An explicit approximation for super-linear stochastic functional differential equations

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Abstract

Since it is difficult to implement implicit schemes on the infinite-dimensional space, we aim to develop the explicit numerical method for approximating super-linear stochastic functional differential equations (SFDEs). Precisely, borrowing the truncation idea and linear interpolation we propose an explicit truncated Euler-Maruyama (EM) scheme for SFDEs, and obtain the boundedness and convergence in $L^p$ ($p \geq 2$). We also yield the convergence rate with $1/2$ order. Different from some previous works [21, 40], we release the global Lipschitz restriction on the diffusion coefficient. Furthermore, we reveal that numerical solutions preserve the underlying exponential stability. Moreover, we give several examples to support our theory.

Keywords. Stochastic functional differential equation; Truncated Euler-Maruyama scheme; Boundedness; Strong convergence; Exponential stability.

1 Introduction

In recent years, stochastic functional differential equations (SFDEs) have been used in a wide spectrum of application, including economics, mechanics, biological sciences, and control theory among others; see [1, 14, 15, 25, 26, 27, 30, 33]. The theory of SFDEs has been investigated extensively; see [7, 10, 20, 22, 25, 28, 31, 34]. Since it is almost impossible to get the explicit solutions of SFDEs, numerical solutions become the viable alternative. For stochastic delay differential equations (SDDEs), which form a special and important class of SFDEs, a good summary and detailed discussion on the numerical
In this paper we focus on the numerical method of nonlinear SFDE
\[
\begin{align*}
\mathrm{d}x(t) &= f(x_t)\mathrm{d}t + g(x_t)\mathrm{d}B(t), \quad t > 0, \\
x(t) &= \xi(t), \quad t \in [-\tau, 0],
\end{align*}
\] (1.1)

where $\tau > 0$ is a constant, $f : C([-\tau, 0]; \mathbb{R}^n) \rightarrow \mathbb{R}^n$ and $g : C([-\tau, 0]; \mathbb{R}^n) \rightarrow \mathbb{R}^{n\times d}$ are local Lipschitz continuous, $B(\cdot)$ is a $d$-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ satisfying the usual conditions (it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets), the initial data $\xi(\cdot) \in C([-\tau, 0]; \mathbb{R}^n)$, and $x_t = \{x_t(\theta) : \theta \in [-\tau, 0]\}$ is $C([-\tau, 0]; \mathbb{R}^n)$-valued stochastic process on $t \geq 0$, in which $x_t(\theta) = x(t + \theta)$. Our primary objectives are to construct an explicit numerical algorithm for nonlinear SFDE (1.1) and further establish the finite-time strong convergence and long-time stability theory.

Numerical algorithms are well investigated for SFDEs with globally Lipschitz continuous coefficients, such as Euler-Maruyama (EM) method [21], $\theta$-EM method [3], semi-implicit EM method [9], split-step backward EM method [13]. More often than not, SFDEs arising in practice are highly nonlinear. As a result, some implicit numerical procedures have been developed to combat the nonlinearity, such as the backward EM method [42], the split-step theta method for stochastic delay integro differential equations (SDIDEs) [41]. Compared with stochastic differential equations (SDEs), it is more difficult to implement implicit schemes for nonlinear SFDEs on the infinite-dimensional space. To tackle this difficulty, the backward EM scheme [42] is implicit only with respect to the current state $x(t)$, and [41] required that the integrand $h(\cdot, \cdot)$ in the integro term $\int_{t-\tau}^t h(s, x(s))\mathrm{d}s$ is globally Lipschitz. Although the strong convergence in [41, 42] have been studied, their convergence rate are still open. Owing to the simple algebraic structures and cheap computational cost, a truncated EM scheme with order less than $1/2$ was proposed [40] for nonlinear SFDEs with one-sided Lipschitz drift coefficient and global Lipschitz diffusion coefficient. It is necessary to point out that above methods are not applicable for a large class of SFDEs with the super-linear diffusion term, such as the stochastic volatility model

\[
\begin{align*}
\mathrm{d}x(t) &= (1 + 4x(t) - 4x^3(t))\mathrm{d}t + 2\int_{-1/2}^0 x^2(t + \theta)\mathrm{d}\theta\mathrm{d}B(t), \quad t > 0, \\
x(t) &= t - 1, \quad t \in [-1/2, 0].
\end{align*}
\] (1.2)

This SFDE may be regarded as the functional version of the stochastic volatility model introduced by Hutzenthaler and Jentzen [12, p.89, (4.63)] and its more general versions have been discussed in [6, 30]. Naturally, under less restrictive conditions, to construct an efficient explicit numerical scheme and to establish the strong convergence theory for nonlinear SFDE (1.1) become our top priority.

Moreover, the stability analysis of the numerical methods attracts more and more attentions. Although stability of numerical solutions for SDIDEs has been investigated widely [9, 13, 16, 19, 29, 41], less research has been done for SFDEs. To our best knowledge, there are only a few papers so far. For example, Wu, Mao and Kloeden [36] examined the almost surely exponential stability of the EM method for linear SFDEs. Zhou, Xie and Fang [43] utilized the backward EM scheme to approximate the almost surely exponential stability of nonlinear SFDEs, and Liu and Deng [18] went a further step to study
this using the complete backward EM scheme. As far as we know, it is still open how to
design an explicit scheme to approximate the stability of super-linear SFDEs. Therefore,
the other aim of this paper is to look for an explicit numerical method to capture the
exponential stability of super-linear SFDEs.

Borrowing the truncation idea from [17] and linear interpolation, we propose an ex-
plicit truncated EM scheme for nonlinear SFDEs, and obtain the strong convergence and
stability theory. Our main contributions are highlighted as follows:

• Under flexible Khasminskii-type conditions ((A1) and (A2)), the explicit truncated
EM scheme for super-linear SFDEs is proposed. It is proved that numerical solutions
are bounded and converge to the exact solutions in the $p$th moment ($p \geq 2$).

• The optimal rate of convergence, $1/2$ order, is also yielded under additional polyno-
mial growth coefficient conditions. Compared with the previous works [21, 40], the
global Lipschitz restriction on the diffusion coefficient is released.

• It is examined that the explicit numerical solutions preserve the long-time asymp-
totic properties of the exact ones, including the exponential stability in $L^p$ and with
probability 1 ($\mathbb{P} - 1$).

The rest of the paper is organized as follows. Section 2 introduces some notations and
preliminary results on the exact solutions of SFDE (1.1). Section 3 proposes an explicit
truncated EM scheme for SFDE (1.1) and studies the strong convergence and convergence
rate. Section 4 goes a further step to investigate the exponential stability of numerical
solutions in $L^p$ and $\mathbb{P} - 1$. Section 5 provides several examples to illustrate our results.

2 Preliminary results

Throughout this paper, the following notations are used. Let $\mathbb{N}$ denote the set of the
non-negative integers. Let $|\cdot|$ denote both the Euclidean norm in $\mathbb{R}^n$ and the trace norm
in $\mathbb{R}^{n \times d}$. Let $\lfloor a \rfloor$ denote the integer part of the real number $a$. Let $a \vee b = \max\{a, b\}$
and $a \wedge b = \min\{a, b\}$ for real numbers $a$, $b$. Let $A^T$ denote the transpose of the vector
or matric $A$. Let $1_A(\cdot)$ denote the indicator function of the set $A$. Define $x/|x| = 0$
when $|x| = 0$. Denote by $\mathcal{C} := C([-\tau, \ 0]; \mathbb{R}^n)$ the space of all continuous functions $\phi(\cdot)$
from $[-\tau, \ 0]$ to $\mathbb{R}^n$ equipped with the norm $\|\phi\| = \sup_{-\tau \leq \theta \leq 0} |\phi(\theta)|$. Let $\mathbb{R}_+ = [0, \infty)$
and denote by $C^2(\mathbb{R}_+; \mathbb{R}_+)$ the space of all continuously twice differentiable nonnegative
functions defined on $\mathbb{R}_+$. For any $V \in C^2(\mathbb{R}_+; \mathbb{R}_+)$, define an operator $\mathcal{L}V : \mathcal{C} \to \mathbb{R}$ by

$$\mathcal{L}V(\phi) = V_x(\phi(0))f(\phi) + \frac{1}{2} \text{trace}(g^T(\phi)V_{xx}(\phi(0))g(\phi)),$$

where

$$V_x(x) = \left( \frac{\partial V(x)}{\partial x_1}, \cdots, \frac{\partial V(x)}{\partial x_n} \right), \quad V_{xx}(x) = \left( \frac{\partial^2 V(x)}{\partial x_i \partial x_j} \right)_{n \times n}.$$

Denote by $\mathcal{W} := \mathcal{W}([-\tau, \ 0]; \mathbb{R}_+)$ the space of all bounded, Borel-measurable functions $\rho(\cdot)$
from $[-\tau, \ 0]$ to $\mathbb{R}_+$ such that $\int_{-\tau}^{0} \rho(\theta) d\theta = 1$. In addition, $L$ denotes a generic positive
constant, independent of $m$, and $\triangle$ (used below), which may take different values at
different appearances.
For convenience we impose the following hypotheses.

\((A1)\). For each \( \ell > 0 \), there exists a \( R_\ell > 0 \) such that

\[
|f(\phi) - f(\bar{\phi})| \vee |g(\phi) - g(\bar{\phi})| \leq R_\ell \left( |\phi(0) - \bar{\phi}(0)| + \frac{1}{\tau} \int_{-\tau}^{0} |\phi(\theta) - \bar{\phi}(\theta)| d\theta \right)
\]

for all \( \phi, \bar{\phi} \in \mathcal{C} \) with \( \|\phi\| \vee \|\bar{\phi}\| \leq \ell \).

\((A2)\). There exist positive constants \( p, \varrho, a_1, a_2, a_3 \) satisfying \( p \geq 2, a_2 > a_3 \), and a function \( \rho_1(\cdot) \in \mathcal{W} \) such that for any \( \phi \in \mathcal{C} \),

\[
2(\phi(0))^T f(\phi) + (p - 1)|g(\phi)|^2 \\
\leq a_1 \left( 1 + |\phi(0)|^2 + \frac{1}{\tau} \int_{-\tau}^{0} |\phi(\theta)|^2 d\theta \right) - a_2 |\phi(0)|^{2+\epsilon} + a_3 \int_{-\tau}^{0} |\phi(\theta)|^{2+\epsilon} \rho_1(\theta) d\theta. \tag{2.1}
\]

**Remark 2.1** If there exists a function \( \tilde{\rho}(\cdot) \in \mathcal{W} \) such that

\[
2(\phi(0))^T f(\phi) + (p - 1)|g(\phi)|^2 \\
\leq a_1 \left( 1 + |\phi(0)|^2 + \int_{-\tau}^{0} |\phi(\theta)|^2 \tilde{\rho}(\theta) d\theta \right) - a_2 |\phi(0)|^{2+\epsilon} + a_3 \int_{-\tau}^{0} |\phi(\theta)|^{2+\epsilon} \rho_1(\theta) d\theta,
\]

due to

\[
\int_{-\tau}^{0} |\phi(\theta)|^2 \tilde{\rho}(\theta) d\theta \leq L \int_{-\tau}^{0} |\phi(\theta)|^2 d\theta,
\]

(2.1) follows by choosing \( a_1 := a_1(\tau L \vee 1) \).

Now we prepare some results on the exact solution to end this section.

**Theorem 2.1** Assume that \((A1)\) and \((A2)\) hold. Then SFDE (1.1) with the initial data \( \xi \in \mathcal{C} \) has a unique global solution \( x(\cdot) \) on \([\tau, \infty)\) satisfying

\[
\sup_{-\tau \leq t < \infty} \mathbb{E}|x(t)|^p \leq L. \tag{2.2}
\]

For any constant \( m > \|\xi\| \) and \( T > 0 \),

\[
\mathbb{P}\{\delta_m \leq T\} \leq \frac{L_1}{m^p}, \tag{2.3}
\]

where

\[
\delta_m := \inf \{ t \geq 0 : |x(t)| \geq m \}, \tag{2.4}
\]

and the constant \( L_1 \) depends on \( T \) but is independent of \( m \).

**Proof.** Under \((A1)\) and \((A2)\), in view of [20, Theorem 2], SFDE (1.1) with the initial data \( \xi \in \mathcal{C} \) admits a unique global solution. Moreover, for any \( \epsilon > 0 \) and \( \phi \in \mathcal{C} \), using
and the Young inequality yields
\[
2(\phi(0))^T f(\phi) + (p-1)|g(\phi)|^2 \\
\leq (a_1 + (a_1 \varepsilon^{-\frac{2}{T}}) (\varepsilon^{-\frac{2}{T}}) |\phi(0)|^2) + (a_1 \varepsilon^{-\frac{2}{T}} \phi(\theta)) + a_2 |\phi(0)|^{2+\varepsilon} \\
+ a_3 \int_{-\tau}^0 |\phi(\theta)|^{2+\varepsilon} \rho_1(\theta) d\theta \\
\leq (a_1 + 2a_1^{\frac{2+\varepsilon}{2}} \varepsilon^{-\frac{2}{T}}) - (a_2 - \varepsilon) |\phi(0)|^{2+\varepsilon} + a_3 \int_{-\tau}^0 |\phi(\theta)|^{2+\varepsilon} \rho_1(\theta) d\theta + \varepsilon \left(1 \int_{-\tau}^0 |\phi(\theta)|^2 d\theta \right)^{\frac{2+\varepsilon}{2}}.
\]

Letting \(\varepsilon = (a_2 - a_3)/4\) and applying the Hölder inequality leads to
\[
2(\phi(0))^T f(\phi) + (p-1)|g(\phi)|^2 \\
\leq L - (a_2 - a_3) |\phi(0)|^{2+\varepsilon} + a_3 \int_{-\tau}^0 |\phi(\theta)|^{2+\varepsilon} \rho_1(\theta) d\theta + \frac{a_2 - a_3}{4} \left(1 \int_{-\tau}^0 |\phi(\theta)|^2 d\theta \right)^{\frac{2+\varepsilon}{2}}.
\]

Define \(V(x) = |x|^p\). Applying the Young inequality and the Hölder inequality we derive
\[
\mathcal{L}V(\phi) \leq \frac{p}{2} |\phi(0)|^{p-2} (2(\phi(0))^T f(\phi) + (p-1)|g(\phi)|^2) \\
\leq L - \frac{a_2 - a_3}{2} - \frac{(p-2)}{p+\varrho}(a_3 + \frac{a_2 - a_3}{4}) |\phi(0)|^{p+\varepsilon} \\
+ \frac{p(2+\varrho)a_3}{2(p+\varrho)} \int_{-\tau}^0 |\phi(\theta)|^{p+\varrho} \rho_1(\theta) d\theta + \frac{p(2+\varrho)(a_2 - a_3)}{8(p+\varrho)} \left(1 \int_{-\tau}^0 |\phi(\theta)|^{p+\varrho} d\theta \right).
\]

Note that
\[
\frac{p(2+\varrho)a_3}{2(p+\varrho)} + \frac{p(2+\varrho)(a_2 - a_3)}{8(p+\varrho)} = \frac{p}{2} (a_3 - \frac{(p-2)a_3}{p+\varrho} + \frac{a_2 - a_3}{4}) - \frac{(p-2)(a_2 - a_3)}{4(p+\varrho)} \\
< \frac{p}{2} (a_2 - a_3) - \frac{(p-2)}{p+\varrho}(a_3 + \frac{a_2 - a_3}{4})
\]

Then by similar arguments in the proof of \([20]\) Theorem 3], the required assertion \((2.2)\) follows. Similarly, by virtue of the Dynkin formula and \((2.5)\) we derive that for any \(T > 0\),
\[
\mathbb{E}|x(T \wedge \delta_m)|^p = |\xi(0)|^p + \mathbb{E} \int_0^{T \wedge \delta_m} \mathcal{L}V(x_t) dt \leq \|\xi\|^p + LT \leq L_1,
\]
where the constant \(L_1\) depends on \(T\) but is independent of \(m\). Therefore,
\[
m^p \mathbb{P}\{\delta_m \leq T\} = \mathbb{E}(|x(\delta_m)|^p 1_{\{\delta_m \leq T\}}) \leq \mathbb{E}|x(T \wedge \delta_m)|^p \leq L_1,
\]
which implies the desired assertion \((2.3)\). \(\square\)

3 Numerical scheme and convergence in \(L^p\)

The purpose of this section is to construct an explicit scheme for SFDE \((1.1)\) and further to establish the strong convergence. Under \((A1)\), we choose a strictly increasing continuous
function $\Gamma(\cdot)$ from $[1, +\infty)$ to $\mathbb{R}_+$ such that $\lim_{l \to +\infty} \Gamma(l) = +\infty$ and

$$
\sup_{\phi \neq \bar{\phi}} \left( \frac{|f(\phi) - f(\bar{\phi})|}{(\Psi(\phi, \bar{\phi}))^{1/2}} \vee \frac{|g(\phi) - g(\bar{\phi})|}{\Psi(\phi, \bar{\phi})} \right) \leq \Gamma(l), \quad l \geq 1, \quad (3.1)
$$

where $\phi, \bar{\phi} \in \mathcal{C}$, and

$$
\Psi(\phi, \bar{\phi}) := |\phi(0) - \bar{\phi}(0)|^2 + \frac{1}{\tau} \int_{-\tau}^{0} |\phi(\theta) - \bar{\phi}(\theta)|^2 d\theta.
$$

Denote by $\Gamma^{-1}(\cdot)$ the inverse function of $\Gamma(\cdot)$ from $[\Gamma(1), +\infty)$ to $\mathbb{R}_+$. For a stepsize $\Delta \in (0, 1]$ which is a fraction of $\tau$, say $\Delta = \tau/N$ for some integer $N \geq \tau$, define a truncation mapping $\Lambda_{\Gamma}^{\Delta, \lambda} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$
\Lambda_{\Gamma}^{\Delta, \lambda}(x) = \left( |x| \wedge \Gamma^{-1}(K\Delta^{-\lambda}) \right) \frac{x}{|x|}, \quad (3.2)
$$

where $K := \Gamma(1) \vee |f(0)| \vee |g(0)|^2$, with a slight abuses of notation $0$ denotes the zero element in $\mathcal{C}$, namely the function $\phi(\theta) \equiv 0$ for $\theta \in [-\tau, 0]$, and $\lambda \in (0, 1/2)$. Let $t_k = k\Delta$ for any $k = -N, -N + 1, \ldots$, and define the truncated EM scheme by

$$
\begin{cases}
\dot{Y}^\Delta(t_k) = \xi(t_k), \quad k = -N, -N + 1, \ldots, 0, \\
Y^\Delta(t_k) = \Lambda_{\Gamma}^{\Delta, \lambda}(\dot{Y}^\Delta(t_k)), \quad k = -N, -N + 1, \ldots, \\
\dot{Y}^\Delta(t_{k+1}) = Y^\Delta(t_k) + f(\dot{Y}^\Delta_{tk})\Delta + g(\dot{Y}^\Delta_{tk})\Delta B_k, \quad k = 0, 1, \ldots,
\end{cases} \quad (3.3)
$$

where $\Delta B_k = B(t_{k+1}) - B(t_k)$, $\dot{Y}^\Delta_{tk} = \{\dot{Y}^\Delta_{tk}(\theta) : -\tau \leq \theta \leq 0\}$ is a $\mathcal{C}$-valued random variable defined by

$$
\dot{Y}^\Delta_{tk}(\theta) = Y^\Delta(t_{k+j}) + \frac{\theta - j\Delta}{\Delta} (Y^\Delta(t_{k+j+1}) - Y^\Delta(t_{k+j}))
$$

$$
= \frac{(j+1)\Delta - \theta}{\Delta} Y^\Delta(t_{k+j}) + \frac{\theta - j\Delta}{\Delta} Y^\Delta(t_{k+j+1}), \quad (3.4)
$$

for $j\Delta \leq \theta \leq (j+1)\Delta$, $j = -N, -N + 1, \ldots, -1$.

