CONVOLUTIONS WITH PROBABILITY DENSITIES
AND APPLICATIONS TO PDES

SORIN G. GAL

Abstract. The purpose of this paper is to introduce several new
convolution operators, generated by some known probability dens-
ities. By using the inverse Fourier transform and taking inverse steps
(in the analogues of the classical procedures used for, e.g., the heat
or Laplace equations), we deduce the initial and final value problems
satisfied by the new convolution integrals.

AMS 2000 Mathematics Subject Classification: 44A35, 35A22,
35C15.

Key words and phrases: Probability densities, convolution integrals,
Fourier transform, initial value problem, final value problem.

1. Introduction

It is well known the fact that the classical Gauss-Weierstrass, Poisson-
Cauchy and Picard convolution singular integrals are based on convo-
lutions with the standard normal density function $\frac{1}{\sqrt{\pi}} \cdot e^{-x^2}$, standard
Cauchy density function $\frac{1}{\pi} \cdot \frac{1}{1+x^2}$ and Laplace density function $\frac{1}{2} \cdot e^{-|x|}$,
respectively. Their approximation properties are studied, for example, in
[1], [3]. Also, by using the Fourier transform method, it is known that
the solutions of the initial value problems for the heat equation and
Laplace equation are exactly the Gauss-Weierstrass and Poisson-Cauchy
convolution singular integrals, respectively, see, e.g., [6], p. 23. On the
other hand, in our best knowledge, the initial value problem and the par-
tial differential equation corresponding to the Picard singular integral, is
missing from mathematical literature. The main aim of the present pa-
er is somehow inverse: introducing convolution singular integrals based
on some known probability densities, we use the inverse Fourier trans-
form in order to find the partial differential equations (initial and final
value problems) satisfied by these integrals, including the Picard singular
integral.
2. Definitions of Convolution Operators

In this section we introduce several convolution operators, based on some well-known densities of probability. If \( d(t, x) \geq 0 \) with \( t > 0 \) and \( x \in \mathbb{R} \) is a probability density, that is \( \int_{-\infty}^{+\infty} d(t, x)dx = 1 \), then our definitions are based on the general known formula

\[
O_t(f)(x) = d(t, \cdot) * f(\cdot) = \int_{-\infty}^{+\infty} f(u) \cdot d(t, x-u)du = \int_{-\infty}^{+\infty} f(x-v) \cdot d(t, v)dv.
\] (2.1)

**Definition 2.1**

(i) For the Maxwell-Boltzmann type probability density (see, e.g., [8], p. 104 and pp. 148-149)

\[
d(t, x) = \frac{1}{\sqrt{2\pi}} \cdot \frac{x^2 e^{-x^2/(2t^2)}}{t^3}, x \in \mathbb{R}, t > 0
\]

and \( f : \mathbb{R} \to \mathbb{R} \), we can formally define the Maxwell-Boltzmann convolution operator

\[
S_t(f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x-v) \cdot \frac{v^2 e^{-v^2/(2t^2)}}{t^3}dv, t > 0, x \in \mathbb{R}. \] (2.2)

(ii) For the Laplace type probability density (see, e.g., [2])

\[
d(t, x) = \frac{1}{2t} e^{-|x|/t}, t > 0, x \in \mathbb{R}
\]

and \( f : \mathbb{R} \to \mathbb{R} \), we can formally define the classical Picard convolution operator

\[
P_t(f)(x) = \frac{1}{2t} \int_{-\infty}^{+\infty} f(x-v) \cdot e^{-|v|/t}dv, t > 0, x \in \mathbb{R}. \] (2.3)

(iii) For the exponential probability density (see, e.g., [2], [7])

\[
d(t, x) = \frac{te^{-t|x|}}{2}, x \in \mathbb{R}, t > 0
\]

and \( f : \mathbb{R} \to \mathbb{R} \), we can formally define the exponential convolution operator

\[
E_t(f)(x) = \int_{-\infty}^{+\infty} f(x-v) \cdot \frac{t e^{-t|x|}}{2}dv, t > 0, x \in \mathbb{R}. \] (2.4)
(iv) For any \( n \in \mathbb{N} \), \( P_t(f)(x) \) can be generalized to the so called Jackson type generalization of the Picard singular integral defined by (see, e.g., [3])

\[
P_{n,t}(f)(x) = -\frac{1}{2t} \int_{-\infty}^{+\infty} \left( \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} f(x + kv) e^{-|v|/t} \right) dv
\]

\[
= \int_{-\infty}^{+\infty} f(x - u) \left[ \sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} \cdot \frac{1}{k} \cdot \frac{e^{-|u|/kt}}{2t} \right] du, t > 0, x \in \mathbb{R}.
\]

