Geometric Interpretations of Quandle Homology

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Abstract

Geometric representations of cycles in quandle homology theory are given in terms of colored knot diagrams. Abstract knot diagrams are generalized to diagrams with exceptional points which, when colored, correspond to degenerate cycles. Bounding chains are realized, and used to obtain equivalence moves for homologous cycles. The methods are applied to prove that boundary homomorphisms in a homology exact sequence vanish.

1 Introduction

A quandle is a set with a self-distributive binary operation (defined below) whose definition was motivated from knot theory. A (co)homology theory was defined in [2] for quandles, which is a modification of rack (co)homology defined in [1]. State-sum invariants using quandle cocycles as weights are defined [2] and computed for important families of classical knots and knotted surfaces [3]. Quandle homomorphisms and virtual knots are applied to this homology theory [4]. The invariants were applied to study knots, for example, in detecting non-invertible knotted surfaces [5].

On the other hand, knot diagrams colored by quandles can be used to study quandle homology groups. This viewpoint was developed in [5, 11, 12] for rack homology and homotopy. In this paper, we generalize this approach to quandle homology theory. For this purpose, we generalize abstract knot diagrams [13], and represent cycles and boundaries geometrically by colored abstract knot diagrams. Equivalence moves are given in terms of colored knot diagrams.

The paper is organized as follows. In Section 2 the definition of quandle homology is recalled. Abstract knot diagrams are generalized in Section 3. Colorings of generalized
diagrams are defined (Section 1), and used to represent cycles of quandle homology (Section 2). Equivalence moves for colored diagrams are given in Section 3. In Section 4 the fundamental quandles for abstract knots are discussed. The boundary homomorphisms in an exact sequence are shown to be trivial (Section 8). Examples appear throughout the paper.

2 Definitions of Quandle (Co)Homology

2.1 Definition. A quandle, $X$, is a set with a binary operation $*$ such that

(I. idempotency) for any $a \in X$, $a * a = a$,

(II. right-invertibility) for any $a, b \in X$, there is a unique $c \in X$ such that $a = c * b$, and

(III. self-distributivity) for any $a, b, c \in X$, we have $(a * b) * c = (a * c) * (b * c)$.

A rack is a set with a binary operation that satisfies (II) and (III). Racks and quandles have been studied in, for example, [1], [7], [13], [17], and [18].

A map $f : X \to Y$ between two quandles (resp. racks) $X, Y$ is called a quandle homomorphism if $f(a * b) = f(a) * f(b)$ for any $a, b \in X$. A quandle homomorphism is a (quandle or rack) isomorphism if it is bijective. An isomorphism between the same quandle (or rack) is an automorphism.

2.2 Examples. Any set $X$ with the operation $x * y = x$ for any $x, y \in X$ is a quandle called the trivial quandle. The trivial quandle of $n$ elements is denoted by $T_n$.

Any group $G$ is a quandle by conjugation as operation: $a * b = b^{-1}ab$ for $a, b \in G$. Any subset of $G$ that is closed under conjugation is also a quandle. For example, the set, $QS(5)$, of non-identity elements of the symmetric group on three letters is a quandle. Similarly the set

$$QS(6) = \{(1234) = a, (1243) = b, (1324) = c, (1342) = B, (1423) = C, (1432) = A\}$$

of 4-cycles in the symmetric group on four letters is a quandle.

Let $n$ be a positive integer. For elements $i, j \in \{0, 1, \ldots, n - 1\}$, define $i * j = 2j - i$ where the sum on the right is reduced mod $n$. Then $*$ defines a quandle structure called the dihedral quandle, $R_n$. This set can be identified with the set of reflections of a regular $n$-gon with conjugation as the quandle operation. We also represent the elements of $R_3$ by $\alpha, \beta$, and $\gamma$, where the quandle multiplication is given by $x * y = z$ where $z \neq x, y$ when $x \neq y$ and $x * x = x$, for $x, y, z \in \{\alpha, \beta, \gamma\}$.

As an exercise the reader may check that there is a quandle homomorphism $p : QS(6) \to R_3$ given by $p(a) = p(A) = \alpha$, $p(b) = p(B) = \beta$, and $p(c) = p(C) = \gamma$.

Any $\Lambda = \mathbb{Z}[T, T^{-1}]$-module $M$ is a quandle with $a * b = Ta + (1 - T)b$, $a, b \in M$, called an Alexander quandle. Furthermore for a positive integer $n$, a mod-$n$ Alexander quandle $\mathbb{Z}_n[T, T^{-1}]/(h(T))$ is a quandle for a Laurent polynomial $h(T)$. The mod-$n$ Alexander quandle is finite if the coefficients of the highest and lowest degree terms of $h$ are $\pm 1$.

See [1], [7], [13], or [18] for further examples.

2
2.3 Remark. Let $X$ denote a quandle. From Axiom II, each element $b \in X$ defines a bijection $S(b) : X \to X$ with $aS(b) = a*b$. The bijection is an automorphism by Axiom III. For a word $w = b_1^{e_1} \ldots b_n^{e_n}$ where $b_1, \ldots, b_n \in X; e_1, \ldots, e_n \in \{\pm 1\}$, we define $a*w = aS(w)$ by $aS(b_1)^{e_1} \ldots S(b_n)^{e_n}$. An automorphism of $X$ is called an inner-automorphism of $X$ if it is $S(w)$ for a word $w$. (The notation $S(b)$ follows Joyce’s paper [13] and $a*w (= a^w)$ follows Fenn-Rourke [1].)

Let $C^R_n(X)$ be the free abelian group generated by $n$-tuples $(x_1, \ldots, x_n)$ of elements of a quandle $X$. Define a homomorphism $\partial_n : C^R_n(X) \to C^R_{n-1}(X)$ by

$$\partial_n(x_1, x_2, \ldots, x_n) = \sum_{i=2}^{n} (-1)^i [(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) - (x_1*x_i, x_2*x_i, \ldots, x_{i-1}*x_i, x_{i+1}, \ldots, x_n)]$$

for $n \geq 2$ and $\partial_n = 0$ for $n \leq 1$. Then $C^R_n(X) = \{C^R_n(X), \partial_n\}$ is a chain complex.

Let $C^D_n(X)$ be the subset of $C^R_n(X)$ generated by $n$-tuples $(x_1, \ldots, x_n)$ with $x_i = x_{i+1}$ for some $i \in \{1, \ldots, n-1\}$ if $n \geq 2$; otherwise let $C^D_n(X) = 0$. If $X$ is a quandle, then $\partial_n(C^D_n(X)) \subset C^D_{n-1}(X)$ and $C^D_0(X) = \{C^R_0(X), \partial_n\}$ is a sub-complex of $C^R_0(X)$. Put $C^Q_n(X) = C^R_n(X)/C^D_n(X)$ and $C^Q_0(X) = \{C^Q_0(X), \partial'_n\}$, where $\partial'_n$ is the induced homomorphism. Henceforth, all boundary maps will be denoted by $\partial_n$.

For an abelian group $G$, define the chain and cochain complexes

$$C^W_n(X; G) = C^W_n(X) \otimes G, \quad \partial = \partial \otimes \text{id};$$

$$C^W_n(X; G) = \text{Hom}(C^W_n(X), G), \quad \delta = \text{Hom}(\partial, \text{id})$$

in the usual way, where $W = D, R, Q$.

2.4 Definition. The $n$th rack homology group and the $n$th rack cohomology group of a rack/quandle $X$ with coefficient group $G$ are

$$H^R_n(X; G) = H_n(C^R_n(X; G)), \quad H^R_0(X; G) = H^0(C^R_n(X; G)).$$

The $n$th degeneration homology group and the $n$th degeneration cohomology group of a quandle $X$ with coefficient group $G$ are

$$H^D_n(X; G) = H_n(C^D_n(X; G)), \quad H^D_0(X; G) = H^0(C^D_n(X; G)).$$

The $n$th quandle homology group and the $n$th quandle cohomology group of a quandle $X$ with coefficient group $G$ are

$$H^Q_n(X; A) = H_n(C^Q_n(X; G)), \quad H^Q_0(X; A) = H^0(C^Q_n(X; G)).$$

The homology group of a rack in the sense of [8] is $H^R_n(X; G)$ and the cohomology of a quandle used in [2] is $H^Q_0(X; A)$. Refer to [3], [8], [10], [12] for some calculations and
applications of the rack homology groups, and to \cite{2, 3} for those of quandle cohomology
groups.

The cycle and boundary groups (resp. cocycle and coboundary groups) are denoted by
$Z_n^W(X; G)$ and $B_n^W(X; G)$ (resp. $Z_n^0(X; G)$ and $B_n^0(X; G)$), so that

$$H_n^W(X; G) = Z_n^W(X; G)/B_n^W(X; G), \quad H_n^0(X; G) = Z_n^0(X; G)/B_n^0(X; G)$$

where $W$ is one of $D, R, Q$. We will omit the coefficient group $G$ if $G = \mathbb{Z}$ as usual.

Here we are almost exclusively interested in quandle homology or cohomology.

