Hawking–Page phase transition of four-dimensional de-Sitter spacetime with nonlinear source

Yun-Zhi Du\textsuperscript{1,2,a}, Huaifan Li\textsuperscript{1,2,b}, Li-Chun Zhang\textsuperscript{1,c}

\textsuperscript{1} Department of Physics, Shanxi Datong University, Datong 037009, China
\textsuperscript{2} Institute of Theoretical Physics, Shanxi Datong University, Datong 037009, China

Received: 3 January 2022 / Accepted: 29 March 2022
© The Author(s), under exclusive licence to Società Italiana di Fisica and Springer-Verlag GmbH Germany, part of Springer Nature 2022

Abstract The interplay between a dS black hole horizon and cosmological horizon in a dS spacetime introduces distinctive thermodynamic behaviors (for example the well-known upper bounds of mass and entropy (Dinsmore et al. in Class. Quant. Grav. 37(5), 2020)). Based on this point, we present the Hawking–Page (HP) phase transition of the four-dimensional dS spacetime with nonlinear charge correction when the effective pressure is fixed, and analyze the effects of different effective pressures and nonlinear charge corrections on HP phase transition. The evolution of this system undergoing the HP phase transition is also investigated. We find that the coexistent curve of HP phase transition is a closed one with two different branches. That indicates there exist the upper bounds of the HP temperature and HP pressure, which is completely distinguished with that in AdS spacetime. With decreasing the distance between two horizons, the dS spacetime at the coexistent curve of HP phase transition is going along with different branches. Furthermore, we also explore the influences of charge and nonlinear charge correction on the coexistent curve.

1 Introduction

Since the discovery of the Hawking-Page (HP) phase transition [1], the phase transition in a spacetime with an/a AdS/dS black hole has been widely investigated in the extended phase space [2–20]. The HP phase transition of an AdS spacetime gave the evolution of spacetime with different phases. Namely, for an AdS spacetime with the increasing of temperature the dominant configuration is from the pure thermal radiation phase, then to the coexistent phase with an AdS black hole and thermal radiation, and finally to a stable black hole. This HP phase transition was explained by Witten [21] as a confinement/deconfinement phase transition in gauge theory. And it could also be understood as a solid/liquid phase transition [17] by regarding the cosmological constant as pressure 

\[ P = -\frac{\Lambda}{8\pi} = \frac{(n-1)(n-2)}{16\pi l^2}, \]

whose conjugate variable is the thermodynamic volume.

For the Schwarzschild-AdS black holes with hyperbolic horizons, which are the thermally stable, the HP phase transition does not emerge. While for Schwarzschild-AdS black holes with spherical horizon, there exists the HP phase transition between the pure thermal radiation in AdS spacetime and stable larger black holes. Subsequently the authors in [22] had extended to the charged AdS (i.e., Reissner-Nordstrom-AdS) black hole. The HP phase transition in the Einstein–Gauss–Bonnet gravity also was investigated [23,24]. Currently, there are some researches on the HP phase transition in AdS spacetimes. Therefore it is a natural question whether HP phase transition can survive in dS spacetimes. In Ref. [25–28], the authors revealed the relationship of HP phase transition properties in dS black holes and their specific boundary in different extended phase spaces by putting black holes into a spherical cavity. However, the boundary of dS black hole is artificially added, which will lead to lose its universality. Based on this issue, we will investigate the HP phase transition of a dS spacetime with nonlinear source through considering the interplay between dS black hole horizon and cosmological horizon.

In nature, most physical systems are nonlinear, so the nonlinear field theories are of interest to different branches of mathematical physics. The nonlinear electrodynamics (NLED) have the richer structures, and they can reduce to linear Maxwell theory (LMT) in special case. Since in LMT there are various limitations [29–31], especially the limitation about the radiation propagation inside specific materials [32–35], NLED should be considered more. In addition, NLED objects can remove both of the big bang and black hole singularities [36]. Recently, the authors [39,40] checked the first law of thermodynamics for the \( n+1 \)-dimensional topological static black hole with the mentioned NLED and analyzed the effect of the nonlinear charge correction on the thermodynamic properties of black hole. In this work, we will present the corresponding properties of HP phase transition in this spacetime. And a unique phenomena will be exhibited in coexistent curve of HP phase transition.