(v) Starting from the well known Gauss-Weierstrass operator \( W_t(f)(x) = \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{+\infty} f(x - v) e^{-v^2/t} dv \), we can define its Jackson type generalization by (see, e.g., [3])

\[
W_{n,t}(f)(x) = -\frac{1}{2C^*(t)} \int_{-\infty}^{+\infty} \left( \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} f(x + kv) e^{-v^2/t} \right) dv
\]

\[
= \int_{-\infty}^{+\infty} f(x - u) \left[ \sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} \cdot \frac{1}{k} \cdot \frac{e^{-u^2/(kt)}}{2C^*(t)} \right] du, t > 0, x \in \mathbb{R},
\]

where \( C^*(t) = \int_{0}^{\infty} e^{-u^2/2t} du = \frac{\sqrt{\pi t}}{2} \). Therefore, for any \( n \in \mathbb{N} \), we can write

\[
W_{n,t}(f)(x) = \int_{-\infty}^{+\infty} f(x - u) \left[ \sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} \cdot \frac{1}{k} \cdot \frac{e^{-u^2/(kt)}}{\sqrt{\pi t}} \right] du.
\]

3. Applications to PDE

Concerning the convolution operators defined in Section 2, we can state the following applications to PDE.

**Theorem 3.1.** (i) Suppose that \( f, f', f'', f''' : \mathbb{R} \rightarrow \mathbb{R} \) are bounded and uniformly continuous on \( \mathbb{R} \). The solution of the initial value problem

\[
\frac{\partial u}{\partial t}(x, t) = t^3 \frac{\partial^4 u}{\partial x^4}(x, t) - t^2 \frac{\partial^3 u}{\partial x^2 \partial t}(x, t) + 3t \frac{\partial^2 u}{\partial x^2}(x, t),
\]

\[
\lim_{s \searrow 0} u(x, s) = f(x), t > 0, x \in \mathbb{R},
\]

is \( u(x, t) := S_t(f)(x) \).
(ii) Suppose that \( f, f', f'' : \mathbb{R} \to \mathbb{R} \) are bounded and uniformly continuous on \( \mathbb{R} \). The solution of the initial value problem

\[
\frac{\partial u}{\partial t}(x, t) = t^2 \frac{\partial^3 u}{\partial x^3 \partial t}(x, t) + 2t \frac{\partial^2 u}{\partial x^2 \partial t}(x, t), \quad \lim_{s \to 0} u(x, s) = f(x), \quad t > 0, \quad x \in \mathbb{R}
\]

is \( u(x, t) := P_t(f)(x) \).

(iii) Suppose that \( f, f', f'' : \mathbb{R} \to \mathbb{R} \) are bounded and uniformly continuous on \( \mathbb{R} \). The solution of the final value problem

\[
\frac{\partial u}{\partial t}(x, t) = \frac{1}{t^2} \frac{\partial^3 u}{\partial x^3 \partial t}(x, t) - \frac{2}{t^3} \frac{\partial^2 u}{\partial x^2 \partial t}(x, t), \quad \lim_{s \to \infty} u(x, s) = f(x), \quad t > 0, \quad x \in \mathbb{R}
\]

is \( u(x, t) := E_t(f)(x) \).

