Instability of fermions with respect to topology fluctuations

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Abstract

We extend our sum over topologies formula to fermions. We show that fermionic fields display an instability with respect to topology fluctuations. We present some phenomenological arguments for a modification of the action in the case of fermions and discuss possible applications.

1 Introduction

As it was recently demonstrated in Ref. [1], when we take into account for spacetime foam effects, all divergencies and inconsistencies in quantum field theories do disappear (in the full agreement with that was expected [2]). This remarkable feature however concerns only the bosonic sector of the field theory. When we try to use the same arguments (i.e., the same technique to account for the spacetime foam) in the case of fermions, we find that inconsistencies only do increase. In the present paper we analyse this feature and show the way to overcome such a difficulty.

It turns out that fermionic degrees of freedom are unstable with respect to topology fluctuations. In the first place such an instability means only that there should take place a phase transition which makes fermions to be stable with respect to topology changes. Yet, we do not know the exact mechanism (and the exact theory) of the stabilization and the particular topology which makes fermions to be stable. However, we point out that this instability is actually not a new result. It was first found by Banks et. al. in Ref. [3] where it was shown that wormholes drive the free fermion mass toward the cut-off scale. If the cut-off scale is absent, then the mass goes to infinity and therefore fermions do not propagate. The equivalence of these two statements is seen directly from the definition of the true Green function which has the form $G(p) = N(p) / (\gamma p - m)$ and which disappears in both cases as $m \to \infty$ and as $N(p) \to 0$. 

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Let us recall first our sum over topologies formula \[1\]. We use the euclidean formulation, e.g., see Ref. \[4\]. Then the euclidean path integral for the expectation value of an observable \(B\) is

\[
\langle B \rangle = \frac{\sum B e^{-S}}{\sum e^{-S}}
\]

where \(S\) is the euclidean action and the sum is taken over all field configurations and all topologies of the euclidean spacetime. The path integral is then taken in the two steps. First, we integrate over all field configurations keeping a specific class of topologies fixed and then sum over different topological classes, so that the partition function can be presented as

\[
Z = \sum e^{-S} = \sum e^{-S_{\text{eff}}}
\]

where \(S_{\text{eff}}\) is an independent effective action for each particular topological class and \(N\) defines the topological class.

An arbitrary topology of space can be accounted for by the bias of sources \[5\]. Indeed, in considering very small scales in particle physics we use an extrapolation of spatial relationships which are well-tested only at laboratory scales. Therefore, if the topological structure of the actual Universe does not match properly that of the extrapolated coordinate space we naturally should observe some discrepancy. It is convenient to describe such a discrepancy as follows.

Consider the euclidean coordinate space \(\mathbb{R}^4\) and let \(H\) be the Hilbert space for a free particle which moves in \(\mathbb{R}^4\) (i.e., \(H\) is merely the space of functions on \(\mathbb{R}^4\)). Let \(\{g_k(x)\}\) be an arbitrary basis in \(H\). Physically, the basis represents a set of eigenvectors for a complete set of observables. In our case we can consider a scalar (without the spin) particle and use the coordinate representation, i.e., \(g_k(x) = \delta(x_k - x)\) is the set of eigenvectors for the position operator \(\hat{X}g_k = x_k g_k\). The basis is supposed to be normalized \((g_k, g_p) = \delta_{kp}\) and complete \(\sum g_k^* (x) g_k (x') = \delta(x - x')\), where \(x, x' \in \mathbb{R}^4\).

