INFECTION IN $\alpha$-BROWNIAN BRIDGE
BASED ON KARHUNEN-LOÈVE EXPANSIONS

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Abstract. We study a simple decision problem on the scaling parameter in the $\alpha$-Brownian bridge $X^{(\alpha)}$ on the interval $[0,1]$: given two values $\alpha_0, \alpha_1 \geq 0$ with $\alpha_0 + \alpha_1 \geq 1$ and some time $0 \leq T \leq 1$ we want to test $H_0 : \alpha = \alpha_0$ vs. $H_1 : \alpha = \alpha_1$ based on the observation of $X^{(\alpha)}$ until time $T$. The likelihood ratio can be written as a functional of a quadratic form $\psi(X^{(\alpha)})$ of $X^{(\alpha)}$. In order to calculate the distribution of $\psi(X^{(\alpha)})$ under the null hypothesis, we generalize the Karhunen-Lo`eve Theorem to positive finite measures on $[0,1]$ and compute the Karhunen-Lo`eve expansion of $X^{(\alpha)}$ under such a measure. Based on this expansion, the distribution of $\psi(X^{(\alpha)})$ follows by Smirnov’s formula.

1. Introduction

We consider the stochastic differential equation

$$
(1) \quad dX_t^{(\alpha)} = dW_t - \frac{\alpha X_t^{(\alpha)}}{1-t} dt, \quad X_0^{(\alpha)} = 0, \quad 0 \leq t < 1,
$$

where $\alpha \geq 0$ and $W = (W_t)_{t \in [0,1]}$ is standard Brownian motion. We assume that $W$ is defined on the probability space $(C([0,1]), \mathcal{C}, \mathbb{P})$, where $C([0,1])$ is the space of continuous functions on the interval $[0,1]$ equipped with the supremum norm, $\mathcal{C}$ denotes the Borel sets, and $\mathbb{P}$ is the Wiener measure. Let $(\mathcal{F}_t)_{t \in [0,1]}$ be the natural filtration induced by $W$. The unique strong solution of (1) is given by $X^{(\alpha)} = (X_t^{(\alpha)})_{t \in [0,1]}$ with

$$
(2) \quad X_t^{(\alpha)} = \int_0^t \left( \frac{1-t}{1-s} \right)^\alpha dW_s, \quad 0 \leq t < 1.
$$

For $\alpha > 0$ we have $\lim_{t \to 1} X_t^{(\alpha)} = 0$ almost surely and thus $X^{(\alpha)}$ has an extension on $[0,1]$ with $X_1^{(\alpha)} = 0$. The process $X^{(\alpha)}$ is called the $\alpha$-Brownian bridge with scaling parameter $\alpha$.

The $\alpha$-Brownian bridge is a mean reverting process, i.e., if $X^{(\alpha)}$ deviates from its mean 0 at some time $0 < t < 1$, it is forced to return to 0. The scaling parameter $\alpha$ determines how strong this force is. In order to examine
Figure 1. The influence of α to the “expected future” $E[X_t^{(\alpha)} | \mathcal{F}_s]$ for different values of α.

this behavior further we compute the “expected future”: for $0 \leq s \leq t \leq 1$ we have

$$E[X_t^{(\alpha)} | \mathcal{F}_s] = E \left[ \int_0^t \left( \frac{1-t}{1-x} \right)^\alpha \, dW_x \bigg| \mathcal{F}_s \right]$$

$$= \left( \frac{1-t}{1-s} \right)^\alpha \int_0^s \left( \frac{1-s}{1-x} \right)^\alpha \, dW_x$$

$$= \left( \frac{1-t}{1-s} \right)^\alpha X_s^{(\alpha)}.$$

Again, we see that the smaller the scaling parameter α is the more the process will deviate from its mean 0. In the case $\alpha = 0$ we obtain standard Brownian motion, i.e., $X^{(1)} = W$ and $E[X_t^{(0)} | \mathcal{F}_s] = X_s^{(0)}$, and in the case $\alpha = 1$ we obtain the usual Brownian bridge with $X_0^{(1)} = X_1^{(1)} = 0$ and

$$E[X_t^{(1)} | \mathcal{F}_s] = \frac{1-t}{1-s} X_s^{(1)}.$$

In this paper we assume that the scaling parameter α is unknown and, given two different values $\alpha_0, \alpha_1 \geq 0$ with $\alpha_0 + \alpha_1 \geq 1$ and some time $0 \leq T \leq 1$, we want to test

$$H_0 : \alpha = \alpha_0 \quad \text{vs.} \quad H_1 : \alpha = \alpha_1,$$

based on an observed trajectory of $X^{(\alpha)}$ until time $T$, i.e., the decision should be based on the information in $\mathcal{F}_T$. When deciding problem (3) we can make two types of error. Rejecting the hypothesis $H_0$ though $\alpha = \alpha_0$ is true we make an error of the first kind, whereas keeping the hypothesis $H_0$ though $\alpha = \alpha_1$ is true we make an error of the second kind. Our aim is to find that
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decision which minimizes the probability of making an error of the second kind, given that the probability of making an error of the first kind is not larger than \( q \) for some \( 0 \leq q \leq 1 \). The Neyman–Pearson Lemma yields the most powerful test (see [6] or other introductory texts on statistical decision theory): let \( \mathbb{P}^{(\alpha)} \) be the induced measure of \( X^{(\alpha)} \) on the filtered measurable space \( (C([0, 1]), \mathcal{C}, \mathcal{F}_t)_{t \in [0, 1]} \) (note in particular that \( \mathbb{P}^{(0)} = \mathbb{P} \) and let \( \mathbb{P}^{(\alpha)}_t \) denote the restriction of the probability measure \( \mathbb{P}^{(\alpha)} \) to the \( \sigma \)-algebra \( \mathcal{F}_t \), \( t \in [0, 1] \). Assume \( T < 1 \) (the case \( T = 1 \) is treated separately in Section 2.2). Then we have to decide according to the following rule:

\[
\text{reject } H_0 \text{ if } \varphi_{\alpha_0, \alpha_1}(T) > c_{\alpha_0, \alpha_1, T}(q),
\]

where \( \varphi_{\alpha_0, \alpha_1}(T) := d\mathbb{P}^{(\alpha_1)}_T / d\mathbb{P}^{(\alpha)}_T \) is the likelihood ratio at time \( T \) and \( c_{\alpha_0, \alpha_1, T}(q) \) is chosen such that

\[
\mathbb{P}^{(\alpha)}(\varphi_{\alpha_0, \alpha_1}(T) > c_{\alpha_0, \alpha_1, T}(q)) = q.
\]

Knowing the distribution of \( \varphi_{\alpha_0, \alpha_1}(T) \) under \( \mathbb{P}^{(\alpha)} \) is thus crucial in finding the optimal decision in the statistical decision problem [3].

In Section 2.1 we will show

**Proposition 1.** The likelihood ratio \( \varphi_{\alpha_0, \alpha_1}(T) \) is given by

\[
\varphi_{\alpha_0, \alpha_1}(T) = \exp \left( (\alpha_0 - \alpha_1)(\psi_{\alpha_0, \alpha_1}(T) + \ln(1 - T))/2 \right),
\]

where

\[
\psi_{\alpha_0, \alpha_1}(T) = \frac{(X_T^{(\alpha)})^2}{1 - T} + (\alpha_0 + \alpha_1 - 1) \int_0^T \frac{(X_s^{(\alpha)})^2}{(1 - s)^2} \, ds.
\]

According to (4) it is enough to determine the distribution of \( \psi_{\alpha_0, \alpha_1}(T) \) under \( \mathbb{P}^{(\alpha)} \). Then the distribution of \( \varphi_{\alpha_0, \alpha_1}(T) \) follows by simple transformations.

