SOME CLASS OF PARABOLIC SYSTEMS APPLIED TO IMAGE PROCESSING

LEKBIR AFRAITES
Faculté des Sciences et Techniques, Université Sultan Moulay Slimane
B.P. 523 Beni-Mellal, Morocco

ABDELGHAFOUR ATLAS
Ecole Nationale des Sciences Appliquées de Safi, Université Cadi Ayyad
Route Sidi Bouzid B.P. 63, Safi, Morocco

FAHD KARAMI AND DRISS MESKINE
Ecole Supérieure de Technologie d’Essaouira, Université Cadi Ayyad
B.P. 383 Essaouira El Jadida, Essaouira, Morocco

(Communicated by Doron Levy)

Abstract. In this paper, we are interested in the mathematical and numerical study of a variational model derived as Reaction-Diffusion System for image denoising. We use a nonlinear regularization of total variation (TV) operator’s, combined with a decomposition approach of $H^{-1}$ norm suggested by Guo and al. ([19],[20]). Based on Galerkin’s method, we prove the existence and uniqueness of the solution on Orlicz space for the proposed model. At last, compared experimental results distinctly demonstrate the superiority of our model, in term of removing noise while preserving the edges and reducing staircase effect.

1. Introduction. In recent years, many mathematicians have been attracted by image processing and computer vision. Several field of application of the image processing, we quote that the fundamental problem in image processing is the image restoration. Their methods has grown with the massive production of digital images and movies, often taken in poor conditions, which gives a noisy images. Image denoising refers to the process of the recovering an image contaminated by the noise. The common starting point is an image which is a collection of information about $N$ pixels. Generally, noised images can be modeled as

$$f(i) = u(i) + n(i) \quad i = 1, ..., N,$$

where $f(i)$ represents the $i^{th}$ pixel of the observed noisy image, $u(i)$ is the true image and $n(i)$ the noise which often considered the stationary Gaussian with zero mean and variance $\sigma^2$.

2010 Mathematics Subject Classification. 35K57, 94A08, 46E30.

Key words and phrases. Image denoising, reaction-diffusion system, Orlicz space, Galerkin method, staircase effect.

This work was supported in part by the PPR CNRST : Modèles Mathématiques appliqués à l’environnement, à l’imagerie médicale et aux Biosystèmes and by Hassan II Academy of Sciences and Technology (Morocco), project “Méthodes mathématiques et outils de modélisation et simulation pour le cancer”.

* Corresponding author: Abdelghafour Atlas.
The challenge of image denoising is to remove the noise while maintaining and recovering the features and details of image as much as possible.

Over decades, the nonlinear diffusion and partial differential equation based variational models have become popular and useful tools for image processing (see [3], [5], [9], [29], [30] and references mentioned therein). Generally, the restored (denoised) image \( u : \Omega \rightarrow \mathbb{R} \) is computed from the following minimization problem:

\[
\min_u F(u) = \int_{\Omega} \Psi(x, |\nabla u|) dx + \lambda ||f - u||_X^2 \tag{1}
\]

where \( \Omega \subset \mathbb{R}^2 \) is an open and bounded domain, \( X \) is a Banach space, \( \Psi : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R} \) is a given function, \( \lambda > 0 \) is a weight parameter, \( ||f - u||_X^2 \) is the fidelity term and \( \int_{\Omega} \Psi(x, |\nabla u|) dx \) is a regularizing term to remove the noise. Notice that a different choice of \( \Psi \) and \( X \) correspond to different models. In the case where \( X = L^2(\Omega) \), the problem (1) is formally equivalent to the following nonlinear equation

\[
u - \frac{1}{2\lambda} \text{div} \left( \Phi(x, |\nabla u|) \frac{\nabla u}{|\nabla u|} \right) = f \tag{2}
\]

where for a.e. \( x \in \Omega \), \( \Psi(x, s) = \int_0^s \Phi(x, t) dt \).

Perona and Malik (case \( \Phi(x, s) = \frac{s}{1 + s^2} \) a.e. in \( \Omega \)) have presented the first nonlinear partial differential equations model for denoising in a class of models commonly referred to as anisotropic diffusion. By choosing \( \Phi(x, s) = 1 \) a.e. in \( \Omega \), Rudin et al. (ROF) in [30] have proposed an alternative model called total variation model, that can protect the details of image better in the course of denoising. When the parameter \( \lambda \) is too small, the smaller details of image are destroyed. That is why, Meyer [27] proposed a new minimization problem, changing in (1) the \( L^2 \) norm of \( (f - u) \) by a weaker norm, more appropriate to present textured or oscillatory patterns, but, the drawback of this model, we cannot express directly the associated Euler-Lagrange equation with respect to \( u \). To overcome this problem, Osher et al. in [28], has proposed an alternative practical approach that changing this weaker norm by \( H^{-1} \). Since the \( H^{-1} \) norm is defined as \( ||.|||_{H^{-1}}^2 = \int_\Omega |\nabla \Delta^{-1}(|.|)|^2 \), the minimization problem (1) in this case is formally associated to Euler-Lagrange equation

\[
\begin{aligned}
\text{div} \left( \frac{\nabla u}{|\nabla u|} \right) &= 2\lambda \Delta^{-1}(f - u) \quad \text{in } \Omega \\
\nabla u \cdot \vec{n} &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\tag{3}
\]

We recall that the Total variation model, is not well defined at locations where \( |\nabla u| = 0 \) due to the presence of the term \( \frac{1}{|\nabla u|} \) and it is known that the solution of this problem is defined in the non standard variational sense (see [4] and [13]). Although the numerical simulations were done by regularized \( |\nabla u| \) with \( \varepsilon + |\nabla u| \).

In current work, we propose a reaction diffusion system, by using a nonlinear regularization of total variation model combined with a decomposition suggested
Some class of parabolic systems applied to image processing 1673
by Guo et al. ([19],[20]), which is written as:

\[
\begin{align*}
\frac{\partial u}{\partial t} - \text{div} \left( \Phi_{\gamma}(x,|\nabla u|) \frac{\nabla u}{|\nabla u|} \right) + 2\lambda w &= 0 \quad \text{in } Q := (0,T) \times \Omega \\
\frac{\partial w}{\partial t} &= \Delta w - (f - u) \quad \text{in } Q := (0,T) \times \Omega \\
\Phi_{\gamma}(x,|\nabla u|) \frac{\nabla u}{|\nabla u|} \cdot \vec{n} &= \nabla w \cdot \vec{n} = 0 \quad \text{in } \Sigma := (0,T) \times \partial \Omega \\
u(0,x) &= f, \quad w(0,x) = 0 \quad \text{in } \Omega,
\end{align*}
\]

where \( \Phi_{\gamma} \) is defined by

\[
\Phi_{\gamma}(x,s) = \begin{cases} 
B(x,s) & \text{if } s \leq \varepsilon \\
\frac{B(x,\varepsilon) \log^\gamma(1+s)}{\varepsilon \log^\gamma(1+\varepsilon)} & \text{if } s > \varepsilon
\end{cases} \quad \text{a.e. } x \in \Omega. \tag{5}
\]