(iv) Suppose that \( f, f', f'' : \mathbb{R} \to \mathbb{R} \) are bounded and uniformly continuous on \( \mathbb{R} \). We have

\[
P_{n,t}(f)(x) = \sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} \cdot u_k(x, t),
\]

where \( u_k(x, t) = P_{kt}(f)(x) = \frac{1}{2kt} \cdot \int_{-\infty}^{+\infty} f(x-u) e^{-|u|/(kt)} du, \ k = 1, \ldots, n+1 \) are solutions of the initial value problems (for \( t > 0 \) and \( x \in \mathbb{R} \))

\[
\frac{\partial u_k}{\partial t}(x, t) = k^2 t^2 \frac{\partial^3 u_k}{\partial x^3 \partial t}(x, t) + 2k^2 t \cdot \frac{\partial^2 u_k}{\partial x^2 \partial t}(x, t), \quad \lim_{s \to 0} u_k(x, s) = f(x).
\]

(v) Suppose that \( f, f', f'' : \mathbb{R} \to \mathbb{R} \) are bounded and uniformly continuous on \( \mathbb{R} \). We have

\[
W_{n,t}(f)(x) = \sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} \cdot u_k(x, t),
\]

where \( u_k(x, t) = \frac{1}{\sqrt{k}} W_{kt}(f)(x) = \frac{1}{k \sqrt{\pi t}} \cdot \int_{-\infty}^{+\infty} f(x-u) e^{-u^2/(kt)} du, \ k = 1, \ldots, n+1 \) are solutions of the initial value problems

\[
\frac{\partial u_k}{\partial t}(x, t) = k^2 \frac{\partial^2 u_k}{\partial x^2 \partial t}(x, t), \quad \lim_{s \to 0} u_k(x, s) = \frac{1}{\sqrt{k}} f(x), \quad t > 0, \quad x \in \mathbb{R}.
\]

**Proof.** Since for the convolution operator given by (2.1), in general we have \( d(t, v) \geq 0 \), for all \( t > 0 \) and \( v \in \mathbb{R} \), by the standard method we easily get

\[
|O_t(f)(x) - f(x)| \leq \int_{-\infty}^{+\infty} |f(x-v) - f(x)|d(t, v)dv
\]
\[ \leq \int_{-\infty}^{+\infty} \omega_1(f; |v|) d(t, v)dv \leq 2\omega_1(f; \varphi(t))_R, \]

where \( \omega_1(f; \delta)_R = \sup\{|f(x) - f(y)|; x, y \in \mathbb{R}, |x - y| \leq \delta\} \) and \( \varphi(t) = \int_{-\infty}^{+\infty} |v| \cdot d(t, v)dv \).

Evidently that this method is useful only if \( \varphi(t) < +\infty \) for all \( t > 0 \).

In order to deduce the PDE equations satisfied by various convolution operators, we will need the concepts of Fourier transform of a function \( g \), defined by

\[ F(g)(\xi) = \hat{g}(\xi) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{+\infty} g(x)e^{-ix \xi}dx, \text{ if } \int_{-\infty}^{+\infty} |g(x)|dx < +\infty, \]

and of inverse Fourier transform defined by

\[ F^{-1}(\hat{g})(x) = g(x) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{+\infty} \hat{g}(\xi)e^{ix \xi}d\xi. \]

(i) By making the change of variable \( v = \sqrt{2}ts \), we get

\[ \varphi(t) = \frac{1}{t^3} \cdot \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^{+\infty} v^3 e^{-v^2/(2t^2)} dv = \frac{1}{t^3} \cdot \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^{+\infty} (2\sqrt{2}t^3 s^3)e^{-s^2}(\sqrt{2}t)ds \]

\[ = \frac{4\sqrt{2}}{\sqrt{\pi}} \cdot t \int_0^{+\infty} s^3 e^{-s^2} ds = \frac{2\sqrt{2}}{\sqrt{\pi}} t < 2t, \]

which immediately implies

\[ |S_t(f)(x) - f(x)| \leq 4\omega_1(f; t)_R, t > 0, x \in \mathbb{R}. \]

Taking into account the uniform continuity of \( f \), the above inequality immediately implies that \( \lim_{t \to 0} S_t(f)(x) = f(x) \), for all \( x \in \mathbb{R} \). Therefore we may take, by convention, \( S_0(f)(x) = f(x) \), for all \( x \in \mathbb{R} \).