We introduce the measure \( \mu_{\alpha_0, \alpha_1, T} \) by

\[
\mu_{\alpha_0, \alpha_1, T}(ds) := \frac{\delta_T(ds)}{1 - T} + \frac{(\alpha_0 + \alpha_1 - 1)\mathbb{I}(s \leq T)ds}{(1 - s)^2},
\]

where \( \delta_T \) denotes the point measure at \( T \) and \( \mathbb{I} \) the indicator function. By the assumption \( \alpha_0 + \alpha_1 \geq 1 \) it follows that \( \mu_{\alpha_0, \alpha_1, T} \) is a positive measure. Let \( L_2(\mu_{\alpha_0, \alpha_1, T}) \) denote the space of functions on \([0, 1]\) that are square integrable with respect to the measure \( \mu_{\alpha_0, \alpha_1, T} \). From (5) we see that \( \psi_{\alpha_0, \alpha_1}(T) \) is the squared \( L_2 \)-norm of \( X^{(\alpha)} \) under the measure \( \mu_{\alpha_0, \alpha_1, T} \), i.e.,

\[
\psi_{\alpha_0, \alpha_1}(T) = \|X^{(\alpha)}\|^2_{L_2(\mu_{\alpha_0, \alpha_1, T})}.
\]

The covariance function \( R^{(\alpha)}(s, t) := \mathbb{E}[X_s^{(\alpha)} X_t^{(\alpha)}] \) of \( X^{(\alpha)} \) is given by

\[
R^{(\alpha)}(s, t) = \frac{(1 - s)^{\alpha}(1 - t)^{\alpha}}{1 - 2\alpha}(1 - (1 - (s \wedge t))^{1-2\alpha})
\]
for $\alpha \neq 1/2$ and

$$R^{(\alpha)}(s, t) = -\sqrt{(1 - s)(1 - t) \ln(1 - (s \wedge t))}$$

for $\alpha = 1/2$, where $s \wedge t$ denotes the minimum of $s$ and $t$. With $R^{(\alpha_0)}$ we associate the integral operator $A_{R^{(\alpha_0)}}$ defined by

$$A_{R^{(\alpha_0)}} = \int_0^1 R^{(\alpha_0)}(t, s)e(s)(s)\mu_{\alpha_0, \alpha_1, T}(ds).$$

For $T < 1$, we have

$$\int_0^1 \int_0^1 |R^{(\alpha_0)}(t, s)|^2 \mu_{\alpha_0, \alpha_1, T}(dt) \mu_{\alpha_0, \alpha_1, T}(ds) < \infty.$$

Hence, by the Cauchy-Schwartz inequality,

$$\|A_{R^{(\alpha_0)}}e\|^2_{L_2(\mu_{\alpha_0, \alpha_1, T})} = \int_0^1 \left| \int_0^1 R^{(\alpha_0)}(t, s)e(s)\mu_{\alpha_0, \alpha_1, T}(ds) \right|^2 \mu_{\alpha_0, \alpha_1, T}(dt) \leq \|e\|^2_{L_2(\mu_{\alpha_0, \alpha_1, T})} \int_0^1 \int_0^1 |R^{(\alpha_0)}(t, s)|^2 \mu_{\alpha_0, \alpha_1, T}(ds) \mu_{\alpha_0, \alpha_1, T}(dt) < \infty$$

for $e \in L_2(\mu_{\alpha_0, \alpha_1, T})$ which implies that $A_{R^{(\alpha_0)}}$ is a linear and bounded operator from $L_2(\mu_{\alpha_0, \alpha_1, T})$ to $L_2(\mu_{\alpha_0, \alpha_1, T})$ with

$$\|A_{R^{(\alpha_0)}}\|^2 \leq \int_0^1 \int_0^1 |R^{(\alpha_0)}(t, s)|^2 \mu_{\alpha_0, \alpha_1, T}(ds) \mu_{\alpha_0, \alpha_1, T}(dt) < \infty.$$

Moreover, from (10) it follows that $A_{R^{(\alpha_0)}}$ is compact, the symmetry of $R^{(\alpha_0)}$ implies the self-adjointness of $A_{R^{(\alpha_0)}}$, and since $R^{(\alpha_0)}$ is non-negative definite it follows that $A_{R^{(\alpha_0)}}$ is positive. Hence, its eigenvalues $(\lambda_k)_{k=1}^\infty$ are real and non-negative and an application of a generalized version of the Karhunen-Loève Theorem (see Section 3.1) yields the following series expansion of $X^{(\alpha_0)}$:

$$X^{(\alpha_0)}_t = \sum_{k=1}^\infty Z_k e_k(t),$$

where $(e_k)_{k=1}^\infty$ is the sequence of corresponding orthonormalized eigenfunctions of the eigenvalues $(\lambda_k)_{k=1}^\infty$ and $(Z_k)_{k=1}^\infty$ is a sequence of independent normal random variables with $\mathbb{E}Z_k^2 = \lambda_k$. The convergence in (11) is almost surely uniform in $t$ for all $t \in [0, T]$.

From the bi-orthogonality in (11), i.e., independent random variables $Z_k$ and orthogonal eigenfunctions $e_k$, we obtain the following distributional equivalence for $\psi_{\alpha_0, \alpha_1}(T)$ under $\mathbb{P}^{(\alpha_0)}$: by (7) and (11) we have

$$\psi_{\alpha_0, \alpha_1}(T) = \|X^{(\alpha_0)}\|^2_{L_2(\mu_{\alpha_0, \alpha_1, T})} = \sum_{k=1}^\infty Z_k^2 = \sum_{k=1}^\infty \lambda_k N_k^2,$$
where \( \equiv_d \) means equality in distribution and \((N_k)_{k=1}^\infty\) is an i.i.d. sequence of standard normal random variables.

Random variables of the form
\[
Q_r = \sum_{k=1}^r \nu_k N_k^2
\]
with \( \nu_k > \nu_l \geq 0 \) for \( k < l \) and \( N_k \) as above were studied by Smirnov in \([9]\).

In \([8]\) it was proven that the formula found by Smirnov extends to \( r = \infty \) whenever \( \sum_{k=1}^\infty \nu_k < \infty \). Namely, it was shown that
\[
\mathbb{P}(Q_\infty \leq x) = 1 - \frac{1}{\pi} \sum_{k=1}^\infty (-1)^{k+1} \int_{1/\nu_{2k-1}}^{1/\nu_{2k}} \frac{e^{-xu/2}}{u \sqrt{|F(u)|}} du,
\]
where \( F \) is the real valued function
\[
F(u) = \prod_{l=1}^\infty (1 - \nu_l u).
\]

In Theorem \([3]\) we calculate the eigenvalues \((\lambda_k)_{k=1}^\infty\) of the operator \( A_{R(\alpha_0)} \). In the case \( \alpha_0, \alpha_1 \geq 1/2 \) these are given by the positive zeros of the function
\[
F_{\alpha_0, \alpha_1, T}(\lambda) := \tan(\beta(\lambda) \ln(1 - T)) + \lambda \beta(\lambda)/(1 + \lambda/2 - \lambda \alpha_0),
\]
where \( \beta(\lambda) = \sqrt{(\alpha_0 + \alpha_1 - 1)/\lambda - \alpha_0(\alpha_0 - 1)} - 1/4 \). In the general case \( \alpha_0, \alpha_1 \geq 0 \) (but with \( \alpha_0 + \alpha_1 \geq 1 \)) a further eigenvalue \( \lambda_0 \) in addition to the zeros of the function \( F_{\alpha_0, \alpha_1, T} \) can appear (see Theorem \([3]\)). We show that \( \sum_{k=1}^\infty \lambda_k < \infty \) in Proposition \([2]\). Then, according to \([12]\), the distribution function of \( \psi_{\alpha_0, \alpha_1, T}(T) \) under \( \mathbb{P}^{(\alpha_0)} \) is given by \([13]\) with \( \nu_k = \lambda_k \). Finally, from \([4]\) we obtain the following

**Theorem 1.** If \( \alpha_0 < \alpha_1 \), then \( \varphi_{\alpha_0, \alpha_1}(T) \leq (1 - T)^{(\alpha_0 - \alpha_1)/2} \) and the distribution function of \( \varphi_{\alpha_0, \alpha_1}(T) \) under \( \mathbb{P}^{(\alpha_0)} \) is given by \( \mathbb{P}^{(\alpha_0)}(\varphi_{\alpha_0, \alpha_1}(T) \leq x) = D_{\alpha_0, \alpha_1, T}(x) \), where
\[
D_{\alpha_0, \alpha_1, T}(x) = \frac{1}{\pi} \sum_{k=1}^\infty (-1)^{k+1} \int_{1/\nu_{2k}}^{1/\nu_{2k-1}} \frac{(1 - T)^{u/2} e^{u/(\alpha_1 - \alpha_0)}}{u \sqrt{|F(u)|}} du
\]
with \( F(u) = \prod_{l=1}^\infty (1 - \lambda_l u) \). In the case \( \alpha_0 > \alpha_1 \) we have \( \varphi_{\alpha_0, \alpha_1}(T) \geq (1 - T)^{(\alpha_0 - \alpha_1)/2} \) and \( \mathbb{P}^{(\alpha_0)}(\varphi_{\alpha_0, \alpha_1}(T) \leq x) = 1 - D_{\alpha_0, \alpha_1, T}(x) \).

**Remark 1.** We may as well define the \( \alpha \)-Brownian bridge on an interval \([0, S]\). Let \( X_{\alpha, S} = (X_{t, \alpha, S})_{t \in [0, 1]} \) be the strong solution of the stochastic differential equation
\[
dX_{t, \alpha, S} = dW_t - \frac{\alpha X_{t, \alpha, S}}{S - t} dt, \quad X_{0, \alpha, S} = 0, \quad 0 \leq t < S.
\]
Then, for $\alpha > 0$, we have $\lim_{t \to S} X_t^{(\alpha,S)} = 0$. The $\alpha$-Brownian bridge is self-similar. Namely,

$$\left( X_t^{(\alpha,S)} \right)_{t \in [0,S]} = d \left( \sqrt{S} X_t^{(\alpha,1)} \right)_{t \in [0,S]}.$$ 

From this self-similarity the results in this paper easily extend to $\alpha$-Brownian bridges on an interval $[0,S]$. However, we do not pursue this case further.

