λφ⁴ Kink and sine-Gordon Soliton in the GUP Framework

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Abstract
We consider λφ⁴ kink and sine-Gordon soliton in the presence of a minimal length uncertainty proportional to the Planck length. The modified Hamiltonian contains an extra term proportional to \(p^4\) and the generalized Schrödinger equation is expressed as a forth-order differential equation in quasiposition space. We obtain the modified energy spectrum for the discrete states and compare our results with 1-loop resummed and Hartree approximations for the quantum fluctuations. We finally find some lower bounds for the deformations parameter so that the effects of the minimal length have the dominant role.

Keywords: Generalized uncertainty principle; Minimal length; Topological defects.

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1 Introduction

At high energy limit the Heisenberg algebra will be modified by adding small corrections to the canonical commutation relation in the form of the Generalized Uncertainty Principle (GUP) [1]. From these modifications a short distance structure is obtained that characterized by a finite minimal uncertainty \((ΔX)_{\text{min}}\) in the position measurement. String theory, loop quantum gravity, noncommutative geometry, and black-hole physics all suggest the existence of such a minimal measurable length of the order of the Planck length \(\ell_P = \sqrt{\frac{G\hbar}{c^3}} \approx 10^{-35}\text{m}\). The presence of this minimal observable length modifies all the Hamiltonians and many papers have appeared in literature to address the effects of GUP on various physical systems [2–24].

A linear and dispersionless relativistic wave equation holds two features, namely (i) keeping the shape and velocity of a single packet and (ii) keeping the asymptotic shape and velocity of several
packets even after collisions \cite{23}. However, there are much more complicated wave equations in many branches of physics that contain nonlinear terms, dispersive terms, and several coupled wave fields. Solitary waves are certain special solutions of nonlinear wave equations that look like pulses of energy traveling without dissipation and with uniform velocity. The solitary waves whose energy density profiles are asymptotically restored to their original shapes and velocities are known as solitons. However, Kink solutions of $\lambda \phi^4$ theory are solitary waves but not solitons. At the classical level they resemble extended particles, i.e., localized and finite-energy objects. On the other hand, the solutions of the sin-Gordon system are solitary waves and also soliton. This model has been used in the study of a wide range of phenomena such as propagation of crystal dislocations, of magnetic flux in Josephson lines and two-dimensional models of elementary particles \cite{26,27}.

Here, we study $\lambda \phi^4$ kink and sin-Gordon soliton in the GUP framework given by the following generalized uncertainty relation \cite{13}

$$\Delta X \Delta P \geq \frac{\hbar}{2} (1 + \beta (\Delta P)^2 + \gamma),$$

(1)

where $\beta$ is the GUP parameter and $\gamma = \beta \langle P \rangle^2$. This inequality implies the existence of a minimum observable length proportional to the square root of the deformation parameter, i.e., $(\Delta X)_{\text{min}} = \hbar \sqrt{\beta}$. We obtain the effects of this minimal length on kink and soliton energy spectrums and compare our results with the ones obtained in 1-loop resummed and Hartree approximations \cite{28,29}. This suggests some lower bounds on the GUP parameter for each case.

This paper is organized as follows: In section 2, we state the generalized uncertainty principle and its first order representation in quasiposition space. In section 3, we find $\lambda \phi^4$ kink modified energy spectrum using the first order perturbation theory and compare our results with 1-loop resummed and Hartree approximations. In section 4, we outline the energy correction in Hartree approximation for the sine-Gordon soliton and obtain the GUP correction to its discrete energy level. Finally, we present our conclusions in section 5.
2 The Generalized Uncertainty Principle

Let us consider the following deformed commutation relation

\[ [X, P] = i\hbar (1 + \beta P^2), \tag{2} \]

which leads to the generalized uncertainty relation \([1]\) and \(\beta = \beta_0/(M_Pc)^2\). Here \(\beta_0\) is of the order of unity and \(M_P\) is the Planck mass. We can define \([20]\)

\[ X = x, \tag{3} \]
\[ P = \frac{\tan(\sqrt{\beta} p)}{\sqrt{\beta}}, \tag{4} \]

to exactly satisfy the above modified uncertainty principle, where \(x\) and \(p\) obey the canonical commutation relations, i.e., \([x, p] = i\hbar\). Moreover, \(p\) can be interpreted as the momentum operator at low energies \(p = -i\hbar \frac{\partial}{\partial x}\) and \(P\) as the momentum operator at high energies. The general form of the Hamiltonian

