ON FINITE ENERGY SOLUTIONS OF FRACTIONAL ORDER EQUATIONS OF THE CHOQUARD TYPE

YUTIAN LEI
Jiangsu Key Laboratory for NSLSCS
School of Mathematical Sciences
Nanjing Normal University, Nanjing 210023, China

ABSTRACT. Finite energy solutions are the important class of solutions of the Choquard equation. This paper is concerned with the regularity of weak finite energy solutions. For nonlocal fractional-order equations, an integral system involving the Riesz potential and the Bessel potential plays a key role. Applying the regularity lifting lemma to this integral system, we can see that some weak integrable solution has the better regularity properties. In addition, we also show the relation between such an integrable solution and the finite energy solution. Based on these results, we prove that the weak finite energy solution is also the classical solution under some conditions. Finally, we point out that the least energy with the critical exponent can be represented by the sharp constant of some inequality of Sobolev type though the ground state solution cannot be found.

1. Introduction. In this paper, we consider the nonlocal static Schrödinger equation

\[(id - \Delta)^{\alpha/2} u = u^{p-1}(\frac{1}{|x|^{\gamma}} * u^p), \quad u > 0 \text{ in } R^n \]

and the corresponding energy functional

\[E(u) = \frac{1}{2} \int_{R^n} |(id - \Delta)^{\alpha/4} u|^2 dx - \frac{1}{2p} \int_{R^n} u^p(\frac{1}{|x|^{\gamma}} * u^p) dx.\]

Here \(n \geq 3, \alpha, \gamma \in (0, n), p > 1\) and \(id\) is the identity operator. We are concerned with the finite energy solutions of (1.1) and the ground state energy with some critical exponent.

Eq. (1.1) can be viewed as the static equation which the solitary solution of the evolutional equation satisfies. When \(\alpha = 2\), (1.1) becomes the classical Choquard equation

\[-\Delta u + u = u^{p-1}(\frac{1}{|x|^\gamma} * u^p), \quad u > 0 \text{ in } R^n.\]

When \(\alpha = 1\), (1.1) is the Dirac-Schrödinger equation, and \((id - \Delta)^{1/2}\) is the kinetic energy operator in the relativity theory. When \(\gamma = n - 2\), (1.1) and (1.2) are the Schrödinger-Poisson-type equations.

These equations play an important role in the study of the global existence and scatter in time for the solutions of the mass-critical Hartree equation (cf. Eq. (6) in

2010 Mathematics Subject Classification. 35J91, 35Q55, 45E10, 45G05.

Key words and phrases. Choquard equation, finite energy solution, integrable solution, Bessel potential.

The research was supported by NSF of China (No. 11471164, 11871278, 11671209).
In general, such a problem is often referred to as Choquard-Pekar equation, which is widely used as the self-consistent field approximation for the same physical system. In particular, it comes up with an approximation to Hartree-Fock theory of a plasma or in the Hartree theory of describing a non-relativistic many-boson systems. It also appears as a Hartree equation for the helium atom. The readers can see [1], [2], [8], [9], [11], [18], [27] and [36].

In 1977, Lieb [24] proved the existence and uniqueness (up to translation) of the least energy solutions of equation (1.2). Afterwards, Lions [28] showed the existence of a sequence of radially symmetric solutions of this equation. In 2010, Ma and Zhao [31] used the method of moving planes to obtain the radial symmetry of positive solutions. Furthermore, they classified the positive solutions of (1.2) without the restriction of the least energy. For (1.2), more systematical results on the existence, the smoothness and the decay estimates can be seen in [33] and [34].

In this paper, we are concerned with the relation between the classical solutions and the weak solutions. This relation is helpful to understand the well-posedness of the fractional order PDE (1.1) and the corresponding system of integral equations.

Different from the nonlocal operator of equations in [13] and [35], the fractional order differential operator in (1.1) is related to the Bessel potential. A positive function \( u \in H^{\alpha/2}(\mathbb{R}^n) \) is called a \( H^{\alpha/2} \)-weak solution of (1.1), if for all \( \phi \in C_0^\infty(\mathbb{R}^n) \), the following equality makes sense

\[
\int_{\mathbb{R}^n} (id - \Delta)^{\alpha/4} u(id - \Delta)^{\alpha/4} \phi dx = \int_{\mathbb{R}^n} u^{p-1}(|x|^{-\gamma} * u^p) \phi dx. \tag{1.3}
\]

Here, the left hand side can be defined by \( Re \int_{\mathbb{R}^n} (1 + 4\pi^2|\xi|^2)^{\alpha/2} \hat{u} \hat{\phi} d\xi \). Since \( C_0^\infty(\mathbb{R}^n) \) is dense in \( H^{\alpha/2}(\mathbb{R}^n) \), the test function \( \phi \) can be taken in \( H^{\alpha/2}(\mathbb{R}^n) \). When \( \alpha = 2 \), (1.3) shows that \( u \) is a usual \( H^1 \)-weak solution. A positive function \( u \in L^2(\mathbb{R}^n) \) is called a \( L^2 \)-weak solution of (1.1), if for all \( \phi \in C_0^\infty(\mathbb{R}^n) \), there holds

\[
\int_{\mathbb{R}^n} u(id - \Delta)^{\alpha/2} \phi dx = \int_{\mathbb{R}^n} u^{p-1}(|x|^{-\gamma} * u^p) \phi dx. \tag{1.4}
\]

Similarly, the left hand side can still be defined by \( Re \int_{\mathbb{R}^n} (1 + 4\pi^2|\xi|^2)^{\alpha/2} \hat{u} \hat{\phi} d\xi \), and the test function can be taken in \( H^\alpha(\mathbb{R}^n) \). Clearly, \( H^{\alpha/2} \)-weak solutions are \( L^2 \)-weak solutions.

Since some solutions of (1.2) have better regularity (cf. [10] and [34]), we can consider the classical solutions. To define the classical solutions of (1.1), we assume that \( \alpha (= 2m) \) is even, and

\[
(1 + |x|)^{-\gamma} u^p(x) \in L^1(\mathbb{R}^n). \tag{1.5}
\]

Condition (1.5) ensures that the improper integral \( v(x) := |x|^{-\gamma} u^p(x) \) is locally uniformly convergent (cf. the argument in §2).

Let \( m > 0 \) be an integer. A positive function \( u \in C^{2m}(\mathbb{R}^n) \) is called a classical solution of (1.1) with \( \alpha = 2m \), if \( u \) satisfies (1.5), and (1.1) holds pointwise in any compact subset of \( \mathbb{R}^n \).

The fractional order equation arises in many branches of sciences such as phase transitions, flame propagation, stratified materials and others. In particular, it can be understood as the infinitesimal generator of a stable Levy process. The readers can see [5], [6], [22] and the references therein. A useful method to study the fractional order equation (1.1) is the integral equations method, which turns a given fractional order equation into its equivalent integral equation (cf. [12] and [31]), and
then various properties of the original equation can be obtained by investigating the integral equation, see [19], [20] and references therein.

To handle the convolution in (1.1), we introduce the following integral system

\[
\begin{align*}
&u = g_\alpha * (u^{p-1}v), \\
v = |x|^{-\gamma} * (u^p),
\end{align*}
\]

(1.6)

where \( \alpha \in (0, n) \), and \( g_\alpha \) is the Bessel kernel (cf. Chapter 5 in [37] or Chapter 2 in [40])

\[ g_\alpha(x) = \frac{1}{(4\pi)^{n/2}\Gamma(\alpha/2)} \int_0^\infty \exp(-\frac{\pi}{t} |x|^2) - \frac{t}{4\pi} t^{(\alpha-n)/2} \frac{dt}{t}. \]

In particular, when \( \alpha = 2 \), (1.6) is equivalent to the Choquard equation (1.2).

**Remark 1.1.** In general, it is easy to verify that the \( C^2(R^n) \)-solution of (1.6) with \( \alpha = 2 \) satisfies (1.2). On the contrary, when the PDE is linear, the solution must be unique (cf. Lemma 9.11 in [26]). This unique solution has the integral form as (1.6) by a simple calculation. For the nonlinear PDE, it is nontrivial to verify that the solution satisfies the corresponding integral equation. For the Lane-Emden-type equation, Chen, Li and Ou [7] pointed out the equivalence between the PDE and the corresponding integral equation. In addition, [4] gave a proof for the Lane-Emden system with \( \alpha = 2 \).

In §2, we prove that if \( u \) is an \( L^2 \)-weak solution of (1.1), then there exists \( c > 0 \) such that \( cu \) satisfies (1.6).

First we present a necessary condition of existence of weak solutions of (1.1).

**Theorem 1.1.** If (1.1) has \( L^2 \)-weak solutions satisfying \( u^p v \in L^1(R^n) \) or \( H^{\alpha/2} \)-weak solutions, then \( p \in I_1 \) with

\[ I_1 := \left( \frac{2n - \gamma}{n}, \frac{2n - \gamma}{n - \alpha} \right). \]

**Remark 1.2.** When \( \alpha = 2 \), Theorem 2 in [34] shows the same conclusion. Applying the integral system (1.6), we can establish the following theorem, which shows that some integrable solutions have many good regularity properties. Write

\[ I_2 := \begin{cases} 
(3n - \gamma - \beta, \min\left\{ \frac{n + 3\beta + \gamma}{2(n - \beta)}, \frac{5n - 3\beta - \gamma}{2(n - \beta)} \right\}), & \text{when } n - \beta < \gamma; \\
(2 + \frac{2(\beta - \gamma)}{n - \gamma - \beta}, 2 + \frac{2\beta}{n - \gamma - \beta}), & \text{when } n - \beta \geq \gamma.
\end{cases} \]

**Theorem 1.2.** Assume that \((u, v)\) is a pair of positive solutions of system (1.6).

If \( u \in L^{2n/(n+\beta)}(R^n) \) for some \( p \in I_2 \) and some \( \beta \in (0, \alpha) \), then

(R1) \( u \in L^s(R^n) \) for any \( s \geq 1 \).

