Bounding the detection efficiency threshold in Bell tests using multiple copies of the two-qubit maximally entangled state

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In this paper we investigate the critical efficiency of detectors in order to see Bell nonlocality using multiple copies of the two-qubit maximally entangled state and local Pauli measurements which act in the corresponding qubit subspaces. It is known that for the two-qubit maximally entangled state a symmetric detection efficiency of 82.84% can be tolerated using the Clauser-Horne-Shimony-Holt (CHSH) Bell test. We show that this threshold can be lowered by using multiple copies of the two-qubit maximally entangled state. We get the upper bounds 80.86%, 73.99% and 69.29% on the symmetric detection efficiency threshold for two, three and four copies of the state, where the respective number of measurements per party are 4, 8 and 16. However, in the case of four copies the result is partly due to a heuristic method. In order to get the corresponding Bell inequalities we made use of linear programming for two copies of the state and convex optimization based on Gilbert algorithm for three and four copies of the state.

I. INTRODUCTION

Quantum theory predicts that there exist correlations in nature which cannot be simulated with classical resources. In particular measurements performed on separated parts of entangled systems can produce outcomes whose associated correlations cannot be explained by any local classical model. These strong correlations can be witnessed by the violation of Bell inequalities [1].

Bell violation has been demonstrated in a number of laboratory experiments, the first pioneering experiment having performed by Aspect and coworkers [2]. However, many loopholes may open in Bell experiments due to technical imperfections [3]. In fact loophole-free Bell violations became possible only recently [4–7]. These experiments give so far the strongest evidence that nature is nonlocal.

One of the main technical difficulties in achieving a loophole-free Bell violation is due to the finite detection efficiency of the detectors [8]: some of the emitted systems can produce outcomes whose associated correlations cannot be explained by any local classical model. These strong correlations can be witnessed by the violation of Bell inequalities [1].

The first detection loophole free Bell tests performed in 2015 all violated the Clauser-Horne-Shimony-Holt (CHSH) Bell inequality [10]. Indeed, this is the simplest bipartite Bell inequality, which consists of two settings ($m = 2$) with two outcomes ($o = 2$) per party, and the required detection efficiency to see Bell violation is $2(\sqrt{2} - 1) \approx 0.8284$ using a two-qubit maximally entangled state $|\varphi_+\rangle = \frac{|0, 0\rangle + |1, 1\rangle}{\sqrt{2}}$.

Here we assumed that all detectors have the same detection efficiency $\eta$. Using Bell inequalities with the two-qubit maximally entangled state, involving more than two inputs only minor improvements were reported. To the best of our knowledge, the smallest reported symmetric detection efficiency threshold is 0.8214 in the case of two measurement outcomes and maximally entangled two-qubit systems [13]. This efficiency is provided by the four-setting Bell inequality $A_5$ from the list of Avis et al. [14]. Exactly the same threshold was obtained by Massar et al. [15] investigating four-setting Bell inequalities using a geometric approach.

In the present work we consider Bell inequalities with finite detection efficiencies using multiple copies of the two-qubit state $|\varphi_+\rangle$. In particular, we show that using $n$ copies of the state $|\varphi_+\rangle$ (or any other two-qubit maximally entangled state), one can lower the critical value 0.8214 by using Bell inequalities with $m = 2^n$ settings and $o = 2^n$ outcomes for $n \geq 2$. Let us denote by $\eta^{(n)}_{sym}$ the lowest possible symmetric detection efficiency threshold in this $n$-copy scenario. In the present work, we give upper bounds on this value. We in particular get the values of $\eta^{(2)}_{sym} \leq 0.8086$ and $\eta^{(3)}_{sym} \leq 0.7399$ for two and three copies, respectively. For more than three copies it is getting hard to get exact bounds, but our numerics strongly supports the upper bound $\eta^{(4)}_{sym} \leq 0.6929$. We will also consider the case, when one
party’s detectors have unit efficiency, in which case we denote the respective threshold by $\eta^{(n)}_{\text{sym}}$. We also obtain upper bounds in this case.

In addition, we upperbound $\eta^{(n)}_{\text{sym}}$ analytically and we find that the $2/3$ value can be surpassed by $n = 13$ copies of the singlet state. This also demonstrates how difficult it is to considerably reduce the required detection efficiency to see Bell violation in this multiple-copy scenario. In a distinct scenario, Massar [17] obtained the threshold values $\eta_{\text{sym}} \simeq D^{3/4}2^{-0.003x}D$ for a special family of Bell inequalities using $(D \times D)$-dimensional maximally entangled states. If we consider this high dimensionally entangled state as multiple copies of the maximally entangled state (1), we find that $n$ should be larger than 10 (and $m$ larger than $2^{1024}$) in order to obtain $\eta_{\text{sym}} < 2/3$.

Another approach to the detection efficiency problem, particularly fruitful in photonic experiments [5, 6] is to use partially entangled states instead of the singlet state. With this modification in the state (and measurements) the detection efficiency threshold can be lowered for the CHSH inequality from 0.8284 down to $2/3$, the latter value being the so-called Eberhard limit [18]. However, in order to reach this value, both the state and the applied measurements have to be fine tuned, as task which is experimentally challenging. Namely, the state has to be a pure state close to the product state and the measurement observables should almost commute [18, 19]. On a related note, recent experimental study showed that almost product states are very fragile to get high rate of random bits [20] from Bell violations (a task based on Bell nonlocality initiated in Refs. [21, 22]).

In this study we take a different route. We consider a measurement setup with anticommuting observables on both sides and multiple copies of the two-qubit maximally entangled state (1) (equivalent to the singlet state up to local change of basis). This gives the threshold $\eta_{\text{sym}} = 2\sqrt{2} - 2 \simeq 0.8284$ for one copy given by the CHSH inequality [10]. In the next section, we reproduce this known value [11, 12]. Then we will show that this value can be lowered by taking multiple copies of the singlet state and performing anticommuting qubit measurements on halves of the singlet states.

Let us mention that for two copies of the singlet state the lowest detection efficiency threshold found so far corresponds to the $I_{1422}$ inequality [13], giving the threshold efficiency of $\eta_{\text{sym}} = 0.7698$ [23]. However, in this case the measurements act nontrivially on the four-dimensional space. More recent studies using Bell inequalities with multiple outcomes and higher dimensional maximally entangled states could not find improvement on this value [24]. However, as we mentioned, our present result shows that $n = 3$ copies of the two-qubit maximally entangled state can tolerate a detection efficiency of 0.7399 beating the above limit of 0.7698.

The structure of the paper is as follows. In section III we introduce two-party correlation-type Bell inequalities (and the CHSH inequality in particular) and discuss their detection efficiency thresholds. In section IV we give upper bounds for the detection efficiency thresholds of the iterated version of the CHSH expression. In Secs. V and VI our geometric approaches based on linear programming and convex optimization are presented, respectively. The paper finishes with discussions in Sec. VII.

