Near-wall extension of a non-equilibrium, omega-based Reynolds stress model

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Abstract.
In this paper, the development of a new $\omega$-based Reynolds stress model that is consistent with asymptotic analysis in the near wall region and with rapid distortion theory in homogeneous turbulence is reported. The model is based on the SSG/LRR-$\omega$ model developed by Eisfeld (2006) with three main modifications. Firstly, the near wall behaviors of the redistribution, dissipation and diffusion terms are modified according to the asymptotic analysis and a new blending function based on low Reynolds number is proposed. Secondly, an anisotropic dissipation tensor based on the Reynolds stress inhomogeneity (Jakirlic et al., 2007) is used instead of the original isotropic model. Lastly, the SSG redistribution term, which is activated far from the wall, is replaced by Speziale’s non-equilibrium model (Speziale, 1998).

Nomenclature

$\bar{\phi} = \text{ensemble average of } \phi.$
$\tilde{\phi} = \text{mass-weighted average of } \phi.$
$\rho = \text{flow density.}$
$t = \text{time.}$
$x_i = \text{position vector.}$
$U_i = \text{velocity.}$
$u' = \text{Favre fluctuating velocity.}$
$P = \text{pressure.}$
$\tau_{ij} = \text{viscous stress tensor.}$
$\tau = \text{dynamic viscosity.}$
$C_p = \text{specific heat capacity at constant pressure.}$
$T = \text{temperature.}$
$C_v = \text{specific heat capacity at constant volume.}$
$k = \text{turbulence kinetic energy.}$
e = \text{internal energy.}$
h = \text{internal enthalpy.}$
$\phi(t) = \text{turbulent variables}$
$\lambda = \text{thermal conductivity.}$
$\delta_{ij} = \text{Kronecker delta.}$
$R = \text{specific gas constant of air.}$
$R_{ij} = \text{Reynolds stress.}$
$S_{ij}^s = \text{traceless strain rate tensor.}$
$D^{(k)} = \text{diffusion of turbulence kinetic energy.}$
$D_{ij} = \text{turbulent diffusion.}$
$\epsilon = \text{turbulence energy dissipation rate.}$
$\epsilon = \text{specific dissipation rate.}$
$\Pi_{ij} = \text{Reynolds stress redistribution.}$
$P_{ij} = \text{Reynolds stress production.}$
1. Introduction

After decades of research and development, second moment closure has reached the level where it can be used more widely in the CFD community. While common Reynolds stress models perform better than the eddy viscosity models in many flows (Aupoix, 2009), they did not show superiority in certain cases such as in flows under adverse pressure gradient and in the presence of flow separation. This is partially because most of the advanced Reynolds stress models employ an $\epsilon$ equation which is known to perform poorly under adverse pressure gradient (Aupoix, 2009). Wilcox (1993) illustrated that an $\omega$-based Reynolds stress model provides better results thus making the $\omega$ equation an ideal choice for the near wall flow.

However, there are several shortcomings in using the $\omega$ equation. The biggest problem is that $\omega$-based turbulence models were shown to be very sensitive to the free-stream value of $\omega$ (Wilcox, 1993). This issue can be overcome by adding a cross-diffusion term into the $\omega$ equation, thus, transforming the length scale equation into an $\epsilon$ equation far from the wall. The SST model from Menter (1994) is among the most successful models to date utilizing this type of length scale equation. Recently, Eisfeld extended the use of this equation and its blending function into Reynolds stress modelling (Eisfeld, 2006). His model is a combination of the Wilcox’s stress-$\omega$ model (Wilcox, 1993) in the near wall region and the SSG-$\epsilon$ model (Sarkar et al., 1991) elsewhere.

Another drawback of an $\omega$-based turbulence model is that the model does not satisfy the asymptotic analysis in the near wall region. For example, Taylor series expansion showed that the turbulence kinetic energy $k$ tends to 0 like $y^2$ near a wall while $k$ actually tends to 0 like $y^{3/23}$ in the original Wilcox’s stress-$\omega$ model (Wilcox, 1993). This compares unfavorably to advanced $\epsilon$-based Reynolds stress models which are capable of predicting the correct asymptotic behavior of the Reynolds stress components in the near wall region (Jakirlic et al., 2007). For certain flows where the Reynolds stress components interact strongly with the mean strain rates, such as flows with strong streamline curvature, the ability to predict the correct near wall behavior may become essential.

