Triangle-free induced subgraphs of polarity graphs

Jared Loucks∗ Craig Timmons†

Abstract

Given a finite projective plane Π and a polarity θ of Π, the corresponding polarity graph is the graph whose vertices are the points of Π. Two distinct vertices p and p′ are adjacent if p is incident to θ(p′). Polarity graphs have been used in a variety of extremal problems, perhaps the most well-known being the Turán number of the cycle of length four. We investigate the problem of finding the maximum number of vertices in an induced triangle-free subgraph of a polarity graph. Mubayi and Williford showed that when Π is the projective geometry PG(2, q) and θ is the orthogonal polarity, an induced triangle-free subgraph has at most \( \frac{1}{2}q^2 + O(q^{3/2}) \) vertices. We generalize this result to all polarity graphs, and provide some interesting computational results that are relevant to an unresolved conjecture of Mubayi and Williford.

1 Introduction

Let Π = (\( \mathcal{P}, \mathcal{L}, \mathcal{I} \)) be a finite projective plane. A polarity θ of Π is a bijection of order two that maps \( \mathcal{P} \) to \( \mathcal{L} \), maps \( \mathcal{L} \) to \( \mathcal{P} \), and has the property that for any point \( p \) and line \( l \),

\[ p \mathcal{I} l \quad \text{if and only if} \quad \theta(l) \mathcal{I} \theta(p). \]

Polarities in projective planes have a rich history in finite geometry. For further discussion, we recommend Hughes and Piper ([13], Chapter 12) or Dembowski ([6], Chapter 3). Given a finite projective plane Π and a polarity θ of Π, the corresponding polarity graph, denoted \( G(\Pi, \theta) \), is the graph whose vertex set is \( \mathcal{P} \). Two distinct vertices \( p \) and \( p' \) are adjacent if and only if \( p \mathcal{I} \theta(p') \).

Let \( q \) be a power of a prime. The special case when the plane Π is PG(2, q) and θ is the polarity that maps a subspace to its orthogonal complement appears frequently in combinatorics. This graph, which we denote by \( ER_q \), was introduced to the graph theory community by Erdős, Rényi [8], Brown [4], and Erdős, Rényi, and Sós [9]. Since then, \( ER_q \) has appeared in many different contexts such as Ramsey theory, spectral and structural graph theory, and Turán problems. For instance, \( ER_q \) has the maximum
number of edges among all graphs with \( q^2 + q + 1 \) vertices that have no cycle of length four. This was proved by Füredi \[10\] and is one of the most important results concerning bipartite Turán problems. In fact, any polarity graph \( G(\Pi, \theta) \) where \( \Pi \) has order \( q \) has this same property provided that the number of absolute points of \( \theta \) is \( q + 1 \). Such a polarity is called orthogonal. A classical result of Baer \[3\] states that any polarity of a projective plane of order \( q \) has at least \( q + 1 \) absolute points. Thus, orthogonal polarities are the ones that have the fewest number of absolute points.

A consequence of its significance in graph theory is that different properties of \( ER_q \) have been studied. In \[8, 9\] it is shown that \( ER_q \) has \( \frac{1}{2} q^2 (q + 1) \) edges, has diameter 2, and does not contain a cycle of length four. In general, this is true for any polarity graph \( G(\Pi, \theta) \) for which \( \theta \) is orthogonal. The automorphism group of \( ER_q \) was determined by Parsons \[18\], and then again by Bachratý and Širáň \[2\] who provided simpler proofs. The independence number and chromatic number of \( ER_q \) was studied in \[11, 12, 17, 20\] and \[19\], respectively.

In this note, we consider the following problem of Mubayi and Williford \[17\].

**Problem 1.1** Determine the maximum number of vertices in an induced subgraph of \( ER_q \) that contains no cycle of length three.

One of the motivations behind Problem 1.1 comes from Turán theory. Let us write \( \text{ex}(n, \{C_3, C_4\}) \) for the maximum number of edges in an \( n \)-vertex graph with no cycle of length 3 or 4. Note that such a graph has girth at least 5. The incidence graph of a projective plane has girth at least 5. Erdős \[7\] has conjectured that this construction is asymptotically best possible; that is

\[
\text{ex}(n, \{C_3, C_4\}) = \frac{1}{2\sqrt{2}} n^{3/2} + o(n^{3/2})
\]  

(1)

It was recently conjectured by Allen, Keevash, Sudakov, and Verstraete \[1\] that \( (1) \) can be improved. More precisely, if \( z(n, C_4) \) is the maximum number of edges in an \( n \)-vertex bipartite graph with no cycle of length 4, then Allen et. al. conjecture that

\[
\liminf_{n \to \infty} \frac{\text{ex}(n, \{C_3, C_4\})}{z(n, C_4)} > 1
\]

The best known lower bound on \( \text{ex}(n, \{C_3, C_4\}) \) comes from an induced triangle free subgraph of \( ER_q \) and shows that for infinitely many \( n \),

\[
\text{ex}(n, \{C_3, C_4\}) > z(n, C_4) + \frac{1}{8} n + O(\sqrt{n}).
\]

This construction is due to Parsons \[18\] and will be discussed momentarily.

