Bishop–Phelps–Bollobás property for positive operators when the domain is $C_0(L)$

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Abstract

Recently it was introduced the so-called Bishop–Phelps–Bollobás property for positive operators between Banach lattices. In this paper we prove that the pair $(C_0(L), Y)$ has the Bishop–Phelps–Bollobás property for positive operators, for any locally compact Hausdorff topological space $L$, whenever $Y$ is a uniformly monotone Banach lattice with a weak unit. In case that the space $C_0(L)$ is separable, the same statement holds for any uniformly monotone Banach lattice $Y$. We also show the following partial converse of the main result. In case that $Y$ is a strictly monotone Banach lattice, $L$ is a locally compact Hausdorff topological space that contains at least two elements and the pair $(C_0(L), Y)$ has the Bishop–Phelps–Bollobás property for positive operators then $Y$ is uniformly monotone.

Keywords Banach space · Operator · Bishop–Phelps–Bollobás theorem · Bishop–Phelps–Bollobás property for positive operators

Mathematics Subject Classification Primary 46B20; Secondary 46B42

1 Introduction

Bishop–Phelps theorem [7] states that every continuous linear functional on a Banach space can be approximated (in norm) by norm attaining functionals. Bollobás proved a “quantitative version” of that result [8]. In order to state such result, we denote by $B_X$, $S_X$ and $X^*$ the
closed unit ball, the unit sphere and the topological dual of a Banach space $X$, respectively. If $X$ and $Y$ are both real or both complex Banach spaces, $L(X, Y)$ denotes the space of (bounded linear) operators from $X$ to $Y$, endowed with its usual operator norm.

**Bishop–Phelps–Bollobás theorem** (see [9, Theorem 16.1], or [10, Corollary 2.4]) Let $X$ be a Banach space and $0 < \varepsilon < 1$. Given $x \in B_X$ and $x^* \in B_{X^*}$ with $|1 - x^*(x)| < \frac{\varepsilon^2}{4}$, there are elements $y \in B_X$ and $y^* \in B_{X^*}$ such that $y^*(y) = 1$, $\|y - x\| < \varepsilon$ and $\|y^* - x^*\| < \varepsilon$.

In 2008 the study of extensions of the Bishop–Phelps–Bollobás theorem for operators was initiated by Acosta et al. [3]. The survey [1] contains many results on that topic. Some of the recent studies related to BPB property can be found in [11, 12, 15] and the references therein.

**Definition 1.1** [3, Definition 1.1] Let $X$ and $Y$ be either real or complex Banach spaces. The pair $(X, Y)$ is said to have the Bishop–Phelps–Bollobás property for positive operators if for every $0 < \varepsilon < 1$ there exists $0 < \eta(\varepsilon) < \varepsilon$ such that for every $S \in S_{L(X, Y)}$, if $x_0 \in S_X$ satisfies $\|S(x_0)\| > 1 - \eta(\varepsilon)$, then there exist an element $u_0 \in S_X$ and an operator $T \in S_{L(X, Y)}$ satisfying the following conditions

$$
\|T(u_0)\| = 1, \quad \|u_0 - x_0\| < \varepsilon, \quad \|T - S\| < \varepsilon.
$$

Recently in [4], the authors introduced a version of Bishop–Phelps–Bollobás property for positive operators between two Banach lattices. Let us mention that the only difference between this property and the previous one is that the new property the operators appearing in Definition 1.1 are positive. In the same paper it is shown that the pairs $(c_0, L_1(\mu))$ and $(L_\infty(\mu), L_1(\nu))$ have the Bishop–Phelps–Bollobás property for positive operators for any positive measures $\mu$ and $\nu$ (see [4, Theorems 1.7 and 1.6]). The paper [5] contains some extensions of those results. More precisely, it is proved that the pair $(c_0, Y)$ has the Bishop–Phelps–Bollobás property for positive operators whenever $Y$ is a uniformly monotone Banach lattice (see [5, Corollary 3.3]). It is also shown that the pair $(L_\infty(\mu), Y)$ has the Bishop–Phelps–Bollobás property for positive operators for any positive measure $\mu$ if $Y$ is a uniformly monotone Banach lattice with a weak unit (see [5, Corollary 2.6]).

