Finite-Length Analyses for Source and Channel Coding on Markov Chains

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Abstract

We study finite-length bounds for source coding with side information for Markov sources and channel coding for channels with conditional Markovian additive noise. For this purpose, we propose two criteria for finite-length bounds. One is the asymptotic optimality and the other is the efficient computability of the bound. Then, we derive finite-length upper and lower bounds for coding length in both settings so that their computational complexity is efficient. To discuss the first criterion, we derive the large deviation bounds, the moderate deviation bounds, and second order bounds for these two topics, and show that these finite-length bounds achieve the asymptotic optimality in these senses. For this discussion, we introduce several kinds of information measure for transition matrices.

Index Terms

Channel Coding, Markov Chain, Finite-Length Analysis, Source Coding

I. INTRODUCTION

Recently, finite-length analyses for coding problems are attracting a considerable attention [1]. This paper focuses on finite-length analyses for the source coding with side-information for Markov sources and the channel coding for channels with conditional Markovian additive noise. Although the main purpose of this paper is finite-length analyses, this paper also develops a unified approach to investigate these topics including the asymptotic analyses. Since this discussion spreads so many subtopics, we explain them separately in the introduction.

A. Two criteria for finite-length bounds

For an explanation of the motivations of this paper, we start with two criteria for finite-length bounds while the problems treated in this paper are not restricted to channel coding. Until now, so many types of finite-length achievability bounds have been proposed. For example, Verdú and Han derived a finite-length bound by using the information spectrum approach in order to derive the general formula [3] (see also [4]), which we call the information-spectrum bound. One of the authors and Nagaoka derived a bound (for the classical-quantum channel) by relating the error probability to the binary hypothesis testing [5, Remark 15] (see also [6]), which we call the hypothesis testing bound. Polyanskiy et. al. derived the RCU (random coding union) bound and the DT (dependence testing) bound [1]. Also, Gallager’s bound [7] is known as an efficient bound to derive the exponential decreasing rate.

Here, we focus on two important criteria for finite-length bounds:

(C1) Computational complexity for the bound, and
(C2) Asymptotic optimality for the bound.

First, we consider the first criterion, i.e., the computational complexity for the bound. For the BSC, the computational complexity of the RCU bound is \(O(n^2)\) and that of the DT bound is \(O(n)\) [8]. However, the computational complexities of these bounds is much larger for general DMCs or channels with memory. It is known that the...
hypothesis testing bound can be described as a linear programming (eg, see [9], [10], and can be efficiently computed under certain symmetry. However, the number of variables in the linear programming grows exponentially in the block length, and it is difficult to compute in general. The computation of the information-spectrum bound depends on the evaluation of a tail probability. The information-spectrum bound is less operational than the hypothesis testing bound in the sense of the hierarchy introduced in [9], and the computational complexity of the former is much smaller than that of the latter. However the computation of a tail probability is still not so easy unless the channel is a DMC. For DMCs, computational complexity of Gallager’s bound is \( O(1) \) since the Gallager function is additive quantity for DMCs. However, this is not the case if there is a memory. Consequently, there is no bound that is efficiently computable for the Markov chain so far. The situation is the same for source coding with side-information.

Next, let us consider the second criterion, i.e., asymptotic optimality. So far, three kinds of asymptotic regimes have been studied in the information theory [1], [2], [12], [13], [14], [15], [16]:

- The large deviation regime in which the error probability \( \epsilon \) asymptotically behaves like \( e^{-n r} \) for some \( r > 0 \),
- The moderate deviation regime in which \( \epsilon \) asymptotically behaves like \( e^{-n^{1-2t}} \) for some \( r > 0 \) and \( t \in (0, 1/2) \), and
- The second order regime in which \( \epsilon \) is a constant.

We shall claim that a good finite-length bound should be asymptotically optimal at least one of the above mentioned three regimes. In fact, the information spectrum bound, the hypothesis testing bound, and the DT bound are asymptotically optimal in the moderate deviation regime and the second order regime; the Gallager bound is asymptotically optimal in the large deviation regime; and the RCU bound is asymptotically optimal in all the regimes. Recently, for DMC, Yang-Meng derived efficiently computable bound for low density parity check (LDPC) codes [17], which is asymptotically optimal in the moderate deviation regime and the second order regime.

### B. Main Contribution for Finite-Length Analysis

To derive finite-length achievability bounds on the problems, we basically use the exponential type bound. In source coding with side-information, the exponential type upper bounds on error probability \( \bar{P}_e(M_n) \) for a given message size \( M_n \) are described by using conditional Rényi entropies as follows (cf. Lemma 13 and Lemma 14):

\[
\bar{P}_e(M_n) \leq \inf_{-\frac{1}{2} \leq \theta \leq 0} M_n^{\frac{\theta}{1+\theta}} e^{-\frac{\theta}{1+\theta} H^\dagger_{1+\theta}(X^n|Y^n)}
\]

and

\[
\bar{P}_e(M_n) \leq \inf_{-1 \leq \theta \leq 0} M_n^{\theta} e^{-\theta H_{1+\theta}(X^n|Y^n)}.
\]

Here, \( H^\dagger_{1+\theta}(X^n|Y^n) \) is the conditional Rényi entropy introduced by Arimoto [18], which we shall call upper conditional Rényi entropy (cf. [12]). On the other hand, \( H^\dagger_{1+\theta}(X^n|Y^n) \) is the conditional Rényi entropy introduced in [19], which we shall call the lower conditional Rényi entropy. Although there are several other definitions of conditional Rényi entropies, we will only use these two in this paper; see [20], [21] for extensive review on conditional Rényi entropies.

Although the above mentioned conditional Rényi entropies are additive for i.i.d. random variables, they are not additive for Markov chains, which is a difficulty to derive finite-length bounds for Markov chains. In general, it is not easy to evaluate the conditional Rényi entropies for Markov chains. Thus, we consider two assumptions on transition matrices (see Assumption 1 and Assumption 2 of Section II). Without Assumption 1 it should be noted that even the conditional entropy rate is difficult to be evaluated. Under Assumption 1 we introduce the lower conditional Rényi entropy for transition matrices \( H^\dagger_{1+\theta}(X|Y) \) (cf. [47]). Then, we evaluate the lower conditional

\[\text{[In the case of quantum channel, the bound is described as a semi-definite programming.}]\]

\[\text{[The Gallager bound for finite states channels was considered in [11] Section 5.9], but a closed form expression for the exponent was not derived.]}\]

\[\text{[The Gallager bound and the RCU bound are asymptotically optimal in the large deviation regime only up to the critical rate.]}\]

\[\text{[For channel coding, it corresponds to the Gallager bound.]}\]
Rényi entropy for the Markov chain in terms of its transition matrix counterpart. More specifically, we derive an approximation

$$H^1_{1+\theta}(X^n|Y^n) = nH^2_{1+\theta}(X|Y) + O(1),$$

where an explicit form of $O(1)$ is also derived. This evaluation gives finite-length bounds under Assumption 1. Under more restrictive assumption, i.e., Assumption 2, we also introduce the upper conditional Rényi entropy for a transition matrix $H^1_{1+\theta}(X|Y)$ (cf. (55)). Then, we evaluate the upper Rényi entropy for the Markov chain in terms of its transition matrix counterpart. More specifically, we derive an approximation

$$H^1_{1+\theta}(X^n|Y^n) = nH^2_{1+\theta}(X|Y) + O(1),$$

where an explicit form of $O(1)$ is also derived. This evaluation gives finite-length bounds that are tighter than those obtained under Assumption 1.

We also derive converse bounds by using the change of measure argument for Markov chains developed by the authors in the accompanying paper on information geometry [22]. For this purpose, we further introduce two-parameter conditional Rényi entropy and its transition matrix counterpart (cf. (13) and (59)). This novel information measure includes the lower conditional Rényi entropy and the upper conditional Rényi entropy as special cases. To clarify the relation among bounds based on these quantities, we numerically calculate the upper and lower bounds for the optimal coding rate in source coding with Markovian source as Figs. 3 and 4. Thanks to the second criterion (C2), this calculation shows that our finite-length bounds are very close to the optimal value. Although this numerical calculation contains the case with the huge size $n = 1 \times 10^5$, its calculation is not so difficult because their calculation complexity behaves as $O(1)$. That is, this calculation shows the advantage of the first criterion (C1).

Here, we would like to remark on terminologies. There are a few ways to express exponential type bounds. In statistics or the large deviation theory, we usually use the cumulant generating function (CGF) to describe exponents. In information theory, we use the Gallager function or the Rényi entropies. Although these three terminologies are essentially the same and are related by change of variables, the CGF and the Gallager function are convenient for some calculations since they have good properties such as convexity. However, they are merely mathematical functions. On the other hand, the Rényi entropies are information measures including Shannon’s information measures as special cases. Thus, the Rényi entropies are intuitively familiar in the field of information theory. The Rényi entropies also have an advantage that two types of bounds (e.g., (157) and (166)) can be expressed in a unified manner. For these reasons, we state our main results in terms of the Rényi entropies while we use the CGF and the Gallager function in the proofs. For readers’ convenience, the relation between the Rényi entropies and corresponding CGFs are summarized in Appendices A and B.

C. Main Contribution for Channel Coding

It is known that there is an intimate relationship between channel coding and source coding with side-information (e.g., [24], [25]). In particular, for an additive channel, the error probability of channel coding by a linear code can be related to the corresponding source coding problem with side information [24]. Chen et al. also showed that the error probability of source coding with side-information by a linear encoder can be related to the error probability of a dual channel coding problem and vice versa [27] (see also [28]). Since those dual channels can be regarded as additive channels conditioned by state-information, we call those channels conditional additive channel.

As a similar symmetric channel, a regular channel [29] is known.

In this paper, we mainly discuss a conditional additive channel, in which, the additive noise is operated subject to a distribution conditioned with an additional output information, and propose a method to convert a regular channel into a conditional additive channel so that our treatment covers regular channels. Additionally, we show that the BPSK-AWGN channel is included in conditional additive channels. Thus, by using aforementioned duality between channel coding and source coding with side-information, we can evaluate the error probability of channel coding for regular channels.

By the same reason as source coding with side-information, we assume two assumptions, Assumption 1 and Assumption 2 on the noise process of a conditional additive channel. It should be noted that the Gilbert-Elliott
channel [30], [31] with state-information available at the receiver can be regarded as a conditional additive channel such that the noise process is a Markov chain satisfying both Assumption 1 and Assumption 2 (see Example 6). Thus, we believe that Assumption 1 and Assumption 2 are quite reasonable assumptions.

D. Asymptotic bounds and asymptotic optimality for finite-length bounds

For asymptotic analyses of the large deviation and the moderate deviation regimes, we derive the characterizations by using our finite-length achievability and converse bounds, which implies that our finite-length bounds are tight in the large deviation regime and the moderate deviation regime. We also derive the second order rate. Although the second order rate can be derived by application of the central limit theorem to the information spectrum bound, the variance involves the limit with respect to the block length because of memory. In this paper, we derive a single letter form of the variance by using the conditional Rényi entropy for transition matrices.

As we will see in Theorem 11, Theorem 12, Theorem 13, Theorem 14, Theorem 22, Theorem 23, Theorem 24, and Theorem 25, our asymptotic results have the same forms as the counterparts of the i.i.d. case (cf. [7], [11], [2], [12], [13], [14]) when the information measures for distributions in the i.i.d. case are replaced by the information measures for transition matrices introduced in this paper.

To see the asymptotic optimality for finite-length bounds, we summarize the relation between the asymptotic results and the finite-length bounds in Table I. In the table, the computational complexity of the finite-length bounds are also described. "Solved" indicates that those problems are solved up to the critical rates. "Ass. 1" and "Ass. 2" indicate that those problems are solved under Assumption 1 or Assumption 2. "O(1)" indicates that both the achievability part and the converse part of those asymptotic results are derived from our finite-length achievability bounds and converse bounds whose computational complexities are O(1). "Tail" indicates that both the achievability part and the converse part of those asymptotic results are derived from the information-spectrum type achievability bounds and converse bounds whose computational complexities depend on the computational complexities of tail probabilities.

Exact computations of tail probabilities are difficult in general though it may be feasible for a simple case such as an i.i.d. case. One way to approximately compute tail probabilities is to use the Berry-Esséen theorem [34, Theorem 16.5.1] or its variant [35]. This direction of research is still continuing [36, 37], and an evaluation of the constant was done in [37] though it is not clear how much tight it is. If we can derive a tight Berry-Esséen type bound for the Markov chain, we can derive a finite-length bound that is asymptotically tight in the second order regime. However, the approximation errors of Berry-Esséen type bounds converge only in the order of 1/√n, and cannot be applied when ε is rather small. Even in the cases such that exact computations of tail probabilities are possible, the information-spectrum type bounds are looser than the exponential type bounds when ε is rather small, and we need to use appropriate bounds depending on the size of ε. In fact, this observation was explicitly clarified in [38] for the random number generation with side-information. Consequently, we believe that our exponential type finite-length bounds are very useful. It should be also noted that, for source coding with side-information and channel coding for regular channels, even the first order results have not been revealed as long as the authors know, and they are clarified in this paper [39].

| Problem                        | First Order | Large Deviation | Moderate Deviation | Second Order |
|--------------------------------|-------------|-----------------|--------------------|--------------|
| SC with SI                     | Solved (Ass. 1) | Solved∗ (Ass. 2), O(1) | Solved (Ass. 1), O(1) | Solved (Ass. 1), Tail |
| CC for Conditional Additive Channels | Solved (Ass. 1) | Solved∗ (Ass. 2), O(1) | Solved (Ass. 1), O(1) | Solved (Ass. 1), Tail |

7 For the large deviation regime, we only derive the characterizations up to the critical rate.
8 An alternative way to derive a single letter characterization of the variance for the Markov chain was shown in [32, Lemma 20]. It should be also noted that a single letter characterization can be derived by using the fundamental matrix [33]. The single letter characterization of the variance in [12, Section VII] and [2, Section III] has an error, which is corrected in this paper.
9 General formulae for those problems were known [3], [4], but single-letter expressions for Markov sources or channels were not clarified in the literature.
E. Related Works on Markov chains

Since related works concerning the finite-length analysis has been reviewed in Section I-A, we only review related works concerning the asymptotic analysis here. There are some studies on Markov chains for the large deviation regime [39], [40], [41]. The derivation in [39] uses the Markov type method. A drawback of this method is that it involves a term that stems from the number of types, which is not important for the asymptotic analysis but is crucial for the finite-length analysis. Our achievability is derived by a similar approach as in [40], [41], i.e., the Perron-Frobenius theorem, but our derivation separates the single-shot part and the evaluation of the Rényi entropy, and thus is more transparent. Also, the converse part of [40], [41] is based on the Shannon-McMillan-Breiman limiting theorem and does not yield finite-length bounds.

For the second order regime, Polyanskiy et al. studied the second order rate (dispersion) of the Gilbert-Elliott channel [42]. Tomamichel and Tan studied the second order rate of channel coding with state-information such that the state-information may be a general source, and derived a formula for the Markov chain as a special case [32]. Kontoyiannis studied the second order variable length source coding for the Markov chain [43]. In [44], Kontoyiannis-Verdú derived the second order rate of lossless source coding under overflow probability criterion.

For channel coding of i.i.d. case, Scarlett et al. derived a saddle-point approximation, which unifies all the three regimes [45], [46].

F. Organization of Paper

In Section II we introduce information measures and their properties that will be used in Section III and Section IV. Then, source coding with side-information and channel coding will be discussed in Section III and Section IV respectively. As we mentioned above, we state our main result in terms of the Rényi entropies, and we use the CGFs and the Gallager function in the proofs. We explain how to cover the continuous case in Remarks 1 and 6. In Appendices A and B the relation between the Rényi entropies and corresponding CGFs are summarized. The relation between the Rényi entropies and the Gallager function are explained as necessary. Proofs of some technical results are also shown in the rest of appendices.

G. Notations

For a set $X$, the set of all distributions on $X$ is denoted by $\mathcal{P}(X)$. The set of all sub-normalized non-negative functions on $X$ is denoted by $\bar{\mathcal{P}}(X)$. The cumulative distribution function of the standard Gaussian random variable is denoted by

$$\Phi(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{x^2}{2} \right] dx.$$  (5)

Throughout the paper, the base of the logarithm is $e$.

II. INFORMATION MEASURES

Since this paper discusses the second order optimality, we need to discuss the central limit theorem for the Markovian process. For this purpose, we usually employ advanced mathematical methods from probability theory. For example, the paper [47, Theorem 4] showed the Markov version of the central limit theorem by using a martingale stopping technique. Lalley [48] employed regular perturbation theory of operators on the infinite dimensional space [49, Ch. 7, #1, Ch. 4, #3, and Ch. 3, #5]. The papers [50], [51], [52] Lemma 1.5 of Chapter 1] employed the spectral measure while it is hard to calculate the spectral measure in general even in the finite state case. Further, the papers [50], [53], [54], [55] showed the central limit theorem by using the asymptotic variance, but they did not give any computable expression of the asymptotic variance without the infinite sum. In summary, to derive the central limit theorem with the variance of computable form, these papers need to use very advanced mathematics beyond calculus and linear algebra.

To overcome this problem, we employ the method used in our recent paper [23]. The paper [23] employed the method based on the cumulant generating function for transition matrices, which the Perron eigenvalue of a specific non-negative-entry matrix. Since a Perron eigenvalue can be explained in the framework of linear algebra, the method can be described with elementary mathematics. To employ this method, we need to define the information measure
in a way similar to the cumulant generating function for transition matrices. That is, we define the information measures for transition matrices, e.g., the conditional Rényi entropy for transition matrices, etc, by using Perron eigenvalues.

Fortunately, these information measures for transition matrices are very useful even for large deviation type evaluation and finite-length bounds. For example, our recent paper [23] derived finite-length bounds for simple hypothesis testing for Markovian chain by using the cumulant generating function for transition matrices. Therefore, using these information measures for transition matrices, this paper derives finite-length bounds for source coding and channel coding with Markov chains, and discusses their asymptotic bounds with large deviation, moderate deviation, and second order type.

Since they are natural extensions of information measures for single-shot setting, we first review information measures for single-shot setting in Section II-A. Next, we introduce information measures for transition matrices in Section II-B. Then, we show that information measures for Markov chains can be approximated by information measures for transition matrices generating those Markov chains in Section II-C.

A. Information measures for Single-Shot Setting

In this section, we introduce conditional Rényi entropies for the single-shot setting. For more detailed review of conditional Rényi entropies, see [21]. For a correlated random variable $(X, Y)$ on $\mathcal{X} \times \mathcal{Y}$ with probability distribution $P_{XY}$ and a marginal distribution $Q_Y$ on $\mathcal{Y}$, we introduce the conditional Rényi entropy of order $1 + \theta$ relative to $Q_Y$ as

$$H_{1+\theta}(P_{XY}|Q_Y) := -\frac{1}{\theta} \log \sum_{x,y} P_{XY}(x,y)^{1+\theta} Q_Y(y)^{-\theta},$$

where $\theta \in (-1, 0) \cup (0, \infty)$. The conditional Rényi entropy of order 0 relative to $Q_Y$ is defined by the limit with respect to $\theta$. When $\mathcal{Y}$ is singleton, it is nothing but the ordinary Rényi entropy, and it is denoted by $H_{1+\theta}(X) = H_{1+\theta}(P_X)$ throughout the paper.

One of important special cases of $H_{1+\theta}(P_{XY}|Q_Y)$ is the case with $Q_Y = P_Y$, where $P_Y$ is the marginal of $P_{XY}$. We shall call this special case the lower conditional Rényi entropy of order $1 + \theta$ and denote

$$H_{1+\theta}^\downarrow(X|Y) := H_{1+\theta}(P_{XY}|P_Y)$$

as

$$= -\frac{1}{\theta} \log \sum_{x,y} P_{XY}(x,y)^{1+\theta} P_Y(y)^{-\theta}.$$  

We have the following property, which follows from the correspondence between the conditional Rényi entropy and the cumulant generating function (cf. Appendix B).

**Lemma 1** We have

$$\lim_{\theta \to 0} H_{1+\theta}^\downarrow(X|Y) = H(X|Y)$$

and

$$V(X|Y) := \text{Var} \left[ \log \frac{1}{P_{X|Y}(X|Y)} \right]$$

as

$$= \lim_{\theta \to 0} \frac{2}{\theta} \left[ H(X|Y) - H_{1+\theta}^\downarrow(X|Y) \right].$$

**Proof:** (9) follows from the relation in [293] and the fact that the first-order derivative of cumulant generating function is the expectation. (11) follows from (293), (9), and (294). 

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10 This notation was first introduce in [55].
The other important special cases of \( H_{1+\theta}(P_{XY}|Q_Y) \) is the measure maximized over \( Q_Y \). We shall call this special case the upper conditional Rényi entropy of order \( 1 + \theta \) and denote\(^{11}\)

\[
H_{1+\theta}^{\uparrow}(X|Y) := \max_{Q_Y \in \mathcal{P}(Y)} H_{1+\theta}(P_{XY}|Q_Y) 
\]

(12)

\[
= H_{1+\theta}(P_{XY}|P_{Y}^{(1+\theta)})
\]

(13)

\[
= -\frac{1 + \theta}{\theta} \log \sum_y P_Y(y) \left[ \sum_x P_{X|Y}(x|y)^{1+\theta} \right]^{\frac{1}{1+\theta}},
\]

(14)

where

\[
P_{Y}^{(1+\theta)}(y) := \frac{\left[ \sum_x P_{XY}(x,y)^{1+\theta} \right]^{\frac{1}{1+\theta}}}{\sum_{y'} \left[ \sum_x P_{XY}(x,y')^{1+\theta} \right]^{\frac{1}{1+\theta}}}.\]

(15)

For this measure, we also have properties similar to Lemma 1. This lemma will be proved in Appendix C.