It is important that in particle physics the actual physical space \(V_{\text{phys}}\) admits an embedding into the coordinate (extrapolated) space \(\mathbb{R}^4\) i.e., \(V_{\text{phys}} \subset \mathbb{R}^4\). However in general there always exists some discrepancy between the actual space \(V_{\text{phys}}\) and the coordinate space \(\mathbb{R}^4\). One may easily imagine some irregular distribution for \(V_{\text{phys}} \subset \mathbb{R}^4\) (i.e., \(V_{\text{phys}}\) includes also ”voids”). In terms of the scalar particle this means that some eigenvalues \(x_k\) for the position operator are absent. This also means that some states in \(H\) cannot be physically realized for actual physical particles and fields and, therefore, we have to restrict the space of states \(H\) onto the space of physically admissible states \(H_{\text{phys}} = \hat{P}H\), where \(\hat{P} = (\hat{P})^2\) is a projection operator. In the basis of eigenvectors the projection operator \(\hat{P}\) takes the diagonal form \((f_j, \hat{P} f_k) = P_{jk} = N_k \delta_{jk}\) with eigenvalues \(N_k = 0, 1\). Thus, an arbitrary (physically realizable) field is biased and can be presented

\[1\] We recall that when extrapolating to extremely large scales (in astrophysics) we should consider the universal covering on which such a simple feature \(V_{\text{phys}} \subset \mathbb{R}^4\) disappears. See for detail Ref. \[1\].
as $\psi_{\text{phys}} = \hat{P}^{1/2} \psi = \sum \sqrt{N_k} a_k f_k(x)$. We see that topological structure of the space $V_{\text{phys}}$ is one-to-one encoded by the bias (projection) operator $\hat{P}$ which can be described by its kernel, i.e., by a two point function $N(x,x')$ (which in general represents a generalized function or a distribution). In particular, all physical observables acquire the structure $\hat{O}_{\text{phys}} = \hat{P}^{1/2} \hat{O} \hat{P}^{1/2}$, while the physical space $V_{\text{phys}}$ of the system represents the space of eigenvalues $x_k \in V_{\text{phys}}$ of the biased position operator of the scalar particle $\hat{X}_{\text{phys}} = \hat{P}^{1/2} \hat{X} \hat{P}^{1/2}$. We also point out that from the point of view of the mathematical coordinate space (i.e., $R^4$) the space $H_{\text{phys}}$ is not complete, i.e., $\sum N_k f_k^* (x) f_k (x') = N(x,x') = \hat{P}^{1/2} \delta (x-x') \hat{P}^{1/2} \neq \delta (x-x')$. Thus, we see that the function $N(x,x')$ replaces the delta function (i.e., the standard unit operator).

Consider now the Green function\(^2\) for a scalar wave equation in $R^4$

$$(-\Box + m^2) G(x,y) = 4 \pi \delta(x-y).$$

When we consider the actual physical space $V_{\text{phys}} \subset R^4$ this equation transforms as follows

$$(-\Box + m^2) G(x,y) = 4 \pi N(x,y), \quad (3)$$

where $N(x,y)$ is the bias (or projection) operator introduced above. Let us return to the path integral (2). Consider a particular virtual topology of space which is one-to-one defined by specifying $N(x,y)$. It is clear that due to symmetries of $R^4$ the action $S$ in (1)-(2) has the same value for all physical spaces which can be obtained by rotations and transitions of the coordinate system in $R^4$. Thus, upon averaging out over possible orientations and transitions the bias acquires always the structure $\tilde{N}(x,y) = N(|x-y|)$ and for the Green function we find

$$G(x,y) = \int \frac{d^4k}{(2\pi)^4} \frac{N(k)}{k^2 + m^2} \exp\{ik(x-y)\}, \quad (4)$$

where $N(k)$ is the Fourier transform for the bias $\tilde{N}(x)$. It is important that in the case of homogeneous and isotropic background space the Green functions for fermions can be found directly from the scalar Green function

$$S_F(x-y) = - (\gamma \hat{p} + m) G(x-y) \quad (5)$$

where $\hat{p}^\alpha = -i \partial^\alpha$. We recall that in the euclidian space the Dirac matrixes have the property $\{\gamma_\alpha \gamma_\beta\} = -2 \delta_{\alpha\beta}$ and therefore this Green function obeys the equation

$$(\gamma \hat{p} - m) S_F(x,y) = 4 \pi N(x-y).$$

We point out that upon averaging out over orientations and transitions the same bias $N(k)$ describes already not a particular topology but rather a particular class of equivalent topologies which are denoted by $N$ in the sum (2).