\[ a \text{ e-mail: duyzh13@lzu.edu.cn} \]
\[ b \text{ e-mail: huaifan999@sxdtdx.edu.cn (corresponding author)} \]
\[ c \text{ e-mail: zhlc2969@163.com} \]
This work is organized as follows: in Sect. 2, we review the thermodynamic quantities of the four-dimensional dS spacetime with nonlinear source and analyze the effect of different nonlinear charge corrections and different effective pressures on thermodynamic quantities. Then in Sect. 3, we investigate the Gibbs free energy to explore the evolution of this system and discuss the property of HP phase transition. Furthermore the coexistent curve of HP phase transition temperature and HP phase transition pressure is presented. We also analyze the influence of the nonlinear charge correction on HP phase transition. Finally, a brief summary is given in Sect. 4.

2 The thermodynamical quantities of the four-dimensional de-Sitter spacetime with nonlinear source

The static spherically symmetric black hole solution in a four-dimensional spacetime with nonlinear source was given as [37–40]

\[ ds^2 = -f(r)dt^2 + f^{-1}dr^2 + r^2d\Omega_2^2, \quad f(r) = k - \frac{2M}{r} - \frac{\Lambda r^2}{3} + \frac{q^2}{r^2} - \frac{2\alpha}{5r^6}. \]

Here \( M \) and \( q \) are the black hole mass and charge, and the last term in Eq. (2) indicates the effect of the nonlinearity. For similarity, we redefine the nonlinear source term as \( 2\phi = \frac{q^2}{r^2} \) and recall \( 2\phi \) as the nonlinear charge correction. In the following, we mainly focus on the solution with \( \Lambda > 0 \) and \( k = 1 \), i.e., the de-Sitter spacetime with a black hole.

In this system, there are three horizons: the dS black hole inner (\( r_{in+} \)) and outer ones (\( r_+ \)), and the cosmology one(\( r_c \)). Here we only focus on the black hole outer and cosmological horizons and in the following we call the black hole outer horizon as the black hole horizon. And these two horizons are satisfied with the expression \( f(r_{+}) = 0 \). The radiation temperatures at two horizons were given in Refs. [29,37,38]. Since there are two different Harking温度 on the black hole outer horizon and the cosmology one, we cannot directly regard the space between the black hole outer and cosmological horizons as an ordinary thermodynamic system in equilibrium. While considering the gravity effect between two horizons’ space, i.e., the interplay of two horizons should be introduced, the dS spacetime with the space between two horizons can be treated as a thermodynamic system affected by gravitational effects, where the thermodynamical laws are still held on. From the view of the whole dS spacetime, two horizons are not independent with each other, because they are in gravity field which will lead to the interaction between them. Therefore, due to the gravity effect the black hole will always have a temperature higher than that of the cosmological horizon. When regarding the four-dimensional dS spacetime with nonlinear source as an ordinary thermodynamic system in thermodynamic equilibrium and considering the correlations of the two horizons, the effective thermodynamic quantities (\( T_{eff}, P_{eff}, V, S, \Phi_{eff} \)) can be calculated. Here we point out the entropy is not only the sum of two horizons, it also contains the connection between two horizons.

Considering the connection between the black hole and cosmological horizons, the corresponding first law of black hole thermodynamics is given by [39]

\[ dM = T_{eff}dS - P_{eff}dV + \Phi_{eff}dq. \]

The thermodynamic volume is the one between the black hole and cosmological horizons [41]

\[ V = \frac{V_2r_+^4(1-x^3)}{3x^2}, \quad S = \frac{V_2r_+^2F(x)}{4x^2}, \]

with \( V_2 = \frac{2\pi^{3/2}}{\Gamma(3/2)}, \quad x = \frac{r_+}{r_c}, \quad \) and

\[ F(x) = \frac{8}{5}(1-x^3)^{\frac{7}{2}}\frac{x}{5(1-x^3)} + 1 + x^2 = \hat{f}(x) + 1 + x^2. \]

Note that the total entropy is not only the sum of entropy at two horizons, and \( \hat{f}(x) \) in \( F(x) \) represents the extra contribution from the correlations of two horizons.