The parameters \( \varepsilon, \lambda \) and \( \gamma \) are nonnegative and

\[
t \to B(x,t) \in [0, +\infty[ \quad \text{is an increasing convex function satisfying}
\]

\[
0 < \mu \leq B(x,\varepsilon) \in L^\infty(\Omega) \quad \text{and } \lim_{t \to 0} \frac{B(x,t)}{t} = 0 \quad \text{for a.e. } x \in \Omega. \tag{6}
\]

A typical example of such \( B \) are \( B(x,t) = t^2 \exp(t) \) or \( t^\alpha(x) \log^\gamma(1+t) \) for a.e. \( x \in \Omega \), where \( \alpha \) is a given continuous function. When \( \gamma \) is close to 0, we have

\[
\Phi_{\gamma}(x,|\nabla u|) \frac{\nabla u}{|\nabla u|} \sim \frac{\nabla u}{|\nabla u|} \quad \text{in } |\nabla u| > \varepsilon,
\]

and when \( |\nabla u| \leq \varepsilon \), \( \Phi_{\gamma}(x,|\nabla u|) \frac{\nabla u}{|\nabla u|} \) behaves like \( \exp(|\nabla u|) \nabla u \) or \( \nabla u^{\alpha(x)-2} \nabla u \).

That is, at edges where \( |\nabla u| \) is large, the proposed operators can be seen as a regularization of TV operator. Our model is well posed in a weak sense and numerical simulations substantially reduces the staircase effect.

For the mathematical analysis, it is reasonable to study the solutions of our problem in the Orlicz space. For this, we give some basic definitions and preliminaries needed to state the results in the next section. In Section 3, we show the existence and the uniqueness result of solutions for the proposed model (4). Finally, section 4 is devoted to numerical results and comments to improve our model.

2. Preliminaries. Let \( M : \mathbb{R}^+ \to \mathbb{R}^+ \) be an N-function, i.e. \( M \) is continuous, convex, with \( M(t) > 0 \) for \( t > 0 \), \( \frac{M(t)}{t} \to 0 \) as \( t \to 0 \) and \( \frac{M(t)}{t} \to \infty \) as \( t \to \infty \).

Equivalently, \( M \) admits the representation: \( M(t) = \int_0^t a(\tau)d\tau \) where \( a : \mathbb{R}^+ \to \mathbb{R}^+ \) is non-decreasing, right continuous, with \( a(0) = 0 \), \( a(t) > 0 \) for \( t > 0 \) and \( a(t) \to \infty \) as \( t \to \infty \). The N-function \( \overline{M} \) conjugate to \( M \) is defined by \( \overline{M}(t) = \int_0^t \overline{a}(\tau)d\tau \), where \( \overline{a} : \mathbb{R}^+ \to \mathbb{R}^+ \) is given by \( \overline{a}(t) = \sup\{s : a(s) \leq t\} \) (see [2], [23] and [24]).
The N-function $M$ is said to satisfy the $\Delta_2$ condition if, for some $k > 0$:

$$M(2t) \leq k M(t) \quad \text{for all } t \geq 0,$$

(7)

when this inequality holds only for $t \geq t_0 > 0$, $M$ is said to satisfy the $\Delta_2$ condition near infinity.

Let $\Omega$ be an open subset of $\mathbb{R}^N$. The Orlicz class $L_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real-valued measurable functions $u$ on $\Omega$ such that:

$$\int_{\Omega} M(u(x)) \, dx < +\infty \quad (\text{resp. } \int_{\Omega} M(u(x)) \, dx < +\infty \text{ for some } \lambda > 0).$$

Note that $L_M(\Omega)$ is a Banach space under the norm

$$\|u\|_{M, \Omega} = \inf \{ \lambda > 0 : \int_{\Omega} M\left(\frac{u(x)}{\lambda}\right) \, dx \leq 1 \}$$

and $L_M(\Omega)$ is a convex subset of $L_M(\Omega)$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\Omega$ is denoted by $E_M(\Omega)$. The equality $E_M(\Omega) = L_M(\Omega)$ holds if and only if $M$ satisfies the $\Delta_2$ condition, for all $t$ or for $t$ large according to whether $\Omega$ has infinite measure or not.

The dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing

$$\int_{\Omega} u(x)v(x) \, dx,$$

and the dual norm on $L_{\overline{M}}(\Omega)$ is equivalent to $\|\cdot\|_{\overline{M}, \Omega}$. The space $L_M(\Omega)$ is reflexive if and only if $M$ and $\overline{M}$ satisfy the $\Delta_2$ condition, for all $t$ or for $t$ large, according to whether $\Omega$ has infinite measure or not.

We now focus on the Orlicz-Sobolev space. $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of all functions $u$ such that $u$ and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp. $E_M(\Omega)$). This is a Banach space under the norm

$$\|u\|_{1,M, \Omega} = \sum_{|\alpha| \leq 1} \|D^\alpha u\|_{M, \Omega}.$$

Thus $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of $N + 1$ copies of $L_M(\Omega)$. Denoting this product by $\Pi L_M$, we will use the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. The space $W^1_0L_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $D(\Omega)$ in $W^1L_M(\Omega)$ and the space $W^1_0E_M(\Omega)$ as the $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closure of $D(\Omega)$ in $W^1L_M(\Omega)$. Let $W^{-1}L_{\overline{M}}(\Omega)$ (resp. $W^{-1}E_{\overline{M}}(\Omega)$) denote the space of distributions on $\Omega$ which can be written as sums of derivatives of order $\leq 1$ of functions in $L_{\overline{M}}(\Omega)$ (resp. $E_{\overline{M}}(\Omega)$). It is a Banach space under the usual quotient norm.

If the open set $\Omega$ has the segment property, then the space $D(\Omega)$ is dense in $W^1_0L_M(\Omega)$ for the modular convergence and for the topology $\sigma(\Pi L_M, \Pi L_{\overline{M}})$ (cf. [18], [17]). Consequently, the action of a distribution in $W^{-1}L_{\overline{M}}(\Omega)$ on an element of $W^1_0L_M(\Omega)$ is well defined.

Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$, $T > 0$ and set $Q = \Omega \times [0, T]$. Let $m \geq 1$ be an integer and let $M$ be an N-function. For each $\alpha \in \mathbb{N}^N$, denote by $D^\alpha_x$ the distributional derivative on $Q$ of order $\alpha$ with respect to the variable $x \in \mathbb{R}^N$. The inhomogeneous Orlicz-Sobolev spaces are defined as follows

$$W^{m,x}L_M(Q) = \{ u \in L_M(Q) : D^\alpha_x u \in L_M(Q) \forall |\alpha| \leq m \}$$

and

$$W^{m,x}E_M(Q) = \{ u \in E_M(Q) : D^\alpha_x u \in E_M(Q) \forall |\alpha| \leq m \}.$$
The last space is a subspace of the first one, and both are Banach spaces under the norm
\[ \|u\| = \sum_{|\alpha| \leq m} \|D^\alpha_x u\|_{M,Q}. \]
We can easily prove that they form a complementary system when \( \Omega \) satisfies
the segment property. These spaces are considered as subspaces of the product
space \( \Pi L_M(Q) \) which have as many copies as there is \( \alpha \)-order derivatives, \(|\alpha| \leq m\).
We shall also consider the weak topologies \( \sigma(\Pi L_M, \Pi E_{\overline{M}}) \) and \( \sigma(\Pi L_M, \Pi L_{\overline{M}}) \).
If \( u \in W^{m,x}L_M(Q) \) then the function : \( t \mapsto u(t) = u(t,.) \) is defined on \([0,T]\) with
values in \( W^{m}L_M(\Omega) \). If, further, \( u \in W^{m,x}E_M(Q) \) then the concerned function is a
\( W^mE_M(\Omega) \)-valued and is strongly measurable. Furthermore the following imbedding
holds: \( W^{m,x}E_M(Q) \subset L^1(0,T;W^{m}E_M(\Omega)) \). The space \( W^{m,x}L_M(Q) \) is not in
general separable, if \( u \in W^{m,x}L_M(Q) \), we can not conclude that the function
\( u(t) \) is measurable on \([0,T]\). However, the scalar function \( t \mapsto \|u(t)\|_{M,\Omega} \) is in
\( L^1(0,T) \). The space \( W^{m,x}_0E_M(Q) \) is defined as the (norm) closure in \( W^{m,x}E_M(Q) \)
of \( D(Q) \). We can easily show as in [17] that when \( \Omega \) has the segment property then
each element \( u \) of the closure of \( D(Q) \) with respect of the weak * topology
\( \sigma(\Pi L_M, \Pi E_{\overline{M}}) \) is limit, in \( W^{m,x}L_M(Q) \), of some subsequence \( (u_i) \subset D(Q) \) for the
modular convergence; i.e., there exists \( \lambda > 0 \) such that for all \(|\alpha| \leq m\),
\[ \int_Q M\left( \frac{D^2 u_i - D^2 u}{\lambda^2} \right) dx \, dt \to 0 \quad \text{as} \quad i \to \infty, \]
this implies that \((u_i)\) converges to \( u \) in \( W^{m,x}L_M(Q) \) for the weak topology \( \sigma(\Pi L_M, \Pi E_{\overline{M}}) \). Consequently
\[ D(Q)^{\sigma(\Pi L_M, \Pi E_{\overline{M}})} = D(Q)^{\sigma(\Pi L_M, \Pi L_{\overline{M}})}, \]
this space will be denoted by \( W^{m,x}_0L_M(Q) \). Furthermore, \( W^{m,x}_0E_M(Q) = W^{m,x}_0 \]
\( L_M(Q) \cap \Pi E_M \). Poincaré’s inequality also holds in \( W^{m,x}_0L_M(Q) \) i.e. there is a
constant \( C > 0 \) such that for all \( u \in W^{m,x}_0L_M(Q) \) one has
\[ \sum_{|\alpha| \leq m} \|D^\alpha_x u\|_{M,Q} \leq C \sum_{|\alpha| = m} \|D^\alpha_x u\|_{M,Q}. \]
Thus both sides of the last inequality are equivalent norms on \( W^{m,x}_0L_M(Q) \).
We have then the following complementary system
\[ \begin{pmatrix} W^{m,x}_0L_M(Q) & F \\ W^{m,x}_0E_M(Q) & F_0 \end{pmatrix} \]
\( F \) being the dual space of \( W^{m,x}_0E_M(Q) \). It is also, except for an isomorphism,
the quotient of \( \Pi L_{\overline{M}} \) by the polar set \( W^{m,x}_0E_M(Q)^\perp \), and will be denoted by \( F = W^{-m,x}L_{\overline{M}}(Q) \) and it is shown that
\[ W^{-m,x}L_{\overline{M}}(Q) = \{ f = \sum_{|\alpha| \leq m} D^\alpha_x f_\alpha : f_\alpha \in L_{\overline{M}}(Q) \}. \]
This space will be equipped with the usual quotient norm
\[ \|f\| = \inf \sum_{|\alpha| \leq m} \|f_\alpha\|_{\overline{M},Q} \]
where the infimum is taken on all possible decomposition
\[ f = \sum_{|\alpha| \leq m} D^\alpha_x f_\alpha, \quad f_\alpha \in L_{\overline{M}}(Q). \]
The space $F_0$ is then given by

$$F_0 = \left\{ f = \sum_{|\alpha| \leq m} D_\alpha^m f_\alpha : f_\alpha \in E_{\mathcal{MF}}(Q) \right\}$$

and is denoted by $F_0 = W^{-m,x}E_{\mathcal{MF}}(Q)$.

To prove our existence Theorem, we need the following corollary introduced in [1].

**Corollary 1.** Let $M$ be an $N$–function and $(u_n)$ be a sequence of $W^{1,x}L_M(Q)$ such that $u_n \rightharpoonup u$ weakly in $W^{1,x}L_M(Q)$ for $\sigma(\prod L_M, \prod E_{\mathcal{MF}})$ and $\frac{\partial u_n}{\partial t}$ is bounded in $W^{-1,x}L_{M'}(Q) + M(Q)$, where $M(Q)$ is the space defined on $Q$. Then $u_n \rightarrow u$ strongly in $L^1_{\text{loc}}(Q)$.

3. **Existence.** For simplicity, we use the function $a_\gamma : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ defined by

$$a_\gamma(x, \xi) = \frac{\Phi_\gamma(x, |\xi|)\xi}{|\xi|} \text{ for any } \xi \in \mathbb{R}^N.$$  

Thanks to (5) and (6), we see that $a_\gamma$ is a Caratheodory function (i.e is measurable in $x \in \Omega$ for all $\xi \in \mathbb{R}^N$ and continuous in $\xi \in \mathbb{R}^N$ for a.e. $x \in \Omega$) with $a_\gamma(., 0) = 0$.

