Reconstruction formula for a 3-d phaseless inverse scattering problem for the Schrödinger equation

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Abstract

The inverse scattering problem of the reconstruction of the unknown potential with compact support in the 3-d Schrödinger equation is considered. Only the modulus of the scattering complex valued wave field is known, whereas the phase is unknown. It is shown that the unknown potential can be reconstructed via the inverse Radon transform. Therefore, a long standing problem posed in 1977 by K. Chadan and P.C. Sabatier in their book “Inverse Problems in Quantum Scattering Theory” is solved.

Keywords: phaseless inverse scattering, Schrödinger equation, reconstruction formula, Radon transform

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1 Introduction

In this publication a long standing problem posed by Chadan and Sabatier in 1977 in chapter 10 of their classical book [3] is addressed. We consider a 3-d inverse scattering problem for the Schrödinger equation with a compactly supported unknown potential in the frequency domain. Unlike the common approach, we assume that only the modulus of the scattering field is known, whereas the phase is unknown. The main result of this paper is a reconstruction formula, which claims that this problem can be solved via the inversion of the Radon transform. To the best knowledge of the authors, the result of this paper represents the first rigorous reconstruction formula for a phaseless inverse scattering problem without an assumption about superpositions of signals caused by some separate targets, some of which are known. It has been experienced by many people from their CT scans in hospitals that the inverse Radon transform usually provides quite high quality images. In this regard, we also refer to the book of Natterer [18]. Therefore, the reconstruction formula of this paper paves the way for future effective computations of
some applied problems. An interesting applied example is in imaging of nano structures, see section 4 in the paper of Khachaturov [11].

The reason which has prompted Chadan and Sabatier to pose the phaseless inverse scattering problem for the Schrödinger equation in [3] is that in the quantum scattering in the frequency domain one is measuring the differential scattering cross section. The latter is the modulus of the scattered complex valued wave field, see page 8 in the book of Newton [20]. However, the phase is not measured. On the other hand, the entire inverse scattering theory in the frequency domain is based on the assumption that both the modulus and the phase are measured outside of the support of a scatterer, see, e.g. books of Chadan and Sabatier [3], Isakov [8], Newton [20] as well as papers of Novikov [21, 22].

Because of a number of its important applications, there are many publications about the problem of phase reconstruction. As some examples, we refer to Aktosun and Sacks [1], Berk and Majkrzak [2], Dobson [4], Feinup [5], Gerth, Hofman, Birkholz, Koke and Steinmeyer [7], Ivanyshyn, Kress and Serranho [9], Ivanyshyn and Kress [10], Ladd and Palmer [16], Nazarchuk, Hryiniv and Synyavsky [19], and Ruhlandt, Krenkel, Bartels and Salditt [25].

In the recent preprint of Novikov [23] another reconstruction formula is obtained for the phaseless inverse scattering problem for the Schrödinger equation. We now point to the main difference between our result and the one of [23]. In the inversion formulae of Theorem 2.1 of [23] three measurements are considered: one from the unknown potential and two more for the case when that target potential is complemented by two other compactly supported potentials, which are known and whose supports do not intersect with the support of the target potential. This means that superpositions of signals scattered by three separate targets are considered in [23]. On the other hand, we consider measurements of the modulus of the scattered wave field generated only by a single compactly supported potential. The inverse Radon transform is not used in [23], and the method of the proof of the main result (Theorem 1) here is significantly different from the one in [23].

In the recent work of Klibanov [12] uniqueness theorems for the 3-d phaseless inverse scattering problem for the Schrödinger equation were proved, also see the work [13] for a similar result for the acoustic equation. Uniqueness of the reconstruction of a complex valued function with compact support from the modulus of its Fourier transform was proved in [14, 15]. However, proofs in papers [12]-[15] are not constructive.

In section 2 we formulate the problem and the main result. In section 3 we prove the main result. In section 4 we prove a certain lemma, which is formulated in section 3.

2 The Main Result

Let $B > 0$ be a number and $\Omega = \{|x| < B\} \subset \mathbb{R}^3$ be the ball of the radius $B$ with the center at $\{0\}$. Denote the corresponding sphere $S = \{|x| = B\}$. Let the potential $q(x) , x \in \mathbb{R}^3$ be a real valued function such that

\begin{align}
q(x) & \in C^4 (\mathbb{R}^3), \\
q(x) & \geq 0, \forall x \in \Omega, \\
q(x) & = 0 \text{ for } x \in \mathbb{R}^3 \setminus \Omega.
\end{align}
Let \( x^0 \) be the position of the point source. As the forward problem, we consider the following

\[
\Delta_x u + k^2 u - q(x) u = -\delta (x - x^0), \quad x \in \mathbb{R}^3, \tag{2.4}
\]

\[
u (x, x^0, k) = O \left( \frac{1}{|x - x^0|} \right), \quad |x| \to \infty, \tag{2.5}
\]

\[
\sum_{j=1}^{3} \frac{x_j - x_j^0}{|x - x^0|} \partial_{x_j} u (x, x^0, k) + iku (x, x^0, k) = o \left( \frac{1}{|x - x^0|} \right), \quad |x| \to \infty. \tag{2.6}
\]

Here the frequency \( k > 0 \) and conditions \ref{eq:2.5}, \ref{eq:2.6} are valid for every fixed source position \( x^0 \). Theorem 1 and Theorem 3.3 of the paper of Vainberg [26], Theorem 6 of Chapter 9 of the book of Vainberg [27] as well as Theorem 6.17 of the book of Gilbarg and Trudinger [6] guarantee that for each pair \((k, x^0) \in (0, \infty) \times \mathbb{R}^3\) there exists a unique solution \( u (x, x^0, k) \) of the problem \ref{eq:2.4}, \ref{eq:2.5}, \ref{eq:2.6} such that it can be represented in the form

\[
u (x, x^0, k) = u_0 (x, x^0, k) + u_{sc} (x, x^0, k), \tag{2.7}
\]

\[
u_0 = \exp \left( -\frac{ik |x - x^0|}{4\pi |x - x^0|} \right), \quad u_{sc} \in C^4 \left( \{ |x - x^0| \geq \eta \} \right), \forall \eta > 0, \forall \beta \in (0, 1). \tag{2.8}
\]

For any number \( a \in \mathbb{R} \) consider the plane \( P_a = \{ x_3 = a \} \). Consider the disk \( Q_a = \overline{\Omega} \cap P_a \) and let \( S_a = S \cap P_a \) be its boundary. Clearly \( Q_a \neq \emptyset \) for \( a \in (-B, B) \) and \( Q_a = \emptyset \) for \( |a| \geq B \). Denote \( 0_a = (0, 0, a) \in Q_a \) the orthogonal projection of the origin on the plane \( P_a \). We have

\[
\Omega = \bigcup_{a=-B}^{B} Q_a, \partial \Omega := S = \bigcup_{a=-B}^{B} S_a.
\]

In our inverse problem we assume that the modulus \( |u_{sc}| \) of the scattered wave is measured for all pairs \( x^0, x \) running along the circle \( S_a \) for every \( a \in (-R, R) \) and for all frequencies \( k > 0 \).

