WILD EXAMPLES OF RECTIFIABLE SETS

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Abstract. We study the geometry of sets based on the behavior of the Jones function, \( J_E(x) = \int_0^1 \beta_{E,2}(x,r)^2 \frac{dr}{r} \). We construct two examples of countably 1-rectifiable sets in \( \mathbb{R}^2 \) with positive and finite \( \mathcal{H}^1 \)-measure for which the Jones function is nowhere locally integrable. These examples satisfy different regularity properties: one is connected and one is Ahlfors regular. Both examples can be generalized to higher-dimension and co-dimension.

1. Introduction

In his solution to the Analyst’s Traveling Salesman Problem \([6]\), Peter Jones introduced a local gauge of flatness which has been generalized by David and Semmes \([4]\) to measures and higher dimensions. These families of local gauges of flatness are called the Jones \( \beta \)-numbers, and they have come to dominate the landscape in quantitative techniques relating rectifiability, potential theory, and boundedness of singular integrals. See, for example the landmark book \([5]\).

For a set \( E \subset \mathbb{R}^d \), \( 1 \leq p < \infty \), and an integer \( 1 \leq n \leq d-1 \), we write \( \mu = \mathcal{H}^n \mathbb{L} E \) and define the Jones \( \beta \)-numbers as follows,

\[
\beta_{E,p}^n(x,r) = \left( \inf_{L \subset \mathbb{R}^d \text{ an } n\text{-plane}} \int_{B(x,r)} \left( \frac{\text{dist}(y,L)}{r} \right)^p d\mu(y) \frac{r^n}{r^n} \right)^{\frac{1}{p}}.
\]

We also write \( \beta_{\mu,p}^n(x,r) \) for \( \beta_{E,p}^n(x,r) \), when \( \mu = \mathcal{H}^n \mathbb{L} E \) is understood. If \( p = \infty \), the \( \beta \)-numbers are defined in terms of the sup-norm instead of the \( L^p \)-norm.

In addition to generalizing the Jones \( \beta \)-numbers, \([4]\) also introduced the notion of uniform rectifiability. A set \( E \subset \mathbb{R}^d \) is said to be Ahlfors \( n \)-regular if there exists \( 0 < c < C < \infty \) such that \( cr^n \leq \mathcal{H}^n(E \cap B(x,r)) \leq Cr^n \) for all \( x \in E \) and all \( 0 < r < \text{diam}(E) \). An \( n \)-Ahlfors regular \( E \subset \mathbb{R}^d \) is said to be uniformly \( n \)-rectifiable if there exist finite constants \( \theta, \Lambda > 0 \) such that for all \( x \in E \) and all \( 0 < r < \text{diam}(E) \) there is a Lipschitz mapping \( g : B(0,r) \subset \mathbb{R}^n \to \mathbb{R}^d \) with \( \text{Lip}(g) \leq \Lambda \) such that \( \mathcal{H}^n(E \cap B(x,r) \cap g(B(0,r))) \geq \theta r^n \).

In \([4]\) the authors show that an \( n \)-Ahlfors regular set \( E \subset \mathbb{R}^d \), is \( n \)-uniformly rectifiable if and only if the Jones \( \beta \)-numbers satisfy the following Carleson condition for some \( 1 \leq p < \frac{2n}{n-2} \),

\[
(1.1) \quad C_{E,p}^n(x,R) := \int_{B(x,R)} \int_0^R \beta_{E,p}^n(y,r)^2 \frac{dr}{r} d\mu(y) \leq cR^n \quad \text{for all } x \in E, \ R > 0
\]
A set $E \subset \mathbb{R}^d$ is said to be countably $n$-rectifiable if there are Lipschitz maps $f_i : \mathbb{R}^n \to \mathbb{R}^d$ with $i = 1, 2, \ldots$, such that
$$H^n(\mathbb{R}^d \setminus \cup_i f_i(\mathbb{R}^n)) = 0.$$

Recently, Tolsa [9] and Azzam and Tolsa [2] show as a special case of their results that $E$ is countably $n$-rectifiable if and only if

$$J^n_E(x, 1) = \int_0^1 \beta^n_{E;2}(x, r)^2 \frac{dr}{r} < \infty \quad \text{for } H^n - \text{a.e. } x \in E$$

where $J^n_E(x, 1)$ is the Jones function at $x$ and scale $1$. See also [7] and [3].

In this paper, we show that sets which satisfy (1.2) can fail to satisfy (1.1) as dramatically as possible.

**Theorem 1.1.** There exists a rectifiable curve (of finite length), $K_0 \subset \mathbb{R}^2$, such that for $\mu = H^1 \ll K_0$, for any $x \in K_0$, and any $\delta > 0$
$$\int_{B_\delta(x)} \int_0^\delta \beta^1_{\mu;2}(y, r)^2 \frac{dr}{r} d\mu(y) = \infty$$

The set $K_0$ arises from unions of modifications of approximations to snowflake-like sets. Note that by the Analyst’s Traveling Salesman theorem [6]
$$\int_{\mathbb{R}^2} \int_0^\infty \beta^1_{\mu,\infty}(y, r)^2 \frac{dr}{r} d\mu(y) < \infty,$$

which prevents $K_0$ from being upper regular at a generic point.

**Theorem 1.2.** There is a 1-Ahlfors regular, countably 1-rectifiable set $A_0$ contained in the unit cube in $\mathbb{R}^2$ such that for $\mu = H^1 \ll A_0$, for every $x \in A_0$, and for every $\delta > 0$
$$\int_{B_\delta(x)} \int_0^\delta \beta^1_{A_0;2}(y, r)^2 \frac{dr}{r} d\mu(y) = \infty.$$

The set $A_0$, whose construction was initially motivated by the machinery introduced in [10], is created from scaled unions of approximations to the 4-corner Cantor set. Ultimately the presentation was simpler using the framework of self-similar sets.

**Remark 1.3.** These examples can be used to create higher-dimensional ones by taking Cartesian products with finite intervals. That is, if $A \in \{K_0, A_0\}$ for any positive integer $n < d$, define $E' = A \times [0, 1]^{n-1} \subset \mathbb{R}^{n+1}$. Embedding $E'$ into the first $(n + 1)$-dimensions of $\mathbb{R}^d$ preserves the properties of $A$. In particular, it is standard that defining $\beta$-numbers over cubes (with sides parallel to the axes in $\mathbb{R}^d$) instead of balls leads to an equivalent definition of the $\beta$-numbers. Consequently finiteness of $C^n_{E;2}(x, R)$ is equivalent to the finiteness of $C^n_{A;2}(x', R)$ where $x'$ is the orthogonal projection of $x$ into $\mathbb{R}^2$.

2. **Proof of Theorem**

For the remainder of this paper, we only consider $E \subset \mathbb{R}^2$ and the $\beta$-numbers when $p = 2$. As such, we write $\beta_E, \beta_\mu, C_E$, and $C_\mu$ in place of $\beta_{E;2, \mu;2}, C_{E;2, \mu;2}$, and $C_{\mu;2}$. Moreover, for any set $L \subset \mathbb{R}^2$ we write $B_r(L) = \{x : \text{dist}(x, L) < r\}$ and $B_r = B_r(\{0\})$. 
We begin by stating two basic properties of the Jones $\beta$-numbers. The first controls how fast the $\beta$-numbers can shrink by relating the $\beta$-numbers at comparable scales. This property is often called “doubling,” though we have chosen to scale by the number 3. The second property shows how the $\beta$-numbers behave under rescaling.

