DISTANCES FROM POINTS TO PLANES

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Abstract. We prove that if $E \subset \mathbb{F}_q^d$, $d \geq 2$, $F \subset \text{Graff}(d - 1, d)$, the set of affine $d - 1$-dimensional planes in $\mathbb{F}_q^d$, then $|\Delta(E, F)| \geq q$ if $|E||F| > q^{d+1}$, where $\Delta(E, F)$ the set of distances from points in $E$ to lines in $F$. In dimension three and higher this significantly improves the exponent obtained by Pham, Phuong, Sang, Vinh and Valculescu ([5]).

1. Introduction

The Erdős-Falconer distance problem in $\mathbb{F}_q^d$ is to determine how large $E \subset \mathbb{F}_q^d$ needs to be to ensure that
$$\Delta(E) = \{|x - y| : x, y \in E\},$$
with $|x| = x_1^2 + x_2^2 + \ldots + x_d^2$, is the whole field $\mathbb{F}_q$, or at least a positive proportion thereof. Here and throughout, $\mathbb{F}_q$ denotes the field with $q$ elements and $\mathbb{F}_q^d$ is the $d$-dimensional vector space over this field.

The distance problem in vector spaces over finite fields was introduced by Bourgain, Katz and Tao in [1]. In the form described above, it was introduced by the second listed author of this paper and Misha Rudnev ([4]), who proved that $\Delta(E) = \mathbb{F}_q$ if $|E| > 2q^{d+1}$. It was shown in [3] that this exponent is essentially sharp for general fields when $d$ is odd. When $d = 2$, it was proved in [2] that if if $E \subset \mathbb{F}_q^2$ with $|E| \geq cq^4$, then $|\Delta(E)| \geq C(c)q$. We do not know if improvements of the $\frac{d+1}{2}$. exponent are possible in even dimensions $\geq 4$. We also do not know if improvements of the $\frac{d+1}{2}$ exponent are possible in any even dimension if we wish to conclude that $\Delta(E) = \mathbb{F}_q$, not just a positive proportion.

More generally, let $\text{Graff}(k, d)$ denote the set of $k$-dimensional affine planes in $\mathbb{F}_q^d$. In this paper we shall focus on distances from points in subsets of $\text{Graff}(0, d) = \mathbb{F}_q^d$ to $d - 1$-dimensional planes in subsets of $\text{Graff}(d - 1, d)$. The set of distances from points to points (see e.g. [4]) can be defined as the set of equivalence classes of two-point configurations where two pairs $(x, y)$ and $(x', y')$ are equivalent if there exists a
translation \( \tau \in \mathbb{F}_q^d \) and a rotation \( \theta \in O_d(\mathbb{F}_q^d) \) that takes one pair to the other. In the case of points and \( d-1 \)-dimensional planes in \( \mathbb{F}_q^d \), we may similarly define \((x, h)\) and \((x', h')\) to be equivalent, where \( x \)'s are points and \( h \)'s are planes, if after translation \( x \) to \( x' \), there exists a rotation \( \theta \in O_d(\mathbb{F}_q) \) that takes \( h \) to \( h' \). Denote the resulting set of equivalence classes by \( \Delta(E, F) \).

Before stating our main results, we need to say a few words about the parameterization of \((d-1)\)-dimensional planes in \( \mathbb{F}_q^d \). A \((d-1)\)-dimensional plane in \( \mathbb{F}_q^d \) can be expressed in the form

\[
H_{v, t} = \{ y \in \mathbb{F}_q^d : y \cdot v = t \},
\]

where we should think of \( v \) as a normal vector to the plane and \( t \) as the distance to the origin. Note that the notion of distance from a point to a plane described above only makes sense if \( ||v|| \neq 0 \). We shall henceforth refer to planes with this property as non-degenerate planes. See Lemma 2.1 below.

**Definition 1.1.** We say that \( V \subset \mathbb{F}_q^d \) is a direction set if given \( x \in \mathbb{F}_q^d, x \neq \vec{0}, \) there exists \( v \in V \) and \( t \in \mathbb{F}_q^* \) such that \( x = tv \).