The effective temperature, effective pressure, and mass were shown as the following in Ref. [19, 20]

\[ T_{eff} = \frac{f_2(x)}{f_1(x)r_+} - \frac{q^2f_5(x)}{f_1(x)r_+^2}, \]

\[ P_{eff} = -\frac{f_5(x)}{f_4(x)r_+} + \frac{q^2f_6(x)}{f_4(x)r_+^2}, \]

\[ M = \frac{V_2r_+(1-x^2)}{8\pi(1-x^3)} \left[ k + \frac{q^2(1+x^2)}{r_+^2} - \frac{2\phi}{5} \right]. \]
with

\[ f_1(x) = \frac{4\pi (1 + x^4)}{1 - x}, \quad f_2(x) = (1 + x + x^2)(1 + x^4) - 2x^3, \quad f_3(x) = \frac{8\pi (1 + x^4)}{x(1 - x)}, \]

\[ f_4(x) = 2\phi \left[ 5(1 + x + x^2)(1 + x^4) + 2x^3(1 + x + x^2 + x^3 + x^4) \right] + k \left[ (1 - 3x^2)(1 + x + x^2 + 4x^3(1 + x)) \right], \]

\[ f_5(x) = kx(1 + x)F''/2 - \frac{2\phi F'(1 + x)}{1 + x + x^2} - \phi F'(1 + x)(1 + x^2)(1 + x^4) \]

\[ = \frac{2\phi F(5 + 10x + 15x^2 + 12x^3 + 9x^4 + 6x^5 + 3x^6)}{5(1 + x + x^2)} \]

\[ f_6(x) = \frac{(1 + 2x + 3x^2)F'}{1 + x + x^2} + x(1 + x)(1 + x^2)F''/2. \]

For an isobaric process, from Eq. (7) we find that the horizon radius \( r_+ \) satisfies the following form

\[ r_+ = r_p = \sqrt{\frac{-f_2(x) + \sqrt{f_2^2(x) - 4q^2 P_{\text{eff}} f_4(x) f_5(x)}}{2 P_{\text{eff}} f_4(x)}}. \] (9)

In the following, we mainly focus on the thermodynamic properties of the four-dimensional dS spacetime with the nonlinear source undergoing the isobaric process. According to Eq. (9), the relations between \( r_+ \) and \( x \) for different effective pressures and nonlinear charge corrections are exhibited in Fig. 1. It is obviously that for the dS spacetime with the fixed nonlinear charge correction and effective pressure, the radius of black hole horizon is decreasing monotonously with \( x \) (0.44 \( \leq x \leq 0.73 \)). And it is decreasing with the effective pressure, and increasing with the nonlinear charge correction.

For the given nonlinear charge correction and effective pressure, the effective temperature and entropy as functions of \( r_+ \) are shown in Figs. 2 and 3, respectively. From Figs. 2 and 3, we know that for the fixed horizon \( r_+ \) the effective temperature increases with the effective pressure and decreases with the nonlinear charge correction. Furthermore, it is not a monotonic function with \( r_+ \) for fixed nonlinear charge correction. Note that there exists the minimal effective temperature \( T_{\text{eff}}^0 \) for this system with fixed
Fig. 3 For the given effective pressure $P_{\text{eff}} = 1.459 \times 10^{-4}$, the effective temperature $T_{\text{eff}}$ and entropy $S$ are both as a function of $r_+$ for the parameters $k = 1$, $q = 1$. The nonlinear charge correction $\phi = 0$ (dashed red thick lines), 0.002 (dashed blue thick lines), 0.005 (thin green lines), respectively.

Effective pressure and nonlinear charge correction. When $T_{\text{eff}} < T_{\text{eff}}^0$, there is no dS black hole, while there exists a pair of dS black holes when $T_{\text{eff}} > T_{\text{eff}}^0$. From Figs. 2 and 3, we can see that the entropy increases monotonously with $r_+$. And it increases with the effective pressure and decreases with the nonlinear charge correction. These properties are similar to that of the effective temperature. Especially for $T_{\text{eff}} > T_{\text{eff}}^0$, the heat capacity at constant pressure is negative for the smaller dS black holes, and it is positive for the bigger ones. That indicates in dS spacetime with $T_{\text{eff}} > T_{\text{eff}}^0$ there exist the stable bigger black holes, instead of the smaller black holes.

