ON LONG TIME BEHAVIOR OF SOLUTIONS OF THE
SCHRÖDINGER-KORTEWEG-DE VRIES SYSTEM.

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ABSTRACT. We are concerned with the decay of long time solutions of the
initial value problem associated to the Schrödinger-Korteweg-de Vries system.
We use recent techniques in order to show that solutions of this system decay
to zero in the energy space. The result is independent of the integrability of
the equations involved and it does not require any size assumptions.

1. INTRODUCTION

We consider the Schrödinger-Korteweg-de Vries system,

\[
\begin{aligned}
&i \partial_t u + \partial^2_x u = \alpha uv + \beta |u|^2, \quad x, t \in \mathbb{R}, \\
&\partial_t v + \partial^3_x v + v \partial_x v = \gamma \partial_x \left( |u|^2 \right), \\
&u(x, 0) = u_0(x), v(x, 0) = v_0(x),
\end{aligned}
\]

where \( u = u(x, t) \) is a complex-valued function, \( v = v(x, t) \) is a real-valued function
and \( \alpha, \beta, \gamma \) are real constants.

The system in (1.1) appears as a particular case (under appropriate transforma-
tions) of the more general system

\[
\begin{aligned}
&i \partial_t S + i c_S \partial_x S + \partial^2_x S = \alpha S L + \gamma |S|^2 S, \\
&\partial_t L + c_L \partial_x L + \nu P(D_x)L + \lambda \partial_x L^2 = \beta \partial_x |S|^2,
\end{aligned}
\]

where \( S \) is a complex-valued function representing the short wave, \( L \) is a real-valued function representing the long wave and \( P(D_x) \) is a differential operator with constant coefficients and \( c_S, c_L, \nu, \lambda, \beta, \alpha, \gamma \) are real parameters, see [3, 23] and references therein. This system has received considerable attention because of the
broad diversity of physical settings in which it arises. For instance, as modelling
the internal gravity-wave packet and the capillary-gravity interaction wave see [11, 
13, 15]). Furthermore, when \( \gamma = 0 \) the previous system has been derived as a model
for the resonant ion-sound/Langmuir wave interaction in plasma physics under the
assumption that the ion-sound wave is unidirectional (see [25, 31]). Moreover,
this system appears in the general theory of water wave interaction in a nonlinear
medium. Finally, this system also occurs as a model for the motion of two fluids
under capillary-gravity waves in a deep water flow (see [12]) or the motion of two
fluids under a shallow water flow (see [12]).

Key words and phrases. Asymptotic behavior, decay of solutions, Schrödinger, Korteweg-de Vries.
The Schrödinger-Korteweg-de Vries system (1.1) has been shown not to be a completely integrable system (see [5]). Even though their solutions satisfy the following conserved quantities,

\[ I_1[t] := \int_{\mathbb{R}} |u|^2 \, dx = I_1[0], \]
\[ I_2[t] := \int_{\mathbb{R}} \left\{ \alpha \gamma v |u|^3 - \frac{\alpha}{6} v^3 + \frac{\beta \gamma}{2} |u|^4 + \frac{\alpha}{2} |\partial_x u|^2 + \gamma |\partial_x u|^2 \right\} \, dx = I_2[0], \] (1.2)
\[ I_3[t] := \int_{\mathbb{R}} \{ \alpha v^2 + 2 \gamma \text{Im} (\bar{u} \partial_x u) \} \, dx = I_3[0]. \]

The IVP (1.1) has been extensively studied from the view point of local and global well-posedness. This is mainly because of the wide research developed for the famous Korteweg-de Vries (KdV) equation

\[ v_t + v_{xxx} + vv_x = 0 \]

and the cubic Schrödinger (NLS) equation

\[ iu_t + u_{xx} + |u|^2 u = 0, \]

for optimal results see [17], [19], [37]. For further references see [6], [24], [34], and [35]. This has served of source of inspiration for several authors to study the Schrödinger-Korteweg-de Vries system (1.1) and also for the coupled system (1.2) = (0, −3) was treated in [14] by Z. Guo and Y. Wang. In [38] Wu extended the local results in [7] for \( \beta = 0 \) in (1.1).

Regarding global well-posedness, the conserved quantities were used in [36] to extend the local theory globally for initial data in \( H^{s+\frac{3}{5}}(\mathbb{R}) \times H^s(\mathbb{R}) \) for \( s \in \mathbb{Z}^+. \) Global well-posedness in the energy space \( H^1(\mathbb{R}) \times H^1(\mathbb{R}) \) was established in [7]. Pecher in [32] proved global results in \( H^s(\mathbb{R}) \times H^s(\mathbb{R}), s > 3/5, \) for \( \beta = 0 \) and \( s > 2/3 \) for \( \beta \neq 0 \) by using the I-method. Finally, Wu ([38]) shows global well-posedness in \( H^s(\mathbb{R}) \times H^s(\mathbb{R}), s > 1/2, \) for any \( \beta \in \mathbb{R}. \)

We also notice that local/global well-posedness results in weighted Sobolev spaces are also known for this system ([23]).

For further reference we present the local and global well-posedness theory obtained in [7]

**Theorem A (7).** Let \( k \geq 0 \) and \( s > -\frac{3}{4}. \) Then for any \((u_0, v_0) \in H^k(\mathbb{R}) \times H^s(\mathbb{R})\) provided:

i) \( k - 1 \leq s \leq 2k - \frac{1}{2} \) for \( k \in [0, \frac{1}{2}]. \)
(ii) \( k - 1 \leq s < k + \frac{1}{2} \) for \( k \in (\frac{1}{2}, \infty) \),

there exist a positive time \( T = T(\|u_0\|_{H^k}, \|v_0\|_{H^s}) \) and a unique solution \((u(t), v(t))\) of the IVP (1.1), satisfying

\[
\varphi_T(t)u \in X^{k,\frac{1}{2}+} \quad \text{and} \quad \varphi_T(t)v \in Y^{k,\frac{1}{2}+},
\]

\[
u \in C([0, T]; H^k(\mathbb{R})) \quad \text{and} \quad v \in C([0, T]; H^s(\mathbb{R})).
\]

Moreover, the map \((u_0, v_0) \mapsto (u(t), v(t))\) is locally Lipschitz from \(H^k(\mathbb{R}) \times H^s(\mathbb{R})\) into \(C([0, T]; H^k(\mathbb{R}) \times H^s(\mathbb{R}))\).

The spaces \(X^{k,\frac{1}{2}+}\) and \(Y^{k,\frac{1}{2}+}\) will be defined below.

Regarding the global theory we have.

**Theorem B** ([7]). Let \(\alpha, \beta, \gamma \in \mathbb{R}\) such that \(\alpha \cdot \gamma > 0\) and \((u_0, v_0) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})\). Then, the unique solution provided by Theorem A can be extended for any time \(T > 0\). Moreover,

\[
\sup_t \left( \|u(t)\|_{H^1} + \|v(t)\|_{H^1} \right) \leq \Psi(\|u_0\|_{H^1}, \|v_0\|_{H^1}),
\]

where \(\Psi\) is a function only depending on \(\|u_0\|_{H^1}\) and \(\|v_0\|_{H^1}\).

Next, we comment on solitary wave solutions or ground states for system (1.1). Solitary wave solutions of the form

\[
(u(x, t), v(x, t)) = (e^{ibt} e^{i(x-ct)} \phi(x-ct), \psi(x-ct)),
\]

have been found for the system (1.1). In [1] the authors consider the system with \(\beta = 0\). They show the existence of solitary wave solutions and prove the orbital stability of these solution for the parameters \(\alpha, \gamma < 0\). The existence of ground states for the system with \(\beta \neq 0\) was established in [9] again for the parameters \(\alpha, \gamma < 0\). The stability of these ground states was proved in [2], (see also [8], [10]).