**Phaseless Inverse Scattering Problem.** Suppose that the potential \( q(x) \) satisfies conditions \ref{eq:2.1} - \ref{eq:2.3}. Determine the function \( q(x) \) for \( x \in \Omega \), assuming that the following function \( f (x, x^0, k) \) is known

\[
f (x, x^0, k) = |u_{sc} (x, x^0, k)|, \forall x^0, x \in S_a, x \neq x^0, \forall a \in (-B, B), \forall k \in (0, \infty). \tag{2.9}
\]

**Remark 1.** As to the issue of collecting experimental data, it follows from \ref{eq:2.9} and Theorem 1 that if one wants to image only one 2-d cross-section \( Q_a \) of the potential \( q \), then it is sufficient to run independently both sources \( x^0 \) and detectors \( x \) only around the circle \( S_a \). This is more economical than running them independently around the entire sphere \( S \).

For an arbitrary \( a \in (-B, B) \) and for any pair of points \( x^0, x \in S_a \) let \( L (x, x^0) \) be the interval of the straight line connecting them. Denote \( B_a = \sqrt{B^2 - a^2} \) the radius of the circle \( S_a \). Since our reconstruction formula is based on the inversion of the two-dimensional Radon transform, we now parametrize \( L (x, x^0) \) in the conventional parametrization of the Radon transform [13]. Let \( n \) be the unit normal vector to the line \( L (x, x^0) \) lying in the plane \( P_a \) and pointing outside of the point \( 0_a \). Let \( \alpha \in [0, 2\pi] \) be the angle between \( n \) and the \( x_1 \)-axis. Then \( n = n (\alpha) = (\cos \alpha, \sin \alpha) \) (it is convenient here to discount the third
coordinate of \( n \), which is zero). Let \( s \) be the signed distance of \( L(x, x^0) \) from the point \( 0_a \) \((18\), page 9\). It is clear that there is a one-to-one correspondence between pairs \((x, x^0)\) and \((n(\alpha), s)\),

\[
(x, x^0) \Leftrightarrow (n(\alpha), s); x, x^0 \in S_a \in S_a, \alpha = \alpha (x, x^0) \in (0, 2\pi], s = s (x, x^0) \in (-B_a, B_a).
\]

Hence, we can write

\[
L(x, x^0) = \{ y_a = (y_1, y_2, a) : \langle y, n(\alpha) \rangle = s \},
\]

where \( y = (y_1, y_2) \in \mathbb{R}^2, \langle \_ \rangle \) is the scalar product in \( \mathbb{R}^2 \) and parameters \( \alpha = \alpha (x, x_0) \) and \( s = s (x, x_0) \) are defined as in \((2.10)\).

Consider an arbitrary function \( g = g(y) \in C^4(P_a) \) such that \( g(y) = 0 \) for \( y \in P_a \setminus Q_a \). Hence,

\[
\int_{L(x, x^0)} g(y) \, d\sigma = \int_{\langle y, n(\alpha) \rangle = s} g(y) \, d\sigma,
\]

for all \( x, x^0 \in S_a \), where \( \alpha = \alpha (x, x^0), s = s (x, x^0) \) as in \((2.10)\). In \((2.12)\) \( \sigma \) is the arc length and the parametrization of \( L(x, x^0) \) is given \((2.11)\). Therefore, using \((2.10)-(2.12)\), we can define the 2-d Radon transform \( Rg \) of the function \( g \) as

\[
(Rg)(x, x^0) = (Rg)(\alpha, s) = \int_{\langle y, n(\alpha) \rangle = s} g(y) \, d\sigma,
\]

We are ready now to formulate Theorem 1, which is our main result.

**Theorem 1.** Suppose that the potential \( q(x) \) satisfies conditions \((2.1)-(2.3)\). Let \( u_{sc}(x, x^0, k) \) be the function defined in \((2.8)\). Then for each pair of points \( x, x^0, x \neq x_0 \) the asymptotic behavior of this function is

\[
u_{sc}(x, x^0, k) = \frac{i \exp(-ik|x-x^0|)}{8\pi|x-x^0|k} \left[ \int_{L(x, x^0)} q(\xi) \, d\sigma + O\left(\frac{1}{k}\right) \right], \quad k \to \infty.
\]

Hence, the asymptotic behavior of the function \( f(x, x^0, k) \) defined in \((2.9)\) is

\[
f(x, x^0, k) = \frac{1}{8\pi|x-x^0|k} \left[ (Rq)(x, x^0) + O\left(\frac{1}{k}\right) \right], \quad k \to \infty; \forall x, x^0 \in S_a, x \neq x^0,
\]

for all \( a \in (-B, B) \). Thus, for \((y, a) = (y_1, y_2, a) \in Q_a, a \in (-B, B) \) the reconstruction formula for the function \( q(y_1, y_2, a) \) is

\[
q(y_1, y_2, a) = 8\pi R^{-1}\{ |x - x^0| \lim_{k \to \infty} [k f(x, x^0, k)] \}(y_1, y_2, a), \quad x, x^0 \in S_a.
\]

In \((2.13)\) and \((2.10)\) the operator \((Rq)(x, x^0) = (Rq)(\alpha, s)\) of the 2-d Radon transform is defined as in \((2.13)\) via taking into account \((2.10)\) and \((2.11)\) and the same is true for its inverse \( R^{-1} \).

**Remark 2.** The inversion formula \((2.16)\) follows immediately from \((2.12), (2.13), (2.15)\) and the results of the book of Natterer \((18)\). Thus, we focus below on the proof of \((2.14)\), since \((2.15)\) follows from \((2.14)\) immediately. It is well known how to explicitly construct the operator \( R^{-1} \), see, e.g. \((18)\). Hence, we are not doing this here for brevity.
3 Proof of Theorem 1

We assume everywhere below that conditions of Theorem 1 are satisfied. To prove Theorem 1, we consider first in subsection 3.1 the fundamental solution of a hyperbolic PDE and formulate Lemma 1 about the \( C^2 \)-smoothness of its regular part above the characteristic cone. It is known how to establish the \( C^\infty \)-smoothness of the fundamental solution of a hyperbolic equation with \( C^\infty \)-coefficients, see, e.g. section 2.2 in the book of Romanov [24]. However, we want to use here only the \( C^4 \)-smoothness of the potential \( q(x) \) as in [22]. Hence, the proof of Lemma 1 is quite technical. For this reason, we present that proof in section 4. We prove formula (2.14) in subsection 3.2.