3 Hawking–Page phase transition of the dS spacetime with nonlinear source

As is well known, the black hole horizon and cosmology horizon both have Hawking radiation. These Hawking radiations can be regarded as the background heat bath. When the dS spacetime is in the thermodynamic equilibrium, the dS black hole can be considered to be in the background heat bath. There are the exchanged energy between the dS black hole and the background heat bath, and the Gibbs free energy should be zero. Gibbs free energy is an important thermodynamic quantity to investigate the phase transition in addition to the equal area law. It also can be used to study the HP phase transition, i.e., the HP phase transition emerges as $G = 0$. In this part, we will analyze the properties of HP phase transition in the four-dimensional dS spacetime with nonlinear source in an isobaric process and give the coexistent curve of $T_{\text{eff}}^{HP} - T_{\text{eff}}$.

In the four-dimensional dS spacetime with nonlinear source, the Gibbs free energy reads

$$G(r_+, x) = M - T_{\text{eff}}S + P_{\text{eff}}V. \quad (10)$$

With Eqs. (4), (6), (7), (8), and (9), we have shown the pictures of $G$ with $q = 1$, $k = 1$ for the given effective pressure and the given nonlinear charge correction in Fig. 4. It is obvious that the Gibbs free energy is not the monotonic function of $T_{\text{eff}}$, and there are two branches corresponding to small and large dS black hole phases. And the Gibbs free energy of two branches are obviously increasing with the effective pressure when the effective temperature is fixed, while the Gibbs free energy of two branches is both slightly decreasing with the nonlinear charge correction. It is clear that in Fig. 4 there exist the inflexion points and intersections with the given effective pressures and nonlinear charge corrections. That means there exist the minimum effective temperature $T_{\text{eff}}^0$ (the inflexion point) and the HP effective temperature $T_{\text{eff}}^{HP}$ (the intersection), $T_{\text{eff}}^0 < T_{\text{eff}}^{HP}$. Furthermore, $T_{\text{eff}}^0$ and $T_{\text{eff}}^{HP}$ are both increasing with the effective pressure, while they are slightly deceasing with the nonlinear charge correction.

In order to analyze the different stable phases in dS spacetime for the given effective pressure and nonlinear charge correction, we show the pictures of $G_{P_{\text{eff}}} - T_{\text{eff}}, T_{\text{eff}} - r_+, r_+ - x$, and $T_{\text{eff}} - x$ with $\phi = 0.002$, $P_{\text{eff}} = 1.459 \times 10^{-4}$ in Figs. 5 and 6. The minimum effective temperature and Hawking–Page temperature are denoted by the uppercase letters $B$ and $C$, respectively. And $A$ stands for any temperature of the upper branch in $G_{P_{\text{eff}}} - T_{\text{eff}}$.

From Figs. 2, 3, and 6, we know that the range between $C$ and $B$ stands for the bigger black holes which have the positive heat capacity at constant pressure and are thermodynamically stable, whereas the range between $B$ and $A$ stands for the smaller black holes which have the negative heat capacity at constant pressure and are thermodynamically unstable. There exists a minimum effective temperature, below which no dS black hole can exist. The pure thermal radiation phase characterized by vanishing Gibbs free energy can stably exist when $T_{\text{eff}} < T_{\text{eff}}^0$. From Fig. 3, we can see that with the increasing of $T_{\text{eff}}$ from zero it becomes possible to form a large dS black hole as $T_{\text{eff}} > T_{\text{eff}}^0$, whose Gibbs free energy is larger than that of the thermal radiation phase. Therefore, such bigger dS black holes are thermodynamically metastable. Further increasing the effective temperature to $T_{\text{eff}}^{HP}$, the radiation phase and black hole coexist with the vanishing Gibbs free energy, i.e., the HP phase transition emerges. Above this temperature, the radiation phase is collapsing into a large dS black hole, which is the most stable phase. The metastable bigger dS
black holes that exist for $T_{\text{eff}}^0 < T_{\text{eff}} < T_{\text{eff}}^{\text{HP}}$ are often neglected. For a stable large dS black hole with continuously decreasing effective temperature, it is possible to pass through HP phase transition point and becomes a metastable phase. This process is just like a supercooled liquid phase of water below its freezing point.

