Coherence is a fundamental ingredient for quantum physics and a key resource for quantum information theory. Baumgratz, Cramer, and Plenio established a rigorous framework (BCP framework) for quantifying coherence [1]. The BCP framework has been widely accepted and triggered rapidly growing research for quantifying coherence (recent reviews see [2, 3]).

To construct a coherence measure, we first define the incoherent states and incoherent operations as in [1]. For a fixed orthonormal basis \{\{j\}\}_{j=1}^d of the d-dimensional Hilbert space \(H\), a quantum state \(\sigma\) on \(H\) is called incoherent with respect to \{\{j\}\}_{j=1}^d if \(\sigma\) is diagonal when expressed in \{\{j\}\}_{j=1}^d.\). We denote the set of all incoherent states by \(I\), and the set of density operators by \(D\). A quantum operation \(\Phi\), or called a CPTP (completely positive trace preserving) map, can be expressed by a set of Kraus operators \{\{K_n\}_n\} satisfying \(\sum_n K_n^\dagger K_n = I\). A quantum operation \(\Phi\) is called an incoherent operation (ICPTP) \(\Phi\), if it admits a set of Kraus operators \{\{K_n\}_n\} and \(K_n\sigma K_n^\dagger\) is diagonal for any \(n\) and any incoherent state \(\sigma\). Notice that the definitions of incoherent state, incoherent operation, and also the coherence measure, are all depend on the fixed orthonormal basis \{\{j\}\}_{j=1}^d, we call this basis the reference basis.

The BCP framework consists of the following postulates (C1-C4) that any quantifier of coherence \(C\) should fulfill. (C1) Non-negativity:

\[
C(\rho) \geq 0, \quad C(\rho) = 0 \iff \rho \in I. \tag{1}
\]

(C2) Monotonicity: \(C\) does not increase under the operation of any incoherent operation \(\Phi_I\),

\[
C[\Phi_I(\rho)] \leq C(\rho). \tag{2}
\]

(C3) Strong monotonicity: for any incoherent operation \(\Phi_I = \{K_n\}_n\),

\[
\sum_n \text{tr}(K_n \rho K_n^\dagger) C[\frac{K_n \rho K_n^\dagger}{\text{tr}(K_n \rho K_n^\dagger)}] \leq C(\rho). \tag{3}
\]

(C4) Convexity: \(C\) is a convex function of the state, i.e.,

\[
\sum_n p_n C(\rho_n) \geq C(\sum_n p_n \rho_n), \tag{4}
\]

where \(p_n > 0, \sum_n p_n = 1, \rho_1, \rho_2 \in D\).

We call a quantifier \(C\) satisfying (C1-C4) together a coherence measure. Note that C3+C4 implies C2 [1].

Yu, Zhang, Xu, and Tong proposed condition (C5), and showed that (C1-C4) is equivalent to (C1+C2+C5) [4]. (C5) additivity on block-diagonal states:

\[
C(\rho_1 \rho_1 \oplus \rho_2 \rho_2) = p_1 C(\rho_1) + p_2 C(\rho_2), \tag{5}
\]

where \(p_1 > 0, p_2 > 0, p_1 + p_2 = 1, \rho_1, \rho_2 \in D\).

BCP framework draws strong attention and discussion, but it is not the unique framework for quantifying coherence, and other potential candidates have been investigated (see [5–13] etc).

Thus far, some coherence measures have been found out for different applications and backgrounds, such as relative entropy of coherence [1], the \(l_1\) norm of coherence [1], geometric coherence [14], modified trace norm of coherence [4], robustness of coherence [15, 16], coherence measure via quantum skew information [17], coherence measures based on Tsallis relative entropy [18–20], coherence weight [21]. For a coherence measure defined only for all pure states, it can be extended to mixed states via the convex roof construction [5, 12, 22, 23]. Also, a coherence measure defined on all pure states is determined by its majorization property on the modular square of coefficients of pure states [6–8, 24–26]. Although the convex roof construction and majorization on pure states together provide a powerful way to construct coherence measures, the coherence measures obtained in such way...
generally speaking are only in the form of optimization and hard to get the analytical expressions [2].

