Abstract. We define a tower of affine Temperley-Lieb algebras of type $\tilde{A}_n$. We prove that there exists a unique Markov trace on this tower, this trace comes from the Markov-Ocneanu-Jones trace on the tower of Temperley-Lieb algebras of type $A_n$.

1. Introduction and notations

Let $K$ be an integral domain of characteristic 0. Suppose that $q$ is an invertible element in $K$. Let $v$ be such that $q^2 = v$. For $x, y$ in a given ring we define $V(x, y) := xyx + xy + yx + x + y + 1$.

We denote by $B(\tilde{A}_n)$ (resp. $W(\tilde{A}_n)$) the affine braid (resp. affine Coxeter) group with $n + 1$ generators of type $\tilde{A}_n$, while we denote by $B(A_n)$ (resp. $W(A_n)$) the braid (resp. Coxeter) group with $n$ generators of type $A$. Where $n \geq 0$. Let $W^c(\tilde{A}_n)$ (resp. $W^c(A_n)$) be the set of fully commutative elements in $W(\tilde{A}_n)$ (resp. $W(A_n)$).

Let $2 \leq n$. We define $\tilde{TL}_{n+1}(q)$ to be the algebra with unit given by a set of generators $\{g_{\sigma_1}, \ldots, g_{\sigma_n}, g_{a_{n+1}}\}$, with the following relations [1]:

- $g_{\sigma_i} g_{\sigma_j} = g_{\sigma_j} g_{\sigma_i}$, for $1 \leq i, j \leq n$ and $|i - j| \geq 2$.
- $g_{\sigma_i} g_{a_{n+1}} = g_{a_{n+1}} g_{\sigma_i}$, for $2 \leq i \leq n - 1$ and $|i - j| \geq 2$.
- $g_{\sigma_i} g_{\sigma_{i+1}} g_{\sigma_i} = g_{\sigma_{i+1}} g_{\sigma_i} g_{\sigma_{i+1}}$, for $1 \leq i \leq n - 1$.
- $g_{\sigma_i} g_{\sigma_{n+1}} g_{\sigma_i} = g_{\sigma_{n+1}} g_{\sigma_i} g_{\sigma_{n+1}}$, for $1 = 1, n$.
- $g_{\sigma_i}^2 = (q - 1) g_{\sigma_i} + q$, for $1 \leq i \leq n$.
- $g_{a_{n+1}}^2 = (q - 1) g_{a_{n+1}} + q$.
- $V(g_{\sigma_i}, g_{\sigma_{i+1}}) = V(g_{\sigma_1}, g_{a_{n+1}}) = V(g_{\sigma_n}, g_{a_{n+1}}) = 0$, for $1 \leq i \leq n - 1$.

The set $\{g_w : w \in W^c(\tilde{A}_n)\}$ is well defined in the usual sense of the theory of Hecke algebra and it is a $K$-basis. We set $T_{a_{n+1}}$ (resp $T_{\sigma_i}$ for $1 \leq i \leq n$) to be $\sqrt[q]{q} g_{a_{n+1}}$ (resp $\sqrt[q]{q} g_{\sigma_i}$ for $1 \leq i \leq n$). Hence, $T_w$ is well defined for $w \in W^c(\tilde{A}_n)$, it equals $q^{\frac{|w|}{2}} g_w$. The multiplication associated to the basis $\{T_w : w \in W^c(\tilde{A}_n)\}$, is given as follows:
For \( w, v \in W^c(\tilde{A}_n) \) and \( s \in \{\sigma_1, \ldots, \sigma_n, a_{n+1}\} \).

In what follows we suppose that \( q + 1 \) is invertible in \( K \), we set \( \delta = \frac{1}{2+q+q^{-1}} = \frac{q}{(1+q)^2} \) in \( K \). In view of \([2]\), for \( 1 \leq i \leq n \) we set \( f_{\sigma_i} := \frac{g_{\sigma_i}+1}{q+1} \) and \( f_{a_{n+1}} := \frac{g_{a_{n+1}}+1}{q+1} \). In other terms \( g_{\sigma_i} = (q+1)f_{\sigma_i} - 1 \), and \( g_{a_{n+1}} = (q+1)f_{a_{n+1}} - 1 \). The set \( \{f_w : w \in W^c(A_n)\} \) is well defined and it is a \( K \)-basis for \( \widehat{T}L_{n+1}(q) \).

We define the Temperley-Lieb algebra of type \( A \) with \( n \) generators \( TL_n(q) \), as the subalgebra of \( \widehat{T}L_{n+1}(q) \) generated by \( \{g_{\sigma_1} \ldots g_{\sigma_n}\} \), with \( \{g_w : w \in W^c(A_n)\} \) as \( K \)-basis.

Now for \( TL_0(q) = K \), we consider the following tower:

\[
TL_0(q) \subset TL_1(q) \subset TL_{n-1}(q) \subset TL_n(q) \ldots
\]

**Theorem 1.1.** \([5]\) There exists a unique collection of traces \( (\tau_{n+1})_{0 \leq n} \) on \( (TL_n)_{0 \leq n} \), such that:

1. \( \tau_1(1) = 1 \).
2. For \( 1 \leq n \), we have \( \tau_{n+1}(hT_{\sigma_n}^{\pm 1}) = \tau_n(h) \), for any \( h \) in \( TL_{n-1}(q) \).

The collection \( (\tau_{n+1})_{0 \leq n} \) is called a Markov trace. Moreover, for \( 1 \leq n \), every \( \tau_{n+1} : TL_n(q) \rightarrow K \) verifies

\[
\tau_{n+1}(bT_{\sigma_n}c) = \tau_n(bc) \text{ and } \tau_{n+1}(a) = -\frac{1+q}{2q} \tau_n(a). \text{ As above } T_{\sigma_n} \text{ here is } \sqrt{q}g_{\sigma_n}.
\]

The aim of this paper is to define a Markov trace in the affine case, then to classify such traces. We will see in theorem 4.5 that there is only one "affine" Markov trace.

2. THE TOWER OF AFFINE TEMPERLEY-LIEB ALGEBRAS AND AFFINE MARKOV TRACE

In this section we define a tower of affine Temperley-Lieb algebras, we show that this tower 'surjects' onto the tower of Temperley-Lieb algebras mentioned in the introduction, and we define the affine Markov trace.

We consider the Dynkin diagram of the group \( B(\tilde{A}_n) \). We denote the Dynkin automorphism \( (\sigma_1 \mapsto \sigma_2 \mapsto \ldots \sigma_n \mapsto a_{n+1} \mapsto \sigma_1) \) by \( \psi_{n+1} \):
We have the following injection

\[ G_n : K[B(A_{n-1}^-)] \rightarrow K[B(\tilde{A}_n)] \]
\[ \sigma_i \mapsto \sigma_i \quad \text{for } 1 \leq i \leq n - 1 \]
\[ a_n \mapsto \sigma_n a_{n+1} \sigma_n^{-1} \]

**Proposition 2.1.** The injection \( G_n \) induces the following morphism of algebras:

\[ F_n : \mathcal{T}L_n(q) \rightarrow \mathcal{T}L_{n+1}(q) \]
\[ g_{\sigma_i} \mapsto g_{\sigma_i} \quad \text{for } 1 \leq i \leq n - 1 \]
\[ g_{a_n} \mapsto g_{\sigma_n} g_{a_{n+1}} g_{\sigma_n}^{-1} \cdot \]

**Proposition 2.2.** The following map is a surjection of algebras

\[ E_n : \mathcal{T}L_{n+1}(q) \rightarrow \mathcal{T}L_n(q) \]
\[ g_{\sigma_i} \mapsto g_{\sigma_i} \quad \text{for } 1 \leq i \leq n \]
\[ g_{a_{n+1}} \mapsto g_{\sigma_1} \cdots g_{\sigma_{n-1}} g_{\sigma_n} g_{\sigma_{n-1}}^{-1} \cdots g_{\sigma_1}^{-1} \cdot \]

Moreover, the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{T}L_n(q) & \xrightarrow{F_n} & \mathcal{T}L_{n+1}(q) \\
\downarrow & & \downarrow \\
\mathcal{T}L_{n-1}(q) & \xleftarrow{E_n} & \mathcal{T}L_n(q)
\end{array}
\]

In [4] we can see more details about proposition 2.1 and 2.2. Moreover, it is immediate that \( E_n \) composed with the natural inclusion of \( \mathcal{T}L_n(q) \) into \( \mathcal{T}L_{n+1}(q) \), gives \( \text{Id}_{\mathcal{T}L_n(q)} \).
In view of proposition 2.1 we can consider the tower of affine T-L algebras, (it is not known whether it is a tower of faithful arrows or not):

\[ \hat{TL}_1(q) \xrightarrow{F_1} \hat{TL}_2(q) \xrightarrow{F_2} \hat{TL}_3(q) \rightarrow \ldots \hat{TL}_n(q) \xrightarrow{F_n} \hat{TL}_{n+1}(q) \rightarrow \ldots \]

**Definition 2.3.** We call \((\hat{\tau}_n)_{1 \leq i}\) an affine Markov trace, if every \(\hat{\tau}_n\) is a trace function on \(\hat{TL}_n(q)\) with the following conditions:

- \(\hat{\tau}_1(1) = 1\), (here \(\hat{TL}_1(q) = K\)).
- \(\hat{\tau}_{n+1}(F_n(h)T_{\sigma_n}^{-1}) = \hat{\tau}_n(h)\). for all \(h \in \hat{TL}_n(q)\) and for \(n \geq 1\).
- \(\hat{\tau}_n\) is invariant under the Dynkin automorphism \(\psi_n\) for all \(n\).

**Remark 2.4.** We notice that the second condition gives us that \(\hat{\tau}_{n+1}\left(F_n(h)T_{\sigma_n}^{-1}\right) = \hat{\tau}_n(h)\), which means that:

\[ \hat{\tau}_{n+1}\left(F_n(h)\left[\frac{1}{q^2}T_{\sigma_n} - \frac{q-1}{q\sqrt{q}}\right]\right) = \hat{\tau}_n(h). \]

Thus \(\hat{\tau}_{n+1}(F_n(h)) = -\frac{q+1}{\sqrt{q}}\hat{\tau}_n(h)\).

**Remark 2.5.** The third condition of definition 2.3 is, in fact, not independent, i.e., it results from the first and second conditions. We just have to see that if we have two elements in \(\hat{TL}_n(q)\) (say \(x\) and \(y\)) such that \(\psi_n(x) = y\), then \(F_n(x)\) and \(F_n(y)\) are conjugate in \(\hat{TL}_{n+1}(q)\), by some power of the element \(g_{\sigma_\ldots \sigma_{a_{n+1}}\ldots }\) (we will explain that in the proof of proposition 4.1). Nevertheless, we will keep viewing it as a condition.

**Remark 2.6.** This affine Markov trace do the job topologically, i.e., It gives an invariant for "affine oriented knots" and generalizes, in fact, the Jones invariant, noticing that the set of oriented knots in \(S^3\) injects naturally into the set of "affine oriented knots". For farther details see [4].

3. **On the space of traces on \(\hat{TL}_{n+1}(q)\)**

3.1. **Traces on \(\hat{TL}_2(q)\).** In this subsection we parametrize all the traces on the algebra \(\hat{TL}_2(q)\), which have the same value over the two generators of \(\hat{TL}_2(q)\), i.e., traces which are fix under the action of the diagram automorphism.

