The Orchard relation of a generic symmetric or antisymmetric function

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Abstract: We associate to certain symmetric or antisymmetric functions on the set \( (E_{d+1}) \) of \((d+1)\)-subsets in a finite set \( E \) an equivalence relation on \( E \) and study some of its properties.

1 Definitions and main results

We consider a finite set \( E \) and denote by \( (E_d) \) the set of subsets containing exactly \( d \) elements of \( E \). In the sequel we move often freely from sets to sequences: we identify a subset \( \{x_1, \ldots, x_d\} \in (E_d) \) with the finite sequence \((x_1, \ldots, x_d)\) where the order of the elements is for instance always increasing with respect to a fixed total order on \( E \).

A function \( \varphi : (E_d) \to \mathbb{R} \) is symmetric if

\[
\varphi(x_1, \ldots, x_i, x_{i+1}, \ldots, x_d) = \varphi(x_1, \ldots, x_{i-1}, x_i, x_i+1, \ldots, x_d)
\]

for \( 1 \leq i < d \) and all \( \{x_1, \ldots, x_d\} \in (E_d) \).

Similarly, such a function \( \varphi : (E_d) \to \mathbb{R} \) is antisymmetric if

\[
\varphi(x_1, \ldots, x_i, x_{i+1}, \ldots, x_d) = -\varphi(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, x_i+2, \ldots, x_d)
\]

for \( 1 \leq i < d \) and all \( x_1, \ldots, x_d \in E \).

\( \varphi \) is generic if \( \varphi(x_1, \ldots, x_d) \neq 0 \) for all subsets \( \{x_1, \ldots, x_d\} \in (E_d) \) of \( d \) distinct elements in \( E \).

In the sequel of this paper all functions will be generic. We will mainly be concerned with sign properties of generic symmetric or antisymmetric functions: Given any symmetric generic function \( \sigma : (E_d) \to \mathbb{R}_{>0} \) and a symmetric or antisymmetric generic function \( \varphi : (E_d) \to \mathbb{R}^d \), the two functions

\[
(x_1, \ldots, x_d) \mapsto \varphi(x_1, \ldots, x_d)
\]

and

\[
(x_1, \ldots, x_d) \mapsto \sigma(x_1, \ldots, x_d)\varphi(x_1, \ldots, x_d)
\]
behave similarly with respect to all properties adressed in this paper.

We have also an obvious sign rule: symmetric or antisymmetric functions on $E_d$ behave with respect to multiplication like the elements of the multiplicative group $\{\pm 1\}$ with symmetric functions corresponding to 1 and antisymmetric functions corresponding to $-1$.

We fix now a generic symmetric or antisymmetric function $\phi : (E^d) \rightarrow \mathbb{R}$. Consider two elements $a, b \in E$. A subset $\{x_1, \ldots, x_d\} \in (E\{a,b\})$ not containing $a$ and $b$ separates $a$ from $b$ with respect to $\phi$ if

$$\phi(x_1, \ldots, x_d, a) \phi(x_1, \ldots, x_d, b) < 0$$

(this definition is of course independent of the particular linear order $(x_1, \ldots, x_d)$ on the set $\{x_1, \ldots, x_d\}$).

We denote by $n(a,b) = n_\phi(a,b)$ the number of subsets in $(E\{a,b\})$ separating $a$ from $b$ (with respect to the function $\phi$).

**Proposition 1.1**

(i) If $\phi : (E^d_{d+1}) \rightarrow \mathbb{R}$ is symmetric and generic then

$$n(a,b) + n(b,c) + n(a,c) \equiv 0 \pmod{2}$$

for any subset $\{a,b,c\}$ of 3 distinct elements in $E$.

(ii) If $\phi : (E^d_{d+1}) \rightarrow \mathbb{R}$ is antisymmetric and generic then

$$n(a,b) + n(b,c) + n(a,c) \equiv \left(\frac{\#(E) - 3}{d - 1}\right) \pmod{2}$$

for any subset $\{a,b,c\}$ of 3 distinct elements in $E$.