Let us now return to Problem 1.1 Mubayi and Williford \[17\] showed that for any \( q \), the maximum number of vertices in an induced triangle-free subgraph of \( ER_q \) is at most

\[
\frac{1}{2} q^2 + q^{3/2} + O(q).
\]

Using an approach based on finite geometry, we generalize this upper bound to all polarity graphs.
Theorem 1.2  Let $\Pi$ be a projective plane of order $q$, $\theta$ be a polarity of $\Pi$, and $G(\Pi, \theta)$ be the corresponding polarity graph. If $H$ is an induced triangle-free subgraph of $G(\Pi, \theta)$, then

$$|V(H)| \leq \frac{1}{2}(q^2 + q + 1) + \sqrt{q \left(\frac{q^2 + q + 1}{q + 1}\right)}.$$  

As for lower bounds, Parsons [18] showed that when $q$ is a power of an odd prime, $ER_q$ contains an induced triangle-free subgraph on $\binom{q}{2}$ vertices if $q \equiv 1(\text{mod} \ 4)$, and on $\binom{q+1}{2}$ vertices if $q \equiv 3(\text{mod} \ 4)$. By the above mentioned result of Mubayi and Williford [17], the construction of Parsons is asymptotically best possible. The following was conjectured in [17] and asserts that one cannot do better than Parsons’ construction.

Conjecture 1.3 (Mubayi, Williford [17])  Let $q$ be a power of an odd prime. The maximum number of vertices in an induced triangle-free subgraph of $ER_q$ containing no absolute points is $\binom{q}{2}$ if $q \equiv 1(\text{mod} \ 4)$, and $\binom{q+1}{2}$ if $q \equiv 3(\text{mod} \ 4)$.

We remark that the reason for excluding absolute points is that in any polarity graph, a vertex that is an absolute point will not lie in a triangle. We prove this in the next section and it is a known result.

Our computational results show that if Conjecture 1.3 is true, then one must assume some lower bound on $q$ as the conjecture fails for small values of $q$. These new lower bounds are summarized in the following table where we write $f(ER_q)$ for the maximum number of vertices in an induced triangle-free subgraph of $ER_q$ that contains no absolute points. Those values marked with a * indicate an improvement over Parsons’ construction.

| $q$    | $f(ER_q)$ |
|--------|-----------|
| 3      | $= 6$     |
| 5*     | $= 16$    |
| 7*     | $\geq 30$|
| 9*     | $\geq 46$|
| 11     | $\geq 66$|
| 13*    | $\geq 80$|

For comparison with Conjecture 1.3 our lower bound for 7, 9, and 13 exceeds the conjectured bound by 2, 10, and 2, respectively. The lower bound for 5 was done by a simple brute force search argument but for larger $q$, such a search is impossible. A Mathematica [21] notebook file giving these lower bounds is available on the second listed author’s website [15].

When $q$ is a power of 2, Mattheus, Pavese, and Storme [16] recently proved that $ER_q$ contains an induced subgraph of girth at least 5 with $\frac{q(q+1)}{2}$ vertices. This answers a question of Mubayi and Williford [17]. Another polarity graph of interest is the unitary polarity graph $U_q$. If $q$ is an even power of a prime, the graph $U_q$ has the same vertex set as $ER_q$. Let us write $(x_0, x_1, x_2)$ for a vertex in $ER_q$ where $(x_0, x_1, x_2)$ is a nonzero
vector, and two 3-tuples represent the same vertex if one is a nonzero multiple of the other. Two distinct vertices \((x_0, x_1, x_2)\) and \((y_0, y_1, y_2)\) are adjacent if
\[x_0 \sqrt{q} y_0 + x_1 \sqrt{q} y_1 + x_2 \sqrt{q} y_2 = 0.\]

Despite this relatively simple algebraic condition for adjacency, we were unable to find a triangle-free induced subgraph of \(U_q\) with \(\frac{1}{2}q^2 - o(q^2)\) vertices. In general, we conclude our introduction with the following question which generalizes one asked in [17].