In this paper, we provide two classes of coherence measures based on the sandwiched Rényi relative entropy in Section II, the geometric coherence [14] is a special case of them. Also in Section III, we discuss whether or not one can get a new coherence measure through a function of a given coherence measure, and coherence measures for qubit states.

II. COHERENCE MEASURES BASED ON SANDWICHED RÉNYI RELATIVE ENTROPY

In this section, we propose two classes of coherence measures based on the sandwiched Rényi relative entropy.

Theorem 1. For \( \alpha \in [\frac{1}{2}, 1), \rho \in \mathcal{D} \),

\[
C_{s,1,\alpha}(\rho) = 1 - \max_{\sigma \in \mathcal{I}} \left\{ \frac{\ln tr[\rho^{\frac{1-\alpha}{2}} \sigma \rho^{\frac{1-\alpha}{2}}]^{\frac{1}{1-\alpha}}}{\alpha - 1} \right\},
\]
(6)
is a coherence measure.

Proof. For \( \alpha \in [\frac{1}{2}, 1), \sigma, \rho \in \mathcal{D} \), the sandwiched Rényi relative entropy was defined as [27, 28]

\[
F_{\alpha}(\sigma||\rho) = \ln \frac{tr[\rho^{\frac{1-\alpha}{2}} \sigma \rho^{\frac{1-\alpha}{2}}]}{\alpha - 1}.
\]
(7)

Note that \( F_{\alpha}(\sigma||\rho) \) can be defined for \( \alpha > 0 \) [27, 28], but in Theorem 1 we only consider the case of \( \alpha \in [\frac{1}{2}, 1) \).

It is shown that [28, 29] for \( \alpha \in [\frac{1}{2}, 1) \),

\[
F_{\alpha}(\sigma||\rho) \geq 0, \text{ and equality iff } \sigma = \rho.
\]
(8)

This is equivalent to

\[
tr[\rho^{\frac{1-\alpha}{2}} \sigma \rho^{\frac{1-\alpha}{2}}] \leq 1, \text{ and equality iff } \sigma = \rho,
\]
(9)

and further equivalent to

\[
\{tr[\rho^{\frac{1-\alpha}{2}} \sigma \rho^{\frac{1-\alpha}{2}}]\}^{\frac{1}{1-\alpha}} \leq 1, \text{ and equality iff } \sigma = \rho.
\]
(10)

This says that \( C_{s,1,\alpha}(\rho) \) satisfies (C1).

For \( \alpha \in [\frac{1}{2}, 1) \), it has been shown that [28, 30] for \( \sigma, \rho \in \mathcal{D} \), and any CPTP map \( \Phi \),

\[
F_{\alpha}(\Phi(\sigma)||\Phi(\rho)) \leq F_{\alpha}(\sigma||\rho).
\]
(11)

This implies

\[
tr[\Phi(\rho)^{\frac{1-\alpha}{2}} \Phi(\sigma)(\Phi(\rho)^{\frac{1-\alpha}{2}})] \geq tr[(\rho^{\frac{1-\alpha}{2}} \sigma \rho^{\frac{1-\alpha}{2}})^{\alpha}]]\frac{1}{1-\alpha}.
\]
(12)

\[
\{tr[\Phi(\rho)^{\frac{1-\alpha}{2}} \Phi(\sigma)(\Phi(\rho)^{\frac{1-\alpha}{2}})]\}^{\frac{1}{1-\alpha}} \geq \{tr[(\rho^{\frac{1-\alpha}{2}} \sigma \rho^{\frac{1-\alpha}{2}})^{\alpha}]]\frac{1}{1-\alpha}.
\]
(13)