At the HP phase transition points, the horizon of dS black hole as a function of $x$ reads

$$r_+ = r_G = q \left( \frac{1-x^4}{2(1-x^4)} + \frac{\pi F(x)f_2(x)}{x^2 f_1(x)} + \frac{4\pi (1-x^3)f_6(x)}{3x^2 f_1(x)} - \frac{x^2}{2(1-x^3)} \right).$$  \hspace{1cm} (11)$$

Substituting the above equation into Eqs. (6) and (7), we can obtain the curves of $T_{\text{eff}}^{\text{HP}} - x$ and $P_{\text{eff}}^{\text{HP}} - x$ with the given charge and nonlinear charge correction in Fig. 7. Unlike in AdS black holes, there exist the upper bound of the HP temperature and HP pressure, i.e., $T_{\text{eff}}^{\text{HP}}_{\text{max}}$ and $P_{\text{eff}}^{\text{HP}}_{\text{max}}$, which are remarked by the uppercase letter $E$. And the ranges of $x$ in diagrams of $T_{\text{eff}}^{\text{HP}} - x$ and $P_{\text{eff}}^{\text{HP}} - x$ are both from zero (marked by $O$) to the maximum $x_{\text{max}}$ (marked by $N$). The corresponding effective temperature and effective pressure are both zero at $O$ and $E$. Furthermore, the values of $x$ at $T_{\text{eff}}^{\text{HP}}_{\text{eff}}$ and $P_{\text{eff}}^{\text{HP}}_{\text{eff}}$ are the same ($x = x_0$). For any fixed HP temperature lower than $T_{\text{eff}}^{\text{HP}}_{\text{eff}}$, there are two values of $x$: $x = x_1$ and $x = x_2$. However, the corresponding HP pressures for $x = x_1$ and $x = x_2$ are different: $P_{\text{eff}}^{\text{HP}}_{\text{eff}2} < P_{\text{eff}}^{\text{HP}}_{\text{eff}1} < P_{\text{eff}}^{\text{HP}}_{\text{eff}max}$. In order to illustrate this character, the coexistence curve of $T_{\text{eff}}^{\text{HP}} - P_{\text{eff}}^{\text{HP}}$ is shown in Fig. 8. It is very interesting that the curve of $T_{\text{eff}}^{\text{HP}} - P_{\text{eff}}^{\text{HP}}$ is a closed one with two different
Fig. 7 $P_{HP}^{\text{eff}}$ and $T_{HP}^{\text{eff}}$ as the functions of $x$. The parameters set to $k = 1$, $q = 1$, and $\phi = 0.002$.

Fig. 8 $P_{HP}^{\text{eff}}$ as the function of $T_{HP}^{\text{eff}}$. The parameters set to $k = 1$, $q = 1$, and $\phi = 0.002$.

Fig. 9 The effective pressure $P_{HP}^{\text{eff}}$ as functions of $T_{HP}^{\text{eff}}$ with different charges and different nonlinear charge corrections for $k = 1$. In the left with $\phi = 0.002$, the charge set to $q = 0.8$ (dashed red thick line), $q = 1$ (dashed blue thick line), and $q = 1.2$ (green thin line), respectively. In the right $q = 1$, the nonlinear charge correction set to $\phi = 0$ (dashed red thick line), $\phi = 0.002$ (dashed blue thick line), and $\phi = 0.005$ (green thin line), respectively.

branches. And the upper branch responds to the process between $O$ and $E$, and the lower is that between $N$ and $E$. That means with the increasing ratio $x$ from zero to $x_0$ (i.e., the distance $d = \frac{r_e(1-x)}{x}$ between two horizons decreases from $\infty$ to $r_e(1-x_0)$), the state with the stable coexistence of dS black hole and radiation phase is going along with $O \rightarrow E$. And the effective temperature and effective pressure are both increasing from zero to the maximum. Further decreasing the distance between two horizons until to $\frac{1-x_{\text{max}}}{x}$, the spacetime is going along with $E \rightarrow N$, and the effective temperature and effective pressure are both decreasing from maximum to zero.

In addition, the effects of the charge and nonlinear charge correction on the coexistent curve are displayed by Fig. 9. We find that for the given nonlinear charge correction, $T_{HP}^{\text{eff}}$ and $P_{HP}^{\text{eff}}$ are both sensitive to the charge, and the maximum of them are both significantly decreasing with the increasing of charge (see Fig. 3). However for the given charge, the nonlinear charge correction has little effect on the coexistence (see Fig. 3).
4 Discussions and conclusions

In this paper, we mainly have analyzed the HP phase transition of the four-dimensional topological dS spacetime with the nonlinear charge correction, which can be regarded as an ordinary thermodynamic system in the thermodynamic equilibrium.