We have: \(\hat{TL}_2(q) = \hat{H}_2(q)\), it is generated by two elements: \(g_{\sigma_1}, g_{a_2}\), with only the Hecke quadratic relations. That is:

\[ g_{\sigma_1}^2 = (q - 1)g_{\sigma_1} + q, \text{ and } g_{a_2}^2 = (q - 1)g_{a_2} + q. \]
Making the same changes as in the introduction, we set $T_{\sigma_1} := \sqrt{q} g_{\sigma_1}$, and $T_{\sigma_2} := \sqrt{q} g_{\sigma_2}$. Hence $T_w = (\sqrt{q})^{(w)} g_w$ for any $w \in W(A)$. The set $\{T_w; w \in W(A)\}$ is another $K$-basis of $H_{\mu}(q)$. The multiplication law of the new basis takes the form:

$$T_{\sigma_1}^2 = \sqrt{q}(q - 1)T_{\sigma_1} + q^2 \quad \text{thus:} \quad T_{\sigma_1}^{-1} = \frac{1}{q^2}(T_{\sigma_1} - \sqrt{q}(q - 1)).$$

$$T_{\sigma_2}^2 = \sqrt{q}(q - 1)T_{\sigma_2} + q^2 \quad \text{thus:} \quad T_{\sigma_2}^{-1} = \frac{1}{q^2}(T_{\sigma_2} - \sqrt{q}(q - 1)).$$

We consider the basis changes mentioned in the introduction. $\widetilde{T\bar{L}}_2(q)$ is generated by $f_{\sigma_1}$ and $f_{\sigma_2}$ with relations $f_{\sigma_1}^2 = f_{\sigma_1}$ and $f_{\sigma_2}^2 = f_{\sigma_2}$. Moreover, $\widetilde{T\bar{L}}_2(q)$ has $\{f_w; w \in W(A)\}$ as a $K$-basis. The aim is to parametrize all traces over this algebra, which are invariant under the action of the Dynkin automorphism $\psi_2$, which exchanges $T_{\sigma_1}$ and $T_{\sigma_2}$, (that is exchanging $f_{\sigma_1}$ and $f_{\sigma_2}$). Clearly, any trace has the same value on $f_{\sigma_1}$ and $f_{\sigma_2}$ is invariant under the Dynkin automorphism $\psi_2$.

**Proposition 3.1.** Let $A_0, A_1$ and $\alpha_i$ be arbitrary elements in the ground field for $1 \leq i$. Then, there exists a unique trace $t$ on $\widetilde{T\bar{L}}_2(q)$, invariant by the action of $\psi_2$, such that:

$A_0 = t(1),\ A_1 = t(f_{\sigma_1})$ and $\alpha_s = t((f_{\sigma_1a_2})^s)$.

**Proof.** We start by the existence. Let $t$ be the linear function given by:

$$t : \widetilde{T\bar{L}}_2(q) \longrightarrow K$$

$$t(1) = A_0$$

$$t(f_{\sigma_1}) = t(f_{\sigma_2}) = A_1$$

$$t((f_{\sigma_1a_2})^s) = t((f_{a_2\sigma_1})^s) = t((f_{\sigma_1a_2})^s f_{\sigma_1}) = t((f_{a_2\sigma_1})^s f_{a_2}) = \alpha_s.$$  

Where $A_0, A_1$ and $\alpha_i$ are arbitrary elements in the ground field for $1 \leq i$.

We show that this linear function is a trace. First we see that $t$ is, by definition, invariant under Dynkin automorphism $\psi_2$. In order to show that $t$ is a trace, we show that $t(xy) = t(yx)$ for any $x$ and $y$ in $\widetilde{T\bar{L}}_2(q)$. The way to do so, is to show that it is true when $x$ is any element of the left column, and $y$ is any element of the right column, in the following table:

| Column 1 | Column 2 |
|----------|----------|
| $[1](f_{\sigma_1a_2})^k$ | $[1'](f_{\sigma_1a_2})^h$ |
| $[2](f_{a_2\sigma_1})^k$ | $[2'](f_{a_2\sigma_1})^h$ |
| $[3](f_{\sigma_1a_2})^k f_{\sigma_1}$ | $[3'](f_{\sigma_1a_2})^h f_{\sigma_1}$ |
| $[4](f_{a_2\sigma_1})^k f_{a_2}$ | $[4'](f_{a_2\sigma_1})^h f_{a_2}$ |
The only cases to consider are [1-2'], [1-3'], [1-4'] and [3-4'], up to applying $\psi_2$.

[1-2']:

Here, $t(xy) = t((f_{\sigma_1 a_2})^k(f_{a_2 \sigma_1})^h) = t((f_{\sigma_1 a_2})^k(f_{a_2 \sigma_1})^{h-1}) = t((f_{\sigma_1 a_2})^k(f_{\sigma_1 a_2})^{h-1}f_{\sigma_1})$

$$= t((f_{\sigma_1 a_2})^{k+h-1}f_{\sigma_1}) = \alpha_{k+h-1},$$

while, $t(yx) = t((f_{a_2 \sigma_1})^h(f_{\sigma_1 a_2})^k) = t((f_{a_2 \sigma_1})^h(f_{\sigma_1 a_2})^{k-1}) = t((f_{a_2 \sigma_1})^h(f_{a_2 \sigma_1})^{k-1}f_{a_2})$

$$= \alpha_{k+h-1}.$$

[1-3']:

Here, $t(xy) = t((f_{\sigma_1 a_2})^k(f_{\sigma_1 a_2})^h f_{\sigma_1}) = t((f_{\sigma_1 a_2})^{k+h}f_{\sigma_1})$, which is equal to $\alpha_{k+h}$,

while, $t(yx) = t((f_{\sigma_1 a_2})^h f_{\sigma_1}(f_{\sigma_1 a_2})^k) = t((f_{\sigma_1 a_2})^{h+k}) = \alpha_{k+h}.$

[1-4']:

Here, $t(xy) = t((f_{\sigma_1 a_2})^k(f_{a_2 \sigma_1})^h f_{a_2}) = t((f_{\sigma_1 a_2})^{k+h} f_{a_2}) = t((f_{\sigma_1 a_2})^{k+h}) = \alpha_{k+h},$

while, $t(yx) = t((f_{a_2 \sigma_1})^h f_{a_2}(f_{\sigma_1 a_2})^k) = t(f_{a_2}(f_{\sigma_1 a_2})^h(f_{\sigma_1 a_2})^k) = t(f_{a_2}(f_{\sigma_1 a_2})^{h+k}) = \alpha_{k+h}.$

[3-4']:

We see that: $t(xy) = t((f_{\sigma_1 a_2})^k f_{\sigma_1}(f_{a_2 \sigma_1})^h f_{a_2}) = t((f_{\sigma_1 a_2})^{k+h+1}) = \alpha_{k+h+1},$

with, $t(yx) = t((f_{a_2 \sigma_1})^h f_{a_2}(f_{\sigma_1 a_2})^k f_{\sigma_1}) = t((f_{a_2 \sigma_1})^{h+k+1}) = \alpha_{k+h+1}.$

Now, we end the proof by showing the uniqueness. Let $t$ be a $\psi_2$-invariant trace on $\tilde{TL}_2(q)$. We have necessarily $t(f_{\sigma_1}) = t(f_{a_2})$, since $t$ is a $\psi_2$-invariant, call this value $A_1$. For every $s \geq 1$ we have $t((f_{\sigma_1 a_2})^s) = t((f_{a_2 \sigma_1})^s)$, since $t$ is a trace, call this value $\alpha_s$. Finally, we have $\alpha_s = t((f_{\sigma_1 a_2})^s f_{\sigma_1}) = t((f_{a_2 \sigma_1})^s f_{a_2})$, since $t$ is a trace, and $f_{a_2}$, $f_{\sigma_1}$ are idempotent. Call $t(1) = A_0$, thus, $t$ is uniquely determined by $A_0$, $A_1$ and $\alpha_s$, for $i \geq 1$. □
3.2. **Traces on** \(\widehat{TL}_3(q)\). In this subsection, we parametrize all the traces over \(\widehat{TL}_3(q)\), which are invariant under the action of the Dynkin automorphism \(\psi_3\).

The affine Temperley-Lieb algebra in three generators \(g_{\sigma_1}, g_{\sigma_2}\) and \(g_{a_3}\) can be presented by those generators with the relations of Hecke algebra, together with:

\[
V(g_{\sigma_1}, g_{\sigma_2}) = V(g_{\sigma_1}, g_{a_3}) = V(g_{\sigma_2}, g_{a_3}) = 0.
\]

We consider the same change of generators as in the case of \(\widehat{TL}_2(q)\). Hence, \(f_{\sigma_i} = \frac{g_{\sigma_i} + 1}{q+1}\) with \(g_{\sigma_i} = (q + 1)f_{\sigma_i} - 1\) for \(i = 1, 2\) the same for \(f_{a_3}\). \(\widehat{TL}_3(q)\) is presented by these three generators and the following relations:

\[
\begin{align*}
  f_{\sigma_i}^2 &= f_{\sigma_i} & \text{for } i = 1, 2 & \text{and } f_{a_3}^2 &= f_{a_3}; \\
  f_{\sigma_i}f_{a_3}f_{\sigma_i} &= \delta f_{\sigma_i} & \text{and } f_{a_3}f_{\sigma_i}f_{a_3} &= \delta f_{a_3}; \\
  f_{\sigma_1}f_{\sigma_2}f_{\sigma_1} &= \delta f_{\sigma_1} & \text{and } f_{\sigma_2}f_{\sigma_1}f_{\sigma_2} &= \delta f_{\sigma_2}.
\end{align*}
\]

Here we will use the \(K\)-basis \(\{f_w; w \in W^c(\tilde{A}_2)\}\).

**Lemma 3.2.** Let \(h\) and \(k\) be two positive integers. Then:

\[
\begin{align*}
  \left(f_{\sigma_2\sigma_1a_3}\right)^h \left(f_{\sigma_1\sigma_2a_3}\right)^k &= \begin{cases} \\
    \delta^3h \left(f_{\sigma_2\sigma_1a_3}\right)^{k-h} & \text{for } h < k. \\
    \delta^{3k-1} f_{\sigma_2a_3} \left(f_{\sigma_1\sigma_2a_3}\right)^{k-h} & \text{for } h \geq k.
  \end{cases}
\end{align*}
\]

\[
\begin{align*}
  \left(f_{\sigma_1\sigma_2a_3}\right)^h \left(f_{\sigma_2\sigma_1a_3}\right)^k &= \begin{cases} \\
    \delta^3h \left(f_{\sigma_1\sigma_2a_3}\right)^{k-h} & \text{for } h > k. \\
    \delta^{3h-1} f_{\sigma_1a_3} \left(f_{\sigma_2\sigma_1a_3}\right)^{k-h} & \text{for } h \leq k.
  \end{cases}
\end{align*}
\]

**Proof.** By induction, with a direct computation the lemma follows. \(\square\)

Now we parametrize all the traces on \(\widehat{TL}_3(q)\), which are invariant by the Dynkin automorphism \(\psi_3\). We know that any element of the \(K\)-basis \(\{f_w; w \in W^c(\tilde{A}_2)\}\) can be written as follows see \([3]\):
Lemma 3.3. Let $k$ be a positive integer, then for any $w$, such that $l(w) = 3k$, the element $f_w$ is the image, under some power of the Dynkin automorphism $\psi_3$, of one of the following elements $(f_{\sigma_2\sigma_1}a_3)^k$ or $(f_{\sigma_1\sigma_2}a_3)^k$. Similarly for any $u$ of length $3k+1$ (resp. $3k+2$), the element $f_u$ is the image under a power of $\psi_3$ of one of the following elements $(f_{\sigma_2\sigma_1}a_3)^kf_{\sigma_2}$ or $(f_{\sigma_1\sigma_2}a_3)^kf_{\sigma_1}$.

Proof. The proof is direct, by induction over $k$. □

Proposition 3.4. For $1 \leq i$, let $B_0, B_1, B_2$ and $\beta_i$ be in $K$. Then, there exists a unique, $\psi_3$-invariant, trace over $\widetilde{TL}_3(q)$, say $s$, such that: $B_0 = s(1)$, $B_1 = s(f_{\sigma_1})$, $B_2 = s(f_{\sigma_2})$, $\beta_1 = s(f_{\sigma_1\sigma_2})$, $\beta_k = s((f_{\sigma_1\sigma_2}a_3)^k f_{\sigma_1})$ and $\beta_k = \frac{1}{\delta}s((f_{\sigma_1\sigma_2}a_3)^k f_{\sigma_1})$.

