Anomalous finite-size effect in superconducting Josephson junction arrays

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We report large-scale simulations of the resistively-shunted Josephson junction array in strip geometry. As the strip width increases, the voltage first decreases following the dynamic scaling ansatz proposed by Minnhagen et al. [Phys. Rev. Lett. \textbf{74}, 3672 (1995)], and then rises towards the asymptotic value predicted by Ambegaokar et al. [Phys. Rev. Lett. \textbf{40}, 783 (1978)]. The nonmonotonic size-dependence is attributed to shortened life time of free vortices in narrow strips, and points to the danger of single-scale analysis applied to a charge-neutral superfluid state.

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Dissipation of the supercurrent through superfluid and superconducting films is caused by the flow of free vortices transverse to the current. This phenomenon was analyzed in detail by Ambegaokar, Halperin, Nelson, and Siggia (AHNS) \cite{AHNS}. In the context of a uniform superconducting film in zero magnetic field, their theory predicts a power-law current-voltage (I-V) relationship with a temperature-dependent exponent. Despite the relatively simple construct of the theory, its experimental and numerical verification has remained controversial \cite{AHNS, AHNS-2, AHNS-3}. It has been noted that boundary and finite-size effects can dominate the measured voltage drop across the sample at sufficiently low temperatures, making it difficult to achieve an unambiguous comparison between theory and experiments.

An alternative theory of vortex flow dissipation which also yields nonlinear I-V curves in the superconducting state derives from the dynamic scaling hypothesis \cite{AHNS, AHNS-2, AHNS-3}. For superconducting films and Josephson junction arrays (JJA), the two approaches differ in their predictions of the exponent \(a(T)\) characterizing the power-law behavior \(V \sim I^a\) below the Kosterlitz-Thouless-Berezinskii (KTB) transition temperature \(T_{KT}\). According to the AHNS theory,

\begin{equation}
    a_{AHNS}(T) = \chi + 1, \tag{1}
\end{equation}

where \(\chi = \pi J_R/k_B T\) and \(J_R\) is the renormalized spin-wave stiffness. On the other hand, the dynamic scaling analysis of Minnhagen et al. \cite{AHNS-2} suggests

\begin{equation}
    a_{MWJO}(T) = 2\chi - 1. \tag{2}
\end{equation}

The two expressions coincide at \(T_{KT}\) where \(\chi = 2\), but the difference grows rapidly as one moves to low temperatures. While there seem to be ample numerical support for the Minnhagen et al. scaling \cite{AHNS-2, AHNS-3, AHNS-4}, agreement with the AHNS theory has also been reported \cite{AHNS}. To rationalize the two scenarios, Bormann \cite{Bormann} made an interesting suggestion that strong current creates a dense set of vortex-antivortex pairs, thereby invalidating the AHNS treatment. Nevertheless, the AHNS theory should still apply at sufficiently weak currents. This picture has not been borne out by a recent numerical study \cite{Bormann} which shows persistently larger value of \(a\) than Eq. (1) predicts.

In this Letter, we present I-V results from large-scale simulations of the resistively shunted Josephson junction (RSJ) array. A rectangular strip geometry is used to decouple boundary effects introduced by the current leads from finite-size effects arising from vortex/antivortex motion transverse to the current \(I\). The main finding of our work is the existence of three distinct regimes as the strip width \(L_y\) is varied. For \(L_y < l_b \sim I^{-1}\), one is in the linear-response regime where the near-equilibrium dynamic scaling hypothesis applies. In this regime, the voltage \(V\) decreases as \(L_y\) increases, and reaches a minimum value predicted by Minnhagen et al. When \(L_y\) exceeds a second length scale \(l_r \sim I^{-\chi/2}\), \(V\) saturates to the asymptotic value predicted by the AHNS theory. The intermediate regime \(l_b < L_y < l_r\) is characterized by a rising \(V\) as \(L_y\) increases. This behavior is explained in terms of “self-recombination” of vortex-antivortex pairs under periodic boundary conditions.

