Existence and regularity of positive solutions of quasilinear elliptic problems with singular semilinear term

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Abstract

This paper deals with existence and regularity of positive solutions of singular elliptic problems on a smooth bounded domain with Dirichlet boundary conditions involving the Φ-Laplacian operator. The proof of existence is based on a variant of the generalized Galerkin method that we developed inspired on ideas by Browder [4] and a comparison principle. By using a kind of Moser iteration scheme we show $L^\infty(\Omega)$-regularity for positive solutions.

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1 Introduction

This paper concerns existence and regularity of solutions to the singular elliptic problem

$$-\text{div}(\phi(|\nabla u|)\nabla u) = \frac{a(x)}{u^\alpha} \text{ in } \Omega, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

(1.1)

where $\Omega \subset \mathbb{R}^N$, with $N \geq 2$, is a bounded domain with smooth boundary $\partial\Omega$, $a$ is a non-negative function, $0 < \alpha < \infty$ and $\phi : (0, \infty) \to (0, \infty)$ is of class $C^1$ and satisfies

$(\phi_1)$  (i) $t\phi(t) \to 0$ as $t \to 0$,  (ii) $t\phi(t) \to \infty$ as $t \to \infty$,

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(\phi_2) \quad t\phi(t) \text{ is strictly increasing in } (0, \infty),

(\phi_3) \text{ there exist } \ell, m \in (1, N) \text{ such that }

\ell - 1 \leq \frac{(t\phi(t))'}{\phi(t)} \leq m - 1, \quad t > 0.

We extend \( s \mapsto s\phi(s) \) to \( \mathbb{R} \) as an odd function. It follows that the function

\[ \Phi(t) = \int_0^t s\phi(s)ds, \quad t \in \mathbb{R} \]

is even and it is actually an \( N \)-function. Due to the nature of the operator

\[ \Delta_{\phi}u := \text{div}(\phi(|\nabla u|)\nabla u) \]

we shall work in the framework of Orlicz and Orlicz-Sobolev spaces namely \( L_{\Phi}(\Omega), L_{\bar{\Phi}}(\Omega) \) and \( W^{1}_{0,\Phi}(\Omega) \).

We recall some basic notation on these spaces along with bibliographical references in the Appendix.

In the last years many research papers have been devoted to the study of singular problems like (1.1). In [21], Karlin & Nirenberg studied the singular integral equation

\[ u(x) = \int_0^1 G(x, y) \frac{1}{u(y)^\alpha} dy, \quad 0 \leq x \leq 1, \]

where \( \alpha > 0 \) and \( G(x, y) \) is a suitable potential. In [9], Crandall Rabinowitz & Tartar, addressed a class of singular problems which included as a special case, the model problem

\[ -\Delta u = a(x) u^\alpha \text{ in } \Omega, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega, \quad (1.2) \]

where \( \alpha > 0 \) and \( a : \Omega \to [0, \infty) \) is a suitable \( L^1 \)-function. A broad literature on problems like (1.2) is available to date. We would like to mention [22, 34, 36] and their references. We would like to refer the reader to the very recent papers by Orsina & Petitta [29], Canino, Sciunzi & Trombetta [5] for the problem

\[ -\Delta u = \frac{\mu}{u^\alpha} \text{ in } \Omega, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega. \]

In [29] \( \mu \) is a nonnegative bounded Radon measure while in [5] \( \mu \) is an \( L^1 \) function. Other kinds of operators have been addressed and we mention Chu-Wenjie [7] and De Cave [11] for problems involving the p-Laplacian like

\[ -\text{div}(|\nabla u|^{p-2} \nabla u) = \frac{a(x)}{u^\alpha} \text{ in } \Omega, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega; \]

Qihu Zhang [35] and Liu, Zhang & Zhao [26] for \( p(x) \)-Laplacian operator,

\[ -\text{div}(|\nabla u|^{p(x)-2} \nabla u) = \frac{a(x)}{u^\alpha} \text{ in } \Omega, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega; \]
Boccardo & Orsina [3] and Bocardo & Casado-Díaz [2] for the problem
\[-\text{div}(M(x)\nabla u) = \frac{a(x)}{u^\alpha} \text{ in } \Omega, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega,\]
where \(M\) is a suitable matrix, Lazer & McKeena [24]; Goncalves & Santos [16], Hu & Wang [20] for problems involving the Monge-Ampère operator, e.g.,
\[\det(D^2u) = \frac{a(x)}{(-u)^\gamma} \text{ in } \Omega, \quad u < 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega,\]
where \(a \in C^\infty(\Omega), \quad a > 0 \text{ and } \gamma > 1.\)
To the best of our knowledge singular problems like (1.1) in the presence of the operator \(\Delta \Phi\) were never studied and the main results of this paper (see Section 2) namely Theorems 2.1, 2.2 as well as Corollary 2.1 are new.

Other problems which are special cases of (1.1) are
\[-\Delta_p u - \Delta_q u = a(x)u^{-\alpha} \text{ in } \Omega, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega, \quad (1.3)\]
where \(\phi(t) = t^{p-2} + t^{q-2}\) with \(1 < p < q < N,\)
\[-\sum_{i=1}^N \Delta_{pi} u = a(x)u^{-\alpha} \text{ in } \Omega, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega. \quad (1.4)\]
where \(\phi(t) = \sum_{j=1}^N t^{p_j-2}, \quad 1 < p_1 < p_2 < \ldots < p_N < \infty \text{ and } \sum_{j=1}^N \frac{1}{p_j} > 1,\)
\[-\text{div}(a(|\nabla u|^p)|u|^{p-2}\nabla u) = a(x)u^{-\alpha} \text{ in } \Omega, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega. \quad (1.5)\]
where \(\phi(t) = a(t^p)t^{p-2}, \quad 2 \leq p < N \text{ and } a : (0, \infty) \to (0, \infty) \text{ is a suitable } C^1(\mathbb{R}^+)-\text{function.}\)
We also refer the reader to the paper [27], where the operator \(\Delta \Phi\) is employed. The operator \(\Delta \Phi\) appears in applied mathematics, for instance in Plasticity, see e.g. Fukagai and Narukawa [14] and references therein. We refer the reader to [31] for problems involving general operators.

2 Main Results

In this work, for each \(x \in \Omega,\) we set \(d(x) = \inf_{y \in \partial \Omega} |x - y|\). Our first result is.

**Theorem 2.1** Assume that \((\phi_1) - (\phi_3)\) and \(a \in L^1(\Omega)\) hold. Then there is \(u\) such that \(u^{(\alpha-1)\ell}/\ell \in W_0^{1,\ell}(\Omega), \quad u \geq Cd \text{ a.e. in } \Omega, \text{ for some } C > 0,\) and:
(i) \(u \in W_0^{1,\Phi}(\Omega),\) and
\[\int_\Omega \phi(|\nabla u|)\nabla u \nabla \varphi dx = \int_\Omega \frac{a(x)}{u^\alpha} \varphi dx, \quad \varphi \in W_0^{1,\Phi}(\Omega), \quad (2.1)\]
provided additionally that either \(ad^{-\alpha} \in L^\varphi(\Omega)\) or \(0 < \alpha \leq 1\) and \(a \in L^{\ell^*(\ell^*+\alpha-1)}(\Omega),\)
\((ii)\) \(u \in W_{loc}^{1,\Phi}(\Omega)\), and
\[
\int_{\Omega} \phi(|\nabla u|) \nabla u \nabla \varphi dx = \int_{\Omega} \frac{a(x)}{u^\alpha} \varphi dx, \quad \varphi \in C_0^\infty(\Omega)
\] (2.2)
provided in addition that \(\alpha \geq 1\).

Next we will present some regularity results:

**Corollary 2.1** Under the conditions of the above Theorem, we have that:

\((i)\) \(u \in C(\Omega)\) if \(a \in L^\infty(\Omega)\),

\((ii)\) \(u \in L^\infty(\Omega)\) if either \(a \in L^q(\Omega) \cap L^{q^*/(q^*+\alpha-1)}(\Omega)\) and \(0 < \alpha \leq 1\) or \(a \in L^q(\Omega)\) and \(\alpha > 1\), where \(N/\ell < q \leq q(\alpha)\) with
\[
q(s) := \begin{cases} 
\ell^*/s & \text{if } 0 < s \leq 1, \\
(\ell^* + (\alpha - 1)\ell^*/\ell)/s & \text{if } s > 1,
\end{cases}
\] (2.3)

\((iii)\) there exists an only solution \(u \in W_{0}^{1,\Phi}(\Omega)\) of Problem \((\text{1.1})\) in the sense of \((\text{2.1})\).

We are going to take advantage of our techniques to show existence results to the singular-convex problem
\[-\text{div}(\phi(|\nabla u|)\nabla u) = \frac{a(x)}{u^\alpha} + b(x)u^\gamma \text{ in } \Omega, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega,\]
(2.4)
where \(\alpha, \gamma > 0\).

**Theorem 2.2** Assume \((\phi_1) - (\phi_3)\) and \(0 \leq \gamma < \ell - 1\). Assume in addition that \(ad^{-\alpha} \in L^{\tilde{\Phi}}(\Omega)\) and \(0 \leq b \in L^\sigma(\Omega)\) for some \(\sigma > \ell/(\ell - \gamma - 1)\). Then problem \((\text{2.4})\) admits a weak solution \(u \in W_{0}^{1,\Phi}(\Omega)\) such that \(u \geq Cd\) in \(\Omega\) for some constant \(C > 0\). Besides this, \(u \in L^\infty(\Omega)\) if \(b \in L^\infty(\Omega)\), and either \(a \in L^q(\Omega) \cap L^{q^*/(q^*+\alpha-1)}(\Omega)\) with \(0 < \alpha \leq 1\) or \(a \in L^q(\Omega)\) with \(\alpha > 1\), where \(N/\ell < q \leq q(\alpha + \gamma)\) and \(q(s)\) was defined in \((\text{2.3})\).