\(^2\)We recall that the complete true Green functions are defined by (1) as $G(x,y) = \langle \phi(x)\phi(y) \rangle = \langle 0 | T \phi(x)\phi(y) | 0 \rangle$. 

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Moreover, if we consider every particular topological class, then the basic property of the projection operator $\hat{P} = (\hat{P})^2$ (which distinguishes a particular topological structure) transforms into $(\hat{P})^2 \leq \hat{P}$ and therefore eigenvalues $N_k = 0, 1$ transforms in general into numbers $N (k) \leq 1$. However we replace further the sum over topological classes (i.e., over all possible functions $N (k) \leq 1$) with an equivalent problem that is the sum over multi-valued fields with $N_k$ having the values $N_k = 0, 1$. We recall that multi-valued fields give not more, than a specific representation for the generalized statistics (or the Green statistics [6]) see for details Ref. [1] and references therein.

Let us return to the path integral (2). The homogeneous and isotropic character of the background (i.e., of the coordinate) space $R^4$ leads to the fact that the multi-valued character of fields is more convenient to describe in the Fourier representation ($\phi = \frac{1}{(2\pi)^2} \int d^4k \phi_k e^{ikx}$) that is to replace a single-valued field $\phi_k$ with a set of fields $\phi^j_k$ where $j = 1, 2, ..., N (k)$, while the bias function $N (k)$ acquires the meaning of the number of such fields. Then the euclidean action for a field of an arbitrary spin which in the standard picture (in the Planckian units where $M_{pl} = 1$) takes the form

$$S = \int \left[ Tr \left( \phi^* \hat{A} \phi \right) + V (\phi) \right] d^4x, \quad (6)$$

transforms into $S_0 + S_{int} (\phi)$. Here the linear part of the action takes the structure

$$S_0 = L^4 \int \sum_{j=1}^{N(k)} Tr \left( \phi^{*j}_k A (k) \phi^j_k \right) \frac{d^4k}{(2\pi)^4}, \quad (7)$$

where $A (k) = k^2 + m^2$ in the case of scalar particles and $A (k) = \gamma k - m$ in the case of fermions. The sign $Tr$ denotes the trace over all additional components of the field $\phi$. In the path integral the non-linear term $S_{int} (\phi)$ is accounted for by perturbations.

We recall that in this expression the values of the number of fields $N (k)$ depend on scales under consideration and, therefore, the result for the mean cutoff function depends on the choice of the continuation used. As it was explained previously in Ref. [1] in astrophysical problems we use the universal covering and the number of fields takes values $N (k) = 0, 1, 2, ...,$, while in particle physics discussed previously the number of fields can take only two possible values $N (k) = 0, 1$.

The physical sense has only the sum of fields ($\bar{\phi}_k = \sum_{j=1}^{N(k)} \phi^j_k$), and therefore the generating functional is taken as (e.g., see the standard books Ref. [7])

$$Z [J] = \exp \left\{ -S_{int} \left( \frac{\delta}{\delta J} \right) \right\} \sum_{N(k)} \bar{Z} [J, N],$$

where

$$\bar{Z} [J, N] = \int D [\phi] D [\phi^*] \exp \left\{ -S_0 (\phi) + L^4 \int J^* (k) \bar{\phi}_k d^4k \right\}$$

$$\bar{Z} [J, N] = \int D [\phi] D [\phi^*] \exp \left\{ -S_0 (\phi) + L^4 \int J^* (k) \bar{\phi}_k d^4k \right\}$$

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\[
\bar{Z} [N] \exp \left\{ L^4 \int Tr \left[ J^* (k) A^{-1} (k) J (k) \right] \frac{N (k) d^4 k}{(2 \pi)^4} \right\} \quad (8)
\]

and for \( \tilde{Z} [N] \) we have (the sign + stands for fermions, while minus stands for bosons)

\[
\tilde{Z} [N] = \exp \left\{ \pm L^4 \int N (k) \frac{d^4 k}{(2 \pi)^4} \ln \frac{\det A (k)}{\pi} \right\}. \quad (9)
\]

