New solvable matrix models III

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Abstract

We present a family of matrix models whose perturbation series in the coupling constants are written as the series in projective Schur functions over strict partitions and are examples of the tau functions of the two-component BKP hierarchy. The case of the usage of the two-component KP hierarchy is also reviewed and generalized.

To Andrey D. Mironov on his 60’s birthday

1 Introduction

This note was initiated by the work on the generalized Kontsevich model [40] and further discussions with the authors. The perturbation theory series for the partition function of this model was written in a very compact form as a sum over strict partitions of a pair of projective Schur functions. It was followed by the work [1] where the similar series appeared for the different model (BGW model). The sources of interest to series in projective Schur functions can be found in [31], [65], [43], [1], [41], [2], [39], [41], [42], [38]. There are some earlier works on the series, see [62], [51], [52], [36], [33], [13]. On the projective Schur functions and the representation theory of the supersymmetric group $q(N)$ and the symmetric group $S_n$, see [57], [19], [18], [9], [58]. The appearance of these functions in the integrable models was presented in [66], [46].

If a matrix integral is a tau function as a function of its coupling constants I call it solvable. As far as I know the first solvable (in this sense) matrix model was presented in the preprint of [10] and other examples in [26] and [27], [28], [24], and see the dissertation [37] of Andrey Mironov. Then we should point out the work [32]. I marked the family as the third one (“New solvable matrix models III”) because it is the direct continuation of [47] and also of [4] where the solvable cases were selected.

Here I present and compare two families of solvable matrix integrals. The first one is a slight generalization of earlier works [47], [16], [53] and in this sense it is not so much new. It is related to the KP hierarchy of integrable equations. The second family is completely new and is related to the KP hierarchy on the root system B (The BKP hierarchy which was introduced in [8]).

Notations

We suppose that the reader is familiar with the notions of partitions (we shall use Greek symbols $\lambda$, $\mu$ for them), strict partitions ($\alpha, \beta, \gamma, \delta$), the Schur functions $s_{\lambda}$, the projective Schur functions $Q_{\alpha}$. We denote the set of all partitions by $P$, the set of strict partition $DP$. The definitions can be found in [35] or, in a brief way in the Appendix. The details about KP and BKP hierarchies may be found in [20], [8] and [21], see also the textbooks [45], [17].

The Schur function can be written either as the symmetric polynomial in eigenvalues of a matrix $X$, in this case we write it as $s_{\lambda}(X)$ or as the polynomial in power sum variables $p = (p_1, p_2, \ldots)$, then we write it as $s_{\lambda}(p)$. Then $s_{\lambda}(X) = s_{\lambda}(p(X))$ where $p_m = \text{tr}X^m$.

Throughout the text, the matrix size is denoted by $N$.

The projective function as the polynomial in power sum variables $p_{\text{odd}} = (p_1, p_3, \ldots)$ is written as $Q_{\alpha}(p_{\text{odd}})$. If it is written as the symmetric polynomial of the eigenvalues of a matrix $X$, then we write it as $Q_{\alpha}(X) = Q_{\alpha}(p_{\text{odd}}(X))$, then $p_{2m-1} = 2\text{tr}X^{2m-1}$ (pay attention on the factor 2 which is different

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from the case of the Schur functions $s_{\lambda}$). The same notations will be applied to tau functions related to the series in the Schur functions (which are 2KP tau functions) and to the series in the projective Schur functions (these are 2BKP tau functions).

Consider an (infinite) matrix $A = \{A_{ij}, \ i, j \geq 0\}$. For a partitions $\lambda = (\lambda_1, \ldots, \lambda_N)$ and $\mu = (\mu_1, \ldots, \mu_N)$ whose length do not exceed $N$ (and can be less than $N$) we introduce the following notation for the determinant of the $N \times N$ submatrix selected by the partitions $\lambda$ and $\mu$:

$$A_{\lambda, \mu}(N) = \det [A_{N+\lambda_i-N+N+\mu_j-j}]_{i,j=1,\ldots,N}, \quad \lambda, \mu \in P$$

In what follows, $N$ is the matrix size and we suppose $N \geq 1$.

Next, for a given pair of strict partitions $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\beta = (\beta_1, \ldots, \beta_k)$ we shall use the notation for the determinant of the submatrix with entries selected by $\alpha$ and $\beta$:

$$A_{\alpha, \beta} := \det [A_{\alpha_i, \beta_j}]_{i,j=1,\ldots,k}, \quad \alpha, \beta \in DP$$

We use notations

$$\Delta(x) = \prod_{i<j \leq N} (x_i - x_j), \quad \Delta^*(x) = \prod_{i<j \leq N} \frac{x_i - x_j}{x_i + x_j}$$

**Hypergeometric tau functions and determinantal formulas.** Here we follow [50]. Such tau functions appeared in [27] in a different form. Let $r$ be a function on the lattice $Z$. Consider the following series

$$1 + r(n+1)x + r(n+1)r(n+2)x^2 + r(n+1)r(n+2)r(n+3)x^3 + \cdots =: \tau_r(n; x, 1)$$

Here $n$ is an arbitrary integer. This is just a Taylor series for a function $\tau_r$ with a given $n$ written in a form that is as close as possible to typical hypergeometric series. Here $n$ is just a parameter that will be of use later. If as $r$ we take a rational (or trigonometric) function, we get a generalized (or basic) hypergeometric series. Indeed, take

$$r(n) = \prod_{i=1}^{p} \frac{a_i + n}{n!} \prod_{i=1}^{q} \frac{b_i + n}{n}$$

We obtain

$$\tau_r(n; x) = \sum_{m \geq 0} \frac{\prod_{i=1}^{p} (a_i + n)_m}{m! \prod_{i=1}^{q} (b_i + n)_m} x^m = {\,}_pF_q \left( \begin{array}{c} a_1 + n, \ldots, a_p + n \\ b_1 + n, \ldots, b_q + n \end{array} \mid x \right)$$

where

$$(a)_n = a(a+1) \cdots (a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$$

is the Pochhammer symbol.

One can prove the following formula which express a certain series over partitions in terms of [3] (see for instance [50])

$$\tau_r(n; X, Y) := \sum_{\lambda} \tau_r(n)s_{\lambda}(X)s_{\lambda}(Y) = \frac{c_n}{c_{n-N}} \frac{\det [\tau_r(n-N+1; x_i y_j, 1)]_{i,j \leq N}}{\Delta(x)\Delta(y)}$$

where $c_k = \prod_{i=0}^{k-1} (r(i))^{1-k}$ and where

$$r_{\lambda}(n) := \prod_{(i,j) \in \lambda} r(n + j - i), \quad r_{(0)}(n) = 1$$

where the product ranges over all nodes of the Young diagram $\lambda$ is the so-called content product (which has the meaning of the generalized Pochhammer symbol related to $\lambda$). Notice that in case $N = 1$ we can write $\tau_r(n; x, y) = \tau_r(n; xy, 1)$. Let us test the formula for the simplest case $r \equiv 1$:

$$\det [1 - X \otimes Y] = \sum_{\lambda} s_{\lambda}(X)s_{\lambda}(Y) = \frac{\det [(1 - x_i y_j)^{-1}]_{i,j \leq N}}{\Delta(x)\Delta(y)}$$
Sometimes we also use infinite sets of power sums $p^i = (p^{(i)}_1, p^{(i)}_2, \ldots)$ and instead of matrices $X$ and $Y$ set:

$$\tau_r(n; p^1, p^2) := \sum_{\lambda} r_{\lambda}(n)s_{\lambda}(p^1)s_{\lambda}(p^2)$$

(9)

An example $r \equiv 1$:

$$\tau_1(n; p^1, p^2) := e^{\sum_{m>0} \frac{1}{m} p^{(1)}_m p^{(2)}_m}$$

In case the function $r$ has zeroes, there exists a determinantal representation. Suppose $r(0)$, then $r_{\lambda}(n) = 0$ if $\ell(\lambda) > n$. Then

$$\tau_r(n; p^1, p^2) = \sum_{\ell(\lambda) \leq n} r_{\lambda}(n)s_{\lambda}(p^1)s_{\lambda}(p^2) = c_n\det \left[ \frac{\partial^{a+b}_{p^{(1)}_i p^{(2)}_i} \tau_r(1; p^1, p^2)}{\partial_{p^{(1)}_i} \partial_{p^{(2)}_i}} \right]_{i,a=0,\ldots,n-1,}$$

(10)

where $c_k = \prod_{i=1}^{k-1} (r(i))^{i-k}$, and where

$$\tau_r(1; p^1, p^2) = 1 + \sum_{m>0} r(1) \cdots r(m)s_{(m)}(p^1)s_{(m)}(p^2),$$

see [47].

In addition there is the following formula [49]:

$$\tau_r(n; X, p) = \sum_{\lambda} r_{\lambda}(n)s_{\lambda}(X)s_{\lambda}(p) = \frac{\det \left[ x_i^{n-k} \tau_r(n-k+1; x_i, p) \right]_{i,k \leq N}}{\det \left[ x_i^{n-k} \right]}$$

(11)

where

$$\tau_r(n; x_i, p) = 1 + \sum_{m>0} r(n) \cdots r(n+m)x_i^m s_{(m)}(p)$$

(12)

Let us test it for $r \equiv 1$, $p = p_\infty := (1, 0, 0, \ldots)$:

$$e^{\tau^{1X}} = \sum_{\lambda} s_{\lambda}(X)s_{\lambda}(p_\infty) = \frac{\det \left[ x_i^{n-k} e^{x_i} \right]_{i,k \leq N}}{\det \left[ x_i^{n-k} \right]}$$

(13)

There are similar series

$$\sum_{\lambda} r_{\lambda}(n)s_{\lambda}(X)$$

which can be written as a Pfaffian [54], however we will not use them in the present text.

### 2 New solvable families of matrix models

In this subsection general formulas for new solvable families of matrix models are written down which will be further applied for certain ensembles of random matrices. Namely new types of coupling are introduced.

The cases of unitary and of Hermitian matrices are considered in the next sections.

**Solvable family related to the two-component KP hierarchy** The first one is related to the two-component KP hierarchy of integrable equations and it is just a generalization of a number of well-known matrix integrals whose partition function is a double series in the Schur functions over partitions. Actually it is a slight generalization of earlier works [47], [16], [53] and in this sense it is not so much new. This solvable family could be written as follows:

$$\int \tau_{A^1}(p^1, X^1)\tau_r(N; XY, 1_N)\tau_{A^2}(Y^1, p^2)\det (v(X)u(Y)) d\Omega_N(X, Y) = \tau_{A^3}(p^1, p^2)$$

(14)

where the relation

$$A^3 = A^1g^{-v, u}A^2$$

(15)
which can be called the composition rule, is satisfied.