**Remark 2.1** We note that:

\((a)\) solutions of both Theorems can be found by variational arguments in some particular cases,

\((b)\) if \(\Psi\) is an \(N\)-function such that \(\Phi < \Psi << \Phi_*\), then the conditions
\[ad^{-\alpha} \in L^{\tilde{\Phi}}(\Omega)\] and \(a \in L^{\tilde{\Phi}}_{\text{loc}}(\Omega)\)
could be used in our results, instead of
\[ad^{-\alpha} \in L^{\tilde{\Phi}}(\Omega)\] and \(a \in L^\infty_{\text{loc}}(\Omega)\), respectively.
3 A family of Auxiliary Problems

In this section, we are going to “regularize” problem (2.4) by considering a perturbation by small $\epsilon > 0$ of the singular term in (2.4). Of course a regularized form of problem (1.1) corresponds to $b = 0$. Let us consider

$$
\begin{aligned}
-\Delta_{\Phi} u &= \frac{a_{\epsilon}(x)}{(u + \epsilon)^{\alpha}} + b_{\epsilon}(x)u^{\gamma} \quad \text{in } \Omega \\
\|u\| > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega
\end{aligned}
$$

(3.1)

for each $\epsilon > 0$ given, where the $L_{\infty}(\Omega)$-functions are defined by

$$
a_{\epsilon}(x) = \min\{a(x), 1/\epsilon\}, \quad b_{\epsilon}(x) = \min\{b(x), 1/\epsilon\}, \quad x \in \Omega.
$$

Consider the map $A := A_{\epsilon} : W_{0}^{1, \Phi}(\Omega) \times W_{0}^{1, \Phi}(\Omega) \rightarrow \mathbb{R}$, defined by

$$
A(u, \varphi) := \int_{\Omega} \left[ \phi(|\nabla u|)\nabla u \nabla \varphi - \frac{a_{\epsilon}(x)\varphi}{|u| + \epsilon^{\alpha}} - b_{\epsilon}(x)(u^{+})^{\gamma} \varphi \right] dx,
$$

(3.2)

Thus, finding a weak solution of (3.1) means to find $u \in W_{0}^{1, \Phi}(\Omega)$ such that

$$
A(u, \varphi) = 0 \quad \text{for each } \varphi \in W_{0}^{1, \Phi}(\Omega).
$$

(3.3)

**Proposition 3.1** For each $u \in W_{0}^{1, \Phi}(\Omega)$, the functional $A(u, \cdot)$ is linear and continuous. In particular, the operator $T := T_{\epsilon} : W_{0}^{1, \Phi}(\Omega) \rightarrow W^{-1, \tilde{\Phi}}(\Omega)$ defined by

$$
\langle T(u), \varphi \rangle = A(u, \varphi), \quad u, \varphi \in W_{0}^{1, \Phi}(\Omega)
$$

is linear and continuous, and satisfies

$$
\|T(u)\|_{W^{-1, \tilde{\Phi}}} \leq 2\|\phi(|\nabla u|)\nabla u\|_{\tilde{\Phi}} + C_{\epsilon} \|a_{\epsilon}\|_{\tilde{\Phi}} + C \|b_{\epsilon}|u|^{\gamma}\|_{\tilde{\Phi}}.
$$

(3.4)

**Proof:** Let $u, \varphi \in W_{0}^{1, \Phi}(\Omega)$. We shall use below the Hölder inequality and the embedding $W_{0}^{1, \Phi}(\Omega) \hookrightarrow L_{\Phi}(\Omega)$:

$$
|A(u, \varphi)| \leq \int_{\Omega} \left[ \phi(|\nabla u|)|\nabla u||\nabla \varphi| + \frac{a_{\epsilon}(x)|\varphi|}{\epsilon^{\alpha}} + b_{\epsilon}(x)(u^{+})^{\gamma}|\varphi| \right] dx
$$

$$
\leq 2\|\phi(|\nabla u|)\nabla u\|_{\tilde{\Phi}}\|\varphi\| + \frac{2}{\epsilon^{\alpha}} \|a_{\epsilon}\|_{\tilde{\Phi}}\|\varphi\|_{\Phi} + 2\|b_{\epsilon}|u|^{\gamma}\|_{\tilde{\Phi}}\|\varphi\|_{\Phi}
$$

$$
\leq (2\|\phi(|\nabla u|)\nabla u\|_{\tilde{\Phi}} + \frac{C}{\epsilon^{\alpha}} \|a_{\epsilon}\|_{\tilde{\Phi}} + C \|b_{\epsilon}|u|^{\gamma}\|_{\tilde{\Phi}})\|\varphi\|.
$$

(3.5)

It is enough to show that $\|b_{\epsilon}|u|^{\gamma}\|_{\tilde{\Phi}} < \infty$. Indeed, by using the embedding $L_{\Phi}(\Omega) \hookrightarrow L^{\ell}(\Omega)$ and $\gamma \in (0, \ell - 1)$ it follows by Lemma 7.2 that

$$
\int_{\Omega} \Phi(b_{\epsilon}(x)|u|^{\gamma}) dx \leq \max\{\|b_{\epsilon}\|_{L^{r_{1}}}, \|b_{\epsilon}\|_{L^{m_{1}}}{1}} \int_{\Omega} \Phi(|u|^{\gamma}) dx
$$

$$
\leq C \left( \int_{u \leq 1} + \int_{u \geq 1} \right) \Phi(|u|^{\gamma}) dx
$$

$$
\leq C \left( |\Omega| + \int_{u \geq 1} |u|^{r_{1}} dx \right) \leq C \left( |\Omega| + \int_{u \geq 1} |u|^{\ell} dx \right)
$$

$$
\leq C \left( |\Omega| + \int_{\Omega} |u|^{\ell} dx \right) \leq C \left( |\Omega| + \|u\|^{\ell} \right),
$$

(3.6)
where $C = C(b, \Phi, \epsilon) > 0$ is a constant. So $A(u, \cdot)$ is linear and continuous. The claims about $T$ are now immediate.

By proposition (3.1) the problem of finding a weak solution of (3.1) reduces to find $u = u_\epsilon \in W^{1, \Phi}_0(\Omega) \setminus \{0\}$ such that $T(u_\epsilon) = 0$.

4 Applied Generalized Galerkin Method

In order to find $u = u_\epsilon \in W^{1, \Phi}_0(\Omega) \setminus \{0\}$ such that $T(u_\epsilon) = 0$, we shall employ a Galerkin like method inspired in arguments found in Browder [4]. We are going to constrain the operator $T$ to finite dimensional subspaces. As a first step take a $\omega \in W^{1, \Phi}_0(\Omega)$ such that $a\omega \neq 0$ and $a\omega \in L^1(\Omega)$, (4.7)

Let $F \subset W^{1, \Phi}_0(\Omega)$ be a finite dimensional subspace such that $\omega \in F$. Now, consider the map $T_F : F \rightarrow F'$ given by $T_F = I_F' \circ T \circ I_F$, where

$$I_F : (F, \| \cdot \|) \rightarrow (W^{1, \Phi}_0(\Omega), \| \cdot \|), \ I_F(u) = u$$

and let $I_F'$ be the adjoint of $I_F$. So, we have that $T_F = T|_F$, because

$$\langle T_F u, v \rangle = \langle I_F' \circ T \circ I_F u, v \rangle = \langle T \circ I_F u, I_F v \rangle = \langle Tu, v \rangle, \ u, v \in F,$$

that is,

$$\langle T_F(u), v \rangle := \int_\Omega \left[ \phi(|\nabla u|) \nabla u \nabla v - \frac{a_\epsilon(x)v}{(|u| + \epsilon)^\alpha} - b_\epsilon(x)(u^+)^\gamma \right] dx, \ u, v \in F. \quad (4.8)$$

The result below, which is a consequence of the Brouwer Fixed Point Theorem (see [25]), will play a central role in solving the finite dimensional equation $T_F(u) = 0$.

**Proposition 4.1** Assume that $S : \mathbb{R}^s \rightarrow \mathbb{R}^s$ is a continuous map such that $(S(\eta), \eta) > 0$, $|\eta| = r$ for some $r > 0$, where $(\cdot, \cdot)$ is the usual inner product in $\mathbb{R}^s$ and $|\cdot|$ is its corresponding norm. Then, there is $\eta_0 \in B_r(0)$ such that $S(\eta_0) = 0$.

**Proposition 4.2** The operator $T_F$ is continuous.

**Proof:** Let $(u_n) \subseteq F$ be a sequence such that $u_n \rightarrow u$ in $F$. Since, the operator $\Delta_{\Phi} : W^{1, \Phi}_0(\Omega) \rightarrow W^{-1, \Phi}(\Omega)$ given by

$$\langle -\Delta_{\Phi} u, v \rangle := \int_\Omega \phi(|\nabla u|) \nabla u \nabla vdx, \ u, v \in W^{1, \Phi}_0(\Omega),$$

is continuous (see [13] Lemma 3.1]), we have that $\Delta_{\Phi}|_F$ is also continuous. To finish our proof, it remains to show that $T_F - \Delta_{\Phi}|_F$ is continuous. By applying Lemma [7,8] and the embedding $L_\Phi(\Omega) \hookrightarrow L^\ell(\Omega)$, it follows, by eventually passing to a subsequence, that
(1) \( u_n \to u \) a.e. in \( \Omega \);

(2) there is \( h \in L^\ell(\Omega) \) such that \(|u_n| \leq h\).

Then for each \( v \in W^{1,\Phi}_0(\Omega) \),

\[
\frac{a_\epsilon(x)v}{(|u_n| + \epsilon)^\alpha} \to \frac{a_\epsilon(x)v}{(|u| + \epsilon)^\alpha}, \quad b_\epsilon(x)(u_n^+)\gamma v \to b_\epsilon(x)(u^+)\gamma v \text{ a.e. in } \Omega.
\]

On the other hand, since \( \widetilde{\Phi} \) is increasing, we obtain

\[
\widetilde{\Phi} \left( \frac{a_\epsilon(x)}{(|u_n| + \epsilon)^\alpha} - \frac{a_\epsilon(x)}{(|u| + \epsilon)^\alpha} \right) \leq \widetilde{\Phi} \left( \frac{a_\epsilon(x)}{(|u_n| + \epsilon)^\alpha} + \frac{a_\epsilon(x)}{(|u| + \epsilon)^\alpha} \right) \leq \widetilde{\Phi} \left( \frac{2a_\epsilon(x)}{\epsilon^\alpha} \right) \in L^1(\Omega),
\]

because \( 0 \leq a_\epsilon \leq 1/\epsilon \). So, by Lebesgue’s Theorem,

\[
\int_\Omega \widetilde{\Phi} \left( \frac{a_\epsilon(x)}{(|u_n| + \epsilon)^\alpha} - \frac{a_\epsilon(x)}{(|u| + \epsilon)^\alpha} \right) dx \to 0,
\]

and as a consequence of \( \widetilde{\Phi} \in \Delta_2 \), we have

\[
\left\| \frac{a_\epsilon(x)}{(|u_n| + \epsilon)^\alpha} - \frac{a_\epsilon(x)}{(|u| + \epsilon)^\alpha} \right\|_\Phi \to 0.
\]

By applying the Hölder inequality, we find that

\[
\left| \int_\Omega \left( \frac{a_\epsilon(x)}{(|u_n| + \epsilon)^\alpha} - \frac{a_\epsilon(x)}{(|u| + \epsilon)^\alpha} \right) vdx \right| \leq 2 \left\| \frac{a_\epsilon(x)}{(|u_n| + \epsilon)^\alpha} - \frac{a_\epsilon(x)}{(|u| + \epsilon)^\alpha} \right\|_\Phi \|v\| \to 0
\]

for each \( v \in W^{1,\Phi}_0(\Omega) \). Estimating as in (4.9), we have

\[
\widetilde{\Phi} \left( b_\epsilon|\left( u_n^+\right)^\gamma - (u^+)^\gamma | \right) \leq \Phi \left( 2|b_\epsilon|_\infty \frac{(u_n^+)^\gamma + (u^+)^\gamma}{2} \right) \leq C \Phi((u_n^+)^\gamma) + \Phi((u^+)^\gamma) \leq C \|u|_\ell + |h|_\ell + 2 \in L^1(\Omega),
\]

for some \( C = C(a, \Phi, \epsilon) > 0 \). Arguing as above, we obtain

\[
\int_\Omega b_\epsilon(x)[(u_n^+)^\gamma - (u^+)^\gamma]vdx \to 0
\]

showing that \( T_F \) is continuous.
Proposition 4.3 There exists $0 \neq u = u_F = u_{\epsilon,F} \in F$ such that $T_F(u) = 0$ for each $\epsilon > 0$ sufficiently small.