(R2) Both \( u(x) \) and \( v(x) \) are bounded, and \( \lim_{|x| \to \infty} u(x) = 0 \).

(R3) Both \( u(x) \) and \( v(x) \) are Lipschitz continuous.

(R4) Moreover, if \( \alpha > 1 \), then \( u \in C^\infty(R^n) \).

**Remark 1.3.** These integrability and the regularity are helpful to estimate the decay rates of \( u \) when \( |x| \to \infty \) (cf. [14], [33] and [34]). For other integral system, the integrability of the solutions is also essential to estimate the asymptotic rates. The readers can see [3] and [16] for the system involving the Riesz potentials, and see [29] and [38] for the system involving the Wolff potentials.

According to Theorem 1.2, we define integrable solutions. Let \( \beta \) be an arbitrary given number in \( (0, \alpha) \). A positive solution \( u \) is called an integrable solution of (1.1) or (1.6), if \( u \in L^{2n/(n+\beta)}(R^n) \). In particular, a positive weak (or classical) solution \( u \) is called an integrable solution of (1.2), if \( u \in L^{2n/(n+\beta)}(R^n) \) for some \( \beta \in (0, 2] \).
We now introduce another important class of weak solutions. A positive weak solution $u$ is called a finite energy solution of (1.2), if $u \in L^2(\mathbb{R}^n) \cup L^2(\mathbb{R}^n)$ and $w^p(|x|^{-\gamma} * u^p) \in L^1(\mathbb{R}^n)$.

The following theorem implies the reason of introduction of finite energy solutions of (1.2). It also shows that the definition of $H^1$-weak solutions makes sense.

**Theorem 1.3.** Assume $u > 0$ solves (1.2) in the classical sense. Then $u \in H^1(\mathbb{R}^n)$ if and only if $u$ is a finite energy solution. Moreover, $p \in I_1$ with $\alpha = 2$ and $\|\nabla u\|^2_2 + \|u\|^2_2 = \|u^p v\|_1$.

In general, we often use $\|u\|^2_2 + \|u\|^2_2 < \infty$ to define the finite energy solution. Theorem 1.3 shows that the definition above is still reasonable. Noting that (1.6) has no gradient term, we generalize the way of definition of finite energy solution of (1.2) to (1.6). Namely, a positive solution $u$ is called a finite energy solution of (1.1) or (1.6), if $u \in L^2(\mathbb{R}^n) \cup L^{n^*}(\mathbb{R}^n)$ and $w^p v \in L^1(\mathbb{R}^n)$. Here $n^* = \frac{2n}{n-\alpha}$.

Theorem 1.2 implies that the integrable solutions have good regularity. We are concerned naturally when the weak finite energy solutions are the integrable solutions. The following result implies the relation between the integrable solutions and the finite energy solutions.

**Theorem 1.4.** (1) If $u$ is an $L^2$-weak integrable solution or a classical integrable solution of (1.1) for some $p \in I_1 \cap I_2$ and some $\beta \in (0, \alpha]$, then $u$ is a finite energy solution.

(2) If $u$ is an $L^2$-weak finite energy solution or an $H^{\alpha/2}$-weak solution of (1.1) with $p \in I_1$, then it is an integrable solution. Moreover, if $u$ is a classical finite energy solution of (1.2), then it is also an integrable solution.

**Remark 1.4.** It is easy to see that $I_1 \cap I_2$ is not empty.

Theorem 1.3 shows that the classical positive solution of (1.2) is a $H^1$-weak solution as long as it is a finite energy solution. Moreover, it is also a $L^2$-weak solution. On the contrary, the following theorem shows that a $L^2$-weak solution $u$ is a classical solution, as long as $u$ is an integrable solution.

**Theorem 1.5.** If $u$ is an $L^2$-weak finite energy solution of (1.1) for some $p \in I_1 \cap I_2$ and some $\beta \in (0, \alpha]$, then $u$ satisfies (R1)-(R4) and (1.5). In particular, $u$ is a classical solution when $\alpha = 2m$ with $m = 1, 2, \ldots$.

**Remark 1.5.** By Theorems 1.4 (2) and 1.5, we see that if $p \in I_1 \cap I_2$, then $L^2$-weak finite energy solutions and $H^{\alpha/2}$-weak solutions of (1.1) satisfy (R1)-(R4). In particularly, when $\alpha = 2$, this conclusion also holds for (1.2). In addition, the classical finite energy solutions of (1.2) still satisfy (R1)-(R4).

Finally, we consider the ground states of the functional $E(u)$ in $H^{\alpha/2}(\mathbb{R}^n) \setminus \{0\}$, where the first term is defined by $\int_{\mathbb{R}^n} |i\delta - \Delta|^{\alpha/4} u^2 dx$. The minimizers are called the least energy solutions of (1.1). Clearly, they are $H^{\alpha/2}$-weak solutions.

In some subcritical case, the ground states can represent the sharp constant of the Gagliardo-Nirenberg inequality (cf. [39], or Lemma 8.4.2 in [2]). Naturally, in the critical case, we expect to shed light on the relation between the ground states and the extremal functions of some Sobolev-type inequality.

When $p$ is the critical exponent $\frac{2n}{n-\alpha}$, according to Theorem 1.1, $E(u)$ has no minimizer in $H^{\alpha/2}(\mathbb{R}^n) \setminus \{0\}$. However, $E(u)$ has lower bound by the Hardy-Littlewood-Sobolev inequality (see the argument in §5).
On the other hand, when $\alpha \in (0, n/2)$, the functional
\[
E_*(u) = \int_{\mathbb{R}^n} |(-\Delta)^{\alpha/4} u|^2 dx \cdot \left[ \int_{\mathbb{R}^n} (u^p v)(x)dx \right]^{-\frac{n-\alpha}{2n-\gamma}},
\]
has minimizer in $H^{\alpha/2}(\mathbb{R}^n) \setminus \{0\}$ (cf. [7], [23], [25]):
\[
U_*(x) = a\left(\frac{b}{b^2 + |x-x_0|^2}\right)^{(n-\alpha)/2}, \quad a, b > 0 \quad \text{and} \quad x_0 \in \mathbb{R}^n.
\]
Here $v(x) = |x|^{-\gamma} \ast w^p(x)$, $p = \frac{2n-\gamma}{n-\alpha}$, and
\[
\int_{\mathbb{R}^n} |(-\Delta)^{\alpha/4} u|^2 dx := \int_{\mathbb{R}^n} (2\pi |\xi|)^\alpha |\hat{u}(\xi)|^2 d\xi.
\]

By the ideas in [2] and [15], we have the following theorem.

**Theorem 1.6.** Assume $\alpha \in (0, n/2)$ and $p = \frac{2n-\gamma}{n-\alpha}$, then
\[
\inf\{E(u); u \in H^{\alpha/2}(\mathbb{R}^n) \setminus \{0\}\} = \frac{n + \alpha - \gamma}{2(2n - \gamma)} [E_*(U_*)]^{(2n-\gamma)/(n+\alpha-\gamma)}.
\]

Theorem 1.6 shows the relation between the energy functionals involving the Riesz potential and the Bessel potential in critical case.

The proofs of Theorems 1.1 and 1.2 are given in Sections 2 and 3, respectively. In Section 4, we give the proofs of Theorems 1.3, 1.4 and 1.5. Theorem 1.6 is proved in Section 5.

2. **Liouville theorems.** When $u \in L^2(\mathbb{R}^n)$ satisfies (1.6), by the Fourier transformation, we have
\[
\hat{u}(\xi) = \frac{(u^{p-1} v)^\wedge(\xi)}{(1 + 4\pi^2 |\xi|^2)^{\alpha/2}}.
\]
Thus, for any $\phi \in C_0^{\infty}(\mathbb{R}^n)$, there holds
\[
Re \int_{\mathbb{R}^n} (1 + 4\pi^2 |\xi|^2)^{\alpha/2} \hat{u} \hat{\phi} d\xi = Re \int_{\mathbb{R}^n} (u^{p-1} v)^\wedge(\hat{\phi} d\xi = \int_{\mathbb{R}^n} u^{p-1} v \phi dx. \quad (2.1)
\]
In view of (1.4), $u$ is a $L^2$-weak solution of (1.1).

When $\alpha = 2m$ and $u \in C^{2m}(\mathbb{R}^n)$ satisfies (1.6), there exists $x_0 \in S^{n-1}$ such that
\[
\int_{\mathbb{R}^n} \frac{u^p(y)dy}{(1 + |y|)^\gamma} \leq C \int_{\mathbb{R}^n} \frac{u^p(y)dy}{|x_0 + |y||} \leq C \int_{\mathbb{R}^n} \frac{u^p(y)dy}{|x_0 - y|^\gamma} \leq C v(x_0) < \infty.
\]
This shows (1.5) holds.

We claim that (1.5) implies $v$ is locally uniformly convergent. In fact, for $x \in B_{R/2}(0)$,
\[
v(x) = \int_{R^n \setminus B_R(0)} \frac{u^p(y)dy}{|x-y|^\gamma} + \int_{B_R(0) \setminus B_{|x|/2}(x)} \frac{u^p(y)dy}{|x-y|^\gamma} + \int_{B_{|x|/2}(x)} \frac{u^p(y)dy}{|x-y|^\gamma} := J_1(x) + J_2(x) + J_3(x).
\]
When $y \in R^n \setminus B_R(0)$, $|x - y| > |y| - R/2 \geq c(1 + |y|)$. Thus, by (1.5),
\[
J_1(x) \leq C \int_{R^n \setminus B_R(0)} \frac{u^p(y)dy}{(1 + |y|)^\gamma} \leq C.
\]
When \( y \in B_R(0) \setminus B_{\frac{R}{2}}(x) \), \( |x-y| \geq |y|-|x| \geq |y|-2|x-y| \). Hence \( |x-y| \geq |y|/3 \). Thus, \( \gamma < n \) leads to

\[
J_2(x) \leq C \int_{B_R(0)} \frac{u^p(y)dy}{|y|^{\gamma}} \leq C.
\]

When \( y \in B_{\frac{R}{2}}(x) \), from \( \gamma < n \) it follows

\[
J_3(x) \leq \|u\|_{C(B_R(0))} \int_{B_{\frac{R}{2}}(0)} \frac{dy}{|y|^{\gamma}} \leq C.
\]

Combining these estimates and noting that \( R \) is an arbitrary constant, we know that \( v(x) \) is locally uniformly convergent.

