Free Energy and Entropy for
Semi-classical Black Holes in
the Canonical Ensemble

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Abstract

We consider the thermodynamics of a black hole coupled to thermal radiation in a spatially finite (spherical) region. Thermodynamic state functions are derived in the canonical ensemble, defined by elements of radius $r_o$ and boundary temperature $T(r_o)$. Using recent solutions of the semi-classical back reaction problem, we compute the $O(\hbar)$ corrections to the mass of the black hole, thermal energy, the entropy and free energy due to the presence of hot conformal scalars, massless spinors and U(1) gauge quantum fields in the vicinity of the hole. The free energy is particularly important for assessing under what conditions the nucleation of black holes from hot flat space is likely to occur.

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1 Introduction

A Schwarzschild black hole in empty space radiates quanta possessing a temperature characterized by the mass $M$ of the hole. At large distances from the hole ($r \gg M$), the temperature of the radiation approaches $T_\infty = \frac{1}{8\pi M}$. Studies seeking to relate the thermal properties of black holes to quantum gravity typically omit the radiation from consideration. Nevertheless, it is clear that black holes are neither thermally nor mechanically isolated from their surroundings and in deducing the thermodynamic properties of black holes, the presence of the radiation field should be taken into account. One way in which to do so is to solve the semi-classical Einstein equation

$$G_{\mu \nu} = 8\pi \langle T_{\mu \nu} \rangle,$$

(1)

taking as source term the gravitationally induced renormalized stress tensor $\langle T_{\mu \nu} \rangle$ characterizing the thermal radiation. The calculation of the modified spacetime metric is the back reaction problem, and it is expected that the new metric gives a better approximation to the spacetime geometry associated with thermal equilibrium than one satisfying the source-free ($\langle T_{\mu \nu} \rangle = 0$) Einstein equation. In equilibrium, the ensemble must be time independent, thus the perturbed metric must be static, as well as spherically symmetric, as the stress tensors employed here are renormalized on a Schwarzschild background. From knowledge of the modified metric, one can calculate the $O(\hbar)$ correction to the black hole temperature $[1]$, the entropy $[2]$ and the effective potential $[3]$. In the present paper, we use solutions of the back reaction problem obtained in $[1]$, $[2]$ and $[3]$ in order to compute the fractional corrections to the mass, the thermodynamic energy, the free energy and the entropy for a black hole in the canonical ensemble arising from its interaction with scalar, spinor and U(1) gauge quantum fields.

An important feature in this analysis is the use of finite spatial bound-
aries. Indeed, it was recognized in [1] that the back reaction problem (1) has no definite solution unless the system composed of black hole plus radiation is enclosed in a spatially bounded region or “box”. This is due to the fact that the stress tensors renormalized on a black hole background spacetime are asymptotically constant, so the corrections to the metric do not remain uniformly small for sufficiently large radius. Physically, this means the radiation in a box that is too large would collapse onto the hole thereby producing a larger one. Yet, there is considerably more to the introduction of finite spatial boundaries than just as a means for ensuring perturbative validity of the solution. As pointed out in [3] there are additional advantages to be gained by employing boundary conditions at a spatially finite location, as opposed to spatial infinity. In the first instance, as already noted, self-gravitating thermodynamic systems are not usually asymptotically flat. In particular, in the canonical ensemble, the system boundary is to be maintained at a constant temperature, and this is achieved by coupling the boundary to an external heat reservoir. Such an arrangement obviously fails to satisfy asymptotic flatness (for the region exterior to the system possesses a constant energy density), but with finite spatial boundaries there is no need to assume asymptotic flatness in the spatial directions. Secondly, the usual thermodynamic limit requiring infinite spatial extent does not exist for equilibrium self-gravitating systems at finite temperature. This follows since the system is unstable to gravitational collapse, or recollapse if a black hole is already present. This in practice presents no problem since physically, one only requires that the system can in principle become sufficiently large so that the fluctuations become negligible.

This instability is reflected in the formally negative heat capacity, which in turn, implies a divergent canonical partition function [3]. Enclosing the black hole in a box has the effect of stabilizing the hole yielding a system with
a positive heat capacity and allows one to derive a meaningful, convergent partition function. The key to these results is the fact that the black hole energy and temperature at a finite radius are not inversely related, as they are at spatial infinity, because of the blueshifting of temperature in a static gravitational field.

Given the advantages of invoking finite spatial boundaries, the choice must be made in specifying the type of boundary data which is to be fixed. In the present work, we consider the canonical ensemble, in which the temperature is held fixed at the cavity wall. The elements of this ensemble thus consist of spherical cavities of radius $r_o$ and temperature $T(r_o)$ enclosing a black hole at their centers and filled with thermal radiation. The boundary temperature is held fixed by coupling the cavity wall to a large external heat reservoir. At first sight, this might seem like an artificial arrangement, however the heat reservoir is intended to represent the natural universe and the boundary conditions mimic the implantation of the hole and its equilibrating radiation into the universe.

One of the main objectives in this paper is the calculation of a thermodynamic potential appropriate for a black hole coupled to a thermal radiation field. This motivated by the fact that the nucleation of a black hole from hot flat space (gravitons and other massless fields on a flat background geometry) can be viewed as a phase transition. In the canonical ensemble the temperature and volume (area of the cavity wall) are fixed, so the relevant potential is the Helmholtz free energy $F = E - TS$, $E$ is the thermal energy and $S$ is the entropy. This is related to the canonical partition function by $\beta F = -\log(Z)$, $\beta = T^{-1}$. In a phase transition, the value of $F$ should decrease and if the free energy is a (local) minimum, the system is in a state of (meta) stable equilibrium. To ascertain the likelihood of nucleation, one compares the free energies evaluated for hot flat space and black
hole configurations. Such an analysis has been addressed in [6] for a free black hole.

