Properties of open quantum walks on $\mathbb{Z}$

Ilya Sinayskiy and Francesco Petruccione

National Institute for Theoretical Physics and School of Chemistry and Physics, University of KwaZulu-Natal, Westville, Durban 4001, South Africa
E-mail: petruccione@ukzn.ac.za

Received 2 May 2012
Accepted for publication 27 July 2012
Published 30 November 2012
Online at stacks.iop.org/PhysScr/T151/014077

Abstract
The connection between the asymptotic behavior of the open quantum walk and the spectrum of generalized quantum coins was studied. In the case of simultaneously diagonalizable transition operators, an exact expression for the probability distribution of the position of the walker for an arbitrary number of steps was found. For a large number of steps, the probability distribution consists of, maximally, two ‘soliton’-like solutions and a certain number of Gaussian distributions. The number of different contributions to the final probability distribution is equal to the number of distinct absolute values in the spectrum of the transition operators. The presence of the zeros in the spectrum is an indicator of the ‘soliton’-like solutions in the probability distribution.

PACS numbers: 03.65.Yz, 05.40.Fb, 02.50.Ga

(Some figures may appear in color only in the online journal)

1. Introduction

It is well known that the mathematical concept of classical random walks [1, 2] has found wide applications in physics [2], computer science [3], economics [4] and biology [5]. Basically, the trajectory of a random walk consists of a sequence of random steps on some underlying graph [2]. Recently, an extension of the concept of random walk to the quantum domain was performed. Quantum walks can be introduced in a discrete time [6] or in a continuous time [7] way. For a classical random walk, the probability distribution of the position of the walker is given by the transition rates between the vertices of the graph. For the quantum case [8], the probability distribution of the walker is defined not only by the transition rates between the nodes of the graph, but also by the dynamics of the internal degrees of freedom of the ‘walker’. The resulting interference effects is what makes these walks truly quantum.

Unitary quantum walks have found wide application as a tool for the formulation of quantum computing algorithms [9]. Although experimental implementation of any quantum concept is usually difficult owing to unavoidable decoherence and dissipation effects [10], the realization of unitary quantum walks has been reported. Implementations with negligible effect of decoherence and dissipation were realized in optical lattices [11], with photons in waveguide lattices [12], with trapped ions [13] and free single photons in space [14].

During the last few years, attempts have been made to include the effects of decoherence and dissipation in the description of the quantum walks [15]. Typically, in these approaches decoherence is treated as an extra modification of the unitary quantum walk scheme, the effect of which needs to be minimized and eliminated.

Recently, a formalism for discrete time open quantum walks (OQWs), which is exclusively based on the non-unitary dynamics induced by the environment, was introduced [16]. The formalism suggested is similar to the formalism of quantum Markov chains [17] and rests upon the implementation of appropriate completely positive maps [10, 18]. It was shown that the formalism of the OQWs includes the classical random walk, and through a physical realization procedure a connection to the unitary quantum walk was established.

OQWs show a rich dynamical behavior [16]. The aim of this paper is to study in detail the dynamics of the probability distribution for the position of the OQW. OQWs were shown to have a probability distribution which is represented by a sum of Gaussian distributions and singular ‘soliton’-like distributions. In particular, we will show that there exists a connection between the number of the Gaussian distributions in the probability distribution and the dimension of the internal degree of freedom of the walker. Furthermore, we will formulate a condition for the appearance of the ‘soliton’-like solution in the dynamics of the walker.
The paper is structured as follows. In section 2, we briefly revise formalism of OQWs. In section 3, we study the connection between asymptotic properties of the walk and the structure of the transition operators. In section 4 we present the conclusions.

2. Open quantum walks

We study a random walk on a set of vertices $\mathcal{V}$ with oriented edges $\{(i,j); i,j \in \mathcal{V}\}$. The number of nodes is considered to be finite or countably infinite. The space of states corresponding to the dynamics on the graph specified by the set of nodes $\mathcal{V}$ will be denoted by $\mathcal{K} = \mathbb{C}^\mathcal{V}$. If $\mathcal{V}$ is an infinite countable set, the space of states $\mathcal{K}$ will be any separable Hilbert space with an orthonormal basis $\{|i\rangle\}_{i \in \mathcal{V}}$ indexed by $\mathcal{V}$. The internal degrees of freedom of the quantum walker, e.g. the spin, angular momenta or $n$-energy levels, will be described by a separable Hilbert space $\mathcal{H}$ attached to each vertex of the graph. So, any state of the quantum walker will be described on the direct product of the Hilbert spaces $\mathcal{H} \otimes \mathcal{K}$.