To the best of our knowledge, the $\alpha$-Brownian bridge was first studied in [4], where it was used to model the arbitrage profit associated with a given futures contract in the absence of transaction costs. In [2] sample path properties of $X^{(\alpha)}$ and the maximum-likelihood estimator of $\alpha$ where studied. In [3] Laplace transforms of $X^{(\alpha)}$ are calculated. In particular, the Laplace transform of $\psi_{\alpha_0,\alpha_1}(T)$ follows from Theorem 21 in [3]. In [1], the Karhunen-Loève expansion of $X^{(\alpha)}$ under the Lebesgue measure was computed. The decision problem [3] was studied before in [11] under the assumption that $\alpha_0, \alpha_1 > 1/2$ and that the time of decision $T$ is close to 1.

An approximation of the distribution of $\varphi_{\alpha_0,\alpha_1}(T)$ under $\mathbb{P}^{(\alpha_0)}$ was derived by means of large deviations. We improve those results by allowing a more general setting for the parameters $\alpha_0, \alpha_1$, and $T$, and by providing exact formulas for the distribution of the likelihood ratio under $H_0$.

The rest of the paper is organized as follows. In Section 2 we calculate the likelihood ratio $\varphi_{\alpha_0,\alpha_1}(T)$ and study the cases $T = 1$ and $\alpha_0 + \alpha_1 = 1$. We exclude these cases in the later sections. In Section 3 we generalize the Karhunen-Loève Theorem and calculate the Karhunen-Loève expansion of $X^{(\alpha_0)}$ under the measure $\mu_{\alpha_0,\alpha_1,T}$. In Section 4 we briefly comment on how the approach of this paper extends to other processes, such as the Ornstein-Uhlenbeck process. Finally, in Section 5 we give some remaining proofs we did not give in earlier sections for the sake of readability.

2. Preliminary results and special cases

2.1. The likelihood ratio process. We prove Proposition [1] i.e., we show

$$\varphi_{\alpha_0,\alpha_1}(T) = \exp \left( (\alpha_0 - \alpha_1)(\psi_{\alpha_0,\alpha_1}(T) + \ln(1-T))/2 \right),$$

where

$$\psi_{\alpha_0,\alpha_1}(T) = \frac{(X_T^{(\alpha)})^2}{1-T} + (\alpha_0 + \alpha_1 - 1) \int_0^T \frac{(X_s^{(\alpha)})^2}{(1-s)^2} ds.$$ 

Proof of Proposition [1]. Under $\mathbb{P}_T^{(0)}$ we have

$$dX_t^{(\alpha)} = dW_t, \quad X_0^{(\alpha)} = 0, \quad t \leq T,$$

i.e., $X^{(\alpha)}$ is Brownian motion. Under $\mathbb{P}_T^{(\alpha)}$ we get

$$dX_t^{(\alpha)} = dW_t - \frac{\alpha X_t^{(\alpha)}}{1-t} dt, \quad X_0^{(\alpha)} = 0, \quad t \leq T,$$

$$\ln\left( \frac{dX_t^{(\alpha)}}{dW_t} \right) = \alpha \int_0^t \frac{X_s^{(\alpha)}}{1-s} ds.$$ 

From this, it follows that

$$\varphi_{\alpha_0,\alpha_1}(T) = \exp \left( (\alpha_0 - \alpha_1)(\psi_{\alpha_0,\alpha_1}(T) + \ln(1-T))/2 \right).$$
and thus
\[ dW_t = dX_t^{(a)} + \frac{\alpha X_t^{(a)}}{1-t} \, dt. \]

For \( 0 \leq t \leq T \), define \( M_t^{(a)} \) by
\begin{align*}
(14) \quad M_t^{(a)} &= \exp \left( - \int_0^t \frac{\alpha X_s^{(a)}}{1-s} \, dW_s - \frac{1}{2} \int_0^t \left( \frac{\alpha X_s^{(a)}}{1-s} \right)^2 \, ds \right) \\
                         &= \exp \left( \alpha \int_0^t \frac{X_s^{(a)}}{1-s} \, dX_s^{(a)} + \frac{\alpha^2}{2} \int_0^t \frac{(X_s^{(a)})^2}{(1-s)^2} \, ds \right).
\end{align*}

The process \( (M_t^{(a)})_{t \in [0,T]} \) is a martingale with respect to \( (\mathcal{F}_t)_{t \in [0,T]} \) and \( \mathbb{P}_T^{(a)} \) and thus, by Girsanov’s Theorem, \( X^{(a)} \) is a Brownian motion on \( \mathcal{F}_T \) with respect to the measure \( \mathbb{Q} \) defined by \( d\mathbb{Q} = M_T^{(a)} \, d\mathbb{P}_T^{(a)} \). Hence, \( \mathbb{Q} = \mathbb{P}_T^{(0)} \) on \( \mathcal{F}_T \) and thus
\begin{align*}
(15) \quad \frac{d\mathbb{P}_T^{(a)}}{d\mathbb{P}_T^{(0)}} &= (M_T^{a})^{-1}.
\end{align*}

It follows
\begin{align*}
(16) \quad \varphi_{\alpha_0,\alpha_1}(T) &= \frac{d\mathbb{P}_T^{(\alpha_1)}}{d\mathbb{P}_T^{(\alpha_0)}} = M_T^{\alpha_0} / M_T^{\alpha_1}.
\end{align*}

In order to calculate \( M_T^{(a)} \) set \( Y_t = X_t^{(a)}/(1-t), 0 \leq t \leq T \). Then, by Itô’s formula,
\begin{align*}
\frac{dY_t}{(1-t)^2} &= \frac{dX_t^{(a)}}{1-t} + \frac{X_t^{(a)}}{(1-t)^2} \, dt, \quad Y_0 = 0, \quad 0 \leq t \leq T,
\end{align*}
and it follows by partial integration that
\begin{align*}
\int_0^T \frac{X_s^{(a)}}{1-s} \, dX_s^{(a)} &= \int_0^T Y_s \, dX_s^{(a)} \\
&= Y_T X_T^{(a)} - Y_0 X_0^{(a)} - \int_0^T X_s^{(a)} \, dY_s - \int_0^T dX_s^{(a)} \cdot dY_s \\
&= \frac{(X_T^{(a)})^2}{1-T} - \int_0^T \frac{X_s^{(a)}}{1-s} \, dX_s^{(a)} - \int_0^T \frac{(X_s^{(a)})^2}{(1-s)^2} \, ds - \int_0^T \frac{ds}{1-s},
\end{align*}
and thus that
\begin{align*}
(17) \quad \int_0^T \frac{X_s^{(a)}}{1-s} \, dX_s^{(a)} &= \frac{1}{2} \left( \frac{(X_T^{(a)})^2}{1-T} - \int_0^T \frac{(X_s^{(a)})^2}{(1-s)^2} \, ds + \ln(1-T) \right).
\end{align*}

Plugging (17) into (14), we obtain
\begin{align*}
(18) \quad M_T^{(a)} &= \exp \left( \alpha \frac{(X_T^{(a)})^2}{2(1-T)} - \frac{\alpha}{2} (1-\alpha) \int_0^T \frac{(X_s^{(a)})^2}{(1-s)^2} \, ds + \frac{\alpha}{2} \ln(1-T) \right).
\end{align*}
Finally, plugging (18) into (16) yields the desired result. □

2.2. The case $T = 1$. From (15) and (18) it follows that the maximum-likelihood estimator of $\alpha$ based on $\mathfrak{F}_T$ is given by

$$
\hat{\alpha}_T = \left( -\frac{(X^{(\alpha)})^2}{1-T} + \int_0^T \frac{(X^{(\alpha)})^2}{(1-s)^2} ds - \ln(1-T) \right) / \left( 2 \int_0^T \frac{(X^{(\alpha)})^2}{(1-s)^2} ds \right).
$$

It was shown in [3] that $\hat{\alpha}_T$ is a strongly consistent estimator for $\alpha$, i.e., we have $\lim_{T \to 1} \hat{\alpha}_T = \alpha$, $\mathbb{P}^{(\alpha)}$-almost surely. Hence, at time $T = 1$ we can test (3) without any risk of making an error of the first or the second kind. Therefore, in the remaining part of the paper we assume that $T < 1$.