We begin in Section II by deriving the mass corrections for a black hole in a finite cavity of radius \( r_o \) valid to \( O(\epsilon) \) in the back reaction, where \( \epsilon \simeq (l_P r_o)^2 \), and \( l_P \) is the Planck length. The action \( I \) of the equilibrium metric is calculated in Section III and depends on the trace anomalies of the stress tensors used in (1). The thermodynamic energy, entropy and the free energy of the combined system comprised of black hole plus radiation are derived assuming that the partition function contains \( I \) as its leading term. The \( O(\epsilon) \) fractional corrections to the energy and entropy are calculated to reveal the spin dependence of the back reaction. In Section IV we discuss some aspects of black hole nucleation based on the results of these calculations. The explicit solutions of the back reaction (1) used here are collected in an Appendix.

2 Schwarzschild Mass in a Spherical Cavity

It is well-known that the equilibrium temperature distribution of a static, self-gravitating system is not constant, but is blueshifted, indicating that temperature is actually scale-dependent and is thus, not a purely intensive thermodynamic variable. This distribution for a Schwarzschild black hole including quantum mechanical back reaction is given by \([1],[2]\) \((w = \frac{2M}{r})\)

\[
T(r) = (8\pi M)^{-1} (1 - w)^{-1/2} \left[ 1 - \epsilon \{ \rho(w) - nK^{-1} - \frac{w}{2}(1 - w)^{-1} \mu(w) \} \right],
\]

(2)

(see the Appendix for the definitions of \( \rho, n \) and \( \mu \) and \( K \)) where \( \epsilon = (\frac{M_P}{M})^2 < 1 \), \( M_P = \hbar^{1/2} \) is the Planck mass (in units where \( G = c = k_B = 1 \)). An important finding is that the Schwarzschild mass is double-valued in the canonical ensemble \([1],[2]\). Indeed, since the local temperature is fixed at
the boundary radius \( r_o \) of the spherical cavity, we can invert (2) and solve for \( M = M(r_o, T(r_o)) \). By squaring (2) and remembering that (we henceforth drop the subscript “o” from the remainder of this discussion)

\[
|\epsilon F(w)| \equiv |\epsilon \{\rho - nK^{-1} - \frac{w}{2}(1-w)^{-1}\mu\}| < 1, \quad (3)
\]

we obtain the equation, valid to \( O(\epsilon) \)

\[
w^3 - w^2 + \sigma^2 = 2\epsilon\sigma^2 F(w), \quad (4)
\]

where \( \sigma = \frac{1}{4\pi rT} \geq 0 \). Although this is a nontrivial transcendental equation in \( w \) (\( F \) contains positive and negative powers of \( w \) and a logarithm), we can solve for approximate solutions by perturbing around the exact solution which has been computed for \( \epsilon = 0 \). In the absence of any back reaction, it is known \( \Re \) that there are no positive solutions when \( rT < \sqrt{27}/8\pi \approx 0.207 \).

When \( rT \geq \sqrt{27}/8\pi \) there are two real nonnegative solutions of the form

\[
w_1 = \frac{2M_1(r, T)}{r} = \frac{1}{3}[1 - 2\cos(\frac{\alpha}{3} + \frac{\pi}{3})], \quad (5)
w_2 = \frac{2M_2(r, T)}{r} = \frac{1}{3}[1 + 2\cos(\frac{\alpha}{3})], \quad (6)
\]

\[
\cos(\alpha) = 1 - \frac{27}{2}\sigma^2, \quad (7)
\]

\[
0 \leq \alpha \leq \pi, \quad (8)
\]

with \( M_2 \geq M_1 \). For \( rT = \sqrt{27}/8\pi \) \((\alpha = \pi)\), \( 3M_2 = 3M_1 = r \) indicating the cavity wall coincides with the circular photon orbit, while for \( rT \to \infty \) \((\alpha \to 0)\) we have \( M_2 \to \frac{r}{2} \) and \( M_1 \to 0 \) \( (\text{see Fig. 1}) \). For the heavier mass branch \((M_2)\), the cavity radius can run from the black hole horizon out to the circular photon orbit. The mass increases as the wall is pulled in. For the lighter mass, \( M_1 \) decreases as \( rT \to \infty \). By virtue of the parametrization of the solutions (5) and (6) in terms of the angle \( \alpha \), the entire semi-infinite two-dimensional parameter domain \( \sqrt{27}/8\pi \leq rT < \infty \) of the \( r-T \) plane can be compactified to a finite one-dimensional interval.
This makes it particularly easy to see, at a glance, how changes in the cavity radius and wall temperature affect the value of the black hole mass consistent with the given boundary conditions. Thus, for given \( r > 0 \) and \( T > 0 \) such that \( rT \geq \sqrt{27}/8\pi \), one calculates \( \alpha \) through (7) and then the values of \( M_1 \) or \( M_2 \) can be read off from the respective branch curves in Fig. 1. Moreover, this plot also indicates the allowed range in the variable \( w = 2M/r \), which appears in all our back-reaction formulae. For example, for the heavier mass solution \( (M_2) \) we see immediately that \( 2/3 \leq w_2 \leq 1 \), whereas \( 0 \leq w_1 \leq 2/3 \) for the \( M_1 \) branch. This is important, for it means when we come to evaluate some function of \( w \), we should restrict the domain of that function to correspond to the range of \( w \). This summarizes the thermal and radial dependence of the mass of the vacuum (that is, without the back reaction) black hole in the canonical ensemble. We will see shortly how the back-reaction modifies the allowed range of \( w_{1,2} \).

