THE BOUNDARY LERCH ZETA-FUNCTION AND SHORT CHARACTER SUMS À LA Y. YAMAMOTO

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Dedicated to Professor Dr. Masanobu Kaneko on his sixtieth birthday
with great respect and friendship

Abstract. As has been pointed out by Chakraborty et al (Seeing the invisible: around generalized Kubert functions. Ann. Univ. Sci. Budapest. Sect. Comput. 47 (2018), 185–195), there have appeared many instances in which only the imaginary part—the odd part—of the Lerch zeta-function was considered by eliminating the real part. In this paper we shall make full use of (the boundary function aspect of) the $q$-expansion for the Lerch zeta-function, the boundary function being in the sense of Wintner (On Riemann's fragment concerning elliptic modular functions. Amer. J. Math. 63 (1941), 628–634). We may thus refer to this as the ‘Fourier series–boundary $q$-series’, and we shall show that the decisive result of Yamamoto (Dirichlet series with periodic coefficients. Algebraic Number Theory. Japan Society for the Promotion of Science, Tokyo, 1977, pp. 275–289) on short character sums is its natural consequence. We shall also elucidate the aspect of generalized Euler constants as Laurent coefficients after a brief introduction of the discrete Fourier transform. These are rather remote consequences of the modular relation, i.e. the functional equation for the Lerch zeta-function or the polylogarithm function. That such a remote-looking subject as short character sums is, in the long run, also a consequence of the functional equation indicates the ubiquity and omnipotence of the Lerch zeta-function—and, a fortiori, the modular relation (S. Kanemitsu and H. Tsukada. Contributions to the Theory of Zeta-Functions: the Modular Relation Supremacy. World Scientific, Singapore, 2014).

1. Introduction

Compared with its counterpart, the Hurwitz zeta-function (1.6), the Lerch zeta-function (1.4) is less well known, the existing monograph [36] notwithstanding. Recently there has been a new representation-theoretic interpretation of the Lerch zeta-function, cf. e.g. [35]. In the last few decades, the most fundamental and influential works related to the Lerch zeta-function are [16], [43], [47], and [71], which are partly incorporated in [10]. We shall describe these toward the end of this section. In the paper [9] the ubiquity of the Lerch zeta-function, especially the monologarithm $\ell_1(x)$ (1.22) of the complex exponential argument, has been pursued.

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In this paper we shall show the ubiquity and omnipotence of the (boundary) Lerch zeta-function by proving above all things (Sections 2 and 3) that even Yamamoto’s decisive results on short character sums, in the long run, are consequences of the modular relation, i.e. the functional equation for the Lerch zeta-function. We shall also mention the closed expressions for the Laurent coefficients, called generalized Euler constants, of \( D(s, f) \) (2.3), Dirichlet series with periodic coefficients \( f \), and sum up almost all results culminating [9] by way of the basis of the vector space \( D(M) \), the space of all \( D(s, f) \). The Dirichlet character \( \chi \) and the Dirichlet \( L \)-function \( L(s, \chi) \) are typical examples of the periodic \( f \) and \( D(s, f) \).

As Corollary 6 shows, we see again the interplay of the Hurwitz formula—functional equation—for the Lerch and Hurwitz zeta-functions.

Below we assemble various known facts about the Lerch and Hurwitz zeta-functions that lie scattered around in the literature.

In general the polylogarithm function of order \( s \) is defined by

\[
\text{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s},
\]

(1.1)

cf. [40], [49], [55, pp. 114–127], [71], etc. for \( s \in \mathbb{C}, |z| < 1 \) or \( \text{Re} \ s > 1, |z| \leq 1 \).

The domain of convergence—the unit disc in (1.1)—is mapped onto the upper half-plane by the familiar transformation

\[
q = e^{2\pi i \tau}, \quad \tau \in \mathcal{H},
\]

(1.2)

with \( \mathcal{H} \) denoting the upper half-plane \( \text{Im} \ \tau > 0 \). Under (1.2),

\[
\ell_s(\tau) = \text{Li}_s(e^{2\pi i \tau}) = \sum_{n=1}^{\infty} \frac{e^{2\pi in\tau}}{n^s},
\]

(1.3)

which is absolutely convergent in \( \mathcal{H} \) for all \( s \in \mathbb{C} \). Equation (1.3) is the \( q \)-expansion, i.e. the Laurent expansion around \( \infty \). Nowadays, it is often referred to as the ‘Fourier series’ since it is a consequence of the periodicity \( \ell_s(\tau + 1) = \ell_s(\tau) \).

The unit circle corresponds to the boundary of \( \mathcal{H} \), the real line \( \text{Im} \ \tau = 0, \ \tau = x \in \mathbb{R} \). In this case, the polylogarithm function of order \( s \) with complex exponential argument, i.e. (1.3) with \( z = e^{2\pi ix}, x \in \mathbb{R}, \sigma > 1 \), is best known as the Lerch zeta-function and denoted by the same symbol

\[
\ell_s(x) = \sum_{n=1}^{\infty} \frac{e^{2\pi inx}}{n^s}.
\]

(1.4)

The right-hand side of (1.4) is absolutely convergent for \( \sigma = \text{Re} \ s > 1 \) and is a boundary function of (1.3) and we refer to this as the (boundary) Lerch zeta-function. The reader is referred to [70] for boundary functions in which a looser condition of almost convergence is assumed. In our case it is analyticity and it produces rich results.

For \( s = 1 \), the series on the right of (1.4) is uniformly convergent in an interval not containing an integer and defines the polylogarithm function of order one, which is indeed the boundary function given as the genuine Fourier series—the monologarithm function (1.22) below; cf. [41]. We refer to (1.3) as the Lerch zeta-function with (1.4) as its boundary function.
Thus the Lerch zeta-function incorporates three aspects—modular function (1.3), the boundary function as the zeta-function (1.4), and the monologarithm function (1.4) with \( s = 1 \).

For fixed \( s \in \mathbb{C} \), the Lerch zeta-function (1.3) is a one-valued analytic function on the \( \tau \)-plane with slit along the negative imaginary axis and its translations by integers, i.e.

\[
C_1 = \mathbb{C} - \{ n + iy \mid n \in \mathbb{Z}, \ y \leq 0 \}. \tag{1.5}
\]

To prove this, the standard Schwarz lemma is not enough and we need its generalization; cf. [61, p. 155]. Then \( \ell_s(\tau) \) is reflected relative to the segment \((0, 1)\) to the lower strip and then by periodicity to the cut plane \( C_1 \).

The boundary Lerch zeta-function (1.4) has its counterpart, the Hurwitz zeta-function

\[
\zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(n + x)^s}, \quad \sigma > 1. \tag{1.6}
\]

This is continued meromorphically over the whole plane with a simple pole at \( s = 1 \).