Since the Bessel kernel \( g_{2m} \) is a fundamental solution of \( (id - \Delta)^m w = 0 \), we get

\[
(id - \Delta)^m u(x) = \int_{R^n} [(id - \Delta)^m g_{2m}(x-y)]u^{p-1}(y)v(y)dy
\]

\[
= \int_{R^n} \delta(x-y)u^{p-1}(y)v(y)dy = u^{p-1}(x)v(x).
\]

Here \( \delta \) is a Dirac function at 0. This implies that \( u \) solves (1.1) with \( \alpha = 2m \) in the classical sense.

The argument above shows that \( u \) solves (1.1) if it is a solution of (1.6). On the contrary, we have the following result.

**Theorem 2.1.** Assume \( u \in L^2(R^n) \) is a weak solution of (1.1), then \( u \) satisfies (1.6) up to a positive multiplicative constant. In addition, if \( u \) is a classical solution of (1.2), then it also solves (1.6) with \( \alpha = 2 \).

**Proof.** **Step 1.** When \( u \in L^2(R^n) \) solves (1.1) in the weak sense, by definition, \( u \) satisfies

\[
Re \int_{R^n} (1 + 4\pi^2|\xi|^2)^{\alpha/2} \hat{u} \hat{\phi} d\xi = \int_{R^n} u^{p-1}v\phi dx, \quad \forall \phi \in H^\alpha(R^n). \tag{2.2}
\]

In fact, if \( u \) is a \( H^{\alpha/2} \)-weak solution, it is still true.

Let \( g_\alpha \) be the Bessel kernel. For any \( \psi \in C_0^\infty(R^n) \), we set \( \Psi(x) = (g_\alpha * \psi)(x) \). Clearly, \( \Psi \in H^\alpha(R^n) \), and

\[
\hat{\Psi} = (\psi * g_\alpha)^w = \hat{\psi} \hat{g}_\alpha = (1 + 4\pi^2|\xi|^2)^{-\alpha/2} \hat{\psi}. \tag{2.3}
\]

Taking the test function \( \phi = \Psi \) in (2.2), we get

\[
Re \int_{R^n} (1 + 4\pi^2|\xi|^2)^{\alpha/2} \hat{u} \hat{\Psi} d\xi = \int_{R^n} u^{p-1}(x)v(x) \int_{R^n} \psi(y)g_\alpha(x-y)dy dx. \tag{2.4}
\]

Inserting (2.3) into the left hand side of (2.4) and using the Parseval formula (cf. Theorem 5.3 in [26]), we have

\[
Re \int_{R^n} (1 + 4\pi^2|\xi|^2)^{\alpha/2} \hat{u}(\xi) \bar{\hat{\Psi}}(\xi)d\xi = Re \int_{R^n} \hat{u}(\xi) \bar{\hat{\psi}}(\xi)d\xi = \int_{R^n} u(x)v(x) dx. \tag{2.5}
\]

Exchanging the order of the variants of the right hand side of (2.4), from (2.5) we obtain

\[
\int_{R^n} u(x)v(x) dx = \int_{R^n} \psi(x) \int_{R^n} g_\alpha(x-y)u^{p-1}(y)v(y)dy dx.
\]
for all $\psi \in C_0^\infty (\mathbb{R}^n)$. This shows that $u$ satisfies (1.6) almost everywhere.

**Step 2.** Next, we consider the case that $u \in C^2 (\mathbb{R}^n)$ is a classical solution of (1.2).

Fix $x_0 \in \mathbb{R}^n$. Denote $B_r (x_0)$ by $B_r$. Let $\phi_r (x)$ solve

$$
\left\{
\begin{array}{ll}
-\Delta \phi (x) + \phi (x) = \delta (x-x_0), & x \in B_r (x_0); \\
\phi > 0 \text{ in } B_r (x_0), & \phi = 0, \quad \text{on } \partial B_r (x_0),
\end{array}
\right.
$$

where $\delta$ is a Dirac function at the origin. Since Lemma 9.11 in [26] shows that the Bessel kernel $g_2$ is the unique fundamental solution, in arbitrary compact subset of $\mathbb{R}^n \setminus \{x_0\}$, there hold

$$
\lim_{r \to \infty} \phi_r (x) = c_0 g_2 (x-x_0), \quad \text{a.e.,}
$$

and

$$
\lim_{r \to \infty} \nabla \phi_r (x) = c_0 \nabla g_2 (x-x_0), \quad \text{a.e.,}
$$

where $c_0$ is a positive constant. In addition, by (2.6), we get

$$
\partial_\nu \phi_r \leq 0, \quad \text{on } \partial B_r,
$$

where $\nu$ is the unit outward norm vector on $\partial B_r$.

Multiplying (1.2) by $\phi_r$ and integrating on $B_r$, we have

$$
\int_{B_r} u^{p-1} \phi_r \, dx = \int_{\partial B_r} u \partial_\nu \phi_r \, ds - \int_{B_r} u (id-\Delta) \phi_r \, dx
$$

$$
= \int_{\partial B_r} u \partial_\nu \phi_r \, ds + u (x_0).
$$

Letting $r \to \infty$, using the Fatou lemma, from (2.7) and (2.9), we obtain

$$
c_0 \int_{\mathbb{R}^n} g_2 (x-x_0) u^{p-1} (x) v (x) \, dx \leq u (x_0).
$$

We claim that there exists a subsequence $r_j \to \infty$ such that

$$
I_j := \int_{\partial B_{r_j}} u \partial_\nu \phi_r \, ds \to 0.
$$

In fact, when $x \in B_1$ and $y \notin B_1$, $|x-y| \leq |x-x_0| + |y-x_0| \leq 1 + |y-x_0| \leq 2 |y-x_0|$. Thus, we have $v (x) \geq c \int_{R^n \setminus B_1} |y-x_0|^{-\gamma} u^p (y) \, dy$ for $x \in B_1$. From this result and (2.11), we see that

$$
\frac{u (x_0)}{c_0} \geq \int_{B_1} g_2 (x-x_0) u^{p-1} (x) v (x) \, dx
$$

$$
\geq c \int_{B_1} g_2 (x-x_0) u^{p-1} (x) \, dx \int_{R^n \setminus B_1} \frac{u^p (y) \, dy}{|y-x_0|^{\gamma}} \geq c \int_{R^n \setminus B_1} \frac{u^p (y) \, dy}{|y-x_0|^{\gamma}}.
$$

Hence, when $R \to \infty$, $\int_{B_2 R \setminus B_R} \frac{u^p (y) \, dy}{|y-x_0|^\gamma} \to 0$. Thus, we can find a subsequence $r_j$ of $R$ such that

$$
\lim_{r_j \to \infty} r_j^{1-\gamma} \int_{\partial B_{r_j}} u^p \, ds = 0.
$$
Using the Hölder inequality, we get

\[ I_j^p = (\int_{\partial B_{r_j}} u |\partial_x \phi| ds)^p \]
\[ \leq (\int_{\partial B_{r_j}} u^p ds)(\int_{\partial B_{r_j}} |\partial_x \phi|^\frac{p}{p-1} ds)^{p-1} \]
\[ \leq C r_j^{\gamma - 1} (r_j^{1-\gamma} \int_{\partial B_{r_j}} u^p(y) ds)(\int_{\partial B_{r_j}} |\nabla \phi|^\frac{p}{p-1} ds)^{p-1}. \]  

By (2.6.4) in [40], we have

\[ (\int_{\partial B_{r_j}} |\nabla g_2|^\frac{p}{p-1} ds)^{p-1} \leq \| \nabla g_2 \|_{L^\infty}^p |\partial B_{r_j}|^{p-1} \]
\[ \leq C r_j^{-p(n-1) e^{-Cpr}_{1/p}} \leq C r_j^{1-n e^{-Cpr}}, \]

where \( c, C > 0 \) are independent of \( r \). Combining this result with (2.8), from (2.13) and (2.14), we deduce that \( I_j \to 0 \) when \( r_j \to \infty \). Eq. (2.12) is verified.

Once (2.12) is true, letting \( r = r_j \to \infty \) in (2.10) and using (2.7), we can see that \( u \) satisfies (1.6) at \( x_0 \) up to a positive multiplicative constant. Since \( x_0 \) is an arbitrary point in \( R^n \), we can complete the proof.

**Proposition 2.2.** Assume \( u \in L^2(R^n) \) solves (1.6), then \( u \in H^{\alpha/2}(R^n) \) if and only if \( u^p v \in L^1(R^n) \). In addition, \( \| u \|^2_{H^{\alpha/2}(R^n)} = \| u^p v \|_{L^1(R^n)} \).

**Proof.** From (1.6), we have \( \hat{u}(\xi) = \hat{g}_\alpha(\xi)(1 + 4\pi^2|\xi|^2)^{\frac{\alpha}{2}} \hat{u}(\xi) = (u^p v)^{\alpha}(\xi) \). Multiplying by \( \bar{\hat{u}} \) and using the Parseval formula, we get

\[ \int_{R^n} (1 + 4\pi^2|\xi|^2)^{\frac{\alpha}{2}} |\hat{u}|^2 d\xi = Re \int_{R^n} (u^p v)^{\alpha} \hat{u} d\xi = \int_{R^n} u^p v dx. \]

Therefore, the proof is complete.

