CASTELNUOVO-MUMFORD REGULARITY FOR COMPLEXES AND WEAKLY KOSZUL MODULES

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Abstract. Let \( A \) be a noetherian AS regular Koszul quiver algebra (if \( A \) is commutative, it is essentially a polynomial ring), and \( \text{gr} \, A \) the category of finitely generated graded left \( A \)-modules. Following Jørgensen, we define the Castelnuovo-Mumford regularity \( \text{reg}(M^\bullet) \) of a complex \( M^\bullet \in D^b(\text{gr} \, A) \) in terms of the local cohomologies or the minimal projective resolution of \( M^\bullet \). Let \( A^! \) be the quadratic dual ring of \( A \). For the Koszul duality functor \( G : D^b(\text{gr} \, A) \rightarrow D^b(\text{gr} \, A^!) \), we have \( \text{reg}(M^\bullet) = \max \{ i \mid H^i(G(M^\bullet)) \neq 0 \} \). Using these concepts, we interpret results of Martinez-Villa and Zacharia concerning weakly Koszul modules (also called componentwise linear modules) over \( A^! \). As an application, refining a result of Herzog and Römer, we show that if \( J \) is a monomial ideal of an exterior algebra \( E = \bigwedge \langle y_1, \ldots, y_d \rangle \), \( d \geq 3 \), then the \((d-2)\text{nd}\) syzygy of \( E/J \) is weakly Koszul.

1. Introduction

Let \( S = K[x_1, \ldots, x_d] \) be a polynomial ring over a field \( K \). We regard \( S \) as a graded ring with \( \text{deg} \, x_i = 1 \) for all \( i \). The following is a well-known result.

Theorem 1.1 (c.f. [4]). Let \( M \) be a finitely generated graded \( S \)-module. For an integer \( r \), the following conditions are equivalent.

1. \( H^i_m(M)_j = 0 \) for all \( i, j \in \mathbb{Z} \) with \( i + j > r \).
2. The truncated module \( M_{\geq r} := \bigoplus_{i \geq r} M_i \) has an \( r \)-linear free resolution.

Here \( m := (x_1, \ldots, x_d) \) is the irrelevant ideal of \( S \), and \( H^i_m(M) \) is the \( i \)th local cohomology module.

If the conditions of Theorem 1.1 are satisfied, we say \( M \) is \( r \)-regular. For a sufficiently large \( r \), \( M \) is \( r \)-regular. We call \( \text{reg}(M) = \min \{ r \mid M \text{ is } r \text{-regular} \} \) the Castelnuovo-Mumford regularity of \( M \). This is a very important invariant in commutative algebra.

Let \( A \) be a noetherian AS regular Koszul quiver algebra with the graded Jacobson radical \( m := \bigoplus_{i \geq 1} A_i \). If \( A \) is commutative, \( A \) is essentially a polynomial ring. When \( A \) is connected (i.e., \( A_0 = K \)), it is the coordinate ring of a “noncommutative projective space” in noncommutative algebraic geometry. Let \( \text{gr} \, A \) be the category of finitely generated graded left \( A \)-modules and their degree preserving maps. (For a graded ring \( B \), \( \text{gr} \, B \) means the similar category for \( B \).) The local cohomology module \( H^i_m(M) \) of \( M \in \text{gr} \, A \) behaves pretty much like in the commutative case. For example, we have “Serre duality theorem” for the derived category \( D^b(\text{gr} \, A) \). See [10, 23] and Theorem 2.7 below. By virtue of this duality, we can show that Theorem 1.1 also holds for bounded complexes in \( \text{gr} \, A \).
Theorem 1.2. For a complex $M^\bullet \in D^b(\text{gr } A)$ and an integer $r$, the following conditions are equivalent.

1. $H^i_m(M^\bullet)_j = 0$ for all $i, j \in \mathbb{Z}$ with $i + j > r$.

2. The truncated complex $(M^\bullet)_{\geq r}$ has an $r$-linear projective resolution.

Here $(M^\bullet)_{\geq r}$ is the subcomplex of $M^\bullet$ whose $i$th term is $(M^\bullet)_{\geq r-i}$.

For a sufficiently large $r$, the conditions of the above theorem are satisfied. The regularity $\text{reg}(M^\bullet)$ of $M^\bullet$ is defined in the natural way. When $A$ is connected, Jørgensen ([10]) has studied the regularity of complexes, and essentially proved the above result. See also [9, 15]. (Even in the case when $A$ is a polynomial ring, it seems that nobody had considered Theorem 1.2 before [10].) But his motivation and treatment are slightly different from ours.

For $M^\bullet \in D^b(\text{gr } A)$, set $\mathcal{H}(M^\bullet)$ to be a complex such that $\mathcal{H}(M^\bullet)^i = H^i(M)$ for all $i$ and the differential maps are zero. Then we have $\text{reg}(\mathcal{H}(M^\bullet)) \geq \text{reg}(M^\bullet)$. The difference $\text{reg}(\mathcal{H}(M^\bullet)) - \text{reg}(M^\bullet)$ is a theme of the last section of this paper.

Let $A^!$ be the quadratic dual ring of $A$. For example, if $S = K[x_1, \ldots, x_d]$ is a polynomial ring, then $S^!$ is an exterior algebra $E = \bigwedge \langle y_1, \ldots, y_d \rangle$. It is known that $A^!$ is always Koszul, finite dimensional, and selfinjective. The Koszul duality functors $\mathcal{F} : D^b(\text{gr } A^!) \to D^b(\text{gr } A)$ and $\mathcal{G} : D^b(\text{gr } A) \to D^b(\text{gr } A^!)$ give a category equivalence $D^b(\text{gr } A^!) \cong D^b(\text{gr } A)$ (see [2]). It is easy to check that

$$\text{reg}(M^\bullet) = \max \{ i \mid H^i(\mathcal{G}(M^\bullet)) \neq 0 \}$$

for $M^\bullet \in D^b(\text{gr } A)$.

Let $\text{gr } A^{\text{op}}$ be the category of finitely generated graded right $A$-modules. The above results on $\text{gr } A$ also hold for $\text{gr } A^{\text{op}}$. Moreover, we have

$$\text{reg}(\mathbf{R}\text{Hom}_A(M^\bullet, D^\bullet)) = -\min \{ i \mid H^i(\mathcal{G}(M^\bullet)) \neq 0 \}$$

for $M^\bullet \in D^b(\text{gr } A)$. Here $D^\bullet$ is a balanced dualizing complex of $A$, which gives duality functors between $D^b(\text{gr } A)$ and $D^b(\text{gr } A^{\text{op}})$.

Let $B$ be a noetherian Koszul algebra. For $M \in \text{gr } B$ and $i \in \mathbb{Z}$, $M_{(i)}$ denotes the submodule of $M$ generated by the degree $i$ component $M_i$ of $M$. We say $M$ is weakly Koszul if $M_{(i)}$ has a linear projective resolution for all $i$. This definition is different from the original one given in [13], but they are equivalent. (Weakly Koszul modules are also called “componentwise linear modules” by some commutative algebraists.) Martinez-Villa and Zacharia proved that if $N \in \text{gr } A^!$ then the $i$th syzygy $\Omega_i(N)$ of $N$ is weakly Koszul for $i \gg 0$. For $N \in \text{gr } A^!$, set

$$\text{lpd}(N) := \min \{ i \in \mathbb{N} \mid \Omega_i(N) \text{ is weakly Koszul} \}.$$

Let $N \in \text{gr } A^!$ and $N' := \text{Hom}_{A^!}(N, A^!)$ be the dual of $N$. In Theorem 1.4 we show that $N$ is weakly Koszul if and only if $\text{reg}(\mathcal{H} \circ \mathcal{F}^{\text{op}}(N')) = 0$, where $\mathcal{F}^{\text{op}} : D^b(\text{gr } A^{\text{op}}) \to D^b(\text{gr } A^{\text{op}})$ is the Koszul duality functor. (Since $\text{reg}(\mathcal{F}^{\text{op}}(N')) = 0$, we have $\text{reg}(\mathcal{H} \circ \mathcal{F}^{\text{op}}(N')) \geq 0$ in general.) Moreover, we have

$$\text{lpd}(N) = \text{reg}(\mathcal{H} \circ \mathcal{F}^{\text{op}}(N'))$$

(Corollary 1.7). As an application of this formula, we refine a result of Herzog and Römer on monomial ideals of an exterior algebra. Among other things, in
Proposition 4.15, we show that if \( J \) is a monomial ideal of an exterior algebra \( E = \bigwedge \langle y_1, \ldots, y_d \rangle \), \( d \geq 3 \), then \( \text{lpd}(E/J) \leq d - 2 \).

Finally, we remark that Herzog and Iyengar ([8]) studied the invariant \( \text{lpd} \) and related concepts over noetherian commutative (graded) local rings. Among other things, they proved that \( \text{lpd}(N) \) is always finite over some “nice” local rings (e.g., complete intersections whose associated graded rings are Koszul).

2. Preliminaries

Let \( K \) be a field. The ring \( A \) treated in this paper is a (not necessarily commutative) \( K \)-algebra with some nice properties. More precisely, \( A \) is a noetherian AS regular Koszul quiver algebra. If \( A \) is commutative, it is essentially a polynomial ring. But even in this case, most results in §4 and a few results in §3 are new. (In the polynomial ring case, many results in §3 were obtained in [3].) So one can read this paper assuming that \( A \) is a polynomial ring.

We sketch the definition and basic properties of graded quiver algebras here. See [3] for further information.

Let \( Q \) be a finite quiver. That is, \( Q = (Q_0, Q_1) \) is an oriented graph, where \( Q_0 \) is the set of vertices and \( Q_1 \) is the set of arrows. Here \( Q_0 \) and \( Q_1 \) are finite sets. The path algebra \( KQ \) is a positively graded algebra with grading given by the lengths of paths. We denote the graded Jacobson radical of \( KQ \) by \( J \). That is, \( J \) is the ideal generated by all arrows. If \( I \subset J^2 \) is a graded ideal, we say \( A = KQ/I \) is a graded quiver algebra. Of course, \( A = \bigoplus_{i \geq 0} A_i \) is a graded ring such that the degree \( i \) component \( A_i \) is a finite dimensional \( K \)-vector space for all \( i \). The subalgebra \( A_0 \) is a product of copies of the field \( K \), one copy for each element of \( Q_0 \). If \( A_0 = K \) (i.e., \( Q \) has only one vertex), we say \( A \) is connected. Let \( R = \bigoplus_{i \geq 0} R_i \) be a graded algebra with \( R_0 = K \) and \( \dim_K R_i =: n < \infty \). If \( R \) is generated by \( R_1 \) as a \( K \)-algebra, then it can be regarded as a graded quiver algebra over a quiver with one vertex and \( n \) loops. Let \( \mathfrak{m} := \bigoplus_{i \geq 1} A_i \) be the graded Jacobson radical of \( A \). Unless otherwise specified, we assume that \( A \) is left and right noetherian throughout this paper.

