Complexity of the universal theory of bounded residuated distributive lattice-ordered groupoids

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Abstract. We prove that the universal theory and the quasi-equational theory of bounded residuated distributive lattice-ordered groupoids are both EXPTIME-complete. Similar results are proven for bounded distributive lattices with a unary or binary operator and for some special classes of bounded residuated distributive lattice-ordered groupoids.

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1. Introduction

By a bounded residuated distributive lattice-ordered groupoid, or brdg, for short, we mean a bounded distributive lattice with binary operations $\circ$, \ and / satisfying the following property:

$$x \circ y \leq z \iff y \leq x \backslash z \iff x \leq z / y.$$  

These algebras fall within the general framework of ‘residuated lattices’, which are studied as algebraic semantics for various substructural logics. The class of brdg’s, in particular, is the algebraic semantics for Distributive Nonassociative Full Lambek Calculus with Bounds. This logic and the corresponding class of brdg’s, as well as the unbounded versions thereof, have been investigated in connection with categorial grammars and context-free languages by Buszkowski and Farulewski [6], who prove decidability of the universal theory.

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of the class of brdg’s via the finite embeddability property—see [11] and [6]. A conEXPTIME upper bound for the universal theory of brdg’s is obtained by Haniková and Horčík in [18]. In the present article, we improve upon the result from [18] and obtain EXPTIME-completeness of the universal theory of brdg’s.

The EXPTIME upper bound is obtained by considering the complementary problem to the universal theory, namely, satisfiability of quantifier-free first-order formulas, to which the methods of [28] are applied (see also [26]). A quantifier-free first-order formula is satisfiable in a brdg if, and only if, it is satisfiable in a ‘partial brdg’ with cardinality not greater than the size of the formula. By a partial brdg we mean an algebraic structure with partially defined operations that is a partial substructure of a (full) brdg. We obtain here a characterization of partial brdg’s that allows us to describe an exponential-time algorithm for identifying partial brdg’s up to a given size. For a given formula, we then test for satisfiability in each partial brdg whose size does not exceed the size of the formula, which can be done in time exponential in the size of the formula.

In order to obtain the EXPTIME lower bound for brdg’s, we consider the class of bounded distributive lattices with operators, or bdo’s, for short, by which we mean bounded distributive lattices with unary operation ♦ that distributes over joins and satisfies ♦0 = 0. The lower bound is obtained by reduction from a two person corridor tiling problem that is known to be EXPTIME-hard [7]. Starting with an instance of the two person corridor tiling problem, say T, following [7], we construct a formula φT in the language of a modal logic with the universal modality (see [17]) such that T is a ‘good’ instance if, and only if, φT is satisfiable in a Kripke model. We then construct a quantifier-free first-order formula ϕT in the language of bdo’s and show that ϕT is satisfiable in a bdo if, and only if, φT is satisfiable in a Kripke model, establishing the EXPTIME lower bound for satisfiability in bdo’s. Lastly, we reduce satisfiability in bdo’s to satisfiability in brdg’s, proving EXPTIME-completeness for the latter decision problem. As a consequence, we also obtained EXPTIME-completeness for satisfiability in bdo’s. The EXPTIME-completeness of the universal theory of these classes then follows.

We also consider some subclasses of brdg’s that correspond to well-known extensions of the associated logic. In particular, we consider brdg’s whose operation ◦ satisfies any combination of the following properties: commutative, decreasing, square-increasing and unital, corresponding to the structural rules of exchange, weakening, contraction and having a truth constant. The method used here for obtaining the EXP-Time upper bound for brdg’s is extended to all these cases. The EXPTIME lower bound proof, however, is extended to the commutative case only.

The Nonassociative Lambek Calculus was introduced in [21] as the logic of sentence structure, or categorical grammar, see, e.g., [29, 23, 24]. Models for this logic are ‘residuated ordered groupoids’, that is, posets with operations ◦, \ and / satisfying the above property. When the poset is a lattice, such structures are algebraic models for the Nonassociative Full Lambek Calculus. Such algebraic structures have been considered in a different context, namely,
as residuated lattices (although associativity of $\circ$ is usually assumed). For a recent overview of the field, see, e.g., [14].

We mention some decidability and complexity results for classes of algebras closely related to $brdg$’s. Firstly, consider the case where the operation $\circ$ is associative and commutative. This class has an undecidable universal theory, which follows from the undecidability of the word problem for this class [27]. The equational theory for this class, however, is decidable [16]. If associativity is assumed, but not commutativity, the undecidability of the universal theory is proved in [12], whereas the equational theory is proved decidable in [20]. In [13], cut-elimination and finite model property are proved for various extensions of Distributive Full Lambek Calculus, proving decidability of the equational theories of the corresponding classes of algebras.

Next, we mention cases in which the lattice is not required to be distributive. This class of ‘lattice-ordered residuated groupoids’ has an undecidable universal theory [8], whereas its equational theory is in $\text{PSPACE}$ [6]. If associativity of $\circ$ is assumed, the undecidability of the universal theory is proved in [19]. If the associative, commutative and unital properties are assumed, undecidability of the universal theory follows from [22] (see [2]), and the $\text{PSPACE}$-completeness of the equational theory is proved in [22]. If the lattice operations are dropped altogether, that is, we consider residuated ordered groupoids, the (quasi-)equational theory is in $\text{P}$ [5], and its associative extension has an $\text{NP}$-complete equational theory [25].

Lastly, suppose the assumption of ‘integrality’ is added, that is, the decreasing and unital properties hold and the greatest element is the unit for the $\circ$ operation. In the case without the distributivity axiom, the universal theory is decidable. This was shown in [3], where the corresponding class of algebras is shown to have the finite embeddability property, or $\text{fep}$, which implies decidability of the universal theory. The result also holds if $\circ$ is associative and/or commutative and if the lattice operations are dropped. In the distributive lattice case, the $\text{fep}$ is obtained in [13] for various classes of integral $brdg$’s, including also the associative and/or commutative cases.

2. Bounded residuated distributive lattice-ordered groupoids

In this section we describe the main classes of algebras that are considered in this paper and give some of the properties we shall require in later sections. We also describe the relational representations for these classes that form the basis of the constructions used for the characterizations of partial algebras in Section 4 and for establishing the lower bound complexity in Section 6. The relational representations and related results are obtained from [10].

We assume familiarity with standard universal algebraic and lattice theoretic notions, as can be found in [4] and [9], for example, and we follow standard notational conventions.

The algebraic structures we consider all have an underlying bounded distributive lattice structure. Recall that the natural order associated with a distributive lattice is defined by the following condition: $x \leq y$ iff $x \land y = x$. 

We shall include the order relation $\leq$ in the type of our algebras as this will be useful for the characterization of partial structures. We use 0 and 1 to denote the least and greatest elements, respectively, of a lattice. We recall the following notions. Let $D = \langle D, \wedge, \vee, 0, 1, \leq \rangle$ be a bounded distributive lattice. A filter of $D$ is a subset $F \subseteq D$ that is upward closed, i.e., $a \in F$ and $a \leq b$ imply $b \in F$, and closed under meets, i.e., $a, b \in F$ imply $a \wedge b \in F$. A prime filter of $D$ is a filter $F$ such that $1 \in F$, $0 \notin F$ and, whenever $a \vee b \in F$, either $a \in F$ or $b \in F$ (equivalently, $D \setminus F$ is closed under joins).

By a bounded residuated distributive lattice-ordered groupoid, $\text{brdg}$ for short, we mean an algebra $A = \langle A, \wedge, \vee, \circ, \setminus, /, 0, 1, \leq \rangle$, where \( A, \wedge, \vee, 0, 1, \leq \) is a bounded distributive lattice, $\circ, \setminus$, and $\div$ are binary operations and $A$ satisfies the following residuation property:

\[ x \circ y \leq z \iff y \leq x \setminus z \iff x \leq z / y. \]  

(2.1)

Notice that any $\text{brdg}$ satisfies $0 \circ x = 0 = x \circ 0$ and

\[ x \circ (y \vee z) = (x \circ y) \vee (x \circ z) \quad \text{and} \quad (y \vee z) \circ x = (y \circ x) \vee (z \circ x), \]  

(2.2)

that is, $\circ$ distributes over joins. In addition, $\circ$ is order-preserving in both co-ordinates, $\setminus$ is order-preserving in its second co-ordinate and order-reversing in its first, while $\div$ is order-preserving in its first co-ordinate and order-reversing in its second. The following properties are also satisfied by $\text{brdg}$'s:

\[ x \circ (x \setminus y) \leq y \]  

(2.3)

\[ y \leq x \setminus (x \circ y) \]  

(2.4)

\[ x \leq ((x \circ y) \vee z) / y \]  

(2.5)

The class of all $\text{brdg}$'s, which we denote by $\text{BRDG}$, is a variety as it has an equational axiomatization given by the bounded distributive lattice identities, together with the identities in (2.2), (2.3) and (2.5) [15].

By a $\text{brdg}$-frame we mean a structure $\mathfrak{F} = \langle P, \leq, R \rangle$, where $P$ is a nonempty set, $\leq$ is a partial order on $P$, and $R$ is a ternary relation on $P$ that satisfies

\[ (\forall x, x', y, z \in P)(R(x, y, z) \text{ and } x' \leq x \Rightarrow R(x', y, z)) \]  

(2.6)

\[ (\forall x, y, y', z \in P)(R(x, y, z) \text{ and } y' \leq y \Rightarrow R(x, y', z)) \]  

(2.7)

\[ (\forall x, y, z, z' \in P)(R(x, y, z) \text{ and } z \leq z' \Rightarrow R(x, y, z')). \]  

(2.8)

Given a $\text{brdg}$ $A$, by its associated $\text{brdg}$-frame $\mathfrak{F}_A$, we mean the structure obtained as follows. Let $P$ be the set of prime filters of (the underlying distributive lattice of) $A$ and $R$ the ternary relation on $P$ that satisfies

\[ (\forall x, x', y, z \in P)(R(x, y, z) \text{ and } x' \leq x \Rightarrow R(x', y, z)) \]  

(2.6)

\[ (\forall x, y, y', z \in P)(R(x, y, z) \text{ and } y' \leq y \Rightarrow R(x, y', z)) \]  

(2.7)

\[ (\forall x, y, z, z' \in P)(R(x, y, z) \text{ and } z \leq z' \Rightarrow R(x, y, z')). \]  

(2.8)

Given a $\text{brdg}$ $A$, by its associated $\text{brdg}$-frame $\mathfrak{F}_A$, we mean the structure obtained as follows. Let $P$ be the set of prime filters of (the underlying distributive lattice of) $A$ and $R$ the ternary relation on $P$ defined by:

\[ R(F, G, H) \Leftrightarrow (\forall a, b \in A)(a \in F \text{ and } b \in G \Rightarrow a \circ b \in H). \]  

(2.9)

Then, set $\mathfrak{F}_A = \langle P, \leq, R \rangle$. It is straightforward to check that $\mathfrak{F}_A$ satisfies (2.6–2.8), hence it is a $\text{brdg}$-frame.

Lemma 2.1 [10]. Let $A$ be a $\text{brdg}$ and $\mathfrak{F}_A = \langle P, \leq, R \rangle$ its associated $\text{brdg}$-frame. Then, for $F, G, H \in P$, the following are equivalent:

\[ \text{...} \]
Proof. Let $F$, $G$ and $H$ be prime filters and suppose that $\mathcal{R}(F,G,H)$. If $a, b \in A$ with $a \in F$ and $a \backslash b \in G$, then $a \circ (a \backslash b) \in H$. By (2.3), $a \circ (a \backslash b) \leq b$, so $b \in H$, as required for (ii). Conversely, suppose (ii) holds and let $a, b \in A$ with $a \in F$ and $b \in G$. By (2.4), $b \leq a \backslash (a \circ b)$, so $a \backslash (a \circ b) \in G$, and hence $a \circ b \in H$, as required for $\mathcal{R}(F,G,H)$. Thus, (i) and (ii) are equivalent. A similar proof shows that (i) and (iii) are equivalent. □

Lemma 2.2 [10]. Let $A$ be a $\text{brdg}$ and $\mathfrak{F}_A = \langle \mathcal{P}, \subseteq, \mathcal{R} \rangle$ its associated $\text{brdg}$-frame. Let $a, b \in A$.