Proof. For the existence, we consider the following linear map, we can show, using lemma 3.3, that it is indeed a $\psi_3$-invariant trace.

$s$ is given as follows, $s : \widetilde{TL}_3(q) \rightarrow K$

$s(1) = B_0,$

$s(f_{\sigma_1}) = s(f_{\sigma_2}) = s(f_{a_3}) = B_1,$

$s(f_u) = B_2$ for any $u$ in $W^c(\tilde{A}_2)$ with $l(u) = 2,$

and $s(f_v) = \begin{cases} 
\beta_k & \text{when } l(v) = 3k \text{ or } l(v) = 3k + 1, \\
\delta \beta_k & \text{when } l(v) = 3k + 2.
\end{cases}$

Where $\beta_k$ (for $1 \leq k$), $B_0, B_1$ and $B_2$ are arbitrary in the field $K$. While for the uniqueness, we follow the steps of the proof of proposition 3.1

□
3.3. Markov elements. We consider \( F_n : \overline{TL}_n(q) \rightarrow \overline{TL}_{n+1}(q) \) of proposition 2.1. In this subsection we set \( F := F_n \). We give a definition of Markov elements in \( \overline{TL}_{n+1}(q) \) for \( 2 \leq n \). Then we show that any trace over \( \overline{TL}_{n+1}(q) \) is uniquely determined by its values on those elements.

**Definition 3.5.** For \( F \) as above, and \( 2 \leq n \). a Markov element in \( \overline{TL}_{n+1}(q) \) is any element of the form \( Ag_{\sigma_n}^\epsilon B \). Where \( A \) and \( B \) are in \( F(\overline{TL}_n(q)) \) and \( \epsilon \in \{0, 1\} \).

The aim of this subsection is to prove the following theorem.

**Theorem 3.6.** Let \( \tau_{n+1} \) be any trace over \( \overline{TL}_{n+1}(q) \) for \( 2 \leq n \). Then, \( \tau_{n+1} \) is uniquely defined by its values on the Markov elements in \( \overline{TL}_{n+1}(q) \).

The proof of theorem 3.6 is divided into two parts. In the first we show some general facts, in the second we prove the above theorem for \( 3 \leq n \), while for \( n = 2 \) we will not give the proof, as it is pretty long and will not be needed in this paper.

**Part 1**

In this part, we suppose that \( \tau_{n+1} \) is any trace on \( \overline{TL}_{n+1}(q) \). We will apply \( \tau_{n+1} \) to \( \overline{TL}_{n+1}(q) \) assuming that \( 2 \leq n \), and show that \( \tau_{n+1} \) is uniquely determined on \( \overline{TL}_{n+1}(q) \) by its values on the positive powers of \( g_{\sigma_n\sigma_{n-1}\ldots\sigma_1a_{n+1}} \), in addition to its values on Markov elements. From now on we denote by \( w \): an arbitrary element in \( W^c(\hat{A}_n) \).

**Lemma 3.7.** In \( \overline{TL}_{n+1}(q) \) we have:

\[
(1) \quad g_{\sigma}(g_{\sigma_{n-1}\ldots\sigma_1a_{n+1}})^k = (q-1)(g_{\sigma_{n-1}\ldots\sigma_1a_{n+1}})^k + \sum_{i=1}^{k-1} f_i(g_{\sigma_{n-1}\ldots\sigma_1a_{n+1}})^i \\
+ A \left(g_{\sigma_{n-1}\sigma_{n-2}\ldots\sigma_1F(t_{a_n})}\right)^k g_{\sigma} \prod_{j=0}^{k-1} \psi_j \left[ F((t_{a_n})^{-1}) \right],
\]

\[
(2) \quad (g_{\sigma_{n-1}\ldots\sigma_1a_{n+1}})^k g_{\sigma} = (q-1)(g_{\sigma_{n-1}\ldots\sigma_1a_{n+1}})^k + \sum_{i=1}^{k-1} h_i(g_{\sigma_{n-1}\ldots\sigma_1a_{n+1}})^i \\
+ A \prod_{j=0}^{k-1} \phi_j \left[ (\sigma_{n-1})^{-1} \right] g_{\sigma} \left(g_{\sigma_{n-1}\sigma_{n-2}\ldots\sigma_1F(t_{a_n})}\right)^k,
\]

with \( A \) in the ground field, \( f_i, h_i \) in \( F(\overline{TL}_n(q)) \) and \( \phi^{-1} = \psi \).
Proof.

\[
g_{\sigma_n} \left( g_{\sigma_n \sigma_{n-1} \ldots \sigma_{1} a_{n+1}} \right)^k = (q - 1) \left( g_{\sigma_n \sigma_{n-1} \ldots \sigma_{1} a_{n+1}} \right)^k + qg_{\sigma_{n-1} a_{n-2} \ldots \sigma_{1} F(t_{a_{n}}) g_{\sigma_{n-1} F((t_{a_{n}})^{-1}) (g_{\sigma_n \sigma_{n-1} \ldots \sigma_{1} a_{n+1}})^{k-1}} = (q - 1) \left( g_{\sigma_n \sigma_{n-1} \ldots \sigma_{1} a_{n+1}} \right)^k + qg_{\sigma_{n-1} a_{n-2} \ldots \sigma_{1} F(t_{a_{n}}) g_{\sigma_{n-1} \ldots \sigma_{1} a_{n+1}} \left( g_{\sigma_n \sigma_{n-1} \ldots \sigma_{1} a_{n+1}} \right)^{k-1} \psi^{k-1} \left[ F((t_{a_{n}})^{-1}) \right]}.\]

So, by induction on \( k \), (1) follows. In the very same way we deal with (2), by noticing that: \( g_{a_{n+1}} g_{\sigma_n} = g_{\sigma_n}^{-1} F(t_{a_{n}}) g_{\sigma_n}^2 = (q - 1) g_{a_{n+1}} + qg_{\sigma_n}^{-1} F(t_{a_{n}}). \)

A main result in [3] is to give a general form for “fully commutative braids”, from which we deduce that any element of the basis \( TL_{n+1}(q) \), which is not a positive power of \( g_{\sigma_n \sigma_{n-1} \ldots \sigma_{1} a_{n+1}} \), is either of the form

\[c(g_{\sigma_n \sigma_{n-1} \ldots \sigma_{1} a_{n+1}})^k g_{\sigma_n \sigma_{n-1} \ldots \sigma_i}\]

or of the form

\[g_{\sigma_{i_0} \ldots \sigma_{2} \sigma_{1} a_{n+1}} (g_{\sigma_n \sigma_{n-1} \ldots \sigma_{1} a_{n+1}})^k d g_{\sigma_n \sigma_{n-1} \ldots \sigma_i}\]

where \( c \) and \( d \) are in \( F(TL_n(q)) \), \( 1 \leq i \leq n + 1 \) and \( 0 \leq i_0 \leq n - 1 \).

By lemma 3.7 \( c(g_{\sigma_n \sigma_{n-1} \ldots \sigma_{1} a_{n+1}})^k g_{\sigma_n \sigma_{n-1} \ldots \sigma_i} \) is of the form:

\[\sum_{j=1}^{j=h} c_j (g_{\sigma_n \sigma_{n-1} \ldots \sigma_{1} a_{n+1}})^j + M.\]

Where \( h \leq k \), \( c_j \) is in \( F(TL_n(q)) \) for any \( j \) and \( M \) is a Markov element.

Now we deal with the second form:

\[\tau_{n+1} \left( g_{\sigma_{i_0} \ldots \sigma_{2} \sigma_{1} a_{n+1}} c(g_{\sigma_n \sigma_{n-1} \ldots \sigma_{1} a_{n+1}})^k g_{\sigma_n \sigma_{n-1} \ldots \sigma_i} \right) = \tau_{n+1} \left( g_{\sigma_n \sigma_{n-1} \ldots \sigma_i g_{\sigma_{i_0} \ldots \sigma_{2} \sigma_{1} a_{n+1}} c(g_{\sigma_n \sigma_{n-1} \ldots \sigma_{1} a_{n+1}})^k \right) \]

For any possible value for \( i_0 \) or \( i \), we see that:

\[g_{\sigma_n \sigma_{n-1} \ldots \sigma_i g_{\sigma_{i_0} \ldots \sigma_{2} \sigma_{1} a_{n+1}} c(g_{\sigma_n \sigma_{n-1} \ldots \sigma_{1} a_{n+1}})^k = c' g_{\sigma_n (g_{\sigma_n \sigma_{n-1} \ldots \sigma_{1} a_{n+1}})^s c''},\]

where \( c', c'' \) are in \( F(TL_n(q)) \) and \( s \leq k + 1 \). By lemma 3.7 we see that this element is of the form:

\[\sum_{j=1}^{j=h} f_j (g_{\sigma_n \sigma_{n-1} \ldots \sigma_{1} a_{n+1}})^j + M,\]
where \( h \leq k + 1 \), \( f_j \) is in \( F(\hat{T}L_n(q)) \) for any \( j \) and \( M \) is a Markov element.

Hence, we see that in order to define \( \tau_{n+1} \) uniquely it is enough to have its values on Markov elements and its values on \( \Omega(g_{\sigma_n\sigma_{n-1}\ldots\sigma_1a_{n+1}}^k) \), where \( 1 \leq k \) (since if \( k \) is equal to 0 then we are again in the case of a Markov element) and \( \Omega \) is in \( F(\hat{T}L_n(q)) \).

**Lemma 3.8.** Let \( 2 \leq n \) then \( \tau_{n+1} \) is uniquely defined by its values on Markov elements, in addition to its values on \( (g_{\sigma_n\sigma_{n-1}\ldots\sigma_1a_{n+1}}^k) \), with \( 0 \leq k \).

**Proof.** In order to determine \( \tau_{n+1} = \tau_{n+1}(h(g_{\sigma_n\sigma_{n-1}\ldots\sigma_1a_{n+1}}^k)) \), with a positive \( k \) and an arbitrary \( h \) in \( F(\hat{T}L_n(q)) \), it is enough to treat \( \tau_{n+1}(F(t_x)(g_{\sigma_n\sigma_{n-1}\ldots\sigma_1a_{n+1}}^k)) \), with \( x \) in \( W^c(A_{n-1}^-) \), but the fact that \( \tau_{n+1} \) is a trace, in addition to the fact that \( g_{\sigma_n\sigma_{n-1}\ldots\sigma_1a_{n+1}} \) acts as a Dynkin automorphism on \( F(\hat{T}L_n(q)) \), authorizes us to suppose that \( x \) has a reduced expression which ends with \( \sigma_{n-1} \).

Now we show by induction on \( l(x) \), that \( \tau_{n+1}(F(t_x)(g_{\sigma_n\sigma_{n-1}\ldots\sigma_1a_{n+1}}^k)) \) is a sum of values of \( \tau_{n+1} \) over \( (g_{\sigma_n\sigma_{n-1}\ldots\sigma_1a_{n+1}}^k) \), elements of the form \( h(g_{\sigma_n\sigma_{n-1}\ldots\sigma_1a_{n+1}}^i) \) with \( i < k \) and Markov elements, (of course with coefficients in the ground ring which might be zeros).

