IRS-aided MIMO Systems over Double-scattering Channels: Impact of Channel Rank Deficiency

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Abstract—Intelligent reflecting surfaces (IRSs) are promising enablers for next-generation wireless communications due to their reconfigurability and high energy efficiency in improving poor propagation condition of channels, e.g., limited scattering environment. However, most existing works assumed full-rank channels requiring rich scatters, which may not be available in practice. To analyze the impact of rank-deficient channels and mitigate the ensued performance loss, we consider a large-scale IRS-aided MIMO system with statistical channel state information (CSI), where the double-scattering channel is adopted to model rank deficiency. By leveraging random matrix theory (RMT), we first derive a deterministic approximation (DA) of the ergodic rate with low computational complexity and prove the existence and uniqueness of the DA parameters. Then, we propose an alternating optimization algorithm for maximizing the DA with respect to phase shifts and signal covariance matrices. Numerical results will show that the DA is tight and our proposed method can effectively mitigate the performance loss induced by channel rank deficiency.

I. INTRODUCTION

While the demand for wireless capacity will continue to grow and be met by 5G networks, the emergence of the Internet of Everything (IoE), connecting billions of people and potentially trillions of machines, already calls for the development of 6G [1]. The development of 6G is far from trivial. Many technical challenges must be tackled and stringent requirements must be met that are beyond the reach of current systems, such as extremely high bit rates of up to 1 Tbps and massive connections reaching at least 10^6 devices/km^2. Recently, intelligent reflecting surfaces (IRSs) have been proposed as promising and disruptive enabling technology for high-capacity future wireless communications systems, due to its ability to customize wireless propagation environment [2]. With smartly controlled phase shifters, IRSs can change the end-to-end signal propagation direction and improve coverage with low energy consumption because of its passive nature. Given that two reflecting channels need to be estimated without active radio frequency chains at IRSs, there are significant challenges and overhead in acquiring instantaneous channel state information (CSI). More importantly, it is impractical to adaptively change the phase shifts according to instantaneous CSI [3]. To address this issue, many works considered the analysis and design of IRS-aided wireless systems with statistical CSI [4]–[6]. In [4] and [5], the authors first derived an accurate deterministic approximation (DA) of the minimum signal-to-interference-plus-noise-ratio (SINR) and ergodic rate, respectively, by utilizing random matrix theory (RMT). Then, based on the DA, the minimum SINR of a multiuser multiple-input-single-output (MISO) system was maximized in [4] by designing the optimal linear precoder, while the phase shifts and the covariance matrix of the transmitted signal were jointly optimized to maximize the ergodic rate for a multiple-input-multiple-output (MIMO) system in [5]. In [6], a closed-form expression for the ergodic rate of a MISO system with distributed IRSs was obtained and an efficient phase shift design algorithm was proposed to maximize the ergodic rate. However, all these works assumed full-rank models, which does not reflect the real propagation environment in IRS-aided systems.

In realistic scenarios, IRSs are typically deployed to help wireless transmission with poor propagation conditions, e.g., limited scattering, which leads to a low-rank channel matrix. The rank deficiency not only severely degrades the capacity of MIMO channels but also increases the spatial correlation [7]. The double-scattering channel was proposed to characterize more realistic channel scenarios [8], [9], and was used to model the poor scattering channel in millimeter wave (mmWave) systems [10]. The rank deficiency, the spatial correlation, and the signal attenuation can be reflected by the number of scatterers, the correlation matrices, and the large-scale fading coefficients in the model, respectively [11].

In this paper, we consider deploying an IRS to assist a point-to-point MIMO communication system over double-scattering fading channels, where the direct link between the transmitter and the receiver is blocked. Our objective is to investigate the impact of the channel rank deficiency and maximize the ergodic rate by optimizing the phase shifts at the IRS and the signal covariance matrix at the transmitter with statistical CSI. To achieve this goal, a DA of the ergodic rate is first derived by capitalizing on the Stieltjes and Shannon transform [12]. The DA is proved to be tight in the asymptotic regime and is also shown to be accurate even in low dimensions by numerical results. Meanwhile, we prove the existence and uniqueness of the DA parameters. We then propose an alternating optimization (AO) algorithm to maximize the DA with respect to the phase shift matrix and signal covariance matrix, which has been shown to be efficient in solving IRS.
optimization problems [13], [14]. The main contributions of this paper include:

1. We evaluate and optimize the ergodic rate of IRS-aided communication systems over a double-scattering channel, which is the first attempt in the literature.
2. We investigate the impact of rank deficiency on system performance and show that the proposed algorithm can significantly mitigate the influence of rank deficiency.