### 2.5 Example.

For $QS(6)$ (defined in \cite{2, 3}), we have computed using MATHEMATICA that
$H_3^Q(QS(6); \mathbb{Z}) = \mathbb{Z}_{24}$. Similarly we have $H_3^Q(R_3; \mathbb{Z}) = \mathbb{Z}_3$. We will illustrate in Example \ref{7.5}
that the homomorphism $p : QS(6) \to R_3$ induces a surjection $\mathbb{Z}_{24} \to \mathbb{Z}_3$. Many other
calculations are found in \cite{2, 3, 4, 12}.

### 3 Generalized Knot Diagrams

In \cite{2} a general framework for colored diagrams was sketched. Such diagrams are useful in
representing rack homology classes and homotopy classes of maps into the classifying space
of a rack. Here we review, amplify, and generalize some of their constructions to the case
of quandle homology classes. Abstract knot diagrams are defined in \cite{15} and the relation
to virtual knots \cite{16} is established in \cite{15}. We extend their definition to include arcs and
surfaces with boundary. First we generalize crossings of classical knot diagram to higher
dimensions.

In the top row and the bottom three rows of Fig. 1, $k$-crossing 1-diagrams are indicated
for $k = 0, 1, 2$. In Fig. 2 $k$-crossing 2-diagrams are indicated for $k = 0, 1, 2, \text{ and } 3$. We define
such diagrams in general.

#### 3.1 Definition.

Fix a non-negative integer $n$. Let $E$ be the $(n + 1)$-disk $[-1, 1]^{n+1} = \{(x_1, \ldots, x_{n+1}) | -1 \leq x_i \leq 1, i = 1, \ldots, n+1\}$, and for $j \in \{1, \ldots, n+1\}$, let $E_j$ be the $n$-disk
in $E$ determined by $x_j = 0$. A 0-crossing $n$-diagram is the disk $E$. A 1-crossing $n$-diagram
(or simply 1-crossing diagram) $\Sigma_1$ is the pair of $E$ and the $n$-disk $E_1$. Let $k$ be an integer in
the range $\{1, \ldots, n+1\}$. A $k$-crossing $n$-diagram (or simply $k$-crossing diagram) $\Sigma_k$ is the
pair of $E$ and the union of the disks $E_1 \cup \cdots \cup E_k$ with crossing information that is indicated
by removing from $E_j$ (for $j = 2, \ldots, k$) an open $\varepsilon$-neighborhood of $(E_1 \cup \cdots \cup E_{j-1}) \cap E_j$.
For example, when $n = 1$ a 2-crossing 1-diagram is the standard depiction of a classical
knot crossing. When $n = 2$, a 2-crossing 2-diagram is the broken surface depiction \cite{2} of
pair of planes in $\mathbb{R}^4$ that project to intersect along a double curve as depicted in Fig. 2 third
entry from top. A 3-crossing 2-diagram is a broken diagram of a triple point as depicted in
Fig. 2 bottom two entries.

The removal of the $\varepsilon$-neighborhood is a convention for indicating relative height in a
projection from $(n + 2)$-dimensional space. Thus in $E \times [0, 1]$, we re-embed $E_k$ as $\tilde{E}_k = E_k \times \{\frac{n+2-k}{n+2}\}$. Then the lift, $\tilde{E}_k$, projects to $E_k$ under the projection $E \times [0, 1] \to E$. The
| Crossing type | Handle Index | 0-Handle | 1-Handle | 2-Handle |
|---------------|--------------|----------|----------|----------|
| 0-Crossing    |              | ![Diagram](image1) | ![Diagram](image2) | ![Diagram](image3) |
| Endpoint      |              | ![Diagram](image4) | ![Diagram](image5) | ![Diagram](image6) |
| 1-Crossing    |              | ![Diagram](image7) | ![Diagram](image8) | ![Diagram](image9) |
| 2-Crossing    |              | ![Diagram](image10) | ![Diagram](image11) | ![Diagram](image12) |

Figure 1: A table of handles for abstract 1-knot diagrams
| Crossing Type | Handle Index | 0-Handle | 1-Handle | 2-Handle | 3-Handle |
|---------------|--------------|----------|----------|----------|----------|
| 0-Crossing    |              | ![0-Handle](image1) | ![1-Handle](image2) | ![2-Handle](image3) | ![3-Handle](image4) |
| 1-Crossing    |              | ![1-Handle](image2) | ![2-Handle](image3) | ![3-Handle](image4) | ![4-Handle](image5) |
| 2-Crossing    |              | ![2-Handle](image3) | ![3-Handle](image4) | ![4-Handle](image5) | ![5-Handle](image6) |
| 3-Crossing    |              | ![3-Handle](image4) | ![4-Handle](image5) | ![5-Handle](image6) | ![6-Handle](image7) |

Figure 2: A table of handles for abstract 2-knot diagrams, crossings
relative position in the $[0, 1]$ factor then is determined by the number of regions on the sheet $E_k$.

Such diagrams, $(E, \cup E_j)$, are oriented. For example, when $n = 1$, orientations of arcs are indicated by arrows along the arc. The orientation of $E$ is indicated by a short arrow normal to the arc $E_j$ where tangent followed by normal is the standard orientation of the disk. The normal arrow may be omitted from the figure when the orientation of the disk $E$ is that of the plane of the paper. When $n = 2$, we assume that the disks $E_j$ are oriented in a counterclockwise fashion and only indicate the normal arrow.

In higher dimensions, signs of $(n+1)$-crossing $n$-diagrams are determined by normal vectors as follows. Let the normal to $E_j$ be denoted by $\nu_j$. Then the $(n+1)$-tuple $(\nu_1, \ldots, \nu_{n+1})$ determines an orientation of the disk $E$. The $(n+1)$-crossing point is positive if and only if this orientation agrees with the standard orientation. Thus if the normal to $E_j$ is the standard basis vector $e_j = (0, \ldots, 1, \ldots, 0)$ (where the non-zero entry is in the $j$th coordinate), then the $(n + 1)$-crossing point is positive.

3.2 Definition. There are $2^k - 1 = 1 + 2 + \cdots + 2^{k-1}$ components in $\Sigma_k$. We call them the $n$-regions of $\Sigma_k$. The disk $E_j$ is called the level $j$ sheet of $\Sigma_k$. The $k$-crossing point set (or simply $k$-crossing) in $E$ is the intersection $E_1 \cap E_2 \cap \cdots \cap E_k$, which is a disk of dimension $n + 1 - k$ (or of codimension $k$). The complement of $E_1 \cup \cdots \cup E_k$ in $E$ has $2^k$ components called $(n + 1)$-regions.

3.3 Remark. Observe that in the level $j$ sheet at a $k$-crossing we have a $(j - 1)$-crossing diagram; this diagram is the orthogonal projection of the previous levels onto the $j$th level.

Next we introduce endpoint, branch point, and hem diagrams. These exceptional diagrams will be useful to us in describing degenerate chains in quandle homology.

3.4 Definition. An endpoint diagram is a 2-disk with an arc embedded; one end of the arc is in the interior of the disk, the other end is on the boundary, and the arc intersects the boundary transversely. In our illustrations, we indicate the interior point as a fat vertex. See Fig. 1 the second and third pictures from top.

3.5 Definition. A branch point diagram consists of a neighborhood of a branch point (also called Whitney umbrella, a generic singularity of surface maps) in a 3-ball in which over and under crossing information is indicated by removing a thin open triangle along the double point arc as depicted in Figure 3, the top and second top pictures. The vertex of the triangle is at the branch point which is in the interior of the 3-ball. The boundary of the branch point neighborhood on the boundary of the 3-ball is the diagram of the unknot with one crossing.

3.6 Definition. A hemmed 1-crossing diagram is a 2-dimensional half-disk embedded in a 3-ball as depicted in Fig. 3 the second bottom pictures. A part of the boundary of the disk is in the interior of the 3-ball, called the hem.
### 3.7 Definition. A hemmed 2-crossing diagram consists of two disks immersed (with crossing information) in the 3-ball with the boundary of one of these (the under-sheet) contained in the interior of the 3-ball. Crossing information is depicted in the bottom of Fig. 3 by removing a thin rectangular neighborhood of the double point arc in the under-sheet — the horizontal sheet in the illustration. The under-sheet of a hem 2-crossing diagram is the sheet that has an interior hem boundary (depicted by a thick line segment). It is important to note that at a hemmed 2-crossing diagram it is the hemmed surface that is the under-sheet.

An endpoint diagram, a branch point diagram, or a hemmed 1- or 2-crossing diagram is also called an exceptional diagram.

In Figs. 1, 2, and 3, the collection of endpoint diagrams, branch point diagrams, crossing diagrams in low dimensions, and the handles that they correspond to are indicated.

### 3.8 Definition. A generalized abstract 0-knot diagram consists of a collection of vertices (points) embedded in a closed 1-manifold. Let \( n = 1 \) or \( n = 2 \). A generalized abstract \( n \)-knot diagram is the image of a generic map \( f : M \to N \) with crossing information indicated where:

1. \( f(M) \cap \partial N = \emptyset \);

2. Local crossing information indicated as follows: Each point of \( N \) has a neighborhood \( B \), such that \( B \cap f(M) \) is a \( k \)-crossing 0-diagram \( (k = 0, \ldots, n+1) \), or an exceptional diagram.