3.1 The fundamental solution a hyperbolic equation

We now consider the following Cauchy problem

\[
\begin{align*}
&w_{tt} = \Delta_x w - q(x)w + 4\pi \delta(x - x^0, t), \quad (x, t) \in \mathbb{R}^4, \\
&w|_{t<0} = 0.
\end{align*}
\]

For an arbitrary \( T > 0 \) denote

\[ G(x^0, T) = \{(x, t) : 0 < |x - x^0| < t \leq T\}. \]

For \( t > |x - x^0| \) consider the ellipsoid \( E(x, x^0, t) \),

\[ E(x, x^0, t) = \{\xi \in \mathbb{R}^3 : |x - \xi| + |x^0 - \xi| = t\}. \]

It was shown in §1 of Chapter 7 of the book of Lavrentiev, Romanov and Shishatskii [17] that the function \( w(x, x^0, t) = 0 \) for \( t < |x - x^0| \) and it can be represented as

\[ w(x, x^0, t) = w_0(x, x^0, t) + \tilde{w}(x, x^0, t) H(t - |x - x^0|), \]

where \( H(t) \) is the Heaviside function and functions \( w_0 \) and \( \tilde{w} \) are defined by the following formulae:

\[ w_0(x, x^0, t) = \frac{\delta(t - |x - x^0|)}{|x - x^0|}, \]

\[ \tilde{w}(x, x^0, t) = w_1(x, x^0, t) + \sum_{n=2}^{\infty} w_n(x, x^0, t), \]

\[
\begin{align*}
\text{where } w_1(x, x^0, t) &= -\frac{1}{4\pi(t^2 - \rho^2)} \int_{E(x,x^0,t)} r^2 q(\xi) d\omega, \\
&= -\frac{1}{4\pi \rho} \int_{(t-\rho)/2}^{(t+\rho)/2} 2\pi q(\xi) d\varphi dr, \quad t > \rho,
\end{align*}
\]
\begin{equation}
    w_n(x,x^0,t) = -\frac{1}{4\pi} \int_0^t \frac{1}{\tau^2 - \rho^2} \left[ \int_{E(x,x^0,\tau)} \tau^3 q(\xi) w_{n-1}(\xi,x^0,\tau - |x - \xi|) d\omega \right] d\tau
\end{equation}

\begin{equation}
    = -\frac{1}{4\pi \rho} \int_0^t \left[ \int_{E(x,x^0,\tau)} r q(\xi) w_{n-1}(\xi,x^0,\tau + r) d\varphi dr \right] d\tau,
\end{equation}

where \( \rho = |x - x^0| \), \( r = |\xi - x^0| \), \( d\omega = \sin \theta d\theta d\varphi \) and \( \xi \) is given by the formulae

\begin{align*}
    \xi &= x^0 + r \nu(\theta, \varphi) A(\theta, \varphi), \quad x = x^0 + \rho(\theta, \varphi), \\
    \nu(\theta, \varphi) &= (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta), \\
    A(\theta, \varphi) &= \begin{pmatrix}
        -\cos \theta \cos \psi & -\cos \theta \sin \psi & \sin \theta \\
        \sin \psi & -\cos \psi & 0 \\
        \sin \theta \cos \psi & \sin \theta \sin \psi & \cos \theta
    \end{pmatrix}.
\end{align*}

Here \( \rho, \theta \in [0, \pi] \) and \( \psi \in [0, 2\pi] \) are spherical coordinates of the vector \( x - x^0 \) with the center at \( \{x^0\} \). Next, \( r, \theta \) and \( \varphi \) are spherical coordinates of the vector \( \xi - x^0 \) with respect to the new coordinates system \( \xi_1, \xi_2, \xi_3 \). The center of this new system is \( \{x^0\} \), the axis \( \xi_3 \) passes through points \( x^0 \) and \( x \), and axis \( \xi_1, \xi_2 \) are orthogonal both to \( \xi_3 \) and to each other and the axis \( \xi_1 \) lays in the plane passing through the axis \( \xi_3 \) and \( \xi_3 \). The orientation of the system \( \xi_1, \xi_2, \xi_3 \) is the same as the orientation of the system \( \xi_1, \xi_2, \xi_3 \). Further, \( \theta \in [0, \pi] \) is the angle between the vector \( \xi - x^0 \) and the axis \( \xi_3 \). In the ellipsoid \( E(x,x^0,t) \) variables \( r \) and \( \theta \) are connected via

\begin{equation}
    r = \frac{t^2 - \rho^2}{2(t - \rho \cos \theta)}, \quad \theta = \arccos \left( \frac{2tr - t^2 + \rho^2}{2r \rho} \right).
\end{equation}

It was shown in [17] that the series (3.5) converges uniformly in \( G(x^0,T) = \{(x,t) : |x - x^0| \leq t \leq T\} \) for any \( T > 0 \) and, moreover,

\begin{equation}
    \lim_{t \to |x - x^0|+} w_1(x,x^0,t) = -\frac{1}{2|x - x^0|} \int_{L(x,x^0)} q(\xi) d\sigma,
\end{equation}

\begin{equation}
    \lim_{t \to |x - x^0|+} w_n(x,x^0,t) = 0, \quad n \geq 2.
\end{equation}

Hence,

\begin{equation}
    \lim_{t \to |x - x^0|+} \tilde{w}(x,x^0,t) = -\frac{1}{2|x - x^0|} \int_{L(x,x^0)} q(\xi) d\sigma = -\frac{1}{2} \int_0^1 q(x^0 + z(x - x^0))dz.
\end{equation}

Lemma 1 and Corollary 1 guarantee a certain smoothness of the function \( \tilde{w} \).

**Lemma 1.** For any \( T > 0 \) and for any \( x^0 \in \mathbb{R}^3 \) functions \( \partial_t^k \tilde{w}(x,x^0,t) \in C\left(G(x^0,T)\right) \) for \( k = 0, 1, 2 \).

**Corollary 1.** The function \( \Delta \tilde{w} \in C\left(G(x^0,T)\right) \).
Proof. By (3.1), (3.4) and (3.5)
\[ \Delta x \tilde{w} = \tilde{w}_{tt} + q(x) \tilde{w} \] for \( t > |x - x^0| \).