\[ H = P^2 + V(X) = \frac{\tan^2(\sqrt{\beta} p)}{\beta} + V(x), \tag{5} \]

to first order of the GUP parameter can be written as

\[ H = H_0 + H_1, \tag{6} \]

where \(H_0 = p^2 + V(x)\) and \(H_1 = \frac{2}{\beta} \partial^4\). Using the expression for the Hamiltonian, the generalized form of the Schrödinger equation in quasiposition space is given by (\(\hbar = 1\))

\[ -\frac{\partial^2 \psi(x)}{\partial x^2} + \frac{2\beta}{3} \frac{\partial^4 \psi(x)}{\partial x^4} + V(x) \psi(x) = E \psi(x), \tag{7} \]

which has an extra term in comparison to ordinary Schrödinger equation because of the deformed commutation relation \([2]\). Since this equation is a 4th-order differential equation, it is not an easy task in general to solve it in quasiposition space. Hence, we implement the perturbation method in order to obtain the solutions.
3 $\lambda \phi^4$ kink in the GUP framework

The $\lambda \phi^4$ theory with the Lagrangian density

$$L(x,t) = \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - V(\phi),$$  \hspace{1cm} (8)

where $V(\phi) = (\lambda/4) \left( \phi^2 - \frac{m^2}{\lambda} \right)^2$ leads to the following equation of motion

$$\partial_\mu \partial^\mu \phi + m^2 \phi - \lambda \phi^3 = 0.$$  \hspace{1cm} (9)

Here, we focus on one of the solutions of the static equation, namely, the classical kink solution $\phi_{\text{kink}}(x) = \pm \left( m/\sqrt{\lambda} \right) \tanh \left[ \left( m/\sqrt{2} \right) x \right]$, which interpolates between the two degenerate vacuum states $\phi = \pm m/\sqrt{\lambda}$. The fluctuation equation for $\eta(x)$ around this solution is

$$\left[ -\frac{\partial^2}{\partial x^2} + m^2 \left( 3 \tanh^2 \left( \frac{m x}{\sqrt{2}} \right) - 1 \right) \right] \eta_n(x) = \omega_n^2 \eta_n(x),$$  \hspace{1cm} (10)

which besides its zero mode

$$\eta_0(x) = \sqrt{\frac{3m}{4 \sqrt{2}}} \text{sech}^2 \left( \frac{m x}{\sqrt{2}} \right), \quad \omega_0^2 = 0,$$  \hspace{1cm} (11)

contains another bound state

$$\eta_1(x) = \sqrt{\frac{3m}{2 \sqrt{2}}} \text{sech} \left( \frac{m x}{\sqrt{2}} \right) \text{sech}^2 \left( \frac{m x}{\sqrt{2}} \right), \quad \omega_1^2 = \frac{3}{2} m^2.$$  \hspace{1cm} (12)

In quantum theory we construct a set of approximate harmonic oscillator states around the point $\phi_{\text{kink}}(x)$ in field space and we expect the energies of these states to be the kink particle associate with the lowest energy level. Using Eqs. (6) and (10) we get the expression for the Hamiltonian of the kink system in the presence of GUP as

$$H = p^2 + \frac{2}{3} \beta p^4 + m^2 \left( 3 \tanh^2 \left( \frac{m x}{\sqrt{2}} \right) - 1 \right),$$  \hspace{1cm} (13)

where $H_0 = p^2 + m^2 \left( 3 \tanh^2 \left( \frac{m x}{\sqrt{2}} \right) - 1 \right)$ is the Hamiltonian of the unperturbed system. Now, we apply the perturbation theory to calculate the kink energy corrections to $O(\beta)$ as

$$\Delta \omega_0^2 = \frac{2}{3} \beta \langle \eta_0 | p^4 | \eta_0 \rangle = \frac{8}{21} \beta m^4,$$  \hspace{1cm} (14)
and
\[ \Delta \omega_1^2 = \frac{2}{3} \beta \langle \eta | p^4 | \eta \rangle = \frac{31}{42} \beta m^4. \]  
(15)

On the other hand, based on Ref. [28], by knowing the correlation function \( G(x) \) of the following equation
\[ \left[ -\frac{\partial^2}{\partial x^2} + m^2 \left( 3 \tanh^2 \left( \frac{mx}{\sqrt{2}} \right) - 1 \right) + 3\lambda G(x) \right] \tilde{\eta}_k(x) = \tilde{\omega}^2_k \tilde{\eta}_k(x), \]  
(16)
the modified kink energy eigenvalues can be obtained in Hartree and 1-loop resummed approximations.