Firstly we reviewed the thermodynamic quantities of the four-dimensional dS spacetime with the nonlinear charge correction and in isobaric processes gave the corresponding behaviors of the effective temperature, entropy, and the dS black hole horizon with the certain range of $x$ ($x \equiv r_+ / r_-$). We found in an isobaric process the effective temperature is not a monotonously function with $r_+$, and the smaller dS black holes are of the negative heat capacity and the bigger ones are with $C_\text{Peff} > 0$. That means in dS spacetime with $T_\text{eff} > T_\text{eff}^0$ there exist the stable bigger dS black holes, instead of the smaller dS black holes. Furthermore, there exists the minimal effective temperature $T_\text{eff}^0$ for the dS black hole with fixed effective pressure and nonlinear charge correction. When $T_\text{eff} < T_\text{eff}^0$, no dS black hole can survive, while a pair of dS black holes emerge when $T_\text{eff} > T_\text{eff}^0$.

Then we investigated the thermodynamic property of HP phase transition with different effective pressure and different nonlinear charge correction. The results showed that there are two branches in the diagram of $G_\text{Peff} \sim T_\text{eff}$: the upper one stands for the unstable smaller black holes, and the lower one is the stable bigger black holes. Furthermore, there both exist the minimum effective temperature and the HP temperature, $T_\text{eff}^0 < T_\text{eff}^H$. While compared with the pure thermal radiation phase characterized by vanishing Gibbs free energy, the dS spacetime is in favor of the thermal radiation phase, not of the bigger black holes when $T_\text{eff}^0 < T_\text{eff} < T_\text{eff}^H$. At $T_\text{eff} = T_\text{eff}^H$, the HP phase transition emerges. Above this temperature, the thermal radiation phase is collapsing into a bigger black hole, which is the most stable phase. In addition, the Gibbs free energy is sensitive to the effective pressure, not the nonlinear charge correction.

Finally, we presented the coexistence curve of $T_\text{eff}^H - P_\text{eff}^H$ in Fig. 8. It is very interesting and unique that the coexistent curve is a closed one with two different branches. And the HP temperature and HP pressure both are bounded from zero to the maximum. That is fully different from the AdS black hole, whose temperature and pressure at HP phase transition point are both from zero to infinity. We can regard the coexistent curve of HP phase transition as a difference between dS spacetime embedded with black hole and AdS black hole. Furthermore with the decreasing of the distance between two horizons from $\infty$ to $\frac{1-x}{\lambda_{\text{max}}}^\frac{r_+}{r_-}$, the spacetime with the coexistent state is going along with the above branch. And the effective temperature and effective pressure are both decreasing from zero to the maximum. Further decreasing the distance between two horizons until to $\frac{1-x_{\text{min}}}{\lambda_{\text{max}}}^\frac{r_+}{r_-}$, the spacetime is going along with the lower branch, and the effective temperature and effective pressure are both decreasing from maximum to zero.

Acknowledgements We would like to thank Prof. Ren Zhao and Meng-Sen Ma for their indispensable discussions and comments. This work was supported by the National Natural Science Foundation of China (Grant No. 11750106, 11475108, 12075143), the Scientific Innovation Foundation of the Higher Education Institutions of Shanxi Province (Grant Nos. 2020L0471, 2020L0472), and the Science Technology Plan Project of Datong City, China (Grant Nos. 20202153).