We now couple the hole to the ambient thermal radiation in the cavity and ask for the fractional corrections to the mass of the hole arising from the back reaction. If (3) holds over some range in \( w \), we can obtain approximate solutions to (4) by assuming an ansatz of the form

\[
\tilde{w} = w + \delta w, \quad \left| \frac{\delta w}{w} \right| < 1
\]

for the perturbed solutions. Inserting (9) into (4) and retaining only the \( O(\epsilon) \) and \( O(\delta w) \) terms yields

\[
\delta w = \frac{2\epsilon \sigma^2 F(w)}{w(3w - 2)}.
\]

The approximate solutions of (4) expressed in terms of the angle \( \alpha \) are thus

\[
\tilde{w}_{1,2}(\alpha) = w_{1,2}(\alpha) + \frac{4\epsilon_{1,2}(1 - \cos(\alpha))F(w_{1,2}(\alpha))}{w_{1,2}(\alpha)(3w_{1,2}(\alpha) - 2)},
\]

where we have used (7) and (9) and \( w_{1,2} \) refers to one of the solutions (5) or (6). In determining the size of the mass perturbations, it should be noted
that \( \epsilon \) itself depends on the angle \( \alpha \):

\[
\epsilon_2 = \left( \frac{M_{Pl}}{M_2} \right)^2 = \left( \frac{l_{Pl}}{r \left[ 1 + 2 \cos \left( \frac{\alpha}{2} \right) \right]} \right)^2
\]

(with a similar expression for \( \epsilon_1 \)). Thus, in varying the product \( rT \) or \( \alpha \), we can hold \( \epsilon < 1 \) to a fixed value by varying \( r \) accordingly. So, for example, we can maintain \( M_2 \) constant as \( rT \to \infty \) by simultaneously decreasing \( r \) as indicated in Fig. 1. This corresponds to a curve of constant mass in the \( r - T \) plane. On the other hand, if \( r \) is fixed, then varying \( \alpha \) corresponds to varying the temperature at the cavity wall. In any case, we can ensure that \( \epsilon < 1 \) by taking the cavity radius to be \( r \geq 3l_{Pl} \) for the heavier mass solution. The pure perturbation term \( \Delta w \equiv \frac{2a^2F}{w(du - 2)} \) from (11) is plotted in Fig. 2 and Fig. 3 for the \( M_2 \) and \( M_1 \) mass branches, respectively. There is a striking qualitative as well as quantitative contrast among the three types of back reaction we are considering here. First, it should be noted that in all cases there is a divergence in \( \Delta w \) as \( \alpha \to \pi \) corresponding to the wall radius approaching the circular photon orbit (coincidentally, the specific heat at constant wall area also diverges at \( r = 3M_1 \), both with and without taking the back reaction into account). This pole is manifest in (11) and simply means that the mass perturbation cannot be extrapolated reliably to this limit. Without having the exact solution(s) of (4), it is difficult to judge whether this pole is a significant feature of the back reaction or is merely an artifact of the approximation. We can only say that reliable results will require calculations beyond \( O(h) \). Nevertheless, there is a wide range in \( \alpha \) for which the perturbation term \( \Delta w \) is truly small and hence consistent with the expansion (9). This range depends, as indicated by the Figures 2 and 3, on the spin of the quantum field. While the spinor tends to increase the mass of \( M_2 \) as \( \alpha \to \pi \), the scalar and gauge particles tend to lower it. The perturbations valid for the lower mass branch behave in quite the opposite
manner. That is, while the spinor back reaction now tends to lower the
mass $M_1$ from its vacuum value, the scalar and vector boson tend to increase
it. Apart from the pole at $w_{1,2} = 2/3$, all three curves for the $M_1$ branch
diverge as $\alpha \to 0$. However, this divergence is well understood, and is due to
the fact that the functions characterizing the metric perturbations ($\rho$ and $\mu$)
grow without bound as $r \to \infty$ \[1, 2, 3, 4\]. This in turn is a consequence of
the fact that the stress tensors renormalized on a Schwarzschild background
are asymptotically constant, as stated previously in the Introduction. So,
in this small $\alpha$ regime, a natural cut-off is provided by the cavity radius.
Referring back to the $M_2$ branch plus perturbations, we see that the lower
limit in the range in $w_2$ is extended towards 0 by the effects of the scalar
and gauge boson back reactions. Clearly, as $\alpha \to \pi$, these perturbations
become arbitrarily large and negative and would result in an $M_2 < 0$, which
is unphysical, as well as being outside the domain of perturbative validity
(for $|\Delta w| > 1$). The spinor tends to extend the upper limit in the range of $w_2$
to be greater than 1, but this would translate into a cavity radius inside the
black hole event horizon. Thus, this perturbation must be cut off before $\alpha$
reaches $\pi$. In the case of the lower mass branch, provided we remain within
the range of perturbative validity, we note that the spinor perturbation tends
to lower this mass while the scalar and vector tend to increase it. Except
for the behavior close to $\alpha = \pi$, the conformal scalar field contributes a
negligible amount to the mass corrections for both branches in comparison
to the spinor and gauge boson. Moreover, these latter two fields give rise
to competing effects, as the signs of their associated mass perturbations are
opposite. This sign difference becomes all the more important when effects
of multiple-field back reaction are treated. Multiple particle species arise,
for example, in gauge theories of particle physics. The actual number of
(fundamental) scalars, spinors and gauge bosons depends on the symmetry
group and the dimensionality of the group representations.

