb{Z} L$, where $L$ is an integral lattice, we have $V[L] = V_L$ ([FLM], [DL]). Then we may view $V[L]$ as a generalization of $V_L$. In [DLM2], we were mainly interested in the case $V = L(k, 0)$ associated to an affine algebra $\hat{g}$ with a positive integral level $k$. Because $L(k, 0)$ has only finitely many irreducible modules up to equivalence, each irreducible $V$-module $V^{(\alpha)}$ in $V[L]$ is not multiplicity-free. Having noticed this we proved that $V[L]$ has a quotient algebra $\overline{V[L]}$ such that every irreducible $V$-module in $\overline{V[L]}$ is multiplicity-free and that $V[L]$ and $\overline{V[L]}$ contain the same number of non-isomorphic irreducible $V$-modules. An irreducible $V$-module $W$ was also extended to $W[L]$ on which $V[L]$ acts and it was proved that $W[L]$ is in general a twisted $V[L]$-module with respect to a certain automorphism of $V[L]$. With this result, under the assumption of complete reducibility of $V$-modules, all irreducible $V[L]$-modules were classified and a complete reducibility theorem for $V[L]$-modules was proved.

In this paper, to generalize Feigin and Miwa’s construction we apply the results of [DLM2] by taking $V = L(k, 0) \otimes M_h(1, 0)$ and by choosing $h'$ and $L$ appropriately, depending on the type of $g$. In this case, each irreducible $V$-module in $V[L]$ is multiplicity-free and $V[L]$ is a simple algebra. On the other hand, since the category of $V$-modules is not semisimple, the results of [DLM2] for the complete reducibility of $V[L]$-modules do not apply to this case directly. The complete reducibility of $V[L]$-modules is proved here. We also naturally extend intertwining operators for modules in the category of $V$-modules to intertwining operators for modules in the category of $V[L]$-modules. Using this result we are able to derive a formula of fusion rules for $V[L]$-modules in terms of fusion rules for $V$-modules.

This paper is organized as follows: In Section 2, we recall the classification of irreducible modules for certain vertex operator algebras and recall a construction of simple currents. Most part of this section is preliminary and the only new result is about the fusion rules for simple currents for $L(k, 0)$. In Section 3 we recall and refine some of the results of [DLM2] on the extended vertex (super)algebra $V[L]$. We furthermore study
the multiplicity-free case. In Section 4, we apply the results of Section 3 to construct extended vertex operator (super)algebras associated to an affine algebra \( \hat{g} \).

## 2 Vertex operator algebras and simple currents

The extended vertex operator (super) algebras we shall construct are based on vertex operator algebras \( L_g(\ell, 0) \), \( M_h(1, 0) \), \( V_L \) and their representations. For this reason, in this section we shall recall the relevant information about these algebras and we also recall from [Li4] a construction of simple currents for a certain type of vertex operator algebras, including \( L_g(\ell, 0) \), \( M_h(1, 0) \), \( V_L \). For the vertex operator algebra \( L_g(\ell, 0) \) associated to an affine algebra \( \hat{g} \) (not type \( E_8^{(1)} \)) with a positive integral level \( \ell \), we prove that the equivalence classes of the simple currents form an abelian group isomorphic to \( P^\vee / Q^\vee \), where \( P^\vee \) and \( Q^\vee \) are the co-weight and co-root lattices of \( g \).

### 2.1 Vertex operator algebras \( L_g(\ell, 0) \), \( M_h(1, 0) \) and \( V_L \)

We shall use standard definitions and notations as given in [FHL] and [FLM] and we also use [K] and [H] as our references for (Kac-Moody) Lie algebras. Following [DL] we use the term “vertex (super)algebra” for an object that satisfies all the axioms defining the notion of vertex operator (super)algebra except the two grading restrictions.

Let \( g \) be a finite-dimensional simple Lie algebra, \( h \) a Cartan subalgebra, and \( \langle \cdot, \cdot \rangle \) the normalized killing form such that the square length of a long root is 2. Let

\[
\hat{g} = g \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c
\]

be the affine Lie algebra.

Let \( \ell \) be a complex number such that \( \ell \neq -h^\vee \), where \( h^\vee \) is the dual Coxeter number of \( g \). Let \( \mathbb{C}_\ell \) be the one-dimensional \((g \otimes \mathbb{C}[t] + \mathbb{C}c)\)-module on which \( c \) acts as scalar \( \ell \) and \( g \otimes \mathbb{C}[t] \) acts as zero. Form the generalized Verma \( \hat{g} \)-module

\[
M_g(\ell, 0) = U(\hat{g}) \otimes_{U(g \otimes \mathbb{C}[t] + \mathbb{C}c)} \mathbb{C}_\ell.
\]

It was well known (cf. [FF], [FZ], [Li1], [MP]) that \( M_g(\ell, 0) \) has a natural vertex operator algebra structure. Furthermore, the category of weak \( M_g(\ell, 0) \)-modules in the sense that all the axioms defining the notion of module except those involving grading hold is canonically equivalent to the category of restricted (cf. [K]) \( \hat{g} \)-modules of level \( \ell \) in the sense that for every vector \( w \) of the module, \((g \otimes t^n \mathbb{C}[t])w = 0 \) for \( n \) sufficiently large.

**Remark 2.1** More generally, let \( W \) be an arbitrary vector space. An element \( a(z) \) of \((\text{End } W)[[z, z^{-1}]]\) is called a vertex operator if \( a(z)w \in W((z)) \) for every \( w \in W \). Two
vertex operators $a(z)$ and $b(z)$ are said to be \textit{mutually local} if there exists a nonnegative integer $N$ such that

$$(z_1 - z_2)^N[a(z_1), b(z_2)] = 0$$

(cf. [DL], (1.4)). It was proved in [Li1] (Corollary 3.2.11) that any set of mutually local vertex operators on $W$ automatically generates a vertex algebra, which is a canonical vector subspace of $(\text{End } W)[[z, z^{-1}]]$, and that $W$ is a natural module for this vertex algebra.

For $\lambda \in \mathfrak{h}^*$, denote by $L_{\mathfrak{g}}(\ell, \lambda)$ the irreducible highest weight $\hat{\mathfrak{g}}$-module of level $\ell$ with highest weight $\lambda$. Each $L_{\mathfrak{g}}(\ell, \lambda)$ is an irreducible $\hat{\mathfrak{g}}(\ell, 0)$-module possibly with infinite-dimensional homogeneous subspaces.

Denote by $\theta$ the highest long root of $\mathfrak{g}$. For a positive integer $\ell$, set

$$P_\ell = \{ \lambda \in P_+ \mid \langle \lambda, \theta \rangle \leq \ell \},$$

where $P_+$ is the set of dominant integral weights of $\mathfrak{g}$. Then $L(\ell, \lambda)$ is an integrable $\hat{\mathfrak{g}}$-module if and only if $\lambda \in P_\ell$ [K]. The following result was known (cf. [DL, Proposition 13.17], [FZ, Theorem 3.1.3], [Li1, Propositions 5.2.4 and 5.2.5], [MP, Theorems 5.9, 5.14 and 5.15]):

\textbf{Proposition 2.2} Let $\ell$ be a positive integer. Then (1) The set of irreducible $L_{\mathfrak{g}}(\ell, 0)$-modules is exactly the set of irreducible highest weight integrable (or standard) $\hat{\mathfrak{g}}$-modules of level $\ell$. (2) Every $L_{\mathfrak{g}}(\ell, 0)$-module is completely reducible.

The following stronger result was obtained in [DLM1] (Theorem 3.7):

\textbf{Proposition 2.3} Let $\ell$ be a positive integer. Then every weak $L_{\mathfrak{g}}(\ell, 0)$-module is a direct sum of irreducible highest weight integrable (or standard) $\hat{\mathfrak{g}}$-modules of level $\ell$. In particular, every irreducible weak $L_{\mathfrak{g}}(\ell, 0)$-module is an (ordinary) $L_{\mathfrak{g}}(\ell, 0)$-module.

A vertex operator algebra with the property that every weak module is a direct sum of irreducible (ordinary) modules is said to be \textit{regular} [DLM1].

Let $\mathfrak{h}$ be a finite-dimensional vector space equipped with a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$, i.e., a finite-dimensional abelian Lie algebra equipped with a nondegenerate symmetric invariant bilinear form. Let $\mathfrak{h} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}c$ be the affine Lie algebra. We have

$$\hat{\mathfrak{h}} = \hat{\mathfrak{h}}_z \oplus \mathfrak{h},$$

where $\hat{\mathfrak{h}}_z = \sum_{n \neq 0} \mathfrak{h} \otimes t^n + \mathbb{C}c$ is a Heisenberg algebra and $\mathfrak{h}$ is central. Similar to the construction of $M_{\mathfrak{g}}(\ell, 0)$ we construct a space $M_{\mathfrak{h}}(1, 0)$, and just like $M_{\mathfrak{g}}(\ell, 0)$, $M_{\mathfrak{h}}(1, 0)$ is a
vertex operator algebra. For any \( \alpha \in \mathfrak{h}^* (= \mathfrak{h}) \), let \( C e^\alpha \) be a one-dimensional \((\mathfrak{h} \otimes \mathbb{C}[t] + \mathbb{C}c)\)-module on which \( \mathfrak{h} \otimes t \mathbb{C}[t] \) acts as zero, \( c \) acts as 1 and \( h = h(0) \) acts as scalar \( \langle \alpha, h \rangle \) for \( h \in \mathfrak{h} \). Form the induced module

\[
M_h(1, \alpha) = U(\hat{\mathfrak{h}}) \otimes_{U(\mathfrak{h} \otimes \mathbb{C}[t] + \mathbb{C}c)} C e^\alpha \simeq M_h(1, 0) \otimes C e^\alpha \quad \text{(linearly).} \tag{2.6}
\]

As in the case of \( M_g(\ell, 0) \), \( M_h(1, \alpha) \) is an irreducible \( M_h(1, 0) \)-module. Furthermore, from [FLM] the lowest \( L(0) \)-weight of \( M_h(1, \alpha) \) is

\[
\Delta_\alpha = \frac{1}{2} \langle \alpha, \alpha \rangle. \tag{2.7}
\]

On the other hand, clearly every irreducible \( M_h(1, 0) \)-module is isomorphic to \( M_h(1, \alpha) \) for some \( \alpha \). It follows from the complete reducibility of certain \( \hat{\mathfrak{h}} \)-modules ([LW], [K]) that an \( M_h(1, 0) \)-module on which \( \mathfrak{h} \) semisimply acts is completely reducible. In general, an \( M_h(1, 0) \)-module may not be completely reducible.

Let \( P \) be a rational lattice of finite rank with the \( \mathbb{Z} \)-bilinear form \( \langle \cdot, \cdot \rangle \). Set

\[
\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} P, \tag{2.8}
\]

and extend \( \langle \cdot, \cdot \rangle \) to a \( \mathbb{C} \)-bilinear form on \( \mathfrak{h} \).

Denote by \( \mathbb{C}[P] \) the group algebra. Set

\[
V_P = \mathbb{C}[P] \otimes M_h(1, 0), \tag{2.9}
\]

equipped with the standard \( M_h(1, 0) \)-module (or \( \hat{\mathfrak{h}} \)-module) structure. That is, \( V_P \) is a direct sum of irreducible \( M_h(1, 0) \)-modules \( M_h(1, \alpha) \simeq C e^\alpha \otimes M_h(1, 0) \) for \( \alpha \in P \).

It was proved in [FLM] (cf. [B]) that when \( P = L \) is even and positive-definite, \( V_L \) has a natural simple vertex operator algebra structure which extends the \( M_h(1, 0) \)-module structure. (It follows from Proposition 3.22 that such a vertex operator algebra structure is unique up to equivalence.) Furthermore, let \( L^o \) be the dual lattice of \( L \). Then \( V_{L^o} \) is a natural \( V_L \)-module and \( V_{\beta+L} \) is an irreducible \( V_L \)-module for \( \beta \in L^o \).

**Proposition 2.4** (1) Let \( \{\beta_1, \ldots, \beta_r\} \) be a complete set of representatives of cosets of \( L \) in \( L^o \). Then \( \{V_{\beta_1+L}, \ldots, V_{\beta_r+L}\} \) is a complete set of representatives of equivalent classes of irreducible \( V_L \)-modules. (2) Every \( V_L \)-module is completely reducible. (3) \( V_L \) is regular, i.e., every weak \( V_L \)-module is a direct sum of irreducible (ordinary) \( V_L \)-modules.

The assertion (1) was proved in [FLM] and [D1], (2) was proved in [Guo] and (3) was proved in [DLM1].

**Remark 2.5** For a general rational lattice \( P \), \( V_P \) is not a vertex (operator) algebra because of the involvement of non-local vertex operator. A notion of generalized vertex algebra was introduced in [DL] with a generalized Jacobi identity as one of the main axioms and it was proved in [DL, Theorem 5.1 and Remark 9.11] that \( V_P \) is a generalized vertex algebra.
2.2 Simple currents and a construction

We first recall from [FHL] the definition of an intertwining operator.

**Definition 2.6** Let $V$ be a vertex operator algebra and let $W_1, W_2$ and $W_3$ be $V$-modules. An **intertwining operator** of type $(W_3, W_1, W_2)$ is a linear map $I$ from $W_1$ to $(\text{Hom}(W_2, W_3))\{z\}$, (where for a vector space $U$, $U\{z\}$ is defined to be the vector space of $U$-valued formal series in $z$ with arbitrary complex powers of $z$), satisfying the following properties: for $w_1 \in W_1$, $w_2 \in W_2$,

$$I(w_1, z)w_2 \in z^{\gamma_1}W_3[[z]] + \cdots + z^{\gamma_n}W_3[[z]] \quad (2.10)$$

for some (finitely many) complex numbers $\gamma_1, \ldots, \gamma_n$, and for $v \in V$, $w_1 \in W_1$,

$$[L(-1), I(w_1, z)] = \frac{d}{dz}I(w_1, z), \quad (2.11)$$

$$z_0^{-1}\delta \left( \frac{z_1 - z_2}{z_0} \right) Y(v, z_1)I(w_1, z_2) - z_0^{-1}\delta \left( \frac{z_2 - z_1}{z_0} \right) I(w_1, z_2)Y(v, z_1)$$

$$= z_2^{-1}\delta \left( \frac{z_1 - z_0}{z_2} \right) I(Y(v, z_0)w_1, z_2). \quad (2.12)$$

All intertwining operators of type $(W_3, W_1, W_2)$ form a vector space denoted by $I_{W_3W_1W_2}$. The dimension of $I_{W_3W_1W_2}$ is called the **fusion rule**, denoted by $N_{W_3W_1W_2}$. Clearly, the fusion rule only depends on the equivalence class of each $W_i$.

The following are among the immediate consequences of the Jacobi identity (2.12):

$$[v_n, I(w_1, z_2)] = \sum_{i \geq 0} \binom{n}{i} z_2^{i} I(v_i w_1, z_2) \quad (2.13)$$

(the **commutator formula**) and

$$I(v_n w_1, z_2) = \text{Res}_{z_1} ((z_1 - z_2)^n Y(v, z_1)I(w_1, z_2) - (-z_2 + z_1)^n I(w_1, z_2)Y(v, z_1)) \quad (2.14)$$

(the **iterate formula**) for $n \in \mathbb{Z}$. Conversely, the commutator and iterate formulas imply the Jacobi identity.

**Remark 2.7** If $V = L_\mathfrak{g}(\ell, 0)$ and $W_1 = L_\mathfrak{g}(\ell, \lambda)$, for $w_1 \in L(\lambda)$ (the lowest $L(0)$-weight subspace of $L_\mathfrak{g}(\ell, \lambda)$), the commutator formula (2.13) gives

$$[a_n, I(w_1, z_2)] = z_2^n I(aw_1, z_2) \quad (2.15)$$

for $a \in \mathfrak{g} \subset L_\mathfrak{g}(\ell, 0)$ and for $n \in \mathbb{Z}$. In many literatures such as [TK] (and [FM]), an intertwining operator was defined on $L(\lambda)$ with the properties (2.11) and (2.13) as the defining axioms. However, it can be proved (cf. [TK], [Li3, Li6]) that the two definitions give rise to the same fusion rules.
Let $V$ be a vertex operator algebra. We denote by $\text{Irr}(V)$ the set of equivalence classes of irreducible modules and for an irreducible $V$-module $W$, denote by $[W]$ the equivalence class. $V$ is said to be quasi-rational [MS] if all fusion rules associated with irreducible modules are finite and if for any $[W_1], [W_2] \in \text{Irr}(V)$, $N^{{[W_3]}\, [W_1]}_{[W_2]} = 0$ for all but finitely many $[W_3] \in \text{Irr}(V)$.

For a quasi-rational vertex operator algebra $V$, the Verlinde algebra $A(V)$ is defined to be an algebra (over $\mathbb{C}$) with $\text{Irr}(V)$ as a basis and with the fusion rules as the structural constants, i.e.,

$$[W_i] \cdot [W_j] = \sum_{[W_k] \in \text{Irr}(V)} N^{{[W_k]}\, [W_i]}_{[W_j]} [W_k].$$

When $V$ is simple, it is easy to show that $[V]$ is the unit. It follows immediately from [FHL, HL2] that $A(V)$ is commutative. Under certain conditions, Huang [Hua1] established the associativity, but for a general $V$ the associativity is still an unsolved problem.

Let $V = L_{sl(2)}(k, 0)$ with a positive integer $k$. It is well known ([GW], [TK], [FZ]) that the Verlinde algebra has the following relations:

$$[L(k, i)] \cdot [L(k, j)] = \sum_{r = \max(i-j, j-i)}^{\min(i+j, 2k-i-j)} [L(k, r)].$$

The following definition is due to Schellekens and Yankielowicz [SY].

**Definition 2.8** An irreducible $V$-module $W$ is called a simple current if the associated matrix of the left multiplication of $[W]$ with respect to the standard basis of the Verlinde algebra is a permutation. The order of the associated matrix as a group element is called the order of $W$.

