Representing Semilattices as Relations

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1 Note for readers

- The first few sections provide background concerning order-theory and semilattices.
- The new category $\text{Dep}$ is introduced and studied in Sections 4 and 5.
  - Its objects $\mathcal{G}$ are the relations between finite sets. Its morphisms $\mathcal{G} \to \mathcal{H}$ are those relations $\mathcal{R}$ factoring via relational composition through $\mathcal{G}$ (resp. $\mathcal{H}$) on the left (resp. right).
  - In subsection 4.2 we prove it is categorically equivalent to $\text{JSL}_f$.
- Section 5 describes the tensor product and tight tensor product of finite join-semilattices, also in $\text{Dep}$.
- Section 6 extends the main result from binary relations to symmetric relations (undirected graphs) and from finite join-semilattices to finite De Morgan algebras.
- Finally, the Appendix describes and proves a number of relevant categorical dualities and free constructions.

2 Conventions and background

2.1 Conventions regarding relations and functions

It is worth clarifying the definition of functions and relations because:

(a) algorithms requires specific representations.

(b) it avoids ‘clutter’ e.g. we don’t want to distinguish between a functional relation and a function.

After these basic definitions we introduce notation to avoid a cumbersome presentation.

Definition 2.1.1 (Relations and functions).

1. The cartesian product of two sets $X$ and $Y$ is defined $X \times Y := \{(x, y) : x \in X, y \in Y\}$.

2. A relation is a triple $(\mathcal{R}, X, Y)$ where $\mathcal{R} \subseteq X \times Y$ is any subset. Then $X$ is called the domain of $\mathcal{R}$ (also denoted $\mathcal{R}_d$) whereas $Y$ is called the codomain of $\mathcal{R}$ (also denoted $\mathcal{R}_c$).

3. (a) The identity relation on a set $X$ is defined $(\Delta_X, X, X)$ where $\Delta_X := \{(x, x) : x \in X\}$.

   (b) The converse of a relation $(\mathcal{R}, X, Y)$ is the relation $(\mathcal{R}^c, Y, X)$ where $\mathcal{R}^c := \{(y, x) : (x, y) \in \mathcal{R}\}$.

   (c) The complement of a relation $(\mathcal{R}, X, Y)$ is the relation $(\mathcal{R}^\complement, X, Y)$ where $\mathcal{R}^\complement := (X \times Y) \setminus \mathcal{R}$.

4. For any relation $(\mathcal{R}, X, Y)$, subset $S \subseteq X$ and domain element $d \in X$, define:

   \[ \mathcal{R}[S] := \{y \in Y : \exists x \in S, (x, y) \in \mathcal{R}\} \quad \mathcal{R}[\{d\}] := \mathcal{R}[\{d\}] \]

   i.e. the image of a subset of the domain, and the image of a domain element.
5. Given any two compatible relations \((R, X, Y)\) and \((S, Y, Z)\) then their composite relation is defined:

\[
(R, X, Y); (S, Y, Z) := \{(x, z) \in X \times Z, \exists y \in Y.((x, y) \in R \text{ and } (y, z) \in S)\}, X, Z
\]

6. A relation \((R, X, Y)\) is functional if \(\forall x \in X. \exists! y \in Y. (x, y) \in R\). Then a function is another word for a functional relation, so the identity relation \((\Delta_X, X, X)\) is also \(X\)'s identity function, written \(id_X\).

7. For any \(X\), its powerset \(\mathcal{P}X\) is the set containing precisely the subsets \(S \subseteq X\) including the empty set \(\emptyset\). If \(X\) is finite let \(|X| \in \mathbb{N} := \{0, 1, 2, \ldots\}\) denote its number of elements.

8. For each set \(X\), subset \(S \subseteq X\) and element \(z \in X\), we define:

\[
\overline{S} := X \setminus S = \{x \in X : x \notin S\} \quad \overline{z} := \{z\} = X \setminus \{z\} \quad \neg X := \{(S, \overline{S}) : S \in \mathcal{P}X\}, \mathcal{P}X, \mathcal{P}X
\]

i.e. the relative-complement of subsets or elements, and the involutive relative-complement function.

9. The notions of injective, surjective and bijective functions are as usual. These concepts only apply to functions so e.g. if we say a relation is injective we mean that it is an injective function. For any function \(f : X \to Y\) its preimage function \((f^{-1}, \mathcal{P}Y, \mathcal{P}X)\) has action \(f^{-1}(S) = \{x \in X : f(x) \in S\}\).

10. Given a relation \((R, X_1, X_2)\) and subsets \(Y_i \subseteq X_i\) for \(i = 1, 2\) then its restriction \((R, X_1, X_2)|_{Y_1 \times Y_2}\) is the relation \((R \cap Y_1 \times Y_2, Y_1, Y_2)\).

We now list standard notational conventions which we shall henceforth adopt.

**Notation 2.1.2 (Relations and functions).**

1. For any relation \((R, X, Y)\) let \(R_s := X ((s)ource = \text{domain})\) and \(R_t := Y ((t)arget = \text{codomain})\). We may denote a relation by the symbol \(R\) as long as \(R_s\) and \(R_t\) have been specified, either by defining them directly, or via words such as ‘the relation \(R \subseteq X \times Y\)’ where it is understood that \(R_s = X\) and \(R_t = Y\).

2. To indicate a relation \((R, X, Y)\) or \(R \subseteq X \times Y\) we may also write \(R : X \to Y\). This is more usual for functions, but is perfectly acceptable for relations too.

3. \(R(x, y)\) indicates that \((x, y) \in R\). Sometimes it is more natural to write \(x R y\), for example if \(R\) is a partial order. The converse relation of \(R\) may be written as \(\bar{R}\) or \(R^-\), observing that \(R_s = \bar{R}_t\) and \(R_t = \bar{R}_s\).

4. We denote general relations by upper-case calligraphic symbols e.g. \(R\), \(S\) and \(T\), and general functions by lower-case standard type symbols e.g. \(f\), \(g\), \(h\). Then we may write \(f(x) = y\) to mean \(f(x, y)\) where the latter \(y\) is necessarily unique. Since we understand functions to be functional relations there will be some overlap in symbols, but usually only if a certain relation turns out (or is restricted) to be functional.

5. By the above remarks, given relations \(R \subseteq X \times Y\) and \(S \subseteq Y \times Z\) we may write their composite relation as \(R; S \subseteq X \times Z\). Following the usual convention, the composite of two functions \(f : X \to Y\) and \(g : Y \to Z\) is written as \(g \circ f : X \to Z\) i.e. the other way around to relational composition.

6. We write the restriction of a relation as \(R \cap Y_1 \times Y_2\) or alternatively as \(R|_{Y_1 \times Y_2}\).

### 2.2 Order Theory

We shall need various basic concepts from order theory i.e. posets, join-semilattices, lattices, bounded lattices, De Morgan algebras (which needn’t be distributive), distributive lattices, boolean lattices and algebras, join and meet-irreducible elements, join and meet-prime elements, and also closure and interior operators on an arbitrary poset. We also prove a number of (mostly) standard results e.g. every finite join-semilattice is a finite lattice in a unique way, a finite lattice is distributive iff every join-irreducible element is join-prime, and we also describe the canonical order-isomorphism between join and meet-irreducibles of a finite distributive lattice.

**Definition 2.2.1 (Basic order theory).**
1. A poset is a pair \( P = (P, \leq_P) \) where \( P \) is a set and the relation \( \leq_P \subseteq P \times P \) is reflexive, transitive and anti-symmetric. An order relation \( R \subseteq X \times X \) with these three properties. A poset \( (P, \leq_P) \) is finite if \( P \) is a finite set.

2. Given a poset \( P \), then its opposite poset is defined \( P^{op} := (P, \leq_{P^{op}}) \) where \( \leq_{P^{op}} := \bar{\leq}_P \) is the converse relation, and is more usually written as \( \geq_P \). A monotone (or monotonic) function from \( P \) to \( Q \) is a function \( f : P \to Q \) such that \( p_1 \leq_P p_2 \implies f(p_1) \leq f(p_2) \) for all elements \( p_1, p_2 \in P \). We may indicate monotone morphisms by writing \( f : (P, \leq_P) \to (Q, \leq_Q) \). We also have the opposite monotone morphism \( f^{op} : P^{op} \to Q^{op} \) which acts in the same way i.e. \( f^{op}(p) = f(p) \) for all elements \( p \in P \), see Note 2.2.2 below.

3. We’ll use the other standard symbols and their converses i.e. \( <_P \) means strictly less than (with converse \( >_P \)), \( \preceq_P \) means not less than or equal to (with converse \( \succeq_P \)), \( \preceq_P \) means not strictly less than (with converse \( \succeq_P \)). We also have the irreflexive and symmetric incomparability relation \( \parallel_P := \preceq_P \cap \preceq_P \) which also equals \( \parallel_{P^{op}} \). Finally, \( P \)’s covering relation \( \prec_P \subseteq P \times P \) is defined:

\[
p_1 \prec_P p_2 : \iff p_1 <_P p_2 \quad \text{and} \quad \nexists p \in P. (p_1 \prec_P p <_P p_2).
\]

4. A chain is a non-empty totally ordered poset \( P \) i.e. such that \( \parallel_P = \emptyset \). An antichain is a non-empty poset \( P \) where distinct elements are incomparable i.e. \( \parallel_P = P \times P \setminus \Delta_P \). A poset which is either empty or an antichain is called a discrete poset. Let us denote the 2-chain by \( 2 := (\{0, 1\}, \Delta_2 \cup \{(0, 1)\}) \) and the 2-antichain by \( 2_a := (\{0, 1\}, \Delta_2) \).

We say that \( P = (P, \leq_P) \) is a subposet of \( Q = (Q, \leq_Q) \) if \( P \subseteq Q \) and \( \leq_P = \leq_Q \cap P \times P \). Then a subposet must inherit the order, so that \( 2_a \) is not a subposet of \( 2 \). If a chain \( (P, \leq_P) \) is finite then its length is defined \( |P| - 1 \) e.g. the two element chain \( 2 \) has length 1. Then the length \( l(P) \in \N \cup \{\omega\} \) of an arbitrary poset \( P \) is defined as the supremum of the lengths of all finite chains arising as subposets of \( P \). That is, if the length of such chains is bounded then it is the maximum length of any chain, otherwise it is \( \omega \).

5. A subset \( S \subseteq P \) of a poset \( P \) is up-closed (or upwards-closed) if whenever \( p \in S \) and \( p \leq_P p' \) then \( p' \in S \). A subset \( S \subseteq P \) of a poset \( P \) is down-closed (or downwards-closed) if it is up-closed in \( P^{op} \). Equivalently, the up-closed (resp. down-closed) sets of \( P \) are precisely those sets of the form \( f^{-1}(\{1\}) \) (resp. \( f^{-1}(\{0\}) \)) for monotone functions \( f : P \to 2 \).

6. Given a poset \( P = (P, \leq_P) \) then its join-irreducible elements \( J(P) \subseteq P \) and meet-irreducible elements \( M(P) \subseteq P \) are defined as follows:

\[
J(P) := \{ p \in P : \exists q \in P. q \prec_P p \} \\
M(P) := \{ p \in P : \exists q \in P. p \prec_P q \}
\]

where \( \exists ! \) should be read as ‘there exists a unique’. If \( P \) has a minimum element \( \perp_P \in P \) then its atoms are defined \( At(P) := \{ p \in P : \perp_P <_P p \} \) i.e. those elements covering \( \perp_P \). On the other hand, if \( P \) has a maximum element \( \top_P \in P \) then its coatoms are defined \( CoAt(P) := \{ p \in P : p \prec_P \top_P \} \) i.e. those elements covered by \( \top_P \). These are order-dual concepts i.e. \( M(P) = J(P^{op}) \) holds generally, and \( CoAt(P) = At(P^{op}) \) holds whenever \( P \) has a top element.

See Lemma 2.2.3.5 below for the way we usually think about join/meet-irreducibles.

7. A join-semilattice \( Q = (Q, \vee_Q, \perp_Q) \) is a commutative and idempotent monoid i.e. \( Q \) is the carrier set, \( \vee_Q : Q \times Q \to Q \) is the associative binary operation and \( \perp_Q \in Q \) is the unit. Equivalently, a join-semilattice is a poset \( (Q, \leq_Q, \vee_Q, \perp_Q) \) where all finite suprema exist, the empty supremum being \( \perp_Q \) and the supremum of \( \{q_1, q_2\} \) being \( q_1 \vee_Q q_2 \). In particular, one can define \( x \leq_Q y : \iff x \vee_Q y = y \). We usually describe \( \perp_Q \) as the bottom element or empty join, \( \vee_Q \) as the binary join, and the suprema of finitely many elements as joins. The latter are often denoted via the symbol \( \vee_Q \) which inductively generalises \( \perp_Q \) and \( \vee_Q \).

8. A lattice \( L = (L, \vee_L, \wedge_L) \) is a poset \( (L, \leq_L) \) with binary joins \( \vee_L \) and binary meets \( \wedge_L \). A bounded lattice \( L = (L, \vee_L, \wedge_L, \perp_L, \top_L) \) is a poset \( (L, \leq_L) \) with all finite joins (suprema) and all finite meets (infima). Notice that the join-structure is always written before the meet-structure. We may also speak of a lattice with bottom or lattice with top. Finite lattices always have a bottom and top, although they may not preserved by the morphisms under consideration.

Bounded lattices may be equationally axiomatised by specifying two commutative idempotent monoids \( (\vee, \perp) \) and \( (\wedge, \top) \) (hence join-semilattices), as well as the absorption laws. The latter laws ensure that their respective order relations are the converse of one another.
9. Given a bounded lattice \( \mathcal{L} \) and elements \( x, y \in \mathcal{L} \) then \( y \) is a complement of \( x \) if:

\[
x \land \mathcal{L} y = \bot_{\mathcal{L}} \quad \text{and} \quad x \lor \mathcal{L} y = \top_{\mathcal{L}}.
\]

There exist lattices where an element may have no complement (the midpoint of a 3-chain) or many of them (add a bottom and top to a 3-antichain). However, an element of a distributive lattice may have at most one complement by Lemma 2.2.3.9 below.

10. A distributive lattice \( \mathcal{D} = (D, \lor_{\mathcal{D}}, \land_{\mathcal{D}}) \) is a lattice where the two distributive laws hold i.e.

\[
x \land (y \lor z) = (x \land y) \lor (x \land z) \quad \text{and} \quad x \lor (y \land z) = (x \lor y) \land (x \lor z)
\]

In practice we’ll mostly deal with bounded lattices which are distributive. A boolean lattice \( \mathcal{B} \) is a complemented bounded distributive lattice i.e. for every element \( x \in \mathcal{B} \) there exists \( y \in \mathcal{B} \) such that \( x \land \mathcal{B} y = \bot_{\mathcal{B}} \) and \( x \lor \mathcal{B} y = \top_{\mathcal{B}} \). Elements of distributive lattices may have at most one complement, see Lemma 2.2.3.9 below. A boolean algebra \( \mathbb{A} = (A, \lor_A, \land_A, \top_A, \bot_A, \neg_A) \) is a boolean lattice \( (A, \lor_A, \land_A, \top_A) \) endowed with its unique complement operation \( \neg_A : A \to A \). Equivalently they may be equationally axiomatised via the equational axiomatisation of lattices, the distributive laws and the two complement laws i.e. \( x \land \neg x \approx \bot \) and \( x \lor \neg x \approx \top \). In particular, the two De Morgan laws follow from the latter equational axiomatisation of boolean algebras, which we also verify syntactically in Lemma 2.2.3.9 below.

11. Regarding morphisms, recall that we’ve already defined the notion of ‘monotonic function’ between posets in item (2) above.

(a) Given join-semilattices \( \mathcal{Q}_i = (Q_i, \lor_{Q_i}, \land_{Q_i}) \) for \( i = 1, 2 \) then a join-semilattice morphism \( f : \mathcal{Q}_1 \to \mathcal{Q}_2 \) is a monotonic function \( f : (Q_1, \leq_{Q_1}) \to (Q_2, \leq_{Q_2}) \) which also preserves the bottom and binary join. Equivalently they are the monoid morphisms (since join-preservation implies monotonicity), and if both join-semilattices are finite they are precisely those functions \( f : Q_1 \to Q_2 \) preserving all joins.

(b) Given lattices \( \mathcal{L}_i = (L_i, \lor_{\mathcal{L}_i}, \land_{\mathcal{L}_i}) \) for \( i = 1, 2 \) then a lattice morphism \( f : \mathcal{L}_1 \to \mathcal{L}_2 \) is a monotonic function \( f : (L_1, \leq_{\mathcal{L}_1}) \to (L_2, \leq_{\mathcal{L}_2}) \) which also preserves the binary join and binary meet. A bounded lattice morphism between bounded lattices is a lattice morphism which additionally preserves the top and bottom element.

(c) Finally, a boolean algebra morphism \( f : \mathbb{A}_1 \to \mathbb{A}_2 \) is a bounded lattice morphism between their underlying boolean lattices \( (A_i, \lor_{A_i}, \land_{A_i}, \top_{A_i}, \bot_{A_i}, \neg_{A_i}) \) for \( i = 1, 2 \). Such morphisms automatically preserve negation via the uniqueness of complements in boolean algebras, see Lemma 2.2.3.9 below. One can define boolean algebra morphisms in a number of equivalent ways e.g. via preservation of bottom, meet and complement.

12. The opposite poset construction restricts to lattices, bounded lattices, boolean algebras, and also finite join-semilattices. Furthermore it preserves distributivity and the existence of complements.

(a) Each bounded lattice \( \mathcal{L} \) has an opposite bounded lattice:

\[
(L, \lor_{\mathcal{L}}, \land_{\mathcal{L}}, \top_{\mathcal{L}})^{\mathcal{L}} := (L, \land_{\mathcal{L}}, \lor_{\mathcal{L}}, \bot_{\mathcal{L}})
\]

We also have the opposite lattice. Observe that \( \mathcal{L}^{\mathcal{L}} \)’s underlying poset is the opposite of \( \mathcal{L} \)’s underlying poset.

(b) If a possibly bounded lattice \( \mathcal{L} \) is distributive then so is \( \mathcal{L}^{\mathcal{L}} \), since the two distributive laws arise from one another by swapping meets and joins. Likewise, if a bounded lattice \( \mathcal{L} \) is a boolean lattice then so is \( \mathcal{L}^{\mathcal{L}} \) by inspecting the two complementation laws.

(c) Each boolean algebra \( \mathbb{A} \) has an opposite boolean algebra i.e.

\[
(A, \lor_A, \land_A, \top_A, \bot_A, \neg_A)^{\mathcal{L}} := (A, \land_A, \lor_A, \top_A, \bot_A, \neg_A)
\]

That negation is well-defined follows because the two standard equational laws expressing complementation are order-dual statements. \( \mathbb{A}^{\mathcal{L}} \)’s underlying poset is the opposite of \( \mathbb{A} \)’s underlying poset.
(d) Each finite join-semilattice \( Q \) is a lattice in a unique fashion i.e.

\[
\top_Q := \bigvee Q \quad q_1 \wedge_Q q_2 := \bigvee \{ q \in Q : q \leq_Q q_1, q \leq_Q q_2 \}
\]

Thus in the finite case we also have the opposite join-semilattice defined \( Q^{\text{op}} := (Q, \wedge_Q, \top_Q) \), noting that \( Q^{\text{op}} \)'s underlying poset is the opposite of \( Q \)'s underlying poset. The existence of meets fails for infinite join-semilattices e.g.

(i) Concerning boundedness, given any infinite set \( X \) then the join-semilattice \( (P_f X, \cup, \varnothing) \) of finite subsets has no top element.

(ii) Below we depict the ‘usual’ example of a join-semilattice which fails to have binary meets.

That is, \( Q \) has the order relation:

\[
\leq_Q = \{ 1 \} \times Q \cup \{(x_i, x_j) : 0 \leq i \leq j \} \cup \{(y_i, y_j) : 0 \leq i \leq j \} \cup (X \cup Y) \times \{\alpha, \beta\} \cup Q \times \{\top\}
\]

where \( X = \{ x_i : i \in \mathbb{N} \} \) and \( Y = \{ y_i : i \in \mathbb{N} \} \). It has all finite joins, whereas \( \alpha \wedge \beta \) doesn’t exist.

Recall that given any monotone function \( f : P \to Q \) we also have the opposite monotone function \( f^{\text{op}} : P^{\text{op}} \to Q^{\text{op}} \) which acts in exactly the same way as \( f \). Then this restricts to bounded lattice morphisms and also boolean algebra morphisms. However it does not restrict to join-semilattice morphisms between finite join-semilattices, since the latter needn’t preserve the top or the binary meet.

13. A preorder is a relation \( \mathcal{R} \subseteq X \times X \) which is reflexive and transitive. There is an associated equivalence relation \( \mathcal{E} \subseteq X \times X \) defined \( \mathcal{E}(x, y) : \iff \mathcal{R}(x, y) \wedge \mathcal{R}(y, x) \). Then the poset induced by the preorder \( \mathcal{R} \) consists of the \( \mathcal{E} \)-equivalence-classes equipped with the well-defined order relation \( [x]_{\mathcal{E}} \leq_{\mathcal{R}\backslash\mathcal{E}} [y]_{\mathcal{E}} : \iff \mathcal{R}(x, y) \).

14. A monotonic function \( f : P \to Q \) is an order-embedding if:

\[
p_1 \leq_P p_2 \iff f(p_1) \leq_Q f(p_2) \quad \text{for every } p_1, p_2 \in P.
\]

Every order-embedding defines an injective function, although there are injective monotone maps which are not order-embeddings e.g. from the 2-antichain to the 2-chain. However, every injective join-semilattice morphism \( f : Q \to R \) defines an order-embedding \( f : (Q, \leq_Q) \to (R, \leq_R) \), as we now show:

\[
f(q_1) \leq_R f(q_2) \iff f(q_1) \lor_R f(q_2) = f(q_2) \iff f(q_1 \lor_Q q_2) = f(q_2) \iff q_1 \lor_Q q_2 = q_2 \iff q_1 \leq_Q q_2.
\]

15. We say that a join-semilattice \( Q \) is a join-semilattice retract of a join-semilattice \( R \) if there exist join-semilattice morphisms:

\[
 Q \xrightarrow{e} R \quad \text{such that } r \circ e = \text{id}_Q.
\]

Note that \( e \) is necessarily injective and hence an order-embedding, whereas \( r \) is necessarily surjective.

16. We list various specific algebras and some associated terminology, where \( Z \) is any set.

(a) \( \mathcal{P}Z := (\mathcal{P}Z, \cup, \varnothing) \) is called a powerset join-semilattice, observing that \( (\mathcal{P}Z)^{\text{op}} = (\mathcal{P}Z, \cap, Z) \). Furthermore \( \mathcal{P}_f Z := (\mathcal{P}_f Z, \cup, \varnothing) \) is called the free join-semilattice on \( Z \), observing that it needn’t have a top if \( Z \) is infinite. Usually \( Z \) will be finite, in which case these two concepts coincide.
(b) \( P_dZ := (PZ, \cup, \emptyset, \cap, Z) \) is called the powerset bounded distributive-lattice, and \( P_bZ := (PZ, \cup, \emptyset, \cap, Z, \neg Z) \) is called the powerset boolean-algebra.

(c) If \( Q \) is a finite join-semilattice then it possesses a unique bounded lattice structure by (12).d above. Then we say that:
   i. \( Q \) is a boolean join-semilattice if its associated bounded lattice is boolean,
   ii. \( Q \) is a distributive join-semilattice if its associated lattice is distributive, noting that it will also be bounded by finiteness.

(d) \( \forall Z := (\{\emptyset, Z\} \cup \{\top : z \in Z\}, \cup, \emptyset) \) may be viewed as the join-semilattice version of an antichain. That is, its order structure amounts to viewing \( Z \) as the \( \preceq \)-antichain \( \{\top : z \in Z\} \), and then adding a bottom \( \emptyset \) and top \( Z \) to obtain a lattice.

Note 2.2.2 (Order opposite \((-)^{op} \) versus categorical opposite \((-)^{op} \)).
The opposite poset/monotone map construction \((-)^{op} \) is distinct from the standard categorical notation \((-)^{op} \). Actually, they do align by viewing posets as thin skeletal categories (so that the opposite poset is the opposite category), and moreover the functors between these categories are the monotone maps (so that the opposite functor corresponds to the opposite monotone map). However, we’ll also use \( f^{op} : Y \to X \) to indicate a morphism in the opposite of other categories, so we distinguish the notations to avoid confusion.

We now list some basic facts concerning these definitions.

Lemma 2.2.3 (Standard order-theoretic results).
1. If a poset \( P \) has all finite suprema then it is the underlying poset of a unique join-semilattice \( Q \).
2. If a poset \( P \) has all finite suprema and infima then it is the underlying poset of a unique bounded lattice \( L \).
3. Viewed as their underlying posets, the finite lattices, the finite bounded lattices, and the finite join-semilattices coincide.
4. If a poset \( P \) has finite length then \( \preceq_P \) is the reflexive transitive closure of \( \prec_P \).
5. Let \( L \) be a locally finite lattice i.e. every interval is finite.
   (a) If \( L \) has a bottom element then \( J(L) \) contains precisely those elements which are not a finite join of other elements. Equivalently:
   \[ x \in J(L) \iff (i) \ x \not\in \bot_L, \text{ and (ii) } \forall x_1, x_2 \in L. \ x = x_1 \lor_L x_2 \implies \exists i. x = x_i. \]
   (b) If \( L \) has a top element then \( M(L) \) contains precisely those elements which are not a finite meet of other elements. Equivalently:
   \[ x \in M(L) \iff (i) \ x \not\in \top_L, \text{ and (ii) } \forall x_1, x_2 \in L. \ x = x_1 \land_L x_2 \implies \exists i. x = x_i. \]
   (c) \( L \) has both a bottom and top element iff it is a finite lattice (hence bounded), in which case both of the above statements hold.
6. Each finite lattice \( L \) is join-generated by \( J(L) \) and every join-generating set \( S \subseteq Q \) contains \( J(L) \). Order-dually, \( L \) is meet-generated by \( M(L) \) and every meet-generating set contains \( M(L) \).
7. Given any finite join-semilattice \( Q \) and elements \( q_1, q_2 \in Q \), the following statements hold:
   \[ q_1 \leq_Q q_2 \iff \forall j \in J(Q). [j \leq_Q q_1 \implies j \leq_Q q_2] \]
   \[ \iff \forall m \in M(Q). [q_2 \leq_Q m \implies q_1 \leq_Q m] \]
8. The atoms of any boolean lattice \( L \) are precisely \( J(L) \), the co-atoms are precisely \( M(L) \).
9. Let \( D \) be a bounded distributive lattice.
   (a) Each element \( d \in D \) can have at most one complement.
(b) If every element \( d \in D \) has a complement then it is a boolean lattice.

Concerning the second point: syntactically speaking, the equational axiomatisation of boolean algebras described in Definition 2.2.1.9 above is correct. That is, the De Morgan laws are deducible from it.

10. For any finite lattice \( \mathcal{L} \) the following statements are equivalent.
   (a) \( \mathcal{L} \) is distributive.
   (b) For all \( j \in J(\mathcal{L}) \) and \( x_1, x_2 \in L \) we have:
       \[ j \leq \mathcal{L} x_1 \lor \mathcal{L} x_2 \iff j \leq \mathcal{L} x_1 \text{ or } j \leq \mathcal{L} x_2 \]
       noting that one may equivalently restrict to \( x_1, x_2 \in L \setminus \{ \bot \}, \mathcal{T} \).
   (c) For all \( j \in J(\mathcal{L}) \) and subsets \( X \subseteq L \):
       \[ j \leq \mathcal{L} \bigvee X \iff \exists x \in X, j \leq \mathcal{L} x \]
   (d) For all \( j \in J(\mathcal{L}) \) and subsets \( X \subseteq J(\mathcal{L}) \):
       \[ j \leq \mathcal{L} \bigvee X \iff \exists x \in X, j \leq \mathcal{L} x \]

11. A lattice \( \mathcal{L} \) is distributive iff \( M_3 \) and \( N_5 \) do not arise as sublattices, recalling that:

\[ \begin{array}{ccc}
   & * & \\
   \\ & & \\
   & * & \\
   & & \\
   & * & \\
   M_3 & & N_5
\end{array} \]

12. If \( D \) is a finite distributive lattice and \( h : D \to 2 \) is any function, the following statements are equivalent:
   (a) \( h \) defines a bounded distributive lattice morphism of type \( D \to 2 \).
   (b) \( h^{-1}(\{1\}) = \uparrow_D j \) for some \( j \in J(D) \).
   (c) \( h^{-1}(\{0\}) = \downarrow_D m \) for some \( m \in M(D) \).

13. For every finite distributive lattice \( D \) we have the bijective monotone and order-reflecting function:

\[ \tau_D : (J(D), \leq_D) \to (M(D), \leq_D), \quad \tau_D^{-1} : (M(D), \leq_D) \to (J(D), \leq_D) \]

\[ \tau_D(j) := \bigvee_D \downarrow_D j \quad \tau_D^{-1}(m) := \bigwedge_D \uparrow_D m \]

14. For every finite distributive lattice \( D \) we have:

\[ \xi_D : J(D) \times M(D) = (\leq_D \upharpoonright J(D) \times J(D)) \;
\tau_D \]

where \( \tau_D : J(D) \to M(D) \) is the canonical bijection from the previous statement. This is equivalent to either of the statements below:

\[ \forall j_1, j_2 \in J(D), (j_1 \xi_D \tau_D(j_2) \iff j_2 \leq_D j_1) \quad \forall j \in J(D), m \in M(D), (j \xi_D m \iff \tau_D^{-1}(m) \leq_D j) \]

15. Given any finite join-semilattice \( Q \) then the following statements are equivalent:
   (a) \( Q \) is a distributive join-semilattice.
   (b) \( Q \) is a join-semilattice retract of a finite boolean join-semilattice.
   (c) \( Q \) is a join-semilattice retract of a finite distributive join-semilattice.
Proof.

1. Given \( P = (P, \leq_P) \) with all finite suprema we have the join-semilattice \( Q := (P, \vee_P, \wedge_P) \) where \( \leq_Q = \leq_P \). Regarding uniqueness, given \( Q = (Q, \vee_Q, \wedge_Q) \) where \( \leq_Q = \leq_P \), the ordering uniquely determines the finite joins.

2. Same argument as previous item, noting that the ordering also uniquely determines the finite meets.

3. Any finite lattice \( L \) is bounded, so we can take its underlying join-semilattice structure \( Q \) i.e. the binary join and the bottom element, which determines the ordering. Conversely any finite join-semilattice defines a lattice with the same ordering, because we have all finite joins, hence all joins, hence all meets too.

4. If \( P \) has finite length then any \( p \leq_P p' \) is witnessed by a finite chain of covers \( p = p_1 \leq_P \cdots \leq_P p_n = p' \), so that \( \leq_P \) is the reflexive transitive closure of \( \leq_P \).

5. Let \( L \) be a locally finite lattice.

(a) Let \( j \in J(L) \) so that it covers precisely one element \( x \in L \). If \( j = \bigvee_{i \in I} x_i \) for some (finite) set \( I \) then there must exist \( i \in I \) such that \( x \leq_L x_i \), and since \( x_i \leq_L j \) we deduce that \( x_i = j \). Conversely, suppose we have \( j \in L \) which is not the finite join of any other elements. Since \( L \) is locally finite and has a bottom element \( \bot_L \), we have the finite set \( \{ \bot_L, j \} \) whose join \( x := \bigvee_L \bot_L, j \) cannot be \( j \), so that \( x <_L j \). Given any \( y <_L j \) then since \( y <_L j \) we have \( y \leq_L x \) so that \( y = x \). We also have the inductive description i.e. instead of finite joins we consider empty joins and binary joins.

(b) Follows from (a) by order-duality, noting that \( M(L) = J(L^{op}) \) in every lattice – see Definition 2.2.1.6.

(c) Every finite lattice (automatically bounded) is locally finite, and every bounded locally finite lattice \( L \) is finite since \( L = [\bot_L, \top_L] \) is finite. Then both (a) and (b) apply.

6. Concerning irreducibles, instead of finite lattices \( L \) we may consider finite join-semilattices \( Q \). Recall the finite join-semilattice \( P \cdot J(Q) = (P \cdot J(Q), \cup, \emptyset) \). There is a surjective join-semilattice morphism \( f : (P \cdot J(Q), \cup, \emptyset) \to Q \) defined \( f(S) := \bigvee_Q S \), as we now verify. It is clearly a well-defined function, and also a morphism because:

\[
\begin{align*}
  f(\bot_{P \cdot J(Q)}) &= f(\emptyset) = \bigvee_Q \emptyset = \bot_Q \\
  f(S_1 \vee_{P \cdot J(Q)} S_2) &= f(S_1 \cup S_2) = \bigvee_Q S_1 \cup S_2 = (\bigvee_Q S_1) \vee_Q (\bigvee_Q S_2) = f(S_1) \vee_Q f(S_2)
\end{align*}
\]

using generalised associativity. By standard universal algebra we obtain a sub join-semilattice \( f(P \cdot J(Q)) \subseteq Q \). Concerning surjectivity, suppose for a contradiction that we have some \( q \in Q \) such that \( q \notin f(P \cdot J(Q)) \). Since the latter is the closure of \( J(Q) \) under \( Q \)-joins, we deduce by (5).(a) that \( q \in J(Q) \) which is a contradiction. Thus \( J(Q) \) join-generates \( Q \).

To see that every join-generating subset contains \( J(Q) \), assume that \( Q \) is join-generated by \( S \subseteq Q \) i.e. \( (S)Q = Q \). For a contradiction assume there exists \( j \in J(Q) \) such that \( j \notin S \). By (5).(a) we know that \( j \) is not the join of other elements, yielding the contradiction \( j \notin Q \). The statements involving meet-irreducibles follow by order-duality i.e. apply the above statements to \( Q^{op} \).

7. The first equivalence follows because each \( q \in Q \) is the join of those join-irreducibles below it by (6). The second equivalence follows by order-duality.

8. Let \( A \) be any possibly infinite boolean algebra. We have \( At(A) \subseteq J(A) \) using the definitions. Take any join-irreducible \( j \in J(A) \) and for a contradiction assume \( j \notin At(A) \), hence \( \bot_A \leq_A a \leq_A j \) for a unique \( a \in A \). Observe that:

\[
(\neg_A a \land_A j) \lor_A a = (\neg_A a \lor_A a) \land_A (j \lor_A a) = \top_A \land_A j = j
\]

using a distributive law, a complement law and a unit law. Because \( j \) covers precisely one element we deduce that \( j = \neg_A a \land_A j \) and hence \( a \leq_A j \leq_A \neg_A a \), so that \( a = a \land_A \neg_A a = \bot_A \), this being a contradiction.

9. Let \( D = (D, \lor_D, \land_D, \top_D, \bot_D) \) be a bounded distributive lattice.

(a) Given any \( d \in D \) suppose we have two complements \( d_1, d_2 \in D \), then:

\[
d_1 = d_1 \land_D \top_D = d_1 \land_D (d \lor_D d_2) = (d_1 \land_D d) \lor_D (d_1 \land_D d_2) = \top_D \lor_D (d_1 \land_D d_2) = d_1 \land_D d_2
\]

so that \( d_1 \leq_D d_2 \), and by the symmetric argument \( d_1 = d_2 \).
(b) We’ll show how to equationally deduce \(-x \wedge y \approx \neg x \vee \neg y\). Firstly, \((x \wedge y) \wedge (\neg x \vee \neg y) \approx 1\) is deduced using a distributive law and a complement law twice followed by idempotence, whereas \((x \wedge y) \vee (\neg x \vee \neg y) \approx \top\) is deduced using the other distributive law and the other complement law twice followed by idempotence. One can then instantiate a general procedure i.e. the syntactic version of the uniqueness of complements in boolean lattices, having already specified their existence via the complement laws. Given \(x \wedge y \approx 1\) and \(x \vee y \approx \top\), one (i) applies \(-x \vee (-)\) to deduce \(-x \wedge y \approx \neg x\), (ii) applies \(-x \wedge (-)\) to deduce \(-x \vee y \approx \neg x\) and therefore \(y \approx y \wedge (\neg x \vee y) \approx \neg x \wedge y\) using an absorption law. Combining (i) and (ii) yields \(y \approx \neg x \wedge y \approx \neg x\). Then applying this general procedure to the earlier equalities we deduce \(-(x \wedge y) \approx \neg x \vee \neg y\).

10. (i) First suppose that (a) holds i.e. \(L = \mathbb{D}\) is a finite distributive lattice. We’ll show that (b), (c) and (d) all hold and are equivalent.

Concerning (b), suppose \(j \in J(\mathbb{D})\), \(d_1, d_2 \in D\) are such that \(j \leq_D d_1 \vee_D d_2\). By distributivity \(j = j \wedge_D (d_1 \vee_D d_2) = (j \wedge_D d_1) \vee_D (j \wedge_D d_2)\), so by join-irreducibility \(\exists i. j = j \wedge_D d_i\) and hence \(j \leq_D d_i\), as required. The more general formulation (c) involving any subset \(X \subseteq D\) follows by induction:

i. If \(X = \emptyset\) recall that \(j \neq_D \bot_D\) by definition.

ii. If \(X = \{d\} \cup Y\) where \(|Y| < |X|\) then we have \(j \leq_D d \vee_D \bigvee Y\), so that either \(j \leq_D d\) and we are done, or \(j \leq_D \bigvee Y\) and we may apply induction.

Finally, (c) is equivalent to the more specific condition (d) i.e. we restrict to subsets \(X \subseteq J(\mathbb{D}) \subseteq D\). This follows because:

\[ \bigvee_D X = \bigvee_J(D) \cap \downarrow_J X \quad \text{for any subset } X \subseteq D \]

i.e. every element in a finite join-semilattice arises as the join of those join-irreducibles below it.

(ii) The proof of the previous item shows that (b), (c) and (d) are all equivalent, even without knowing that the finite lattice \(L\) is distributive. Then it suffices to show that (b) implies (a). We’ll achieve this by embedding \(L\) into a set-theoretic bounded distributive lattice, recalling that sublattices of bounded distributive lattices are distributive. So define:

\[ e : L \to (\mathcal{P}J(L), \cup, \emptyset, \cap, J(L)) \quad \text{with action} \quad e(x) := J(L) \cap \downarrow_L X. \]

This is a well-defined function and also injective because elements of a finite (join-semi)lattice arise as the join of those join-irreducibles beneath them. We have \(e(\bot_L) = \emptyset\) and \(e(\top_L) = J(L)\), and also \(e(x_1 \wedge_L x_2) = e(x_1) \cap e(x_2)\) by virtue of the defining property of meets. Finally we use (b) to show preservation of joins:

\[ e(x_1 \wedge_L x_2) = \{ j \in J(L) : j \leq_L x_1 \vee_L x_2 \} = \{ j \in J(L) : \exists i. j \leq_L x_i \} = e(x_1) \cup e(x_2). \]

11. See [Grä98, Chapter II, Theorem 1].

12. (a) \(\implies\) (b).

Assume (a) and let \(X := h^{-1}(\{1\})\). Then \(X\) is upwards-closed (since \(h\) is monotone) and closed under meets (since \(h\) preserves meets), so that \(X = \uparrow_D \wedge_D X\) by finiteness. Moreover \(d := \bigwedge_D X \in J(\mathbb{D})\) because if \(d = d_1 \vee_D d_2\) then \(h(d_1) \vee h(d_2) = h(d_1) \vee h(d_2)\), thus \(h(d_1) = 1\) so for some \(i\), hence \(d_i \leq_D d_1 \vee_D d_2 = d \leq_D d_i\).

(a) \(\implies\) (c).

Given \(h : \mathbb{D} \to 2\) then we also have the bounded distributive lattice morphism \(h^\text{op} : \mathbb{D}^\text{op} \to 2^\text{op}\) with the very same action. Thus we also have \(g := \text{swap} \circ h^\text{op} : \mathbb{D}^\text{op} \to 2\). Applying the previous argument we deduce that \(g^{-1}(\{1\}) = \uparrow_D j = \downarrow_D j\) for some \(j \in J(\mathbb{D}^\text{op}) = M(\mathbb{D})\). Finally observe that \(h^{-1}(\{0\}) = g^{-1}(\{1\})\).

(b) \(\implies\) (a).

We have a function \(h : D \to 2\) such that \(h(d) = 1\) iff \(j \leq_D d\) where \(j \in J(\mathbb{D})\). Firstly \(h(\bot_D) = 0\) because \(j\) is join-irreducible, and also \(h(\top_D) = 1\) by virtue of being the top element. Moreover \(h(d_1 \wedge_D d_2) = 1\) if \(j \leq_D d_1 \wedge_D d_2\) iff \(\forall i. j \leq_D d_i\) iff \(\forall i. h(d_i) = 1\). Finally \(h(d_1 \vee_D d_2) = 1\) iff \(h(d_1) = 1\) or \(h(d_2) = 1\) by Lemma 2.2.3.10.

(c) \(\implies\) (a).

We have a function \(h : D \to 2\) such that \(h(d) = 0\) iff \(d \leq_D m\). Therefore \(g := \text{swap} \circ h : D \to 2\) is such that \(g(d) = 1\) iff \(m \leq_D d\) where \(m \in J(\mathbb{D}^\text{op})\). By the previous statement we deduce that \(g\) has type \(\mathbb{D}^\text{op} \to 2\), so that \(h = \text{swap} \circ g^\text{op}\) has type \(\mathbb{D} \to 2\).
13. The functions are well-defined, bijective and the inverse of each other by the previous statement, since distinct join/meet-irreducibles have distinct principal up/downsets. Finally,

\[ j_1 \leq_D j_2 \iff \exists_D j_2 \leq \exists_D j_1 \iff \exists_D j_1 \leq \exists_D j_2 \iff \exists_D \exists_D j_1 \leq_D \exists_D \exists_D j_2 \iff \tau_D(j_1) \leq_D \tau_D(j_2). \]

14. The following calculation:

\[ \mathbb{D} \mid_{(J(D) \times M(D))} : \tau_D^{-1}(j_1, j_2) \iff \exists m \in M(D). (j_1 \nleq_D m \text{ and } j_2 = \tau_D^{-1}(m)) \]

proves that:

\[ \mathbb{D} \mid_{(J(D) \times M(D))} : \tau_D^{-1} = \mathbb{D} \mid_{(J(D) \times J(D))} \]

so post-composing the bijection yields:

\[ \mathbb{D} \mid_{(J(D) \times M(D))} : \tau_D^{-1} = \mathbb{D} \mid_{(J(D) \times J(D))} : \tau_D. \]

15. Let \( Q \) be a finite join-semilattice.

(a) If \( Q \) is distributive then we have the retract:

\[ e : Q \to \mathbb{P}J(Q) \]

\[ e(a) := \{ j \in J(Q) : j \leq_Q a \} \]

\[ r : \mathbb{P}J(Q) \to Q \]

\[ r(S) := \bigvee_Q S \]

That \( e \) is well-defined follows because join-irreducibles in finite distributive lattices are join-prime (see (10) above), and \( r \) is well-defined by freeness of \( \mathbb{P}J(Q) \) (or is easily directly verified). That \( r \circ e = id_Q \) follows because elements of a finite join-semilattice are the join of those join-irreducibles beneath them.

(b) To finish the proof, it suffices to establish that:

\[ \text{if } Q \text{ is a join-semilattice retract of a finite distributive join-semilattice } R \text{ then it is distributive.} \]

By assumption \( r \circ e = id_Q \) for some injective morphism \( e : Q \to R \) and surjective morphism \( r : R \to Q \). If \( S \subseteq R \) is the closure of \( e(Q) \) under binary \( R \)-meets, then \( S \) is also closed under binary \( R \)-joins because \( e(Q) \) is closed under them and \( R \) is distributive. Now, since \( \downarrow_R = e(\downarrow_Q) \in e(Q) \subseteq S \), it follows that \( S \) defines a sub join-semilattice \( e : S \to R \) which is also a sublattice of \( (R, \lor, \land_R) \) and hence distributive. Since the join-semilattice morphism \( r' := r \circ e : S \to Q \) is surjective because \( e(Q) \subseteq S \), it suffices to establish that \( r' \) preserves binary meets.

Given any \( s_1, s_2 \in S \) then by construction there exist subsets \( X_i \subseteq Q \) such that \( s_i = \land_R e[X_i] \) for \( i = 1, 2 \). Now, for any subset \( X \subseteq Q \) and element \( x \in X \) we have:

\[ \land_Q X = r \circ e(\land_Q X) \leq_Q r(\land_Q e[X]) \leq_Q r \circ e(x) = x \]

using monotonicity, and since \( x \) is arbitrary it follows that \( r(\land_Q e[X]) = \land_Q X \). Finally,

\[ r'(s_1 \land_Q s_2) = r(\land_Q e[X_1]) \land_R e[X_2]) = \land_Q (X_1 \lor \land_Q X_2) \]

\[ \text{associativity} \]

\[ \text{see above} \]

\[ r'(s_1) \land_Q r'(s_2) \]

\[ \text{see above} \]
We shall also need the concepts of a closure operator and interior operator on a poset.

**Definition 2.2.4.** Let $P = (P, \leq_P)$ be any poset.

1. A closure operator on $P$ is a function $\text{cl} : P \to P$ such that:
   
   (a) $x \leq_P \text{cl}(x)$, 
   
   (b) $x \leq_P y$ implies $\text{cl}(x) \leq_P \text{cl}(y)$, 
   
   (c) $\text{cl} \circ \text{cl} = \text{cl}$.

   for all $x, y \in P$. That is, a closure operator is a monotone endomorphism $P \to P$ which is extensive (property 1) and idempotent (property 3). Its fixpoints are those $P$ where $\text{cl}[P] \subseteq P$ and are called the closed elements.

2. An interior operator on $P$ is a function $\text{in} : P \to P$ such that:

   (a) $\text{in}(x) \leq_P x$, 
   
   (b) $x \leq_P y$ implies $\text{in}(x) \leq_P \text{in}(y)$, 
   
   (c) $\text{in} \circ \text{in} = \text{in}$.

   Only the first property is different: the co-extensive property. Fixpoints of $\text{in}$ are those where $\text{in}[P] \subseteq P$ and are called the open elements.

Observe that a closure operator on $P$ is really the same thing as an interior operator on $P^{\text{op}}$.

**Lemma 2.2.5 (Open and closed elements as substructures).**

Take any closure operator $\text{cl} : P \to P$ and interior operator $\text{in} : P \to P$.

1. $\text{cl}[P]$ is closed under all meets that exist in $P$. In particular, it contains $\top_P$ if the latter exists.

2. If $P$ has a lattice structure $\mathcal{L}$ then $\text{cl}[P]$ forms a sub-join-semilattice of $\mathcal{L}^{\text{op}}$. Moreover, if $P$ is finite and has a join-semilattice structure $\mathcal{Q}$ then $\text{cl}[P]$ is the carrier of a sub join-semilattice of $\mathcal{Q}^{\text{op}}$.

3. $\text{in}[P]$ is closed under all joins that exist in $P$. In particular, it contains $\bot_P$ if the latter exists.

4. If $P$ has a lattice structure $\mathcal{L}$ then $\text{in}[P]$ forms a sub-join-semilattice of $\mathcal{L}$. Moreover, if $P$ is finite and has a join-semilattice structure $\mathcal{Q}$ then $\text{in}[P]$ is the carrier of a sub join-semilattice of $\mathcal{Q}$.

**Proof.** Regarding the first statement, suppose that $x_i \in \text{cl}[P]$ for all $i \in I$ and that $z = \bigwedge_{i \in I} x_i$ exists in $P$. Then $\text{cl}(z) \leq_P \text{cl}(x_i)$ for all $i \in I$ by monotonicity of $\text{cl}$. Thus $\text{cl}(z) \leq_P \bigwedge_{i \in I} x_i = z$, so by the extensivity of $\text{cl}$ we deduce that $\text{cl}(z) = z$. The second statement follows immediately from the first. The final statements are order-duals of the first two. 

**Note 2.2.6 (Closure/interior operators needn’t preserve meets/joins).**

Although the meet of closed sets is closed whenever it exists, $\text{cl}$ needn’t preserve meets i.e. we may have $\text{cl}(x \land y) \neq \text{cl}(x) \land \text{cl}(y)$. For example, let $X = \{x, y_1, y_2, z\}$ have four distinct elements, take the two binary relations $R_1 = \{(x, y_1), (y_2, z)\} \subseteq X \times X$ for $i = 1, 2$, and let $\text{cl}$ construct the transitive closure on the respective inclusion-ordered lattice of binary relations. Then although $\text{cl}(R_1 \cap R_2) = \emptyset$ and $\text{cl}(R_1) \cap \text{cl}(R_2) = \{(x, z)\}$ are both closed under transitivity, they are not equal. Of course, this also means that interior operators needn’t preserve joins.

**Lemma 2.2.7 (Properties of adjoint monotone morphisms).**

Given two monotone functions $f : P \to Q$ and $g : Q \to P$ such that:

$$f(p) \leq_P q \iff p \leq_P g(q) \quad \text{for all } p \in P \text{ and } q \in Q$$

then the following statements hold:

1. $g \circ f : P \to P$ is a closure operator and $f \circ g : Q \to Q$ is an interior operator.
2. For any subset \( X \subseteq P \) such that \( \bigvee_P X \) exists in \( P \), we have:

\[
f(\bigvee_P X) = \bigvee_Q f[X]
\]

so in particular the latter join exists in \( Q \).

3. For any subset \( Y \subseteq Q \) such that \( \bigwedge_Q Y \) exists in \( Q \), we have:

\[
g(\bigwedge_Q Y) = \bigwedge_P g[Y]
\]

so in particular the latter meet exists in \( P \).

Proof.

1. Defining \( \text{cl} := g \circ f \) then it is certainly a monotone morphism \( P \to P \). Regarding extensivity, \( p \leq_P \text{cl}(p) \) iff \( p \leq_Q g(f(p)) \) iff \( f(p) \leq_Q f(p) \) and hence always holds. Similarly \( f \circ g \) is co-extensive because \( f \circ g(q) \leq_Q q \) iff \( g(q) \leq_Q g(q) \). Regarding the idempotence of \( \text{cl} \), applying monotonicity to extensivity yields \( \text{cl}(p) \leq_P \text{cl} \circ \text{cl}(p) \) and the converse follows by the co-extensivity of \( f \circ g \) and the monotonicity of \( g \):

\[
g \circ f \circ g \circ f = g(f \circ g(f(p))) \leq_P g(f(p)).
\]

Thus \( \text{cl} \) is a well-defined closure operator. Regarding the interior operator \( \text{in} := f \circ g \), since \( q \leq_{Q^\text{op}} f(p) \) iff \( g(q) \leq_{\text{cl}} p \) we can apply the above argument to deduce that \( f^{\text{op}} \circ g^{\text{op}} : Q^\text{op} \to Q^\text{op} \) is a closure operator on \( Q^\text{op} \), hence \( \text{in} \) is an interior operator on \( Q \).

2. Given any \( X \subseteq P \) such that \( \bigvee_P X \) exists in \( P \) we are going to show that \( \bigvee_Q f[X] \) exists in \( Q \), and in fact equals \( q_X := f(\bigvee_P X) \). For every \( x \in X \) we have \( f(x) \leq_Q q_X \) by monotonicity i.e. it is an upper-bound for \( f[X] \subseteq Q \). Given any other \( q_0 \in Q \) such that \( \forall x \in X. f(x) \leq_Q q_0 \), then by adjointness we have \( x \leq_P g(q_0) \) and hence \( \bigvee_P X \leq_P g(q_0) \). Applying monotonicity and the co-extensivity of \( f \circ g \) proved in (1), we deduce that:

\[
q_X = f(\bigvee_P X) \leq_Q f(g(q_0)) \leq_Q q_0.
\]

3. This follows from (2) via order-duality i.e. \( g(q) \leq_{Q^\text{op}} p \iff q \leq_{Q^\text{op}} f(p) \) for every \( (p, q) \in P \times Q \).

\[\square\]

**Note 2.2.8.** This instantiates a well-known categorical result i.e. given an adjunction \( G \dashv F : \mathcal{C} \to \mathcal{D} \) between categories then \( G \circ F \) is the functorial component of a monad (closure operator) and \( F \circ G \) is the functorial component of a comonad (interior operator).

### 3 Finite join-semilattices and their self-duality

**Definition 3.0.1** (The category of finite join-semilattices). \( \text{JSL}_f \) is the category whose objects are the finite join-semilattices \( Q = (Q, \vee_Q, \bot_Q) \) and whose morphisms \( f : Q \to R \) are the join-semilattice morphisms between them, see Definition 2.2.1.10. Composition is the usual functional composition, and the identity morphism \( \text{id}_Q \) is the identity function \( \Delta_Q \).

Thus \( \text{JSL}_f \) consists of all finite join-semilattices with its usual algebra homomorphisms. Viewing the finite join-semilattices as the finite commutative and idempotent monoids, the latter are precisely the monoid morphisms. Alternatively they may be described as those functions preserving all joins i.e. such that:

\[
f(\bigvee_Q S) = \bigvee_R f[S] \quad \text{(for all } S \subseteq Q \text{)}
\]

due to the finiteness of the join-semilattices involved. We are going to describe the self-duality of \( \text{JSL}_f \), which is of fundamental importance to our approach. It restricts two distinct dualities:
1. The self-duality of complete join-semilattices i.e. complete lattices equipped with those functions which preserve all joins. This variety (with infinitary signature) consists of the Eilenberg-Moore algebras for the powerset functor $\mathcal{P} : \text{Set} \to \text{Set}$ [Mac71].

2. The Stone-type duality between the variety of join-semilattices with bottom (with infinitary signature) and the Stone topological join-semilattices [Joh82].

But it also follows directly from the adjoint functor theorem restricted to posets i.e. thin skeletal categories. Each finite join-semilattice has a bottom element and binary joins, hence all joins by finiteness, hence all meets by completeness. Each $\mathsf{JSL}_f$-morphism $f : Q \to R$ preserves all joins (= colimits), thus by the adjoint functor theorem it has a unique left adjoint i.e. a function $f_* : R \to Q$ such that:

$$f(q) \leq_R r \iff q \leq_Q f_*(r) \quad \text{(for all } q \in Q, r \in R)$$

which preserves all meets i.e. sends meets in $R$ to meets in $Q$. It follows that:

$$f_*(r) = \bigvee_{q \in Q} f^{-1}(\downarrow_R r) = \bigvee_{q \in Q} \{ q \in Q : f(q) \leq_R r \} \quad \text{(for all } r \in R)$$

and defines a $\mathsf{JSL}_f$-morphism of type $R^{\text{op}} \to Q^{\text{op}}$.

**Theorem 3.0.2.** $\mathsf{JSL}_f$ is self-dual via the functor $\mathsf{OD}_j : \mathsf{JSL}_f^{\text{op}} \to \mathsf{JSL}_f$ defined:

$$\mathsf{OD}_j Q := Q^{\text{op}} \quad \mathsf{OD}_j f^{\text{op}} := \lambda r \in R. \bigvee_{Q} f^{-1}(\downarrow_R r) : R^{\text{op}} \to Q^{\text{op}}$$

with natural isomorphism $\text{rep} : I d_{\mathsf{JSL}_f} \Rightarrow \mathsf{OD}_j \circ \mathsf{OD}_j^{\text{op}}$ whose components are the identity morphisms $\text{rep}_Q := id_Q$.

**Proof.** We first verify that $\mathsf{OD}_j$ is a well-defined functor. Recall that each finite join-semilattice $Q = (Q, \vee_Q, \bot_Q)$ defines a finite lattice $\mathcal{L} = (Q, \vee_Q, \bot_Q, \land_Q, \top_Q)$ in a unique fashion, so that $\mathsf{OD}_j Q = Q^{\text{op}} = (Q, \land_Q, \top_Q)$ is a well-defined finite join-semilattice. We have already explained why $\mathsf{OD}_j f^{\text{op}} = f_*$ is a well-defined $\mathsf{JSL}_f$-morphism of type $R^{\text{op}} \to Q^{\text{op}}$, but let us directly verify this anyway. It is certainly a well-defined function because all joins exist in $Q$, so let us verify the ‘adjointness’ i.e. $f(q) \leq_R r$ iff $q \leq_Q f_*(r)$, for any $q \in Q$ and $r \in R$.

1. ($\Rightarrow$) by definition of $f_*$.

2. ($\Leftarrow$) because if $q$ is the join of all $q_i \in Q$ such that $f(q_i) \leq_R r$ then $f(q) = f(\vee_i q_i) = \vee_i f(q_i) \leq_R r$.

We now use this to verify preservation of the bottom element and binary join:

$$f_*(\bot_{\mathcal{L}^{\text{op}}}) = f(\top_{\mathcal{L}}) = \bigvee_{Q} f^{-1}(\downarrow_R \top_{\mathcal{L}}) = \bigvee_{Q} f^{-1}(R) = \bigvee_{Q} Q = \top_{Q} = \bot_{\mathcal{L}^{\text{op}}}

\begin{align*}
\quad f_*(r_1 \land_{\mathcal{L}^{\text{op}}} r_2) &= f_*(r_1 \land_Q r_2) \\
&= f_* f^{-1}(\downarrow_R r_1 \land_Q r_2) \\
&= f_* f^{-1}(\downarrow_R r_1 \cap \downarrow_R r_2) \\
&= f_* f^{-1}(\downarrow_R r_1) \land f^{-1}(\downarrow_R r_2) \\
&= f_* \{ q \in Q : f(q) \leq_R r_1, r_2 \} \\
&= f_* \{ q \in Q : q \leq_Q f_*(r_1), f_*(r_2) \} \\
&= f_*(r_1) \land_Q f_*(r_2) \\
&= f_*(r_1) \land_{\mathcal{L}^{\text{op}}} f_*(r_2)
\end{align*}

Thus $\mathsf{OD}_j$’s action is well-defined. Regarding the preservation of identity morphisms:

$$\mathsf{OD}_j(id_Q^{\text{op}})_* = (id_Q)_* = \lambda q \in Q. \bigvee_{Q} f^{-1}(\downarrow_{\mathcal{L}^{\text{op}}} q) = \lambda q \in Q. \bigvee_{Q} \downarrow_{\mathcal{L}} q = \lambda q \in Q. q = id_{\mathsf{OD}_j Q}$$
Finally let us verify that $\text{rep} : \text{Id}_{\text{JSL}_f} \to \text{OD}_j \circ \text{OD}_j^{\text{op}}$ where $\text{rep}_Q = \text{Id}_Q$ is a natural isomorphism i.e. for all morphisms $f : Q \to R$ we must show the following diagram commutes:

\[
\begin{array}{c}
Q \\
\downarrow f
\end{array}
\begin{array}{c}
\text{rep}_Q \\
\text{OD}_j \circ \text{OD}_j^{\text{op}} f
\end{array}
\begin{array}{c}
R \\
\downarrow \text{rep}_R
\end{array}
\]

or equivalently that $(f_*)_* = f$. This already follows from the uniqueness of adjoints, but we’ll verify it anyway:

$$(f_*)_* \begin{array}{ll} = \lambda q \in Q. \lambda r \in R. f(q) \leq_R r & \text{by adjoint relationship} \\
= \lambda q \in Q. \lambda r \in R. (f_*)^{-1}(f(q) \leq_R r) & \text{for every } q \in Q \text{ and } r \in R
\end{array}$$

Let us make a few basic observations concerning these adjoint morphisms.

**Lemma 3.0.3.**

1. For all JSL$_f$-morphisms $f : Q \to R$ we have the adjoint relationship:

$$f(q) \leq_R r \iff q \leq_Q f_*(r) \quad \text{for every } q \in Q \text{ and } r \in R$$

2. The adjoint of an isomorphism between finite join-semilattices acts like its inverse. That is, if $f : Q \to R$ is a JSL$_f$-isomorphism then $f_* = (f^{-1})^{\text{op}}$, where it is permissible to take the order-dual monotone mapping because we are dealing with bounded lattice isomorphisms. Moreover:

$$(f^{-1})_* = (f_*)^{-1} = f^{\text{op}}$$

3. The image function is the adjoint of the preimage function. That is, for any function $f : X \to Y$ between finite sets, the adjoint of $\mathcal{P}f : (\mathcal{P}X, \cup, \emptyset) \to (\mathcal{P}Y, \cup, \emptyset)$ is the preimage $f^{-1} : (\mathcal{P}Y, \cap, Y) \to (\mathcal{P}X, \cap, X)$.

4. Each JSL$_f(Q, R)$ admits a join-semilattice structure, the ordering $\leq_{(Q, R)}$ being the pointwise-ordering. The mapping $f \mapsto f_*$ defines a JSL$_f$-isomorphism from $(\text{JSL}_f(Q, R), \leq_{(Q, R)})$ to $(\text{JSL}_f(R^{\text{op}}, Q^{\text{op}}), \leq_{(R^{\text{op}}, Q^{\text{op}})})$.

5. A JSL$_f$-morphism is a section iff its adjoint is a retract.

**Proof.**

1. See the proof of Theorem 3.0.2.

2. Given a JSL$_f$-isomorphism $f : Q \to R$ then the adjoint $f_*: R^{\text{op}} \to Q^{\text{op}}$ has action:

$$f_*(r) = \bigvee_{Q} f^{-1}(\downarrow_{R} r) = \bigvee_{Q} \downarrow_{Q} f^{-1}(r) = f^{-1}(r)$$
using the fact that \( f^{-1} : \mathbb{R} \rightarrow \mathbb{Q} \) is a monotone bijection. Then it has the same typing and action as \((f^{-1})^{\text{op}}\), so these are the same \( \text{JSL}_f \)-morphisms. Consequently:

\[
(f_*)^{-1} = ((f^{-1})^{\text{op}})^{-1} = ((f^{-1})^{-1})^{\text{op}} = f^{\text{op}} \quad (f^{-1})_* = ((f^{-1})^{-1})^{\text{op}} = f^{\text{op}}
\]

where in the left derivation we have used the general fact that inverses commute with \((-)^{\text{op}}\).

3. We calculate:

\[
(Pf)_*(S) = \bigvee_{(PX,Y,z)} (Pf)^{-1}((P(Y,z)) S) \\
= \bigcup (Pf)^{-1}\{(K \in Y : Z \subseteq S)\} \\
= \bigcup\{K \in X : f[X] \subseteq S\} \\
= f^{-1}(S)
\]

since the \( f \)-preimage of \( S \) is the largest subset whose image under \( f \) lies inside \( S \).

4. The bottom element is \( \lambda r \in R\downarrow_\mathbb{R} \), and the pointwise-join of morphisms is again a \( \text{JSL}_f \)-morphism:

\[
f_1 \vee_{(\mathbb{Q},\mathbb{R})} f_2(\downarrow_\mathbb{Q}) = f_1(\downarrow_\mathbb{Q}) \vee_{\mathbb{R}} f_2(\downarrow_\mathbb{R}) = \downarrow_\mathbb{R} \vee_{\mathbb{R}} \downarrow_\mathbb{R} = \downarrow_\mathbb{R}
\]

\[
f_1 \vee_{(\mathbb{Q},\mathbb{R})} f_2(q_1 \vee_{\mathbb{Q}} q_2) = f_1(q_1 \vee_{\mathbb{Q}} q_2) \vee_{\mathbb{R}} f_2(q_1 \vee_{\mathbb{Q}} q_2) = f_1(q_1) \vee_{\mathbb{R}} f_2(q_1) \vee_{\mathbb{R}} f_1(q_2) \vee_{\mathbb{R}} f_2(q_2) = f_1 \vee_{(\mathbb{Q},\mathbb{R})} f_2(q_1) \vee_{\mathbb{R}} f_1 \vee_{(\mathbb{Q},\mathbb{R})} f_2(q_2)
\]

The mapping \( f \mapsto f_* \) is bijective by the self-duality theorem. Given \( f \leq_{(\mathbb{Q},\mathbb{R})} g \) we first show that \( f_* \leq_{(\mathbb{Q},\mathbb{R})} g_* \). Given any \( r \in \mathbb{R} \), then \( g_*(r) \) is the \( \mathbb{Q} \)-join of all elements \( a \in \mathbb{Q} \) such that \( g(a) \leq_{\mathbb{R}} r \), and since \( f(a) \leq_{\mathbb{Q}} g(a) \leq_{\mathbb{R}} r \) we deduce that \( f_*(r) \) is the \( \mathbb{Q} \)-join of a larger set. Therefore \( g_*(r) \leq_{\mathbb{Q}} f_*(r) \) and thus \( f_*(r) \leq_{\mathbb{Q},\mathbb{R}} g_*(r) \), and since \( r \) was arbitrary we have \( f_* \leq_{(\mathbb{R},\mathbb{Q},\mathbb{R})} g_* \). This proves monotonicity. Order-reflection follows by applying the adjunction in the opposite direction i.e. by the same argument \( f_* \leq_{(\mathbb{R},\mathbb{Q},\mathbb{R})} g_* \) implies that \( f = (f_*)^* = (g_*)^* = g \) using the naturality of \( \text{rep} \).

5. Recall that an algebra morphism \( s : \mathbb{Q} \rightarrow \mathbb{R} \) is a section (resp. \( r : \mathbb{R} \rightarrow \mathbb{Q} \) is a retract) if there exists an algebra morphism \( r : \mathbb{R} \rightarrow \mathbb{S} \) (resp. \( s : \mathbb{S} \rightarrow \mathbb{R} \)) such that \( r \circ s = id_\mathbb{Q} \). Then since \( s_* \circ r_* = (r \circ s)_* = (id_\mathbb{Q})_* = id_\mathbb{Q} \) the statement is clear.

The fourth point above and its proof lead naturally to the following standard definition.

**Definition 3.0.4.** To any two finite join-semilattices \( \mathbb{Q} \) and \( \mathbb{R} \) we associate the finite join-semilattice:

\[
\text{JSL}_f[\mathbb{Q},\mathbb{R}] = (\text{JSL}_f(\mathbb{Q},\mathbb{R}), \bigvee_{(\mathbb{Q},\mathbb{R})}, \downarrow_{(\mathbb{Q},\mathbb{R})})
\]

where \( f_1 \vee_{(\mathbb{Q},\mathbb{R})} f_2 := \lambda q \in \mathbb{Q}. f_1(q) \vee_{\mathbb{R}} f_2(q) \) and also \( \downarrow_{(\mathbb{Q},\mathbb{R})} := \lambda q \in \mathbb{Q}. \downarrow_{\mathbb{R}} \).

That is, the join is the pointwise-join and the bottom is the constantly bottom map.

**Lemma 3.0.5.**

1. The self-duality of join-semilattices restricts to a join-semilattice isomorphism:

\[
\text{JSL}_f[\mathbb{Q},\mathbb{R}] \xrightarrow{(-)} \text{JSL}_f[\mathbb{R},\mathbb{Q}^{\text{op}}]
\]

for each \( \mathbb{Q}, \mathbb{R} \in \text{JSL}_f \).

2. Any \( \text{JSL}_f \)-morphism \( \theta : \mathbb{R} \rightarrow \mathbb{S} \) induces a join-semilattice morphism:

\[
\text{JSL}_f[\mathbb{Q},\mathbb{R}] \xrightarrow{\theta \circ (-)} \text{JSL}_f[\mathbb{Q},\mathbb{S}] \quad \text{with action} \quad g \mapsto \theta \circ g
\]
Proof. The first statement follows from the statement and proof of Lemma 3.0.3.4 above. Regarding the second statement, we certainly have a well-defined function and:

\[
\theta \circ (-)(\mathcal{JSL}_f[Q,R]) = \theta \circ (-)(\lambda q \in Q, \mathcal{JSL}_f[Q,R] f) = \lambda q \in Q, (\theta \circ g(q) \lor \theta \circ f(q))
\]

Recall that the elements of a possibly infinite join-semilattice \( \mathcal{Q} \) biject with the join-semilattice morphisms of type \( 2 \rightarrow \mathcal{Q} \) i.e. consider the action on the latter on \( \tau_2 = 1 \). Restricting to the finite level, the self-duality yields a correspondence between elements of \( \mathcal{Q} \) and the ideals \( \text{idl}_{\mathcal{Q}^{op}}(q) \) i.e. the morphisms of type \( \mathcal{Q}^{op} \rightarrow 2 \).

**Definition 3.0.6** (Elements and ideals as morphisms).

Let \( \mathcal{Q} \) be any finite join-semilattice.

1. Each element \( q \in Q \) has an associated join-semilattice morphism:

\[
el_Q(q) := \lambda b \in \{0,1\}.(b = 1) ? q : \perp_Q : 2 \rightarrow \mathcal{Q}
\]

and we define the join-semilattice \( \text{Elem}(Q) := \mathcal{JSL}_f[2, \mathcal{Q}] \).

2. Each element \( q_0 \in Q \) has an associated join-semilattice morphism:

\[
\text{idl}_Q(q_0) := \lambda q \in Q. (q \leq q_0) ? 0 : 1 : \mathcal{Q} \rightarrow 2
\]

and we define the join-semilattice \( \text{Ideel}(Q) := \mathcal{JSL}_f[\mathcal{Q}, 2] \). □

**Lemma 3.0.7.** For each finite join-semilattice \( \mathcal{Q} \) the following statements hold.

1. We have the join-semilattice isomorphism:

\[
el_Q(-) : Q \rightarrow \text{Elem}(Q) = \mathcal{JSL}_f[2, \mathcal{Q}]
\]

\[
el_Q(q) := \lambda b \in 2. b ? q : \perp_Q : \mathcal{Q} \rightarrow \mathcal{Q}
\]

\[
el_Q^{-1}(h : 2 \rightarrow \mathcal{Q}) := h(1)
\]

2. We have the join-semilattice isomorphism:

\[
\text{idl}_Q(-) : \mathcal{Q}^{op} \rightarrow \text{Ideel}(Q) = \mathcal{JSL}_f[\mathcal{Q}, 2]
\]

\[
\text{idl}_Q(q_0) = \lambda q \in Q. (q \leq q_0) ? 0 : 1 : \mathcal{Q} \rightarrow \mathcal{Q}
\]

\[
\text{idl}_Q^{-1}(h : \mathcal{Q} \rightarrow 2) := \lor_Q h^{-1}([0])
\]

3. Regarding the adjoints of these special morphisms,

\[
(el_Q(q))_* = \mathcal{Q}^{op} \xrightarrow{\text{idl}_{\mathcal{Q}^{op}}(q)} 2 \xrightarrow{\text{swap}^{-1}} 2^{op}
\]

\[
(idl_Q(q))_* = 2^{op} \xrightarrow{\text{swap}} 2 \xrightarrow{el_{\mathcal{Q}^{op}}(q)} \mathcal{Q}^{op}
\]

Proof:

1. A join-semilattice morphism \( f : 2 \rightarrow \mathcal{Q} \) must map \( 0 = \perp_2 \) to \( \perp_\mathcal{Q} \) and may send \( 1 \) to any element of \( \mathcal{Q} \). Thus \( el_Q(-) \) is a well-defined bijective function, and preserves joins because:

\[
el_Q(\perp_\mathcal{Q}) = \lambda b \in 2.b \perp_\mathcal{Q} \perp_\mathcal{Q} = \lambda b \perp_\mathcal{Q} = \perp_{\text{Elem}(Q)}
\]

\[
el_Q(q_1 \lor Q q_2) = \lambda b \in 2.b \lor q_1 \lor Q q_2 : \perp_\mathcal{Q} \lor \perp_\mathcal{Q} = el_Q(q_1 \lor Q q_2) \lor_{\text{Elem}(Q)} el_Q(q_2)
\]

The correctness of its inverse is clear.
2. Each $\text{idl}_Q(q_0) : Q \to 2$ is a well-defined join-semilattice morphism because it is the composite:

$$\text{idl}_Q(q_0) = Q \xrightarrow{(\text{el}_Q(q_0))_{\text{op}}} 2^{\text{op}} \xrightarrow{\text{swap}} 2$$

where swap is the unique join-semilattice morphism of type $2^{\text{op}} \to 2$ (it flips the bit). To see this, let us describe the action of $(\text{el}_Q(q_0))_{\text{op}}$.

$$q \mapsto \begin{cases} 1 & \text{if } q \leq q_0 \\ 0 & \text{otherwise} \end{cases}$$

Applying swap yields the desired action. It also follows from (1) that $\text{idl}_Q(\cdot)$ is a bijection, and regarding preservation of $Q^{\text{op}}$-joins:

$$\text{idl}_Q(\top_Q) = \lambda q \in Q, (q \leq \top_Q) ? 0 : 1 = \lambda q \in Q, 0 = 1_{\text{idl}(Q)}$$

$$\text{idl}_Q(q_1 \land_Q q_2) = \lambda q \in Q, (q \leq q_1 \land q_2) ? 0 : 1$$

3. Follows from the proof of the previous statement, noting that swap : $2^{\text{op}} \to 2$ is self-adjoint, swap$^{-1} : 2 \to 2^{\text{op}}$ is self-adjoint, and they are the same underlying functions (although distinct $\text{JSL}_f$-morphisms).

We shall spend the rest of this section discussing embeddings and quotients of finite join-semilattices. Later on we shall again consider the structural properties of $\text{JSL}_f$ e.g. we define the tensor product and prove its universality using the category $\text{Dep}$.

**Lemma 3.0.8.** Let $f : Q \to R$ be any $\text{JSL}_f$-morphism.

1. $f$ is a monomorphism iff it is injective.

2. $f$ is an epimorphism iff it is surjective.

3. $f$ is injective iff $f_*$ is surjective, and equivalently $f$ is surjective iff $f_*$ is injective.

4. $f$ is an isomorphism iff it is monic and epic iff it is bijective.

**Proof.**

1. That $f$ is monic means precisely that $f \circ \alpha = f \circ \beta$ implies $\alpha = \beta$ for any $\text{JSL}_f$-morphisms $\alpha, \beta : S \to Q$. Given that $f$ is injective then $f$ is monic because $f(\alpha(q)) = f(\beta(q))$ implies $\alpha(q) = \beta(q)$. Conversely if $f$ is monic and $f(q_1) = f(q_2)$ then $f \circ \text{el}_Q(q_1) = f \circ \text{el}_Q(q_2)$ and hence $\text{el}_Q(q_1) = \text{el}_Q(q_2)$, so that $q_1 = q_2$.

2. That $f$ is epic means precisely that $\alpha \circ f = \beta \circ f$ implies $\alpha = \beta$ for any $\text{JSL}_f$-morphisms $\alpha, \beta : R \to S$. Given that $f$ is surjective then $f$ is epic because $\alpha(r) = \alpha(f(q)) = \beta(f(q)) = \beta(r)$ by choosing a suitable $q$. Conversely assume $f$ is epic, so that $f$ is the dual of an injective function by using Theorem 3.0.2 and the previous statement. Then it suffices to show that $f_* : R^{\text{op}} \to Q^{\text{op}}$ is surjective whenever $f : Q \to R$ is injective. So given any $q \in Q$ we need to find some $r \in R$ such that $f_*(r) = q$, and the obvious choice is $r := f(q)$.

$$f_*(f(q)) = \bigvee_{q \in Q} f^{-1}(f(q)) = \bigvee_{q \in Q} \{ q' \in Q : f(q') \leq_R f(q) \}$$

Certainly $q$ is one of the summands. Conversely if $f(q') \leq_R f(q)$ then $f(q' \lor_R q) = f(q') \lor_R f(q) = f(q)$, so by injectivity $q \lor_R q' = q$ and thus $q' \leq q$. Therefore $f_*(f(q)) = q$ and we are finished.

3. $f$ is injective iff $f$ is $\text{JSL}_f$-monic by the first statement, iff $f_*$ is $\text{JSL}_f$-epic by the duality of Theorem 3.0.2, iff $f_*$ is surjective by the second statement. Since $f = (f_*)_*$ by the naturality of $\text{rep}$ we obtain the other statement.
4. That $f$ is an isomorphism means that there exists a $JSL_f$-morphism $g : R \rightarrow Q$ such that $g \circ f = id_Q$ and $f \circ g = id_R$. Then if $f$ is an isomorphism it is split-monic hence monic hence injective, and split-epic hence epic hence surjective. Thus $f$ is bijective. Conversely suppose $f$ is injective and surjective, hence bijective. Then its functional inverse is a well-defined $JSL_f$-morphism, a well-known fact that holds in any variety of algebras.

\[ \square \]

**Note 3.0.9.** Although a bijective homomorphism defines an algebra isomorphism in any variety of algebras, this fails in the ordered setting. For example, a bijective monotone function from a discretely ordered two element set to a 2-chain does not have a monotone inverse. Moreover, algebra homomorphisms can be epic and yet not surjective, as is the case in the variety of distributive lattices. For example, each of two embeddings of a 3-chain into a 4-element boolean algebra are not surjective. However they are both epic using the fact that complements in distributive lattices are unique whenever they exist, see Lemma 2.2.3.9. \[ \blacksquare \]

We have more to say regarding injective and surjective join-semilattice morphisms. Let us start with some negative results and their duals.

**Lemma 3.0.10.** Let $f : Q \rightarrow R$ be any join-semilattice morphism between finite join-semilattices.

1. If $f$’s restriction to $J(Q) \subseteq Q$ is injective then $f$ need not be injective.
2. If $f$’s restriction to $M(Q) \subseteq Q$ is injective then $f$ need not be injective.
3. Moreover even if both these restrictions are injective then $f$ needn’t be.

*Proof.*

1. As a counter-example, first recall that the join-semilattice $M_3$ is obtained by adding a new top and bottom element to the discrete poset with elements $X = \{x_1, x_2, x_3\}$. Then we have the join-semilattice morphism $f : \mathbb{P}X \rightarrow M_3$ where $f(\{x_i\}) = x_i$ for each of three join-irreducibles (atoms). It is clearly injective on the join-irreducibles, yet maps every meet-irreducible (coatom) to $\top_{M_3}$.

2. We illustrate a counter-example below.

![Counter-example diagram](image)

It is easily seen to be a well-defined join-semilattice morphism $f : Q \rightarrow R$ i.e. we are essentially extending the identity function on $2^2$ with an identification. Then it is injective on the meet-irreducibles $\{m_1, m_2, m_3\}$ but it is not an injective function.

3. The third statement follows from the second example above, noting that $J(Q) = \{m_2, m_3, \top_Q\}$. \[ \blacksquare \]

We now dualise the above observations item-by-item. Recall that the ideal associated to an element $q_i \in Q$ of a join-semilattice $Q$ is the join-semilattice morphism $idl_Q(q_i) : Q \rightarrow 2$ defined $\lambda q \in Q. (q \leq q_i) \uparrow 0 : 1$. Then one says $f$ separates a collection of ideals $\{idl_Q(q_i) : i \in I\}$ if whenever $q_i \neq q_j$ then $idl_Q(q_i) \circ f \neq idl_Q(q_j) \circ f$.

**Lemma 3.0.11.** Let $f : Q \rightarrow R$ be any join-semilattice morphism between finite join-semilattices.

1. If $f$ separates the ideals $\{idl_Q(m) : m \in M(Q)\}$ then $f$ needn’t be surjective.
2. If $f$ separates the ideals $\{idl_Q(j) : j \in J(Q)\}$ then $f$ needn’t be surjective.
3. If $f$ separates the ideals $\{idl_Q(q) : q \in J(Q) \cup M(Q)\}$ then $f$ needn’t be surjective.

Now for some positive results and their dual statements. This time we shall start with the surjective morphisms.
Lemma 3.0.12. A morphism \( f : Q \to R \) of finite join-semilattices is surjective iff \( J(R) \subseteq f[J(Q)] \).

Proof. Generally speaking, an algebra homomorphism is surjective iff the image of any subset generating the domain generates the codomain. Assume that \( f \) is surjective. By Lemma 2.2.3.10 we know (i) \( J(Q) \) generates \( Q \), and moreover (ii) \( J(R) \) is contained in any subset generating \( R \), thus in particular \( J(R) \subseteq f[J(Q)] \). Conversely the latter inclusion implies \( f \) is surjective via (i).

Dualising yields the following characterisation of embeddings.

Lemma 3.0.13. A morphism \( f : Q \to R \) of finite join-semilattices is injective iff:

\[
\forall m \in M(Q), \exists m_r \in M(R). \forall j_q \in J(Q). (f(j_q) \leq_R m_r) \iff j_q \leq_Q m_q
\]

Proof. \( f : Q \to R \) is injective iff \( f_* : R^{op} \to Q^{op} \) is surjective by Lemma 3.0.8.3, or equivalently:

\[
M(Q) = J(Q^{op}) \subseteq f_*[J(R^{op})] = f_*[M(R)]
\]

by Lemma 3.0.12. Then we observe that:

\[
f_*[M(R)] = \{f_*(m_r) : m_r \in M(R)\}
\]

\[
= \{\forall Q \{q \in Q : q \leq_Q f_*(m_r)\} : m_r \in M(R)\}
\]

\[
= \{\forall Q \{j_q \in J(Q) : j_q \leq_Q f_*(m_r)\} : m_r \in M(R)\}
\]

\[
= \{\forall Q \{j_q \in J(Q) : f(j_q) \leq_R m_r\} : m_r \in M(R)\}
\]

by adjoint relationship

We also have the following related well-known facts.

Lemma 3.0.14. Let \( Q \) be any finite join-semilattice.

1. Given any surjective join-semilattice morphism \( \sigma : PZ \to Q \) then \( |J(Q)| \leq |Z| \).

2. Given any injective join-semilattice morphism \( e : Q \to PZ \) we have \( |M(Q)| \leq |Z| \).

Proof. The first statement holds by Lemma 3.0.12 we know that \( \sigma[J(PZ)] \supseteq J(Q) \) and therefore \( |J(Q)| \leq |J(PZ)| = |Z| \). The second statement follows from the first by the self-duality of \( JSL_f \) and the fact that surjections dualise injections via Lemma 3.0.8. That is, given \( e \) we obtain the surjective morphism \( e_* : (PZ)^{op} \to Q^{op} \) and thus also \( e_* \circ (-Z)^{-1} : PZ \to Q^{op} \), so that \( |M(Q)| = |J(Q^{op})| \leq |Z| \).

3.1 Congruence lattices of finite join-semilattices

Definition 3.1.1 (Congruence and subalgebra lattices).

Let \( Q \) be a finite join-semilattice.

1. A congruence of \( Q \) (also called a \( Q \)-congruence) is an equivalence relation \( R \subseteq Q \times Q \) closed under the rule:

\[
\frac{R(q_1,q_2), R(q_3,q_4)}{R(q_1 \lor Q p q_3, q_2 \lor Q q_4)} \quad \text{for every } q_1,q_2,q_3,q_4 \in Q.
\]

Letting \( Con(Q) \) be the collection of all \( Q \)-congruences and ordering by inclusion yields:

\[
CON(Q) := (Con(Q), \lor_{CON(Q)}, \Delta_Q, \cap, Q \times Q) \quad \text{i.e. the bounded lattice of } Q \text{-congruences}.
\]

For general universal algebraic reasons, it is a sub bounded lattice of the lattice of all equivalence relations on \( Q \). In particular, the binary join \( \lor_{CON(Q)} \) constructs the transitive closure of the binary union. Given any relation \( S \subseteq Q \times Q \), let:

\[
GC_Q(S) := \bigcap \{R \in Con(Q) : S \subseteq R\}
\]

be the \( Q \)-congruence generated by \( S \).
In the case where \( S = \{(q_1, q_2)\} \) is a singleton we instead write \( PC_Q^{\{q_1,q_2\}} \) (which equals \( PC_Q^{q_1,q_2} \)), these being the principal \( Q \)-congruences. By universal algebra, the principal congruences where \( q_1 \neq q_2 \) are precisely the join-irreducible elements of \( CON(Q) \). On the other hand, we also have the meet-irreducible \( Q \)-congruences:

\[
\mathcal{MC}_Q^\cap := (\{q\} \times (\{q\} q) \cup \{(q) q \} \times (\{q\} q) \subseteq Q \times Q \quad \text{for each } q \in Q \setminus \{q\}.
\]

We also permit \( \mathcal{MC}_Q^\cap \) under the above definition, observing that it equals \( \tau_{CON(Q)} \) and thus is the maximum \( Q \)-congruence or alternatively the trivial \( Q \)-congruence.

2. The \( Q \)-subalgebras also define a finite inclusion-ordered bounded lattice:

\[
\text{SUB}(Q) := (\text{Sub}(Q), \vee_{\text{SUB}(Q)}, \{1\}_Q, \cap, Q)
\]

where \( \text{Sub}(Q) := \{S : (S, \vee, 1_Q) \subseteq Q\} \subseteq \mathcal{P} Q \). Notice that we collect the underlying sets of \( Q \)'s subalgebras, rather than the subalgebras themselves. The binary join \( \vee_{\text{SUB}(Q)} \) constructs all possibly-empty finite joins of the binary union i.e. the elements of the \( Q \)-subalgebra generated by the binary union.

Recall the usual notation for generated subalgebras i.e. \( \langle X \rangle_Q \subseteq Q \) is the sub join-semilattice generated by \( X \subseteq Q \). Let us denote the carrier of this algebra by \( GS_Q(X) \), so it is the closure of \( X \subseteq Q \) under all possibly-empty \( Q \)-joins. In the case where \( X = \{q\} \) is a singleton we have:

\[
GS_Q(\{q\}) = \{1_Q, q\}.
\]

Excluding the 0-generated subalgebra with carrier \( GS_Q(\{1\}_Q) = \{1_Q\} = 1_{\text{SUB}(Q)} \), it follows by universal algebra that these 1-generated subalgebras are precisely the join-irreducible elements of \( \text{SUB}(Q) \). In fact they are clearly atoms so that \( \text{SUB}(Q) \) is atomistic: every element is a join of atoms. Finally, for each \( q_1, q_2 \in Q \) we have the \( Q \)-subalgebra:

\[
\mathcal{MS}_Q^{q_1,q_2} \subseteq Q \quad \text{with carrier} \quad \mathcal{MS}_Q(q_1, q_2) := \{q \in Q : q \leq q_1 \leftrightarrow q \leq q_2\} \subseteq Q.
\]

Observing that \( \mathcal{MS}_Q(q, q) = Q = \tau_{\text{SUB}(Q)} \), then the meet-irreducible \( Q \)-subalgebras are those where \( q_1 \neq q_2 \). \( \square \)

The above definitions will soon be clarified. Let us start by describing the bounded lattice isomorphism between \( Q^{\text{op}} \)-subalgebras and \( Q \)-congruences, after which we provide a Corollary describing the connection in terms of \( Q \)-quotients. Then in Lemma 3.1.4 we'll describe the irreducible \( Q \)-congruences and \( Q \)-subalgebras, and the action of the bounded lattice isomorphisms upon them.

**Theorem 3.1.2** (Representing congruences as subalgebras and conversely).

For each finite join-semilattice \( Q \) we have the bounded lattice isomorphism:

\[
c2s_Q : (CON(Q))^{\text{op}} \rightarrow \text{SUB}(Q^{\text{op}}) \quad \text{c2s}_Q(\mathcal{R}) := \{ \forall_Q[q]_\mathcal{R} : q \in Q \}
\]

\[
s2c_Q(S)(q_1, q_2) : \iff \forall s \in S. (q_1 \leq s \Leftrightarrow q_2 \leq s)
\]

where \( s2c_Q = c2s_Q^{-1} \).

**Proof.** By universal algebra, the \( Q \)-congruences \( \mathcal{R} \in CON(Q) \) are precisely the kernels \( \ker f \) of all surjective join-semilattice morphisms \( f : Q \rightarrow \mathcal{R} \) where \( \mathcal{R} \in JSL_f \). Let us recall that:

\[
\ker f := \{(q_1, q_2) \in Q \times Q : f(q_1) = f(q_2)\}.
\]

Certainly each such kernel is a \( Q \)-congruence. Conversely we have the canonical surjective function \( \llbracket - \rrbracket_\mathcal{R} : Q \rightarrow Q\setminus \mathcal{R} \) because \( \mathcal{R} \) is an equivalence relation, and this actually defines a join-semilattice morphism \( Q \rightarrow Q\setminus \mathcal{R} = (Q\setminus \mathcal{R}, \vee_{Q\setminus \mathcal{R}}, \llbracket 1_Q \rrbracket_\mathcal{R}) \) where of course \( \llbracket q_1 \rrbracket_\mathcal{R} \vee_{Q\setminus \mathcal{R}} \llbracket q_2 \rrbracket_\mathcal{R} := \llbracket q_1 \vee q_2 \rrbracket_\mathcal{R} \). Importantly, we note that every \( \mathcal{R} \)-equivalence class is non-empty and closed under binary \( Q \)-joins. Then by finiteness \( \forall Q[q]_\mathcal{R} \in [q]_\mathcal{R} \) i.e. each \( \mathcal{R} \)-equivalence class always contains a maximum element.

Given any \( Q \)-congruence \( \mathcal{R} \), take the adjoint of its associated canonical surjective morphism:

\[
\llbracket - \rrbracket_\mathcal{R} : Q \rightarrow Q\setminus \mathcal{R}
\]

\[
(\llbracket - \rrbracket_\mathcal{R})^\text{op} : (Q\setminus \mathcal{R})^{\text{op}} \rightarrow Q^{\text{op}}
\]

20
The latter is necessarily injective by Lemma 3.0.8, so define \( S_\mathcal{R} := (\llbracket \cdot \mathcal{R} \rrbracket)_* : \llbracket \mathcal{Q} \rrbracket \mathcal{R} \subseteq \mathcal{Q}^{\text{op}} \) to be the image of this embedding. To understand its elements, consider the following calculation:

\[
\begin{align*}
(\llbracket \cdot \mathcal{R} \rrbracket)_* (\llbracket q \mathcal{R} \rrbracket) & = \bigvee Q (q' \in Q : [q']\mathcal{R} \leq \llbracket q \mathcal{R} \rrbracket) \quad \text{by definition} \\
& = \bigvee Q (q' \in Q : [q']\mathcal{R} \land \llbracket q \mathcal{R} \rrbracket = \llbracket q \mathcal{R} \rrbracket) \\
& = \bigvee Q (q' \in Q : [q']\mathcal{R} \land \llbracket q \mathcal{R} \rrbracket = \llbracket q \mathcal{R} \rrbracket) \\
& = \bigvee Q (q' \in Q : [q']\mathcal{R} \land \llbracket q \mathcal{R} \rrbracket = \llbracket q \mathcal{R} \rrbracket) \\
& = \bigvee Q (q' \in Q : [q']\mathcal{R} \land \llbracket q \mathcal{R} \rrbracket = \llbracket q \mathcal{R} \rrbracket) \quad \text{by well-definedness} \\
& = \bigvee Q (q' \in Q : [q']\mathcal{R} \land \llbracket q \mathcal{R} \rrbracket) \\
& = \bigvee Q (q' \in Q : [q']\mathcal{R} \land \llbracket q \mathcal{R} \rrbracket) \quad \text{by maximality} \\
& = \bigvee Q (q' \in Q : [q']\mathcal{R} \land \llbracket q \mathcal{R} \rrbracket).
\end{align*}
\]

Thus \( S_\mathcal{R} \) is obtained by taking the maximum element from each \( \mathcal{R} \)-equivalence class. It follows that \( c_{2s_Q} \) is a well-defined function. For injectivity it suffices to show that \( \llbracket \cdot \mathcal{R} : \mathcal{Q} \to \mathcal{Q} \mathcal{R} \) and the (surjective) adjoint of \( \iota : S_\mathcal{R} \to \mathcal{Q} \) have the same kernel. We have \( \iota \circ \beta = (\llbracket \cdot \mathcal{R} \rrbracket)_* \) for some isomorphism \( \beta \), and thus \( \beta_* \circ \iota_* = \llbracket \cdot \mathcal{R} \rrbracket \) where \( \beta_* = (\beta^{-1})^{\text{op}} \) is also an isomorphism. It follows that \( \ker \iota_* = \mathcal{R} \), as required. Concerning surjectivity, take any sub join-semilattice \( \iota : S \to \mathcal{Q}^{\text{op}} \) and define \( \mathcal{R}_S := \ker \iota_* \). We are going to show that \( c_{2s_{\mathcal{R}_S}}(\mathcal{R}_S) = S \). First observe,

\[
\iota_*(q) = \bigvee \{ s \in S : s \leq \llbracket q \mathcal{R} \rrbracket \} = \bigwedge \{ s \in S : q \llbracket q \mathcal{R} \rrbracket \}
\]

so if we assume \( \iota_*(q_1) = \iota_*(q_2) \) for any fixed \( q_1, q_2 \in \mathcal{Q} \), then if \( q_1 \leq \llbracket q \mathcal{R} \rrbracket \mathcal{S} \) we have \( q_2 \leq \llbracket q \mathcal{R} \rrbracket \mathcal{S} \) and symmetrically \( q_2 \leq \llbracket q \mathcal{R} \rrbracket \mathcal{S} \) implies \( q_1 \leq \llbracket q \mathcal{R} \rrbracket \mathcal{S} \). It follows that:

\[
\mathcal{R}_S(q_1, q_2) \iff \forall s \in S. (q_1 \leq \llbracket q \mathcal{R} \rrbracket \mathcal{S} \iff q_2 \leq \llbracket q \mathcal{R} \rrbracket \mathcal{S})
\]

also because the latter condition implies that \( \iota_*(q_1) \) and \( \iota_*(q_2) \) have the same summands. Since \( \iota_*(s) = s \) for each \( s \in S \), it follows that \( \iota_*(q) = \iota_*(\iota_*(q)) \) and hence every \( \mathcal{R}_S \)-equivalence class contains some \( s \in S \). Furthermore they may contain at most one element of \( S \) via anti-symmetry. Then it follows that:

\[
c_{2s_{\mathcal{R}_S}}(\mathcal{R}_S) = \{ \bigvee_{q} \llbracket q \mathcal{R}_S \rrbracket : q \in \mathcal{Q} \} = S
\]

because if \( s \in \llbracket q \mathcal{R}_S \rrbracket \) then it is necessarily the maximum element relative to \( \leq \llbracket q \mathcal{R} \rrbracket \). Then we have proved that \( c_{2s_Q} \) is bijective and have also described its inverse \( s_{2c_Q} \) as desired. To establish that they are bounded lattice isomorphisms we'll show that \( s_{2c_Q} \) preserves and reflects the given orderings.

1. Assuming \( S_1 \subseteq S_2 \subseteq \mathcal{Q}^{\text{op}} \) we need to show that \( s_{2c_Q}(S_2) \subseteq s_{2c_Q}(S_1) \). Since \( S_1 \subseteq S_2 \) this follows immediately by restricting the universal quantification from \( S_2 \) to \( S_1 \).

2. Now suppose that \( S_2 \subseteq S_2 \) i.e. for every \( (q_1, q_2) \in \mathcal{Q} \times \mathcal{Q} \) we know:

\[
\forall s \in S_2. (q_1 \leq \llbracket q \mathcal{R} \rrbracket \mathcal{S} \iff q_2 \leq \llbracket q \mathcal{R} \rrbracket \mathcal{S}) \quad \text{implies} \quad \forall s \in S_1. (q_1 \leq \llbracket q \mathcal{R} \rrbracket \mathcal{S} \iff q_2 \leq \llbracket q \mathcal{R} \rrbracket \mathcal{S}).
\]

For a contradiction assume \( S_1 \not\subseteq S_2 \) so we have some \( s_1 \in S_1 \cap \overline{S_2} \), and define \( s_2 := \bigwedge \{ s \in S_2 : s_1 \leq \llbracket q \mathcal{R} \rrbracket \} \) observing that \( s_2 \in S_2 \) (because \( S_2 \subseteq \mathcal{Q}^{\text{op}} \)) and also \( s_1 \leq s_2 \). Setting \( (q_1, q_2) := (s_1, s_2) \), one can see that the premise of the above correspondence holds. Instantiating the deduced conclusion with \( s := s_1 \) we obtain \( s_1 \leq \llbracket q \mathcal{R} \rrbracket \mathcal{S} \iff s_2 \leq \llbracket q \mathcal{R} \rrbracket \mathcal{S} \), and hence derive the contradiction \( s_1 = s_2 \in S_2 \).

\[\square\]

**Corollary 3.1.3** (Subalgebra/quotient correspondence).

*For any \( S \subseteq \mathcal{Q}^{\text{op}} \) and any \( \mathcal{Q} \)-congruence \( \mathcal{R} \) there are associated isomorphisms:

\[
\begin{align*}
1. \quad \alpha : \mathcal{S}^{\mathcal{R}} & \to \mathcal{Q}/\mathcal{T} \\
\alpha(s) & := \llbracket s \mathcal{T} \rrbracket \\
\alpha^{-1}(\llbracket q \mathcal{T} \rrbracket) & := \bigwedge \{ s \in \mathcal{S} : q \leq \llbracket q \mathcal{R} \rrbracket \} \\
T & := s_{2c_Q}(S) \in \text{Con}(\mathcal{Q}). \\
2. \quad \beta : (\llbracket \mathcal{Q}/\mathcal{R} \rrbracket)^{\text{op}} & \to \mathcal{R} \\
\beta(\llbracket q \mathcal{R} \rrbracket) & := \bigvee Q [q] \mathcal{R} \\
\beta^{-1}(r) & := \llbracket r \mathcal{R} \rrbracket \\
R & := \mathcal{Q}^{\text{op}} \text{ has carrier } c_{2s_{\mathcal{R}}} \mathcal{R} \mathcal{C}.
\end{align*}
\]

*Proof.* Fixing any \( S \subseteq \mathcal{Q}^{\text{op}} \) and \( \mathcal{R} \in \text{Con}(\mathcal{Q}) \) let us verify the claimed isomorphisms \( \alpha \) and \( \beta \).
1. The carrier of the subalgebra \( S \) yields the \( \mathbb{Q} \)-congruence \( T := s2c_{\mathbb{Q}}(S) \), and the inclusion join-semilattice morphism \( \iota : S \to \mathbb{Q}^{\mathrm{op}} \) yields the \( \mathbb{Q} \)-congruence \( \ker \iota \). It follows directly from the proof of Theorem 3.1.2 that these two kernels coincide. Consequently there exists a unique \( JS\mathcal{L}_f \)-isomorphism such that:

\[
\begin{array}{ccc}
\mathbb{Q} & \xrightarrow{\iota^*} & \mathbb{Q} \setminus T \\
\downarrow \alpha & & \downarrow \alpha \\
\mathbb{Q} & \xrightarrow{\iota^*} & \mathbb{Q}^{\mathrm{op}}
\end{array}
\]

using the appropriate homomorphism theorem from universal algebra.