In particular, for a suitable parameter \(c^*\), \(\alpha \in (-\frac{1}{6}, 0)\) and \(c = 4c^* - \frac{1}{6}\alpha(1 + 6\alpha)\), the authors in [9] found the following explicit solution for the system (1.1),

\[
\phi(x) = \frac{\sqrt{2c^*(1 + 6\alpha)}}{\cosh(\sqrt{c^*} x)}, \quad \text{and} \quad \psi(x) = \frac{12c^*}{\cosh^2(\sqrt{c^*} x)}.
\]

In this work, we are interested in the asymptotic behaviour of solutions to the system (1.1). As far as we know there are not results regarding this issue for the global solutions obtained in [7] for this system. We will show decay properties for solutions of the system (1.1) as time evolves.

Our work is inspired by recent results obtained for several dispersive equations and systems of dispersive equations and the techniques implemented to establish those results. We shall mention the works of Muñoz and Ponce for the generalised KdV equation ([28]) and the Benjamin-Ono (BO) equation ([29]). Where the authors show the decay of the solutions to these equations. In particular, their results rule out the possibility to have breathers solutions. The methods used were motivated by the works of Kowalczyk, Martel and Muñoz [20]-[21]. In [22], Linares, Mendez and Ponce used the approach developed in [29] to study the the decay of solutions to the dispersion generalized BO equation. New techniques were introduced by Muñoz, Ponce and Saut [30] to investigate the long time behaviour issue for
solutions to the Intermediate Long Wave (IWL) equation. Extensions to higher dimensional model as the Zakharov-Kusnetsov (ZK) equation were made by Mendez, Muñoz, Poblete and Pozo in [27].

Martinez in [26] considered the 1-dimensional Zakharov system

\[
\begin{cases}
  iu_t + u_{xx} = nu, & x, t \in \mathbb{R}, \\
  n_{tt} - n_{xx} = (|u|^2)_{xx}.
\end{cases}
\]  

(1.7)

Under some smallness assumptions it is proved that in compact sets the solutions of (1.7) decay to zero as time tends to infinity. In addition, the author established decay in far field regions along curves.

2. Main Results

Our first result regards the decay of global solutions in \( H^1(\mathbb{R}) \times H^1(\mathbb{R}) \) in the \( L^2 \)-norm. More precisely,

**Theorem 2.1.** Let \( \alpha, \gamma < 0 \). Suppose that \((u_0, v_0) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})\). Then, the corresponding solution \((u, v)\) to (1.1) with initial condition \((u_0, v_0)\) satisfies

\[
\liminf_{t \uparrow \infty} \int_{\Omega_{p_1}(t)} v^2(t, x) \, dx = \liminf_{t \uparrow \infty} \int_{\Omega_{p_1}(t)} |u|^2(t, x) \, dx = 0,
\]

where

\[
\Omega_{p_1}(t) = \{ x \in \mathbb{R} \mid |x| \lesssim t^{p_1} \} \text{ with } 0 < p_1 < \frac{2}{3}.
\]

As a corollary of Theorem 2.1 we obtain the following result in the non-centered case.

**Corollary 2.2.** Let \( \alpha, \gamma \) be real numbers such that \( \alpha, \gamma < 0 \). Suppose that \((u_0, v_0) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})\). Then, the corresponding solution \((u, v)\) to (1.1) with initial condition \((u_0, v_0)\) satisfies

\[
\liminf_{t \uparrow \infty} \int_{\Gamma_{p_1}(t)} (v(t, x))^2 \, dx = \liminf_{t \uparrow \infty} \int_{\Gamma_{p_1}(t)} |u(t, x)|^2 \, dx = 0,
\]

where

\[
\Gamma_{p_1}(t) = \{ x \in \mathbb{R} \mid |x - t^m| \lesssim t^{p_1} \}, \quad 0 < p_1 < \frac{2}{3} \text{ and } 0 < m < 1 - \frac{p_1}{2}.
\]

The second main result in this work tells us about the decay of the first derivatives in the \( L^2 \)-norm of solutions in \( H^1(\mathbb{R}) \times H^1(\mathbb{R}) \) of the IVP (1.1).

**Theorem 2.3.** Let \( \alpha, \gamma \) be real numbers such that \( \alpha, \gamma < 0 \). Suppose that \((u_0, v_0) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})\). Then, the corresponding solution \((u, v)\) to (1.1) with initial condition \((u_0, v_0)\) satisfies:

(i) If \( \beta > 0 \), then

\[
\liminf_{t \uparrow \infty} \int_{\Omega_{p_1}(t)} (\partial_x v(t, x))^2 \, dx = \liminf_{t \uparrow \infty} \int_{\Omega_{p_1}(t)} |\partial_x u(t, x)|^2 \, dx = 0,
\]

and

\[
\liminf_{t \uparrow \infty} \int_{\Omega_{p_1}(t)} |u(t, x)|^4 \, dx = 0,
\]

where

\[
\Omega_{p_1}(t) = \{ x \in \mathbb{R} \mid |x| \lesssim t^{p_1} \} \text{ with } 0 < p_1 < \frac{2}{3}.
\]
(ii) If $\beta \leq 0$, then
\[
\liminf_{t \uparrow \infty} \int_{\Omega_{p_1}(t)} (\partial_x v(t, x))^2 \, dx = \liminf_{t \uparrow \infty} \int_{\Omega_{p_1}(t)} |\partial_x u(t, x)|^2 \, dx = 0,
\]
where $\Omega_{p_1}(t)$ is as in (2.4).

Some remarks are in order.

Remark 2.1. Combining the results in Theorem 2.1 and Theorem 2.3 we guarantee the decay in the energy space of solutions of the IVP (1.1) for the parameters $\alpha, \gamma < 0$.

Remark 2.2. The approach we follows is closer to the one introduced in [27]. This method shows to be independent of the integrability of the equation and does not need size restriction.

Remark 2.3. The results above rule out the possibility to have small breather solutions.

Remark 2.4. The idea of the proof of Corollary 2.2 follows the argument used in [27]. Thus, it is enough to define the functional
\[
\mathcal{J}(t) := \frac{1}{\eta(t)} \int_{\mathbb{R}} u(t, x) \varphi_a \left( \frac{x - t^m}{\lambda_1(t)} \right) \phi_b \left( \frac{x - t^m}{\lambda_2(t)} \right) \, dx
\]
and then proceed with the analysis described in the proof of Theorem 2.1.

Remark 2.5. The case when the parameters $\alpha$ and $\gamma$ are both positive is open. More precisely, following the notation introduced in the proof of Theorem 2.1 (see Section 3) we obtain by means of energy estimates the bound
\[
\int_{\{t \geq 1\}} \frac{1}{\eta(t)\lambda_1(t)} \int_{\mathbb{R}} \left( \frac{v^2}{4} - \gamma |u|^2 \right) \varphi_a \left( \frac{x}{\lambda_1(t)} \right) \phi_b \left( \frac{x}{\lambda_2(t)} \right) \, dx < \infty, \tag{2.5}
\]
where $a, b > 0$. However, since there is not a defined sign in the inequality (2.5) we cannot decide if there is some decay of the $L^2$-mass norms of $u, v$.

We observe from the comments regarding solitary waves that in this case the existence of such solutions is unknown.