(i) If $a \in \mathcal{P}$ and $a \circ b \in H$, then there exist $F, G \in \mathcal{P}$ such that $a \in F$, $b \in G$, and $\mathcal{R}(F,G,H)$.

(ii) If $G \in \mathcal{P}$ and $a \backslash b \notin G$, then there exist $F, H \in \mathcal{P}$ such that $a \in F$, $b \notin H$, and $\mathcal{R}(F,G,H)$.

(iii) If $F \in \mathcal{P}$ and $b \backslash a \notin F$, then there exist $G, H \in \mathcal{P}$ such that $a \in G$ and $b \notin H$ and $\mathcal{R}(F,G,H)$.

Let $\mathfrak{F} = \langle \mathcal{P}, \leq, R \rangle$ be a $\text{brdg}$-frame. We construct a $\text{brdg}$ $A_{\mathfrak{F}}$ associated with $\mathfrak{F}$ as follows. First, for arbitrary $X, Y \subseteq P$, define the following sets:

$$X \circ Y = \{ z \in P \mid (\exists x, y \in P) (x \in X \text{ and } y \in Y \text{ and } R(x,y,z)) \},$$

$$X \backslash Y = \{ y \in P \mid (\forall x, z \in P) (R(x,y,z) \text{ and } x \in X \Rightarrow z \in Y) \},$$

$$Y \backslash X = \{ x \in P \mid (\forall y, z \in P) (R(x,y,z) \text{ and } y \in X \Rightarrow z \in Y) \}.$$

Let $\mathcal{U}(P)$ be the set of all upclosed subsets of $P$. Then $\langle \mathcal{U}(P), \cap, \cup, \emptyset, P, \subseteq \rangle$ is a bounded distributive lattice. That $\circ, \backslash$ and $/$ are operations on $\mathcal{U}(P)$ follows directly from (2.6–2.8), and (2.1) follows from the definitions of the operations. Thus, $\langle \mathcal{U}(P), \cap, \cup, \emptyset, \backslash, /, \emptyset, P, \subseteq \rangle$ is a $\text{brdg}$, which we denote by $A_{\mathfrak{F}}$.

Let $A$ be a $\text{brdg}$ and $\mathfrak{F}_A = \langle \mathcal{P}, \subseteq, \mathcal{R} \rangle$ its associated $\text{brdg}$-frame. For each $a \in A$, let $\mu(a) = \{ F \in \mathcal{P} : a \in F \}$. Then $\mu$ is an embedding of the algebra $A$ into the algebra $A_{\mathfrak{F}}$ [10]. Thus, every $\text{brdg}$ can be represented as a subalgebra of a $\text{brdg}$ $A_{\mathfrak{F}}$ constructed out of a $\text{brdg}$-frame $\mathfrak{F}$.

By a bounded distributive lattice with (normal) binary operator, $\text{bdbo}$ for short, we mean an algebra $A = \langle A, \wedge, \vee, \circ, 0, 1, \leq \rangle$, where $\langle A, \wedge, \vee, 0, 1, \leq \rangle$ is a bounded distributive lattice, $\circ$ is a binary operation and $A$ satisfies $x \circ 0 = 0 = 0 \circ x$ and (2.2). The class of all $\text{bdbo}$’s, which we denote by $\text{BDBO}$, is a variety.

Clearly, the $\backslash, /$-free reduct of any $\text{brdg}$ is a $\text{bdbo}$. Moreover, every $\text{bdbo}$ can be embedded into a $\text{brdg}$ as follows. Let $A$ be a $\text{bdbo}$. We construct the frame $\mathfrak{F}_A = \langle \mathcal{P}, \subseteq \mathcal{R} \rangle$ as above, where $\mathcal{P}$ is the set of prime filters of $A$ and $\mathcal{R}$ is defined as in (2.9). The conditions (2.6–2.8) hold in this case as well, so $\mathfrak{F}_A$ is a $\text{brdg}$-frame. Thus, we may construct the $\text{brdg}$ $A_{\mathfrak{F}_A}$ into which $A$ embeds by the map $\mu$.

By a bounded distributive lattice with (normal) operator, $\text{bdo}$ for short, we mean an algebra $A = \langle A, \wedge, \vee, \diamond, 0, 1, \leq \rangle$, where $\langle A, \wedge, \vee, 0, 1, \leq \rangle$ is a bounded
distribution lattice and ♦ is a unary operation such that A satisfies ♦ 0 = 0 and ♦(x ∨ y) = ♦ x ∨ ♦ y. We note that ♦ is order-preserving. The class of all bdo's is a variety, which we denote by BDO.

By a bdo-frame we mean a structure ₩ = ⟨P, ⊆, R⟩, where P is a non-empty set, ⊆ is a partial order on P and R is a binary relation on P such that the following holds:

\[(∀ x, x', y ∈ P)(R(x, y) \text{ and } x ≤ x' ⇒ R(x', y)). \tag{2.10} \]

Given a bdo A, we define the bdo-frame associated with A as follows. Let ₩A = ⟨P, ⊆, R⟩, where P is the set of prime filters of A and R is the binary relation on P defined by:

\[R(F, G) ⇔ (∀ a ∈ A)(a ∈ G ⇒ ♦ a ∈ F). \tag{2.11} \]

It is straightforward to check that (2.10) is satisfied by ₩A, so ₩A is a bdo-frame.

**Lemma 2.3 [10].** Let A be a bdo and ₩A = ⟨P, ⊆, R⟩ its associated bdo-frame. If F ∈ P and a ∈ A such that ♦ a ∈ F, then there exists G ∈ P such that a ∈ G and R(F, G).

Given a bdo-frame ₩ = ⟨P, ⊆, R⟩, we define the bdo A ₩ associated with it as follows. For an arbitrary X ⊆ P, define the set ♦ X as follows:

\[♦ X = \{x ∈ P \mid (∃ y ∈ P)(y ∈ X \text{ and } R(x, y))\}. \tag{2.12} \]

It follows from (2.10) that ♦ is an operation on ₱(P), the set of all upclosed subsets of P. From the definition of ♦ it follows that ⟨₱(P), ∩, ∪, ♦, ⊆, P, ⊆⟩ is a bdo; we denote this algebra by A ₩.

Let A be a bdo and ₩A = ⟨P, ⊆, R⟩ its associated bdo-frame. For each a ∈ A, let μ(a) = {F ∈ P : a ∈ F}. Clearly, μ(a) ∈ ₱(P). Moreover, μ is an embedding of A into A ₩A [10]. Thus, every bdo can be represented as a subalgebra of a bdo A ₩.

**Lemma 2.4.** Let A = ⟨A, ∧, ∨, ♦, 0, 1, ⊆⟩ be a bdo and let ◦ be the binary operation on A defined by x ◦ y := ♦(x ∧ y). Then, A ◦ = ⟨A, ∧, ∨, ◦, 0, 1, ⊆⟩ is a bdo.

**Proof.** We have 0 ◦ x = ♦(0 ∧ x) = ♦ 0 = 0 and, similarly, x ◦ 0 = 0. We derive x ◦ (y ∨ z) = ♦(x ∧ (y ∨ z)) = ♦((x ∧ y) ∨ (x ∧ z)) = ♦(x ∧ y) ∨ ♦(x ∧ z) = (x ◦ y) ∨ (x ◦ z). Similarly, we may derive (y ∨ z) ◦ x = (y ◦ x) ∨ (z ◦ x). \[\square\]

We note, for later, that the operation x ◦ y := ♦(x ∧ y), defined in the above lemma, is commutative.

**Lemma 2.5.** Let A = ⟨A, ∧, ∨, ◦, 0, 1, ⊆⟩ be a bdbo and let ♦ be the unary operation on A defined by ♦ x := x ◦ 1. Then, A ♦ = ⟨A, ∧, ∨, ♦, 0, 1, ⊆⟩ is a bdo.

**Proof.** Indeed, ♦(x ∨ y) = (x ∨ y) ◦ 1 = (x ◦ 1) ∨ (y ◦ 1) = ♦ x ∨ ♦ y and ♦ 0 = 0 ◦ 1 = 0. \[\square\]
3. Satisfiability and universal theories

We recall here the notion of satisfiability of a quantifier-free first-order formula in a structure and its relevance to the universal theory of a class of structures. We then recall the notions of a partial structure and satisfiability in partial structures, and how this relates to satisfiability in full structures. For more detailed background on the partial structure approach to satisfiability, we refer the reader to [28].

The languages we consider in this paper consist of a finite set containing operation symbols, constant symbols and the relation symbol \( \leq \), together with an \emph{arity} function \( \text{ar} \) that assigns to each operation symbol \( \delta \) a natural number \( \text{ar}(\delta) \). If \( \sigma \) denotes a language, then we use \( \sigma_{\text{op}} \) and \( \sigma_{\text{con}} \) to denote the sets of all operation and constant symbols, respectively. The relation symbol \( \leq \) is assumed to be binary. We write \( \sigma_\equiv \) to denote the language \( \sigma \) augmented with the equality symbol. The set of \( (\sigma-)\text{terms} \) over a set of variables is constructed in the usual way from constants, variables and operation symbols.

Let \( \sigma \) be a language of the above type and \( \mathcal{K} \) a class of \( \sigma \)-structures. We use \textbf{and}, \textbf{or}, \textbf{not}, \( \Rightarrow \) for (meta-)logical connectives. Let \( \varphi(x_1, \ldots, x_n) \) be a quantifier-free first-order formula in the language \( \sigma_\equiv \), i.e., a Boolean combination of atomic formulas of the form \( s = t \) or \( s \leq t \), where \( s \) and \( t \) are terms. Recall that \( \varphi(x_1, \ldots, x_n) \) is satisfiable in a structure \( \mathcal{A} \) of type \( \sigma \) if there exists a valuation \( v : \{x_1, \ldots, x_n\} \to \mathcal{A} \) such that \( \varphi \) is true in \( \mathcal{A} \) under \( v \). Then, \( \varphi(x_1, \ldots, x_n) \) is satisfiable in \( \mathcal{K} \) if there exists \( \mathcal{A} \in \mathcal{K} \) such that \( \varphi \) is satisfiable in \( \mathcal{A} \). We refer to the problem of deciding if a given \( \varphi \) is satisfiable in \( \mathcal{K} \) as satisfiability in \( \mathcal{K} \).

By a \textbf{universal sentence} we mean a first-order sentence of the form \( \Phi := (\forall x_1, \ldots, x_n) \varphi(x_1, \ldots, x_n) \) where \( \varphi \) is a quantifier-free first-order formula (in the language \( \sigma_\equiv \)). For such a sentence, we write \( \mathcal{K} \models \Phi \) to mean that \( \varphi \) is true in every \( \mathcal{A} \in \mathcal{K} \) under every valuation. By the \textbf{universal theory} of \( \mathcal{K} \), we mean the set of all universal sentences \( \Phi \) for which \( \mathcal{K} \models \Phi \). Observe that \( \mathcal{K} \not\models \Phi \) means that there exists \( \mathcal{A} \in \mathcal{K} \) and valuation \( v \) such that \( \varphi \) is not true in \( \mathcal{A} \) under \( v \), i.e., \textbf{not} \( \varphi \) is true in \( \mathcal{A} \) under \( v \). Thus, \( \mathcal{K} \models \Phi \) iff \textbf{not} \( \varphi \) is not satisfiable in \( \mathcal{K} \). The complexity of the universal theory of \( \mathcal{K} \), therefore, yields the complexity of satisfiability in \( \mathcal{K} \), and vice versa.