These are connected by the Hurwitz formula (i.e. the functional equation for the Hurwitz zeta-function), with \( x = 1 \) being the limiting case: for \( \sigma > 1, \ 0 < x \leq 1 \),

\[
\zeta(1 - s, x) = \frac{\Gamma(s)}{(2\pi)^s} \left( e^{-\pi i s/2} \ell_s(x) + e^{\pi i s/2} \ell_s(1 - x) \right), \tag{1.7}
\]

while its reciprocal is

\[
\ell_{1-s}(x) = \frac{\Gamma(s)}{(2\pi)^s} \left( e^{\pi i s/2} \zeta(s, x) + e^{-\pi i s/2} \zeta(s, 1 - x) \right), \quad 0 < x < 1. \tag{1.8}
\]

By (1.8), the boundary Lerch zeta-function \( \ell_{1-s}(x) \) is continued meromorphically over the whole plane with \( s = 0 \) a plausible singular point. However, it is also a regular point since the pole of \( \Gamma(s) \) is cancelled by the factor \( \zeta(s, x) + \zeta(s, 1 - x) = 0 \) by (1.20); cf. [32, pp. 145–147], [10, pp. 65–84], [36], etc.

Both of these reduce to the Riemann zeta-function for \( x \in \mathbb{Z} \) for the former and \( x = 1 \) for the latter:

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{1.9}
\]

valid for \( \sigma > 1 \) in the first instance. This is continued meromorphically over the whole plane with a simple pole at \( s = 1 \) by way of the functional equation

\[
\pi^{-s/2} \Gamma\left( \frac{s}{2} \right) \zeta(s) = \pi^{-(1-s)/2} \Gamma\left( \frac{1-s}{2} \right) \zeta(1-s), \tag{1.10}
\]

which is a special case and indeed is equivalent to (1.8). Most of the known zeta-functions satisfy a functional equation of this kind, and relations equivalent to it, the modular relations, have been developed in [31].

Let

\[
B_{\chi}(x) = \sum_{a=0}^{\chi} \binom{\chi}{a} B_a x^{\chi-a}
\]

be the \( \chi \)th Bernoulli polynomial of degree \( \chi \) with \( B_0(x) = 1 \) and let \([x]\) be the integer part of \( x \).
For \( \kappa \in \mathbb{N} \), the \( \kappa \)th periodic Bernoulli polynomial \( \bar{B}_\kappa(x) \) is defined by

\[
\bar{B}_\kappa(x) = B_\kappa([x]) = \sum_{a=0}^{\kappa} \binom{\kappa}{a} B_a \{x\}^{\kappa-a},
\]

(1.11)

where \( \{x\} = x - [x] \) indicates the fractional part.

The following is well known and taken as a heaven-sent fact that it has the Fourier expansion (cf. [71])

\[
\bar{B}_\kappa(x) = -\frac{\kappa!}{(2\pi i)^\kappa} \sum_{n=-\infty}^{\infty} e^{2\pi i n x} n^{\kappa-1},
\]

(1.12)

where the prime on the summation sign means that \( n=0 \) is excluded and the summation is taken in symmetric sense.

In [10, (3.7), pp. 48–50] there is a list of expressions for the Hurwitz and reverse Hurwitz formulas. One of them reads

\[
\zeta(s, x) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}} \sum_{n=1}^{\infty} \frac{\sin(\pi n x + (\pi s)/2)}{n^{1-s}},
\]

(1.13)

which in particular gives the Fourier series for \( \zeta(0, x) = -\bar{B}_1(x) \); more generally, cf. [10, (1.1)]. Hence (1.12) may be viewed as a consequence of the modular relation (1.13).

The periodic Bernoulli polynomial has its counterpart connected by the special case \((0 < x < 1)\) of the Lerch zeta-function (1.4):

\[
\ell_\kappa(x) = \frac{(2\pi i)^{\kappa-1}}{\kappa!} (A_\kappa(x) - \pi i \bar{B}_\kappa(x)),
\]

(1.14)

where \( \bar{B}_\kappa(x) \) is as in (1.11) and \( A_\kappa(x) \) is the counterpart, the \( \kappa \)th Clausen function. The first Clausen function is given by (1.23) and higher-order ones are obtained by integration:

\[
A_\kappa(x) = \frac{\kappa!}{2(2\pi i)^{\kappa-1}} \sum_{n=1}^{\infty} \frac{e^{2\pi i n x} + (-1)^{\kappa-1} e^{-2\pi i n x}}{n^{\kappa}},
\]

\[
= \frac{\kappa!}{2(2\pi i)^{\kappa-1}} \sum_{n=-\infty}^{\infty} \frac{\text{sgn}(n)e^{2\pi i n x}}{n^{\kappa}},
\]

(1.15)

where the sign function \( \text{sgn}(n) \) is 1 for \( n > 0 \), -1 for \( n < 0 \) and 0 for \( n = 0 \).

However, using the analytic function \( \ell_\kappa(x) \), the situation is much more transparent. Indeed, from (1.14) we have natural expressions

\[
\bar{B}_\kappa(x) = -\frac{\kappa!}{(2\pi i)^{\kappa}} \left( \ell_\kappa(x) - (-1)^{\kappa-1} \ell_\kappa(1-x) \right)
\]

(1.16)

and

\[
A_\kappa(x) = \frac{\kappa!}{2(2\pi i)^{\kappa-1}} \left( \ell_\kappa(x) + (-1)^{\kappa-1} \ell_\kappa(1-x) \right)
\]

(1.17)

which are in conformity with the functional equations (parity relation)

\[
A_\kappa(x) = (-1)^{\kappa-1} A_\kappa(1-x), \quad \bar{B}_\kappa(x) = (-1)^{\kappa} \bar{B}_\kappa(1-x).
\]

(1.18)
Since we have the formula
\[ \zeta(-l, x) = -\frac{1}{l+1} \tilde{B}_{l+1}(x), \]  
(1.19)
the parity relation (1.18) entails
\[ \zeta(-l, x) + (-1)^l \zeta(-l, 1-x) = 0, \quad \zeta(0, x) + \zeta(0, 1-x) = 0. \]  
(1.20)
This is a generalization of trivial zeros of the Riemann zeta-function.

Our novel viewpoint is that it is the very definition (1.4), as the boundary function of the 'q-expansion', of the Lerch zeta-function that gives the Fourier expansion (1.12) and (1.15) when substituted in (1.16) and (1.17) and that, for finding radial limits, use of the Lerch zeta-function is in the very nature of things since it does is the limit function.

For the latter, as has been noticed \[9\], there are many instances of radial limits in which the odd part, the first periodic Bernoulli polynomial, of the polylogarithm function of order one appears as a result of eliminating the real part, the log sin function; cf. e.g. \[65\] and \[66\]. The polylogarithm function of order one is indeed the monologarithm function, the ordinary logarithm function extended to the circle of convergence \(|z| = 1, z \neq 1\),
\[ -\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n} = L_1(z). \]
(1.21)
The series is absolutely convergent for \(|z| < 1\) and uniformly convergent for \(|z| = 1, z \neq 1\).

We assemble the identities for \(\ell_1(s)\) of which use has been made:
\[ \sum_{n=1}^{\infty} \frac{\cos(2\pi nx)}{n} + i \sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{n} = \ell_1(x) = -\log(1 - e^{2\pi ix}) \]
\[ = \sum_{n=1}^{\infty} \frac{e^{2\pi inx}}{n} = A_1(x) - \pi i \tilde{B}_1(x), \]  
(1.22)
\(0 < x < 1\), where
\[ A_1(x) = -\log 2i|\sin \pi x| = \sum_{n=1}^{\infty} \frac{\cos(2\pi nx)}{n} \]  
(1.23)
is its real part, the first Clausen function (or the log gamma function) and the imaginary part is (1.16) with \(x = 1\), which reads
\[ x - [x] - \frac{1}{2} = \tilde{B}_1(x) = -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{n} \]  
(1.24)
for \(x \notin \mathbb{Z}\), where \([x]\) indicates the greatest integer function.