The goal of this paper is to obtain a far reaching extension of those results. To be precise, we prove that the pair $(C_0(L), Y)$ has the Bishop-Phelps–Bollobás property for positive operators, for any locally compact Hausdorff topological space $L$, whenever $Y$ is a uniformly monotone Banach lattice with a weak unit. If $C_0(L)$ is separable, the same statement holds for any uniformly monotone Banach lattice $Y$. Further we show that these results are optimal in case that $Y$ is strictly monotone. That is, if $Y$ is a strictly monotone Banach lattice and $L$ is a locally compact Hausdorff topological space that contains at least two elements, if the pair $(C_0(L), Y)$ has the Bishop–Phelps–Bollobás property for positive operators then $Y$ is uniformly monotone.

**2 The results**

**Definition 2.1** [4, Definition 1.3] Let $X$ and $Y$ be Banach lattices. The pair $(X, Y)$ is said to have the Bishop–Phelps–Bollobás property for positive operators if for every $0 < \varepsilon < 1$ there exists $0 < \eta(\varepsilon) < \varepsilon$ such that for every $S \in S_{L(X, Y)}$, such that $S \geq 0$, if $x_0 \in S_X$ satisfies $\|S(x_0)\| > 1 - \eta(\varepsilon)$, then there exist an element $u_0 \in S_X$ and a positive operator $T \in S_{L(X, Y)}$ satisfying the following conditions

$$
\|T(u_0)\| = 1, \quad \|u_0 - x_0\| < \varepsilon \quad \text{and} \quad \|T - S\| < \varepsilon.
$$
In order to refine the result, we recall the following version of [2, Definition 1.3], which was introduced in [4].

**Definition 2.2** [4, Definition 1.3] Let $X$ and $Y$ be Banach lattices and $M$ a subspace of $L(X, Y)$. The subspace $M$ is said to have the Bishop–Phelps–Bollobás property for positive operators if for every $0 < \varepsilon < 1$ there exists $0 < \eta(\varepsilon) < \varepsilon$ such that for every $S \in S_M$, such that $S \geq 0$, if $x_0 \in S_X$ satisfies $\|S(x_0)\| > 1 - \eta(\varepsilon)$, then there exist an element $u_0 \in S_X$ and a positive operator $T \in S_M$ satisfying the following conditions

$$\|T(u_0)\| = 1, \quad \|u_0 - x_0\| < \varepsilon \quad \text{and} \quad \|T - S\| < \varepsilon.$$

We will use the notions of Banach function space and strictly monotone and uniformly monotone Banach lattice, that we recall now.

In case that $(\Omega, \mu)$ is a measure space, we denote by $L^0(\mu)$ the space of (equivalence classes of $\mu$-a.e. equal) real valued measurable functions on $\Omega$. We say that a Banach space $X$ is a Banach function space on $(\Omega, \mu)$ if $X$ is an ideal in $L^0(\mu)$ and whenever $x, y \in X$ and $|x| \leq |y|$ a.e., then $\|x\| \leq \|y\|$.

**Definition 2.3** Let $X$ be a real Banach lattice. $X$ is strictly monotone if for any $x, y \in X^+$ such that $x \leq y$ and $x \neq y$, it is satisfied that $\|x\| < \|y\|$. The Banach lattice $X$ is uniformly monotone, if for every $0 < \varepsilon < 1$, there is $0 < \delta \leq \varepsilon$ satisfying the following property

$$x, y \in X^+, \|x\| = 1, \|x + y\| \leq 1 + \delta \Rightarrow \|y\| \leq \varepsilon.$$

The following characterization of uniform monotonicity can be found in [5, Proposition 4.2].

**Proposition 2.4** Let $Y$ be a Banach lattice. The following conditions are equivalent.

1. $Y$ is uniformly monotone.
2. For every $0 < \varepsilon < 1$, there is $\eta(\varepsilon) > 0$ satisfying

$$u \in Y, \ v \in S_Y, \ 0 \leq u \leq v \ \text{and} \ \|v - u\| > 1 - \eta(\varepsilon) \Rightarrow \|u\| \leq \varepsilon.$$

3. For every $0 < \varepsilon < 1$, there is $\eta(\varepsilon) > 0$ satisfying

$$u, v \in Y, \ 0 \leq u \leq v \ \text{and} \ \|v - u\| > (1 - \eta(\varepsilon))\|v\| \Rightarrow \|u\| \leq \varepsilon\|v\|.$$

