A Diameter-Revealing Proof of the Bondy-Lovász Lemma

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Abstract

We present a strengthened version of a lemma due to Bondy and Lovász. This lemma establishes the connectivity of a certain graph whose nodes correspond to the spanning trees of a 2-vertex-connected graph, and implies the \( k = 2 \) case of the Győri-Lovász Theorem on partitioning of \( k \)-vertex-connected graphs. Our strengthened version constructively proves an asymptotically tight \( O(|V|^2) \) bound on the worst-case diameter of this graph of spanning trees.

Keywords: Bondy-Lovász Lemma, Győri-Lovász Theorem, graph diameter, graph partitioning, \( st \)-numbering

1 Introduction

The Győri-Lovász Theorem [6,10] asserts that a \( k \)-vertex-connected graph \( G = (V,E) \), for any distinct \( u_1, \ldots, u_k \in V \) and \( n_1 + \cdots + n_k = |V| \), can be partitioned into \( k \) vertex-disjoint connected subgraphs where the \( i \)-th subgraph

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consists of exactly \( n_i \) vertices including \( u_i \). In the case \( k = 2 \), Lovász [10] provided an elegant proof based on a lemma due to Bondy and Lovász that a certain graph (of exponential or even superexponential size) is connected. The vertices of this graph are the spanning trees of \( G \); for a specified vertex \( a \in V \), two spanning trees are adjacent if their intersection contains a tree on \(|V| - 1\) vertices including \( a \). The proof in [10] establishes only an exponential upper bound on the diameter of this graph, leaving unresolved the question of whether the graph has polynomial diameter.

In this paper, we present a strengthened version of the Bondy–Lovász lemma that constructively proves an \( O(|V|^2) \) bound on the worst-case diameter of this graph of spanning trees. We also show that this bound is asymptotically tight.

**Algorithmic motivation for our results.** One motivation for our results stems from the challenge of understanding the computational complexity of the Győri–Lovász Search Problem: given a \( k \)-vertex-connected graph, find a spanning forest composed of \( k \) trees with specified root vertices and sizes. This problem is known to be solvable in polynomial time when \( k = 2 \) [10] or \( k = 3 \) [11], but for \( k > 3 \) it is only known to belong to the complexity class PLS [2]. Lovász’s polynomial-time algorithm in the case \( k = 2 \) stems from his proof that the graph of spanning trees is connected: the method of proof yields a polynomial-time algorithm that essentially performs bisection search\(^1\) on the (potentially exponentially long) path linking two spanning trees. Our quadratic upper bound on the diameter of the graph of spanning trees yields a different algorithm for the \( k = 2 \) case of the Győri–Lovász Search Problem, based on a sequential search of a polynomially long path. The algorithm defines \( G^+ \) to be the 2-vertex-connected graph obtained from the given graph \( G \) by adding a vertex \( a \) and edges \((u_1, a)\) and \((u_2, a)\). For \( i \in \{1, 2\} \) let \( T_i \) be a spanning tree of \( G^+ \) obtained by deleting the vertex \( u_i \) from \( G^+ \), taking any spanning tree of the resulting graph, and reattaching \( u_i \) as a leaf of that tree. By Theorem [11] below, there is a polynomial-time algorithm to compute a path \( P \) in the graph of spanning trees of \( G^+ \) (rooted at \( a \)) such that \( P \) starts at \( T_1 \), ends at \( T_2 \), and has length \( O(|V|^2) \). For any

\(^1\)The algorithm iteratively splits the path into two subpaths and recurses on one of the subpaths, but unlike in bisection search, the two subpaths are not necessarily of equal size. Nevertheless a different progress measure can be used to prove that the number of iterations of the search process is at most the number of vertices of \( G \).
spanning tree of \( G^+ \) rooted at \( a \), let \( N_1(T) \) denote the number of vertices in the subtree rooted at \( u_1 \), excluding \( u_2 \) and its descendants. Any pair of adjacent trees \( T, T' \) satisfy \( |N_1(T) - N_1(T')| \leq 1 \). Since \( N_1(T_1) = 1 \) and \( N_1(T_2) = |V(G)| \), as \( T \) ranges over the trees in path \( P \) the value \( N_1(T) \) must take every value in the range \( \{1, 2, \ldots, n\} \). Therefore, a brute-force search of the \( O(|V|^2) \) trees that constitute \( P \) is assured of finding a tree \( T \) with \( N_1(T) = n_1 \). Deleting \( a \) from \( T \), and disconnecting \( u_2 \) from its parent if that parent is not \( a \), one obtains a spanning forest of \( G \) whose two components have sizes \( n_1 \) and \( |V(G)| - n_1 \) and roots \( u_1 \) and \( u_2 \), respectively.