It was Walfisz \[64, 63\] who first pointed out that the Fourier series (1.24) is a consequence of the functional equation for the Riemann zeta-function.

As has been announced, we give a brief description of the situation surrounding the Lerch zeta-function.

In Deninger \[16\], cf. Meyer \[46, pp. 526–557\], it has been shown that the best and most proper approach to the determination of Laurent coefficients of a zeta-function is to use the reciprocal of the Hurwitz formula, i.e. to appeal to the functional equation (1.8).
expanding the analytic function on the right-hand side into power series around \( s = 0 \), which involves higher derivatives of the Hurwitz zeta-function. We call this the Deninger–Meyer method. The Hurwitz zeta-function and its derivatives can be characterized as a principal solution of a difference equation. This establishes the closed formula for the derivative of the Dirichlet \( L \)-function \( L(s, \chi) \) at \( s = 1 \) leading to the generalized Lerch–Chowla–Selberg formula for a real quadratic field beyond that of an imaginary quadratic field. Thus, this gives the evaluation of \( L^{(k)}(1, \chi) \) for \( k = 0, 1 \). Higher-order derivatives are computed in [28] and then more extensively in [8]; cf. [10, pp. 177–191]. This procedure is reminiscent of Stark’s method of evaluating the value at \( s = 0 \) rather than at \( s = 1 \) of \( L \)-functions, [12], [56], [57], etc.

Milnor [47] gives a very clear description of the two-dimensional vector space of Kubert functions and, above all things, elucidates the functional equation for the Hurwitz and the Lerch zeta-functions as the relation between basis elements.

Yamamoto [71] established the vector space structure of periodic arithmetic functions or the corresponding Dirichlet series, and evaluated the special values \( L(k, \chi) \) by providing sound basics for the Lerch zeta-function. The main results are the expressions of short character sums in terms of these special values; cf. Section 2 below. The method of discrete Fourier transform (DFT) (or finite Fourier series) has also been developed therein; cf. Section 4 below.

As a continuation of these, the second author [28] applied the Deninger–Meyer method to evaluate the three types of quantities \( L(k, \chi) \), \( L^{(k)}(1, \chi) \) and the generalized Euler constants \( \gamma_k(a, M) \); cf. Section 4 below.

2. Short character sums à la Yamamoto

Short character sums have been the objective of an enormous amount of research. Among others, Lerch made an important contribution to short character sums. In [38, p. 400] he introduced the short character sum (III) \( S(x, D) = \sum_{a=1}^{x} (D/a) - (D/x) \Delta \) and obtained some results on it, where \( D \) is the fundamental discriminant positive or negative, \( \Delta = |D| \) and \( 0 \leq x (< 1) \). The main aim is the class number expression in terms of the Dirichlet series for \( L(1, \chi) \) with various weights, and notably \( \arctan x \). Lerch [39, pp. 381–384] contains character sums with log sine weight.

Then there are works of [27] which was generalized by [3], cf. also [2], [4], [5] and others, most of which are concerned with character sums with power weight. In a similar spirit, Akiyama studied the divisibility of the class number of quadratic fields [1]. There are statements about this in Dickson [17, Vol. 3]. Some of the literature on divisibility are [67] and [68]. We shall return to this problem elsewhere.

Yamamoto [71] unified all the existing results on short character sums including those with log sine weight by his theory of Dirichlet series with periodic coefficients and Fourier expansion, terminating further research. In [45, pp. 137–148] as well as [10, Section 7.3, pp. 151–156] there are descriptions of Yamamoto’s theory using Fourier series. The results are stated for imprimitive characters, too, although, in Yamamoto, characters are restricted to primitive ones.

We fix \( M \in \mathbb{N} \) once and for all and let \( C(M) \) denote the space of all periodic arithmetic functions with period \( M \), where an arithmetic function is one defined only for integer
arguments:

\[ f : \mathbb{Z} \to \mathbb{C}, \quad f(n + M) = f(n). \]  

(2.1)

Then \( C(M) \) forms a vector space of dimension \( M \) with inner product

\[ (f, g) = \sum_{a=1}^{M} f(a) \overline{g}(a). \]  

(2.2)

In considering periodic functions of period \( M \) we may choose any interval of length \( M \), e.g. \( 0 \leq a \leq M - 1 \). As in [71], let \( D(s, f) \) denote the Dirichlet series with the coefficients \( f(n) \in C(M) \),

\[ D(s, f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}, \]  

(2.3)

absolutely convergent in some half-plane. These functions have been studied by [22], [25], [52], [54] et al. The space \( D(M) \) of all periodic Dirichlet series \( D(s, f) \) of period \( M \) forms a vector space of dimension \( M \) canonically isomorphic to \( C(M) \). In [10, Section 4.3, pp. 72–76] there is a description of the vector space structure with emphasis on the generalized Eisenstein formula as a relation between bases. One of the most important bases is the Lerch zeta-functions \( \ell_s(x) \), and Yamamoto made use of the real and imaginary parts of \( \ell_s(x) \) with \( s \) a positive integer. For more details on \( D(M) \) we refer to [10, Chs 3, 4, pp. 47–84] and Section 4 below.

Independently of Yamamoto [71], Szmidt, Urbanowicz and Zagier [58] found another method for treating short character sums with power weight, which is expounded in [45, pp. 137–148]. The main result of [58] is [45, Theorem 5.7, p. 140] expressing the short sum \( S_k(\chi) \) in (2.22) in terms of generalized Bernoulli numbers. Since this is a form of an associated Lambert series, we are convinced that the Szmidt–Urbanowicz–Zagier theorem is a consequence of the functional equation. This stimulated further research on short character sums and [30], [51], [33], [29], etc. have been published, the last two being concerned with congruences.

A natural question arises asking for a relation between Szmidt–Urbanowicz–Zagier and Yamamoto’s results. Here we can give an answer that Yamamoto’s results, in the long run, are also consequences of the functional equation. The argument hinges on (2.7), which contains the ‘Fourier series’ for the periodic Bernoulli polynomials arising from the functional equation. This is given in [10, Proposition 7.5, p. 165], generalizing Walfisz’s result, cf. the remark immediately after (1.24). Our work [10, Sections 7.8 and 7.9, pp. 168–175] gives the Fourier series for the Clausen functions through the Kummer Fourier series for the log gamma function (cf. the end of this paper). We treat the character sum with the Clausen function weight (3.5) as well as the chopping power weight (2.22) as a \( q \)-expansion of the Lerch zeta-function.