II. BELL INEQUALITIES AND DETECTION EFFICIENCIES

The CHSH inequality.—We use the maximally entangled two-qubit state (1), locally equivalent to the Bell singlet state $\langle 0,1 | - | 1,0 \rangle /\sqrt{2}$, and we let Alice’s and Bob’s respective measurements be

$$M_{a|x} = (1 + (-1)^{a}\vec{a}_x \cdot \vec{\sigma})/2,$$

$$M_{b|y} = (1 + (-1)^{b}\vec{b}_y \cdot \vec{\sigma})/2,$$

where the outputs are labelled by $a, b = 0, 1$ and the inputs by $x, y = 0, 1$, and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices. With the state (1) and measurement directions (i.e. Bloch vectors) $\vec{a}$ for Alice and $\vec{b}$ for Bob we have the following correlations:

$$P(a, b|x, y) = \frac{1}{4} \left(1 + (-1)^{a\oplus b}\vec{a}_x \cdot \vec{b}_y \right),$$

where $\vec{b} = (b_1, -b_2, b_3)$ and $a \oplus b$ means addition of bits $a$ and $b$ modulo 2.

In the special case of orthogonal unit Bloch vectors

$$\vec{a}_0 = (1, 0, 0),$$

$$\vec{a}_1 = (0, 0, 1)$$

on Alice’s side and

$$\vec{b}_0 = (1, 0, 1)/\sqrt{2},$$

$$\vec{b}_1 = (1, 0, -1)/\sqrt{2}$$

on Bob’s side, we get the following statistics

$$P(a, b|x, y) = \frac{1}{4} \left(1 + (-1)^{a\oplus b}(-1)^{xy}\sqrt{2}/2 \right),$$

where $a, b, x, y$ are assumed to take values in $\{0, 1\}$. These correlations give rise to the symmetric detection efficiency threshold $\eta_{\text{sym}} = 2\sqrt{2} - 2 \simeq 0.8284$ [11, 12]. In order to reproduce this value, let us first consider the more general case where Alice detects her particle with efficiency $\eta_A$...
and Bob detects his particle with efficiency \( \eta_B \) for each of
their input settings. In the special symmetric case we have
\( \eta_{sym} = \eta_A = \eta_B \).

Let the CHSH inequality \([10]\) be written in the form \([25]\):

\[
CHSH = P(00\mid 00) + P(11\mid 00) + P(00\mid 01) + P(11\mid 01) + P(00\mid 10) + P(11\mid 10) + P(01\mid 11) + P(10\mid 11) \leq 3, \tag{8}
\]

where \( L = 3 \) is the local bound, which can be achieved with
deterministic strategies. Such a strategy is as follows. Alice outputs
\( a = 1 \) for every \( x = 0, 1 \) and Bob outputs \( b = 1 \) for
every \( y = 0, 1 \). That is we have the correlations

\[
P_L(a, b\mid x, y) = \delta_{a,1}\delta_{b,1}, \tag{9}
\]

where \( \delta_{i,j} \) is the Kronecker delta function:

\[
\delta_{i,j} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise}. \end{cases} \tag{10}
\]

Note that the above form \((9)\) is a less usual form of the
CHSH inequality, however it is equivalent to it up to relabelling
the measurement outcomes. By plugging \((7)\) into \((8)\) we get the
quantum value \( Q = (2 + \sqrt{2}) \sim 3.4142 \) for the
CHSH expression. This value gives the maximum violation of the
CHSH inequality \((8)\) as it has been shown by Tsirelson \([25]\). However, this value can only be attained in the ideal case when both Alice and Bob’s
detectors are perfect, that is, their detectors have unit efficiency
\( \eta_A = \eta_B = 1 \). Next we consider the case of finite efficiencies in particular focusing on two limiting cases, \( \eta_{sym} = \eta_A = \eta_B \) and \( \eta_{asym} = \eta_A, \eta_B = 1 \). In case of non-detection let Alice and Bob agree to output the value corresponding to the deterministic strategy above \((9)\) giving the local bound \(3\). Then we have to distinguish between four cases according to the detection and non-detection events of Alice and Bob detectors:

1. Both Alice’s and Bob’s detectors fire, which happens with probability \( \eta_A \eta_B \), in which case we have \( CHSH = Q = 2 + \sqrt{2} \).

2. Only Alice’s detectors fire, which happens with probability \( \eta_A (1 - \eta_B) \), in which case we have the correlations \( P(a, b\mid x, y) = (1/2) \delta_{b,1} \) resulting in \( CHSH = M_A = 2 \). In that case let Bob’s detector output \( b = 1 \) for every \( y = 0, 1 \).

3. Only Bob’s detectors fire. This happens with probability \( (1 - \eta_A) \eta_B \), and we have \( P(a, b\mid x, y) = \delta_{a,1}(1/2) \) resulting in \( CHSH = M_B = 2 \). In that case Alice’s detector outputs \( a = 1 \) for every \( x = 0, 1 \) in case of non-detection.

4. Neither detector fires which happens with probability \( (1 - \eta_A)(1 - \eta_B) \). In this case the statistics \((9)\) gives the local bound \( L = 3 \), that is \( CHSH = X = 3 \).

Then we have the resulting detection efficiency dependent Bell inequality:

\[
I(\eta_A, \eta_B) = \eta_A \eta_B Q + \eta_A (1 - \eta_B) M_A + (1 - \eta_A) \eta_B M_B + (1 - \eta_A)/(1 - \eta_B) X \leq L, \tag{11}
\]

where \( M_A (M_B) \) denotes the Bell value in the case if only Alice’s (Bob’s) detector fires, and \( X \) denotes the Bell value in the case of neither detector fires. Whenever this inequality is violated, it entails the violation of the original Bell inequality with \( \eta_A \) and \( \eta_B \) detection efficiencies. From \((11)\) and the assumption that \( X = L \) we get the threshold efficiency for the symmetric case:

\[
\eta_{sym} = \frac{2L - M_A - M_B}{Q - L - M_A - M_B}. \tag{12}
\]

On the other hand, the asymmetric case results in the threshold:

\[
\eta_{asym} = \frac{L - M_A}{Q - M_A}. \tag{13}
\]

Note that this latter inequality does not depend on \( X \). For the standard single-copy CHSH case, we have the parameters \( Q = 2 + \sqrt{2}, M_A = M_B = 2 \), and \( L = 3 \), which by substitution into \((12)\) give the values

\[
\eta_{sym} = 2(\sqrt{2} - 1) \simeq 0.8284, \\
\eta_{asym} = 1/\sqrt{2} \sim 0.7071, \tag{14}
\]

reproducing the well-known thresholds \([11][12]\).

Correlation-type Bell inequalities.— The CHSH inequality discussed above is a special type of correlation inequality. Indeed, if we define the two-party correlations

\[
E_{x,y} = P(00\mid xy) + P(11\mid xy) - P(01\mid xy) - P(10\mid xy), \tag{15}
\]

we can bring \((8)\) into this form, which can be written as

\[
CHSH = E_{0,0} + E_{0,1} + E_{1,0} + E_{1,1} \leq 2, \tag{16}
\]

where \(2\) is the local bound. Let us now consider generic correlation type Bell inequalities, in which case we can express the Bell inequality as

\[
I = \sum_{x=1}^{m} \sum_{y=1}^{m} M_{x,y} E_{x,y} \leq L, \tag{17}
\]

where \( M_{x,y} \) are the Bell coefficients, \( m \) is the number of settings per party and \( L \) is the local bound.