Let \( X \) and \( Y \) be sets. By a \textbf{partial function} from \( X \) to \( Y \) we mean a function \( \tau : X' \to Y \) where \( X' \) is a subset of \( X \) called the \textbf{domain} of \( \tau \) and denoted by \( \text{dom}(\tau) \). For any \( a \in X \), we say that \( \tau(a) \) is \textit{defined} iff \( a \in \text{dom}(\tau) \). Let \( A \) be a non-empty set and \( n \) a natural number such that \( n \geq 1 \). An \( n \)-ary \textbf{partial operation} on \( A \) is a partial function \( \tau \) from \( A^n \) to \( A \); \( n \) is called the \textbf{arity} of \( \tau \).

Let \( \sigma \) be a language of the type described above. A structure \( B \) is called a \textbf{partial} \( \sigma \)-\textbf{structure} if it consists of the following: a nonempty set \( B \), an \( \text{ar}(\delta) \)-ary partial operation \( \delta^B \) on \( B \), for each \( \delta \in \sigma_{\text{op}} \), a constant \( c^B \in B \) for each \( c \in \sigma_{\text{con}} \), and a binary relation \( \leq^B \) on \( B \).

If \( B \) is a partial \( \sigma \)-structure then for every term \( t(x_1, \ldots, x_n) \) there is an associated partial function \( t^B \) from \( B^n \) to \( B \). For \((a_1, \ldots, a_n) \in B^n \), \( t^B(a_1, \ldots, a_n) \)
is defined iff \( t \) can be evaluated in \( \mathcal{B} \) under the valuation \( v(x_i) = a_i \) (that is, if all partial operations required for the valuation of \( t \) in \( \mathcal{B} \) under \( v \) are defined).

Let \( \mathcal{B} \) be a partial \( \sigma \)-structure, \( \varphi(x_1, \ldots, x_n) \) a quantifier-free first-order \( \sigma_\text{=} \)-formula and \( v : \{ x_1, \ldots, x_n \} \to B \) a valuation. We say that \( \varphi(x_1, \ldots, x_n) \) is satisfied in \( \mathcal{B} \) under \( v \) if \( t^\mathcal{B}(v(x_1), \ldots, v(x_n)) \) is defined for every term \( t \) occurring in \( \varphi \) and \( \varphi \) is true in \( \mathcal{B} \) under \( v \). We say that \( \varphi \) is satisfiable in \( \mathcal{B} \) iff \( \varphi \) is satisfied in \( \mathcal{B} \) under some valuation.

Let \( \mathcal{A} \) be a \( \sigma \)-structure and \( \mathcal{B} \) a partial \( \sigma \)-structure. We say that \( \mathcal{B} \) is a partial substructure of \( \mathcal{A} \) if \( B \subseteq A \), \( \delta^\mathcal{B}(a_1, \ldots, a_{\text{ar}(\delta)}) = \delta^\mathcal{A}(a_1, \ldots, a_{\text{ar}(\delta)}) \) for all \( \delta \in \sigma_{\text{op}} \) and \( (a_1, \ldots, a_{\text{ar}(\delta)}) \in \text{dom}(\delta^\mathcal{B}) \), \( c^\mathcal{B} = c^\mathcal{A} \) for all \( c \in \sigma_{\text{con}} \) and \( \leq^\mathcal{B} = \leq^\mathcal{A} \cap (B^2) \).

Let \( \varphi \) be a quantifier-free first-order \( \sigma_\text{=} \)-formula. Let \( |\text{Op}(\varphi)| \) be the total number of occurrences of all operation symbols in \( \varphi \) and let \( |\text{Var}(\varphi)| \) be the number of distinct variables occurring in \( \varphi \). The size of \( \varphi \) is then defined as \( s(\varphi) := |\text{Op}(\varphi)| + |\text{Var}(\varphi)| + |\sigma_{\text{con}}| \).

**Theorem 3.1** [28]. Let \( \mathcal{K} \) be a class of \( \sigma \)-structures and \( \varphi \) a quantifier-free first-order \( \sigma_\text{=} \)-formula. Then \( \varphi \) is satisfiable in \( \mathcal{K} \) if, and only if, \( \varphi \) is satisfiable in some partial substructure \( \mathcal{B} \) of some member of \( \mathcal{K} \), with \( |B| \leq s(\varphi) \).

If we have an algorithm for identifying the partial substructures of members of \( \mathcal{K} \) up to a given size amongst the set of all partial \( \sigma \)-structures, then, by the above theorem, we have an algorithm for deciding satisfiability in \( \mathcal{K} \). In addition, we obtain an upper bound for the complexity of satisfiability in \( \mathcal{K} \) if the complexity of identifying the partial substructures of members of \( \mathcal{K} \) is known. In that case, we also have an upper bound for the complexity of the universal theory of \( \mathcal{K} \).

Notice that, conversely, if we can decide satisfiability in \( \mathcal{K} \), then we can decide if a given partial structure is a substructure of some member of \( \mathcal{K} \). To see this, suppose one can decide satisfiability in \( \mathcal{K} \), and let \( \mathcal{B} \) be a partial structure in the language of \( \mathcal{K} \) with domain \( B = \{ a_1, \ldots, a_n \} \). Construct a quantifier-free first-order formula \( \varphi \) in the language of \( \mathcal{K} \) describing \( \mathcal{B} \) as follows: use \( \{ x_1, \ldots, x_n \} \) as variables representing the elements of \( \mathcal{B} \) and set \( \varphi \) as the conjunction of the following literals: \( x_i \neq x_j \) for \( i \neq j \); for every \( k \)-ary function symbol \( \delta \) in the language and each \( i_1, \ldots, i_k \), if \( \delta^\mathcal{B}(a_{i_1}, \ldots, a_{i_k}) = a_j \), include \( \delta(x_{i_1}, \ldots, x_{i_k}) = x_j \); for every \( k \)-ary relation symbol \( R \) in the language and each \( i_1, \ldots, i_k \), include \( R(x_{i_1}, \ldots, x_{i_k}) \) if \( R^\mathcal{B}(a_{i_1}, \ldots, a_{i_k}) \), and not \( R(x_{i_1}, \ldots, x_{i_k}) \), otherwise. Clearly, \( \varphi \) is satisfiable in \( \mathcal{B} \) under the valuation \( v(x_i) = a_i \). Moreover, for any \( \mathcal{A} \) in \( \mathcal{K} \), \( \varphi \) is satisfiable in \( \mathcal{A} \) if, and only if, \( \mathcal{B} \) is a partial substructure of \( \mathcal{A} \). Thus, \( \varphi \) is satisfiable in \( \mathcal{K} \) if, and only if, \( \mathcal{B} \) is a partial substructure of some member of \( \mathcal{K} \).

### 4. Partial brdg’s

We now provide a characterization of partial brdg’s. We start by giving characterizations of partial bounded (distributive) lattices, as obtained in [28]. Let \( \sigma_{\text{bl}} \) be the language containing two binary operation symbols \( \land \) and \( \lor \),
two constant symbols 0 and 1, and a binary relation symbol \( \leq \). Let \( \mathbb{B} = \langle B, \wedge^B, \vee^B, \circ^B, \setminus^B, 0^B, 1^B, \leq^B \rangle \) be a partial \( \sigma^{bf} \)-structure. Then, \( \mathbb{B} \) is a partial substructure of a bounded lattice, i.e., a partial bounded lattice, and there exists a set \( \mathbb{F} \) of prime filters required for (D) must also form the basis for a partial bounded distributive lattice, i.e., a partial bounded distributive lattice, if \( \leq^B \) is a partial order on \( B \) with \( 0^B \) and \( 1^B \) as bounds, that is, they are the least and greatest elements of \( B \) with respect to \( \leq^B \); and the partial operations \( \wedge^B \) and \( \vee^B \) are compatible with \( \leq^B \) in the following sense: for all \( a, b \in B \),

- if \( a \wedge^B b \) is defined, then \( a \wedge^B b \) is the greatest lower bound of \( \{a, b\} \) with respect to \( \leq^B \),
- if \( a \vee^B b \) is defined, then \( a \vee^B b \) is the least upper bound of \( \{a, b\} \) with respect to \( \leq^B \).

We may characterize partial bounded distributive lattices as follows. Let \( \mathbb{B} \) be a partial bounded lattice. By a prime filter of \( \mathbb{B} \) we mean a subset \( f \) of \( B \) such that, for all \( a, b \in B \),

\[
1^B \in f \text{ and } 0^B \notin f; \tag{4.1}
\]

if \( a \in f \) and \( a \leq^B b \), then \( b \in f \); \tag{4.2}

if \( a \in f \) and \( b \in f \) and \( a \wedge^B b \) is defined, then \( a \wedge^B b \in f \); \tag{4.3}

if \( a \notin f \) and \( b \notin f \) and \( a \vee^B b \) is defined, then \( a \vee^B b \notin f \). \tag{4.4}

By [28], a partial \( \sigma^{bf} \)-structure \( \mathbb{B} \) is a partial substructure of a bounded distributive lattice, i.e., a partial bounded distributive lattice, if it is a partial bounded lattice and there exists a set \( \mathbb{F} \) of prime filters of \( \mathbb{B} \) such that the following holds:

\[
(D) \ (\forall a, b \in B)[a \not\leq^B b \Rightarrow (\exists f \in \mathbb{F})(a \in f \text{ and } b \notin f)].
\]

We use \( \sigma^{brdg} \) to denote the language of \( \text{brdg} \)’s, that is \( \sigma^{brdg} \) contains binary operation symbols \( \wedge, \vee, \circ, \setminus \) and \( / \), constant symbols 0 and 1 and a binary relation symbol \( \leq \). By a partial \( \text{brdg} \) we shall mean a partial \( \sigma^{brdg} \)-structure that is a partial substructure of a \( \text{brdg} \).

In the following theorem we characterize partial \( \text{brdg} \)’s. Clearly, a partial \( \text{brdg} \) must contain a partial bounded distributive lattice; as we show, the set \( \mathbb{F} \) of prime filters required for (D) must also form the basis for a \( \text{brdg} \)-frame.

**Theorem 4.1.** Let \( \mathbb{B} = \langle B, \wedge^B, \vee^B, \circ^B, \setminus^B, /^B, 0^B, 1^B, \leq^B \rangle \) be a partial \( \sigma^{brdg} \)-structure. Then, \( \mathbb{B} \) is a partial \( \text{brdg} \) if, and only if, its \( \sigma^{bf} \)-reduct is a partial bounded lattice and there exists a set \( \mathbb{F} \) of prime filters of \( \mathbb{B} \) such that (D) holds, as well as

\[
(M_o) \ (\forall h \in \mathbb{F})(\forall (a, b) \in \text{dom}(\circ^B))[a \circ^B b \in h \Rightarrow (\exists f, g \in \mathbb{F})(a \in f \text{ and } b \in g \text{ and } R^B(f, g, h))],
\]

\[
(M_l) \ (\forall g \in \mathbb{F})(\forall (a, b) \in \text{dom}(\setminus^B))[a \setminus^B b \notin g \Rightarrow (\exists f, h \in \mathbb{F})(a \in f \text{ and } b \notin h \text{ and } R^B(f, g, h))],
\]

\[
(M_l) \ (\forall f \in \mathbb{F})(\forall (a, b) \in \text{dom}(/^B))[a /^B b \notin f \Rightarrow (\exists g, h \in \mathbb{F})(a \in g \text{ and } b \notin h \text{ and } R^B(f, g, h))],
\]

\[
(M_l) \ (\forall f \in \mathbb{F})(\forall (a, b) \in \text{dom}(\leq^B))[a \leq^B b \notin f \Rightarrow (\exists g, h \in \mathbb{F})(a \in g \text{ and } b \notin h \text{ and } R^B(f, g, h))].
\]
where

\[ R_B(f, g, h) \equiv (\forall (a, b) \in \text{dom}(\sigma_B))(a \in f \text{ and } b \in g \Rightarrow a \cdot_B b \in h) \text{ and } \\
(\forall (a, b) \in \text{dom}(\sigma_B))(a \in f \text{ and } a \cdot_B b \in g \Rightarrow b \in h) \text{ and } \\
(\forall (a, b) \in \text{dom}(\sigma_B))(b \cdot_B a \in f \text{ and } a \in g \Rightarrow b \in h). \]

Proof. Suppose \( B \) is a partial \( \text{brdg}\); say \( B \) is a partial substructure of a \( \text{brdg} \( A \). Since \( \leq_B = \leq_A \cap (B^2) \), it follows that \( \leq_B \) is a partial order with \( 0_B \) and \( 1_B \) as bounds, and that \( \wedge_B \) and \( \lor_B \) are compatible with \( \leq_B \). Thus, the \( \sigma^{brd}_B \)-reduct of \( B \) is a partial bounded lattice. We show that there exists a set \( \mathcal{F} \) of prime filters of \( B \) that satisfies (D), (M\(_0\)), (M\(_1\)) and (M\(_f\)). Let \( \mathcal{F} := \{ F \cap B \mid F \text{ is a prime filter of } A \} \).