Let \( \text{Gr} A \) (resp. \( \text{Gr} A^{op} \)) be the category of graded left (resp. right) \( A \)-modules and their degree preserving \( A \)-homomorphisms. Note that the degree \( i \) component \( M_i \) of \( M \in \text{Gr} A \) (or \( M \in \text{Gr} A^{op} \)) is an \( A_0 \)-module for each \( i \). Let \( \text{gr} A \) (resp. \( \text{gr} A^{op} \)) be the full subcategory of \( \text{Gr} A \) (resp. \( \text{Gr} A^{op} \)) consisting of finitely generated modules. Since we assume that \( A \) is noetherian, \( \text{gr} A \) and \( \text{gr} A^{op} \) are abelian categories. In the sequel, we will define several concepts for \( \text{Gr} A \) and \( A \). But the corresponding concepts for \( \text{Gr} A^{op} \) and \( A^{op} \) can be defined in the same way.

For \( n \in \mathbb{Z} \) and \( M \in \text{Gr} A \), set \( M_{\geq n} := \bigoplus_{i \geq n} M_i \) to be a submodule of \( M \), and \( M_{\leq n} := \bigoplus_{i \leq n} M_i \) to be a graded \( K \)-vector space. The \( n \)th shift \( M(n) \) of \( M \) is defined by \( M(n)_i = M_{n+i} \). Set \( \sigma(M) := \sup \{ i \mid M_i \neq 0 \} \) and \( \iota(M) := \inf \{ i \mid M_i \neq 0 \} \). If \( M = 0 \), we set \( \sigma(M) = -\infty \) and \( \iota(M) = +\infty \). Note that if \( M \in \text{gr} A \) then \( \iota(M) > -\infty \). For a complex \( M^* \) in \( \text{Gr} A \), set

\[
\sigma(M^*) := \sup \{ \sigma(H^i(M^*)) + i \mid i \in \mathbb{Z} \}
\]

and

\[
\iota(M^*) := \inf \{ \iota(H^i(M^*)) + i \mid i \in \mathbb{Z} \}.
\]
For \( v \in Q_0 \), we have the idempotent \( e_v \) associated with \( v \). Note that \( 1 = \sum_{v \in Q_0} e_v \). Set \( P_v := A e_v \) and \( v P := e_v A \). Then we have \( A A = \bigoplus_{v \in Q_0} P_v \) and \( A A = \bigoplus_{v \in Q_0} (v P) \). Each \( P_v \) and \( v P \) are indecomposable projectives. Conversely, any indecomposable projective in \( Gr A \) (resp. \( Gr A^{\text{op}} \)) is isomorphic to \( P_v \) (resp. \( v P \)) for some \( v \in Q_0 \) up to degree shifting. Set \( K_v := P_v/(m P_v) \) and \( \nu K := P_v/(v P m) \). Each \( K_v \) and \( \nu K \) are simple. Conversely, any simple object in \( Gr A \) (resp. \( Gr A^{\text{op}} \)) is isomorphic to \( K_v \) (resp. \( \nu K \)) for some \( v \in Q_0 \) up to degree shifting.

We say a graded left (or right) \( A \)-module \( M \) is \textit{locally finite} if \( \dim_K M_i < \infty \) for all \( i \). If \( M \in \text{gr} A \), then it is locally finite. Let \( I f A \) (resp. \( I f A^{\text{op}} \)) be the full subcategory of \( Gr A \) (resp. \( Gr A^{\text{op}} \)) consisting of locally finite modules.

Let \( C^b(Gr A) \) be the category of bounded cochain complexes in \( Gr A \), and \( D^b(Gr A) \) its derived category. We have similar categories for \( Gr A^{\text{op}} \), \( I f A \), \( I f A^{\text{op}} \), \text{gr} \( A \), and \( \text{gr} A^{\text{op}} \). For a complex \( M^\bullet \) and an integer \( p \), let \( M^\bullet[p] \) be the \( p \)-th translation of \( M^\bullet \). That is, \( M^\bullet[p] \) is a complex with \( M^i[p] = M^{i+p} \). Since \( D^b(\text{gr} A) \cong D^b_{\text{gr} A}(Gr A) \cong D^b_{\text{gr} A}(lf A) \), we freely identify these categories. A module \( M \) can be regarded as a complex \( \cdots \rightarrow 0 \rightarrow M \rightarrow 0 \rightarrow \cdots \) with \( M \) at the 0-th term. We can regard \( Gr A \) as a full subcategory of \( C^b(Gr A) \) and \( D^b(Gr A) \) in this way.

For \( M, N \in Gr A \), set \( \text{Hom}_{Gr A}(M, N) := \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{Gr A}(M, N(i)) \) to be a graded \( K \)-vector space with \( \text{Hom}_{Gr A}(M, N)_i = \text{Hom}_{Gr A}(M, N(i)) \). Similarly, we can also define \( \text{Hom}^\bullet_{Gr A}(M^\bullet, N^\bullet) \), \( \text{RHom}_{Gr A}(M^\bullet, N^\bullet) \), and \( \text{Ext}^i_{Gr A}(M^\bullet, N^\bullet) \) for \( M^\bullet, N^\bullet \in D^b(Gr A) \).

If \( V \) is a \( K \)-vector space, \( V^\bullet \) denotes the dual vector space \( \text{Hom}_K(V, K) \). For \( M \in \text{Gr} A \) (resp. \( M \in \text{Gr} A^{\text{op}} \)), \( M^\vee := \bigoplus_{i \in \mathbb{Z}} (M_i)^* \) has a graded \textit{right} (resp. \textit{left}) \( A \)-module structure given by \( (fa)(x) = f(ax) \) (resp. \( (af)(x) = f(xa) \)) and \( (M^\vee)_i = (M_{-i})^* \). If \( M \in I f A \), then \( M^\vee \in I f A^{\text{op}} \) and \( M^\vee \cong M \). In other words, \( (\ -)^\vee \) gives exact duality functors between \( I f A \) and \( I f A^{\text{op}} \), which can be extended to duality functors between \( C^b(I f A) \) and \( C^b(I f A^{\text{op}}) \), or between \( D^b(I f A) \) and \( D^b(I f A^{\text{op}}) \). In this paper, when we say \( W \) is an \( A-A \) bimodule, we always assume that \( (aw)a' = a(wa') \) for all \( w \in W \) and \( a, a' \in A \). If \( W \) is a graded \( A-A \) bimodule, then so is \( W^\vee \).

It is easy to see that \( I_v := (v P)^\vee \) (resp. \( I_v := (P_v)^\vee \)) is injective in \( Gr A \) (resp. \( Gr A^{\text{op}} \)). Moreover, \( I_v \) and \( I_v \) are graded injective hulls of \( K_v \) and \( \nu K \) respectively. In particular, the \( A-A \) bimodule \( A^\vee \) is injective both in \( Gr A \) and in \( Gr A^{\text{op}} \).

Let \( W \) be a graded \( A-A \) bimodule. For \( M \in Gr A \), we can regard \( \text{Hom}_{Gr A}(M, W) \) as a graded \textit{right} \( A \)-module by \( (fa)(x) = f(xa) \). We can also define \( \text{RHom}_{Gr A}(M^\bullet, W) \in D^b(Gr A^{\text{op}}) \) and \( \text{Ext}^i_{Gr A^0}(M^\bullet, W) \in Gr A^{\text{op}} \) for \( M^\bullet \in D^b(Gr A) \) in this way. Similarly, for \( M^\bullet \in D^b(Gr A^{\text{op}}) \), we can make \( \text{RHom}_{Gr A^0}(M^\bullet, W) \) and \( \text{Ext}^i_{Gr A^{\text{op}}}(M^\bullet, W) \) (bounded complex of) graded left \( A \)-modules. For \( M \in Gr A \), we can regard \( \text{Hom}_{Gr A}(W, M) \) as a graded \textit{left} \( A \)-module by \( (af)(x) = f(xa) \).
For the functor $\text{Hom}_A(-, W)$, we mainly consider the case when $W = A$ or $W = A^\vee$. But, we have $\text{Hom}_A(-, A^\vee) \cong (-)^\vee$. To see this, note that

$$(M^\vee)_i = \text{Hom}_K(M_{-i}, K) = \bigoplus_{v \in Q_0} \text{Hom}_K(e_v M_{-i}, K)$$

$$\cong \bigoplus_{v \in Q_0} \text{Hom}_K(e_v M_{-i}, K_v)$$

$$\cong \text{Hom}_{A_0}(M_{-i}, A_0).$$

Via the identification $(A^\vee)_0 \cong (A_0)^* \cong A_0$, $f \in (M^\vee)_i \cong \text{Hom}_{A_0}(M_{-i}, A_0)$ gives a morphism $f^\prime : M_{\geq -i} \to A^\vee(i)$ in Gr $A$. Since $\text{Hom}_{\text{Gr} A}(M/M_{\geq -i}, A^\vee(i)) = 0$ and $A^\vee$ is injective, the short exact sequence $0 \to M_{\geq -i} \to M \to M/M_{\geq -i} \to 0$ induces a unique extension $f^\prime^\vee : M \to A^\vee(i)$ of $f^\prime$. From this correspondence, we have $\text{Hom}_A(M, A^\vee) \cong M^\vee$.

Let $P^\bullet$ be a right bounded complex in gr $A$ such that each $P^i$ is projective. We say $P^\bullet$ is minimal if $d(P^i) \subset \mathfrak{m} P^{i+1}$ for all $i$. Here $d$ is the differential map. Any complex $M^\bullet \in C^b(\text{gr} A)$ has a minimal projective resolution, that is, we have a minimal complex $P^\bullet$ of projective objects and a graded quasi-isomorphism $P^\bullet \to M^\bullet$. A minimal projective resolution of $M^\bullet$ is unique up to isomorphism. We denote a graded module $A/\mathfrak{m}$ by $A_0$. Set $\beta^{i,j}(M^\bullet) := \dim_K \text{Ext}^{-i}_A(M^\bullet, A_0)_{-j}$. Let $P^\bullet$ be a minimal projective resolution of $M^\bullet$, and $P^i := \bigoplus_{l=1}^m T_{i,l}$ an indecomposable decomposition. Then we have

$$\beta^{i,j}(M^\bullet) = \# \{ l \mid T_{i,l}(j) \cong P_v \text{ for some } v \}.$$ 

We can also define $\beta^{i,j}(M^\bullet)$ as the dimension of $\text{Tor}_A^i(A_0, M^\bullet)_j$. This definition must be much more familiar to commutative algebraists. Note that $\beta^{i,j}(-)$ is an invariant of isomorphism classes of the derived category $D^b(\text{gr} A)$. Note that these facts on minimal projective resolutions also hold over any noetherian graded algebra.

**Definition 2.1.** Let $A$ be a (not necessarily noetherian) graded quiver algebra. We say $A$ is **Artin-Schelter regular** (AS-regular, for short), if

- $A$ has finite global dimension $d$.
- $\text{Ext}_A^i(K_v, A) = \text{Ext}_A^i(K_v, A) = 0$ for all $i \neq d$ and all $v \in Q_0$.
- There are a permutation $\delta$ on $Q_0$ and an integer $n_v$ for each $v \in Q_0$ such that $\text{Ext}_A^d(K_v, A) \cong \delta(v) K(n_v)$ (equivalently, $\text{Ext}_A^{d}(\nu K, A) \cong K_{\delta^{-1}(v)}(n_v)$) for all $v$.