We now recall a construction of simple currents from [Li4]. In the following, one may think of $V$ as one of, or more general, any tensor product of the following vertex operator algebras:

$$L_\ell(0, 0), \quad M_h(1, 0), \quad V_L, \quad L_\ell(0, 0) \otimes M_h(1, 0), \quad L_\ell(0, 0) \otimes V_L.$$

Let $\alpha \in V_{(1)}$ satisfying the following conditions:

$$L(n)\alpha = \delta_{n,0}\alpha, \quad \alpha(n)\alpha = \delta_{n,1}\gamma \mathbf{1} \quad \text{for } n \in \mathbb{Z}_+, \quad (2.18)$$

where $Y(\alpha, z) = \sum_{n \in \mathbb{Z}} \alpha(n) z^{-n-1}$, i.e., $\alpha(n) = \alpha_n$, and $\gamma$ is a fixed complex number. Notice that condition (2.18) implies that $\alpha$ is a primary vector and that operators $\alpha(n)$ satisfy the Heisenberg algebra relation

$$[\alpha(m), \alpha(n)] = m\gamma \delta_{m+n,0} \quad \text{for } m, n \in \mathbb{Z}. \quad (2.19)$$
Furthermore, assume that $\alpha(0)$ acts semisimply on $V$. It is clear that $e^{2\pi i \alpha(0)}$ is an automorphism of $V$ and that $e^{2\pi i \alpha(0)} = 1$ if and only if $\alpha(0)$ has only integral eigenvalues on $V$. If each $\alpha(0)$ has only rational eigenvalues on $V$ and if $V$ is finitely generated, then $e^{2\pi i \alpha(0)}$ is of finite order. Define
\[
\Delta(\alpha, z) = z^{\alpha(0)} \exp \left( \sum_{k=1}^{\infty} \frac{\alpha(k)}{-k} (-z)^{-k} \right).
\]
This is a well defined element of $(\text{End } W\{z\})$ for any weak $V$-module $W$ on which $\alpha(0)$ semisimply acts. The following result is a special case of Proposition 5.4 of [Li2].

**Proposition 2.9** Let $\alpha, \Delta(\alpha, z)$ be given as before. Assume that $\alpha(0)$ has only integral eigenvalues on $V$. Let $W$ be any (irreducible) weak $V$-module. Set
\[
(W^{(\alpha)}, Y_{\alpha}(\cdot, z)) = (W, Y(\Delta(\alpha, z), z)).
\]
Then $(W^{(\alpha)}, Y_{\alpha})$ carries the structure of an (irreducible) weak $V$-module.

As a convention, by $V$-module $W^{(\alpha)}$ we mean the $V$-module $(W^{(\alpha)}, Y_{\alpha})$.

Recall the following result from [Li4] (Proposition 2.5):

**Proposition 2.10** Let $\alpha, \Delta(\alpha, z)$ be as in Proposition 2.9. Let $W_1$ and $W_2$ be weak $V$-modules and $f$ be a $V$-homomorphism from $W_1$ to $W_2$. Then $f$ is also a $V$-homomorphism from $W_1^{(\alpha)}$ to $W_2^{(\alpha)}$.

**Remark 2.11** In view of Propositions 2.9 and 2.10, we obtain a canonical functor $F_\alpha$ from the category of weak $V$-modules to itself in the obvious way. Since
\[
\Delta(\alpha, z)\Delta(-\alpha, z) = \Delta(-\alpha, z)\Delta(\alpha, z) = 1,
\]
we easily see that $F_{-\alpha}$ is the inverse functor of $F_\alpha$. Therefore, $F_\alpha$ is an isomorphism.

The following result [Li4, Proposition 2.4] is a generalization of Proposition 2.9.

**Proposition 2.12** Let $\alpha, \Delta(\alpha, z)$ be given as in Proposition 2.9. Let $I$ be an intertwining operator of type $\left( \begin{array}{c} W_3 \\ W_1 W_2 \end{array} \right)$. Define
\[
I_\alpha(w_1, z)w_2 = I(\Delta(\alpha, z)w_1, z)w_2
\]
for $w_1 \in W_1$, $w_2 \in W_2$. Then $I_\alpha$ is an intertwining operator of type $\left( \begin{array}{c} W_3^{(\alpha)} \\ W_1 W_2^{(\alpha)} \end{array} \right)$.

The following results [Li4, Corollary 2.12, Theorem 2.15, Proposition 3.2] give a construction of simple currents:
Theorem 2.13 Let $V$ be a simple vertex operator algebra and let $\alpha \in V_{(1)}$ be such that (2.18) holds and such that $\alpha(0)$ has only integral eigenvalues on $V$. If for each irreducible (ordinary) $V$-module $W$, the weak module $W^{(\alpha)}$ is an (ordinary) $V$-module, then $V^{(\alpha)}$ is a simple current. Furthermore, for any irreducible $V$-module $W$,

$$[W] \cdot [V^{(\alpha)}] = [W^{(\alpha)}].$$

(2.24)

The following lemma gives more information about $V^{(\alpha)}$.

Lemma 2.14 Let $V, \alpha$ be as in Theorem 2.13. Then $L(0)$ acts semisimply on $V^{(\alpha)}$ with eigenvalues in $\frac{1}{2}\gamma + \mathbb{Z}$, where $\alpha(1)\alpha = \gamma 1$.

Proof. From [Li5, (3.18)], we have

$$\Delta(\alpha, z)\omega = \omega + \alpha z^{-1} + \frac{1}{2}\alpha(1)\alpha z^{-2} = \omega + \alpha z^{-1} + \frac{1}{2}\gamma 1 z^{-2},$$

(2.25)

where $\omega$ is the Virasoro element. Then

$$Y_\alpha(\omega, z) = Y(\Delta(\alpha, z)\omega, z) = Y(\omega, z) + z^{-1}Y(\alpha, z) + \frac{1}{2}\gamma z^{-2}.$$  (2.26)

In terms of components we have

$$L_\alpha(m) = L(m) + \alpha(m) + \frac{1}{2}\gamma \delta_{m,0}$$

(2.27)

for $m \in \mathbb{Z}$, where $Y_\alpha(\omega, z) = \sum_{m \in \mathbb{Z}} L_\alpha(m)z^{-m-2}$. Since $L(0)$ and $\alpha(0)$ act semisimply on $V$ with integral eigenvalues, $L_\alpha(0)$ acts semisimply on $V$ with eigenvalues in $\frac{1}{2}\gamma + \mathbb{Z}$. That is, $L(0)$ acts semisimply on $V^{(\alpha)}$ with eigenvalues in $\frac{1}{2}\gamma + \mathbb{Z}$. \hspace{1cm} \square

Since any irreducible weak module is an ordinary module for a regular vertex operator algebra, from Theorem 2.13 we immediately have:

Corollary 2.15 Let $V$ be a regular vertex operator algebra and let $\alpha \in V_{(1)}$ be given as in Theorem 2.13. Then $V^{(\alpha)}$ is a simple current. In particular, this is true when $V$ is a tensor product from the following algebras:

$$L_\varrho(\ell, 0), \quad V_L, \quad L_\varrho(\ell, 0) \otimes V_L,$$

where $\ell$ is a positive integer and $L$ is a positive-definite even lattice. \hspace{1cm} \square

The following result was obtained in [Li4]:

Proposition 2.16 Let $L$ be a positive definite even lattice. (1) For $\beta \in L^\circ$, as a $V_L$-module, $V_L^{(\beta)}$ is isomorphic to $V_{L+\beta}$. (2) Every irreducible $V_L$-module is a simple current. (3) The Verlinde algebra is canonically isomorphic to the group algebra $\mathbb{C}[L^\circ/L]$. 

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Previously, intertwining operators for $V_L$ were explicitly constructed, fusion rules were calculated and (3) was proved in [DL]. (Of course, (3) implies (2).)

Though vertex operator algebra $M_{h}(1,0)$ is not regular, the same proof of Proposition 2.16 gives the following result:

**Proposition 2.17** (1) For $h \in \mathfrak{h}$, as an $M_{h}(1,0)$-module, $M_{h}(1,0)^{(h)}$ is isomorphic to $M_{h}(1,h)$. (2) Every irreducible $M_{h}(1,0)$-module is a simple current. (3) The Verlinde algebra is canonically isomorphic to the group algebra $\mathbb{C}[\mathfrak{h}]$. 

**Remark 2.18** Suppose that $V = V^1 \otimes V^2$ is a tensor product vertex operator algebra. Then $V^1_{(1)}$ and $V^2_{(1)}$ are canonical subspaces of $V_{(1)}$. Suppose $\alpha = \alpha' + \alpha''$ where $\alpha^1 \in V^1_{(1)}$, $\alpha^2 \in V^2_{(1)}$. Then $\alpha$ satisfies (2.18) if and only if both $\alpha^1$ and $\alpha^2$ satisfy (2.18). Furthermore, $\alpha(0)$ acting on $V$ has only integral eigenvalues if and only if $\alpha^i(0)$ acting on $V^i$ has only integral eigenvalues for $i = 1, 2$. Let $W = W_1 \otimes W_2$, where $W_1, W_2$ are $V_1$ and $V_2$-modules, respectively. Since $[\alpha^1(m), \alpha^2(n)] = 0$ for $m, n \in \mathbb{Z}$, we have

$$\Delta(\alpha, z) = \Delta(\alpha^1, z)\Delta(\alpha^2, z).$$

Then we have

$$W^{(a)} = W^1_{1}^{(\alpha^1)} \otimes W^2_{2}^{(\alpha^2)}.$$ (2.29)

Let $\mathfrak{g}, \langle \cdot, \cdot \rangle$ be as in Section 2.1. Let $\{e_i, f_i \mid i = 1, \ldots, n\}$ be the Chevalley generators with simple roots $\alpha_1, \ldots, \alpha_n$ and simple coroots $\alpha_1^\vee, \ldots, \alpha_n^\vee$. Let $\lambda_i (i = 1, \ldots, n)$ be the fundamental weights for $\mathfrak{g}$. Let $Q = \bigoplus_{i=1}^{n} \mathbb{Z}\alpha_i$ be the root lattice and let $P = \bigoplus_{i=1}^{n} \mathbb{Z}\lambda_i$ be the weight lattice.

Let $h^{(1)}, \ldots, h^{(n)} \in \mathfrak{h}$ be the fundamental co-weights, i.e.,

$$\alpha_i(h^{(j)}) = \delta_{i,j} \quad \text{for } i, j = 1, \ldots, n. \quad (2.30)$$

Set

$$Q^\vee = \mathbb{Z}\alpha_1^\vee + \cdots + \mathbb{Z}\alpha_n^\vee,$$ (2.31)

$$P^\vee = \mathbb{Z}h^{(1)} + \cdots + \mathbb{Z}h^{(n)},$$ (2.32)

the co-root lattice and the co-weight lattice.

Let

$$\theta = \sum_{i=1}^{n} a_i \alpha_i$$ (2.33)

be the highest long root. Then

$$\theta(h^{(i)}) = a_i \quad \text{for } i = 1, \ldots, n. \quad (2.34)$$
We shall need to know which $a_i$ equal 1. The following is a list for such $a_i$ (cf. [K]):

$$
A_n : a_1, \ldots, a_n \\
B_n : a_1 \\
C_n : a_n \\
D_n : a_1, a_{n-1}, a_n \\
E_6 : a_1, a_5 \\
E_7 : a_6.
$$

(2.35)

**Remark 2.19** Simple roots for type $E$ ($E_6$, $E_7$, $E_8$) were numbered differently in [H] and [K]. Here, we use the numbering system of [K].

Let $\Lambda_0, \ldots, \Lambda_n$ be the fundamental weights of $\hat{\mathfrak{g}}$ [K]. Then each $\lambda_i$ for $1 \leq i \leq n$ is naturally extended to $\Lambda_i$. From the Dynkin diagram ([K], TABLE Aff 1) we find that $a_i = 1$ if and only if the vertices 0 and $i$ are in the same orbit under the automorphism group of the affine Dynkin diagram. We point out that if $a_i = 1$, then $\Lambda_i$ is of level one.

The following proposition was proved in [Li4] (Proposition 3.5, Remark 3.8):

**Proposition 2.20** Let $\ell$ be a complex number with $\ell \neq -h^\vee$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. If the coefficient $a_i$ of $\alpha_i$ in $\theta$ is 1, then as an $L(\ell, 0)$-module

$$L(\ell, 0)^{(h^\vee)} \simeq L(\ell, \ell \lambda_i).$$

(2.36)

Furthermore, if $\ell$ is a positive integer, $L(\ell, \ell \lambda_i)$ is a simple current for $L(\ell, 0)$.

**Remark 2.21** It was known ([FG], [F]) that $L(\ell, \ell \lambda_i)$ with $a_i = 1$ are all the simple currents except the level 2 simple current $L(\Lambda_7)$ for $\mathfrak{g}$ of type $E_8$.

**Remark 2.22** The element $\Delta(h^{(i)}, z)$ gives rise to an automorphism $\psi_i$ of affine Lie algebra $\hat{\mathfrak{g}}$ via

$$
\psi_i(Y(a, z)) = Y_{h^{(i)}}(a, z) = Y(\Delta(h^{(i)}, z)a, z)
$$

(2.37)

for $a \in \mathfrak{g}$, where $Y(a, z) = \sum_{n \in \mathbb{Z}} a(n)z^{-n-1}$ is the generating series of $a$. That is,

$$
\psi_i(\alpha_i^\vee(n)) = \alpha_i^\vee(n) + \delta_{n,0} \ell, \quad \psi_i(e_i(n)) = e_i(n+1), \quad \psi_i(f_i(n)) = f_i(n-1); \quad (2.38) \\
\psi_i(\alpha_j^\vee(n)) = \alpha_j^\vee(n), \quad \psi_i(e_j(n)) = e_j(n), \quad \psi_i(f_j(n)) = f_j(n) \quad \text{for } j \neq i, n \in \mathbb{Z}. \quad (2.39)
$$

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and

$$\psi_i(f_\theta(n)) = f_\theta(n - 1) \quad \text{for } n \in \mathbb{Z}. \quad (2.40)$$

The vector $1$ in $L(\ell, 0)^{(h^{(i)})}$ is a highest weight vector of weight $\ell \Lambda_i$. More general automorphisms of this type was recently studied in \cite{FS}. Note that the composition of the representation on $L(\ell, 0)$ with the corresponding Dynkin diagram automorphism of $\hat{g}$ also gives $L(\ell, \ell \Lambda_i)$. But $\psi_i$ are not Dynkin diagram automorphisms. Dynkin diagram automorphisms played important roles in \cite{FG}, \cite{F}, \cite{SY} and \cite{FM}.

**Remark 2.23** For $\alpha \in h$, from (2.25) we have

$$L_\alpha(m) = L(m) + \alpha(m) + \frac{1}{2} \ell(\alpha, \alpha) \delta_{m,0} \quad (2.41)$$

for $m \in \mathbb{Z}$, recall that $Y_\alpha(\omega, z) = \sum_{m \in \mathbb{Z}} L_\alpha(m) z^{-m-2}$ with $\omega$ being the Virasoro element. Then the $L_{h^{(i)}}(0)$-weight of $1$ is $\frac{1}{2} \ell(h^{(i)}, h^{(i)})$. Since under the new action $Y(\Delta(h^{(i)}, z), z)$ on $L(\ell, 0)$, $1$ is still a highest weight vector for $\hat{g}$, the lowest $L(0)$-weight of $L(\ell, \ell \Lambda_i)$ is $\frac{1}{2} \ell(h^{(i)}, h^{(i)})$. Of course, there is a formula for the lowest weight of any irreducible module (cf. \cite{DL}).

From now on we assume that $k$ is a positive integer. Let $h \in h$. Note that for any root vector $e_\alpha$ of $g$ and for $m \in \mathbb{Z},$

$$[h(0), e_\alpha(m)] = \alpha(h) e_\alpha(m). \quad (2.42)$$

Since $h(0)1 = 0$ and $L(k, 0)$ is generated by $\hat{g}$ from $1$, using (2.42) we see that $h(0)$ has only integral eigenvalues on $L(k, 0)$ if and only if $h \in P^\vee$.

Define a map $\pi$ from $P^\vee$ to the Verlinde algebra $\mathcal{V}(L(k, 0))$ of $L(k, 0)$ by

$$\pi(\alpha) = [L(k, 0)^{(\alpha)}] \quad \text{for } \alpha \in P^\vee. \quad (2.43)$$

The map $\pi$ naturally extends to a linear map from the group algebra $\mathbb{C}[P^\vee]$ to $\mathcal{V}(L(k, 0))$. We abuse the notation $\pi$ for this extension also.

**Proposition 2.24** The linear map $\pi$ is an algebra homomorphism.

**Proof.** For $\alpha, \beta \in P^\vee$, since $[\alpha(r), \beta(s)] = 0$ for $r, s \geq 0$, we have

$$\Delta(\alpha + \beta, z) = \Delta(\alpha, z) \Delta(\beta, z), \quad (2.44)$$

recall (2.20). Then we get

$$L(k, 0)^{\alpha+\beta} \cong (L(k, 0)^{(\alpha)})^{(\beta)} \quad (2.45)$$
as $L(k,0)$-modules. In view of Theorem 2.13 we have

$$[L(k,0)^{(\alpha)}] \cdot [L(k,0)^{(\beta)}] = [(L(k,0)^{(\alpha)})^{(\beta)}] = [L(k,0)^{(\alpha+\beta)}].$$

(2.46)

Thus $\pi$ is an algebra homomorphism. □

It follows immediately that $\pi(P^\vee)$ is an abelian group. The following result gives important kernel elements of $\pi$ as a group homomorphism on $P^\vee$.

**Proposition 2.25** We have

$$[L(k,0)^{(\alpha)}] = [L(k,0)] \quad \text{for } \alpha \in Q^\vee.$$  \hspace{1cm} (2.47)

Furthermore, for any irreducible $L(k,0)$-module $W$, we have

$$[W^{(\alpha)}] = [W] \quad \text{for } \alpha \in Q^\vee.$$  \hspace{1cm} (2.48)

**Proof.** It suffices to prove (2.47) for $\alpha = \alpha_i^\vee$. For $1 \leq i \leq n$, set $r = \frac{2}{\langle \alpha_i, \alpha_i \rangle}$, a positive integer. From [H] or [K] we have

$$\langle \alpha_i^\vee, \alpha_i^\vee \rangle = \frac{4}{\langle \alpha_i, \alpha_i \rangle} = 2r.$$ \hspace{1cm} (2.49)

Since

$$[e_i(1), f_i(-1)] = \alpha_i^\vee(0) + \langle e_i, f_i \rangle c = \alpha_i^\vee(0) + \frac{1}{2} \langle \alpha_i^\vee, \alpha_i^\vee \rangle c = \alpha_i^\vee(0) + rc,$$ \hspace{1cm} (2.50)

$L_i := \mathbb{C} e_i(1) + \mathbb{C} f_i(-1) + \mathbb{C} (\alpha_i^\vee + rc)$ is a subalgebra of $\hat{g}$ isomorphic to $sl(2)$. Furthermore, $L(k,0)$, being an integrable $\hat{g}$-module, is an integrable $L_i$-module. Clearly, $1$ is a highest weight vector of weight $rk$ for $L_i$. Then $f_i(-1)^{rk} 1 \neq 0$. From (2.25) we have

$$L_{\alpha_i^\vee}(0) = L(0) + \alpha_i^\vee(0) + \frac{1}{2} k \langle \alpha_i^\vee, \alpha_i^\vee \rangle = L(0) + \alpha_i^\vee(0) + kr,$$ \hspace{1cm} (2.51)

where $Y_{\alpha_i^\vee}(\omega, z) = \sum_{n \in \mathbb{Z}} L_{\alpha_i^\vee}(n) z^{-n-2}$. Then

$$L_{\alpha_i^\vee}(0) f_i(-1)^{rk} 1 = (L(0) + \alpha_i^\vee(0) + kr) f_i(-1)^{rk} 1 = (kr - 2kr + kr) f_i(-1)^{rk} 1 = 0.$$ \hspace{1cm} (2.52)

That is, $f_i(-1)^{rk} 1$ is a non-zero element of $L(k,0)^{(\alpha_i^\vee)}$ of weight zero.