**Proof.** Consider first a subset $\{x_1, \ldots, x_d\}$ not intersecting $\{a,b,c\}$. Such a subset separates no pair of elements in $\{a,b,c\}$ if

$$\phi(x_1, \ldots, x_d, a), \phi(x_1, \ldots, x_d, b) \text{ and } \phi(x_1, \ldots, x_d, c)$$

all have the same sign. Otherwise, consider a reordering $\{a', b', c'\} = \{a, b, c\}$ such that $\phi(x_1, \ldots, x_d, a') \phi(x_1, \ldots, x_d, b') < 0$ and $\phi(x_1, \ldots, x_d, a') \phi(x_1, \ldots, x_d, c') < 0$. The subset $\{x_1, \ldots, x_d\}$ contributes in this case 1 to $n(a', b')$, $n(a', c')$ and 0 to $n(b', c')$. Such a subset $\{x_1, \ldots, x_d\} \in (E\{a,b,c\})$ yields hence always an even contribution (0 or 2) to the sum $n(a,b) + n(a,c) + n(b,c)$.

Consider now a subset $\{x_1, \ldots, x_d-1\} \in (E\{a,b,c\})$. We have to understand the contributions of the sets

- $\{x_1, \ldots, x_{d-1}, c\}$ to $n(a,b)$
- $\{x_1, \ldots, x_{d-1}, b\}$ to $n(a,c)$
- $\{x_1, \ldots, x_{d-1}, a\}$ to $n(b,c)$.
Since the product of the six factors
\[ \varphi(x_1, \ldots, x_{d-1}, c, a) \varphi(x_1, \ldots, x_{d-1}, c, b) \]
\[ \varphi(x_1, \ldots, x_{d-1}, b, a) \varphi(x_1, \ldots, x_{d-1}, b, c) \]
\[ \varphi(x_1, \ldots, x_{d-1}, a, b) \varphi(x_1, \ldots, x_{d-1}, a, c) \]
is always positive (respectively negative) for a generic symmetric (respectively antisymmetric) function, such a subset \( \{x_1, \ldots, x_{d-1}\} \) yields an even contribution to \( n(a, b) + n(a, c) + n(b, c) \) in the symmetric case and an odd contribution in the antisymmetric case.

Proposition 1.1 follows now from the fact that \( \binom{E \setminus \{a,b,c\}}{d-1} \) has \( \binom{2(E)-3}{d-1} \) elements.

Given a generic symmetric or antisymmetric function \( \varphi: \binom{E}{d+1} \to \mathbb{R} \) on some finite set \( E \) we set \( x \sim y \) if either \( x = y \in E \) or if
\[ n(x, y) \equiv 0 \pmod{2} \quad \text{for symmetric } \varphi \]
respectively
\[ n(x, y) \equiv \binom{2(E)-3}{d-1} \pmod{2} \quad \text{for antisymmetric } \varphi . \]

We call the relation \( \sim \) defined in this way on the set \( E \) the Orchard relation.

**Theorem 1.2** The Orchard relation is an equivalence relation having at most two classes.

**Proof.** Reflexivity and symmetry are obvious. Transitivity follows easily from Proposition 1.1.

If \( a \not\sim b \) and \( b \not\sim c \) then \( n(a, b) + n(b, c) \) is even. It follows then from Proposition 1.1 that \( a \sim c \). □

**Example.** A tournament is a generic antisymmetric function \( \binom{E}{1+1} \to \{\pm 1\} \). It encodes for instance orientations of all edges in the complete graph with vertices \( E \) and can be summarized by an antisymmetric matrix \( A \) with coefficients in \( \{\pm 1\} \).

Given such a matrix \( A \) with coefficients \( a_{i,j} \), \( 1 \leq i, j \leq n \), we have
\[ n_A(i, j) = \frac{n - 2 - \sum_k a_{i,k}a_{j,k}}{2} . \]

This implies \( i \sim_A j \) if and only if
\[ \sum_k a_{i,k}a_{j,k} \equiv n \pmod{4} \]
for \( i \neq j \). In the language of tournaments (cf. for instance [4]), this result can be restated in terms of score vectors: Two elements \( i \) and \( j \) are Orchard
equivalent if and only if the corresponding coefficients of the score vector (counting the number of 1’s in line $i$ respectively $j$) have the same parities.