Now, we prove the following main result:

**Theorem 3.1.** Let $u_0 \in L^2(\Omega)$, $f \in L^2(Q)$ be given and $A(t) = t \log^\gamma(1 + t)$. Then for every $\lambda > 0$, there exists a unique solution

$$(u, w) \in W^{1,x}L_A(Q) \times L^2\left(0, T; H^1(\Omega)\right) \cap \left[ L^\infty\left(0, T; L^2(\Omega)\right) \right]^2$$

such that

$$-\left\langle \left( \frac{\partial \phi}{\partial t}, u \right) \right\rangle_Q + \int_0^T \int_\Omega a_\gamma(x, \nabla u)\nabla \phi dx dt + \int_\Omega u\phi dx \bigg|_0^\tau + 2\lambda \int_0^T \int_\Omega w\phi dx dt = 0,$$

and

$$\left\langle \left( \frac{\partial \phi}{\partial t}, w \right) \right\rangle_Q + \int_0^T \int_\Omega \nabla w\nabla \phi dx dt + \int_\Omega w\phi dx \bigg|_0^\tau = \int_0^T \int_\Omega (f - u)\phi dx dt,$$

for every $\tau \in (0, T]$ and every test-functions

$$\phi \in W^{1,x}L_A(Q) \cap L^2(Q), \quad \frac{\partial \phi}{\partial t} \in W^{-1,x}L_{\mathcal{MF}}(Q) + L^2(Q)$$

and

$$\varphi \in L^2\left(0, T; H^1(\Omega)\right), \quad \frac{\partial \varphi}{\partial t} \in L^2\left(0, T; (H^1(\Omega))^\prime\right).$$

Where $<.,.>$ resp. $<<.,.>>$ denotes the duality bracket between $L^2\left(0, T; (H^1(\Omega)^\prime\right)$ and $L^2\left(0, T; (H^1(\Omega))\right)$. (resp. between $W^{1,x}L_A(Q) \cap L^2(Q)$ and $W^{-1,x}L_{\mathcal{MF}}(Q) + L^2(Q)$).

**Proof of Theorem 3.1.** To prove the existence of a weak solution, we shall use the nonlinear Galerkin method. The idea is to solve the problem in a finite-dimensional space firstly and we are looking estimates that allows us to pass to the limit. We decompose the proof of Theorem 3.1 into four parts: first, we write the approximate
solution, then we give a priori estimates, after we pass to the limit and finally we close the demonstration by proving the uniqueness of the solution.

**First Step. Approximate solution.** We choose a sequence \( \{ \phi_1, \phi_2, \ldots \} \) in \( D(\Omega) \) such that \( \bigcup_{m=1}^{\infty} \mathcal{V}_m \) with \( \mathcal{V}_m = \text{span}\{ \phi_1, \phi_2, \ldots, \phi_m \} \) is dense in \( H^s(\Omega) \) with \( s \) large enough such that \( H^s(\Omega) \) is continuously embedded in \( C^1(\bar{\Omega}) \). We consider the following sequence for approximating solutions of the problem (4):

\[
u^m(x,t) = \sum_{k=1}^{m} c_k^m(t) \phi_k(x), \quad w^m(x,t) = \sum_{k=1}^{m} d_k^m(t) \phi_k(x),
\]

(10)

where \( c_k^m, d_k^m : [0, T] \to \mathbb{R} \) are supposed to be measurable bounded functions. For the initial conditions, we choose the coefficients as

\[
c_k^m(0) := \int_{\Omega} f(x) \phi_k(x) \, dx, \quad d_k^m(0) := 0
\]

such that as \( m \to \infty \), we have

\[
u^m(.,0) \to u_0, \quad w^m(.,0) \to w_0 \quad \text{in} \quad L^2(\Omega).
\]

(11)

where \( \nu^m(.,0) := \sum_{k=1}^{m} c_k^m(0) \phi_k(.) \) and \( w^m(.,0) := 0 \).

The coefficients \( c_k^m, d_k^m \) are obtained from the following system of ordinary differential equations

\[
\int_{\Omega} u_i^m \phi_k dx + \int_{\Omega} a_\gamma(x, \nabla u^m) \nabla \phi_k dx + 2\lambda \int_{\Omega} w^m \phi_k dx = 0, 
\]

(12)

\[
\int_{\Omega} w_i^m \phi_k dx + \int_{\Omega} \nabla w^m \nabla \phi_k dx + \int_{\Omega} (f - u^m) \phi_k dx = 0,
\]

(13)

for \( k \in \{1, 2, \ldots, m\} \). Now, let \( R \) large enough, such that

\[
\left( c_1^m(0), \ldots, c_m^m(0), d_1^m(0), \ldots, d_m^m(0) \right) \in B(0, R) \subset \mathbb{R}^{2m}.
\]

It is not difficult to see that, for \( \tau > 0 \) the functions

\[
F_k : \quad [0, \tau] \times B(0, R) \to \mathbb{R}
\]

\[
t, c_1^m, \ldots, c_m^m, d_1^m, \ldots, d_m^m \quad \to \quad -\int_{\Omega} a_\gamma(x, \sum_{j=1}^{m} c_j^m \nabla \phi_j) \nabla \phi_k dx - 2\lambda \int_{\Omega} \sum_{j=1}^{m} d_j^m \phi_j \phi_k dx
\]

and

\[
G_k : \quad [0, \tau] \times B(0, R) \to \mathbb{R}
\]

\[
t, c_1^m, \ldots, c_m^m, d_1^m, \ldots, d_m^m \quad \to \quad -\int_{\Omega} \sum_{j=1}^{m} d_j^m \nabla \phi_j \nabla \phi_k dx + \int_{\Omega} (\sum_{j=1}^{m} c_j^m \phi_j - f) \phi_k dx
\]

are a Caratheodory functions. In addition, we have for \( j \in \{1, 2, \ldots, m\} \)

\[
|F_j(t, c_1^m, \ldots, c_m^m, d_1^m, \ldots, d_m^m)| \leq C_1(m, |\Omega|) M_1(t),
\]

\[
|G_j(t, c_1^m, \ldots, c_m^m, d_1^m, \ldots, d_m^m)| \leq C_2(m, |\Omega|) M_2(t),
\]

(14)

where for \( i = 1, 2, M_i(t) \in L^1(0, \tau) \) and \( C_i(m, |\Omega|) \) is a constant which depends on \( m, f \) and \( |\Omega| \). Then, thanks to the existence result of ordinary differential equations
(cf. [22]), the system
\[
\begin{aligned}
&\left\{ \begin{array}{l}
\left( \begin{array}{c}
c^m_k(t) \\
d^m_k(t)
\end{array} \right)' = F_k \left( t, c^m_1(t), ..., c^m_m(t), d^m_1(t), ..., d^m_m(t) \right), \\
\left( \begin{array}{c}
c^m_k(t) \\
d^m_k(t)
\end{array} \right)' = G_k \left( t, c^m_1(t), ..., c^m_m(t), d^m_1(t), ..., d^m_m(t) \right),
\end{array} \right.
\end{aligned}
\]
(15)
for \( k \in \{1, 2, ..., m\} \), has a continuous solution \( \left( c^m_k(t), d^m_k(t) \right)_{k=1}^m \) on an interval \((0, \tau')\), \( \tau' > 0 \) and may depend on \( m \). Using a standard arguments, It is not difficult to show, that the local solution constructed above can be extended to the whole interval \([0, T]\) independent of \( m \). To passing to the limit in (12)-(13) and proving the existence of \( u(x, t) \) and \( w(x, t) \), we need the following a priori estimates lemma.