Now, in order to deduce the PDE satisfied by \( S_t(f)(x) \), we write it in the form

\[ S_t(f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y) \cdot \frac{(x - y)^2 e^{-(x-y)^2/(2t^2)}}{t^3} dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y) \cdot \hat{g}_t(y - x) dy. \]

Here, by using standard reasonings/calculation (or the WolframAlpha soft of calculation), we obtain

\[ g_t(\xi) = F^{-1} \left[ w^2 e^{-w^2/(2t^2)} / t^3 \right](\xi, t) = e^{-t^2 \xi^2 / 2} (1 - t^2 \xi^2), \]
which implies

\[
S_t(f)(x) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{+\infty} f(y) \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ix(y-x)}e^{-t^2\xi^2/2(1-t^2\xi^2)}d\xi \right]dy
\]

\[
= \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{+\infty} \left[ \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{+\infty} e^{-iy\xi}f(y)dy \right] e^{ix\xi}e^{-t^2\xi^2/2(1-t^2\xi^2)}d\xi
\]

\[
= \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{+\infty} e^{ix\xi}\hat{f}(\xi)e^{-t^2\xi^2/2(1-t^2\xi^2)}d\xi
\]

\[
= \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{+\infty} e^{ix\xi}\hat{u}(\xi,t)d\xi := u(x,t),
\]

where

\[
\hat{u}(\xi,t) = \hat{f}(\xi) \cdot e^{-t^2\xi^2/2(1-t^2\xi^2)}.
\]

This is equivalent to \( \hat{u}(\xi,t) \cdot \frac{e^{t^2\xi^2/2}}{1-t^2\xi^2} = \hat{f}(\xi) \), which is equivalent to

\[
\frac{\partial}{\partial t} \left[ \hat{u}(\xi,t) \cdot \frac{e^{t^2\xi^2/2}}{1-t^2\xi^2} \right] = \frac{\partial \hat{u}}{\partial t}(\xi,t) \cdot \frac{e^{t^2\xi^2/2}}{1-t^2\xi^2} + \hat{u}(\xi,t) \cdot \left( \frac{e^{t^2\xi^2/2}}{1-t^2\xi^2} \right)' = 0.
\]

Note that the above relation can be evidently written under the form

\[
\frac{\partial}{\partial t} \left[ \hat{u}(\xi,t) \cdot \frac{1}{F_w^{-1}(d(t,w))(\xi,t)} \right] = 0,
\]

where \(d(t,x)\) is the Maxwell-Boltzmann type probability density in Definition 2.1, (i), entering in the formula for \(S_t(f)(x)\).

After simple calculation, the above formula is formally equivalent to (of course for \(1 \neq t^2\xi^2\))

\[
\frac{\partial \hat{u}}{\partial t}(\xi,t) + t^2 \left( -\xi^2 \cdot \frac{\partial \hat{u}}{\partial t}(\xi,t) \right) - 3t[-\xi^2 \hat{u}(\xi,t)] - t^3 \cdot [\xi^4 \hat{u}(\xi,t)] = 0.
\]

Now, taking into account that

\[
\frac{\partial \hat{u}}{\partial t}(\xi,t) = \hat{u}(\xi,t), \quad \frac{\partial^2 \hat{u}}{\partial x^2}(\xi,t) = -\xi^2 \hat{u}(\xi,t), \quad \frac{\partial^4 \hat{u}}{\partial x^4}(\xi,t) = \xi^4 \hat{u}(\xi,t),
\]

and replacing above, we obtain

\[
F \left( \frac{\partial u}{\partial t} + t^2 \frac{\partial^3 u}{\partial x^2 \partial t} - 3t \frac{\partial^2 u}{\partial x^2} - t^3 \frac{\partial^4 u}{\partial x^4} \right)(\xi,t) = 0,
\]

that is

\[
\frac{\partial u}{\partial t}(x,t) = t^3 \frac{\partial^4 u}{\partial x^4}(x,t) - t^2 \frac{\partial^3 u}{\partial x^2 \partial t}(x,t) + 3t \frac{\partial^2 u}{\partial x^2}(x,t).
\]
Finally, following the above steps in inverse order, we arrive at the conclusion in the statement.

(ii) By [1], p. 142, Corollary 3.4.2, it was obtained

\[ |f(x) - P_t(f)(x)| \leq C\omega_2(f; t)_\mathbb{R}. \]

Therefore, it is immediate that \( \lim_{t \to 0^+} P_t(f)(x) = f(x) \), for all \( x \in \mathbb{R} \).