**Main Example.** A finite set $\mathcal{P} = \{P_1, \ldots, P_n\} \subset \mathbb{R}^d$ of $n > d$ points in real affine space $\mathbb{R}^d$ is *generic* if the affine span of any subset containing $(d + 1)$ points in $\mathcal{P}$ is all of $\mathbb{R}^d$. Such a generic set $\mathcal{P}$ is endowed with a generic antisymmetric function by restricting

$$\varphi(x_0, \ldots, x_d) = \det(x_1 - x_0, x_2 - x_0, \ldots, x_d - x_0)$$

to $\binom{n}{d+1}$. The Orchard relation partitions hence a generic subset $\mathcal{P} \subset \mathbb{R}^d$ into two (generally non-empty) subsets. Its name originates from the fact that the planar case ($d = 2$) yields a natural rule to plant trees of two different species at specified generic locations in an orchard, see [1] and [2].

**Proposition 1.3** Given a finite set $E$ let $\varphi$ and $\psi$ be two generic symmetric or antisymmetric functions on $\binom{E}{d+1}$.

(i) If the numbers

$$\varphi(x_0, \ldots, x_d) \psi(x_0, \ldots, x_d)$$

have the same sign for all $\{x_0, \ldots, x_d\} \in \binom{E}{d+1}$ then the two Orchard relations $\sim_\varphi$ and $\sim_\psi$ induced by $\varphi$ and $\psi$ coincide.

(ii) If there exists exactly one subset $\mathcal{F} = \{x_0, \ldots, x_d\} \in \binom{E}{d+1}$ such that

$$\varphi(x_0, \ldots, x_d) \psi(x_0, \ldots, x_d) < 0$$

then the restrictions of $\sim_\varphi$ and $\sim_\psi$ to the two subsets $\mathcal{F}$ and $E \setminus \mathcal{F}$ coincide but $a \sim_\varphi b \iff a \not\sim_\psi b$ for $a \in \mathcal{F}$ and $b \in E \setminus \mathcal{F}$.

We call two symmetric or antisymmetric functions $\varphi$ and $\psi$ satisfying the condition of assertion (ii) above *flip-related*. Colouring the equivalence classes of an Orchard relation with two distinct colours, one can express assertion (ii) by the statement that changing a generic (symmetric or antisymmetric) function by a flip switches the colours in the flip-set $\mathcal{F} = \{x_0, \ldots, x_d\}$ and leaves the colours of the remaining elements unchanged.

Assertion (i) shows that we can restrict our attention to symmetric or antisymmetric functions from $\binom{E}{d+1}$ into $\{\pm 1\}$ when studying properties of the Orchard relation.

**Proof of Proposition 1.3.** Assertion (i) is obvious.

For proving assertion (ii) it is enough to remark that the numbers $n_\varphi(a, b)$ and $n_\psi(a, b)$ of separating sets (with respect to $\varphi$ and $\psi$) are identical if either $\{a, b\} \subset \mathcal{F}$ or $\{a, b\} \subset E \setminus \mathcal{F}$ and they differ by exactly one in the remaining cases. □
2 An easy characterisation in the symmetric case

In this section we give a different and rather trivial description of the Orchard relation in the symmetric case.

Given a generic symmetric function $\varphi : \binom{E}{d+1} \rightarrow \mathbb{R}$ on some finite set $E$ we consider the function

$$
\mu(x) = \sharp(\{\{x_1, \ldots, x_d\} \in \binom{E \setminus \{x\}}{d} \mid \varphi(x, x_1, \ldots, x_d) > 0\})
$$

from $E$ to $\mathbb{N}$.

**Theorem 2.1** Two elements $x, y \in E$ are Orchard equivalent with respect to $\varphi$ if and only if $\mu(x) \equiv \mu(y) \pmod{2}$.

**Proof.** The result holds if $\varphi$ is the constant function

$$
\varphi(x_0, \ldots, x_d) = 1
$$

for all $\{x_0, \ldots, x_d\} \in \binom{E}{d+1}$.

