Order of Meromorphic Maps and Rationality
of the Image Space

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(Received Mar. 23, 2011)

Abstract. Let \( \iota : C^2 \hookrightarrow S \) be a compactification of the two dimensional complex space \( C^2 \). By making use of Nevanlinna theoretic methods and the classification of compact complex surfaces K. Kodaira proved in 1971 ([2]) that \( S \) is a rational surface. Here we deal with a more general meromorphic map \( f : C^n \to X \) into a compact complex manifold \( X \) of dimension \( n \), whose differential \( df \) has generically rank \( n \). Let \( \rho_f \) denote the order of \( f \). We will prove that if \( \rho_f < 2 \), then every global symmetric holomorphic tensor must vanish; in particular, if \( \dim X = 2 \) and \( X \) is kähler, then \( X \) is a rational surface. Without the kähler condition there is no such conclusion, as we will show by a counter-example using a Hopf surface. This may be the first instance that the kähler or non-kähler condition makes a difference in the value distribution theory.

1. Introduction and main results.

Let \( X \) be a compact hermitian manifold with metric form \( \omega \). Let \( f : C^n \to X \) be a meromorphic map (cf. [4] for this section in general). If the differential \( df \) is generically of maximal rank, \( f \) is said to be differentiably non-degenerate. We set

\[
\alpha = dd^c ||z||^2
\]

for \( z = (z_j) \in C^n \), where \( d^c = (i/4\pi)(\bar{\partial} - \partial) \) and \( ||z||^2 = \sum_{j=1}^{n} |z_j|^2 \). We use the notation:

\[
B(r) = \{ z \in C^n : ||z|| < r \}, \quad S(r) = \{ z \in C^n : ||z|| = r \} \quad (r > 0).
\]

We define the order function of \( f \) with respect to \( \omega \) by

\[
T_f(r; \omega) = \int_{1}^{r} \frac{dt}{t^{2n-1}} \int_{B(t)} f^* \omega \wedge \alpha^{n-1}.
\]
Then the (upper) order is defined by

\[ \rho_f = \lim_{r \to \infty} \frac{\log T_f(r; \omega)}{\log r}. \]

It is easy to see that \( \rho_f \) is independent of the choice of the metric (form \( \omega \)) on \( X \).

**Example 1.3.**

(i) If \( X = P^n(C) \) and \( f \) is rational, then \( \rho_f = 0 \).

(ii) Let \( X \) be a compact torus. If \( f : C^n \to X \) is non-constant, then \( \rho_f \geq 2 \). If \( \lambda : C^n \to X \) (dim \( X = n \)) is the universal covering map, then \( \rho_{\lambda} = 2 \).

A compact complex manifold which is bimeromorphic to \( P^n(C) \) is called a rational variety. A two-dimensional compact complex manifold is called a complex surface. If it admits a kähler metric, it is called a kähler surface.

The main result of this paper is the following:

**Main Theorem 1.4.** Let \( X \) be a kähler surface. Assume that there is a differentiably non-degenerate meromorphic map \( f : C^2 \to X \). If \( \rho_f < 2 \), then \( X \) is rational.

The kähler condition is necessary by the following:

**Theorem 1.5.** There is a Hopf surface \( S \) for which there is a differentiably non-degenerate holomorphic map \( f : C^2 \to S \) with \( \rho_f = 1 \).

Let \( \Omega^k_X \) denote the sheaf of germs of holomorphic \( k \)-forms over a complex manifold \( X \). We denote by \( S^l\Omega^k_X \) its \( l \)-th symmetric tensor power. In particular, \( K_X = \Omega^n_X \) (\( n = \text{dim } X \)) denotes the canonical bundle over \( X \).