**Remark 2.2.** The AS regularity is a very important concept in non-commutative algebraic geometry. In the original definition, it is assumed that an AS regular algebra $A$ is connected and there is a positive real number $\gamma$ such that $\dim_K A_n < n^\gamma$ for $n \gg 0$, while some authors do not require the latter condition. We also remark that Martinez-Villa and coworkers called rings satisfying the conditions of Definition 2.1 **generalized Auslander regular algebras** in [6, 11].
Definition 2.3. For an integer \( l \in \mathbb{Z} \), we say \( M^\bullet \in \text{gr} \ A \) has an \( l \)-linear (projective) resolution, if
\[
\beta^i,j(M^\bullet) \neq 0 \Rightarrow i + j = l.
\]
If \( M^\bullet \) has an \( l \)-linear resolution for some \( l \), we say \( M^\bullet \) has a linear resolution.

Definition 2.4. We say \( A \) is Koszul, if the graded left \( A \)-module \( A_0 \) has a linear resolution.

In the definition of the Koszul property, we can regard \( A_0 \) as a right \( A \)-module. (We get the equivalent definition.) That is, \( A \) is Koszul if and only if any simple graded left (or, right) \( A \)-module has a linear resolution.

Lemma 2.5. If \( A \) is noetherian, AS-regular, Koszul, and has global dimension \( d \), then \( \text{Ext}^d_A(K_v,A) \cong \delta(v)K(d) \) and \( \text{Ext}^d_{A^e}(vK,A) \cong K_{\delta^{-1}(v)}(d) \) for all \( v \). Here \( \delta \) is the permutation of \( Q_0 \) given in Definition 2.1.

Proof. Since \( A \) is Koszul, \( P^{-d} \) of a minimal projective resolution \( P^\bullet : 0 \to P^{-d} \to \cdots \to P^0 \to 0 \) of \( K_v \) is generated by its degree \( d \)-part (more precisely, \( P^{-d} = P_{\delta(v)}(-d) \)).

In the rest of this paper, \( A \) is always a noetherian AS-regular Koszul quiver algebra of global dimension \( d \).

Example 2.6. (1) A polynomial ring \( K[x_1,\ldots,x_d] \) is clearly a noetherian AS-regular Koszul (quiver) algebra of global dimension \( d \). Conversely, if a regular noetherian graded algebra is connected and commutative, it is a polynomial ring.

(2) Let \( K\langle x_1,\ldots,x_d \rangle \) be the free associative algebra, and \((q_{i,j})\) a \( d \times d \) matrix with entries in \( K \) satisfying \( q_{i,j}q_{j,i} = q_{i,i} = 1 \) for all \( i,j \). Then the quotient ring \( A = K\langle x_1,\ldots,x_n \rangle/\langle x_jx_i - q_{i,j}x_ix_j \mid 1 \leq i,j \leq d \rangle \) is a noetherian AS-regular Koszul algebra with global dimension \( d \). This fact must be well-known to specialists, but we will sketch a proof here for the reader’s convenience. Since \( x_1,\ldots,x_d \in A_1 \) form a regular normalizing sequence with the quotient ring \( K = A/(x_1,\ldots,x_d) \), \( A \) is a noetherian ring with a balanced dualizing complex by \cite{[15]} Lemma 7.3]. It is not difficult to construct a minimal free resolution of the module \( K = A/\mathfrak{m} \), which is a “\( q \)-analog” of the Koszul complex of a polynomial ring \( K[x_1,\ldots,x_d] \). So \( A \) is Koszul and has global dimension \( d \). Since \( A \) has finite global dimension and admits a balanced dualizing complex, it is AS-regular (c.f. \cite{[15]} Remark 3.6 (3)).

Artin, Tate and Van den Bergh (\cite{[1]}) classified connected AS regular algebras of global dimension 3. (Their definition of AS regularity is stronger than ours. See Remark 2.2.) All of the algebras they listed are noetherian (\cite{[1]} Theroem 8.1)). But some are Koszul and some are not.

(3) A preprojective algebra is an important example of non-connected AS regular algebras. See \cite{[16]} and the references cited there for the definition of this algebra and further information. The preprojective algebra \( A \) of a finite quiver \( Q \) is a graded quiver algebra over the inverse completion \( \overline{Q} \) of \( Q \). If the quiver \( Q \) is connected (of course, it does not mean \( A \) is connected), then \( A \) is (almost) always an AS regular algebra of global dimension 2, but it is not Koszul in some cases, and not noetherian
in many cases. Let $G$ be the bipartite graph of $Q$ in the sense of [6, §3]. If $G$ is Euclidean, then $A$ is a noetherian AS-regular Koszul algebra of global dimension 2.

For $M \in \text{Gr} A$, set

$$\Gamma_m(M) = \lim_{\to} \text{Hom}_A(A/m^n, M) = \{ x \in M \mid A_n x = 0 \text{ for } n \gg 0 \} \in \text{Gr} A.$$ 

Then $\Gamma_m(\cdot)$ gives a left exact functor from $\text{Gr} A$ to itself. So we have a right derived functor $R\Gamma_m : D^b(\text{Gr} A) \to D^b(\text{Gr} A)$. For $M^\bullet \in D^b(\text{Gr} A)$, $H^i_m(M^\bullet)$ denotes the $i^{th}$ cohomology of $R\Gamma_m(M^\bullet)$, and we call it the $i^{th}$ local cohomology of $M^\bullet$. It is easy to see that $H^i_m(M^\bullet) = \lim Ext_A^i(A/m^n, M^\bullet)$. Similarly, we can define $R\Gamma_m^{\text{op}} : D^b(\text{Gr} A^{\text{op}}) \to D^b(\text{Gr} A^{\text{op}})$ and $H^i_m^{\text{op}} : D^b(\text{Gr} A^{\text{op}}) \to \text{Gr} A^{\text{op}}$ in the same way.

Let $I \in \text{Gr} A$ be an indecomposable injective. Then $\Gamma_m(I) \neq 0$, if and only if $I \cong I_v(n)$ for some $v \in Q_0$ and $n \in \mathbb{Z}$, if and only if $\Gamma_m(I) = I$. Similarly, $\text{Hom}_A(A_0, I) \neq 0$ if and only if $I \cong I_v(n)$ for some $v \in Q_0$ and $n \in \mathbb{Z}$. In this case, $\text{Hom}_A(A_0, I) = K_v(n)$. The same is true for an indecomposable injective $I \in \text{Gr} A^{\text{op}}$.

Let $I^\bullet$ be a minimal injective resolution of $A$ in $\text{Gr} A$. Since $A$ is AS regular, $I^d = 0$ for all $i > d$, $\Gamma_m(I^i) = 0$ for all $i < d$, and $\Gamma_m(I^d) = A^\vee(d)$. Hence $R\Gamma_m(A) \cong A^\vee(d)[-d]$ in $D^b(\text{gr} A)$. By the same argument as [23, Proposition 4.4], we also have $R\Gamma_m(A) \cong A^\vee(d)[-d]$ in $D^b(\text{gr} A^{\text{op}})$. It does not mean that $H^i_m(A) \cong A^\vee(d)$ as $A$-$A$ bimodules. But there is an $A$-$A$ bimodule $L$ such that $L \otimes_A H^i_m(A) \cong A^\vee(d)$ as $A$-$A$ bimodules. The underlying additive group of $L$ is $A$, but the bimodule structure is given by $A \times L \times A \ni (a, l, b) \mapsto \phi(a)lb \in A = L$ for $A$ (a fixed) $K$-algebra automorphism $\phi$ of $A$. In particular, $L \cong A$ as left $A$-modules and as right $A$-modules (separately). Note that $\phi(e_v) = e_{\delta(v)}$ for all $v \in Q_0$, where $\delta$ is the permutation on $Q_0$ appeared in Definition 2.1. If $A$ is commutative, then $\phi$ is the identity map.

We give a new $A$-$A$ bimodule structure $L'$ to the additive group $A$ by $A \times L' \times A \ni (a, l, b) \mapsto a \phi(b) \in A = L'$. Then $L' \cong \text{Hom}_A(L, A)$. Set $D^\bullet := L'(-d)[d]$. Note that $D^\bullet$ belongs both $D^b(\text{gr} A)$ and $D^b(\text{gr} A^{\text{op}})$. We have $H_i^m(D^\bullet) = H^i_{m}^{\text{op}}(D^\bullet) = 0$ for all $i \neq 0$ and $H^0_m(D^\bullet) = H_0^{\text{op}}(D^\bullet) = A^\vee$ as $A$-$A$ bimodules by the same argument as [23, §4]. Thus (an injective resolution of) $D^\bullet$ is a balanced dualizing complex of $A$ in the sense of [23] (the paper only concerns connect rings, but the definition can be generalized in the obvious way).

Easy computation shows that $\text{Hom}_A(P_v, L') \cong \delta^{-1}(v)P$ and $\text{Hom}_{A^{\text{op}}}(P_v, L') \cong P_{\delta(v)}$ for all $v \in Q_0$. Since $R\text{Hom}_A(M^\bullet, D^\bullet)$ (resp. $R\text{Hom}_{A^{\text{op}}}(M^\bullet, D^\bullet)$) for $M^\bullet \in \text{gr} A$ (resp. $M^\bullet \in \text{gr} A^{\text{op}}$) can be computed by a projective resolution of $M^\bullet$, $R\text{Hom}_A(\cdot, D^\bullet)$ and $R\text{Hom}_{A^{\text{op}}}(\cdot, D^\bullet)$ give duality functors between $D^b(\text{gr} A)$ and $D^b(\text{gr} A^{\text{op}})$. (Of course, we can also prove this by the same argument as [23, Proposition 3.4].)

**Theorem 2.7** (Yekutieli [23, Theorem 4.18], Martinez-Villa [11, Proposition 4.6]). For $M^\bullet \in D^b(\text{gr} A)$, we have

$$R\Gamma_m(M^\bullet)^\vee \cong R\text{Hom}_A(M^\bullet, D^\bullet).$$
In particular,

\[(H^i_m(M^*))_j^* \cong \text{Ext}^{-i}_m(M^*, D^*_{-j}).\]

Proof. The above result was proved by Yekutieli in the connected case. (In some sense, Martinez-Villa proved a more general result than ours, but he did not concern complexes.) But, the proof of [23, Theorem 4.18] only uses formal properties such as \(A\) is noetherian, \(\text{RHom} \cong (\text{RHom}_{\text{op}}(\text{RHom}_{\text{op}}(\text{RHom}_{\text{op}}(-, D^*)), D^*) \cong \text{Id},\) and \(\text{RHom}_{\text{op}}(\text{RHom}_{\text{op}}(\text{RHom}_{\text{op}}(-, D^*)), D^*) \cong A^\vee.\) So the proof also works in our case.

\[\Box\]

**Definition 2.8 (Jorgensen, [10]).** For \(M^* \in D^b(\text{gr} A)\), we say

\[\text{reg}(M^*) := \sigma(H^i_\text{max}(M^*)) = \sup \{i + j \mid H^i_\text{max}(M^*)_j \neq 0\}\]

is the *Castelnuovo-Mumford regularity* of \(M^*\).

By Theorem 2.7 and the fact that \(\text{RHom}(M^*, D^*) \in D^b(\text{gr} A^\text{op})\), we have \(\text{reg}(M^*) < \infty\) for all \(M^* \in D^b(\text{gr} A)\).

