New confining optical media generated by Darboux transformations

Rubén Razo, and Sara Cruz y Cruz*

Instituto Politécnico Nacional, UPIITA, Av. Instituto Politécnico Nacional 2580, La Laguna Ticomán, C.P. 07340, Ciudad de México, Mexico

E-mail: *sgcruzc@ipn.mx

Abstract. The parametric Darboux transformation is applied to the paraxial Helmholtz equation for graded-index media in order to generate new refractive index profiles depending also on the longitudinal coordinate. The particular case of the parabolic medium is considered, and the construction of new deformed quadratic profiles exhibiting self-focusing properties is addressed. Some examples illustrate our formalism.

1. Introduction

The development of theoretical and experimental graded index optics in the last decades has enabled the control of light propagation processes and diffraction phenomena (see [1–3]). In particular, the study and design of optical media with refractive index profiles such that the propagating beam undergoes no distortion have been extensively considered. These include, for instance, the parabolic and hyperbolic secant profiles characterized by their self-focusing and collimating properties [4–7]. On the other hand, the Darboux transformation has been widely used in many areas of physics, such as soliton theory [8] and quantum mechanics [9] for the generation of new exactly solvable models. A parametric version of the Darboux transformation has been also considered in [10–13] allowing the construction of time-dependent exactly solvable potentials in quantum mechanics. An equivalent approach is introduced in [14] for the design of $PT$ symmetric optical wave guides.

In this work we apply the parametric Darboux transformation, as stated in [11, 12], to the paraxial Helmholtz equation in order to construct new exactly solvable models of confining optical media. The so obtained refractive index distributions are dependent on the transversal as well as the longitudinal coordinates and exhibit self-focusing properties as they are deformations of the quadratic profile. The corresponding modal fields are easily determined by the application of the corresponding intertwining operator.

2. The quadratic refractive index profile

Consider a graded index optical medium with refractive index $n = n(x, z)$ depending on a single transversal coordinate $x$ and, possibly, on the longitudinal coordinate $z$. It is well known that in the weakly-guiding approximation $n(x, z)$ can be written in the form

$$n(x, z) = n_0 + \Delta n(x, z),$$
where \( n_0 \) is the reference refractive index and \( \Delta n(x, z) = n(x, z) - n_0 \ll n_0 \), is a small, variable fluctuation around the constant value \( n_0 \). In the weakly-guiding regime the paraxial approximation is valid. The electric field amplitude \( E(x, z) \) of a transversal monochromatic wave has the form

\[
E(x, z) = U(x, z)e^{ik_0n_0z},
\]

where \( k_0 \) is the wave number in free space, and \( U(x, z) \) satisfies the paraxial Helmholtz equation

\[
\left\{ \frac{i}{k_0} \frac{\partial}{\partial z} + \frac{1}{k_0^2n_0} \frac{\partial^2}{\partial x^2} + n(x, z) - n_0 \right\} U(x, z) = 0. \tag{1}
\]

In this section we are interested in the propagation of localized wave-packets in a confining optical optical medium whose refractive index present a parabolic profile in the transverse coordinate

\[
n^2(x, z) = n^2(x) = n_0^2 \left( 1 - \Omega^2 x^2 \right). \tag{2}
\]

In this case \( n_0 \) is the refractive index of the medium at the optical axis -that is chosen as the \( z \)-axis-, and the parameter \( \Omega \) encodes the self-focusing properties of the material. In some cases it is assumed that this parameter is small enough so that the quantity \( \Omega^2 x^2 \ll 1 \) for the values of \( x \) of interest [15]. Under this condition the refractive index can be written in the weakly-guiding approximation as

\[
n(x) \approx n_0 \left( 1 - \frac{1}{2} \Omega^2 x^2 \right). \tag{3}
\]

The paraxial Helmholtz equation now reads

\[
\frac{i}{k_0} \frac{\partial U}{\partial z} = \left\{ -\frac{1}{k_0^2n_0} \frac{\partial^2}{\partial x^2} + \frac{n_0}{2} \Omega^2 x^2 \right\} U(x, z). \tag{4}
\]