In order to deduce the PDE satisfied by \( P_t(f)(x) \), we reason exactly as in the above case (i). Indeed, by standard calculation (or by making use of the WolframAlpha program), we get

\[
F_w^{-1}[e^{-|w|t}/(2t)](\xi, t) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{1 + t^2\xi^2}
\]

and similar reasonings with those in the case (i), immediately leads to

\[
P_t(f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ix\xi} \hat{u}(\xi, t) d\xi := u(x, t),
\]

where

\[ \hat{u}(\xi, t) = \hat{f}(\xi) \cdot \frac{1}{t^2\xi^2 + 1}. \]

In fact, directly as in the proof of Theorem 3.1, (i), we can write

\[
\frac{\partial}{\partial t} \left[ \hat{u}(\xi, t) \cdot \frac{1}{F_w^{-1}(d(t, w))(\xi, t)} \right] = 0,
\]

where \( d(t, x) \) is the Laplace type probability density in Definition 2.1, (ii), entering in the formula for \( P_t(f)(x) \).

Therefore,

\[
\frac{\partial}{\partial t} \left[ \hat{u}(\xi, t) \cdot (1 + t^2\xi^2) \right] = \frac{\partial \hat{u}}{\partial t}(\xi, t) + t^2\xi^2 \frac{\partial \hat{u}}{\partial \xi}(\xi, t) + 2t\xi^2 \hat{u}(\xi, t) = 0,
\]

which immediately leads to

\[
\frac{\partial u}{\partial t}(x, t) = t^2 \frac{\partial^3 u}{\partial x^2 \partial t}(x, t) + 2t \frac{\partial^2 u}{\partial x^2}(x, t).
\]

Following the above steps, now from the end to the beginning, we arrive at the conclusion in the statement.

(iii) Firstly, we observe that \( E_t(f)(x) = P_{1/t}(f)(x) \), for all \( t > 0 \) and \( x \in \mathbb{R} \). The, by (ii) we immediately get

\[
|E_t(f)(x) - f(x)| = |P_{1/t}(f)(x) - f(x)| \leq 2\omega_1 \left( f; \frac{1}{t} \right)_\mathbb{R}.
\]

7
Then, again by standard calculation (or by using WolframAlpha), we have $F_w^{-1}(e^{-|w|t})(\xi, t) = \frac{t}{\sqrt{2\pi}} \cdot \frac{t^2}{t^2 + \xi^2}$. This immediately implies

$$F^{-1}(d(t, w))(\xi, t) = \frac{t}{2} \cdot F_w^{-1}(e^{-|w|t})(\xi, t) = \frac{1}{\sqrt{2\pi}} \cdot \frac{t^2}{t^2 + \xi^2}.$$ 

It follows $F^{-1}(\frac{1}{d(t, w)(\xi, t)}) = \sqrt{2\pi} \cdot \frac{\xi^2 + t^2}{t^2} = \sqrt{2\pi} \left(1 + \frac{\xi^2}{t^2}\right)$.

Therefore, denoting $u(x, t) = E_t(f)(x)$, by the method used at the above points, we arrive at the PDE

$$\frac{\partial}{\partial t} \left(\hat{u}(\xi, t) \cdot \left(1 + \frac{\xi^2}{t^2}\right)\right) = \hat{u}(\xi, t) \left(1 + \frac{\xi^2}{t^2}\right) + \hat{u}(\xi, t) \left(-\frac{2\xi^2}{t^3}\right) = 0.$$

This immediately leads to the following PDE, satisfied by $u(x, t) = E_t(f)(x)$

$$\frac{\partial u}{\partial t}(x, t) = \frac{1}{t^2} \cdot \frac{\partial^3 u}{\partial x^2 \partial t}(x, t) - \frac{2}{t^3} \cdot \frac{\partial^2 u}{\partial x^2}(x, t).$$

Since $E_t(f)(x) = P_{1/t}(f)(x)$, it follows that $\lim_{t \to \infty} E_t(f)(x) = f(x)$, for all $x \in \mathbb{R}$.

Following the above steps in inverse order, we arrive at the conclusion in the statement.