This functional has the structure \( \tilde{Z} [N] = \prod_k Z_k^{N(k)} \)

where \( Z_k \) is given by the standard single-field expressions

\[
Z_k = (\det A (k) / \pi)^{\pm 1}
\]

and the sum over possible values \( N (k) = 0, 1 \) gives

\[
Z = \sum_N \tilde{Z} [N] = \prod_k \left( \sum_{N=0,1} Z_k^{N(k)} \right) = \prod_k (1 + Z_k), \quad (10)
\]

while for the mean cutoff we find from (11)

\[
N (k) = \frac{Z_k}{1 + Z_k} \quad (11)
\]

This expression straightforwardly generalizes on an arbitrary set of fields which gives

\[
\ln Z_k = \frac{1}{2} \sum_{\alpha \in F} \ln \left( \frac{k^2 + m_{\alpha,F}^2}{\pi} \right) - \frac{1}{2} \sum_{\alpha \in B} \ln \left( \frac{k^2 + m_{\alpha,B}^2}{\pi} \right), \quad (12)
\]

where \( F \) and \( B \) stands for fermions and bosons respectively and the sum is taken over all fields and helicity states.

When we restrict ourself with the bosonic sector only (i.e., with the second sum in (12), as in Ref. [1]), then from (12) we find that on the mas-shell (as \( k^2 + m_{\alpha,B}^2 = 0 \) for, at least, any particular particle \( m_{\alpha,B} \)) \( Z_k \to \infty \) and the cutoff (11) reduces to \( N (k) \to 1 \). In the limit \( k \to \infty \) (at very small planckian scales) \( Z_k \ll 1 \) and the cutoff acquires the \( N (k) \sim 1/k^g \to 0 \) (where \( g \) is the total number of bosonic degrees of freedom). All these features disappear when we take into account for the fermionic sector. Indeed, in the case of fermions every closed loop in Feynman diagrams includes the additional multiplier \(-1\) and therefore fermions give contribution to (12) with the opposite sign. Therefore, on the mas-shell for any particular fermion, as \( k^2 + m_{\alpha,F}^2 = 0 \), we get \( Z_k = 0 \) and \( N (k) = 0 \). Then from (4), (5) we find that \( S_F (k) \) does not contain poles (singularities ) and therefore such fermions cannot propagate. By other words dynamics in the fermion sector disappears. Moreover, analogously the functions

\footnote{The mas-shell requires considering the analytic continuation to the Minkowski space.}
\(Z_k, \overline{N}(k)\) acquire a pathologic behavior at very small scales \(Z_k \sim k^{F-B}\) as \(k \to \infty\) whose behavior depends now on the difference between the number of fermionic and bosonic degrees of freedom. In particular, in theories where the number of fermions exceeds that of bosons \(F - B > 0\) and therefore the cutoff is absent (i.e., \(\overline{N}(k) \to 1\) as \(k \to \infty\)).

Origin of such an anomalous behavior is quite clear. First we recall that the action for fermions has not a clear classical analogue (the formal analogue is a two-level system or Grassman fields). Therefore, in the case of fermions the standard expressions for the action or the energy density admit an ambiguity, i.e., they admit a shift on an arbitrary function. While topology is fixed and cannot change, such a shift renormalizes merely the cosmological constant and gives no contribution to any observables in particle physics (save gravitational physics). However, when topology may change (or fluctuate) such a shift becomes an issue; for it defines the resulting stable topology of the actual physical space.

Consider the energy for a particular field mode which in the standard picture has the form

\[E = \varepsilon_k \left(n_k \pm \frac{1}{2}\right)\]