Given two generic symmetric functions $\varphi, \psi$ related by a flip with respect to the set $F = \{x_0, \ldots, x_d\} \in \binom{E}{d+1}$ we have

$$
\mu_{\varphi}(x) = \mu_{\psi}(x)
$$

if $x \not\in F$ and

$$
\mu_{\varphi}(x) = \mu_{\psi}(x) \pm 1
$$

otherwise. Proposition 1.3 implies hence the result since any generic symmetric function can be related by a finite number of flips to the constant function. \(\square\)

3 Reducing $d$

Let $\varphi : \binom{E}{d+1} \rightarrow \mathbb{R}$ be a generic symmetric or antisymmetric function. Consider the function

$$
R\varphi : \binom{E}{d} \rightarrow \mathbb{R}
$$

defined by

$$
R\varphi(x_1, \ldots, x_d) = \prod_{x \in E \setminus \{x_1, \ldots, x_d\}} \varphi(x, x_1, \ldots, x_d).
$$

$R\varphi$ is generic symmetric if $\varphi$ is generic symmetric.

For $\varphi$ generic antisymmetric, the function $R\varphi$ is generic symmetric if $\sharp(E) \equiv d \pmod{2}$ and $R\varphi$ is generic antisymmetric otherwise.

Dependencies of the Orchard relations associated to $\varphi$ and $R\varphi$ are described by the following result.
**Proposition 3.1** Let \( \varphi : \binom{E}{d+1} \rightarrow \mathbb{R} \) be a generic symmetric or antisymmetric function.

(i) If \( d \equiv 0 \pmod{2} \) then the Orchard relation of \( R\varphi \) is trivial (i.e. \( x \sim_{R\varphi} y \) for all \( x, y \in E \)).

(ii) If \( d \equiv 1 \pmod{2} \) then the Orchard relations \( \sim_\varphi \) and \( \sim_{R\varphi} \) coincide on \( E \).

The main ingredient of the proof is the following lemma.

**Lemma 3.2** Let \( \varphi, \psi : \binom{E}{d+1} \rightarrow \mathbb{R} \) be two generic symmetric or antisymmetric functions which are flip-related with respect to the set \( F = \{x_0, \ldots, x_d\} \in \binom{E}{d+1} \). Then

\[
R\varphi(y_1, \ldots, y_d) \ R\psi(y_1, \ldots, y_d) < 0
\]

if \( \{y_1, \ldots, y_d\} \subset F \) and

\[
R\varphi(y_1, \ldots, y_d) \ R\psi(y_1, \ldots, y_d) > 0
\]

otherwise.

**Proof of Lemma 3.2** If \( \{y_1, \ldots, y_d\} \not\subset F \) then \( \varphi(x, y_1, \ldots, y_d) = \psi(x, y_1, \ldots, y_d) \) for all \( x \in E \setminus \{y_1, \ldots, y_d\} \) and hence \( R\varphi(y_1, \ldots, y_d) = R\psi(y_1, \ldots, y_d) \). Otherwise, exactly one factor of the product yielding \( R\varphi(y_1, \ldots, y_d) \) changes sign with respect to the factors yielding \( R\psi(y_1, \ldots, y_d) \).

**Proof of Proposition 3.1** We consider first the case where \( \varphi : \binom{E}{d+1} \rightarrow \mathbb{R} \) is generic and symmetric.

Proposition 3.1 holds then for the constant symmetric application \( \varphi : \binom{E}{d+1} \rightarrow \{\pm 1\} \).

Two generic symmetric functions \( \varphi, \psi \) on \( \binom{E}{d+1} \) which are flip-related with respect to \( F = \{x_0, \ldots, x_d\} \) give rise to \( R\varphi \) and \( R\psi \) which are related through \( d + 1 \) flips with respect to all \( d + 1 \) elements in \( \binom{F}{d} \) by Lemma 3.2. Proposition 1.3 implies hence the result since an element of \( E \setminus F \) is contained in no element of \( \binom{F}{d} \) and since all elements of \( F \) are contained in exactly \( d \) such sets.