Perhaps the most important point to be drawn from all this is that it is possible for the roles of \( M_2 \) and \( M_1 \) to be dynamically switched, that is, although at leading order we have \( M_2 \geq M_1 \), the effects of the back reaction may result in a level crossing such that \( M_1 \geq M_2 \). Thermodynamically, it is the heavier of the two masses which gets nucleated \(^3\), so the back reaction must provide an important ingredient in assessing the likelihood of the nucleation of black holes from hot flat space.

### 3 Action, Thermal Energy and Entropy

In deriving gravitational thermodynamics from a Euclidean path integral,

\[
Z = \int d\mu[g, \phi] e^{-I[g, \phi]}.
\]  

one expects the dominant contribution to the canonical partition function to come from metrics \( g \) and fields \( \phi \) that are near a background metric \( g^{(0)} \) and background fields \( \phi^{(0)} \), respectively. These background fields are solutions of classical field equations. First-order quantum effects can be incorporated in \( Z \) by expanding the action in a Taylor series about the background fields

\[
g = g^{(0)} + \tilde{g}, \quad \phi = \phi^{(0)} + \tilde{\phi},
\]

so that

\[
I[g, \phi] = I[g^{(0)}] + I_2[\tilde{g}] + I_2[\tilde{\phi}] + \text{higher order terms},
\]

where \( I_2 \) is quadratic in the field fluctuations, and we have set the background matter (or radiation) fields to zero, this corresponding to the case at hand (i.e., Hawking radiation is quantum mechanical, not classical). As is well known, the functional integration of \( I_2 \) with respect to \( \tilde{g} \) and \( \tilde{\phi} \) leads to determinants of differential operators which can be exponentiated formally
to yield, when added to $I[g^{(0)}]$, the one-loop effective action. However, from
the point-of-view of the back reaction of quantum fields on the background
gometry $g^{(0)}$, the solutions of (1), which is a semi-classical field equation,
contain, by construction, the effects of the quantum fields as represented by
$< T_{\mu\nu} >$. Therefore, given a solution

$$g = g^{(0)} + h \Delta g$$

(16)
of (1) (we must distinguish $\Delta g$ from $\tilde{g}$, since the former obtains from solv-
ing the differential equation (1), whereas the latter represents an arbitrary
fluctuation to be integrated out) we assume that $Z$ contains the action of the
semi-classical metric as its leading term:

$$Z \approx e^{-I[g^{(0)} + h \Delta g]}.$$  

(17)

Moreover, since (1) can be obtained from an variational principle [15], the
form of the action $I$ for the semi-classical metric (16) is identical to that
Corresponding to the classical background metric.

The action is given by [9]

$$I = I_1 - I_{\text{subtract}}$$

(18)

where

$$I_1 = -\frac{1}{16\pi} \int_{2M}^{r_o} \int_{0}^{\beta_s} d^4 x \sqrt{g} R + \frac{1}{8\pi} \int_{S^1 \times S^2} d^3 x \sqrt{\gamma} K.$$  

(19)

The four space metric (16) is given in the Appendix. The boundary at $r_o = \text{const.}$ is the product $S^1 \times S^2$ of the periodically identified Euclidean time
with the unit two-sphere of area $A = 4\pi r_o^2$. The proper length around the
$S^1$ coordinate is $\beta_s = 8\pi M$. The trace of the boundary extrinsic curvature
is $K$ and $\gamma_{i,j}$ denotes the induced three-metric. The volume term in (19) is
sensitive to the trace anomaly $< T^\mu_{\mu} >$, as can be seen immediately by taking
the trace of equation (1). These anomalies have been evaluated exactly for
the conformal scalar \[10\] and the U(1) gauge boson \[11\] while an analytic approximation has been given for the massless spinor case \[12\]. They are

\[
<T_\mu^\mu>_{\text{scalar}} = \frac{\varepsilon \sqrt{w}}{\pi KM^2},
\]

(20)

\[
<T_\mu^\mu>_{\text{spinor}} = \frac{7\sqrt{w}}{4\pi KM^2},
\]

(21)

\[
<T_\mu^\mu>_{\text{vector}} = \frac{-13\sqrt{w}}{\pi KM^2}.
\]

(22)

As these are all of order \(O(\varepsilon)\), we can replace \(\sqrt{g} \rightarrow r^2 \sin \theta\) under the integral. Calculation of the volume term is immediate and yields the result

\[
\text{volume term} = \frac{\varepsilon 128\pi CM^2}{3K}(1 - w_0^3),
\]

(23)

where \(C \in (1, \frac{7}{4}, -13)\) is a spin dependent constant and \(w_0 = \frac{2M}{r_o}\). When integrating (20-22), \(M\) is of course held constant, as it depends only on the upper endpoint of the integration interval, while \(w = 2M(r_o, T(r_o))/r\) varies from 1 to \(w_0 < 1\).

