Coded Data Rebalancing for Distributed Data Storage Systems with Cyclic Storage

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Abstract

We consider replication-based distributed storage systems in which each node stores the same quantum of data and each data bit stored has the same replication factor across the nodes. Such systems are referred to as balanced distributed databases. When existing nodes leave or new nodes are added to this system, the balanced nature of the database is lost, either due to the reduction in the replication factor, or the non-uniformity of the storage at the nodes. This triggers a rebalancing algorithm, that exchanges data between the nodes so that the balance of the database is reinstated. The goal is then to design rebalancing schemes with minimal communication load. In a recent work by Krishnan et al., coded transmissions were used to rebalance a carefully designed distributed database from a node removal or addition. These coded rebalancing schemes have optimal communication load, however, require the file-size to be at least exponential in the system parameters. In this work, we consider a cyclic balanced database (where data is cyclically placed in the system nodes) and present coded rebalancing schemes for node removal and addition in such a database. These databases (and the associated rebalancing schemes) require the file-size to be only cubic in the number of nodes in the system. We bound the advantage of our node removal rebalancing scheme over the uncoded scheme, and show that our scheme has a smaller communication load. In the node addition scenario, the rebalancing scheme presented is a simple uncoded scheme, which we show has optimal load. Finally, we derive a lower bound for the single node-removal rebalancing for the specific choice of data placements specified by our achievable rebalancing schemes, and show that our achievable rebalancing loads are within a multiplicative gap from the lower bound obtained.

I. INTRODUCTION

In replication-based distributed storage systems, the available data is stored in a distributed fashion in the storage nodes with some replication factor. Doing this helps prevent data loss in case of node failures, and also provides for greater data availability and thus higher throughputs. In [1], replication-based distributed storage systems in which (A) each bit is available in the same number of nodes (i.e., the replication factor of each bit is the same) and (B) each node stores the same quantum of data, were referred to as balanced distributed databases. In such databases,
when a storage node fails, or when a new node is added, the ‘balanced’ nature of the database is disturbed (i.e., the properties (A) or (B) do not hold anymore); this is known as data skew. Data skew results in various issues in the performance of such distributed databases (see Section 1 of [1]). Correcting such data skew requires some communication between the nodes of the database. In distributed systems literature, this communication phase is known as data rebalancing (see, for instance, [2]–[5]). In traditional distributed storage systems, an uncoded rebalancing phase is initiated, where uncoded bits are exchanged between the nodes to recreate the balanced conditions in the new collection of nodes. Clearly, a primary goal in such a scenario would be to minimize the communication load on the network during the rebalancing phase.

The rebalancing problem was formally introduced in the information theoretic setting in [1] by Krishnan et al. The idea of coded data rebalancing was presented in [1], based on principles similar to the landmark paper on coded caching [6]. In coded data rebalancing, coded data bits are exchanged between the nodes; the decoding of the required bits can then be done by using the prior stored data available. In [1], coded data rebalancing schemes were presented to rectify the data skew and reinstate the replication factor, in case of a single node removal or addition, for a carefully designed balanced database. The communication loads for these rebalancing schemes were characterized and shown to offer multiplicative benefits over the communication required for uncoded rebalancing. Information theoretic converse results for the communication loads were also presented in [1], proving the optimality of the achievable loads. These results were extended to the setting of decentralized databases in [7], where each bit of the file is placed in some random subset of the $K$ nodes.

While Krishnan et al. [1] present an optimal scheme for the coded data rebalancing problem, the centralized database design in [1] requires that the number of segments in the data file be a large function of the number of nodes (denoted by $K$) in the system. In fact, as $K$ grows, the number of file segments, and thus also the file size, have to grow exponentially in $K$. Thus, this scheme would warrant a high level of coordination to construct the database and perform the rebalancing. Because of these reasons, the scheme in [1] could be impractical in real-world settings. Motivated by this, in the present work, we study the rebalancing problem for cyclic balanced databases, in which each segment of the data file is placed in a consecutive set of nodes, in a wrap-around fashion. For such cyclic placement, the number of segments of the file could be as small as linear in $K$. Constructing such cyclic databases and designing rebalancing schemes for them also may not require much coordination, owing to the simplicity of the cyclic placement technique. Such cyclic storage systems have been proposed for use in distributed systems [8], [9], as well as in recent works on information theoretic approaches to private information retrieval [10] and distributed computing [11].

We now describe the organization and contributions of this work. Section II sets up the formalism and gives the main result (Theorem 1) of this work on rebalancing for single node removal or addition in a cyclic balanced database. Sections III and IV are devoted to proving this main result. In Section III, we present a coded data rebalancing algorithm (Algorithm 1) for node removal in balanced cyclic databases. Algorithm 1 chooses between two coded transmission schemes (Scheme 1 and Scheme 2) based on the system parameters. Each of the two schemes has lower communication load than the other in certain parameter regimes determined by the values of $K$ and the replication factor $r$. We bound the advantage of our schemes over the uncoded scheme, and show that
the minimum of their communication loads is always strictly smaller than the uncoded rebalancing scheme, which does not permit coded transmissions. Further, the segmentation required for the scheme is only quadratic in \( K \), and the size of the file itself is required only to be cubic in \( K \) (thus much smaller than that of [1]). In Section IV, we present a rebalancing scheme for addition of a single node to the cyclic database, and show that its load is optimal. In Section V, we derive a lower bound (Theorem 2) for rebalancing load for the single node removal scenario, and show that our achievable rebalancing loads are within a multiplicative gap from this lower bound. In Section VI, we show the numerical comparisons of the lower bound with the rebalancing load given in Theorem 1 and the optimal load from [1]. We conclude this work in Section VII with a short discussion on future work.

Notation: For a non-negative integer \( n \), we use \([n]\) to denote the set \( \{1, \ldots, n\} \). We also define \([0] \triangleq \emptyset\). Similarly, for a positive integer \( n \), \((n)\) denotes the set \( \{0,1,\ldots,n-1\} \). To describe operations involving wrap-arounds, we define two operators. For positive integers \( i, j, K \) such that \( i, j \leq K \), \( i \overset{K}{\oplus} j = \begin{cases} i+j, & \text{if } i+j \leq K \\ i+j-K, & \text{if } i+j > K \end{cases} \). Similarly, \( i \overset{K}{\ominus} j = \begin{cases} i-j, & \text{if } i-j > 0 \\ i-j+K, & \text{if } i-j \leq 0 \end{cases} \). We also extend these operations to sets. For \( A \subseteq [K] \) and \( i \in [K] \), we use \( i \overset{K}{\oplus} A \) to denote the set \( \{i \overset{K}{\oplus} a : a \in A\} \). Similarly, \( i \overset{K}{\ominus} A \) denotes \( \{i \overset{K}{\ominus} a : a \in A\} \).

For a binary vector \( X \), we use \(|X|\) to denote its size. The concatenation of two binary vectors, \( X_1 \) and \( X_2 \), is denoted by \( X_1 | X_2 \). In a directed graph \( G \), the subgraph induced by a subset \( A \) of its vertices is the graph with \( A \) as its vertices and all the edges in \( G \) between vertices in \( A \) as its edges.

II. Rebalancing Schemes for Cyclic Databases

In this section, we give the formal definition of cyclic databases and present our main result (Theorem 1) on rebalancing schemes for node removal and addition for such databases. Towards that end, we recall the formal system model and other relevant definitions from [1].

Consider a binary file \( W \) consisting of a set of \( N \) equal-sized segments where the \( i \)th segment is denoted by \( W_i \), where \(|W_i| = T\) bits. The system consists of \( K \) nodes indexed by \([K]\) and each node \( k \in [K] \) is connected to every other node in \([K] \setminus \{k\}\) via a bus link that allows a noise-free broadcast between the nodes. A distributed database of \( W \) across nodes indexed by \([K]\) is a collection of subsets \( D = \{D_k \subseteq \{W_i : i \in [N]\} : k \in [K]\} \), such that \( \bigcup_{k \in [K]} D_k = \{W_i : i \in [N]\} \), where \( D_k \) denotes the set of segments stored at node \( k \). We will denote the set of nodes where \( W_i \) is stored as \( S_i \). The replication factor of the segment \( W_i \) is then \(|S_i|\). The distributed database \( D \) is said to be \( r \)-balanced, if \(|S_i| = r, \forall i\), and \(|D_k| = \frac{N}{r}, \forall k\). That is, each segment is stored in \( r \) nodes, and each node stores an equal \( \frac{1}{r} \)-fraction of the \( N \) segments. We may assume without loss of generality that \( 2 \leq r \leq K-1 \), since if \( r = 1 \) no rebalancing is possible from node removal, and if \( r = K \) no rebalancing is required.

When a node \( k \) is removed from the \( r \)-balanced database, the replication factor of the segments \( D_k \) stored in node \( k \) drops by one, thus disturbing the ‘balanced’ state of the database. If a new empty node \( K+1 \) is added to the database, once again, the new database is not balanced. To reinstate the balanced state, a rebalancing scheme is initiated. Formally, a rebalancing scheme is a collection of transmissions between the nodes present, such that upon decoding the transmissions, the final database denoted by \( D' \) (on nodes \([K] \setminus \{k\}\) in case of node removal, or on
nodes \([K + 1]\) in case of node addition) is another \(r\)-balanced database. Let \(X_k' = \phi_k(D_k')\) be the transmission by node \(k'\) during the rebalancing phase, where \(\phi_k\) represents some encoding function. The communication loads of rebalancing (denoted by \(L_{\text{rem}}(r)\) for the case of node removal, \(L_{\text{add}}(r)\) for node addition) are then defined as

\[
L_{\text{rem}}(r) = \sum_{k' \in [K] \setminus \{k\}} \frac{|X_k'|}{T},
\]

\[
L_{\text{add}}(r) = \sum_{k \in [K]} \frac{|X_k|}{T}.
\]

Now, it is easy to see that the rebalancing load for node removal using the uncoded scheme is \(L_u(r) = r\). The normalized communication loads are then defined as \(\xi_{\text{rem}}(r) = L_{\text{rem}}(r)/N\) and \(\xi_{\text{add}}(r) = L_{\text{add}}(r)/N\). The optimal normalized communication loads for the node removal and addition scenarios are denoted by \(\xi_{\text{rem}}^*(r)\) and \(\xi_{\text{add}}^*(r)\) respectively. Here, the optimality is by minimization across all possible initial and target (final) databases, and all possible rebalancing schemes. In [1], it was shown\(^1\) that \(\xi_{\text{rem}}^*(r) \geq \frac{r}{K(r-1)}\) and \(\xi_{\text{add}}^*(r) \geq \frac{r}{K+1}\). Further, schemes for rebalancing were presented for node removal and addition for a carefully designed database which required \(N = \frac{(K+1)!}{r}\) and \(T = r - 1\), which achieve these optimal loads. Observe that, in these achievable schemes, the file size \(NT\) grows (at least) exponentially in \(K\) as \(K\) grows, for any fixed replication factor \(r\), which is one of the main drawbacks of this result. Therefore, our interest lies in databases where \(N\) and \(T\) are small. Towards this end, we now define cyclic databases.

**Definition 1.** A distributed database is an \(r\)-balanced cyclic database if \(N = K\) and a segment labelled \(W_i\) is stored precisely in the nodes \(S_i = \{i \oplus K <r>\}\).

Fig. 1 depicts an \(r\)-balanced cyclic balanced database on \(K\) nodes as defined above. In this work, we present rebalancing schemes for node removal and addition, for such a cyclic database on \(K\) nodes. Specifically, we prove the following result.

**Theorem 1.** For an \(r\)-balanced cyclic database having \(K\) nodes and \(r \in \{3, \ldots, K-1\}\), if the segment size \(T\) is divisible by \(2(K^2 - 1)\), then rebalancing schemes for node removal and addition exist which achieve the respective

\(^1\)The size of the file in the present work is \(NT\) bits; whereas in [1], the notation \(N\) represents the file size in bits, thus absorbing both the segmentation and the size of each segment. The definitions of communication loads in [1] are also slightly different, involving a normalization by the storage size of the removed (or added) node. The results of [1] are presented here according to our current notations and definitions.
communication loads

\[ L_{\text{rem}}(r) = \frac{K - r}{(K - 1)} + \min \left( L_1(r), L_2(r) \right) , \]

\[ L_{\text{add}}(r) = \frac{rK}{K + 1} , \]

where, \( L_1(r) = \frac{(K-r)(2r-1)}{(K-1)} \) and \( L_2(r) = \frac{1}{2(K-1)} \left( K(r-1) + \left\lceil \frac{r^2-2r}{2} \right\rceil \right) \).

Also, the following relationship holds between \( L_{\text{rem}}(r) \) and the load \( L_u(r) \) of the uncoded rebalancing scheme for node removal

\[ \frac{L_{\text{rem}}(r)}{L_u(r)} < \min \left( 2 \left( 1 - \frac{r-1}{K-1} \right), \left( \frac{1}{2} + \frac{1}{2r} + \frac{r}{4(K-1)} \right) \right) , \]

where the RHS term is strictly smaller than 1. Further, the rebalancing scheme for node addition is optimal (i.e., \( L_{\text{add}}(r)/N = \Sigma_{\text{add}}^*(r) = \frac{K}{K+1} \)).

**Remark 1.** In the proof of the node removal part of Theorem 1, we assume that the target database is also cyclic. For the case of \( r = 2 \), with this target database, our rebalancing scheme does not apply, as coding opportunities do not arise. Hence, we restrict our result to the scenario of \( r \in \{3, \ldots, K-1\} \).

Theorem 1 is proved via Sections III and IV. In Section III, we prove the result in Theorem 1 regarding the node removal scenario. We present a rebalancing algorithm, the core of which is a transmission phase in which coded subsegments are communicated between the nodes. In the transmission phase, the algorithm chooses between two schemes, Scheme 1 and Scheme 2. Scheme 1 has the communication load \( \frac{K - r}{K - 1} + L_1(r) \) and Scheme 2 has the load \( \frac{K - r}{K - 1} + L_2(r) \). We identify a threshold value for \( r \), denoted \( r_{\text{th}} \), beyond which Scheme 1 is found to be performing better than Scheme 2, as shown by the following claim; and thus the rebalancing algorithm chooses between the two schemes based on whether \( r \geq r_{\text{th}} \) or otherwise. The proof of the below claim is in Appendix A.

