New phenomena in the containment problem for simplicial arrangements

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Abstract

In this note we consider two simplicial arrangements of lines and ideals $I$ of intersection points of these lines. There are 127 intersection points in both cases and the numbers $t_i$ of points lying on exactly $i$ configuration lines (points of multiplicity $i$) coincide. We show that in one of these examples the containment $I^{(3)} \subseteq I^2$ holds, whereas it fails in the other. We also show that the containment fails for a subarrangement of 21 lines. The interest in the containment relation between $I^{(3)}$ and $I^2$ for ideals of points in $\mathbb{P}^2$ is motivated by a question posted by Huneke around 2000. Configurations of points with $I^{(3)} \not\subseteq I^2$ are quite rare. Our example reveals two particular features: All points are defined over $\mathbb{Q}$ and all intersection points of lines are involved. In all examples found before now it was always ideals of points with multiplicity $i \geq 3$ and non-containment was a consequence of the product of the lines not being in $I^2$. In contrast, our examples show that having some points of multiplicity 2 does not always prevent non-containment, and that non-containment can occur even when the product of the lines is in $I^2$.

Keywords simplicial arrangements, arrangements of lines, containment problem, symbolic powers

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1 Introduction

The following problem has attracted a lot of attention in the last two decades.

Containment Problem.

Determine all pairs of positive integers $(m, r)$ such that the containment

$$I^{(m)} \subseteq I^r$$

holds for all homogeneous ideals $I \subseteq \mathbb{K}[x_0, \ldots, x_N]$ in the ring of polynomials over a field $\mathbb{K}$.

In 2000 Ein, Lazarsfeld and Smith [8] in characteristic zero and Hochster and Huneke [14] in positive characteristic discovered that the containment (1) holds provided $m \geq Nr$.

Theorem 1.1 (Ein-Lazarsfeld-Smith, Hochster-Huneke). Let $I \subseteq \mathbb{K}[x_0, \ldots, x_N]$ be a homogeneous ideal. Then there is

$$I^{(m)} \subseteq I^r$$
for all $m \geq Nr$.

This ground-breaking result prompted a natural question about the optimality of the bound $m \geq Nr$. A number of examples suggested the following conjectural improvement (see [1, Conjecture 8.4.2], or [12, Conjecture 4.1.1], or [2, Conjecture 1.1])

**Conjecture 1.2.** Let $I$ be a homogeneous ideal. Ten

$$I^{(m)} \subseteq I^r$$

for all $m \geq Nr - (N - 1)$.

The first non-trivial case is $N = 2$ and $r = 2$. Then there is always $I^{(4)} \subset I^2$ and it is very easy to give examples with $I^{(2)} \nsubseteq I^2$. Huneke asked around 2000 if

$$I^{(3)} \subseteq I^2$$

holds for all ideals defining points in $\mathbb{P}^2$.

This is not the case. The first non-containment example was announced in [7] and soon after additional non-containment examples were discovered and described in [4], [13], [19], [15], [16], [10].

Such examples are quite rare and they all follow the same pattern, in particular they are related to line arrangements. More precisely, let $\mathcal{L} = \{L_1, \ldots, L_s\}$ be an arrangement of lines in $\mathbb{P}^2$ and let $\mathcal{P} = \{P_1, \ldots, P_t\}$ be the set of all points contained in at least 2 lines from $\mathcal{L}$. Let $I$ be the ideal of those points which are contained in at least 3 lines. By the Zariski-Nagata Theorem [9, Theorem 3.14] the product

$$f = l_1 \cdot \ldots \cdot l_s \in I^{(3)}$$

and sometimes it happens that $f \nsubseteq I^2$ (here $l_i$ is the equation of $L_i$).