Yamamoto defined the chopping weight function \((0 \leq k \in \mathbb{Z})\)

\[ w(x) = w_k(x) = \begin{cases} x^k, & 0 \leq x < \alpha, \\ 0, & \alpha \leq x < 1, \end{cases} \]  

(2.4)

where we may assume

\[ 0 < \alpha \leq 1. \]  

(2.5)
He obtained explicit expressions for short character sums by way of the Fourier series for the periodic extension \( \tilde{w} \) of \( w \). We use the same symbol \( w \) for the periodic extension. We shall show that the chopping weight function can be expressed in terms of periodic Bernoulli polynomials, which have expressions (1.16) in terms of \( \ell_{\frac{x}{1}}(x) \), and thence that using the \( q \)-series for \( \ell_{\frac{x}{1}}(x) \) leads very naturally to the Fourier series for \( w \), whereby we confirm the ubiquity of the Lerch zeta-function.

**Theorem 1.** We have the expression

\[
w(x) = \frac{1}{k+1} \left( \bar{B}_{k+1}(x) - \bar{B}_{k+1}(x - \alpha) \right) + \frac{1}{k+1} \sum_{\chi=1}^{k} \left( \frac{k+1}{\chi} \right) \bar{B}_\chi(x - \alpha) \alpha^{k+1-\chi}
\]

or

\[
w(x) = \frac{k!}{(2\pi i)^{k+1}} \sum_{\chi=1}^{k} \left( \frac{k+1}{\chi} \right) \frac{x!}{(2\pi i)^{x}} \left( \ell_{\chi}(x - \alpha) - (-1)^{x-1} \ell_{\chi}(1 - x + \alpha) \right) \alpha^{k+1-\chi}
\]

\[
+ \frac{\alpha^{k+1}}{k+1}.
\]

**Corollary 1.** We have the Fourier expansion

\[
w(x) = \sum_{n=0}^{\infty} a_n(x),
\]

where

\[
a_0 = \frac{\alpha^{k+1}}{k+1},
\]

\[
a_n = a_n(x) = -\sum_{\chi=1}^{k} \frac{k! \alpha^{k+1-\chi}}{(2\pi i)^{x}(k - \chi + 1)!} \frac{e^{2\pi i n(x - \alpha)} - (-1)^{x-1} e^{-2\pi i (x - \alpha)}}{n^\chi}
\]

\[
+ \frac{k! e^{2\pi i n x} - (-1)^{k} e^{-2\pi i nx} - (e^{2\pi i n(x-\alpha)} - (-1)^{k} e^{-2\pi i (x-\alpha)})}{n^{k+1}}
\]

for \( n \geq 1 \).

**Proof.** The proof follows by substituting the Fourier series (1.4) for \( \ell_{\chi}(x) \).

**Remark 1.** Yamamoto has as a consequence of the ordinary Fourier expansion theorem

\[
w(x) = \sum_{n=-\infty}^{\infty} f_n e^{2\pi i n x},
\]
where \( f_0 = \alpha^{k+1}/(k+1) \), the sum is taken in symmetric sense and for \( n \neq 0 \)

\[
f_n = -\sum_{\kappa=1}^k \frac{k!\alpha^{k-\kappa+1}}{(2\pi i)^\kappa(k-\kappa+1)!} \frac{e^{-2\pi i n \alpha}}{n^\kappa} + \frac{k!}{(2\pi i)^{k+1}} \frac{1 - e^{-2\pi i n \alpha}}{n^{k+1}}. \tag{2.11}
\]

From our reasoning, the sum (2.10) extending over \(-\infty\) to \(\infty\) is just an abbreviation of the difference of two Fourier series of the form

\[
\ell_\alpha(x) - (-1)^{\kappa-1} \ell_\alpha(1-x) = \sum_{n=1}^{\infty} \frac{e^{2\pi i n x}}{n^\kappa} + \sum_{n=1}^{\infty} \frac{e^{-2\pi i n x}}{(-n)^\kappa}, \tag{2.12}
\]

and there is no need to take it in symmetric sense. Indeed, in the final result of Yamamoto, use is made of Corollary 1 rather than the Fourier series (2.10).

**Lemma 1.** For \( 0 < \alpha \leq 1 \)

\[
\tilde{B}_{k+1}(x) - \tilde{B}_{k+1}(x - \alpha) = \sum_{r=0}^{k} \binom{k+1}{r} \tilde{B}_r(x - \alpha) \alpha^{k+1-r}
+ \begin{cases} 
-(k+1)[x]^k, & 0 \leq \{x\} \leq \alpha, \\
0, & \alpha < \{x\}.
\end{cases} \tag{2.13}
\]

**Proof.** Writing

\[
B_{k+1}(x) = B_{k+1}(x - \alpha + \alpha)
\]

and applying the Addition Theorem, we obtain

\[
B_{k+1}(x) = \sum_{r=0}^{k+1} \binom{k+1}{r} B_r(x - \alpha) \alpha^{k+1-r},
\]

whence, in particular,

\[
\tilde{B}_{k+1}(x) = B_{k+1}([x]) = \sum_{r=0}^{k+1} \binom{k+1}{r} B_r([x] - \alpha) \alpha^{k+1-r}. \tag{2.14}
\]

We label the two cases in (2.4) as follows.

**Case 1.** \( 0 < \{x\} < \alpha \).

**Case 2.** \( \alpha \leq \{x\} < 1 \).

We note that if \( m \in \mathbb{Z} \) and \( 0 \leq \beta < 1 \), then

\[
[m + \beta] = m. \tag{2.15}
\]

Hence in view of \( [x - \alpha] = [[x] + \{x\} - \alpha] \) we have

\[
[x - \alpha] = \begin{cases} 
[x] - 1, & \text{Case 1}, \\
[x], & \text{Case 2}.
\end{cases} \tag{2.16}
\]

For Case 1, \( [x - \alpha] = [[x] + \{x\} - \alpha] = ([x] - 1 + \{x\} - \alpha) \), and we have Hence it follows that for Case 1 \( \{x\} - \alpha = x - [x] - \alpha = x - \alpha - ([x - \alpha] + 1) = \{x - \alpha\} - 1 \); and for Case 2 \( \{x\} - \alpha = \{x - \alpha\} \).
Hence for \( r \geq 1 \)
\[
B_r([x] - \alpha) = \begin{cases} 
B_r([x - \alpha] - 1), & \text{Case 1,} \\
B_r([x - \alpha]) = \bar{B}_r(x - \alpha), & \text{Case 2.} 
\end{cases}
\] (2.17)

Applying the formula
\[
B_r(y) - B_r(y - 1) = r(y - 1)^{r-1},
\] (2.18)

(2.17) in Case 1 becomes
\[
B_r([x - \alpha]) - r([x - \alpha] - 1)^{r-1} = \bar{B}_r(x - \alpha) - r([x] - \alpha)^{r-1},
\]
so that (2.17) amounts to
\[
B_r([x] - \alpha) = \bar{B}_r(x - \alpha) + \begin{cases} 
r([x] - \alpha)^{r-1}, & \text{for Case 1,} \\
0, & \text{for Case 2.} 
\end{cases}
\] (2.19)

Substituting (2.19) in (2.14), we conclude that
\[
\bar{B}_{k+1}(x) - \bar{B}_{k+1}(x - \alpha) = \sum_{r=0}^{k} \binom{k+1}{r} \bar{B}_r(x - \alpha)\alpha^{k+1-r} + S,
\] (2.20)

where
\[
S = \begin{cases} 
- \sum_{r=1}^{k+1} \binom{k+1}{r} r([x] - \alpha)^{r-1} \alpha^{k+1-r}, & \text{for Case 1,} \\
0, & \text{for Case 2.} 
\end{cases}
\] (2.21)

In Case 2, the equality in the lemma holds true by (2.20).