On the other hand, with $L(k,0)$ being regular, every irreducible module, in particular, $L(k,0)^{(\alpha_i^\vee)}$, is an integrable $\hat{g}$-module of level $k$, which is unitary. Thus the lowest weights of $L(k,0)^{(\alpha_i^\vee)}$ are nonnegative. Consequently, the lowest weight of $L(k,0)^{(\alpha_i^\vee)}$ is $0$. Because $L(k,0)$ is the only irreducible $L(k,0)$-module with $0$ being the lowest weight, we must have $L(k,0)^{(\alpha_i^\vee)} \simeq L(k,0)$ as $L(k,0)$-modules.

For an irreducible $L(k,0)$-module $W$, using the first part and Theorem 2.13 we get

$$[W] = [W] \cdot [V] = [W] \cdot [V^{(\alpha)}] = [W^{(\alpha)}] \quad \text{for } \alpha \in Q^\vee.$$ \hspace{1cm} (2.53)

This completes the proof. □

The following result generalizes Proposition 2.20 with $L = Q$:
**Theorem 2.26** The algebra homomorphism \( \pi \) gives rise to an algebra isomorphism from the group algebra \( \mathbb{C}[P^\vee/Q^\vee] \) onto \( \pi(\mathbb{C}[P^\vee]) \). Furthermore,

\[
\pi(P^\vee) = \{ [L(k, k\lambda_i)] \mid a_i = 1 \}. \quad (2.54)
\]

**Proof.** In view of Proposition 2.25, \( \pi \) gives rise to an algebra homomorphism \( \bar{\pi} \) from \( \mathbb{C}[P^\vee/Q^\vee] \) onto \( \pi(\mathbb{C}[P^\vee]) \). From [H] (Section 13.1), we have

\[
|P/Q| = n + 1, 2, 4, 3, 2, 1, 1 \quad (2.55)
\]

for \( g \) of type \( A_{n+1}, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2 \), respectively. Note that \( B_n \) and \( C_n \) are dual to each other and the others are self-dual. Then

\[
|P^\vee/Q^\vee| = |P/Q| \quad \text{for all types.}
\]

On the other hand, from Proposition 2.20, all \( [L(k, k\lambda_i)] \) for \( i \) with \( a_i = 1 \), which are distinct basis elements \( \mathcal{V}(L(k,0)) \), are images of \( \pi \). Then it follows immediately that (2.54) holds and \( \bar{\pi} \) is an algebra isomorphism. \( \square \)

The group structure of \( P/Q \) for the nontrivial cases was given in [H] (Exercise 4 on page 71). Then we have:

For \( A_{n+1} \), \( P^\vee/Q^\vee \simeq \mathbb{Z}/(n + 1)\mathbb{Z} \) with \( h^{(1)} + Q^\vee \) as a generator such that

\[
mh^{(1)} + Q^\vee = h^{(\bar{m})} + Q^\vee, \quad (2.56)
\]

where \( \bar{m} \) is the least nonnegative residue of \( m \) modulo \( n + 1 \).

For \( D_n \) with \( n \) being odd, \( P^\vee/Q^\vee \simeq \mathbb{Z}/4\mathbb{Z} \) with \( h^{(n)} + Q^\vee \) as a generator such that

\[
2h^{(n)} + Q^\vee = h^{(1)} + Q^\vee, \quad 3h^{(n)} + Q^\vee = h^{(n-1)} + Q^\vee. \quad (2.57)
\]

For \( D_n \) with \( n \) being even, \( P^\vee/Q^\vee \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) with \( h^{(n-1)} + Q^\vee \) and \( h^{(n)} + Q^\vee \) as generators such that

\[
h^{(1)} + Q^\vee = h^{(n-1)} + h^{(n)} + Q^\vee. \quad (2.58)
\]

Then with Theorem 2.26 we immediately have the following results which have been known to physicists:

**Corollary 2.27** The following fusion algebra relations hold:

For \( A_{n+1} \), for \( 0 \leq i, j \leq n \),

\[
[L(k, k\lambda_i)] \cdot [L(k, k\lambda_j)] = [L(k, k\lambda_{i+j})]. \quad (2.59)
\]

For \( B_n \),

\[
[L(k, k\lambda_i)] \cdot [L(k, k\lambda_1)] = [L(k, 0)]. \quad (2.60)
\]
For $C_n$, 
\[ [L(k, k\lambda_n)] \cdot [L(k, k\lambda_n)] = [L(k, 0)]. \quad (2.61) \]

For $D_n$ with odd $n$,
\[ [L(k, k\lambda_n)]^2 = [L(k, k\lambda_1)], \quad [L(k, k\lambda_n)]^3 = [L(k, k\lambda_{n-1})], \quad [L(k, k\lambda_n)]^4 = [L(k, 0)]. \quad (2.62) \]

For $D_n$ with even $n$,
\[ [L(k, k\lambda_1)]^2 = [L(k, k\lambda_{n-1})]^2 = [L(k, k\lambda_n)]^2 = [L(k, 0)], \quad (2.63) \]
\[ [L(k, k\lambda_{n-1})] \cdot [L(k, k\lambda_n)] = [L(k, k\lambda_1)]. \quad (2.64) \]

For $E_6$,
\[ [L(k, k\lambda_1)]^2 = [L(k, k\lambda_5)], \quad [L(k, k\lambda_1)]^3 = [L(k, 0)]. \quad (2.65) \]

For $E_7$,
\[ [L(k, k\lambda_1)]^2 = [L(k, k\lambda_6)]^2 = [L(k, 0)]. \quad \square \quad (2.66) \]

We shall need the number $\langle h^{(i)}, h^{(i)} \rangle$ and the explicit expression of $h^{(i)}$ in terms of $\alpha_j^\vee$.

The expression of each $\lambda_i$ in terms of simple roots $\alpha_j$ was known ([H], page 69, Table 1). Suppose that for $1 \leq i \leq n$,
\[ \lambda_i = a_{i1}\alpha_1 + \cdots + a_{in}\alpha_n, \quad (2.67) \]
\[ h^{(i)} = b_{i1}\alpha_1^\vee + \cdots + b_{in}\alpha_n^\vee. \quad (2.68) \]

Then
\[ a_{ij} = \lambda_i(h^{(j)}) = b_{ji}. \quad (2.69) \]

Furthermore, from [H] we get
\[ \langle h^{(i)}, \alpha^\vee_j \rangle = \langle h^{(i)}, t_{\alpha_j} \rangle \frac{2}{\langle \alpha_j, \alpha_j \rangle} = \alpha_j(h^{(i)}) \frac{2}{\langle \alpha_j, \alpha_j \rangle} = \delta_{ij} \frac{2}{\langle \alpha_j, \alpha_j \rangle}. \quad (2.70) \]

Then
\[ \langle h^{(i)}, h^{(i)} \rangle = b_{ii} \frac{2}{\langle \alpha_i, \alpha_i \rangle} = a_{ii} \frac{2}{\langle \alpha_i, \alpha_i \rangle}. \quad (2.71) \]

With a glance of Table Aff in [K] we see that if $a_i = 1$, $\alpha_i$ is a long root, hence $\langle \alpha_i, \alpha_i \rangle = 2$. Therefore, we have obtained:

**Lemma 2.28** If $\lambda_i = a_{i1}\alpha_1 + \cdots + a_{in}\alpha_n$ for $1 \leq i \leq n$, then
\[ h^{(i)} = a_{i1}\alpha_1^\vee + \cdots + a_{ni}\alpha_i^\vee, \quad (2.72) \]
\[ \langle h^{(i)}, h^{(i)} \rangle = a_{ii} \frac{2}{\langle \alpha_i, \alpha_i \rangle}. \quad (2.73) \]

In particular, if $a_i = 1$, we have $\langle h^{(i)}, h^{(i)} \rangle = a_{ii}. \quad \square \quad (2.68) \]
3 Extension of vertex operator algebras by simple currents

We shall first recall some of the results from [DLM2] on $V[L]$, an extension of a certain vertex operator algebra $V$ by simple currents $V^{(\alpha)}$ parametrized by $\alpha \in L$, where $L$ is a lattice carrying an intrinsic structure of $V$. We then extend and refine those results. In Section 3.3, we concentrate a special class of $V[L]$. We classify all irreducible $V[L]$-modules, prove a complete reducibility theorem and we give a formula of fusion rules in the category of $V[L]$-modules in terms of fusion rules in the category of $V$-modules.

3.1 Extension of algebras

We shall first establish some basic assumptions which will remain in force throughout this section.

Let $V$ be a simple vertex operator algebra. As in Section 2, one may think of $V$ as a tensor product from the following vertex operator algebras: $L_{\ell}(0), M_{1}(0), V_{L}$.

Let $H$ be a (finite-dimensional) subspace of $V_{(1)}$ satisfying the following conditions:

$$L(n)h = \delta_{n,0}h, \quad h(n)h' = B(h,h')\delta_{n,1}1 \quad \text{for } n \in \mathbb{Z}_{+}, \ h, h' \in H,$$

where $Y(h, z) = \sum h(n)z^{-n-1}$ for $h \in H$, and $B(\cdot, \cdot)$ is assumed to be a nondegenerate symmetric bilinear form on $H$. We then identify $H$ with its dual $H^{*}$. We also assume that for any $h \in H$, $h(0)$ acts semisimply on $V$. Then

$$V = \bigoplus_{\alpha \in H} V^{(0, \alpha)}, \quad \text{where } V^{(0, \alpha)} = \{ v \in V \mid h(0)v = B(\alpha, h)v \ \text{for } h \in H \}. \quad (3.2)$$

Set

$$P = \{ \alpha \in H \mid V^{(0, \alpha)} \neq 0 \}. \quad (3.3)$$

As $V$ is simple, $P$ is a subgroup of $H$ (cf. [LX]). Then $P$ equipped with the bilinear form $B$ is a lattice.

Let $L$ be a subgroup of $H$ such that for each $\alpha \in L$, $\alpha(0)$ acting on $V$ has only integral eigenvalues. This amounts to $L \subset P^{o}$, where

$$P^{o} = \{ h \in H \mid B(h, \alpha) \in \mathbb{Z} \ \text{for } \alpha \in P \} \quad (3.4)$$

is the dual lattice of $P$. Note that the rank of $P$ may be less than dim $H$.

Let $W$ be a $V$-module. By Proposition $2.9$, for $\alpha \in L$, we have a (weak) $V$-module

$$(W^{(\alpha)}, Y_{\alpha}(\cdot, z)) := (W, Y(\Delta(\alpha, z), z)). \quad (3.5)$$
For convenience, we reformulate the construction of the $V$-module $W^{(a)}$ as follows: Set

$$W^{(a)} = \mathbb{C}e^a \otimes W \simeq W \text{ (linearly)},$$

where $e^a$ is a symbol for now and $\mathbb{C}e^a$ is a one-dimensional vector space with $e^a$ as a pre-chosen basis element. Then define

$$Y_a(v, z)(e^a \otimes w) = e^a \otimes Y(\Delta(\alpha, z)v, z)w \quad \text{for } v \in V, \ w \in W. \quad (3.7)$$

Set

$$W[L] = \bigoplus_{\alpha \in L} W^{(a)} = \mathbb{C}[L] \otimes W,$$

equipped with the direct sum $V$-module structure. Now, the symbol $e^a$ in the definition of $W^{(a)}$ is considered as an element of the group algebra $\mathbb{C}[L]$.

For $\alpha \in L$, we define a linear endomorphism $\psi_\alpha$ of $W[L]$ by

$$\psi_\alpha(e^\beta \otimes w) = e^{a+\beta} \otimes w \quad \text{for } \beta \in L, \ w \in W. \quad (3.9)$$

Then

$$\psi_0 = 1, \quad \psi_{\alpha+\beta} = \psi_\alpha \psi_\beta \quad \text{for } \alpha, \beta \in L. \quad (3.10)$$

That is, $\psi$ gives rise to a representation of $L$ on $W[L]$.

For $\alpha \in L$, we set $([LW], [FLM])$

$$E^\pm(\alpha, z) = \exp \left( \sum_{n=1}^{\infty} \frac{h(\pm n)}{\pm n} z^{\pm n} \right). \quad (3.11)$$

Then

$$\Delta(\alpha, z) = z^{\alpha(0)} E^+(\alpha, -z). \quad (3.12)$$

Next, we extend the domain of $Y_\alpha$ from $V$ to $V[L]$.

**Definition 3.1** For $u \in V^{(\alpha)}$, $v \in V^{(\beta)}$ with $\alpha, \beta \in L$, we define

$$Y_\alpha(u, z)v = \psi_{\alpha+\beta} E^-(\alpha, z) Y(\psi_\alpha \Delta(\beta, z)u, z) \Delta(\alpha, -z) \psi_\beta(v) \in V^{(\alpha+\beta)} \{z\}. \quad (3.13)$$

We then define a linear map $\tilde{Y}(\cdot, z)$ from $V[L]$ to $(\text{End } V[L])\{z\}$ via $\tilde{Y}(u, z) = Y_\alpha(u, z)$ for $u \in V^{(\alpha)}$.\(^2\)

\(^2\)The map $\psi_\alpha$ here is the inverse of the map $\psi_\alpha$ in [DLM2]
Note that the $E^-(\alpha, z)$ defined in [DLM2] is the $E^-(\alpha, z)$ defined in (3.11) ([LW], [FLM]). Then the definition of $Y_\alpha$ is exactly the same as the one defined in [DLM2].

For $\alpha \in P$, $h \in H$, set

$$V^{(\alpha, h)} = \psi_\alpha(V^{(0, h)}). \quad \text{(3.14)}$$

Then

$$V^{(\alpha)} = \bigoplus_{h \in P} V^{(\alpha, h)}. \quad \text{(3.15)}$$

**Definition 3.2** We define $\mathbb{C}$-valued functions $\eta$ and $C$ on $(L \times H) \times (L \times H)$ by

$$\eta((\alpha_1, h_1), (\alpha_2, h_2)) = -B(\alpha_1, \alpha_2) - B(\alpha_1, h_2) - B(\alpha_2, h_1) \in \mathbb{C}, \quad \text{(3.16)}$$

$$C((\alpha_1, h_1), (\alpha_2, h_2)) = e^{(B(\alpha_1, h_2) - B(\alpha_2, h_1))\pi i} \in \mathbb{C}^\times \quad \text{(3.17)}$$

for $(\alpha_i, h_i) \in L \times H$, $i = 1, 2$.

Then we have ([DLM2], Theorem 3.5):

**Theorem 3.3** Let $u \in V^{(\alpha_1, h_1)}$, $v \in V^{(\beta, h_2)}$, $w \in V^{(\gamma, h_3)}$ with $\alpha, \beta, \gamma \in L$, $h_1, h_2, h_3 \in P$. Then

$$z_0^{-1}\delta \left( \frac{z_1 - z_2}{z_0} \right) \eta((\alpha_1, h_1), (\beta, h_2)) \tilde{Y}(u, z_1)\tilde{Y}(v, z_2)w$$

$$- C((\alpha_1, h_1), (\beta, h_2))z_0^{-1}\delta \left( \frac{z_2 - z_1}{z_0} \right) \eta((\alpha_1, h_1), (\beta, h_2)) \tilde{Y}(v, z_2)\tilde{Y}(u, z_1)w$$

$$= z_2^{-1}\delta \left( \frac{z_1 - z_0}{z_2} \right) \eta((\alpha_1, h_1), (\gamma, h_3)) \tilde{Y}(\tilde{Y}(u, z_0)v, z_2)w. \quad \text{(3.18)}$$

Furthermore, for all $v \in V[L]$,

$$[L(-1), \tilde{Y}(v, z)] = \frac{d}{dz}\tilde{Y}(v, z). \quad \text{(3.19)}$$

Now we express $\tilde{Y}$ more explicitly.

**Lemma 3.4** For $\alpha, \beta \in L$, $u, v \in V$, we have

$$\tilde{Y}(e^\alpha \otimes u, z)(e^\beta \otimes v) = e^{\alpha + \beta} \otimes z^{B(\alpha, \beta)}E^{-}(\alpha, z)Y(\Delta(\beta, z)u, z)E^+(\alpha, z)(-z)^\alpha(0)v. \quad \text{(3.20)}$$

In particular,

$$\tilde{Y}(e^\alpha \otimes 1, z)(e^\beta \otimes v) = e^{\alpha + \beta} \otimes z^{B(\alpha, \beta)}E^{-}(\alpha, z)E^+(\alpha, z)(-z)^\alpha(0)v. \quad \text{(3.21)}$$
Proof. From Lemma 3.2 of [DLM2] we have

$$\psi_{-\alpha}\Delta(\beta, z) = z^{B(\alpha, \beta)}\Delta(\beta, z)\psi_{-\alpha}. \quad (3.22)$$

Note that the map \(\psi_{\alpha}\) defined here is the map \(\psi_{-\alpha}\) defined in [DLM2]. Then (3.20) follows immediately. \(\square\)

Remark 3.5 Let \(G = L \times P\) be the product group. Suppose that there exists a positive integer \(T\) such that \(\eta\) restricted on \(G\) is \(\frac{1}{T}Z\)-valued. The original theorem states that \((V[L], 1, \omega, \tilde{Y}, T, G, \eta(\cdot, \cdot), C(\cdot, \cdot))\) is a generalized vertex algebra in the sense of [DL]. This result is similar to Theorem 5.1 of [DL], which states that if \(L\) is a rational lattice, then \(V_L\) has a canonical generalized vertex algebra structure. In fact, by taking \(V = M_h(1, 0)\) with \(h = C \otimes_{\mathbb{Z}} L\), we have \(P = 0\) and \(V[L] = V_L\).