In this work, we attempt to modify the SSG/LRR-$\omega$ model in order to achieve consistency with asymptotic analysis. Following the suggestions in previous works by Wilcox (1993) and Lai & So (1990), the dissipation term, the diffusion term and the slow part of the redistribution term are modified in the vicinity of a solid wall. Furthermore, an anisotropic formulation based on the Reynolds stress inhomogeneity (Jakirlic et al., 2007) is used for the dissipation tensor. Far from the wall, the SSG redistribution model is replaced by Speziale’s non-equilibrium model (Speziale, 1998) which is consistent with rapid distortion theory in homogeneous turbulence. Finally, Menter’s blending function is changed in order to include the effect of low Reynolds number in the near wall region. The details of the model formulation are discussed in the subsequent sections.

2. Turbulence Modeling

2.1. Reynolds-averaged Navier-Stokes framework

We consider a set of equations representing mass, momentum and energy conservations within the framework of Reynolds-averaged Navier-Stokes (RANS) computation:

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_k} \left( \bar{\rho} \bar{U}_k \right) = 0,$$

$$\frac{\partial \left( \bar{\rho} \bar{U}_i \right)}{\partial t} + \frac{\partial}{\partial x_k} \left( \bar{\rho} \bar{U}_i \bar{U}_k + \bar{\rho} u''_i u''_k \right) = -\frac{\partial \bar{P}}{\partial x_i} + \frac{\partial \tau_{ik}}{\partial x_k},$$
\[
\frac{\partial \left( \bar{\rho} \tilde{E} \right)}{\partial t} + \frac{\partial}{\partial x_k} \left( \bar{\rho} H \tilde{U}_k + \bar{\rho} \tilde{u}_i u_k \tilde{U}_i \right) = \frac{\partial}{\partial x_k} \left( \bar{\tau}_{ik} \tilde{U}_i \right) - \frac{\partial}{\partial x_k} \left( \bar{q}_k + \bar{q}_k^{(t)} \right) + \bar{\rho} D^{(k)}. \tag{3}
\]

The total energy and the total enthalpy are defined below:

\[
\tilde{E} = \bar{e} + \bar{U}_k \bar{U}_k/2 + \bar{\varepsilon} = C_v \bar{T}, \tag{4}
\]

\[
\tilde{H} = \bar{h} + \bar{U}_k \bar{U}_k/2 + \bar{\varepsilon} = C_p \bar{T}. \tag{5}
\]

We assume that gas is ideal (\(\bar{P} = \bar{\rho} R \bar{T}\)) and calorically perfect (\(C_p\) and \(C_v\) are constants) and fluid is Newtonian so that the viscous stress tensor can be given as:

\[
\bar{\tau}_{ij} = 2\bar{\mu} \bar{S}^*_{ij}; \quad \bar{S}^*_{ij} = \frac{1}{2} \left( \frac{\partial \tilde{U}_i}{\partial x_j} + \frac{\partial \tilde{U}_j}{\partial x_i} - \frac{2}{3} \frac{\partial \tilde{U}_k}{\partial x_k} \delta_{ij} \right). \tag{6}
\]

In Eqn. 6, the dynamic viscosity \(\bar{\mu}\) is computed using Sutherland’s law. The laminar and turbulent heat fluxes in Eqn. 3 are modeled based on Fourier type heat conduction:

\[
\bar{q}_i = -\lambda \frac{\partial \bar{T}}{\partial x_i}; \quad \bar{q}_i^{(t)} = \bar{\rho} \bar{h} u_k \tilde{u}_k = -\lambda^{(t)} \frac{\partial \bar{T}}{\partial x_i}, \tag{7}
\]

in which the heat conductivity is computed via the dynamic viscosity and a constant Prandtl number:

\[
\bar{\lambda} = \frac{C_p \bar{\mu}}{Pr}; \quad \lambda^{(t)} = \frac{C_p \bar{\mu}^{(t)}}{Pr^{(t)}}. \tag{8}
\]

\(\bar{\mu}^{(t)}\) is determined by an eddy viscosity model scheme. The turbulent diffusion term \(\bar{\rho} D^{(k)}\) in Eqn. 3 is modeled as half of the trace of the Reynolds stress diffusion tensor (Eisfeld, 2006):

\[
\bar{\rho} D^{(k)} = \frac{\bar{\rho} D_{kk}}{2}. \tag{9}
\]

A wide variety of turbulence models for compressible fluid flow, ranging from one-equation eddy viscosity model to full Reynolds stress model, are available. These models provide the eddy viscosity \(\bar{\mu}^{(t)}\), the Reynolds stresses \(\bar{R}_{ij} = \bar{u}_i \bar{u}_j\) and the diffusion term \(D_{kk}\) to close the set of equations above. In this work we focus on the SSG/LRR-\(\omega\) Reynolds stress model (Eisfeld, 2006) and its near-wall extension. In addition we also compute the test case with Menter’s SST k-\(\omega\) (Menter, 1994) model for comparison purposes.