**Lemma 2** We have

\[
\lim_{\theta \rightarrow 0} H_{1+\theta}^{\uparrow}(X|Y) = H(X|Y)
\]

(16)

and

\[
\lim_{\theta \rightarrow 0} \frac{2}{\theta} \left[ H(X|Y) - H_{1+\theta}^{\uparrow}(X|Y) \right] = \mathbb{V}(X|Y).
\]

(17)

When we derive converse bounds, we need to consider the case such that the order of the Rényi entropy and the order of conditioning distribution defined in (15) are different. For this purpose, we introduce two-parameter conditional Rényi entropy:

\[
H_{1+\theta,1+\theta'}(X|Y)
\]

:= \( H_{1+\theta}(P_{XY}|P_{Y}^{(1+\theta')}) \)

(18)

\[
= -\frac{1}{\theta} \log \sum_y P_Y(y) \left[ \sum_x P_{X|Y}(x|y)^{1+\theta} \right] \left[ \sum_x P_{X|Y}(x|y)^{1+\theta'} \right]^{\frac{\theta'}{1+\theta'}} + \frac{\theta'}{1+\theta'} H_{1+\theta'}^{\uparrow}(X|Y).
\]

(19)

Next, we investigate some properties of the measures defined above, which will be proved in Appendix D.

**Lemma 3**

1. For fixed \( Q_Y \), \( \theta H_{1+\theta}(P_{XY}|Q_Y) \) is a concave function of \( \theta \), and it is strict concave iff. \( \text{Var} \left[ \log \frac{Q_Y(y)}{P_{XY}(X,Y)} \right] > 0 \).

2. For fixed \( Q_Y \), \( H_{1+\theta}^{\uparrow}(P_{XY}|Q_Y) \) is a monotonically decreasing\(^{12}\) function of \( \theta \).

3. The function \( \theta H_{1+\theta}^{\uparrow}(P_{XY}|Q_Y) \) is a concave function of \( \theta \), and it is strict concave iff. \( \mathbb{V}(X|Y) > 0 \).

4. \( H_{1+\theta}^{\uparrow}(X|Y) \) is a monotonically decreasing function of \( \theta \).

5. The function \( \theta H_{1+\theta}^{\downarrow}(X|Y) \) is a concave function of \( \theta \), and it is strict concave iff. \( \mathbb{V}(X|Y) > 0 \).

6. \( H_{1+\theta}^{\downarrow}(X|Y) \) is a monotonically decreasing function of \( \theta \).

7. For every \( \theta \in (-1, 0) \cup (0, \infty) \), we have \( H_{1+\theta}^{\uparrow}(X|Y) \leq H_{1+\theta}^{\downarrow}(X|Y) \).

8. For fixed \( \theta' \), the function \( \theta H_{1+\theta,1+\theta'}(X|Y) \) is a concave function of \( \theta \), and it is strict concave iff. \( \mathbb{V}(X|Y) > 0 \).

9. For fixed \( \theta' \), \( H_{1+\theta,1+\theta'}^{\uparrow}(X|Y) \) is a monotonically decreasing function of \( \theta \).

10. We have

\[
H_{1+\theta,1}(X|Y) = H_{1+\theta}^{\downarrow}(X|Y).
\]

(21)

\(^{11}\)For \( -1 < \theta < 0 \), (14) can be proved by using the Hölder inequality, and, for \( 0 < \theta \), (13) can be proved by using the reverse Hölder inequality \(^{57}\) Lemma 8).

\(^{12}\)Technically, \( H_{1+\theta}(P_{XY}|Q_Y) \) is always non-increasing and it is monotonically decreasing iff. strict concavity holds in Statement 1.

Similar remarks are also applied for other information measures throughout the paper.
11) We have
\[ H_{1+\theta,1+\theta}(X|Y) = H_{1+\theta}^\dagger(X|Y). \] (22)

12) For every \( \theta \in (-1, 0) \cup (0, \infty) \), \( H_{1+\theta,1+\theta}(X|Y) \) is maximized at \( \theta' = \theta \).

We can also derive explicit forms of the conditional Rényi entropies of order 0.

**Lemma 4** We have
\[ \lim_{\theta \to -1} H_{1+\theta}(P_{XY}|Q_Y) = H_0(P_{XY}|Q_Y) \]
\[ = \log \sum_y Q_Y(y)|\text{supp}(P_{X|Y}(\cdot|y))|, \] (23)
\[ \lim_{\theta \to -1} H_{1+\theta}^\dagger(X|Y) = H_0^\dagger(X|Y) \]
\[ = \log \max_{y \in \text{supp}(P_Y)} |\text{supp}(P_{X|Y}(\cdot|y))|, \] (25)
\[ \lim_{\theta \to -1} H_{1+\theta}^\downarrow(X|Y) = H_0^\downarrow(X|Y) \]
\[ = \log \sum_y P_Y(y)|\text{supp}(P_{X|Y}(\cdot|y))|. \] (28)

**Proof:** See Appendix 4.

From Statement 12 of Lemma 3, \( \frac{d[\theta H_{1+\theta}(P_{XY}|Q_Y)]}{d\theta} \) is monotonically decreasing. Thus, we can define the inverse function \( \theta(a) = \theta^Q(a) \) of \( \frac{d[\theta H_{1+\theta}(P_{XY}|Q_Y)]}{d\theta} \) by
\[ \frac{d[\theta H_{1+\theta}(P_{XY}|Q_Y)]}{d\theta} \bigg|_{\theta=\theta(a)} = a \] (29)
for \( \underline{a} < a \leq \overline{a} \), where \( \underline{a} = aQ := \lim_{\theta \to -\infty} \frac{d[\theta H_{1+\theta}(P_{XY}|Q_Y)]}{d\theta} \) and \( \overline{a}Q := \lim_{\theta \to -1} \frac{d[\theta H_{1+\theta}(P_{XY}|Q_Y)]}{d\theta} \). Let
\[ R(a) = R^Q(a) := (1 + \theta(a))a - \theta(a)H_{1+\theta}(P_{XY}|Q_Y). \] (30)

Since
\[ R'(a) = (1 + \theta(a)), \] (31)
\( R(a) \) is a monotonic increasing function of \( \underline{a} < a \leq R(\overline{a}) \). Thus, we can define the inverse function \( a(R) = a^Q(R) \) of \( R(a) \) by
\[ (1 + \theta(a(R)))a(R) - \theta(a(R))H_{1+\theta(a(R))}(P_{XY}|Q_Y) = R \] (32)
for \( R(a) < R \leq H_0(P_{XY}|Q_Y) \).

For \( \theta H_{1+\theta}^\dagger(X|Y) \), by the same reason as above, we can define the inverse functions \( \theta(a) = \theta^\dagger(a) \) and \( a(R) = a^\dagger(R) \) by
\[ \frac{d[\theta H_{1+\theta}^\dagger(X|Y)]}{d\theta} \bigg|_{\theta=\theta(a)} = a \] (33)
and
\[ (1 + \theta(a(R)))a(R) - \theta(a(R))H_{1+\theta(a(R))}^\dagger(X|Y) = R, \] (34)

\(^{13}\)Throughout the paper, the notations \( \theta(a) \) and \( a(R) \) are reused for several inverse functions. Although the meanings of those notations are obvious from the context, we occasionally put superscript \( Q, \dagger \) or \( ^\dagger \) to emphasize that those inverse functions are induced from corresponding conditional Rényi entropies. This definition is related to Legendre transform of the concave function \( \theta \mapsto \theta H_{1+\theta}^\dagger(X|Y) \).
for \( R(a) < R \leq H_0^1(X|Y) \). For \( \theta H_{1+\theta}^1(X|Y) \), we also introduce the inverse functions \( \theta(a) = \theta^+(a) \) and \( a(R) = a^+(R) \) by

\[
\frac{d\theta H_{1+\theta}^1(X|Y)}{d\theta}
\bigg|_{\theta = \theta(a)} = a
\]  

(35)

and

\[
(1 + \theta(a(R)))a(R) - \theta(a(R))H_{1+\theta(a(R))}^1(X|Y) = R
\]  

(36)

for \( R(a) < R \leq H_0^1(X|Y) \).

**Remark 1** Here, we discuss the possibility for extension to the continuous case. Since the entropy on the continuous diverges, we cannot extend the information quantities to the case when \( \mathcal{X} \) is continuous. However, it is possible to extend these quantities to the case when \( \mathcal{Y} \) is continuous but \( \mathcal{X} \) is a discrete finite set. In this case, we prepare a general measure \( \mu \) (like the Lebesgue measure) on \( \mathcal{Y} \) and probability density function \( p_Y \) and \( q_Y \) such that the distributions \( P_Y \) and \( Q_Y \) are given as \( p_Y(y)\mu(dy) \) and \( q_Y(y)\mu(dy) \), respectively. Then, it is sufficient to replace \( \sum, Q(y) \), and \( P_{XY}(x,y) \) by \( \int_Y \mu(dy) \), \( P_{XY}(x,y)p_Y(y) \), and \( q_Y(y) \), respectively. Hence, in the \( n \)-independent and identical distributed case, these information measures are given as \( n \) times of the original information measures.

One might consider the information quantities for transition matrices given in the next subsection to this continuous case. However, it is not so easy because it needs a continuous extension of the Perron eigenvalue.

**B. Information Measures for Transition Matrix**

Let \( \{W(x, y|x', y')\} \) be an ergodic and irreducible transition matrix. The purpose of this section is to introduce transition matrix counterparts of those measures in Section II-A. For this purpose, we first need to introduce some assumptions on transition matrices:

**Assumption 1 (Non-Hidden)** We say that a transition matrix \( W \) is *non-hidden* (with respect to \( \mathcal{Y} \)) if\(^{14}\)

\[
\sum_x W(x, y|x', y') = W(y|y')
\]  

(37)

for every \( x' \in \mathcal{X} \) and \( y, y' \in \mathcal{Y} \). This condition is equivalent to the existence of the following decomposition of \( W(x, y|x', y') \):

\[
W(x, y|x', y') = W(y|y')W(x|x', y', y).
\]  

(38)

**Assumption 2 (Strongly Non-Hidden)** We say that a transition matrix \( W \) is *strongly non-hidden* (with respect to \( \mathcal{Y} \)) if, for every \( \theta \in (-1, \infty) \) and \( y, y' \in \mathcal{Y} \),

\[
W_\theta(y|y') := \sum_x W(x, y|x', y')^{1+\theta}
\]  

(39)

is well defined, i.e., the right hand side of (39) is independent of \( x' \).

**Remark 2** Assumption 2 has another expression as follows. Assumption 2 holds if and only if, for every \( x' \neq \tilde{x}' \), there exists a permutation \( \pi_{x'x} \) on \( \mathcal{X} \) such that \( W(x|x'|, y'|y) = W(\pi_{x'x}(x)|\tilde{x}', y') \).

Now, we fix an element \( x_0 \in \mathcal{X} \), and transform a sequence of random numbers \( (X_1, Y_1, X_2, Y_2, \ldots, X_n, Y_n) \) to the sequence of random numbers \( (X'_1, Y'_1, X'_2, Y'_2, \ldots, X'_n, Y'_n) := (X_1, Y_1, \pi_{x_0,x}(X_2), Y_2, \ldots, \pi_{x_0,x}(X_n), Y_n) \). Then, the reason for the name “non-hidden” is the following. In general, the random variable \( Y \) is subject to a hidden Markov process. However, when the condition (37) holds, the random variable \( Y \) is subject to a Markov process. Hence, we call the condition (37) non-hidden.
letting $W'(x|y', y) := W(x|x_0, y', y)$, we have $P_{X_i, Y_i|X_{i-1}, Y_{i-1}} = W'(y'_i|y'_{i-1}) W(x'_i|y'_i, y_{i-1})$. That is, essentially, the transition matrix of this case can be written by the transition matrix $W(y'_i|y'_{i-1}) W'(x'_i|y'_i)$. So, the transition matrix can be written by using the positive-entry matrix $W_{X_i}(y'_i|y'_{i-1}) := W(y'_i|y'_{i-1}) W'(x'_i|y'_i, y_{i-1})$.

Since the part “if” is trivial, we show the part “only if” as follow. By noting (38), Assumption 2 can be rephrased as

$$\sum_{x} W(x|x', y', y)^{1+\theta}$$

(40)

do not depend on $x'$ for every $\theta \in (-1, \infty)$. Furthermore, this condition can be rephrased as follows. For $x' \neq \hat{x}'$, if the largest values of $\{W(x|x', y')\}_{x \in X}$ and $\{W(x|\hat{x}', y')\}_{x \in X}$ are different, say the former is larger, then $\sum_x W(x|x', y')^{1+\theta} > \sum_x W(x|\hat{x}', y')^{1+\theta}$ for sufficiently large $\theta$, which contradict the fact that (40) does not depend on $x'$. Thus, the largest values of $\{W(x|x', y')\}_{x \in X}$ and $\{W(x|\hat{x}', y')\}_{x \in X}$ must coincide. By repeating this argument for the second largest value of $\{W(x|x', y')\}_{x \in X}$ and $\{W(x|\hat{x}', y')\}_{x \in X}$ and so on, we find Assumption 2 implies that for every $x' \neq \hat{x}'$, there exists a permutation $\pi_{x', \hat{x}'}$ on $X$ such that $W(x|x', y', y) = W(\pi_{x', \hat{x}'}(x)|\hat{x}', y', y)$.

The followings are non-trivial examples satisfying Assumption 1 and Assumption 2.

**Example 1** Suppose that $X = Y$ are a module. Let $P$ and $Q$ be transition matrices on $X$. Then, the transition matrix given by

$$W(x, y|x', y') = Q(y|y') P(x-y|x'-y')$$

(41)

satisfies Assumption 1. Furthermore, if transition matrix $P(z|z')$ can be written as

$$P(z|z') = P_Z(\pi_{z'}(z))$$

(42)

for permutation $\pi_{z'}$ and a distribution $P_Z$ on $X$, then transition matrix $W$ defined by (41) satisfies Assumption 2 as well.

**Example 2** Suppose that $X$ is a module, and $W$ is (strongly) non-hidden with respect to $Y$. Let $Q$ be a transition matrix on $Z = X$. Then, the transition matrix given by

$$V(x, y, z|x', y', z') = W(x-z, y|x'-z', y) Q(z|z')$$

(43)

is (strongly) non-hidden with respect to $Y \times Z$.

The following is also an example satisfying Assumption 2 which describes a noise process of an important class of channels with memory (cf. Example 6).

**Example 3** Let $X = Y = \{0, 1\}$. Then, let

$$W(y|y') = \begin{cases} 1 - qy' & \text{if } y = y' \\ qy' & \text{if } y \neq y' \end{cases}$$

(44)

for some $0 < q_0, q_1 < 1$, and let

$$W(x|x', y', y) = \begin{cases} 1 - py & \text{if } x = 0 \\ py & \text{if } x = 1 \end{cases}$$

(45)

for some $0 < p_0, p_1 < 1$. By choosing $\pi_{x', \hat{x}'}$ to be the identity, this transition matrix satisfies the condition given in Remark 2 that is equivalent to Assumption 2.

First, we introduce information measures under Assumption 1. In order to define a transition matrix counterpart of (7), let us introduce the following tilted matrix:

$$\tilde{W}_\theta(x, y|x', y') := W(x, y|x', y')^{1+\theta} W(y|y')^{-\theta}.$$
Here, we should notice that the tilted matrix $\tilde{W}_\theta$ is not normalized, i.e., is not a transition matrix. Let $\lambda_\theta$ be the Perron-Frobenius eigenvalue of $\tilde{W}_\theta$ and $\tilde{P}_{\theta,X Y}$ be its normalized eigenvector. Then, we define the lower conditional Rényi entropy for $W$ by

$$H_{1+\theta}^{\downarrow W}(X|Y) := -\frac{1}{\theta} \log \lambda_\theta,$$

where $\theta \in (-1, 0) \cup (0, \infty)$. For $\theta = 0$, we define the lower conditional Rényi entropy for $W$ by

$$H^W(X|Y) = H_{1}^{\downarrow W}(X|Y)$$

$$:= \lim_{\theta \to 0} H_{1+\theta}^{\downarrow W}(X|Y),$$

and we just call it the conditional entropy for $W$. In fact, the definition of $H^W(X|Y)$ above coincide with

$$-\sum_{x',y'} P_{0,X Y}(x',y') \sum_{x,y} W(x,y|x',y') \log \frac{W(x,y|x',y')}{W(y|y')},$$

where $P_{0,X Y}$ is the stationary distribution of $W$ (cf. [58, Eq. (30)]). For $\theta = -1$, $H_{0}^{\downarrow W}(X|Y)$ is also defined by taking the limit. When $Y$ is singleton, the Rényi entropy $H_{1+\theta}^{\downarrow W}(X)$ for $W$ is defined as a special case of $H_{1+\theta}^{\downarrow W}(X|Y)$.

As a counterpart of (11), we also define

$$V^W(X|Y) := \frac{2}{\theta} \left[ H^W(X|Y) - H_{1+\theta}^{\downarrow W}(X|Y) \right].$$

**Remark 3** When transition matrix $W$ satisfies Assumption [2], $H_{1+\theta}^{\downarrow W}(X|Y)$ can be written as

$$H_{1+\theta}^{\downarrow W}(X|Y) = -\frac{1}{\theta} \log \lambda'_\theta,$$

where $\lambda'_\theta$ is the Perron-Frobenius eigenvalue of $W_\theta(y|y')W(y|y')^{-\theta}$. In fact, for the left Perron-Frobenius eigenvector $\tilde{Q}_\theta$ of $W_\theta(y|y')W(y|y')^{-\theta}$, we have

$$\sum_{x,y} \tilde{Q}_\theta(y) W(x,y|x',y')^{1+\theta} W(y|y')^{-\theta} = \lambda'_\theta \tilde{Q}_\theta(y'),$$

which implies that $\lambda'_\theta$ is the Perron-Frobenius eigenvalue of $\tilde{W}_\theta$. Consequently, we can evaluate $H_{1+\theta}^{\downarrow W}(X|Y)$ by calculating the Perron-Frobenius eigenvalue of $|X| \times |Y|$ matrix instead of $|X||Y| \times |X||Y|$ matrix when $W$ satisfies Assumption [2]

Next, we introduce information measures under Assumption [2]. In order to define a transition matrix counterpart of (12), let us introduce the following $|Y| \times |Y|$ matrix:

$$K_\theta(y|y') := W_\theta(y|y') \frac{\sum_{x,y} x K_\theta(y|x) W(x,y|x',y')^{1+\theta} W(y|y')^{-\theta}}{\sum_{x,y} x K_\theta(y|x) W(x,y|x',y')^{1+\theta} W(y|y')^{-\theta}},$$

where $W_\theta$ is defined by [39]. Let $\kappa_\theta$ be the Perron-Frobenius eigenvalue of $K_\theta$. Then, we define the upper conditional Rényi entropy for $W$ by

$$H_{1+\theta}^{\uparrow W}(X|Y) := -\frac{1+\theta}{\theta} \log \kappa_\theta,$$

where $\theta \in (-1, 0) \cup (0, \infty)$. For $\theta = -1$ and $\theta = 0$, $H_{1+\theta}^{\uparrow W}(X|Y)$ is defined by taking the limit. We have the following properties, which will be proved in Appendix [2]

**Lemma 5** We have

$$\lim_{\theta \to 0} H_{1+\theta}^{\uparrow W}(X|Y) = H^W(X|Y)$$

15Since the limiting expression in (51) coincides with the second derivative of the CGF (cf. [298]), and since the second derivative of the CGF exists (cf. [22, Appendix D]), the variance in (51) is well defined.
and
\[
\lim_{\theta \to 0} \frac{2 \left[ H^W(X|Y) - H_{1+\theta}^W(X|Y) \right]}{\theta} = V^W(X|Y).
\] (57)

Now, let us introduce a transition matrix counterpart of \([18]\). For this purpose, we introduce the following \(|\mathcal{Y}| \times |\mathcal{Y}|\) matrix:
\[
N_{\theta,\theta'}(y|y') := W_\theta(y|y')W_{\theta'}(y'|y')^{1-\theta'}. \tag{58}
\]
Let \(\nu_{\theta,\theta'}\) be the Perron-Frobenius eigenvalue of \(N_{\theta,\theta'}\). Then, we define the two-parameter conditional Rényi entropy by
\[
H_{1+\theta,1+\theta'}^W(X|Y) := -\frac{1}{\theta} \log \nu_{\theta,\theta'} + \frac{\theta'}{1+\theta'} H_{1+\theta}^W(X|Y). \tag{59}
\]

**Remark 4** Although we defined \(H_{1+\theta}^W(X|Y)\) and \(H_{1+\theta}^\uparrow W(X|Y)\) by \((47)\) and \((55)\) respectively, we can alternatively define these measures in the same spirit as the single-shot setting by introducing a transition matrix counterpart of \(H_{1+\theta}(P_{XY}|Q_Y)\) as follows. For the marginal \(W(y|y')\) of \(W(x,y|x',y')\), let \(\mathcal{Y}_W^2 := \{(y, y') : W(y|y') > 0\}\). For another transition matrix \(V\) on \(\mathcal{Y}\), we define \(\mathcal{Y}_V^2\) in a similar manner. For \(V\) satisfying \(\mathcal{Y}_W^2 \subset \mathcal{Y}_V^2\), we define
\[
H_{1+\theta}^{W|V}(X|Y) := -\frac{1}{\theta} \log \lambda_{\theta}^{W|V}
\]
for \(\theta \in (-1, 0) \cup (0, \infty)\), where \(\lambda_{\theta}^{W|V}\) is the Perron-Frobenius eigenvalue of
\[
W(x, y|x', y')^{1+\theta}V(y|y')^{-\theta}. \tag{61}
\]
By using this measure, we obviously have
\[
H_{1+\theta}^W(X|Y) = H_{1+\theta}^{W|V}(X|Y). \tag{62}
\]
Furthermore, under Assumption 2 we can show that
\[
H_{1+\theta}^\uparrow W(X|Y) = \max_V H_{1+\theta}^{W|V}(X|Y) \tag{63}
\]
holds (see Appendix C for the proof), where the maximum is taken over all transition matrices satisfying \(\mathcal{Y}_W^2 \subset \mathcal{Y}_V^2\).