**Theorem 2.3.** If \( p \not\in I_1 \), then (1.1) has no \( H^{\alpha/2}(R^n) \)-weak solution.

**Proof.** Assume that \( u \) is an \( H^{\alpha/2}(R^n) \)-weak solution of (1.1). Taking \( \phi = u \) in (1.3), we have

\[ \int_{R^n} (1 + 4\pi^2|\xi|^2)^{\alpha/2} |\hat{u}(\xi)|^2 d\xi = \int_{R^n} (u^p v)(x) dx. \]  

In addition, (1.3) implies that the Gateaux derivative of the functional

\[ E(v) = \frac{1}{2} \int_{R^n} \left( 1 + 4\pi^2|\xi|^2 \right)^{\alpha/2} |\hat{\phi}(\xi)|^2 d\xi - \frac{1}{2p} \int_{R^n} v^p |x|^{-\gamma} \ast v^p dx \]

is zero at \( v = u \). Since \( E(v) \) is a \( C^1 \)-functional, the Frechet derivative of \( E(v) \) is also zero at \( v = u \). Thus, the Pohozaev identity \( \left[ \frac{d}{du} E(u(\xi)) \right]_{u=1} = 0 \) holds. By virtue of

\[ E(u(\xi)) = \frac{\mu^{n-\alpha}}{2} \int_{R^n} (\mu^2 + 4\pi^2|\xi|^2)^{\alpha/2} |\hat{u}(\xi)|^2 d\xi - \frac{\mu^{2n-\gamma}}{2p} \int_{R^n} (u^p v)(y) dy, \]

the Pohozaev identity leads to

\[ (n - \alpha) \int_{R^n} (1 + 4\pi^2|\xi|^2)^{\alpha/2} |\hat{u}(\xi)|^2 d\xi + \alpha \int_{R^n} (1 + 4\pi^2|\xi|^2)^{(\alpha-2)/2} |\hat{u}(\xi)|^2 d\xi \]
\[ = \frac{2n - \gamma}{p} \int_{R^n} (u^p v) dy. \]
Combining with (2.15), we get

\[
\alpha \int_{R^n} (1 + 4\pi^2|\zeta|^2)^{\alpha/2} |\hat{u}(\zeta)|^2 d\zeta = \left[\frac{2n-\gamma}{p} - (n - \alpha)\right] \int_{R^n} u^p v dx,
\]

(2.16)

\[-\alpha \int_{R^n} (1 + 4\pi^2|\zeta|^2)^{\alpha/2} 4\pi^2|\hat{u}(\zeta)|^2 d\zeta = \left[\frac{2n-\gamma}{p} - n\right] \int_{R^n} u^p v dx.
\]

(2.17)

Noting that the left hand side of (2.16) is positive and that of (2.17) is negative, we can see \( p \in I_1 \) easily.

\[\square\]

**Corollary 2.4.** If \( p \notin I_1 \), then neither (1.1) has \( L^2 \)-weak solution satisfying \( u^p v \in L^1(R^n) \) nor (1.6) has positive solution in \( H^{\alpha/2}(R^n) \).

**Proof.** (1) If \( u \in H^{\alpha/2}(R^n) \) is a positive solution of (1.6), Proposition 2.2 shows that (2.15) still holds. In addition, (2.1) implies that \( u \) is a \( H^{\alpha/2} \)-weak solution of (1.1), and hence it is also a critical point of functional \( E(u) \). By the same argument in the proof of Theorem 2.3, \( p \in I_1 \).

(2) If \( u \) is an \( L^2 \)-weak solution of (1.1) satisfying \( u^p v \in L^1(R^n) \), by Theorem 2.1 we can find \( c > 0 \) such that \( cu \) satisfies (1.6). Combining with \( u^p v \in L^1(R^n) \) we know that \( u \in H^{\alpha/2}(R^n) \) by Proposition 2.2. Therefore, \( cu \) is a positive solution of (1.6) in \( H^{\alpha/2}(R^n) \). The argument in (1) implies \( p \in I_1 \).

\[\square\]

3. Regularity of integrable solutions. Assume

\[n \geq 3, \alpha \in (0, n), \gamma \in (0, n), p > 1. \quad (3.1)\]

**Theorem 3.1.** Assume \( u, v \) are positive solutions of system (1.6) with (3.1). If \( u \in L^{\frac{2(n-\alpha)}{n+\beta-\gamma}}(R^n) \) for some \( p \in I_2 \) and some \( \beta \in (0, \alpha] \), then

(R1) \( u \in L^s(R^n) \) for any \( s \geq 1 \).

(R2) Both \( u(x) \) and \( v(x) \) are bounded, and \( \lim_{|x| \to \infty} u(x) = 0 \).

(R3) Both \( u(x) \) and \( v(x) \) are Lipschitz continuous.

(R4) Moreover, if \( \alpha > 1 \), then \( u \in C^\infty(R^n) \).

**Proof.** When \( \alpha + \gamma \leq n \), Theorem 1.1 in [17] shows that for \( p \in (2 + \frac{2(\beta-\gamma)}{n+\gamma-\beta}, 2 + \frac{2\beta}{n-\gamma-\beta}) \), the conclusions of Theorem are true. We only need to give the proof under the condition \( \alpha + \gamma > n \).

**Step 1.** We prove (R1) in two cases.

**Case I.** \( n - \beta < \gamma \). Now, we assume \( p \in \left( \frac{3n-\gamma-\beta}{2(n-\beta)}, \min\left\{ \frac{n+3\beta+\gamma}{2(\beta+\gamma)}, \frac{5n-3\beta-\gamma}{2(n-\beta)} \right\} \right) \).

For \( A > 0 \), set

\[u_A(x) = u(x), \quad \text{for } \, u(x) > A \quad \text{or} \quad |x| > A; \quad u_A(x) = 0, \quad \text{otherwise.} \]

Similarly, we can also define \( v_A \).

Take \( s > \frac{n}{n-\beta} \) and \( r \) satisfying

\[\frac{1}{r} = \frac{1}{s} - \frac{n-\gamma-\beta}{2n}. \quad (3.2)\]
Let $f \in L^r(\mathbb{R}^n)$ and $g \in L^s(\mathbb{R}^n)$. Write

$$T_1f(x) := \int_{\mathbb{R}^n} g_\alpha(x - y)u_{A}^{-1}(y)f(y)dy,$$

$$T_2g(x) := \int_{\mathbb{R}^n} |x - y|^{-\gamma}u_{A}^{-1}(y)g(y)dy,$$

$$F(x) := \int_{\mathbb{R}^n} g_\alpha(x - y)(u - u_A)^{p-1}(y)(v - v_A)(y)dy,$$

$$G(x) := \int_{\mathbb{R}^n} |x - y|^{-\gamma}(u - u_A)^{p}dy.$$ 

Define the operator $T$

$$T(g, f) = (T_1f, T_2g)$$

with the norm $\|T(f, g)\|_{s \times r} = \|T_1f\|_s + \|T_2g\|_r$.

Using (3.2), the Hardy-Littlewood-Sobolev inequality (cf. Theorem 1 of Chapter 5 in [37]) and the Hölder inequality, we get

$$\|T_1f\|_s \leq C\|I_{\beta}(u_{A}^{-1}f)\|_s \leq C\|u_{A}^{-1}f\|_s^{\frac{n}{n-\beta}} \leq C\|u_{A}\|_{\frac{p-1}{2n(p-1)}}\|f\|_r, \quad (3.3)$$

$$\|T_2g\|_r \leq C\|I_{n-\gamma}(u_{A}^{-1}g)\|_r \leq C\|u_{A}^{-1}g\|_r^{\frac{n}{n+\gamma}} \leq C\|u_{A}\|_{\frac{p-1}{2n(p-1)}}\|g\|_s. \quad (3.4)$$

Here $I_{\beta}(f) := c|x|^{\beta-n}*f$ is the Riesz potential of nonnegative functions $f$, where $c$ only depends on $n$ and $\beta$ (cf. Section 1 of Chapter 5 in [37]).

Noting $u \in L^{2n(p-1)/(n-\gamma+\beta)}(\mathbb{R}^n)$, we see that $C\|u_{A}\|_{2n(p-1)/(n-\gamma+\beta)} \leq \frac{1}{2}$ when $A$ is suitably large. Then for any $r, s$ satisfying

$$\frac{1}{s} \in (0, \frac{n-\beta}{n}), \quad \frac{1}{r} \in \left(\frac{\beta+\gamma-n}{2n}, \frac{n-\beta+\gamma}{2n}\right), \quad (3.5)$$

the operator $T$ is a contracting map from $L^s(\mathbb{R}^n) \times L^r(\mathbb{R}^n)$ to itself.

Noticing the integrable solution $u \in L^{s_0}(\mathbb{R}^n)$, where $s_0 = \frac{2n(p-1)}{n+\gamma}$, we take $r_0$ such that $(\frac{1}{s_0}, \frac{1}{r_0})$ satisfies (3.2). Applying the Hardy-Littlewood-Sobolev inequality to $v = |x|^{-\gamma}*u^p$, we get $v \in L^{s_0}(\mathbb{R}^n)$ with $r_0 = \frac{2n(p-1)}{\beta(p-2)(n-\gamma)}$. Noting $p \in \left(\frac{2n-\gamma-\beta}{2n-\beta}, \frac{n+3\beta+\gamma}{2n}\right)$, we see that $r_0, s_0$ also satisfy (3.5). Therefore, $T$ is also a contraction map from $L^{s_0}(\mathbb{R}^n) \times L^{s_0}(\mathbb{R}^n)$ to itself.