It is very convenient to work with a "canonical" direction set provided by the following simple observation.

**Lemma 1.2.** Let \( S_t = \{ x \in \mathbb{F}_q^d : ||x|| = t \} \). Let \( \gamma \in \mathbb{F}_q^* \) be a non-square. Define \( V_\gamma = S_0 \cup S_1 \cup S_\gamma \). Then \( V_\gamma \) is a direction set.

To prove this, choose \( x \) such that \( ||x|| = 0 \). Then \( x \in S_0 \). Now choose \( x \) such that \( ||x|| = t^2 \) for some \( t \neq 0 \). Then \( \left( \frac{x}{t} \right)^2 + \left( \frac{x}{t} \right)^2 = 1 \), so \( x = tv \) with \( v \in S_1 \). Finally, suppose that \( ||x|| = u \) where \( u \) is not a square in \( \mathbb{F}_q^* \). To see that \( x = tv \) for some \( v \in S_\gamma \), it is enough to check that \( u\gamma^{-1} \) is a square in \( \mathbb{F}_q^\ast \). Moreover, it is enough to prove that a product of two non-squares is a square. To see this, let \( \phi : \mathbb{F}_q^\ast \to \mathbb{F}_q^\ast \) given by \( \phi(x) = ux \), where \( u \) is a non-square. The image of a square is certainly a non-square since otherwise \( u \) would be forced to be a square. It follows that an image of a non-square is a square since exactly half the elements of \( \mathbb{F}_q^\ast \) are squares. This completes the proof of Lemma 1.2.

Our main result is the following.

**Theorem 1.3.** Let \( E \subset \mathbb{F}_q^d, d \geq 2, \) and \( F \) be a subset of non-degenerate planes in \( \text{Graff}(d - 1, d), d \geq 2 \). Let \( \gamma \) be a non-square in \( \mathbb{F}_q \). Suppose that \( |E| |F| > q^{d+1} \). Then \( |\Delta(E, F)| > q \). More precisely,

\[
|\Delta(E, F)| \geq \frac{|E|^2 |F|^2}{2|E|^2 |F|^2 q^{-1} + 2q^{d-1} |E||F|} \cdot \max_{||v||=1, \gamma} \sum_t F(v, t).
\]
When $d = 2$, a better exponent was obtained by Pham, Phuong, Sang, Vinh and Valculescu ([5]). They proved that the conclusion of Theorem 1.3 holds in $\mathbb{F}_q^2$ if $|E||F| > Cq^{\delta/3}$.

It is not clear if it is possible to weaken the $|E||F| > q^{d+1}$ assumption in higher dimensions. It is not difficult to see that we cannot do better than assuming $|E||F| > q^d$. To see this, take $q = p^2$, $p$ prime, let $E = \mathbb{F}_p^d$ and $F$ be the set of all $(d-1)$-dimensional affine planes in $\mathbb{F}_p^d$. Then $|E| \approx |F| \approx q^d$ while $\Delta(E, F) = p$.

2. Proof of Theorem 1.3

We begin with a couple simple algebraic observations that make working with $\Delta(E, F)$ much easier. Given $F \subset \text{Graff}(d-1,d)$, we write the indicator function of $F$ in the form $F(v, t)$, where each plane in $\text{Graff}(d-1,d)$ is parameterized by $(v, t) \in V_\gamma \times \mathbb{F}_q$, where $V_\gamma$ is as in Lemma 1.2. For a point $x \in E$ and a plane $F(v, t) \in F$, the distance function between them, denoted by $d[x, F(v, t)]$, is defined by

$$d[x, F(v, t)] := \frac{(x \cdot v - t)^2}{||v||}.$$ 

In the following lemma, we show that the size of $\Delta(E, F)$ is at least the number of distinct non-zero distances between points in $E$ and planes in $F$.