where the sign +/- stands for bosons/fermions, \(n_k = 0, 1, 2, \ldots\) in the case of bosons, and \(n_k = 0, 1\) in the case of fermions. The ground state of the mode has the energy \(E_0 = \pm \varepsilon_k / 2\). Topology changes allow to remove some of such oscillators and therefore such a process accompanies always with a decrease/increase of the total energy by the factor \(\Delta E = \pm \sum_k \varepsilon_k / 2\) where the sum is taken over those oscillators which are "removed". It is quite natural to expect that when all modes are absent, then the field itself (and therefore the physical space itself) is absent. In this case there are no observables related to the field at all and such a case is convenient to imagine as an absolute vacuum (absolute ground state). Such a feature holds indeed in the case of bosonic oscillators and it is tempting to suppose that creation of the physical space requires to spent some energy. However, we see that in the case of fermionic systems the minimum is always reached when all oscillators are present, i.e., the bias has the form \(N_k = 1\), while the true vacuum (i.e., when the field is totally absent \(N_k = 0\)) has bigger energy and therefore is unstable\(^4\). All our knowledge in physics teaches us that any unstable situation leads to some phase transition upon which the system has to be stable.

Let us re-define the action \(\mathcal{S}_0\) in the form

\[\mathcal{S}_0 = L^4 \int \sum_{j=1}^{N(k)} \left[ \text{Tr} \left( \phi_j^* A(k) \phi_j \right) + \lambda(k) \right] \frac{d^4k}{(2\pi)^4},\]

\(^4\)such an instability is more prominent in the astrophysics where we have to use the universal covering \(\mathbb{H}\) and the number of fermionic modes can be \(N(k) = 0, 1, 2, \ldots\). Then the instability leads to the formation of the more and more number of fermionic modes i.e., \(N(k) \to \infty\). In this case the minimum energy is \(E \to -\infty\).
where $\lambda(k)$ is yet an arbitrary function. In the case of bosons (since we can fix the action principle by the classical limit) we may expect that $\lambda(k) = \text{const}$, while in the case of fermions this function requires an additional consideration. The presence of such an additional function leads to the following modification of (12)

$$\ln Z_k = \ln Z_k^F + \ln Z_k^B,$$

where

$$\ln Z_k^F(B) = \sum_{\alpha \in F(B)} \left[ \lambda_{\alpha,F}(B) (k) \pm \frac{1}{2} \ln \left( \frac{k^2 + m_{\alpha,F}^2}{\pi} \right) \right]. \quad (13)$$

It is convenient to re-write this in a more symmetric manner as follows

$$\ln Z_k^F(B) = \sum_{\alpha \in F(B)} \frac{1}{2} \ln \left( \frac{\mu_{\alpha,F}^F(k)}{k^2 + m_{\alpha,F}^2} \right), \quad (14)$$

where parameters $\mu_{\alpha,F}^F$ are defined by comparing (13) and (14).

As it was discussed in Ref. [1] in the case of bosons the most natural choice is $\mu_{\alpha,B}^F(k) = \mu$ (i.e., $\lambda(k) = \text{const}$) where $\mu$ characterizes the scale of the actual cutoff. Analogously, we may expect that upon the phase transition (i.e., when fermions become stable) such a parameter takes the value of the same actual cutoff $\mu_{\alpha,F}^F(k) = \mu$. Indeed, let us consider some coarse graining in the phase space, i.e., each value $k_j$ will correspond to some volume $\Delta k^3$, so that there will be a sufficiently big number of states in an every coarse grained state $k_j$. Then the difference between fermions and bosons should disappear (upon the coarse graining, more than one fermion can occupy the same quantum state $k_j$) and, therefore, we may expect that fermions and bosons should give a symmetric contribution to the cutoff function $\mathcal{N}(k)$

By other words in particle physics the cutoff function always acquires the structure (see for details Ref. [1])

$$\mathcal{N}(k) = \frac{\mu^g}{(\mu^g + k^2 \alpha_0 (k^2 + m_1^2)^{\alpha_1} \cdots (k^2 + m_n^2)^{\alpha_n})}, \quad (15)$$

where $\mu$ is the cutoff scale, $g = \sum 2\alpha_n$ is the total number of fields (both fermions and bosons) we have to retain, and the cutoff parameter $\mu$ can be defined via the total (observational) cosmological constant term.

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\footnote{We understand that such arguments are far from being rigorous though. However in the absence of the complete theory for topology changes, we may use only such phenomenological arguments. At least the only rigorous criteria here may come from experiments.}
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