(iv) Concerning the approximation properties of $P_{n, t}(f)(x)$, in [3] it was obtained the estimate

$$|f(x) - P_{n, t}(f)(x)| \leq \sum_{k=1}^{n+1} k! \cdot \binom{n+1}{k} \cdot \omega_{n+1}(f; t) \mathbb{R},$$

where $\omega_{n+1}(f; \delta) = \sup_{0 \leq h \leq \delta} \{|\Delta_{n+1}^h f(x); x \in \mathbb{R}\}$, with $\Delta_{n+1}^h = \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} f(x + jh)$. This immediately implies $\lim_{t \to 0} P_{n, t}(f)(x) = f(x)$, for all $x \in \mathbb{R}$.

In order to deduce the PDE satisfied by $P_{n, t}(f)(x)$, since $F_w^{-1}$ is linear operator and since known calculation (or by using the WolframAlpha software) give $F_w^{-1}(e^{-|w|/t})(\xi, t) = \frac{t}{\sqrt{2\pi}} \cdot \frac{1}{t^2 \xi^2 + 1}$, replacing here $t$ by $kt$, we easily obtain

$$F_w^{-1} \left[\sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} \cdot \frac{1}{2kt} e^{-|w|/(kt)}\right] = \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} \cdot \frac{kt}{k^2 t^2 \xi^2 + 1}$$

8
\[
= \frac{1}{\sqrt{2\pi}} \cdot \sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} \cdot \frac{1}{k^2 t^2 \xi^2 + 1}.
\]

Therefore, denoting \( u(x, t) := P_{n,t}(f)(x) \) we immediately get the differential equation
\[
\frac{\partial}{\partial t} \left[ \hat{u}(\xi, t) \cdot \frac{1}{\sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} \cdot \frac{1}{k^2 t^2 \xi^2 + 1}} \right] = 0,
\]
which is equivalent to
\[
\sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} \left[ \frac{\partial \hat{u}}{\partial t}(\xi, t) \cdot \frac{1}{k^2 t^2 \xi^2 + 1} + \hat{u}(\xi, t) \cdot \frac{2k^2 t^2 \xi^2}{(k^2 t^2 \xi^2 + 1)^2} \right] = 0.
\]

It is worth noting that denoting
\[
u_k(x, t) = P_{k,t}(f)(x) = \frac{1}{2kt} \cdot \int_{-\infty}^{+\infty} f(x - u) e^{-|u|/(kt)} du,
\]
we can write
\[
P_{n,t}(f)(x) = \sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} \cdot \nu_k(x, t),
\]
where reasoning as above for \( P_t(f)(x) \), we easily obtain
\[
\frac{\partial \nu_k}{\partial t}(\xi, t) = k^2 t^2 \frac{\partial^3 \nu_k}{\partial x^2 \partial t}(\xi, t) + 2k^2 t \frac{\partial^2 \nu_k}{\partial x^2}(\xi, t)
\]
and which implies
\[
\frac{\partial \nu_k}{\partial t}(x, t) = k^2 t^2 \frac{\partial^3 \nu_k}{\partial x^2 \partial t}(x, t) + 2k^2 t \frac{\partial^2 \nu_k}{\partial x^2}(x, t), \quad x \in \mathbb{R}, \quad t > 0, \quad k = 1, \ldots, n+1,
\]
with \( \nu_k(x, 0) = f(x) \), for all \( x \in \mathbb{R} \), \( k = 1, \ldots, n+1 \).

(v) Concerning the approximation properties of \( W_{n,t}(f)(x) \), reasoning as in \([3]\), we get the estimate
\[
|f(x) - W_{n,t}(f)(x)| \leq C_n \cdot \omega_{n+1}(f; \sqrt{t})_\mathbb{R},
\]
where \( C_n > 0 \) is a constant independent of \( f, t \) and \( x \). This immediately implies that \( \lim_{t \to 0^+} W_{n,t}(f)(x) = f(x) \), for all \( x \in \mathbb{R} \).