For any ICPTP map \( \Phi_1 \), there exists \( \sigma^* \in \mathcal{I} \), such that

\[
\max_{\sigma \in \mathcal{I}} \{tr[(\rho^{\frac{1-\alpha}{2}} \sigma \rho^{\frac{1-\alpha}{2}})^{\alpha}]]\frac{1}{1-\alpha} \}
\]

\[
= \{tr[(\rho^{\frac{1-\alpha}{2}} \sigma^* \rho^{\frac{1-\alpha}{2}})^{\alpha}]]\frac{1}{1-\alpha} \}
\]

\[
\leq \{tr[(\Phi_1(\rho))^{\frac{1-\alpha}{2}} \Phi_1(\sigma^*)(\Phi_1(\rho)^{\frac{1-\alpha}{2}})^{\alpha}]]\frac{1}{1-\alpha} \}
\]

\[
\leq \max_{\sigma \in \mathcal{I}} \{tr[(\Phi_1(\rho))^{\frac{1-\alpha}{2}} \sigma(\Phi_1(\rho)^{\frac{1-\alpha}{2}})^{\alpha}]]\frac{1}{1-\alpha} \}.
\]
(14)

This proves that \( C_{s,1,\alpha}(\rho) \) satisfies (C2).

Next we prove \( C_{s,1,\alpha}(\rho) \) satisfies (C5). Suppose \( \rho \) is block-diagonal in the reference basis \( \{|j\}\}_{j=1}^{2} \),

\[
\rho = p_1 \rho_1 \oplus p_2 \rho_2,
\]
(15)

with \( p_1 > 0, p_2 > 0, p_1 + p_2 = 1, \rho_1, \rho_2 \in \mathcal{D} \).

Let

\[
\sigma = q_1 \sigma_1 \oplus q_2 \sigma_2,
\]
(16)

with \( \sigma_1, \sigma_2 \) diagonal states having the same rows (columns) with \( \rho_1, \rho_2 \) respectively, \( q_1 \geq 0, q_2 \geq 0, q_1 + q_2 = 1 \).

It follows that

\[
\max_{\sigma \in \mathcal{I}} \{tr[(\rho^{\frac{1-\alpha}{2}} \sigma \rho^{\frac{1-\alpha}{2}})^{\alpha}]]\frac{1}{1-\alpha} \}
\]

\[
= \max_{\{p_1^{1-\alpha} q_1^\alpha \}} \max_{\{p_2^{1-\alpha} q_2^\alpha \}} \max_{\{\sigma_1 \sigma_2 \}} \{tr[(\rho_1^{1-\alpha} \sigma_1^{1-\alpha} \rho_2^{1-\alpha})^{\alpha}]]\frac{1}{1-\alpha} \}
\]

\[
= \max_{\{p_1^{1-\alpha} q_1^\alpha \}} \max_{\{p_2^{1-\alpha} q_2^\alpha \}} \{tr[(\rho_1^{1-\alpha} \sigma_1^{1-\alpha} \rho_2^{1-\alpha})^{\alpha}]]\frac{1}{1-\alpha} \}
\]

\[
= (p_1^{1-\alpha} p_2^{1-\alpha} t_1 t_2 (p_1^{1-\alpha} t_1^{1-\alpha} + p_2^{1-\alpha} t_2^{1-\alpha}))^{1-\alpha},
\]
(17)

where we have denoted

\[
t_1 = \max_{\sigma_1} tr[(\rho_1^{1-\alpha} \sigma_1^{1-\alpha} \rho_2^{1-\alpha})^{\alpha}]]\frac{1}{1-\alpha} \}
\]
(18)

\[
t_2 = \max_{\sigma_2} tr[(\rho_2^{1-\alpha} \sigma_2^{1-\alpha} \rho_2^{1-\alpha})^{\alpha}]]\frac{1}{1-\alpha} \}
\]
(19)

and have used the Hölder inequality in Appendix A (note that \( t_1 > 0, t_2 > 0 \)).

Consequently,

\[
\max_{\sigma \in \mathcal{I}} \{tr[(\rho^{\frac{1-\alpha}{2}} \sigma \rho^{\frac{1-\alpha}{2}})^{\alpha}]]\frac{1}{1-\alpha} \}
\]

\[
= \max_{\{p_1^{1-\alpha} q_1^\alpha \}} \max_{\{p_2^{1-\alpha} q_2^\alpha \}} \{tr[(\rho_1^{1-\alpha} \sigma_1^{1-\alpha} \rho_2^{1-\alpha})^{\alpha}]]\frac{1}{1-\alpha} \}
\]

\[
= p_1 p_2 t_1^{1-\alpha} t_2^{1-\alpha} (p_1^{1-\alpha} t_1^{1-\alpha} + p_2^{1-\alpha} t_2^{1-\alpha})
\]
(20)

This shows that \( C_{s,1,\alpha}(\rho) \) satisfies (C5). \( \square \)

We remark that when \( \alpha = 2, C_{s,1,2}(\rho) \) corresponds to the geometric coherence [14].