**Theorem 2.9 (Jorgensen, [10]).** If \(M^* \in C^b(\text{gr} A)\), then

\[(2.1) \quad \text{reg}(M^*) = \max \{i + j \mid \beta^{i,j}(M^*) \neq 0\}.\]

When \(A\) is a polynomial ring and \(M^*\) is a module, the above theorem is a fundamental result obtained by Eisenbud and Goto [4]. In the non-commutative case, under the assumption that \(A\) is connected but not necessarily regular, this has been proved by Jorgensen [10, Corollary 2.8]. (If \(A\) is not regular, we have \(\text{reg}(A) > 0\) in many cases. So one has to assume that \(\text{reg} A = 0\) there.) In our case (i.e., \(A\) is AS-regular), we have a much simpler proof. So we will give it here. This proof is also different from one given in [4].

**Proof.** Set \(Q^* := \text{Hom}^*(P^*, L'(−d)[d])\). Here \(P^*\) is a minimal projective resolution of \(M^*\), and \(L'(−d)[d]\) is the \(A\)-\(A\) bimodule given in the construction of the dualizing complex \(D^*\). Recall that \(\text{Hom}^A(P_v, L') \cong \delta^{-1(v)}P\) for all \(v \in Q_0\). Let \(s\) be the right hand side of (2.1), and \(l\) the minimal integer with the property that \(\beta^{s,l}(M^*) \neq 0\). Then \(l(Q^{−d−l}) = l − s + d,\) and \((Q^{−d−l+1})_k = 0\) (Note that \(\beta^{l−1,m}(M^*) = 0\) for all \(m \geq s − l + 1\)). Since \(Q^*\) is a minimal complex, we have

\[0 \neq H^{−d−l}(Q^*)_{−s−d} = \text{Ext}^{−d−l}_A(M^*, D^*)_{−s−d} = (H^{d+l}_m(M^*)_{−t+s−d})^*.\]

Thus \(\text{reg}(M^*) \geq \max \{i + j \mid \beta^{i,j}(M^*) \neq 0\}\).

On the other hand, if \(H^{d+l}_m(M^*)_{−t+s−d} = 0\), we have that \(\beta^{i−t−l}(M^*) \neq 0\) for some \(i \geq r\) by an argument similar to the above. Hence \(\text{reg}(M^*) \leq \max \{i + j \mid \beta^{i,j}(M^*) \neq 0\}\), and we are done.

For \(M^* \in D^b(\text{gr} A)\), set \(\mathcal{H}(M^*)\) to be the complex such that \(\mathcal{H}(M^*)^i = H^i(M)\) for all \(i\) and all differential maps are zero.

**Lemma 2.10.** We have \(\beta^{i,j}(\mathcal{H}(M^*)) \geq \beta^{i,j}(M^*)\) for all \(M^* \in D^b(\text{gr} A)\) and all \(i, j \in \mathbb{Z}\). In particular, \(\text{reg}(\mathcal{H}(M^*)) \geq \text{reg}(M^*)\).

The difference between \(\text{reg}(M^*)\) and \(\text{reg}(\mathcal{H}(M^*))\) can be arbitrary large. In the last section, we will study the relation between this difference and a work of Martinez-Villa and Zacharia [13].
Proof. The assertion easily follows from the spectral sequence
\[ E_2^{p,q} = \text{Ext}_q^p(H^{-q}(N^*), A_0) \longrightarrow \text{Ext}_q^{p+q}(N^*, A_0). \]
\[ \square \]

For a complex \( M^* \in C^b(\text{gr} \ A) \) and an integer \( r \), \( (M^*)_\geq r \) denotes the subcomplex of \( M^* \) whose \( i \)-th term is \( (M^*)_i \geq (r-i) \). Even if \( M^* \cong N^* \) in \( D^b(\text{gr} \ A) \), we have \( (M^*)_\geq r \neq (N^*)_\geq r \) in general.

In the module case, the following is a well-known property of Castelnuovo-Mumford regularity.

**Proposition 2.11.** Let \( M^* \in C^b(\text{gr} \ A) \). Then \( (M^*)_\geq r \) has an \( r \)-linear resolution if and only if \( r \geq \text{reg}(M^*) \).

To prove the proposition, we need the following lemma.

**Lemma 2.12.** For a module \( M \in \text{gr} \ A \) with \( \dim_K M < \infty \), we have \( H^0_m(M) = M \) and \( H^1_m(M) = 0 \) for all \( i \neq 0 \). In particular, \( \text{reg}(M) = \sigma(M) \) in this case.

**Proof.** If \( P^* \) is a minimal projective resolution of \( M^* \) in \( A_0 \), then \( I^* := (P^*)^* \) is a minimal injective resolution of \( M^* \). Since each indecomposable summand of \( I^* \) is isomorphic to \( I_v(n) \) for some \( v \in Q_0 \) and \( n \in \mathbb{Z} \), we have \( \Gamma_m(I^*) = I^* \).

**Proof of Proposition 2.11.** For a complex \( T^* \in D^b(\text{gr} \ A) \), it is easy to see that \( \iota(T^*) = \min \{ i + j \mid \beta_i^j(T^*) \neq 0 \} \). In particular, \( \iota(T^*) \leq \text{reg}(T^*) \). Hence \( T^* \) has an \( i \)-linear projective resolution if and only if \( \iota(T^*) = \text{reg}(T^*) = l \).

Consider the short exact sequence of complexes
\[ 0 \rightarrow (M^*)_\geq r \rightarrow M^* \rightarrow M^*/(M^*)_\geq r \rightarrow 0, \]
and set \( N^* := M^*/(M^*)_\geq r \). Note that \( \dim_K H^i(N) < \infty \) for all \( i \). By Lemmas 2.12 and 2.10, we have
\[ r > \sigma(N^*) = \max \{ \text{reg}(H^i(N^*)) + i \mid i \in \mathbb{Z} \} = \text{reg}(H(N^*)) \geq \text{reg}(N^*). \]
By the long exact sequence of \( \text{Ext}^*_A(\_ , A_0) \) induced by (2.2), we have
\[ r \leq \iota((M^*)_\geq r) \leq \text{reg}((M^*)_\geq r) \leq \max \{ \text{reg}(N^*) + 1, \text{reg}(M^*) \} \leq \max \{ r, \text{reg}(M^*) \}. \]
Moreover, if \( r < \text{reg}(M^*) \) then we have \( \text{reg}(N^*) + 1 < \text{reg}(M^*) \) and \( \text{reg}((M^*)_\geq r) = \text{reg}(M^*) > r \). Hence \( (M^*)_\geq r \) has an \( r \)-linear resolution if and only if \( r \geq \text{reg}(M^*) \).

The following is one of the most basic results on Castelnuovo-Mumford regularity (see [4]). Jørgensen [5] proved the same result for \( M \in \text{gr} \ A \).

Let \( S = K[x_1, \ldots, x_d] \) be a polynomial ring. If \( M \in \text{gr} \ S \) satisfies \( H^0_m(M)_{\geq r+1} = 0 \) and \( H^i_m(M)_{r+1-i} = 0 \) for all \( i \geq 1 \), then \( r \geq \text{reg}(M) \) (i.e., \( H^i_m(M)_{\geq r+1-i} = 0 \) for all \( i \geq 1 \)).

The similar result also holds for \( M^* \in D^b(\text{gr} \ A) \). Since a minor adaptation of the proof of [4, Theorem 2.4] also works for complexes, we leave the proof to the reader.
Proposition 2.13. If \( M^\bullet \in D^b(\text{gr} A) \) with \( t := \max\{ i \mid H^i(M^\bullet) \neq 0 \} \) satisfies

- \( H^i_m(M^\bullet)_{\geq r+1-i} = 0 \) for all \( i \leq t \)
- \( H^i_m(M^\bullet)_{r+1-i} = 0 \) for all \( i > t \),

then \( r \geq \text{reg}(M^\bullet) \) (i.e., \( H^i_m(M^\bullet)_{\geq r+1-i} = 0 \) for all \( i > t \)).

3. Koszul duality

In this section, we study the relation between the Castelnuovo-Mumford regularity of complexes and the Koszul duality. For precise information of this duality, see [2, §2]. There, the symbol \( A \) (resp. \( A^! \)) basically means a finite dimensional (resp. noetherian) Koszul algebra. This convention is opposite to ours. So the reader should be careful.

Recall that \( A = KQ/I \) is a graded quiver algebra over a finite quiver \( Q \). Let \( Q^\text{op} \) be the opposite quiver of \( Q \). That is, \( Q_0^\text{op} = Q_0 \) and there is a bijection from \( Q_1 \) to \( Q_1^\text{op} \) which sends an arrow \( \alpha : v \to u \) in \( Q_1 \) to the arrow \( \alpha^\text{op} : u \to v \) in \( Q_1^\text{op} \).

Consider the bilinear form \( \langle -,- \rangle : (KQ)_2 \times (KQ^\text{op})_2 \to K \) defined by

\[
\langle \alpha\beta, \gamma\delta^\text{op} \rangle = \begin{cases} 
1 & \text{if } \alpha = \delta \text{ and } \beta = \gamma, \\
0 & \text{otherwise}
\end{cases}
\]

for all \( \alpha, \beta, \gamma, \delta \in Q_1 \). Let \( I^\perp \subset KQ^\text{op} \) be the ideal generated by

\[ \{ y \in (KQ^\text{op})_2 \mid \langle x, y \rangle = 0 \text{ for all } x \in I_2 \}. \]

We say \( KQ^\text{op}/I^\perp \) is the quadratic dual ring of \( A \), and denote it by \( A^! \). Clearly, \((A^!)_0 = A_0 \). Since \( A \) is Koszul, so is \( A^! \) and \((A^!)_1 \cong A_1 \). Since \( A \) is AS regular, \( A^! \) is a finite dimensional selfinjective algebra with \( A = \bigoplus_{i=0}^d A_i \) by [12, Theorem 5.1]. If \( A \) is a polynomial ring, then \( A^! \) is the exterior algebra \( \bigwedge (A_1)^* \).

Since \( A^! \) is selfinjective, \( D_{A^!} := \text{Hom}_{A^!}(-, A^!) \) and \( D_{(A^!)^\text{op}} := \text{Hom}_{A^!}(-, A^!)^\text{op} \) give exact duality functors between \( \text{gr } A^! \) and \( \text{gr } (A^!)^\text{op} \). They induce duality functors between \( D^b(\text{gr } A^!) \) and \( D^b(\text{gr } (A^!)^\text{op}) \), which are also denoted by \( D_{A^!} \) and \( D_{(A^!)^\text{op}} \). It is easy to see that \( D_{A^!}(N) \cong \text{Hom}_K(N, K)(-d) \).

We say a complex \( F^\bullet \in C(\text{gr } A^!) \) is a projective (resp. injective) resolution of a complex \( N^\bullet \in C^b(\text{gr } A^!) \), if each term \( F^i \) is projective (= injective), \( F^\bullet \) is right (resp. left) bounded, and there is a graded quasi isomorphism \( F^\bullet \to N^\bullet \) (resp. \( N^\bullet \to F^\bullet \)). We say a projective (or, injective) resolution \( F^\bullet \in C^b(\text{gr } A^!) \) is minimal if \( d^i(F^i) \subset nF^{i+1} \) for all \( i \), where \( n \) is the graded Jacobson radical of \( A^! \). (The usual definition of a minimal injective resolution is different from the above one. But they coincide in our case.) A bounded complex \( N^\bullet \in C^b(\text{gr } A^!) \) has a minimal projective resolution and a minimal injective resolution, and they are unique up to isomorphism. If \( F^\bullet \) is a minimal projective (resp. injective) resolution of \( N^\bullet \) then \( D_{A^!}(F^\bullet) \) is a minimal injective (resp. projective) resolution of \( D_{A^!}(N^\bullet) \).