A complete set of square integrable solutions of this equation is conformed by the Hermite-Gaussian modes in the parabolic medium [16]

\[
U_n(x, z) = \left( \frac{2}{\pi} \right)^{1/4} \frac{1}{\sqrt{2^n n! w(z)}} e^{-i(n+\frac{1}{2})\chi(z)} e^{ik_0n_0x^2} e^{-\frac{x^2}{w^2(z)}} H_n \left( \frac{\sqrt{2}}{w(z)} x \right), \tag{5}
\]

where \( H_n(\zeta) \) is the Hermite polynomial of degree \( n \) and the functions \( w(z) \), \( R(z) \) and \( \chi(z) \) are, respectively, the beam width, radius of curvature and Gouy phase shift given by [16, 17]

\[
w(z) = w_0 \left[ \cos^2(\Omega z) + \frac{1}{(\Omega z R)^2} \sin^2(\Omega z) \right]^{1/2}, \quad \frac{1}{R(z)} = \frac{d}{dz} \ln w(z), \quad \chi(z) = \frac{2}{k_0n_0} \int_{-d}^{d} \frac{dt}{w^2(t)}, \tag{6}
\]

with \( w_0 \) the minimum width and \( z_R \) the Rayleigh range (see, e.g., [15, 18]). Observe that the transversal pattern of the corresponding intensity distribution is given by the Hermite-Gaussian profile, while the propagation behavior along the longitudinal coordinate is completely encoded in the width function \( w(z) \). In our case, this is an oscillating function of period \( \frac{\pi}{R} \) that indicates that the spreading beam is periodically re-focused due to the confining properties of the medium (see Figure 1).

In the limit \( \Omega \to 0 \) we recover the Hermite-Gaussian modes of a homogeneous medium of refractive index \( n_0 \) (see Figure 1(a)), for which [18, 19]

\[
w(z) = w_0 \sqrt{1 + \left( \frac{z}{z_R} \right)^2}.
\]

On the other hand, if we take \( \Omega = \frac{1}{z_R} \), we obtain a collimated beam of constant width \( w_0 \), as shown in Figure 1(c).
Figure 1. Intensity distributions for the Hermite-Gaussian mode $U_2(x,z)$ for the values of $\Omega$ indicated in each case. The transversal and longitudinal coordinates are measured, respectively, in units of $w_0$ and $z_R$.

3. Darboux transformations and the paraxial Helmholtz equation

Consider again the paraxial Helmholtz equation (1) for a refractive index profile $n(x,z)$. Consider also the paraxial Helmholtz equation for a different profile $m(x,z)$

$$\left\{ \frac{i}{k_0} \frac{\partial}{\partial z} + \frac{1}{k_0^2 n_0} \frac{\partial^2}{\partial x^2} + m(x,z) - n_0 \right\} W(x,z) = 0,$$

with $W(x,z)$ the corresponding electric field amplitude of the transversal electromagnetic mode associated to this medium.

Let us assume that there exist a transformation operator $L$ of the form (compare to [10–12])

$$L = L_1(z) \left( \frac{\partial}{\partial x} + L_0(x,z) \right),$$

where $L_0$ and $L_1$ are two functions to be determined in such a way that $L$ intertwines the paraxial Helmholtz operators corresponding to equations (1) and (7), i.e.,

$$L \left\{ \frac{i}{k_0} \frac{\partial}{\partial z} + \frac{1}{k_0^2 n_0} \frac{\partial^2}{\partial x^2} + n(x,z) - n_0 \right\} = \left\{ \frac{i}{k_0} \frac{\partial}{\partial z} + \frac{1}{k_0^2 n_0} \frac{\partial^2}{\partial x^2} + m(x,z) - n_0 \right\} L.$$

The functions $L_0$ and $L_1$ satisfy the following set of equations

$$m(x,z) - n(x,z) = -\frac{i}{k_0} \frac{d}{dz} (\ln L_1(z)) - \frac{1}{k_0^2 n_0} \frac{\partial}{\partial x} L_0(x,z),$$

$$\frac{\partial}{\partial x} n(x,z) = \left\{ \frac{i}{k_0} \frac{\partial}{\partial z} - \frac{1}{2k_0^2 n_0} L_0(x,z) \frac{\partial}{\partial x} + \frac{1}{2k_0^2 n_0} \frac{\partial^2}{\partial x^2} \right\} L_0(x,z).$$

The introduction of a transformation function $u(x,z)$ such that

$$L_0(x,z) = -\frac{\partial}{\partial x} (\ln u(x,z)),$$

reduces equation (11) to the paraxial Helmholtz equation

$$\left\{ \frac{i}{k_0} \frac{\partial}{\partial z} + \frac{1}{k_0^2 n_0} \frac{\partial^2}{\partial x^2} + n(x,z) - n_0 + C(z) \right\} u(x,z) = 0,$$

where $C(z) = \frac{1}{2k_0^2 n_0} \frac{\partial}{\partial x} L_0(x,z) \frac{\partial}{\partial x} + \frac{1}{2k_0^2 n_0} \frac{\partial^2}{\partial x^2}$. 


where $C(z)$ is an integration constant that is not involved in the dynamics and that can be set to zero without loss of generality. The functions $L_1$ and $m$ can be expressed in terms of $u$ as follows

$$L_1(z) = \exp \left[ \frac{1}{k_0 n_0} \int dz \, \text{Im} \left( \frac{\partial^2}{\partial x^2} \ln u(x, z) \right) \right],$$

$$m(x, z) = n(x, z) + \frac{1}{2k_0 n_0} \left( \frac{\partial^2}{\partial x^2} \ln |u(x, z)|^2 \right).$$

If the function $u(x, z)$ is chosen to be nodeless, then this new refractive index profile will be non-singular. Furthermore, if we consider real valued refractive profiles $n(x, z)$ and $m(x, z)$, then equation (10) leads to

$$\frac{\partial^3}{\partial x^3} \left( \ln \frac{u(x, z)}{u^*(x, z)} \right) = 0,$$

where the symbol $^*$ stands for complex conjugation.