The rest of this paper is organized as follows. In Section II, we introduce the double-scattering channel model and formulate the problem. In Section III, an asymptotic approximation of the ergodic rate is presented. In Section IV, an algorithm is proposed to design the optimal signal covariance and phase shift matrix based on the derived approximation. Numerical results in Section V will validate the tightness of the approximation and the effectiveness of the proposed algorithm. Finally, Section VI concludes the paper. The notations used throughout the paper are listed in the footnote1 below.

II. IRS-AIDED COMMUNICATION

As shown in Fig. 1, we consider the downlink of an IRS-aided MIMO communication system, where there is a base station (BS) with \( n_T \) transmitter antennas and a user with \( n_R \) receive antennas. The IRS is deployed to establish favorable communication links for the user that would otherwise be blocked. The IRS is assumed to be implemented by \( n_L \) passive phase shifters. Therefore, the received signal \( y \) at the receiver is given by

\[
y = H_i \Phi H_2 s + n,
\]

where \( s \in \mathbb{C}^{n_T} \) represents the transmitted signal and \( n \in \mathbb{C}^{n_R} \) denotes the additive white Gaussian noise with variance \( \sigma^2 \). \( H_i \in \mathbb{C}^{n_R \times n_L} \) represents the channel matrix from the IRS to user and \( H_2 \in \mathbb{C}^{n_L \times n_T} \) represents the channel matrix from the BS to IRS. \( \Phi = \text{diag} \left( \phi_1, \phi_2, \ldots, \phi_{n_L} \right) \) with \( \phi_i = e^{j\theta_i}, i = 1, 2, \ldots, n_L \) represents the phase shifts implemented at the IRS.

The normalized ergodic rate of the MIMO channel is given by

\[
R(\Phi, Q) = \frac{1}{n_R} \mathbb{E} \left[ \log \det \left( I + \frac{1}{\sigma^2} H_1 \Phi H_2 Q H_i^H \Phi^H H_1^H \right) \right],
\]

in bits per second per Hz per antenna, where \( Q = \mathbb{E} \left[ ss^H \right] \) represents the covariance matrix of the transmitted signal and the expectation is taken over all random matrices, e.g., \( H_1, H_2 \).

A. Channel Model

In this paper, we consider the double-scattering channel [8], which is more general and commonly adopted to model the rank deficiency issue in mmWave systems. With the double-scattering model, the channel matrix can be written as

\[
H_i = R_{i \times i}^1 X_i S_i^1 D_i^1, i = 1, 2,
\]

where \( R_i \in \mathbb{H}^{n_R \times n_i}, S_i \in \mathbb{H}^{n_i \times n_i} \) and \( D_i \in \mathbb{H}^{n_D \times n_D} \) represent the transmit, scatterer, and receive correlation matrices, respectively, and \( n_S_i \) denotes the number of scatters. The rank deficiency is reflected by \( n_S_i < \min \{ n_R, n_D_i \} \). Consequently, we have \( n_R = n_R, n_L = n_D_i = n_S_i, n_D_2 = n_T \). \( X_i \in \mathbb{C}^{n_R \times n_S_i} \) and \( Y_i \in \mathbb{C}^{n_S_i \times n_D_i} \) are independent and their entries \( X_{i,j} \) and \( Y_{i,j} \) are independent and identically distributed (i.i.d.) complex zero-mean Gaussian random variables with variance \( \frac{1}{n_S^2} \) and \( \frac{1}{n_D^2} \), respectively.

B. Problem Formulation

Our objective is to maximize the normalized ergodic rate with respect to the phase shift matrix \( \Phi \) and signal covariance matrix \( Q \) under energy constraints, which can be formulated as

\[
\mathcal{P}1: \max_{Q, \Phi} R(\Phi, Q),
\]

s.t. \( \text{Tr} (Q) \leq n_T P, Q \succeq 0 \), \( \Phi = \text{diag} \left\{ \phi_1, \ldots, \phi_{n_L} \right\}, |\phi_i| = 1, i = 1, \ldots, n_L \).