In order to have vertex (super)algebras \(V[L]\), we shall restrict ourselves to special \(L\). We have already assumed that \(L \subset P^0\), or what is equivalent to, \(\alpha(0)\) acting on \(V\) has only integral eigenvalues for every \(\alpha \in L\). Now we furthermore assume that \((L, B)\) is an integral lattice. Then

\[B(\alpha, \alpha), B(\alpha, \beta) \in \mathbb{Z} \quad \text{for} \quad \alpha \in L, \beta \in P. \quad (3.23)\]

Recall from Lemma 2.14 that for \(\alpha \in L\), the weights of \(V(\alpha)\) are contained in \(\frac{1}{2}B(\alpha, \alpha) + \mathbb{Z}\). Then \(V[L]\) is \(\frac{1}{2}\mathbb{Z}\)-graded by \(L(0)\)-weights. Furthermore, the function \(\eta\) restricted to \((L \times P) \times (L \times P)\) is \(\mathbb{Z}\)-valued and

\[C((\alpha_1, h_1), (\alpha_2, h_2)) = (-1)^{B(\alpha_1, h_2) - B(\alpha_2, h_1)}, \quad (3.24)\]

(recall (3.16) and (5.17)). Then we have ([DLM2], (5.9)):

Corollary 3.6 Assume that \(L \subset P^0\) and \((L, B)\) is an integral lattice. For \(a \in V(\alpha), b \in V(\beta)\) with \(\alpha, \beta \in L\), we have

\[z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)\tilde{Y}(a, z_1)\tilde{Y}(b, z_2) - (-1)^{B(\alpha, \beta)}z_0^{-1}\delta\left(\frac{z_2 - z_1}{z_0}\right)\tilde{Y}(b, z_2)\tilde{Y}(a, z_1)\]

\[= z_0^{-1}\delta\left(\frac{z_2 - z_0}{z_2}\right)\tilde{Y}(\tilde{Y}(a, z_0)b, z_2). \quad (3.25)\]

Remark 3.7 Set

\[L^e = \{\alpha \in L \mid B(\alpha, \alpha) \in 2\mathbb{Z}\}. \quad (3.26)\]

Clearly, \(L^e\) is a subgroup of \(L\) of index 2. If \(L^e \subset 2L^0\), i.e.,

\[B(\alpha, \beta) \in 2\mathbb{Z} \quad \text{for} \quad \alpha \in L^e, \beta \in L, \quad (3.27)\]
we easily see that $V[L] = \oplus_{\beta \in L} V^{(\beta)}$ is a vertex superalgebra with

$$V[L]^0 = V[L^e] = \oplus_{\alpha \in L^e} V^{(\alpha)}, \quad V[L]^1 = \oplus_{\alpha \in L-L^e} V^{(\alpha)}.$$  \hfill (3.28)

In particular, if $L$ is of rank one, clearly (3.27) holds, hence $V[L]$ is a vertex (super)algebra. However, without assuming (3.27), $V[L]$ equipped with the vertex operator map $\tilde{Y}$ may not be a vertex superalgebra. Even if $L$ is even, $V[L]$ may not be a vertex algebra unless $B$ is $2\mathbb{Z}$-valued. Corollary 3.13 of [DLM2], which states that $V[L]$ equipped with $\tilde{Y}$ is a vertex superalgebra if $L$ is integral (without the condition (3.27), is incorrect.

As in [FLM] and [DL] for $V_L$, we shall need a 2-cocycle $\epsilon$ on $L$. Suppose that $(L, B)$ is an integral lattice of finite rank. Let $\{\alpha_1, \ldots, \alpha_d\}$ be a $\mathbb{Z}$-basis for $L$. Define $\epsilon$ to be the (uniquely determined) $\{\pm 1\}$-valued multiplicative function on $L \times L$ such that

$$\epsilon(\alpha_i, \alpha_j) = (-1)^{B(\alpha_i, \alpha_i) + B(\alpha_i, \alpha_j)B(\alpha_j, \alpha_j)}$$ if $i \geq j$, and 1 otherwise \hfill (3.29)

(cf. [DL], [FLM]). Note that $\epsilon(\alpha_i, \alpha_i) = 1$ because $B(\alpha_i, \alpha_i) + B(\alpha_i, \alpha_i)B(\alpha_i, \alpha_i)$ is even. Then

$$\epsilon(\alpha, \beta)\epsilon(\beta, \alpha)^{-1} = (-1)^{B(\alpha, \beta) + B(\alpha, \alpha)B(\beta, \beta)}$$ for $\alpha, \beta \in L$. \hfill (3.30)

Define a vertex operator map $Y$ on $V[L]$ by

$$Y(e^\alpha \otimes u, z)(e^\beta \otimes v) = \epsilon(\alpha, \beta)\tilde{Y}(e^\alpha \otimes u, z)(e^\beta \otimes v)$$ \hfill (3.31)

for $\alpha, \beta \in L$, $u, v \in V$. By Lemma 3.4,

$$Y(e^\alpha \otimes u, z)(e^\beta \otimes v) = \epsilon(\alpha, \beta) e^{\alpha+\beta} \otimes z^{B(\alpha, \beta)} E^-(\alpha, z)Y(\Delta(\beta, z)u, z)E^+(-\alpha, z)E^{(0)}v. \hfill (3.32)$$

From Corollary 3.3 we immediately obtain:

**Proposition 3.8** Suppose that $L \subset P^o$ and that $(L, B)$ is an integral lattice of finite rank. Let $\epsilon$ be the $\{\pm 1\}$-valued multiplicative function on $L \times L$ defined in (3.29). Then $V[L]$ equipped with the vertex operator map $Y$ defined in (3.32) is a vertex (super)algebra with the even and odd parts being defined in (3.28). \hfill \Box

**Remark 3.9** It was proved in [DL] (Theorem 6.7 and Remarks 6.17 and 12.38) that if $L$ is an integral lattice, $V_L$ is a vertex superalgebra.

Since we in this paper are mainly interested in vertex operator (super)algebras, for the rest of Section 3 we shall assume that $L \subset P^o$ and $(L, B)$ is integral, and we fix the multiplicative function $\epsilon$. 

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3.2 Extensions of modules and intertwining operators

We continue Section 3.1 to study the extension of an irreducible $V$-module, following [DLM2]. The extension $W[L]$ of an irreducible $V$-module $W$ is in general a twisted $V[L]$-module with respect to an automorphism of $V[L]$. We shall also study the extension of an intertwining operator.

Let $W$ be an irreducible $V$-module. By definition, homogeneous subspaces of $W$ are finite-dimensional. Since $[L(0), h(0)] = 0$ for $h \in H$, $H$ preserves each homogeneous subspace of $W$ so that there exist $0 \neq w \in W$, $h \in H = H^*$ such that $h(0)w = B(h, h')w$ for $h' \in H$. Since $H$ acts semisimply on $V$ (by assumption) and $w$ generates $W$ by $V$ (from the irreducibility of $W$), $H$ also acts semisimply on $W$. For any $h \in H$, we define

$$W^{(0, h)} = \{w \in W \mid h(0)w = B(h, h')w \text{ for } h' \in H\}.$$  

(3.33)

Set

$$P(W) = \{h \in H \mid W^{(0, h)} \neq 0\}.$$  

(3.34)

Since $W$ is irreducible, $P(W)$ is an irreducible $P(V)$-set. Then $P(W) = h + P(V)$ for any $h \in P(W)$.

**Definition 3.10** Let $W$ be an irreducible $V$-module and $h \in P(W)$. Define a linear endomorphism $\sigma_W$ of $V[L]$ by

$$\sigma_W(a) = e^{-2\pi i B(\alpha, h)} a \quad \text{for } a \in V(\alpha) \quad \text{with } \alpha \in L.$$  

(3.35)

Because $L \subset P(V)^0$ and $P(W) = h + P(V)$, $\sigma_W$ is well defined, i.e., it does not depend on the choice of $h$.

**Lemma 3.11** The defined linear endomorphism $\sigma_W$ of $V[L]$ is an automorphism of the vertex (super)algebra and $\sigma_W = 1$ if and only if $\alpha(0)$ has only integral eigenvalues on $W$ for every $\alpha \in L$, i.e., $P(W) \subset L^\circ$. Furthermore, if $V$ is finitely generated, $\sigma_W$ is of finite order if and only if $\alpha(0)$ has rational eigenvalues on $W$ for every $\alpha \in L$, or equivalently, $B(\alpha, h) \in \mathbb{Q}$ for $\alpha \in L$.  

**Proof.** In view of (3.13), clearly, $\sigma_W$ is an automorphism of the vertex (super)algebra and $\sigma_W = 1$ if and only if $\alpha(0)$ has only integral eigenvalues on $W$ for every $\alpha \in L$. Furthermore, when $V$ is finitely generated, $\sigma_W$ is of finite order if and only if $\alpha(0)$ has rational eigenvalues on $W$ for every $\alpha \in L$, or equivalently, $B(\alpha, h) \in \mathbb{Q}$ for $\alpha \in L$.  

Recall that $W[L] = \mathbb{C}[L] \otimes W = \oplus_{\alpha \in L} W^{(\alpha)}$.

**Definition 3.12** Let $W$ be an irreducible $V$-module. For $a \in V(\alpha)$, $w \in W(\beta)$, $\alpha, \beta \in L$, we define $Y_{W[L]}(a, z)w \in W^{(\alpha + \beta)}\{z\}$ by

$$Y_{W[L]}(a, z)w = \epsilon(\alpha, \beta) \psi_{\alpha + \beta} E^-(\alpha, z) Y(\psi_\alpha \Delta(\beta, z)a, z) \Delta(\alpha, -z) \psi_{-\beta} w.$$  

(3.36)
(cf. (3.13)).

In terms of the notion of twisted module as defined in [Le], [D2] and [FFR] we have ([DLM2], Theorem 3.6, Corollary 3.14):

** Proposition 3.13** Let $W$ be an irreducible $V$-module such that $α(0)$ has rational eigenvalues on $W$ for every $α ∈ L$. Then the following twisted Jacobi identity holds for $u ∈ V^{(α)}$, $v ∈ V^{(β)}$, $w ∈ W^{(γ, h)}$,

$$ z_0^{-1}δ\left(\frac{z_1 - z_2}{z_0}\right)Y(u, z_1)Y(v, z_2)w - (-1)^{B(α, α)B(β, β)}z_0^{-1}δ\left(\frac{z_2 - z_1}{z_0}\right)Y(v, z_2)Y(u, z_1)w $$

$$ = z_2^{-1}\left(\frac{z_1 - z_0}{z_2}\right)^{B(α, h)}\delta\left(\frac{z_1 - z_0}{z_2}\right)Y(Y(u, z_0)v, z_2)w. $$

(3.37)

Moreover, $W[L]$ is a $σ_W$-twisted $V[L]$-module. In particular, if $P(W) ⊂ L^0$, i.e., $α(0)$ acting on $W$ has only integral eigenvalues for $α ∈ L$, $W[L]$ is an (untwisted) $V[L]$-module.

Next, we prove a functorial property.

** Proposition 3.14** Let $σ$ be an automorphism of $V[L]$ of finite order such that

$$ σ(V^{(α)}) = V^{(α)} \text{ for } α ∈ L, \quad (3.38) $$

$$ σ(v) = v \quad \text{for } v ∈ V = V^{(0)}. \quad (3.39) $$

Let $M$ be a $σ$-twisted weak $V[L]$-module which is a direct sum of irreducible $V$-modules and let $W$ be an irreducible $V$-submodule of $M$. Then $σ = σ_W$ and there is a canonical $V[L]$-homomorphism $φ[L]$ from $W[L]$ to $M$ extending the embedding $φ$ of $W$ into $M$.

** Proof.** The following arguments are essentially the ones of [DLM2], Corollary 3.15 and Lemma 4.3.

Note that for $α ∈ L$, $Y_{W[L]}$ restricted to $V^{(α)} × W$ is a nonzero intertwining operator of type $\left(\frac{W^{(α)}}{V^{(α)}W}\right)$. Since $Y_M(\cdot, z)φ$ restricted to $V^{(α)} × W$ is an intertwining operator of type $\left(\frac{M}{V^{(α)}W}\right)$ and $[V^{(α)}] · [W] = [W^{(α)}]$, it follows from Schur lemma (cf. [FHL]) that there exists a unique $V$-homomorphism $φ_α$ from $V^{(α)}$ to $M$ such that

$$ Y_M(a, z)w = φ_α(Y_{W[L]}(a, z)w) $$

(3.40)

for $a ∈ V^{(α)}$, $w ∈ W$. From the definition of a twisted module we have

$$ z^{r_α}Y_M(a, z)w ∈ M((z)), \quad φ_α(z^{B(α, h)}Y_{W[L]}(a, z)w) ∈ M((z)). $$

(3.41)

Then it follows from (3.40) that $r_α - B(α, h) ∈ Z$, hence $σ(a) = σ_W(a)$. Thus $σ = σ_W$.

Define a $V$-homomorphism $φ[L]$ from $W[L]$ to $M$ by $φ[L] = φ_α$ on $V^{(α)}$ for $α ∈ L$. Now we show that $φ[L]$ is a $V[L]$-homomorphism.

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Let $w \in W$, $a \in V^{(\alpha)}$, $b \in V^{(\beta)}$ with $\alpha, \beta \in L$. Let $k_0$ be a positive integer such that

$$
(z_0 + z_2)^{k_0 + B(a, h)} Y_{W[L]}(a, z_0 + z_2) Y_{W[L]}(b, z_2) w = (z_2 + z_0)^{k_0 + B(a, h)} Y_{W[L]}(Y(a, z_0) b, z_2) w, \tag{3.42}
$$

$$
(z_0 + z_2)^{k_0 + B(a, h)} Y_{M}(a, z_0 + z_2) Y_{M}(b, z_2) \phi(w) = (z_2 + z_0)^{k_0 + B(a, h)} Y_{M}(Y(a, z_0) b, z_2) \phi(w). \tag{3.43}
$$

Then using (3.40) we get

$$
(z_0 + z_2)^{k_0 + B(a, h)} \phi_{\alpha + \beta} Y_{W[L]}(a, z_0 + z_2) Y_{W[L]}(b, z_2) w = (z_2 + z_0)^{k_0 + B(a, h)} \phi_{\alpha + \beta} Y_{M}(Y(a, z_0) b, z_2) w,
$$

that is,

$$
\phi[L] Y(a, z_0 + z_2) Y_{W[L]}(b, z_2) w = Y_{M}(a, z_0 + z_2) \phi[L] Y_{M}(b, z_2) w. \tag{3.46}
$$

Since $W^{(\beta)}$ is linearly spanned by $b_n W$ for $b \in V^{(\beta)}$, $n \in \mathbb{Z}$, we have

$$
\phi[L] (Y_{W[L]}(a, z) u) = Y_{M}(a, z) u \tag{3.47}
$$

for $a \in V^{(\alpha)}$, $u \in W^{(\beta)}$. Thus $\phi[L]$ is a $V[L]$-homomorphism. □

Our next result gives a characterization of the equivalence relation on (twisted) $V[L]$-modules $W[L]$ in terms of the equivalence of $V$-modules:

**Proposition 3.15** Let $W_1$ and $W_2$ be irreducible $V$-modules on which $\alpha(0)$ has rational eigenvalues for each $\alpha \in L$. Then $\sigma_{W_1} = \sigma_{W_2}$ and $W_1[L] \simeq W_2[L]$ if and only if $W_2 \simeq W_1^{(\alpha)}$ for some $\alpha \in L$.

**Proof.** The “only if” part is clear. Note that $\sigma_{W^{(\alpha)}} = \sigma_W$ for any irreducible $V$-module $W$ and $\alpha \in L$ because $P(W^{(\alpha)}) = \alpha + P(W)$. Assume $W_2 \simeq W_1^{(\alpha)}$ for some $\alpha \in L$. Then $\sigma_{W_1} = \sigma_{W_2}$. Let $\phi$ be a $V$-isomorphism from $W_2$ to $W_1^{(\alpha)} \subset W_1[L]$. It follows from Proposition 3.13 that $\phi$ extends to a $V[L]$-homomorphism $\phi[L]$ from $W_2[L]$ into $W_1[L]$ with $\phi[L](W_2^{(\beta)}) = W_1^{(\alpha + \beta)}$ for $\beta \in L$. With each $W^{(\beta)}$ being an irreducible $V$-module, $\phi[L]$ is an isomorphism. □

Next, we shall extend an intertwining operator $I$ in the category of $V$-modules to an intertwining operator $I[L]$ in the category of $V[L]$-modules.
**Definition 3.16** Let $W_1, W_2$ and $W_3$ be irreducible $V$-modules and $I$ be an intertwining operator of type $(w_3^{W_3/W_1W_2})$. We define a linear map

$$I[L] : W_1[L] \to (\text{Hom}(W_2[L], W_3[L]))\{w\}$$

by

$$I[L](a, z)w = \epsilon(\alpha, \beta)\psi_{\alpha+\beta}E^{-}(\alpha, z)\Delta(\beta, z)\psi_{-\beta}(w)$$

(cf. (3.13) and (3.36)) for $a \in W_1^{(\alpha)}$, $w \in W_2^{(\beta)}$ with $\alpha, \beta \in L$.

The same proof of Theorem 3.5 in [DLM2] gives:

$$I[L](L(-1)a, z) = \frac{d}{dz}I[L](a, z) \quad \text{for } a \in W_1[L],$$

and

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)\left(\frac{z_1 - z_2}{z_0}\right)^{\eta((\alpha, h), (\beta, h_1))}Y_W(a, z_1)I[L](b, z_2)u$$

$$- C((\alpha, h), (\beta, h_1))z_0^{-1}\delta\left(\frac{z_2 - z_1}{z_0}\right)\left(\frac{z_2 - z_1}{z_0}\right)^{\eta((\alpha, h), (\beta, h_1))}I[L](b, z_2)Y_W(a, z_1)u$$

$$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)\left(\frac{z_2 + z_0}{z_1}\right)^{\eta((\alpha, h), (\gamma, h_2))}I[L](Y(a, z_0)b, z_2)u$$

(3.51)

for $a \in V^{(\alpha, h)}$, $b \in W_1^{(\beta, h_1)}$, $u \in W_2^{(\gamma, h_2)}$, with $\alpha, \beta, \gamma \in L$, $h \in P$, $h_1 \in P(W_1)$, $h_2 \in P(W_2)$. If the extensions $W_i[L]$ are (untwisted) $V[L]$-modules, then $P(W_i) \subset P^n$, so that $\eta$ and $C$ have integer values. Then we conclude:

**Proposition 3.17** Let $W_1, W_2, W_3$ be irreducible $V$-modules such that for $\alpha \in L$, $\alpha(0)$ has only integral eigenvalues on $W_i$ for $i = 1, 2, 3$, or what is equivalent to, the extensions $W_i[L]$ are (untwisted) $V[L]$-modules. Let $I$ be an intertwining operator of type $(w_3^{W_3/W_1W_2})$ in the category of $V$-modules. Then $I[L]$ is an intertwining operator of type $(w_3^{W_3[L]/W_1[L]W_2[L]})$ in the category of $V[L]$-modules. \(\square\)

Note that various $V$-submodules $V^{(\alpha)}$ of $V[L]$ may be $V$-isomorphic to each other. It was proved in [DLM2] that $V[L]$ contains an ideal $I$ such that each irreducible $V$-module $V^{(\alpha)}$ is of multiplicity-one in the quotient algebra $V[L] \mod I$. In the following we present an abstract reformulation of this result.