For \( l(x) = 0 \) the property is true. Take \( l(x) > 0 \), and let \( x = z\sigma_{n-1} \) be a reduced expression, hence:

\[
\tau_{n+1}(F(t_x)(g_{\sigma_n\sigma_{n-1}\ldots\sigma_1a_{n+1}}^k)) = \tau_{n+1}(F(t_{z})(F(t_{\sigma_{n-1}})g_{\sigma_n\sigma_{n-1}\ldots\sigma_1a_{n+1}}(g_{\sigma_n\sigma_{n-1}\ldots\sigma_1a_{n+1}}^k-1)))
\]

\[
= \tau_{n+1}(F(t_{z})g_{\sigma_n\sigma_{n-1}}g_{\sigma_n}g_{\sigma_n-2\ldots\sigma_2a_{n+1}}(g_{\sigma_n\sigma_{n-1}\ldots\sigma_1a_{n+1}}^k-1)).
\]

This is equal to the following sum:

\[
- \tau_{n+1}(F(t_{z})(g_{\sigma_n\sigma_{n-1}\ldots\sigma_1a_{n+1}}^k))
\]

\[
- \tau_{n+1}(F(t_{z})g_{\sigma_n\sigma_{n-1}}g_{\sigma_n-2\ldots\sigma_2a_{n+1}}(g_{\sigma_n\sigma_{n-1}\ldots\sigma_1a_{n+1}}^k-1))
\]

\[
- \tau_{n+1}(F(t_{z})g_{\sigma_n\sigma_{n-2}}g_{\sigma_n\sigma_{n-1}\ldots\sigma_1a_{n+1}}^k-1))
\]

\[
- \tau_{n+1}(F(t_{z})g_{\sigma_n\sigma_{n-2}}g_{\sigma_n\sigma_{n-1}}}g_{\sigma_n\sigma_{n-2\ldots\sigma_2a_{n+1}}^k-1))
\]

\[
- \tau_{n+1}(F(t_{z})g_{\sigma_n\sigma_{n-2\ldots\sigma_2a_{n+1}}^k-1)).
\]
Now we apply the induction hypothesis to the first term. The second and the third terms are equal to:

\[
\tau_{n+1} \left( F(t)g_{\sigma_{n-1}}g_{\sigma_{n-2}}F(t_{an})g_{\sigma_n}F((t_{an})^{-1}) \left( g_{\sigma_{n-1}\sigma_1a_{n+1}} \right)^{k-1} \right) \\
+ \tau_{n+1} \left( F(t)g_{\sigma_{n-2}}F(t_{an})g_{\sigma_n}F((t_{an})^{-1}) \left( g_{\sigma_{n-1}\sigma_1a_{n+1}} \right)^{k-1} \right),
\]

which is equal to:

\[
\tau_{n+1} \left( \psi^{1-k}\left[ F((t_{an})^{-1}) \right] F(t)g_{\sigma_{n-1}}g_{\sigma_{n-2}}F(t_{an}) \left( g_{\sigma_n}g_{\sigma_{n-1}\sigma_1a_{n+1}}^{k-1} \right) \right) \\
+ \tau_{n+1} \left( \psi^{1-k}\left[ F((t_{an})^{-1}) \right] F(t)g_{\sigma_{n-2}}F(t_{an}) \left( g_{\sigma_n}g_{\sigma_{n-1}\sigma_1a_{n+1}}^{k-1} \right) \right).
\]

The fourth and the fifth terms are equal to:

\[
\tau_{n+1} \left( F(t)g_{\sigma_{n-1}}g_{\sigma_{n-2}}F(t_{an}) \left( g_{\sigma_n}g_{\sigma_{n-1}\sigma_1a_{n+1}}^{k-1} \right) \right) \\
+ \tau_{n+1} \left( F(t)g_{\sigma_{n-2}}F(t_{an}) \left( g_{\sigma_n}g_{\sigma_{n-1}\sigma_1a_{n+1}}^{k-1} \right) \right).
\]

Thus, lemma 3.7 tells us that the property is true for those four terms. This step is to be applied repeatedly, to the powers of \( g_{\sigma_{n-1}\sigma_1a_{n+1}} \) down to an element of the form \( \tau_{n+1}(h(g_{\sigma_{n-1}\sigma_1a_{n+1}})^1) \), arriving to the sum of:

\[
\tau_{n+1}(g_{\sigma_{n-1}\sigma_1a_{n+1}})
\]

and

\[
\tau_{n+1}(h'g_{\sigma_{n-1}\sigma_1a_{n+1}}),
\]

which is the sum of values of \( \tau_{n+1} \) on Markov elements, since \( h, h' \in F(TL_n(q)) \). \( \square \)

We end this part by the following lemma:

**Lemma 3.9.** Let \( 1 \leq k \). Then \( g_{\sigma_{n-1}\sigma_1a_{n+1}}^k \) is a sum of two kinds of elements:

1. \( g_{\sigma_n}(g_{\sigma_{n-1}\sigma_2a_{1}}F(t_{an}))^j g_{\sigma_n}h, \) with \( j \leq k \).
2. \( (g_{\sigma_{n-1}\sigma_2a_{1}}F(t_{an}))^i g_{\sigma_n}f, \) with \( i < k \).

with \( h, f \ in F(TL_n(q)) \) and \( 2 \leq n \).

Moreover, in the first type we have one, and only one element, with \( j = k \), in which we have:
Proof. Suppose that $k = 1$. Then,
\[ g_{\sigma_n}g_{\sigma_{n-1}\ldots\sigma_{1\alpha_{n+1}}} = g_{\sigma_n}\left(g_{\sigma_{n-1}\sigma_{n-2}\ldots\sigma_1}F(t_{\alpha_n})\right)\sigma_nF(t_{\alpha_n})^{-1}, \]
the property is true.

Suppose the property is true for $k - 1$, then, with $2 \leq k$, we have:
\[ (g_{\sigma_n}g_{\sigma_{n-1}\ldots\sigma_{1\alpha_{n+1}}})^k = (g_{\sigma_n}g_{\sigma_{n-1}\ldots\sigma_{1\alpha_{n+1}}})^{k-1}g_{\sigma_n}g_{\sigma_{n-1}\ldots\sigma_{1\alpha_{n+1}}}. \]

We apply the property to $(g_{\sigma_n}g_{\sigma_{n-1}\ldots\sigma_{1\alpha_{n+1}}})^{k-1}$, which gives two cases:

1. $g_{\sigma_n}\left(g_{\sigma_{n-1}\sigma_{n-2}\ldots\sigma_1}F(t_{\alpha_n})\right)^{j'}g_{\sigma_n}hg_{\sigma_n}g_{\sigma_{n-1}\ldots\sigma_{1\alpha_{n+1}}}$, with $j' \leq k - 1$ which is:
\[ qg_{\sigma_n}\left(g_{\sigma_{n-1}\sigma_{n-2}\ldots\sigma_1}F(t_{\alpha_n})\right)^{j'+1}g_{\sigma_n}F((t_{\alpha_n})^{-1})\psi^{-1}[h] + (q - 1)g_{\sigma_n}g_{\sigma_n}g_{\sigma_{n-1}\ldots\sigma_{1\alpha_{n+1}}}\psi^{-1}\left([g_{\sigma_{n-1}\sigma_{n-2}\ldots\sigma_1}F(t_{\alpha_n})]^{j'}\right)\psi^{-1}[h]. \]

Since, $j' + 1 \leq k$, the first term is clear to be of the first type, while the second term is equal to:
\[ (q - 1)g_{\sigma_n}g_{\sigma_{n-1}\ldots\sigma_{1\alpha_{n+1}}}F(t_{\alpha_n})g_{\sigma_n}F((t_{\alpha_n})^{-1})\psi^{-1}\left([g_{\sigma_{n-1}\sigma_{n-2}\ldots\sigma_1}F(t_{\alpha_n})]^{j'}\right)\psi^{-1}[h] + (q - 1)^2g_{\sigma_n}g_{\sigma_{n-1}\ldots\sigma_{1\alpha_{n+1}}}g_{\sigma_n}g_{\sigma_{n-1}\ldots\sigma_{1\alpha_{n+1}}}\psi^{-1}\left([g_{\sigma_{n-1}\sigma_{n-2}\ldots\sigma_1}F(t_{\alpha_n})]^{j'}\right)\psi^{-1}[h]. \]

Here, the first term is of the second type (with $i = 1 < k$), and the second term is of the first type (with $j = 1$).

2. $(g_{\sigma_{n-1}\sigma_{n-2}\ldots\sigma_1}F(t_{\alpha_n}))^{i'}g_{\sigma_n}g_{\sigma_n}g_{\sigma_{n-1}\ldots\sigma_{1\alpha_{n+1}}}$, with $i' < k - 1$, which is:
\[ q\left(g_{\sigma_{n-1}\sigma_{n-2}\ldots\sigma_1}F(t_{\alpha_n})\right)^{i'+1}g_{\sigma_n}F((t_{\alpha_n})^{-1})\psi^{-1}[f] + (q - 1)g_{\sigma_n}g_{\sigma_{n-1}\ldots\sigma_1}F(t_{\alpha_n})g_{\sigma_n}F((t_{\alpha_n})^{-1})\psi^{-1}\left([g_{\sigma_{n-1}\sigma_{n-2}\ldots\sigma_1}F(t_{\alpha_n})]^{i'}\right)\psi^{-1}[f]. \]
Since $i' + 1 < k$, the first term is of the second type, while the second term is of the first type $j = 1$. The lemma is proven.

(By induction over $k$ again, the last formula is easy).

\[\square\]

**Part 2**

In this part we treat theorem 3.6 when $n \geq 3$. As said at the beginning of Part 1, for $n \geq 3$, and by sending the “fully commutative braids” onto $\widehat{TL}_{n+1}(q)$, we get that any element of the basis of $\widehat{TL}_{n+1}(q)$ is a linear combination of two kinds of elements, namely:

\[I = F_n(t_u)(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^k g_{\sigma_n\sigma_{n-1}..\sigma_s},\]

\[II = g_{\sigma_{i_0}..\sigma_1a_{n+1}} F_n(t_u)(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^k g_{\sigma_n\sigma_{n-1}..\sigma_s},\]

here, $u$ is in $W^c(A_{n-1}^-)$, where $1 \leq s \leq n + 1$ with $0 \leq i_0 \leq n - 1$ and $0 \leq k$.

By lemma 3.7 we see that:

\[\left(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}}\right)^k g_{\sigma_n} = (q - 1)\left(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}}\right)^k\]

\[+ \sum_{i=1}^{i=k-1} h_i \left(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}}\right)^i\]

\[+ A \prod_{j=0}^{j=k-1} \phi^j (\sigma_{n-1})^{-1} g_{\sigma_n} \left(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1\sigma_{n+1}} F(t_{a_n})\right)^k g_{\sigma_{n-1}..\sigma_s},\]

but, $I = F_n(t_u)(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^k g_{\sigma_n} g_{\sigma_{n-1}..\sigma_s}$, that is:

\[I = (q - 1) F_n(t_u)(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^k g_{\sigma_n\sigma_{n-1}..\sigma_s}\]

\[+ \sum_{i=1}^{i=k-1} h_i F_n(t_u)(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^i g_{\sigma_n\sigma_{n-1}..\sigma_s}\]

\[+ A \prod_{j=0}^{j=k-1} F_n(t_u) \phi^j (\sigma_{n-1})^{-1} g_{\sigma_n} \left(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1\sigma_{n+1}} F(t_{a_n})\right)^k g_{\sigma_{n-1}..\sigma_s}.\]

Using the action of $g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}}$ on $F_n(\widehat{TL}_n(q))$, we see that:

\[I = \sum_{i=1}^{i=k} F_n(t_{b_i}) (g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^i + \sum_j F_n(t_{b_j}) g_{\sigma_n} F_n(t_{d_j}),\]
where $b_j$, $c_j$ and $d_i$ are in $W^c(A_{n-1}^\sim)$, for every $i$ and $j$.

Now, we see, as well, that:

$$II = g_{\sigma_{i_0}..\sigma_2\sigma_1}a_{n+1}F_n(t_u)\left(g_{\sigma_{n-1}..\sigma_1a_{n+1}}\right)^k g_{\sigma_{n-1}..\sigma_s}$$

$$= g_{\sigma_{i_0}..\sigma_2\sigma_1}F_n(t_{a_n})g_{\sigma_{n-1}..\sigma_1a_{n+1}}\left(g_{\sigma_{n-1}..\sigma_s}\right)^k \psi^k \left\{F_n(t_{a_n})F_n(t_u)\right\} g_{\sigma_{n-1}..\sigma_s},$$

since $a_{n+1} = F_n(t_{a_n})g_{\sigma_{a_n}}F_n(t_{a_n})$.