Example 1. For pure state \( \rho = |\psi\rangle \langle \psi| \),

\[
C_{s,1,\alpha}(\psi) = 1 - \max_j \{|\langle j|\psi|\rangle|^{\frac{2}{1-\alpha}} \}
\]
(21)
Proof. Suppose $\sigma = \sum_j \sigma_j |j\rangle \langle j|$ is an incoherent state,

$$tr[(\rho^{\frac{1}{2}-(1)} \sigma^{\frac{1}{2}+(1)})^\alpha]$$

$$= tr[(\psi\langle \psi | \sum_j \sigma_j |j\rangle \langle j| \psi\rangle)^\alpha]$$

$$= (\sum_j \sigma_j |\langle j|\psi\rangle|^2)^\alpha, \quad (22)$$

then we can get the result.

**Theorem 2.** For $\alpha \in [\frac{1}{2}, 1) \cup (1, \infty)$,

$$C_{s,\alpha}(\rho) = \min_{\sigma \in \mathcal{L}} \frac{\{tr[(\sigma^{\frac{1}{2}-(1)} \rho^{\frac{1}{2}+(1)})^\alpha]\}}{\alpha - 1}, \quad (23)$$

$sup(\rho) \subset sup(\sigma)$ when $\alpha > 1$,

is a coherence measure.

**Proof.** Theorem 2 can be proved in the similar way of proof for Theorem 1 with minor modification. □

Note that since the quantum fidelity

$$F(\rho, \sigma) = tr[(\rho^{\frac{1}{2}-(1)} \sigma^{\frac{1}{2}+(1)})^\frac{1}{2}],$$

then $C_{s,\frac{1}{2}}(\rho) = C_{s,1}(\rho)$ again coresponds to the geometric coherence [14].

We remark that a coherence quantifier based on sandwiched Rényi relative entropy was also investigated in [6, 31], but that quantifier is not a coherence measure in the sense that satisfying (C1-C4), i.e., under the BCP framework.

**Example 2.** For pure state $\rho = |\psi\rangle \langle \psi|,

$$C_{s,\alpha}(|\psi\rangle) = \frac{\{tr[|\psi\rangle \langle \psi |^{\frac{1}{2}-(1)} \rho^{\frac{1}{2}+(1)}]^\alpha\}}{\alpha - 1} - 1, \quad (28)$$

Proof. Suppose $\sigma = \sum_j \sigma_j |j\rangle \langle j|$ is an incoherent state,

$$tr[(\sigma^{\frac{1}{2}-(1)} \rho^{\frac{1}{2}+(1)})^\alpha]$$

$$= tr[(\psi\langle \psi | \sum_j \sigma_j |j\rangle \langle j| \psi\rangle)^\alpha]$$

$$= (\sum_j \sigma_j |\langle j|\psi\rangle|^2)^\alpha.$$  

Using the Hölder inequality (Appendix A), we then get the result.

Examples 1 and 2 show that $C_{s,\alpha}$ and $C_{s,1,\beta}$ are not equivalent even $\alpha = \beta$.

**III. LINEARIZATION THEOREM AND COHERENCE MEASURES FOR QUBIT STATES**

One might ask that for given coherence measure $C$, whether or not there exists a function $f$ such that $f(C(\rho))$ still is a coherence measure. The answer of this question is essentially no. We have the following theorem.

**Theorem 3.** (Linearization Theorem) Given a coherence measure $C$ defined on $d$-dimensional quantum states with $d > 2$, the function $f : [0, \infty) \rightarrow [0, \infty)$, makes $f(C(\rho))$ also a coherence measure, if and only if there must exists $\lambda > 0$, such that $f(x) = \lambda x$.