For \( k > 2 \), Lovász’s topological proof \([10]\) of the Győri-Lovász Theorem is based on constructing a topological space that generalizes the graph of spanning trees used in the \( k = 2 \) case and satisfies a topological connectivity property, defined in terms of reduced homology groups, that generalizes the connectedness of the graph of spanning trees. (See \([3]\) Theorem 29) for a precise formulation of the relevant topological connectivity property.) Lovász’s proof does not lead directly to a polynomial-time algorithm because the topological space defined in the proof is composed of a potentially (super)exponential number of polyhedral cells. Unlike in the \( k = 2 \) case, it is not known whether bisection search (or a higher-dimensional generalization thereof) can be used to search this (super)exponentially large cell complex in polynomial time. However, if the cell complex could be “sparsified” in polynomial time by extracting a subcomplex, composed of only polynomially many cells, that satisfies the same topological connectivity property as in Lovász’s proof, then brute-force search over the vertices of that subcomplex would solve the Győri-Lovász Search Problem in polynomial time. Our Theorem \([14]\) implements this computationally efficient sparsification procedure when \( k = 2 \); the subcomplex in that case is the path \( P \) defined above. We hope this may motivate investigation into the existence of efficient sparsification procedures when \( k > 2 \), although constructing such a sparsification, if it is even possible, would almost assuredly require more sophisticated mathematics than the methods deployed in the proof of Theorem \([14]\).

**Related work.** There exist alternative proofs of (generalizations of) the Győri-Lovász Theorem. Hoyer and Thomas \([7]\) presented an alternative exposition of Győri’s proof; Idzik \([8]\) presented a proof in the same spirit as Győri’s to give a slightly stronger conclusion: given a partition of \( G \) into connected subgraphs \( V_1, \ldots, V_k \) each containing \( u_1, \ldots, u_k \), if \( V_1 \) has more
than one vertex, then there is another partition $V'_1, \ldots, V'_k$ (again, each containing $u_1, \ldots, u_k$) such that $V'_1$ has one fewer vertices than $V_1$, $V'_k$ is a proper superset of $V_k$, and $|V'_i| = |V_i|$ for all $i = 2, \ldots, k-1$. Chen et al. [3] proved a version with vertex weights by generalizing Lovász’s topological proof; Chandran et al. [2], among other results, rederived the vertex-weighted generalization using a proof similar to Győri’s, obtaining an $O^*(4^n)$-time algorithm for constructing the partition.

2 Upper bound

**Definition 1.** For $G = (V, E)$ with a specified vertex $a \in V$, two spanning trees of $G$ are adjacent if their intersection contains a tree on $|V| - 1$ vertices including $a$.

From now on, we will consider spanning trees as rooted at $a$. Let $n := |V|$. We assume that $n \geq 2$.

**Observation 1.** Two spanning trees $T_A$, $T_B$ are adjacent if and only if $T_B$ can be obtained by detaching some leaf $v \neq a$ of $T_A$ from its current parent and attaching it to some vertex.

**Theorem 1.** Let $G = (V, E)$ be a 2-vertex-connected graph and let $a$ be a specified vertex of $G$. For any two spanning trees $T, T'$ of $G$, there is a sequence of at most $O(n^2)$ trees beginning with $T$ and ending with $T'$, such that every pair of consecutive trees in the sequence are adjacent. Moreover, this path can be found in polynomial time.

**Proof.** Recall that an st-numbering of a graph $G$ with respect to an edge $(s, t)$ is a numbering of the vertices of $G$ as $v_1, \ldots, v_n$ such that $s = v_1, t = v_n$, and every vertex $v_i \neq s, t$ has two neighbors $v_j, v_k$ such that $j < i < k$. It is well-known that every 2-vertex-connected graph has an st-numbering with respect to every one of its edges [9]. Let us choose an arbitrary edge incident to the distinguished vertex $a$, and let $v_1, v_2, \ldots, v_n$ be an st-numbering with respect to this edge, such that $v_1 = a$. The st-numbering can be found in polynomial time [9, 5, 1, 12, 1].