The remainder of this paper is structured as follows, Section 2 will be devoted to prove Theorem 2.1 and Section 3 will present the proof of Theorem 2.3. Before leaving this section we will set the notation we employ in this manuscript.

Notation. For $k, s \in \mathbb{R}$ and $b \in (0, 1)$ we let $X^{k,s}$ and $Y^{s,b}$ be the completion of $S(\mathbb{R}^2)$ with respect to the norms
\[
\|f\|_{X^{k,s}} = \left( \int \int |\xi|^{2k} |\tau + |\xi|^2|^{2b} |\hat{f}(\xi, \tau)|^2 \, d\tau d\xi \right)^{1/2}
\]
and
\[
\|f\|_{Y^{s,b}} = \left( \int \int |\xi|^{2s} |\tau - |\xi|^2|^{2b} |\hat{f}(\xi, \tau)|^2 \, d\tau d\xi \right)^{1/2}
\]
where $\langle \cdot \rangle = (1 + |\xi|^2)^{1/2}$ and $\hat{f}(\xi, \tau)$ denotes the Fourier transform in the $x, t$ variables.
3. Proof of Theorem 2.1

Proof of Theorem 2.1. We follow the argument of proof presented in [27]. Then, we will consider the following weighted functions:

\[ \varphi(x) = \frac{2}{\pi} \arctan(e^x), \quad x \in \mathbb{R}, \]  

(3.1) and

\[ \varphi'(x) = \frac{1}{\pi \cosh(x)} \sim e^{-|x|}, \quad |\varphi''(x)| \lesssim \varphi'(x), \quad x \in \mathbb{R}. \]  

(3.2)

For \( a > 0 \), we define \( \varphi_a(x) := a \varphi \left( \frac{x}{a} \right) \) and \( \phi_a(x) := \varphi' \left( \frac{x}{a} \right) \).

For \( p_1, r_1, r_2, q_1, q_2 > 0 \) we set

\[ \lambda_1(t) = \frac{t^{p_1}}{\ln^{q_1} t}, \quad \lambda_2(t) = (\lambda_1(t))^{p_2} \quad \text{and} \quad \eta(t) = t^{r_1} \ln^r t, \]  

(3.3)

whence

\[ p_1 + r_1 = 1, \quad p_2 > 1 \quad \text{and} \quad r_2 = 1 + q_1. \]

Then

\[ \frac{\lambda_1(t)}{\lambda_1(t)} \sim \frac{1}{t}, \quad \frac{\lambda_2(t)}{\lambda_2(t)} \sim \frac{1}{t} \quad \text{and} \quad \frac{\eta'(t)}{\eta(t)} \sim \frac{1}{t} \quad \text{for} \quad t \gg 1. \]

For \( v = v(t, x) \) solution to (1.1) we define the functional

\[ J(t) := \frac{1}{\eta(t)} \int_{\mathbb{R}} v(t, x) \varphi_a \left( x \frac{x}{\lambda_1(t)} \right) \phi_b \left( x \frac{x}{\lambda_2(t)} \right) dx, \]  

(3.4)

where \( a, b > 0 \).

We claim that \( J \) is well defined. In this sense, we have by Hölder’s inequality

\[ |J(t)| \leq \frac{(\lambda_2(t))^{1/2} \|v(t)\|_2}{\eta(t)} \left\| \varphi_a \left( \frac{x}{\lambda_1(t)} \right) \right\|_\infty. \]

Thus, by (1.2)

\[ |J(t)| \lesssim \frac{(\lambda_2(t))^{1/2} \|v(t)\|_2}{\eta(t)} \]  

\[ \lesssim \frac{(\lambda_2(t))^{1/2} \|v(t)\|_2}{\eta(t)} \left( \frac{I_3[0]}{|a|} + \frac{2\gamma}{\alpha} \|u_0\|_2 \|\partial_x u(t)\|_2 \right). \]

Next, from Theorem 1 we get

\[ \sup_{t \gg 1} |J(t)| < \infty, \]

whenever

\[ 0 < p_1 \leq \frac{2}{p_2 + 2}. \]  

(3.5)

In the following we will suppress the dependence on the variables in order to simplify the notation unless it would be necessary.
Next, let
\[
\frac{d}{dt} J(t) = \frac{1}{\eta(t)} \int_{\mathbb{R}} \partial_t v \phi_a \left( \frac{x}{\lambda_1(t)} \right) \phi_b \left( \frac{x}{\lambda_2(t)} \right) dx
\]
\[- \frac{\lambda'_1(t)}{\eta(t) \lambda_1(t)} \int_{\mathbb{R}} v \phi_a \left( \frac{x}{\lambda_1(t)} \right) \phi'_b \left( \frac{x}{\lambda_2(t)} \right) dx
\]
\[- \frac{\lambda'_2(t)}{\eta(t) \lambda_2(t)} \int_{\mathbb{R}} v \phi_a \left( \frac{x}{\lambda_1(t)} \right) \phi'_b \left( \frac{x}{\lambda_2(t)} \right) dx
\]
\[- \frac{\eta'(t)}{\eta^2(t)} \int_{\mathbb{R}} v \phi_a \left( \frac{x}{\lambda_1(t)} \right) \phi_b \left( \frac{x}{\lambda_2(t)} \right) dx.
\]
(3.6)

For \( A_1 \) we combine (1.1) and integration by parts whence we obtain
\[
A_1(t) = - \frac{\gamma}{\lambda_1(t) \eta(t)} \int_{\mathbb{R}} |u|^2 \phi'_a \left( \frac{x}{\lambda_1(t)} \right) \phi_b \left( \frac{x}{\lambda_2(t)} \right) dx
\]
\[- \frac{\gamma}{\eta(t) \lambda_2(t)} \int_{\mathbb{R}} |u|^2 \phi_a \left( \frac{x}{\lambda_1(t)} \right) \phi'_b \left( \frac{x}{\lambda_2(t)} \right) dx
\]
\[- \frac{1}{2 \eta(t) \lambda_1(t)} \int_{\mathbb{R}} u'^2 \phi'_a \left( \frac{x}{\lambda_1(t)} \right) \phi'_b \left( \frac{x}{\lambda_2(t)} \right) dx
\]
\[- \frac{1}{2 \eta(t) \lambda_2(t)} \int_{\mathbb{R}} u'^2 \phi_a \left( \frac{x}{\lambda_1(t)} \right) \phi'_b \left( \frac{x}{\lambda_2(t)} \right) dx
\]
\[+ \frac{3}{\eta(t) \lambda_1^2(t) \lambda_2(t)} \int_{\mathbb{R}} u \phi''_a \left( \frac{x}{\lambda_1(t)} \right) \phi_b \left( \frac{x}{\lambda_2(t)} \right) dx
\]
\[+ \frac{3}{\eta(t) \lambda_1(t) \lambda_2^2(t)} \int_{\mathbb{R}} v \phi'_a \left( \frac{x}{\lambda_1(t)} \right) \phi''_b \left( \frac{x}{\lambda_2(t)} \right) dx
\]
\[+ \frac{1}{\eta(t) \lambda_2^3(t)} \int_{\mathbb{R}} v \phi_a \left( \frac{x}{\lambda_1(t)} \right) \phi'''_b \left( \frac{x}{\lambda_2(t)} \right) dx
\]
(3.7)

Up to a constant, the term \( A_{1,1} \) is the quantity to be estimated after integrating in time. Instead, for \( A_{1,2} \) we have the following bound
\[
|A_{1,2}(t)| \lesssim \gamma \|u_0\|_{L^2} \frac{1}{\lambda_2(t) \eta(t)} \in L^1 \left( \{ t \gg 1 \} \right), \quad (3.8)
\]
since \( p_2 > 1 \).