References

1. S.W. Hawking, D.N. Page, Commun. Math. Phys. 87, 577–588 (1983)
2. J. Dinsmore, P. Draper, D. Kastor, Y. Qiu, J. Traschen, Class. Quant. Grav. 37, 5 (2020). arXiv:1907.00248
3. R.-G Cai, L.-M Cao, L. Li, and R.-Q Yang, JHEP 9 (2013) 1, arXiv:1306.6233
4. S.-W. Wei, Y.-X. Liu, Phys. Rev. D 87, 044014 (2013). arXiv:1209.1707
5. M.M. Caldarelli, G. Cognola, D. Klemm, Class. Quant. Grav. 17, 399 (2000). arXiv:9908022
6. D. Kubiznak, R.B. Mann, JHEP 1207, 033 (2012). arXiv:1205.0559
7. S.-W. Wei, Y.-X. Liu, Phys. Rev. Lett 115, 111302 (2015)
8. R. Banerjee, D. Roychowdhury, JHEP 11, 004 (2011). arXiv:1109.2433
9. S.H. Hendi, R.B. Mann, S. Panahiyan, B. Eslam Panah, Phys. Rev. D 95, 021501 (2017). arXiv:1702.00432
10. K. Bhattacharyya, B.R. Majhi, S. Samanta, Phys. Rev. D 96, 084037 (2017)
11. X.-X. Zeng, L.-F. Li, Phys. Lett. B 764, 100 (2017)
12. S.H. Hendi, Z.S. Taghadomi, C. Corda, Phys. Rev. D 97, 084039 (2018). arXiv:1803.10767
13. J.-L. Zhang, R.-G. Cai, H.-W. Yu, Phys. Rev. D 91, 044028 (2015). arXiv:1502.01428
14. P. Cheng, S.-W. Wei, Y.-X. Liu, Phys. Rev. D 94, 024025 (2016). arXiv:1603.08694
15. D.-C. Zou, Y.-Q. Liu, R.-H. Yue, Eur. Phys. J. C 77, 365 (2017). arXiv:1702.08118
16. B.P. Dolan, Class. Quant. Grav. 31, 135012 (2014). arXiv:1308.2672
17. N. Altamirano, D. Kubiznak, R.B. Mann, Phys. Rev. D 88, 101502 (2013). arXiv:1306.5756
18. N. Altamirano, D. Kubiznak, R.B. Mann, Z. Sherkatghanad, Class. Quant. Grav. 31, 042001 (2014). arXiv:1308.2672
19. Y.-Z. Du, R. Zhao, L.-C. Zhang, Eur. Phys. J. C 82, 4 (2022). arXiv:2104.10300
20. Y. Zhang, W.-Q. Wang, Y.-B. Ma, J. Wang, Adv. H. E. Phys. 2020, 7263059 (2020). arXiv:2004.06796
21. E. Witten, Adv. Theor. Math. Phys. 2, 505 (1998). arXiv:hep-th/9803131
22. A. Chamblin, R. Emparan, C.V. Johnson, R.C. Myers, Phys. Rev. D 60, 064018 (1999). arXiv:hep-th/9902170
23. S.-W. Wei, Y.-X. Liu, Phys. Rev. D 102, 104011 (2020). arXiv:2006.11503
24. Y.Y. Wang, B.Y. Su, N. Li, Phys. Dark. Univ. 31, 100769 (2021). arXiv:2008.01985
25. W.-B. Zhao, G.-R. Liu, N. Li, Eur. Phys. J. Plus 136, 981 (2021). arXiv:2012.13921
26. B.-Y Su, N. Li, Nucl. Phys. B 979, 115782 (2022). arXiv:2105.06670
27. F. Simovic, R.B. Mann, JHEP 2019, 136 (2019). arXiv:1904.04871
28. P. Wang, H.-W Wu, H.-T Yang, and F.-Y Yao, JHEP 2020, 154 (2020), arXiv:2006.14349
29. W. Heisenberg, H. Euler, Z. Phys. 98, 714 (1936). arXiv:hep-th/0605038
30. H. Yajima, T. Tamaki, Phys. Rev. D 63, 064007 (2001). arXiv:gr-qc/0005016
31. J. Schwinger, Phys. Rev. 82, 664 (1951)
32. V.A. De Lorenzi, M.A. Souza, Phys. Lett. B 512, 417 (2001)
33. V.A. De Lorenzi, R. Klippert, Phys. Rev. D 65, 064027 (2002)
34. M. Novello et al., Class. Quantum Gravit. 20, 859 (2003)
35. M. Cavaglia, S. Das, R. Maartens, Class. Quantum Gravit. 20, L205 (2003)
36. S.H. Hendi, M. Momennia, Eur. Phys. J. C 75, 54 (2015). arXiv:1501.04863
37. S.H. Hendi, R. Naderi, Phys. Rev. D 91, 024007 (2015)
38. Y. Zhang, L.-C. Zhang, R. Zhao, Mod. Phys. Lett. A 31, 1950254 (2019). arXiv:1910.14223
39. H.-H. Zhao, L.-C. Zhang, Fang Liu, Commun. Theor. Phys. 73(9), 095401 (2021). arXiv:1704.05167
40. M.S. Ali, S.G. Ghosh, Eur. Phys. J. Plus 137(4), 486 (2022). arXiv:1906.11284