Let us show that we can slightly beat \( \eta_{sym} \) value in \((14)\) if \( m > 2 \) settings are available. For traceless observables on the two-qubit maximally entangled state we have \( M_A = M_B = 0 \) in Eq. \((11)\). Hence for a correlation-type Bell
inequality \[^{(17)}\] with two-qubit maximum quantum value \(Q\) and local maximum \(L\) we get the thresholds

\[
\eta_{sym} = \frac{2}{(Q/L) + 1}, \\
\eta_{asym} = \frac{L}{Q}
\] (18)

using formulas \[^{(12)}\][\(^{(13)}\)]. Based on the relation of the maximal two-qubit quantum violation of correlation Bell inequalities with the Grothendieck constant of order three, \(K_G(3)\), \[^{(27)}\][\(^{(28)}\)], we have

\[
1.4359 \leq \frac{Q}{L} \leq 1.4644,
\] (19)

where the upper bound is due to Ref. \[^{(29)}\], and the lower bound comes from Ref. \[^{(30)}\]. Based on the left-hand side value in \(^{(19)}\), we get \(\eta_{sym} \leq 0.8211\) and \(\eta_{asym} \leq 0.6964\). Note that \(\eta_{sym}\) is slightly lower than 0.8214 corresponding to the threshold of the \(A_5\) inequality using the singlet state. The \(A_5\) inequality is however not a correlation-type Bell inequality.

III. MULTIPLE COPIES OF THE CHSH EXPRESSION

In this section we consider detection efficiency thresholds for the iterated version of the CHSH inequality \[^{(31)}\][\(^{(32)}\)]. For two copies we have the expression

\[
\text{CHSH}_2(a_1, a_2, b_1, b_2, x_1, x_2, y_1, y_2) = \text{CHSH}(a_1, b_1, x_1, y_1)\text{CHSH}(a_2, b_2, x_2, y_2),
\] (20)

where \(\text{CHSH}\) is defined by the expression \[^{(8)}\]. This setup is depicted in Fig. 1. Similarly we define the \(n\)th iterated version by the following product:

\[
\text{CHSH}_n(a_1, \ldots, a_n, b_1, \ldots, b_n, x_1, \ldots, x_n, y_1, \ldots, y_n) = \prod_{i=1}^{n} \text{CHSH}(a_i, b_i, x_i, y_i),
\] (21)

where \(a_i\) (\(b_i\)) corresponds to Alice’s (Bob’s) output for the \(i\)th copy (\(j\)th copy). Also \(x_i\) (\(y_j\)) corresponds to Alice’s (Bob’s) input for the \(i\)th copy (\(j\)th copy). Therefore the corresponding \(\text{CHSH}_n\) inequality has \(m = 2^n\) inputs and \(o = 2^n\) outputs.

Quantum value.—Let us now look at the quantum value of the \(\text{CHSH}_n\) expression. We investigate the setup with multiple copies of the two-qubit maximally entangled state and measurements acting on these two-qubit states. In this case the probabilities factorize. For instance in the case of the double CHSH scenario above we get the joint correlations

\[
P(a_1, a_2, b_1, b_2|x_1, x_2, y_1, y_2) = P(a_1, b_1|x_1, y_1) \times P(a_2, b_2|x_2, y_2),
\] (22)

where the \(P(a_i, b_i|x_i, y_i)\) distribution is given by \(^{(7)}\). For \(n\) copies we have the quantum correlation:

\[
P(a_1, \ldots, a_n, b_1, \ldots, b_n|x_1, \ldots, x_n, y_1, \ldots, y_n) = P(a_1, b_1|x_1, y_1) \ldots P(a_n, b_n|x_n, y_n) = \prod_{i=1}^{n} P(a_i, b_i|x_i, y_i).
\] (23)

This scenario has \(m = 2^n\) inputs and \(o = 2^n\) outputs.

Let us now calculate the quantum value \(Q^{(n)}\) of the \(n\)th iterated CHSH expression \(\text{CHSH}_n\). Let us denote the local bound by \(L^{(n)}\). First we look at the double CHSH inequality \((n = 2)\). \(\text{CHSH}_2\) is a product of two CHSH expressions and since the probabilities factorize (see Eq. \(^{(22)}\)), we have

\[
Q^{(2)} = Q(\text{CHSH}) \times Q(\text{CHSH}) = Q^2 = (2 + \sqrt{2})^2.
\] (24)

So the quantum value simply doubles. Similarly, for \(n\) copies, we have \(Q^{(n)} = (2 + \sqrt{2})^n\). It is also known \[^{(32)}\] that this value is the maximum Tsirelson bound for the \(\text{CHSH}_n\) inequality, that is, this bound holds true even in the presence of arbitrary quantum resources (and in particular if the probabilities do not factorize such as in Eq. \(^{(22)}\)).

Local bound.—When we calculate the local bound, however, the probabilities may not factorize with respect to each
copy as in Eq. (23). For \( n = 2 \) they are as follows:

\[
P_L(a_1, a_2, b_1, b_2|x_1, x_2, y_1, y_2) = \sum_\lambda P_A(a_1, a_2|x_1, x_2, \lambda)P_B(b_1, b_2|y_1, y_2, \lambda)q(\lambda),
\]

where \( \lambda \) is a shared random variable with \( \sum_\lambda q(\lambda) = 1 \), and \( P_A \) and \( P_B \) are arbitrary conditional probability functions (labeled by \( \lambda \)) on Alice’s and Bob’s side. Hence in general it is allowed to exploit (classical) strategies in between the different copies. Say, on Alice’s side, \( a_1 \) may not only depend on \( x_1 \) but also on \( x_2 \). Indeed, it turns out that the local value \( L^{(2)} \) for the Bell expression \( \text{CHSH}_2 \) is \( L^{(2)} = 10 \), which is greater than \( L^2 = 3^2 \). To attain this value, the parties can use the following local deterministic strategies \( P_A(a_1, a_2|x_1, x_2) \) and \( P_B(b_1, b_2|y_1, y_2) \) in Eq. (25). Note in this case the local strategies \( P_A \) and \( P_B \) do not depend on \( \lambda \) and their actual forms are as follows

\[
P_A(1|00) = P_A(1|01) = P_A(1|10) = P_A(1|11) = 1,
\]

\[
P_B(1|00) = P_B(1|01) = P_B(1|10) = P_B(1|11) = 1
\]

for Alice and Bob’s respective functions. The value of \( L^{(2)} = 10 \) attainable with the above strategy has been obtained by Barrett et al. [31] and Aaronson independently. Similarly, for the three-copy case \( \text{CHSH}_3 \) we get \( L^{(3)} = 31 \). Note this value is higher than \( L(\text{CHSH}) \times L(\text{CHSH}_2) = 30 \). The value of \( L^{(3)} = 31 \) is due to S. Aaronson and B. Toner as noted in Cleve et al. [32]. For \( n > 3 \) empirical values are available: \( L^{(4)} = 100 \), \( L^{(5)} = 310 \), and \( L^{(6)} = 1000 \), which have been recently found in Ref. [33]. Also, the following analytical upper bound holds for \( L^{(n)} \):

\[
L^{(n)} \leq (1 + \sqrt{5})^n,
\]

which asymptotically becomes an equality for large \( n \). The upper bound [27] is due to A. Ambainis building on Ref. [34].

Detection efficiencies.—We now upperbound the \( \eta_{\text{sym}} \) and \( \eta_{\text{asym}} \) thresholds in the function of \( n \) copies for the CHSH_\( n \) expression. In particular, let us denote by \( \eta_{\text{sym}}^{(n)} \) and \( \eta_{\text{asym}}^{(n)} \) the detection efficiency thresholds, which can be obtained by \( n \)-copies of the singlet state and anticommuting Pauli measurements. The \( n \)-copy distribution [33] corresponds to this scenario. Note that in this case we do not fix the Bell inequality in advance. Hence, any upper bound on the detection efficiency thresholds we obtain in this section for the special CHSH_\( n \) expression will hold true for the general case as well.