The term \( A_{1,3} \) corresponds to the contribution coming from the KdV dynamics that we want to estimate after integrating in time. Nevertheless, for \( A_{1,4} \) the situation is quite different and we shall proceed in a different manner.
By using Hölder’s inequality we get
\[ |A_{1,4}(t)| \lesssim_{\gamma, \alpha, \|u_0\|_{L^2}, \|u\|_{L^\infty H^1}} \frac{1}{\eta(t) \lambda_2(t)} \in L^1(\{t \gg 1\}), \tag{3.9} \]
since \( p_2 > 1 \).
Next, we have
\[ |A_{1,5}(t)| \lesssim_{\alpha, \gamma, \|u_0\|_{L^2}, \|u\|_{L^\infty H^1}} \frac{1}{\eta(t) \lambda_1^{5/2}(t)} \in L^1(\{t \gg 1\}), \tag{3.10} \]
since \( p_1 > 0 \).
For \( A_{1,6} \) the we obtain the following bound
\[ |A_{1,6}(t)| \lesssim_{\alpha, \gamma, \|u_0\|_{L^2}, \|u\|_{L^\infty H^1}} \frac{1}{\eta(t) \lambda_1^{3/2}(t) \lambda_2(t)} \in L^1(\{t \gg 1\}) \tag{3.11} \]
whenever \( p_1, p_2 > 0 \).
For \( A_{1,7} \) we have
\[ |A_{1,7}(t)| \lesssim_{\alpha, \gamma, \|u_0\|_{L^2}, \|u\|_{L^\infty H^1}} \frac{1}{\eta(t) \lambda_1^{5/2}(t) \lambda_2^{1/2}(t)} \in L^1(\{t \gg 1\}), \tag{3.12} \]
since \( p_2 > 1 \).
The last term coming from \( A_1 \) corresponds to \( A_{1,8} \) whence we obtain after applying Hölder’s inequality
\[ |A_{1,8}(t)| \lesssim_{\alpha, \gamma, \|u_0\|_{L^2}, \|u\|_{L^\infty H^1}} \frac{1}{\eta(t) \lambda_1^{1/2}(t) \lambda_2^{1/2}(t)} \in L^1(\{t \gg 1\}), \tag{3.13} \]
since \( p_2 > 1 \).
Next, we estimate \( A_2 \). By Young’s inequality we have the following: For \( \epsilon > 0 \),
\[
|A_2(t)| = \left| -\frac{\lambda'_1(t)}{\eta(t) \lambda_1(t)} \int_{\mathbb{R}} u \varphi_a' \left( \frac{x}{\lambda_1(t)} \right) \left( \frac{x}{\lambda_1(t)} \right) \phi_b \left( \frac{x}{\lambda_2(t)} \right) \, dx \right|
\leq \frac{1}{4\epsilon} \left| \frac{\lambda'_1(t)}{\eta(t) \lambda_1(t)} \right| \int_{\mathbb{R}} u^2 \varphi_a' \left( \frac{x}{\lambda_1(t)} \right) \phi_b \left( \frac{x}{\lambda_2(t)} \right) \, dx
+ \epsilon \left| \frac{\lambda'_1(t)}{\eta(t) \lambda_1(t)} \right| \int_{\mathbb{R}} \varphi_a' \left( \frac{x}{\lambda_1(t)} \right) \left| \frac{x}{\lambda_1(t)} \right|^2 \phi_b \left( \frac{x}{\lambda_2(t)} \right) \, dx,
\]
whence after taking \( \epsilon = |\lambda'_1(t)| > 0 \) we obtain
\[ |A_2(t)| \leq \frac{1}{4\eta(t) \lambda_1(t)} \int_{\mathbb{R}} u^2 \varphi_a' \left( \frac{x}{\lambda_1(t)} \right) \phi_b \left( \frac{x}{\lambda_2(t)} \right) \, dx
+ \frac{(\lambda'_1(t))^2}{\eta(t)} \| \phi_b \|_{L^\infty} \| (\cdot)^2 \varphi_a(\cdot) \|_{L^1}.
\]
Note that the first term in the r.h.s is up to constant the quantity to be estimated after integrating in time. Instead, the second term in the r.h.s satisfies
\[ \frac{(\lambda'_1(t))^2}{\eta(t)} \in L^1(\{t \gg 1\}), \]
whenever \( 0 < p_1 < \frac{2}{3} \).
The situation is more delicate when we try to estimate $A_3$, so that we will consider the following auxilar function $\theta(t) = t^{1-p_1}$, then by Young's inequality we obtain
\[
|A_3(t)| = \left| -\frac{\chi_2^\prime(t)}{\eta(t)\lambda_2(t)} \int_{\mathbb{R}} v \varphi_a \left( \frac{x}{\lambda_1(t)} \right) \left( \frac{x}{\chi_2(t)} \right) \phi_b \left( \frac{x}{\lambda_2(t)} \right) \, dx \right| \\
\approx \frac{\chi_2^\prime(t)\theta(t)}{\eta(t)\lambda_2(t)} \int_{\mathbb{R}} v^2 \left| \varphi_a \left( \frac{x}{\chi_1(t)} \right) \right|^2 \, dx \\
+ \left| \frac{\chi_2^\prime(t)}{\eta(t)(\theta(t))\lambda_2(t)} \int_{\mathbb{R}} \left| \frac{x}{\chi_2(t)} \right|^2 \left| \phi_b \left( \frac{x}{\lambda_2(t)} \right) \right|^2 \, dx \right| \\
\lesssim_{\alpha, \gamma, \|u_0\|_2, \|u\|_{L^\infty}} t^{1/2} \left| \frac{\theta(t)}{t\eta(t)\theta(t)} \right| + \left| \frac{\chi_2(t)}{t\eta(t)\theta(t)} \right| \in L^1 (\{t \gg 1\}),
\]
since $0 < p_1 \leq \frac{2}{p_2+2}$ and $r_2 > 1$.

To handle $A_4$ we combine Hölder’s inequality, Theorem 1, (1.2) and (3.5) to obtain
\[
A_4(t) \leq \left| \frac{\eta^\prime(t)(\lambda_2(t))^{1/2}}{\eta^2(t)} \right| \|v(t)\|_2 \|\phi_a\|_2 \|\varphi\|_\infty \\
\lesssim_{\gamma, \alpha, \|u_0\|_2} \left| \frac{\eta^\prime(t)(\lambda_2(t))^{1/2}}{\eta^2(t)} \right| \in L^1 (\{t \gg 1\}),
\]
since $0 < p_1 \leq \frac{2}{p_2+2}$ and $r_2 > 1$.

Finally, we get after integrating in time that
\[
\int \left\{ \begin{array}{l}
\int_{\{t \gg 1\}} \frac{1}{\eta(t)\lambda_1(t)} \int_{\mathbb{R}} v^2 \varphi_a \left( \frac{x}{\chi_1(t)} \right) \phi_b \left( \frac{x}{\lambda_2(t)} \right) \, dx \, dt \\
+ \int_{\{t \gg 1\}} \frac{1}{\eta(t)\lambda_1(t)} \int_{\mathbb{R}} \left| u \right|^2 \varphi_a \left( \frac{x}{\chi_1(t)} \right) \phi_b \left( \frac{x}{\lambda_2(t)} \right) \, dx \, dt < \infty,
\end{array} \right.
\]
for any $a, b > 0$.