**Proposition 2.1.** Let $E \subset \mathbb{R}^2$ have $\dim_H(E) = 1$.

1. For any ball $B_{\tau}(y) \subset B_{3\tau}(x)$,
   \[ \beta_E(y, r)^2 \leq 3\beta_E(x, 3r)^2 \]

2. The $\beta$-numbers have the following scaling property. If $E^x,t = tE + z$ then
   \[ \beta_{E^x,t}(x, r)^2 = \beta_E \left( \frac{2^x - z}{t}, \frac{1}{t} \right)^2. \]
   Consequently, $C_{E^x,t}(z, r) = tC_E(0, t^{-1} r)$.

To construct a 1-rectifiable set that is connected (hence Ahlfors lower-regular) for which the Jones function is locally non-integrable, we modify approximations to the Koch snowflake. This set will not be upper regular. Recall some facts about the standard approximation to the Koch snowflake.

**Definition 2.2.** Let $I \subset \mathbb{R}^2$ be a line segment, and fix $0 < \alpha < \pi/2$. Define $P(I)$ as the set which results from the following operation

1. Divide $I$ into three equal subintervals, $I_{\text{left}} \cup I_{\text{center}} \cup I_{\text{right}}$.
2. Over the middle interval, $I_{\text{center}}$, construct an isosceles triangle with angles $\alpha$ and base $I_{\text{center}}$.
3. Delete $I_{\text{center}}$, the base of the isosceles triangle.

We define

\[ S(I) = \overline{P(I) \setminus I}, \]

and call $S(I)$ the bump. If $q_I$ is the orthogonal projection onto the line containing $I$ and $q_I^1$ is the orthogonal projection onto $I^1$, then height($S(I)$) = diam($\{q_I^1(S(I))\}$) and width($S(I)$) = diam($\{q_I(S(I))\}$) = $\frac{1}{3} H^1(I)$. We shall abuse our notation slightly by saying that for a collection of line segments, $E$, the set $P(E)$ is obtained by applying $P$ to each maximal line segment contained in $E$.

If $I = [0, 1] \times \{0\}$ and $\alpha = \frac{\pi}{3}$, the standard approximations to the Koch snowflake are given by $\{P^k(I)\}_{k=1}^{\infty}$, where $P^k$ denotes applying $P$ iteratively $k$ times. We emphasize a few properties about deformations under the operation $P$.

**Proposition 2.3.** For any finite line segment $I \subset \mathbb{R}^2$ and positive integer $n$,

\[ \text{height}(S(I)) = \frac{\tan(\alpha)}{6} |I| \]

\[ H^1(S(I)) = \frac{\sec(\alpha)}{3} |I| \]

\[ H^1(P^n(E)) = \left( \frac{\sec(\alpha) + 2}{3} \right)^n H^1(E) \]

When $\tau = \frac{1}{20} \min \left\{ \frac{\tan(\alpha)}{6}, \frac{1}{3} \right\}$, there exists $c_0 = c(\alpha)$ such that for all lines $L$

\[ H^1(S(I) \setminus B_\tau(L)) \geq c_0 H^1(S(I)). \]

**Proof.** (2.2) and (2.3) follow from planar geometry. The $n = 1$ case for (2.4) follows by adding back in the unchanged intervals $I_{\text{left}}$ and $I_{\text{right}}$, which have total length
\[\frac{2}{3}|I|\]. The geometric nature of the definition of \(P\) allows us to then iterate this to achieve (2.4).

To verify (2.5) we proceed by contradiction. Suppose no such constant \(c_0\) exists. Then, there exists a sequence of lines intersecting \(S\) segment, \(I\)
\[\text{Definition 2.4.}\]
Define \(P_j(I)\) to be the set operation defined on line-segments by
\[P_j(I) = P^{j-1}(S(I)) \bigcup (I \setminus I_{\text{center}}),\]
recalling the definition of \(S(I)\) can be found in (2.1). Loosely speaking, for any line segment, \(I\), \(P_j(I)\) is the set that replaces the center of \(I\) with a jth approximation of the Koch curve.

\[\text{Corollary 2.5.}\]
For any line segment \(I \subset \mathbb{R}^2\) and positive integer \(n\)
\[\text{(2.6)}\]
\[\mathcal{H}^1(P_n(I)) = \frac{2}{3}|I| + \left(\frac{\sec(\alpha) + 2}{3}\right)^{n-1}\frac{\sec(\alpha)}{3}|I|,\]

Moreover, if \(\alpha \leq \pi/3\),
\[\text{dist}_{\mathcal{H}}(I, P^n(I)) \leq \frac{\tan(\alpha)}{12}|I|,\]

\[\text{Proof.}\] Equations (2.3) and (2.4) verify (2.6). Indeed,
\[\mathcal{H}^1(P^{n-1}(S(I))) = \left(\frac{\sec(\alpha) + 2}{3}\right)^{n-1}\mathcal{H}^1(S(I)) = \left(\frac{\sec(\alpha) + 2}{3}\right)^{n-1}\frac{\sec(\alpha)}{3}|I|,\]

The restriction to \(\alpha \leq \pi/3\) ensures the longest line segment of \(P^n(I)\) has length at most \(3^{-1}\). Consequently, (2.2) guarantees
\[\text{dist}_{\mathcal{H}}(P^n(I), I) \leq \sum_{i=1}^{n}\text{dist}_{\mathcal{H}}(P^i(I), P^{i-1}(I)) \leq \sum_{i=1}^{n}3^{-i}\text{height}(S(I)) \leq \frac{\tan(\alpha)}{12}|I|,\]

\[\text{Definition 2.6.}\] Now, let \(n\) be a natural number to be chosen later and \(E_0 = I = [0, 1] \times \{0\}\). We define \(E_1 = P_n(I)\). For \(k \geq 2\) inductively define
\[\text{(2.8)}\]
\[E_k = P_{kn}\left([0, 3^{-(k-1)}] \times \{0\}\right) \bigcup \left(\left\{3^{-(k-1)}, 1\right\} \times \mathbb{R}\right) \cap E_{k-1}.\]

Notably, for all integers \(j\) the operation \(P_j\) applied to \([0, 3^{-(k-1)}] \times \{0\}\) leaves the segment \([0, 3^{-k}] \times \{0\}\) untouched. Consequently, the sequence of sets \(\{E_k\}\) are defined by replacing the “next” triadic interval with a scaled approximation of the Koch snowflake. The fact that each triadic strip \([3^{-k}, 3^{-(k-1)}] \times \mathbb{R}\) is only modified once in the sequence of sets \(E_k\) is ensures the Hausdorff dimension of the final set remains 1.
Lemma 2.7 (Base Set). Fix $0 < \frac{\alpha}{3}$ and any integer $n$ satisfying

$$3^{-1} \left( \frac{\sec(\alpha) + 2}{3} \right)^{n} < 1 < 3^{-1} \left( \frac{\sec(\alpha) + 2}{3} \right)^{2n}.\tag{2.9}$$

Then the sequence of sets $E_k$ from (2.1) converge to a compact and connected Borel set $E_\infty$ in the Hausdorff topology on compact subsets. Furthermore, $E_\infty$ satisfies:

1. $\mathcal{H}^1(E_\infty) < \infty$
2. For all $\delta > 0$, $C_{E_\infty}(0, \delta) = +\infty$.