**Claim 1.** Let \( r_{\text{th}} = \left\lceil \frac{2K+2}{3} \right\rceil \). If \( r \geq r_{\text{th}} \), then \( \min(L_1(r), L_2(r)) = L_1(r) \) (thus, Scheme 1 has a smaller load) and if \( r < r_{\text{th}} \), we have \( \min(L_1(r), L_2(r)) = L_2(r) \) (thus, Scheme 2 has a lower communication load).

A comparison of these schemes is shown in Fig. 2 for the case of \( K = 15 \) (as \( r \) varies), along with the load \( L_u(r) \) of the uncoded rebalancing scheme. We observe that the minimum file size required in the case of cyclic databases in conjunction with the above schemes is \( NT = 2(K^2-1)K \), i.e., it is cubic in \( K \) and thus much smaller than the file size requirement for the schemes of [1].

In Section IV, we show a rebalancing scheme for node addition, as given by Theorem 1. We also calculate the communication load of this scheme, and show that it is optimal. In Section V, we derive a lower bound for the node-removal rebalancing load, for the specific placement scheme used in the achievability scheme.

### III. Proof of Theorem 1: Rebalancing for Single Node Removal in Cyclic Databases

In Subsection III-A, we provide some intuition for the rebalancing algorithm, which covers both transmission schemes, Scheme 1 and Scheme 2. Then, in Subsection III-B, we describe how the algorithm works for two example parameters (one for Scheme 1, and another for Scheme 2). In Subsection III-C, we formally describe the complete details of the rebalancing algorithm. In Subsection III-D, we prove the correctness of two transmission schemes.
Fig. 2: For $K = 15$, the figure shows comparisons of communication loads of Scheme 1 and Scheme 2 with the load of uncoded transmission scheme, for varying $r$. Note that all curves are relevant only for $r \in \{3, \ldots, K-1\}$. We see that the minimum of the loads of the two schemes is always less than the uncoded load. Further, by Claim 1, any integer value of $r \geq 11$, we see that Scheme 1 has smaller load than Scheme 2, and the reverse is true otherwise.

Remark 2. Note that throughout this section, we describe the scheme when the node $K$ is removed from the system. A scheme for the removal of a general node can be extrapolated easily by permuting the labels of the subsegments. Further details are provided in Subsection III-C (see Remark 3).

A. Intuition for the Rebalancing Scheme

Consider an $r$-balanced cyclic database as shown in Fig. 1. Without loss of generality, consider that the node $K$ is removed. Now the segments that were present in node $K$, i.e., $D_K = \{W_{K-r+1}, \ldots, W_K\}$, no longer have replication factor $r$. In order to restore the replication factor of these segments, we must reinstate each bit in these segments via rebalancing into a node where it was not present before. We fix the target database post-rebalancing to also be an $r$-balanced cyclic database. Recall that $S_i = \{i \oplus_K (r)\}$ represents the nodes where $W_i$ was placed in the initial database. We represent the $K-1$ file segments in this target cyclic database as $\tilde{W}_i : i \in [K-1]$, and the nodes where $\tilde{W}_i$ would be placed is denoted as $\tilde{S}_i = \{i \oplus_{K-1} (r)\}$. This target database is depicted in Fig. 3.

Our rebalancing algorithm involves three phases: (a) a splitting phase where the segments in $D_K$ are split into subsegments, (b) a transmission phase in which coded subsegments are transmitted, and (c) a merge phase, where
the decoded subsegments are merged with existing segments, and appropriate deletions are carried out, to create the target database. Further, the algorithm will choose one of two transmission schemes, Scheme 1 and Scheme 2, in the transmission phase. Our discussion here pertains to both these schemes.

The design of the rebalancing algorithm is driven by two natural motives: (a) move the subsegments as minimally as possible, and (b) exploit the available coding opportunities. Based on this, we give the three generic principles below.

- **Principle 1**: The splitting and merging phases are unavoidable for maintaining the balanced nature of the target database and reducing the communication load. In our merging phase, the target segment \( \hat{W}_j : j \in [K - 1] \) is constructed by merging, (a) either a subsegment of \( W_j \) or the complete \( W_j \), along with (b) some other subsegments of the segments in \( D_K \).

- **Principle 2**: Particularly, for each \( W_i \in D_K \), we seek to split \( W_i \) into subsegments and merge these into those \( \hat{W}_j : j \in [K - 1] \) such that \( |\hat{S}_j \cap S_i| \) is as large as possible, while trying to ensure the balanced condition of the target database. Observe that the maximum cardinality of such intersection is \( r - 1 \). We denote the subsegment of segment \( W_i \) which is to be merged into \( \hat{W}_j \), and thus to be placed in the nodes \( \hat{S}_j \setminus S_i \), as \( W_i^{\hat{S}_j \setminus S_i} \). As making \( |\hat{S}_j \cap S_i| \) large reduces \( |\hat{S}_j \setminus S_i| \), we see that this principle reduces the movement of subsegments during rebalancing.

- **Principle 3**: Because of the structure of the cyclic placement, there exist ‘nice’ subsets of nodes whose indices are separated (cyclically) by \( K - r \), which provide coding opportunity. In other words, there is a set of subsegments of segments in \( D_K \), each of which is present in all-but-one of the nodes in any such ‘nice’ subset, and is to be delivered to the remaining node. Transmitting the XOR-coding of these subsegments ensures successful decoding at the respective nodes they are set to be delivered to (given by the subsegments’ superscripts), because of this ‘nice’ structure.

We shall illustrate the third principle, which guides the design of our transmission schemes, via the examples and the algorithm itself. We now elaborate on how the first two principles are reflected in our algorithms. Consider the segment \( W_{K-r+1} \). We call this a corner segment of the removed node \( K \). Following Principles 1 and 2, this segment \( W_{K-r+1} \) will be split into \( \lceil \frac{K - r + 2}{2} \rceil \) subsegments, out of which one large subsegment is to be merged into \( \hat{W}_{K-r+1} \) (as \( |S_{K-r+1} \cap \hat{S}_{K-r+1}| = r - 1 \)). In order to maintain a balanced database, the remaining \( \lceil \frac{K - r + 2}{2} \rceil - 1 \) are to be merged into the \( \lfloor \frac{K-r}{2} \rfloor \) target segments \( \hat{W}_{K-r+1}, \ldots, \hat{W}_{K-r} \), and additionally into the segment \( \hat{W}_{K-r+1} \) if \( K - r \) is odd. The other corner segment of \( K \) is \( W_K \), for which a similar splitting is followed. One large
subsegment of $W_K$ will be merged into $\tilde{W}_{K-1}$ (again, as $|S_K \cap \tilde{S}_{K-1}| = r - 1$) and the remaining $\lceil \frac{K-r+2}{2} \rceil - 1$ will be merged into $\lceil \frac{K-r+2}{2} \rceil$ target segments $\tilde{W}_1, \ldots, \tilde{W}_{\frac{K-r+1}{2}}$, and additionally into the segment $\tilde{W}_{\frac{K-r+1}{2}}$ if $K - r$ is odd.

Now, consider the segment $W_{K-r+2}$. This is not a corner segment, hence we refer to this as a middle segment. This was available in the nodes $S_{K-r+2} = \{K - r + 2, \ldots, K, 1\}$. Following Principles 1 and 2, this segment $W_{K-r+2}$ will split into two: one to be merged into $\tilde{W}_{K-r+1}$ (for which $\tilde{S}_{K-r+1} = \{K - r + 1, \ldots, K - 1, 1\}$) and $\tilde{W}_{K-r+2}$ (for which $\tilde{S}_{K-r+2} = \{K - r + 2, \ldots, K - 1, 2\}$). Observe that $|S_{K-r+2} \cap \tilde{S}_{K-r+1}| = r - 1$ and $|S_{K-r+2} \cap \tilde{S}_{K-r+2}| = r - 1$. In the same way, each middle segment $W_{K-r+i+1} : i \in [r-2]$ is split into two subsegments, and will be merged into $\tilde{W}_{K-r+i}$ and $\tilde{W}_{K-r+i+1}$ respectively.

B. Examples

We now provide two examples illustrating our rebalancing algorithm, one corresponding to each of the two transmissions schemes.

Example illustrating Scheme 1: Consider a database with $K = 8$ nodes satisfying the $r$-balanced cyclic storage condition with replication factor $r = 6$. A file $W$ is thus split into segments $W_1, \ldots, W_8$ such that the segment $W_1$ is stored in nodes $\{1 \oplus_8 (6)\}$, $W_2$ in nodes $\{2 \oplus_8 (6)\}$, $W_3$ in nodes $\{3 \oplus_8 (6)\}$, and so on.

Node 8 is removed from the system and its contents, namely $W_3, W_4, W_5, W_6, W_7, W_8$, must be restored. The rebalancing algorithm performs the following steps.

Splitting: The splitting is guided by Principles 1 and 2. Each node splits the segments it contains into subsegments as follows:

- $W_3$ is a corner segment with respect to the removed node 8. Thus, it is split into two subsegments. The larger is labelled $W_3^{(1)}$ and is of size $\frac{10T}{14}$. This is to be merged into $\tilde{W}_3$ since $|S_3 \cap \tilde{S}_3| = 5 = (r - 1)$. The other segment is labelled $W_3^{(2)}$ and is of size $\frac{4T}{14}$. As before, the idea is to merge this into $\tilde{W}_2$ to maintain a balanced database.

- The other corner segment $W_8$ is handled similarly. It is split into two subsegments labelled $W_8^{(7)}$ and $W_8^{(6)}$ of sizes $\frac{10T}{14}$ and $\frac{4T}{14}$ respectively.

- $W_4$ is a middle segment for node 8. It is split into two subsegments labelled $W_4^{(2)}$ and $W_4^{(3)}$ of sizes $\frac{10T}{14}$ and $\frac{4T}{14}$ respectively. The intent once again is to merge $W_4^{(2)}$ into $\tilde{W}_4$ and $W_4^{(3)}$ into $\tilde{W}_3$ since both $|S_4 \cap \tilde{S}_3| = |S_4 \cap \tilde{S}_4| = 5 = (r - 1)$. The remaining middle segments are treated similarly.

- $W_5$ into two subsegments labelled $W_5^{(3)}$ and $W_5^{(4)}$ of sizes $\frac{8T}{14}$ and $\frac{6T}{14}$ respectively.

- $W_6$ into two subsegments labelled $W_6^{(4)}$ and $W_6^{(5)}$ of sizes $\frac{6T}{14}$ and $\frac{8T}{14}$ respectively.

- $W_7$ into two subsegments labelled $W_7^{(5)}$ and $W_7^{(6)}$ of sizes $\frac{4T}{14}$ and $\frac{10T}{14}$ respectively.

The superscript represents the set of nodes to which the subsegment is to be delivered.

Coding and Transmission: Now, to deliver these subsegments, nodes make use of coded broadcasts. The design of these broadcasts are guided by Principle 3. We elucidate the existence of the ‘nice’ subsets given in Principle 3 using a matrix form (referred to as matrix $M$) in Figure 4. We note that this representation is similar to the combinatorial structure defined for coded caching in [12], called a placement delivery array. Consider a submatrix.
Fig. 4: The matrix $M$ for $K = 8$, $r = 6$. The rows correspond to subsegments and the columns correspond to nodes. Entry $M_{i,j} = ‘s’$ if the $i^{th}$ subsegment is contained in the $j^{th}$ node. $M_{i,j} = ‘∗’$ if the $i^{th}$ subsegment must be delivered to the $j^{th}$ node. For each shape enclosing an entry, the row and column corresponding each entry with that shape gives a valid XOR-coded transmission.

of $M$ described by distinct rows $i_1, \ldots, i_l$ and distinct columns $j_1, \ldots, j_{l+1}$. If this submatrix is equivalent to the $l \times (l+1)$ matrix

$$
\begin{bmatrix}
s & * & \ldots & * & *
\end{bmatrix}
$$

up to some row/column permutation, then each of the nodes $j_1, \ldots, j_l$ can decode their required subsegment from the XOR of the $i_1^{th}, \ldots, i_l^{th}$ subsegments which can be broadcasted by the $(l+1)^{th}$ node. Our rebalancing algorithm makes use of this property to design the transmissions. To denote the submatrices we make use of shapes enclosing each requirement (represented using an ‘$s$’) in the matrix. For each shape, the row and column corresponding to each ‘$s$’ result in a XOR-coded transmission. Before such XOR-coding, padding the ‘shorter’ subsegments with 0s to match the length of the longest subsegment would be required. There are other $s$ entries in the matrix $M$ which are not covered by matrices of type (1). These will correspond to uncoded broadcasts. Thus, we get the following transmissions from the matrix

- Node 1 pads $W_6^{(5)}$ and $W_4^{(3)}$ to size $\frac{127T}{14}$ and broadcasts $W_8^{(7)} \oplus W_6^{(5)} \oplus W_4^{(3)}$.
- Similarly, Node 7 pads $W_5^{(3)}$ and $W_7^{(5)}$ and broadcasts $W_7^{(1)} \oplus W_5^{(3)} \oplus W_7^{(5)}$.
- Node 1 pads $W_5^{(4)}$ to size $\frac{45T}{14}$ and broadcasts $W_5^{(4)} \oplus W_7^{(6)}$.
- Similarly, Node 7 pads $W_6^{(4)}$ and broadcasts $W_4^{(2)} \oplus W_6^{(4)}$.
- Finally, Node 1 broadcasts $W_8^{(6)}$ and Node 7 broadcasts $W_3^{(2)}$. 
Merging and Relabelling: To restore the cyclic storage condition, each node segments that must be stored in it in the final database. These newly formed subsegments are to be merged into the system. The contents of node 6 are split into subsegments as per Principles 1 and 2.

\[
W_4 = W_6/W_8 \quad 1 + \frac{2T}{14} - \frac{8T}{7}
\]
is obtained at nodes \{1, 2, 3, 4, 5, 6\}. Similarly, Nodes 3 and 5 can recover \(W_4^{(3)}\) and \(W_6^{(5)}\) respectively.

\[
W_7 = W_8/W_7 \quad \frac{4T}{7}
\]
is obtained at nodes \{3, 4, 5, 6, 7\}. Similarly, Node 6 can recover \(W_7^{(6)}\).

\[
W_8 = W_6/W_8 \quad \frac{8T}{7}
\]
is obtained at nodes \{4, 5, 6, 7, 1, 2\}. Similarly, Node 4 can recover \(W_8^{(4)}\).