It remains to compute \( S \). We transform it as
\[
S = -(k + 1) \sum_{r=1}^{k+1} \binom{k}{r-1} ([x] - \alpha)^{r-1} \alpha^{k-(r-1)}
= -(k + 1) \sum_{l=0}^{k} \binom{k}{l} ([x] - \alpha)^{l} \alpha^{k-l}
= -(k + 1)([x] - \alpha + \alpha)^{k} = -(k + 1)[x]^{k}.
\]

Hence in Case 1, this and (2.20) prove the equality in the lemma. \( \square \)

Let \( k \geq 0, M \in \mathbb{N} \) be fixed integers and \( \alpha = r/u, u, r \in \mathbb{Z}, 0 < r \leq u \). Let \( \chi \) denote a primitive Dirichlet character modulo \( M \) and define the (short) character sum
\[
S_\alpha = S^k_\alpha(\chi) = \frac{1}{M^k} \sum_{1 \leq \alpha \leq M} \chi(a) \alpha^k = \sum_{1 \leq \alpha \leq \alpha M} \chi(a) w \left( \frac{a}{M} \right),
\] (2.22)

which differs slightly from Yamamoto’s definition in which for \( a = 0 \) or \( a = \alpha M \), the corresponding term is to be halved. But we may exclude the case \( a = 0 \) since \( \chi(0) = 0 \). Hence the only difference is \( \frac{1}{2} \chi(\alpha M) w(\alpha) \) which is assumed to be zero.

We use similar notation as in Yamamoto:
\[
\eta = e^{2\pi i / u}, \quad b_\alpha(n) = (-1)^{\alpha+1} \chi(-1) \eta^\alpha - \eta^{-\alpha} \quad (1 \leq \alpha \leq k),
\]
\[
b_{k+1}(n) = (-1)^{k+1} \chi(-1)(1 - \eta^{nr}) + 1 - \eta^{-nr}.
\] (2.23)
THEOREM 2. We have the closed form for $S_\alpha = S^k_{r/u}$:

$$S_\alpha = k! \tau(\chi) \sum_{x=1}^{k} \frac{\chi^k - \chi + 1}{(2\pi i)^k (k - \alpha + 1)!} \sum_{n=1}^{\infty} \frac{b_n(n) \bar{\chi}(n)}{n^\alpha} + \frac{k! \tau(\chi)}{2\pi i} \sum_{n=1}^{\infty} \frac{b_{k+1}(n) \bar{\chi}(n)}{n^{k+1}},$$

(2.24)

where $\tau(\chi)$ means the normalized Gauss sum

$$\tau(\chi) = \sum_{a=1}^{M} \bar{\chi}(a) \varepsilon_a(1),$$

(2.25)

where $\varepsilon_a(1) = e^{2\pi i a/M}$ is an additive character defined in (4.1).

The following examples are taken partially from [45, Remark 5.3, p. 146] with additions which are not stated in Yamamoto.

Example 2.1. Case $k = 0$. From Yamamoto [71, Example 5.1]

$$S_\alpha = S^0_\alpha = \tau(\chi) \frac{\pi}{2\pi i} \sum_{n=1}^{\infty} \frac{b_1(n) \bar{\chi}(n)}{n},$$

(2.26)

and

$$b_1(n) = -\chi(-1)(1 - \eta^{nr}) + 1 - \eta^{nr} = \begin{cases} \eta^{nr} - \eta^{-nr}, & \chi(-1) = 1, \\ 2 - \eta^{nr} - \eta^{-nr}, & \chi(-1) = -1. \end{cases}$$

Hence for $r/u = \alpha = 1/1$, we have $r = u$ and $\eta' = 1$. Hence we cover the orthogonality of characters

$$S^0_{1/1} = \sum_{a=1}^{M} \chi(a) = 0.$$ 

Similarly, for $\alpha = \frac{1}{4}$ and $\chi$ even, we have $b_1(n) = 2i\chi_4(n)$, where $\chi_4$ is the unique primitive character modulo 4 so that

$$S^0_{1/4} = \sum_{a=1}^{[M/4]} \chi(a) = \frac{\pi}{2\pi i} L(1, \chi_4 \bar{\chi}),$$

(2.27)

which leads to (2.31).

In the case of $\chi$ odd, we need a formula for an alternating sum,

$$\sum_{n} (-1)^n a_n = \sum_{2|n} a_n - \sum_{2|n} a_n = 2 \sum_{2|n} a_n - \sum_{n} a_n.$$  

(2.28)

Since

$$b_1(n) = 2 - 2 \begin{cases} (-1)^{n/2}, & n \equiv 0 \mod 2, \\ 0, & n \equiv 1 \mod 2, \end{cases}$$

it follows that

$$S^0_{1/4} = \frac{\pi}{2\pi i} \left( 2L(1, \bar{\chi}) - \bar{\chi}(2) \sum_{n=1}^{\infty} \frac{(-1)^n \bar{\chi}(n)}{n} \right).$$
By (2.28), the last sum is \((\bar{\chi}(2) - 1)L(1, \bar{\chi})\), whence

\[
S_{\frac{1}{4}}^{0} = \frac{\tau(\chi)}{2\pi i} (2 - \bar{\chi}(2) + \bar{\chi}(2))L(1, \bar{\chi}).
\] (2.29)

The 1/4th sum in the following form

\[
h(-4d) = 2S_{\frac{1}{4}}^{0}(\chi_d)
\] (2.30)
is due to Dirichlet [17, Vol. 3, (5), (ii), p. 101], cf. [4, Theorem 3.7] whose special case with \(p \equiv 1 \mod 4\) has been essentially used in Chowla [13, p. 58] and also in [72]. The latter authors used it to deduce from a form of the class number formula (2.34) their main ingredient being in the following disguised form:

\[
S_{\frac{1}{4}}^{0}(\chi \cdot p) = \frac{\tau(\chi \cdot p)}{2\pi} L(1, \chi p).
\] (2.31)

**Example 2.2.** Case \(k = 1\). From Yamamoto [71, Example 5.1]

\[
S_{\alpha} = S_{\alpha}^{1} = \frac{\alpha \tau(\chi)}{2\pi i} \sum_{n=1}^{\infty} \frac{b_{1}(n) \bar{\chi}(n)}{n} + \frac{\tau(\chi)}{(2\pi i)^{2}} \sum_{n=1}^{\infty} \frac{b_{2}(n) \bar{\chi}(n)}{n^{2}},
\] (2.32)

and

\[
b_{1}(n) = \chi(-1)\eta^{nr} - \eta^{-nr},
b_{2}(n) = \chi(-1)(1 - \eta^{nr}) + 1 - \eta^{-nr} = \begin{cases} 
\eta^{nr} - \eta^{-nr}, & \chi(-1) = -1, \\
2 - \eta^{nr} - \eta^{-nr}, & \chi(-1) = 1.
\end{cases}
\]

Hence for \(\alpha = 1/1\), we have

\[
b_{1}(n) = \begin{cases} 
-2, & \chi(-1) = -1, \\
0, & \chi(-1) = 1,
\end{cases}
b_{2}(n) = 0.
\]

Hence

\[
S_{1/1}^{1} = \sum_{a=1}^{M} \chi(a)a = \begin{cases} 
-(\tau(\chi)/\pi i)L(1, \bar{\chi}), & \chi(-1) = -1, \\
0, & \chi(-1) = 1,
\end{cases}
\]

which gives (2.35) below.