The derivation of the detection-efficiency dependent inequalities follows the standard procedure based on grouping of the non-detection event with one of the outcomes for each setting. This approach has been discussed in Sec. II on the example of the CHSH inequality. Similarly to that case we associate the non-detection outcome with the particular outcome for which the local deterministic strategy outputs 1 and gives the maximum local value of the Bell inequality. Let us first discuss the two-copy case CHSH_2 described by [29], which we generalize later for higher \( n \). We now compute the relevant quantities for this scenario.

The quantum value \( Q^{(2)} = Q^2 = (2 + \sqrt{2})^2 \) is due to Eq. (24). We show that \( M_A^{(2)} = 2^2 \) and \( M_B^{(2)} = 2^2 \), the cases when only Alice’s and Bob’s detectors fires, respectively.

**Proof.**—If only Alice’s detector fires, we have

\[
P(a_1, a_2, b_1, b_2|x_1, x_2, y_1, y_2) = (1/4)P_B(b_1, b_2|y_1, y_2),
\]

that is, the probability distribution does not depend on Alice’s outputs \( a_1 \) and \( a_2 \). Using this probability distribution we get

\[
M_A^{(2)} = \sum_{b_1, b_2} P_B(b_1, b_2|y_1, y_2) = 4
\]

for CHSH_2 in [29]. This value of 4 traces back to two features of the double CHSH expression: (i) all nonzero Bell coefficients are 1, and (ii) for each input \( (x_1, x_2) \) of Alice, input \( (y_1, y_2) \) of Bob and for each output \( (a_1, a_2) \) of Alice there occurs a single nonzero coefficient for Bob’s output. That is, the Bell inequality corresponds to a so-called unique game [35]. Note, this value does not depend on the actual deterministic strategy to be used in the case of the non-detection event. Similarly, we get \( M_B^{(2)} = 2^2 \) in the case when Bob’s detector fires.

For general \( n \) relying on the two features (i) and (ii) above we get \( M_A^{(n)} = M_B^{(n)} = 2^n \).

By plugging the above numbers into (12) and (13) we get upper bounds on the required detection efficiencies \( \eta_{\text{sym}}^{(n)} \) and \( \eta_{\text{asym}}^{(n)} \) to see Bell nonlocality. In particular, we have \( Q^{(n)} = (2 + \sqrt{2})^n \), \( M_A^{(n)} = M_B^{(n)} = 2^n \) and for \( L^{(n)} \) the upper bound value in Eq. (27) was used. Then we get the following upper bounds for \( n \geq 1 \)

\[
\eta_{\text{sym}}^{(n)} \leq \frac{2L^{(n)} - M_A^{(n)} - M_B^{(n)}}{Q^{(n)} + L^{(n)} - M_A^{(n)} - M_B^{(n)}} \leq \frac{2(1 + \sqrt{5})^n}{2n - 2(\sqrt{2} + 1)^n},
\]

and

\[
\eta_{\text{asym}}^{(n)} \leq \frac{L^{(n)} - M_A^{(n)}}{Q^{(n)} - M_A^{(n)}} \leq \frac{2^n - (1 + \sqrt{5})^n}{2n - \sqrt{2}^n}.
\]

Let us show that in arriving these bounds it is valid to use an upper bound for \( L^{(n)} \). Indeed, we have \( Q^{(n)} > L^{(n)} \), \( L^{(n)} > M_A^{(n)} \) and \( L^{(n)} > M_B^{(n)} \). Then both [12] and
TABLE I. Table for the detection efficiency thresholds of the CHSH$_n$ expression. (First column) Number of copies $n$. (Second column) The dimension $d$ of the bipartite $(d \times d)$-dimensional system. (Third column) Upper bound on the symmetric detection efficiency threshold $\eta_{\text{sym}}^{(n)}$. (Fourth column) Upper bound on the asymmetric detection efficiency threshold $\eta_{\text{asym}}^{(n)}$.

| $n$ | $d$ | $\eta_{\text{sym}}^{(n)} \leq$ | $\eta_{\text{asym}}^{(n)} \leq$ |
|-----|-----|-------------------------------|-------------------------------|
| 1   | 2   | 0.8284                        | 0.7071                        |
| 2   | 4   | 0.8787                        | 0.7836                        |
| 3   | 8   | 0.8994                        | 0.7233                        |
| 4   | 16  | 0.8772(0.8240)                | 0.7813(0.7097)                |
| 5   | 32  | 0.8555(0.7832)                | 0.7475(0.6436)                |
| 6   | 64  | 0.8328(0.7622)                | 0.7135(0.6158)                |
| 7   | 128 | 0.8093                        | 0.6796                        |
| 8   | 256 | 0.7853                        | 0.6464                        |
| 9   | 512 | 0.7610                        | 0.6142                        |
| 10  | 1024| 0.7367                        | 0.5832                        |
| 11  | 2048| 0.7125                        | 0.5534                        |
| 12  | 4096| 0.7367                        | 0.6142                        |
| 13  | 8192| 0.6647                        | 0.4978                        |
| 20  | 2^{20}| 0.5101                      | 0.3424                        |
| 50  | 2^{50}| 0.1284                      | 0.0686                        |
| 100 | 2^{100}| 0.0094                     | 0.0047                        |

In order to take into account finite detection efficiencies in this method, we modify the probability distribution $P(ab|xy)$ instead of the Bell coefficients. To take care of inconclusive events due to the finite detection efficiencies, Alice simply chooses for every input $x = (x_1, x_2, \ldots, x_n)$ the last output $a = (a_1, a_2, \ldots, a_n) = (1, 1, \ldots, 1)$ in case of non-detection. Similarly, for every input $y = (y_1, y_2, \ldots, y_n)$ Bob outputs $b = (b_1, b_2, \ldots, b_n) = (1, 1, \ldots, 1)$ in case of non-detection. As a result the probability values from the last outcome for Alice and Bob are modified according to

$$
P_{\eta_A \eta_B}(ab|xy) = \eta_A \eta_B P(ab|xy),$$

$$P_{\eta_A}(a|x) = \eta_A P_{\eta_A}(a|x),$$

$$P_{\eta_B}(b|y) = \eta_B P_{\eta_B}(b|y)$$

for every $(x, y)$ and $(a, b)$ except for outputs $a = (a_1, a_2, \ldots, a_n) = (1, 1, \ldots, 1)$ and $b = (b_1, b_2, \ldots, b_n) = (1, 1, \ldots, 1)$. Above $P_{\eta_A}$ and $P_{\eta_B}$ are the marginal distri-
tions for Alice and Bob defined as follows
\[
P^A(a|x) = \sum_b P(ab|x) \quad \text{for all } y,
\]
\[
P^B(b|y) = \sum_a P(ab|y) \quad \text{for all } x. \tag{33}
\]

Note that the set of probabilities (32) completely determines the probability distribution, as the missing probabilities corresponding to the last 2^n-th outcome of Alice and Bob are completely determined by (32) due to the non-signalling conditions of the probabilities (33). Hence from the set of distributions (32) we can build up the full set of probabilities:
\[
\tilde{P}_{\eta_A,\eta_B} \equiv \{P_{\eta_A,\eta_B}(ab|xy)\}_{a,b,x,y}. \tag{34}
\]

Given the above set (34), our task is to decide if this probability distribution can be described by a local model or not. For a pair of fixed (\(\eta_A, \eta_B\)) detection efficiencies, this is a feasibility problem, which in turn casts as a linear programming (LP) task.