![Table of Handles](image)

**Figure 3**: A table of handles for abstract 2-knot diagrams, branch points and end curves.
The image $f(M)$ is called the \textit{universe of the diagram}. If the manifold $M$ is closed, then the diagram is said to be \textit{closed}. Thus when $n = 2$, a closed diagram may have branch points but does not have hems.

Abstract diagrams will often be denoted by $K = [f : M \to N]$. This is a slight abuse of notation since we are considering the image $f(M) \subset N$ with crossing information.

Abstract diagrams are depicted in Figs. 4, 5, 7, and 11.

\textbf{3.9 Remark.} In [14] abstract link diagrams were introduced in which the image $f(M)$ is a deformation retract of the ambient space $N$ (see also [15]). Here we do not need to use this strong condition. Moreover, a generalized abstract diagram (of a closed manifold $M$) in the current sense gives rise to an abstract diagram in the sense of [14] by taking a regular neighborhood of the image. Figure 5 is a strict generalization of abstract diagrams defined in [14], and Fig. 4 is an abstract diagram in any sense. Throughout the sequel, generalized abstract diagrams are called abstract diagrams for simplicity.

Abstract diagrams can be constructed via handle theoretic techniques. In Figs. 1, 2, and 3, crossing diagrams and exceptional diagrams are classified as handles, and attaching regions are indicated.

\textbf{3.10 Remark.} Let $K$ be an abstract 1-knot diagram in $N$. Then $K$ determines an embedded arc $\hat{K}$ in $N \times [0, 1]$ by lifting each over-arc to a higher level (in the $[0, 1]$ factor) than the under-arc at each double point of $K$, as is typically done in classical knot theory. In the present convention endpoints are embedded in $N \times \{0\}$.
3.11 Example. An abstract diagram of a closed curve is indicated in Fig. 4. The 1-manifold is $S^1$, and the manifold $N$ is a thrice punctured torus. There are three 2-crossing 1-diagrams as its subdiagrams, the top two are positive and the bottom one is negative. The 0-handles of $N$ are these 2-crossings depicted as lightly shaded squares. The unshaded bands are 1-handles that are 1-crossing 1-diagrams.

3.12 Example. An abstract diagram of an arc in a planar surface is indicated in Fig. 5. The diagram has two positive crossing points and two endpoints.

3.13 Remark. An abstract closed 2-knot diagram is constructed from copies of $k$-crossing 2-diagrams ($k = 1, 2, 3$) and branch point diagrams as follows. The crossing diagrams are identified along their boundaries so that the arcs of double points either form closed loops or end at 3-crossing diagrams or branch point diagrams. Thus in the boundary of $N$, there is, at most, a collection of simple closed curves. These then are capped-off by adding 2-handles of the form 1-crossing 2-diagrams. Moreover, when crossing diagrams agree at points of their boundaries the broken and unbroken sheets match broken and unbroken sheets, respectively (see Fig. 4, Fig. 8 bottom two pictures, and Fig. 19). It is also required that orientations match when crossing diagrams are glued.

3.14 Remark. An abstract 2-knot diagram represents a surface $M$ embedded in $N \times [0, 1]$ by filling in the surface along the broken sheets in the $[0, 1]$ direction (see Fig. 5 for details on filling in broken surface diagrams). In this way, the surface represented by the diagram is embedded in $N \times [0, 1]$.

3.15 Example. In Fig. 7, a part of a closed abstract 2-knot diagram is indicated (the relation to the arc diagram in the lower right will be explained subsequently). The diagram has two 3-crossing points (triple points) and two branch points. The arcs of double points that end in the front of the diagram are to be glued to the back of the diagram via two 1-handles each of which is a 2-crossing diagram. In this way, the 3-manifold that contains the diagram is a handle-body, and the boundary contains 5 simple closed curves. These are attaching regions for 2-handles that are 1-crossing diagrams.
Figure 6: Broken sheets are glued together consistently

Figure 7: An abstract 2-knot diagram
### 3.16 Remark.
Abstract diagrams can be defined similarly in higher dimensions using \(k\)-crossing \(n\)-diagrams, as in \(\mathbb{R}\). However, the branch point set is more subtle in higher dimensions.

### 4 Coloring Abstract Diagrams

Abstract diagrams, when colored by quandles, represent homology classes. Colored abstract 1- and 2-knot diagrams are of particular interest.

#### 4.1 Definition.
For \(n = 0, 1, 2\) and \(k \leq n + 1\), a coloring of a \(k\)-crossing \(n\)-diagram is defined as an assignment of quandle elements to the points, arcs or regions of the diagram such that the following color condition at the crossing is satisfied. The condition is depicted in Fig. 8. In this figure, an under-arc (or lower sheet of a 2-dimensional region) is colored with a quandle element, say \(q_1\). The arc (or region) that is so colored is the arc away from which the normal to the over-arc (or region) points. The over-arc is colored with \(q_2\). And the remaining under arc (or region) is colored with \(q_1 * q_2\). In case \(n = 0\) only the point is colored with \(q_1\). In case the crossing point is a triple point, then the middle sheet has two colors \((q_2 \text{ and } q_2 * q_3)\) as described above, and the lowest sheet has four colors \((q_1, q_1 * q_3, q_1 * q_2, \text{ and } (q_1 * q_2) * q_3)\).

A branch point diagram is colored by assigning a single quandle element to the 2-dimensional region of the diagram.

Let \(n = 1, 2\). A (quandle) coloring of a closed abstract \(n\)-knot diagram is an assignment of quandle elements to the connected regions of the diagram such that the restriction to each branch point (when \(n = 2\)) or \(k\)-crossing diagram is a coloring. More precisely, let \(\mathcal{R}\) be the set of arcs (for \(n = 1\)) or regions (for \(n = 2\)) of an abstract \(n\)-knot diagram \((n = 1, 2)\). A coloring is a map \(C : \mathcal{R} \to X\), where \(X\) is a quandle, such that \(\{C(r) | r \in \mathcal{R}\}\) satisfies...
the above mentioned conditions at each $k$-crossing $n$-diagram in the given abstract $n$-knot diagram.

4.2 Examples. Figure 4 indicates a coloring of an abstract 1-knot diagram by $R_3$. Figure 7 indicates a coloring of an abstract 2-knot diagram by $R_3$.

In Fig. 9 shadow colorings of $k$-crossing $n$-diagrams for all $n = 0, 1, 2$, and $1 \leq k \leq n + 1$ are depicted. We define these case by case.

4.3 Definition. For any $n$, a shadow coloring of a 0-crossing diagram is an assignment of a quandle element to the $(n + 1)$-dimensional region of the diagram. A shadow coloring of a 1-crossing diagram is an assignment of three quandle elements to the constituents of the diagram, as follows. The $(n + 1)$-dimensional region away from which the normal of the 1-crossing receives a quandle element $q_0$, the $n$-dimensional region (1-crossing) receives the color $q_1$, and the $(n + 1)$-dimensional region into which the normal points receives color $q_0 \ast q_1$.

In general, a shadow coloring of a $k$-crossing $n$-diagram is an assignment of colors to the $n$ and $(n + 1)$-dimensional regions that satisfies the following conditions. Along the $n$-dimensional regions of the diagram, it is a coloring of the $k$-crossing diagram. Each point in an $(n + 1)$-dimensional region of the diagram is assigned a quandle element, and if two such regions are separated by an $n$-dimensional region, then any point in the $n$-dimensional region has a neighborhood homeomorphic to a 1-crossing $n$-diagram. In this case, the
A shadow coloring of an endpoint diagram is the assignment of the same color to the arc and the adjacent region. See Fig. 10 (1). A shadow coloring of a branch point diagram is the assignment of a single quandle element, say $b$, to the 2-dimensional faces of the diagram. There are three 3-dimensional regions in a neighborhood of the branch point. One such region is assigned the color $a$; the other two are assigned the color $a \ast b$ or $(a \ast b) \ast b$, and the assignment is determined by the condition that normal vectors point into regions that receive quandle products. See Fig. 10 (2) and (3).

A shadow coloring of a hem 1-crossing diagram is an assignment of a single quandle element to the region and the disk. See Fig. 10 (4). A shadow coloring of a hem 2-crossing diagram is an assignment of an element $a$ on the disk and the region away from which the normal of the disk dividing the hem, the dividing disk receives $b$, and then, the region and the disk into which the normal points receive the element $a \ast b$. See Fig. 10 (5).

Let $n = 1, 2$. A shadow coloring of an abstract $n$-knot diagram is an assignment of quandle elements to the $n$ and $(n + 1)$-dimensional regions of the diagram such that the restriction to each endpoint, branch point, hem or $k$-crossing diagram is a shadow coloring.

A shadow coloring is, as before, a map $C : R \rightarrow X$ where $R$ is the set of arcs ($n = 1$) or regions ($n = 2$) and complementary ($n + 1$)-dimensional regions.