2.2. SSG/LRR-\(\omega\) Reynolds stress model

The transport equation of the Reynolds stresses can be derived by multiplying the momentum equation with the fluctuating velocity components and time averaging the product (Wilcox, 1993):

\[
\partial \left( \bar{\rho} \tilde{R}_{ij} \right) \partial t + \partial \left( \bar{\rho} H \tilde{U}_k \tilde{R}_{ij} + \bar{\rho} \tilde{u}_i \tilde{u}_k \tilde{R}_{ij} \right) = \partial \left( \bar{\rho} \tilde{\tau}_{ik} \tilde{R}_{ij} \right) - \partial \left( \bar{\rho} \tilde{q}_k \tilde{R}_{ij} \right) + \bar{\rho} \tilde{D}^{(k)}. \tag{10}
\]
\[
\frac{\partial}{\partial t} \left( \tilde{\rho} \tilde{R}_{ij} \right) + \frac{\partial}{\partial x_k} \left( \tilde{\rho} \tilde{U}_k \tilde{R}_{ij} \right) = \tilde{\rho} P_{ij} + \rho \Pi_{ij} + \tilde{\rho} \epsilon_{ij} + \tilde{\rho} D_{ij},
\]

where \( P_{ij} \), \( \Pi_{ij} \), \( \epsilon_{ij} \) and \( D_{ij} \) are the production, pressure-strain correlation, dissipation and diffusion of the Reynolds stresses, respectively. Compressibility effects, other than the variation of density, are neglected in this formulation. All terms in Eqn. 10 require modeling except the production term:

\[
\tilde{\rho} P_{ij} = -\tilde{\rho} \tilde{R}_{ik} \frac{\partial \tilde{U}_j}{\partial x_k} - \tilde{\rho} \tilde{R}_{jk} \frac{\partial \tilde{U}_i}{\partial x_k}.
\]

The pressure-strain correlation term in Eqn. 10 represents a process in which the energy is redistributed between the Reynolds stress components (George, 2010), hence, it is also commonly referred to as "redistribution term". Since this term does not appear in the transport equation of the eddy viscosity models and it is of the same order of magnitude as the production term (George, 2010), modeling this term has received a great amount of attention. The redistribution term is modeled here as follows:

\[
\tilde{\rho} \Pi_{ij} = -C_1 \tilde{\rho} \epsilon_{ij} + \frac{1}{2} C_1^* \tilde{\rho} P_{kk} \tilde{b}_{ij} + C_2 \tilde{\rho} \epsilon \left( \tilde{b}_{ik} \tilde{b}_{kj} - \frac{1}{3} \tilde{b}_{mn} \tilde{b}_{mn} \delta_{ij} \right) + C_3 \tilde{\rho} \epsilon \left( \tilde{b}_{ik} \tilde{S}_{jk} + \tilde{b}_{jk} \tilde{S}_{ik} - \frac{2}{3} \tilde{b}_{mn} \tilde{S}_{mn} \delta_{ij} \right) + C_5 \tilde{\rho} \epsilon \left( \tilde{b}_{ik} \tilde{W}_{jk} + \tilde{b}_{jk} \tilde{W}_{ik} \right)
\]

The first two terms on the right hand side are commonly referred to as "slow pressure-strain/return-to-isotropy" term while the last three terms are commonly named "rapid pressure-strain" term (Aupoix, 2009). The formulation in Eqn. 12 is based on the non-linear SSG model (Sarkar et al., 1991), however, if \( C_1^* \), \( C_2 \) and \( C_3^* \) are set to zero, the LRR model formulation (Reece et al., 1975) can be retrieved (Eisfeld, 2006). It should be noted that the wall-reflection term is neglected in this formulation.

Rotta’s isotropic model is used for the dissipation term:

\[
\tilde{\rho} \epsilon_{ij} = \frac{2}{3} C_6 \tilde{\rho} \tilde{k} \omega \delta_{ij},
\]

where \( C_6 \) is a constant and equal to 0.09.