**Question 1.4** Given a projective plane \(\Pi\) of order \(q\) and a polarity \(\theta\) of \(\Pi\), is it always possible to find a triangle-free subgraph of \(G(\Pi, \theta)\) with \(\frac{1}{2}q^2 - o(q^2)\) vertices?

The rest of this note is organized as follows. In Section 2 we prove Theorem 1.2. In Section 3 we discuss some of our computational results and make some additional remarks.

## 2 Proof of Theorem 1.2

Throughout this section, \(\Pi\) is a projective plane of order \(q\), \(\theta\) is a polarity of \(\Pi\), and \(G(\Pi, \theta)\) is the corresponding polarity graph.

The first lemma is known but a proof is included for completeness.

**Lemma 2.1** No absolute point of \(\theta\) is in a triangle in \(G(\Pi, \theta)\).

**Proof.** Suppose \(p_1\) is an absolute point that lies in a triangle and the other vertices of the triangle are \(p_2\) and \(p_3\). It must be the case that all three of \(p_1, p_2,\) and \(p_3\) are incident to \(\theta(p_1)\). However, \(p_1\) is incident to \(\theta(p_3)\) and \(p_2\) is incident to \(\theta(p_3)\). As the line through any pair of points is unique, \(\theta(p_1) = \theta(p_3)\) which implies \(p_1 = p_3\), a contradiction. \(\blacksquare\)

**Lemma 2.2** If \(p\) is a vertex of \(G(\Pi, \theta)\) and \(p\) is not an absolute point of \(\theta\), then the vertices adjacent to \(p\) can be partitioned into two sets \(A_p\) and \(B_p\) such that

1. the set \(A_p\) is a (possibly empty) subset of the absolute points of \(\theta\), and
2. the vertices in \(B_p\) induce a matching in \(G(\Pi, \theta)\), and no vertex in \(B_p\) is an absolute point of \(\theta\).

**Proof.** Since \(\Pi\) has order \(q\), there are exactly \(q + 1\) lines that \(p\) is incident to. These lines can be written as \(\theta(p_1), \theta(p_2), \ldots, \theta(p_{q+1})\) for some \(p_1, p_2, \ldots, p_{q+1} \in \mathcal{P}\). By definition, we have that \(p\) is adjacent to \(p_1, p_2, \ldots, p_{q+1}\) in the graph \(G(\Pi, \theta)\). Note that no \(p_i\) is equal to \(p\) since \(p\) is not an absolute point. By relabeling if necessary, we may assume that \(p_1, p_2, \ldots, p_c\) are not absolute points, and that \(p_{c+1}, p_{c+2}, \ldots, p_{q+1}\) are absolute points. Let \(A_p = \{p_{c+1}, p_{c+2}, \ldots, p_{q+1}\}\) and \(B_p = \{p_1, p_2, \ldots, p_c\}\). We have that \(A_p\) is a subset of the absolute points and that \(B_p\) contains no absolute points. To finish the proof of the lemma, we must show that the vertices in \(B_p\) induce a matching.
Let \( p_i \in B_p \) so \( p \) is incident to \( \theta(p_i) \). There is exactly one line \( l \in \mathcal{L} \) such that \( p \) and \( p_i \) are both incident to \( l \). There must be a \( j \in \{1, 2, \ldots, q + 1\} \) such that \( l = \theta(p_j) \) and so \( p, p_i, \) and \( p_j \) form a triangle. By Lemma 2.1, \( p_j \) cannot be an absolute point so \( j \in \{1, 2, \ldots, c\} \). If \( j = i \), then \( p_i \) is an absolute point, but \( p_i \in B_p \) and \( B_p \) contains no absolute points. Therefore, \( j \neq i \) and the vertices \( p, p_i, \) and \( p_j \) are all distinct. Because there is exactly one line \( l \) with both \( p \) and \( p_i \) incident to \( l \), \( p_i \) uniquely determines \( p_j \) and so the vertices in \( B_p \) induce a matching.

The next result is the well-known Expander Mixing Lemma.

**Theorem 2.3** Let \( G \) be a \( d \)-regular graph, possibly with loops where a loop adds one to the degree of a vertex. If \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \) are the eigenvalues of the adjacency matrix of \( G \) and \( \lambda = \max_{2 \leq i \leq n} |\lambda_i| \), then for any sets \( X, Y \subseteq V(G) \),

\[
|e(X, Y) - \frac{d|X||Y|}{n}| \leq \lambda \sqrt{|X||Y|}
\]

where \( e(X, Y) = |\{(x, y) \in X \times Y : \{x, y\} \in E(G)\}| \).