**Lemma 2.5** [5, Lemma 2.4] Let $Y$ be a uniformly monotone Banach function space and $0 < \varepsilon < 1$. Assume that $f_1$ and $f_2$ are positive elements in $Y$ such that

$$\|f_1 + f_2\| \leq 1 \quad \text{and} \quad \frac{1}{1 + \delta (\frac{\varepsilon}{4})} \leq \|f_1 - f_2\|,$$

where $\delta$ is the function satisfying the definition of uniform monotonicity for $Y$. Then there are two positive functions $h_1$ and $h_2$ in $Y$ with disjoint supports satisfying that

$$\|h_1 + h_2\| = 1 \quad \text{and} \quad \|h_i - f_i\| < \varepsilon \quad \text{for} \quad i = 1, 2.$$

**Proposition 2.6** Let $L$ be a locally compact Hausdorff space and $Y$ be a Banach lattice. Assume that $S_1$ and $S_2$ are positive operators in $L(C_0(L), Y)$ and $g_1$ and $g_2$ are positive elements in $C_0(L)$. Then the operator $U : C_0(L) \rightarrow Y$ defined by

$$U(f) = S_1(fg_1) + S_2(fg_2) \quad (f \in C_0(L))$$

satisfies $\|U\| = \|S_1(g_1) + S_2(g_2)\|$. 
Proof If \( f \in B_{C_0(L)} \), we have
\[
0 \leq |f| g_i \leq g_i, \quad i = 1, 2.
\]
Since \( S_i \) is positive for \( i = 1, 2 \), we have that
\[
0 \leq U(|f|) \leq S_1(g_1) + S_2(g_2),
\]
so
\[
\|U(f)\| \leq \|U(|f|)\| \leq \|S_1(g_1) + S_2(g_2)\|.
\]
As a consequence,
\[
\|U\| \leq \|S_1(g_1) + S_2(g_2)\|.
\]

Given \( \varepsilon > 0 \), the subset \( K \) given by
\[
K = \{ t \in L : g_1(t) \geq \varepsilon \} \cup \{ t \in L : g_2(t) \geq \varepsilon \}
\]
is compact. By Urysohn lemma there is a function \( g \in C_0(L) \) such that \( 0 \leq g \leq 1 \) and such that \( g(K) = \{ 1 \} \). So we have that
\[
\|g_i - gg_i\|_{\infty} \leq \varepsilon, \quad i = 1, 2.
\]
Therefore
\[
\|U\| \geq \|U(g)\| = \|S_1(gg_1) + S_2(gg_2)\| \geq \|S_1(g_1) + S_2(g_2)\| - \varepsilon(\|S_1\| + \|S_2\|).
\]

By taking limit as \( \varepsilon \to 0 \) we obtain that \( \|U\| \geq \|S_1(g_1) + S_2(g_2)\| \) and the proof is finished. \( \square \)

Lemma 2.7 Let \( L \) be a locally compact Hausdorff space and \( Y \) be a uniformly monotone Banach lattice. Let \( \delta \) be the function satisfying the definition of uniform monotonicity for the Banach lattice \( Y \) and \( 0 < \eta < 1 \). Assume that \( f_0 \in S_{C_0(L)} \) and \( S \in S_{L(C_0(L), Y)} \), \( S \geq 0 \) and
\[
\|S(f_0)\| > \frac{1}{1 + \delta(\eta^2)}.
\]
Define the sets \( A_1, A_2, B_1 \) and \( B_2 \) by
\[
A_1 = \left\{ t \in L : -1 \leq f_0(t) < -1 + \frac{\eta}{2} \right\}, \quad A_2 = \left\{ t \in L : -1 + \frac{\eta}{2} \leq f_0(t) < -1 + \eta \right\},
\]
\[
B_1 = \left\{ t \in L : -1 + \frac{\eta}{2} < f_0(t) \leq 1 \right\}, \quad B_2 = \left\{ t \in L : 1 - \eta < f_0(t) \leq 1 - \frac{\eta}{2} \right\}.
\]
There are positive functions \( g_1 \) and \( g_2 \) in \( B_{C_0(L)} \) satisfying the following assertions
(a) \( g_1|_{A_1} = 1 \) and \( g_1|_{L \setminus (A_1 \cup A_2)} = 0 \).
(b) \( g_2|_{B_1} = 1 \) and \( g_2|_{L \setminus (B_1 \cup B_2)} = 0 \).
(c) \( \|g_1 + f_0g_1\|_{\infty} \leq \eta \) and so \( \|S(g_1) + S(f_0g_1)\| \leq \eta \).
(d) \( \|f_0g_2 - g_2\|_{\infty} \leq \eta \), so \( \|S(f_0g_2) - S(g_2)\| \leq \eta \).
(e) \( \|S(h)\| \leq 2\eta \) for every element \( h \in B_{C_0(L)} \) such that \( h|_{A_1 \cup B_1} = 0 \). As a consequence \( \|S(fg_1 + fg_2 - f)\| \leq 2\eta \|f\|_{\infty} \) for every \( f \in C_0(L) \).
Proof We can define the functions \( g_1 \) and \( g_2 \) as follows