Next, we investigate some properties of the information measures introduced in this section. The following lemma is proved in Appendix D.

**Lemma 6**
1. The function \(\theta H_{1+\theta}^W(X|Y)\) is a concave function of \(\theta\), and it is strict concave iff. \(V^W(X|Y) > 0\).
2. \(H_{1+\theta}^W(X|Y)\) is a monotonically decreasing function of \(\theta\).
3. The function \(\theta H_{1+\theta}^\uparrow W(X|Y)\) is a concave function of \(\theta\), and it is strict concave iff. \(V^W(X|Y) > 0\).
4. \(H_{1+\theta}^\uparrow W(X|Y)\) is a monotonically decreasing function of \(\theta\).
5. For every \(\theta \in (-1, 0) \cup (0, \infty)\), we have \(H_{1+\theta}^W(X|Y) \leq H_{1+\theta}^\uparrow W(X|Y)\).
6. For fixed \(\theta'\), the function \(\theta H_{1+\theta,1+\theta'}^W(X|Y)\) is a concave function of \(\theta\), and it is strict concave iff. \(V^W(X|Y) > 0\).
7. For fixed \(\theta'\), \(H_{1+\theta,1+\theta'}^W(X|Y)\) is a monotonically decreasing function of \(\theta\).
8. We have
\[
H_{1+\theta,1}^W(X|Y) = H_{1+\theta}^W(X|Y). \tag{64}
\]
9. We have
\[
H_{1+\theta,1+\theta'}^W(X|Y) = H_{1+\theta}^\uparrow W(X|Y). \tag{65}
\]

\footnote{\(\)Although we can also define \(H_{1+\theta}^{W|V}(X|Y)\) even if \(\mathcal{Y}_W^2 \subset \mathcal{Y}_V^2\) is not satisfied (see \([22]\) for the detail), for our purpose of defining \(H_{1+\theta}^W(X|Y)\) and \(H_{1+\theta}^\uparrow W(X|Y)\), other cases are irrelevant.}
10) For every \( \theta \in (-1, 0) \cup (0, \infty) \), \( H_{1+\theta,1+\theta}^W(X|Y) \) is maximized at \( \theta' = \theta \), i.e.,

\[
\left. \frac{d[H_{1+\theta,1+\theta}^W(X|Y)]}{d\theta} \right|_{\theta' = \theta} = 0.
\] (66)

From Statement 10 of Lemma 6 of (66), \( \frac{d[H_{1+\theta,1+\theta}^W(X|Y)]}{d\theta} \) is monotonically decreasing. Thus, we can define the inverse function \( \theta(a) = \theta^{\dagger}(a) \) of \( \frac{d[H_{1+\theta,1+\theta}^W(X|Y)]}{d\theta} \) by

\[
\left. \frac{d[H_{1+\theta,1+\theta}^W(X|Y)]}{d\theta} \right|_{\theta = \theta(a)} = a \] (67)

for \( a < a \leq \overline{a} \), where \( \underline{a} := \lim_{\theta \to \infty} \frac{d[H_{1+\theta,1+\theta}^W(X|Y)]}{d\theta} \) and \( \overline{a} := \lim_{\theta \to -1} \frac{d[H_{1+\theta,1+\theta}^W(X|Y)]}{d\theta} \). Let

\[ R(a) := (1 + \theta(a))a - \theta(a)H_{1+\theta(a)}^+(X|Y). \] (68)

Since

\[ R'(a) = (1 + \theta(a)), \] (69)

\( R(a) \) is a monotonic increasing function of \( \underline{a} < a < R(\overline{a}) \). Thus, we can define the inverse function \( a(R) = a^{\dagger}(R) \) of \( R(a) \) by

\[ (1 + \theta(a(R)))a(R) - \theta(a(R))H_{1+\theta(a(R))}^+(X|Y) = R \] (70)

for \( R(\underline{a}) < R < H_{1,1+\theta}^+(X|Y) \), where \( H_{0,1}^+(X|Y) := \lim_{\theta \to -1} H_{1+\theta}^+(X|Y) \).

For \( \theta H_{1+\theta}^+(X|Y) \), by the same reason, we can define the inverse function \( \theta(a) = \theta^{\dagger}(a) \) by

\[
\left. \frac{d[H_{1+\theta,1+\theta}^W(X|Y)]}{d\theta} \right|_{\theta = \theta(a)} = \left. \frac{d[H_{1+\theta}^+(X|Y)]}{d\theta} \right|_{\theta = \theta(a)} = a, \] (71)

and the inverse function \( a(R) = a^{\dagger}(R) \) of

\[ R(a) := (1 + \theta(a))a - \theta(a)H_{1+\theta(a)}^+(X|Y) \] (72)

by

\[ (1 + \theta(a(R)))a(R) - \theta(a(R))H_{1+\theta(a(R))}^+(X|Y) = R, \] (73)

for \( R(\underline{a}) < R < H_{1,1+\theta}^+(X|Y) \), where \( H_{0,1}^+(X|Y) := \lim_{\theta \to -1} H_{1+\theta}^+(X|Y) \). Here, the first equality in (71) follows from (66).

Since \( \theta \to \theta H_{1+\theta}^+(X|Y) \) is concave, and \(-1 \leq \theta(R) \leq 0 \) for \( H^W(X|Y) \leq R \leq H_{0,1}^+(X|Y) \), we can prove the following.

**Lemma 7** The function \( \theta(R) \) defined in (67) satisfies

\[
\sup_{-1 \leq \theta \leq 0} [-\theta R + \theta H_{1+\theta}^+(X|Y)] = -\theta(R)R + \theta(R)H_{1+\theta(R)}^+(X|Y) \] (74)

for \( H^W(X|Y) \leq R \leq H_{0,1}^+(X|Y) \).

Furthermore, we can show the following.

**Lemma 8** The function \( \theta(a(R)) \) defined by (70) satisfies

\[
\sup_{-1 \leq \theta \leq 0} \frac{-\theta R + \theta H_{1+\theta}^+(X|Y)}{1 + \theta} = -\theta(a(R))a(R) + \theta(a(R))H_{1+\theta(a(R))}^+(X|Y) \] (75)
Taking the derivative, (67) implies that

\[
H^{\downarrow, W}_{1+\theta}(X|Y) = -\frac{\theta R}{1+\theta} + \frac{1}{2} V^{W}(X|Y)\theta + o(\theta),
\]  

(77)

and

\[
H^{\downarrow, W}_{1+\theta}(X|Y) = H^{W}(X|Y) - \frac{1}{2} V^{W}(X|Y)\theta + o(\theta),
\]  

(78)

for \( H^{W}(X|Y) \leq R \leq H^{\downarrow, W}_{0}(X|Y) \), and the function \( \theta(a(R)) \) defined in (75) satisfies

\[
\sup_{-\theta \leq 0} \frac{-\theta R + \theta H^{\downarrow, W}_{1+\theta}(X|Y)}{1+\theta} = -\theta(a(R))a(R) + \theta(a(R))\sup_{0} H^{\downarrow, W}_{1+\theta(a(R))}(X|Y)
\]  

(76)

for \( H^{W}(X|Y) \leq R \leq H^{\downarrow, W}_{0}(X|Y) \).

**Proof:** See Appendix [1].

**Remark 5** As we can find from [49], [51], and Lemma [5] both the conditional Rényi entropies expand as

\[
H^{\downarrow, W}_{1+\theta}(X|Y) = H^{W}(X|Y) - \frac{1}{2} V^{W}(X|Y)\theta + o(\theta),
\]  

(77)

and

\[
H^{\downarrow, W}_{1+\theta}(X|Y) = H^{W}(X|Y) - \frac{1}{2} V^{W}(X|Y)\theta + o(\theta)
\]  

(78)

around \( \theta = 0 \). Thus, the difference of these measures significantly appear only when \( |\theta| \) is rather large. For the transition matrix of Example 3 with \( q_{0} = q_{1} = 0.1 \), \( p_{0} = 0.1 \), and \( p_{1} = 0.4 \), we plotted the values of the information measures in Fig. 1. Although the values at \( \theta = -1 \) coincide in Fig. 1, note that the values at \( \theta = -1 \) may differ in general.

In Example 1, we have mentioned that transition matrix \( P \) in (41) satisfies Assumption [2] when transition matrix \( W \) is given by (42). In this case, we can find that

\[
H^{\downarrow, W}_{1+\theta}(X|Y) = H^{\downarrow, W}_{1+\theta}(X|Y)
\]  

(79)

\[
= H_{1+\theta}(P_Z),
\]  

(80)

i.e., the two kinds of conditional Rényi entropies coincide.

Now, let’s consider asymptotic behavior of \( H^{\downarrow, W}_{1+\theta}(X|Y) \) around \( \theta = 0 \). When \( \theta(a) \) is close to 0, we have

\[
\theta(a)H^{\downarrow, W}_{1+\theta(a)}(X|Y) = \theta(a)H^{W}(X|Y) - \frac{1}{2} V^{W}(X|Y)\theta(a)^{2} + o(\theta(a)^{2}).
\]  

(81)

Taking the derivative, (67) implies that

\[
a = H^{W}(X|Y) - V^{W}(X|Y)\theta(a) + o(\theta(a)).
\]  

(82)

Hence, when \( R \) is close to \( H^{W}(X|Y) \), we have

\[
R = (1 + \theta(a(R))\theta(a) - \theta(a(R))H^{\downarrow, W}_{1+\theta(a(R))}(X|Y)
\]  

(83)

\[
= H^{W}(X|Y) - \left(1 + \frac{\theta(a(R))}{2}\right)\theta(a(R))V^{W}(X|Y) + o(\theta(a(R)),
\]  

(84)
i.e.,
\[
\theta(a(R)) = \frac{-R + H^W(X|Y)}{V^W(X|Y)} + o\left(\frac{R - H^W(X|Y)}{V^W(X|Y)}\right).
\] (85)

Furthermore, (81) and (82) imply
\[
-\theta(a(R))a(R) + \theta(a(R))H_{1+\theta(a(R))}^W(X|Y) = V^W(X|Y)\frac{\theta(a(R))^2}{2} + o(\theta(a(R))^2)
\] (86)
\[
= \frac{V^W(X|Y)}{2} \left(\frac{R - H^W(X|Y)}{V^W(X|Y)}\right)^2 + o\left(\frac{R - H^W(X|Y)}{V^W(X|Y)}\right)^2.
\] (87)
\[
(88)
\]

C. Information Measures for Markov Chain

Let \((X, Y)\) be the Markov chain induced by transition matrix \(W\) and some initial distribution \(P_{X_1Y_1}\). Now, we show how information measures introduced in Section II-B are related to the conditional Rényi entropy rates. First, we introduce the following lemma, which gives finite upper and lower bounds on the lower conditional Rényi entropy.

Lemma 9 Suppose that transition matrix \(W\) satisfies Assumption [1] Let \(v_\theta\) be the eigenvector of \(W^T\) with respect to the Perron-Frobenius eigenvalue \(\lambda_\theta\) such that \(\min_{x, y} v_\theta(x, y) = 1\). Let \(w_\theta(x, y) := P_{X_1Y_1}(x, y)^1 + \theta P_{Y_1}(y)^{-\theta}\). Then, for every \(n \geq 1\), we have
\[
(n - 1)\theta H_{1+\theta}^W(X|Y) + \delta(\theta) \leq \theta H_{1+\theta}^W(X^n|Y^n) \leq (n - 1)\theta H_{1+\theta}^W(X|Y) + \overline{\delta}(\theta),
\] (89)

where
\[
\overline{\delta}(\theta) := -\log\left(v_\theta w_\theta\right) + \log\max_{x, y} v_\theta(x, y),
\] (90)
\[
\overline{\delta}(\theta) := -\log\left(v_\theta w_\theta\right),
\] (91)
and \(\langle v_\theta|w_\theta\rangle\) is defined as \(\sum_{x, y} v_\theta(x, y) w_\theta(x, y)\).

Proof: It follows from (297) and Lemma 26. From Lemma 9, we have the following.

Theorem 1 Suppose that transition matrix \(W\) satisfies Assumption [1] For any initial distribution, we have
\[
\lim_{n \to \infty} \frac{1}{n} H_{1+\theta}^W(X^n|Y^n) = H_{1+\theta}^W(X|Y),
\] (92)
\[
\lim_{n \to \infty} \frac{1}{n} H(X^n|Y^n) = H^W(X|Y).
\] (93)

We also have the following asymptotic evaluation of the variance, which follows from Lemma 27 in Appendix A

Theorem 2 Suppose that transition matrix \(W\) satisfies Assumption [1] For any initial distribution, we have
\[
\lim_{n \to \infty} \frac{1}{n} \mathbb{V}(X^n|Y^n) = \mathbb{V}^W(X|Y).
\] (94)

Theorem 2 is practically important since the limit of the variance can be described by a single letter characterized quantity. A method to calculate \(\mathbb{V}^W(X|Y)\) can be found in [23].

Next, we show the lemma that gives finite upper and lower bound on the upper conditional Rényi entropy in terms of the upper conditional Rényi entropy for the transition matrix.

\[\text{Since the eigenvector corresponding to the Perron-Frobenius eigenvalue for an irreducible non-negative matrix has always strictly positive entries [59, Theorem 8.4.4, p. 508], we can choose the eigenvector \(v_\theta\) satisfying this condition.}\]
Lemma 10 Suppose that transition matrix $W$ satisfies Assumption 2. Let $v_{\theta}$ be the eigenvector of $K_{\theta}^T$ with respect to the Perro-Frobenius eigenvalue $\kappa_{\theta}$ such that $\min_{y} v_{\theta}(y) = 1$. Let $w_{\theta}$ be the $|\mathcal{Y}|$-dimensional vector defined by

$$w_{\theta}(y) := \left[ \sum_{x} P_{X,Y}(x,y)^{1+\theta} \right]^{\frac{1}{1+\theta}}. \quad (95)$$

Then, we have

$$(n-1) \frac{\theta}{1+\theta} H_{1+\theta}^W(X|Y) + \overline{\xi}(\theta) \leq \frac{\theta}{1+\theta} H_{1+\theta}(X^n|Y^n) \leq (n-1) \frac{\theta}{1+\theta} H_{1+\theta}^W(X|Y) + \overline{\xi}(\theta), \quad (96)$$

where

$$\overline{\xi}(\theta) := -\log(v_{\theta}|w_{\theta}) + \log \max_y v_{\theta}(y), \quad \overline{\xi}(\theta) := -\log(v_{\theta}|w_{\theta}). \quad (97)$$

Proof: See Appendix [1].

From Lemma 10, we have the following.

Theorem 3 Suppose that transition matrix $W$ satisfies Assumption 2. For any initial distribution, we have

$$\lim_{n \to \infty} \frac{1}{n} H_{1+\theta}^W(X^n|Y^n) = H_{1+\theta}^W(X|Y). \quad (99)$$

Finally, we show the lemma that gives finite upper and lower bounds on the two-parameter conditional Rényi entropy in terms of the two-parameter conditional Rényi entropy for the transition matrix.

Lemma 11 Suppose that transition matrix $W$ satisfies Assumption 2. Let $v_{\theta,\theta'}$ be the eigenvector of $N_{\theta,\theta'}^T$, with respect to the Perro-Frobenius eigenvalue $v_{\theta,\theta'}$ such that $\min_{y} v_{\theta,\theta'}(y) = 1$. Let $w_{\theta,\theta'}$ be the $|\mathcal{Y}|$-dimensional vector defined by

$$w_{\theta,\theta'}(y) := \left[ \sum_{x} P_{X,Y}(x,y)^{1+\theta} \right] \left[ \sum_{x} P_{X,Y}(x,y)^{1+\theta'} \right]^{\frac{1}{1+\theta'}}. \quad (100)$$

Then, we have

$$(n-1) \theta H_{1+\theta,1+\theta'}^W(X|Y) + \overline{\zeta}(\theta,\theta') \leq \theta H_{1+\theta,1+\theta'}(X^n|Y^n) \leq (n-1) \theta H_{1+\theta,1+\theta'}^W(X|Y) + \overline{\zeta}(\theta,\theta'), \quad (101)$$

where

$$\overline{\zeta}(\theta,\theta') := -\log(v_{\theta,\theta'}|w_{\theta,\theta'}) + \log \max_y v_{\theta,\theta'}(y) + \theta \overline{\xi}(\theta'), \quad (102)$$

$$\overline{\zeta}(\theta,\theta') := -\log(v_{\theta,\theta'}|w_{\theta,\theta'}) + \theta \overline{\xi}(\theta'), \quad (103)$$

for $\theta > 0$ and

$$\overline{\zeta}(\theta,\theta') := -\log(v_{\theta,\theta'}|w_{\theta,\theta'}) + \log \max_y v_{\theta,\theta'}(y) + \theta \overline{\xi}(\theta'), \quad (104)$$

$$\overline{\zeta}(\theta,\theta') := -\log(v_{\theta,\theta'}|w_{\theta,\theta'}) + \theta \overline{\xi}(\theta') \quad (105)$$

for $\theta < 0$.

Proof: We can write

$$\theta H_{1+\theta,1+\theta'}(X^n|Y^n) \quad (106)$$

$$= -\log \sum_{y^n} \left[ \sum_{x^n} P_{X^n,Y^n}(x^n,y^n)^{1+\theta} \right] \left[ \sum_{x^n} P_{X^n,Y^n}(x^n,y^n)^{1+\theta'} \right]^{\frac{1}{1+\theta'}} + \frac{\theta \theta'}{1+\theta'} H_{1+\theta'}^W(X^n|Y^n). \quad (107)$$

The second term is evaluated by Lemma 10. The first term can be evaluated almost the same manner as Lemma 10.
From Lemma [11], we have the following.

**Theorem 4** Suppose that transition matrix $W$ satisfies Assumption [2]. For any initial distribution, we have

$$
\lim_{n \to \infty} \frac{1}{n} H_{1+\theta,1+\theta'}(X^n|Y^n) = H_{1+\theta,1+\theta'}(X|Y).
$$

(108)

## III. SOURCE CODING WITH FULL SIDE-INFORMATION

In this section, we investigate the source coding with side-information. We start this section by showing the problem setting in Section III-A. Then, we review and introduce some single-shot bounds in Section III-B. We derive finite-length bounds for the Markov chain in Section III-C. Then, in Sections III-F and III-E, we show the asymptotic characterization for the large deviation regime and the moderate deviation regime by using those finite-length bounds. We also derive the second order rate in Section III-D.

The results shown in this section are summarized in Table II. The checkmarks ✓ indicate that the tight asymptotic bounds (large deviation, moderate deviation, and second order) can be obtained from those bounds. The marks ✓* indicate that the large deviation bound can be derived up to the critical rate. The computational complexity "Tail" indicates that the computational complexities of those bounds depend on the computational complexities of tail probabilities. It should be noted that Theorem 8 is derived from a special case ($Q_Y = P_Y$) of Theorem 5. The asymptotically optimal choice is $Q_Y = P_Y^{1+\theta}$, which corresponds to Corollary [1]. Under Assumption [1] we can derive the bound of the Markov case only for that special choice of $Q_Y$, while under Assumption [2] we can derive the bound of the Markov case for the optimal choice of $Q_Y$.

### TABLE II

**SUMMARY OF THE BOUNDS FOR SOURCE CODING WITH FULL SIDE-INFORMATION.**

| Achiev./Conv. | Markov | Single Shot | $P_s$/$\bar{P}_s$ | Complexity | Large Deviation | Moderate Deviation | Second Order |
|---------------|--------|-------------|-----------------|-------------|-----------------|--------------------|-------------|
| Achievability | Theorem [5] (Ass. 1) | Lemma [14] | $P_s$ | $O(1)$ | ✓ | | |
| | Theorem [9] (Ass. 2) | Lemma [13] | $P_s$ | $O(1)$ | ✓ ✓ * | ✓ | |
| | Lemma [12] | $P_s$ | Tail | ✓ | ✓ | |
| Converse | Theorem [8] (Ass. 1) | (Theorem [5]) | $P_s$ | $O(1)$ | ✓ | | |
| | Theorem [10] (Ass. 2) | Corollary [1] | $P_s$ | $O(1)$ | ✓ ✓ * | ✓ | |
| | Lemma [17] | $P_s$ | Tail | ✓ | ✓ | |

### A. Problem Formulation

A code $\Psi = (e,d)$ consists of one encoder $e : \mathcal{X} \to \{1, \ldots, M\}$ and one decoder $d : \{1, \ldots, M\} \times \mathcal{Y} \to \mathcal{X}$. The decoding error probability is defined by

$$
P_s[\Psi] = P_s[\Psi|P_{XY}] = \Pr\{X \neq d(e(X), Y)\}.
$$

(109)

(110)

For notational convenience, we introduce the infimum of error probabilities under the condition that the message size is $M$:

$$
P_s(M) = \inf_{\Psi} P_s[\Psi|P_{XY}].
$$

(111)

(112)

For theoretical simplicity, we focus on a randomized choice of our encoder. For this purpose, we employ a randomized hash function $F$ from $\mathcal{X}$ to $\{1, \ldots, M\}$. A randomized hash function $F$ is called two-universal hash when $\Pr\{F(x) = F(x')\} \leq \frac{1}{M}$ for any distinctive $x$ and $x'$ [60]; the so-called bin coding [61] is an example of two-universal hash function. In the following, we denote the set of two-universal hash functions by $\mathcal{F}$. Given an encoder $f$ as a function from $\mathcal{X}$ to $\{1, \ldots, M\}$, we define the decoder $d_f$ as the optimal decoder by $\arg\min_{d} P_s[(f, d)]$. Then,
we denote the code \((f, d_f)\) by \(\Psi(f)\). Then, we bound the error probability \(P_s[\Psi(F)]\) averaged over the random function \(F\) by only using the property of two-universality. In order to consider the worst case of such schemes, we introduce the following quantity:

\[
\bar{P}_s(M) = \bar{P}_s(M|P_{XY}) := \sup_{F \in \mathcal{F}} \mathbb{E}_F[\bar{P}_s[\Psi(F)]] ,
\]

When we consider \(n\)-fold extension, the source code and related quantities are denoted with the superscript \((n)\). For example, the quantities in (112) and (114) are written to be \(\bar{P}_s^{(n)}(M)\) and \(\bar{P}_s^{(n)}(M)\), respectively. Instead of evaluating them, we are often interested in evaluating

\[
M(n, \varepsilon) := \inf \{ M_n : \bar{P}_s^{(n)}(M_n) \leq \varepsilon \},
\]

\[
\bar{M}(n, \varepsilon) := \inf \{ M_n : \bar{P}_s^{(n)}(M_n) \leq \varepsilon \}
\]

for given \(0 \leq \varepsilon < 1\).