The local set \(\mathcal{L}\) for a finite number of inputs \(m\) and outputs \(o\) is a polytope, the so-called Bell polytope, which is the convex hull of a finite number of points defined by its vertices. The vertices are given by the local deterministic strategies \(P_D(ab|xy) = D_A(a|x)D_B(b|y)\), where \(D_A(a|x) \) and \(D_B(b|y)\) are deterministic response functions of Alice and Bob, respectively. Alice and Bob each has \(o^m\) such functions, so there are altogether \(o^{2m}\) deterministic strategies,
\[
\tilde{P}_D^{(\lambda)} := \{P_D^{(\lambda)}(ab|xy)\}_{a,b,x,y}, \tag{35}
\]
where \(\lambda = (1, \ldots, o^2m)\). Each strategy \(\lambda\) translates to a vertex of the \(o^2m^2\)-dimensional Bell polytope \(\mathcal{L}\). Any point inside this polytope is a convex combination of vertices \(\tilde{P}_D^{(\lambda)}\) with some positive weights \(q(\lambda)\).

For the special case of \(n\) copies of the probability distribution (34), we have the Bell scenario \(m = o = 2^n\). Let us denote in this case the corresponding polytope by \(\mathcal{L}^{(n)}\). In particular, if the probability point \(\tilde{P}_{\eta_A,\eta_B}\) lies outside the Bell polytope, then it cannot be written as a convex combination of the vertices of the Bell polytope. In this case we can find the hyperplane which separates the polytope \(\mathcal{L}^{(n)}\) from the point \(\tilde{P}_{\eta_A,\eta_B}\). This plane is identified as the Bell expression \(C\) underlying the probability point (34):
\[
\sum_{a,b,x,y} C_{a,b,x,y}P_L(a,b|x,y) \leq 0, \tag{36}
\]
where \(C_{a,b,x,y}\) are the Bell coefficients and \(\tilde{P}_L = \{P_L(ab|xy)\}\) is any local distribution satisfying the locality conditions (25). In geometric terms, \(\tilde{P}_L\) can be any point situated inside the Bell polytope \(\mathcal{L}^{(n)}\).

In order to obtain the Bell expression \(C\) below the point (34), we choose fixed parameters \(\eta_A\) and \(\eta_B\), and we solve an LP task as follows:
\[
Q \equiv \max_{C_{a,b,x,y}} \sum_{a,b,x,y} C_{a,b,x,y}P_{\eta_A,\eta_B}(a,b|x,y) \quad \text{s.t. } \sum_{a,b,x,y} C_{a,b,x,y}P_D^{(\lambda)}(a,b|x,y) \leq 0 \quad \text{for all } \lambda,
\]
\[
C_{a,b,x,y} \leq 1 \quad \text{for all } a, b, x, y, \tag{37}
\]
where index \(\lambda\) runs over all local deterministic strategies \(\tilde{P}_D^{(\lambda)}\) in Eq. (35) and the conditions in the last line take care of bounding the coefficients \(C_{a,b,x,y}\) from above. These coefficients are our optimization variables. As we mentioned, there are \((o^m)(o^m)\) different local deterministic strategies \(\tilde{P}_D^{(\lambda)}\). Hence in the \(n\)-copy case, we have \(o = m = 2^n\), which amounts to \(2^{2^{2^n}}\) strategies. In our special case of \(n = 2\), there are \(2^{2^2}\) such strategies. This number of vectors have to be provided as an input to the LP, which complexity is feasible in a standard desktop computer.

We note that in the implementation of the algorithm instead of solving the LP in (37), we solved a task where the full set of probabilities (34) (having dimension \(16 \times 16\)) are replaced by the smaller set (32), which has dimension \(13 \times 13\). The corresponding objective function in the optimization (37) is given by
\[
\sum_{a,x} C_A^{a|x}P^A(a|x) + \sum_{b,y} C_B^{b|y}P^B(b|y) + \sum_{a,b,x,y} C_{a,b,x,y}P_{\eta_A,\eta_B}(a,b|x,y), \tag{38}
\]
where the sum for outputs \(a\) and \(b\) runs over the first three outputs, that is, \((a_1, a_2) = (0, 0), (0, 1), (1, 0)\) and \((b_1, b_2) = (0, 0), (0, 1), (1, 0)\). However, the sum for inputs \(x\) and \(y\) runs over all inputs, that is, \((x_1, x_2) = (0, 0), (0, 1), (1, 0), (1, 1)\) for Alice, and similarly for Bob. We used Mosek [36] to perform this LP task, which returned solution to this LP problem within a few seconds.

If the solution of the linear program (37) above gives \(Q > 0\), it indicates that the point (34) with given \((\eta_A, \eta_B)\) values lies outside the polytope. In the symmetric case we specify \(\eta_{sym} = \eta_A = \eta_B\), and our aim is to choose the smallest such \(\eta_{sym}\) such that \(Q > 0\). We do the same for the asymmetric case, \(\eta_{asym} = \eta_A \neq \eta_B = 1\). In the limit of the smallest such \(\eta_{sym}\) and \(\eta_{asym}\), the point (34) lies on the boundary of the local set. In the actual computations, we choose \(\eta_{sym}\) and \(\eta_{asym}\) such that \(Q\) is slightly larger than zero. We next give detailed results for \(n = 2\) copies running the above LP problem (37) for both the symmetric and asymmetric detection efficiencies.

**Symmetric detection efficiencies for two copies.**—Let us focus on the symmetric case for \(n = 2\). Here we are left
with a single parameter, \( \eta_{sym} = \eta_A = \eta_B \). Given the probability distribution \( \tilde{P}_{a|x,y} \) in (37) our task is to find \( \eta_{sym} \) such that the solution \( Q \) is some small number \( \epsilon \) (we in particular set \( \epsilon = 0.001 \)). As a solution of the LP (37), we get the following form of the Bell inequality \( I_{sym} \leq 0 \),

\[
I_{sym} = \sum_{a=1}^{3} \sum_{b=1}^{4} C_{a|x} A_p A(x) + \sum_{b=1}^{4} C_{b|y} P_B(b) + \sum_{a=1}^{3} \sum_{b=1}^{4} \sum_{x,y=1}^{4} C_{a|b, x, y} P(a|x, y),
\]

(39)

where Alice’s marginal coefficients are \( C_{11}^{A} = C_{11}^{A} = C_{13}^{A} = C_{14}^{A} = C_{21}^{A} = C_{22}^{A} = C_{23}^{A} = C_{24}^{A} = -2, \) \( C_{31}^{A} = C_{32}^{A} = C_{33}^{A} = C_{34}^{A} = -1 \) and all other entries for \( C^{A} \) are zero. Also, Bob has \( C^{B} = C^{A} \). On the other hand, the \( 13 \times 13 \) matrix \( C \) is as follows:

(40)

where an element \( C_{a,b,x,y} \) has been written above as the element \((a,b)\) of the \( 3 \times 3 \) submatrix at coordinate \((x,y)\).