Similar to (3.3) and (3.4), for any $r, s$ satisfying $r > \frac{n}{\beta}$, $s > \frac{n}{n-\beta}$, we have

$$\|F\|_s \leq C\|(u - u_A)^{p-1}(v - v_A)\|_s^{\frac{n}{n+\gamma}}, \quad \|G\|_r \leq C\|(u - u_A)^p\|_r^{\frac{n}{n+\gamma}}.$$ 

In view of the definitions of $u_A$ and $v_A$, we see $F \in L^s(\mathbb{R}^n)$ and $G \in L^r(\mathbb{R}^n)$.

Let $X = L^{s_0}(\mathbb{R}^n) \times L^{s_0}(\mathbb{R}^n)$, $Y = L^s(\mathbb{R}^n) \times L^r(\mathbb{R}^n)$. Since $(u, v)$ solves the equation

$$(g, f) = T(g, f) + (F, G),$$

by the regularity lifting Lemma 2.1 in [16] (see also Theorem 3.3.1 in [3]), we know that $(u, v) \in L^s(\mathbb{R}^n) \times L^r(\mathbb{R}^n)$ for all $r, s$ satisfying (3.5).

Choose $s_1 = \frac{n}{n-\beta}, \frac{1}{r_1}$ with small $\epsilon > 0$. By the Hölder inequality, $\|u^{p-1}v\|_1 \leq \|u\|_{s_1}^{p-1}\|v\|_{r_1}$, where $\frac{1}{r_1} = 1 - \frac{p-1}{s_1}$. Noting $p \in \left(\frac{3n-\beta-\gamma}{2(n-\beta)}, \frac{5n-3\beta-\gamma}{2(n-\beta)}\right)$, we see that $\frac{1}{r_1}$ belongs to the interval of (3.5). Therefore, $\int_{\mathbb{R}^n} u^{p-1}vdx < \infty$. This result, together
with (1.6), implies
\[
\int_{\mathbb{R}^n} u(x)dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g_{\alpha}(x-y)u^{p-1}(y)v(y)dydx \\
\leq \int_{\mathbb{R}^n} g_{\alpha}(x)dx \int_{\mathbb{R}^n} u^{p-1}(x)v(x)dx < \infty.
\]

Namely, \( u \in L^1(\mathbb{R}^n) \).

Next, we prove \( u \in L^r(\mathbb{R}^n) \) for any \( \frac{1}{r} \in [\frac{n-\beta}{n}, 1) \). To do this, we can choose \( \varepsilon \) sufficiently small such that \( \frac{\varepsilon}{1+\varepsilon} \in (0, \frac{n-\beta}{n}) \). Therefore, \( u \in L^{(t-1+\varepsilon)/\varepsilon}(\mathbb{R}^n) \).

Applying the Hölder inequality, we get
\[
\int_{\mathbb{R}^n} u^t dx = \int_{\mathbb{R}^n} u^{(1-\varepsilon)+(t-1+\varepsilon)} dx \leq \|u\|_1^{1-\varepsilon}\|u\|^{t-1+\varepsilon}_{(t-1+\varepsilon)/\varepsilon} < \infty.
\]

**Case II.** \( n-\beta \geq \gamma \). Now, we assume \( p \in (2 + \frac{2(\beta-\gamma)}{n+\gamma-\beta}, 2 + \frac{2\beta}{n-\gamma-\beta}) \).

Take \( r \) satisfying \( \frac{1}{r} \in (0, \gamma/n) \), and \( s \) satisfying (3.2). By the same argument in Case I, for all \( r, s \) satisfying
\[
\frac{1}{r} \in (0, \frac{\gamma}{n}), \quad \frac{1}{s} \in (\frac{n-\gamma-\beta}{2n}, \frac{n+\gamma-\beta}{2n}),
\]
\( T \) is still a contraction from \( L^s(\mathbb{R}^n) \times L^r(\mathbb{R}^n) \) to itself.

In view of \( p \in (\frac{2n}{n-\beta+\gamma}, \frac{2(n-\gamma)}{n-\beta-\gamma}) \), we see that \( s_0 \) and \( r_0 \) satisfy (3.6). Thus, \( T \) is also contraction from \( L^{s_0}(\mathbb{R}^n) \times L^{r_0}(\mathbb{R}^n) \) to itself.

Using Lemma 2.1 in [16], we also get \( (u, v) \in L^s(\mathbb{R}^n) \times L^r(\mathbb{R}^n) \) for all \( s, r \) satisfying (3.6).

Next, we improve the integrability interval of \( u \) from \( (\frac{n-\gamma-\beta}{2n}, \frac{n+\gamma-\beta}{2n}) \) to \( (0, \frac{n-\beta}{n}) \).

In fact, applying the Hardy-Littlewood-Sobolev inequality and the Hölder inequality to
\[
u(x) = \int_{\mathbb{R}^n} g_{\alpha}(x-y)u^{p-1}(y)v(y)dy,
\]
we obtain
\[
\|u\|_{\xi_j} \leq C\|I_{\sigma}(u^{p-1})\|_{\xi_j} \leq C\|u^{p-1}\|_{\frac{n+\gamma-\beta}{n-\beta-\gamma}} \leq C\|u\|_{\xi_{j-1}}^p \|v\|_r,
\]
for \( j = 1, 2, \ldots, \) where \( r > \frac{n}{\beta}, \sigma \in (0, \alpha] \), and
\[
\frac{1}{\xi_j} < 1 - \frac{\sigma}{n}, \quad \frac{1}{\xi_j} = \frac{p-1}{\xi_{j-1}} + 1 - \frac{\sigma}{n}.
\]

Let \( \frac{1}{\xi_0} \in (\frac{n-\gamma-\beta}{2n}, \frac{n+\gamma-\beta}{2n}) \). Noting \( \frac{1}{r} \in (0, \frac{\gamma}{n}) \), from (3.7) and (3.8) we deduce \( u \in L^{u_0}(\mathbb{R}^n) \) for all
\[
\frac{1}{\xi_1} = \max\{0, \frac{n-\gamma-\beta}{2n}(p-1) - \frac{\alpha}{n}\}, \min\{\frac{n+\gamma-\beta}{2n}(p-1) + \frac{\gamma}{n}, \frac{n-\beta}{n}\}.
\]

In view of \( p \in (2 + \frac{2(\beta-\gamma)}{n+\gamma-\beta}, 2 + \frac{2\beta}{n-\gamma-\beta}) \), this new interval covers the original one in (3.6).

From (3.8) it follows \( \frac{1}{\xi_j} - \frac{1}{\xi_{j-1}} = \frac{p-2}{\xi_{j-1}} + 1 - \frac{\sigma}{n} \). By virtue of \( p \in (2 + \frac{2(\beta-\gamma)}{n+\gamma-\beta}, 2 + \frac{2\beta}{n-\gamma-\beta}) \), \( \frac{1}{\xi_j} \) is increasing as \( \frac{1}{\xi_0} = \frac{n+\gamma-\beta}{2n} \), \( \frac{1}{\xi_1} = \frac{n-\gamma-\beta}{2n} \), and \( \sigma = \alpha \), and \( \frac{1}{\xi_j} \) is decreasing as \( \frac{1}{\xi_0} = \frac{n-\gamma-\beta}{2n} \), \( \frac{1}{r} = \frac{\gamma}{n} \), and \( \sigma = \alpha \), where \( \epsilon > 0 \) is sufficiently small. We can deduce by induction that the integrability intervals are larger and larger.
By finite steps, we claim that \( u \in L^s(R^n) \) for all \( \frac{1}{s} \in (0, \frac{n-\beta}{n}) \). Otherwise, the limit \( \lim_{j \to \infty} \frac{1}{\xi_j} \) must exist and belong to \( (0, \frac{n-\beta}{n}) \). We denote the limit by \( L \).

When \( \frac{1}{\xi_j} \) is increasing, there holds

\[
\frac{1}{\xi_j} = \frac{p-1}{\xi_{j-1}} + \frac{\gamma - \epsilon}{\xi_{j-1}} - \frac{n}{\xi_{j-1}}, \quad \frac{1}{\xi_0} = \frac{n + \gamma - \beta - \epsilon}{2n}.
\]

Let \( j \to \infty \), then \( L = \frac{n + \gamma - \beta - \epsilon}{2n} \), which implies \( p < 2 \). Noting \( p > 2 - \frac{2\gamma}{n + \gamma - \beta} \), we see \( L < \frac{1}{\xi_0} \), which contradicts with the monotonicity as \( \epsilon \) is sufficiently small.

When \( \frac{1}{\xi_j} \) is decreasing, there holds

\[
\frac{1}{\xi_j} = \frac{p-1}{\xi_{j-1}} + \frac{\epsilon}{\xi_{j-1}} - \frac{\alpha}{\xi_{j-1}}, \quad \frac{1}{\xi_0} = \frac{n - \gamma + \beta + \epsilon}{2n}.
\]

This lead to \( L = \frac{n - \gamma + \beta + \epsilon}{2n} \) which implies \( p > 2 \). From \( p < 2 + \frac{2\beta}{n - \gamma - \beta} \), we deduce that the value of \( L \) contradicts with \( L < \frac{1}{\xi_0} \) as \( \epsilon \) is sufficiently small.

Finally, similar to the proof of Theorem 3.5 in [17], we also get \( u \in L^s(R^n) \) for all \( s \geq 1 \).