By definition \( \iota^*(q) = \mathbb{V}_{\mathbb{Q}}\{ s \in S : s \leq \mathbb{Q} q \} = \mathbb{W}_{\mathbb{Q}}\{ s \in S : q \leq \mathbb{Q} s \} \) so that \( \iota^*(s) = s \) for any \( s \in S \). Thus \( \alpha(s) = \alpha(\iota^*(s)) = [s]_T \) as expected, and finally \( \alpha^{-1}(\mathbb{Q} q) = \iota^*(q) = \mathbb{W}_{\mathbb{Q}}\{ s \in S : q \leq \mathbb{Q} s \} \).

2. The \( \mathbb{Q} \)-congruence \( \mathcal{R} \) yields the subalgebra \( \iota : \mathbb{R} \to \mathbb{Q}^{\mathrm{op}} \) with carrier \( R := c2s_{\mathbb{Q}}(\mathcal{R}) \). By the proof of Theorem 3.1.2 we know that the latter is precisely the image of the embedding \( \{1\}_\mathcal{R}^* : (\mathbb{Q} \setminus \mathcal{R})^{\mathrm{op}} \to \mathbb{Q}^{\mathrm{op}} \), so the action of the latter defines a \( JS\mathcal{L}_f \)-isomorphism \( \beta \) as follows:

\[
\begin{array}{ccc}
\mathbb{Q} \setminus \mathcal{R} & \xrightarrow{\iota^*} & \mathbb{Q}^{\mathrm{op}} \\
\downarrow \beta & & \downarrow \beta \\
\mathbb{Q}^{\mathrm{op}} & \xrightarrow{\iota^*} & \mathbb{Q}^{\mathrm{op}}
\end{array}
\]

Then \( \beta([q]_\mathcal{R}) = \iota(\beta([q]_\mathcal{R})) = ([1]_\mathcal{R})^*(q) = \mathbb{V}_{\mathbb{Q}}[q]_\mathcal{R} \), where the final equality was established in the proof of the Theorem. Finally, since we always know that \( \mathbb{V}_{\mathbb{Q}}[q]_\mathcal{R} \) actually lies inside \([q]_\mathcal{R}\), it follows that \( \beta^{-1}(r) = [r]_\mathcal{R} \).

\[\square\]

**Lemma 3.1.4.** Concerning the isomorphism \( c2s_{\mathbb{Q}} : (\mathbb{CON}(\mathbb{Q}))^{\mathrm{op}} \to \mathbb{SUB}(\mathbb{Q}^{\mathrm{op}}) \) from Theorem 3.1.2.

1. For any \( q_1, q_2 \in \mathbb{Q} \), the isomorphism \( c2s_{\mathbb{Q}} \) acts as follows:

\[
\begin{array}{ccc}
\mathbb{P}_{\mathbb{Q}}^{q_1, q_2} & \mapsto & \mathbb{M}_{\mathbb{Q}^{\mathrm{op}}}(q_1, q_2) \\
\mathbb{P}_{\mathbb{Q}}^{\bot, q} & \mapsto & \mathbb{M}_{\mathbb{Q}^{\mathrm{op}}}(q, \bot) \\
\mathbb{P}_{\mathbb{Q}}^{\top, q} & \mapsto & \mathbb{M}_{\mathbb{Q}^{\mathrm{op}}}(q, \top) \\
\mathbb{M}_{\mathbb{Q}^{\mathrm{op}}}(q_1) & \mapsto & \mathbb{G}_{\mathbb{Q}^{\mathrm{op}}}(\{q_1\})
\end{array}
\]

Finally, for any relation \( S \subseteq \mathbb{Q} \times \mathbb{Q} \) we have:

\[
c2s_{\mathbb{Q}}(\mathbb{GC}_S(\mathbb{Q})) = \{ q \in \mathbb{Q} : \forall (q_1, q_2) \in S, (q_1 \leq q \iff q_2 \leq q) \}
\]

2. For any \( q_1, q_2 \in \mathbb{Q} \), the isomorphism \( s2c_{\mathbb{Q}^{\mathrm{op}}} \) acts as follows:

\[
\begin{array}{ccc}
\mathbb{M}_{\mathbb{Q}}(q_1, q_2) & \mapsto & \mathbb{P}_{\mathbb{Q}^{\mathrm{op}}}(q_1, q_2) \\
\mathbb{M}_{\mathbb{Q}}(q_1, \bot) & \mapsto & \mathbb{P}_{\mathbb{Q}^{\mathrm{op}}}(q_1, \bot) \\
\mathbb{M}_{\mathbb{Q}}(q_1, \top) & \mapsto & \mathbb{P}_{\mathbb{Q}^{\mathrm{op}}}(q_1, \top) \\
\mathbb{G}_{\mathbb{Q}}(\{q_1\}) & \mapsto & \mathbb{M}_{\mathbb{Q}^{\mathrm{op}}}(q_1, q_2)
\end{array}
\]

Finally, for any subset \( X \subseteq \mathbb{Q} \) we have:

\[
s2c_{\mathbb{Q}^{\mathrm{op}}}(\mathbb{G}_{\mathbb{Q}}(X)) = \{(q_1', q_2') \in \mathbb{Q} \times \mathbb{Q} : \forall x \in X (x \leq q_1' \iff x \leq q_2') \} = \mathbb{U}\{K((\uparrow_{\mathbb{Q}} V_{\mathbb{Q}} A) \cap \uparrow_{\mathbb{Q}} X \setminus A) : A \subseteq X\}
\]

recalling that \( K(Z) := Z \times Z \).
3. Concerning irreducible elements,

\[
J(\text{CON}(Q)) = \{PC^1_q : q_1 \neq q_2 \in Q\} \quad \text{and} \quad M(\text{CON}(Q)) = \{MC^1_q : q \in Q\setminus \{q_0\}\}
\]

\[
J(\text{SUB}(Q)) = \{GS_q(q) : q \in Q\setminus \{q_0\}\} \quad \text{and} \quad M(\text{SUB}(Q)) = \{MS_q(q_1, q_2) : q_1 \neq q_2 \in Q\}
\]

and consequently:

\[
|J(\text{CON}(Q))| = |M(\text{SUB}(Q))| = \frac{1}{2} |Q| \cdot (|Q|-1) \quad \text{and} \quad |M(\text{CON}(Q))| = |J(\text{SUB}(Q))| = |Q|-1
\]

Proof.

1. Fixing any \(q_1, q_2 \in Q\), we are going to establish that \(c2Q(\text{PC}^1_q) = MS_{q_0}(q_1, q_2)\). Observe that:

\[
MS_{q_0}(q_1, q_2) = \{q \in Q : q_1 \leq Q q \Leftrightarrow q_2 \leq Q q\}
\]

using Definition 3.1.1 and the fact that the ordering is reversed. Let us first verify that \(S := MS_{q_0}(q_1, q_2)\) defines a sub join-semilattice of \(Q^\text{op}\). Certainly \(\bot_{Q^\text{op}} = q_0 \in S\), so given any \(s_1, s_2 \in S\) we need to show that \(s_1 \land_{Q^\text{op}} s_2 \in S\). To this end, define the predicates \(\phi(s) := (q_1 \leq Q s \land q_2 \leq Q s)\) and \(\psi(s) := (q_1 \not\leq Q s \land q_2 \not\leq Q s)\), and proceed case-by-case:

(a) if \(\phi(s_1) \land \phi(s_2)\) then \(\phi(s_1 \land_{Q_{Q^\text{op}}} s_2)\),

(b) if \(\phi(s_1) \land \psi(s_2)\) then \(\psi(s_1 \land_{Q_{Q^\text{op}}} s_2)\) else we obtain at least one of the contradictions \(q_1, q_2 \leq Q s_2\),

(c) finally if \(\psi(s_1) \land \psi(s_2)\) then \(\psi(s_1 \land_{Q_{Q^\text{op}}} s_2)\) lest we obtain contradictions \(q_i \leq Q s_j\).

and we are done. Now, we are going to establish that:

\[
\mathcal{R}(q_1, q_2) \iff c2Q(\mathcal{R}) \leq MS_{q_0}(q_1, q_2) \quad \text{for any } Q\text{-congruence } \mathcal{R}.
\]

This suffices because the principal \(Q\)-congruence generated by \((q_1, q_2)\) is characterised by the property that \(\text{PC}^1_q \subseteq \mathcal{R} \iff \mathcal{R}(q_1, q_2)\), so via the order-isomorphism we’d deduce that \(c2Q(\text{PC}^1_q) = MS_{q_0}(q_1, q_2)\). Using the definition of \(c2Q\) and \(MS_{q_0}(q_1, q_2)\), the desired equivalence can be rewritten as follows:

\[
\mathcal{R}(q_1, q_2) \iff \forall q \in Q (q_1 \leq Q \sqrt{q} \iff q_2 \leq Q \sqrt{q})
\]

and we also recall that \(\sqrt{q} \in \sqrt{Q}\) for every \(Q\)-congruence \(\mathcal{R}\) and every element \(q \in Q\).

(a) \((\Rightarrow)\) Assume \(\mathcal{R}(q_1, q_2)\). Recalling the join-semilattice morphism \((\sqrt{\cdot}_Q) : (Q\text{-})^\text{op} \to Q^\text{op}\) from Theorem 3.1.2, its monotonicity informs us that:

\[
(\ast) \quad \forall q_a, q_b \in Q. (\sqrt{q}_a \leq Q \sqrt{q}_b \Rightarrow \sqrt{q}_a \leq Q \sqrt{q}_b)
\]

also using the description of its action from the proof of Theorem 3.1.2. If we assume that \(q_1 \leq Q \sqrt{q}_R\) for any fixed \(q \in Q\), then we have \(\sqrt{q}_1 \leq Q \sqrt{q}_R\) \(\sqrt{V_Q[\sqrt{q}_R]} = \sqrt{q}_R\) via the monotonicity of \(\sqrt{\cdot}_R\), and consequently \(V_Q[q_1] \leq Q V_Q[q]_R\) by applying \((\ast)\). Thus we have:

\[
q_2 \leq \leq Q V_Q[q_2]_R = V_Q[q_1]_R \quad \text{since } \mathcal{R}(q_1, q_2)
\]

\[
\leq Q V_Q[q]_R \quad \text{by above}
\]

Via the symmetric argument when assuming \(q_2 \leq Q \sqrt{q}_R\), we are done.

(b) \((\Leftarrow)\) Conversely, assume that \(q_1 \leq Q V_Q[q]_R \iff q_2 \leq Q \sqrt{q}_R\) for every \(q \in Q\). Then the two particular cases where \(q := q_1\) and \(q := q_2\) yield:

\[
q_1 \leq Q \sqrt{q_2} \quad \text{and} \quad q_2 \leq Q \sqrt{q_1}.
\]

By the monotonicity of \(\sqrt{\cdot}_R\) we deduce \(\sqrt{q}_1 = \sqrt{q}_2\), so that \(\mathcal{R}(q_1, q_2)\) as required.
Having proved the first claim of (1), the next two claims follow because they are instantiations of the first where \( q_2 := \bot_Q \) and \( q_2 := \top_Q \), respectively. Concerning the third claim, we point out that \( MS\_Q(Q_1, \tau_Q) \) necessarily
contains \( \tau_Q \) by well-definedness, and whenever \( q \neq \tau_Q \) then \( q_2 := \tau_Q \sharp Q \). Regarding the fourth claim, let us verify that:

\[
c_{2S_Q}(MC_Q^J) \cong GS_Q \{q\} \quad \text{for every } q \in Q.
\]

Indeed, since \( R := MC_Q^J = K(\downarrow q) \cup K(\uparrow q) \) we deduce that:

(a) If \( q = \tau_Q \) then \( Q \backslash R = [\{\top_R\}] \) and hence by definition \( c_{2S_Q}(R) = \{V_Q[\tau_Q]_R\} = \{\tau_R\} \) as required.

(b) If \( q \neq \tau_Q \) then \( Q \backslash R = [\{q\}]_R \) where \( [q]_R = \downarrow q \) and \( [\tau_Q] \), so that \( c_{2S_Q}(R) = \{q, \tau_R\} \).

As for the fifth and final claim, it follows directly from the first:

\[
c_{2S_Q}(GC_Q(S)) = c_{2S_Q}(V_{CON(Q)}(PC_Q^{q_1q_2}) : S(q_1, q_2))
\]

\[
= \cap\{c_{2S_Q}(PC_Q^{q_1q_2}) : S(q_1, q_2)\} \quad \text{apply order-isomorphism}
\]

\[
= \cap\{\{q \in Q : q_1 \leq q \leftrightarrow q_2 \leq q\}: S(q_1, q_2)\} \quad \text{by first claim}
\]

\[
= \{q \in Q : \forall(q_1, q_2) \in S.q_1 \leq q \leftrightarrow q_2 \leq q\}
\]

Here we have used the fact that every \( Q \)-congruence \( R \) is the \( CON(Q) \)-join of the principal \( Q \)-congruences it contains. This follows because whenever \( R(q_1, q_2) \) we necessarily have \( PC_Q^{q_1q_2} \subseteq R \) by definition of principal congruences.

2. The second statement mirrors the first, and is mostly directly deducible from it by virtue of the isomorphisms at hand. However, additional information is provided by describing e.g. the principal \( Q^{op} \)-congruences explicitly.

On the other hand, all of these descriptions can be readily verified by directly unwinding the definitions. The final claim follows because:

\[
s_{2S_Q}(GS_Q(X)) = s_{2S_Q}(V_{SUB(Q)}(GS_Q(\{x\}) : x \in X))
\]

\[
= \cap\{c_{2S_Q}(GS_Q(\{x\})) : x \in X\}
\]

\[
= \{x \in X \in MC_Q^{op} : x \in X\} \quad \text{by first claim}
\]

\[
= \{x \in X \in K(\top_Q x) \cup K(\downarrow Q x) : x \in X\} \quad \text{by set-theoretic distributivity}
\]

\[
= \cup\{K(\top_Q V_1 A) \cup K(\downarrow Q X \backslash A) : A \subseteq X\} \quad \text{see below}
\]

Regarding the final equality, observe that:

\[
K(I \cap K(J) = I \times I \cap J \times J = (I \cap J) \times (I \cap J) = K(I \cap J)
\]

and also the general equalities:

\[
\uparrow_Q x_1 \cap \uparrow_Q x_2 = \uparrow_Q (x_1 \lor Q x_2)
\]

\[
\uparrow_Q x_1 \cap \uparrow_Q x_2 = \uparrow_Q x_1 \cup \uparrow_Q x_2 = \uparrow_Q \{x_1, x_2\}.
\]

3. The description of \( J(CON(Q)) \) follows by universal algebra i.e. is a general statement concerning the lattice of congruences of a finite algebra. Likewise, the description of \( J(SUB(Q)) \) follows for the subalgebra lattice of any (possibly infinite) algebra. Then the descriptions of the meet-irreducibles follow via (1) and (2), seeing as \( c_{2S_Q} : (CON(Q))^{op} \rightarrow SUB(Q^{op}) \) is a bounded lattice isomorphism, and hence induces bijections between join/meet-irreducibles.

\[\square\]

4 The category \textbf{Dep}

4.1 Introducing \textbf{Dep} and its self-duality

We describe a category and its self-duality. We’ll prove it is equivalent to \( JSL_\_f \) in the next section. It is based on the work of Moshier and Jipsen [Jip12] (see here). We reuse their notation \((-)^J \) and \((-)^J \), and our category \textbf{Dep} is a variation of Moshier’s category \textbf{Ctx} restricted to finite relations.
**Definition 4.1.1** (The category Dep). Its objects are the relations between finite sets \( G \subseteq \mathcal{G}_s \times \mathcal{G}_t \). Its morphisms \( \mathcal{R}: G \to H \) are those relations \( \mathcal{R} \subseteq G_s \times H_t \) such that the Rel-diagram:

\[
\begin{array}{c}
G_s \xrightarrow{\mathcal{R}} H_s \\
\downarrow \mathcal{R} \quad \downarrow \mathcal{R} \\
G_t \xrightarrow{\mathcal{R}^{-1}} H_t
\end{array}
\]

commutes for some relations \( \mathcal{R}_t \) and \( \mathcal{R}_1^{-1} \). Equivalently, a morphism \( \mathcal{R}: G \to H \) is a relation \( \mathcal{R} \subseteq G_s \times H_t \) which factors through \( G \) (on the left) and \( H \) (on the right).

The identity morphism \( \text{id}_G : G \to G \) is the relation \( G \subseteq G_s \times G_t \). Given \( \mathcal{R}, S: G \to H \) and \( \mathcal{S}: H \to I \), their composite \( \mathcal{R} ; \mathcal{S} \subseteq G_s \times I_t \) is defined by the following commuting Rel-diagram:

\[
\begin{array}{c}
G_s \xrightarrow{\mathcal{R} ; \mathcal{S}} I_s \\
\downarrow \mathcal{R} \quad \downarrow \mathcal{S} \\
G_t \xrightarrow{\mathcal{R} ; \mathcal{S}^{-1}} I_t
\end{array}
\]

e.g. \( \mathcal{R} ; \mathcal{S} \) is the relational composite \( \mathcal{R}_t ; \mathcal{S} \).

**Example 4.1.2** (Dep-morphisms).

1. **Dep-morphisms are closed under converse and union.**
   Given \( \mathcal{R} : G \to H \) then \( \mathcal{R} : H \to G \) by taking the converse of the commutative square, which actually swaps the witnessing relations. We have \( \emptyset : G \to H \) via empty witnessing relations. Given \( \mathcal{R}, \mathcal{S} : G \to H \) then \( \mathcal{R} ; \mathcal{S} : G \to H \) by (i) unioning the respective witnessing relations, (ii) the bilinearity of relational composition w.r.t. union.

2. **Bipartite graph isomorphisms \( \beta : G_1 \to G_2 \) induce Dep-isomorphisms.**
   Suppose we have a bipartite graph isomorphism \( \beta : G_1 \to G_2 \) where each \( G_i = (V_i, \mathcal{E}_i) \), so \( \mathcal{E}_1(x, y) \iff \mathcal{E}_2(\beta(x), \beta(y)) \). Given any bipartition \( (X, Y) \) of \( G_1 \) we obtain a bipartition \( (\beta[X], \beta[Y]) \) of \( G_2 \). Changing notation provides the Dep-morphism below left:

\[
\begin{array}{c}
Y \xrightarrow{\beta[Y]} \beta[Y] \\
\xleftarrow{\mathcal{R}} \mathcal{R} \xrightarrow{\beta[X]} \beta[X] \\
\xleftarrow{\mathcal{S}} \mathcal{S} \xrightarrow{\beta[y]} X
\end{array}
\]

where each \( \mathcal{G}_i := \mathcal{E}_i|_{X \times Y} \). The bijective inverse \( \beta^{-1} = \tilde{\beta} \) provides witnessing relations in the opposite direction i.e. the Dep-morphism \( \mathcal{S} : \mathcal{G}_2 \to \mathcal{G}_1 \) above right. These morphisms are mutually inverse: \( \mathcal{G}_1 \) is Dep-isomorphic to \( \mathcal{G}_2 \).

3. **The canonical quotient poset of a preorder defines a Dep-isomorphism.**
   Let \( \mathcal{G} \subseteq X \times X \) be a transitive and reflexive relation. There is a canonical way to construct a poset \( \mathcal{P} = (X/\mathcal{E}, \subseteq_{\mathcal{P}}) \) via the equivalence relation \( \mathcal{E}(x_1, x_2) \iff \mathcal{G}(x_1, x_2) \wedge \mathcal{G}(x_2, x_1) \), where \( \lceil x \rceil \mathcal{E} \subseteq \lceil x \rceil \mathcal{P} : \iff \mathcal{G}(x_1, x_2) \).
   Consider the Rel-diagram:

\[
\begin{array}{c}
X \xrightarrow{\mathcal{G}} \{ \mathcal{G}[x] : x \in X \} \xrightarrow{\bigcup \lceil \mathcal{P} \rceil \mathcal{E}} \{ \lceil \mathcal{P} \rceil \mathcal{E} \mathcal{G}[x] \} \xrightarrow{\lceil \mathcal{P} \rceil \mathcal{E} \mathcal{G}[x]} X/\mathcal{E} \\
\xleftarrow{\mathcal{G}} \xleftarrow{\mathcal{G}} \xrightarrow{\mathcal{G}} \\
X \xrightarrow{\lambda x, \mathcal{G}[x]} \{ \mathcal{G}[x] : x \in X \} \xrightarrow{\bigcup \lceil \mathcal{P} \rceil \mathcal{E}} \{ \lceil \mathcal{P} \rceil \mathcal{E} \mathcal{G}[x] \} \xrightarrow{\lceil \mathcal{P} \rceil \mathcal{E} \mathcal{G}[x]} X/\mathcal{E}
\end{array}
\]

\[1\] We use the converse relation \( \mathcal{R}_t^{-1} \) to make the self-duality of this category ‘nicer’ later on.
Note that \(\mathcal{G}[x]\) is the ‘upwards closure’ i.e. the union of the upwards closure \(\uparrow_P [x]_\mathcal{E}\), whereas \(\mathcal{G}[x]\) is the ‘downwards closure’ in a similar manner. The left square commutes for completely general reasons, defining the \(\text{Dep}\)-morphism:

\[
\mathcal{R}(x_1, \mathcal{G}[x_2]) : \iff \exists x \in X. [\mathcal{G}(x_1, x) \text{ and } \mathcal{G}(x, x_2)] \iff \mathcal{G}(x_1, x_2).
\]

The right square involves bijections via (i) identifying elements of \(P\) with principal up/downsets, (ii) the disjointness of equivalence classes. It also commutes:

\[
\begin{align*}
\cup \uparrow [x_1]_\mathcal{E} \notin \cup \downarrow [x_2]_\mathcal{E} & \iff \cup \uparrow [x_1]_\mathcal{E} \cap \cup \downarrow [x_2]_\mathcal{E} \neq \emptyset \\
& \iff \exists x \in X. [\mathcal{G}(x_1, x) \leq_P [x]_\mathcal{E} \leq_P [x_2]_\mathcal{E}]
\end{align*}
\]

In fact, \(\mathcal{R} : \mathcal{G} \rightarrow \mathcal{G}\) is an instance of the natural isomorphism \(\text{red}_\mathcal{G}\) from Theorem 4.2.10 further below, and the right square defines a \(\text{Dep}\)-isomorphism by Example 2 above. Thus \(\mathcal{G} \equiv \leq_P\), although whenever \(|X| > |X/\mathcal{E}|\) this isomorphism cannot arise from a bipartite graph isomorphism.

4. **Monotonicity can be characterised by \(\text{Dep}\)-morphisms.**

Given finite posets \(P\) and \(Q\), a function \(f : P \rightarrow Q\) is monotonic iff the following \(\text{Rel}\)-diagram commutes:

\[
\begin{array}{ccc}
P & \xrightarrow{f} & Q \\
\downarrow{\leq_P} & & \downarrow{\leq_Q} \\
P & \xrightarrow{f} & Q
\end{array}
\]

as the reader may verify. Actually, \(f\) is monotonic iff \(\leq_Q \circ f \leq_P\) is a \(\text{Dep}\)-morphism.

5. **Biclique edge-coverings amount to \(\text{Dep}\)-monos.**

Generally speaking, \(\text{Dep}\)-morphisms represent two edge-coverings of a bipartitioned graph. A single edge-covering amounts to a \(\text{Dep}\)-mono of a special kind:

\[
\begin{array}{ccc}
\mathcal{G}_t & \xrightarrow{\Delta_{\mathcal{G}_t}} & \mathcal{G}_t \\
\downarrow{\mathcal{G}} & & \downarrow{\mathcal{H}} \\
\mathcal{G}_s & \xrightarrow{\Delta_{\mathcal{G}_s}} & \mathcal{H}_s
\end{array}
\]

i.e. morphisms \(\mathcal{G} : \mathcal{G} \rightarrow \mathcal{H}\) where additionally \(\mathcal{G}_t = \mathcal{H}_t\). Later we’ll see that any mono \(\mathcal{R} : \mathcal{G} \rightarrow \mathcal{I}\) induces such a \(\mathcal{G} : \mathcal{G} \rightarrow \mathcal{H}\) where \(|\mathcal{H}_s| \leq |\mathcal{I}_s|\) and \(|\mathcal{H}_t| \leq |\mathcal{I}_t|\).

6. **Biclique edge-coverings amount to \(\text{Dep}\)-epis.**

Analogous to the previous example, a single edge-covering can be represented as a \(\text{Dep}\)-epi \(\mathcal{G} : \mathcal{H} \rightarrow \mathcal{G}\) where \(\mathcal{G}_s = \mathcal{H}_s\). This will follow from self-duality i.e. epis are precisely the converses of monos.

\[\text{Lemma 4.1.3.} \text{ Dep is a well-defined category}\]

**Proof.** \(\text{id}_\mathcal{G} : \mathcal{G} \rightarrow \mathcal{G}\) is well-defined via witnesses \(\Delta_{\mathcal{G}_t} : \mathcal{G} = \mathcal{G} ; \Delta_{\mathcal{G}_s} : \mathcal{G} = \mathcal{G}\. Each composite \(\mathcal{R} ; \mathcal{S}\) is well-defined via the composite witnesses, see the diagram in Definition 4.1.1. Composites are independent of the witnesses of their components since \(\mathcal{R} ; \mathcal{S} = \mathcal{R} ; \mathcal{S} = \mathcal{R} ; \mathcal{S}\). Composition is associative because the respective composition of \(\text{Rel}\)-diagrams is associative. Finally, given \(\mathcal{R} : \mathcal{G} \rightarrow \mathcal{H}\) then \(\text{id}_\mathcal{G} ; \mathcal{R} = \Delta_{\mathcal{G}_t} ; \mathcal{R} = \mathcal{R} \) and similarly \(\mathcal{R} ; \text{id}_\mathcal{H} = \mathcal{R} ; \Delta_{\mathcal{H}_t} = \mathcal{H}\). \(\square\)

We’ll introduce further notation and auxiliary results. In particular, we’ll prove that each \(\text{Dep}\)-morphism \(\mathcal{R}\) has canonical inclusion-maximum witnesses.

**Definition 4.1.4 ((-)^t and (-)^s).** Given any relation \(\mathcal{R} \subseteq \mathcal{R}_s \times \mathcal{R}_t\) between finite sets we define two functions:

\[
\begin{align*}
\mathcal{R}^t : P\mathcal{R}_s & \rightarrow P\mathcal{R}_t \\
\mathcal{R}^t(X) & := P\mathcal{R}_t \cap \{x \in \mathcal{R}_s : \mathcal{R}[x] \subseteq Y\}
\end{align*}
\]
Then $R^!$ is called the $R$-image function whereas $R^\|$ is called the $R$-preimage function. They induce a closure operator and an interior operator (co-closure operator) as follows:

\[
\begin{align*}
\text{cl}_R & := R^! \circ R^\| : (P R_s, \subseteq) \to (P R_s, \subseteq) \\
\text{in}_R & := R^\| \circ R^! : (P R_t, \subseteq) \to (P R_t, \subseteq)
\end{align*}
\]

See Definition 2.2.4 for background.

**Note 4.1.5.** $R^\|$ is called the $R$-preimage function because it generalises the usual preimage function of a function. That is, given any function (= functional relation) $f : X \to Y$ then $f^!(B) := \{ x \in X : f[x] \subseteq B \} = \{ x \in X : f(x) \subseteq B \}$.

**Note 4.1.6.** The operators $(-)^!$ and $(-)^\|$ faithfully represent relational composition as functional composition.

1. $(-)^!$ defines an equivalence functor (in fact, isomorphism) from the category of finite sets and relations $\text{Rel}_f$ to the full subcategory of $\text{JSL}_f$ with objects $PX = (PX, \cup, \emptyset)$ where $X$ is a finite set.

2. $(-)^\|$ defines an equivalence functor (in fact, isomorphism) from $\text{Rel}_f^{op}$ to the full subcategory of $\text{JSL}_f$ with objects $(PX)^{op} = (PX, \cap, X)$ where $X$ is a finite set.

**Lemma 4.1.7** (Relating $(-)^!$ and $(-)^\$).

Let $G, H$ be relations between finite sets, $R \subseteq G_s \times H_t$, $S \subseteq H_s \times I_t$ any relations and $X$ any finite set.

1. We have the adjoint relationship:

\[
(\uparrow \downarrow) \quad R^!(X) \subseteq Y \iff X \subseteq R^\|(Y) \quad \text{for all subsets } X \subseteq G_s, Y \subseteq H_t
\]

hence they actually define adjoint $\text{JSL}_f$-morphisms:

\[
\begin{align*}
R^! & : (P G_s, \cup, \emptyset) \to (P H_t, \cup, \emptyset) \\
R^\| & : (P H_t, \cap, H_t) \to (P G_s, \cap, G_s)
\end{align*}
\]

2. $\text{cl}_R$ is a well-defined closure operator and $\text{in}_R$ is a well-defined interior operator.

3. The following labelled equalities hold:

\[
\begin{align*}
(\uparrow \Delta) & \quad \Delta^!_X = id_{P X} & \Delta^\|_X = id_{P X} & \quad (\downarrow \Delta) \\
(\uparrow \circ) & \quad (R; S)^! = S^! \circ R^! & \quad (R; S)^\| = R^\| \circ S^\| & \quad (\downarrow \circ) \\
(\uparrow \uparrow) & \quad R^! \circ R^! \circ R^! = R^! & \quad R^\| \circ R^\| \circ R^\| = R^\| & \quad (\downarrow \downarrow) \\
(\neg \uparrow \neg) & \quad \neg G_s \circ R^! \circ \neg G_s = R^\| & \quad \neg G_s \circ R^\| \circ \neg G_s = R^! & \quad (\neg \downarrow \neg)
\end{align*}
\]

The rules $(\neg \uparrow \neg)$ and $(\neg \downarrow \neg)$ are referred to as ‘De Morgan dualities’.

4. We have two sets of four equivalent statements:

\[
\begin{array}{c|c}
\text{equivalent statements} & \text{equivalent statements} \\
\hline
R^! = R^\| \circ \text{cl}_G & R^\| = \text{in}_H \circ R^! \\
R^\| = \text{cl}_G \circ R^! & R^! = \text{in}_H \circ R^\|
\end{array}
\]

**Proof:**

1. $R^!(X) \subseteq Y \iff R[X] \subseteq Y \iff \forall x \in X. R[x] \subseteq Y \iff X \subseteq R^\|(Y)$ establishes the adjunction. Thus $R^!$ preserves all colimits = joins in $(P G_s, \subseteq) =$ unions, and also $R^\|$ preserves all limits = meets in $(P H_t, \subseteq) =$ intersections.

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2. First observe that \( \mathcal{R}^! \) defines a monotone function on \((\mathcal{P}\mathcal{R}_s, \subseteq)\), and \( \mathcal{R}^\dagger \) defines a monotone function on \((\mathcal{P}\mathcal{R}_t, \subseteq)\). Then this follows from (1) via Lemma 2.2.7.

3. Regarding the topmost rules:
   \[ \Delta_X^! = \lambda A \subseteq X. \Delta_X[A] = id_{\mathcal{P}X} = \lambda A \subseteq X. \{ x \in X : \{ x \} \subseteq A \} = \Delta_X^\dagger \]

Next we prove \((\uparrow \circ)\) and \((\downarrow \uparrow)\):
   \[ (\mathcal{R}; \mathcal{S})^!(X) = (\mathcal{R}; \mathcal{S})[X] = (\mathcal{S}[\mathcal{R}[X]]) = \mathcal{S}^! \circ \mathcal{R}^!(X) \]
   \[ \mathcal{R}^! \circ \mathcal{R}^\dagger \circ \mathcal{R}^!(X) = \mathcal{R} \circ \mathcal{R}^! \circ \mathcal{R}^!(X) \quad = \mathcal{R}[\mathcal{R} \circ \mathcal{R}^! \circ \mathcal{R}^!(X)] = \mathcal{R}[X] = \mathcal{R}^!(X) \]

and now \((\downarrow \circ)\) and \((\downarrow \uparrow)\):
   \[ (\mathcal{R}; \mathcal{S})^!(Z) = \{ x \in X : \mathcal{R}; \mathcal{S}[x] \subseteq Z \} \]
   \[ = \{ x \in X : \mathcal{S}[\mathcal{R}[x]] \subseteq Z \} \]
   \[ = \{ x \in X : \mathcal{R}[x] \subseteq \mathcal{S}^!(Z) \} \]

by adjoint relationship
   \[ \mathcal{R}^! \circ \mathcal{R}^\dagger \circ \mathcal{R}^!(Y) = \mathcal{R} \circ \mathcal{R}^! \circ \mathcal{R}^!(Y) \]
   \[ = \mathcal{R} \circ \mathcal{R}^!(\mathcal{R} \circ \mathcal{R}^! \circ \mathcal{R}^!(Y)) \]
   \[ = \{ x \in X : \mathcal{R}[x] \subseteq \{ y \in Y : \exists x \in X. y \in \mathcal{R}[x] \subseteq Y \} \} \]
   \[ = \{ x \in X : \mathcal{R}[x] \subseteq Y \} \]
   \[ = \mathcal{R}^!(Y) \]

Finally we prove the ‘De Morgan dualities’. Firstly, \((\neg \uparrow \neg)\) holds because:
   \[ \neg g_r \circ \mathcal{R}^! \circ \neg g_r(X) = \mathcal{R}[X] \]
   \[ = \{ h_t \in \mathcal{H}_t : \neg \exists g_s \in \mathcal{X} \mathcal{R}(g_s, h_t) \} \]
   \[ = \{ h_t \in \mathcal{H}_t : \neg \exists g_s \in \mathcal{X} \mathcal{R}(h_t, g_s) \} \]
   \[ = \{ h_t \in \mathcal{H}_t : \neg \exists g_s \in \mathcal{X} \mathcal{R}(h_t) \subseteq X \} \]
   \[ = \mathcal{R}^!(X) \]

and \((\neg \downarrow \neg)\) follows by setting \( \mathcal{R} := \mathcal{R} \) and cancelling involutions.

4. If the left-hand set of four statements are equivalent, then so are the right-hand set of four statements. This follows by substituting \( \mathcal{R} \mapsto \mathcal{R} \). Also, in the left-hand statements, the last two follow from the first two by applying De Morgan duality. Then it suffices to prove that the first two statements on the left are equivalent. The pointwise-inclusion-orderings \( \mathcal{R}^! \leq \mathcal{C}_G \circ \mathcal{R}^! \) and \( \mathcal{R}^\dagger \leq \mathcal{R}^! \circ \mathcal{C}_G \) always hold because \( \mathcal{C}_G \) is extensive and both \( \mathcal{R}^! \) and \( \mathcal{R}^\dagger \) are monotone. We must prove that:
   \[ \mathcal{G}^! \circ \mathcal{G}^\dagger \circ \mathcal{R}^! \leq \mathcal{R}^! \quad \iff \quad \mathcal{R}^! \circ \mathcal{G}^! \circ \mathcal{G}^\dagger \leq \mathcal{R}^! \]

(a) \((\Rightarrow)\) Applying the adjoint relationship yields \( \mathcal{R}^! \circ \mathcal{G}^! \circ \mathcal{G}^\dagger \leq id_{\mathcal{P}H_s}, \) so precomposing with the monotone function \( \mathcal{R}^! \) yields \( \mathcal{R}^! \circ \mathcal{G}^! \circ \mathcal{G}^\dagger \circ \mathcal{R}^! \circ \mathcal{R}^! \leq \mathcal{R}^!, \) finally observe that:
   \[ \mathcal{R}^! \circ \mathcal{G}^! \circ \mathcal{G}^\dagger \leq \mathcal{R}^! \circ \mathcal{G}^! \circ \mathcal{G}^\dagger \circ \mathcal{R}^! \circ \mathcal{R}^! \leq \mathcal{R}^! \]

because \( \mathcal{R}^! \circ \mathcal{R}^! \) is a closure operator by (2) and hence extensive, and \( \mathcal{R}^! \circ \mathcal{G}^! \circ \mathcal{G}^\dagger \) is monotone.

(b) \((\Leftarrow)\) Applying the adjoint relationship yields \( \mathcal{C}_G \leq \mathcal{R}^! \circ \mathcal{R}^! \), so precomposing with \( \mathcal{R}^! \) we obtain:
   \[ \mathcal{G}^! \circ \mathcal{G}^\dagger \circ \mathcal{R}^! \leq \mathcal{R}^! \circ \mathcal{R}^! \circ \mathcal{R}^! = \mathcal{R}^! \]

where the final equality is by \((\uparrow \uparrow)\).

\(\Box\)

We are now ready to formalise the canonical maximum witnesses of \( \text{Dep-morphisms} \). We’ll also prove an important functional characterisation of \( \text{Dep-morphisms} \).
Definition 4.1.8. The component relations of a Dep-morphism \( \mathcal{R} : \mathcal{G} \to \mathcal{H} \) are defined:

\[
\mathcal{R}_- := \{(g_s, h_s) \in \mathcal{G}_s \times \mathcal{H}_s : h_s \in \mathcal{H}^i(\mathcal{R}[g_s])\}
\]

\[
\mathcal{R}_+ := \{(h_t, g_t) \in \mathcal{H}_t \times \mathcal{G}_t : g_t \in \mathcal{G}^i(\mathcal{R}[h_t])\}
\]

Example 4.1.9 (Component relations of \( id_{\mathcal{S}_p} \)). Given a finite poset \( P \) we compute the component relations of the identity-morphism \( id_{\mathcal{S}_p} \). Firstly:

\[
\leq_p (\leq_p[p_s]) = (\leq_p \circ P)(p_s) = \{p \in P : \leq_p p \leq p \leq_p p_s\} = \leq_p[p_s]
\]

\[
\leq_p[(\leq_p[p_s])] = \leq_p(\leq_p \circ P)(p_s) = \leq_p[p_s] = \leq_p[p_t]
\]

where (*) follows from the 1st line. Consequently \( \leq_p[-] = \leq_p \) and \( \leq_p[-] = \leq_p \). These are witnesses because:

\[
\leq_p ; \leq_p = \leq_p = \leq_p ; \leq_p = \leq_p
\]

by reflexivity and transitivity. Concerning maximality, if \( \mathcal{R}_t \in P \times P \) satisfies \( \mathcal{R}_t ; \leq_p = \leq_p \) then \( \mathcal{R}_t \in \leq_p \) because if \( \mathcal{R}_t(p_1, p_2) \) then \( p_1 \leq_p p_2 \) by reflexivity of order-relations. Similarly if \( \leq_p = \leq_p ; \mathcal{R}_r \) then \( \mathcal{R}_r \in \leq_p \).

Lemma 4.1.10 (Morphism characterisation and maximum witnesses).

1. A relation \( \mathcal{R} \subseteq \mathcal{G}_s \times \mathcal{H}_t \) defines a Dep-morphism \( \mathcal{G} \to \mathcal{H} \) iff

\[
\mathcal{R}^\dagger \circ \mathcal{G} = \mathcal{R}^\dagger = \mathcal{H} \circ \mathcal{R}^\dagger,
\]

or equivalently \( \mathcal{R}^\dagger \circ \mathcal{G} = \mathcal{R}^\dagger = \mathcal{H} \circ \mathcal{R}^\dagger \).

2. Each Dep-morphism \( \mathcal{R} : \mathcal{G} \to \mathcal{H} \) has the maximum witness \( (\mathcal{R}_-, \mathcal{R}_+) \) i.e.

(a) \( \mathcal{R}_- ; \mathcal{H} = \mathcal{R} = \mathcal{G} ; \mathcal{R}_+ \), and

(b) for any \( (\mathcal{R}_t, \mathcal{R}_r) \) such that \( \mathcal{R}_t ; \mathcal{H} = \mathcal{R} = \mathcal{G} ; \mathcal{R}_r \) we have both \( \mathcal{R}_t \in \mathcal{R}_- \) and \( \mathcal{R}_r \in \mathcal{R}_+ \).

Proof.

i. Let us prove half of the first statement. Assuming that \( \mathcal{R} : \mathcal{G} \to \mathcal{H} \) is a Dep-morphism then we have some witnessing relations \( (\mathcal{R}_t, \mathcal{R}_r) \) such that \( \mathcal{R}_t ; \mathcal{H} = \mathcal{R} = \mathcal{G} ; \mathcal{R}_r \). Consequently:

\[
\mathcal{R}^\dagger \circ \mathcal{G} = (\mathcal{G} ; \mathcal{R}_+) \circ \mathcal{G} = \mathcal{G} \circ \mathcal{G} \quad \text{by assumption and definition}
\]

\[
\mathcal{R}^\dagger = (\mathcal{G} ; \mathcal{R}_+) \circ \mathcal{G}^\dagger = (\mathcal{G} \circ \mathcal{G}) \circ \mathcal{G}^\dagger = \mathcal{G} \circ \mathcal{G}^\dagger \quad \text{by Lemma 4.1.7}(\dagger \circ)
\]

\[
\mathcal{R}^\dagger = (\mathcal{R}_+ \circ \mathcal{G}) \circ \mathcal{G}^\dagger \quad \text{by assumption}
\]

\[
\mathcal{H} \circ \mathcal{R}^\dagger = \mathcal{H} \circ \mathcal{H}^\dagger \circ \mathcal{R}_- \quad \text{by assumption and definition}
\]

\[
\mathcal{H} \circ \mathcal{H}^\dagger \circ \mathcal{R}_- = \mathcal{H} \circ \mathcal{H}^\dagger \circ \mathcal{H}_- \quad \text{by Lemma 4.1.7}(\dagger \circ)
\]

\[
\mathcal{H} \circ \mathcal{H}^\dagger \circ \mathcal{R}_- \quad \text{by Lemma 4.1.7}(\dagger \circ)
\]

\[
\mathcal{H} \circ \mathcal{R}^\dagger \quad \text{by assumption}
\]

ii. Before proving the other half of the first statement, let us first prove the second statement i.e. we again assume \( \mathcal{R} : \mathcal{G} \to \mathcal{H} \) is a Dep-morphism and now know that \( \mathcal{R}^\dagger \circ \mathcal{G} = \mathcal{R}^\dagger = \mathcal{H} \circ \mathcal{R}^\dagger \). We first show that the 'associated
component relations’ \((\mathcal{R}_-, \mathcal{R}_+)\) witness the fact that \(\mathcal{R}\) is a morphism.

\[
\begin{align*}
\mathcal{R}_-; \mathcal{H}(g_s, h_t) & \iff \exists h_s \in \mathcal{H}_s, \exists h_t \in \mathcal{H}_t(\mathcal{R}[g_s]) \text{ and } \mathcal{H}(h_s, h_t) & \text{ by definition of } \mathcal{R}_- \\
& \iff h_t \in \mathcal{H}^1 \circ \mathcal{R}^1(\{g_s\}) & \text{ by definition of } \mathcal{H}^1 \\
& \iff h_t \in \mathcal{R}^1(\{g_s\}) & \text{ since } \mathcal{R}^1 = \text{in}_\mathcal{H} \circ \mathcal{R}^1 \\
& \iff \mathcal{R}(g_s, h_t) & \\
\mathcal{G}; \mathcal{R}_+(g_s, h_t) & \iff \exists g_t \in \mathcal{G}_t, \exists g_t \in \mathcal{G}_t(\mathcal{R}[h_t]) & \text{ by definition of } \mathcal{R}_+ \\
& \iff \exists g_t \in \mathcal{G}_t, \exists g_t \in \mathcal{G}_t(\mathcal{R}[h_t]) & \text{ by definition of } \mathcal{G}_t \\
& \iff g_s \in \mathcal{R}^1(\{h_t\}) & \text{ since } \mathcal{R}^1 = \text{in}_\mathcal{G} \circ \mathcal{R}^1 \\
& \iff g_s \in \mathcal{R}^1(\{h_t\}) \\
& \iff \mathcal{R}(g_s, h_t)
\end{align*}
\]

The penultimate equivalence follows because we know that \(\mathcal{R}^1 \circ \text{cl}_\mathcal{G} = \mathcal{R}^1\) and may apply Lemma 4.1.7.4. To show that \((\mathcal{R}_-, \mathcal{R}_+)\) is maximum, take any other witnesses \(\mathcal{R}_t; \mathcal{H} = \mathcal{R} = \mathcal{G}; \mathcal{R}_+').\ Then:

\[
\begin{align*}
\mathcal{R}_t(g_s, h_t) & \iff \forall h_t \in \mathcal{H}_t, \exists h_s \in \mathcal{R}[g_s] & \text{ since } \mathcal{R}_t; \mathcal{H} = \mathcal{R} \\
& \iff \mathcal{H}(h_t, g_s) \in \mathcal{R}[g_s] & \text{ by definition of } \mathcal{H}_t \\
& \iff h_s \in \mathcal{R}_t(\mathcal{R}[g_s]) & \text{ by definition of } \mathcal{R}_t \\
& \iff \mathcal{R}_- (g_s, h_s) & \text{ by definition of } \mathcal{R}_-
\end{align*}
\]

\[
\begin{align*}
\mathcal{R}_+(h_t, g_t) & \iff \mathcal{R}_+(g_t, h_t) & \text{ by definition of } \mathcal{R}_+ \\
& \iff \forall g_t \in \mathcal{G}_t, \exists g_t \in \mathcal{G}_t(\mathcal{R}[h_t]) & \text{ since } \mathcal{R} = \mathcal{G}; \mathcal{R}_+ \\
& \iff \mathcal{G}(h_t, g_t) \in \mathcal{G}[g_t] & \text{ by definition of } \mathcal{G}_t \\
& \iff \mathcal{G}[g_t] \in \mathcal{R}[h_t] & \text{ by definition of } \mathcal{G}_t \\
& \iff g_t \in \mathcal{R}_+(\mathcal{R}[h_t]) & \text{ by definition of } \mathcal{R}_+ \\
& \iff \mathcal{R}_+(h_t, g_t) & \text{ by definition of } \mathcal{R}_+
\end{align*}
\]

iii. Let us prove the remaining part of the first statement:

given a relation \(\mathcal{R} \subseteq \mathcal{G}_s \times \mathcal{H}_t\) such that \(\mathcal{R}^1 \circ \text{cl}_\mathcal{G} = \mathcal{R}^1 = \text{in}_\mathcal{H} \circ \mathcal{R}^1\) we must establish that \(\mathcal{R}\) defines a \(\text{Dep}\)-morphism of type \(\mathcal{G} \to \mathcal{H}\).

Even though we don’t yet know that \(\mathcal{R}\) is a \(\text{Dep}\)-morphism, we can apply Definition 4.1.8 to obtain the two relations \((\mathcal{R}_-, \mathcal{R}_+).\) Then we can reuse the first proof in (ii) above to deduce that \(\mathcal{R} = \mathcal{R}_-; \mathcal{H}\). Furthermore we can also prove that \(\mathcal{R} = \mathcal{G}; \mathcal{R}_+\) because the assumption \(\mathcal{R}^1 \circ \text{cl}_\mathcal{G} = \mathcal{R}^1\) implies that \(\mathcal{R}^1 = \text{in}_\mathcal{G} \circ \mathcal{R}^1\) by Lemma 4.1.7.4.

iv. Finally, the first statement can be weakened to \(\mathcal{R}^1 \circ \text{cl}_\mathcal{G} = \text{in}_\mathcal{H} \circ \mathcal{R}^1\) because this already implies both composites are equal to \(\mathcal{R}^1\). Indeed, since \(\text{cl}_\mathcal{G}\) is a closure operator and \(\text{in}_\mathcal{G}\) is an interior operator,

\[
\text{in}_\mathcal{H} \circ \mathcal{R}^1 \subseteq \mathcal{R}^1 \subseteq \mathcal{R}^1 \circ \text{cl}_\mathcal{G}
\]

so we can replace the inclusions by equalities.

\[
\square
\]

Here is yet another useful result.

**Lemma 4.1.11** (Computing composites). For any \(\text{Dep}\)-morphisms \(\mathcal{R} : \mathcal{G} \to \mathcal{H}\) and \(S : \mathcal{H} \to \mathcal{I},\)

\[
\begin{align*}
(\uparrow \downarrow) & \quad (\mathcal{R} \downarrow \mathcal{S})^1 = S^1 \circ \mathcal{H}^1 \circ \mathcal{R}^1 \\
(\downarrow \uparrow) & \quad (\mathcal{R} \downarrow \mathcal{S})^1 = \mathcal{R}^1 \circ \mathcal{H}^1 \circ S^1.
\end{align*}
\]

Finally, \(\mathcal{R}_-; S = \mathcal{R}_-; S_+; \mathcal{I} = \mathcal{R}_+; S = \mathcal{G}; \mathcal{R}_+; S_+ = \mathcal{R}; S_+; S_+\).

**Proof.** Recalling that \(\mathcal{R}\) has canonical witnesses \((\mathcal{R}_-, \mathcal{R}_+)\), let us prove \((\uparrow \downarrow)\).

\[
(\mathcal{R} \downarrow \mathcal{S})^1 = (\mathcal{R}_-; \mathcal{S})^1 & \quad \text{by definition} \\
= S^1 \circ \mathcal{R}^1 & \quad \text{by } (\uparrow \circ) \\
= S^1 \circ \mathcal{H}^1 \circ \mathcal{H}^1 \circ \mathcal{R}^1 & \quad \text{by Lemma 4.1.10} \\
= \mathcal{R}^1 \circ \mathcal{H}^1 \circ (\mathcal{R}_-; \mathcal{H})^1 & \quad \text{by } (\uparrow \circ) \\
= \mathcal{R}^1 \circ \mathcal{R}^1 & \quad \text{since } \mathcal{R} = \mathcal{R}_-; \mathcal{H}
\]

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We infer (↓ ↓) because (↑ ↓) is an equality of JSL-f-morphisms, so we can take adjoints, flipping the composition and also the direction of the arrows ↑ and ↓. The final claim follows by the definition of Dep-composition.

**Definition 4.1.12.** The self-duality \((-)^{\vee} : \text{Dep}^{\text{op}} \to \text{Dep}\) takes the converse of both objects and morphisms, and moreover flips the component relations.

\[
G^{\vee} := \tilde{G} \quad \begin{array}{c}
\mathcal{R} : \mathcal{G} \to \mathcal{H} \\
(\mathcal{R}^{\text{op}})^{\vee} := \mathcal{R} : \mathcal{H} \to \mathcal{G}
\end{array}
\]

\[
(\mathcal{R}^{\vee})_- = \mathcal{R}_+ \\
(\mathcal{R}^{\vee})_+ = \mathcal{R}_-
\]

**Theorem 4.1.13** (Self-duality of Dep). \((-)^{\vee} : \text{Dep}^{\text{op}} \to \text{Dep}\) is a well-defined equivalence functor with respective natural isomorphism:

\[
\alpha : \text{id}_{\text{Dep}} \Rightarrow (\dashv^\vee)^\text{op} \circ ((\dashv)^\vee) \quad \alpha_G := \text{id}_G = G
\]

**Proof.** \((-)^{\vee}\)'s action on objects is certainly well-defined. Regarding its action on morphisms, given a morphism \(\mathcal{R} : \mathcal{G} \to \mathcal{H}\) we have a relation \(\mathcal{R} \subseteq G_s \times H_s\) so that \(\tilde{\mathcal{R}} \subseteq \tilde{H}_s \times \tilde{G}_s\) has the correct type \(\tilde{H} \to \tilde{G}\). To establish that \(\tilde{\mathcal{R}}\) is a well-defined morphism we must show that:

\[
\tilde{\mathcal{R}}^l \circ \text{cl}_H = \tilde{\mathcal{R}}^l = \text{in}_G \circ \tilde{\mathcal{R}}^l
\]

by Lemma 4.1.10. But by the same Lemma we already know that \(\mathcal{R}^l \circ \text{cl}_G = \mathcal{R}^l = \text{in}_H \circ \mathcal{R}^l\), so by Lemma 4.1.7.4 we deduce that the above equivalent statements hold. Preservation of identity morphisms follows because:

\[
(id_G)^{\vee} = (G : \mathcal{G} \to \mathcal{G})^{\vee} = \tilde{G} : \tilde{G} \to \tilde{G} = id_{\tilde{G}} = id_G.
\]

Next we show preservation of composition, i.e. given compatible Dep-morphisms \(\mathcal{R} : \mathcal{G} \to \mathcal{H}\) and \(\mathcal{S} : \mathcal{H} \to \mathcal{I}\) we must show that \(((\mathcal{R} ; \mathcal{S})^{\text{op}})^{\vee} = \mathcal{S}^{\vee} ; \mathcal{R}^{\vee}\). We first point out the typing \((\mathcal{R} ; \mathcal{S})^{\vee} : \mathcal{I} \to \mathcal{G}\) so that \((\mathcal{R} ; \mathcal{S})^{\vee} \subseteq I_s \times \tilde{G}_s = I_s \times \tilde{G}_s\). Then we calculate as follows:

\[
\begin{align*}
((\mathcal{R} ; \mathcal{S})^{\vee})^l &= \neg_{\mathcal{G}} \circ (\mathcal{S}^{\vee} ; \mathcal{R}^{\vee})^l \circ \neg_{\mathcal{I}} \\
&= \neg_{\mathcal{G}} \circ (\mathcal{S}^l ; \mathcal{R}^l) \circ \neg_{\mathcal{I}} \\
&= (\neg_{\mathcal{G}} \circ \mathcal{S}^l \circ \neg_{\mathcal{H}}) \circ (\neg_{\mathcal{H}} \circ \mathcal{R}^l \circ \neg_{\mathcal{I}}) \\
&= \neg_{\mathcal{G}} \circ \mathcal{S}^l \circ \neg_{\mathcal{H}} \circ \mathcal{R}^l \\
&= (\mathcal{S}^l ; \mathcal{R}^l) \\
&= (\mathcal{S}^{\vee} ; \mathcal{R}^{\vee})^l
\end{align*}
\]

where we have implicitly used the fact that \(\tilde{\mathcal{R}}\) and \(\tilde{\mathcal{S}}\) are well-defined morphisms. Then \((-)^{\vee}\) is a well-defined functor. Each component \(\alpha_G = id_G\) is certainly an isomorphism. Naturality comes down to the equality \(\alpha_{\mathcal{G}} ; \tilde{\mathcal{R}}^l = \mathcal{R} ; \alpha_{\mathcal{H}}\) for each morphism \(\mathcal{R} : \mathcal{G} \to \mathcal{H}\), which follows because Dep is a well-defined category and relational converse is involutive.

Finally we establish that \((\tilde{\mathcal{R}}_+ ; \tilde{\mathcal{R}}_-) = (\mathcal{R}_+ ; \mathcal{R}_-).\) Since \((\mathcal{R}_+ ; \mathcal{R}_-)\) is a witness for \(\mathcal{R}\) we have \(\mathcal{R}_+ ; \mathcal{H} = \mathcal{R} ; G\), \(\mathcal{R}_-.\). Applying relational converse yields \(\mathcal{R} = \mathcal{R} ; H\), \(\mathcal{R}_-.\) so that \(\tilde{\mathcal{R}}\) has the witness \((\mathcal{R}_+ ; \mathcal{R}_-).\). By maximality of \(\mathcal{R}\)'s associated components we deduce that \(\mathcal{R}_+ \subseteq \mathcal{R}_-\) and \(\mathcal{R}_- \subseteq \mathcal{R}_+.\) The reverse inclusions follow by the symmetric argument i.e. by starting with \(\mathcal{R}\)'s associated components.

**4.2 Dep is categorically equivalent to JSLf**

Each \(\mathcal{G}\) has an associated interior operator \(\text{in}_G\) by Definition 4.1.4. Let \(O(\mathcal{G}) \subseteq \mathcal{P}G_t\) be its fixpoints, which we will also refer to as the \(\mathcal{G}\)-open sets.

**Definition 4.2.1** (Equivalence functors between Dep and JSLf).

1. \(\text{Open} : \text{Dep} \to \text{JSL}_f\) is defined:

\[
\text{OpenG} := (O(\mathcal{G}), \cup, \emptyset) \\
\text{Open} \mathcal{R} := (\mathcal{R} : \mathcal{G} \to \mathcal{H}) \\
\text{Open} := \lambda Y. \text{cl}_{\mathcal{H}}[Y] : \text{Open}(G) \to \text{Open}(H)
\]

recalling \(\mathcal{R}_+\) from Definition 4.1.8. Equivalently \(\text{Open}_R := \lambda Y \in O(\mathcal{G}). \mathcal{R}^l \circ \mathcal{G}^l(Y)\) (see below).
2. \( \text{Pirr} : \text{JSL}_f \to \text{Dep} \) is defined:

\[
\text{Pirr} Q := \xi_Q \subseteq J(Q) \times M(Q)
\]

\[
f : Q \to \mathbb{R}
\]

\[
\text{Pirr} := \{(j, m) \in J(\mathbb{R}) \times M(\mathbb{R}) : f(j) \leq_R m\} : \text{Pirr} Q \to \text{Pirr} \mathbb{R}
\]

with component relations:

\[
(Pirr f)_\cdot := \{(j_1, j_2) \in J(Q) \times J(R) : j_2 \leq_R f(j_1)\}
\]

\[
(Pirr f)_\cdot := \{(m_1, m_2) \in M(R) \times M(Q) : f_\cdot(m_1) \leq_Q m_2\}.
\]

It constructs the \textit{poset of irreducibles} introduced by Markowsky [Mar75].

The two definitions of \text{Open} above are consistent.

**Lemma 4.2.2.** For any \text{Dep}-morphism \( R : G \to H \) and \( Y \in O(G) \) we have \( R^+_\cdot [Y] = R^+ \circ G^\cdot (Y) \).

**Proof.**

\[
\text{Open} R(Y) = R^+_\cdot [Y]
\]

\[
= (R^+_\cdot \circ G^\cdot) Y \quad \text{\textit{Y is G-open}}
\]

\[
= (G^\cdot \circ R^+_\cdot \circ G^\cdot) (Y) \quad \text{\textit{by (\cdot \circ)}}
\]

\[
= R^+ \circ G^\cdot (Y) \quad \text{\textit{by Lemma 4.1.10.2}}
\]

\( \Box \)

**Example 4.2.3.**

1. \( \text{Open} \Delta_X = \mathbb{P} X = (\mathbb{P} X, \cup, \varnothing) \) is a boolean semilattice. Recalling that every relation \( R \subseteq X \times Y \) defines a \text{Dep}-morphism \( R : \Delta_X \to \Delta_Y \), then \( \text{Open} R : \mathbb{P} X \to \mathbb{P} Y \) has action:

\[
\text{Open} R(A) := R^+ \circ (\Delta_X)^\cdot (A) = R^+ (A).
\]

2. By (1), \( \text{Open} \) sends identity relations to boolean join-semilattices. This generalizes to bijections i.e. bijective functional relations. But there exist non-functional relations with this property too:

\[
\begin{array}{c|c}
G \subseteq X \times Y & \text{Open} G \\
\hline
y_1 \quad y_2 \quad y_3 \\
\hline
x_1 \quad x_2 \quad x_3 \\
\end{array}
\]

\[
\begin{array}{c|c}
\{y_1, y_2, y_3\} & \{y_1, y_2\} \quad \{y_3\} \\
\hline
\{y_1\} \quad \{y_2\} \\
\end{array}
\]

3. Recall that each boolean semilattice \( Q = (Q, \lor_Q, \bot_Q) \) has a unique bijective complementation operation:

\[
\neg_Q : Q \to Q \quad \neg_Q (q) := \lor_Q \neg Q q.
\]

It turns out that \( \text{Pirr} Q = \xi_Q \subseteq J(Q) \times M(Q) \) is precisely the domain/codomain restriction \( \neg_Q : \text{At}(Q) \to \text{CoAt}(Q) \).

This restriction is also bijective: an atom \( a \) is not less than or equal to a coatom \( c \) iff \( c = \neg_Q a \).

Before proving the well-definedness of \( \text{Open} \) and \( \text{Pirr} \) we provide a number of helpful results.

**Definition 4.2.4** (The finite lattice of \( G \)-open sets and its isomorphic lattice of \( G \)-closed sets).

Let \( G \) be a relation between finite sets.
1. Define two sets of subsets:
   \[ O(G) := O(\text{in}_G) \subseteq P_G \] the \( G \)-open sets.
   \[ C(G) := C(\text{cl}_G) \subseteq P_G^\text{op} \] the \( G \)-closed sets.

2. There are two inclusion-ordered bounded lattice structures on these sets:
   \[ O(G) := (O(G), \cup, \varnothing, \wedge_O(G), \text{in}_G(G_t)) \] where \( Y_1 \wedge_O(G) Y_2 := \text{in}_G(Y_1 \cap Y_2) \),
   \[ C(G) := (C(G), \vee_C(G), \text{cl}_G(\varnothing), \wedge_G) \] where \( X_1 \vee_C(G) X_2 := \text{cl}_G(X_1 \cup X_2) \),
   recalling that \( \text{in}_G(G_t) = G[G_t] \) and \( \text{cl}_G(\varnothing) = G^i(\varnothing) \).

3. There is a lattice isomorphism \( \theta_G : C(G) \to O(G) \):
   \[ \theta_G(X) := G^i(X) = G[X] \quad \theta_G^{-1}(Y) := G^i(Y) \]

4. There is a self-inverse lattice isomorphism \( \kappa_G : (C(G))^{\text{op}} \to O(G) \) with action \( \kappa_G(X) := \overline{X} \).

\[ \text{Lemma 4.2.5 (The bounded lattices of } G\text{-open/closed sets and their irreducibles).} \]

1. The bounded lattices \( O(G) \) and \( C(G) \) are well-defined.

2. Each \( \theta_G : C(G) \to O(G) \) and \( \kappa_G : (C(G))^{\text{op}} \to O(G) \) are well-defined bounded lattice isomorphisms.

3. The \( G \)-open sets and \( G \)-closed sets can be described as follows:
   \[ O(G) = \{ G[S] : S \subseteq G_t \} \quad \text{i.e. the closure of } \{ G[g_s] : g_s \in G_t \} \text{ under unions.} \]
   \[ C(G) = \{ G^i(S) : S \subseteq G_t \} \quad \text{i.e. the closure of } \{ G^i(\varnothing) : g_t \in G_t \} \text{ under intersections.} \]

   Finally, we have the following inclusions:
   \[ J(O(G)) \subseteq \{ G[g_s] : g_s \in G_t \} \quad M(O(G)) \subseteq \{ \text{in}_G(\varnothing) : g_t \in G_t \}. \]

**Proof.**

1. \( O(G) \) is the ‘standard’ bounded inclusion-ordered lattice one obtains from an interior operator defined on the underlying poset of a bounded lattice. In detail, \( \text{in}_G \) is defined on \( (P_G, \subseteq) \) and the latter has all joins = unions, hence the \( G \)-open sets \( O(G) \) are closed under all possibly-empty unions (see Lemma 2.2.5.3). Since \( (O(G), \cup, \varnothing) \) is a finite join-semilattice it is also a bounded lattice in a unique way. The induced top is \( \text{in}_G(G_t) = G^i \circ G^i(G_t) = G^i(G_t) = G[G_t] \). The induced meet is:
   \[ Y_1 \wedge Y_2 := \bigcup \{ Y \in O(G) : Y \subseteq Y_1 \cap Y_2 \} = \text{in}_G(Y_1 \cap Y_2) \]
   where \( \subseteq \) follows because \( Y = \text{in}_G(Y) \subseteq \text{in}_G(Y_1 \cap Y_2) \) by monotonicity, and \( \supseteq \) follows by co-extensivity. The argument concerning \( C(G) \) is analogous, noting that \( \bigwedge_C(G) = \text{cl}_G(\varnothing) = G^i \circ G^i(\varnothing) = G^i(\varnothing) \).

2. We first show that both \( \theta_G : C(G) \to O(G) \) and \( \theta_G^{-1} : O(G) \to C(G) \) are well-defined functions.
   \[ \theta_G(X) = \theta_G(\text{cl}_G(X)) \quad X \text{ is } G\text{-closed} \]
   \[ = G^i \circ G^i \circ G^i(X) \text{ by definition} \]
   \[ = \text{in}_G(G^i(X)) \text{ by definition} \]

   Then for any \( G \)-open set \( Y \) and \( G \)-closed set \( X \) we have:
   \[ \theta_G \circ \theta_G^{-1}(Y) = G^i \circ G^i(Y) = \text{in}_G(Y) = Y \quad \theta_G^{-1} \circ \theta_G(X) = G^i \circ G^i(X) = \text{cl}_G(X) = X \]
   so they are mutually inverse bijections. Finally they inherit monotonicity from \( G^i \) and \( G^i \), so they are order-isomorphisms and hence also bounded lattice isomorphisms.
Next we show that \( \kappa_\mathcal{G} : (C(\mathcal{G}))^{\text{op}} \to \mathcal{O}(\mathcal{G}) \) is a well-defined bounded lattice isomorphism. Since relative complement is involutive and flips the inclusion-ordering, we need only show it is a well-defined surjective function.

\[
\kappa_\mathcal{G}(X) = \neg_\mathcal{G}(X) = \neg_\mathcal{G} \circ \mathcal{G}^1 \circ \mathcal{G}^1(X) \quad \text{since } X \text{ is } \mathcal{G}\text{-closed}
\]

\[
= \mathcal{G}^1 \circ \mathcal{G}^1 \circ \neg_\mathcal{G}(X) \quad \text{by De Morgan duality}
\]

\[
= \mathcal{G}^1 \circ \mathcal{G}^1 \circ \mathcal{G}^1(X) \quad \text{by definition}
\]

\[
\kappa^{-1}_\mathcal{G}(Y) = \neg_\mathcal{G}(Y) = \neg_\mathcal{G} \circ \mathcal{G}^1 \circ \mathcal{G}^1(Y) \quad \text{since } Y \text{ is } \mathcal{G}\text{-open}
\]

\[
= \mathcal{G}^1 \circ \mathcal{G}^1 \circ \neg_\mathcal{G}(Y) \quad \text{by De Morgan duality}
\]

\[
= \mathcal{G}^1 \circ \mathcal{G}^1 \circ \mathcal{G}^1(Y) \quad \text{by definition}
\]

The first equality establishes well-definedness and the second surjectivity i.e. each \( \mathcal{G}\)-open set \( Y \) is the relative complement of the \( \mathcal{G}\)-closed set \( \mathcal{C}(\mathcal{G})(\overline{Y}) \).

3. We establish \( \mathcal{O}(\mathcal{G}) = \{ \mathcal{G}[S] : S \subseteq \mathcal{G}_s \} \). Given any \( \mathcal{G} \in \mathcal{O}(\mathcal{G}) \) then since \( \theta_\mathcal{G} : C(\mathcal{G}) \to \mathcal{O}(\mathcal{G}) \) is bijective we deduce \( \mathcal{G}[\mathcal{G}[X]] \) for some \( X \in C(\mathcal{G}) \subseteq \mathcal{P}\mathcal{G}_s \). Conversely, given any subset \( S \subseteq \mathcal{G}_s \),

\[
\mathcal{G}[S] = \mathcal{G}^1(S) \equiv \mathcal{G}^1 \circ \mathcal{G}^1 \circ \mathcal{G}^1(S) = \mathcal{C}(\mathcal{G})(\overline{S})
\]

Every \( \mathcal{G}[S] \) is clearly a possibly-empty union of the sets \( \mathcal{G}[s] \). Next we show that \( C(\mathcal{G}) = \{ \mathcal{G}^1(S) : S \subseteq \mathcal{G}_t \} \). Given any \( \mathcal{G} \in C(\mathcal{G}) \) then since \( \theta^{-1}_\mathcal{G} \) is bijective we deduce \( \mathcal{G} = \mathcal{G}^1(Y) \) for some \( \mathcal{G} \in \mathcal{O}(\mathcal{G}) \subseteq \mathcal{P}\mathcal{G}_t \). Conversely, given any \( S \subseteq \mathcal{G}_t \) then:

\[
\mathcal{G}^1(S) \equiv \mathcal{G}^1 \circ \mathcal{G}^1 \circ \mathcal{G}^1(S) = \mathcal{C}(\mathcal{G})(\overline{S})
\]

where the marked equality follows by \((11)_1\). Recalling that \( \mathcal{G}^1 \) preserves all possibly-empty intersections (it is right adjoint to \( \mathcal{G}^1 \)), it follows that every \( \mathcal{G}^1(S) \) is a possibly-empty intersections of the special sets \( \mathcal{G}^1(\mathcal{G}[t]) \).

The inclusion \( J(\mathcal{O}(\mathcal{G})) \subseteq \{ \mathcal{G}[g_s] : g_s \in \mathcal{G}_s \} \) follows because the latter sets join-generate \( \mathcal{O}(\mathcal{G}) \), and thus must contain the join-irreducibles by Lemma 2.2.3.6. Concerning the inclusion \( M(\mathcal{O}(\mathcal{G})) \subseteq \{ \mathcal{C}(\mathcal{G})(\overline{t}) : \overline{t} \in \mathcal{G}_t \} \), the preceding inclusion informs us that every \( J \in J(\mathcal{O}(\mathcal{G})) \) takes the form \( \mathcal{G}[g_t] \). Then the composite bounded lattice isomorphism:

\[
\mathcal{O}(\mathcal{G}) \xrightarrow{\kappa_\mathcal{G}} (C(\mathcal{G}))^{\text{op}} \xrightarrow{\theta_{\mathcal{G}}^{\text{op}}} (\mathcal{O}(\mathcal{G}))^{\text{op}}
\]

necessarily restricts to a bijection \( J(\mathcal{O}(\mathcal{G})) \to M(\mathcal{O}(\mathcal{G})) \), with action:

\[
\theta_{\mathcal{G}}^{\text{op}} \circ \kappa^{-1}_\mathcal{G}(\mathcal{G}[g_t]) = \theta_\mathcal{G}(\mathcal{G}^1(\mathcal{G}[g_t]))
\]

\[
= \mathcal{G}^1 \circ \mathcal{G}^1(\mathcal{G}[g_t])
\]

\[
= \mathcal{G}^1 \circ \mathcal{G}^1(\overline{g_t}) \quad \text{by De Morgan duality}
\]

\[
= \mathcal{C}(\mathcal{G})(\overline{g_t})
\]

and we are finished. \( \square \)

We have associated two isomorphic finite bounded lattices \( \mathcal{O}(\mathcal{G}) \) and \( C(\mathcal{G}) \) to each relation \( \mathcal{G} \). Next we describe two pairs of adjoint morphisms between the underlying join-semilattices of these bounded lattices, parametric in any \( \mathcal{R}\)-morphisms.

**Lemma 4.2.6.** Let \( \mathcal{R} : \mathcal{G} \to \mathcal{H} \) be any \( \mathcal{R}\)-morphisms.

1. For all subsets \( X_1 \subseteq \mathcal{G}_s \) and \( \mathcal{H}\)-closed subsets \( X_2 \subseteq \mathcal{H}_s \):

\[
\mathcal{H}^1 \circ \mathcal{R}^1(X_1) \subseteq X_2 \iff X_1 \subseteq \mathcal{R}^1 \circ \mathcal{H}^1(X_2).
\]

2. Restricting the domain and codomain yields the adjoint \( \mathcal{J}\text{S}\mathcal{L}^1\)-morphisms:

\[
\mathcal{R}^1 \circ \mathcal{H}^1 : (C(\mathcal{H}), \cap, \mathcal{H}_s) \to (C(\mathcal{G}), \cap, \mathcal{G}_s)
\]

\[
\mathcal{H}^1 \circ \mathcal{R}^1 : (C(\mathcal{G}), \vee_{\mathcal{G}(\mathcal{G})}, \mathcal{C}(\mathcal{G}))) \to (C(\mathcal{H}), \vee_{\mathcal{H}(\mathcal{H})}, \mathcal{C}(\mathcal{H})))
\]

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3. For all subsets $Y_1 \subseteq \mathcal{H}_t$ and $\mathcal{G}$-open subsets $Y_2 \subseteq \mathcal{G}_t$:

$$R^! \circ G^!(Y_2) \subseteq Y_1 \iff Y_2 \subseteq G^! \circ R^!(Y_1)$$

4. Restricting the domain and codomain yields the adjoint $\mathcal{JSL}_f$-morphisms:

$$R^! \circ G^! : (O(\mathcal{G}), \cup, \emptyset) \to (O(\mathcal{H}), \cup, \emptyset)$$

$$G^! \circ R^! : (O(\mathcal{H}), \wedge, \mathcal{O}(\mathcal{H})), \mathcal{in}_R(\mathcal{H}_t)) \to (O(\mathcal{G}), \wedge, \mathcal{O}(\mathcal{G})), \mathcal{in}_G(\mathcal{G}_t))$$

**Proof.**

1. Regarding the first statement:

$$X_1 \subseteq R^! \circ H^!(X_2) \iff R^!(X_1) \subseteq H^!(X_2) \iff H^! \circ H^! \circ R^!(X_1) \subseteq H^! \circ R^!(X_2)$$

by Lemma 4.1.10.1

$$H^! \circ R^!(X_1) \subseteq H^! \circ H^! \circ R^!(X_2)$$

since $X_2$ is closed.

2. As for the second statement, $R^! \circ H^!$ sends all subsets (in particular $H$-closed subsets) to $G$-closed subsets because (i) $R^! = \mathcal{cl}_G \circ R^!$ by Lemma 4.1.10.1 and Lemma 4.1.7.4, and (ii) every subset of the form $G^!(S)$ is $G$-closed by Lemma 4.2.5.3. Furthermore $H^! \circ R^!$ sends all subsets (in particular $G$-closed subsets) to $H$-closed subsets because $H^! = \mathcal{cl}_H \circ H^!$ by (1.1). Thus when restricted to closed subsets in both their domain and codomain they define an adjunction. Consequently, the right adjoint $R^! \circ H^!$ preserves all meets (= intersections) and the left adjoint preserves all joins (which needn’t be unions).

3. (3) and (4) follow from (1) and (2) by applying duality. Take the dual morphism $R^\vee = \bar{R} : \bar{H} \to \bar{G}$ and apply the first statement, yielding:

$$X_1 \subseteq \bar{R}^! \circ \bar{G}^!(X_2) \iff \bar{G}^! \circ \bar{R}^!(X_1) \subseteq X_2$$

for all subsets $X_1 \subseteq H_2 = H_t$ and all $G$-closed subsets $X_2 \subseteq \bar{G}_s = \bar{G}_t$. Since relative complement defines a lattice isomorphism $\kappa_{\bar{G}} : (C(\bar{G}))^{op} \to O(G)$, we can substitute both $X_1 := \bar{Y}_1$ and $X_2 := \bar{Y}_2$ and rewrite so that:

$$\neg H_t \circ \bar{R}^! \circ \bar{G}^!(Y_2) \subseteq Y_1 \iff \bar{Y}_2 \subseteq \neg G_t \circ \bar{G}^! \circ \bar{R}^! \circ \neg H_t(Y_1)$$

for all subsets $Y_1 \subseteq H_t$ and all $G$-open subsets $Y_2 \subseteq \bar{G}_t$. Applying De Morgan duality yields the desired equivalence. Regarding the fourth statement, this follows via the restriction of $\bar{R}$ via the second statement.

The De Morgan dualities in Lemma 4.1.7.3 extend to the closure and interior operators associated to $G$. Furthermore they satisfy the usual characterisations as intersections/unions of closed/open sets, and in particular $\mathcal{cl}_\text{Pirr}Q$ is the ‘usual’ closure structure associated to a finite lattice with underlying join-semilattice $Q$.

**Lemma 4.2.7.**

1. For each relation $G$:

$$\text{in}_G = \neg G_t \circ \mathcal{cl}_G \circ \neg G_s$$

$$\text{in}_G(Z) = \bigcup \{ Y \in O(G) : Y \subseteq Z \}$$

for any subset $Z \subseteq G_t$.

$$\mathcal{cl}_G = \neg G_s \circ \text{in}_G \circ \neg G_t$$

$$\mathcal{cl}_G(Z) = \bigcap \{ X \in C(G) : Z \subseteq X \}$$

for any subset $Z \subseteq G_s$.

Moreover,

$$Y \subseteq \text{in}_G(\overline{Y}) \iff g_t \notin Y$$

for every $G$-open $Y \subseteq \mathcal{G}_t$ and every $g_t \in \mathcal{G}_t$.

2. For each finite join-semilattice $Q$:

$$\mathcal{cl}_{\text{Pirr}Q}(S) = \{ j \in J(Q) : j \le Q \bigvee Q S \}$$

$$\mathcal{in}_{\text{Pirr}Q}(S) = \{ m \in M(Q) : \wedge Q S \le Q m \}$$

$$\mathcal{cl}_{(\text{Pirr}Q)^\vee}(S) = \{ m \in M(Q) : \wedge Q S \le Q m \}$$

$$\mathcal{in}_{(\text{Pirr}Q)^\vee}(S) = \{ j \in J(Q) : j \le Q \bigvee Q S \}$$

recalling that $\text{Pirr}Q := \{ s \in J(Q) \times M(Q)$.  

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Proof.

1. The two left-hand statements follow by applying De Morgan duality i.e. \((\neg \uparrow \neg)\) and \((\neg \downarrow \neg)\). Next:

\[
\begin{align*}
\bigcup \{ Y \in O(G) : Y \subseteq Z \} &= \bigcup \{ G[g_s] : g_s \in G_s, \ G[g_s] \subseteq Z \} \\
&= \{ g_t \in G_t : \exists g_s, g_t \in G[g_s] \subseteq Z \} \\
&= G^t \circ G^i(Z) \\
&= \mathsf{in}_G(Z)
\end{align*}
\]

The description of \(\mathsf{cl}_G(Z)\) follows by (i) the De Morgan duality above, and (ii) because \(\bar{G}\)-open sets are the relative complements of \(G\)-closed sets by Lemma 4.2.5.2. The final equivalence follows because \(Y \subseteq \mathfrak{T}\) iff \(g_t \notin Y\), and moreover \(\mathsf{in}_G\) is co-extensive, monotone and idempotent.

2. Regarding the second statement, for all subsets \(S \subseteq J(Q)\) we have:

\[
\mathsf{cl}_{\mathsf{Pirr}Q}(S) = \mathsf{cl}_{\mathsf{Pirr}Q}(S) = \mathsf{cl}_{\mathsf{Pirr}Q}(S) = \mathsf{cl}_{\mathsf{Pirr}Q}(S) = \mathsf{cl}_{\mathsf{Pirr}Q}(S)
\]

Next observe \(\mathsf{Pirr}Q^\circ = \mathsf{Pirr}Q^\circ\), so that:

\[
\mathsf{cl}_{(\mathsf{Pirr}Q)^\circ}(S) = \{ m \in J(Q^\circ) : m \leq \bigvee S \} = \{ m \in M(Q) : \bigwedge S \leq m \}
\]

The descriptions of the interior operators follow by the De Morgan duality exhibited in the first statement.

\[
\square
\]

Lemma 4.2.8. \(\mathsf{Open} : \mathsf{Dep} \rightarrow \mathcal{JSL}_f\) is a well-defined faithful functor.

Proof. \(\mathsf{Open} = (O(G), \cup, \sigma)\) is a finite join-semilattice. Regarding its action on morphisms, recall that \(\mathsf{Open}R(Y) := R^I \circ G^i(Y)\) for every \(\bar{G}\)-open \(Y\). Then by Lemma 4.2.6.4 this is a well-defined \(\mathcal{JSL}_f\)-morphism of the desired type. \(\mathsf{Open}\) preserves identity morphisms because:

\[
\mathsf{Open}id_G = \mathsf{Open}(G) = \lambda Y. G^i(Y) = \lambda Y. G^i(Y) = \mathsf{id}_{\mathsf{Open}G}
\]

since \(Y\) is \(\bar{G}\)-open. Next we show preservation of composition i.e. for any \(R : G \rightarrow \mathcal{H}\) and \(S : \mathcal{H} \rightarrow \mathcal{I}\):

\[
\mathsf{Open}S \circ \mathsf{Open}R = \mathsf{Open}(R \upharpoonright S).
\]

We have not reversed the sense of the morphisms, the difference in ordering is due to the different order in which one writes functional and \(\mathsf{Dep}\)-composition, the latter in keeping with relational composition. It follows via:

\[
\begin{align*}
\mathsf{Open}(R \upharpoonright S)(Y) &= \mathsf{Open}(R \upharpoonright S)(Y) \\
&= \mathsf{Open}(R \upharpoonright S)(Y) \\
&= \mathsf{Open}(R \upharpoonright S)(Y)
\end{align*}
\]

We now establish that \(\mathsf{Open}\) is faithful. Given \(R, S : G \rightarrow \mathcal{H}\) such that \(\mathsf{Open}R = \mathsf{Open}S\) then for all \(\bar{G}\)-open sets \(Y\) we have \(\mathcal{R}_+[Y] = \mathcal{S}_+[Y]\). Then for each \(g_s \in G_s\) we have:

\[
\begin{align*}
\mathcal{R}[g_s] &= (\mathcal{G}; \mathcal{R}_+)[g_s] \quad \text{by Lemma 4.1.10.2} \\
&= \mathcal{R}_+[g_s] \\
&= \mathcal{S}_+[g_s] \\
&= \mathcal{S}[g_s] \quad \text{since } G[g_s] \text{ is } G\text{-open} \\
&= (\mathcal{G}; \mathcal{S}_+)[g_s] \quad \text{by Lemma 4.1.10.2}
\end{align*}
\]

so that \(\mathcal{R} = \mathcal{S}\) as required. \(\square\)
Lemma 4.2.9. \( \mathsf{Pirr} : \mathsf{JSL}_f \to \mathsf{Dep} \) is a well-defined functor.

Proof. \( \mathsf{Pirr} \) is clearly well-defined on objects. Take any join-semilattice morphism \( f : Q \to R \) and recall that:

\[
\mathsf{Pirr} f := \{(j,m) \in J(Q) \times M(R) : f(j) \leq_R m\}.
\]

We must show that it is a \( \mathsf{Dep} \)-morphism of type \( \mathsf{Pirr} Q \to \mathsf{Pirr} R \). By Lemma 4.2.7.2 we know that:

\[
\mathsf{cl}_{\mathsf{Pirr} Q}(S) = \{ j \in J(Q) : j \leq_Q S \} \quad \text{and also} \quad \mathsf{in}_{\mathsf{Pirr} R}(S) = \{ m \in M(R) : \bigwedge_R S \leq_R m \}.
\]

Then we calculate:

\[
\begin{align*}
(\mathsf{Pirr} f)^1 \circ \mathsf{cl}_{\mathsf{Pirr} Q}(S) &= (\mathsf{Pirr} f)^1(\{ j \in J(Q) : j \leq_Q S \}) \\
&= \{ m \in M(R) : \forall j \in J(Q). [j \leq_Q S \text{ and } f(j) \leq_R m] \} \\
&= \{ m : \forall j. [\forall j \leq_R S \Rightarrow f(j) \leq_R m] \} \\
&= \{ m : \forall j. [\forall j \leq_R S \Rightarrow j \leq_R f(m)] \} \\
&= \{ m : \forall j \in \mathsf{V}_Q S \leq_R f(m) \} \\
&= \{ m : \forall j \in \mathsf{V}_Q S \leq_R f(m) \} \\
&= \{ m : f(\mathsf{V}_Q S) \leq_R m \} \\
&= \{ m : f(\mathsf{V}_Q J) \leq_R m \} \\
&= \{ m \in M(R) : \bigwedge_R f[J[S] \leq_R m \} \\
&= \{ m : \exists j \in S.f(j) \leq_R m \} \\
&= (\mathsf{Pirr} f)^1(S)
\end{align*}
\]

\[
\begin{align*}
\mathsf{in}_{\mathsf{Pirr} R} \circ (\mathsf{Pirr} f)^1(S) &= \mathsf{in}_{\mathsf{Pirr} R}(\{ m_1 \in M(R) : \exists j \in S.f(j) \leq_R m_1 \}) \\
&= \{ m_2 \in M(R) : \forall j. [m_1 \in M(R) \cap \exists j \in S.f(j) \leq_R m_1] \leq_R m_2 \} \\
&= \{ m_2 \in M(R) : \forall j. [m_1 \in M(R) \cap \exists j \in S.f(j) \leq_R m_1] \leq_R m_2 \} \\
&= \{ m_2 \in M(R) : \forall j. [m_1 \in M(R) \cap \exists j \in S.f(j) \leq_R m_1] \leq_R m_2 \} \\
&= \{ m_2 \in M(R) : \forall j. [f(m) \leq_R m] \} \\
&= \{ m_2 \in M(R) : \exists j \in S.f(j) \leq_R m \} \\
&= (\mathsf{Pirr} f)^1(S)
\end{align*}
\]

Then \( \mathsf{Pirr} f : \mathsf{Pirr} Q \to \mathsf{Pirr} R \) is a well-defined \( \mathsf{Dep} \)-morphism. Concerning preservation of identity morphisms:

\[
\mathsf{Pirrid}_{\mathsf{Q}} = \{(j,m) \in J(Q) \times M(Q) : j \leq_Q m\} = \mathsf{Pirr} Q : \mathsf{Pirr} Q \to \mathsf{Pirr} Q = \mathsf{id}_{\mathsf{Pirr} Q}.
\]

Next, given any join-semilattice morphisms \( f : Q \to R \) and \( g : R \to S \) we must show that \( \mathsf{Pirr}(g \circ f) = \mathsf{Pirr} f ; \mathsf{Pirr} g \).

First we verify the definitions of \( \mathsf{Pirr} f \)'s component relations.

\[
\begin{align*}
(\mathsf{Pirr} f)(j_1, j_2) &\iff j_2 \in (\mathsf{Pirr} R)^1(\mathsf{Pirr} f(j_1)) \\
&\iff \mathsf{Pirr} R[j_2] \subseteq \{ m_1 \in M(R) : f(j_1) \leq_R m_1 \} \\
&\iff \{ m \in M(R) : f(j_1) \leq_R m \} \subseteq \{ m \in M(R) : j_2 \leq_R m \} \\
&\iff \forall m \in M(R). [f(j_1) \leq_R m \Rightarrow j_2 \leq_R m] \\
&\iff j_2 \leq_R f(j_1)
\end{align*}
\]

\[
\begin{align*}
(\mathsf{Pirr} f)_+(m_1, m_2) &\iff m_2 \in (\mathsf{Pirr} R^+)^1((\mathsf{Pirr} f)^+[m_1]) \\
&\iff \mathsf{Pirr} R^+[m_2] \subseteq (\mathsf{Pirr} f)^+[m_1] \\
&\iff \{ j \in J(Q) : j \leq_Q m_2 \} \subseteq \{ j \in J(Q) : f(j) \leq_R m_1 \} \\
&\iff \{ j \in J(Q) : f(j) \leq_R m_1 \} \subseteq \{ j \in J(Q) : j \leq_Q m_2 \} \\
&\iff \forall j \in J(Q). [f(j) \leq_R m_1 \Rightarrow j \leq_Q m_2] \\
&\iff f_+(m_1) \leq_Q m_2
\end{align*}
\]

Then finally:

\[
\begin{align*}
\mathsf{Pirr} \cdot \mathsf{Pirr} g &= \mathsf{Pirr} : (\mathsf{Pirr} g)^+ \\
&= \{(j_q, m_s) \in J(Q) \times M(S) : \exists m_r \in M(R). [f(j_q) \leq_R m_r \land g_+(m_s) \leq_R m_r] \} \\
&= \{(j_q, m_s) \in J(Q) \times M(S) : \forall m_r \in M(R). [g_+(m_s) \leq_R m_r \Rightarrow f(j_q) \leq_R m_r] \} \\
&= \{(j_q, m_s) \in J(Q) \times M(S) : f(j_q) \leq_R g_+(m_s) \} \\
&= \{(j_q, m_s) \in J(Q) \times M(S) : g \circ f(j_q) \leq_S m_s \} \\
&= \mathsf{Pirr}(g \circ f)
\end{align*}
\]

□

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We are now finally ready to prove the categorical equivalence of JSL \(_f\) and Dep. We do this very explicitly i.e. the two natural isomorphisms and their inverses are provided, as well as the associated component relations of the Dep-isomorphisms.

**Theorem 4.2.10 (Dep is equivalent to JSL\(_f\)).** The functors Open : Dep \(\rightarrow\) JSL\(_f\) and Pirr : JSL\(_f\) \(\rightarrow\) Dep define an equivalence of categories, with respective natural isomorphisms:

\[
\begin{align*}
\text{rep} : \text{Id}_{\text{JSL}_f} &\Rightarrow \text{Open} \circ \text{Pirr} \\
\text{rep}_Q &:= \lambda q \in Q. \{ m \in M(Q) : q \not\in m \} \\
\text{rep}_Q^{-1} &:= \lambda Y. \land_Q M(Q) \setminus Y \\
\text{red} : \text{Id}_{\text{Dep}} &\Rightarrow \text{Pirr} \circ \text{Open} \\
\text{red}_G &:= \{(g_s, Y) \in G_s \times M(\text{Open} G) : G[g_s] \not\in Y \} \\
\text{red}_G^{-1} &:= \xi \in J(\text{Open} G) \times G_t
\end{align*}
\]

where \(\text{red}_G\) and its inverse have associated component relations:

\[
\begin{align*}
(\text{red}_G)_* &:= \{(g_s, X) \in G_s \times J(\text{Open} G) : X \subseteq G[g_s]\} \\
(\text{red}_G)_+ &:= \xi \subseteq M(\text{Open} G) \times G_t
\end{align*}
\]

**Proof.**

1. We verify that \(\text{rep}\) is a natural isomorphism. Each \(\text{rep}_Q\) is a well-defined function because:

\[
\text{in}_{\text{PirrQ}}(\text{rep}_Q(q)) = \text{in}_{\text{PirrQ}}(\{(m \in M(Q) : q \notin m\}) = \{m' \in M(Q) : \land_Q \{ m : q \leq m \} \not\in m' \} \quad \text{by Lemma 4.2.7}
\]

\[
= \{m' \in M(Q) : q \not\in m'\} = \text{rep}_Q(q)
\]

It is well-known that \(\text{rep}_Q\) defines a JSL\(_f\)-isomorphism, usually described as an embedding into \((PM(Q), \cup, \emptyset)\). We verify this explicitly: \(\text{rep}_Q(\bot_Q) = \{ m \in M(Q) : \bot_Q \not\in m \} = \emptyset = \emptyset_{\text{PirrQ}}, \) and:

\[
\text{rep}_Q(q_1 \cup q_2) = \{ m \in M(Q) : q_1 \cup q_2 \not\in m \} = \{ m \in M(Q) : q_1 \not\in m \text{ or } q_2 \not\in m \} = \text{rep}_Q(q_1) \cup \text{rep}_Q(q_2).
\]

Next we show that \(\text{rep}_Q\) is bijective. It is injective because distinct elements \(q_1 \not\in q_2\) necessarily have distinct sets of meet irreducibles above them, as they are the respective meet of them, so their complements relative to \(M(Q)\) are also distinct. For surjectivity, we first observe that for any subset \(Y \subseteq M(Q)\) we have:

\[
\land_Q Y = \lor_Q \{ q \in Q : \forall m \in Y, q \not\in m \} = \lor_Q \{ j \in J(Q) : \forall m \in Y, j \not\in m \} = \lor_Q \{ m \in M(Q) : j \not\in m \} = \land_Q (\text{PirrQ})^{-1}(Y).
\]

Then for any Pirr\(_Q\)-open \(Y \subseteq M(Q)\) we now show that \(\text{rep}_Q(\land_Q M(Q) \setminus Y) = Y\).

\[
\text{rep}_Q(\land_Q Y) = \text{rep}_Q(\lor_Q \uparrow \downarrow_Y) = \{ m \in M(Q) : \lor_Q \uparrow \downarrow_Y (Y) \not\in m \} = \{ m \in M(Q) : \exists j \in \uparrow \downarrow_Y (Y), j \not\in m \} = \{ m \in M(Q) : \exists j \in J(Q), (\uparrow \downarrow_Y j) \not\in m \text{ and } j \not\in m \} = \uparrow \downarrow_Y \circ \downarrow \downarrow_Y (Y) = Y.
\]

Since \(Y\) is Pirr\(_Q\)-open

Thus we have shown that each \(\text{rep}_Q : Q \rightarrow \text{Open}(\text{PirrQ})\) is a JSL\(_f\)-isomorphism, and furthermore the inverse is necessarily \(\text{rep}_Q^{-1}(Y) = \land_Q M(Q) \setminus Y\) by the above argument. Then it only remains to prove naturality i.e.

\[
\begin{array}{ccc}
Q & \xrightarrow{\text{rep}_Q} & (O(\uparrow \downarrow \downarrow | J(Q) \times M(Q)), \cup, \emptyset) \\
\downarrow f & & \downarrow \text{Open(Pirr)} \\
\emptyset & \xrightarrow{\text{rep}_{\emptyset}} & (O(\uparrow \downarrow \downarrow | J(\emptyset) \times M(\emptyset)), \cup, \emptyset)
\end{array}
\]
for all JS$_f$-morphisms $f : Q \to R$. Unwinding the definitions, $\text{rep}_R \circ f(q) = \{m \in M(R) : f(q) \not\in m\}$ and:

$$\text{Open}(\text{Pirr}) \circ \text{rep}_Q(q) = \text{Open}(\text{Pirr})((m_1 \in M(Q) : q \not\in m_1))$$

$$= (\text{Pirr})_*\{m_1 \in M(Q) : q \not\in m_1\}$$

$$= \{m \in M(R) : \exists \, m_1 \in M(Q), (q \not\in m_1 \land f_s(m) \not\in m_1)\}$$

$$= \{m : \forall m_1 \in M(Q), (f_s(m) \not\in m_1) \Rightarrow q \not\in m_1\}$$

$$= \{m : q \not\in f_s(m)\}$$

$$= \{m : f(q) \not\in m\}$$

via adjoints

2. We verify that \textit{red} is a natural isomorphism. Let $Q := \text{Open} G$. We start by showing that each $\text{red}_G = \{(g_s,Y) \in G_s \times M(Q) : G[g_s] \not\subseteq Y\}$ is a well-defined Dep-morphism of type $G \to \text{Pirr} Q$.

$$\text{red}_G^t \circ \text{cl}_G(S) = \{(Y \in M(Q) : \exists g_s \in \text{cl}_G(S), G[g_s] \not\subseteq Y)$$

$$= \{(Y \in M(Q) : \exists g_s \in \text{cl}_G(S), g_s \not\in G(Y)\}$$

$$= \{(Y \in M(Q) : G^t \circ \text{cl}_G(S) \not\subseteq Y)\}$$

$$= \{Y \in M(Q) : \exists s \in S, G^t \circ \text{cl}_G(S) \not\subseteq Y\}$$

$$= \text{red}_G^t(S)$$

$$\text{in}_{\text{Pirr} Q} \circ \text{red}_G^t(S) = \text{in}_{\text{Pirr} Q}((Y \in M(Q) : \exists s \in S, G[s] \not\subseteq Y)$$

$$= \text{in}_{\text{Pirr} Q}((Y \in M(Q) : G[S] \not\subseteq Y)$$

$$= \{Y \in M(Q) : \exists s \in S, G[S] \not\subseteq Y\}$$

$$= \text{red}_G^t(S)$$

To show that $\text{red}_G$ is an isomorphism, we first show that its proposed inverse $\text{red}_G^{-1} := \hat{e} \subseteq J(Q) \times G$ is a well-defined Dep-morphism of type $\text{Pirr} Q \to G$.

$$\hat{e}^t \circ \text{cl}_{\text{Pirr} Q}(S) = \{g_t \in G_t : \exists Y \in J(Q), g_t \in Y\}$$

$$= \{g_t \in G_t : \exists Y \in J(Q), [Y \subseteq \text{cl}_{\text{Pirr} Q}(S) \land g_t \in Y]\}$$

$$= \{g_t \in G_t : \exists Y \in J(Q), g_t \in Y \subseteq \text{cl}_{\text{Pirr} Q}(S)\}$$

$$= \{g_t \in G_t : g_t \in \text{cl}_{\text{Pirr} Q}(S)\}$$

$$= \hat{e}^t(S)$$

Furthermore $\text{in}_G \circ \hat{e}^t(S) = \text{in}_G(J(S) = \bigcup S = \hat{e}^t(S)$ because $S \subseteq J(Q)$ is a collection of $G$-open sets. Now we show they that these two morphisms are the inverse of one another.

$$(\text{red}_G \circ \text{red}_G^{-1})(S) = (\text{red}_G^{-1})^t \circ (\text{Pirr} Q)^t \circ (\text{red}_G)^t(S)$$

$$= \hat{e}^t((X \in J(Q) : X \subseteq \text{cl}_{\text{Pirr} Q}(S) \subseteq \text{Pirr} Q)$$

$$= \hat{e}^t([X : \forall Y \in M(Q), G[S] \subseteq Y \Rightarrow X \subseteq Y])$$

$$= \hat{e}^t([X : X \subseteq G[S]])$$

$$= \text{in}_{\text{Pirr} Q}(S)$$

$$= \text{id}_G^t(S)$$

$$(\text{red}_G^{-1} \circ \text{red}_G)^t(S) = \text{red}_G^t \circ G^t \circ (\text{red}_G^{-1})^t(S)$$

$$= \text{red}_G^t(G^t[\bigcup S])$$

$$= \{Y \in M(Q) : \exists g_s \in G^t[\bigcup S], G[g_s] \not\subseteq Y\}$$

$$= \{Y : G^t \circ G^t[\bigcup S] \not\subseteq Y\}$$

$$= \{Y : \exists S \subseteq Y\}$$

$$= \text{in}_{\text{Pirr} Q}(S)$$

$$= \text{id}_{\text{Pirr} Q}(S)$$
Thus each $\text{rep}_G$ is indeed a Dep-isomorphism with inverse $\text{rep}^{-1}_G$. Let us also verify one of $\text{rep}_G$’s associated components:

$$(\text{red}_G)_* = \{(Y,g_t) \in M(\text{Open}G) \times G_t : g_t \in \overleftarrow{G}(\text{red}_G[Y])\}$$

by definition

$$ \{ (Y,g_t) : \overleftarrow{G}[g_t] \subseteq \{ g_s \in G_s : \overrightarrow{G}[g_s] \not\subseteq Y \} \}$$

$$ \{ (Y,g_t) : \forall g_s \in G_s, (G[g_s] \subseteq Y \Rightarrow g_t \in \overrightarrow{G}[g_s]) \}$$

by Lemma 4.2.7.1

$$ \{ (Y,g_t) : g_t \notin Y \}$$

by Lemma 4.2.7.1

$$ \forall \bar{\epsilon} \subseteq M(\text{Open}G) \times G_t$$

It remains to verify naturality i.e. the diagram below on the left commutes for all Dep-morphisms $\mathcal{R} : G \to H$.