2.3. The case $\alpha_0 + \alpha_1 = 1$. We will now study the case $\alpha_0 + \alpha_1 = 1$. From Proposition [1] we know

$$
\varphi_{\alpha_0,\alpha_1}(T) = \exp \left( \frac{(\alpha_0 - \alpha_1)}{2} \left( \frac{(X^{(\alpha)})^2}{1-T} + \ln(1-T) \right) \right).
$$

If $\alpha_0 < \alpha_1$ then $\varphi_{\alpha_0,\alpha_1}(T) \leq (1-T)^{(\alpha_0 - \alpha_1)/2}$ and since $X^{(\alpha)}_T$ is normally distributed with mean 0 and variance $R^{(\alpha)}(T,T)$ it follows $\mathbb{P}^{(\alpha)}(\varphi_{\alpha_0,\alpha_1}(T) \leq x) = D_{\alpha_0,\alpha_1,T}(x)$, where

$$
D_{\alpha_0,\alpha_1,T}(x) = 2 - 2\Phi \left( \sqrt{(1-T)\left( \frac{2\ln(x)}{\alpha_0 - \alpha_1} - \ln(1-T) \right) / R^{(\alpha)}(T,T)} \right),
$$

and

$$
\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-y^2/2)dy
$$

is the distribution function of the standard normal distribution. In the case $\alpha_0 > \alpha_1$ we obtain $\varphi_{\alpha_0,\alpha_1}(T) \geq (1-T)^{(\alpha_0 - \alpha_1)/2}$ and then $\mathbb{P}^{(\alpha)}(\varphi_{\alpha_0,\alpha_1}(T) \leq x) = 1 - D_{\alpha_0,\alpha_1,T}(x)$.

In the particular case $\alpha_0 = 0$ and $\alpha_1 = 1$ we would like to distinguish Brownian bridge from Brownian motion. We have

$$
\varphi_{0,1}(T) = \exp \left( -\frac{1}{2} \left( \frac{(X^{(\alpha)})^2}{1-T} + \ln(1-T) \right) \right)
$$

$$
= \exp \left( -\frac{(X^{(\alpha)})^2}{2(1-T)} \right) / \sqrt{1-T}.
$$

Moreover, $R^{(0)}(T,T) = T$ and thus

$$
\mathbb{P}^{(0)}(\varphi_{0,1}(T) \leq x) = 2 - 2\Phi(\sqrt{(1-T)(-2\ln(x) - \ln(1-T))/T})
$$

for $x \leq 1/\sqrt{1-T}$.

In the remaining part of the paper we assume that $\alpha_0 + \alpha_1 > 1$. 
3. A Karhunen-Loève expansion of $X^{(\alpha)}$

Let $Y = (Y_t)_{t \in [a,b]}$ be a centered continuous Gaussian process indexed by a compact interval $[a,b]$ with covariance function $R(s,t) = \mathbb{E}Y_sY_t$. Let $(\lambda_k)_{k=1}^\infty$ and $(e_k)_{k=1}^\infty$ be the eigenvalues and corresponding orthonormalized eigenfunctions of the operator $A_R : L_2([a,b]) \to L_2([a,b])$ defined by

$$(A_Re)(t) = \int_a^b R(s,t)e(s)ds, \quad e \in L_2([a,b]).$$

The Karhunen-Loève Theorem (see Theorem 34.5.B in [7]) implies that $Y$ has a series expansion of the form

$$Y_t = \sum_{k=1}^\infty Z_ke_k(t) \quad \text{with} \quad Z_k = \int_a^b Y_se_k(s)ds,$$

where $(Z_k)_{k=1}^\infty$ is a sequence of independent centered normal distributed random variables with $EZ_k^2 = \lambda_k$ and the convergence in (20) is almost surely uniform in $t \in [a,b]$.

We extend this result by replacing the Lebesgue measure on the interval $[a,b]$ by any positive finite Borel measure $\nu$ on $[a,b]$. Then we calculate the Karhunen-Loève expansion of $X^{(\alpha)}$ under the measure $\mu_{\alpha_0,\alpha_1,T}$ defined in (6).

3.1. A generalized Karhunen-Loève Theorem. Consider a continuous centered Gaussian process $Y = (Y_t)_{t \in [a,b]}$ and let $R(s,t) = \mathbb{E}Y_sY_t$ be the covariance function of $Y$. Let $\nu$ be a positive finite Borel measure on $[a,b]$ with support $C \subset [a,b]$. Then we have $R \in L_2([a,b]^2, \nu \otimes \nu)$ and the linear operator $A_R : L_2([a,b], \nu) \to L_2([a,b], \nu)$ defined by

$$(A_Re)(t) = \int_a^b R(s,t)e(s)\nu(ds)$$

is bounded, compact, self-adjoint, and positive definite. Hence, the eigenvalues $(\lambda_k)_{k=1}^\infty$ of $A_R$ are real and non-negative and we get the spectral decomposition

$$A_R e = \sum_{k=1}^\infty \lambda_k \langle e_k, e \rangle e_k,$$

where $e_k$ is the corresponding orthonormalized eigenfunction of the eigenvalue $\lambda_k$ and $\langle \cdot, \cdot \rangle$ denotes the scalar product of $L_2([a,b], \nu)$.

**Theorem 2** (Generalized Karhunen-Loève theorem). We have

$$Y_t = \sum_{k=1}^\infty Z_ke_k(t) \quad \text{with} \quad Z_k = \int_a^b Y_se_k(s)\nu(ds),$$

where the convergence is almost surely uniform in $t$ for all $t \in C$. Moreover, the $(Z_k)_{k=1}^\infty$ form a sequence of independent centered normal distributed random variables with $EZ_k^2 = \lambda_k$. 

Remark 2. We may even replace the interval \([a, b]\) by a topological Hausdorff space \(E\) and consider Gaussian processes indexed by \(E\). However, we do not need this generality and thus we do not pursue this case further.

The following proof is a modified version of the proof of the Karhunen-Loève Theorem in [7].

Proof. From (21) and (22) it follows for all \(e \in L^2([a, b], \nu)\) that

\[
\int_a^b R(s, t)e(s)\nu(ds) = \sum_{k=1}^{\infty} \lambda_k \langle e_k, e \rangle e_k(t)
\]

\[
= \sum_{k=1}^{\infty} \lambda_k \int_a^b e_k(s)e(s)\nu(ds)e_k(t)
\]

\[
= \int_a^b \left( \sum_{k=1}^{\infty} \lambda_k e_k(s)e_k(t) \right) e(s)\nu(ds),
\]

and thus

\[
(24) \quad R(s, t) = \sum_{k=1}^{\infty} \lambda_k e_k(s)e_k(t)
\]

for all \(s, t \in C\). Moreover, by Mercer’s theorem (see Theorem 3.a.1 in [5]), the convergence in (24) is uniform in \(s, t \in C\).

We introduce

\[
Y^{(n)}_t = \sum_{k=1}^{n} Z_k e_k(t)
\]

with \(Z_k\) as in (23). Then

\[
(25) \quad E[Y_t - Y^{(n)}_t]^2 = E[Y_t]^2 - 2E[Y_t Y^{(n)}_t] + E[Y^{(n)}_t]^2
\]

\[
= R(t, t) - 2 \sum_{k=1}^{n} e_k(t)E[Y_t Z_k] + \sum_{k,l=1}^{n} e_k(t)e_l(t)E[Z_k Z_l].
\]

Let \(\delta_{i,j}\) be the Kronecker symbol. We have

\[
E[Y_t Z_k] = \int_a^b E[Y_t Y_s]e_k(s)\nu(ds) = \int_a^b \sum_{l=1}^{\infty} \lambda_l e_l(s)e_l(t)e_k(s)\nu(ds)
\]

\[
= \sum_{l=1}^{\infty} \lambda_l \delta_{l,k} e_l(t) = \lambda_k e_k(t)
\]
and, in a similar way,

\[
\mathbb{E}[Z_k Z_l] = \int_a^b \int_a^b \mathbb{E}[Y_s Y_t] e_k(s) e_l(t) \nu(ds) \nu(dt)
\]

\[
= \int_a^b \int_a^b \sum_{m=1}^{\infty} \lambda_m e_m(s) e_k(s) e_l(t) \nu(ds) \nu(dt)
\]

\[
= \sum_{m=1}^{\infty} \lambda_m \int_a^b e_m(s) e_k(s) \nu(ds) \int_a^b e_m(t) e_l(t) \nu(dt)
\]

\[
= \sum_{m=1}^{\infty} \lambda_m \delta_{m,k} \delta_{m,l} = \lambda_k \delta_{k,l}.
\]

By (25), it follows

\[
E[Y_t - Y_{t^{(n)}}]^2 = R(t, t) - 2 \sum_{k=1}^{n} e_k(t) \lambda_k e_k(t) + \sum_{k=1}^{n} e_k(t) e_k(t) \lambda_k
\]

\[
= R(t, t) - \sum_{k=1}^{n} \lambda_k e_k^2(t)
\]

and hence, by (24), \(E[Y_t - Y_{t^{(n)}}]^2 \to 0\) as \(n \to \infty\) uniform in \(t\) for \(t \in C\).