**Step 2.** Based on the result of Step 1, by the corresponding argument in [17], we can also obtain (R2)-(R4).

4. Regularity of finite energy solutions.

**Theorem 4.1.** Assume \( u > 0 \) solves (1.2) in the classical sense. Then \( u \in H^1(R^n) \) if and only if \( u \in L^2(R^n) \cup L^2(R^n) \) and \( u^p v \in L^1(R^n) \). Moreover, \( p \in I_1 \) and 

\[
||\nabla u||_2^2 + ||u||_2^2 = ||u^p v||_1.
\]

**Proof.** Denote \( B_r(0) \) by \( B_r \). Multiplying (1.2) by \( u \) and integrating on \( B_r \), we have

\[
\int_{B_r} (|\nabla u|^2 + u^2) dx - \int_{\partial B_r} u \partial_x u ds = \int_{B_r} u^p v dx.
\]

**Necessity.** If \( u \in H^1(R^n) \), then \( u \in L^2(R^n) \cap L^2(R^n) \). Hence, there exists a sequence \( r_j \to \infty \) such that \( r_j \int_{\partial B_{r_j}} (|\nabla u|^2 + u^2) ds \to 0 \). Applying this result and the H"older inequality, we get

\[
\int_{\partial B_{r_j}} u \partial_x u ds \leq \left( \int_{\partial B_{r_j}} |\nabla u|^2 ds \right)^{\frac{1}{2}} \left( \int_{\partial B_{r_j}} u^2 ds \right)^{\frac{1}{2}} \leq C \left( \int_{\partial B_{r_j}} |\nabla u|^2 ds \right)^{\frac{1}{2}} \left( \int_{\partial B_{r_j}} u^2 ds \right)^{\frac{1}{2}} r_j^{-1} \to 0
\]

as \( r_j \to \infty \). Inserting this into (4.1) and letting \( r = r_j \to \infty \), we have 

\[
||u^p v||_1 = ||\nabla u||_2^2 + ||u||_2^2 < \infty.
\]

**Sufficiency.** If \( u \in L^2(R^n) \cup L^2(R^n) \) and \( u^p v \in L^1(R^n) \), we claim \( u \in H^1(R^n) \).

Take smooth function \( \zeta(x) \) satisfying

\[
\begin{cases}
\zeta(x) = 1, & \text{for } |x| \leq 1; \\
\zeta(x) \in [0, 1], & \text{for } |x| \in [1, 2]; \\
\zeta(x) = 0, & \text{for } |x| \geq 2.
\end{cases}
\]

Define the cut-off function \( \zeta_R(x) = \zeta(\frac{x}{R}) \).
Multiplying (1.2) by \( u\zeta^2_R \) and integrating on \( D := B_{3R}(0) \) (here \( R > 1 \)), we have
\[
-\int_D \zeta_R^2 (u\Delta u + u^2) dx = \int_D u^p v\zeta_R^2 dx.
\]
Integrating by parts, we obtain
\[
\int_D (|\nabla u|^2 + u^2)\zeta_R^2 dx + 2 \int_D u\zeta_R \nabla u \nabla \zeta_R dx = \int_D u^p v\zeta_R^2 dx. \tag{4.3}
\]
Applying the Cauchy inequality, we get
\[
|\int_D u\zeta_R \nabla u \nabla \zeta_R dx| \leq \delta \int_D |\nabla u|^2 \zeta_R^2 dx + C \int_D u^2 |\nabla \zeta_R|^2 dx \tag{4.4}
\]
for any \( \delta \in (0, 1/2) \).

When \( u \in L^2(\mathbb{R}^n) \), we can find \( C > 0 \) which is independent of \( R \) such that
\[
\int_D u^2 |\nabla \zeta_R|^2 dx \leq C. \tag{4.5}
\]
When \( u \in L^{2^*}(\mathbb{R}^n) \), (4.5) is still true. In fact, by the Hölder inequality,
\[
\int_D u^2 |\nabla \zeta_R|^2 dx \leq (\int_D u^{2^*} dx)^{1-2/n} (\int_D |\nabla \zeta_R|^n dx)^{2/n} \leq C.
\]
Noting \( u^p \in L^1(\mathbb{R}^n) \), from (4.3)-(4.5) we deduce \( \int_D (|\nabla u|^2 + u^2)\zeta_R^2 dx \leq C \).

Letting \( R \to \infty \) yields \( \int_{\mathbb{R}^n} (|\nabla u|^2 + u^2) dx < \infty \). The claim is verified.

Now, (4.2) still holds, and hence (4.1) shows \( \|u\|_k^2 + \|u\|_2^2 = \|u^p\|_1 \).

Finally, according to Theorem 2.1, if \( u \) is an classical solutions of (1.2), we can find \( c > 0 \) such that \( cu \) solves (1.6). By Theorem 2.3, we know that \( u \in H^1(\mathbb{R}^n) \) implies \( p \in I_1 \).

**Theorem 4.2.** If \( u \) is an \( L^2 \)-weak integrable solution or a classical integrable solution of (1.1) for some \( p \in I_1 \cap I_2 \) and some \( \beta \in (0, \alpha] \), then it is a finite energy solution.

**Proof.** First, Theorem 2.1 shows that there exists \( c > 0 \) such that \( cu \) solves (1.6). Thus, Theorem 3.1 shows that \( u \in L^\alpha(\mathbb{R}^n) \) for all \( \frac{1}{2} \in [0, 1] \), which implies \( u \in L^2(\mathbb{R}^n) \cap L^{\alpha^*}(\mathbb{R}^n) \). In addition, by the Hölder inequality and the integrability results, we obtain \( \|u^p\|_1 \leq \|u^p\|_k \|u\|_{r_0} < \infty \) by taking \( \frac{1}{k} = 1 - \frac{1}{r_0} \). Here \( r_0 \) is the constant in the proof of Theorem 3.1.

**Theorem 4.3.** If \( u \) is an \( L^2 \)-weak finite energy solution or an \( H^{\alpha/2} \)-weak solution of (1.1) with \( p \in I_1 \), then it is an integrable solution. Moreover, if \( u \) is a classical finite energy solution of (1.2), then it is also an integrable solution.

**Proof.** If \( u \) is an \( H^{\alpha/2} \)-weak solution of (1.1), then
\[
u \in L^t(\mathbb{R}^n), \quad \forall t \in [2, \alpha^*]. \tag{4.6}
\]
In view of \( p \in I_1 \), we get \( 2(n-\gamma) < 2n(p-1) < \alpha^*(n-\gamma+\alpha) \). Therefore, we can find some \( \beta \in (0, \alpha] \) such that \( \frac{2n(p-1)}{n-\gamma+\alpha} \in [2, \alpha^*] \). Thus, (4.6) implies \( u \in L^{\frac{2n(p-1)}{n-\gamma+\alpha}}(\mathbb{R}^n) \).

If \( u \) is an \( L^2 \)-weak finite energy solution of (1.1), then by Theorem 2.1 and Proposition 2.2, \( u \) is also an \( H^{\alpha/2} \)-weak solution. By the same argument above, \( u \) is still an integrable solution.

If \( u \) is a classical finite energy solution of (1.2), then by Theorem 4.1, \( u \in H^1(\mathbb{R}^n) \), and hence it is also an integrable solution.
Theorem 5.1. Assume terms in $E$ be expressed by the sharp constant of (5.3).

Proof. First, Theorem 2.1 implies that there exists $c > 0$ such that $cu$ satisfies the integral system (1.6). Since $u$ is an integrable solution which is implied by Theorem 3.1, shows that (R4) holds true and $u \in L^t(R^n)$ for all $t \in [1, \infty]$. Therefore,

$$
\int_{R^n} (1 + |x|)^{-\gamma} u^p(x) dx 
\leq \|u\|_\infty^p \int_{B_R(0)} \frac{dx}{(1 + |x|)^\gamma} + \|u\|_k^p \int_{R^n \setminus B_R(0)} |x|^{-\gamma k'} dx^{1/k'} < \infty.
$$

Here $0 < k < \frac{n}{n-\gamma}$ and $\frac{1}{k'} = 1 - \frac{1}{k}$. Eq. (1.5) is verified. \((\Box)\)

5. Infimum of critical energy. Consider the infimum of the following energy functional in $H^{\alpha/2}(R^n) \setminus \{0\}$

$$
E(u) = \frac{1}{2} \int_{R^n} (1 + 4\pi^2 |\xi|^{n/2}) |\hat{u}(\xi)|^2 d\xi - \frac{1}{2p} \int_{R^n} (u^p v)(x) dx.
$$

The minimizers are $H^{\alpha/2}$-weak solutions of (1.1). When $p$ is the critical exponent $\frac{2n-\gamma}{n-\alpha}$, according to Theorem 2.3, $E(u)$ has no minimizer in $H^{\alpha/2}(R^n) \setminus \{0\}$.

On the other hand, the radial function

$$
U_\alpha(x) = a \left( \frac{b}{b^2 + |x-x_0|^2} \right)^{(n-\alpha)/2}, \quad a, b > 0 \quad \text{and} \quad x_0 \in R^n
$$

is an extremal function of the following Hardy-Littlewood-Sobolev inequality

$$
\int_{R^n} \int_{R^n} \frac{u^{2n/(n-\alpha)}(x) u^{2n/(n-\alpha)}(y)}{|x-y|^\gamma} dx dy \leq C(\int_{R^n} u^{2n/(n-\alpha)} dx)^{(2n-\gamma)/n}, \quad (5.1)
$$

where $u$ is an arbitrary nonnegative measurable function. In addition, when $\alpha \in (0, n/2)$, according to the classification results in [7] and [23], the radial function $U_\alpha$ is also the extremal function of the Sobolev inequality

$$
(\int_{R^n} |u|^{2n/(n-\alpha)} dx)^{(n-\alpha)/n} \leq C \int_{R^n} |(-\Delta)^{\alpha/4} u|^2 dx, \quad \forall u \in H^{\alpha/2}(R^n). \quad (5.2)
$$

Combining (5.1) and (5.2), we have

$$
e(\int_{R^n} \int_{R^n} \frac{u^{2n/(n-\alpha)}(x) u^{2n/(n-\alpha)}(y)}{|x-y|^\gamma} dx dy)^{\frac{2n-\gamma}{\alpha}} \leq \int_{R^n} |(-\Delta)^{\alpha/4} u|^2 dx. \quad (5.3)
$$

Thus, $U_\alpha$ is still the extremal function of the functional

$$
E_\alpha(u) = \int_{R^n} (2\pi |\xi|)^{\alpha} |\hat{u}(\xi)|^2 d\xi \cdot [\int_{R^n} (u^{2n/(n-\alpha)} v)(x) dx]^{-\frac{2n-\gamma}{2n-\gamma}}
$$

in $H^{\alpha/2}(R^n) \setminus \{0\}$, where $v(x) = |x|^{-\gamma} * u^{2n/(n-\alpha)}$. The corresponding minimum can be expressed by the sharp constant of (5.3).