By lemma 3.7, we see that $II$ is equal to:

$$(q-1)g_{\sigma_{i_0}..\sigma_2\sigma_1}F_n(t_{a_n})\left(g_{\sigma_{n-1}..\sigma_1a_{n+1}}\right)^k \psi^k \left\{F_n(t_{a_n})F_n(t_u)\right\} g_{\sigma_{n-1}..\sigma_s}$$

$$+ \sum_{i=1}^{i=k-1} g_{\sigma_{i_0}..\sigma_2\sigma_1}F_n(t_{a_n})f_i\left(g_{\sigma_{n-1}..\sigma_1a_{n+1}}\right)^i \psi^k \left\{F_n(t_{a_n})F_n(t_u)\right\} g_{\sigma_{n-1}..\sigma_s}$$

$$+ Ag_{\sigma_{i_0}..\sigma_2\sigma_1}F_n(t_{a_n})\left(g_{\sigma_{n-1}..\sigma_1F(t_{a_n})}\right)^k g_{\sigma_{n-1}..\sigma_s} \prod_{j=0}^{j=k-1} \psi^j \left\{F(t_{a_n})^{-1}\right\} \psi^k \left\{F_n(t_{a_n})F_n(t_u)\right\} g_{\sigma_{n-1}..\sigma_s},$$

which is equal to:

$$+ \sum_{i=1}^{i=k} \left\{F_n(t_{x_i})\left(g_{\sigma_{n-1}..\sigma_1a_{n+1}}\right)^i g_{\sigma_{n-1}..\sigma_s}\right\}$$

$$+ Ag_{\sigma_{i_0}..\sigma_2\sigma_1}F_n(t_{a_n})\left(g_{\sigma_{n-1}..\sigma_1F(t_{a_n})}\right)^k g_{\sigma_{n-1}..\sigma_s} \prod_{j=0}^{j=k-1} \psi^j \left\{F(t_{a_n})^{-1}\right\} \psi^k \left\{F_n(t_{a_n})F_n(t_u)\right\} g_{\sigma_{n-1}..\sigma_s},$$

where $x_i$ is in $W^c(A_{n-1}^\sim)$ for all $i$.

Now we repeat the same step as for $I$, to get the next corollary.

**Corollary 3.10.** Let $3 \leq n$. Let $w$ be in $W^c(A_n)$ .

Then there exist $0 \leq k$ and $1 \leq s \leq n + 1$. There exist $x_i$, $y_i$ and $z_i$ in $W^c(A_{n-1}^\sim)$ such that:

$$g_w = \sum_{i=1}^{i=k} \left\{F_n(t_{x_i})\left(g_{\sigma_{n-1}..\sigma_1a_{n+1}}\right)^i \sum_{j} \left\{F_n(t_{y_j})g_{\sigma_{a_n}F(t_{z_j})}\right\} g_{\sigma_{n-1}..\sigma_s}\right\}.$$
We keep using \( g_{\sigma_i} \) (resp. \( t_{\sigma_i} \)) as generators of \( \tilde{T}L_n(q) \) (resp. \( \tilde{T}L_{n+1}(q) \)). We use \( e_{\sigma_i} \) for \( \tilde{T}L_{n-1}(q) \). With a simple computation, we see that \( g_{\sigma_n} \) commutes with \( F_nF_{n-1}(e_{\sigma_i}) \), for \( 1 \leq i \leq n - 2 \), and with \( F_nF_{n-1}(e_{\sigma_{n-1}}) \), hence it commutes with every element in \( F_nF_{n-1}(\tilde{T}L_{n-1}(q)) \).

Lemma 3.8 and lemma 3.9 confirm that \( \tau_{n+1} \) is uniquely defined over \( \tilde{T}L_{n+1}(q) \) by its values on \( g_{\sigma_n}(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n}))^kg_{\sigma_n}h \), for a positive \( k \) and an arbitrary \( h \) in \( F(\tilde{T}L_n(q)) \). In other terms: \( \tau_{n+1} \) is uniquely defined over \( \tilde{T}L_{n+1}(q) \) by its values over \( g_{\sigma_n}(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n}))^kg_{\sigma_n}F_{n}(t_{v}) \), with a positive \( k \) and an arbitrary \( v \) in \( W^c(A_{n-1}) \).

Set \( I := \tau_{n+1}\left(g_{\sigma_n}(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n}))^kg_{\sigma_n}F_{n}(t_{v})\right) \),

by corollary 3.10 we see that:

\[
\begin{align*}
t_v &= \sum_{i=1}^{h} F_{n-1}(e_{x_i})(t_{\sigma_{n-1}\sigma_{n-2}..\sigma_1a_n})^i \\
&+ \sum_j F_{n-1}(e_{y_j})t_{\sigma_{n-1}}F_{n-1}(e_{z_j})t_{\sigma_{n-1}\sigma_{n-2}..\sigma_s} \\
&+ \sum_j F_{n-1}(e_{y'_j})t_{\sigma_{n-1}}F_{n-1}(e_{z'_j}),
\end{align*}
\]

where \( 0 \leq h \) and \( 1 \leq s \leq n - 1 \). With \( x_i \), \( y_i \), \( z_i \), \( y'_i \) and \( z'_i \) are in \( W^c(A_{n-2}) \).

We have added the third term \( C \) to the two terms of corollary 3.10, because we had to take into account here, the case of \( s = n + 1 \), i.e., \( g_{\sigma_{n+1}} = 1 \) for \( W^c(A_{n-1}) \).

For terms of Type (A), we see that:

\[
I_1 := \tau_{n+1}\left(g_{\sigma_n}(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n}))^kg_{\sigma_n}F_{n}(F_{n-1}(e_{y'_j})t_{\sigma_{n-1}}F_{n-1}(e_{z'_j})))\right)
\]

\[
= \tau_{n+1}\left((g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n}))^kF_n(F_{n-1}(e_{y'_j}))g_{\sigma_n}F_{n}(t_{\sigma_{n-1}})g_{\sigma_n}F_{n}(F_{n-1}(e_{z'_j})))\right)
\]

\[
= \tau_{n+1}\left((g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n}))^kF_n(F_{n-1}(e_{y'_j}))g_{\sigma_n}g_{\sigma_{n-1}}g_{\sigma_n}F_{n}(F_{n-1}(e_{z'_j})))\right)
\]

\[
= \tau_{n+1}\left((g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n}))^kF_n(F_{n-1}(e_{y'_j}))g_{\sigma_{n-1}}g_{\sigma_n}g_{\sigma_{n-1}}F_{n}(F_{n-1}(e_{z'_j})))\right).
\]
which is clearly, the sum of values of $\tau_{n+1}$ on Markov elements, and elements in $F_n(\overline{TL}_n(q))$.

For terms of **Type (B)**, we see that:

$$I_2 := \tau_{n+1} \left[ g_{\sigma_n} \left( g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1} F(t_{a_n}) \right)^k g_{\sigma_n} F_n \left( F_{n-1}(e_{y_j}) t_{\sigma_{n-1}F_{n-1}(e_{z_j})t_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}} \right) \right]$$

$$= \tau_{n+1} \left[ g_{\sigma_n} \left( g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1} F(t_{a_n}) \right)^k g_{\sigma_n} F_n \left( F_{n-1}(e_{y_j}) t_{\sigma_{n-1}F_{n-1}(e_{z_j})t_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}} \right) \right]$$

$$= \tau_{n+1} \left[ g_{\sigma_n} F_n \left( e_{\sigma_{n-2}..\sigma_1} \right) (g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1} F(t_{a_n}))^k g_{\sigma_n} F_n \left( F_{n-1}(e_{y_j}) t_{\sigma_{n-1}F_{n-1}(e_{z_j})t_{\sigma_{n-1}} \sigma_{n-2}..\sigma_1} \right) \right].$$

Now, we set $m \cdot F := F_m F_{m-1}..F_r$.

We call $\delta$ the image of $F_{n-1}(e_{\sigma_{n-2}..\sigma_1})$ under the action of $\left( g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1} F(t_{a_n}) \right)^k$, thus:

$$I_2 = \tau_{n+1} \left[ g_{\sigma_n} \left( g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1} F(t_{a_n}) \right)^k g_{\sigma_n} F_n \left( \delta \left( \binom{n}{n-1} F(e_{y_j}) \right) t_{\sigma_{n-1} \left( \binom{n}{n-1} F(e_{z_j}) \right)} g_{\sigma_{n-1}} \right) \right]$$

$$= \tau_{n+1} \left[ g_{\sigma_n} \left( F_n \left( t_{\sigma_{n-1}\sigma_{n-2}..\sigma_1 a_n} \right) \right)^k g_{\sigma_n} F_n \left( \delta \left( \binom{n}{n-1} F(e_{y_j}) \right) t_{\sigma_{n-1} \left( \binom{n}{n-1} F(e_{z_j}) \right)} g_{\sigma_{n-1}} \right) \right].$$

Now consider $(t_{\sigma_{n-1}\sigma_{n-2}..\sigma_1 a_n})^k$. We apply lemma 3.9 to this element in $\overline{TL}_n(q)$, hence, it is the sum of two kind of elements: (1) Markov elements (2) elements of the form $t_{\sigma_{n-1}(e_{\sigma_{n-2}..\sigma_1 a_n})^j} t_{\sigma_{n-1}} \delta$, where $j \leq k$, and $\delta$ in $F_{n-1}(\overline{TL}_{n-1}(q))$. In the case (1) we are done. If we are in case (2), then we apply the lemma 3.9 on $(e_{\sigma_{n-2}..\sigma_1 a_{n-1}})^j$. We keep going in the same manner, by applying lemma 3.9 repeatedly (in fact $n-2$ times), we arrive to:

$$t_{\sigma_{n-1} t_{\sigma_{n-2}..\sigma_2} \left( F_{n-1} F_{n-2}..F_2(2g_{\sigma_1 a_2})^j \right) t_{\sigma_2}..t_{\sigma_{n-2}} t_{\sigma_{n-1} \lambda}}$$

$$= t_{\sigma_{n-1} t_{\sigma_{n-2}..\sigma_2} \left( \binom{n}{n-1} F(2g_{\sigma_1 a_2})^j \right) t_{\sigma_2}..t_{\sigma_{n-2}} t_{\sigma_{n-1} \lambda}},$$

where $\lambda$ is in $\binom{n}{n-1} F(\overline{TL}_{n-1}(q))$. We get:

$$I_2 = \tau_{n+1} \left[ g_{\sigma_n} F_n \left( t_{\sigma_{n-1} t_{\sigma_{n-2}..\sigma_2} \left( \binom{n}{n-1} F(2g_{\sigma_1 a_2})^j \right) t_{\sigma_2}..t_{\sigma_{n-2}} t_{\sigma_{n-1} \lambda}} \right) g_{\sigma_n} F_n \left( \delta \right) \right]$$

$$\left( \binom{n}{n-1} F(e_{y_j}) \right) t_{\sigma_{n-1} \left( \binom{n}{n-1} F(e_{z_j}) \right)} g_{\sigma_{n-1}} \right]$$

$$= \tau_{n+1} \left[ g_{\sigma_n} g_{\sigma_{n-1}} g_{\sigma_{n-2}..g_{\sigma_2} \left( \binom{n}{n-1} F(2g_{\sigma_1 a_2})^j \right) g_{\sigma_2}..g_{\sigma_{n-2}} g_{\sigma_{n-1}} F_n \left( \lambda \delta \right) \right]$$

$$\left( \binom{n}{n-1} F(e_{y_j}) \right) t_{\sigma_{n-1} \left( \binom{n}{n-1} F(e_{z_j}) \right)} g_{\sigma_{n-1}} \right].$$
We set $M' := F_n\left(\lambda \delta \right)^{\alpha_{n-1} F(e_{y_1})} F_{\sigma_n-1}^{\alpha_{n-1} F(e_{z_j})}$, which is a Markov element in $\overline{TL}_{n-1}(q)$. Hence, we have:

$$
\tau_{n+1} \left[ g_{\sigma_n} g_{\sigma_{n-1}} g_{\sigma_{n-2}} \cdots g_{\sigma_2} \left( F^{(n-1)}(2^g_{\sigma_{1a_2}}) \right) g_{\sigma_2} \cdots g_{\sigma_{n-2}} g_{\sigma_{n-1}} g_{\sigma_n} M' g_{\sigma_{n-1}} \right]
$$

$$
= \tau_{n+1} \left[ g_{\sigma_{n-1}} g_{\sigma_n} g_{\sigma_{n-2}} \cdots g_{\sigma_2} \left( F^{(n-1)}(2^g_{\sigma_{1a_2}}) \right) g_{\sigma_2} \cdots g_{\sigma_{n-2}} g_{\sigma_{n-1}} g_{\sigma_n} M' \right].
$$

We apply the TL relations. The cases corresponding to 1 and $g_{\sigma_{n-1}}$ are obvious.