To describe the dynamics of the quantum walker for each edge $(i,j)$, we introduce a bounded operator $B^i_j \in \mathcal{H}$. This operator describes the change in the internal degree of freedom of the walker due to the “jump” from node $j$ to node $i$ (see figure 1). By imposing for each $j$ that

$$\sum_i B^i_j \dagger B^i_j = I, \quad (1)$$

we make sure that for each node of the graph $j \in \mathcal{V}$ there is a corresponding completely positive map on the positive operators of $\mathcal{H}$:

$$\mathcal{M}^j(\tau) = \sum_i B^i_j \tau B^i_j \dagger. \quad (2)$$

The operators $B^i_j$ act only on $\mathcal{H}$ and do not perform transitions from node $j$ to node $i$; they can be extended to operators $M^i_j \in \mathcal{H} \otimes \mathcal{K}$ acting on total Hilbert space in the following way:

$$M^i_j = B^i_j \otimes |i\rangle \langle j|. \quad (3)$$

It is clear that, if the condition expressed in equation (1) is satisfied, then $\sum_j M^i_j \dagger M^i_j = 1$. This condition defines a completely positive map for density matrices on $\mathcal{H} \otimes \mathcal{K}$, i.e.

$$\mathcal{M}(\rho) = \sum_i \sum_j M^i_j \rho M^i_j \dagger. \quad (4)$$

The above map defines the discrete time OQW. It is easy to see that for an arbitrary initial state the density matrix $\sum_{i,j} \rho_{i,j} \otimes |i\rangle \langle j|$ will take a diagonal form after just one step of the open quantum random walk equation (4). Hence, in the following, we will assume that the initial state of the system has the form

$$\rho = \sum_i \rho_i \otimes |i\rangle \langle i|, \quad (5)$$

with

$$\sum_j \text{Tr}(\rho_j) = 1. \quad (6)$$

It is straightforward to give an explicit iteration formula for the OQW from step $n$ to step $n+1$.

$$\rho^{[n+1]} = \mathcal{M}(\rho^{[n]}) = \sum_i \rho_i^{[n] + 1} \otimes |i\rangle \langle i|, \quad (7)$$

where

$$\rho_i^{[n+1]} = \sum_j B^i_j \rho_j^{[n]} B^i_j \dagger. \quad (8)$$

The above iteration formula gives a clear physical meaning to the mapping that we introduced: the state of the system on site $i$ is determined by the conditional shift from all connected sites $j$, which are defined by the explicit form of the generalized quantum coin operators $B^i_j$. Also, one can see that $\text{Tr}(\rho^{[n+1]}) = \sum_i \text{Tr}(\rho^{[n]}) = 1$.

3. Properties of transition operators

OQWs, even on the line, show rich dynamical behavior [16]. In this paper we concentrate on the particular case of OQWs on $\mathbb{Z}$ with transition between neighboring nodes (see figure 2(a)). In this case, the general expression for the OQW on the graph reads as

$$\rho^{[n+1]} = \mathcal{M}(\rho^{[n]}) = \sum_i \rho_i^{[n+1]} \otimes |i\rangle \langle i|, \quad (9)$$

where

$$\rho_i^{[n+1]} = B^i_{-1} \rho_i^{[n]} B^i_{-1} \dagger + B^i_{+1} \rho_i^{[n]} B^i_{+1} \dagger. \quad (10)$$

In this paper we will analyze a homogeneous OQW (which implies that $\forall i, B^i_{+1} = B$ and $B^i_{-1} = C$) with simultaneously diagonalizable generalized coin operators, i.e. $[B, C] = 0$.

As the first example of the open walk on $\mathbb{Z}$, let us consider an OQW with a two-dimensional Hilbert space for the generalized quantum coins, i.e. $B, C \in \mathbb{C}^2$. Specifically, we consider the transition operators $B$ and $C$ defined as

$$B = \begin{pmatrix} 1 & 0 \\ 0 & \cos \theta \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 0 & \sin \theta \end{pmatrix}. \quad (11)$$
It is clear that the above $B$ and $C$ satisfy the normalization condition, i.e. $B \dagger B + C \dagger C = I$. If, initially, the walker is in the node 0,

$$\rho_0 = \left( \begin{array}{c|c} p & z \\ \hline z^* & q \end{array} \right) \otimes |0\rangle \langle 0|,$$  \hspace{1cm} (12)

then the probability distribution of the position of the walker after $n$ steps has the following form:

$$P_k^{[n]} = \text{Tr}([k] \rho^{[n]} |k\rangle \langle k|) = p \delta_{n,k} + q \left( \frac{n}{2} \right) \frac{1}{2} \frac{1}{n!} \frac{1}{2}^n \frac{1}{n!} \cos^2 \frac{\theta}{2} (\sin \theta)^{n-k} (\cos \theta)^{n+k}.$$

where the non-zero values of the probability distribution correspond to $|k| \leq n$ and $n \pm k$ should be even. The last condition means that after an odd number of steps only odd nodes can be populated and after an even number of steps only even ones.