### B. Single Shot Bounds

In this section, we review existing single shot bounds and also show novel converse bounds. For the information measures used below, see Section II.

By using the standard argument on information-spectrum approach, we have the following achievability bound.

**Lemma 12 (Lemma 7.2.1 of [4])** The following bound holds:

\[
\bar{P}_s(M) \leq \inf_{\gamma \geq 0} \left[ P_{XY} \left\{ \log \frac{1}{P_X|Y(x|y)} > \gamma \right\} + e^\gamma \frac{1}{M} \right].
\]

Although Lemma 12 is useful for the second-order regime, it is known to be not tight in the large deviation regime. By using the large deviation technique of Gallager, we have the following exponential type achievability bound.

**Lemma 13 ([62])** The following bound holds:

\[
\bar{P}_s(M) \leq \inf_{-\frac{1}{\theta} \leq \theta \leq 0} \frac{M^{\theta} e^{-\theta H^1_{1+\theta}(X|Y)}}{1+\theta}.
\]

Although Lemma 13 is known to be tight in the large deviation regime for i.i.d. sources, \(H^1_{1+\theta}(X|Y)\) for Markov chains can only be evaluated under the strongly non-hidden assumption. For this reason, even though the following bound is looser than Lemma 13 it is useful to have another bound in terms of \(H^1_{1+\theta}(X|Y)\), which can be evaluated for Markov chains under the non-hidden assumption.

**Lemma 14** The following bound holds:

\[
\bar{P}_s(M) \leq \inf_{-1 \leq \theta \leq 0} \frac{\theta^\theta e^{-\theta H^1_{1+\theta}(X|Y)}}{1+\theta}. \tag{119}
\]

**Proof:** To derive this bound, we change variable in (118) as \(\theta = \frac{\theta'}{1-\theta'}\). Then, \(-1 \leq \theta' \leq 0\), and we have

\[
M^{\theta'} e^{-\theta' H^1_{1+\theta'}(X|Y)} \leq M^{\theta'} e^{-\theta' H^1_{1+\theta'}(X|Y)},
\]

where we used Lemma 28 in Appendix C. \(\square\)

When \(Y\) is singleton, we have the following bound, which is tighter than Lemma 13.

**Lemma 15 ((2.39) [63])** The following bound holds:

\[
P_s(M) \leq \inf_{-1 < \theta \leq 0} \frac{\theta^\theta e^{-\theta H^1_{1+\theta}(X)}}{1+\theta}. \tag{120}
\]

\(\text{Note that the Gallager function and the upper conditional Rényi entropy are related by (313).}\)
For converse part, we first have the following bound, which is very close to the operational definition of source coding with side-information.

**Lemma 16 ([64])** Let \( \{ \Omega_y \}_{y \in Y} \) be a family of subsets \( \Omega_y \subset \mathcal{X} \), and let \( \Omega = \bigcup_{y \in Y} \Omega_y \times \{ y \} \). Then, for any \( Q_Y \in \mathcal{P}(Y) \), the following bound holds:

\[
P_\gamma(M) \geq \min_{\{ \Omega_y \}} \left\{ P_{XY}(\Omega^c) : \sum_y Q_Y(y)|\Omega_y| \leq M \right\}.
\]

(121)

Since Lemma 16 is close to the operational definition, it is not easy to evaluate Lemma 16. Thus, we derive another bound by loosening Lemma 16, which is more tractable for evaluation. Slightly weakening Lemma 16, we have the following.

**Lemma 17 ([4], [5])** For any \( Q_Y \in \mathcal{P}(Y) \), we have\(^{19}\)

\[
P_\gamma(M) \geq \sup_{\gamma \geq 0} \left[ P_{XY} \left\{ \log \frac{Q_Y(y)}{P_{XY}(x, y)} > \gamma \right\} - \frac{M}{\gamma} \right].
\]

(122)

By using the change-of-measure argument, we can also derive the following converse bound.

**Theorem 5** For any \( Q_Y \in \mathcal{P}(Y) \), we have

\[
- \log P_\gamma(M) \leq \inf_{\gamma > 0} \left[ (1+s) \tilde{\theta} \left\{ H_{1+\tilde{\theta}}(P_{XY}|Q_Y) - H_{1+(1+s)\tilde{\theta}}(P_{XY}|Q_Y) \right\} 
- (1+s) \log \left( 1 - 2e^{-\frac{s}{1+(1+s)(1+\tilde{\theta})(1+(1+s)\tilde{\theta})}} \right) \right] / s 
\]

(123)

\[
\leq \inf_{\gamma > 0} \left[ (1+s) \tilde{\theta} \left\{ H_{1+\tilde{\theta}}(P_{XY}|Q_Y) - H_{1+(1+s)\tilde{\theta}}(P_{XY}|Q_Y) \right\} 
- (1+s) \log \left( 1 - 2e^{\tilde{\theta} [\theta(a(R))-\tilde{\theta} a(R)] \tilde{\theta} a(R) \tilde{\theta} a(R) H_{1+\tilde{\theta}}(P_{XY}|Q_Y) + \tilde{\theta} H_{1+\tilde{\theta}}(P_{XY}|Q_Y) \right) \right] / s,
\]

(124)

(125)

(126)

where \( R = \log M \), and \( \theta(a) = \theta^Q(a) \) and \( a(R) = a^Q(R) \) are the inverse functions defined in (29) and (32), respectively.

*Proof:* See Appendix K.

In particular, by taking \( Q_Y = P_Y^{(1+\theta(a(R)))} \) in Theorem 5, we have the following.

**Corollary 1** We have

\[
- \log P_\gamma(M) \leq \inf_{\gamma > 0} \left[ (1+s) \tilde{\theta} \left\{ H_{1+\tilde{\theta}}(X|Y) - H_{1+(1+s)\tilde{\theta}}(X|Y) \right\} 
- (1+s) \log \left( 1 - 2e^{\theta[a(R)-\tilde{\theta} a(R)] \tilde{\theta} a(R) (X|Y) + \tilde{\theta} H_{1+\tilde{\theta}}(X|Y) \right) \right] / s,
\]

(128)

(129)

(130)

where \( \theta(a) = \theta^\dagger(a) \) and \( a(R) = a^\dagger(R) \) are the inverse functions defined in (35) and (36).

**Remark 6** Here, it is better to discuss the possibility for extension to the continuous case. As explained in Remark 11, we can define the information quantities to the case when \( Y \) is continuous but \( X \) is a discrete finite set. The discussions in this subsection still hold even in this continuous case. In particular, in the \( n \)-i.i.d. extension case with this continuous setting, Lemma 13 and Corollary 1 hold when the information measures are replaced by \( n \) times of the single-shot information measures.

\(^{19}\)In fact, a special case for \( Q_Y = P_Y \) correspond to Lemma 7.2.2 of [4]. A bound that involve \( Q_Y \) was introduced in [5] for channel coding, and it can be regarded as a source coding counterpart of that result.
C. Finite-Length Bounds for Markov Source

In this subsection, we derive several finite-length bounds for Markovian source with a computable form. Unfortunately, it is not easy to evaluate how tight those bounds are only with their formula. Their tightness will be discussed by considering the asymptotic limit in the remaining subsections of this section. Since we assume the irreducibility for the transition matrix describing the Markovian chain, the following bound hold with any initial distribution.

To derive a lower bounds on \(- \log \bar{P}_s(M_n)\) in terms of the Rényi entropy of transition matrix, we substitute the formula for the Rényi entropy given in Lemma 9 into Lemma 14. Then, we can derive the following achievability bound.

**Theorem 6 (Direct, Ass. 1)** Suppose that transition matrix \(W\) satisfies Assumption 1. Let \(R := \frac{1}{n} \log M_n\). Then, for every \(n \geq 1\), we have

\[
- \log \bar{P}_s(n)(M_n) \geq \sup_{-1 \leq \theta \leq 0} \left[ -\theta nR + (n - 1)\theta H_{1+\theta}^W(X|Y) + \delta(\theta) \right],
\]

where \(\delta(\theta)\) is given by (131).

When \(\mathcal{Y}\) is singleton, from Lemma 15 and a special case of Lemma 9 we have the following achievability bound.

**Theorem 7 (Direct, Singleton)** Let \(R := \frac{1}{n} \log M_n\). Then, for every \(n \geq 1\), we have

\[
- \log \bar{P}_s(n)(M_n) \geq \sup_{-1 \leq \theta \leq 0} \frac{-n\theta R + (n - 1)\theta H_{1+\theta}^W(X) + \delta(\theta)}{1 + \theta}.
\]

To derive an upper bound on \(- \log \bar{P}_s(M_n)\) in terms of the Rényi entropy of transition matrix, we substitute the formula for the Rényi entropy in Lemma 10 to Lemma 13. Then, we have the following converse bound.

**Theorem 8 (Converse, Ass. 1)** Suppose that transition matrix \(W\) satisfies Assumption 1. Let \(R := \frac{1}{n} \log M_n\). For any \(H^W(X|Y) < R < H_{1+\theta}^W(X|Y)\), we have

\[
- \log \bar{P}_s(n)(M_n) \leq \inf_{\theta(a(R)) > 0} \left[ (n - 1)(1 + s)\tilde{\theta} \left( H_{1+\theta}^W(X|Y) - H_{1+\theta}^{W_{\tilde{\theta}(a(R))}}(X|Y) \right) \right] + \delta_1,
\]

\[
- (1 + s) \log \left( 1 - 2e^{-(n-1)\tilde{\theta}a(R)-\tilde{\theta}(a(R))-\tilde{\theta}(a(R))H_{1+\theta}^{W_{\tilde{\theta}(a(R))}}(X|Y)+\tilde{\theta}H_{1+\theta}^{W}(X|Y)+\delta_2) \right) /s,
\]

where \(\theta(a) = \tilde{\theta}(a)\) and \(a(R) = \tilde{\theta}(a(R))\) are the inverse functions defined by (67) and (70) respectively,

\[
\delta_1 := (1 + s)\tilde{\theta}((1 + s)\tilde{\theta}),
\]

\[
\delta_2 := \frac{\theta(a(R)) - \tilde{\theta}R - (1 + \tilde{\theta}\tilde{\theta}(a(R)) + (1 + \theta(a(R)))\tilde{\theta}\tilde{\theta}}{1 + \theta(a(R))},
\]

and \(\tilde{\theta}(\cdot)\) and \(\tilde{\theta}(\cdot)\) are given by (90) and (91), respectively.

**Proof:** We first use (125) of Theorem 5 for \(Q_{Y^n} = P_{Y^n}\) and Lemma 9. Then, we restrict the range of \(\tilde{\theta}\) as \(-1 < \tilde{\theta} < \theta(a(R))\) and set \(\bar{\theta} = \frac{\theta(a(R)) - \tilde{\theta}}{1 + \tilde{\theta}}\). Then, we have the assertion of the theorem.

Next, we derive tighter bounds under Assumption 2. To derive a lower bound on \(- \log \bar{P}_s(M_n)\) in terms of the Rényi entropy of transition matrix, we substitute the formula for the Rényi entropy in Lemma 10 to Lemma 13. Then, we have the following achievability bound.

**Theorem 9 (Direct, Ass. 2)** Suppose that transition matrix \(W\) satisfies Assumption 2. Let \(R := \frac{1}{n} \log M_n\). Then we have

\[
- \log \bar{P}_s(n)(M_n) \geq \sup_{-\frac{1}{2} \leq \theta \leq 0} \frac{-\theta nR + (n - 1)\theta H_{1+\theta}^{W}(X|Y)}{1 + \theta} + \xi(\theta),
\]
where \( \xi(\theta) \) is given by (98).

Finally, to derive an upper bound on \( -\log P_s(M_n) \) in terms of the Rényi entropy for transition matrix, we substitute the formula for the Rényi entropy in Lemma 11 to Theorem 5 for \( Q_{Y^n} = P_{Y^n}^{(1+\theta(a(R)))} \). Then, we can derive the following converse bound.

**Theorem 10 (Converse, Ass. 2)** Suppose that transition matrix \( W \) satisfies Assumption 2. Let \( R := \frac{1}{n} \log M_n \). For any \( H^W(X|Y) < R < H^W_0(X|Y) \), we have

\[
-\log P_s(n)(M_n) \leq \inf_{\tilde{\theta} > 0} \left[ (n - 1)(1 + s)\tilde{\theta} \left\{ H^W_{1+\tilde{\theta},1+\theta(a(R))}(X|Y) - H^W_{1,(1+s)\tilde{\theta},1+\theta(a(R))}(X|Y) \right\} + \delta_1 \right.
\]

\[
- (1 + s) \log \left( 1 - 2e^{(n-1)((\theta(a(R)) - \tilde{\theta})\theta(a(R)) - \theta(a(R))H^W_{1+\theta(a(R)),W}(X|Y) + \theta(a(R))H^W_{1+\tilde{\theta},1+\theta(a(R))}(X|Y))} + \delta_2 \right) / s,
\]

where \( \theta(a) = \theta^+(a) \) and \( a(R) = a^+(R) \) are the inverse functions defined by (71) and (74) respectively,

\[
\delta_1 := (1 + s)\zeta(\tilde{\theta}, \theta(a(R))) - \zeta((1 + s)\tilde{\theta}, \theta(a(R))),
\]

\[
\delta_2 := \left( \theta(a(R)) - \tilde{\theta} \right) R - (1 + \tilde{\theta})\zeta(\theta(a(R)), \theta(a(R))) + (1 + \theta(a(R)))\zeta(\tilde{\theta}, \theta(a(R))),
\]

and \( \zeta(\cdot, \cdot) \) and \( \overline{\zeta}(\cdot, \cdot) \) are given by (102)-(105).

*Proof:* We first use (125) of Theorem 5 for \( Q_{Y^n} = P_{Y^n}^{(1+\theta(a(R)))} \) and Lemma 11. Then, we restrict the range of \( \tilde{\theta} \) as \( -1 < \tilde{\theta} < \theta(a(R)) \) and set \( \theta = \frac{\theta(a(R)) - \tilde{\theta}}{1 + \tilde{\theta}} \). Then, we have the assertion of the theorem. \( \blacksquare \)

**D. Second Order**

By applying the central limit theorem to Lemma 12 (cf. 65, Theorem 27.4, Example 27.6) and Lemma 17 for \( Q_Y = P_Y \), and by using Theorem 2, we have the following.

**Theorem 11** Suppose that transition matrix \( W \) on \( X \times Y \) satisfies Assumption 1. For arbitrary \( \varepsilon \in (0, 1) \), we have

\[
M(n, \varepsilon) = \bar{M}(n, \varepsilon) + o(\sqrt{n}) = nH^W(X|Y) + \sqrt{V^W(X|Y)}\sqrt{n} + o(\sqrt{n}).
\]

*Proof:* The central limit theorem for Markovian process cf. 65, Theorem 27.4, Example 27.6 guarantees that the random variable \( -\log P_{X^n|Y^n}(X^n|Y^n) - nH^W(X|Y)) / \sqrt{n} \) asymptotically obeys the normal distribution with average 0 and the variance \( V^W(X|Y) \), where we use Theorem 2 to show that the limit of the variance is given by \( V^W(X|Y) \). Let \( R = \sqrt{V^W(X|Y)}\Phi^{-1}(1 - \varepsilon) \). Substituting \( M = e^{nH^W(X|Y) + \sqrt{\gamma}R} \) and \( \gamma = nH^W(X|Y) + \sqrt{\gamma}R + n\varepsilon^2 \) in Lemma 12 we have

\[
\lim_{n \to \infty} P_s(n) \left( e^{nH^W(X|Y) + \sqrt{\gamma}R} \right) \leq \varepsilon.
\]

On the other hand, substituting \( M = e^{nH^W(X|Y) + \sqrt{\gamma}R} \) and \( \gamma = nH^W(X|Y) + \sqrt{\gamma}R + n\varepsilon^2 \) in Lemma 17 for \( Q_Y = P_Y \), we have

\[
\lim_{n \to \infty} P_s(n) \left( e^{nH^W(X|Y) + \sqrt{\gamma}R} \right) \geq \varepsilon.
\]

Combining (145) and (146), we have the statement of the theorem. \( \blacksquare \)

From the above theorem, the (first-order) compression limit of source coding with side-information for a Markov source under Assumption 1 is given by

\[
\lim_{n \to \infty} \frac{1}{n} \log M(n, \varepsilon) = \lim_{n \to \infty} \frac{1}{n} \log \bar{M}(n, \varepsilon)
\]

\[
= H^W(X|Y).
\]

\( ^{20} \) Although the compression limit of source coding with side-information for a Markov chain is known more generally 65, we need Assumption 1 to get a single letter characterization.
for any \( \varepsilon \in (0, 1) \). In the next subsections, we consider the asymptotic behavior of the error probability when the rate is larger than the compression limit \( H^W(X|Y) \) in the moderate deviation regime and the large deviation regime, respectively.

### E. Moderate Deviation

From Theorem 6 and Theorem 8 we have the following.

**Theorem 12** Suppose that transition matrix \( W \) satisfies Assumption 1. For arbitrary \( t \in (0, 1/2) \) and \( \delta > 0 \), we have

\[
\lim_{n \to \infty} -\frac{1}{n^{1-2t}} \log P^{(n)}_s \left( e^{nH^W(X|Y) + n^{-t} \delta} \right) = \lim_{n \to \infty} -\frac{1}{n^{1-2t}} \log \bar{P}^{(n)}_s \left( e^{nH^W(X|Y) + n^{-t} \delta} \right) = \frac{\delta^2}{2V^W(X|Y)}. \tag{149}
\]

**Proof:** We apply Theorem 6 and Theorem 8 to the case with \( R = H^W(X|Y) + n^{-t} \delta \), i.e., \( \theta(a(R)) = -n^{-1} \frac{\delta}{V^W(X|Y)} + o(n^{-t}) \). For the achievability part, from (88) and Theorem 6 we have

\[
-\log P^{(n)}_s (M_n) \geq \sup_{-1 \leq \theta \leq 0} \left[ -\theta nR + (n-1)\theta H^W_{1+\theta}(X|Y) \right] + \inf_{-1 \leq \theta \leq 0} \delta(\theta) \geq n^{1-2t} \frac{\delta}{2V^W(X|Y)} + o(n^{1-2t}). \tag{151}
\]

To prove the converse part, we fix arbitrary \( s > 0 \) and choose \( \tilde{\theta} \) to be \(-n^{-t} \frac{\delta}{V^W(X|Y)} + n^{-2t}\). Then, Theorem 8 implies that

\[
\limsup_{n \to \infty} -\frac{1}{n^{1-2t}} \log P_s(M_n) \leq \limsup_{n \to \infty} n^{2t} \frac{1+s}{s} \tilde{\theta} \left\{ H^W_{1+\tilde{\theta}}(X|Y) - H^W_{1+(1+s)\tilde{\theta}}(X|Y) \right\} = \limsup_{n \to \infty} n^{2t} \frac{1+s}{s} \tilde{\theta} \frac{dH^W_{1+\tilde{\theta}}(X|Y)}{d\theta} \bigg|_{\theta=\tilde{\theta}} = (1+s) \frac{\delta^2}{2V^W(X|Y)}. \tag{155}
\]

**Remark 7** In the literatures [13], [67], the moderate deviation results are stated for \( \epsilon_n \) such that \( \epsilon_n \to 0 \) and \( n\epsilon_n^2 \to \infty \) instead of \( n^{-t} \) for \( t \in (0, 1/2) \). Although the former is slightly more general than the latter, we employ the latter formulation in Theorem 12 since the order of convergence is clearer. In fact, \( n^{-t} \) in Theorem 12 can be replaced by general \( \epsilon_n \) without modifying the argument of the proof.

### F. Large Deviation

From Theorem 6 and Theorem 8 we have the following.

**Theorem 13** Suppose that transition matrix \( W \) satisfies Assumption 1. For \( H^W(X|Y) < R \), we have

\[
\liminf_{n \to \infty} -\frac{1}{n} \log \bar{P}^{(n)}_s \left( e^{nR} \right) \geq \sup_{-1 \leq \theta \leq 0} \left[ -\theta R + \theta H^W_{1+\theta}(X|Y) \right]. \tag{156}
\]

On the other hand, for \( H^W(X|Y) < R < H^W_{0}(X|Y) \), we have

\[
\limsup_{n \to \infty} -\frac{1}{n} \log P^{(n)}_s \left( e^{nR} \right) \leq -\theta(a(R))a(R) + \theta(a(R))H^W_{1+\theta(a(R))}(X|Y) \leq \sup_{-1 \leq \theta \leq 0} \left[ -\theta R + \theta H^W_{1+\theta}(X|Y) \right]. \tag{158}
\]
Proof: The achievability bound (156) follows from Theorem 9. The converse part (157) is proved from Theorem 8 as follows. We first fix $s > 0$ and $-1 < \bar{\theta} < \theta(a(R))$. Then, Theorem 8 implies

$$\limsup_{n \to \infty} -\frac{1}{n} \log P_s^n(e^{nR}) \leq \frac{1 + s}{s} \tilde{\theta} \left\{ H_{1+\hat{\theta}}^W(X|Y) - H_{1+(1+s)\hat{\theta}}^W(X|Y) \right\}. \quad (159)$$

By taking the limit $s \to 0$ and $\bar{\theta} \to \theta(a(R))$, we have

$$\frac{1 + s}{s} \tilde{\theta} \left\{ H_{1+\hat{\theta}}^W(X|Y) - H_{1+(1+s)\hat{\theta}}^W(X|Y) \right\} = \frac{1}{s} \left( \tilde{\theta} H_{1+\hat{\theta}}^W(X|Y) - (1 + s)\tilde{\theta} H_{1+(1+s)\hat{\theta}}^W(X|Y) \right) + \tilde{\theta} H_{1+\hat{\theta}}^W(X|Y) \quad (160)$$

$$\to -\tilde{\theta} \frac{d[\tilde{\theta} H_{1+\hat{\theta}}^W(X|Y)]}{d\theta} \bigg|_{\theta=\bar{\theta}} + \tilde{\theta} H_{1+\hat{\theta}}^W(X|Y) \quad (161)$$

$$\to -\tilde{\theta}(a(R)) \frac{d[\tilde{\theta} H_{1+\hat{\theta}}^W(X|Y)]}{d\theta} \bigg|_{\theta=\bar{\theta}} + \theta(a(R)) H_{1+\hat{\theta}(a(R))}^W(X|Y) \quad (162)$$

$$= -\tilde{\theta}(a(R))a(R) + \theta(a(R)) H_{1+\hat{\theta}(a(R))}^W(X|Y) \quad (163)$$

Thus, (157) is proved. The alternative expression (158) is derived via Lemma 8.