**Lemma 2.1.** Let $F \subset \text{Graff}(d-1,d)$ be parameterized as above, with coordinates $(v, t) \in V_\gamma \times \mathbb{F}_q$, where $||v|| \neq 0$. Then

$$|\Delta(E, F)| \geq \# \left\{ \frac{(x \cdot v - t)^2}{||v||} \neq 0 : x \in E; (v, t) \in F \right\}.$$ 

**Proof.** To prove this lemma, it is enough to indicate that for $x, x' \in E$ and $(v, t), (v', t') \in F$, if $d[x, F(v, t)] = d[x', F(v', t')]$, then there is a rotation $\theta$ such that the translation from $x$ to $x'$ followed by $\theta$ takes the plane $F(v, t)$ to $F(v', t')$. Indeed, since $d[x, F(v, t)] = d[x', F(v', t')]$, we have

$$\frac{(x \cdot v - t)^2}{||v||} = \frac{(x' \cdot v' - t')^2}{||v'||}.$$ 

This implies that $||v||/||v'||$ is a square. From this we deduce, just as in the proof of Lemma 1.2 above that either both $||v||$ and $||v'||$ are squares or they are both non-squares. Since we are only considering $||v||$ and $||v'||$ that are equal to 1 or $\gamma$, we conclude that $||v|| = ||v'||$. From the equation (2.1), we have $x \cdot v - t = \pm (x' \cdot v' - t')$. Without loss of generality, we assume that $x' = 0$. Since $||v|| = ||v'|| \neq 0$, there
exists a rotation $\theta \in O_d(\mathbb{F}_q)$ such that $\theta v = \pm v'$. Thus we have the following

\[
\{ \theta(y - x) : y \cdot v = t \} = \{ z : (\theta^{-1}z + x) \cdot v = t \} \\
= \{ z : z \cdot \theta v = t - x \cdot v \} \\
= \{ z : \pm z \cdot v' = \pm t' \} \\
= \{ z : z \cdot v' = t' \}.
\]

In other words, the translation from $x$ to $x'$ followed by the rotation $\theta$ about $x'$ takes the plane $F(v, t)$ to the plane $F(v', t')$. This concludes the proof of the lemma. □

Before proving Theorem 1.3, we need to review the Fourier transform of functions on $\mathbb{F}_q^d$. Let $\chi$ be a non-trivial additive character on $\mathbb{F}_q$. For a function $f : \mathbb{F}_q \to \mathbb{C}$, we define

\[
\hat{f}(m) = q^{-d} \sum_{x \in \mathbb{F}_q^d} \chi(-x \cdot m) f(x).
\]

It is clear that

\[
f(x) = \sum_{m \in \mathbb{F}_q^d} \chi(x \cdot m) \hat{f}(m),
\]

and

\[
\sum_{m \in \mathbb{F}_q^d} |\hat{f}(m)|^2 = q^{-d} \sum_{x \in \mathbb{F}_q^d} |f(x)|^2.
\]

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. It follows from Lemma 2.1 that it suffices to prove that

\[
\# \left\{ \frac{(x \cdot v - t)^2}{||v||} : x \in E; (v, t) \in F \right\} \geq \frac{|E|^2 |F|^2}{2q^{-1}|F|^2 |E|^2 + 2q^{d-1} \max_{v \in V} F(v, t) \cdot |E||F|}.
\]

For $r \in \mathbb{F}_q$, let

\[
\nu(r) := \sum_{(x \cdot v - t)^2 = r ||v||} E(x) F(v, t).
\]

By the Cauchy-Schwartz inequality,

\[
|E|^2 |F|^2 = \left( \sum_r \nu(r) \right)^2 \leq \sum_r \nu^2(r) \cdot \# \left\{ \frac{(x \cdot v - t)^2}{||v||} : x \in E; (v, t) \in F \right\}.
\]

This implies that

\[
\# \left\{ \frac{(x \cdot v - t)^2}{||v||} : x \in E; (v, t) \in F \right\} \geq \frac{|E|^2 |F|^2}{\sum_{r \in \mathbb{F}_q} \nu(r)^2}.
\]
We now are going to show that

\[
\sum_{r \in \mathbb{F}_q} \nu(r)^2 \leq 2q^{-1}|F|^2|E|^2 + 2q^{d-1}|F||E| \cdot \max_{v \in V} \sum_t F(v, t).
\]