The diffusion term is modeled based on generalized gradient diffusion hypothesis (Eisfeld, 2006):

\[
\tilde{\rho} D_{ij} = \frac{\partial}{\partial x_k} \left[ \left( \tilde{\rho} \delta_{kl} + D^{(GGD)} \tilde{\rho} \tilde{R}_{kl} \right) \frac{\partial \tilde{R}_{ij}}{\partial x_l} \right].
\]

In the above equations, the turbulence kinetic energy and the dissipation rate are defined as:

\[
\tilde{k} = \frac{\tilde{R}_{kk}}{2}; \quad \epsilon = C_6 \tilde{k} \omega.
\]
\[ \tilde{b}_{ij} = \frac{R_{ij}}{2k} - \frac{\delta_{ij}}{3} \; ; \; II = \tilde{b}_{ij}b_{ij}. \]  

(16)

The strain rate tensor and the rotation rate tensor are computed by:

\[ \tilde{S}_{ij} = \frac{1}{2} \left( \frac{\partial \tilde{U}_i}{\partial x_j} + \frac{\partial \tilde{U}_j}{\partial x_i} \right) ; \quad \tilde{W}_{ij} = \frac{1}{2} \left( \frac{\partial \tilde{U}_i}{\partial x_j} - \frac{\partial \tilde{U}_j}{\partial x_i} \right). \]  

(17)

We employ Menter’s baseline \( \omega \)-equation to compute the turbulence length-scale:

\[ \frac{\partial (\bar{\rho} \omega)}{\partial t} + \frac{\partial (\bar{\rho} \tilde{U}_i \omega)}{\partial x_i} = -\alpha_{\omega} \bar{\rho} \omega \tilde{U}_k \frac{\partial \tilde{U}_i}{\partial x_k} - \beta_{\omega} \bar{\rho} \omega^2 + \frac{\partial}{\partial x_k} \left[ \left( \tilde{\mu} + \frac{\sigma_{\omega}}{\omega} \right) \frac{\partial \omega}{\partial x_k} \right] + \sigma_{\omega} \bar{\rho} \omega_{\text{max}} \left( \frac{\partial \omega}{\partial x_k} \frac{\partial \tilde{k}}{\partial x_k} \right) 0, \]  

(18)

where the four terms on the right hand side represent the production, destruction, diffusion and cross-diffusion of the turbulence dissipation, respectively.

The coefficients in the Reynolds stress and \( \omega \) equations are blended using the following relationship:

\[ \phi = F \phi^{LRR} + (1 - F) \phi^{SSG}; \quad F = \tanh \left( \zeta^4 \right), \]  

(19)

with

\[ \zeta = \min \left[ \max \left( \frac{\sqrt{k}}{C_\mu \omega d}, \frac{500 \mu}{C_\mu \omega d^2} \right) ; \frac{4 \sigma_{\omega}^{(SSG)} \bar{\rho} \tilde{k}}{\rho \sigma_{\omega}^{(SSG)} d^2} \right]. \]  

(20)

The numerical values of these coefficients are given in the table 1 and table 2. It can be seen that the SSG/LRR-\( \omega \) model is equivalent to a combination of Wilcox’s stress-\( \omega \) model (Wilcox, 1993) in the near wall region and the SSG-\( \epsilon \) model (Sarkar et al., 1991) elsewhere.

**Table 1: Closure coefficients in the redistribution and diffusion terms.** 

|       | \( C_1 \) | \( C_1^* \) | \( C_2 \) | \( C_3 \) | \( C_3^* \) | \( C_4 \) | \( C_5 \) | \( D_{GGD}^{(GGD)} \) |
|-------|-----------|-------------|-----------|-----------|-----------|-----------|-----------|----------------|
| SSG   | 3.4       | 1.8         | 4.2       | 0.8       | 1.3       | 1.25      | 0.40      | \( C_s/C_\mu \)  |
| LRR   | 3.6       | 0.0         | 0.0       | 0.0       | 0.0       | \( (18C_{LRR} + 12) / 11 \) | \( (14C_{LRR} + 20) / 11 \) | 0.5           |

It should be noted that \( \alpha_{\omega} \) in Table 2 is computed from the following relationship:

\[ \alpha_{\omega} = \frac{\beta_{\omega}}{C_\mu} - \frac{\sigma_{\omega} \kappa^2}{\sqrt{C_\mu}}, \]  

(21)

in which \( \kappa \) is the von Karman constant. This relationship was derived from the log law of the wall and is used in turbulence modeling to fix the slope of the log line through \( \kappa \) and to define the balance in the logarithmic region.
Table 2: Closure coefficients for the $\omega$-equation.