Proof: Let $s := \dim F$ be the dimension of the subspace $F$, and set $F = \langle e_1, e_2, \ldots, e_s \rangle$. That is, each $u \in F$ is uniquely expressed as

$$u = \sum_{j=1}^{s} \xi_j e_j, \quad \xi = (\xi_1, \xi_2, \ldots, \xi_s) \in \mathbb{R}^s.$$

Set $|\xi| := \|u\|$ and consider the map $i = i_F : (\mathbb{R}^s, |.|) \rightarrow (F, \|.|)$ given by $i(\xi) = u$.

So, it follows by Proposition 4.2 and the fact that $i$ is an isometry that the operator

$$S_F : \mathbb{R}^s \rightarrow \mathbb{R}^s$$

is continuous as well, where $i'$ is the adjoint of $i$.

Besides this, by setting $u := i(\xi)$ for $\xi \in \mathbb{R}^s$, it follows from $(\phi_3)$ and the embeddings $W_0^1, \Phi(\Omega) \hookrightarrow L_\Phi(\Omega) \hookrightarrow L^\ell(\Omega) \hookrightarrow L^{\gamma+1}(\Omega)$ that

$$(S_F \xi, \xi) = (i' \circ T_F \circ i(\xi), \xi) = \langle T_F(u), u \rangle$$

$$\geq \ell \int_\Omega \Phi(|\nabla u|)dx - \frac{1}{\epsilon^\alpha} a_{\epsilon}(x)||u||_\Phi - |b_{\epsilon}|_{\infty}||u||^{\gamma+1}_{\gamma+1}$$

(4.12)

for some positive constants $C_1 = C_1(\epsilon)$ and $C_2 = C_2(\epsilon)$. So, we can choose an $r_0 = r_0(\epsilon) > 1$ such that $\ell r_0^\ell - C_1 r_0 - C_2 r_0^{\gamma+1} > 0$. More specifically, for each $\xi$ such that $|\xi| = r_0$, we have $(S_F \xi, \xi) > 0$.

By the above, it follows from Proposition 4.1 that there exists a $\xi_F \in \overline{B}_{r_0}(0)$ such that $S_F(\xi_F) = 0$, that is, letting $u = u_F = i(\xi_F)$, it follows from (4.12), that

$$\langle T_F(u), v \rangle = (S_F(\xi_F), v) = 0$$

for all $v \in F$.

where $v = i(\eta)$, and hence $T_F(u) = 0$. As a consequence of this, we have

$$\int_\Omega \left[ \phi(|\nabla u|)\nabla u \nabla v - \frac{a_{\epsilon}(x)v}{(|u| + \epsilon)\alpha} - b_{\epsilon}(x)(u^+)\gamma v \right] dx = 0$$

for all $v \in F$.

We claim that $u = u_{\epsilon} \neq 0$ for enough small $\epsilon > 0$. Indeed, otherwise by taking $v = w$ and using Lebesgue’s Theorem, we obtain

$$\int_\Omega a(x)wdx = \lim_{\epsilon \rightarrow 0} \int_\Omega a_{\epsilon}(x)wdx = 0,$$

but this is impossible by (4.7). This ends the proof of Proposition 4.3.

The result below is a direct consequence of the Proposition proved just above.
Corollary 4.1 The number $r_0 > 0$ and the function $u_F \in F$ found above satisfy: $\|u_F\| \leq r_0$, $T_F(u_F) = 0$, and $r_0 > 0$ does not depend on subspace $F \subset W_0^{1,\Phi}(\Omega)$ with $0 < \dim F < \infty$. Besides this, we can choose it independent of $\epsilon > 0$ as well if $0 < \alpha \leq 1$, $a \in L^{r/(\ell + \alpha - 1)}(\Omega)$, and $b \in L^{\sigma}(\Omega)$ for some $\sigma > \ell/((\ell - \gamma - 1))$.

Proof: The first part of it was proved above. To show that $r_0$ does not depend on $\epsilon > 0$, we just redo the estimatives in (4.12) by using the hypotheses on $a$ and $b$.

Our aim below is to build a non-zero vector $u_\epsilon \in W_0^{1,\Phi}(\Omega)$ such that $T(u_\epsilon) = 0$, where $T$ was given by Proposition 3.1. This will provide us with some $u_\epsilon \in W_0^{1,\Phi}(\Omega)$ such that

$$
\int_{\Omega} \left[ \phi(|\nabla u|) \nabla u \nabla \varphi - \frac{a_\epsilon(x)\varphi}{(|u| + \epsilon)^\alpha} - b_\epsilon(x)(u^+)^{\gamma \varphi} \right] dx = 0, \quad \varphi \in W_0^{1,\Phi}(\Omega). \tag{4.13}
$$

In this direction we have

Lemma 4.1 There is a non-zero vector $u_\epsilon \in W_0^{1,\Phi}(\Omega)$ such that $T(u_\epsilon) = 0$ or equivalently (4.13) holds true.

Proof: Let $w$ as in (4.7) and set

$$
A = \left\{ F \subset W_0^{1,\Phi}(\Omega) \mid F \text{ is a finite dimensional subspace of } W_0^{1,\Phi}(\Omega) \text{ and } \omega \in F \right\},
$$

We assume that $A$ is partially ordered by set inclusion. Take $F_0 \in A$ and set

$$
V_{F_0} = \left\{ u_F \in F \mid F \in A, \ F_0 \subset F, \ T_F(u_F) = 0 \text{ and } \|u_F\| \leq r_0 \right\}.
$$

Note that by Proposition 4.3 and Corollary 4.1, $V_{F_0} \neq \emptyset$. Since $V_{F_0} \subset B_{r_0}(0)$, then $\overline{V}_{F_0} \subset B_{r_0}(0)$, where $\overline{V}_{F_0}$ denotes the weak closure of $V_{F_0}$. As a matter of fact, $\overline{V}_{F_0}$ is weakly compact. Consider the family

$$
\mathcal{B} := \left\{ \overline{V}_F \mid F \in A \right\}.
$$

Claim. $\mathcal{B}$ has the finite intersection property.

Indeed, let $\{\overline{V}_{F_1}, \overline{V}_{F_2}, \ldots, \overline{V}_{F_p}\}$ be a finite subfamily of $\mathcal{B}$ and set

$$
F := \text{span}\{F_1, F_2, \ldots, F_p\}.
$$

By the very definition of $V_{F_i}$, we have that $u_F \in \overline{V}_{F_i}$, $i = 1, 2, \ldots, p$, that is

$$
\bigcap_{i=1}^{p} \overline{V}_{F_i} \neq \emptyset.
$$

This ends the proof of the Claim.

Since $B_{r_0}$ is weakly compact, it follows that (cf. [28 Thm. 26.9])

$$
W := \bigcap_{F \in A} \overline{V}_F \neq \emptyset.
$$
Let $u_\epsilon \in W$. Then $u_\epsilon \in \nabla^i_F$ for each $F \in \mathcal{A}$.

Take $F_0 \in \mathcal{A}$ such that $\text{span}\{\omega, u_\epsilon\} \subset F_0$. Since $u_\epsilon \in \nabla^i_{F_0}$, it follows by [12 Thm. 1.5] and the definition of $\nabla^i_{F_0}$ that there are sequences $(u_n) = (u_{n, \epsilon}) \subset F_0$ and $(F_n) = (F_{n, \epsilon}) \subset \mathcal{A}$ such that $u_n \rightarrow u_\epsilon$ in $W^{1, \Phi}(\Omega)$, $u_n \in F_n$, $\|u_n\| \leq r_0$, $F_0 \subset F_n$, and

$$
\int_{\Omega} \phi(|\nabla u_n|) \nabla u_n \nabla v dx = \int_{\Omega} \left( \frac{a_\epsilon(x)}{|u_n| + \epsilon} + b_\epsilon(x)(u_n^+)\gamma \right) \nabla u_n \nabla v dx
$$

for each $v \in F_n$.

Now, by eventually taking subsequences and using $W^{1, \Phi}_0(\Omega) \xrightarrow{\text{comp}} L_\Phi(\Omega)$, we obtain that $u_n \rightarrow u_\epsilon$ in $L_\Phi(\Omega)$, $u_n \rightarrow u_\epsilon$ a.e. in $\Omega$ and $(|u_n|)$ is bounded away by some function in $L_\Phi(\Omega)$.

Set $v_n = u_n - u_\epsilon$ and note that $v_n \in F_n$, because $u_n \in F_n$ and $u_\epsilon \in F_0 \subset F_n$ in (4.14). Then

$$
\lim (-\Delta_\Phi(u_n), u_n - u_\epsilon) = \lim \int_{\Omega} \left( \frac{a_\epsilon(x)}{|u_n| + \epsilon} + b_\epsilon(x)(u_n^+)\gamma \right) (u_n - u_\epsilon) dx
\leq \lim \int_{\Omega} \left( \frac{a_\epsilon(x)}{\epsilon^\alpha} + b_\epsilon(x)|u_n|^\gamma \right) |u_n - u_\epsilon| dx. \tag{4.15}
$$

As $W^{1, \Phi}_0(\Omega) \xrightarrow{\text{comp}} L_\Phi(\Omega)$, we have

$$
\left| \int_{\Omega} \frac{a_\epsilon(x)}{\epsilon^\alpha} (u_n - u_0) dx \right| \leq \frac{1}{\epsilon^\alpha} \|a_\epsilon\|_{\Phi} \|u_n - u_\epsilon\|_{\Phi} \rightarrow 0.
$$

Recalling that $\gamma < \ell - 1$, $W^{1, \Phi}_0(\Omega) \hookrightarrow L^\ell(\Omega)$ and $(u_n)$ is bounded in $L^\ell(\Omega)$, we get

$$
\int_{\Omega} b_\epsilon(x)|u_n|^\gamma |u_n - u_\epsilon| dx \leq |b_\epsilon|_\infty \left( \int_{\Omega} |u_n|^\frac{2\ell}{\ell - 1} dx \right)^\frac{\ell - 1}{\ell} |u_n - u_\epsilon|_\ell
\leq |b_\epsilon|_\infty \left( |\Omega| + \int_{\Omega} |u_n|^\ell dx \right)^\frac{\ell - 1}{\ell} |u_n - u_\epsilon|_\ell \rightarrow 0.
$$

Now, by using the facts above, it follows from (4.14) that

$$
\lim (-\Delta_\Phi(u_n), u_n - u_\epsilon) \leq 0,
$$

and a consequence of this, we have that $u_n \rightarrow u_\epsilon$ in $W^{1, \Phi}_0(\Omega)$, because $-\Delta_\Phi$ satisfies the condition $(S_+)$ (see [6 Prop. A.2]).