4.4 Example. Figure 11 indicates a shadow quandle coloring by $R_3$ of an abstract 0-knot diagram. The small flag-like arrows at the 1-crossing points indicate the orientation of the vertices. When this arrow coincides with the tangent direction of the circle (as in the figure), the crossing is positive; otherwise the crossing is negative.

4.5 Example. Figure 12 indicates a shadow quandle coloring by $R_3$ of the abstract 1-knot diagram that was given in Fig. 5. The boundary of the surface is abbreviated and is not drawn in Fig. 12 for simplicity. In the figure, colors on regions are indicated by letters in squares.
Figure 11: A shadow colored abstract 0-knot diagram

Figure 12: A shadow colored abstract 1-knot diagram
4.6 Remark. Shadow colorings were defined in [9] and used in [21] to prove that the right and left trefoil knots are distinct. The notions of coloring and shadow coloring of \( k \)-crossing \( n \)-diagrams extend for all \( n = 1, 2, \ldots \), and for all \( k = 0, \ldots, n + 1 \) (see [8]).

To see that shadow colorings exist, we prove the following lemmas.

4.7 Lemma. Let \( C \) be a coloring by a quandle \( X \) of an oriented \( k \)-crossing \( n \)-diagram, \( (k \leq n) \), which is obtained from an oriented \( (k + 1) \)-crossing \( n \)-diagram by ingoring the level \( k + 1 \) sheet, \( E_{k+1} \). Let \( D \) be an \( n \)-region on \( E_{k+1} \), and let \( x \in X \). Then there is a unique coloring \( C' \) by \( X \) of the \((k + 1)\)-crossing diagram which restricts to \( C \) and such that \( C'(D) = x \).

Proof. Let \( D_j \), \( j = 1, \ldots, 2^{k+1} \), be the \( n \)-regions on \( E_{k+1} \). Let \( D = D_1 \), and pick points \( p_j \in D_j \). For any fixed \( j \), let \( \gamma \) be a path in the level \( k + 1 \) sheet (the image of a continuous map from \([0, 1]\) to the level \( k + 1 \) sheet) from \( p_1 \) to \( p_j \).

Assume without loss of generality that the path is in general position with the level \( g \) sheets for all \( g \), so that it meets the sheets in finitely many points. Let \( s_1, \ldots, s_r \) be the intersection points, and let \( c_i \), \( i = 1, \ldots, r \) be \(+1\) (resp. \(-1\)) if the path goes in the same (resp. the opposite) direction as the normal of the sheet at \( p_i \). Let \( C(p_i) = c_i \) (precisely speaking, the color of the disk in which the point \( p_i \) lies). Then define the color \( C'(D_j) = x \cdot w \) where \( w \) is the word \( c_{i_1}^*c_{i_2}^* \cdots c_{i_r}^* \). This definition is made in such a way that the condition of coloring is satisfied at each intersection point along the path \( \gamma \), and in remains to be proved that the color thus defined does not depend on the choice of the path \( \gamma \).

Let \( \gamma_i \), \( i = 0, 1 \), be such two paths. Since the level \( k + 1 \) sheet is simply connected, there is a homotopy between them. Such a homotopy is a map from a 2-disk to the level \( k + 1 \) sheet, whose image is denoted by \( U \). Assume without loss of generality that \( U \) is in general position with the level \( g \) sheets for all \( g(< k) \). Then the intersection between \( U \) and the level \( g \) sheets for all \( g(< k) \) is generically immersed 1-manifold with boundary, i.e., arcs with transverse double points. When the \( \varepsilon \)-neighborhood in the definition of the crossing diagram is removed from these arcs, then we obtain a classical knot diagram on \( U \) with boundary of arcs lying on \( \partial U \). Give the color by \( X \) on the regions and arcs in \( U \) by using the rule of the shadow color depicted in Fig. [10]. Then the color is well-defined on the diagram on \( U \), proving that the color at the end point of \( \gamma_i \), \( i = 0, 1 \), coincide. \( \square \)

4.8 Lemma. Let \( K = [f : M^n \rightarrow N^{n+1}] \) be an abstract \( n \)-knot diagram, and let \( C \) be a coloring by a quandle \( X \). Suppose \( N \) is simply connected. Then for any \((n + 1)\)-region \( R \) and \( x \in X \), there is a shadow color \( C' \) for \( K \) such that \( C' \) restricts \( C \) and \( C'(R) = x \).

Proof. This is proved by defining colors using paths as in Lemma [4.7] and the proof is similar. The condition \( \pi_1(N) = \{1\} \) guarantees the existence of a homotopy between paths.

In the case \( n = 1 \) and \( 2 \), we included end/branch points and hem 1- and 2-crossings, in addition to \( k \)-crossing \( n \)-diagrams. For these diagrams, a similar argument applies, using the definition of shadow colors for these diagrams. \( \square \)

4.9 Extended Remark. Let \( SC(1) \) denote the collection of shadow colored abstract 1-knot diagrams, and let \( C(2) \) denote the collection of colored abstract closed 2-knot diagrams.
Figure 13: The correspondence between shadow colored 1-knot diagrams and colored 2-knot diagrams

We have maps $D : C(2) \to SC(1)$ and $I : SC(1) \to C(2)$ that are defined as follows (see also Fig. 13 for a local description).

Consider a colored closed abstract 2-knot diagram, $K^2 = [f : M^2 \to N^3]$. The map $D$ assigns to $K^2$ the following shadow colored 1-knot diagram. The abstract 1-knot diagram is the lower decker set in a regular neighborhood of the lower decker set in the surface $M$ — the lower decker points form the 1-crossings and 2-crossings. Here, lower decker set (see [5]) means the preimage of the double point set of the surface $M$ that are in the under sheet and represented as the broken sheet in the abstract 2-knot diagram. The 2-crossing points of $D(K^2)$ correspond to the triple points of the abstract 2-knot diagram, and the endpoint diagrams correspond to branch points (see [5] for details). We color the arcs of $D(K^2)$ with the quandle elements that appear on the regions that contain the corresponding upper sheets. At a 2-crossing of $D(K^2)$, the over-arc is colored by the color on the upper sheet, the under arcs are colored by the colors on the two portions of the corresponding middle sheet. The 2-dimensional regions $D(K^2)$ are colored by the quandle elements that are on the pieces of the surface that contain the arc diagram.

On the other other hand, the map $I$ assigns to an abstract shadow colored 1-knot diagram, $K^1$, a colored abstract 2-knot diagram as follows. Decompose the abstract 1-knot diagram into crossing and endpoint diagrams. Construct an abstract 2-knot diagram using the local correspondence between $k$-crossing 1-diagrams and $(k + 1)$-crossing 2-diagrams, and between endpoints and branch points that is depicted in Fig. 13. Colors on the surface are determined as in the figure. In this way, a surface with double points on the boundary is constructed. The colors on the double points on the boundary just above and below the
surface of the arc diagram agree. Thus these double points can be joined together as they were in Fig. 7.

Given $K^1$, the 1-knot diagram $\mathcal{D}(\mathcal{I}(K^1))$ differs from $K^1$ in that in addition to $K^1$, it contains unknotted unlinked colored components that may not be found in $K^1$.

In particular, given a shadow colored classical knot diagram, we obtain a colored abstract 2-knot diagram. If $K$ is a knot, the 2-knot diagram consists of a sphere (in the plane of the knot diagram) and a torus of the form $K \times S^1$. The 2-knot diagram is in $S^2 \times S^1$. We call this the suspension of a classical knot. See Fig. 14.

A similar construction applies to give maps $\mathcal{D} : C(1) \rightarrow SC(0)$ and $\mathcal{I} : SC(0) \rightarrow C(1)$ between the sets, $C(1)$, of colored abstract closed 1-knot diagrams and SC(0) of shadow colored 0-knot diagrams. The essence of this construction is found in [12] (see also Fig. 15).

5 Representing Homology Classes

In Greene [12] (see also [4, 14]), rack cycles are represented by colored knot diagrams. We generalize this method to quandle cycles, using end/branch point and hem diagrams. Specifically, for a fixed quandle 2- or 3-cycle we construct a colored or shadow colored diagram that represents the cycle. Moreover, if a cycle is a boundary, we construct an abstract 2-knot diagram in a 3-manifold, the boundary of which is the given abstract 1-knot diagram that represents the cycle. Care will be taken at endpoint diagrams to make this precise. Finally, we represent 4-cycles by shadow colored abstract 2-knot diagrams with hems. We do not pursue boundaries in this case.

5.1 Lemma. (1) There is a one-to-one correspondence between the set of $(n + 1)$-tuples of quandle elements and the set of quandle colored positive $(n + 1)$-crossing $n$-diagrams.

(2) There is a one-to-one correspondence between the set of $(n + 1)$-tuples of quandle elements and shadow colored positive $n$-crossing $n$-diagrams.
Proof. The correspondence (1) for \( n = 1 \) is illustrated in Fig. 16 (A). The correspondence (2) for \( n = 1 \) is illustrated in Fig. 16 (B), where the correspondence between the shadow colors and the lower decker set is illustrated. The correspondence (1) for \( n = 2 \) is illustrated in the center of Fig. 17. The correspondence (2) for \( n = 2 \) is illustrated in Fig. 18.