By Lemma 1 the right hand side of this equation belongs to \( C(G(x^0, T)) \). Hence, the assertion of this Corollary is true. □

3.2 Proof of (2.14)

First, we show that functions \( \partial_k t w(x, x^0, t), k = 0, 1, 2 \) and \( \Delta_x w(x, x^0, t) \) decay exponentially as \( t \to \infty \) and \( x \) remains in a bounded domain. To do this, we refer to Lemma 6 of Chapter 10 of the book of Vainberg [27] as well as to Remark 3 after that lemma. It follows from these results as well as from Lemma 1 and Corollary 1 that for every \( R > 0 \) and domain \( D(x^0, R) = \{ x \in \mathbb{R}^3 : |x - x^0| < R \} \) there exist numbers \( C_2 = C_2 > 0, c_2 > 0, t_0 > 0 \) depending only on \( q, x^0, R \) such that for \( k = 0, 1, 2 \)

\[ |\partial_k w(x, x^0, t)|, |\Delta_x w(x, x^0, t)| \leq C_2 e^{-c_2 t} \] for all \( t \geq t_0 \) and for all \( x \in D(x^0, R) \). (3.12)

By (3.12) we can apply Fourier transform with respect to \( t \) to functions \( \partial_k t w(x, x^0, t), \Delta_x w(x, x^0, t) \). Let

\[ v(x, x^0, k) = \frac{1}{4\pi} \int_0^\infty w(x, x^0, t) e^{-ikt} dt, \forall x, x^0 \in \mathbb{R}^3, x \neq x^0, \forall k \in \mathbb{R}. \] (3.13)

Using again the same results of references [6, 26, 27] as ones cited in section 2, we obtain that

\[ u(x, x^0, k) = v(x, x^0, k), \forall x, x^0 \in \mathbb{R}^3, x \neq x^0, \forall k \in \mathbb{R}. \] (3.14)

Consider now the asymptotic behavior of the function \( u_{sc}(x, x^0, k) = u(x, x^0, k) - u_0(x, x^0, k) \) as \( k \to \infty \). Clearly

\[ u_{sc}(x, x^0, k) = \frac{1}{4\pi} \int_{|x - x^0|}^\infty \tilde{w}(x, x^0, t) e^{-ikt} dt, \forall x, x^0 \in \mathbb{R}^3, x \neq x^0, \forall k \in \mathbb{R}. \] (3.14)

Using (3.9), Lemma 1 and integration by parts, we obtain

\[ \int_{|x - x^0|}^\infty \tilde{w}(x, x^0, t) e^{-ikt} dt = \]

\[ -i \exp \left( -\frac{i k |x - x^0|}{k} \right) \left[ \frac{1}{8\pi |x - x^0|} \int_{L(x,x^0)} q(\xi) d\sigma + \frac{i}{k} \partial_t \tilde{w}(x, x^0, |x - x^0|^+) \right] \]

\[ - \frac{1}{k^2} \int_{|x - x^0|}^\infty \partial_t^2 \tilde{w}(x, x^0, t) e^{-ikt} dt. \]
Hence, by (3.14) the asymptotic behaviour of the function \( u_{sc}(x, x^0, k) \) is

\[
u_{sc}(x, x^0, k) = \frac{i \exp(i k |x-x^0|)}{8\pi k |x-x^0|} \left[ \int_{L(x,x^0)} q(\xi) d\sigma + O\left( \frac{1}{k} \right) \right], \quad k \to \infty.
\] (3.15)

Since formula (3.15) coincides with formula (2.14), then the proof of (2.14) is complete. Next, since by Remark 2 the validity of Theorem 1 follows from (2.14), then the proof of this theorem is complete as well. \( \Box \)

4 Proof of Lemma 1

Denote

\[
q_0 = \|q\|_{C(\Omega)}, q_2 = \|q\|_{C^2(\Omega)}^2, q_4 = \|q\|_{C^4(\Omega)}.
\] (4.1)

It follows from (3.9) and (3.11) that the function \( w_1(x, x^0, t) \in C\left(G(x^0, T)\right) \). First, we prove that the function \( w_1 \) given by formula (3.6) has the first derivative \( \partial_t w_1 \) which belongs to \( C\left(G(x^0, T)\right) \). Change variables \( r \leftrightarrow z \) in the second integral (3.6) as

\[
r = \frac{1}{2}(t - \rho + 2z\rho).
\] (4.2)

Then

\[
w_1(x, x^0, t) = -\frac{1}{4\pi} \int_0^{2\pi} \int_0^r q(\xi) d\varphi dz,
\] (4.3)

where \( \xi \in E(x, x^0, t) \) is

\[
\xi = x^0 + \frac{1}{2}(t - \rho + 2z\rho)v(\theta, \varphi)A(\vartheta, \psi), \quad \theta = \arccos\left(\frac{\rho - t + 2z\rho}{t - \rho + 2z\rho}\right).
\] (4.4)

Using (4.3), we obtain

\[
\partial_t w_1(x, x^0, t) = -\frac{1}{4\pi} \int_0^{2\pi} \int_0^r \nabla q(\xi) \cdot \partial_t \xi d\varphi dz.
\] (4.5)

Using (4.4), we calculate \( \partial_t \xi \) as

\[
\partial_t \xi = \frac{1}{2}v(\theta, \varphi)A(\vartheta, \psi) + \frac{1}{2}(t - \rho + 2z\rho)v_\theta(\theta, \varphi)A(\vartheta, \psi)\partial_t \theta
\]

\[
= \frac{1}{2}v(\theta, \varphi)A(\vartheta, \psi) + \frac{2\rho z(1 - z)}{(t - \rho + 2z\rho)\sin \theta} v_\theta(\theta, \varphi)A(\vartheta, \psi),
\] (4.6)

where \( v_\theta(\theta, \varphi) = \partial_\theta v(\theta, \varphi) \). In (4.6) we have used the following formula:

\[
\partial_t \theta = \frac{4\rho z(1 - z)}{(t - \rho + 2z\rho)^2 \sin \theta}.
\] (4.7)
Indeed, by (4.4)
\[\cos \theta = \frac{\rho - t + 2zt}{t - \rho + 2z\rho}.\]

Hence,
\[-\sin \theta \partial_t \theta = \frac{(2z - 1)(t - \rho + 2z\rho) - (\rho - t + 2zt)}{(t - \rho + 2z\rho)^2} = \frac{-4z\rho(1 - z)}{(t - \rho + 2z\rho)^2},\]

which proves (4.7).