|       | $\alpha_\omega$ | $\beta_\omega$ | $\sigma_\omega$ | $\sigma_d$ |
|-------|-----------------|----------------|-----------------|-----------|
| SSG   | 0.44            | 0.0828         | 0.856           | 2$\sigma_\omega$ |
| LRR   | 0.556           | 0.075          | 0.5             | 0         |

2.3. Near wall modifications based on asymptotic analysis

Consider incompressible flow for simplicity, the velocity fluctuations in the near wall region can be expanded as follows:

$$
\begin{align*}
    u'(x, y, z, t) &= b_u y + c_u y^2 + d_u y^3 + \ldots, \\
    v'(x, y, z, t) &= c_v y^2 + d_v y^3 + \ldots, \\
    w'(x, y, z, t) &= b_w y + c_w y^2 + d_w y^3 + \ldots
\end{align*}
$$

(22)

The zeroth order terms in Eqn. 22 are null because of the no-slip condition. The first order term in the $v'$ equation also vanishes due to the continuity equation. Therefore, the wall normal velocity fluctuation is suppressed faster in the presence of the wall and turbulence tends toward a two component limit. It can also be seen that $u'u'$ and $w'w'$ evolve like $y^2$ in the near wall region while $v'v'$ evolves like $y^4$ and is negligible. The turbulence kinetic energy is contributed mainly by $u'u'$ and $w'w'$ and thus, also proportional to $y^2$. Similarly, the off-diagonal component $u'w'$ behaves like $y^2$ while the other two components $u'v'$ and $v'w'$, in which the normal velocity fluctuation is included, behaves like $y^3$.

The balance near the wall of the Wilcox' stress-$\omega$ yields an unexpected result in which $k$ evolves like $y^{2.25}$ in the near wall region. Wilcox pointed out that in order to achieve asymptotic consistency, $C_\mu/\beta_w$ must tend to 1/3 as $y$ tends to 0 (Wilcox, 1993). It was further suggested that $\beta_w$ should be kept constant which means $C_\mu$ should approach $\beta_w/3$ near the wall. This can be taken into account by using a blending function similar to that in the SST model:

$$
C_{\mu,\text{new}} = f_w \frac{\beta_w}{3} + (1 - f_w)C_\mu,
$$

(23)

$$
f_w = \min \left( \exp \left[ -\left( \frac{Re_t}{12} \right)^2 \right], 1 \right) \frac{F}{1 + F},
$$

(24)

where $Re_t = \frac{\rho k}{\mu}$ is the turbulent Reynolds number. Here, $f_w$ is designed to be unity in the near wall region and null far from the wall. While the near wall behavior is governed by both the low Reynolds number effect and the wall blocking effect (Aupoix, 2009), only the former is considered in this formulation. The addition of $F$ in Eqn. 24 is a safeguard to make sure that $f_w$ is zero in the free-stream flow where the turbulence kinetic energy and the turbulent Reynolds number may become null.

The above modification will make sure that $k$ is asymptotically consistent when approaching the wall. However, in order to achieve the same condition for all Reynolds stress components, more changes have to be considered. Lai & So (1990) performed a careful analysis of the near wall behavior of all terms in Eqn. 10 and concluded that both the dissipation term and the slow pressure-strain correlation term require modifications to be asymptotically consistent.

Asymptotically consistent models for $\epsilon_{ij}$ were proposed along the suggestions by several authors mentioned in Lai & So (1990). However, these models normally require the wall normal
vector which is difficult to obtain in an unstructured scheme. A simpler model similar to that of Jakirlic et al. (2007) is used here in which the dissipation component is given as below:

$$\epsilon_{ij} = f_s \overline{u'_i u'_j} \epsilon + (1 - f_s) \frac{2}{3} \delta_{ij},$$

(25)

where $f_s = 1 - \sqrt{A}$. $A = 1 - 9/8 (\text{II} - \text{III})$ is the flatness parameter and $\text{III} = \tilde{b}_{ik} \tilde{b}_{kj} \tilde{b}_{ki}$ is the third invariant of the Reynolds stress tensor. $A$ is zero in the two component limit (e.g. the near wall region) and is unity when turbulence is isotropic. It should be noted that this formulation does not provide correct asymptotic behavior of the $\epsilon_{22}, \epsilon_{12}$ and $\epsilon_{23}$ components. However, this fact causes only a slight imbalance and a marginal effect on the governing equations (Jakirlic et al., 2007). It can be seen that in the near wall region, the dissipation component is anisotropic and proportional to the Reynolds stress component while Rotta’s isotropic model is recovered in the far-field (see Eqn. 13).