Merging and Relabelling: To restore the cyclic storage condition, each node \(k \in [K - 1]\) merges and relabels all segments that must be stored in it in the final database. These newly formed subsegments are \(\hat{W}_1, \ldots, \hat{W}_7\).

\[
\hat{W}_1 = W_1/W_8^{(6)} \quad \text{of size } 1 + \frac{2T}{14} - \frac{8T}{7}
\]
is obtained at nodes \{1, 2, 3, 4, 5, 6\}. Similarly, Node 2 contains \(W_6\) and can hence recover \(W_8^{(4)}\). Similarly, Node 6 can recover \(W_7^{(6)}\).

\[
\hat{W}_2 = W_2/W_8^{(2)} \quad \text{of size } 1 + \frac{2T}{14} - \frac{8T}{7}
\]
is obtained at nodes \{2, 3, 4, 5, 6, 7\}. Similarly, Node 4 contains \(W_6\) and can hence recover \(W_8^{(4)}\). Similarly, Node 4 can recover \(W_6^{(4)}\).

Decoding: The uncoded subsegments are directly received by the respective nodes. The nodes present in the superscript of the XORED subsegment proceed to decode their respective required subsegment as follows.

- From the transmission \(W_8^{(7)} \oplus W_6^{(5)} \oplus W_4^{(3)}\), Node 7 contains \(W_6, W_4\) and can hence recover \(W_8^{(7)}\) by XORing away the other subsegments. Similarly, Nodes 3 and 5 can recover \(W_4^{(3)}\) and \(W_6^{(5)}\) respectively.

- From the transmission \(W_3^{(1)} \oplus W_5^{(3)} \oplus W_7^{(5)}\), Node 1 contains \(W_5, W_7\) and can recover \(W_3^{(1)}\). Similarly, Nodes 3 and 5 can recover \(W_5^{(3)}\) and \(W_7^{(5)}\) respectively.

- From the broadcast \(W_5^{(4)} \oplus W_7^{(6)}\), Node 4 contains \(W_7\) and can hence recover \(W_5^{(4)}\). Similarly, Node 6 can recover \(W_7^{(6)}\).

- From the broadcast \(W_4^{(2)} \oplus W_6^{(4)}\). Node 2 contains \(W_6\) and can hence recover \(W_4^{(2)}\). Similarly, Node 4 can recover \(W_6^{(4)}\).

\[
\frac{2T}{14} + \frac{22T}{14} = \frac{24T}{7}
\]
...
• Similarly, the other corner segment $W_6$ is split into three subsegments labelled $W_6^{(5)}$, $W_6^{(3)}$, and $W_6^{(3,4)}$ of sizes $\frac{7T}{10}$, $\frac{2T}{10}$, and $\frac{T}{10}$, and are to be merged with $\tilde{W}_5$, $\tilde{W}_1$, and $\tilde{W}_2$ respectively.

• $W_5$ is a middle segment of node 6. It is split into two subsegments labelled $W_5^{(1)}$ and $W_5^{(4)}$ of size $\frac{5T}{10}$ each. $W_5^{(1)}$ is to be merged into $\tilde{W}_5$, and $W_5^{(4)}$ into $\tilde{W}_4$, since both $|S_5 \cap \tilde{S}_5| = |S_5 \cap \tilde{S}_4| = 2 = (r - 1)$.

\[
\begin{array}{cccccc}
\text{Nodes} & 1 & 2 & 3 & 4 & 5 \\
\hline
W_4^{(1)} & \mathbb{S} & - & - & * & \bigcirc \\
W_5^{(2)} & \mathbb{S} & \mathbb{S} & - & - & * \\
W_5^{(4)} & * & - & - & \mathbb{S} & \bigcirc \\
W_6^{(5)} & * & * & - & - & \mathbb{S} \\
W_7^{(2,3)} & - & s & s & * & - \\
W_8^{(3)} & - & s & s & * & - \\
W_9^{(3)} & * & s & - & - & - \\
W_6^{(3,4)} & * & * & s & s & - \\
\end{array}
\]

Fig. 5: Matrix $M$ for $K = 6$, $r = 3$ case. The rows of this matrix $M$ correspond to subsegments and the columns correspond to nodes. Entry $M_{i,j} = \ast$ if the $i$th subsegment is contained in the $j$th node. $M_{i,j} = s$ if the $i$th subsegment must be delivered to the $j$th node. For each shape enclosing an entry, the row and column corresponding each entry with that shape lead to a XOR-coded transmission.

Coding and Transmission: As before, we make use of the placement matrix shown in Fig. 5 to explain how nodes perform coded broadcasts as per Principle 3. Consider the submatrix denoted by the circles in Figure 5. This submatrix described by columns 1, 4, 5 and rows 1, 3 means that each of the subsegments corresponding to these rows are present in all but one of the nodes corresponding to these columns. Further, node 5 contains both of these subsegments, and thus node 5 can broadcast the XOR of them and each of the nodes 1, 4 can recover the respective subsegments denoted by the rows. Further, those $s$ entries in the matrix not covered by any shape lead to uncoded broadcasts. Following these ideas, we get the following transmissions.

- Node 1 pads $W_6^{(2)}$ to size $\frac{7T}{10}$ and broadcasts $W_6^{(2)} \oplus W_6^{(5)}$.
- Similarly, Node 4 pads $W_6^{(4)}$ and broadcasts $W_4^{(1)} \oplus W_5^{(4)}$.
- Finally, Node 1 broadcasts $W_6^{(3)}$, $W_6^{(3,4)}$ and Node 4 broadcasts $W_4^{(3)}$, $W_4^{(2,3)}$.

The total communication load incurred in performing these broadcasts is $\frac{1}{T} (2 \cdot \frac{7T}{10} + 2 \cdot \frac{T}{10} + 2 \cdot \frac{2T}{10}) = 2$. Note that the uncoded rebalancing load for node removal is 3.

Decoding: The uncoded broadcast subsegments are received as-is by the superscript nodes. With respect to any XOR-coded subsegment, the nodes present in the superscript of the subsegment can decode the subsegment, due to the careful design of the broadcasts as per Principle 3. For this example, we have the following.

- From the transmission $W_6^{(2)} \oplus W_6^{(5)}$, node 2 can decode $W_6^{(2)}$ by XORing away $W_6^{(5)}$ and similarly node 5 can decode $W_6^{(5)}$.
- Similarly, from $W_4^{(1)} \oplus W_5^{(4)}$, nodes 1 and 4 can recover $W_4^{(1)}$ and $W_5^{(4)}$ respectively.
Merging and Relabelling: To restore the cyclic storage condition, each node \( k \in [K - 1] \) merges and relabels all segments that must be stored in it in the final database. These newly formed subsegments are \( \tilde{W}_1, \ldots, \tilde{W}_5 \).

- Observe that \( W_1 \) and \( W_6^{(3)} \) are available at nodes \( \{1, 2, 3\} \), from either the prior storage or due to decoding.

  Thus, the segment \( \tilde{W}_1 = W_1 W_6^{(3)} \) of size \( 1 + \frac{2T}{10} + \frac{T}{10} = \frac{6T}{5} \) is obtained and stored at nodes \( \{1, 2, 3\} \). Similarly, we have the other merge operations as follows.

- \( \tilde{W}_2 = W_2 W_4^{(2,3)} W_6^{(3,4)} \) of size \( 1 + \frac{T}{10} + \frac{T}{10} = \frac{6T}{5} \) is obtained at nodes \( \{2, 3, 4\} \).
- \( \tilde{W}_3 = W_3 W_4^{(3)} \) of size \( 1 + \frac{2T}{10} + \frac{T}{10} = \frac{6T}{5} \) is obtained at nodes \( \{3, 4, 5\} \).
- \( \tilde{W}_4 = W_4^{(1)} W_5^{(4)} \) of size \( \frac{T}{10} + \frac{5T}{10} = \frac{6T}{5} \) is obtained at nodes \( \{4, 5, 1\} \).
- \( \tilde{W}_5 = W_5^{(2)} W_6^{(5)} \) of size \( \frac{T}{10} + \frac{7T}{10} = \frac{6T}{5} \) is obtained at nodes \( \{5, 1, 2\} \).

After merging and relabelling, each node keeps only the required segments mentioned previously and discards any extra data present. Since each node now stores 3 segments each of size \( \frac{6T}{5} \), the total data stored is still \( 18T = rNT \). Thus the cyclic storage condition is satisfied.

C. Algorithm for Rebalancing for Node-Removal

In this section, following the intuition built in Subsection III-A, we give our complete rebalancing algorithm (Algorithm 1) for removal of a node from an \( r \)-balanced cyclic database on \( K \) nodes (with \( r \in \{3, \ldots, K - 1\} \)). We thus prove the node-removal result in Theorem 1. Algorithm 1 initially invokes the SPLIT routine (described in Algorithm 2) which gives the procedure to split segments into subsegments. Each subsegment’s size is assumed to be an integral multiple of \( \frac{T}{2(K-1)} \). This is without loss of generality, by the condition on the size \( T \) of each segment as in Theorem 1. This splitting scheme is also illustrated in the figures Fig. 6-8. Guided by Claim 1, based on the value of \( r \), Algorithm 1 selects between two routines that correspond to the two transmission schemes: SCHEME 1 and SCHEME 2. These schemes are given in Algorithm 3 and 4. We note that, since the sizes of the subsegments may not be the same after splitting, appropriate zero-padding (up to the size of the larger subsegment) is done before the XOR operations are performed in the two schemes. Finally, the MERGE routine (given in Algorithm 5) is run at the end of Algorithm 1. This merges the subsegments and relabels the merged segments as the target segments, thus resulting in the target \( r \)-balanced cyclic database on \( K - 1 \) nodes.

Remark 3. Note that, for ease of understanding, we describe the algorithms for the removal of node \( K \) from the system. The scheme for the removal of a general node \( i \) can be obtained as follows. Consider the set of permutations \( \phi_i : [K] \to [K] \), where \( \phi_i(j) = j \circ_K (K - i) \), for \( i \in [K] \). If a node \( i \) is removed instead of \( K \), we replace each label \( j \) in the subscript and superscript of the subsegments in our Algorithms 2-5 with \( \phi_i(j) \). Due to the cyclic nature of both the input and target databases, we naturally obtain the rebalancing scheme for the removal of node \( i \).
Algorithm 1 Rebalancing Scheme for Node Removal from Cyclic Database

1: procedure TRANSMIT
2:   SPLIT() ▷ Call SPLIT
3:   if $r \geq r_{th} = \lceil \frac{2K+2}{3} \rceil$ then
4:     SCHEME 1() ▷ Call SCHEME 1
5:   else
6:     SCHEME 2() ▷ Call SCHEME 2
7:   end if
8:   MERGE() ▷ Call MERGE
9: end procedure

Fig. 6: For each $i \in [r-2]$, $W_{K-r+1+i}$ is split into two parts labelled $W^{(i+1)}_{K-r+1+i}$ and $W^{(i+K-r)}_{K-r+1+i}$ of sizes $\frac{K+2r-2}{2(K-1)}$ and $\frac{K-r+2}{2(K-1)}$ respectively.

Fig. 7: Let $p = \lfloor \frac{K-r}{2} \rfloor$. When $K - r$ is odd, $W_{K-r+1}$ is split into $p + 2$ parts labelled $W^{(1)}_{K-r+1}$, $W^{(r)}_{K-r+1}$, $W^{((K-r-p)\min(r,p+1))}_{K-r+1}$, and $W^{((K-r+1-j)\min(r,j))}_{K-r+1}$ for $j = 1, \ldots, p$; of sizes $\frac{K+r-2}{2(K-1)}$, $\frac{1}{2(K-1)}$, $\frac{2}{2(K-1)}$, $\frac{2}{2(K-1)}$, $\frac{2}{2(K-1)}$, $\frac{2}{2(K-1)}$, $\frac{2}{2(K-1)}$, $\frac{2}{2(K-1)}$, $\frac{2}{2(K-1)}$, respectively. Similarly, $W_K$ is split into $p + 2$ parts labelled $W^{(1)}_{K}$, $W^{(r)}_{K}$, $W^{((r+p)\min(r,p+1))}_{K}$, and $W^{((r+1-j)\min(r,j))}_{K}$ for $j = 1, \ldots, p$; of sizes $\frac{K+r-2}{2(K-1)}$, $\frac{1}{2(K-1)}$, $\frac{2}{2(K-1)}$, $\frac{2}{2(K-1)}$, $\frac{2}{2(K-1)}$, $\frac{2}{2(K-1)}$, $\frac{2}{2(K-1)}$, $\frac{2}{2(K-1)}$, $\frac{2}{2(K-1)}$, respectively.