The notations are as follows:

Two infinite sets \( p^1 = (p_1^1, p_2^1, \ldots) \) are the sets of the coupling constants. The measure \( d\Omega_N(X, Y) \) on the space of \( N \times N \) matrices \( X \) and \( Y \) is defined by the choice of matrix ensembles. Ensembles of unitary, of Hermitian, of complex and of normal matrices \( X, Y \) should be separately considered. The determinant \( \det(u(X)v(Y)) \) (where \( u \) and \( v \) are given functions) can be considered part of the measure, but we want to highlight it. The term which pairs the matrices \( X \) and \( Y \), namely \( \tau_r \), which is labeled by a function \( r \) on the lattice \( Z \), is the series \([5]\) of the following form

\[
\tau_r(N; XY, I_N) := \sum_{\lambda} s_{\lambda}(XY) s_{\lambda}(I_N) \prod_{(i,j) \in \lambda} r(N + j - i)
\]

where \( I_N \) is the identity matrix and \( s_{\lambda}(XY) \) is the Schur function indexed by a partition \( \lambda \). The sum over all partitions and actually is restricted by the condition that the number of parts of each \( \lambda \) (called the length \( \ell(\lambda) \) of \( \lambda \)) does not exceed \( N \) because otherwise \( s_{\lambda} \) vanishes. The product in the right hand side over nodes with coordinates \( i, j \) of the Young diagram of \( \lambda \) is called content product. \(1\) The main examples of the \( X-Y \) pairing given by \([16]\) are

\[
e^{cXY} = \sum_{\lambda} s_{\lambda}(XY) s_{\lambda}(p_{\infty}) c^{[\lambda]}, \quad p_{\infty} = (1,0,0,\ldots)
\]

\[
\det(1-cXY)^{-a} = \sum_{\lambda} s_{\lambda}(XY) s_{\lambda}(p_{\infty}) c^{[\lambda]} \prod_{(i,j) \in \lambda} (a + j - i),
\]

\[
pFq \left( \begin{array}{c} N + a_1, \ldots, N + a_p \\ N + b_1, \ldots, N + b_q \end{array} \right) {}_2F_1 | XY \right) = \sum_{\lambda} \prod_{i=1}^{p} (N + a_\lambda_i)^{\lambda} \prod_{i=1}^{q} (N + b_\lambda_i)^{\lambda} \cdot s_{\lambda}(XY) s_{\lambda}(p_{\infty})
\]

related respectively to \( r(x) = c e^{-a}, r(x) = c e^x \) and to \( r(x) = \frac{(a_1 + x) \cdots (a_p + x)}{(b_1 + x) \cdots (b_q + x)} \) and \( pFq \) is the hypergeometric function of matrix argument \([64]\). In \([14]\) \( A^1, A^2, A^3 \) are three infinite matrices which define respectively tau functions \( \tau_{A^1}, \tau_{A^2}, \tau_{A^3} \) which are tau functions of the two-component KP hierarchy and can be written in form

\[
\tau_{A^1}(N; p^1, X) = \sum_{\lambda, \mu} s_{\lambda}(p^1) s_{\mu}(X) A^1_{\lambda, \mu}(N)
\]

\[
\tau_{A^2}(N; p^2, Y) = \sum_{\lambda, \mu} s_{\lambda}(p^2) s_{\mu}(Y) A^2_{\lambda, \mu}(N)
\]

\[
\tau_{A^3}(N; p^1, p^2) = \sum_{\lambda, \mu} s_{\lambda}(p^1) s_{\mu}(p^2) A^3_{\lambda, \mu}(N)
\]

where \( A^i_{\lambda, \mu}(N) \) are defined as in \([1]\), see \([60]\) for the fermionic representation of such two-component KP tau function. In what follows if possible \( I \) will omit the argument \( N \).

**Remark 1.** In the case when each of the matrices \( X, Y \) belongs to one of the groups \( U(N), GL_N(C) \) and when, in addition to this, both matrices \( A^1, A^2 \) are diagonal, such a model was presented in \([4]\) and is related to the graph \( c \) resembling the number 8, see Fig 1 below.

**Remark 2.** In the Appendix we present examples of tau function \([17], [18]\) obtained as \( \mathbb{N}_1 \)-matrix integrals over unitary and complex matrices from the works \([4], [48]\). In this case tau function \([19]\) considered in the right hand side of \([14]\) below is equal to a \( (n_1 + n_2 + 2) \)-matrix integral.

The matrix \( A^3 \) is defined by the choice of \( A^1 \) and \( A^2 \) and is equal to the product of three (infinite) matrices, see \([15]\) where \( g^{u,v,w} \) is the matrix of moments:

\[
g^{u,v,w}_{r,i,j} = K_N \int \frac{dx}{2\pi} \frac{dy}{2\pi} \tau_r(1; xy, 1) v(x) u(y) d\Omega_1(x, y)
\]

\(^1\)It is of use in the representation theory of the symmetric group, in the context of soliton theory, see a short review in \([11]\).
where the bar denotes the complex conjugation and where

$$K_N = \frac{1}{\prod_{k=1}^{N-1} \prod_{i=1}^{k} \tau(i)}$$

(21)

Here \(d\Omega_1(x, y)\) is \(d\Omega_N(X, Y)\) taken for \(N = 1\) and

$$\tau_r(1; x y, 1) = \sum_{m \geq 0} x^m y^m r(1) \cdots r(m)$$

(22)

see [16]. The integration measure is defined by the choice of matrix ensemble.

The main example of self-coupling terms \(\tau_A^1\) and \(\tau_A^2\) is the case \(A^1 = A^2 = I_\infty\) (identity matrix):

$$\tau_A^1(p^1, X) = e^{\sum_m \frac{1}{m!} p_{(1)}^{(m)} \text{tr} X^m}, \quad \tau_A^2(p^2, Y) = e^{\sum_m \frac{1}{m!} p_{(2)}^{(m)} \text{tr} Y^m}$$

(23)

The tau functions \(\tau_A^1\) and \(\tau_A^2\) which depend on the choice of \(A^i\) and of \(p^i, i = 1, 2\) describe the self-coupling of matrices \(X\) and \(Y\) respectively. Tau function \(\tau_r\) describes the coupling between \(X\) and \(Y\) and depend on the choice of the function \(r\).

**Solvable models related to the two-component BKP hierarchy** In this article I am going to show that the simple replacement of \(\tau_r(N; X Y, I_N)\) by \(\tau_r(N; X^2 Y^2, I_N)\) and the replacements of 2KP tau functions \(\tau_A^i, i = 1, 2, 3\) by 2BKP tau functions \(\tau_A^B\):

$$\tau_A^B(p^1_{\text{odd}}, X^\dagger) = \sum_{\alpha,\beta \in \text{DP}} 2^{-\ell(\alpha)} Q_\alpha(p^1_{\text{odd}}) Q_\beta(X^\dagger) A^1_{\alpha,\beta}$$

(24)

$$\tau_A^B(p^2_{\text{odd}}, Y^\dagger) = \sum_{\alpha,\beta \in \text{DP}} 2^{-\ell(\alpha)} Q_\alpha(p^2_{\text{odd}}) Q_\beta(Y^\dagger) A^2_{\alpha,\beta}$$

(25)

$$\tau_A^B(p^1_{\text{odd}}, p^2_{\text{odd}}) = \sum_{\alpha,\beta \in \text{DP}} 2^{-\ell(\alpha)} Q_\alpha(p^1_{\text{odd}}) Q_\beta(p^2_{\text{odd}}) A^3_{\alpha,\beta}$$

(26)

where \(A^a_{\alpha,\beta}\) are defined by [2] applied to \(A = A^a, a = 1, 2, 3\). See [90] for the fermionic representation of such two-component KP tau function. Let us presents another family of matrix models:

$$\int \tau_A^B(p^1_{\text{odd}}, X^\dagger) \tau_r(N; X^2 Y^2, I_N) \tau_A^B(Y^\dagger, p^2_{\text{odd}}) \det(v(X) u(Y)) \, d\Omega_N(X, Y)$$

(27)

$$= \tau_A^B(p^1_{\text{odd}}, p^2_{\text{odd}})$$

(28)

with the same composition rule [15] with the replacement of \(\tau_r(1; x y, 1)\) by \(\tau_r(1; x^2 y^2, 1)\) in [20] and where now \(p^i = (p^i_1, p^i_3, p^i_5, \ldots)\), \(\tau_A^B, i = 1, 2\) is a pair of 2BKP tau functions, \(\tau_r\) is given by [16] and \(\tau_A^B(p^1_{\text{odd}}, p^2_{\text{odd}})\) is the 2BKP tau function which is written as the double series over strict partitions in projective Schur functions with explicitly written coefficients. Now we have

$$\tilde{g}^{r, u, v} = K_N \int \int x^3 y^3 \tau_r(1; x^2 y^2, 1) v(x) u(y) \, d\Omega_1(x, y)$$

(29)

where

$$\tau_r(1; x^2 y^2, 1) = \sum_{m \geq 0} x^{m^2} y^{2m} r(1) \cdots r(m)$$

(30)

instead of [20, 22].

It can be written also as fermionic vacuum expectation value, see [91], and can be presented in the pfaffian form. We will consider few different matrix ensembles, each type of matrices defines the choice of the integration measure \(d\Omega_N\).

One of the differences between families [27] and [14] is that in the last case the ensembles of complex and of normal matrices are included, and it is not possible to do in version [27].

The examples of self-coupling terms \(\tau_{1,2}^B\) are

$$\tau_A^B(p^1_{\text{odd}}, X) = e^{\sum_{m > 0, \text{odd}} \frac{1}{m!} p_{(1)}^{(m)} \text{tr} X^m}, \quad \tau_A^B(p^2_{\text{odd}}, Y) = e^{\sum_{m > 0, \text{odd}} \frac{1}{m!} p_{(2)}^{(m)} \text{tr} Y^m}$$

(31)
compare to \(23\).

The examples of the \(X-Y\) pairing are

\[
e^{c_{\alpha}X^2Y^2} - \det(1 - cX^2Y^2)^{-\alpha}, \quad pF_q \left( \begin{array}{c} N + a_1, \ldots, N + a_p \\ N + b_1, \ldots, N + b_q \end{array} \mid X^2Y^2 \right) \quad (32)
\]

**Remark 3.** Instead of tau function of two-component tau function \(\tau_{\alpha}^{(i)}(X, p^i)\) (where \(i\) is either 1 or 2) one can consider a one component tau function in form \(\sum_{\alpha} Q_\alpha(X)B_\alpha\), where \(B_\alpha\) is a pfaffian (for instance, see \[54\]) and have the meaning of the “Cartan coordinate” of the related point in the isotropic grassmannian \[17\]. Then in the right hand side of \(27\) we get a certain one-component BKP tau function. The similar remark is true for the right hand side of \(14\) where ones can take a certain Sato series \(\sum_{\alpha} \sigma_\alpha(X)\pi_\alpha\) (where \(\pi_\alpha\) is the “Plucker coordinate”) instead of \(\tau_{\alpha}(X, p^1)\) in the left hand side.

**Remark 4.** Thus, we have three group of parameters to construct the families \(14, 14\) and \(27\), these are matrices \(A^1, A^2\) and functions \(\tau, v, u\). We have the transformation group \(A^1 \rightarrow A^1S_1, A^2 \rightarrow S_2^{-1}A^2, g^{v,u} \rightarrow S_1^{-1}g^{v,u}S_2\) to get equivalent models as we see from \(15\). The choice of possible \(g^{v,u}\) is rather restricted by the type of measure in \(20\) and in \(29\).

**Remark 5.** Notice that zeroes of \(r\) do not allow to consider \(20, 28\) as a particular case of \(22, 27\).

**Chain matrix models**

The use of tau functions in both sides of \(14\) and also of \(27\) allows to consider multimagix models since tau functions under the integral can be treated as multimatrix integrals themselves. The chain ensemble of \(2n\) matrices is obtained as follows:

In the 2KP case:

\[
\int \tau_{A^1}(p^1, X_1)K(X_1, \ldots, X_{2n-1})\tau_{r,2(n-1)}(N; X_{2n-1}X_{2n})\tau_{A^2}(X_{2n}, p^2)d\Omega = \tau_{A^{2n+1}}(p^1, p^2) \quad (33)
\]

where

\[
K(X_1, \ldots, X_{2n-1}) = \prod_{i=1}^{n-1} \tau_{r,i}(N; X_{2i-1}X_{2i})\tau_{A^{i+1}}(X^\dagger_{2i}, X^\dagger_{2i+1})
\]

(we put \(K = 1\) for \(n = 1\)) and

\[
d\Omega = \prod_{i, odd} \det(v(X_i)u(X_{i+1}))d\Omega_N(X_i, X_{i+1})
\]

The composition rule is

\[
A^{2n+1} = A^1 g^{(1),v(1)},u(1) A^2 g^{(2),v(2)},u(2) \ldots A^{2n-1} g^{(2n-1),v(2n-1)},u(2n-1) A^{2n} \quad (34)
\]

This chain model contains the chain models considered in \[4\] and in \[47, 16\] (Appendix).

**Remark 6.** In the appendix we present

In the 2BKP case is obtained by the replacement of each \(\tau_{r,i}(N; X_{2i-1}X_{2i})\) by \(\tau_{r,i}(N; X^\dagger_{2i-1}X^\dagger_{2i})\):

\[
\int \tau_{A^1}(p^{odd}, X_1)K(X_1, \ldots, X_{2n-1})\tau_{r,2(n-1)}(N; X^\dagger_{2n-1}X^\dagger_{2n})\tau_{A^{2n}}(X_{2n}, p^{odd})d\Omega = \tau_{A^{2n+1}}(p^{odd}, p^{odd}) \quad (35)
\]

where \(d\Omega\) is the same as in the previous case, where

\[
K(X_1, \ldots, X_{2n-1}) = \prod_{i=1}^{n-1} \tau_{r,i}(N; X^\dagger_{2i-1}X^\dagger_{2i})\tau_{A^{i+1}}(X^\dagger_{2i}, X^\dagger_{2i+1})
\]

and the same composition rule \(34\).