Note that a possible multiplicative constant will not change the Bell inequality. In fact, we doubled the Bell coefficients \( C, C^{A} \) and \( C^{B} \) coming from the solution of the LP task to get integer values. Also notice the symmetry of the matrix \( C \) with respect to transposition.

From this inequality, we can calculate analytically the critical value of \( \eta_{sym} \) in the two-copy case, which we denote by \( \eta_{sym}^{(2)} \). To this end, we apply formula (12) for the calculation of \( \eta_{sym} \), given the quantum bound, the \( L \) local bound, and \( M_A(M_B) \) value if only Alice’s (Bob’s) detectors fire and the \( X \) value (if neither detectors fire), respectively.

We get \( Q = 4(\sqrt{2} - 1) \) by substituting (40) into (39), where the probabilities are given by the tensor product (22). The local bound \( L \) on the other hand is \( L = 0 \). This value can be achieved by a deterministic strategy where for every \( x \) and \( y \) the fourth output \((a = 1,1)\) and \( b = (1,1) \) is given deterministically. Hence \( X = L = 0 \) and the corresponding distribution is \( P_L(a|x,y) = \delta_{a,4}\delta_{b,4} \) for every input \((x,y)\). Indeed, we have chosen the last outcome for the non-detection event to obtain (32). In case of Alice’s non-detection result, we get \( P(ab|x,y) = (1/4)\delta_{b,4} \),

which entails \( M_A = -14/4 \) for the Bell expression in (39). Similarly, in case of Bob’s non-detection result, we get \( P(ab|x,y) = \delta_{a,4}(1/4) \), which entails \( M_B = -14/4 \) for the Bell expression in (39). Putting these values together, we have

\[
\eta_{sym}^{(2)} = \frac{2L - M_A - M_B}{Q + L - M_A - M_B} = \frac{28\sqrt{2} - 21}{23} \approx 0.8086.
\]

(41)

We would like to stress that this value is an exact upper bound for \( \eta_{sym}^{(2)} \), in the case of Pauli measurements performed on two copies of the singlet state. This value is shown in Fig. 2 with blue cross for \( n = 2 \).

Asymmetric detection efficiencies for two copies.—Let us now consider the asymmetric case for \( n = 2 \), in which case we have \( \eta_{sym}^{(2)} = \eta_A \) and \( \eta_B = 1 \). We solve the LP (37) for \( \eta_{sym}^{(2)} \) such that the solution \( Q \) is a small number \( \epsilon \), where we set \( \epsilon = 0.001 \) in the actual computations. As a result, we get the Bell inequality \( I_{asym} \leq 0 \) defined similarly to (39), where

\[
I_{asym} = \sum_{a=1}^{3} \sum_{x=1}^{4} C_{a|x} A_p A(x) + \sum_{b=1}^{4} C_{b|y} P_B(b) + \sum_{a,b,x,y} C_{a|b, x, y} P(a|x, y),
\]

(42)

Multiplying all Bell coefficients by three to get integer values we find

\[
C_{11}^{B} = C_{12}^{B} = C_{21}^{B} = C_{4|1}^{B} = C_{4|3}^{B} = -1,
\]

\[
C_{2|1}^{B} = C_{3|1}^{B} = -2
\]

(43)

and

\[
C_{11}^{A} = C_{21}^{A} = C_{3|1}^{A} = -3,
\]

\[
C_{2|2}^{A} = C_{4|3}^{A} = C_{A|3}^{A} = -3,
\]

\[
C_{1|4}^{A} = C_{2|4}^{A} = C_{3|4}^{A} = -2
\]

(44)

and all other coefficients appearing in \( C^{A} \) and \( C^{B} \) are zero. In addition, we have the matrix \( C \):

\[
\begin{pmatrix}
1 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

(45)
where an element $C_{abxy}$ is written as the element $(a, b)$ of the $3 \times 3$ submatrix at coordinate $(x, y)$. Using inequality (42), we can give the analytical value for the critical value of $\eta_{\text{asym}}^{(2)}$ in the two-copy case, which we denote by $\eta_{\text{asym}}^{(2)}$. We apply formula (13) for the calculation of $\eta_{\text{asym}}^{(2)}$ given the parameters $Q$ quantum bound, $L$ local bound, and $M_A$ which latter value stands for the quantum value if only Alice’s detectors fire. To this end, we substitute $\eta_{\text{asym}}^{(2)}$ into (42), where the probabilities are given by the tensor product (22), and we get:

$$Q = (9/2)(1 + \sqrt{2}) - 9. \quad (46)$$

The local bound for (45) is $L = 0$, which can be achieved by a deterministic strategy where for every input $(x, y)$ the output is given by the fourth outcome (i.e., $a = b = (1, 1)$). That is, we have the local distribution $P_3(ab|xy) = \delta_a \delta_b \delta_{a,b}$ for every $(x, y)$. According to (32), we have chosen this particular outcome for the non-detection event, and then we get $P(ab|xy) = (1/4) \delta_a \delta_b$ if Alice’s detector fires, which results in the value of $M_A = -9/4$. Putting these together, we arrive at

$$\eta_{\text{asym}}^{(2)} = \frac{L - M_A}{Q - M_A} = (1 + 2\sqrt{2})/7 \approx 0.5469. \quad (47)$$

We can compare this value with the lowest known critical value $\eta_{\text{asym}} = 0.6520$ among four-setting two-outcome Bell inequalities [37]. This value in particular corresponds to the $A_{44}$ inequality from the list of Bell inequalities in Ref. [14].

V. GEOMETRIC APPROACH BASED ON GILBERT ALGORITHM

Unfortunately, the LP (37) used in the preceding section for $n > 2$ is not feasible on a standard desktop computer. This is mainly due to the very large number of vectors corresponding to the different deterministic strategies which we have to provide as an input into the LP problem. Note that for $n = 3$ we already have $2^{48}$ different strategies, where each strategy translates to a vector with 4096 entries.

However, for the $n \geq 3$ case we can use an iterative algorithm, the so-called Gilbert algorithm [38] to obtain bounds on $\eta_{\text{sym}}^{(n)}$ and $\eta_{\text{asym}}^{(n)}$. This algorithm avoids the issue of inputting all the deterministic strategies in the LP, and it provides us the underlying Bell inequality, too. For $n \leq 3$ our method gives correct upper bounds, whereas in the case of $n = 4$ the calculated bound partly relies on heuristic numerical computations. However, we are still confident about the validity of the obtained bounds in this case as well.

The values $\eta_{\text{sym}}^{(n)}$ and $\eta_{\text{asym}}^{(n)}$ we will obtain in this section for $n = 3$ and $n = 4$ are considerably lower than the thresholds corresponding to the case $n = 2$ in section IV and also much lower than the ones coming from the iterated CHSH, inequalities for $n = 3$. We conjecture that the obtained values for $n = 3$ and $n = 4$ are close to the ones which could have been obtained by linear programming (provided the computations were feasible). Here we will also give the Bell matrices $C$ obtained by Gilbert’s distance method, which we will provide as auxiliary data files due to the large size of them.