Since $\frac{1}{\eta(t)\lambda_1(t)} = \frac{1}{\eta(t)} \notin L^1 (\{t \gg 1\})$ we can guarantee that there exist a sequence of positive time $(t_n)_{n \geq 1}$ with $t_n \uparrow \infty$ as $n$ goes to infinity such that
\[
\lim_{n \to \infty} \int_{\Omega_{p_1}(t_n)} v^2(t_n, x) \, dx = \lim_{n \to \infty} \int_{\Omega_{p_1}(t_n)} \left| u \right|^2(t_n, x) \, dx = 0,
\]
where $\Omega_{p_1}(t)$ is as in (2.1).

\[\square\]

4. PROOF OF THEOREM 2.3

Proof. We start by defining the functional
\[
I(t) = \frac{\theta}{2\eta(t)} \int_{\mathbb{R}} v^2(t, x) \varphi_1 \left( \frac{x}{\lambda_1(t)} \right) \, dx + \frac{\mu}{\eta(t)} \text{Im} \int_{\mathbb{R}} u(t, x) \bar{u}(t, x) \varphi_1 \left( \frac{x}{\lambda_1(t)} \right) \, dx,
\]
where $\theta, \mu, l$ are real constant to be chosen.
We require \( I \) to be well defined. Indeed, the Cauchy-Schwarz inequality yield
\[
\sup_{t \gg 1} |I(t)| \leq \frac{\theta}{2} \|v\|_2^2 \|\varphi_l \left( \frac{x}{\lambda_1(t)} \right)\|_\infty + |\mu| \|u_0\|_2 \|\partial_x u\|_{L^\infty_t L^2_x} \|\varphi_l \left( \frac{x}{\lambda_1(t)} \right)\|_\infty. \tag{4.2}
\]

On the other hand, from (1.2) we get
\[
\|v(t)\|_2^2 \leq \frac{1}{|\alpha|} |I_3[0]| + 2 \frac{|\gamma|}{|\alpha|} \int_{\mathbb{R}} \left| \text{Im} \left( u(t, x) \partial_x u(t, x) \right) \right| dx \leq \frac{1}{|\alpha|} |I_3[0]| + 2 \frac{|\gamma|}{|\alpha|} \|u_0\|_2 \|\partial_x u(t)\|_2, \quad \gamma, \alpha \neq 0. \tag{4.3}
\]

Therefore, combining (4.2) and (4.3) it follows that
\[
\sup_{t \gg 1} |I(t)| < \infty.
\]

Next, combining integration by parts and (1.1) we obtain
\[
\frac{d}{dt} I(t) = \frac{-\theta \gamma}{\eta(t)} \int_{\mathbb{R}} \partial_x v |u|^2 \varphi_l \left( \frac{x}{\lambda_1(t)} \right) dx - \frac{\theta \gamma}{\eta(t) \lambda_1(t)} \int_{\mathbb{R}} v |u|^2 \varphi_l' \left( \frac{x}{\lambda_1(t)} \right) dx
\]
\[
\begin{align*}
+ & \frac{\theta}{3 \eta(t) \lambda_1(t)} \int_{\mathbb{R}} v^3 \varphi_l' \left( \frac{x}{\lambda_1(t)} \right) dx - \frac{3 \theta}{2 \eta(t) \lambda_1(t)} \int_{\mathbb{R}} (\partial_x v)^2 \varphi_l' \left( \frac{x}{\lambda_1(t)} \right) dx \\
+ & \frac{\theta}{2 \eta(t) \lambda_1^2(t)} \int_{\mathbb{R}} v^2 \varphi_l''' \left( \frac{x}{\lambda_1(t)} \right) dx - \frac{\theta \lambda_1'(t)}{2 \eta(t) \lambda_1(t)} \int_{\mathbb{R}} v^2 \varphi_l' \left( \frac{x}{\lambda_1(t)} \right) \left( \frac{x}{\lambda_1(t)} \right) dx \\
- & \frac{\theta \gamma(t)}{2 \eta^2(t)} \int_{\mathbb{R}} v^2 \varphi_l \left( \frac{x}{\lambda_1(t)} \right) dx \frac{\mu^2(t)}{\eta^2(t)} \int_{\mathbb{R}} u \partial_x u \varphi_l \left( \frac{x}{\lambda_1(t)} \right) dx \\
- & \frac{2 \mu}{\eta(t) \lambda_1(t)} \int_{\mathbb{R}} |\partial_x u|^2 \varphi_l' \left( \frac{x}{\lambda_1(t)} \right) dx + \frac{\alpha \mu}{\eta(t)} \int_{\mathbb{R}} \partial_x v |u|^2 \varphi_l \left( \frac{x}{\lambda_1(t)} \right) dx \\
- & \frac{\mu \lambda_1'(t)}{\eta(t) \lambda_1(t)} \int_{\mathbb{R}} u \partial_x u \varphi_l' \left( \frac{x}{\lambda_1(t)} \right) \left( \frac{x}{\lambda_1(t)} \right) dx + \frac{\mu}{2 \eta(t) \lambda_1^2(t)} \int_{\mathbb{R}} |u|^2 \varphi_l''' \left( \frac{x}{\lambda_1(t)} \right) dx \\
+ & \frac{\mu \lambda_1'(t)}{\eta(t) \lambda_1(t)} \int_{\mathbb{R}} u \partial_x u \varphi_l' \left( \frac{x}{\lambda_1(t)} \right) \left( \frac{x}{\lambda_1(t)} \right) dx.
\end{align*}
\tag{4.4}
\]

First, we chose \( \theta > 0 \) and \( \mu = \frac{\alpha \theta}{\eta} \) note that this last choice is well defined according to our hypothesis, then
\[
B_1 + B_{10} = 0.
\]
Next, we focus our attention in to provide the required upper bounds for the remainder terms.

In this sense, we have for $B_2$

$$|B_2(t)| = \left| -\frac{\theta \gamma}{\eta(t)\lambda_1(t)} \int_{\mathbb{R}} v |u|^2 \varphi_i' \left( \frac{x}{\lambda_1(t)} \right) \, dx \right| \leq \frac{\theta |\gamma|}{\eta(t)\lambda_1(t)} \int_{\mathbb{R}} v^2 \varphi_i' \left( \frac{x}{\lambda_1(t)} \right) \, dx + \frac{\theta |\gamma|}{\eta(t)\lambda_1(t)} \int_{\mathbb{R}} |u|^4 \varphi_i' \left( \frac{x}{\lambda_1(t)} \right) \, dx \quad (4.5)$$

$$= B_{2,1}(t) + B_{2,2}(t).$$

Notice that if we choose $l > 0$, satisfying

$$\frac{1}{a} + \frac{1}{b} \leq \frac{1}{l}$$

we obtain by (3.14) that $B_{2,1} \in L^1 (\{ t \gg 1 \})$, while $B_{2,2}$ remains to be estimated. Later on, we will provide the required arguments to bound this term.

Note that, combining the properties of $\varphi$, i.e.

$$|\varphi'''(x)| \lesssim |\varphi'(x)| \quad \text{for all} \quad x \in \mathbb{R},$$

we get

$$|B_5(t)| \lesssim \frac{\theta}{\eta(t)\lambda_1^3(t)} \int_{\mathbb{R}} v^2 |\varphi_i'''(x)| \left( \frac{x}{\lambda_1(t)} \right) \, dx \leq \frac{1}{\eta(t)\lambda_1^3(t)} \int_{\mathbb{R}} v^2 \varphi_i(\lambda_1(t)) \, dx \in L^1 (\{ t \gg 1 \}),$$

since $p_1 > 0$.