By the Itô-Nisio Theorem (see Theorem 2.4 in [10]), the convergence in quadratic mean implies the convergence almost surely.

\[\square\]

3.2. A Karhunen-Loève expansion of the \(\alpha\)-Brownian bridge. Let \((\lambda'_k)_{k=1}^{\infty}\) be the decreasing sequence of zeros of the function

\[
F_{\alpha_0, \alpha_1, T}(\lambda) = \tan(\beta(\lambda) \ln(1 - T)) + \lambda \beta(\lambda)/(1 + \lambda/2 - \lambda \alpha_0),
\]

where \(\beta(\lambda) = \sqrt{(\alpha_0 + \alpha_1 - 1)/\lambda - \alpha_0(\alpha_0 - 1) - 1/4}\). Denote by \(a_0 \land a_1\) and \(a_0 \lor a_1\) the minimum and maximum of \(a_0\) and \(a_1\). We prove the following theorem.

**Theorem 3.** Define

\[
c := \frac{1/2 - (a_0 \lor a_1)}{(1/2 - (a_0 \lor a_1)) \ln(1 - T) + 1} + 1/2.
\]

(i) If \(a_0 \land a_1 \geq c\), then, under \(\mathbb{P}^{(\alpha_0)}\), the sequence of decreasing eigenvalues in the Karhunen-Loève expansion of \(X^{(\alpha)}\) under the measure \(\mu_{\alpha_0, \alpha_1, T}\) is given by \((\lambda'_k)_{k=1}^{\infty} = (\lambda'_k)_{k=1}^{\infty}\). The corresponding normed eigenfunctions are given by

\[
e_k(t) = \rho_k \sqrt{1 - t} \sin(\beta(\lambda_k) \ln(1 - t)),
\]

where \(\rho_k\) is chosen such that \(e_k\) is normal in the \(L_2(\mu_{\alpha_0, \alpha_1, T})\)-norm.
(ii) If $\alpha_0 \land \alpha_1 < c$, then, under $\mathbb{P}^{(\alpha_0)}$, the Karhunen-Loève expansion of $X^{(\alpha)}$ under the measure $\mu_{\alpha_0,\alpha_1,T}$ contains in addition to the eigenvalues from (i) a further term $\lambda_0$ with

$$\lambda_0 = \frac{\alpha_0 + \alpha_1 - 1}{\alpha_0(\alpha_0 - 1) + 1/4 - \sigma_0^2} > \lambda_1,$$

where $\sigma_0$ is the unique zero of the function

$$G(\sigma) = (\sigma + \alpha_0 - 1/2)(\sigma + \alpha_1 - 1/2) - (1 - T)^2(\sigma - \alpha_0 + 1/2)(\sigma - \alpha_1 + 1/2)$$

with $0 < \sigma_0 < 1/2 - \alpha_0$. The corresponding eigenfunction is given by

$$e_0(t) = \rho_0 \sqrt{1 - t} \left( (1 - t)^{\sigma_0} - (1 - t)^{-\sigma_0} \right),$$

where $\rho_0$ is chosen such that $e_0$ is normal in the $L_2(\mu_{\alpha_0,\alpha_1,T})$-norm.

**Remark 3.** Note that the constant $c$ is always less than or equal to $1/2$. Hence, if $\alpha_0, \alpha_1 \geq 1/2$ then the first part of Theorem 3 will always apply.

The proof of Theorem 3 in full requires some simple but lengthy auxiliary calculations. They are organized as Lemmas 1-3 and moved to Section 5 for the sake of readability. Moreover, we will carry out the proof only in the case $\alpha_0 \neq 1/2$. The case $\alpha_0 = 1/2$ leads to almost exactly the same calculations.

**Proof for $\alpha_0 \neq 1/2$.** Let $\lambda$ be a non-zero eigenvalue of the integral operator associated with the kernel $R^{(\alpha_0)}$ and the measure $\mu_{\alpha_0,\alpha_1,T}$, i.e., we consider the equation (with $t \in [0,T]$)

$$\lambda e(t) = \int_0^1 R^{(\alpha_0)}(t,s)e(s)\mu_{\alpha_0,\alpha_1,T}(ds) = (1 - T)^{-1}R^{(\alpha_0)}(t,T)e(T) + (\alpha_0 + \alpha_1 - 1) \int_0^t R^{(\alpha_0)}(t,s)e(s)/(1 - s)^2 ds + (\alpha_0 + \alpha_1 - 1) \int_t^T R^{(\alpha_0)}(t,s)e(s)/(1 - s)^2 ds,$$

where $0 \neq e \in L_2(\mu_{\alpha_0,\alpha_1,T})$. The eigenfunctions $e$ are twice continuously differentiable since so is $R^{(\alpha_0)}(t,s)$. Differentiating both sides of (27) twice with respect to $t$ gives the second order differential equation

$$\lambda e''(t) = \lambda e(t)\alpha_0(\alpha_0 - 1)(1 - t)^{-2} - (\alpha_0 + \alpha_1 - 1)e(t)(1 - t)^{-2}$$

or equivalently

$$\frac{(1 - t)^2}{\lambda}e''(t) - (\alpha_0(\alpha_0 - 1) - (\alpha_0 + \alpha_1 - 1)/\lambda)e(t) = 0$$

(28)
with boundary conditions $e(0) = 0$ and
\begin{equation}
\lambda e(T) = (1 - T)^{-1} R^{(\alpha_0)}(T, T)e(T) + (\alpha_0 + \alpha_1 - 1) \int_0^T R^{(\alpha_0)}(T, s)e(s)/(1 - s)^2 ds.
\end{equation}

The general solution of (28) is
\[ e(t) = y(\ln(1 - t)) = (1 - t)^{1/2} \left( \rho(1 - t)^\sigma + \rho(1 - t)^{-\sigma} \right), \]
where $\sigma^2 = \sigma^2(\lambda)$ is given by
\[ \sigma^2 = \alpha_0(\alpha_0 - 1) - (\alpha_0 + \alpha_1 - 1)/\lambda + 1/4. \]
In fact, setting $e(t) = y(\ln(1 - t))$ in (28) together with the substitution $s = \ln(1 - t)$ yields
\[ 0 = (1 - t)^2 \frac{\partial^2}{\partial t^2} y(\ln(1 - t)) - (\sigma^2 - 1/4)y(\ln(1 - t)) \]
\[ = y''(\ln(1 - t)) + y'(\ln(1 - t)) - (\sigma^2 - 1/4)y(\ln(1 - t)) \]
\[ = y''(s) - y'(s) - (\sigma^2 - 1/4)y(s). \]
The characteristic polynomial
\[ \chi(r) = r^2 - r - (\sigma^2 - 1/4) \]
has roots $r_{1,2} = 1/2 \pm \sigma$. Hence, $y$ is in general given by
\[ y(s) = \rho \exp(r_1 s) + \rho \exp(r_2 s), \]
and thus, the solution of (28) is
\[ e(t) = y(\ln(1 - t)) = \rho(1 - t)^{1/2+\sigma} + \rho(1 - t)^{1/2-\sigma}. \]
The boundary condition $e(0) = 0$ yields $\rho = -\rho$ and thus
\[ e(t) = \rho(1 - t)^{1/2} \left( (1 - t)^\sigma - (1 - t)^{-\sigma} \right). \]
By Lemma 1, the boundary condition (29) is fulfilled whenever
\begin{equation}
0 = \lambda \sigma \left( (1 - T)^\sigma + (1 - T)^{-\sigma} \right) + (1 + \lambda/2 - \lambda \alpha_0)((1 - T)^\sigma - (1 - T)^{-\sigma}).
\end{equation}
Equality is obviously given for $\sigma^2 = 0$. But then $e = 0$ and $\lambda$ with $\sigma^2(\lambda) = 0$ is thus not an eigenvalue.
In the case $\sigma^2 < 0$ we have $\lambda < (\alpha_0 + \alpha_1 - 1)/(1/4 + \alpha_0(\alpha_0 - 1))$ and $\sigma = i\beta$ with
\[ \beta = \sqrt{(\alpha_0 + \alpha_1 - 1)/\lambda - \alpha_0(\alpha_0 - 1) - 1/4} \in \mathbb{R}. \]
It follows that
\[ (1 - t)^\sigma + (1 - t)^{-\sigma} = \exp(i\beta \ln(1 - t)) + \exp(-i\beta \ln(1 - t)) \]
\[ = 2 \cos(\beta \ln(1 - t)), \]
\[ (1 - t)^\sigma - (1 - t)^{-\sigma} = 2i \sin(\beta \ln(1 - t)), \]
and thus
\[ e(t) = \rho' \sqrt{1 - t} \sin(\beta \ln(1 - t)). \]
Then finding the solutions of (30) is equivalent to finding the zeros in
\[ F_{\alpha_0, \alpha_1, \tau}(\lambda) = \tan(\beta \ln(1 - T)) + \lambda \beta / (1 + \lambda/2 - \lambda \alpha_0) \]
with \( \beta \neq 0 \).