Remark 8 For $R \leq R_c$, where (cf. (72) for the definition of $R(a)$),

$$R_c := R \left( \frac{d[\tilde{\theta} H_{1+\hat{\theta}}^W(X|Y)]}{d\theta} \bigg|_{\theta=-\frac{1}{2}} \right). \quad (174)$$
Fig. 2. The description of the transition matrix.

is the critical rate, we can rewrite the lower bound in (165) as (cf. Lemma 8)

$$
\sup_{-\frac{1}{2} \leq \theta \leq 0} \left( -\theta R + \theta H_{1+\theta}^W(X|Y) \right) = -\theta(a(R))a(R) + \theta(a(R))H_{1+\theta(a(R))}^W(X|Y).
$$

Thus, the lower bound and the upper bound coincide up to the critical rate.

**Remark 9** When \( Y \) is singleton, from Theorem 7 and a special case of (157), we can derive

$$
\lim \inf_{n \to \infty} \frac{1}{n} \log \bar{P}_s^{(n)}(e^{nR}) \geq \sup_{-1 \leq \theta \leq 0} \left( -\theta R + \theta H_{1+\theta}^W(X) \right)
$$

and

$$
\lim \sup_{n \to \infty} \frac{1}{n} \log P_s^{(n)}(e^{nR}) \leq \sup_{-1 < \theta \leq 0} \left( -\theta R + \theta H_{1+\theta}^W(X) \right).
$$

for \( H^W(X) < R < H_0^W(X) \). Thus, we can recover the results in [39], [40] by our approach.

**G. Numerical Example**

In this section, to demonstrate the advantage of our finite-length bound, we numerically evaluate the achievability bound in Theorem 7 and a special case of the converse bound in Theorem 8 for singleton \( Y \). Thanks to the criterion (C2), our numerical calculation shows that our upper finite-length bounds is very close to our lower finite-length bounds when the size \( n \) is sufficiently large. Thanks to the criterion (C1), we could calculate both bounds with huge size \( n = 1 \times 10^5 \) because the calculation complexity behaves as \( O(1) \).

We consider a binary transition matrix \( W \) given by Fig. 2 i.e.,

$$
W = \begin{bmatrix}
1 - p & q \\
p & 1 - q
\end{bmatrix}.
$$

In this case, the stationary distribution is

$$
\tilde{P}(0) = \frac{q}{p + q},
$$

$$
\tilde{P}(1) = \frac{p}{p + q}.
$$

The entropy is

$$
H^W(X) = \frac{q}{p + q}h(p) + \frac{p}{p + q}h(q),
$$

where \( h(\cdot) \) is the binary entropy function. The tilted transition matrix is

$$
W_\theta = \begin{bmatrix}
(1 - p)^{1+\theta} & q^{1+\theta} \\
p^{1+\theta} & (1 - q)^{1+\theta}
\end{bmatrix}.
$$
Fig. 3. A comparison of the bounds for $p = 0.1$, $q = 0.2$, and $\varepsilon = 10^{-3}$. The horizontal axis is the block length $n$ and the vertical axis is the rate $R$ (nats). The red curve is the achievability bound in Theorem 7. The blue curve is the converse bound in Theorem 8. The purple line is the entropy $H^W(X)$.

The Perron-Frobenius eigenvalue is
\[
\lambda_\theta = \frac{(1 - p)^{1+\theta} + (1 - q)^{1+\theta} + \sqrt{((1 - p)^{1+\theta} - (1 - q)^{1+\theta})^2 + 4p^{1+\theta}q^{1+\theta}}}{2}
\]  
and its normalized eigenvector is
\[
\tilde{P}_\theta(0) = \frac{q^{1+\theta}}{\lambda_\theta - (1 - p)^{1+\theta} + q^{1+\theta}},
\]
\[
\tilde{P}_\theta(1) = \frac{\lambda_\theta - (1 - p)^{1+\theta}}{\lambda_\theta - (1 - p)^{1+\theta} + q^{1+\theta}}.
\]
The normalized eigenvector of $W^T_\rho$ is also given by
\[
\hat{P}_\theta(0) = \frac{p^{1+\theta}}{\lambda_\theta - (1 - p)^{1+\theta} + p^{1+\theta}},
\]
\[
\hat{P}_\theta(1) = \frac{\lambda_\theta - (1 - p)^{1+\theta}}{\lambda_\theta - (1 - p)^{1+\theta} + p^{1+\theta}}.
\]

From these calculations, we can evaluate the bounds in Theorem 7 and Theorem 8. For $p = 0.1$, $q = 0.2$, the bounds are plotted in Fig. 3 for fixed error probability $\varepsilon = 10^{-3}$. Although there is a gap between the achievability bound and the converse bound for rather small $n$, the gap is less than approximately 5% of the entropy rate for $n$ larger than 10000. We also plotted the bounds in Fig. 4 for fixed block length $n = 10000$ and varying $\varepsilon$. The gap between the achievability bound and the converse bound remains approximately 5% of the entropy rate even for $\varepsilon$ as small as $10^{-10}$.

IV. CHANNEL CODING

In this section, we investigate the channel coding with a conditional additive channel. The former part of this section discusses general properties of the channel coding with a conditional additive channel. The latter part of this section discusses properties of the channel coding when the conditional additive noise of the channel is Markovian. We start this section by showing the problem setting in Section IV-A by introducing a conditional additive channel. Section IV-B gives a canonical method to convert a regular channel to a conditional additive channel. Section IV-C gives a method to convert a BPSK-AWGN channel to a conditional additive channel. Then, we show some single-shot achievability bounds in Section IV-D and single-shot converse bounds in Section IV-E.

As the latter part, we derive finite-length bounds for the Markov noise channel in Section IV-F. Then, in Sections IV-I and IV-H we show the asymptotic characterization for the large deviation regime and the moderate deviation regime by using those finite-length bounds. We also derive the second order rate in Section IV-G.
The results shown in this section for the Markovian conditional additive noise are summarized in Table III. The checkmarks ✓ indicate that the tight asymptotic bounds (large deviation, moderate deviation, and second order) can be obtained from those bounds. The marks ✓* indicate that the large deviation bound can be derived up to the critical rate. The computational complexity “Tail” indicates that the computational complexities of those bounds depend on the computational complexities of tail probabilities. It should be noted that Theorem 18 is derived from a special case \( Q_Y = P_Y \) of Theorem 16. The asymptotically optimal choice is \( Q_Y = P_Y (1 + \theta) \). Under Assumption 1, we can derive the bound of the Markov case only for that special choice of \( Q_Y \), while under Assumption 2 we can derive the bound of the Markov case for the optimal choice of \( Q_Y \). Furthermore, Theorem 18 is not asymptotically tight in the large deviation regime in general, but it is tight if \( Y \) is singleton, i.e., the channel is additive. It should be also noted that Theorem 20 does not imply Theorem 18 even for the additive channel case since Assumption 2 restricts the structure of transition matrices even when \( Y \) is singleton.

### Table III

**Summary of the finite-length bounds for channel coding.**

| Ach./Conv. | Markov | Single Shot | \( P_c / \bar{P}_c \) | Complexity | Large Deviation | Moderate Deviation | Second Order |
|------------|--------|-------------|----------------|--------------|-----------------|-------------------|-------------|
| Achievability | Theorem 17 (Ass. 1) | Lemma 21 | \( P_c \) | O(1) | ✓ | ✓ | ✓ |
| | Theorem 19 (Ass. 2) | Lemma 20 | \( P_c \) | O(1) | ✓* | ✓ | ✓ |
| | Theorem 21 (Additive) | Lemma 22 | \( P_c \) | O(1) | ✓* | ✓ | ✓ |
| | Lemma 19 | \( P_c \) | Tail | ✓ | ✓ | ✓ |
| Converse | Theorem 18 (Ass. 1) | (Theorem 16) | \( P_c \) | O(1) | ✓ | ✓ | ✓ |
| | Theorem 20 (Ass. 2) | Theorem 16 | \( P_c \) | O(1) | ✓* | ✓ | ✓ |
| | Theorem 18 (Additive) | (Theorem 16) | \( P_c \) | O(1) | ✓* | ✓ | ✓ |
| | Lemma 24 | \( P_c \) | Tail | ✓ | ✓ | ✓ |

### A. Formulation for conditional additive channel

1) **Single-shot case:** We first present the problem formulation by the single shot setting. For a channel \( P_{B|A}(b|a) \) with input alphabet \( \mathcal{A} \) and output alphabet \( \mathcal{B} \), a channel code \( \Psi = (e, d) \) consists of one encoder \( e : \{1, \ldots, M\} \to \mathcal{A} \) and one decoder \( d : \mathcal{B} \to \{1, \ldots, M\} \). The average decoding error probability is defined by

\[
P_c[\Psi] := \sum_{m=1}^{M} \frac{1}{M} P_{B|A}(\{b : d(b) \neq m\}|e(m)). \tag{188}
\]
For notational convenience, we introduce the error probability under the condition that the message size is $M$:

$$P_c(M) := \inf_{\Psi} P_c(\Psi).$$  \hfill (189)

Assume that the input alphabet $A$ is the same set as the output alphabet $B$ and they equals an additive group $\mathcal{X}$. When the transition matrix $P_{B|A}(b|a)$ is given as $P_{\mathcal{X}}(b - a)$ by using a distribution $P_{\mathcal{X}}$ on $\mathcal{X}$, the channel is called additive.

To extend the concept of additive channel, we consider the case when the input alphabet $A$ is an additive group $\mathcal{X}$ and the output alphabet $B$ is the product set $\mathcal{X} \times \mathcal{Y}$. When the transition matrix $P_{B|A}(x, y|a)$ is given as $P_{\mathcal{X}\mathcal{Y}}(x - a, y)$ by using a distribution $P_{\mathcal{X}\mathcal{Y}}$ on $\mathcal{X} \times \mathcal{Y}$, the channel is called conditional additive. In this paper, we are exclusively interested in the conditional additive channel. As explained in Subsection IV-B, a channel is a conditional additive channel if and only if it is a regular channel in the sense of [29]. When we need to explicitly express the underlying distribution of the noise, we denote the average decoding error probability by $P_c[\Psi|P_{\mathcal{X}\mathcal{Y}}]$.

2) $n$-fold extension: When we consider $n$-fold extension, the channel code is denoted with subscript $n$ such as $\Psi_n = (e_n, d_n)$. The error probabilities given in (188) and (189) are written with the superscript $(n)$ as $P_c^{(n)}(\Psi_n)$ and $P_c^{(n)}(M_n)$, respectively. Instead of evaluating the error probability $P_c^{(n)}(M_n)$ for given $M_n$, we are also interested in evaluating

$$M(n, \varepsilon) := \sup \left\{ M_n : P_c^{(n)}(M_n) \leq \varepsilon \right\}$$  \hfill (190)

for given $0 \leq \varepsilon \leq 1$.

When the channel is given as a conditional distribution, the channel is given by

$$P_{B^n|A^n}(x^n, y^n|a^n) = P_{\mathcal{X}\mathcal{Y}^n}(x^n - a^n, y^n),$$  \hfill (191)

where $P_{\mathcal{X}\mathcal{Y}^n}$ is a noise distribution on $\mathcal{X}^n \times \mathcal{Y}^n$.

For the code construction, we investigate the linear code. For an $(n, k)$ linear code $C_n \subset \mathcal{A}^n$, there exists a parity check matrix $f_n : \mathcal{A}^n \to \mathcal{A}^{n-k}$ such that the kernel of $f_n$ is $C_n$. That is, given a parity check matrix $f_n : \mathcal{A}^n \to \mathcal{A}^{n-k}$, we define the encoder $I_{\text{Ker}(f_n)} : C_n \to \mathcal{A}^n$ as the imbedding of the kernel $\text{Ker}(f_n)$. Then, using the decoder $d_{f_n} := \text{argmin} P_c[(I_{\text{Ker}(f_n)}, d)],$ we define $\Psi(f_n) = (I_{\text{Ker}(f_n)}, d_{f_n})$.

Here, we employ a randomized choice of a parity check matrix. In particular, instead of a two-universal hash function, we focus on linear two-universal hash functions, because the linearity is required in the above relation with source coding. So, denoting the set of linear two-universal hash functions from $\mathcal{A}^n$ to $\mathcal{A}^{n-k}$ by $F_l$, we introduce the quantity:

$$\bar{P}_c(n, k) := \sup_{F_n \in F_l} \mathbb{E}_{F_n} \left[ P_c^{(n)}(\Psi(F_n)) \right].$$  \hfill (192)

Taking the infimum over all linear codes associated with $F_n$ (cf. (113)), we obviously have

$$P_c^{(n)}(|A|^k) \leq \bar{P}_c(n, k).$$  \hfill (193)

When we consider the error probability for conditionally additive channels, we use notation $\bar{P}_c(n, k|P_{\mathcal{X}\mathcal{Y}})$ so that the underlying distribution of the noise is explicit. We are also interested in characterizing

$$k(n, \varepsilon) := \sup \left\{ k : \bar{P}_c(n, k) \leq \varepsilon \right\}$$  \hfill (194)

for given $0 \leq \varepsilon \leq 1$.

### B. Conversion from regular channel to conditional additive channel

This subsection shows that a channel is a regular channel in the sense of [29] if and only if it is conditional additive. Then, we see that a binary erasure symmetric channel is an example of a regular channel.

We assume that the input alphabet $A$ has an additive group structure. Let $P_{\mathcal{X}}$ be a distribution on the output alphabet $B$. Let $\pi_a$ be a representation of the group $A$ on $B$, and let $G = \{ \pi_a : a \in A \}$. A regular channel [29] is defined by

$$P_{B|A}(b|a) = P_{\mathcal{X}}(\pi_a(b)).$$  \hfill (195)
The group action induces orbit
\[ \text{Orb}(b) := \{ \pi_a(b) : a \in A \} \quad \text{(196)} \]
The set of all orbits constitute a disjoint partition of \( B \). A set of representatives of the orbits is denoted by \( \overline{B} \), and let \( \varpi : B \to \overline{B} \) be the map to the representatives.

**Example 4 (Binary Erasure Symmetric Channel)** Let \( A = \{0, 1\} \), \( B = \{0, 1, ?\} \), and
\[
P_{\bar{X}}(b) = \begin{cases} 
1 - p - p' & \text{if } b = 0 \\
p & \text{if } b = 1 \\
p' & \text{if } b = ?
\end{cases} \quad \text{(197)}
\]
Then, let
\[
\pi_0 = \begin{bmatrix} 0 & 1 & ? \\ 0 & 1 & ? \end{bmatrix}, \quad \pi_1 = \begin{bmatrix} 0 & 1 & ? \\ 1 & 0 & ? \end{bmatrix} \quad \text{(198)}
\]
The channel defined in this way is a regular channel (see Fig. 5). In this case, there are two orbits: \( \{0, 1\} \) and \( \{?\} \).

Let \( B = X \times Y \) and \( P_{\bar{X}} = P_{XY} \) for some joint distribution on \( X \times Y \). Now, we consider a conditional additive channel, whose transition matrix \( P_{B|A}(x, y|a) \) is given as \( P_{XY}(x - a, y) \). When the group action is given by \( \pi_a(x, y) = (x - a, y) \), the above conditional additive channel given as a regular channel. In this case, there are \( |Y| \) orbits and the size of each orbit is \( |X| \) respectively. This fact shows that any conditional additive channel is written as a regular channel.

Conversely, we show that any regular channel is written as a conditional additive channel. For this purpose, we convert a regular channel to a conditional additive channel as follows.

We first explain the construction for single shot channel. For random variable \( \bar{X} \sim P_{\bar{X}} \), let \( Y = \bar{B} \) and \( Y = \varpi(\bar{X}) \) be the random variable describing the representatives of the orbits. For each orbit \( \text{Orb}(y) \), fix an element \( 0_y \in \text{Orb}(y) \). Then, let
\[
\text{Stb}(0_y) := \{ a \in A : \pi_a(0_y) = 0_y \} \quad \text{(199)}
\]
be the stabilizer subgroup of \( 0_y \). Let \( A/\text{Stb}(0_y) \) be a set of coset representatives of the coset \( A/\text{Stb}(0_y) \), and let
\[
\vartheta_y : A \to A/\text{Stb}(0_y) \quad \text{(200)}
\]
\footnote{Since \( A \) is an Abelian group, the stabilizer group actually does not depend on the choice \( 0_y \in \text{Orb}(y) \).}
be the map to the coset representatives. Then, we can define the bijective map
\[ \iota_y : \text{Orb}(y) \ni \pi_{-\bar{a}}(0_y) \mapsto \bar{a} \in \mathcal{A}/\text{Stb}(0_y). \]

Let \( \mathcal{X} = \mathcal{A} \) and \( P_{X|Y}(\cdot|y) \) be the distribution on \( \mathcal{A} \) defined by
\[ P_{X|Y}(x|y) := \frac{P_X(\iota_{y}^{-1}(\vartheta_y(x)))}{P_X(\text{Orb}(y))} \frac{1}{|\text{Stb}(0_y)|}. \]

When the output from the real channel is \( b \), the output from the virtual channel is defined by
\[ (\iota_{\bar{w}}(b) + A', \bar{w}(b)) \]
where \( A' \) is randomly chosen from \( \text{Stb}(0_{\bar{w}(b)}). \)

**Theorem 15** The virtual channel defined by (203) is the conditional additive channel such that the output is given by \((a + X, Y) \sim P_{XY}, \) where \( P_{XY} \) is defined from \( P_Y \) and \( P_{X|Y} \) of (202).

**Proof:** When the input to the real channel is \( a \), note that the output can be written as \( \pi_{-a}(\tilde{X}), \) where \( \tilde{X} \sim P_{\tilde{X}}. \) By noting that \( Y = \bar{w}(\pi_{-a}(\tilde{X})) \sim P_Y, \) the output of the virtual channel is written as
\[ (\iota_{\bar{w}}(\pi_{-a}(\tilde{X})) + A', Y) = (\iota_{\bar{w}}(\pi_{-\vartheta_Y(a)}(\tilde{X})) + A', Y) \]
\[ = (\vartheta_Y(\vartheta_Y(a)) + \iota_{\bar{w}}(\tilde{X})) + A', Y) \]
\[ = (a + \iota_{\bar{w}}(\tilde{X}) + A'', Y), \]
where (205) follows from the fact that
\[ \pi_{-\vartheta_Y(a)}(\tilde{X}) = \pi_{-\vartheta_Y(a)}(\pi_{-\iota_{\bar{w}}}(\tilde{X}))(0_Y) \]
\[ = \pi_{-\vartheta_Y(a)}(\vartheta_Y(\tilde{X}))(0_Y), \]
and we set \( A'' = \vartheta_Y(\vartheta_Y(a)) + \iota_{\bar{w}}(\tilde{X}) - a - \iota_{\bar{w}}(\tilde{X}) + A' \) in (206). Since \( A'' \) is the uniform random variable on \( \text{Stb}(0_Y), \) the joint distribution of \((\iota_{\bar{w}}(\tilde{X}) + A'', Y) \) is \( P_{XY}. \) Thus, we have the statement of the theorem. \( \blacksquare \)

**Example 5 (Binary Erasure Symmetric Channel Revisited)** We convert the regular channel of Example 4 to a conditional additive channel. Let us label the orbit \( \{0, 1\} \) as \( y = 0 \) and \( \{?\} \) as \( y = 1. \) Let \( 0_0 = 0 \) and \( 0_1 = ? \). Then, \( \text{Stb}(0_0) = \{0\} \) and \( \text{Stb}(0_1) = \{0, 1\} \). The map \( \vartheta_0 \) is the identity map, and \( \vartheta_1 \) is the trivial map defined by \( \vartheta_1(a) = 0. \) The map \( \iota_y \) is given by \( \iota_0(0) = 0, \iota_0(1) = 1, \) and \( \iota_1(? = ?) = 0. \) The distribution \( P_Y \) is given by \( P_Y(0) = 1 - p' \) and \( P_Y(1) = p'. \) The conditional distribution \( P_{X|Y} \) is given by
\[ P_{X|Y}(x|0) = \begin{cases} \frac{1 - p - p'}{1 - p} & \text{if } x = 0 \\ \frac{p}{1 - p'} & \text{if } x = 0 \end{cases} \]
and \( P_{X|Y}(0|1) = P_{X|Y}(1|1) = \frac{1}{2}. \)

When we consider \( n \)-th extension, a channel is given by
\[ P_{B^n|A^n}(b^n|a^n) = P_{X^n}(\pi_{a^n}(b^n)), \]
where \( n \)-th extension of the group action is defined by \( \pi_{a^n}(b^n) = (\pi_{a_n}(b_{1}), \ldots, \pi_{a_n}(b_{n})). \)

Similarly, for \( n \)-fold extension, we can also construct the virtual conditional additive channel. More precisely, for \( X^n \sim P_{X^n}, \) we set \( Y^n = \bar{w}(X^n) = (\bar{w}(X_{1}), \ldots, \bar{w}(X_{n})) \) and
\[ P_{X^n|Y^n}(a^n|y^n) := \frac{P_{X^n}(\iota_{\bar{w}}^{-1}(\vartheta_{\bar{y}}(x^n)))}{P_{X^n}(\text{Orb}(y^n))} \frac{1}{|\text{Stb}(0_{y^n})|}, \]

and
\[ P_{X^n|Y^n}(0^n|1^n) = P_{X^n|Y^n}(1^n|1^n) = \frac{1}{2}. \]
where
\[
\text{Orb}(y^n) := \text{Orb}(y_1) \times \cdots \times \text{Orb}(y_n),
\]
\[
\partial_{y^n}(x^n) := (\partial_{y_1}(x_1), \ldots, \partial_{y_n}(x_n)),
\]
\[
t_{y^n}(b^n) := (t_{y_1}(b_1), \ldots, t_{y_n}(b_n)),
\]
\[
\text{Stb}(0_{y^n}) := \text{Stb}(0_{y_1}) \times \cdots \times \text{Stb}(0_{y_n}).
\]

Since the conversion to the virtual channel in (203) is reversible, we can assume that the channel is a conditional additive from the beginning without loss of generality.