Note that each \( F \cap B \in \mathcal{F} \) is a prime filter of \( B \) since \( a \wedge_B b = a \wedge_A b \) and \( a \lor_B b = a \lor_B b \) whenever \( a \wedge_B b = a \wedge_B b \) and \( a \lor_B b = a \lor_B b \) are defined. To verify (D), let \( a, b \in B \) and assume that \( a \not\leq_B b \); hence also \( a \not\leq_B b \). By properties of distributive lattices, there exists a prime filter \( F \) of \( A \) such that \( a \in F \) and \( b \not\in F \). Then, \( f := F \cap B \in \mathcal{F} \) is a witness to the satisfaction of (D).

To verify (M\(_0\)), (M\(_1\)) and (M\(_f\)), we first make the following observation. Recall that \( \mathfrak{F}_A = \langle \mathcal{P}, \subseteq, \mathcal{R} \rangle \), where \( \mathcal{P} \) is the set of prime filters of \( A \) and \( \mathcal{R} \) is defined as in (2.9), is the \( \text{brdg} \)-frame associated with \( A \). For \( F, G, H \in \mathcal{P} \), let \( f := F \cap B \), \( g := G \cap B \) and \( h := H \cap B \). Then, if \( \mathcal{R}(F, G, H) \), then \( R_B(f, g, h) \). (4.5)

To see this, let \( \mathcal{R}(F, G, H) \) and suppose, first, that \( (c, d) \in \text{dom}(\sigma_B) \), \( c \in f \) and \( d \in g \). Then \( c \in F \) and \( d \in G \). As \( B \) is a partial substructure of \( A \), \( c \cdot_B d = c \cdot_A d \in H \). Also, \( c \cdot_B d \in B \), hence \( c \cdot_B d \in H \). Next, suppose that \( (c, d) \in \text{dom}(\sigma_B) \), \( c \in f \) and \( c \cdot_B d \in g \). Since \( c \cdot_B d = c \cdot_B d \), we have \( c \cdot_B d \in G \); also \( c \in F \), so \( c \cdot_B (c \cdot_B d) \in H \). By (2.3), therefore, we have \( d \in H \), hence \( d \in h \), as required. The third clause of the definition of \( R_B \) follows similarly.

To see that (M\(_0\)) holds, assume that \( h \in \mathcal{F} \), \( (a, b) \in \text{dom}(\sigma_B) \) and \( a \cdot_B b \in h \). Then \( h = H \cap B \), for some prime filter \( H \) of \( A \) and \( a \cdot_B b = a \cdot_B b \in H \). In view of Lemma 2.2(i), there exist prime filters \( F \) and \( G \) of \( A \) such that \( a \in F \), \( b \in G \), and \( \mathcal{R}(F, G, H) \). Thus, for \( f := F \cap B \) and \( g := G \cap B \) we have \( a \in f \), \( b \in g \) and \( R_B(f, g, h) \), by (4.5). That (M\(_1\)) and (M\(_f\)) hold follows in a similar way from Lemma 2.2(ii) and (iii), respectively.

Next, assume that \( B \) is a partial \( \sigma^{brd}_B \)-structure satisfying the requirements of the theorem. We construct a \( \text{brdg} \) into which \( B \) can be embedded, thereby showing that \( B \) is isomorphic to a substructure of a \( \text{brdg} \). Observe that the structure \( \mathfrak{F}_B = \langle \mathcal{F}, \subseteq, R_B \rangle \) satisfies (2.6–2.8), hence it is a \( \text{brdg} \)-frame. Let \( \mathfrak{A}_B = \langle \mathcal{F}, \cup, \cap, \cdot, \wedge, \lor, \mathcal{R}, \mathcal{F}, \subseteq \rangle \) be the \( \text{brdg} \) associated with this \( \text{brdg} \)-frame. For each \( a \in B \), let \( \mu(a) = \{ f \in \mathcal{F} \mid a \in f \} \). We show that \( \mu \) is an embedding of \( B \) into \( \mathfrak{A}_B \). For \( a, b \in B \), if \( a \leq_B b \), then \( \mu(a) \subseteq \mu(b) \), and if \( a \not\leq_B b \), then, by (D), there exists \( f \in \mathcal{F} \) with \( a \in f \) and \( b \not\in f \), so \( \mu(a) \not\subseteq \mu(b) \). This shows that \( a \leq_B b \iff \mu(a) \subseteq \mu(b) \), and also that \( \mu \) is one-to-one. In addition, by (4.1–4.4), \( \mu(0_B) = \mathcal{R}, \mu(1_B) = \mathcal{F}, \mu(a \wedge_B b) = \mu(a) \cap \mu(b) \) for all \( (a, b) \in \text{dom}(\wedge_B) \), and
\[ \mu(a \lor^B b) = \mu(a) \cup \mu(b) \text{ for all } (a, b) \in \text{dom}(\lor^B). \]  
For \((a, b) \in \text{dom}(\land^B), \) we have
\[ \mu(a \land^B b) = \{ h \in \mathcal{F} \mid a \land^B b \in h \} \text{ and} \]
\[ \mu(a) \cup \mu(b) = \{ h \in \mathcal{F} \mid (\exists f, g \in \mathcal{F}) (a \in f \text{ and } b \in g \text{ and } R^B(f, g, h)) \}. \]

That \(\mu(a \land^B b) = \mu(a) \cup \mu(b)\) follows directly from \((M_\lor)\) and the definition of \(R^B.\) That \(\mu(a /^B b) = \mu(a) \cup \mu(b)\) for \((a, b) \in \text{dom}(\land^B)\) and \(\mu(a /^B b) = \mu(a) / \mu(b)\) for \((a, b) \in \text{dom}(\lor^B)\) follows easily from \((M_\lor), (M_\land)\) and the definition of \(R^B.\)

\[ \square \]

5. Complexity of satisfiability in \(BRDG:\) Upper bound

We now prove that satisfiability in \(BRDG\) is in \(\text{EXPTIME}.\) We describe an algorithm that, given a quantifier-free first-order \(\sigma^B\)-formula \(\varphi,\) determines whether there exists a \(\text{brdg}\) \(A\) and a valuation \(v\) such that \(\varphi\) is true in \(A\) under \(v,\) that is, if \(\varphi\) is satisfiable in \(BRDG.\) In view of Theorem 3.1, \(\varphi\) is true in some \(\text{brdg}\) under some valuation if, and only if, \(\varphi\) is true in a partial \(\text{brdg}\) whose cardinality does not exceed \(s(\varphi) = |\text{Var}(\varphi)| + |\text{Op}(\varphi)| + 2,\) under some valuation. The algorithm described below establishes if such a partial \(\text{brdg}\) and valuation exist for \(\varphi.\)

We start by describing an algorithm for determining if a given (finite) partial \(\sigma^B\)-structure is a partial \(\text{brdg}.\) Let \(\mathbb{B}\) be a partial \(\sigma^B\)-structure with finite \(B.\) To check whether \(\mathbb{B}\) is a partial \(\text{brdg},\) using Theorem 4.1, we carry out the following steps:

1. Check that \(\leq^B\) is a partial order on \(B\) with bounds \(0^B\) and \(1^B\) and that \(\land^B\) and \(\lor^B\) are compatible with \(\leq^B.\)
2. Check whether there exists a set \(\mathcal{F}\) of prime filters of \(\mathbb{B}\) that satisfies conditions \((M_\land), (M_\lor), (M_f)\) and \((D).\)

Step (1) can be done in time polynomial in \(|B|;\) if the conditions hold we proceed to step (2), otherwise the algorithm terminates with the negative answer. We now describe how step (2) can be carried out in time exponential in \(|B|). First, generate the set \(\mathcal{F}_0\) of all prime filters of \(\mathbb{B.\) To that end, we check, for every \(S \subseteq B,\) if it satisfies conditions \((4.1–4.4).\) As checking these conditions for a given \(S \subseteq B\) can be done in time polynomial in \(|B|),\) this step takes time \(O(r(|B|) \times 2^{|B|})\) for some polynomial \(r.\)

Next, for each \(f \in \mathcal{F}_0,\) we check that the following three properties hold:

\[(\forall(a, b) \in \text{dom}(\land^B))[a \land^B b \in f] \]
\[\Rightarrow (\exists f', f'' \in \mathcal{F}_0)(a \in f' \text{ and } b \in f'' \text{ and } R^B(f', f'', f))\],

\[(\forall(a, b) \in \text{dom}(\lor^B))[a \lor^B b \notin f] \]
\[\Rightarrow (\exists f', f'' \in \mathcal{F}_0)(a \in f' \text{ and } b \notin f'' \text{ and } R^B(f', f, f''))\],

\[(\forall(a, b) \in \text{dom}(\land^B))[a /^B b \notin f] \]
\[\Rightarrow (\exists f', f'' \in \mathcal{F}_0)(a \in f' \text{ and } b \notin f'' \text{ and } R^B(f, f', f''))\],

where \(R^B\) is as defined in Theorem 4.1. Notice that checking \(R^B(f, g, h)\) for prime filters \(f, g, h,\) is polynomial in \(|B|\) and, thus, checking whether the above
properties hold, for a given \( f \), can be done in time \( \mathcal{O}(q(|B|) \times 2^{2|B|}) \), for some polynomial \( q \). If any of the above properties fails for \( f \), that is, no suitable \( f' \), \( f'' \) are found, we eliminate \( f \) from \( \mathcal{F}_0 \). Otherwise, we retain \( f \) in \( \mathcal{F}_0 \) and repeat the process with the next element of \( \mathcal{F}_0 \).

Having, in the above way, traversed the entire \( \mathcal{F}_0 \), we obtain the resultant set \( \mathcal{F}_1 \) of prime filters. If we had eliminated at least one \( f \in \mathcal{F}_0 \), but not all of them, the above procedure is then repeated on \( \mathcal{F}_1 \). Traversing the set \( \mathcal{F}_i \) to obtain the set \( \mathcal{F}_{i+1} \) in this way is repeatedly carried out until either we obtain the empty set or no eliminations have been done on the latest pass, resulting in the non-empty set \( \mathcal{F} \) of prime filters of \( \mathcal{B} \). At each traversal, we go through a list of length at most \( 2^{|B|} \), and the entire number of traversals does not exceed \( 2^{|B|} \), as at each one (except possibly the last) we eliminate at least one element of the remaining list. Therefore, this procedure can be carried out in time \( \mathcal{O}(q(|B|) \times 2^{4|B|}) \), which is \( \mathcal{O}(2^5|B|) \).

If the empty set has been produced, the algorithm terminates with the negative answer. Otherwise, the set \( \mathcal{F} \) satisfies the properties \((M_0), (M_1)\) and \((M_2)\), and this is the largest set of prime filters of \( \mathcal{B} \) that satisfies these properties (notice that the ordering of the prime filters in \( \mathcal{F}_0 \) does not affect the outcome). For such an \( \mathcal{F} \), we proceed to check if it satisfies \((D)\). Note that if \((D)\) does not hold for \( \mathcal{F} \), then it does not hold for any subset of \( \mathcal{F} \). If \((D)\) holds, then \( \mathcal{B} \) is a partial \( \text{brdg} \); if not, the negative answer is returned. Checking \((D)\) can be done in time \( \mathcal{O}(|B|^2 \times 2^{|B|}) \), hence checking whether \( \mathcal{B} \) is a partial \( \text{brdg} \) can be done in time \( \mathcal{O}(2^5|B|) \).