Applying $\text{Open}$ to this diagram yields the diagram above on the right, and since $\text{Open}$ is faithful it suffices to show the latter commutes. In fact it is an instance of rep’s naturality because $\text{Open}\text{red}_G = \text{rep}_\text{Open}G$ for any relation $G$, as we now show.

$$\text{Open}\text{red}_G = \lambda Y \circ \text{O}(G). (\text{red}_G)_*[Y]$$

by above calculation

$$ = \lambda Y, \bar{\epsilon} [Y]$$

$$ = \lambda Y, \bar{\epsilon} \subseteq [Y]$$

$$ = \lambda Y, \{ M \in M(\text{Open}G) : \exists g_t \in Y, g_t \notin M \}$$

$$ = \lambda Y, \{ M : Y \notin M \}$$

$$ = \lambda Y, \{ M : Y \notin \text{Open}_G M \}$$

$$ = \text{rep}_\text{Open}G$$

3. Having proved the main result, we finally mechanically verify the associated component relations of $\text{red}_G$ and its inverse $\text{red}^{-1}_G$, starting with $(\text{red}_G)_- \subseteq G_s \times J(Q)$.

$$(\text{red}_G)_-(g_s,X) \iff G_s \times J(Q) : X \in \overleftarrow{G}[g_s]$$

$$ \iff \overleftarrow{G}[X] \subseteq \{ Y \in M(Q) : \overrightarrow{G}[g_s] \not\subseteq Y \}$$

$$ \iff \forall Y \in M(Q). (X \notin Y \Rightarrow \overrightarrow{G}[g_s] \not\subseteq Y)$$

$$ \iff X \subseteq \overrightarrow{G}[g_s]$$

In (2) we established that $(\text{red}_G)_+ = \bar{\epsilon} \subseteq M(Q) \times G_t$, so next consider $(\text{red}^{-1}_G)_- \subseteq J(Q) \times G_s$:

$$(\text{red}^{-1}_G)_-(X,g_s) \iff g_s \in G^i(\text{red}^{-1}_G[X])$$

$$ \iff \overleftarrow{G}[g_s] \subseteq \bar{\epsilon}[X]$$

$$ \iff \overrightarrow{G}[g_s] \subseteq X$$

and finally $(\text{red}^{-1}_G)_+ \subseteq G_t \times M(Q)$:

$$(\text{red}^{-1}_G)_+(g_t,Y) \iff Y \in (\bar{\epsilon}^{-1})^i((\text{red}^{-1}_G)[g_t])$$

$$ \iff \bar{\epsilon}^{-1} [Y] \subseteq \{ X \in J(Q) : g_t \in X \}$$

$$ \iff \forall X \in J(Q). (X \notin g_t \Rightarrow g_t \in X)$$

$$ \iff \forall X \in J(Q). (g_t \notin X \Rightarrow X \subseteq Y)$$

$$ \iff \forall X \in J(Q). (X \in \text{in}_G(Y) \Rightarrow X \subseteq Y)$$

by Lemma 4.2.7.1
Then we have proved the claimed equivalence. It will be helpful to further clarify the fullness of $\text{Open}$.

**Lemma 4.2.11 (Explicit fullness of $\text{Open}$).**

*Given any JSL$_f$-morphism $f : \text{Open}\mathcal{G} \to \text{Open}\mathcal{H}$ then $f = \text{Open}\mathcal{R}$ where the Dep-morphism:*

\[
\mathcal{R} : \mathcal{G} \to \mathcal{H} \quad \text{is defined} \quad \mathcal{R}(g, h_t) : \iff h_t \in f(\mathcal{G}[g_s]).
\]

*Proof.* Consider:

\[
\begin{array}{cccc}
\mathcal{G} & \xrightarrow{\mathcal{S}_t} & M(\text{Open}\mathcal{G}) & \xrightarrow{\text{Pirrf}_t} & M(\text{Open}\mathcal{H}) & \xrightarrow{\mathcal{T}_t} & \mathcal{H} \\
\xrightarrow{\text{Pirrf}_t} & \mathcal{J}(\text{Open}\mathcal{G}) & \xrightarrow{\mathcal{T}_t} & \mathcal{H}
\end{array}
\]

where the respective relations are defined:

\[
\begin{align*}
\mathcal{S}_t(g_s, X) : & \iff X \subseteq \mathcal{G}[g_s] \\
\mathcal{T}_t(Y, h_t) : & \iff \mathcal{H}[h_s] \subseteq \mathcal{X}
\end{align*}
\]

The left and right squares commute because:

\[
\begin{align*}
\mathcal{S}_t; \text{Pirrf}_t\mathcal{G} = \{(g_s, Y) \in \mathcal{G}_s \times M(\text{Open}\mathcal{G}) : \mathcal{G}[g_s] \not\subseteq Y\} = \mathcal{G}; \mathcal{S}_t \\
\mathcal{T}_t; \mathcal{H} = \{(X, h_t) \in J(\text{Open}\mathcal{H}) \times \mathcal{H}_t : h_t \in \mathcal{X}\} = \text{Pirrf}_t\mathcal{H}; \mathcal{T}_t,
\end{align*}
\]

as the reader may verify. Composing together these three Dep-morphisms yields $\mathcal{R} := \mathcal{S}_t; \text{Pirrf}_t; \mathcal{T}_t : \mathcal{G} \to \mathcal{H}$, where:

\[
\begin{align*}
\mathcal{R}(g_s, h_t) & \iff \exists X \in J(\text{Open}\mathcal{G}), Y \in M(\text{Open}\mathcal{H}).(X \subseteq \mathcal{G}[g_s] \text{ and } \text{Pirrf}(X, Y) \text{ and } \text{in}_t\mathcal{H}(h_t) \subseteq Y) \\
& \iff \exists X, Y. (X \subseteq \mathcal{G}[g_s] \text{ and } f(X) \not\subseteq Y \text{ and } \text{in}_t\mathcal{H}(h_t) \subseteq Y) \\
& \iff f(\mathcal{G}[g_s]) \not\subseteq \text{in}_t\mathcal{H}(h_t) \quad \text{by Lemma 4.2.7.1.}
\end{align*}
\]

Regarding the marked equivalence, $\implies$ holds because $f$ is monotonic and hence preserves inclusions, and the converse follows because $\mathcal{G}[x]$ is a union of join-irreducibles and $\text{in}_t\mathcal{H}(h_t)$ is a meet of meet-irreducibles. Then $\mathcal{R}$ is a well-defined Dep-morphism and:

\[
\begin{align*}
\text{Open}\mathcal{R}(\mathcal{G}[g_s]) & = \mathcal{R}^1 \circ \mathcal{G}^1(\mathcal{G}[g_s]) \\
& = \mathcal{R}(g_s) \\
& = f(\mathcal{G}[g_s]).
\end{align*}
\]

Thus $f = \text{Open}\mathcal{R}$ because their action on join-irreducibles is the same.

We finish off this subsection by using the above equivalence theorem to characterise all morphisms between finite boolean and distributive join-semilattices.

**Theorem 4.2.12 (Characterisation of JSL$_f$-morphisms between boolean and distributive join-semilattices).**

1. Each finite boolean join-semilattice $Q$ is isomorphic to $\mathcal{PZ} = \text{Open}\Delta_2$ for some finite set $Z$.

2. The JSL$_f$-morphisms $\mathcal{PZ}_1 \to \mathcal{PZ}_2$ are precisely the functions $\mathcal{R}^1$ where $\mathcal{R} \subseteq Z_1 \times Z_2$ is an arbitrary relation.

3. Each finite distributive join-semilattice $Q$ is iso to $(\mathcal{UP}(P), \cup, \emptyset) = \text{Open} \leq_{\text{P}}$ for some finite poset $P = (P, \leq_{\text{P}})$.

4. The Dep-morphisms $\mathcal{R} : \leq_{\text{P}} \to \leq_{\text{Q}}$ are precisely those relations $\mathcal{R} \subseteq P \times Q$ such that:

\[
\forall p \in P, \mathcal{R}[p] \in \mathcal{UP}(Q) \quad \text{and} \quad \forall q \in Q, \mathcal{R}[q] \in \mathcal{DN}(P).
\]

Moreover, the JSL$_f$-morphisms $\text{Open} \leq_{\text{P}} \to \text{Open} \leq_{\text{Q}}$ are precisely the functions $\mathcal{R}^1|_{\mathcal{UP}(P) \times \mathcal{UP}(Q)}$ where $\mathcal{R} \subseteq P \times Q$ satisfies the above two conditions.
Proof.

1. Recall that a finite join-semilattice $Q$ is said to be boolean if its associated bounded lattice is. Then by Lemma 2.2.3.13 and the fact that $(J(Q), M(Q)) = (At(Q), CoAt(Q))$ by Lemma 2.2.3.8, we have:

$$\text{Pirr}Q = ↓_{At(Q) \times CoAt(Q)} (↓_{Q\rightarrow At(Q) \times At(Q)}); \tau_Q$$

where $\tau_Q : At(Q) \rightarrow CoAt(Q)$ is the canonical bijection. Since atoms are incomparable we see that $\text{Pirr}Q \subseteq At(Q) \times CoAt(Q)$ is a functional composite of bijections and hence a bijection itself. It follows that every $X \subseteq CoAt(Q)$ is Pirr-open, and we may use the canonical JSL$_f$-isomorphism:

$$Q \xrightarrow{rePQ} \text{OpenPirr}Q = FC0At(Q) = \text{Open}JSL_{CoAt(Q)}.$$

2. Every relation $R \subseteq Z_1 \times Z_2$ between finite sets defines a Dep-morphism of type $R : \Delta Z_1 \rightarrow \Delta Z_2$. Then since $\text{Open}\Delta Z = \mathbb{P}Z$ and by the equivalence theorem, the JSL$_f$-morphisms of type $\mathbb{P}Z_1 \rightarrow \mathbb{P}Z_2$ are precisely the functions:

$$\text{Open}R = \lambda S \in \mathbb{P}Z_1. R \circ \Delta^1_{Z_1}(S) = \lambda S \subseteq Z_1. R[S] = R^1,$$

where $R \subseteq Z_1 \times Z_2$ is arbitrary.

3. A finite join-semilattice $Q$ is distributive if its associated lattice is. By Lemma 2.2.3.13,

$$\text{Pirr}Q = ↓_{J(Q) \times M(Q)} (↓_{Q \rightarrow J(Q)}); \tau_Q$$

where $\tau_Q : J(Q) \rightarrow M(Q)$ is the canonical order-isomorphism. For brevity, let $P := (J(Q), \leq Q \cap J(Q) \times J(Q))$ so we have the bipartite graph isomorphism:

$$\begin{array}{ccc}
J(Q) & \xrightarrow{\tau_Q} & M(Q) \\
\leq_{P^op} & \downarrow & \xrightarrow{\text{Pirr}Q} \\
J(Q) & \xrightarrow{\Delta_{J(Q)}} & J(Q)
\end{array}$$

This witnesses a Dep-morphism $R := \text{Pirr}Q : \leq_{P^op} \rightarrow \text{Pirr}Q$ and we now show that $\text{Open}R$ is a JSL$_f$-isomorphism. First observe that:

(a) The $\leq_{P^op}$-open sets $Y \subseteq J(Q)$ are precisely the down-closed subsets $Dn(P)$.

(b) Given any $Y \in Dn(P)$,

$$\leq_{P^op} (Y) = \{ j \in J(Q) : \leq_{P^op} [j] \subseteq Y \} = \{ j \in J(Q) : J_P j \subseteq Y \} = Y,$$

and similarly $\leq_{P^op}^{-1} (Y) = J_P Y = Y$.

Then the join-semilattice morphism $\text{Open}R$ has action:

$$\text{Open}R(Y) = \text{Pirr}Q^! \circ \leq_{P^op} (Y) = \text{Pirr}Q[Y] = \tau_Q[\leq_{P^op} [X]] = \tau_Q[Y]$$

It is injective because $\tau_Q$ is, and surjective because $\text{Pirr}Q[X] = \tau_Q[\leq_{P^op} [X]]$ for every $X \subseteq J(Q)$, so that every $\text{Pirr}Q$-open set is the image of some $\leq_{P^op}$-open set. Then we have the composite isomorphism:

$$\text{Open} \xrightarrow{\text{Open}R} \text{OpenPirr}Q \xrightarrow{rePQ} Q.$$

4. First observe that for every finite poset $P = (P, \leq_P)$ we have:

$$\text{Open}P = (U_P(P), \cup, \emptyset)$$

since the $P$-open sets are precisely the images $\leq_P [X]$ where $X \subseteq P$. By Lemma 4.1.10.1 and Lemma 4.1.7.4, the Dep-morphisms $R : \leq_P \rightarrow \leq_Q$ are precisely those relations $R \subseteq P \times Q$ such that:

$$R^1 = R^! \circ \text{cl}_{\leq_P} \quad \text{and} \quad \bar{R}^1 = \bar{R}^! \circ \text{cl}_{\leq_P}.$$
Regarding these closure operators, we have:

\[
\begin{align*}
\text{cl}_{\leq_p} &= \chi_p^{\leq} \circ \chi_p^{\geq} \\
&= \lambda S \subseteq P, \, \chi_p^{\leq} (\uparrow_p S) \\
&= \lambda S \subseteq P, \{p \in P : \uparrow_p p \subseteq \uparrow_p S\} \\
&= \lambda S \subseteq P, \uparrow_p S
\end{align*}
\]

and thus \(\text{cl}_{\leq_p}\) constructs the up-closure in \(P\), so that \(\text{cl}_{\leq_p} = \text{cl}_{\leq_{\text{op}p}}\) constructs the down-closure in \(Q\). Now, by monotonicity and the fact that downwards-closure preserves unions, the above equalities may equivalently be written:

\[
\forall p \in P. \, \mathcal{R}^{\uparrow_p} = \subseteq_p \subseteq \mathcal{R}[p] \quad \text{and} \quad \forall q \in Q. \, \mathcal{R}^{\downarrow_q} = \subseteq_q \subseteq \mathcal{R}[q].
\]

The left-hand equality says that whenever \(p \leq_p p'\) and \(\mathcal{R}(p', q)\) then \(\mathcal{R}(p, q)\), or equivalently that \(\mathcal{R}[q]\) is downwards-closed in \(P\) for every \(q \in Q\). As for the right-hand equality, it equivalently asserts that \(\mathcal{R}[p]\) is up-closed in \(Q\) for every \(p \in P\).

Finally let us apply the categorical equivalence, so that the \(\text{JSL}_f\)-morphisms of type \(\text{Open} \leq_p \rightarrow \text{Open} \leq_q\) are precisely those of the form \(\text{Open}\mathcal{R}\) where \(\mathcal{R}\) is restricted as above. Concerning its action,

\[
\text{Open}\mathcal{R}(Y) = \mathcal{R}^\dagger \circ \subseteq_p (Y) = \mathcal{R}^\dagger(Y)
\]

because \(\{p \in P : \uparrow_p p \subseteq Y\} = Y\) whenever \(Y \subseteq U_p(P)\). In conclusion, \(\text{Open}\mathcal{R} = \mathcal{R}^\dagger \cap U_p(\subseteq_p) \times U_p(\subseteq_q)\) where it is both necessary and sufficient that the relation \(\mathcal{R} \subseteq P \times Q\) satisfies the claimed conditions.

\[\Box\]

**Note 4.2.13.**

1. Concerning Lemma 4.2.12.1, the proof can be contrasted with another method i.e. use the duality between finite boolean algebras and finite sets, and view the representing boolean algebra isomorphism as a \(\text{JSL}_f\)-isomorphism, see Theorem 7.1.5 in the Appendix.

2. Concerning Lemma 4.2.12.3, an alternative proof would use Birkhoff’s duality between finite bounded distributive lattices and finite posets, viewing the representing bounded distributive lattice isomorphism as a \(\text{JSL}_f\)-isomorphism – see Theorem 7.1.3 in the Appendix.

3. Regarding the bipartite graph isomorphism in Lemma 4.2.12.3, such isomorphisms always induce \(\text{Dep}\)-isos – see Example 4.1.2.2.

\[\blacksquare\]

### 4.3 The equivalence \(\text{JSL}_f \cong \text{Dep}\) without using irreducibles

In this short subsection we describe a functor \(\text{Nleq}\) which is naturally isomorphic to \(\text{Pirr} : \text{JSL}_f \rightarrow \text{Dep}\). On objects, \(\text{Nleq}Q = \leq_Q \subseteq Q \times Q\) is the full unrestricted relation i.e. makes no mention of join/meet-irreducibles. We’ll describe the equivalence between \(\text{JSL}_f\) and \(\text{Dep}\) in terms of \(\text{Nleq}\) and \(\text{Open}\) i.e. explicitly describe their respective natural isomorphisms.

**Lemma 4.3.1.** Let \(Q\) be any finite join-semilattice.

1. \(\text{cl}_{\leq_Q}(S) = \{q \in Q : q \leq_Q \forall Q S\}\).

2. We have the \(\text{Dep}\)-isomorphism \(\mathcal{E}_Q : \text{Pirr}Q \rightarrow \leq_Q\) defined:

\[
\begin{align*}
\mathcal{E}_Q &= \{(j, q) \in J(Q) \times Q : j \leq_Q q\}, \\
(\mathcal{E}_Q)_- &= \{(j, q) \in J(Q) \times Q : q \leq_Q j\}, \\
(\mathcal{E}_Q)_+ &= \{(q, m) \in Q \times M(Q) : q \leq_M m\}
\end{align*}
\]

3. Such isomorphisms always induce \(\text{Dep}\)-isos – see Example 4.1.2.2.

**Proof.**
1. This follows by a simple calculation:

\[
\text{cl}_{\leq Q}(S) = \frac{1}{g} \circ \frac{1}{g}(S)
\]

\[
= \frac{1}{g}(\{q \in Q : \exists s \in S, s \leq Q q\})
\]

\[
= \{q \in Q : \exists s \in S, s \leq Q q' \mid q' \leq Q q\}
\]

\[
= \{q \in Q : \forall s \in S, s \leq Q q' \mid q' \leq Q q\}
\]

\[
= \{q \in Q : \forall q' \in Q, (\forall q \leq Q q' \Rightarrow q \leq Q q')\}
\]

\[
= \{q \in Q : q \leq Q \forall q S\}
\]

2. \(E_Q\) is a well-defined \textsf{Dep}-morphism because \((E_Q)_\ast : \leq Q = E_Q = \text{Pirr}\mathcal{Q} ; (E_Q)_\ast\) as is easily verified. These are \(E_Q\)'s associated component relations because each \((E_Q)_\ast [j]\) is closed via \(\text{cl}_{\leq Q}\) (see (1)), and each \((E_Q)_\ast [q]\) is closed via \(\text{cl}_{(\text{Pirr}\mathcal{Q})}\) (see Lemma 4.2.7.2). Similarly \(Z^{-1}_Q\) is a well-defined \textsf{Dep}-morphism and its associated components are correct, observing that they are \textit{not} the converses of \(E_Q\)'s components. Finally:

\[
E_Q ; E_Q^{-1} = E_Q ; (E_Q^{-1})_*
\]

\[
= \{(j, m) \in J(Q) \times M(Q) : \exists q \in Q, (j \notin Q q \land m \leq Q q)\}
\]

\[
= \{(j, m) \in J(Q) \times M(Q) : \forall q \in Q, (m \leq Q q \Rightarrow j \leq Q q)\}
\]

\[
= \{(j, m) \in J(Q) \times M(Q) : j \notin Q m\}
\]

\[
= \text{id}_{\text{Pirr}\mathcal{Q}}
\]

\[
E_Q^{-1} ; E_Q = (E_Q^{-1})_* ; E_Q
\]

\[
= \{(q_1, q_2) \in Q \times Q : \exists m \in M(Q), (q_1 \notin Q m \land q_2 \leq Q m)\}
\]

\[
= \{(q_1, q_2) \in Q \times Q : \forall m \in M(Q), (q_2 \leq Q m \Rightarrow q_1 \leq Q m)\}
\]

\[
= \{(q_1, q_2) \in Q \times Q : q_1 \notin Q q_2\}
\]

\[
= \text{id}_{\leq Q}
\]

using the definition of \textsf{Dep}-composition.

\[
\square
\]

\textbf{Definition 4.3.2} (The equivalence functor \(N\text{leq} : \text{JSL}_f \rightarrow \text{Dep}\)).

The functor \(N\text{leq} : \text{JSL}_f \rightarrow \text{Dep}\) is defined:

\[
N\text{leq}Q := \leq Q \in Q \times Q
\]

\[
f : Q \rightarrow R
\]

\[
\text{Nleq}f := \{(q, r) \in Q \times R : f(q) \notin R r\} : \leq Q \rightarrow \leq R
\]

We also have the natural isomorphism \(E : \text{Pirr} \rightarrow N\text{leq}\) whose components \(E_Q\) are described in Lemma 4.3.1.2. \(\blacksquare\)

\textbf{Lemma 4.3.3}.

1. \(N\text{leq} : \text{JSL}_f \rightarrow \text{Dep}\) is a well-defined functor.

2. \(E : \text{Pirr} \rightarrow N\text{leq}\) is a well-defined natural isomorphism.

\textbf{Proof}. Given any join-semilattice morphism \(f : Q \rightarrow R\), let us show that:

\[
N\text{leq}f = E_Q^{-1} ; \text{Pirr}f ; E_R
\]

Before doing so, we first compute:

\[
(\text{E}_R)_\ast ; (\text{Pirr}f)_\ast = \{(r, m_q) \in R \times M(R) : \exists m_r \in M(R), (r \leq R m_r \land f_\ast(m_r) \leq R m_q)\}
\]

\[
= \{(r, m_q) \in R \times M(R) : f_\ast(r) \leq R m_q\}
\]

Regarding the final equality, \(\leq\) follows because \(f_\ast : R^{\text{op}} \rightarrow Q^{\text{op}}\) also defines a monotonic function from \((R, \leq R)\) to \((Q, \leq Q)\),
and $\sqsupseteq$ follows because $M(R) = J(\mathbb{R}^\mathbb{R})$ so that $r$ arises as a $\vee_{Pirr}$-join of join-irreducibles $m_r$.

\[
\begin{align*}
\mathcal{E}_Q^{-1}\downarrow Pirr \downarrow \mathcal{E}_R &= (\mathcal{E}_Q^{-1} \downarrow Pirr \downarrow \mathcal{E}_R)_+, \\
&= \mathcal{E}_Q^{-1} \downarrow (Pirr \downarrow \mathcal{E}_R)_+, \\
&= (\mathcal{E}_Q^{-1} \downarrow Pirr \downarrow \mathcal{E}_R)_+ \\
&= \mathcal{E}_Q^{-1} \downarrow ((\mathcal{E}_R)_+), \\
&= \mathcal{E}_Q^{-1} \downarrow (((\mathcal{E}_R)_+), (Pirr \downarrow \mathcal{E}_R)_+) \\
&= \mathcal{E}_Q^{-1} \downarrow \{(r, m) \in R \times M(Q) : f_*(r) \leq_R m_q \}\) \\
&= \{(g, r) \in Q \times R : \exists m \in M(Q). (q \leq_Q m \land f_*(r) \leq_R m) \} \\
&= \{(g, r) \in Q \times R : \neg \forall m \in M(Q). (f_*(r) \leq_R m) \} \\
&= \{(g, r) \in Q \times R : q \leq_Q f_*(r) \} \\
&= \{(g, r) \in Q \times R : f(q) \leq_R r \}
\end{align*}
\]

Thus the action of $\mathsf{Nleq}$ is well-defined. In fact for completely general reasons it inherits functoriality from $\mathsf{Pirr}$. Firstly we have \(\mathsf{Nleq}d_0 = \mathcal{E}_Q^{-1} \downarrow Pirrd_0 \downarrow \mathcal{E}_R = \mathcal{E}_Q^{-1} \downarrow \mathcal{E}_R = id_{\mathsf{Nleq}Q} \), and secondly:

\[
\begin{align*}
\mathsf{Nleq}(g \circ f) &= \mathcal{E}_Q^{-1} \downarrow Pirr(g \circ f) \downarrow \mathcal{E}_R \\
&= \mathcal{E}_Q^{-1} \downarrow Pirr \downarrow \mathcal{E}_R \\
&= (\mathcal{E}_Q^{-1} \downarrow Pirr \downarrow \mathcal{E}_R)_+ \downarrow (\mathcal{E}_R)_+ \\
&= \mathsf{Nleq}f \downarrow \mathsf{Nleq}g \\
\end{align*}
\]

Finally, the fact that each $\mathcal{E}_Q$ is a $\mathsf{Dep}$-isomorphism and $\mathsf{Nleq}f = \mathcal{E}_Q^{-1} \downarrow Pirr \downarrow \mathcal{E}_R$ immediately implies that $\mathcal{E} : \mathsf{Pirr} \rightarrow \mathsf{Nleq}$ defines a natural isomorphism. \(\square\)

**Theorem 4.3.4** (Equivalence between $\mathsf{JSL}_f$ and $\mathsf{Dep}$ involving $\mathsf{Nleq}$).

The functors $\mathsf{Nleq} : \mathsf{JSL}_f \rightarrow \mathsf{Dep}$ and $\mathsf{Open}$ define an equivalence of categories with associated natural isomorphisms:

\[
\begin{align*}
\alpha : id_{\mathsf{JSL}_f} &\Rightarrow \mathsf{Open} \circ \mathsf{Nleq} \\
\beta : id_{\mathsf{Dep}} &\Rightarrow \mathsf{Nleq} \circ \mathsf{Open}
\end{align*}
\]

where $\beta_Q$ and its inverse have associated components:

\[
\begin{align*}
(\beta_Q)_- &:= \{(g, X) \in \mathcal{G}_s \times O(\mathcal{G}) : X \subseteq \mathcal{G}[g_s] \subseteq X \} \\
(\beta_Q)_+ &:= \{(X, g_s) \in O(\mathcal{G}) \times \mathcal{G}[g_s] \subseteq X \}
\end{align*}
\]

Proof. We'll combine Theorem 4.2.10 with the natural isomorphism $\mathcal{E} : \mathsf{Pirr} \Rightarrow \mathsf{Nleq}$. That is, we define:

\[
\begin{align*}
\alpha := id_{\mathsf{JSL}_f} &\Rightarrow \mathsf{Open} \circ \mathsf{Pirr} \circ \mathsf{Open} \circ \mathsf{Nleq} \\
\beta := id_{\mathsf{Dep}} &\Rightarrow \mathsf{Pirr} \circ \mathsf{Open} \circ \mathsf{Nleq} \circ \mathsf{Open}
\end{align*}
\]

Since they are built from natural isomorphisms and functors, they are themselves natural isomorphisms i.e. we have an equivalence of categories. Let us now verify their action:

\[
\begin{align*}
\alpha_Q(q) &= \mathsf{Open} \mathcal{E}_Q \circ \mathsf{rep}_Q(q) \\
&= \mathsf{Open} \mathcal{E}_Q \{m \in M(Q) : q \leq_Q m\} \\
&= \mathcal{E}_Q \{m \in M(Q) : q \leq_Q m\} \\
&= \{q' \in Q : \exists m \in M(Q). (q' \leq_Q m \land q \leq_Q m)\} \\
&= \{q' \in Q : \neg \forall m \in M(Q). (q' \leq_Q m \Rightarrow q \leq_Q m)\} \\
&= \{q' \in Q : q \leq_Q q'\} \\
&= \mathcal{Q} q
\end{align*}
\]

Then since we know $\alpha_Q$ is an isomorphism it follows that $\alpha_Q^{-1}$ is the inverse. Recalling that $\leq_{\mathsf{Open}Q}$ is the inclusion...
relation on the $\mathcal{G}$-open sets $O(\mathcal{G})$, then:

$$\beta_{\mathcal{G}} = \text{red}_\mathcal{G} \uplus E_{\text{open G}}$$

$$= \text{red}_\mathcal{G} \uplus (E_{\text{open G}})^{-1}$$

$$= \{(g_s, Y) \in \mathcal{G}_s \times O(\mathcal{G}) : \exists M \in M(\text{Open G}).(\mathcal{G}[g_s] \not\subseteq \text{open G} M \wedge Y \not\subseteq \text{open G} M)\}$$

$$= \{(g_s, Y) \in \mathcal{G}_s \times O(\mathcal{G}) : \forall M \in M(\text{Open G}).(Y \not\subseteq \text{open G} M \Rightarrow \mathcal{G}[g_s] \not\subseteq \text{open G} M)\}$$

$$= \{(g_s, Y) \in \mathcal{G}_s \times O(\mathcal{G}) : \mathcal{G}[g_s] \not\subseteq Y\}$$

$$\beta_{\mathcal{G}}^{-1} = \mathcal{E}_{\text{open G}}^{-1} \uplus \text{red}_\mathcal{G}^{-1}$$

$$= \mathcal{E}_{\text{open G}}^{-1} \uplus (\text{red}_\mathcal{G}^{-1})^{-1}$$

$$= \{(Y, g_t) \in O(\mathcal{G}) \times \mathcal{G}_t : \exists M \in M(\text{Open G}).(Y \not\subseteq \text{open G} M \wedge \text{in}_\mathcal{G}(\mathcal{G}[g_t]) \not\subseteq \text{open G} Y)\}$$

$$= \{(Y, g_t) : \forall M \in M(\text{Open G}).(\text{in}(\mathcal{G}[g_t]) \not\subseteq \text{open G} Y \Rightarrow Y \not\subseteq \text{open G} M)\}$$

$$= \{(Y, g_t) : Y \not\subseteq \text{in}(\mathcal{G}[g_t])\}$$

$$= \{\{Y, g_t) : g_t \in Y\}$$

$$\in \mathcal{G} \times \mathcal{G}_t$$

by Lemma 4.2.7.1

The descriptions of $\beta_{\mathcal{G}}$ and $\beta_{\mathcal{G}}^{-1}$’s associated components follows via similar simple computations. 

### 4.4 Dep as a canonical construction

In this subsection we provide an alternative description of Dep.

We introduce the category $\text{Cover}$ which is essentially the arrow category of $\text{Rel}_f$. Its hom-sets admit a natural closure structure, so that $\text{Dep}$ is the restriction of $\text{Cover}$ to closed morphisms.

This closure structure is really just a more detailed explanation of the maximum $\mathcal{R}$-witnesses ($\mathcal{R}_-, \mathcal{R}_+$), revealing that their construction is functorial in nature. Given what we already know, it is not hard to prove. However it is useful because it allows us to work with morphisms ‘modulo closure’ in a precise sense.

**Motivation 4.4.1.** Of particular importance is the following basic fact. Given a finite set of $\text{Dep}$-endomorphisms $\{(\mathcal{R}_a, \mathcal{R}_a) : \mathcal{G} \to \mathcal{G} : a \in \Sigma\}$ then:

$$\{(\mathcal{R}_a, \mathcal{R}_a) : \mathcal{G} \to \mathcal{G} : a \in \Sigma\}$$

is the closure of $\{(\mathcal{R}_a, \mathcal{R}_a, \mathcal{R}_a, \mathcal{R}_a, \mathcal{R}_a) : \mathcal{G} \to \mathcal{G} : a \in \Sigma\}$

That is, we may use the usual relational composition in each component and then close once. The restriction to endomorphisms is unnecessary. The reason we emphasise it stems from our interest in nondeterministic acceptance of regular languages. Later on, the endomorphisms $\{(\mathcal{R}_a, \mathcal{R}_a) : \mathcal{G} \to \mathcal{G} : a \in \Sigma\}$ will be viewed as the $a$-transitions of two classical nondeterministic automata, one with states $\mathcal{G}_s$ (the ‘lower one’) and the other with states $\mathcal{G}_t$ (the ‘upper one’). These paired nondeterministic automata naturally accept a single regular language, using only the definition of $\text{Dep}$-composition. Then using the above fact, this language is precisely the language accepted by the lower nfa, or equivalently the reverse of the language accepted by the upper nfa. That is, our ‘categorial’ notion of language acceptance corresponds to the classical notion of nondeterministic acceptance.

**Definition 4.4.2** (The category $\text{Cover}$). The objects of $\text{Cover}$ are the relations between finite sets i.e. the objects of $\text{Dep}$. A morphism $(\mathcal{R}_l, \mathcal{R}_r) : \mathcal{G} \to \mathcal{H}$ is a pair of relations $\mathcal{R}_l \subseteq \mathcal{G}_s \times \mathcal{H}_s$ and $\mathcal{R}_r \subseteq \mathcal{H}_t \times \mathcal{G}_t$ such that:

$$\mathcal{R}_l : \mathcal{H} = \mathcal{G} \cap \mathcal{R}_r^-$$

Then $id_{\mathcal{G}} = (\Delta_{\mathcal{G}_s}, \Delta_{\mathcal{G}_t})$ and composition is defined $(\mathcal{R}_l, \mathcal{R}_r) ; (S_l, S_r) = (\mathcal{R}_l ; S_l, S_r) \cap \mathcal{R}_r$. 

**Definition 4.4.3** (Dep-morphism associated to a $\text{Cover}$-morphism). A $\text{Cover}$-morphism $(\mathcal{R}_l, \mathcal{R}_r) : \mathcal{G} \to \mathcal{H}$ has an associated $\text{Dep}$-morphism $\mathcal{R} : \mathcal{G} \to \mathcal{H}$, namely $\mathcal{R}_l : \mathcal{H} = \mathcal{G} \cap \mathcal{R}_r^-$. 

**Note 4.4.4** ($\text{Cover}$ is isomorphic to the arrow category of $\text{Arr(Rel)}_f$). Arguably the most natural category whose objects are relations between finite sets is the arrow category $\text{Arr(Rel)}_f$ of the category of finite sets and relations $\text{Rel}_f$. It is isomorphic to $\text{Cover}$ by (i) reversing the type of $\mathcal{R}_+$, and (ii) changing composition appropriately (use pairwise relational composition). In fact, $\text{Arr(Rel)}_f$ corresponds to the comma category $\text{Id}_{\text{Rel}_f} \downarrow \text{Id}_{\text{Rel}_f}$, whereas $\text{Cover}$ corresponds to $\text{Id}_{\text{Rel}_f} \downarrow (-)^* : \text{Rel}_f^{op} \to \text{Rel}_f$. Small comma categories always arise as natural pullbacks in $\text{Cat}$, the category of small categories.
Cover is a well-defined category by Note 4.4.4 above. Given \((R_l, R_r) : \mathcal{G} \rightarrow \mathcal{H}\) s.t. \(R_l : \mathcal{G} \rightarrow \mathcal{R}_l\), taking the relational converse yields \(R_r : \mathcal{G} \rightarrow \mathcal{R}_r\) i.e. a Cover-morphism \((R_r, R_l) : \mathcal{H} \rightarrow \hat{\mathcal{G}}\). This defines a self-duality and acts in the same way as \(\text{Dep}\)'s self-duality (Definition 4.1.12); we denote it by the same symbol.

**Lemma 4.4.5 (Self-duality of Cover).** Cover is well-defined and self-dual via \((-)^\vee : \text{Cover}^{op} \rightarrow \text{Cover},\)

\[
\mathcal{G}^\vee := \hat{\mathcal{G}} \quad (R_l, R_r) : \mathcal{G} \rightarrow \mathcal{H} \quad (R_l, R_r)^\vee := (R_r, R_l) : \mathcal{H} \rightarrow \hat{\mathcal{G}}
\]

with witnessing natural isomorphism \(\alpha : \text{id}_{\text{Cover}} = (-)^\vee \circ ((-)^\vee)^{op}\) with action \(\alpha_G := \text{id}_G = (\Delta_{\mathcal{G}}, \Delta_{\mathcal{G}})\).

Cover’s hom-sets admit a natural ordering i.e. pairwise inclusion. We now define a natural closure operator uniformly on each such poset.

**Definition 4.4.6 (The poset \text{Cover}(\mathcal{G}, \mathcal{H}).** For each pair of relations \((\mathcal{G}, \mathcal{H})\) we define the finite poset:

\[
\text{Cover}(\mathcal{G}, \mathcal{H}) := (\text{Cover}(\mathcal{G}, \mathcal{H}), \leq_{\text{Cover}(\mathcal{G}, \mathcal{H})))
\]

where \((R_1, R_2) \leq_{\text{Cover}(\mathcal{G}, \mathcal{H})); S_1, S_2) \iff R_1 \subseteq S_1 \text{ and } R_2 \subseteq S_2.

This poset of morphisms admits a natural closure operator \(\text{cl}_{\mathcal{G}, \mathcal{H}} : \text{Cover}(\mathcal{G}, \mathcal{H}) \rightarrow \text{Cover}(\mathcal{G}, \mathcal{H})\) defined:

\[
\text{cl}_{\mathcal{G}, \mathcal{H}}(R_l, R_r) := (R_l^*, R_r^*) \text{ where:}
\]

\[
R_l^* := \{(g_s, h_s) \in \mathcal{G}_s \times \mathcal{H}_s ; h_s \in \text{cl}_H(R_l[g_s])\} \\
R_r^* := \{(h_t, g_t) \in \mathcal{H}_t \times \mathcal{G}_t ; g_t \in \text{cl}_G(R_r[h_t])\}
\]

using the closure operators \(\text{cl}_H = H^i \circ H^l\) and \(\text{cl}_G = G^i \circ G^l\) from Definition 4.1.4.

Each finite poset \text{Cover}(\mathcal{G}, \mathcal{H}) is actually a finite lattice: the bottom is \((\mathcal{G}, \mathcal{G}) : \mathcal{G} \rightarrow \mathcal{H}\) and the join is pairwise binary union (the meet structure is induced). We now prove that these closure operators are well-defined and construct the associated components \((R_l, R_r)\). Furthermore they naturally interact with the self-duality and compositional structure.

**Lemma 4.4.7.**

1. For any Cover-morphism \((R_l, R_r) : \mathcal{G} \rightarrow \mathcal{H}\) we have:

\[
(R_l, R_r) \leq_{\text{Cover}(\mathcal{G}, \mathcal{H})} (R_l^*, R_r^*) \\
R_l^*; \mathcal{H} = R_l; \mathcal{H} \\
R_r^* = \mathcal{G}; R_r^* = \mathcal{G}; (R_r^*)^\vee
\]

so that \(\text{cl}_{\mathcal{G}, \mathcal{H}}(R_l, R_r) : \mathcal{G} \rightarrow \mathcal{H}\) is also a Cover-morphism.

2. \(\text{cl}_{\mathcal{G}, \mathcal{H}}\) is a well-defined closure operator on the finite poset \text{Cover}(\mathcal{G}, \mathcal{H}).

3. The closure of a Cover-morphism \((R_l, R_r) : \mathcal{G} \rightarrow \mathcal{H}\) can be described in the following three ways.
   i. The components \((R_l, R_r)\) of its associated Dep-morphism \(\mathcal{R}\).
   ii. The pairwise union of all Cover-morphisms \((S_l, S_r) : \mathcal{G} \rightarrow \mathcal{H}\) such that \(S_l; \mathcal{H} = R_l; \mathcal{H}\).
   iii. The pairwise union of all Cover-morphisms \((S_l, S_r) : \mathcal{G} \rightarrow \mathcal{H}\) such that \(\mathcal{G}; S_r^* = \mathcal{G}; R_r^*\).

4. The closure operators \(\text{cl}_{\mathcal{G}, \mathcal{H}}\) commute with Cover’s self-duality i.e.

\[
\text{cl}_{\mathcal{G}, \mathcal{H}}(R_l, R_r)^\vee = (\text{cl}_{\mathcal{G}, \mathcal{H}}(R_l, R_r))^\vee
\]

5. The closure operators \(\text{cl}_{\mathcal{G}, \mathcal{H}}\) are well-behaved w.r.t. Cover-composition i.e.

\[
\text{cl}_{\mathcal{G}, \mathcal{H}}(R_l, R_r) = \text{cl}_{\mathcal{G}, \mathcal{H}}(S_l, S_r) \\
\text{cl}_{\mathcal{G}, \mathcal{H}}(S_l, S_r) = \text{cl}_{\mathcal{G}, \mathcal{H}}(T_l, T_r) \\
\text{cl}_{\mathcal{G}, \mathcal{H}}(S_l, S_r) = \text{cl}_{\mathcal{G}, \mathcal{H}}(T_l, T_r)
\]

for all appropriately typed morphisms \((R_l, R_r), (S_l, S_r)\) and \((T_l, T_r)\).
Proof.

1. The left statement follows because $cl_H$ and $cl_0$ are extensive. The central and right statement follow because for all $g_s \in G_s$ and $h_t \in H_t$,

$$R_i; H[g_s] = H[R_i^*[g_s]]$$
$$= H[cl_H(R_i[g_s])]$$
$$= H^i \circ H^i \circ H^i(R_i[g_s])$$
$$= H^i(R_i[g_s])$$

Then $cl_G,H(R_i, R_r)$ is a well-defined Cover-morphism using the fact that $(R_i, R_r)$ is.

2. That $cl_G,H$ is a well-defined function follows from the previous statement. That it is monotonic, extensive and idempotent follows because $cl_H$ and $cl_0$ possess these properties.

3. Given a Cover-morphism $(R_i, R_r)$ let $R$ be its associated Dep-morphism and $(R_-, R_r)$ the latter’s associated component relations. Then:

$$R_-[g_s] = H^i(R[g_s])$$
$$= H^i \circ (R_i; H)^i({g_s})$$
$$= H^i \circ H^i \circ R_i^i({g_s})$$
$$= cl_H(R_i[g_s])$$

for all $g_s \in G_s$ and $h_t \in H_t$. This proves the first statement. By Lemma 4.1.10.2 we know that $(R_-, R_r)$ is the union of all Cover-morphisms $(S_i, S_r) : G \to H$ such that $S_i; H = R = G; S_r$. Then the second and third statement follow by $R_i; H = R$ and $R = G; R_i^r$ respectively.

4. Follows from the definitions:

$$cl_R^-, G^r((R_i, R_r)^r) = cl_{R_i; G}(R_i, R_r) = (R_r^*, R_r^r) = (R_i^*, R_i^r) = (cl_G,H(R_i, R_r))^r$$

5. To prove the first rule, assume we have Cover-morphisms $(R_i, R_r), (S_i, S_r) : G \to H$ with the same closure, and also a Cover-morphism $(T_i, T_r) : H \to I$. We need only show that $(R_i; T_r)^* = (S_i; T_i)^*$ because the other component relation is uniquely determined. Then we calculate:

$$(R_i; T_r)^*[g_s]$$
$$= cl_T(R_i; T_i[g_s])$$
$$= T^i \circ T^i \circ R_i^r({g_s})$$

for all $g_s \in G_s$. The second rule follows by dualising (using (4)), applying the first rule, and dualising again. 

Each closure operator $cl_G,H$ induces an equivalence relation on its respective hom-set i.e. the kernel:

$$ker cl_G,H \subseteq Cover(G, H) \times Cover(G, H)$$

which relates those morphisms with the same closure. Then by Lemma 4.4.7.5 these relations are collectively compatible with Cover-composition and thus induce a ‘quotient category’. We denote the composition of morphisms in this category by ‘$\cdot$’ i.e. the same symbol we use to denote Dep-composition. This is warranted because these two categories are isomorphic.
Definition 4.4.8 (The category Cover/\text{cl}). It has the same objects as Cover, whereas its hom-sets are:

\[
\text{Cover/\text{cl}} (\mathcal{G}, \mathcal{H}) := \text{Cover}(\mathcal{G}, \mathcal{H}) / \ker \text{cl}_{\mathcal{G}, \mathcal{H}}
\]

i.e. the equivalence classes of \text{Cover}-morphisms relative to \ker \text{cl}_{\mathcal{G}, \mathcal{H}}. Let us denote the associated surjective canonical maps by \[\]_{\mathcal{G}, \mathcal{H}} : \text{Cover}(\mathcal{G}, \mathcal{H}) \to \text{Cover/\text{cl}}(\mathcal{G}, \mathcal{H}). Then identity morphisms and composition are defined:

\[
\text{id}_{\mathcal{G}} := \text{id}_{\mathcal{G}} = [\Delta_{\mathcal{G}, \mathcal{G}}]_{\mathcal{G}, \mathcal{G}}
\]

\[
([R_l, R_r])_{\mathcal{G}, \mathcal{H}} ; (\mathcal{S}_l, \mathcal{S}_r) := [(R_l, R_r); (\mathcal{S}_l, \mathcal{S}_r)]_{\mathcal{G}, \mathcal{I}}
\]

We also define two identity-on-objects functors:

\[
\mathcal{I} : \text{Cover/\text{cl}} \to \text{Dep}
\]

\[
\mathcal{I}\mathcal{G} := \mathcal{G}
\]

and also the composite functor \text{cl} := \mathcal{I} \circ [\mathcal{G}] : \text{Cover} \to \text{Dep}. Recalling that \text{Dep}-morphisms \mathcal{R} may be identified with their associated components \((R_-, R_+)\), we may abuse notation by equivalently defining:

\[
\text{cl}(R_l, R_r) := \text{cl}_{\mathcal{G}, \mathcal{H}}(R_l, R_r) = (R_-, R_+)
\]

\[\square\]

Theorem 4.4.9 (Dep as a quotient category of Cover).

1. \text{Cover/\text{cl}} is a well-defined category and \[\] : \text{Cover} \to \text{Cover/\text{cl}} is a well-defined functor.

2. \mathcal{I} : \text{Cover/\text{cl}} \to \text{Dep} is a well-defined isomorphism of categories.

3. \text{cl} : \text{Cover} \to \text{Dep} is a well-defined functor and preserves the ordering on morphisms i.e.

\[
(R_l, R_r) \leq_{(\mathcal{G}, \mathcal{H})} (\mathcal{S}_l, \mathcal{S}_r) \implies \mathcal{R} \subseteq \mathcal{S} \quad \text{(or equivalently } (R_-, R_+) \leq_{\mathcal{G}, \mathcal{H}} (S_-, S_+))
\]

Proof.

1. Follows via Lemma 4.4.7.5, also see section II.8 on 'Quotient functors' in MacClane’s book.

2. \mathcal{I}'s action on objects and morphisms is well-defined, noting that elements of the same equivalence class induce the same \text{Dep}-morphism by definition. Concerning preservation of identity morphisms:

\[
\mathcal{I}\text{id}_{\mathcal{G}} = \mathcal{I}[\Delta_{\mathcal{G}, \mathcal{G}}]_{\mathcal{G}, \mathcal{G}} = \mathcal{I}[\Delta_{\mathcal{G}, \mathcal{G}}]_{\mathcal{G}, \mathcal{G}} = [\Delta_{\mathcal{G}, \mathcal{G}}]_{\mathcal{G}, \mathcal{G}} \quad \text{a } \mathcal{G}-\text{witness, Lemma 4.4.7.3}
\]

and regarding preservation of composition:

\[
\mathcal{I}([R_l, R_r])_{\mathcal{G}, \mathcal{H}} ; (\mathcal{S}_l, \mathcal{S}_r) = \mathcal{I}((R_l, R_r); (\mathcal{S}_l, \mathcal{S}_r))_{\mathcal{G}, \mathcal{H}} \quad \text{by definition by Corollary ??}
\]

Next, \mathcal{I} is faithful because distinct equivalence classes induce distinct \text{Dep}-morphisms. It is full by passing from \mathcal{R} : \mathcal{G} \to \mathcal{H} to \([(R_-, R_+)]_{\mathcal{G}, \mathcal{H}}). Finally it acts like the identity on objects, so we have an isomorphism of categories.

3. Consequently the composite functor \text{cl} := \mathcal{I} \circ [\mathcal{G}] is well-defined. Then it preserves the natural ordering on morphisms: given \mathcal{R}_l \subseteq \mathcal{S}_l then \mathcal{R} = \mathcal{R}_l; \mathcal{H} \subseteq \mathcal{S}_l; \mathcal{H} = \mathcal{S} because relational composition is monotonic separately in each argument.

\[\square\]

We now deduce an important property, viewing \text{Dep}-morphisms as components \((R_-, R_+)\).

Corollary 4.4.10. For any \(n \geq 0\) and any chain of \text{Dep}-morphisms \((\mathcal{G}_i, \mathcal{G}_{i+1}) : \mathcal{G}_i \to \mathcal{G}_{i+1})_{1 \leq i \leq n},

\[
(R^1_-, R^1_+) \mathrel{\top} \cdots \mathrel{\top} (R^n_-, R^n_+) = \text{cl}(\mathcal{G}_1, \mathcal{G}_{n+1})(\mathcal{G}_1, \mathcal{G}_{i+1})(\cdots (R^1_-, R^1_+); (R^n_-, R^n_+))
\]

By the usual convention, the case \(n = 0\) is the fact that \((\mathcal{G}_-, \mathcal{G}_+) = \text{cl}(\mathcal{G}, \mathcal{G})(\Delta_{\mathcal{G}_-, \mathcal{G}_+})

Proof. This is simply the action of \text{cl} : \text{Cover} \to \text{Dep} on composite morphisms.

\[\square\]
4.5 Dedekind-MacNeille completions

Definition 4.5.1 (Dedekind-MacNeille completion of finite posets).
Given any finite poset \( P \) then:

1. its **Dedekind-MacNeille completion** is the finite join-semilattice \( \mathbb{D}(P) := \text{Open} \downarrow_{P} = (\downarrow_{P}, \cup, \emptyset) \).
2. its associated **canonical order-embedding** is defined:

\[
\epsilon_{P} : P \rightarrow (\downarrow_{P}, \subseteq) \quad \text{where} \quad \epsilon_{P}(p) := \downarrow_{P}[p] = \uparrow_{P}p.
\]

noting that \( \downarrow_{P}[p] = \uparrow_{P}[p] = \downarrow_{P}[\uparrow_{P}p] = \uparrow_{P}p \).

\[ \blacksquare \]

Theorem 4.5.2 (Dedekind-MacNeille embedding for finite posets).
\( \epsilon_{P} : P \rightarrow U\mathbb{D}(P) \) is a well-defined order-embedding, and preserves all meets and joins which exist in \( P \).

Proof. \( \epsilon_{P} \) is a well-defined function because \( \mathbb{D}(P) \) has carrier \( O(\downarrow_{P}) = \{ \downarrow_{P}[X] : X \subseteq P \} \). Then:

\[
p_{1} \leq_{P} p_{2} \iff \uparrow_{P}p_{2} \subseteq \uparrow_{P}p_{1} \iff \uparrow_{P}p_{1} \subseteq \uparrow_{P}p_{2} \iff \epsilon_{P}(p_{1}) \leq_{\text{open}_{P}} \epsilon_{P}(p_{2}).
\]

so that \( \epsilon_{P} \) is an order-embedding. Next, given that \( \bigvee_{P}X \) exists we’ll show that \( \epsilon_{P} \) preserves this join:

\[
\bigvee_{\text{open}_{P}} \epsilon[X] = \bigcup_{p \in P} \uparrow_{P}p = \bigcap_{p \in P} \uparrow_{P}p = \bigcap_{p} \bigvee_{P}X = \epsilon_{P}(\bigvee_{P}X).
\]

Finally suppose that \( \bigwedge_{P}X \) exists. Recalling Definition 4.2.4.3, the join-semilattice of open sets \( \text{Open} \downarrow_{P} \) is isomorphic to the join-semilattice of closed sets \( (C(\downarrow_{P}), \vee, P) \) whose meet is intersection. This isomorphism acts on the embedding image as follows:

\[
\downarrow_{P}[p] \rightarrow \downarrow_{P}[\downarrow_{P}[p]] = \{ p' \in P : \downarrow_{P}[p'] \subseteq \downarrow_{P}[p] \} = \downarrow_{P}p
\]

where in the marked equality we recall that \( \epsilon_{P} \) is an order-embedding. Then since:

\[
\bigwedge_{(C(\downarrow_{P}), \vee, P)} \{ \downarrow_{P}p : p \in X \} = \bigcap_{p \in X} \downarrow_{P}p = \downarrow_{P}\bigwedge_{P}X
\]

applying the inverse join-semilattice isomorphism we deduce that \( \epsilon_{P} \) preserves the meet \( \bigwedge_{P}X \).

\[ \blacksquare \]

4.6 Canonical embeddings and quotients

Every finite join-semilattice \( Q \) arises canonically as a quotient of \( P\text{J}(Q) \). It also embeds into \( P\text{M}(Q) \). In particular, we have the join-preserving morphisms:

\[
\epsilon_{Q} : Q \rightarrow P\text{M}(Q) \quad \quad \sigma_{Q} : P\text{J}(Q) \rightarrow Q \quad \quad \sigma_{Q}(S) := \bigvee_{Q}S.
\]

In this subsection we:

1. Explain that these two constructions are adjoint.
2. Prove a ‘tight extension lemma’ involving them.
3. Show how canonical embeddings/quotients can be defined parametric in a relation \( \mathcal{G} \), generalising \( \epsilon_{Q} \) and \( \sigma_{Q} \).

Lemma 4.6.1 (Adjoint relationships involving \( \epsilon_{Q} \) and \( \sigma_{Q} \)). For every finite join-semilattice \( Q \) we have the commuting diagram:

\[
\begin{array}{ccc}
P\text{J}(Q) & \xrightarrow{\sigma_{Q}} & P\text{M}(Q) \\
\downarrow{\text{J}(Q)} & & \downarrow{\text{M}(Q)} \\
(F\text{J}(Q))^\text{op} & \xrightarrow{(\epsilon_{Q})^\text{op}} & (P\text{M}(Q))^\text{op} \\
\end{array}
\]

Diagram 50.
Equivalently, we have the three equalities:

(a) \((\text{Pirr}Q)^! = \varepsilon_Q \circ \sigma_Q\)

(b) \(\sigma_Q = (\varepsilon_Q^\#)^{-1} \circ (\neg_{J(Q)})^{-1}\)

(c) \(\varepsilon_Q = \neg_M(Q) \circ (\sigma_Q^\#)\).

Proof.

(a) Recall \(\text{Pirr}Q = \mathcal{J}_Q \subseteq J(Q) \times M(Q)\) and observe \(\varepsilon_Q \circ \sigma_Q(\{j\}) = \{m \in M(Q) : j \not\leq Q m\} = \mathcal{J}_Q[j]\) for all \(j \in J(Q)\).

(b) First observe that \(\varepsilon_Q^\# : Q^{op} \rightarrow PM(Q^{op}) = \mathcal{J}(Q)\) has action:

\[\varepsilon_Q^\#(q) = \{m \in M(Q^{op}) : q \not\leq Q m\} = \{j \in J(Q) : j \not\leq Q m\}\]

Then for any subset \(X \subseteq J(Q)\) we calculate:

\[(\varepsilon_Q^\#)^{-1}(X) = \sigma_Q(X)\]

as required.

(c) The third equality follows from the second i.e. (i) reassign \(Q \rightarrow Q^{op}\), (ii) take the adjoints of both sides recalling that \((\neg_{J(Q^{op})})^{-1}\) is self-adjoint, and (iii) post-compose by the isomorphism \(\neg_{J(Q^{op})} = \neg_M(Q)\).

\[\square\]

Lemma 4.6.2 (Tight extension lemma).

1. Each join-semilattice morphism \(f : \mathcal{P}Z \rightarrow Q\) has a canonical compatible morphism:

\[
\begin{array}{ccc}
\mathcal{P}Z & \xrightarrow{\mathcal{J}f^!} & \mathcal{P}M(Q) \\
\downarrow{f} & & \downarrow{\sigma_Q} \\
\mathcal{P}Q & \xrightarrow{\varepsilon_Q} & \mathcal{P}Z
\end{array}
\]

where \(\mathcal{J}f := \{(z, j) \in Z \times J(Q) : j \leq Q f(z)\}\).

2. Each join-semilattice morphism \(f : Q \rightarrow \mathcal{P}Z\) has a canonical extension:

\[
\begin{array}{ccc}
\mathcal{P}M(Q) & \xrightarrow{\mathcal{M}f^!} & \mathcal{P}Z \\
\downarrow{\varepsilon_Q} & & \downarrow{f} \\
\mathcal{P}Q & & \mathcal{P}Z
\end{array}
\]

where \(\mathcal{M}f := \{(m, z) \in M(Q) \times Z : f_*(z) \leq_Q m\}\).

Proof.

1. Recalling that \(\sigma_Q(S) := \bigvee Q S\), we have:

\[\sigma_Q \circ \mathcal{J}f^!(\{z\}) = \sigma_Q(\mathcal{J}f[z]) = \bigvee_Q \{j \in J(Q) : j \leq Q f(\{z\})\} = f(\{z\})\]

for each \(z \in Z\), because every element is the join of those join-irreducibles beneath it. Thus commutativity follows by the freeness of \(\mathcal{P}Z\).
2. The second statement follows from the first directly. That is, given \( f \) then we define \( g \equiv f_s \circ (\neg Z)^{-1} : PZ \to Q^{\text{op}} \) where the self-adjoint (and self-inverse as a function) isomorphism \((\neg Z)^{-1} = ((\neg Z)^{-1})_s : PZ \to (PZ)^{\text{op}}\) takes the relative complement. Applying the first statement yields \( \sigma_{Q^{\text{op}}} \circ \mathcal{J} g^! = g = f_s \circ (\neg Z)^{-1} \) where \( \mathcal{J}g \subseteq Z \times M(Q) \). Equivalently \((\neg Z)^{-1} \circ f = (\mathcal{J} g^!)_s \circ (\sigma_{Q^{\text{op}}})_s \) by taking adjoints, so post-composing with \((\neg Z)^{-1}\) yields:

\[
\begin{align*}
    f &= \neg Z \circ (\mathcal{J} g^!)_s \circ (\sigma_{Q^{\text{op}}})_s, \\
    &= \neg Z \circ \mathcal{J} g^! \circ (\sigma_{Q^{\text{op}}})_s, \\
    &= (\mathcal{J} g^!)^\dagger \circ \neg M(Q) \circ (\sigma_{Q^{\text{op}}})_s \quad \text{by De Morgan duality}, \\
    &= (\mathcal{J} g^!)^\dagger \circ \epsilon_Q \quad \text{by Lemma 4.6.1.(b)}.
\end{align*}
\]

Finally we have \( \mathcal{J} g^! = Mf \) because:

\[
\mathcal{J} g^! [m] = \{ z \in Z : m \leq Q^{\text{op}} g((z)) \} = \{ z \in Z : f_s(\tau) \leq Q m \} = Mf[m]
\]

for all \( m \in M(Q) \).

\[\square\]

We now define ‘similar’ join-semilattice morphisms for any bipartite graph \( \mathcal{G} \).

**Definition 4.6.3 (Canonical embedding and quotient arising from a bipartite graph).**

For each bipartite graph \( \mathcal{G} \) take the unique (surjection, inclusion) factorisation of the JSL \( f \)-morphism \( \mathcal{G}^! : PG_s \to PG_t \):

\[
\begin{array}{ccc}
P G_s & \xrightarrow{\mathcal{G}^!} & P G_t \\
\sigma_G & \downarrow & \downarrow \iota_G \\
\text{Open} G & \xrightarrow{\text{Open} G} & \text{Open} G
\end{array}
\]

where necessarily \( \sigma_{\mathcal{G}}(X) \equiv \mathcal{G}[X] \) and \( \iota_{\mathcal{G}}(X) \equiv X \)

recalling that \( \text{Open} G = (O(\mathcal{G}), \cup, \emptyset) \) consists precisely of the sets \( \mathcal{G}[X] \) where \( X \subseteq \mathcal{G}_t \) by Lemma 4.2.5.3.

We shall see that \( \epsilon_Q \) and \( \iota_{\mathcal{G} \cap \mathcal{Q}} \) are the ‘same maps’, but we also have the maps \( \iota_{\mathcal{G}} \) for arbitrary \( \mathcal{G} \). We use the symbol ‘\( e \)’ because \( \epsilon_Q \) is an embedding which is never an inclusion, whereas \( s_{\mathcal{G}} \) is an inclusion so we use the symbol ‘\( i \)’. Likewise the surjective join-semilattice morphisms \( \sigma_Q \) and \( \sigma_{\mathcal{G} \cap \mathcal{Q}} \) are essentially the same concepts. This will clarify the sense in which \( \epsilon_Q \) and \( \sigma_Q \) are ‘canonical’ morphisms.

**Note 4.6.4.** One could also view \( \mathcal{G} \) as the join-semilattice morphism \( \mathcal{G}^! : (PG_s, \cap, \mathcal{G}_t) \to (PG_s, \cap, \mathcal{G}_s) \) and take the unique (surjection, inclusion) factorisation. The induced factor is then \((\mathcal{C}(\mathcal{G}), \cap, \mathcal{G}_s)\) recalling that \( \mathcal{C}(\mathcal{G}) \) consists of all sets \( \mathcal{G}^!(Y) \) where \( Y \subseteq \mathcal{G}_t \) by Lemma 4.2.5.3. All our subsequent results can be rephrased in terms of these factorisations via the bounded lattice isomorphisms from Lemma 4.2.5.2:

\[
\begin{align*}
    \theta_{\mathcal{G}} : (C(\mathcal{G}), \vee_{C(\mathcal{G})}, \mathcal{G}^!(\emptyset), \cap, \mathcal{G}_s) &\to (O(\mathcal{G}), \cup, \emptyset, \wedge_{O(\mathcal{G})}, \mathcal{G}[\mathcal{G}_s]) \quad \text{where} \quad \theta_{\mathcal{G}}(X) \equiv \mathcal{G}[X] \\
    \kappa_{\mathcal{G}} : (C(\mathcal{G}), \cap, \mathcal{G}_s, \vee_{C(\mathcal{G})}, \mathcal{G}^!(\emptyset)) &\to (O(\mathcal{G}), \cup, \emptyset, \wedge_{O(\mathcal{G})}, \mathcal{G}[\mathcal{G}_s]) \quad \text{where} \quad \kappa_{\mathcal{G}}(X) \equiv \overline{X} \quad \kappa_{\mathcal{G}}^{-1}(Y) \equiv \overline{Y}
\end{align*}
\]

However, the very same isomorphisms allow us to suppress the closure semilattices.

\[\square\]

Just as the morphisms \( \epsilon_Q \) and \( \sigma_Q \) collectively satisfy an adjoint relationship, so too do the morphisms \( \iota_{\mathcal{G}} \) and \( \sigma_{\mathcal{G}} \). In order to describe it, we first need explicit notation for a certain composite isomorphism.

**Definition 4.6.5 (Isomorphism representing the order-dual of \( \text{Open} G \)).**

For each bipartite graph \( \mathcal{G} \subseteq \mathcal{G}_s \times \mathcal{G}_t \) we have the join-semilattice isomorphism:

\[
\begin{align*}
    \partial_{\mathcal{G}} : (\text{Open} G)^{\text{op}} &\xrightarrow{\left(\theta_{\mathcal{G}}^{-1}\right)^{\text{op}}} (C(\mathcal{G}), \cap, \mathcal{G}_s) \xrightarrow{\kappa_{\mathcal{G}}} \text{Open} \mathcal{G} \quad \text{with action} \quad \partial_{\mathcal{G}}(X) = \mathcal{G}^{!}(\overline{X}) = \overline{\mathcal{G}[X]} \\
    \partial_{\mathcal{G}}^{-1} : \text{Open} \mathcal{G} &\xrightarrow{\kappa_{\mathcal{G}}^{-1}} (C(\mathcal{G}), \cap, \mathcal{G}_s) \xrightarrow{\theta_{\mathcal{G}}^{\text{op}}} (\text{Open} G)^{\text{op}} \quad \text{with action} \quad \partial_{\mathcal{G}}^{-1}(X) = \mathcal{G}[\overline{X}]
\end{align*}
\]

where well-definedness follows by restricting Lemma 4.2.5.2 i.e. the bounded lattice isomorphisms also described in Note 4.6.4 directly above, and also De Morgan duality.

\[\square\]
The above isomorphisms are collectively closed under adjoints, and also collectively relate the components of the canonical natural isomorphism \( rep : \text{Id}_{\text{JSL}_f} \Rightarrow \text{Open} \circ \text{Pirr}. \)

**Lemma 4.6.6 (Basic properties of the isomorphisms \(\partial_G\)).**

1. For every bipartite graph \(G\) we have:
   \[
   (\partial_G)_* = \partial_G \quad \text{and} \quad (\partial_G^{-1})_* = (\partial_G^{-1})
   \]

2. For every finite join-semilattice \(Q\) we have:
   \[
   (\text{Open} \circ \text{PirrQ})^{\text{op}} \circ \text{rep}_{\text{Q}^{\text{op}}} = (\text{Open} \circ \text{Pirr})^{\text{op}} \circ \partial_{\text{PirrQ}} \circ \text{Open} \circ \text{PirrQ}
   \]

**Proof.**

1. By Lemma 3.0.3.2 \((\partial_G)_* = (\partial_G^{-1})^{\text{op}} : (\text{Open}G)^{\text{op}} \Rightarrow \text{Open}G\), which has the same type as \(\partial_G\) and acts in the same way. Similarly we have \((\partial_G^{-1})_* = \partial_G^{\text{op}}\) which has the same type as \(\partial_G^{-1}\) and acts in the same way.

2. We first verify the triangle on the right. Its typing is correct, so consider its action:
   \[
   \partial_{\text{PirrQ}^{\text{op}}} \circ \text{rep}_{\text{Q}^{\text{op}}} (q) = \partial_{\text{PirrQ}^{\text{op}}} (\text{rep}_{\text{Q}^{\text{op}}} (q)) = \partial_{\text{PirrQ}^{\text{op}}} (\{j \in J(Q) : j \leq Q q\}) = \text{PirrQ} (\{j \in J(Q) : j \leq Q q\}) = \{m \in M(Q) : \exists j \in J(Q), j \leq Q q \text{ and } j \preceq Q m\} = \{m \in M(Q) : \forall j \in J(Q), j \leq Q q \implies j \preceq Q m\} = \text{rep}_{\text{Q}^{\text{op}}} (q)
   \]
   Thus \(\text{rep}_{\text{Q}^{\text{op}}} = \partial_{\text{PirrQ}^{\text{op}}} \circ \text{rep}_{\text{Q}^{\text{op}}}\) so by the standard law of composite inverses:
   \[
   \text{rep}_{\text{Q}^{\text{op}}}^{-1} = (\text{rep}_{\text{Q}^{\text{op}}}^{-1} \circ \partial_{\text{PirrQ}^{\text{op}}}^{-1} \circ \text{rep}_{\text{Q}^{\text{op}}}^{-1})^{-1} \circ \partial_{\text{PirrQ}^{\text{op}}}^{-1} \circ \text{rep}_{\text{Q}^{\text{op}}}^{-1}
   \]
   also using Lemma 3.0.3.2.

Recalling that \(\text{Pirr}_f, = (\text{Pirr}_f)^{\text{op}} = (\text{Pirr}_f)^{\text{op}}\) for any \(\text{JSL}_f\)-morphism \(f\), the correspondence between adjoints in the other direction is captured precisely by the isomorphisms \(\partial_G\). To see this, first recall that:

\[
\text{OD}_f : \text{JSL}_f^{\text{op}} \rightarrow \text{JSL}_f \quad \text{and} \quad (-)^{\text{op}} : \text{Dep}^{\text{op}} \rightarrow \text{Dep}
\]

are the self-duality functors on their respective categories.

**Theorem 4.6.7 (\(\partial\) defines a natural isomorphism).**

The isomorphisms \(\partial_G\) collectively define a natural isomorphism:

\[
\partial : \text{OD}_f \circ \text{Open}^{\text{op}} \Rightarrow \text{Open} \circ (-)^{\text{op}}
\]

Consequently, for each \(\text{Dep}\)-morphism \(\mathcal{R} : \mathcal{G} \rightarrow \mathcal{H}\) we have:

\[
(\text{Open} \circ \mathcal{R})_* = \partial_{\mathcal{G}}^{-1} \circ \text{Open} \circ \mathcal{R} \circ \partial_{\mathcal{H}}
\]

**Proof.** We already know that each \(\partial_G : (\text{Open}G)^{\text{op}} \rightarrow (\text{Open}G)^{\text{op}}\) is well-defined \(\text{JSL}_f\)-isomorphism. Observe that the functors \(\text{OD}_f \circ \text{Open}^{\text{op}}\) and \(\text{Open} \circ (-)^{\text{op}}\) both have type \(\text{Dep}^{\text{op}} \rightarrow \text{JSL}_f\). Then we need to verify that the following diagram commutes:

\[
\begin{array}{ccc}
\text{OD}_f \circ \text{Open}^{\text{op}} \mathcal{H} & \xrightarrow{\partial_{\mathcal{H}}} & \text{Open} \circ (-)^{\text{op}} (\mathcal{H}) = \text{Open} \mathcal{H} \\
(\text{Open} \circ \mathcal{R})_* & \downarrow & \downarrow \\
\text{OD}_f \circ \text{Open}^{\text{op}} \mathcal{G} & \xrightarrow{\partial_{\mathcal{G}}} & \text{Open} \circ (-)^{\text{op}} (\mathcal{G}) = \text{Open} \mathcal{G}
\end{array}
\]
or equivalently that \((\text{Open} \mathcal{R})_* = \partial^{-1}_G \circ \text{Open} \mathcal{R} \circ \partial_H\), where the latter has action:

\[
\partial^{-1}_G \circ \text{Open} \mathcal{R} \circ \partial_H(Y) = \partial^{-1}_G \circ \text{Open} \mathcal{R}(\mathcal{H}[\tilde{Y}]) \quad \text{by definition}
\]

\[
= \partial^{-1}_G \circ \mathcal{R}^{-1} \circ \mathcal{H}[\tilde{Y}] \quad \text{by definition}
\]

\[
= \partial^{-1}_G \circ \mathcal{H} \circ \mathcal{R}^{-1}[Y] 
= \partial^{-1}_G \circ \mathcal{R}^{-1}[\tilde{Y}]
\]

\[
= \mathcal{G}[\mathcal{R}[\tilde{Y}]] \quad \text{by definition}
\]

\[
= \mathcal{G}^\dagger \circ \neg_{\mathcal{G}} \circ \mathcal{R}^\dagger \circ \neg_{\mathcal{H}}(Y) 
= \mathcal{G}^\dagger \circ \mathcal{R}^\dagger(Y) \quad \text{by De Morgan duality}
\]

Finally observe that this morphism was already described in Lemma 4.2.6.4, where it was shown to have adjoint \(\lambda Y \in \mathcal{O}(\mathcal{G}).\mathcal{R}^\dagger \circ \mathcal{G}^\dagger(Y) = \text{Open} \mathcal{R}\), so we are done.

\[\square\]

**Lemma 4.6.8** (Adjoint relationships involving \(\iota_{\mathcal{G}}\) and \(\sigma_{\mathcal{G}}\)).

*For every bipartite graph \(\mathcal{G}\) we have the commuting diagram:*

\[
\begin{array}{c}
\text{PG}_s \\
\downarrow_{(\iota_{\mathcal{G}})^{-1}} \\
(\text{PG}_s)^{\text{op}} \\
\downarrow_{(\iota_{\mathcal{G}})^{-1}} \\
\text{Open} \mathcal{G} \\
\downarrow_{\partial_{\mathcal{G}}} \\
\text{Open} \mathcal{G} \\
\downarrow_{(\text{Open} \mathcal{G})^{\text{op}}} \\
\iota_{\mathcal{G}} \\
\downarrow_{(\sigma_{\mathcal{G}})_*} \\
(\text{PG}_t)^{\text{op}} \\
\downarrow_{(\sigma_{\mathcal{G}})_*} \\
\text{PG}_t \\
\downarrow_{(\text{PG}_t)^{\text{op}}} \\
\end{array}
\]

*Equivalently, we have the three equalities:*

\[\begin{align*}
(\text{a}) \quad & \mathcal{G}^\dagger = \iota_{\mathcal{G}} \circ \sigma_{\mathcal{G}} \\
(\text{b}) \quad & \sigma_{\mathcal{G}} = \partial_{\mathcal{G}} \circ (\iota_{\mathcal{G}})_* \circ (\neg_{\mathcal{G}})^{-1} \\
(\text{c}) \quad & \iota_{\mathcal{G}} = \neg_{\mathcal{G}} \circ (\sigma_{\mathcal{G}})_* \circ \partial_{\mathcal{G}}^{-1}
\end{align*}\]

**Proof.**

(a) This is the unique (surjection,inclusion) factorisation described in Definition 4.6.3.

(b) For any subset \(X \subseteq \mathcal{G}_s\) we have:

\[
\partial_{\mathcal{G}} \circ (\iota_{\mathcal{G}})_* \circ (\neg_{\mathcal{G}})^{-1}(X) = \partial_{\mathcal{G}} \circ (\iota_{\mathcal{G}})_*(X)
\]

\[
= \partial_{\mathcal{G}}(\{Y \in \mathcal{O}(\mathcal{G}) : \iota_{\mathcal{G}}(Y) \leq \mathcal{G}_s, X\})
\]

\[
= \partial_{\mathcal{G}}(\{Y \in \mathcal{O}(\mathcal{G}) : Y \leq X\})
\]

\[
= \partial_{\mathcal{G}}(\mathcal{G}(X)) \quad \text{by Lemma 4.2.7.1}
\]

\[
= \mathcal{G}^\dagger \circ \neg_{\mathcal{G}} \circ \mathcal{G}^\dagger \circ \neg_{\mathcal{G}}(X) \quad \text{action of } \partial_{\mathcal{G}} \text{ and } \mathcal{G}
\]

\[
= \mathcal{G}^\dagger \circ \mathcal{G}^\dagger(X) \quad \text{by De Morgan duality}
\]

\[
= \mathcal{G}^\dagger(X) \quad \text{by (1↑↑)}
\]

(c) Instantiate the previous statement by assigning \(\mathcal{G} \rightarrow \tilde{\mathcal{G}}\) and take the adjoints of both sides to obtain:

\[\begin{align*}
(\sigma_{\mathcal{G}})_* = (\neg_{\mathcal{G}})_*^{-1} \circ \iota_{\mathcal{G}} \circ (\partial_{\mathcal{G}})_* = (\neg_{\mathcal{G}})_*^{-1} \circ \iota_{\mathcal{G}} \circ \partial_{\mathcal{G}}
\end{align*}\]

using Lemma 4.6.6.1. The statement follows by post-composing with \(\neg_{\mathcal{G}}\) and pre-composing with \(\partial_{\mathcal{G}}^{-1}\).

\[\square\]

Finally we explain the relationship between \(\varepsilon_{\mathcal{G}}\) and \(\iota_{\mathcal{G}}\), and also \(\sigma_{\mathcal{G}}\) and \(\sigma_{\mathcal{Q}}\).
Lemma 4.6.9 (The relationship between \( \iota_G \) and \( e_Q \)). For every finite join-semilattice \( Q \) and bipartite graph \( G \),

\[
\begin{array}{ccc}
\text{OpenPirr}Q & \xrightarrow{\text{rep}_Q} & PM(Q) \\
\text{rep}_Q & \Downarrow & e_Q \\
Q & \xrightarrow{\iota_G} & G \\
\end{array}
\]

where:

1. \( \text{rep}_Q(q) := \{ m \in M(Q) : q \preceq_M m \} \) is a component of the natural isomorphism \( \text{rep} : \text{id}_{\text{JSL}} \Rightarrow \text{Open} \circ \text{Pirr} \),

2. the morphism \( M_{\iota_G}^{-1} \) is the canonical extension of \( \iota_G \) as defined in Lemma 4.6.2.2, so that:

\[
M_{\iota_G}(X,g_t) : \iff (\iota_G)_* (\overline{g_t}) \preceq_{\text{Open}G} X \iff \text{in}_G(\overline{g_t}) \subseteq X
\]

Proof. First observe that the inclusion \( \iota_{\text{Pirr}Q} : \text{OpenPirr}Q \rightarrow PM(Q) \) is well-typed because \( \text{Pirr}Q \), \( Q \) is well-typed. Then the left triangle clearly commutes because the embedding \( e_Q \) and the isomorphism \( \text{rep}_Q \) act in the same way. The second triangle commutes via the tight extension Lemma 4.6.2.2. The final \( \iff \) holds because \( (\iota_G)_* (\overline{g_t}) = \text{in}_G(\overline{g_t}) \) by Lemma 4.2.7.1.

Lemma 4.6.10 (The relationship between \( \sigma_G \) and \( e_Q \)). For every finite join-semilattice \( Q \) and bipartite graph \( G \),

\[
\begin{array}{ccc}
\text{OpenPirr}Q & \xrightarrow{\text{rep}_Q^{-1}} & Q \\
\text{rep}_Q^{-1} & \Downarrow & \sigma_Q \\
\text{PJ}(Q) & \xrightarrow{\text{J}\sigma_G^{-1}} & \text{PJ}(\text{Open}G) \\
\text{J}\sigma_G^{-1} & \Downarrow & \sigma_G \\
\text{Open}G & \xrightarrow{\text{OPEN}_{\text{G}}^{-1}} & G \\
\end{array}
\]

where:

1. \( \text{rep}_Q^{-1}(S) := \wedge_Q M(Q)/S \) is a component of the natural isomorphism \( \text{rep}^{-1} : \text{id}_{\text{JSL}} \Rightarrow \text{Open} \circ \text{Pirr} \),

2. the morphism \( \text{J}\sigma_G^{-1} \) is the canonical morphism compatible with \( \sigma_G \) as defined in Lemma 4.6.2.1, so that:

\[
\text{J}\sigma_G(g_s,Y) : \iff Y \preceq_{\text{Open}_G} \sigma_G(\{ g_s \}) \iff Y \subseteq G[g_s]
\]

Proof. Although the first triangle is easily shown by considering the action, it can also be formally derived using the above results:

\[
\sigma_Q = (e_Q)_* \circ \overline{\text{J}(Q)}^{-1}
\]

by Lemma 4.6.1.(b)

\[
= (\text{rep}_Q)_* \circ (\text{Pirr}_Q)_* \circ \overline{\text{J}(Q)}^{-1}
\]

by Lemma 4.6.9

\[
= (\text{rep}_Q)_* \circ (\iota_{\text{Pirr}Q})_* \circ \overline{\text{J}(Q)}^{-1}
\]

by Lemma 4.6.8.(b)

\[
= (\text{rep}_Q)_* \circ \partial_{\text{Pirr}Q}^{-1} \circ \sigma_{\text{Pirr}Q} \circ \overline{\text{J}(Q)}^{-1}
\]

by Lemma 4.6.6.2

Finally, the right triangle follows via the tight extension Lemma 4.6.2.1.

4.7 Monos, Epis and Isos

We begin with characterisations of \( \text{Dep} \)'s monomorphisms and epimorphisms.

Lemma 4.7.1. Let \( R : G \rightarrow H \) be any \( \text{Dep} \)-morphism.

1. The following statements are equivalent.

(a) \( R \) is monic.

(b) \( \text{Open}R \) is injective.
(4.2.10) and that it suffices to show that we know that

\[ \text{Lemma 4.7.2} \quad \text{(Adjoints and inverses commute)} \]

**Proof.**

1. (a \iff b): Follows because \( \text{Open} \) is an equivalence functor by Theorem 4.2.10 and \( JSL_f \)-monos are precisely the injective morphisms by Lemma 3.0.8.1.

2. The following statements are equivalent.

   (a) \( \mathcal{R} \) is epic.
   
   (b) \text{Open}\( \mathcal{R} \) is surjective.
   
   (c) Any of the four equivalent statements holds:

\[
\begin{align*}
\text{in}_\mathcal{R} &= \text{in}_\mathcal{H} \\
\mathcal{H}^i \circ \text{in}_\mathcal{R} &= \mathcal{H}^i
\end{align*}
\]

They could be re-written in terms of \( \text{cl}_\mathcal{R}, \text{cl}_\mathcal{H} \) and \((-)^i\) using De Morgan duality.

Proof:

1. (a \iff b): Follows because \( \text{Open} \) is an equivalence functor by Theorem 4.2.10 and \( JSL_f \)-monos are precisely the injective morphisms by Lemma 3.0.8.1.

(c) We always have \( \text{cl}_\mathcal{G} \leq \text{cl}_\mathcal{R} \) because:

\[
\begin{align*}
\text{cl}_\mathcal{R} &= \mathcal{R}^i \circ \mathcal{R}^i = \mathcal{R}^i \circ \mathcal{R}^i \circ \text{cl}_\mathcal{G} = \text{cl}_\mathcal{R} \circ \text{cl}_\mathcal{G} \geq \text{cl}_\mathcal{G}
\end{align*}
\]

using Lemma 4.1.10 and that \( \text{cl}_\mathcal{R} \) is extensive. We always have \( \mathcal{G}^i \leq \mathcal{G}^i \circ \text{cl}_\mathcal{R} \) because \( \text{cl}_\mathcal{R} \) is extensive and \( \mathcal{G}^i \) is monotonic. Finally \( \text{cl}_\mathcal{R} \leq \text{cl}_\mathcal{G} \) if \( \mathcal{G}^i \circ \text{cl}_\mathcal{R} \leq \mathcal{G}^i \) via the usual adjoint relationship.

(b \iff c): First assume (c). By Lemma 4.2.2 it suffices to show that \( \mathcal{R}^i \circ \mathcal{G}^i \) is injective on the restricted domain \( O(\mathcal{G}) \subseteq \mathcal{P}\mathcal{G}_i \). Given any \( Y_1, Y_2 \in O(\mathcal{G}) \) then:

\[
\begin{align*}
&\mathcal{R}^i \circ \mathcal{G}^i(Y_1) = \mathcal{R}^i \circ \mathcal{G}^i(Y_2) \quad \text{by assumption} \\
&\Rightarrow \mathcal{R}^i \circ \mathcal{R}^i \circ \mathcal{G}^i(Y_1) = \mathcal{R}^i \circ \mathcal{R}^i \circ \mathcal{G}^i(Y_2) \quad \text{apply function} \\
&\Rightarrow \text{cl}_\mathcal{G} \circ \mathcal{G}^i(Y_1) = \text{cl}_\mathcal{G} \circ \mathcal{G}^i(Y_2) \quad \text{by assumption} \\
&\Rightarrow \mathcal{G}^i(Y_1) = \mathcal{G}^i(Y_2) \quad \text{by (\text{↓↓})} \\
&\Rightarrow \text{in}_\mathcal{G}(Y_1) = \text{in}_\mathcal{G}(Y_2) \quad \text{apply function} \\
&\Rightarrow Y_1 = Y_2 \quad \text{by openness}
\end{align*}
\]

so that \( \text{Open}\( \mathcal{R} \) is injective. Conversely, assuming that \( \text{Open}\( \mathcal{R} \) is injective it suffices to establish that \( \mathcal{G}^i \circ \mathcal{R}^i \circ \mathcal{R}^i = \mathcal{G}^i \). By Lemma 4.2.2 we know that \( \mathcal{R}^i \circ \mathcal{G}^i \) restricts to an injection on \( O(\mathcal{G}) \subseteq \mathcal{P}\mathcal{G}_i \). We have the equalities:

\[
\mathcal{R}^i \circ \mathcal{G}^i \circ \mathcal{R}^i = \mathcal{R}^i \circ \mathcal{G}^i \circ \mathcal{G}^i \circ \mathcal{R}^i \circ \mathcal{R}^i = \mathcal{R}^i \circ \mathcal{R}^i \circ \mathcal{R}^i
\]

using (\text{↓↓}) and also \( \mathcal{R}^i \circ \text{cl}_\mathcal{G} = \mathcal{R}^i \) because \( \mathcal{R} \) is a Dep-morphism. Putting them together yields:

\[
\mathcal{R}^i \circ \mathcal{G}^i \circ \mathcal{R}^i \circ \mathcal{R}^i(X) = \mathcal{R}^i \circ \mathcal{G}^i(\mathcal{G}^i(X))
\]

for any \( X \subseteq \mathcal{G}_s \). Then by injectivity and the fact that the \( \mathcal{G} \)-image of any subset is \( \mathcal{G} \)-open (Lemma 4.2.5.3), we deduce that \( \mathcal{G}^i \circ \mathcal{R}^i \circ \mathcal{R}^i = \mathcal{G}^i \).

2. Since the \( JSL_f \)-epis are precisely the surjective morphisms we have (a \iff b). That (c \iff c) follows from the previous argument and De Morgan duality. Furthermore (b \iff c) follows from the previous statement and duality. For example, \( \mathcal{R} : \mathcal{G} \rightarrow \mathcal{H} \) is epic iff its dual \( \mathcal{R}^\circ : \mathcal{H} \rightarrow \mathcal{G} \) is monic iff \( \text{cl}_\mathcal{R} = \text{cl}_\mathcal{H} \) iff \( \text{in}_\mathcal{R} = \text{in}_\mathcal{H} \) by De Morgan duality i.e. Lemma 4.2.7.2.

The following result should be compared to Lemma 3.0.3.2.

**Lemma 4.7.2** (Adjoints and inverses commute). If \( \mathcal{R} : \mathcal{G} \rightarrow \mathcal{H} \) is a Dep-isomorphism then \( (\mathcal{R}^{-1})^\circ = (\mathcal{R}^\circ)^{-1} \).
Proof. The functoriality of \((-)^\dagger\) : \text{Dep}^{op} \to \text{Dep} informs us that \(\mathcal{R}^\dagger \circ (\mathcal{R}^{-1})^\dagger = (\mathcal{R}^{-1} \circ \mathcal{R})^\dagger = \text{id}_{\mathcal{H}} = \text{id}_{\mathcal{H}^\dagger}\) and similarly \((\mathcal{R}^{-1})^\dagger \circ \mathcal{R}^\dagger = (\mathcal{R} \circ \mathcal{R}^{-1})^\dagger = \text{id}_{\mathcal{G}^\dagger} = \text{id}_{\mathcal{G}}\).

\[\square\]

**Lemma 4.7.3 (Dep-isomorphisms via components).**

Given \text{Dep}-morphisms \(\mathcal{R} : \mathcal{G} \to \mathcal{H}\) and \(\mathcal{S} : \mathcal{H} \to \mathcal{G}\) then t.f.a.e.

1. \(\mathcal{R}\) is a \text{Dep}-isomorphism with inverse \(\mathcal{S}\).

2. We know either \(\mathcal{R} : \mathcal{G} \to \mathcal{H}\) or \(\mathcal{S} : \mathcal{H} \to \mathcal{G}\). We also know either \(\mathcal{S} \circ \mathcal{R} = \mathcal{H}\) or \(\mathcal{R} \circ \mathcal{S} = \mathcal{G}\).

Proof. Their equivalence follows by considering the diagrams:

\[
\begin{array}{c}
\mathcal{G}_s \xrightarrow{\mathcal{R}_-} \mathcal{H}_s \\
\mathcal{G}_t \xrightarrow{\mathcal{R}_+} \mathcal{H}_t
\end{array} \quad \begin{array}{c}
\mathcal{H}_s \xrightarrow{\mathcal{S}_-} \mathcal{G}_s \\
\mathcal{H}_t \xrightarrow{\mathcal{S}_+} \mathcal{G}_t
\end{array}
\]

which commute because \(\mathcal{R}\) and \(\mathcal{S}\) are \text{Dep}-morphisms. In the left diagram, any composite from \(\mathcal{G}_s\) to \(\mathcal{G}_t\) equals \(\mathcal{R} \circ \mathcal{S}\). Thus the latter equals \(\text{id}_{\mathcal{G}} = \mathcal{G}\) iff \(\mathcal{R} \circ \mathcal{S} = \mathcal{G}\) or alternatively \(\mathcal{S} \circ \mathcal{R} = \mathcal{H}\). Likewise in the right diagram.

In the previous result, we assumed the candidate inverse was already known to be a \text{Dep}-morphism. We then relied upon knowing some of the component relations. The following result avoids the component relations, and makes no assumptions concerning the candidate inverse.

**Lemma 4.7.4 (Dep-isomorphisms via functional compositions).**

Given any \text{Dep}-morphism \(\mathcal{R} : \mathcal{G} \to \mathcal{H}\) and relation \(\mathcal{S} \subseteq \mathcal{H} \times \mathcal{G}\), the following statements are equivalent.

1. \(\mathcal{R}\) is a \text{Dep}-isomorphism with inverse \(\mathcal{S}\).

2. The following four equations hold:

   \[
   \begin{align*}
   (a) \quad & \mathcal{R}^\dagger \circ \mathcal{H}^\dagger = \mathcal{G}^\dagger \circ \mathcal{S}^\dagger \\
   (b) \quad & \mathcal{S}^\dagger \circ \mathcal{G}^\dagger = \mathcal{H}^\dagger \circ \mathcal{R}^\dagger \\
   (c) \quad & \mathcal{R}^\dagger \circ \mathcal{G}^\dagger = \mathcal{H}^\dagger \circ \mathcal{S}^\dagger \\
   (d) \quad & \mathcal{S}^\dagger \circ \mathcal{H}^\dagger = \mathcal{G}^\dagger \circ \mathcal{R}^\dagger.
   \end{align*}
   \]

Proof.

- \((1 \Rightarrow 2)\): Assume that \(\mathcal{R}\) is an isomorphism with inverse \(\mathcal{S} : \mathcal{H} \to \mathcal{G}\), so that \(\mathcal{S}\) is also a \text{Dep}-isomorphism. We first show that (a) holds.

\[
\begin{align*}
\mathcal{R}^\dagger \circ \mathcal{H}^\dagger &= (\mathcal{R} \circ \mathcal{H})^\dagger \circ \mathcal{H}^\dagger \quad \text{associated component} \\
&= \mathcal{R}^\dagger \circ (\mathcal{H} \circ \mathcal{R})^\dagger \circ \mathcal{H}^\dagger \quad \text{by } (\dagger) \\
&= \mathcal{R}^\dagger \circ \mathcal{H}^\dagger \circ \mathcal{R}^\dagger \circ \mathcal{H}^\dagger \quad \text{by definition} \\
&= \mathcal{R}^\dagger \circ \mathcal{H}^\dagger \circ \mathcal{R}^\dagger \circ \mathcal{H}^\dagger \quad \text{by definition} \\
&= \mathcal{G}^\dagger \circ \mathcal{S}^\dagger \quad \text{by definition} \\
&= \mathcal{G}^\dagger \circ \mathcal{S}^\dagger \quad \text{by Lemma 4.7.3}.
\end{align*}
\]

Since \(\mathcal{S}\) is also a \text{Dep}-isomorphism we obtain (a) for it, which is actually (b). Finally \(\mathcal{R}^\dagger\) and \(\mathcal{S}^\dagger\) are also \text{Dep}-isomorphisms so we obtain (a) for each of them, and applying De Morgan duality yields (c) and (d) respectively.

- \((2 \Rightarrow 1)\): The calculation:

\[
\begin{align*}
\mathcal{S}^\dagger \circ \mathcal{H}^\dagger &= \mathcal{S}^\dagger \circ \mathcal{H}^\dagger \circ \mathcal{R}^\dagger \circ \mathcal{S}^\dagger \\
&= \mathcal{G}^\dagger \circ \mathcal{R}^\dagger \circ \mathcal{H}^\dagger \circ \mathcal{R}^\dagger \circ \mathcal{S}^\dagger \quad \text{by (d)} \\
&= \mathcal{G}^\dagger \circ \mathcal{G}^\dagger \circ \mathcal{S}^\dagger \quad \text{by (a)} \\
&= \mathcal{G}^\dagger \circ \mathcal{S}^\dagger \quad \text{by Lemma 4.7.3}.
\end{align*}
\]
4.1.10

Lemma 4.7.6. Every bipartite graph

\[ (R \circ S) \circ H = S \circ H \circ R \]

by (\( \dagger \))

\[ (R \circ S) \circ H = S \circ H \circ (R \circ S) \]

by (d)

\[ = S \circ (H \circ R) \]

by (b)

Then since \( S \circ (H \circ R) \subseteq S \circ G \), by co-extensivity, extensitivity and monotonicity, we deduce that \( (R \circ S) \circ H = S \circ (H \circ R) \) and hence \( R \circ S = id_H \).

A useful class of isomorphisms arises from bijections.

Definition 4.7.5 (Bipartite Dep-isomorphisms). A bipartite Dep-isomorphism \( R : G \to H \) is a Dep-morphism witnessed by bijections i.e. \( \gamma : R = H \gamma \) for injective functions \( f : G_\gamma \to H_\gamma \) and \( g : H_\gamma \to G_\gamma \).

Lemma 4.7.6. Every bipartite Dep-isomorphism is a Dep-isomorphism.

Proof. If \( R : G \to H \) has bijective witnesses \( (\lambda, \gamma) \), it has an inverse \( R^{-1} : H \to G \) via witnesses \( (\lambda^{-1}, \gamma^{-1}) \).

Note 4.7.7 (Dep-objects as bipartite graphs).
1. Any relation \( G \subseteq G_s \times G_t \) can be viewed as an undirected bipartite graph with vertices \( G_s + G_t \) (disjoint union), edges \( E(e_1(g_s), e_2(g_t)) \subseteq \gamma_1(G_s, G_t) \) (and no others) and bipartition \( (e_1[G_s], e_2[G_t]) \). The latter is a pair and is sometimes called an ordered bipartition. The dual Dep-object \( \tilde{G} \subseteq G_t \times G_s \) yields the same bipartite graph modulo-isomorphism, yet has a distinct ordered bipartition \( (e_2[G_t], e_1[G_s]) \) unless \( G_s = G_t = \emptyset \).

2. A bipartite Dep-isomorphism \( R : G \to H \) has bijective witnesses \( (f, g) \). They induce a bipartite graph isomorphism \( f + \tilde{g} \) between the underlying undirected bipartite graphs. Since the isomorphisms must respect the bipartitions, not every bipartite graph isomorphism arises in this way.

Example 4.7.8 (A bipartite Dep-isomorphism). The relation \( G \subseteq X \times Y \) below on the left,

is bipartite Dep-isomorphic to \( P \) where \( 3 \) is the 3-chain. We’ve depicted the associated components of the Dep-isomorphism, which are not functional.

Example 4.7.9 (A non-bipartite Dep-isomorphism). If \( G = \{x_1, x_2\} \times \{y\} \), the canonical Dep-isomorphism \( red_G : G \to P \) has associated components:

It cannot be a bipartite Dep-iso because \( 2 > |G| \).

Nevertheless we have the following clarifying result.
Lemma 4.7.10 (Bipartite graph isomorphism by restriction). Every $G$ has a domain/codomain restriction which is bipartite $\text{Dep}$-isomorphic to $\text{PirrOpen}_G$.

Proof. By Lemma 4.2.5.3 we know that:

$$J(\text{Open}G) \subseteq \{G[g_s] : g_s \in G_s\} \quad M(\text{Open}G) \subseteq \{\text{in}_G(\overline{M}) : g_t \in G_t\}.$$

Then for each $X \in J(\text{Open}G)$ choose $j_X \in G_s$ such that $G[j_X] = X$, and for each $Y \in M(\text{Open}G)$ choose $m_Y \in G_t$ such that $\text{in}_G(\overline{M}) = Y$. These chosen elements are necessarily distinct e.g. if $X \neq Y$ then $j_X \neq j_Y$, and induce both a restriction $G'$ of $G$ and a pair of relations $(R_l, R_r)$ as follows:

$$J := \{j_X : X \in J(\text{Open}G)\} \quad M := \{m_Y : Y \in M(\text{Open}G)\} \quad G' := G \cap J \times M$$

$$R_l := \{(j_X, X) : X \in J(\text{Open}G)\} \subseteq G_s \times J(\text{Open}G) \quad G' \subseteq G \cap J \times M$$

We now establish that $(R_l, R_r) : G' \to \text{PirrOpen}_G$ is a bipartite $\text{Dep}$-isomorphism. Clearly $R_l$ and $R_r$ are bijective functions. Further recall that $\text{PirrOpen}_G := \mathfrak{g} \subseteq J(\text{Open}G) \times M(\text{Open}G)$ i.e. the domain/codomain restriction of the binary relation $\mathfrak{g}$ on $G_s$. Then:

$$R_l : \mathfrak{g} \subseteq \{j_X, Y\} \quad \iff \quad G[j_X] \in \text{in}_G(\overline{M})$$

$$\iff \quad j_X \notin G^l \circ G' \circ G^l(\overline{M})$$

$$\iff \quad \exists g_t \in G_t \subseteq \{G[j_X, g_t] \land R_r(Y, g_t)\}$$

$$\iff \quad G[j_X, m_Y]$$

Thus $R_l; \mathfrak{g} \subseteq G' \subseteq R_l; \mathfrak{g}$ as required.

Next we describe certain degenerate yet useful isomorphisms.

Lemma 4.7.11 (Isomorphisms via join/meet generators). Given any finite join-semilattice $S$ and any subsets $J(S) \subseteq X \subseteq Q$ and $M(S) \subseteq Y \subseteq Q$, consider the domain/codomain restriction $G := \mathfrak{s} \subseteq |X \times Y|$. Then $R$ and $S$ are mutually inverse $\text{Dep}$-isomorphisms:

$$R : G \to \text{Pirr}S \quad R := \mathfrak{r} \subseteq |X \times M(S)| \quad R-(x, j) : \iff \quad j \subseteq x \quad R-(m, y) : \iff \quad m \subseteq y$$

$$S : \text{Pirr}S \to G \quad S := \mathfrak{s} \subseteq |J(S) \times Y| \quad S-(x, j) : \iff \quad x \subseteq s \quad S-(y, m) : \iff \quad y \subseteq s \cdot m.$$

Proof. We first verify that the following diagram commutes:

$$\begin{array}{c}
Y \xrightarrow{R} M(S) \xrightarrow{S} Y \\
\downarrow \quad \downarrow \quad \downarrow \\
X \xrightarrow{R} J(S) \xrightarrow{S} X
\end{array}$$

That is, $\mathfrak{r} \subseteq R$ and $\mathfrak{s} \subseteq S$ follow by Lemma 2.2.3.7 and $M(S) \subseteq Y$. Similarly, $\mathfrak{s} \subseteq S$ and $\mathfrak{s} \subseteq S$ follow by Lemma 2.2.3.7 and $J(S) \subseteq X$. Thus both $R$ and $S$ are well-defined $\text{Dep}$-morphisms and the reader can verify that $(\mathfrak{r}, \mathfrak{r})$ and $(\mathfrak{s}, \mathfrak{s})$ are the associated components. Finally we verify they are mutually inverse:

$$R \subseteq S \subseteq (x, y) \quad \iff \quad \exists m \in M(S). (x \notin s \quad y \subseteq m)$$

$$\iff \quad \neg \forall m \in M(S). (y \subseteq m \Rightarrow x \subseteq s m)$$

$$\iff \quad x \notin s \quad y$$

$$S \subseteq R \subseteq (j, m) \quad \iff \quad \exists y \in Y. (j \notin s \quad y \subseteq s)$$

$$\iff \quad \neg \forall y \in Y. (y \subseteq s \Rightarrow j \subseteq s)$$

$$\iff \quad j \subseteq s m.$$
Example 4.7.12. For each finite join-semilattice \( S \) we have \( \sharp_S \cong \sharp_S |_{J(S),M(S)} = \text{Pirr}S \) inside \( \text{Dep} \).

The previous Lemma permits us to extend \( \text{Pirr}S \)'s domain/codomain whilst remaining isomorphic. Similarly we may extend \( \text{Pirr}f \) up to isomorphism.