The Dirichlet class number formula for an imaginary quadratic field in finite form reads (cf. [15, p. 53])

\[
h(d) = \frac{1}{|d|} S_{1/1}^{1}(\chi_d),
\] (2.33)
in view of

\[
L(1, \chi_d) = -\frac{\pi}{|d|^{3/2}} S_{1/1}^{1}(\chi_d),
\] (2.34)

whose general form for an odd primitive character \(\chi \mod q\) is

\[
L(1, \chi) = -\frac{\pi i}{\tau(\chi)q} S_{1/1}^{1}(\chi).
\] (2.35)

It follows that

\[
S_{1/1}^{1}(\chi) = \sum_{a=1}^{M} \chi(a)a \neq 0
\] (2.36)
in view of \(L(1, \chi) \neq 0\) for a non-principal primitive character \(\chi\). See [59, pp. 171–172] for remarks on the Prime Number Theorem (PNT) for an arithmetic progression and the non-vanishing of \(L(1, \chi_i)\). It is also to be remarked that in [26, p. 14] Iwasawa states that there is no elementary proof of (2.36). Indeed, this is one of many equivalent formulations of Chowla’s problem which asks for an elementary proof of the non-vanishing of \(L(1, f)\); cf. e.g. [14]. There is an enormous number of papers devoted to the inverse Chowla problem to the effect that certain sets of functions at roots of unity are to be proved linearly independent from the non-vanishing of \(L(1, \chi)\); cf. e.g. [21]. Combining

\[(1 - 2\chi(2))S_{1/1,1} = \chi(2)NS_{1/2,0}\]  

(2.37)

and (2.34) gives rise to

\[S_{1/2}(\chi_d) = (2 - \chi_d(2))h(d),\]  

(2.38)

which is [4, Corollary 3.4].

3. Short character sums with Clausen function weight

Yamamoto also considers the short character sums with Fourier series that is conjugate to (2.4). Instead of considering the conjugate Fourier series, we define the weight function verbatim to Theorem 1 as follows.

**Definition 1.** We define

\[
\tilde{w}(x) = \tilde{w}_k(x) = -\frac{1}{k+1}(A_{k+1}(x) - A_{k+1}(x - \alpha)) + \frac{1}{k+1} \sum_{\chi=1}^{k} \left( \frac{k+1}{\chi} \right) A_{\chi}(x - \alpha)\alpha^{k+1-x}
\]  

(3.1)

or

\[
\tilde{w}(x) = -\pi i \frac{k!}{(2\pi i)^{k+1}} 
\times \left( \ell_{k+1}(x) + (-1)^k \ell_{k+1}(1-x) - (\ell_{k+1}(x - \alpha) + (-1)^k \ell_{k+1}(1-x - \alpha)) \right) 
+ \pi i \sum_{\chi=1}^{k} \frac{k!}{(2\pi i)^{k}(k+1-\chi)!} \left( \ell_{\chi}(x - \alpha) + (-1)^{x-1} \ell_{\chi}(1-x - \alpha) \right)\alpha^{k+1-x}.
\]  

(3.2)

We have the following similarly to Corollary 1.

**Corollary 2.** We have the Fourier expansion

\[
\tilde{w}(x) = \sum_{n=1}^{\infty} \tilde{a}_n(x),
\]  

(3.3)
where
\[ \tilde{a}_n = a_n(x) \]
\[ = \pi i \sum_{\kappa=1}^{k} \frac{k! \lambda^{k+1-\kappa}}{(2\pi i)^{\kappa}(k + 1 - \kappa)!} e^{2\pi in(x-\alpha)} - (-1)^{\kappa} e^{-2\pi in(x-\alpha)} \]
\[ - \pi i \frac{k!}{(2\pi i)^{k+1}} e^{2\pi inx} - (-1)^{k+1} e^{-2\pi inx} - (e^{2\pi in(x-\alpha)}(-1)^{k+1} e^{-2\pi in(x-\alpha)}) \]
\[ \frac{n^{k+1}}{n^{k+1}}. \]  
(3.4)

Remark 2. Here \( \tilde{a}_n \) is different from Yamamoto’s coefficient by \( \pi i : \tilde{a}_n = \pi ia_n \).

Let \( \chi \) denote a primitive Dirichlet character modulo \( M \) and define the (short) character sum
\[ T_\alpha = T_\alpha^k(\chi) = \sum_{1 \leq a < \chi M} \chi(a) \tilde{w}\left(\frac{a}{M}\right). \]  
(3.5)

**Theorem 3.** We have the closed form for \( T_\alpha = T_{\alpha/u}^k \):
\[ \frac{1}{\pi i} T_\alpha = k! \tau(\chi) \sum_{\kappa=1}^{k} \frac{\alpha^{k+1-\kappa}}{(2\pi i)^{\kappa}(k + 1 - \kappa)!} \sum_{n=1}^{\infty} \tilde{b}_\chi(n) \tilde{\chi}(n) \frac{n^{k+1}}{n^{k+1}} \]
\[ - \frac{k!}{(2\pi i)^{k+1}} \sum_{n=1}^{\infty} \tilde{b}_{k+1}(n) \tilde{\chi}(n) \frac{n^{k+1}}{n^{k+1}}, \]  
(3.6)

where similarly to (2.23) we have
\[ \eta = e^{2\pi i/a}, \quad \tilde{b}_\chi(n) = (-1)^{\kappa} \chi(-1) n^{\eta} - \eta^{nr} \quad (1 \leq \kappa \leq k), \]
\[ \tilde{b}_{k+1}(n) = (-1)^{k} \chi(-1)(1-\eta^{nr}) + 1 - \eta^{nr}. \]  
(3.7)

**Example 3.1.** The \( k = 0 \) case reads as in Yamamoto [71] as follows:
\[ \tilde{w}^0(x) = -(A_1(x) - A_1(x-\alpha)), \]
\[ T_0^\alpha = -\sum_{a=1}^{M} \chi(a) \left( A_1\left(\frac{a}{M}\right) - A_1\left(\frac{a}{M} - \alpha\right) \right) = U_0 - U_0^\alpha, \]
say, where
\[ U_0^\kappa = -\sum_{a=1}^{M} \chi(a) A_1\left(\frac{a}{M} - \alpha\right), \quad U_0 = U_0^\alpha. \]

Moreover,
\[ U_0 = -\sum_{a=1}^{M} \chi(a) A_1\left(\frac{a}{M}\right) = \begin{cases} \tau(\chi)L(1, \chi), & \chi(-1) = 1, \\ 0, & \chi(-1) = -1, \end{cases} \]  
(3.8)

so that
\[ U_0^\alpha = T_0^\alpha + \begin{cases} \tau(\chi)L(1, \chi), & \chi(-1) = 1, \\ 0, & \chi(-1) = -1. \end{cases} \]  
(3.9)