We next estimate how many candidate structures \( \mathcal{B} \) we need to check. As noticed above, if \( \varphi \) is true in a \( \text{brdg} \) under some valuation, it is true in a partial \( \text{brdg} \) with cardinality not greater than \( s(\varphi) \). Thus, we need to check all structures \( \mathcal{B} \) with \( |B| \leq s(\varphi) \) and all valuations in such structures. Each such structure can be encoded using matrices corresponding to the operations and the partial order, each of which has size \( \mathcal{O}(s(\varphi)^2) \). Each entry in the matrices can take on \( \mathcal{O}(s(\varphi)) \) values (which includes an ‘undefined’ value). Thus, the total number of structures to be checked is \( \mathcal{O}((k_1s(\varphi))^{k_2s(\varphi)^2}) \), which is \( \mathcal{O}(2^{k_3s(\varphi)^3}) \) (where \( k_1, k_2, k \) are positive constants). Checking if \( \varphi \) holds in a structure \( \mathcal{B} \) under a given valuation can be done in time \( \mathcal{O}(s(\varphi)) \). Checking if \( \varphi \) holds in \( \mathcal{B} \) under some valuation requires considering all valuations; this can be done in time \( \mathcal{O}(s(\varphi) \times |B|^{\text{Var(\varphi)}}) \), which is \( \mathcal{O}(2^{s(\varphi)^3}) \).

Thus, the algorithm runs in time \( \mathcal{O}(2^{k_3s(\varphi)^3} \times 2^{5s(\varphi)} \times 2^{s(\varphi)^3}) \), that is, in time exponential in \( s(\varphi) \). This establishes the following result.

**Theorem 5.1.** Satisfiability in \( \text{BRDG} \) is in \( \text{EXPTIME} \).

We use Theorem 5.1 to obtain \( \text{EXPTIME} \) upper bounds for satisfiability in \( \text{BDBO} \) and \( \text{BDO} \). The languages of \( \text{bdbo} \)’s and \( \text{bdo} \)’s are denoted by \( \sigma^{\text{bdbo}} \) and \( \sigma^{\text{bdo}} \), respectively.

**Lemma 5.2.** Let \( \varphi \) be a quantifier-free first-order \( \sigma^{\text{bdbo}} \)-formula. Then \( \varphi \) is satisfiable in a \( \text{bdbo} \) if, and only if, it is satisfiable in a \( \text{brdg} \).
Proof. If \( \varphi \) holds in a \( \text{brdg} \) \( A \) under valuation \( v \), then it also holds in the \( \setminus, / \)-free reduct of \( A \), which is a \( \text{bdbo} \), under the same valuation. Conversely, suppose \( \varphi \) holds in a \( \text{bdbo} \) \( A \) under valuation \( v \). As shown in Section 2, \( A \) embeds into the \( \text{brdg} \) \( A_3 ) \) by the map \( \mu \). Then, \( \varphi \) holds in \( A_3 \) under the valuation \( v' = \mu \circ v \).

The following result now follows immediately from Theorem 5.1.

**Corollary 5.3.** Satisfiability in BDBO is in EXPTIME.

Next, we consider satisfiability in \( \text{bdo}'s \). For any quantifier-free first-order \( \sigma_{bdo}\)-formula \( \varphi \), let \( \varphi^o \) be the formula obtained by replacing in \( \varphi \) all terms of the form \( \Diamond t \) by \( t \circ 1 \). Then \( \varphi^o \) is a quantifier-free first-order formula in the language \( \sigma_{bdbo} \) (and also in the language \( \sigma_{brdg} \)).

**Lemma 5.4.** Let \( \varphi \) be a quantifier-free first-order \( \sigma_{bdo}\)-formula. Then \( \varphi \) is satisfiable in a \( \text{bdo} \) if and only if, \( \varphi^o \) is satisfiable in a \( \text{bdbo} \).

Proof. Suppose that \( \varphi \) holds in a \( \text{bdo} \) \( A \) under an assignment \( v \). Define the operation \( \circ \) on \( A \) by \( x \circ y := \Diamond(x \land y) \). Then, in view of Lemma 2.4, the algebra \( A^o = \langle A, \land, \lor, \circ, 0, 1 \rangle \) is a \( \text{bdbo} \). To verify that \( \varphi^o \) holds in \( A^o \) under \( v \), it suffices to notice that every term of the form \( t \circ 1 \) occuring in \( \varphi^o \) evaluates to \( \Diamond(t \land 1) \), which equals \( \Diamond t \).

Conversely, suppose that \( \varphi^o \) holds in a \( \text{bdbo} \) \( A \) under an assignment \( v \). Define an operation \( \Diamond \) on \( A \) by \( \Diamond x := x \circ 1 \). Then, in view of Lemma 2.5, the algebra \( A^{\Diamond} = \langle A, \land, \lor, \Diamond, 0, 1 \rangle \) is a \( \text{bdo} \). To verify that \( \varphi \) holds in \( A^{\Diamond} \) under \( v \), it suffices to notice that every term of the form \( \Diamond t \) occuring in \( \varphi \) evaluates to \( t \circ 1 \). \( \square \)

**Corollary 5.5.** Satisfiability in \( \text{BDO} \) is in EXPTIME.

Proof. Observe that constructing the formula \( \varphi^o \) from a quantifier-free first-order \( \sigma_{bdo}\)-formula \( \varphi \) can be done in polynomial time. Thus, the result follows from Lemma 5.4 and Corollary 5.3. \( \square \)

### 6. Complexity of satisfiability in \( \text{BRDG} \): Lower bound

In this section, we establish the EXPTIME lower bound for satisfiability in \( \text{BDO} \). By embedding \( \text{bdo}'s \) into \( \text{bdbo}'s \) and \( \text{brdg}'s \) we obtain similar results for the classes BDBO and BRDG. Consequently, we obtain EXPTIME-completeness for satisfiability in \( \text{BDO} \), BDBO and BRDG, and also for the universal theories of these classes.

The lower bound for \( \text{BDO} \) is established by a reduction from an EXPTIME-hard two person corridor tiling problem from [7]. The reduction proceeds in two steps. Starting with an instance, say \( T \), of the two person corridor tiling problem, following [7], we construct a formula \( \phi_T \) in the language of the logic \( L^M \), which is a modal logic with the universal (or, global) modality \( [\forall] \) (see [17]). We use the fact that \( T \) is a ‘good’ instance iff \( \phi_T \) is satisfiable in a Kripke model. The argument is similar to [7] (see also [1], Theorem 6.52).
Then, we construct a quantifier-free first-order $\sigma^{bdo}_{=}\varphi_T$ formula that is satisfiable in a $bdo$ if, and only if, $\varphi_T$ is satisfiable in a Kripke model.

We start by describing the syntax and semantics of $L[\forall]$. The language $\sigma^{m\ell u}$ of $L[\forall]$ consists of the binary operation symbols $\land$ and $\lor$, the unary operation symbols $\neg$, $\diamond$ and $[\forall]$, and the constant symbol $\top$. Formulas, or terms, are defined in the usual way. We also write $\Box \varphi$ for $\neg \Diamond \neg \varphi$.

Satisfiability of $\sigma^{m\ell u}$-formulas is defined in terms of Kripke semantics. For more background on Kripke semantics we refer the reader to [1, Chapter 2]. A Kripke frame is a pair $\langle W, R \rangle$, where $W$ is a non-empty set of worlds and $R$ is a binary accessibility relation on $W$. A Kripke model is a pair $\mathcal{M} = \langle \mathfrak{F}, V \rangle$, where $\mathfrak{F}$ is a Kripke frame and $V$ is a valuation function that assigns to every variable a subset of $W$. The satisfaction relation between Kripke models $\mathcal{M}$, worlds $w$, and $\sigma^{m\ell u}$-formulas $\phi$ is defined as follows:

- $\mathcal{M}, w \models p \iff w \in V(p)$, for every variable $p$;
- $\mathcal{M}, w \models \top$ always holds;
- $\mathcal{M}, w \models \neg \phi_1 \iff \mathcal{M}, w \not\models \phi_1$;
- $\mathcal{M}, w \models \phi_1 \land \phi_2 \iff \mathcal{M}, w \models \phi_1$ and $\mathcal{M}, w \models \phi_2$;
- $\mathcal{M}, w \models \phi_1 \lor \phi_2 \iff \mathcal{M}, w \models \phi_1$ or $\mathcal{M}, w \models \phi_2$;
- $\mathcal{M}, w \models \Diamond \phi_1 \iff R(w, v)$ and $\mathcal{M}, v \models \phi_1$, for some $v \in W$;
- $\mathcal{M}, w \models [\forall] \phi_1 \iff \mathcal{M}, v \models \phi_1$, for every $v \in W$.

We say that a $\sigma^{m\ell u}$-formula $\phi$ is satisfiable in a Kripke model if there exists a Kripke model $\mathcal{M}$ and a world $w$ such that $\mathcal{M}, w \models \phi$.

We now describe the two person corridor tiling game from [7] on which the two person corridor tiling problem is based. Our description of the game closely follows that in [1, Section 6.8]. The game is played by two players, Eloise and Abelard, who are placing tiles of types $T_1, \ldots, T_{s+1}$ into a grid, or corridor, comprised of $n$ columns of infinite height, according to the rules described below. The boundaries of the grid are delimited by tiles of a special type $T_0$. Each side of every tile $t$ is coloured; the colors are denoted by left($t$), right($t$), up($t$), and down($t$). If $t$ is of type $T_0$, then left($t$) = right($t$) = up($t$) = down($t$). A tile of type $T_{s+1}$, like those of type $T_0$, is special—it is a “winning” tile.

At the start of a play of the game, the initial $n$ tiles are already in place in the first row of the grid. The players then take turns to place tiles into the grid. Eloise goes first in every play of the game. The rules of placement are as follows. The grid has to be filled continuously, from left to right and from bottom to top. The tile $t$ being placed has to match the colours of the neighbouring tiles already in place, including the boundary tile of type $T_0$, in the sense that left($t$) = right($t'$) for the tile $t'$ to the left of $t$, down($t$) = up($t''$) for the tile $t''$ straight below $t$, and right($t$) = left($t'''$) for the tile $t'''$, if it exists, to the right of $t$ (such a tile would always be of type $T_0$). If a winning tile of type $T_{s+1}$ is placed into the first column of a row, Eloise wins. Otherwise, Abelard wins (in particular, he wins if the play goes on indefinitely or if one of the players cannot make a move).

A winning strategy for a player is defined as usual. An instance, say $T$, of this corridor tiling game consists of the set $\{T_0, \ldots, T_{s+1}\}$ of tile types, the
number \( n \) of columns in the corridor and the types \( T_{I_1}, \ldots, T_{I_n} \) of the tiles placed in the first row.

The problem of deciding, for a given instance \( T \) of the two person corridor tiling game, if Eloise has a winning strategy in \( T \) is called the two person corridor tiling problem; it is known to be \textsc{EXPTIME}-hard [7].

Given an instance \( T \) of the two person corridor tiling game, we construct a \( \sigma^{m\ell u} \)-formula \( \phi_T \) such that \( \phi_T \) is satisfiable in a Kripke model if, and only if, Eloise has a winning strategy in \( T \). We follow closely the construction used in [1, Theorem 6.52] for Propositional Dynamic Logic.