**Second Step. A priori estimates**

**Lemma 3.2.** Let \((u^m, w^m)\) be a solution of the problem (12)-(13). Then, we have
\[
\begin{aligned}
&|u^m|^2_{L^\infty(0,T,L^2(\Omega))} + \frac{2\mu}{\varepsilon \log^2(1 + \varepsilon)} \int_0^T \int_\Omega |\nabla u^m| \log(1 + |\nabla u^m|) dx dt \leq C \\
&|w^m|^2_{L^\infty(0,T,L^2(\Omega))} + \int_0^T \int_\Omega |\nabla w^m|^2 dx dt \leq C,
\end{aligned}
\]
where the constant \( C \) is independent of \( m \).

**Proof.** Let \( \tau < T \), taking \( u^m \chi_{(0,\tau)} \) (respectively \( w^m \chi_{(0,\tau)} \)) as a test function in (12) (respectively in (13)), we obtain
\[
\begin{aligned}
\frac{1}{2} \int_\Omega u^m(\tau)^2 dx + \int_0^\tau \int_\Omega a_\gamma(x, \nabla u^m) \nabla u^m dx dt + 2\lambda \int_0^\tau \int_\Omega w^m u^m dx dt \\
= \frac{1}{2} \int_\Omega u^m(0)^2 dx,
\end{aligned}
\]
(17)
Multiplying (18) by \( 2\lambda \) and adding the previous equality, we have
\[
\begin{aligned}
\frac{1}{2} \int_\Omega u^m(\tau)^2 dx + \lambda \int_\Omega w^m(\tau)^2 dx + \int_0^\tau \int_\Omega a_\gamma(x, \nabla u^m) \nabla u^m dx dt \\
+ 2\lambda \int_0^\tau \int_\Omega |\nabla w^m|^2 dx dt + 2\lambda \int_0^\tau \int_\Omega f w^m dx dt = \frac{1}{2} \int_\Omega u^m(0)^2 dx + \lambda \int_\Omega w^m(0)^2 dx.
\end{aligned}
\]
(18)
Using the initial condition \( u^m(x, 0) = f(x) \) and \( w(x, 0) = 0 \), then the equation becomes
\[
\begin{aligned}
\frac{1}{2} \int_\Omega u^m(\tau)^2 dx + \lambda \int_\Omega w^m(\tau)^2 dx + \int_0^\tau \int_\Omega a_\gamma(x, \nabla u^m) \nabla u^m dx dt \\
+ 2\lambda \int_0^\tau \int_\Omega |\nabla w^m|^2 dx dt + 2\lambda \int_0^\tau \int_\Omega f w^m dx dt = \frac{1}{2} \int_\Omega f^2 dx.
\end{aligned}
\]
(19)
On the one hand, applying Young’s inequality, we have
\[
\left| \int_0^\tau \int_\Omega f w^m dxdt \right| \leq \frac{1}{2} \int_0^\tau \int_\Omega f^2 dx + \frac{1}{2} \int_0^\tau \int_\Omega (w^m)^2 dxdt \\
\leq \frac{T}{2} \int_\Omega f^2 dx + \frac{1}{2} \int_0^\tau \int_\Omega (w^m)^2 dxdt. \tag{20}
\]

On the other hand, thanks to the definition of \(a_\gamma\), we see that
\[
\int_0^\tau \int_\Omega a_\gamma(x, \nabla u^m) \nabla u^m dxdt \\
= \int_{|\nabla u^m|\leq \varepsilon} B(x, |\nabla u^m|) dxdt + \int_{|\nabla u^m|> \varepsilon} \frac{B(x, \varepsilon)}{\varepsilon \log (1 + |\nabla u^m|)} \log (1 + |\nabla u^m|) |\nabla u^m| dxdt \\
\geq -T \int_\Omega B(x, \varepsilon) dx + \frac{\mu}{\varepsilon \log (1 + \varepsilon)} \int_0^\tau \int_\Omega |\nabla u^m|^{\gamma} \log (1 + |\nabla u^m|) |\nabla u^m| dxdt.
\]

Using (20) and (21), the equation (19) becomes
\[
\frac{1}{2} \int_\Omega u^m(\tau)^2 dx + \lambda \int_\Omega w^m(\tau)^2 dx + \frac{\mu}{\varepsilon \log (1 + \varepsilon)} \int_0^\tau \int_\Omega |\nabla u^m|^{\gamma} \log (1 + |\nabla u^m|) |\nabla u^m| dxdt \\
+ 2\lambda \int_0^\tau \int_\Omega |\nabla w^m|^{2} dxdt \\
\leq \frac{1}{2} \int_\Omega f^2 dx + T \int_\Omega B(x, 1) dx + \lambda T \int_0^\tau \int_\Omega f^2 dx + \lambda \int_0^\tau \int_\Omega (w^m)^2 dxdt. \tag{21}
\]

Now, setting \(\Theta_n(\tau) = \int_\Omega w^m(\tau)^2 dx\), we observe that
\[
0 \leq \Theta_n(\tau) \leq T \int_\Omega B(x, \varepsilon) dx + \left( \frac{1}{2} + \lambda T \right) \int_\Omega f^2 dx + \lambda \int_0^\tau \Theta_n(t) dt.
\]

Using Gronwall’s inequality, we get
\[
0 \leq \Theta_n(\tau) \leq \left[ T \int_\Omega B(x, \varepsilon) dx + \left( \frac{1}{2} + \lambda T \right) \int_\Omega f^2 dx \right] \exp (\lambda \tau), \quad \forall \tau \in [0, T]
\]
which implies that
\[
\int_\Omega w^m(\tau)^2 dx \leq C,
\]
where the constant \(C\) depends only on \(T\) and \(\lambda\). Consequently, we have
\[
\max_{0<\tau<T} \int_\Omega w^m(\tau)^2 dx \leq C \quad \text{and} \quad \int_0^T \int_\Omega |\nabla w^m|^2 dx dt \leq C.
\]

Then, we deduce that \(w^m\) is bounded in \(L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))\). Finally, using (3) to obtain the assertion of Lemma 3.2.