Next,

$$|B_6(t)| \lesssim \frac{1}{t^r \eta(t)} \frac{I_3[0]}{I_3^{t+1}} + \frac{\|u_0\|_2 \|\nabla_x u\|_{L^\infty L^2}}{t^{r+1}} \in L^1 (\{ t \gg 1 \}),$$

since $r_1 > 0$.

Also,

$$|B_7(t)| \lesssim \frac{1}{t^r \eta(t)} \frac{I_3[0]}{I_3^{t+1}} + \frac{\|u_0\|_2 \|\nabla_x u\|_{L^\infty L^2}}{t^{r+1}} \in L^1 (\{ t \gg 1 \}),$$

and

$$|B_8(t)| \lesssim \frac{\|u_0\|_2 \|u\|_{L^\infty H^1}}{t^{r+1}} \in L^1 (\{ t \gg 1 \}).$$

Concerning the term $B_{12}$ we have that

$$|B_{12}(t)| \lesssim \frac{1}{\eta(t)\lambda_1^3(t)} \int_{\mathbb{R}} |u|^2 \varphi_i' \left( \frac{x}{\lambda_1(t)} \right) \, dx \in L^1 (\{ t \gg 1 \}),$$

since $p_1 > 0$.

We end the first set of estimates bounding $B_{13}$. Indeed,

$$|B_{13}(t)| = \left| \frac{\mu \lambda_1(t)}{\eta(t)\lambda_1(t)} \Im \int_{\mathbb{R}} u \partial_x u \varphi_i' \left( \frac{x}{\lambda_1(t)} \right) \left( \frac{x}{\lambda_1(t)} \right) \, dx \right| \leq \mu \frac{\lambda_1(t)}{\eta(t)\lambda_1(t)} \int_{\mathbb{R}} |u| |\nabla_x u| |\varphi_i'(x)| \left( \frac{x}{\lambda_1(t)} \right) \left( \frac{x}{\lambda_1(t)} \right) \, dx \leq \mu \frac{\lambda_1(t)}{\eta(t)\lambda_1(t)} \int_{\mathbb{R}} |u|^2 \varphi_i' \left( \frac{x}{\lambda_1(t)} \right) \left( \frac{x}{\lambda_1(t)} \right) \, dx \leq \mu \frac{\lambda_1(t)}{\eta(t)\lambda_1(t)} \int_{\mathbb{R}} |\nabla_x u|^2 \varphi_i' \left( \frac{x}{\lambda_1(t)} \right) \left( \frac{x}{\lambda_1(t)} \right) \, dx + \mu \frac{\lambda_1(t)}{\eta(t)\lambda_1(t)} \int_{\mathbb{R}} |\nabla_x u|^2 \varphi_i' \left( \frac{x}{\lambda_1(t)} \right) \left( \frac{x}{\lambda_1(t)} \right) \, dx.$$
whence we obtain after taking $\epsilon = \frac{|\lambda'(t)|}{2} > 0$, the bound

$$|B_{13}(t)| \leq \frac{\mu}{2\eta(t)\lambda_1(t)} \int_{\mathbb{R}} |\partial_x u|^2 \varphi' \left( \frac{x}{\lambda_1(t)} \right) \, dx$$

$$+ \frac{\mu}{2} \frac{(\lambda'_1(t))^2}{\eta(t)\lambda_1(t)} \int_{\mathbb{R}} |u|^2 \varphi' \left( \frac{x}{\lambda_1(t)} \right) \left| \frac{x}{\lambda_1(t)} \right|^2 \, dx.$$  

Thus,

$$\frac{(\lambda'(t))^2}{\eta(t)\lambda_1(t)} \int_{\mathbb{R}} |u|^2 \varphi' \left( \frac{x}{\lambda_1(t)} \right) \left| \frac{x}{\lambda_1(t)} \right|^2 \, dx \lesssim \|u_0\|_2 \frac{(\lambda'_1(t))^2}{\eta(t)\lambda_1(t)} \in L^1 (\{t \gg 1\}),$$

since $0 < p_1 < 1$. The remainder term will be estimated after integrating in time.

Finally, we bound the terms that require a different approach to the one used for the previous terms.

Note that, for $\epsilon > 0$ the following inequality holds

$$|B_3(t)| = \left| \frac{\theta}{3\eta(t)\lambda_1(t)} \int_{\mathbb{R}} v^3 \varphi' \left( \frac{x}{\lambda_1(t)} \right) \, dx \right|$$

$$\leq \frac{\theta}{3\eta(t)\lambda_1(t)} \int_{\mathbb{R}} v^6 \varphi' \left( \frac{x}{\lambda_1(t)} \right) \, dx + \frac{3\theta}{12(4\epsilon)^{1/3} \eta(t)\lambda_1(t)} \int_{\mathbb{R}} v^2 \varphi' \left( \frac{x}{\lambda_1(t)} \right) \, dx$$

$$= B_{3,1}(t) + B_{3,2}(t). \quad (4.6)$$

We shall stress that from (3.14) it is clear that independently of $\epsilon$, the term $B_{3,2} \in L^1 (\{t \gg 1\})$.

Next, we focus our attention on $B_{3,1}$. In this sense, we proceed with a modification of the argument in [16], so, we consider a function $\chi : \mathbb{R} \rightarrow \mathbb{R}$ satisfying $\chi \equiv 1$ on $[0, 1]$ and $\chi \equiv 0$ on $(-\infty - 1) \cup [2, \infty)$.

Thus, if we set $\chi_n(x) := \chi(x - n)$, then we get

$$\int_{\mathbb{R}} v^6 \varphi' \left( \frac{x}{\lambda_1(t)} \right) \, dx \leq \sum_{n \in \mathbb{Z}} \left( \int_{\mathbb{R}} v^6 \chi_n^6 \, dx \right) \left( \sup_{x \in [n, n+1]} \varphi' \left( \frac{x}{\lambda_1(t)} \right) \right). \quad (4.7)$$

Next, by the Gagliardo-Nirenberg-Sobolev inequality in its optimal form

$$\sum_{n \in \mathbb{Z}} \|v \chi_n\|^6_{\infty} \leq C_{opt} \sum_{n \in \mathbb{Z}} \left( \int_{\mathbb{R}} |\partial_x (v \chi_n)|^2 \, dx \right) \|v \chi_n\|^4_2$$

$$\leq C_{opt} \|v\|_2^4 \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} |\partial_x (v \chi_n)|^2 \, dx$$

$$\leq 2C_{opt} \|v\|_2^2 \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} (\partial_x v)^2 \chi_n^2 \, dx + 2C_{opt} \|v\|_2^2 \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} (v \partial_x \chi_n)^2 \, dx \quad (4.8)$$

$$\leq 2C_{opt} \rho^2 \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} (\partial_x v)^2 \chi_n^2 \, dx + 2C_{opt} \rho^2 \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} (v \phi_n)^2 \, dx,$$

where

$$\rho := \frac{|I_3[0]}{\alpha} + \frac{2\gamma}{\alpha} \|u_0\|_2 \|u\|_{L^\infty H^1},$$

and $\phi_n(x) := \phi(x - n)$, being $\phi$ a $C^\infty (\mathbb{R})$ function such that $\phi \equiv 1$ on $[-1, 2]$ and $\phi \equiv 0$ on $(-\infty, -2) \cup [3, \infty)$.  