The key tool for the proof of the Main Theorem 1.4 is:

**Theorem 1.6.** Let \( X \) be an \( n \)-dimensional compact complex manifold. Assume that there exists a differentiably non-degenerate meromorphic map \( f : C^m \to X \) (\( m \geq n \)) with \( \rho_f < 2 \). Then for arbitrary \( l_k \geq 0 \) with \( \sum_{k=1}^n l_k > 0 \)

\[ H^0(X, S^{l_1}\Omega^1_X \otimes \cdots \otimes S^{l_n}\Omega^n_X) = \{0\}. \]

**Remark 1.7.** So far by our knowledge, the above theorems are the first instance that the kähler or non-kähler condition makes a difference in the value distribution theory.
2. Proof of the Main Theorem.

2.1. Proof of Theorem 1.6.
Assume the existence of an element
\[ \tau \in H^0(X, S^1 \Omega_X^1 \otimes \cdots \otimes S^n \Omega_X^n) \setminus \{0\}. \]
We take a hermitian metric \( h \) on \( X \) with the associated form \( \omega \). There are induced hermitian metrics on the symmetric powers of the bundles \( \Omega_X^k \) and their tensor products which by abuse of notation are again denoted by \( h \). Let \( \| \tau \|_h \) denote the norm of \( \tau \) with respect to \( h \). Then there is a constant \( c_1 > 0 \) such that
\[ \| \tau \|_h \leq c_1. \] (2.1)
We denote by \( \xi_\Lambda \) the coefficient functions of \( f^* \tau \) with respect to the standard coordinate system \((z_1, \ldots, z_m)\) on \( \mathbb{C}^m \). Since \( f \) is meromorphic, \( f^* \tau \) is obviously holomorphic outside the indeterminacy set \( I_f \). Because \( \text{codim}(I_f) \geq 2 \) and because \( f^* \tau \) is a section in a globally defined vector bundle, it extends holomorphically to \( I_f \). Thus we may regard \( f^* \tau \) as being holomorphic on \( \mathbb{C}^n \) and the coefficient functions \( \xi_\Lambda \) with respect to the flat frames generated by \( dz_1, \ldots, dz_n \) are holomorphic as well. We set
\[ \| f^* \tau \|^2_{\mathbb{C}^m} = \sum_\Lambda |\xi_\Lambda|^2 \neq 0. \] (2.2)
We define a function \( \zeta \) on \( \mathbb{C}^m \) by
\[ f^* \omega \wedge \alpha^{m-1} = \zeta \alpha^m. \]
Since \( f \) is differentiably non-degenerate, \( f^* \tau \neq 0 \). By (2.1) there are positive constants \( c_2 \) and \( c_3 \) such that
\[ \zeta \geq c_2 \| f^* \tau \|_{\mathbb{C}^m}^{2c_3}. \] (2.3)
By (2.2) \( \| f^* \tau \|_{\mathbb{C}^m}^{2c_3} \) is plurisubharmonic. Since \( f^* \tau \neq 0 \) is holomorphic, it follows that
\[ \int_{S(1)} \| f^* \tau \|_{\mathbb{C}^m}^{2c_3} \gamma = c_4 > 0, \]
where
\[ \gamma = \frac{1}{r^{2m-1}} d^c \|z\|^2 \wedge \alpha^{m-1}, \quad (2.4) \]

induced on \( S(r) \) with \( r = 1 \). Since the plurisubharmonicity of \( \|f^* \tau\|^{2c_3}_{C_m} \) implies \( dd^c \|f^* \tau\|^{2c_3}_{C_m} \geq 0 \) as currents,

\[
\int_{S(r)} \|f^* \tau\|^{2c_3}_{C_m} \gamma - \int_{S(s)} \|f^* \tau\|^{2c_3}_{C_m} \gamma = \int_s^r \frac{dt}{t^{2m-1}} \int_{B(t)} dd^c \|f^* \tau\|^{2c_3}_{C_m} \wedge \alpha^{m-1} \geq 0
\]

for \( r > s > 0 \) (cf. [4]). Thus,

\[
\int_{S(r)} \|f^* \tau\|^{2c_3}_{C_m} \gamma
\]