The novelty of our non-containment example is that whereas the ideal of points is determined by lines, it is not their product which sits in $I^{(3)} \setminus I^2$. More precisely, our main results are the following

**Theorem A.** There exists an arrangement of 31 lines which intersect in the total of 127 points such that for the ideal $I$ of these 127 points there is

$$I^{(3)} \nsubseteq I^2.$$ 

Moreover, there is an element $f$ of degree 33 in $I^{(3)}$, which is not contained in $I^2$ and which is a product of

- 21 of arrangement lines and
- an irreducible curve of degree 12.

**Theorem B.** There exists an arrangement of 21 lines which intersect in the total of 115 points such that for the ideal $I$ of these 115 points there is

$$I^{(3)} \nsubseteq I^2.$$ 

Moreover, there is an element $f$ of degree 31 in $I^{(3)}$, which is not contained in $I^2$ and which is a product of

- all arrangement lines and
- an irreducible curve of degree 10.

A number of elementary but tedious calculations is omitted. Instead we provide a Singular script [17] which provides easy verification of our claims.
2 Preliminaries

In this section we define the basic object we are interested in and state the central conjecture in the field, which motivated our research here.

Let \( I \subseteq \mathbb{K}[x_0, \ldots, x_N] \) be a homogeneous ideal in the ring of polynomials over a field \( \mathbb{K} \).

**Definition 2.1.** (Symbolic power) For \( m \geq 1 \), the \( m \)-th symbolic power of \( I \) is the ideal

\[
I^{(m)} = \mathbb{K}[\mathbb{P}^N] \cap \left( \bigcap_{p \in \text{Ass}(I)} (I^m)_p \right),
\]

where the intersection is taken over all associated primes of \( I \).

Symbolic powers of ideals are of geometric interest due to Zariski-Nagata theorem [9, Theorem 3.14].

**Theorem 2.2.** (Zariski-Nagata) Let \( I \) be a radical homogeneous ideal, and let \( \text{char}(\mathbb{K}) = 0 \). For \( m \geq 1 \)

\[
I^{(m)} = \{ f \in \mathbb{K}[x_0, \ldots, x_N] : f \text{ vanishes to order } \geq m \text{ in all points } P \in V(I) \}.
\]

In the situation when \( V(I) \) is a finite set of points \( P_1, \ldots, P_t \in \mathbb{P}^N \), the symbolic power is particularly easy to compute:

\[
I^{(m)} = \bigcap_{i=1}^{t} I(P_i)^m.
\]

An arrangement of lines \( \mathcal{A} \) is a finite set of mutually distinct lines \( L_1, \ldots, L_s \).
An arrangement of lines determines a finite set of points \( \mathcal{P} = \{ P_1, \ldots, P_t \} \) in \( \mathbb{P}^2 \), where at least 2 of arrangement lines intersect. For \( i \geq 2 \), we denote by \( t_i(\mathcal{A}) \) the number of points in \( \mathcal{P} \) where exactly \( i \) lines from \( \mathcal{A} \) intersect. These numbers define the \( t \)-vector \( t(\mathcal{A}) \) of \( \mathcal{A} \)

\[
t(\mathcal{A}) = (t_2(\mathcal{A}), t_3(\mathcal{A}), \ldots, t_s(\mathcal{A})).
\]

It is a basic combinatorial invariant of \( \mathcal{A} \).

For line arrangements defined over \( \mathbb{R} \), the following property has been distinguished.

**Definition 2.3** (Simplicial arrangement). We say that an arrangement \( \mathcal{A} \) of real lines is simplicial if every connected component of its complement \( \mathbb{F}^2(\mathbb{R}) \setminus \mathcal{A} \) is a triangle.

It is expected, but not known, if (apart of 3 obvious infinite families described in [11]) there are only finitely many sporadic examples. A list of such examples was constructed by Grünbaum in [11] and extended recently by Cuntz in [3].

3 Simplicial arrangements \( \mathcal{A}(31, 2) \) and \( \mathcal{A}(31, 3) \)

The arrangements we study here come from [11], where they are called \( \mathcal{A}(31, 2) \) and \( \mathcal{A}(31, 3) \).