Lemma 4.7.13. Let \( f : S \to T \) be a JSL\(_f\)-morphism and fix join/meet-generating subsets \((X_S,Y_S)\) and \((X_T,Y_T)\). Then using isomorphisms from Lemma 4.7.11:

\[
\sharp_S : \sharp_S |_{X_S \times Y_S} \to \text{Pirr}S \quad \sharp_T : \text{Pirr}T \to \sharp_T |_{X_T \times Y_T}
\]

we have the following commuting \( \text{Dep} \)-diagram where \( \mathcal{R}(s,t) : \iff f(s) \sharp_S t \):

\[
\begin{array}{ccc}
\sharp_S |_{X_T \times Y_T} & \xrightarrow{\sharp_T |_{J(S) \times J(T)}} & \text{Pirr}T \\
\uparrow & & \downarrow_{\text{Pirr}f} \\
\sharp_S |_{X_S \times Y_S} & \xrightarrow{\sharp_S |_{J(S) \times J(T)}} & \text{Pirr}S \\
\end{array}
\]

Proof. By Lemma 4.7.11 we know \((\sharp_S |_{X_S \times J(S)})^{-1} = \sharp_S |_{X_S \times J(S)}\) and also \((\sharp_T |_{J(T) \times Y_T})_* = (\sharp_S |_{M(T) \times Y_T})^*\), so composing the compatible arrows in \( \text{Dep} \) yields:

\[
\sharp_S |_{X_S \times J(S)} : \text{Pirr}f : \sharp_S |_{M(T) \times Y_T} |_{(x_s,y_t)} \iff \exists j \in J(S). \exists m \in M(S). (j \leq x_s \land f(j) \sharp_T m \land y_t \leq T m) \quad (*)
\]

Assuming (*) we’ll show \( \mathcal{R}(x_s,y_t) \). If \( f(x_s) \leq m \) then \( f(j) \leq f(x_s) \) by monotonicity, yielding contradiction \( f(j) \leq m \), so we know \( f(x_s) \not\leq m \). Thus \( f(x_s) \not\leq y_t \) for otherwise we obtain the contradiction \( f(x_s) \not\leq y_t \leq m \).

Conversely, if \( f(x_s) \not\leq T y_t \) then \( x_s \not\leq \sharp_S y_t \) for otherwise \( f(1_S) = \sharp_S y_t \) is a contradiction. So some \( j \in J(S) \) satisfies \( j \leq x_s \). If every such join-irreducible satisfied \( f(j) \leq y_t \), then since \( f \) preserves joins we’d infer the contradiction \( f(x_s) \leq y_t \). Thus \( f(j) \not\leq y_t \) for some \( j \leq x_s \). Finally since \( f(j) \not\leq T y_t \) we know \( y_t \not\leq T T \) hence \( \exists m \in M(T) \) with \( y_t \leq T m \).

If \( f(j) \leq T m \) for every such meet-irreducible, we’d infer the contradiction \( f(j) \leq T m \) by the definition of meets.

5 Tensors and tight tensors

5.1 Hom-functors, irreducible morphisms and the tensor product

We now investigate the join-semilattice of morphisms JSL\(_f\)(\( Q, R \)). These are the morphisms JSL\(_f\)(\( Q, R \)) equipped with the pointwise-join and the constantly \( \bot \) map. It is extended to a functor in the standard way. We describe its meet-irreducible elements, and in some cases its join-irreducible elements. This is achieved by considering certain special morphisms. We then define the tensor product of finite join-semilattices as a composite functor, whose action on objects is \( Q \otimes R := \text{JSL}_f(Q, R)^{op})^{op} \). Bimorphisms are introduced and some basic properties of the tensor product are proved.

However, we leave the proof of the universality of the tensor product until the next subsection. We do this because one can prove it in an elegant way using \( \text{Dep} \). There is a pre-existing inductively defined notion of ‘bi-ideal’ which has been used to define the tensor product of finite join-semilattices [GW05]. A bi-ideal over a pair of finite join-semilattices \( (Q, R) \) is precisely the same thing as the relative complement of a \( \text{Dep} \)-morphism of type \( \sharp_Q \cong Q \times Q \rightarrow \sharp_R \cong R \times R \).

Definition 5.1.1 (Internal hom-functor). For any pair of finite join-semilattices \( (Q, R) \) recall by Definition 3.0.4 that JSL\(_f\)(\( Q, R \)) is the join-semilattice of join-semilattice morphisms JSL\(_f\)(\( Q, R \)). This extends to a functor JSL\(_f\)(\( -,- \)) : JSL\(_f\)^{op} \times JSL\(_f\) \to JSL\(_f\) as follows:

\[
f : Q_2 \to Q_1 \quad g : R_1 \to R_2 \quad JSL\(_f\)(f^{op},g) := \lambda h.g \circ h \circ f : JSL\(_f\)(Q_1, R_1) \to JSL\(_f\)(Q_2, R_2)
\]

We refer to this functor as the \textit{internal hom-functor}.
Lemma 5.1.2. JSL\(_f[\cdot, \cdot] : \text{JSL}^{op}_f \times \text{JSL}_f \to \text{JSL}_f\) is a well-defined functor.

Proof. It suffices to show its action is well-defined, since for general categorical reasons we have the well-defined functor JSL\(_f[\cdot, \cdot] : \text{JSL}^{op}_f \times \text{JSL}_f \to \text{Set}\) with the same underlying action as JSL\(_f[\cdot, \cdot]\. Each JSL\(_f[Q, R] is a well-defined finite join-semilattice by Lemma 3.0.4. Given \(f : Q_2 \to Q_1\) and \(g : R_1 \to R_2\) it remains to show that the action of JSL\(_f[f^{op}, g]\) preserves the pointwise join-structure on JSL\(_f(Q_1, R_1)\).

\[
g \circ \bot_{\text{JSL}_f[Q, R]} \circ f = \lambda q_2 \in Q_2, g(\bot_{\text{JSL}_f[Q, R]}(f(q_2))) = \lambda q_2 \in Q_2, \bot_{R_2} = \bot_{\text{JSL}_f[Q_2, R_2]}\]

\[
g \circ (h_1 \lor_{\text{JSL}_f[Q, R]} h_2) \circ f = \lambda q_2 \in Q_2, g(h_1 \lor_{\text{JSL}_f[Q, R]} h_2)(f(q_2)) = \lambda q_2 \in Q_2, g(h_1(f(q_2))) \lor_{R_2} g(h_2(f(q_2))) = (g \circ h_1 \circ f) \lor_{\text{JSL}_f[Q_2, R_2]} (g \circ h_2 \circ f)\]

using the fact that \(g\) is a join-semilattice morphism. \(\square\)

Lemma 5.1.3. Although \(\land_{\text{JSL}_f[Q, R]}\) needn’t be pointwise, the ordering \(\leq_{\text{JSL}_f[Q, R]}\) is. Moreover:

\[
\bigwedge_{\text{JSL}_f[Q, R]} \{f_i : i \in I\}(q) \leq_{\text{JSL}_f[Q, R]} \bigwedge_{\text{JSL}_f[Q, R]} \{f_i(q) : i \in I\}
\]

for any morphisms \((f_i : Q \to R)_{i \in I}\) and \(q \in Q\).

Proof. We have the injective join-semilattice morphism:

\[
e : \text{JSL}_f[Q, R] \to R^Q \quad e(f) := (f(q))_{q \in Q}
\]

because the join in \(S := \text{JSL}_f[Q, R]\) is constructed pointwise. Then \(\leq_S\) is the pointwise-ordering because injective join-semilattice morphisms are order-embeddings. Furthermore since \(e\) is monotonic we have \(e(\land_S \{f_i : i \in I\}) \leq_{\text{JSL}_f[Q, R]} \land_{\text{JSL}_f[Q, R]} e(f_i)\) which is precisely the claim above. Finally, Example 5.1.4 below provides a specific example where the meet is not constructed pointwise. \(\square\)

Example 5.1.4. The meet in JSL\(_f[Q, R]\) needn’t be pointwise. Let \(Q = R = M_3\) and consider the two endomorphisms:

\[
\begin{array}{ccc}
  x_1 & \searrow & 1 \\
  \downarrow & \nearrow & 1 \\
  x_2 & \swarrow & x_3 \\
\end{array}
\]

\(f_1(q) := \begin{cases} 1 & \text{if } q = x_1 \\ q & \text{otherwise} \end{cases}\)

\(f_3(q) := \begin{cases} 1 & \text{if } q = x_3 \\ q & \text{otherwise} \end{cases}\)

Let \(g := f_1 \land_{\text{JSL}_f[Q, Q]} f_3\) denote their meet. By Lemma 5.1.3 we have \(g(x_1) = g(x_3) = \bot_Q\), and hence \(g(\top_Q) = g(x_1 \lor_Q x_3) = g(x_1) \lor_Q g(x_3) = \bot_Q\). Thus \(g\) is the constantly bottom map, so that \(g(x_2) \leq_Q x_2 = f_1(x_2) \land_Q f_2(x_2)\). \(\blacksquare\)

For any pair of finite join-semilattices \((Q, R)\) we are going to define two types of special morphisms, where the first meet-generate JSL\(_f[Q, R]\) generally, and second join-generate this join-semilattice as long as \(Q\) or \(R\) is distributive. A subset of the former will later provide the join-irreducible elements of the tensor product, whereas a subset of the latter will induce the join-irreducible elements of the tight tensor product. First recall the following basic constructions from Definition 3.0.6 and Lemma 3.0.7.

1. We have the join-semilattice \(2 = (\{0, 1\}, \lor_2, 0)\) where \(\lor_2\) is the boolean function OR, or equivalently \(\max(b_1, b_2)\).

   There is a unique join-semilattice isomorphism of type \(\text{swap} : 2^{op} \to 2\). It flips the bit, so that \(\text{swap}(b) = 1 - b\).

2. We have the join-semilattice isomorphisms:

\[
e_{\text{Q}}(-) : \text{Q} \to \text{Elem}(Q) = \text{JSL}_f[2, Q] \quad e_{\text{Q}}(q) := \lambda b \in \{0, 1\}. b ? q : \bot_Q
\]

\[
idl_{\text{Q}} : \text{Q}^{op} \to \text{Idem}(Q) = \text{JSL}_f[Q, 2] \quad idl_{\text{Q}}(q_0) := \lambda q \in Q. (q \leq_q q_0) \land_q 0 : 1
\]

For what follows, it will be helpful to define similar structures involving a three element chain.

Definition 5.1.5 (Elements and ideals relative to the three-chain).
1. Define the join-semilattice $3 = \{0, 1, 2\}, \lor_3, 0\}$ where $x_1 \lor_3 x_2 := \text{max}(x_1, x_2)$. There is a unique join-semilattice isomorphism with typing $\text{rot} : 3^{op} \to 3$. It rotates around 1, so that $\text{rot}(x) = 2 - x$. It is self-adjoint, as is its inverse $\text{rot}^{-1} : 3 \to 3^{op}$.

2. For each finite join-semilattice $Q$ define the finite join-semilattice:

\[
\text{Trel}_\text{el}(Q) = (\{ f \in JL_f (3, Q) : f(1_3) = 1_Q \}, \lor \text{Trel}_\text{el}(Q), 1 \text{Trel}_\text{el}(Q))
\]

where $\lor \text{Trel}_\text{el}(Q)$ constructs the pointwise-join in $Q$, and $1 \text{Trel}_\text{el}(Q) = \lambda x. (x = 2) ? 1_Q : 1_Q$. That is, $\text{Trel}_\text{el}(Q)$ inherits the join structure from $JL_f[3, Q]$ but has a different bottom element. There is an associated join-semilattice isomorphism:

\[
el_3^3(-) : Q \to \text{Trel}_\text{el}(Q) \quad \nel_3^3(q) := \lambda x \in \{0, 1, 2\}. \begin{cases} 1_Q &\text{if } x = 0 \\ q &\text{if } x = 1 \\ 1_Q &\text{if } x = 2 \end{cases}
\]

3. For each finite join-semilattice $Q$, define the finite join-semilattice:

\[
el_{3^{op}}(Q) := (\{ f \in JL_f (Q, 3) : f(1_3) = 1_Q \}, \lor \nel_{3^{op}}(Q), 1 \nel_{3^{op}}(Q))
\]

where $\lor \nel_{3^{op}}(Q)$ is the pointwise-join inside $3$, and $1 \nel_{3^{op}}(Q) = \lambda q \in Q. (q = 1_Q) ? 0 : 1$. Thus $\nel_{3^{op}}(Q)$ inherits the join-structure from $JL_f[Q, 3]$ yet has a different bottom element. There is an associated join-semilattice isomorphism:

\[
el_3^3(-) : Q^{op} \to \nel_{3^{op}}(Q) \quad \nel_3^3(q_0) := \lambda q \in Q. \begin{cases} 0 &\text{if } q = 1_Q \\ 1 &\text{if } 1_Q < q \leq q_0 \\ 2 &\text{if } q < q_0 \end{cases}
\]

which provides a concrete description of this join-semilattice.

Lemma 5.1.6.

1. $\nel_3^3(-) : Q \to \text{Trel}_\text{el}(Q)$ is a well-defined join-semilattice isomorphism.

2. $\nel_{3^{op}}(-) : Q^{op} \to \nel_{3^{op}}(Q)$ is a well-defined join-semilattice isomorphism.

3. For each $q \in Q$ we have:

\[ (\nel_3^3(q))_* = \text{rot}^{-1} \circ \nel_{3^{op}}^3(q) \quad (\nel_{3^{op}}^3(q))_* = \nel_3^3(q) \circ \text{rot} \]

Proof.

1. $\text{Trel}_\text{el}(Q)$ is a well-defined join-semilattice because the top-preserving morphisms $f : 3 \to Q$ are closed under pointwise joins, and there is at least such morphism $1 \text{Trel}_\text{el}(Q) = \nel_3^3(1_Q) = \lambda x. (x = 2) ? 1_Q : 1_Q$. Since the only parameter is the value of $f(1)$ and this may be freely chosen, it follows that $\nel_3^3(-)$ is a bijection. We’ve already observed that the bottom is preserved, and clearly $\nel_{3^{op}}^3(q_0 \lor Q q_2)$ is the pointwise binary join of $(\nel_3^3(q_i))_{i=1,2}$, and thus also their binary join inside $\text{Trel}_\text{el}(Q)$.

2. We first show that $\nel_{3^{op}}(Q)$ is a well-defined join-semilattice. Fixing any morphism $f : Q \to 3$, then $f_*(1_3) = 1_Q$ iff $f_*(\text{rot}^{-1}) : 3 \to Q^{op}$ preserves the top element. Using the bijective correspondence between adjoints and the fact that rot is self-adjoint, it follows that the elements of $\nel_{3^{op}}(Q)$ are precisely those of the form $\text{rot} \circ (\nel_{3^{op}}^3(q_0))_*$, where $q_0 \in Q$. Since $\nel_{3^{op}}(Q^{op})$ is closed under pointwise binary joins, so are their adjoints by Lemma 3.0.5, as is their post-composition with the fixed morphism rot by applying the functor $JL_f[\text{rot}^{op}, -]$. We have a bottom element because $\text{Trel}_\text{el}(Q^{op})$ has one. Regarding its description, we first compute rot $\circ (\nel_{3^{op}}^3(q_0))_*$ in general.

\[
(\nel_{3^{op}}^3(q_0))_* = \lambda q \in Q. \lor_3 \{ x \in \{0, 1, 2\} : \nel_{3^{op}}^3(q_0)(x) \leq q \}
= \lambda q \in Q. \lor_3 \{ x \in \{1, 2\} : q \leq \nel_{3^{op}}^3(q_0)(x) \}
= \lambda q \in Q. \begin{cases} 2 &\text{if } q \leq \nel_{3^{op}}^3(q_0)(2) = 1_Q \\ 1 &\text{if } 1_Q < q \leq \nel_{3^{op}}^3(q_0)(1) = q_0 \\ 0 &\text{if } q < q_0 \end{cases}
\]
and hence:

\[
\text{rot} \circ (\text{idl}_Q^3(q_0))_* = \begin{cases} 
0 & \text{if } q = \bot_Q \\
1 & \text{if } \bot_Q \leq q \leq q_0 \\
2 & \text{if } q \not\leq q_0
\end{cases}
\]

Thus we have the bottom element \(\text{rot} \circ (\text{idl}_Q^3(\bot_Q)) = \lambda q \in Q. (q = \bot_Q) ? 0 : 1\), and have also described a join-semilattice isomorphism:

\[
\text{s.JSL.}(\text{Q}^{op}) \rightarrow \text{Tr.idl}(Q)\text{ and action }\text{idl}_Q^3(q_0) \rightarrow \text{idl}_Q^3(\lambda q).
\]

Then precomposing with the isomorphism \(\text{idl}_Q^3(-) : Q^{op} \rightarrow \text{s.JSL.}(Q^{op})\) from (1) yields the desired join-semilattice isomorphism.

3. Follows by the previous statement, where it is proved that \(\text{idl}_Q^3(q) = \text{rot} \circ (\text{idl}_Q^3(q))_*\).

\[\square\]

We now define various special morphisms as compositions of element morphisms and ideal morphisms.

**Definition 5.1.7 (Special morphisms).**

To any pair \((Q, R)\) of finite join-semilattices and elements \((q_0, r_0) \in Q \times R\), we associate two \(\text{JSL}_f\)-morphisms:

\[
\uparrow_{Q,R}^{q_0,r_0} : Q \xrightarrow{\text{idl}_Q(q_0)} 2 \xrightarrow{\text{idl}_R(r_0)} R \\
\downarrow_{Q,R}^{q_0,r_0} : Q \xrightarrow{\text{idl}_Q(q_0)} 3 \xrightarrow{\text{idl}_R(r_0)} R
\]

\[
\text{idl}_Q^3(q_0) \mapsto (q_0, r_0) \xrightarrow{\text{idl}_R^3(r_0)} (q_0, r_0)
\]

\[
\text{idl}_Q^3(q_0) \mapsto (q_0, r_0) \xrightarrow{\text{idl}_R^3(r_0)} (q_0, r_0)
\]

**Note 5.1.8 (Intuition regarding special morphisms).**

1. We often think of the special morphisms \(\uparrow_{Q,R}^{q,r} : Q \rightarrow R\) as ‘approximations from below’ i.e. we imagine constructing arbitrary morphisms \(Q \rightarrow R\) as pointwise joins of these special morphisms. If \((q, r) \in M(Q) \times J(R)\) then these morphisms are join-irreducible in \(\text{JSL}_f[Q, R]\). In the special case where \(Q\) or \(R\) is distributive every join-irreducible morphism takes this form. However this fails in general, and the restriction to those morphisms they join-generate (i.e. pointwise-join-generate) yields the previously studied concept of ‘tight morphism’, and also a subfunctor of \(\text{JSL}_f[\lambda, -]\) satisfying a universal property relative to tight morphisms. We shall investigate this carefully later on.

2. The special morphisms \(\downarrow_{Q,R}^{q,r} : Q \rightarrow R\) may be thought of as ‘approximations from above’. They are used extensively over the next two subsections. As we shall see, they are precisely the meet-irreducible morphisms in \(\text{JSL}_f[Q, R]\), so in particular every morphism \(Q \rightarrow R\) arises as a meet (which is rarely pointwise) of these special morphisms. For the moment, observe that \(\downarrow_{Q,R}^{q,r} : Q \rightarrow R\) is the largest element of \(\text{JSL}_f[Q, R]\) extending:

\[
\text{idl}_R(r) \circ \text{idl}_{[\bot_Q, q]}(\bot_Q) : [\bot_Q, q] \rightarrow R
\]

where \([\bot_Q, q] \subseteq Q\) is the interval sub join-semilattice. We should also mention that the equality:

\[
\downarrow_{Q,R}^{q,r} = \uparrow_{Q,R}^{\bot_Q, \bot_R} \vee \text{JSL}_f[Q, R] \uparrow_{Q,R}^{q, r}
\]

holds generally. However, this relationship will not be used or proved until the section concerning tight morphisms, although one could already deduce it from Lemma 5.1.9.6 below.

**Lemma 5.1.9 (Properties of special morphisms).**

*Fix any finite join-semilattices \(Q, R\) and elements \((q_0, r_0), (q_1, r_1) \in Q \times R*.*
1. We have the following symmetric equalities involving $\uparrow_{Q,R}^{\top}$ and $\downarrow_{Q,R}^{\top}$:

\[
\begin{align*}
\uparrow_{Q,R}^{\top} & = \uparrow_{Q,R}^{\top} \in JSL_f[Q,R] \\
\downarrow_{Q,R}^{\top} & = \downarrow_{Q,R}^{\top} \in JSL_f[Q,R] \\
\uparrow_{Q,R}^{\top} \land_{Q,R}^{\top} & = \uparrow_{Q,R}^{\top} \land_{Q,R}^{\top} \in JSL_f[Q,R] \\
\downarrow_{Q,R}^{\top} \land_{Q,R}^{\top} & = \downarrow_{Q,R}^{\top} \land_{Q,R}^{\top} \in JSL_f[Q,R] \\
\uparrow_{Q,R}^{\top} \lor_{Q,R}^{\top} & = \uparrow_{Q,R}^{\top} \lor_{Q,R}^{\top} \in JSL_f[Q,R] \\
\downarrow_{Q,R}^{\top} \lor_{Q,R}^{\top} & = \downarrow_{Q,R}^{\top} \lor_{Q,R}^{\top} \in JSL_f[Q,R]
\end{align*}
\]

2. Given $q_0 \neq \top$ and $r_1 \neq \bot$, then:

\[
\uparrow_{Q,R}^{q_0,r_0} \leq \downarrow_{Q,R}^{q_1,r_1} \iff q_1 \leq q_0 \text{ and } r_0 \leq r_1
\]

Moreover $\uparrow_{Q,R}^{\top} = \downarrow_{Q,R}^{\top} = \uparrow_{Q,R}^{\bot}$ explains the remaining cases.

3. We also have the following equalities involving $\downarrow_{Q,R}^{\bot}$ only:

\[
\begin{align*}
\downarrow_{Q,R}^{\top} & = \downarrow_{Q,R}^{\top} \land_{Q,R}^{\top} = \downarrow_{Q,R}^{\top} \land_{Q,R}^{\top} \in JSL_f[Q,R] \\
\downarrow_{Q,R}^{\bot} & = \downarrow_{Q,R}^{\bot} \land_{Q,R}^{\bot} = \downarrow_{Q,R}^{\bot} \land_{Q,R}^{\bot} \in JSL_f[Q,R] \\
\downarrow_{Q,R}^{\top} & = \downarrow_{Q,R}^{\top} \land_{Q,R}^{\top} = \downarrow_{Q,R}^{\top} \land_{Q,R}^{\top} \in JSL_f[Q,R] \\
\downarrow_{Q,R}^{\bot} & = \downarrow_{Q,R}^{\bot} \land_{Q,R}^{\bot} = \downarrow_{Q,R}^{\bot} \land_{Q,R}^{\bot} \in JSL_f[Q,R]
\end{align*}
\]

4. Given $\bot \neq q_0$ and $r_1 \neq \top$, then:

\[
\downarrow_{Q,R}^{\top} \leq \downarrow_{Q,R}^{\bot} \iff q_1 \leq q_0 \text{ and } r_0 \leq r_1
\]

Moreover the remaining cases are explained by the previous statement.

5. If $q_0 \leq q_1$ then the meet $\downarrow_{Q,R}^{q_0,r_0} \land_{Q,R}^{q_1,r_1}$ is constructed pointwise in $R$. In fact,

\[
\downarrow_{Q,R}^{q_0,r_0} \land_{Q,R}^{q_1,r_1} = \lambda q \in Q. (\downarrow_{Q,R}^{q_0,r_0} (q) \land _{Q,R}^{q_1,r_1} (q)) = \land_{Q,R}^{\top} \{ q \mid \downarrow_{Q,R}^{q_0,r_0} (q) \neq \bot \}
\]

6. Regarding the relationship between the two different types of special morphisms:

\[
\downarrow_{Q,R}^{\top} \leq \downarrow_{Q,R}^{\bot} \iff q_1 \leq q_0 \text{ or } r_0 \leq r_1
\]

Proof.

1. The top two equalities follow via very similar calculations:

\[
\begin{align*}
\uparrow_{Q,R}^{q_0,r_0} & = (\mathsf{el}_Q^{\top}(r_0) \circ \mathsf{idl}_Q^{\top}(q_0)) \circ (\mathsf{idl}_R^{\top}(r_0)) \\
& = (\mathsf{idl}_Q^{\top}(q_0)) \circ (\mathsf{idl}_R^{\top}(r_0)) \\
& = (\mathsf{idl}_Q^{\top}(q_0) \circ \mathsf{rot}^{-1} \circ \mathsf{idl}_R^{\top}(r_0)) \circ \mathsf{swap}^{-1} \circ \mathsf{idl}_R^{\top}(r_0) \\
& = \uparrow_{Q,R}^{q_0,r_0}
\end{align*}
\]

The other equalities also follow by considering the join-semilattice isomorphisms:

\[
\begin{align*}
\mathsf{el}_Q^{\top} : Q \to JSL_f[2, Q] & \quad \mathsf{idl}_Q^{\top} : Q^{\text{op}} \to JSL_f[Q, 2] \\
\mathsf{el}_Q^{\bot} : Q \to JSL_f[2, Q] & \quad \mathsf{idl}_Q^{\bot} : Q^{\text{op}} \to JSL_f[Q, 2]
\end{align*}
\]
which are necessarily also bounded lattice isomorphisms. Combined with the fact that \( JSL_f \)-composition is bilinear we obtain all the other equalities e.g.

\[
\downarrow_{Q,R}^{q_0 \land q_1 \rightarrow 0} = \downarrow_{Q,R}^{q_0}(r_0) \circ \downarrow_{Q,R}^{q_1}(q_0 \land q_1)
\]

and the final line of equalities follows by preservation of top elements.

2. We have \( \uparrow_{Q,R}^{q_0 \rightarrow 0} \leq_{JSL_f[Q,R]}^{q_1 \rightarrow r_1} \) if and only if the following two statements hold:

(a) for all \( q \lesssim Q \) we have \( q \lesssim Q \) (since by assumption \( r_0 \neq \top_R \)),

(b) for all \( q \lesssim Q \) we have \( r_0 \leq_R r_1 \).

Then (a) is equivalent to \( q_1 \leq Q \) and, since we assume \( q_0 \neq \top_Q \), (b) implies and thus is equivalent to \( r_0 \leq_R r_1 \).

3. It suffices to prove the left-hand equalities, since the others follow via \( (\downarrow_{Q,R}^q) = (\downarrow_{Q,R}^q)^{op} \) proved in (1), recalling that taking adjoints defines a join-semilattice isomorphism \( JSL_f[Q,R] \cong JSL_f[Q,R^{op}, Q^{op}] \).

Regarding the first equality,

\[
\downarrow_{Q,R}^{q_0 \land q_1 \rightarrow 0} = \downarrow_{Q,R}^{q_0}(r_0) \circ \downarrow_{Q,R}^{q_1}(q_0 \land q_1)
\]

using the explicit description of \( \downarrow_{Q,R}^{q_0}(q) \) and the fact that \( \downarrow_{Q,R}^{q_0}(r_0) \) preserves \( \top_3 \). Finally, we certainly have:

\[
\uparrow_{Q,R}^{q_0 \lor q_1 \rightarrow 0} = \uparrow_{Q,R}^{q_0}(q_0 \lor q_1)
\]

for some morphism \( h : Q \rightarrow R \). Then there must exist \( q \leq Q \) such that \( r_0 \leq_R h(q) \), so by monotonicity and join-preservation \( r_0 \leq_R h(q) \), which contradicts the fact that:

\[
h(q_i) \leq_R (\downarrow_{Q,R}^{q_0 \rightarrow 0} \land_{JSL_f[Q,R]} \downarrow_{Q,R}^{q_1 \rightarrow r_1})(q_i) \leq_R \downarrow_{Q,R}^{q_0 \rightarrow 0} \land_{JSL_f[Q,R]} \downarrow_{Q,R}^{q_1 \rightarrow r_1}(q_i)
\]

for \( i = 0, 1 \).

4. Letting \( f_i := \downarrow_{Q,R}^{q_i} \) for \( i = 0, 1 \), then \( f_0 \leq_{JSL_f[Q,R]} f_1 \) if and only if the following two statements hold:

(a) \( \forall q \lesssim Q : \downarrow_{Q,R}^{q_0}(q) = \top_R \)

(b) \( \forall \bot \lesssim q \leq Q : q \leq_R r_0 \leq \downarrow_{Q,R}^{q_0} \)

Since by assumption \( r_1 \neq \top_R \), (a) is equivalent to \( \forall q \in Q : q \lesssim Q \rightarrow q \lesssim Q \) or equivalently \( q_1 \leq Q q_0 \). Then in the presence of (a), statement (b) is equivalent to:

\[
\forall \bot \lesssim q \leq Q : q \leq q_0 \circ r_0 \leq q_1
\]

Since \( q_0 \neq \top_Q \) by assumption, the latter is equivalent to \( r_0 \leq_R r_1 \) and we are done.

5. Consider the morphism:

\[
h := \bigvee_{JSL_f[Q,R]} \{ \downarrow_{Q,R}^{q_0 \land q_1 \rightarrow r_1}, \downarrow_{Q,R}^{q_0 \rightarrow r_1}, \downarrow_{Q,R}^{q_1 \rightarrow r_1} \} : Q \rightarrow R
\]

and also define the function \( g : Q \rightarrow R \) as:

\[
g(x) := \downarrow_{Q,R}^{q_0 \rightarrow 0}(x) \land_{JSL_f[Q,R]} \downarrow_{Q,R}^{q_1 \rightarrow r_1}(x) \quad \text{for each } x \in Q.
\]
By Lemma 5.1.3 it suffices to establish that \( g = h \). Recalling our assumption that \( q_0 \leq Q q_1 \), we can never have \( x \leq Q q_0 \) and \( x \not\leq Q q_1 \). Then \( g \) has action:

\[
g(x) = \begin{cases} 
\top & \text{if } x \not\leq Q q_1 \text{ (hence also } x \not\leq Q q_0) \\
r_1 & \text{if } \bot \not< Q x \leq Q q_1 \text{ and } x \not\leq Q q_0 \\
r_0 \land Q r_1 & \text{if } \bot \not< Q x \leq Q q_0 \text{ (hence also } \bot \not< Q x \leq Q q_1) \\
\bot & \text{if } x = \bot 
\end{cases}
\]

One can directly verify that \( g(x) = h(x) \) for each of the four disjoint cases.

6. We calculate:

\[
\mathbf{JSL}_{[Q,R]}^{(r_0, r_1)} \not\leq Q \mathbf{JSL}_{[Q,R]}^{(q_1, r_1)} \iff \exists j_q \in J(Q), \exists j_q \not\leq Q q_0 \text{ and } r_0 \not\leq R \mathbf{JSL}_{[Q,R]}^{(r_1, r_1)} (j_q)
\]

\[
\iff \exists j_q \in J(Q), \exists j_q \not\leq Q q_0 \text{ and } j_q \leq Q q_1 \text{ and } r_0 \not\leq R r_1
\]

\[
\iff \exists j_q \in J(Q), \exists j_q \not\leq Q q_0 \text{ and } j_q \leq Q q_1
\]

\[
\iff \forall j_q \in J(Q), j_q \leq Q q_1 \Rightarrow j_q \not\leq Q q_0 \text{ and } r_0 \not\leq R r_1
\]

\[
\iff q_1 \not\leq Q q_0 \text{ and } r_0 \not\leq R r_1 
\]

\[\square\]

We now describe the meet-irreducible elements of \( \mathbf{JSL}_{[Q,R]} \), and also some of its join-irreducibles. In the special case that either \( Q \) or \( R \) is distributive then the latter will be precisely the join-irreducible morphisms.

**Lemma 5.1.10** (Meet and join-irreducible homomorphisms).

Let \( Q \) and \( R \) be finite join-semilattices.

1. \( \mathbf{JSL}_{[Q,R]}^{j,m} \) is meet-irreducible in \( \mathbf{JSL}_{[Q,R]} \) whenever \( j \in J(Q) \) and \( m \in M(R) \), in fact:

\[
M(\mathbf{JSL}_{[Q,R]}) = \{ \mathbf{JSL}_{[Q,R]}^{j,m} : j \in J(Q), m \in M(R) \}
\]

so that \( |M(\mathbf{JSL}_{[Q,R]})| = |J(Q)| \cdot |M(R)| \).

2. Concerning the morphisms \( \mathbf{JSL}_{[Q,R]}^{m,j} : Q \to R \) where \( m \in M(Q) \) and \( j \in J(R) \).

   (a) They are always join-irreducible in \( \mathbf{JSL}_{[Q,R]} \).

   (b) If \( Q \) is distributive then:

\[
J(\mathbf{JSL}_{[Q,R]}) = \{ \mathbf{JSL}_{[Q,R]}^{m,j} : m \in M(Q), j \in J(R) \} \quad (\ast)
\]

so that \( |J(\mathbf{JSL}_{[Q,R]})| = |J(Q)| \cdot |J(R)| \).

   (c) If \( R \) is distributive then \((\ast)\) again holds, so that \( |J(\mathbf{JSL}_{[Q,R]})| = |M(Q)| \cdot |M(R)| \).

   (d) If neither \( Q \) nor \( R \) are distributive then these morphisms needn’t join-generate \( \mathbf{JSL}_{[Q,R]} \). We may have \( |J(\mathbf{JSL}_{[Q,R]})| > |J(Q)|^2 \) where \( |J(Q)| = |M(Q)| \).

**Proof.**

1. Let \( S = \mathbf{JSL}_{[Q,R]} \). We first show that every morphism \( f : Q \to R \) arises as an \( S \)-meet of the morphisms \( \mathbf{JSL}_{[Q,R]}^{j,m} : Q \to R \) where \( (j,m) \in J(Q) \times M(R) \). Indeed, consider:

\[
g := \bigwedge_{\leq S} \{ \mathbf{JSL}_{[Q,R]}^{j,m} : (j,m) \in J(Q) \times M(R), f(j) \leq R m \}
\]

First of all, \( f \leq S g \) because \( f \leq S \mathbf{JSL}_{[Q,R]}^{j,m} \) for each summand \( \mathbf{JSL}_{[Q,R]}^{j,m} \) above. To see this, observe that if \( \bot \not< Q q \leq Q j \) then \( f(q) \leq R f(j) \leq R m \) using the monotonicity of \( f \). Now, by Lemma 5.1.3 we know that:

\[
g(q) \leq R \bigwedge_{\leq R} \{ \mathbf{JSL}_{[Q,R]}^{j,m} (q) : (j,m) \in J(Q) \times M(Q), f(j) \leq R m \} \quad \text{for each } q \in Q
\]

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and consequently for every \( j_0 \in J(Q) \) we have:

\[
g(j_0) \leq \bigwedge \{ i_{Q,R}^{j_0,m} (j_0) : f(j_0) \leq m \in M(R) \} = \bigwedge \{ m \in M(R) : f(j_0) \leq m \} = f(j_0)
\]

and it follows that \( g \leq f \). Thus every morphism \( Q \to R \) arises as the \( S \)-meet of these special morphisms, and hence every meet-irreducible in \( S \) is one of these morphisms. Then to show that every \( i_{Q,R}^{j,m} \) is meet-irreducible, it suffices to establish that they are not meets of other such special morphisms. To this end, first observe that \( i_{Q,R}^{j_1,m_1} \leq i_{Q,R}^{j_2,m_2} \) if and only if \( j_2 \leq j_1 \) and \( m_1 \leq m_2 \) by Lemma 5.1.9.4. Now, fix any \( f := i_{Q,R}^{j,m} \) where \( (j,m) \in J(Q) \times M(R) \) and consider the morphisms:

\[
g_j := i_{Q,R}^{j,m} \text{ where } g_j := \bigvee \{ j' \in J(Q) : j' \leq j \},
\]

\[
g_m := i_{Q,R}^{j,m} \text{ where } g_m := \bigwedge \{ m' \in M(R) : m \leq m' \}.
\]

Then we have \( g_j \leq g_m \) by join-irreducibility and \( m \leq g_j \) by meet-irreducibility. Using Lemma 5.1.9.4:

(a) \( f \leq g_1, g_2 \) and hence \( f \leq g_1 \land g_2 \).

(b) Whenever \( f \leq i_{Q,R}^{j,m} \) for some \( (j,m) \in J(Q) \times M(R) \) then either \( (j \leq j) \) and \( m \leq m \) or \( (j \leq j) \) and \( m \leq m \), and consequently \( g_1 \land g_2 \leq i_{Q,R}^{j,m} \).

It follows that to establish the meet-irreducibility of \( f \) we can show that \( f \neq g_1 \land g_2 \). Since \( g_j \leq g \) we may apply Lemma 5.1.9.5 to deduce that \( g_1 \land g_2 \) is constructed pointwise, hence:

\[(g_1 \land g_2)(j) = g_1(j) \land g_2(j) = \bigwedge \{ m \in M(R) : m \leq m \} = f(j)\]

as required. Finally, these maps are pairwise distinct so \( |M(S)| = |J(Q)| \cdot |M(R)| \).

2. Again let \( S := \text{JSL}_f(Q,R) \) and now consider the special morphisms \( i_{Q,R}^{m,j} : Q \to R \) where \( m \in M(Q) \) and \( j \in J(R) \).

(a) To see that they are join-irreducible, suppose that \( i_{Q,R}^{m,j} = f \circ \text{JSL}_f(Q,R) \) \( g \). Since \( m \) is meet-irreducible it has a unique cover \( m \leq x \), and since \( j = f(x) \lor g(x) \) is join-irreducible we may assume w.l.o.g. that \( f(x) = j \).

Seeing as \( f \leq i_{Q,R}^{m,j} \) it follows that for any \( q \leq m \) we have \( f(q) = \bot_R \), and for any \( q \leq m \) we have \( f(q) \leq j \). Now, fix any \( q \leq m \) and observe that \( m \lessdot q \lor \forall \delta \) because equality yields a contradiction. Thus \( x \lessdot q \lor \forall \delta \) and hence:

\[j = f(x) \leq f(q \lor \forall \delta) = f(q) \lor f(\forall \delta) = f(q) \lor f(\forall \delta) = f(q) \]

using the monotonicity of \( f \), preservation of joins and also \( f(m) = \bot_R \). Hence \( f = i_{Q,R}^{m,j} \) and we are done.

(b) Assuming that \( Q \) is distributive, let us show that the \( i_{Q,R}^{m,j} \) join-generate \( S \). Given any join-semilattice morphism \( f : Q \to R \) define the morphism:

\[g := \bigvee \{ i_{Q,R}^{m,j} : j \in J(Q), j_0 \in J(R), j_0 \leq f(j) \}
\]

where \( j_0 := \bigvee \{ q \in Q : j \lessdot q \} \in M(Q) \) is the meet-irreducible corresponding to \( j \) under the canonical bijection from Lemma 2.2.3.13. To establish \( g \leq f \) we’ll show that every summand \( i_{Q,R}^{m,j} \leq f \) i.e. whenever \( q \leq m \) we must show that \( j_0 \leq f(q) \). By construction \( j_0 \leq f(j) \) and the canonical bijection informs us that \( j = \bigvee \{ q \in Q : j \lessdot q \} \), hence \( j \leq q \) and thus \( j_0 \leq f(j) \leq f(q) \) using the monotonicity of \( f \). To establish the converse \( f \leq g \) it suffices to show that:

\[f(j_0) \leq \bigvee \{ i_{Q,R}^{m,j_0} : j_0 \in J(Q), j_0 \leq f(j_0) \} = \bigvee \{ j_0 \in J(Q) : j_0 \leq f(j_0) \}
\]

which follows because \( f(j_0) \) is the \( R \)-join of those join-irreducibles beneath it. Then the \( i_{Q,R}^{m,j} \) are precisely the join-irreducibles in \( S \). Since they are pairwise distinct the number of join-irreducibles is exactly \( |M(Q)| \cdot |J(R)| = |J(Q)| \cdot |J(R)| \), recalling that \( |J(Q)| = |M(Q)| \) in a distributive lattice via the canonical bijection.
(c) Now instead assume that \( R \) is distributive. By Lemma 3.0.5 we know that \( \text{JSL}_f[Q, R] \cong \text{JSL}_f[R^{\text{op}}, Q^{\text{op}}] \) where the action of this join-semilattice isomorphism takes the adjoint. Since distributive lattices are closed under taking the order-dual, we may apply the previous statement. This then translates back to the desired statement via Lemma 5.1.9.4. We finally deduce that:

\[
|J(\text{JSL}_f[Q, R])| = |J(\text{JSL}_f[R^{\text{op}}, Q^{\text{op}}])| = |J(Q^{\text{op}})| \cdot |M(R)| = |M(Q)|
\]

(d) Let \( Q = R \) be \( M_3 \) with three atoms \( x_1, x_2, x_3 \). Then the identity morphism \( id_Q : Q \to Q \) does not arise as a join of the special morphisms \( i^{m,j}_{Q,Q} \). To see this, observe that the latter sends \( m \) to \( \bot_Q \), and the other two atoms to \( j \). Thus none of them are pointwise below \( id_Q \), so it cannot arise as a join of them. In fact, none of the six isomorphisms of \( Q \) are join-generated by these special morphisms. By (a) every \( i^{m,j}_{Q,Q} \) is join-irreducible, in fact they are atoms: if \( f : Q \to Q \) sends more than one atom to \( \bot_Q \) then it sends everything to \( \bot_Q \). The remaining join-irreducibles are also atoms: send one atom to \( \bot_Q \) and the others to distinct atoms. Thus \( |J(\text{JSL}_f[Q, Q])| = 3 \cdot (3^2) = 27 > 3 \cdot 3 = |J(Q)|^2 \), where \( |J(Q)| = |M(Q)| \) by symmetry.

In the rest of this subsection we define the tensor product functor, the associated notion of bimorphism and prove some basic properties. The tensor product \( Q \otimes R \) is defined as a composite functor built from one copy of \( \text{JSL}_f[-, -] : \text{JSL}_f^{\text{op}} \times \text{JSL}_f \to \text{JSL}_f \) and two copies of the self-duality functor \( \text{OD}_j : \text{JSL}_f^{\text{op}} \to \text{JSL}_f \). The associated bimorphisms are actually mappings \( (q, r) \mapsto i^{q,r}_{Q,R} \) or \( \lambda q \), so these special morphisms play a prominent role. In particular, the above Lemmas concerning irreducible homomorphisms directly provide descriptions of irreducible elements inside \( Q \otimes R \). In the next subsection we’ll describe the tensor product in a different way i.e. in terms of so-called bi-ideals, a concept that arises naturally from \( \text{Dep} \).

**Definition 5.1.11** (Tensor product of finite join-semilattices).

The **tensor product** functor \( \otimes : \text{JSL}_f \times \text{JSL}_f \to \text{JSL}_f \) is the composite functor:

\[
\text{JSL}_f \times \text{JSL}_f \xrightarrow{(\text{JSL}_f[-, -])^{\text{op}}} \text{JSL}_f^{\text{op}} \xrightarrow{\text{OD}_j} \text{JSL}_f
\]

It also has canonically associated functions for each pair \((Q, R)\),

\[
\beta_{Q,R} : Q \times R \to \text{JSL}_f(Q, R^{\text{op}})
\]

where

\[
\beta_{Q,R}(q_0, r_0) := i^{q_0,r_0}_{Q,R} = \lambda q_0 \circ i^{q_0} = \lambda q_0 \in Q,
\]

\[
\begin{cases}
\tau_R & \text{if } q = \bot_Q \\
\rho_R & \text{if } q < q \leq q_0 \\
\bot_R & \text{if } q \not\leq q_0
\end{cases}
\]

observing that the \( \tau_R \) and \( \rho_R \) are ‘switched’ because we work with \( R^{\text{op}} \).

**Note 5.1.12** (The tensor product in more detail). Regarding its action on objects,

\[
Q \otimes R = (\text{JSL}_f[Q, R^{\text{op}}])^{\text{op}} = (\text{JSL}_f(Q, R^{\text{op}}), \nu_{Q \otimes R}, \bot_{Q \otimes R})
\]

where \( \nu_{Q \otimes R} \) is defined as the binary meet in \( \text{JSL}_f[Q, R^{\text{op}}] \), and

\[
\bot_{Q \otimes R} = \tau_{\text{JSL}_f[Q, R^{\text{op}}]} = \lambda q \in Q. (q = \bot_Q) : \tau_R = \bot_R.
\]

Observe \( \bot_{Q \otimes R} = \beta_{Q,R}(\bot_Q, r) = \beta_{Q,R}(q, \bot_R) \) for any \( q \in Q, r \in R \) by Lemma 5.1.9.3, this being the bilinearity condition for bottom elements.

Since \( - \otimes - \) is defined as a composite of well-defined functors, we have:

**Lemma 5.1.13.** \( \otimes : \text{JSL}_f \times \text{JSL}_f \to \text{JSL}_f \) is a well-defined functor.

Each function \( \beta_{Q,R} \) is ‘well-defined’ in the sense that it defines a bilinear mapping i.e. a **bimorphism**.
Definition 5.1.14 (Bimorphisms). For any triple of finite join-semilattices \((Q, R, S)\), a bimorphism (or bilinear mapping) from \((Q, R)\) to \(S\) is a function \(\beta : Q \times R \to S\) such that:

1. \(\beta(\bot_Q, r) = \beta(q, \bot_R) = \bot_S\) for any \(q \in Q\) and \(r \in R\).
2. \(\beta(q_1 \lor q_2, r) = \beta(q_1, r) \lor \beta(q_2, r)\) for any \(q_1, q_2 \in Q\) and \(r \in R\).
3. \(\beta(q, r_1 \lor r_2) = \beta(q, r_1) \lor \beta(q, r_2)\) for any \(q \in Q\) and \(r_1, r_2 \in R\).

Each \(\text{JSL}_f\)-morphism \(f : Q \otimes R \to S\) induces the bimorphism \(\beta : Q \times R \to S\) from \((Q, R)\) to \(S\) with action:

\[
\beta_f(q, r) := f(\beta_{Q \otimes R}(q, r))
\]

Finally, let \(\text{BiMor}(Q, R, S)\) be the set of all bimorphisms from \((Q, R)\) to \(S\).

Lemma 5.1.15 (Basic properties of \(Q \otimes R\)).

Let \((Q, R)\) be finite join-semilattices, and recall that \(\beta_{Q,R}(q, r) = \downarrow_{Q,R}^q r\).

1. Each function \(\beta_{Q,R} : Q \times R \to \text{JSL}_f(Q, R^{\text{op}})\) is a well-defined bilinear mapping from \((Q, R)\) to \(Q \otimes R\).

2. Concerning irreducibles.
   
   (a) \(J(Q \otimes R) = \{\beta_{Q,R}(q, r) : q \in J(Q), r \in J(R)\}\) hence \(|J(Q \otimes R)| = |J(Q)| \cdot |J(R)|\).
   
   (b) If \(Q\) or \(R\) are distributive then:
   
   \[
   M(Q \otimes R) = \{m_q \cdot m_r : m_q \in M(Q), m_r \in M(R)\}
   \]
   
   hence \(|M(Q \otimes R)| = |M(Q)| \cdot |M(R)|\).

   Thus by (a) the images \(\beta_{Q,R}[J(Q) \times J(R)] \subseteq \beta_{Q,R}[Q \times R]\) both join-generate \(Q \otimes R\).

3. \(\beta_{Q,R}\) almost defines an order-embedding of \((Q \times R, \leq_{Q \otimes R})\) into \(Q \otimes R\). That is, for any \((q_1, r_1), (q_2, r_2) \in Q \times R\) such that \(q_1 \neq \bot_Q\) and \(r_1 \neq \bot_R\),

   \[
   \beta_{Q,R}(q_1, r_1) \leq_{Q \otimes R} \beta_{Q,R}(q_2, r_2) \iff (q_1, r_1) \leq_{Q \times R} (q_2, r_2)
   \]

   Also, the implication \(\Leftarrow\) holds without restriction i.e. \(\beta_{Q,R}\) defines a monotone map from \(Q \times R\) to \(Q \otimes R\).

4. For any join-semilattice morphism \(f : Q \otimes R \to S\), \(\beta_f\) is a well-defined bilinear mapping from \((Q, R)\) to \(S\).

Proof:

1. \(\beta_{Q,R}\) is a well-defined function because each \(\downarrow_{Q,R}^q r\) is a well-defined join-semilattice morphism of type \(Q \to R^{\text{op}}\) by Lemma 5.1.9.1. Concerning bilinearity, we already observed that \(\beta(\bot_Q, r) = \beta(q, \bot_R) = \downarrow_{Q \otimes R}\) in Note 5.1.12. The other conditions follow directly from Lemma 5.1.9.3, since \(\vee_{Q \otimes R} = \wedge_{\text{JSL}_f(Q, R)}\) and \(\wedge_{R^{\text{op}}} = \vee_{R}\).

2. These statements follow directly from Lemma 5.1.10 i.e. our description of join-irreducibles and meet-irreducibles in \(\text{JSL}_f(Q, R)\).

   (a) This is the first statement, since \(J(Q \otimes R) = M(\text{JSL}_f(Q, R^{\text{op}}))\) and also \(M(R^{\text{op}}) = J(R)\).
   
   (b) We are using the second statement, since \(M(Q \otimes R) = J(\text{JSL}_f(Q, R^{\text{op}}))\). If \(Q\) is distributive their cardinality is \(|J(Q)| \cdot |J(R^{\text{op}})| = |M(Q)| \cdot |M(R)|\) recalling that \(|J(Q)| = |M(Q)|\). On the other hand, if \(R\) is distributive their cardinality is \(|M(Q)| \cdot |M(R^{\text{op}})| = |M(Q)| \cdot |M(R)|\) since \(|J(Q)| = |M(Q)|\).

3. Unwinding the definitions, we have:

   \[
   \beta_{Q,R}(q_1, r_1) \leq_{Q \otimes R} \beta_{Q,R}(q_2, r_2) \iff \downarrow_{Q,R}^{q_1} r_1 \leq_{\text{JSL}_f(Q, R^{\text{op}})} \downarrow_{Q,R}^{q_2} r_2
   \]

   Then our assumptions that \(\bot_Q \neq q_1\) and \(r \neq \bot_R\) are precisely those from Lemma 5.1.9.4. Thus the above holds iff \(q_1 \leq_Q q_2\) and \(r \leq_R r_2\) (or equivalently \(r_1 \leq_R r_2\)).

   Finally if \(q_1 = \bot_Q\) or \(r_1 = \bot_R\) then by bilinearity \(\beta_{Q,R}(q_1, r_1) = \downarrow_{Q \otimes R} \beta_{Q,R}(q_2, r_2)\). Hence the original implication \(\Leftarrow\) holds without restriction.
4. Given any join-semilattice morphism \( f : Q \otimes R \to S \) we must verify that \( \beta_f := \lambda(q, r) \in Q \times R. f(\beta_{Q,R}(q, r)) \) is bilinear. This follows immediately via (1) i.e. that \( \beta_{Q,R} \) is bilinear.

\[
\begin{align*}
    f(\beta_{Q,R}(1_Q, r)) &= f(1_{Q \otimes R}) = 1_S \\
    f(\beta_{Q,R}(q, 1_R)) &= f(1_{Q \otimes R}) = 1_S \\
    f(\beta_{Q,R}(q_1 \lor q_2, r)) &= f(\beta_{Q,R}(q_1, r) \lor_{Q \otimes R} \beta_{Q,R}(q_2, r)) = f(\beta_{Q,R}(q_1, r)) \lor_{S} f(\beta_{Q,R}(q_2, r))
\end{align*}
\]

where preservation of joins in the right parameter follows symmetrically.

\[
\blacksquare
\]

5.1.1 Universality of the tensor product via Dep and bi-ideals

In order to prove the universality of the tensor product, we’ll describe the latter in terms of the category Dep. This amounts to (and explains) the ‘bi-ideals’ approach of Fraser [Fra78] and more recently of Grätzer and Wehrung [GW05]. We proceed as follows.

1. One has the inclusion-ordered join-semilattice of Dep-morphisms:

\[ \text{Dep}[\mathcal{G}, \mathcal{H}] := (\text{Dep}(\mathcal{G}, \mathcal{H}), \cup, \emptyset : \mathcal{G} \to \mathcal{H}) \]

and moreover \( \text{JSL}_f[Q, R] \) is isomorphic to \( \text{Dep}[\xi_Q, \xi_R] \). Here we have chosen to use \( \xi_Q \) rather than its restriction \( \text{Pirr}Q \), recalling that this is permissible via the natural isomorphism \( \xi : \text{Pirr} \Rightarrow \text{Nil} \text{eq} \) – see Lemmas 4.3.1 and 4.3.3. More importantly, it makes the connection with bi-ideals clearer.

2. Recall that \( Q \otimes R = (\text{JSL}_f[Q, R]^{\text{op}})^{\text{op}} \). Let us express this in terms of Dep.

   (a) Start with the sub join-semilattice \( \text{Dep}[\xi_Q, \xi_R] \subseteq \text{P}(Q \times R) \).

   (b) To obtain the opposite join-semilattice we take the pointwise relative complements inside \( Q \times R \) and order by inclusion.

   (c) Such relative complements correspond to taking the complement relation, and are necessarily closed under intersections. This inclusion-ordered join-semilattice is denoted by:

\[ \text{Bld}(Q, R) := (\text{Bld}(Q, R), \lor_{\text{Bld}(Q, R)}, \bot_{\text{Bld}(Q, R)}) \]

By construction we have \( Q \otimes R \cong \text{Bld}(Q, R) \) i.e. another description of the tensor product.

3. Importantly we have two different descriptions of \( \text{Bld}(Q, R) \).

   (a) The first comes directly from the equivalence between \( \text{JSL}_f \) and Dep. That is, the elements of \( \text{Bld}(Q, R) \) are precisely the relations \( \mathcal{R}(q, r) \iff r \leq f(q) \) for some join-semilattice morphism \( f : Q \to R^{\text{op}} \).

   (b) The second is the pre-existing notion of bi-ideal [Fra78, GW05]: a subset \( \mathcal{R} \subseteq Q \times R \) lies in \( \text{Bld}(Q, R) \) iff it is closed under the following rules:

   i. \( \mathcal{R}(1_Q, r) \) for all \( r \in R \), and \( \mathcal{R}(q, 1_R) \) for all \( q \in Q \).

   ii. \( \mathcal{R} \) is downwards-closed inside \( Q \times R \).

   iii. \( \mathcal{R} \) is closed under ‘lateral joins’ i.e.

\[ (\mathcal{R}(q, r_1) \land \mathcal{R}(q, r_2) \implies \mathcal{R}(q, r_1 \lor_R r_2)) \quad (\mathcal{R}(q_1, r) \land \mathcal{R}(q_2, r) \implies \mathcal{R}(q_1 \lor_q q_2, r)) \]

4. Having established the latter correspondence, we’ll prove that the tensor product of finite join-semilattices is universal [Fra78].

So let us begin by describing the join-semilattice structure of Dep’s hom-sets, and also its top element.

**Lemma 5.1.16.** Let \( \mathcal{G} \) and \( \mathcal{H} \) be relations between finite sets.

1. We have the Dep-morphism \( \emptyset : \mathcal{G} \to \mathcal{H} \) with associated components:

\[
\begin{align*}
\emptyset_- &:= \mathcal{K}(G_s, \text{cl}_H(\emptyset)) \subseteq G_s \times H_s \\
\emptyset_+ &:= \mathcal{K}(H_t, \text{cl}_G(\emptyset)) \subseteq H_t \times G_t
\end{align*}
\]

i.e. we connect everything to the respective isolated elements.

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2. Given \( \text{Dep-morphisms } R, S : \mathcal{G} \to \mathcal{H} \), their union defines a \( \text{Dep-morphism } R \cup S : \mathcal{G} \to \mathcal{H} \), and:

\[
(R \cup S)_- := \{(g_s, h_s) \in G_s \times H_s : h_s \in cl_H((R_- \cup S_-)[g_s])\}
\]

\[
(R \cup S)_+ := \{(h_t, g_t) \in H_t \times G_t : g_t \in cl_G((R_+ \cup S_+)[h_t])\}
\]

are its associated component relations.

3. We have the \( \text{Dep-morphism } \tau_{\text{Dep}[R, S]} = \mathcal{K}(\hat{\mathcal{G}}[G], \mathcal{H}[H]) : \mathcal{G} \to \mathcal{H} \) with associated components:

\[
(\tau_{\text{Dep}[R, S]})_- := \emptyset_\mathcal{G} \cup \mathcal{K}(\hat{\mathcal{G}}[G], \mathcal{H}[H]) \subseteq G_s \times H_s
\]

\[
(\tau_{\text{Dep}[R, S]})_+ := \emptyset_\mathcal{H} \cup \mathcal{K}(\mathcal{H}[H], \mathcal{G}[G]) \subseteq H_t \times G_t
\]

\[\tag*{Proof.}\]

1. \( \emptyset : \mathcal{G} \to \mathcal{H} \) is a \( \text{Dep-morphism} \) via the witnesses \( \emptyset ; \mathcal{H} = \emptyset = \emptyset = \mathcal{G} ; \emptyset \). Closing these witnesses, the negative component \( \emptyset_- \) sends every \( g_s \) to \( \text{cl}_H(\emptyset) \), whereas the positive component sends every \( h_t \) to \( \text{cl}_H(\emptyset) \).

2. We have \( R_- ; \mathcal{H} = R = \mathcal{G} ; R_+ \) and \( S_- ; \mathcal{H} = S = \mathcal{G} ; S_+ \). Then since (i) relational composition preserves unions separately in each component, (ii) relational converse preserves unions, we deduce that \( R \cup S \) is a well-defined \( \text{Dep-morphism} \) via the witnesses \( (R_- \cup S_-, R_+ \cup S_+) \). Closing these witnesses point-image-wise yields the associated components.

3. \( \tau_{\text{Dep}[G, H]} := \hat{\mathcal{G}}[G] \times \mathcal{H}[H] \) defines a \( \text{Dep-morphism of type } \mathcal{G} \to \mathcal{H} \) via the witnesses:

\[
\begin{array}{ccc}
G_t & \xrightarrow{\mathcal{K}(G_t, H_t)} & H_t \\
\mathcal{G} & \xrightarrow{\mathcal{K}(\hat{G}[G], \hat{H}[H])} & \mathcal{H}
\end{array}
\]

so let us compute the negative component:

\[
(\tau_{\text{Dep}[G, H]})(g_s, h_s) \iff h_s \in H^t(\tau_{\text{Dep}[\hat{G}, \hat{H}]}[g_s])
\]

\[
\iff \mathcal{H}[h_s] \subseteq (\mathcal{G}[G_t] \times \mathcal{H}[H_t])[g_s]
\]

\[
\iff (g_s \in \text{cl}_G(\emptyset) \text{ and } \mathcal{H}[h_s] \subseteq \emptyset) \text{ or } (g_s \notin \text{cl}_G(\emptyset) \text{ and } \mathcal{H}[h_s] \subseteq \mathcal{H}[H_s])
\]

\[
\iff (g_s, h_s) \in \text{cl}_G(\emptyset) \times \text{cl}_H(\emptyset) \cup \mathcal{G}[G_t] \times \mathcal{H}[H_t]
\]

As for the positive component, recall that it is the negative component of the dual morphism. Since relational converse preserves inclusions it also preserves the largest morphism, thus:

\[
(\tau_{\text{Dep}[G, H]})(g_s, h_s) = (\hat{\mathcal{G}} \to \mathcal{H} \cup \mathcal{K}(\mathcal{H}[H_s], \mathcal{G}[G_s]) = \emptyset_\mathcal{H} \cup \mathcal{K}(\mathcal{H}[H_s], \mathcal{G}[G_s])
\]

\[\tag*{\Box}\]

Now we repackage the preceding Lemma as a Definition.

**Definition 5.1.17** (Join-semilattice structure on \( \text{Dep} \)'s hom-sets).

For each bipartite graph \( \mathcal{G} \) and \( \mathcal{H} \) define the finite join-semilattice:

\[
\text{Dep}[\mathcal{G}, \mathcal{H}] := (\text{Dep}(\mathcal{G}, \mathcal{H}), \cup, \emptyset : \mathcal{G} \to \mathcal{H})
\]

which is well-defined by Lemma 5.1.16. Observe that it is ordered by inclusion. It extends to a functor \( \text{Dep}[-, -] : \text{Dep}^{op} \times \text{Dep} \to JSL_f \), whose action on morphisms is as follows:

\[
R : \mathcal{G} \to \mathcal{H} \quad S : \mathcal{G}' \to \mathcal{H}'
\]

\[
\text{Dep}[R^{op}, S] := \lambda \mathcal{T} \mathcal{R} \vdash \mathcal{T} ; \mathcal{S} : \text{Dep}[\mathcal{H}, \mathcal{G}'] \to \text{Dep}[\mathcal{G}, \mathcal{H}']
\]

\[\tag*{\blacksquare}\]
Lemma 5.1.18. \( \text{Dep}[\cdot, \cdot] : \text{Dep}^{op} \times \text{Dep} \to \text{Dep} \) is a well-defined functor. In particular:

\[
\begin{align*}
(\emptyset : \mathcal{G} \to \mathcal{G}_1) \uplus R &= \emptyset : \mathcal{G} \to \mathcal{G}_2 \\
\mathcal{R} \uplus (\varnothing : \mathcal{G}_2 \to \mathcal{H}) &= \varnothing : \mathcal{G}_1 \to \mathcal{H} \\
(R_1 \cup R_2) \uplus \mathcal{R} &= (R_1 \uplus \mathcal{R}) \cup (R_2 \uplus \mathcal{R}) \\
\mathcal{R} \uplus (S_1 \cup S_2) &= (\mathcal{R} \uplus S_1) \cup (\mathcal{R} \uplus S_2)
\end{align*}
\]

for any \( \text{Dep} \)-morphisms \( \mathcal{R} : \mathcal{G}_1 \to \mathcal{G}_2, (\mathcal{R}_i : \mathcal{G} \to \mathcal{G}_1)_{i \in \{1, 2\}} \) and \( (S_i : \mathcal{G}_2 \to \mathcal{H})_{i \in \{1, 2\}} \).

**Proof.** Each \( \text{Dep}[\mathcal{G}, \mathcal{H}] \) is a well-defined join-semilattice by Lemma 5.1.16. Take any \( \text{Dep} \)-morphisms \( \mathcal{R} : \mathcal{G} \to \mathcal{H} \) and \( \mathcal{S} : \mathcal{G} \to \mathcal{H} \). Then functoriality follows if \( \text{Dep}[\mathcal{R}_{op}, \mathcal{S}] = \lambda \mathcal{T} : \mathcal{R} \uplus \mathcal{T} \uplus \mathcal{S} \) is a well-defined join-semilattice morphism. The bottom element is preserved:

\[
\mathcal{R} \uplus (\varnothing \uplus \mathcal{S}) = \mathcal{R} \uplus (\varnothing \uplus \mathcal{S}) = \mathcal{R} \uplus \mathcal{S} = (\mathcal{R} \uplus \mathcal{H}) : (\mathcal{G} \uplus \mathcal{H}) \\
\]

recalling that \( \text{cl}_\mathcal{R}(\varnothing) \) is the set of isolated elements in \( \mathcal{H} \). Next, \( \mathcal{R} \uplus (\mathcal{T}_1 \cup \mathcal{T}_2) = (\mathcal{R} \uplus \mathcal{T}_1) \cup (\mathcal{R} \uplus \mathcal{T}_2) \) because:

\[
(\mathcal{R} \uplus (\mathcal{T}_1 \cup \mathcal{T}_2))^!(X) = (\mathcal{T}_1 \cup \mathcal{T}_2)^!(\mathcal{G}) \circ (\mathcal{G})^! \circ (\mathcal{R}^!)(X) = \bigcup_{i=1, 2} (\mathcal{T}_i \circ (\mathcal{G})^! \circ (\mathcal{R}^!)(X) = \bigcup_{i=1, 2} (\mathcal{R}_i \circ (\mathcal{T}_i)^!(X)
\]

Since the self-duality of \( \text{Dep} \) is relational converse (which preserves unions), we immediately deduce that \( (\mathcal{T}_1 \cup \mathcal{T}_2) \uplus \mathcal{S} = (\mathcal{T}_1 \uplus \mathcal{S}) \cup (\mathcal{T}_2 \uplus \mathcal{S}) \). Thus \( \text{Dep}[\mathcal{R}_{op}, \mathcal{S}] \) preserves binary unions, as desired. \( \square \)

**Note 5.1.19** (Isomorphisms between join-semilattices of morphisms).

Using the equivalence functors \( \text{Pirr} \) and \( \text{Open} \) and also their respective natural isomorphisms \( \text{rep}_\mathcal{Q} \) and \( \text{red}_\mathcal{G} \), one can describe explicit join-semilattice isomorphisms:

\[
\text{JSL}_J[\mathcal{Q}, \mathcal{R}] \cong \text{Dep}[\text{Pirr}_J \mathcal{Q}, \text{Pirr}_J \mathcal{R}] \quad \text{and} \quad \text{Dep}[\mathcal{G}, \mathcal{H}] \cong \text{JSL}_J[\text{Open}_J \mathcal{G}, \text{Open}_J \mathcal{H}]
\]

By Theorem 4.3.4 we also know that \( \text{Nleq} \) and \( \text{Open} \) define an equivalence of categories, yielding the join-semilattice isomorphisms \( \text{JSL}_J[\mathcal{Q}, \mathcal{R}] \cong \text{Dep}[\text{Nleq}_J \mathcal{Q}, \text{Nleq}_J \mathcal{R}] \) described directly below. \( \blacksquare \)

Lemma 5.1.20. For each pair of finite join-semilattices \( (\mathcal{Q}, \mathcal{R}) \) we have the join-semilattice isomorphism:

\[
\varrho_{\mathcal{Q}, \mathcal{R}} : \text{JSL}_J[\mathcal{Q}, \mathcal{R}] \to \text{Dep}[\varrho_{\mathcal{Q}}, \varrho_{\mathcal{R}}] \\
\varrho_{\mathcal{Q}, \mathcal{R}}(f) := \{(q, r) \in \mathcal{Q} \times \mathcal{R} : f(q) \varrho_{\mathcal{Q}} r\} \\
\varrho_{\mathcal{Q}, \mathcal{R}}^{-1}(\mathcal{R}) := \varrho_{\mathcal{Q}} \Lambda_{\mathcal{R} \uplus \mathcal{Q}} \varrho_{\mathcal{R}}(q)
\]

where \( \varrho_{\mathcal{Q}} \subseteq \mathcal{Q} \times \mathcal{Q} \) and \( \varrho_{\mathcal{R}} \subseteq \mathcal{R} \times \mathcal{R} \). In particular, \( f \leq_{\text{JSL}_J[\mathcal{Q}, \mathcal{R}]} g \iff \varrho_{\mathcal{Q}, \mathcal{R}}(f) \leq \varrho_{\mathcal{Q}, \mathcal{R}}(g) \).

**Proof.** Recall Theorem 4.3.4 i.e. the categorical equivalence via functors \( \text{Nleq} : \text{JSL}_J \to \text{Dep} \) and \( \text{Open} \), where \( \text{Nleq}_J \mathcal{Q} := \varrho_{\mathcal{Q}} \subseteq \mathcal{Q} \times \mathcal{Q} \) and \( \text{Nleq}_J \mathcal{R} := \{(q, r) \in \mathcal{Q} \times \mathcal{Q} : f(q) \varrho_{\mathcal{Q}} r\} \). Fixing \( (\mathcal{Q}, \mathcal{R}) \) then \( \text{Nleq} \) restricts to the bijective function \( \varrho_{\mathcal{Q}, \mathcal{R}}(f) = \text{Nleq}_J f \), and clearly \( \varrho_{\mathcal{Q}, \mathcal{R}}^{-1} \) is its functional inverse. Then it suffices to establish that this bijection \( \varrho_{\mathcal{Q}, \mathcal{R}} \) defines an order-embedding i.e.

\[
f \leq_{\text{JSL}_J[\mathcal{Q}, \mathcal{R}]} g \iff \text{Nleq}_J f \subseteq \text{Nleq}_J g
\]

Regarding \((\Rightarrow)\), by assumption \( f(q) \leq_{\mathcal{R}} g(q) \) for all \( q \in \mathcal{Q} \). Then \( \text{Nleq}_J(f, g) \) means that \( f(q) \varrho_{\mathcal{R}} r \) hence \( g(q) \varrho_{\mathcal{R}} r \) (else contradiction), so that \( \text{Nleq}_J(g, f) \). Concerning \((\Leftarrow)\), suppose that \( \text{Nleq}_J f \subseteq \text{Nleq}_J g \). Then:

\[
f(q) = \bigwedge_{\mathcal{R}} \text{Nleq}_J f[q] \varrho_{\mathcal{R}} \bigwedge_{\mathcal{R}} \text{Nleq}_J g[q] = g(q)
\]

because \( \text{Nleq}_J f[q] \subseteq \text{Nleq}_J g[q] \) i.e. we have fewer summands. \( \square \)

This permits an alternative description of the tensor product \( \mathcal{Q} \otimes \mathcal{R} \). The name ‘bi-ideal’ already exists in the literature, and the following definition will be shown to coincide with the pre-existing notion.

**Definition 5.1.21** (Bi-ideals over a pair of finite join-semilattices).
Given a pair of finite join-semilattices $(Q, R)$, define:

$$\text{BId}(Q, R) := \{ R \subseteq Q \times R : R \in \text{Dep}(\mathcal{JSL}[Q, R]) \}$$

and call them the bi-ideals over $(Q, R)$. Note that $\mathcal{JSL}[Q, R]$ is closed under relative complements. In words, a bi-ideal over $(Q, R)$ is the collection of Dep-morphisms of type $\mathcal{JSL}[Q, R] \to \mathcal{JSL}[Q, R]$. Ordering them by inclusion uniquely determines a join-semilattice:

$$\text{Bld}(Q, R) := \text{Bld}(Q, R) \cap \text{Bld}(Q, R)$$

where $\cap \text{Bld}(Q, R) = \cap$ and $\cap \text{Bld}(Q, R) = Q \times R$. Then the join constructs the intersection of all bi-ideals containing the summands as subsets, and we also have the explicit description $\cap \text{Bld}(Q, R) = \{ 1 \} \times R \cup Q \times \{ 1 \}$.

**Note 5.1.22.** $\text{Bld}(Q, R)$'s meets are intersections because $\text{Dep}(\mathcal{JSL}[Q, R])$ is closed under arbitrary unions by Lemma 5.1.16. Concerning the bottom element, it is necessarily the relative complement of:

$$\text{Bld}(Q, R) \cap \text{Bld}(Q, R)$$

$$\text{Bid}(Q, R) := \{ R \subseteq Q \times R : R \in \text{Dep}(\mathcal{JSL}[Q, R]) \}$$

Thus

$$Q \times R = \text{Bld}(Q, R) \cap \text{Bld}(Q, R)$$

Concerning the condition $r \leq \mathcal{JSL}[Q, R]_{\mathcal{JSL}[Q, R]}(q)$,

- If $q = 1$, then $r \leq 1 \subseteq \mathcal{JSL}[Q, R]_{\mathcal{JSL}[Q, R]}(q)$ which never holds.
- If $q \neq 1$, then $r \leq 1 \subseteq \mathcal{JSL}[Q, R]_{\mathcal{JSL}[Q, R]}(q)$ holds iff $r = 1$.

We may reinterpret the tensor product $Q \otimes R$ as the collection of bi-ideals over $(Q, R)$ ordered by inclusion.

**Lemma 5.1.23** (Tensor product as join-semilattice of bi-ideals).

*We have the join-semilattice isomorphism:*

$$\text{bid}_{Q, R} : Q \otimes R \to \text{Bld}(Q, R)$$

$$\text{bid}_{Q, R}(h : Q \to R_{\mathcal{JSL}[Q, R]}) := \{ (q, r) \in Q \times R : r \leq h(q) \}$$

**Proof.** Recalling that $Q \otimes R := (\mathcal{JSL}[Q, R]_{\mathcal{JSL}[Q, R]})_{\mathcal{JSL}[Q, R]}$, first observe:

$$\text{bid}_{Q, R} = -Q \otimes R \circ \text{U}_{Q, R}$$

where $\text{U}_{Q, R} : \mathcal{JSL}[Q, R]_{\mathcal{JSL}[Q, R]} \to \text{Dep}(\mathcal{JSL}[Q, R])$ is the isomorphism from Lemma 5.1.20. Thus $\text{bid}_{Q, R}$ is bijective, and:

$$h_1 \leq Q \otimes R h_2 \iff h_1 \leq \mathcal{JSL}[Q, R]_{\mathcal{JSL}[Q, R]} h_2 \iff \text{U}_{Q, R}(h_2) \leq \text{U}_{Q, R}(h_1) \quad \text{by Lemma 5.1.20}$$

so it is an order-isomorphism. The description of its inverse is immediate.

Let us now prove that bi-ideals correspond to the classical concept.

**Lemma 5.1.24** (Inductive description of bi-ideals).

*A relation $R \subseteq Q \times R$ defines a bi-ideal over $(Q, R)$ iff the following three statements hold:*

(a) $\cap \text{Bld}(Q, R) \subseteq R$.

(b) $R$ is down-closed inside $Q \times R$ i.e.

$$R(q_1, r_1), q_2 \leq Q q_1, r_2 \leq R r_1 \quad R(q_2, r_2)$$

$$R(q_1, r_1)$$

$$R(q_2, r_2)$$
(c) $\mathcal{R}$ is closed under ‘lateral joins’ i.e.

$$
\begin{align*}
\mathcal{R}(q, r_1) & \subseteq \mathcal{R}(q, r_2) \quad \text{and} \quad \mathcal{R}(q_1, r) \subseteq \mathcal{R}(q_2, r) \\
\mathcal{R}(q, r_1 \lor_R r_2) & \subseteq \mathcal{R}(q_1 \lor_R q_2, r)
\end{align*}
$$

Proof.
1. We first show that every bi-ideal $\mathcal{R} \in \text{Bld}(Q, R)$ satisfies the above three statements. (a) is immediate by well-definedness of the inclusion-ordered join-semilattice $\text{Bld}(Q, R) = (\text{Bld}(Q, R), \lor_{\text{Bld}(Q, R)}, \bot_{\text{Bld}(Q, R)})$. Next, by Lemma 5.1.23 there exists a join-semilattice morphism $h : Q \to \mathcal{R}^{\text{op}}$ such that $\mathcal{R} = \{(q, r) \in Q \times R : r \leq_R h(q)\}$. Thus (b) holds because if $(q_2, r_2) \leq_{Q \times R} (q_1, r_1)$ then:

$$
q_2 \leq Q q_1 \implies h(q_2) \leq_R h(q_1)
$$

noting that $q_2 \leq Q q_1$ implies $h(q_2) \leq_R h(q_1)$. Finally, (c) also follows easily. That is:

$$
\begin{align*}
\mathcal{R}(q, r_1) \land \mathcal{R}(q, r_2) & \implies r_1, r_2 \leq_R f(q) \implies r_1 \lor_R r_2 \leq_R h(q) \implies \mathcal{R}(q, r_1 \lor_R r_2) \\
\mathcal{R}(q_1, r) \land \mathcal{R}(q_2, r) & \implies r \leq Q f(q_1), f(q_2) \implies r \leq Q f(q_1 \lor_R q_2) \implies \mathcal{R}(r, q_1 \lor_R q_2)
\end{align*}
$$

2. Conversely take any relation $\mathcal{R} \subseteq Q \times R$ satisfying the three statements above. By Lemma 5.1.23 it suffices to construct a join-semilattice morphism $h : Q \to \mathcal{R}^{\text{op}}$ such that $\mathcal{R}(q, r) \iff r \leq_R h(q)$, so define:

$$
h : Q \to R \quad h(q) := \bigvee_R \mathcal{R}[q]
$$

Then using (a) we have $h(\bot_Q) = \forall_R R = \top_R = \bot_{\mathcal{R}^{\text{op}}}$, so it remains to prove preservation of joins i.e.

$$
x := \bigvee_R \mathcal{R}[q_1 \lor_R q_2] = \bigvee_R \mathcal{R}[q_1] \land_R (\bigvee_R \mathcal{R}[q_2]) =: y
$$

for any fixed $q_1, q_2 \in Q$. First observe that:

$$
\mathcal{R}[q_1 \lor_R q_2] = \{r \in R : \mathcal{R}(q_1 \lor_R q_2, r)\} = \{r \in R : \mathcal{R}(q_1, r) \land \mathcal{R}(q_2, r)\} = \mathcal{R}[q_1] \land \mathcal{R}[q_2]
$$

because $\mathcal{R}(q_1 \lor_R q_2, r) \iff \mathcal{R}(q_1, r) \land \mathcal{R}(q_2, r)$ follows by downwards-closure (b), and closure under lateral joins (c). Then since $\mathcal{R}[q_1] \land \mathcal{R}[q_2] \subseteq \mathcal{R}[q_1]$ for $i = 1, 2$ we deduce that $x \leq_R y$.

In order to prove $y \leq_R x$, observe that $\mathcal{R}[q] \in \forall_R \mathcal{R}[q]$ for every $q \in Q$. This follows because $\mathcal{R}[q] = \{r \in R : \mathcal{R}(q, r)\}$ is finite, so we can apply closure under lateral joins in the second component iteratively to deduce $\mathcal{R}(q, \forall_R \mathcal{R}[q])$. Thus $\mathcal{R}(q_1, \forall_R \mathcal{R}[q_1])$ and hence $\mathcal{R}(q_1, \forall_R \mathcal{R}[q_1] \land_R \forall_R \mathcal{R}[q_2])$ for $i = 1, 2$ by downwards-closure. Applying closure under lateral joins in the first component yields:

$$
\mathcal{R}(q_1 \lor_R q_2, \forall_R \mathcal{R}[q_1] \land_R \forall_R \mathcal{R}[q_2])
$$

and hence $\forall_R \mathcal{R}[q_1] \land_R \forall_R \mathcal{R}[q_2] \subseteq \forall_R \mathcal{R}[q_1 \lor_R q_2]$ as required.

\[\square\]

Corollary 5.1.25. For any collection of bi-ideals $S \subseteq \text{Bld}(Q, R)$ we have:

$$
\bigvee_{\text{Bld}(Q, R)} S = \bot_{\text{Bld}(Q, R)} \cup \bigcup_{n \geq 0} S_n
$$

where $S_0 := \cup S$ and, for each $n \geq 0$, $S_{n+1}$ is the downwards-closure in $Q \times R$ of all lateral-joins of $S_n$.

Proof. Denote the left-hand-side by $X$ and the right-hand-side by $Y \subseteq Q \times R$. Then since $\bot_{\text{Bld}(Q, R)} \subseteq X$ and $\forall_R \in S, \mathcal{R} \subseteq X$ (i.e. $\cup S \subseteq X$), it follows by Theorem 5.1.24 that every $S_n \subseteq X$, so that $\cup S \subseteq Y \subseteq X$. Then it remains to show that $Y$ is a bi-ideal, and we'll again use Theorem 5.1.24. Certainly $\bot_{\text{Bld}(Q, R)} \subseteq Y$, and the union of down-closed sets is down-closed. Since each $S_n \subseteq S_{n+1}$ we also deduce closure under lateral joins e.g. given $(q, r_1) \in S_m$ and $(q, r_2) \in S_n$ then $(q, r_i) \in S_{\max(m, n)}$ for $i = 1, 2$, hence $(q, r_1 \lor_R r_2) \in S_{\max(m, n)} \subseteq Y$. 

\[\square\]
Proof. We first explain why the four descriptions of $f$’s action are equivalent. The first equality follows by Lemma 5.1.23 i.e. the definition of bid$_{Q,R}$. The second equality follows by the calculation:

$$
\beta_{Q,R}(q,r) \leq_{Q \otimes R} h \iff 1_{Q,R} \leq \beta_{Q,R}(q,r) \otimes h \iff h \leq \text{JSL}_f[Q \otimes R ; (Q \otimes R)] \iff \forall q' \in Q, h(q') \leq_{Q \otimes R} h(q) \iff \forall q' \in Q, h(q') \leq h(q) \iff h \leq_{Q \otimes R} h(q) \iff h \leq_{Q \otimes R} h(q).
$$

The marked equality follows because if $x_n = \forall S \{ \beta(q,r) : (q,r) \in S \}$ then we have $x_n \leq x_0$ for every $n \geq 0$. That is, adding lateral joins and taking the down-closure can be ‘mirrored’ inside $x_0$ using the bilinearity of $\beta$, since we may add the appropriate summands without altering the value of $x_0$.

Finally, using the third description of $f$ we’ll show that $f(\beta_{Q,R}(q_0,r_0)) = \beta(q_0,r_0)$:

$$
f(\beta_{Q,R}(q_0,r_0)) = \forall S \{ \beta(q,r) : (q,r) \leq_{Q \otimes R} \beta_{Q,R}(q_0,r_0) \} = \forall S \{ \beta(q,r) : (q,r) \leq_{Q \otimes R} \beta_{Q,R}(q_0,r_0) \} \text{ bilinearity of } \beta = \forall S \{ \beta(q,r) : (q,r) \leq_{Q \otimes R} \beta_{Q,R}(q_0,r_0) \} \text{ by Lemma 5.1.15.3}.
$$

Note that since $\beta_{Q,R}[Q \times R]$ join-generates $Q \otimes R$ by Lemma 5.1.15.2, there can only be one join-semilattice morphism extending $\beta$ in this way.

Lemma 5.1.27.

1. We have the following natural isomorphisms.

(a) $i_Q : Q \to 2 \otimes Q$ i.e. ‘the unit’ arises by applying $(-)^{op}$ to the element-morphism:

$$
Q^{op} \xrightarrow{\text{incl}(-)} \text{JSL}_f[2, Q^{op}] = (2 \otimes Q)^{op}
$$

(b) $\pi_{Q,R,S} : (Q \otimes R) \otimes S \to Q \otimes (R \otimes S)$ i.e. ‘associativity’ arises by applying $(-)^{op}$ to the universality of the tensor product:

$$
((Q \otimes R) \otimes S)^{op} = \text{JSL}_f[Q \otimes R, (S^{op})] \to \text{JSL}_f[Q, \text{JSL}_f(R, S^{op})] = (Q \otimes (R \otimes S))^{op}
$$
(c) \( \tau_{\mathbb{Q}, R} : \mathbb{Q} \otimes R \rightarrow R \otimes \mathbb{Q} \) i.e. ‘commutativity’ arises by applying \((-)^{\text{op}}\) to the duality isomorphism between internal-homs:

\[
(\mathbb{Q} \otimes \mathbb{R})^{\text{op}} = \text{JSL}_f[\mathbb{Q}, \mathbb{R}^{\text{op}}] \rightarrow \text{JSL}_f[\mathbb{C}^{\text{op}}, \mathbb{R}] = (\mathbb{R} \otimes \mathbb{Q})^{\text{op}}
\]

(d) \( d_{\mathbb{Q}, \mathbb{R}, S} : (\mathbb{Q} \times \mathbb{R}) \otimes S \rightarrow (\mathbb{Q} \otimes S) \times (\mathbb{R} \otimes S) \) i.e. ‘distributivity’ arises by applying \((-)^{\text{op}}\) to the universality of the (co)product:

\[
((\mathbb{Q} \times \mathbb{R}) \otimes \mathbb{S})^{\text{op}} = \text{JSL}_f[\mathbb{Q} \times \mathbb{R}, \mathbb{S}^{\text{op}}] \rightarrow \text{JSL}_f[\mathbb{Q}, \mathbb{S}^{\text{op}}] \times \text{JSL}_f[\mathbb{R}, \mathbb{S}^{\text{op}}] = ((\mathbb{Q} \otimes \mathbb{S}) \times (\mathbb{R} \otimes \mathbb{S}))^{\text{op}}
\]

2. Given \(|\mathbb{Q}|, |\mathbb{R}| \geq 2\), then:

(a) \( \mathbb{Q} \) and \( \mathbb{R} \) are boolean join-semilattices iff \( \mathbb{Q} \otimes \mathbb{R} \) is a boolean join-semilattice,

(b) \( \mathbb{Q} \) and \( \mathbb{R} \) are distributive join-semilattices iff \( \mathbb{Q} \otimes \mathbb{R} \) is a distributive join-semilattice.

Proof.

1. ok

2. (a) If \( \mathbb{Q} \) and \( \mathbb{R} \) are boolean then iteratively apply \( d_{\mathbb{Q}, \mathbb{R}, S} \). Conversely, if (w.l.o.g.) \( \mathbb{Q} \) is not boolean then there exists \( j_1 \in J(\mathbb{Q}) \) which is not an atom, and by assumption some \( j_2 \in J(\mathbb{R}) \). Then since \( \perp_{\mathbb{Q}} \leq_{\mathbb{Q}} x \leq_{\mathbb{Q}} j_1 \),

\[
\perp_{\mathbb{Q} \otimes \mathbb{R}} \leq_{\mathbb{Q} \otimes \mathbb{R}} \perp_{\mathbb{Q}} \otimes_{\mathbb{Q} \otimes \mathbb{R}} j_2 \leq_{\mathbb{Q} \otimes \mathbb{R}} j_1 \otimes_{\mathbb{Q} \otimes \mathbb{R}} j_2
\]

thus the latter join-irreducible element is not an atom.

(b) If \( \mathbb{Q} \) and \( \mathbb{R} \) are distributive then their join-irreducibles are join-prime, and since:

\[
\perp_{\mathbb{Q} \otimes \mathbb{R}} \leq_{\mathbb{Q} \otimes \mathbb{R}} \perp_{\mathbb{Q}} \otimes_{\mathbb{Q} \otimes \mathbb{R}} \perp_{\mathbb{R}} \leq_{\mathbb{Q} \otimes \mathbb{R}} \perp_{\mathbb{Q} \otimes \mathbb{R}} \otimes_{\mathbb{Q} \otimes \mathbb{R}} \perp_{\mathbb{R}}
\]

it follows that every join-irreducible in \( \mathbb{Q} \otimes \mathbb{R} \) is join-prime, hence the latter is distributive. Conversely if (w.l.o.g.) \( \mathbb{Q} \) is not distributive then there exists \( j \in J(\mathbb{Q}) \) which is not join-prime. By fixing \( j' \in J(\mathbb{R}) \) one can show that the join-irreducible \( \perp_{\mathbb{Q} \otimes \mathbb{R}} \) is not join-prime, hence \( \mathbb{Q} \otimes \mathbb{R} \) is not distributive.

\[\square\]

Example 5.1.28 (Morphisms obtained via bilinearity).

1. Evaluation map \( \text{evl} : \text{JSL}_f[\mathbb{Q}, \mathbb{R}] \otimes \mathbb{Q} \rightarrow \mathbb{R} \).

2. Internal composition \( \text{cmp} : \text{JSL}_f[\mathbb{R}, \mathbb{S}] \otimes \text{JSL}_f[\mathbb{Q}, \mathbb{R}] \rightarrow \text{JSL}_f[\mathbb{Q}, \mathbb{S}] \).

3. Approximation from above \( \text{tig} : \mathbb{R} \otimes \mathbb{Q}^{\text{op}} \rightarrow \text{JSL}_f[\mathbb{Q}, \mathbb{R}] \).

Note 5.1.29 (Addendum).

1. Theorem 5.1.26 actually defines a natural isomorphism:

\[
\text{JSL}_f[\mathbb{Q} \otimes \mathbb{R}, \mathbb{S}] \cong \text{JSL}_f[\mathbb{Q}, \text{JSL}_f[\mathbb{R}, \mathbb{S}]]
\]

2. Fraser also has a characterisation of:

\[
\beta_{\mathbb{Q}, \mathbb{R}}(q, r) \leq_{\mathbb{Q} \otimes \mathbb{R}} \bigvee_{q \in \mathbb{Q}} \{\beta_{\mathbb{Q}, \mathbb{R}}(q, r_i) : i \in I\}
\]

i.e. it holds iff there exists a lattice term \( \phi \) in variables \( I \) such that:

\[
q \leq_{\mathbb{Q}} [\phi[i \mapsto q_i]]_{\mathbb{Q}} \quad \text{and} \quad r \leq_{\mathbb{R}} [\phi^d[i \mapsto r_i]]_{\mathbb{R}}
\]

where \( \phi^d \) is obtained from \( \phi \) by swapping the joins/meets.

\[\blacksquare\]
5.2 Tight morphisms and tight tensors

In this subsection we define tight join-semilattice morphisms. We describe their join/meet-irreducibles and define:

1. the tight hom-functor $Tf[-,-]$ which is a subfunctor of $JSL_f[-,-]$.
2. the tight tensor product $\otimes : JSL_f \times JSL_f \to JSL_f$.

In the next subsection we’ll describe the synchronous product functor $\odot : Dep \times Dep \to Dep$, which may also be viewed as the Kronecker product of binary matrices over the boolean semiring. We shall then prove that the tight tensor product and the synchronous product are essentially the same concepts. In the final subsection we’ll consider the notions of ‘tightness’ inside $Dep$ and prove the universal property of the synchronous product, and hence also of the tight tensor product.

Definition 5.2.1 (Tight join-semilattice morphisms).

A $JSL_f$-morphism $f : Q \to R$ is tight if it factors through some $PZ \in JSL_f$ i.e.

$$f = Q \xrightarrow{\alpha} PZ \xrightarrow{\beta} R$$

for $JSL_f$-morphisms $\alpha, \beta$. Equivalently, $f$ factors through some boolean join-semilattice inside $JSL_f$. ■

Then every morphism from or to $PZ \cong 2^Z \cong 2^{|Z|}$ is tight, as are the special morphisms:

$$i^q,r_{Q,R} = Q \xrightarrow{id_r(q)} 2 \xrightarrow{el_r} R.$$

Furthermore each special morphism $i^q,r_{Q,R}$ is also tight, see Corollary 5.2.6 below. Before characterising tight morphisms, the following Lemma provides plenty of non-examples.

Lemma 5.2.2. A $JSL_f$-isomorphism $f : Q \to R$ is tight iff both $Q$ and $R$ are distributive.

Proof. Given a tight $JSL_f$-isomorphism $f : Q \to R$ then:

$$id_Q = f^{-1} \circ f = Q \xrightarrow{\alpha} PZ \xrightarrow{\beta} R \xrightarrow{f^{-1}} Q$$

for some morphisms $\alpha, \beta$. Thus $Q$ is a join-semilattice retract of a boolean join-semilattice, so by Lemma 2.2.3.15 we deduce that $Q$ is distributive. Hence $R$ is also distributive, since $JSL_f$-isomorphisms are also lattice isomorphisms. Conversely, suppose that $Q$ and $R$ are distributive. Then again by Lemma 2.2.3.15 we know that $Q$ is a join-semilattice retract of some $PZ$, so that $f = f \circ id_Q = f \circ r \circ e$ is tight. ■

We also briefly observe that tight morphisms are closed under tensor products.

Lemma 5.2.3. If $(f_i : Q_i \to R_i)_{i=1,2}$ are tight then $f_1 \otimes f_2 : Q_1 \otimes Q_2 \to R_1 \otimes R_2$ is tight.

Proof. Follows because the tensor product of boolean join-semilattices is boolean, see Lemma 5.1.27.2. ■

Note 5.2.4. Let us recall some basic terminology, used in the proof of Lemma 5.2.5 directly below. For any finite set $Z$ we have the join-semilattice:

$$2^Z = (\text{Set}(Z,2), \vee_{2^Z}, 1_{2^Z})$$

whose elements are all functions $Z \to 2 = \{0,1\}$, whose join is the pointwise join inside $2$, and whose bottom element is necessarily $\lambda z \in Z.0$. Every finite boolean join-semilattice is isomorphic to such an algebra, since $PZ = (PZ, \cup, \emptyset) \cong 2^Z$ via the mapping:

$$S \subseteq Z \mapsto \lambda z \in Z.(z \in S) ? 1 : 0$$

i.e. a subset is sent to its indicator function. ■

Lemma 5.2.5 (Characterisation of tight morphisms).

For any $JSL_f$-morphism $f : Q \to R$, the following statements are equivalent.

1. $f$ is tight.
2. $f$ factors through some distributive join-semilattice inside $JSL_f$.

3. $f$ is a $JSL_f[Q,R]$-join of morphisms $\uparrow_{Q,R}^{q,r}: Q \to R$ where $q \in Q$ and $r \in R$.

4. $f$ is a $JSL_f[Q,R]$-join of morphisms $\uparrow_{Q,R}^{m,j}: Q \to R$ where $m \in M(Q)$ and $j \in J(R)$.

Proof.

1. (1 $\iff$ 2):
   Certainly (1) implies (2). Conversely, suppose $f = Q \xrightarrow{\alpha} D \xrightarrow{\beta} R$ for some finite join-semilattice $D$ which is distributive. By Lemma 2.2.3.15 we know every finite distributive lattice arises as the join-semilattice retract of a finite boolean join-semilattice, so that:

   $$id_D = D \xrightarrow{\alpha} PZ \xrightarrow{r} D$$

   and thus $f = (\beta \circ r) \circ (s \circ \alpha)$ implies (1).

2. (1 $\iff$ 3):
   $f : Q \to R$ is tight iff we have morphisms $\alpha, \beta$ such that:

   $$f = Q \xrightarrow{\alpha} 2Z \xrightarrow{\beta} R$$

   for some finite set $Z$. Since the coproduct and product coincide in $JSL_f$ (in fact also in $JSL$), we equivalently have morphisms $(\alpha_z : Q \to 2)_{z \in Z}$ and $(\beta_z : 2 \to R)_{z \in Z}$ such that:

   $$f = [\beta_z]_{z \in Z} \circ (\alpha_z)_{z \in Z}$$

   where

   $$\alpha_z(q) = \lambda z \in Z, \alpha_z(q)$$

   and

   $$[\beta_z]_{z \in Z}(q) = \lambda \beta_z(\delta(z)) : z \in Z$$

   so that:

   $$f(q) = \bigvee_R \{\alpha_z(\delta(z)) : z \in Z\}$$

   Thus tight morphisms are precisely the joins of the special morphisms $\uparrow_{Q,R}^{q,r}$.

3. (3 $\iff$ 4):
   It suffices to show that every special morphism $\uparrow_{Q,R}^{q,r}$ arises as a possibly-empty join of the special morphisms $\uparrow_{Q,R}^{m,j}$ where $m \in M(Q)$ and $j \in J(R)$. Recall the equalities from Lemma 5.1.9.1. First of all:

   $$\uparrow_{Q,R}^{q,r} = \uparrow_{JSL_f[Q,R]} = \uparrow_{Q,R}^{q,r}$$

   so that these morphisms arise as the empty-join. Finally if $q \neq \top_Q$ then it arises as a non-empty meet of meet-irreducibles, and if $r \neq \bot_R$ then it arises as a non-empty join of join-irreducibles, so by Lemma 5.1.9.1,

   $$\uparrow_{Q,R}^{q,r} = \bigwedge_{Q,R}(m \in M(Q), q \leq m, r)$$

   $\uparrow_{JSL_f[Q,R]} = \bigvee_{JSL_f[Q,R]}(m \in M(Q), q \leq m, r)$

   $\uparrow_{JSL_f[Q,R]} = \bigvee_{JSL_f[Q,R]}(m \in M(Q), j \in J(R), q \leq m, j \leq r)$

   □

Corollary 5.2.6. The special morphisms $\uparrow_{Q,R}^{q,r}$ are always tight. In fact,

$$\uparrow_{Q,R}^{q,r} = \uparrow_{JSL_f[Q,R]}$$

Proof. Each $\uparrow_{Q,R}^{q,r}$ is certainly tight, since by definition it factors through 2. Each $\uparrow_{Q,R}^{q,r}$ is tight because by definition it factorises through the distributive lattice 3, so we may apply Lemma 5.2.5. In particular, viewing 3 as a join-semilattice retract of $2^{\{1,2\}}$ leads to the above equality, which we now verify directly.

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5.1.1 $\uparrow^{q_0,r_0} (q)$ equals $\bot_R$ whenever $q \leq Q q_0$, otherwise it is $T_R$.

5.2.5 $\uparrow^{q_0,r_0} (q)$ equals $\bot_R$ if $q = \bot_Q$, otherwise it is $r_0$.

Thus their join is precisely the morphism $\downarrow^{q_0,r_0}_{Q,R} : Q \to R$. \hfill \Box

To understand why the tight morphisms are a particularly natural subclass of the JSL$_f$-morphisms, first observe that they determine a subfunctor of JSL$_f[-,-]$.

**Definition 5.2.7** (Tight hom-functor).

Given finite join-semilattices $Q$, $R$, first let $T_i(Q,R) \subseteq JSL_f(Q,R)$ be the subset of tight morphisms. Then we define the finite join-semilattice:

$$T_i(Q,R) := (T_i(Q,R), \lor_{T_i(Q,R)}, \bot_{T_i(Q,R)}) \subseteq JSL_f(Q,R)$$

whose join is necessarily the pointwise-join and whose bottom is necessarily $\bot_{JSL_f(Q,R)} = \lambda q \in Q \cdot \bot_R$. This extends to a functor $T_i[-,-] : JSL_f^{op} \times JSL_f \to JSL_f$ as follows:

$$f : Q_2 \to Q_1, \quad g : R_1 \to R_2 \quad \overline{T_i[f^{op}, g]} := \lambda h. g \circ h \circ f : T_i[Q_1, R_1] \to T_i[Q_2, R_2]$$

this being precisely the same way that JSL$_f[-,-]$ acts, see Definition 5.1.1. \hfill \Box

**Lemma 5.2.8.** $T_i[-,-] : JSL_f^{op} \times JSL_f \to JSL_f$ is a well-defined functor.

**Proof.** This follows from the well-definedness of JSL$_f[-,-]$ and the following two observations.

1. Each $T_i(Q,R)$ is well-defined sub join-semilattice via Lemma 5.2.5.3 noting also that $\bot_{JSL_f(Q,R)} = \uparrow_{Q,R}^{r_0} T_R$.

2. Tight morphisms are closed under pre/post-composition by arbitrary JSL$_f$-morphisms, since the factorisation through a boolean join-semilattice is preserved. \hfill \Box

Then we immediately have the following important fact:

**Corollary 5.2.9.** Whenever $Q$ or $R$ are distributive then:

$$T_i(Q,R) = JSL_f(Q,R)$$

**Proof.** Every JSL$_f$-morphism $Q \to R$ such that either $Q$ or $R$ are distributive is tight by Lemma 5.2.5. \hfill \Box

That tight morphisms are closed under composition with arbitrary morphisms is now further clarified.

**Lemma 5.2.10** (Composing special morphisms with arbitrary morphisms).

Take any JSL$_f$-morphisms $f : Q \to R$ and $g : R \to S$ and fix any elements $(q,r,s) \in Q \times R \times S$.