The pairing of the neighboring matrices \(X_{2i}\) and \(X_{2i+1}\) is given by the choice of \(\tau_{A^1}(X_{2i}, X_{2i+1})\) and the pairing of \(X_{2i-1}\) and \(X_{2i}\) is given by \(\tau_{r,i}(X^\dagger_{2i-1}, X^\dagger_{2i})\) of form \(16\) defined by the choice of functions \(r^{(1)}, r^{(2)}, \ldots\). This chain is different from the one discussed in \[25\] and from the one discussed in \[4\].

The mixed chain where both 2KP and 2BKP tau functions are used is natural in this context.
Remark 7. As it is well-known, the replacements
\[ \tau(p^1, p^2) \rightarrow e^{\sum_{a=0}^n (a_m p^1_a + b_m p^2_a)} \tau(p^1, p^2), \quad \tau^B(p^1, p^2) \rightarrow e^{\sum_{a=0}^n (a_m p^1_a + b_m p^2_a)} \tau^B(p^1, p^2) \]  
with any sets of parameters \( \{a_m\} \) and \( \{b_m\} \) convert a tau function to a tau function in both 2KP and 2BKP cases. This is taken into account by the freedom in the choice of factors \( \det(v(X)) \) and \( \det(u(Y)) \) in the integration measure.

2.1 Technical tools
There is a number of useful lemmas:

Lemma 1. \( [47] \)
\[ \int_{\Omega_N} \tau_r(U X U^\dagger Y, Y) d\alpha \bigl( Z, Z^* \bigr) = K_N \frac{\det \tau_r(x_i y_j, 1)_{i,j}}{\Delta(x) \Delta(y)} \]  
where \( K_N \) is given by \( [21] \) and where \( \tau_r(U X U^\dagger Y, Y) \) was defined in \( [16] \) and where \( \int_{\Omega_N} d\alpha = 1 \).

Lemma 2. \( [47] \)
\[ \int_{\mathcal{G}L_N(C)} \tau_r(N; Z X Z^\dagger Y, p_\infty) d\Omega_N(Z, Z^*) = K_N \frac{\det \tau_r(x_i y_j, 1)_{i,j}}{\Delta(x) \Delta(y)} \]  
where
\[ d\Omega_N(Z, Z^*) = C_N e^{-\text{tr} Z Z^\dagger} \prod d\Re Z_{i,j} d\Im Z_{i,j}, \]  
where \( \tau_r(N; Z X Z^\dagger Y, p_\infty) \) was defined in \( [73] \) and \( \int_{\mathcal{G}L_N(C)} d\Omega_N(Z, Z^*) = 1 \).

Remark 8. A very simple technical idea to go from (14) to (27) is to use the fact that if we replace the sets \( \{x_i, y_i\} \) by \( \{x_i', y_i'\} \), then the result of the multiplication of the both left hand sides in (47) and (38) by the integration measure of certain matrix ensembles \( d\Omega_N \) results in the factor \( \Delta(x) \Delta(y) \) which is typical for the multifold integrals related to BKP, see Appendix in \( [51] \) and \( [13] \) and \( [55] \). These ensembles are ensemble of two unitary matrices, the ensemble of two Hermitian matrices and some others. However it fails for ensembles of complex and of normal matrices.

Next:

Lemma 3. For \( \lambda, \mu \in \mathbb{P} \) we have
\[ < s_\lambda, s_\mu >_{\Delta^{r,v,u}} := \int s_\lambda(X) s_\mu(Y) \Delta(x) \Delta(y) \prod_{i=1}^N \tau_r(1;x_i y_i, 1) v(x_i) u(y_i) d\Omega_1(x_i, y_i) = g^{r,v,u}_{\lambda, \mu} \]  
where
\[ g^{r,v,u}_{\lambda, \mu}(N) = \det \left[ g^{r,v,u}_{\lambda_i, -i + N, \mu_i - i + N} \right]_{i,j} \]  
see \( [7] \), and \( g^{r,v,u}_{i,j} \) is given by \( [20] \).

Lemma 4. For \( \alpha, \beta \in \mathbb{P} : \)
\[ < Q_\alpha, Q_\beta >_{\Delta^{r,v,u}} := \int Q_\alpha(X) Q_\beta(Y) \Delta^*(x) \Delta^*(y) \prod_{i=1}^N \tau_r(1;x_i y_i, 1) v(x_i) u(y_i) d\Omega_1(x_i, y_i) = g^{r,v,u}_{\alpha, \beta} \]  
where
\[ g^{r,v,u}_{\alpha, \beta} = \det \left[ g^{r,v,u}_{\alpha_i, \beta_j} \right]_{i,j} \]  
and \( g^{r,v,u}_{i,j} \) is given by \( [20] \).
Lemma 5. Let \( A = \{A_{i,j}, i,j \geq 0\} \) and \( B = \{B_{i,j}, i,j \geq 0\} \) be \(\{\text{finite}\}\) matrices and their product \( C = AB \) exists. Then
\[
C_{\alpha,\beta} := \sum_{\gamma \in \mathcal{D}^P} A_{\alpha,\gamma} B_{\gamma,\beta}
\]
where we use notation [3].

This is complete set of tools to prove relations [14], [27], [74], [76], [33], [35] for ensembles of unitary and ensembles of Hermitian matrices.

2.2 Both matrices are unitary

A. Preliminary In [67] the following equality was proven
\[
\int_{U_N \times U_N} e^{tr U_1 U_2 + \sum_{m > 0} \frac{1}{m!} t^m r^{(m)}_{A^1} U_1^{-m} + \sum_{m > 0} \frac{1}{m!} t^m r^{(m)}_{A^2} U_2^{-m}} dU_1 dU_2 = \sum_{\lambda} s_\lambda(p^1) s_\lambda(p^2) \prod_{(i,j) \in \lambda} \frac{1}{N + j - i}
\]
where \( s_\lambda \) denotes the Schur function [35] indexed by a partition \( \lambda \). The series over partitions in the right hand side is an example of the family called KP hypergeometric tau functions, see [27], [50] and can be presented as the determinant of the hypergeometric functions \( _1 F_1 \) for a special choice of \( p^1, p^2 \).

Remark 9. In the approach of work [4] this matrix integral is related to the graph (c) on Fig 1 resembling number 8, where we have 3 faces (decorated with three 'monodromies' \( U_1 U_2, U_1^2, U_2^2 \) under the integral), two ribbon edges (two 'random matrices' \( U_1^2, U_2^2 \)) and one vertex (decorated with the 'monodromy' \( \Omega \) in the example).

This is contained in the family [14] where \( X,Y \in U_N, A^1 = A^2 = I_N \), \( u = v = 1 \) and \( r(x) = x^{-1} \).

Another example is the choice \( A^1 = A^2 = I_N \) and \( \tau_r(N; U_1 U_2, I_N) = \det(I_N - c U_1 U_2)^{\alpha} \) as it was suggested in [16] and studied in [3].

After some work (see the next paragraph for details) we obtain
\[
J_{U_N \times U_N}^{A^1, A^2, r, v, u} (p^1, p^2) := \int_{U_N \times U_N} \tau_{A^1}(p^1, U_1^\dagger) \tau_r(N; U_1 U_2, I_N) \tau_{A^2}(U_2^\dagger, p^2) \det(v(X) u(Y)) dU_1 dU_2
\]
\[
= \sum_{\mu, \nu \in \mathcal{P}} A^1_{\lambda, \mu}(N) g^{r,v,u}_{\lambda, \mu}(N) A^2_{\mu, \nu}(N)
\]
where \( \tau_{A^i}, i = 1, 2 \) are given by [17], [18], \( \tau_r(N; U_1 U_2, I_N) \) is given by [16], \( g^{r,v,u}_{\lambda, \mu}(N) \) is given by [41] where according to [20] we have
\[
g^{r,v,u}_{i,j} = -\frac{1}{2\pi i} \oint \oint x^{-1} y^{-1} \tau_r(1; xy, 1) v(x) u(y) \frac{dx}{x} \frac{dy}{y}
\]

where \( \tau_r(1; xy, 1) \) is given by (22). The family is parametrized by the choice of \( A_i, i = 1, 2 \) and by the choice of functions \( r, v, u \) and one keeps in mind Remark 4. Thanks to Lemma 5 the relation (15) is fulfilled.

Here we present different family of solvable integrals and our main example which is similar to (45) is the Example 3. For us it is more suitable to change \( U_1, U_2 \) respectively to \( X, Y \) below.

B. New family. We will study the integral
\[
J_{U_N \times U_N}^{A^1, A^2, r, v, u} (p^{1, \text{odd}}, p^{2, \text{odd}}) := \int_{U_N \times U_N} \tau_{A^1}(p^{1, \text{odd}}, X^\dagger) \tau_r(N; X^2 Y^2, I_N) \tau_{A^2}(Y^\dagger, p^{2, \text{odd}}) \det(v(X) u(Y)) d\Omega_N(X, Y)
\]
where \( d\Omega_N(X,Y) \) is the product of Haar measures \( d_x d_y \) on \( U_N \). Using the standard separation of variables in \( X, Y \in U_N \) see [34] \( X = U_1 \text{diag}(x_i)U_1^{-1}, Y = U_2 \text{diag}(y_j)U_2^{-1} \), where \( U_1, U_2 \in U_N \), we can write
\[
d_x X d_y Y = \left| \Delta(x) \Delta(y) \right|^2 \prod_{i=1}^N dx_i dy_i dU_1 dU_2
\]
Since \( X^2 = U_1 \text{diag}(x_i^2)U_1^{-1}, Y^2 = U_2 \text{diag}(y_j^2)U_2^{-1} \), then, using Lemma [1] we obtain
\[
\int_{U_N \times U_N} \tau_r(N; X^2 Y^2, I_N) dU_1 dU_2 = K_N \frac{\det \left[ \tau_r(N; x_i^2 y_j^2, 1) \right]_{i,j}}{\Delta(x^2)\Delta(y^2)} \tag{49}
\]
Bearing in mind Remark [8] we see that the whole integral [48] is the integral over \( 2N \) eigenvalues \( \{x_i\} \) and \( \{y_i\} \):
\[
I_{U_N \times U_N}^{A^1, A^2, r, v, u} = \sum_{\alpha, \beta \in \text{DP}} Q_{\alpha}(p_{\text{odd}}) Q_{\beta}(p_{\text{odd}}) A_{\alpha} A_{\beta} <Q_{\gamma}, Q_{\delta}>_{U_N \times U_N}^{r, v, u}
\]
where
\[
<Q_{\gamma}, Q_{\delta}>_{U_N \times U_N}^{r, v, u} = \frac{1}{\sqrt{2\pi}} \int dx dy \Delta(x)\Delta(y) \sum_{i=1}^N \tau_r(1; x_i^2 y_i^2, 1) \tilde{v}(x_i) \tilde{u}(y_i) d\Omega_1(x_i, y_i) = \tilde{g}_{\gamma, \delta}^{r, v, u}
\]
where according to Lemma [4] the entries of the matrix \( g^{r, v, u} \) are
\[
g^{r, v, u}_{i,j} = \frac{1}{2\pi} \int dx dy \Delta(x)\Delta(y) \sum_{i=1}^N \tau_r(1; x^2 y^2) u(x) v(y) dxdy \tag{50}
\]
The factor \((xy)^\frac{1}{2} N(1-N)\) appeared thanks to the ratio \( \frac{\left| \Delta(x)\Delta(y) \right|^2}{\Delta(x^2)\Delta(y^2)} \) where \(|x_i| = |y_i| = 1\). We get
\[
I_{U_N \times U_N}^{A^1, A^2, r, v, u}^{(p_{\text{odd}}, p_{\text{odd}})} = \sum_{\alpha, \beta \in \text{DP}} Q_{\alpha}(p_{\text{odd}}) Q_{\beta}(p_{\text{odd}}) A_{\alpha} A_{\beta} <Q_{\gamma}, Q_{\delta}>_{U_N \times U_N}^{r, v, u} \tag{51}
\]

**Example 1.** Take \( A^1 = A^2 = I_\infty \). Choose \( \tau_r(1; x^2 y^2, 1) = e^{x^2 y^2} \). Then \( g_{2i,2j} = \delta_{i,j} \frac{1}{2\pi} \) and \( g_{2i-1,2j} = 0 = g_{2i,2j-1} \) for each pair \( i, j \).

**Example 2.** Take \( A^1 = A^2 = I_\infty \). Choose \( \tau_r(1; x^2 y^2, 1) = (1 + x^2 y^2)^n \). Then \( g_{2i,2j} = \delta_{i,j} a(a-1) \cdots (a-i+1) \frac{1}{2\pi} \) and \( g_{2i-1,2j} = 0 = g_{2i,2j-1} \) for each pair \( i, j \).