Fig. 8: Let $p = \lfloor \frac{K-r}{2} \rfloor$. When $K - r$ is even, $W_{K-r+1}$ is split into $p + 1$ parts labelled $W^{(1)}_{K-r+1}$, and $W^{((K-r+1-j)\min(r,j))}_{K-r+1}$ for $j = 1, \ldots, p$; of sizes $\frac{K+r-2}{2(K-1)}$, $\frac{2}{2(K-1)}$, $\frac{2}{2(K-1)}$, $\frac{2}{2(K-1)}$, $\frac{2}{2(K-1)}$, $\frac{2}{2(K-1)}$, $\frac{2}{2(K-1)}$, $\frac{2}{2(K-1)}$, respectively. Similarly, $W_K$ is split into $p + 1$ parts labelled $W^{(1)}_{K}$, and $W^{((r+1-j)\min(r,j))}_{K}$ for $j = 1, \ldots, p$; of sizes $\frac{K+r-2}{2(K-1)}$, $\frac{2}{2(K-1)}$, $\frac{2}{2(K-1)}$, $\frac{2}{2(K-1)}$, $\frac{2}{2(K-1)}$, $\frac{2}{2(K-1)}$, $\frac{2}{2(K-1)}$, $\frac{2}{2(K-1)}$, $\frac{2}{2(K-1)}$, respectively.
Algorithm 2 Splitting Scheme

1: procedure Split
2: for each $i \in [r - 2]$ do
3: Split $W_{K-r+i+1}$ into subsegments labelled $W_{K-r+1+i}^B$ for $B \in \{\{i + 1\}, \{i + K - r\}\}$, where the size of the subsegment is
\[
\frac{K + r - 2i - 2}{2(K-1)}, \quad \text{if } B = \{i + 1\}
\]
\[
\frac{K - r + 2i}{2(K-1)}, \quad \text{if } B = \{i + K - r\}
\]
4: end for
5: if $K - r$ is odd then
6: Let $p = \left\lfloor \frac{K-r}{2} \right\rfloor$
7: Split $W_{K-r+1}$ into $p + 2$ subsegments labelled $W_{K-r+1}^B$, for $B \in \{\{1\}, (K - r - p) \oplus_{K-1} \langle \min(r, p + 1) \rangle \} \cup \{(\langle K - r + 1 - j \rangle \oplus_{K-1} \langle \min(r, j) \rangle) : j \in [p]\}$ where the size of the subsegment is
\[
\frac{K + r - 2}{2(K-1)}, \quad \text{if } B = \{1\}
\]
\[
\frac{1}{2(K-1)}, \quad \text{if } B = \{(K - r - p) \oplus_{K-1} \langle \min(r, p + 1) \rangle\}
\]
\[
\frac{2}{2(K-1)}, \quad \text{otherwise}
\]
8: Split $W_{K}$ into $p + 2$ subsegments labelled $W_{K}^B$, for $B \in \{\{K - 1\}, (r + p) \oplus_{K-1} \langle \min(r, p + 1) \rangle\} \cup \{(\langle r - 1 + j \rangle \oplus_{K-1} \langle \min(r, j) \rangle) : j \in [p]\}$ where the size of the subsegment is
\[
\frac{K + r - 2}{2(K-1)}, \quad \text{if } B = \{K - 1\}
\]
\[
\frac{1}{2(K-1)}, \quad \text{if } B = \{(r + p) \oplus_{K-1} \langle \min(r, p + 1) \rangle\}
\]
\[
\frac{2}{2(K-1)}, \quad \text{otherwise}
\]
9: else
10: Let $p = \frac{K-r}{2}$
11: Split $W_{K-r+1}$ into $p + 1$ subsegments labelled $W_{K-r+1}^B$, for $B \in \{1\} \cup \{(\langle K - r + 1 - j \rangle \oplus_{K-1} \langle \min(r, j) \rangle) : j \in [p]\}$ where the size of the subsegment is
\[
\frac{K + r - 2}{2(K-1)}, \quad \text{if } B = \{1\}
\]
\[
\frac{2}{2(K-1)}, \quad \text{otherwise}
\]
12: Split $W_{K}$ into $p + 1$ subsegments labelled $W_{K}^B$, for $B \in \{K - 1\} \cup \{(\langle r - 1 + j \rangle \oplus_{K-1} \langle \min(r, j) \rangle) : j \in [p]\}$ where the size of the subsegment is
\[
\frac{K + r - 2}{2(K-1)}, \quad \text{if } B = \{K - 1\}
\]
\[
\frac{2}{2(K-1)}, \quad \text{otherwise}
\]
13: end if
14: end procedure
Algorithm 3 Transmission Scheme 1

1: procedure SCHEME 1
2:   for each \( i = 1, \ldots, K - r \) do
3:     Node 1 broadcasts \( \bigoplus_{j=0}^{\lfloor r - i - 1 \rfloor} W_{K+1-i-j(K-r)}^{K-r-i-j(K-r)} \).
4:     Node \( K - 1 \) broadcasts \( \bigoplus_{j=0}^{\lfloor r - i - 1 \rfloor} W_{K-r+i+j(K-r)}^{i+j(K-r)} \).
5:   end for
6:   Node 1 broadcasts all subsegments of \( W_K \) except \( W_K^{(K-1)} \).
7:   Node \( K - 1 \) broadcasts all subsegments of \( W_{K-r+1} \) except \( W_{K-r+1}^{(1)} \).
8: end procedure

Algorithm 4 Transmission Scheme 2

1: procedure SCHEME 2
2:   for each \( i = 2, \ldots, r - 1 \) do
3:     Node 1 broadcasts \( W_{K-r+i}^{(i)} \oplus W_{K-r+i+1}^{(K-r+i)} \).
4:   end for
5:   Node \( K - 1 \) broadcasts \( W_{K-r+1}^{(1)} \oplus W_{K-r+2}^{(K-r+1)} \).
6:   Node 1 broadcasts all subsegments of \( W_K \) except \( W_K^{(K-1)} \).
7:   Node \( K - 1 \) broadcasts all subsegments of \( W_{K-r+1} \) except \( W_{K-r+1}^{(1)} \).
8: end procedure
uncoded transmissions are directly received. Thus, we now check only the decoding involved for XOR-coded segment are available at that node. Finally, we check that the target database is the cyclic database on also check that the merging scheme is successful, i.e., at any node, all the subsegments to be merged into a target subsegment can be decoded at the corresponding 'superscript' nodes where it is meant to be delivered. We must and broadcasts given in the transmissions schemes are correct. For checking the decoding, we must check that each subsegments, contain all respective subsegments according to the design of the initial storage. Thus, the XOR-coding merging. It is straightforward to check that the nodes that broadcast any transmission, whether coded or uncoded nodes.

\[ \text{Algorithm 5 Merging and Relabelling} \]

1: \textbf{procedure} Merge
2: \hspace{1em} for each \( i = 1, \ldots, r - 1 \) do
3: \hspace{2em} Each node in \( \{(K - r + i) \boxplus_{K-1} \langle r \rangle\} \) performs the concatenation \( \tilde{W}_{K-r+i} = W^{\{(r-1+i)\boxplus_{K-1}(min(r,i))\}}_{K-r+i} \).
4: \hspace{1em} end for
5: \hspace{1em} if \( K - r \) is even then
6: \hspace{2em} for each \( i = 1, \ldots, \frac{K-r}{2} \) do
7: \hspace{3em} Each node in \( \{(i \boxplus_{K-1} \langle r \rangle)\} \) performs the concatenation \( \tilde{W}_i = W_i|W^{\{\text{min}(r,K-r-i+1)\}}_{K-r+i} \).
8: \hspace{2em} end for
9: \hspace{2em} for each \( i = \frac{K-r}{2} + 1, \ldots, K - r \) do
10: \hspace{3em} Each node in \( \{(i \boxplus_{K-1} \langle r \rangle)\} \) performs the concatenation \( \tilde{W}_i = W_i|W^{\{\text{min}(r,K-r-i+1)\}}_{K-r+i} \).
11: \hspace{2em} end for
12: \hspace{2em} else
13: \hspace{3em} for each \( i = 1, \ldots, \frac{K-r-1}{2} \) do
14: \hspace{4em} Each node in \( \{(i \boxplus_{K-1} \langle r \rangle)\} \) performs the concatenation \( \tilde{W}_i = W_i|W^{\{\text{min}(r,K-r-i+1)\}}_{K-r+i} \).
15: \hspace{3em} end for
16: \hspace{3em} for each \( i = \frac{K-r+1}{2} + 1, \ldots, K - r \) do
17: \hspace{4em} Each node in \( \{(i \boxplus_{K-1} \langle r \rangle)\} \) performs the concatenation \( \tilde{W}_i = W_i|W^{\{\text{min}(r,K-r-i+1)\}}_{K-r+i} \).
18: \hspace{3em} end for
19: \hspace{2em} Each node in \( \{(\frac{K-r+1}{2}) \boxplus_{K-1} \langle r \rangle\} \) performs the concatenation \( \tilde{W}_{K-r+1} = W_{\frac{K-r+1}{2}}|W^{\{(r+p)\boxplus_{K-1}(min(r,p+1))\}}_{K-r+1} \).
20: \hspace{1em} end if
21: \textbf{end procedure}

Note: Once the target segments \( \tilde{W}_1, \ldots, \tilde{W}_{K-1} \) are recovered at the required nodes, any extra bits present at the node are discarded.

D. Correctness of Algorithm 1

To check the correctness of the scheme, we have to check the correctness of the encoding, decoding, and the merging. It is straightforward to check that the nodes that broadcast any transmission, whether coded or uncoded subsegments, contain all respective subsegments according to the design of the initial storage. Thus, the XOR-coding and broadcasts given in the transmissions schemes are correct. For checking the decoding, we must check that each subsegment can be decoded at the corresponding ‘superscript’ nodes where it is meant to be delivered. We must also check that the merging scheme is successful, i.e., at any node, all the subsegments to be merged into a target segment are available at that node. Finally, we check that the target database is the cyclic database on \( K - 1 \) nodes.

Now, we focus on checking the decoding of the transmissions in both Scheme 1 and Scheme 2. Clearly, all uncoded transmissions are directly received. Thus, we now check only the decoding involved for XOR-coded
transmissions, for the two schemes.

- **Decoding for Scheme 1:** For each \( i \in [K-r] \), two broadcasts \( \bigoplus_{j=0}^{\lfloor \frac{r-1}{K-r} \rfloor} W_{K+1-i-j(K-r)} \) and \( \bigoplus_{j=0}^{\lfloor \frac{r-1}{K-r} \rfloor} W_{K-r+i+j(K-r)} \) are made. Consider the first broadcast. Let \( J = \{0, \ldots, \lfloor \frac{r-1}{K-r} \rfloor \} \). For some \( j \in J \), consider the segment \( W_{K+1-i-j(K-r)} \). For any \( j' \in J \setminus \{j\} \), we claim that node \( K - i - j'(K-r) \) contains the segment \( W_{K+1-i-j(K-r)} \). Going through all possible \( j, j' \) would then mean that all the segments in this first XOR-coded broadcast can be decoded at the respective superscript-nodes. Now, for node \( K - i - j'(K-r) \) to contain the segment \( W_{K+1-i-j(K-r)} \), the following condition must be satisfied:

- **Condition (A):** \( K - i - j'(K-r) \in S_{K+1-i-j(K-r)} = \{(K + 1 - i - j(K-r)) \bowtie_K \{r\}\}. \)

To remove the wrap-around, we simplify Condition (A) into two cases based on the relation between \( j \) and \( j' \). For Condition (A) to hold, it is easy to check that one of the following pairs of inequalities must hold.

1) if \( j < j' \), \( K + 1 - i - j(K-r) \leq 2K - i - j'(K-r) \leq K + 1 - i - j(K-r) + r - 1 \)

2) if \( j > j' \), \( K + 1 - i - j(K-r) \leq K - i - j'(K-r) \leq K + 1 - i - j(K-r) + r - 1. \)

Consider the first inequality. First we prove that when \( j < j' \), \( K + 1 - i - j(K-r) \leq 2K - i - j'(K-r) \). To show this, we consider the following sequence of equations,

\[
(2K - i - j'(K-r)) - (K + 1 - i - j(K-r)) = K - 1 - (j' - j)(K-r).
\]

\[
\begin{align*}
&\geq K - 1 - \left( \frac{r-1-i}{K-r} \right) (K-r).
&\geq K - r + i
&\geq K - r + 1
&\geq K - (K-1) + 1
&\geq 0,
\end{align*}
\]

where (a) holds as the maximum value of \( j' - j \) is equal to \( \lfloor \frac{r-1-i}{K-r} \rfloor \), (b) holds as the minimum value of \( i \) is 1, and (c) holds as the maximum value of \( r \) is \( K-1 \).

Similarly,

\[
(K + 1 - i - j(K-r) + r - 1) - (2K - i - j'(K-r)) = r - K + (j' - j)(K-r)
\]

\[
\begin{align*}
&\geq r - K + (K-r)
&\geq 0,
\end{align*}
\]

where (a) holds as the minimum value of \( j' - j \) is equal to 1.

Now, for the second inequality, we first prove that when \( j > j' \), \( K + 1 - i - j(K-r) \leq K - i - j'(K-r) \). To show this, we consider the following sequence of equations

\[
(K - i - j'(K-r)) - (K + 1 - i - j(K-r)) = (j - j')(K-r) - 1
\]

\[
\begin{align*}
&\geq K - r - 1
&\geq K - (K-1) - 1
\end{align*}
\]
\[ \geq 0, \]

where (a) holds as the minimum value of \( j - j' \) is equal to 1 and (b) holds as the maximum value of \( r \) is \( K - 1 \). Similarly,

\[
(K + 1 - i - j(K - r) + r - 1) - (K - i - j'(K - r)) = r - (j - j')(K - r)
\]  
\[
\geq r - \left( \frac{r - 1 - i}{K - r} \right)(K - r)
\]  
\[
\geq 1 + i
\]  
\[
(b) \geq 0,
\]

where (a) holds as the maximum value of \( j - j' \) is equal to \( \lceil \frac{r - 1 - i}{K - r} \rceil \) and (b) holds as the minimum value of \( i \) is 1.

Hence, all the inequalities for both the cases are true. Similar arguments hold for the second broadcast as well.

**Decoding for Scheme 2:** For each \( i \in [r] \), a broadcast \( W_{K-r+i}^{(i)} \oplus W_{K-r+i+1}^{(K-r+i)} \) is made (c.f. Lines 3,5 in Algorithm 4). Now, node \( i \) contains the subsegment \( W_{K-r+i+1} \) since \( (K-r+i+1) \equiv_K (r-1) = i \). Similarly, node \( K-r+i \) clearly contains the subsegment \( W_{K-r+i} \). Thus, node \( i \) can decode \( W_{K-r+i}^{(i)} \), and node \( K-r+i \) can decode \( W_{K-r+i+1}^{(K-r+i)} \). Thus, we have verified the correctness of the transmission schemes.

**Checking the merging phase:** Initially, for each \( i \in [K] \), \( W_i \) is stored at the nodes \( \{i \oplus_K \langle r \rangle \} \). After the transmissions are done, \( \tilde{W}_i \) is obtained by merging some subsegments with \( W_i \), for each \( i \in \{1, \ldots, K\} \). Now, to verify that the merging can be done correctly, we need to show that all these subsegments are present at the nodes \( \{i \oplus_{K-1} \langle r \rangle \} \) after the transmissions are done.

- For each \( i \in [r-1] \), consider the segment \( \tilde{W}_{K-r+i} \) which is obtained by merging \( W_{K-r+i}^{(i)} \) with \( W_{K-r+i+1}^{(K-r+i)} \) (c.f. Algorithm 5 Line 2 and 3). We observe that \( W_{K-r+i}^{(i)} \) was present in \( \{(K-r+i) \oplus_{K-1} \langle r \rangle \}\{i\} \) before rebalancing and was decoded by node \( i \) during the rebalancing process. Similarly, \( W_{K-r+i+1}^{(K-r+i)} \) was present in \( \{(K-r+i) \oplus_{K-1} \langle r \rangle \}\{K-r+i\} \) before rebalancing and was decoded by node \( K-r+i \) during the rebalancing process.