The numerical results for the eigenvalues in 1-loop resummed approximation are \( \tilde{\omega}^2_0 = 0.5\lambda \) and \( \tilde{\omega}^2_1 = 1.71\lambda \) [28], which lead to the energy shifts
\[ \Delta \tilde{\omega}^2_0 = 0.5\lambda, \quad \Delta \tilde{\omega}^2_1 = 0.21\lambda. \]  
(17)

Also in Hartree approximation we have \( \tilde{\omega}^2_0 = 0.33\lambda \) and \( \tilde{\omega}^2_1 = 1.29\lambda \) [28], so we find
\[ \Delta \tilde{\omega}^2_0 = 0.33\lambda, \quad \Delta \tilde{\omega}^2_1 = -0.21\lambda. \]  
(18)

Therefore, the comparison between the above results and ones obtained in the GUP scenario shows that for
\[ \beta > 1.32 \frac{\lambda}{m}, \]  
(19)
the effects of the minimal length or the discreetness of space are more important than the effects that come from considering the quantum fluctuations.

### 4 sine-Gordon soliton in the GUP framework

The sine-Gordon system is defined by a single scalar field \( \phi(x,t) \) in (1+1) dimensions governed by the Lagrangian density [25]
\[ \mathcal{L}(x,t) = \frac{1}{2} (\partial_{\mu} \phi) (\partial^{\mu} \phi) + \frac{m^2}{g^2} (\cos g\phi - 1), \]  
(20)
which gives rise to the following sine-Gordon equation
\[ \partial_{\mu} \partial^{\mu} \phi + \frac{m^2}{g} \sin (g\phi) = 0. \]  
(21)
One of the static localized solutions, namely,

$$\phi_{sol} = \frac{4}{g} \arctan(e^{mx}),$$

(22)
is called soliton. The equation for the fluctuations around this solution is expressed as the following Schrödinger-like equation \[31\]

$$\left[ -\frac{\partial^2}{\partial x^2} + m^2 \left( 1 - \frac{2}{\cosh^2(mx)} \right) \right] \eta_n(x) = \omega_n^2 \eta_n(x),$$

(23)

that only has one discrete mode, i.e., the zero mode of translation

$$\eta_0(x) = \sqrt{\frac{m}{2}} \text{sech}(mx), \quad \omega_0^2 = 0.$$  

(24)

Now we consider this system in the GUP framework where the modified Hamiltonian takes the form

$$H = p^2 + \frac{2\beta}{3} p^4 + m^2 \left( 1 - \frac{2}{\cosh^2(mx)} \right).$$

(25)

The extra term $\frac{2}{3}\beta p^4$ in the Hamiltonian results in a positive shift in the zero mode energy as

$$\Delta \omega_0^2 = \frac{2\beta}{3} \langle \eta_0 | p^4 | \eta_0 \rangle = \frac{14}{45} \beta m^4.$$ 

(26)

Moreover, in the Hartree approximation, the zero mode energy correction is given by \[29\]

$$\Delta \tilde{\omega}_0^2 = 30^{-2/3} g^{4/3}. $$

(27)

So the comparison between these two results shows that for

$$\beta > 0.33 \frac{g^{4/3}}{m^4},$$

(28)

the gravitational effects that modifies the ordinary uncertainty principle are more dominant than the quantum fluctuations.

5 Conclusions

In this paper, we have considered a GUP framework that admits a nonzero minimal position uncertainty. Due to the presence of this minimal length a term proportional to $p^4$ is added to the Hamiltonians of all
quantum mechanical systems. Using the perturbation theory we have obtained the effects of this extra term on the $\lambda \phi^4$ kink and the sine-Gordon soliton energy spectrum. We have compared our results with the ones that obtained for the quantum fluctuations in 1-loop resummed and Hartree approximations. We showed that for the $\lambda \phi^4$ kink the effects of GUP are more important than 1-loop resummed and Hartree approximations for $\beta > 1.32\lambda/m^4$. Also, for the sine-Gordon case, the GUP energy correction is more dominant than the Hartree approximation for $\beta > 0.33(g^{1/3}/m)^4$.

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