1. We have the equalities:

$$\uparrow^{r,s} \circ f \uparrow^{q,r} = \uparrow^{q,s} \circ f \uparrow^{q,r} = \uparrow^{q,r} f \uparrow^{q,r} = \uparrow^{q,r} g(r)$$

2. If additionally $f^{-1}(\{ r \}) = \{ q \}$ and $g(\top_R) = \top_S$ then we have the equalities:

$$\uparrow^{r,s} \circ f \downarrow^{q,r} = \uparrow^{q,r} f \downarrow^{q,r} = \uparrow^{q,r} g(r)$$

**Proof.**
1. To see that the left equality holds, consider the action:

\[\iota_{R,S}^{r,s} \circ f(q) = \begin{cases} 
\downarrow S & \text{if } f(q) \leq_R r \\
\uparrow S & \text{otherwise}
\end{cases} \quad \text{for each } q \in Q\]

and recall that \( f(q) \leq_R r \iff q \leq_Q f_*(r) \). Regarding the right equality:

\[g \circ \iota_{Q,R}^{q,r} = ((\iota_{Q,R}^{q,r})_* \circ g_*)* = (\iota_{Q,R}^{q,r, op} \circ g_*)* \quad \text{by Lemma 5.1.9.1} \]

\[\iota_{R,S}^{r,s} \circ f = (\iota_{R,S}^{r,s} \vee \iota_{R,S}^{r,s}) \circ f = (\iota_{R,S}^{r,s} \circ f) \vee (\iota_{R,S}^{r,s} \circ f) = \iota_{R,S}^{f_*(r),s} \vee \iota_{R,S}^{f_*(r),s} \quad \text{by left equality}
\]

Finally by Lemma 5.1.9.1

2. Concerning the left equality,

\[\iota_{R,S}^{r,s} \circ f = (\iota_{R,S}^{m,j} \vee \iota_{R,S}^{r,s}) \circ f = (\iota_{R,S}^{m,j} \circ f) \vee (\iota_{R,S}^{r,s} \circ f) = \iota_{R,S}^{f_*(r),s} \vee \iota_{R,S}^{f_*(r),s}\]

Now, since \( f_*(\downarrow_R) = V_Q \{q \in Q : f(q) \leq_R \downarrow_R \} = V_Q \{\downarrow_Q \} = \downarrow_Q \) by assumption, the above equals \( \iota_{R,S}^{f_*(r),s} \) as desired. Finally, a similar argument yields the right equality—this time using \( g(\uparrow_R) = \uparrow_R \).

The irreducible tight morphisms are easier to describe than the irreducible morphisms (Lemma 5.1.10).

**Lemma 5.2.11 (Irreducible tight morphisms).** For all finite join-semilattices \( Q, R \),

\[J(Ti[Q, R]) = \{ j^{m,j}_{Q,R} : m \in M(Q), j \in J(R) \} \quad \text{and} \quad M(Ti[Q, R]) = \{ j^{m,j}_{Q,R} : j \in J(Q), m \in M(R) \}\]

and hence \(|J(Ti[Q, R])| = |M(Q)| \cdot |J(R)| \) and \(|M(Ti[Q, R])| = |J(Q)| \cdot |M(R)|\).

**Proof.**

1. Regarding join-irreducibles, Lemma 5.2.5.4 informs us that every join-irreducible tight morphism takes the form \( \iota_{Q,R}^{m,j} \) where \( m \in M(Q) \) and \( j \in J(R) \). Finally by Lemma 5.1.10.2 we know that each such morphism is join-irreducible in \( JSL_{J}[Q, R] \), and hence also in the sub join-semilattice \( Ti[Q, R] \).

2. Concerning meet-irreducibles, recall that every \( j^{q,r}_{Q,R} \) lies in \( Ti[Q, R] \) by Corollary 5.2.6. It turns out we can completely reuse the proof of Lemma 5.1.10.1. That is, every tight morphism \( f : Q \to R \) arises as the meet:

\[\bigwedge \{ j^{m,j}_{Q,R} : j \in J(Q), m \in M(R), f(j) \leq_R m \}\]

because the proof only used (i) the pointwise-ordering (again \( Ti[Q, R] \) order-embeds into \( R^Q \)), (ii) the fact that \( f \) is a join-semilattice morphism, and (iii) the usual properties of join/meet-irreducibles in \( Q \) and \( R \). The proof that these special morphisms do not arise as meets of other such morphisms uses only (i) their relative pointwise ordering, and (ii) the fact that if \( q_0 \leq_R q_1 \) then the \( JSL_{J}[Q, R] \)-meet of \( j^{q_0,r}_{Q,R} \) and \( j^{q_1,r}_{Q,R} \) is constructed pointwise. The latter point continues to hold in our setting i.e. their \( Ti[Q, R] \)-meet is constructed pointwise. To see this, observe that Lemma 5.1.9.5 actually shows that the pointwise meet is:

\[\bigvee_{JSL_{J}[Q, R]} \{ j^{q_0,r}_{Q,R}, j^{q_1,r}_{Q,R} \} \quad \text{this being a tight morphism.} \]

**Definition 5.2.12 (Tight tensor product).**
The **tight tensor product** functor \( \otimes_t : JSL_f \times JSL_f \to JSL_f \) is defined as the composite functor:

\[
JSL_f \times JSL_f \xrightarrow{\otimes_t^{op} \times \text{id}_{JSL_f}} JSL_f^{op} \times JSL_f \xrightarrow{\text{Ti}[-,-]} JSL_f
\]

There are associated canonical functions:

\[
\beta^t_{Q,R} : Q \times R \to \text{Ti}(Q^{op}, R)
\]

where \( \beta^t_{Q,R}(q_0, r_0) := \downarrow_{Q^{op}, R}^{q_0, r_0} \circ \text{id}_{Q^{op}}(q_0) = \begin{cases} \uparrow_R & \text{if } q_0 \leq Q q \\ r_0 & \text{if } q_0 \nleq Q q. \end{cases} \)

\[
\bullet
\]

Since \( Q \otimes_t R = \text{Ti}[Q^{op}, R] \), observe that Lemma 5.2.11 immediately implies the following important statement.

**Lemma 5.2.13** (Irreducibles in tight tensor products).

For all finite join-semilattices \( Q, R \),

\[
J(Q \otimes_t R) = \{ q_1 \cdot q_2 : q_1 \in J(Q), q_2 \in J(R) \} \quad M(Q \otimes_t R) = \{ q_1 \cdot m_2 : m_1 \in M(Q), m_2 \in M(R) \}.
\]

Therefore \( |J(Q \otimes_t R)| = |J(Q)| \cdot |J(R)| \) and \( |M(Q \otimes_t R)| = |M(Q)| \cdot |M(R)| \).

We also have the following basic result.

**Lemma 5.2.14.** \( \otimes_t : JSL_f \times JSL_f \to JSL_f \) preserves embeddings: \( f \otimes_t g \) is injective whenever both \( f \) and \( g \) are.

**Proof.** Given morphisms \( f : Q_1 \to Q_2 \) and \( g : R_1 \to R_2 \) then:

\[
f \otimes_t g : \text{Ti}[Q_1^{op}, R_1] \to \text{Ti}[Q_2^{op}, R_2] \quad f \otimes_t g(h) := g \circ h \circ f_+.
\]

Recall that \( g \) is injective iff it is \( JSL_f \)-monic, and \( f \) is injective iff \( f_+ \) is \( JSL_f \)-epic. Thus if \( f \otimes_t g(h_1) = f \otimes_t g(h_2) \) we immediately deduce that \( h_1 = h_2 \).

\[
\square
\]

### 5.2.1 Tight morphisms: some more examples

**Lemma 5.2.15** (Constant morphisms are tight).

For each pair of finite join-semilattices \( (Q, R) \) and element \( r_0 \in R \), the constant morphism:

\[
\lambda q \in Q, \begin{cases} \uparrow_R & \text{if } q = \uparrow_R \\ r_0 & \text{otherwise} \end{cases} : Q \to R
\]

is a tight morphism.

**Proof.** This is simply the special morphism \( \uparrow_{Q,R}^{\downarrow_{Q,R}} \).

\[
\square
\]

Recall that for every finite distributive join-semilattice \( Q \) we have the canonical order-isomorphism \( \tau_Q : J(Q) \to M(Q) \) between join/meet-irreducibles, see Lemma 2.2.3.13. It extends naturally to a (tight) endomorphism of \( Q \).

**Lemma 5.2.16** (Special endomorphisms of distributive join-semilattices).

If \( Q \) is a finite distributive join-semilattice,

\[
\bigvee_{\tau_{[Q,Q]}(Q)} \{ t^{q,q}_Q : q \in Q \} = J(Q) \cdot M(Q) = \tau_Q \quad \bigwedge_{\tau_{[Q,Q]}(Q)} \{ t^{q,q}_Q : q \in Q \} = id_Q
\]

recalling that \( \text{Ti}[Q, Q] = JSL_f[Q, Q] \) because \( Q \) is distributive (Corollary 5.2.9).
Proof. For any join-irreducible \( j \in J(Q) \) we have:
\[
(V_{\tau(Q)} \{ \tau_{\nu_q} (q) : q \in Q \}) (j) = V_{\tau(q)} (q : j \leq q)
\]
by definition of \( \tau_{\nu,q} \) and see Lemma 2.2.3.13.

Regarding the second equality, first observe that \( \text{id}_{Q} \leq \nu_{Q} q \) for every \( q \in Q \). The converse follows because:
\[
\bigwedge_{\tau(Q)} \{ \nu_{Q} q q : q \in Q \} \leq \bigwedge_{Q} \{ \nu_{Q} q q : q \in Q \} = \bigwedge_{Q} \{ q : q \leq q \} = q'
\]
for each \( q' \in Q \), where the first inequality follows by Lemma 5.1.3.

\[\Box\]

**Lemma 5.2.17** (Comparing tight morphisms to arbitrary morphisms).

Take any finite join-semilattice \( Q \) and any pair \((m, j) \in M(Q) \times J(Q)\).

1. The following statements hold:
   \[\exists q \in Q \setminus \{ \tau_{Q} \}. \{ \nu_{Q} q q \leq \nu_{m,j} q \} \iff m \leq q j \quad \exists q \in Q \setminus \{ \tau_{Q} \}. \{ \nu_{Q} q q \leq \nu_{m,j} q \} \iff j \leq q m\]

2. For any JSL \(_{f}\)-morphism \( f : Q \rightarrow Q \) we have:
   \[f \leq \nu_{m,j} q q \iff f(j) \leq q m \iff \text{Pirr} f(j, m)\]

   and for any tight JSL \(_{f}\)-morphism \( g : Q \rightarrow Q \) we have:
   \[\nu_{m,j} q q \leq g \iff (\nu_{Q_{\cdot} q q})(j) \leq \nu_{m,j} q q \iff (\nu_{Q_{\cdot} q q})(j) \leq q m \iff \text{Pirr} (\nu_{Q_{\cdot} q q})(m, j)\]

**Proof.**

1. Consider the left-hand equality and assume its left-hand side. Since \( q \neq \tau_{Q} \) by assumption and also \( j \neq \tau_{Q} \) by join-irreducibility, we may apply Lemma 5.1.9.2. Thus \( \nu_{Q} q \leq \nu_{Q} q \) if and only if \( m \leq q j \) which certainly implies \( m \leq q j \). Conversely, if \( m \leq q j \) then again by Lemma 5.1.9.2 we have \( \nu_{Q} q \leq \nu_{Q} q \), where the former is applicable because \( m \neq \tau_{Q} \) by meet-irreducibility. The right-hand equality follows by a symmetric argument, using Lemma 5.1.9.4. For the second part of the argument one finds that \( j \leq q m \) implies \( \nu_{Q} q q \leq \nu_{m,j} q q \).

2. We calculate:
   \[
f \leq \nu_{m,j} q q \iff \forall j' \in J(Q). (j' \leq j \Rightarrow f(j') \leq q m) \quad \text{using monotonicity of } f
   \]
   \[
   \iff f(j) \leq q m \quad \text{by definition of } \text{Pirr}
   \]
   recalling that the pointwise ordering is determined by the restriction to join-irreducibles. Regarding the final claim, take any tight morphism \( g : Q \rightarrow Q \) and consider the composite isomorphism:
   \[
   \alpha := (\text{Ti}[Q, Q])_{op} \nu_{Q, q q} \text{Ti}[Q_{op}, Q_{op}] (-)_{*} \text{Ti}[Q, Q]
   \]
   using \( \nu_{Q, q q} \) from Theorem 5.3.10. The latter informs us that \( \alpha(\nu_{m,j} q q) = (\nu_{Q_{\cdot} q q})(j) \leq q m \) hence:
   \[
   \nu_{m,j} q q \leq g \iff (\nu_{Q_{\cdot} q q})(j) \leq q m \iff \text{Pirr} (\nu_{Q_{\cdot} q q})(m, j) \leq q m
   \]
   using the order-isomorphism and the previous claim. The final equivalence follows by definition of \( \text{Pirr} \), recalling that \( \text{Pirr}(h_{*}) = (\text{Pirr} h)^{-1} \) holds generally.

\[\Box\]
5.3 Tight tensors are essentially synchronous products

In order to better understand the tight tensor product, we'll describe essentially the same functor inside Dep. This turns out to be the synchronous product of binary relations, and corresponds to the Kronecker product of binary matrices over the boolean semiring \[\text{[Wat01]}\].

**Definition 5.3.1** (Synchronous product functor). The *synchronous product functor* \(\boxplus: \text{Dep} \times \text{Dep} \to \text{Dep}^\ast\) is defined on objects as follows:

\[
\mathcal{G} \boxplus \mathcal{H} \subseteq (G_s \times H_s) \times (G_t \times H_t) \quad \mathcal{G} \boxplus \mathcal{H}((g_s, h_s), (g_t, h_t)) \Leftrightarrow \mathcal{G}(g_s, g_t) \land \mathcal{H}(h_s, h_t).
\]

Its action on morphisms is the same i.e. given \(\text{Dep}\)-morphisms \(\mathcal{R} : \mathcal{G} \to \mathcal{H}\) and \(\mathcal{R}' : \mathcal{G}' \to \mathcal{H}'\) then:

\[
\mathcal{R} \boxplus \mathcal{R}' : \mathcal{G} \boxplus \mathcal{G}' \to \mathcal{H} \boxplus \mathcal{H}'
\]
views the parameters \(\mathcal{R} \subseteq G_s \times H_t\) and \(\mathcal{R}' \subseteq G'_s \times H'_t\) as binary relations and constructs the relation \(\mathcal{R} \boxplus \mathcal{R}' \subseteq (G_s \times G'_s) \times (H_t \times H'_t)\) as above. Similarly, the associated component morphisms are:

\[
(\mathcal{R} \boxplus \mathcal{R}')_- := \mathcal{R}_- \boxplus \mathcal{R}'_-, \quad (\mathcal{R} \boxplus \mathcal{R}')_+ := \mathcal{R}_+ \boxplus \mathcal{R}'_+
\]

Note 5.3.2 (Kronecker product of binary matrices).

Given an \(m \times n\) binary matrix \(M\), and also an \(m' \times n'\) binary matrix \(N\), then their Kronecker product (over the boolean semiring) is obtained by replacing each 1 in \(M\) by a copy of \(N\), and each 0 in \(M\) by the \(m' \times n'\) zero-matrix. More formally, it is the \((m \times m') \times (n \times n')\) binary matrix \(M \otimes N\) where the indices are ordered lexicographically; and:

\[
(M \otimes N)_{(i,i'),(j,j')} := M_{i,j} \land N_{i',j'}
\]

Then the Kronecker product of binary matrices is the synchronous product of their corresponding indicator relations, endowed with the lexicographic ordering.

Before proving that this functor is well-defined we will prove a number of basic properties e.g. synchronous products preserve bicliques (Cartesian-products), and also \(\mathcal{R} \boxplus \mathcal{S}\) is reduced iff both \(\mathcal{R}\) and \(\mathcal{S}\) are reduced, as long as none of the domains/codomains of \(\mathcal{R}\) and \(\mathcal{S}\) are empty.

**Lemma 5.3.3.** Let \(\mathcal{R} \subseteq X \times Y\) and \(\mathcal{R}' \subseteq X' \times Y'\) be any relations between finite sets.

1. Given any biclique \(A \times A' \subseteq X \times X'\), then:

\[
(\mathcal{R} \boxplus \mathcal{R}')^{-1}(A \times A') = \mathcal{R} \boxplus \mathcal{R}'[A \times A'] = \mathcal{R}[A] \times \mathcal{R}'[A']
\]

and we have the special case \(\mathcal{R} \boxplus \mathcal{R}'[(x, x')] = \mathcal{R}[x] \times \mathcal{R}'[x']\).

2. Given any biclique \(B \times B' \subseteq Y \times Y'\) then:

\[
(\mathcal{R} \boxplus \mathcal{R}')^{-1}(B \times B') = \mathcal{R}^{-1}(B) \times (\mathcal{R}')^{-1}(B')
\]

3. \((\mathcal{R} \boxplus \mathcal{R}')^{-1} = \mathcal{R}^{-1} \boxplus (\mathcal{R}')^{-1}\)

4. \(\mathcal{R} \boxplus \mathcal{R}'\) is strict iff both \(\mathcal{R}\) and \(\mathcal{R}'\) are strict.

**Proof.**

1. We calculate:

\[
\mathcal{R} \boxplus \mathcal{R}'[A \times A'] = \{(y, y') \in Y \times Y' : \exists (x, x') \in A \times A', \mathcal{R}(x, y) \land \mathcal{R}'(x', y')\}
\]

\[
= \{(y, y') \in Y \times Y' : (\exists x \in A, \mathcal{R}(x, y)) \land (\exists x' \in A', \mathcal{R}'(x', y'))\}
\]

\[
= \{(y, y') \in Y \times Y' : y \in \mathcal{R}[A] \land y \in \mathcal{R}'[A']\}
\]

\[
= \mathcal{R}[A] \times \mathcal{R}'[A']
\]
Thus in either case we deduce that $S = \emptyset$.

Otherwise we fix some $x \in X$. If the second statement fails we can find $y_1 \in Y$ such that $\mathcal{R}[y_1] = \mathcal{R}[Z]$, so that:

$$(\mathcal{R} \circ \mathcal{S})^4[Z \times \{y_1\}] = \mathcal{R}[Z] \times \mathcal{S}[y_1] \subseteq \mathcal{R}[Z] \times \mathcal{S}[\{y_1\}] = \mathcal{R} \circ \mathcal{S}[\{y_1\}]$$

whereas $(y_1, y_2) \notin Z \times \{y_2\}$. 

Thus in either case we deduce that $\mathcal{R} \circ \mathcal{S}$ is not reduced.

\[\Box\]

**Lemma 5.3.4.** Take any relations $\mathcal{R} \subseteq X_1 \times Y_1$ and $\mathcal{S} \subseteq X_2 \times Y_2$ such that each of the four sets $X_1$, $Y_1$, $X_2$, $Y_2$ defining the domains/codomains are non-empty. Then $\mathcal{R} \circ \mathcal{S}$ is reduced iff both $\mathcal{R}$ and $\mathcal{S}$ are reduced.

Proof. In the first half of the proof we do not use the non-emptiness assumption. Suppose that $\mathcal{R} \subseteq X_1 \times Y_1$ and $\mathcal{S} \subseteq X_2 \times Y_2$ are reduced i.e. satisfy the two statements from Lemma ??4. If either of them is $\emptyset \subseteq \emptyset \times \emptyset$ then so is $\mathcal{R} \circ \mathcal{S}$ and thus is reduced. Otherwise, given any $(x_1, x_2) \in X_1 \times X_2$ and any subset $A \subseteq X_1 \times X_2$ we must show that:

$$\mathcal{R}[x_1] \times \mathcal{S}[x_2] = \mathcal{R} \circ \mathcal{S}[A] = \bigcup_{(a_1, a_2) \in A} \mathcal{R}[a_1] \times \mathcal{S}[a_2]$$

implies that $(x_1, x_2) \in A$

where $(a_1, a_2) \in C_{y_2}$ also.

Then since $\mathcal{S}$ is reduced there exists $y_2 \in \mathcal{S}[x_2]$ such that $Z_{y_2} = \mathcal{S}[x_2]$, so that $\mathcal{R}[x_1] \times \mathcal{S}[x_2] = \mathcal{R}[a_1] \times \mathcal{S}[a_2]$ for some $(a_1, a_2) \in C_{y_2} \subseteq A$. Finally since $\mathcal{R}$ and $\mathcal{S}$ are reduced we have $x_1 = a_1$ and $x_2 = a_2$.

Regarding the converse, assuming that $\mathcal{R}$ is not reduced we’ll show that $\mathcal{R} \circ \mathcal{S}$ is not reduced. First observe that if $X_1 = Y_2 = \emptyset$ and $X_2$, $Y_1$ are non-empty then both $\mathcal{R}$ and $\mathcal{S}$ fail to be strict (hence cannot be reduced by Lemma ??), whereas $\mathcal{R} \circ \mathcal{S} = \emptyset \subseteq \emptyset \times \emptyset$ is reduced. This explains our assumption that every set $X_1$, $Y_1$, $X_2$, $Y_2$ is non-empty. If $\emptyset = \mathcal{S} \subseteq X_2 \times \emptyset$ then $\mathcal{R} \circ \mathcal{S} = \emptyset$ with non-empty domain/codomain, so it is not strict and thus is not reduced. Otherwise we fix some $(x_2, y_2) \in \mathcal{S}$. Since $\mathcal{R}$ is not reduced one of two statements in Lemma ??4 fails.

1. If the first statement fails we find $x_1 \notin Z \subseteq X_1$ such that $\mathcal{R}[x_1] = \mathcal{R}[Z]$, so that:

$$\mathcal{R} \circ \mathcal{S}[Z \times \{x_2\}] = \mathcal{R}[Z] \times \mathcal{S}[x_2] = \mathcal{R}[x_1] \times \mathcal{S}[x_2] = \mathcal{R} \circ \mathcal{S}[\{x_1, x_2\}]$$

whereas $(x_1, x_2) \notin Z \times \{x_2\}$.

2. If the second statement fails we can find $y_1 \notin Z \subseteq Y_1$ such that $\mathcal{R}[y_1] = \mathcal{R}[Z]$, so that:

$$(\mathcal{R} \circ \mathcal{S})^4[Z \times \{y_2\}] = \mathcal{R} \circ \mathcal{S}[Z \times \{y_2\}] = \mathcal{R}[Z] \times \mathcal{S}[y_2] = \mathcal{R}[y_1] \times \mathcal{S}[y_2] = \mathcal{R} \circ \mathcal{S}[\{y_1, y_2\}]$$

whereas $(y_1, y_2) \notin Z \times \{y_2\}$.

Thus in either case we deduce that $\mathcal{R} \circ \mathcal{S}$ is not reduced.

\[\Box\]

**Lemma 5.3.5.** $\circ : \text{Dep} \times \text{Dep} \to \text{Dep}$ is a well-defined functor.
Proof. Certainly its action on objects is well-defined. Given Dep-morphisms \( R : G \to H \) and \( R' : G' \to H' \) then \( R \otimes R' \) is a well-defined Dep-morphism of type \( G \otimes G' \to H \otimes H' \) via the witnesses:

\[
\begin{array}{c}
G_x \times G'_x \\
\xymatrix{ G \times G' \ar[r]^{(R \otimes R')_x} & H \times H' \ar[l]_{R \otimes R'} }
\end{array}
\]

That is, consider the following basic calculations:

\[
\begin{align*}
\mathcal{H} \otimes H'[(R_- \otimes R'_-)[(g_s, g'_s)]] & = \mathcal{H} \otimes H'[R_-[g_s] \times R'_-[g'_s]] & \text{preserves biclique} \\
& = \mathcal{H}[R_-[g_s]] \times H'[R'_-[g'_s]] & \text{preserves biclique} \\
& = R_-[g_s] \times R'_-[g'_s] & \text{components are witnesses} \\
& \equiv (R \otimes R')[(g_s, g'_s)] & \text{preserves biclique}
\end{align*}
\]

\[
\begin{align*}
(R_+ \otimes R'_+) \circ [G \otimes G'((g_s, g'_s))] & = (R_+ \otimes R'_+) \circ [G[g_s] \times G'[g'_s]] & \text{preserves biclique} \\
& = R_+ \circ (R'_+) \circ [G[g_s] \times G'[g'_s]] & \text{by Lemma 5.3.3.3} \\
& = R_+[g_s] \times R'_+[g'_s] & \text{preserves biclique} \\
& \equiv (R \circ R')[(g_s, g'_s)] & \text{components are witnesses} \\
& \equiv (R \otimes R')[g_s] & \text{preserves biclique}
\end{align*}
\]

Next we establish that these witnesses are the associated components.

\[
\begin{align*}
(R \otimes R')_-[(g_s, g'_s)] & = c_{H \otimes H'}(R_- \otimes R'_-)[(g_s, g'_s)] & \text{close witness} \\
& = (H \otimes H')^1 \circ (H \otimes H')^1 \circ (R_-[g_s] \times R'_-[g'_s]) & \text{preserves biclique} \\
& = (H \otimes H')^1(H[R_-[g_s]] \times H'[R'_-[g'_s]]) & \text{preserves biclique} \\
& = (H \otimes H')^1(R_-[g_s] \times R'_-[g'_s]) & \text{components are witnesses} \\
& = H'[R_-[g_s]] \times (H')^1(R'_-[g'_s]) & \text{preserves biclique} \\
& = R_-[g_s] \times R'_-[g'_s] & \text{by definition} \\
& \equiv (R \otimes R')_-[(g_s, g'_s)] & \text{preserves biclique}
\end{align*}
\]

and the proof that \((R \otimes R')_+ = R_+ \otimes R'_+\) is similar. Regarding preservation of identity morphisms:

\[
id_G \otimes id_H = (G : G \to G) \otimes (H : H \to H) = G \otimes H : G \otimes H \to G \otimes H = id_{G \otimes H}
\]

To prove preservation of Dep-composition, we first establish that:

\[
\begin{array}{c}
G \otimes G' \xrightarrow{(R \otimes S) \otimes R'} H \otimes H' \xrightarrow{id_H} H \otimes H' \\
\xymatrix{ G \otimes G' \ar[r]^{(R \otimes S) \otimes R'} & H \otimes H' \ar[l]_{id_H} }
\end{array}
\]

We prove this using the characterisation of Dep-morphisms from Lemma 4.1.10, and also the functional description of Dep-composition from Corollary ??.

\[
\begin{align*}
(R \otimes R') \circ (S \otimes id_H)[(g_s, g'_s)] & = (S \otimes id_H)^1 \circ (H \otimes H')^1 \circ (R \otimes R')^1(\{(g_s, g'_s)\}) & \text{preserves biclique} \\
& = (S \otimes id_H)^1 \circ (H \otimes H')^1 \circ (R[g_s] \times R'[g'_s]) & \text{preserves biclique} \\
& = (S \otimes H)^1 \circ (H \otimes H')^1 \circ (R[g_s] \times (H')^1(R'[g'_s])) & \text{preserves biclique} \\
& = S \circ H^1 \circ R^1[(g_s) \times (H')^1(R'[g'_s])] & \text{preserves biclique} \\
& = S \circ H^1 \circ R^1[(g_s) \times R'[g'_s]] & \text{preserves biclique} \\
& = (R \otimes S)[g_s] \times R'[g'_s] & \text{in } H \circ R' = R' \\
& \equiv ((R \otimes S) \otimes R')[(g_s, g'_s)] & \text{preserves biclique}
\end{align*}
\]

Thus we also have the symmetric statement \( R \otimes (R' \circ S') = (R \otimes R') \circ (id_H \otimes S') \). Then we calculate:

\[
\begin{align*}
(R \otimes S) \circ (R' \circ S') & = ((R \otimes S) \circ R') \circ (id_H \circ S') & \text{right preservation} \\
& = ((R \circ R') \circ (S \otimes id_H)) \circ (id_Z \circ S') & \text{left preservation} \\
& = (R \circ R') \circ ((S \otimes id_H) \circ (id_Z \circ S')) & \text{associativity} \\
& = (R \circ R') \circ (S \circ S') & \text{see below}
\end{align*}
\]

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Regarding the final statement, we have:

\[(S \otimes \text{id}_{H'}) \circ (\text{id}_{X} \otimes S')[(h_s, h_s')] = (\text{id}_{X} \otimes S') \circ (I \otimes H') \circ (S \otimes \text{id}_{H'})[(h_s, h_s')]\]

\[= (\text{id}_{X} \otimes S') \circ (S[h_s] \times H'[h_s'])\]

\[= (\text{id}_{X} \otimes S')((S[h_s]) \times ((S') \circ \text{cl}_H([h_s']))\]

We now prove the main result of this subsection. It is further clarified via its corollaries further below.

**Theorem 5.3.6** (The synchronous product is essentially the tight tensor product).

We have the natural isomorphism:

\[T_S : \text{Pirr}( \otimes_t \otimes) \to (\text{Pirr} \otimes) \circ (\text{Pirr}) \quad T_{S, Q, R} : \text{Pirr}(Q \otimes R) \to \text{Pirr} \otimes \text{Pirr} \]

\[T_{S, Q, R}(m, (m_1, m_2)) \iff \text{Pirr}(Q) \otimes \text{Pirr}(R)((m_1, m_2)) \iff m_1 \leq Q m_1 \text{ and } m_2 \leq R m_2\]

with associated components:

\[T_{S, Q, R}((m_1, m_2), (j_1, j_2)) \iff \text{Pirr}(Q) \circ \text{Pirr}(R)((j_1, j_2)) \iff j_1 \leq Q j_1 \text{ and } j_2 \leq R j_2\]

\[T_{S, Q, R}((m_1, m_2), (m_3, m_4)) \iff \text{Pirr}(Q) \circ \text{Pirr}(R)((m_3, m_4)) \iff m_1 \leq Q m_3 \text{ and } m_2 \leq R m_4\]

Its inverse is described in Note 5.3.7 directly below.

**Proof.** Although the notation is somewhat cumbersome, the proof that each \(T_{S, Q, R}\) is a Dep-isomorphism is relatively simple. Importantly, we shall show that \(\text{Pirr}(Q \otimes R)\) is bipartite graph isomorphic to \(\text{Pirr} \otimes \text{Pirr}\). Then one can see that \(T_{S, Q, R}\) and its components are really just the Dep-identity-morphism \(\text{id}_{\text{Pirr} \otimes \text{Pirr}} = \text{Pirr} \otimes \text{Pirr}\) modulo relabelling, recalling that \((\text{Pirr} \otimes \text{Pirr})_+ = (\text{Pirr})_+ \circ \text{Pirr}_+\) and similar for the positive component (see Definition 5.3.1). First let:

\[G := \text{Pirr}(Q \otimes R) = \text{Pirr} T \left[ \text{Q}^\text{op}, R \right] \quad \text{and} \quad H := \text{Pirr} \otimes \text{Pirr}\]

and recall that by Lemma 5.2.13:

\[G_s = J(T_i[Q^\text{op}, R]) = \{ \uparrow_{m_1, m_2}^{j_1, j_2} : (j_1, j_2) \in J(Q) \times J(R) \}\]

\[G_i = M(T_i[Q^\text{op}, R]) = \{ \uparrow_{m_1, m_2}^{j_1, j_2} : (m_1, m_2) \in M(Q) \times M(R) \}\]

\[H_s = J(Q) \times J(R) \quad H_i = M(Q) \times M(R)\]

Clearly \(|G_s| = |H_s|\) and \(|G_i| = |H_i|\), and moreover:

\[G(i_{m_1, m_2}^{j_1, j_2}) \iff \uparrow_{m_1, m_2}^{j_1, j_2} \leq_{\text{JSL}[Q^\text{op}, R]} \uparrow_{m_1, m_2}^{j_1, j_2}\]

\[H((j_1, j_2), (m_1, m_2)) \iff (j_1 \leq Q m_1 \text{ and } j_2 \leq R m_2)\]

recalling that \(T_i[Q^\text{op}, R]\) is a sub join-semilattice of \(\text{JSL}_i[Q^\text{op}, R]\) and hence inherits the pointwise ordering. There is an obvious candidate for a bipartite graph isomorphism i.e. send \(\uparrow_{m_1, m_2}^{j_1, j_2}\) to \((j_1, j_2)\), and send \(\uparrow_{m_1, m_2}^{j_1, j_2}\) to \((m_1, m_2)\). To verify its correctness, we need to show that for any fixed \((j_1, j_2) \in J(Q) \times J(R)\) and \((m_1, m_2) \in M(Q) \times M(R)\):

\[\uparrow_{m_1, m_2}^{j_1, j_2} \leq_{\text{JSL}[Q^\text{op}, R]} \uparrow_{m_1, m_2}^{j_1, j_2} \iff (j_1 \leq Q m_1 \text{ or } j_2 \leq R m_2)\]

which follows immediately by Lemma 5.1.9.6.
Having established this bipartite graph isomorphism between $\text{Pirr}(Q \otimes_t F)$ and $\text{Pirr}Q \otimes \text{Pirr}F$, it follows that each $\mathcal{T}S_{Q,R}$ is a well-defined Dep-isomorphism. That is, $\mathcal{T}S_{Q,R}$ is constructed by starting with the Dep-isomorphism $id_{\text{Pirr}Q \otimes \text{Pirr}F} = \text{Pirr}Q \otimes \text{Pirr}F$ and applying a bipartite graph isomorphism to its domain. The description of the associated components also follow from this.

It remains to show naturality i.e. given morphisms $(f_i : Q_i \to R_i)_{i=1,2}$ we must show that:

$$
\begin{array}{c}
\text{Pirr}(Q_1 \otimes Q_2) \\
\downarrow \text{Pirr}(f_1 \otimes f_2) \\
\text{Pirr}(R_1 \otimes R_2)
\end{array}
\xrightarrow{\mathcal{T}S,_{Q_2}}
\begin{array}{c}
(\text{Pirr}Q_1) \otimes (\text{Pirr}Q_2) \\
(\text{Pirr}f_1 \otimes \text{Pirr}f_2) \\
(\text{Pirr}R_1) \otimes (\text{Pirr}R_2)
\end{array}
$$

Let us first calculate:

$$
\mathcal{T}S_{Q_1,Q_2} \downarrow \text{Pirr} f_1 \otimes \text{Pirr} f_2 = \mathcal{T}S_{Q_1,Q_2} : \text{Pirr} f_1 \otimes \text{Pirr} f_2)^* \\
= \mathcal{T}S_{Q_1,Q_2} : \text{Pirr} f_1 \otimes \text{Pirr} f_2)^* \\
= \mathcal{T}S_{Q_1,Q_2} : \text{Pirr} f_1 \otimes \text{Pirr} f_2)^*
$$

so that:

$$
\forall i = 1,2, \exists m_i \in M(Q_i), (j_i \in Q_i, m_i \in Q_i) \text{ and } (f_i)_* (m_i) \leq Q_i, (m_i') \\
\forall i = 1,2, \forall m_i \in M(Q_i), (f_i)_* (m_i) \leq Q_i, (m_i') \\
\forall i = 1,2, \forall m_i \in M(Q_i), (f_i)_* (m_i) \leq Q_i, (m_i') \\
\forall i = 1,2, \forall m_i \in M(Q_i), (f_i)_* (m_i) \leq Q_i, (m_i')
$$

We now compute the other composite Dep-morphism:

$$
\text{Pirr}(f_1 \otimes f_2) : \mathcal{T}S_{R_1,R_2} = \mathcal{T}S_{R_1,R_2} : (\text{Pirr} f_1 \otimes \text{Pirr} f_2)^*
$$

in three steps.

1. The first relation $\text{Pirr}(f_1 \otimes f_2) \leq J(Q_1 \otimes Q_2) \times M(R_1 \otimes R_2)$ has definition:

$$
\text{Pirr}(f_1 \otimes f_2)(t_{Q_1,Q_2}^{Q_1',Q_2'} (m_1, m_2)) \\
\iff f_1 \otimes f_2(t_{Q_1,Q_2}^{Q_1',Q_2'} (m_1, m_2)) = f_1 \otimes f_2(t_{Q_1,Q_2}^{Q_1',Q_2'} (m_1, m_2)) \\
\iff f_2 = f_1 \otimes f_2(t_{Q_1,Q_2}^{Q_1',Q_2'} (m_1, m_2))
$$

2. The second relation $(\mathcal{T}S_{R_1,R_2})^*_ (m_1, m_2) \leq M(R_1 \otimes R_2) = (M(R_1) \times M(R_2))$ has definition:

$$(\mathcal{T}S_{R_1,R_2})^*_ (m_1, m_2) \iff m_1 \leq R_1, m_1 \text{ and } m_2 \leq R_2, m_2
$$

as per the statement of this theorem.

3. Composing yields all pairs $(t_{Q_1,Q_2}^{Q_1',Q_2'} (m_1, m_2))$ s.t. $\exists (m_1', m_2') \in M(R_1) \times M(R_2)$ satisfying:

(a) $f_1 \circ t_{Q_1,Q_2}^{Q_1',Q_2'} (m_1, m_2') \leq \text{JSL}_{(R_1 \otimes R_2)} \downarrow R_1 \otimes R_2$

(b) $m_1 \leq R_1$ and $m_2 \leq R_2$

By Lemma 5.1.9.4 the latter condition is equivalent to

$$
(\mathcal{T}S_{R_1,R_2})^*_ (m_1, m_2) \iff m_1 \leq R_1, m_2 \leq R_2
$$

where it is important that $R_1^{op}$ reverses the ordering. Consequently:

$$
\text{Pirr}(f_1 \otimes f_2) : (\mathcal{T}S_{R_1,R_2})^*_ (m_1, m_2) \\
\iff \exists (m_1', m_2') \in M(R_1) \times M(R_2) \iff (m_1, m_2) \leq \downarrow R_1 \otimes R_2
$$

$$
\iff \neg \forall m_1 \in M(R_1), m_2 \in M(R_2) \iff (m_1, m_2) \leq \downarrow R_1 \otimes R_2 \\
\iff \exists (m_1', m_2') \in M(R_1) \times M(R_2) \iff (m_1, m_2) \leq \downarrow R_1 \otimes R_2
$$

$$
\iff f_2 \circ t_{Q_1,Q_2}^{Q_1',Q_2'} (m_1, m_2') \leq \text{JSL}_{(R_1 \otimes R_2)} \downarrow R_1 \otimes R_2
$$

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Having described the two sides of the naturality square, we need to prove their equality. By the above descriptions it suffices to prove that:
\[ f_2 \circ \downarrow_{Q,R}^{	ext{f}_1 j_1 j_2} \circ (f_1) \leq \downarrow_{R_1}^{	ext{m}_1 \text{m}_2} \iff (f_1(j_1) \leq_{R_1} \text{m}_1 \text{ or } f_2(j_2) \leq_{R_2} \text{m}_2) \]
for any \((j_1, j_2) \in J(Q_1) \times J(Q_2)\) and \((\text{m}_1, \text{m}_2) \in M(\overline{R}_1) \times M(\overline{R}_2)\). By Lemma 5.2.10.1 this amounts to:
\[ \downarrow_{R_1}^{	ext{f}_1 j_1 j_2} f_2(j_2) \leq \downarrow_{R_1}^{	ext{m}_1 \text{m}_2} \iff (f_1(j_1) \leq_{R_1} \text{m}_1 \text{ or } f_2(j_2) \leq_{R_2} \text{m}_2) \]
which follows immediately by Lemma 5.1.9.6.

**Note 5.3.7.** The natural inverse \(TS^{-1} : (\text{Pirr}^{-}) \oplus (\text{Pirr}^{-}) \Rightarrow \text{Pirr}(\ominus t^{-})\) and its associated components are defined:
\[
\begin{align*}
TS^{-1}_{Q,R}((j_1, j_2), \downarrow_{R_1}^{	ext{m}_1 \text{m}_2}) & \iff j_1 \not\in Q \text{m}_1 \text{ and } j_2 \not\in R \text{m}_2 \\
(TS^{-1}_{Q,R})_{-}((j_1, j_2), \downarrow_{Q,R}^{	ext{j}_3 \text{j}_4}) & \iff j_3 \not\in Q \text{ and } j_4 \not\in R \\
(TS^{-1}_{Q,R}) \downarrow_{Q,R}^{	ext{m}_1 \text{m}_2}, (m_3, m_4)) & \iff m_1 \not\in Q \text{m}_3 \text{ and } m_2 \not\in R m_4
\end{align*}
\]
This follows from the proof above i.e. apply the bipartite graph isomorphism to the codomain of \(id_{\text{PirrQ} \oplus \text{PirrR}}\).

That the synchronous product and the tight tensor product are ‘equivalent concepts’ is now further clarified i.e. we describe certain composite natural isomorphisms. Recall that definition \(Q \oplus_t R = T[iQ^{\text{op}}, R]\).

**Corollary 5.3.8.**
We have the composite natural isomorphisms:

1. \(Q \oplus_t R \xrightarrow{\text{rep}_Q \otimes R} \text{OpenPirr}(Q \oplus_t R) \xrightarrow{\text{Open}TS^{-1}_{Q,R}} \text{Open}(\text{PirrQ} \otimes \text{PirrR})\)
   with action \(f \mapsto \{(m_q, m_r) \in M(Q) \times M(R) : f(m_q) \not\in R \text{m}_r\}\)
   and inverse action \(Y \mapsto \lambda q \in Q, \forall \{\Lambda_{m_r} \in M(R) : (m_q, m_r) \not\in Y\} : q \not\in Q m_q \in M(Q)\}.

Moreover the action on join/meet-irreducibles is as follows:
\[
\begin{align*}
\uparrow_{Q,R}^{j_3, j_4} & \Rightarrow \text{PirrQ} \otimes \text{PirrR}[(j_q, j_r) = \text{PirrQ}[j_q] \times \text{PirrR}[j_r] \\
\downarrow_{Q,R}^{m_q, m_r} & \Rightarrow \text{inPirrQ} \otimes \text{PirrR}((m_q, m_r))
\end{align*}
\]

2. \(G \otimes H \xrightarrow{\text{red}_G \oplus H} (\text{PirrOpenG}) \oplus (\text{PirrOpenH}) \xrightarrow{TS^{-1}_{\text{OpenG}, \text{OpenH}}} \text{Pirr}(\text{OpenG} \otimes_t \text{OpenH})\)
   where we relate \((g_s, h_s)\) to \(\downarrow_{\text{OpenG}, \text{OpenH}}^{\text{inG}(\text{G}), \text{inH}(\text{H})}\)
   iff we have \(G(g_s, g_t)\) and \(H(h_s, h_t)\).

Regarding the relation directly above, recall that every meet-irreducible in \(\text{OpenG}\) arises as \(\text{inG}(\text{G})\) for some \(g_t \in G_t\). However not all \(g_t \in G_t\) need yield a meet-irreducible in this way, unless \(G\) is reduced.

**Proof.**

1. We begin by showing that:
\[
\text{rep}_{Q \otimes_t R}(f : Q^{\text{op}} \rightarrow R) = \{ \downarrow_{Q^{\text{op}}, R}^{m_1 m_2} : (m_1, m_2) \in M(Q) \times M(R), f(m_1) \not\in R m_2 \}
\]
Recall by definition that \(\text{rep}_{Q \otimes_t R}(f)\) contains all those meet-irreducibles \(m \in M(Q \otimes_t R)\) such that \(f \not\in Q \otimes_t R m\). Now, by Lemma 5.2.11 we know these meet-irreducibles are precisely \(\downarrow_{Q^{\text{op}}, R}^{m_1 m_2}m \in M(Q) \times M(R)\). Then it only remains to show that \(f \not\in Q \otimes_t R m\). We calculate:
\[
\begin{align*}
& f \leq_{Q \otimes_t R} \downarrow_{Q^{\text{op}}, R}^{m_1 m_2} \iff \forall q \in Q, q \leq_{Q^{\text{op}}} m_1 \Rightarrow f(q) \leq_{R} m_2) \text{ since } q \leq_{R} T R \\
& \iff \forall q \in Q, (m_1 \leq_{Q} q \Rightarrow f(q) \leq_{R} m_2) \\
& \iff f(m_1) \leq_{R} m_2 \text{ since } f : Q^{\text{op}} \rightarrow R \text{ monotonic}
\end{align*}
\]
In order to apply $\text{Open} \mathcal{T} \mathcal{S}_{Q, R}$, first recall that $(\mathcal{T} \mathcal{S}_{Q, R})_* : M(Q \otimes_l R) \times (M(Q) \times M(R))$ has definition:

$$(\mathcal{T} \mathcal{S}_{Q, R})_* (m_q, m_r) \iff m_q \leq Q m_1 \text{ and } m_r \leq R m_2$$

Then to finally understand why:

$$\text{Open} \mathcal{T} \mathcal{S}_{Q, R} [\text{rep}_{Q \otimes_l R}(f)] = \{(m_q, m_r) \in M(Q) \times M(R) : f(m_q) \not\leq_R m_r\}$$

observe that $f(m_1) \not\leq_R m_2$ and $m_q \leq Q m_1$ and $m_r \leq R m_2$ imply that $f(m_q) \not\leq_R m_r$, by making use of the ‘order-reversing’ monotonicity of $f : Q^{\text{op}} \to R$.

The inverse action follows because every $q \in Q$ arises as a meet of those meet-irreducibles above it and $f : Q^{\text{op}} \to R$ sends $Q$-meets to $R$-joins. The description of the action on join/meet-irreducibles is ‘the natural one’ in the sense that (i) we know the join/meet-irreducibles of $Q \otimes_l R = \mathcal{T}[(Q^{\text{op}}, R)]$ via Lemma 5.2.13, and (ii) we know the join/meet-irreducibles of $\text{Open}(\text{Pirr}Q \otimes \text{Pirr}R)$ via Lemma 4.2.5.3 and the fact that $\text{Pirr}Q \otimes \text{Pirr}R$ is reduced via Lemma 5.3.4. Nevertheless, let us directly verify these claims.

(a) First the action on join-irreducibles:

$$\alpha_{Q, R}(\lambda_{Q, R} (m_q, m_r)) = \{(m_q, m_r) \in M(Q) \times M(R) : \Delta_{Q, R} (m_q) \not\leq_R m_r\}$$

by definition

$$\alpha_{Q, R}(\lambda_{Q, R} (m_q, m_r)) = \{(m_q, m_r) \in M(Q) \times M(R) : m_q \not\leq_Q m_q, m_r \not\leq_R m_r\}$$

(b) Finally we verify the action on meet-irreducibles in terms of the inverse action. For brevity let $\mathcal{G}_Q = \text{Pirr}Q$ and $\mathcal{G}_R = \text{Pirr}R$.

$$\alpha_{Q, R}^\text{inv} (\lambda_{Q, R}^\text{inv} (m_q, m_r))$$

$$= \lambda q \in Q. \mathcal{V}_R (\lambda m' \in M(R) : (m'_q, m'_r) \in \text{in}_{\mathcal{G}_Q \otimes \mathcal{G}_R}((m_q, m_r)) : q \leq Q m'_q \in M(Q))$$

$$= \lambda q \in Q. \mathcal{V}_R (\lambda m' \in M(R) : (m'_q, m'_r) \in \text{cl}_{\mathcal{G}_Q \otimes \mathcal{G}_R}((m_q, m_r)) : q \leq Q m'_q \in M(Q))$$

$$= \lambda q \in Q. \mathcal{V}_R (\lambda m' \in M(R) : m_q \leq Q m'_q, m_r \leq R m'_r : q \leq Q m'_q \in M(Q))$$

Here we have used De Morgan duality applied to interior/closure operators, the fact that synchronous products preserve biclques (see Lemma 5.3.3), and also that e.g. $\mathcal{G}_Q = (\text{Pirr}Q)^* = \text{Pirr}Q^{\text{op}}$ so that:

$$m'_q \in \text{cl}_{\mathcal{G}_Q}((m_q)) \iff m'_q \in \text{cl}_{\text{Pirr}Q^{\text{op}}}((m_q)) \iff m'_q \leq_{Q^{\text{op}}} m_q \iff m_q \leq_{Q} m'_q$$

using Lemma 4.2.7.2 in the second equivalence.

2. First recall the canonical isomorphism:

$$\text{red}_G : G \to \text{Pirr} \text{Open} G \quad \text{red}_G \subseteq \mathcal{G}_s \times M(\text{Open} G) \quad \text{red}_G(g_s, Y) : \iff \mathcal{G}[g_s] \not\subseteq Y$$

and consequently:

$$\text{red}_G \otimes \text{red}_H \subseteq (\mathcal{G}_s \times \mathcal{H}_s) \times (M(\text{Open} G) \times M(\text{Open} H))$$

$$\text{red}_G \otimes \text{red}_H ((g_s, h_s), (Y_1, Y_2)) \iff \mathcal{G}[g_s] \not\subseteq Y_1 \text{ and } \mathcal{H}[h_s] \not\subseteq Y_2$$

We already described the positive component of $\mathcal{T} S^{-1}$ in Note 5.3.7 above,

$$(\mathcal{T} S^{-1}_{\text{Open} G, \text{Open} H})_* \subseteq (M(\text{Open} G) \times M(\text{Open} H)) \times (\text{Open} G \otimes \text{Open} H)$$

$$(\mathcal{T} S^{-1}_{\text{Open} G, \text{Open} H})_* ((Y_1, Y_2), (Y_{1'}, Y_{1''})) \iff Y_G \subseteq Y_1 \text{ and } Y_H \subseteq Y_2$$
also recalling that $\leq_{\text{open}}$ is inclusion of sets. Then:

$$
\begin{align*}
& (\text{red}_Q \otimes \text{red}_R) \diamond T_\text{open}^{-1} \text{open}_H ((g_s, h_s), Y_G \cdot Y_H) \\
\iff & (\text{red}_Q \otimes \text{red}_R) : (T_\text{open}^{-1} \text{open}_H)^* ((g_s, h_s), Y_G \cdot Y_H) \\
\iff & \exists (Y_1, Y_2) \in M(\text{open}_G) \times M(\text{open}_H). (G[g_s] \not\subseteq Y_1 \text{ and } H[h_s] \not\subseteq Y_2 \text{ and } Y_G \subseteq Y_1 \text{ and } Y_H \subseteq Y_2) \\
\iff & G[g_s] \not\subseteq \text{in}_G (\overline{Y}) \\
& \qquad \iff g_s \not\subseteq G^\dagger \circ G^1 \circ \overline{G}(\overline{Y}) \quad \text{by adjointness} \\
& \qquad \iff g_s \not\subseteq G^1(\overline{Y}) \quad \text{by } (\uparrow \downarrow 1) \\
& \qquad \iff G[g_s] \not\subseteq \overline{Y} \quad \text{by adjointness} \\
& \qquad \iff G(g_s, \overline{Y})
\end{align*}
$$

where the final equivalence follows easily using basic properties of sets. Finally, recall by Lemma 4.2.5.3 that every meet-irreducible $Y_G \in M(\text{open}_G)$ equals $\text{in}_G(\overline{Y})$ for some $g_s \in G_s$, and similarly $Y_H = \text{in}_H(\overline{H})$ for some $h_t \in H_t$. Then since:

$$
G[g_s] \not\subseteq \text{in}_G (\overline{Y}) \iff g_s \not\subseteq G^1 \circ \overline{G}(\overline{Y}) \quad \text{by adjointness}
$$

we are finished.

We also have the following basic result.

**Corollary 5.3.9.** The synchronous product functor preserves monos.

**Proof.** By Corollary 5.3.8.2 we have the natural isomorphism $\dashv - \cong \text{Pirr}(\text{open} - \otimes \text{open} -)$. Equivalence functors preserve monos, and by Lemma 5.2.14 the tight tensor product preserves them too.

The following Theorem describes the equality $G \ominus H^\circ = G \otimes H$ inside JSL, where it becomes a non-trivial natural isomorphism. Importantly, it also provides a method to compute meets inside tight tensor products.

**Theorem 5.3.10 (Tight tensor products are isomorphic to their De Morgan dual).**

We have the natural isomorphism:

$$
(Q^\text{op} \otimes \text{r}^\text{op})^\text{op} \overset{\nu_{Q,R}}{\longrightarrow} Q \otimes \text{r} \overset{\nu_{Q,R}^{-1}}{\longrightarrow} (Q^\text{op} \otimes \text{r}^\text{op})^\text{op}
$$

In particular, $\nu_{Q,R}^{-1} = \nu_{Q,R}^\text{op}$ and furthermore:

$$
\begin{align*}

\nu_{Q,R}(\text{f} \in \text{op}_{\text{r}} \cap \text{open}_G) & = \lambda q \in Q. \text{v}_G \{ j \in J(R) : f_j(j) \not\subseteq Q \} \\

\nu_{Q,R}^{-1}(q) & = \lambda q \in Q. \text{v}_G \{ m \in M(R) : q \not\subseteq Q \cdot m(q) \}
\end{align*}
$$

for any $(q, r) \in Q \times R$ and index set $I$.

**Proof.** We define $\nu_{Q,R}$ as the following composite natural isomorphism:

$$
( (Q^\text{op} \otimes \text{r}^\text{op})^\text{op} \overset{\alpha_{Q,R,\text{op}}^\text{op}}{\longrightarrow} (\text{op} (\text{Pirr}Q^\text{op} \otimes \text{Pirr}R^\text{op}))^\text{op} \overset{\partial (\text{Pirr}Q \otimes \text{Pirr}R)^\text{op}}{\longrightarrow} (\text{Pirr}Q \otimes \text{Pirr}R)^\text{op}
$$

where:

$$
\begin{align*}

\alpha_{Q,R,\text{op}}(h) & : = \{ (j_q, r_j) \in J(Q) \times J(R) : r_j \not\subseteq R \cdot h(j_q) \} \\

\alpha_{Q,R}^{-1}(Y) & : = \lambda q \in Q. \text{v}_R \{ \text{m}_\text{r} \in M(R) : \{ (\text{m}_q, \text{m}_r) \not\subseteq Q \cdot (\text{m}_q \in M(Q)) \} \\

\partial (\text{Pirr}Q \otimes \text{Pirr}R)^\text{op}(X) & : = \text{Pirr}Q \otimes \text{Pirr}R[X]
\end{align*}
$$

is from Corollary 5.3.8.1

see above

is from Definition 4.6.5

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Then \( \nu_{Q,R} \) is a well-defined natural isomorphism, and moreover:

\[
\nu_{Q_{\text{op}}, R_{\text{op}}} = (\alpha_{\text{op}, Q_{\text{op}}, R_{\text{op}}} \circ \delta_{\text{op}}(\mathbb{P}_{\text{Pirr}} \circ \mathbb{P}_{\text{Pirr}}))_{\circ} \circ \alpha_{Q, R} \quad \text{apply } (-)_{\circ}
\]

\[
= (\alpha_{\text{op}, Q_{\text{op}}, R_{\text{op}}} \circ \delta_{\text{op}}(\mathbb{P}_{\text{Pirr}} \circ \mathbb{P}_{\text{Pirr}}))_{\circ} \circ \alpha_{Q, R} \quad \text{apply } \mathcal{G}
\]

\[
= (\alpha_{\text{op}, Q_{\text{op}}, R_{\text{op}}} \circ \delta_{\text{op}}(\mathbb{P}_{\text{Pirr}} \circ \mathbb{P}_{\text{Pirr}}))_{\circ} \circ \alpha_{Q, R} \quad \text{see below}
\]

\[
= \nu_{Q,R}
\]

by rule of composite inverses.

The marked equality follows because for any relation \( \mathcal{G} \) we have:

\[
\delta_{\mathcal{G}} = ((\delta_{\mathcal{G}})_{\circ})_{-1} \quad \text{by Lemma 3.0.3.2 and } (\delta_{\mathcal{G}})_{\circ} = \delta_{\mathcal{G}} \text{ by Lemma 4.6.6.1.}
\]

Using this fact, one can readily verify that the description of \( \nu_{Q,R}^{-1} \)'s action follows from that of \( \nu_{Q,R} \), so it remains to prove that the latter is correct. To this end, we first show that:

\[
\nu_{Q,R}(\uparrow_{Q_{\text{op}},R_{\text{op}}}^{j_q,j_r}) = \uparrow_{Q_{\text{op}},R_{\text{op}}}^{j_q,j_r}
\]

for each pair of join-irreducibles \( (j_q, j_r) \in J(Q) \times J(R) \).

So let \( h : Q_{\text{op}, R_{\text{op}}} : Q \to R_{\text{op}} \), and also define \( \mathcal{G} = \mathbb{P}_{\text{Pirr}} \cap \mathbb{P}_{\text{Pirr}} \). To compute \( \nu_{Q,R}(h) \) consider the action of the first two composites:

\[
\partial_{\mathcal{G}} \circ \alpha_{Q_{\text{op}}, R_{\text{op}}}((h)_{\circ}) = \partial_{\mathcal{G}}(\{(j_q', j_r') : j_q' \leq j_q \text{ and } j_r' \leq j_r\})
\]

\[
\mathcal{G}(\{(j_q', j_r') : j_q' \leq j_q \text{ and } j_r' \leq j_r\})
\]

\[
= \{((m_q, m_r) : j_q \leq j_q' \text{ and } j_r \leq j_r')\}
\]

\[
= \{((m_q, m_r) : j_q \leq j_q' \text{ and } j_r \leq j_r')\}
\]

\[
= \partial_{\mathcal{G}}(\{(j_q', j_r') : j_q' \leq j_q \text{ and } j_r' \leq j_r\})
\]

\[
= \alpha_{Q,R}(\uparrow_{Q_{\text{op}}, R_{\text{op}}}^{j_q,j_r})
\]

where the final step follows from Corollary 5.3.8.1. Thus \( \nu_{Q,R}(h) = \uparrow_{Q_{\text{op}}, R_{\text{op}}}^{j_q,j_r} \), and we now use this to derive the action on an arbitrary tight morphism \( f : Q \to R_{\text{op}} \).

\[
\nu_{Q,R}(f) = \nu_{Q,R}(\Lambda_{T[Q_{\text{op}}, R_{\text{op}}} \{ \uparrow_{Q_{\text{op}}, R_{\text{op}}}^{j_q,j_r} : f \leq \uparrow_{Q_{\text{op}}, R_{\text{op}}}^{j_q,j_r} \})
\]

\[
= \nu_{Q,R}(\Lambda_{T[Q_{\text{op}}, R_{\text{op}}} \{ \uparrow_{Q_{\text{op}}, R_{\text{op}}}^{j_q,j_r} : j_q \leq f_j(j_q) \})
\]

\[
= \mathbb{V}_{\mathbb{P}_{\text{Pirr}}}(\{ \uparrow_{Q_{\text{op}}, R_{\text{op}}}^{j_q,j_r} : j_r \leq f_j(j_q) \})
\]

\[
= \lambda q \in Q, \mathbb{V}_{R}(j_r \leq J(R) : \exists j_q \in J(Q) (j_q \leq q \text{ and } j_r \leq f_j(j_q)))
\]

\[
= \lambda q \in Q, \mathbb{V}_{R}(j_r : \exists j_q \in J(Q) (j_q \leq q \text{ and } f_j(j_q) \leq j_r))
\]

\[
= \lambda q \in Q, \mathbb{V}_{R}(j_r : j_q \leq f_j(j_r))
\]

\[
= \lambda q \in Q, \mathbb{V}_{R}(j_r : j_r \leq f_j(j_r))
\]

Regarding the description of \( \nu_{Q,R} \)'s action on meets of special morphisms, given any \( (q, r) \in Q \times R \) we have:

\[
\nu_{Q,R}(\uparrow_{Q_{\text{op}}, R_{\text{op}}}^{q,r}) = \lambda q' \in Q, \mathbb{V}_{R}(j \in J(R) : (\uparrow_{Q_{\text{op}}, R_{\text{op}}}^{q,r}).j \leq q')
\]

\[
= \lambda q' \in Q, \mathbb{V}_{R}(j \in J(R) : (\uparrow_{Q_{\text{op}}, R_{\text{op}}}^{q,r}).j \leq q')
\]

\[
= \lambda q' \in Q, \mathbb{V}_{R}(j \in J(R) : j_r \leq q \text{ and } f_j(j_q) \leq j_r)
\]

\[
= \lambda q' \in Q, \mathbb{V}_{R}(j \in J(R) : j \leq q \text{ and } q \leq q')
\]

\[
= \lambda q' \in Q, \mathbb{V}_{R}(j \in J(R) : f_j(j_r) \leq q)
\]

\[
= \uparrow_{Q_{\text{op}}, R_{\text{op}}}^{q,r}
\]

and the first claim follows because meets are sent to joins. Applying \( \nu_{Q,R}^{-1} = \nu_{Q_{\text{op}}, R_{\text{op}}} \) yields:

\[
\wedge_{\mathbb{V}_{R}(\{ \uparrow_{Q_{\text{op}}, R_{\text{op}}}^{q,r} : i \in I \})} = \nu_{Q_{\text{op}}, R_{\text{op}}} \mathbb{V}_{R}(\{ \uparrow_{Q_{\text{op}}, R_{\text{op}}}^{q,r} : i \in I \})
\]

and relabelling yields the final claim.
Lemma 5.3.11. Given any finite join-semilattices \( Q, R \) consider the relation \( \mathcal{G} := \text{Pirr}Q \otimes \text{Pirr}R \). Then:

\[
O(\mathcal{G}) = \{ o_{\mathcal{G}}(f) : f \in \text{Ti}[Q^{\text{op}}, R] \}
\]

where \( o_{\mathcal{G}}(f) := \{(m_q, m_r) \in M(Q) \times M(R) : f(m_q) \leq_R m_r \} \),

\[
C(\mathcal{G}) = \{ c_{\mathcal{G}}(g) : g \in \text{Ti}[Q^{\text{op}}, R] \}
\]

where \( c_{\mathcal{G}}(g) := \{(j_q, j_r) \in J(Q) \times J(R) : j_r \leq_R g(j_q) \} \),

and the generic isomorphism \( \theta_{\mathcal{G}} : C(\mathcal{G}) \to O(\mathcal{G}) \) has action:

\[
\theta_{\mathcal{G}}(c_{\mathcal{G}}(g)) = o_{\mathcal{G}}(\nu_{Q,R}(g)) = \{(m_q, m_r) \in M(Q) \times M(R) : 1_{\mathcal{Q}, Q^{\text{op}}}^{m_q, m_r} \leq_{\text{JSL}_{Q, Q^{\text{op}}}} g \}
\]

\[
\theta_{\mathcal{G}}^{-1}(o_{\mathcal{G}}(f)) = c_{\mathcal{G}}(\nu_{Q,R}^{-1}(f)) = \{(j_q, j_r) \in J(Q) \times J(R) : 1_{\mathcal{Q}, Q^{\text{op}}}^{j_q, j_r} \leq_{\text{JSL}_{Q, Q^{\text{op}}}} f \}
\]

where \( \nu_{Q,R} \) is the natural isomorphism from Theorem 5.3.10.

Proof.

1. The description of \( O(\mathcal{G}) \) follows directly from Corollary 5.3.8.1. Using the bounded lattice isomorphism \( \kappa_{\mathcal{G}} \), the elements of \( C(\mathcal{G}) \) are precisely the relative complements of the \( \mathcal{G} \)-open sets. Now, since:

\[
\hat{\mathcal{G}} = (\text{Pirr}Q \otimes \text{Pirr}R)^{\text{op}} = (\text{Pirr}Q)^{\text{op}} \otimes (\text{Pirr}R)^{\text{op}} = \text{Pirr}(Q^{\text{op}}) \otimes \text{Pirr}(R^{\text{op}})
\]

we deduce that \( O(\hat{\mathcal{G}}) \) consists precisely of those sets \( \{(j_q, j_r) \in M(Q^{\text{op}}) \times M(R^{\text{op}}) : g(j_q) \leq_{R^{\text{op}}} j_r \} \) where \( g : Q \to R^{\text{op}} \) is tight, so the \( \mathcal{G} \)-closed sets are those of the form:

\[
\{(j_q, j_r) \in J(Q) \times J(R) : j_r \leq_R g(j_q) \}
\]

as required.

2. We finally verify the description of the bounded lattice isomorphism \( \theta_{\mathcal{G}} = \lambda X \in C(\mathcal{G}), \mathcal{G}[X] \) and also its inverse. Recall that elements of \( C(\mathcal{G}) \) take the form \( c_{\mathcal{G}}(g) \) where \( g : Q \to R^{\text{op}} \), and also that:

\[
\nu_{Q,R}(g) = \lambda q \in Q, \bigvee_{R} j_r \in J(R) : g_* \{ j_r \} \leq_R q \}
\]

see Theorem 5.3.10. Then we calculate:

\[
\theta_{\mathcal{G}}(c_{\mathcal{G}}(g)) = \mathcal{G}\{(j_q, j_r) : j_r \leq_R g(j_q) \}
\]

\[
= \mathcal{G}\{(j_q, j_r) : j_q \leq_R g_* (j_r) \}
\]

\[
= \{(m_q, m_r) : \exists j_r \leq_R g_* (j_r) \}
\]

\[
= \{(m_q, m_r) : \exists j_r \leq_R g_* (j_r) \}
\]

by definition

Regarding the marked equality, \( \Rightarrow \) follows because if \( g_* (j_r) \leq_R m_q \) we derive the contradiction \( j_q \leq_R g_* (j_r) \leq_R m_q \). Also, \( \Leftarrow \) follows because if \( \forall j_r \in J(R) (j_q \leq_R m_q \Rightarrow j_q \leq_R g_* (j_r)) \) then by converting to contrapositives we derive the contradiction \( g_* (j_r) \leq_R m_q \). Concerning the alternative description of \( \theta_{\mathcal{G}}(c_{\mathcal{G}}(g)) \),

\[
\exists j_r \leq_R m_r \text{ and } g_* (j_r) \leq_R m_q \iff 1_{\mathcal{Q}, Q^{\text{op}}}^{m_q, m_r} \leq_{\text{JSL}_{Q, Q^{\text{op}}}} g \iff 1_{\mathcal{Q}, Q^{\text{op}}}^{m_q, m_r} \leq_{\text{JSL}_{Q, Q^{\text{op}}}} g
\]

where the final equivalence uses the generic order-isomorphism \( \text{JSL}_{f}[Q, Q^{\text{op}}] \equiv \text{JSL}_{f}[Q, R^{\text{op}}] \) i.e. take the adjoint, recalling that \( (1_{\mathcal{Q}, Q^{\text{op}}}^{m_q, m_r})_* = 1_{\mathcal{Q}, R^{\text{op}}}^{m_q, m_r} \) by Lemma 5.1.9.1.

It remains to describe the action of the inverse \( \theta_{\mathcal{G}}^{-1} = \lambda Y \in O(\mathcal{G}), \mathcal{G}^i(Y) \). First recall that by De Morgan duality
\( G^i = \neg g_i \circ \check{g}^i \circ \neg g_i \). Then for any tight morphism \( f : Q^{op} \to R \) we calculate:

\[
\theta_G^{-1}(o_G(f)) = \theta_G^{-1}(c_G(f)) = \theta_G^{-1}(\{ (m_q, m_r) : f(m_q) \leq_R m_r \}) = \theta_G^{-1}(\{ (m_q, m_r) : m_r \leq_{R=\tau} f(m_q) \}) = \theta_G^{-1}(c_G(f)) = \theta_G^{-1}(\nu_{Q^{op}}(f)) = \{(j_q, j_r) : j_r \leq_{R} \nu_{Q^{op}}(f)(j_q) \} \]

see below

\[
= \{(j_q, j_r) : j_r \leq_R \nu_{Q^{op}}(f)(j_q) \} \quad \text{recalling \((-)^{op} \) has same action}
\]

\[
= \{(j_q, j_r) : j_r \leq_R \nu_{Q^{op}}(f)(j_q) \} \quad \text{by definition}
\]

To understand the marked equality, use the fact that \( \mathcal{G} = \mathcal{P}\text{rr}(Q^{op}) \otimes \mathcal{P}\text{rr}(R^{op}) \). Regarding the alternative description, we'll make use of the previous alternative description:

\[
\theta_G^{-1}(o_G(f)) = \theta_G^{-1}(c_G(f)) = \{(j_q, j_r) : j_r \leq_R f(\tau_Q(j_q)) \} \quad \text{see above}
\]

\[
\{(j_q, j_r) : j_r \leq_R f(\tau_Q(j_q)) \} \quad \text{see above}
\]

\[
\{(j_q, j_r) : j_r \leq_R f(\tau_Q(j_q)) \} \quad \text{by definition}
\]

Corollary 5.3.12. In the notation of Lemma 5.3.11 where \( \mathcal{G} := \mathcal{P}\text{rr}Q \otimes \mathcal{P}\text{rr}R \), the following statements hold.

1. If \( Q \) is distributive then:

\[
\theta_G(c_G(g)) = \{(m_q, m_r) \in M(Q) \times M(R) : g(\tau_Q^{-1}(m_q)) \leq_R m_r \}
\]

\[
\theta_G^{-1}(o_G(f)) = \{(j_q, j_r) \in J(Q) \times J(R) : j_r \leq_R f(\tau_Q(j_q)) \}
\]

Recalling that \( \tau_Q = \lambda_j \in J(Q) \) \((\forall Q \exists j \in M(Q))\) is the canonical order-isomorphism from Lemma 2.2.3.13.

2. If \( R \) is distributive then:

\[
\theta_G(c_G(g)) = \{(m_q, m_r) \in M(Q) \times M(R) : g_\tau(\tau_Q^{-1}(m_q)) \leq_Q m_q \}
\]

\[
\theta_G^{-1}(o_G(f)) = \{(j_q, j_r) \in J(Q) \times J(R) : j_q \leq_R f_\tau(j_r) \}
\]

3. If both \( Q, R \) are distributive and \( f : Q \to R \) is any JSL-f-morphism, we deduce the equivalence:

\[
j_r \leq_R f(\tau_Q^{-1}(m_q)) \iff f_\tau(j_r) \leq_Q m_q \quad \text{for every } (m_q, j_r) \in M(Q) \times J(R).
\]

Proof.

1. Regarding the description of \( \theta_G^{-1} \), by Lemma 5.3.11 we know that:

\[
\theta_G^{-1}(o_G(f)) = \{(j_q, j_r) \in J(Q) \times J(R) : \tau_{Q \otimes Q^{op}}(j_q, j_r, f) \}
\]

\[
\{(j_q, j_r) \in J(Q) \times J(R) : \forall \nu \in Q^+, [j_q \leq_Q \nu \Rightarrow j_r \leq_R f(\nu)] \}
\]

where the final equality follows because morphisms \( Q^{op} \to R \) are determined by their action on \( J(Q^{op}) = M(Q) \).

Let us prove that:

\[
\forall m_q \in M(Q), [j_q \leq_Q m_q \Rightarrow j_r \leq_R f(m_q)] \iff j_q \leq_R f(\tau_Q(j_q))
\]

Then \( \Rightarrow \) follows because \( j_q \leq_{Q^{op}} \tau_Q(j_q) = \nu \in Q^+, \forall j \in M(Q) \), since the subset \( \nu \in Q^+ \) is both down-closed and closed under joins (using join-primeness).

Conversely, if \( j_q \leq_Q m_q \) then certainly \( m_q \leq_Q \tau_Q(j_q) \) and hence \( j_q \leq_R f(\tau_Q(j_q)) \exists m_q \leq_R f(m_q) \) using monotonicity of \( f : Q^{op} \to R \).

To understand \( \theta_G(c_G(g)) \), recall that \( \mathcal{G} = \mathcal{P}\text{rr}(Q^{op}) \otimes \mathcal{P}\text{rr}(R^{op}) \) where \( Q^{op} \) is also distributive. Then:

\[
\theta_G(c_G(g)) = \theta_G^{-1}(o_G(f)) = \{(m_q, m_r) \in M(Q) \times M(R) : m_r \leq_{R^{op}} f(\tau_Q(m_q)) \}
\]

where in the final equality we use the fact that \( \tau_Q^{op} \) acts like the inverse of \( \tau_Q \).

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2. Let \( \mathcal{H} := \text{Pirr}\mathcal{R} \otimes \text{Pirr}\mathcal{Q} \). Taking the converse relation defines bounded lattice isomorphisms \( \mathcal{O}(\mathcal{G}) \cong \mathcal{O}(\mathcal{H}) \) and \( \mathcal{C}(\mathcal{G}) \cong \mathcal{C}(\mathcal{H}) \). Using the previous statement, we have:

\[
\begin{align*}
\phi(q) & \rightarrow (\phi(q))^\prime = \phi_{\mathcal{R}}(q) \\
\phi(f) & \rightarrow (\phi(f))^\prime = \phi_{\mathcal{R}}(f)
\end{align*}
\]

\[\{(m_r, j_r) : f_{\mathcal{R}}(\tau_{\mathcal{R}^\prime}(m_r)) \leq m_q \} \iff \{(m_r, j_r) : f_{\mathcal{R}}(\tau_{\mathcal{R}}(j_r)) \leq m_q \}\]

and rewriting yields:

\[f_{\mathcal{R}}(\tau_{\mathcal{R}^\prime}(m_q)) \iff f_{\mathcal{R}}(\tau_{\mathcal{R}}(j_r)) \leq m_q\]

The other bijection involving closed sets yields the same equivalence.

\[\square\]

**Note 5.3.13.** Let us provide some basic examples of the equivalence:

\[\forall m_q \in M(\mathcal{Q}), \ j_r \in J(\mathcal{R}). \ (j_r \leq_R f(\tau_{\mathcal{Q}^\prime}(m_q))) \iff f_{\mathcal{R}}(\tau_{\mathcal{R}}(j_r)) \leq m_q\]

which holds for any \( \text{JSL} \) morphism \( f : \mathcal{Q} \to \mathcal{R} \) between distributive join-semilattices \( \mathcal{Q} \) and \( \mathcal{R} \). Take any relation \( R \subseteq X \times Y \) between finite sets and let \( f := R^1 : \mathcal{P}X \to \mathcal{P}Y \), so that \( f_{\mathcal{R}} = R^1 \). Noting that \( M(\mathcal{Q}) = \{x : x \in X\} \) and \( J(\mathcal{R}) = \{y : y \in Y\} \), and moreover that the canonical order-isomorphisms \( \tau_{\mathcal{P}X} \) and \( \tau_{\mathcal{P}Y} \) take the relative complement, the equivalence becomes:

\[\{y\} \subseteq R^1(\{x\}) \iff R^1(\{x\}) \subseteq \{y\}\]

Applying De Morgan duality one sees this is the equivalence \( y \in R[x] \iff x \in R^1(\{y\}) = R[y] \).

We finish off with a clean set-theoretic description of the tensor product of finite distributive join-semilattices, which is also the tight tensor product. Let us agree that a ‘set-theoretic bounded distributive lattice’ is one which arises as a sub bounded lattice of some \( \mathcal{P}Z = (\mathcal{P}Z, \cup, \varnothing, \cap, Z) \), in which case \( Z \) is uniquely determined.

**Theorem 5.3.14** (Representing the tensor and tight tensor product of distributive join-semilattices).

*Let \( D_1 \) and \( D_2 \) be finite distributive join-semilattices.*

1. **Their tight tensor product and tensor product are isomorphic:**

\[\text{JSL}_f[D_1^{\mathcal{P}Z}, D_2] \cong D_1 \otimes D_2 \xrightarrow{\nu_{D_1,D_2}^1} D_1 \otimes D_2 \cong (\text{JSL}_f[D_1^{\mathcal{P}Z}, D_2])^{\mathcal{P}Z}\]

using the isomorphism from Theorem 5.3.10.

2. **If each \( D_i \) defines a set-theoretic bounded distributive lattice over \( Z_i \), then:**

\[\text{trep}_{D_1,D_2} : D_1 \otimes D_2 \to \{\{j_1 \times j_2 : (j_1, j_2) \in J(D_1) \times J(D_2)\}\}_{P(Z_1 \times Z_2)}\]

\[\text{trep}_{D_1,D_2}(f) := \bigcup\{\{j_1 \times j_2 : (j_1, j_2) \in J(D_1) \times J(D_2), \ j_2 \leq f(\tau_{D_1}(j_1))\}\}\]

defines a join-semilattice isomorphism. Regarding \( \text{trep}_{D_1,D_2} \)'s codomain \( D \):

(a) it defines a set-theoretic bounded distributive lattice over \( Z_1 \times Z_2 \),
(b) its associated canonical bilinear mapping has action \( (d_1,d_2) \mapsto d_1 \times d_2 \),
(c) its irreducible elements are:

\[J(D) = \{j_1 \times j_2 : (j_1, j_2) \in J(D_1) \times J(D_2)\} \quad M(D) = \{m_1 \times m_2 : (m_1, m_2) \in M(D_1) \times M(D_2)\}\]

with associated canonical order-isomorphism \( \tau_D(j_1 \times j_2) := \tau_{D_1}(j_1) \times Z_2 \cup Z_1 \times \tau_{D_2}(j_2) \).
Proof.

1. Recall one of the characterisations of tight morphisms from Lemma 5.2.5 i.e. they are those \( \text{JSL}_f \)-morphisms which factor through a distributive join-semilattice. Then:

\[
\text{JSL}_f[D_1^{op}, D_2] = \text{Ti}[D_1^{op}, D_2] \quad \text{see above}
\]
\[
= D_1 \otimes D_2 \quad \text{by definition}
\]
\[
\cong (D_1^{op} \otimes D_2^{op})^{op} \quad \text{via } \nu_{D_1, D_2}^1 \text{ from Theorem 5.3.10}
\]
\[
= (\text{Ti}[D_1, D_2^{op}])^{op} \quad \text{by definition}
\]
\[
= (\text{JSL}_f[D_1, D_2^{op}])^{op} \quad \text{see above}
\]
\[
= D_1 \otimes D_2 \quad \text{by definition}
\]

2. For notational convenience, define the induced posets \( P_i := (J(D_i), \leq_{D_i}, |J(D_i) \times J(D_i)|) \) for \( i = 1, 2 \). We are going to construct \( \text{tre}P_{D_1, D_2} \) as a composite isomorphism:

\[
\xymatrix{
\mathbb{D} \otimes_{\mathbb{D}} \mathbb{D} \ar[r]^{\alpha_{D_1, D_2}} \ar[d]_{\Pi \mathbb{D}} & \mathbb{D} \ar[d]_{\Delta_{J(D)}} \ar[r]^{	au_D^{-1}} & J(D) \ar[d]_{\gamma_D} & \mathcal{R}_D := \mathcal{D} \supseteq J(D) \times J(D) : \Pi \mathbb{D} \rightarrow \mathcal{D} \supseteq J(D) \times J(D) \ar[l]_{\beta} \ar[r] & \beta_{P_1, \otimes P_2} = (\text{Dn}(P_1 \times P_2), \cup, \emptyset).}
\]

Here we are using the canonical order-isomorphism \( \tau_D \) between join/meet-irreducibles, see Lemma 2.2.3.14. This commutative square witnesses a \( \text{Dep} \)-isomorphism \( \mathcal{R}_D \) shown above. To provide some clarification, \( \text{Open} \mathcal{R}_D \) is the well-known isomorphism representing \( \mathbb{D} \cong \text{OpenPirr} \mathbb{D} \) as down-closed sets of join-irreducibles. These \( \text{Dep} \)-isomorphisms induce the \( \text{JSL}_f \)-isomorphism:

\[
\beta := \text{Open} \left( \mathcal{R}_{D_1} \otimes \mathcal{R}_{D_2} \right) : \text{Open} \left( \Pi \mathbb{D}_1 \otimes \Pi \mathbb{D}_2 \right) \rightarrow \text{Open} \left( \gamma_{P_1, \otimes P_2} \right)
\]

since functors preserve isomorphisms. To see that \( \beta \)'s codomain is correct, observe that the synchronous product of two order relations is the order relation of the product of their respective posets, and also that the open sets of an order relation are precisely the up-closed subsets of its corresponding poset, hence:

\[
\text{Open} \left( \gamma_{P_1, \otimes P_2} \right) = (\text{Dn}(P_1 \times P_2), \cup, \emptyset).}
\]

To compute the action of \( \beta \circ \alpha_{D_1, D_2} \), let \( \mathcal{G} := \Pi \mathbb{D}_1 \otimes \Pi \mathbb{D}_2 \) and recall that:

\[
\beta(Y) = \text{Open} \left( \mathcal{R}_{D_1} \otimes \mathcal{R}_{D_2} \right)(Y) = (\mathcal{R}_{D_1} \otimes \mathcal{R}_{D_2})^i \circ \mathcal{G}^i(Y) \quad \text{see Definition 4.2.1.1.}
\]
\[
\alpha_{D_1, D_2}(f) = \{(m_1, m_2) \in M(D_1) \times M(D_2) : f(m_1) \leq_{D_2} m_2 \} = \mathcal{O}_f \quad \text{using notation of Lemma 5.3.11.}
\]

Thus:

\[
\beta \circ \alpha_{D_1, D_2}(f) = (\mathcal{R}_{D_1} \otimes \mathcal{R}_{D_2})^i \circ \mathcal{G}^i(\mathcal{O}_f) \quad \text{recall Definition 4.2.4}
\]
\[
= (\mathcal{R}_{D_1} \otimes \mathcal{R}_{D_2})^i \circ (\mathcal{R}_{D_1} \otimes \mathcal{R}_{D_2})^i \circ \mathcal{G}^i(\mathcal{O}_f) \quad \text{by Corollary 5.3.12}
\]
\[
= (\mathcal{R}_{D_1} \otimes \mathcal{R}_{D_2})^i \circ \{(j_1, j_2) \in J(D_1) \times J(D_2) : j_2 \leq_{D_2} f(\tau_{D_1}(j_1)) \} \quad \text{by definition of } \tau_D
\]
\[
= \{(j_1, j_2) : j_2 \leq_{D_2} f(\tau_{D_1}(j_1)) \} \quad \text{see below}
\]
\[
= \{(j_1, j_2) : j_2 \subseteq f(\tau_{D_1}(j_1)) \} \quad \text{\( \leq_{D_2} \) is inclusion}
\]

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4.7.1

(a) To see that \( D \) is set-theoretically given \( (j'_1, j'_2) \leq_{P_1 \times P_2} (j_1, j_2) \) then:
\[
j'_2 \leq_{D_1} j_2 \leq_{D_2} f(\tau_{D_1}(j_1)) \leq_{D_2} f(\tau_{D_1}(j'_1))
\]

using the monotonicity of \( \tau_{D_1} \) and \( f : D_1^{\text{op}} \to D_2 \).

It remains to describe the isomorphism \( \delta \). Firstly, for each \( D_i \) consider the diagram below:

\[
\begin{array}{ccc}
J(D_i) & \xrightarrow{\bar{\varepsilon}} & Z_i \\
\downarrow{\varepsilon} & & \downarrow{\Delta Z_i} \\
J(D_i) & \xrightarrow{\bar{\varepsilon}} & Z_i
\end{array}
\]

which commutes because \( D_i \) is inclusion-ordered, and hence witnesses the \( \text{Dep} \)-morphism \( S_i \) above. Recall that \( S_i \) is monic iff \( \text{cl}_{S_i} = \text{cl}_{\tau_i} \) by Lemma 4.7.1. The closure operator induced by an order relation constructs the upwards closure in the respective poset, as one may verify. Then since:

\[
\text{cl}_{S_i}(X) = (\bar{\varepsilon})^i \circ (\bar{\varepsilon})^j(X) = (\bar{\varepsilon})^i(\bigcup X) = \{ j \in J(D_i) : j \subseteq \bigcup X \} = \downarrow_{D_i} X \quad j \text{ is join-prime, see Lemma 2.2.3.2}
\]

we deduce that each \( S_i \) is monic. Consider the induced join-semilattice morphism:

\[
\gamma := \text{Open}(S_1 \uplus S_2) : (Dn(P_1 \times P_2), \cup, \emptyset) \mapsto \text{Open}(\Delta Z_1 \uplus \Delta Z_2) = \mathcal{P}(Z_1 \times Z_2)
\]

It is injective because the synchronous product functor preserves monos by Corollary 5.3.9, and also \( \text{Open} \) is an equivalence functor. Then the isomorphism \( \delta \) is defined by restricting \( \gamma \)'s codomain to its embedded image:

\[
\begin{array}{ccc}
(Dn(P_1 \times P_2), \cup, \emptyset) & \xrightarrow{\gamma} & \mathcal{P}(Z_1 \times Z_2) \\
\downarrow{\delta} & & \downarrow{((j_1 \times j_2) : (j_1, j_2) \in P_1 \times P_2)}_{\mathcal{P}(Z_1 \times Z_2)} \\
\langle \downarrow \bigcup (j_1 \times j_2) : (j_1, j_2) \in P_1 \times P_2 \rangle_{\mathcal{P}(Z_1 \times Z_2)} & \xrightarrow{\gamma} & \mathcal{P}(Z_1 \times Z_2)
\end{array}
\]

To understand \( \delta \) and its codomain, we describe \( \gamma \)'s action on any downset \( X \in Dn(P_1 \times P_2) \):

\[
\gamma(X) = (S_1 \uplus S_2)^j (X) = S_1 \uplus S_2 \bigcup X \quad \text{since } X \text{ is down-closed}
\]

\[
= \bar{\varepsilon}_1 \uplus \bar{\varepsilon}_2 \bigcup X \quad \text{equals the union}
\]

Since \( \delta \)'s domain is union-generated by the principal downsets, its codomain is correct. Consequently, the composite isomorphism \( \text{trep}_{D_1, D_2} := \delta \circ \beta \circ \alpha_{D_1, D_2} \) has the desired action:

\[
\text{trep}_{D_1, D_2}(f) = \delta \circ \beta \circ \alpha_{D_1, D_2}(f)
\]

\[
= \delta((j_1, j_2) : j_1 \in P_1 \times P_2 : f(\tau_{D_1}(j_1)))
\]

\[
= \bigcup (j_1 \times j_2) : (j_1, j_2) \in P_1 \times P_2 \subseteq f(\tau_{D_1}(j_1))
\]

Finally we verify the claimed properties of \( \text{trep}_{D_1, D_2} \)'s codomain distributive join-semilattice, denoted \( D \).

(a) To see that \( D \) is set-theoretic, we'll show that \( \gamma \) defines a distributive lattice morphism between the two set-theoretic distributive join-semilattices. The top element is preserved because \( Z_1 \times Z_2 \) equals the union over all \( j_1 \times j_2 \)'s. It preserves binary meets iff it preserves meets of join-irreducibles (apply distributivity twice), and finally:

\[
\gamma(\downarrow_{P_1 \times P_2} (j_1, j_2) \cap \downarrow_{P_1 \times P_2} (j'_1, j'_2)) = \gamma(\downarrow_{P_1 \times P_2} ((j_1 \cap j'_1), (j_2 \cap j'_2)))
\]

\[
= (j_1 \cap j'_1) \times (j_2 \cap j'_2)
\]

\[
= (j_1 \times j_2) \cap (j'_1 \times j'_2)
\]

since binary intersections and products commute.
(b) The canonical bilinear map associated to the tensor product $\mathbb{D}_1 \otimes \mathbb{D}_2$ is the function:

$$\beta_{\mathbb{D}_1, \mathbb{D}_2} : D_1 \times D_2 \to \text{JSL}_f(D_1, \mathbb{D}_1^{\text{op}}) \quad \beta_{\mathbb{D}_1, \mathbb{D}_2}(d_1, d_2) := t_{\mathbb{D}_1, \mathbb{D}_2}^{d_1, d_2}$$

recalling that $\mathbb{D}_1 \otimes \mathbb{D}_2 = (\text{JSL}_f(D_1, \mathbb{D}_1^{\text{op}}))^{\text{op}}$, see Definition 5.1.11. Since the tensor product of distributive join-semilattices is isomorphic to their tight tensor product by (1), we may equivalently view $\mathbb{D}$ as the tensor product of $\mathbb{D}_1$ and $\mathbb{D}_2$. Then $\mathbb{D}$‘s associated canonical bilinear map arises by composing with the isomorphism as follows:

$$D_1 \times D_2 \xrightarrow{\beta_{\mathbb{D}_1, \mathbb{D}_2}} \text{JSL}_f(D_1, \mathbb{D}_1^{\text{op}}) \xrightarrow{\nu_{\mathbb{D}_1, \mathbb{D}_2}} \text{JSL}_f(D_1^{\text{op}}, D_2) \xrightarrow{\text{trep}_{\mathbb{D}_1, \mathbb{D}_2}} D$$

\[ (d_1, d_2) \mapsto t_{\mathbb{D}_1^{\text{op}}, \mathbb{D}_2}^{d_1, d_2} \mapsto \text{trep}_{\mathbb{D}_1, \mathbb{D}_2}(t_{\mathbb{D}_1^{\text{op}}, \mathbb{D}_2}^{d_1, d_2}) \]

also using Theorem 5.3.10. It only remains to simplify the latter:

\[ \text{trep}_{\mathbb{D}_1, \mathbb{D}_2}(t_{\mathbb{D}_1^{\text{op}}, \mathbb{D}_2}^{d_1, d_2}) = \bigcup \{ j_1 \times j_2 : j_1 \notin \tau_{\mathbb{D}_1}(j_1) \} \]

\[ = \bigcup \{ j_1 \times j_2 : j_1 \notin \tau_{\mathbb{D}_1}(j_1) \} = \bigcup \{ j_1 \times j_2 : j_1 \notin \tau_{\mathbb{D}_1}(j_1) \} \]

\[ \text{by definition} \]

\[ = \bigcup \{ j_1 \times j_2 : \exists j_1' \text{ such that } j_1' \notin \tau_{\mathbb{D}_1}(j_1) \} \]

\[ \text{Thus } M(\mathbb{D}) \text{ contains precisely the elements } \overline{m_1} \times \overline{m_2} \text{ where } (m_1, m_2) \in M(\mathbb{D}_1) \times M(\mathbb{D}_2). \]

5.4 Tightness inside $\text{Dep}$ and the universality of the tight tensor product

We now define the $\text{Dep}$-correspondents of various concepts above i.e. tight join-semilattice morphisms, the special join-irreducible join-semilattice morphisms $t_{Q,R}^{m,j} : Q \to R$ which join-generate them, and also the tight hom-functor. It is also worth describing the correspondents of the special morphisms $j_{Q,R}^{m,j} : Q \to R$, seeing as they are both the join-irreducibles of $Q \otimes R = (\text{JSL}_f(Q, R))^{\text{op}}$ by Lemma 5.1.15, and also the meet-irreducibles of $Q^{\text{op}} \otimes R = T_{\mathbb{Q}, \mathbb{R}}$ by Lemma 5.2.13. On the other hand, it is the correspondents of $t_{Q,R}^{m,j}$ which take the leading role in this final subsection.

**Definition 5.4.1 (Tight $\text{Dep}$-morphisms, basic bicliques and basic independents).**

1. A $\text{Dep}$-morphism $\mathcal{R} : \mathcal{G} \to \mathcal{H}$ is **tight** if it factors through an identity relation i.e. $\mathcal{R} = S \cup T$ for some $\text{Dep}$-morphisms $S : \mathcal{G} \to \Delta_Z$, $T : \Delta_Z \to \mathcal{H}$ and some finite set $Z$.

2. Let $\text{Ti}(\mathcal{G}, \mathcal{H}) \subseteq \text{Dep}(\mathcal{G}, \mathcal{H})$ be the subset of tight morphisms. Then we have the join-semilattice:

$$\text{Ti}[\mathcal{G}, \mathcal{H}] := (\text{Ti}(\mathcal{G}, \mathcal{H}), \cup, \emptyset) \subseteq \text{Dep}[\mathcal{G}, \mathcal{H}]$$

This extends to a functor $\text{Ti}[-,-] : \text{Dep}^{\text{op}} \times \text{Dep} \to \text{JSL}_f$ whose action on morphisms is:

$$\mathbb{R} : \mathcal{G} \to \mathcal{H} \quad S : \mathcal{G}' \to \mathcal{H}'$$

$$\text{Ti}[\mathbb{R}, S] = \lambda \mathcal{R} : \mathcal{G} \to \mathcal{H} ; \mathcal{T} : \mathcal{H} \to \mathcal{H}' ; \mathcal{T} : \text{Ti}[\mathcal{H}, \mathcal{G}'] \to \text{Ti}[\mathcal{G}, \mathcal{H}']$$

this being precisely the way that $\text{Dep}[-,-]$ acts, see Definition 5.1.17.
3. Given relations $\mathcal{G}$, $\mathcal{H}$ and elements $(g_t, h_s) \in \mathcal{G}_t \times \mathcal{H}_s$, there is an associated $\text{Dep}$-morphism:

$$t^{g_t, h_s}_{\mathcal{G}, \mathcal{H}} := \mathcal{K}(\tilde{\mathcal{G}}[g_t], \mathcal{H}[h_s]) : \mathcal{G} \rightarrow \mathcal{H}$$

$$\left( t^{g_t, h_s}_{\mathcal{G}, \mathcal{H}} \right)_- := \emptyset - \cup \mathcal{K}(\tilde{\mathcal{G}}[g_t], \text{cl}_H((h_s)))$$

$$\left( t^{g_t, h_s}_{\mathcal{G}, \mathcal{H}} \right)_+ := \emptyset + \cup \mathcal{K}(\mathcal{H}[h_s], \text{cl}_G((g_t)))$$

Call them basic bicliques, where distinct pairs $(g_t, h_s) \neq (g'_t, h'_s)$ may induce the same morphism.

4. Given relations $\mathcal{G}$, $\mathcal{H}$ and elements $(g_s, h_t) \in \mathcal{G}_s \times \mathcal{H}_t$, there is an associated $\text{Dep}$-morphism:

$$t^{g_s, h_t}_{\mathcal{G}, \mathcal{H}} := \text{In}_G(\mathcal{G}[g_s]) \cap \text{cl}_H(\{h_t\})$$

$$\left( t^{g_s, h_t}_{\mathcal{G}, \mathcal{H}} \right)_- := (\text{In}_G(\mathcal{G}[g_s]) \cap \text{cl}_H(\{h_t\}) = \emptyset - \cup \text{In}_G(\mathcal{G}[g_s]) \times \mathcal{H}_t \cup \tilde{\mathcal{G}}[g_t] \times \mathcal{H}(h_t)$$

$$\left( t^{g_s, h_t}_{\mathcal{G}, \mathcal{H}} \right)_+ := (\text{In}_G(\mathcal{G}[g_s]) \cap \text{cl}_H(\{h_t\}) \cup \mathcal{G}[g_t] \times \mathcal{H}[h_t] \cap \tilde{\mathcal{G}}(g_t)$$

Call them basic independents, where distinct pairs $(g_s, h_t) \neq (g'_s, h'_t)$ may induce the same morphism.

**Example 5.4.2** (Understanding the special $\text{Dep}$-morphisms).

1. Each basic biclique $t^{g_t, h_s}_{\mathcal{G}, \mathcal{H}} = \mathcal{K}(\tilde{\mathcal{G}}[g_t], \mathcal{H}[h_s]) : \mathcal{G} \rightarrow \mathcal{H}$ is well-defined via the witnesses:

   ![Diagram](https://via.placeholder.com/150)

   Indeed, $g_t$ may be viewed as the apex of a ‘cone’ with base $\tilde{\mathcal{G}}[g_t]$, and $h_s$ may be viewed as the apex of an upside-down cone with base $\mathcal{H}[h_s]$. Connecting each cone’s apex to the other cone’s base yields the diagram above on left, whose support is shown on the right. Closing these witnesses yields the components described earlier. As suggested by the notation, these morphisms correspond to the special morphisms $t^{g_t, h_s}_{\mathcal{G}, \mathcal{H}}$ under the equivalence functors $\text{Pirr}$ and $\text{Open}$, see Lemma 5.4.3.5 below. Importantly, a $\text{Dep}$-morphism is tight iff it is a union of basic bicliques.

2. Regarding the previous example, it seems natural to ‘flip’ the two cones upside-down i.e. we take $g_s \in \mathcal{G}_s$ as the apex of an upside-down cone with base $\tilde{\mathcal{G}}[g_s]$, and $h_t \in \mathcal{H}_t$ as the apex of a cone with base $\mathcal{H}[h_t]$. However, gluing apexes to bases need not yield witnessing relations, as indicated by either of the two dashed arrows shown above.

3. It turns out that the ‘natural’ choice for a $\text{Dep}$-morphism $\mathcal{G} \rightarrow \mathcal{H}$ depending on elements $(g_s, h_t) \in \mathcal{G}_s \times \mathcal{H}_t$ is the basic independent morphism:

$$t^{g_s, h_t}_{\mathcal{G}, \mathcal{H}} = \text{In}_G(\mathcal{G}[g_s]) \cap \text{cl}_H(\{h_t\})$$

Let us first explain its alternate description from the Definition:

$$t^{g_s, h_t}_{\mathcal{G}, \mathcal{H}} = \text{In}_G(\mathcal{G}[g_s]) \cap \text{cl}_H(\{h_t\})$$

$$= \tilde{\mathcal{G}}[g_t] \times \mathcal{H}[h_t] \cap (\text{cl}_G(\{g_s\}) \times \mathcal{H}_t \cup \mathcal{G}_s \times \text{cl}_H(\{h_t\}))$$

$$= \tilde{\mathcal{G}}[g_t] \times (\text{In}_G(\mathcal{G}[g_s]) \times \mathcal{H}_t \cup \mathcal{G}_s \times \text{In}_H(\{h_t\}))$$

$$= \text{In}_G(\mathcal{G}[g_s]) \times \mathcal{H}[h_t] \cup \tilde{\mathcal{G}}[g_t] \times \text{In}_H(\{h_t\})$$

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Here we have used Lemma 5.1.16.3 i.e. that $\uparrow^{\text{Dep}}_{G, H}$ relates everything which is not isolated, and also De Morgan duality and the fact that binary intersections commute with binary products. That the basic independent $\downarrow^{g_s,h_t}_{G, H}$ is a well-defined Dep-morphism follows because it is a union of basic bicliques:

$$\text{in}_G(\mathcal{G}) \times \mathcal{H}[h_s] = \mathcal{G}\mathcal{H}^{\downarrow \left(\uparrow^{\mathcal{G}}(\mathcal{H})\right)} = \mathcal{H}\mathcal{G}^{\downarrow \left(\uparrow^{\mathcal{H}}(\mathcal{G})\right)}$$

We have the equalities:

$$\mathcal{G}[G_t] \times \text{in}_H(\mathcal{H}_t) = \mathcal{G}\mathcal{H}^{\downarrow \left(\uparrow^{\mathcal{G}}(\mathcal{H})\right)} = \mathcal{H}\mathcal{G}^{\downarrow \left(\uparrow^{\mathcal{H}}(\mathcal{G})\right)}$$

It is slightly tedious to verify the associated components and their alternative descriptions. They simplify in the case that $\mathcal{G}$ and $\mathcal{H}$ are strict, which is easily enforced. Of course, $\downarrow^{g_s,h_t}_{G, H}$ corresponds to the special morphisms $\mathcal{G}^m_{Q,R}$ as we prove below. Furthermore, their description as a binary union of $\mathcal{G}[G_t] \times \text{in}_H(\mathcal{H}_t)$ and $\text{in}_G(\mathcal{G}) \times \mathcal{H}[h_s]$ corresponds to an equality we’ve already seen i.e.

$$\downarrow^{g_s,h_t}_{Q,R} = \downarrow^{g_s,h_t}_{Q,R} \lor \text{JSL}_f(Q,R) \lor \downarrow^{g_s,h_t}_{Q,R}$$

see Corollary 5.2.6. As mentioned above, we will not have much use for the basic independents in this subsection. However we make one more observation. Recall that the morphisms $\downarrow^{g_s,h_t}_{Q,R}$ are precisely the meet-

irreducibles of $\text{JSL}_f(Q,R)$ by Lemma 5.1.10. Analogously, the basic independents $\downarrow^{g_s,h_t}_{G, H}$ are precisely the meet-

irreducibles of $\text{Dep}[G, H]$, as long as both $\mathcal{G}$ and $\mathcal{H}$ are reduced.