Proof. The existence of the limiting compact set $E_\infty$ follows from precompactness of sets contained in $B_{10}$ in the Hausdorff distance and (2.7) which ensures that $\text{dist}_{\mathcal{H}}(E_{k+1}, E_k) \sim 3^{-k}$. Connected follows since $E_\infty \setminus B_{3^{-k}}(0) = E_k \setminus B_{3^{-k}}(0)$ is connected for each $k$.

To see that $E_\infty$ has finite length we write $\mathcal{H}^1(E_k) = \mathcal{H}^1(E_k \setminus B_{3^{-k}}) + \mathcal{H}^1(E_k \cap B_{3^{-k}})$. Since $E_k \setminus B_{3^{-k}} = E_{k-1} \setminus B_{3^{-(k-1)}}$ and $\mathcal{H}^1(E_{k-1} \setminus B_{3^{-(k-1)}}) = \mathcal{H}^1(E_{k-1}) - 3^{-k}$, (2.6) implies,

$$\mathcal{H}^1(E_k) - \mathcal{H}^1(E_{k-1}) = 3^{-k} \left( \frac{\sec(\alpha) + 2}{3} \right)^{nk} \sec(\alpha) - 1.\tag{2.10}$$

Since $\mathcal{H}^1(E) = 1$, iteration yields

$$\mathcal{H}^1(E_k) = 1 + \sum_{i=1}^{k} 3^{-i} \left( \frac{\sec(\alpha) + 2}{3} \right)^{ni} \sec(\alpha) - 1.$$

In particular, $\lim_{k \to \infty} \mathcal{H}^1(E_k) \leq \infty$ whenever $n$ satisfies the lower bound from (2.9). Moreover $\mathcal{H}^1(E_\infty) = \lim_{k \to \infty} \mathcal{H}^1(E_k)$ since for all $j \geq k$,

$$\mathcal{H}^1(E_j \Delta E_k) \leq 2 \sum_{i=k+1}^{\infty} 3^{-i} \left( \frac{\sec(\alpha) + 2}{3} \right)^{ni} \sec(\alpha),$$

which decays to zero as $k \to \infty$. Hence, (2.10) holds for $E_\infty$ and $0 < \mathcal{H}^1(E_\infty) < \infty$.

It only remains to show $C_{E_\infty}(0, \delta) = +\infty$ for all $\delta > 0$. To this end, we first note that when $r = r(n, \alpha) = 3^{-1} \left( \frac{\sec(\alpha) + 2}{3} \right)^{n}$,

$$\mathcal{H}^1(E_\infty \cap B_{3^{-k}}(0)) = 3^{-k} + \sec(\alpha) \frac{r^{k+1}}{1 - r} - 3^{-r} \frac{(k+1)}{1 - 3^{-k}}.\tag{2.11}$$

Indeed, by (2.10) and the trick used to prove (2.10)

$$\mathcal{H}^1(E_\infty \cap B_{3^{-k}}(0)) = 3^{-k} + \sum_{i=k+1}^{\infty} 3^{-i} \left( \sec(\alpha) \left( \frac{\sec(\alpha) + 2}{3} \right)^{ni} - 1 \right).$$

Claim: With $\tau$ as in Proposition 2.3 and $\alpha \leq \pi/3$, there exists a constant $c_1$ and integer $j_0$ independent of $k$ such that for any line $L$, and all $k$ such that $nk - 1 - j_0 \geq 0$,

$$\mathcal{H}^1 \left( \left( E_\infty \setminus B_{\frac{\pi}{2\pi \tau}}(L) \right) \cap B_{3^{-k}} \right) \geq c_1 3^{-k} \left( \frac{\sec(\alpha) + 2}{3} \right)^{nk - 1 - j_0}.\tag{2.12}$$

\footnote{Note that for instance, $\alpha = \pi/3$ and $n \in \{2, 3\}$ satisfies (2.9).}
Proof of Claim. Writing $I' = [0, 1] \times \{0\}$, we will in fact scale by $3^k$ and show the stronger result that
\[
\mathcal{H}^1 \left( \left( \mathcal{P}_{nk}(I') \setminus B_{\frac{1}{53^k}}(L) \right) \cap B_{3^k} \right) \geq c_1 3^k \left( \frac{\sec(\alpha) + 2}{3} \right)^{nk - 1 - j_0} |I'|.
\]

To do so, we find a line segment $J \subset S(I') \setminus B_r(L)$ such that $J$ has an endpoint in common with one of the two line segments of $S(I')$ and $|J| = 3^{-j_0} \mathcal{H}^1(S(I'))/2$, where $j_0$ to be chosen later is independent of $L$. This specific choice of length and endpoint ensure that $P^{nk-1-j_0}(J) \subset \mathcal{P}_{nk}(I')$. Moreover, the choice of $j_0$ will both guarantee that $|J|$ is large enough and that $P^{nk-1-j_0}(J)$ remains outside of $B_{\tau(2)}(L)$, hence verifying Claim 2.

To find $J$, we note that the simple shape of $S(I')$ guarantees that $S(I') \setminus B_r(L)$ has at most 4 maximal line segments. Hence, there exists a maximal line segment $K_L \subset S(I') \setminus B_r(L)$ with $\mathcal{H}^1(K_L) \geq \frac{1}{2} \mathcal{H}^1(S(I') \setminus B_r(L))$. If $K_L$ is parallel to $L$, let $x_L$ denote either endpoint of $K_L$. Otherwise, let $x_L$ denote the unique endpoint of $K_L$ that is not contained in $B_{\tau}(L)$. Define $J$ to be the unique subset of $K_L$ of length $3^{-j_0} \sec(\alpha) |I'|$ with endpoint $x_L$. Now, define $j_0$ as the smallest integer such that
\[
3^{-j_0} < \min \left\{ \frac{c_0}{4}, \left( \frac{\tan(\alpha)}{12} \cdot \frac{\sec(\alpha)}{6} |I'| \right)^{-1} \frac{\tau}{2} \right\},
\]
where $c_0$ is as in Proposition 2.3. The first condition ensures that $J \subset K_L$ and guarantees that the first constraint on $j_0$ is independent of $L$ and $k$. The second constraint combined with (2.3) and (2.7) ensure that $\text{dist}_{\mathcal{H}}(P^{nk-1-j_0}(J), J) \leq \frac{\tau}{2}$. Moreover, choosing $j_0$ to be the smallest admissible integer, and guarantees that $|J| = 3^{-j_0} \sec(\alpha) |I'| \geq c' |I'|$ where $c'$ is independent of $L$ and $k$. Finally, (2.4) completes the proof of the Claim since
\[
\mathcal{H}^1 \left( \mathcal{P}_{nk}(I') \setminus B_{\tau(2)}(L) \right) \geq \mathcal{H}^1(P^{nk-1-j_0}(J)) \geq c_1 \left( \frac{\sec(\alpha) + 2}{3} \right)^{nk - 1 - j_0} |I'|,
\]
where $c_1$ depends only on $\alpha$.

