Quantum Field Theory on Quantum Spacetime

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Abstract. Condensed account of the Lectures delivered at the Meeting on Noncommutative Geometry in Field and String Theory, Corfu, September 18 - 20, 2005.

1. Introduction.
Can Quantum Mechanics and General Relativity merge in a unified consistent theory? Opinions range from the positive assertion of String Theory to the opposite extreme; yet a similar question is in a sense still open also for Quantum Mechanics and Special Relativity, since no exactly solved nontrivial model of interacting Quantum Field Theory on Minkowski space is yet known. However Quantum Mechanics and Special Relativity do meet in one basic principle, the principle of locality, which by itself (rather surprisingly, with unreasonable effectiveness) implies most of the conceptual and formal structure of Quantum Field Theory (e.g., the existence of a gauge group and the global gauge principle based only on local observables, cf [1, 2]). Do Quantum Mechanics and General Relativity meet in a basic principle, valid at all scales, which at scales which are large compared to the Planck scale reduces to locality, and is equally rigid and fruitful of structural consequences? Following the lessons of Einstein and Heisenberg, which marked the birth of Modern Physics, can it be operationally grounded?

Unfortunately we know too little of Physics at Planck scale, where "operational" might loose meaning; yet in this spirit it is possible at least to draw limitations on the very meaning of Spacetime, which lead to Quantum Spacetime. The theory of interacting Quantum Fields on Quantum Spacetime can then be tackled; the various formulations which are equivalent to one another on the classical Minkowski space are now distinct; there is no Osterwalder - Schrader connection between Minkowkian and Euclidian Theories, which might well be complelety unrelated, in particular as far as divergences are concerned; causality is lost, as expected, but it is still unclear whether interaction of fields on Quantum Spacetime is reconciliable with Lorentz invariance. We summarise merits and defects of some of the approaches.

2. The Basic model.
At large scales spacetime is a pseudo Riemaniann manifold locally modelled on Minkowski space. But the concurrence of the principles of Quantum Mechanics and of Classical General Relativity renders this picture untenable in the small.

Those theories are often reported as hardly reconcilable, but they do meet at least in a single principle, which concerns just the background geometry, the Principle of Gravitational Stability against localization of events formulated in [3, 4, 5]:

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The gravitational field generated by the concentration of energy required by the Heisenberg Uncertainty Principle to localise an event in spacetime should not be so strong to hide the event itself to any distant observer - distant compared to the Planck scale.

Already at a semiclassical level, this Principle leads to Spacetime Uncertainty Relations, that were proposed and shown to be implemented by Commutation Relations between coordinates, thus turning Spacetime into Quantum Spacetime [3, 4]. The word “Quantum” is very appropriate here, to stress that noncommutativity does not enter just as a formal generalization, but is strongly suggested by a compelling physical reason (unlike the very first discussions of possible noncommutativity of coordinates in the pre-renormalisation era, by Heisenberg, Snyder and Yang, where noncommutativity was regarded as a curious, in itself physically doubtful, regularization device).

Such an analysis leads to the following conclusions:

(i) There is no a priori lower limit on the precision in the measurement of any single coordinate (it is worthwhile to stress once more that the apparently opposite conclusions, still often reported in the literature, are drawn under the implicit assumption that all the space coordinates of the event are simultaneously sharply measured);

(ii) The Space Time Uncertainty Relations emerging from the Principle of Gravitational Stability against localization of events, in their weak form and disregarding the contributions to the source in Einstein Equations due to the average energy momentum density in generic quantum states, can be implemented by covariant commutation relations between the coordinates, which define a fully Poincare' covariant Basic Model of Quantum Spacetime. This situation parallels that of Non Relativistic Quantum Mechanics, where position and momentum become operators, but they are acted upon by the classical Galilei Group. Indeed the quantum features of spacetime should show up only in the small, at Planck scale, while the Poincare group describes the global motions of spacetime, and should act accordingly the same way both in the small and in the large, where of course the classical Poincare' group governs the scene. In particular we can say that spacetime coordinates are operators, but their translation parameters are numbers.