C. Conversion of BPSK-AWGN Channel to Conditional Additive Channel

Although we only considered finite input/output sources and channels throughout the paper, in order to demonstrate the utility of the conditional additive channel framework, let us consider the additive white Gaussian noise (AWGN) channel with binary phase shift keying (BPSK) in this section. Let \( A = \{0,1\} \) be the input alphabet of the channel, and let \( B = \mathbb{R} \) be the output alphabet of the channel. For an input \( a \in A \) and Gaussian noise \( Z \) with mean 0 and variance \( \sigma^2 \), the output of channel is given by \( B = (-1)^a + Z \). Then, the conditional probability density function of this channel is given as
\[
P_{B|A}(b|a) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(b-(-1)^a)^2}{\sigma^2}}. \tag{216}
\]

Now, to define a conditional additive channel, we choose \( Y := \mathbb{R}_+ \) and define the probability density function \( p_Y \) on \( Y \) with respect to the Lebesgue measure and the conditional distribution \( P_{X|Y}(x|y) \) as
\[
p_Y(y) := \frac{1}{\sqrt{2\pi\sigma}} (e^{-\frac{(y-1)^2}{\sigma^2}} + e^{-\frac{(y+1)^2}{\sigma^2}}) \tag{217}
\]
\[
P_{X|Y}(0|y) := \frac{e^{-\frac{(y-1)^2}{\sigma^2}}}{e^{-\frac{(y-1)^2}{\sigma^2}} + e^{-\frac{(y+1)^2}{\sigma^2}}} \tag{218}
\]
\[
P_{X|Y}(1|y) := \frac{e^{-\frac{(y+1)^2}{\sigma^2}}}{e^{-\frac{(y-1)^2}{\sigma^2}} + e^{-\frac{(y+1)^2}{\sigma^2}}} \tag{219}
\]
for \( y \in \mathbb{R}_+ \). When we define \( b := (-1)^x y \in \mathbb{R} \) for \( x \in \{0,1\} \) and \( y \in \mathbb{R}_+ \), we have
\[
p_{XY|A}(y, x|a) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(y-(-1)^a x)^2}{\sigma^2}} = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{((-1)^a x y - (-1)^a)^2}{\sigma^2}} = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(b-(-1)^a)^2}{\sigma^2}}. \tag{220}
\]

The relations (216) and (220) show that the AWGN channel with BPSK is given as a conditional additive channel in the above sense.

By noting this observation, as explained in Remark 6, the single-shot achievability bounds in Section III-B are also valid for continuous \( Y \). Also, the discussions for the single-shot converse bounds in Subsection IV-B hold even for continuous \( Y \). So, the bounds in Subsections IV-D and IV-E are also applicable to the BPSK-AWGN channel.

In particular, in the \( n \) memoryless extension of the BPSK-AWGN channel, the information measures for the noise distribution are given as \( n \) times of the single-shot information measures for the noise distribution. Even in this case, the upper and lower bounds in Subsections IV-D and IV-E are also applicable by replacing the information measures by \( n \) times of the single-shot information measures. Therefore, we obtain finite-length upper and lower bounds of the optimal coding length for the memoryless BPSK-AWGN channel. Furthermore, even though the additive noise is not Gaussian, when the probability density function \( p_Z \) of the additive noise \( Z \) satisfies the symmetry \( p_Z(z) = p_Z(-z) \), the BPSK channel with the additive noise \( Z \) can be converted to a conditional additive channel in the same way.
D. Achievability Bound Derived by Source Coding with Side-Information

In this subsection, we give a code for a conditional additive channel from a code of source coding with side-information in a canonical way. In this construction, we see that the decoding error probability of the channel code equals that of the source code.

When the channel is given as the conditional additive channel with conditional additive noise distribution \( P_{X^n|Y^n} \) as (191) and \( X = A \) is the finite field \( \mathbb{F}_q \), we can construct a linear channel code from a source coder with full side-information whose encoder and decoder are \( f_n \) and \( d_n \) as follows. That is, we assume the linearity for the source encoder \( f_n \). Let \( C_n(f_n) \) be the kernel of the linear encoder \( f_n \) of the source coder. Suppose that the sender sends a codeword \( c_n \in C_n(f_n) \) and \( (e_n + X^n, Y^n) \) is received. Then, the receiver computes the syndrome \( f_n(e_n + X^n) = f_n(X^n) \), estimates \( X^n \) from \( f_n(X^n) \) and \( Y^n \), and subtracts the estimate from \( e_n + X^n \). That is, we choose the channel decoder \( d_n \) as

\[
\tilde{d}_n(x^n, y^n) := x^n - d_n(f_n(x^n), y^n).
\]  

We succeeded in decoding in this channel coding if and only if \( d_n(f_n(X^n), Y^n) \) equals \( X^n \). Thus, the error probability of this channel code coincides with that of the source code for the correlated source \( (X^n, Y^n) \). In summary, we have the following lemma, which was first pointed out in [27].

**Lemma 18 ([27 (19)])** Given a linear encoder \( f_n \) and a decoder \( d_n \) for source coding with side-information with distribution \( P_{X^n|Y^n} \), let \( I_{\text{Ker}(f_n)} \) and \( \tilde{d}_n \) be channel encoder and decoder induced from \( (f_n, d_n) \). Then, the error probability of channel coding for conditionally additive channel with noise distribution \( P_{X^n|Y^n} \) satisfies

\[
P_c^n(I_{\text{Ker}(f_n)}, \tilde{d}_n)|P_{X^n|Y^n}| = P_c^n(f_n, d_n)|P_{X^n|Y^n}|.
\]  

Furthermore, taking the infimum for \( F_n \) chosen to be a linear two-universal hash function, we also have

\[
P_c(n, k) = \sup_{F_n \in F_0} \mathbb{E}_{F_n} \left[ P_c^n(f_n, d_n) \right] = \sup_{F_n \in F_0} \mathbb{E}_{F_n} \left[ P_c^n(I_{\text{Ker}(f_n)}, \tilde{d}_n) \right] = \mathbb{P}^{-1}((p^{n-k})) \leq \sup_{F_n \in F} \mathbb{E}_{F_n} \mathbb{P}^{-1}(e^n(k)).
\]  

By using this observation and the results in Section III-B, we can derive the achievability bounds. By using the conversion argument in Section IV-B, we can also construct a channel code for a regular channel from a source code with full side-information. Although the following bounds are just specialization of known bounds for conditional additive channels, we review these bounds here to clarify correspondence between the bounds in source coding with side-information and channel coding.

From Lemma 12 and (223), we have the following.

**Lemma 19 ([3])** The following bound holds:

\[
\overline{P}_c(n, k) \leq \inf_{\gamma \geq 0} \left[ P_{X^n|Y^n} \left\{ \log \frac{1}{P_{X^n|Y^n}(x^n|y^n)} > \gamma \right\} + \frac{e^\gamma}{|A|^{n-k}} \right].
\]  

From Lemma 13 and (223), we have the following exponential type bound.

**Lemma 20 ([7])** The following bound holds:

\[
\overline{P}_c(n, k) \leq \inf_{-\frac{n}{2} \leq \theta \leq 0} |A|^\frac{n(k-n)}{2} e^{-\frac{\theta^2}{2}} H^\gamma_{1+k}(X^n|Y^n).
\]  

From Lemma 14 and (223), we have the following slightly loose exponential bound.

**Lemma 21 ([4], [68])** The following bound holds:

\[
\overline{P}_c(n, k) \leq \inf_{-1 \leq \theta \leq 0} |A|^\theta(n-k) e^{-\theta H^\gamma_{1+k}(X^n|Y^n)}.
\]  

---

22In fact, when we additionally impose the linearity to the random function \( F \) in the definition [14] for the definition of \( \overline{P}_c(M|P_{X^n|Y^n}) \), the result in [27] implies that the equality in (223) holds.

23The bound (226) was derived in the original Japanese edition of [4], but it is not written in the English edition [4]. The quantum analogue was derived in [68].
Lemma 24 When \( \mathcal{Y} \) is singleton, i.e., the virtual channel is additive, we have the following special case of Lemma 20.

**Lemma 22** ([7]) Suppose that \( \mathcal{Y} \) is singleton. Then, the following bound holds:

\[
\bar{P}_c(n, k) \leq \inf_{\frac{1}{2} \leq \theta \leq 0} |A|^{\frac{\binom{n(k-n)}{2}}{\theta + 1} e^{-\frac{\theta}{\theta + 1} H_{1+\theta}(X^n)}}.
\] (227)

**E. Converse Bound**

In this subsection, we show some converse bounds. The following is the information spectrum type converse shown in [5].

**Lemma 23** ([5] Lemma 4) For any code \( \Psi_n = (e_n, d_n) \) and any output distribution \( Q_{B^n} \in \mathcal{P}(B^n) \), we have

\[
P_c^{(n)}[\Psi_n] \geq \sup_{\gamma \geq 0} \left[ \sum_{m=1}^{M_n} \frac{1}{M_n} P_{B^n|A^n} \left\{ \log \frac{P_{B^n|A^n}(b^n|e_n(m))}{Q_{B^n}(b^n)} < \gamma \right\} \right] - \frac{e^\gamma}{M_n}.
\] (228)

When a channel is a conditional additive channel, we have

\[
P_{B^n|A^n}(a^n + x^n, y^n|a^n) = P_{X^nY^n}(x^n, y^n).
\] (229)

By taking the output distribution \( Q_{B^n} \) as

\[
Q_{B^n}(a^n + x^n, y^n) = \frac{1}{|A|^n} Q_{Y^n}(y^n)
\] (230)

for some \( Q_{Y^n} \in \mathcal{P}(Y^n) \), we have the following bound.

**Lemma 24** When a channel is a conditional additive channel, for any distribution \( Q_{Y^n} \in \mathcal{P}(Y^n) \), we have

\[
P_c^{(n)}(M_n) \geq \sup_{\gamma \geq 0} \left[ P_{X^nY^n} \left\{ \log \frac{Q_{Y^n}(y^n)}{P_{X^nY^n}(x^n, y^n)} > n \log |A| - \gamma \right\} \right] - \frac{e^\gamma}{M_n}.
\] (231)

**Proof:** By noting (229) and (230), the first term of the right hand side of (228) can be rewritten as

\[
\sum_{m=1}^{M_n} \frac{1}{M_n} P_{B^n|A^n} \left\{ \log \frac{P_{B^n|A^n}(b^n|e_n(m))}{Q_{B^n}(b^n)} < \gamma \right\}
\] (232)

\[
= \sum_{m=1}^{M_n} \frac{1}{M_n} P_{X^nY^n} \left\{ \log \frac{P_{X^nY^n}(e_n(m)|x^n, y^n)}{Q_{B^n}(b^n)} - n \log |A| > \gamma \right\}
\] (233)

\[
= P_{X^nY^n} \left\{ \log \frac{Q_{Y^n}(y^n)}{P_{X^nY^n}(x^n, y^n)} > n \log |A| - \gamma \right\},
\] (234)

which implies the statement of the lemma.

By a similar argument as in Theorem 5 we can also derive the following converse bound.

**Theorem 16** For any \( Q_{Y^n} \in \mathcal{P}(Y^n) \), we have

\[
- \log P_c^{(n)}(M_n) \leq \inf_{\theta \in \mathbb{R}, \theta \geq 0} \left[ (1 + s) \tilde{\theta} \left\{ H_{1+\tilde{\theta}}(P_{X^nY^n}|Q_{Y^n}) - H_{1+(1+s)\tilde{\theta}}(P_{X^nY^n}|Q_{Y^n}) \right\} \right] / s
\] (235)

\[
-(1 + s) \log \left( 1 - 2e^{-\theta R + (\tilde{\theta} + 1 + \tilde{\theta}) H_{1+\tilde{\theta}+\tilde{\theta}}(P_{X^nY^n}|Q_{Y^n})} \right) / s
\] (236)

\[
\leq \inf_{\theta \in \theta(a(R))} \left[ (1 + s) \tilde{\theta} \left\{ H_{1+\tilde{\theta}}(P_{X^nY^n}|Q_{Y^n}) - H_{1+(1+s)\tilde{\theta}}(P_{X^nY^n}|Q_{Y^n}) \right\} \right] / s
\] (237)

\[
-(1 + s) \log \left( 1 - 2e^{\theta(a(R))-\tilde{\theta} a(R)-\theta(a(R)) H_{1+\theta(a(R))}(P_{X^nY^n}|Q_{Y^n})} \right) / s,
\] (238)

where \( R = n \log |A| - \log M_n \), and \( \theta(a) \) and \( a(R) \) are the inverse functions defined in (29) and (52) respectively.

**Proof:** See Appendix [L].
F. Finite-Length Bound for Markov Noise Channel

From this section, we address conditional additive channel whose conditional additive noise us subject to Markovian chain. Here, the input alphabet \( A^n \) equals the additive group \( X^n = \mathbb{F}_q^n \) and the output alphabet \( B^n \) is \( X^* \mathcal{Y}^n \). That is, the transition matrix describing the channel is given by using a transition matrix \( W \) on \( X^* \mathcal{Y}^n \) and an initial distribution \( Q \) as

\[
P_{B^n | A^n}(x^n + a^n, y^n | a^n) = Q(x_1, y_1) \prod_{i=2}^{n} W(x_i, y_i | x_{i-1}, y_{i-1}).
\]

(240)

As in Section II-B, we consider two assumptions on the transition matrix \( W \) of the noise process \( (X, Y) \), i.e., Assumption 1 and Assumption 2. We also use the same notations as in Section II-B.

Example 6 (Gilbert-Elliot channel with state-information available at the receiver) The Gilbert-Elliot channel [30], [31] is characterized by a channel state \( Y^n \) on \( \mathcal{Y}^n = \{0, 1\}^n \), and an additive noise \( X^n \) on \( \mathcal{X}^n = \{0, 1\}^n \). The noise process \( (X^n, Y^n) \) is a Markov chain induced by the transition matrix \( W \) introduced in Example 3. For the channel input \( a^n \), the channel output is given by \( (a^n + X^n, Y^n) \) when the state-information is available at the receiver. Thus, this channel can be regarded as a conditional additive channel, and the transition matrix of the noise process satisfies Assumption 2.

Proofs of the following bounds are almost the same as those in Section III-C and thus omitted. From Lemma 21 and Lemma 9 we can derive the following achievability bound.

Theorem 17 (Direct, Ass. 1) Suppose that the transition matrix \( W \) of the conditional additive noise satisfies Assumption 1. Let \( R := \frac{\nu - k}{n} \log |A| \). Then we have

\[
-\log \tilde{P}_c(n,k) \geq \sup_{-1 < \delta \leq 0} \left[ -\theta nR + (n-1)\theta H_{1+\delta}^{1+W}(X|Y) + \delta(\theta) \right].
\]

(241)

From Theorem 16 for \( Q_{Y^n} = P_{Y^n} \) and Lemma 9 we have the following converse bound.

Theorem 18 (Converse, Ass. 1) Suppose that transition matrix \( W \) of the conditional additive noise satisfies Assumption 1. Let \( R := \log |A| - \frac{1}{n} \log M_n \). If \( H_W^W(X|Y) < R < H_0^{1+W}(X|Y) \), then we have

\[
-\log \tilde{P}_c(n,M_n) \leq \inf_{\frac{\nu - k}{n} \leq \theta(a(R))} \left[ (n-1)(1+s)\tilde{\theta} \left( H_{1+\tilde{\theta}}^{1+W}(X|Y) - H_{1+(1+s)\tilde{\theta}}^{1+W}(X|Y) \right) + \delta_1 \right] + \delta_2
\]

(242)

(243)

where \( \theta(a) = \theta^{\downarrow}(a) \) and \( a(R) = a^{\downarrow}(R) \) are the inverse functions defined by (67) and (70) respectively, and

\[
\delta_1 := (1+s)\tilde{\delta}(\tilde{\theta}) - \tilde{\delta}(1+s)\tilde{\theta},
\]

(244)

\[
\delta_2 := \frac{(\theta(a(R)) - \theta)(1+\theta)\tilde{\delta}(\theta(a(R))) + (1 + \theta(a(R)))\tilde{\delta}(\theta)}{1 + \theta(a(R))}
\]

(245)

Next, we derive tighter bounds under Assumption 2. From Lemma 20 and Lemma 10 we have the following achievability bound.

Theorem 19 (Direct, Ass. 2) Suppose that the transition matrix \( W \) of the conditional additive noise satisfies Assumption 2. Let \( R := \frac{\nu - k}{n} \log |A| \). Then we have

\[
-\log \tilde{P}_c(n,k) \geq \sup_{-1 < \theta \leq 0} \frac{-\theta nR + (n-1)\theta H_{1+\theta}^{1+W}(X|Y) + \delta(\theta)}{1 + \theta}.
\]

(246)

By using Theorem 16 for \( Q_{Y^n} = P_{Y^n}^{(1+\theta(a(R)))} \) and Lemma 11 we can derive the following converse bound.
Theorem 20 (Converse, Ass. 2) Suppose that the transition matrix $W$ of the conditional additive noise satisfies Assumption 2 Let $R := \log |\mathcal{A}| - \frac{1}{n} \log M_n$. If $H_W(X|Y) < R < H_{0}^{\uparrow}W(X|Y)$, we have

$$
-\log P^{(n)}_c(M_n) 
\leq \inf_{\epsilon > 0} \left[ (n-1)(1+s) \tilde{\theta} \left\{ H_{1+\hat{\theta},1+\theta(a(R))}^W(X|Y) - H_{1+1+s,1+\theta(a(R))}^W(X|Y) \right\} + \delta_1 
- (1+s) \log \left( 1 - 2e^{-(n-1)}[(\theta(a(R)) - \tilde{\theta}(\theta(a(R)), \theta(a(R))) + (1 + \theta(a(R))) \tilde{\theta}(\theta(a(R))) \right) / s \right],
$$

where $\theta(a) = \theta^\dagger(a)$ and $a(R) = a^\dagger(R)$ are the inverse functions defined by (71) and (73) respectively, and

$$
\delta_1 := (1+s)\zeta(\tilde{\theta}, \theta(a(R))) - \zeta((1+s)\tilde{\theta}, \theta(a(R))), \quad \delta_2 := (\theta(a(R)) - \tilde{\theta})R - (1 + \tilde{\theta})\zeta(\theta(a(R)), \theta(a(R))) + (1 + \theta(a(R))) \tilde{\theta}(\theta(a(R))).
$$

Finally, when $\gamma$ is singleton, i.e., the channel is additive, we can derive the following achievability bound from Lemma 22.

Theorem 21 (Direct, Singleton) Let $R := \frac{n-k}{n} \log |\mathcal{A}|$. Then we have

$$
-\log \tilde{P}(n,k) \geq \sup_{-1/2 \leq \theta \leq 0} \frac{-\theta nR + (n-1)\theta H_{1+\theta}^W(X) + \delta(\theta)}{1 + \theta}.
$$

Remark 10 Our treatment for Markovian conditional additive channel covers Markovian regular channels because Markovian regular channel can be reduced to Markovian conditional additive channel as follows. Let $\tilde{X} = \{\tilde{X}^n\}_{n=1}^\infty$ be a Markov chain on $\mathcal{B}$ whose distribution is given by

$$
P_{\tilde{X}^n}(\tilde{x}^n) = Q(\tilde{x}_1) \prod_{i=2}^{n} \tilde{W}(\tilde{x}_i | \tilde{x}_{i-1})
$$

for a transition matrix $\tilde{W}$ and an initial distribution $Q$. Let $(X^n, Y^n) = \{(X^n, Y^n)\}_{n=1}^\infty$ be the noise process of the conditional additive channel derived from the noise process $\tilde{X}$ of the regular channel by the argument of Section IV-B. Since we can write

$$
P_{X^nY^n}(x^n, y^n) = Q(y_1^{\uparrow} \theta_{y_1}(x_1)) \frac{1}{[\text{Stb}(0_{y_1})]} \prod_{i=2}^{n} \tilde{W}(y^{-1}_i(\theta_{y_i}(x_i))) \frac{1}{[\text{Stb}(0_{y_i})]} \tilde{W}(y_{i-1}(\theta_{y_{i-1}}(x_{i-1}))) \frac{1}{[\text{Stb}(0_{y_{i-1}})]},
$$

the process $(X, Y)$ is also a Markov chain. Thus, the regular channel given by $\tilde{X}$ is reduced to the conditional additive channel given by $(X, Y)$.

G. Second Order

To discuss the asymptotic performance, we introduce the quantity

$$
C := \log |\mathcal{A}| - H_W(X|Y).
$$

By applying the central limit theorem (cf. [65, Theorem 27.4, Example 27.6]) to Lemma 19 and Lemma 24 for $Q_{Y^n} = P_{Y^n}$, and by using Theorem 2 we have the following.