For the terms corresponding to $g_{\sigma_{n-1}} g_{\sigma_n}$, we have:

$$
\tau_{n+1} \left[ g_{\sigma_{n-1}} g_{\sigma_n} g_{\sigma_{n-2}} \cdots g_{\sigma_2} \left( F^{(n-1)}(2^g_{\sigma_{1a_2}}) \right) g_{\sigma_2} \cdots g_{\sigma_{n-2}} g_{\sigma_{n-1}} g_{\sigma_n} M' \right]
$$

$$
= \tau_{n+1} \left[ g_{\sigma_{n-1}} g_{\sigma_n} g_{\sigma_{n-2}} \cdots g_{\sigma_2} \left( F^{(n-1)}(2^g_{\sigma_{1a_2}}) \right) g_{\sigma_2} \cdots g_{\sigma_{n-2}} g_{\sigma_{n-1}} g_{\sigma_n} M' \right].
$$

We are done, since it is a sum of values of $\tau_{n+1}$ on Markov elements, and elements in $F_n(\overline{TL}_n(q))$. (the same for the term corresponding to $g_{\sigma_n}$).

For the terms corresponding to $g_{\sigma_n} g_{\sigma_{n-1}}$, we have:

$$
\tau_{n+1} \left[ g_{\sigma_n} g_{\sigma_{n-1}} g_{\sigma_{n-2}} \cdots g_{\sigma_2} \left( F^{(n-1)}(2^g_{\sigma_{1a_2}}) \right) g_{\sigma_2} \cdots g_{\sigma_{n-2}} g_{\sigma_{n-1}} g_{\sigma_n} M' \right],
$$

which is the case of term (A), since $M'$ is a Markov element.

For terms of Type (C), we see that:

$$
I_3 := \tau_{n+1} \left[ g_{\sigma_n} \left( g_{\sigma_{n-1} \sigma_{n-2} \cdots \sigma_1} F_{n}^k \left( t_{a_n} \right) \right) g_{\sigma_n} F_n \left( F_{n-1}(e_{x_i}) \left( t_{\sigma_{n-1} \sigma_{n-2} \cdots \sigma_1 a_n} \right)^i \right) \right]
$$

$$
= \tau_{n+1} \left[ g_{\sigma_n} F_n \left( (t_{\sigma_{n-1} \sigma_{n-2} \cdots \sigma_1 a_n})^k \right) g_{\sigma_n} F_n \left( F_{n-1}(e_{x_i}) \left( t_{\sigma_{n-1} \sigma_{n-2} \cdots \sigma_1 a_n} \right)^i \right) \right].
$$

Call $\gamma$ the image of $F_{n-1}(e_{x_i})$ under the action of $(t_{\sigma_{n-1} \sigma_{n-2} \cdots \sigma_1 a_n})^i$. Thus:

$$
I_3 = \tau_{n+1} \left[ g_{\sigma_n} F_n \left( (t_{\sigma_{n-1} \sigma_{n-2} \cdots \sigma_1 a_n})^k \right) g_{\sigma_n} F_n \left( (t_{\sigma_{n-1} \sigma_{n-2} \cdots \sigma_1 a_n})^i \gamma \right) \right].
$$

As we have seen in the case (B), $(t_{\sigma_{n-1} \sigma_{n-2} \cdots \sigma_1 a_n})^k$ can be written as some of elements of the form:

$$
t_{\sigma_{n-1} t_{\sigma_{n-2} \cdots t_{\sigma_2} \left( F^{(n-1)}(2^g_{\sigma_{1a_2}}) \right) t_{\sigma_2} \cdots t_{\sigma_{n-2}} t_{\sigma_{n-1}}},
$$
where \( j \leq k \), and \( \lambda \) is in \( \frac{n}{n-1}F(\overline{T\overline{L}}_{n-1}(q)) \).

Call \( \eta \) the image of \( \lambda \) under the action of \((t_{\sigma_{n-1}\sigma_{n-2}..\sigma_1})^i\).

The determination of \( I_3 \) can be reduced to computing the following value:

\[
\tau_{n+1} \left[ \left( g_{\sigma_n}g_{\sigma_{n-1}}g_{\sigma_{n-2}}..g_{\sigma_2} \left( \binom{n-1}{2} F(2g_{\sigma_1\sigma_2})^i \right) g_{\sigma_2}..g_{\sigma_{n-2}}g_{\sigma_{n-1}} g_{\sigma_n} \left( t_{\sigma_{n-1}\sigma_{n-2}..\sigma_1} \right)^i \eta \right) \right].
\]

We repeat the same algorithm to \((t_{\sigma_{n-1}\sigma_{n-2}..\sigma_1})^i\). Hence, we get some \( l \leq i \), and some \( \Delta \) in \( \frac{n}{n-1}F(\overline{T\overline{L}}_{n-1}(q)) \), such that we are reduced to compute:

\[
\tau_{n+1} \left[ \left( g_{\sigma_n}g_{\sigma_{n-1}}g_{\sigma_{n-2}}..g_{\sigma_2} \left( \binom{n-1}{2} F(2g_{\sigma_1\sigma_2})^i \right) g_{\sigma_2}..g_{\sigma_{n-2}}g_{\sigma_{n-1}} \left( t_{\sigma_{n-1}\sigma_{n-2}..\sigma_2} \right)^l \left( g_{\sigma_{n-2}}g_{\sigma_{n-1}} .. g_{\sigma_2} \right)^l \right) \right].
\]

We see, after using the T-L relations, that the terms corresponding to 1 and \( g_{\sigma_{n-1}} \) are values of \( \tau_{n+1} \) on Markov elements.

The term corresponding to \( g_{\sigma_{n-1}}g_{\sigma_n} \) is:

\[
\tau_{n+1} \left[ \left( g_{\sigma_n}g_{\sigma_{n-1}}g_{\sigma_{n-2}}..g_{\sigma_2} \left( \binom{n-1}{2} F(2g_{\sigma_1\sigma_2})^i \right) g_{\sigma_2}..g_{\sigma_{n-2}}g_{\sigma_{n-1}} \left( t_{\sigma_{n-1}\sigma_{n-2}..\sigma_2} \right)^l \right) g_{\sigma_{n-2}}g_{\sigma_{n-1}} \eta \right].
\]

The term in square brackets is clearly a Markov element (the same thing with the term corresponding to \( g_{\sigma_n} \)).
The term corresponding to \( g_\sigma g_{\sigma_{n-1}} \) is:

\[
\tau_{n+1} \left[ g_\sigma g_{\sigma_{n-1}} g_{\sigma_{n-2}} \cdots g_{\sigma_2} \left( \frac{n-1}{2} F(g_{\sigma_2})^j \right) g_{\sigma_2} \cdots g_{\sigma_{n-2}} g_{\sigma_{n-1}} g_{\sigma_{n-2}} \cdots g_{\sigma_2} \right. \\
\left. \left( \frac{n-1}{2} F(g_{\sigma_2})^j \right) g_{\sigma_2} \cdots g_{\sigma_{n-2}} g_{\sigma_{n-1}} \right] \\
= \tau_{n+1} \left[ g_\sigma g_{\sigma_{n-1}} g_{\sigma_n} g_{\sigma_{n-2}} \cdots g_{\sigma_2} \left( \frac{n-1}{2} F(g_{\sigma_2})^j \right) g_{\sigma_2} \cdots g_{\sigma_{n-2}} g_{\sigma_{n-1}} g_{\sigma_{n-2}} \cdots g_{\sigma_2} \right. \\
\left. \left( \frac{n-1}{2} F(g_{\sigma_2})^j \right) g_{\sigma_2} \cdots g_{\sigma_{n-2}} g_{\sigma_{n-1}} \right].
\]

It is a Markov element, theorem 3.6 follows.

4. **Affine Markov Trace: Existence and Uniqueness**

4.1. **Existence.** Now, consider the following commutative diagram:

\[
\begin{array}{ccccccc}
\widehat{TL}_1(q) & \rightarrow & \widehat{TL}_2(q) & \rightarrow & \ldots & \rightarrow & \widehat{TL}_n(q) & \rightarrow & \widehat{TL}_{n+1}(q) \\
\downarrow & & \downarrow & & \ldots & & \downarrow & & \downarrow \\
TL_1(q) & \leftarrow & TL_2(q) & \leftarrow & \ldots & \leftarrow & TL_{n-1}(q) & \leftarrow & TL_n(q) \\
\end{array}
\]

Set \( \rho_{n+1} \) to be the trace over \( \widehat{TL}_{n+1}(q) \) induced by \( \tau_{n+1} \) over \( TL_n(q) \) for \( 0 \leq n \).

**Proposition 4.1.** Under the above notations, we have:

- \( \rho_{n+1}(F_n(h)T_{\sigma_n}^{\pm1}) = \rho_n(h) \) for all \( h \in \widehat{TL}_n(q) \). Where \( 1 \leq n \).
- \( \rho_i \) is invariant the action of \( \phi_i \) for all \( i \).

**Proof.** We have: \( \rho_{n+1}(F_n(h)T_{\sigma_n}^{\pm1}) \) equals \( \tau_{n+1} \left( e'_n(F_n(h)) e'_n(T_{\sigma_n}^{\pm1}) \right) \).

Hence, \( \rho_{n+1}(F_n(h)T_{\sigma_n}^{\pm1}) = \tau_{n+1} \left( x_n(e'_{n-1}(h)) T_{\sigma_n}^{\pm1} \right) = \tau_n(e'_{n-1}(h)) = \rho_n(h) \).
We made use of the fact that the following diagram commutes, together with the fact that $\tau_n$ is a Markov trace:

For the second statement, we show that $\rho_n(h) = \rho_n([h])$, where $[h]$ is the image of $h$ under $\phi_n^{-1}$. So we start from $\rho_n(h) = \tau_n(e'_n-1(h))$. But since $\tau_n$ is the $n$-th Markov trace, we have $\tau_n(e'_n-1(h)) = -\sqrt{q} \tau_n+1(e'_n(F_n(h)))$, which is equal to $-\sqrt{q} \tau_n+1(F_n([h]))$, since the diagram $T$ commutes, this term is equal to $-\sqrt{q} \tau_n+1(F_n([h])))$.

Now we consider the same steps in the opposite direction, that is:

$$-\sqrt{q} \rho_n+1(F_n([h])) = -\sqrt{q} \tau_n+1(\rho_n([h])).$$

\[\square\]

**Corollary 4.2.** With the above notations, in the sense of definition 2.3: $(\rho_i)_{1 \leq i}$ is an affine Markov trace over $(\tilde{TL}_i(q))_{1 \leq i}$.