Whenever $nk - 1 - j_0 \geq 0$, (2.12) implies (2.13)
\[
\beta_{E_{\infty}}(0, 3^{-k})^2 \geq \frac{1}{3^k} \left( \frac{\tau}{3} \right)^2 c_1 3^k \left( \frac{\sec(\alpha) + 2}{3} \right)^{nk - 1 - j_0}.
\]

Fix $\delta > 0$ and any integer $k_3$ such that $3^{-k_3} < \delta$ and $nk_3 - 1 - j_0 > 0$. Then, with $\mu = \mathcal{H}^1 \cap E_{\infty}$, repeated applications of Proposition 2.1, (2.13), and (2.11) yield
\[
\int_{\mathcal{B}_{\delta}(0)} \int_0^3 \beta_\mu(x, r)^2 \frac{dr}{r} d\mu(x) \geq \ln(3)3^{-2} \sum_{k=k_3}^{\infty} \mu(B_{3^{-k} - (k+2)})^2 \beta_\mu(0, 3^{-k(k+2)})^2
\]
\[
\geq \ln(3)3^{-2} \sum_{k=k_3}^{\infty} \left( 3^{-k} + \sec(\alpha) \frac{r^{k+1}}{1-r} - 3^{-k-1} \frac{\sec(\alpha)}{1-3^{-1}} \right) \left( c_2 \left( \frac{2}{3} + \frac{\sec(\alpha)}{3} \right)^{nk} \right).
\]
Due to the lower bound in (2.9), this sum diverges if and only if
\[
\sum_{k=k_3}^{\infty} \left( \frac{\sec(\alpha)}{1-r} \right) \left( 1 - \frac{r^{k+1}}{3^{k+1} - 3^k} \right) = \sum_{k=k_3}^{\infty} \left( \frac{\sec(\alpha)}{1-r} \right) \left( 2 + \frac{\sec(\alpha)}{3} \right)^{nk} = \sum_{k=k_3}^{\infty} \left( \frac{\sec(\alpha)}{1-r} \right) \left( 2 + \frac{\sec(\alpha)}{3} \right)^{nk}.
\]
That for any \( Theorem 2.8. \) There exists a connected set, \( K_0 \subset \mathbb{R}^2 \) of finite \( H^1 \)-measure such that for any \( x \in K_0 \) and \( \delta > 0 \)

\[
C_{K_0}(x, \delta) = \infty.
\]

Proof of \( Theorem 1.1. \) Let \( \{r_i\}_{i=1}^{\infty} \) be a sequence of positive numbers such that \( \sum_i r_i \leq 1 \). Let \( E^{x,r} \subset \mathbb{R}^2 \) be the set \( E^{x,r} = rE_\infty + x \). We construct \( K_0 \) as the union of a countable collection of nested sets \( \{\Gamma_i\} \).

Let \( \Gamma_0 = E_\infty \). Now, let \( \{x_{i,j}\}_{j=1}^{N_i} \) be a maximal \( 2^{i-1} \)-separated collection of points in \( \Gamma_0 \). Let

\[
\Gamma_1 = \Gamma_0 \cup \bigcup_{j=1}^{N_1} E^{x_{i,j}, r_{i,j}}.
\]

Suppose that we have defined \( \Gamma_{i-1} \), some positive integers \( \{N_i\}_{i=1}^{\infty} \) and a collection of points \( \{x_{\ell,j} \in \Gamma_{i-2} \mid 1 \leq \ell \leq i-1, 1 \leq j \leq N_i\} \) that form a maximal \( 2^{i-1} \)-net for \( \Gamma_{i-2} \). Then choose \( N_i \in \mathbb{N} \) and points \( \{x_{\ell,j}\}_{1 \leq j \leq N_i} \subset \Gamma_{i-1} \) so that \( \{x_{\ell,j} \in \Gamma_{i-1} \mid 1 \leq \ell \leq i, 1 \leq j \leq N_i\} \) is a maximal \( 2^{i-1} \)-net in \( \Gamma_{i-1} \). Then define \( \Gamma_i \) by

\[
\Gamma_i = \Gamma_{i-1} \cup \left( \bigcup_{j=1}^{N_i} E^{x_{i,j}, r_{i,j}} \right).
\]

We claim that \( K_0 = \bigcup_{i=0}^{\infty} \Gamma_i \) is the desired set. First note that since each \( \Gamma_i \) is rectifiable, \( K_0 \) is rectifiable. Moreover, \( \{x_{i,j}\}_{j=1}^{N_i} \subset \Gamma_{i-1} \) for all \( i \) ensures \( K_0 \) inherits connectivity from \( E_\infty \). Furthermore, since \( \{\Gamma_i\} \) is a nested sequence increasing to \( K_0 \) and \( \sum_i r_i \leq 1 \),

\[
H^1(K_0) = H^1 \left( E_\infty \cup \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{N_i} E^{x_{i,j}, r_{i,j}} \right) \leq H^1(E_\infty) \left( 1 + \sum_{i=1}^{\infty} r_i \right) \leq 2H^1(E_\infty).
\]

It only remains to show that for \( x \in K_0 \) and \( \delta > 0 \) that \( C_{K_0}(x, \delta) = \infty \). To this end, fix \( x \in K_0 \), and \( \delta > 0 \). By definition of \( K_0 \), there exists \( \ell_0 \) such that \( x \in \Gamma_{\ell_0} \). Then, by the net property of the points \( \{x_{i,j}\} \), it follows that for \( \ell \geq \ell_0 \) large enough that \( 2^{\ell-1} < \delta/4 \), there exists \( i \leq \ell \) with \( x_{i,j} \in \Gamma_{\ell-1} \cap B(x, \delta/2) \subset K_0 \cap B(x, \delta/2) \). Writing \( \mu = H^1 \cap K_0 \) and \( \mu_{i,j} = H^1 \cap E^{x_{i,j}, r_{i,j}/2} \) it follows from monotonicity of the integral that

\[
(2.14) \quad \int_{B_{\delta/2}(x)} \int_{y}^{\delta/2} \beta_{K_0;2}(y, r)^2 \frac{dr}{r} \, d\mu(y) \geq \int_{B_{\delta/2}(x_{i,j})} \int_{y}^{\delta/2} \beta_{\mu_{i,j};2}(y, r)^2 \frac{dr}{r} \, d\mu_{i,j}(y),
\]

or equivalently \( C_{K_0}(x, \delta) \geq C_{\mu_{i,j}}(x_{i,j}, \delta/2) \). Recalling that \( E^{x,t} = tE_\infty + z \), we use (2.14), Proposition 2.1(2), and Lemma 2.7 to conclude

\[
C_{K_0}(x, \delta) \geq C_{E^{x_{i,j}, r_{i,j}/2}} \left( x_{i,j}, \frac{\delta}{2} \right) = \frac{r_{i,j}}{N_i} C_{E_\infty} \left( 0, \frac{\delta N_i}{2r_{i,j}} \right) = \infty.
\]

Since \( x \in K_0 \) and \( \delta > 0 \) are arbitrary this finishes the proof.
Remark 2.9. Since $K_0$ from Theorem 1.1 is connected, $\mathcal{H}^1(K_0) = \mathcal{H}^1(K_0) < \infty$ and $K_0$ is compact, see [8, Lemma 3.4, 3.5]. Thus $K_0$ is a rectifiable curve by Wazewski’s theorem, see [8, Lemma 3.7] or [1, Theorem 4.4].

3. Proof of Theorem 1.2

To produce the desired set $A_0$, we use approximations of the 4-corner Cantor set to produce a base set that has precise control on the $\beta$-numbers at the origin, then we carefully iterate this set “on itself” in order to preserve Ahlfors regularity.

3.1. Approximations to the 4-corner Cantor set. Consider the following sequence of approximations to the 4-corner cantor set, by sets of positive and finite $\mathcal{H}^1$-measure.