Configurations \( \mathcal{A}(31, 2) \) and \( \mathcal{A}(31, 3) \) are non isomorphic simplicial arrangements of 31 lines with the total number of 127 intersection points. Moreover we have

\[
t_2 = 54, t_3 = 42, t_4 = 21, t_5 = 6, t_6 = 1, t_8 = 3
\]

and all other \( t_i = 0 \).
3.1 Configuration $\mathcal{A}(31, 3)$

This configuration can be realized in the following way. We begin with ten lines:

$$x \pm auz = 0, \ y \pm bz = 0,$$

where $u = \frac{\sqrt{3}}{2}$, $a = 0, 1, 2, 4$ and $b = 0, 1$. These ten lines are visualized in Figure 1.

\[\begin{array}{c|c|c|c}
 & x = -2uz & x = 0 & x = 2uz \\
\hline
y = z & & & \\
y = 0 & & & \\
y = -z & & & \\
\hline
x = -4uz & x = -uz & x = uz & x = 4uz
\end{array}\]

Figure 1. 10 initial lines in the construction of $\mathcal{A}(31, 3)$.

Then we rotate these lines by $60^\circ$ and $120^\circ$ around the point $(0 : 0 : 1)$. In this way, we obtain 30 lines. The last line is the line at infinity $z = 0$. As a result we obtain a configuration of lines indicated in Figure 2. Taking the product of linear forms defining the ten initial lines we obtain the following polynomial

$$F_{10} = x^7y^3 - x^7yz^2 - \frac{63}{4}x^5y^3z^2 + \frac{63}{4}x^5yz^4 + \frac{189}{4}x^3y^3z^4 - \frac{189}{4}x^3yz^6 - 27xy^3z^6 + 27xyz^8$$

and taking the product of all 31 lines we get a polynomial $F_{31}$ of degree 31. We are interested in the Jacobian ideal $\text{Jac}(F_{31})$ defined by this polynomial. The radical $I_{\mathcal{A}(31, 3)}$ of this ideal describes all 127 intersection points among arrangement lines.

By construction the arrangement is invariant under the group $G = \mathbb{Z}_3 \times \mathbb{Z}_2$ generated by the rotation matrix $A = \begin{bmatrix} \frac{1}{2} & -u & 0 \\ u & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and the reflection $B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, which together generate the dihedral group $D_6$. This group acts on the set of 127 intersection points so that, there are the following orbits
Figure 2. Configuration $A(31,3)$. The line $z = 0$, which lies at infinity, is not shown. Intersection points of lines which belong to the dotted circle form the orbit represented by point $(9 : 2u : 4u)$.

| length of orbit | number of orbits | representing point |
|----------------|-----------------|-------------------|
| 1              | 1               | $(0 : 0 : 1)$     |
| 6              | 9               | $(0 : 1 : 1)$, $(1 : 0 : u)$, $(2u : 0 : 1)$, $(0 : 2 : 1)$, $(4u : 0 : 1)$, $(0 : 4 : 1)$, $(u : 0 : 1)$, $(8u : 0 : 1)$, $(1 : 0 : 1)$ |
| 12             | 6               | $(6u : 1 : 4)$, $(9 : 2u : 4u)$, $(4u : 1 : 1)$, $(15 : 6u : 4u)$, $(6u : 1 : 1)$, $(10u : 1 : 1)$ |

It is helpful to consider the sub-arrangement $B_{21}$ consisting of 7 lines:

$$x \pm auz = 0, \ y \pm bz = 0,$$

where $a = 1, 4$, $b = 0, 1$ and images of these lines under $A$ and $A^2$. This 21 lines intersect altogether in 115 points, with multiplicities $t_2 = 72$, $t_3 = 40$ and $t_4 = 3$. The difference between the 127 and 115 points is one full orbit represented by point $(9 : 2u : 4u)$.