Actually the moment matrix \( g \) is diagonal in these examples where \( u(x)v(y) \) is a function of the product \( xy \). For more general we take different choice, for possible corrections to \( v, u \) see also Remark [7] below.

**Example 3.** Perhaps, the simplest example is as follows. We take each \( A^v \) be identity matrices, \( v = u \equiv 1 \) and \( r(j) = j^{-1} \) which gives
\[
\int_{U_N \times U_N} \frac{\left( \prod_{i=1}^N 2^{\alpha_i^0} \right)^{\frac{1}{2}}}{\prod_{i=1}^N \alpha_i !} d_x d_y Y = \sum_{\alpha \in \text{DP}} 2^{-\ell(\alpha)} Q_{2\alpha}(p_{\text{odd}}) Q_{2\alpha}(p_{\text{odd}}) \prod_{i=1}^{\ell(\alpha)} \frac{1}{\alpha_i !}
\]
The right hand side is a special (degenerate) case of the BKP hypergeometric tau function studied in [51].

**Remark 10.** It follows from the last formula that in the considered case we get (up to an overall factor)
\[
<Q_{\alpha}(U_1^1), Q_{\beta}(U_2^1)>_{U_N \times U_N} = \int_{U_N \times U_N} 2^{-\ell(\alpha)-\ell(\beta)} Q_{\alpha}(U_1^1) \sum_{\gamma} 2^{-\ell(\gamma)} Q_{\gamma}(U_1^2 U_2^2) Q_{\gamma}(\frac{1}{2^{p_{\text{odd}}}}) d_x d_y Y \tag{52}
\]
One may ask: is the following formula correct
\[
\int_{U_N} Q_{2\alpha}(U_1^1) Q_{\beta}(U_2^1) A d_x d_U = \delta_{\alpha,\beta} Q_{\alpha}(A) \prod_{i=1}^{\ell(\alpha)} \alpha_i ?
\]
We do not know.
Example 4.
\[
\int_{\mathbb{R}^N \times \mathbb{R}^N} e^{\sum_{m>0} \theta_m \left( p_m^{(1)} x^m + p_m^{(2)} y^m \right)} \det(1 - c X^2 Y^2)^{-n} d_x d_y d_x d_y
\]
\[
= \sum_{\alpha \in D_p} Q_{2\alpha}(p_{3\alpha}) Q_{2\alpha}(p_{2\alpha}) \prod_{i=1}^{\ell(\alpha)} \frac{1}{\alpha_i !}
\]

2.3 Both matrices are Hermitian

A. 2KP case. The interesting 2KP cases were written in various papers \[10\], \[47\], \[29\].

In the same way anti-Hermitian matrices are treated.

This case can be considered in the same way as the previous one. It is natural to take the following measure on the space $\mathcal{H}_N$ of Hermitian $N \times N$ matrices:
\[
d\Omega_N(X) = C_N e^{-w_1 \text{tr} X^2} \prod_{i<j \leq N} dX_{i,j} \prod_{i<j \leq N} d\Re X_{i,j}
\]
\[
d\Omega_N(Y) = C_N e^{-w_2 \text{tr} Y^2} \prod_{i<j \leq N} dY_{i,j} \prod_{i<j \leq N} d\Re Y_{i,j}
\]

where $w_i > 0$ are parameters. We take $d\Omega_N(X,Y) = d\Omega_N(X) d\Omega_N(Y)$. We have $X = U_1 \text{diag}(x_1) U_1^{-1}$, $Y = U_2 \text{diag}(y_1) U_2^{-1}$, where $U_1, U_2 \in \mathbb{U}_n$. Then we have \[34\]:
\[
d\Omega_N(X) = c_N (w_1; v) (\Delta(x))^2 \prod_{i=1}^{N} e^{-w_1 x_i^2} dx, dU
\]
\[
d\Omega_N(Y) = c_N (w_2; u) (\Delta(y))^2 \prod_{i=1}^{N} e^{-w_2 y_i^2} dy, dU
\]

where we suppose that $\int d\Omega_N(X) = \int d\Omega_N(Y) = 1$.

Repeating the calculation from the previous subsection we obtain
\[
J_{\mathcal{H}_N}^{A^1, A^2, r, v, u, w_1, w_2}(p, q) := \int_{\mathcal{H}_N \times \mathcal{H}_N} \tau_i r_i N, XY, I_N) \tau_i A_i r_i N, p^2 \det(v(X) u(Y)) d\Omega_N(X) d\Omega_N(Y)
\]
\[
= \sum_{\lambda \in \Delta, \mu \in \Omega \leq N} s_{\lambda} (p^1) s_{\mu} (p^2) \sum_{\lambda \leq \rho \leq \Omega} A_{\lambda, \rho}^i (N) g_{\lambda, \rho}^{r, v, u, w_1, w_2} (N) A_{\rho, \mu}^i (N)
\]

where $\tau_i, i = 1, 2$ are given by \[17\], \[18\], $\tau_{XY, I_N}$ is given by \[16\], $g_{\lambda, \rho}^{r, v, u} (N)$ is given by \[41\], where according to \[20\] we have
\[
g_{i,j}^{r, v, u, w_1, w_2} = C \int \int x^i y^j \tau_r (1; x y, 1) v(x) u(y) e^{-w_1 x^2 - w_2 y^2} dx dy
\]

where $\tau_r (1; x y, 1)$ is given by \[22\]. The family is parametrized by the choice of $A^i, i = 1, 2$ and by the choice of functions $r, v, u$ and one keeps in mind Remark \[4\]. Thanks to Lemma \[5\] the relation \[15\] is fulfilled.

Let us present well-known models.

Example 5. Two-matrix model. Take $w_1 = w_2 = 0$, $v = u = 1$ and $r(j) = c \sqrt{-1} j^{-1}$ that is
\[
\tau_r (N; XY, I_N) = e^{c \sqrt{-1} \text{tr} (XY)}, \quad \tau_r (1; x y, 1) = e^{c \sqrt{-1} x y}
\]

Then $g_{i,j}^{r, v, u, w_1, w_2} = e^{c \delta_{i,j}}$
\[
J_{\mathcal{H}_N}^{A^1, A^2, r, v, u, w_1, w_2}(p, q) = \sum_{\lambda \in \Delta, \mu \in \Omega \leq N} s_{\lambda} (p^1) s_{\mu} (p^2) c_{\lambda} \prod_{(i,j) \in \lambda} (N + j - i)
\]

This model can be treated as Hermitian-anti-Hermitian two matrix model \[12\] if we put $p^{(2)}_m \to (-1)^{i} m p^{(2)}_m$ and keep $c$ to be real.
Example 6. Take $A^1 = A^2 = I_{\infty}$ and $w_1 = w_2 = 0$, $r(j) = c\sqrt{-1}j^{-1}$ as in the previous example. We can use the fact $w_2rY^2 + c\sqrt{-1}\text{tr}(XY) - (\sqrt{2}w_2)^{-1}c^2\text{tr}X^2 = w_2 \sum_{i,j} \tilde{Y}^2$ where $\tilde{Y}$ is the matrix with entries $\tilde{Y}_{ij} = Y_{ij} + c\sqrt{-1}\text{Tr}X_{ij}$. After the Gaussian integration over the shifted variables $\{Y_{ij}, 0 \leq i, j \leq N\}$, we obtain

$$\int_{\mathcal{H}_N} \sum_{m > 0} \frac{1}{2m} p_m^{(1)} u X^m + (\sqrt{2}w_2)^{-1}c^2 \text{tr} X^2 \ d\Omega(X) = \sum_{\ell, \lambda \leq N} s_{\lambda}(p^1) s_{\lambda}(0, w_2, 0, 0, \ldots)c^{N} \prod_{i,j} (N + j - i)$$

Unitary-Hermitian two matrix model. The same method can obviously be applied to mixed ensemble of interacting Hermitian and of unitary matrices, namely to the ensemble \(14\), where $X \in \mathcal{U}_N$ and $Y \in \mathcal{H}_N$, for example:

$$\int_{\mathcal{U}_N \times \mathcal{H}_N} \sum_{m > 0} \frac{1}{2m} (p_m^{(1)} u X^m + p_m^{(2)} \text{tr} X^m) + \text{ctr}(U, X) \ detU^{-k} \ detX^n \ dU \ d\Omega(X) = \sum_{\ell, \lambda \leq N} s_{\lambda}(p^1) s_{\lambda}(p^2)g_{\lambda, \mu}(N)$$

where $g_{\lambda, \mu}(N) = \det \{g_{N+\lambda, -i, N+\mu, -j}\}_{i,j}$, see \(1\) and \(20\) where $g_{ij} = \frac{(2\pi)^{-\frac{3}{2}}}{\sqrt{-1}} \int dx \ e^{\gamma y, x - k - i} e^{n} e^{-\frac{1}{2}y^2} dy$. The right-hand side of \(57\) has the form of the Takasaki series \(59\), \(60\) for the TL tau function.

Remark 11. Let us note that there is the direct analogue of the model $\mathcal{J}_{\mathcal{U}_N \times \mathcal{H}_N}^{A^1, A^2, r,v,u,w_1,w_2}(p^1, p^2)$ where the unitary matrices $U_1, U_1^\dagger$ and $U_2, U_2^\dagger$ are replaced by the complex matrices $Z_1, Z_1^\dagger$ and $Z_2, Z_2^\dagger$. This model of complex matrices together with the model of unitary matrices \(43\) can be solved via character expansion as it was done, say, in \(3\). In both cases of unitary complex and mixed unitary-complex matrix ensembles such models are related to the graph on Figure 1 (c), see Appendix. However, this method does work for unitary matrix models $\mathcal{J}_{\mathcal{U}_N \times \mathcal{H}_N}^{A^1, A^2, r,v,u,w_1,w_2}(p^1, p^2)$, written down in \(3\) and it does not directly work for the both Hermitian ensembles $\mathcal{J}_{\mathcal{H}_N \times \mathcal{H}_N}^{A^1, A^2, r,v,u,w_1,w_2}(p^1, p^2)$ and $\mathcal{J}_{\mathcal{H}_N \times \mathcal{H}_N}^{A^1, A^2, r,v,u,w_1,w_2}(p^1, p^2)$.

B. 2BKP case

$$\mathcal{J}_{\mathcal{H}_N \times \mathcal{H}_N}^{A^1, A^2, r,v,u,w_1,w_2}(p^1, p^2) :=$$

$$\int_{\mathcal{H}_N \times \mathcal{H}_N} \tau^A_{\mathcal{H}}(p_{\text{odd}}, X) \tau_r(N; X^2 Y^2, I_N) \tau^A_{\mathcal{H}}(Y, p^2) \ det(r(X) u(Y)) \ d\Omega(X) d\Omega(Y)$$

$$= \sum_{\alpha, \beta \in \mathcal{D}} Q_{\alpha}(p_{\text{odd}}) Q_{\beta}(p_{\text{odd}}) \sum_{\gamma, \delta \in \mathcal{D}} A_{\alpha, \gamma} A_{\beta, \delta}$$

where $\tau_{\mathcal{H}}, i = 1, 2$ are given by \(24\), \(25\), $\tau_r(N; X^2 Y^2, I_N)$ is given by \(16\), $g_{\lambda, \mu}(N)$ is given by \(41\) where according to \(20\) we have

$$g_{i,j} = \int x^i y^j \tau_r(1; x, y) v(x) u(y) e^{-\frac{1}{2}x^2 - \frac{1}{2}y^2} \ dx \ dy$$

where $\tau_r(1; x, y)$ is given by \(22\). The family is parametrized by the choice of $A^i$, $i = 1, 2$ and by the choice of functions $r, v, u$ and one keeps in mind each Remark \(4\). Thanks to Lemma \(5\) the relation \(15\) is fulfilled.