Hence,
\[\nabla q(\xi) \cdot \partial_t \xi = Q_1 \cos \varphi + Q_2 \sin \varphi + Q_3 (2z - 1),\]

where functions \(Q_j = Q_j(\xi, t, z, \theta, \vartheta, \psi), \ j = 1, 2, 3\) are:

\[
\begin{align*}
Q_1 &= \left[ \frac{2\rho z(1 - z) \cot \theta}{(t - \rho + 2z\rho)} + \frac{1}{2} \sin \theta \right] \sum_{k=1}^{3} (\partial_{\xi_k} q) a_{1k}, \\
Q_2 &= \left[ \frac{2\rho z(1 - z) \cot \theta}{(t - \rho + 2z\rho)} + \frac{1}{2} \sin \theta \right] \sum_{k=1}^{3} (\partial_{\xi_k} q) a_{2k}, \\
Q_3 &= \frac{1}{2} \sum_{k=1}^{3} (\partial_{\xi_k} q) a_{3k},
\end{align*}
\]

where \(a_{jk}(\vartheta, \psi)\) are elements of the matrix \(A\). We now calculate integrals over the ellipsoid \(E(x, x^0, t)\) separately for each term \(Q_j, j = 1, 2, 3\). Using the integration by parts with respect to \(\varphi\), we obtain

\[
I_1(x, x^0, t) = -\frac{1}{4\pi} \int_{0}^{2\pi} \int_{0}^{1} Q_1 \cos \varphi d\varphi dz = \frac{1}{4\pi} \int_{0}^{2\pi} \int_{0}^{1} (\partial_{\varphi} Q_1) \sin \varphi d\varphi dz
\]

\[
= \frac{1}{4\pi} \int_{0}^{1} \int_{0}^{2\pi} \sum_{m=1}^{3} (\partial_{\xi_m} Q_1)(\partial_{\varphi} \xi_m) a_{1k} \sin \varphi d\varphi dz
\]

\[
= \frac{1}{4\pi} \int_{0}^{1} \int_{0}^{2\pi} \left[ \rho z(1 - z) \cos \theta + \frac{(t - \rho + 2z\rho)}{4} \sin^2 \theta \right] \times \sum_{k,m=1}^{3} (\partial_{\xi_k,\xi_m}^2 q) a_{1k} \sin \varphi (a_{2m} \cos \varphi - a_{1m} \sin \varphi) d\varphi dz.
\]

Similarly,

\[
I_2(x, x^0, t) = -\frac{1}{4\pi} \int_{0}^{1} \int_{0}^{2\pi} Q_2 \sin \varphi d\varphi dz
\]

\[
= -\frac{1}{4\pi} \int_{0}^{1} \int_{0}^{2\pi} \left[ \rho z(1 - z) \cos \theta + \frac{(t - \rho + 2z\rho)}{4} \sin^2 \theta \right] \times \sum_{k,m=1}^{3} (\partial_{\xi_k,\xi_m}^2 q) a_{2k} \cos \varphi (a_{2m} \cos \varphi - a_{1m} \sin \varphi) d\varphi dz.
\]
Consider now the integral with $Q_3$. Since $(2z - 1)dz = d(z^2 - z)$, then the integration by parts with respect to $z$ leads to

$$I_3(x, x^0, t) = -\frac{1}{4\pi} \int_0^{2\pi} Q_3(2z - 1) d\varphi dz$$

$$= -\frac{\rho}{8\pi} \int_0^{2\pi} \int_0^{\rho} z(1 - z) \sum_{k,m=1}^{3} (\partial^2_{\xi_k,\xi_m} q) a_{3k}$$

$$\times [\sin \theta(a_{1m} \cos \varphi + a_{2m} \sin \varphi) + a_{3m} \cos \theta] d\varphi dz. \quad (4.11)$$

Hence, using (4.5), (4.8)-(4.11), we obtain

$$\partial_t w_1(x, x^0, t) = -\frac{\rho}{8\pi} \int_0^{2\pi} \int_0^{\rho} \sum_{k,m=1}^{3} (\partial^2_{\xi_k,\xi_m} q) \left\{ z(1 - z) \cos \theta + \frac{(t - \rho + 2z\rho)}{4} \sin^2 \theta \right\}$$

$$\times \left[ b_{0km} + b_{1km} \cos(2\varphi) + b_{2km} \sin(2\varphi) \right]$$

$$+ z(1 - z) a_{3k} [\sin \theta(a_{1m} \cos \varphi + a_{2m} \sin \varphi) + a_{3m} \cos \theta] \right\} d\varphi dz, \quad (4.12)$$

where

$$b_{0km} = a_{1ka_{1m}} + a_{2k} a_{2m}, \quad b_{1km} = a_{2ka_{2m}} - a_{1k} a_{1m}, \quad b_{2km} = -a_{2k} a_{1m} - a_{1k} a_{2m}. \quad (4.13)$$

Thus, it follows from (4.12) and (4.13) that the function $\partial_t w_1 \in C \left( G(x^0, T) \right)$ and the following estimate holds

$$|\partial_t w_1(x, x^0, t)| \leq C q_2 \text{ in } G(x^0, T),$$

where the number $q_2$ is defined in (4.1). Here and below $C = C(T)$ is a positive constant depending only on $T$. Since $\cos \theta \to 1$ and $\xi \to x^0 + z(x - x^0)$, as $t \to |x - x^0|^+$, then

$$\lim_{t \to |x - x^0|^+} \partial_t w_1(x, x^0, t) = -\frac{\rho}{8} \sum_{k,m=1}^{3} \int_0^{1} z(1 - z)(\partial^2_{\xi_k,\xi_m} q)(x^0 + z(x - x^0))(b_{0km} + a_{3k} a_{3m}) dz,$$

$$= -\frac{\rho}{8} \int_0^{1} z(1 - z)(\Delta q)(x^0 + z(x - x^0)) dz.$$

In the latter equality we use the fact that

$$b_{0km} + a_{3k} a_{3m} = \sum_{j=1}^{3} a_{jk} a_{jm} = \delta_{km},$$

where $\delta_{km}$ is the Kronecker delta, since the matrix $A$ is orthogonal.

Consider now functions $w_n(x, x^0, t), n \geq 2$ given by (4.7). We show that each function $w_n(x, x^0, t) \in C \left( G(x^0, T) \right)$ and the following estimate holds

$$|w_n(x, x^0, t)| \leq \frac{q_0^n T^{n-1}(t - \rho)^{n-1}}{4(n - 1)!}, \quad n \geq 2, \quad (4.14)$$

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where the number \( q_0 \) is defined in (4.11). First, we observe that the same estimate is valid for \( n = 1 \). Indeed, taking into account that by (3.6) and (3.8) \( r \leq (\tau + \rho)/2 \leq T \) for \( \xi \in E(x,x^0,\tau) \), we obtain

\[
|w_1(x,x^0,t)| \leq \frac{1}{4\pi} \int_0^1 \int_{\partial T} q_0 \varphi \, dz = \frac{q_0}{2},
\]

\[
|w_2(x,x^0,t)| \leq \frac{1}{4\pi} \int_0^t \left( \int_0^{2\pi} \int_0^q T q_0 \frac{q_0}{2} \phi \, dz \right) \, dr = \frac{T q_0^2}{4} (t - \rho),
\]

\[
|w_3(x,x^0,t)| \leq \frac{1}{4\pi} \int_0^t \left( \int_0^{2\pi} \int_0^q T q_0 \frac{q_0}{2} (t - \tau + \rho) \phi \, dz \right) \, dr \leq \frac{T^2 q_0^3 (t - \rho)^2}{4 \pi}.
\]  

(4.15)