- For each \( i \in \{1, \ldots, \lceil \frac{K-r}{2} \rceil \} \), \( \tilde{W}_i \) is obtained by merging \( W_i \) and \( W_i^{(r-1+i) \oplus_{K-1} \langle \min(r,i) \rangle} \) (c.f. Algorithm 5 Lines 7 and 14). Each node in \( \{i \oplus_{K-1} \langle r \rangle \} \) that does not contain \( W_K \) can obtain \( W_K^{(r-1+i) \oplus_{K-1} \langle \min(r,i) \rangle} \) using the broadcasts made in Lines 6-7 of Algorithms 3 and 4. Thus, \( \tilde{W}_i \) can be obtained at the nodes \( \{i \oplus_{K-1} \langle r \rangle \} \).

- We use similar arguments for the target segments \( \tilde{W}_i \) for each \( i \in \{\lceil \frac{K-r}{2} \rceil + 1, \ldots, K-r\} \) and \( \tilde{W}_{K-r+i} \) if \( K-r \) is odd.

**Checking the target database structure:** As we have already checked the placement of the new subsegments in the previous bullet, we only have to show that the sizes of all the segments after rebalancing are equal. For this, we look at how the new segments are formed by merging some subsegments with the older segment.

- The size of the target segment \( \tilde{W}_{K-r+i} \), for each \( i \in \{1, \ldots, r-1\} \), is \( \frac{K+r-2(i-1)-2}{2(K-1)} + \frac{K-r+2i}{2(K-1)} = \frac{K}{K-1} \) (c.f. Algorithm 5 - Line 2 and 3).
The size of the target segment $\tilde{W}_i$, for each $i \in [\lfloor \frac{K-r}{2} \rfloor]$, is $1 + \frac{2}{2(K-1)} = \frac{K}{K-1}$ (c.f. Algorithm 5 - Line 6, 7, 13, and 14).

- For even $K - r$, the size of the target segment $\tilde{W}_i$, for each $i \in \{\frac{K-r}{2} + 1, \ldots, K-r\}$, is $1 + \frac{2}{2(K-1)} = \frac{K}{K-1}$ (c.f. Algorithm 5 - Line 9 and 10).

- For odd $K - r$, the size of the target segment $\tilde{W}_i$, for each $i \in \{\frac{K-r+1}{2} + 1, \ldots, K-r\}$, is $1 + \frac{2}{2(K-1)} = \frac{K}{K-1}$ (c.f. Algorithm 5 - Line 16 and 17).

- The size of the target segment $\tilde{W}_{\frac{K-r+1}{2}}$ is $1 + \frac{1}{2(K-1)} + \frac{1}{2(K-1)} = \frac{K}{K-1}$ (c.f. Algorithm 5 - Line 19).

We can see that the sizes of all the segments after rebalancing are the same, i.e., $\frac{K}{K-1}$. This completes the verification of the correctness of our rebalancing algorithm, which assures that the target database is an $r$-balanced cyclic database on $K-1$ nodes.

E. Communication Load of Algorithm 1

We now calculate the communication loads of the two schemes. For the uncoded broadcasts in both schemes corresponding to lines 6, 7 in both Algorithms 3 and 4, the communication load incurred is $2 \left(\frac{K-r-1}{2} + \frac{2}{2(K-1)} + \frac{1}{2(K-1)}\right) = \frac{K-r}{(K-1)}$, when $K - r$ is odd, and $2 \left(\frac{K-r}{2} - \frac{1}{2(K-1)}\right) = \frac{K-r}{(K-1)}$ when $K - r$ is even, respectively. Now, we analyse the remainder of the communication loads of the two schemes.

1) Scheme 1: The coded broadcasts made in Scheme 1 are $\bigoplus_{j=0}^{\lfloor \frac{K-r}{2} \rfloor} W_{\lfloor K-i-j(K-r) \rfloor}^{K+1-i-j(K-r)}$ and $\bigoplus_{j=0}^{\lfloor \frac{K-r}{2} \rfloor} W_{\lfloor K-r+i+j(K-r) \rfloor}^{i+j(K-r)}$. The size of each subsegment involved is $\frac{K-r-2(i+j(K-r)+1)+2}{2(K-1)}$. Thus, the subsegment having the maximum size is the one corresponding to $j = 0$ having size $\frac{K-r-2i}{2(K-1)}$. Similarly, each subsegment involved in $\bigoplus_{j=0}^{\lfloor \frac{K-r}{2} \rfloor} W_{\lfloor K-i-j(K-r) \rfloor}^{K+1-i-j(K-r)}$ is of size $\frac{K-r+2(r-i-j(K-r))}{2(K-1)}$. Thus, the maximum size is the one corresponding to $j = 0$ having size $\frac{K-r+2r-2i}{2(K-1)}$. Therefore, the communication load is

$$L_1(r) = 2 \cdot \sum_{i=1}^{\lfloor \frac{K-r}{2} \rfloor} \frac{K-r-2i}{2(K-1)}$$
$$= \left(\frac{1}{K-1}\right) \frac{(K-r)(K+r) - (K-r)(K-r+1)}{(K-1)}$$
$$= \frac{(K-r)(2r-1)}{(K-1)}$$.

2) Scheme 2: The coded broadcasts made in Scheme 2 involve $W_{\lfloor K-r+i \rfloor}^{i+1} \oplus W_{\lfloor K-r+i+1 \rfloor}^{K-r+i+1}$ for each $i \in [r-1]$ (c.f. Algorithm 4 Lines 2-5). Since the smaller subsegment of each pair is padded to match the size of the larger, the cost involved in making the broadcast depends on the larger subsegment. Now, the size of these subsegments are $\frac{K-r-2j+2}{2(K-1)}$ and $\frac{K-r+2(j+1)+2}{2(K-1)}$ respectively (c.f. Figures 6, 7, 8). For the first subsegment to be larger, $\frac{K-r-2j+2}{2(K-1)} \geq \frac{K-r+2(j+1)+2}{2(K-1)}$. It is easy to verify that this occurs when $j \leq \frac{r}{2} - 1$. We separate our analysis into cases based on the parity of $r$.

For odd $r$, we have,

$$L_2(r) = \sum_{j=0}^{\frac{r-1}{2}} \frac{K-r-2j-2}{2(K-1)} + \sum_{j=\frac{r+1}{2}}^{r-2} \frac{K-r+2(j+1)}{2(K-1)}$$
over the uncoded scheme

We then have the following sequence of equations.

\[
L_1(r) = \frac{1}{r} \left( \frac{K - r}{K - 1} + \frac{(K - r)(2r - 1)}{K - 1} \right)
\]

where (a) is obtained by changing the variable \( j' = (r - 2) - j \).

Similarly, for even \( r \), we have,

\[
L_2(r) = \frac{1}{2(K - 1)} \left( 2K + r - 2 - 2j - 2 \right) + \frac{1}{2(K - 1)} \left( 2K - r + 2j + 1 \right)
\]

where (a) is obtained by changing the variable \( j' = (r - 2) - j \).

The total communication load is therefore \( L_{\text{tota}}(r) = \frac{K - r}{K - 1} + \min(L_1(r), L_2(r)) \).

F. Advantage of Algorithm I over the uncoded scheme

In this subsection, we bound the advantage of the rebalancing schemes presented in this work, over the uncoded scheme, in which the nodes simply exchange all the data which was available at the removed node via uncoded transmissions. We know that the load of the uncoded scheme is \( L_u(r) = \frac{r}{K - 1} = r \). Consider the ratio of the communication load of Scheme 1 to that of the uncoded scheme. We then have the following sequence of equations,

\[
\frac{K - r + L_1(r)}{L_u(r)} = \frac{1}{r} \left( \frac{K - r}{K - 1} + \frac{(K - r)(2r - 1)}{K - 1} \right)
\]

\[
= \frac{K - r}{r(K - 1)} \left( 1 + 2r - 1 \right)
\]

\[
= 2(K - r) \frac{(K - 1) - (r - 1)}{K - 1} = 2 \left( \frac{(K - 1) - (r - 1)}{K - 1} \right)
\]
\[= 2 \left(1 - \frac{r - 1}{K - 1}\right).\]

Now we do the same for Scheme 2.

\[
\frac{K - r}{K - 1} + \frac{L_2(r)}{L_u(r)} = \frac{1}{r} \left(\frac{K - r}{K - 1} + \frac{r - 1}{2(K - 1)} \left(K + \frac{r - 1}{2}\right)\right)
\]
\[
= \frac{1}{2r(K - 1)} \left(2K - 2r + rK - K + \frac{(r - 1)^2}{2}\right)
\]
\[
= \frac{1}{2r(K - 1)} \left(K(r + 1) + \frac{(r - 1)^2 - 4r}{2}\right)
\]
\[
< \frac{1}{2r(K - 1)} \left(K(r + 1) + \frac{r^2}{2} - 4r\right)
\]
\[
\leq \frac{1}{2r(K - 1)} \left(K(r + 1) + \frac{r^2}{2} - 4r\right) + \frac{r - 4}{4(K - 1)}
\]
\[
\leq \frac{1}{2} + \frac{1}{2(K - 1)} \left(\frac{K - r}{r} + \frac{r}{2}\right) \leq \frac{1}{2} + \frac{1}{2r} + \frac{r}{4(K - 1)},
\]

where the last inequality follows by using the fact that \(K - r \leq K - 1\). Observe that \(\frac{1}{2} + \frac{1}{2r} + \frac{r}{4(K - 1)} < 1\), as \(r \geq 3\) and \(K - 1 \geq r\). As the choice of the scheme selected for transmissions is based on the minimum load of Scheme 1 and Scheme 2, we have the result in the theorem. This completes the proof of the node-removal part of Theorem 1.

IV. Proof of Theorem 1: Rebalancing for Single Node Addition in Cyclic Databases

We consider the case of a \(r\)-balanced cyclic database system when a new node is added. Let this new empty node be indexed by \(K + 1\). For this imbalance situation, we present a rebalancing algorithm (Algorithm 6) in which nodes split the existing data segments and broadcast appropriate subsegments, so that the target database which is an \(r\)-balanced cyclic database on \(K + 1\) nodes, can be achieved. Each subsegment’s size is assumed to be an integral multiple of \(\frac{1}{K+1}\). This is without loss of generality, by the condition on the size \(T\) of each segment as in Theorem 1. Since node \(K + 1\) starts empty, there are no coding opportunities; hence, the rebalancing scheme uses only uncoded transmissions. We show that this rebalancing scheme achieves a normalized communication load of \(\frac{r}{K+1}\), which is known to be optimal from the results of [1]. This proves the node-addition part of Theorem 1.
Algorithm 6 Rebalancing Scheme for Single Node Addition

1: procedure DELIVERY SCHEME
2:  for each $i \in [K]$ do
3:    Split $W_i$ into two subsegments, labelled $\tilde{W}_i$ of size $\frac{K}{K+1}$ and $W_i\{K+1\cup[\min(r-1,i-1)]$ of size $\frac{1}{K+1}$.
4:    Node $i$ broadcasts $W_i\{K+1\cup[\min(r-1,i-1)]$.
5:  end for
6:  Each node in $\{(K+1)\oplus_{K+1}\langle r \rangle\}$ initializes the segment $\tilde{W}_{K+1}$ as an empty vector.
7:  for each $i \in [K]$ do
8:    Each node in $\{(K+1)\oplus_{K+1}\langle r \rangle\}$ performs the concatenation $\tilde{W}_{K+1} = \tilde{W}_{K+1}|W_i\{K+1\cup[\min(r-1,i-1)]$.
9:  end for
10:  for each $i = (K-r+2)$ to $K$ do
11:    Node $i$ transmits $\tilde{W}_i$ to node $K + 1$.
12:  end for
13:  for each $i = 1$ to $r$ do
14:    Node $i$ discards $\tilde{W}_{K-r+1+i}$.
15:  end for
16: end procedure

Note: Once the target segments $\tilde{W}_1, \ldots, \tilde{W}_{K+1}$ are recovered at the required nodes, any extra bits present at the node are discarded.

A. Correctness of Algorithm 6

We verify the correctness of the rebalancing algorithm, i.e., we check that the target $r$-balanced cyclic database on $K + 1$ nodes is achieved post-rebalancing. To achieve the target database, each target segment $\tilde{W}_i$, for $i \in [K + 1]$, must be of size $\frac{K}{K+1}$ and stored exactly in $i \ominus_{K+1}\langle r \rangle$. Consider the segments $\tilde{W}_i$ for each $i \in [K-r+1]$. Since, these were a part of $W_i$, they are already present exactly at nodes $\{i \ominus_{K+1}\langle r \rangle\}$. Also, the nodes where $\tilde{W}_i$ is present but $\tilde{W}_i$ must not be present are given by $S_i \setminus \tilde{S}_i = \{K + 1\}$. This is performed in lines 10-12 of Algorithm 6. Also, the nodes where $\tilde{W}_i$ is present but $\tilde{W}_i$ must not be present are given by $S_i \setminus \tilde{S}_i = \{i - K + r - 1\}$. This node discards $\tilde{W}_i$ in lines 13-15 of Algorithm 6. Finally, $\tilde{W}_{K+1}$ must be present in nodes $\{(K+1)\ominus_{K+1}\langle r \rangle\}$. This can be obtained by the those nodes from the broadcasts made in line 4 in Algorithm 6 and the concatenation performed in lines 7-9.

Finally, It is easy to calculate that each target segment $\tilde{W}_i : i \in [K+1]$ is of size $\frac{K}{K+1}$. Thus, the final database satisfies the cyclic storage condition.
B. Communication Load of Algorithm 6

For broadcasting each $W_i^{(K+1)\cup [\min(r-1,i-1)]}$ of size $\frac{1}{K+1}$, $\forall i \in [K]$, the communication load incurred is $K \cdot \frac{1}{K+1} = \frac{K}{K+1}$. Now, to transmit $\tilde{W}_{K-r+2} \ldots \tilde{W}_K$ to node $K+1$, the communication load incurred is $(r-1) \cdot \frac{K}{K+1}$. Hence, the total communication load is $L_{add}(r) = r \cdot \frac{K}{K+1}$.

V. LOWER BOUND FOR REBALANCING LOAD FOR NODE-REMOVAL

We now obtain a lower bound for the number of transmitted bits required for node-removal rebalancing. In [1], a lower bound on the rebalancing load was obtained, which is applicable for any $r$-balanced database (not necessarily cyclic), which is $\frac{K}{r}$ with the cyclic database constraint in our case, obtaining a general lower bound on the rebalancing load that is applicable across all choices for the data placements at the surviving nodes seems difficult, as the number of such choices is prohibitively large. Hence, we proceed to give a lower bound in Subsection V-B, only for the specific data placement as indicated by the superscripts of the split subsegments in Algorithm 2. In Subsection V-C, we show that the achievable loads of Theorem 1 are within a multiplicative gap from the lower bound shown in Subsection V-B.