So, passing to a subsequence if necessary, we have

(1) $\nabla u_n \rightarrow \nabla u_\epsilon$ a.e. in $\Omega$,

(2) there is $h \in L_\Phi(\Omega)$ such that $|\nabla u_n| \leq h$. 


Since $\varphi \in W_0^{1,\Phi}(\Omega)$, it follows of the fact that $t\phi(t)$ is nondecreasing in $[0, \infty)$ and (2), that
\[
|\phi(|\nabla u_n|)\nabla u_n\nabla \varphi| \leq \phi(|\nabla u_n|)|\nabla u_n||\nabla \varphi| \leq \phi(h)|\nabla \varphi|
\leq \Phi(h)h + \Phi(|\nabla \varphi|) \leq \Phi(2h) + \Phi(|\nabla \varphi|) \in L^1(\Omega),
\]
that is, it follows by the Lebesgue Theorem, that
\[
\int_\Omega \phi(|\nabla u_n|)\nabla u_n\nabla \varphi dx \longrightarrow \int_\Omega \phi(|\nabla u_\varepsilon|)\nabla u_\varepsilon\nabla \varphi dx.
\]
Now, by passing to the limit in (4.14) and using the above informations, we get that $u_\varepsilon$ satisfies (4.13), that is, $T_\varepsilon(u_\varepsilon) = T(u_\varepsilon) = 0$ for each $\varepsilon > 0$, since $\varphi \in W_0^{1,\Phi}(\Omega)$ was taken arbitrarily. By arguments as in the proof of Proposition 4.3 we infer that $u_\varepsilon \neq 0$.

**Lemma 4.2** The function $u_\varepsilon \in C^{1,\alpha_\varepsilon}(\overline{\Omega})$, for some $0 < \alpha_\varepsilon \leq 1$, and it is a solution of (3.1).

**Proof:** By Lemma 4.1, it remains to show that $u_\varepsilon > 0$. Set $-u_\varepsilon^*$ as a test function in (4.13). So, it follows by Remark 7.1 (see Appendix), that
\[
\ell \int_\Omega \Phi(|\nabla u_\varepsilon^-|)dx \leq \int_\Omega \phi(|\nabla u_\varepsilon^-|)|\nabla u_\varepsilon^-|^2dx
= - \int_\Omega \frac{a_\varepsilon(x)}{|u_\varepsilon^-| + \varepsilon} u_\varepsilon^- dx,
\]
which implies that $u_\varepsilon^- \equiv 0$. So, $u_\varepsilon$ satisfies
\[
\int_\Omega \phi(|\nabla u_\varepsilon|)\nabla u_\varepsilon\nabla \varphi = \int_\Omega \frac{a_\varepsilon(x)}{|u_\varepsilon + \varepsilon_\varepsilon|} \varphi dx + \int_\Omega b_\varepsilon(x)u_\varepsilon^\gamma \varphi dx, \varphi \in W_0^{1,\Phi}(\Omega).
\]
Finally, for each $p \in (m, \ell^*)$, it follows that
\[
|f(x,t)| := \frac{a_\varepsilon(x)}{|t| + \varepsilon} + b_\varepsilon(x)(t^\gamma - 1) \leq C_\varepsilon(1 + |t|^{p-1}) \text{ and } \lim_{t \to 0} \frac{t^p}{\Phi_\varepsilon(\lambda t)} = 0
\]
for each $\varepsilon > 0$ given. So by Corollary 3.1, $u_\varepsilon \in C^{1,\alpha_\varepsilon}(\overline{\Omega})$ for some $0 < \alpha_\varepsilon \leq 1$. Now, by summing up the term $u_\varepsilon \phi(u_\varepsilon)$ to both sides of (4.17) and applying Proposition 5.2 we infer that $u_\varepsilon > 0$. In conclusion, $u_\varepsilon$ is a solution of (3.1).}

**5 Comparison of Solutions and Estimates**

Let $n \geq 1$ be an integer and take $\varepsilon = 1/n$. Let $u_n \in W_0^{1,\Phi}(\Omega) \cap C^{1,\alpha_n}(\overline{\Omega})$, for some $\alpha_n \in (0,1]$, denotes the solution of (3.1), both for $b = 0$ and $b \geq 0$ not identically null, given by Lemma 4.2 that is,
\[
-\Delta \phi u_n = \frac{a_n(x)}{u_n + 1/n} + b_n(x)u^\gamma \text{ in } \Omega, \ u_n > 0 \text{ in } \Omega, \ u_n = 0 \text{ on } \partial \Omega
\]
We have the following result on comparison of solutions.
Lemma 5.1 The following inequalities hold:

(i) \( u_n + 1/n \geq u_1 \) for each integer \( n \geq 1 \)

(ii) \( u_1 \geq Cd \) in \( \Omega \) for some \( C > 0 \) which independs of \( n \).

Proof. First, we consider \( b = 0 \) in (5.18), that is,

\[- \Delta \Phi u_n = \frac{a_n(x)}{(u_n + 1/n)^\alpha} \text{ in } \Omega, \quad u_n > 0 \text{ in } \Omega, \quad u_n = 0 \text{ on } \partial \Omega. \tag{5.19}\]

So, by (5.19) we have

\[ \text{div} (\phi(|\nabla u_1|) \nabla u_1) - \frac{a_1(x)}{(u_1 + 1)^\alpha} \geq 0 \text{ in } \Omega, \tag{5.20}\]

in the weak sense. On the other hand, since

\[ \frac{a_n(x)}{(w_n + 1/n)^\alpha} \geq \frac{a_1(x)}{((w_n + 1/n) + 1)^\alpha} \text{ in } \Omega. \]

we get by (5.19) that

\[ \text{div} (\phi(|\nabla (u_n + 1/n)|) \nabla (u_n + 1/n)) - \frac{a_1(x)}{((u_n + 1/n) + 1)^\alpha} \leq 0 \text{ in } \Omega, \tag{5.21}\]

in the weak sense, (test functions are taken non-negative).

By applying Theorem 2.4.1 in [30] to (5.20) and (5.21), we obtain \( u_n + 1/n \geq u_1 \).

Now, since \( \partial \Omega \) is smooth, it follows by [15, Lemma 14.16] that the distance function \( x \mapsto d(x) \) satisfies

\[ d \in C^2(\overline{\Omega}), \ d > 0 \text{ on } \overline{\Omega}_\delta \text{ and } \frac{\partial d}{\partial \eta} < 0 \text{ on } \overline{\Omega} \setminus \Omega_\delta, \]

where \( \Omega_\delta = \{ x \in \overline{\Omega} \mid d(x) > \delta \} \) for some \( \delta > 0 \), and \( \eta \) stands for the exterior unit normal to \( \partial \Omega \).

Now, since \( u_1 \in W_0^{1,\Phi}(\Omega) \cap C^{1,\alpha_1}(\overline{\Omega}) \) is a solution of

\[- \Delta \Phi u = \frac{a_1(x)}{(u + 1)^\alpha} \text{ in } \Omega, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega, \tag{5.22}\]

it follows by [33, Lemma 4.2] that

\[ \frac{\partial u_1}{\partial \eta} < 0 \text{ on } \overline{\Omega} \setminus \Omega_\delta. \]

So there is a constant \( C > 0 \) such that

\[ \frac{\partial u_1}{\partial \eta} \leq C \frac{\partial d}{\partial \eta} \text{ on } \overline{\Omega} \setminus \Omega_\delta, \]

and as a consequence

\[ Cd(x) \leq u_1(x) \text{ for } x \in \Omega. \tag{5.23}\]
This ends the proof of Lemma 5.1 for $b = 0$. If $b$ is not identically null, we redo the above proof by considering (5.20) and obtaining (5.21) as a consequence of $b$ be non-negative. This ends the proof.

We have the following estimates.

**Lemma 5.2** Let $u_n \in C^{1,\alpha_n}(\Omega)$ be a solution of (5.19). Then there is a constant $C > 0$ such that

$$
\|[(u_n + 1/n)^{\alpha - 1} - (1/n)^{\alpha - 1}]\|_{1, \ell} \leq C, \text{ for all integer } n \geq 1.
$$

where $\| \cdot \|_{1, \ell}$ above is the norm of $W_0^{1, \ell}$.

**Proof.** At first notice that

$$
u_n, [(u_n + 1/n)^{\alpha} - (1/n)^{\alpha}] \in W_0^{1, \Phi}(\Omega) \cap C^{1,\alpha_n}(\Omega) \subset W_0^{1, \ell}(\Omega).
$$

By estimating, we get

$$
\ell \alpha \Phi(1) \int_{|\nabla u_n| \geq 1} |\nabla u_n|^{\ell}(u_n + 1/n)^{\alpha - 1} dx \leq \ell \alpha \int_{\Omega} \Phi(|\nabla u_n|)(u_n + 1/n)^{\alpha - 1} dx
$$

Applying Remark 7.1 and Lemma 7.1 (both in the Appendix), and using $[(u_n + 1/n)^{\alpha} - (1/n)^{\alpha}]$ as a test function in (5.19), we find

$$
\ell \alpha \Phi(1) \int_{|\nabla u_n| \leq 1} |\nabla u_n|^{\ell}(u_n + 1/n)^{\alpha - 1} dx \leq \ell \alpha \int_{\Omega} \Phi(|\nabla u_n|)(u_n + 1/n)^{\alpha - 1} dx
$$

When $\alpha \leq 1$, it follows from Lemma 5.1 that

$$
\ell \alpha \Phi(1) \int_{|\nabla u_n| \leq 1} |\nabla u_n|^{\ell}(u_n + 1/n)^{\alpha - 1} dx \leq \ell \alpha \int_{\Omega} \Phi(|\nabla u_n|)(u_n + 1/n)^{\alpha - 1} dx
$$

which is finite, by a well known result, cf. Lazer and McKenna [23].