The points in this orbit are now contained each in only one of the 21 lines. In order to get an element in $f^{(3)}_{A(31,3)}$ we need to complete the 21 lines by a divisor $\Gamma$ vanishing in these 12 points to order 2 and passing through the remaining 72 points, which are double points for $B_{21}$.

To this end we consider $X = \mathbb{P}^2/G$. The ring of invariant polynomials $\mathbb{K}[x, y, z]^G$ is generated by

$$f_1 = z, \ f_2 = x^2 + y^2, \ f_3 = 11x^6 + 15x^4y^2 + 45x^2y^4 + 9y^6.$$
Using Moliens’s Theorem (see [18], Theorem 2.2.1), we see that the space of invariant polynomials of degree 12 has dimension 12.

Since vanishing to order 2 at a smooth point of $X$ imposes 3 conditions and the 72 points split into 4 orbits of order 6 and 4 orbits of order 12, counting conditions

$$12 - 3 - 4 - 4 = 1 > 0$$

we conclude that the desired divisor $\Gamma$ exists (it is invariant under $G$, so it pulls back from $X$). Computing with Singular, we are able to identify the equation of $\Gamma$:

$$
\begin{align*}
\frac{2093688}{17}f_1^{12} - \frac{9398511}{34}f_1^{10}f_2 + \frac{2995218}{17}f_1^8f_2^2 - \frac{6448513}{1088}f_1^6f_2^3 \\
+ \frac{18708003}{4352}f_1^4f_2^4 + \frac{1258659}{4352}f_1^2f_2^5 - \frac{493695}{4352}f_2^6 + \frac{2121309}{1088}f_2^6f_3 \\
- \frac{402561}{4352}f_1^2f_2f_3 - \frac{158697}{4352}f_1^2f_2^2f_3 + \frac{2619}{128}f_2^3f_3 - \frac{3979}{4352}f_2^3,
\end{align*}
$$

in terms of the invariant generators $f_1, f_2, f_3$.

Considering the equation of $\Gamma$ in the ring $\mathbb{K}[f_1, f_2, f_3]$, it is easy to check that there is just one singular point, which is locally simple crossing. This implies that $\Gamma$ is irreducible.

For the non-containment $I_{A(31, 3)}^{(3)} \nsubseteq I_{A(31, 3)}^2$ we used Singular. We do not have a theoretical proof. Summing up claims in this section, we see that Theorem A is proved.

At the end of this section we want to underline another interesting observation about curve from set $I_{A(31, 3)}^{(3)} \setminus I_{A(31, 3)}^2$ indicated on Figure 5. Let us denote this curve by $F$.

We claim first that $F$ is irreducible. Assume to the contrary that $F$ decomposes $F = G \cup H$. Then the intersection points of $G$ and $H$ are singular points of $F$. But $F$ has exactly 12 nodes (all lying on a conic) and no other singularities. Since the intersection subscheme of $G$ and $H$ has length $\deg G \cdot \deg H$ and $\deg G + \deg H = 12$, we see that it cannot be $\deg G \cdot \deg H = 12$, where this 12 is length of the singularity locus of $F$. Thus we can easily calculate the arithmetic genus, which is

$$p_a(F) = \binom{12 - 1}{2} - 12 \binom{2}{2} = 43.$$

This is the first known example of the curve, which form an element from the set $I^{(3)} \setminus I^2$ and which is not rational at the same time.

### 3.2 Configuration $A(31, 2)$

This configuration is very similar to $A(31, 3)$. It can be realized starting with lines

$$x \pm auz = 0, \quad y \pm bz = 0,$$

where $a = 0, 1, 2, 3$ and $b = 0, 1$. These lines are visualized in Figure 3.

Rotating again by $60^\circ$ and $120^\circ$ and taking the line at infinity we obtain the configuration presented in Figure 4.