In the case \( \sigma^2 > 0 \) we have \( \lambda > (\alpha_0 + \alpha_1 - 1)/(1/4 + \alpha_0(\alpha_0 - 1)) \). Expressing \( \lambda \) in terms of \( \sigma^2 \),
\[ \lambda = \frac{\alpha_0 + \alpha_1 - 1}{\alpha_0(\alpha_0 - 1) + 1/4 - \sigma^2}, \]
we are looking for the non-zero solutions of
\[
0 = \frac{\sigma(\alpha_0 + \alpha_1 - 1)}{\alpha_0(\alpha_0 - 1) + 1/4 - \sigma^2}((1 - T)^\sigma + (1 - T)^{-\sigma}) \\
+ (1 + \frac{(\alpha_0 + \alpha_1 - 1)(1/2 - \alpha_0)}{\alpha_0(\alpha_0 - 1) + 1/4 - \sigma^2})(\sigma - (1 - T)^{-\sigma})
\]
(31) \[ = \frac{(1 - T)^{-\sigma}}{(\alpha_0 - 1/2)^2 - \sigma^2} G(\sigma), \]
where
\[ G(\sigma) = (\sigma + \alpha_0 - 1/2)(\sigma + \alpha_1 - 1/2) - (1 - T)^{2\sigma}(\sigma - \alpha_0 + 1/2)(\sigma - \alpha_1 + 1/2). \]
The first factor in (31) is never equal to zero and thus we are looking for the non-zero zeros of \( G \). Without loss of generality we may assume that \( \alpha_0 < \alpha_1 \) which implies \( \alpha_1 > 1/2 \) because of the assumption \( \alpha_0 + \alpha_1 - 1 \geq 0 \). Then if
\[ \alpha_0 > \frac{1/2 - \alpha_1}{(1/2 - \alpha_1) \ln(1 - T) + 1} + 1/2, \]
it follows \( G(\sigma) > 0 \) for all \( \sigma > 0 \) by Lemma 2. On the other hand, if
\[ \alpha_0 < \frac{1/2 - \alpha_1}{(1/2 - \alpha_1) \ln(1 - T) + 1} + 1/2 < 1/2, \]
then by Lemma 3, \( G'(0) < 0, G(\sigma) > 0 \) for \( \sigma \geq 1/2 - \alpha_0 \), and \( G''(\sigma) > 0 \) for \( 0 \leq \sigma \leq 1/2 - \alpha_0 \), i.e., \( G \) is strictly convex. This implies that \( G \) has a unique zero with \( 0 < \sigma < 1/2 - \alpha_0 \).

We now show the summability of the eigenvalues \( (\lambda_k)_{k=1}^\infty \).

**Proposition 2.** There is a constant \( c > 0 \) such that \( k^2 \lambda_k \to c \) as \( k \) tends to infinity. In particular \( \sum_{k=1}^\infty \lambda_k < \infty \).

**Proof.** According to Theorem 3 the \( \lambda_k \)'s are (possibly except for one) given by the zeros of the function
\[ (32) \quad F_{\alpha_0, \alpha_1, \tau}(\lambda) = \tan(\beta(\lambda) \ln(1 - T)) + \lambda \beta(\lambda)/(1 + \lambda/2 - \lambda \alpha_0), \]
where \( \beta(\lambda) = \sqrt{(\alpha_0 + \alpha_1 - 1)/\lambda - \alpha_0(\alpha_0 - 1) - 1/4} \). We know that \( \lambda_k \to 0 \) and thus that
\[ \lambda_k \beta(\lambda_k)/(1 + \lambda_k/2 - \lambda_k \alpha_0) \to 0 \]
as \( k \) tends to infinity. Hence, for small \( \lambda \) the zeros of \( F_{\alpha_0,\alpha_1,T}(\lambda) \) are essentially given by the zeros of \( \tan(\beta(\lambda) \ln(1 - T)) \). Those are given by \( \lambda'_k \) such that \( \beta(\lambda'_k) \ln(1 - T) = -k\pi, \ k = 1,2,\ldots \), which implies that

\[
\lambda'_k = \frac{\alpha_0 + \alpha_1 - 1}{\alpha_0(\alpha_0 - 1) + 1/4 + k^2\pi^2/(\ln(1 - T))^2}.
\]

The result follows with \( c = (\alpha_0 + \alpha_1 - 1)(\ln(1 - T))^2/\pi^2 \).

\[\square\]

4. Ornstein-Uhlenbeck Processes

The approach described in Section 1 is not restricted to \( \alpha \)-Brownian bridges but may be applied to other cases as well. We briefly study the case of Ornstein-Uhlenbeck processes. Proofs are omitted since they follow the same paths as the proofs of Proposition 1 and Theorem 3. Moreover, in order to emphasize the analogy to Section 1 we use the same notation as there.

We consider the stochastic differential equation

\[
dX^{(\alpha)}_t = dW_t - \alpha X^{(\alpha)}_t dt, \quad X^{(\alpha)}_0 = 0, \quad 0 \leq t < \infty,
\]
where \( \alpha \geq 0 \) and \( W = (W_t)_{t \in [0,\infty)} \) is standard Brownian motion. Let \( \mathcal{F}_t \) be the induced filtration of \( (W_s)_{s \in [0,t]} \). Given two different values \( \alpha_0, \alpha_1 \geq 0 \) and some time \( 0 < T < \infty \), we want to test

\[H_0 : \alpha = \alpha_0 \quad \text{vs.} \quad H_1 : \alpha = \alpha_1,\]

based on an observed trajectory of \( X^{(\alpha)} \) until time \( T \). Again, our aim is to find that decision which minimizes the probability of making an error of the second kind, given that the probability of making an error of the first kind is not larger than \( q \) for some \( 0 \leq q \leq 1 \). Let \( \mathbb{P}^{(\alpha)}_T \) be the induced measure of \( (X^{(\alpha)}_t)_{t \in [0,T]} \) on the measurable space \( (C([0,T]),\mathcal{C}) \). Then we have to decide according to the following rule:

reject \( H_0 \) if \( \varphi_{\alpha_0,\alpha_1}(T) > c_{\alpha_0,\alpha_1,T}(q) \),

where \( \varphi_{\alpha_0,\alpha_1}(T) := \frac{d\mathbb{P}^{(\alpha_1)}_T}{d\mathbb{P}^{(\alpha_0)}_T} \) is the likelihood ratio at time \( T \) and \( c_{\alpha_0,\alpha_1,T}(q) \) is chosen such that

\[
\mathbb{P}^{(\alpha_0)}(\varphi_{\alpha_0,\alpha_1}(T) > c_{\alpha_0,\alpha_1,T}(q)) = q.
\]

Analogous to Proposition 1 we obtain

\[
\varphi_{\alpha_0,\alpha_1}(T) = \exp \left( (\alpha_0 - \alpha_1)(\psi_{\alpha_0,\alpha_1}(T) - T)/2 \right),
\]

where

\[
\psi_{\alpha_0,\alpha_1}(T) = (X^{(\alpha)}_T)^2 + (\alpha_0 + \alpha_1) \int_0^T (X^{(\alpha)}_s^2) ds.
\]

Introducing the measure

\[
\mu_{\alpha_0,\alpha_1,T}(ds) := \delta_T(ds) + (\alpha_0 + \alpha_1)\mathbb{I}(s \leq T) ds,
\]
we obtain
\[ \psi_{\alpha_0,\alpha_1}(T) = \|X^{(\alpha)}\|^2_{L^2(\mu_{\alpha_0,\alpha_1}, T)}. \]

The covariance function of \( X^{(\alpha)} \) is given by
\[ R^{(\alpha)}(s, t) := \mathbb{E}[X^{(\alpha)}_s X^{(\alpha)}_t] = \frac{1}{2\alpha}(e^{-\alpha|s-t|} - e^{-\alpha(s+t)}). \]

With \( R^{(\alpha_0)} \) we associate the integral operator \( A_{R^{(\alpha_0)}} : L^2(\mu_{\alpha_0,\alpha_1}, T) \to L^2(\mu_{\alpha_0,\alpha_1}, T) \) defined by
\[ (A_{R^{(\alpha_0)}} e)(t) = \int_0^\infty R^{(\alpha_0)}(t, s) e(s)\mu_{\alpha_0,\alpha_1, T}(ds). \]

Analogous to Theorem 3 and Proposition 2 we obtain

**Theorem 4.** Define
\[ c := e^{-2\alpha_0 T} \left( \frac{1}{2\alpha_0 T} + \frac{1 - \alpha_0}{2} \right) + 1 - \alpha_0. \]

(i) If \( \alpha_1 \geq c \), then the sequence \( (\lambda_k)_{k=1}^\infty \) of decreasing eigenvalues of the operator \( A_{R^{(\alpha_0)}} \) is given by the zeros of the function
\[ F_{\alpha_0,\alpha_1, T}(\lambda) = \tan(\beta(\lambda)T) \]
\[ + \frac{\lambda\beta(\lambda)(1 - e^{-2\alpha_0 T})}{(\alpha_0 + \alpha_1)(e^{-2\alpha_0 T} - 1 + 2\alpha_0 \lambda) - \lambda\alpha_0(1 + e^{-2\alpha_0 T})}, \]
where \( \beta(\lambda) = \sqrt{(\alpha_0 + \alpha_1)/\lambda - \alpha_0^2} \). The corresponding normed eigenfunctions are given by
\[ e_k(t) = \rho_k \sin(\beta(\lambda_k)t), \]
where \( \rho_k \) is chosen such that \( e_k \) is normal in the \( L^2(\mu_{\alpha_0,\alpha_1, T}) \)-norm.