The challenge in solving \( \mathcal{P}1 \) comes from two aspects. First, the objective function is an expectation form over four random matrices, which is usually evaluated by numerical methods with a huge amount of computation payload. Second, the feasible set of \( \mathcal{P}1 \) is highly non-convex due to the uni-modular constraints on phase shifts.

III. AN ASYMPTOTIC APPROXIMATION FOR ERGODIC RATE WITH DOUBLE-SCATTERING CHANNEL

Since there are two random matrices in (3), it is challenging to evaluate (2), not only in closed forms but also via numerical methods. Therefore, an accurate and computationally-efficient approximation is desired. In this section, we leverage random matrix theory to derive the DA of the normalized ergodic rate. First of all, we present two assumptions, based on which the DA is developed.

**Assumption 1:** By letting \( \alpha_{R_i, S_j} = \frac{n_R}{n_S^2}, \alpha_{S_i, D_j} = \frac{n_S}{n_D^2}, \) then \( 0 < \lim \inf \alpha_{R_i, S_j} < \infty, 0 < \lim \inf \alpha_{S_i, D_j} < \infty, i, j = \ldots \)
This assumption indicates that the scale of the IRS and the numbers of antennas and scatters grow in the same order.

**Assumption 2:** \( \lim \sup \| R_i \| < \infty, \lim \sup \| S_i \| < \infty \) and \( \lim \sup \| T_i \| < \infty \), for \( i = 1, 2 \), where

\[
\begin{align*}
T_1 &= \Phi^H D \Phi, \\
T_2 &= D_1^\frac{1}{2} Q D_2^\frac{1}{2}.
\end{align*}
\]

The normalized ergodic rate in (2) can be written as

\[
R(\Phi, Q) = \frac{1}{n_{R_1}} \mathbb{E} \left[ \log \det \left( I + \frac{1}{\sigma^2} B \right) \right],
\]

where \( B = H' H' \), \( H' = H_1' H_2' \) and \( H_1' = R_1^\frac{1}{2} X_i S_i^\frac{1}{2} Y_i T_i^\frac{1}{2} \). The matrix \( \Phi^H \) can be omitted because \( Y_i \Phi^H \) has the same statistical properties as \( Y_i \) due to the unitary-invariant attribute of the Gaussian distribution. According to Shannon transform [15], we have

\[
R(\Phi, Q) = \int_{\mathbb{Z}} \frac{1}{z} - \mathbb{E}[m_B(z)] dz = R(\sigma^2),
\]

where

\[
m_B(z) = \frac{1}{n_{R_1}} \text{Tr} \left( (zI + B)^{-1} \right)
\]

is the Stieltjes transform of the limiting spectral distribution (LSD) of matrix \( B \) for \( z \in \mathbb{C} - \mathbb{R}^- \).

Hence, it is essential to derive an asymptotic approximation of \( \mathbb{E}[m_B(z)] \) and we have the following theorem regarding the approximation.

**Theorem 1:** The DA of \( \mathbb{E}[m_B(z)] \) is given by

\[
\mathbb{E}[m_B(z)] \xrightarrow{n_{R_1} \rightarrow \infty} \frac{1}{n_{R_1}} \text{Tr} \left( (zI + h_2 h_3 h_4 h_5 R_1)^{-1} \right),
\]

where \( h_i, i = 1, 2, ..., 5 \), satisfy the following system of equations with a unique positive solution

\[
\begin{align*}
h_1 &= \frac{1}{n_{R_1}} \text{Tr} \left( R_1 (zI + h_2 h_3 h_4 h_5 R_1)^{-1} \right), \\
h_2 &= \frac{1}{n_{S_1}} \text{Tr} \left( S_1 (I + \alpha R_1, S_1 h_1 h_3 h_4 h_5 S_1)^{-1} \right), \\
h_3 &= \frac{1}{n_{T_1}} \text{Tr} \left( T_1^\frac{1}{2} R_1 T_1^\frac{1}{2} (I + \alpha R_1, T_1 h_1 h_2 h_3 h_5 T_1^\frac{1}{2} R_2 T_2^\frac{1}{2})^{-1} \right), \\
h_4 &= \frac{1}{n_{S_2}} \text{Tr} \left( S_2 (I + \alpha R_1, S_2 h_1 h_3 h_4 h_5 S_2)^{-1} \right), \\
h_5 &= \frac{1}{n_{T_2}} \text{Tr} \left( T_2 (I + \alpha R_1, T_1 h_2 h_3 h_4 T_2)^{-1} \right).
\end{align*}
\]