Let \( G^\circ(\Pi, \theta) \) be the graph obtained from \( G(\Pi, \theta) \) by adding one loop to each absolute point. It is known that the eigenvalues of \( G^\circ(\Pi, \theta) \) are \( q + 1 \) with multiplicity 1, and all others have magnitude at most \( \sqrt{q} \). For any subset of vertices \( J \subseteq V(G^\circ(\Pi, \theta)) \), we have by Theorem 2.3

\[
|e(J, J) - \frac{(q + 1)|J|^2}{q^2 + q + 1}| \leq \sqrt{q}|J|.
\]

We now have all of the tools that we need in order to prove Theorem 1.2.

**Proof of Theorem 1.2.** Let \( J \subset V(G(\Pi, \theta)) \) and assume that \( J \) contains no absolute points, and that the subgraph induced by \( J \) contains no triangles. Since \( J \) contains no absolute points, the number of edges in \( G(\Pi, \theta) \) whose endpoints are in \( J \) is the same as the number of edges in \( G^\circ(\Pi, \theta) \) whose endpoints are in \( J \). By Theorem 2.3

\[
e(J, J) \geq \frac{(q + 1)|J|^2}{q^2 + q + 1} - \sqrt{q}|J|.
\]

Note that \( e(J, J) = \sum_{v \in J} d_J(v) \), where \( d_J(v) \) is the number of neighbors of \( v \) in \( J \). Let \( v \in J \). By Lemma 2.2, since \( J \) contains no absolute points, all of the vertices adjacent to \( v \) are contained in \( B_v \), and therefore induce a matching in \( G(\Pi, \theta) \). Since \( J \) contains no triangles, \( d_J(v) \leq \frac{|B_v|}{2} \leq \frac{q^2 + q + 1}{2} \). Combining this inequality with 2), we get that

\[
\frac{(q + 1)|J|^2}{q^2 + q + 1} - \sqrt{q}|J| \leq e(J, J) \leq \sum_{v \in J} d_J(v) \leq |J| \left( \frac{q + 1}{2} \right).
\]

Solving this inequality for \( |J| \) yields

\[
|J| \leq \frac{1}{2}(q^2 + q + 1) + \sqrt{q} \left( \frac{q^2 + q + 1}{q + 1} \right)
\]

completing the proof of the theorem.
3 Concluding Remarks

We begin this section by giving a brief description of how our computational results were obtained. A close look at the proof of Theorem 1.2 suggests that one way to find a large set $J$ that induces a triangle-free graph is to choose an independent set $I$ of size $q$, and then for each vertex $v \in I$, we choose one vertex from each triangle in the neighborhood $v$ and put it into $J$. This would give a set of size about $\frac{1}{2}q^2$ which is the size we are aiming for, but of course we need to avoid triangles. This is the main difficulty. Our lower bounds for $q \geq 7$ are, more or less, obtained by following this approach. More details are provided in [15].

In any polarity graph $G(\Pi, \theta)$, the neighborhood of a vertex induces a graph of maximum degree 1, otherwise we find a cycle of length four. If $\Pi$ has order $q$, then this provides a trivial lower bound of $\frac{1}{2}(q + 1)$ on the number of vertices in an induced triangle-free subgraph but there may be absolute points in this set. Regardless, this lower bound can be improved by considering the hypergraph $\mathcal{H}(\Pi, \theta)$ whose vertex set is the vertices of $G(\Pi, \theta)$ that are not absolute points. The edges of $\mathcal{H}(\Pi, \theta)$ are the triangles in $G(\Pi, \theta)$. Since a polarity has at most $\frac{q^3}{2} + 1$ absolute points (see [13]), $\mathcal{H}(\Pi, \theta)$ has at least $q^2 + q - \frac{q^3}{2}$ vertices. Furthermore, each vertex in $G(\Pi, \theta)$ is in at most $\frac{q+1}{2}$ triangles and no two triangles share an edge. This implies that $\mathcal{H}(\Pi, \theta)$ has maximum degree $\frac{q+1}{2}$ and maximum codegree 1. By a result of Duke, Lefmann, and Rödl [5], there is positive constant $c$, not depending on $\Pi$ or $\theta$, such that the independence number of $\mathcal{H}(\Pi, \theta)$ is at least $cq^{3/2}\sqrt{\log q}$. By definition, such a set induces a triangle-free graph in $G(\Pi, \theta)$. This argument was pointed out to the second author by Jacques Verstraëte.