Once the demanded subsegments are fixed, the data exchange problem involved in the transmission phase of the rebalancing algorithm becomes a index coding problem [13]. Any such index coding problem can be represented [14] using a corresponding directed bipartite graph $G$ with left-vertices as the nodes and right vertices as the demanded subsegments. The definition of the edges in this graph is as follows.

- For each node $i$, a single solid edge points from the vertex corresponding to the demanded subsegment by node $i$ to the vertex corresponding to node $i$.
- For each node $i$, dashed edges point from the vertex corresponding to node $i$ to the vertices corresponding to the subsegments that are present in node $i$.

A lower bound for index coding problem was then derived in [14] by extracting an acyclic induced subgraph from $G$ by performing some pruning operations. We will use the same principles to find a lower bound for the rebalancing problem.

Recall the matrix $M$ used in the examples presented in Section III, which captures the index coding problem corresponding to the rebalancing algorithms (Algorithm 3 and 4). The rows of this matrix $M$ correspond to subsegments and the columns correspond to nodes. Recall that the $(i,j)$ entry $M_{i,j}$ is ‘*’ if the $i^{th}$subsegment is contained in the $j^{th}$node. Further, $M_{i,j} = ‘s’$ if the $i^{th}$subsegment must be delivered to the $j^{th}$node. Denote by $G_M$ the directed bipartite graph corresponding to the index coding problem defined by $M$. We illustrate the matrix and the corresponding graph for the case of $K = 5$ nodes and $r = 3$ as shown in Fig. 9a, and the corresponding bipartite graph shown in Fig. 9b.

For general $K, r$, we order the rows of the matrix $M$ in the following way.

- The first $2r-2$ rows of the matrix $M$ are arranged as given below. We will call the submatrix corresponding to these rows as $M_U$ (upper submatrix).
  - The largest subsegment of the corner segment $W_{K-r+1}$, denoted by $W_{K-r+1}^{(1)}$, is placed at the top.
(a) Matrix $M$ for $K = 5$, $r = 3$ case.

(b) Corresponding directed bipartite graph.

Fig. 9: The matrix and directed bipartite graph representing the index coding problem arising out of the choice of the demanded subsegments according to the splitting algorithm in Section III, for $K = 5$ and $r = 3$.

- Recall that for each $i \in [r - 2]$, the middle segment $W_{K-r+i+1}$ is split into two subsegments, i.e., $W_{K-r+i+1}^{(i+1)}$ (the first subsegment) and $W_{K-r+i+1}^{(i+K-r)}$ (the second subsegment). The first subsegments of the middle segments are placed next in ascending order of the segment index. Note that these are in ascending order of their respective sizes. The second subsegments of the middle segments are placed next in ascending order of the segment index. Note that these are in descending order of their respective sizes, as per Algorithm 2.

- The largest subsegment of the corner segment $W_K$ denoted by $W_K^{(K-1)}$ is placed at the bottom.

- Based on the two ways of splitting as described in Algorithm 2, the number of remaining rows in the matrix $M$ is either $K - r$ (when $K - r$ is even) or $K - r + 1$ (when $K - r$ is odd). These remaining rows are arranged according to the order given below. We will call the submatrix corresponding to these rows as $M_L$ (lower submatrix).

- First, the smaller subsegments of the corner segment $W_{K-r+1}$ are placed in subsequent rows in the ascending order of their superscripts (note that the list of superscripts varies according to the two ways of splitting).

- Subsequently, the smaller subsegments of the the corner segment $W_K$ are arranged in ascending order of their superscripts.

Fig. 10 shows an example for the matrix $M$, with the submatrices $M_U$ and $M_L$ as described.

Consider a submatrix $M'$ of $M$ of size $|R| \times |C|$, corresponding to a subset $R$ of rows and a subset of columns $C$ of $M$, such that $|R| \geq |C|$. It is easy to see that the subgraph of $G_M$ induced by the subsegments given by $R$, and the nodes given by $C$ has edges as per the entries of this submatrix $M'$. We therefore denote it as $G_{M'}$. The following lemma gives the property to be satisfied by $M'$ for $G_{M'}$ to be an acyclic induced subgraph of $G_M$. 


Lemma 1. Let $M'$ be the submatrix of $M$ obtained by picking some subset of its rows and columns. The graph $G_{M'}$ corresponding to $M'$ will be an acyclic induced subgraph of $G_M$, if the following property is true in $M'$ (upon applying some permutations to its columns):

- Property A: For each $i \in [R]$, let $j_{\text{min},i} \in [C]$ represent the smallest $j \in [C]$ such that the $j$th column of $M'$ contains a 's' in the $i$th row. Then $M'(i,j) \neq \ast$, for any $j > j_{\text{min},i}$.

Proof: Let $\tilde{j} \triangleq \max\{j_{\text{min},i} : \forall i \in [R]\}$. Without loss of generality, we assume $j_{\text{min},1} \leq j_{\text{min},2} \leq \cdots \leq j_{\text{min},|R|} = \tilde{j}$. Otherwise, we apply a suitable permutation on the rows such that this is true. Consider the submatrix $A$ comprised of the first $\tilde{j}$ columns of $M'$. For convenience, we index the subsegments (with respect to the rows of $A$, from top-to-bottom), as $b_1, \ldots, b_{|R|}$ and the storage nodes (corresponding to the columns) as $1, \ldots, \tilde{j}$. In the subgraph $G_A$ of $G_{M'}$ induced by the matrix $A$, by the given property, for every row in $[R]$ there is no ‘dashed’ edge (indicating side-information) to any subsegment $b_i$ from any node $j \in [\tilde{j}]$ such that $j > j_{\text{min},i}$. This immediately rules out the possibility of a cycle existing in $G_A$, given that $G_A$ is a finite graph. The remaining $(|C| - \tilde{j})$ columns in $M'$ which are not in $A$ do not contain any *'s and therefore results in no further dashed edges in $G_{M'}$. Hence, $G_{M'}$ is acyclic as well.

A. Algorithm for finding a submatrix of $M$ satisfying Property A

Algorithm 7 takes the matrix $M$ as input and generates a submatrix $M'$ that satisfies Property A of Lemma 1 as its output.

Algorithm 7 Finding a submatrix that satisfies Property A

Input: Matrix $M$

Output: A submatrix $M'$ of $M$ that satisfies Property A of Lemma 1

1: $M' \leftarrow$ empty matrix
2: $t \leftarrow \min(r - 1, K - r)$
3: for each row in the first $t$ rows in $M_U$ do
4: Include the entries in the first $K - r$ columns of this row as the entries in a new row of $M'$
5: end for
6: for each row in $M_L$ do
7: if this row does not have an ‘s’ in any column in the range $[K - 1] \backslash [K - r]$ then
8: Include the entries in the first $K - r$ columns of this row as the entries in a new row of $M'$.
9: end if
10: end for
11: Return $M'$

Example 1. We illustrate the above algorithm using this example. Consider the matrix $M$ for $K = 6, r = 3$ case given in Fig. 10.
Fig. 10: Matrix $M$ for $K = 6, r = 3$ case.

| Subsegments   | $1$ | $2$ | $3$ | $4$ | $5$ |
|---------------|-----|-----|-----|-----|-----|
| $W_4^{(1)}$   | $s$ | $-$ | $-$ | $*$ | $*$ |
| $W_5^{(2)}$   | $*$ | $s$ | $-$ | $-$ | $*$ |
| $W_5^{(4)}$   | $*$ | $-$ | $-$ | $s$ | $*$ |
| $W_6^{(5)}$   | $*$ | $*$ | $-$ | $-$ | $s$ |
| $W_4^{(2,3)}$ | $-$ | $s$ | $s$ | $*$ | $*$ |
| $W_4^{(3)}$   | $-$ | $-$ | $s$ | $*$ | $*$ |
| $W_6^{(3)}$   | $*$ | $*$ | $s$ | $-$ | $-$ |
| $W_6^{(3,4)}$ | $*$ | $*$ | $s$ | $s$ | $-$ |

From $M_U$, the number of rows picked will be $\min(r - 1, K - r) = \min(2, 3) = 2$. So, the first 2 rows and first 3 columns of $M_U$ are included in $M'$. From $M_L$, all the rows except the one corresponding to the subsegment $W_6^{(3,4)}$ (has an ‘$s$’ in the 4th column) will be picked. Therefore, Algorithm 7 will output the following submatrix $M'$ of $M$.

$$M' = \begin{bmatrix}
W_4^{(1)} & s & - & - \\
W_5^{(2)} & * & s & - \\
W_4^{(2,3)} & - & s & s \\
W_4^{(3)} & - & - & s \\
W_6^{(3)} & * & * & s
\end{bmatrix}$$

It is easy to see that $M'$ satisfies Property A. The induced acyclic subgraph $G_{M'}$ corresponding to $M'$ is shown in Fig. 11.
From the algorithm, we can clearly identify the \( \min(r-1, K-r) \times (K-r) \) submatrix of \( M_U \) included in \( M' \). The following claim gives the submatrix of \( M_L \) included in \( M' \). This claim will also enable us to show the lower bound on the rebalancing load for the specific choice of demands as given by \( M \).

**Claim 2.** The number of successive rows of \( M_L \) (starting from the top-most row) picked by Algorithm 7 to construct matrix \( M' \) is given as follows.

- **When \( K - r \) is even:**
  
  \[
  \begin{align*}
  \frac{K-r}{2}, & \quad r \geq \frac{K+1}{2} \\
  \frac{3K-5r+2}{2}, & \quad \text{if } r < \frac{K+1}{2} \text{ and } r > \frac{K+2}{3} \\
  K-r, & \quad \text{otherwise.}
  \end{align*}
  \]

- **When \( K - r \) is odd:**
  
  \[
  \begin{align*}
  \left\lfloor \frac{K-r}{2} \right\rfloor + 1, & \quad r \geq \frac{K+1}{2} \\
  \left\lfloor \frac{K-r}{2} \right\rfloor + K - 2r + 2, & \quad \text{if } r < \frac{K+1}{2} \text{ and } r > \frac{K+1}{3} \\
  K-r+1, & \quad \text{otherwise.}
  \end{align*}
  \]

**Proof:** Recall that the rows of \( M_L \) only correspond to the smaller subsegments of the corner segments, namely \( W_{K-r+1} \) and \( W_K \). Splitting is done differently for these segments when \( K - r \) is even and when \( K - r \) is odd (see Algorithm 2). Thus, the rows of the submatrix of \( M_L \) chosen to be included in \( M' \) is different for two cases, as we now elaborate.

1) **\( K - r \) is even:** We look at the subsegments of \( W_{K-r+1} \) and \( W_K \) separately. Let \( p = \frac{K-r}{2} \).

   a) \( W_{K-r+1} \): From Algorithm 2, we know that the smaller subsegments of \( W_{K-r+1} \) are labeled as \( W_{K-r+1}^B \). For \( B \in \{(K-r+1-j)\ominus K-1 \min(r,j) : j \in [p]\} \). Recall that each set \( B \) contains the indices of the nodes where the corresponding subsegment is demanded, i.e., \( M \) has an ‘s’ precisely in those columns, in the row corresponding to subsegment \( W_{K-r+1}^B \). It is easy to see that, across all the \( B \)’s, the largest element is \( K-r \) (for \( j = 1 \)) and the smallest element is \( \frac{K-r}{2} + 1 \) (for \( j = p \)). Thus, following the if condition in Algorithm 7, all the subsegments of \( W_{K-r+1} \) will be included in \( M' \). Therefore, the total number of rows corresponding to the subsegments of \( W_{K-r+1} \) picked to be included in \( M' \) is \( p = \frac{K-r}{2} \).

   b) \( W_K \): Similarly, the smaller subsegments of \( W_K \) are labeled as \( W_K^B \), for \( B \in \{\{(r-1+j)\ominus K-1 \min(r,j) : j \in [p]\} \). It is easy to see that, across all the \( B \)’s, the largest element is \( r + \frac{K-r}{2} - 1 \) (for \( j = p \)) and the smallest element is \( r \) (for \( j = 1 \)). Recall that Algorithm 7 will include those rows corresponding to each such \( B \) which has its largest element to be at most \( K-r \). Let us first consider the case when the smallest element overall (considering all the \( B \)’s) is \( \geq K-r+1 \), i.e., \( r \geq K-r+1 \), which implies \( r \geq \frac{K+1}{2} \). In this case, no subsegment \( W_K^B \) qualifies as a row-index include-able in \( M' \), i.e., the number of rows picked from \( M_L \) would be zero. Note that in all the other cases where at least one subsegment is included, we would have \( r < \frac{K+1}{2} \). Next, let us consider the case when the largest element overall is \( \leq K-r \), i.e., \( r + \frac{K-r}{2} - 1 \leq K-r \), which implies \( r \leq \frac{K+2}{3} \). In this case, all the subsegments would be included, i.e., the number of rows picked would be \( p = \frac{K-r}{2} \). For the other cases, let \( x \) be a non-negative integer that \( r \leq r + x = K-r < r + \frac{K-r}{2} - 1 \), i.e., \( x = K-2r \), which implies that the number of subsegments (i.e., number of rows to be picked) to be included would be \( x+1 = K-2r+1 \).
2) $K - r$ is odd: Again, we look at the subsegments of $W_{K-r+1}$ and $W_K$ involved in $M_L$ separately. Let $p = \lfloor \frac{K-r}{2} \rfloor$.

a) $W_{K-r+1}$: All the $p$ subsegments are labeled in the same way as in the case of $K - r$ being even, except that there is an additional subsegment labeled as $W_{K-r+1}^B$ where $B = \{(K-r-p) \mod K-1 \langle \min(r, p+1) \rangle \}$. Using similar arguments as in the previous case, we will have that no such $B$ will have an element $> K-r$, which implies that all the subsegments of $W_{K-r+1}$ will be included. Therefore, the total number of rows picked is $p + 1 = \lfloor \frac{K-r}{2} \rfloor + 1$.

b) $W_K$: For $W_K$ as well, apart from the $p$ subsegments mentioned as in the case of $K - r$ being even, there is an additional subsegment labeled as $W_K^B$, where $B = \{(r+p) \mod K-1 \langle \min(r, p+1) \rangle \}$. Note that as in the previous case, the smallest element overall is $r$. So, if $r \geq K - r + 1$, i.e., $r \geq K + 1$, no subsegments would be included, i.e., the number of rows picked from $M_L$ to be included in $M'$ is zero. Also, the largest element overall (considering all the $B$'s) is equal to $r + \lfloor \frac{K-r}{2} \rfloor$, which corresponds to the subsegment of size $\frac{1}{2(K-1)}$. So, if $r + \lfloor \frac{K-r}{2} \rfloor \leq K - r$, i.e., $r \leq \frac{K+1}{3}$, then all the subsegments of $W_K$ would be included, i.e., the number of rows picked would be $\lfloor \frac{K-r}{2} \rfloor + 1$. For other cases, i.e., when $r > \frac{K+1}{3}$, the number of rows picked would be $K - 2r + 1$, using similar arguments as in the previous case.