It follows from (5.24) and (5.25), that

$$
\int_{\Omega} \left| \nabla \left( (u_n + 1/n)^{\alpha - 1/\ell} \right) \right|^\ell dx \leq \left( \frac{\alpha + \ell - 1}{\ell} \right)^\ell \frac{1}{\ell \alpha \Phi(1)} (\|a\|_1 + D),
$$
because
\[
\left| \nabla \left( (u_n + 1/n)^{\alpha + \ell - 1}/\ell \right) \right| \ell = \left( \frac{\alpha + \ell - 1}{\ell} \right)^\ell |\nabla u_n|^\ell (u_n + 1/n)^{\alpha - 1}.
\]

Hence, \([(u_n + 1/n)^{(\alpha + \ell - 1)/\ell} - (1/n)^{(\alpha + \ell - 1)/\ell}]\) is bounded in \(W^{1,\ell}_0(\Omega)\).

When \(\alpha > 1\), we have
\[
\ell \alpha \Phi(1) \int_{|\nabla u_n| \leq 1} |\nabla u_n|^\ell (u_n + 1/n)^{\alpha - 1} dx \leq \ell \alpha \Phi(1) \left[ |\Omega| + \int_{u_n > 1} (u_n + 1/n)^{\alpha - 1} dx \right].
\]

Summing up (5.24) and (5.26), we obtain a positive constant \(C\) such that
\[
\int_\Omega \left| \nabla \left( (u_n + 1/n)^{\alpha + \ell - 1}/\ell \right) \right| \ell dx \leq C \left( 1 + \int_{u_n > 1} (u_n + 1/n)^{\alpha - 1} dx \right).
\]

Now, by picking \(\epsilon\) such that \(0 < \epsilon < \ell - \ell((\alpha - 1)/\alpha + \ell - 1)\), it follows from (5.27), using \(u_n > 1\) and of the embbeding \(W^{1,\ell}_0(\Omega) \hookrightarrow L^\ell(\Omega) \hookrightarrow L^{\ell - \epsilon}(\Omega)\), that
\[
\left\| \nabla \left( (u_n + 1/n)^{\alpha + \ell - 1}/\ell \right) \right\|_\ell \leq C \left( 1 + \int_{u_n > 1} (u_n + 1/n)^{\alpha + \ell - 1}/\ell \left( (u_n + 1/n)^{\alpha + \ell - 1}/\ell \right)^{\ell - \epsilon} dx \right)
\]
\[
\leq C \left( 1 + \left\| \nabla \left( (u_n + 1/n)^{\alpha + \ell - 1}/\ell \right) \right\|_\ell^{\ell - \epsilon} \right),
\]
for some \(C > 0\). That is, \([(u_n + 1/n)^{(\alpha + \ell - 1)/\ell} - (1/n)^{(\alpha + \ell - 1)/\ell}]\) is bounded in \(W^{1,\ell}_0(\Omega)\) as well. This ends the proof of Lemma 5.2.

\[\square\]

6 Proof of The Main Results

We begin proving Theorem 2.1 that treats about existence of positive solution to the pure singular problem.
6.1 Pure Singular Problem - Existence of Solutions

**Proof of (i) of Theorem 2.1** Assume first that $ad^{-\alpha} \in L_\Phi(\Omega)$. Since $u_n \in W_0^{1,\Phi}(\Omega)$ satisfies (5.19), it follows from Remark 7.1, Lemma 7.1, (5.23) and Hölder inequality, that

$$\ell \zeta_0(\|\nabla u_n\|_\Phi) \leq \ell \int_\Omega \Phi(|\nabla u_n|) dx \leq \int_\Omega \phi(|\nabla u_n|) |\nabla u_n|^2 dx$$

$$= \int_\Omega \frac{a_n(x)}{(u_n + \frac{1}{n})^\alpha} u_n dx \leq C \int_\Omega \frac{a(x)}{d^\alpha} |u_n| dx$$

$$= C \left( \int_\Omega \frac{a(x)}{d^\alpha} |u_n| dx + \int_{\Omega_\delta} \frac{a(x)}{d^\alpha} |u_n| dx \right)$$

$$\leq C \int_\Omega |u_n| dx + C \int_\Omega \frac{a(x)}{d^\alpha} |u_n| dx$$

$$\leq C \|u_n\|_\Phi + 2C \left\| \frac{a}{d^\alpha} \right\|_\Phi \|u_n\|_\Phi,$$

where we used $a_n \leq a$ just above. It follows from our assumptions and from $W_0^{1,\Phi}(\Omega) \hookrightarrow L_\Phi(\Omega)$, that $(u_n) \subset W_0^{1,\Phi}(\Omega)$ is bounded. If $0 < \alpha \leq 1$ and $a \in L^{\ell/(\ell+\alpha-1)}(\Omega)$, then the boundedness of $(u_n)$ in $W_0^{1,\Phi}(\Omega)$ is a consequence of Corollary 4.1.

So, in both cases, up to subsequences, there exist $u \in W_0^{1,\Phi}(\Omega)$ and $\theta \in L_\Phi(\Omega)$ such that

1. $u_n \rightharpoonup u$ in $W_0^{1,\Phi}(\Omega)$,
2. $u_n \to u$ in $L_\Phi(\Omega)$,
3. $u_n \to u$ a.e. in $\Omega$,
4. $0 \leq u_n \leq \theta$.

As a first consequence of these facts, it follows from Lemma 5.1 and (3) that $u \geq Cd$ a.e. in $\Omega$.

Now, by using $u_n - u$ as a test function in (5.19) and following similar arguments as in (4.15), we get

$$\langle -\Delta_\Phi u_n, u_n - u \rangle \leq \left| \int_\Omega \frac{a_n(x)}{(u_n + 1/n)^\alpha} (u_n - u) dx \right|$$

$$\leq \left[ C + 2 \left\| \frac{a}{d^\alpha} \right\|_\Phi \right] \|u_n - u\|_\Phi$$

(6.2)

for some $C > 0$ independent of $n$. Since, the operator $-\Delta_\Phi$ is of the type $S_+$, it follows from (2) and (6.2) that $u_n \to u$ in $W_0^{1,\Phi}(\Omega)$.

To finish our proof, given $\varphi \in W_0^{1,\Phi}(\Omega)$, it follows from Lemma 5.1 that

$$\left| \frac{a_n}{(u_n + 1/n)^\alpha} \varphi \right| \leq \frac{a}{d^\alpha} \left( \frac{d}{u_n + 1/n} \right)^\alpha |\varphi| \leq C \frac{a}{d^\alpha} |\varphi| \in L^1(\Omega),$$

that is, by passing to the limit in (5.19), we obtain that $u$ is a solution of (1.1). This ends our proof.

We were not able to employ the above arguments in the proof of (ii) of Theorem 2.1 because in such case we do not know if $a/d^\alpha$ belongs to $L_\Phi(\Omega)$, that is, the sequence $(u_n)$ likely is not
bounded in $W^{1,\Phi}_0(\Omega)$. Instead, it was possible to show that $(u_n)$ is bounded in $W^{1,\Phi}_{loc}(\Omega)$.

This was done by applying Lemma 5.2.

**Proof of (ii) of Theorem 2.1** Given $U \subset \subset \Omega$, let $\delta_U = \min\{d(x) / x \in U\} > 0$. So, it follows from Lemma 5.1 that

$$u_n + 1/n \geq C\delta_U := C_U > 0 \text{ in } U,$$

that is, for $n > 1$ enough big, we can take $(u_n + 1/n - C_U)^+$ as a test function in (5.19), to obtain

$$\int_U \phi(|\nabla u_n|) |\nabla u_n|^2 \leq \int_{u_n + 1/n \geq C_U} \phi(|\nabla u_n|) |\nabla u_n|^2 dx$$

$$\leq \int_{u_n + 1/n \geq C_U} \frac{a(x)}{(u_n + 1/n)^{\alpha - 1}} dx$$

$$\leq \frac{1}{C^\alpha_U} \int_\Omega adx < \infty,$$

because $a \in L^1(\Omega)$, and $\alpha \geq 1$.

So, it follows from Remark 7.1 and Lemma 7.1 that $(u_n) \subset W^{1,\Phi}(U)$ is bounded. That is, there exist $(u_{n_1}^U), u^U \in W^{1,\Phi}(U)$ such that $u_{n_1}^U \rightharpoonup u^U$ in $W^{1,\Phi}(U)$, $u_{n_1}^U \rightarrow u^U$ in $L^1(\Omega)$, $u_{n_1}^U(x) \rightarrow u^U(x)$ a.e. in $U$. In particular, it follows from Lemma 5.1 and of the pointwise convergence that $u \geq C d$ a.e. in $U$.

Hence, by using a Cantor diagonalization argument applied to an exhaustion $U_k$ of $\Omega$ with $U_k \subset \subset U_{k+1} \subset \subset \Omega$, we show that there is $u \in W^{1,\Phi}_{loc}(\Omega)$ such that $u_k \rightarrow u$ in $W^{1,\Phi}_{loc}(\Omega)$ and $u \geq C d$ a.e. in $\Omega$.

Now, we are going to show that this $u$ satisfies the equation in (1.1). Given $\varphi \in C_0^\infty(\Omega)$, let $\Theta \subset \subset \Omega$ be the support of $\varphi$. So, by very above informations, we have that

(a) $u_n \rightharpoonup u$ in $W^{1,\Phi}(\Theta)$, (b) $u_n \rightarrow u$ in $L^{1,\Phi}(\Theta)$, (c) $u_n(x) \rightarrow u(x)$ a.e. in $\Theta$.

and there exists $\theta \in L^{1,\Phi}(\Theta)$ such that $u_n \leq \theta$ in $\Theta$.