**Proof:** Please refer to Appendix A.

Note that \( R_i, S_i, T_i, i = 1, 2 \) can be arbitrary positive semi-definite matrices. Based on Theorem 1 and (7), we further have the following result for the DA of \( R(\Phi, Q) \).

**Theorem 2:** The DA of \( R(\Phi, Q) \) is given by

\[
R(\Phi, Q) \xrightarrow{n_{R_1} \rightarrow \infty} \mathcal{R}(\Phi, Q),
\]

where \( \mathcal{R}(\Phi, Q) \) is given in (12) at the bottom of the next page.

**Proof:** Please refer to Appendix B.

### IV. Optimization of the Signal Covariance Matrix and Phase Shift Matrix

In this section, based on the derived DA in (12), we propose an AO algorithm to jointly optimize the phase shift matrix \( \Phi \) and signal covariance matrix \( Q \) to maximize the ergodic rate.

#### A. Problem Reformulation

We adopt the DA form derived in (12) as an asymptotic approximation of the objective function in \( P1 \). Hence, we re-state the DA optimization problem as follows:

\[
P2: \max_{Q, \Phi} \mathcal{R}(\Phi, Q),
\]

subject to

\[
\begin{align*}
\text{s.t.} & \quad \text{Tr}(Q) \leq n_{D_2} P, \quad Q \succeq 0, \\
\Phi &= \text{diag} \{ \phi_1, ..., \phi_{n_L} \}, \quad |\phi_i| = 1, i = 1, ..., n_L.
\end{align*}
\]

Due to the non-convexity of \( P2 \), we resort to the AO technique to jointly optimize \( \Phi \) and \( Q \). Specifically, an \( (n_L + 1) \)-block AO approach is developed, where the correlation matrix of the transmitted signal and the phase shifts of the IRS are optimized alternately.

#### B. Signal Covariance Matrix Optimization

With a given \( \Phi \), \( P2 \) is the maximization of a concave function with convex constraints and it becomes

\[
P3: \max_{Q} \frac{1}{n_{R_1}} \log \det (I + \alpha R_1, D_1 h_1 h_2 h_3 h_4 D_2 Q),
\]

subject to

\[
\begin{align*}
\text{s.t.} & \quad \text{Tr}(Q) \leq n_{D_2} P, \quad Q \succeq 0,
\end{align*}
\]

In fact, \( P3 \) can be directly solved by the standard water-filling method, which is also adopted in [16] to design optimum transmission direction. The optimal solution is given by

\[
\hat{Q} = U Q D U^H,
\]

where the unitary matrix \( U \) and the diagonal matrix \( D_2 \) satisfy \( D_2 = U Q D U^H \). The diagonal matrix \( \Lambda_Q = \left( \frac{1}{\alpha R_1, D_1 h_1 h_2 h_3 h_4 D_2} \right)^+, \mu \) is determined by the constraint \( \text{Tr}(Q) = \text{Tr}(\Lambda_Q) = n_{D_2} P \).

#### C. Phase Shift Matrix Optimization

It can be observed from (10) and (5) that all \( h_i \)'s are related to the phase shifts. By fixing \( Q \), \( P2 \) is transformed to the following sub-problem with respect to \( \Phi \)

\[
P4: \max_{\Phi} \mathcal{R}(\Phi, Q) = \max_{\Phi} G(\theta),
\]

subject to

\[
\begin{align*}
\Phi &= \text{diag} \{ \phi_1, ..., \phi_{n_L} \} = \text{diag} \{ \exp(\theta) \}, \\
\theta_i &\in [0, 2\pi], \quad i = 1, ..., n_L.
\end{align*}
\]