Two distinct behaviors can be identified. The first is a trapped state which propagates in the positive direction and the second is a binomially distributed part, which for a large number of steps becomes Gaussian. One can calculate the mean and variance of the binomial distribution,

$$\mathbb{E}(x_n) = \cos (2 \theta) n, \quad \text{Var}(x_n) = \sin (2 \theta) \sqrt{n},$$

where $x_n$ denotes the binomial distribution part of the position of the walker after $n$ steps.

We consider now the more general case of an OQW with an $n$-dimensional Hilbert space for the transition operators. Again, we require the generalized quantum coin operators to be simultaneously diagonalizable. This implies that the generalized coins $B$ and $C$ can be written in the form

$$B = \sum_{i=1}^{n} \lambda_i |b_i\rangle \langle b_i|, \quad C = \sum_{i=1}^{n} \phi_i |b_i\rangle \langle b_i|.$$

The normalization condition $B \dagger B + C \dagger C = I$ implies that $\forall i, |\lambda_i|^2 + |\phi_i|^2 = 1$. In this case the probability distribution of the position of the walker after $n$ steps has the form

$$P_k^{[n]} = \text{Tr}([k] \rho^{[n]} |k\rangle \langle k|) = \sum_{i=1}^{n} p_i \left( \frac{n}{2 \pi k} \right)^{(n-k)/2} (\lambda_i^2)^{(n+k)/2}.$$

where the coefficients $p_i$ are the populations of the corresponding levels of the initial state in the basis $|b_i\rangle$, i.e.

$$p_i = |\langle b_i, 0|\rho_0|b_i, 0\rangle|^2.$$  \hspace{1cm} (16)

The mean and variance of each binomial distribution can be calculated explicitly,

$$\mathbb{E}_n(x_n) = n \left( |\lambda_i|^2 - |\phi_i|^2 \right),$$

$$\text{Var}_n(x_n) = 2 |\lambda_i||\phi_i|\sqrt{n}.$$  \hspace{1cm} (18)

The explicit knowledge of the probability distribution allows us to describe the asymptotic behavior of the OQW on $\mathbb{Z}$. It is clear that for the generalized quantum coins $B$ and $C$ that are simultaneously diagonalizable, there are two cases. Firstly, if all coefficients $\lambda_i$ are different and have value between 0 and 1, i.e.

$$\forall i \neq j, |\lambda_i| \neq |\lambda_j|, \quad 0 < |\lambda_i| < 1,$$

then for a large number of steps the probability distribution consists of $n$ Gaussian distributions, where $n$ is the dimension of the Hilbert space of the generalized quantum coins. If some of the eigenvalues of the generalized quantum coins are degenerate, then the number of Gaussians in the distributions is given by the number of distinct eigenvalues of the operators $B$ and $C$. Secondly, if one or more of eigenvalues $|\lambda_i|$ are equal to 0 or 1, then for a large number of steps the probability distribution contains ‘soliton’-like distributions in the probability distribution. If one of the $\lambda_i \equiv 0$ (which means that the corresponding $|\phi_i \rangle \equiv 1$), then the probability distribution contains a ‘soliton’-like solution propagating ballistically in the negative direction; if one of the $|\lambda_i \rangle = 1$, then a ‘soliton’-like solution propagates in the positive direction (see figures 2(b)–(d)).

It is clear that in the case considered here $[B, C] = 0$ the probability distribution can have $m$ components, i.e. Gaussians and solitons, where $m$ is the number of distinct absolute values of eigenvalues of transition operators $B$ and $C$. This includes, maximally, two ‘soliton’-like solutions propagating in opposite directions.
4. Conclusions

In conclusion, we have given a brief review of the OQW formalism and studied the connection between the asymptotic behavior of the OQW and the spectrum of the generalized quantum coins. In particular, we have considered the case of the OQW on the line with simultaneously diagonalizable transition operators $B$ and $C$. We found exact expressions for the probability distribution of the position of the walker for an arbitrary number of steps. We have shown that for a large number of steps the probability distribution consists of, maximally, two ‘soliton’-like solutions and a certain number of Gaussian distributions. We also provided an explicit expression for the mean and variance for each of the Gaussians as a function of the spectrum of the generalized quantum coin operators. We found that a number of different contributions to the final probability distribution is equal to the number of distinct absolute values in the spectrum of the transition operators. We have shown that the presence of zeros in the spectrum is an indicator of the ‘soliton’-like solutions in probability distribution. In future work, we plan to extend the present analysis to the case of the generic transition operators.