For \( N^\bullet \in D^b(\text{gr } A^!) \), set

\[
\mu^{i,j}(N^\bullet) := \dim_K \text{Ext}^i_{A^!}(A_0, N^\bullet)_j.
\]
Then $\mu^{i,j}(N^\bullet)$ measures the size of a minimal injective resolution of $N^\bullet$. More precisely, if $F^\bullet$ is a minimal injective resolution of $N^\bullet$, and $F^i := \bigoplus_{l=1}^m T^{i,l}$ is an indecomposable decomposition, then we have

$$
\mu^{i,j}(N^\bullet) = \# \{ l \mid \text{soc}(T^{i,l}) = (T^{i,l})_j \} \quad = \# \{ l \mid T^{i,l}(j) \text{ is isomorphic to a direct summand of } A^l(d) \}.
$$

Let $V$ be a finitely generated left $A_0$-module. Then $\text{Hom}_{A_0}(A^1, V)$ is a graded left $A^1$-module with $(af)(a') = f(a'a)$ and $\text{Hom}_{A_0}(A^1, V)_i = \text{Hom}_{A_0}((A^1)_i, V)$. Since $A^1$ is selfinjective, we have $\text{Hom}_{A_0}(A^1, A_0) \cong A^1(d)$. Hence $\text{Hom}_{A_0}(A^1, V)$ is a projective (and injective) left $A^1$-module for all $V$. If $V$ has degree $i$ (e.g., $V = M_i$ for some $M \in \text{gr} A$), then we set $\text{Hom}_{A_0}(A^1, V)_j = \text{Hom}_{A_0}(A^1_{j-i}, V)$.

For $M^\bullet \in C^b(\text{gr} A)$, let $G(M^\bullet) := \text{Hom}_{A_0}(A^1, M^\bullet) \in C^b(\text{gr} A)$ be the total complex of the double complex with $G(M^\bullet)^{i,j} = \text{Hom}_{A_0}(A^1, M^j_i)$ whose vertical and horizontal differentials $d'$ and $d''$ are defined by

$$
d'(f)(x) = \sum_{\alpha \in Q_1} \alpha f(\alpha^{op} x), \quad d''(f)(x) = \partial M^\bullet(f(x))
$$

for $f \in \text{Hom}_{A_0}(A^1, M^j_i)$ and $x \in A^1$. The gradings of $G(M^\bullet)$ is given by

$$
G(M^\bullet)^p_q := \bigoplus_{p=i+j, q=-l-j} \text{Hom}_{A_0}((A^1)_l, M^j_i).
$$

Each term of $G(M^\bullet)$ is injective. For a module $M \in \text{gr} A$, $G(M)$ is a minimal complex. Thus we have

$$
\mu^{i,j}(G(M)) = \begin{cases} 
\dim_K M_i & \text{if } i + j = 0, \\
0 & \text{otherwise}.
\end{cases}
$$

Similarly, for a complex $N^\bullet \in C^b(\text{gr} A)$, we can define a new complex $F(N^\bullet) := A \otimes_A N^\bullet \in C^b(\text{gr} A)$ as the total complex of the double complex with $F(N^\bullet)^{i,j} = A \otimes_A N^j_i$ whose vertical and horizontal differentials $d'$ and $d''$ are defined by

$$
d'(a \otimes x) = \sum_{\alpha \in Q_1} a\alpha \otimes \alpha^{op} x, \quad d''(a \otimes x) = a \otimes \partial_N^\bullet(x)
$$

for $a \otimes x \in A \otimes_A N^l_i$. The gradings of $F(N^\bullet)$ is given by

$$
F(N^\bullet)^p_q := \bigoplus_{p=i+j, q=-l-j} A_l \otimes_A N^j_i.
$$

Clearly, each term of $F(N^\bullet)$ is a projective $A$-module. For a module $N \in \text{gr} A_1$, $F(N)$ is a minimal complex. Hence we have

$$
\beta^{i,j}(F(N)) = \begin{cases} 
\dim_K N_i & \text{if } i + j = 0, \\
0 & \text{otherwise}.
\end{cases}
$$

It is well-known that the operations $F$ and $G$ define functors $F : D^b(\text{gr} A^1) \to D^b(\text{gr} A)$ and $G : D^b(\text{gr} A) \to D^b(\text{gr} A^1)$, and they give an equivalence $D^b(\text{gr} A) \cong D^b(\text{gr} A^1)$ of triangulated categories. This equivalence is called the Koszul duality.
When $A$ is a polynomial ring, this equivalence is called *Bernstein-Gelfand-Gelfand correspondence*. See, for example, [3].

Note that $(A^{\text{op}})^{1} \cong (A')^{\text{op}}$. We have the functors $\mathcal{F}^{\text{op}} : D^{b}(\text{gr}(A'))^{\text{op}} \to D^{b}(\text{gr} A^{\text{op}})$ and $\mathcal{G}^{\text{op}} : D^{b}(\text{gr} A^{\text{op}}) \to D^{b}(\text{gr}(A')^{\text{op}})$ giving $D^{b}(\text{gr} A^{\text{op}}) \cong D^{b}(\text{gr}(A')^{\text{op}})$.

**Proposition 3.1** (c.f. [3 Proposition 2.3]). *In the above situation, we have\[ \beta^{i,j}(M^{*}) = \dim_{K} H^{i+j}(\mathcal{G}(M^{*}))_{-j} \] and \[ \mu^{i,j}(N^{*}) = \dim_{K} H^{i+j}(\mathcal{F}(N^{*}))_{-j}. \]

*Proof.* While the assertion follows from Proposition 3.4 below, we give a direct proof here. We have\[
\Ext_{A^{\text{op}}}^{i}(A_{0}, N^{*})_{j} \cong \text{Hom}_{D^{b}(\text{gr} A^{\text{op}})}(A_{0}, N^{*}[i][j]) \\
\cong \text{Hom}_{D^{b}(\text{gr} A^{\text{op}})}(\mathcal{F}(A_{0}), \mathcal{F}(N^{*}[i])(j)) \\
\cong \text{Hom}_{D^{b}(\text{gr} A^{\text{op}})}(A, \mathcal{F}(N^{*})(i+j)(-j)) \\
\cong H^{i+j}(\mathcal{F}(N^{*}))_{-j}.
\]

Since $\mu^{i,j}(N^{*}) = \dim_{K} \Ext_{A^{\text{op}}}^{i}(A_{0}, N^{*})_{j}$, the second equation of the proposition follows. We can prove the first equation by a similar argument. But in this time we use the contravariant functor $D_{A'} \circ \mathcal{G} : D^{b}(\text{gr} A) \to D^{b}(\text{gr}(A')^{\text{op}})$ and the fact that $D_{A'} \circ \mathcal{G}(A_{0}) \cong D_{A'}(A'(d)) \cong A'(-d)$. \(\square\)

**Corollary 3.2.** $\text{reg}(M^{*}) = \max\{i \mid H^{i}(\mathcal{G}(M^{*})) \neq 0\}$.

*Proof.* Follows Theorem 2.9 and Proposition 3.1. \(\square\)

Recall that $D_{A} := \mathbf{R}\text{Hom}_{A}(\mathcal{A}, \mathcal{A})$ is a duality functor from $D^{b}(\text{gr} A)$ to $D^{b}(\text{gr} A^{\text{op}})$.

**Proposition 3.3.** $\text{reg}(D_{A}(M^{*})) = -\min\{i \mid H^{i}(\mathcal{G}(M^{*})) \neq 0\}$.

*Proof.* Let $L'$ be the $A$-$A$ bimodule given in the construction of the dualizing complex $\mathcal{D}^{*}$. Note that $D_{A}(M^{*}) \cong \text{Hom}_{A}^{*}(P^{*}, L'(-d)[d]) =: Q^{*}$ for a projective resolution $P^{*}$ of $M^{*}$. Since $D_{A}(P_{j}) = \delta_{-j(0)}P(-d)[d]$, $Q^{*}$ is a complex of projectives. And $Q^{*}$ is a minimal complex if and only if $P^{*}$ is. Hence $\beta^{i-d,-j+d}(D_{A}(M^{*})) = \beta^{i,j}(M^{*})$. Therefore, the assertion follows from Proposition 3.1. \(\square\)

We can refine Proposition 3.1 using the notion of *linear strands* of projective (or injective) resolutions, which was introduced by Eisenbud et. al. (See [3 §3].) First, we will generalize this notion to our rings. Let $B$ be a noetherian Koszul algebra (e.g. $B = A$ or $A'$) with the graded Jacobson radical $\mathfrak{m}$, and $P^{*}$ a *minimal* projective resolution of a bounded complex $M^{*} \in D^{b}(\text{gr} B)$. Consider the decomposition $P^{i} := \bigoplus_{j \in \mathbb{Z}} P^{i,j}$ such that any indecomposable summand of $P^{i,j}$ is isomorphic to a summand of $B(-j)$. For an integer $l$, we define the $l$-*linear strand* $\text{proj. lin}_{l}(M^{*})$ of a projective resolution of $M^{*}$ as follows: The term $\text{proj. lin}_{l}(M^{*})^{i}$ of cohomological degree $i$ is $P^{i,l-i}$ and the differential $P^{i,l-i} \to P^{i+1,l-i-1}$ is the corresponding component of the differential $P^{i} \to P^{i+1}$ of $P^{*}$. So the differential of $\text{proj. lin}_{l}(M^{*})$ is represented by a matrix whose entries are elements in $A_{i}$. Set $\text{proj. lin}(M^{*}) := \bigoplus_{l \in \mathbb{Z}} \text{proj. lin}_{l}(M^{*})$. It is obvious that $\beta^{i,j}(M^{*}) = \beta^{i,j}(\text{proj. lin}(M^{*}))$ for all $i, j$. 


Using spectral sequence argument, we can construct \( \text{proj} \cdot \text{lin}(M^\bullet) \) from a (not necessarily minimal) projective resolution \( Q^\bullet \) of \( M^\bullet \). Consider the \( m \)-adic filtration \( Q^\bullet = F_0Q^\bullet \supset F_1Q^\bullet \supset \cdots \) of \( Q^\bullet \) with \( F_pQ^i = m^pQ^i \) and the associated spectral sequence \( \{ E_r^* \}, d_r \). The associated graded object \( \text{gr}_M := \bigoplus_{p \geq 0} m^pM/m^{p+1}M \) of \( M \in \text{gr} B \) is a module over \( \text{gr}_M B = \bigoplus_{p \geq 0} m^p/m^{p+1}B \). Since \( m^pM \) is a graded submodule of \( M \), we can make \( \text{gr}_M M \) a graded module using the original grading of \( M \) (so \( \text{gr}_M M_i \) is not \( m^pM/m^{p+1}M \) here). Under the identification \( \text{gr}_M B \) with \( B \), we have \( \text{gr}_M M \cong M \) in general. But if each indecomposable summand \( N \) of \( M \) is generated by \( N_i(N) \) then \( \text{gr}_M M \cong M \). Since \( Q^t \) is a projective \( B \)-module, \( Q_0^t := \bigoplus_{p+q=t} E_0^{p,q} = \bigoplus_{p \geq 0} m^pQ^t/m^{p+1}Q^t = \text{gr}_M Q^t \) is isomorphic to \( Q^t \). The maps \( d_0^{p,q} : E_0^{p,q} \to E_0^{p,q+1} \) make \( Q_0^t \) a cochain complex of projective \( \text{gr}_M B \)-modules. Consider the decomposition \( Q^t = P^t \oplus C^t \), where \( P^t \) is minimal and \( C^t \) is exact. (We always have such a decomposition.) If we identify \( Q^t_0 \) with \( Q^t = P^t \oplus C^t \), the differential \( d_0 \) of \( Q^t_0 \) is given by \( (0, d_{C^t}) \). Hence we have \( Q_1^t = \bigoplus_{p+q=t} E_1^{p,q} \cong P^t \).