In general, let \( X \) be a quandle and consider an \((n+1)\)-tuple \((x_1, \ldots, x_{n+1}) \in X^{n+1}\). Recall that \( E_j \) is the coordinate \( n \)-disk in \([-1, 1]^{n+1}\) whose \( j \)th coordinate is 0. Color the region of \( E_j \) at which the \( \ell \)th coordinates are less than \(-\varepsilon\) for \( \ell < j \) with \( x_{n+2-j} \). Color the remaining regions of \( E_j \) in such a way that the quandle condition holds at each double point of the diagram.

Such a coloring can be uniquely extended to a shadow coloring by coloring the \((n+1)\)-region of \( E \) at which all the coordinates are less than \(-\varepsilon\) by \( x_0 \). Thus associated to an \((n+1)\)-tuple there is a colored \((n+1)\)-crossing diagram or a shadow colored \( n \)-diagram.

By choosing the color \( x_{n+2-j} \) from the region of \( E_j \) at which the \( \ell \)th coordinates are less than \(-\varepsilon\) for \( \ell < j \), we construct an \((n+1)\)-tuple. A shadow colored diagram works similarly. \( \square \)

5.2 Scholium. The boundary \( \partial(x_1, \ldots, x_{n+1}) \) of a generating rack chain \((x_1, \ldots, x_{n+1}) \in X^{n+1}\) corresponds to the boundary of the quandle colored \((n+1)\)-crossing diagram.

Proof. The correspondence is illustrated in the Figs. 16 (A) and 17. We leave the rest of the details to the reader. \( \square \)

5.3 Remark. Colored and shadow colored abstract knot diagrams represent chains — formal-sums of signed \((n+1)\)-tuples (or \((n+2)\)-tuples) of quandle elements corresponding to the 0-dimensional crossings as in Lemma 5.1. Recall that the 0-dimensional crossings of an abstract \( n \)-knot diagram are the \((n+1)\)-crossing points. Each such (shadow) colored crossing point in the diagram has an associated sign. The chain determined by the diagram...
\[ \delta(x,y) = y - y + x \cdot y - x \]

Figure 16: A geometric representation of a generating 2-chain and its boundary

Figure 17: A geometric representation of a generating 3-chain and its boundary
is the sum of these signed \((n+1)\)-tuples (or \((n+2)\)-tuples) taken over all the 0-dimensional crossings. This is the \textit{chain represented by a (shadow) colored diagram.}

An abstract diagram (of a closed manifold \(M\)) that has no exceptional points has its 1-dimensional crossing set closed. Therefore the sum of the representative chain is a rack cycle in this case [8].

In low dimensions, shadow colored exceptional points or colored branch points represent degenerate chains. Specifically, a shadow colored endpoint or a colored branch point represents a chain of the form \(\pm(a, a)\), a shadow colored branch point represents a chain of the form \(\pm(a, b, b)\) and a shadow colored hem 2-crossing point represents a chain \(\pm(a, a, b)\). The signs are determined by the direction of the arrow along the arc that originates or terminates at the exceptional point.

Thus a shadow colored abstract 0-knot diagram represents a quandle 2-cycle (for example, Fig. 11), as does a colored closed abstract 1-knot diagram (for example, Fig. 4). A shadow colored abstract 1-knot diagram represents a \textit{quandle} 3-cycle as does a colored closed abstract 2-knot diagram (for example, Fig. 5). And a shadow colored abstract 2-knot diagram represents a \textit{quandle} 4-cycle. For example the 2-twist-spun trefoil may be shadow colored by \(R_3\) (see [2], and apply Lemma 12). This is the context of Theorem 5.5 (1a) and (1b).

**5.4 Examples.** The \(R_3\) 2-chain \((\alpha, \beta) + (\gamma, \alpha) + (\beta, \gamma)\), represented by the diagram in Fig. 11 is a cycle in \(Z_2^Q(R_3)\).

The \(R_3\) 2-chain represented by the colored abstract 1-knot diagram in Fig. 4 is the cycle \((\alpha, \beta) + (\beta, \gamma) - (\beta, \alpha) \in Z_2^Q(R_3; \mathbb{Z})\).

It is known [2] that \(H_2^Q(R_3; \mathbb{Z}) = 0\), so that the above examples are in fact boundaries.

The 3-chain represented by the colored abstract 2-knot diagram in Fig. 5 is the cycle \((\alpha, \beta, \gamma) + (\alpha, \gamma, \alpha) \in Z_3^Q(R_3; \mathbb{Z})\). The corresponding (in the sense of [4,9]) shadow colored abstract 1-knot diagram is depicted at the right bottom of the figure, which is the same as Fig. 12. It is known [3] that \(H_3^Q(R_3; \mathbb{Z}) \cong \mathbb{Z}_3\), and the above 3-cycle is a generator.
5.5 Theorem. Let $X$ denote a quandle.

(1a) Let $n = 1, 2$. Any colored closed abstract $n$-knot diagram represents a quandle $(n+1)$-cycle in $Z_{n+1}^3(X;\mathbb{Z})$.

(1b) Let $n = 1, 2, 3$. Any shadow colored abstract $(n-1)$-knot diagram (possibly with exceptional points if $n = 2, 3$) represents a quandle $(n+1)$-cycle in $Z_{n+2}^Q(X;\mathbb{Z})$.

(2a) Let $n = 1, 2$. Let $\eta \in Z_{n+1}^Q(X;\mathbb{Z})$. Then there is a colored $n$-knot diagram, $D_n^\eta$, that represents $\eta$.

(2b) Let $n = 1, 2, 3$. Let $\eta \in Z_{n+1}^Q(X;\mathbb{Z})$. Then there is a shadow colored $(n-1)$-knot diagram (possibly with exceptional points for $n = 2, 3$), $SD_{n-1}^\eta$, that represents $\eta$.

Proof. Statements (1a) and (1b) (for $n = 2, 3$), follow from Remark 5.3. For $n = 1$ statement (1b) is also easy: A shadow colored 0-knot diagram represents a rack 2-cycle which is also a quandle 2-cycle.

Consider statement (1b) for $n = 2$, the endpoints of shadow arc diagrams represent chains of the form $(a, a)$ which are trivial in quandle homology. Thus shadow colored arc diagrams represent quandle 2-cycles.

Branch points of shadow colored abstract 2-knot diagrams represent chains of the form $(a, b, b)$ while hem 2-crossing points represent chains of the form $(a, a, b)$. Both such chains are trivial in quandle homology. This proves (1b) for $n = 3$.

Consider a quandle 2-cycle $\eta = \sum_j \epsilon_j \vec{x}_j \in Z_{3}^3(X;\mathbb{Z})$ where $\vec{x}_j = (x_1^j, x_2^j)$ and $\epsilon_j = \pm 1$. For each $(x_1^j, x_2^j)$, construct a colored 2-crossing 1-diagram as in Lemma 5.1 (1); the sign of the crossing is $\epsilon_j$. The boundary of such a chain is $\partial(x_1^j, x_2^j) = x_1^j - x_2^j \ast x_2^j$. Since $\eta$ is a cycle the sum of these boundary terms adds to 0. Thus we can interconnect the under-arcs to form a collection of simple closed curves of under-arcs. The over-arcs each can be joined end-to-end locally to form the abstract diagram as depicted in Fig. 1 right top. For example, Fig. 2 illustrates an abstract diagram that represents the cycle $(\alpha, \beta) + (\beta, \gamma) - (\beta, \alpha) \in Z^3(R_3)$. By Remark 14, a desired shadow colored 0-knot diagram for (2b) is obtained from the above constructed representative 1-knot diagram. Thus statements (2a) and (2b) hold for $n = 1$. Note that the way canceling pairs are joined together is not unique.

Now suppose that $\eta = \sum_j \epsilon_j \vec{x}_j \in Z_{3}^3(X;\mathbb{Z})$ (where $\vec{x}_j = (x_0^j, x_1^j, x_2^j)$ and $\epsilon_j = \pm 1$) is a 3-cycle. We represent each $\epsilon_j \vec{x}_j$ by a shadow colored 2-crossing 1-diagram as in Lemma 5.1 (2); the sign of the crossing is $\epsilon_j$. The boundary of such a chain is the sum of four 2-chains. Each 2-chain is represented as a shadow colored 0-knot diagram. Take the cartesian product of such a diagram with the unit interval to form a shadow colored 1-crossing 1-knot diagram. Such a diagram is a 1-handle the attaching region of which is identified with the segments on the boundary of the squares that represent the chains $\vec{x}_j$. Thus we can attach 1-handles to these squares. Since $\eta$ is a quandle cycle, the boundary of squares that are not attached to these 1-handles have the same color assigned to the arc and the region. Cap each of these boundary segments by a colored endpoint diagram. This constructs a colored abstract 1-knot diagram with endpoints representing $\eta$. To construct an abstract 2-knot diagram that represents $\eta$, we follow the procedure outlined in Remark 4.9. This gives statement (2a) and (2b) for $n = 2$.