Continuing this way, we obtain estimate (4.14) by the method of mathematical induction. Differentiating formula (3.7) with respect to \( t \), we obtain

\[
\partial_t w_n(x,x^0,t) = -\frac{1}{4\pi \rho} \int_{E(x,x^0,t)} r q_0 (\xi) w_n-1(\xi,x^0,r) d\phi dr
\]

\[
- \frac{1}{4\pi \rho} \int_0^t \left[ \int_{E(x,x^0,\tau)} r q_0 (\xi) \partial_t w_n-1(\xi,x^0,t - \tau + r) d\phi dr \right] d\tau, \quad n \geq 2.
\]

(4.16)

If \( n = 2 \), then the latter formula implies that the function \( \partial_t w_2(x,x^0,t) \) is continuous in \( G(x^0,T) \) and

\[
\lim_{t \to |x-x^0|^+} \partial_t w_2(x,x^0,t) = -\frac{1}{2} \int_0^1 z q(x^0 + z(x-x^0),x^0,z) \, dz
\]

\[
= \frac{1}{4} \int_0^1 z q(x^0 + z(x-x^0)) \left( \int_0^1 q(x^0 + z(x-x^0)) \, dz \right)\, dz.
\]

Hence, \( \partial_t w_2 \in C \left( G(x^0,T) \right) \). Also, we obtain the following estimate from the above discussion

\[
|\partial_t w_2(x,x^0,t)| \leq \frac{T q_0}{2} \left( \frac{q_0}{2} + C T q_2 \right) \leq \frac{T q_0}{4} C_1 q_2, \quad \text{in} \ G(x^0,T),
\]

(4.17)

where \( C_1 = 1 + 2TC \). If \( n \geq 3 \), then the first integral in formula (4.16) vanishes, which follows immediately from estimate (4.14). Indeed, by this estimate \( w_n-1(\xi,x^0,|\xi-x^0|) = 0 \) for \( n \geq 3 \). As to the second integral in (4.16), it obviously tends to zero as \( t \to |x-x^0|^+ \).

Thus, by (4.16)

\[
\lim_{t \to |x-x^0|^+} \partial_t w_n(x,x^0,t) = 0, \quad n \geq 3.
\]

(4.18)

Continuing estimates (4.17) for \( n = 3,4,\ldots \) and taking into account (4.16) and (4.18), we obtain by the method of mathematical induction that all functions \( \partial_t w_n \in C \left( G(x^0,T) \right) \), \( n \geq 1 \), and the following estimates are valid

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\[ |\partial_t w_n(x, x^0, t)| \leq \frac{C_1 q_2 T^{n-1} q_0^{n-1} (t - \rho)^{n-2}}{4(n - 2)!}, \quad n \geq 3. \]

Hence, the series
\[ \sum_{n=1}^{\infty} \partial_t w_n(x, x^0, t) \]
converges uniformly in \( G(x^0, T) \) and its sum \( \partial_t \tilde{w} \in C\left(G(x^0, T)\right) \). Thus, we have proved the assertion of Lemma 1 for functions \( \tilde{w} \) and \( \partial_t \tilde{w} \).

Now we prove the assertion of this lemma for the second derivative \( \partial_t^2 \tilde{w}(x, x^0, t) \). Since the proof of this fact is quite similar to the previous one, we omit some details for brevity.

We represent the formula for the first derivative of the function \( w_1 \) in the form:
\[ \partial_t w_1(x, x^0, t) = -\rho \frac{\mathcal{R}_1 + \mathcal{R}_2}{8\pi} \int_0^{2\pi} \int_0^1 S d\varphi dz, \]
where
\[ S = S_1 \cos \theta + S_2 \sin \theta + S_3 \sin^2 \theta, \]
where functions \( S_j = S_j(\xi, t, z, \varphi, \rho, \psi) \) are defined by:
\[ S_1 = \rho z (1 - z) \sum_{k,m=1}^3 (\partial^2_{\xi_k, \xi_m} q) [\delta_{km} + b_{1km} \cos(2\varphi) + b_{2km} \sin(2\varphi)], \]
\[ S_2 = \rho z (1 - z) \sum_{k,m=1}^3 (\partial^2_{\xi_k, \xi_m} q) a_{3k} [(a_{1m} \cos \varphi + a_{2m} \sin \varphi)], \]
\[ S_3 = \frac{(t - \rho + 2z\rho)}{4} \sum_{k,m=1}^3 (\partial^2_{\xi_k, \xi_m} q) [b_{0km} + b_{1km} \cos(2\varphi) + b_{2km} \sin(2\varphi)]. \]

Hence,
\[ \partial_t^2 w_1(x, x^0, t) = -\rho \frac{\mathcal{R}_1 + \mathcal{R}_2}{8\pi} \int_0^{2\pi} \int_0^1 \left[ \nabla S \cdot \partial_t \xi + (\partial_t S_3) \sin^2 \theta + (S_2 \cos \theta - S_1 \sin \theta + S_3 \sin(2\theta)) \partial_t \theta \right] d\varphi dz. \]

We have
\[ \nabla S \cdot \partial_t \xi = \sum_{i=1}^3 \left[ \frac{2\rho z (1 - z) \coth \theta}{(t - \rho + 2z\rho)} + \frac{1}{2} \sin \theta \right] (\partial_{\xi_i} S) \times [a_{1i} \cos \varphi + a_{2i} \sin \varphi] + \frac{2z - 1}{2} \sum_{i=1}^3 (\partial_{\xi_i} S) a_{3i}. \]

Hence, we represent \( \partial_t^2 w_1(x, x^0, t) \) as
\[ \partial_t^2 w_1(x, x^0, t) = -\rho \frac{\mathcal{R}_1 + \mathcal{R}_2}{8\pi} \int_0^{2\pi} \int_0^1 [R_1 + R_2] d\varphi dz, \quad (4.19) \]
where

\[
R_1 = \left[ \frac{2 \rho z (1 - z) \cos \theta}{(t - \rho + 2z \rho)} \sum_{i=1}^{3} (\partial_{\xi_i} (S_2 + S_3 \sin \theta)) + \frac{1}{2} \sin \theta \sum_{i=1}^{3} (\partial_{\xi_i} S) \right] [a_{1i} \cos \varphi + a_{2i} \sin \varphi]
\]

\[
+ \frac{2z - 1}{2} \sum_{i=1}^{3} (\partial_{\xi_i} S) a_{3i} + \frac{1}{4} \sum_{k,m=1}^{3} (\partial^2_{\xi_k, \xi_m} q) [b_{0km} + b_{1km} \cos (2 \varphi) + b_{2km} \sin (2 \varphi)] \sin^2 \theta
\]

\[
+ \frac{4 \rho z (1 - z)}{(t - \rho + 2z \rho)^2} (-S_1 + 2S_3 \cos \theta),
\]

\[
R_2 = \frac{2 \rho z (1 - z) \cos \theta}{(t - \rho + 2z \rho) \sin \theta} \left[ \cos \theta \sum_{i=1}^{3} (\partial_{\xi_i} S_1) [a_{1i} \cos \varphi + a_{2i} \sin \varphi] + \frac{2S_2}{(t - \rho + 2z \rho)} \right].
\]