Now, in order to deduce the PDE satisfied by \( W_{n,t}(f)(x) \), since \( F^{-1}_w \) is linear operator and since (by, e.g., WolframAlpha software) we have
\( F^{-1}_w(e^{-w^2/2})(\xi, t) = \frac{\sqrt{t} e^{-t\xi^2/4}}{\sqrt{2}} \), replacing here \( t \) by \( kt \), we easily obtain
\[
F^{-1}_w(e^{-w^2/(kt)}) (\xi, t) = \frac{\sqrt{kt e^{-kt\xi^2/4}}}{\sqrt{2}} \]
and
\[
F^{-1}_w \left[ \sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} \cdot \frac{1}{\sqrt{n+k}} e^{-w^2/(kt)} \right] (\xi, t)
\]
\[
= \frac{1}{\sqrt{2\pi}} \cdot \sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} \cdot \frac{1}{\sqrt{k}} e^{-kt\xi^2/4}
\]
Therefore, denoting \( u(x, t) := W_{n,t}(f)(x) \) we immediately get the differential equation
\[
\frac{\partial}{\partial t} \left[ \hat{u}(\xi, t) \cdot \frac{1}{\sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} \cdot \frac{1}{\sqrt{k}} e^{-kt\xi^2/4}} \right] = 0,
\]
which is equivalent to
\[
\sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} \cdot \frac{\partial \hat{u}(\xi, t) \cdot e^{-kt\xi^2/4} + \hat{u}(\xi, t) \cdot \frac{k\xi^2}{4} \cdot e^{-kt\xi^2/4}}{\partial \xi} \]

It is worth noting that denoting
\[
u_k(x, t) = \frac{1}{\sqrt{k}} W_{kt}(f)(x) = \frac{1}{k \sqrt{\pi t}} \cdot \int_{-\infty}^{+\infty} f(x - u) e^{-u^2/(kt)} du,
\]
we can write
\[
W_{n,t}(f)(x) = \sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} \cdot u_k(x, t),
\]
where reasoning as above but now for \( W_t(f)(x) \), we easily obtain
\[
\frac{\partial \hat{\nu}_k}{\partial t}(\xi, t) = \frac{k}{4} \cdot \frac{\partial^2 \hat{\nu}_k}{\partial x^2}(\xi, t)
\]
and which implies
\[
\frac{\partial u_k}{\partial t}(x, t) = \frac{k}{4} \cdot \frac{\partial^2 u_k}{\partial x^2}(x, t), \quad x \in \mathbb{R}, \ t > 0, \ k = 1, ..., n + 1,
\]
with \( u_k(x, 0) = f(x) \), for all \( x \in \mathbb{R}, \ k = 1, ..., n + 1. \) \( \square \)
Remark 3.2 The methods in this paper could be used to make analogous studies for the convolutions with other known probability densities, like the Rayleigh probability density (shortly written Rayleigh p.d.), Gumbel p.d., logistic p.d., Johnson p.d., Fréchet p.d., Gompertz p.d., Lévy p.d., Lomax p.d. and so on.

Remark 3.3. It would be also of interest to use the methods in this paper to the case of the corresponding complex convolutions, based on the ideas and results in the books [4] and [5].

References

[1] Butzer, P. L. and Nessel, R. J., *Fourier Analysis and Approximation, Vol. 1, One-Dimensional Theory*, Pure And Applied Mathematics, Academic Press, New York and London, 1971.

[2] Everitt, B. S. and Skrondall, A., *The Cambridge Dictionary of Statistics*, Fourth edition, Cambridge University Press, Cambridge, 2010.

[3] Gal, S. G., Degree of approximation of continuous functions by some singular integrals, *Rev. D’Analyse Numér. Théor. L’Approx.* (Cluj), **XXVII**, No. 2 (1998), 251–261.

[4] Gal, S. G., *Approximation by Complex Bernstein and Convolution Type Operators*, World Scientific Publ., Singapore, 2009.

[5] Gal, C. G., Gal, S. G. and Goldstein, J. A., *Evolution Equations with a Complex Spatial Variable*, World Scientific, Singapore, 2014.

[6] Goldstein, J. A., *Semigroups of Linear Operators and Applications*, Oxford University Press, 1985.

[7] Johnson, N. L., Kotz S. and Balakrishnan, N., *Continuous Univariate Distributions*, vol. 1, second ed., John Wiley Publ., New York, 1994.

[8] Papoulis, A., *Probability, Random Variables, and Stochastic Processes*, 2nd ed., New York: McGraw-Hill, New York, 1984, pp. 104 and 148-149.

University of Oradea
Department of Mathematics and Computer Science
Str. Universității 1
410087 Oradea, Romania
E-mail address: galso@uoradea.ro