Let $E_0 = [0, 1) \times \{0\}$ and inductively define

$$E_k = \sum_{(i, j) \in \{0, 1\}^2} p_{ij} \cdot 2^{-2}E_{k-1} \quad \text{where} \quad p_{ij} = \left(\frac{i}{2^k}, \frac{j}{2^k}\right).$$

The word similarity is used to refer to any mapping that can be written as a composition of scalings, rotations, reflections, and translations. Throughout the rest of the paper, we say that two sets are similar if one is the image of the other by a similarity. In reality the similarities we discuss can always be written as a scaling and translation, as in (3.1).

We let $\Delta$ denote the collection of tetradic half-open cubes in $\mathbb{R}^2$, that is

$$\Delta = \{[a2^{-2k}, (a+1)2^{-2k}) \times [b2^{-2k}, (b+1)2^{-2k}) \mid a, b, k \in \mathbb{Z}\}.$$ 

For some $Q \in \Delta$, we let $\ell(Q)$ denote the sidelength of $Q$. We partition the tetradic cubes into cubes of fixed sidelength by defining $\Delta^i = \{Q \in \Delta \mid \ell(Q) = 2^{-2i}\}$.

In general, for a set $E \subset \mathbb{R}^2$ we the length of $E$ and respectively height of $E$ by

$$\ell(E) = \text{diam}\{\pi_x(E)\} \quad \text{and} \quad h(E) = \text{diam}\{\pi_y(E)\}$$

where $\pi_x$ and $\pi_y$ denote the orthogonal projection onto the horizontal and vertical axes. In particular, for a cube $Q$, this notion of length coincides with its sidelength.

Definition 3.1 (Clusters and sub-clusters). Any set which is similar to any $E_k$ or $E_k \cup [0, 1) \times \{0\}$ for $k \in \mathbb{N}$ will be called a cluster.

Moreover, for fixed $k \in \mathbb{N}$, we will call $E_k$ the 0th sub-cluster of $E_k$ and the $2^{2k}$ line segments that make up $E_k$ are called the $k$th subclusters of $E_k$. For $\ell \in \{1, \ldots, k-1\}$, the $2^{2\ell}$ sets contained in $E_k$ which are similar to $E_{k-\ell}$ are called the $\ell$th sub-clusters of $E_k$.

Definition 3.2 (Root points). We associate to each cluster and each cube a root point. The root point of a cluster $E$ is the lower-most and left-most point in the cluster. Since a sub-cluster is itself a cluster, the notion of a root point extends to sub-clusters. For a cluster $E$, we let $x_E$ denote its root point. For a tetradic cube $Q \in \Delta$ we let $x_Q$ denote the lower-most and left-most point of $Q$ and call $x_Q$ the root point of $Q$.

Proposition 3.3. For fixed non-negative integer $k$, the set $E_k$ has the following properties.
E \cdot 2^j \cdot \text{separation. For the vertical separation, we only verify that the vertical separation is because the first subclusters are contained in the strips } R \times \{a2^{-2k}\} \text{ for some } a \in \mathbb{N}_0. \\
(2) If j \geq 0 \text{ is an integer and if } Q \in \Delta^j \text{ is such that } Q \cap E_k \text{ is non-empty, then}

\begin{align*}
Q \cap E_k &= \begin{cases} 
x_Q + [0, \ell(Q)] \times \{0\} & j \geq k \\
x_Q + 2^{-2j}E_{k-j} & j \leq k
\end{cases}
\end{align*}

(3) Each } E_k \text{ is Ahlfors regular with regularity constant independent of } k. \\
(4) For 0 \leq j \leq k \text{ an integer, the } j\text{th subcluster of } E_k \text{ has } \mathcal{H}^1\text{-measure } 2^{-2j}. \\
(5) For 1 \leq j \leq k \text{ an integer, the } j\text{th subclusters of } E_k \text{ are } 2 \cdot 2^{-2j}\text{-separated horizontally and at least } 2 \cdot 2^{-2j}\text{-separated vertically. In fact, they are}

\left(3 - \frac{3}{2} \sum_{i=1}^{k-j} 2^{-2i}\right) \cdot 2^{-2j}\text{-separated vertically.} \\
(6) If J \subset E_k \text{ is a connected component, then } J \text{ is a vertical distance of } 3 \cdot 2^{-2k} \text{ from the nearest connected component } J' \text{ of } E_k. \\
(7) There exists a universal constant } c > 0 \text{ such that if } k \geq 2 \text{ and } \mu_k = \mathcal{H}^1 \mathbb{L} E_k, \text{ then for all } x \in E_k,

\int_{6 \cdot 2^{-2k}}^1 \beta_{\mu_k}(x, r)^2 \frac{dr}{r} \geq c(k-2)

Proof. (1) follows immediately from [3.1] since each } p_{ij} \in 2^{-2}\mathbb{Z}^2. \\
To see (2), we first note that the case } j = 0 \text{ is clear for any } k \in \mathbb{N}. \text{ Further, the case } k = 0 \text{ is clear for all } j \in \mathbb{N}. \text{ To proceed inductively suppose that } [3.2] \text{ holds for all } k \in \mathbb{N} \text{ when } j = \ell - 1. \text{ We will show it holds for all } k \in \mathbb{N} \text{ when } j = \ell. \text{ Indeed, suppose that } Q \in \Delta^\ell \text{ has non-empty intersection with } E_{\ell}. \text{ Let } x_Q \text{ be the root of } Q. \text{ Choose } p \in \{p_{ij}\}_{(i,j) \in \{0,3\}^2} \text{ such that } Q \subset p + [0, 2^{-2}]^2. \text{ Then,}

4(Q \cap E_k - p) = (4Q - 4p) \cap (4E_k - 4p) = \hat{Q} \cap E_{k-1} \text{ where } \hat{Q} := 4Q - 4p \in \Delta^{\ell-1}. \text{ By the inductive assumption,}

\hat{Q} \cap E_{k-1} = \begin{cases} 
x_Q + [0, \ell(\hat{Q})] \times \{0\} & \ell - 1 \geq k - 1 \\
x_Q + 2^{-2(\ell-1)}E_{(\ell-1)-(\ell-1)} & \ell - 1 \leq k - 1
\end{cases}

\text{Translating and scaling this back to what this means about } Q \cap E_k \text{ verifies the induction.}

(3) follows from (1) and (2) since these imply that \(\mathcal{H}^1(Q \cap E_k) / \ell(Q) = 1\) for tetradic cubes } Q \text{ with } \ell(Q) \leq 1 \text{ that intersect } E_k. \text{ This suffices since any ball contains a tetradic cube of comparable sidelength and is contained in } 4^2 \text{ tetradic cubes of comparable sidelength.} \\
(4) is equivalent to showing that } E_k \text{ is made of } 2^k \text{ intervals, each of length } 2^{-2k}. \\
(5) The horizontal separation is verified by an argument similar to the vertical separation. For the vertical separation, we only verify that the vertical separation is at least } 2 \cdot 2^{-2j}. \text{ Indeed, this follows since } E_{\ell} \text{ is contained in the horizontal strips } \mathbb{R} \times [0, 1/4] \cup [3/4, 1] \text{ for all } \ell. \text{ Then, the scaling from [3.1] ensures that the } j\text{th subclusters, which arise by applying [3.1] } j \text{ times to the sets } E_{k-j} \text{ are vertically } 2 \cdot 2^{-2j} = \frac{1}{2} \cdot 2^{-2(j-1)} \text{-separated. The reason the height-bound can be improved, is because the } j\text{th subclusters are actually contained in smaller strips. See for instance, } E_1, \text{ where the first subclusters are contained in lines, and } E_2 \text{ where the first subclusters are contained in the strips } \mathbb{R} \times [0, \frac{3}{16}] \cup [\frac{12}{16}, \frac{15}{16}].
(6) follows from the fact that vertically-closest connected components in $E_k$ come from the connected components of $E_1$ which are $3 \cdot 2^{-2}$ separated. After being scaled by $2^{-2}$ in $E_k$, another $(k - 1)$ times the separation is reduced to a distance of $3 \cdot 2^{-2k}$ as claimed. This coincides with the precise formula in (5) and could be considered as a base case for induction on $j$ for the interested reader.