So, by using $\varphi(u_n - u)$ as a test function in (5.19), $L^{1,\Phi}(\Theta) \hookrightarrow L^1(\Theta)$, and (b) above, we obtain

$$\left| \int_\Theta \phi(|\nabla u_n|) \nabla u_n \nabla (\varphi(u_n - u)) \right| dx \leq \frac{1}{C^\alpha_d} \int_\Theta a_n \varphi(u_n - u) dx$$

$$\leq C\varphi \|a\|_{L^{1,\Phi}(\Theta)} \|u_n - u\|_{L^{1,\Phi}(\Theta)} \rightharpoonup 0,$$

where $\Theta \subset \subset \Omega$ is the support of $\varphi$. That is,

$$\int_\Theta \phi(|\nabla u_n|) \nabla u_n \nabla (u_n - u) \varphi dx = \int_\Theta \phi(|\nabla u_n|) \nabla u_n \nabla \varphi(u_n - u) dx + o_n(1).$$

Besides this, it follows from Holder’s inequality, (b) above and the property $\tilde{\Phi}(\varphi(t)) \leq \Phi(2t)$
for $t > 0$, that
\[
\left| \int_{\Theta} \phi(|\nabla u_n|) \nabla u_n \nabla \varphi(u_n - u) \right| \, dx \leq C_{\varphi} \int_{\Theta} \phi(|\nabla u_n|) |\nabla u_n - u| \, dx \\
\leq C_{\varphi} \|\phi(|\nabla u_n|)\|_{L_\infty(\Theta)} \|u_n - u\|_{L_\infty(\Theta)} \to 0 \\
\leq C_{\varphi} \|u_n - u\|_{L_\infty(\Theta)} \to 0,
\]
and using this information in (6.3), we obtain that
\[
\int_{\Omega} \phi(|\nabla u_n|) \nabla u_n \nabla (u_n - u) \varphi \, dx = o_n(1). \tag{6.6}
\]
Besides this, we note that
\[
\left| \int_{\Theta} \phi(|\nabla u|) \nabla u \nabla (u_n - u) \varphi \, dx \right| \leq \left| \int_{\Theta} \phi(|\nabla u|) \nabla u \nabla [\varphi(u_n - u)] \varphi \, dx \right| \\
+ \left| \int_{\Theta} \phi(|\nabla u|) \nabla u \nabla \varphi(u_n - u) \, dx \right|,
\]
and the first integral on the right side goes to zero, due to (a) above, and the second one converges to zero due to (b) above. That is,
\[
\left| \int_{\Theta} \phi(|\nabla u|) \nabla u \nabla (u_n - u) \varphi \, dx \right| \to 0. \tag{6.7}
\]
So, it follows from (4.13) and (4.15), that
\[
\int_{\Theta} \left( \phi(|\nabla u_n|) \nabla u_n - \phi(|\nabla u|) \nabla u, \nabla u_n - \nabla u \right) \varphi \, dx \to 0, \tag{6.8}
\]
and a consequence of this together with the Lemma 6 in [10], we have that $\nabla u_n(x) \to \nabla u(x)$ a.e. in $\Theta$, that is,
\[
\phi(|\nabla u_n(x)|) \nabla u_n(x) \to \phi(|\nabla u(x)|) \nabla u(x) \text{ a.e. in } \Theta.
\]
In addition, since $(\phi(|\nabla u_n|) \nabla u_n) \subset (L_\infty(\Theta))^N$ is bounded, it follows from Lemma 2 in [17] that
\[
\phi(|\nabla u_n|) \nabla u_n \to \phi(|\nabla u|) \nabla u \text{ in } (W^{1,\Phi}(\Theta))^N.
\]
Now, passing to limit in (5.19), we obtain that $u \in W^{1,\Phi}_{loc}(\Omega)$ satisfies
\[
\int_{\Omega} \phi(|\nabla u|) \nabla u \nabla \varphi \, dx = \int_{\Omega} \frac{a(x)}{u^\alpha} \varphi \, dx.
\]
Besides this, it follows from Lemma 5.2 that
\[
u_n \to v \text{ in } W^{1,\ell}\_{0}(\Omega),
\]
that is, $u^{(\alpha-\ell)/\ell} \in W^{1,\ell}\_{0}(\Omega)$ as well. This ends our proof.}

Below, we take advantage of the former arguments to show existence of solutions to Problem 2.4. The greatest effort is done to show $L^\infty$-regularity of its solutions.
6.2 Convex Singular Problem - Regularity of Solutions

Proof of Theorem 2.2: Since \( 0 < \gamma < \ell - 1 \) and \( 0 \leq a \in L^q(\Omega) \) for some \( q > \ell/(\ell - \gamma - 1) \), it follows by arguments similar to those used in the proof of Theorem 2.1 that there exist both a sequence of approximating solutions still denoted by \( (u_n) \) and a corresponding solution \( u \in W_0^{1,\Phi}(\Omega) \) to problem (2.4) such that \( u \geq C \theta_0 \) in \( \Omega \) for some constant \( C > 0 \).

Claim. \( u \in L^\infty(\Omega) \).

The proof of this Claim uses arguments driven by a Moser Iteration Scheme. Parts of our argument were motivated by reading [19]. However our proof in the present paper is self-contained. In order to show the Claim, set

\[
\beta_1 := (\ell + \alpha - 1)q' > 0, \quad \beta_k := \beta_k + \beta_1, \quad \beta_{k+1} := \frac{\ell^*}{\ell q} \beta_k, \quad \delta := \ell^*/(q' \ell),
\]

where \( 1/q' + 1/q = 1 \).

We point out that \( \delta > 1 \) because \( q > N/\ell \). In addition,

\[
\beta_k = \frac{2\delta^k - \delta^k - 1}{\delta - 1} \beta_1, \quad \beta_{k+1} = \frac{2\delta^k - \delta^k - 1}{\delta - 1} \beta_1,
\]

and since \( \delta > 1 \), \( \beta_k \nearrow \infty \).

Now, taking \( k_0 \) such that \( \beta_{k_0}, \beta_{k_0} + q'(\alpha - 1) > 1 \), we have that \( u_n^{\beta_k/(q' + \alpha)} \) is a test function for each \( k \geq k_0 \) and using it in (6.13), we obtain

\[
\frac{\beta_k}{q'} \int_{\Omega} \phi(|\nabla u_n|) |\nabla u_n|^{2 \beta_k + \alpha - 1} dx \leq \int_{\Omega} \left( \frac{a_n u_n^{2 \beta_k + \alpha}}{(u_n + 1/n)^\alpha} + b u_n^{\beta_k + \alpha + \gamma} \right) dx
\]

\[
\leq \int_{\Omega} \left( a u_n^{\beta_k} + b u_n^{\beta_k + \alpha + \gamma} \right) dx
\]

\[
\leq \|a\|_q \|u_n\|_{\beta_k} + \|b\|_\infty \|u_n\|_{\beta_k} \|u_n^{\beta_k + \gamma}\|_q.
\]

We claim that \( \|u_n^{\beta_k + \gamma}\|_q \) is bounded. Indeed, if \( (\alpha + \gamma)q \leq 1 \), it follows that \( \alpha \leq 1 \), because \( q > N/\ell \). In this case, it follows from Corollary 4.11 that \( u_n \) is bounded in \( W_0^{1,\Phi}(\Omega) \). In particular, there exists \( \theta_0 \in L^1(\Omega) \) such that \( u_n \leq \theta_0 \), that is,

\[
\|u_n^{\alpha + \gamma}\|_q \leq (|\Omega| + \|\theta_0\|_1)^{\frac{1}{q'}}.
\]

If \( (\alpha + \gamma)q > 1 \) we distinguish between the two cases: \( \alpha > 1 \) and \( \alpha \leq 1 \).

In the case \( \alpha > 1 \), we find by using that \( ((u_n + 1/n)^{\ell + (\alpha - 1)/\ell}) \) is bounded in \( W^{1,\ell}(\Omega) \) and \( W^{1,\ell}(\Omega) \hookrightarrow L^{\ell^*}(\Omega) \) that

\[
\|u_n\|_{L^{\ell^* + (\alpha - 1)/\ell^*}} = \left( \int_{\Omega} u_n^{\ell^* + (\alpha - 1)/\ell^*} dx \right)^{\frac{1}{\ell^*}} = \|u_n^{(\ell + (\alpha - 1)/\ell)}\|_{\ell^*} \leq C,
\]
that is, by using our assumption \( q \leq q(\alpha + \gamma) \), it follows from its definition (see (2.3) for this) that \((\alpha + \gamma)q \leq \ell^* + (\alpha - 1)\ell / \ell\). So,

\[
\|u_n\|_{(\alpha+\gamma)q} \leq C,
\]

because \(L^{\ell^*+(\alpha-1)\ell / \ell} \hookrightarrow L^{(\alpha+\gamma)q}(\Omega)\).

In the case \( \alpha \leq 1 \), again we have that \( u_n \) is bounded in \( W_0^1,\Phi(\Omega) \). So, it follows from \( W_0^1,\Phi(\Omega) \hookrightarrow L^{(\gamma+\alpha)q}(\Omega) \), see (2.3) again, that

\[
\|u_n^{\alpha+\gamma}\|_q = \|u_n\|_{(\alpha+\gamma)q} \leq \kappa \|u_n\|^{\alpha+\gamma} \leq C,
\]

for some \( \kappa, C > 0 \).

Thus, in both cases, it follows from (6.11) and the estimates just above that there exists a constant \( c_0 > 0 \) such that

\[
\frac{\beta_k}{q'} \int_\Omega \phi(|\nabla u_n|)|\nabla u_n|^2 u_n^{\beta_k+\alpha-1} dx \leq \left( \|a\|_q + \|b\|_\infty c_0 \right) \|u_n\|_{\beta_k}^\beta.
\]

(6.13)

On the other hand, it follows by Lemma 7.1 that

\[
\frac{\beta_k}{q'} \int_\Omega \phi(|\nabla u_n|)|\nabla u_n|^2 u_n^{\beta_k+\alpha-1} dx \geq \frac{\ell \Phi(1)}{q'} \beta_k \int_{|\nabla u_n| \geq 1} |\nabla u_n|\|u_n\|_{\beta_k}^\beta dx
\]

and so it follows from (6.13) and (6.14), that

\[
\frac{\ell \Phi(1)}{q'} \beta_k \int_\Omega |\nabla u_n| u_n^{\beta_k+\alpha-1} dx \leq \frac{\ell \Phi(1)}{q'} \beta_k \int_{|\nabla u_n| < 1} |\nabla u_n|^{\ell} u_n^{\beta_k+\alpha-1} dx
\]

\[+ \left( \|a\|_q + \|b\|_\infty c_0 \right) \|u_n\|_{\beta_k}^{\beta_k} \]

\[\leq \frac{\ell \Phi(1)}{q'} \beta_k \int_\Omega u_n^{\beta_k+\alpha-1} dx \]

\[+ \left( \|a\|_q + \|b\|_\infty c_0 \right) \|u_n\|_{\beta_k}^{\beta_k}.
\]

(6.15)

Our next objective is to show that

\[
\int_\Omega |\nabla u_n|^{\ell} u_n^{\beta_k+(\alpha-1)q'} dx \leq B \|u_n\|_{\beta_k}^{\beta_k},
\]

(6.16)

for some constant \( B > 0 \). To do this, we are going to consider two cases again: \( \alpha \leq 1 \) and \( \alpha > 1 \).