### 3.1 Uniform moment boundedness of numerical solutions

We begin with proving that the numerical solutions preserve the uniform boundedness of exact ones in $L^p$. Under (A2) we can fix a constant $\varepsilon_0 \in (0, 1)$ such that

$$
\frac{a_2 + a_3}{2} < a_2(1 - \varepsilon_0)^\frac{\xi}{2}. \quad (3.5)
$$

For any $\phi \in \mathcal{C}$, define

$$
\Phi(\phi) = 1 + (1 - \varepsilon_0)|\phi(0)|^2 + \frac{\varepsilon_0}{\tau} \int_{-\tau}^{0} |\phi(\theta)|^2 d\theta. \quad (3.6)
$$
By \((3.1)\)–\((3.4)\) we derive that for any \(k \in \mathbb{N}\),
\[
|f(\bar{Y}_{t_k}^\Delta)| \leq \Gamma(1 - (K \Delta^{-\lambda})) (|Y^\Delta(t_k)|^2 + \frac{1}{\tau} \int_{-\tau}^{0} |\dot{Y}_{t_k}^\Delta(\theta)|^2 d\theta)^{1/2} + |f(0)|
\]
\[
\leq 2K\Delta^{-\lambda} (1 + |Y^\Delta(t_k)|^2 + \frac{1}{\tau} \int_{-\tau}^{0} |\dot{Y}_{t_k}^\Delta(\theta)|^2 d\theta)^{1/2}
\]
\[
\leq 2K\Delta^{-\lambda}((1 - \varepsilon_0) \wedge \varepsilon_0)^{-1/2} (1 + (1 - \varepsilon_0)|Y^\Delta(t_k)|^2 + \frac{\varepsilon_0}{\tau} \int_{-\tau}^{0} |\dot{Y}_{t_k}^\Delta(\theta)|^2 d\theta)^{1/2},
\]
which implies
\[
(\Phi(\bar{Y}_{t_k}^\Delta))^{-1}|f(\bar{Y}_{t_k}^\Delta)|^2 \leq L\Delta^{-2\lambda}.
\] (3.7)

Similarly,
\[
(\Phi(\bar{Y}_{t_k}^\Delta))^{-1}|g(\bar{Y}_{t_k}^\Delta)|^2 \leq L\Delta^{-\lambda}.
\] (3.8)

Due to the definition of \(\Phi\), for the desired assertion \(\sup_{\Delta \in [0,1]} \sup_{k \geq -N} \mathbb{E}|Y^\Delta(t_k)|^p \leq L\), it is sufficient to prove
\[
\sup_{\Delta \in [0,1]} \sup_{k \geq -N} \mathbb{E}(\Phi(\bar{Y}_{t_k}^\Delta))^{\frac{p}{2}} \leq L.
\] (3.9)

In fact, it follows from \((3.3)\) that
\[
|\bar{Y}^\Delta(t_{k+1})|^2 = |Y^\Delta(t_k) + f(\bar{Y}_{t_k}^\Delta)\bar{\Delta} + g(\bar{Y}_{t_k}^\Delta)\Delta B_k|^2 = |Y^\Delta(t_k)|^2 + \varphi_k,
\] (3.10)

where
\[
\varphi_k := |f(\bar{Y}_{t_k}^\Delta)|^2 \Delta^2 + |g(\bar{Y}_{t_k}^\Delta)\Delta B_k|^2 + 2(Y^\Delta(t_k))^T f(\bar{Y}_{t_k}^\Delta)\Delta
\]
\[
+ 2(Y^\Delta(t_k))^T g(\bar{Y}_{t_k}^\Delta)\Delta B_k + f^T(\bar{Y}_{t_k}^\Delta) g(\bar{Y}_{t_k}^\Delta) \Delta B_k.
\]

By virtue of \((3.3)\) and \((3.4)\), we derive from the convex property of \(u(x) = x^2\),
\[
\frac{1}{\tau} \int_{-\tau}^{0} |\bar{Y}_{t_{k+1}}^\Delta(\theta)|^2 d\theta = \frac{1}{\tau} \int_{-\tau}^{0} |\bar{Y}_{t_k}^\Delta(\theta)|^2 d\theta + \frac{1}{\tau} \int_{-\tau}^{0} |\dot{Y}_{t_{k+1}}^\Delta(\theta)|^2 d\theta - \frac{1}{\tau} \int_{-\tau}^{0} |\dot{Y}_{t_k}^\Delta(\theta)|^2 d\theta
\]
\[
\leq \frac{1}{\tau} \int_{-\tau}^{0} |\bar{Y}_{t_k}^\Delta(\theta)|^2 d\theta + \frac{1}{\tau} \int_{-\tau}^{0} |-\bar{\Delta} Y^\Delta(t_k) + \Delta \bar{\Delta} Y^\Delta(t_{k+1})|^2 d\theta
\]
\[
\leq \frac{1}{\tau} \int_{-\tau}^{0} |\bar{Y}_{t_k}^\Delta(\theta)|^2 d\theta + \frac{\Delta^2}{2\tau} |Y^\Delta(t_k)|^2 + \frac{\Delta^2}{2\tau} |Y^\Delta(t_{k+1})|^2
\]
\[
\leq \frac{1}{\tau} \int_{-\tau}^{0} |\bar{Y}_{t_k}^\Delta(\theta)|^2 d\theta + \frac{\Delta^2}{2\tau} |Y^\Delta(t_k)|^2 + \frac{\Delta^2}{2\tau} |Y^\Delta(t_{k+1})|^2 + f(\bar{Y}_{t_k}^\Delta)\bar{\Delta} + g(\bar{Y}_{t_k}^\Delta)\Delta B_k
\]
\[
\leq \frac{1}{\tau} \int_{-\tau}^{0} |\bar{Y}_{t_k}^\Delta(\theta)|^2 d\theta + \psi_k,
\] (3.11)

where
\[
\psi_k := \frac{2\Delta}{\tau} |Y^\Delta(t_k)|^2 + \frac{3\Delta^3}{2\tau} |f(\bar{Y}_{t_k}^\Delta)|^2 + \frac{3\Delta^3}{2\tau} |g(\bar{Y}_{t_k}^\Delta)\Delta B_k|^2.
\]
Hence, inserting (3.10) and (3.11) into (3.6) yields

\[ \Phi(\bar{Y}_{t_{k+1}}) \leq 1 + (1 - \varepsilon_0)|\bar{Y}_{t_{k+1}}(\theta)|^2 + \varepsilon_0 \int_{-\tau}^{\tau} |\bar{Y}_{t_{k+1}}(\theta)|^2 d\theta \]

\[ \leq \Phi(\bar{Y}_{t_k}) + (1 - \varepsilon_0)\varphi_k + \varepsilon_0 \psi_k = \Phi(\bar{Y}_{t_k})(1 + \Theta_k), \quad (3.12) \]

where

\[ \Theta_k := (\Phi(\bar{Y}_{t_k}))^{-1}(1 - \varepsilon_0)\varphi_k + \varepsilon_0 \psi_k, \]

and it is easy to see that \( \Theta_k > -1 \) a.s. Making use of [38, Lemma 3.3] and (3.12) we derive

\[ \mathbb{E}\left((\Phi(\bar{Y}_{t_{k+1}}))^{\frac{\ell}{2}}|\mathcal{F}_{t_k}\right) \leq (\Phi(\bar{Y}_{t_k}))^{\frac{\ell}{2}} \left(1 + \frac{p}{2} \mathbb{E}(\Theta_k|\mathcal{F}_{t_k}) + \frac{p(p - 2)}{8} \mathbb{E}(\Theta_k^2|\mathcal{F}_{t_k}) \right) + \mathbb{E}(\Theta_k^2 P_{\ell}(\Theta_k)|\mathcal{F}_{t_k}), \quad (3.13) \]

where \( \ell \in \mathbb{N} \) satisfying \( 2\ell < p \leq 2(\ell + 1) \), and \( P_{\ell}(\cdot) \) is an \( \ell \)-th-order polynomial. In order to estimate \( \mathbb{E}(\Phi(\bar{Y}_{t_{k+1}}))^{\frac{\ell}{2}} \), we prepare several elementary inequalities on \( \Theta_k \).

**Lemma 3.1** Assume that (A1) holds. Then for any \( k \in \mathbb{N} \),

\[ \mathbb{E}(\Theta_k|\mathcal{F}_{t_k}) \leq (1 - \varepsilon_0)\Phi(\bar{Y}_{t_k}))^{-1}(2(Y_{t_{k+1}})^T f(\bar{Y}_{t_k}) + |g(\bar{Y}_{t_k})|^2) \Delta + L \Delta, \quad (3.14) \]

\[ \mathbb{E}(\Theta_k^2|\mathcal{F}_{t_k}) \leq 4(1 - \varepsilon_0)^2(\Phi(\bar{Y}_{t_k}))^{-2}|(Y_{t_{k+1}})^T g(\bar{Y}_{t_k})|^2 \Delta + L \Delta, \quad (3.15) \]

\[ \mathbb{E}(\Theta_k^2 P_{\ell}(\Theta_k)|\mathcal{F}_{t_k}) \leq L \Delta. \quad (3.16) \]

**Proof.** We use the similar technique as in the proof of [32, Theorem 3.1]. The fact that \( B_k \) is independent of \( \mathcal{F}_{t_k} \) leads to

\[ \mathbb{E}((A\Delta B_k)|\mathcal{F}_{t_k}) = 0, \quad \mathbb{E}(|A\Delta B_k|^2|\mathcal{F}_{t_k}) = |A|^2 \Delta, \quad A \in \mathbb{R}^{n \times d}. \]

This implies

\[ (\Phi(\bar{Y}_{t_k}))^{-1}\mathbb{E}(\varphi_k|\mathcal{F}_{t_k}) = (\Phi(\bar{Y}_{t_k}))^{-1} \left(2(Y_{t_{k+1}})^T f(\bar{Y}_{t_k}) + |g(\bar{Y}_{t_k})|^2\right) \Delta + (\Phi(\bar{Y}_{t_k}))^{-1}|f(\bar{Y}_{t_k})|^2 \Delta^2. \]

It follows from (3.7) and \( \lambda \in (0, 1/2) \) that

\[ (\Phi(\bar{Y}_{t_k}))^{-1}\mathbb{E}(\varphi_k|\mathcal{F}_{t_k}) = (\Phi(\bar{Y}_{t_k}))^{-1} \left(2(Y_{t_{k+1}})^T f(\bar{Y}_{t_k}) + |g(\bar{Y}_{t_k})|^2\right) \Delta + L \Delta. \quad (3.17) \]

One observes

\[ \mathbb{E}((A\Delta B_k)^{2i-1}|\mathcal{F}_{t_k}) = 0, \quad \mathbb{E}(|A\Delta B_k|^{2i}|\mathcal{F}_{t_k}) \leq L \Delta^i, \quad A \in \mathbb{R}^{1 \times d}, \quad i \in \mathbb{N} \text{ with } i \geq 2, \]

These together with (3.7) and (3.8) imply

\[ (\Phi(\bar{Y}_{t_k}))^{-2}\mathbb{E}(\varphi_k^2|\mathcal{F}_{t_k}) \leq 4(\Phi(\bar{Y}_{t_k}))^{-2}|(Y_{t_{k+1}})^T g(\bar{Y}_{t_k})|^2 \Delta + L \Delta, \quad (3.18) \]
For any positive integer $i$,
\[
(\Phi(\tilde{Y}_{k}^{\Delta}))^{-i} \mathbb{E}(\psi_{k}^{j} | \mathcal{F}_{k}) \leq L \Delta, \quad j \in \mathbb{N} \text{ with } j \geq 3.
\] (3.19)

Similarly, for any positive integer $i$ and $j$,
\[
(\Phi(\tilde{Y}_{k}^{\Delta}))^{-(i+j)} \mathbb{E}(|\varphi_{k}^{i} | \psi_{k}^{j} | \mathcal{F}_{k}) \leq L \Delta.
\] (3.20)

Recalling the definition of $\Theta_{k}$, we derive from (3.17) and (3.20) that
\[
\mathbb{E}(\Theta_{k}^{2} | \mathcal{F}_{k}) \leq (1 - \varepsilon_{0}) (\Phi(\tilde{Y}_{k}^{\Delta}))^{-1} (2(Y^{\Delta}(t_{k})^{T} f(\tilde{Y}_{k}^{\Delta}) + |g(\tilde{Y}_{k}^{\Delta})|^{2}) \Delta + L \Delta.
\]

Using (3.18), (3.20), (3.21), and the nonnegativity of $\psi_{k}$ yields
\[
\mathbb{E}(\Theta_{k}^{2} | \mathcal{F}_{k}) \leq 4(1 - \varepsilon_{0})^{2} (\Phi(\tilde{Y}_{k}^{\Delta}))^{-2} |(Y^{\Delta}(t_{k}))^{T} g(\tilde{Y}_{k}^{\Delta})|^{2} \Delta + L \Delta.
\]

By (3.19)–(3.21) we obtain that for any integer $j \geq 3$ and $c \in \mathbb{R},$
\[
c \mathbb{E}(\Theta_{k}^{2} | \mathcal{F}_{k}) \leq (\Phi(\tilde{Y}_{k}^{\Delta}))^{-j} \mathbb{E}\left( c \varphi_{k}^{j} + c \sum_{i=1}^{j} \frac{j!}{i!(j-i)!} |\varphi_{k}^{i} | \psi_{k}^{j-i} | \mathcal{F}_{k} \right) \leq L \Delta,
\]
which implies that (3.16) holds. Therefore the desired results follows. \qed

To prove the moment boundedness of the numerical solutions we give the following inequality.