(7) Throughout the proof of (7), we fix integers $1 \leq j < k$ and $k \geq 2$.

**Claim 1:** For all $x \in E_k$ there exists some $x' \in E_j$ with

$$\text{dist}(x, x') \leq 2^{-2j}$$

**Proof of Claim 1.** Note that the scaling in (3.1) ensures that for some $\ell$, we know that every $x \in E_{\ell+1}$ is within a distance $3 \cdot 2^{-2(\ell+1)}$ of a point in $E_{\ell}$. Iterating verifies the claim by showing for $x \in E_k$ there exists $x' \in E_j$ such that

$$\text{dist}(x, x') \leq \sum_{\ell=j+1}^{k} 3 \cdot 2^{-2\ell} \leq 3 \sum_{\ell=j+1}^{\infty} 2^{-2\ell} = 4 \cdot 2^{-2(j+1)}.$$

**Claim 2:** There exists $c$ independent of $j$ such that for all $5 \cdot 2^{-2j} \leq r \leq 11 \cdot 2^{-2j}$ and all $x' \in E_j$, $$\beta_{\mu,2}(x', r)^2 \geq c$$

**Proof of Claim 2.** Let $J \subset E_j$ be the connected component containing $x'$. By (4)-(6) of this proposition, it follows that for $r \geq 5 \cdot 2^{-2j} = \sqrt{(3 \cdot 2^{-2j})^2 + (4 \cdot 2^{-2j})^2}$, the ball $B_r(x')$ contains $J$ and 3 other connected components of $E_j$. Consequently, there are two horizontal lines $L^u$ and $L^d$ such that $B_r(x') \cap (L^u \cup L^d)$ contains at least 4 connected components of $E_j$. Part (1) of this proposition ensures,

$$\min\{\mu_j(L^u \cap B_r(x')), \mu_j(L^d \cap B_r(x'))\} \geq 2 \cdot 2^{-2j}.$$ 

Moreover, part (6) ensures that the distance between $L^u$ and $L^d$ is $3 \cdot 2^{-2j}$, which combined with (3.4) forces that any line $L$ satisfies,

$$\mu_j\left(\left\{y \in B_r(x') \mid \text{dist}(y, L) \geq 3 \cdot 2^{-2j-1}\right\}\right) \geq 2 \cdot 2^{-2j}.$$ 

Finally, recalling $5 \cdot 2^{-2j} \leq r \leq 11 \cdot 2^{-2j}$, (3.5) implies

$$\int_{B_r(x')} \left(\frac{\text{dist}(y, L)}{r}\right)^2 \frac{d\mu_j(y)}{r} \geq \left(\frac{3 \cdot 2^{-2j}}{2}\right) \left(\frac{2 \cdot 2^{-2j}}{r}\right) \geq c$$

which verifies Claim 2.

**Claim 3:** There exists $c'$ such that for all $x \in E_k$ and all integers $1 \leq j < k$ and $\rho$ such that $6 \cdot 2^{-2j} \leq \rho \leq 12 \cdot 2^{-2j}$,

$$\beta_{\mu,2}^1(x, \rho)^2 \geq c'.$$

**Proof of Claim 3.** Claim 1 ensures that for all $5 \cdot 2^{-2j} \leq r \leq 11 \cdot 2^{-2j}$ there exists $x' \in E_j$ such that $B_r(x') \subset B_{\rho}(x)$. As in Claim 2, fix lines $L^d$ and $L^u$ such that $B_r(x) \cap (L^u \cup L^d)$ contains at least 4 connected components of $E_j$. Choose $a$ so that $L^d = \mathbb{R} \times \{a\}$ and $L^u = \{a + (0, 3 \cdot 2^{-2j})\} + \mathbb{R} \times \{0\}$. Moreover, suppose the left-most connected component of $L^u$ has right-most endpoint with $x$-value equal to $c_1$. Define $L_u = \{c_1 + 2^{-2j}\} \times \mathbb{R}$ and $L_h = a + 2^{-2j}$. By Proposition 3.3, 5.6,
Figure 1. When \( j = k - 2 \), the picture displays a subcluster of equal length for \( E_j \) and \( E_k \) on the left and right respectively. In \( E_k \), the line \( L_v \) and its neighborhood \( N_v \) are in green, whereas the line \( L_h \) and its neighborhood \( N_h \) are drawn where it would pass through both \( E_j \) and \( E_k \) the neighborhoods \( N_v = B_2 - 2j(L_v) \) and \( N_h = B_2 - 2j(L_h) \) are disjoint from \( E_\ell \) for all \( \ell \geq j \). See Figure 1.

Consequently, for any line \( L \) the neighborhood \( B_2 - 2j - 1(L) \) can intersect at most 4 of the “quadrants” made by the neighborhoods of \( N_v \) and \( N_L \). Making a generous estimate since the ball may cut-off part of one of the quadrants in Figure 1, we conclude

\[
\mu_k \left( \{ y \in B_r(x') \mid \text{dist}(y, L) \geq 2^{-2j-2} \} \right) \geq 2^{-2j-2}
\]

where the measure-bound comes Proposition 3.3(1). Since \( B_r(x') \subset B_\rho(x) \) and \( 1 \leq \frac{\rho}{r} \leq C < \infty \) Claim 3 follows from (3.7) analogously to how Claim 2 followed from (3.5).

Finally, we verify (7) because

\[
\int_{6.2^{-2j}}^{1} \beta \mu_k(x, \rho)^2 \frac{d\rho}{\rho} \geq \sum_{j=2}^{k} \int_{6.2^{-2j}}^{11.2^{-2j}} \rho^c \frac{d\rho}{\rho} = c(k - 2)
\]

We construct \( \Sigma_0 \) from approximations to the 4-corner Cantor set by first defining

\[
E(n) = (2^{-2n}, 0) + 2^{-2n} E_{2^{2n}} \quad \text{and} \quad \Sigma_0 = \bigcup_n E(n) \cup \{(0, 1) \times \{0\}\}.
\]

Proposition 3.4. \( \Sigma_0 \) has the following properties.

1. \( 0 < \mathcal{H}^1(\Sigma_0) < \infty \) and \( \Sigma_0 \) is countably 1-rectifiable.
2. If \( j \geq 0 \) is an integer and \( \mathcal{Q} \in \Delta^j \) is such that \( \mathcal{Q} \cap \Sigma_0 \neq \emptyset \), then

\[
Q \cap \Sigma_0 = \begin{cases} 
\Sigma_0 \cap [0, \ell(Q)]^2 & x_Q = (0,0) \\
Q + 2^{-2j} E_k & \text{for some } k \neq (0,0) \text{ and } \pi_y(x_Q) \neq 0 \\
Q + 2^{-2j} E_k \cup [0, \ell(E_k)] \times \{0\} & \text{for } \pi_y(x_Q) = 0.
\end{cases}
\]
(3) \( C_{\Sigma_0}(0, \delta) = +\infty \) for all \( \delta > 0 \).