The calculation of the boundary contribution to the action is slightly more involved. The determinant of the induced three-metric is

\[
\sqrt{\gamma} = g_{tt}^{1/2}(w) r^2 \sin \theta |_{r=r_o}.
\]

(24)

The trace of the extrinsic curvature tensor is

\[
\mathcal{K} = \left[\frac{2}{r} g_{rr}^{-1/2} + \frac{1}{2} g_{rr}^{-1/2} \frac{\partial}{\partial r} \log(g_{tt}) \right] |_{r=r_o}.
\]

(25)

Substituting the metric components from Appendix (45,46) into (25) and performing the integrations indicated in (19) yields

\[
\text{boundary term} = (8\pi Mr)(1 - w) \left[1 + \varepsilon\{\rho - nK^{-1} - w(1 - w)^{-1}\mu\} \right] + 4\pi M^2 \left[1 + \varepsilon\{\rho - nK^{-1} + \mu + \frac{32\pi M^2}{\varepsilon w^3} < T_r^r >\} \right], \quad (26)
\]
where use of the equations of motion A(47-48) has been made in order to arrive at this final expression. Lastly, the subtraction term is just \( I_1 \) evaluated for a flat four-metric having the same period for the \( S^1 \) coordinate:

\[
I_{\text{subtract}} = -\beta r,
\]

(27)

where \( \beta = T^{-1}(r) \) is the inverse local temperature. Putting all this back together, we obtain

\[
I = \frac{\epsilon 128\pi C M^2}{3K}(1 - w^3) - (8\pi M r)(1 - w) \left[ 1 + \epsilon \{ \rho - nK^{-1} - w(1 - w)^{-1} \mu \} \right] - (4\pi M^2) \left[ 1 + \epsilon \{ \rho - nK^{-1} + \mu + \frac{32\pi M^2}{\epsilon w^3} \langle T^r_r \rangle \} \right] + \beta r,
\]

(28)

where it is understood that this is evaluated at \( r = r_o \), and that the functions \( \mu, \rho \) and \( \langle T^r_r \rangle \), as well as the constants \( n, C \) are spin dependent (see the Appendix).

We are now ready to calculate various thermodynamic state functions of interest. In particular, the thermal energy \( E \) in the canonical ensemble is

\[
E = - \left( \frac{\partial \log(Z)}{\partial \beta} \right)_A \approx \left( \frac{\partial I}{\partial \beta} \right)_A,
\]

(29)

while the free energy \( F \) and the entropy \( S \) are

\[
F = -\beta^{-1} \log(Z) \approx \beta^{-1} I, \text{ and}
\]

\[
S = \beta E + \log(Z) \approx \beta E - I.
\]

(30)

(31)

In calculating \( E \), there are two possible ways to proceed. On can, for example, compute the derivative in (29) directly. Although straightforward, this is somewhat involved due to the fact that the black hole mass \( M \) appearing in \( I \) is a function of the temperature (and cavity radius) and hence, so are \( \epsilon \).
and \( w = \frac{2M}{r} \), as well. The action must therefore be regarded as a function of \( T \) (or \( \beta \)) and \( r \), and thus

\[
\left( \frac{\partial I}{\partial \beta} \right)_A = \left( \frac{\partial I}{\partial M} + \frac{2}{r} \frac{\partial I}{\partial w} - \frac{2\epsilon}{M} \frac{\partial I}{\partial \epsilon} \right) \left( \frac{\partial M}{\partial \beta} \right)_A.
\] (32)

As such, a useful ingredient in these calculations is the quantity \( \left( \frac{\partial M}{\partial \beta} \right)_A \) which can be derived directly by first differentiating \( \beta = T^{-1} \) (using (2)) partially with respect to \( M \) at fixed radius \( r \), and then inverting the expression so obtained, remembering to expand the inverse derivative only to \( O(\epsilon) \). We find, following this procedure, that

\[
\left( \frac{\partial M}{\partial \beta} \right)_A = \frac{(1 - w)^{1/2}}{8\pi(1 - \frac{3}{2}w)} \left\{ 1 + \epsilon \left( \frac{1 - \frac{1}{2}w}{1 - \frac{3}{2}w} \right) F(w) + \frac{\epsilon}{(1 - \frac{3}{2}w)} \left[ \frac{w}{2} (1 - w)^{-1} \mu + \frac{16\pi M^2}{\epsilon w^2} < T^r_r > \right] \right\}
\] (33)

where \( F(w) = \rho - nK^{-1} - \frac{w}{2} (1 - w)^{-1} \mu(w) \). The stress tensor enters here because we have employed the semiclassical equations of motion (see the Appendix) to eliminate the derivatives of the metric functions. Attention should be brought to the pole at \( w = \frac{2}{3} \) (\( r = 3M \)), as we might expect this to lead to singularities in some thermodynamical quantities at the circular photon orbit. A similar pole was found in the expressions for the mass perturbations (11). In fact, the specific heat at constant cavity wall area \( C_A = -\beta^2 \left( \frac{\partial E}{\partial \beta} \right)_A \) indeed diverges at \( r = 3M \), a result which was already known for a vacuum (\( \epsilon = 0 \)) black hole \(^3\). The pole in \( C_A \) can be understood as arising from the singularity in (33).