Let us first briefly describe Gilbert’s distance algorithm [38] which is a popular numerical method for collision detection problems (i.e., it detects collisions between rigid convex bodies). In particular, this algorithm estimates the distance between a point $\vec{P}$ and an arbitrary convex set $S$ in some finite-dimensional Euclidean space $\mathbb{R}^d$ via calls to an oracle which performs linear optimizations over the set $\mathcal{S}$. The running time and the convergence properties of this algorithm are very favourable. These properties have been analysed in detail in Ref. [39] along with several applications in quantum information. Similar method has been used recently in Ref. [40] to discriminate nonlocal correlations and further applications in entanglement detection have been appeared in Refs. [41, 42].

In our particular case the point $\vec{P}_{\text{sym}}^{(n)}$ is defined by the probability distribution (34) for given $\eta_A$ and $\eta_B$ values. Let us first focus on the symmetric case, $\eta = \eta_A = \eta_B$, in which case we get a one-parameter family of points $\vec{P}(\eta)$. We fix $\eta$ such a way that presumably $\vec{P}(\eta)$ lies outside the local Bell polytope $L^{(n)}$ (we can take for instance $\eta$ as the best upper bound so far for $\eta_{\text{sym}}^{(n)}$). The vertices of the Bell polytope are defined by the deterministic vectors $\vec{P}_D^{(n)}$. For $n$ copies, we have $m = o = 2^n$, and there are $N = o^2 = 2^{2n+1}$ corners of this polytope in dimension $D = (om)^2 = 2^{4n}$.

We run Gilbert’s distance algorithm, where the inputs to the problem are the target point $\vec{P}(\eta)$ for fixed $\eta$ and the vertex description $\vec{P}_D^{(n)}$ of the Bell local polytope $L^{(n)}$. However, it is important to keep in mind that this algorithm does not require to store all these data in the computer memory unlike the linear programming algorithm discussed in Sec IV. This is a big advantage of the Gilbert method over the LP-based method, as we have already seen that in the case of $n = 3$ copies the number of vertices $N = 2^{2n+1} = 2^{48}$ is too large to be stored in the computer memory.

Gilbert algorithm outputs (an estimate to) the distance between the point $\vec{P}(\eta)$ and the polytope $L^{(n)}$ by providing a separating hyperplane with normal vector $\vec{C}$ between $\vec{P}(\eta)$ and the polytope $L^{(n)}$. This hyperplane is identified with the matrix $C$ of Bell coefficients we are looking for. The algorithm in our particular case is defined as follows [39].

**Inputs:** the vector $\vec{P}(\eta)$ specified by the number of copies $n$
and $\eta$, and the description of the polytope $\mathcal{L}^{(n)}$. The steps are as follows.

1. Set $k = 0$ and an $\epsilon$ value (typically small), and pick an arbitrary point $\vec{P}_k$ inside the polytope $\mathcal{L}^{(n)}$.

2. Given the point $\vec{P}_k$ and the target point $\vec{P}(\eta)$, run an oracle which maximizes the overlap $(\vec{P}(\eta) - \vec{P}_k) \cdot \vec{P}_D^{(k)}$ over all vertices $\vec{P}_D^{(k)} \in \mathcal{L}$, $\lambda = 1, \ldots, N$, where $N = 2^{2n+1}$. Let us denote the index of the local deterministic point returned by the oracle by $k'$, and the corresponding point by $\vec{P}_D^{(k')}$.

3. Find the point $\vec{P}_{k+1}$ as a convex combination of $\vec{P}_k$ and $\vec{P}_D^{(k')}$ that minimizes the distance $\|\vec{P}(\eta) - \vec{P}_{k+1}\|$. 

4. Let $k = k + 1$ and go to Step 2 until the distance $\|\vec{P}(\eta) - \vec{P}_{k+1}\| \leq \epsilon$.

Output: $\vec{C} \equiv \vec{P}(\eta) - \vec{P}_k$.

The Bell matrix $\vec{C}$ is then identified with the returned solution vector $\vec{C}$. Let us add remarks on possible modifications of the above algorithm.

In step 2 we have to maximize the overlap $(\vec{P}(\eta) - \vec{P}_k) \cdot \vec{P}_D^{(k)}$ over $N = 2^{2n+1}$ deterministic vectors. This number is exponential in the number of measurement settings $m = 2^n$. In fact this maximization task is an NP-hard problem and it seems unlikely to find an efficient solution to it in the generic case. Therefore instead of the exact enumeration method we resort to a heuristic search. The description of this method can be found in Refs. [29, 39] for the specific case of two outcomes and in Refs. [33, 34] for more than two outcomes.

The returned vector $\vec{C}$ has entries $C_{a,b,x,y}$, where $a, b, x, y = (1, \ldots, 2^n)$. This $\vec{C}$ corresponds to a separating hyperplane, which separates the point $\vec{P}(\eta)$ from the Bell polytope $\mathcal{L}^{(n)}$. From $\vec{C}$ we can produce the matrix $\mathcal{C}$, where an element $C_{a,b,x,y}$ is written as the element $(a, b)$ of the $2^n \times 2^n$ submatrix at coordinate $(x, y)$. However, note that the oracle in step 2 has a heuristic nature. Therefore we also run a brute force computation enumerating all $2^{2n+1}$ strategies to check that the local bound for the Bell expression $\mathcal{C}$ is given correctly. We carried out this check for $n = 2$ and $n = 3$, but the case of $n = 4$ is computationally hard to tackle, hence in this latter case our result is partly based on heuristic computation.

We can add to step 3 a modification introduced in Ref. [39]. In this case, in finding the point $\vec{P}_{k+1}$ we not only keep $\vec{P}_k$ but the previous $m$ points $\vec{P}_{k-1}, \vec{P}_{k-2}, \ldots, \vec{P}_{k-m}$ as well and find a convex combination of all of them to minimize the distance to $\vec{P}(\eta)$. This optimization can be carried out efficiently for not too large $m$ by solving a linear least squares problem. We set the value $m$ in the range $m = 20, \ldots, 100$ in our actual computations.

In addition, we may build a symmetrization procedure in step 3. Here we use the fact that the distribution $P(ab|xy)$ in formula (23) is invariant under the simultaneous permutation of Alice and Bob devices. Hence in the case of $n = 2$ we can switch simultaneously Alice and Bob devices without changing the distribution $P(ab|xy)$. For $n$ copies we have $n!$ such permutations of the devices. We impose this symmetry on the Bell functional $\mathcal{C}$ as well. Namely, for $n = 2$ let us have $C_{a,b,x,y} = C_{b,a,x,y}$ for all $a, b, x, y$, where we define

\[ C_{a,b,x,y} \equiv C_{a_1,a_2,b_1,b_2,x_1,x_2,y_1,y_2}, \]

\[ C_{b,a,x,y} \equiv C_{a_2,a_1,b_1,b_2,x_1,x_2,y_1,y_2}. \] 

(48)

Then from $\vec{P}_D^{(k')}$ in step 3 of the above algorithm we form the symmetrized vector

\[ \frac{P_D^{(k')} + \vec{P}_D^{(k')}}{2}, \] 

(49)

where the components of $\vec{P}_D^{(k')}$ are given by

\[ \vec{P}_D^{(k')}(a, b|x, y) \equiv P_D^{(k')}(a_2, a_1, b_2, b_1|x_2, x_1, y_2, y_1). \] 

(50)

Note that the symmetrized vector (49) in step 2 gives the same overlap with $(\vec{P}(\eta) - \vec{P}_k)$ as $\vec{P}_D^{(k')}$ does. On the other hand, it provides us a Bell matrix $\mathcal{C}$ at the end of the procedure, which possesses the required symmetry $\mathcal{C} = \vec{C}$. For $n > 2$, the symmetrization is performed similarly to the above procedure. In the general case of $n$ copies, there are $n!$ different possible permutations of the devices all of which have to be taken into account in the symmetrization task.