**Proof.** (1) \( \Sigma_0 \) has positive and finite mass due, Proposition [3,3][1]) and the geometric scaling in [3.8]. It is also the countable union of countably 1-rectifiable sets by Proposition 3.3(1).

(2) The case when \( x_Q = (0, 0) \) is clear. Suppose \( x_Q \neq (0, 0) \). There exists unique \( a, b \) such that

\[
(3.10) \quad x_Q = (a 2^{-2j}, b 2^{-2j}).
\]

If \( j = 0, Q \cap \Sigma_0 \neq \emptyset, \) and \( \Sigma_0 \subset [0, 1]^2 \) forces \( a = b = 0 \). Therefore, \( j \geq 1 \). Since \( h(E_{2n}) < \ell(E_{2n}) \) and the \( E(n) \) only use a translation in the positive horizontal direction of \( E_{2n} \) and a homogeneous scaling, it follows that \( \Sigma_0 \cap Q \neq \emptyset \) implies \( 0 \leq b < a \) and that \( a \geq 1 \). Since, \( \ell(Q) = 2^{-2j} \) it follows that \( a 2^{-2j} \geq \ell(Q) \).

Comparing the translation and scaling sizes in (3.1), \( a \geq 2^j \ell(Q) \) implies

\[
(3.11) \quad \Sigma_0 \cap Q = \left\{ \begin{array}{ll}
Q \cap E(n) & , b \geq 1 \\
Q \cap (E(n) \cup [0, \ell(E(n))) \times \{b\}) & , b = 0
\end{array} \right.
\]

for some specific \( n \leq j \). For simplicity of writing, assume we’re in the first case. Then, \( 2^{2n} (Q \cap E(n) - (2^{-2n}, 0)) = \left( 2^{2n} (Q - (2^{-2n}, 0)) \right) \cap E_{2n} \) or equivalently

\[
(3.12) \quad Q \cap E(n) = (2^{-2n}, 0) + 2^{-2n} \left( 2^{2n} (Q - (2^{-2n}, 0)) \right) \cap E_{2n}.
\]

In light of (3.12), it follows that (3.2) implies the 2nd case of (3.9) since \( 2^{2n} (Q - (2^{-2n}, 0)) \in \Delta_{j^{-n}} \) and \( n \leq j \). Analogously the \( b = 0 \) case corresponds to the 3rd case of (3.9).

(3) Fix \( \delta > 0 \). Choose \( N \) so that \( 1 \cdot 2^{-2N} < \delta/2 \), so that for all \( n \geq N, E(n) \subset B_\delta(0) \). Then, with \( \mu = \mathcal{H}^1 \ll \Sigma_0 \) and \( \mu_n = \mathcal{H}^1 \ll E(n) \), it follows from Proposition 3.3(1), Proposition 2.1(2), and the scaling in (3.8) that

\[
C_{\Sigma_0}(0, \delta) \geq \sum_{n \geq N} \int_{E(n)} \int_0^{2^{-2n}} \beta_{\mu_n, 2}^1(x, r) \frac{dr}{r} d\mu_n(x) \geq \sum_{n \geq N} c(2^{2n} - 2) \mathcal{H}^1(E(n)),
\]

which diverges and completes the proof. \( \square \)

We wish to iterate \( \Sigma_0 \) densely along itself while being careful to maintain Ahlfors upper- and lower-regularity. This is attained by scaling, and being careful where we iterate.

**Definition 3.5** (Tail points). We say a point \( y \) is a tail point of \( E \) if \( 0 < \mathcal{H}^1(E) < \infty \) and there exists a tetradic number \( r \) and \( \delta > 0 \) such that

\[
y + r \Sigma_0 \cap B_\delta \subseteq E.
\]

Note, if \( y \in B_\delta(x) \) is a tail point of a set \( E \), then \( C_E(x, \delta) \equiv \infty \). See Claim 1 of Theorem 1.2.

**Definition 3.6** (Iterative construction). Let \( \Sigma_0 \) be as above. Supposing that \( \Sigma_{i-1} \) has been defined, we define a (possibly empty) special collection of tetradic points,

\[
(3.13) \quad D^i = \left\{ x \in 2^{-2i} \mathbb{Z}^2 \mid x + (0, 2^{-2i})^2 \cap \Sigma_{i-1} = x + [0, 2^{-2i}) \times \{0\} \right\},
\]
and define $\Sigma_i$ by

$$\Sigma_i = \Sigma_{i-1} \cup \left\{ \bigcup_{x \in D^i} x + 2^{-8i} \Sigma_0 \right\}.$$  \hfill (3.14)

Define,

$$A_0 = \bigcup_{j \in \mathbb{N}} \Sigma_j.$$  \hfill (3.15)

**Proposition 3.7.** The sets $\{\Sigma_j\}_{j=0}^\infty$ and $\{D^j\}_{j=1}^\infty$ as in Definition 3.6 have the following properties:

1. $\Sigma_{i-1} \subset \Sigma_i$ for all $j \geq 1$.
2. $\Sigma_i$ is contained in countably many horizontal line segments with tetradi heights.
3. $D^j$ is non-empty infinitely often.
4. If $I$ is a connected component of $\Sigma_j$ then $\partial I \subset \ell(I)\mathbb{Z}^2$.
5. $\Sigma_j$ contains no connected component of length at least $2^{-2j}$ that contain no tail point.

**Proof.** Indeed, (1) follows from (3.14).

(2) Follows by induction. For $\Sigma_0$ it follows from Proposition 3.3 (1) combined with the scaling in (3.8). For general $\Sigma_j$ induction holds due to the fact that each scaled copy of $\Sigma_0$ in (3.14) has a tail point on the dyadic lattice $D^i$ which is coarser than the tetradi scaling factor of $\Sigma_0$.

(3) follows from (2). (5) follows from (4) and the definition of $D^j$ in (3.13).

(4) If $I$ is a connected component of $\Sigma_j$ then there exists $y \in D^i$ some $i \leq j$ such that $I$ is a connected component of $y + 2^{-8i} \Sigma_0$. But then, $2^{8i}(I - y)$ is a connected component of $\Sigma_0$. Since $y \in 2^{-2i}\mathbb{Z}^2$, Propositions 3.3 (1) and 3.4 (2) ensure $\partial (2^{8i}(I - y)) \in 2^{8i}\ell(I)\mathbb{Z}^2$ which verifies (4).

\hfill $\Box$

**Definition 3.8 (Associated cubes).** Any cluster (or subcluster) $E$ has associated to it the dyadic cube $Q_E = x_E + [0, \ell(E)]^2$. In particular, by Proposition 3.3 (5) it follows that if clusters $E, E'$ are disjoint with $\ell(E) = \ell(E')$, then $Q_E, Q_{E'}$ are disjoint cubes. Moreover, for some cluster $E$, the root point of $Q_E$ and the root point of $E$ coincide.