Alternatively, we can make use of the fact that the thermal energy (29) is identical to the quasilocal energy, as demonstrated in \(^{14}\) for gravitational systems possessing arbitrary static and spherically symmetric metrics. For such metrics, the quasilocal energy is given by \(^{13}\)

\[
E = r - r [ g^{rr}(r) ]^{1/2}.
\] (34)
For the case at hand, we have

$$E = E^{(0)} + \epsilon \Delta E,$$

with

$$E^{(0)} = r - r(1 - w)^{\frac{1}{2}}, \text{ and }$$

$$\Delta E = \frac{rw}{2}(1 - w)^{-\frac{1}{2}}\mu(w),$$

where $E^{(0)}$ is the quasilocal energy corresponding to a vacuum Schwarzschild black hole and $\Delta E$ the correction due to the back reaction.

To get a feeling for the nature of the correction term, we plot the (scaled) ratios $K \frac{\Delta E}{E^{(0)}}$ in Fig. 4 ($K = 3840\pi$ is a constant appearing in the metric functions $\rho$ and $\mu$). The necessity of employing a finite spatial boundary becomes vividly apparent upon examining the large-$r$ limit of these ratios. Inspection of the functions $\mu(w)$ in A(53,55,57) shows that $\mu \rightarrow r^3$, whereas $E^{(0)} \rightarrow M$. Thus, for any fixed value of $\epsilon < 1$, the contribution to the thermal energy coming from the back reaction eventually dominates if $r$ is unbounded. This translates into large energy fluctuations and the consequent instability of the system. The relative corrections for the fermion and conformal scalar are plotted in Fig. 4. While the fermion contribution is positive definite, the conformal scalar perturbation exhibits a minimum at roughly $r \approx 2.44M$ and passes through zero at $r \approx 3M$. This perturbation is negative from $2M < r < 3M$ and is due to the fact that the effective mass function $\mu(w)$ is negative in the same interval. The renormalized stress tensors typically violate all the classical energy conditions and the violation of the weak energy condition is what is responsible for the negative energy correction seen here. Similar behavior is exhibited by the gauge boson, as shown in Fig. 5. There, the effect is much larger than in the two former cases. (All these corrections have been scaled by the large constant $K$ for improved visibility.)
In a similar fashion, the entropy (31) of the combined system of black hole plus radiation can be split into two contributions:

\[ S = S^{(0)} + \epsilon \Delta S, \]  

(37)

where

\[ S^{(0)} = 4\pi M^2 \]  

(38)

is the Bekenstein-Hawking entropy of a Schwarzschild black hole and the correction term is

\[ \Delta S = 4\pi M^2 \left( \rho(w) - nK^{-1} + \mu(w) + \frac{32\pi M^2}{\epsilon w^3} < T^\nu_\nu > - \frac{128\pi CM^2}{3K} (1-w^2) \right). \]  

(39)

The (scaled) fractional correction \( K(\frac{\Delta S}{S^{(0)}}) \) is plotted in Fig. 6 for the scalar and fermion and in Fig. 7 for the gauge boson back reactions. Using the explicit forms for the functions appearing in (39) and the indicated component of the stress tensors which are readily available from [10, 11, 12], we have

\[ K(\frac{\Delta S}{S^{(0)}}) = \frac{2}{3} w^{-3} + 2w^{-2} + 6w^{-1} - 8 \log(w) - 10w - 6w^2 + 22w^3 - \frac{44}{3}, \]  

(40)

for the conformal scalar field,

\[ K(\frac{\Delta S}{S^{(0)}}) = \frac{7}{8} \left( \frac{4}{3} w^{-3} + 4w^{-2} + 12w^{-1} - 16 \log(w) - \frac{180}{7} w - \frac{124}{7} w^2 + \frac{92}{3} w^3 - \frac{32}{7} \right), \]  

(41)

for the massless spin-1/2 fermion, and

\[ K(\frac{\Delta S}{S^{(0)}}) = \frac{4}{3} w^{-3} + 4w^{-2} + 12w^{-1} - 16 \log(w) + 420w - 52w^2 + \frac{332}{3} w^3 - 496, \]  

(42)

for the \( U(1) \) gauge boson. These corrections have the desirable feature that \( \Delta S = 0 \) at \( w = 1 \) (\( r = 2M \)). This means one can think of adding layer upon layer of entropy, associated with the hole and a given \( < T^\mu_\mu > \) beginning at
the horizon $r = 2M$ and ending at the cavity wall $r = r_o$. At $w = 1$, with no “room” for the fields to contribute anything further, one then obtains only the Bekenstein-Hawking entropy $\frac{1}{4}A_H h^{-1}$, as would be expected. Thus, one must regard $\Delta S$ as arising from both the radiation fields and their effects on the gravitational field. These corrections are also of the same order as the naive flat space entropy. Again, as is by now familiar, we come across an apparent spin dependence in the correction terms. While the fermion contribution is positive definite, the fractional entropy corrections coming from the conformal scalar and U(1) gauge boson are negative in the ranges $2M \leq r \lesssim 3M$ and $2M \leq r \lesssim 5.5M$, respectively. It should be emphasized, of course, that the total net system entropy is positive definite for all $r \geq 2M$. Nevertheless, it is unexpected that the presence of spin-0 and spin-1 fields should tend to diminish the entropy in the neighborhood of the horizon. We must ascribe this phenomenon to the type of boundary conditions employed. When microcanonical (fixed energy) boundary conditions are used, these fractional corrections are positive definite in all three spin cases using the same set of back reaction solutions [2].