Detection efficiencies using Gilbert method for multiple copies.— Below we give our computational results on the upper bounds for $\eta^{(n)}_{\text{sym}}$ and $\eta^{(n)}_{\text{asym}}$ using Gilbert method discussed above. This includes the $\mathcal{C}$ matrices obtained for $n = 2, 3$ and 4 copies both for the symmetric and asymmetric detection efficiencies.

We used MATLAB for all the calculations in this paper. The routines test_sym_n.m and test_asym_n.m test certain properties of the Bell matrices $\mathcal{C}$. The Bell matrices in the different scenarios are named by $\text{Csym}_n.m$ and $\text{Casymp}_n.m$, where $n$ stands for the number of copies $n = 2, 3, 4$ and $\text{sym}$/asym denotes the case of symmetric/asymmetric detection efficiency. We provide these MATLAB routines and data files as an ancillary file.

The routines test_sym_n.m and test_asym_n.m define the $n$ copies of the quantum state $\lvert 1 \rangle$ and the measurement operators $\lvert 2 \rangle$, which build up the $n$-copy statistics. From this, the routines compute the following quantities appearing in the formulas (12, 13).
\( Q \): the quantum value evaluated for the Bell expression \( C \).

\( M_A \) (\( M_B \)): the quantum value evaluated for the Bell expression \( C \) where only Alice’s (Bob’s) detectors fire.

\( L \): local bound for \( C \). For \( n = 2, 3 \) this value is exact, whereas in the case of \( n = 4 \) this defines a lower bound, however, we conjecture that it gives the exact value.

\( X \): the value corresponding to the special deterministic strategies on Alice’s and Bob’s side (i.e. when neither detectors fire), where the last outcome is outputted in the case of non-detection event.

The above listed values give \( \eta_{\text{asym}}^{(n)} \) according to the formula (13). On the other hand, we get \( \eta_{\text{asym}}^{(n)} \) solving the quadratic equation

\[
\eta^2 Q + \eta (1 - \eta) (M_A + M_B) (1 - \eta)^2 X = L, \tag{51}
\]

for \( \eta \).

| \( n \) | \( n = 2 \) | \( n = 3 \) | \( n = 4 \) |
|------|---------|---------|---------|
| \( Q \)  | 7411.71 | 2562.88 | 88170.96 |
| \( M_A \) | 1097.25 | 9024.25 | 35297   |
| \( M_B \) | 1097.25 | 9024.25 | 35297   |
| \( X \)   | 5579   | 18949   | 60869   |
| \( L \)   | 5580   | 18979   | 65996   |
| \( \eta_{\text{asym}}^{(n)} \) | \( \leq 0.8091 \) | \( \leq 0.7399 \) | \( \leq 0.6929 \) |

TABLE II. Table for calculating upper bounds on the symmetric detection efficiency thresholds \( \eta_{\text{asym}}^{(n)} \) for \( n = 2, 3 \) and 4 copies. The corresponding Bell expression \( C \) has \( m = 2^n \) inputs and \( o = 2^n \) outputs. Let us note that \( C \) is invariant under party exchange, hence \( M_A \) and \( M_B \) are the same. The values for \( X \) and \( L \) are integers since the Bell coefficients are rounded to integers. The value \( L \) for \( n = 4 \) is a lower bound, however it is conjectured to be exact. Hence the upper bound to \( \eta_{\text{asym}}^{(n)} \) is a conjectured value as well.

We obtain Table II for the case of symmetric detection efficiency. Let us remark that for \( n = 2 \) the result \( \eta_{\text{asym}}^{(2)} \leq 0.8091 \) is consistent with the exact value \( \eta_{\text{asym}}^{(2)} = 0.8086 \) obtained with the LP-based algorithm in Sec. IV. On the other hand, in the case of \( n = 4 \) we had to resort to a heuristic numerical search to obtain \( L \), hence the obtained value is only a lower bound to \( L \). We still have good confidence about the value due to the efficient numerical procedure used (see [33] [44]). Let us also mention that despite the enormous number of distinct strategies \((2^{128})\) a branch-and-bound type algorithm may still allow to tackle this problem similarly to the two-party two-outcome problem used in Ref. [40]. On the other hand, Table III shows the asymmetric case.

We note that similarly to the symmetric case the 2-copy \((n = 2)\) upper bound result 0.5562 validates the usage of Gilbert method. This value is consistent with the numerically exact value 0.5469 computed with LP in Sec. IV. Note that the \( \eta_{\text{asym}}^{(4)} \) value falls below 1/2 corresponding to the \( \eta_{\text{asym}} \) bound for Bell measurements with two inputs and an arbitrary number of outputs. Indeed, any Bell test with \( N \) inputs cannot tolerate \( \eta_{\text{asym}} \) smaller than 1/\( N \) [45]. Note also the decreasing value of upper bounds for \( \eta_{\text{asym}}^{(n)} \) by increasing \( n \).

VI. DISCUSSION

In this paper we investigated the critical efficiency of detectors to observe Bell nonlocality using multiple copies of the two-qubit maximally entangled state and local Pauli observables in the corresponding qubit subspaces. The measurements applied give the Tsirelson bound of the CHSH inequality for each copy of the state. We showed that the threshold \( \approx 82.84\% \) corresponding to the CHSH-Bell test with the two-qubit maximally entangled state can be considerably lowered by using multiple copies of them. To this end, we firstly studied analytically the \( n \)th iterative version of the CHSH inequality and found that the \( \eta_{\text{sym}}^{(n)} \) and \( \eta_{\text{asym}}^{(n)} \) thresholds of the overall Bell inequality tend to zero with increasing \( n \). Then we made use of linear programming in the case of \( n = 2 \) copies and Gilbert algorithm in the case of \( n = 3 \) and \( n = 4 \) copies in order to obtain Bell inequalities which overcome the performance of the \( n \)th iterated CHSH inequality. Namely, in the symmetric case we found the upper bounds 80.86\%, 73.99\% and 69.29\% on \( \eta_{\text{sym}}^{(n)} \) using \( n = 2, 3, 4 \) respective copies of the maximally entan-
gled two-qubit state. In the asymmetric case (where one of the parties has unit detection efficiency) the upper bounds $54.69\%$, $51.83\%$ and $45.84\%$ have been obtained on $\eta^{(n)}_{\text{asy}}$ for $n = 2, 3$ and 4. The number of measurements and number of outcomes per party for $n = 2, 3$ and $n = 4$ are $4, 8$ and $16$, respectively.

We note that the above values for $n = 4$ are in the same range as the emblematic Eberhard thresholds $\eta^{\text{sym}} = 66.67\%$ and $\eta^{\text{asy}} = 1/2$ corresponding to two partially entangled two-qubit states \cite{massar1993}. However, in contrast to Eberhard’s result, we used multiple maximally entangled Bell pairs. Both cases have their own advantages regarding technological implementation and we believe that our setup may offer a promising alternative way to obtain loophole-free Bell violation beside Eberhard’s setup used in the experiments of Refs. \cite{cavalcanti1999, eberhard1993}.

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