is monotone increasing in \( r > 0 \). Then,

\[
\frac{1}{r^{2m-1}} \int_{S(r)} \|f^* \tau\|^{2c_3}_{C_m} d^c \|z\|^2 \wedge \alpha^{m-1} = \int_{S(r)} \|f^* \tau\|^{2c_3}_{C_m} \gamma \geq \int_{S(1)} \|f^* \tau\|^{2c_3}_{C_m} \gamma = c_4
\]

for \( r > 1 \), so that

\[
\int_{S(r)} \|f^* \tau\|^{2c_3}_{C_m} d^c \|z\|^2 \wedge \alpha^{m-1} \geq c_4 r^{2m-1}, \quad r > 1.
\]

Therefore

\[
\int_{B(r)} \|f^* \tau\|^{2c_3}_{C_m} \alpha^m \geq \int_1^r c_4 t^{2m-1} dt = \frac{c_4}{2m} (r^{2m} - 1), \quad r > 1.
\]

We deduce from this that

\[
T_f(r, \omega) = \int_1^r \frac{dt}{t^{2m-1}} \int_{B(t)} \zeta \alpha^m \geq c_2 \int_1^r \frac{dt}{t^{2m-1}} \int_{B(t)} \|f^* \tau\|^{2c_3}_{C_m} \alpha^m
\]

\[
\geq \frac{c_2 c_4}{2m} \int_1^r \left( t - \frac{1}{t^{2m-1}} \right) dt = \frac{c_2 c_4}{4m} r^2 + C_m(r),
\]

where \( C_1(r) = O(\log r) \) and \( C_m(r) = O(1) \) for \( m \geq 2 \). Thus,

\[
\rho_f = \lim_{r \to \infty} \frac{\log T_f(r, \omega)}{\log r} \geq 2.
\]
This is a contradiction. □

**Corollary 2.5.** If \( X \) in Theorem 1.6 is 1-dimensional, then \( X \) is biholomorphic to \( \mathbb{P}^1(C) \).

**2.2. Proof of the Main Theorem 1.4.**

There is a fine classification theory of complex surfaces (cf. Kodaira [2], Barth-Peters-Van de Ven [1]). According to it, we know the following fact, where \( b_1(X) = \dim H_1(X, \mathcal{R}) \) denotes the first Betti number of \( X \).

**Theorem 2.6** (Kodaira [68, Theorem 54]). If a complex surface \( X \) satisfies \( b_1(X) = 0 \) and \( H^0(X, K_X^l) = \{0\} \) for all \( l > 0 \), then \( X \) is rational.

This enables us to prove Theorem 1.4 as follows. By Theorem 1.6 \( \dim H^0(X, \Omega_X^1) = 0 \). Due to the kähler assumption, we have \( b_1(X) = 2 \dim H^0(X, \Omega_X^1) = 0 \). Moreover, \( H^0(X, K_X^l) = \{0\} \) for all \( l > 0 \) again by Theorem 1.6. It follows from Theorem 2.6 that \( X \) is rational. □

**3. Proof of Theorem 1.5.**

Let \( \lambda \in \mathbb{C} \) with \( |\lambda| > 1 \). Then a Hopf surface \( S \) is defined as the quotient of \( \mathbb{C}^2 \setminus \{(0, 0)\} \) under the \( \mathbb{Z} \)-action given by \( n : (x, y) \mapsto (\lambda^n x, \lambda^n y) \). Such a surface \( S \) is known to be diffeomorphic to \( S^1 \times S^3 \). As a consequence \( b_1(S) = 1 \) and \( S \) is not kähler.

Now

\[
\omega = \frac{i}{2\pi} \left( dx \wedge d\bar{x} + dy \wedge d\bar{y} \right) = \frac{dd^c \|(x, y)\|^2}{\|(x, y)\|^2}
\]

is a positive \((1, 1)\)-form on \( \mathbb{C}^2 \setminus \{(0, 0)\} \) which is invariant under the above given \( \mathbb{Z} \)-action. Therefore it induces a positive \((1, 1)\)-form on the quotient surface \( S \) which by abuse of notation is again denoted by \( \omega \).