4 Black Hole Nucleation

The free energy in (30) can be used to determine the likelihood of the nucleation of semi-classical black holes from hot flat space. Since the free energy is least when the system is in a state of thermodynamical equilibrium, the idea is to evaluate $F$ for different phases and then identify the phase with the minimum value for $F$. Since $I = \beta F$ and $\beta > 0$, we can also search for minima in the action. Hot flat space is defined as massless quantum fields on a flat background geometry. Given the flat space radiation entropy $S_{HFS} = \frac{4}{3}aT^4V$ and the thermal energy $E_{HFS} = aT^4V$, where $a = \pi^2/15h^3$,
the free energy for hot flat space is
\[ F_{HFS} = E_{HFS} - TS_{HFS} = -\frac{a}{3}T^4V < 0 \] (43)
and is negative. The corresponding action is
\[ I_{HFS} = \beta F_{HFS} = -\frac{a}{3}\beta^{-3}V. \] (44)
This is to be compared with the action of a semi-classical black hole, Eq.(28).
This is easiest to do for the high temperature limit where the mass perturbations for the \( M_2 \) branch are unimportant (see, e.g., Fig. 2). In this regime, we can approximate \( M_2 \) as
\[ M_2 \approx \frac{r}{2}[1 - \left(\frac{\beta}{4\pi r}\right)^2], \] (45)
to obtain an estimate for \( I(M_2) \):
\[ I(M_2) \approx \beta r - \pi r^2 - \frac{\beta^2}{8\pi} + \hbar \frac{b\pi}{K} \left(\frac{\beta}{4\pi r}\right)^2, \] (46)
with \( b = 96, -32 \) or 2432 for the conformal scalar, spinor or gauge boson, respectively. It is clear that for \( T \to \infty \), the action for hot flat space can be made arbitrarily negative (and therefore its free energy) so hot flat space should be the dominant phase at high temperature. This conclusion holds as well in the limit of zero back reaction [4], but the point here is to determine whether the back reaction tends to enlarge or diminish this phase. In fact, relative to the vacuum black hole action (or free energy), we see that the scalar and U(1) back reactions tend to extend (in temperature) the HFS phase, as these correction terms are positive, while the fermion tends to diminish this phase. For the low temperature limit, \( \beta \to \infty \) and \( I_{HFS} \to 0 \). However, in this regime (\( \alpha \to \pi \) in Figures 2 and 3) the mass perturbations become important and a higher-order (or non-perturbative) treatment of \( F \) is probably required to settle the issue.
Independently from the temperature dependence, we can explore the $r$-dependence of the action. For the vacuum black hole, the action is negative from $2 \leq r/M \lesssim 2.25$, as can be seen from Fig. 8. This implies $F < 0$ in the same interval. Turning on the back reaction leads to corrections which are presented in Fig. 9 for the massless spinor and conformal scalar (scaled up by the factor $K$). The fermion appears to make the action (and hence the free energy) more positive relative to the vacuum case, but the scalar induces a localized negative component to the action in the range $2 \leq r/M \lesssim 3$. The U(1) gauge boson component is even more striking (Fig. 10), exhibiting a substantial negative correction (roughly 60 times deeper than the scalar) which extends out to $r/M \approx 30$. Qualitatively then, we conclude that the lowest order back reaction makes the free energy more negative than the vacuum case, and thereby increases the possibility for nucleation.

5 Discussion

Lowest order solutions of the semi-classical back reaction problem have been used in this paper to calculate the $O(\hbar)$ contribution to the black hole’s mass, entropy, thermal energy and free energy in the canonical ensemble. We have calculated the separate contributions coming from spin 0, $\frac{1}{2}$ and 1 quantum fields as there are important qualitative distinctions among the different spins. Already at lowest order we find evidence for competing effects between the half-integer and integer spin field back-reactions in the spatial region $2M \leq r \lesssim 3M$. This is important, because this is also the same interval in which the specific heat $C_A$ is positive definite and where, consequently, the semi-classical black hole is (locally) stable. The algebraic sign differences between the fermion and scalar and gauge boson fractional corrections become all the more important if the effect of multiple species...
are taken into account, such as would arise when coupling the black hole to multiplets associated with gauge theories of particle physics. For example, the standard model contains 45 fermions, 12 gauge fields and 4 scalars, whereas the minimal SU(5) grand unified model contains 45 fermions, 24 gauge fields and 34 scalar particles. These integers can be used to estimate the back reaction induced by multiplet fields, as discussed in [3, 4].

The canonical boundary conditions employed here require a static boundary held at a fixed temperature. This situation could be dispensed with by adopting conditions of fixed pressure and temperature. In this case, the appropriate thermodynamic potential is the Gibbs potential \( G = F - pA \). This function can be obtained from an appropriate action, as shown in [14], by adding the boundary term \( pA \) to the canonical action. To make the system dynamic, a time-dependent ambient temperature can be introduced by embedding the black hole plus radiation into a Friedmann-Robertson-Walker cosmology. The external heat reservoir is thus realized in a natural way. The boundary separating the black hole and radiation from the cosmological background becomes dynamic, behaving as a domain wall. The embedding could be carried out using the techniques of spacetime surgery. At a given time (or equivalently, a given ambient temperature), one excises a spherical region from the FRW background and in its place, grafts in one of the modified black hole metrics (16) having the same instantaneous radius as the excised region and temperature as the FRW spacetime. The two spacetimes are joined using the standard junction conditions. If the background expansion is not too rapid, one might argue that the system is in quasithermal equilibrium with the ambient spacetime.