In constructing \( \phi_T \), we use the following variables:

- \( p_1, \ldots, p_n \), to represent the grid position into which a tile is to be placed in the current round of the play;
- \( c_i(T_u) \), where \( i \in \{0, \ldots, n+1\} \) and \( u \in \{0, \ldots, s+1\} \), to assert that the tile placed in the topmost row of column \( i \) is of type \( T_u \);
- \( e \), to represent whose turn it is to make a move in the current round of the play (\( e \) for Eloise; \( \neg e \) for Abelard);
- \( w \), to assert that the current position is a winning one for Eloise, i.e., she has a winning strategy starting from the current position;
- \( q_1, \ldots, q_m \) for representing the number of rounds in the game, using a binary encoding, where \( m = \lceil \log_2(s+2)n+2 \rceil \) (see below).

The following \( \sigma^{m\ell u} \)-formula describes the initial position of a play:

\[
\text{Init} = e \land p_1 \land c_0(T_0) \land c_1(T_{I_1}) \land \cdots \land c_n(T_{I_n}) \land c_{n+1}(T_0).
\]

We now describe the rules of the game with \( \sigma^{m\ell u} \)-formulas.

Tiles are placed in exactly one of the columns 1 though \( n \):

\[
R_1 = [\forall] \left( (p_1 \lor \cdots \lor p_n) \land \bigwedge_{i=1}^{n} \bigwedge_{j \neq i} (\neg p_i \lor \neg p_j) \right).
\]

In every column, a tile of exactly one type has been previously placed:

\[
R_2 = [\forall] \bigwedge_{i=0}^{n+1} \bigwedge_{u=0}^{s+1} (c_i(T_0) \lor \cdots \lor c_i(T_{s+1})) \land [\forall] \bigwedge_{i=0}^{n+1} \bigwedge_{u=0}^{s+1} \bigwedge_{v \neq u} (\neg c_i(T_u) \lor \neg c_i(T_v)).
\]

Columns 0 and \( n+1 \) always contain a tile of type \( T_0 \):

\[
R_3 = [\forall] (c_0(T_0) \land c_{n+1}(T_0)).
\]

The tiles are always placed in successive positions:

\[
R_4 = [\forall] ((\neg p_1 \lor \Box p_2) \land (\neg p_2 \lor \Box p_3) \land \cdots \land (\neg p_n \lor \Box p_1)).
\]

In a column where no tile is being placed, nothing changes once a move has been made:

\[
R_5 = [\forall] \bigwedge_{i=0}^{n+1} \bigwedge_{u=0}^{s+1} (p_i \lor (\neg c_i(T_u) \lor \Box c_i(T_u)) \land (c_i(T_u) \lor \Box \neg c_i(T_u))).
\]

Players alternate in their moves:

\[
R_6 = [\forall] ((\neg e \lor \Box \neg e) \land (e \lor \Box e)).
\]
Players only place tiles that match the ones placed to the left and below:

\[ \forall i \left( \neg (p_i \land c_{i-1}(T') \land c_i(T'')) \lor \bigvee_{C(T',T'',T)} c_i(T) \right), \]

where \( C(T',T'',T) \) holds if, and only if, \( \text{right}(T') = \text{left}(T) \) and \( \text{up}(T'') = \text{down}(T) \). In column \( n \), players only place tiles that match the boundary tile to the right:

\[ \forall \left( \neg p_n \lor \bigvee_{\text{right}(T)=\text{left}(T_0)} c_n(T) \right). \]

At his turn, Abelard can place any tile permitted by the rules of the game:

\[ \forall \left( \neg (\neg e \land p_i \land c_{i-1}(T'') \land c_i(T')) \lor \bigwedge_{C(T',T'',T)} \Diamond c_i(T) \right) \]

\[ \land \forall \left( \neg (\neg e \land p_n \land c_{n-1}(T'') \land c_n(T')) \lor \bigwedge_{C(T',T'',T),\text{right}(T)=\text{left}(T_0)} \Diamond c_n(T) \right). \]

We next say, using \( \sigma^{m\ell u} \)-formulas, that Eloise has a winning strategy in a play. The initial position is a winning position for Eloise. At all other positions, one of the following holds:

- a winning tile of type \( T_{n+1} \) has been placed in column 1;
- if Eloise is to move at the current position, then there exists a move to a winning position for Eloise;
- if Abelard is to move at the current position, then he can make a move and all his moves result in a winning position for Eloise.

Thus, we define

\[ \text{Win} = w \land \forall \left( \neg w \lor c_1(T_{s+1}) \lor (e \land \Diamond w) \lor (\neg e \land \Diamond \top \land \Box w) \right). \]

The play does not run forever. First, notice that, since the number of tile types is finite, the play runs forever if, and only if, a row in the tiling is repeated in the course of the play. Second, observe that such a repetition is bound to occur once the play has gone on for \( N = (s + 2)^{n+2} \) rounds. We can represent all the numbers 0 through \( N \) in binary using propositional variables \( q_1, \ldots, q_m \), where \( m = \lceil \log_2 N \rceil \). Let

\[ I_0 = q_1 \lor \left( \Box q_1 \land \bigwedge_{j=2}^m \left( (\neg q_j \lor \Box q_j) \land (q_j \lor \Box \neg q_j) \right) \right), \]

\[ I_1 = \neg \left( \neg q_{i+1} \land \bigwedge_{j=1}^i q_j \right) \]

\[ \lor \left( \Box q_{i+1} \land \bigwedge_{j=1}^i \Box \neg q_j \land \bigwedge_{k=i+2}^m \left( (\neg q_k \lor \Box q_k) \land (q_k \lor \Box \neg q_k) \right) \right). \]
Now, let

\[ F = \neg q_m \land \cdots \land \neg q_1 \land \left[ \forall \right] \left( I_0 \land \bigwedge_{i=1}^{m-1} I_1^i \right) \land \left[ \forall \right] \left( \neg q_m \lor \cdots \lor \neg q_1 \lor \Box \neg w \right). \]

Finally, define

\[ \phi_T = \text{Init} \land R_1 \land R_2 \land R_3 \land R_4 \land R_5 \land R_6 \land R_7 \land R_8 \land R_9 \land \text{Win} \land F. \]

**Theorem 6.1.** Let \( T \) be an instance of the two person corridor tiling game. Then, Eloise has a winning strategy in \( T \) if, and only if, \( \phi_T \) is satisfiable in a Kripke model.

**Proof.** Similar to the proof of Theorem 6.52 from [1]. \( \square \)

Next, given an instance \( T \) of the two person corridor tiling game, we construct a quantifier-free first-order \( \sigma^{\text{bdo}} \)-formula \( \varphi_T \) (based on \( \phi_T \)) such that \( \varphi_T \) is satisfiable in a \( \text{bdo} \) if, and only if, \( \phi_T \) is satisfiable in a Kripke model. The formula \( \varphi_T \) is a conjunction of two parts: (1) \( T r(\phi_T) \), the ‘translation’ of \( \phi_T \) and (2) \( \zeta_T \), the conjunction of identities that ‘simulate’ the occurrences of \( \neg \) and \( \Box \) in \( \phi_T \) (which is needed since the language \( \sigma^{\text{bdo}} \) contains neither \( \neg \) nor \( \Box \)).

First, distribute \( \neg \) in \( \phi_T \) over \( \lor \) and \( \land \) and eliminate double negations (call the resultant formula \( \phi_T \) to avoid notational clutter). Then, \( \neg \) in \( \phi_T \) only applies to variables. Next, perform the following steps:

1. For every variable \( p \) in \( \phi_T \), introduce a fresh variable \( p' \) and add to \( \zeta_T \) the identities \( p \lor p' = 1 \) and \( p \land p' = 0 \).
2. For every subformula of \( \phi_T \) of the form \( \Box p \), introduce a fresh variable \( b(p) \) and add to \( \zeta_T \) the identities \( b(p) \lor \Diamond p' = 1 \) and \( b(p) \land \Diamond p' = 0 \).
3. For every subformula of \( \phi_T \) of the form \( \neg \Box p \), introduce a fresh variable \( bn(p) \) and add to \( \zeta_T \) the identities \( bn(p) \lor \Diamond p = 1 \) and \( bn(p) \land \Diamond p = 0 \).
4. For every subformula of \( \phi_T \) of the form \( \Box (p_1 \lor \cdots \lor p_k) \), introduce a fresh variable \( bd(p_1, \ldots, p_k) \) and add to \( \zeta_T \) the identities
   \[ bd(p_1, \ldots, p_k) \lor \Diamond (p_1' \land \cdots \land p_k') = 1 \quad \text{and} \quad bd(p_1, \ldots, p_k) \land \Diamond (p_1' \land \cdots \land p_k') = 0. \]
5. For every subformula of \( \phi_T \) of the form \( \Box \bigwedge_{i=1}^{k} \bigvee_{j=i}^{l} p_{i,j} \), introduce a fresh variable \( bcd(p_{1,1}, \ldots, p_{k,l}) \), and add to \( \zeta_T \) the identities
   \[ bcd(p_{1,1}, \ldots, p_{k,l}) \lor \Diamond \bigwedge_{i=1}^{k} \bigvee_{j=i}^{l} p_{i,j}' = 1 \quad \text{and} \quad bcd(p_{1,1}, \ldots, p_{k,l}) \land \Diamond \bigwedge_{i=1}^{k} \bigvee_{j=i}^{l} p_{i,j}' = 0. \]

Construct a formula \( T r(\phi_T) \) as follows. Put \( \phi_T \) into the form

\[ \hat{\phi}_T = \chi \land \left[ \forall \right] \psi_1 \land \cdots \land \left[ \forall \right] \psi_m. \]

This can be done by bringing together all the conjuncts of \( \phi_T \) that do not have occurrences of \( \left[ \forall \right] \) into a single conjunction \( \chi \). Notice that none of \( \chi, \psi_1, \ldots, \psi_m \) contain occurrences of \( \left[ \forall \right] \) since \( \left[ \forall \right] \) never occurs within the scope of a modal
connective. For every formula in \( \{ \chi, \psi_1, \ldots, \psi_m \} \), recursively define the translation \( \cdot^* \) of \( \sigma^{\text{mfn}} \)-formulas into \( \sigma^{\text{bdo}} \)-terms, as follows:

\[
\begin{align*}
p^* &= p, \text{ where } p \text{ is a variable outside of the scope of } \neg \text{ and } \Box; \\
(\neg p)^* &= p', \text{ where } p \text{ is a variable outside of the scope of } \Box; \\
(\phi_1 \land \phi_2)^* &= (\phi_1)^* \land (\phi_2)^*, \text{ where } \phi_1 \land \phi_2 \text{ is outside of the scope of } \Box; \\
(\phi_1 \lor \phi_2)^* &= (\phi_1)^* \lor (\phi_2)^*, \text{ where } \phi_1 \lor \phi_2 \text{ is outside of the scope of } \Box; \\
(\langle \phi \rangle)^* &= \langle \phi^* \rangle; \\
(\Box p)^* &= b(p); \\
(\Box (p_1 \lor \cdots \lor p_k))^* &= bd(p_1, \ldots, p_k); \\
\left( \bigwedge_{i=1}^{k} \bigvee_{j=1}^{l} \beta_{i,j} \right)^* &= bcd(p_{1,1}, \ldots, p_{k,i}).
\end{align*}
\]

Finally, let \( T_r(\phi_T) \) be the following quantifier-free \( \sigma^{\text{bdo}} \)-formula:

\[
(\text{not}(x^* = 0)) \text{ and } \psi_1^* = 1 \text{ and } \ldots \text{ and } \psi_m^* = 1.
\]

Lemma 6.2. Let \( \phi_T \) and \( \varphi_T \) be formulas constructed, as above, from an instance \( T \) of the two person corridor tiling game. Then, \( \varphi_T \) is satisfiable in a \( \text{bdo} \) if, and only if, \( \varphi_T \) is satisfiable in a Kripke model.