Let \( \alpha \) and \( \gamma \) be as in (1.1) and (2.4), respectively. We claim that the holomorphic map \( f : \mathbb{C}^2 \to S \) induced by

\[
(z, w) \mapsto (z, 1 + zw)
\]

is of order 1. By definition this means

\[
\rho_f = \lim_{r \to \infty} \frac{\log T_f(r, \omega)}{\log r} = 1,
\]
i.e.,

$$\lim_{r \to \infty} \frac{1}{\log r} \log \int_1^r \frac{dt}{t^3} \int_{B(t)} f^* \omega \wedge \alpha = 1.$$ 

Note that

$$f^* \omega \wedge \alpha = \frac{1 + |z|^2 + |w|^2}{2(|z|^2 + |1 + zw|^2)} \alpha^2.$$

We define

$$I_r = \int_{S(r)} \frac{r^2}{|z|^2 + |1 + zw|^2} dV, \quad r = \|(z, w)\|.$$ 

Here $dV$ is the euclidean volume element on $S(r)$, and therefore a constant multiple of $r^3 \gamma$. It is sufficient to show

$$I_r = O(r^{2+\varepsilon}), \quad \forall \varepsilon > 0, \quad \text{and} \quad r^2 = O(I_r). \quad (3.1)$$

Indeed, assume that this holds. Because of $\lim_{r \to \infty} (1 + r^2)/r^2 = 1$, (3.1) is equivalent to the assertion

$$I'_r = O(r^{2+\varepsilon}), \quad \text{and} \quad r^2 = O(I'_r).$$

with

$$I'_r = \int_{S(r)} \frac{1 + r^2}{|z|^2 + |1 + zw|^2} dV.$$ 

From this we first obtain

$$\int_{B(r)} \frac{1 + r^2}{|z|^2 + |1 + zw|^2} \alpha^2 = O\left(\int I'_r \, dr\right) = O(r^{3+\varepsilon}), \quad \forall \varepsilon > 0,$$

implying

$$T_f(r) = \frac{1}{2} \int_1^r \frac{dt}{t^3} \int_{B(t)} \frac{1 + r^2}{|z|^2 + |1 + zw|^2} \alpha^2 = O(r^{1+\varepsilon}), \quad \forall \varepsilon > 0,$$
and
\[ \rho_f = \lim_{r \to \infty} \frac{\log T_f(r)}{\log r} \leq 1. \]

In the same way from the second estimate of (3.1) we get the opposite estimate \( \rho_f \geq 1 \), and therefore \( \rho_f = 1 \). Hence it suffices to show (3.1).

We define
\[ \eta = \frac{r^2}{|z|^2 + |1 + zw|^2}. \]

Thus we have to show
\[ I_r = \int_{S(r)} \eta dV = O(r^{2+\varepsilon}). \]

We set
\[ \eta = \frac{r^2}{\phi(z, w)}, \quad \phi(z, w) = |z|^2 + |1 + zw|^2. \]

3.1. Geometric estimates.

For \((z, w) \in S(r)\) let \(\theta \in [0, 2\pi)\) such that \(e^{i\theta}|zw| = zw\). Let \(K > 0\), \(-\infty < \lambda < 1\) and \(\mu \geq 0\). We set
\[ \Omega_{K, \lambda, \mu} = \{ (z, w) \in S(r) : |z| \leq Kr^\lambda, |\sin \theta| \leq r^{-\mu} \}. \]

We need some volume estimates.