Due to the uni-modular constraint for each phase shift, \( P4 \) is a non-convex problem with regard to the reflecting matrix \( \Phi \). However, we can find a suboptimal solution by resorting to the gradient method. First, we determine the gradient of \( \mathcal{R}(\Phi, Q) \) with respect to \( \phi_i \)'s as

\[
\frac{\partial G(\theta)}{\partial \theta_i} = \frac{\alpha R_1, D_1 h_1 h_2 h_3 h_4}{n_{R_1}} \text{Tr}[D_1(R_2 \otimes F) \times (I + \alpha R_1, D_1 h_1 h_2 h_3 h_4 D_1 \Phi R_2 \Phi^H)^{-1}].
\]
In (17), \( F \in \mathbb{C}^{nL \times nL} \) is given by
\[
[F]_{p,q} = \begin{cases} 
je^{j(\theta_p - \theta_q)}, & p = i, \\
-je^{j(\theta_p - \theta_q)}, & q = i, \\
0, & \text{otherwise.}
\end{cases}
\]

Here, we use the backtrack line search method \([17]\) to find the step size \( \gamma \) such that
\[
G(\theta + \gamma \nabla G(\theta)) \geq G(\theta) + c\gamma \|\nabla G(\theta)\|,
\]
where \( \nabla G(\theta) = \left( \frac{G(\theta)}{\partial \theta_1}, \frac{G(\theta)}{\partial \theta_2}, ..., \frac{G(\theta)}{\partial \theta_{nL}} \right)^T \) and \( 0 < c < 1 \) is a constant.

**D. AO Algorithm**

**Algorithm 1** Alternating Optimization on Phase Shift Matrix \( \Phi \) and Signal Covariance Matrix \( Q \)

**Input:** \( \theta^{(0)}, Q^{(0)} \), and set \( t = 0 \).

1: repeat
2: \hspace{1em} Obtain optimal \( Q^{(t+1)} \) with given \( \theta^{(t)} \) by (15).
3: \hspace{1em} Compute \( h_i, i = 1, ..., 5 \) according to (10) with given \( \theta^{(t)} \) and \( Q^{(t+1)} \).
4: \hspace{1em} Compute \( \nabla G(\theta^{(t)}) \) by (17).
5: \hspace{1em} Find the step size \( \gamma \) by backtrack line search.
6: \hspace{1em} \( \theta^{(t+1)} = \theta^{(t)} + \gamma G(\theta^{(t)}) \).
7: \hspace{1em} \( t \leftarrow t + 1 \)
8: until Convergence.

**Output:** \( \theta^t, Q^t \).

Based on the results in Sections IV-B and IV-C, the overall AO algorithm is summarized in **Algorithm 1**. It is worth noting that the solution \( Q \) in \( P3 \) is optimal for its concavity and the \( R \) in \( P4 \) will converge as it is monotonically increasing. Therefore, the AO algorithm is guaranteed to converge.

**V. NUMERICAL RESULTS**

In this section, numerical results are provided to validate the effectiveness of the proposed methods. For a double-scattering channel \([9]\), the \((l,m)\)-th entry of the correlation matrix is given by
\[
[C(\phi, N, d)]_{l,m} = \frac{1}{N} \sum_{n=1}^{N} \exp \left( j2\pi d(l-m)\sin \left( \frac{n\phi}{1-N} \right) \right),
\]
where \( \phi \) represents the angular spread of the signals, \( d \) is the antenna spacings, and \( N_s \) is the number of the scatters. Thus, we have \( R = C(\phi_R, n_s, d_r) \), \( S = C(\phi_s, n_s, d_s) \), and \( D = C(\phi_t, n_s, d_l) \) in (3). The key parameters are set as: \( \phi_R = \frac{\pi}{16}, \phi_s = \frac{\pi}{16}, d_r = d_r = d_s = 25 \).

In Fig. 2, we show the comparison between the DA in (12) and Monte-Carlo simulation results. The parameters are set as \( Q^{(0)} = I, \phi^{(0)} = \text{diag}(e^{2\pi j}), l = 0, 1, ..., n_L - 1, n_{R_1} = n_{D_2} = n_{S_1} = n_{S_2} = 5, n_{R_1} = n_{S_1} = n_{T_1} = \{5, 15, 25, 75\}, i = 1, 2 \). The number of realizations in Monte-Carlo simulation is 2000. It can be observed from Fig. 2 that the derived DA can approximate the normalized ergodic rate accurately and the approximation remains tight even for the low dimension case \((n_L = 5)\).