**Lemma 3.2** Assume that (A2) holds. Then for any $\phi \in \mathcal{C}$ and $c > 0,$
\[
c (\Phi(\phi))^{\frac{p-2}{2}} (2(\phi(0))^{T} f(\phi) + (p - 1)|g(\phi)|^{2})
\]
\[
\leq L - \alpha_{1} |\phi(0)|^{p+\rho} + \frac{\alpha_{2}}{\tau} \int_{-\tau}^{0} |\phi(\theta)|^{p+\rho} d\theta + \alpha_{3} \int_{-\tau}^{0} |\phi(\theta)|^{p+\rho} \rho_{1}(\theta) d\theta,
\] (3.22)

where $\alpha_{i}, \ i = 1, 2, 3,$ are positive constants given by (3.26), and satisfy $\alpha_{2} + \alpha_{3} < \alpha_{1}.$

**Proof.** For any $\phi \in \mathcal{C},$ making use of (2.1) and (3.6) leads to
\[
(\Phi(\phi))^{\frac{p-2}{2}} (2(\phi(0))^{T} f(\phi) + (p - 1)|g(\phi)|^{2})
\]
\[
\leq a_{1} (\Phi(\phi))^{\frac{p-2}{2}} (1 + |\phi(0)|^{2} + \frac{1}{\tau} \int_{-\tau}^{0} |\phi(\theta)|^{2} d\theta) - a_{2} (\Phi(\phi))^{\frac{p-2}{2}} |\phi(0)|^{2+\rho}
\]
\[
+ a_{3} (\Phi(\phi))^{\frac{p-2}{2}} \int_{-\tau}^{0} |\phi(\theta)|^{2+\rho} \rho_{1}(\theta) d\theta
\]
\[
\leq a_{1} ((1 - \varepsilon_{0}) \wedge \varepsilon_{0})^{-1} (\Phi(\phi))^{\frac{p}{2}} - a_{2} (1 - \varepsilon_{0})^{\frac{p-2}{2}} |\phi(0)|^{p+\rho}
\]
\[
+ a_{3} (\Phi(\phi))^{\frac{p-2}{2}} \int_{-\tau}^{0} |\phi(\theta)|^{2+\rho} \rho_{1}(\theta) d\theta.
\]
Applying the Young inequality yields that for any $c > 0$, \[
c(\Phi(\phi))^\frac{q}{p} + \frac{(1 - \varepsilon_0)p}{2} (\Phi(\phi))^\frac{p-2}{p} (2(\phi(0))^T f(\phi) + (p-1)|g(\phi)|^2) \]
\[
\leq L(\Phi(\phi))^\frac{q}{p} - \frac{p a_2 (1 - \varepsilon_0)^{\frac{p}{2}}}{2} |\phi(0)|^{p+\varrho} + \frac{p a_3 (p - 2)}{2(p + \varrho)} (\Phi(\phi))^\frac{p+\varrho}{2} \]
\[
+ \frac{p a_3 (2 + \varrho)}{2(p + \varrho)} \left( \int_{-\tau}^{0} |\phi(\theta)|^{2+\varrho} \rho_1(\theta) d\theta \right)^\frac{p+\varrho}{2+\varrho} \]
\[
\leq L - \frac{p a_2 (1 - \varepsilon_0)^{\frac{p}{2}}}{2} |\phi(0)|^{p+\varrho} + \frac{p a_3 (p - 2)}{2(p + \varrho)} \left( \int_{-\tau}^{0} |\phi(\theta)|^{2+\varrho} \rho_1(\theta) d\theta \right)^\frac{p+\varrho}{2+\varrho}. \tag{3.23} \]

By virtue of (3.5), choose a $\kappa \in (0, 1)$ sufficiently small such that \[
\frac{(a_2 + a_3)(1 + \kappa)}{2} < a_2 (1 - \varepsilon_0)^{\frac{p}{2}}. \tag{3.24} \]

According to [22, p.211, Lemma 4.1] we know that for any $a, b \in \mathbb{R}$, and $q \geq 1$, there is a constant $\tilde{L} = \bar{L}(\kappa) > 1$ such that \[
|a + b|^q \leq \tilde{L}|a|^q + (1 + \kappa)|b|^q. \]

The above inequality together with the convex property of $u(x) = x^\frac{p+\varrho}{2}$ implies \[
(\Phi(\phi))^\frac{p+\varrho}{2} \leq L + (1 + \kappa)((1 - \varepsilon_0)|\phi(0)|^2 + \frac{\varepsilon_0}{\tau} \int_{-\tau}^{0} |\phi(\theta)|^2 d\theta)^\frac{p+\varrho}{2} \]
\[
\leq \tilde{L} + (1 + \kappa) \left( (1 - \varepsilon_0)|\phi(0)|^{p+\varrho} + \varepsilon_0 \left( \frac{1}{\tau} \int_{-\tau}^{0} |\phi(\theta)|^2 d\theta \right)^\frac{p+\varrho}{2} \right). \tag{3.25} \]

Inserting (3.25) into (3.23) and using the Hölder inequality we arrive at \[
c(\Phi(\phi))^\frac{q}{p} + \frac{(1 - \varepsilon_0)p}{2} (\Phi(\phi))^\frac{p-2}{p} (2(\phi(0))^T f(\phi) + (p-1)|g(\phi)|^2) \]
\[
\leq L - \frac{p}{2} a_2 (1 - \varepsilon_0)^{\frac{p}{2}} + (1 + \kappa)(1 - \varepsilon_0) \left( \frac{a_3 (p - 2)}{p + \varrho} + \frac{a_2 - a_3}{2} \right) |\phi(0)|^{p+\varrho} \]
\[
+ \frac{p (1 + \kappa) \varepsilon_0}{2} \left( \frac{a_3 (p - 2)}{p + \varrho} + \frac{a_2 - a_3}{2} \right) \left( \frac{1}{\tau} \int_{-\tau}^{0} |\phi(\theta)|^2 d\theta \right)^\frac{p+\varrho}{2} \]
\[
+ \frac{p a_3 (2 + \varrho)}{2(p + \varrho)} \left( \int_{-\tau}^{0} |\phi(\theta)|^{2+\varrho} \rho_1(\theta) d\theta \right)^\frac{p+\varrho}{2+\varrho} \]
\[
\leq L - \alpha_1 |\phi(0)|^{p+\varrho} + \alpha_2 \int_{-\tau}^{0} |\phi(\theta)|^{p+\varrho} d\theta + \alpha_3 \int_{-\tau}^{0} |\phi(\theta)|^{p+\varrho} \rho_1(\theta) d\theta, \]

where \[
\alpha_1 := \frac{p}{2} a_2 (1 - \varepsilon_0)^{\frac{p}{2}} - (1 + \kappa)(1 - \varepsilon_0) \left( \frac{a_3 (p - 2)}{p + \varrho} + \frac{a_2 - a_3}{2} \right), \]
\[
\alpha_2 := \frac{p (1 + \kappa) \varepsilon_0}{2} \left( \frac{a_3 (p - 2)}{p + \varrho} + \frac{a_2 - a_3}{2} \right), \]
\[
\alpha_3 := \frac{a_3 p (2 + \varrho)}{2(p + \varrho)}. \tag{3.26} \]

It follows from (3.24) that $0 < \alpha_2 + \alpha_3 < \alpha_1$. The proof is therefore complete. \qed
Theorem 3.1 Assume that (A1) and (A2) hold. Then,

$$\sup_{\Delta \in (0,1]} \sup_{k \geq -N} \mathbb{E}|Y^\Delta(t_k)|^p \leq L. \quad (3.27)$$

**Proof.** For any $\Delta \in (0,1]$ and $i \in \mathbb{N}$, putting (3.14)–(3.16) into (3.13) leads to

$$\mathbb{E}\left( \left( \Phi(Y^\Delta_{t_{i+1}}) \right)^{\frac{p}{2}} | \mathcal{F}_{t_i} \right) \leq (1 + L\Delta)(\Phi(Y^\Delta_{t_{i}}))^{\frac{p}{2}} + \frac{(1 - \varepsilon_0)p}{2} (\Phi(Y^\Delta_{t_{i}})\left(2(Y^\Delta(t_i))^T f(Y^\Delta_{t_{i}}) + (p - 1)|g(Y^\Delta_{t_{i}})|^2 \right) \Delta
$$

$$\leq (1 + L\Delta)(\Phi(Y^\Delta_{t_{i}}))^{\frac{p}{2}} + \frac{(1 - \varepsilon_0)p}{2} (\Phi(Y^\Delta_{t_{i}})\left(2(Y^\Delta(t_i))^T f(Y^\Delta_{t_{i}}) + (p - 1)|g(Y^\Delta_{t_{i}})|^2 \right) \Delta. \quad (3.28)$$

Due to $\alpha_2 + \alpha_3 < \alpha_1$ in Lemma 3.2 we can choose a positive constant $\sigma$ such that

$$e^{\sigma\tau}(\alpha_2 + \alpha_3) \leq \alpha_1.$$

An application of Lagrange’s mean value theorem yields $e^{\sigma t_{i+1}} - e^{\sigma t_i} \leq e^{\sigma t_i+1}\sigma\Delta$. This, along with (3.28) implies that

$$e^{\sigma t_{i+1}} \mathbb{E}\left( \left( \Phi(Y^\Delta_{t_{i+1}}) \right)^{\frac{p}{2}} | \mathcal{F}_{t_i} \right) \leq e^{\sigma t_i}(\Phi(Y^\Delta_{t_{i}}))^{\frac{p}{2}} + e^{\sigma t_{i+1}}\Delta \left( L - \alpha_1|Y^\Delta(t_i)|^{p+\varepsilon} + \frac{\alpha_2}{\tau} \int_{-\tau}^{0} |Y^\Delta_t(\theta)|^{p+\varepsilon} d\theta + \alpha_3 \int_{-\tau}^{0} |Y^\Delta_t(\theta)|^{p+\varepsilon} \rho_1(\theta) d\theta \right) . \quad (3.29)$$

For any $k \in \mathbb{N}$, taking expectations in the above inequality and summing from $i = 0$ to $k$ derives

$$e^{\sigma t_{k+1}} \mathbb{E}\left( \left( \Phi(Y^\Delta_{t_{k+1}}) \right)^{\frac{p}{2}} \right) \leq (\Phi(Y^\Delta_0))^{\frac{p}{2}} + L\Delta \sum_{i=0}^{k} e^{\sigma t_{i+1}} + \Delta \sum_{i=0}^{k} e^{\sigma t_{i+1}} \mathbb{E}\left(- \alpha_1|Y^\Delta(t_i)|^{p+\varepsilon} + \frac{\alpha_2}{\tau} \int_{-\tau}^{0} |Y^\Delta_t(\theta)|^{p+\varepsilon} d\theta + \alpha_3 \int_{-\tau}^{0} |Y^\Delta_t(\theta)|^{p+\varepsilon} \rho_1(\theta) d\theta \right). \quad (3.29)$$
It follows from (3.4) and the convex property of \( u(x) = x^{p+\theta} \) that

\[
\int_{-\tau}^{0} |Y_{t_i}^{\Delta}(\theta)|^{p+\theta} d\theta = \frac{1}{\tau} \sum_{i=0}^{k} e^{\sigma_{t_{i+1}}} \int_{-\tau}^{0} |Y_{t_i}^{\Delta}(\theta)|^{p+\theta} d\theta \\
\leq e^{\sigma_{t_{i+1}}} \sum_{j=-N}^{j=-1} \sum_{i=0}^{k} e^{\sigma_{t_{i+1}}} |Y^\Delta(t_{i+j})|^{p+\theta} + e^{\sigma_{t_{i+1}}} |Y^\Delta(t_{i+j+1})|^{p+\theta} \\
\leq e^{\sigma_{t_{i+1}}} N \sup_{-N \leq i \leq -1} |\xi(t_i)|^{p+\theta} + e^{\sigma_{t_{i+1}}} \sum_{i=0}^{k} e^{\sigma_{t_{i+1}}} |Y^\Delta(t_i)|^{p+\theta}. \tag{3.30}
\]

Similarly,

\[
\sum_{i=0}^{k} e^{\sigma_{t_{i+1}}} \int_{-\tau}^{0} |Y_{t_i}^{\Delta}(\theta)|^{p+\theta} \rho_1(\theta) d\theta \\
\leq e^{\sigma_{t_{i+1}}} \sum_{j=-N}^{j=-1} \int_{j\Delta}^{(j+1)\Delta} \left( \frac{(j+1)\Delta - \theta}{\Delta} \right) \rho_1(\theta) d\theta \sum_{i=0}^{k} e^{\sigma_{t_{i+1}}} |Y^\Delta(t_{i+j})|^{p+\theta} \\
+ e^{\sigma_{t_{i+1}}} \sum_{j=-N}^{j=-1} \int_{j\Delta}^{(j+1)\Delta} \left( \frac{\theta - j\Delta}{\Delta} \right) \rho_1(\theta) d\theta \sum_{i=0}^{k} e^{\sigma_{t_{i+1}}} |Y^\Delta(t_{i+j+1})|^{p+\theta} \\
\leq e^{\sigma_{t_{i+1}}} \sum_{j=-N}^{j=-1} \int_{j\Delta}^{(j+1)\Delta} \left( \frac{(j+1)\Delta - \theta}{\Delta} \right) \rho_1(\theta) d\theta \sum_{v=-N}^{k} e^{\sigma_{t_{i+1}}} |Y^\Delta(t_v)|^{p+\theta} \\
+ e^{\sigma_{t_{i+1}}} \sum_{j=-N}^{j=-1} \int_{j\Delta}^{(j+1)\Delta} \left( \frac{\theta - j\Delta}{\Delta} \right) \rho_1(\theta) d\theta \sum_{v'=-N}^{k} e^{\sigma_{t_{i+1}}} |Y^\Delta(t_{v'})|^{p+\theta} \\
\leq e^{\sigma_{t_{i+1}}} N \sup_{-N \leq i \leq -1} |\xi(t_i)|^{p+\theta} + e^{\sigma_{t_{i+1}}} \sum_{i=0}^{k} e^{\sigma_{t_{i+1}}} |Y^\Delta(t_i)|^{p+\theta}. \tag{3.31}
\]
Inserting (3.30) and (3.31) into (3.29) we arrive at
\[
e^{\sigma t_{k+1}}E(\Phi(\bar{Y}_{t_{k+1}}))^{\frac{p}{p+\rho}} \\
\leq (\Phi(\bar{Y}_0))^{\frac{p}{p+\rho}} + L\Delta \sum_{i=0}^{k} e^{\sigma t_{i+1}} + (\alpha_2 + \alpha_3)e^{\sigma \tau} \frac{\sup_{-N \leq i \leq -1} |\xi(t_i)|^{p+\theta}}{\rho} \\
- (\alpha_1 - e^{\sigma \tau}(\alpha_2 + \alpha_3))\Delta \sum_{i=0}^{k} e^{\sigma t_{i+1}}E|Y^{\Delta}(t_i)|^{p+\theta} \\
\leq L + L\Delta \frac{e^{\sigma t_{k+2}} - e^{\sigma \Delta}}{e^{\sigma \Delta} - 1} \leq L + L\frac{e^{\sigma t_{k+2}}}{\sigma} ,
\]
which implies that (3.27) holds. The proof is therefore complete. □

3.2 Convergence in \( L^p \)

In this subsection we will investigate the strong convergence of the truncated EM scheme (3.3). For convenience, define an auxiliary process \( Z^{\Delta}(\cdot) \) by
\[
\begin{cases}
Z^{\Delta}(t) = Y^{\Delta}(t_k) + f(Y_{t_k}^{\Delta})(t - t_k) + g(Y_{t_k}^{\Delta})(B(t) - B(t_k)), \ t \in [t_k, t_{k+1}), k \in \mathbb{N}, \\
Z^{\Delta}(t) = \frac{t_{k+1} - t}{\Delta} Y^{\Delta}(t_k) + \frac{t - t_k}{\Delta} Y^{\Delta}(t_{k+1}), \ t \in [t_k, t_{k+1}], k = -N, \cdots , -1. 
\end{cases}
\]
(3.32)

Obviously,
\[
\lim_{t \uparrow t_k} Z^{\Delta}(t) = \bar{Y}^{\Delta}(t_k), \ k = 1, 2, \cdots , \\
Z^{\Delta}(t_k) = Y^{\Delta}(t_k), \ k = -N, -N + 1, \cdots .
\]

For any constant \( m > \| \xi \| \) and \( \Delta \in (0, 1] \), define
\[
\gamma_{m,\Delta} = \inf \{ t \geq 0 : |Z^{\Delta}(t)| > m \}.
\]
(3.33)

Choose a \( \Delta_1 \in (0, 1] \) sufficiently small such that
\[
\Gamma^{-1}(K\Delta_1^{-\lambda}) \geq m,
\]
which implies that for any \( \Delta \in (0, \Delta_1] \), \( Z^{\Delta}(\cdot) \) is continuous on \([-\tau, \gamma_{m,\Delta}]\). Let \( \eta_{m,\Delta} := [\gamma_{m,\Delta}/\Delta] \). For any \( t \in [t_k, t_{k+1}) \) with \( k \in \mathbb{N} \), one has
\[
[(t \wedge \gamma_{m,\Delta})/\Delta] = k \wedge \eta_{m,\Delta}.
\]

Define
\[
\bar{Y}_{t_k}^{\Delta} = \sum_{k=0}^{\infty} \bar{Y}_{t_k}^{\Delta} 1_{[t_k, t_{k+1})}(t), \ \forall \ t \geq 0.
\]
(3.34)

Hence it follows from (3.32)-(3.34) that for any \( \Delta \in (0, \Delta_1] \),
\[
Z^{\Delta}(t \wedge \gamma_{m,\Delta}) = Y^{\Delta}(t_{k \wedge \eta_{m,\Delta}}) + f(\bar{Y}_{t_{k \wedge \eta_{m,\Delta}}}^{\Delta})(t \wedge \gamma_{m,\Delta} - t_{k \wedge \eta_{m,\Delta}}) \\
+ g(\bar{Y}_{t_{k \wedge \eta_{m,\Delta}}}^{\Delta})(B(t \wedge \gamma_{m,\Delta}) - B(t_{k \wedge \eta_{m,\Delta}})) \\
= Y^{\Delta}(0) + \int_{0}^{t \wedge \gamma_{m,\Delta}} f(\bar{Y}_{s}^{\Delta})ds + \int_{0}^{t \wedge \gamma_{m,\Delta}} g(\bar{Y}_{s}^{\Delta})dB(s).
\]
(3.35)
Now we start with estimating the probability that $Z^\Delta(t)$ remains within the bounded set $\{x \in \mathbb{R}^n : |x| \leq m\}$ for $t \in [-\tau, T]$.

**Lemma 3.3** Assume that (A1) and (A2) hold. Then for any $\Delta \in (0, \Delta_1]$ and $T > 0$,

$$
\mathbb{P}\{\gamma_m, \Delta \leq T\} \leq \frac{L_2}{m^p},
$$

where the constant $L_2$ depends on $T$ but is independent of $m$ and $\Delta$.