First we note that \((\sin \theta)/\theta \geq 2/\pi\) for all \(\theta \in [0, \pi/2]\), because \(\sin\) is concave on \([0, \pi/2]\). It follows that for every \(C \in ]0, 1]\) we have the following bound for the Lebesgue measure:
\[ \text{vol} \left( \{ \theta \in [0, 2\pi] : |\sin \theta| \leq C \} \right) \leq 4(C\pi/2) = 2C\pi. \quad (3.2) \]

Second we define a map \(\zeta : C^2 \to C \times R^2\) as follows:
\[ \zeta : (z, w) \mapsto (z, r \arg(zw), r), \]

where \(r = \|(z, w)\| = \sqrt{|z|^2 + |w|^2}\).
To compute the Jacobian $J$ of this map we set the coordinates so that $z = x + iy$, $w = u + iv$, and write

$$r = \sqrt{x^2 + y^2 + u^2 + v^2},$$

$$\zeta : (x, y, u, v) \mapsto (x, y, r(\arg z + \arg w), r) \in \mathbb{R}^4.$$ 

Then the Jacobian $J$ takes the following form:

$$J = \begin{vmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & \frac{u}{r}(\arg z + \arg w) + r\frac{\partial}{\partial u}\arg w & \frac{u}{r} \\ 0 & 0 & \frac{v}{r}(\arg z + \arg w) + r\frac{\partial}{\partial v}\arg w & \frac{v}{r} \end{vmatrix} = r\frac{\partial}{\partial u}\arg w \frac{u}{r} \frac{\partial}{\partial v}\arg w \frac{v}{r}.$$ 

An easy practice of computation implies that “$J \equiv -1$”.

Furthermore the gradient $\nabla(r)$ is of length one and normal on the level set $S(r)$. Correspondingly the map $\zeta$ is volume preserving and $S(r)$ has the same volume as its image

$$\zeta(S(r)) = \{z \in C : |z| \leq r\} \times [0, 2\pi r] \times \{r\}, \quad (3.3)$$

namely $2\pi^2 r^3$.

Similarly the euclidean volume of $\Omega_{K,\lambda,\mu}$ agrees with the euclidean volume of

$$\zeta(\Omega_{K,\lambda,\mu}) = \{z \in C : |z| \leq Kr^\lambda\} \times \{\theta r : \theta \in [0, 2\pi), |\sin \theta| \leq r^{-\mu}\} \times \{r\}.$$ 

Using (3.2) it follows that for $r \geq 1$ the volume of $\Omega_{K,\lambda,\mu}$ is bounded by

$$\pi(Kr^\lambda)^2 \cdot 2r^{-\mu}\pi r = 2K^2\pi^2 r^{2\lambda+1-\mu}.$$ 

In particular,

$$\text{vol}(\Omega_{K,\lambda,\mu}) = O(r^{2\lambda+1-\mu}). \quad (3.4)$$
3.2. Arithmetic estimates.

Besides the Landau $O$-symbols, we also use the notation “$\gtrsim$”: If $f, g$ are functions of a real parameter $r$, then $f(r) \gtrsim g(r)$ indicates that

$$\liminf_{r \to +\infty} \frac{f(r)}{g(r)} \geq 1.$$ 

Similarly $f \sim g$ indicates

$$\lim_{r \to +\infty} \frac{f(r)}{g(r)} = 1.$$ 

In the sequel, we will work with domains $\Omega \subset S(r)$ (i.e. for each $r > 0$ some subset $\Omega = \Omega_r \subset S(r)$ is chosen). In this context, given functions $f, g$ on $\mathbb{C}^2$ we say “$f(z, w) \gtrsim g(z, w)$ holds on $\Omega$” if for every sequence $(z_n, w_n) \in \mathbb{C}^2$ with

$$\lim\sup_{n \to \infty} \left\| (z_n, w_n) \right\| = +\infty$$ 

and $(z_n, w_n) \in \Omega_r (r = \left\| (z_n, w_n) \right\|)$ we have

$$\liminf_{n \to \infty} \frac{f(z_n, w_n)}{g(z_n, w_n)} \geq 1.$$ 

We develop some estimates for $\phi(z, w) = |z|^2 + |1 + zw|^2$. Fix $\mu > 0, -\infty < \lambda < 1$.