The maps \( d_1^{p,q} : E_1^{p,q} = m^pP^t/m^{p+1}P^t \to E_1^{p,q+1} = m^{p+1}P^{t+1}/m^{p+2}P^{t+1} \) make \( Q_1^t \) a cochain complex of projective \( \text{gr}_M B(\cong B) \)-modules whose differential is the “linear component” of the differential \( d_{P^t} \) of \( P^t \). Thus the complex \( (Q_1^t, d_1) \) is isomorphic to \( \text{proj} \cdot \text{lin}(M^\bullet) \).

Since \( A^i \) is selfinjective, we can consider the linear strands of an injective resolution. More precisely, starting from a minimal injective resolution of \( N^\bullet \in D^b(\text{gr} A^i) \), we can construct its \( l \)-linear strand \( \text{inj} \cdot \text{lin}(N^\bullet) \) in a similar way. Here, if \( I^i \) is the cohomological degree \( i \)th term of \( \text{inj} \cdot \text{lin}(N^\bullet) \), then the socle of \( I^i \) coincides with \( (I^i)_{t-i} \). In other words, any indecomposable summand of \( I^i \) is isomorphic to a summand of \( A^i(i = l + d) \). Set \( \text{inj} \cdot \text{lin}(N^\bullet) = \bigoplus_{l \in \mathbb{Z}} \text{inj} \cdot \text{lin}(N^\bullet) \). This complex can also be constructed using spectral sequences.

We have that \( D_{A^i} \cdot (\text{inj} \cdot \text{lin}(N^\bullet)) \cong \text{proj} \cdot \text{lin}(D_{A^i} \cdot (N^\bullet)) \) and \( D_{A^i} \cdot (\text{proj} \cdot \text{lin}(N^\bullet)) \cong \text{inj} \cdot \text{lin}(D_{A^i} \cdot (N^\bullet)) \).

**Proposition 3.4** (c.f. [3] Corollary 3.6). For \( M^\bullet \in D^b(\text{gr} A) \) and \( N^\bullet \in D^b(\text{gr} A^i) \), we have

\[
\text{proj} \cdot \text{lin}(\mathcal{F}(N^\bullet)) = \mathcal{F}(\mathcal{H}(N^\bullet)) \quad \text{and} \quad \text{inj} \cdot \text{lin}(\mathcal{G}(M^\bullet)) = \mathcal{G}(\mathcal{H}(M^\bullet)).
\]

More precisely,

\[
\text{proj} \cdot \text{lin}(\mathcal{F}(N^\bullet)) = \mathcal{F}(\mathcal{H}^1(N^\bullet))[-l] \quad \text{and} \quad \text{inj} \cdot \text{lin}(\mathcal{G}(M^\bullet)) = \mathcal{G}(\mathcal{H}^1(M^\bullet))[-l].
\]

**Proof.** Set \( Q^\bullet = \mathcal{F}(N^\bullet) \). Note that \( Q^\bullet \) is a (non minimal) complex of projective modules. We use the above spectral sequence argument (and the notation there). Then the differential \( d_0^t : Q_0^t \cong \mathcal{F}^t(N^\bullet) \to Q_0^{t+1} \cong \mathcal{F}^{t+1}(N^\bullet) \) is given by \( \pm \partial_{N^\bullet} \). Thus

\[
Q_1^t \cong \bigoplus_{t=i+j} A \otimes_{A^i} H^i(N^\bullet)_j = \bigoplus_{t=i+j} \mathcal{F}^j(H^i(N^\bullet)),
\]

and the differential of \( Q_1^t \) is induced by that of \( \mathcal{F}(N^i) \). Hence we can easily check that \( Q_1^t \), which can be identified with \( \text{proj} \cdot \text{lin}(\mathcal{F}(N^\bullet)) \), is isomorphic to \( \mathcal{F}(\mathcal{H}(N^\bullet)) \cong \bigoplus_{l \in \mathbb{Z}} \mathcal{F}(\mathcal{H}^l(N^\bullet))[-l] \). We can prove the statement for \( \text{inj} \cdot \text{lin}(\mathcal{G}(M^\bullet)) \) in the same way. □
4. Weakly Koszul Modules

Let $B$ be a noetherian Koszul algebra (e.g. $B = A$ or $A^1$) with the graded Jacobson radical $m$. For $M \in \text{gr} B$ and an integer $i$, $M_{(i)}$ denotes the submodule of $M$ generated by its degree $i$ component $M_i$.

**Proposition 4.1.** In the above situation, the following are equivalent.

(1) $M_{(i)}$ has a linear projective resolution for all $i$.

(2) $H^i(\text{proj. lin}(M)) = 0$ for all $i \neq 0$.

(3) All indecomposable summands of $\text{gr}_m M$ have linear resolutions as $B(\cong \text{gr}_m B)$-modules.

**Proof.** This result was proved in [20, Proposition 4.9] under the assumption that $B$ is a polynomial ring. (Römer also proved this for a commutative Koszul algebra. See [18, Theorem 3.2.8].) In this proof, only the Koszul property of a polynomial ring is essential, and the proof also works in our case. But, to refer this, the reader should be careful with the following points.

(a) In [20], the grading of $\text{gr}_m M$ is given by a different way. There, $(\text{gr}_m M)_i = m^i M/m^{i+1} M$. It is easy to see that $\text{gr}_m M$ has a linear resolution in this grading if and only if the condition (3) of the proposition is satisfied in our grading.

(b) In the proof of [20, Proposition 4.9], the regularity $\text{reg}(N)$ of $N \in \text{gr} B$ is an important tool. Unless $B$ is AS regular, one cannot define $\text{reg}(N)$ using the local cohomologies of $N$. But if we set $\text{reg}(N) := \sup \{i + j \mid \beta^{i,j}(N) \neq 0\}$, then everything works well. It is not clear whether $\text{reg}(N) < \infty$ for all $N \in \text{gr} B$ (c.f. [10]). But modules appearing in the argument similar to the proof of [20, Proposition 4.9] have finite regularities.

(c) In the proof of [20 Proposition 4.9], a few basic properties of the Castelnuovo-Mumford regularity (over a polynomial ring) are used. But $\text{reg}(N)$ of $N \in \text{gr} B$ also has these properties, if we define $\text{reg}(N)$ as (b). For example, if $N \in \text{gr} B$ satisfies $\dim_K N < \infty$, then $\text{reg}(N) = \sigma(N)$. This can be proved by induction on $\dim_K N$. Using the short exact sequence $0 \to N_{> r} \to N \to N/N_{> r} \to 0$, we can also prove that $N_{> r}$ has an $r$ linear resolution if and only if $r \geq \text{reg}(N)$ (see also Proposition 2.11).

(d) For the implication $(2) \Rightarrow (3)$, [20] refers another paper. But this implication can be proved by spectral sequence argument, since $\text{proj. lin}(M)$ can be constructed using a spectral sequence as we have seen in the previous section. □

**Definition 4.2** ([5, 13]). In the above situation, we say $M \in \text{gr} B$ is weakly Koszul, if it satisfies the equivalent conditions of Proposition 4.1.

**Remark 4.3.** (1) If $M \in \text{gr} B$ has a linear resolution, then it is weakly Koszul.

(2) The notion of weakly Koszul modules was first introduced by Green and Martinez-Villa [5]. But they used the name “strongly quasi Koszul modules”. Weakly Koszul modules are also called “componentwise linear modules” by some commutative algebraists (see [7]).
Theorem 4.4. Let $0 \neq N \in \text{gr } A^1$ and set $N' := D_{A^1}(N)$. Then the following are equivalent.

(1) $N$ is weakly Koszul.

(2) $H^i(F^{\text{op}}(N'))$ has a $(-i)$-linear projective resolution for all $i$.

(3) $\text{reg}(H \circ F^{\text{op}}(N')) = 0$.

(4) $\text{reg}(H \circ F^{\text{op}}(N')) \leq 0$.

Proof. Since $\iota(H \circ F^{\text{op}}(N')) \geq 0$ (i.e., $\iota(H^i(F^{\text{op}}(N'))) \geq -i$ for all $i$), the equivalence among (2), (3) and (4) follows from Proposition 2.11. So it suffices to prove (1) $\iff$ (4). Since $D_{(A')^{\text{op}}}(\text{inj. lin}(N')) \cong \text{proj. lin}(N)$, $N$ is weakly Koszul if and only if $H^i(\text{inj. lin}(N')) = 0$ for all $i > 0$. By Proposition 3.4, we have

$$\text{inj. lin}(N') = \text{inj. lin}(G^{\text{op}} \circ F^{\text{op}}(N')) = G^{\text{op}} \circ H \circ F^{\text{op}}(N').$$

Therefore, by Corollary 3.2, $H^i(\text{inj. lin}(N')) = 0$ for all $i > 0$ if and only if the condition (4) holds. \hfill $\square$

Remark 4.5. Martinez-Villa and Zacharia proved that if $N$ is weakly Koszul then there is a filtration

$$U_0 \subset U_1 \subset \cdots \subset U_p = N$$

such that $U_{i+1}/U_i$ has a linear resolution for each $i$ (see [13, pp. 676–677]). We can interpret this fact using Theorem 4.4 in our case.

Let $N \in \text{gr } A^1$ be a weakly Koszul module. Set $N' := D_{A^1}(N)$ and $T^\bullet := F^{\text{op}}(N')$. Assume that $N$ does not have a linear resolution. Then $H^i(T^\bullet) \neq 0$ for several $i$. Set $n = \min\{i \mid H^i(T^\bullet) \neq 0\}$. Consider the truncation

$$\sigma_{\geq n}T^\bullet : \cdots \longrightarrow 0 \longrightarrow \text{im } d^n \longrightarrow T^{n+1} \longrightarrow T^{n+2} \longrightarrow \cdots$$

of $T^\bullet$. Then we have $H^i(T^\bullet) = H^i(\sigma_{\geq n}T^\bullet)$ for all $i > n$ and $H^i(\sigma_{\geq n}T^\bullet) = 0$ for all $i \leq n$. We have a triangle

$$(4.1) \quad H^n(T^\bullet)[-n] \longrightarrow T^\bullet \longrightarrow \sigma_{\geq n}T^\bullet \longrightarrow H^n(T^\bullet)[-n + 1].$$

By Theorem 4.4, $H^n(T^\bullet)[-n]$ has a 0-linear resolution. On the other hand,

$$0 = \text{reg}(H(\sigma_{\geq n}T^\bullet)) \geq \text{reg}(\sigma_{\geq n}T^\bullet) \geq \iota(\sigma_{\geq n}T^\bullet) \geq 0.$$

Hence $\sigma_{\geq n}T^\bullet$ also has a 0-linear resolution. Therefore, both $D_{(A')^{\text{op}}} \circ G^{\text{op}}(\sigma_{\geq n}T^\bullet)$ and $D_{(A')^{\text{op}}} \circ G^{\text{op}}(H^n(T^\bullet)[-n])$ are acyclic complexes (that is, the $i$th cohomology vanishes for all $i \neq 0$). Set

$$U := H^0(D_{(A')^{\text{op}}} \circ G^{\text{op}}(\sigma_{\geq n}T^\bullet)) \quad \text{and} \quad V := H^0(D_{(A')^{\text{op}}} \circ G^{\text{op}}(H^n(T^\bullet)[-n])).$$

Since $N = D_{(A')^{\text{op}}} \circ G^{\text{op}}(T^\bullet)$, the triangle (4.1) induces a short exact sequence

$$0 \longrightarrow U \longrightarrow N \longrightarrow V \longrightarrow 0 \text{ in gr } A^1.$$

It is easy to see that $V$ has a linear resolution. Since $H \circ F^{\text{op}} \circ D_{A^1}(U) = H(\sigma_{\geq n}T^\bullet)$, $U$ is weakly Koszul by Theorem 4.4. Repeating this procedure, we can get the expected filtration.