Let $T^+$ be the “canonical” spanning tree constructed as follows: $v_n$ is a child of $v_1$; every vertex other than $v_1$ and $v_n$ is a child of its highest-numbered neighbor. It is easy to show by induction that $T^+$ is a uniquely defined spanning tree.
It suffices to prove the theorem only for $T = T^+$. In constructing a sequence of spanning trees beginning with $T^+$ and ending with an arbitrary spanning tree $T'$, we identify “milestones” $T_1 = T^+, T_2, \ldots, T_{n-1}, T_n = T'$ where each pair of consecutive milestones are joined by a sequence of $O(n)$ spanning trees, each adjacent to the next one in the sequence. First, we define $S_1, \ldots, S_n$ that are connected subgraphs of $T'$ containing $a$. Note that the vertex set of $S_k$, $V(S_k)$, uniquely determines $S_k$. Our construction will satisfy $S_1 \subset \cdots \subset S_n$, where $S_1$ is the singleton tree $\{a\}$, $S_{k+1}$ contains $S_k$ and one other vertex, and $S_n = T'$. In particular, among all $(u, v) \in T'$ such that $u \in S_k$ and $v \notin S_k$, choose $(u_k^*, v_k^*)$ in which $v_k^*$ has the highest number; $V(S_{k+1}) := V(S_k) \cup \{v_k^*\}$.

The spanning tree $T_k$ is defined to be a supergraph of $S_k$. In $T_k$, every vertex $v$ in $V \setminus V(S_k)$ becomes a child of its highest-numbered neighbor unless $v = v_n$. If $v = v_n$, $v$ becomes a child of $v_1$. It is easy to see that $T_k$ is indeed a tree. We have $T_1 = T^+$ and $T_n = S_n = T'$.

Now, for $1 \leq k < n$, we present an algorithm that produces a sequence of $O(n)$ spanning trees beginning with $T_k$ and ending with $T_{k+1}$ such that every pair of consecutive trees are adjacent. First, for each $v \in V \setminus V(S_k)$ in the ascending order of the st-numbering, if $v \neq v_k^*$, we detach $v$ from its current parent and attach it to its lowest-numbered neighbor; if $v = v_k^*$, we detach $v$ from its current parent, attach it to $u_k^*$, and stop processing further vertices in $V \setminus V(S_k)$. Note that $v \neq v_n$ in the first case. Then, for every vertex $v$ that was reattached in the first loop except for the last one $v_k^*$, in the reverse order (i.e., descending order of the numbering), detach $v$ from its current parent and attach it to its highest-numbered neighbor. The algorithm outputs the snapshot of the current spanning tree after each reattachment.

We claim that every vertex that was reattached during this process was a leaf at the time of detachment; then, this algorithm produces a sequence of $O(n)$ spanning trees where every pair of consecutive spanning trees are adjacent. All of these $O(n)$ spanning trees contain $S_k$, because the vertices in $S_k$ are never detached by the algorithm. In the second loop, every detached vertex is attached back to its parent in $T^+$, except for $v_k^*$ that is now attached to $u_k^*$, thus, the last spanning tree produced by the algorithm is $T_{k+1}$.

To complete the proof, it remains to verify the claim that every vertex $v$ that was reattached during this process was a leaf at the time of detachment. We implicitly use induction on the number of iterations of the algorithm.

In an iteration of the first loop, suppose $v_i$ gets reattached but was not a leaf. Let $v_j$ be its arbitrary child in the tree before the reattachment. Observe
that \( v_j \notin S_k \), since \( v_i \notin S_k \) and \( S_k \) is a connected subtree contained in all the spanning trees. Suppose \( j > i \); then \( v_j \) has not been considered by the algorithm yet and therefore its parent in the initial tree \( T_k \) also is \( v_i \). Since \( v_i \neq a, v_j \neq v_n \). Since \( v_j \notin S_k \), from the definition of \( T_k \), \( v_i \) is the highest-numbered neighbor of \( v_j \). This implies \( i > j \), leading to contradiction. Now suppose \( i > j \); then \( v_j \notin S_k \) must have already been reattached by the algorithm to its lowest-numbered neighbor. This implies \( i < j \), yielding contradiction again.

In the second loop, reattachments are undone in the exactly opposite order, except for \( v_k^* \); thus, if \( v \) is not a leaf in an iteration of the second loop, the only possibility is when \( v_k^* \) is its child. However, \( u_k^* \), the new parent of \( v_k^* \), is in \( S_k \), whereas \( v \notin S_k \).

Finally, observe that all the above constructions can be performed in polynomial time. \( \square \)

3 Lower bound

Now we exhibit a family of graphs for which the diameter of the graph of spanning trees is \( \Omega(|V|^2) \).