(i) For all $z, w$: $\phi \geq |z|^2$.

(ii) If $(z, w) \in S(r)$ and $|z| \leq 1/2r$, then

$$|w| \leq r \implies |zw| \leq \frac{1}{2} \implies |1 + zw| \geq \frac{1}{2}$$

and therefore $\phi \geq 1/4$.

(iii) For $|z| \leq r^\lambda$ we have $|w| \sim r$, i.e., for fixed $\lambda, \mu$ and any choice of $(z_r, w_r) \in S(r)$ with $|z_r| \leq r^\lambda$ we have $\lim_{r \to \infty} |w_r|/r = 1$.

(iv) For $|z| \geq 3/2r$ and $|z| \leq r^\lambda$ we have that $\phi \gtrsim (1/9)|zw|^2$, because $|w| \sim r$ and $|zw| \gtrsim 3/2$ (equivalently, $1 \lesssim (2/3)|zw|$), implying $|1 + zw| \geq |zw| - 1 \gtrsim (1/3)|zw|$.

(v) For all $z, w$, $\phi \geq |\Re(1 + zw)|^2 = (|zw| \sin \theta)^2$.

3.3. Putting things together.

We are going to prove first the claim

"$I(r) = O(r^{2+\epsilon}), \forall \epsilon > 0$"
by dividing $S(r)$ into regions $A$, $B$, $C$, $D_{-2}$, $D_{-1}$, $D_0$, $D_1$, $E$, $F$, each of which is investigated separately.

- **Region A** consists of those points with $|z| \leq 1/2r$, i.e., $A = \Omega_{1/2,-1,0}$. The volume $\text{vol}(A)$ is thus of order $O(r^{-1})$. Due to (ii) the integrand $\eta$ is bounded by $\eta|_A = O(r^2)$. It follows that

$$\int_A \eta \, dV \leq \text{vol}(A) \cdot \sup_{(z,w) \in A} \eta(z,w) = O(r).$$

Hence the contribution of $A$ to the integral $I_r = \int_{S(r)} \eta \, dV$ is bounded by $O(r)$.

- **Region B** consists of those points with $1/2r \leq |z| \leq 3/2r$ and $|\sin \theta| < 1/r$. Thus $B \subset \Omega_{3/2,-1,1}$. Due to (3.4) this implies $\text{vol}(B) = O(r^{-2})$. For the integrand $\eta|_B$ we have the bound $\eta|_B = O(r^4)$ (using (i) and $|z| \geq 1/2r$). Hence

$$\int_B \eta \, dV \leq \text{vol}(B) \cdot \sup_{(z,w) \in B} \eta(z,w) = O(r^2);$$

i.e., the contribution of $B$ to the integral $I_r$ is bounded by $O(r^2)$.

- **Region C** consists of those points with $1/2r \leq |z| \leq 3/2r$ and $|\sin \theta| > 1/r$. Since $|w| \sim r$, $1/2 \lesssim |zw| \lesssim 3/2$ We take the volume-compatible parameter $\psi = r\theta$ due to (3.3). Then $1/r < |\sin \psi/r| < \psi/r$, and so $\psi > 1$. Therefore

$$J_r := \int_{1<\psi<2\pi r, |\sin \psi/r|>1/r} \eta \, d\psi$$

$$= \int_{1<\psi<2\pi r, |\sin \psi/r|>1/r} \frac{2r^2}{(\sin \psi/r)^2} d\psi = O(r^4).$$

Here in fact we have that there is a constant $c > 1$ such that

$$\frac{r^4}{c} \leq J_r \leq cr^4.$$

Therefore it follows that

$$\frac{r^2}{c} \leq \int_C \eta \, dV = \int_{1/2r \leq |z| \leq 3/2r} J_r \frac{i}{2} dz \wedge d\bar{z} \leq c'r^2,$$  \hspace{1cm} (3.5)
where $c'$ is a positive constant. Thus the contribution of $C$ to the integral $I_r$ is bounded by $O(r^2)$.