Consider an (abstract) vertex operator (super)algebra $U = \oplus_{g \in G} V^g$ graded by a (finite or infinite) abelian group $G$ satisfying the following conditions:

(C1) $V^0$ is a vertex operator subalgebra and $V^g$ for $g \in G$ are simple currents for $V^0$.
(C2) For $u \in V^g$, $v \in V^h$ with $g, h \in G$, $u_nv \in V^{g+h}$ for $n \in \mathbb{Z}$. 

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(C3) For \( g, h \in G, u_n v \neq 0 \) for some \( u \in V^g, v \in V^h, n \in \mathbb{Z} \).

It is easy to see that under these conditions, \( U \) is a simple \( G \)-graded algebra, i.e., there is no nontrivial \( G \)-graded ideal. From Conditions (2) and (3), \( Y \) restricted to \( V^g \times V^h \) is a nonzero intertwining operator of type \( (V^{g+h}_{V^g V^h}) \). Then from Condition (1) we have

\[
[V^g] \cdot [V^h] = [V^{g+h}] \quad \text{for } g, h \in G. \tag{3.52}
\]

Set

\[
G_0 = \{ g \in G \mid V^g \simeq V^0 \text{ as } V^0\text{-modules} \}. \tag{3.53}
\]

Using (3.52) by a routine argument we easily get (cf. [DLM2], Lemma 3.7):

**Lemma 3.18** The defined subset \( G_0 \) of \( G \) is a subgroup and for \( g, h \in G \), \( [V^g] = [V^h] \) if and only if \( g - h \in G_0 \). \( \square \)

For each \( g_0 \in G_0 \), fix a \( V^0\)-isomorphism \( f_{g_0} \) from \( V^0 \) to \( V^{g_0} \). We particularly define \( f_0 = 1 \). Let \( g_0 \in G_0, h \in G \). Then \( Y(\cdot, z) \circ f_{g_0} \) is a nonzero intertwining operator of type \( (V^{g_0+h}_{V^h V^0}) \). On the other hand, for any \( V^0\)-isomorphism \( \psi \) from \( V^h \) to \( V^{g_0+h} \), \( \psi \circ Y(\cdot, z) \) is a nonzero intertwining operator of type \( (V^{g_0+h}_{V^h V^0}) \). Because \( V^h, V^0, V^{g_0+h} \) are simple currents and

\[
[V^{g_0+h}] = [V^g] \cdot [V^h] = [V^0] \cdot [V^h] = [V^h],
\]

there exists a unique \( V^0\)-isomorphism \( f_{g_0,h} \) from \( V^h \) to \( V^{g_0+h} \) such that

\[
f_{g_0,h}(Y(u, z)v) = Y(u, z)f_{g_0}(v) \quad \text{for } v \in V^0, u \in V^h. \tag{3.54}
\]

Define a \( V^0\)-endomorphism \( \bar{f}_{g_0} \) of \( U \) via \( \bar{f}_{g_0} = f_{g_0,h} \) on \( V^h \) for \( h \in G \).

Next we define \( I \) to be the linear span of

\[
\bar{f}_{g_0}(u) - u \quad \text{for } g_0 \in G_0, u \in U. \tag{3.55}
\]

**Lemma 3.19** The defined subspace \( I \) is an ideal of \( U \) with \( I \cap V^0 = 0 \). Furthermore, \( I = 0 \) if and only if \( G_0 = 0 \).

**Proof.** Let \( g_0 \in G_0, h, h' \in G \) and let \( u \in V^h, v \in V^{h'} \). Then

\[
\bar{f}_{g_0}(u) = \text{Res}_{z_2} \bar{f}_{g_0}(Y(u, z_2)1) = \text{Res}_{z_2} Y(u, z_2)\bar{f}_{g_0}(1). \tag{3.56}
\]

Since

\[
Y(v, z)\bar{f}_{g_0}(1) = \bar{f}_{g_0}(Y(v, z)1) \in U[[z]], \tag{3.57}
\]

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we have (cf. [Li2])
\[
Y(v, z_0 + z_2)Y(u, z_2)f_{g_0,h}(1) = Y(Y(v, z_0)u, z_2)f_{g_0,h}(1).
\] (3.58)

Using (3.56)-(3.58) we obtain
\[
Y(v, z_0)(\bar{f}_{g_0}(u) - u) = \text{Res}_{z_2} Y(v, z_0 + z_2)Y(u, z_2)f_{g_0,h}(1) - Y(v, z_0)u = \text{Res}_{z_2} Y(Y(v, z_0)u, z_2)f_{g_0,h}(1) - Y(v, z_0)u = f_{g_0,h}(Y(v, z_0)u) - Y(v, z_0)u = \bar{f}_{g_0}(Y(v, z_0)u) - Y(v, z_0)u.
\] (3.59)

It follows immediately that \( I \) is an ideal.

Clearly, \( I \neq 0 \) if \( G_0 \neq \{0\} \), so \( I = 0 \) implies \( G_0 = 0 \). If \( G_0 = 0 \), with \( f_0 = 1 \) from the definition of \( I \) we have \( I = 0 \). □

**Proposition 3.20** The algebra \( U \) is simple if and only if \( G_0 = 0 \). Furthermore, the quotient algebra \( \bar{U} = U/I \) is simple.

**Proof.** From Lemma 3.19, \( U \) is not simple if \( G_0 \neq 0 \). Now we prove that \( U \) is simple if \( G_0 = \{0\} \). In view of Lemma 3.18, all \( V^g \) for \( g \in G \) are non-isomorphic irreducible \( V^0 \)-modules. Then any nonzero ideal of \( U \) as a \( V^0 \)-module must be a sum of some \( V^g \). Then it follows from the conditions (1)-(3) that any nonzero ideal of \( U \) must be \( U \). That is, \( U \) is simple.

Note that \( \bar{U} = U/I \) is a vertex operator (super)algebra graded by group \( G/G_0 \) with all the conditions (1)-(3) being satisfied. Furthermore, \( \bar{U} \) is a direct sum of non-isomorphic simple current \( V^0 \)-modules. From Part one, \( \bar{U} \) must be simple. □

Applying Proposition 3.20 to \( V[L] \) we immediately have (cf. [DLM2], Corollary 3.13):

**Corollary 3.21** Let \( V, L \) be as before. Set
\[
L_0 = \{ \alpha \in L \mid V^{(\alpha)} \simeq V \text{ as } V\text{-modules} \}.
\] (3.60)

Then \( V[L] \) has an ideal \( I \) such that \( V[L]/I \) is simple with \( V \) as a subalgebra and such that as a \( V \)-module
\[
V/I \simeq \bigoplus_{\alpha \in S} V^{(\alpha)},
\] (3.61)

where \( S \) is a complete set of representatives of cosets of \( L_0 \) in \( L \). □
3.3 Multiplicity-free extension $V[L]$

In this subsection we consider extended vertex (super)algebra $V[L]$ in which each $V$-module $V^{(\alpha)}$ is multiplicity-free. We shall classify all irreducible $V[L]$-modules in terms of irreducible $V$-modules and determine the fusion rules of $V[L]$-modules by the fusion rules of $V$-modules. Our concrete examples we shall construct in Section 4 is of this type, so that all the results of this subsection apply to those examples.

Throughout this subsection we assume that for any irreducible $V$-module $W$ and for $\alpha, \beta \in L$, $W^{(\alpha)} \simeq W^{(\beta)}$ as $V$-modules if and only if $\alpha = \beta$.

We shall need the following result:

**Proposition 3.22** The vertex (super)algebra $V[L]$ is simple. Furthermore, if $Y_1$ and $Y_2$ are two simple vertex operator (super)algebra structures on $V[L]$ extending the $V$-module structure $Y_V$, then vertex operator (super)algebras $(V[L], Y_1)$ and $(V[L], Y_2)$ are isomorphic.

**Proof.** First, we prove that $V[L]$ is simple. Notice that in the definition (3.13) of the vertex operator map $Y$, $\psi_{\alpha+\beta}, \psi_{-\alpha}, \psi_{-\beta}, E^{-}(\alpha, z), \Delta(\beta, z)$ and $\Delta(\alpha, -z)$ are invertible elements and that

$$Y(a, z)b \in V^{(\alpha+\beta)} \quad \text{for} \quad a \in V^{(\alpha)}, \ b \in V^{(\beta)}$$

(cf. (3.13)). Since $V$ is simple, $Y(u, z)v \neq 0$ for $0 \neq u, v \in V$ ([DL], Proposition 11.9, or [FHL], Remark 5.4.6). Then it follows from Proposition 3.20 immediately that $V[L]$ is simple.

For $\alpha, \beta \in L$, because $V^{(\alpha)}$ and $V^{(\beta)}$ are simple currents, there exists $\epsilon'(\alpha, \beta) \in \mathbb{C}^\times$ such that

$$Y_2(a, z)b = \epsilon'(\alpha, \beta)Y_1(a, z)b \quad \text{for} \quad a \in V^{(\alpha)}, \ b \in V^{(\beta)}.$$  \hspace{1cm} (3.62)

It follows from weak associativity of vertex operators (cf. (3.43)) that

$$\epsilon'(\alpha, \beta + \gamma)\epsilon'(\beta, \gamma) = \epsilon'(\alpha, \beta)\epsilon'(\alpha + \beta, \gamma)$$ \hspace{1cm} (3.63)

for $\alpha, \beta \in L$. That is, $\epsilon'$ is a ($\mathbb{C}^\times$-valued) 2-cocycle on $L$. We also have

$$\epsilon'(0, \alpha) = \epsilon'(\alpha, 0) = 1.$$ \hspace{1cm} (3.64)

Since $V = V^{(0)}$ is even for both superalgebra structures, each structure corresponds a sublattice $L_i$ of $L$ of index 2 with $V[L_i]$ being the even parts.

Now we claim that $L_1 = L_2$. Otherwise, suppose $L_1 - L_2 \neq \emptyset$ and let $\beta \in L_1 - L_2$. Then we have the following skew-symmetry

$$Y_1(a, z)b = e^{zL(-1)}Y_1(b, -z)a, \quad \hspace{1cm} (3.65)$$

$$Y_2(a, z)b = -e^{zL(-1)}Y_2(b, -z)a \quad \hspace{1cm} (3.66)$$
for $a, b \in V^{(\beta)}$. Since both $Y_1$ and $Y_2$ extend the $V$-module structure, the two vertex superalgebra structures have the same Virasoro vector. Consequently,

$$\epsilon'(\beta, \beta) = -\epsilon'(\beta, \beta),$$

(3.67)

which is impossible because $\epsilon'(\beta, \beta) \neq 0$.

With $L_1 = L_2$, using the skew-symmetry we obtain

$$\epsilon'(\alpha, \beta) = \epsilon'(\beta, \alpha) \quad \text{for} \quad \alpha, \beta \in L.$$  

(3.68)

It follows from the proof of Propositions 5.1.2 and 5.2.3 in [FLM] (with $\mathbb{Z}/s\mathbb{Z}$ being replaced by $\mathbb{C}^\times$) that $\epsilon'$ is a 2-coboundary. Then the two vertex superalgebra structures on $V[L]$ are equivalent.  

\begin{remark}
More generally, let $G$ be a (finite or infinite) abelian group and let $V$ be a simple vertex operator algebra and $V[G] = \oplus_{g \in G} V^g$ be a $V$-module with each $V^g$ being an irreducible $V$-submodule. Furthermore, assume that each $V^g$ is a simple current of $V$. Then the set of equivalence class of simple vertex operator (super)algebra structures on $V[G]$ which extend the $V$-module one-to-one corresponds to the set of equivalence classes of symmetric $\mathbb{C}^\times$-valued 2-cocycles of $G$.

Similar to Proposition 3.22 we have (cf. [DLM2], Lemma 4.2):

\begin{proposition}
Let $W$ be an irreducible $V$-module. Then $W[L]$ is irreducible.  
\end{proposition}

The following theorem gives the complete reducibility for every $V[L]$-module under certain conditions:

\begin{theorem}
Assume that there is a sublattice $L_1$ of $L$ such that $V[L_1]$ is regular and that every irreducible $V[L_1]$-module is a direct sum of irreducible $V$-modules. Let $\sigma$ be an automorphism of $V[L]$ such that $\sigma$ fixes $V[L_1]$ point-wise. Then any $\sigma$-twisted weak $V[L]$-module is a direct sum of irreducible $\sigma$-twisted $V[L]$-modules of type $W[L]$ with $\sigma = \sigma_W$. In particular, $V[L]$ is regular.

\end{theorem}

\begin{proof}
Let $M$ be a $\sigma$-twisted weak $V[L]$-module. Since $\sigma$ fixes $V[L_1]$ point-wise, $M$ is a weak $V[L_1]$-module. With $V[L_1]$ being regular, $M$ is a direct sum of irreducible (ordinary) $V[L_1]$-modules. With the assumption that each irreducible $V[L_1]$-module is a direct sum of irreducible $V$-modules, $M$ is a direct sum of irreducible $V$-modules. Let $W$ be an irreducible $V$-submodule of $M$. By Proposition 3.14, $\sigma = \sigma_W$ and there exists a $V[L]$-homomorphism $\phi[L]$ from $W[L]$ to $M$ extending the embedding $\phi$ of $W$ into $M$. Since $W[L]$ is $V[L]$-irreducible (Proposition 3.24), the $V[L]$-submodule of $M$ generated
by $W$, which is the image of $\phi[L]$, is an irreducible $\sigma$-twisted $V[L]$-module. Therefore, $M$ is a direct sum of irreducible $\sigma$-twisted $V[L]$-modules of type $W[L]$.

Let $W_1, W_2, W_3$ be irreducible $V$-modules such that $\sigma_{W_1} = 1$, or equivalently, the extensions $W_1[L], W_2[L]$ and $W_3[L]$ are $V[L]$-modules. For each $\alpha \in L$, with $W_3^{(\alpha)}$ being a $V$-submodule of $W_3[L]$, by Proposition 3.14, there is a $V[L]$-homomorphism $g_\alpha$ from $W_3^{(\alpha)}[L]$ to $W_3[L]$. Then with Proposition 3.17, we obtain a natural linear map $f_{\alpha}$ from $I_{W_1W_2}^{W_3[L]}$ to $I_{W_1[L]W_2[L]}^{W_3[L]}$ defined by

$$f_{\alpha}(I) = g_\alpha \circ I[L].$$

The next result gives a precise connection between the fusion rules for $V$-modules and the fusion rules for $V[L]$-modules:

**Theorem 3.26** Let $W_1, W_2, W_3$ be irreducible $V$-modules such that $\sigma_{W_1} = 1$, which implies that $W_1[L], W_2[L], W_3[L]$ are irreducible $V[L]$-modules. In addition, we assume that $V$ is quasi-rational. Then $f = \prod_{\alpha \in L} f_{\alpha}$ is a linear isomorphism from $\prod_{\alpha \in L} I_{W_1W_2}^{W_3^{(\alpha)}}$ to $I_{W_1[L]W_2[L]}^{W_3[L]}$. In particular,

$$N_{W_1[L]W_2[L]}^{W_3[L]} = \sum_{\alpha \in L} N_{W_1W_2}^{W_3^{(\alpha)}}.$$

**Proof.** Since $W_3^{(\alpha)} \simeq W_3^{(\beta)}$ only if $\alpha = \beta$, clearly, $f_{\alpha}$ is one-to-one. On the other hand, if $\mathcal{Y}$ is an intertwining operator of type $(W_3^{(\alpha)}_{W_1W_2}[L])$, then by restricting $\mathcal{Y}$ to $W_1 \times W_2$ we have an intertwining operator $I$ of type $(W_3^{(\alpha)}_{W_1W_2})$ for a unique $\alpha \in L$. It is clear that $\mathcal{Y} = I[L]$. This completes the proof.

We now describe the Verlinde algebra of $V[L]$ in terms of the Verlinde algebra of $V$ explicitly. Let $\mathcal{A}(V)$ be the Verlinde algebra of $V$. The Verlinde algebra $\mathcal{A}(V[L])$ with a basis $[W[L]]$ for $[W] \in \mathcal{A}(V)$ with $\sigma_W = 1$. First, we have:

**Lemma 3.27** All $[W]$ with $\sigma_W = 1$ linearly span a subalgebra $A(V, L)$ of $\mathcal{A}(V)$.

**Proof.** Suppose that $[W_1], [W_2] \in \text{Irr}(V)$ with $\sigma_{W_1} = \sigma_{W_2} = 1$. Let $h_1 \in P(W_1), h_2 \in P(W_2)$. Then $\sigma_{W_1} = \sigma_{W_2} = 1$ amount to $h_1, h_2 \in \mathcal{O}$. Let $\mathcal{Y}$ be a nonzero intertwining operator of type $(W_3_{W_1W_2})$ for some irreducible $V$-module $W_3$. Then $h_1 + h_2 \in P(W_3)$ because for $h \in H, w(i) \in W^{(0,h)}_i$,

$$h(0)\mathcal{Y}(w(1), z)w(2) = \mathcal{Y}(h(0)w(1), z)w(2) + \mathcal{Y}(w(1), z)h(0)w(2) = B(h, h_1 + h_2)\mathcal{Y}(w(1), z)w(2).$$

Since $h_1 + h_2 \in \mathcal{O}$, $\sigma_{W_3} = 1$. Then $[W_3] \in A(V, L)$. The proof is complete.
Define a subspace \( R \) of \( A(V, L) \) linearly spanned by
\[
[W] - [W^{(\alpha)}] \quad \text{for } \alpha \in L.
\] (3.72)

Then \( R \) is an two-sided ideal of \( A(V, L) \). Indeed, let \( W_1 \) and \( W_2 \) be irreducible \( V \)-modules with \( \sigma_{W_i} = 1 \) for \( i = 1, 2 \). For \( \alpha \in L \), by Proposition 2.10,
\[
I_{W_1 W}^{W_2} \cong I_{W_1 W^{(\alpha)}}^{W_2^{(\alpha)}}.
\] (3.73)

Thus
\[
[W_1] \cdot ([W] - [W^{(\alpha)}]) = \sum_{[W_2] \in \text{Irr}(V)} N_{W_1 W}^{W_2} ([W_2] - [W_2^{(\alpha)}]) \in R.
\] (3.74)

Since \( A(V) \) is a commutative algebra, \( R \) is a two-sided ideal. Furthermore, by Proposition 3.13, \([W_1] = [W_2]\) in the quotient algebra \( A(V, L)/R \) if and only if \([W_1[L]] = [W_2[L]]\) in \( A(V[L]) \). Then in view of Theorem 3.26 we immediately have:

**Proposition 3.28** The subspace \( R \) is a two-sided ideal of \( A[V, L] \) and the Verlinde algebra \( A(V[L]) \) is canonically isomorphic to the quotient algebra of \( A(V, L) \) modulo \( R \). \( \square \)

### 4 Extended vertex operator (super)algebras of affine types

In this section we shall specialize the vertex (super)algebra \( V[L] \) constructed in Section 3 from a pair \((V, L)\) to obtain extensions of vertex operator algebras associated with affine Lie algebras \( \hat{g} \). In the case \( g = sl(2) \), we shall obtain Feigin-Miwa’s extended vertex operator (super)algebras \( A_k \) \([FM]\). To apply the results of Section 3 we need to define \( V \) and \( L \) explicitly and check the necessary conditions. For each type, \( V \) will be the tensor product vertex operator algebra \( L_{\hat{g}}(k, 0) \otimes M_{h'}(1, 0) \) where \( h' \) is a 1 or 2-dimensional vector space equipped with a nondegenerate symmetric bilinear form. When defining \( h' \), we follow two basic principles: (1) To include all the \( L(k, 0) \)-simple currents in the construction of \( V[L] \). (2) To make \( \dim h' \), or equivalently, to make the rank of \( V[L] \) as small as possible. After \( h' \) is chosen, we still have plenty of choices for the bilinear form on \( h' \). Another principle to follow is to make \( V[L] \) as large as possible. We shall define \( h' \) and \( L \) type by type.