**Definition 3.9.** We associate to the base set $\Sigma_0$ the following family of cubes

$$Q_{\Sigma_0} = \{[0, 2^{-2i}]^2 : i \geq 0\} \cup \{Q_E : E \text{ is a subcluster of } E(n) \subset \Sigma_0, n \geq 1\}.$$  \hfill (3.16)

By similarity, for any $y \in D^i$ we associate to $y + 2^{-8i} \Sigma_0$ the family of cubes

$$Q_y = \left(y + 2^{-8i} Q_{\Sigma_0}\right) \cup \left(y + \{[0, 2^{-2k}]^2 : i \leq k\}\right).$$  \hfill (3.17)

We will let

$$Q = \bigcup_{i \geq 0} \bigcup_{y \in D^i} Q_y$$

which we stratify by scale in the following sense

$$Q^i = \{Q \in Q \mid \ell(Q) = 2^{-2i}\}$$

and we enumerate the elements $Q^i$ so that

$$Q^i = \{Q_j\}_{j=1}^{N(i)}.$$  \hfill (3.19)

Finally, for $Q \in Q$ and any positive integer $\ell$ we let $C_\ell(Q) = \{Q' \in Q \mid \ell(Q') = 2^{-2\ell}\ell(Q)\}$, and call $C_\ell(Q)$ the $\ell$th descendent cubes of $Q$. 

Lemma 3.10. For all i ≥ 0 and all cubes, Q_j^i ∈ Q^i, Σ_i ∩ Q_j^i is similar to one of the following:

1. (2^{−2k}Σ_0 ∪ [0, 1) × {0}) ∩ [0, 1)^2 for some integer k.
2. E ∩ Q_E for some sub-cluster E ⊂ E(n) for some integer n ≥ 1.

This follows immediately from the explicit definition of cubes.

Lemma 3.11. Q^j ⊆ Δ^j and for all Q ∈ Δ^j, then either Σ ∩ Q_j = ∅ or Q ∈ Q_j.

This follows from an induction argument similar to the proofs of Propositions 3.3 (1) and 3.4 (2). The key observation in the induction is that the scaling in (3.14) ensures that all tail points added in the jth stage have root points in tetradic/dyadic cubes that are coarser than the length of the scaled copy of Σ_0 being added.

Corollary 3.12. The cubes Q have the following nice properties:

1. Each collection Q_i is a disjoint collection of cubes, and for any Q ∈ Q and any integer ℓ ≥ 0, C_(ℓ)(Q) is a disjoint collection of subcubes of Q.
2. For all non-negative integers i and j,

(3.21) Σ_i ⊆ ∪_{Q ∈ Q_i} Q

(3.22) Σ_i ∩ Q_0 = Σ := \bigcap \left( \cup_{Q ∈ C_i(Q)} Q \right)

Proof of Theorem 1.2. Indeed, by Lemma 3.4 (1), Σ_0 is 1-rectifiable, and A_0 is a countable union of scaled translations of Σ_0 so A_0 is 1-rectifiable.

Next, we show that A_0 is 1-Ahlfors regular. Indeed, it suffices to show that there exists 0 < c ≤ C < ∞ independent of i such that for any j ≥ 0, Q ∈ Δ^j, and Q ∩ A_0 ≠ ∅,

(3.23) c|Q| ≤ H^1(Q ∩ A_0) ≤ C|Q|.

We do this by showing similar bounds for \( \frac{H^1(Q ∩ Σ_j)}{|Q|} \) for cubes Q ∈ Δ^j that intersect Σ_j, and then proving that not too much additional mass is added to the cube Q.

Due to Lemma 3.11, the condition that Q ∈ Δ^j and Q ∩ A_j ≠ ∅ is equivalent to Q ∈ Q_j. Since Q ∈ Q_j, Lemma 3.10 characterizes what Q ∩ Σ_j looks like and we conclude

(3.24) ℓ(Q) ≤ H^1(Q ∩ Σ_j) ≤ 3ℓ(Q),

by considering each of the three cases in Lemma 3.10 Indeed, each cube either contains its entire bottom portion, or contains a cluster E with ℓ(E) = ℓ(Q). In either case this implies the lower bound in (3.24). On the other hand, we know that a rough upper-bound is to assume that Q ∩ Σ_j contains a cluster with a line segment at the bottom, and contains Σ_0 scaled by 2^{−2k}, then by Proposition 3.3 the upper bound in (3.24) follows.

It remains to show that (3.24) implies (3.23). Due to Proposition 3.7 (1), the lower-bound in (3.23) is inherited directly from (3.24). The upper-bound follows with the additional observation that for ℓ ≥ j,

\[ H^1(Q ∩ Σ_{ℓ+1} \setminus Σ_j) \leq |D_{ℓ+1}| 2^{−8(ℓ+1)} H^1(Σ_0) \leq 2^{−4(ℓ+1)} H^1(Σ_0). \]

Summing over ℓ ≥ j verifies (3.23). It is a standard argument to go from Ahlfors regularity in tetradic/dyadic cubes to in balls, see for instance the brief description
in the proof of Proposition 3.3(3). Since the cubes in \( Q \) are all the tetradic cubes with non-empty intersection with \( A_0 \), we have regularity in tetradic cubes.

Finally, to see that \( C_{A_0}(x, \delta) = \infty \) it suffices to show the following claim.

Claim 1- If \( x \in A_0 \) and \( \delta > 0 \), then there is a tail point in \( A_0 \cap B_{\delta/2}(x) \).

Briefly assuming that Claim 1 holds, the fact that \( C_{A_0}(x, \delta) = \infty \) for all \( x \in A_0 \) and \( \delta > 0 \) follows since if \( y \) is the tail point in \( B_{\delta/2}(x) \) then, by Proposition 3.4(3) and monotonicity of integrals of non-negative functions:

\[
C_{A_0}(x, \delta) \geq C_{A_0}(y, \delta/2) \geq C_{\Sigma_0}(0, \epsilon_y) = \infty,
\]

where \( \epsilon_y > 0 \) is some scale dependent on which \( D^i \) the tail point \( y \) is in.

To verify Claim 1, fix \( x \) and \( \delta \) as in the claim. Adopting the convention that \( \Sigma_{-1} = \emptyset \) fix \( i_0 \) such that \( x \in \Sigma_{i_0} \setminus \Sigma_{i_0-1} \). Choose \( k \) to be the smallest natural number such that \( \text{diam}(2^{-8k}\Sigma_0) \leq \delta/4 \).

Case 1- \( B_{\delta/4}(x) \cap \Sigma_k \) contains a tail. Since \( \Sigma_k \subset A_0 \) in this case the claim holds.

Case 2- Otherwise, choose \( k_0 > k \) such that

\[
\begin{aligned}
\left\{ (\Sigma_{k_0-1} \setminus \Sigma_k) \cap B_{\delta/4}(x) = \emptyset \right. \\
\left. (\Sigma_{k_0} \setminus \Sigma_k) \cap B_{\delta/4}(x) \neq \emptyset, \right.
\end{aligned}
\]

that is \( k_0 \) is the first stage after \( k \) where something new is added to the ball \( B_{\delta/4}(x) \).

The way something new is added to the ball \( B_{\delta/4}(x) \) in the \( k_0 \)-th stage is if there exists \( y \) such that,

\( \{y + 2^{-8k_0}\Sigma_0\} \cap \{\Sigma_{k_0} \cap B_{\delta/4}(x)\} \neq \emptyset \).

But then, \( y \) is a tail point of \( \Sigma_{k_0} \) and consequently of \( A_0 \). By our choice of \( k \), we conclude

\[ |x - y| < \text{diam}(2^{-4k_0}\Sigma_0) + \delta/4 \leq \delta/2. \]

Hence the tail point \( y \) is indeed in \( B_{\delta/2}(x) \). So, by Proposition 2.1(2)

\[
C_{A_0}(x, \delta) \geq C_{A_0}(y, \delta/2) \geq cC_{\Sigma}(0, \delta') = \infty.
\]

This completes the theorem.

\[ \square \]

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