Example 7. Take $A^1 = A^2 = I_{\infty}$ and $w_1 = w_2 = 0$, $r = v = u \equiv 1$ and $r(j) = c\sqrt{-1}j^{-1}$ that is

$$\tau_r(N; X^2 Y^2, I_N) = e^{c\sqrt{-1}\text{tr}(X^2 Y^2)}, \quad \tau_r(1; x^2 y^2, 1) = e^{c\sqrt{-1}\text{tr}(X^2 Y^2)}$$

Then

$$g_{i,j} = \begin{cases} \left(\frac{1}{2}\right)^\frac{1}{4} \delta_{i,j}, & \text{both } i, j \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

We obtain

$$\int_{\mathcal{H}_N \times \mathcal{H}_N} e^{c\sqrt{-1}\text{tr}(X^2 Y^2) + \sum_{m > 0, n > 0} \frac{1}{2}(p_m^{(1)} \text{tr} X^m + p_m^{(2)} \text{tr} Y^m)} \ d\Omega(X) d\Omega(Y)$$

$$= \sum_{n \in \mathcal{D}} \frac{2^{-\ell(n)} Q_{2n}(p_{\text{odd}}) Q_{2n}(p_{\text{odd}})}{\prod_{i=1}^{\ell(n)} \alpha_i!}$$

11
3 Other ensembles

Two-matrix model where one matrix is real skew-symmetric and the other is skew-symmetric Hermitian. Let us consider the following two-matrix integral where $X \in \mathfrak{S}_N$ with eigenvalues $\pm x_i \sqrt{-1}$ and $Y$ is skew-symmetric Hermitian (the set of these $N \times N$ matrices we denote $\mathfrak{S}HN$), then its eigenvalues are $\pm y_i$ without $\sqrt{-1}$:

\[
\int_{\mathfrak{S}_N \times \mathfrak{S}HN} e^{\sum_{m>0} \frac{i}{\pi} \left( p_1^{(1)} \text{tr} X^{2m} + p_2^{(2)} \text{tr} Y^{2m} + \text{tr}(XY) \right)} d\omega(X) d\Omega(Y) = C \sum_{\lambda \in \mathcal{P}} s_\lambda(p_1) s_\lambda(p_2) \ell(\lambda) \prod_{i=1}^{\ell(\lambda)} (2(\lambda_i - i + N))!
\]

which is the KP tau function of hypergeometric type (9) where $r(i) = (2i)(2i + 1)$, therefore it can be presented in the determinantal form, see Section 1. The last equality (64) is derived using (112) and (111) in Appendix C in the similar way as it was done for Hermitian-anti-Hermitian two-matrix model in [12].

Two-matrix model where both matrices are skew-symmetric. Let us use the following result of Brezin-Hikami [6]

\[
\int_{\mathfrak{S}_N} e^{\text{tr} OXO^\dagger Y} d\omega = \frac{1}{\Delta(x^2) \Delta(y^2)} \begin{cases} \det [2 \cosh x_{ij}]_{1,1 \leq i,j \leq m}, & N = 2m \\ \det [2 \sinh x_{ij}]_{1,1 \leq i,j \leq m}, & N = 2m + 1 \end{cases}
\]

where

\[
d_O = \ldots
\]

Some other ensembles can be studied in the same way. It will be written down in a more detailed version.

Then we get the following 2KP tau function

\[
I_{2N}(p_1, p_2) := \int_{\mathfrak{S}_N \times \mathfrak{S}_N} \tau_A^1(X) e^{\text{tr} XY} \tau_A^2(Y, p_2) d\omega(X) d\omega(Y)
\]

where

\[
d_\omega = \ldots
\]

and

\[
g_{ij} = \ldots
\]

Three matrix models One can add the auxiliary complex matrix $Z \in \mathbb{G}\mathfrak{l}N(\mathbb{C})$ to the ensemble of the matrices $X$ and $Y$ and consider the three matrix model as follows. Firstly we remind the measure on the space of complex $N \times N$ matrices:

\[
d\Omega_N(Z, Z^\dagger) = C_N e^{-\text{tr} ZZ^\dagger} \prod d\mathcal{R}Z_{i,j} d\mathcal{R}Z_{i,j},
\]

where $C_N$ is chosen by $\int d\Omega_N(Z, Z^\dagger) = 1$. Let us take the same choice of $\tau_A^i$, $i = 1, 2, 3$ and of functions $r, v, u$ as in (14). And consider

\[
\tau_r(N; X, Y, p_\infty) = \sum_{\lambda \in \mathcal{P}} s_\lambda(X, Y) s_\lambda(p_\infty) \prod_{(i,j) \in \lambda} r(N + j - i)
\]

where $p_\infty = (1, 0, 0, \ldots)$. 

12
We have
\[
\int \tau_{A^1}(p^1, X^1) \tau_{r}(N; ZXZ^\dagger Y, p_\infty) \tau_{A^2}(Y^\dagger, p^2) \det (v(X)u(Y)) \, d\Omega_N(X, Y) \, d\Omega(Z, Z^\dagger)
\]
\[= \tau_{A^2}(p^1, p^2) \quad (74)\]

However, let us notice that the auxiliary matrix \(Z\) does not add coupling constants to the model and the right hand side is the same as in (14) and in this sense it is equivalent model.

**One-matrix model and 1BKP** We get
\[
\int_{\mathcal{B}_N \times GL_N(C)} e^{\sum_{m > 0, odd} \frac{1}{m} (p^{(1)}\text{tr}X^m + p^{(2)}\text{tr}X^{-m})} \tau_{r}(N; X^2Y^2, p_\infty) \, d\Omega(X) = \frac{1}{\Delta(y^2)} \int_{\mathbb{R}^N} \bigl(\Delta^*(x)\bigr)^2 \prod_{i=1}^N e^{\sum_{m > 0, odd} \frac{1}{m} (p^{(1)}x^m + p^{(2)}x^{-m})} \tau_{r}(1; x_i^2y_i^2, 1) e^{-wx_i^2} \, dx_i
\]
which is related to the Ising model, see Appendix in [13] written by H. Braden.

**Three matrix models and 2BKP** Adding the auxilliary complex matrix \(Z \in GL_N(C)\) as in the 2KP case (74) one obtains
\[
\int_{\mathcal{B}_N \times GL_N(C)} e^{\sum_{m > 0, odd} \frac{1}{m} (p^{(1)}\text{tr}X^m + p^{(2)}\text{tr}X^{-m})} \tau_{r}(N; ZXZ^\dagger Y^2, p_\infty) \, d\Omega(X) \, d\Omega(Z, Z^\dagger) = \int_{\mathbb{R}^N} \bigl(\Delta^*(x)\bigr)^2 \prod_{i=1}^N e^{\sum_{m > 0, odd} \frac{1}{m} (p^{(1)}x^m + p^{(2)}x^{-m})} \tau_{r}(1; x_i^2y_i^2, 1) e^{-wx_i^2} \, dx_i
\]
which is related to the Ising model, see Appendix in [13] written by H. Braden.

**Multimatrix models** As it was mentioned one can use multimatrix models considered in [4] to present the terms \(\tau_{r}(N; X^2y^2, 1_\infty)\) (or, the term ) describing the pairing of matrices \(X\) and \(Y\).

Actually any embedded graph drawn on the complex plane can be used provided that what we called “the spectrum of the star” is nontrivial for a single “star” results in certain tau function. By the term ”star” we called the top of the graph, blown up to a small circle. Each such vertex has a monodromy: the product of the source matrices when walking around the boundary of the star. For details, see the article [4], also [48]. One can construct infinitely many graphs, each being related to a solvable matrix integral, we present few examples.

It is possible to construct infinitely many graphs equipped with source matrices, each of which specifies a precisely solvable matrix integral, we give several examples of such models.

**Example 8.** The \(n\)-gon gives two monodromies \(U_1 \cdots U_n X^2\) and \(Y^2 U_n^\dagger \cdots U_1^\dagger\) when passing around inside and outside, and \(X^2Y^2\) is the monodromy around a chosen vertex:
\[
\tau_{r}(N; X^2Y^2, 1_N) = \int_{U_N \cdots \times U_N} e^{tr(U_1 \cdots U_n X^2 \cdots Y^2 U_n^\dagger \cdots U_1^\dagger)} \prod_{i=1}^n du_i
\]
\[= \tau_{A^2}(p^1, p^2) \quad (75)\]
\[ \sum_{\lambda} s_\lambda(X^2Y^2)s_\lambda(\mathbb{I}_N)((N)_{\lambda})^{-n-1} \]  

Thus we obtain

\[ K_N \frac{\det \left[ \tau_r(1; \mathbb{I}_N, 1) \right]_{\lambda,\lambda}}{\Delta(x^2)\Delta(y^2)} \]  

\[ \int_{ GL_N(C) \times \cdots \times GL_N(C) } e^{ \text{tr}(Z_1 A_1 + \cdots + Z_n A_n) } e^{ \text{tr}(Z_1^\dagger B_1 + \cdots + Z_n^\dagger B_n) } \prod_{i=1}^{n} d\Omega(Z_i, Z_i^\dagger) \]  

**Example 9.**

\[ \tau_r(N; X^2Y^2, I_N) = \int_{ GL_N(C) \times \cdots \times GL_N(C) } e^{ \text{tr}(Z_1 A_1 + \cdots + Z_n A_n) } e^{ \text{tr}(Z_1^\dagger B_1 + \cdots + Z_n^\dagger B_n) } \prod_{i=1}^{n} d\Omega(Z_i, Z_i^\dagger) \]  

where \( A_1, \ldots, A_n, B_1, \ldots, B_n \) are any matrices independent of \( \{ Z_i, Z_i^\dagger g_i, i = 1, \ldots, n \} \) conditioned by

\[ A_1 B_1 \cdots A_n B_n = X^2Y^2 \]  

**Example 10.**

\[ \tau_r(N; X^2Y^2, I_N) = \int_{ GL_N(C) \times GL_N(C) } e^{ \text{tr}(Z_1 A_1 + Z_1^\dagger B_1 + \cdots + Z_n A_n + Z_n^\dagger B_n) } \prod_{i=1}^{n} d\Omega(Z_i, Z_i^\dagger) \]  

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A Partitions. The Schur polynomials

We recall that a nonincreasing set of nonnegative integers \( \lambda_1 \geq \cdots \geq \lambda_k \geq 0 \), we call partition \( \lambda = (\lambda_1, \ldots, \lambda_k) \), and \( \lambda_k \) are called parts of \( \lambda \). The sum of parts is called the weight \( |\lambda| \) of \( \lambda \). The number of nonzero parts of \( \lambda \) is called the length of \( \lambda \), it will be denoted \( \ell(\lambda) \). See [35] for details. Partitions will be denoted by Greek letters: \( \lambda, \mu, \ldots \)
The set of all partitions is denoted by \( \mathbb{P} \). The set of all partitions with odd parts is denoted OP. Partitions with distinct parts are called strict partitions, we prefer letters \( P \). Partitions \( \lambda, \mu, \ldots \in \mathbb{P} \).

In particular, \( \{p, q\} \) are called power sum variables. For \( \lambda = (\lambda_1, \lambda_2, \ldots) \in \mathbb{P} \) we define

\[
 s_\lambda(p) = \det \left[ s_{\lambda_i - 1 + j}(p) \right]_{i,j \geq 0} \tag{86}
\]

where the variables \( p = (p_1, p_2, p_3, \ldots) \) are called power sum variables. For \( \lambda = (\lambda_1, \lambda_2, \ldots) \in \mathbb{P} \) we define

\[
 s_\lambda(p) = \det \left[ s_{\lambda_i - 1 + j}(p) \right]_{i,j \geq 0} \tag{86}
\]

If we put \( p_m = p_m(X) = \text{tr} X^m \) where \( X \) is a matrix we write \( s_\lambda(p(X)) = s_\lambda(X) \).

To define the projective Schur function \( P \), \( \lambda \in \mathbb{P} \) at the first step we introduce the set of elementary Schur functions \( s_m(p) \) by

\[
e_{\sum m > 0} p_m x^m = \sum_{m \geq 0} x^m s_m(p) \tag{87}
\]

where

\[
e_{\sum m > 0} p_m x^m = \sum_{m \geq 0} x^m q_m(p_{\text{odd}}) \tag{87}
\]

where now \( p_{\text{odd}} = (p_1, p_3, \ldots) \). Next we defined the following skew symmetric matrix

\[
 Q_{ij}(p_{\text{odd}}) := \begin{cases} q_i(p_{\text{odd}})q_j(p_{\text{odd}}) + 2 \sum_{k=1}^{\infty} (-1)^k q_{i+k}(p_{\text{odd}})q_{j-k}(p_{\text{odd}}) & \text{if } (i, j) \neq (0, 0), \\ 0 & \text{if } (i, j) = (0, 0). \end{cases} \tag{88}
\]

In particular \( Q_{ij}(p_{\text{odd}}) = Q_{ij}(p_{\text{odd}}) = q_i(p_{\text{odd}}) \) for \( j \geq 1 \). For a strict partition \( \alpha = (\alpha_1, \ldots, \alpha_{2r}) \) where \( \alpha_{2r} \geq 0 \) the projective Schur function is defined

\[
 Q_\alpha(p_{\text{odd}}) := \text{Pf} \left[ Q_{\alpha_i \alpha_j}(p_{\text{odd}}) \right]_{1 \leq i, j \leq 2r}, \quad Q_\emptyset := 1 \tag{89}
\]

Let \( X \) be a matrix. If we put \( p_m = p_m(X) = \text{tr} (X^m - (-X)^m) \) which we call odd power sum variables, \( m \) odd we write \( Q_\alpha(p_{\text{odd}}(X)) = Q_\alpha(X) \).