**Proof.** We use the similar techniques in proofs of Lemma 3.1 and Theorem 3.1. Let $\Delta \in (0, \Delta_1]$ and $T > 0$. We simply write $\gamma = \gamma_{m, \Delta}, \eta = \eta_{m, \Delta}$. For any integer $i \in \mathbb{N}$, if $\eta \geq i + 1$, it follows from $(3.3)$ that

$$
Y^\Delta(t_{i+1}) = \hat{Y}^\Delta(t_{i+1}) = \hat{Y}^\Delta(t_i) + f(\hat{Y}^\Delta)\Delta + g(\hat{Y}_i^\Delta)\Delta B_i.
$$

If $\eta < i + 1$, it is obvious that $\eta \leq i$. One has

$$
Y^\Delta(t_{i+1}) = Y^\Delta(t_i) = Y^\Delta(t_i) = Y^\Delta(t_i) = Y^\Delta(t_i) = Y^\Delta(t_i).
$$

The above two cases implies that for any $i \in \mathbb{N}$,

$$
|Y^\Delta(t_{i+1})|^2 = |Y^\Delta(t_i)|^2 + (f(\hat{Y}^\Delta)\Delta + g(\hat{Y}_i^\Delta)\Delta B_i)1_{\{\eta \geq i + 1\}}
$$

where $\varphi_i$ is given by the equation below $(3.10)$. In a similar way as $(3.11)$ was shown, we obtain

$$
\frac{1}{\tau} \int_{-\tau}^0 |\dot{Y}^\Delta_{i+1}(\theta)|^2 d\theta = \frac{1}{\tau} \int_{-\tau}^0 |\dot{Y}^\Delta_{i+1}(\theta)|^2 d\theta + \frac{1}{\tau} \int_{-\tau}^0 |\dot{Y}^\Delta_{i+1}(\theta)|^2 d\theta 1_{\{\eta \geq i + 1\}}
$$

where $\psi_i$ is given by the equation below $(3.11)$. Those implies

$$
\Phi(Y^\Delta_{i+1}) \leq 1 + (1 - \varepsilon_0)\|\dot{Y}^\Delta(t_{i+1})\|^2 + \varepsilon_0 \int_{-\tau}^0 |\dot{Y}^\Delta_{i+1}(\theta)|^2 d\theta
$$

$$
\leq \Phi(Y^\Delta_{i+1}) + (1 - \varepsilon_0)\varphi_i + \varepsilon_0 \psi_i 1_{\{\eta \geq i + 1\}}
$$

where $\Theta_i$ is given by the equation below $(3.12)$. In a similar way as $(3.13)$ was shown, we can get

$$
\mathbb{E}\left(\left(\Phi(Y^\Delta_{i+1})\right)^2 \bigg| \mathcal{F}_{t_{i+1}}\right)
$$

$$
\leq \left(\Phi(Y^\Delta_{i+1})\right)^2 \left(1 + \frac{p}{2} \mathbb{E}\left(\Theta_i 1_{\{\eta \geq i + 1\}} \bigg| \mathcal{F}_{t_{i+1}}\right) + \frac{p(p - 2)}{8} \mathbb{E}\left(\Theta_i^2 1_{\{\eta \geq i + 1\}} \bigg| \mathcal{F}_{t_{i+1}}\right)
$$

$$
+ \mathbb{E}\left(\Theta_i^2 P_t(\Theta_i) 1_{\{\eta \geq i + 1\}} \bigg| \mathcal{F}_{t_{i+1}}\right)\right).
$$

(3.37)
Note that \[ \triangle B_i 1_{\{\eta \geq i+1\}} = B(t_{i+1 \wedge \eta}) - B(t_i \wedge \eta). \]

An application of the Doob martingale stopping time theorem \cite[p.11, Theorem 3.3]{22} yields

\[
\mathbb{E}(A \triangle B_i 1_{\{\eta \geq i+1\}} | \mathcal{F}_{t_i \wedge \eta}) = 0,
\]

\[
\mathbb{E}\left(|A \triangle B_i|^2 1_{\{\eta \geq i+1\}} | \mathcal{F}_{t_i \wedge \eta}\right) = |A|^2 \triangle 1_{\{\eta \geq i+1\}}, \quad \forall A \in \mathbb{R}^{n \times d}.
\] (3.38)

Similarly,

\[
\mathbb{E}\left((A \triangle B_i)^{2j-1} 1_{\{\eta \geq i+1\}} | \mathcal{F}_{t_i \wedge \eta}\right) = 0,
\]

\[
\mathbb{E}\left(|A \triangle B_i|^{2j} 1_{\{\eta \geq i+1\}} | \mathcal{F}_{t_i \wedge \eta}\right) \leq L \triangle^j 1_{\{\eta \geq i+1\}},
\] (3.39)

where \( A \in \mathbb{R}^{1 \times d}, j \in \mathbb{N} \) with \( j \geq 2 \). Using the similar technique as in the proof of Lemma 3.1 it follows from (3.7), (3.8), (3.38), and (3.39) that

\[
\mathbb{E}(\Theta_i 1_{\{\eta \geq i+1\}} | \mathcal{F}_{t_i \wedge \eta}) \leq \left((1 - \varepsilon_0) (\Phi(Y_{t_i}))^{-1} (2(\triangle Y(t_i))^T f(\hat{Y}_{t_i}^\Delta) + |g(\hat{Y}_{t_i}^\Delta)|^2) \Delta + L \Delta\right) 1_{\{\eta \geq i+1\}},
\] (3.40)

\[
\mathbb{E}(\Theta_i^2 1_{\{\eta \geq i+1\}} | \mathcal{F}_{t_i \wedge \eta}) \leq \left(4(1 - \varepsilon_0)^2 (\Phi(Y_{t_i}))^{-2} |(\triangle Y(t_i))^T g(\hat{Y}_{t_i}^\Delta)|^2 \Delta + L \Delta\right) 1_{\{\eta \geq i+1\}},
\] (3.41)

and

\[
\mathbb{E}(\Theta_i^3 P_k(\Theta_i) 1_{\{\eta \geq i+1\}} | \mathcal{F}_{t_i \wedge \eta}) \leq L \triangle 1_{\{\eta \geq i+1\}}.
\] (3.42)

Plugging (3.40), (3.42) back into (3.37) and using (3.22), we derive

\[
\mathbb{E}\left(\left(\Phi(Y_{t_{i+1} \wedge \eta})\right)^{\frac{p}{2}} | \mathcal{F}_{t_{i \wedge \eta}}\right) \leq (1 + L \Delta) (\Phi(Y_{t_{i \wedge \eta}}))^{\frac{p}{2}} + \frac{(1 - \varepsilon_0) p \Delta}{2} 1_{\{\eta \geq i+1\}} (\Phi(Y_{t_i}))^{\frac{p-2}{2}} (2(\triangle Y(t_i))^T f(\hat{Y}_{t_i}^\Delta)
\]

\[
+ (p-1) |g(\hat{Y}_{t_i}^\Delta)|^2)^2)
\]

\[
\leq (1 + L \Delta) (\Phi(Y_{t_{i \wedge \eta}}))^{\frac{p}{2}} + \Delta 1_{\{\eta \geq i+1\}} \left(- \alpha_1 (\triangle Y(t_i))^p \tau + \frac{\alpha_2}{\tau} \int_{-\tau}^{0} |\hat{Y}_{t_i}(\theta)|^{p+\varrho} d\theta
\]

\[
+ \alpha_3 \int_{-\tau}^{0} |\hat{Y}_{t_i}(\theta)|^{p+\varrho} \rho_1(\theta) d\theta\right).
\] (3.43)

Taking expectations and summing both sides of (3.43) from \( i = 0 \) to \( k \) we obtain

\[
\mathbb{E}\left(\Phi(Y_{t_{i+1} \wedge \eta})\right)^{\frac{p}{2}} \leq \left(\Phi(Y_{t_0})\right)^{\frac{p}{2}} + L \Delta \sum_{i=0}^{k} \mathbb{E}\left(\Phi(Y_{t_{i \wedge \eta}})\right)^{\frac{p}{2}}
\]

\[
\leq \alpha_1 (\triangle Y(t_i))^p \tau + \frac{\alpha_2}{\tau} \int_{-\tau}^{0} |\hat{Y}_{t_i}(\theta)|^{p+\varrho} d\theta
\]

\[
+ \alpha_3 \int_{-\tau}^{0} |\hat{Y}_{t_i}(\theta)|^{p+\varrho} \rho_1(\theta) d\theta\right).
\] (3.44)
One notices that \(1_{\{\eta > i+1\}} \leq 1_{\{\eta > i+j+1\}}\) for any \(j \in \{-N,-N+1, \ldots, 0\}\). In the similar ways as (3.30) and (3.31) were shown, we derive
\[
\frac{1}{\tau} \sum_{i=0}^{k} \int_{-\tau}^{0} |\bar{Y}_{t_i}^\Delta(\theta)|^{p+\rho} \, d\theta 1_{\{\eta \geq i+1\}}
\]
and
\[
sup_{-N \leq i \leq -1} |\xi(t_i)|^{p+\rho} + \sum_{i=0}^{k} |Y^\Delta(t_i)|^{p+\rho} 1_{\{\eta \geq i+1\}} \leq N \sup_{-N \leq i \leq -1} |\xi(t_i)|^{p+\rho} \sum_{i=0}^{k} |Y^\Delta(t_i)|^{p+\rho} 1_{\{\eta \geq i+1\}}.
\]
Inserting (3.45) and (3.46) into (3.44) and using \(\alpha_2 + \alpha_3 < \alpha_1\) leads to
\[
\mathbb{E}(\Phi(\bar{Y}_{t_{(k+1)\wedge \eta}}))^{\frac{p}{2}} \leq L + L\Delta \sum_{i=0}^{k} \mathbb{E}(\Phi(\bar{Y}_{t_{i\wedge \eta}}))^{\frac{p}{2}}.
\]
Applying the discrete Gronwall inequality [24, p.56, Theorem 2.5] yields that for any \(T > 0\),
\[
\sup_{\Delta \in (0,\Delta_1]} \sup_{0 \leq k \Delta \leq T} \mathbb{E}(\Phi(\bar{Y}_{t_{k\wedge \eta}}))^{\frac{p}{2}} \leq \bar{L},
\]
where the constant \(\bar{L}\) depends on \(T\) but is independent of \(m\) and \(\Delta\). Furthermore, for any \(t \in [0,T]\) there exists a \(k \in \mathbb{N}\) such that \(t \in [t_k, t_{k+1})\). Using the first equality of (3.35) and the Burkholer-Davis-Gundy inequality yields
\[
\mathbb{E}|Z^\Delta(t \wedge \gamma)|^p \leq 3^{p-1} \mathbb{E}|Y^\Delta(t_{k\wedge \eta})|^p + 3^{p-1} \mathbb{E}|f(\bar{Y}_{t_{k\wedge \eta}})|^p |\Delta|^p \left(3^p \Delta \left(\frac{p+1}{2(p-1)p-1}\right)^{\frac{p}{2}} \mathbb{E}|g(\bar{Y}_{t_{k\wedge \eta}})|^p\right)^{\frac{p}{2}}.
\]
Recalling the definition of \(\Phi\) and making use of (3.7), (3.8), and (3.47) leads to
\[
\mathbb{E}|Z^\Delta(t \wedge \gamma)|^p \leq L \mathbb{E}|Y^\Delta(t_{k\wedge \eta})|^p + L \mathbb{E}(\Phi(\bar{Y}_{t_{k\wedge \eta}}))^{\frac{p}{2}} |\Delta|^{p(1-\lambda)} + L \mathbb{E}(\Phi(\bar{Y}_{t_{k\wedge \eta}}))^{\frac{p}{2}} |\Delta|^{\lambda(p+1)\frac{p+1}{2}} \leq L \sup_{0 \leq k \Delta \leq T} \mathbb{E}(\Phi(\bar{Y}_{t_{k\wedge \eta}}))^{\frac{p}{2}} \leq L_2.
\]
which implies
\[ \mathbb{P}\{ \gamma \leq T \} \leq \frac{\mathbb{E}|Z^\triangle(T \wedge \gamma)|^p}{m^p} \leq \frac{L_p}{m^p}. \]
The proof is therefore complete. \( \square \)

Let
\[ \vartheta_{m, \triangle} := \delta_m \wedge \gamma_{m, \triangle}, \]
where \( \delta_m \) and \( \gamma_{m, \triangle} \) are given by (2.4) and (3.33), respectively. For any \( t \geq -\tau \), define
\[ Y^\triangle(t) = \sum_{k=-N}^{\infty} Y^\triangle(t_k) 1_{[t_k, t_{k+1})}(t). \tag{3.48} \]

**Lemma 3.4** Assume that (A1) and (A2) hold. Then for any \( m > \| \xi \|, \tilde{p} > 0, \) and \( T > 0, \)
\[ \lim_{\triangle \to 0^+} \mathbb{E} \left( |x(T) - Y^\triangle(T)|^{\tilde{p}} 1_{\{ \vartheta_{m, \triangle} > T \}} \right) = 0. \]

**Proof.** Define
\[ f_m(\phi) = f \left( (\| \phi \| \wedge m) \frac{\phi}{\| \phi \|} \right), \quad g_m(\phi) = g \left( (\| \phi \| \wedge m) \frac{\phi}{\| \phi \|} \right). \]
By virtue of (A1) we derive that for any \( \phi, \bar{\phi} \in \mathcal{C}, \)
\[ |f_m(\phi) - f_m(\bar{\phi})| \vee |g_m(\phi) - g_m(\bar{\phi})| \leq R_m \left( |\phi(0) - \bar{\phi}(0)| + \frac{1}{\tau} \int_{-\tau}^{0} |\phi(\theta) - \bar{\phi}(\theta)| d\theta \right), \]
which implies
\[ |f_m(\phi)| \vee |g_m(\phi)| \leq (2R_m \vee |f(0)| \vee |g(0)|)(1 + \| \phi \|). \tag{3.49} \]
Consider the following SFDE
\[ d\bar{x}(t) = f_m(\bar{x}_t) dt + g_m(\bar{x}_t) dB(t) \tag{3.50} \]
with the initial data \( \xi \in \mathcal{C} \). For each \( \triangle \in (0, \triangle_1] \), we make use of the EM scheme to approximate SFDE (3.50) and denote by \( \{y^\triangle(k\triangle)\}_{k \geq -N} \) the discrete EM solution (see [21 (2.3)]). Define the piecewise constant EM process \( \{y^\triangle(t)\}_{t \geq -\tau} \) by
\[ y^\triangle(t) = \sum_{k=-N}^{\infty} y^\triangle(k\triangle) 1_{[k\triangle, (k+1)\triangle)}(t), \]
and denote by \( \{z^\triangle(t)\}_{t \geq -\tau} \) the corresponding continuous EM solution (see [21 (2.7)]). Let \( T > 0 \). In a similar way as [21 Theorem 2.5] were shown, we derive that for any \( \tilde{p} > 0, \)
\[ \lim_{\triangle \to 0^+} \mathbb{E} \left( \sup_{0 \leq t \leq T} |\bar{x}(t) - z^\triangle(t)|^{\tilde{p}} \right) = 0. \tag{3.51} \]
For any $k \in \mathbb{N}$, denote by $\hat{y}_{k, \triangle}(\cdot)$ the $\mathcal{C}$-valued linear interpolation of $y^\triangle((k-N)\triangle), \cdots, y^\triangle(k\triangle)$ (see [21 (2.4)]). Applying [21 Lemma 3.2] yields

$$
\mathbb{E}\left(\sup_{0 \leq \triangle \leq T} \|\hat{y}_{k, \triangle}\|^\bar{p}\right) \leq \|\xi\|^\bar{p} + \mathbb{E}\left(\sup_{0 \leq t \leq T} |z^\triangle(t)|^\bar{p}\right) \leq \bar{L},
$$

(3.52)

where the positive constant $\bar{L}$ depends on $T$ but is independent of $\triangle$. It is straightforward to see that for any $t \in [t_k, t_{k+1})$ with $k \in \mathbb{N},$

$$
|z^\triangle(t) - y^\triangle(t)|^\bar{p} = |f(\hat{y}_{k, \triangle})(t - t_k) + g(\hat{y}_{k, \triangle})(B(t) - B(t_k))|^\bar{p} \\
\leq 2\bar{p} |f(\hat{y}_{k, \triangle})|^\bar{p}\triangle^\bar{p} + 2\bar{p}|g(\hat{y}_{k, \triangle})|^\bar{p}|B(t) - B(t_k)|^\bar{p}.
$$

According to (3.49) yields

$$
|z^\triangle(t) - y^\triangle(t)|^\bar{p} \leq L \left(1 + \|\hat{y}_{k, \triangle}\|^\bar{p}\right) (\Delta^\bar{p} + |B(t) - B(t_k)|^\bar{p}).
$$