Now, to estimate the approximate solution, we state this following lemma:

**Lemma 3.3.** For every \(m \in \mathbb{N}\), we have \(\frac{\partial u^m}{\partial t}\) is bounded in \(W^{-1,\varepsilon} L^\infty(Q) + L^2(Q)\) and \(\frac{\partial w^m}{\partial t}\) is bounded in \(L^2(0, T; (H^1(\Omega))')\).
Proof. Combining (19) with the estimates (16) of Lemma 3.2, we have
\[ 2\lambda \int_0^\tau \int_\Omega |\nabla w_m|^2 \, dx \, dt + \int_0^\tau \int_\Omega a_\gamma(x, \nabla u_m) \nabla u_m \, dx \, dt \leq C, \tag{22} \]
where \( C \) is a constant not depending on \( m \). Now, taking \( \phi \in (E_A(Q))^N \) and \( \varphi \in (L^2(Q))^N \) satisfying \( ||\phi||_{A,Q} = 1 \) and \( ||\varphi||_{L^2(Q)} = 1 \), then we have
\[ \int_0^\tau \int_\Omega \left( a_\gamma(\cdot, \nabla u_m) - a_\gamma(\cdot, \phi) \right) (\nabla u_m - \phi) \, dx \, dt \geq 0, \]
which gives
\[ \int_0^\tau \int_\Omega a_\gamma(\cdot, \nabla u_m) \phi \, dx \, dt \leq \int_0^\tau \int_\Omega a_\gamma(\cdot, \nabla u_m) \nabla u_m \, dx \, dt + \int_0^\tau \int_\Omega a_\gamma(\cdot, \phi) (\nabla u_m - \phi) \, dx \, dt. \]
Using (22), we easily see that
\[ \left| \int_0^\tau \int_\Omega a_\gamma(\cdot, \nabla u_m) \phi \, dx \, dt \right| \leq C. \]
Therefore, we have \( a_\gamma(\cdot, \nabla u_m) \) is bounded sequence in \( L^A(Q) \) and so \( \frac{\partial u_m}{\partial t} \) is a bounded in \( W^{-1,x} L^A(Q) + L^2(Q) \) and similarly one can prove that \( \frac{\partial w_m}{\partial t} \) is bounded in \( L^2(0,T; (H^1(\Omega))^') \), which complete the proof of lemma. \( \square \)

**Third Step. Passage to the limit.** Thanks to the Lemma 3.2, there exist a subsequence of \((u_m, w_m)\) will be noted also by \((u_m, w_m)\) such that
\[ u_m \rightharpoonup u \text{ weakly in } W^{1,x} L_A(Q) \text{ for } \sigma(\prod L_A, \prod E_A) \]
\[ w_m \rightharpoonup w \text{ weakly in } L^2(0,T; H^1(\Omega)). \tag{23} \]
Using lemma 3.3, there exists \( \chi \in (L^A(\Omega))^N \) such that for a subsequence
\[ \nabla w_m \rightarrow \nabla w \text{ weakly in } L^2(Q) \text{ and } a_\gamma(\cdot, \nabla u_m) \rightarrow \chi \text{ weakly in } L^A(Q) \text{ for } \sigma(\prod L_A, \prod E_A). \tag{24} \]
Consequently, by applying the corollary 1, we get
\[ u_m \rightarrow u \text{ and } w_m \rightarrow w \text{ strongly in } L^1(Q). \tag{25} \]
Letting \( m \rightarrow \infty \) in (12)-(13) and for all \( (\phi, \varphi) \in C^1([0,\tau], \mathcal{D}(\Omega)) \times C^1([0,\tau], \mathcal{D}(\Omega)), \)
we have
\[ -\int_0^\tau \int_\Omega u_\phi \, dx \, dt + \int_0^\tau \int_\Omega x \nabla \phi \, dx \, dt + \int_0^\tau \int_\Omega u \phi \, dx \mid_0^\tau + 2\lambda \int_0^\tau \int_\Omega w \phi \, dx \, dt = 0, \tag{26} \]
\[ -\int_0^\tau \int_\Omega w_\varphi \, dx \, dt + \int_0^\tau \int_\Omega \nabla w \nabla \varphi \, dx \, dt + \int_0^\tau \int_\Omega w \varphi \, dx \mid_0^\tau = \int_0^\tau \int_\Omega (f - u) \varphi \, dx \, dt. \tag{27} \]
Taking \((u^m - u^n)\chi(0, \tau)\) (respectively \((w^m - w^n)\chi(0, \tau)\)) as a test function in (12) (respectively in (13)), we obtain

\[
\int_{0}^{\tau} \int_{\Omega} (u^m - u^n)_{t} (u^m - u^n) \, dx \, dt + \int_{0}^{\tau} \int_{\Omega} (w^m - w^n)_{t} (w^m - w^n) \, dx \, dt + \int_{0}^{\tau} \int_{\Omega} \left( a_{\gamma}(x, \nabla u^m) - a_{\gamma}(x, \nabla u^n) \right) \nabla(u^m - u^n) \, dx \, dt
\]

\[
+ 2\lambda \int_{0}^{\tau} \int_{\Omega} |\nabla(u^m - u^n)|^2 \, dx \, dt = 0.
\]

Using the monotonicity of \(a_{\gamma}\), then we have

\[
\int_{\Omega} |u^m(\tau) - u^n(\tau)|^2 \, dx + \int_{\Omega} |w^m(\tau) - w^n(\tau)|^2 \, dx \leq \int_{\Omega} |u^m(0) - u^n(0)|^2 \, dx - \int_{\Omega} |w^m(0) - w^n(0)|^2 \, dx.
\]

Consequently \(u^m\) and \(w^m\) are a Cauchy sequences in \(C\left(0, T; L^2(\Omega)\right)\), then

\[(u^m, w^m) \to (u, w) \quad \text{in} \quad C\left(0, T; L^2(\Omega)\right) \times C\left(0, T; L^2(\Omega)\right).
\]

Now, let us to prove that \(a_{\gamma}(x, \nabla u) = \chi\) a.e. in \(Q\). Indeed, we proceed as [16], let \(v^k \in D(\Omega \times \mathbb{R})\) be a regularization the prolongation of \(u\) such that

\[v^k \to u \quad \text{weakly in} \quad W^{1,\gamma}L_{A}(Q) \quad \text{for} \quad \sigma(\prod L_{A}, \prod E_{\gamma})\]

\[\frac{\partial v^k}{\partial t} \to \frac{\partial u}{\partial t} \quad \text{in} \quad W^{-1,\gamma}L_{A}(Q) + L^2(Q).
\]

As in [16] (Proof of Theorem 2), we have \(v^k \to u\) in \(C([0, T], L^2(\Omega))\) and

\[\lim_{k \to \infty} \lim_{m \to \infty} \int_{Q} \frac{\partial u^m}{\partial t} (v^k - u^m) \, dx \, dt \leq 0.
\]

Letting \(m \to \infty\) and \(k \to \infty\) in (26) and using (31) with the previous convergence results, we obtain

\[\lim_{m \to \infty} \int_{0}^{\tau} \int_{\Omega} a_{\gamma}(x, \nabla u^m) \nabla(u^m - u) \, dx \, dt \leq 0.
\]

By using the standard Minty Browder arguments, we obtain \(a_{\gamma}(x, \nabla u) = \chi\) a.e. in \(Q\). To finish the proof of (8) and (9), let now \(\phi \in W^{1,\gamma}L_{A}(Q) \cap L^2(Q)\), and \(\psi \in L^2\left(0, T; H^1(\Omega)\right)\) such that

\[\frac{\partial \phi}{\partial t} \in W^{-1,\gamma}L_{A}(Q) + L^2(Q) \quad \text{and} \quad \frac{\partial \psi}{\partial t} \in L^2\left(0, T; (H(\Omega))'\right).
\]