Now, we show that the matrix $M'$ output from Algorithm 7 indeed satisfies Property A of Lemma 1.

**Lemma 2.** Algorithm 7 outputs a submatrix $M'$ of $M$ that satisfies Property A.

**Proof:** We first look at the rows picked from $M_U$. It is clear that the subsegment $W_{K-r+1}^{\{1\}}$ is picked. The first row has an ‘$s$’ in the first column and a ‘$*$’ in each column only in the range $[K-r+1, K-1]$ due to cyclic placement. But these columns $[K-r+1, K-1]$ are not included in $M'$ (only few rows and the first $K-r$ columns of $M$ are included). So, this row readily satisfies Property A. The next $\min(r-1, K-r) - 1$ rows of $M_U$ correspond to the subsegments of the middle segments denoted by $W_{K-r+i+1}^{\{i+1\}}$, for $i \in [r-2]$. These subsegments are present only in the nodes $[K-r+i+1, K-1] \cup [i]$. So, the corresponding rows would have a ‘$*$’ in these columns and an ‘$s$’ in column $i+1$. Therefore, these $\min(r-1, K-r) - 1$ rows which are picked by Algorithm 7 to be included in $M'$ will satisfy property A.

Now, we look at the rows picked from $M_L$. We know that the rows of $M_L$ are indexed by the smaller subsegments of the corner segments $W_{K-r+1}$ and $W_K$. From the proof of Claim 2, we know that the rows corresponding to the subsegments of $W_{K-r+1}$ have an ‘$s$’ only in the column range $[K-r+1, K-r]$. Furthermore, since these rows have a ‘$*$’ only within the range $[K-r+1, K-1]$, it follows that all these rows readily satisfy Property A. For the rows indexed by any subsegment of $W_K$, there is a ‘$*$’ only in the columns $[r-1]$, as it is present in these nodes. Furthermore, from proof of Claim 2, we know that the rows corresponding to its smaller subsegments can have an ‘$s$’ in a column with index greater than or equal to $r$. Therefore, all these rows satisfy Property A as well.
B. Lower Bound for Rebalancing Load for specific choice of demands given by $M$

Using Lemmas 1 and 2, we can now prove the lower bound for the rebalancing load for single node removal, for the case of any target database that is achieved using the matrix $M$ for the choices for the demands at the surviving nodes.

**Theorem 2.** The rebalancing load, with the demands in the delivery scheme as given by the matrix $M$, satisfies the following bounds.

1) if $r \geq \frac{K+1}{2}$: \[ \frac{2r(K-r)}{2(K-1)}. \]

2) if $r < \frac{K+1}{2}$ and $K - r$ is even: \[ \begin{cases} \frac{2K+r(K-5)+2}{2(K-1)}, & \text{if } r > \frac{K+2}{3} \\ \frac{K+r(K-2)}{2(K-1)}, & \text{otherwise}. \end{cases} \]

3) if $r < \frac{K+1}{2}$ and $K - r$ is odd: \[ \begin{cases} \frac{2K+r(K-5)+2}{2(K-1)}, & \text{if } r > \frac{K+1}{3} \\ \frac{K+r(K-2)}{2(K-1)}, & \text{otherwise}. \end{cases} \]

**Proof:** From Lemmas 1 and 2, we see that we only have to sum the sizes of the subsegments included in the matrix $M'$.

First, let’s calculate the sum of the sizes of the subsegments included in $M'$ from the submatrix $M_U$ of $M$. From algorithm 7, we know that there are two cases for $M_U$: i) $r - 1 \geq K - r$, i.e., $r \geq \frac{K+1}{2}$ and ii) $r - 1 < K - r$, i.e., $r < \frac{K+1}{2}$. Furthermore, as mentioned in the proof of Lemma 2, the subsegments included in $M'$ corresponding to the rows of $M_U$ are $W_{K-r+1}^{(i+1)}$ and the subsegments of the middle segments denoted by $W_{K-r+i+1}^{(i+1)}$, $i \in [r-2]$.

1) $r \geq \frac{K+1}{2}$: From Algorithm 2, we know that the size of the subsegment $W_{K-r+1}^{(i+1)}$ is $\frac{K+r-2-2i}{2(K-1)}$ and the size of the subsegment $W_{K-r+i+1}^{(i+1)}$, $i \in [r-2]$ is $\frac{K+r-2-2i}{2(K-1)}$. It is clear that these form an arithmetic progression (including $\frac{K+r-2}{2(K-1)}$). In this case, we have to sum over the first $K - r$ terms. Therefore, we have

\[
\begin{align*}
\frac{K + r - 2}{2(K-1)} + \frac{K + r - 4}{2(K-1)} + \cdots + \frac{K + r - 2(K-r)}{2(K-1)} &= \frac{(K + r)(K-r) - 2(1+2+\cdots+(K-r))}{2(K-1)} \\
&= \frac{(K + r)(K-r) - (K - r + 1)(K-r)}{2(K-1)} \\
&= \frac{(K-r)(2r-1)}{2(K-1)}.
\end{align*}
\]

2) $r < \frac{K+1}{2}$: In this case, we have to sum over the first $r - 1$ terms of the arithmetic progression. Therefore, we have

\[
\begin{align*}
\frac{K + r - 2}{2(K-1)} + \frac{K + r - 4}{2(K-1)} + \cdots + \frac{K + r - 2(r-1)}{2(K-1)} &= \frac{(K + r)(r-1) - 2(1+2+\cdots+(r-1))}{2(K-1)} \\
&= \frac{(K + r)(r-1) - r(r-1)}{2(K-1)} \\
&= \frac{K(r-1)}{2(K-1)}.
\end{align*}
\]

So, for $M_U$, the sum is

\[
\begin{cases} \frac{(K-r)(2r-1)}{2(K-1)}, & \text{if } r \geq \frac{K+1}{2} \\ \frac{K(r-1)}{2(K-1)}, & \text{otherwise}. \end{cases}
\]
Now, we calculate the sum of the sizes of the subsegments included from the submatrix $M_L$. From Claim 2, we know exactly how many rows were picked from the submatrix $M_L$. We have the following cases for $M_L$.

1) $K - r$ is even: We look at the subsegments of $W_{K-r+1}$ and $W_K$ separately.

   a) $W_{K-r+1}$: From Algorithm 2, we know that all the smaller subsegments of $W_{K-r+1}$ are of size $\frac{2}{2(K-1)}$.

      From part (1a) in proof of Claim 2, we know that $\frac{K-r}{2}$ subsegments are included, so the sum of the sizes of all the subsegments is $\frac{2}{2(K-1)} \cdot \frac{K-r}{2} = \frac{K-r}{2(K-1)}$.

   b) $W_K$: Similarly, all the smaller subsegments of $W_K$ are of size $\frac{2}{2(K-1)}$. From part (1b) in proof of Claim 2, we know that there are three cases for $W_K$. The sum of the sizes of the subsegments included for the three cases are as follows.

      i) $r \geq \frac{K+1}{2}$: No subsegments are included, so the sum is 0.

      ii) $r < \frac{K+1}{2}$ and $r > \frac{K+2}{3}$: $(K - 2r + 1)$ subsegments are included, so the sum is $\frac{2}{2(K-1)} \cdot (K - 2r + 1) = \frac{2}{2(K-1)} \cdot (K - 2r + 1) = \frac{2}{2(K-1)}$.

      iii) $r < \frac{K+1}{2}$ and $r \leq \frac{K+2}{3}$: All $\frac{K-r}{2}$ subsegments are included, so the sum is $\frac{2}{2(K-1)} \cdot \frac{K-r}{2} = \frac{2}{2(K-1)}$.

      So, when $K - r$ is even, the total sum is

      $\begin{align*}
      \begin{cases}
      \frac{K-r}{2(K-1)}, & \text{if } r \geq \frac{K+1}{2} \\
      \frac{3(K-5r+2)}{2(K-1)}, & \text{if } r < \frac{K+1}{2} \text{ and } r > \frac{K+2}{3} \\
      \frac{2(K-r)}{2(K-1)}, & \text{otherwise.}
      \end{cases}
      \end{align*}$

2) $K - r$ is odd: Again, we look at the subsegments of $W_{K-r+1}$ and $W_K$ separately. Let $p = [\frac{K-r}{2}]$.

   a) $W_{K-r+1}$: As mentioned in part (2a) in the proof of Claim 2, we know that all the subsegments of $W_{K-r+1}$ are included in $M'$. Also, other than the $p$ subsegments, there is an additional subsegment. From Algorithm 2, we know that this additional subsegment is of size $\frac{1}{2(K-1)}$ and all the other $p$ subsegments are of size $\frac{2}{2(K-1)}$. Therefore, the sum of the sizes of the subsegments included is $\frac{2}{2(K-1)} \cdot [\frac{K-r}{2}] + \frac{1}{2(K-1)} = \frac{2}{2(K-1)} \cdot \frac{K-r}{2} + \frac{1}{2(K-1)} = \frac{K-r}{2(K-1)}$.

   b) $W_K$: Similarly, for $W_K$ as well, other than the $p$ subsegments of size $\frac{2}{2(K-1)}$, there is an additional subsegment of size $\frac{1}{2(K-1)}$. From part (2b), in proof of Claim 2, we know that there are three cases for $W_K$. The sum of the sizes of the subsegments included for the three cases are as follows.

      i) $r \geq \frac{K+1}{2}$: No subsegments are included, so the sum is 0.

      ii) $r < \frac{K+1}{2}$ and $r > \frac{K+2}{3}$: $(K - 2r + 1)$ subsegments are included, so the sum is $\frac{2}{2(K-1)} \cdot (K - 2r + 1) = \frac{2}{2(K-1)} \cdot (K - 2r + 1) = \frac{2}{2(K-1)}$.

      iii) $r < \frac{K+1}{2}$ and $r \leq \frac{K+2}{3}$: All $\frac{K-r}{2}$ subsegments are included, so the sum is $\frac{2}{2(K-1)} \cdot \frac{K-r}{2} = \frac{2}{2(K-1)} \cdot \frac{K-r}{2} = \frac{2}{2(K-1)}$.

      So, when $K - r$ is odd, the total sum is

      $\begin{align*}
      \begin{cases}
      \frac{K-r}{2(K-1)}, & \text{if } r \geq \frac{K+1}{2} \\
      \frac{3(K-5r+2)}{2(K-1)}, & \text{if } r < \frac{K+1}{2} \text{ and } r > \frac{K+2}{3} \\
      \frac{2(K-r)}{2(K-1)}, & \text{otherwise.}
      \end{cases}
      \end{align*}$

On summing for all the cases, we get the expressions as given in the Theorem.
Remark 4. The lower bound of Theorem 2 is derived using the index coding lower bound in [14]. However, in this work, the assumption is that there is a central server which contains all the requested data, which is responsible for the delivery of the coded symbols. In our case, no surviving node has all the data in the removed node. Nevertheless, the lower bound in Theorem 2 still applies.

C. Optimality of Lower Bound from Theorem 2 up to a multiplicative factor

We now show that the achievable rebalancing load for node-removal from Theorem 1 is within a multiplicative factor of the lower bound given in Theorem 2. Towards that end, we first prove the following inequality.

Claim 3. The lower bounds given in Theorem 2 are greater than \( r(K - r) \) for all values of \( K \geq 4 \) and all \( r \in \{3, \ldots, K - 1\} \).

Proof: We prove the claim for the three cases given in Theorem 2 separately. For simplicity, we ignore the denominator \( \frac{1}{2(K - 1)} \), which is common in all expressions.

1) if \( r \geq \frac{K + 1}{2} \): The numerator of the lower bound is \( 2r(K - r) \), which is clearly greater than \( r(K - r) \).

2) if \( r < \frac{K + 1}{2} \) and \( K - r \) is even: There are two subcases in this case.

a) if \( r > \frac{K + 2}{3} \): The numerator is \( 2K + r(K - 5) + 2 \). We have the following.

\[
2K + r(K - 5) + 2 > 2r + r(K - 5) + 2 \quad (\because K > r)
\]
\[
> r(K - 3) + 2
\]
\[
> r(K - r). \quad (\because r \geq 3)
\]

b) if \( r \leq \frac{K + 2}{3} \): The numerator is \( K + r(K - 2) \). We have the following.

\[
K + r(K - 2) > K + r(K - r) \quad (\because r \geq 3)
\]
\[
> r(K - r).
\]

3) if \( r < \frac{K + 1}{2} \) and \( K - r \) is odd: The expressions for the lower bound are same as the previous case, so no proof is needed.

Now, we show the multiplicative gap between the lower bound and the rebalancing load of the achievable schemes.

Theorem 3. The achievable rebalancing load for node-removal from Theorem 1 is within a multiplicative gap of 13 from the corresponding lower bound expressions in Theorem 2.

Proof: For simplicity, we once again ignore the denominator \( \frac{1}{2(K - 1)} \) throughout.

Let \( \tilde{L}_0(r) = 2(K - r) \). Let \( \tilde{L}_1(r) = 2(K - 1)L_1(r) \) and \( \tilde{L}_2(r) = 2(K - 1)L_2(r) \), where \( L_1(r) \) and \( L_2(r) \) are as in Theorem 1. By Claim 3 and Theorem 1, it is sufficient to prove that \( \tilde{L}_0(r) + \tilde{L}_i(r) < 13r(K - r) \), for \( i = 1, 2 \), under the respective regimes \( r \geq \lceil \frac{2K + 2}{3} \rceil \), and \( r < \lceil \frac{2K + 2}{3} \rceil \).
Let us first consider the case when \( r \geq \lceil \frac{2K+2}{3} \rceil \), where Scheme 1 is applicable. In this case, we have from the statement of Theorem 1 \( \tilde{L}_0(r) + \tilde{L}_1(r) = 2(K - r) + 2(K - r)(2r - 1) = 4r(K - r) \). Hence, the statement of the present claim holds in this case.