If $\eta = \sum_j \epsilon_j \vec{x}_j$ (where $\vec{x}_j = (x_0^j, x_1^j, x_2^j, x_3^j)$ and $\epsilon_j = \pm 1$) is a 4-cycle, then we construct a shadow colored abstract 2-knot diagram that may have hems following a procedure
analogous to the ones above. Specifically, for each chain $\vec{x}_j$ we construct a shadow colored triple point diagram where the colors on the regions away from which normals point is $x_i^j$. The boundary of the chain $\vec{x}_j$ corresponds to the shadow colored 2-crossing diagrams on the boundary of the colored cubes. We attach shadow colored 1-handles between pairs of square faces to cancel like terms. Since $\eta$ is a quandle cycle, its boundary consists of terms of the form $(a, a, b)$ and $(a, b, b)$. There are faces that remain representing such terms. In case the term is of the form $(a, b, b)$, we attach a branch point as in Fig. 19. In case the term is the form $(a, a, b)$, then we extend the over-sheet to create a hem diagram. The double point of the hem diagram represents this chain. This completes the proof of (2b) for $n = 3$.

5.6 Remark. We included condition (1) of Definition 3.8 for expository convenience. Otherwise boundary points might be confused with hems or endpoints. In the following theorem, we use abstract diagrams with actual boundaries — diagrams for which condition (2) is satisfied, but not necessarily condition (1). The notions of colored and shadow colored diagrams extend to diagrams with actual boundary in a straight-forward way.

In ordinary homology theory, boundaries of complexes correspond to boundary terms of chains. This is our motivation of extending the abstract diagrams to those with boundaries, as will be seen in the following theorem. The exceptional points that are defined above are not regarded as boundaries in quandle homology, as they were introduced to represent degenerate chains, instead of boundary terms.

For example, an abstract 1-knot diagram with actual boundary $[f : M \to N]$ has two types of endpoints. One is the endpoint diagram defined already. The actual boundary end point lies on the boundary $\partial N$. When a diagram is shadow colored by a quandle, the former (the endpoint diagram) corresponds to a degenerate chain, and the latter corresponds to the sum of the boundary terms in $\partial(x_0, x_1, x_2)$. See the right-side of Fig. 20, for example.

In case a shadow colored 2-knot diagram with actual boundary has a hem, the hem may have a boundary point in the form of an endpoint diagram, but the hem itself is not
regarded as the actual boundary.

5.7 Theorem. Let \( n = 1, 2 \). If the given cycle, \( \eta \in Z_{n+1}^Q(X; \mathbb{Z}) \), is a boundary, so \( \eta = \partial \nu \), for \( \nu \in C_{n+2}^Q(X; \mathbb{Z}) \), then \( \partial \nu \) (or \( S \partial \nu - 1 \)) is the boundary of a colored \((n+1)\)-knot diagram, \( \partial \nu \), (or shadow colored \( n \)-knot diagram, \( S \partial \nu \),) with actual boundary (and with hems when appropriate) that represents \( \nu \).

Proof. Suppose that the quandle \((n+1)\)-cycle \( \eta = \partial \nu \). In general, there is a colored abstract \((n+1)\)-knot diagram (or shadow colored abstract \( n \)-knot diagram) with actual boundary that represents \( \nu \). For example, take the disjoint union of colored \((n+2)\)-crossing \((n+1)\)-diagrams (or shadow colored \((n+1)\)-crossing \( n \)-diagrams) that represent the chains which constitute \( \nu \). Let \((S)D_{\nu}\) denote such a diagram, where the parenthesis represents that it is either a colored diagram (without \( S \)) or shadow colored diagram (with \( S \)). We also use the notation \((S)D_{\nu} = [g : M_{\nu} \rightarrow N_{\nu}]_{C_{\nu}}\) to specify the map \( g \). Similarly, let \((S)D_{\eta} = [f : M_{\eta} \rightarrow N_{\eta}]_{C_{\eta}}\) denote a given colored diagram that represents \( \eta \).

The sum of the (shadow) colored crossings of \( \partial (S)D_{\nu} \) may differ from the sum of the (shadow) colored crossings on \((S)D_{\eta}\) in that either may include degenerate chains or canceling terms. That is, there can be crossings that represent chains of the form \((a, a)\) in case \( n = 1 \), or chains of one of the forms \((a, a, b)\) or \((a, b, b)\) if \( n = 2 \). Or there may be a canceling pair of (shadow) colored crossings.

Take the product \((S)D_{\eta} \times [0, 1]\) which is \( g(M_{\eta}) \times [0, 1] \subset N_{\eta} \times [0, 1] \) with the given coloring \((\times[0,1])\), where the original \((S)D_{\eta}\) is regarded as embedded in \( N_{\eta} \times \{0\} \). For each pair of canceling (shadow) colored crossings, join them by a (shadow) colored 1-handle (with coloring induced from the crossings), and cancel them. Such 1-handles are attached on \( N_{\eta} \times \{1\} \), and give a new diagram \( [g' : M'_{\eta} \rightarrow N'_{\eta}] \) with a (shadow) color such that \( \partial N'_{\eta} \) consists two pieces, one of which \( \partial_1 N'_{\eta} \) contains the original \( g(M_{\eta}) \), and the other \( \partial_2 N'_{\eta} \) contains \( g(M_{\eta}) \) with all canceling pairs eliminated. Perform the same process on \( \partial N_{\nu} \) (we continue to use the same notation \( \partial \nu \) after this process), so that we now assume that \( \partial_2 N'_{\eta} \) and \( \partial N_{\nu} \) do not contain canceling pairs. Next we cancel degenerate chains, case by case.

Case 1. Suppose that \( \eta \) is a 2-cycle represented by a shadow colored 0-knot diagram, \( S \partial \eta = [f : M_{\eta} \rightarrow N_{\eta}]_{C_{\eta}} \). Then \( M_{\eta} \) is a 0-dimensional manifold, \( N_{\eta} \) is a 1-dimensional manifold. We call the components of \( M_{\eta} \) oriented vertices. A 2-chain \((a, b)\) is represented by a shadow colored 1-crossing 0-diagram where \( b \) is the color of the vertex and \( a \) is the color on the arc away from which the normal arrow to the vertex points. The description of \( \partial S \partial \nu \) is the same. For each degenerate vertex (shadow colored as \((a, a)\)), we construct a shadow colored endpoint diagram with color \( a \) on the 2-dimensional region and color \( a \) on the edge. The orientation of the arc is determined by the sign of the 0-crossing. Each such endpoint diagram is a 0-handle that is disjoint from \( N_{\eta} \times [0, 1] \) and \( N_{\nu} \). Now attach a 1-handle (in the guise of a shadow colored 1-crossing 1-diagram) between the degenerate crossing and the endpoint diagram. In this way, we may assume that there are no degenerate 2-chains among the 1-crossings representing the summands of \( \eta \) and \( \partial \nu \). See Fig. [2] for an example.

Case 2. Suppose that \( \eta \) is a 2-cycle that is represented by a colored 1-diagram. Each degenerate colored 2-crossing in \( D_{\eta} \) or \( \partial D_{\nu} \) is a crossing at which all the colors are the same;
Figure 20: Cobordism for shadow colored 0-diagrams

thus a degenerate crossing represents a chain for the form \((a, a)\). For each such crossing we construct a colored branch point diagram that functions as a 0-handle. The color on the surface is \(a\). This handle is attached to the diagram via a 1-handle in the guise of a colored 2-crossing 2-diagram. Therefore, as above we can assume that there are no degenerate 2-chains among the crossings representing the summands \(\eta\) and \(\partial \nu\). See Fig. [9].

Case 3. Suppose \(\eta\) is a 3-cycle that is represented by a shadow colored 1-knot diagram, \(SD_{\eta}\). The degenerate chains are of the form \((a, a, b)\) and \((a, b, b)\). The former are represented by colored crossings with color \(a\) on one face, color \(a\) on one of the lower arcs, and color \(b\) on the upper arc. The latter are represented by shadow colored crossings where \(a\) is the color on one face, and \(b\) is the color on all edges. For each degenerate chain of the form \((a, a, b)\), we insert a shadow colored hem 2-crossing diagram as a 0-handle disjoint from \(N'_{\eta}\) and \(N_{\nu}\). For each degenerate chain of the form \((a, b, b)\) we insert a shadow colored branch point diagram as a 0-handle. Then the 0-handles are attached to either \(\partial_{1} N'_{\eta}\) or \(\partial N_{\nu}\) via 1-handles that are 2-crossing 2-diagrams. The 1-handles are attached to the 0-handles at the crossing diagram on the boundary. Thus in this case, we may assume that there are no degenerate 3-chains among the crossings representing the summands of \(\eta\) and \(\partial \nu\).

Next, we consider the cases when the colored crossings on the resulting manifolds (that we again denote by \(\partial_{1} N'_{\eta}\) and \(\partial N_{\nu}\)) are in one-to-one correspondence. In each case, we can attach 1-handles between pairs of similarly (shadow) colored crossings. The 1-handles are of the form: (a) shadow colored \(n\)-crossing \(n\)-diagrams in case 1 and 3; (b) colored \((n+1)\)-crossing \((n+1)\)-diagrams in case 2.