**Lemma 5.4.3 (Tightness inside Dep).**

*Let $\mathcal{G}$, $\mathcal{H}$ be any relations between finite sets.*

1. A Dep-morphism $\mathcal{R}$ is tight iff $\text{Open}\mathcal{R}$ is tight, a JSL$_f$-morphism $f$ is tight iff $\text{Pirrf}$ is tight.

2. $\text{Ti}[\mathcal{G}, \mathcal{H}]$ is a well-defined join-semilattice.

3. $\text{Ti}[\cdot, -]: \text{Dep}^{op} \times \text{Dep} \to \text{JSL}_f$ is a well-defined functor.

4. Basic bicliques and basic independents are well-defined tight Dep-morphisms.

5. We have the equalities:

$$\left(\mathcal{G}[G_t] \times \mathcal{H}[h_s]\right)^\lor \uparrow^{h_s, g_t}_{H, G} = \left(\mathcal{H}[h_t] \times \mathcal{G}[G_s]\right)^\lor \downarrow^{h_t, g_s}_{H, G}$$

$$\text{Open} \uparrow^{g_s, h_t}_{G, H} = \text{in}_G(\mathcal{H}) \times \text{in}_H(\mathcal{G})$$

$$\text{Pirrf} \downarrow^{g_r, h_s}_{Q,R} = \bigcup \downarrow^{m, j}_{\text{Pirrf}_Q, \text{Pirrf}_R} : q \leq m \in M(Q), J(R) \ni j \leq R$$

for any $(g_t, h_s) \in \mathcal{G}_t \times \mathcal{H}_s$, $(g_s, h_t) \in \mathcal{G}_s \times \mathcal{H}_t$ and any finite join-semilattices $Q$, $R$ with $(q, r) \in Q \times R$.

6. A Dep-morphism $\mathcal{R}: \mathcal{G} \to \mathcal{H}$ is tight iff it is a union of basic bicliques of type $\mathcal{G} \to \mathcal{H}$.

7. We have the equalities:

$$\mathcal{R} \downarrow^{g_s, h_t}_{G, H} = \mathcal{K}(\mathcal{R}[g_t], \mathcal{H}[h_s])$$

for any elements $(g_t, h_s) \in \mathcal{G}_t \times \mathcal{H}_s$, and any Dep-morphisms $\mathcal{R}: \mathcal{F} \to \mathcal{G}, \mathcal{S}: \mathcal{H} \to \mathcal{I}$.

8. For every Dep-morphism $\mathcal{R}: \mathcal{G} \to \mathcal{H}$ we have:

$$\mathcal{R} \in \downarrow^{g_s, h_t}_{G, H} \iff \mathcal{R}(g_s, h_t).$$
Proof.

1. If $R : G \rightarrow H$ is tight via $R = S \uparrow T$ then $\text{Open}R = \text{Open}T \circ \text{Open}S$ and hence $\text{Open}R$ is tight, since $\text{Open}\Delta_Z = \mathbb{P}Z$. On other hand, if $f : Q \rightarrow R$ is tight via $f = h \circ g$ then $\text{Pirrf} = \text{Pirrg} \circ \text{Pirrh}$ is tight, since $\text{Pirrf}Z$ is bipartite graph isomorphic to $\Delta_Z$.

2. This follows from the previous statement, since tight JSL-$f$-morphisms are closed under joins (= pointwise joins), and $\text{Open}$ induces a join-semilattice isomorphism $\text{Dep}[G, H] \cong \text{JSL}_f[\text{Open}G, \text{Open}H]$. However, we choose to provide an explicit proof.

3. This follows from the well-definedness of $\text{Dep}[\cdot, \cdot]$ (see Lemma 5.1.18) and the following two observations.

(a) Each $\text{Tf}[G, H] \subseteq \text{Dep}[G, H]$ is well-defined sub join-semilattice by the previous statement.

(b) Tight $\text{Dep}$-morphisms are closed under pre/post-composition by arbitrary $\text{Dep}$-morphisms, since the factorisation through an identity relation is preserved.

4. That they are well-defined $\text{Dep}$-morphisms is proved in Example 5.4.2 above. Since basic independents are unions of basic bicliques, it suffices to show the latter are tight. This follows because:

5. (a) Regarding the topmost equalities,

(b) As for the central equalities, consider the left one and let $f := \text{Open} \uparrow^g \downarrow^h \circ \text{Open}G : \text{Open}G \rightarrow \text{Open}H$, recalling:

Now recall that $\leq_{\text{Open}G}$ is inclusion of sets, and a $G$-open set $Y$ satisfies $Y \notin \text{in}_G(\mathbb{T})$ iff $g_t \in Y$ by Lemma 4.2.7.1. Furthermore $\varnothing = \varnothing_0$ because $Y$ is $G$-open i.e. each element $g_t \in Y$ must be contained in some ‘neighbourhood’ $G[g_t]$. Consequently:

$$f(Y) = (\uparrow^g \downarrow^h)_2^\uparrow [Y] = \begin{cases} \varnothing_0[Y] = \varnothing = 1_{\text{Open}G} & \text{if } Y \leq_{\text{Open}G} \text{in}_G(\mathbb{T}) \\ \varnothing_1[Y] & \text{otherwise} \end{cases}$$
as required. Regarding the right equality, let \( f := \text{open} \circ \gamma_{g,h} : \text{open} \to \text{open} \) and recall:

\[
(\gamma_{h,h})_+ \circ \sigma^\star = \sigma^\star \cup \mathcal{G} \times \text{in}^\star(\tilde{h}_t) \cup \tilde{G}^\star(\tilde{g}_s) \times \mathcal{H}[H_s]
\]

Certainly \( \sigma^\star[Y] = \emptyset \) as before, and furthermore:

\[
Y \cap \tilde{G}^\star(g_s) \neq \emptyset \iff Y \cap \tilde{G}^\star(g_s) \neq \emptyset \quad \text{by De Morgan duality}
\]

Further recalling that \( \leq_{\text{open} \mathcal{H}} \) is inclusion, \( \emptyset = \bot_{\text{open} \mathcal{G}} \) and \( \mathcal{H}[H_s] = \top_{\text{open} \mathcal{H}} \), we obtain the desired action:

\[
f(Y) = (\gamma_{h,h})_+^\star[Y] = \begin{cases} 
\top_{\text{open} \mathcal{H}} & \text{if } Y = \bot_{\text{open} \mathcal{G}} \\
\text{in}^\star(\tilde{h}_t) & \text{if } \emptyset \leq_{\text{open} \mathcal{G}} Y \leq_{\text{open} \mathcal{G}} g_s \\
\top_{\text{open} \mathcal{H}} & \text{if } Y \leq_{\text{open} \mathcal{G}} g_s
\end{cases}
\]

(c) Concerning the final equality, let \( \mathcal{R} := \text{Pirr} \downarrow_{\mathcal{Q}, \mathcal{R}} = \{(j,m) \in J(\mathcal{Q}) \times M(\mathcal{R}) : \gamma_{m,j} \leq_{\mathcal{R}} m\} \). Recall that:

\[
\gamma_{m,j} = \bigvee_{JSLJ(\mathcal{Q}, \mathcal{R})} \{q \leq_{\mathcal{Q}} m \in M(\mathcal{Q}), J(\mathcal{R}) \ni j \leq_{\mathcal{R}} r \} \quad \text{by Lemma 5.1.9.1}
\]

and hence \( \text{Pirr} \downarrow_{\mathcal{Q}, \mathcal{R}} \) is the union of the \( \text{Pirr} \gamma_{m,j} \)'s, using the generalisation of Lemma 5.1.20. So fix any \( m \in M(\mathcal{Q}) \) and \( j \in J(\mathcal{R}) \), finally observing that:

\[
\gamma_{m,j} = \{(j',m') \in M(\mathcal{Q}) \times J(\mathcal{R}) : \gamma_{m,j} \leq_{\mathcal{R}} m', j' \leq_{\mathcal{Q}} m \} = \{(j',m') \in M(\mathcal{Q}) \times J(\mathcal{R}) : j' \leq_{\mathcal{Q}} m \text{ and } j \leq_{\mathcal{R}} m' \} \quad \text{by definition of } \gamma_{m,j}
\]

By Lemma 5.2.5, a JSLJ-morphism \( f : \mathcal{Q} \to \mathcal{R} \) is tight iff it arises as a join of the special morphisms \( \gamma_{m,j} \) where \( m \in M(\mathcal{Q}) \) and \( j \in J(\mathcal{R}) \). In particular, if \( \mathcal{Q} = \text{open} \mathcal{G} \) and \( \mathcal{R} = \text{open} \mathcal{H} \) then by Lemma 4.2.5.3, \( m = \text{in}_\mathcal{Q}(\tilde{g}_s) \) for some \( \tilde{g}_s \in \mathcal{G} \). By Lemma 5.1.20, a \( \text{open} \mathcal{H} \)-morphism \( \mathcal{R} : \mathcal{G} \to \mathcal{H} \) is tight iff \( \text{open} \mathcal{R} \) is tight, and also \( \text{open} \mathcal{G} \) induces a join-semilattice isomorphism \( \text{open} \mathcal{G} \colon [\mathcal{G}, \mathcal{H}] \cong JSLJ(\text{open} \mathcal{G}, \text{open} \mathcal{H}) \). Then by the left-central equality in (5), \( \mathcal{R} \) is tight iff it arises as a union of basic biquadric.

7. We calculate:

\[
\mathcal{R} \downarrow_{\mathcal{Q}, \mathcal{H}} = \mathcal{R} \downarrow_{\mathcal{Q}} \cup \mathcal{R} \downarrow_{\mathcal{H}} = \mathcal{R} \downarrow_{\mathcal{Q}} \cup \mathcal{R} \downarrow_{\mathcal{H}}
\]

The other equality follows by duality and the top-left equality in (5), recalling that \( (\mathcal{S}^\star)_+ = \mathcal{S}^\star \).

8. We have:

\[
\begin{align*}
\mathcal{R} \subseteq \mathcal{Q}^\star_{\mathcal{G}, \mathcal{H}} & \iff \text{open} \mathcal{R} \subseteq \text{open} \mathcal{Q}^\star_{\mathcal{G}, \mathcal{H}} \quad \text{open preserves ordering} \\
\text{open} \mathcal{R} \subseteq \mathcal{Q}^\star_{\mathcal{G}, \mathcal{H}} & \iff \text{open} \mathcal{R} \subseteq \mathcal{Q}^\star_{\mathcal{G}, \mathcal{H}} \quad \text{by (5)} \\
\text{open} \mathcal{R} \subseteq \mathcal{Q}^\star_{\mathcal{G}, \mathcal{H}} & \iff \text{open} \mathcal{R} \subseteq \mathcal{Q}^\star_{\mathcal{G}, \mathcal{H}} \quad \text{by Lemma 5.2.17.2} \\
\mathcal{R}[g_s] \subseteq \mathcal{H}[h_t] & \iff \mathcal{R}[g_s] \subseteq \mathcal{H}[h_t] \quad \text{since } \mathcal{R} = \mathcal{G} : \mathcal{R}^\star_+ \\
h_t \notin \mathcal{R}[g_s] & \iff h_t \notin \mathcal{R}[g_s] \quad \text{by Lemma 4.2.7.1} \\
\mathcal{R}(g_s, h_t) & \iff \mathcal{R}(g_s, h_t)
\end{align*}
\]
We may now use Theorem 5.3.10 to obtain a natural isomorphism \( v : \mathcal{O}_D \circ \text{Ti}[-,-] \Rightarrow \text{Ti}([-)^\vee,-] \) between join-semilattices of tight \( \text{Dep} \)-morphisms.

**Theorem 5.4.4** (Dual isomorphism between join-semilattices of tight \( \text{Dep} \)-morphisms).

We have the natural isomorphism:

\[
v_{G,H} : (\text{Ti}[G,H])^\text{op} \to \text{Ti}[\check{G},\check{H}]
\]

\[
v_{G,H}(R) := \overline{\text{R}}_+ ; \check{H}
\]

and in particular \( v_{G,H}^{-1} = v_{\check{G},\check{H}}^\text{op} \). Furthermore:

\[
v_{G,H}(g_{s,h_s}^*) = g_{s,h_s}^* \text{ for every } (g_s,h_s) \in G_s \times H_s.
\]

\[
v_{G,H}(g_{t,h_t}^*) = g_{t,h_t}^* \text{ for every } (g_t,h_t) \in G_t \times H_t.
\]

**Proof.** We construct \( v_{G,H} \) as a composition of natural isomorphisms:

\[
\begin{align*}
\alpha_{G,H} & : \text{Ti}[G,H] \cong \text{Ti}[[G,H]]^\text{op} \cong \text{Ti}[\check{G},\check{H}] \\
\beta & : \text{Ti}[[G,H]]^\text{op} \cong \text{Ti}[\check{G},\check{H}]
\end{align*}
\]

Let us start with an arbitrary tight \( \text{Dep} \)-morphism \( \mathcal{R} : G \to \check{H} \).

1. \( \alpha_{G,H} \) applies \( \text{Open} \). It is a natural isomorphism because:

\[
\text{Dep}[G,H] \cong \text{JSL}_{f,[\text{Open}G,\text{Open}H]} \text{ restricts to the respective subfunctors of tight morphisms,}
\]

see Lemma 5.4.3.1. In particular \( \alpha_{G,H}(\mathcal{R}) = \text{Open}\mathcal{R} \).

2. \( (\text{Ti}[id_{\text{Open}G},\partial_{\check{H}}^{-1}])^\text{op} \) post-composes with \( \partial_{\check{H}}^{-1} \), and thus:

\[
\beta(\text{Open}\mathcal{R}) = \text{Open}G \xrightarrow{\text{Open}\mathcal{R}} \text{Open}\check{H} \xrightarrow{\partial_{\check{H}}^{-1}} (\text{Open}\check{H})^\text{op}
\]

recalling that \( \partial_{\check{H}}^{-1}(X) = \check{H}[-X] \) – see Definition 4.6.5.

3. \( v_{\text{Open}G,\text{Open}H} \) instantiates the natural isomorphism from Theorem 5.3.10, so that:

\[
f := v_{\text{Open}G,\text{Open}H} (\partial_{\check{H}}^{-1} \circ \text{Open}\mathcal{R}) = \lambda Y \in O(G). \bigcup \{ \check{H}[h_s] : h_s \in H_s, (\partial_{\check{H}}^{-1} \circ \text{Open}\mathcal{R})(\check{H}[h_s]) \notin Y \}
\]

Here we have used a slightly modified description of \( v_{G,R}(f : Q \to R^\text{op}) \) from the statement of the Theorem i.e. one can replace \( j_s \in J(R) \) by any join-generating set. Then let us simplify the above comprehension:

\[
\begin{align*}
(\partial_{\check{H}}^{-1} \circ \text{Open}\mathcal{R})(\check{H}[h_s]) \notin Y & \iff (\text{Open}\mathcal{R}), \circ (\partial_{\check{H}}^{-1})_*(H[h_s]) \notin Y \text{ for Lemma 4.6.6.1} \\
& \iff (\text{Open}\mathcal{R}), \circ \partial_{\check{H}}^{-1}(H[h_s]) \notin Y \text{ for Lemma ??} \\
& \iff \partial_{\check{H}}^{-1} \circ \text{Open}\mathcal{R} \circ \partial_{\check{H}} \circ \partial_{\check{H}}^{-1}(H[h_s]) \notin Y \text{ by definition} \\
& \iff \partial_{\check{H}}^{-1} \circ \text{Open}\mathcal{R} \circ \partial_{\check{H}}(H[h_s]) \notin Y \text{ since } (\check{R})_+ = \check{R}_- \text{ generally} \\
& \iff \check{G}^1 \circ \check{R}_+ \circ \check{R} \circ \partial_{\check{H}} \circ \partial_{\check{H}}^{-1}(H[h_s]) \notin Y \text{ recall } \check{R} : G \to \check{H} \\
& \iff \check{R}^1 \circ \check{R} \circ \partial_{\check{H}}(h_s) \notin Y \text{ by De Morgan duality} \\
& \iff \check{R}^1 \circ \check{R}(h_s) \notin \check{G}^1(Y) \text{ standard adjoint}
\end{align*}
\]

Consequently, we have:

\[
f = \lambda Y \in O(G). \bigcup \{ \check{H}[h_s] : h_s \in H_s, \check{R}(h_s) \notin \check{G}(Y) \}
\]
4. $\text{Ti}[\partial_g^{-1}, id_{\text{Open}\mathcal{H}}]$ pre-composes with $\partial_g^{-1}$, yielding the tight morphism:

$$
\begin{align*}
g &:= \{Y \in O(\mathcal{G}^\ast) : \mathcal{H}[h_s] : h_s \in \mathcal{H}_s, R_x(h_s) \notin cl_g(Y)\} \\
&= \{Y \in O(\mathcal{G}^\ast) : \mathcal{H}[h_s] : h_s \in \mathcal{H}_s, \text{in}_g(Y) \notin \bar{R}[h_s]\} \\
&: \text{Open}\mathcal{G} \to \text{Open}\mathcal{H}
\end{align*}
$$

using De Morgan duality at the level of closure and interior operators.

5. Finally we need to apply $(\alpha_{\mathcal{G}^\ast, \mathcal{H}}^\ast)^{-1}$, this being the inverse of the natural isomorphism $\text{Ti}[\mathcal{G}^\ast, \mathcal{H}] \to \text{Ti}[\text{Open}\mathcal{G}, \text{Open}\mathcal{H}]$ whose action applies $\text{Open}$. Then we seek the necessarily unique tight Dep-morphism $\mathcal{S} := \mathcal{G}^\ast \to \mathcal{H}$ such that $\text{Open}\mathcal{S} = g$, this being a relation $\mathcal{S} \subseteq \mathcal{G}_s \times \mathcal{H}_t$. Observing that $g(\mathcal{G}[g_s]) = \mathcal{S}^\ast[\mathcal{G}[g_s]] = \mathcal{G}^\ast; \mathcal{S}[g_s] = \mathcal{S}[g_s]$, it follows that for every $g_t \in \mathcal{G}_t$ we have:

$$
\begin{align*}
\mathcal{S}[g_t] = g(\mathcal{G}[g_t]) &= \mathcal{H}[h_s] : h_s \in \mathcal{H}_s, \text{in}_g(\mathcal{G}[g_t]) \notin \bar{R}[h_s] \\
&= \mathcal{H}[h_s] : h_s \in \mathcal{H}_s, \mathcal{G}[g_t] \notin \bar{R}[h_s] \\
&= \mathcal{H}[h_s] : h_s \in \mathcal{H}_s, g_t \notin \mathcal{G}^\ast \circ \bar{R} \{(h_s)\} \\
&: \text{via usual adjoint}
\end{align*}
$$

Recalling the definition of $\mathcal{R}$’s associated positive component (see Definition 4.1.8), we deduce that:

$$
\begin{align*}
\mathcal{S}(g_t, h_t) &\iff \exists h_s \in \mathcal{H}_s, [\mathcal{H}(h_s, h_t) \text{ and } \bar{\mathcal{R}}_s(h_s, g_t)] \\
&\iff \exists h_s \in \mathcal{H}_s, [\bar{\mathcal{R}}_s(g_t, h_s) \text{ and } \mathcal{H}(h_s, h_t)] \\
&\iff \bar{\mathcal{R}}_s; \mathcal{H}(g_t, h_t)
\end{align*}
$$

since converse commutes with complement.

To prove that $\nu_{\mathcal{G}, \mathcal{H}}^{-1} = \nu_{\mathcal{G}^\ast, \mathcal{H}}^\ast$, observe that the latter is a well-defined join-semilattice isomorphism of the correct type. It will be helpful to verify that $\nu_{\mathcal{G}, \mathcal{H}}$ sends basic bicliques to basic independents i.e.

$$
\nu_{\mathcal{G}, \mathcal{H}}(g_t, h_t) = \nu_{\mathcal{G}^\ast, \mathcal{H}}(g_t, h_t)
$$

for every $(g_t, h_t) \in \mathcal{G}_t \times \mathcal{H}_t$.

Then let us consider the composite action:

$$
\begin{align*}
\nu_{\mathcal{G}, \mathcal{H}}(g_t, h_t) &\xrightarrow{\alpha_{\mathcal{G}, \mathcal{H}}} \text{in}_g(\bar{\mathcal{R}}_s), \mathcal{H}[h_t] \\
&\xrightarrow{\text{Open}\mathcal{G} \circ \text{Open}\mathcal{H}} \text{in}_g(\mathcal{G}[g_t]), \mathcal{H}[h_t] \\
&\xrightarrow{\text{by Lemma 5.4.3.5}} \text{Open}\mathcal{G}, \mathcal{H}
\end{align*}
$$

recalling that $\partial_g^{-1}(X) := \mathcal{G}[\bar{X}]$. By essentially the same proof we also have $\nu_{\mathcal{G}, \mathcal{H}}(\downarrow g_t, h_t) = \nu_{\mathcal{G}^\ast, \mathcal{H}}(g_t, h_t)$ for every $(g_s, h_s) \in \mathcal{G}_s \times \mathcal{H}_s$, noting that isomorphisms satisfy the extra conditions listed in Lemma 5.2.10.2. It follows that $\nu_{\mathcal{G}^\ast, \mathcal{H}}^\ast$ has the same action as $\nu_{\mathcal{G}, \mathcal{H}}^{-1}$, so they are the same morphisms.

Recall that $(\cdot)^\ast : \text{Dep}^{\text{op}} \to \text{Dep}$ is the self-duality functor, whose action on both objects and morphisms takes the relational converse. We now prove the universal property of the synchronous product i.e.

$$
tight morphisms \mathcal{G} \otimes \mathcal{H} \to \mathcal{I} \text{ naturally biject with tight morphisms } \mathcal{G} \to \mathcal{H}^\ast \otimes \mathcal{I}.
$$
This bijection cannot hold without the tightness assumption. To see why, let \( G = \Delta_{(0)} \) so that \( G \otimes H \cong H \). Applying \( \text{Open} \), we see that morphisms of type \( G \otimes H \rightarrow I \) biject (also as a join-semilattice) with \( \text{JSL}_f[\text{Open}H, \text{Open}I] \). Similarly since \( \text{Open}\Delta_{(0)} \cong 2 \), the morphisms of type \( G \rightarrow H \otimes I \) biject with \( \text{Open}(H \otimes I) \cong \text{Ti}[\text{Open}H, \text{Open}I] \) using Theorem 5.3.6. Since join-semilattice morphisms needn’t be tight by Lemma 5.2.2, it follows that we cannot drop the tightness assumption. Moving on, the relevant natural isomorphism has a very natural action:

\[
\text{given a relation } R \subseteq (G \otimes H)_s \times I_t = (G_s \times H_s) \times I_t, \text{ re-tuple each element } ((g_s, h_s), i_t) \mapsto (g_s, (h_s, i_t)),
\]
yielding a relation \( R' \subseteq G_s \times (H_s \times I_t) = G_s \times (H \otimes I)_t \) of the desired type.

The interpretation of this natural isomorphism inside \( \text{JSL}_f \) is:

\[
\text{tight morphisms } Q \otimes rtup \rightarrow S \text{ naturally biject with tight morphisms } Q \rightarrow \text{Ti}[R, S],
\]
which will follow by combining Theorem 5.3.6 with the result we are just about to prove.

**Theorem 5.4.5** (The synchronous product is universal w.r.t. tight morphisms).

We have the natural isomorphism:

\[
rtup : \text{Ti}[\cdot \otimes \cdot, \cdot] \Rightarrow \text{Ti}[\cdot, (\cdot)' \otimes \cdot] \quad \text{rtup}_{G, H, I} : \text{Ti}[G \otimes H, I] \rightarrow \text{Ti}[G, \hat{H} \otimes I]
\]

The associated components of its component’s action are described in Note 5.4.6 below, and its natural inverse re-tuples in the other direction.

**Proof.** For the first part of the proof, fix any \( G, H, I \), and also any tight \( \text{Dep} \)-morphism \( R : G \otimes H \rightarrow I \), and define the relation:

\[
S := rtup_{G, H, I}(R) = \{(g_s, (h_s, i_t)) \in G_s \times (H_s \times I_t) : R((g_s, h_s), i_t)\}
\]

We begin by verifying that \( S \) is a well-defined tight \( \text{Dep} \)-morphism of type \( G \rightarrow \hat{H} \otimes I \). Since \( R \) is tight it is a union of basic bicliques by Lemma 5.4.3.

1. First consider the base case where \( R \) is a basic biclique:

\[
R = \{g_{(g_s, h_s), i_t} = (\hat{G}[g_t] \times \hat{H}[h_t]) \times I_s : G_s \times H_s \times I_t \}
\]

Then \( S = R_{G, H, I} = \{g_{(g_s, h_s), i_t} \in (\hat{G} \otimes \hat{H})_s \times I_t : R((g_s, h_s), i_t)\} \) is a basic biclique and thus is well-defined and tight.

2. Generally speaking, if \( R \) is a union of basic bicliques then observe that \( rtup_{G, H, I} \) preserves unions i.e. re-tupling preserves \( \emptyset \) and also binary unions of relations. Then by the previous point \( S \) is a union of basic bicliques, and is therefore well-defined and tight.

Then each \( rtup_{G, H, I} \) is a well-defined function, and also preserves the join-semilattice structure: \( \emptyset \) and binary union. Since re-tupling can be undone, we know that \( rtup_{G, H, I} \) is an injective join-semilattice morphism. Furthermore the preceding argument implies surjectivity, since every basic biclique \( S : G \rightarrow \hat{H} \otimes I \) arises from a basic biclique \( R \).

It remains to prove naturality i.e. that the following diagram commutes inside \( \text{JSL}_f \):

\[
\begin{array}{ccc}
\text{Ti}[G \otimes H, I] & \xrightarrow{rtup_{G, H, I}} & \text{Ti}[G, \hat{H} \otimes I] \\
\downarrow\text{Ti}((R \otimes S)^{op}, T) & & \downarrow\text{Ti}[(R \otimes S)^{op}, T] \\
\text{Ti}[G' \otimes H', I'] & \xrightarrow{rtup_{G', H', I'}} & \text{Ti}[G', (H')^{op} \otimes I']
\end{array}
\]

for any \( \text{Dep} \)-morphisms \( R : G' \rightarrow G, S : H' \rightarrow H \) and \( T : I \rightarrow I' \). In other words, for every tight \( \text{Dep} \)-morphism \( R' : G \otimes H \rightarrow I \), we must establish that:

\[
R \circ rtup_{G, H, I}(R') = rtup_{G', H', I'}(R' \circ (R \otimes S) \circ T)
\]
Since \( \mathcal{R} \) preserves unions of Dep-morphisms separately in each component by Lemma 5.1.18, and each component of \( \text{rtup} \) preserves such unions, it suffices to consider the special case where \( \mathcal{R}' = \text{rtup}_{\mathcal{G},\mathcal{H},\mathcal{I}}^{(g_r,h_i)} \) is a basic biclique. We finally use Lemma 5.4.3.7 to prove this.

\[
\mathcal{R} \downarrow \text{rtup}_{\mathcal{G},\mathcal{H},\mathcal{I}}(\mathcal{R}') \downarrow (\tilde{\mathcal{S}} \otimes \mathcal{T})
\]

definition of \( \mathcal{R}' \)

\[
= \mathcal{R} \downarrow \text{rtup}_{\mathcal{G},\mathcal{H},\mathcal{I}}^{(g_r,h_i)}(\mathcal{S} \otimes \mathcal{T})
\]

\( \text{rtup} \) preserves basic bicliques

\[
= \mathcal{R} \downarrow \text{rtup}_{\mathcal{G},\mathcal{H},\mathcal{I}}^{(g_r,h_i)}(\tilde{\mathcal{S}} \otimes \mathcal{T})
\]

by Lemma 5.4.3.7

\[
= (\mathcal{U}\{\text{rtup}_{\mathcal{G},\mathcal{H},\mathcal{I}}^{(g_r,h_i)}(g_r,g_i)\}) \downarrow (\tilde{\mathcal{S}} \otimes \mathcal{T})
\]

\( \text{rtup} \) preserves unions of morphisms

\[
= \mathcal{U}\{\text{rtup}_{\mathcal{G},\mathcal{H},\mathcal{I}}^{(g_r,h_i)}(g_r,g_i)\} \downarrow (\tilde{\mathcal{S}} \otimes \mathcal{T})
\]

by Lemma 5.4.3.7

\[
= \text{rtup}_{\mathcal{G},\mathcal{H},\mathcal{I}}((\mathcal{R} \otimes \mathcal{S}) \downarrow \mathcal{G}^{(g_r,h_i)})(\tilde{\mathcal{S}} \otimes \mathcal{T})
\]

repeating above reasoning

\[
= \text{rtup}_{\mathcal{G},\mathcal{H},\mathcal{I}}(\mathcal{R} \otimes \mathcal{S}) \downarrow \mathcal{R}' \downarrow \mathcal{T}
\]

definition of \( \mathcal{R}' \)

\[\square\]

**Note 5.4.6** (Associated components of \( \text{rtup}_{\mathcal{G},\mathcal{H},\mathcal{I}} \)'s action).

The components of \( \text{rtup} \) are join-semilattice isomorphisms \( \text{rtup}_{\mathcal{G},\mathcal{H},\mathcal{I}} \), sending tight Dep-morphisms \( \mathcal{R} : \mathcal{G} \otimes \mathcal{H} \to \mathcal{I} \) to tight Dep-morphisms \( \mathcal{R} : \mathcal{G} \to \mathcal{H} \otimes \mathcal{I} \) by re-tupling each element of \( \mathcal{R} \). We now describe the associated components of the morphism \( \text{rtup}_{\mathcal{G},\mathcal{H},\mathcal{I}}(\mathcal{R}) \).

\[
(\text{rtup}_{\mathcal{G},\mathcal{H},\mathcal{I}}(\mathcal{R}))_{(g_r,h_i)}(g_r,g_i) \iff \forall h_s. \forall i_s. \{\mathcal{H}(h_s,h_t) \land \mathcal{I}(s_i,i_t) \Rightarrow \mathcal{R}((g_s,h_s),i_t)\}
\]

Here we have simply used the definition of the associated components i.e. Definition 4.1.8.

\[\square\]

**Theorem 5.4.7** (The tight tensor product is universal w.r.t. tight morphisms).

We have the natural isomorphism:

\[
\text{ut} : \text{Ti}[\mathcal{Q},\mathcal{R}] \cong \text{Ti}[\mathcal{Q},\mathcal{R}]
\]

with action:

\[
\text{ut}_{\mathcal{Q},\mathcal{R},S} : \text{Ti}[\mathcal{Q},\mathcal{R},S] \to \text{Ti}[\mathcal{Q},\mathcal{R},S]
\]

Finally, the action on join/meet-irreducibles is as follows:

\[
\text{ut}_{\mathcal{Q},\mathcal{R},S}^{\text{m,q} \iff \text{m,q} \iff \text{m,q}}(q,r) = (q,r) \iff \forall h_s. \forall i_s. \mathcal{H}(h_s,h_t) \land \mathcal{I}(s_i,i_t) \Rightarrow \mathcal{R}((g_s,h_s),i_t)
\]

\[\square\]

**Proof.** We shall make use of the following natural isomorphisms.

1. By Corollary 5.3.8 of Theorem 5.3.6 we have the natural isomorphism:

   \[\alpha_{\mathcal{Q},\mathcal{R}} : \mathcal{Q} \otimes \mathcal{R} \to \text{Open}(\mathcal{Pirr}(\mathcal{Q} \otimes \mathcal{Pirr}(\mathcal{R}))\]

   \[\alpha_{\mathcal{Q},\mathcal{R}}(f : \mathcal{Q} \to \mathcal{R}) := \{ (m_q,m_r) \in M(\mathcal{Q}) \times M(\mathcal{R}) : f(m_q) \not\in m_r \} \]

   \[\alpha_{\mathcal{Q},\mathcal{R}}^{-1}(Y) := \lambda q \in Q. \forall r \in \mathcal{R}. \{ m_r \in M(\mathcal{R}) : (m_q,m_r) \not\in Y \} : q \leq q \in M(\mathcal{Q}) \]

   witnessing the fact that the tight tensor product is essentially the synchronous product of binary relations.

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2. We have the inverse of the re-tupling natural isomorphism from Theorem 5.4.5:

\[ rtup_{G,H,I}^{-1} : Ti[G, \tilde{H} \otimes I] \rightarrow Ti[G \otimes H, I] \]

\[ rtup_{G,H,I}^{-1}(R) := \{ ((g_s, h_s), i_t) : R(g_s, (h_s, i_t)) \} \]

which re-tuples in the ‘other direction’. This is the universal property of synchronous products w.r.t. tight Dep-morphisms, this being the Dep-version of the natural isomorphism we are trying to describe.

3. We’ll also use auxiliary natural isomorphisms:

\[ p_{Q,G} : Ti[Q, \text{Open} G] \rightarrow Ti[\text{Pirr} Q, G] \]
\[ f : Q \rightarrow \text{Open} G \implies \text{Pirr} f \circ \text{red}_{G}^{-1} : \text{Pirr} Q \rightarrow G \]
\[ q_{G,S} : Ti[G, \text{Pirr} S] \rightarrow Ti[\text{Open} G, S] \]
\[ R : G \rightarrow \text{Pirr} S \implies \text{rep}_{S}^{-1} \circ \text{Open} R : \text{Open} G \rightarrow S \]

which are correct because tightness is preserved by these equivalence functors, and tight morphisms are closed under composition with arbitrary morphisms. Further recall that:

\[ \text{red}_{G}^{-1} := \bar{\epsilon} = \{ (X, g_t) \in J(\text{Open} G) \times G_t : g_t \in X \} \]
\[ \text{rep}_{S}^{-1} := \lambda Y \in \text{Open} \text{Pirr} S. \bigwedge_{S} M(S) \backslash Y \]

Then \( ut_{Q,R,S}^{-1} \) is defined as the composite natural isomorphism:

\[
\begin{align*}
\text{Ti}[Q, \text{Ti}[R, S]] & \xrightarrow{\text{Ti}[\text{id}_{Q}, \text{rep}_{Q} \otimes \text{id}_{S}]} \text{Ti}[Q, \text{rep}_{Q} \otimes \text{id}_{S}] \\
\text{Ti}[Q, \text{Open}((\text{Pirr} R)^\circ \text{Pirr} S)] & \xrightarrow{\text{Ti}[\text{id}_{Q}, \text{rep}_{Q} \otimes \text{id}_{S}]} \text{Ti}[Q, \text{Pirr} Q \otimes \text{Pirr} R, \text{Pirr} S] \\
\text{Ti}[Q \otimes R, S] & \xrightarrow{\text{Ti}[\alpha_{Q,R,S}, \text{id}_{S}]} \text{Ti}[\text{Open}(\text{Pirr} Q \otimes \text{Pirr} R), S]
\end{align*}
\]

which has action:

\[ h : Q \rightarrow \text{Ti}[R, S] \implies \text{rep}_{S}^{-1} \circ \text{Open}(\text{rtup}_{\text{Pirr} Q, \text{Pirr} R, \text{Pirr} S}^{-1}(\text{Pirr}(\alpha_{Q,R,S} \circ h) \circ \text{red}_{(\text{Pirr} R)^\circ \text{Pirr} S}^{-1})) \circ \alpha_{Q,R} \]

Now, since \( h \) is tight it arises as the pointwise-join of special morphisms:

\[ \uparrow_{Q,\text{Ti}[R,S]}^{m_{Q}} \quad \uparrow_{R,S}^{m_{R,S}} \quad \text{where } m_{Q} \in M(Q) \text{ and } \uparrow_{R,S}^{m_{R,S}} \in J(\text{Ti}[R, S]) \].

Then let us consider the action of the mapping \( ut_{Q,R,S}^{-1} \) upon these specific morphisms. For brevity it will be helpful to first set some basic notation:

\[ G_{Q} := \text{Pirr} Q \quad G_{R} := \text{Pirr} R \quad G_{S} := \text{Pirr} S \]

and we are going to split the computation of \( ut_{Q,R,S}^{-1}(\uparrow_{Q,\text{Ti}[R,S]}^{m_{Q}} \uparrow_{R,S}^{m_{R,S}}) \) into parts.
1. Let us begin with the following simplification:

\[
\text{Pirr}(\alpha_{R^S,m_R,J_S} \circ \text{rtup}_{Q,T_0[S]}^{m_q,\alpha_{Q,T[R,S]}})
\]

\[
= \text{Pirr}(\uparrow_{Q_0,0}(\text{rtup}_{Q,T_0[S]}^{m_q,\alpha_{Q,T[R,S]}}))
\]

by Lemma 5.2.10

\[
= \bigcup \{\text{rtup}_{Q_0,G_{Q,T_0[S]}(G_{Q,G_S})}^{m_q,m_g} : m_q \leq Q, m'_q \leq \mathcal{G}_{R'[Q,G_S]} \leq \alpha_{R^S,m_R,J_S}(m_q, m_g)\}
\]

by Lemma 5.4.3.5

Concerning the marked equality, \( \mathcal{G}_R \otimes \mathcal{G}_S = \text{Pirr}^{op} \otimes \text{Pirr}S \) is reduced so that \( \text{rtup}_{Q,T_0[S]}^{m_q,\alpha_{Q,T[R,S]}} \) is the closure of a bipartite graph isomorphism. It turns out that post-composing with it bijectively relabels \( \mathcal{G}_{R'[Q,G_S]} \) with \( (m_r, j_s) \).

2. Continuing, we have:

\[
\text{Open}(\text{rtup}_{Q_0,G_{Q,R}G_S}^{-1}(\text{Pirr}(\alpha_{Q,T[R,S]} \circ h) \circ \text{rtup}_{Q,T_0[S]}^{m_q,\alpha_{Q,T[R,S]}})) = \text{Open}(\text{rtup}_{Q_0,G_{Q,R}G_S}^{-1}(\text{rtup}_{Q_0,G_{Q,R}G_S}^{m_q,\alpha_{Q,T[R,S]}}))
\]

see below

\[
\text{Open}(\text{rtup}_{Q_0,G_{Q,R}G_S}^{-1}(\text{rtup}_{Q_0,G_{Q,R}G_S}^{m_q,\alpha_{Q,T[R,S]}}))
\]

using definitions

\[
\text{Open}(\text{rtup}_{Q_0,G_{Q,R}G_S}^{-1}(\text{rtup}_{Q_0,G_{Q,R}G_S}^{m_q,\alpha_{Q,T[R,S]}})) = \text{Open}(\text{rtup}_{Q_0,G_{Q,R}G_S}^{-1}(\text{rtup}_{Q_0,G_{Q,R}G_S}^{m_q,\alpha_{Q,T[R,S]}}))
\]

by Lemma 5.4.3.5

Concerning the marked equality, \( \mathcal{G}_R \otimes \mathcal{G}_S = \text{Pirr}^{op} \otimes \text{Pirr}S \) is reduced so that \( \text{rtup}_{Q,T_0[S]}^{m_q,\alpha_{Q,T[R,S]}} \) is the closure of a bipartite graph isomorphism. It turns out that post-composing with it bijectively relabels \( \mathcal{G}_{R'[Q,G_S]} \) with \( (m_r, j_s) \).

3. Then it remains to simplify:

\[
\text{rep}_{S^{-1}}(\mathcal{G}_S[j_s]) = \bigcup S \{s \in S \mathcal{G}_S[j_s] = \bigcup S \{m \in S : m \leq m_s \}
\]

In conclusion, we have shown that:

\[
\text{rep}_{S^{-1}}(\mathcal{G}_S[j_s]) = \bigcup S \{m \in Q, m_r \leq m_s \}
\]

and have thus established its action on join-irreducibles as desired. Now, to verify its general action:

\[
\text{ut}_{Q,R,S}(f) = \lambda q \in Q. \lambda r \in R. f(\uparrow_{Q,W,R}^{m_q,\alpha_{Q,W,R}})
\]

it suffices to establish this when \( f \) is join-irreducible because (i) \( \text{ut}_{Q,R,S} \) preserves joins \( \cup_i f_i \), and (ii) we can absorb the \( \vee_i \)-joins into \( \text{JSL}_f[T[Q^R, R], S] \)-joins of the \( f_i \)'s. Thus we need to prove that:

\[
\uparrow_{Q,W,R}^{m_q,\alpha_{Q,W,R}}(q)(r) = \uparrow_{Q,W,R}^{m_q,\alpha_{Q,W,R}}(q)(r)
\]

for every \( (q, r) \in Q \times R \).

Indeed:

\[
\uparrow_{Q,W,R}^{m_q,\alpha_{Q,W,R}}(q)(r) = \begin{cases} j_s & \text{if } q \not\in Q, r \not\in R, r \not\in m_r \\ \uparrow_{Q,W,R}^{m_q,\alpha_{Q,W,R}} & \text{otherwise} \end{cases}
\]

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using Lemma 5.1.9.6, whereas:
\[
\uparrow_{Q,T\uparrow[S,R]}^{m_s,m_r,j_s,j_r}(q)(r) = \begin{cases} 
\uparrow_{[R,S]} & \text{if } q \not\approx_m m_q \\
\downarrow_{[R,S]} & \text{otherwise}
\end{cases} \quad \begin{cases} 
\downarrow_{S} & \text{if } q \not\approx_Q m_q \text{ and } r \not\approx_Q m_r \\
\uparrow_{S} & \text{otherwise}
\end{cases}
\]
as required. Having verified the action of \(ut_{Q,R,S}^1\), the action of the inverse \(ut_{Q,R,S}^{-1}\) follows using the fact that \(f : Tri[m_R, R] \to S\) preserves joins. Then it only remains to verify the action of \(ut_{Q,R,S}\) on meet-irreducibles:
\[
\uparrow_{Q,R,S}^{j_s,j_r,m_s,m_r}(q)(r) = \begin{cases} 
\uparrow_{S} & \text{if } f_{Q,R} = \uparrow_{Q,R} \\
\downarrow_{S} & \text{if } \downarrow_{Q,R} < \uparrow_{Q,R} \leq \uparrow_{j_s,j_r} \\
\uparrow_{S} & \text{if } f_{Q,R} \not\approx \downarrow_{Q,R} \\
\downarrow_{S} & \text{if } q = \uparrow_Q \text{ or } r = \downarrow_R \\
\uparrow_{S} & \text{if } q < \uparrow_Q q \leq \uparrow_Q j_q \text{ and } \downarrow_R < r \leq \downarrow_R j_r \\
\downarrow_{S} & \text{if } q < \uparrow_Q j_q \text{ and } r \not\approx_R j_r
\end{cases}
\]
whereas:
\[
\downarrow_{Q,T\uparrow[S,R]}^{j_s,j_r,m_s,m_r}(q)(r) = \begin{cases} 
\uparrow_{S} & \text{if } q = \uparrow_Q \\
\downarrow_{S} & \text{if } \downarrow_{Q,R} < q \leq \downarrow_Q j_q \\
\uparrow_{S} & \text{if } q \not\approx_Q j_q \text{ and } \downarrow_R < r \leq \downarrow_R j_r \\
\downarrow_{S} & \text{if } q \not\approx_Q j_q \text{ and } r \not\approx_R j_r
\end{cases}
\]
and we are finally finished. □

6 Reduced undirected graphs and De Morgan algebras

So far, our main result amounts to a categorical equivalence:

\[
\text{binary relations} \quad \approx \quad \text{finite join-semilattices}
\]

where isomorphism classes of \textit{reduced} relations correspond to isomorphism classes of finite lattices.

We are going to extend this to a categorical equivalence:

\[
\text{undirected graphs} \quad \approx \quad \text{finite de morgan algebras}
\]

where isomorphism classes of \textit{reduced} undirected graphs correspond to isomorphism classes of finite de morgan algebras.

Denoting the latter categories \(UG\) and \(SAI\), respectively, the variety \(SAI\) consists of join-semilattices equipped with a \textbf{S}elf\textbf{-A}djoint \textbf{I}nvolutive morphism i.e. De Morgan algebras where distributivity is not assumed.

Here’s a brief summary of our approach:

- \(UG\) is a category whose objects are pairs of relations \((G, E)\) between finite sets where \(E\) is symmetric and satisfies \(E = G; H\) for some \(H\). The category \(UG_m\) has objects \((G, E)\) where instead \(E = H; G\) for some \(H\).

- There are respective equivalent categories of algebras \(SAJ\) and \(SAM\), i.e. the two different ways of extending \(JSL\) with a single self-adjoint morphism.

- \(SAI\) and \(SAM\) are varieties whose intersection is the variety of De Morgan algebras \(SAI\).

- The corresponding intersection of \(UG\) and \(UG_m\) amounts to \(UG\) i.e. a category whose isomorphism classes are those undirected graphs \((V, E)\) where \(E[v] = E[S] \implies v \in S\).
6.1 Preliminary definitions

**Definition 6.1.1 (Undirected graphs).**

1. A relation \( R \subseteq X \times X \) is **symmetric** if \( R = R^\top \) i.e. it equals its converse relation.

2. An **undirected graph** (or just **graph**) is a pair \((V,E)\) where \( V \) is a possibly-empty finite set and \( E \subseteq V \times V \) is a symmetric relation. Then vertices may have self-loops i.e. \( E(v,v) \) is permissable.

3. An undirected-graph \((V,E)\) is **irreflexive** if \( E \cap \Delta_V = \emptyset \), and **reflexive** if \( \Delta_V \subseteq E \).

4. A **bipartite** undirected graph \((V,E)\) satisfies:

\[
E = E|_{U \times U} \cup E|_{U^\top \times U}
\]

for some subset \( U \subseteq V \).

Then the set \( \{U,U^\top\} \) is called a **bipartition for** \( E \) and the pair \((U,U^\top)\) is called an **ordered bipartition for** \( E \).

**Example 6.1.2 (Visualising undirected-graphs).** Here are 3 examples, depicted classically and as a typed relation.

| undirected graph | symmetric relation \( E \subseteq V \times V \) | binary relation |
|------------------|-----------------------------------------------|-----------------|
| \[\begin{array}{c}
  x & \rightarrow & y & \rightarrow & z \\
  y & \downarrow & & & \uparrow \\
  z & \downarrow & & & \uparrow \\
  \end{array}\] | \{(x,x),(x,y),(y,x),(z,z)\}  \\
where \( V = \{x,y,z\} \) | \[\begin{array}{ccc}
  x & \rightarrow & y & \rightarrow & z \\
  \end{array}\] |
| \[\begin{array}{c}
  x & \downarrow & y & \downarrow & z \\
  \end{array}\] | \{(u,v) \in V \times V: u \neq v\}  \\
where \( V = \{x,y,z\} \) | \[\begin{array}{c}
  \uparrow & \rightarrow & \downarrow \\
  x & \rightarrow & y & \rightarrow & z \\
  \end{array}\] |
| \[\begin{array}{c}
  x_0 & \rightarrow & x_1 & \leftarrow & x_2 \\
  x_3 & \rightarrow & x_4 \\
  x_5 & \leftarrow & x_4 \\
  \end{array}\] | \{(x_i,x_j) \in V \times V: j = i \pm 1 \text{ (mod 6)}\}  \\
where \( V = \{x_i: 0 \leq i < 6\} \) | \[\begin{array}{cccc}
  x_0 & \rightarrow & x_1 & \leftarrow & x_2 \\
  \uparrow & \rightarrow & \downarrow & \rightarrow & \downarrow \\
  x_3 & \rightarrow & x_4 & \rightarrow & x_5 \\
  \end{array}\] |

The 2nd and 3rd examples are irreflexive and the latter is also bipartite as witnessed by the bipartition:

\( \{(x_0,x_2,x_4),\{x_1,x_3,x_5\}\} \).

Finally, observe that every bipartite graph is irreflexive.

**Definition 6.1.3 (Basic graph-theoretic notions).** Let \((V,E)\) be a graph.

1. A graph \((S,E_0)\) is a **subgraph of** \((V,E)\) if \( S \subseteq V \) and \( E_0 \subseteq E \), in which case we write \((S,E_0) \subseteq (V,E)\). Such a subgraph is **induced** if \( E_0 = E|_{S \times S} \).

2. Given any finite sets \( X \) and \( Y \) we define specific sets constructed from them.

   (a) \( K_u(X,Y) := X \times Y \cup Y \times X \) is called an **undirected biclique**.

   (b) If \( X \cap Y = \emptyset \) then \( K_u(X,Y) \) is called a **reflexive undirected biclique**.

   (c) \( K_r(X) := X \times X \) is called a **reflexive undirected clique**.

   (d) \( K_i(X) := (X \times X) \setminus \Delta_X \) is called an **irreflexive undirected clique**.

Irreflexive undirected bicliques and irreflexive undirected cliques are the standard notion of ‘biclique’ and ‘clique’ in an undirected graph without self-loops. Since we permit self-loops we have additional concepts.

3. The equivalence classes \( S \subseteq V \) of the reflexive transitive closure of \( E \) are the **connected components**. A graph is **connected** if it has precisely one connected component, and thus cannot be empty.
4. The *neighbourhood function* \( N_\mathcal{E} : V \to \mathcal{P}V \) is defined \( N_\mathcal{E}(v) := \mathcal{E}[v] \), and the *degree function* \( \deg_\mathcal{E} : V \to \mathbb{N} \) is defined \( \deg_\mathcal{E}(v) := |N_\mathcal{E}(v)| \). Then for each vertex \( v \in V \), its *neighbourhood* is \( N_\mathcal{E}(v) \) and its *degree* is \( \deg_\mathcal{E}(v) \).

5. A graph’s associated *adjacency matrix* \( \text{Adj}(V, \mathcal{E}) \) is the function \( f : V \times V \to 2 \) where \( 2 := \{0, 1\} \) and:
\[
f := \lambda(u, v) \in V \times V. \mathcal{E}(u, v) ? 1 : 0.
\]
It is usually depicted as a \(|V| \times |V|\) binary matrix whose rows and columns are indexed by \( V \).

6. There are also some important special types of graphs.

   (a) A *path* is a connected irreflexive graph \((V, \mathcal{E})\) with \( v_0 \neq v_1 \in V \) such that \( \deg_\mathcal{E} = \lambda v \in V. v \in \{v_0, v_1\} \equiv 1 : 2 \).

   This is the usual notion of a path with at least two vertices and no repetitions. Its *length* is the number of edges \( |\mathcal{E}| = |V| - 1 \). Finally, for each \( n \geq 2 \) we have the path \( P_n := (\{0, 1, \ldots, n - 1\}, \mathcal{E}) \) where \( \mathcal{E}(i, j) : \iff j = i \pm 1 \), so that \( P_n \) has length \( n - 1 \).

   (b) A *cycle* is a connected irreflexive graph \((V, \mathcal{E})\) where every vertex has degree 2.

   This corresponds to the usual notion of a cycle with at least three distinct vertices. Its *length* is the number of edges \( |\mathcal{E}| = |V| \). A cycle is *odd* if its length is, otherwise it is *even*. For each \( n \geq 3 \) we have the cycle \( C_n := (\{0, 1, 2, \ldots, n - 1\}, \mathcal{E}) \) where \( \mathcal{E}(i, j) : \iff j = i \pm 1 \) (mod \( n \)), so that \( C_n \) has length \( n \).

Recall the standard notion of morphism between undirected graphs.

**Definition 6.1.4** (Graph morphisms and isomorphisms).

1. Given graphs \((V_1, \mathcal{E}_1)\) for \( i = 1, 2 \), an *undirected graph morphism* (or *graph morphism*) \( f : (V_1, \mathcal{E}_1) \to (V_2, \mathcal{E}_2) \) is a function \( f : V_1 \to V_2 \) such that \( \mathcal{E}_1; f \subseteq f; \mathcal{E}_2 \), or equivalently: \( \mathcal{E}_1(v_1, v_2) \Rightarrow \mathcal{E}_2(f(v_1), f(v_2)) \) for all \( v_1, v_2 \in V \).

2. A *graph isomorphism* is a graph morphism \( f : (V_1, \mathcal{E}_1) \to (V_2, \mathcal{E}_2) \) such that \( f : V_1 \to V_2 \) is bijective and satisfies \( \mathcal{E}_1; f = f; \mathcal{E}_2 \). Equivalently, \( f \) is bijective and \( \mathcal{E}_1(v_1, v_2) \iff \mathcal{E}_2(f(v_1), f(v_2)) \) for all \( v_1, v_2 \in V \).

Now for our first non-standard concept.

**Definition 6.1.5** (Reduced undirected graph). \((V, \mathcal{E})\) is *reduced* if its edge-relation is reduced i.e.
\[
\forall v \in V, S \subseteq V. \mathcal{E}[v] = \mathcal{E}[S] \Rightarrow v \in S.
\]
This coincides with the previous notion because \( \mathcal{E} \) is symmetric.

**Example 6.1.6** (Reduced graphs).

1. The complete graph \( K_V = (V, \Delta_V) \) is reduced iff \(|V| \neq 1\). The case \(|V| = 0\) is the empty graph and is reduced. If \(|V| \geq 2\) then the neighbourhoods \( \{v : v \in V\} \) are not unions of others. The case \(|V| = 1\) is an isolated point and is not reduced.

2. The complete bipartite graph \( K_{X,Y} \) is reduced iff \(|X|, |Y| \leq 1\) i.e. if it is the empty graph or a single-edge. In all other cases we have two distinct vertices with the same neighbourhood.

3. For each finite set \( V \), the reflexive graph \((V, V \times V)\) is reduced iff \(|V| \leq 1\). That is, the only reduced examples are the empty graph and a single self-loop.

4. The 0-regular graphs are disjoint unions of isolated vertices, and only the empty disjoint union is reduced i.e. the empty graph.

5. The 1-regular graphs are disjoint unions of self-loops and single-edges. They are all reduced, and correspond to the finite boolean De Morgan algebras (see below).
6. The 2-regular graphs are disjoint unions of cycles \((C_n)_{n\geq 3}\) (see Definition 6.1.3) and also paths \((P_n)_{n\geq 2}\) with additional self-loops at each distinct end. There are precisely two connected 2-regular graphs which are not reduced i.e.

\[
\begin{array}{c}
\begin{tabular}{c}
\text{\includegraphics{path.png}}
\end{tabular}
\end{array}
\quad \text{and} \quad \begin{array}{c}
\begin{tabular}{c}
\text{\includegraphics{cycle.png}}
\end{tabular}
\end{array}
\]

That is, \(C_4\) is not reduced because diagonally opposite elements have the same neighbourhoods, and the two vertices of the other graph have the same neighbourhood. We have already seen the two smallest non-empty reduced 2-regular graphs:

\[
\begin{array}{c}
\begin{tabular}{c}
\text{\includegraphics{example.png}}
\end{tabular}
\end{array}
\]

Their corresponding De Morgan algebras have the same lattice structure \((M_3)\) yet with different involutions, see Example ?? below.

7. Let us restrict to irreflexive graphs (forbidding self-loops) and fix \(m \geq 2\). Then an \(m\)-regular irreflexive graph \(G\) is reduced iff the complete bipartite graph \(K_{2,m}\) is not an induced subgraph. Indeed, the neighbourhoods are \(m\)-element sets, so the graph can only fail to be reduced if two distinct vertices have the same neighbourhood.

- \(K_{m,m}\) is an \(m\)-regular irreflexive graph which is not reduced, the case \(m = 2\) yields \(K_{2,2} \cong C_4\) as above.
- Since \(K_{2,m}\) has a 4-cycle, any \(m\)-regular graph with strictly greater \(girth\) is reduced. For example, the girth of the 3-regular Petersen graph is known to be 5.

8. Recall that \(P_n\) denotes the path with \(n \geq 0\) edges. A path \(P_n\) is reduced iff \(n = 1\) or \(n \geq 3\). That is, the only non-reduced paths are the isolated point and the path with two edges and three vertices. In the latter case, the two endpoints have the same neighbourhood i.e. the central point.

Here is another non-standard notion.

**Definition 6.1.7 (Self-dual bipartite graphs).**

A connected bipartite graph \((V,E)\) is **self-dual** if there exists a graph isomorphism \(\theta : (V,E) \to (V,E)\) such that \(\theta[U] = \overline{U}\), where \(\{U,\overline{U}\}\) is its unique bipartition. Then an arbitrary bipartite graph is **self-dual** if all of its connected components are.

**Example 6.1.8 (Self-dual bipartite graphs).**

1. Every even cycle \(C_{2n}\) is a connected bipartite graph and is also self-dual via the graph isomorphism \(\theta(i) := i + 1 \pmod{2n}\).

2. Although every path \(P_n\) is a connected bipartite graph, it is self-dual iff \(n\) is even. Indeed, the only non-identity graph isomorphism \(\theta : P_n \to P_n\) has action \(\theta(i) = (n - 1) - i\), so that \(\theta\) switches the parity iff \(n - 1\) is odd iff \(n\) is even.

3. Any graph \((V,\emptyset)\) with no edges is a bipartite graph. If \(V = \emptyset\) then it is self-dual because it has no connected components. However if \(V \neq \emptyset\) then it is not self-dual: each connected component \(\{\ast\},\emptyset\) has unique bipartition \(\{\ast\},\emptyset\) and we cannot have \(f[\emptyset] = \{\ast\}\).

4. Consider the following connected bipartite graph:

\[
\begin{array}{c}
\begin{tabular}{c}
\text{\includegraphics{example2.png}}
\end{tabular}
\end{array}
\]

It is self-dual and has two distinct witnessing graph isomorphisms. The first reflects along the centered horizontal axis, whereas the second additionally reflects along the centered vertical axis.
5. Later on we'll see that the ‘self-dual bipartite graphs’ corresponds to the symmetric finite lattices i.e. one which is isomorphic to its order-dual. For example, we derived the previous example from the lattice $\mathcal{L}$:

![Diagram](image)

which has two distinct automorphisms.

Later we'll also need **polarities** which are defined in terms of the operators $(-)^{\dagger}$ and $(-)^{\ddagger}$.

**Definition 6.1.9** (Polarities). Each relation $R \subseteq R_s \times R_t$ between finite sets yields functions:

\[
R^\ddagger := \neg R_s \circ \check{\check{R}} : \mathcal{P} R_s \to \mathcal{P} R_t \quad R^\dagger := \check{\check{R}} \circ \neg R_t : \mathcal{P} R_t \to \mathcal{P} R_s
\]

recalling that $\check{\check{R}} \subseteq R_s \times R_t$ is the complement relation, and $\neg : \mathcal{P} X \to \mathcal{P} X$ constructs the relative complement. We refer to these functions as the **polarities** of $R$.

The polarity $(-)^\ddagger$ is a ‘De Morgan dual’ of $(-)^\dagger$ involving the complement relation rather than the converse. The up/down arrows are not flipped because the complement of a relation does not alter its type. We now show that polarities correspond to the classical concept, and prove some basic related equalities.

**Lemma 6.1.10** (Basic properties of polarities).

Let $R \subseteq R_s \times R_t$ be any relation between finite sets.

1. **The mappings $(-)^\ddagger$ and $(-)^\dagger$ correspond to the ‘standard polarities’:**

\[
R^\ddagger (X) = \bigcap_{x \in X} R[x] = \{ y \in R_t : \forall x \in X. R(x, y) \} \\
R^\dagger (Y) = \bigcap_{y \in Y} \check{R} [y] = \{ x \in R_s : \forall y \in Y. R(x, y) \}
\]

2. **The polarities define adjoint join-semilattice morphisms:**

\[
R^\ddagger = (R^\dagger)_* : \mathcal{P} R_s \to (\mathcal{P} R_t)^{\text{op}} \quad R^\dagger = (R^\ddagger)_* : \mathcal{P} R_t \to (\mathcal{P} R_s)^{\text{op}}
\]

$Y \subseteq R^\ddagger (X) \iff R^\dagger (X) \leq (\mathcal{P} R_s)^{\text{op}} Y \iff X \leq \mathcal{P} R_t \check{R}^\dagger (Y) \iff X \subseteq R^\dagger (Y)$

for all $X \subseteq R_s$, $Y \subseteq R_t$. Then both $R^\ddagger$ and $R^\dagger$ send arbitrary unions to intersections.

3. **We have the equalities:**

\[
R^\ddagger \circ R^\dagger = \text{cl}_{\check{\check{R}}} = \lambda X \subseteq R_s. \{ r_s \in R_s : \bigcap_{x \in X} R[x] \subseteq R[r_s] \} \\
R^\dagger \circ R^\ddagger = \text{cl}_{\check{R}} = \lambda Y \subseteq R_t. \{ r_t \in R_t : \bigcap_{y \in Y} \check{R} [y] \subseteq \check{R}[r_t] \}
\]

**Proof:**

1. **We calculate:**

\[
R^\ddagger (X) = \neg R_s \circ \check{\check{R}} (X) = \bigcup_{x \in X} \check{R}[x] = \bigcap_{x \in X} R[x]
\]

since $\check{R}[x] = \check{R}[x]$. Furthermore:

\[
R^\dagger (Y) = \check{\check{R}} \circ \neg R_t (Y) \\
= \neg R_t \circ (\check{R}^\dagger)^* (Y) \quad \text{by DeMorgan duality} \\
= \check{R}^\dagger (Y) \quad \text{by definition} \\
= \bigcap_{y \in Y} \check{R} [y] \quad \text{by previous equality}
\]
2. The polarities actually define composite join-semilattice morphisms:

\[
\mathcal{R}^\downarrow = \mathcal{P}G_s \xrightarrow{\neg} \mathcal{P}G_t \xrightarrow{\neg} (\mathcal{P}G_t)^{\text{op}} \quad \mathcal{R}^\uparrow = \mathcal{P}G_t \xrightarrow{\neg} \mathcal{P}G_t \xrightarrow{\neg} (\mathcal{P}G_s)^{\text{op}}
\]

They are adjoint because \((\mathcal{R}^\downarrow, \mathcal{R}^\uparrow)\) are adjoint by Lemma 4.1.7, and each \(\neg^{\text{op}} : \mathcal{P}X \to (\mathcal{P}X)^{\text{op}}\) is self-adjoint.

3. That \(\mathcal{R}^\parallel \circ \mathcal{R}^\parallel = \text{cl}_{\mathcal{R}}\) follows because \(\neg_{\mathcal{R}_t}\) is involutive, and the second description follows by (1). Finally:

\[
\mathcal{R}^\parallel \circ \mathcal{R}^\parallel = \neg_{\mathcal{R}_t} \circ \mathcal{R}^\parallel \circ \mathcal{R}^\parallel \circ \neg_{\mathcal{R}_t} = (\neg_{\mathcal{R}_t})^4 \circ (\neg_{\mathcal{R}_t})^1 = \text{cl}_{\mathcal{R}} \quad \text{by definition}
\]

and the second description again follows by (1).

6.2 The Varieties SAJ, SAM and SAI

We will soon define three varieties (equationally-defined classes of algebras) extending JSL.

- the finite algebras of SAJ amount to a finite join-semilattice \(Q\) with a self-adjoint morphism \(Q \to Q^{\text{op}}\).
- the finite algebras of SAM amount to a finite join-semilattice \(Q\) with a self-adjoint morphism \(Q^{\text{op}} \to Q\).
- SAI = SAJ ∩ SAM is essentially the variety of De Morgan algebras i.e. bounded lattices equipped with an involutive endofunction satisfying the De Morgan laws.

**Definition 6.2.1** (The three varieties SAJ, SAM and SAI). In each case we’ll extend JSL’s signature \(\{ \bot : 0, \lor : 2\}\) with a unary operation \(\sigma\) satisfying the equation:

\[(\text{Rev}\sigma) \quad \sigma(x \lor y) \preceq \sigma(x)\]

where \(\phi \preceq \psi\) is syntactic sugar for the equation \(\phi \lor \psi \equiv \psi\).

1. **SAJ** extends JSL with a single unary operation satisfying (Rev\(\sigma\)) and:

\[(\text{Ex}\sigma^2) \quad x \preceq \sigma\sigma(x)\]

where ‘Ex’ stands for **extensive**.

2. **SAM** extends JSL with a single unary operation satisfying (Rev\(\sigma\)) and:

\[(\text{Cx}\sigma^2) \quad \sigma\sigma(x) \preceq x\]

where ‘Cx’ stands for **co-extensive**.

3. **SAI** extends JSL with a single unary operation satisfying (Rev\(\sigma\)) and:

\[(\text{Inv}\sigma) \quad \sigma\sigma(x) \approx x\]

where ‘Inv’ stands for **involutive**.

We view them as categories in the usual sense: the objects are the algebras and a morphism \(f : (Q_1, \sigma_1) \to (Q_2, \sigma_2)\) is function \(f : Q_1 \to Q_2\) which preserves the three basic operations. Equivalently, \(f\) defines a JSL-morphism \(Q_1 \to Q_2\) such that \(f(\sigma_1(q)) = \sigma_2(f(q))\) for every \(q \in Q\).

These three varieties are related to one another as follows.

\(^2\text{(Rev) stands for order-reversing because it is equivalent to the rule } x \preceq \preceq y \Rightarrow \sigma(y) \preceq \sigma(x).\)
Lemma 6.2.2 (Basic observations concerning SAJ, SAM and SAI).

1. \((Q, \sigma) \in SAJ\) iff \((Q^{op}, \sigma) \in SAM\).

2. \(SAI = SAJ \cap SAM\).

3. In SAJ, SAM and SAI the equation \(\sigma \sigma(x) \approx \sigma(x)\) holds.

4. The equation \(x \leq \sigma(\bot)\) holds in SAJ but not SAM. The equation \(\sigma(\bot) \leq x\) holds in SAM but not SAJ.

5. \((Q, \sigma) \in SAJ\) iff \(\sigma \circ \sigma\) defines a closure operator on \((Q, \leq_Q)\). Similarly, \((Q, \sigma) \in SAM\) iff \(\sigma \circ \sigma\) defines an interior operator on \((Q, \leq_Q)\).

Proof.

1. \((Ex^2)\) is the order-dual of \((Cx^2)\).

2. \((Inv)\) holds iff both \((Ex^2)\) and \((Cx^2\)) hold.

3. In SAJ we can apply \((Rev)\) to \((Ex^2)\) to deduce that \(\sigma(\sigma(\sigma(x)) \leq \sigma(x)\), whereas \(\sigma(x) \leq \sigma(\sigma(\sigma(x)))\) arises from \((Ex^2)\) and the substitution rule. We have the order-dual argument in SAM, and in SAI we apply substitution to \((Inv)\).

4. In SAJ we have \(x \leq \sigma(\sigma(x)\) by \((Ex^2)\) and applying \((Rev)\) to \(\bot \leq \sigma(x)\) yields \(\sigma(\sigma(x)) \leq \sigma(\bot)\), so that \(x \leq \sigma(\bot)\). This fails in SAM e.g. take any \(Q \in JSL\) with at least two elements and define \(\sigma := \lambda q \in Q \cdot \bot_Q\). Finally, in SAM applying \((Rev)\) twice yields \(\sigma(\bot) \leq \sigma(x)\) and applying \((Inv)\) yields \(\sigma(x) \leq x\). This fails in SAJ e.g. take any join-semilattice \(Q\) with a distinct bottom and top element and define \(\sigma := \lambda q \in Q \cdot \top_Q\).

5. Given \((Q, \sigma) \in SAJ\) then certainly \(x \leq_Q \sigma(\sigma(x))\) holds by \((Ex^2)\) i.e. \(\sigma \circ \sigma\) is extensive. Monotonicity follows by applying \((Rev)\) twice (viewed as a rule), whereas idempotence follows using \(\sigma(\sigma(x)) \approx \sigma(x)\) from (3). The proof for SAM is completely analogous.

\(\square\)

Example 6.2.3 (BA forms a full subcategory of SAI). Observe that every possibly infinite boolean algebra \(\mathbb{A}\) arises as an SAI-algebra ((\(\mathbb{A}, \wedge, \vee, \bot, \top\)), \(\neg\)), and moreover the SAI-morphisms between such algebras are precisely the boolean algebra morphisms. That is, BA forms a full subcategory of SAI, and also of SAJ and SAM.

Example 6.2.4 (Characterising algebras built on a finite boolean join-semilattice). Let \(Z\) be a finite set.

1. \((PZ, \sigma) \in SAI\) iff \(\sigma = \neg_Z \circ \theta^\dagger\) for some involutive function \(\theta : Z \rightarrow Z\).

These algebras are well-defined i.e. \((Rev)\) holds because \(\theta^\dagger\) preserves the inclusion-ordering and \(\neg_Z\) flips it, whereas \((Inv)\) holds because:

\[
\sigma \circ \sigma = \neg_Z \circ \theta^\dagger \circ \neg_Z \circ \theta^\dagger = \hat{\theta}^\dagger \circ \neg_Z \circ \neg_Z \circ \theta^\dagger = \hat{\theta} \circ \theta^\dagger = \theta^{-1} \circ \theta \quad \text{by De Morgan duality}
\]

\[
= \hat{\theta} \circ \theta^\dagger \quad \text{since \(\theta\) bijective}
\]

\[
= \hat{id}_{PZ}.
\]

Conversely, take any SAI-algebra \((PZ, \sigma)\). By the characterisation in Lemma 6.4.3 further below:

\(\sigma\) defines a self-adjoint JSL-isomorphism \(PZ \rightarrow (PZ)^{op}\).

The JSL-morphisms of type \(PZ \rightarrow PZ\) are precisely the functions \(R^1\) where \(R \subseteq Z \times Z\) is an arbitrary relation. Recalling the self-inverse JSL-isomorphism \(\neg_Z : PZ \rightarrow (PZ)^{op}\), the JSL-morphisms of type \(PZ \rightarrow (PZ)^{op}\) are precisely the functions \(\neg_Z \circ R^1\). Thus \(\sigma = \neg_Z \circ R^1\) where \(R^1\) is bijective because \(\sigma\) and \(\neg_Z\) are, so that \(R\) is a bijective function (each singleton must be seen). Since \(\neg_Z\) is self-adjoint,

\[
\sigma_* = (\neg_Z \circ R^1)_* = R^1 \circ (\neg_Z)_* = R^1 \circ \neg_Z \quad \text{so that}\]

\[
R^1 \circ \neg_Z = \neg_Z \circ R^1.
\]

Thus \(\neg_Z \circ R^1 = R^1\) by De Morgan duality, so \(R = \hat{R}\). Then \(R\) is a self-inverse bijection i.e. an involutive function \(\theta : Z \rightarrow Z\).
2. \((PZ, \sigma) \in \text{SAJ}\) iff \(\sigma = \neg_Z \circ R^!\) for some symmetric relation \(R \subseteq Z \times Z\).

(\(\text{Rev}\sigma\)) is satisfied because \(R^!\) preserves the inclusion-ordering and \(\neg_Z\) flips it. As for (\(\text{Ex}\sigma^2\)),
\[
\neg_Z \circ R^! \circ \neg_Z \circ R^! = R^! \circ \neg_Z \circ R^! = R^! \circ R^! = \text{cl}_R
\]

using De Morgan duality and symmetry, which suffices because closure operators are extensive. Conversely, take any \(\text{SAJ}\)-algebra \((PZ, \sigma)\). By the characterization in Lemma 6.4.1.1 below:
\[
\sigma \text{ defines a self-adjoint } \text{JSL-morphism } PZ \rightarrow (PZ)^{\text{op}}.
\]

Repeating the reasoning in the previous example, we know that \(\sigma = \neg_Z \circ R^!\) for some relation \(R \subseteq Z \times Z\). Then by \(\sigma\)'s self-adjointness we deduce that \(R^! \circ \neg_Z = \neg_Z \circ R^!\) and thus \(R^! = \text{R}^!\) by De Morgan duality, so that \(R\) is symmetric as required.

3. \((PZ, \sigma) \in \text{SAM}\) iff \(\sigma = R^! \circ \neg Z\) for some symmetric relation \(R \subseteq Z \times Z\).

This follows by the previous example and \((PZ, \sigma) \in \text{SAM} \iff ((PZ)^{\text{op}}, \sigma) \in \text{SAJ}\). In more detail, the latter \(\text{SAJ}\)-algebras necessarily take the form:
\[
\sigma = (PZ)^{\text{op}} \xrightarrow{\sigma_0} PZ \xrightarrow{\sigma} (PZ)^{\text{op}} \xrightarrow{\sigma} PZ
\]

where \((PZ, \sigma_0)\) is a \(\text{SAJ}\)-algebra. Then we immediately deduce that:
\[
\sigma = \neg_Z \circ (\neg_Z \circ R^!) = R^! \circ \neg_Z
\]

where \(R\) is symmetric, and every symmetric relation is permissible.

4. We explain how the above \(\text{SAI}\)-algebras correspond to undirected graphs i.e. \((PZ, \sigma) \in \text{SAI}\) where \(\sigma = \neg_Z \circ \theta^!\) for some involutive function \(\theta : Z \rightarrow Z\). The ‘equivalent’ undirected graph is the relation \(\text{Pirr} \sigma \subseteq J(PZ) \times J(PZ)\).

It is symmetric because:
\[
\text{Pirr}\sigma(\{z_1\}, \{z_2\}) \iff \sigma(\{z_1\}) \not\subseteq \{z_2\} \iff \{z_2\} \not\subseteq \neg Z \circ \theta^!(\{z_1\}) \iff z_2 \notin \theta[z_1] \iff z_2 \not\in \theta(z_1)
\]

and \(\theta\) is an involutive function i.e. a functional relation which is bijective and symmetric. Here are three examples of these undirected graphs:

That is, they are disjoint unions of self-loops and single-edge-paths. Note that \((PZ, \neg_Z)\) corresponds to the graph consisting of \(|Z|\) self-loops i.e. \(\Delta_{J(PZ)}\).

5. We’ll characterise the \textit{distributive} \(\text{SAJ}_f\), \(\text{SAM}_f\) and \(\text{SAI}_f\)-algebras in Theorem 3.0.2 further below. The respective undirected graphs \((V, E)\) are precisely those satisfying \(E = \theta; \preceq_P\) for some finite poset \(P\) and involutive order-isomorphism \(\theta : P \rightarrow P^{\text{op}}\).

\[\Box\]

6.3 Adjointness and self-adjointness

Before reinterpreting the above finite algebras, let us first clarify the notion of adjoint morphism. Recall that adjoint \(\text{JSL}_f\)-morphisms \(0\text{D}_f f^{\text{op}} := f^\ast\) arise via the action of the self-duality functor \(0\text{D}_f : \text{JSL}_f^{\text{op}} \rightarrow \text{JSL}_f\) from Theorem 3.0.2. We’ll also consider the infinite case. The notion of \textit{self-adjoint morphism} in \(\text{JSL}_f\) and \(\text{Dep}\) will also be defined and compared.

\textbf{Definition 6.3.1} (Adjoints of \(\text{JSL}\)-morphisms between possibly infinite algebras).
Given a JSL-morphism \( f : Q \to R \) where \( Q \) and \( R \) define bounded lattices (so that \( Q^{\text{op}} \) and \( R^{\text{op}} \) are well-defined), then we say that \( f \) has an adjoint if there exists a JSL-morphism \( g : R^{\text{op}} \to Q^{\text{op}} \) such that:

\[
 f(q) \leq_R r \iff q \leq_Q g(r) \quad \text{for all } q \in Q \text{ and } r \in R.
\]

We also say that \( f \) has adjoint \( g \).

**Lemma 6.3.2.** If \( f : Q \to R \) is a JSL-morphism between bounded lattices then t.f.a.e.

(a) \( f \) has an adjoint.

(b) The function \( f_* : = \lambda r \in R. \bigvee_Q \{ q \in Q : f(q) \leq_R r \} : R \to Q \) is well-defined.

(c) \( f \) has the unique adjoint \( f_* : R^{\text{op}} \to Q^{\text{op}}. \)

**Proof.**

- (a \( \implies \) b): Suppose \( f \) has an adjoint i.e. we have a JSL-morphism \( g : R^{\text{op}} \to Q^{\text{op}} \) satisfying the adjoint relationship between the two orderings. Then for all \( r \in R \) we have:

\[
 g(r) = \bigvee_R \{ q \in Q : q \leq_Q g(r) \} = \bigvee_R \{ q \in Q : f(q) \leq_R r \} = f_*(r)
\]

using the adjoint relationship, so \( f_* = g \) is a well-defined function.

- (b \( \implies \) a): Given \( f_* \) is a well-defined function we first establish \( \forall q \in Q, r \in R. \ (f(q) \leq_R r \iff q \leq_Q f_*(r)) \). The implication \( \Rightarrow \) is immediate. Conversely if \( q \leq_Q f_*(r) \) then by monotonicity and join-preservation:

\[
 f(q) \leq_R f(f_*(r)) = f(\bigvee_Q \{ q_0 \in Q : f(q_0) \leq_R r \}) = \bigvee_Q \{ f(q_0) : q_0 \in Q, f(q_0) \leq_R r \} \leq_R r
\]

as required. To see that \( f_* \) defines a JSL-morphism of type \( R^{\text{op}} \to Q^{\text{op}} \), observe \( f_*(\tau_R) = \bigvee_Q Q = \tau_Q \) and:

\[
 f_*(r_1 \land_R r_2) = \bigvee_Q \{ q \in Q : f(q) \leq_R r_1 \land_R r_2 \}
\]

- (a \( \iff \) c): Inspecting the proof of (a \( \implies \) b) we see that if \( f \) has adjoint \( g \) then \( g = f_* \) and hence is unique. The converse is immediate.

**Definition 6.3.3 (Self-adjoint morphisms in JSL\(_f\) and Dep).**

A JSL\(_f\)-morphism \( f : Q \to R \) is self-adjoint if \( f = f_* \), so we must have \( R = Q^{\text{op}} \). Likewise, a Dep-morphism \( R : \mathcal{G} \to \mathcal{H} \) is self-adjoint if \( R^\wedge = R \), which means precisely that \( R \) is a symmetric relation, and also \( \mathcal{H} = \check{\mathcal{G}} \).

These two concepts are two sides of the same coin.

**Lemma 6.3.4 (Self-adjointness in Dep and JSL\(_f\)).**

1. A Dep-morphism \( R : \mathcal{G} \to \mathcal{H} \) is self-adjoint iff \( \mathcal{H} = \check{\mathcal{G}} \) and \( R = \check{R} \).

2. Given any Dep-morphism \( R : \mathcal{G} \to \check{\mathcal{G}} \) t.f.a.e.

   a. \( R : \mathcal{G} \to \check{\mathcal{G}} \) is self-adjoint.
   b. \( R = R^\wedge \).
   c. \( \partial_{\check{\mathcal{G}}}^{-1} \circ \text{Open}_R \) is a self-adjoint JSL\(_f\)-morphism.
   d. \( \text{Open}_R \circ \partial_{\mathcal{G}}^{-1} \) is a self-adjoint JSL\(_f\)-morphism.

3. Given any JSL\(_f\)-morphism \( f : Q \to Q^{\text{op}} \) t.f.a.e.
(a) \( f \) is self-adjoint.
(b) \( \text{Nleq} \) is a self-adjoint \( \text{Dep} \)-morphism.
(c) \( \text{Pirrf} \) is a self-adjoint \( \text{Dep} \)-morphism.
(d) \( (\text{Pirrf})_+ = (\text{Pirrf})_+ = \{(j, m) \in J(Q) \times M(Q) : f(j) \leq Q m\}. \)

Proof.

1. A \( \text{Dep} \)-morphism \( \mathcal{R} : \mathcal{G} \to \mathcal{H} \) is self-adjoint if \( \mathcal{R} = \mathcal{R}^\vee \) recalling that \( (-)^\vee : \text{Dep}^{op} \to \text{Dep} \) is the self-duality functor. Since \( \mathcal{R}^\vee = \mathcal{R} : \mathcal{H} \to \tilde{\mathcal{G}} \), this holds iff \( \mathcal{R} = \tilde{\mathcal{R}} \) and \( \mathcal{G} = \mathcal{H} \), in which case \( \tilde{\mathcal{G}} = \mathcal{H} \) follows.

2. \( (a \iff b) \): If \( \mathcal{R} \) is self-adjoint then recall that \( \mathcal{R}_- = (\mathcal{R}^\vee)_+ \) holds generally. Conversely, recall the associated components always satisfy \( \mathcal{R}_- : \tilde{\mathcal{G}} = \mathcal{R} = \mathcal{G} ; \mathcal{R}^\vee_+ \), so that if \( \mathcal{R}_- = \mathcal{R}_+ \) we deduce \( \mathcal{R} = \tilde{\mathcal{R}} \).

\( (a \iff c) \): Given any \( \text{Dep} \)-morphism \( \mathcal{R} : \mathcal{G} \to \tilde{\mathcal{G}} \) which is self-adjoint i.e. \( \mathcal{R} = \tilde{\mathcal{R}} \), then:

\[
(\partial_{\tilde{\mathcal{G}}}^{-1} \circ \text{Open} R)_* = (\text{Open} R)_* \circ (\partial_{\mathcal{G}}^{-1})_* \]
\[
= (\text{Open} R)_* \circ \partial_{\mathcal{G}}^{-1} \]
\[
= \partial_{\mathcal{G}}^{-1} \circ \text{Open} R \circ \partial_{\mathcal{G}} \circ \partial_{\mathcal{G}}^{-1} \]
\[
= \partial_{\mathcal{G}}^{-1} \circ \text{Open} R \]
\[
\implies \partial_{\mathcal{G}}^{-1} \circ \text{Open} R = \partial_{\mathcal{G}}^{-1} \circ \text{Open} R^\vee \]  

Conversely, if \( \partial_{\mathcal{G}}^{-1} \circ \text{Open} R \) is self-adjoint we can reuse the above calculation to deduce that:

\[
\partial_{\mathcal{G}}^{-1} \circ \text{Open} R = \partial_{\mathcal{G}}^{-1} \circ \text{Open} R^\vee .
\]

Cancelling the isomorphism yields \( \text{Open} R = \text{Open} R^\vee \) and hence \( \mathcal{R} = \mathcal{R}^\vee \) by faithfulness.

\( (a \iff d) \): Again suppose that \( \mathcal{R} = \tilde{\mathcal{R}} \) and calculate:

\[
(\text{Open} R \circ \partial_{\mathcal{G}})_* = (\partial_{\mathcal{G}})_* \circ (\text{Open} R)_* \]
\[
= \partial_{\mathcal{G}} \circ (\text{Open} R)_* \]
\[
= \partial_{\mathcal{G}} \circ \partial_{\mathcal{G}}^{-1} \circ \text{Open} R \circ \partial_{\mathcal{G}} \]
\[
= \text{Open} R \circ \partial_{\mathcal{G}} \]
\[
\implies \text{Open} R = \text{Open} R^\vee \]  

Conversely we deduce as in the previous item that \( \text{Open} R = \text{Open} R^\vee \) so that \( \mathcal{R} = \mathcal{R}^\vee \) by faithfulness.

3. \( (a \iff b) \): Given any self-adjoint \( \text{JSL} \)-morphism \( f : Q \to Q^{op} \), first observe the typing \( \text{Nleq} f : \text{Nleq} Q \to \text{Nleq} Q^{op} = (\text{Nleq} Q)^\vee \). It is a symmetric relation because:

\[
\text{Nleq} f(q_1, q_2) \iff f(q_1) \leq_{Q^{op}} q_2 \quad \text{by definition}
\]
\[
\iff q_1 \leq_{Q} f(q_2) \quad \text{since } f = f^*.
\]
\[
\iff f(q_2) \leq_{Q^{op}} q_1
\]
\[
\implies \text{Nleq} f(q_2, q_1).
\]

Conversely, suppose that \( \text{Nleq} f = (\text{Nleq} f)^\vee \) so that:

\[
q_2 \leq_{Q} f_1(q_1) \iff f_1(q_1) \leq_{Q^{op}} q_2 \iff \overline{\text{Nleq}}_{f_1}(q_1, q_2) \iff \overline{\text{Nleq}} f_2(q_2, q_1) \iff q_1 \leq_{Q} f_2(q_2).
\]

Then it follows that \( f^* = f \) e.g. by Lemma 6.3.2.

\( (b \iff c) \): Follows because we have the natural isomorphism \( \mathcal{E} : \text{Pirrf} \Rightarrow \text{Nleq} \), see Lemma 4.3.1.2.

\( (c \iff d) \): Follows by the equivalence of (2).a and (2).b, observing that \( (\text{Pirrf})_+ (j, m) \iff m \leq_{Q^{op}} f(j) \) holds generally, see Definition 4.2.1.

\[\Box\]

**Note 6.3.5** (Explicit description of the associated component of a self-adjoint morphism). For any self-adjoint \( \text{Dep} \)-morphism \( \mathcal{E} : \mathcal{G} \to \tilde{\mathcal{G}} \) we have \( \mathcal{E}_- = \mathcal{E}_+ \) by the Lemma above. In Lemma 6.5.8 further below we’ll prove a more explicit description i.e. if \( \mathcal{E} : \mathcal{G} \to \tilde{\mathcal{G}} \) is self-adjoint then \( \mathcal{E}_- = \mathcal{E}_+ \colon \mathcal{G} \subseteq \mathcal{G}_+ \times \mathcal{G}_- \).
6.4 Interpreting the finite algebras of the three varieties

We now reinterpret the finite algebras of SAJ and SAM, and also all SAI algebras.

**Lemma 6.4.1 (Interpretation of finite SAJ and SAM algebras).**

Fix any finite join-semilattice \(Q \in JSL_f\) and endofunction \(\sigma : Q \to Q\).

1. \((Q, \sigma) \in SAJ_f\) iff \(\sigma\) defines a self-adjoint \(JSL_f\)-morphism of type \(Q \to Q^{op}\), or equivalently:

\[
q_2 \leq_0 \sigma(q_1) \iff q_1 \leq_0 \sigma(q_2) \quad \text{for all } q_1, q_2 \in Q.
\]

2. \((Q, \sigma) \in SAM_f\) iff \(\sigma\) defines a self-adjoint \(JSL_f\)-morphism of type \(Q^{op} \to Q\), or equivalently:

\[
\sigma(q_1) \leq \sigma(q_2) \iff \sigma(q_2) \leq q_1 \quad \text{for all } q_1, q_2 \in Q.
\]

**Proof.**

1. Given a self-adjoint \(JSL_f\)-morphism \(\sigma : Q \to Q^{op}\) then it certainly defines a monotone morphism \((Q, \leq_0) \to (Q, \geq_0)\), hence \((\text{Rev}\sigma)\) holds. The self-adjoint relationship informs us that:

\[
q_2 \leq_0 \sigma(q_1) \iff \sigma(q_1) \leq_0 q_2 \iff q_1 \leq_0 \sigma(q_2).
\]

Consequently \((\text{Ex}\sigma^2)\) holds, because for every \(x \in Q\) we have \(\sigma(x) \leq_0 \sigma(x) \iff x \leq_0 \sigma(\sigma(x))\). Conversely, suppose we have a function \(\sigma : Q \to Q\) satisfying \((\text{Rev}\sigma)\) and \((\text{Ex}\sigma^2)\). Then \(\forall q_1, q_2 \in Q\) we have:

\[
q_2 \leq_0 \sigma(q_1) \implies \sigma(q_1) \leq_0 q_2 \implies q_1 \leq_0 \sigma(q_2)
\]

using the order-reversing monotonicity of \(\sigma\) and also \(q_1 \leq_0 \sigma(\sigma(q_1))\). Then by symmetry we have:

\[
\sigma(q_1) \leq_0 q_2 \iff q_1 \leq_0 \sigma(q_2)
\]

for all \(q_1, q_2 \in Q\).

By Lemma 2.2.7.2 and the fact that \(Q\) has all finite joins, we deduce that \(\sigma\) defines a \(JSL_f\)-morphism of type \(Q \to Q^{op}\). Finally, the above equivalence informs us that \(\sigma\) is self-adjoint i.e. \(\sigma = \sigma_\ast\).

2. We have \((Q, \sigma) \in SAM_f\) if and only if \((Q^{op}, \sigma) \in SAJ_f\) by Lemma 6.2.2, so apply (1).

\(\square\)

**Note 6.4.2 (Interpretation of infinite SAJ and SAM algebras).**

The above interpretation does not extend to infinite algebras e.g. because \(Q^{op}\) needn’t be a well-defined join-semilattice in the infinite case.

1. Given any join-semilattice \(Q \in JSL\) then we have the SAM-algebra \((Q, \sigma)\) where \(\sigma(q) := \bot_Q\) for all \(q \in Q\). Indeed, \((\text{Rev}\sigma)\) holds trivially for this constant map, as does \((\text{Cex}\sigma^2)\) because \(\sigma(\sigma(x)) = \bot_Q \leq_0 x\). Thus there are SAM-algebras whose join-semilattice has no top and/or fails to have binary meets, see Definition 2.2.1.12.d.

2. Concerning SAJ-algebras \((Q, \sigma)\), it so happens that \(Q\) always has the top element \(\sigma(\bot_Q)\) because:

\[
\bot_Q \leq_0 \sigma(q) \iff q \leq_0 \sigma(\bot_Q)
\]

via the adjoint relationship. However, \(Q\) needn’t have binary meets e.g. let \(Q\) be the join-semilattice depicted in Definition 2.2.1.12.d and define \(\sigma(q) := 1\) for all \(q \in Q\).

\(\blacksquare\)

**Lemma 6.4.3 (Interpretation of arbitrary SAI-algebras).**

1. If \((Q, \sigma) \in SAI\) then \(Q\) is a bounded lattice and \(\sigma\) defines a self-adjoint \(JSL\)-isomorphism \(Q \to Q^{op}\) (hence bounded lattice isomorphism) and also its inverse. Furthermore, \(Q\) has the following meet structure:

\[
\top_Q = \sigma(\bot_Q) \quad q_1 \land_Q q_2 = \sigma(q_1) \lor_Q \sigma(q_2).
\]

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2. Given any $Q \in \text{JSL}$ and function $\sigma : Q \rightarrow Q$ then t.f.a.e.

(a) $(Q, \sigma) \in \text{SAI}$. 
(b) $\sigma$ defines a self-adjoint $\text{JSL}$-isomorphism $Q \rightarrow Q^{\text{op}}$. 
(c) $\sigma$ defines a self-adjoint $\text{JSL}$-morphism of type $Q \rightarrow Q^{\text{op}}$ and $Q^{\text{op}} \rightarrow Q$. 
(d) $\sigma$ is involutive and defines a $\text{JSL}$-morphism of type $Q \rightarrow Q^{\text{op}}$. 

Furthermore, in (b) and (d) one may replace $Q$ with $Q^{\text{op}}$ with $Q^{\text{op}} \rightarrow Q$.

3. Every $\text{SAI}$-morphism defines a bounded lattice morphism.

Proof.

1. Let $(Q, \sigma)$ be an $\text{SAI}$-algebra. Given $q_2 \leq Q \sigma(q_1)$ then applying $\sigma$ yields $q_1 = \sigma(\sigma(q_1)) \leq Q \sigma(q_2)$, so that:

$$
(\ast) \quad \sigma(q_1) \leq Q \sigma(q_2) \iff q_2 \leq Q \sigma(q_1) \iff q_1 \leq Q \sigma(q_2) \quad \text{for all } q_1, q_2 \in Q.
$$

Since $Q$ has all finite joins, we may apply Lemma 2.2.7.2 to deduce that $\sigma(\bigvee Q \ X) = \bigwedge Q \sigma[X]$ for every finite subset $X \subseteq Q$ i.e. these particular meets exist in $Q$. But since $\sigma$ is involutive it is bijective, hence $Q$ has all finite meets. It follows that $Q$ is a bounded lattice and $\sigma$ defines a join-semilattice isomorphism $\sigma : Q \rightarrow Q^{\text{op}}$.

Now, since $\sigma$ is involutive it is bijective and thus a $\text{JSL}$-isomorphism by universal algebra, and also a bounded lattice isomorphism because $Q$ and $Q^{\text{op}}$ are bounded lattices. Again by involutiveness $\sigma^{-1}(q) = \sigma(q)$. Observe that the join-semilattice isomorphism $\sigma : Q \rightarrow Q^{\text{op}}$ between bounded lattices is self-adjoint by $(\ast)$. Concerning $Q$’s meet structure, since $\sigma : Q \rightarrow Q^{\text{op}}$ is a bounded lattice isomorphism we have $\top Q = \sigma(\bot Q)$, and finally:

$$
\sigma(\sigma(q_1) \vee Q \sigma(q_2)) = \sigma(\sigma(q_1)) \land Q \sigma(\sigma(q_2)) = q_1 \land Q q_2.
$$

2. 

- $(a \iff b)$: The implication $\Rightarrow$ follows by (1). Conversely, $(\text{Rever} \sigma)$ follows by taking the underlying monotone map of $\sigma : Q \rightarrow Q^{\text{op}}$ whereas $(\text{Inv} \sigma)$ holds because for every $q \in Q$ we have:

$$
\sigma(q) = \sigma_{\ast}(q) \quad \text{by self-adjointness}
$$

$$
= \bigvee Q \{q' \in Q : \sigma(q') \leq Q q\} \quad \text{by definition}
$$

$$
= \bigvee Q \{q' \in Q : q \leq Q \sigma(q')\}
$$

$$
= \bigvee Q \{q' \in Q : q' \leq Q \sigma^{-1}(q)\} \quad \text{apply isomorphism}
$$

$$
= \sigma^{-1}(q).
$$

- $(a \iff c)$: Regarding $\Rightarrow$, this follows because $a \Rightarrow b$, and the inverse of a self-adjoint isomorphism is itself self-adjoint. Conversely, taking an underlying monotone map yields $(\text{Rever} \sigma)$, and concerning involutiveness we can use the two adjoint relations to deduce that for every $q \in Q$:

$$
\sigma(q) \leq Q \sigma(q) \iff q \leq Q \sigma(\sigma(q)) \quad \text{and} \quad \sigma(q) \leq Q \sigma(q) \iff \sigma(\sigma(q)) \leq Q q.
$$

- $(a \iff d)$: First of all, $\Rightarrow$ follows because $a \Rightarrow b$. The converse is immediate by taking the underlying monotone morphism.

3. Follows because the top and binary meet are definable in terms of $\bot$, $\vee$ and $\sigma$ by (1), hence are preserved by algebra homomorphisms.

\[\square\]

Corollary 6.4.4 \textit{(SAI} is the variety of De Morgan algebras).

By identifying $(Q, \sigma) \in \text{SAI}$ with the tuple $(Q, \vee Q, \bot Q, \land Q, \top Q, \sigma)$, the category $\text{SAI}$ is precisely the variety of De Morgan algebras i.e. bounded lattices $L$ equipped with a unary operation $\sigma : L \rightarrow L$ satisfying the equations:

$$
\sigma \sigma(x) \approx x \quad \sigma(x \lor y) \approx \sigma(x) \land \sigma(y) \quad \sigma(x \land y) \approx \sigma(x) \lor \sigma(y).
$$
Proof. Given \((Q, \sigma) \in \text{SAI}\) then the induced tuple \((Q, \vee_{Q}, \bot_{Q}, \wedge_{Q}, \top_{Q}, \sigma)\) is a de morgan algebra by \((\text{Inv}\sigma)\) and Lemma 6.4.3.1. Conversely, any de morgan algebra defines an \(\text{SAI}\)-algebra because \((\text{Inv}\sigma)\) by assumption, and moreover \(\sigma(x \vee y) = \sigma(x) \wedge \sigma(y)\) implies \((\text{Rev}\sigma)\):

\[
x \leq y \iff x \vee y = y \implies \sigma(x \vee y) = \sigma(y) \iff \sigma(x) \lor \sigma(y) = \sigma(y) \iff \sigma(y) \leq \sigma(x).
\]

The homomorphisms of de morgan algebras are the bounded lattice morphisms which preserve the unary operation, and using Lemma 6.4.3.3 they are precisely the \(\text{SAI}\)-morphisms.

\[\Box\]

**Corollary 6.4.5** (Order-dual algebras).

1. Given \((Q, \sigma) \in \text{SAJ}_f\) then \((Q^{op}, \sigma) \in \text{SAJ}_f\) if \((Q, \sigma) \in \text{SAI}_f\).
2. Given \((Q, \sigma) \in \text{SAM}_f\) then \((Q^{op}, \sigma) \in \text{SAM}_f\) if \((Q, \sigma) \in \text{SAI}_f\).
3. Given \((Q, \sigma) \in \text{SAI}\) then \(\sigma\) defines an \(\text{SAI}\)-isomorphism \((Q, \sigma) \to (Q^{op}, \sigma)\) and also its inverse.

*Proof.*

1. Given \((Q, \sigma) \in \text{SAI}\) then \((Q^{op}, \sigma) \in \text{SAI}\) because \((\text{Inv}\sigma)\) continues to hold, as does \((\text{Rev}\sigma)\) by considering the opposite monotone morphism. Conversely, if \((Q^{op}, \sigma) \in \text{SAJ}_f\) then \(\sigma\) defines a self-adjoint morphism of type \(Q \to Q^{op}\) and \(Q^{op} \to Q\) and hence an \(\text{SAI}_f\)-algebra by Lemma 6.4.3.2.

2. Follows because \((Q, \sigma) \in \text{SAM}\) if \((Q^{op}, \sigma) \in \text{SAJ}_f\).

3. \(\sigma\) defines a \(\text{JSL}_f\)-isomorphism \(Q \to Q^{op}\) by Lemma 6.4.3, and also an \(\text{SAI}_f\)-morphism because \(\sigma \circ \sigma = \sigma \circ \sigma\).

\[\Box\]

We can also characterise the finite \(\text{SAJ}_f\) and \(\text{SAM}_f\)-algebras in terms of the finite de morgan algebras.