B Tau functions and free fermions

The tau functions [17] - [19] of the two-component KP hierarchy which are of use in our construction have the following fermionic vacuum expectation value

\[
 \tau_A(N, p^1, p^2) = \langle N, -N \mid e^{\sum_{i,j \geq 0} \alpha_i^A \phi_i^{(1)}(p^1) \phi_{j-1}^{(2)}(p^2)} \mid 0, 0 \rangle, \quad \alpha = 1, 2, 3 \tag{90}
\]

where \( p^a = (p_1^a, p_2^a, p_3^a, \ldots) \). We have \( p_m^a = m t_m^a \) where \( t_m^a \) are the higher times of the two-component KP hierarchy. Tau functions [17] - [19] can be written as

\[
 \sum_{N > 0} \kappa^N \tau_A^B(N, p_{\text{odd}}^1, p_{\text{odd}}^2) = \langle 0 \mid e^{\sum_{i,j \geq 0} \alpha_i^B \phi_i^{(1)}(p_{\text{odd}}^1) \phi_{j-1}^{(2)}(p_{\text{odd}}^2)} \mid 0, 0 \rangle, \quad \alpha = 1, 2, 3 \tag{91}
\]

which we can call the determinantal 2BKP tau functions and where odd indexed power sums \( p^a = (p_1^a, p_3^a, p_5^a, \ldots) \) are related to the BKP higher times \( \{t_m^a\} \) by \( t_m^a = \frac{m}{2} p_m^a, \) \( m \) odd.
Such tau function are rather simple examples of tau functions of two-component hierarchies. Tau function $\tau_{r}$ and its fermionic representation was defined in [49] and was called hypergeometric tau function. It can be also expressed as the following two-component KP tau function

$$\tau_r(N; p, p^{2}) = K_N(N, -N|e^{\sum_{i \in Z} e^{-Ti} \phi_{i}^{(1)}(p^{2})\phi_{-i}^{(2)}(p^{2})}|0, 0)$$

(92)

where $e^{-Ti}, i \in Z$ is related to $r(i), i \in Z$ by $r(i) = e^{T_{i-1}-T_{i}}$. (Earlier in different form it was introduced in [27].)

Notations are explained below.

**Charged and neutral fermions** [20]. The creation and annihilation Fermi modes satisfy the anticommutation relations:

$$[\psi_{a}^{\dagger}, \psi_{b}^{\dagger}] = [\psi_{a}^{\dagger}, \psi_{b}^{\dagger}] = 0, \quad [\psi_{a}^{\dagger}, \psi_{b}^{\dagger}] = \delta_{a,b}\delta_{j_1 j_2}.$$  

(93)

where the superscripts $a$ and $b$ take values 1 and 2 for two-component fermions. The Fermi fields $\psi_{a}^{n}(z)$, $\psi_{b}^{n}(z)$ and the Fermi modes are related by $\psi_{a}^{n}(z) = \sum_{i \in Z} z^{i} \psi_{a}^{n}(z) = \sum_{i \in Z} z^{i} \psi_{a}^{n}(z)$, One can introduce 'time dependent' Fermi fields as $\psi_{a}^{n}(z, p^{2}) = e^{\sum_{m \geq 0} \frac{1}{m!} z^{m} \phi_{m}(z)} \psi_{a}^{n}(z)$, $\psi_{b}^{n}(z, p^{2}) = e^{-\sum_{m \geq 0} \frac{1}{m!} z^{m} \phi_{m}(z)} \psi_{b}^{n}(z)$ and the related 'time dependent' Fermi modes $\psi_{a}^{n}(z, p^{2})$ by

$$\psi_{a}^{n}(z, p^{2}) = \sum_{i \in Z} z^{i} \psi_{a}^{n}(p^{2}), \quad \psi_{b}^{n}(z, p^{2}) = \sum_{i \in Z} z^{i} \psi_{a}^{n}(p^{2})$$

(94)

There are left and right vacuum vectors:

$$\langle 0 | \psi_{a}^{n}_{i-1} = \langle 0 | \psi_{a}^{n} = 0 = \psi_{b}^{n}(0) = \psi_{b}^{n}_{i-1}|0 \rangle, \quad i < 0$$

(95)

We need the left vacuum vectors with Dirac sea levels $N$ and $-N$ for, respectively, first and second components of Fermi modes which we will define as follows:

$$\langle N, -N| \psi_{a}^{n}_{i-1} = \langle N, -N| \psi_{a}^{n} = 1 = \langle N, -N| \psi_{b}^{n}_{i-1} = \langle N, -N| \psi_{b}^{n}, \quad i < N, j < -N$$

(96)

(97)

The creation and annihilation modes of neutral Fermi fields satisfy the anticommutation relations:

$$[\phi_{a}, \phi_{b}] = (-1)^{i} \delta_{a,b} \delta_{k,0}$$

(98)

where the superscripts $a$ and $b$ take values 1 and 2 for two-component neutral fermions. In particular $\phi_{a}^{2} = \frac{1}{2}$, $a = 1, 2$. The Fermi fields $\phi_{a}^{n}(z)$ and the Fermi modes are related by $\phi_{a}^{n}(z) = \sum_{i \in Z} z^{i} \phi_{a}^{n}$. There are left and right vacuum vectors:

$$\langle 0 | \phi_{a}^{n}_{i-1} = 0 = \phi_{b}^{n}(0), \quad i < 0$$

(99)

and as one can verify we get

$$\langle 0 | \phi_{a}^{n}_{i-1} = 0 = \phi_{a}^{n}(0), \quad i < 0$$

(100)

The neutral Fermi modes $\phi_{a}$ can be related to the charged ones by

$$\phi_{a}^{n} := \psi_{a}^{n} + (-1)^{i} \psi_{b}^{n}_{i-1}, \quad j \in Z$$

(101)

In particular we have $\phi_{a}^{0}(0) = \frac{1}{\sqrt{2}} \phi_{a}^{0}(0)$ and $0|\phi_{a}^{0} = \frac{1}{\sqrt{2}} (0|\psi_{a}^{0}$.

Let us use the following notations. For $\lambda = (\lambda_{1}, \ldots, \lambda_{k}) \in P, \lambda_{k} > 0$

$$\Psi_{\lambda, p, N} := \prod_{i=1}^{k} \psi_{\lambda_{i}+N}^{1/2} e_{i}^{2} \psi_{i}^{1/2}$$

(102)

where $\lambda = (\alpha|\beta)$. Note that $|\lambda\rangle = \Psi_{\alpha, \beta}|0\rangle$, $\langle \lambda| = 0|\Psi_{\alpha, \beta}$.

For $\alpha = (\alpha_{1}, \ldots, \alpha_{k}) \in \text{DP}$ introduce

$$\Phi_{\alpha} := \Phi_{\alpha}^{1} \cdots \phi_{\alpha_{k}}$$

$$\Phi_{-\alpha} := (-1)^{\sum_{i=1}^{k} \alpha_{i}} \Phi_{\alpha}^{1} \cdots \phi_{-\alpha_{1}}$$

$$\Phi(X) = \phi(x_{N}) \cdots \phi(x_{1}), \quad \Phi^{*}(Y) = \phi(-y_{N}^{-1}) \cdots \phi(-y_{1}^{-1})$$

(103)

(104)

(105)
We have Sato formula:
\[ s_\lambda(x)\Delta(x) = \langle\langle 0 | \psi^\dagger(x_1) \cdots \psi^\dagger(x_N) \Psi_{\alpha,\beta} | N \rangle = \langle N | \Psi_{\alpha,\beta}^\dagger \psi(x_1) \cdots \psi(x_N) | 0 \rangle \] (106)
and formula obtained in [66]:
\[ Q_\alpha(x)\Delta^*(x) = \langle\langle 0 | \phi(-x_1) \cdots \phi(-x_N) \Phi_{\alpha} | 0 \rangle = \langle 0 | \Phi_{-\alpha} \phi(x_1) \cdots \phi(x_N) | 0 \rangle \] (107)
\[
\frac{2^N}{N!} \int \prod_{i=1}^N d\Omega_{x_i,y_i}(x_i,y_i) \phi(x_N) \cdots \phi(x_1)|0\rangle\langle 0 | = \sum_{\alpha,\beta \in DP} \Phi_{\alpha} | 0 \rangle g_{\alpha,\beta} | \Phi_{-\beta} \rangle (108)
\]

Examples of tau functions

Vacuum tau functions Notice that
\[ e^{\sum m t_m tr X^m} \text{ and } e^{-tr XY} \]
can be viewed as simplest (“vacuum”) Toda lattice tau function!

Sato-Takasaki series 2KP (-Toda lattice) tau function, (Sato,Takasaki,Takebe):
\[ \tau_N(t,t') = \sum_{\lambda,\mu} s_\lambda(t) g_{\lambda,\mu}(N) s_\mu(t') \]
where
- sum ranges over all possible pairs of partitions \( \lambda, \mu \)
- \( s_\lambda, s_\mu \) are the Schur function (polynomials in KP higher times)
- \( g_{\lambda,\mu} \) is the determinant of a certain matrix (initial data for the TL solution)

Example
\[ e^{\sum m t_m tr X^m} = \sum_\lambda s_\lambda(t) s_\lambda(X) \]
\[ e^{tr XY} = \sum_\lambda s_\lambda(XY) s_\lambda(1,0,0,\ldots) \]

C Matrix ensembles

Unitary matrices. The Haar measure on \( U_N \) in the explicit form is written as
\[ dU = \frac{1}{(2\pi)^N} \prod_{1 \leq i < k \leq N} |e^{\theta_i} - e^{\theta_k}|^2 \prod_{i=1}^N d\theta_i, \quad -\pi < \theta_1 < \cdots < \theta_N \leq \pi \] (109)

Hermitian matrices. This case can be considered in the same way as the previous one. It is natural [34] to take the following measure on the space \( H_N \) of Hermitian \( N \times N \) matrices:
\[ d\Omega_{w,X}(X) = C_N \prod_{i<j \leq N} e^{-w(R_{X_{i,j}})^2} dR_{X_{i,j}} \prod_{i<j \leq N} e^{-w(G_{X_{i,j}})^2} dG_{X_{i,j}} \]
\[ = c_N (\Delta(X))^2 \prod_{i=1}^N e^{-w x^2_i} dx_i dU \]
where \( w > 0 \) is a parameter and where we use \( X = U \text{diag}(s_{x_1} \ldots s_{x_N}) U^{-1}, \quad U \in U_N. \)

Remark 12. In many applied problems in which random matrices are used, it is very convenient to assume that the parameter \( w \) is proportional to the size of the matrices \( N \).
Orthogonal matrices. The Haar measure on \( \mathcal{O}_N \) is

\[
d_\omega(X) = \frac{1}{\pi^{n(n-1)/2}} \prod_{i<j} (\cos(\theta_i) - \cos(\theta_j))^2 \prod_{i=1}^n, \quad N = 2n
d_\omega(X) = \frac{1}{\pi^{n(n-1)/2}} \prod_{i<j} (\cos(\theta_i) - \cos(\theta_j))^2 \prod_{i=1}^n \sin^2 \frac{\theta_i}{2} d\theta_i, \quad N = 2n+1
\]

(110)

where \( d\theta_i, \quad 0 \leq \theta_i < \cdots < \theta_n \leq \pi \). The prefactors are chosen to provide \( \int_{\mathcal{O}_N} d_\omega O = 1 \).

By \( \mathcal{S}_N \) we denote the space of \( N \times N \) real skew-symmetric matrices. Recall that the eigenvalues of real skew fields are purely imaginary; the eigenvalues occur in pairs \( \pm x_i \sqrt{-1}, i = 1, \ldots, n \) in the case \( N = 2n \) while in the case \( N = 2n+1 \) there is an additional eigenvalue \( \pm \).