Let $N \in \text{gr } A^1$ and $\cdots \longrightarrow f_2 \longrightarrow P^{-1} \longrightarrow f_1 \longrightarrow P^0 \longrightarrow f_0 \longrightarrow N \longrightarrow 0$ its minimal projective resolution. For $i \geq 1$, we call $\Omega_i(N) := \ker(f_{i-1})$ the $i$th syzygy of $N$. Note that $\Omega_i(N) = \text{im}(f_i) = \text{coker}(f_{i+1})$. 


By the original definition of a weakly Koszul module given in \[5\] [13], if \( N \in \text{gr} \ A^i \) is weakly Koszul then so is \( \Omega_i(N) \) for all \( i \geq 1 \).

**Definition 4.6** (Herzog-Römer, [18]). For \( 0 \neq N \in \text{gr} \ A^1 \), set
\[
\text{lpd}(N) := \inf \{ i \in \mathbb{N} | \Omega_i(N) \text{ is weakly Koszul} \},
\]
and call it the linear part dominates of \( N \).

Since \( A \simeq (A^1)^{\infty} \) is a noetherian ring of finite global dimension, \( \text{lpd}(N) \) is finite for all \( N \in \text{gr} \ A^1 \) by [13 Theorem 4.5].

**Theorem 4.7.** Let \( N \in \text{gr} \ A^1 \) and set \( N' := D_{A^1}(N) \). Then we have
\[
\text{lpd}(N) = \text{reg}(H \circ \mathcal{F}^{\text{op}}(N')) = \max \{ \text{reg}(H^i(\mathcal{F}^{\text{op}}(N'))) + i | i \in \mathbb{Z} \}.
\]

**Proof.** Note that \( P^* := D_{(A^1)^{\text{op}}} \circ \mathcal{G}^{\text{op}} \circ \mathcal{F}^{\text{op}}(N') \) is a projective resolution of \( N \), and \( Q^* := D_{(A^1)^{\text{op}}} \circ \mathcal{G}^{\text{op}} \circ \mathcal{F}^{\text{op}}(N')_i \) is the truncation \( \cdots \to P^{-i-1} \to P^{-i} \to 0 \to \cdots \) of \( P^* \) for each \( i \geq 1 \). Hence we have \( H^j(Q^*) = 0 \) for all \( j \neq -i \) and there is a projective module \( P \) such that \( H^{-i}(Q^*) \cong \Omega_i(N) \oplus P \). Since \( P \) is weakly Koszul, \( \Omega_i(N) \) is weakly Koszul if and only if so is \( Q := H^{-i}(Q^*) \). We have
\[
\text{proj.\,lin}(Q)[i] \cong D_{(A^1)^{\text{op}}} \circ \mathcal{G}^{\text{op}} \circ H(\mathcal{F}^{\text{op}}(N')_i).
\]

By Theorem 4.4, \( Q \) is weakly Koszul if and only if \( H(\mathcal{F}^{\text{op}}(N')_i) \) has an \( i \)-linear resolution, that is, \( H^j(\mathcal{F}^{\text{op}}(N')_i) \) has an \((i-j)\)-linear resolution for all \( j \). But there is some \( L \in \text{gr} \ (A^1)^{\text{op}} \) such that \( L = L_{i-j} \) and \( H^j(\mathcal{F}^{\text{op}}(N')_i) \cong H^j(\mathcal{F}^{\text{op}}(N')_{i-j}) \). Note that \( L \) has an \((i-j)\)-linear resolution. Therefore, \( H^j(\mathcal{F}^{\text{op}}(N')_i) \) has an \((i-j)\)-linear resolution if and only if so does \( H^j(\mathcal{F}^{\text{op}}(N')_{i-j}) \). Summing up the above facts, we have that \( \Omega_i(N) \) is weakly Koszul if and only if \( (H \circ \mathcal{F}^{\text{op}}(N'))_i \) has an \( i \)-linear resolution. By Proposition 2.11, the last condition is equivalent to the condition that \( i \geq \text{reg}(H \circ \mathcal{F}^{\text{op}}(N')) \).

**Remark 4.8.** Assume that \( A \) is noetherian, Koszul, and has finite global dimension, but not necessarily AS regular. Then \( A^1 \) is a finite dimensional Koszul algebra, but not necessarily selfinjective. Even in this case, \( \mathcal{G}(M^*) \) for \( M^* \in D^b(\text{gr} \ A) \) is a complex of injective \( A^1 \)-modules, and the results in §3 and Theorem 4.7 also hold. But now we should set \( \text{reg}(M^*) := \sup \{ i + j | \beta_{i,j}(M^*) \neq 0 \} \) for \( M^* \in D^b(\text{gr} \ A) \) (local cohomology is not helpful to define the regularity). Since \( A \) is noetherian and has finite global dimension, we have \( \text{reg}(M^*) < \infty \) for all \( M^* \). In particular, we have \( \text{lpd}(N) < \infty \) for all \( N \in \text{gr} \ A^1 \) (if \( A \) is right noetherian) as proved in [13 Theorem 4.5].

If \( \text{lpd}(N) \geq 1 \) for some \( N \in \text{gr} \ A^1 \), then \( \sup \{ \text{lpd}(T) | T \in \text{gr} \ A^1 \} = \infty \). In fact, if \( \Omega_{-i}(N) \) is the \( i \)-th cosyzygy of \( N \) (since \( A^1 \) is selfinjective, we can consider cosyzygies), then \( \text{lpd}(\Omega_{-i}(N)) > i \). But Herzog and Römer proved that if \( J \) is a monomial ideal of an exterior algebra \( E = \Lambda \langle y_1, \ldots, y_d \rangle \) then \( \text{lpd}(E/J) \leq d - 1 \) (c.f. [13 §3.3]). We will refine their results using Theorem 4.7.

In the sequel, we regard the polynomial ring \( S = K[x_1, \ldots, x_d] \), \( d \geq 1 \), as an \( \mathbb{N}^d \)-graded ring with \( \deg x_i = (0, \ldots, 0, 1, 0, \ldots, 0) \) where 1 is at the \( i \)-th position.
Similarly, the exterior algebra \( E = S^d = \wedge \langle y_1, \ldots, y_d \rangle \) is also an \( \mathbb{N}^d \)-graded ring. Let \( \ast \text{Gr} S \) be the category of \( \mathbb{Z}^d \)-graded \( S \)-modules and their degree preserving \( S \)-homomorphisms, and \( \ast \text{gr} S \) its full subcategory consisting of finitely generated modules. We have a similar category \( \ast \text{gr} E \) for \( E \). For \( S \)-modules and graded \( E \)-modules, we do not have to distinguish left modules from right modules. Since \( \mathbb{Z}^d \)-graded modules can be regarded as \( \mathbb{Z} \)-graded modules in the natural way, we can discuss \( \text{reg}(M^\bullet) \) for \( M^\bullet \in D^b(\ast \text{gr} S) \) and \( \text{lpd}(N) \) for \( N \in \ast \text{gr} E \).

Note that \( D_E(-) = \bigoplus_{a \in \mathbb{Z}^d} \text{Hom}_{\ast \text{gr} E}(-, E(a)) \) gives an exact duality functor from \( \ast \text{gr} E \) to itself. Sometimes, we simply denote \( D_E(N) \) by \( N' \). Set \( 1 := (1, 1, \ldots, 1) \in \mathbb{Z}^d \). Then \( D_S^\bullet := S(1)[d] \in D^b(\ast \text{gr} S) \) is a \( \mathbb{Z}^d \)-graded normalized dualizing complex and \( D_S(-) := \mathcal{R} \text{Hom}_S(-, D_S^\bullet) = \bigoplus_{a \in \mathbb{Z}^d} \mathcal{R} \text{Hom}_{\ast \text{gr} S}(-, D_S^\bullet(a)) \) gives a duality functor from \( D^b(\ast \text{gr} S) \) to itself. As shown in [21, Theorem 4.1], we have the \( \mathbb{Z}^d \)-graded Koszul duality functors \( F^\star \) and \( G^\star \) giving an equivalence \( D^b(\ast \text{gr} S) \cong D^b(\ast \text{gr} E) \). These functors are defined in the same way as in the \( \mathbb{Z} \)-graded case.

For \( a = (a_1, \ldots, a_d) \in \mathbb{Z}^d \), set \( \text{supp}(a) := \{ i \mid a_i > 0 \} \subset [d] := \{ 1, \ldots, d \} \). We say \( a \in \mathbb{Z}^d \) is squarefree if \( a_i = 0, 1 \) for all \( i \in [d] \). When \( a \in \mathbb{Z}^d \) is squarefree, we sometimes identify \( a \) with \( \text{supp}(a) \). For example, if \( F \subset [d] \), then \( S(-F) \) means the free module \( S(-a) \), where \( a \in \mathbb{N}^d \) is the squarefree vector with \( \text{supp}(a) = F \).

**Definition 4.9** ([20]). We say \( M \in \ast \text{gr} S \) is squarefree, if \( M \) has a presentation of the form
\[
\bigoplus_{F \subset [d]} S(-F)^{n_F} \to \bigoplus_{F \subset [d]} S(-F)^{m_F} \to M \to 0
\]
for some \( m_F, n_F \in \mathbb{N} \).

The above definition seems different from the original one given in [20], but they coincide. Stanley-Reisner rings (that is, the quotient rings of \( S \) by squarefree monomial ideals) and many modules related to them are squarefree. Here we summarize basic properties of squarefree modules. See [20, 21] for further information. Let \( \text{Sq}(S) \) be the full subcategory of \( \ast \text{gr} S \) consisting of squarefree modules. Then \( \text{Sq}(S) \) is closed under kernels, cokernels, and extensions in \( \ast \text{gr} S \). Thus \( \text{Sq}(S) \) is an abelian category. Moreover, we have \( D^b(\text{Sq}(S)) \cong D^b_{\text{Sq}(S)}(\ast \text{Gr} S) \). If \( M \) is squarefree, then each term in a \( \mathbb{Z}^d \)-graded minimal free resolution of \( M \) is of the form \( \bigoplus_{F \subset [d]} S(-F)^{n_F} \). Hence we have \( \text{reg}(M) \leq d \). Moreover, \( \text{reg}(M) = d \) if and only if \( M \) has a summand which is isomorphic to \( S(-1) \).