For each $\bar{p} > 1$, applying the Hölder inequality, the Burkholder-Davis-Gundy inequality, and using (3.52), we get

$$
\lim_{\triangle \rightarrow 0^+} \mathbb{E}\left(\sup_{0 \leq t \leq T} |z^\triangle(t) - y^\triangle(t)|^\bar{p}\right) \\
\leq L \lim_{\triangle \rightarrow 0^+} \mathbb{E}\left(\sup_{0 \leq k, \triangle \leq T, k\triangle \leq t < (k+1)\triangle} \left(1 + \|\hat{y}_{k, \triangle}\|^\bar{p}\right) (\Delta^\bar{p} + |B(t) - B(t_k)|^\bar{p})\right) \\
\leq L \lim_{\triangle \rightarrow 0^+} \left(1 + \mathbb{E}\left(\sup_{0 \leq k, \triangle \leq T} \|\hat{y}_{k, \triangle}\|^2\bar{p}\right)\right)\frac{1}{2} \\
\times \left(\Delta^{2\bar{p}} + \mathbb{E}\left(\sup_{0 \leq k, \triangle \leq T, k\triangle \leq t < (k+1)\triangle} |B(t) - B(t_k)|^{2\bar{p}}\right)\right)^{\frac{1}{2}} \\
\leq L \lim_{\triangle \rightarrow 0^+} \left(\Delta^{2\bar{p}} + \left\lfloor \frac{T}{\triangle} \right\rfloor \Delta^{\bar{p}} (\left\lfloor \frac{T}{\triangle} \right\rfloor + 1)\right)^{\frac{1}{2}} = 0.
$$

This, along with the Lyapunov inequality implies that for any $\bar{p} > 0,$

$$
\lim_{\triangle \rightarrow 0^+} \mathbb{E}\left(\sup_{0 \leq t \leq T} |z^\triangle(t) - y^\triangle(t)|^\bar{p}\right) = 0.
$$

(3.53)

Utilizing (3.51) and (3.53) we have that for any $\bar{p} > 0,$

$$
\lim_{\triangle \rightarrow 0^+} \mathbb{E}\left(\sup_{0 \leq t \leq T} |\bar{x}(t) - y^\triangle(t)|^\bar{p}\right) = 0.
$$

(3.54)

According to $m > \|\xi\|$ yields

$$
x(t \wedge \vartheta_{m, \triangle}) = \bar{x}(t \wedge \vartheta_{m, \triangle}), \ \forall \ t \geq -\tau, \ \text{a.s.},
$$

(3.55)

and

$$
Y^\triangle(t \wedge \vartheta_{m, \triangle}) = y^\triangle(t \wedge \vartheta_{m, \triangle}), \ \forall \ t \geq -\tau, \ \triangle \in (0, \triangle_1], \ \text{a.s.}
$$

(3.56)
By virtue of (3.54)–(3.56) we derive that for any $\tilde{p} > 0$,
\[
\lim_{\Delta \to 0^+} \mathbb{E}\left( |x(T) - Y^\Delta(T)|^{\tilde{p}} \right) = \lim_{\Delta \to 0^+} \mathbb{E}\left( |\bar{x}(T \wedge \theta_{m,\Delta}) - y^\Delta(T \wedge \theta_{m,\Delta})|^\tilde{p} \mathbf{1}_{\{\theta_{m,\Delta} > T\}} \right)
\leq \lim_{\Delta \to 0^+} \mathbb{E}\left( |\bar{x}(T \wedge \theta_{m,\Delta}) - y^\Delta(T \wedge \theta_{m,\Delta})|^\tilde{p} \right) \leq \lim_{\Delta \to 0^+} \mathbb{E}\left( \sup_{0 \leq t \leq T} |\bar{x}(t) - y^\Delta(t)|^{\tilde{p}} \right) = 0.
\]

The proof is complete. \(\square\)

**Theorem 3.2** Assume that (A1) and (A2) hold. Then for any $T > 0$ and $q \in (0, p)$,
\[
\lim_{\Delta \to 0^+} \mathbb{E}|x(T) - Y^\Delta(T)|^q = 0.
\]

**Proof.** Let $m > \|\xi\|$, $\Delta \in (0, \triangle_1]$. For any $T > 0$, $q \in (0, p)$, and $\epsilon_1 > 0$, applying the Young inequality yields
\[
\mathbb{E}|x(T) - Y^\Delta(T)|^q = \mathbb{E}\left(|x(T) - Y^\Delta(T)|^q \mathbf{1}_{\{\theta_{m,\Delta} \leq T\}}\right) + \mathbb{E}\left(|x(T) - Y^\Delta(T)|^q \mathbf{1}_{\{\theta_{m,\Delta} > T\}}\right)
\leq \frac{q\epsilon_1}{p} \mathbb{E}|x(T) - Y^\Delta(T)|^p + \frac{p - q}{pe_1^{q/(p-q)}} \mathbb{P}\left(\theta_{m,\Delta} \leq T\right)
+ \mathbb{E}\left(|x(T) - Y^\Delta(T)|^q \mathbf{1}_{\{\theta_{m,\Delta} > T\}}\right). \tag{3.57}
\]

For any $\epsilon > 0$, in view of Theorem 2.1 and Theorem 3.1, choose an $\epsilon_1 > 0$ sufficiently small such that
\[
\frac{q\epsilon_1}{p} \mathbb{E}|x(T) - Y^\Delta(T)|^p \leq \frac{2^{p-1}q\epsilon_1}{p} \mathbb{E}|x(T)|^p + \sup_{\Delta \in [0,1]} |Y^\Delta(T)|^p < \frac{\epsilon}{3}. \tag{3.58}
\]

It follows from (2.3) and (3.36) that
\[
\frac{p - q}{pe_1^{q/(p-q)}} \mathbb{P}\left(\theta_{m,\Delta} \leq T\right) \leq \frac{p - q}{pe_1^{q/(p-q)}} \left( \mathbb{P}\left\{\delta_m \leq T\right\} + \mathbb{P}\left\{\gamma_{m,\Delta} \leq T\right\} \right) \leq \frac{(L_1 + L_2)(p - q)}{m^p pe_1^{q/(p-q)}}.
\]

Choose an $m$ sufficiently large such that
\[
\frac{(L_1 + L_2)(p - q)}{m^p pe_1^{q/(p-q)}} < \frac{\epsilon}{3},
\]

which implies
\[
\frac{p - q}{pe_1^{q/(p-q)}} \mathbb{P}\left(\theta_{m,\Delta} \leq T\right) < \frac{\epsilon}{3}. \tag{3.59}
\]

Inserting (3.58) and (3.59) into (3.57) and using Lemma 3.4 we derive
\[
\lim_{\Delta \to 0^+} \mathbb{E}|x(T) - Y^\Delta(T)|^q = 0.
\]

The proof is complete. \(\square\)
3.3 Convergence rate in $L^p$

To study the convergence rate of the truncated EM scheme, we require the following notations and assumptions. By $\mathcal{U} := \mathcal{U}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}_+)$, denote the space of all continuous functions $U(\cdot, \cdot)$ from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}_+$ such that for each $\ell > 0$, there exists a $\bar{R}_\ell > 0$ for which

$$U(x, \bar{x}) \leq \bar{R}_\ell |x - \bar{x}|, \quad \forall \ x, \ \bar{x} \in \mathbb{R}^n \text{ with } |x| \vee |\bar{x}| \leq \ell. \quad (3.60)$$

(A3). There are positive constants $\mu \geq 1/2$ and $a_4$ such that the initial data $\xi$ satisfies

$$|\xi(s_1) - \xi(s_2)| \leq a_4 |s_1 - s_2|^{\mu}, \quad \forall \ s_1, \ s_2 \in [\tau, \ 0].$$

(A4). There exist positive constants $\bar{q}, \bar{p}, a_5$ satisfying $2 \leq \bar{q} < \bar{p}$, and functions $U(\cdot) \in \mathcal{U}$, $\rho_2(\cdot) \in \mathcal{W}$ such that for any $\phi, \bar{\phi} \in \mathcal{C}$,

$$|\phi(0) - \bar{\phi}(0)|^{q-2} \left(2(\phi(0) - \bar{\phi}(0))^T (f(\phi) - f(\bar{\phi})) + (\bar{p} - 1)|g(\phi) - g(\bar{\phi})|^2\right)\notag$$

$$\leq a_5 (|\phi(0) - \bar{\phi}(0)|^q + \frac{1}{\tau} \int_{-\tau}^0 |\phi(\theta) - \bar{\phi}(\theta)|^q d\theta - U(\phi(0), \bar{\phi}(0)) + \int_{-\tau}^0 U(\phi(\theta), \bar{\phi}(\theta)) \rho_2(\theta) d\theta. \quad (3.61)$$

(A5). There is a pair of positive constants $a_6, r$ such that for any $\phi, \bar{\phi} \in \mathcal{C}$,

$$|f(\phi) - f(\bar{\phi})| \leq a_6 \left( |\phi(0) - \bar{\phi}(0)| + \frac{1}{\tau} \int_{-\tau}^0 |\phi(\theta) - \bar{\phi}(\theta)| d\theta \right) \notag$$

$$\times \left( 1 + |\phi(0)|^r + |\bar{\phi}(0)|^r + \frac{1}{\tau} \int_{-\tau}^0 |\phi(\theta)|^r d\theta + \frac{1}{\tau} \int_{-\tau}^0 |\bar{\phi}(\theta)|^r d\theta \right),$$

and

$$|g(\phi) - g(\bar{\phi})|^2 \leq a_6 \left( |\phi(0) - \bar{\phi}(0)|^2 + \frac{1}{\tau} \int_{-\tau}^0 |\phi(\theta) - \bar{\phi}(\theta)|^2 d\theta \right) \notag$$

$$\times \left( 1 + |\phi(0)|^r + |\bar{\phi}(0)|^r + \frac{1}{\tau} \int_{-\tau}^0 |\phi(\theta)|^r d\theta + \frac{1}{\tau} \int_{-\tau}^0 |\bar{\phi}(\theta)|^r d\theta \right). \quad (3.61)$$

Remark 3.1 Under (A2), (A4), and (A5), by virtue of (3.1) we may take the function $\Gamma(\cdot)$ by:

$$\Gamma(l) = \sqrt{2} a_6 (1 + 4l^r), \quad \forall \ l \geq 1.$$ 

Then,

$$\Gamma^{-1}(l) = \left(\frac{l}{4\sqrt{2} a_6} - \frac{1}{4}\right)^{\frac{1}{r}}, \quad l \geq 5\sqrt{2} a_6. \quad (3.62)$$

Define

$$\lambda = \frac{\bar{q} r}{2(p - \bar{q})}. \quad (3.63)$$

Obviously, if $\bar{q} < p/(r + 1)$, then $0 < \lambda < 1/2.$
We will estimate the error between the exact solution \( \{x(t)\}_{t \geq -\tau} \) and the auxiliary process \( \{Z^\triangle(t)\}_{t \geq -\tau} \), and further obtain the convergence rate of the numerical solution \( \{Y^\triangle(t)\}_{t \geq -\tau} \). For this purpose, we begin with proving that the moment of \( Z^\triangle(t) \) is bounded.

**Lemma 3.5** Assume that (A1) and (A2) hold. Then

\[
\sup_{\triangle \in (0,1]} \sup_{t \geq -\tau} \mathbb{E}|Z^\triangle(t)|^p \leq L.
\]

**Proof.** Let \( \triangle \in (0,1] \). For any \( t \in [t_k, t_{k+1}) \) with \( k \in \mathbb{N} \), making use of (3.32) yields

\[
\mathbb{E}|Z^\triangle(t)|^p \leq 3^{p-1}(\mathbb{E}|Y^\triangle(t_k)|^p + \mathbb{E}|f(Y^\triangle_{t_k})|^p \triangle^p + \mathbb{E}|g(Y^\triangle_{t_k})|^p \triangle^\frac{p}{2}).
\]

One notices from (3.7) and (3.8) that

\[
\mathbb{E}|Z^\triangle(t)|^p \leq L(1 + \triangle^{p(1-\lambda)} + \triangle^{p(1-\lambda)} \int_{-\tau}^{0} |Y^\triangle(t_k)|^p \lambda^\frac{1}{p} d\lambda)
\]

\[
\leq L(1 + \sup_{\triangle \in (0,1]} \sup_{k \geq -\tau} \mathbb{E}|Y^\triangle(t_k)|^p).
\]

Therefore, the desired assertion follows from (3.27). \( \square \)

For any \( t \geq 0 \), define

\[
Z^\triangle(t) = Z^\triangle(t + \theta), \quad \forall \theta \in [-\tau, 0].
\]

Choose a \( \bar{\triangle} \in (0,1] \) sufficiently small such that \( \Gamma^{-1}(K\bar{\triangle}^{-\lambda}) \geq \|\xi\| \).

**Lemma 3.6** Assume that (A2)--(A5) hold with \( p \geq (2 \vee r) \). Then for any \( \triangle \in (0, \bar{\triangle}] \),

\[
\sup_{t \geq 0} \sup_{\theta \in [-\tau,0]} \mathbb{E}|Z^\triangle_t(\theta) - Y^\triangle_t(\theta)|^{\frac{2p}{p+2}} \leq L\triangle^{\frac{p}{p+2}}.
\]  

(3.64)

**Proof.** Let \( \triangle \in (0, \bar{\triangle}] \). For any \( t \geq 0 \) there exists a \( k \in \mathbb{N} \) such that \( t \in [t_k, t_{k+1}) \). For any \( \theta \in [-\tau, 0) \), there exists a \( j \in \{-N, \cdots, -1\} \) such that \( \theta \in [t_j, t_{j+1}) \). Clearly,

\[
t + \theta \in [t_{k+j}, t_{k+j+2}), \quad t_k + \theta \in [t_{k+j}, t_{k+j+1}).
\]

According to the range of \( t + \theta \), we divide the proof into five cases.

**Case 1.** If \( t + \theta \in [t_{k+j}, t_{k+j+1}) \subset [-\tau, 0) \), by virtue of \( \triangle \in (0, \bar{\triangle}] \) we get

\[
Z^\triangle_t(\theta) - Y^\triangle_t(\theta) = Z^\triangle(t + \theta) - Y^\triangle(t_k + \theta)
\]

\[
= \frac{t_{k+j+1} - (t + \theta)}{\triangle} \xi(t_{k+j}) + \frac{(t + \theta) - t_{k+j+1}}{\triangle} \xi(t_{k+j+1})
\]

\[
- \frac{t_{k+j+1} - (t_k + \theta)}{\triangle} \xi(t_{k+j}) + \frac{(t_k + \theta) - t_{k+j+1}}{\triangle} \xi(t_{k+j+1})
\]

\[
= \frac{t - t_k}{\triangle} (\xi(t_{k+j+1}) - \xi(t_{k+j})).
\]

Making use of (A3), we derive

\[
\mathbb{E}|Z^\triangle_t(\theta) - Y^\triangle_t(\theta)|^{\frac{2p}{p+2}} \leq L\triangle^{\frac{p}{p+2}}.
\]
Case 2. If $t + \theta \in [t_{k+j+1}, t_{k+j+2}) \subset [-\tau, 0)$, we have

$$Z_t^\Delta(\theta) - \bar{Y}_t(\theta) = \frac{t_{k+j+2} - (t + \theta)}{\Delta} \xi(t_{k+j+1}) + \frac{(t + \theta) - t_{k+j+1}}{\Delta} \xi(t_{k+j+2})$$

$$- \frac{(t_{k+j+1} - (t + \theta))}{\Delta} \xi(t_{k+j}) + \frac{(t_k + \theta) - t_{k+j}}{\Delta} \xi(t_{k+j+1})$$

$$= \frac{(t + \theta) - t_{k+j+1}}{\Delta} \left( \xi(t_{k+j+2}) - \xi(t_{k+j+1}) \right)$$

$$+ \frac{t_{k+j+1} - (t_k + \theta)}{\Delta} \left( \xi(t_{k+j+1}) - \xi(t_{k+j}) \right).$$

It follows from (A3) that

$$\mathbb{E} \left| Z_t^\Delta(\theta) - \bar{Y}_t(\theta) \right|^{2p} \leq L \Delta^{\frac{p}{2}}.$$

Case 3. If $t + \theta \in [t_{k+j}, t_{k+j+1}) \subset [0, \infty)$, making use of (3.4), (3.32), and (3.34), we derive

$$|Z_t^\Delta(\theta) - \bar{Y}_t(\theta)| = |Z^\Delta(t + \theta) - \bar{Y}(t_k + \theta)|$$

$$= |Y^\Delta(t_{k+j}) + f(Y_{t_{k+j}}^\Delta)(t + \theta - t_{k+j}) + g(Y_{t_{k+j}}^\Delta)(B(t + \theta) - B(t_{k+j}))$$

$$- \left( \frac{j + 1}{\Delta} \theta - \frac{j}{\Delta} \theta Y^\Delta(t_{k+j}) + \frac{\theta - j \Delta}{\Delta} Y^\Delta(t_{k+j+1}) \right) |$$

$$\leq |f(Y_{t_{k+j}}^\Delta)(t + \theta - t_{k+j}) + g(Y_{t_{k+j}}^\Delta)(B(t + \theta) - B(t_{k+j}))|$$

$$+ \frac{\theta - j \Delta}{\Delta} |Y^\Delta(t_{k+j+1}) - Y^\Delta(t_{k+j})|.$$  \hspace{1cm} (3.65)

Using the techniques in the proof of \textit{[17 (7.21)]} and together with \textit{[3.3]} we get

$$|Y^\Delta(t_{k+j+1}) - Y^\Delta(t_{k+j})| \leq \left| \bar{Y}^\Delta(t_{k+j+1}) - Y^\Delta(t_{k+j}) \right|$$

$$\leq \left| f(Y_{t_{k+j}}^\Delta) \Delta + g(Y_{t_{k+j}}^\Delta) \Delta B_{k+j} \right|. \hspace{1cm} (3.66)$$

Substituting (3.66) into (3.65) leads to

$$\mathbb{E} \left| Z_t^\Delta(\theta) - \bar{Y}_t(\theta) \right|^{2p} \leq L \mathbb{E} \left( |f(Y_{t_{k+j}}^\Delta)|^{\frac{2p}{r+2}} \Delta^{\frac{2p}{r+2}} + |g(Y_{t_{k+j}}^\Delta)|^{\frac{2p}{r+2}} \Delta^{\frac{2p}{r+2}} \right). \hspace{1cm} (3.67)$$