**Lemma 6.4.6** \((\text{SAJ}_f\) and \(\text{SAM}_f\)-algebras as extensions of finite de morgan algebras).

1. Given any \((Q, \sigma) \in \text{SAJ}_f\) then we have \((\sigma[Q], \sigma[Q] \times \sigma[Q]) \in \text{SAI}_f\), in fact:

\[
\sigma = \begin{pmatrix} Q \rightarrow \sigma[Q] \rightarrow Q \end{pmatrix} \sigma_{|\sigma[Q] \times \sigma[Q]} \rightarrow Q^{op}, \quad Q^{op}.
\]

2. Given any \((Q, \sigma) \in \text{SAM}_f\) then \((\sigma[Q], \sigma[Q] \times \sigma[Q]) \in \text{SAI}_f\) and moreover:

\[
\sigma = \begin{pmatrix} Q^{op} \rightarrow \sigma[Q^{op}] \rightarrow Q^{op} \end{pmatrix} \sigma_{|\sigma[Q] \times \sigma[Q]} \rightarrow Q^{op}, \quad Q^{op}.
\]

3. Consequently,

- \((Q, \sigma) \in \text{SAJ}_f\) if \((R, \sigma_0) \in \text{SAI}_f\) and a \(\text{JSL}_f\)-morphism \(\alpha : Q \to R\) such that \(\sigma = \alpha \circ \sigma_0 \circ \alpha\).
- \((Q, \sigma) \in \text{SAM}_f\) if \((R, \sigma_0) \in \text{SAI}_f\) and a \(\text{JSL}_f\)-morphism \(\beta : R \to Q\) such that \(\sigma = \beta \circ \sigma_0 \circ \beta\).

*Proof.*

1. By Lemma 6.4.1 we know \(\sigma\) defines a join-semilattice morphism \(\sigma : Q \to Q^{op}\). By the (surjection, inclusion) factorisation we have the join-semilattice inclusion-morphism \(\sigma[Q] \to Q^{op}\) which restricts to a join-semilattice endomorphism \(\sigma[Q] \times \sigma[Q]\) of type \(\sigma[Q] \to \sigma[Q]\). By restriction it satisfies \((\text{Rev}\sigma)\), whereas for any \(\sigma(x) \in \sigma[Q]\),

\[
\sigma_{|\sigma[Q] \times \sigma[Q]} \circ \sigma_{|\sigma[Q] \times \sigma[Q]} \circ \sigma_{|\sigma[Q] \times \sigma[Q]}(x) = \sigma\sigma\sigma(x) = \sigma(x)
\]

by Lemma 6.2.2.3, so that \((\text{Inv}\sigma)\) holds. Consequently \((\sigma[Q], \sigma[Q] \times \sigma[Q])\) is a finite de morgan algebra.

Next, the surjective join-semilattice morphism \(\sigma[Q] \times \sigma[Q]\) arises from the other part of \(\sigma\)'s (surjection, inclusion) factorisation. Then the respective composite is a well-defined join-semilattice morphism of type \(Q \to Q^{op}\). Its action is that of \(\sigma\) because:

\[
(\sigma_{|\sigma[Q] \times \sigma[Q]} \circ \sigma_{|\sigma[Q] \times \sigma[Q]} \circ \sigma_{|\sigma[Q] \times \sigma[Q]}(q_0)) = (\sigma_{|\sigma[Q] \times Q})(\sigma(q_0)) = V_Q\{q \in Q : \sigma_{|\sigma[Q] \times Q}(q) \leq Q^{op} \sigma\sigma\sigma(q_0)\} = V_Q\{q \in Q : \sigma\sigma\sigma(q_0) \leq Q^{op} \sigma\sigma\sigma(q_0)\}
\]

by adjoint relationship

\[
\sigma\sigma\sigma(q_0) = \sigma(q_0)\]

by Lemma 6.2.2.3.
2. Follows because \((Q, \sigma) \in \SAM_f\) iff \((Q^{\text{op}}, \sigma) \in \SAI_f\).

3. Take any \((R, \sigma_0) \in \SAI_f\) and any \(\JSL_f\)-morphism \(\alpha : Q \to R\). Then \((Q, \alpha \circ \sigma_0 \circ \alpha) \in \SAJ_f\) because \(\alpha \circ \sigma_0 \circ \alpha : Q \to Q^{\text{op}}\) is a self-adjoint morphism because \(\sigma_0 : R \to R^{\text{op}}\) is. Conversely by (1) every \(\SAJ_f\)-algebra arises in this way, in fact we may assume \(\alpha\) is surjective. The second item follows analogously.

It is worth mentioning a related result.

**Lemma 6.4.7** (Lifting \(\JSL_f\)-quotients and embeddings to \(\SAJ_f\) and \(\SAM_f\)).

Fix any finite De Morgan algebra \((Q, \sigma)\).

1. Each surjective \(\JSL_f\)-morphism \(\psi : R \to Q\) defines a \(\SAJ\)-morphism:

\[
\psi : (R, \psi \circ \sigma \circ \psi) \to (Q, \sigma)
\]

2. Each injective \(\JSL_f\)-morphism \(e : Q \to R\) defines a \(\SAM\)-morphism:

\[
e : (Q, \sigma) \to (R, e \circ \sigma \circ e_*)
\]

**Proof.**

1. \((R, \psi \circ \sigma \circ \psi) \in \SAJ_f\) by Lemma 6.4.6.3. To establish that \(\psi : (R, \sigma_R) \to (Q, \sigma)\) is a well-defined \(\SAJ\)-morphism we must show that \(\psi(\psi \circ \sigma \circ \psi (r)) = \sigma(\psi (r))\) for each \(r \in R\). This follows because for every \(q \in Q\),

\[
\psi(\psi_*(q)) = \psi(\bigvee_R \{r \in R : \psi(r) \leq_Q q\}) \text{ by definition}
\]

\[
= \bigvee_Q \{\psi(r) : r \in R, \psi(r) \leq_Q q\} \text{ by join-preservation}
\]

\[
= \sigma(\psi(q)) \text{ by surjectivity.}
\]

2. \((R, e \circ \sigma \circ e_*) \in \SAM_f\) by Lemma 6.4.6.3. It remains to establish that \(e : (Q, \sigma) \to (R, \sigma_R)\) is a well-defined \(\SAM\)-morphism. Then for every \(q \in Q\) we must show that \(e(\sigma(q)) = e \circ \sigma \circ e_*(e(q))\), which follows because:

\[
e_*(e(q)) = \bigvee_Q \{q' \in Q : e(q') \leq_R e(q)\} \text{ by definition}
\]

\[
= \bigvee_Q \{q' \in Q : q' \leq_Q q\} \text{ injective \(\JSL\)-morphisms are order embeddings}
\]

\[
= q.
\]

We finish off with an explicit description of the free one-generated algebras. They are finite in each case, whereas the two-generated algebras are already infinite. In fact, a free De Morgan algebra on \(X\) amounts to a free bounded lattice on \(X + X\) equipped with a natural involution.

**Proposition 6.4.8** (Free one-generated \(\SAJ\), \(\SAM\) and \(\SAI\)-algebras).

1. The free one-generated \(\SAI\)-algebra may be depicted as follows:

![Diagram](image)

More generally, given any set \(X\) then the free \(X\)-generated \(\SAI\)-algebra arises as the free bounded lattice on generators \(X + X\) with inductively defined unary operation:

\[
\sigma_X(l(x)) := r(x) \quad \sigma_X(r(x)) := l(x)
\]

\[
\sigma_X(\phi \land \psi) := \sigma_X(\phi) \lor \sigma_X(\psi) \quad \sigma_X(\phi \lor \psi) := \sigma_X(\phi) \land \sigma_X(\psi).
\]

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2. The free one-generated SAJ-algebra \((Q, \sigma)\) is depicted below:

![Diagram of the free one-generated SAJ-algebra](image)

The boxed elements show that the image \(\sigma[Q] \subseteq Q^{\text{op}}\) is a free SAJ-algebra on the generator \(\sigma(x)\).

3. The free one-generated SAM-algebra \((Q, \sigma)\) may be depicted as follows:

![Diagram of the free one-generated SAM-algebra](image)

The boxed elements show that the image \(\sigma[Q] \subseteq Q\) is a free SAJ-algebra on the generator \(\sigma(x)\).

**Proof.**

1. The depicted finite de Morgan algebra is a well-defined bounded lattice because:

\[
\sigma(\bot) = \top, \quad \sigma(x \lor \sigma(x)) = x \land \sigma(x)
\]

i.e. we have the bounded lattice structure by Lemma 6.4.3. Then it is closed under the involution \(\sigma\) and defines a finite de Morgan algebra. Since no additional relations were assumed this is a free one-generated algebra.

Regarding the more general statement, take any set \(X\) and let:

\[Q_X := F(X + X)\]

be a free bounded lattice on generators \(X + X\) i.e. two copies of \(X\).

We may view its elements as equivalence classes of bounded lattice terms in variables \(l(x), r(x)\) for \(x \in X\). Then \(\sigma_X\) is a well-defined involutive bounded lattice isomorphism \(Q_X \rightarrow Q_X^{\text{op}}\). This follows by the symmetry of the usual equational presentation of bounded lattices, and the fact that we may bijectively relabel variables, so that \(\phi \equiv \psi \iff \sigma_X(\phi) \equiv \sigma_X(\psi)\). Then given \((Q, \sigma) \in \text{SAI}\) and elements \(e, l : X \rightarrow Q\), we have a unique bounded lattice morphism \(\alpha : Q_X \rightarrow Q\) where:

\[
\alpha(l(x)) := e(l(x)) \quad \alpha(r(x)) := \sigma(e(l(x))),
\]

via the universal property of free bounded lattices since \(Q\) is a bounded lattice. It remains to establish that \(\alpha\) preserves the unary operation. First consider the base case:

\[
\alpha(\sigma_X(l(x))) = \alpha(r(x)) = \sigma(e(l(x))) = \sigma(\alpha(l(x))) \quad \text{for each} \ x \in X.
\]
As for the inductive case, assuming that \( \alpha(\sigma_X(\phi)) = \sigma(\alpha(\phi)) \) holds for all \( \phi \in \Phi \), then:

\[
\begin{align*}
\alpha(\sigma_X(\sigma_X(\phi))) &= \alpha(\phi) & \sigma_X \text{ involutive} \\
&= \sigma(\alpha(\sigma_X(\phi))) & \text{by induction, } \sigma \text{ involutive.}
\end{align*}
\]

\[
\begin{align*}
\alpha(\sigma_X(\sqcup_{Q_X} \Phi)) &= \alpha(\sqcup_{Q_X} \sigma_X[\Phi]) & \sigma_X : Q_X \to Q_X^\text{op} \text{ a bounded lattice morphism} \\
&= \sqcup_{Q_X} \sigma[\alpha(\Phi)] & \text{by induction} \\
&= \sigma(\alpha(\sqcup_{Q_X} \Phi)) & \text{repeating reasoning in reverse.}
\end{align*}
\]

2. (Sketch) Ignoring the terms we have a well-defined join-semilattice, which is actually distributive. One may verify that \( \sigma \) satisfies the rules (Rev\( \sigma \)) and (Ex\( \sigma^2 \)), so we have a well-defined finite SAJ-algebra. Now view the elements as their respective term modulo the equational axioms of SAJ. The join-structure is compatible using the fact that \( \sigma(\bot) \) is the top element by Lemma 6.2.2.4. To see that the unary operation is compatible one verifies that the action is derivable from the equational laws. It suffices to verify a subset of them.

- \( \sigma(\bot) \equiv \sigma(\bot) \) trivially.
- \( \sigma(\sigma(\bot)) \equiv \sigma(\bot) \) by Lemma 6.2.2.3.
- \( \sigma(x \vee \sigma(\bot)) \equiv \sigma(x) \). Indeed, since \( x \leq x \vee \sigma(\bot) \) we obtain \( \sigma(x \vee \sigma(\bot)) \leq \sigma(x) \) via (Rev\( \sigma \)). Conversely, \( \sigma(x) \leq \sigma(x \vee \sigma(\bot)) \iff x \vee \sigma(\bot) \leq \sigma(x) \) via the adjoint relationship, and hence holds using (Ex\( \sigma^2 \)) and by applying (Rev\( \sigma \)) twice.
- \( \sigma(x \vee \sigma(x \vee \sigma(x))) \equiv \sigma(x) \). Firstly since \( x \leq x \vee \sigma(x \vee \sigma(x)) \), applying (Rev\( \sigma \)) yields half of the desired equality. Conversely, we need to establish that:

\[
\sigma(x) \leq \sigma(x \vee \sigma(x \vee \sigma(x))).
\]

Applying the adjoint relationship this is equivalent to \( x \vee \sigma(x \vee \sigma(x)) \leq \sigma(x) \). Then \( x \leq \sigma(x) \) is (Ex\( \sigma^2 \)) and finally \( \sigma(x \vee \sigma(x)) \leq \sigma(x) \) follows by applying (Rev\( \sigma \)) to \( \sigma(x) \leq x \vee \sigma(x) \).

3. (Sketch) Follows by the method used in (2), noting that \( \sigma(\bot) \equiv \bot \) in SAM by Lemma 6.2.2.4. That the action of \( \sigma \) is witnessed by various equational proofs is easier than in (2). One only needs to use \( \sigma \sigma \sigma(x) \equiv \sigma(x) \) and \( \sigma(x \vee \sigma(\bot)) \equiv \sigma(\bot) \equiv \bot \) via (Rev\( \sigma \)).

\[
\square
\]

Corollary 6.4.9. If \(|X| > 1\) then the free SAJ, SAM and SAI-algebra on \( X \) are infinite.

\textbf{Proof.} By Proposition 6.4.8.1, a free SAJ-algebra on \( X \) is a free bounded lattice on \( X + X \) generators equipped with an involution. It is well-known that the free bounded lattice on 3 generators \( \{x, y, z\} \) is infinite e.g. we have the strict \(<\)-chain where \( \phi_0 := x \) and \( \phi_{n+1} := x \vee (y \wedge (z \vee (x \vee (y \wedge (z \wedge \phi_n)))))) \) for all \( n \geq 0 \). Then if \(|X| > 1\) it follows that the free SAJ-algebra on \( X \) is infinite. Finally, the free \( X \)-generated SAJ and SAM-algebra have the free \( X \)-generated SAJ-algebra as a quotient, so they are themselves infinite.

\[
\square
\]

6.5 The categories UG\(_j\), UG\(_m\) and UG

\textbf{Definition 6.5.1} (The three categories corresponding to the varieties).

The compositional structure of the categories below is inherited from Dep.

1. UG\(_j\)'s objects are pairs \((\mathcal{G}, \mathcal{E})\) where:
   
   (a) \( \mathcal{G} \subseteq \mathcal{G}_x \times \mathcal{G}_r \) is an arbitrary relation between finite sets,
   
   (b) \( \mathcal{E} \subseteq \mathcal{G}_x \times \mathcal{G}_s \) is symmetric and defines a Dep-morphism of type \( \mathcal{G} \rightarrow \mathcal{G} \).

   Its morphisms \( \mathcal{R} : (\mathcal{G}, \mathcal{E}_1) \rightarrow (\mathcal{H}, \mathcal{E}_2) \) are those Dep-morphisms \( \mathcal{R} : \mathcal{G} \rightarrow \mathcal{H} \) such that:

   \[
   \mathcal{R}^1 \circ \mathcal{E}_1^1 = \mathcal{H}^1 \circ (\mathcal{E}_2^1 \circ \mathcal{R})^1.
   \]

2. UG\(_m\)'s objects are pairs \((\mathcal{G}, \mathcal{E})\) where:
(a) $\mathcal{G} \subseteq \mathcal{G}_s \times \mathcal{G}_t$ is an arbitrary relation between finite sets,
(b) $\mathcal{E} \subseteq \mathcal{G}_s \times \mathcal{G}_t$ is symmetric and defines a Dep-morphism of type $\mathcal{G} \to \mathcal{G}$.

Its morphisms $\mathcal{R} : (\mathcal{G}, \mathcal{E}_1) \to (\mathcal{H}, \mathcal{E}_2)$ are those Dep-morphisms $\mathcal{R} : \mathcal{G} \to \mathcal{H}$ such that:

$$\mathcal{E}^i_2 \circ \mathcal{R}^i = (\mathcal{R}^\downarrow \mathcal{E}_1)^i \circ \mathcal{G}^i.$$

3. UG’s objects are the undirected graphs $(V, \mathcal{E})$ i.e. $V$ is a finite set and $\mathcal{E} \subseteq V \times V$ is a symmetric relation. Its morphisms $\mathcal{R} : (V_1, \mathcal{E}_1) \to (V_2, \mathcal{E}_2)$ are those Dep-morphisms $\mathcal{R} : \mathcal{E}_1 \to \mathcal{E}_2$ such that:

$$\mathcal{R}^i \circ \mathcal{E}^i_1 = \mathcal{E}^i_2 \circ \mathcal{R}^i \quad \text{or equivalently} \quad \mathcal{E}^i_2 \circ \mathcal{R}^i = \mathcal{R}^i \circ \mathcal{E}^i_1$$

by De Morgan duality.

\begin{proof}
By Lemma 6.3.4.1, for any relation $\mathcal{E}$, requiring it is symmetric and defines a Dep-morphism $\mathcal{G} \to \mathcal{G}$ is equivalent to requiring it defines a self-adjoint Dep-morphism $\mathcal{G} \to \mathcal{G}$.
\end{proof}

\begin{note}[Concerning the additional constraints on Dep-morphisms] These constraints will be seen to capture the preservation of the unary operation at the algebraic level. We do not in general know how to interpret these conditions in a more intuitive fashion.
\end{note}

Before proving well-definedness we describe UG-morphisms via a single equation i.e. without the underlying assumption they are Dep-morphisms.

\begin{lemma}[Characterisation of UG-morphisms]
Given undirected graphs $(V_i, \mathcal{E}_i)_{i=1,2}$, a relation $\mathcal{R} \subseteq V_1 \times V_2$ defines a UG-morphism $\mathcal{R} : (V_1, \mathcal{E}_1) \to (V_2, \mathcal{E}_2)$ iff:

$$\mathcal{R}^i = \mathcal{E}^i_2 \circ \mathcal{R}^i \circ \mathcal{E}^i_1.$$

\end{lemma}

\begin{proof}
A UG-morphism $\mathcal{R} : (V_1, \mathcal{E}_1) \to (V_2, \mathcal{E}_2)$ is a Dep-morphism hence $\mathcal{R}^i = \mathcal{R}^i \circ \mathcal{cl}_{\mathcal{E}_1}$ by Lemma 4.1.10. Since $\mathcal{R}^i \circ \mathcal{E}^i_1 = \mathcal{E}^i_2 \circ \mathcal{R}^i$, precomposing with $\mathcal{E}^i_1$ yields the desired equality. Conversely suppose $\mathcal{R} \subseteq V_1 \times V_2$ satisfies $\mathcal{R}^i = \mathcal{E}^i_2 \circ \mathcal{R}^i \circ \mathcal{E}^i_1$. It follows that $\mathcal{R}^i \circ \mathcal{cl}_{\mathcal{E}_1} = \mathcal{R}^i = \mathcal{in}_{\mathcal{E}_1} \circ \mathcal{R}^i$ by using $(\uparrow \downarrow)$ twice i.e. $\mathcal{R} : \mathcal{E}_1 \to \mathcal{E}_2$ is a Dep-morphism. Precomposing the assumed equality yields $\mathcal{R}^i \circ \mathcal{E}^i_1 = \mathcal{E}^i_2 \circ \mathcal{R}^i \circ \mathcal{E}^i_1$ via Lemma 4.1.10 and Lemma 4.1.7.4.
\end{proof}

\begin{lemma}
UG$_j$, UG$_m$ and UG are well-defined categories.
\end{lemma}

\begin{proof}
1. We’ll show that the UG$_j$-morphisms are closed under the compositional structure of Dep. For each $(\mathcal{G}, \mathcal{E}) \in$ UG$_j$, the Dep identity-morphism $id_{\mathcal{G}} = \mathcal{G} : \mathcal{G} \to \mathcal{G}$ defines a UG$_j$-morphism $(\mathcal{G}, \mathcal{E}_\mathcal{G}) \to (\mathcal{G}, \mathcal{E}_\mathcal{G})$ because:

$$\mathcal{G}^i \circ (\mathcal{E}_\mathcal{G} \circ (id_{\mathcal{G}})) = \mathcal{G}^i \circ (\mathcal{E}_\mathcal{G} \circ id_{\mathcal{G}}) = \mathcal{G}^i \circ \mathcal{E}^i_\mathcal{G}.$$

Finally, given any composite $(\mathcal{G}, \mathcal{E}_\mathcal{G}) \xrightarrow{\mathcal{R}} (\mathcal{H}, \mathcal{E}_\mathcal{H}) \xrightarrow{\mathcal{S}} (\mathcal{I}, \mathcal{E}_\mathcal{I})$ we calculate:

$$\mathcal{I}^i \circ (\mathcal{E} \circ (\mathcal{E}_S \circ (\mathcal{R} \circ \mathcal{S}))^i = \mathcal{I}^i \circ (\mathcal{E} \circ \mathcal{E}_S \circ \mathcal{R} \circ \mathcal{S})$$

by functorality

$$= \mathcal{I}^i \circ (\mathcal{E} \circ \mathcal{E}_S \circ \mathcal{R} \circ \mathcal{S})$$

by $(\downarrow \uparrow)$

$$= \mathcal{S}^i \circ \mathcal{E}^i_\mathcal{S} \circ \mathcal{H}^i \circ \mathcal{R}^i$$

by assumption

$$= \mathcal{S}^i \circ (\mathcal{E} \circ \mathcal{S} \circ \mathcal{R})$$

by $(\downarrow \uparrow)$

$$= \mathcal{S}^i \circ \mathcal{H}^i \circ \mathcal{R}^i \circ (\mathcal{E}_\mathcal{S} \circ \mathcal{R})$$

by Lemma 4.1.10

$$= \mathcal{S}^i \circ \mathcal{H}^i \circ \mathcal{R}^i \circ \mathcal{E}^i_\mathcal{G}$$

by assumption

$$= (\mathcal{R} \circ \mathcal{S})^i \circ \mathcal{E}^i_\mathcal{G}$$

by $(\uparrow \downarrow)$.
\end{proof}
2. Given any UG\textsubscript{m}-object \((\mathcal{G}, \mathcal{E})\), the Dep identity-morphism \(id_\mathcal{G} = \mathcal{G}\) defines a UG\textsubscript{m}-morphism \((\mathcal{G}, \mathcal{E}) \to (\mathcal{G}, \mathcal{E})\):

\[
(((id_\mathcal{G})^\vee \mathcal{E})^\dagger \circ \mathcal{G}^\dagger) \circ (id_\mathcal{G})^\dagger \circ (\mathcal{E})^\dagger = \mathcal{E}_\mathcal{G}^\dagger \circ \mathcal{G}^\dagger.
\]

Given any composite \((\mathcal{G}, \mathcal{E}) \xrightarrow{R} (\mathcal{H}, \mathcal{E}_2) \xrightarrow{S} (I, \mathcal{E}_I)\) we calculate:

\[
(((R \circ S)^\vee \mathcal{E})^\dagger \circ \mathcal{G}^\dagger = (S \circ R^\vee \mathcal{E})^\dagger \circ \mathcal{G}^\dagger \quad \text{by functorality}
\]

\[
= S^\dagger \circ H^\dagger \circ (R \circ \mathcal{E})^\dagger \circ \mathcal{G}^\dagger \quad \text{by (1 \dagger)}
\]

\[
= S^\dagger \circ H^\dagger \circ \mathcal{E}_2^\dagger \circ R^\dagger \quad \text{by assumption}
\]

\[
= (S \circ \mathcal{E}_2)^\dagger \circ R^\dagger \quad \text{by (1 \dagger)}
\]

\[
= (S \circ \mathcal{E}_2)^\dagger \circ H^\dagger \circ R^\dagger \quad \text{using Lemma 4.1.7.4}
\]

\[
= \mathcal{E}_I^\dagger \circ S^\dagger \circ H^\dagger \circ R^\dagger \quad \text{by assumption}
\]

\[
= \mathcal{E}_I^\dagger \circ (R \circ S)^\dagger. \quad \text{by (1 \dagger)}
\]

3. Each UG-object \((V, \mathcal{E})\) induces a well-defined UG\textsubscript{j}-object \((\mathcal{E}, \mathcal{E})\) because \(id_\mathcal{E} = \mathcal{E} : \mathcal{E} \to \mathcal{E} = \mathcal{E}\). In fact, a Dep-morphism \(R : \mathcal{E}_1 \to \mathcal{E}_2\) defines a UG\textsubscript{j}-morphism iff \(R : (\mathcal{E}_1, \mathcal{E}_1) \to (\mathcal{E}_2, \mathcal{E}_2)\) is a UG\textsubscript{j}-morphism, which follows because the constraint on UG\textsubscript{j}-morphisms:

\[
\mathcal{R}^\dagger \circ \mathcal{E}_2^\dagger = H^\dagger \circ (E_2 \circ \mathcal{R})^\dagger \quad \text{becomes} \quad \mathcal{R}^\dagger \circ \mathcal{E}_2^\dagger \circ \mathcal{E}_2^\dagger = (E_2 \circ \mathcal{R})^\dagger = \mathcal{E}_2^\dagger \circ \mathcal{R}^\dagger.
\]

Thus UG is isomorphic to a full subcategory of the well-defined category UG\textsubscript{j}, and hence is itself a well-defined category.

\[\square\]

**Lemma 6.5.6** (Basic observations concerning the UG, UG\textsubscript{j} and UG\textsubscript{m}-objects).

1. The UG-objects are precisely the undirected graphs.

2. \((\mathcal{G}, \mathcal{E}) \in UG\textsubscript{j}\) iff \((\mathcal{G}, \mathcal{E}) \in UG\textsubscript{m}\).

3. The UG\textsubscript{j}-objects are precisely the pairs \((\mathcal{G}, \mathcal{E})\) where \(\mathcal{E}\) defines a self-adjoint Dep-morphism \(\mathcal{G} \to \mathcal{G}\).

4. The UG\textsubscript{m}-objects are precisely the pairs \((\mathcal{G}, \mathcal{E})\) where \(\mathcal{E}\) defines a self-adjoint Dep-morphism \(\mathcal{G} \to \mathcal{G}\).

Next a simple yet important characterisation. Recall the standard polarity \(\mathcal{E}^\dagger : SV \to SV\) from Definition 6.1.9. It has action \(\mathcal{E}^\dagger(X) = \bigcap_{x \in X} \mathcal{E}[x]\) by Lemma 6.1.10.

**Lemma 6.5.7** (Characterisation of the UG\textsubscript{j}-objects and the UG\textsubscript{m}-objects).

1. Given any bipartite graph \(\mathcal{G} \subseteq V \times \mathcal{G}\) and symmetric relation \(\mathcal{E} \subseteq V \times V\) t.f.a.e.

   a. \((\mathcal{G}, \mathcal{E}) \in UG\textsubscript{j}\).

   b. \(\mathcal{E} = \mathcal{G}; \mathcal{H}\) for some relation \(\mathcal{H} \subseteq \mathcal{G} \times V\).

   c. \(\forall (v_1, v_2) \in \mathcal{E}\) there exists \(g_r \in \mathcal{G}\) such that \(v_1 \in \mathcal{G}[g_r]\) and \(\forall v \in \mathcal{G}[g_r], \mathcal{E}(v, v_2)\).

   d. \(\mathcal{E} = \bigcup_{g \in \mathcal{G}} \mathcal{G}[g_r] \times \mathcal{E}^\dagger(\mathcal{G}[g_r])\).

   e. \(\mathcal{E}^\dagger = \text{in}_\mathcal{G} \circ \mathcal{E}\).

2. Given any bipartite graph \(\mathcal{G} \subseteq \mathcal{G} \times V\) and symmetric relation \(\mathcal{E} \subseteq V \times V\) t.f.a.e.

   a. \((\mathcal{G}, \mathcal{E}) \in UG\textsubscript{m}\).

   b. \(\mathcal{E} = \mathcal{H}; \mathcal{G}\) for some relation \(\mathcal{H} \subseteq V \times \mathcal{G}\).

   c. \(\forall (v_1, v_2) \in \mathcal{E}\) there exists \(g_s \in \mathcal{G}\) such that \(v_1 \in \mathcal{G}[g_s]\) and \(\forall v \in \mathcal{G}[g_s], \mathcal{E}(v, v_2)\).

   d. \(\mathcal{E} = \bigcup_{g \in \mathcal{G}} \mathcal{G}[g_s] \times \mathcal{E}^\dagger(\mathcal{G}[g_s])\).

   e. \(\mathcal{E}^\dagger = \text{in}_\mathcal{G} \circ \mathcal{E}\).
Proof. The second collection of equivalent statements follows from the first because \((G, E) \in UG_m \iff (\tilde{G}, \tilde{E}) \in UG_j\), and moreover \(E = \tilde{G} : H \iff \tilde{E} = H : \tilde{G}\) since \(E\) is symmetric. We verify the first collection of equivalences.

- \((a \iff b)\): Given a Dep-morphism \(E : G \to \tilde{G}\) then \(E = G; \tilde{E}\) so we may choose \(H := \tilde{E}^\perp\). Conversely, if \(E = G; H\) then since \(E\) is symmetric we have witnesses \(H; \tilde{G} = E = G; \tilde{H}\), hence \(E\) defines a Dep-morphism \(G \to \tilde{G}\).

- \((b \iff c)\): Suppose that \(E = G; H\). Then given \((v_1, v_2) \in E\) there exists \(g_t \in G_t\) such that \(G(v_1, g_t)\) and \(H(g_t, v_2)\). Thus \(v_1 \in \tilde{G}[g_t]\) and for any \(v \in \tilde{G}[g_t]\) we have \(G(v, g_t) \land H(g_t, v_2)\) and hence \(E(v, v_2)\).

For the other implication, suppose that \((c)\) holds and define:
\[
H := \{(g_t, v) \in G_t \times V : \forall u \in \tilde{G}[g_t].E(u, v)\}.
\]

Then whenever \(G(v_1, g_t) \land H(g_t, v_2)\) we deduce \(E(v_1, v_2)\) by instantiating \(u := v_1\), so that \(G; H \subseteq E\). For the converse inclusion, if \(E(v_1, v_2)\) then by assumption there exists \(g_t \in G_t\) such that \(G(v_1, g_t) \land H(g_t, v_2)\).

- \((c \iff d)\): For any \(G \subseteq V \times G_t\) we have \(U_{g_t \in G_t} \tilde{G}[g_t] \times \tilde{E}^\perp(\tilde{G}[g_t]) \subseteq E\), seeing as \(\tilde{E}^\perp(\tilde{G}[g_t])\) consists precisely of those vertices \(v \in V\) which are adjacent in \(E\) to every \(u \in \tilde{G}[g_t]\). To see that \((c)\) is equivalent to the converse inclusion, observe that \(\forall v \in \tilde{G}[g_t].E(v, v_2)\) holds iff \(v_2 \in \tilde{E}^\perp(\tilde{G}[g_t]) = \bigcap_{v \in \tilde{G}[g_t]} E[v]\).

- \((b \iff e)\): Since open sets are closed under unions, \((e)\) is equivalent to \(\forall v \in V.\tilde{E}[v] \subseteq O(\tilde{G})\). But this in turn is equivalent to assuming \(E = H; \tilde{G}\) for some relation \(H\).

Recall \(E_\perp = E^\perp\) for any self-adjoint Dep-morphism \(E : G \to \tilde{G}\).

**Lemma 6.5.8** (Associated component of self-adjoint Dep-morphisms). If \(E \subseteq V \times V\) is a self-adjoint morphism \(E : G \to \tilde{G}\) then:
\[
E_\perp = E^\perp = \overline{\overline{E}}; \tilde{G} \subseteq V \times G_t
\]
and also \(E_\perp[g_t] = E_\perp^\perp(\tilde{G}[g_t])\) for every \(g_t \in G_t\).

**Proof.** By Lemma 6.3.4.2 it suffices to establish the equality \(E_\perp = \overline{\overline{E}}; \tilde{G}\); the other claim will follow on the way. First recall that for any Dep-morphism \(R : H_1 \to H_2\), its associated components \((R_\perp, R^\perp)\) are the maximum witnesses by Lemma 4.1.10.2. That is, whenever \(R_\perp; H_2 = \tilde{R} = H_1; R^\perp\) then \((R_\perp, R^\perp)\) pairwise include into \((R_\perp, R^\perp)\). Applied to \(R = E\) we deduce that \(E_\perp = E^\perp\) is the largest relation \(S \subseteq G_\times G_t\) such that \(E = S; \tilde{G}\). Since \((G, E) \in UG_j\) via our assumption, it follows by Lemma 6.5.7.1 that:
\[
E = \bigcup_{g_t \in G_t} E^\perp(\tilde{G}[g_t]) \times \tilde{G}[g_t] \quad \text{or equivalently} \quad E = S; \tilde{G}\text{ where } S(v, g_t) : \iff v \in E^\perp(\tilde{G}[g_t]).
\]

Then by maximality we have \(E^\perp(\tilde{G}[g_t]) \subseteq E_\perp[g_t]\) for every \(g_t \in G_t\), whereas the converse inclusions follow because if \(v \notin E^\perp(\tilde{G}[g_t])\) then \(\{v\} \times \tilde{G}[g_t] \not\subseteq E\). Finally, we can rewrite these equalities by recalling the original definition of polarities i.e. as the ‘de morgan dual’ \(E^\perp = \neg V \circ \neg E\).

\[
E_\perp(v, g_t) \iff v \in E_\perp[g_t] \iff v \in \overline{\overline{E^\perp(\tilde{G}[g_t])}} \quad \text{by above reasoning}
\]
\[
\iff v \in \overline{\overline{\overline{V \circ \overline{E}(\tilde{G}[g_t])}}} \quad \text{by definition of } (-)^\perp
\]
\[
\iff v \in \overline{V \circ \overline{E}(\tilde{G}[g_t])} \quad \text{using } (\uparrow)
\]
\[
\iff v \in \overline{\overline{\overline{\overline{E}(\tilde{G}[g_t])}}} \quad \text{property of complement relations}
\]
\[
\iff \overline{\overline{\overline{G; E}}}(v, g_t) \quad \text{take converse, see below.}
\]

For the final step recall that the complement and converse of arbitrary relations commute, and \(E\) is symmetric.
6.6 \( UG_j, UG_m \) and \( UG \) – some structural lemmas

We now prove a number of useful lemmas. These results mirror certain properties of the finite algebras of \( SAJ \), \( SAM \) and \( SAI \). They will be easier to understand once the categorical equivalences have been proved.

**Lemma 6.6.1** (The diagonals of \( UG_j \) and \( UG_m \) are equal and isomorphic to \( UG \)).

1. \( UG \) is isomorphic to the full subcategory of \( UG_j \) with objects \((E, E)\) where \( E = \bar{E} \). The witnessing identity-on-morphisms functor has action \( E \mapsto (\bar{E}, \bar{E}) \).

2. \( UG \) is isomorphic to the full subcategory of \( UG_m \) with objects \((E, E)\) where \( E = \bar{E} \). The witnessing identity-on-morphisms functor has action \( E \mapsto (\bar{E}, \bar{E}) \).

**Proof.**

1. We already observed this in the proof of Lemma 6.5.5.3.

2. As above, also because the constraint on \( UG_m \)-morphisms:

\[ \mathcal{E}_2 \circ \mathcal{R}^i = (\bar{\mathcal{R}} \circ \mathcal{E}_j)^i \circ \mathcal{G}^i \quad \text{becomes} \quad \mathcal{E}_2 \circ \mathcal{R}^i = (\bar{\mathcal{R}} \circ \text{id}_{\mathcal{E}_j})^i \circ \mathcal{E}_1 \circ \mathcal{R}^i \]

which is one of the two equivalent constraints on \( UG \)-morphisms.

**Lemma 6.6.2** (Reflection of \( \text{Dep} \)-isomorphisms).

The three forgetful functors from \( UG_j, UG_m \) and \( UG \) to \( \text{Dep} \) reflect isomorphisms.

1. If \( \mathcal{R} : (\mathcal{G}, \mathcal{E}_1) \to (\mathcal{H}, \mathcal{E}_2) \) is a \( UG_j \)-morphism and a \( \text{Dep} \)-isomorphism then its inverse is a \( UG_j \)-morphism.

2. If \( \mathcal{R} : (\mathcal{G}, \mathcal{E}_1) \to (\mathcal{H}, \mathcal{E}_2) \) is a \( UG_m \)-morphism and a \( \text{Dep} \)-isomorphism then its inverse is a \( UG_m \)-morphism.

3. If \( \mathcal{R} : (V_1, \mathcal{E}_1) \to (V_2, \mathcal{E}_2) \) is a \( \text{UG} \)-morphism and a \( \text{Dep} \)-isomorphism then its inverse is a \( \text{UG} \)-morphism.

**Proof.**

1. By assumption we have a \( \text{Dep} \)-isomorphism \( \mathcal{R} : \mathcal{G} \to \mathcal{H} \) with inverse \( \mathcal{S} : \mathcal{H} \to \mathcal{G} \), and:

\[
\begin{align*}
\mathcal{R}^i \circ \mathcal{E}_1^i &= \mathcal{H}^i \circ (\mathcal{E}_2 \circ \bar{\mathcal{R}})^i = \mathcal{H}^i \circ \mathcal{E}_2^i \circ \bar{\mathcal{H}} \circ \bar{\mathcal{R}}^i.
\end{align*}
\]

Then we have:

\[
\begin{align*}
(\mathcal{S}^i)^i \circ \mathcal{A} &= (\mathcal{R} ; \mathcal{S}^i)^i \circ \mathcal{E}_1^i = \mathcal{G}^i \circ \mathcal{E}_1^i \\
(\mathcal{S}^i)^i \circ \mathcal{B} &= (\mathcal{H} ; \mathcal{S}^i)^i \circ \mathcal{E}_2^i \circ \bar{\mathcal{H}} \circ \bar{\mathcal{R}}^i = \mathcal{S}^i \circ \mathcal{E}_2^i \circ \bar{\mathcal{H}} \circ \bar{\mathcal{R}}^i
\end{align*}
\]

using Lemma 4.7.3 and an associated component of \( \mathcal{S} \). Hence:

\[
\begin{align*}
\mathcal{G}^i \circ \mathcal{E}_1^i \circ \bar{\mathcal{R}}^i \circ \bar{\mathcal{H}}^i &= \mathcal{S}^i \circ \mathcal{E}_2^i \circ \bar{\mathcal{H}}^i \circ \mathcal{C} \circ \bar{\mathcal{R}} \circ \bar{\mathcal{H}}^i \quad \text{see above} \\
&= \mathcal{S}^i \circ \mathcal{E}_2^i \circ \bar{\mathcal{H}}^i \circ \mathcal{C} \circ \bar{\mathcal{H}} \circ \bar{\mathcal{R}}^i \quad \text{since \( \mathcal{R}^v \) monic, see Lemma 4.7.1} \\
&= \mathcal{S}^i \circ \mathcal{E}_2^i \circ \bar{\mathcal{H}}^i \quad \text{by (11)} \\
&= \mathcal{S}^i \circ \mathcal{E}_2^i \circ \text{in}_{\bar{\mathcal{H}}} \quad \text{see Lemma 4.1.10.1 and Lemma 4.1.7.4.} \\
\end{align*}
\]

Furthermore since \( \mathcal{R} \) is an isomorphism we deduce that \( \mathcal{R}^i \circ \mathcal{H}^i = \mathcal{G}^i \circ \mathcal{S}^i \) by Lemma 4.7.4, or equivalently \( \bar{\mathcal{R}}^i \circ \bar{\mathcal{H}}^i = \bar{\mathcal{G}}^i \circ \bar{\mathcal{S}}^i \) by de morgan duality. Thus:

\[
\begin{align*}
\mathcal{S}^i \circ \mathcal{E}_2^i &= \mathcal{G}^i \circ \mathcal{E}_1^i \circ \bar{\mathcal{G}}^i \circ \bar{\mathcal{S}}^i = \mathcal{G}^i \circ (\mathcal{E}_1 \circ \bar{\mathcal{S}})^i
\end{align*}
\]

as required.
2. We have a Dep-isomorphism \( \mathcal{R} : \mathcal{G} \to \mathcal{H} \) with inverse \( \mathcal{S} : \mathcal{H} \to \mathcal{G} \), so:

\[
\mathcal{E}_2^i \circ \mathcal{R}^i = (\mathcal{R} \circ \mathcal{E}_1^i) \circ \mathcal{G}^i = \mathcal{R}^i \circ \mathcal{G}^i \circ \mathcal{E}_1^i \circ \mathcal{G}^i.
\]

Then we have:

\[
A \circ S^i = E^i_2 \circ (S_{\cdot; \mathcal{R}}) = E^i_2 \circ \mathcal{H}^i
\]

\[
B \circ S^i = \mathcal{R}^i \circ \mathcal{G}^i \circ E^i_1 \circ (S_{\cdot; \mathcal{G}})^i = \mathcal{R}^i \circ \mathcal{G}^i \circ E^i_1 \circ \mathcal{S}^i
\]

using Lemma 4.7.3 and an associated component of \( \mathcal{S} \). Hence:

\[
\mathcal{G}^i \circ \mathcal{R}^i \circ \mathcal{E}^i_2 \circ \mathcal{H}^i = \mathcal{G}^i \circ \mathcal{E}^i_2 \circ \mathcal{S}^i \quad \text{see above}
\]

\[
= \mathcal{E}^i_1 \circ \mathcal{S}^i \quad \text{since } \mathcal{R}^\vee \text{ epic, see Lemma 4.7.1}
\]

\[
= \mathcal{G}^i \circ \mathcal{E}^i_1 \circ \mathcal{S}^i \quad \text{by } (\uparrow \downarrow)
\]

\[
= \mathcal{E}^i_1 \circ \mathcal{S}^i \quad \text{see Lemma 4.1.10.1 and Lemma 4.1.7.4.}
\]

Moreover since \( \mathcal{G}^i \circ \mathcal{R}^i = \mathcal{S}^i \circ \mathcal{H}^i \) by Lemma 4.7.4, or equivalently \( \mathcal{G}^i \circ \mathcal{R}^i = \mathcal{S}^i \circ \mathcal{H}^i \) by De Morgan duality,

\[
\mathcal{E}^i_1 \circ \mathcal{S}^i = \mathcal{S}^i \circ \mathcal{H}^i \circ \mathcal{E}^i_2 \circ \mathcal{H}^i = (\mathcal{S}_{\cdot; \mathcal{E}_2})^i \circ \mathcal{H}^i
\]

as required.

3. Follows because \( \mathcal{U}G \) is isomorphic to the full subcategory of \( \mathcal{U}G_j \) with objects \( (\mathcal{E}, \mathcal{E}) \) where \( \mathcal{E} \) is a symmetric relation, so we can apply (1). \( \square \)

**Lemma 6.6.3** (Graph isomorphisms induce \( \mathcal{U}G \)-isomorphisms).

Each undirected graph isomorphism \( f : (V, \mathcal{E}_1) \to (V, \mathcal{E}_2) \) induces the \( \mathcal{U}G \)-isomorphism:

\[
f; \mathcal{E}_2 = \mathcal{E}_1; f : (V, \mathcal{E}_1) \to (V, \mathcal{E}_2).
\]

**Proof.** The equality \( f; \mathcal{E}_2 = \mathcal{E}_1; f \) provides a Dep-isomorphism \( \mathcal{R} := f; \mathcal{E}_2 = \mathcal{E}_1; f \) of type \( \mathcal{E}_1 \to \mathcal{E}_2 \). By Lemma 6.6.2 it suffices to show that \( \mathcal{R} \) defines a \( \mathcal{U}G \)-morphism \( (V, \mathcal{E}_1) \to (V, \mathcal{E}_2) \) i.e. \( f^i \circ \mathcal{E}_1^i = \mathcal{E}_2^i \circ f^i \). We certainly know \( f^i \circ \mathcal{E}_1^i = \mathcal{E}_2^i \circ f^i \), and applying De Morgan Mularity yields:

\[
\mathcal{E}_2^i \circ f^i = (f^{-1})^i = f^i \quad \text{by bijectivity.} \square
\]

**Lemma 6.6.4** (\( \mathcal{U}G \)-isomorphisms of reduced graphs). Given reduced graphs \( (V_i, \mathcal{E}_i) \),

\[
\mathcal{R} : (V_1, \mathcal{E}_1) \to (V_2, \mathcal{E}_2) \text{ is a } \mathcal{U}G \text{-isomorphism iff there exists a graph isomorphism } f : (V_1, \mathcal{E}_1) \to (V_2, \mathcal{E}_2)
\]

such that \( f; \mathcal{E}_2 = \mathcal{R} = \mathcal{E}_1; f \).

**Proof.** Recall that the usual graph isomorphisms \( f : (V_1, \mathcal{E}_1) \to (V_2, \mathcal{E}_2) \) are precisely those bijective functions \( f : V_1 \to V_2 \) such that \( f; \mathcal{E}_2 = \mathcal{E}_1; f \). Given such an \( f \) we obtain the \( \mathcal{U}G \)-isomorphism \( \mathcal{R} := f; \mathcal{E}_2 \) by Lemma 6.6.3. Conversely, suppose that \( \mathcal{R} : (V_1, \mathcal{E}_1) \to (V_2, \mathcal{E}_2) \) is a \( \mathcal{U}G \)-isomorphism between reduced graphs. Since \( \mathcal{U}G \) inherits the compositional structure of \( \text{Dep} \) we know that \( \mathcal{R} : \mathcal{E}_1 \to \mathcal{E}_2 \) is a Dep-isomorphism between reduced relations. Then by Lemma 4.7.10:

\[
\begin{array}{ccc}
V_1 & \xrightarrow{f_\circ} & V_2 \\
\mathcal{E}_1 & \xrightarrow{\mathcal{R}} & \mathcal{E}_2 \\
V_1 & \xrightarrow{f_1} & V_2
\end{array}
\]

for some bijections \( f_\circ \) and \( f_1 \). Now, since \( \mathcal{R} \) is a \( \mathcal{U}G \)-morphism we have \( \mathcal{R}^i \circ \mathcal{E}^i_1 = \mathcal{E}^i_2 \circ \mathcal{R}^i \). Moreover:

\[
\mathcal{R}^i = (\mathcal{E}_1; f_\circ)^i \equiv (f_\circ^i \circ \mathcal{E}^i_1) \quad \mathcal{R}^i = ((f_1; \mathcal{E}_2)^i)^i \equiv (f_1^i \circ \mathcal{E}^i_2)^i \equiv \mathcal{E}^i_2 \circ f_1^i
\]

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where the marked equality follows because \((f_{\ell})^i = (f_{\ell}^{-1})^i = f_{\ell}^i\) since \(f_{\ell}\) is bijective. Substituting into the known equality yields:

\[ f_{u}^i \geq f_{u}^i \circ \text{in}_{\mathcal{E}_1} = \text{cl}_{\mathcal{E}_2} \circ f_{i}^i \geq f_{i}^i \]

using the pointwise inclusion-ordering. But since \(f_{i}^i\) and \(f_{u}^i\) preserve singleton sets this implies \(f_{u} = f_{i}\).

**Corollary 6.6.5** (Automorphism groups of reduced graphs).

1. Two reduced graphs are \(\text{UG}\)-isomorphic iff they are graph isomorphic.
2. The \(\text{UG}\)-automorphism group of a reduced graph is isomorphic to its classical automorphism group.

**Proof.**

1. Immediate by Lemma 6.6.4.

2. Fix any reduced graph \((V, \mathcal{E})\). The elements of the two automorphism groups biject via Lemma 6.6.4, via \(f \mapsto f; \mathcal{E}\). The identity function \(id_V\) and is sent to \(id_V; \mathcal{E} = \mathcal{E}\) i.e. the \(\text{UG}\) identity morphism. Concerning composition:

\[
\begin{array}{ccc}
V & \xrightarrow{f} & V \\
\downarrow{\mathcal{E}} & & \downarrow{\mathcal{E}} \\
V & \xrightarrow{g} & V
\end{array}
\]

we have \(f; g \mapsto f; g; \mathcal{E} = (f; \mathcal{E}); (g; \mathcal{E})\) by the usual rules of \(\text{Dep}\)-composition.

**Lemma 6.6.6** (Isomorphism correspondence between \(\text{UG}_j\) and \(\text{UG}_m\)).

\(\mathcal{R} : (\mathcal{G}, \mathcal{E}_1) \to (\mathcal{H}, \mathcal{E}_2)\) is a \(\text{UG}_j\)-isomorphism iff \(\mathcal{R}^\vee : (\mathcal{H}, \mathcal{E}_1) \to (\mathcal{G}, \mathcal{E}_2)\) is a \(\text{UG}_m\)-isomorphism.

**Proof.** Let \(\mathcal{R} : (\mathcal{G}, \mathcal{E}_1) \to (\mathcal{H}, \mathcal{E}_2)\) be a \(\text{UG}_j\)-isomorphism. \(\mathcal{R} : \mathcal{G} \to \mathcal{H}\) is a \(\text{Dep}\)-isomorphism because the compositional structure of \(\text{UG}\) is inherited from \(\text{Dep}\). Then \(\mathcal{R}^\vee : \mathcal{H}^\vee \to \mathcal{G}^\vee\) is also a \(\text{Dep}\)-isomorphism because the self-duality functor \((-)^\vee : \text{Dep}^{\text{op}} \to \text{Dep}\) preserves isos (as do all functors). By Lemma 6.6.2 it remains to show \(\mathcal{R}^\vee\) defines a \(\text{UG}_m\)-morphism of type \((\mathcal{H}, \mathcal{E}_1) \to (\mathcal{G}, \mathcal{E}_2)\) i.e.

\[
\mathcal{E}_1^i \circ \tilde{\mathcal{R}}^i = (\mathcal{R}^\vee \mathcal{E}_2)^i \circ \tilde{\mathcal{H}}^i \quad \text{(iii)} \quad \mathcal{R}^i \circ \mathcal{H}^i \circ \mathcal{E}_1^i \circ \mathcal{R}^i.
\]

Since \(\mathcal{R}\) is a \(\text{UG}_j\)-morphism by assumption,

\[
\begin{align*}
\Rightarrow \quad \mathcal{R}^i \circ \mathcal{E}_1^i &= \mathcal{H}^i \circ (\mathcal{E}_2 \circ \tilde{\mathcal{R}})^i \\
&= \mathcal{H}^i \circ \mathcal{E}_1^i \circ \tilde{\mathcal{H}}^i \circ \tilde{\mathcal{R}}^i \quad \text{by (iii)} \\
&= \text{cl}_\mathcal{R} \circ \mathcal{E}_1^i \circ \tilde{\mathcal{R}}^i = \tilde{\mathcal{R}}^i \circ \mathcal{H}^i \circ \mathcal{E}_1^i \circ \mathcal{H}^i \circ \text{cl}_\mathcal{R} \quad \text{pre/post compose with \(\tilde{\mathcal{R}}^i / \mathcal{R}^i\)} \\
&= \tilde{\mathcal{R}}^i \circ \mathcal{H}^i \circ \mathcal{E}_1^i \circ \mathcal{H}^i \circ \text{cl}_\mathcal{R} \\
&= \mathcal{E}_1^i \circ \tilde{\mathcal{R}}^i = \mathcal{R}^i \circ \mathcal{H}^i \circ \mathcal{E}_2^i \circ \mathcal{H}^i \quad \text{since \(\mathcal{E}_1 \circ \mathcal{G} \to \mathcal{G}, \) also (\(\uparrow \downarrow\)).}
\end{align*}
\]

Conversely given any \(\text{UG}_m\)-isomorphism \(\mathcal{R} : (\mathcal{G}, \mathcal{E}_1) \to (\mathcal{H}, \mathcal{E}_2)\) it suffices to show \(\mathcal{R}^\vee\) defines a \(\text{UG}_j\)-isomorphism of type \((\mathcal{H}, \mathcal{E}_2) \to (\mathcal{G}, \mathcal{E}_1)\). Reusing previous reasoning, we need only show that the \(\text{Dep}\)-isomorphism \(\mathcal{R}^\vee\) is a \(\text{UG}_m\)-morphism i.e.

\[
\tilde{\mathcal{R}}^i \circ \mathcal{E}_2^i = (\mathcal{R}^\vee \circ \mathcal{E}_1)^i = \mathcal{G}^i \circ \mathcal{E}_1^i \circ \mathcal{R}^i.
\]

where now \(\mathcal{R}\) is a \(\text{UG}_m\)-morphism by assumption:

\[
\begin{align*}
\Rightarrow \quad \mathcal{E}_2^i \circ \mathcal{R}^i &= (\mathcal{R}^\vee \circ \mathcal{E}_1)^i \circ \mathcal{G}^i \\
&= \mathcal{E}_2^i \circ \mathcal{R}^i = \tilde{\mathcal{R}}^i \circ \mathcal{E}_1^i \circ \mathcal{G}^i \quad \text{by (iii)} \\
\Rightarrow \quad \mathcal{R}^i \circ \mathcal{E}_2^i \circ \text{in}_{\mathcal{R}} = \text{in}_{\mathcal{R}} \circ \mathcal{G}^i \circ \mathcal{E}_1^i \circ \mathcal{G}^i \circ \mathcal{R}^i \quad \text{pre/post compose by \(\tilde{\mathcal{R}}^i / \mathcal{R}^i\)} \\
\Rightarrow \quad \mathcal{R}^i \circ \mathcal{E}_2^i \circ \text{in}_{\mathcal{G}} = \mathcal{G}^i \circ \mathcal{E}_1^i \circ \mathcal{G}^i \circ \mathcal{R}^i \quad \text{\(\mathcal{R}\) and \(\mathcal{R}^\vee\) epic, see Lemma 4.7.1} \\
\Rightarrow \quad \mathcal{R}^i \circ \mathcal{E}_2^i = \mathcal{G}^i \circ \mathcal{E}_1^i \circ \mathcal{G}^i \circ \mathcal{R}^i \quad \text{since \(\mathcal{E}_2 \circ \mathcal{H} \to \mathcal{H}, \) also (\(\uparrow \downarrow\)).}
\end{align*}
\]
Lemma 6.6.7 (The inverse of a UG-isomorphism is its converse).

1. \( \mathcal{R} : (V_1, \mathcal{E}_1) \to (V_2, \mathcal{E}_2) \) is a UG-isomorphism iff \( \tilde{\mathcal{R}} : (V_2, \mathcal{E}_2) \to (V_1, \mathcal{E}_1) \) is.

2. If \( \mathcal{R} \) is a UG-isomorphism then \( \mathcal{R}^{-1} = \tilde{\mathcal{R}} \).

Proof.

1. By Lemma 6.6.1 the diagonals of \( \text{UG}_j \) and \( \text{UG}_m \) are (i) the same full subcategory, and (ii) categorically isomorphic to \( \text{UG} \) via the identity-on-morphisms functor where \( (V, \mathcal{E}) \mapsto (\mathcal{E}, \mathcal{E}) \). Then a UG-isomorphism \( \mathcal{R} : (V_1, \mathcal{E}_1) \to (V_2, \mathcal{E}_2) \) defines a UG\(_j\)-isomorphism \( \tilde{\mathcal{R}} : (\mathcal{E}_1, \mathcal{E}_1) \to (\mathcal{E}_2, \mathcal{E}_2) \). Applying Lemma 6.6.6 we obtain the UG\(_m\)-isomorphism \( \tilde{\mathcal{R}} : (\mathcal{E}_2, \mathcal{E}_2) \to (\mathcal{E}_1, \mathcal{E}_1) \) (since \( \mathcal{E}_1 \cong \mathcal{E}_1 \)), yielding a UG-isomorphism \( \tilde{\mathcal{R}} : (V_2, \mathcal{E}_2) \to (V_1, \mathcal{E}_1) \).

2. Let \( \mathcal{R} : (V_1, \mathcal{E}_1) \to (V_2, \mathcal{E}_2) \) be a UG-isomorphism. By (1) we have the UG-isomorphism \( \tilde{\mathcal{R}} : (V_2, \mathcal{E}_2) \to (V_1, \mathcal{E}_1) \). Then:

\[
(R \circ \tilde{R})^j = \tilde{R}^j \circ \mathcal{E}_2^j \circ \mathcal{R}^j \quad \text{by } (\mathbb{c} \downarrow)
\]

\[
= \tilde{R}^j \circ \mathcal{R}^j \circ \mathcal{E}_1^j \quad \text{since } \mathcal{R} \text{ a UG-morphism}
\]

\[
= \text{id} \circ \mathcal{E}_1^j = \mathcal{E}_1^j \quad \text{by } (\mathbb{c} \uparrow),
\]

and consequently \( \mathcal{R} \circ \tilde{\mathcal{R}} = \text{id}_{\mathcal{E}_1} \). By a symmetric argument one can prove \( \tilde{\mathcal{R}} \circ \mathcal{R} = \text{id}_{\mathcal{E}_2} \) too.

\[\square\]

Lemma 6.6.8 (Isomorphic graphs induce UG\(_j\) and UG\(_m\)-isomorphisms).

Fix any graph isomorphism \( f : (V_1, \mathcal{E}_1) \to (V_2, \mathcal{E}_2) \).

1. Each \( (\mathcal{G}, \mathcal{E}_1) \in \text{UG}_j \) has an associated UG\(_j\)-isomorphism \( \mathcal{G} : (\mathcal{G}, \mathcal{E}_1) \to (f^{-1}; \mathcal{G}, \mathcal{E}_2) \).

2. Each \( (\mathcal{G}, \mathcal{E}_2) \in \text{UG}_m \) has an associated UG\(_m\)-isomorphism \( \mathcal{G} : (\mathcal{G}; f, \mathcal{E}_1) \to (\mathcal{G}, \mathcal{E}_2) \).

Proof.

1. To see \( (f^{-1}; \mathcal{G}, \mathcal{E}_2) \in \text{UG}_j \) observe that \( \mathcal{E}_2 = f^{-1}; \mathcal{E}_1 ; f = f^{-1}; \mathcal{G}; \mathcal{E}_1 ; f \) and apply Lemma 6.5.7. Next, \( \mathcal{G} \) defines a Dep-isomorphism \( \mathcal{G} \to f^{-1}; \mathcal{G} \) via the following commuting diagram with bijective witnesses:

To show that \( \mathcal{G} \) defines a UG\(_j\)-isomorphism of the desired type, it suffices to establish that it is a UG\(_j\)-morphism by Lemma 6.6.2. Since \( f \) is a graph isomorphism we deduce \( \mathcal{E}_2^j \circ f^j = f^j \circ \mathcal{E}_1^j \) by Lemma 6.6.3, noting that \( \tilde{f}^j = f^j \). Then we calculate:

\[
(f^{-1}; \mathcal{G})^j \circ (\mathcal{E}_2 \circ \tilde{R})^j = \mathcal{G}^j \circ (f^{-1})^j \circ \mathcal{E}_2^j \circ ((f^{-1}; \mathcal{G})^{-1})^j \circ \tilde{R}^j \quad \text{by } (\mathbb{c} \downarrow) \quad \text{and } (\mathbb{c} \downarrow)
\]

\[
= \mathcal{G}^j \circ (f^{-1})^j \circ \mathcal{E}_2^j \circ (\mathcal{G}; f)^j \circ \tilde{G}^j \quad \tilde{f}^{-1} = f \quad \text{and } \tilde{R} = \mathcal{G}
\]

\[
= \mathcal{G}^j \circ (f^{-1})^j \circ \mathcal{E}_1^j \circ \tilde{G}^j \quad \text{by earlier equality}
\]

\[
= \mathcal{G}^j \circ \mathcal{E}_1^j \quad f; f^{-1} = \Delta_{V_1} \quad \text{and } \mathcal{E}_1^j \circ \text{id}_\mathcal{G} = \mathcal{E}_1^j.
\]

2. Given \( (\mathcal{G}, \mathcal{E}_2) \in \text{UG}_m \) then \( (\tilde{\mathcal{G}}, \mathcal{E}_2) \in \text{UG}_j \) so by (1) we have the UG\(_j\)-isomorphism \( \tilde{\mathcal{G}} : (\tilde{\mathcal{G}}, \mathcal{E}_2) \to (f^{-1}; \tilde{\mathcal{G}}, \mathcal{E}_1) \). Then by Lemma 6.6.6 we obtain the desired UG\(_m\)-isomorphism \( \mathcal{G} : (\mathcal{G}; f, \mathcal{E}_1) \to (\mathcal{G}, \mathcal{E}_2) \) since \( (f^{-1}; \mathcal{G})^{-1} = \mathcal{G}; f \).

\[\square\]

Lemma 6.6.9 (Lifting certain Dep-epis and monos to UG\(_j\) and UG\(_m\)).
Let \((V, \mathcal{E})\) be an undirected graph.

1. Given \((H, \mathcal{E}) \in \text{UG}_j\) then any \text{Dep}-morphism \(H : \mathcal{G} \to \mathcal{H}\) defines a \(\text{UG}_j\)-morphism \((\mathcal{G}, \mathcal{E}) \to (H, \mathcal{E})\).

2. Given \((\mathcal{G}, \mathcal{E}) \in \text{UG}_m\) then any \text{Dep}-morphism \(\mathcal{G} : \mathcal{G} \to \mathcal{H}\) defines a \(\text{UG}_m\)-morphism \((\mathcal{G}, \mathcal{E}) \to (H, \mathcal{E})\).

**Proof.**

1. For clarity let \(R : \mathcal{G} \to \mathcal{H}\) where \(R = H\). Then given that \((H, \mathcal{E}) \in \text{UG}_j\) we deduce that \((\mathcal{G}, \mathcal{E}) \in \text{UG}_j\) because \(E = H; E_+ = R; E_+ = \mathcal{G}; R_+; E_+\), so we can apply Lemma 6.5.7. Finally we calculate:

\[
\begin{align*}
\mathcal{H}^i \circ (\mathcal{E} \circ R)^i & = \mathcal{H}^i \circ \mathcal{H}^i \circ \mathcal{H}^i \circ R^i & \text{since } \mathcal{E} : \mathcal{H} \to \mathcal{H}^i \\
& = \mathcal{H}^i \circ \mathcal{H}^i \circ \mathcal{H}^i \circ R^i & \text{since } R = \mathcal{H} \\
& = \mathcal{H}^i \circ \mathcal{H}^i \circ \mathcal{H}^i & \text{since } \mathcal{E}^i \circ \text{in}_R = \mathcal{E}^i \\
& = \mathcal{R}^i \circ \mathcal{E}^i & \text{since } R = \mathcal{H}.
\end{align*}
\]

2. For clarity let \(R : \mathcal{G} \to \mathcal{H}\) where \(R = \mathcal{G}\). Given that \((\mathcal{G}, \mathcal{E}) \in \text{UG}_m\) then we deduce \((H, \mathcal{E}) \in \text{UG}_m\) because \(E = \mathcal{G}; E_+ = R; E_+ = \mathcal{G}; R_+; E_+\). Finally we calculate:

\[
\begin{align*}
(\mathcal{R} \circ \mathcal{E})^i & = \mathcal{R}^i \circ \mathcal{E}^i & \text{since } \mathcal{E} : \mathcal{G} \to \mathcal{G} \\
& = \mathcal{G}^i \circ \mathcal{E}^i & \text{since } R = \mathcal{G} \\
& = \mathcal{E}^i \circ \mathcal{G}^i & \text{since } \mathcal{E}^i \circ \mathcal{E}^i = \mathcal{E}^i \\
& = \mathcal{E}^i \circ \mathcal{R}^i & \text{since } R = \mathcal{G}.
\end{align*}
\]

\[\square\]

### 6.7 The three categorical equivalences

**Definition 6.7.1 (The equivalence functors).**

\[
\begin{align*}
\text{UG}_j & \xleftarrow{\cong} \text{SAJ} \quad \text{Open}_j \quad \text{Pirr}_j \quad \text{UG}_m & \xleftarrow{\cong} \text{SAM} \quad \text{Open}_m \quad \text{Pirr}_m \quad \text{UG} & \xleftarrow{\cong} \text{SAI} \quad \text{Open}_g \quad \text{Pirr}_g
\end{align*}
\]

1. **Action on objects:**

\[
\begin{align*}
\text{Open}_j(\mathcal{G}, \mathcal{E}) & := (\text{Open}_j, \partial_{\mathcal{G}}^{-1} \circ \text{Open}_\mathcal{E}) & \text{Pirr}_j(Q, \sigma) & := (\text{Pirr}_Q, \text{Pirr}_\sigma) \\
\text{Open}_m(\mathcal{G}, \mathcal{E}) & := (\text{Open}_\mathcal{G}, \text{Open}_\mathcal{E} \circ \partial_{\mathcal{G}}) & \text{Pirr}_m(Q, \sigma) & := (\text{Pirr}_Q, \text{Pirr}_\sigma) \\
\text{Open}_g(V, \mathcal{E}) & := (\text{Open}_\mathcal{E}, \partial_{\mathcal{E}}) & \text{Pirr}_g(Q, \sigma) & := (J(Q), \text{Pirr}_\sigma).
\end{align*}
\]

For clarity,

Regarding the functors on the left,

for \(\text{Open}_j\) \(\mathcal{E}\) is viewed as a (self-adjoint) \text{Dep}-morphism of type \(\mathcal{G} \to \mathcal{G}\) when applying \text{Open}.

for \(\text{Open}_m\) \(\mathcal{E}\) is viewed as a (self-adjoint) \text{Dep}-morphism of type \(\mathcal{G} \to \mathcal{G}\) when applying \text{Open}.

for \(\text{Open}_g\) \(\mathcal{E}\) is a viewed as a (symmetric) binary relation when applying \text{Open}.

Regarding the functors on the right,

for \(\text{Pirr}_j\) \(\sigma\) is viewed as a (self-adjoint) \text{JSL}_f\-morphism of type \(\text{Q} \to \text{Q}^{\text{op}}\) when applying \text{Pirr}.

for \(\text{Pirr}_m\) \(\sigma\) is viewed as a (self-adjoint) \text{JSL}_f\-morphism of type \(\text{Q}^{\text{op}} \to \text{Q}\) when applying \text{Pirr}.

for \(\text{Pirr}_g\) \(\sigma\) is viewed as a (self-adjoint) \text{JSL}_f\-morphism of type \(\text{Q} \to \text{Q}^{\text{op}}\) when applying \text{Pirr}.

2. **Action on morphisms:**

...
– $\text{Open}_i$, $\text{Open}_m$ and $\text{Open}_g$ act as $\text{Open}$ on the underlying Dep-morphism.
– $\text{Pirr}_j$ and $\text{Pirr}_m$ act as $\text{Pirr}$ on the underlying join-semilattice morphism.
– Finally, for any $\text{SAl}_f$-morphism $f : (Q, \sigma_1) \to (R, \sigma_2)$,

$$\text{Pirr}_g f := \text{Pirr}(\sigma_2 \circ f) \downarrow \equiv \text{Pirr} f; \sigma_2|_{J(R) \times J(R)} : (J(Q), \text{Pirr} \sigma_1) \to (J(R), \text{Pirr} \sigma_2)$$

where the asserted equality is proved below.

\begin{example}[Complete graphs.]
Consider $(V, \mathcal{E})$ where $\mathcal{E} := \Delta_V$. Applying $\text{Open}_{\mathcal{E}}$ yields the De Morgan algebra $(\text{Open}_{\mathcal{E}}, \partial_{\mathcal{E}})$. Recall $\text{Open}_{\mathcal{E}} = (O(\mathcal{E}), \cup, \circ)$ where $O(\mathcal{E}) := \{\mathcal{E}[X] : X \subseteq V\} = \{(\{v : v \in V\})_{v \in V}\}$, and:

$$\partial_{\mathcal{E}} = \lambda Y. \mathcal{E}[\overline{Y}] = \lambda Y. \begin{cases} \overline{\overline{\emptyset}} & \text{if } Y = \overline{\emptyset} \\ \emptyset & \text{if } Y = \emptyset \\ \overline{Y} & \text{if } Y = Y. \end{cases}$$

Then $O(\mathcal{E}) = \{\emptyset\}$ if $|V| \leq 1$ and is $\{\emptyset, V\} \cup \{\overline{\{v : v \in V\}}\}$ otherwise. Graphically:

\[
\begin{array}{c}
\emptyset & v_1 & v_2 & \ldots & v_{n-1} & v_n \\
\emptyset & \emptyset & \emptyset & \ldots & \emptyset & \emptyset \\
\end{array}
\]

They are well-defined De Morgan algebras and non-distributive whenever $|V| \geq 3$. Applying $\text{Pirr}_{\mathcal{E}}$ yields the graph $(J(M_V), \text{Pirr}\sigma)$ with vertices $J(M_V) = \{\overline{v} : v \in V\} \subseteq M_V$ and symmetric relation $\text{Pirr}\sigma \subseteq J(M_V) \times J(M_V)$,

$$\text{Pirr}\sigma(\overline{v_1}, \overline{v_2}) : \iff \overline{v_2} \notin \partial_{\mathcal{E}}(\overline{v_1}) \iff \overline{v_1} \notin \overline{v_2} \iff v_1 \neq v_2 \iff \mathcal{E}(v_1, v_2).$$

That is, these De Morgan algebras lead back to the complete graphs.
\end{example}

\begin{example}[Chains as undirected graphs.]
Chains are important examples of distributive lattices. Recall:

$$\mathcal{C}_n := (C_n, \text{max}, 0) \quad \text{where} \quad C_n := \{0, \ldots, n\}$$

has $n + 1$ elements whereas its Hasse diagram has $n$ edges, and by definition its length is $n$. We denote its underlying poset by $\mathcal{C}_n := (C_n, \leq_{\mathcal{C}_n})$. The join-semilattice morphisms $\mathcal{C}_{m+1} \to \mathcal{C}_n$ naturally biject with the monotone morphisms $\mathcal{C}_m \to \mathcal{C}_n$ via the free construction $F_j : \text{Poset}_f \to \text{JSL}_f$, see Definition 7.2.3 in the Appendix. Every chain $\mathcal{C}_n$ extends to a finite De Morgan algebra in precisely one way:

$$\sigma : C_n \to C_n \quad \sigma(x) := n - x.$$ 

The two equational axioms defining $\text{SAl}_f$ are satisfied because $x \leq_{\mathcal{C}_n} y$ implies $n - y \leq_{\mathcal{C}_n} n - x$, and moreover $\sigma(\sigma(x)) = n - (n - x) = x$. It is unique because $\mathcal{C}_n$ has only one automorphism. Here is $\mathcal{C}_n$ for $0 \leq n < 5$,

\[
\begin{array}{c}
0 & 1 & 2 & 3 & 4 \\
0 & 1 & 2 & 3 & 4 \\
0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 1 \\
\end{array}
\]

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Concerning their equivalent undirected graphs:

\[ \text{Pirr}_g(C_n, \sigma) = (C_n \setminus \{0\}, E) \]

where \[ E(x, y) : \iff y \notin c_n \sigma(x) \iff y > n - x \iff x + y > n. \]

We now depict \( \text{Pirr}_g C_n \) for \( 0 \leq n < 7 \),

They are planar graphs, and so is the next graph in the sequence. However, \( \text{Pirr}_g C_n \) is non-planar for all \( n \geq 8 \) because \( \{|\frac{n}{2}, \ldots, n\}| \geq 5 \) forms a clique, so we may apply Kuratowski’s theorem.

For brevity let \( C_n^\sigma = (C_n, \lambda x. n - x) \) for each \( n \geq 0 \). Whenever \( m = \alpha \cdot n \) i.e. \( m \) divides \( n \), there is an associated injective de morgan algebra morphism:

\[ f_{m, n} : C_m^\sigma \rightarrow C_n^\sigma \]

\[ f_{m, n}(k) := k \cdot \frac{n}{m}. \]

i.e. it defines a join-semilattice morphism and preserves the involution:

\[ 0 \cdot \frac{n}{m} = 0 \quad k \cdot \text{max}(x, y) = \text{max}(k \cdot x, k \cdot y) \quad (m - k) \cdot \frac{n}{m} = n - k \cdot \frac{n}{m}. \]

Then the corresponding \( \text{UG} \)-monomorphism \( \text{Pirr}_g f_{m, n} : (C_m \setminus \{0\}, E_m) \rightarrow (C_n \setminus \{0\}, E_n) \) is the relation:

\[ \text{Pirr}_g f_{m, n} \subseteq (C_m \setminus \{0\}) \times (C_n \setminus \{0\}) \]

where \( \text{Pirr}_g f_{m, n}(x, y) \iff 1 < n \cdot \frac{x}{m} + \frac{y}{n} \)

which follows by unwinding the definitions.

To prove well-definedness of the functors we’ll make use of the following Lemma. Recall that the diagonals of \( \text{UG}_j \) and \( \text{UG}_m \) are equal and isomorphic to \( \text{UG} \) by Lemma 6.6.1. We now provide isomorphisms between the two images \( \text{Pirr}_j[\text{SAI}_j \rightarrow \text{SAJ}_j] \) and \( \text{Pirr}_m[\text{SAI}_j \rightarrow \text{SAM}_f] \) and this diagonal. After proving functoriality we’ll be able to rephrase this result as two natural isomorphisms.

**Lemma 6.7.4** (The diagonals are isomorphic to the images of \( \text{SAI}_f \subseteq \text{SAJ}_f, \text{SAM}_f \)).

Take any finite de morgan algebra \((Q, \sigma) \in \text{SAI}_f\).

1. Viewing \( \sigma \) as a JSL-isomorphism \( Q \rightarrow Q^{op} \) we have the \( \text{UG}_j \)-isomorphism \( \text{Pirr}Q : (\text{Pirr}_j, \text{Pirr}_j) \rightarrow (\text{Pirr}_Q, \text{Pirr}_Q) \) with inverse \( \text{Pirr}_Q. \)

2. Viewing \( \sigma \) as a JSL-isomorphism \( Q^{op} \rightarrow Q \) we have the \( \text{UG}_m \)-isomorphism \( \text{Pirr} : (\text{Pirr}_j, \text{Pirr}_j) \rightarrow (\text{Pirr}_Q, \text{Pirr}_Q) \) with inverse \( \text{Pirr}. \)

**Proof.**

1. Given \((Q, \sigma) \in \text{SAI}_f\) then since \( \sigma_* = \sigma \) we deduce that \( \text{Pirr}_Q \) defines a self-adjoint \( \text{Dep} \)-morphism \( \text{Pirr}Q \rightarrow (\text{Pirr}Q)^\ast \) by Lemma 6.3.4.3. Thus \((\text{Pirr}Q, \text{Pirr}Q) \in \text{UG}_j\), by definition, and also \((\text{Pirr}_Q, \text{Pirr}_Q) \in \text{UG}_j\) because \( \text{Pirr}_Q \) is symmetric. To see that \( \text{Pirr}_Q \) defines a \( \text{Dep} \)-morphism of type \( \text{Pirr}_j \rightarrow \text{Pirr}Q \), first recall:

\[ \text{Pirr}_Q(j_1, j_2) : \iff \sigma(j_1) \notin Q \sigma(j_2) \iff j_2 \notin Q \sigma(j_1) \quad \text{and} \quad \text{Pirr}_Q(j, m) : \iff j \notin Q m. \]
Since $\sigma : Q \to Q^{\text{op}}$ is a join-semilattice isomorphism, it restricts to bijections $\sigma|_{J(Q) \times M(Q)}$ and $\sigma|_{M(Q) \times J(Q)}$, which are the inverse of one another because $\sigma$ is involutive. Then we have:

$$\text{Pirr}\sigma : \sigma|_{J(Q) \times M(Q)}(j, m) \iff \text{Pirr}\sigma(j, \sigma(m)) \iff \sigma(m) \leq Q \sigma(j) \iff j \leq Q m \iff \text{Pirr}Q(j, m),$$

where the marked equality follows because $\sigma$ defines an order-isomorphism $(Q, \leq Q) \to (Q, \geq Q)$. Then the following diagram of relations commutes:

$$
\begin{array}{cccc}
J(Q) & \xrightarrow{\sigma|_{J(Q) \times M(Q)}} & M(Q) & \xrightarrow{\sigma|_{M(Q) \times J(Q)}} & J(Q) \\
\text{Pirr}\sigma & \downarrow & \text{Pirr}\sigma & \downarrow & \text{Pirr}\sigma \\
J(Q) & \xrightarrow{\Delta_{J(Q)}} & J(Q) & \xrightarrow{\Delta_{J(Q)}} & J(Q)
\end{array}
$$

It follows that $\text{Pirr}Q : \text{Pirr}\sigma \to \text{Pirr}Q$ is a $\text{Dep}$-isomorphism with inverse $\text{Pirr}\sigma$. Then by Lemma 6.6.9 and also Lemma 6.6.2 it defines a $\text{UG}_{J}$-isomorphism $(\text{Pirr}\sigma, \text{Pirr}\sigma) \to (\text{Pirr}Q, \text{Pirr}\sigma)$ with the same inverse.

2. Let $(Q, \sigma) \in \text{SAI}_{f}$ and view $\sigma$ as a self-adjoint isomorphism $Q^{\text{op}} \to Q$. Since $(Q^{\text{op}}, \sigma) \in \text{SAI}_{f}$ we may apply (1), yielding the $\text{UG}_{J}$-isomorphism:

$$\text{Pirr}\sigma : (\text{Pirr}Q^{\text{op}}, \text{Pirr}\sigma) \to (\text{Pirr}\sigma, \text{Pirr}\sigma) \quad \text{with inverse } \text{Pirr}Q^{\text{op}}.$$ 

By Lemma 6.6.6 we obtain the $\text{UG}_{m}$-isomorphism:

$$\text{Pirr}\sigma : (\text{Pirr}\sigma, \text{Pirr}\sigma) \to (\text{Pirr}Q, \text{Pirr}\sigma) \quad \text{with inverse } \text{Pirr}Q,$$

also using the fact that $(\text{Pirr}\sigma)^{-} = \text{Pirr}\sigma$ and $(\text{Pirr}Q^{\text{op}})^{-} = \text{Pirr}Q$.

We now prove well-definedness of the functors under consideration. That their action on objects is well-defined follows via Lemma 6.3.4 i.e. the correspondence between self-adjointness in $\text{JSL}_{f}$ and $\text{Dep}$. Concerning their action on morphisms, well-definedness follows via mostly mindless computations. However, in the case of $\text{Pirr}_{g}$ we make crucial use of the above Lemma. Notice that this is the only functor whose action on morphisms is not inherited from the underlying equivalence functors $\text{Pirr}$ and $\text{Open}$.

**Lemma 6.7.5.** The six functors from Definition 6.7.1 above are well-defined.

**Proof.**

1. We first show that:

$$\text{Open}_{j} : \text{UG}_{J} \to \text{SAJ}_{f} \quad \text{and} \quad \text{Open}_{m} : \text{UG}_{m} \to \text{SAM}_{f} \quad \text{and} \quad \text{Open}_{g} : \text{UG} \to \text{SAI}_{f}$$

are well-defined. Consider their action on objects:

$$\text{Open}_{j}(G, E) := (\text{Open}G, \partial_{g}^{-1} \circ \text{Open}E) \quad \text{Open}_{m}(G, E) := (\text{Open}G, \text{Open}E \circ \partial_{g}) \quad \text{Open}_{g}(V, E) := (\text{Open}E, \partial_{E})$$

The left and central actions are well-defined by Lemma 6.3.4.2 parts (c) and (d). Regarding the rightmost, first apply the identity-on-morphisms categorical isomorphism $(V, E) \to (E, E)$ from Lemma 6.6.1.1, and subsequently $\text{Open}_{g}$. Then observe that $\partial_{E}^{-1} \circ \text{Open}id_{E} = \partial_{E}^{-1}$ acts the same as $\partial_{E}$ because $E = E$. Concerning the action on morphisms, we consider each functor in turn.

2. Concerning $\text{Open}_{g}$, take any $\text{UG}_{J}$-morphism $R : (G, E_{1}) \to (H, E_{2})$ and consider the well-defined join-semilattice morphism $\text{Open}_{g}R = \text{Open}R : \text{Open}G \to \text{Open}H$. To see that it is a $\text{SAJ}$-morphism we must establish that:

$$\text{Open}R(\sigma_{\text{Open}_{g}(G, E_{1})}(Y)) = \sigma_{\text{Open}_{g}(H, E_{2})}(\text{Open}R(Y)) \quad \text{for every } Y \in \text{O}(G_{1}) \subseteq \text{P}G_{1},$$

or more explicitly:

$$\text{Open}R(\partial_{g}^{-1} \circ \text{Open}_{g}E_{1}(Y)) = \partial_{H}^{-1} \circ \text{Open}_{g}E_{2}(\text{Open}R(Y)).$$
Since the $\mathcal{G}$-open sets are precisely those of the form $\mathcal{G}[X]$, we may equivalently show that $\forall X \subseteq \mathcal{G}_s$,

$$\text{Open}(\partial_1^{-1}(\text{Open}E_1(\mathcal{G}(X)))) \overset{?}{=} \partial_1^{-1} \circ \text{Open}E_2(\text{Open}(\mathcal{G}(X)))$$

by defn

$$\text{Open}(\partial_1^{-1}(\mathcal{E}_1(\mathcal{G}(X)))) = \partial_1^{-1}(\text{Open}E_2(\mathcal{R}_1(\mathcal{G}(X))))$$

by defn

$$\mathcal{E}_1 = \mathcal{G}; (\mathcal{E}_1)_+$$

by defn

$$\mathcal{R} = \mathcal{G}; \mathcal{R}_+$$

Then since $X$ is an arbitrary subset this amounts to our assumption $\mathcal{R}_1 \circ \mathcal{E}_1 = \mathcal{H} \circ (\mathcal{E}_2 \upharpoonright \mathcal{R})$. Hence $\text{Open}$'s action on both objects and morphisms is well-defined. It preserves the compositional structure because it acts in the same way as $\text{Open}$ : $\text{Dep} \to \text{JSL}_f$, and the compositional structure in both $\text{JSL}_f$ and $\text{SAJ}_f$ is functional.

3. Next consider $\text{Open}_m$ i.e. take any $\text{UG}_m$-morphism $\mathcal{R} : (\mathcal{G}, \mathcal{E}_1) \to (\mathcal{H}, \mathcal{E}_2)$ and consider the well-defined $\text{JSL}$-morphism $\text{Open}_m \mathcal{R} = \text{Open} \circ \mathcal{R} : \mathcal{G} \to \mathcal{H}$. To see that it is a $\text{SAM}$-morphism we must establish:

$$\text{Open}(\sigma_{\text{Open}_m(\mathcal{G}, \mathcal{E}_1)}(Y)) = \sigma_{\text{Open}_m(\mathcal{H}, \mathcal{E}_2)}(\text{Open}(Y)))$$

for every $Y \in O(\mathcal{G}_1) \subseteq PG_1$, or more explicitly:

$$\text{Open}(\text{Open}E_2 \circ \partial_\mathcal{G}(Y)) = \text{Open}E_2 \circ \partial_\mathcal{H}(\text{Open}(Y)).$$

Since the $\mathcal{G}$-open sets are precisely those of the form $\mathcal{G}[X]$, we may equivalently show that $\forall X \subseteq \mathcal{G}_s$,

$$\text{Open}(\partial_1^{-1}(\text{Open}E_1(\mathcal{G}(X)))) \overset{?}{=} \partial_1^{-1} \circ \text{Open}E_2(\text{Open}(\mathcal{G}(X)))$$

by defn

$$\text{Open}(\partial_1^{-1}(\mathcal{E}_1(\mathcal{G}(X)))) = \partial_1^{-1}(\text{Open}E_2(\mathcal{R}_1(\mathcal{G}(X))))$$

by defn

$$\mathcal{E}_1 = \mathcal{G}; (\mathcal{E}_1)_+$$

by defn

$$\mathcal{R} = \mathcal{G}; \mathcal{R}_+$$

Since $X$ was arbitrary this amounts to our assumed condition. Then $\text{Open}_m$'s action on both objects and morphisms is well-defined. It preserves the compositional structure because it acts in the same way as $\text{Open}$, and the compositional structure in both $\text{JSL}_f$ and $\text{SAM}_f$ is functional.

4. Finally consider $\text{Open}_e$ i.e. take any $\text{UG}_e$-morphism $\mathcal{R} : (\mathcal{V}_1, \mathcal{E}_1) \to (\mathcal{V}_2, \mathcal{E}_2)$. Then this relation defines a $\text{UG}_e$-morphism $\mathcal{R} : (\mathcal{E}_1, \mathcal{E}_1) \to (\mathcal{E}_2, \mathcal{E}_2)$, so by (2) we deduce that $\text{Open}\mathcal{R}$ defines a $\text{SAJ}_f$-morphism of type:

$$(\text{Open} \mathcal{G}, \partial_{\mathcal{E}_1}) = (\text{Open} \mathcal{G}, \partial_{\mathcal{E}_1}^{-1} \circ \text{Open} \text{id}_{\mathcal{E}_1}) \to (\text{Open} \mathcal{G}, \partial_{\mathcal{E}_2}^{-1} \circ \text{Open} \text{id}_{\mathcal{E}_2}) = (\text{Open} \mathcal{G}, \partial_{\mathcal{E}_2})$$

recalling that $\partial_{\mathcal{E}}$ and $\partial_{\mathcal{E}}^{-1}$ have the same action whenever $\mathcal{E} \equiv \mathcal{E}$. Then since each $\partial_{\mathcal{E}}$ is an isomorphism this is actually a $\text{SAJ}_f$ morphism by Lemma 6.4.3. As before, functoriality follows from that of $\text{Open}$.

5. It remains to show that the three functors:

$$\text{Pirr}_f : \text{SAJ}_f \to \text{UG}_j$$

and $\text{Pirr}_m : \text{SAM}_f \to \text{UG}_m$ and $\text{Pirr}_g : \text{SAJ}_f \to \text{UG}$

are well-defined. Their action on objects:

$$\text{Pirr}_f(\mathcal{Q}, \sigma) := (\text{Pirr}_\mathcal{Q}, \text{Pirr}\sigma)$$

$$\text{Pirr}_m(\mathcal{Q}, \sigma) := (\text{Pirr}_\mathcal{Q}, \text{Pirr}\sigma)$$

$$\text{Pirr}_g(\mathcal{Q}, \sigma) := (J(\mathcal{Q}), \text{Pirr}\sigma)$$

is well-defined by Lemma 6.3.4.3.c, recalling that if $\sigma : \mathcal{Q} \to \mathcal{Q}^{op}$ then $\text{Pirr}\sigma : \text{Pirr}\mathcal{Q} \to \text{Pirr}(\mathcal{Q}^{op}) = (\text{Pirr}\mathcal{Q})^\circ$. So now consider their action on morphisms.
6. Take any \( \text{SAJ}_f \)-morphism \( f: (Q, \sigma_Q) \to (R, \sigma_R) \) and consider \( \text{Pirr}_f := \text{Pirrf} \). We know \( f(\sigma_Q(q)) = \sigma_R(f(q)) \) for every \( q \in Q \), and must establish the equality:

\[
(A) := \quad \mathcal{R}^! \circ (\text{Pirr}\sigma_Q)^! \overset{?}{=} (\text{Pirr}\mathcal{R})^! \circ (\text{Pirr}\sigma_R; \mathcal{R})^! =: (B)
\]

where we define \( \mathcal{R} := \text{Pirrf} \). We’ll achieve this by showing that these two functions have the same action.

(A) Given any subset \( X \subseteq J(Q) \), we calculate:

\[
\mathcal{R}^! \circ (\text{Pirr}\sigma_Q)^!(X) = \mathcal{R}^!((\{j \in J(Q) : \text{Pirr}\sigma_Q[j] \subseteq X\}) = \mathcal{R}^!((\{j : \forall j' \in J(Q), [j' \not\leq \sigma_Q(j) \Rightarrow j' \not\in X\}))
\]

by definition of \( (\cdot)^! \)

\[
= \mathcal{R}^!((\{j : \forall j' \in J(Q), [j' \not\leq \sigma_Q(j) \Rightarrow j' \not\in X\})) = \mathcal{R}^!((\{j : \forall j' \in J(Q), [j' \not\leq \sigma_Q(j) \Rightarrow j' \not\in X\}))
\]

by definition of \( \text{Pirr}\sigma_Q \)

\[
= \mathcal{R}^!((\{j : \forall j' \in J(Q), [j' \not\leq \sigma_Q(j) \Rightarrow j' \not\in X\})) = \mathcal{R}^!((\{j : \forall j' \in J(Q), [j' \not\leq \sigma_Q(j) \Rightarrow j' \not\in X\}))
\]

by definition of \( \mathcal{R}^! \)

\[
= \{m \in M(R) : \exists j \in J(Q).[f(j) \not\leq m \text{ and } \mathcal{V}_Q \mathcal{X} \not\leq m \sigma_Q(j)]\}
\]

by definition of \( \mathcal{R} \)

\[
= \{m \in M(R) : \exists j \in J(Q).[f(j) \not\leq m \text{ and } \mathcal{V}_Q \mathcal{X} \not\leq m \sigma_Q(j)]\}
\]

by definition of \( \mathcal{R}^! \)

\[
= \{m \in M(R) : \exists j \in J(Q).[f(j) \not\leq m \text{ and } \mathcal{V}_Q \mathcal{X} \not\leq m \sigma_Q(j)]\}
\]

by definition of \( \text{Pirrf} \)

\[
= \{m \in M(R) : \exists j \in J(Q).[f(j) \not\leq m \text{ and } \mathcal{V}_Q \mathcal{X} \not\leq m \sigma_Q(j)]\}
\]

take adjoints, \( \sigma_Q \) self-adjoint

\[
= \{m \in M(R) : \exists j \in J(Q).[f(j) \not\leq m \text{ and } \mathcal{V}_Q \mathcal{X} \not\leq m \sigma_Q(j)]\}
\]

f a \( \text{SAJ} \)-morphism.

(B) Then let us consider the other action:

\[
(\text{Pirr}\mathcal{R})^! \circ (\text{Pirr}\sigma_R; \mathcal{R})^!(X)
\]

\[
= (\text{Pirr}\mathcal{R})^! \circ (\text{Pirr}\sigma_R; \mathcal{R})^!(X)
\]

by definition of \( \mathcal{R} = (\text{Pirrf})^- = \text{Pirrf} \).

\[
= (\text{Pirr}\mathcal{R})^! \circ \text{Pirr}(f \circ \sigma_R)^!(X)
\]

by definition of \( (\cdot)^! \)

\[
= (\text{Pirr}\mathcal{R})^! \circ \text{Pirr}(f \circ \sigma_R)^!(X)
\]

by definition of \( \text{Pirr} \).

\[
= (\text{Pirr}\mathcal{R})^! \circ \text{Pirr}(f \circ \sigma_R)^!(X)
\]

by definition of \( (\cdot)^! \)

\[
= (\text{Pirr}\mathcal{R})^! (\{j_r \in J(R) : f(m) \not\leq j_r \text{ and } \mathcal{V}_Q \mathcal{X} \not\leq \sigma_R(f(m))\})
\]

by definition of \( (\cdot)^! \)

\[
= (\text{Pirr}\mathcal{R})^! (\{j_r : \forall j_r \in J(Q), [j_r \not\leq \sigma_R(f(m)) \Rightarrow j_r \not\leq m]\})
\]

by definition of \( (\cdot)^! \)

\[
= (\text{Pirr}\mathcal{R})^! (\{j_r : \forall j_r \in J(Q), [j_r \not\leq \sigma_R(f(m)) \Rightarrow j_r \not\leq m]\})
\]

by definition of \( (\cdot)^! \)

\[
= (\text{Pirr}\mathcal{R})^! (\{j_r : \forall j_r \in J(Q), [j_r \not\leq \sigma_R(f(m)) \Rightarrow j_r \not\leq m]\})
\]

by definition of \( (\cdot)^! \)

\[
= (\text{Pirr}\mathcal{R})^! (\{j_r : \forall j_r \in J(Q), [j_r \not\leq \sigma_R(f(m)) \Rightarrow j_r \not\leq m]\})
\]

by definition of \( (\cdot)^! \)

\[
= (\text{Pirr}\mathcal{R})^! (\{j_r : \forall j_r \in J(Q), [j_r \not\leq \sigma_R(f(m)) \Rightarrow j_r \not\leq m]\})
\]

by definition of \( (\cdot)^! \)

\[
= (\text{Pirr}\mathcal{R})^! (\{j_r : \forall j_r \in J(Q), [j_r \not\leq \sigma_R(f(m)) \Rightarrow j_r \not\leq m]\})
\]

by definition of \( (\cdot)^! \)

Thus \( \text{Pirr}_f \)’s action on objects and morphisms is well-defined. Then it is a well-defined functor because it acts as \( \text{Pirr} \) on morphisms, and \( \text{UG}_j \) inherits the compositional structure of \( \text{Dep} \).

7. Next, take any \( \text{SAM}_f \)-morphism \( f: (Q, \sigma_Q) \to (R, \sigma_R) \) and consider \( \text{Pirr}_m f := \text{Pirrf} \). We know that \( f \) preserves the unary operations and must establish:

\[
(A) := \quad (\text{Pirr}\sigma_R)^! \circ \mathcal{R}^! \overset{?}{=} (\mathcal{R}; \text{Pirr}\sigma_Q)^! \circ (\text{Pirr}\sigma_R)^! =: (B)
\]

where we define \( \mathcal{R} := \text{Pirrf} \). Then let us simplify their actions.

(A) Given any subset \( Y \subseteq J(Q) \), we calculate:

\[
(\text{Pirr}\sigma_R)^! \circ \mathcal{R}^!(Y)
\]

\[
= \{m \in M(R) : \text{Pirr}\sigma_R(m) \subseteq \mathcal{R}[Y]\}
\]

by definition of \( (\cdot)^! \)

\[
= \{m \in M(R) : \forall m' \in M(R). (\text{Pirr}\sigma_R(m, m') \Rightarrow m' \in \mathcal{R}[Y]\})
\]

by definition of \( \text{Pirr}\sigma_R, \mathcal{R} \)

\[
= \{m \in M(R) : \forall m' \in M(R). (\text{Pirr}(m) \not\leq m' \Rightarrow m' \not\in \mathcal{R}[Y]\})
\]

by definition of \( \text{Pirrf} \)

\[
= \{m \in M(R) : \forall m' \in M(R). (\text{Pirr}(m) \not\leq m' \Rightarrow m' \not\in \mathcal{R}[Y]\})
\]

by definition of \( \text{Pirrf} \)

\[
= \{m \in M(R) : \text{Pirr}(m) \not\leq \mathcal{R}[Y]\}
\]

by definition of \( \text{Pirrf} \)

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(B) Let us consider the other action.

\[
(\mathcal{R}_\sigma \circ \text{Pir} \sigma_Q) \circ (\text{Pir} \sigma_Q)(Y)
\]

functorality of \text{Pirr}

\[
= (\text{Pir} \sigma_Q \circ f)(\text{Pir} \sigma_Q)(Y)
\]
definition of \((-)\)

\[
= (m \in M(Q) : \text{Pir} \sigma_Q(m) \preceq \text{Pir} \sigma_Q(Y))
\]
definition of \text{Pirr}

\[
= (m \in M(Q) : \exists m'. \forall m \in M(Q), (\text{Pir} \sigma_Q(m) \preceq m' \rightarrow m' \in \text{Pir} \sigma_Q(Y))
\]
definition of \text{PirrQ}

\[
= (m \in M(Q) : \forall m'. (\forall y \in Y, y \preceq m' \rightarrow \exists m \in \text{Pir} \sigma_Q(Y))
\]
definition of \text{PirrQ}

Thus \text{Pirr}_m's action on objects and morphisms is well-defined. Then it is a well-defined functor because it acts as \text{Pirr} on morphisms, and \text{UG}_m inherits the compositional structure of \text{Dep}.

8. Finally we consider the action of \text{Pirr}_g on \text{SAI}_f-morphisms \( f : (Q, \sigma_Q) \rightarrow (R, \sigma_R) \). The two different descriptions of its action are equivalent because:

\[
\text{Pirr}(\sigma_R \circ f)(j_q, j_r) \iff \sigma_R \circ f(j_q) \preceq \sigma_R \circ j_r \quad \text{by definition}
\]

\[
j_r \preceq f(j_q) \preceq j_r \quad \text{self-adjoint}
\]

where the final step uses the fact that the \text{JSL}_f-isomorphism \( \sigma : R \rightarrow R^{op} \) restricts to a bijection \( \sigma|_{M(R) \times J(R)} \). Next, since \( f \) is also a \text{SAI}_f-morphism and \text{Pirr}_f is well-defined by (6), \( \text{Pirr}_f : (\text{Pirr}_Q, \text{Pirr}(\sigma_Q)) \rightarrow (\text{Pirr}_R, \sigma_R) \) is a well-defined \text{UG}_f-morphism. Then using Lemma 6.7.4 we have the well-defined \text{UG}_f-morphism:

\[
\begin{align*}
(\text{Pirr}_Q, \text{Pirr}(\sigma_Q)) & \xrightarrow{R_{\sigma_Q}} (\text{Pirr}_Q, \text{Pirr}(\sigma_Q)) \\
& \xrightarrow{\text{Pirr}_f} (\text{Pirr}_R, \text{Pirr}(\sigma_R)) \\
& \xrightarrow{(R_{\sigma_R})^{-1}} (\text{Pirr}_R, \text{Pirr}(\sigma_R))
\end{align*}
\]

formed by pre/post-composing with \text{SAI}_f-isomorphisms \( R_{\sigma_Q} = \text{Pirr}_Q \) and \( R_{\sigma_R}^{-1} = \text{Pirr}_Q \). This composite is actually \text{Pirr}_g f by the following calculation:

\[
\begin{align*}
(R_{\sigma_Q} \circ \text{Pirr}_f \circ R_{\sigma_R})^{-1} &= (R_{\sigma_R})^{-1} \circ \text{Pirr}_R \circ \text{Pirr}_f \circ (R_{\sigma_Q})^{-1} \\
&= (R_{\sigma_R})^{-1} \circ \text{Pirr}_R \circ (R_{\sigma_Q})^{-1} \circ \text{Pirr}_f \circ (R_{\sigma_Q})^{-1} \circ \text{Pirr}_R \circ \text{Pirr}_f \circ (R_{\sigma_Q})^{-1} \\
&= \text{Pirr}_f \circ \text{Pirr}_R \circ (R_{\sigma_Q})^{-1} \circ \text{Pirr}_f \circ (R_{\sigma_Q})^{-1} \circ \text{Pirr}_R \circ \text{Pirr}_f \\
&= \text{Pirr}_g f
\end{align*}
\]

Then \text{Pirr}_g is a well-defined functor using the functorality of \text{Pirr}, the uniform nature of the isomorphisms \( R_{\sigma} \), and the fact that \text{UG} is isomorphic to the diagonal of \text{UG}_f.

Having proved functorality we can now capture previous concepts as natural isomorphisms.