\[
\begin{align*}
g_1(t) &= \begin{cases} 
1 & \text{if } -1 < f_0(t) < -1 + \frac{\eta}{2} \\
-\frac{2}{\eta} (f_0(t) + 1 - \eta) & \text{if } -1 + \frac{\eta}{2} < f_0(t) < -1 + \eta \\
0 & \text{if } -1 + \eta \leq f_0(t) \leq 1,
\end{cases} \\
g_2(t) &= \begin{cases} 
0 & \text{if } -1 \leq f_0(t) \leq 1 - \eta \\
\frac{2}{\eta} (f_0(t) - 1 + \eta) & \text{if } 1 - \eta < f_0(t) \leq 1 - \frac{\eta}{2} \\
1 & \text{if } 1 - \frac{\eta}{2} < f_0(t) \leq 1.
\end{cases}
\end{align*}
\]

It is immediate to check that \( g_1 \) and \( g_2 \) are positive functions in \( BC_{c} (\mathbb{L}) \) satisfying the conditions stated in \( a, \ b, \ c \) and \( d \). If \( h \in BC_{c} (\mathbb{L}) \) satisfies \( h|_{A_{1} \cup B_{1}} = 0 \) we clearly have that \( |f_0| + \frac{\eta}{2} |h| \in \mathbb{L} \). By using that \( \mathbb{S} \) is a positive operator and the assumption we have that

\[
\begin{align*}
\| S(|f_0| + \frac{\eta}{2} |h|) \| & \leq 1 \\
& < \| S(f_0) \| (1 + \delta(\eta^2)) \\
& \leq \| S(\| f_0 \|) \| (1 + \delta(\eta^2)).
\end{align*}
\]

By using the uniform monotonicity of \( Y \) we obtain that \( \| S(\frac{\eta}{2} |h|) \| \leq \eta^2 \) and so

\[
\| S(h) \| \leq \| S(|h|) \| \leq 2 \eta.
\]

Finally notice that \( (g_1 + g_2 - 1)(A_{1} \cup B_{1}) = \{0\} \) and \( \| g_1 + g_2 - 1 \|_{\infty} \leq 1 \). So from the previous part we conclude for every \( f \in C_{0} (\mathbb{L}) \), \( \| S(f g_1 + f g_2 - f) \| \leq 2 \eta \| f \|_{\infty} \). □

**Theorem 2.8** The pair \( (C_{0} (\mathbb{L}), Y) \) has the Bishop-Phelps–Bollobás property for positive operators, for any locally compact Hausdorff topological space \( \mathbb{L} \), whenever \( Y \) is a uniformly monotone Banach function space. The function \( \eta \) satisfying Definition 2.1 depends only on the function \( \delta \) appearing in Definition 2.3.