It follows from (3.61) that

$$|g(Y_{t_{k+j}}^\Delta)|^{\frac{2p}{r+2}} \leq L |g(Y_{t_{k+j}}^\Delta) - g(0)|^{\frac{2p}{r+2}} + L |g(0)|^{\frac{2p}{r+2}}$$

$$\leq L \left( |Y^\Delta(t_{k+j})|^2 \right)^{\frac{2p}{r+2}} + \frac{1}{\tau} \int_{-\tau}^{0} |Y_{t_{k+j}}^\Delta(\theta)|^2 d\theta |y^\Delta(t_{k+j})|^r + \frac{1}{\tau} \int_{-\tau}^{0} |Y_{t_{k+j}}^\Delta(\theta)|^r d\theta |y^\Delta(t_{k+j})|^2 + L.$$
Using $p \geq (2 \lor r)$ and applying the Hölder inequality yields
\[
|g(\bar{Y}_{t_k+j})|^2 \leq L(1 + |Y_{t_k+j}|)^p + \frac{1}{\tau} \int_{-\tau}^{0} |\bar{Y}_{t_k+j}(\theta)|^p d\theta. \tag{3.68}
\]
Inserting (3.68) into (3.67), and then using (3.7) and $\lambda \in (0, 1/2)$, we have
\[
E|Z_t^\triangle(\theta) - \bar{Y}_t^\triangle(\theta)|^\frac{2p}{\tau+2} \leq L\triangle^\frac{p}{\tau+2}.
\]

**Case 5.** Using $p \geq (2 \lor r)$. The proof is therefore complete.

Combining the above five cases together, we get
\[
\sup_{t \geq 0} \sup_{\theta \in [-\tau, 0]} E|Z_t^\triangle(\theta) - \bar{Y}_t^\triangle(\theta)|^\frac{2p}{\tau+2} \leq L\triangle^\frac{p}{\tau+2}.
\]
In addition,
\[
E|Z_t^\triangle(0) - \bar{Y}_t^\triangle(0)|^\frac{2p}{\tau+2} = E\left|f(\bar{Y}_{t_k}^\triangle)(t - t_k) + g(\bar{Y}_{t_k}^\triangle)(B(t) - B(t_k))\right|^\frac{2p}{\tau+2} \leq L\triangle^\frac{p}{\tau+2}.
\]
The proof is therefore complete. \(\square\)

For any $\triangle \in (0, \bar{\triangle}]$, let $\bar{\theta}_{\triangle} := \delta_{\Gamma^{-1}(\triangle^{-1})} \wedge \gamma_{\Gamma^{-1}(\triangle^{-1}), \triangle}$.  

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Lemma 3.7 Assume that (A2), (A4), and (A5) hold with \( \tilde{q} \in (0, p/(r + 1)) \). Then for any \( \triangle \in (0, \bar{\triangle}] \) and \( T > 0 \),

\[
\mathbb{E}(|x(T) - Z^{\triangle}(T)|^{\tilde{q}} 1_{\{\vartheta \leq T\}}) \leq L_3 \triangle^{\tilde{q}/2},
\]

where the constant \( L_3 \) depends on \( T \) but is independent of \( \triangle \).

Proof. Using the Young inequality we derive

\[
\frac{\tilde{q} \triangle^{\tilde{q}/2}}{p} \mathbb{E}|x(T) - Z^{\triangle}(T)|^{p} \leq \frac{p - \tilde{q}}{p \triangle^{\frac{q^2}{2(p - q)}}} (\mathbb{E}|x(T)|^{p} + \mathbb{E}|Z^{\triangle}(T)|^{p}) \leq L_3 \triangle^{\tilde{q}/2}.
\]

In view of Theorem 2.1 and Lemma 3.5, we get

\[
\frac{p - \tilde{q}}{p \triangle^{\frac{q^2}{2(p - q)}}} \mathbb{P}(\vartheta \leq T) \leq \frac{p - \tilde{q}}{p \triangle^{\frac{q^2}{2(p - q)}}} \left( \mathbb{P}(\delta_{\Gamma^{-1}(K\Delta^{-\lambda})} \leq T) + \mathbb{P}(\gamma_{\Gamma^{-1}(K\Delta^{-\lambda}), \vartriangle} \leq T) \right) \\
\leq \frac{p - \tilde{q}}{p \triangle^{\frac{q^2}{2(p - q)}}} \left( L_1 + L_2 \frac{L_3 + L_2}{\Gamma^{-1}(K\Delta^{-\lambda})} \right).
\]

This, along with (3.62) and (3.63) implies that

\[
\frac{p - \tilde{q}}{p \triangle^{\frac{q^2}{2(p - q)}}} \mathbb{P}(\vartheta \leq T) \leq L_3 \triangle^{\frac{q^2}{2(p - q)}} - L_3 \triangle^{\frac{q^2}{2(p - q)}} = L_3 \triangle^{\tilde{q}/2},
\]

where the constant \( L_3 \) depends on \( T \) but is independent of \( \triangle \). \( \square \)

Lemma 3.8 Assume that (A2)–(A5) hold with

\[3r + 2 \leq p \text{ and } \tilde{q} \in [2, \frac{2p}{3r + 2}] \cap [2, \bar{p}).\]

Then for any \( \triangle \in (0, \bar{\triangle}] \) and \( T > 0 \),

\[
\mathbb{E}|x(T) - Z^{\triangle}(T)|^{\tilde{q}} \leq L_4 \triangle^{\tilde{q}/2}, \tag{3.69}
\]

where the constant \( L_4 \) depends on \( T \) but is independent of \( \triangle \).

Proof. For any \( \triangle \in (0, \bar{\triangle}] \) and \( T > 0 \), by Lemma 3.7 we get

\[
\mathbb{E}|x(T) - Z^{\triangle}(T)|^{\tilde{q}} = \mathbb{E}(|x(T) - Z^{\triangle}(T)|^{\tilde{q}} 1_{\{\vartheta \leq T\}}) + \mathbb{E}(|x(T) - Z^{\triangle}(T)|^{\tilde{q}} 1_{\{\vartheta > T\}}) \\
\leq L_3 \triangle^{\tilde{q}/2} + \mathbb{E}(|x(T) - Z^{\triangle}(T)|^{\tilde{q}} 1_{\{\vartheta > T\}}).
\]

We only require to show

\[
\mathbb{E}(|x(T) - Z^{\triangle}(T)|^{\tilde{q}} 1_{\{\vartheta > T\}}) \leq \hat{L} \triangle^{\tilde{q}/2}, \tag{3.70}
\]

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where the constant \( \hat{L} \) depends on \( T \) but is independent of \( \triangle \). By virtue of (1.1) and (3.32), we obtain

\[
x(T \land \vartheta_\triangle) - Z^\triangle(T \land \vartheta_\triangle) = \int_0^{T \land \vartheta_\triangle} (f(x_t) - f(Y^\triangle_t)) \, dt + \int_0^{T \land \vartheta_\triangle} (g(x_t) - g(Y^\triangle_t)) \, dB(t).
\]

Applying the Itô formula yields

\[
\mathbb{E}|x(T \land \vartheta_\triangle) - Z^\triangle(T \land \vartheta_\triangle)|^q \leq \frac{q}{2} \mathbb{E} \int_0^{T \land \vartheta_\triangle} |x(t) - Z^\triangle(t)|^{q-2} \left( 2(x(t) - Z^\triangle(t))^T (f(x_t) - f(Y^\triangle_t)) + (q - 1)|g(x_t) - g(Y^\triangle_t)|^2 \right) dt.
\]

(3.71)

In view of \( \bar{q} < \bar{p} \) and applying the Young inequality, we derive

\[
2(x(t) - Z^\triangle(t))^T (f(x_t) - f(Y^\triangle_t)) + (q - 1)|g(x_t) - g(Y^\triangle_t)|^2 \leq 2(x(t) - Z^\triangle(t))^T (f(x_t) - f(Z^\triangle_t)) + (\bar{p} - 1)|g(x_t) - g(Z^\triangle_t)|^2 + 2|x(t) - Z^\triangle(t)||f(Z^\triangle_t) - f(Y^\triangle_t)| + L|g(Z^\triangle_t) - g(Y^\triangle_t)|^2.
\]

(3.72)

Inserting (3.72) into (3.71) and using (A4) leads to

\[
\mathbb{E}|x(T \land \vartheta_\triangle) - Z^\triangle(T \land \vartheta_\triangle)|^q \leq \frac{q}{2} \mathbb{E} \int_0^{T \land \vartheta_\triangle} |x(t) - Z^\triangle(t)|^{q-2} \left( |x(t) - Z^\triangle(t)||f(Z^\triangle_t) - f(Y^\triangle_t)| + |g(Z^\triangle_t) - g(Y^\triangle_t)|^2 \right) dt + \frac{q}{2} \mathbb{E} \int_0^{T \land \vartheta_\triangle} U(x(t), Z^\triangle(t)) \, dt + \frac{q}{2} \mathbb{E} \int_0^{T \land \vartheta_\triangle} U(x(t), Z^\triangle_t) \rho_2(\theta) \, d\theta \, dt + I,
\]

(3.73)

where

\[
I := L \mathbb{E} \int_0^{T \land \vartheta_\triangle} |x(t) - Z^\triangle(t)|^{q-2} |x(t) - Z^\triangle(t)||f(Z^\triangle_t) - f(Y^\triangle_t)| + |g(Z^\triangle_t) - g(Y^\triangle_t)|^2 \, dt.
\]

By changing the integration order and using (A3) shows

\[
\frac{1}{\tau} \int_{-\tau}^0 \int_0^\tau |x_t(\theta) - Z^\triangle_t(\theta)|^q d\theta d\tau = \frac{1}{\tau} \int_0^{T \land \vartheta_\triangle} |x(s) - Z^\triangle(s)|^q ds dt \\
\leq \frac{1}{\tau} \int_{-\tau}^0 \int_0^{\tau + s} |x(s) - Z^\triangle(s)|^q ds dt + \frac{1}{\tau} \int_0^{T \land \vartheta_\triangle} \int_s^{\tau + s} |x(s) - Z^\triangle(s)|^q dt ds \\
\leq \frac{1}{\tau} \int_{-\tau}^0 \int_0^{\tau + s} |\xi(s) - (\frac{t_{k_s + 1} - s}{\triangle} x(t_{k_s}) + \frac{s - t_{k_s}}{\triangle} x(t_{k_s + 1}))|^q dt ds \\
+ \frac{1}{\tau} \int_0^{T \land \vartheta_\triangle} \int_s^{\tau + s} |x(s) - Z^\triangle(s)|^q dt ds \\
\leq L \triangle \frac{q}{2} + \int_0^{T \land \vartheta_\triangle} |x(s) - Z^\triangle(s)|^q ds,
\]

(3.74)
where $k_s := \lfloor s/\triangle \rfloor$. Similarly, it follows from (3.60) and (A3) that
\[
\int_0^{T\wedge \triangle} \int_{-\tau}^0 U(x_t(\theta), Z_t^\triangle(\theta)) \rho_2(\theta) d\theta dt \leq L \triangle^{3/2} + \int_0^{T\wedge \triangle} U(x(s), Z^\triangle(s)) ds. \tag{3.75}
\]
Inserting (3.74) and (3.75) into (3.73) yields
\[
\mathbb{E} |x(T \wedge \vartheta) - Z^\triangle(T \wedge \vartheta)|^q
\leq L \triangle^{3/2} + L \mathbb{E} \int_0^T |x(t \wedge \vartheta) - Z^\triangle(t \wedge \vartheta)|^q dt + I, \tag{3.76}
\]
We aim to estimate $I$. Making use of the Young inequality we get
\[
I \leq L \mathbb{E} \int_0^T |x(t \wedge \vartheta) - Z^\triangle(t \wedge \vartheta)|^q dt + L \mathbb{E} \int_0^T (|f(Z^\triangle_t)| - f(\bar{Y}^\triangle_t)|^q
+ |g(Z^\triangle_t) - g(\bar{Y}^\triangle_t)|^q) dt. \tag{3.77}
\]
By (A5), the Hölder inequality, and $2 \leq q \leq 2p/(3r + 2)$, we derive
\[
\mathbb{E}(|f(Z^\triangle_t) - f(\bar{Y}^\triangle_t)|^q + |g(Z^\triangle_t) - g(\bar{Y}^\triangle_t)|^q)
\leq L \mathbb{E} \left( |Z^\triangle_t(0) - \bar{Y}^\triangle_t(0)|^q + \frac{1}{r} \int_{-\tau}^0 |Z^\triangle_t(\theta) - \bar{Y}^\triangle_t(\theta)|^q d\theta
\times (1 + |Z^\triangle(t)|^{qr} + |\bar{Y}^\triangle(t)|^{qr}) \right)
\leq L \left( \mathbb{E} \left( |Z^\triangle_t(0) - \bar{Y}^\triangle_t(0)|^q + \frac{1}{r} \int_{-\tau}^0 |Z^\triangle_t(\theta) - \bar{Y}^\triangle_t(\theta)|^q d\theta
+ |\bar{Y}^\triangle(t)|^{qr} \right) + \mathbb{E} |Z^\triangle_t(0)|^p + \frac{1}{r} \int_{-\tau}^0 \mathbb{E} |Z^\triangle_t(\theta)|^p d\theta
+ \mathbb{E} |\bar{Y}^\triangle(t)|^p \right) ^{2q/(q+r)} \tag{3.77}
\]
Using Theorem 3.1 Lemma 3.5 and Lemma 3.6, we derive
\[
\mathbb{E}(|f(Z^\triangle_t) - f(\bar{Y}^\triangle_t)|^q + |g(Z^\triangle_t) - g(\bar{Y}^\triangle_t)|^q)
\leq L \left( \mathbb{E} \left( |Z^\triangle_t(0) - \bar{Y}^\triangle_t(0)|^{2q/(q+r)} + \frac{1}{r} \int_{-\tau}^0 \mathbb{E} |Z^\triangle_t(\theta) - \bar{Y}^\triangle_t(\theta)|^{2q/(q+r)} d\theta \right)
+ \mathbb{E} |Z^\triangle_t(0)|^p + \frac{1}{r} \int_{-\tau}^0 \mathbb{E} |Z^\triangle_t(\theta)|^p d\theta
+ \mathbb{E} |\bar{Y}^\triangle(t)|^p \right) ^{2q/(q+r)} \tag{3.77}
\]
Together with (3.77) we arrive at
\[
I \leq L \mathbb{E} \int_0^T |x(t \wedge \vartheta) - Z^\triangle(t \wedge \vartheta)|^q dt + L \triangle^{q/2}. \tag{3.78}
\]
Inserting (3.78) into (3.76) and
\[
\mathbb{E} |x(T \wedge \vartheta) - Z^\triangle(T \wedge \vartheta)|^q \leq L \triangle^{q/2} + L \mathbb{E} \int_0^T |x(t \wedge \vartheta) - Z^\triangle(t \wedge \vartheta)|^q dt.
\]
The required assertion (3.70) follows by applying the Gronwall inequality. \qed
**Theorem 3.3** Assume that (A2)–(A5) hold with

\[ 3r + 2 \leq p \quad \text{and} \quad \bar{q} \in \left[ 2, \frac{2p}{3r + 2} \right] \cap \left[ 2, \bar{p} \right). \]

Then for any \( \triangle \in (0, \bar{\Delta}] \) and \( T > 0 \),

\[ \mathbb{E}\left| x(T) - Y^\triangle(T) \right|^{\bar{q}} \leq L_5 \bar{\Delta}^{\frac{n}{2}}, \]

where the constant \( L_5 \) depends on \( T \) but is independent of \( \triangle \).

**Proof.** By virtue of Lemma 3.6 and Lemma 3.8 yields

\[ \mathbb{E}\left| x(T) - Y^\triangle(T) \right|^{\bar{q}} \leq \mathbb{E}\left| x(T) - Z^\triangle(T) \right|^{\bar{q}} + \mathbb{E}\left| Z^\triangle_T(0) - \tilde{Y}^\triangle_T(0) \right|^{\bar{q}} \leq L_5 \bar{\Delta}^{\frac{n}{2}}, \]

where the constant \( L_5 \) depends on \( T \) but is independent of \( \triangle \). \( \square \)

**Remark 3.1** In fact, the results on the convergence and convergence rate are still hold as \( a_2 = a_3 = 0 \) in (A2).

### 4 Exponential stability

In this section, we focus on approximating the exponential stability of SFDE (1.1). We begin with the exponential stability of the exact solutions in \( L^p \) and \( P - 1 \). Then we approximate the long-time behaviors by the truncated numerical solutions. Without loss of generality we assume that \( f(0) = 0 \), \( g(0) = 0 \), and give the following hypothesis.