Lai & So (1990) showed that the redistribution term of the LRR type should go to zero at the wall like $y^n$ with $n > 0$ therefore the behavior of the $C_1$ and $C_2$ terms in the original LRR model need to be corrected. This is because the terms involving $C_1$ and $C_2$ are computed from $\epsilon$ which is finite at the wall. In order to mimic the near wall behavior of turbulence, at least to the lowest order, these terms must be null at the wall. Since $C_2$ is already zero in the original SSG/LRR-$\omega$ formulation, only $C_1$ needs to be modified as follows:

$$C_{1,\text{new}} = (1 - f_w)C_1.$$ 

(26)

Further numerical experimentations indicated that $C_4$ should also be modified to obtain a better prediction of $\tilde{R}_{31}$ in the near wall region:

$$C_{4,\text{new}} = f_w C_{4,\text{new}} + (1 - f_w)C_4,$$

(27)

where $C_{4,\text{new}} = 1.7$. In Eqn. 26 and Eqn. 27, $C_1$ and $C_4$ are computed from Eqn. 19. Here, the pressure-strain correlation can be seen as having three layers: an asymptotically consistent model in the vicinity of the wall, the LRR model in the middle part of the boundary layer and the SSG model elsewhere.

The diffusion terms in Eqn. 10 and Eqn. 18 deserve some attention. In the original formulation, $\sigma_{LRR}^{\omega}$ is chosen as 0.5 which is a good value to provide the balance in the logarithmic region (see Eqn. 21). However, direct numerical simulations (DNS) of boundary layer and channel flow indicate that $1/\sigma_{\omega}$ varies between 1.6 in the near wall region and 0.8 at the edge of the boundary layer layer (Aupoix, 2009). This means that average value of $\sigma_{\omega}$ is about 0.83 (equivalent to $1/\sigma_{\omega} = 1.2$) which is significantly larger than 0.5. We found that with the near wall modification of $C_{\mu}$ in Eqn. 23, in order to get the correct balance of the logarithmic region in Eqn. 21, $\sigma_{\omega}^{LRR}$ should now be 0.86. Thus, it can be concluded that by forcing asymptotic consistency near the wall, we have achieved better agreement with DNS data. Similarly, $D^{(GGD)}$ in the near wall region is set to 1.875 instead of 0.5. This new value is in better agreement with DNS data in which $D^{(GGD)}$ is in between 1.25 and 2.5 (Aupoix, 2009).

Finally, we set $C_{LRR}$ and $\beta_{\omega}^{LRR}$ to 0.52 and 0.0708 instead of the values used in the original SSG/LRR-$\omega$ model (see Table 1 and Table 2). While $C_{LRR}$ is reverted to the original value proposed in the LRR model, the new value of $\beta_{\omega}^{LRR}$ is recently proposed by Wilcox and was found to be better for flows under adverse pressure gradient (Wilcox, 2008). These coefficients are summarized in Table 3.
Table 3: Closure coefficients used in the new model.

| β  | ω  | σ  | C_{LRR} |
|----|----|----|---------|
| LRR| 0.0708 | 0.86 | 1.875  | 0.52   |

2.4. Non-equilibrium modifications

Another known drawback of the current redistribution model is that they do not agree with rapid distortion theory (RDT) in flows that are far from equilibrium. It is thus desired to have a model that is identical to the current model in equilibrium flow and is consistent with RDT in the non-equilibrium limit. Speziale proposed a non-equilibrium version of the SSG model in order to achieve this goal by a regularization based on Padé approximation (Speziale, 1998). The model was shown to provide good prediction of homogeneous shear flow and plane strain turbulence in the RDT limit while maintaining agreement for equilibrium turbulent flows. We adopt this model for the SSG part and the coefficients are modified as functions of strain and vorticity rates (Speziale, 1998) as follows:

\[
C_{1,mod}^* = \frac{1.8 + 0.225\eta^6}{1 + 0.0625\eta^6 + 0.5\zeta^8}; \quad C_{3,mod}^* = \frac{1.3 + 8.84\eta^8}{1 + 9.02\eta^8}, \tag{28}
\]

\[
C_{4,mod}^* = \frac{1.25 + 6.33\eta^6}{1 + 1.52\eta^6 + 0.1\zeta^2}; \quad C_{5,mod}^* = \frac{0.4 + 0.114\eta^6}{1 + 0.285\eta^6 + 0.5\zeta^8}, \tag{29}
\]