#### 4.1 A complete reducibility theorem for a certain family of \( V[L] \)

In Section 3, for a general pair \((V, L)\), under a certain assumption we proved a complete reducibility theorem (Theorem 3.25) for extended algebra \( V[L] \). In this section, we shall
consider a family of $V[L]$ such that the assumption of Theorem 3.25 holds. All the extended algebras we shall construct later belong to this family, so the complete reducibility theorem holds for all of them.

Let $g = g_1 \oplus \cdots \oplus g_r$ be a semisimple Lie algebra with a Cartan subalgebra

$$h = h_1 + \cdots + h_r,$$

(4.1)

where $g_i$ are simple factors with Cartan subalgebras $h_i$, equipped with the normalized Killing forms. Let $Q^\vee = Q_1^\vee + \cdots + Q_r^\vee$ and $P^\vee = P_1^\vee + \cdots + P_r^\vee$ be the coroot lattice and coweight lattice of $g$, respectively, where $Q_i^\vee$ and $P_i^\vee$ are the coroot lattice and coweight lattice of $g_i$.

Let $k = (k_1, \ldots, k_r)$ be an $r$-tuple of nonnegative integers. Set

$$L_g(k, 0) = L_{g_1}(k_1, 0) \otimes \cdots \otimes L_{g_r}(k_r, 0),$$

(4.2)

equipped with the standard tensor product vertex operator algebra structure. Set

$$P_k = \{ (\lambda^1, \ldots, \lambda^r) | \lambda^i \in P_{k_i}(g_i) \},$$

(4.3)

where $P_{k_i}(g_i)$ stands for $P_{k_i}$ for the Lie algebra $g_i$. Then $L_g(k, 0)$ is regular [DLM1], i.e., every weak module is a direct sum of irreducible (ordinary) modules $L_g(k, \lambda)$, where

$$L_g(k, \lambda) = L_{g_1}(k_1, \lambda^1) \otimes \cdots \otimes L_{g_r}(k_r, \lambda^r)$$

(4.4)

every weak module is a direct sum of irreducible (ordinary) modules $L_g(k, \lambda)$, where

Let $h'$ be a finite-dimensional vector space equipped with a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$. Associated to $h'$ is the vertex operator algebra $M_{h'}(1, 0)$. Set

$$V = L_g(k, 0) \otimes M_{h'}(1, 0),$$

(4.5)

equipped with the standard tensor product vertex operator algebra structure. The algebra $V$ can be considered as the vertex operator algebra associated to the affine algebra of the reductive Lie algebra $g + h'$.

Set

$$H = h + h' \subset g + h' = V(1).$$

(4.6)

Then $H$ is a Cartan subalgebra of the reductive Lie algebra $g + h'$. Clearly, (3.1) holds and

$$B(h_1, h_2) = \delta_{i,j} k_i \langle h_1, h_2 \rangle \quad \text{for } h_1 \in h_i, \ h_2 \in h_j$$

(4.7)

because $h_1(1)h_2 = h_1(1)h_2(-1)1 = \delta_{i,j} k_i \langle h_1, h_2 \rangle 1$ (and $h_1(1)h_2 = B(h_1, h_2)1$). We also have

$$B(h'_1, h'_2) = \langle h'_1, h'_2 \rangle \quad \text{for } h'_1, h'_2 \in h'.$$

(4.8)
For \( h \in H \), we write
\[
h = h^1 + \cdots + h^r + h', \quad h = h'' + h',
\]
where \( h^i \in h_i, \ h' \in h', \) and \( h'' \in h \).

For \( \lambda = (\lambda^1, \ldots, \lambda^r), \) \( \gamma \in h' \) with \( \lambda^i \in P_{k_i}(g_i) \), we set
\[
W(\lambda, \gamma) = L_{\theta_1}(k_1, \lambda^1) \otimes \cdots \otimes L_{\theta_r}(k_r, \lambda^r) \otimes M_{h'}(1, \gamma).
\]
(4.10)

Then from [FHL] all such \( W(\lambda, \gamma) \) form a complete set of representatives of equivalence classes of irreducible \( V \)-modules.

Let \( L \) be a subgroup of \( H \) such that \( (L, B) \) is an integral lattice of finite rank. To have a vertex (super)algebra \( V[L] \), we shall also need the condition that \( \alpha(0) \) has only integral eigenvalues on \( V \) for every \( \alpha \in L \). For \( \alpha \in L \), since \( \alpha'(0) \) acts as zero on \( V \), \( \alpha(0) \) has only integral eigenvalues on \( V \) if and only if \( \alpha''(0) \) has only integral eigenvalues on \( L_{\theta}(k, 0) \).

Then we immediately have:

**Lemma 4.1** For \( \alpha \in L \), \( \alpha(0) \) has only integral eigenvalues on \( V \) if and only if \( \alpha'' \in P^\vee \), the coweight lattice of \( g \). □

Set
\[
L' = \{ \alpha' | \alpha \in L \}, \quad L'' = \{ \alpha'' | \alpha \in L \}.
\]
(4.11)

**Lemma 4.2** Assume that the projection of \( L \) onto \( L' \) is one-to-one. Then for \( \lambda \in P_k, \) \( \gamma \in h' \) and for \( \alpha, \beta \in L \), \( W(\lambda, \gamma)^{(\alpha)} \simeq W(\lambda, \gamma)^{(\beta)} \) if and only if \( \alpha = \beta \).

**Proof.** Because \( \Delta(\alpha, z) = \Delta(\alpha'', z)\Delta(\alpha', z) \) and \( \Delta(\alpha'', z) = 1 \) on \( M_{h'}(1, \gamma) \) and \( \Delta(\alpha', z) = 1 \) on \( L_{\theta}(k, \lambda) \) for \( \alpha \in L \), we have
\[
W(\lambda, \gamma)^{(\alpha)} = L_{\theta}(k, \lambda)^{(\alpha'')} \otimes M_{h'}(1, \gamma)^{(\alpha')} \simeq L_{\theta}(k, \lambda)^{(\alpha'')} \otimes M_{h'}(1, \gamma + \alpha').
\]
(4.12)

We knew \( M_{h'}(1, \gamma + \alpha') \simeq M_{h'}(1, \gamma + \beta') \) if and only if \( \alpha' = \beta' \). Then it follows immediately. □

Now we have:

**Proposition 4.3** Let \( V, H \) be defined as in (4.5) and (4.6) and let \( L \) be a subgroup of \( H \) such that \( (L, B) \) is an integral lattice of finite rank. Assume that \( L'' \subset P^\vee \), the projection of \( L \) onto \( L' \) is one-to-one and \( L' \) is a positive definite lattice. Then \( V[L] \) equipped with the vertex operator map \( Y \) defined in (3.13) is a simple vertex operator (super)algebra.
In particular, every weak $V$-irreducible (ordinary) $\sigma$-non-isomorphic irreducible $V$-modules isomorphic to $\alpha$ exists a positive integer $d$ the lowest weight at least $\dim \tau$ grading restrictions follows immediately.

Theorem 4.4 Let $V,L$ be as in Proposition 4.3 with all the assumptions. In addition we assume that $\dim h' = \text{rank } L'$. Let $\sigma$ be an automorphism of $V[L]$ of finite order which fixes $V$ point-wise. Then every $\sigma$-twisted weak $V[L]$-module is a direct sum of irreducible (ordinary) $\sigma$-twisted $V[L]$-modules isomorphic to $W(\lambda,\gamma)[L]$ with $\sigma = \sigma_{W(\lambda,\gamma)}$. In particular, every weak $V[L]$-module is a direct sum of irreducible (ordinary) $V[L]$-modules isomorphic to $W(\lambda,\gamma)[L]$ for $\lambda \in P_k$, $\gamma \in (L')^o$ (the dual of $L'$) with

$$\lambda^1(\alpha^1) + \cdots + \lambda^r(\alpha^r) + \langle \gamma, \alpha' \rangle \in \mathbb{Z} \quad \text{for } \alpha \in L.$$ (4.14)

Proof. Denote by $o(\sigma)$ the order of $\sigma$. In view of Lemma 4.2, $V^{(\alpha)}$ for $\alpha \in L$ are non-isomorphic irreducible $V$-modules. Then $\sigma(V^{(\alpha)}) = V^{(\alpha)}$ for $\alpha \in L$ and $\sigma$ acts on $V^{(\alpha)}$ as a scalar, which is an $o(\sigma)$-th root of unity. Therefore, $\sigma$ acts trivially on $V^{(m \circ (\sigma))}$ for $\alpha \in L$, $m \in \mathbb{Z}$.

Because $L_{\theta}(k,0)$ has only finitely many irreducible modules up to equivalence, there exists a positive integer $d_1$ such that as $L_{\theta}(k,0)$-modules,

$$L_{\theta}(k,0)^{(d_1 \alpha''')} \simeq L_{\theta}(k,0) \quad \text{for all } \alpha \in L.$$ (4.15)

Let $d_2$ be another positive integer such that $d_2 L'$ is an even lattice. Set $d = o(\sigma)d_1d_2$ and $L_1 = dL$. Then

$$V[L_1] = \oplus_{\alpha \in L} V^{(d \alpha)}.$$ (4.16)

Then $V[L_1]$ is a simple vertex operator subalgebra of $V[L]$ and $\sigma$ fixes $V[L_1]$ point-wise. Furthermore, as a $V$-module,

$$V^{(d \alpha)} \simeq L_{\theta}(k,0)^{(d \alpha''')} \otimes M_{h'}(1,d \alpha') \simeq L_{\theta}(k,0) \otimes M_{h'}(1,d \alpha')$$ (4.17)
for $\alpha \in L$, hence
\[ V[L_1] \simeq L_\theta(k, 0) \otimes V_{dL'} \tag{4.18} \]
(Here we used the fact that $\dim h' = \text{rank } L'$.) Note that $L_\theta(k, 0) \otimes V_{dL'}$ is a natural simple vertex operator algebra, which is regular. It follows from Proposition 3.22 that $V[L_1]$ is regular. Clearly, each irreducible $V[L_1]$-module is a direct sum of irreducible $V$-modules. Then it follows from Theorem 3.25 immediately that every $\sigma$-twisted weak $V[L]$-module is a direct sum of irreducible (ordinary) $\sigma$-twisted $V[L]$-modules of type $W(\lambda, \gamma)$ with $\sigma = \sigma_{W(\lambda, \gamma)}$.

From Lemma 3.11, $\sigma_{W(\lambda, \gamma)} = 1$ if and only if for $\alpha \in L$, $\alpha(0)$ has only integral eigenvalues on $W(\lambda, \gamma)$. With $(\lambda^1, \ldots, \lambda^r, \gamma)$ being an $H$-weight of $W(\lambda, \gamma)$, we see that $\sigma_{W(\lambda, \gamma)} = 1$ if and only if
\[ \sum_{i=1}^{r} \lambda_i(\alpha_i') + \langle \gamma, \alpha' \rangle \in \mathbb{Z} \quad \text{for } \alpha \in L, \tag{4.19} \]
which furthermore implies that $\gamma \in (L')^o$ because $\lambda \in P_k$. This completes the proof. $\square$

**Remark 4.5** From the proof of Theorem 4.4, one can easily see that the regularity result still holds for $V[L]$ if we replace $L_\theta(k, 0)$ by any regular vertex operator algebra $U$.

Because
\[ W(\lambda, \gamma)^{(\alpha)} = L(k, \lambda)^{(\alpha')} \otimes M_{h'}(1, \gamma + \alpha') \]
and $M_{h'}(1, \gamma) \simeq M_{h'}(1, \gamma')$ if and only if $\gamma = \gamma'$, in view of Proposition 3.15, we see that $W(\lambda, \gamma)[L] \simeq W(\lambda', \gamma')[L]$ if and only if there is $\alpha \in L$ such that
\[ \gamma' = \gamma + \alpha', \quad L(k, \lambda')^{(\alpha'')} \simeq L(k, \lambda). \tag{4.20} \]

To describe explicitly the equivalence relation on the set of $W(\lambda, \gamma)[L]$, or to get a complete set of equivalence classes of irreducible $V[L]$-modules, we need to know $L(k, \lambda')^{(\alpha'')}$ as a $\hat{g}$-module. Of course, from Theorem 2.13,
\[ [L(k, \lambda')^{(\alpha'')}] = [L(k, \lambda')] \cdot [L(k, 0)^{(\alpha''})]. \tag{4.21} \]

Nevertheless, in view of Proposition 3.28 we immediately have:

**Proposition 4.6** The subspace $A$ of $V(L(k, 0)) \otimes \mathbb{C}[(L')^o]$, linearly spanned by
\[ [L(k, \lambda)] \otimes e^\gamma \tag{4.22} \]
for $\lambda \in P_k$, $\gamma \in (L')^o$ satisfying (4.14), is a subalgebra. Furthermore, the Verlinde algebra $V(V[L])$ is canonically isomorphic to the quotient algebra of $A$ modulo the relations:
\[ [L(k, \lambda)] \otimes e^\gamma - [L(k, \lambda)^{(\alpha'')} \otimes e^{\gamma + \alpha'} \tag{4.23} \]
for $\alpha \in L$. $\square$
4.2 Extended vertex operator (super)algebras $A_k$ of type $sl(n+1)$

Starting from this subsection we shall work on the setting of Section 4.1 and we shall consider a simple Lie algebra $g$, i.e., $r = 1$. For $g$ of each type, we take

$$V = L(k, 0) \otimes M_W(1, 0)$$

(4.24)

and we define $A_k(g)$ to be the extended algebra $V[L]$ for a certain $L$. We shall case by case define the pair $(h', \langle \cdot, \cdot \rangle)$ and the lattice $L$, and then verify that $(L, B)$ is an integral lattice, $L'' \subset P^\vee$, $L'$ is positive-definite and the projection of $L$ onto $L'$ is one-to-one, so that Proposition 4.3 and Theorem 4.4 hold.

In this subsection we shall consider the case $g = sl(n+1)$. For a fixed positive integer $k$, $L(k, 0)$ has $n+1$ simple currents $L(k, k\lambda_i)$ for $i = 0, \ldots, n$. By Corollary 2.27 the equivalence classes of the $(n+1)$ simple currents form a cyclic group of order $(n+1)$ with $[L(k, k\lambda_1)]$ as a generator.

Recall that $h^{(i)} \in h$ with $\alpha_j(h^{(i)}) = \delta_i, 1$ for $i, j = 1, \ldots, n$. From [H] (Table 1 on page 69) and Lemma 2.28, we have

$$h^{(i)} = \frac{1}{n+1} \left( (n+1-i)\alpha_1^\vee + 2(n+1-i)\alpha_2^\vee + \cdots + (i-1)(n+1-i)\alpha_{i-1}^\vee \right)$$

$$+ \frac{1}{n+1} \left( i(n+1-i)\alpha_i^\vee + i(n-i)\alpha_{i+1}^\vee + \cdots + i\alpha_n^\vee \right),$$

(4.25)

$$\langle h^{(i)}, h^{(i)} \rangle = \frac{i(n+1-i)}{n+1}. $$

(4.26)

Define $h' = \mathbb{C}\alpha'$ to be a one-dimensional vector space equipped with a symmetric bilinear form $\langle \cdot, \cdot \rangle$ such that

$$\langle \alpha', \alpha' \rangle = \frac{k}{n+1}. $$

(4.27)

Set

$$L = \mathbb{Z}\alpha, \quad \text{where} \quad \alpha = h^{(1)} + \alpha'.$$

(4.28)

Because

$$B(\alpha, \alpha) = B(h^{(1)}, h^{(1)}) + B(\alpha', \alpha') = \frac{kn}{n+1} + \frac{k}{n+1} = k \in \mathbb{Z},$$

(4.29)

$(L, B)$ is an integral lattice. Clearly, $L'' = \mathbb{Z}h^{(1)} \subset P^\vee$, $L'$ is positive-definite and the projection of $L$ onto $L'$ is one-to-one. By Proposition 4.3, we have a simple vertex operator (super)algebra $V[L]$.

**Definition 4.7** We define $A_k(sl(n+1))$ to be the simple vertex operator (super)algebra $V[L]$ with $V$ and $L$ being defined in (4.24) and (4.28).
Remark 4.8 There are many ways to define \( \langle \alpha', \alpha' \rangle \) such that \( V[L] \) is a vertex operator superalgebra. For examples, one may define \( h' \) with \( \langle \alpha', \alpha' \rangle = 1 - \frac{nk}{n+1} \), where \( nk \) is the least nonnegative residue of \( nk \) modulo \( n + 1 \). One may also define \( h' \) with \( \langle \alpha', \alpha' \rangle = s + \frac{1}{n+1} \), where \( s \) is any nonnegative integer.