Second case: \( \varphi : \binom{E}{d+1} \rightarrow \mathbb{R} \) generic and antisymmetric. This case is slightly more involved. As in the symmetric case, we prove the result for a particular function \( \varphi \) and use the fact that flips of \( \varphi \) affect the Orchard relation \( \sim_{R\varphi} \) only for odd \( d \). This shows that it is enough to prove that \( \sim_{R\varphi} \) is trivial for a particular function \( \varphi \) in the case of even \( d \) and that \( \sim_{R\varphi} \) and \( \sim_\varphi \) coincide (for a particular generic antisymmetric function \( \varphi \)) in the case of odd \( d \).

We consider now the set \( E = \{1, \ldots, n\} \) endowed with the generic antisymmetric function \( \varphi : \binom{E}{d+1} \rightarrow \{\pm 1\} \) defined by

\[
\varphi(i_0, \ldots, i_d) = 1
\]
for all $1 \leq i_0 < i_1 < \ldots < i_d \leq n$.

Each element of $({}^E\setminus_{d-1}^{i+1})$ separates then $i$ from $i+1$ with respect to the generic function $R\varphi$. We have indeed

$$R\varphi(j_1, \ldots, j_{d-1}, i) = \varphi(i+1, j_1, \ldots, j_{d-1}, i) \prod_{j \in E\setminus\{j_1, \ldots, j_{d-1}, i, i+1\}} \varphi(j, j_1, \ldots, j_{d-1}, i)$$

$$= -\varphi(i, j_1, \ldots, j_{d-1}, i+1) \prod_{j \in E\setminus\{j_1, \ldots, j_{d-1}, i, i+1\}} \varphi(j, j_1, \ldots, j_{d-1}, i+1)$$

$$= -R\varphi(j_1, \ldots, j_{d-1}, i+1)$$

showing that the number $n_{R\varphi}(i, i+1)$ of sets separating $i$ from $i+1$ equals $\binom{n-2}{d-1}$.

The proof splits now into four cases according to the parities of $n$ and $d$.

If $n \equiv d \equiv 0 \pmod{2}$, then $R\varphi$ is symmetric and $\binom{n-2}{d-1}$ is even (recall that

$$\left(\sum_{i=0}^{d-1} \nu_i 2^i \right) \equiv \prod_{i=0}^{d-1} \left(\nu_i \kappa_i \right) \pmod{2}$$

for $\nu_i, \kappa_i \in \{0, 1\}$, cf. for instance Exercise 5.36 in Chapter 5 of [3]). Since $n_{R\varphi}(i, i+1) = \binom{n-2}{d-1}$ is even for all $i < n$, the Orchard relation $\sim_{R\varphi}$ associated to the symmetric function $R\varphi$ is trivial.

If $n \equiv 1 \pmod{2}$, $d \equiv 0 \pmod{2}$, then $R\varphi$ is antisymmetric. We have then $\binom{n-3}{d-1} \equiv 0 \pmod{2}$ and thus $\binom{n-3}{d-2} \equiv \binom{n-3}{d-2} + \binom{n-3}{d-1} \equiv \binom{n-2}{d-1} \pmod{2}$ which implies again the triviality of the Orchard relation $\sim_{R\varphi}$ since we have $n_{R\varphi}(i, i+1) = \binom{n-2}{d-1} \equiv \binom{n-3}{d-2} \pmod{2}$ which shows $i \sim_{R\varphi} (i+1)$ for all $i$.

If $n \equiv d \equiv 1 \pmod{2}$ then $R\varphi$ is symmetric. Since $\binom{n-3}{d-2} \equiv 0 \pmod{2}$ we have $\binom{n-2}{d-1} = \binom{n-3}{d-2} + \binom{n-3}{d-1} \equiv \binom{n-3}{d-1} \equiv \binom{n-3}{d-2} + \binom{n-2}{d-1} \pmod{2}$ proving that the Orchard relations $\sim_{\varphi}$ and $\sim_{R\varphi}$ coincide.