To illustrate the performance loss caused by rank-deficiency, we compare the ergodic rate of double-scattering channel with different ranks and that of the full-rank Rayleigh channel in Fig. 3. The transmit and receive correlation matrices for the Rayleigh channel are identical to those for the double-scattering channel, i.e., \( R_t \) and \( T_i \), \( i = 1, 2 \), which are full-rank. Simulation results are also included and denoted with markers. The parameters are set as \( n_{R_1} = n_{T_2} = n_{R_2} = n_{T_2} = 15 \) and \( n_{S_1} = n_{S_2} = 3, 7, 15 \). We observe that the ergodic rate degrades rapidly as \( n_{S_1} \) and \( n_{S_2} \) decreases, which corresponds to a more severe rank deficiency.

In Fig. 4, the effectiveness of the proposed optimization algorithm is illustrated when \( n_{R_1} = n_{T_1} = n_{R_2} = n_{T_2} = 9 \) and \( n_{S_1} = n_{S_2} = 3, 5, 9 \) with \( c = 0.0005 \). Simulation results are also included and denoted with markers. By applying the proposed AO algorithm, the ergodic rate loss caused by rank-deficient channels can be efficiently compensated. For example, the optimized rate of low-rank channel \((n_{S_1} = 3)\) outperforms the full-rank case \((n_{S_1} = 9)\) when SNR is lower than 20 dB.

**VI. CONCLUSIONS**

In this paper, we investigated the impact of channel rank-deficiency on IRS-aided systems by considering the double-scattering channel. Given the statistical CSI, we maximized the ergodic rate by optimizing the transmit signal covariance.
and phase shift matrices. For that purpose, we first derived a DA of the ergodic rate, which is accurate and computationally efficient. Then, an AO algorithm was proposed to obtain the optimal transmit covariance matrix and phase shifts. Numerical results demonstrated not only the accuracy of the evaluation but also the effectiveness of our proposed algorithm in compensating the ergodic rate loss due to the channel rank-deficiency.

**APPENDIX A**

**PROOF OF THEOREM 1**

**Proof:** The proof includes two parts. First we derive the DA form of $E[m_B(z)]$ through an iterative approach. Then, we show the existence and uniqueness of the solution of (10) using the standard interference function theory [18].

Here we use an iterative approach to erase the randomness of each $X_i$ and $Y_i$, $i=1,2$, $j=1,2$. Let $\tilde{T} = S_1^2Y_1T_1^2R_1^2X_2S_2^2Y_2T_2S_2^2Y_2^H \tilde{R}_2^2Y_1^H S_1^2$ and consider $\tilde{T}$ as a deterministic matrix. Under our assumptions in section III, by theorem 1 in [19], we have:

$$m_{\tilde{B}|\tilde{T}}(z) \xrightarrow{a.s.} \frac{1}{n_{R_1}} \mathrm{Tr} (\Theta_1(z)),$$

$$\Theta_1(z) = (zI + \tilde{\beta}_1 R_1)^{-1},$$

$$\tilde{\beta}_1 = \frac{1}{n_{R_1}} \mathrm{Tr} (\tilde{T} (I + \beta_1 \alpha \rho_{s_1} \tilde{R}_1 \tilde{T})^{-1}).$$

Since the remaining randomness lies in matrix $\tilde{T}$, by letting $\tilde{T} = T_1^2 R_1^2 X_2 S_2^2 Y_2 T_2 S_2^2 Y_2^H R_2^2 Y_1^H S_1^2$ and referring to Theorem 1 in [19], we have:

$$m_{\tilde{B}|\tilde{T}}(z) \xrightarrow{a.s.} \frac{1}{n_{R_1}} \mathrm{Tr} (\Theta_2(z)),$$

$$\Theta_2(z) = (zI + \tilde{\beta}_2 S_1)^{-1},$$

$$\tilde{\beta}_2 = \frac{1}{n_{S_1}} \mathrm{Tr} (\tilde{S}_1 \Theta_2(z)),$$

$$\tilde{\beta}_2 = \frac{1}{n_{T_1}} \mathrm{Tr} (\frac{1}{T} (I + \beta_2 \sigma_{s_1,t_1} \tilde{T})^{-1}).$$