**Definition 6.7.6** (Natural isomorphisms involving the diagonals).

1. The functors \( I_j, I_m, \text{Diag}_j \) and \( \text{Diag}_m \).

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We have the functors:
\[
I_j : SAI_f \to SAJ_f \quad I_m : SAI_f \to SAM_f \quad \text{identity on objects and morphisms},
\]
\[
\text{Diag}_j : UG \to UG_j \quad \text{Diag}_m : UG \to UG_m \quad \text{identity on morphisms}.
\]
That is, \( I_j \) and \( I_m \) are the full-inclusion functors whereas the action of the full functors \( \text{Diag}_j \) and \( \text{Diag}_m \) on objects is \( (V, \mathcal{E}) \to (\mathcal{E}, \mathcal{E}) \).

2. The image of \((Q, \sigma) \in SAI_f \subseteq SAJ_f\) under \(Pirr_j\) is naturally \(UG_j\)-isomorphic to \((Pirr\sigma, Pirr\sigma)\).

We have the natural isomorphism:
\[
\begin{align*}
  r_j : \text{Diag}_j \circ Pirr_g \Rightarrow Pirr_j \circ I_j & \\
  r_{j(Q, \sigma)} := Pirr_Q : (Pirr\sigma, Pirr\sigma) \to (PirrQ, Pirr\sigma) & \\
  r_{j(Q, \sigma)}^{-1} := Pirr\sigma : (PirrQ, Pirr\sigma) \to (Pirr\sigma, Pirr\sigma)
\end{align*}
\]
noting that \( \sigma \) is viewed as a join-semilattice morphism \(Q \to Q^{op}\) when applying \(Pirr\).

\textbf{Note 6.7.7} (Concerning a certain asymmetry in our approach).

Just as we have the natural isomorphism \(r_j : \text{Diag}_j \circ Pirr_g \Rightarrow Pirr_j \circ I_j\) there is another natural isomorphism \(rm : \text{Diag}_m \circ Pirr_g \Rightarrow Pirr_m \circ I_m\) defined:
\[
rm_{(Q, \sigma)} := Pirr_Q : (Pirr\sigma, Pirr\sigma) \to (PirrQ, Pirr\sigma_m)
\]
with inverse \(Pirr\sigma_j\).

Here \(\sigma_j : Q \to Q^{op}\) and \(\sigma_m : Q^{op} \to Q\) are the two join-semilattice isomorphisms whose underlying function is \(\sigma\). Both join-semilattice morphisms are necessary because we choose to view \(SAI_f\)-algebras as morphisms \(Q \to Q^{op}\) when applying \(Pirr_g\), whereas the \(SAM_f\)-algebras are necessarily viewed as morphisms \(Q^{op} \to Q\). We will not need to use \(rm\) in what follows.

\textbf{Lemma 6.7.8}.

1. The functors and natural isomorphism \(r_j : \text{Diag}_j \circ Pirr_g \Rightarrow Pirr_j \circ I_j\) from Definition 6.7.6 are well-defined.

2. We have the following equalities:
\[
\begin{align*}
  \text{Open}_j \circ \text{Diag}_j & = I_j \circ \text{Open}_g \\
  \text{Open}_m \circ \text{Diag}_m & = I_m \circ \text{Open}_g.
\end{align*}
\]

\textbf{Proof}.

1. The fully-faithful inclusion-functors \(I_j\) and \(I_m\) are well-defined because \(SAI_f = SAJ_f \cap SAM_f\). Recall that the full subcategories of \(UG_j\) and \(UG_m\) consisting of objects \((\mathcal{E}, \mathcal{E})\) are actually equal, and also categorically isomorphic to \(UG\) by Lemma 6.6.1. It follows that \(\text{Diag}_j\) and \(\text{Diag}_m\) are well-defined fully-faithful functors, since they act in the same way as this categorical isomorphism.

Next, the components \(r_{j(Q, \sigma)}\) are well-defined \(UG_j\)-isomorphisms by Lemma 6.7.4 (which also specifies the inverses) where the typing is correct because:
\[
\begin{align*}
  \text{Diag}_j \circ Pirr_g(Q, \sigma) & = \text{Diag}_j(J(Q), Pirr\sigma) = (Pirr\sigma, Pirr\sigma) \\
  Pirr_j \circ I_j(Q, \sigma) & = Pirr_j(Q, \sigma) = (PirrQ, Pirr\sigma).
\end{align*}
\]
Concerning naturality we must verify that for every \(SAI_f\)-morphism \(f : (Q, \sigma_1) \to (R, \sigma_2)\) the following square commutes inside \(UG_j\):
\[
\begin{array}{ccc}
  (Pirr\sigma_1, Pirr\sigma_1) & \xrightarrow{r_{j(Q, \sigma_1)}} & (Pirr\sigma_1, Pirr\sigma_1) \\
  \downarrow & & \downarrow Pirr_j \\
  (Pirr\sigma_2, Pirr\sigma_2) & \xrightarrow{r_{j(R, \sigma_2)}} & (Pirr\sigma_2, Pirr\sigma_2)
\end{array}
\]
Firstly \((Pirr\sigma ; Pirr f)^! = (Pirr f)^! \circ (PirrQ)^! \circ (Pirr\sigma)^! = (Pirr f)^!\) and secondly:
\[
\begin{align*}
  (Pirr(\sigma_2 \circ f)^! ; Pirr\sigma)^! & \\
  = (Pirr\sigma)^! \circ (Pirr\sigma_2)^! \circ (Pirr(\sigma_2 \circ f))^! & \text{by } (\dagger) \\
  = (Pirr\sigma)^! \circ (Pirr\sigma_2)^! \circ (Pirr f)^! \circ (Pirr\sigma_2)^! & \text{functorality of } Pirr \\
  = (Pirr\sigma)^! \circ (Pirr\sigma_2)^! \circ (Pirr f)^! \circ (Pirr\sigma)^! \circ (Pirr\sigma_2)^! & \text{by } (\dagger) \\
  = (Pirr\sigma)^! \circ (Pirr\sigma_2)^! \circ (Pirr f)^! & (Pirr\sigma)^! = \text{cl}_{Pirr\sigma_2} \circ (Pirr\sigma)^! \\
  = (Pirr f)^! & (Pirr)^! = \text{in}_{Pirr\sigma_2} \circ (Pirr f)^!.
\end{align*}
\]
2. One can directly verify that their action on objects and morphisms are the same, recalling that the underlying function of $\partial E^1$ and $\partial ^\triangleright$ are equal because $E$ is symmetric.

We now finally prove the three categorical equivalences.

**Theorem 6.7.9** (Categorical equivalence between $\text{SAJ}_f$ and $\text{UG}_j$).

The functors $\textit{Open}$ and $\textit{Pirr}_j$ define an equivalence of categories with respective natural isomorphisms:

\[
\begin{align*}
\text{jrep} : \text{Id}_{\text{SAJ}_f} &\Rightarrow \text{Open}_j \circ \text{Pirr}_j \\
\text{jrep}_{(Q, \sigma)} : (Q, \sigma) &\rightarrow (\text{OpenPirr}_Q, \partial^{-1} \circ \text{OpenPirr}_Q) \\
\text{jrep}_{(Q, \sigma)}(q) &:= \text{rep}_Q(q) = \{ m \in M(Q) : q \not\in Q \} \\
\text{jrep}_{(Q, \sigma)}(Y) &:= \text{rep}^{-1}_Q(Y) = \wedge_0 M(Q) \setminus Y.
\end{align*}
\]

\[
\begin{align*}
\text{jred} : \text{Id}_{\text{UG}_j} &\Rightarrow \text{Pirr}_j \circ \text{Open}_j \\
\text{jred}_{(G, E)} : (G, E) &\rightarrow (\text{Pirr}_G \text{Open}_G, \text{Pirr} (\partial^{-1}_G \circ \text{Open}_G)) \\
\text{jred}_{(G, E)} &:= \text{red}_G = \{(v, Y) \in V \times M(\text{Open}_G) : G[v] \not\in Y \} \\
\text{jred}_{(G, E)} &:= \text{red}^{-1}_G = \xi \subseteq J(\text{Open}_G) \times G.
\end{align*}
\]

The associated components of the latter natural isomorphisms follow from Theorem 4.2.10.

**Proof.**

1. Regarding $\text{jrep}$, we already know that $\text{rep} : \text{Id}_{\text{SAJ}_f} \Rightarrow \text{OpenPirr}$ defines a natural isomorphism by Theorem 4.2.10.

Then since $\text{OpenPirr}_j f = \text{OpenPirr} f$ as functions, it suffices to show that for each $(Q, \sigma) \in \text{SAJ}_f$,

\[
\text{rep}_Q(\sigma(q)) = \partial^{-1}_Q \circ \text{OpenPirr}_Q(\text{rep}_Q(q))
\]

for every $q \in Q$, i.e. the respective unary operations are preserved. The LHS equals $\{ m \in M(Q) : \sigma(q) \not\in Q \}$, so let us consider the other side:

\[
\begin{align*}
\partial^{-1}_Q \circ \text{OpenPirr}_Q(\text{rep}_Q(q)) \\
= \partial^{-1}_Q((\text{Pirr}_Q)_* \{ \{ m \in M(Q) : q \not\in Q \} \}) \\
= \text{Pirr}_Q^{\dagger} \circ \text{OpenPirr}_Q(\{ m \in M(Q) : q \not\in Q \}) \\
= \text{Pirr}_Q^{\dagger} \circ \text{OpenPirr}_Q(\{ m \in M(Q) : q \not\in Q \}) \\
= \text{Pirr}_Q^{\dagger} \circ \text{OpenPirr}_Q(\{ m \in M(Q) : q \not\in Q \}) \\
= \{ m \in M(Q) : \exists j \in J(Q), [j, \not\in Q \} \\
= \{ m \in M(Q) : \forall j \in J(Q), [j, \not\in Q \} \\
= \{ m \in M(Q) : \sigma(j) \not\in Q \}
\end{align*}
\]

which completes the proof that $\text{jrep}$ is natural. The description of its inverse is immediate, and also instantiates $\text{rep}^{-1}$ from Theorem 4.2.10.

2. Concerning $\text{jred}$, we know that $\text{red} : \text{Id}_{\text{Dep}} \Rightarrow \text{Pirr}_G \text{Open}_G$ defines a natural isomorphism by Theorem 4.2.10. Since $\text{Pirr}_j \text{Open}_j \mathcal{R} = \text{Pirr}_G \text{Open}_G \mathcal{R}$ as Dep-morphisms it suffices to show that for each $(G, E)$ we have:

\[
\text{red}_G \circ \mathcal{E}^j = (\text{Pirr}_G \text{Open}_G)^{\dagger} \circ (\text{Pirr}(\partial^{-1}_G \circ \text{Open}_G) ; \text{red}_G)^{\dagger}
\]

Regarding the LHS, for every subset $X \subseteq G_s$ we have:

\[
\begin{align*}
\text{red}_G \circ \mathcal{E}^j(X) \\
= \{ Y \in M(\text{Open}_G) : \exists [g_s \in G_s, \in \mathcal{E}^j(X) \text{ and } G[g_s] \not\in Y] \} \\
= \{ Y \in M(\text{Open}_G) : \exists [g_s \in G_s, \in \mathcal{E}^j(X) \text{ and } G[g_s] \not\in Y] \} \\
= \{ Y \in M(\text{Open}_G) : \mathcal{E}^j(X) \subseteq \mathcal{E}^j(Y) \} \\
= \{ Y \in M(\text{Open}_G) : \mathcal{E}^j(X) \not\subseteq \mathcal{E}^j(Y) \}.
\end{align*}
\]

by $(\uparrow \downarrow)$
As for the RHS, we first simplify a sub-term:

\[
\begin{align*}
\text{Pirr}(\partial_g^{-1} \circ \text{OpenE}) & \circ \text{red}_g(Y, g_s) \\
\iff & \text{Pirr}(\partial_g^{-1} \circ \text{OpenE}); (\text{red}_g)^1(Y, g_s) \\
& \text{Dep-composition} \\
& (S^*)_\ast = S. \text{ generally} \\
\iff & \exists Y'. (\partial_g^{-1} \circ \text{OpenE}(Y') \ni (\text{OpenG})_{\ast} Y' \text{ and } Y' \ni G[g_s]) \\
& \text{definition of Pirr, red}_g \\
\iff & \exists Y'. (Y' \ni G[g_s]) \\
& \text{OpenG inclusion-ordered} \\
\iff & \forall Y'. (Y' \ni G[g_s]) \Rightarrow Y' \ni \partial_g^{-1} \circ \text{OpenE}(Y')) \\
\iff & G[g_s] \ni \partial_g^{-1} \circ \text{OpenE}(Y) \\
\iff & \text{OpenE}(Y) \ni \partial_g(G[g_s]) \\
\iff & \text{definition of } \partial_g, (\sim \uparrow \sim) \\
\iff & g_s \in \text{in}_g(Y) \\
& \text{by Lemma 4.2.7.1}
\end{align*}
\]

and finally simplify its action:

\[
\begin{align*}
(\text{PirrOpenG})^1 & \circ (\text{Pirr}(\partial_g^{-1} \circ \text{OpenE}) \circ \text{red}_g)^1(X) \\
= & (\text{PirrOpenG})^1 \circ (\text{Pirr})(\partial_g^{-1} \circ \text{OpenE}) \circ \text{red}_g(X) \\
= & \{ Y \in \text{M(OpenE)} : (\exists \ Y' \ni X \ni Y \ni \text{OpenE}(X')) \} \\
= & \{ Y : \forall X'. (X' \ni \text{OpenE}(X') \ni X \Rightarrow X' \ni Y) \} \\
= & \{ Y : (\text{OpenE})^1(X) \ni Y \} \\
= & \{ Y : G^1 \circ \partial^1(Y) \ni Y \} \\
& \text{using Lemma 4.2.6.4}
\end{align*}
\]

as required. The description of its inverse follows because red$^{-1}_g$ is the inverse Dep-isomorphism by Theorem 4.2.10, and thus is also the inverse UG$_f$-isomorphism by Lemma 6.6.6.

Next we prove the equivalence SAM$_f \cong$ UG$_m$, making use of SAJ$_f \cong$ UG$_f$. As was the case with the latter equivalence, the respective natural isomorphisms lift directly from the fundamental equivalence JSL$_f \cong$ Dep.

**Theorem 6.7.10 (Categorical equivalence between SAM$_f$ and UG$_m$).**

The functors Open$_m$ and Pirr$_m$ define an equivalence of categories with witnessing natural isomorphisms:

\[
\begin{align*}
\text{mrep} & : \text{Id}_{\text{SAM}_f} \Rightarrow \text{Open}_m \circ \text{Pirr}_m \\
\text{mrep}_{(Q, \sigma)} & : (Q, \sigma) \rightarrow (\text{OpenPirr}Q, \text{OpenPirr} \circ \partial_{\text{Pirr}Q}) \\
\text{mrep}_{(Q, \sigma)}(q) & := \text{rep}_Q(q) = \{ m \in \text{M}(Q) : q \ni_0 m \} \\
\text{mrep}_{(Q, \sigma)}^1(Y) & := \text{rep}_Q^1(Y) = \wedge_0 Q \ni M(Q) \backslash Y.
\end{align*}
\]

\[
\begin{align*}
\text{mred} & : \text{Id}_{\text{UG}_m} \Rightarrow \text{Pirr}_m \circ \text{Open}_m \\
\text{mred}_{(Q, E)} & : (Q, E) \rightarrow (\text{PirrOpenG}, \text{PirrOpenE} \circ \partial_{\text{Pirr}Q}) \\
\text{mred}_{(Q, E)} & := \text{red}_Q := \{ (g_s, Y) \in G_s \times \text{M(OpenG)} : G[g_s] \ni Y \} \\
\text{mred}_{(Q, E)}^1(Y) & := \text{red}_Q^1 = \partial_{\text{Pirr}Q}^{-1} = \partial_{\text{Pirr}Q}^{-1} \ni J(\text{OpenG}) \times Y
\end{align*}
\]

The associated components of the latter natural isomorphisms follow from Theorem 4.2.10.

**Proof.**

1. Fix any $(Q, \sigma) \in \text{SAM}_f$, so we have a respective join-semilattice morphism $\sigma : Q^\circ \rightarrow Q$. Then $(Q^\circ, \sigma) \in \text{SAJ}_f$ so by Theorem 6.7.9 we have the SAJ$_f$-isomorphism: $\text{Jrep}_{(Q^\circ, \sigma)} : (Q^\circ, \sigma) \rightarrow (\text{OpenPirr}Q^\circ, \partial_{\text{Pirr}Q^\circ}) \circ \text{OpenPirr}$. We can apply the opposite construction to join-semilattice isomorphisms, yielding the SAM$_f$-isomorphism:

\[
\text{Jrep}_{(Q^\circ, \sigma)}^{-1} : (Q, \sigma) \rightarrow ((\text{OpenPirr}Q^\circ)^\circ, \partial_{\text{Pirr}Q^\circ}^{-1} \circ \text{OpenPirr}).
\]

Now, we are going to compose the above isomorphism with the following SAM-isomorphism:

\[
\partial_{\text{Pirr}Q^\circ}^{-1} : ((\text{OpenPirr}Q^\circ)^\circ, \partial_{\text{Pirr}Q^\circ}^{-1} \circ \text{OpenPirr}) \rightarrow (\text{OpenPirr}Q, \text{OpenPirr} \circ \partial_{\text{Pirr}Q}).
\]

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Its typing is well-defined because \( \text{Open}_m \text{Pirr}_m Q \in \text{SAM}_f \) and it is a join-semilattice isomorphism by construction – see Definition 4.6.3. The unary operation is preserved because:

\[
\partial_{\text{Pirr}Q^{-\omega}}(\partial_{\text{Pirr}Q^{-\omega}} \circ \text{OpenPirr}\sigma(Y)) = \text{OpenPirr}\sigma(Y)
\]

\[
\text{OpenPirr}\sigma \circ \partial_{\text{Pirr}Q^{-\omega}}(\partial_{\text{Pirr}Q^{-\omega}}(Y)) = \text{OpenPirr}\sigma \circ \partial_{\text{Pirr}Q^{-\omega}}(\partial_{\text{Pirr}Q^{-\omega}}(Y)) = \text{OpenPirr}\sigma(Y)
\]

where in the second equality we use the fact that \( \partial_{\text{Pirr}Q^{-\omega}} = (\partial_{\text{Pirr}Q})^{-}\) acts in the same way as \( \partial_{\text{Pirr}Q}^{-1} \). Then we define the component \( \text{SAM}_f \)-isomorphisms as follows:

\[
\text{mrep}_{(Q, \sigma)} := \partial_{\text{Pirr}Q^{-\omega}} \circ \text{jrep}_{(Q, \sigma)}^{\text{op}} : (Q, \sigma) \to \text{Open}_m \text{Pirr}_m (Q, \sigma)
\]

which actually act in the same way as \( \text{rep}_Q \):

\[
\text{mrep}_{(Q, \sigma)}(q) = \partial_{\text{Pirr}Q^{-\omega}} \circ \text{jrep}_{(Q, \sigma)}^{\text{op}}(q) = \partial_{\text{Pirr}Q^{-\omega}} \circ \{ j \in J(Q) : j \not\leq_Q q \}
\]

\[
= \text{Pirr}Q\{ j \in J(Q) : j \not\leq_Q q \} = \{ m \in M(Q) : \exists j \in J(Q), [j \not\leq_Q q \text{ and } j \not\leq_Q m] \}
\]

\[
= \{ m \in M(Q) : \forall j \in J(Q), [j \not\leq_Q q \Rightarrow j \not\leq_Q m] \} = \{ m \in M(Q) : q \not\leq_Q m \} = \text{rep}_Q(q).
\]

Then since \( \text{rep} : \text{Id}_{\text{JSL}_f} \Rightarrow \text{OpenPirr} \) is natural and \( \text{SAM}_f \) is built on top of \( \text{JSL}_f \), it follows that \( \text{mrep} : \text{Id}_{\text{SAM}_f} \Rightarrow \text{Open}_m \text{Pirr}_m \) is a natural isomorphism. The description of the component inverses is immediate.

2. Fixing \( (G, E) \in \text{UG}_m \) we have the self-adjoint \( \text{Dep} \)-morphism \( E : \tilde{G} \to G \). Then we have \( (\tilde{G}, E) \in \text{UG}_j \) and thus the \( \text{UG}_j \)-isomorphism:

\[
\text{jred}^{-1}_{\tilde{G}} : (\text{Pirr} \text{Open}\tilde{G}, \text{Pirr}(\partial_{\tilde{G}}^{-1} \circ \text{Open}E)) \to (\tilde{G}, E)
\]

by Theorem 6.7.9. Consequently by Lemma 6.6.6 we have the \( \text{UG}_m \)-isomorphism:

\[
(\text{jred}^{-1}_{\tilde{G}})^\gamma : (G, E) \to ((\text{Pirr} \text{Open}\tilde{G})^\gamma, \text{Pirr}(\partial_{\tilde{G}}^{-1} \circ \text{Open}E)) = (\text{Pirr} \text{Open}\tilde{G})^\gamma, \text{Pirr}(\partial_{\tilde{G}}^{-1} \circ \text{Open}E)).
\]

We are going to compose the above isomorphism with the \( \text{UG}_m \)-isomorphism:

\[
\text{Pirr}_m \theta \quad \text{where} \quad \theta := \partial_{\tilde{G}} : ((\text{Open}\tilde{G})^{\text{op}}, \partial_{\tilde{G}}^{-1} \circ \text{Open}E) \to (\text{Open}\tilde{G}, \text{Open}E \circ \partial_{\tilde{G}}).
\]

To see that \( \theta \) is a \( \text{SAM}_f \)-isomorphism, observe that it is a join-semilattice isomorphism by construction. It preserves the unary operation because \( \partial_{\tilde{G}}(\partial_{\tilde{G}}^{-1} \circ \text{Open}E(Y)) = \text{Open}E(Y) \) and also:

\[
\text{Open}E \circ \partial_{\tilde{G}}(\partial_{\tilde{G}}(Y)) = \text{Open}E \circ \tilde{G}^\gamma \circ -_V \circ G^\gamma \circ \neg_{\tilde{G}_s}(Y) \quad \text{by definition of } \partial.
\]

\[
= \text{Open}E \circ \tilde{G}^\gamma \circ \tilde{G}_s(Y) \quad \text{by de morgan duality}
\]

\[
= \text{Open}E \circ \text{in}_{\tilde{G}}(Y)
\]

\[
= \text{Open}E(\tilde{G}_s(Y)) \quad \text{since } Y \in O(\tilde{G}).
\]

Then we define the component \( \text{UG}_m \)-isomorphisms as follows:

\[
\text{mred}_{(G, E)} := (\text{jred}^{-1}_{\tilde{G}})^\gamma \circ \text{Pirr}_m \theta : (G, E) \to \text{Pirr}_m \text{Open}_m (G, E).
\]

To compute this composite, first observe that \( \text{jred}^{-1}_{\tilde{G}}(\tilde{G})^{-1} = (\varepsilon)^{-1} = \varepsilon \in V \times J(\text{Open}\tilde{G}) \) and moreover:

\[
(\text{Pirr}_m \partial_{\tilde{G}})_+ \subseteq M(\text{Open}\tilde{G}) \times J(\text{Open}\tilde{G}) \quad (\text{Pirr}_m \partial_{\tilde{G}})(Y, X) \iff (\partial_{\tilde{G}})(Y) \leq_{(\text{Open}\tilde{G})^{-\omega}} X \iff X \subseteq \tilde{G}[\overline{Y}],
\]

using Definition 4.2.1 and the fact that \( (\partial_{\tilde{G}})_+ = \partial_{\tilde{G}} \) by Lemma 4.6.6. We now show that the underlying relation
\[ m_{\text{red}}(G, \mathcal{E}) \subseteq V \times M(\text{Open}_{\mathcal{G}}) \text{ is actually } \text{red}_{\mathcal{G}}. \]

\[ m_{\text{red}}(G, \mathcal{E})(v, Y) \iff \left( (\text{red}^{-1}_{G, \mathcal{E}}) \cdot \text{Pirr}_{m} \right)(v, Y) \]

Finally we use the first equivalence \( \text{SAJ}_{f} \cong \text{UG}_{m} \) to prove the categorical equivalence between finite de morgan algebras and our category of undirected graphs. The construction of the witnessing natural isomorphisms follows quickly, whereas the explicit description of their components requires some computation. Unlike the previous two equivalences, the natural isomorphisms do not lift directly from the fundamental equivalence \( JSL_{f} \cong \text{Dep} \).

**Theorem 6.7.11 (Categorical equivalence between \( \text{SAI}_{f} \) and \( \text{UG} \)).**

The functors \( \text{Open}_{g} \) and \( \text{Pirr}_{g} \) define an equivalence of categories with respective natural isomorphisms:

\[ \text{grep} : \text{ld}_{\text{SAI}} \Rightarrow \text{Open}_{g} \circ \text{Pirr}_{g} \]

\[ \text{grep}_{(Q, \sigma)} : (Q, \sigma) \rightarrow (\text{OpenPirr}_{\sigma}, \partial_{\text{Pirr}_{\sigma}}) \]

\[ \text{grep}_{(Q, \sigma)}(q) := \{ j \in J(Q) : j \notin \sigma(q) \} \]

\[ \text{grep}_{(Q, \sigma)}^{-1}(Y) := \sigma(V_{Q}(J(Q)) \setminus Y) = \land_{Q} M(Q) \setminus \sigma(Y). \]

\[ \text{gred} : \text{ld}_{\text{UG}} \Rightarrow \text{Pirr}_{g} \circ \text{Open}_{g} \]

\[ \text{gred}_{(V, \mathcal{E})} : (V, \mathcal{E}) \rightarrow (J(\text{Open}_{\mathcal{E}}), \text{Pirr}_{\mathcal{E}}^{-1}) \]

\[ \text{gred}_{(V, \mathcal{E})} \vdash \epsilon \subseteq V \times J(\text{Open}_{\mathcal{E}}) \]

\[ \text{gred}_{(V, \mathcal{E})} \vdash \epsilon \subseteq J(\text{Open}_{\mathcal{E}}) \times V. \]

Furthermore, the components of the latter natural isomorphisms have the following associated components.

\[ \left( \text{gred}_{(V, \mathcal{E})}^{-1} \right)_{+} = \left( \text{gred}_{(V, \mathcal{E})}^{-1} \right)_{-} = \left\{ (v, X) \in V \times J(\text{Open}_{\mathcal{E}}) : X \subseteq \mathcal{E}[v] \right\}, \]

\[ \left( \text{gred}_{(V, \mathcal{E})}^{-1} \right)_{+} = \left( \text{gred}_{(V, \mathcal{E})}^{-1} \right)_{-} = \left\{ (X, v) \in J(\text{Open}_{\mathcal{E}} \times \mathcal{E}[v] : v \subseteq X \right\}. \]

**Proof.**

1. Recall the inclusion functor \( I_{j} : \text{SAI}_{f} \rightarrow \text{SAJ}_{f} \), the natural isomorphism \( r_{j}^{-1} : \text{Pirr}_{j} \circ I_{j} \Rightarrow \text{Diag}_{j} \circ \text{Pirr}_{g} \) from Definition 6.7.6, and also the equality \( \text{Open}_{j} \circ \text{Diag}_{j} = I_{j} \circ \text{Open}_{g} \) from Lemma 6.7.8.2. Using the natural isomorphism \( \text{jrep} \) from Theorem 6.7.9, we obtain the composite natural isomorphism:

\[ I_{j} \stackrel{\text{jrep}_{j}}{=} \text{Open}_{j} \circ \text{Pirr}_{j} \circ I_{j} \text{Open}_{j} \circ r_{j}^{-1} \circ \text{Diag}_{j} \circ \text{Pirr}_{g} = I_{j} \circ \text{Open}_{g} \circ \text{Pirr}_{g}. \]

Then since \( I_{j} \) is a fully-faithful inclusion functor we further obtain the natural isomorphism:

\[ \text{grep} : \text{ld}_{\text{SAI}} \Rightarrow \text{Open}_{g} \circ \text{Pirr}_{g} \]

\[ \text{grep}_{(Q, \sigma)} := \text{Open}_{j} r_{j}^{-1}_{(Q, \sigma)} \circ \text{jrep}_{(Q, \sigma)} : (Q, \sigma) \rightarrow (\text{OpenPirr}_{\sigma}, \partial_{\text{Pirr}_{\sigma}}). \]

It remains to explain the codomain of the above components, and also simplify their action. For each \( (Q, \sigma) \in \text{SAI}_{f} \subseteq \text{SAJ}_{f} \), the first isomorphism is:

\[ \alpha := \text{jrep}_{(Q, \sigma)} : (Q, \sigma) \rightarrow (\text{OpenPirr}_{Q}, \partial_{\text{Pirr}_{Q}}^{-1} \circ \text{OpenPirr}_{Q}) \]

\[ \alpha(q) := \text{rep}_{Q}(q) = \{ m \in M(Q) : q \notin \mathcal{E}_{Q} \}. \]
Concerning the second isomorphism, let us recall the $\text{UG}_j$-isomorphism $r_j^{-1} : \Pi\sigma : (\Pi\sigma, \Pi\sigma) \rightarrow (\Pi\sigma, \Pi\sigma)$. Confusingly, the symmetric relation $\Pi\sigma \in J(\Pi\sigma, \Pi\sigma)$ arises in a number of different ways.

- It is the underlying relation of the $\text{UG}_j$-isomorphism $r_j^{-1}$.
- In the domain $(\Pi\sigma, \Pi\sigma)$ it is understood as a Dep-morphism $\Pi\sigma : \Pi\sigma \rightarrow (\Pi\sigma)$.
- In the codomain $(\Pi\sigma, \Pi\sigma)$ it is the bipartite graph in the first component, whereas the second component is understood as the Dep-morphism $\Pi\sigma : \Pi\sigma \rightarrow (\Pi\sigma)$. Recall that since $\Pi\sigma$ is symmetric, this is actually the Dep-identity-morphism $id_{\Pi\sigma}$.

Applying the equivalence functor $\text{Open}$ yields the second isomorphism, with typing:

$$\text{Open}_j r_j^{-1}_{(\Pi\sigma, \Pi\sigma)} : (\text{Open}_j \Pi\sigma, \text{Open}_j \Pi\sigma) \rightarrow (\text{Open}_j \Pi\sigma, \text{Open}_j \Pi\sigma)$$

where the final equality follows because $\Pi\sigma$ is symmetric. Concerning its action, since $\text{Open}_j$ acts as $\text{Open}$ on the underlying Dep-morphism we have:

$$\beta := \text{Open}_j r_j^{-1}_{(\Pi\sigma, \Pi\sigma)} = \text{Open} \mathcal{R} \text{ where } \mathcal{R} := \Pi\sigma : \Pi\sigma \rightarrow (\Pi\sigma).$$

Then by definition of $\text{Open} : \text{Dep} \rightarrow \text{JSL}_f$, for each $Y \in \text{O}(\Pi\sigma)$ we have $\beta(Y) = \mathcal{R}_* [Y]$ where the relation $\mathcal{R}_* \subseteq J(\mathcal{Q}) \times M(\mathcal{Q})$ satisfies:

$$\mathcal{R}_* (j, m) \iff (\Pi\sigma)[j] \subseteq \mathcal{R} [j] \text{ by definition of } (-),$$

$$\mathcal{R}_* (j, m) \iff (\Pi\sigma)[j] \subseteq \mathcal{R} [j] \text{ symmetric}$$

$$\mathcal{R}_* (j, m) \iff \forall j' \in J(\mathcal{Q}).j' \not\in \mathcal{Q}_0 m \Rightarrow j' \not\in \mathcal{Q}_0 \sigma(j) \text{ definition of } \Pi\sigma \text{ and } \Pi\sigma$$

$$\mathcal{R}_* (j, m) \iff \sigma(j) \not\in \mathcal{Q}_0 m.$$

Consequently,

$$\beta(Y) = \{ j \in J(\mathcal{Q}) : \exists m \in Y. \sigma(j) \not\in \mathcal{Q}_0 m \}$$

$$\beta(Y) = \{ j \in J(\mathcal{Q}) : \exists m \in Y. \sigma(m) \not\in \mathcal{Q}_0 j \} \text{ since } \sigma : \mathcal{Q} \rightarrow \mathcal{Q}^\mathcal{op}$$

$$\beta(Y) = J(\mathcal{Q}) \cap \mathcal{Q}_0 \sigma[Y].$$

Then we can finally compute the composite action for each $q_0 \in Q$ as follows,

$$\text{grep}_{(\Pi\sigma, \Pi\sigma)}(q_0) = \beta \circ \alpha(q_0)$$

$$\text{grep}_{(\Pi\sigma, \Pi\sigma)}(q_0) = \beta(\{ m \in M(\mathcal{Q}) : q_0 \not\in \mathcal{Q}_0 m \})$$

$$\text{grep}_{(\Pi\sigma, \Pi\sigma)}(q_0) = J(\mathcal{Q}) \cap \mathcal{Q}_0 \{ \sigma(m) : m \in M(\mathcal{Q}), q_0 \not\in \mathcal{Q}_0 m \}$$

$$\text{grep}_{(\Pi\sigma, \Pi\sigma)}(q_0) = J(\mathcal{Q}) \cap \mathcal{Q}_0 \{ j \in J(\mathcal{Q}), q_0 \not\in \mathcal{Q}_0 \sigma(q_0) \}$$

$$\text{grep}_{(\Pi\sigma, \Pi\sigma)}(q_0) = \{ j \in J(\mathcal{Q}), q_0 \not\in \mathcal{Q}_0 \sigma(q_0) \} \text{ since } \sigma : \mathcal{Q} \rightarrow \mathcal{Q}^\mathcal{op}$$

$$\text{grep}_{(\Pi\sigma, \Pi\sigma)}(q_0) \text{ since already up-closed.}$$

The descriptions of its inverse $\text{grep}^{-1}_{(\Pi\sigma, \Pi\sigma)}$ follow immediately, recalling that $\sigma$ sends arbitrary joins to arbitrary meets and restricts to a bijection $\sigma_{J(\mathcal{Q}) \times M(\mathcal{Q})}$.

2. Using the natural isomorphism $jred$ we obtain the composite natural isomorphism:

$$\text{Diag}_j \xrightarrow{jred_{\text{Diag}_j}} \Pi\sigma_j \circ \text{Open}_j \circ \text{Diag}_j = \Pi\sigma_j \circ I_j \circ \text{Open}_j \xrightarrow{r_j^{-1}_{\text{Open}_j}} \text{Diag}_j \circ \Pi\sigma \circ \text{Open}_j$$

also using the equality $\text{Open}_j \circ \text{Diag}_j = I_j \circ \text{Open}_j$ and the natural isomorphism $r_j$, see Lemma 6.7.8. Recall that $\text{Diag}_j : \text{UG} \rightarrow \text{UG}_j$ is a fully-faithful identity-on-morphisms functor. Its action on objects is $(V, \mathcal{E}) \rightarrow (V, \mathcal{E})$, with inverse-action $(\mathcal{E}, \mathcal{E}) \rightarrow (\mathcal{E}, \mathcal{E})$ where $V_\mathcal{E}$ is the source = target of the symmetric relation $\mathcal{E}$. Consequently we obtain the natural isomorphism:

$$\text{grep}_j : jred_{(\mathcal{E}, \mathcal{E})} \circ r_j^{-1}_{\text{Open}_j} : (V, \mathcal{E}) \rightarrow (J(\text{Open}_j), \text{Pirr}^{-1}_{\mathcal{E}}).$$

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noting that $V_{\text{Pirr}^{-1}} = J(\text{Open}_\mathcal{E})$. Let us simplify its action inside $\text{UG}_j$, so that the $\text{UG}$-morphism $\text{gred}(v, \mathcal{E})$ will be the underlying $\text{Dep}$-morphism. The component $\text{UG}_j$-isomorphisms are:

$$R := \text{gred}(v, \mathcal{E}) = \text{red}_\mathcal{E} : (\mathcal{E}, \mathcal{E}) \to (\text{Pirr}\mathcal{Open}_\mathcal{E}, \text{Pirr}\partial_\mathcal{E}^{-1} \circ \text{Open}_\mathcal{E}id_{\mathcal{E}}),$$

$$S := \text{r}^{-1}_{\text{Open}_\mathcal{E}} = \text{Pirr}\partial_\mathcal{E}^{-1} : (\text{Pirr}\mathcal{Open}_\mathcal{E}, \text{Pirr}\partial_\mathcal{E}^{-1}) \to (\text{Pirr}\partial_\mathcal{E}^{-1}, \text{Pirr}\partial_\mathcal{E}^{-1}).$$

Recall the canonical relation $R = \text{red}_\mathcal{E} \subseteq V \times M(\mathcal{Open}_\mathcal{E})$ and its associated components from Theorem 4.2.10. Furthermore the canonical join-semilattice isomorphism $\partial_\mathcal{E}^{-1} : \mathcal{Open}_\mathcal{E} \to (\mathcal{Open}_\mathcal{E})^{\text{op}}$ has action $\mathcal{Y} \mapsto \mathcal{E}[\mathcal{Y}]$. We can now finally show that the underlying relation,

$$R \triangleright S \subseteq \mathcal{E} \times (\text{Pirr}\mathcal{Open}_\mathcal{E}) = V \times J(\mathcal{Open}_\mathcal{E})$$

is in fact the element-of relation:

$$R \triangleright S(v, X) \iff (\text{red}_\mathcal{E})_\mathcal{E}; S(v, X)$$

$$\iff \exists X' \in J(\mathcal{Open}_\mathcal{E}), (X' \in \mathcal{E}[v] \text{ and } \text{Pirr}\partial_\mathcal{E}^{-1}(X', X))$$

$$\iff \exists X' \in J(\mathcal{Open}_\mathcal{E}), (X' \in \mathcal{E}[v] \text{ and } X' \notin \mathcal{E}[-X])$$

$$\iff \neg \forall X' \in J(\mathcal{Open}_\mathcal{E}). (X' \in \mathcal{E}[v] \Rightarrow X' \notin \mathcal{E}[-X])$$

$$\iff \mathcal{E}[v] \not\subseteq \mathcal{E}[\mathcal{X}]$$

$$\iff v \notin \text{cl}_\mathcal{E}(\mathcal{X})$$

$$\iff v \in \text{in}_\mathcal{E}(\mathcal{X}) = X$$

Concerning the inverses $\text{gred}^{-1}(v, \mathcal{E})$; by Lemma 6.6.7 they are precisely the converse-element-of relations $\in \subseteq J(\mathcal{Open}_\mathcal{E}) \times V$.

3. It remains to verify our description of the associated components. Since $\text{gred}^{-1}(v, \mathcal{E}) = (\text{gred}(v, \mathcal{E}))^\vee$ as $\text{Dep}$-morphisms, the negative/positive component of $\text{gred}^{-1}(v, \mathcal{E})$ is actually the positive/negative component of $\text{gred}(v, \mathcal{E})$ respectively. Finally,

$$\text{gred}^{-1}(v, \mathcal{E})-(v, X) \iff \text{Pirr}\partial_\mathcal{E}^{-1}[X] \subseteq [v]$$

$$\iff \forall X' \in J(\mathcal{Open}_\mathcal{E}). (X' \notin \mathcal{E}[\mathcal{X}] \Rightarrow v \in X')$$

$$\iff \forall X' \in J(\mathcal{Open}_\mathcal{E}). (X' \notin \mathcal{E}[\mathcal{X}] \Rightarrow X' \notin \text{in}_\mathcal{E}(\mathcal{X}))$$

$$\iff \forall X' \in J(\mathcal{Open}_\mathcal{E}). (X' \in \text{in}_\mathcal{E}(\mathcal{X}) \Rightarrow X' \subseteq \mathcal{E}[\mathcal{X}])$$

$$\iff \text{in}_\mathcal{E}(\mathcal{X}) \subseteq \mathcal{E}[\mathcal{X}]$$

$$\iff \mathcal{E}[\mathcal{X}] \subseteq \text{cl}_\mathcal{E}(\mathcal{X})$$

$$\iff \text{in}_\mathcal{E}(\mathcal{X}) \subseteq \mathcal{E}[v]$$

$$\iff X \subseteq \mathcal{E}[v]$$

$$\iff \mathcal{E}[v] \subseteq X.$$  

Just as we described the fullness of $\text{Open}$ explicitly in Lemma 4.2.11, we now do the same for our new equivalence functors $\text{Open}_j$, $\text{Open}_m$, and $\text{Open}_n$. Recall that each of these three functors acts in the same way as $\text{Open}$.

Lemma 6.7.12 (Explicit fullness of $\text{Open}_j$, $\text{Open}_m$ and $\text{Open}_n$).

Fix any bipartite graph $\mathcal{G}$.

1. Each $(\text{Open}_\mathcal{G}, \sigma) \in \text{SAJ}_f$ arises as $\text{Open}_j(\mathcal{G}, \mathcal{E})$ where $\mathcal{E}(g, g') : \iff \mathcal{G}[g'] \not\subseteq \sigma(\mathcal{G}[g]).$

2. Each $(\text{Open}_\mathcal{G}, \sigma) \in \text{SAM}_f$ arises as $\text{Open}_m(\mathcal{G}, \mathcal{E})$ where $\mathcal{E}(g, g') : \iff \sigma(\text{in}_\mathcal{G}(g')) \not\subseteq \text{in}_\mathcal{G}(g').$

3. Each $(\text{Open}_\mathcal{G}, \sigma) \in \text{SAI}_f$ arises as $\text{Open}_n(\mathcal{G}, \mathcal{E})$ where $\mathcal{E}(g, g') : \iff \sigma(\text{in}_\mathcal{G}(g')) \not\subseteq \text{in}_\mathcal{G}(g'),$ as in (2).

4. Consequently, each $\text{SAJ}_f$, $\text{SAM}_f$ or $\text{SAI}_f$-morphism of type $f : (\text{Open}_\mathcal{G}_1, \sigma_1) \to (\text{Open}_\mathcal{H}, \sigma_2)$ arises by applying $\text{Open}_j$, $\text{Open}_m$, or $\text{Open}_n$ respectively, using the explicit fullness Lemma 4.2.11 and the above three statements.
Proof.

1. Given \((\text{Open}G, \sigma) \in \text{SAJ}_f\) then consider the join-semilattice morphism:

\[
f := \text{Open}G \xrightarrow{\sigma} (\text{Open}G)^{\text{op}} \xrightarrow{\partial_G} \text{Open}G.
\]

By the explicit fullness Lemma 4.2.11, we have \(f = \text{Open}E\) where the Dep-morphism \(E : G \to \tilde{G}\) is defined:

\[
E(g_s, g_s') : \iff g_s' \in f(G[gs]) \iff g_s' \in \neg g_s \circ G^! \circ \sigma(G[gs]) \iff G[g_s'] \notin \sigma(G[gs]).
\]

To see that \((G, E) \in \text{UG}_j\), observe that \(\partial_G^{-1} \circ \text{Open}E = \sigma\) is self-adjoint by assumption, and consequently \(E : G \to \tilde{G}\) is self-adjoint by Lemma 6.3.4.2.

2. Given \((\text{Open}G, \sigma) \in \text{SAM}_f\) then consider the join-semilattice morphism:

\[
f := \text{Open}G \xrightarrow{\partial_G} (\text{Open}G)^{\text{op}} \xrightarrow{\sigma} \text{Open}G.
\]

By the explicit fullness Lemma 4.2.11 we have \(f = \text{Open}E\) where the Dep-morphism \(E : G \to G\) is defined:

\[
E(g_t, g_t') : \iff g_t' \in f(G[gt]) \iff g_t' \in \sigma(G^! \circ \neg g_t \circ G[gt]) = \sigma(\text{in}_G(\text{op}_t)) \iff g_t' \in \neg \text{in}_G(\text{op}_t).
\]

To see that \((G, E) \in \text{UG}_m\), observe that \(\text{Open}E \circ \partial_G = \sigma\) is self-adjoint by assumption, and consequently \(E : G \to \tilde{G}\) is self-adjoint by Lemma 6.3.4.2.

3. Given \((\text{Open}G, \sigma) \in \text{SAI}_f \subseteq \text{SAM}_f\) then by (2) we have the self-adjoint Dep-morphism \(E : \tilde{G} \to G\) defined:

\[
E(g_t, g_t') : \iff \sigma(\text{in}_G(\text{op}_t)) \notin \text{in}_G(\text{op}_t) \iff g_t' \in \sigma(\text{in}_G(\text{op}_t)).
\]

Then given the u-graph \((G_t, E)\) we'll show that \(\text{Open}_E(G_t, E) = (\text{Open}E, \partial_E) \cong (\text{Open}G, \sigma)\).

- To establish \(\text{Open}(E \subseteq G_t \times G_t) = \text{Open}G\), observe \(X \subseteq G_t, E[X] \in O(\tilde{G})\) because \(E : \tilde{G} \to G\) is a Dep-morphism. We have \(E[gt] = \sigma(\text{in}_G(\text{op}_t))\), and every meet-irreducible in \(\text{Open}G\) takes the form \(\text{in}_G(\text{op}_t)\). Since \(\sigma : \text{Open}G \to (\text{Open}G)^{\text{op}}\) is an isomorphism, every join-irreducible in \(\text{Open}G\) arises as some \(E[gt]\).
- To see that \(\partial_E = \sigma\) as functions, by the previous item we have the typing \(\partial_E : (\text{Open}G)^{\text{op}} \to \text{Open}G\) recalling that \(\tilde{E} = E\). Then let us calculate:

\[
\partial_E(\text{in}_G(\text{op}_t)) = E[\text{cl}_G(\{gt\})] \quad \text{by De Morgan duality}
\]
\[
= E[\text{in}_G(\{gt\})] \quad \text{since } E : \tilde{G} \to G \text{ a Dep-morphism}
\]
\[
= \sigma(\text{in}_G(\text{op}_t)).
\]

Consequently \(\partial_E\) and the isomorphism \(\sigma : \text{Open}G \to (\text{Open}G)^{\text{op}}\) have the same action on meet-irreducible elements of \(\text{Open}G\), hence they have the same action on all elements.

4. Follows because \(\text{Open}_j, \text{Open}_m\) and \(\text{Open}_e\) inherit their action from \(\text{Open}\).

\(\square\)

### 6.8 Various interesting results

Recall the notion of tight morphism i.e. Definition 5.2.1.

**Theorem 6.8.1** (Characterisation of self-adjoint tight morphisms).

For each \(\text{JSL}_f\)-morphism \(\sigma : Q \to Q^{\text{op}}\) the following statements are equivalent.

a. \(\sigma\) is self-adjoint and tight.
b. \( \sigma \) arises as a join (= pointwise-join) of special morphisms:

\[
\sigma = \bigvee_{JSL_f[Q,Q^\op]} \{ {}^\ast q_0 q_1 : \mathcal{R}(q_0, q_1) \}
\]

where \( \mathcal{R} \subseteq Q \times Q \) is a symmetric relation.

c. \((Q, \sigma)\) is a \(SAJ\)-algebra and ‘factorises through’ a boolean \(SAI_f\)-algebra i.e.

\[
\sigma = Q \xrightarrow{\alpha} \text{Open} \xrightarrow{\sigma_0} (\text{Open} \leq P) \xrightarrow{\alpha^\op} Q^\op
\]

for some \(JSL_f\)-morphism \(\alpha : Q \rightarrow P \) and \(SAI_f\)-algebra \((P, \sigma_0)\).

d. \((Q, \sigma)\) is a \(SAJ\)-algebra and ‘factorises through’ a distributive \(SAI_f\)-algebra i.e.

\[
\sigma = Q \xrightarrow{\alpha} \text{Open} \leq P \xrightarrow{\sigma_0} (\text{Open} \leq P) \xrightarrow{\alpha^\op} Q^\op
\]

for some \(JSL_f\)-morphism \(\alpha : Q \rightarrow \text{Open} \leq P\) and \(SAI_f\)-algebra \((\text{Open} \leq P, \sigma_0)\).

Proof.

- \((a \Rightarrow b)\): \(\sigma\) is tight so by Lemma 5.2.5,

\[
\sigma = \bigvee_{JSL_f[Q,Q^\op]} \{ {}^\ast q_0 q_1 : \mathcal{R}(q_0, q_1) \}
\]

where \( \mathcal{R} := \{(q_0, q_1) : {}^\ast q_0 q_1 \leq \sigma\} \).

Finally if \( {}^\ast q_0 q_1 \leq \sigma \) then applying adjoints:

\[
\sigma^\ast = ({}^\ast q_0 q_1)^\ast \leq \sigma^\ast = \sigma
\]

by Lemma 5.1.9.1 and the self-adjointness of \( \sigma \). Thus \( \mathcal{R} \) is a symmetric relation.

- \((b \Rightarrow a)\): It is tight because joins of these special morphisms are tight by Lemma 5.2.5. It is self-adjoint because adjoints preserve joins and \(({}^\ast q_0 q_1)^\ast = {}^\ast q_0 q_1\).

- \((a \Rightarrow c)\): Since \((a \Rightarrow b)\) we know \(\sigma = \bigvee \{ {}^\ast q_0 q_1 : \mathcal{R}(q_0, q_1) \}\) for some symmetric relation \( \mathcal{R} \subseteq Q \times Q \). Also, \(\sigma : Q \rightarrow Q^\op\) is self-adjoint and hence defines a finite \(SAJ\)-algebra \((Q, \sigma)\) (see Lemma 6.4.1). It suffices to establish that the following diagram commutes:

\[
\begin{array}{ccc}
Q & \xrightarrow{\sigma} & Q^\op \\
\alpha \downarrow & & \downarrow \alpha^\ast \\
FR & \xrightarrow{\sigma_0} & (FR)^\op
\end{array}
\]

where:

1. \(\alpha := \bigvee_{JSL_f[Q,FR]} \{ {}^\ast \mathcal{R}(q_0, q_1) : \mathcal{R}(q_0, q_1) \}\) is a join of special morphisms.
2. \(\sigma_0\) is the composite isomorphism:

\[
FR \xrightarrow{\theta'} FR \xrightarrow{\theta^\ast} (FR)^\op
\]

where the involutive function \(\mathcal{R} \xrightarrow{\theta} \mathcal{R}\) is defined \(\theta(q_0, q_1) := (q_1, q_0)\), so that \(\theta'(S) := \tilde{S}\) can be viewed as constructing converse relations. Observe that it defines a finite \(\text{SAI}\)-algebra \((FR, \sigma_0)\) by Example 6.2.4.

- The adjoint of \(\alpha\) has description:

\[
\alpha^\ast = \bigvee_{JSL_f[(FR)^\op, Q^\op]} \{ {}^\ast \mathcal{R}(q_0, q_1) : \mathcal{R}(q_0, q_1) \}
\]

by Lemma 5.1.9.1 and Lemma 3.0.5.
To see that the diagram commutes, first observe:

\[
\sigma_0 \circ \alpha = \sigma_0 \circ (\vee_{\text{JSL}}_{\{Q,P\}} \{ \uparrow_{Q,P} \} : \mathcal{R}(q_0,q_1))
\]

\[
= \vee_{\text{JSL}}_{\{Q,P\}} \{ \uparrow_{Q,P} \} : \mathcal{R}(q_0,q_1)
\]

\[
= \vee_{\text{JSL}}_{\{Q,P\}} \{ \uparrow_{Q,P} \} : \mathcal{R}(q_0,q_1)
\]

by Lemma 5.2.10.1

Then to see:

\[
\sigma = \vee_{\text{JSL}}_{\{Q,P\}} \{ \uparrow_{Q,P} \} : \mathcal{R}(q_0,q_1)
\]

observe that the joins distribute over composition by bilinearity, the relevant internal compositions being:

\[
\uparrow_{Q,P} \circ \uparrow_{Q,P} = \uparrow_{Q,P} \circ \uparrow_{Q,P}
\]

by Lemma 5.2.10.1

By the symmetry of \( \mathcal{R} \) we deduce that their join over all \( \mathcal{R}(q_0,q_1) \) is indeed \( \sigma \).

- \((c \rightarrow a)\): If \( \langle Q, \sigma \rangle \) is a finite SAJ-algebra then \( \sigma : Q \to Q^{\text{op}} \) is self-adjoint by Lemma 6.4.1. Furthermore \( \sigma \) is tight because \( Q \) is boolean.

- \((c \rightarrow d)\): Immediate because boolean join-semilattices are distributive. In particular we can choose the discrete poset \( P := (Z, \Delta_Z) \).

\( (d \Rightarrow c) \): Fix the respective finite poset \( (P, \leq_P) \) and consider the following diagram inside \( \text{JSL}_P \).

The top square commutes by assumption. By Theorem ??A \( \sigma_0 \) has action \( \lambda Y \in O(\leq_P) \to \) \( P \to P \). Then the triangle makes sense and commutes by construction. Certainly \( \sigma_1 \) is a join-semilattice isomorphism. To see that it is involutive, first recall that for any bijection \( f : P \to P \) we have \( f^t \circ \neg_P = \neg_P \circ f^t \).\(^3\) Then:

\[
\sigma_1 \circ \sigma_1 = \neg_P \circ \theta \circ \neg_P \circ \theta = \neg_P \circ \neg_P \circ \theta \circ \theta = \text{id}_P
\]

since both \( \neg_P \) and \( \theta \) are involutive. Thus by Lemma 6.5.7 we deduce that \( (FP, \sigma_1) \) is a well-defined SAJ-algebra. It remains to show that the central square commutes. That is, for every \( Y \in O(\leq_P) \) we must show that:

\[
\sigma_0(Y) \cong \iota(Y)
\]

or equivalently that \( \iota_* (\sigma_0(Y)) = \sigma_0(Y) \) as we now show:

\[
\iota_* (\sigma_0(Y)) = \cup \{ X \in O(\leq_P) : \sigma_0(X) \subseteq \sigma_0(Y) \} \quad \text{by definition of adjoints}
\]

\[
= \cup \{ X \in O(\leq_P) : X \subseteq Y \} \quad \text{since } \sigma_0 \text{ an isomorphism}
\]

\[
= Y \quad \text{since } Y \in O(\leq_P).
\]

\(^3\)Although easily directly verified, this also follows because \( f^t : FP \to FP \) is a join-semilattice isomorphism, thus a bounded lattice isomorphism, and thus a boolean algebra isomorphism.
7 Appendix

Consider the following standard categories:

| category   | objects        | morphisms                      |
|------------|----------------|--------------------------------|
| Set<sub>f</sub> | finite sets    | functions                      |
| Poset<sub>f</sub> | finite posets | + preserves order              |
| JSL<sub>f</sub> | finite join-semilattices with bottom | + preserve all joins          |
| DL<sub>f</sub> | finite distributive lattices      | + preserve all meets          |
| BA<sub>f</sub> | finite boolean algebras           | + preserve negation          |
| DL<sub>f</sub><sup>op</sup> | finite distributive lattices      | functions preserving all joins |
| BA<sub>f</sub><sup>op</sup> | finite boolean algebras           | functions preserving all joins |

where composition is functional composition in each case. Each such category is equivalent to a possibly non-full subcategory of JSL<sub>f</sub>. We now describe many dualities, including Birkhoff’s between posets and distributive lattices.

| duality | functors | natural isomorphisms |
|---------|----------|---------------------|
| Set<sub>f</sub><sup>op</sup> \(\overset{\text{Pred}}{\longrightarrow}\) BA<sub>f</sub> | \(\text{Pred}X := \mathbb{P}_X\) \(\text{Pred}f^{op} := \lambda S \subseteq Y. f^{-1}(S)\) \(\text{At}A := \text{At}(A)\) \(\text{At}f^{op} := \lambda b \in \text{At}(\mathbb{E}). \bigwedge_\mathbb{A} f^{-1}(\uparrow b)\) | \(\alpha : \text{Id}_{\text{Set}} \Rightarrow \text{At} \circ \text{Pred}^{op}\) \(\alpha_X : X \rightarrow \text{At}(\mathbb{P}_X)\) \(\alpha_X(x) := \{x\}\) \(\beta : \text{Id}_{BA} \Rightarrow \text{Pred} \circ \text{At}^{op}\) \(\beta_A(a) := \text{At}(A) \cap \downarrow A\) |
| Poset<sub>f</sub><sup>op</sup> \(\overset{\text{Up}}{\longrightarrow}\) DL<sub>f</sub> | \(\text{Up}P := (\mathbb{U}p(P), \cup, \emptyset, \cap, P)\) \(\text{Up}f^{op} := \lambda X. f^{-1}(X)\) \(\text{Ji}D := (J(D), \leq_{D^{op}})\) \(\text{Ji}f^{op} := \lambda j \in J(E). \bigwedge_D f^{-1}(\uparrow j)\) | \(\alpha : \text{Id}_{\text{Poset}} \Rightarrow \text{Ji} \circ \text{Up}^{op}\) \(\alpha_P : P \\rightarrow (\mathbb{P}_1(P), \emptyset)\) \(\alpha_P(p) := \uparrow p\) \(\beta : \text{Id}_{\text{DL}} \Rightarrow \text{Up} \circ \text{Ji}^{op}\) \(\beta_D(d) := (\mathbb{J}(D) \cap \downarrow D, \leq_{D^{op}}, \cup, \emptyset, \cap, D)\) |
| JSL<sub>f</sub><sup>op</sup> \(\overset{\text{OD}_j}{\longrightarrow}\) JSL<sub>f</sub> | \(\text{OD}_j Q := Q^{op}\) \(\text{OD}_j f := \lambda r \in R. \bigvee_Q f^{-1}(| r |)\) | \(\alpha : \text{Id}_{\text{JSL}} \Rightarrow \text{OD}_j \circ \text{OD}_j^{op}\) \(\alpha_Q = \text{id}_Q\) |
| (DL<sub>f</sub><sup>op</sup>) \(\overset{\text{OD}_d}{\longrightarrow}\) DL<sub>f</sub> | \(\text{OD}_d\) restricts \(\text{OD}_j\) | \(\alpha : \text{Id}_{\text{DL}} \Rightarrow \text{OD}_d \circ \text{OD}_d^{op}\) \(\alpha_d = \text{id}_d\) |
| (BA<sub>f</sub><sup>op</sup>) \(\overset{\text{OD}_b}{\longrightarrow}\) BA<sub>f</sub> | \(\text{OD}_b\) restricts \(\text{OD}_d\) | \(\alpha : \text{Id}_{\text{BA}} \Rightarrow \text{OD}_b \circ \text{OD}_b^{op}\) \(\alpha_b = \text{id}_b\) |

The third entry is the self-duality of finite join-semilattices proved in Theorem 3.0.2. The fourth and fifth entries follow because distributive lattices and boolean algebras are stable under order-dualisation, see Definition 2.2.1.12. The first entry is the well-known duality between finite boolean algebras and finite sets, which restricts Birkhoff’s famous duality between finite distributive lattices with bounded lattice morphisms and finite posets with monotone functions. Lemma 7.1.2 below proves that Up and Ji are well-defined functors, Theorem 7.1.3 proves Birkhoff duality and Theorem 7.1.5 restricts it to boolean algebras and finite sets.

Concerning these five dualities, exactly seven categories are mentioned at the beginning of this subsection. Modulo categorical equivalence these seven categories are closed under taking the formal dual category. They are also related to one another via the free constructions:

\[
\begin{array}{c c c c c c}
\text{Set}_f & \overset{F_s}{\longrightarrow} & \text{Poset}_f & \overset{F_s}{\longrightarrow} & \text{JSL}_f & \overset{F_s}{\longrightarrow} & \text{DL}_f & \overset{F_s}{\longrightarrow} & \text{BA}_f \\
\downarrow U_s & & \downarrow U_s & & \downarrow U_s & & \downarrow U_s & & \downarrow U_s
\end{array}
\]

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### 7.1 Birkhoff duality and its restriction to boolean algebras

**Definition 7.1.1** (Equivalence functors between \( \text{Poset}^{op} \) and \( \text{DL} \)).

\[
\begin{align*}
\text{Up} &: \text{Poset}^{op} \rightarrow \text{DL} \\
\text{Ji} &: \text{DL}^{op} \rightarrow \text{Poset}
\end{align*}
\]

where:

\[
\begin{align*}
\text{Up} &= \{(p, \cup, \varnothing, \cap, X) \in \text{DL} \mid X \in \text{Poset}\} \\
\text{Ji} &= \{(D, \wedge, \top) \in \text{DL} \mid D \in \text{Poset}\}
\end{align*}
\]

**Lemma 7.1.2.** \( \text{Up} : \text{Poset}^{op} \rightarrow \text{DL} \) and \( \text{Ji} : \text{DL}^{op} \rightarrow \text{Poset} \) are well-defined functors.

*Proof.* \( \text{Up} \) is a set-theoretic distributive lattice and the restricted preimage function \( \text{Up}^{op} \) preserves all unions and intersections. It preserves the compositional structure because the preimage functor does i.e. \((g \circ f)^{-1} = f^{-1} \circ g^{-1}\).
JiD is clearly a well-defined poset so take any $\text{DL}_f$-morphism $f : \mathbb{D} \to \mathbb{E}$. Then for $\text{Ji}f^{\text{op}}$ to be a well-defined function we need to show that $\wedge_{\mathbb{D}} f^{-1}(\uparrow_{\mathbb{E}} j) \in \mathbb{E}(\mathbb{D})$ whenever $j \in \mathbb{E}(\mathbb{E})$. By Lemma 2.2.3.12 it suffices to show that $f^{-1}(\uparrow_{\mathbb{E}} j) \subseteq D$ arises as $\theta^{-1}(\{1\})$ for some $\text{DL}_f$-morphism $\mathbb{D} \to \mathbb{2}$. Then since $d \in f^{-1}(\uparrow_{\mathbb{E}} j)$ iff $j \leq_{\mathbb{E}} f(d)$ we consider $\theta := \lambda d \in D. (j \leq_{\mathbb{E}} f(d)) ? 1 : 0$ as follows.

1. $\theta_{\uparrow_{\mathbb{D}}} = 0$ because $f(\uparrow_{\mathbb{D}}) = \uparrow_{\mathbb{E}}$ and $j$ is join-irreducible by assumption.
2. $\theta_{\tau_{\mathbb{D}}} = 1$ because $f(\tau_{\mathbb{D}}) = \tau_{\mathbb{E}}$.
3. $\theta(d_1 \wedge_{\mathbb{D}} d_2) = 1$ iff $j \leq_{\mathbb{E}} f(d_1) \wedge_{\mathbb{D}} f(d_2)$ iff $\theta(d_1) = 1$ and $\theta(d_2) = 1$.
4. $\theta(d_1 \vee_{\mathbb{D}} d_2) = 1$ iff $j \leq_{\mathbb{E}} f(d_1) \vee_{\mathbb{D}} f(d_2)$, iff $\theta(d_1) = 1$ or $\theta(d_2) = 1$ by Lemma 2.2.3.10.

Next, $\text{Ji}f^{\text{op}}$ is monotonic because:

\[ j_1 \leq_{\mathbb{E}} j_2 \implies j_2 \leq_{\mathbb{E}} j_1 \implies \uparrow_{\mathbb{E}} j_1 \leq_{\mathbb{E}} \uparrow_{\mathbb{E}} j_2 \implies f^{-1}(\uparrow_{\mathbb{E}} j_1) \leq_{\mathbb{D}} f^{-1}(\uparrow_{\mathbb{E}} j_2) \implies \wedge_{\mathbb{D}} f^{-1}(\uparrow_{\mathbb{E}} j_2) \leq_{\mathbb{D}} f^{-1}(\uparrow_{\mathbb{E}} j_1) \]

and thus $\text{Ji}f^{\text{op}}(j_1) \leq_{\mathbb{D}} \text{Ji}f^{\text{op}}(j_2)$ recalling that $\text{JiD}$ restricts $\mathbb{D}^{\text{op}}$. Regarding the compositional structure:

\[
\text{Ji} id_{\mathbb{D}} = \lambda j. \wedge_{\mathbb{D}} id_{\mathbb{D}}(\uparrow_{\mathbb{D}} j) = \lambda j. \uparrow_{\mathbb{D}} j = \lambda j. j = id_{\mathbb{D}}
\]

\[
\text{Ji}(g \circ f)^{\text{op}} = \lambda j. \wedge_{\mathbb{D}} (g \circ f)^{-1}(\uparrow_{\mathbb{E}} j) = \lambda j. \wedge_{\mathbb{D}} f^{-1} \circ g^{-1}(\uparrow_{\mathbb{E}} j) = \lambda j. \wedge_{\mathbb{D}} f^{-1}(\uparrow_{\mathbb{E}} \Lambda_{\mathbb{E}} g^{-1}(\uparrow_{\mathbb{E}} j)) \text{ see below} = \lambda j. \text{Ji} \circ \text{Ji}^{\text{op}}(j) = \lambda j. \text{Ji}^{\text{op}}(j).
\]

The marked equality holds because $g^{-1}(\uparrow_{\mathbb{E}} j)$ is an upset one-generated by its $\mathbb{E}$-meet, see the argument further above.

**Theorem 7.1.3 (Birkhoff Duality).**

Up and $\text{Ji}^{\text{op}}$ define an equivalence between $\text{Poset}^{\text{op}}$ and $\text{DL}_f$ with natural isomorphisms:

\[
\begin{align*}
\alpha : \text{Id}_{\text{Poset}_f} &\to \text{Ji} \circ \text{Up}^{\text{op}} \\
\beta : \text{Id}_{\text{DL}_f} &\to \text{Up} \circ \text{Ji}^{\text{op}}
\end{align*}
\]

\[
\begin{align*}
\alpha_{\text{P}} : \text{P} &\to (\text{Pr}_1(\text{P}), \uparrow) \\
\beta_{\text{D}} : \text{D} &\to (\text{Dn}(\text{J}(\text{D})), \uparrow, \wedge, \uplus, \cdot, \cap, \uplus, \cap)
\end{align*}
\]

\[
\begin{align*}
\alpha_{\text{P}}(p) &:= \uparrow_{\text{P}} p \\
\beta_{\text{D}}(d) &:= \{ j \in \text{J}(\text{D}) : j \leq_{\text{D}} d \}
\end{align*}
\]

**Proof.** Observe that $\text{Ji} \circ \text{Up}^{\text{op}} = \text{Ji}(\text{Up}(\text{P}), \uplus, \wedge, \uplus, \cap)$ is the collection of $\text{P}$-principal-upsets $\text{Pr}_1(\text{P})$ ordered by reverse inclusion. Then $\alpha_{\text{P}}$ is the well-known poset isomorphism sending $p$ to its principal upset $\uparrow_{\text{P}} p$, recalling that $p_1 \leq_{\text{P}} p_2$ iff $\uparrow_{\text{P}} p_2 \subseteq \uparrow_{\text{P}} p_1$. Regarding naturality, we must verify that the square:

\[
\begin{array}{ccc}
\text{P} & \xrightarrow{f} & \text{Q} \\
\downarrow{\alpha_{\text{P}}} & & \downarrow{\alpha_{\text{Q}}} \\
(\text{Pr}_1(\text{P}), \uparrow) & \xrightarrow{\text{Up}^{\text{op}} f} & (\text{Pr}_1(\text{Q}), \uparrow)
\end{array}
\]

commutes for all monotone maps $f : \text{P} \to \text{Q}$. To this end, let $g := \text{Up}^{\text{op}} : \text{Up} \text{Q} \to \text{Up} \text{P}$ recalling its action $g(Y) = f^{-1}(Y)$. Then we calculate:

\[
\text{Ji} \circ \text{Up}^{\text{op}} f(\uparrow_{\text{P}} p) = \text{Ji} g^{\text{op}}(\uparrow_{\text{P}} p) = \Lambda_{\text{Up} \text{D}} g^{-1}(\uparrow_{\text{Up} \text{P}}(\uparrow_{\text{P}} p)) = \Lambda_{\text{Up} \text{D}} g^{-1}(\{ X \in \text{Up} \text{P} : p \in X \}) = \Lambda_{\text{Up} \text{Q}} = \uparrow_{\text{Q}} f(p)
\]

for any $p \in \text{P}$, which proves naturality.
Next, $\beta_D$ is well-typed because:

$$\text{Up} \circ J^{op}\mathcal{D} = \text{Up}(J(\mathcal{D}), \leq_{\mathcal{D}^{op}}) = (Up(J(\mathcal{D}), \leq_{\mathcal{D}^{op}}), \cup, \cap, J(\mathcal{D})) = (Dn(J(\mathcal{D}), \leq_{\mathcal{D}}), \cup, \cap, J(\mathcal{D}))$$

Thus its action is well-defined. $\beta$ is injective because each element of a finite join-semilattice (or distributive lattice) is the join of those join-irreducibles beneath it, and thus is uniquely determined by them. For $\beta$ to be surjective we must show that distinct down-closed sets of join-irreducibles yield distinct elements. This follows by applying Lemma 2.2.3.10. That is, if $\mathcal{V}_D X = \mathcal{V}_D Y$ where $X, Y \in Dn(J(\mathcal{D}))$ then for each $j \in X$ we have $j \leq_{\mathcal{D}} \mathcal{V}_D Y$ and hence $\exists j' \in Y, j \leq_{\mathcal{D}} j'$, and thus $j \in Y$ by downwards-closure. Then $X \subseteq Y$ and by the symmetric argument $X = Y$, so that $\beta$ is bijective. It is a bounded distributive lattice morphism i.e. preservation of bottom, top and binary meet follow easily, whereas preservation of binary join follows by Lemma 2.2.3.10. Concerning naturality, we must show the following square commutes:

$$\begin{array}{ccc}
\mathcal{D} & \xrightarrow{f} & \mathcal{E} \\
\beta_D \downarrow & & \downarrow \beta_E \\
(Dn(J(\mathcal{D}), \leq_{\mathcal{D}}), \cup, \cap, J(\mathcal{D})) & \xrightarrow{\text{Up} \circ J^{op} f} & (Dn(J(\mathcal{E}), \leq_{\mathcal{E}}), \cup, \cap, J(\mathcal{E}))
\end{array}$$

for every bounded distributive lattice morphism $f : \mathcal{D} \rightarrow \mathcal{E}$. If we let $X = \beta_D(d) = \downarrow_D d \cap J(\mathcal{D})$, then:

$$\text{Up} \circ J^{op} f(X) = \text{Up}(Jf^{op})^{op}(X) = \{j \in J(\mathcal{E}) : Jf^{op}(j) \in X\} = \{j \in J(\mathcal{E}) : \wedge_D f^{-1}(\uparrow_{\mathcal{E}} j) \subseteq \downarrow_D d \cap J(\mathcal{D})\} = \{j \in J(\mathcal{E}) : \wedge_D f^{-1}(\uparrow_{\mathcal{E}} j) \leq_{\mathcal{D}} d\}$$

whereas $\beta_E \circ f(d) = \{j \in J(\mathcal{E}) : j \leq_{\mathcal{E}} f(d)\}$. Thus it suffices to show that:

$$j \leq_{\mathcal{E}} f(d) \iff \bigwedge_D f^{-1}(\uparrow_{\mathcal{E}} j) \leq_{\mathcal{D}} d$$

for all $j \in J(\mathcal{E})$ and $d \in D$. This is actually an instance of an adjoint relationship inside JSL$_f$. That is, given $f : \mathcal{D} \rightarrow \mathcal{E}$ then we have the underlying join-semilattice morphism $U_{\text{dm}} f : (\mathcal{D}, \wedge_D, \top_D) \rightarrow (\mathcal{E}, \wedge_{\mathcal{E}}, \top_{\mathcal{E}})$ i.e. restrict to the underlying meet structure. Then observing that:

$$\bigwedge_D f^{-1}(\uparrow_{\mathcal{E}} j) = \bigvee_{\mathcal{D}^{op}} f^{-1}(\downarrow_{\mathcal{E}^{op}} j) = (U_{\text{dm}} f)^*(j)$$

we may instantiate Lemma 3.0.3.1 to obtain:

$$j \leq_{\mathcal{E}} f(d) \iff U_{\text{dm}} f(d) \leq_{\mathcal{E}^{op}} j \iff d \leq_{\mathcal{D}^{op}} (U_{\text{dm}} f)^*(j) \iff \bigwedge_D f^{-1}(\uparrow_{\mathcal{E}} j) \leq_{\mathcal{D}} d$$

which completes the proof. \hfill \Box

**Definition 7.1.4 (Equivalence functors between Set$_{op}$ and BA$_f$.)**

\begin{align*}
\text{Pred} : \text{Set}_{op} & \rightarrow \text{BA}_f & \text{Pred} X := \mathbb{P}_b X \\
\text{At} : \text{BA}_{f^{op}} & \rightarrow \text{Set}_f & \text{At} \mathbb{B} := \text{At}(\mathbb{E})
\end{align*}

\begin{align*}
\text{Pred} X & := \mathbb{P}_b X \\
& \xrightarrow{f : X \rightarrow Y} \mathbb{P}_b Y \\
\text{Pred} f^{op} & := \lambda x. f^{-1}(x) : \mathbb{P}_b X \rightarrow \mathbb{P}_b Y \\
\text{At} \mathbb{B} & := \text{At}(\mathbb{E}) \\
\text{At} f^{op} & := \lambda a. \bigwedge_{\mathcal{E}} f^{-1}(\uparrow_{\mathbb{B}} a) : \text{At}(\mathbb{E}) \rightarrow \text{At}(\mathbb{E})
\end{align*}

**Theorem 7.1.5 (Duality between finite boolean algebras and finite sets).**

\text{Pred} and \text{At}$_{op}$ define an equivalence between Set$_{op}$ and BA$_f$ with natural isomorphisms:

\begin{align*}
\alpha : \text{Id}_{\text{Set}_f} & \Rightarrow \text{At} \circ \text{Pred}^{op} & \alpha_X : X \rightarrow \text{At}(\mathbb{P}_b X) & \alpha_X(x) := \{x\} \\
\beta : \text{Id}_{\text{BA}_f} & \Rightarrow \text{Pred} \circ \text{At}^{op} & \beta_\mathbb{B} : \mathbb{B} \rightarrow \mathbb{P}_b \text{At}(\mathbb{E}) & \beta_\mathbb{B}(b) := \{a \in \text{At}(\mathbb{B}) : a \leq_{\mathbb{B}} b\}
\end{align*}
Proof. This follows by restricting Theorem 7.1.3 i.e. Birkhoff duality. That is, we have the commuting diagram:

\[
\begin{array}{ccc}
\text{Set}_f^{op} & \xrightarrow{\text{Up}_f} & \text{DL}_f \\
\downarrow & & \downarrow \text{Ji}_f^{op} \\
\text{Set}_f^{op} & \xrightarrow{\text{At}_f^{op}} & \text{BA}_f
\end{array}
\]

where:

1. \( I : \text{Set}_f \to \text{Poset}_f \) is the fully faithful functor defined \( IX = (X, \Delta_X) \) and \( If = f \).

2. \( \text{At}_f : \text{BA}_f \to \text{DL}_f \) is the fully faithful forgetful functor.

Certainly \( I \) is fully faithful because the monotone maps from a discrete poset \((X, \leq_X)\) to a discrete poset \((Y, \leq_Y)\) are precisely the functions \( f : X \to Y \), and clearly \( U_\leq \) is faithful. To see that \( U_\leq \) is full recall that bounded distributive lattice morphisms between boolean algebras are boolean algebra morphisms, since by Lemma 2.2.3.9 complements in distributive lattices are unique whenever they exist.

That the diagram above commutes is easily verified i.e. observe that the definitions of \( \text{Pred} \) and \( \text{Up}_f \) align, as do \( \text{At}_f \) and \( \text{Ji}_f \). Then \( \alpha \) and \( \beta \) are natural isomorphisms because they restrict the corresponding natural isomorphisms witnessing Birkhoff duality. \(\square\)

### 7.2 Free constructions between sets, posets, join-semilattices, distributive lattices and boolean algebras

**Definition 7.2.1** (Free poset on a set). Let \( U_\leq : \text{Poset}_f \to \text{Set}_f \) be the forgetful functor which forgets the ordering i.e. \( U_\leq P := P \) and \( U_\leq f := f \). Further define:

\[ F_\leq : \text{Set}_f \to \text{Poset}_f \quad F_\leq X := (X, \Delta_X) \quad f : X \to Y \quad F_\leq f := \lambda x. f(x) : (X, \Delta_X) \to (Y, \Delta_Y) \]

**Lemma 7.2.2** (Free poset on a set).

\( F_\leq : \text{Set}_f \to \text{Poset}_f \) is left adjoint to the forgetful functor \( U_\leq : \text{Poset}_f \to \text{Set}_f \) via natural transformations:

\[ \eta : \text{id}_{\text{Set}_f} \Rightarrow U_\leq \circ F_\leq \quad \eta_X : X \to X \quad \eta_X(x) := x \]

\[ \varepsilon : F_\leq \circ U_\leq \Rightarrow \text{id}_{\text{Poset}_f} \quad \varepsilon_P : (P, \Delta_P) \to P \quad \varepsilon_P(p) := p \]

**Proof.** Each \( \eta_X \) is a well-defined function and each \( \varepsilon_P \) is a well-defined monotone function. Although they are both bijective, \( \varepsilon_P \) is not a \( \text{Poset}_f \)-isomorphism whenever \( |P| \geq 2 \). Naturality is obvious by inspecting the required commutative squares:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow \eta_X & & \downarrow \eta_Y \\
X & \xrightarrow{f} & Y
\end{array}
\]

\[
\begin{array}{ccc}
(P, \Delta_P) & \xrightarrow{g} & (Q, \Delta_Q) \\
\downarrow \varepsilon_P & & \downarrow \varepsilon_Q \\
P & \xrightarrow{g} & Q
\end{array}
\]

for all functions \( f : X \to Y \) and monotone functions \( g : P \to Q \). Finally, the counit-unit equations are also immediate:

\[ \varepsilon_{F_\leq X} \circ F_\leq \eta_X = \lambda x. x = \text{id}_{F_\leq X} \quad U_\leq \varepsilon_P \circ \eta_U \circ P = \lambda p. p = \text{id}_{U_\leq P} \]

\(\square\)

**Definition 7.2.3** (Free join-semilattice on a poset). Let \( U_\vee : \text{JSL}_f \to \text{Poset}_f \) be the forgetful functor which takes the underlying ordering i.e. \( U_\vee Q = (Q, \leq_Q) \) and \( U_\vee f := f \). Furthermore define:

\[ F_\vee : \text{Poset}_f \to \text{JSL}_f \quad F_\vee P := (\text{Dn}(P), \cup, \emptyset) \quad f : P \to Q \quad F_\vee f := \lambda X. f[X] : F_\vee P \to F_\vee Q \]

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Theorem 7.2.4 (Free join-semilattice on a poset).
\( F_\nu : \text{Poset}_f \to \text{JSL}_f \) is left adjoint to the forgetful functor \( U_\nu : \text{JSL}_f \to \text{Poset}_f \) via natural transformations:

\[
\begin{align*}
\eta &: \text{Id}_{\text{Poset}_f} \Rightarrow U_\nu \circ F_\nu, \\
\eta_p &: \text{P} \to (\text{Dn}(\text{Q}), \leq), \quad \eta_p(p) := \downarrow_p \text{P} \\
\varepsilon &: F_\nu \circ U_\nu \Rightarrow \text{Id}_{\text{JSL}_f}, \\
\varepsilon_Q &: (\text{Dn}(\text{Q}), \cup, \varnothing) \to \text{Q}, \quad \varepsilon_Q(S) := \bigvee_Q S
\end{align*}
\]

Proof. We first verify that \( F_\nu \) is a well-defined functor. Its action on objects is well-defined because the downsets \( \text{P} \) contain \( \varnothing \) and are union-closed. Concerning its action on morphisms:

\[
F_\nu f(\downarrow_{\text{P}} p) = \downarrow_Q f(\downarrow_{\text{P}} p) = \downarrow_Q f(\varnothing) = \varnothing
\]

\[
F_\nu f(A_1 \vee_{\text{P}} A_2) = \downarrow_Q f[A_1 \cup A_2] = \downarrow_Q f[A_1] \cup \downarrow_Q f[A_2] = F_\nu f(A_1) \vee_{\text{Q}} F_\nu f(A_2)
\]

Each \( \eta_p \) is monotone because \( p \leq \text{Q} \) implies \( \downarrow_p \text{P} \leq \downarrow_p \text{Q} \). Concerning naturality we must verify that:

\[
\begin{array}{cc}
P & Q \\ 
\downarrow \downarrow & \downarrow \\
\eta_p & \varepsilon_Q
\end{array}
\]

i.e. \( \downarrow_Q f(\downarrow_p \text{P}) = \downarrow_Q f(p) \) which follows by the monotonicity of \( f \). Each \( \varepsilon_Q \) is a join-semilattice morphism:

\[
\varepsilon_Q(\varnothing) = \bigvee_Q \varnothing = \varnothing_Q
\]

\[
\varepsilon_Q(S_1 \cup S_2) = \bigvee_Q S_1 \cup S_2 = \bigvee_Q S_1 \vee \bigvee_Q S_2 = \varepsilon_Q(S_1) \vee \varepsilon_Q(S_2)
\]

and for naturality we must verify that:

\[
\begin{array}{cc}
\text{Dn}(\text{P}, \leq) & \text{Dn}(\text{Q}, \leq) \\ 
\varepsilon_Q & \varepsilon_R
\end{array}
\]

i.e. \( f(\bigvee_Q S) = \bigvee_R f[S] \) which follows because (i) \( f \) preserves arbitrary joins, (ii) adding smaller elements has no effect. Then it only remains to verify the counit-unit equations:

\[
\varepsilon_{F_\nu, p} \circ F_\nu \eta_p(A) = \varepsilon_{F_\nu, p}(\downarrow_{F_\nu, U_\nu, F_\nu} \{ \downarrow_{F_\nu} p : p \in A \}) = \bigvee_{F_\nu, p} \downarrow_{F_\nu, U_\nu, F_\nu} \{ \downarrow_{F_\nu} p : p \in A \} = \bigcup \{ S \in \text{Dn}(\text{P}) : \exists p \in A. S \subseteq \downarrow_{F_\nu} p \} = A
\]

since \( A \) downclosed

\[
U_\nu \varepsilon_Q \circ \eta_{U_\nu, Q}(q) = \varepsilon_Q(\downarrow_Q q) = \bigvee_Q q = q
\]

We are now going to describe the free distributive lattice on a finite join-semilattice. Let us first define the relevant functor \( F_\lambda \).

Definition 7.2.5 (Free distributive lattice on a join-semilattice). Let \( U_\lambda : \text{DL}_f \to \text{JSL}_f \) be the forgetful functor which takes the underlying join-semilattice structure i.e. \( U_\lambda \text{D} := (D, \vee_D, \lambda_D) \) and \( U_\lambda f := f \). Further define:

\[
F_\lambda : \text{JSL}_f \to \text{DL}_f \\
F_\lambda \text{Q} := (\text{Dn}(\text{Q}), \cup, \varnothing, \cap, \text{Q}) \\
f : \text{Q} \to \text{R}
\]

Lemma 7.2.6. \( F_\lambda \) equals the composite functor:

\[
\begin{array}{ccc}
\text{JSL}_f & \overset{\text{op}}{\to} & \text{JSL}_f^{\text{op}} \\
\overset{U_\nu^{\text{op}}}{\longrightarrow} & & \overset{U_\nu^{\text{op}}}{\longrightarrow} \\
\text{Poset}_f^{\text{op}} & \overset{\text{op}}{\longrightarrow} & \text{DL}_f
\end{array}
\]

and is thus a well-defined functor.
Proof. We have:

$$\text{Up} \circ U^{op}_v \circ \mathcal{D}^{op}_j Q = \text{Up} \circ U^{op}_v (Q^{op})$$
$$= \text{Up}(Q, \geq_Q)$$
$$= (U p(Q, \geq_Q), \cup, \emptyset, \cap, Q)$$
$$= (D n(Q, \leq_Q), \cup, \emptyset, \cap, Q)$$
$$= F_\alpha Q$$

and furthermore $\text{Up} \circ U^{op}_v \circ \mathcal{D}^{op}_j f = \text{Up} f_* = (f_*)^{-1}$ with domain $F_\alpha Q$ and codomain $F_\alpha \mathbb{R}$.

\[ \square \]

Theorem 7.2.7 (Free distributive lattice on a join-semilattice).

$F_\alpha : \text{JSL}_f \to \text{DL}_f$ is left adjoint to the forgetful functor $U_\alpha : \text{DL}_f \to \text{JSL}_f$ with associated natural transformations:

$$\eta : \text{ld}_{JSL} \Rightarrow U_\alpha \circ F_\alpha \quad \eta_Q : Q \to (D n(Q), \cup, \emptyset) \quad \eta_Q(q) := \sqcap_Q q$$
$$\varepsilon : F_\alpha \circ U_\alpha \Rightarrow \text{ld}_{DL} \quad \varepsilon_D : (D n(\mathbb{D}), \cup, \emptyset, \cap, D) \to \mathbb{D} \quad \varepsilon_D(S) := \sqcap_D S \cap M(\mathbb{D})$$

Proof. Each $\eta_Q$ is a well-defined join-semilattice morphism because:

$$\eta_Q(q) = \sqcap_Q q$$
$$\eta_Q(q_1 \cup Q q_2) = \sqcap_Q (q_1 \cup q_2)$$
$$\eta_Q(q_1 \cap Q q_2) = \{ q \in Q : q_1 \leq_Q q \text{ and } q_2 \leq_Q q \}$$
$$\eta_Q(q_1 \cap Q q_2) = \{ q \in Q : q_1 \leq_Q q \text{ or } q_2 \leq_Q q \}$$

For naturality we must show that:

$$Q \xrightarrow{f} \mathbb{R}$$
$$\eta_Q \downarrow \quad \eta \downarrow$$
$$(D n(Q), \cup, \emptyset) \xrightarrow{U_\alpha F_\alpha f} (D n(\mathbb{R}), \cup, \emptyset)$$

commutes for all join-semilattice morphisms $f : Q \to \mathbb{R}$. Observing that $\eta_\mathbb{R} \circ f(q) = \sqcap_\mathbb{R} f(q)$, we calculate:

$$U_\alpha F_\alpha f \circ \eta_Q(q) = F_\alpha f(\sqcap_Q q)$$
$$= (f_*)^{-1}(\sqcap_Q q)$$
$$= \{ r \in \mathbb{R} : q \leq r, f_*(r) \}$$
$$= \{ r \in \mathbb{R} : f(q) \leq r \}$$

by adjoint relationship

as required.

Next we show that $\varepsilon_D : (D n(\mathbb{D}), \cup, \emptyset, \cap, D) \to \mathbb{D}$ is a well-defined bounded distributive lattice morphism:

1. $\varepsilon_D(\sqcup_{D, U, U_{\sqcup}}) = \varepsilon_D(\emptyset) = \sqcap_D \emptyset \cap M(\mathbb{D}) = \sqcap_D M(\mathbb{D}) = \sqcap_D$.
2. $\varepsilon_D(\sqcup_{D, U, U_{\sqcup}}) = \varepsilon_D(D) = \sqcap_D \emptyset = \sqcap_D$.

3. Regarding meet-preservation:

$$\varepsilon_D(X_1 \sqcap_{F_\alpha, U_\alpha, \sqcap} X_2) = \varepsilon_D(X_1 \cap X_2)$$
$$= \sqcap_D(X_1 \cap X_2)$$
$$= \sqcap_D(X_1 \sqcup X_2) \cap M(\mathbb{D})$$
$$= \sqcap_D(X_1 \sqcup X_2) \cap M(\mathbb{D})$$
$$= \sqcap_D(X_1 \cap M(\mathbb{D})) \cup (X_2 \cap M(\mathbb{D}))$$
$$= \varepsilon_D(X_1) \sqcup \varepsilon_D(X_2)$$

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4. Regarding join-preservation:

\[ \varepsilon_D(X_1 \cup X_2) = \bigwedge_D X_1 \cap X_2 \cap M(D) \]  
\[ = \forall_D \{ d \in D : \forall m \in X_1 \cap X_2 \cap M(D), d \leq_D m \} \]  
\[ = \varepsilon_D(X_1) \vee_D \varepsilon_D(X_2) = (\bigwedge_D X_i \cap M(D)) \vee_D (\bigwedge_D X_i \cap M(D)) \]  
\[ = \bigwedge_D \{ m_1 \vee_D m_2 : m_i \in X_i \cap M(D), i = 1, 2 \} \]

using distributivity in the final equality. Then (B) \( \leq (A) \) because \( X_1 \cap X_2 \cap M(D) \subseteq X_i \cap M(D) \) for \( i = 1, 2 \). To understand why (A') \( \leq (B') \), first observe that each \( X_i \) is up-closed inside \( D \), as is their intersection. Thus given any elements \( m_i \in X_i \cap M(D) \) (where \( i = 1, 2 \)) we have \( m_1 \vee_D m_2 \in X_1 \cap X_2 \). Furthermore any meet-irreducible above \( m_1 \vee_D m_2 \) lies in \( X_1 \cap X_2 \cap M(D) \). Thus any \( d \in D \) which lies below every meet-irreducible in \( X_1 \cap X_2 \) also lies below \( m_1 \vee_D m_2 \), since the latter is the meet of those meet-irreducibles above it.

Concerning the counit-unit equations, we first need to show that:

\[ (Dn(Q), \cup, \varnothing, \cap, Q) \quad (Dn(Dn(Q), \leq), \cup, \varnothing) \quad (Dn(Q), \cup, \varnothing, \cap, Q) \]

1. The first map has action \( F_\varnothing \eta_0(X) = (f_\ast)^{-1}(X) = \{ Y \in Dn(Q) : \bigwedge Q \uparrow \in X \} \), using the following calculation:

\[ (\eta_0)_\ast(Y) = \forall Q \{ q \in Q : \eta_0(q) \subseteq Y \} \]
\[ = \forall Q \{ q \in Q : \uparrow q \subseteq Y \} \]
\[ = \forall Q \{ q \in Q : \downarrow q \subseteq \bigwedge Q \} \]
\[ = \forall Q \{ q \in Q : q \subseteq \bigwedge Q \} \]
\[ = \bigwedge Q \downarrow \]

2. Regarding the second map, we first observe that:

\[ M(F_\varnothing Q) = M(Dn(Q), \cup, \varnothing, \cap, Q) = \{ \uparrow Q q : q \in Q \} \]

which holds because:

\( a \) If \( \uparrow Q q = X_1 \cap X_2 \) then \( \uparrow Q q = \overline{X_1} \cup \overline{X_2} \). Since each \( \overline{X_i} \) is \( Q \)-upclosed \( \exists i. \uparrow Q q \subseteq \overline{X_i} \), hence \( \uparrow Q q \subseteq X_i \subseteq \uparrow Q q \).

\( b \) Every downset is an intersection of these sets, since every upset arises as a union of principal upsets.

Then the second map has action:

\[ \varepsilon_{F_\varnothing Q}(S) = \bigwedge_{F_\varnothing Q} S \cap M(F_\varnothing Q) \]
\[ = \bigcap \{ \uparrow Q q : q \in Q \} \]

3. Composing we obtain:

\[ \varepsilon_{F_\varnothing Q} \circ F_\varnothing \eta_0(X) = \varepsilon_{F_\varnothing Q}(\{ Y \in Dn(Q) : \bigwedge Q \uparrow \in X \}) \]
\[ = \bigcap \{ \uparrow Q q : q \in Q, \bigwedge Q \uparrow q \notin X \} \]
\[ = \bigcap \{ \uparrow Q q : q \in Q, \bigwedge Q \uparrow q \notin X \} \]
\[ = \bigcap \{ \uparrow Q q : q \in X \} \]
\[ = \bigcap \{ \uparrow Q q : q \in \overline{X} \} \]
\[ = \bigcap \{ \uparrow Q q : q \in X \} \]
\[ = X \]  

Regarding the final step, we already observed that every down-closed set arises as an intersection of sets \( \uparrow Q q \).
Finally we show the other counit-unit equation holds:

\[
\begin{array}{ccc}
(D, \vee_D, 1_D) & (Dn(D), \cup, \emptyset) & (D, \vee_D, 1_D) \\
\begin{array}{c}
U_\lambda \mathcal{D} \\
\eta_{U_\lambda \mathcal{D}} \\
\varepsilon_{U_\lambda \mathcal{D}} \\
\id_{U_\lambda \mathcal{D}}
\end{array} & \begin{array}{c}
U_\lambda \circ F_\lambda \circ U_\lambda \mathcal{D} \\
\Upsilon_{U_\lambda \mathcal{D}} \\
U_\lambda \circ \varepsilon_{U_\lambda \mathcal{D}}
\end{array} & U_\lambda \mathcal{D} \\
\end{array}
\]

which follows because:

\[
U_\lambda \varepsilon_{U_\lambda \mathcal{D}} \circ \eta_{U_\lambda \mathcal{D}}(d) = \varepsilon_{U_\lambda \mathcal{D}}(\lambda_D d) = \bigwedge_{\mathcal{D}} \lambda_D d \cap M(\mathcal{D}) = \bigwedge_{\mathcal{D}} \uparrow_D d \cap M(\mathcal{D}) = \bigwedge_{\mathcal{D}} \{ m \in M(\mathcal{D}) : d \leq_M m \} = d
\]

since every element is the meet of those meet-irreducibles above it.

**Definition 7.2.8** (Free boolean algebra on a distributive lattice). Let \( U_\cdot : \mathbb{B}_f \to \mathcal{D}_f \) be the forgetful functor where \( U_\cdot : (B, \vee_B, \wedge_B, \neg_B, \top_B) \) and \( U_\cdot f := f \). Further define:

\[ F_\cdot : \mathcal{D}_f \to \mathbb{B}_f \quad F_\cdot(\mathcal{D}) := \mathbb{P}_b J(\mathcal{D}) \quad F_\cdot f := \lambda X. (U_{dm} f)^{-1}(X) : F_\cdot J(\mathcal{D}) \to F_\cdot J(\mathcal{E}) \]

where \( U_{dm} f \) takes the underlying join-semilattice morphism between the meet structures i.e.

\[ U_{dm} f : (D, \wedge_D, \top_D) \to (E, \wedge_E, \top_E) \]

so that \( F_\cdot f(X) = \{ j \in J(E) : \wedge_D f^{-1}(\uparrow j) \in X \} \).

**Lemma 7.2.9.** \( F_\cdot \) equals the composite functor:

\[ \mathcal{D}_f \xrightarrow{\mathcal{J}_{\mathcal{D}}^{op}} \mathcal{P}_{\mathcal{J}_{\mathcal{D}}^{op}} \xrightarrow{\mathcal{U}_{\mathbb{P}_b J(\mathcal{D})}^{op}} \mathcal{P}_{\mathbb{E}_b J(\mathcal{E})} \xrightarrow{\text{Pred}} \mathbb{B}_f \]

and is thus a well-defined functor.

**Proof.** We have:

\[ \text{Pred} \circ U_\mathcal{D}^{op} \circ \mathcal{J}_{\mathcal{D}}^{op} \mathcal{D} = \text{Pred} U_\mathcal{D} \mathcal{J}_{\mathcal{D}}^{op} (J(\mathcal{D}), \leq_{\mathcal{D}, \mathcal{D}}) = \text{Pred} J(\mathcal{D}) = \mathbb{P}_b J(\mathcal{D}) = F_\cdot(\mathcal{D}) \]

Furthermore given any \( \mathcal{D}_f \)-morphism \( f : \mathcal{D} \to \mathcal{E} \) we have:

\[
\begin{aligned}
\text{Pred} \circ U_\mathcal{D}^{op} \circ \mathcal{J}_{\mathcal{D}}^{op} f &= \text{Pred} \lambda j \in J(\mathcal{E}), \wedge_{\mathcal{D}} f^{-1}(\uparrow j) \\
&= \text{Pred} \lambda j \in J(\mathcal{E}), \vee_{\mathcal{D}} f^{-1}(\downarrow_{\mathcal{E}} j) \\
&= \text{Pred} \lambda j \in J(\mathcal{E}), (U_{dm} f)_*(j) \\
&= \lambda X \in J(\mathcal{D}), (U_{dm} f)^{-1}(X) \\
&= F_\cdot f
\end{aligned}
\]

**Theorem 7.2.10** (Free boolean algebra on a distributive lattice).

\( F_\cdot : \mathcal{D}_f \to \mathbb{B}_f \) is left adjoint to the forgetful functor \( U_\cdot : \mathbb{B}_f \to \mathcal{D}_f \) with associated natural transformations:

\[
\begin{array}{c}
\eta : \text{Id}_{\mathcal{D}_f} \Rightarrow U_\cdot \circ F_\cdot \\
\varepsilon : F_\cdot \circ U_\cdot \Rightarrow \text{Id}_{\mathbb{B}_f}
\end{array}
\]

**Proof.** To see that each \( \eta_\mathcal{D} \) is a well-defined bounded distributive lattice morphism (which needn’t be an isomorphism), observe that it is a codomain extension of the canonical representation of \( \mathcal{D} \) from Theorem 7.1.3 i.e. Birkhoff duality. In order to prove naturality:

\[
\begin{array}{ccc}
\mathcal{D} & \xrightarrow{f} & \mathcal{E} \\
\eta_\mathcal{D} & \downarrow & \eta_\mathcal{E} \\
\mathbb{P}_d J(\mathcal{D}) & \xrightarrow{U_\cdot F_\cdot f} & \mathbb{P}_d J(\mathcal{E})
\end{array}
\]

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we calculate as follows:

\[ U \circ \eta_D(d) = F \circ (J \cap \downarrow_D d) \]
\[ = \{ j \in J : (U dm f) \circ (j) \leq \downarrow_D d \} \]
\[ = \{ j \in J : (U dm f) \circ (j) \leq f(d) \} \]
\[ = \{ j \in J : d \leq E f(d) \} \]
\[ \text{see proof of Theorem 7.1.3} \]
\[ = J \cap \downarrow_E f(d) \]
\[ = \eta_E \circ f(d) \]

Each \( \varepsilon_B \) is well-defined boolean algebra morphism because it is the inverse of a canonical isomorphism from Theorem 7.1.5 i.e. the duality between finite sets and finite boolean algebras. Thus naturality also follows.

Finally we verify the counit-unit equations. Firstly, for any \( X \subseteq J(D) \) we have:

\[ \varepsilon_{F \circ D} \circ F \circ \eta_D(X) = \varepsilon_{F \circ D} \circ (U dm \eta_D)^{-1}(X) \]
\[ = \varepsilon_{F \circ D}(\{ j \in J(D) : \Lambda_D \eta_D^{-1}(\uparrow_D j) \in X \}) \]
\[ = \varepsilon_{F \circ D}(\{ \{ j \} : j \in J(D), \Lambda_D \{ d \in D : \eta_D(d) \ni j \} \in X \}) \]
\[ = \bigvee_{F \circ D}(\{ j \} : j \in J(D), \Lambda_D \{ d \in D : j \in (J(D) \cap \downarrow_D d) \} \in X \}
\[ = \bigcup(\{ j \} : j \in J(D), j \in X \}
\[ = X \]

and finally:

\[ U \circ \eta_B \circ \eta_B(b) = \varepsilon_B(J(U \circ B) \cap \downarrow_B B) = \bigvee_B \operatorname{At}(B) \cap \downarrow_B B = B \]

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