**\( (A2') \).** There exist positive constants \( p, \bar{q}, b_i (1 \leq i \leq 4) \) satisfying \( p \geq 2 \), \( b_1 > b_2 \), \( b_3 > b_4 \), and functions \( \rho_3(\cdot) \), \( \rho_4(\cdot) \in \mathcal{W} \) such that for any \( \phi \in \mathcal{C} \),

\[
2(\phi(0))^T f(\phi) + (p - 1)|\varphi(\phi)|^2 \\
\leq -b_1|\phi(0)|^2 + b_2 \int_{-\tau}^{0} |\phi(\theta)|^2 \rho_3(\theta) \text{d}\theta - b_3|\phi(0)|^{2+\bar{q}} + b_4 \int_{-\tau}^{0} |\phi(\theta)|^{2+\bar{q}} \rho_4(\theta) \text{d}\theta. \tag{4.1}
\]

Owing to \( b_1 > b_2 \) and \( b_3 > b_4 \) in (A2’), we can fix a positive constant \( \nu \) such that

\[
\frac{p}{2}(b_1 - b_2 e^{\nu \tau}) - \nu > 0 \quad \text{and} \quad b_3 - b_4 e^{\nu \tau} > 0. \tag{4.2}
\]

By similar arguments in the proof of [20, Theorem 3 and Theorem 4] we give stability results of the exact solutions.

**Theorem 4.1** Assume that (A1) and (A2’) hold. Then the solution \( x(t) \) to SFDE (1.1) with the initial data \( \xi \in \mathcal{C} \) satisfies

\[
\limsup_{t \to \infty} \frac{1}{t} \log(\mathbb{E}|x(t)|^p) \leq -\nu,
\]

\[
\limsup_{t \to \infty} \frac{1}{t} \log |x(t)| \leq -\frac{\nu}{p}, \quad \text{a.s.}
\]
Now we start with the stability analysis of the numerical solutions. Due to (4.2) we can fix a constant $\varepsilon_1 \in (0, 1)$ sufficiently small such that
\[
\frac{p}{2} \left( (1 - \varepsilon_1)^{\frac{p}{2}} b_1 - b_2 e^{\nu \tau} \right) - \nu (1 + \varepsilon_1 e^{\nu \tau}) \geq 0,
\]
\[
(1 - \varepsilon_1)^{\frac{p}{2}} b_3 - b_4 e^{\nu \tau} \geq 0.
\]
(4.3)

For any $\phi \in C$, define
\[
\Phi_1(\phi) := (1 - \varepsilon_1)|\phi(0)|^2 + \frac{\varepsilon_1}{\tau} \int_{-\tau}^{0} |\phi(\theta)|^2 \, d\theta.
\]

In the similar ways as (3.7) and (3.8) were shown, we get that for any $k \in \mathbb{N}$,
\[
|f(\bar{Y}_{t_k})|^2 \leq L \Delta^{-2}\Phi_1(\bar{Y}_{t_k}),
\]
(4.4)
\[
|g(\bar{Y}_{t_k})|^2 \leq L \Delta^{-\lambda}\Phi_1(\bar{Y}_{t_k}).
\]
(4.5)

The following inequality plays a key role in the proof of the stability of the numerical solutions.

**Lemma 4.1** Assume that (A2') holds. Then for any $\phi \in C$,
\[
\nu (\Phi_1(\phi))^{\frac{p}{2}} + \frac{(1 - \varepsilon_1)p}{2} (\Phi_1(\phi))^{\frac{p^2}{2}} (2(\phi(0))^T f(\phi) + (p - 1)|g(\phi)|^2)
\]
\[
\leq -\beta_1 |\phi(0)|^p + \frac{\beta_2}{\tau} \int_{-\tau}^{0} |\phi(\theta)|^p \, d\theta + \beta_3 \int_{-\tau}^{0} |\phi(\theta)|^p \rho_3(\theta) \, d\theta
\]
\[
- \beta_4 |\phi(0)|^{p+\varepsilon} + \frac{\beta_5}{\tau} \int_{-\tau}^{0} |\phi(\theta)|^{p+\varepsilon} \, d\theta + \beta_6 \int_{-\tau}^{0} |\phi(\theta)|^{p+\varepsilon} \rho_4(\theta) \, d\theta,
\]
(4.6)

where $\beta_i$ ($1 \leq i \leq 6$) are positive constants given by (4.11), and satisfy
\[
e^{\nu \tau} (\beta_2 + \beta_3) < \beta_1 \quad \text{and} \quad e^{\nu \tau} (\beta_5 + \beta_6) < \beta_4.
\]

**Proof.** For any $\phi \in C$, making use of (4.1) and the Young inequality leads to
\[
(\Phi_1(\phi))^{\frac{p-2}{2}} (2(\phi(0))^T f(\phi) + (p - 1)|g(\phi)|^2)
\]
\[
\leq - (1 - \varepsilon_1)^{\frac{p-2}{2}} b_1 |\phi(0)|^p + b_2 (\Phi_1(\phi))^{\frac{p-2}{2}} \int_{-\tau}^{0} |\phi(\theta)|^2 \rho_3(\theta) \, d\theta
\]
\[
- (1 - \varepsilon_1)^{\frac{p-2}{2}} b_3 |\phi(0)|^{p+\varepsilon} + b_4 (\Phi_1(\phi))^{\frac{p-2}{2}} \int_{-\tau}^{0} |\phi(\theta)|^{2+\varepsilon} \rho_4(\theta) \, d\theta
\]
\[
\leq - (1 - \varepsilon_1)^{\frac{p+\varepsilon}{2}} b_1 |\phi(0)|^p + \frac{(p - 2)b_2}{p} (\Phi_1(\phi))^{\frac{p}{2}} + \frac{2b_2}{p} \left( \int_{-\tau}^{0} |\phi(\theta)|^2 \rho_3(\theta) \, d\theta \right)^{\frac{p}{2}}
\]
\[
- (1 - \varepsilon_1)^{\frac{p+\varepsilon}{2}} b_3 |\phi(0)|^{p+\varepsilon} + \frac{(p - 2)b_4}{p + \varepsilon} (\Phi_1(\phi))^{\frac{p+\varepsilon}{2}}
\]
\[
+ \frac{(2 + \varepsilon)b_4}{p + \varepsilon} \left( \int_{-\tau}^{0} |\phi(\theta)|^{2+\varepsilon} \rho_4(\theta) \, d\theta \right)^{\frac{p+\varepsilon}{2}},
\]
(4.7)
Applying the H"older inequality yields

\[
\left( \int_{-\tau}^{0} |\phi(\theta)|^2 \rho_3(\theta) d\theta \right)^{\frac{p}{2}} \leq \int_{-\tau}^{0} |\phi(\theta)|^p \rho_3(\theta) d\theta
\]

\[
\left( \int_{-\tau}^{0} |\phi(\theta)|^{2+e} \rho_4(\theta) d\theta \right)^{\frac{p+e}{p+e}} \leq \int_{-\tau}^{0} |\phi(\theta)|^{p+e} \rho_4(\theta) d\theta.
\]  

(4.8)

Combining (4.7) and (4.8) we derive

\[
\nu \Phi_1(\phi) \geq \frac{(1-\varepsilon)}{p} (\Phi_1(\phi))^\frac{p-2}{p} (2\phi(0))^T f(\phi) + (p-1)|g(\phi)|^2
\]

\[
\leq \frac{(1-\varepsilon)}{2p} b_1 \nu \Phi_1(\phi) + (\nu + \frac{(p-2)b_2}{2}) \Phi_1(\phi) \frac{p}{p+e}
\]

\[
+ \frac{(p-2)b_4}{2(p+\rho)} \Phi_1(\phi) \frac{p+e}{p+e} + \frac{(2+\rho)b_4}{2(p+\rho)} \int_{-\tau}^{0} |\phi(\theta)|^{p+e} \rho_4(\theta) d\theta,
\]  

(4.9)

where the positive constant \( \nu \) is given by (4.2). Using the convex property of \( u(x) = x^a \ (a > 1) \) and the H"older inequality yields

\[
\Phi_1(\phi) \leq (1-\varepsilon) |\phi(0)|^p + \varepsilon \left( \frac{1}{\tau} \right) \int_{-\tau}^{0} |\phi(\theta)|^2 d\theta \frac{p}{p+e}
\]

\[
\leq (1-\varepsilon) |\phi(0)|^p + \frac{\varepsilon_1}{\tau} \int_{-\tau}^{0} |\phi(\theta)|^p d\theta,
\]

\[
\Phi_1(\phi) \frac{p}{p+e} \leq (1-\varepsilon) |\phi(0)|^{p+e} + \frac{\varepsilon_1}{\tau} \int_{-\tau}^{0} |\phi(\theta)|^{p+e} d\theta.
\]  

(4.10)

Inserting (4.10) into (4.9) yields

\[
\nu \Phi_1(\phi) \frac{p}{2} \Phi_1(\phi) \frac{p-2}{p} (2\phi(0))^T f(\phi) + (p-1)|g(\phi)|^2
\]

\[
\leq - \beta_1 |\phi(0)|^p + \frac{\beta_2}{\tau} \int_{-\tau}^{0} |\phi(\theta)|^p d\theta + \beta_3 \int_{-\tau}^{0} |\phi(\theta)|^p \rho_3(\theta) d\theta
\]

\[
- \beta_4 |\phi(0)|^{p+e} + \frac{\beta_5}{\tau} \int_{-\tau}^{0} |\phi(\theta)|^{p+e} d\theta + \beta_6 \int_{-\tau}^{0} |\phi(\theta)|^{p+e} \rho_4(\theta) d\theta,
\]

where

\[
\beta_1 := \frac{(1-\varepsilon)^2 p b_2}{2} - (1-\varepsilon)(\nu + \frac{(p-2)b_2}{2}),
\]

\[
\beta_2 := \varepsilon \left( \frac{1}{\tau} \right) (\nu + \frac{(p-2)b_2}{2}), \quad \beta_3 := b_2,
\]

\[
\beta_4 := \frac{p}{2} (1-\varepsilon)^2 p b_3 - \frac{(1-\varepsilon)(p-2)b_4}{p+\rho},
\]

\[
\beta_5 := \frac{(p-2)b_4 \varepsilon \left( \frac{1}{\tau} \right)}{2(p+\rho)}, \quad \beta_6 := \frac{(2+\rho)b_4}{2(p+\rho)}.
\]  

(4.11)
It follows from (4.3) that
\[ e^{\nu_T}(\beta_2 + \beta_3) < \beta_1 \quad \text{and} \quad e^{\nu_T}(\beta_5 + \beta_6) < \beta_4. \]
The proof is complete. \( \Box \)

**Theorem 4.2** Assume that (A1) and (A2') hold. Then for any \( \nu_1 \in (0, \nu) \), there exists a \( \hat{\Delta} \in (0, 1] \) such that for any \( \Delta \in (0, \hat{\Delta}) \),

\[
\limsup_{k \to \infty} \frac{1}{k} \log(\mathbb{E}[Y^{\Delta}(t_k)]) \leq -\nu_1, \tag{4.12}
\]

\[
\limsup_{k \to \infty} \frac{1}{k} \log |Y^{\Delta}(t_k)| \leq -\frac{\nu_1}{p}, \text{ a.s.} \tag{4.13}
\]

**Proof.** This proof uses the similar techniques as Theorem 3.1. For any \( \ell \in (0, 1] \) and \( i \in \mathbb{N} \), in a similar way as (3.13) was shown, we derive

\[
\mathbb{E}\left((\ell + \Phi_1(Y_{t_i}^{\Delta}))^{\frac{p}{2}} | \mathcal{F}_{t_i}\right) \leq \mathbb{E}\left((\ell + (1 - \varepsilon_1)|f(Y_{t_i}^{\Delta})|\Delta^2 + |g(Y_{t_i}^{\Delta})|\Delta B_i|^2 + 2(Y^{\Delta}(t_i)) f(Y_{t_i}^{\Delta})\Delta\right.
\]

\[+ 2(Y^{\Delta}(t_i))^T g(Y_{t_i}^{\Delta}) \Delta B_i + 2f^T(Y_{t_i}^{\Delta}) g(Y_{t_i}^{\Delta}) \Delta B_i \Delta + \varepsilon_1 \left(\frac{2\Delta}{\tau} |Y^{\Delta}(t_i)|^2\right.\]

\[+ \frac{3\Delta^3}{2\tau} |f(Y_{t_i}^{\Delta})|^2 + \frac{3\Delta^3}{2\tau} |g(Y_{t_i}^{\Delta})|\Delta B_i|^2\biggr) .
\]

In a similar way as Lemma 3.1 was shown, and making use of (4.4) and (4.5), we obtain

\[
\mathbb{E}\left(\hat{\Theta}_i | \mathcal{F}_{t_i}\right) \leq (1 - \varepsilon_1)(\ell + \Psi_1(Y_{t_i}^{\Delta}))^{-1}(2(Y^{\Delta}(t_i))^T f(Y_{t_i}^{\Delta}) + |g(Y_{t_i}^{\Delta})|\Delta + L \Delta^{2-\lambda},
\]

\[
\mathbb{E}\left(\hat{\Theta}_i^2 | \mathcal{F}_{t_i}\right) \leq 4(1 - \varepsilon_1)^2(\ell + \Psi_1(Y_{t_i}^{\Delta}))^{-2} \left| (Y^{\Delta}(t_i))^T g(Y_{t_i}^{\Delta}) \right|^2 + L \Delta^{2-\lambda},
\]

\[
\mathbb{E}\left(\hat{\Theta}_i^4 P_\ell(\hat{\Theta}_i) | \mathcal{F}_{t_i}\right) \leq L \Delta^{2-\lambda} .\tag{4.15}
\]

Inserting (4.15) into (4.14) and taking expectations leads to

\[
\mathbb{E}(\ell + \Phi_1(Y_{t_i}^{\Delta}))^{\frac{p}{2}} = \mathbb{E}\left(\mathbb{E}(\ell + \Phi_1(Y_{t_i}^{\Delta}))^{\frac{p}{2}} | \mathcal{F}_{t_i}\right)
\]

\[
\leq (1 + L \Delta^{2-\lambda}) \mathbb{E}(\ell + \Phi_1(Y_{t_i}^{\Delta}))^{\frac{p}{2}} + \frac{(1 - \varepsilon_1)p \Delta}{2} \mathbb{E}\left((\ell + \Phi_1(Y_{t_i}^{\Delta}))^{\frac{p-2}{2}} ((\ell + \Phi_1(Y_{t_i}^{\Delta}))
\]

\[\times (2(Y^{\Delta}(t_i))^T f(Y_{t_i}^{\Delta}) + |g(Y_{t_i}^{\Delta})|^2) + (1 - \varepsilon_1)(p - 2) |(Y^{\Delta}(t_i))^T g(Y_{t_i}^{\Delta})|^2)\right)\]

\[
\leq (1 + L \Delta^{2-\lambda}) \mathbb{E}(\ell + \Phi_1(Y_{t_i}^{\Delta}))^{\frac{p}{2}} + \frac{(1 - \varepsilon_1)p \Delta}{2} \mathbb{E}\left((\ell + \Phi_1(Y_{t_i}^{\Delta}))^{\frac{p-2}{2}} (2(Y^{\Delta}(t_i))^T f(Y_{t_i}^{\Delta})
\]