After all of the crossings that represent summands of \(\eta\) have been glued to the chains representing \(\nu\), we attach (shadow) colored 2-handles in the guise of

(1) shadow colored 0-crossing 0-diagrams in case 1.
(2) 0-crossing 1-diagrams in case 2.
(3) shadow colored 1-crossing 2-diagrams in case 3.

This completes the proof. \(\square\)
6 Equivalence of Colored and Shadow Colored Diagrams

So far we have introduced abstract diagrams, colorings, and shadow colorings thereof. Here we discuss moves to such diagrams.

6.1 Theorem. For $i = 0, 1$, let $SD_i$ denote shadow colored abstract 0-knot diagrams where the color set is the quandle $X$. Suppose that $[SD_0] = [SD_1] \in H^Q_2(X; \mathbb{Z})$. Then $SD_0$ can be obtained from $SD_1$ by a finite sequence of moves taken from those depicted in Fig. 21.

Proof. Let $SD_i$ be represented by maps with colorings $[f_i : M_i \to N_i]_{C_i}$ as in the proof of the preceding theorem. By Theorem 5.3, $[f_0 : M_0 \to N_0]_{C_0}$ and $[f_1 : M_1 \to N_1]_{C_1}$ cobound a shadow colored abstract 1-knot diagram $M$ in a 2-manifold $N$ with actual boundary. Let $F : N \to [0, 1]$ be a smooth function such that $F^{-1}(i) = N_i$, for $i = 0, 1$. We may assume (after a small perturbation if necessary) that $F$ satisfies the following conditions:

1) $F$ is transverse at 0 and 1.
2) $F$ is generic on $M, N, \partial M, \partial N$, and on all the self intersections and singularities of $M$.

Thus $F$ has isolated Morse critical points on all the sets listed in (2), at distinct critical values. By taking the inverse images $F^{-1}(h - \epsilon)$ and $F^{-1}(h + \epsilon)$ at every critical value $h$, we obtain a sequence of moves. Hence we obtain the result by classifying these Morse critical points as follows.

They are classified into two categories: Reidemeister moves for 0-dimensional knot diagrams as singular sets and critical points of $M$ (Fig. 21 left 3 figures), and Morse critical points of $N$ (Fig. 21 right 2 figures). The left 3 figures correspond to the endpoints of $M$, maxima/minima of $M$, and transverse double points of $M$, respectively. The right 2 figures correspond to the maxima/minima of $N$, and saddle points of $N$, respectively. These exhaust generic critical points of $f$, and the theorem follows. □
Figure 22: Moves for shadow colored diagrams, Part I (Reidemeister moves)

| Boundary of surfaces | Empty diagram | a |
|----------------------|---------------|---|
|                      | a             | a |

| Interior of surfaces | Empty diagram | a |
|----------------------|---------------|---|
|                      | a             | a |

| Boundary of knots    | a             | a |
|----------------------|---------------|---|

| Interior of knots    | a             | a |
|----------------------|---------------|---|

Figure 23: Moves for shadow colored diagrams, Part II (Morse critical points)
6.2 Theorem. Let \( X \) be a quandle. For \( i = 0, 1 \), let \( \mathcal{SD}_i = \{ f_i : M_i \to N_i \}_{\mathcal{C}_i} \) denote shadow colored 1-knot diagrams that represent the same homology class in \( H_3^Q(X; \mathbb{Z}) \). Then \( [f_1 : M_1 \to N_1]_{\mathcal{C}_1} \) can be obtained from \( [f_0 : M_0 \to N_0]_{\mathcal{C}_0} \) by a finite sequence of moves taken from those depicted in Figs. 22 and 23.

Proof. By Theorem 5.5, \( [f_0 : M_0 \to N_0]_{\mathcal{C}_0} \) and \( [f : M_1 \to N_1]_{\mathcal{C}_1} \) cobound a shadow colored abstract 2-knot diagram \( M \) in a 3-manifold \( N \) with actual boundary. Let \( F : N \to [0,1] \) be a smooth function such that \( F^{-1}(i) = N_i \), for \( i = 0, 1 \). We may assume (after a small perturbation if necessary) that \( F \) satisfies the following conditions.

1. \( F \) is transverse at 0 and 1.
2. \( F \) is generic on \( M, N, \partial M, \partial N \), and on all the self intersections and singularities of \( M \).

Thus \( F \) has Morse isolated critical points on all the sets listed in (2), at distinct critical values. The proof proceeds as follows in a similar way as in the preceding theorem.

The singularities and critical points of \( M \) are listed in Fig. 23 and are Reidemeister moves. From top to bottom they represent the intersection of the boundary points and interior of \( M \), branch points, maxima/minima of double curves of \( M \), and triple points, respectively.

The Morse critical points as handle moves are listed in Fig. 23. From top to bottom, they are critical points of \( \partial N, \text{Int} N, \partial M \), and \( \text{Int} M \). The critical points of \( \partial N \) are maxima/minima (the left entry) or saddle points (the right). From the point of view of the boundary 1-manifold, they correspond to handles of indices 0/2 and 1, respectively. The critical points of \( \text{Int} N \) are similar, and depicted in the second row left and right. The critical points of \( \partial M \) are the creation or deletion of a pair of points. There are two types, left and right, of these 0/1-handles, in relation to interior points. The bottom entry in the figure illustrates the critical points of the interior of \( M \). The Theorem follows as these exhaust generic singularities and critical points.

6.3 Scholium. Colored abstract closed 1-knot diagrams represent the same 2-dimensional quandle homology class if and only if one is obtained from the other by a finite number of moves that are taken from those depicted in Figs. 23 and 24 where:

1. those moves that involve endpoints are excluded; and
2. colors on the 2-dimensional regions are ignored.

Proof. Imitate the proof of Theorem 6.2 replacing shadow colored with colored throughout. Since \( M_i \) are closed, we may assume that the cobounded surface \( M \) has no hems. Thus the moves involving endpoints are excluded.

6.4 Remark. For higher dimensions, the authors expect similar theorems. The classifications of moves for boundaries, however, become subtle. For one dimensional higher case, for example, the moves include Roseman moves (see for example [5]) with quandle colors, Morse critical points of the ambient space \( N \) and \( \partial N \), and singularities and critical points involving hems will provide the set of moves. The moves for hems are depicted in Fig. 27.
6.5 Discussion. Abstract 1-knot diagrams that have no endpoints correspond to virtual knot diagrams (see \cite{[15]}). In \cite{[15]}, a set of moves to abstract knot diagrams was given, and it was shown that up to these moves the set of abstract diagrams is equivalent to virtual knots up to virtual Reidemeister moves. Also, the correspondence was made by thickening the virtual knots and making them abstract knots. However, the moves to abstract knots are not to be confused with virtual Reidemeister moves. Virtual Reidemeister moves for thickened virtual knots are not equivalence moves for abstract knots thus obtained. For example, type II moves to thickened virtual knots do not necessarily preserve shadow coloring. We defined moves to colored and shadow colored diagrams to avoid diagrams that are type II equivalent but that have different quandles. Therefore, quandle homology provides an invariant of colored diagrams under (shadow) colored Reidemeister moves listed above.

7 The Quandle of an Abstract Knot and Examples

The (fundamental) quandle of an abstract \(n\)-knot diagram is defined in \cite{[15]}. It is generated by the \(n\)-regions of the diagram; the relations in the quandle can be read from the 2-crossings. It generalizes the knot quandle of classical knots (\cite{[13, 18]}). See \cite{[6, 7]} for Wirtinger presentations of knot quandles defined from knot diagrams, which are similar to Wirtinger presentations of knot groups. In this case, arcs of knot diagrams represent generators, and crossings give relations of Wirtinger form. Let \(Q(K)\) represent such a quandle for a knot diagram \(K\).

7.1 Example. If \(K_0\) is the diagram illustrated in Fig. 4, then its quandle has presentation

\[ Q(K_0) = \langle x, y, z : x \ast y = z, y \ast z = x, y \ast x = z \rangle. \]

The quandle \(\langle x, y, z : x \ast y = z, y \ast z = x, y \ast x = z \rangle\) is isomorphic to the 3-element dihedral quandle \(R_3\). The abstract knot \(K_0\) represents the 2-cycle \((x, y) + (y, z) - (y, x) \in \mathbb{Z}_2^Q(Q(K_0))\).

7.2 Proposition. Every abstract closed 1-knot diagram \(K\) represents a cycle \([K] \in \mathbb{Z}_2^Q(Q(K))\).

The abstract knot \(K_0\) is a boundary since the dihedral quandle, \(R_3\), has trivial 2-dimensional homology.

7.3 Question. Under what circumstances is the cycle \([K]\) a boundary?

7.4 Discussion and Example. On the other hand, we may define a shadow quandle for an abstract diagram. We can show that the 3-dimensional homology class that is represented by a shadow coloring of Fig. 4 is non-trivial as follows. The 5-element quandle \(QS(5)\) (consisting of non-identity permutations in the symmetric group on 3 letters) colors the diagram non-trivially. Transpositions go on the arcs and 3-cycles go on the 2-dimensional regions. By a \textsc{Mathematica} calculation, we have determined that the 3-cycle represented thereby is not a boundary.
7.5 Example. The example depicted in Fig. 24 represents a 3-cycle over the quandle $QS(6)$. The homomorphism $p : QS(6) \to R_3$ defined in 2.2 induces a map on homology. The corresponding $R_3$ shadow colored diagram represents a trivial 3-cycle. We can show, by mean of a Mathematica calculation, that the $QS(6)$ colored cycle is non-trivial.