**Proof** Assume that \( Y \) is a Banach function space on a measure space \( (\Omega, \mu) \). Let \( 0 < \varepsilon < 1 \) and \( \delta \) be the function satisfying the definition of uniform monotonicity for the Banach function space \( Y \). Choose a real number \( \eta \) such that \( 0 < \eta = \eta(\varepsilon) < \frac{\varepsilon}{2} \) and satisfying also

\[
\frac{1}{1 + \delta(\frac{\varepsilon}{2})} < \frac{1}{1 + \delta(\eta^2)} - 4 \eta.
\]

Assume that \( f_0 \in S_{C_{0} (\mathbb{L})} \), \( S \in S_{L(C_{0} (\mathbb{L}), Y)} \) and \( S \) is a positive operator such that

\[
\| S(f_0) \| > \frac{1}{1 + \delta(\eta^2)}.
\]

Hence we can apply Lemma 2.7 and so there are positive functions \( g_1 \) and \( g_2 \) in \( BC_{c} (\mathbb{L}) \) satisfying all the conditions stated in Lemma 2.7, therefore

\[
\begin{align*}
\| S(g_1) - S(g_2) \| & \geq \| S(f_0 g_1) + S(f_0 g_2) \| - \| S(g_1) + S(f_0 g_1) \| - \| S(f_0 g_2) - S(g_2) \| \\
& \geq \| S(f_0) \| - \| S(f_0 g_1 + f_0 g_2 - f_0) \| - 2 \eta \\
& \geq \frac{1}{1 + \delta(\eta^2)} - 4 \eta \\
& > \frac{1}{1 + \delta(\frac{\varepsilon}{2})}.
\end{align*}
\]
Since $S$ is a positive operator and $\|S(g_1) + S(g_2)\| \leq 1$, in view of (2.1) we can apply Lemma 2.5. Hence there are two positive functions $h_1$ and $h_2$ in $Y$ satisfying the following conditions

\[
\|h_1 - S(g_1)\| < \frac{\varepsilon}{6}, \quad \|h_2 - S(g_2)\| < \frac{\varepsilon}{6},
\]  

\[
\text{supp} \ h_1 \cap \text{supp} \ h_2 = \emptyset \quad \text{and} \quad \|h_1 + h_2\| = 1.
\]  

Hence

\[
\|S(g_1)\chi_{\Omega \setminus \text{supp} \ h_1}\| = \|(h_1 - S(g_1))\chi_{\Omega \setminus \text{supp} \ h_1}\|
\leq \|h_1 - S(g_1)\|
< \frac{\varepsilon}{6} \quad \text{(by (2.2))}
\]  

and

\[
\|S(g_2)\chi_{\Omega \setminus \text{supp} \ h_2}\| = \|(h_2 - S(g_2))\chi_{\Omega \setminus \text{supp} \ h_2}\|
\leq \|h_2 - S(g_2)\|
< \frac{\varepsilon}{6} \quad \text{(by (2.2))}.
\]  

Now we define the operator $U : C_0(L) \rightarrow Y$ as follows

\[
U(f) = S(fg_1)\chi_{\text{supp} \ h_1} + S(fg_2)\chi_{\text{supp} \ h_2} \quad (f \in C_0(L)).
\]  

Since $Y$ is a Banach function space and $S \in L(C_0(L), Y)$, $U$ is well defined and belongs to $L(C_0(L), Y)$. The operator $U$ is positive since $g_1$ and $g_2$ are positive elements in $C_0(L)$ and $S$ is a positive operator. For any $f \in B_{C_0(L)}$ we have that

\[
\|(U - S)(f)\| = \|S(fg_1)\chi_{\text{supp} \ h_1} + S(fg_2)\chi_{\text{supp} \ h_2} - S(f)\|
= \|S(fg_1)\chi_{\text{supp} \ h_1} + S(fg_2)\chi_{\text{supp} \ h_2} - S(fg_1 + fg_2) + S(fg_1 + fg_2 - f)\|
\leq \|S(g_1)\chi_{\Omega \setminus \text{supp} \ h_1}\| + \|S(g_2)\chi_{\Omega \setminus \text{supp} \ h_2}\| + \|S(fg_1 + fg_2 - f)\|
< \frac{\varepsilon}{3} + 2\eta < \frac{\varepsilon}{2} \quad \text{(by (2.4), (2.5) and item (e) of Lemma (2.7))}.
\]  

Hence

\[
\|\|U\| - 1\| < \frac{\varepsilon}{2},
\]  

so $U \neq 0$.