\[+ (p - 1) |g(Y_{t_i}^{\Delta})|^2\biggr). \tag{4.16}
\]
It is straightforward to see from \( \epsilon \in (0, 1] \) that
\[
(\epsilon + \Phi_1(\tilde{Y}_{t_{i+1}}))^{\frac{p}{2}} \leq (1 + \Phi_1(\tilde{Y}_{t_{i+1}}))^{\frac{p}{2}},
\]
and
\[
(\epsilon + \Phi_1(\tilde{Y}_{t_{i}}))^{\frac{p-2}{2}} (2(Y^\Delta(t_i))^T f(\tilde{Y}_{t_{i}}) + (p-1)|g(\tilde{Y}_{t_{i}})|^2) \Delta \\
\leq (1 + \Phi_1(\tilde{Y}_{t_{i}}))^{\frac{p-2}{2}} (2|Y^\Delta(t_i)||f(\tilde{Y}_{t_{i}})| + (p-1)|g(\tilde{Y}_{t_{i}})|^2) \Delta.
\]
By the definition of \( \Phi_1 \) and Theorem 3.1 we have
\[
\mathbb{E}(1 + \Phi_1(\tilde{Y}_{t_{i+1}}))^{\frac{p}{2}} \leq L(1 + \sup_{\Delta \in (0,1]} \sup_{j \geq -N} \mathbb{E}|Y^\Delta(t_j)|^p) \leq L.
\]
By (4.4), (4.5), and Theorem 3.1 we derive
\[
\mathbb{E}(1 + \Phi_1(\tilde{Y}_{t_{i+1}}))^{\frac{p-2}{2}} (2|Y^\Delta(t_i)||f(\tilde{Y}_{t_{i}})| + (p-1)|g(\tilde{Y}_{t_{i}})|^2) \Delta \\
\leq L \Delta^{1-\lambda} \mathbb{E}(1 + \Phi_1(\tilde{Y}_{t_{i}}))^{\frac{p}{2}} \\
\leq L(1 + \sup_{\Delta \in (0,1]} \sup_{j \geq -N} \mathbb{E}|Y^\Delta(t_j)|^p) \leq L.
\]
Hence according to (4.16) and using the dominated convergence theorem yields
\[
\mathbb{E}(\Phi_1(\tilde{Y}_{t_{i+1}}))^{\frac{p}{2}} = \lim_{\epsilon \to 0^+} \mathbb{E}(\epsilon + \Phi_1(\tilde{Y}_{t_{i+1}}))^{\frac{p}{2}} \\
\leq (1 + L \Delta^{2-2\lambda}) \mathbb{E}(\Phi_1(\tilde{Y}_{t_{i}}))^{\frac{p}{2}} + \frac{(1 - \epsilon_1)p\Delta}{2} \mathbb{E}\left((\Phi_1(\tilde{Y}_{t_{i}}))^{\frac{p-2}{2}} (2(Y^\Delta(t_i))^T f(\tilde{Y}_{t_{i}}) \\
+ (p-1)|g(\tilde{Y}_{t_{i}})|^2)\right).
\]
For any \( \nu_1 \in (0, \nu) \), choose a \( \hat{\Delta} \in (0, 1] \) sufficiently small such that
\[
L \hat{\Delta}^{1-2\lambda} + \nu_1 \leq \nu.
\]
Making use of \( e^{\nu_1 t_{i+1}} - e^{\nu_1 t_i} \leq e^{\nu_1 t_i} \nu_1 \Delta \) we derive that for any \( \Delta \in (0, \hat{\Delta}] \),
\[
e^{\nu_1 t_{i+1}} \mathbb{E}(\Phi_1(\tilde{Y}_{t_{i+1}}))^{\frac{p}{2}} \\
\leq (e^{\nu_1 t_i} + e^{\nu_1 t_{i+1}} \nu_1 \Delta) \mathbb{E}(\Phi_1(\tilde{Y}_{t_{i}}))^{\frac{p}{2}} + e^{\nu_1 t_{i+1}} \Delta \mathbb{E}\left(L \hat{\Delta}^{1-2\lambda} (\Phi_1(\tilde{Y}_{t_{i}}))^{\frac{p}{2}} + \frac{(1 - \epsilon_1)p}{2} \\
\times (\Phi_1(\tilde{Y}_{t_{i}}))^{\frac{p-2}{2}} (2(Y^\Delta(t_i))^T f(\tilde{Y}_{t_{i}}) + (p-1)|g(\tilde{Y}_{t_{i}})|^2)\right) \\
\leq e^{\nu_1 t_i} \mathbb{E}(\Phi_1(\tilde{Y}_{t_{i}}))^{\frac{p}{2}} + e^{\nu_1 t_{i+1}} \Delta \mathbb{E}\left((\Phi_1(\tilde{Y}_{t_{i}}))^{\frac{p}{2}} + \frac{(1 - \epsilon_1)p}{2} (\Phi_1(\tilde{Y}_{t_{i}}))^{\frac{p-2}{2}} \\
\times (2(Y^\Delta(t_i))^T f(\tilde{Y}_{t_{i}}) + (p-1)|g(\tilde{Y}_{t_{i}})|^2)\right).
\]
This, along with (4.6) implies that
\[
e^{\nu_1 t_{i+1}} \mathbb{E}(\Phi_1(\tilde{Y}_{t_{i+1}}))^{\frac{p}{2}} - e^{\nu_1 t_i} \mathbb{E}(\Phi_1(\tilde{Y}_{t_{i}}))^{\frac{p}{2}} \\
\leq e^{\nu_1 t_{i+1}} \Delta \mathbb{E}\left(-\beta_1|Y^\Delta(t_i)|^p + \frac{\beta_2}{\tau} \int_{-\tau}^{0} |\tilde{Y}_{t_{i}}(\theta)|^p d\theta + \beta_3 \int_{-\tau}^{0} |\tilde{Y}_{t_{i}}(\theta)|^p \rho_3(\theta) d\theta \\
- \beta_4 |Y^\Delta(t_i)|^{p+\rho} + \frac{\beta_5}{\tau} \int_{-\tau}^{0} |\tilde{Y}_{t_{i}}(\theta)|^{p+\rho} d\theta + \beta_6 \int_{-\tau}^{0} |\tilde{Y}_{t_{i}}(\theta)|^{p+\rho} \rho_4(\theta) d\theta\right).
\]
Summing the above inequality on both sides from \( i = 0 \) to \( k \) yields
\[
e^{\nu_{t_{k+1}}} \mathbb{E}(\Phi_1(Y_{t_{k+1}}))^{\frac{k}{2}} \\
\leq (\Phi_1(\bar{Y}_{0}^\triangle))^{\frac{k}{2}} + \Delta \sum_{i=0}^{k} e^{\nu_{t_{i+1}}} \mathbb{E}\left(-\beta_1 |Y_{t_i}^\triangle|^p + \frac{\beta_2}{\tau} \int_{-\tau}^{0} |\bar{Y}_{t_i}^\triangle(\theta)| \, d\theta \right) \\
+ \beta_3 \int_{-\tau}^{0} |\bar{Y}_{t_i}^\triangle(\theta)| \, d\theta - \beta_4 |Y_{t_i}^\triangle|^p + \beta_5 \int_{-\tau}^{0} |\bar{Y}_{t_i}^\triangle(\theta)| \, d\theta + \beta_6 \int_{-\tau}^{0} |\bar{Y}_{t_i}^\triangle(\theta)| \, d\theta.
\]

In the similar ways as (3.30) and (3.31) were shown, we have
\[
\sum_{i=0}^{k} e^{\nu_{t_{i+1}}} \left( \frac{1}{\tau} \int_{-\tau}^{0} |\bar{Y}_{t_i}^\triangle(\theta)| \, d\theta \right) \leq L + e^{\nu_{t_{1}}} \sum_{i=0}^{k} e^{\nu_{t_{i+1}}} |Y_{t_i}^\triangle|^p, \\
\sum_{i=0}^{k} e^{\nu_{t_{i+1}}} \int_{-\tau}^{0} |\bar{Y}_{t_i}^\triangle(\theta)| \, d\theta \leq L + e^{\nu_{t_{1}}} \sum_{i=0}^{k} e^{\nu_{t_{i+1}}} |Y_{t_i}^\triangle|^p, \\
\sum_{i=0}^{k} e^{\nu_{t_{i+1}}} \left( \frac{1}{\tau} \int_{-\tau}^{0} |\bar{Y}_{t_i}^\triangle(\theta)| \, d\theta \right) \leq L + e^{\nu_{t_{1}}} \sum_{i=0}^{k} e^{\nu_{t_{i+1}}} |Y_{t_i}^\triangle|^p + \beta_5 \int_{-\tau}^{0} |\bar{Y}_{t_i}^\triangle(\theta)| \, d\theta + \beta_6 \int_{-\tau}^{0} |\bar{Y}_{t_i}^\triangle(\theta)| \, d\theta.
\]

Inserting (4.18) into (4.17) and then using
\[
e^{\nu_{\tau}} (\beta_2 + \beta_3) < \beta_1 \quad \text{and} \quad e^{\nu_{\tau}} (\beta_5 + \beta_6) < \beta_4,
\]
we derive
\[
e^{\nu_{\tau}} \mathbb{E}(\Phi_1(\bar{Y}_{t_{k+1}}))^{\frac{k}{2}} \leq L,
\]
which implies that (4.12) holds. By virtue of (4.12), using the similar technique as in the proof of [37, Theorem 3.4] implies the desired assertion (4.13). \( \square \)

**Remark 4.1** The results of exponential stability for the exact solutions and the numerical solutions still hold as \( b_3 = b_4 = 0 \) in (A2').

**Remark 4.2** Our numerical method is also suitable for systems with coefficients \( f(\cdot) \) and \( g(\cdot) \) to be of the form
\[
H(\phi(0), \int_{-\tau_1}^{0} \phi(\theta) \rho_1(\theta) \, d\theta, \ldots, \int_{-\tau_M}^{0} \phi(\theta) \rho_M(\theta) \, d\theta),
\]
where \( M \) is a positive integer, \( \tau_i > 0 \) and \( \rho_i \in \mathcal{W}([-\tau_i, 0]; \mathbb{R}_+) \) for any \( i \in \{1, \ldots, M\} \), \( \tau := \max_{1 \leq i \leq M} \tau_i \) and \( \phi \in C([-\tau, 0]; \mathbb{R}^n) \), and the function \( H: \mathbb{R}^n \times \cdots \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) satisfies the local Lipschitz condition. In such a case, we may rewrite \( \sum_{i=1}^{M} \int_{-\tau}^{0} \phi(\theta) \rho_i(\theta) \, d\theta \) as \( \int_{-\tau}^{0} \phi(\theta) \bar{\rho}(\theta) \, d\theta \), where \( \bar{\rho}(\theta) := \frac{1}{M} \sum_{i=1}^{M} \rho_i(\theta) 1_{[-\tau_i,0]}(\theta) \in \mathcal{W}([-\tau, 0]; \mathbb{R}_+) \).
5 Numerical examples

To verify the efficiency of our explicit scheme we give two examples and some numerical experiments.

Example 5.1 Recall SFDE (1.2) and let $p = 8$. Applying the Hölder inequality yields

$$2(f(\phi), \phi(0)) + (p - 1)|g(\phi)|^2 \leq 2\phi(0) + 8\phi^2(0) - 8\phi^4(0) + 7\int_{-1/2}^0 \phi^4(\theta) \cdot 2d\theta,$$

which implies that (A2) holds. It is straightforward to see that (A3) holds. Let $\bar{p} = 2$, $\bar{q} = 3$, and compute

$$2(\phi(0) - \bar{\phi}(0))^T (f(\phi) - f(\bar{\phi})) + (\bar{p} - 1)|g(\phi) - g(\bar{\phi})|^2$$

\[ \leq 2|\phi(0) - \bar{\phi}(0)|^2 - 8|\phi(0) - \bar{\phi}(0)|^2 (\phi^2(0) + \phi(0)\bar{\phi}(0) + \bar{\phi}^2(0)) \]

$$+ 2\int_{-1/2}^0 |\phi(\theta) - \bar{\phi}(\theta)|^2 |\phi(\theta) + \bar{\phi}(\theta)|^2 \cdot 2d\theta$$

\[ \leq 2|\phi(0) - \bar{\phi}(0)|^2 - 4|\phi(0) - \bar{\phi}(0)|^2 |\phi(0) + \bar{\phi}(0)|^2 \]

$$+ 2\int_{-1/2}^0 |\phi(\theta) - \bar{\phi}(\theta)|^2 |\phi(\theta) + \bar{\phi}(\theta)|^2 \cdot 2d\theta,$$

which implies that (A4) holds. By a direct computation we know that (A5) holds with $a_6 = 6$ and $r = 2$. In view of Remark 3.1 we take $\Gamma(l) = 6\sqrt{2}(1 + 4l^2)$ for any $l \geq 1$. This implies

$$\Gamma^{-1}(l) = \left( \frac{l}{24\sqrt{2}} - \frac{1}{4} \right)^{1/2}, \quad \forall \ l \geq 30\sqrt{2}.$$

By (3.63) we have $\lambda = 1/3$. Together with (3.2) one goes a further step to obtain that for any $\Delta \in (0, 1]$ and $x \in \mathbb{R}^n$,

$$\Lambda_\Gamma^\Delta (x) = \left( |x| \wedge \left( \frac{5}{4\Delta^{1/3}} - \frac{1}{4} \right)^{1/2} \right) \frac{x}{|x|}.$$

Let $\bar{\Delta} = 2^{-5}$. By virtue of Theorem 3.3 truncated EM solution (3.48) has the property that for any $\Delta \in (0, 2^{-5}]$ and $T > 0$,

$$\left( \mathbb{E}|x(T) - Y^\Delta(T)|^2 \right)^{1/2} \leq L\Delta^{1/2}.$$

Since it is impossible to find the closed form of the solutions of SFDE (1.2), we have to regard the truncated EM solution $\{Y^\Delta(t)\}_{t \geq -\tau}$ with the smaller stepsize $\Delta = 2^{-18}$ as the exact solution $\{x(t)\}_{t \geq -\tau}$. To verify the convergence of the truncated EM solution $\{Y^\Delta(t)\}_{t \geq -\tau}$, we simulate the root mean square approximation error $(\mathbb{E}|x(10) - Y^\Delta(10)|^2)^{1/2}$ by using MATLAB soft. In Figure 1 the red dotted line depicts $(\mathbb{E}|x(10) - Y^\Delta(10)|^2)^{1/2}$ as the function of $\Delta$ for 1000 sample points as $\Delta \in \{2^{-5}, 2^{-6}, 2^{-7}, 2^{-8}, 2^{-10}\}$, while the blue line represents the reference line with the slope 1/2.

Example 5.2 Consider the 2-dimensional SFDE

$$\begin{cases}
    dx_1(t) = (-2x_1(t) - 3x_1^3(t) + \int_{-1/4}^0 x_2(t + \theta)d\theta)dt + x_1(t)dB_1(t), & t > 0, \\
    dx_2(t) = (-2x_2(t) - 2x_2^3(t) + \int_{-1/2}^0 x_1^3(t + \theta)d\theta)dt + x_2(t)dB_2(t),
\end{cases}
\quad (5.1)$$
Figure 1: The red dotted line depicts the root mean square approximation error \((\mathbb{E}|x(10) - Y^{\Delta}(10)|^2)^{1/2}\) between exact solution \(x(10)\) and truncated EM solution \(Y^{\Delta}(10)\) for 1000 sample points as \(\Delta \in \{2^{-5}, 2^{-6}, 2^{-7}, 2^{-8}, 2^{-10}\}\), while the blue line represents the reference line with the slope \(1/2\).

with the initial data \((\xi_1(\theta), \xi_2(\theta))^T = (\theta^2, \sin(-\theta + 2))^T\) for any \(\theta \in [-1/2, 0]\). Letting \(p = 2\) and applying the Young inequality, we derive that for any \(\phi(\cdot) = (\phi_1(\cdot), \phi_2(\cdot))^T \in C\),

\[
2(\phi(0))^T f(\phi) + |g(\phi)|^2 \\
\leq -3(|\phi_1(0)|^2 + |\phi_2(0)|^2) - 4(|\phi_1(0)|^4 + |\phi_2(0)|^4) + \frac{1}{2} |\phi_1(0)| \int_{-1/4}^0 |\phi_2(\theta)| \cdot 4d\theta \\
+ |\phi_2(0)| \int_{-1/2}^0 |\phi_1(\theta)|^3 \cdot 2d\theta \\
\leq -\frac{11}{4} |\phi(0)|^2 - \frac{15}{4} (|\phi_1(0)|^4 + |\phi_2(0)|^4) + \frac{1}{4} \int_{-1/4}^0 |\phi_2(\theta)|^2 \cdot 4d\theta + \frac{3}{4} \int_{-1/2}^0 |\phi_1(\theta)|^4 \cdot 2d\theta \\
\leq -\frac{11}{4} |\phi(0)|^2 - \frac{15}{4} |\phi(0)|^4 + \frac{1}{4} \int_{-1/4}^0 |\phi(\theta)|^2 \cdot 4d\theta + \frac{3}{4} \int_{-1/2}^0 |\phi^4(\theta)| \cdot 2d\theta,
\]

which implies that \((A2')\) holds with

\[
b_1 = \frac{11}{4}, \quad b_2 = \frac{1}{4}, \quad b_3 = \frac{15}{4}, \quad b_4 = \frac{3}{4},
\]

\[
\rho_3(\theta) = 41_{[-1/4, 0]}(\theta), \quad \rho_4(\theta) = 2, \quad \forall \theta \in [-\frac{1}{2}, 0].
\]

Choose \(\nu = 2\) in \((4.2)\). In view of Theorem 4.1, the exact solution of SFDE \((5.1)\) satisfies

\[
\limsup_{t \to \infty} \frac{1}{t} \log(\mathbb{E}|x(t)|^2) \leq -2, \\
\limsup_{t \to \infty} \frac{1}{t} \log |x(t)| \leq -1, \quad \text{a.s.}
\]

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For each $l \geq 1$, a direct computation derives
\[
\sup_{\|\phi\| \vee \|\tilde{\phi}\| \leq l} |f(\phi) - f(\tilde{\phi})| \leq (4 + 18l^2)(|\phi(0) - \tilde{\phi}(0)|^2 + 2 \int_{-1/2}^0 |\phi(\theta) - \tilde{\phi}(\theta)|^2 d\theta)^{1/2},
\]
\[
\sup_{\|\phi\| \vee \|\tilde{\phi}\| \leq l} |g(\phi) - g(\tilde{\phi})|^2 \leq |\phi(0) - \tilde{\phi}(0)|^2,
\]
for all $\phi, \tilde{\phi} \in \mathcal{C}$. Recalling (3.1) we take
\[
\Gamma(l) = 4 + 18l^2, \quad \forall \ l \geq 1.
\]
This implies
\[
\Gamma^{-1}(l) = \left( \frac{l}{18} - \frac{2}{9} \right)^{1/2}, \quad \forall \ l \geq 22.
\]
Let $\lambda = 0.001$. Then, define
\[
\Lambda_{\Gamma, \lambda}(x) = \left( |x| \wedge \left( \frac{11}{9\lambda^{0.001}} - \frac{2}{9} \right)^{1/2} \right) \frac{x}{|x|},
\]
By virtue of Theorem 4.2, for any $\nu_1 \in (0, 2)$, there exists a $\hat{\Delta} \in (0, 1]$ such that for any $\Delta \in (0, \hat{\Delta}]$, truncated EM solution (3.48) satisfies
\[
\limsup_{k \to \infty} \frac{1}{k\Delta} \log (\mathbb{E}|Y^\Delta(t_k)|^2) \leq -\nu_1, \quad \text{and} \quad \limsup_{k \to \infty} \frac{1}{k\Delta} \log |Y^\Delta(t_k)| \leq -\frac{\nu_1}{2}, \text{ a.s.}
\]
Figure 2 depicts the sample mean of the truncated EM solution $Y^\Delta(t)$. Figure 3 depicts the sample paths of the EM solution and the truncated EM solution.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{The sample mean of $Y^\Delta(t)$ for 1000 sample points as $\Delta = 2^{-6}$.}
\end{figure}
Figure 3: (Left) The sample path of EM solution $\ln(z(t))$ as $\Delta = 2^{-6}$. (Right) The sample path of truncated EM solution $Y^\Delta(t)$ as $\Delta = 2^{-6}$.

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