where

\[
\eta = \frac{1}{2} \frac{\alpha_3}{\alpha_1} \frac{k}{\epsilon} \left( \tilde{S}_{ij} \tilde{S}_{ij} \right)^{1/2}; \quad \zeta = \frac{\alpha_2}{\alpha_1} \frac{k}{\epsilon} \left( \tilde{W}_{ij} \tilde{W}_{ij} \right)^{1/2}, \tag{30}
\]

\[
\alpha_1 = \left( \frac{4}{3} - C_2 \right) g; \quad \alpha_2 = \frac{1}{2} \left( \frac{4}{3} - C_2 \right) (2 - C_4) g^2, \tag{31}
\]

\[
\alpha_3 = \left( \frac{4}{3} - C_2 \right) (2 - C_3) g^2; \quad g = \left( \frac{1}{2} C_1 + \frac{P_k}{\epsilon} - 1 \right). \tag{32}
\]

It can be seen these new coefficients are identical to the original ones when \( \eta \) and \( \zeta \) approach zero. The constants \( C_1 - C_4 \) in the SSG part are provided in Table 1. \( P_k \) is the production of turbulence kinetic energy and is given by the following equation:

\[
P_k = -\tilde{S}_{ij} \frac{\partial \tilde{U}_i}{\partial x_j}. \tag{33}
\]

3. Numerical Method

We perform the computations using QUADFLOW, a well-validated cell-centered finite volume flow solver (Bramkamp et al., 2004). QUADFLOW solves the coupled system of conservation equations of mass, momentum, total energy and turbulent quantities in their time-dependent, integral formulation:
\begin{equation}
\int_{\Omega} \frac{\partial \vec{W}}{\partial t} \, dV + \oint_{\partial \Omega} \left( \vec{F}_c - \vec{F}_v \right) \cdot \vec{n} dS - \int_{\Omega} \vec{Q} dV = 0. \tag{34}
\end{equation}

in which $\partial \vec{W} / \partial t$ represents the time-derivative of the vector of the conservative variables, $\vec{F}_c$ and $\vec{F}_v$ are the tensors of convective and viscous fluxes, respectively, and $\vec{Q}$ is the vector containing the source terms. The convective fluxes are discretized by an upwind method (AUSMDV, Wada & Liou (1997)) and second-order accuracy in space is achieved by linear Green-Gauss reconstruction. Venkatakrishnan’s limiter (Venkatakrishnan, 1993) is used to prevent spurious oscillations. The viscous fluxes are discretized by quasi-central differencing. The mean flow equations are integrated in time by an explicit five-stage Runge-Kutta scheme or a fully implicit Euler-backward via Newton-Krylov method. The flow is assumed to be quasi-steady. QUADFLOW has been validated extensively with different test cases at various flow conditions (Schieffer et al., 2009). The flow solver was also proven to be suitable for supersonic and hypersonic computations (Nguyen et al., 2009). The original SSG/LRR-$\omega$ model was implemented into the code by Bosco et al. (2009).

In subsonic flows, standard characteristic boundary conditions are applied at the far-field in which the incoming Riemann invariants are set by the free-stream conditions and the outgoing Riemann invariants are determined by quantities being extrapolated from the interior domain. The free-stream turbulence intensity was fixed at 0.5%. At solid walls, the no-slip condition is enforced and isothermal condition is used. The turbulence kinetic energy is set to zero at the wall and the respective length scale is prescribed based on the first grid spacing according to Menter (Menter, 1994).

4. Model Validation

The flow over a flat plate at zero pressure gradient is used here as the test case for model validation because most turbulence models are calibrated to reproduce the log law of the wall. In this particular work, the model was also designed to capture the near wall behavior of the Reynolds stresses in zero pressure gradient flow. For the computations, the grid is identical to that in the work by Eisfeld (Eisfeld, 2006). The inflow boundary is located 2L upstream of the flat plate leading edge where L is the length of the plate. The upper boundary is parallel to the flat plate and is at a distance 0.8L above the plate. The grid consists of 224x160 cells which are densely clustered around the leading edge to resolve the formation of the boundary layer. The transverse grid lines were stretched in the wall normal direction so that the distance from the first cell center to the wall corresponds to $y^+ = 1.85$ at the leading edge and decreases to $y^+ = 0.16$ towards the trailing edge.