Theorem 22 Suppose that the transition matrix $W$ of the conditional additive noise satisfies Assumption 1 For arbitrary $\varepsilon \in (0, 1)$, we have

$$
\log M(n, \varepsilon) = k(n, \varepsilon) \log |\mathcal{A}| = Cn + \sqrt{V_W(X|Y)} \Phi^{-1}(\varepsilon) \sqrt{n} + o(\sqrt{n}).
$$

Proof: It can be proved exactly in the same manner as Theorem 11
From the above theorem, the (first-order) capacity of the conditional additive channel under Assumption 1 is given by

$$\lim_{n \to \infty} \frac{1}{n} \log M(n, \varepsilon) = \lim_{n \to \infty} \frac{1}{n} \log k(n, \varepsilon) \log |\mathcal{A}| = C$$

(258)

for every $0 < \varepsilon < 1$. In the next subsections, we consider the asymptotic behavior of the error probability when the rate is smaller than the capacity in the moderate deviation regime and the large deviation regime, respectively.

H. Moderate Deviation

From Theorem 17 and Theorem 18 we have the following.

**Theorem 23** Suppose that the transition matrix $W$ of the conditional additive noise satisfies Assumption 1. For arbitrary $t \in (0, 1/2)$ and $\delta > 0$, we have

$$\lim_{n \to \infty} \frac{1}{n^{1-2t}} \log P^{(n)}_c \left( e^{nC+n^{1-t}\delta} \right) = \lim_{n \to \infty} \frac{1}{n^{1-2t}} \log \bar{P}^{(n)}_c \left( n, \frac{nC - n^{1-t}\delta}{\log |\mathcal{A}|} \right)$$

$$= \frac{\delta^2}{2V^W(X|Y)}.$$  

(259)

(260)

**Proof:** It can be proved exactly in the same manner as Theorem 12.

I. Large Deviation

From Theorem 17 and Theorem 18 we have the following.

**Theorem 24** Suppose that the transition matrix $W$ of the conditional additive noise satisfies Assumption 1. For $H^W(X|Y) < R$, we have

$$\lim_{n \to \infty} \frac{1}{n} \log \bar{P}^{(n)}_c \left( n, n \left( 1 - \frac{R}{\log |\mathcal{A}|} \right) \right) \geq \sup_{-1 \leq \theta \leq 0} \left[ -\theta R + \theta H^{\downarrow W}_{1+\theta} (X|Y) \right].$$

(261)

On the other hand, for $H^W(X|Y) < R < H^\downarrow_0 W(X|Y)$, we have

$$\lim_{n \to \infty} \frac{1}{n} \log \bar{P}^{(n)}_c \left( e^{n(\log |\mathcal{A}| - R)} \right) \leq -\theta (a(R)) a(R) + \theta (a(R)) H^\downarrow W_{1+\theta (a(R))} (X|Y)$$

$$= \sup_{-1 \leq \theta \leq 0} \frac{-\theta R + \theta H^\downarrow W_{1+\theta} (X|Y)}{1 + \theta}.$$  

(262)

(263)

**Proof:** It can be proved exactly in the same manner as Theorem 13.

Under Assumption 2 from Theorem 19 and Theorem 20 we have the following tighter bound.

**Theorem 25** Suppose that the transition matrix $W$ of the conditional additive noise satisfies Assumption 2. For $H^W(X|Y) < R$, we have

$$\lim_{n \to \infty} \frac{1}{n} \log \bar{P}^{(n)}_c \left( n, n \left( 1 - \frac{R}{\log |\mathcal{A}|} \right) \right) \geq \sup_{-\frac{1}{4} \leq \theta \leq 0} \frac{-\theta R + \theta H^{\downarrow W}_{1+\theta} (X|Y)}{1 + \theta}. $$

(264)

On the other hand, for $H^W(X|Y) < R < H^\downarrow_0 W(X|Y)$, we have

$$\lim_{n \to \infty} \frac{1}{n} \log \bar{P}^{(n)}_c \left( e^{n(\log |\mathcal{A}| - R)} \right) \leq -\theta (a(R)) a(R) + \theta (a(R)) H^\downarrow W_{1+\theta (a(R))} (X|Y)$$

$$= \sup_{-1 \leq \theta \leq 0} \frac{-\theta R + \theta H^\downarrow W_{1+\theta} (X|Y)}{1 + \theta}.$$  

(265)

(266)

**Proof:** It can be proved exactly in the same manner as Theorem 14.

When $\mathcal{Y}$ is singleton, i.e., the channel is additive, from Theorem 21 and (263), we have the following.
Theorem 26 For \( H^W(X) < R \), we have
\[
\liminf_{n \to \infty} -\frac{1}{n} \log \bar{P}_c^{(n)}(n, n(1 - \frac{R}{\log |A|})) \geq \sup_{-\frac{1}{2} \leq \theta \leq 0} -\frac{\theta R + \theta H^W_{1+\theta}(X)}{1 + \theta}.
\] (267)

On the other hand, for \( H^W(X) < R < H^W_0(X) \), we have
\[
\limsup_{n \to \infty} -\frac{1}{n} \log P_c^{(n)}(e^{n(\log |A| - R)}) \leq \sup_{-1 < \theta \leq 0} -\frac{\theta R + \theta H^W_{1+\theta}(X)}{1 + \theta}.
\] (268)

**Proof:** It can be proved in the same manner as Remark 9.

V. DISCUSSION AND CONCLUSION

In this paper, we have developed a unified approach to source coding with side information and channel coding for conditional additive channel for finite-length and asymptotic analyses of Markov chains. In our approach, the conditional Rényi entropies defined for transition matrices play important roles. Although we only illustrated the source coding with side-information and the channel coding for conditional additive channel as applications of our approach, it can be applied to some other problems in information theory such as random number generation problems, as shown in another paper [58].

Our obtained results for the source coding with side information and the channel coding of the conditional additive channel has been extended to the case when the side information is continuous and the joint distribution \( X \) and \( Y \) is memoryless. Since this case covers the BPSK-AWGN channel, it can be expected that it covers the MPSK-AWGN channel. Since such channels are often employed in the real channel coding, it is an interesting future topic to investigate the finite-length bound for these channels. Further, we could not define the conditional Rényi entropy for transition matrices of continuous \( Y \). Hence, our result could not extended to such a continuous case. It is another interesting future topic to extend the obtained result to the case with continuous \( Y \).

**APPENDIX**

A. Preparation for Proofs

When we prove some properties of Rényi entropies or derive converse bounds, some properties of cumulant generating functions (CGFs) become useful. For this purpose, we introduce some terminologies in statistics from [22], [23]. Then, in Appendix B, we show relation between terminologies in statistics and those in information theory. For proofs, see [22], [23].

1) Single-Shot Setting: Let \( Z \) be a random variable with distribution \( P \). Let
\[
\phi(\rho) := \log \mathbb{E}[e^{\rho Z}] = \log \sum_z P(z) e^{\rho z}
\] (269)
be the cumulant generating function (CGF). Let us introduce an exponential family
\[
P_{\rho}(z) := P(z) e^{\rho z - \phi(\rho)}.
\] (270)

By differentiating the CGF, we find that
\[
\phi'(\rho) = \mathbb{E}_{\rho}[Z] := \sum_z P_{\rho}(z) z.
\] (272)

We also find that
\[
\phi''(\rho) = \sum_z P_{\rho}(z) (z - \mathbb{E}_{\rho}[Z])^2.
\] (274)

We assume that \( Z \) is not constant. Then, (274) implies that \( \phi(\rho) \) is a strict convex function and \( \phi'(\rho) \) is monotonically increasing. Thus, we can define the inverse function \( \rho(a) \) of \( \phi'(\rho) \) by
\[
\phi'(\rho(a)) = a.
\] (275)
Let
\[ D_{1+s}(P\|Q) := \frac{1}{s} \log \sum_z P(z)^{1+s} Q(z)^{-s} \] (276)
be the Rényi divergence. Then, we have the following relation:
\[ sD_{1+s}(P_\rho\|P_\rho) = \phi((1 + s)\rho - s\rho) - (1 + s)\phi(\rho) + s\phi(\rho). \] (277)

2) Transition Matrix: Let \( \{W(z'|z')\}_{z',z''} \) be an ergodic and irreducible transition matrix, and let \( \tilde{P} \) be its stationary distribution. For a function \( g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R} \), let
\[ E[g] := \sum_{z,z'} \tilde{P}(z')W(z|z')g(z,z'). \] (278)
We also introduce the following tilted matrix:
\[ W_\rho(z|z') := W(z|z')e^{\rho g(z,z')}. \] (279)
Let \( \lambda_\rho \) be the Perron-Frobenius eigenvalue of \( W_\rho \). Then, the CGF for \( W \) with generator \( g \) is defined by
\[ \phi(\rho) := \log \lambda_\rho. \] (280)

Lemma 25 The function \( \phi(\rho) \) is a convex function of \( \rho \), and it is strict convex iff. \( \phi''(0) > 0 \).

From Lemma 25, \( \phi'(\rho) \) is monotone increasing function. Thus, we can define the inverse function \( \rho(a) \) of \( \phi'(\rho) \) by
\[ \phi'(\rho(a)) = a. \] (281)

3) Markov Chain: Let \( Z = \{Z^n\}_{n=1}^\infty \) be the Markov chain induced by \( W(z|z') \) and an initial distribution \( P_{Z_1} \). For functions \( g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R} \) and \( \tilde{g} : \mathbb{Z} \to \mathbb{R} \), let \( S_n := \sum_{i=2}^n g(Z_i, Z_{i-1}) + \tilde{g}(Z_1) \). Then, the CGF for \( S_n \) is given by
\[ \phi_n(\rho) := \log E[e^{\rho S_n}]. \] (282)
We will use the following finite evaluation for \( \phi_n(\rho) \).

Lemma 26 Let \( v_\rho \) be the eigenvector of \( W_\rho^T \) with respect to the Perron-Frobenius eigenvalue \( \lambda_\rho \) such that \( \min_z v_\rho(z) = 1 \). Let \( w_\rho(z) := P_{Z_1}(z)e^{\rho g(z)} \). Then, we have
\[ (n - 1)\phi(\rho) + \delta_\rho(\rho) \leq \phi_n(\rho) \leq (n - 1)\phi(\rho) + \overline{\delta}_\rho(\rho), \] (283)
where
\[ \delta_\rho(\rho) := \log \langle v_\rho|w_\rho \rangle, \] (284)
\[ \overline{\delta}_\rho(\rho) := \log \langle v_\rho|w_\rho \rangle - \log \max_z v_\rho(z). \] (285)

From this lemma, we have the following.

Corollary 2 For any initial distribution and \( \rho \in \mathbb{R} \), we have
\[ \lim_{n \to \infty} \phi_n(\rho) = \phi(\rho). \] (286)
The relation
\[ \lim_{n \to \infty} \frac{1}{n} E[S_n] = \phi'(0) \] (287)
\[ \lim_{n \to \infty} \frac{1}{n} E[g] = \phi'(0) \] (288)
is well known. Furthermore, we also have the following.

Lemma 27 For any initial distribution, we have
\[ \lim_{n \to \infty} \frac{1}{n} \text{Var} [S_n] = \phi''(0). \] (289)
B. Relation Between CGF and Conditional Rényi Entropies

1) Single-Shot Setting: For correlated random variable \((X, Y)\), let us consider \(Z = \log \frac{Q_Y(Y)}{P_{XY}(X,Y)}\). Then, the relation between the CGF and conditional Rényi entropy relative to \(Q_Y\) is given by

\[
\theta H_{1+\theta}(P_{XY}|Q_Y) = -\phi(-\theta; P_{XY}|Q_Y).
\] (290)

From this, we can also find that the relationship between the inverse functions (cf. (29) and (275)):

\[
\theta(a) = -\rho(a).
\] (291)

Thus, the inverse function defined in (32) also satisfies

\[
(1 - \rho(a(R)))a(R) + \phi(\rho(a(R))); P_{XY}|Q_Y = R.
\] (292)

Similarly, by setting \(Z = \log \frac{1}{P_{X|Y}(X|Y)}\), we have

\[
\theta H^1_{1+\theta}(X|Y) = -\phi(-\theta; P_{XY}|P_Y).
\] (293)

Then, the variance (cf. (1)) satisfies

\[
\mathbb{V}(X|Y) = \phi''(0; P_{XY}|P_Y).
\] (294)

Let \(\phi(\rho, \rho')\) be the CGF of \(Z = \log \frac{P_Y^{0(1-\rho')}(Y)}{P_{XY}(X,Y)}\) (cf. (15) for the definition of \(P_Y^{(1-\rho')}\)). Then, we have

\[
\theta H_{1+\theta,1+\rho'}(X|Y) = -\phi(-\theta, -\theta').
\] (295)

It should be noted that \(\phi(\rho, \rho')\) is a CGF for fixed \(\rho'\), but \(\phi(\rho, \rho)\) cannot be treated as a CGF.

2) Transition Matrix: For transition matrix \(W(x,y|x',y')\), we consider the function given by

\[
g((x,y),(x',y')) := \log \frac{W(y|y')}{W(x,y|x',y')}.\] (296)

Then, the relation between the CGF and the lower conditional Rényi entropy is given by

\[
\theta H^{1,W}_{1+\theta}(X|Y) = -\phi(-\theta).
\] (297)

Then, the variance defined in (51) satisfies

\[
\mathbb{V}^W(X|Y) = \phi''(0).
\] (298)

C. Proof of Lemma 2

We use the following lemma.

**Lemma 28** For \(\theta \in (-1, 0) \cup (0, 1)\), we have

\[
H^1_{1-\theta}(X|Y) \leq H^1_{1+\theta}(X|Y) \leq H^1_{1+\theta}(X|Y).
\] (299)

**Proof**: The left hand side inequality of (299) is obvious from the definition of two Rényi entropies (the latter is defined by taking maximum). The right hand side inequality was proved in [69, Lemma 6].

Now, we go back to the proof of Lemma 2 From (29) and (11), by the Taylor approximation, we have

\[
H^1_{1+\theta}(X|Y) = H(X|Y) - \frac{1}{2} \mathbb{V}(X|Y)\theta + o(\theta).
\] (300)

Furthermore, since \(\frac{1}{1-\theta} = 1 + \theta + o(\theta)\), we also have

\[
H^1_{1-\theta}(X|Y) = H(X|Y) - \frac{1}{2} \mathbb{V}(X|Y)\theta + o(\theta).
\] (301)

Thus, from Lemma 28 we can derive (16) and (17).
D. Proof of Lemma

Statements 1 and 3 follow from the relationships in (290) and (293) and strict convexity of the CGFs. To prove Statement 5, we first prove strict convexity of the Gallager function

\[ E_0(\tau; P_{XY}) := \log \sum_y P_Y(y) \left( \sum_x P_{X|Y}(x|y) \right)^{1+\tau} \]  

(302)

for \( \tau > -1 \). We use the Hölder inequality:

\[ \sum_i a_i^\alpha b_i^\beta \leq \left( \sum_i a_i \right)^\alpha \left( \sum_i b_i \right)^\beta \]  

(303)

for \( \alpha, \beta > 0 \) such that \( \alpha + \beta = 1 \), where the equality holds iff. \( a_i = cb_i \) for some constant \( c \). For \( \lambda \in (0, 1) \), let

\[ 1 + \tau_3 = \lambda (1 + \tau_1) + (1 - \lambda)(1 + \tau_2), \]  

(304)

and

\[ \frac{\lambda (1 + \tau_1)}{1 + \tau_3} + \frac{(1 - \lambda)(1 + \tau_2)}{1 + \tau_3} = 1. \]  

(305)

Then, by applying the Hölder inequality twice, we have

\[
\sum_y P_Y(y) \left( \sum_x P_{X|Y}(x|y) \right)^{1+\tau_3} \]

(306)

\[ = \sum_y P_Y(y) \left( \sum_x P_{X|Y}(x|y) \frac{1}{1+\tau_1} \frac{1}{1+\tau_2} P_{X|Y}(x|y) \right) \]

(307)

\[ \leq \sum_y P_Y(y) \left[ \left( \sum_x P_{X|Y}(x|y) \right)^{\frac{1}{1+\tau_1}} \left( \sum_x P_{X|Y}(x|y) \right)^{\frac{1}{1+\tau_2}} \right]^\lambda \]

(308)

\[ = \sum_y P_Y(y) \left( \sum_x P_{X|Y}(x|y) \right)^{\frac{\lambda (1 + \tau_1)}{1 + \tau_3}} \left( \sum_x P_{X|Y}(x|y) \right)^{\frac{(1 - \lambda)(1 + \tau_2)}{1 + \tau_3}} \]

(309)

\[ \leq \left[ \sum_y P_Y(y) \left( \sum_x P_{X|Y}(x|y) \right)^{\frac{1}{1+\tau_1}} \right]^\lambda \left[ \sum_y P_Y(y) \left( \sum_x P_{X|Y}(x|y) \right)^{\frac{1}{1+\tau_2}} \right]^{1-\lambda}. \]  

(311)

The equality in the second inequality holds iff.

\[ \left( \sum_x P_{X|Y}(x|y) \right)^{1+\tau_1} = c \left( \sum_x P_{X|Y}(x|y) \right)^{1+\tau_2} \quad \forall y \in \mathcal{Y}. \]  

(312)

for some constant \( c \). Furthermore, the equality in the first inequality holds iff. \( P_{X|Y}(x|y) = \frac{1}{\text{supp}(P_{X|Y}(\cdot|y))} \). Substituting this into (312), we find that \( \text{supp}(P_{X|Y}(\cdot|y)) \) is irrespective of \( y \). Thus, both the equalities hold simultaneously iff. \( V(X|Y) = 0 \). Now, since

\[ \theta H_{1+\theta}(X|Y) = -(1 + \theta) E_0 \left( \frac{-\theta}{1 + \theta}; P_{XY} \right), \]  

(313)
we have
\[
\frac{d^2[\theta H_{1+\theta}^2(X|Y)]}{d\theta^2} = -\frac{1}{(1+\theta)^4} E_0^\theta \left( \frac{-\theta}{1+\theta}; P_{XY} \right)
\]
\[
\leq 0
\]
(314)
(315)
for \(\theta \in (-1, \infty)\), where the equality holds iff. \(\forall (X|Y) = 0\).

Statement 7 is obvious from the definitions of the two measures. The first part of Statement 8 follows from (295) and convexity of the CGF, but we need another argument to check the conditions for strict concavity. Since the second term of
\[
\theta H_{1+\theta,1+\theta}(X|Y) = -\log \sum_y P_Y(y) \left[ \sum_x P_{X|Y}(x|y)^{1+\theta_3} \right] \left[ \sum_x P_{X|Y}(x|y)^{1+\theta_2} \right]^{\frac{\theta_3}{1+\theta_2}} \left[ \sum_x P_{X|Y}(x|y)^{1+\theta} \right]^{\frac{\theta_2}{1+\theta_2}}
\]
is linear with respect to \(\theta\), it suffice to show strict concavity of the first term. By using the Hölder inequality twice, for \(\theta_3 = \lambda \theta_1 + (1-\lambda) \theta_2\), we have
\[
\sum_y P_Y(y) \left[ \sum_x P_{X|Y}(x|y)^{1+\theta_3} \right] \left[ \sum_x P_{X|Y}(x|y)^{1+\theta_2} \right]^{\frac{\theta_3}{1+\theta_2}} \left[ \sum_x P_{X|Y}(x|y)^{1+\theta} \right] \leq \sum_y P_Y(y) \left[ \sum_x P_{X|Y}(x|y)^{1+\theta_1} \right] \left[ \sum_x P_{X|Y}(x|y)^{1+\theta_2} \right]^{\frac{\theta_1}{1+\theta_2}} \left[ \sum_x P_{X|Y}(x|y)^{1+\theta} \right]^{\frac{\theta_2}{1+\theta_2}}
\]
\[
\leq \left[ \sum_y P_Y(y) \left[ \sum_x P_{X|Y}(x|y)^{1+\theta_1} \right] \left[ \sum_x P_{X|Y}(x|y)^{1+\theta_2} \right]^{\frac{\theta_1}{1+\theta_2}} \left[ \sum_x P_{X|Y}(x|y)^{1+\theta} \right]^{\frac{\theta_2}{1+\theta_2}} \right] \left[ \sum_x P_{X|Y}(x|y)^{1+\theta} \right]^{\frac{\theta_3}{1+\theta_3}} \left[ \sum_x P_{X|Y}(x|y)^{1+\theta} \right]^{\frac{\theta_2}{1+\theta_2}}
\]
(317)
(318)
(319)
(320)
where both the equalities hold simultaneously iff. \(\forall (X|Y) = 0\), which can be proved in a similar manner as the equality conditions in (308) and (311). Thus we have the latter part of Statement 8.

Statements 10-12 are also obvious from the definitions. Statements 2, 4, 6, 9, 10-12 follows from Statements 1, 3, 5, 8 (cf. (69) Lemma 1).

### E. Proof of Lemma 2

Since (24) and (28) are obvious from the definitions, we only prove (26). We note that
\[
\left[ \sum_y P_Y(y) \left[ \sum_x P_{X|Y}(x|y)^{1+\theta} \right] \right]^{1+\theta}
\]
\[
\leq \left[ \sum_y P_Y(y) \left| \text{supp}(P_{X|Y}(\cdot|y)) \right|^{1+\theta} \right]^{1+\theta}
\]
\[
\leq \max_{y \in \text{supp}(P_Y)} \left| \text{supp}(P_{X|Y}(\cdot|y)) \right|
\]
(321)
(322)
(323)
and
\[
\left[ \sum_y P_Y(y) \left[ \sum_x P_{X|Y}(x|y)^{1+\theta} \right] \right]^{1+\theta}
\]
\[
\geq P_Y(y^*)^{1+\theta} \left[ \sum_x P_{X|Y}(x|y^*)^{1+\theta} \right]^{\theta \to -1} \left| \text{supp}(P_{X|Y}(\cdot|y^*)) \right|
\]
(324)
(325)
(326)
where
\[
y^* := \arg\max_{y \in \text{supp}(P_Y)} |\text{supp}(P_{X|Y}(-|y))|.
\]
(327)

\section*{F. Proof of Lemma 5}
From Lemma 28, Theorem 1, and Theorem 3, we have
\[
H_{\downarrow, W_1}(X|Y) - \theta(X|Y) \leq H_{\uparrow, W_1}(X|Y) \leq H_{1+\theta}(X|Y)
\]
for \(\theta \in (-1, 0) \cup (0, 1)\). Thus, we can prove Lemma 5 in the same manner as Lemma 2.