**Proof.** Since $C(\rho)$ and $f(C(\rho))$ are all coherence measures, then (C1) leads to

$$f(x) \geq 0 \quad \text{and} \quad f(0) = 0 \quad \text{iff} \quad x = 0. \quad (25)$$

Suppose $d > 2$, for the state

$$\rho = p_1 \rho_1 \oplus p_2 \rho_2, \quad (26)$$

with $p_1 > 0, p_2 > 0, p_1 + p_2 = 1, p_1, p_2 \in D, \dim \rho_1 \geq 1, \dim \rho_2 \geq 1, \dim \rho_1 + \dim \rho_2 = d$, since $C(\rho)$ and $f(C(\rho))$ are all coherence measures, then (C5) leads to

$$f[C(\rho_1 p_1 + \rho_2 p_2)]$$

$$= f[p_1 C(\rho_1) + p_2 C(\rho_2)]$$

$$= p_1 f[C(\rho_1)] + p_2 f[C(\rho_2)]. \quad (27)$$

Without loss of generality, suppose $\dim \rho_2 \geq 2$, then there exist $\rho_1, \rho_2$ such that $C(\rho_1) = 0, C(\rho_2) = \mu > 0$. The above equation yields

$$f(p_2 \mu) = p_2 f(\mu), \quad (28)$$

with $f(\mu) > 0$.

Let $p_2 \mu = x$, then

$$f(x) = \frac{x}{\mu} f(\mu) = \lambda x, \lambda > 0. \quad (29)$$

We then complete this proof. □

Since coherence measures $C$ and $\lambda C$ $(\lambda > 0)$ have no essential difference, then from Theorem 3, we say that, it is impossible to get a new coherence measure $f(C(\rho))$ by a function $f$.

The reason of assuming $d > 2$ in Theorem 3 is that (C5) is trivial for $d = 2$. With this in mind, and after some algebra, we have the following Theorem.

**Theorem 4.** Given a coherence measure $C$ for qubit states, the function $f : [0, \infty) \rightarrow [0, \infty)$, makes $f(C(\rho))$ also a coherence measure, if and only if

1. $f(x) \geq 0$ and $f(0) = 0$ iff $x = 0$.
2. $f(x) \geq f(y)$ when $x \geq y$.

For example, for $d = 2$, the coherence of $l_1$ norm [1]

$$C_{l_1} = 2|\rho_{12}| = 2|\langle 1|\rho|2\rangle|,$$

then any function $f$ satisfying (1) and (2) of Theorem 4, makes $f(C_{l_1})$ still a coherence measure, such as geometric coherence for $d = 2$ [14] and coherence formation for $d = 2$ [22].
IV. SUMMARY

In summary, under the BCP framework for quantifying coherence, we proposed two classes of coherence measures based on sandwiched Rényi relative entropy. Our strategy to prove these coherence measures satisfying the (C1-C4) of BCP framework is to prove they satisfy (C1+C2+C5). We also proved that it is essentially impossible to get new coherence measures $f(C(\rho))$ by a function $f$ acting on a given coherence measure $C$, except the case of qubit states.

There are many open questions for future investigations. For example, the monotonicity of $C_{s,\alpha}$, $C_{s,\alpha}$ in $\alpha$, the ordering of magnitude for them and other coherence measures, the operational interpretations for them, potential applications in quantum information processes, and also the counterparts for quantifying coherence of Gaussian states as done in [32, 33].

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APPENDIX A: HÖLDER INEQUALITY

Suppose $\{a_j\}_{j=1}^d, \{b_j\}_{j=1}^d$, are all positive real numbers, then

(1). when $\alpha \in (0, 1)$,

$$\sum_{j=1}^{d} a_j b_j \leq \left( \sum_{j=1}^{d} a_j \right)^{\alpha} \left( \sum_{j=1}^{d} b_j \right)^{1-\alpha},$$

(30)

and equality iff $\frac{a_j}{b_j} = \frac{a_k}{b_k}$ for any $j, k$;

(2). when $\alpha > 1$,

$$\sum_{j=1}^{d} a_j b_j \geq \left( \sum_{j=1}^{d} a_j \right)^{\alpha} \left( \sum_{j=1}^{d} b_j \right)^{1-\alpha},$$

(31)

and equality iff $\frac{a_j}{b_j} = \frac{a_k}{b_k}$ for any $j, k$. 

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