Inequality (5.3) implies that, when $u \in H^{\alpha/2}(R^n)$, the two improper integral terms in $E(u)$ with $p = \frac{2n-\gamma}{n-\alpha}$ are convergent, and $E(u)$ is bounded from below.

Theorem 5.1. Assume $\alpha \in (0, n/2)$ and $p = \frac{2n-\gamma}{n-\alpha}$, then

$$
\inf \{E(u); u \in H^{\alpha/2}(R^n) \setminus \{0\} \} = \frac{n + \alpha - \gamma}{2(2n - \gamma)} \left[ E_\alpha(U_\alpha) \right]^{2(2n-\gamma)/(n+\alpha-\gamma)}.
$$
Proof. Write the scaling function
\[ u^\lambda_{t,s}(x) = e^{t\lambda}u(e^{-s\lambda}x), \]
where \( \lambda \geq 0, \ t \geq 0, \ t^2 + s^2 > 0, \ \mu := 2t + (n - \alpha)s \geq 0, \) and \( \nu := 2t + ns \geq 0. \) Set \( \bar{\mu} = \max\{\mu, \nu\}. \)

By a simple calculation, we have
\[
K(u) := \frac{dE(u_{t,s}^\lambda)}{d\lambda} |_{\lambda=0} = \frac{\mu}{2} \int_{R^n} \left(1 + 4\pi^2|\xi|^2\right)^{\frac{\alpha-2}{2}} |\hat{u}(\xi)|^2 d\xi + \frac{s\alpha}{2} \int_{R^n} 
\]
\[
\int_{R^n} (u^p v)(x) dx
\]
\[
= \frac{\nu}{2} \int_{R^n} \left(1 + 4\pi^2|\xi|^2\right)^{\frac{\alpha-2}{2}} |\hat{u}(\xi)|^2 d\xi - \frac{s\alpha}{2} \int_{R^n} \left(1 + 4\pi^2|\xi|^2\right)^{\frac{\alpha-2}{2}} 4\pi^2|\xi|^2 |\hat{u}(\xi)|^2 d\xi
\]
\[
- \frac{\mu}{2} \int_{R^n} (u^p v)(x) dx.
\]

Similarly, if we set
\[
E_0(u) = \frac{1}{2} \int_{R^n} (2\pi|\xi|)^{\alpha} |\hat{u}(\xi)|^2 d\xi - \frac{1}{2p} \int_{R^n} (u^p v)(x) dx,
\]
then
\[
K_0(u) := \frac{dE_0(u_{t,s}^\lambda)}{d\lambda} |_{\lambda=0} = \frac{\mu}{2} \int_{R^n} (2\pi|\xi|)^{\alpha} |\hat{u}(\xi)|^2 d\xi - \frac{\nu}{2} \int_{R^n} (u^p v)(x) dx.
\]

Write
\[
L(u) := E(u) - \frac{K(u)}{\mu}
\]
\[
= \left\{ \begin{array}{ll}
\frac{s\alpha}{2} \int_{R^n} \left(1 + 4\pi^2|\xi|^2\right)^{\frac{\alpha-2}{2}} 4\pi^2|\xi|^2 |\hat{u}(\xi)|^2 d\xi \\
+ \left(\frac{\mu}{2\nu} - \frac{1}{2p}\right) \int_{R^n} u^p v dx,
\end{array} \quad \mu < \nu; \right.
\]
\[
\frac{s\alpha}{2\mu} \int_{R^n} \left(1 + 4\pi^2|\xi|^2\right)^{\frac{\alpha-2}{2}} |\hat{u}(\xi)|^2 d\xi
\]
\[
+ \left(\frac{1}{2} - \frac{1}{2p}\right) \int_{R^n} u^p v dx,
\]
\[
\mu \geq \nu,
\]
and
\[
L_0(u) := E_0(u) - \frac{K_0(u)}{\nu}
\]
\[
= \left\{ \begin{array}{ll}
\left(\frac{1}{2} - \frac{\mu}{2\nu}\right) \int_{R^n} (2\pi|\xi|)^{\alpha} |\hat{u}(\xi)|^2 d\xi + \left(\frac{\mu}{2\nu} - \frac{1}{2p}\right) \int_{R^n} u^p v dx,
\end{array} \quad \mu < \nu; \right.
\]
\[
\left(\frac{1}{2} - \frac{1}{2p}\right) \int_{R^n} u^p v dx,
\]
\[
\mu \geq \nu.
\]

Noting \( s > 0 \) and \( \frac{\mu}{2\nu} > \frac{1}{2p} \) when \( \mu < \nu \), and \( s \leq 0 \) when \( \mu \geq \nu \), we can see \( L(u), L_0(u) \geq 0 \).
In view of the parameter independence (cf. [15]), we define

\[ m = \inf \{ E(u); K(u) = 0, u \in H^{\alpha/2}(R^n) \setminus \{0\} \}; \]

\[ \tilde{m} = \inf \{ L(u); K(u) \geq 0, u \in H^{\alpha/2}(R^n) \setminus \{0\} \}; \]

\[ m_0 = \inf \{ E_0(u); K_0(u) = 0, u \in H^{\alpha/2}(R^n) \setminus \{0\} \}; \]

\[ \tilde{m}_0 = \inf \{ L_0(u); K_0(u) > 0, u \in H^{\alpha/2}(R^n) \setminus \{0\} \}. \]

Clearly, \( m = \tilde{m} \). Set

\[ F = \{ u \in H^{\alpha/2}(R^n) \setminus \{0\}; K_0(u) > 0 \}, \]

\[ \tilde{F} = \{ u \in H^{\alpha/2}(R^n) \setminus \{0\}; K_0(u) \geq 0 \}, \]

\[ \tilde{F} = \{ u \in H^{\alpha/2}(R^n) \setminus \{0\}; K_0(u) = 0 \}. \]

We claim \( F = \cup_{\lambda > 0} \{ u \in H^{\alpha/2}(R^n) \setminus \{0\}; K_0(u^\lambda) = 0 \} \). Once it holds, then \( m_0 = \tilde{m}_0 \).

In fact, for any \( \lambda > 0 \), if \( K_0(u^\lambda_{t,s}) = 0 \), then

\[ e^{[2t' + (n-\alpha)s']\lambda} \int_{R^n} (2\pi|\xi|)^{\alpha}|\hat{u}(\xi)|^2 d\xi = e^{[2t' + (n-\alpha)s'] \frac{(2n-\gamma)\lambda}{n-\alpha}} \int_{R^n} u^p v dx. \]

This leads to \( K_0(u) > 0 \), and hence \( F \supset \cup_{\lambda > 0} \{ u \in H^{\alpha/2}(R^n) \setminus \{0\}; K_0(u^\lambda) = 0 \} \).

On the other hand, for any \( u \in F \), there holds \( \int_{R^n} (2\pi|\xi|)^{\alpha}|\hat{u}(\xi)|^2 d\xi > \int_{R^n} u^p v dx \).

Thus, we can find \( \lambda_* > 0 \) such that

\[ e^{[2t' + (n-\alpha)s']\lambda_*} \int_{R^n} (2\pi|\xi|)^{\alpha}|\hat{u}(\xi)|^2 d\xi = e^{[2t' + (n-\alpha)s'] \frac{(2n-\gamma)\lambda_*}{n-\alpha}} \int_{R^n} u^p v dx. \]

This shows \( u \in \{ u \in H^{\alpha/2}(R^n) \setminus \{0\}; K_0(u^\lambda_{t,s}) = 0 \} \).

In addition, it is easy to see that \( F \) is dense in \( \tilde{F} \), which implies

\[ \tilde{m}_0 = \inf \{ L_0(u); K_0(u) \geq 0, u \in H^{\alpha/2}(R^n) \setminus \{0\} \}. \] (5.4)

Set \( G = \{ u \in H^{\alpha/2}(R^n) \setminus \{0\}; K(u) \geq 0 \} \). Clearly, \( G \subset \tilde{F} \).