The term \( R_1 \) is obviously bounded. However, \( R_2 \) is unbounded. Hence, we represent \( R_2 \) as

\[
R_2 = \frac{z(1 - z)^2 \cos \theta}{(t - \rho + 2 \rho z) \sin \theta} \sum_{s=1}^{3} [M_s \cos (s \varphi) + N_s \sin (s \varphi)],
\]

where \( M_s = M_s(\xi, t, z, \rho, \vartheta, \psi) \) and \( N_s = N_s(\xi, t, z, \rho, \vartheta, \psi) \) are defined by:

\[
M_1 = \rho z \cos \theta \sum_{k,m,i=1}^{3} (\partial^3_{\xi_k, \xi_m, \xi_i} q) (2 \delta_{km} a_{1i} + b_{1km} a_{1i} + b_{2km} a_{2i})
\]

\[
+ \frac{2 \rho z}{(t - \rho + 2 \rho z)} \sum_{k,m=1}^{3} (\partial^2_{\xi_k, \xi_m} q) a_{3k} a_{1m},
\]

\[
N_1 = \rho z \cos \theta \sum_{k,m,i=1}^{3} (\partial^3_{\xi_k, \xi_m, \xi_i} q) (2 \delta_{km} a_{2i} + b_{2km} a_{1i} - b_{1km} a_{2i})
\]

\[
+ \frac{2 \rho z}{(t - \rho + 2 \rho z)} \sum_{k,m=1}^{3} (\partial^2_{\xi_k, \xi_m} q) a_{3k} a_{2m},
\]

\[
M_2 = N_2 = 0,
\]

\[
M_3 = \rho z \cos \theta \sum_{k,m,i=1}^{3} (\partial^3_{\xi_k, \xi_m, \xi_i} q) (b_{1km} a_{1i} - b_{2km} a_{2i})
\]

\[
N_3 = \rho z \cos \theta \sum_{k,m,i=1}^{3} (\partial^3_{\xi_k, \xi_m, \xi_i} q) (b_{2km} a_{1i} + b_{1km} a_{2i}).
\]

By (4.19)

\[
\partial_{t}w_1(x, x^0, t) = J_1(x, x^0, t) + J_2(x, x^0, t),
\]

\[
J_j(x, x^0, t) = -\frac{\rho}{8\pi} \int_{0}^{\frac{1}{2\pi}} \int_{0}^{2\pi} R_j d\varphi dz, \quad j = 1, 2.
\]

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The function $J_1(x, x^0, t)$ is obviously continuous and bounded for $(x, t) \in \overline{G(x^0, T)}$. We show now that the function $J_2(x, x^0, t)$ has the same properties. We have

$$J_2(x, x^0, t) = \frac{-\rho}{8\pi} \int_0^1 \int_0^{2\pi} \left[ \sum_{s=1}^{3} M_s \cos(s\varphi) + N_s \sin(s\varphi) \right] d\varphi \, dz$$

$$= \frac{\rho}{8\pi} \int_0^1 \int_0^{2\pi} \frac{(1-z)^2 \cot \theta}{(t+2\rho z)} \left( \sum_{s,j=1}^{3} \frac{1}{s} \sin(s\varphi)(\partial_{\xi_j} M_s - \frac{1}{s} \cos(s\varphi)(\partial_{\xi_j} N_s)) \right) d\varphi dz.$$

Hence,

$$J_2(x, x^0, t) = \frac{\rho}{16\pi} \int_0^1 \int_0^{2\pi} (1-z)^2 \left[ \sum_{s,j=1}^{3} \frac{1}{s} \sin(s\varphi) \partial_{\xi_j} (M_s - \frac{1}{s} \cos(s\varphi) \partial_{\xi_j} N_s) \right] d\varphi dz \times (a_{2j} \cos \varphi - a_{1j} \sin \varphi).$$

Since functions $\partial_{\xi_j} M_s$ and $\partial_{\xi_j} N_s$ are bounded for $(x, t) \in \overline{G(x^0, T)}$, the function $J_2(x, x^0, t)$ is also bounded and continuous in $G(x^0, T)$. Thus, the functions $J_1, J_2, \partial^2_tw_1$ are bounded and continuous in $G(x^0, T)$. Simple estimates lead to

$$|\partial^2_tw_1(x, x^0, t)| \leq Cq_k. \quad (4.20)$$

Moreover,

$$\lim_{t \to |x-x^0|} J_1(x, x^0, t) = -\frac{1}{8} \int_0^1 \int_0^{2\pi} \rho z(1-z) \sum_{k,m,i=1}^{3} \left( \partial^3_{\xi_k \xi_m \xi_i} q \right) \times \left[ a_{3k}(a_{1m}a_{1i} + a_{2m}a_{2i}) + (2z - 1)a_{3m}\delta_{km} - 2(1-z)^2 \sum_{k,m=1}^{3} \partial^3_{\xi_k \xi_m \xi_i} q a_{3k}a_{3m} \right] dz,$$

$$\lim_{t \to |x-x^0|} J_2(x, x^0, t) = -\frac{\rho}{32\pi} \int_0^1 \int_0^{2\pi} z(1-z)^2 \left[ \rho z \sum_{k,m,i,j=1}^{3} \left( \partial^4_{\xi_k \xi_m \xi_i \xi_j} q \right) \times \left[ a_{1j}(2a_{1i}\delta_{km} + b_{1km}a_{1i} + b_{2km}a_{2i}) + a_{2j}(a_{2i}\delta_{km} - b_{1km}a_{2i} + b_{2km}a_{2i}) \right] \right] d\varphi dz.$$

Hence, there exists the limit $\lim_{t \to |x-x^0|} \partial^2_tw_1(x, x^0, t)$ and it is continuous with respect to $x$. Thus, we have proven that the function $\partial^2_tw_1 \in C \left( \overline{G(x^0, T)} \right)$. Consider now functions $\partial^2_tw_n$ for $n \geq 2$. Differentiating formula (4.16) with respect to $t$, we obtain

$$\partial^2_tw_n(x, x^0, t) = -\frac{1}{4\pi\rho} \int_0^1 \int_0^{2\pi} r q(\xi) w_{n-1}(\xi, x^0, r) d\varphi dz.$$
\[
\int_{E(x,x^0,t)} rq(\xi) \partial_t w_{n-1}(\xi, x^0, r) d\varphi dr + \int_{E(x,x^0,\tau)} \int \frac{d\tau}{r} \left( rq(\xi) \partial_t^2 w_{n-1}(\xi, x^0, t - \tau + r) d\varphi dr \right), \quad n \geq 2,
\]
where by (4.2) \( r = |\xi - x^0| = (t - \rho + 2z\rho)/2 \). Note that the first integral in (4.21) vanishes for \( n > 2 \) since \( w_{n-1}(x, x^0, |x - x^0|) \) vanishes for \( n > 2 \). Hence, we should calculate the derivative of this integral only for \( n = 2 \).