The same program show that the colored diagram on the left of Fig. 25 represents a generator of $H^Q_3(QS(6); \mathbb{Z}) = \mathbb{Z}_{24}$. The induced map $f_*$ is illustrated to be surjective.

7.6 Remark. Let $K$ be a knot diagram on a compact oriented surface $F$. Then the fundamental shadow quandle $SQ(K)$ is defined as follows. The generators correspond to over-arcs and connected components of $F \setminus$ universe of $K$. The relations are defined for each crossing as ordinary fundamental quandles, and at each arc dividing regions. Specifically, if $a$ and $b$ are generators corresponding to adjacent regions such that the normal points from the region colored $a$ to that colored $b$, and if the arc dividing these regions is colored by $c$, then we have the relation $b = a * c$. This defines a presentation of a quandle, which is called the fundamental shadow quandle of $K$. Two diagrams on $F$ that differ by Reidemeister moves on $F$ have isomorphic fundamental shadow quandles. The shadow colors are regarded as quandle homomorphisms from the fundamental shadow quandle to a quandle $X$. 
8 Boundary Homomorphisms $H^Q_{n+1} \rightarrow H^D_n$

In [4], from a split short exact sequence

$$0 \rightarrow C^D_n(X) \xrightarrow{i} C^R_n(X) \xrightarrow{j} C^Q_n(X) \rightarrow 0,$$

the following homology long exact sequence

$$\cdots \xrightarrow{\partial_i} H^D_n(X;G) \xrightarrow{i_*} H^R_n(X;G) \xrightarrow{j_*} H^Q_n(X;G) \xrightarrow{\partial_i} H^D_{n-1}(X;G) \rightarrow \cdots$$

was constructed. We give an application of Theorem 5.5 to boundary homomorphisms of this exact sequence.

The endpoints of shadow colored arc diagrams are oriented by the orientation of the arcs. Each such endpoint represents a rack 2-chain $\pm (a, a)$ where $a$ is the color on the surrounding region. The sum of these represents the image of the boundary map $\partial_* : H^Q_3(X) \rightarrow H^D_2(X)$ in the long exact homology sequence.

Similarly, each oriented shadow colored branch point represents a rack 3-chain $\pm (a, b, b)$ where $a$ is the color in one of the surrounding regions and $b$ is the color on the local surface. An oriented shadow colored hem 2-crossing diagrams represent 3-chains of the form $(a, a, b)$ where $a$ is the color on one side of the upper sheet, and $b$ is the color on the upper sheet. The image of the boundary map $\partial_* : H^Q_4(X) \rightarrow H^D_4(X)$ is represented as the sum over all branch points and hem 2-crossings of the chains these represent.

In Theorems 8.1 and 8.2, we will use these descriptions of the homomorphisms $\partial_*$ and geometric techniques to show that these boundary maps are trivial.

8.1 Theorem [4]. Let $X$ be a quandle. The boundary homomorphism $\partial_* : H^Q_3(X) \rightarrow H^D_2(X)$ in the long exact sequence of quandle homology is trivial.
Figure 26: The boundary homomorphism is trivial

Proof. We give two proofs.

(1) Let $\eta = \sum_{i=1}^{k} \epsilon_i(x_i, y_i, z_i) \in \mathbb{Z}_3^Q(X)$. Then $\partial \eta = \sum_{j=1}^{h} \epsilon_j(w_j, w_j)$ for some $w_j \in X$ for $j = 1, \ldots, h$, from the definition of quandle cycle groups. Other terms coming out from each $\partial(x_i, y_i, z_i)$ cancel out.

Let $\tau_i$, $i = 1, \ldots, k$, be the 3-crossing diagrams colored with the triple $(x_i, y_i, z_i)$ for each $i$. According to the cancelation of the terms $\partial(x_i, y_i, z_i)$, paste together $\tau_i$. There are boundary crossings (boundaries of $\tau_i$) that are colored by $(w_j, w_j)$. As the double curves form an immersed 1-manifold with boundary, these boundary crossings are paired as $\sum_{g=1}^{h/2}[(a_g, a_g) - (b_g, b_g)]$, where the negative sign for $b_g$ represents that the orientation of the double curve points into the corresponding crossing. As one traces the double curve from the crossing with color $(a_g, a_g)$ to that with $(b_g, b_g)$, the curve under-goes the middle or top sheets at some of $\tau_i$’s. The color changes accordingly, but we have that $a_g$ and $b_g$ belong to the same orbit (as $b_g = a_g * w_g$ for some word $w_g$ in $X$).

On the other hand, one computes that $\partial(a, a, b) = -(a, a) + (a * b, a * b)$, so that $(a_g, a_g) - (b_g, b_g) \in B_2^D(X)$, as desired.

(2) Represent the homology class by a cycle $\eta$ and choose a shadow colored abstract 1-knot diagram to represent $\eta$. Along each arc there are a collection of crossings. These represent the summands of $\eta$. We push the arc back along a parallel as indicated in Fig 26. When pushing backwards we make sure that the endpoint always passes under the crossings. Each 2-crossing that is introduced in the process is shadow colored of the form $(x, x, y)$. Thus the new crossings do not affect the quandle homology class represented by the diagram. The shadow colored endpoints can be pairwise canceled. Thus $\partial_*(\eta) = 0$. This completes the proof. □

8.2 Theorem. Let $X$ be a quandle. The boundary homomorphism $\partial_* : H_4^Q(X) \to H_3^D(X)$ in the long exact sequence of quandle homology is trivial.

Proof. Consider a homology class $[\eta] \in H_4^Q(X)$ and represent it by the cycle $\eta$. Construct
a shadow colored abstract 2-knot diagram $SD_\eta = [f : M \to N]$ with hems to represent $\eta$. We may assume (by attaching handles if necessary) that the ambient 3-manifold, $N$, is closed. Moreover, the boundary of the surface $M$ consist of simple closed curves formed by the hems in the diagram.

The image $\partial_s [\eta]$ is represented on $SD_\eta$ as the collection of shadow colored branch point diagrams (terms of the form $(a, b, b)$) and hemmed 2-crossing diagrams (terms of the form $(a, a, b)$). We first eliminate the terms of the form $(a, a, b)$.

Consider the collection, $B^1$ of hems. These form a closed 1-manifold in the 3-manifold, $N$. The 1-manifold $B^1$ is null homologous in $N$ since it is the boundary of the surface $M$ of $SD_\eta$. Therefore $B^1$ bounds an embedded Seifert surface in $N$. The Seifert surface can be assumed to intersect the surface $M$ in general position. Now decompose the Seifert surface into 1- and 2-handles where the 1-handles are attached along a neighborhood of the hem. We will show that the hem can be eliminated by attaching the Seifert surface to the diagram $SD_\eta$ along the hem. In the process, triple points will be introduced, but these will not affect the quandle homology class represented.

The core disk of a 1-handle of the Seifert surface intersects $M$ at a finite number of points. We push a segment of the hem along this core disk using the move depicted in (A) of Fig. 27 until segments are in a small ball neighborhood that contains no further sheet of the surface. The situation is depicted in (C) left in Fig. 27, and connect the segments as indicated in (C). Having pushed the $B^1$ and joined up segments, we may assume that each component of the hem is unknotted and the collection is unlinked. Then an unknotting disk for a component of $B^1$ intersects the double point set of $SD_\eta$ in a finite number of points. We push the hem across these points introducing triple points, as indicated in Fig. 27 (D),
that represent chains of the form \((a, a, b, c)\). (Recall that the hemmed sheet is always the under-sheet at each hemmed 2-crossing diagram).

But \([\eta \pm (a, a, b, c)] = [\eta] \in H^4_{1,2}(X)\). The number of intersections with the resulting hem and \(M\) is necessarily even since the hem is null homologous. We can push the components further, using the move depicted in Fig. 27 (B), until each bounds a disk that does not intersect the surface \(M\). Now attach these disks to \(M\) to obtain a shadow colored diagram of a closed surface that represents \(\eta\). In this way we may assume that \(\partial_*[\eta]\) consists entirely of terms of the form \((a, b, b)\).

The terms of the form \((a, b, b)\) are represented on the diagram by branch points. There are an even number of these since each arc of double points has two ends. Using the same technique as in [6], we can cancel these in pairs. The process of moving the surface involves quandle colored Roseman moves. The cancelation may introduce shadow colored triple points, but a consecutive pair of two of the four colors adjacent to the triple points introduced agree. Thus the triple points do not affect the quandle homology class that the diagrams represent.

In this way \(\eta\) is represented by a shadow colored diagram of a closed surface without branch points. Thus \(\partial_*[\eta] = 0\). \(\square\)

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