- For $\gamma \in \{-2, -1, 0, 1\}$ let $D_\gamma$ denote the set of those points where $|z| \geq 3/2r$, $|z| \leq r^{1-\varepsilon}$ and $r^{\gamma/2} \leq |z| \leq r^{(\gamma+1)/2}$. For each $\gamma$ the integrand $\eta$ is bounded on $D_\gamma$ by $O(r^{-\gamma})$ (due to (iv)), and the volume $\text{vol}(D_\gamma)$ is bounded by $O(r^{2+\gamma})$, because $D_\gamma \subset \Omega_{1, (\gamma+1)/2, 0}$. Thus the contribution of $D_\gamma$ to the integral $I_r$ is bounded by $O(r^2)$.

- Let $E$ denote the region where $|z| \geq r^{1-\varepsilon}$, $|w| \geq r^{1/2}$. For the integrand we have that $\eta|_E = O(r^{2\varepsilon-1})$ (using (iv)). The volume of $E$ is bounded by the total volume of $S(r)$, i.e., $\text{vol}(E) = O(r^3)$. Together this shows that the contribution of $E$ to $I_r$ is bounded by $O(r^{2+2\varepsilon})$.

- Let $F$ denote the region where $|w| \leq r^{1/2}$. In analogy to (iii) we have $|z| \sim r$. With (i) it follows that $\sup_{(z, w) \in F} \eta(z, w) = O(1)$. On the other hand the volume of $F$ agrees with the volume of $\{ (z, w) \in S(r) : |z| \leq r^{1/2} \}$ which according to (3.4) is bounded by $O(r^2)$. Together this yields that the contribution of $F$ to $I_r$ is bounded by $O(r^2)$.

Thus we have a collection of nine regions ($A$, $B$, $C$, $D_{-2}$, $D_{-1}$, $D_0$, $D_1$, $E$, $F$) covering the sphere $S(r)$. For each such region $\Omega$ we have verified

$$\int_{\Omega} \eta \, dV = O(r^{2+\varepsilon}), \quad \varepsilon > 0.$$ 

This establishes our claim

$$I_r = O(r^{2+\varepsilon}), \quad \varepsilon > 0.$$ 

Furthermore, it follows from (3.5) that

$$r^2 = O(I_r).$$

As a consequence, the holomorphic map $f : \mathbb{C}^2 \to S$ induced by $f : (z, w) \mapsto (z, 1 + zw)$ is of order $\rho_f = 1$. $\square$

4. Problems.

Because of the results presented above it may be interesting to recall some problems (conjectures) from [3, Section 1.4]. An $n$-dimensional compact complex manifold $X$ is said to be unirational if there is a surjective meromorphic map $\phi : \mathbb{P}^n(\mathbb{C}) \to X$; in this case, if $g : \mathbb{C}^n \to \mathbb{P}^n(\mathbb{C})$ is a differentiably non-degenerate meromorphic map with order $\rho_g < 2$, then $\phi \circ g : \mathbb{C}^n \to X$ is differentiably
non-degenerate and has order less than two. Therefore, the rationality and the unirationality of $X$ cannot be distinguished by the existence of a differentiably non-degenerate meromorphic map $f : C^n \to X$ with $\rho_f < 2$.

**Problem 4.1.** Let $X$ be a compact kähler manifold of dimension $n$. If there is a differentiably non-degenerate meromorphic map $f : C^n \to X$ with order $\rho_f < 2$, is $X$ unirational?

At least this is true for $\dim X \leq 2$ by Corollary 2.5 and the Main Theorem 1.4.

**Problem 4.2.** Let $f : C \to X$ be a non-constant entire curve into a projective (or kähler) manifold $X$. If $\rho_f < 2$, then does $X$ contain a rational curve?

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