Proof. Assume that \( \varphi_T \) is true in some \( \text{bdo} \) \( \mathbb{A} = \langle A, \land, \lor, \Diamond, 0, 1, \subseteq \rangle \) under some valuation \( v \). Recall that the \( \text{bdo} \)-frame \( \mathfrak{F}_A \) associated with \( \mathbb{A} \) is the structure \( \langle \mathcal{P}, \subseteq, \mathcal{R} \rangle \), where \( \mathcal{P} \) is the set of prime filters of \( A \) and \( \mathcal{R}(F, G) \) iff \( a \in G \) implies \( \Diamond a \in F \), for all \( a \in A \). If we omit the order relation, then \( \langle \mathcal{P}, \mathcal{R} \rangle \) is a Kripke frame and \( \mathfrak{M} = \langle \mathcal{P}, \mathcal{R}, V \rangle \), where \( V(p) = \{ F \in \mathcal{P} \mid v(p) \in F \} \) for each variable \( p \), is a Kripke model. We show that, for each \( F_0 \in \mathcal{P} \), \( \mathfrak{M}, F_0 \models \varphi_T \) holds for some \( F_0 \in \mathcal{P} \).

As \( v(\chi^*) \neq 0 \), there exists \( F_0 \in \mathcal{P} \) containing all of \( v(e), v(p_1), v(c_0(T_0)), v(c_1(T_{1,1})), \ldots, v(c_n(T_{n,1})), v(w), v(q'_1), \ldots, v(q'_m) \). Therefore, \( \mathfrak{M}, F_0 \models \chi \). It remains to show that, for every subformula of \( \varphi_T \) of the form \( [\forall] \psi \), the formula \( \psi \) is true in \( \mathfrak{M} \) at every \( F \in \mathcal{P} \). We consider one such \( \psi \) as an example, say \( \psi_0 = \neg w \lor (c_1(T_{s+1}) \lor (e \land \Diamond w)) \lor (\neg e \land \Diamond \top \land \Box w) \), which is a subformula of \( \text{Win} \). Let \( F \in \mathcal{P} \); we need to show that \( \mathfrak{M}, F \models \psi_0 \). By assumption, all of the following are true in \( A \) under \( v \) (we henceforth omit \( v \) for ease of notation):

\[
\begin{align*}
w' \lor c_1(T_{s+1}) \lor (e \land \Diamond w) \lor (e' \land \Diamond \top \land b(w)) &= 1; \\
e \lor e' &= 1; \quad w \lor w' &= 1; \quad b(w) \lor \Diamond w' &= 1; \\
e \land e' &= 0; \quad w \land w' &= 0; \quad b(w) \land \Diamond w' &= 0.
\end{align*}
\]

Since \( 1 \in F \), either \( w' \) or \( c_1(T_{s+1}) \) or \( e \land \Diamond w \) or \( e' \land \Diamond \top \land b(w) \) belongs to \( F \). Assume that \( w' \in F \). As \( w \land w' = 0 \), we have \( w \notin F \), i.e., \( F \notin V(w) \), hence \( \mathfrak{M}, F \models \neg w \), and so \( \mathfrak{M}, F \models \psi_0 \). It is straightforward to show that \( \mathfrak{M}, F \models \psi_0 \) when \( c_1(T_{s+1}) \in F \) and when \( e \land \Diamond w \in F \). Assume that \( e' \land \Diamond \top \land b(w) \in F \); thus, \( e' \in F \), \( \Diamond \top \in F \), and \( b(w) \in F \). As \( e \land e' = 0 \), we have \( e \notin F \), hence
\( M, F \models \neg e. \) As \( b(w) \land \Diamond w' = 0, \) we have \( \Diamond w' \notin F. \) Suppose that \( R(F,G) \) for some \( G \in P; \) then, \( w' \notin G, \) and thus \( w \in G, \) as \( w \lor w' = 1; \) therefore, \( M, F \models \Box w, \) and so \( M, F \models \psi_0. \)

Assume, on the other hand, that \( M, w_0 \models \phi_T \) for some Kripke model \( M = \langle W, R, V \rangle \) and some \( w_0 \in W. \) Let \( \leq \) be the trivial partial order on \( W, \) that is \( w \leq z \) if, and only if, \( w = z. \) Then, \( \mathcal{F} = \langle W, \leq, R \rangle \) is a \( \text{bdo} \)-frame and so we can construct its associated \( \text{bdo} \) \( A_{\mathcal{F}} = \langle \mathcal{U}(W), \cup, \cap, \Diamond, \Box, W, \leq \rangle \) as in Section 2. It remains to define a valuation \( v \) that will satisfy \( \varphi_T. \) For the variables of \( \phi_T, \) let \( v(p) = \{ w \in W \mid w \in V(p) \}. \) For the variables not occurring in \( \phi_T, \) but occurring in \( \varphi_T, \) the valuation \( v \) is defined as follows:

\[
\begin{align*}
v(p') &= \{ w \in W \mid w \notin V(p) \}; \\
v(bp) &= \{ w \in W \mid R(w, v) \text{ implies } v \in V(p) \}; \\
v(bn(p)) &= \{ w \in W \mid R(w, v) \text{ implies } v \notin V(p) \}; \\
v(bd(p_1, \ldots, p_k)) &= \{ w \in W \mid R(w, v) \text{ implies } v \in V(p_1) \cup \cdots \cup V(p_k) \}; \\
v(bcd(p_1,1, \ldots, p_k,l)) &= \left\{ w \in W \mid R(w, v) \text{ implies } v \in \bigcap_{i=1}^k \bigcup_{j=1}^l V(p_{i,j}) \right\}.
\end{align*}
\]

It is then straightforward to check that \( \varphi_T \) is true in \( A_{\mathcal{F}} \) under \( v. \) \hfill \( \square \)

Observe that constructing the formula \( \varphi_T \) from an instance \( T \) of the two person corridor tiling game can be done in polynomial time. The above results, therefore, give us a polynomial time reduction of the two person corridor tiling problem to satisfiability in \( \text{BDO}. \) Thus, we have the following.

**Theorem 6.3.** Satisfiability in \( \text{BDO} \) is \( \text{EXPTIME}-\text{hard}. \)

We now apply Theorem 6.3 to obtain similar results for the classes \( \text{BDBO} \) and \( \text{BRDG}. \)

**Corollary 6.4.** Satisfiability in \( \text{BDBO} \) and in \( \text{BRDG} \) are both \( \text{EXPTIME}-\text{hard}. \)

**Proof.** Let \( \varphi_T \) be a quantifier-free first-order \( \sigma_{\text{bdo}} \)-formula constructed from an instance \( T \) of the two-person corridor tiling game, as above. Recall that \( \varphi_T \) is satisfiable in a \( \text{bdo} \) if, and only if, Eloise has a winning strategy in the game instance \( T. \) In polynomial time, we may construct the formula \( \varphi_T^0 \) from \( \varphi_T \) as in Section 5. Recall, from Lemma 5.4, that \( \varphi_T \) is satisfiable in a \( \text{bdo} \) if, and only if, \( \varphi_T^0 \) is satisfiable in a \( \text{bdo}. \) In addition, by Lemma 5.2, \( \varphi_T^0 \) is satisfiable in a \( \text{bdo} \) if, and only if, \( \varphi_T^0 \) is satisfiable in a \( \text{brdg}. \) The result now follows from the \( \text{EXPTIME}-\text{hardness} \) of the tiling problem. \hfill \( \square \)

Recall that, for a class \( K \) and universal sentence \( \Phi, \) we have \( K \models \Phi \) iff \( \text{not} \varphi \) is not satisfiable in \( K, \) where \( \varphi \) is the quantifier-free part of \( \Phi. \) This yields the connection between the complexity of the universal theory of \( K \) and the complexity of satisfiability in \( K. \) Noting also that the complement of an \( \text{EXPTIME}-\text{complete} \) problem is also \( \text{EXPTIME}-\text{complete,} \) we obtain our main result.

**Theorem 6.5.** Satisfiability in each of the classes \( \text{BDO}, \text{BDBO} \) and \( \text{BRDG} \) is \( \text{EXPTIME}-\text{complete}. \) The universal theory of each of the classes \( \text{BDO}, \text{BDBO} \) and \( \text{BRDG} \) is \( \text{EXPTIME}-\text{complete}. \)
Recall that a quasi-identity (or quasi-equation) is a universal sentence of the form \((\forall x_1, \ldots, x_n)(s_1 = t_1 \text{ and } \ldots \text{ and } s_n = t_n \Rightarrow u = v)\), and that the quasi-equational theory of a class \(\mathcal{K}\) is the set of all quasi-identities \(\Phi\) such that \(\mathcal{K} \models \Phi\). Consider a quantifier-free formula \(\varphi_T\) constructed from an instance \(T\) of the two person corridor tiling game, as above. Observe that if the existential quantifiers are added to \(\varphi_T\), then the resulting sentence is the negation of a quasi-identity. Thus, the above proofs establish the lower bound for the quasi-equational theory of each of the classes \(BDO, BDBO\) and \(BRDG\). Thus, we have the following result.

**Corollary 6.6.** The quasi-equational theory of each of the classes \(BDO, BDBO\) and \(BRDG\) is \(\text{EXPTIME}\)-complete.

It follows from Corollary 6.6 that the consequence relation for Distributive Nonassociative Full Lambek Calculus with Bounds is \(\text{EXPTIME}\)-complete.

Since \(P \neq \text{EXPTIME}\) and \(P\) is conventionally considered to be the class of ‘tractable’ problems, the above results show that the universal and quasi-equational theories for each of the classes \(BDO, BDBO\) and \(BRDG\) are ‘intractable’.

### 7. Special classes of \(brdg\)’s

We now consider complexity of universal theories of special classes of \(brdg\)’s that correspond to standard extensions of Distributive Nonassociative Full Lambek Calculus with Bounds. In particular, we consider the following properties of \(\circ\):

1. **commutative**: \(x \circ y = y \circ x\)
2. **decreasing**: \(x \circ y \leq x\) and \(x \circ y \leq y\)
3. **square-increasing**: \(x \leq x \circ x\)
4. **unital**: There exists \(e\) such that \(x \circ e = e \circ x = x\).

There is a correspondence between the properties of \(\circ\) listed above and properties of the relation \(R\) on \(brdg\)-frames \((P, \leq, R)\). The frame conditions corresponding to (P1–P4) are as follows (see, e.g., [10]):

1. \((\forall x, y, z \in P)(R(x, y, z) \Rightarrow R(y, x, z))\)
2. \((\forall x, y, z \in P)(R(x, y, z) \Rightarrow (x \leq z \text{ and } y \leq z))\)
3. \((\forall x \in P) R(x, x, x)\)
4. There exists a set \(E \subseteq P\) such that
   \[(\forall x, y, z \in P)(R(x, y, z) \text{ and } y \in E \Rightarrow x \leq z)\]
   \[(\forall x, y, z \in P)(R(x, y, z) \text{ and } x \in E \Rightarrow y \leq z)\]
   \[(\forall x \in P)[(\exists y \in E)R(x, y, x) \text{ and } (\exists z \in E)R(z, x, x)].\]

If \(\mathbb{A}\) is a \(brdg\) satisfying any subset of the properties (P1–P4), then the associated \(brdg\)-frame \(\mathfrak{F}_\mathbb{A}\) satisfies the corresponding properties in (R1–R4). In addition, if \(\mathfrak{F}\) is a \(brdg\)-frame satisfying any subset of the properties (R1–R4), then the associated \(brdg\) \(\mathbb{A}_\mathfrak{F}\) satisfies the corresponding properties in (P1–P4).
We also note that in the case of (P4), the upward closure of the set \( E \) is the identity element of \( \mathbb{A}_3 \).

We seek to characterize partial algebras for the special classes of brdg’s considered above. We describe the changes required in the statement and proof of Theorem 4.1 in each case.