For \( \lambda \in P_k \), \( \gamma \in \mathbb{C} \), set
\[
W(\lambda, \gamma) = L(k, \lambda) \otimes M_{h'}\left(1, \frac{\gamma}{k}\alpha'\right).
\]
Since \( L = \mathbb{Z}(k(1) + \alpha') \), (4.14) amounts to
\[
\lambda(\alpha(1)) + \frac{\gamma}{n+1} \in \mathbb{Z},
\]
which from (4.25) is equivalent to
\[
n\lambda(\alpha_1') + (n-1)\lambda(\alpha_2') + \cdots + \lambda(\alpha_n') + \gamma \in (n+1)\mathbb{Z}.
\]
Note that (4.32) implies \( \gamma \in \mathbb{Z} \) because \( \lambda \in P_k \). We also see that in general, \( \sigma_{W(\lambda, \gamma)} \) is of finite order if and only if \( \gamma \in \mathbb{Q} \). Then Propositions 3.13 and 3.24 immediately give:

**Proposition 4.9** For \( \lambda \in P_k \), \( \gamma \in \mathbb{Q} \), \( \sigma_{W(\lambda, \gamma)} \) is of finite order and \( W(\lambda, \gamma)[L] \) is an irreducible \( \sigma_{W(\lambda, \gamma)} \)-twisted \( A_k(sl(n+1)) \)-module. In particular, if (4.32) holds, \( W(\lambda, \gamma)[L] \) is an irreducible \( V[L] \)-module. \( \square \)

From Theorem 4.4 we also have:

**Proposition 4.10** Let \( \sigma \) be an automorphism of \( A_k(sl(n+1)) \) of finite order which fixes \( V = L(k,0) \otimes M_{h'}(1,0) \) point-wise. Then every \( \sigma \)-twisted weak \( A_k(sl(n+1)) \)-module is a direct sum of irreducible (ordinary) \( \sigma \)-twisted \( A_k(sl(n+1)) \)-modules \( W(\lambda, \gamma)[L] \) for some \( \lambda \in P_k \), \( \gamma \in \mathbb{Q} \) with \( \sigma = \sigma_{W(\lambda, \gamma)} \). In particular, every weak \( A_k(sl(n+1)) \)-module is a direct sum of \( W(\lambda, \gamma)[L] \) for some \( \lambda \in P_k \), \( \gamma \in \mathbb{Z} \) that satisfy (4.32). \( \square \)

Let us consider the case \( n = 1 \). Then we can make our results more explicit. We have \( P_k = \{0, 1, \ldots, k\} \) and \( h^{(1)} = \frac{1}{2}\alpha_1' \). From Corollary 2.27 we have
\[
L(k,0)^{(2mh^{(1)})} \simeq L(k,0), \quad L(k,0)^{(2m+1)h^{(1)}} \simeq L(k,k)
\]
for \( m \in \mathbb{Z} \). Since
\[
V^{(ma)} = L(k,0)^{(mh^{(1)})} \otimes M_{h'}(1,0)^{(ma')} = L(k,0)^{(mh^{(1)})} \otimes M_{h'}(1, ma'),
\]
it follows from Proposition 3.13 that \( W(i, \gamma)[L] \simeq W(i', \gamma')[L] \) if and only if there exists \( m \in \mathbb{Z} \) such that
\[
L(k,i') \simeq L(k,i)^{(mh^{(1)})}, \quad \frac{\gamma'}{k} = \frac{\gamma}{k} + m.
\]
Recall that \([L(k,k)]: [L(k,i)] = [L(k,k-i)]\). If \(m\) is even, we have \(i' = i\) and \(\gamma' = i' = \gamma + mk\). Then \(W(i,\gamma)[L] = W(i',\gamma')[L]\) if and only if either \(i' = i\) and \(\gamma' \equiv \gamma \mod 2k\), or \(i' = k - i\) and \(\gamma' \equiv \gamma + k \mod 2k\). Then Propositions 4.9 and 4.10 give (cf. [FM], Proposition 3):

**Corollary 4.11** Every weak \(A_k(sl(2))\)-module is completely reducible and

\[
W(i, j)[L] \quad \text{for } 0 \leq i \leq k, \ 0 \leq j \leq k-1 \quad \text{with } i + j \in 2\mathbb{Z}
\]

form a complete set of representatives of equivalence classes of irreducible \(A_k(sl(2))\)-modules. \(\square\)

Note that \(W(i, j)[L]\) was denoted by \(R(i, j)\) in [FM]. Using the fusion rules for \(L(k,0)\) we have the following relations in the Verlinde algebra \(A(V)\):

\[
[W(i_1, j_1)] \cdot [W(i_2, j_2)] = \sum_{i = \min(i_1 - i_2, i_2 - i_1)} \max(i_2 - i_1) [W(i, j_1 + j_2)].
\]

Then in the Verlinde algebra of \(A_k\), we have

\[
[W(i_1, j_1)[L]] \cdot [W(i_2, j_2)[L]] = \sum_{i = \max(i_1 - i_2, i_2 - i_1)} \min(i_1 + i_2, 2k - i_1 - i_2) [W(i, j_1 + j_2)[L]].
\]

Note that when \(j_1 + j_2 \geq k\), we have \(W(i, j_1 + j_2)[L] = W(k - i, j_1 + j_2 - k)[L]\).

**Remark 4.12** Set \(L' = \mathbb{Z} \alpha'\). Then, as a \(V\)-module,

\[
A_k \simeq L(k, 0) \otimes V_{2L'} + L(k, k) \otimes V_{2L' + \alpha'}. \tag{4.39}
\]

Furthermore, using more general fusion rules we get

\[
W(i, \gamma)[L] \simeq L(k, i) \otimes V_{2L' + \frac{\gamma}{2} \alpha'} + L(k, k - i) \otimes V_{2L' + \frac{\gamma + k}{2} \alpha'} \tag{4.40}
\]

for \(i = 0, \ldots, k; \ \gamma \in \mathbb{Q}\). With this, one can easily write down the characters of \(W(i, j)[L]\) in terms of the characters of \(L(k, j)\) and the theta functions of \(2L'\).

**Remark 4.13** For \(g = sl(n + 1)\), from Corollary 2.27 we have

\[
L(k, 0)_{(m)} \simeq L(k, k\lambda_m) \quad \text{for } m \in \mathbb{Z}. \tag{4.41}
\]

Then

\[
A_k(sl(n + 1)) \simeq \prod_{i=0}^n L(k, k\lambda_i) \otimes V_{2(n+1)L' + \alpha'}, \tag{4.42}
\]

as a \(V\)-module. More general fusion rules are needed to express \(W(\lambda, j)[L]\) explicitly.
4.3 Generating property for the extended algebras $A_k(sl(n+1))$

First, we review some properties for a general vertex operator superalgebra $U$. Recall Borcherds’ commutator formula [B]:

$$[u_m, v_n]_\pm = \sum_{i \geq 0} \binom{m}{i} (u_iv)_{m+n-i}$$

(4.43)

for $u, v \in U$ and $m, n \in \mathbb{Z}$, where $[\cdot, \cdot]_\pm$ refers to the super commutator. Thus, the super commutator $[Y(u, z_1), Y(v, z_2)]_\pm$ is uniquely determined by $u_i v$ for $i \geq 0$. From this we have

$$(z_1 - z_2)^r [Y(u, z_1), Y(v, z_2)]_\pm = 0$$

(4.44)

if $r$ is a nonnegative integer such that $u_i v = 0$ for $i \geq r$. For homogeneous vectors $u, v \in U$ and for $m \in \mathbb{Z}$, we have (cf. [FLM])

$$\text{wt } (u_m v) = \text{wt } u + \text{wt } v - m - 1,$$

(4.45)

where $\text{wt } u$ stands for the $L(0)$-weight of $u$.

Let $U = \coprod_{n \in \frac{1}{2} \mathbb{Z}} U(n)$ be such that $U(0) = \mathbb{C} (= \mathbb{C} 1)$ and $U_n = 0$ for $n < 0$. Then

$$[u_m, v_n]_+ = (u_0 v)_{m+n-1} = \delta_{m+n,1} u_0 v$$

(4.46)

for $u, v \in U_{(\frac{1}{2})}$, $m, n \in \mathbb{Z}$, where $u_0 v \in U_{(0)} = \mathbb{C}$. That is, the component operators $u_m$ for $u \in U_{(\frac{1}{2})}$, $m \in \mathbb{Z}$ give rise to a Clifford algebra.

It is well known ([B], [FLM]) that the weight-one subspace $U_{(1)}$ is a Lie algebra with $[u, v] = u_0 v$ and with a symmetric invariant bilinear form $(u, v) = u_1 v \in \mathbb{C}$. We have

$$[u_m, v_n] = (u_0 v)_{m+n} + m \delta_{m+n,0} (u, v)$$

(4.47)

for $u, v \in U_{(1)}$, $m, n \in \mathbb{Z}$. Then operators $u_m$ for $u \in U_{(1)}$, $m \in \mathbb{Z}$ give rise to a natural representation of affine Lie algebra $\hat{U}_{(1)}$.

Now we consider $A_k(sl(n+1))$, which is a vertex operator algebra when $k$ is even and which is a vertex operator superalgebra when $k$ is odd. It is easy to see that vertex operator (super)algebra $A_k(sl(n+1))$ is generated by $V^{(\alpha)}$ and $V^{(-\alpha)}$. Denote by $V_{\text{low}}^{(\beta)}$ the lowest $L(0)$-weight subspace of $V^{(\beta)}$ for $\beta \in L$. Because $V$ as a vertex operator algebra is generated by $\mathfrak{g} + \mathbb{C}\alpha'$ and both $V^{(\alpha)}$ and $V^{(-\alpha)}$ are irreducible $V$-modules, $A_k(sl(n+1))$ is furthermore generated by

$$S := (\mathfrak{g} + \mathbb{C}\alpha') + V_{\text{low}}^{(\alpha)} + V_{\text{low}}^{(-\alpha)}.$$  

(4.48)

Since (Corollary 2.27)

$$V^{(\alpha)} = L(k, k\lambda_1) \otimes M_{\text{h}}^{(1, \alpha')}, \quad V^{(-\alpha)} = L(k, k\lambda_n) \otimes M_{\text{h}}^{(-1, -\alpha')}$$

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we have
\[ V_{\text{low}}^{(0)} = L(k\lambda_1) \otimes e^{\alpha'}, \quad V_{\text{low}}^{(-0)} = L(k\lambda_n) \otimes e^{-\alpha'}. \] (4.49)

From Remark 2.23 and (2.7), we find that the lowest \( L(0) \)-weights of \( V \)-modules \( V^{(\alpha)} \) and \( V^{(-\alpha)} \) are \( \frac{1}{2}B(\alpha, \alpha) = \frac{k}{2} \). Now we are ready to prove our main result of this subsection.

**Proposition 4.14** The algebra \( A_k(sl(n+1)) \) is generated by the subspace
\[ (\mathfrak{g} + \mathbb{C}\alpha') + V_{\text{low}}^{(\alpha)} + V_{\text{low}}^{(-\alpha)}, \]
where \( V_{\text{low}}^{(\alpha)} = L(k\lambda_1) \otimes e^{\alpha'} \) and \( V_{\text{low}}^{(-\alpha)} = L(k\lambda_n) \otimes e^{-\alpha'} \) are of weight \( \frac{k}{2} \). Furthermore, the following relations hold:
\[
Y(u, z_1)Y(v, z_2) = (-1)^k Y(v, z_2)Y(u, z_1), \tag{4.50}
\]
\[
Y(u', z_1)Y(v', z_2) = (-1)^k Y(v', z_2)Y(u', z_1) \tag{4.51}
\]
for \( u, v \in V_{\text{low}}^{(\alpha)}, \ u', v' \in V_{\text{low}}^{(-\alpha)} \) and
\[
u_s v' \in V_{k-s-1}, \tag{4.52}
\]
\[
(z_1 - z_2)k[Y(u, z_1), Y(v', z_2)]_\pm = 0 \tag{4.53}
\]
for \( u \in V_{\text{low}}^{(\alpha)}, \ v' \in V_{\text{low}}^{(-\alpha)}, \ s \in \mathbb{Z} \).

**Proof.** First we calculate the lowest \( L(0) \)-weight of \( V^{(\alpha)} \). From Theorem 2.26 and Corollary 2.27 we have
\[
V^{(\alpha)} = L(k, 0)^{(m\bar{m})} \otimes M_{\bar{m}'}(1, m\alpha') = L(k, k\lambda_{\bar{m}}) \otimes M_{\bar{m}'}(1, m\alpha'),
\]
where \( \bar{m} \) is the least nonnegative residue of \( m \) modulo \( n + 1 \). From Remark 2.23, we see that the lowest weight of \( L(k, k\lambda_{\bar{m}}) \) is \( \bar{m}(n+1-\bar{m})k \). Then the lowest weight of \( V^{(\alpha)} \) is
\[
\frac{\bar{m}(n+1-\bar{m})k}{2(n+1)} + \frac{m^2k}{2(n+1)} = \frac{\bar{m}k}{2} + \frac{(m^2-\bar{m}^2)k}{2(n+1)},
\]
which is at least \( k \) if \( |m| \geq 2 \).

Let \( u, v \in V_{\text{low}}^{(\alpha)} \). Thus \( wt u = wt v = \frac{k}{2} \). Then for \( i \geq 0, u_i v \in V^{(2\alpha)} \) and \( wt (u_i v) = k - i - 1 < k \). Since the lowest weight of \( V^{(2\beta)} \) is at least \( k \), we obtain
\[
u_i v = 0 \quad \text{for } i \geq 0. \tag{4.54}
\]
Then (4.50) follows immediately from (4.43). Similarly, (4.51) holds.

(4.52) directly follows from the definition of the vertex operator map and the weight formula (4.45). Since \( u_s v' \in V^{(0)} = V \) for \( s \in \mathbb{Z} \) and the lowest weight of \( V \) is zero, we have
\[
u_i v' = 0 \quad \text{for } i \geq k. \tag{4.55}
\]
Then (4.53) follows immediately from (4.44). \( \Box \)
Remark 4.15 In the case $k = 1$, $L(\lambda_1)$ is the vector representation of $sl(n+1)$ on $\mathbb{C}^{n+1}$. In this case, the algebra $A_1(sl(n+1))$ is generated by $L(\lambda_1) \otimes e^{\alpha'} + L(\lambda_1)^* \otimes e^{-\alpha'}$, which generates a Clifford algebra. The algebra $A_1(sl(n+1))$ is exactly the spinor representation of $D_{n+1}$ [FFR], which is isomorphic to $L(1,0) + L(1,\lambda_1)$ as a $\hat{D}_{n+1}$-module.

Remark 4.16 When $k = 2$,
\[ L(2\lambda_1)^* \otimes e^{-\alpha'} + (g + \mathbb{C}\alpha') + L(2\lambda_1) \otimes e^{\alpha'} \]
is exactly the weight-one subspace of $A_2(sl(n+1))$ and it has a natural Lie algebra structure with the obvious $\mathbb{Z}$-grading. Using the fact that $L(2\lambda_1)^* \otimes e^{-\alpha'}$ and $L(2\lambda_1) \otimes e^{\alpha'}$ are non-isomorphic irreducible $(g + \mathbb{C}\alpha')$-modules one easily shows that this Lie algebra is simple and of rank $n + 1$. Consider the standard Dynkin diagram embedding of $sl(n+1)$ into $C_{n+1}$. Then we see
\[ C_{n+1} = sl(n+1) + L(2\lambda_1) + L(2\lambda_n). \]
Thus the weight one subspace of $A_2(sl(n+1))$ as a Lie algebra is isomorphic to $C_{n+1}$. (For $n = 1$, this was pointed out in [FM].) Then $A_2(sl(n+1))$ is a vertex operator algebra associated to the affine Lie algebra $\hat{C}_{n+1}$.

Remark 4.17 For $k \geq 3$, since $\text{wt } (u_0v') = k - 1 \geq 2$, $[Y(u, z_1), Y(v', z_2)]_\pm$ involves nonlinear normal ordered products of vertex operators (or fields) $Y(a, z)$ for $a \in g + \mathbb{C}\alpha'$. This type of algebras are commonly referred by physicists as nonlinear $W$-algebras.

Remark 4.18 The following consideration was motivated by [GH1-2] and [Gun]. In the construction of $A_k$, let us define $h' = \mathbb{C}\alpha'$ with $\langle \alpha', \alpha' \rangle = 1 + \frac{k}{n+1}$ and keep the rest unchanged. Then $B(\alpha, \alpha) = 1 + k$. With $(L, B)$ being an integral lattice, $V[L]$ is a vertex operator (super)algebra (cf. Remark 4.8). Furthermore, $V[L]$ is generated by the subspace
\[ V_{\text{low}}^{(-\alpha')} + (g + \mathbb{C}\alpha') + V_{\text{low}}^{(\alpha')} = L(k\lambda_1)^* \otimes e^{-\alpha'} + (g + \mathbb{C}\alpha') + L(k\lambda_1) \otimes e^{\alpha'}, \]
where $L(k\lambda_1) \otimes e^{\alpha'}$ and $L(k\lambda_1)^* \otimes e^{-\alpha'}$ are of weight $\frac{k+1}{2}$. In particular, when $k = 2$, $V[L]$ is a vertex operator superalgebra and $L(2\lambda_1) \otimes e^{\alpha'}$ and $L(2\lambda_1)^* \otimes e^{-\alpha'}$ are of weight $\frac{3}{2}$. In view of this and Remark 4.14, we may view $V[L]$ with $k = 2$ as a superization of the vertex operator algebra $V[L]$ with $k = 2$ defined in Remark 4.14. In [Gun], an $N = 2$ vertex operator superalgebra was constructed from a simple Lie algebra $\gamma$ equipped with a $\mathbb{Z}$-grading such that $\mathfrak{g}_m = 0$ for $|m| > 1$. From Remark 4.14, symplectic Lie algebra $C = C_{n+1}$ is naturally $\mathbb{Z}$-graded with only three homogeneous subspaces being nonzero. A further study on the connection between $V[L]$ and the $N = 2$ vertex operator superalgebra constructed in [Gun] will be conducted in a future paper.
4.4 Extended algebras $A_k$ of type $D_n$

We consider $g$ of $D_n$ type for $n \geq 3$. From Corollary 2.27, the equivalence classes of simple currents $L(k, k\lambda_1)$, $L(k, k\lambda_{n-1})$, $L(k, k\lambda_n)$ and $L(k, 0)$ form a group which is cyclic for an odd $n$ and which is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ for an even $n$. We shall define the extended algebra separately for the two cases.

From [H] and Lemma 2.28 we have

$$h^{(n-1)} = \frac{1}{2}(\alpha_1^\vee + 2\alpha_2^\vee + \cdots + (n-2)\alpha_{n-2}^\vee + \frac{1}{2}n\alpha_{n-1}^\vee + \frac{1}{2}(n-2)\alpha_n^\vee),$$

(4.56)

$$h^{(n)} = \frac{1}{2}(\alpha_1^\vee + 2\alpha_2^\vee + \cdots + (n-2)\alpha_{n-2}^\vee + \frac{1}{2}(n-2)\alpha_{n-1}^\vee + \frac{1}{2}n\alpha_n^\vee)$$

(4.57)

and

$$\langle h^{(n-1)}, h^{(n-1)} \rangle = \langle h^{(n)}, h^{(n)} \rangle = \frac{n}{4}. \quad (4.58)$$

Using the relation $h^{(n-1)} = h^{(n)} + \frac{1}{2}\alpha_{n-1}^\vee - \frac{1}{2}\alpha_n^\vee$ we get

$$\langle h^{(n-1)}, h^{(n)} \rangle = \frac{n-2}{4}. \quad (4.59)$$

Case I, $n$ is odd.