The inflow condition was designed for a Reynolds number of $Re_L = 10^7$ based on the flat plate length (see table 4). The free-stream Mach number is set to 0.3 so that the flow can be considered incompressible. Figure 2 and figure 3 illustrate the performance of the new near wall model in this test case. Similar to the SST model, the new model predicts the skin friction accurately while the SSG/LRR-$\omega$ model slightly underpredicts this quantity. The velocity profiles are plotted in wall units ($u^+$ and $y^+$) at a momentum thickness Reynolds number $Re_\theta = 1410$ on the left side of figure 2. Here, the new model slightly overpredicts the buffer layer but the agreement with DNS data in the wake layer is better than the other models. Regarding the Reynolds stresses, the new model shows significant improvement in the predictions of the streamwise Reynolds stress component in the near wall region. Nevertheless, for this simple flow, the Reynolds shear stress is the most important contribution in the momentum equation and all turbulence models can predict this quantity well.
Table 4: Free-stream and boundary conditions of zero pressure gradient flow on a flat plate.

| $M_\infty$ $[-]$ | $Re_L$ $[-]$ | $T_\infty[K]$ | $\rho_\infty[kg/m^3]$ | $T_w[K]$ |
|------------------|--------------|---------------|---------------------|---------|
| 0.3              | $10^7$       | 285           | 1.7475              | 300     |

Figure 1: Computational grid of turbulent flat plate.

5. Conclusion

In this paper, we carried out asymptotic analysis and introduced a simple near wall correction into an $\omega$-based Reynolds stress model. The new model was shown to be able to capture the near wall anisotropy of the Reynolds stress tensor better than before. Furthermore, the pressure-strain term is switched to a non-equilibrium model far from the wall so that the model is consistent with rapid distortion theory in homogeneous turbulence. Further validations for flows under favorable pressure gradient, adverse pressure gradient and separated flow are planned in future work.

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Figure 3: Predictions of Reynolds stress components in a zero pressure gradient boundary layer.

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References

Aupoix, B. 2009 Introduction to statistical one-point closures. Lecture notes, Ecole de Mecanique des Fluides Numerique, Oleron.

Bosco, A., Reinartz, B. & Müller, S. 2009 Computation of hypersonic shock boundary layer interaction on a double wedge using a differential reynolds stress model. Proceedings of 27th Int. Syposium on Shock Waves (ISSW-27).

Bramkamp, F. D., Lamby, P. & Müller, S. 2004 An adaptive multiscale finite volume solver for unsteady and steady flow computations. Journal of Computational Physics 197, 460–490.

Eisfeld, B. 2006 Computation of complex compressible aerodynamic flows with a reynolds stress turbulence model. Int. Conference on Boundary and Interior Layers.

George, W. K. 2010 Lectures in turbulence for the 21st century. Department of Applied Mechanics, Chalmers University of Technology, Gothenburg, Sweden.

Jakirlic, S., Eisfeld, B., Jester-zürker, R. & Kroll, N. 2007 Near-wall, reynolds-stress model calculations of transonic flow configurations relevant to aircraft aerodynamics. Int. J. Heat and Fluid Flow 28, 602–615.

Lai, Y. G. & So, R. M. C. 1990 On the near-wall turbulent flow modelling. J. Fluid Mech. 221, 641–673.

Menter, F. R. 1994 Two-equation eddy-viscosity turbulence models for engineering applications. AIAA Journal 32, 1598–1605.

Nguyen, T., Behr, M. & Reinartz, B. 2009 Numerical investigation of compressible turbulent boundary layer over expansion corner. AIAA-7371-2009.

Reece, G. J., Launder, B. E. & Rodi, W. 1975 Progress in the development of a reynolds-stress turbulence closure. J. Fluid Mech. 68, 537–566.
SARKAR, S., SPEZIALE, C. G. & GATSKI, T. B. 1991 Modelling the pressure-strain correlation of turbulence: an invariant dynamical systems approach. *J. Fluid Mech.* **227**, 245–272.

SCHIEFFER, G., BALLMANN, J. & BEHR, M. 2009 Validation of advanced turbulence models in quadflow. *Proceedings of the Sixth International Symposium on Turbulence, Heat and Mass Transfer*.

SPEZIALE, C. G. 1998 Non-equilibrium modeling of complex turbulent flows. *Tech. Rep.*, Boston University.

VENKATAKRISHNAN, V. 1993 On the accuracy of limiters and convergence to steady-state solutions. *AIAA-0880-1993*.

WADA, Y. & LIOU, M. S. 1997 An accurate and robust flux splitting scheme for shock and contact discontinuities. *SIAM Journal on Scientific Computing* **18**, 633–648.

WILCOX, D. C. 1993 Chapter 4. In *Turbulence modeling for CFD*. DCW Industries Inc.

WILCOX, D. C. 2008 Formulation of the $k – \omega$ turbulence model revisited. *AIAA Journal* **46**, 2823–2838.