It is also appropriate to emphasise that we are discussing Quantum Minkowski Space, that is we look for quantum aspects in the small which do not change the large scale flat geometry, in view of a discussion of interactions between elementary particles. Covariance under general coordinate transformations is not required. A further step would be to discuss “operationally” how the general covariance principle ought to modify in the Quantum Gravity domain, where, at Planck scale, Einstein gedanken experiment of the freely falling lift would loose meaning,

(iii) In the Basic Model of Quantum Spacetime, the Euclidean distance between two events and the elementary area have both a lower bound of the unit order in Planck units; this is quite compatible, as shown by the model, with Poincare' covariance, and not to be confused with the unlimited accuracy which is in principle allowed in the measurement of a single coordinate.

Similarly, the spectrum of the space 3 - volume operator extends down to zero; while the spacetime 4 - volume operator is Poincare' invariant, with pure point spectrum with a lower bound of the unit order in Planck units [14].

These results, in full agreement with the Spacetime Uncertainty Relations, can be derived from a new interesting interplay of different algebraic, and C* - algebraic structures underlying the universal differential calculus.

(iv) The Basic Model replaces the algebra of continuous functions vanishing at infinity on Minkowsky Space by a noncommutative C* - Algebra E, the enveloping C* - Algebra of the Weyl form of the commutation relations between the coordinates, which turns out to be the C*
- Algebra of continuous functions vanishing at infinity from $\Sigma$ to the C*-Algebra of compact operators. Here $\Sigma$ is the joint spectrum of the commutators, which, due to the uncertainty relations, turns out to be the full Lorentz orbit of the standard symplectic form in 4 dimensions, that is the union of two connected components, each homeomorphic to $SL(2,\mathbb{C})/C^* \simeq TS^2$.

This manifold does survive the large scale limit; thus, QST predicts extradimensions, which indeed manifest themselves in a compact manifold $S^2 \times \{\pm 1\}$ if QST is probed with optimally localised states. The discrete two-point space which thus appears here as a factor reminds of the one postulated in the Connes-Lott theory of the Standard Model [6, 7, 8].

In this light QST looks similar to the phase space of a 2-dimensional Schrödinger particle; and thus naturally divides into cells (of volume governed by the 4-th power of the Planck length); so that, though being continuous and covariant, QST is effectively discretised by its Quantum nature. (Compare the earlier discussion of the "fuzzy sphere" by John Madore).

### 3. Quantum Field Theory.

Quantum Field Theory on the Basic Model of Quantum Spacetime was first developed in [3]; while fully Poincare Covariant Free Field Theory (as Wightman Fields on QST, or as Poincare Covariant nets of von Neumann Algebras labelled by projections in the Borel completion of $E$, which specify "noncommutative regions" in QST) can be explicitly constructed, and its violation of causality computed, all attempts to construct interacting QFT on QST seem to lead sooner or later to violations of Lorentz invariance, besides the inevitable violations of causality.

In the first approach to QFT on QST in [3], a natural prescription was given, not leading to interacting fields, but to a perturbative expansion of the S-Matrix, without manifest violations of unitarity; but interaction required an integration over $\Sigma$, thus breaking Lorentz invariance (there is no finite invariant measure or mean on $\Sigma$), yet preserving spacetime translation and space rotation covariance.

The approach based on Yang-Feldman Equation defines perturbatively covariant interacting fields, but Lorentz invariance will break a) at the level of renormalization b) at the level of asymptotic states [9, 11].

At level a) a fully covariant procedure replaces Wick Products by Quasiplanar Wick Products where one subtracts only terms which are local and divergent on QST [11]; the above problems would remerge with a further finite renormalization which ensures to recover the usual renormalised perturbative expansion in the large scale limit.