Noting

\[ K(u^\lambda_{t,s}) = \frac{\mu}{2} e^{\lambda(2t' + (n-\alpha)s')} \int_{R^n} \left( e^{2s\lambda} + 4\pi^2|\xi|^2 \right)^{\frac{n-\gamma}{2}} |\hat{u}(\xi)|^2 d\xi \]

\[ + \frac{2\alpha}{2} e^{\lambda(2t' + (n-\alpha+2)s')} \int_{R^n} \left( e^{2s\lambda} + 4\pi^2|\xi|^2 \right)^{\frac{n-\gamma}{2}} |\hat{u}(\xi)|^2 d\xi \]

\[ - \frac{\mu}{2} e^{\lambda(2t' + (2n-\gamma)s')} \int_{R^n} u^p v(x) dx, \]

and taking \( t' = \frac{n-\alpha}{2} \) and \( s' = -1 \), we can deduce that

\[ \lim_{\lambda \to +\infty} K(u^\lambda_{(n-\alpha)/2,-1}) = K_0(u). \] (5.5)

Similarly, we also get

\[ \lim_{\lambda \to +\infty} L(u^\lambda_{(n-\alpha)/2,-1}) = L_0(u). \] (5.6)

Clearly, (5.5) shows that \( G \) is dense in \( \tilde{F} \). Combining with (5.4) yields

\[ \tilde{m}_0 = \inf \{ L_0(u); K(u) \geq 0, u \in H^{\alpha/2}(R^n) \setminus \{0\} \}. \]
In addition, (5.6) implies
\[ \inf \{ L_0(u); K(u) \geq 0, u \in H^{\alpha/2}(\mathbb{R}^n) \setminus \{0\} \} = \bar{m}. \]
Therefore, \( m = \bar{m} = \bar{m}_0 = m_0. \)

The argument above shows that \( m = \bar{m} = \bar{m}_0 = m_0. \)

Therefore, \( \bar{m}_0 = \bar{m}. \)

Thus, \( m = m_0 \)
\[ = \inf \left\{ \frac{n + \alpha - \gamma}{2(2n - \gamma)} \int_{\mathbb{R}^n} (2\pi|\xi|)^{\alpha} |\hat{u}(\xi)|^2 d\xi; \int_{\mathbb{R}^n} (2\pi|\xi|)^{\alpha} |\hat{u}(\xi)|^2 d\xi = \int_{\mathbb{R}^n} \bar{u}^p v dx \right\} \]
\[ = \inf \left\{ \frac{n + \alpha - \gamma}{2(2n - \gamma)} \int_{\mathbb{R}^n} (2\pi|\xi|)^{\alpha} |\hat{u}(\xi)|^2 d\xi \left[ \frac{\int_{\mathbb{R}^n} (2\pi|\xi|)^{\alpha} |\hat{u}(\xi)|^2 d\xi}{\int_{\mathbb{R}^n} \bar{u}^p v dx} \right]^{\frac{n-\alpha}{2(n-\gamma)}}, u \in H^\alpha(\mathbb{R}^n) \setminus \{0\} \right\} \]
\[ = \frac{n + \alpha - \gamma}{2(2n - \gamma)} \inf \left\{ \left[ \int_{\mathbb{R}^n} \bar{u}^p v(x) dx \right]^{(n-\alpha)/(2n-\gamma)} \right\}^{\frac{2n-\gamma}{2n-\alpha}}; u \in H^\alpha(\mathbb{R}^n) \setminus \{0\} \]
\[ = \frac{n + \alpha - \gamma}{2(2n - \gamma)} c^\alpha_{\gamma, \alpha} \]

Here \( c_* \) is the sharp constant of (5.3). According to the classification results in [7] and [25], we know \( c_* = E_*(U_*) \). \( \square \)

Acknowledgments. The author would like to thank the referees for their valuable comments. Their suggestions have greatly improved this article. He is also grateful to Dr. Xingdong Tang for many fruitful discussions.

REFERENCES

[1] N. Ackermann, On a periodic Schrödinger equation with nonlocal superlinear part, Math. Z., 248 (2004), 423–443.
[2] T. Cazenave, Semilinear Schrödinger Equations, Courant Lecture Notes in Mathematics, 10. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.
[3] W. Chen and C. Li, Methods on Nonlinear Elliptic Equations, AIMS Book Series on Diff. Equa. Dyn. Sys., Vol. 4, 2010.
[4] W. Chen and C. Li, An integral system and the Lane-Emden conjecture, Discrete Contin. Dyn. Syst., 24 (2009), 1167–1184.
[5] W. Chen, C. Li and Y. Li, A direct method of moving planes for the fractional Laplacian, Adv. Math., 308 (2017), 404–437.
[6] W. Chen, C. Li and G. Li, Maximum principles for a fully nonlinear fractional order equation and symmetry of solutions, Calc. Var. Partial Differential Equations, 56 (2017), Art. 29, 18 pp.
[7] W. Chen, C. Li and B. Ou, Classification of solutions for an integral equation, Comm. Pure Appl. Math., 59 (2006), 330–343.
[8] Y. Chen and H. Gao, The Cauchy problem for the Hartree equations under random influences, J. Differential Equations, 259 (2015), 5192–5219.
[9] S. Cingolani, M. Clapp and S. Secchi, Intertwining semiclassical solutions to a Schrödinger-Newton system, Discrete Contin. Dyn. Syst., 6 (2013), 891–908.
[10] S. Cingolani, M. Clapp and S. Secchi, Multiple solutions to a magnetic nonlinear Choquard equation, Z. Angew. Math. Phys., 63 (2012), 233–248.
[11] S. Cingolani, S. Secchi and M. Squassina, Semi-classical limit for Schrödinger equations with magnetic field and Hartree-type nonlinearities, Proc. Roy. Soc. Edinburgh Sect. A, 140 (2010), 973–1009.
[12] W. Dai, J. Huang, Y. Qin, B. Wang and Y. Fang, Regularity and classification of solutions to static Hartree equations involving fractional Laplacians, *Discrete Contin. Dyn. Syst.*, 39 (2019), 1389–1403.

[13] P. Felmer, A. Quaas and J. Tan, Positive solutions of nonlinear Schrödinger equation with the fractional Laplacian, *Proc. Roy. Soc. Edinburgh Sect. A*, 142 (2012), 1237–1262.

[14] X. Han and G. Lu, Regularity of solutions to an integral equation associated with Bessel potential, *Commun. Pure Appl. Anal.*, 10 (2011), 1111–1119.

[15] S. Ibrahim, N. Masmoudi and K. Nakanishi, Scattering threshold for the focusing nonlinear Klein-Gordon equation, *Anal. PDE*, 4 (2011), 405–460.

[16] C. Jin and C. Li, Qualitative analysis of some systems of integral equations, *Calc. Var. Partial Differential Equations*, 26 (2006), 447–457.

[17] Y. Lei, On the regularity of positive solutions of a class of Choquard type equations, *Math. Z.*, 273 (2013), 883–905.

[18] Y. Lei, Qualitative analysis for the static Hartree-type equations, *SIAM J. Math. Anal.*, 45 (2013), 388–406.

[19] Y. Lei and C. Li, Sharp criteria of Liouville type for some nonlinear systems, *Discrete Contin. Dyn. Syst.*, 36 (2016), 3277–3315.

[20] Y. Lei, C. Li and C. Ma, Asymptotic radial symmetry and growth estimates of positive solutions to weighted Hardy-Littlewood-Sobolev system, *Calc. Var. Partial Differential Equations*, 45 (2012), 43–61.

[21] C. Li and L. Ma, Uniqueness of positive bound states to Schrödinger systems with critical exponents, *SIAM J. Math. Anal.*, 40 (2008), 1049–1057.

[22] C. Li, Z. Wu and H. Xu, Maximum principles and Bocher type theorems, *Proceedings of the National Academy of Sciences*, 115 (2018), 6976–6979.

[23] Y. Li, Remark on some conformally invariant integral equations: The method of moving spheres, *J. Eur. Math. Soc.*, 6 (2004), 153–180.

[24] E. Lieb, Existence and uniqueness of the minimizing solution of Choquard’s nonlinear equation, *Studies in Appl. Math.*, 57 (1976/77), 93–106.

[25] E. Lieb, Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities, *Ann. of Math.*, 118 (1983), 349–374.

[26] E. Lieb and M. Loss, *Analysis*, 2nd edition, American Mathematical Society, Rhode Island, 2001.

[27] E. Lieb and B. Simon, The Hartree–Fock theory for Coulomb systems, *Comm. Math. Phys.*, 53 (1977), 185–194.

[28] P. L. Lions, The Choquard equation and related questions, *Nonlinear Anal.*, 4 (1980), 1063–1072.

[29] C. Ma, W. Chen and C. Li, Regularity of solutions for an integral system of Wolff type, *Adv. Math.*, 226 (2011), 2676–2699.

[30] L. Ma and D. Chen, Radial symmetry and monotonicity for an integral equation, *J. Math. Anal. Appl.*, 342 (2008), 943–949.

[31] L. Ma and L. Zhao, Classification of positive solitary solutions of the nonlinear Choquard equation, *Arch. Rational Mech. Anal.*, 195 (2010), 455–467.

[32] C. Miao, G. Xu and L. Zhao, Global well-posedness and scattering for the mass-critical Hartree equation with radial data, *J. Math. Pures Appl.*, 91 (2009), 49–79.

[33] V. Moroz and J. Van Schaftingen, Nonexistence and optimal decay of supersolutions to Choquard equations in exterior domains, *J. Differential Equations*, 254 (2013), 3089–3145.

[34] V. Moroz and J. Van Schaftingen, Groundstates of nonlinear Choquard equations: Existence, qualitative properties and decay asymptotics, *J. Funct. Anal.*, 265 (2013), 153–184.

[35] X. Shang and J. Zhang, Multi-peak positive solutions for a fractional nonlinear elliptic equation, *Discrete Contin. Dyn. Syst.*, 35 (2015), 3183–3201.

[36] Z. Shen, F. Gao and M. Yang, Multiple solutions for nonhomogeneous Choquard equation involving Hardy-Littlewood-Sobolev critical exponent, *Z. Angew. Math. Phys.*, 68 (2017), Art. 61, 25 pp.

[37] E. Stein, *Singular Integrals and Differentiability Properties of Function*, Princetion Math. Series, Vol. 30, Princetion University Press, Princetion, NJ, 1970.

[38] S. Sun and Y. Lei, Fast decay estimates for integrable solutions of the Lane-Emden type integral systems involving the Wolff potentials, *J. Funct. Anal.*, 263 (2012), 3857–3882.

[39] M. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, *Comm. Math. Phys.*, 87 (1983), 567–576.
[40] W. Ziemer, *Weakly Differentiable Functions*, Graduate Texts in Math. Vol. 120, Springer-Verlag, New York, 1989.

Received December 2017; revised April 2018.

E-mail address: leiyutian@njnu.edu.cn