If \( \alpha \leq 1 \) notice that \( L^{\beta_k}(\Omega) \hookrightarrow L^{\beta_k/\ell + \alpha-1}(\Omega) \). Hence

\[
\int_\Omega u_n^{\beta_k/\ell + \alpha-1} dx = \|u_n\|_{\beta_k/\ell + \alpha-1}^{\beta_k/\ell + \alpha-1} \leq \|\Omega\|^{1-\frac{1}{\ell} + \frac{1-\alpha}{\beta_k}} \|u_n\|_{\beta_k}^{\beta_k/\ell + \alpha-1} \|u_n\|_{\beta_k}^{\alpha-1}.
\]

(6.17)

On the other hand, since \( u_1 \leq u_n \), we have

\[
\|u_1\|_{\beta_k} \leq \|u_n\|_{\beta_k},
\]

(6.18)
and by the embedding $L^{eta_k}(\Omega) \hookrightarrow L^1(\Omega)$ we get
\[
\|u_1\|_1 \leq |\Omega|^{1 - \frac{1}{\beta_k}} \|u_1\|_{\beta_k}.
\] (6.19)

Combining (6.18) and (6.19) we have
\[
\|u_n\|_{\beta_k}^{\alpha - 1} \leq |\Omega|^{(1 - \alpha)(1 - \frac{1}{\beta_k})} \|u_1\|_1^{\alpha - 1}.
\] (6.20)

So by (6.17) and (6.20) we infer that
\[
\int_\Omega \frac{\beta_k}{u_n^{\alpha - 1}} \, dx \leq |\Omega|^{2 - \frac{\alpha}{q'}} \|u_1\|_1^{\alpha - 1} \|u_n\|_{\beta_k}^{\frac{\alpha}{q'}}.
\] (6.21)

Now by applying (6.21) in (6.15), we get
\[
\frac{\ell \Phi(1)}{q'} \beta_k \int |\nabla u_n|^{\frac{\beta_k}{q'} + \alpha - 1} \, dx \leq \frac{\ell \Phi(1)}{q'} |\Omega|^{2 - \frac{\alpha}{q'}} \|u_1\|_1^{\alpha - 1} \|u_n\|_{\beta_k}^{\frac{\alpha}{q'}}
\]
\[
+ (\|a\|_q + \|b\|_{\infty}c_0) \|u_n\|_{\beta_k}^{\frac{\beta_k}{q'}}.
\] (6.22)

Let $\alpha > 1$. By Hölder Inequality, $(\alpha - 1)q < (\alpha + \gamma)q$ and (6.12), we have
\[
\int_\Omega \frac{\beta_k}{u_n^{\alpha - 1}} \, dx \leq \|u_n\|_{\beta_k}^{\frac{\beta_k}{\beta_k}} \left( \int_\Omega u_n^{(\alpha - 1)q} \, dx \right)^\frac{1}{\alpha - 1}
\]
\[
\leq \|u_n\|_{\beta_k}^{\frac{\beta_k}{\beta_k}} \left( |\Omega| + \int_{[u_n \geq 1]} u_n^{(\alpha - 1)q} \, dx \right)^\frac{1}{\alpha - 1}
\]
\[
\leq \|u_n\|_{\beta_k}^{\frac{\beta_k}{\beta_k}} \left( |\Omega| + \|a\|_q \left( \frac{(\alpha - 1)q}{\alpha + \gamma} \right) \right)^\frac{1}{\alpha - 1}
\]
\[
\leq \left( |\Omega| + C \right) \frac{1}{\alpha - 1} \|u_n\|_{\beta_k}^{\frac{\beta_k}{\beta_k}}.
\] (6.23)

Now by applying (6.23) in (6.15), we get
\[
\frac{\ell \Phi(1)}{q'} \beta_k \int |\nabla u_n|^{\frac{\beta_k}{q'} + \alpha - 1} \, dx \leq \frac{\ell \Phi(1)}{q'} |\Omega|^{\frac{\alpha}{q'}} \|u_n\|_{\beta_k}^{\frac{\beta_k}{q'}}
\]
\[
+ (\|a\|_q + \|b\|_{\infty}c_0) \|u_n\|_{\beta_k}^{\frac{\beta_k}{q'}}.
\] (6.24)

So, it follows from (6.22) (the case $\alpha \leq 1$) and (6.24) (the case $\alpha > 1$) that the inequality (6.16) is true for $B > 0$ defined by
\[
B := \left\{ \begin{array}{ll}
\frac{q'}{\ell \Phi(1)} \left( \frac{\ell \Phi(1)}{q'} |\Omega|^{2 - \frac{\alpha}{q'}} \|u_1\|_1^{\alpha - 1} + \|a\|_q + \|b\|_{\infty}c_0 \right), & 0 < \alpha \leq 1, \\
\frac{q'}{\ell \Phi(1)} \left( |\Omega| + C \right)^{\frac{1}{\alpha - 1}} + \|a\|_q + \|b\|_{\infty}c_0, & \alpha > 1.
\end{array} \right.
\]

This shows the inequality (6.16). Now since
\[
\left( \frac{\ell q'}{\beta_k + \beta_1} \right) \ell \int_\Omega |\nabla (u_n^{\frac{\beta_k + \beta_1}{q'}})|^\ell \, dx = \int_\Omega |\nabla u_n|^{\frac{\beta_k + \beta_1}{q'}} \, dx,
\]
it follows from (6.16) and $W^1_0(\Omega) \hookrightarrow L^\ell(\Omega)$, that
\[
\|u_n\|_{\beta_{k+1}} = \left\| \frac{\beta_k^*}{\ell} \right\|_{\ell^*} \leq \mu \ell B \left( \frac{\beta_k^*}{\ell q} \right) \|u_n\|_{\beta_k},
\] (6.25)
for some $\mu > 0$.
Set $F_{k+1} := \beta_{k+1} \ln(\|u_n\|_{\beta_{k+1}})$. So, it follows from the last inequality, that
\[
F_{k+1} \leq \frac{\beta_{k+1} q'}{\beta_k^*} \left( \ell \ln \mu + \ell \ln \left( \frac{\beta_k^*}{\ell q} \right) + \ln B + \frac{\beta_k}{q'} \ln(\|u_n\|_{\beta_k}) \right)
\leq \ell^* \ln (\mu B \beta_k^*) + \frac{\ell^*}{q' \ell} F_k
\leq \lambda_k + \delta F_k,
\] (6.26)
where $\lambda_k := \ell^* \ln (\mu B \beta_k^*)$.

Now, by using (6.9) and (6.10), we can infer that
\[
\lambda_k = b + \ell^* \ln \left( 2\delta^{k-1} + \delta^{k-2} + \ldots + 1 \right),
\]
where $b := \ell^* \ln(\mu B \beta_1)$, that is,
\[
F_k \leq \delta^{k-1} F_1 + \lambda_{k-1} + \delta \lambda_{k-2} + \ldots + \delta^{k-2} \lambda_1.
\]
So,
\[
\frac{F_k}{\beta_k} \leq \frac{\delta^{k-1} F_1 + \lambda_{k-1} + \delta \lambda_{k-2} + \ldots + \delta^{k-2} \lambda_1}{2^k - 1 - 2^{k-1} - \delta \beta_1} = \frac{F_1 + \lambda_{k-1} + \lambda_{k-2} + \ldots + \lambda_1}{2^k - 1 - 2^{k-1} - \delta \beta_1}.
\] (6.27)
Since
\[
\frac{\lambda_n}{\delta^n} = \frac{b}{\delta^n} + \frac{\ell^*}{\delta^n} \ln \left( \frac{2\delta^{n-1} - \delta^{n-1} - 1}{\delta - 1} \right)
\leq \frac{b}{\delta^n} + \frac{\ell^*}{\delta^n} \ln \left( \frac{2\delta^n}{\delta - 1} \right),
\]
it follows from (6.27), that
\[
\frac{F_k}{\beta_k} \leq \frac{F_1 + \frac{b}{\delta-1} + \frac{\ell^*}{\delta-1} \ln \left( \frac{2\delta^{k-1}}{\delta-1} \right) + \ldots + \frac{\ell^*}{\delta-1} \ln \left( \frac{2\delta}{\delta-1} \right)}{2^k - 1 - 2^{k-1} - \delta \beta_1} \leq \frac{F_1 + \frac{b}{\delta-1} + \ell^* \left[ \frac{1}{\delta-1} \ln \left( \frac{2\delta}{\delta-1} \right) + \ldots + \frac{1}{\delta} \ln \left( \frac{2\delta}{\delta-1} \right) \right]}{2^k - 1 - 2^{k-1} - \delta \beta_1} \leq \frac{F_1 + \frac{b}{\delta-1} + \ell^* \left[ \frac{1}{\delta-1} \ln \left( \frac{2\delta}{\delta-1} \right) + \ldots + \frac{1}{\delta} \ln \delta \sum_{n=1}^{\infty} \frac{n}{\delta^n} \right]}{2^k - 1 - 2^{k-1} - \delta \beta_1} \to 0.
\] (6.28)
Now, going back to the definition of $F_k$, we obtain

$$|u_n(x)| \leq \|u_n\|_\infty = \limsup_{k \to \infty} \|u_n\|_{\beta_k} \leq \limsup_{k \to \infty} e^{\frac{F_k}{\beta_k}} \leq e^{\beta_0} \text{ for all } x \in \Omega,$$

and

$$|u(x)| = \lim_{n \to \infty} |u_n(x)| \leq e^{\beta_0} \text{ a.e. } x \in \Omega,$$

because $u_n(x) \to u(x)$ a.e. in $\Omega$, that is, $u \in L^\infty(\Omega)$. This ends our proof. □

**Proof of Corollary 1.1:** First, let $u \in W^{1,\Phi}_0(\Omega)$ be a solution of (1.1). Take $\varphi \in W^{1,\Phi}_0(\Omega)$, and $\varphi_n \in C^\infty(\Omega)$ such that $\varphi_n \to \varphi$ in $W^{1,\Phi}_0(\Omega)$. So, by taking $\sqrt{\theta} + |\varphi_n - \varphi_k| - \epsilon$, for some $\theta \in \mathbb{N}$, as a test function, we obtain

$$0 \leq \int_\Omega \frac{a(x)}{u^\alpha}[\sqrt{\theta} + |\varphi_n - \varphi_k| - \epsilon] dx$$

for every $\epsilon > 0$ given. Making $\epsilon \to 0$, we find that

$$\left(\frac{a\varphi_n}{u^\alpha}\right)$$

is a Cauchy sequence in $L^1(\Omega)$, so that $(a\varphi_n)/u^\alpha \to v \in L^1(\Omega)$. Since, $\varphi_n(x) \to \varphi(x)$ a.e. in $\Omega$, we have that $v = (a\varphi)/u^\alpha$. By hypothesis, $u \in W^{1,\Phi}_0(\Omega)$ satisfies (see (2.2))

$$\int_\Omega \phi(|\nabla u|)|\nabla u| \nabla \varphi_n dx = \int_\Omega \frac{a(x)}{u^\alpha} \varphi_n dx,$$

and, passing to the limit, it follows that

$$\int_\Omega \phi(|\nabla u|)|\nabla u| \nabla \varphi dx = \int_\Omega \frac{a(x)}{u^\alpha} \varphi dx \text{ for all } \varphi \in W^{1,\Phi}_0(\Omega). \quad (6.30)$$