Then, we further derive $\tilde{\beta}_1$ as

$$\tilde{\beta}_1 = \frac{1}{n_{S_1}} \frac{1}{\beta_1 \alpha \rho_{s_1}} \mathrm{Tr} (\tilde{T} (I + \beta_1 \alpha \rho_{s_1} \tilde{T})^{-1})$$

$$= \frac{1}{\beta_1 \alpha \rho_{s_1}} \frac{1}{n_{S_1}} \mathrm{Tr} (\tilde{T} (I + \beta_1 \alpha \rho_{s_1} \tilde{T}))$$

$$= \frac{1}{\beta_1 \alpha \rho_{s_1} \beta_2 \sigma_{s_1,t_1}} \frac{1}{n_{S_1}} \mathrm{Tr} (\tilde{S}_1 \Theta_2 (I + \beta_1 \alpha \rho_{s_1} \tilde{T}))$$

$$= \frac{1}{\beta_1 \alpha \rho_{s_1} \beta_2 \sigma_{s_1,t_1}} \frac{1}{n_{S_1}} \mathrm{Tr} (\tilde{S}_1 \Theta_2 (I + \beta_2 \sigma_{s_1,t_1} \tilde{T}))$$

With the following manipulations,

$$f_1 = \frac{1}{\alpha \rho_{s_1}} f_2 = \frac{\beta_2}{\tilde{\beta}_1}, f_3 = \tilde{\beta}_2,$$

we can obtain

$$f_1 = \frac{1}{n_{R_1}} \mathrm{Tr} (R_1 (zI + f_2 f_3 R_1)^{-1}),$$

$$f_2 = \frac{1}{n_{S_1}} \mathrm{Tr} (S_1 (I + \alpha R_1 \tilde{f}_3 S_1)^{-1}),$$

$$f_3 = \frac{1}{n_{T_1}} \mathrm{Tr} (\tilde{T} (I + \alpha \rho_{s_1,t_1} \tilde{T})^{-1}),$$

and

$$m_{B|T}(z) \xrightarrow{a.s.} \frac{1}{n_{R_1}} \mathrm{Tr} ((zI + f_2 f_3 R_1)^{-1}),$$

while the proof is similar to Theorem 1 in [5]. Following the same approach, we have

$$m_{B|T}(z) \xrightarrow{a.s.} \frac{1}{n_{T_1}} \mathrm{Tr} ((zI + g_2 g_3 T_1^2 R_1^2 T_1)^{-1}),$$

and the following equations,

$$g_1 = \frac{1}{n_{R_1}} \mathrm{Tr} (T_1^2 R_1^2 T_1^2 (zI + g_2 g_3 T_1^2 R_1^2 T_1)^{-1}),$$

$$g_2 = \frac{1}{n_{S_2}} \mathrm{Tr} (S_2 (I + \alpha \rho_{s_2,t_2} T_1^2 g_1 S_2)^{-1}),$$

$$g_3 = \frac{1}{n_{T_2}} \mathrm{Tr} (T_2 (I + \alpha \rho_{s_2,t_2} T_1^2 g_2 T_2)^{-1}).$$

From (31c), we have the following relation between $f_i$ and $g_i$, $i = 1,2,3$, by using the techniques same as (29)

$$f_3 \xrightarrow{a.s.} \frac{1}{n_{R_1} \alpha \rho_{s_1,t_1} \tilde{f}_3 g_1 g_2 g_3}.$$
Next, we will use the standard interference function theory [18] to show that the system of equations in (10) has a unique solution. Define the function \( e(h_1) \) as
\[
e(h_1) := \frac{1}{n_{R_1}} \text{Tr} \left( R_1 (zI + h_2 h_3 h_4 h_5 R_1)^{-1} \right),
\]
where \( h_2, h_3, h_4, h_5 \) satisfy equations (10b) to (10e). First we will show that \( e(h_1) \) is a standard interference function according to its definition in [18]. Specifically, we have the following 3 steps:

i). Positivity: It can be readily checked.