This is the sum considered by Lerch.
4. Discrete Fourier transform and generalized Euler constants

The theory of the discrete Fourier transform (DFT) is stated in much of the literature [20, pp. 101–164], [50], [69, pp. 89–109]. For Fourier analysis on finite groups, cf. [60]. The theory of DFT for arithmetic functions has been developed in [45] in the case of periodic functions and in [44, pp. 109–114] in the case of a finite group; See also [32, Section 8.1], [10, Sections 4.1 and 4.3]. Let

\[ \varepsilon_j(a) = e^{2\pi i ja/M}, \quad 1 \leq j \leq M, \quad (4.1) \]

where \( a \) is an integer variable. Then the set \( \{ \varepsilon_j(a) \mid 1 \leq j \leq M \} \) forms a basis of the vector space \( C(M) \) defined in Section 2. We define the DFT \( \hat{f} \) (or the \( b \)th Fourier coefficient) of \( f \in C(M) \) by

\[ \hat{f}(b) = \frac{1}{\sqrt{M}} \sum_{a=1}^{M} \varepsilon_b(-a)f(a). \quad (4.2) \]

Then the Fourier inversion or Fourier expansion formula holds true:

\[ f(a) = \frac{1}{\sqrt{M}} \sum_{b=1}^{M} \hat{f}(b)\varepsilon_b(-a) = \hat{\hat{f}}(-a). \quad (4.3) \]

Note that (4.3) is the expression of \( f \) with respect to the basis \( \{ \varepsilon_j \} \).

Since

\[ \sum_{n=1}^{\infty} \frac{|f(n)|}{n^\sigma} \ll \zeta(\sigma), \]

the series in (2.3) is absolutely convergent for \( \sigma > 1 \). Let \( D(M) \) denote the set of all Dirichlet series of the form (2.3). Then it forms a vector space of dimension \( M \) canonically isomorphic to \( C(M) \). One of the bases of \( D(M) \) is \( \{ \ell_s(a/M) \mid 1 \leq a \leq M \} \). Hence we have

\[ D(s, f) = \frac{1}{\sqrt{M}} \sum_{a=1}^{M} \hat{f}(-a)\ell_s\left(\frac{a}{M}\right) = \frac{1}{\sqrt{M}} \sum_{a=1}^{M-1} \hat{f}(a)\ell_s\left(\frac{a}{M}\right) + \hat{f}(M)\sqrt{M} \zeta(s). \quad (4.4) \]

It follows that \( D(s, f) \) can be continued meromorphically over the whole plane and that it is entire if and only if

\[ \hat{f}(M) = \frac{1}{\sqrt{M}} \sum_{a=1}^{M} f(a) = 0. \quad (4.5) \]

Let

\[ \zeta(s) = \frac{1}{s - 1} + \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} (s - 1)^k \quad (4.6) \]

be the Laurent expansion of \( \zeta(s) \) around \( s = 1 \), where \( \gamma = \gamma_0 \) indicates the Euler constant. In (4.4), the singular part is separated and we have the Laurent expansion given by the next theorem.
THEOREM 4. [25, Theorem 2] We have

\[
\frac{\hat{f}(M)/\sqrt{M}}{s-1} + \frac{1}{\sqrt{M}} \sum_{k=0}^{\infty} \frac{\gamma_k(f)}{k!} (s-1)^k
\]

\[
= D(s, f) = \frac{\hat{f}(M)/\sqrt{M}}{s-1} + \frac{1}{\sqrt{M}} \sum_{k=0}^{\infty} \frac{1}{k!} \left( \sum_{a=1}^{M-1} \hat{f}(a)(-1)^k \sigma_k^a + \hat{f}(M)\gamma_k \right) (s-1)^k
\]

(4.7)

where

\[
\sigma_k^a = \sum_{n=1}^{\infty} e^{2\pi i n a/M} \frac{\log n}{n} = (-1)^k \frac{\partial^k}{\partial s^k} \ell_1 \left( \frac{a}{M} \right),
\]

(4.8)

and, a fortiori,

\[
\gamma_k(f) = \sum_{a=1}^{M-1} \hat{f}(a) \frac{\partial^k}{\partial s^k} \ell_1 \left( \frac{a}{M} \right) + \hat{f}(M)\gamma_k,
\]

(4.9)

i.e. \( \ell_1(a/M) \) is a generating function of \( \sigma_k^a \).

Proof. Expanding \( n^{-s} = n^{-1} e^{-(s-1) \log n} \) into the power series around \( s = 1 \),

\[
n^{-s} = \frac{1}{n} \sum_{k=0}^{\infty} \frac{(-\log n)^k}{k!} (s-1)^k,
\]

we transform the Lerch zeta-function (1.4) into

\[
\ell_s(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left( \sum_{n=1}^{\infty} e^{2\pi i n x / n} \log^k n \right) (s-1)^k
\]

\[
= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\partial^k}{\partial s^k} \ell_1(x)(s-1)^k,
\]

(4.10)

whence for \( x = a/M \), (4.8) follows. \( \square \)

Remark 3. We defined the \( k \)th generalized Euler constant \( \gamma_k(f) \) by (4.7) for the sake of simpler expressions (4.8) for them. In the literature, the definition varies with different coefficients.

Funakura [22] deduces as his main theorem the Kronecker limit formula for Dirichlet series with periodic coefficients which is compactified as Corollary 3 below, where there appear both real and imaginary parts of the Lerch zeta-function. His method is rather computational, without mentioning Yamamoto [71]. We refer to [25] for a more general treatment.

COROLLARY 3. [22, Theorem 5] We have

\[
\frac{\gamma_0(f)}{\sqrt{M}} = \lim_{s \to 1} \left( D(s, f) - \frac{\hat{f}(M)/\sqrt{M}}{s-1} \right) = \frac{1}{\sqrt{M}} \sum_{a=1}^{M-1} \hat{f}(a) \ell_1 \left( \frac{a}{M} \right) + \frac{\hat{f}(M)}{\sqrt{M}} \gamma,
\]

(4.11)
which may be expressed as
\[
\gamma_0(f) \sqrt{M} = \frac{1}{\sqrt{M}} \sum_{a=1}^{M-1} \hat{f}(a) A_1 \left( \frac{a}{M} \right) - \frac{\pi i}{\sqrt{M}} \sum_{a=1}^{M-1} \hat{f}(a) B_1 \left( \frac{a}{M} \right) + \frac{\hat{f}(M)}{\sqrt{M}} \gamma
\]  
(4.12)

and as
\[
\gamma_0(f) \sqrt{M} = -\frac{1}{\sqrt{M}} \sum_{a=1}^{M-1} \hat{f}(a) \log 2 \sin \frac{\pi a}{M} + \pi \frac{2}{M} \sum_{a=1}^{M-1} f(a) \cot \frac{\pi a}{M} + \frac{\hat{f}(M)}{\sqrt{M}} \gamma.
\]  
(4.13)

**Proof.** Taking the limit as \( s \to 1 \) in (4.7), we deduce (4.11), which amounts to (4.8) with \( k = 0 \). We deduce (4.12) on substituting from (1.22).