If $n \equiv 0 \pmod{2}$, $d \equiv 1 \pmod{2}$, then $R\varphi$ is antisymmetric. The equality $\binom{n-2}{d-1} = \binom{n-3}{d-2} + \binom{n-3}{d-1}$ implies $\binom{n-3}{d-1} \equiv \binom{n-3}{d-2} + \binom{n-2}{d-1} \pmod{2}$. This shows hat the Orchard relations $\sim_{\varphi}$ and $\sim_{R\varphi}$ coincide. \qed

4 Homology

We recall that $R\varphi : \binom{E}{d} \to \mathbb{R}$ is defined by

$$R\varphi(x_1, \ldots, x_d) = \prod_{x \in E\setminus\{x_1, \ldots, x_d\}} \varphi(x, x_1, \ldots, x_d)$$

for a given generic symmetric or antisymmetric function $\varphi : \binom{E}{d+1} \to \mathbb{R}$.

Lemma 4.1 We have

$$R(R\varphi)(x_1, \ldots, x_{d-1}) \in e^{\binom{n}{2}}_{\mathbb{R}^+}$$
where $\epsilon = 1$ if $\varphi$ is generic and symmetric and $\epsilon = -1$ if $\varphi$ is generic and antisymmetric.

**Proof.** Setting $S = \{x_1, \ldots, x_{d-1}\}$ we have

\[
R(R\varphi)(x_1, \ldots, x_{d-1}) = \prod_{y \in E \setminus S} R\varphi(y, x_1, \ldots, x_{d-1}) = \prod_{x \neq y \in E \setminus S} \varphi(x, y, x_1, \ldots, x_{d-1}) \varphi(y, x, x_1, \ldots, x_{d-1})
\]

which is positive if $\varphi$ is symmetric or if $(\#(E) - d + 1) / 2$ is even and negative otherwise.

Writing as in the beginning $[n] = \{1, \ldots, n\}$, the set $\{\pm 1\}^{[n]}$ (endowed with the usual product of functions) of all symmetric generic functions $(E^{d+1}) \to \{\pm 1\}$ is a vector space of dimension $\binom{n}{d+1}$ over the field $\mathbb{F}_2$ of 2 elements. The map $R$ considered above defines group homomorphisms between these vector spaces and the above Lemma allows to define homology groups. These groups are however all trivial except for $d = 0$ since one obtains the ordinary (simplicial) homology with coefficients in $\mathbb{F}_2$ of an $(n - 1)$ dimensional simplex.

## 5 Increasing $d$

This section is a close analogue of section 3.

Given a generic symmetric or antisymmetric function $(E^{d+1}) \to \mathbb{R}$ we define a function $A\varphi : (E^{d+2}) \to \mathbb{R}$ by setting

\[
A\varphi(x_0, \ldots, x_{d+1}) = \prod_{i=0}^{d+1} \varphi(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{d+1})
\]

The function $A\varphi$ is generic symmetric if $\varphi$ is symmetric. For $\varphi$ antisymmetric it is generic symmetric if $d \equiv 0 \pmod{2}$ and generic antisymmetric otherwise.

The dependency between the Orchard relations $\sim_\varphi$ and $\sim_{A\varphi}$ for a generic symmetric or antisymmetric function $\varphi : (E^{d+1}) \to \mathbb{R}$ is described by the following result.

**Proposition 5.1** Let $\varphi : (E^{d+1}) \to \mathbb{R}$ be a generic symmetric or antisymmetric function.

The Orchard relation $\sim_{A\varphi}$ of $A\varphi$ is trivial if $\#(E) \equiv d \pmod{2}$. Otherwise, the Orchard relations $\sim_\varphi$ and $\sim_{A\varphi}$ of $\varphi$ and $A\varphi$ coincide.

The main ingredient of the proof is the following lemma whose easy proof is left to the reader.
Lemma 5.2 Let \( \varphi, \psi : \binom{E}{d+1} \rightarrow \mathbb{R} \) be two generic symmetric or antisymmetric functions which are flip-related with respect to the set \( \mathcal{F} = \{x_0, \ldots, x_d\} \). Then

\[
A\varphi(y_0, \ldots, y_{d+1}) A\psi(y_0, \ldots, y_{d+1}) > 0
\]

if \( \mathcal{F} \not\subset \{y_0, \ldots, y_{d+1}\} \) and

\[
A\varphi(y_0, \ldots, y_{d+1}) A\psi(y_0, \ldots, y_{d+1}) < 0
\]

otherwise.