When \( r < \lceil \frac{2K+2}{3} \rceil \), Scheme 2 is applicable. By Theorem 1, \( \tilde{L}_2(r) \) as two cases depending on the parity of \( r \). We will only show the multiplicative gap for the case when \( r \) is odd. Similar arguments will hold for the case when \( r \) is even, as the expressions for the rebalancing load are almost identical. When \( r \) is odd, we have \( \tilde{L}_0(r) + \tilde{L}_2(r) = 2(K - r) + (r - 1)(K + \frac{r-1}{2}) \). Clearly, \( L_2(r) = (r - 1)(K + \frac{r-1}{2}) < r(K + r) \). For the case \( K \geq 4 \) and \( r < \lceil \frac{2K+2}{3} \rceil \), we will now show that, \( \tilde{L}_0(r) + r(K + r) = 2(K - r) + r(K + r) < 13r(K - r) \). This will complete the proof of the theorem. Consider the following sequence of inequalities.

\[
13r(K - r) - 2(K - r) - r(K + r) > 13r(K - r) - r(K - r) - r(K + r) \quad (\because r \geq 3)
\]
\[
> 12r(K - r) - r(K + r)
\]
\[
= r(12K - 12r - K - r)
\]
\[
= r(11K - 13r)
\]
\[
> r \left(11 - 13 \left(\frac{2K + 2}{3}\right)\right) \quad (\because r < \lceil \frac{2K + 2}{3} \rceil)
\]
\[
= r \left(\frac{7K - 26}{3}\right) > 0 \quad (\because K \geq 4 \text{ and } r > 0).
\]

This completes the proof.

**VI. Numerical Comparisons**

A numerical comparison of the lower bound given in Theorem 2 with the rebalancing load given in Theorem 1, along with the tight lower bound based on the results of [1] \( L_{ren}^*(r)N = \frac{rK}{K(r-1)} = \frac{r}{r-1} \) is shown in Fig. 12, for \( K = 15 \) and \( r \) varying in the range \([3, K - 2]\). Recall from Algorithm 1 that Scheme 1 is applicable for \( r \geq \lceil \frac{2K+2}{3} \rceil = 11 \), otherwise Scheme 2 is applicable. We do not plot the \( r = K - 1 \) scenario in Fig. 12. In this case, the lower bound from Theorem 2 is 1, whereas the tight lower bound from [1] is \( \frac{r}{r-1} = \frac{11}{10} \). This is not a contradiction, because of the fact that Theorem 2 is applicable in the centralized setting, while the result from [1] is not (as in Remark 4).

**VII. Discussion**

We have presented a XOR-based coded rebalancing scheme for the case of node removal and node addition in cyclic databases. Our scheme only requires the file size to be only cubic in the number of nodes in the system, as compared to the scheme in [1] which requires exponential file size. For the node removal case, we present a coded rebalancing algorithm that chooses between the better of two coded transmission schemes in order to reduce the communication load incurred in rebalancing. We showed that the communication load of this rebalancing algorithm is always smaller than that of the uncoded scheme. For the node addition case, we present a simple uncoded scheme which achieves the optimal load. For the specific data placements of the node-removal rebalancing algorithm, we
Fig. 12: For $K = 15$, the figure shows comparisons of the lower bound from Theorem 2 with the rebalancing load given in Theorem 1 (using a large file size), as well as the optimal load achieved by the scheme in [1], for varying $r$.

present a lower bound for the load in Theorem 2, and show that the achievable loads are within a multiplicative factor of this lower bound.

We give a few comments regarding other future directions. Constructing good converse arguments in the cyclic database setting for the minimum communication load required for rebalancing from node-removal seems to be a challenging problem, due to the freedom involved in choosing the target database, the necessity of the target database to be balanced, and also because of the low file size requirement. We also presented in Fig. 2 a numerical comparison of the loads of our schemes with the converse from [1] (which is applicable for all $r$-balanced databases), as well as the lower bound from Theorem 2 for the specific placements dictated by Algorithm 2. There is a large difference between these two lower bounds. However, the conditions of having a constrained file size or balanced target database are not used to show the converse in [1], thus this bound is possibly quite loose for our cyclic placement setting. It would be certainly worthwhile to construct a tight converse for our specific setting that applies to cyclic placement as well as to all possible data placements. Designing rebalancing schemes for node removal with lower communication loads, as well as for other types of placement schemes, is also an interesting future direction.

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[1] P. Krishnan, V. Lalitha, and L. Natarajan, “Coded data rebalancing: Fundamental limits and constructions,” in 2020 IEEE International Symposium on Information Theory (ISIT). IEEE, 2020, pp. 640–645.
First, we can assume $K \geq 4$ without loss of generality, as our replication factor $r$ lies between $\{3, \ldots, K - 1\}$. Consider the expressions in Theorem 1 for $L_1(r)$ and $L_2(r)$ as continuous functions of $r$. Also, consider $L_{2,o}(r) = \frac{K-r}{(K-1)} + \frac{r-1}{2(K-1)} \left( K + r - 1 \right)$ be the continuous function of $r$ which matches with $\frac{K-r}{K-1} + L_2(r)$ in Theorem 1 for odd values of $r \in \{3, \ldots, K-1\}$. Similarly, let $L_{2,e}(r) = \frac{K-r}{(K-1)} + \frac{r-2}{2(K-1)} \left( K + \frac{r}{2} \right) + \frac{K}{2(K-1)}$ be the continuous function of $r$ which matches with $\frac{K-r}{K-1} + L_2(r)$ in Theorem 1 for even values of $r \in \{3, \ldots, K-1\}$.

Let $r_o$ be a real number in the interval $[3 : K - 1]$ such that $L_{2,o}(r_o) = L_1(r_o)$. Similarly, let $r_e$ be a real number in the interval $[3 : K - 1]$ such that $L_{2,e}(r_e) = L_1(r_e)$.

With these quantities set up, the proof then proceeds according to the following steps.

1) Firstly, we show that $L_{2,o}(r) > L_{2,e}(r)$ for $K \geq 4$.

2) We then find the values of $r_o$ and $r_e$, which turn out to be unique. We also show that $r_e > r_o$ and that $\lceil r_e \rceil = \left\lceil \frac{2K+2}{3} \right\rceil = \lceil r_o \rceil + 1$.

3) Then, we shall show that for any integer $r > r_e$, we have $L_1(r) < L_{2,e}(r)$. Further, we will also show that for any integer $r < r_o$, we have $L_{2,o}(r) < L_1(r)$.
It then follows from the above steps that the threshold value is precisely \( r_{th} = \lfloor r_e \rfloor = \lceil \frac{2K+2}{3} \rceil \). We now show the above steps one by one.

1) **Proof of \( L_{2,o}(r) > L_{2,e}(r) \) for \( K \geq 4 \):** We have that

\[
L_{2,o}(r) - L_{2,e}(r) = \frac{r - 1}{2(K - 1)} \left( K + \frac{r - 1}{2} \right) - \frac{r - 2}{2(K - 1)} \left( K + \frac{r}{2} \right) - \frac{K}{2(K - 1)} \\
= \frac{(r - 1)(2K + r - 1) - (r - 2)(2K + r) - 2K}{4(K - 1)} \\
= \frac{2rK - 2K + r^2 - 2r + 1 - 2rK + 4K - r^2 + 2r - 2K}{4(K - 1)} \\
= \frac{1}{4(K - 1)} > 0,
\]

which holds as \( K \geq 4 \). Hence, \( L_{2,o}(r) > L_{2,e}(r) \) for \( K \geq 4 \).

2) **Finding \( r_o \) and \( r_e \) and their relationship:** Calculating \( L_{2,o}(r) - L_1(r) \), we get the following

\[
L_{2,o}(r) - L_1(r) = \frac{r - 1}{2(K - 1)} \left( K + \frac{r - 1}{2} \right) - \frac{(K - r)(2r - 1)}{(K - 1)} \\
= \frac{(r - 1)(2K + r - 1) - 4(K - r)(2r - 1)}{4(K - 1)} \\
= \frac{9r^2 - 6r(K + 1) + 2K + 1}{4(K - 1)}.
\]

Solving for \( r \) from \( L_{2,o}(r) - L_1(r) = 0 \) with the condition that \( r \geq 3 \), we get

\[
r = \frac{2K + 1}{3}.
\]

It is easy to see that \( \frac{2K+1}{3} > 2 \) for \( K \geq 4 \). Also, we can check that \( \frac{2K+1}{3} < (K - 1) \), when \( K \geq 4 \). Thus, we have that \( r_o = \frac{2K+1}{3} \). By similar calculations for \( L_{2,e}(r) - L_1(r) \), we see that \( r_e = \frac{K+1+\sqrt{K^2+1}}{3} \). Further, we observe that \( r_e - r_o = \frac{\sqrt{K^2+1} - K}{3}, \) for \( K \geq 4 \). Hence, \( r_e > r_o \) for \( K \geq 4 \). Also observe that \( [r_e] \leq \lceil \frac{2K+2}{3} \rceil = \lfloor \frac{2K+1}{3} \rfloor + 1 = [r_o] + 1 \), where the first equality holds as \( K \geq 4 \). Now, as \( r_e > r_o \) we have that \( [r_e] > [r_o] \). Thus, we see that \( [r_e] = \lceil \frac{2K+2}{3} \rceil = [r_o] + 1 \).

**Proof of 3):** We now prove that for any integer \( r \) if \( r > r_e \), then \( L_1(r) < L_{2,e}(r) \). Let \( r = r_e + a \), for some real number \( a > 0 \) such that \( r_e + a \) is an integer. We know that \( L_1(r_e) = L_{2,e}(r_e) \). Consider the following sequence of equations.

\[
L_1(r) = L_1(r_e + a) \\
= \frac{K - (r_e + a)}{K - 1} + \frac{(K - (r_e + a))(2(r_e + a) - 1)}{(K - 1)} \\
= \frac{K - r_e}{K - 1} - \frac{a}{K - 1} + \frac{(K - r_e)(2r_e - 1 + 2a)}{K - 1} \\
= \frac{K - r_e}{K - 1} + \frac{(K - r_e)(2r_e - 1)}{K - 1} + \frac{-a - a(2r_e - 1 + 2a) + 2a(K - r_e)}{K - 1} \\
= L_1(r_e) + \frac{2aK - 2a^2 - 4ar_e}{K - 1} \\
= L_{2,e}(r_e) + \frac{2aK - 2a^2 - 4ar_e}{K - 1}.
\]
Similarly, consider the following.

\[ L_{2,e}(r) = L_{2,e}(r_e + a) \]

\[ = \frac{K - (r_e + a)}{(K - 1)} + \frac{(r_e + a) - 2}{2(K - 1)} \left( K + \frac{r_e + a}{2} \right) \]

\[ = \frac{K - r_e}{(K - 1)} - \frac{a}{K - 1} + \frac{(r_e - 2) + a}{2(K - 1)} \left( K + \frac{(r_e + a)}{2} \right) \]

\[ = \frac{K - r_e}{(K - 1)} + \frac{r_e - 2}{2(K - 1)} \left( K + \frac{r_e}{2} \right) - 6a + a^2 + 2aK + 2ar_e \]

\[ = L_{2,e}(r_e) + \frac{-6a + a^2 + 2aK + 2ar_e}{4(K - 1)}. \]

Now, \( L_{2,e}(r) - L_1(r) = \frac{18ar_e - 6aK + 9a^2 - 6a}{4(K - 1)} \]

\[ = \frac{a(18r_e - 6K + 9a - 6)}{4(K - 1)} \]

\[ \geq \frac{a(6\sqrt{K^2 + 1} + 9a)}{4(K - 1)} \]

\[ \geq 0, \]

where (a) holds on substituting \( r_e = \frac{K + 1 + \sqrt{K^2 + 1}}{3} \), (b) holds as \( K, a > 0 \). Therefore, when \( r > r_e \), \( L_1(r) < L_{2,e}(r) \).

We now prove that for any integer \( r \) if \( r < r_o \), then \( L_{2,o}(r) < L_1(r) \). Let \( r = r_o - a \), for some real number \( a < r_o \) such that \( r_o - a \) is an integer. We know that \( L_1(r_o) = L_{2,o}(r_o) \). Consider the following sequence of equations.

\[ L_1(r) = L_1(r_o - a) \]

\[ = \frac{K - (r_o - a)}{(K - 1)} + \frac{(K - (r_o - a))(2(r_o - a) - 1)}{(K - 1)} \]

\[ = \frac{K - r_o}{K - 1} + \frac{a}{K - 1} + \frac{(K - r_o + a)(2r_o - 1 - 2a)}{K - 1} \]

\[ = \frac{K - r_o}{K - 1} + \frac{(K - r_o)(2r_o - 1) - a + a(2r_o - 1 - 2a) - 2a(K - r_o)}{K - 1} \]

\[ = L_1(r_o) + \frac{4ar_o - 2aK - 2a^2}{K - 1} \]

\[ = L_{2,o}(r_o) + \frac{4ar_o - 2aK - 2a^2}{K - 1}. \]

Similarly, consider the following.

\[ L_{2,o}(r) = L_{2,o}(r_o - a) \]

\[ = \frac{K - (r_o - a)}{(K - 1)} + \frac{(r_o - a) - 1}{2(K - 1)} \left( K + \frac{(r_o - a) - 1}{2} \right) \]

\[ = \frac{K - r_o}{(K - 1)} + \frac{a}{K - 1} + \frac{(r_o - 1) - a}{2(K - 1)} \left( K + \frac{(r_o - 1) - a}{2} \right) \]

\[ = \frac{K - r_o}{(K - 1)} + \frac{(r_o - 1)(K + \frac{r_o - 1}{2})}{2(K - 1)} + \frac{6a - 2ar_o + a^2 - 2aK}{4(K - 1)} \]
Now, $L_1(r) - L_{2,o}(r) = \frac{18a r_o - 6aK - 9a^2 - 6a}{4(K - 1)}$

\[= a\frac{(18r_o - 6K - 9a - 6)}{4(K - 1)}\]

\[= a\frac{3a(3r_o - 3a + 3r_o - 2K - 2)}{4(K - 1)}\]

\[\geq a\frac{3a(3 + 3r_o - 2K - 2)}{4(K - 1)}\]

\[\geq a\frac{3a(3r_o - 2K + 1)}{4(K - 1)}\]

\[\geq a\frac{3a(1 + 1)}{4(K - 1)}\]

\[\geq 0,\]

where (a) holds because $r_o - a \geq 1$ and (b) holds on substituting $r_o = \frac{2K+1}{3}$. Therefore, when $r < r_o$, $L_{2,o}(r) < L_1(r)$. The proof is now complete.