**Definition 2.** For \( k \geq 1 \), \( G_k = (V_k, E_k) \) is a graph with \( 4k + 1 \) vertices and the specified vertex \( a = v_0 \), defined as follows:

\[
\begin{align*}
V_k &:= \{v_0, \ldots, v_{4k}\}, \\
E_k &:= \{(v_0, v_1), (v_0, v_2)\} \cup \\
& \quad \left( \cup_{i=0}^{k-1} \{(v_{4i+1}, v_{4i+2}), (v_{4i+2}, v_{4i+3}), (v_{4i+3}, v_{4i+4}), (v_{4i+4}, v_{4i+1})\} \right) \cup \\
& \quad \left( \cup_{i=0}^{k-2} \{(v_{4i+4}, v_{4i+5}), (v_{4i+3}, v_{4i+6})\} \right);
\end{align*}
\]

\( T_k^A \) and \( T_k^B \) are its two spanning trees defined by:

\[
\begin{align*}
e_i &:= (v_i, v_{i+1}), \\
E(T_k^A) &:= \{e_0, \ldots, e_{4k-1}\}, \\
E(T_k^B) &:= \{(v_0, v_1), (v_0, v_2)\} \cup \\
& \quad \left( \cup_{i=0}^{k-1} \{(v_{4i+1}, v_{4i+2}), (v_{4i+2}, v_{4i+3})\} \right) \cup \\
& \quad \left( \cup_{i=0}^{k-2} \{(v_{4i+4}, v_{4i+5}), (v_{4i+3}, v_{4i+6})\} \right).
\end{align*}
\]

It is easy to observe that \( G_k \) is 2-vertex-connected.
Theorem 2. Let \( T_1, \ldots, T_\ell \) be a shortest sequence of spanning trees of \( G_k \) beginning with \( T_1 = T_k^A \) and ending with \( T_\ell = T_k^B \) such that every pair of consecutive trees are adjacent. The sequence length satisfies \( \ell = \Omega(|V_k|^2) \).

Proof. Let \( t_i \) be the smallest \( t \) such that \( e_i \in T_t \) and \( e_i \notin T_{t+1} \); if there is no such \( t \) then \( t_i := \infty \). For \( i \neq j \), \( t_i \neq t_j \) or \( t_i = t_j = \infty \) since otherwise the intersection of \( T_t \) and \( T_{t+1} \) contains at most \( n - 3 \) edges. We have \( t_1 < \infty \) because \( e_1 \in T_1 \) and \( e_1 \notin T_\ell \).

We claim that \( \min\{t_0, \ldots, t_i\} = t_i \) for all \( i = 1, \ldots, 4k - 1 \). Let \( t^* := \min\{t_0, \ldots, t_i\} \). We have \( t^* < \infty \) from \( t_1 < \infty \). Since \( \{e_0, \ldots, e_i\} \subseteq T_t^* \) and therefore every endpoint of \( e_0, \ldots, e_{i-1} \) either has degree at least 2 or is \( v_0 \), we have \( \{e_0, \ldots, e_{i-1}\} \subseteq T_{t^*+1} \). This shows \( t_0, \ldots, t_{i-1} > t^* \), proving the claim. The claim yields \( t_4k - 1 < \cdots < t_1 < \infty \).

For \( 0 \leq i < k \), consider \( T_{4i+3} \). Since \( t_{4i+2} > t_{4i+3} \), \( e_{4i+2} \) is in \( T_{4i+3} \) and \( v_{4i+3} \) is not a leaf in \( T_{4i+3} \); \( v_{4i+4} \) is a leaf with parent \( v_{4i+3} \). Thus, every vertex \( v_j \) for \( j > 4i + 4 \) must be connected to \( v_0 \) through \( v_{4i+3} \); the subtree rooted at \( v_{4i+3} \) contains at least \( 4k - (4i + 3) \) vertices (excluding \( v_{4i+3} \) itself). On the other hand, in \( T_{4i+2} \), \( v_{4i+3} \) is a leaf (this follows from \( t_{4i+1} > t_{4i+2} \) using an argument analogous to the above). Observing that the number of vertices in the subtree rooted at \( v_{4i+3} \) decreases by at most one between each consecutive pair of spanning trees in the sequence, \( t_{4i+2} - t_{4i+3} \geq 4k - (4i + 3) \).

We have \( \ell \geq \sum_{i=0}^{k-1}[4k - (4i + 3)] = \Omega(k^2) \). \( \square \)
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