\section*{G. Proof of (63)}
First, in the same manner as Theorem 1, we can show
\[
\lim_{n \to \infty} \frac{1}{n} H_{1+\theta}(P_{X^n|Y^n}|Q_{Y^n}) = H_{W|V}(X|Y),
\]
(329)
where \(Q_{Y^n}\) is a Markov chain induced by \(V\) for some initial distribution. Then, since \(H_{1+\theta}(P_{X^n|Y^n}|Q_{Y^n}) \leq H_{1+\theta}(X^n|Y^n)\) for each \(n\), by using Theorem 3 we have
\[
H_{1+\theta}(X|Y) \leq H_{1+\theta}(W|V)(X|Y).
\]
(330)
Thus, the rest of the proof is to show that \(H_{1+\theta}(X|Y)\) is attainable by some \(V\).

Let \(\hat{Q}_\theta\) be the normalized left eigenvector of \(K_\theta\), and let
\[
V_\theta(y|y') := \frac{\hat{Q}_\theta(y)}{\kappa_\theta \hat{Q}_\theta(y')} K_\theta(y|y').
\]
(331)
Then, \(V_\theta\) attains the maximum. To prove this, we will show that \(\kappa_\theta^{1+\theta}\) is the Perron-Frobenius eigenvalue of
\[
W(x, y|x', y')^{1+\theta} V_\theta(y|y')^{-\theta}.
\]
(332)
We first confirm that \((\hat{Q}_\theta(y)^{1+\theta} : (x, y) \in \mathcal{X} \times \mathcal{Y})\) is an eigenvector of \(W\) as follows:
\[
\sum_{x, y} \hat{Q}_\theta(y)^{1+\theta} W(x, y|x', y')^{1+\theta} V_\theta(y|y')^{-\theta}
\]
(333)
\[
= \sum_{y} \hat{Q}_\theta(y)^{1+\theta} W_\theta(y|y') \left[ \frac{\hat{Q}_\theta(y)}{\kappa_\theta \hat{Q}_\theta(y')} \frac{1}{1+\theta} \right]^{-\theta}
\]
(334)
\[
= \kappa_\theta^{\theta} \hat{Q}_\theta(y')^\theta \sum_{y} \hat{Q}_\theta(y) W_\theta(y|y')^{1+\theta}
\]
(335)
\[
= \kappa_\theta^{1+\theta} \hat{Q}_\theta(y')^{1+\theta}.
\]
(336)
Since \((\hat{Q}_\theta(y)^{1+\theta} : (x, y) \in \mathcal{X} \times \mathcal{Y})\) is a positive vector and the Perron-Frobenius eigenvector is the unique positive eigenvector, we find that \(\kappa_\theta^{1+\theta}\) is the Perron-Frobenius eigenvalue. Thus, we have
\[
H_{1+\theta}^{W|V}(X|Y) = -\frac{1}{\theta} \log \kappa_\theta
\]
(337)
\[
= H_{1+\theta}^{W}(X|Y).
\]
(338)
Statement 3 follows from (297) and strict convexity of the CGF. Statements 5, 8, 9, and 10 follow from the corresponding statements in Lemma 3, Theorem 1, Theorem 3, and Theorem 4.

Now, we prove Statement 3. For this purpose, we introduce transition matrix counterpart of the Gallager function as follows. Let

$$K_{\tau}(y'|y) := W(y'|y) \left[ \sum_x W(x|x', y', y) \right]^{1+\tau}$$

for $\tau > -1$, which is well defined under Assumption 2. Let $\bar{K}_{\tau}$ be the Perron-Frobenius eigenvalue of $K_{\tau}$, and let $\tilde{Q}_{\tau}$ and $\bar{Q}_{\tau}$ be its normalized right and left eigenvectors. Then, let

$$L_{\tau}(y'|y) := \frac{\hat{Q}_{\tau}(y')}{\bar{K}_{\tau} \tilde{Q}_{\tau}(y')} K_{\tau}(y'|y)$$

be a parametrized transition matrix. The stationary distribution of $L_{\tau}$ is given by

$$Q_{\tau}(y') := \frac{\hat{Q}_{\tau}(y')}{\tilde{Q}_{\tau}(y')Q_{\tau}(y')} Q_{\tau}(y') \quad (341)$$

We prove strict convexity of $E_{\tau}^W(\tau) := \log \bar{K}_{\tau}$ for $\tau > -1$. Then, by the same reason as (314), we can show Statement 3. Let $Q_{\tau}(y, y') := L_{\tau}(y|y')Q_{\tau}(y')$. By the same calculation as Proof of Lemma 13 and Lemma 14, we have

$$\sum_{y, y'} Q_{\tau}(y, y') \left[ \frac{d}{d\tau} \log L_{\tau}(y|y') \right]^2 = -\sum_{y, y'} Q_{\tau}(y, y') \left[ \frac{d^2}{d\tau^2} \log L_{\tau}(y|y') \right].$$

Furthermore, from the definition of $L_{\tau}$, we have

$$-\sum_{y, y'} Q_{\tau}(y, y') \left[ \frac{d^2}{d\tau^2} \log L_{\tau}(y|y') \right] = -\sum_{y, y'} Q_{\tau}(y, y') \left[ \frac{d^2}{d\tau^2} \log L_{\tau}(y|y') \right] - \sum_{y, y'} Q_{\tau}(y, y') \frac{d^2}{d\tau^2} \log K_{\tau}(y|y').$$

Now, we show convexity of $\log \bar{K}_{\tau}(y'|y)$ for each $(y, y')$. By using the Hölder inequality (cf. Appendix D), for $\tau_3 = \lambda \tau_1 + (1 - \lambda) \tau_2$, we have

$$\left[ \sum_x W(x|x', y', y)^{1+\tau_3} \right]^{\lambda(1+\tau_2)} \leq \left[ \sum_x W(x|x', y', y)^{1+\tau} \right]^{\lambda(1+\tau_1)} \left[ \sum_x W(x|x', y', y)^{1+\tau_2} \right]^{(1-\lambda)(1+\tau_2)}.$$ (346)

Thus, $E_{\tau}^W(\tau)$ is convex. To check strict convexity, we note that the equality in (346) holds iff. $W(x|x', y', y) = \frac{1}{\supp(W(\cdot|x', y', y))}$. Since

$$\sum_x W(x|x', y', y)^{1+\theta} = \frac{1}{\supp(W(\cdot|x', y', y))^{\theta}}$$

does not depend on $x'$ from Assumption 2, we have $\supp(W(\cdot|x', y', y)) = C_{yy'}$ for some integer $C_{yy'}$. By substituting this into $\bar{K}_{\tau}$, we have

$$\bar{K}_{\tau}(y'|y) = W(y'|y) C_{yy'}.$$ (348)

\[24\] The concavity of $\theta H_{1+\theta}(X | Y)$ follows from the limiting argument, i.e., the concavity of $\theta H_{1+\phi}(X^n | Y^n)$ (cf. Lemma 3 and Theorem 3). However, the strict concavity does not follow from the limiting argument.
On the other hand, we note that the CGF $\phi(\rho)$ is defined as the logarithm of the Perron-Frobenius eigenvalue of

$$W(x, y|w', y')^{1-\rho}W(y|y')^\rho = W(y|y')^{\frac{1}{\rho}}C_v^{\rho\nu}1[x \in \mathsf{supp}(W(\cdot|x', y', y))]. \quad (349)$$

Since

$$\sum_{x, y} \hat{Q}_\tau(y)W(y|y')^{\frac{1}{\rho}}C_v^{\rho\nu}1[x \in \mathsf{supp}(W(\cdot|x', y', y))] \quad (350)$$

$$= \sum_{y} \hat{Q}_\tau(y)W(y|y')C_v^{\rho\nu} \quad (351)$$

$$= \hat{\kappa}_\tau \hat{Q}_\tau(y'), \quad (352)$$

$\hat{\kappa}_\tau$ is the Perro-Frobenius eigenvalue of (349), and thus we have $E_0^W(\tau) = \phi(\tau)$ when the equality in (346) holds for every $(y, y')$ such that $W(y|y') > 0$. Since $\phi(\tau)$ is strict convex if $V^W(X|Y) > 0$, $E_0^W(\tau)$ is strict convex if $V^W(X|Y) > 0$. Thus, $\theta H_{1+\theta}^W(X|Y)$ is strict concave if $V^W(X|Y) > 0$. On the other hand, from (37), $\theta H_{1+\theta}^W(X|Y)$ is strict concave only if $V^W(X|Y) > 0$.

Statement 3 can be proved by modifying the proof of Statement 8 of Lemma 3 to a transition matrix in a similar manner as Statement 3 of the present lemma.

Finally, Statements 4, 5, 6 follow from Statements 1, 3, 6 (cf. [69, Lemma 1]).

\section{Proof of Lemma 8}

We only prove (75) since we can prove (76) exactly in the same manner by replacing $H_{1+\theta}^W(X|Y)$, $\theta^\dagger(a)$, and $a^\dagger(R)$ by $H_{1+\theta}^W(X|Y)$, $\theta^\dagger(a)$, and $a^\dagger(R)$. Let

$$f(\theta) := \frac{-\theta R + \theta H_{1+\theta}^W(X|Y)}{1 + \theta}. \quad (353)$$

Then, we have

$$f'(\theta) = -R + (1 + \theta) \frac{d[\theta H_{1+\theta}^W(X|Y)]}{d\theta} - \theta H_{1+\theta}^W(X|Y) \quad (354)$$

$$= -R + R \left( \frac{d[\theta H_{1+\theta}^W(X|Y)]}{d\theta} \right) \quad (355)$$

Since $R(a)$ is monotonically increasing and $\frac{d[\theta H_{1+\theta}^W(X|Y)]}{d\theta}$ is monotonically decreasing, we have $f'(\theta) \geq 0$ for $\theta \leq \theta(a(R))$ and $f'(\theta) \leq 0$ for $\theta \geq \theta(a(R))$. Thus, $f(\theta)$ takes its maximum at $\theta(a(R))$. Furthermore, since $-1 \leq \theta(a(R)) \leq 0$ for $H_{1+\theta}^W(X|Y) \leq R \leq H_0^W(X|Y)$, we have

$$\sup_{-1 \leq \theta \leq 0} \frac{-\theta R + \theta H_{1+\theta}^W(X|Y)}{1 + \theta} \quad (356)$$

$$= -\theta(a(R)) R + \theta(a(R)) H_{1+\theta(a(R))}^W(X|Y) \quad (357)$$

$$= -\theta(a(R)) [(1 + \theta(a(R))) a(R) - \theta(a(R)) H_{1+\theta(a(R))}^W(X|Y)] + \theta(a(R)) H_{1+\theta(a(R))}^W(X|Y) \quad (358)$$

$$= -\theta(a(R)) a(R) + \theta(a(R)) H_{1+\theta(a(R))}^W(X|Y), \quad (359)$$

where we substituted $R = R(a(R))$ in the second equality.
J. Proof of Lemma 10

Let $u$ be the vector such that $u(y) = 1$ for every $y \in \mathcal{Y}$. From the definition of $H_{1+\theta}^+(X^n|Y^n)$, we have the following sequence of calculations:

$$e^{-\frac{\theta}{1+\theta}H_{1+\theta}^+(X^n|Y^n)}$$

$$= \sum_{y_1,\ldots,y_n} \left[ \sum_{x_n,\ldots,x_1} P(x_1, y_1) 1+\theta \prod_{i=2}^{n} W(x_i, y_i|x_{i-1}, y_{i-1}) \right] 1+\theta$$

$$= \langle u|K_{\theta}^{n-1}w_0 \rangle$$

$$\leq \langle v_T|K_{\theta}^{n-1}w_0 \rangle$$

$$= \langle (K_{\theta}^T)^{n-1}v_T|w_0 \rangle$$

$$= \kappa_{\theta}^{n-1}\langle v_T|w_0 \rangle$$

$$= e^{-(n-1)\frac{\theta}{1+\theta}H_{1+\theta}^+(X|Y)}\langle v_T|w_0 \rangle,$$

which implies the left hand side inequality, where we used Assumption 2 in (a). On the other hand, we have the following sequence of calculations:

$$e^{-\frac{\theta}{1+\theta}H_{1+\theta}^+(X^n|Y^n)}$$

$$= \langle u|K_{\theta}^{n-1}w_0 \rangle$$

$$\geq \frac{1}{\max_y v_0(y)}\langle v_0|K_{\theta}^{n-1}w_0 \rangle$$

$$= \frac{1}{\max_y v_0(y)}\langle (K_{\theta}^T)^{n-1}v_0|w_0 \rangle$$

$$= \kappa_{\theta}^{n-1}\langle v_0|w_0 \rangle \frac{\max_y v_0(y)}{\max_y v_0(y)}$$

$$= e^{-(n-1)\frac{\theta}{1+\theta}H_{1+\theta}^+(X|Y)}\langle v_0|w_0 \rangle,$$

which implies the right hand side inequality.

K. Proof of Theorem 5

For arbitrary $\tilde{\rho} \in \mathbb{R}$, we set $\alpha := P_{XY}\{X \neq d(e(X), Y)\}$ and $\beta := P_{XY,\tilde{\rho}}\{X \neq d(e(X), Y)\}$, where

$$P_{XY,\tilde{\rho}}(x, y) := P_{XY}(x, y)\rho e^{-\phi(\rho;P_{XY}|Q_Y)}.$$

Then, by the monotonicity of the Rényi divergence, we have

$$sD_{1+s}(P_{XY,\tilde{\rho}}||P_{XY}) \geq \log \left[ \beta^{1+s}\alpha^{-s} + (1 - \beta)^{1+s}(1 - \alpha)^{-s} \right]$$

$$\geq \log \beta^{1+s}\alpha^{-s}.$$  

Thus, we have

$$-\log \alpha \leq \frac{\phi((1 + s)\tilde{\rho};P_{XY}|Q_Y) - (1 + s)\phi(\tilde{\rho};P_{XY}|Q_Y) - (1 + s)\log \beta}{s}.$$  

Now, by using Lemma 17, we have

$$1 - \beta \leq P_{XY,\tilde{\rho}} \left\{ \log \frac{Q_Y(y)}{P_{XY,\tilde{\rho}}(x, y)} \leq \gamma \right\} + \frac{M}{e^\gamma}.$$
We also have, for any $\sigma \leq 0$,
\[
P_{X,Y,\tilde{\rho}} \left\{ \log \frac{Q_Y(y)}{P_{X,Y,\tilde{\rho}}(x,y)} \leq \gamma \right\} \leq \sum_{x,y} P_{X,Y,\tilde{\rho}}(x,y)e^{\sigma \left( \log \frac{Q_Y(y)}{P_{X,Y,\tilde{\rho}}(x,y)} \right) - \gamma}
\]
\[
= e^{-[\sigma \gamma - \phi(\sigma; P_{X,Y,\tilde{\rho}}|Q_Y)]}.
\] (381)

Thus, by setting $\gamma$ so that
\[
\sigma \gamma - \phi(\sigma; P_{X,Y,\tilde{\rho}}|Q_Y) = \gamma - R,
\] (382)
we have
\[
1 - \beta \leq 2e^{-\frac{\sigma R - \phi(\sigma; P_{X,Y,\tilde{\rho}}|Q_Y)}{1 - \sigma}}.
\] (383)

Furthermore, we have the relation
\[
\phi(\sigma; P_{X,Y,\tilde{\rho}}|Q_Y) = \log \sum_{x,y} P_{X,Y,\tilde{\rho}}(x,y)^{1 - \sigma} Q_Y(y)^{\sigma}
\]
\[
= \log \sum_{x,y} \left( P_{X,Y}(x,y)^{1 - \tilde{\rho}} Q_Y(y)^{\tilde{\rho}} e^{-\phi(\tilde{\rho}; P_{X,Y}|Q_Y)} \right)^{1 - \sigma} Q_Y(y)^{\sigma}
\]
\[
= -(1 - \sigma) \phi(\tilde{\rho}; P_{X,Y}|Q_Y) + \log \sum_{x,y} P_{X,Y}(x,y)^{1 - \tilde{\rho}} - (1 - \sigma) Q_Y(y)^{\tilde{\rho} + (1 - \tilde{\rho})}
\]
\[
= \phi(\tilde{\rho} + (1 - \tilde{\rho}); P_{X,Y}|Q_Y) - (1 - \sigma) \phi(\tilde{\rho}; P_{X,Y}|Q_Y).
\] (387)

Thus, by substituting $\tilde{\rho} = -\tilde{\theta}$ and $\sigma = -\tilde{\theta}$, and by using (290), we can derive (125).

Now, we restrict the range of $\tilde{\rho}$ so that $\rho(a(R)) < \tilde{\rho} < 1$, and take
\[
\sigma = \frac{\rho(a(R)) - \tilde{\rho}}{1 - \tilde{\rho}}.
\] (388)

Then, by substituting this into (387) and (387) into (383), we have ($\phi(\rho; P_{X,Y}|Q_Y)$ is omitted as $\phi(\rho)$)
\[
\frac{\sigma R - \phi(\tilde{\rho} + (1 - \tilde{\rho}))}{1 - \sigma} + (1 - \sigma) \phi(\tilde{\rho})
\]
\[
= \frac{(\rho(a(R)) - \tilde{\rho}) R - (1 - \tilde{\rho}) \phi(\rho(a(R))) + (1 - \rho(a(R))) \phi(\tilde{\rho})}{1 - \rho(a(R))}
\]
\[
= \frac{(\rho(a(R)) - \tilde{\rho}) \{ (1 - \rho(a(R))) a(R) + \phi(\rho(a(R))) \} - (1 - \tilde{\rho}) \phi(\rho(a(R))) + (1 - \rho(a(R))) \phi(\tilde{\rho})}{1 - \rho(a(R))}
\]
\[
= (\rho(a(R)) - \tilde{\rho}) a(R) - \phi(\rho(a(R))) + \phi(\tilde{\rho}),
\] (392)

where we used (292) in the second equality. Thus, by substituting $\tilde{\rho} = -\tilde{\theta}$ and by using (290) again, we have (127).

\[\boxed{\text{L. Proof of Theorem 16}}\]

Let
\[
P_{X^n,Y^n,\tilde{\rho}}(x^n, y^n) := P_{X^n,Y^n}(x^n, y^n)^{1 - \tilde{\rho}} Q_Y^n(y^n)^{\tilde{\rho}} e^{-\phi(\tilde{\rho}; P_{X^n,Y^n}|Q_Y^n)},
\] (393)
and let $P_{B^n|A^n,\tilde{\rho}}$ be a conditional additive channel defined by
\[
P_{B^n|A^n,\tilde{\rho}}(a^n + x^n|a^n) = P_{X^n,Y^n,\tilde{\rho}}(x^n, y^n).
\] (394)
We also define the joint distribution of the message, the input, the output, and the decoded message for each channel:

\[
P_{M_n, A^n, B^n, \hat{M}_n}(m, a^n, b^n, \hat{m}) := \frac{1}{M_n} \mathbf{1}[e_n(m) = a^n] P_{B^n|A^n}(b^n|a^n) \mathbf{1}[d_n(b^n) = \hat{m}],
\]

(395)

\[
P_{M_n, A^n, B^n, \hat{M}_n, \tilde{\rho}}(m, a^n, b^n, \hat{m}) := \frac{1}{M_n} \mathbf{1}[e_n(m) = a^n] P_{B^n|A^n, \tilde{\rho}}(b^n|a^n) \mathbf{1}[d_n(b^n) = \hat{m}],
\]

(396)

For arbitrary \( \tilde{\rho} \in \mathbb{R} \), let \( \alpha := P_{M_n, \hat{M}_n}(m \neq \hat{m}) \) and \( \beta := P_{M_n, \hat{M}_n, \tilde{\rho}}(m \neq \hat{m}) \). Then, by the monotonicity of the Rényi divergence, we have

\[
sD_{1+s}(P_{A^n B^n, \tilde{\rho}} \| P_{A^n B^n}) \geq sD_{1+s}(P_{M_n, \hat{M}_n, \tilde{\rho}} \| P_{M_n, \hat{M}_n}) \geq \log \left[ \beta^{1+s} \alpha^{-s} + (1 - \beta)^{1+s} (1 - \alpha)^{-s} \right] \geq \log \beta^{1+s} \alpha^{-s}.
\]

(397)

(398)

(399)

Thus, we have

\[-\log \alpha \leq \frac{sD_{1+s}(P_{A^n B^n, \tilde{\rho}} \| P_{A^n B^n}) - (1 + s) \log \beta}{s}.\]

(400)

Here, we have

\[D_{1+s}(P_{A^n B^n, \tilde{\rho}} \| P_{A^n B^n}) = D_{1+s}(P_{X^n Y^n, \tilde{\rho}} \| P_{X^n Y^n}).\]

(401)

On the other hand, from Lemma 24 we have

\[1 - \beta \leq P_{X^n Y^n, \tilde{\rho}} \left\{ \log \frac{Q_{Y^n}(y^n)}{P_{X^n Y^n, \tilde{\rho}}(x^n, y^n)} \leq n \log |A| - \gamma \right\} + \frac{e^{\gamma}}{e^{n \log |A| - \gamma}}.\]

(402)

Thus, by the same argument as in (379)-(387) and by noting (290), we can derive (237).

Now, we restrict the range of \( \tilde{\rho} \) so that \( \rho(a(R)) < \tilde{\rho} < 1 \), and take

\[\sigma = \frac{\rho(a(R)) - \tilde{\rho}}{1 - \tilde{\rho}}.\]

(403)

Then, by noting (290), we have (239).

\[\Box\]

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