Then, there exists \((\phi^k, \psi^k) \in D(\Omega \times \mathbb{R}) \times D(\Omega \times \mathbb{R})\) such that

\[
\phi^k \to \phi \quad \text{weakly in} \quad W^{1,\gamma}L_{A}(\Omega \times \mathbb{R}) \quad \text{for} \quad \sigma(\prod L_{A}, \prod E_{\gamma}),
\]

\[
\frac{\partial \phi^k}{\partial t} \to \frac{\partial \phi}{\partial t} \quad \text{in} \quad W^{-1,\gamma}L_{A}(\Omega \times \mathbb{R}) + L^2(\Omega \times \mathbb{R}),
\]

\[
\psi^k \to \psi \quad \text{weakly in} \quad L^2\left(\mathbb{R}; H^1(\Omega)\right) \quad \text{and} \quad \frac{\partial \psi^k}{\partial t} \to \frac{\partial \psi}{\partial t} \quad \text{weakly in} \quad L^2\left(\mathbb{R}; (H^1(\Omega))'\right),
\]
where \( \tilde{\phi} \) (respectively \( \tilde{\psi} \)) is a prolongation of \( \phi \) (respectively \( \psi \)) in \( \mathbb{R} \times \Omega \) (see [16] for the existence of \( \phi \) and \( \psi \)). Letting \( \phi^k \) (respectively \( \psi^k \)) as a test function in (26) (respectively (27)) and using the previous estimations, then we obtain (8) and (9).

**Fourth Step. Uniqueness.** The final step in the proof of the theorem 3.1 is to show the uniqueness of the solution to the problem (4). Let us assume that the problem (4) admits two weak solutions \( (u_1, w_1) \) and \( (u_2, w_2) \) and taking \( (u_1 - u_2, w_1 - w_2) \) as a test function in the definition of weak solutions, then we have:

\[
\left\langle \left\langle u_1 - u_2, \frac{\partial (u_1 - u_2)}{\partial t} \right\rangle \right\rangle_{Q^\tau} + \int_0^\tau \int_\Omega \left( a_{\gamma}(x, \nabla u_1) - a_{\gamma}(x, \nabla u_2) \right) \nabla (u_1 - u_2) dx dt
\]

\[+ 2\lambda \int_0^\tau \int_\Omega (w_1 - w_2)(u_1 - u_2) dx dt = 0,
\]

(32)

\[
\left\langle w_1 - w_2, \frac{\partial (w_1 - w_2)}{\partial t} \right\rangle + \int_0^\tau \int_\Omega |\nabla (w_1 - w_2)|^2 dx dt + \int_0^\tau \int_\Omega (w_2 - w_1)(w_1 - w_2) = 0.
\]

(33)

Using the monotonicity of the operator \( a_{\gamma} \) and adding \( 2\lambda \times (33) \) to (32), we obtain

\[
\left\langle \left\langle u_1 - u_2, \frac{\partial (u_1 - u_2)}{\partial t} \right\rangle \right\rangle_{Q^\tau} + 2\lambda \left\langle w_1 - w_2, \frac{\partial (w_1 - w_2)}{\partial t} \right\rangle \leq 0,
\]

(34)

then we have

\[
\frac{1}{2} \int_\Omega (u_1 - u_2)^2 dx + \lambda \int_\Omega (w_1 - w_2)^2 dx \leq 0.
\]

(35)

Consequently \( u_1(x, t) = u_2(x, t) \) and \( w_1(x, t) = w_2(x, t) \) a.e. in \( \Omega \). This completes the proof of the theorem.

\[\Box\]

4. **Numerical aspects and results.** In this section, we present the numerical performance of the proposed model applying to image denoising, in addition to comparative study to some denoising technique described in the literature. For simplicity, the problem (4) can be rewritten as follows:

\[
\begin{cases}
\frac{\partial u}{\partial t} - \text{div} (p(x, t)) + 2\lambda w = 0 & \text{in } Q := (0, T) \times \Omega \\
\frac{\partial w}{\partial t} = \Delta w - (f - u) & \text{in } Q := (0, T) \times \Omega \\
\frac{\partial u}{\partial t} = 0, \quad \frac{\partial w}{\partial n} = 0 & \text{in } \Sigma := (0, T) \times \Gamma \\
u(0) = f, \quad w(0) = 0 & \text{in } \Gamma,
\end{cases}
\]

(36)

where

\[
p(x, t) = \nabla u \exp(|\nabla u|\chi_{|\nabla u| \leq \varepsilon}) + \lambda \frac{\nabla u}{|\nabla u|} \log(1 + |\nabla u|)\chi_{|\nabla u| > \varepsilon}
\]

with \( \lambda = \frac{e}{\log^\gamma(1 + \varepsilon)} \),

the function \( f \) describes the noisy image and \( u(\cdot, t) \) is the image with scale parameter \( t \).

For computing numerically problem (36), we attempt to discretize it by finite difference method. Assuming \( k \) to be the time step size and \( h \) the spatial grid size, we discretize time and space as follows:

\[
t_n = nk, \quad n = 0, 1, 2, ... \\
x_i = ih, \quad i = 0, 1, 2, ..., M,
\]
We denote by \( \text{div}(\cdot) \).

We use the following notations for simplicity:

Approximation:\n\[
\gamma \quad \text{Figs. 1-4) in the restorations provided by our approach and we choose the param-
\]

In experiment result of our proposed method compared with Total Variation model. when \( \gamma = 10 \), the restored image is well improved. Fig (6) illustrate another

In our case, we remark that our model reconstructs sharp edges as effectively as TV model and recovers smooth regions and reducing the staircasing effect(Fig 6a).

Through the above lines, we can obtain \( u_{i,j}^k \) and \( w_{i,j}^k \). The program will stop when it achieves our goal. Most algorithm parameters are chosen heuris-

tically for the algorithms to perform their best. We set \( \lambda = 0.01 \), the time step size \( k = 0.1 \) and the space step size \( h = 1 \). We start by the improvements tests (cf. Figs. 1-4) in the restorations provided by our approach and we choose the parameter \( \gamma = 10^{-6} \). Fig(4) exhibit an example test for the second operator by taking \( \gamma = 10^{-6}, \alpha(x) = 2.00001 \) and \( \varepsilon = 1 \). In the second experiment, we illustrate the influence of \( \gamma \) with respect to the restored image quality(see Fig 5). We remark that when \( \gamma \) is near to 0, the restored image is well improved. Fig (6) illustrate another experiment result of our proposed method compared with Total Variation model.