Skew-symmetric matrices. For any \( X \in \mathcal{S}_N \) there exists such \( O \in \mathcal{O}_N \) that \( X = O \text{diag} \left\{ \begin{pmatrix} 0 & x_1 \\ -x_1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & x_n \\ -x_n & 0 \end{pmatrix} \right\} O^{-1} \in \mathcal{S}_{2n}, \) where \( O \in \mathcal{O}_{2n}, \) \( X = O \text{diag} \left\{ \begin{pmatrix} 0 & x_1 \\ -x_1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & x_n \\ -x_n & 0 \end{pmatrix} \right\} O^{-1} \in \mathcal{S}_{2n+1}, \) where \( O \in \mathcal{O}_{2n+1} \).

The measure on \( \mathcal{S}_N \) is as follows:

\[
d_\omega(N)(X) = \pi^{1\left(n(n-2)\right)} \prod_{i<j} e^{-x_i^2} dx_{i,j}
= \prod_{i<j} (x_i^2 - x_j^2)^2 \prod_{i=1}^n e^{-x_i^2} dx_i, \quad N = 2n
= \prod_{i<j} (x_i^2 - x_j^2)^2 \prod_{i=1}^n e^{-x_i^2} dx_i, \quad N = 2n+1
\]

(111)

where prefactors \( c_1 = 2^{-n} \prod_{i=0}^{n-1} (\Gamma(2+i)\Gamma(2+i))^{-1} \) and \( c_2 = 2^{-n} \prod_{i=0}^{n-1} (\Gamma(2+i)\Gamma(2+i))^{-1} \) provide the normalization \( \int \mathcal{O}_N d_\omega(N)(X) = 1 \), see (17.6.5) in [34].

It is known [？] that

\[
\int_{\mathcal{O}_N} e^{u(OXO^{-1}Y)} d_\omega(O) = \begin{cases}
\prod_{i<j} (\text{det}[2 \cosh(2x_i y_j)])_{i,j}, & N = 2n \\
\prod_{i<j} (\text{det}[2 \sinh(2x_i y_j)])_{i,j}, & N = 2n+1
\end{cases}
\]

(112)

where \( X, Y \in \mathcal{S}_N \) and \( \pm x_i \sqrt{-1} \) and \( \pm y_i \sqrt{-1} \) are eigenvalues of respectively \( X \) and \( Y \).

Complex matrices. The measure on the space of complex matrices is defined as

\[
d\Omega(Z) = c_N^2 \prod_{a,b=1}^N dR_{ab} d\Delta_{ab} e^{-N|Z_{ab}|^2}
\]

(113)

We will consider integrals over \( N \times N \) complex matrices \( Z_1, \ldots, Z_n \) where the measure is defined as

\[
d\Omega(Z_1, \ldots, Z_n) = c_N^2 \prod_{i=1}^n \prod_{a,b=1}^N dR(Z_i)_{ab} d\Delta(Z_i)_{ab} e^{-N|Z_i_{ab}|^2}
\]

(114)

where the integration domain is \( \mathbb{C}^{N^2} \times \cdots \times \mathbb{C}^{N^2} \) and where \( c_N^2 \) is the normalization constant defined via \( \int d\Omega(Z_1, \ldots, Z_n) = 1 \).

The set of \( n \times N \times N \) complex matrices with measure \([114]\) is called the set of \( n \) independent complex Ginibre ensembles. Such ensembles have wide applications in physics and in information transfer theory [？], [？], [？], [？], [？], [？].

D Solvable multi-matrix models, related to embedded graphs drawn on the Riemann sphere [4]

Here we present some examples of tau functions \( \tau_A, \in \mathbb{Z} = 1, 2 \), see [17], [18], which are the building blocks of the family \([14]\) (and also of the chain family \([33]\)) and some examples of the pairing tau functions \( \tau_r(N; XY, I_N) \) and \( \tau_r(N; X^{-1}, I_N) \) which are the building blocks which are the building blocks respectively.

Consider a connected graph \( \Gamma \) on an orientable connected surface \( \Sigma \) without boundary with Euler characteristic e. We require such properties of the graph:

(1) its edges do not intersect. For example, the edges of the graph in figure (1) do not intersect: the fact is that the graph is drawn on a torus, and not on a piece of paper.

20
(2) if we cut the surface $\Sigma$ along the edges of the graph, then the surface will decompose into disks (more precisely: into pieces homeomorphic to disks).

As an example, see Figure 1 which contains all such graphs with 2 edges.

Such a graph is sometimes called a (clean) dessins d’enfants\(^2\)sometimes - a map [30].

Let our graph have $f$ faces, $n$ edges and $v$ vertices, then $e = v - n + f$.

We number all stars (i.e., all vertices of the graph $\Gamma$) with numbers from 1 to $v$, all faces of $\Gamma$ with numbers from 1 to $f$ and all edges of $\Gamma$ with numbers from 1 to $n$ in any way.

We will slightly expand the vertices of the graph and turn them into small disk, which sometimes for the sake of visual clarity we will call stars.

---

\(^2\)With the term dessins d’enfants without the additional “clean” serves for such graph with a bipartite structure.
Let's use the following numbering. If one side of the edge of the tape \(|i|\) is labeled \(i\) then the other side of the same edges is labeled \(-i\). (This doesn't matter which side of the edge \(|i|\) we assign the number \(i\) to, and which - the number \(-i\), but we should fix the numbering we have chosen). The two sides of the edge of the graph \(\Gamma\) are actually arrows looking in opposite directions (In order not to complicate the picture, we do not depict these arrows in our figures). We attribute the number \(-i\). We assign the number \(-i\) to both the edge side and the arrow-segment, in which the side \(-i\) rests. Acting in this way, we give numbers to all segments of all small disks.

We attribute a sequential set of numbers to each star as follows. (This set is defined up to a cyclic permutation, and we will call it a cycle associated with the star). Examples: These are numbers \((1, -1)\) assigned to the star (yellow small disk) in Figure 2 (a). These are the numbers \((k, -j)\) that we attribute to the star on the top in the Figure 2 (c) and the numbers \((-k, j, -i)\) that we attribute to the lower star in the same figure. And to each cycle we ascribe the related cyclic products:

\[
(1, -1) \leftrightarrow C_1 C_{-1},
\]
to the star (yellow small disk) in Figure 2 (a). As for the stars in Figure 2 (c) we obtain

\[
(k, -j) \leftrightarrow C_k C_{-j} \quad (-k, j, -i) \leftrightarrow C_{-k} C_j C_{-i},
\]

Each cycle product we will call the star monodromy. As a result, a star's monodromy is a product of matrices corresponding face monodromies with numbers from 1 to \(i\) for a face with a number \(i\), \(i\) is included and which face cycles and face monodromies: The cycles corresponding to the face \(i\) for an ordinary face (or counterclockwise if the face contains infinity), we collect segment numbers of small disks; for a face with a number \(i\), this ordered collection of ordered numbers is \(f_i\). As in the case of stars, we build cyclic products \(f_i \leftrightarrow W_{i}\). Examples:

\[
f_1 = (1) \leftrightarrow C_1 = W_1 \quad \text{and} \quad f_2 = (-1) \leftrightarrow C_{-1} = W_2
\]

for two faces in Figure 2 (a).

We also introduce dressed monodromies:

\[
\mathcal{L}_X(W_1) = X_1 C_1, \quad \mathcal{L}_X(W_{-1}) = X_{-1} C_{-1}
\]

And for the face in Figure 2 (c): 

\[
f_1 = (-j, -k) \leftrightarrow C_{-j} C_{-k} = W_1 \leftrightarrow \mathcal{L}_X(W_1) = X_{-j} C_{-j} X_{-k} C_{-k}
\]

So, we have two sets of cycles and two sets of monodromies: vertex cycles and vertex monodromies

\[
\sigma_i^{-1} \leftrightarrow W_{i}^*, \quad i = 1, \ldots, v \tag{115}
\]

and face cycles and face monodromies: The cycles corresponding to the face \(i\) will be denoted by \(f_i\), we have

\[
f_i \leftrightarrow W_i \leftrightarrow \mathcal{L}_X(W_i), \quad i = 1, \ldots, f \tag{116}
\]

Let us note that both cycles and monodromies are defined up to the cyclic permutation.

**Remark 13.** Important remark. Please note that each of the matrices \(C_i, i = \pm 1, \ldots, \pm n\) enters the set of monodromies \(W_1, \ldots, W_f\) once and only once. Accordingly, each random matrix \(X_i, i = \pm 1, \ldots, \pm n\) is included once and only once in the set of dressed monodromies \(\mathcal{L}_X[W_1], \ldots, \mathcal{L}_X[W_f]\). This determines the class of matrix models that we will consider and which we will call matrix models of dessins d'enfants.

### E Examples of matrix models

Recall that we introduced the function \(\tau_c(n; P^{(1)}, P^{(2)})\). In what follows we use the conventions:

\[
\tau_r(P^{(1)}, P^{(2)}) := \tau_r(0, P^{(1)}, P^{(2)}) = \sum_{\lambda} r_{\lambda} s_{\lambda}(P^{(1)}) s_{\lambda}(P^{(2)}), \quad r_{\lambda} = r_{\lambda}(0) \tag{117}
\]
It depends on two sets $p^i = (p_1^{(i)}, p_2^{(i)}, \ldots), i = 1, 2, \ldots$, as well as on the choice of an arbitrary function of the variable $r$. As one of their sets, we will choose $p_2 = p(X)$ like in (11), and the second set will be the set of arbitrary parameters. With $r = 1$ we get

$$
\tau_1(p, X) = e^{\sum_{m>0} \frac{1}{m} \text{tr}(X^m)}
$$

(118)

For example, if we take

$$
r(x) = \prod_i (a_i + x) \prod_j (b_j + x),
$$

and in addition $p^1 = (1, 0, 0, \ldots)$ we get the so-called hypergeometric function of the matrix argument:

$$
\binom{p}{q}(a_1, \ldots, a_p | b_1, \ldots, b_q | X) = \sum_{\lambda} \frac{\dim \lambda}{|\lambda|!} s_{\lambda}(X) \prod_i (a_i + x)^{\lambda_i} \prod_j (b_j + x)^{\lambda_j}
$$

(119)

Special cases:

$$
e^{x X} = \sum_{\lambda} s_{\lambda}(X) \frac{\dim \lambda}{|\lambda|!},
$$

(120)

$$
\det(1 - z X)^{-a} = \sum_{\lambda} z^{|\lambda|} (a) s_{\lambda}(X) \frac{\dim \lambda}{|\lambda|!}
$$

(121)

**Integrals.** We have

**Theorem 1.** Suppose $W_1, \ldots, W_f$ and $W_1^*, \ldots, W_v^*$ are dual sets. Let sets $p^i = (p_1^{(i)}, p_2^{(i)}, p_3^{(i)}, \ldots), i = 1, \ldots, \max(f, v)$ be independent complex parameters and $r(i), i = 1, \ldots, \max(f, v)$ be a set of given functions in one variable.

$$
E_{n_1, n_2} \left\{ \mathcal{L} x \left[ \prod_{i=1}^f \tau_{r(i)}(N; p^i, W_i) \det(W_i)^{\alpha_i} \right] \right\}
$$

(122)

$$
= \sum_{\lambda} r_{\lambda} h^{n_1(|\lambda| + a N)} \left( \frac{\dim \lambda}{|\lambda|!} \right)^{-n} \prod_{i=1}^f s_{\lambda}(p^i) \prod_{i=1}^v s_{\lambda}(W_i)^{\alpha_i} \det(W_i)^{\alpha_i},
$$

(123)

**This is an example of the so-called tau function, but we will not use this fact.**
where each $\tau_{\nu,(i)}(p_i,W_i)$ is defined by

$$r_\lambda = ((N)_\lambda)^{-n_2} \left( \frac{(N + \alpha)\lambda}{(N)\lambda} \right)^{n_1} \prod_{i=1}^f r_\lambda^{(i)}(n)$$

Similarly

$$E_{n_1,n_2} \{ \mathcal{E}_X [\tau_{\nu,(i)}(N; p^1_i, W^*_i) \cdots \tau_{\nu,(i)}(N; p^v_i, W^*_i)] \} = \sum_\lambda r_\lambda h^{n_1|\lambda|} \left( \frac{\dim \lambda}{|\lambda|!} \right)^{-n} \prod_{i=1}^f \delta_\lambda(W_i) \prod_{i=1}^v \delta_\lambda(p^i),$$