Indeed, applying Cauchy-Schwarz inequality again gives us

\[
\sum_{r \in \mathbb{F}_q} \nu^2(r) \leq |F| \sum_{x, x', v, t, d[x, F(v, t)] = d[x', F(v, t)]} E(x)E(x')F(v, t)
\]

\[
= |F| \left( \sum_{x \cdot v = x' \cdot v = 0} F(v, t)E(x)E(x') + \sum_{x \cdot v + x' \cdot v - 2t = 0} F(v, t)E(x)E(x') \right) = |F|(I + II).
\]

We now bound $I$ and $II$ as follows.

\[
I = \sum_{x \cdot v = x' \cdot v = 0} F(v, t)E(x)E(x') = q^{-1}|F||E|^2 + q^{-1} \sum_{s \neq 0} \sum_{v, t, x, x'} \chi(sv \cdot (x - x'))F(v, t)E(x)E(x')
\]

\[
= q^{-1}|F||E|^2 + q^{-1} \sum_{s \neq 0} \sum_{v, t, x, x'} \chi(sv \cdot (x - x'))F(v, t)E(x)E(x')
\]

\[
= q^{-1}|F||E|^2 + q^{2d-1} \max_{v \in V} \sum_t F(v, t) \cdot \sum_{z \in \mathbb{F}_q^d} |\widehat{E}(z)|^2
\]

\[
\leq q^{-1}|F||E|^2 + q^{2d-1} \cdot \max_{v \in V} \sum_t F(v, t) \cdot \sum_{z \in \mathbb{F}_q^d} |\widehat{E}(z)|^2
\]

(2.2)\]

\[
= q^{-1}|F||E|^2 + q^{2d-1}|E| \cdot \max_{v \in V} \sum_t F(v, t),
\]
where we used $\sum_{z \in \mathbb{F}_q} |\hat{E}(z)|^2 = q^{-d}|E|$.

$$II = \sum_{x \cdot v - x' \cdot v = 2t} F(v,t) E(x) E(x') = q^{-1}|F||E|^2 + q^{-1}\sum_{s \neq 0} \sum_{v,t,x,x'} \chi(sv \cdot (x + x')) \chi(2st) F(v,t) E(x) E(x')$$

$$= q^{-1}|F||E|^2 + q^{-1}\sum_{s \neq 0} \sum_{v,t,x,x'} \chi(sv \cdot (x + x')) \chi(2st) F(v,t) E(x) E(x')$$

$$= q^{-1}|F||E|^2 + q^{-1}\sum_{s \neq 0} \sum_{v,t} \hat{E}(sv) \hat{E}(sv) \chi(st + st) F(v,t)$$

$$\leq q^{-1}|F||E|^2 + q^{-d-1}\sum_{s \neq 0} \sum_{v,t} |\hat{E}(sv)|^2 F(v,t)$$

$$\leq q^{-1}|F||E|^2 + q^{-d-1} \cdot \max_{v \in V} \sum_t F(v,t) \cdot \sum_{z \in \mathbb{F}_q^d} |\hat{E}(z)|^2$$

(2.3)

$$= q^{-1}|F||E|^2 + q^{-d-1}|E| \cdot \max_{v \in V} \sum_t F(v,t).$$

Putting (2.2) and (2.3) together, we obtain

$$\sum_{r \in \mathbb{F}_q} \nu(r)^2 \leq 2q^{-1}|F|^2|E|^2 + 2q^{-d-1}|F||E| \cdot \max_{v \in V} \sum_t F(v,t).$$

We conclude that

$$\# \left\{ \frac{(x \cdot v - t)^2}{||v||^2} : x \in E; (v,t) \in F \right\} \geq \frac{|E|^2|F|^2}{2q^{-1}|F|^2|E|^2 + 2q^{-d-1} \max_{v \in V} F(v,t) \cdot |E||F|}.$$

Hence,

$$|\Delta(E,F)| \geq \frac{|E|^2|F|^2}{2q^{-1}|F|^2|E|^2 + 2q^{-d-1} \max_{v \in V} F(v,t) \cdot |E||F|}.$$

This concludes the proof once we note that

$$\max_{v \in V} F(v,t) \leq q.$$

□

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