To complete the proof of the uniqueness, let $v \in W^{1,\Phi}_0(\Omega)$ be another solution of (1.1). Now assuming that $u, v \in W^{1,\Phi}_0(\Omega)$, $u \neq v$ satisfy (6.30), setting $\varphi = u - v$ and using the fact that $\Delta_\Phi$ is strictly monotone, we obtain

$$0 \leq \int_\Omega (\phi(|\nabla u|)\nabla u - \phi(|\nabla v|)\nabla v)(\nabla u - \nabla v) dx \quad (6.31)$$

$$= \int_\Omega a(x) \left(\frac{1}{u^\alpha} - \frac{1}{v^\alpha}\right) (u - v) dx < 0,$$

impossible. Now, we proceed to the regularity. First (i). In this case, we have $a_n = a$ for $n$ large enough. So, as a consequence of the Comparison Principle, like at the end of the proof in Lemma 5.1, that $u_{n+1} \geq u_n$. Besides this, if we assume that

$$\Omega_0 := \left\{ x \in \Omega \mid u_{n+1}(x) + \frac{1}{n+1} > u_n(x) + \frac{1}{n} \right\} \subset \subset \Omega,$$
is not empty, then we would obtain \(-\Delta \Phi (u_{n+1} + 1/(n+1)) \leq -\Delta \Phi (u_n + 1/n)\) in \(\Omega_0\), that is, 
\(u_{n+1}(x) + \frac{1}{n+1} \leq u_n(x) + \frac{1}{n}\) in \(\Omega_0\). This is impossible. So, we have
\[
0 \leq u_n - u_k \leq \frac{1}{k} - \frac{1}{n} \quad \text{in} \quad \Omega.
\]

Since \((u_n) \subset C^1(\Omega)\), we obtain that \(u_n\) converges uniformly to \(u\), that is, \(u \in C(\overline{\Omega})\).

Proof of \((ii)\). It just follows from the same arguments that we used to proof Theorem 2.2 by taking \(b = 0\). This ends our proof.

\[\square\]

7 Appendix - On Orlicz-Sobolev spaces

In this section we present for, the reader’s convenience, several results/notation used in the paper. The reader is referred to \([11, 32]\) regarding basics on Orlicz-Sobolev spaces. The usual norm on \(L_\Phi(\Omega)\) is, (Luxemburg norm),
\[
\|u\|_\Phi = \inf \left\{ \lambda > 0 \mid \int_\Omega \Phi \left( \frac{u(x)}{\lambda} \right) dx \leq 1 \right\},
\]
while the Orlicz-Sobolev norm of \(W^{1,\Phi}(\Omega)\) is
\[
\|u\|_{1,\Phi} = \|u\|_\Phi + \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_\Phi.
\]

We denote by \(W^{1,\Phi}_0(\Omega)\) the closure of \(C_\infty(\Omega)\) with respect to the Orlicz-Sobolev norm of \(W^{1,\Phi}(\Omega)\). It remind that
\[
\tilde{\Phi}(t) = \max_{s \geq 0} \{ ts - \Phi(s) \}, \quad t \geq 0.
\]

It turns out that \(\Phi\) and \(\tilde{\Phi}\) are N-functions satisfying the \(\Delta_2\)-condition, (cf. \([32, p\ 22]\)). In addition, \(L_\Phi(\Omega)\) and \(W^{1,\Phi}(\Omega)\) are reflexive and Banach spaces.

Remark 7.1 It is well known that \((\phi_3)\) implies that the condition
\[
(\phi_3)' \quad \ell \leq \frac{\phi(t)t^2}{\Phi(t)} \leq m, \quad t > 0,
\]
is verified. Furthermore, under this condition, \(\Phi, \tilde{\Phi} \in \Delta_2\).

By the Poincaré Inequality (see e.g. \([17]\)), that is, the inequality
\[
\int_\Omega \Phi(u) dx \leq \int_\Omega \tilde{\Phi}(2d_\Omega \|\nabla u\|) dx,
\]
where \(d_\Omega = \text{diam}(\Omega)\), it follows that
\[
\|u\|_\Phi \leq 2d_\Omega \|\nabla u\|_\Phi \quad \text{for all} \quad u \in W^{1,\Phi}_0(\Omega).
\]
As a consequence of this, we have that \( \|u\| := \|\nabla u\|_\Phi \) defines a norm in \( W_0^{1,\Phi}(\Omega) \) that is equivalent to \( \|\cdot\|_{1,\Phi} \). Let \( \Phi_* \) be the inverse of the function
\[
t \in (0, \infty) \mapsto \int_0^t \frac{\Phi^{-1}(s)}{s^{N+1}} ds
\]
which can be extended to \( \mathbb{R} \) by \( \Phi_*(t) = \Phi_*(-t) \) for \( t \leq 0 \).

We say that an \( N \)-function \( \Psi \) grows essentially more slowly (grows more slowly) than \( \Upsilon \), denoted by \( \Psi \ll \Upsilon \) (\( \Psi < \Upsilon \)), if
\[
\lim_{t \to \infty} \frac{\Psi(\lambda t)}{\Phi_*(t)} = 0 \quad \text{for each } \lambda > 0
\]
(\( \Psi(t) \leq \Upsilon(kt) \) for all \( t \geq t_0 \) for some \( k, t_0 > 0 \)).

The imbeddings below (cf. \([1]\)) were used in this paper. First, we have
\[
W_0^{1,\Phi}(\Omega) \hookrightarrow L_\Psi(\Omega) \quad \text{if } \Phi < \Psi \ll \Phi_*,
\]
and in particular,
\[
W_0^{1,\Phi}(\Omega) \hookrightarrow L_\Phi(\Omega),
\]
because \( \Phi \ll \Phi_* \) (cf. \([18, \text{Lemma 4.14}]\)). Furthermore,
\[
W_0^{1,\Phi}(\Omega) \hookrightarrow L_{\Phi_*}(\Omega).
\]
Besides this, It is worth mentioning that if \( (\phi_1) - (\phi_2) \) and \( (\phi_3)' \) are satisfied (cf. \([8, \text{Lemma D.2}]\)), then
\[
L_\Phi(\Omega) \hookrightarrow L^\ell(\Omega).
\]

We used in this text the notation \( L_\Psi^{\psi}(\Omega) \) in the sense that \( u \in L_\Psi^{\psi}(\Omega) \) if and only if \( u \in L_\Psi(\Omega) \) for all \( U \subset \subset \Omega \).

**Lemma 7.1** (cf. \([13]\)) Assume that \( \phi \) satisfies \( (\phi_1) - (\phi_3) \) hold. Set
\[
\zeta_0(t) = \min\{t^\ell, t^m\} \quad \text{and} \quad \zeta_1(t) = \max\{t^\ell, t^m\}, \quad t \geq 0.
\]
Then \( \Phi \) satisfies
\[
\zeta_0(t)\Phi(\rho) \leq \Phi(\rho t) \leq \zeta_1(t)\Phi(\rho), \quad \rho, t > 0,
\]
\[
\zeta_0(\|u\|_\Phi) \leq \int_\Omega \Phi(u) dx \leq \zeta_1(\|u\|_\Phi), \quad u \in L_\Phi(\Omega).
\]

**Lemma 7.2** (cf. \([13]\)) Assume that \( \phi \) satisfies \( (\phi_1) - (\phi_3) \) and \( 1 < \ell, m < N \) hold. Set
\[
\zeta_2(t) = \min\{t^{\tilde{\ell}}, t^{\tilde{m}}\} \quad \text{and} \quad \zeta_3(t) = \max\{t^{\tilde{\ell}}, t^{\tilde{m}}\}, \quad t \geq 0,
\]
where \( \tilde{m} = m/(m-1) \) and \( \tilde{\ell} = \ell/(\ell-1) \). Then
\[
\tilde{\ell} \leq \frac{t^{2\tilde{\ell}}\Phi'(t)}{\Phi(t)} \leq \tilde{m}, \quad t > 0,
\]
\[
\zeta_2(t)\Phi(\rho) \leq \Phi(\rho t) \leq \zeta_3(t)\Phi(\rho), \quad \rho, t > 0,
\]
\[
\zeta_2(\|u\|_\Phi) \leq \int_\Omega \Phi(u) dx \leq \zeta_3(\|u\|_\Phi), \quad u \in L_{\Phi}(\Omega).
\]
Lemma 7.3 Let $\Phi$ be an $N$-function satisfying $\Delta_2$. Let $(u_n) \subset L_\Phi(\Omega)$ be a sequence such that $u_n \to u$ in $L_\Phi(\Omega)$. Then there is a subsequence $(u_{n_k}) \subseteq (u_n)$ such that

(i) $u_{n_k}(x) \to u(x)$ a.e. $x \in \Omega$,

(ii) there is $h \in L_\Phi(\Omega)$ such that $|u_{n_k}| \leq h$ a.e. in $\Omega$.

Proof:(Sketch) We have that $\int_\Omega \Phi(u_n - u)dx \to 0$. By $\boxed{} L_\Phi(\Omega) \hookrightarrow L^1(\Omega)$. So there are a subsequence, we keep the notation, and $\tilde{h} \in L^1(\Omega)$ such that $u_n \to u$ a.e. in $\Omega$ and $\Phi(u_n - u) \leq \tilde{h}$ a.e. in $\Omega$.

Since $\Phi$ is convex, increasing and satisfies $\Delta_2$, we have

\[
\begin{align*}
\Phi(|u_n|) &\leq C\Phi\left(\frac{|u_n - u| + |u|}{2}\right) \\
&\leq \frac{C}{2} \left[\Phi(|u_n - u|) + \Phi(|u|)\right] \\
&\leq \frac{C}{2} \left[\tilde{h} + \Phi(|u|)\right],
\end{align*}
\]

that is,

\[
|u_n| \leq \Phi^{-1}\left(\frac{C}{2} \left(\tilde{h} + \Phi(|u|)\right)\right) := h \in L_\Phi(\Omega),
\]

because $\tilde{h} \in L^1(\Omega)$, $\Phi(|u|) \in L^1(\Omega)$, and

\[
\begin{align*}
\int_\Omega \Phi(h)dx &= \int_\Omega \Phi\left(\Phi^{-1}\left(\frac{K}{2} \left(\tilde{h} + \Phi(|u|)\right)\right)\right)dx \\
&= \int_\Omega \left(\frac{K}{2} \left(\tilde{h} + \Phi(|u|)\right)\right)dx < \infty,
\end{align*}
\]

showing that $h \in L_\Phi(\Omega)$.

\[\square\]

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