4.2. **Uniqueness.** Consider the following algebras homomorphism:

$$F_2 : \tilde{TL}_2(q) \rightarrow \tilde{TL}_3(q)$$

$$g_{\sigma_1} \mapsto g_{\sigma_1}$$

$$g_{a_2} \mapsto g_{\sigma_2}g_{a_3}g_{\sigma_2}.$$
We set $F := F_2$ in order to simplify in what follows. $F$ can be expressed by the following form considering the 'f' generators, we see that $F(f_{a_2}) = F\left(\frac{g_{a_2} + 1}{q+1}\right)$, which is equal to $\frac{1}{q+1}g_{a_2}g_{a_3}g_{\sigma_1}^{-1} + \frac{1}{q+1}$, hence to:

$$\frac{1}{q+1}\left[\left((q + 1)f_{\sigma_1} - 1\right)\left((q + 1)f_{a_3} - 1\right)\left(\frac{1}{q}\left((q + 1)f_{\sigma_1} - 1\right) + \frac{1 - q}{q}\right)\right] + \frac{1}{q+1}.$$  

Thus, we see that:

$$F : \hat{T}L_2(q) \rightarrow \hat{T}L_3(q)$$

$$f_1 \mapsto f_{\sigma_1}$$

$$f_{a_2} \mapsto -\frac{q + 1}{q}f_{a_3}(q) - (q + 1)f_{\sigma_1} + f_{\sigma_1} + f_{a_3}.$$

Notice that $F(f_{a_2})f_{\sigma_1}F(f_{a_2}) = \delta F(f_{a_2})$, and $f_{\sigma_1}f_{\sigma_1}f_{\sigma_1} = \delta f_{\sigma_1}$. Since we are interested with viewing $F(\hat{T}L_2(q))$ in $\hat{T}L_3(q)$, we will investigate in what follows, the elements $(F(f_{\sigma_1}f_{a_2}))^k$ and $(F(f_{a_2}f_{\sigma_1}))^k$, for $k$ a positive integer.

Set $x_1 := F(f_{a_2}f_{\sigma_1}) = f_{\sigma_1}F(f_{a_2}) = -\frac{q + 1}{q}f_{\sigma_1} + f_{\sigma_1} + f_{a_3}$. And for $1 \leq i$, we set:

$$x_i := (-1)^i\left(\frac{q + 1}{q}\right)^i f_{\sigma_1}^i - (q + 1)f_{\sigma_1}^i + f_{\sigma_1} + f_{a_3}.

$$

Notice that $x_1 = 3\delta x_1 + x_2$. It is easy to show that:

$$x_i x_i = \delta^2 x_{i-1} + 2\delta x_i + x_{i+1}, \text{ for } 2 \leq i,$$

thus, for $1 \leq k$, we have $x_k = \sum_{i=1}^{i=k-1} \gamma_i x_i + x_k$, where $\gamma_i$ is a polynomial in $\delta$, for all $i$.

Notice that $x_1 x_j = x_1 x_j$ for $j = 1, 2$. For $j = 1$ it is clear, while for $j = 2$ we have $x_2 = x_1^2 - 3\delta x_1$. Now suppose that $3 \leq j$. We have $x_j = x_1 x_j - \delta^2 x_{j-1} - 2\delta x_j$, hence we see by induction on $j$, that $x_1 x_j = x_1 x_j$, for all $j$.

We define the Q-linear map $\chi : \hat{T}L_3(q) \rightarrow \hat{T}L_3(q)$ which sends 1 to 1, and for any $u = s_1s_2..s_r$ reduced expression of any element $u$ in $W^c(A_2)$, it sends $f_u$ to $f_{s_r s_{r-1}..s_1}$, with $q$ sent to $\frac{1}{q}$.

Set $z_1 := F(f_{a_2}f_{\sigma_1})$. Then

$$z_1 = F(f_{a_2}f_{\sigma_1})f_{\sigma_1} = -\frac{q + 1}{q}f_{\sigma_1}a_3 - (q + 1)f_{\sigma_1}a_3 + f_{\sigma_1} + f_{a_3}.$$
And for $1 \leq i$, we set
\[ z_i := (-1)^i \left( \frac{q+1}{q} \right)^i f_{a_1 a_2} + (-1)^i \left( \frac{q+1}{q} \right)^{i-1} f_{a_2 a_3} f_{a_3 a_1} + (-1)^{i-1} \left( \frac{q+1}{q} \right)^{i-1} f_{a_2 a_3} f_{a_3 a_1}. \]

Notice that $\chi(x_i) = z_i$ for all $i$. Now $\chi(x_1 x_i) = \chi(x_i x_1) = \chi(z_1 z_i)$. We see that $\chi(\delta) = \delta$. Moreover, $z_iz_j = \chi(x_1 x_j) = \chi(\delta^2 x_{i-1} + 2\delta x_i + x_{i+1}) = \delta^2 z_{i-1} + 2\delta z_i + z_{i+1}$. And in the same way, by acting by $\chi$, we find that $z_1^k = \sum_{i=1}^{k-1} \gamma_i z_i + z_k$, where $\gamma_i$ is as above.

Consider $x_if_{\sigma_2}$ for $1 \leq i$, we see that it is equal to:

\[ (-1)^i \left( \frac{q+1}{q} \right)^i f_{a_1 a_2} + (-1)^i \left( \frac{q+1}{q} \right)^{i-1} f_{a_2 a_3} f_{a_1 a_2} + (-1)^{i-1} \left( \frac{q+1}{q} \right)^{i-1} f_{a_2 a_3} f_{a_1 a_2}. \]

which is:

\[ (-1)^i \left( \frac{q+1}{q} \right)^i f_{a_1 a_2} + \delta(-1)^i \left( \frac{q+1}{q} \right)^{i-1} f_{a_2 a_3} f_{a_1 a_2} + (-1)^{i-1} \left( \frac{q+1}{q} \right)^{i-1} f_{a_2 a_3} f_{a_1 a_2}. \]

Hence, $x_if_{\sigma_2} = \left[ (-1)^{i-1} \left( \frac{q+1}{q} \right)^{i-1} \right] f_{a_1 a_2} + \left[ (-1)^{i-1} \left( \frac{q+1}{q} \right)^{i-2} \right] f_{a_2 a_3} f_{a_1 a_2}$. For $1 \leq i$.

In particular $x_1f_{\sigma_2} = -\frac{q+1}{q} f_{a_1 a_2} - (q+1) f_{a_2 a_3} f_{a_3 a_2} + f_{a_1 a_2} + f_{a_2 a_3} f_{a_3 a_2}$, thus,

\[ x_1f_{\sigma_2} = -\frac{1}{q} f_{a_1 a_2} + \frac{1}{q+1} f_{a_1 a_2}. \]

Now we apply $\chi$ to $x_if_{\sigma_2}$. Hence

\[ f_{\sigma_2} z_i = \left[ (-1)^i q(q+1)^{i-1} \right] f_{a_2 a_3} f_{a_3 a_1} + \left[ (-1)^{i-1} \left( \frac{q+1}{q} \right)^{i-2} \right] f_{a_2 a_3} f_{a_3 a_2}. \]

In particular $f_{\sigma_2} z_1 = -q f_{a_2 a_3} f_{a_3 a_1} + \frac{q}{q+1} f_{a_2 a_3} f_{a_3 a_1}$.

Take $t$ to be any $\psi_2$-invariant trace over $T_2(q)$, determined by $A_0, A_1$ and $(\alpha_i)_{1 \leq i}$. Let $s$ be any $\psi_3$-invariant trace over $T_2(q)$, determined by $B_0, B_1, B_2$ and $(\beta_i)_{1 \leq i}$. We show in what follows that there are a unique $t$ and a unique $s$, such that $t$ is the second component and $s$ is the third component of a Markov trace. So, in order to simplify, we set $\hat{\tau}_2 := t$ and $\hat{\tau}_3 := s$. 
At first, being a first component of a Markov trace, forces \( \hat{\tau}_2 \) to have the value 1 over \( T_{\sigma_1} \) and \( T_{\sigma_2} \), but \( f_{\sigma_1} = \frac{1+q}{1+q} = \frac{1}{1+q} + \frac{T_{\sigma_1}}{\sqrt{q(1+q)}} \). Hence, \( A_1 = -\frac{\sqrt{q}}{1+q} \). Moreover, \( \hat{\tau}_2(1) = -\frac{1+q}{\sqrt{q}} \hat{\tau}_1(1) \).
Thus, \( A_0 = -\frac{1+q}{\sqrt{q}} \).

Now, we have:
\[
B_0 = \hat{\tau}_3(1) = -\frac{1+q}{\sqrt{q}} \hat{\tau}_2(1) = \left( -\frac{1+q}{\sqrt{q}} \right)^2,
\]
and \( B_1 = \hat{\tau}_3(f_{\sigma_1}) = -\frac{1+q}{\sqrt{q}} \hat{\tau}_2(f_{\sigma_1}) = \frac{1+q}{\sqrt{q}} \frac{1}{1+q} = 1. \)

**Remark 4.3.** \( \hat{\tau}_3 \) must verify \( \hat{\tau}_3(F(h)T_{\sigma_2}) = \hat{\tau}_2(h) \), for every \( h \) in \( \text{TL}_2(q) \).

But, \( \hat{\tau}_3(F(h)T_{\sigma_2}) = \sqrt{q} \hat{\tau}_3(F(h)g_{\sigma_2}) = \sqrt{q} \hat{\tau}_3 \left( F(h) \left[ (q + 1)f_{\sigma_2} - 1 \right] \right). \)

So, \( \sqrt{q} \hat{\tau}_3 \left( F(h) \left[ (q + 1)f_{\sigma_2} - 1 \right] \right) = \sqrt{q} (q + 1) \hat{\tau}_3(F(h)f_{\sigma_2}) - \sqrt{q} \hat{\tau}_3(F(h)) \)
\[
\sqrt{q} (q + 1) \hat{\tau}_3(F(h)f_{\sigma_2}) + \sqrt{q} \frac{1+q}{\sqrt{q}} \hat{\tau}_2(h).
\]

Hence, our condition becomes
\[
\sqrt{q} (q + 1) \hat{\tau}_3(F(h)f_{\sigma_2}) = -\sqrt{q} \frac{1+q}{\sqrt{q}} \hat{\tau}_2(h) + \hat{\tau}_2(h) = -q \hat{\tau}_2(h).
\]

Thus, we must have
\[
\hat{\tau}_3(F(h)f_{\sigma_2}) = -\frac{\sqrt{q}}{(q + 1)} \hat{\tau}_2(h), \text{ as an } 'f' \text{ equivalent to } \hat{\tau}_3(F(h)T_{\sigma_2}) = \hat{\tau}_2(h).
\]

Now, we have:
\[
B_2 = \hat{\tau}_3(f_{\sigma_1}) = -\frac{\sqrt{q}}{1+q} \hat{\tau}_2(f_{\sigma_1}) = \left( \frac{\sqrt{q}}{1+q} \right)^2.
\]

So, under the assumption that our two traces are the second and the third components of a given Markov trace, we get the following:
\[
A_1 = -\frac{\sqrt{q}}{1+q}, \quad A_0 = -\frac{1+q}{\sqrt{q}}
\]
\[
B_2 = \left( \frac{\sqrt{q}}{1+q} \right)^2, \quad B_1 = 1 \quad \text{and} \quad B_0 = \left( \frac{1+q}{\sqrt{q}} \right)^2.
\]

In particular, we have for all \( 1 \leq i \):
\[
\hat{\tau}_3(x_1^i f_{\sigma_2}) = -\frac{\sqrt{q}}{(q + 1)} \hat{\tau}_2((f_{\sigma_1})^i), \quad \text{and} \quad \hat{\tau}_3(z_1^i f_{\sigma_2}) = -\frac{\sqrt{q}}{(q + 1)} \hat{\tau}_2((f_{\sigma_2})^i).
\]
In other terms, for all \( i \) we have:

\[
\hat{\tau}_3(x_1^i f_{\sigma_2}) = -\frac{\sqrt{q}}{(q + 1)} \alpha_i, \quad \text{and} \quad \hat{\tau}_3(f_{\sigma_2} z_1^i) = -\frac{\sqrt{q}}{(q + 1)} \alpha_i.
\]

Since \( \hat{\tau}_3 \) is determined by \( \beta_i \), we can view this equalities as system of equations in \( \beta_i \) and \( \alpha_i \). In what follows, we show that this system has at most one solution: \((\alpha_i, \beta_i)_{1 \leq i} \).