(ii) If \( \alpha_1 < c \), then the sequence of eigenvalues of the operator \( A_{R^{(\alpha_0)}} \) contains in addition to the eigenvalues from (i) a further term \( \lambda_0 \) with
\[ \lambda_0 = \frac{\alpha_0 + \alpha_1}{\alpha_0^2 - \sigma_0^2} > \lambda_1, \]
where \( \sigma_0 \) is the unique zero of the function
\[ G(\sigma) = \sigma(1 - e^{-2\alpha_0 T})(e^{2\sigma T} + 1) \]
\[ + (\alpha_0^2 - \sigma^2)(e^{-2\alpha_0 T} - 1)(e^{2\sigma T} - 1) \]
\[ + (2\alpha_0(\alpha_0 + \alpha_1) - \alpha_0(1 + e^{-2\alpha_0 T}))(e^{2\sigma T} - 1). \]

with \( 0 < \sigma_0 < \alpha_0 \). The corresponding eigenfunction is given by
\[ e_0(t) = \rho_0 \left( e^{\sigma_0 t} - e^{-\sigma_0 t} \right), \]
where \( \rho_0 \) is chosen such that \( e_0 \) is normal in the \( L^2(\mu_{\alpha_0,\alpha_1, T}) \)-norm.

(iii) The sequence \( (\lambda_k)_{k=1}^\infty \) is summable, i.e., \( \sum_{k=1}^\infty \lambda_k < \infty \).
By Theorem 2, we have the following series expansion of $X^{(\alpha_0)}$:

\begin{equation}
X_t^{(\alpha_0)} = \sum_{k=0}^{\infty} Z_k e_k(t),
\end{equation}

where $(Z_k)_{k=0}^{\infty}$ is a sequence of independent normal random variables with $\mathbb{E} Z_k^2 = \lambda_k$. The convergence in (34) is almost surely uniform in $t$ for all $t \in [0, T]$. It follows that

\begin{equation}
\psi_{\alpha_0, \alpha_1}(T) = \|X^{(\alpha_0)}\|_{L_2(\mu_{\alpha_0, \alpha_1}, T)}^2 = \sum_{k=0}^{\infty} Z_k^2 = d \sum_{k=0}^{\infty} \lambda_k N_k^2,
\end{equation}

where $(N_k)_{k=0}^{\infty}$ is an i.i.d. sequence of standard normal random variables.

Hence, according to (35), the distribution function of $\psi_{\alpha_0, \alpha_1}(T)$ under $\mathbb{P}(\alpha_0)$ is given by (13) with $\nu_k = \lambda_k - 1$. Finally, from (33) we obtain

\textbf{Theorem 5.} If $\alpha_0 < \alpha_1$, then $\varphi_{\alpha_0, \alpha_1}(T) \leq \exp((\alpha_1 - \alpha_0)T/2)$ and the distribution function of $\varphi_{\alpha_0, \alpha_1}(T)$ under $\mathbb{P}(\alpha_0)$ is given by $\mathbb{P}(\alpha_0)(\varphi_{\alpha_0, \alpha_1}(T) \leq x) = D_{\alpha_0, \alpha_1, T}(x)$, where

\begin{equation}
D_{\alpha_0, \alpha_1, T}(x) = \frac{1}{\pi} \sum_{k=0}^{\infty} (-1)^k \int_{1/\lambda_{k+1}}^{1/\lambda_k} e^{-uT/2} u^{(\alpha_1 - \alpha_0)} du
\end{equation}

with $F(u) = \prod_{k=0}^{\infty} (1 - \lambda k u)$. In the case $\alpha_0 > \alpha_1$ we have $\varphi_{\alpha_0, \alpha_1}(T) \geq \exp((\alpha_1 - \alpha_0)T/2)$ and $\mathbb{P}(\alpha_0)(\varphi_{\alpha_0, \alpha_1}(T) \leq x) = 1 - D_{\alpha_0, \alpha_1, T}(x)$.

\section{5. Remaining proofs}

\textbf{Lemma 1.} Assume that $0 \leq \alpha_0 \neq 1/2$ and let

\begin{equation}
e(t) = \rho(1-t)^{1/2}((1-t)^{\sigma} - (1-t)^{-\sigma}),
\end{equation}

where $\sigma^2 = \sigma(\lambda) = \alpha_0(\alpha_0 - 1) - (\alpha_0 + \alpha_1 - 1) + 1/4$. Then, for $0 < T < 1$,

\begin{equation}
\lambda e(T) = \int_0^1 R^{(\alpha_0)}(T, s)e(s)\mu_{\alpha_0, \alpha_1, T}(ds)
\end{equation}

if and only if

\begin{equation}
0 = \lambda \sigma((1-T)^{\sigma} + (1-T)^{-\sigma}) + (1 + \lambda/2 - \lambda \alpha_0)((1-T)^{\sigma} - (1-T)^{-\sigma}).
\end{equation}

\textbf{Proof.} Since we assume $\alpha_0 \neq 1/2$ we can multiply both sides in (37) by $1 - 2\alpha_0$. Moreover, we ignore the constant $\rho \neq 0$ in (36). That is, with the definition of $\mu_{\alpha_0, \alpha_1, T}$ in (6), we consider

\begin{equation}
0 = -\lambda(1-2\alpha_0)\tilde{e}(T) + \frac{(1-2\alpha_0)R^{(\alpha_0)}(T, T)\tilde{e}(T)}{1-T} + (\alpha_0 + \alpha_1 - 1)(1-2\alpha_0)\int_0^T R^{(\alpha_0)}(T, s)\tilde{e}(s)/(1-s)^2 ds
\end{equation}

with

\begin{equation}
\tilde{e}(t) = (1-t)^{\sigma+1/2} - (1-t)^{-\sigma+1/2}.
\end{equation}
Then
\[
(1 - 2\alpha_0) \int_0^T R^{(\alpha_0)}(T, s) \bar{e}(s)/(1 - s)^2 ds
= \frac{(1 - T)^{1/2 - \sigma} - (1 - T)^{2\alpha_0 - 1/2 + \sigma}}{\alpha_0 - 1/2 + \sigma} + \frac{(1 - T)^{2\alpha_0 - 1/2 - \sigma} - (1 - T)^{1/2 + \sigma}}{\alpha_0 - 1/2 - \sigma}
= \frac{(\alpha_0 - 1/2 - \sigma)((1 - T)^{1/2 - \sigma} - (1 - T)^{2\alpha_0 - 1/2 + \sigma})}{(\alpha_0 + \alpha_1 - 1)/\lambda}
+ \frac{(\alpha_0 - 1/2 + \sigma)((1 - T)^{2\alpha_0 - 1/2 - \sigma} - (1 - T)^{1/2 + \sigma})}{(\alpha_0 + \alpha_1 - 1)/\lambda},
\]
where we used \((\alpha_0 - 1/2 + \sigma)(\alpha_0 - 1/2 - \sigma) = (\alpha_0 + \alpha_1 - 1)/\lambda\). Plugging this into (38) and replacing \(\bar{e}(t)\) according to (39) we get
\[
0 = -\lambda(1 - 2\alpha_0) \left( (1 - T)^{1/2 + \sigma} - (1 - T)^{1/2 - \sigma} \right)
+ \lambda(\alpha_0 - 1/2 - \sigma) \left( (1 - T)^{1/2 - \sigma} - (1 - T)^{2\alpha_0 - 1/2 + \sigma} \right)
+ \lambda(\alpha_0 - 1/2 + \sigma) \left( (1 - T)^{2\alpha_0 - 1/2 - \sigma} - (1 - T)^{1/2 + \sigma} \right)
+ \left( (1 - T)^{1/2 + \sigma} - (1 - T)^{1/2 - \sigma} \right) \left( (1 - T)^{2\alpha_0 - 1/2} - (1 - T)^{1/2} \right)
= \lambda \sigma \left( (1 - T)^{\sigma} + (1 - T)^{-\sigma} \right) \left( (1 - T)^{2\alpha_0 - 1/2} - (1 - T)^{1/2} \right)
+ (1 + \lambda(1/2 - \alpha_0)) \left( (1 - T)^{\sigma} - (1 - T)^{-\sigma} \right)
\times \left( (1 - T)^{2\alpha_0 - 1/2} - (1 - T)^{1/2} \right).
\]
We divide by \((1 - T)^{2\alpha_0 - 1/2} - (1 - T)^{1/2}\) and obtain
\[
0 = \lambda \sigma \left( (1 - T)^{\sigma} + (1 - T)^{-\sigma} \right) + (1 + \lambda/2 - \lambda\alpha_0) \left( (1 - T)^{\sigma} - (1 - T)^{-\sigma} \right). \tag{40}
\]