In (Fig 6b), the image restored by the TV-based diffusion reconstructs sharp edges, but the staircasing phenomenon is clearly present ( nose, mouth and cheeks). In our case, we remark that our model reconstructs sharp edges as effectively as TV model and recovers smooth regions and reducing the staircasing effect(Fig 6a).
Figure 1. Original Images

Figure 2. Images corrupted by Gaussian noise with zero mean and variance $\sigma^2 = 0.2$

Figure 3. Restored image with the first operator with $\gamma = 10^{-6}$

Figure 4. Restored image the second operator with $\gamma = 10^{-6}$, $\alpha(x) = 2.00001$ and $\epsilon = 1$
Figure 5. Restored images for different values of $\gamma$.

Figure 6. In this experiment, we present the comparison between restored images with our model and TV model. Prominent stair-casing can be observed on Lena’s face in the image obtained from the ROF model (b). Staircasing has been successfully reduced in the image obtained from our model.
The smooth image \(u\) using our method

The texture of \(u - f\) using our method

The smooth image using TV Model

The texture of \(u - f\) using TV Model

Figure 7. Here, we compare the texture of restored images between our model and TV model. This result shows that texture and fine details are better preserved using the proposed framework than when using TV model.

5. **Conclusion.** This paper describes a model for filtering gray scale images corrupted by independently and identically distributed Gaussian noise. The proposed model, which is based on reaction-diffusion system, reconstructs sharp edges as a TV model, preserves fine details and reduces the staircasing phenomenon during the image denoising.

**REFERENCES**

[1] R. Aboulaich, D. Meskine and A. Souissi, New diffusion models in image processing, *Comput. Math. Appl.*, **56** (2008), 874–882.

[2] R. Adams, *Sobolev Spaces*, Ac. Press, New York, 1975.

[3] L. Alvarez, P.-L. Lions and J.-M. Morel, Image selective smoothing and edge detection by nonlinear diffusion, *SIAM J. Numer. Anal.*, **29** (1992), 845–866.

[4] F. Andreu, C. Ballester, V. Caselles and J. L. Mazón, Minimizing total variation flow, *Differential Integral Equations*, **14** (2001), 321–360.

[5] A. Atlas, F. Karami and D. Meskine, The Perona-Malik inequality and application to image denoising, *Nonlinear Anal. Real World Appl.*, **18** (2014), 57–68.

[6] P. Blomgren, P. Mulet, T. Chan and C. Wong, Total variation image restoration: numerical methods and extensions, in: Proceeding of the 1997 IEEE International Conference on Image Processing, **3** (1997), 384–387.

[7] H. Brezis, Equations et inéquations non linéaires dans les espaces vectoriels en dualité, *Ann.Inst. Fourier*, **18** (1968), 115–175.

[8] Y. Cao, Yin, J. Liu, Qiang and M. Li, A class of nonlinear parabolic- hyperbolic equations applied to image restoration, *Nonlinear Anal. Real World Appl.*, **11** (2010), 253–261.
[9] F. Catté, P. L. Lions, J. M. Morel and T. Call, Image selective smoothing and edge detection by nonlinear diffusion, SIAM J. Numer. Anal., 29 (1992), 182–193.
[10] A. Chambolle and P. L. Lions, Image recovery via total variation minimization and related problems, Numer. Math., 76 (1997), 167–188.
[11] T. F. Chan, S. Esedoglu and F. E. Park, A fourth order dual method for staircase reduction in texture extraction and image restoration problems, 2010 IEEE International Conference on Image Processing, (2010), 4137–4140.
[12] T. F. Chan, S. Esedoglu and F. E. Park, Image decomposition combining staircase reduction and texture extraction, Journal of Visual Communication and Image Representation, 18 (2007), 464–486.
[13] M. G. Crandall and T. M. Liggett, Generation of semi-groups of nonlinear transformations on general Banach spaces, Amer. J. Math., 93 (1971), 265–298.
[14] E. DiBenedetto, Degenerate Parabolic Equations, Universitext, Springer-Verlag, New York, 1993.
[15] C. M. Elliot and S. A. Smitheman, Analysis of the TV regularization and $H^{-1}$ fidelity model for decomposing an image into cartoon plus texture, Commun. Pure Appl. Anal., 6 (2007), 917–936.
[16] A. Elmahi and D. Meskine, Parabolic equations in Orlicz spaces, J. London Math. Soc., 72 (2005), 410–428.
[17] J. P. Gossez, Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients, Trans. Amer. Math. Soc., 190 (1974), 163–205.
[18] J. P. Gossez, Some approximation properties in Orlicz-Sobolev spaces, Studia Math., 74 (1982), 17–24.
[19] Z. Guo, Q. Liu, J. Sun and B. Wu, Reaction-diffusion systems with $p(x)$-growth for image denoising, Nonlinear Anal. Real World Appl., 12 (2011), 2904–2918.
[20] Z. Guo, J. Yin and Q. Liu, On a reaction-diffusion system applied to image decomposition and restoration, Math. Comput. Modelling, 53 (2011), 1336–1350.
[21] P. Harjulehto, P.A. Hasto, V. Latvala, Minimizers of the variable exponent, non-uniformly convex Dirichlet energy, J. Math. Pures Appl., 89 (2008), 174–197.
[22] P. Hartman, Ordinary Differential Equations, 2nd edn. SIAM, Philadelphia, 2002.
[23] M. Krasnosel’skii and Ya. Rutickii, Convex Functions and Orlicz Spaces, Noordhoff Groningen, 1969.
[24] A. Kufner, O. John and S. Fucik, Function Spaces, Academia, Prague, 1977.
[25] R. Landes and V. Mustonen, A strongly nonlinear parabolic initial-boundary value problem, Ark. Mat., 25 (1987), 29–40.
[26] J. L. Lions, Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires, Dunod; Gauthier-Villars, Paris 1969.
[27] Y. Meyer, Oscillating Patterns in Image Processing and Nonlinear Evolution Equations, The Fifteenth Dean Jacques B. Lewis Memorial Lectures, AMS, 2001.
[28] S. Osher, A. Solé and L. Vese, Image decomposition and restoration using total variation minimization and the $H^{-1}$ norm, Multiscale Model. Simul., 1 (2003), 349–370.
[29] P. Perona and J. Malik, Scale space and edge detection using anisotropic diffusion, IEEE, Trans, Pattern anal. Match. Intell., 12 (1990), 629–639.
[30] L. Rudin, S. Osher and E. Fatemi, Nonlinear total variation based noise removal algorithms, Physica D, 60 (1992), 259–268.
[31] L. Vese and S. Osher, Modeling textures with total variation minimization and oscillating patterns in image processing, J. Sci. Comput., 19 (2003), 553–572.
[32] J. Weickert, Anisotropic Diffusion in Image Processing, Teubner-Verlag, Stuttgart, Germany, 1998.

Received January 2015; revised May 2016.
E-mail address: lekibir.affaites@gmail.com
E-mail address: a.atlas@uca.ma
E-mail address: fa.karami@uca.ma
E-mail address: driss.meskine@laposte.net