We begin with the model Hamiltonian of a two-dimensional (2D) JJA in zero magnetic field,

\begin{equation}
    H = -J \sum_{(ij)} \cos(\phi_i - \phi_j). \tag{3}
\end{equation}

Here \(\phi_i\) denotes the phase of the superconducting order parameter on grain \(i\), and \(J\) sets the strength of the Josephson coupling between neighbouring superconducting grains. The summation in Eq. (3) is over all nearest neighbor grain pairs. In the RSJ dynamics, the network is insulated from the substrate, but normal currents are allowed between neighboring grains, in addition to the supercurrent. The dynamics of the phase variables follows the Josephson relations plus the Kirchoff law for current conservation at each node \cite{Bormann}.
\[
\frac{\sigma h}{2e} \sum_{j \in \text{n.n. of } i} \left( \frac{d\phi_i}{dt} - \frac{d\phi_j}{dt} \right) = -2e \frac{\partial H}{\partial \phi_i} + I_{\text{ext}}. \tag{4}
\]

Here \( \sigma \) is the conductivity of normal links, and \( I_{\text{ext}} \) is the external current source which is nonvanishing only at the boundaries of the network. To model the effect of temperature \( T \), a gaussian noise current \( \eta_{ij} \) is added to each link between two adjacent nodes \( i \) and \( j \), with \([\eta_i(t)\eta_{j'j}(t')] = 2\pi k_B T \delta_{ij'} \delta_0(t - t') \).

To integrate the coupled equations (4), one needs to first solve for \( \chi_h/\bar{L} \) for each data set at \( J/k_B T \) = 0, 1, 3, 5, at \( T = 0.8 J/k_B \). We have confirmed this value in the simulation by examining the correlation function of the superconducting order parameter and the helicity modulus (or \( J_R \)), which exhibit universal behavior at the transition. Direct measurement of the I-V curve at this temperature has also confirmed the predicted exponent \( a(T_{KT}) = 3 \).

We now describe the main results of our simulation. Figures 1(a) and 1(b) present the I-V data at \( T = 0.8 J/k_B \) and \( T = 0.7 J/k_B \), respectively, for four different array sizes: \( L_y = 8, 32, 128, \) and 512. Here the current \( I \) is measured in units of \( I_c = 2eJ/h \), and \( V \), the voltage drop per column, in units of \( h/2e \). As the current \( I \) decreases, the finite size effect becomes more and more evident. Examining the data at the smallest \( I \) shown, we see a clear nonmonotonic size dependence: for small values of \( L_y \), \( V \) decreases with increasing \( L_y \). However, this trend is reversed as \( L_y \) is increased further. The solid and dashed lines in the figure correspond to the power-law behavior predicted by AHNS [Eq. (4)] and Minnhagen et al. [Eq. (2)], respectively, using \( \chi = 2.8 \) at \( T = 0.8 J/k_B \) and \( \chi = 3.5 \) at \( T = 0.7 J/k_B \). These values of \( \chi \) are obtained from measurements of the helicity modulus at the respective temperatures, and agree with previous stud-

\[ \text{FIG. 1. Current-voltage curves for arrays of various strip width } L_y \text{ at two temperatures below } T_{KT}: (a) } T = 0.8 J/k_B, \text{ and (b) } T = 0.7 J/k_B. \text{ The value of } L_y \text{ for each data set is given in the legend box. Also shown are the predicted power-laws by AHNS (solid line) and by Minnhagen et al.(dashed line) at each temperature.} \]

\[ \text{FIG. 2. Nonmonotonic finite-size dependence of the voltage on the strip width for three different sets of current and temperature values.} \]
Figure 2 shows three sets of voltage data against transverse array size $L_y$. Data at $T = 0.8J/k_B$ are represented by open ($I = 0.1I_c$) and solid circles ($I = 0.05I_c$), while the set at $T = 0.7J/k_B$ and $I = 0.1I_c$ are presented by open squares. Statistical error of each data point is smaller or comparable to the symbol size. As $I$ decreases, the size $l_m$ where the minimum value $V = V_m$ is reached increases. $l_m$ also varies with $T$, though this dependence may be weak. Since the change of $V$ with $L_y$ is slow in the neighborhood of $l_m$, it may give the false impression that the finite-size effect is already saturated at $l_m$. As our data shows, the true asymptotic value of $V$ is reached only at much larger values of $L_y$. At $T = 0.7J/k_B$ and $I = 0.1I_c$, for example, the asymptotic value is larger than $V_m$ by five-fold.