In the search for induced triangle-free graphs, a related problem arose. Consider the graph $ER_q$ where $q$ is a power of an odd prime. The vertices of $ER_q$ can be partitioned into three sets: the absolute points, the vertices that are adjacent to at least one absolute point, and the vertices not adjacent to any absolute points. This is proved in [18] and [20]. Let us call these sets $A_q$, $S_q$, and $E_q$, respectively. When $q \equiv 1(\text{mod } 4)$, the subgraph induced by $E_q$ is triangle-free, and when $q \equiv 3(\text{mod } 4)$, the subgraph induced by $S_q$ is triangle-free. This is the construction of Parsons [18] which shows Conjecture 1.3 if true, would be best possible. One can ask if this property characterizes $PG(2, q)$? That is, suppose $G(\Pi, \theta)$ is a polarity graph for which the vertex set admits a partition into three sets consisting of the absolute points of $\theta$, the neighbors of the absolute points (which we denote by $S$), and the vertices not adjacent to absolute points (which we denote by $E$). If the subgraph induced by $S$ or by $E$ is triangle-free, then must $\Pi = PG(2, q)$ and $\theta$ be an orthogonal polarity of $PG(2, q)$?

References

[1] P. Allen, P. Keevash, B. Sudakov, J. Verstraëte, Turán numbers of bipartite graphs plus an odd cycle, J. Combin. Theory Ser. B 106 (2014), 134–162.

[2] M. Bachratý, J. Širáň, Polarity graphs revisited, Ars Math. Contemp. 8 (2015), no. 1, 55-67.
[3] R. Baer, Polarities in finite projective planes, *Bull. Amer. Math. Soc.* 52, (1946). 77–93.

[4] W. G. Brown, On graphs that do not contain a Thomsen graph, *Canad. Math. Bull.* 9 1966 281–285.

[5] R. Duke, H. Lefmann, V. Rödl, On uncrowded hypergraphs, *Random Structures Algorithms* 6 (1995), no. 2-3, 209–212.

[6] P. Dembowski, *Finite Geometries*, Springer-Verlag Berlin Heidelberg, Germany, 1968.

[7] P. Erdős, Some recent progress on extremal problems in graph theory, *Congr. Numer.* 14 (1975), 3-14.

[8] P. Erdős, A. Rényi, On a problem in the theory of graphs. (Hungarian) *Magyar Tud. Akad. Mat. Kutató Int. Közl.* 7 1962 623–641 (1963).

[9] P. Erdős, A. Rényi, V. T. Sós, On a problem of graph theory, *Studia Sci. Math. Hungar.* 1 1966 215–235.

[10] Z. Füredi, On the number of edges of quadrilateral-free graphs, *J. Combin. Theory Ser. B* 68 (1996), no. 1, 1–6.

[11] C. Godsil, M. Newman, Eigenvalue bounds for independent sets, *J. Combin. Theory Ser. B* 98 (2008), no. 4, 721–734.

[12] S. Hobart, J. Williford, The independence number for polarity graphs of even order planes, *J. Algebraic Combin.* 38 (2013), no. 1, 57–64.

[13] D. R. Hughes, F. C. Piper, *Projective Planes*, GTM Vol. 6, Springer-Verlag New-York-Berlin, 1973.

[14] F. Lazebnik, J. Verstraëte, On hypergraphs of girth five, *Electron. J. Combin.* 10 (2003), #R25.

[15] J. Loucks, C. Timmons, Supporting Mathematica notebook file available at [http://webpages.csus.edu/craig.timmons/papers](http://webpages.csus.edu/craig.timmons/papers)

[16] S. Mattheus, F. Pavese, L. Storme, On the independence number of graphs related to a polarity, arXiv:1704.00487v1 3 Apr 2017.

[17] D. Mubayi, J. Williford, On the independence number of the Erdős-Rényi and projective norm graphs and a related hypergraph, *J. Graph Theory* 56 (2007), no. 2, 113-127.

[18] T. D. Parsons, Graphs from projective planes, *Aequationes Math.* 14 (1976), no. 1-2, 167-189.
[19] X. Peng, M. Tait, C. Timmons, On the chromatic number of the Erdős-Rényi orthogonal polarity graph, *Electron. J. Combin.* 22 (2015), no. 2, Paper 2.21, 19 pp.

[20] J. Williford, *Constructions in finite geometry with applications to graphs*, PhD Thesis, University of Delaware, 2004.

[21] Wolfram Research, Inc., *Mathematica*, Version 11.0, Champaign, IL (2016).