Denote
\[
Q(x, x^0) = q(x)w_1(x, x^0, |x - x^0|) = -\frac{q(x)}{2} \int_0^1 q(x^0 + z(x - x^0)) dz.
\]
The function \( Q(x, x^0) \in C^4(\mathbb{R}^3 \times \mathbb{R}^3) \). We have
\[
\partial_t \int_0^1 \int_0^{2\pi} rq(\xi)w_1(\xi, x^0, r) d\varphi dz = \frac{1}{2} \int_0^1 \int_0^{2\pi} (t - \rho + 2z\rho) Q(\xi, x^0) dz.
\]
\[
= \frac{1}{2} \int_0^1 \int_0^{2\pi} Q(\xi, x^0) d\varphi dz + \frac{1}{2} \int_0^1 \int_0^{2\pi} (t - \rho + 2z\rho)(\nabla Q(\xi, x^0) \cdot \partial_t \xi) d\varphi dz. \tag{4.22}
\]
The first of the integrals in the second line of (4.22) is a bounded and continuous function in \( G(x^0, T) \) and
\[
\lim_{t \to |x - x^0|} \frac{1}{2} \int_0^1 \int_0^{2\pi} Q(\xi, x^0) d\varphi dz = \pi \int_0^1 Q(x + z(x - x^0), x^0) dz.
\]
The second integral in the second line of (4.22) can be evaluated in the same way as we have done above for the case of the first \( t \)-derivative. More precisely, we use
\[
\nabla Q(\xi, x^0) \cdot \partial_t \xi = Q_1 \cos \varphi + Q_2 \sin \varphi + Q_3 (2z - 1),
\]
where functions \( \overline{Q}_j = \overline{Q}_j(\xi, x^0, t, \theta, \rho, \varphi, \psi), j = 1, 2, 3, \) are:
\[
\overline{Q}_1 = \left[ \frac{2\rho z(1 - z) \cot \theta}{(t - \rho + 2z\rho)} + \frac{1}{2} \sin \theta \right] \sum_{k=1}^3 (\partial_{\xi_k} Q)a_{1k},
\]
\[
\overline{Q}_2 = \left[ \frac{2\rho z(1 - z) \cot \theta}{(t - \rho + 2z\rho)} + \frac{1}{2} \sin \theta \right] \sum_{k=1}^3 (\partial_{\xi_k} Q)a_{2k},
\]
\[
\overline{Q}_3 = \frac{1}{2} \sum_{k=1}^3 (\partial_{\xi_k} Q)a_{3k}.
\]
Next, integrating by parts the integrals containing $Q_j$, we obtain

$$\frac{1}{2} \int_0^1 \int_0^{2\pi} (t - \rho + 2z\rho) \left( \nabla_\xi Q(\xi, x^0) \cdot \partial_t \xi \right) d\varphi dz$$

$$= \frac{\rho}{4} \int_0^1 \int_0^{2\pi} (t - \rho + 2z\rho) \left( \sum_{k=1}^{3} (\partial^2_{\xi_k, \xi_m} Q) \left\{ \left[ z(1 - z) \cos \theta + \left( t - \rho + 2z\rho \right) \frac{1}{4} \sin^2 \theta \right] \times \left[ b_{0km} + b_{1km} \cos(2\varphi) + b_{2km} \sin(2\varphi) \right] 
+ z(1 - z) \sin \theta a_{3k} \left[ (a_{1m} \cos \varphi + a_{2m} \sin \varphi) + a_{3m} \cos \theta \right] \right\} \right) d\varphi dz. $$

Moreover,

$$\lim_{t \to |x_0^0|} \frac{1}{2} \int_0^1 \int_0^{2\pi} (t - \rho + 2z\rho) \left( \nabla_\xi Q(\xi, x^0) \cdot \partial_t \xi \right) d\varphi dz$$

$$= \pi \rho^2 \int_0^1 z^2 (1 - z) (\Delta_\xi Q)(x^0 + z(x - x^0), x^0) dz.$$  

Hence, the first term in the right hand side of (4.21) is a bounded and continuous function in $G(x^0, T)$ for $n = 2$ and it vanishes for $n > 2$. The second term in the right hand side of (4.21) is also a bounded and continuous function in $G(x^0, T)$ for $n = 2, 3$ and vanishes for $n > 3$. Since it was proven above that $\partial^2_t w_1 \in C \left( \overline{G(\xi^0, T)} \right)$, then the third integral in the right hand side of (4.21) is also a bounded and continuous function in $G(x^0, T)$ for $n = 2$. Using estimate (4.20) and the obvious inequalities

$$|\partial_{\xi_k, \xi_m} Q| \leq C q_0 q_2 \leq C q_0 q_4, \quad k, m = 1, 2, 3,$$

we obtain

$$|\partial^2_t w_2(x, x^0, t)| \leq C q_0 q_4.$$  

Next, using (4.21), we obtain a similar estimate for the second derivative of $w_3(x, x^0, t)$:

$$|\partial^2_t w_3(x, x^0, t)| \leq C q_0^2 q_4.$$  

Continuing the above estimates, we obtain

$$|\partial^2_t w_4(x, x^0, t)| \leq \frac{T q_0}{4\pi \rho} \int_0^t d\tau \int_{E(x,x^0,\tau)} C q_0^2 q_4 d\varphi dr = \frac{CT q_0^2 q_4}{2} (t - \rho),$$

$$|\partial^2_t w_5(x, x^0, t)| \leq \frac{T q_0}{4\pi \rho} \int_0^t d\tau \int_{E(x,x^0,\tau)} \frac{CT q_0^3 q_4}{2} \left( t - \frac{\tau + \rho}{2} \right) d\varphi dr \leq \frac{CT^2 q_0^3 q_4 (t - \rho)^2}{2}. $$

Using the mathematical induction method, we get the following estimate

$$|\partial^2_t w_n(x, x^0, t)| \leq \frac{CT^{n-3} q_0^{n-1} q_4 (t - \rho)^{n-3}}{2} \frac{1}{(n - 3)!}, \quad n \geq 4.$$
Hence, all functions $\partial_t^2 w_n \in C \left( G(x^0, T) \right)$, the series

$$
\sum_{n=1}^{\infty} \partial_t^2 w_n(x, x^0, t)
$$

converges uniformly in $G(x^0, T)$ and its sum $\partial_t^2 \tilde{w} \in C \left( G(x^0, T) \right)$. □

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