In the remaining part of the paper, we propose an explanation of the unusual finite-size effect, based on the standard picture of creation and annihilation of free vortices under an applied current $I$. For small $I$, the supercurrent through the network is maintained by an average phase jump $\Delta \phi = I/L_y$ per column. Passage of a free vortex across the strip (i.e., in the $y$-direction) decreases the overall phase advance along the strip by an amount $2\pi$. In a steady-state situation, this loss is compensated by a difference in the phase velocities at the two opposing ends of the strip, and hence a voltage drop across the system. Simple counting then yields the following expression for the average voltage drop per column in a strip of length $L_x \gg 1$ (see, e.g. Ref. [3]),

$$V = \frac{2\pi}{L_x L_y} (N_+ \bar{v}_+ - N_- \bar{v}_-) .$$

(5)

Here $N_\pm$ are the number of $\pm$ vortices in the system, and $\bar{v}_\pm$ are their respective mean velocities in the $y$-direction. Although Eq. (3) does not discriminate between bound and free vortices, it is clear that correlated drift of a vortex-antivortex pair does not contribute to $V$.

In the AHNS theory, the energy of a vortex/antivortex pair of size $R$ and oriented perpendicular to the external current is given by,

$$E(R) = 2E_c + 2\pi J \left( \ln R - \frac{I}{I_c} R \right) ,$$

(6)

where $E_c$ is the core energy of a single vortex. Equation (3) has a maximum $E = E_B = 2E_c + 2\pi J \ln (V_c/I) - 2\pi J$ at $R = l_b = I_c/I$. This is the activation energy for pair breaking. The rate for such processes follows the Arrhenius law,

$$\Gamma \sim e^{-E_B/k_B T} \approx (I/I_c)^{2x} .$$

(7)

In the second step of the argument, one writes down a mean-field rate equation governing the density $\rho = \rho_\pm$ of free vortices,

$$\dot{\rho} = \Gamma - \gamma \rho^2 ,$$

(8)

where $\gamma$ is the diffusion constant. The steady-state vortex density is then given by $\rho_\infty = (\Gamma/\gamma)^{1/2}$. Going back to Eq. (3), one finds

$$V_\infty \sim \rho_\infty \bar{v}_\pm \sim I^{x+1} ,$$

(9)

where we have assumed the drift velocity $\bar{v}_\pm$ of a free vortex to be proportional to $I$.

In the RSJ dynamics, the spin-waves have a wavelength-independent relaxation time, of the order of the basic time unit $\sigma h/2eI_c$. Hence the equilibrium interaction between vortices in (3) is un-retarded in the dynamic case. Equation (3), on the other hand, is based on the assumption that there is no spatial correlation between vortices and antivortices. It turns out that this assumption is violated when the width of the strip is less than a certain characteristic size $l_r = \rho_\infty^{1/2} \sim (I_c/I)^{1/2}$. In a bulk system, there is typically one free vortex/antivortex pair in an area of linear size $l_r$ at any given time. This is no longer the case when $L_y < l_r$. Under periodic boundary conditions in the $y$-direction, a vortex may annihilate with an antivortex created from the same bound pair. Consequently, the life time of the two are shortened, giving rise to a spurious lower voltage as seen in Fig. 2.

To obtain a semi-quantitative estimate of the voltage reduction, let us consider a simplified model of vortex-antivortex recombination. In a coarse-grained description, we take $l_0$ to be the basic unit of length. The time for a single vortex to traverse the width of the strip is $\tau = L_y/\bar{v}_\pm = L_y l_0$. During this period, the typical displacement of the vortex in the $x$-direction is $1 = \tau^{1/2}$. The probability for the vortex to be within a distance $l_0$ from its initial position after one round is thus $p_1 = l_0/\tau = (l_0/L_y)^{1/2}$. This is the probability for a vortex-antivortex pair to recombine in one round across the strip. Assuming $p_1 \ll 1$, the probability for this to happen in two rounds is $p_2 \simeq (l_0/2L_y)^{1/2}$. In general, $p_n \simeq p_1 n^{-1/2}$ provided $p_1 n^{-1/2} < 1$. The survival probability after $n$ rounds is $1 - \sum_{i=1}^n p_i \simeq 1 - p_1 n^{1/2}$. This yields a self-recombination time,

$$\tau_s \simeq p_1^{-2} \tau = L_y^2 .$$

(10)