ii). Monotonicity: This step is to show that \( e(h_1) \) is an increasing function. Assume that \( h_1 > h_1' \). From (10b), we have
\[
1 = g(h_1, h_2, h_3, h_4, h_5) - g(h_1', h_2, h_3, h_4, h_5) = \frac{1}{n_{S_1}} \text{Tr} \left( S_1 (h_2 I + \alpha R_1 S_1 h_2 h_3 h_4 h_5 S_1)^{-1} \right),
\]
where \( g(h_1, h_2, h_3, h_4, h_5) \) is decreasing with \( h_2 \) and \( M := h_2 h_3 h_4 h_5 \). If \( h_2 > h_2' \), there must be an \( M < M' = h_2' h_3' h_4' h_5' \), which implies that \( h_3 > h_3', h_4 > h_4', h_5 > h_5' \) from (10c) to (10e). Therefore, there must be \( h_1 > h_1' \), which is in conflict with our assumption. Hence, there should be \( h_2 < h_2', h_3 < h_3', h_4 < h_4', h_5 < h_5' \) and \( M > M' \), from which we conclude that \( e(h_1) > e(h_1') \).

iii). Scalability: For \( \alpha > 1 \), using the matrix equation \( A^{-1} - B^{-1} = A^{-1} (B - A) B^{-1} \), we have
\[
\alpha e(h_1) - e(\alpha h_1) = \frac{1}{n_{R_1}} \text{Tr} \left( R_1 \left( \frac{z}{\alpha} I + \frac{1}{\alpha h_2 h_3 h_4 h_5} R_1 \right)^{-1} \right)
\times \left( \left( 1 - \frac{z}{\alpha} \right) I + \frac{h_2^2 h_3^2 h_4^2 h_5^2}{h_2 h_3 h_4 h_5} - \frac{1}{\alpha h_2 h_3 h_4 h_5} \right)^{-1} R_1
\times (zI + h_2^2 h_3^2 h_4^2 h_5^2 R_1)^{-1} > 0,
\]
where \( h_0^2 \) is the solution under \( \alpha h_1 \). Since \( \alpha h_1 > h_1 \), from the conclusion in ii), we have \( \alpha h_1 h_2^2 h_3^2 h_4^2 h_5^2 > h_1 h_2 h_3 h_4 h_5 \), which implies that \( h_2 h_3 h_4 h_5 > h_2 h_3 h_4 h_5 \) and (38) holds.

From the above, we know that \( e(h_1) \) is a standard interference function. Based on Theorem 2 in [18], for any initial value of \( h_1^0 > 0 \), the iterative equation \( h_{i+1} = e(h_i) \) converges to the unique positive solution.

**APPENDIX B**

**PROOF OF THEOREM 2**

Proof: We consider the function
\[
\mathcal{R}(x) = \mathcal{R}(x, h_3, h_2, h_3, h_3, h_4, h_5). \tag{39}
\]
By combining (10a) to (10e), we obtain the partial derivatives with respect to \( h_i \), \( i = 1, \ldots, 5 \),
\[
\frac{\partial \mathcal{R}(x, h_3, h_2, h_3, h_3, h_4, h_5)}{\partial h_i} = 4 M \frac{\partial R_1}{h_i} - 4 M \frac{h_i}{h_i} = 0. \tag{40}
\]
Hence, we have
\[
\frac{\partial \mathcal{R}(x)}{\partial x} = \frac{1}{x} \sum_{i=1}^{5} \frac{\partial \mathcal{R}(x, h_3, h_2, h_3, h_3, h_4, h_5)}{\partial h_i} + \frac{\partial \mathcal{R}(x, h_3, h_2, h_3, h_4, h_5)}{\partial x} = \frac{1}{x h_i} \text{Tr} \left( \frac{1}{x} I + h_2 h_3 h_4 h_5 R_1 \right)^{-1} \frac{1}{x} R_1.
\]
Since \( \mathcal{R}(0) = 0 \) and from (7), we have
\[
\mathcal{R}(x) = \frac{1}{x} \mathcal{R}(x, h_3, h_2, h_3, h_3, h_4, h_5). \tag{41}
\]