A simple inspection to the expression (9) allows to conclude that

$$\left\{ \frac{i}{k_0} \frac{\partial}{\partial z} + \frac{1}{k_0^2 n_0} \frac{\partial^2}{\partial x^2} + m(x, z) - n_0 \right\} LU(x, z) = 0.$$

This indicates that we may construct the solutions of the paraxial Helmholtz equation (7) applying the intertwining operator $L$ on the solutions of (1), i.e.,

$$W(x, z) = LU(x, z).$$

Moreover, a direct calculation reveals that $Lu(x, z) = 0$, and that the function

$$W_0(x, z) = \frac{1}{L_1(z)u^*(x, z)}$$

is also a solution of (7).

4. New Darboux-deformed parabolic media

Let us assume that the refractive index $n(x, z)$ is given by (2) and consider the Darboux transformation (10). According to (12)-(15), the functions $L_0$, $L_1$ and $m$ are given in terms of the transformation function $u$ fulfilling the paraxial Helmholtz equation (4) for the parabolic medium. In order to obtain new non-singular profiles $m$, we must choose nodeless solutions to this equation. It can be shown that the general solution can be cast in the form

$$u(x, z) = \frac{1}{\sqrt{w(z)}} e^{-i\alpha \chi(z)} e^{\frac{kn_0}{2R(z)}} g(y(x, z)),$$

where

$$y(x, z) = \sqrt{2} \frac{x}{w(z)},$$

$$g(y) = e^{-y^2} \left\{ A_1 F_1 \left( \frac{1}{4}(1 - 2\alpha), \frac{1}{2}, y^2 \right) + B y_1 F_1 \left( \frac{1}{4}(3 - 2\alpha), \frac{3}{2}, y^2 \right) \right\},$$

$$_1 F_1(a, b, \zeta)$$ stands for the Confluent Hypergeometric Function and $\alpha$, $A$, $B$ are arbitrary constants. Observe that, if $g(y)$ is a real function, then

$$\ln \left( \frac{u(x, z)}{u^*(x, z)} \right) = 2i \left( \frac{kn_0}{2R(z)} x^2 - i\alpha \chi(z) \right),$$
(a) $\alpha = \frac{1}{2}$, (b) $\alpha = -\frac{1}{2}$

**Figure 2.** Refractive index $m(x, z)$ for $\Omega = 0.5$ and the values of $\alpha$ indicated in each case. In (a), the parabolic profile is only shifted by an oscillating function of period $2\pi z R$. In (b) the parabolic profile undergoes an oscillating deformation of the same period induced by the function $g(y)$ in (26). In these plots we have chosen the divergency $\frac{w_0^2}{z R} = \frac{\pi}{6}$ and $n_0 = 1.5$. The transversal and longitudinal coordinates are measured, respectively, in units of $w_0$ and $z R$.

meaning that the reality condition (16) is fulfilled for any real values of $A$ and $B$. Moreover, as

$$\text{Im} \left( \frac{\partial^2}{\partial x^2} u(x, z) \right) = \frac{1}{2i} \frac{\partial^2}{\partial x^2} \ln \left( \frac{u(x, z)}{u^*(x, z)} \right) = k_0 n_0 \left( \frac{d}{dz} \ln w(z) \right),$$

the expression (14) for $L_1(z)$ can be readily evaluated to yield

$$L_1(z) = w(z). \quad (21)$$

As an explicit example, let us assume that $\alpha = \frac{1}{2}$. Then (20) reduces to

$$g(y) = e^{-\frac{y^2}{2}} \left( A - i \frac{\sqrt{\pi}}{2} B \text{ Erf}(iy) \right), \quad (22)$$

where Erf($\zeta$) is the Error Function (see, for instance, [20]). In this case, the only nodeless function that can be constructed requires that $B = 0$. As the functions $L_1$ and $m$ will not depend on $A$, we may set $A = 1$ without loss of generality and then, the resulting function $u(x, z)$ coincides with the zero-order Hermite-Gaussian mode $U_0(x, z)$ given by (5) (compare to [17,21]). After some calculations we obtain

$$L = w(z) \left[ \frac{\partial}{\partial x} - 2i \left( \frac{k_0 n_0}{2R(z)} + \frac{i}{w^2(z)} \right) x \right] \quad (23)$$

$$m(x, z) = n_0 \left( 1 - \frac{1}{2} \Omega^2 x^2 \right) - \frac{4}{w^2(z)}. \quad (24)$$