For \( i = 1 \), we see that we have two equations:

\[
\hat{\tau}_3\left(\frac{-1}{q} f_{\sigma_1 a_3 \sigma_2} + \frac{1}{q + 1} f_{\sigma_1 \sigma_2}\right) = -\frac{\sqrt{q}}{(q + 1)} \alpha_1, \quad \text{and} \quad \hat{\tau}_3\left(-q f_{\sigma_2 a_3 \sigma_1} f_{\sigma_2 \sigma_1}\right) = -\frac{\sqrt{q}}{(q + 1)} \alpha_1,
\]

that is

\[
\frac{-1}{q} \beta_1 + \frac{1}{q + 1} B_2 = -\frac{\sqrt{q}}{(q + 1)} \alpha_1, \quad \text{and} \quad -q \beta_1 + \frac{q}{q + 1} B_2 = -\frac{\sqrt{q}}{(q + 1)} \alpha_1,
\]

that is

\[
\frac{-1}{q} \beta_1 + \frac{q}{(q + 1)^3} = -\frac{\sqrt{q}}{(q + 1)} \alpha_1, \quad \text{and} \quad -q \beta_1 + \frac{q^2}{(q + 1)^3} = -\frac{\sqrt{q}}{(q + 1)} \alpha_1.
\]

Clearly, those two linear equations are independent, hence, they determine a unique solution \((\alpha_1, \beta_1)\). Let us see the equations when \( i = 2 \), we have:

\[
\hat{\tau}_3(x_1^2 f_{\sigma_2}) = -\frac{\sqrt{q}}{(q + 1)} \alpha_2, \quad \text{and} \quad \hat{\tau}_3(f_{\sigma_2} z_1^2) = -\frac{\sqrt{q}}{(q + 1)} \alpha_2.
\]

We see that:

\[
x_1^2 f_{\sigma_2} = 3 \delta x_1 f_{\sigma_2} + x_2 f_{\sigma_2} = 3 \frac{-1}{q} \delta f_{\sigma_1 a_3 \sigma_2} + 3 \frac{1}{q + 1} \delta f_{\sigma_1 \sigma_2} - \frac{(q + 1)}{q^2} f_{\sigma_1 a_3 \sigma_2} - f_{\sigma_1 \sigma_2 a_3} f_{\sigma_1 \sigma_2} = \frac{-3}{(1 + q)^2} f_{\sigma_1 a_3 \sigma_2} + \frac{3}{(1 + q)^3} f_{\sigma_1 \sigma_2} - \frac{(q + 1)}{q^2} f_{\sigma_1 a_3 \sigma_2} - f_{\sigma_1 \sigma_2 a_3} f_{\sigma_1 \sigma_2},
\]

hence, \( \hat{\tau}_3(F(x_1^2 f_{\sigma_2})) = \frac{-3}{(1 + q)^2} \beta_1 + \frac{3}{(1 + q)^3} B_2 - \frac{(q + 1)}{q^2} \beta_2 - \delta \beta_1 \)

\[
= \frac{3}{(1 + q)^3} B_2 - \frac{3 + q}{(1 + q)^2} \beta_1 - \frac{(q + 1)}{q^2} \beta_2.
\]

Now, \( f_{\sigma_2} z_1^2 = \chi(x_1^2 f_{\sigma_2}) \)

\[
= \chi \left( \frac{-3}{(1 + q)^2} f_{\sigma_2 a_3 \sigma_1} + \chi \left( \frac{3}{(1 + q)^3} f_{\sigma_2 \sigma_1} \right) - \chi \left( \frac{(q + 1)}{q^2} f_{\sigma_2 a_3 \sigma_1} - f_{\sigma_2 \sigma_1 a_3} f_{\sigma_2 \sigma_1} \right) \right),
\]

so \( f_{\sigma_2} z_1^2 = \frac{-3q^2}{(1 + q)^2} f_{\sigma_2 a_3 \sigma_1} + \frac{3q^3}{(1 + q)^3} f_{\sigma_2 \sigma_1} - q(q + 1) f_{\sigma_2 a_3 \sigma_1} - f_{\sigma_2 \sigma_1 a_3} f_{\sigma_2 \sigma_1} \).
Now, we apply the trace $\hat{\tau}_3\left(f_{\sigma_2}\sigma_1^2\right) = \frac{-3q^2}{(1+q)^2} \beta_1 + \frac{3q^3}{(1+q)^3} B_2 - q(q+1)\beta_2 - \delta \beta_1$\[= \frac{3q^3}{(1+q)^3} B_2 - \frac{3q^2 + q}{(1+q)^2} \beta_1 - q(q+1)\beta_2.\]

In other terms, we have the two equations:

\[-\frac{(q+1)}{q^2} \beta_2 - \frac{3+q}{(1+q)^2} \beta_1 + \frac{3q}{(1+q)^5} = -\frac{\sqrt{q}}{(q+1)} \alpha_2,\]

\[-q(q+1)\beta_2 - \frac{3q^2 + q}{(1+q)^2} \beta_1 + \frac{3q^4}{(1+q)^5} = -\frac{\sqrt{q}}{(q+1)} \alpha_2.\]

Which indeed determine a unique $(\alpha_2, \beta_2)$ as a solution.

Now, have:

\[x_1^k = \sum_{i=1}^{i=k-1} \gamma_i x_i + x_k,\]

hence, \[x_1^k f_{\sigma_2} = \sum_{i=1}^{i=k-1} \gamma_i x_i f_{\sigma_2} + x_k f_{\sigma_2},\]

thus

\[x_1^k f_{\sigma_2} = \sum_{i=1}^{i=k-1} \gamma_i \left[ (-1)^{i-1} \frac{(q+1)^{i-1}}{q^i} \right] f_{\sigma_1 \sigma_2} + \gamma_i \left[ (-1)^{i-1} (q+1)^{i-2} \right] f_{\sigma_1 \sigma_2} f_{\sigma_1 \sigma_2} \]

\[+ \left[ (-1)^{k-1} \frac{(q+1)^{k-1}}{q^k} \right] f_{\sigma_1 \sigma_2} f_{\sigma_1 \sigma_2} + \left[ (-1)^{k-1} (q+1)^{k-2} \right] f_{\sigma_1 \sigma_2} f_{\sigma_1 \sigma_2} .\]

Now we apply $\hat{\tau}_3$, we get:

\[-\frac{\sqrt{q}}{(q+1)} \alpha_k = \sum_{i=1}^{i=k-1} \gamma_i \left[ (-1)^{i-1} \frac{(q+1)^{i-1}}{q^i} \right] \beta_i + \delta \gamma_i \left[ (-1)^{i-1} (q+1)^{i-2} \right] \beta_{i-1} \]

\[+ \delta \left[ (-1)^{k-1} (q+1)^{k-2} \right] \beta_{k-1} + \left[ (-1)^{k-1} (q+1)^{k-1} \right] \frac{q^k}{q^k} \beta_k.\]
It is clear that the coefficients of $\beta_k$ is not zero, since $\beta_k$ does not appear in:

$$A := \sum_{i=1}^{i=k-1} \gamma_i \left[ (-1)^i \frac{q + 1}{q^i} \right] \beta_i + \delta \gamma_i \left[ (-1)^i \frac{q + 1}{q^{i-2}} \right] \beta_{i-1}$$

$$+ \delta \left[ (-1)^{k-1} (q + 1)^{k-2} \right] \beta_{k-1}.$$ 

Now, we repeat the same steps with $z_i$, namely:

$$z_i^k = \sum_{i=1}^{i=k-1} \gamma_i d_i + d_k,$$

hence, $f_{\sigma_2} z_i^k = \sum_{i=1}^{i=k-1} \gamma_i f_{\sigma_2} d_i + f_{\sigma_2} d_k$.

Thus,

$$f_{\sigma_2} z_i^k = \sum_{i=1}^{i=k-1} \gamma_i \left[ (-1)^i q (q + 1)^{i-1} \right] f_{\sigma_2 a_3 \sigma_1} + \gamma_i \left[ (-1)^{i-1} \frac{q + 1}{q} \right] f_{\sigma_2 \sigma_1} f_{a_3 \sigma_2 \sigma_1}$$

$$+ \left[ (-1)^{k-1} q (q + 1)^{k-1} \right] f_{\sigma_2 a_3 \sigma_1} + \left[ (-1)^{k-1} \frac{q + 1}{q} \right] f_{\sigma_2 \sigma_1} f_{a_3 \sigma_2 \sigma_1}.$$ 

Now we apply $\hat{\tau}_3$, we get:

$$- \frac{\sqrt{q}}{q + 1} \alpha_k = \sum_{i=1}^{i=k-1} \gamma_i \left[ (-1)^i q (q + 1)^{i-1} \right] \beta_i + \gamma_i \delta \left[ (-1)^{i-1} \frac{q + 1}{q} \right] \beta_{i-1}$$

$$+ \delta \left[ (-1)^{k-1} \frac{q + 1}{q} \right] \beta_{k-1} + \left[ (-1)^k q (q + 1)^{k-1} \right] \beta_k.$$ 

The coefficients of $\beta_k$ is not zero, since $\beta_k$ does not appear in

$$B := \sum_{i=1}^{i=k-1} \gamma_i \left[ (-1)^i q (q + 1)^{i-1} \right] \beta_i + \gamma_i \delta \left[ (-1)^{i-1} \frac{q + 1}{q} \right] \beta_{i-1}$$

$$+ \delta \left[ (-1)^{k-1} \frac{q + 1}{q} \right] \beta_{k-1}.$$
In other terms, we have the two following equations, in $\beta_k$ and $\alpha_k$:

$$-\frac{\sqrt{q}}{(q + 1)}\alpha_k = A + \left[ (\frac{1}{q})^{k-1}(q+1)^{k-1}\right]\beta_k,$$

$$-\frac{\sqrt{q}}{(q + 1)}\alpha_k = B + \left[ (\frac{1}{q})^{k}(q+1)^{k-1}\right]\beta_k.$$

Those are two independent linear equations in $\beta_k$ and $\alpha_k$, with non-zero coefficients, by induction over $k$ (that is: assuming that $(\alpha_i, \beta_i)$ is unique for $i < k$ then $(\alpha_k, \beta_k)$ is unique) we get the following corollary.

**Corollary 4.4.** Suppose that $(\tilde{\tau}_i)_{1 \leq i}$ is a Markov trace over the tower of $\tilde{A}$-type T-L algebras, then $\tilde{\tau}_i = \rho_i$ for $i = 1, 2, 3$.

Finally, we sum up the proof of the main theorem: we know, by corollary 4.2 that there exists, at least, one affine Markov trace. Now, corollary 4.4 says that in any given affine Markov trace, the three first components are $\rho_1, \rho_2$ and $\rho_3$ (of corollary 4.2), while 3.6 affirms that a third component in a given Markov trace determines a unique forth component, and so on for any $\tilde{\tau}_i$ with $i \geq 3$. Hence, we get our main theorem:

**Theorem 4.5.** There exists a unique affine Markov trace over the tower of $\tilde{A}$-type Temperley-Lieb algebras, namely $(\rho_i)_{1 \leq i}$.

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**IMJ Université Paris 7**

E-mail address: sadikharbat@math.univ-paris-diderot.fr