A more radical modification of the Wick product is suggested by the very quantum nature of Spacetime, the Quantum Wick Product [10]. It is based on the remark that the usual Wick Product is defined taking the product of field operators at independent points, making the necessary subtractions, and then taking the limit where all differences of independent coordinates tend to zero. But differences of independent coordinates in Quantum Spacetime are quantum variables themselves, obeying, up to a factor of order 1, the same commutation relations as the quantum coordinates; and thus they cannot be set equal to zero. The best one can do is to evaluate, on functions of several independent quantum coordinates, a conditional expectation which essentially evaluates on the functions of each difference variable an optimally localised state, and leaves the barycenter coordinates unchanged. Combined with the usual subtractions this procedure leads to the Quantum Wick Product: its use to define interactions regularises completely QFT in the Ultraviolet; but Lorentz covariance is broken here by the Quantum Wick Product itself (while no integration on $\Sigma$ is needed here - the resulting interaction Hamiltonian turns out to be constant on $\Sigma$ - the very notion of optimally localised state refers to a specific Lorentz frame: it requires that the sum of the squares of the uncertainties of the four quantum coordinates is minimal), however, again, spacetime translation and space rotation covariance are preserved. Moreover, the Adiabatic Limit poses serious problems [10]; see however [20, 21].
Thus Lorentz breaking appears in all the above attempts through the presence of a non trivial centre of the (multiplier algebra of the -) Algebra of QST, whose spectrum is $\Sigma$: no finite invariant integration is possible and renormalization introduces a bad dependence on the points of $\Sigma$.

The problem of Lorenz breaking has received a lot of attention from many viewpoints, quite different from the one advocated here (from Quantum deformation of Poincare’ group, notably studied by J.Lukierski, M.Chaichian, J.Wess and their groups of collaborators, to the recent discussion of modifications of the coproduct in the Field Algebra itself, by the same Authors and T.R.Govindarajan, A. P. Balachandran and others; see other contributions to this Volume, and compare [19]. Related is the discussions of deformed General Covariance, notably by J.Wess and collaborators, see the contributions to this Volume by Julius Wess and Paolo Aschieri).

But most of these approaches assume that the commutators of the coordinates are numbers; the prize to pay is then deformation of the Poincare’ group, if not to give up altogether the principle that laws of nature are the same in all Lorentz frames - as required in String Theory by the presence of external $B - F$ ields.

One might evaluate on QST relativistic interacting quantum fields (supposedly) already given on Minkowski space; in the perturbative picture this would amount to evaluate the ordinary, Minkowskian renormalised interaction density as a function on QST (rather than doing this on each field operator). In this case, the interaction would not be affected by the Quantum nature of Spacetime, which would leave no trace at all in the S-Matrix. Some formal manipulations might just lead to this option.

A common feature of the S - matrix approaches described above, based both on ordinary and Quantum Wick product, is that, while not leading to proper interacting fields on Quantum Spacetime, they lead, through a natural ansatz, to a Gell Mann - Low formula for the S - matrix which agrees with the one which would be derived on Classical Minkowski Space from a non local interaction, obtained from the one given at the start by smearing with a non local kernel, whose shape reflects the quantum nature of Spacetime and depends upon the chosen procedure (in the case of the Quantum Wick product it would be a Dirac delta function of the sum of all variables, reflecting translation covariance, times a Gaussian in all the difference variables, which regularises all the Ultraviolet divergences); such an effective interaction is formally self adjoint and does not lead to any unitarity violation.

The crucial point is that Feynmann rules cannot be freely imposed as a further ansatz, but must be deduced from the Dyson expansion: the time ordering refers to the times of the effective interaction Hamiltonians in the ineraction representation, and not to the time variables in the field operators [3]; thus Feynmann propagators have to be replaced by the Denk - Schweda propagators; but full Feynmann rules can be reformulated in a complete way [15].