The density of free vortices in this case is $\rho_s \simeq \Gamma \tau_s \sim L_y^2 (I/I_c)^{2x}$. From Eq. (3) we obtain,

$$V_s \sim \rho_s \bar{v}_\pm \sim L_y^2 (I/I_c)^{2x+1} ,$$

(11)

which increases with increasing $L_y$.

The typical displacement of the vortex in the $x$-direction over $\tau_s$ is $\tau_s^{1/2} = L_y$. To preempt self-recombination, we need $I L_y^{2/3} \geq 1$, i.e., creation of the second vortex/antivortex pair in the same region in space within the time $\tau_s$. This yields $L_y = (I_c/I)^{3/2} = l_1!$
Therefore Eq. (11) applies in the intermediate regime $l_0 \ll L_y \ll l_1$.

The situation is somewhat different when $L_y < l_0$. In this case, the pair-breaking interaction in (1) due to the external current is always weaker than the vortex-antivortex attraction and hence can be treated as a perturbation. The $I$-$V$ curves can in principle be calculated in a linear response theory [18]. Qualitatively, though, we may estimate the finite-size effect from the equilibrium relaxation time of a vortex-antivortex pair on scale $L_y$,

$$\tau_{eq} \approx L_y^z,$$

where $z = 2\chi - 2$ is the dynamic exponent. We now divide the strip into square segments of $L_y$ columns each. The rate of creating a vortex-antivortex pair of size $L_y$ in such a segment is $\tau_{eq}^{-1}$. In thermal equilibrium, the vortex/antivortex may drift in either direction around the strip and recombine to generate a $\pm 2\pi$ phase jump. This symmetry is broken by the external current which enhances the $-2\pi$ jump (along the current direction) by a factor equal to the ratio between the energy difference of the two possibilities and $k_B T$. This leads to the estimate,

$$V_{Ohm} \sim \frac{1}{\tau_{eq}L_y} \left( \frac{IJL_y}{I_c k_B T} \right) \sim (I/I_c)L_y^{-z}. \quad (13)$$

In the Ohmic regime, $V$ decreases rapidly with increasing $L_y$, as seen in Fig. 2.

At $L_y = l_0$, the two expressions (11) and (13) coincide to give the minimum value $V_m = (I/I_c)^{1+z}$. This is indeed the power-law form obtained by Minnhagen et al. by assuming $l_0$ as the only relevant length scale.

A direct consequence of the self-recombination picture proposed above is the intermittent phase difference jumps across a square array of size less than $l_1$. In the steady state, the phase difference $\Delta \phi$ across the system, averaged in the direction perpendicular to the current, is expected to grow linearly with time. However, if there is only at most one pair of free vortex/antivortex present in the system at any given time, the growth of $\Delta \phi$ is intermittent. This is indeed the case for $L_y < l_0$, where the life time of a free vortex/antivortex pair is less than the time it takes to create the pair. The step-wise growth of $\Delta \phi$ has been observed in a previous numerical study using a square array [8] which we have confirmed. A similar phenomenon occurs in the $L_y < l_0$ regime, although in this case there are occasionally reverse steps. For a long strip, however, $\Delta \phi$ is a much smoother function of time due to averaging along the strip.

In summary, extensive simulations of the Josephson junction array in strip geometry reveal the existence of three size regimes with distinct current-voltage relationships. Arrays of size less than $l_0 \sim I^{-1}$ are in the Ohmic regime and obey dynamic scaling as proposed by Minnhagen et al. Between $l_0$ and a second scale $l_1 \sim I^{-x/2}$, a reversed size-effect is observed. The power-law $I$-$V$ dependence of AHNS is shown to hold in the asymptotic regime. Our study reconciles a long-standing dispute between two theoretical approaches, and provides a framework for systematic analysis of finite-size effects in 2D superfluid and superconducting systems with equal number of vortices and antivortices.

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