**Definition 4.10** (Römer [16]). We say \( N \in \ast \text{gr} E \) is squarefree, if \( N = \bigoplus_{F \subset [d]} N_F \) (i.e., if \( a \in \mathbb{Z}^d \) is not squarefree, then \( N_a = 0 \)).

A monomial ideal of \( E \) is always a squarefree \( E \)-module. Let \( \text{Sq}(E) \) be the full subcategory of \( \ast \text{gr} E \) consisting of squarefree modules. Then \( \text{Sq}(E) \) is an abelian category with \( D^b(\text{Sq}(E)) \cong D^b_{\text{Sq}(E)}(\ast \text{gr} E) \). If \( N \) is a squarefree \( E \)-module, then so is \( D_E(N) \). That is, \( D_E \) gives an exact duality functor from \( \text{Sq}(E) \) to itself. We have functors \( S : \text{Sq}(E) \to \text{Sq}(S) \) and \( E : \text{Sq}(S) \to \text{Sq}(E) \) giving an equivalence \( \text{Sq}(S) \cong \text{Sq}(E) \). Here \( S(N)_F = N_F \) for \( N \in \text{Sq}(E) \) and \( F \subset [d] \), and the multiplication
map \( \mathcal{S}(N)_F \ni z \mapsto x_i z \in \mathcal{S}(N)_{F \cup \{i\}} \) for \( i \not\in F \) is given by \( \mathcal{S}(N)_F = N_F \ni z \mapsto (-1)^{\alpha(i,F)} y_i z \in N_{F \cup \{i\}} = \mathcal{S}(N)_{F \cup \{i\}} \), where \( \alpha(i,F) = \# \{ j \in F \mid j < i \} \). See [16, 21] for detail. Since a free module \( E(a) \) is not squarefree unless \( a = 0 \), the syzygies of a squarefree \( E \)-module are not squarefree.

Proposition 4.11 (Herzog-Römer, [18 Corollary 3.3.5]). If \( N \) is a squarefree \( E \)-module (e.g., \( N = E/J \) for a monomial ideal \( J \)), then we have \( \text{lpd}(N) \leq d - 1 \).

This result easily follows from Theorem 4.7 and the fact that \( H^i(\mathcal{F}^*(N'))(-1) \) is a squarefree \( S \)-module for all \( i \) and \( H^i(\mathcal{F}^*(N')) = 0 \) unless \( 0 \leq i \leq d \). (Recall the remark on the regularity of squarefree modules given before Definition 4.10 and note that \( M := H^d(\mathcal{F}^*(N'))(-1) \) is generated by \( M_0 \).

We also remark that [18 Corollary 3.3.5] just states that \( \text{lpd}(N) \leq d \). But their argument actually proves that \( \text{lpd}(N) \leq d - 1 \). In fact, they showed that

\[
\text{lpd}(N) \leq \text{proj. dim}_S S(N).
\]

But, if \( \text{proj. dim}_S S(N) = d \) then \( S(N) \) has a summand which is isomorphic to \( K = S/(x_1, \ldots, x_d) \) and hence \( N \) has a summand which is isomorphic to \( K = E/(y_1, \ldots, y_d) \). But \( K \in \text{Sq}(E) \) has a linear resolution and irrelevant to \( \text{lpd}(N) \).

To refine Proposition 4.11 we need further properties of squarefree modules.

If \( M^* \in D^b(\text{Sq}(S)) \), then \( \text{Ext}^i_S(M^*, D^*_S) \) is squarefree for all \( i \). Hence \( D^*_S \) gives a duality functor on \( D^b(\text{Sq}(S)) \). On the other hand, \( A := S \circ D_E \circ \mathcal{E} \) is an exact duality functor on \( \text{Sq}(S) \) and it induces a duality functor on \( D^b(\text{Sq}(S)) \). Miller [24 Corollary 4.21] and Römer [17 Corollary 3.7] proved that \( \text{reg}(A(M)) = \text{proj. dim}_S M \) for all \( M \in \text{Sq}(S) \). I generalized this equation to a complex \( M^* \in D^b(\text{Sq}(S)) \) in [22 Corollary 2.10].

Lemma 4.12. Let \( N \in \text{Sq}(E) \) and set \( N' := D_E(N) \). Then we have

\[
\text{reg}(H^i(\mathcal{F}^*(N'))) = -\text{depth}_S(\text{Ext}^{d-i}_S(S(N'), S))
\]

and

\[
\text{lpd}(N) = \max \{ i - \text{depth}_S(\text{Ext}^{d-i}_S(S(N'), S)) \mid 0 \leq i \leq d \}.
\]

Here we set the depth of the 0 module to be \( +\infty \).

If \( M := \text{Ext}^{d-i}_S(S(N'), S) \neq 0 \), then \( \text{depth}_S M \leq \text{dim}_S M \leq i \). Therefore all members in the set of the right side of (4.3) are non-negative or \(-\infty \).

Proof. By Theorem 4.7, 4.3 follows from (4.2). So it suffices to show (4.2). By [21 Proposition 4.3], we have \( \mathcal{F}^*(N') \cong (A \circ D_S \circ S(N'))(1) \). The degree shifting “(1)” does not occur in [21 Proposition 4.3]. But \( E \) is a negatively graded ring there, and we need the degree shifting in the present convention.) Since \( A \) is exact, we have

\[
H^i(\mathcal{F}^*(N')) \cong H^i(A \circ D_S \circ S(N'))(1) \cong A(H^{-i}(D_S \circ S(N')))(1) \\
= A(\text{Ext}^{-i}_S(S(N'), D^*_S))(1).
\]

Recall that \( \text{reg}(A(M)) = \text{proj. dim}_S M \) for \( M \in \text{Sq}(S) \). On the other hand, since \( M \) is finitely generated, the underlying module of \( \text{Ext}^{-i}(M, D^*_S) \) is isomorphic to
Ext_{S}^{d-i}(M, S)$. So (4.12) follows from these facts and the Auslander-Buchsbaum formula.

\begin{proof}

Corollary 4.13. For $N \in \text{Sq}(E)$, $N$ is weakly Koszul (over $E$) if and only if $S(N)$ is weakly Koszul (over $S$).

In [17, Corollary 1.3], it was proved that $N$ has a linear resolution if and only if so does $S(N)$. Corollary 4.13 also follows from this fact and (the squarefree module version of) [7, Proposition 1.5].

Proof. We say $M \in \text{gr} S$ is sequentially Cohen-Macaulay, if for each $i$ Ext_{S}^{i}(M, S)$ is either the zero module or a Cohen-Macaulay module of dimension $d - i$ (c.f. [19, III. Theorem 2.11]). By Lemma 4.12, $N$ is weakly Koszul if and only if $S(N')$ $(\equiv A \circ S(N))$ is sequentially Cohen-Macaulay. By [17, Theorem 4.5], the latter condition holds if and only if $S(N)$ is weakly Koszul.

Many examples of squarefree monomial ideals of $S$ which are weakly Koszul (dually, Stanley-Reisner rings which are sequentially Cohen-Macaulay) are known. So we can obtain many weakly Koszul monomial ideals of $E$ using Corollary 4.13.

Proposition 4.14. For an integer $i$ with $1 \leq i \leq d - 1$, there is a squarefree $E$-module $N$ such that $\text{lpd} N = \text{proj. dim}_S S(N) = i$. In particular, the inequality of Proposition 4.11 is optimal.

Proof. Let $M$ be the $\mathbb{Z}^d$-graded $i$th syzygy of $K = S/\mathfrak{m}$. Note that $M$ is squarefree. We can easily check that $N := D_{E} \circ E(M) \in \text{Sq}(E)$ satisfies the expected condition. In fact, $\text{proj. dim}_S S(N) = \text{proj. dim}_S A(M) = \text{reg} M = i$. On the other hand, since Ext_{S}^{d-1}(S(N'), S) = Ext_{S}^{d-i}(M, S) = K, Ext_{S}^{j}(S(N'), S) = 0$ for all $j \neq d - i$, and depth_{S}(\text{Hom}_{S}(S(N'), S)) = d - i + 1$, we have $\text{lpd} N = i$.

The above result also says that the inequality $\text{lpd}(N) \leq \text{proj. dim}_S S(N)$ of [18, Corollary 3.3.5] is also optimal. But for a monomial ideal $J \subset E$, the situation is different.

Proposition 4.15. If $d \geq 3$, then we have $\text{lpd}(E/J) \leq d - 2$ for a monomial ideal $J$ of $E$.

Proof. If $d = 3$, then easy computation shows that any squarefree monomial ideal $I \subset S$ is weakly Koszul. Hence $J$ is weakly Koszul by Corollary 4.13. So we may assume that $d \geq 4$.

Note that $A \circ S(E/J)$ is isomorphic to a squarefree monomial ideal of $S$. We denote it by $I$. By Lemma 4.12, it suffices to show that depth_{S}(\text{Hom}_{S}(I, S)) \geq 2$ and depth_{S}(\text{Ext}_{S}^{1}(I, S)) \geq 1. Recall that $\text{Hom}_{S}(I, S)$ satisfies Serre’s condition $(S_2)$, hence its depth is at least 2. Since Ext_{S}^{1}(I, S) \approx \text{Ext}_{S}^{2}(S/I, S)$, it suffices to prove that depth_{S}(\text{Ext}_{S}^{2}(S/I, S)) \geq 1.

If $\text{ht}(I) > 2$, then we have $\text{Ext}_{S}^{2}(S/I, S) = 0$. If $\text{ht}(I) = 2$, then $\text{Ext}_{S}^{2}(S/I, S)$ satisfies $(S_2)$ as an $S/I$-module and depth_{S} Ext_{S}^{2}(S/I, S) \geq \min\{2, \dim(S/I)\} \geq 2$. So we may assume that $\text{ht}(I) = 1$. If the heights of all associated primes of $I$ are 1, then $I$ is a principal ideal and $\text{Ext}_{S}^{2}(S/I, S) = 0$ for all $i \neq 1$. So we
may assume that $I$ has an prime of larger height. Then we have ideals $I_1$ and $I_2$ of $S$ such that $I = I_1 \cap I_2$ and the heights of any associated prime of $I_1$ (resp. $I_2$) is 1 (at least 2). Since $I$ is a radical ideal, we have $\text{ht}(I_1 + I_2) \geq 3$. Hence $\text{Ext}_S^2(S/(I_1+I_2),S) = 0$ and $\text{Ext}_S^3(S/(I_1+I_2),S)$ is either the zero module or satisfies $(S_2)$ as an $S/(I_1+I_2)$-module. In particular, if $\text{Ext}_S^2(S/(I_1+I_2),S) \neq 0$ (equivalently, if $\dim(S/(I_1+I_2)) = d-3$) then $\text{depth}_S(\text{Ext}_S^3(S/(I_1+I_2),S)) \geq \min\{2, d-3\} \geq 1$. Note that $\text{depth}_S(\text{Ext}_S^2(S/I_2,S)) \geq 2$. From the short exact sequence
\[ 0 \to S/I \to S/I_1 \oplus S/I_2 \to S/(I_1 + I_2) \to 0 \]
and the above argument, we have the exact sequence
\[ (4.4) \quad 0 \to \text{Ext}_S^2(S/I_2,S) \to \text{Ext}_S^2(S/I,S) \to \text{Ext}_S^3(S/(I_1 + I_2),S). \]
We have $\text{depth}_S(\text{Ext}_S^2(S/I,S)) \geq 1$ by (4.3), since the modules beside this module have positive depth.

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