4. Other models of Quantum Spacetime.

Does the commutator of Quantum coordinates have to be central? This hypothesis, introduced in [3] on mere simplicity grounds, can be removed in a class of more elaborate models, where the commutators commute with one another but not with the coordinates themselves. Yet the centre is again large (and not even translation invariant), actually acted upon freely by the Lorentz group.

While the model is Poincare’ covariant, the irreducible representations break (not only Lorentz invariance, as in the basic model, but also) translation covariance in some direction. The model has however the virtue of implementing the Spacetime Uncertainty Relations not only in the weak form, as reproduced by the Basic Model, but in a somewhat stronger form, closer to that suggested by the original semiclassical argument [13].
This feature might imply a better regularization of interactions: indeed already the one implied by the Basic Model for interactions in the form originally introduced in [3], has been shown to be sufficient to regularise the $\phi^3$ interaction [17]; the present model might well go beyond.

5. A New Scenario.

The Principle of Gravitational Stability ought to be fully used in the very derivation of Space Time Uncertainty Relations, which would then depend also on the energy-momentum density of generic background quantum states; this leads to commutation relations between Spacetime coordinates depending in principle on the metric tensor, and hence, through the gravitational coupling, on the interacting fields themselves. Thus the commutation relations between Spacetime coordinates would appear as part of the equations of motions along with Einstein and matter field Equations.

In other words we may expect that, while Classical General Relativity taught us that Geometry is dynamics, Quantum Gravity might show that also Algebra is dynamics.

This new scenario [12] appears extremely difficult to formalise and implement, but promises most interesting developments. Notably it would be related to the nonvanishing of the Cosmological Constant [12] and might explain Thermodynamical Equilibrium of the early Universe without Inflation; it might relate to the distribution of correlations in the early Universe shown by WMAP3. The key feature of this scenario is that it seems to indicate that commutators of spacetime coordinates, hence the range of acausal effects, should be larger where gravitational forces are stronger and diverge near singularities. Thus at the very beginning, near the Big Bang, the range of acausal effects would have been infinite, establishing thermodynamical equilibrium among well distant regions. Also, the Universe could have started as an effectively zero dimensional system, and if so this fact might have left traces in the spectrum of the Cosmic Microwave Radiation.

Spacetime coordinates and fields would appear as intimately connected features of the quantum texture of spacetime, which cannot be separated from one another, except in some drastic approximation, whose price might well be the breakdown of Lorentz invariance of interactions between Quantum Fields.

Moreover, QST might teach us something about dark matter if, as expected, it implies a minimal size for black holes, where Hawking evaporation would stop. The minimal black holes would be stable, and fill the Universe with a gas which would contribute to the dark matter.

The problem, however, would be displaced to that of their possible formation [22].

6. Phenomenological consequences?

The square of the Planck length appears in the commutator of coordinates, hence Quantum Spacetime corrections to ordinary interactions are to be expected to be, at the lowest order, quadratic in the Planck length; if not higher. The effects are bound to be very small: the difficulties to detect directly gravitational waves (an effect in Classical General Relativity, for which the Universe offers sources of tremendously high energy) do not induce to optimism to see direct effects.

Moreover, as briefly described here above, field interactions on Quantum Spacetime can well be formulated in a spacetime translation and space rotation invariant way, and the lack of a satisfactory formulation which is also Lorentz invariant may well be just a weak point of the routes followed so far, rather than an intrinsic impossibility. Therefore it might well be necessary to look for more subtle tests, rather than for effects due to possible spacetime symmetry breakdown.
But the scenario alluded to in the earlier paragraph might well lead to detect shadows of the Quantum structure of Spacetime in the early Universe.

One difficulty is that, as implicit in the discussion of [3], the $U(1)$ Gauge Theory (QED) on QST becomes effectively a $U(\infty)$ Gauge Theory and gauge invariant quantities essentially disappear; an interesting solution to this difficulty has been proposed by J. Madore, S. Schraml, P. Schupp and J. Wess, introducing the concept of covariant coordinates [16]; see also [18].

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