The multiplicities vector of this configuration is the same as for $A(31, 3)$, i.e., there is

$$t_2 = 54, t_3 = 42, t_4 = 21, t_5 = 6, t_6 = 1, t_8 = 3.$$
Figure 3. 10 initial lines in the construction of $A(31, 2)$.

Figure 4. Configuration $A(31, 2)$. The line $z = 0$ at infinity is not shown.
In particular there are again 127 intersection points of pairs of arrangement lines. However, a quick Singular check shows that now we have
\[ I_{A(31,2)}^{(3)} \subseteq I_{A(31,2)}^2. \]
This shows, once again (see [10]), that the (non)containment property is quite subtle and cannot be decided by looking at the basis combinatorial invariants only.

### 3.3 Arrangement $\mathcal{B}_{21}$

Now we consider more closely the arrangement $\mathcal{B}_{21}$ defined in the previous section. We keep the notation introduced there.

The ideal $I_{\mathcal{B}_{21}}$ defines 115 points. This is the subset of 127 points defined by $I_{A(31,3)}$, the difference being one $G$–orbit, consisting of 12 double points of $\Gamma$.

In order to exhibit an element in $I_{\mathcal{B}_{21}}^{(3)}$, we need to find a divisor vanishing at the 72 points, where only 2 of arrangement lines meet.

Revoking again Molien’s Theorem, we see that the dimension of the space of $G$–invariant polynomials of degree 10 is 9, thus the expected dimension of invariant polynomials vanishing at 4 order 6 and 4 order 12 orbits in which the 72 points split is
\[ 9 - 4 - 4 = 1 > 0. \]
Hence there is a divisor $\Delta$ of degree 10 vanishing at these 72 points. We can express its equation in terms of invariant polynomials:
\[
\begin{align*}
- \frac{38320128}{107} f_1^{10} + \frac{80453952}{107} f_1^6 f_2 - \frac{42393996}{4107} f_1^4 f_2^2 + \frac{50759217}{214} f_1^4 f_2^3 \\
- \frac{20519091}{856} f_1^2 f_2^4 + \frac{67086}{107} f_2^5 - \frac{3811059}{214} f_1^4 f_3 + \frac{1778227}{856} f_1^2 f_2 f_3 - \frac{6089}{107} f_2^2 f_3.
\end{align*}
\]
Since $\Delta$ is smooth, it is irreducible.

The non-containment $I_{\mathcal{B}_{21}}^{(3)} \not\subseteq I_{\mathcal{B}_{21}}^2$ is proved again with the aid of Singular [6].

Summing up the claims of this section, we obtain the proof of Theorem B.

As for the curve in Section 3, we also calculate arithmetic genus for the curve, which is indicated as a solid line on Figure 6. Using any computer algebra program one can check that this curve has only two non-reduced singular points.

In fact in both singular points the length of the singular scheme is 3. Argumenting similarly as at the end of Section 3.1, we see that curve is irreducible and its arithmetic genus is 30.

We conclude this section by noting that the arrangement $\mathcal{B}_{21}$ is not free. Indeed, its characteristic polynomial is
\[ \chi(\mathcal{B}_{21}, t) = 1 - 20t + 144t^2 \]
and it does not split over the integers (see Main Theorem in [20]).

### 4 Realizability over rational numbers

The first non-containment
\[ I^{(3)} \not\subseteq I^2 \]
was the dual Hesse arrangement, see [7]. This arrangement cannot be realized over the reals. The first real non-containment example, the Böröczky arrangement of 12 lines was discovered in [4]. It was realized in [2] and [15] that the Böröczky arrangement can be defined over the rational numbers. Additional examples were provided in [16] and [10]. Such examples are quite rare, so we find it worth to mention that $A(31,3)$ and $B_{21}$ can be both realized over $\mathbb{Q}$. We can be quite explicit here. Table 1 contains equations of all 31 lines, whereas coordinates of their intersection points are provided in Table 2.