Observe that this profile corresponds to the quadratic refractive index $n(x)$ displaced by a $z$-dependent oscillating shift fixed by the width of the beam. In Figure 2(a) it is depicted the refractive index $m(x, z)$ for $\Omega = \frac{1}{2} z R$. Note that the transversal parabolic profile is rigidly shifted by an oscillating displacement of period $\frac{\pi}{2}$.

In order to obtain square integrable solutions to the paraxial Helmholtz equation (7), we consider the complete set of solutions of (1) supplied by the Hermite-Gaussian modes $U_n(x, z)$. It is possible to show that

$$W_n(x, z) = LU_{n+1}(x, z) = 2\sqrt{n+1} \ e^{-i\chi(z)} U_n(x, z) \quad n = 0, 1, 2, \ldots \quad (25)$$
For this particular choice of \( u(x, z) \), the function \( \frac{1}{\sqrt{|w(z)|}} \) is not square integrable. Indeed, the set (25) span the Hilbert space of guided (localized) electromagnetic modes associated to the refractive index \( m(x, z) \).

Let us consider now the value \( \alpha = -\frac{1}{2} \). In this case we have
\[
g(y) = A + \frac{\sqrt{\pi}}{2} B \text{Erf} \left( \frac{\sqrt{2}}{w(z)} x \right),
\]
that leads to the transformation function
\[
\begin{align*}
   u(x, z) &= \frac{1}{\sqrt{w(z)}} e^{-\frac{i}{2} \chi(z)} e^{i \frac{k_0 n_0}{2R(z)} x} \left\{ A + \frac{\sqrt{\pi}}{2} B \text{Erf} \left( \frac{\sqrt{2}}{w(z)} x \right) \right\},
   \\
   m(x, z) &= n_0 \left\{ 1 - \frac{1}{2} \Omega^2 x^2 - \frac{2}{k_0^2 n_0^2 w^2(z)} + \frac{1}{k_0 n_0} \frac{\partial^2}{\partial x^2} \ln \left[ A + \frac{\sqrt{\pi}}{2} B \text{Erf} \left( \frac{\sqrt{2}}{w(z)} x \right) \right] \right\}
\end{align*}
\]

Observe that the first r.h.s. term of (28) coincides with the intertwining operator (23). Then the new mode amplitudes are given by
\[
W_{n+1}(x, z) = LU_n(x, z) = 2 \sqrt{m} e^{-i \chi(z)} U_{n-1}(x, z)
\]
\[
- w(z) \left\{ \frac{4}{w^2(z)} x + \frac{\partial}{\partial x} \left[ A + \frac{\sqrt{\pi}}{2} B \text{Erf} \left( \frac{\sqrt{2}}{w(z)} x \right) \right] \right\} U_n(x, z) \quad n = 0, 1, 2, \ldots (31)
\]

For this choice of \( \alpha \), the function \( W_0(x, z) = \frac{1}{L_{\chi(z)}(x, z)} \) is square integrable so that it must be considered in the complete set of solutions of (7).

In Figure 3 we show the intensity distributions for the first 3 Darboux-deformed Hermite-Gaussian modes. Observe that the maximum of the intensity of the zero-order mode is shifted to the positive values of \( x \) due to the sharp peak that the refractive index exhibits in this region (see Figure 2 (b)). As \( m(x, z) \) decreases faster for positive values of \( x \) that for negative ones, the maxima of the higher order modes are, rather, shifted to these values of \( x \).

5. Summary
We have applied the Darboux transformation to the paraxial Helmholtz equation for graded index media in order to generate new refractive index profiles depending also on the longitudinal variable. The particular case of a parabolic medium for which the wave-packet type solutions are known has been considered in order to construct new deformed confining refractive profiles. Due to the underlying algebraic structure of the parabolic medium problem, the expression for the transversal modes associated to these new profiles can be readily expressed in terms of the Hermite-Gaussian modes. Some examples have been presented in order to illustrate the self-focusing phenomena of transversal modes in these media.

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Figure 3. Intensity distributions for the Darboux-deformed Hermite-Gaussian modes $W_0(x, z)$ (top), $W_1(x, z)$ (middle) and $W_2(x, z)$ (bottom), for $A = -0.9$, $B = 1$ and the values of $\Omega$ indicated in each case. The transversal and longitudinal coordinates are measured, respectively, in units of $w_0$ and $z_R$.

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