Remark 14. We recall the convention $[117]$ In $[123]$ $r_\lambda$ is the content product $[7]$

$$r_\lambda = r_\lambda(0) = \prod_{(i,j) \in \lambda} (j-i)$$

where

$$r(x) = (N + x)^{-n_2} \prod_{i=1}^f r^{(i)}(x)$$

To get examples we choose

- dual sets $W_1, \ldots, W_t \leftrightarrow W^*_1, \ldots, W^*_t$
- the fraction of unitary matrices given by $n_2$
- the set of functions $r^{(i)}$, $i = 1, \ldots, f$
- the sets $p^{(i)}$, $i = 1, \ldots, f$

Remark 15. Important notice. Answers in some cases are further simplified. Let us mark two cases

(i) Firstly, this is the case when the spectrum of the stars has the form

$$\text{Spect } W^*_i = \text{Spect } \mathcal{I}_{N,k_i} = \text{diag}\{1,1,\ldots,1,0,0,\ldots,0\}, \quad i = 1, \ldots, v$$

where $\mathcal{I}_{N,k_i}$ is the matrix with $k_i$ units of the main diagonal. Such star monodromies are obtained in case source matrices have a rank smaller than $N$. Insertion of such matrices in the left hand sides of $[123]$ and $[125]$ corresponds to the integration over rectangular random matrices. One should take into account that

$$\delta_\lambda(\mathcal{I}_{N,k}) = (k)_\lambda \delta_\lambda(p_{\infty}),$$

where we recall the notation

$$(a)_\lambda := (a)_{\lambda_1} (a - 1)_{\lambda_2} \cdots (a - \ell + 1)_{\lambda_\ell}$$

(ii) The case is the specification of the sets $p^i$, $i = 1, \ldots, f$ according to the following

Lemma 6. Denote

$$p_\infty = (1,0,0,\ldots)$$

$$p(a) = (a,a,a,\ldots)$$

$$p(q, t) = (p_1(q, t), p_2(q, t), \ldots), \quad p_m(q, t) = \frac{1 - q^m}{1 - t^m}$$

Then

$$\frac{\delta_\lambda(p(a))}{\delta_\lambda(p_\infty)} = (a)_\lambda, \quad p(a) = (a,a,a,\ldots)$$

where $(a)_\lambda := (a)_{\lambda_1} (a - 1)_{\lambda_2} \cdots (a - \ell + 1)_{\lambda_\ell}$, $(a)_n := (a + 1) \cdots (a + n - 1)$, where $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ is a partition. More generally

$$\frac{\delta_\lambda(p(q, t))}{\delta_\lambda(p(0, t))} = (q, t)_\lambda$$

where $(q, t)_\lambda := (q, t)_{\lambda_1} (q^{-1}; t)_{\lambda_2} \cdots (q, t^{1-\ell})_{\lambda_\ell}$, where $(q, t)_k := (1-q)(1-q) \cdots (1-q t^{n-1})$ is $t$-deformed Pochhammer symbol. $(q, t)_0 = 1$ is implied.
With such specifications, one can diminish the number of the Schur functions in the right-hand side of (123) (or of (125)) and the right hand side can take one of the forms:

\[ \sum_{\lambda} r_{\lambda} s_{\lambda}(A) s_{\lambda}(B) \]  

(133)

\[ \sum_{\lambda} r_{\lambda} s_{\lambda}(A) \]  

(134)

\[ \sum_{\lambda} r_{\lambda} \]  

(135)

For (133) there is a determinantal representation, for (134) there is a Pfaffian representation and (135) can be rewritten as a sum of products.

For instance, one can take \( r(x) = a + x \) and get

\[ (a)_{\lambda} = \frac{\Gamma(a + \lambda_1 - 1) \Gamma(a + \lambda_2 - 2) \cdots \Gamma(a + \lambda_N - N)}{\Gamma(a) \Gamma(a - 1) \cdots \Gamma(a - N + 1)} \]  

(136)

Then we introduce \( h_i = \lambda_i - i + N \) and write

\[ \sum_{\lambda} \frac{(a)_{\lambda}}{(b)_{\lambda}} h_{\lambda} = \sum_{h_1 > \cdots > h_N \geq 0} \prod_{i=1}^{N} \Gamma(b_i - i + 1) \Gamma(h_i + a - N) \frac{\Gamma(a - i + 1) \Gamma(h_i + b - N)}{\Gamma(a - N + 1) \Gamma(a + b - N)} \]  

(137)

where \( \Gamma \) is the gamma-function. (In case the argument of gamma-function turns out to be a nonpositive integer one should keep in mind both the enumerator and denominator). See examples below.

**Example 11.** See Example and Figure 2 (a). Take \( X_1 = Z \) and \( r \) given by (4).

(a) The example of (123) can be chosen as follows

\[ E_{1.0} \left\{ \rho F_{\alpha} \left( \frac{a_1, \ldots, a_p}{b_1, \ldots, b_q} | ZC_1 \right) \rho F_{\alpha'} \left( \frac{a_1', \ldots, a_{p'}'}{b_1', \ldots, b_{q'}'} | Z^T C_{-1} \right) \det \left( ZZ^T \right)^{\alpha} \right\} = \]  

(138)

\[ = \rho F_{\alpha} \left( \frac{a_1, \ldots, a_p, a_1', \ldots, a_{p'}', N + \alpha}{b_1, \ldots, b_q, b_1', \ldots, b_{q'}', N} | C_1 Z C_{-1} \right), \]  

the corresponding determinantal representation see in (11).

See and Figure 2 (b) which is dual to (a) in this Figure.

(b) An example of (123) can be chosen as

\[ E_{1.0} \left\{ \rho F_{\alpha} \left( \frac{a_1, \ldots, a_p, a_1', \ldots, a_{p'}', N + \alpha}{b_1, \ldots, b_q, b_1', \ldots, b_{q'}', N} | ZC_1 Z^T C_{-1} \right) \det \left( ZZ^T \right)^{\delta} \right\} = \]  

(139)

\[ \sum_{\lambda} s_{\lambda}(C_1) s_{\lambda}(C_{-1}) \frac{(N + \alpha)_{\lambda}(N + \beta)_{\lambda}}{((N)_{\lambda})^2} \frac{\prod_i (a_i)_{\lambda} \prod_i (a_i')_{\lambda}}{\prod_i (b_i)_{\lambda} \prod_i (b_i')_{\lambda}}, \]  

(140)

The determinantal representation of the left hand side is given by (6).

(c) Next example

\[ E_{0.1} \left\{ \sum_{m \geq 0} \frac{1}{\rho m \tr \left( U C_1 U^T C_{-1} \right)^m} \right\} = \sum_{\lambda} \frac{s_{\lambda}(p) s_{\lambda}(C_1) s_{\lambda}(C_{-1})}{s_{\lambda}(1_N)} \]  

(141)

It can be written as a determinant either if \( p \) is specialized according to, then it can be written in form, or if any of the matrices \( C_1, C_{-1} \) has the same spectrum as \( 1_{N} \) while the parameters \( p \) are arbitrary. For instance

\[ E_{0.1} \left\{ \sum_{m \geq 0} \frac{1}{\rho m \tr \left( U C_1 U^T \right)^m} \right\} = \sum_{\lambda} s_{\lambda}(p) s_{\lambda}(C_1) \frac{(k)_{\lambda}}{(N)_{\lambda}} \]  

(142)

(d) 

\[ E_{0.1} \left\{ \sum_{m \geq 0} \frac{1}{\rho m \tr \left( (p_1^{(1)} U C_1)^m + p_2^{(2)} U C_{-1} \right)^m} \right\} = \sum_{\lambda} s_{\lambda}(p_1) s_{\lambda}(p_2) s_{\lambda}(C_1 C_{-1}) \frac{(N + \alpha)_{\lambda}}{((N)_{\lambda})^2} \]  

(143)
Example 12. Graph 1 (c) yields
\[
E_{2,0} \left\{ e^{\sum_{m>0} \frac{1}{m+1} \text{tr}(Z_1 C_1 Z_2 C_2)^m} + \sum_{m>0} \frac{1}{m+1} \text{tr}(Z_1 C_1 Z_2 C_2 Z_3 C_3)^m \right\} = \det \left( Z_1 Z_1^* \right)^{\alpha} \sum_{\lambda} s_{\lambda}(p^1) s_{\lambda}(p^2)(a)_\lambda \left( \frac{(N + \alpha)\lambda}{(N)_\lambda} \right)^3
\]

For a determinantal representation see [10].

Example 13. In the case below we use an open chain with \( n \) edges as in Figures 1 (b), 2 (b), 3 (a).
\[
E_{n,0} \left\{ e^{\sum_{m>0} \frac{1}{m+1} \text{tr}(Z_1 C_1 Z_2 C_2 \cdots Z_n C_n Z_1^* C_n^* Z_1^* C_n^* \cdots Z_1^* C_n^*)^m} \right\} = \sum_{\lambda} s_{\lambda}(C_n) s_{\lambda}(C_{n-1}) \frac{s_{\lambda}(p^1)}{s_{\lambda}(p^1 \infty)} \prod_{i=1}^{n-1} \frac{s_{\lambda}(C_{i-1} C_{i-1})}{s_{\lambda}(p^1 \infty)}
\]

Graphs dual to the chain look like in Figure 3 (b).
\[
E_{n,0} \left\{ e^{\sum_{m>0} \frac{1}{m+1} \text{tr}(Z_1 C_1 Z_2 C_2 \cdots Z_n C_n Z_1^* C_n^* Z_1^* C_n^* \cdots Z_1^* C_n^*)^m} \right\} = \sum_{\lambda} s_{\lambda}(p^1) s_{\lambda}(C_1 C_2 Z_n \cdots C_{n-1}) \prod_{i=1}^{n} \frac{s_{\lambda}(p^1)}{s_{\lambda}(p^1 \infty)}
\]

In case we specify each argument of the Schur functions in the right hand side except any two ones according to Remark 8 we can write down a determinantal representation.

Example 14. Our graph is a polygon with \( n \) edges and \( n \) vertices (stars), see Figures 1 (b), 2 (a), 3 (c) and 4 (c) for examples.
\[
E_{m,0} \left\{ e^{\sum_{m>0} \frac{1}{m+1} \text{tr}(Z_1 C_1 Z_2 C_2 \cdots Z_n C_n Z_1^* C_n^* Z_1^* C_n^* \cdots Z_1^* C_n^*)^m} \right\} = \sum_{\lambda} s_{\lambda}(p^1) s_{\lambda}(C_1 C_2 \cdots C_n) \prod_{i=1}^{n} \frac{s_{\lambda}(p^1)}{s_{\lambda}(p^1 \infty)}
\]

To apply determinantal formulas one should use Remark 15.

Example 15. Consider the star-graph with \( n \)-rays which end at other stars (see Figure 4 (a) where \( n = 3 \)).
\[
E_{n,0} \left\{ e^{\sum_{m>0} \frac{1}{m+1} \text{tr}(Z_1 C_1 Z_2 C_2 \cdots Z_n C_n Z_1^* C_n^* Z_1^* C_n^* \cdots Z_1^* C_n^*)^m} \right\} = \sum_{\lambda} s_{\lambda}(p^1) s_{\lambda}(C_1 C_2 Z_n \cdots C_{n-1}) \prod_{i=1}^{n} \frac{s_{\lambda}(p^1)}{s_{\lambda}(p^1 \infty)}
\]

The similar model was studied in [7, 9]. It has the determinantal representation (11) in case all \( W_{\alpha}^* \) except one are of form (126). There is The determinantal representation (4) in case we specialize the set \( \mathbf{p} \) according to Lemma 6 and choose each \( W_{\alpha}^* \) except two be in form (126).

Now, let us choose the dual graph (this is petel graph. see (see Figure 4 (b) where \( n = 3 \)) and consider
\[
E_{n,0} \left\{ \det \left( 1 - Z_1 C_1 Z_2 C_2 \cdots Z_n C_n Z_1^* C_n^* \right)^{\alpha} e^{\sum_{m>0} \frac{1}{m+1} \text{tr}(Z_1 C_1 Z_2 C_2 \cdots Z_n C_n Z_1^* C_n^* \cdots Z_1^* C_n^*)^m} \right\} = \sum_{\lambda} e^{\lambda(a)} s_{\lambda}(p^1) s_{\lambda}(C_1 C_2 \cdots C_n) \prod_{i=1}^{n} \frac{s_{\lambda}(p^1)}{s_{\lambda}(p^1 \infty)}
\]

By Remark 15 we find all cases where the determinantal representations (11) or (11) exist.

Remark 16. Notice the following symmetry: the left hand side produces the same right hand side if we permute \( \alpha \leftrightarrow a - N \).