| $\mathcal{A}(31,3)$ |
|---------------------|
| $x + y + iz = 0$, for $i \in \{0, 2, 3, 4, 5, 6, 8\}$, |
| $2x - y + jz = 0$, for $j \in \{4, 6, 7, 8, 9, 10, 12\}$, |
| $3x + kz = 0$, for $k \in \{8, 10, 11, 12, 13, 14, 16\}$, |
| $x - 2y + lz = 0$, for $l \in \{2, 4, 6\}$, |
| $4x + y + mz = 0$, for $m \in \{14, 16, 18\}$, |
| $5x - y + nz = 0$, for $n \in \{18, 20, 22\}$, |
| $z = 0$ |

Table 1 The equations of lines of $\mathcal{A}(31,3)$.
| double  | (2,-2,-3), (6,-6,-1), (7,-7,-3), (13,-13,-3), (3,-1,-1), (13,-7,-3), (5,1,-1), (11,7,-3), (22,2,-3), (2,6,-1), (17,7,-3), (11,13,-3), (7,2,-3), (3,0,-1), (14,-5,-3), (16,-7,-3), (7,-1,-2), (23,-5,-6), (25,-7,-6), (5,0,-1), (17,-2,-3), (8,7,-3), (10,5,-3), (23,7,-6), (25,5,-6), (9,1,-2), (-22,-8,3), (-22,-5,6), (-26,-7,6), (-22,1,6), (-26,-1,6), (-2,8,3), (-22,7,6), (-26,5,6), (-6,8,1), (-13,14,3), (-13,-8,3), (-11,8,3), (-2,8,1), (-11,14,3), (-8,-22,3), (8,-11,-3), (8,-26,-3), (10,7,-3), (14,7,-3), (-16,22,3), (16,11,3), (16,26,3), (7,0,-2), (23,8,-6), (23,4,-6), (25,4,-6), (25,8,-6), (9,0,-2) |
| triple  | (4,-4,-3), (2,-2,1), (8,-8,3), (4,-4,1), (14,-14,3), (16,-16,3), (3,3,-1), (11,-11,-3), (2,0,1), (16,-10,3), (8,4,-3), (16,-4,3), (6,0,-1), (8,10,-3), (6,2,-1), (20,4,3), (4,4,1), (16,8,3), (8,16,-3), (10,14,-3), (5,3,1), (13,11,3), (8,1,3), (5,-2,1), (16,-1,3), (3,2,1), (-16,-5,3), (-34,8,9), (-38,10,9), (-32,2,9), (-40,-2,9), (-8,5,3), (-34,10,9), (-38,8,9), (-14,-16,3), (-16,-20,3), (-11,-10,3), (-16,-14,3), (-8,14,3), (-8,20,3), (-10,16,3), (-13,10,3) |
| quadruple | (2,1,0), (-1,4,0), (1,5,0), (10,-10,3), (8,-2,3), (14,8,3), (11,-5,3), (11,1,3), (13,-1,3), (16,2,3), (10,8,3), (13,5,-3), (14,10,-3), (10,-1,3), (4,1,1), (11,-2,3), (13,-4,3), (4,1,1), (14,1,3), (11,4,-3), (13,2,-3) |
| quintuple | (10,-4,3), (4,-2,1), (10,2,-3), (14,-2,3), (14,4,-3), (4,2,-1) |
| sextuple | (4,0,-1) |
| octuple | (-1,1,0), (1,2,0), (0,1,0) |

Table 2 The coordinates of points of $A(31,3)$. 
Figure 5. The graph of affine part of polynomial from set $I_{A(31,3)}^{(3)} \setminus I_{A(31,3)}^{2}$, which consists of 21 dashed lines and a curve of degree 12.

Figure 6. The graph of affine part of polynomial from set $I_{B_{21}}^{(3)} \setminus I_{B_{21}}^{2}$, which consists of 21 dashed lines and a curve of degree 10.
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