Acknowledgment

This work is based upon research supported by the South African Research Chair Initiative of the Department of Science and Technology and the National Research Foundation.

References

[1] Feller W 1968 An Introduction to Probability Theory and its Applications vol 1 (New York: Wiley)
[2] Barber M and Ninham B W 1970 Random and Restricted Walks: Theory and Applications (New York: Gordon and Breach)
[3] Motwani R and Raghavan P 1995 Randomized Algorithms (Cambridge: Cambridge University Press)
[4] Malkiel B 1973 A Random Walk Down Wall Street (New York: W W Norton)
[5] Berg H C 1993 Random Walks in Biology (Princeton, NJ: Princeton University Press)
[6] Aharonov Y, Davidovich L and Zagury N 1993 Phys. Rev. A 48 1687
[7] Farhi E and Gutmann S 1998 Phys. Rev. A 58 915
[8] Kempe J 2003 Contemp. Phys. 44 307
[9] Aharonov D, Ambainis A, Kempe J and Vazirani U 2001 Proc. 33rd ACM Symp. on Theory of Computing pp 50–97
[10] Venegas-Andraca S 2008 Quantum Walks for Computer Scientists (San Rafael, CA: Morgan and Claypool)
[11] Konno N 2008 Quantum walks Quantum Potential Theory (Lecture Notes in Mathematics vol 1954) (New York: Springer) p 309
[12] Ambainis D 2008 Lect. Notes Comput. Sci. 4910 1
[13] Childs A M, Farhi E and Gutmann S 2002 Quantum Inform. Process. 1 (1–2) 35
[14] Watrous J 2003 J. Comput. Syst. Sci. 62 376
[15] Childs A M, Cleve R, Deotto E, Farhi E, Gutmann S and Spielman D A 2003 Proc. 35th ACM Symp. on Theory of Computing pp 59–68
[16] Shenvi N, Kempe J and Whaley K B 2003 Phys. Rev. A 67 052307
[17] Farhi E, Goldstone J and Gutmann S 2008 Theory Comput. 4 169
[18] Breuer H-P and Petruccione F 2002 The Theory of Open Quantum Systems (Oxford: Oxford University Press)
[19] Karski M, Forster L, Choi J-M, Steffen A, Alt W, Meschede D and Widera A 2009 Science 325 174
[20] Perets H B, Lahini Y, Pozzi F, Sorel M, Morandotti R and Silberberg Y 2008 Phys. Rev. Lett. 100 170506
[21] Schmitz H, Matjeschik R, Schneider Ch, Glueckert J, Enderlein M, Huber T and Schaeetz T 2009 Phys. Rev. Lett. 103 090504
[22] Zähringer F, Kirchmair G, Gerritsma R, Solano E, Blatt R and Roos C F 2010 Phys. Rev. Lett. 104 100503
[23] Broome M A, Fedrizzi A, Lanyon B P, Kassal I, Aspuru-Guzik A and White A G 2010 Phys. Rev. Lett. 104 153602
[24] Kendon V 2007 Math. Struct. Comput. Sci. 17 1169
[25] Brun T A, Carteret H A and Ambainis A 2003 Phys. Rev. Lett. 91 150602
[26] Romanelli A, Siri R, Ahab G, Auyuanet A and Donangelo R 2005 Physica A 347 137
[27] Love P and Boghosian B 2005 Quantum Inform. Process. 4 335
[28] Srikanth R, Banerjee S and Chandrashekar C M 2010 Phys. Rev. A 81 062123
[29] Attal S, Petruccione F and Sinayskiy I 2012 Phys. Lett. A 376 1545–48
[30] Attal S, Petruccione F, Sabot C and Sinayskiy I 2012 J. Stat. Phys. 147 832
[31] Gudder S 2010 Found. Phys. 40 1566
[32] Gudder S 2008 J. Math. Phys. 49 072105
[33] Kraus K 1983 States, Effects and Operations: Fundamental Notions of Quantum Theory (Berlin: Springer)
[34] Alcide R and Lendi K 1987 Quantum Dynamical Semigroups and Applications (Berlin: Springer)