Before start we shall highlight the following relationship given on compacts sets for the weighted function \( \phi_l \). More precisely,

\[
\sup_{x \in [n,n+1]} \left( \frac{x}{\lambda_1(t)} \right) \leq \max \left\{ e^{-\frac{l}{\lambda_1(t)}}, e^{\frac{l}{\lambda_1(t)}} \right\} \inf_{x \in [n,n+1]} \left( \frac{x}{\lambda_1(t)} \right) . \tag{4.9}
\]

Finally, combining (4.9) and (4.8) we get from (4.7) that

\[
\int_R v^6 \phi_l' \left( \frac{x}{\lambda_1(t)} \right) \, dx \\
\leq 2C_{\text{opt}} \rho^2 \max \left\{ e^{-\frac{l}{\lambda_1(t)}}, e^{\frac{l}{\lambda_1(t)}} \right\} \sum_{n \in \mathbb{Z}} \int_R (\partial_x v)^2 \phi_l' \left( \frac{x}{\lambda_1(t)} \right) \chi_n^2 \, dx \\
+ 2C_{\text{opt}} \rho^2 \max \left\{ e^{-\frac{l}{\lambda_1(t)}}, e^{\frac{l}{\lambda_1(t)}} \right\} \sum_{n \in \mathbb{Z}} \int_R v^2 \phi_l' \left( \frac{x}{\lambda_1(t)} \right) \phi_n^2 \, dx \\
\leq 2C_{\text{opt}} \rho^2 \int_R (\partial_x v)^2 \phi_l' \left( \frac{x}{\lambda_1(t)} \right) \, dx + 2CcC_{\text{opt}} \rho^2 \int_R v^2 \phi_l' \left( \frac{x}{\lambda_1(t)} \right) \, dx.
\]

Therefore,

\[
|B_{3,1}(t)| \leq \frac{2\epsilon C_{\text{opt}} \rho^2}{3\eta(t)\lambda_1(t)} \int_R (\partial_x v)^2 \phi_l' \left( \frac{x}{\lambda_1(t)} \right) \, dx + \frac{2\epsilon C_{\text{opt}} \rho^2}{3\eta(t)\lambda_1(t)} \int_R v^2 \phi_l' \left( \frac{x}{\lambda_1(t)} \right) \, dx.
\]

Note that the second term on the r.h.s is bounded after integrating in time, this is a consequence of (3.14). To handle the first term we notice that this is up to constant the quantity to be estimated after integrating in time. Thus, for our proposes it is enough to take \( \epsilon > \frac{3}{C_{\text{opt}} \rho^2} \).

Next, we decompose as follows: For all \( \epsilon_1 > 0 \),

\[
|B_{11}(t)| = \frac{\mu \beta}{2\eta(t)\lambda_1(t)} \int_R |u|^4 \phi_l' \left( \frac{x}{\lambda_1(t)} \right) \, dx \\
\leq \frac{\epsilon_1 \mu \beta}{2\eta(t)\lambda_1(t)} \int_R |u|^2 \phi_l' \left( \frac{x}{\lambda_1(t)} \right) \, dx + \frac{\mu \beta}{8\epsilon_1 \eta(t)\lambda_1(t)} \int_R |u|^6 \phi_l' \left( \frac{x}{\lambda_1(t)} \right) \, dx \\
= B_{11,1}(t) + B_{11,2}(t). \tag{4.10}
\]

**Remark 4.1.** In the case \( \beta > 0 \), the decomposition (4.10) is not required, since the integral of \( B_{11} \) is obtained by directly from the energy estimate (4.4).

Note that independent of the value of \( \epsilon_1 \), the bound in (3.14) yield

\[
\int_{\{ \epsilon \geq 1 \}} |B_{11,1}(t)| \, dt < \infty.
\]

Thus, by using the same notation as in (4.7) it follows that

\[
\int_R |u|^6 \phi_l' \left( \frac{x}{\lambda_1(t)} \right) \, dx \leq \sum_{n \in \mathbb{Z}} \left( \int_R |u|^6 \lambda_n^6 \, dx \right) \left( \sup_{x \in [n,n+1]} \phi_l' \left( \frac{x}{\lambda_1(t)} \right) \right). \tag{4.11}
\]
whence we get by the Gagliardo-Nirenberg-Sobolev inequality that
\[
\sum_{n \in \mathbb{Z}} \|u \chi_n\|_6^6 \leq C_{\text{opt}} \sum_{n \in \mathbb{Z}} \left( \int_{\mathbb{R}} |\partial_x (u \chi_n)|^2 \, dx \right) \|u \chi_n\|_2^4 \\
\leq C_{\text{opt}} \|u_0\|_2^4 \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} |\partial_x (u \chi_n)|^2 \, dx \\
\leq 2C_{\text{opt}} \|u_0\|_2^4 \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} |\partial_x u|^2 \chi_n^2 \, dx + 2C_{\text{opt}} \|u_0\|_2^4 \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} |u \partial_x \chi_n|^2 \, dx \\
\leq 2C_{\text{opt}} \|u_0\|_2^4 \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} |\partial_x u|^2 \chi_n^2 \, dx + 2C_{\text{opt}} \|u_0\|_2^4 \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} |u \phi_n|^2 \, dx,
\]
where $\phi_n(x) := \phi(x - n)$ being $\phi$ a $C^\infty(\mathbb{R})$ function such that $\phi \equiv 1$ on $[-1, 2]$ and $\phi \equiv 0$ on $(-\infty, -2] \cup [3, \infty)$.

We shall remark that
\[
\sup_{x \in [n, n+1]} \varphi_n' \left( \frac{x}{\lambda_1(t)} \right) \leq \max \left\{ e^{-\frac{x}{\lambda_1(t)}}, e^{\frac{x}{\lambda_1(t)}} \right\} \inf_{x \in [n, n+1]} \varphi_n' \left( \frac{x}{\lambda_1(t)} \right). \tag{4.13}
\]

Finally, combining (4.13) and (4.12) we get from (4.11) that
\[
\int_{\mathbb{R}} |u|^{\beta} \varphi_n' \left( \frac{x}{\lambda_1(t)} \right) \, dx \\
\leq 2C_{\text{opt}} \|u_0\|_2^4 \max \left\{ e^{-\frac{x}{\lambda_1(t)}}, e^{\frac{x}{\lambda_1(t)}} \right\} \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} |\partial_x u|^2 \varphi_n' \left( \frac{x}{\lambda_1(t)} \right) \chi_n^2 \, dx \\
2C_{\text{opt}} \|u_0\|_2^4 \max \left\{ e^{-\frac{x}{\lambda_1(t)}}, e^{\frac{x}{\lambda_1(t)}} \right\} \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} |u|^2 \varphi_n' \left( \frac{x}{\lambda_1(t)} \right) \phi_n^2 \, dx \\
\leq 2C_{\text{opt}} \|u_0\|_2^4 \int_{\mathbb{R}} |\partial_x u|^2 \varphi_n' \left( \frac{x}{\lambda_1(t)} \right) \, dx \\
+ 2C_{\text{opt}} \|u_0\|_2^4 \int_{\mathbb{R}} |u|^2 \varphi_n' \left( \frac{x}{\lambda_1(t)} \right) \, dx.
\]
The same analysis can be applied to estimate the term $B_{2, 2}$.