Finally we define $T = \frac{U}{\|U\|}$. Since $U$ is a positive operator, $T$ is also positive. Of course $T \in S_{L(C_0(L), Y)}$ and also satisfies

\[
\|T - S\| \leq \|T - U\| + \|U - S\|
< \left\| \frac{U}{\|U\|} - U \right\| + \frac{\varepsilon}{2} \quad \text{(by (2.6))}
= \left\| \frac{1}{\|U\|} - 1 \right\| + \frac{\varepsilon}{2}
< \varepsilon \quad \text{(by (2.7))}.
\]
The function $f_1$ given by

$$f_1(t) = \begin{cases} 
1 & \text{if } 1 - \eta < f_0(t) \leq 1 \\
\frac{f_0(t)}{1 - \eta} & \text{if } |f_0(t)| \leq 1 - \eta \\
-1 & \text{if } -1 \leq f_0(t) < -1 + \eta
\end{cases}$$

belongs to $SC_0(L)$ and satisfies that

$$\|f_1 - f_0\|_\infty \leq \eta < \varepsilon. \quad (2.9)$$

We clearly have that

$$U(f_1) = S(g_2)\chi_{\text{supp } h_2} - S(g_1)\chi_{\text{supp } h_1}.$$ 

Since $S \geq 0$ and $g_1$ and $g_2$ are positive functions, in view of Proposition 2.6 we have that

$$\|U\| = \|S(g_1)\chi_{\text{supp } h_1} + S(g_2)\chi_{\text{supp } h_2}\|.$$

Since $h_1$ and $h_2$ have disjoint supports, for each $x \in \Omega$ we obtain that

$$\left| (S(g_1)\chi_{\text{supp } h_1} + S(g_2)\chi_{\text{supp } h_2})(x) \right| = \left| (U(f_1))(x) \right|.$$ 

Since $Y$ is a Banach function space we conclude that

$$\|U\| = \|S(g_1)\chi_{\text{supp } h_1} + S(g_2)\chi_{\text{supp } h_2}\| = \|S(g_1)\chi_{\text{supp } h_1} - S(g_2)\chi_{\text{supp } h_2}\| = \|U(f_1)\|.$$ 

By (2.9) and (2.8), since $T$ attains its norm at $f_1$, the proof is finished \(\square\)

Now we will improve the statement in Theorem 2.8. For that purpose we use operators in an operator ideal, a notion that we recall below.

**Definition 2.9** [13, Definition 9.1] An operator ideal $\mathcal{A}$ is a subclass of the class $L$ of all continuous linear operators between Banach spaces such that for all Banach spaces $X$ and $Y$ its components

$$\mathcal{A}(X, Y) = L(X, Y) \cap \mathcal{A}$$

satisfy

(1) $\mathcal{A}(X, Y)$ is a linear subspace of $L(X, Y)$ which contains the finite rank operators.

(2) The ideal property: If $S \in \mathcal{A}(X_0, Y_0)$, $R \in L(X, X_0)$ and $T \in L(Y_0, Y)$, then the composition $TSR$ is in $\mathcal{A}(X, Y)$.

We used in the previous proof that for a Banach function space $Y$ on a measure space $(\Omega, \mu)$, for any measurable set $A \subset \Omega$, the operator $h \mapsto h\chi_A$ is linear and bounded. So in case that $\mathcal{I}$ is some operator ideal and the operator $S$ in the proof of Theorem 2.8 belongs to $\mathcal{I}(C_0(L), Y)$, then the operator $U$ also belongs to $\mathcal{I}(C_0(L), Y)$. Hence we obtain the following result.

**Corollary 2.10** Under the assumptions of Theorem 2.8, if $\mathcal{I}$ is some operator ideal, then the ideal $\mathcal{I}(C_0(L), Y)$ has the Bishop–Phelps–Bollobás property for positive operators.
Theorem 2.8 is a far reaching extension of [5, Theorems 2.5 and 3.2], where the same result was proved in case that the domain space is a $L_\infty$ or $c_0$ space. Our purpose now is to obtain a version of Theorem 2.8 for some abstract Banach lattices. In order to get this result, we use that every uniformly monotone Banach lattice is order continuous (see [6, Theorem 21, p. 371] and [14, Proposition 1.a.8]). Also any order continuous Banach lattice with a weak unit is order isometric to a Banach function space (see [14, Theorem 1.b.14]). From Theorem 2.8 and the previous argument we deduce the following result.