In the case that \( \mathbb{A} \) is a commutative brdg, we replace \( R^B \) in the statement of Theorem 4.1 by the relation \( R^B_1 \) defined by:

\[
R^B_1(f,g,h) \iff [R^B(f,g,h) \text{ and } R^B(g,f,h)],
\]

where \( R^B \) is as before, that is,

\[
R^B(f,g,h) \iff (\forall (a,b) \in \text{dom}(\circ^B)(a \in f \text{ and } b \in g \Rightarrow a \circ^B b \in h) \text{ and } (\forall (a,b) \in \text{dom}(\backslash^B)(a \in f \text{ and } a \backslash^B b \in g \Rightarrow b \in h) \text{ and } (\forall (a,b) \in \text{dom}(/^B)(b/^B a \in f \text{ and } a \in g \Rightarrow b \in h)).
\]

For the proof, suppose that \( B \) is a partial substructure of a commutative brdg \( \mathbb{A} \). As in the proof of Theorem 4.1, define the set

\[
\mathcal{F} := \{ F \cap B \mid F \text{ is a prime filter of } \mathbb{A} \}.
\]

Then the brdg-frame \( \langle \mathcal{P}, \subseteq, \mathcal{R} \rangle \) associated with \( \mathbb{A} \) satisfies (R1) and it is straightforward to check that (4.5) holds for the relation \( R^B_1 \). It then follows, as in the proof of Theorem 4.1, that (D), (M_\circ), (M_\backslash) and (M_/) hold, with \( R^B_1 \) replacing \( R^B \). Conversely, if we start with a partial \( \sigma^{\text{brdg}} \)-structure \( \mathbb{B} \) satisfying all the requirements of Theorem 4.1 with the relation \( R^B_1 \) replacing \( R^B \), then the structure \( \mathfrak{F} = \langle \mathcal{F}, \subseteq, R^B_1 \rangle \) satisfies (2.6–2.8), hence it is a brdg-frame and it clearly satisfies (R1). Thus, the brdg \( \mathbb{A}_3 \) constructed from \( \mathfrak{F} \) and into which \( \mathbb{B} \) embeds, is commutative.

Commutative brdg’s satisfy \( x \backslash y = y/x \) and so a single binary operation \( \rightarrow \) is usually used in place of \( \backslash \) and \( / \), replacing \( x \backslash y \) and \( y/x \) by \( x \rightarrow y \). With this change the description of partial substructures of commutative brdgs can be written in a simpler form. We retain our notation, however, for uniformity of presentation.

For decreasing brdg’s the required change to Theorem 4.1 is that \( R^B \) is replaced by the relation \( R^B_2 \) defined by:

\[
R^B_2(f,g,h) \iff [R^B(f,g,h) \text{ and } f \subseteq h \text{ and } g \subseteq h].
\]

If \( \mathbb{B} \) is a partial substructure of a decreasing brdg \( \mathbb{A} \), then, in this case, the brdg-frame \( \langle \mathcal{P}, \subseteq, \mathcal{R} \rangle \) associated with \( \mathbb{A} \) satisfies (R2). It then follows that (4.5) holds for the relation \( R^B_2 \) and that (D), (M_\circ), (M_\backslash) and (M_/) hold. Conversely, if we start with a partial \( \sigma^{\text{brdg}} \)-structure \( \mathbb{B} \) satisfying all the requirements of Theorem 4.1 with the relation \( R^B_2 \) replacing \( R^B \), then the structure \( \mathfrak{F} = \langle \mathcal{F}, \subseteq, R^B_2 \rangle \) satisfies (2.6–2.8) and (R2). Thus, the brdg \( \mathbb{A}_3 \) constructed from \( \mathfrak{F} \) and into which \( \mathbb{B} \) embeds, is decreasing.

For square-increasing brdg’s the required change to Theorem 4.1 is to the definition of a prime filter of the partial structure \( \mathbb{B} \). A set \( f \subseteq B \) is now called a prime filter of \( \mathbb{B} \) if it satisfies (4.1–4.4), as well as the following:
If $\mathbb{B}$ is a partial substructure of a square-increasing $\text{brdg} \ A$, then the proofs that $(D)$, $(M_e)$, $(M_{\wedge})$ and $(M_j)$ hold (with respect to the relation $R^\mathbb{B}$) are as before. We need only check that each $F \cap B \in \mathcal{F}$ satisfies (7.1–7.3), which follows from the fact that any prime filter $F$ of a square-increasing $\text{brdg}$ is closed under $\circ$. Conversely, if we start with a partial $\sigma^\text{brdg}$-structure $\mathbb{B}$ satisfying all the requirements of Theorem 4.1 with the additional requirements (7.1–7.3) on elements of $\mathcal{F}$, then it is clear that the structure $\mathcal{F} = \langle \mathcal{F}, \subseteq, R^\mathbb{B} \rangle$ is a $\text{brdg}$-frame for which $R^\mathbb{B}(f, f, f)$ holds for each $f \in \mathcal{F}$, i.e., (R3) holds. Thus, the $\text{brdg} \ A_{\mathcal{F}}$, constructed from $\mathcal{F}$ and into which $\mathbb{B}$ embeds, is square-increasing.

In the case of unital $\text{brdg}$’s, we require that the identity element $e$ be included in the language so that, by our convention, the identity element is an element of every partial substructure. Thus, we define the language $\sigma^{\text{brdg}}$ as $\sigma^{\text{brdg}}$ augmented with a constant symbol $e$ and by a $\text{brdg}$ we mean a $\sigma^{\text{brdg}}$-algebra whose $\sigma^{\text{brdg}}$-reduct is a $\text{brdg}$ and that satisfies $x \circ e = x = e \circ x$. To characterize partial $\text{brdg}$’s, in the statement of Theorem 4.1, we replace $R^\mathbb{B}$ by the relation $R^\mathbb{B}_4$ defined by

$R^\mathbb{B}_4(f, g, h) \iff [R^\mathbb{B}(f, g, h) \text{ and } (\forall a \in B)(a \in f \text{ and } e^\mathbb{B} \in g \Rightarrow a \in h)] \text{ and } (\forall a \in B)(e^\mathbb{B} \in f \text{ and } a \in g \Rightarrow a \in h)].$

In addition, we require that the following condition holds:

$$(M_e) \ (\forall f \in \mathcal{F})[(\exists g \in \mathcal{F})(e^\mathbb{B} \in g \text{ and } R^\mathbb{B}_4(f, g, f)) \text{ and } (\exists h \in \mathcal{F})(e^\mathbb{B} \in h \text{ and } R^\mathbb{B}_4(h, f, f))].$$

For the proof, suppose that $\mathbb{B}$ is a partial substructure of a $\text{brdg} \ A$. Then the $\text{brdg}$-frame $\langle \mathcal{P}, \subseteq, \mathcal{R} \rangle$ associated with $\mathbb{A}$ satisfies (R4). Define the set $\mathcal{F}$ as above. It is straightforward to check that (4.5) holds for the relation $R^\mathbb{B}_4$, from which it follows, as in the proof of Theorem 4.1, that $(D)$, $(M_e)$, $(M_{\wedge})$ and $(M_j)$ hold. To see that $(M_e)$ holds, we require the following result.

**Lemma 7.1** [10]. Let $\mathbb{A}$ be a $\text{brdg}$. For any prime filter $F$ of $\mathbb{A}$, there exists a prime filter $G$ of $\mathbb{A}$ with $e^\mathbb{A} \in G$ and $\mathcal{R}(F, G, F)$ and there exists a prime filter $H$ of $\mathbb{A}$ with $e^\mathbb{A} \in H$ and $\mathcal{R}(H, F, F)$.

Let $f \in \mathcal{F}$, so $f = F \cap B$ for some prime filter $F$ of $\mathbb{A}$. By Lemma 7.1, there exists a prime filter $G$ of $\mathbb{A}$ such that $e^\mathbb{A} \in G$ and $\mathcal{R}(F, G, F)$. Then, for $g := G \cap B$, we have $e^\mathbb{B} = e^\mathbb{A} \in g$, while $R^\mathbb{B}_4(f, g, f)$ follows from (4.5). The required $h$ is obtained similarly.

Conversely, suppose $\mathbb{B}$ is a partial $\sigma^{\text{brdg}}$-structure satisfying the requirements above. Then $\langle \mathcal{F}, \subseteq, R^\mathbb{B}_4 \rangle$ is a $\text{brdg}$-frame; we show that it satisfies (R4) so that the constructed $\text{brdg} \ A_{\mathcal{F}}$ is a $\text{brdg}$. Set $\mathcal{E} := \{g \in \mathcal{F} : e^\mathbb{B} \in g\}$. Suppose $f, g, h \in \mathcal{F}$ are such that $R^\mathbb{B}_4(f, g, h)$ and $g \in \mathcal{E}$, and let $a \in f$. Since $e^\mathbb{B} \in g$, 

$$(\forall(a, b) \in \text{dom}(\circ))(a \in f \text{ and } b \in f \Rightarrow a \circ b \in f),$$

$$(\forall(a, b) \in \text{dom}(\text{closure}))(a \in f \text{ and } a \\text{closure} b \in f \Rightarrow b \in f),$$

$$(\forall(a, b) \in \text{dom}(\text{closure}))(a/f \text{ and } b \in f \Rightarrow a \in f).$$
we have $a \in h$, by definition of $R_4^B$. Thus, $f \subseteq h$. Similarly, if $R_4^B(f, g, h)$ and $f \in E$, then $g \subseteq h$. The third condition in (R4) is immediate from (M_e).

Lastly, the identity element of $\mathbb{A}_3$ is $E$ and the embedding $\mu$, as defined in Theorem 4.1, satisfies $\mu(e_B^f) = \{ f \in F : e_B^f \in f \} = E$.

We note that the above adaptations to the characterizations of partial substructures of BRDG’s satisfying properties (P1–P4) can be combined in obvious ways to provide characterizations of partial substructures of BRDG’s satisfying any subset of the properties (P1–P4). In addition, the EXPTIME-algorithm for satisfiability in BRDG described in Section 5 can be easily adapted to cater for these characterizations. Indeed, for (P1) and (P2) we need only use a different version of $R_B^E$, and for (P3) we need only check that the prime filters satisfy the additional requirements. For (P4), we require relation $R_4^B$ and we must additionally check condition (M_e) for each $f$ during each traversal of the current set $F$. If there does not exist $g \in F$ with $e_B^g \in g$ and $R_B^E(f, g, f)$, then we remove $f$ from $F$; similarly, for $h$. This does not change the exponential running time of the algorithm. Thus, by the methods of Section 5, we get an upper bound of exponential time for the complexity of satisfiability in the class of BRDG’s with any combination of the additional properties (P1–P4). As regards the lower bound, we can only infer the exponential time lower bound for BRDG’s with the commutative property, which follows from the fact that the operation $\circ$ defined in Lemma 2.4 is commutative. Thus, we have the following result.

**Theorem 7.2.** For any subset $Q$ of the properties (P1–P4), let $BRDG^Q$ be the class of all BRDG’s satisfying the properties in $Q$. Then, satisfiability in $BRDG^Q$ and the universal theory of $BRDG^Q$ are both in EXPTIME. In the case where $Q$ contains only (P1), that is, for commutative BRDG’s, satisfiability in $BRDG^Q$ and the universal theory of $BRDG^Q$ are both EXPTIME-complete.

8. Discussion

We conclude by mentioning some unresolved problems closely related to the results presented in this paper.

Theorem 7.2 provides an upper bound for the complexity of the universal theory of each class $BRDG^Q$, where $Q$ is a subset of (P1–P4). Tight complexity bounds for these classes are not known, with three exceptions. One is the case where $Q$ is empty, that is for BRDG; another is where $Q$ contains just the commutative property (P1), as discussed in the previous section. The other is the case where $Q$ contains all properties (P1–P4), or even just (P2–P3), from which (P1) and (P4) follow; in this case, $BRDG^Q$ is the variety of Heyting algebras, whose universal theory is known to be PSPACE-complete (see [28]).

If we remove the bounds 0 and 1 from the language of BRDG’s, we obtain the class of ‘residuated distributive lattice-ordered groupoids’, or rdg’s for short. Since any rdg can be embedded into a brdg (in a straightforward way), the EXPTIME upper bound for the universal theory of BRDG holds for the class of rdg’s as well; however, the proof of the EXPTIME lower bound presented in
this paper requires the bounds, so the tight complexity of the universal theory of this class remains unknown.

Lastly, the complexity of the equational theories of $BRDG$, $BDO$ and $BDBO$, to the best of our knowledge, has not been settled.

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