Therefore, after choosing $\epsilon_1 > 0$, satisfying $\epsilon_1 > 2 \beta C_{\text{opt}} \|u_0\|_2^2$, we get after gathering the estimates corresponding to this step that
\[
\int_{\{t > 1\}} \left( \frac{1}{\eta(t) \lambda_1(t)} \right) \left( |\partial_x u(t, x)|^2 + (\partial_x v(t, x))^2 \right) \varphi_n' \left( \frac{x}{\lambda_1(t)} \right) \, dx \, dt < \infty; \tag{4.14}
\]
that implies
\[
\liminf_{t \to \infty} \int_{\Omega_{\epsilon_1}(t)} |\partial_x u(t, x)|^2 \, dx = \liminf_{t \to \infty} \int_{\Omega_{\epsilon_1}(t)} (\partial_x v(t, x))^2 \, dx = 0,
\]
whenever $0 < p_1 \leq \frac{2}{p_2 + 2}$ for $p_2 > 1$. \hfill \square

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REFERENCES

[1] Albert, J. and Angulo Pava, J., Existence and stability of ground-state solutions of a Schrödinger-KdV system. Proc. Roy. Soc. Edinburgh Sect. A 133 (2003), no. 5, 987–1029.
[2] Albert, J. and Bhattarai, S., Existence and stability of a two-parameter family of solitary waves for an NLS-KdV system. Adv. Differential Equations 18 (2013), no. 11-12, 1129–1164.
[3] Bekiranov D., Ogawa T., and Ponce G., Interaction equatoins for short and long dispersive waves J. Funct. Anal. 158 (1998) 357–388.
[4] Bekiranov D., Ogawa T., and Ponce G., Weak solvability and well-posedness of a coupled Schrödinger-Korteweg de Vries equation for capillary-gravity wave interactions Proc. Amer. Math. Soc. 125 (1997) 2907–2919.
[5] Benilov E.S. and Burtsev S.P., To the integrability of the equations describing the Langmuir-wave-ion-acoustic-wave interaction Phys. Let. A 98 (1983), 256–258.
[6] Bourgain, J., Global solutions of nonlinear Schrödinger equations. American Mathematical Society Colloquium Publications, 46. American Mathematical Society, Providence, RI, 1999. viii+182 pp.
[7] Corcho, A. J. and Linares, F., Well-posedness for the Schrödinger-Korteweg-de Vries system. Trans. Amer. Math. Soc. 359 (2007), no. 9, 4089–4106.
[8] Corcho, A.J., Panthee, M., and Bhattarai, S., Well-Posedness for Multicomponent Schrödinger-gKdV Systems and Stability of SolitaryWaves with Prescribed Mass. J Dyn Diff Equat 30 (2018), 845–88.
[9] Dias, J.P., Figueira, M. and Oliveira, F., Existence of bound states for the coupled Schrödinger-KdV system with cubic nonlinearity. C. R. Math. Acad. Sci. Paris 348 (2010), no. 19-20, 1079–1082.
[10] Dias, J.P., Figueira, M. and Oliveira, F., Well-posedness and existence of bound states for a coupled Schrödinger-gKdV system. Nonlinear Anal. 73 (2010), no. 8, 2686–2698.
[11] Djordjevic V.D. and Redekopp L.G., On two-dimensional packet of capillary-gravity waves, J. Fluid Mech. 79 (1977) 703–714.
[12] Funakoshi M. and Oikawa M., The resonant interaction between a long internal gravity wave and a surface gravity wave packet, J. Phys. Soc. Japan 52 (1983) 1982–1995.
[13] Grimshaw R.H.J., The modulation of an interval gravity-wave packet and the resonance with the mean motion, Stud. Appl. Math. 56 (1977) 241–266.
[14] Guo Z. and Wang Y., On the well-posedness of the Schrödinger-Korteweg-de Vries system, J. Differential Equations 249 (2010) 2500–2520.
[15] Kawahara T., Sugimoto N., and Kakutani T., Nonlinear interaction between short and long capillary-gravity waves, J. Phys. Soc. Japan 39 (1975) 1379–1386.
[16] Kenig, C.E. and Martel, Y., Asymptotic stability of solitons for the Benjamin-Ono equation. Rev. Mat. Iberoam. 25 (2009), no. 3, 909–970.
[17] Kenig, C.E., Ponce G., and Vega L., A bilinear estimate with applications to the KdV equation, J. Amer. Math. Soc. 9 (1996) 573–603.
[18] Kenig C.E., Ponce G., and Vega L., Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle. Comm. Pure Appl. Math. 46 (1993), no. 4, 527–620.
[19] Killip, R. and Visan, M., KdV is well-posed in $H^{-1}$. Ann. of Math. (2) 190 (2019), no. 1, 249–305.
[20] Kowalczyk, M., Martel, Y., and Muñoz, C., Kink dynamics in the $\phi^4$ model: asymptotic stability for odd perturbations in the energy space. J. Amer. Math. Soc. 30 (2017), 769–798.
[21] Kowalczyk, M., Martel, Y., and Muñoz, C., Nonexistence of small, odd breathers for a class of nonlinear wave equations. Letters in Math. Physics, (2017) Vol. 107, Issue 5, 921–931.
[22] Linares, F., Mendez, A.J., and Ponce, G. Asymptotic Behavior of Solutions of the Dispersion Generalized Benjamin–Ono Equation. J Dyn Diff Equat (2020). https://doi.org/10.1007/s10884-020-09843-6.
[23] Linares, F. and Palacios, J.M., Dispersive blow-up and persistence properties for the Schrödinger–Korteweg–de Vries system. Nonlinearity 32 (2019), no. 12, 4996–5016.
[24] Linares F. and Ponce G., Introduction to Nonlinear Dispersive Equations, Universitext. Springer, New York, 2009.
[25] Makhankov V., On stationary solutions of the Schrödinger equation with a self-consistent potential satisfying Boussinesq’s equation, Phys. Lett. A 50 (1974) 42-44.
[26] Martinez, M.E., On the decay problem for the Zakharov and Klein-Gordon Zakharov systems in one dimension. arXiv:2004.01070.
[27] A. Mendez, C. Muñoz, F. Poblete, J. Pozo, On local energy decay for large solutions of the Zakharov-Kuznetsov equation. preprint: arXiv: 2007.04918v1.
[28] Muñoz, C. and Ponce, G., On the asymptotic behavior of solutions to the Benjamin-Ono equation. Proc. Amer. Math. Soc. 147 (2019), no. 12, 5303–5312.
[29] Muñoz, C. and Ponce, G., Breathers and the dynamics of solutions in KdV type equations, Comm. Math. Phys. 367 (2019), 581–598.
[30] Muñoz, C., Ponce, G., and Saut, J-C., On the long time behavior of solutions to the Intermediate Long Wave equation. arXiv:2002.04339.
[31] Nishikawa K., Hojo H., Mima K., and Ikezi H., Coupled nonlinear electron-plasma and ion-acoustic waves, Phys. Rev. Lett., 33 (1974), 148–151.
[32] Pecher, H., The Cauchy problem for a Schrödinger–Korteweg-de Vries system with rough data. Differ. Int. Equ. 18, (2005), 1147–1174.
[33] Satsuma J. and Yajima N., Soliton solutions in a diatomic lattice system, Progr. Theor. Phys., 62 (1979) 370–378.
[34] Sulem, C. and Sulem, P-L., The nonlinear Schrödinger equation. Self-focusing and wave collapse. Applied Mathematical Sciences, 139. Springer-Verlag, New York, 1999. xvi+350 pp.
[35] Tao, T., Nonlinear dispersive equations. Local and global analysis. CBMS Regional Conference Series in Mathematics, 106. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006. xvi+373 pp.
[36] Tsutsumi, M., Well-posedness of the Cauchy problem for a coupled Schrödinger-KdV equation. Math. Sciences Appl., 2 (1993), 513–528.
[37] Tsutsumi Y., $L^2$-solutions for nonlinear Schrödinger equations and nonlinear groups. Ekvacioj, 30 (1987) 115–125.
[38] Wu,Y., The Cauchy problem of the Schrödinger–Korteweg-deVries system. Differ. Int. Equ. 23 (2010), 569–600.

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