In Lemmas 2 and 3 we consider the function
\[
G(\sigma) = (\sigma + \alpha_0 - 1/2)(\sigma + \alpha_1 - 1/2) - (1 - T)^{2\sigma} (\sigma - \alpha_0 + 1/2)(\sigma - \alpha_1 + 1/2)
\]
for \(\sigma \geq 0\).

**Lemma 2.** Assume that \(0 \leq T < 1\), \(\alpha_1 > 1/2\), and
\[
\alpha_0 > \frac{1/2 - \alpha_1}{(1/2 - \alpha_1) \ln(1 - T) + 1} + 1/2.
\]
Then \(G(\sigma) > 0\) for all \(\sigma > 0\).
Proof. We have
\[
G(\sigma) = \alpha_0(\sigma(1 + (1 - T)^{2\sigma}) + (1 - (1 - T)^{2\sigma})(\alpha_1 - 1/2))
\]
\[
+ (1 - (1 - T)^{2\sigma})(\sigma^2 - \alpha_1/2 + 1/4) + \sigma(1 + (1 - T)^{2\sigma})(\alpha_1 - 1)
\]
\[
\geq \left( \frac{1/2 - \alpha_1}{(1/2 - \alpha_1)\ln(1 - T) + 1} + 1/2 \right)
\]
\[
\times (\sigma(1 + (1 - T)^{2\sigma}) + (1 - (1 - T)^{2\sigma})(\alpha_1 - 1/2))
\]
\[
+ (1 - (1 - T)^{2\sigma})(-\alpha_1/2 + 1/4) + \sigma(1 + (1 - T)^{2\sigma})(\alpha_1 - 1)
\]
\[
= -\frac{(\alpha_1 - 1/2)^2}{(1/2 - \alpha_1)\ln(1 - T) + 1} f(\sigma)
\]
with
\[
f(\sigma) = \sigma(1 + (1 - T)^{2\sigma}) \ln(1 - T) + (1 - (1 - T)^{2\sigma}).
\]
It follows \(f(0) = 0\) and
\[
f'(\sigma) = \ln(1 - T)(1 - (1 - T)^{2\sigma} + 2\sigma \ln(1 - T)(1 - T)^{2\sigma})
\]
implying \(f'(0) < 0\). Moreover,
\[
f''(\sigma) = 4\sigma \ln(1 - T)^3(1 - T)^{2\sigma} < 0,
\]
i.e., \(f\) is a strictly concave function and thus, \(f(\sigma) < 0\) for all \(\sigma > 0\). This yields,
\[
G(\sigma) \geq -\frac{(\alpha_1 - 1/2)^2}{(1/2 - \alpha_1)\ln(1 - T) + 1} f(\sigma) > 0
\]
for all \(\sigma > 0\). \(\square\)

Lemma 3. Assume \(0 \leq T < 1, \alpha_1 > 1/2, and \alpha_0 < \frac{1/2 - \alpha_1}{(1/2 - \alpha_1)\ln(1 - T) + 1} + 1/2.\)

Then
(i) \(G(\sigma) > 0\) for \(\sigma \geq 1/2 - \alpha_0\),
(ii) \(G'(0) < 0,\) and
(iii) \(G''(\sigma) > 0\) for \(0 \leq \sigma \leq 1/2 - \alpha_0.\)

Proof. If \((\sigma - \alpha_0 + 1/2)(\sigma - \alpha_1 + 1/2) > 0,\) the estimate \((1 - T)^{2\sigma} < 1\) yields
\[
G(\sigma) > (\sigma + \alpha_0 - 1/2)(\sigma + \alpha_1 - 1/2) - (\sigma - \alpha_0 + 1/2)(\sigma - \alpha_1 + 1/2)
\]
\[
= 2\sigma(\alpha_0 + \alpha_1 - 1) \geq 0.
\]
If \((\sigma - \alpha_0 + 1/2)(\sigma - \alpha_1 + 1/2) < 0\) the estimate \((1 - T)^{2\sigma} > 0\) yields
\[
G(\sigma) > (\sigma + \alpha_0 - 1/2)(\sigma + \alpha_1 - 1/2) \geq 0,
\]
since \((\sigma + \alpha_0 - 1/2) \geq 0\) and \((\sigma + \alpha_1 - 1/2) \geq 0\) by the assumptions. This proves (i).
The derivative of $G$ is given by

$$G'(\sigma) = 2\sigma(1 - (1 - T)^{2\sigma}) + (\alpha_0 + \alpha_1 - 1)(1 + (1 - T)^{2\sigma}) - 2(1 - T)^{2\sigma}\ln(1 - T)(\sigma - \alpha_0 + 1/2)(\sigma - \alpha_1 + 1/2).$$

Hence,

$$G'(0) = 2(\alpha_0 + \alpha_1 - 1) - 2\ln(1 - T)(1/2 - \alpha_0)(1/2 - \alpha_1)
= 2\alpha_0(1 + \ln(1 - T)(1/2 - \alpha_1)) + 2\alpha_1 - 2 - \ln(1 - T)(1/2 - \alpha_1)
> 2\left(\frac{1/2 - \alpha_1}{(1/2 - \alpha_1)\ln(1 - T) + 1} + 1/2\right)(1 + \ln(1 - T)(1/2 - \alpha_1))
+ 2\alpha_1 - 2 - \ln(1 - T)(1/2 - \alpha_1)
= 2(1/2 - \alpha_1) + (1/2 - \alpha_1)\ln(1 - T)
+ 1 + 2\alpha_1 - 2 - \ln(1 - T)(1/2 - \alpha_1)
= 0,$$

which proves (ii).

The second derivative of $G$ equals

$$G''(\sigma) = 2(1 - (1 - T)^{2\sigma}) + 4(1 - T)^{2\sigma}\ln(1 - T)|g(\sigma)$$

with

$$g(\sigma) = 2\sigma - \alpha_0 - \alpha_1 + 1 + \ln(1 - T)(\sigma - \alpha_0 + 1/2)(\sigma - \alpha_1 + 1/2).$$

Assume $\sigma \geq (\alpha_0 + \alpha_1 - 1)/2$. The assumption $\sigma \leq 1/2 - \alpha_0$ implies $\sigma \geq \alpha_1 - 1/2$ and thus $\ln(1 - T)(\sigma - \alpha_0 + 1/2)(\sigma - \alpha_1 + 1/2) \geq 0$. It follows

$$g(\sigma) \geq 2\sigma - \alpha_0 - \alpha_1 + 1 \geq 0.$$

Finally, if $\sigma < (\alpha_0 + \alpha_1 - 1)/2$, then

$$g(0) = 1 - \alpha_0 - \alpha_1 + \ln(1 - T)(1/2 - \alpha_0)(1/2 - \alpha_1)
> 1 - \alpha_0 - \alpha_1 - \frac{\ln(1 - T)(1/2 - \alpha_1)^2}{(1/2 - \alpha_1)\ln(1 - T) + 1}
> 1 - \alpha_0 - \alpha_1 - \frac{\ln(1 - T)(1/2 - \alpha_1)^2}{(1/2 - \alpha_1)\ln(1 - T)}
= 1 - \alpha_0 - \alpha_1 - (1/2 - \alpha_1) = 1/2 - \alpha_0 > 0$$

and

$$g'(\sigma) = 2 + \ln(1 - T)(2\sigma - \alpha_0 - \alpha_1 + 1) \geq 2.$$

Hence $g(\sigma) \geq 0$ for all relevant $0 \leq \sigma \leq 1/2 - \alpha_0$ and thus $G''(\sigma) > 0$ for all $0 \leq \sigma \leq 1/2 - \alpha_0$. This proves (iii). $\square$

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