Proof of Proposition 5.1. Lemma 5.2 shows that \( \sim_{A\varphi} \) is independent of \( \varphi \) if \( \#(E) \equiv d \pmod{2} \). Otherwise, the Orchard relations of \( \varphi \) and \( A\varphi \) behave in a similar way under flips. Indeed, given \( \psi \) which is flip-related with flipset \( \mathcal{F} = \{x_0, \ldots, x_d\} \) to \( \varphi \) the functions \( A\psi \) and \( A\varphi \) are related through \( (\#(E) - (d+1)) \) flips with flipsets \( \mathcal{F} \cup \{x\}, x \in E \setminus \mathcal{F} \). Each element of \( E \setminus \mathcal{F} \) is hence flipped once and each element of \( \mathcal{F} \) is flipped \( \#(E) - (d+1) \) times.

Proposition 1.3 implies hence that \( \sim_{A\varphi} \) is independent of \( \varphi \) if \( 1 \equiv \#(E) - (d+1) \) and that \( \sim_{\varphi} \) and \( \sim_{A\varphi} \) behave similarly under flips otherwise. It is hence enough to proof Proposition 5.1 in a particular case.

If \( \varphi \) is symmetric, then Proposition 5.1 clearly holds for the constant application \( \varphi : \binom{E}{d+1} \rightarrow \{1\} \).

In the antisymmetric case we set \( E = \{1, \ldots, n\} \) and we consider the generic antisymmetric function \( \varphi : \binom{E}{d+1} \rightarrow \{-1, 1\} \) defined by

\[
\varphi(i_0, \ldots, i_d) = 1
\]

for all \( 1 \leq i_0 < i_1 < \ldots < i_d \leq n \). The function \( A\varphi : \binom{E}{d+2} \rightarrow \{-1, 1\} \) is now given by

\[
A\varphi(i_0, \ldots, i_d, i_{d+1}) = 1
\]

for all \( 1 \leq i_0 < i_1 < \ldots < i_d < i_{d+1} \leq n \). The numbers \( n_{A\varphi}(i, i+1) \) of subsets in \( \binom{E \setminus \{i, i+1\}}{d+1} \) separating \( i \) from \( i+1 \) are hence all 0 and we split the discussion into several cases according to the parities of \( n = \#(E) \) and \( d \).

- \( n \equiv d \equiv 0 \pmod{2} \) implies \( A\varphi \) symmetric and hence \( \sim_{A\varphi} \) trivial. Since then \( \binom{n-2}{d-1} \equiv 0 \pmod{2} \) we have also \( \sim_{\varphi} \) trivial.
- \( n \equiv 1 \pmod{2} \), \( d \equiv 0 \pmod{2} \) implies \( A\varphi \) symmetric and hence \( \sim_{A\varphi} \) trivial. We have then \( \binom{n-2}{d-1} \) proving equality of the two Orchard relations \( \sim_{\varphi} \) and \( \sim_{A\varphi} \).
- \( n \equiv d \equiv 1 \pmod{2} \) implies \( A\varphi \) antisymmetric and \( \binom{n-3}{d-3} \equiv 0 \pmod{2} \) thus proving triviality of the Orchard relation \( \sim_{A\varphi} \). \( \square \)
Remark 5.3 One sees easily that the function $A(A\varphi)$ is strictly positive for a generic symmetric or antisymmetric function $\varphi : (e_{d+1}) \to \mathbb{R}$.

This allows the definition of cohomology groups on the set of generic symmetric functions $(e_{d+1}) \to \{\pm 1\}$. The resulting groups are of course not interesting since this boils down once more to the cohomology groups of the $(\sharp(E) - 1)-$dimensional simplex with coefficients in the field $\mathbb{F}_2$ of 2 elements.

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