Lastly, we comment on the semiclassical back reaction program and its possible relation to a (correct) quantum gravity. The solutions of (1) employed here contain the \( O(h) \) corrections to the classical background
Schwarzschild spacetime. One can also ask for the influence of this metric on the ambient fields, that is, by computing the radiation stress tensor over the new, perturbed, metric (16). This source would now contain terms of $O(h^2)$. This source could then be inserted into the right hand side of (1) and one could solve for the "back-back-reaction", and so on. In other words, (1) is to be solved self-consistently for the equilibrium metric. However, while the lowest order solution of (1) is expected to reveal qualitatively reliable information, carrying out this iterative program to higher orders using only (1) may run the risk of bypassing essential features of a putative theory of quantum gravity. This concern is suggested by the fact that the semiclassical field equation (1) is derivable from an variational principle, which includes the renormalized quantum stress tensor as part of a dynamical theory involving gravity [15]. In fact, the variation of that (effective) action leads to the equation

\[ G_{\mu\nu} + \Lambda g_{\mu\nu} + a H^{(1)}_{\mu\nu} + b H^{(2)}_{\mu\nu} = 8\pi G < T_{\mu\nu} >, \]  

(47)

where the tensors $H^{(1)}_{\mu\nu}, H^{(2)}_{\mu\nu}$ are linear combinations of quadratic curvature terms arising in the renormalization of the stress tensor. The (renormalized) constants $\Lambda, a, b$ and $G$ can only be determined by experiment. In particular, in order to avoid conflict with observation, it is necessary to assume both $a$ and $b$ are very small numerically. To recover Einstein’s theory, it is necessary to set them identically to zero. The point to be emphasized here is that quantum field theory indicates that higher derivative terms are to be expected a priori and treatments going beyond lowest order need to address this issue.

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A Appendix

As shown in [1], the (Euclidean) metric of the perturbed black hole can be written as

\[ ds^2 = g_{tt}(w) dt^2 + g_{rr}(w) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \]

where

\[ g_{tt}(w) = (1 - w)[1 + \epsilon \{2(\rho(w) - nK^{-1}) - w(1 - w)^{-1}\mu(w)\}], \]

\[ g_{rr}(w) = (1 - w)^{-1}[1 + \epsilon w(1 - w)^{-1}\mu(w)], \]

and \( K = 3840\pi \). The two functions \( \mu \) and \( \rho \) are solutions of the linearized semiclassical Einstein equations

\[ \epsilon \frac{d\rho}{dw} = -\frac{16\pi M^2}{w^3} (1 - w)^{-1} < T^r_r - T^t_t >, \]

\[ \epsilon \frac{d\mu}{dw} = \frac{32\pi M^2}{w^4} < T^t_t >. \]

These as well as the constant \( n \) and the stress tensors actually depend on the spin of the quantum field interacting with the hole. Denoting with the subscripts \( S, f, V \) the conformal scalar, massless spinor and vector boson back reactions, respectively, these functions are given by

\[ K\mu_S = \frac{1}{2}(\frac{2}{3}w^{-3} + 2w^{-2} + 6w^{-1} - 8 \log(w) - 10w - 6w^2 + 22w^3 - \frac{44}{3}), \]

\[ K\rho_S = \frac{1}{2}(\frac{2}{3}w^{-2} + 4w^{-1} - 8 \log(w) - \frac{40}{3}w - 10w^2 - \frac{28}{3}w^3 + \frac{84}{3}), \]

\[ K\mu_f = \frac{7}{8}(\frac{2}{3}w^{-3} + 2w^{-2} + 6w^{-1} - 8 \log(w) - \frac{90}{7}w - \frac{62}{7}w^2 + \frac{46}{3}w^3 - \frac{16}{7}), \]

\[ K\rho_f = \frac{7}{8}(\frac{2}{3}w^{-2} + 4w^{-1} - 8 \log(w) - \frac{200}{21}w - \frac{50}{7}w^2 - \frac{52}{7}w^3 + \frac{136}{7}). \]
\[ K_{\mu \nu} = \frac{2}{3} w^{-3} + 2w^{-2} + 6w^{-1} - 8 \log(w) + 210w - 26w^2 + \frac{166}{3} w^3 - 248, \quad (57) \]
\[ K_{\rho \nu} = \frac{2}{3} w^{-2} + 4w^{-1} - 8 \log(w) + \frac{40}{3} w + 10w^2 + 4w^3 - 32, \quad (58) \]

while \( n_s = 12, \ n_f = -4 \) and \( n_V = 304 \)

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Figure Captions

Figure 1. The double-valued black hole mass in the canonical ensemble plotted as a function of $\alpha$.

Figure 2. Mass perturbations $\Delta w$ for the (heavy) $M_2$ branch arising from the spinor, conformal scalar and U(1) gauge field back-reaction.

Figure 3. Mass perturbations $\Delta w$ for the (light) $M_1$ branch arising from the spinor, conformal scalar and U(1) gauge field back-reaction.

Figure 4. Black hole fractional thermal energy corrections (scaled up by $K = 3840\pi$) arising from the fermion and conformal scalar back reaction.

Figure 5. Black hole fractional thermal energy corrections (scaled up by $K = 3840\pi$) arising from the gauge field back reaction.

Figure 6. Fractional entropy corrections (scaled up by $K$): massless spinor and conformal scalar cases.

Figure 7. Fractional entropy correction due to the gauge boson.

Figure 8. Action (19) for the vacuum Schwarzschild black hole.

Figure 9. Back reaction contribution to the black hole action coming from the fermion and conformal scalar (scaled by $K$).

Figure 10. Back reaction contribution to the black hole action coming from the U(1) gauge boson (scaled by $K$).
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