Define $h' = C\alpha'$ with $\langle \alpha', \alpha' \rangle = \frac{3nk}{4}$. Set

$$L = \mathbb{Z}\alpha, \quad \text{where } \alpha = h^{(n)} + \alpha'.$$

(4.60)

Then

$$L' = \mathbb{Z}\alpha', \quad L'' = \mathbb{Z}h^{(n)}. \quad (4.61)$$

Since

$$B(\alpha, \alpha) = k\langle h^{(n)}, h^{(n)} \rangle + \langle \alpha', \alpha' \rangle = nk, \quad (4.62)$$

$(L, B)$ is a positive definite integral lattice. By Proposition 4.3 (with the other assumptions being obvious) $V[L]$ is a simple vertex operator (super)algebra. We define $A_k(g)$ to be $V[L]$. Then we have the following results with the same proof as that of Propositions 4.9 and 4.10:

**Proposition 4.19** For $g$ of type $D_n$ with an odd $n$, the extended algebra $A_k(g)$ is regular. Furthermore, for $\lambda \in \Pi_k$, $j \in \mathbb{Q}$, set

$$W(\lambda, j) = L(k, \lambda) \otimes M_{h'}(1, \frac{j}{3nk}\alpha').$$

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Then any irreducible $A_k(g)$-module is isomorphic to $W(\lambda, j)[L]$ for some $\lambda \in P_k, j \in \mathbb{Z}$ with

$$2\lambda(\alpha_1^\vee) + 4\lambda(\alpha_2^\vee) + \cdots + 2(n - 2)\lambda(\alpha_{n-2}^\vee) + (n - 2)\lambda(\alpha_{n-1}^\vee) + n\lambda(\alpha_n^\vee) + j \in 4\mathbb{Z}. \quad (4.63)$$

Furthermore, $A_k(g)$ is generated by

$$(g + C\alpha') + V_{\text{low}}^{(\alpha)} + V_{\text{low}}^{(-\alpha)},$$

where $V_{\text{low}}^{(\alpha)} = L(k\lambda_1) \otimes e^{\alpha'}$ and $V_{\text{low}}^{(-\alpha)} = L(k\lambda_{n-1}) \otimes e^{-\alpha'}$ are of weight $\frac{nk}{2}$, and the relations $(4.50)-(4.53)$ with $k$ being replaced by $nk$ hold. $\square$

Case II: $n$ is even.

Define $h' = C\alpha_1' + C\alpha_2'$ to be a two-dimensional vector space with a symmetric bilinear form $\langle \cdot , \cdot \rangle$ such that

$$\langle \alpha_1', \alpha_1' \rangle = \langle \alpha_1', \alpha_1' \rangle = \frac{3}{4}nk,$$  

$$\langle \alpha_1', \alpha_2' \rangle = \frac{1}{4}k(n-2). \quad (4.65)$$

Define

$L = Za_1 + Za_2,$ \quad (4.66)

where

$$a_1 = h^{(n-1)} + \alpha_1', \quad a_2 = h^{(n)} + \alpha_2'.$$ \quad (4.67)

Then

$L' = Za_1' + Za_2', \quad L'' = Zh^{(n-1)} + Zh^{(n)}.$  

We have

$$B(a_1, a_1) = B(a_2, a_2) = kn,$$ \quad (4.69)

$$B(a_1, a_2) = \frac{1}{4}k(n-2) + \frac{1}{4}k(n-2) = \frac{1}{2}k(n-2).$$ \quad (4.70)

Since $n$ is even, $(L, B)$ is a positive-definite even lattice. Clearly, $L'' = Zh^{(n-1)} + Zh^{(n)} \subset P^\vee$, $L'$ is positive-definite, and the projection of $L$ onto $L'$ is one-to-one. By Proposition 4.3, $V[L]$ is a simple vertex operator algebra. Now we define $A_k(g) = V[L]$, as a simple vertex operator algebra. We just mention that this is a regular vertex operator algebra and a set of generators and relations can be worked out similarly but with some extra work.

**Remark 4.20** Note that $L(k, k\lambda_1)$ is a simple current of order 2 and we have $\langle h^{(1)}, h^{(1)} \rangle = 1$. Let $V = L(k, 0)$ and $L = Zh^{(1)}$. Then in view of Corollary 3.21, $L(k, 0) + L(k, k\lambda_1)$ has a natural simple vertex operator superalgebra structure (cf. Remark 4.15).
4.5 Extended vertex operator (super)algebras $A_k(E_6)$

Let $\mathfrak{g}$ be of type $E_6$. From Section 2.2, for any positive integer $k$, $L(k, k\lambda_1)$ and $L(k, k\lambda_6)$ are (the only) nontrivial simple currents for $L(k, 0)$. From [H] and Lemma 2.28, we have

$$h^{(1)} = \frac{1}{3}(4\alpha_1^\vee + 3\alpha_6^\vee + 5\alpha_2^\vee + 6\alpha_3^\vee + 4\alpha_4^\vee + 2\alpha_5^\vee),$$

$$h^{(5)} = \frac{1}{3}(2\alpha_1^\vee + 3\alpha_6^\vee + 4\alpha_2^\vee + 6\alpha_3^\vee + 5\alpha_4^\vee + 4\alpha_5^\vee)$$

and

$$\langle h^{(1)}, h^{(1)} \rangle = \langle h^{(5)}, h^{(5)} \rangle = \frac{4}{3}. \quad (4.71)$$

Define $\mathfrak{h}' = \mathbb{C}\alpha'$ to be a one-dimensional vector space equipped with the bilinear form $\langle \cdot, \cdot \rangle$ such that

$$\langle \alpha', \alpha' \rangle = \frac{2k}{3}. \quad (4.72)$$

Set

$$L = z\alpha, \quad \text{where } \alpha = h^{(1)} + \alpha'.$$

Then

$$L'' = zh^{(1)} \subset P^\vee, \quad L' = z\alpha'.$$

Since

$$B(\alpha, \alpha) = \frac{4k}{3} + \frac{2k}{3} = 2k, \quad (4.73)$$

($L, B$) is a positive-definite even lattice. By Proposition 4.3 (with the other assumptions being obvious), we have a simple vertex operator algebra $V[L]$. We define $A_k(\mathfrak{g})$ to be the vertex operator algebra $V[L]$. For $\lambda \in P_k$, $j \in \mathbb{Q}$, set

$$W(\lambda, j) = L(k, \lambda) \otimes M_{\mathfrak{h}'}(1, \frac{1}{2k}\alpha'),$$

an irreducible $V$-module. Then we have:

**Proposition 4.21** For $\mathfrak{g}$ of type $E_6$, the extended algebra $A_k(\mathfrak{g})$ is regular and any irreducible module is isomorphic to $W(\lambda, j)[L]$ for some $\lambda \in P_k$, $j \in \mathbb{Z}$ with

$$4\lambda(\alpha_1^\vee) + 3\lambda(\alpha_6^\vee) + 5\lambda(\alpha_2^\vee) + 6\lambda(\alpha_3^\vee) + 4\lambda(\alpha_4^\vee) + 2\lambda(\alpha_5^\vee) + j \in 3\mathbb{Z}. \quad \Box \quad (4.74)$$
Similar to the case $\mathfrak{g} = \mathfrak{sl}(n+1)$, $A_k(E_6)$ as a vertex operator algebra is generated by

$$(\mathfrak{g} + \mathbb{C}\alpha') + V_{\text{low}}^{(\alpha)} + V_{\text{low}}^{(-\alpha)}.$$

We have

$$V_{\text{low}}^{(\alpha)} = L(k, k\lambda_1)_{\text{low}} \otimes e^{\alpha'} = L(k\lambda_1) \otimes e^{\alpha'}, \quad (4.80)$$

$$V_{\text{low}}^{(-\alpha)} = L(k, k\lambda_1)^* \otimes e^{-\alpha'} = L(k\lambda_1)^* \otimes e^{-\alpha'}. \quad (4.81)$$

Since the $L(0)$-weights of $L(k, k\lambda_1)_{\text{low}}$ and $L(k, k\lambda_1)^*_{\text{low}}$ are

$$\frac{1}{2}B(h^{(1)}, h^{(1)}) = \frac{k}{2}(h^{(1)}, h^{(1)}) = \frac{2k}{3},$$

the lowest weights of $V^{(\alpha)}$ and $V^{(-\alpha)}$ are $\frac{2k}{3} + \frac{k}{3} = k$.

For $m \in \mathbb{Z}$, the lowest $L(0)$-weight of $M_{h^*}(1, ma')$ is $\frac{1}{2}(ma', ma') = \frac{km^2}{3}$. Then the lowest $L(0)$-weight of $V^{(ma)}$ is at least $\frac{km^2}{3}$. If $|m| \geq 3$, the lowest $L(0)$-weight of $V^{(ma)}$ is at least $3k$. The lowest $L(0)$-weight of $V^{(2\alpha)}$ is the sum of the lowest $L(0)$-weight of $L(k, 0)^{(2h^{(1)})}$ and $\frac{4k}{3}$. We know that $L(k, 0)^{(2h^{(1)})} \simeq L(k, k\lambda_5)$ whose lowest $L(0)$-weight is $\frac{4k}{3}$. Then the lowest $L(0)$-weight of $V^{(2\alpha)}$ is $\frac{8k}{3}$, which is greater than $2k$. With this information, using the same proof of Proposition 4.14 we immediately have:

**Proposition 4.22** The extended vertex operator algebra $A_k(E_6)$ is generated by

$$(\mathfrak{g} + \mathbb{C}\alpha') + V_{\text{low}}^{(\alpha)} + V_{\text{low}}^{(-\alpha)}, \quad (4.82)$$

where $V_{\text{low}}^{(\alpha)} = L(k\lambda_1) \otimes e^{\alpha'}$ and $V_{\text{low}}^{(-\alpha)} = L(k\lambda_1)^* \otimes e^{-\alpha'}$ are of weight $k$. Furthermore, the relations (4.50)-(4.53) with $k$ being replaced by $2k$ hold. □

**Remark 4.23** When $k = 1$, $V_{\text{low}}^{(-\alpha)} + (\mathfrak{g} + \mathbb{C}\alpha') + V_{\text{low}}^{(\alpha)}$ is exactly the weight-one subspace of $A_1(\mathfrak{g})$, which is a natural Lie algebra with

$$[L(\lambda_1) \otimes e^{\alpha'}, L(\lambda_1) \otimes e^{\alpha'}] = 0, \quad [L(\lambda_1)^* \otimes e^{-\alpha'}, L(\lambda_1)^* \otimes e^{-\alpha'}] = 0, \quad (4.83)$$

$$[L(\lambda_1) \otimes e^{\alpha'}, L(\lambda_1)^* \otimes e^{-\alpha'}] \subset \mathfrak{g} + \mathbb{C}\alpha'. \quad (4.84)$$

These relations give rise to a $\mathbb{Z}$-grading for the Lie algebra. One can easily show that this Lie algebra is simple and of rank 7. Using the standard Dynkin diagram embedding of $E_6$ into $E_7$ we can show that it is really $E_7$.  

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4.6 Extended vertex operator (super)algebras $A_k(E_7)$

Let $\mathfrak{g}$ be of type $E_7$. From Section 2.2, for any positive integer $k$, $L(k, k\lambda_6)$ is a (and the only nontrivial) simple current for $L(k, 0)$. Using [H] (Table 1 on page 69) and Lemma 2.28 we have

$$h^{(6)} = \frac{1}{2} (2\alpha_1^\vee + 3\alpha_7^\vee + 4\alpha_2^\vee + 6\alpha_3^\vee + 5\alpha_4^\vee + 4\alpha_5^\vee + 3\alpha_6^\vee), \quad \langle h^{(6)}, h^{(6)} \rangle = \frac{3}{2}. \quad (4.85)$$

Define $h' = c\alpha'$ to be a one-dimensional vector space equipped with a bilinear form $\langle \cdot, \cdot \rangle$ such that

$$\langle \alpha', \alpha' \rangle = \frac{k}{2}. \quad (4.86)$$

Set

$$L = \mathbb{Z}\alpha, \quad \text{where} \quad \alpha = h^{(6)} + \alpha'. \quad (4.87)$$

Then $L' = \mathbb{Z}\alpha'$ and $L'' = \mathbb{Z}h^{(6)}$. Since $B(\alpha, \alpha) = \frac{3k}{2} + \frac{k}{2} = 2k$, $(L, B)$ is a positive-definite even lattice. By Proposition L.3 (with the other assumptions being obvious), $V[L]$ is a simple vertex operator algebra. We define $A_k(E_7)$ to be the simple vertex operator algebra $V[L]$. For $\lambda \in P_k$, $j \in \mathbb{Q}$, we set

$$W(\lambda, j) = L(k, \lambda) \otimes M_{h'}(1, \frac{j}{k} \alpha'). \quad (4.88)$$

In view of Theorem 4.4 we immediately have:

**Proposition 4.24** For $\mathfrak{g}$ of type $E_7$, the extended algebra $A_k(\mathfrak{g})$ is regular and any irreducible module is isomorphic to $W(\lambda, j)[L]$ for $\lambda \in P_k$, $j \in \mathbb{Z}$ with

$$2\lambda(\alpha_7^\vee) + 3\lambda(\alpha_2^\vee) + 4\lambda(\alpha_3^\vee) + 6\lambda(\alpha_4^\vee) + 5\lambda(\alpha_5^\vee) + 4\lambda(\alpha_6^\vee) + 3\lambda(\alpha_6^\vee) + j \in 2\mathbb{Z}. \quad \square \quad (4.89)$$

The lowest weights of $V^{(\alpha)}$ and $V^{(-\alpha)}$ are $\frac{1}{2}B(\alpha, \alpha) = k$. Since $[L(k, 0)^{(2h^{(6)})}] = [L(k, 0)]$, the lowest weights of $V^{(2\alpha)}$ and $V^{(-2\alpha)}$ are $\frac{1}{2}B(2\alpha', 2\alpha') = k$ also. For $|m| \geq 3$, the lowest weight of $V^{(m\alpha)}$ is at least $\frac{1}{2}B(m\alpha', m\alpha') = \frac{m^2k}{4}$, which is greater than $2k$. With this information we immediately have:

**Proposition 4.25** The vertex operator algebra $A_k(E_7)$ is generated by

$$V_{\text{low}}^{(-2\alpha)} + V_{\text{low}}^{(-\alpha)} + (\mathfrak{g} + C\alpha') + V_{\text{low}}^{(\alpha)} + V_{\text{low}}^{(2\alpha)},$$

where

$$V_{\text{low}}^{(\alpha)} = L(k\lambda_6) \otimes e^{\alpha'}, \quad V_{\text{low}}^{(-\alpha)} = L(k\lambda_6) \otimes e^{-\alpha'}, \quad (4.90)$$

$$V_{\text{low}}^{(2\alpha)} = C \otimes e^{2\alpha'}, \quad V_{\text{low}}^{(-2\alpha)} = C \otimes e^{-2\alpha'}. \quad (4.91)$$

are of weight $k$. \quad \square
Remark 4.26 When \( k = 1 \),
\[
\mathbb{C} \otimes e^{-2\alpha'} + L(\lambda_6) \otimes e^{-\alpha'} + (\mathfrak{g} + \mathbb{C} \alpha') + L(\lambda_6) \otimes e^{\alpha'} + \mathbb{C} \otimes e^{2\alpha'}
\]
is exactly the weight-one subspace of \( A_1(\mathfrak{g}) \). It is a \( \mathbb{Z} \)-graded Lie algebra with the obvious grading. Similarly, we can show that it is \( E_8 \).

4.7 Extended algebras of types \( B_n \) and \( C_n \)

For \( \mathfrak{g} \) of type \( B_n \), \( L(k,k\lambda_1) \) is the only nontrivial simple current and for \( \mathfrak{g} \) of type \( C_n \), \( L(k,k\lambda_n) \) is the only nontrivial simple current. For \( B_n \), from \([H]\) and Lemma 2.28 we have
\[
h^{(1)} = \alpha_1^\vee + \cdots + \alpha_{n-1}^\vee + \frac{1}{2} \alpha_n^\vee, \quad \langle h^{(1)}, h^{(1)} \rangle = 1 \quad (4.92)
\]
and for \( C_n \) we have
\[
h^{(n)} = \frac{1}{2}(\alpha_1^\vee + 2\alpha_2^\vee + \cdots + n\alpha_n^\vee), \quad \langle h^{(n)}, h^{(n)} \rangle = \frac{n}{2}. \quad (4.93)
\]

Remark 4.27 We here correct an error of \([DLM2]\) (Examples 5.12 and 5.13) where it was stated that \( L(k,k\lambda_n) \) was the nontrivial simple current for \( \mathfrak{g} \) of type \( B_n \) and that \( L(k,k\lambda_1) \) was the nontrivial simple current for \( \mathfrak{g} \) of type \( C_n \). For \( \mathfrak{g} \) of type \( B_n \), it follows from \([DLM2]\) or Proposition 3.20 with \( V = L(k,0) \) and \( L = \mathbb{Z} h^{(1)} \) that for any positive integral level \( k \), \( L(k,0) + L(k,k\lambda_1) \) has a natural simple vertex operator (super)algebra structure. However, for \( C_n \), \( L(k,0) + L(k,k\lambda_n) \) is a vertex operator (super)algebra only for a positive integral level \( k \) with \( nk \) being even.

For \( \mathfrak{g} \) of type \( C_n \), we define \( h' = \mathbb{C} \alpha' \) with \( \langle \alpha', \alpha' \rangle = \frac{nk}{2} \). Set
\[
L = \mathbb{Z} \alpha, \quad \text{where } \alpha = h^{(n)} + \alpha'. \quad (4.94)
\]
Then \( L'' = \mathbb{Z} h^{(n)} \) and \( L' = \mathbb{Z} \alpha' \). Furthermore,
\[
B(\alpha, \alpha) = k \langle h^{(n)}, h^{(n)} \rangle + \langle \alpha', \alpha' \rangle = nk. \quad (4.95)
\]
Then \( (L, B) \) is a positive definite integral lattice. Hence \( V[L] \) is a simple vertex operator (super)algebra. Furthermore, we have:

Proposition 4.28 For \( \mathfrak{g} \) of type \( C_n \), \( A_k(\mathfrak{g}) \) is regular and any irreducible \( A_k(\mathfrak{g}) \)-module is isomorphic to \( W(\lambda, j)[L] \) for \( \lambda \in \mathbb{P}_k, \ j \in \mathbb{Z} \) with
\[
\lambda(\alpha_1^\vee) + 2\lambda(\alpha_2^\vee) + \cdots + n\lambda(\alpha_n^\vee) + j \in 2\mathbb{Z}, \quad (4.96)
\]
where
\[ W(\lambda, j) = L(k, \lambda) \otimes M_{h'}(1, j, \frac{1}{nk} \alpha') . \] (4.97)

Furthermore, \( A_k(g) \) is generated by
\[ (g + C\alpha') + V_{\text{low}}^{(\alpha)} + V_{\text{low}}^{(-\alpha)} \]
and the the relations (4.50)-(4.53) with \( k \) being replaced by \( nk \) hold, where \( V_{\text{low}}^{(\alpha)} \) and \( V_{\text{low}}^{(-\alpha)} \) are of weight \( \frac{nk}{2} \). \( \square \)

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