**Corollary 2.11** The pair $(C_0(L), Y)$ has the Bishop–Phelps–Bollobás property for positive operators, for any locally compact Hausdorff topological space $L$ whenever $Y$ is a uniformly monotone Banach lattice with a weak unit. Moreover, the function $\eta$ satisfying Definition 2.1 depends only on the modulus of uniform monotonicity of $Y$.

By using the previous result and the same argument of [5, Corollary 3.3] we obtain the next statement:

**Corollary 2.12** For any locally compact Hausdorff topological space $L$ such that $C_0(L)$ is separable, The pair $(C_0(L), Y)$ has the Bishop–Phelps–Bollobás property for positive operators whenever $Y$ is a uniformly monotone Banach lattice. Moreover, the function $\eta$ satisfying Definition 2.1 depends only on the modulus of uniform monotonicity of $Y$.

Our intention now is to provide a class of Banach lattices for which the previous result is in fact a characterization. The proof of the next result is a refinement of the arguments used in [5, Proposition 4.3], where it is assumed that the domain has a non-trivial $M$-summand.

**Proposition 2.13** Let $Y$ be a strictly monotone Banach lattice and $L$ a locally compact Hausdorff topological space that contains at least two elements. If the pair $(C_0(L), Y)$ has the BPBp for positive operators then $Y$ is uniformly monotone.

**Proof** It suffices to show that $Y$ satisfies condition (2) in Proposition 2.4 in case that the pair $(C_0(L), Y)$ has the BPBp for positive operators.

Let $0 < \varepsilon < 1$. Let us take elements $u$ and $v$ in $Y$ such that

$$0 \leq u \leq v, \quad \|v\| = 1 \quad \text{and} \quad \|v - u\| > 1 - \eta(\varepsilon),$$

where $\eta$ is the function satisfying the definition of BPBp for positive operators for the pair $(C_0(L), Y)$.

Since $L$ contains at least two elements there are $t_i \in L$ and $f_i \in SC_0(L)$ for $i = 1, 2$ such that

$$0 \leq f_i \leq 1, \quad f_i(t_i) = 1, \quad i = 1, 2, \quad \text{and} \quad \text{supp } f_1 \cap \text{supp } f_2 = \emptyset.$$

Define the operator $S$ from $C_0(L)$ to $Y$ given by

$$S(f) = f(t_1)(v - u) + f(t_2)u, \quad (f \in C_0(L)).$$

Since $v - u$ and $u$ are positive elements in $Y$, $S$ is a positive operator from $C_0(L)$ to $Y$.

By using that $v - u$ and $u$ are positive elements in $Y$, for any element $f \in B_{C_0(L)}$ we have that

$$\|S(f)\| \leq \| |f(t_1)| (v - u) + |f(t_2)| u\| \leq \|v\| = 1.$$
Since we also have that $f_1 + f_2 \in SC_0(L)$ and $S(f_1 + f_2) = v \in S_Y$, we obtain that $S \in SL(C_0(L), Y)$. Notice also that $\|S(f_1)\| = \|v - u\| > 1 - \eta(\epsilon)$. By using that the pair $(C_0(L), Y)$ has the BPBP for positive operators and [5, Remark 2.2], there exist a positive operator $T \in SL(C_0(L), Y)$ and a positive element $g \in SC_0(L)$ satisfying

$$\|T(g)\| = 1, \quad \|T - S\| < \epsilon \quad \text{and} \quad \|g - f_1\| < \epsilon.$$  

Hence, if $t \in L \setminus \text{supp } f_1$ we have that $|g(t)| < \epsilon$.

By using that $T$ is a positive and normalized operator we get that

$$1 = \|T(g)\| \leq \|T(g + (1 - \epsilon)f_2)\| \leq 1.$$  

Since $Y$ is strictly monotone we obtain that $T(f_2) = 0$. As a consequence,

$$\|u\| = \|S(f_2)\| = \|(S - T)(f_2)\| \leq \|S - T\| < \epsilon.$$  

In view of Proposition 2.4 we proved that $Y$ is uniformly monotone. 

Taking into account the previous result and Corollary 2.11 we obtain the following characterization.

**Corollary 2.14** Let $Y$ be a strictly monotone Banach lattice and $L$ a locally compact Hausdorff topological space that contains at least two elements. If the pair $(C_0(L), Y)$ has the BPBP for positive operators then $Y$ is uniformly monotone. In case that $Y$ has a weak unit the converse is also true.

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