In Funakura’s statement of the above theorem, there is the term
\[
\frac{\log 2}{M} \left( (M - 1) f(M) - \sum_{b=1}^{M-1} f(b) \right),
\]
which is nothing other than the sum \((\log 2)/\sqrt{M}) \sum_{a=1}^{M-1} \hat{f}(a)\) which arises if the log sin integral is used instead of the Clausen function. To transform (4.12) into Funakura’s form (4.13), we appeal to the reverse Eisenstein formula [42]:
\[
\sum_{a=1}^{M-1} \epsilon_b(-a) B_1 \left( \frac{a}{M} \right) = -\frac{1}{2i} \cot \frac{\pi b}{M}.
\]  
(4.14)

The \( k \)th generalized Euler constant \( \gamma_k(a, M) \) is defined in the first instance by
\[
\gamma_k(a, M) = \lim_{x \to \infty} \left( \sum_{n \equiv a \pmod{M} \leq x} \frac{\log^k n}{n} - \frac{\log^{k+1} x}{M(k+1)} \right).
\]

The most proper way of introducing them is through the Laurent expansion of the partial zeta-function,
\[
\zeta(s; a, M) = \sum_{n=1}^{\infty} \frac{1}{n^s} = M^{-s} \zeta(s, \frac{a}{M}),
\]  
(4.15)

as has been thoroughly developed by [53].

**Corollary 4.** [37, (6)] We have
\[
M \gamma_0(a, M) = \sum_{j=1}^{M-1} \epsilon_a(-j) \ell_1 \left( \frac{j}{M} \right) + \gamma.
\]  
(4.16)

**Proof.** As a special case of (4.4), we have a generating function [53, (19)] for \( \gamma_k(a, M) \):
\[
M \zeta(s; a, M) = \sum_{j=1}^{M-1} \epsilon_a(-j) \ell_s \left( \frac{j}{M} \right) + \zeta(s).
\]  
(4.17)

This implies an expression [53, Proposition 4′] in terms of \( \sigma_k^i \) in (4.8),
\[
M \gamma_k(a, M) = \sum_{j=1}^{M-1} \epsilon_a(-j) \sigma_k^j + \gamma_k.
\]  
(4.18)
which is \[28, \text{formula (26)}\]. Substituting (4.8) in (4.18) proves the formula, which is one genesis of the study of generalized Euler constants.

For a proof of (4.17) we may also argue as follows. Take as \(f\) in (4.4) the characteristic function \(\chi_{a \mod M}\):

\[
f(n) = \chi_{a \mod M} = \begin{cases} 1, & n \equiv a \mod M, \\ 0, & n \not\equiv a \mod M. \end{cases} \tag{4.19}
\]

Then \(\sqrt{M} \hat{f}(n) = \sum_{j=1}^{M} \varepsilon_n(-j) \chi_a(j) = \varepsilon_a(-n)\). Hence (4.4) for \(\gamma_0(\chi_a)\) leads to (4.18).

**COROLLARY 5.** [6] Funakura’s expression

\[
\gamma_0(f) = -\frac{1}{M} \sum_{a=1}^{M} f(a) \psi\left(\frac{a}{M}\right) - \frac{\hat{f}(M)}{\sqrt{M}} \log M \tag{4.20}
\]

leads to

\[
M \gamma_0(a, M) = -\psi\left(\frac{a}{M}\right) - \log M, \tag{4.21}
\]

where \(\psi(s) = \Gamma'(s)/\Gamma(s)\) is the Euler digamma function.

**Proof.** For his proof of Corollary 3, Funakura appealed to the representation, similar to (4.15),

\[
D(s, f) = \frac{1}{M^s} \sum_{a=1}^{M} f(a) \xi\left(s, \frac{a}{M}\right), \tag{4.22}
\]

and used the Laurent constant \(-\psi(x)\) for \(\xi(s, x)\) at \(s = 1\), to arrive at (4.20). From this, Corollary 3 follows from Gauss’ formula (4.23). This is stated by Deninger [16] as another method.

**COROLLARY 6.** The Two expressions for \(\gamma_0(a, M)\) in Corollary 4 (Lerch zeta) and Corollary 5 (Hurwitz zeta) lead to Gauss’ formula for the digamma function at rational arguments [37, p. 135]:

\[
\psi\left(\frac{a}{M}\right) = -\gamma - \log \frac{M}{2} - \frac{\pi}{2} \cot \frac{a}{M} + 2 \sum_{0 < j < M/2} \frac{2\pi aj}{M} \log \sin \frac{\pi j}{M}. \tag{4.23}
\]

**Proof.** Comparing (4.21) and (4.16), we find an expression for the Euler digamma function in terms of the polylogarithm function. Substituting the real and imaginary parts (1.23) and (1.24), we deduce (4.23).

This is a prototype of the equivalence theorem in [24] to the effect that the finite expression for the Dirichlet class number formula is equivalent to (4.23).

Theorem 4 shows the ubiquity of \(\ell_1(x)\). However, the main formula (4.9) is in its first stage. For the Deninger–Meyer method involves its second stage of expressing the derivatives of the Lerch zeta-function by those of the Hurwitz zeta-function.

We may compute the higher-order Laurent coefficients of \(D(s, f)\) around \(s = 1\) by (4.4) by applying the Deninger–Meyer method, i.e. by finding the coefficient \((-1)^k (\partial^k / \partial s^k) \ell_1(x)\)
of \( s^k \) of \( \ell_k(1-s) \) in (1.8):

\[
(-1)^k \frac{\partial^k}{\partial s^k} \ell_1(x) = k! \sum_{a+b+c+d=k} \frac{(-\log 2\pi)^a}{a!} \frac{\Gamma(b)(1)(\pi i/2)^c}{b!} \frac{c!d!}{c!d!} \times \{\zeta^{(d)}(0, x) + (-1)^c \zeta^{(d)}(0, 1-x)\},
\]

(4.24)

where \( a, b, c, d \) run through integers \( \geq 0 \) whose sum is \( k + 1 \). In actual calculation, one can omit those terms which correspond to \( d = 0, 2|c \) by (1.20).

The \( R \)-function or higher-order derivatives of the Hurwitz zeta-function is introduced using the Dufresnoy–Pisot theorem, which is a generalization of the Bohr–Mollerop theorem [7], while in [28] Nörlund’s principal solution [48] is used, which is also used in [19] to introduce a generalized gamma function to express the generalized Euler constants. Its logarithmic derivative appears in Ramanujan’s second notebook [5]. Higher-order derivatives of Dirichlet \( L \)-function have been studied in [9] using the Deninger \( R_k \)-function. In a similar setting, Vignéras [62] uses the Dufresnoy–Pisot theorem to introduce the Barnes multiple gamma function (cf. [55, pp. 49–50]).

After [6] and [37] it was Dilcher [18, 19] and Kanemitsu [28] who developed the theory of generalized Euler constants. Most of these results have been summarized and elucidated in [53]. There are further generalizations and we refer to [11] and references given therein. We may use the Deninger–Meyer method to find Laurent coefficients of a more general class of Dirichlet series, including e.g. [23], in terms of principal solutions to a difference equation, which will be conducted elsewhere.

As can be found, e.g. in [34], there are many relations known among bases and Laurent coefficients, typical ones being

\[
\chi(n) = \tau(\bar{\chi})^{-1} \sum_{j=1}^{M-1} \bar{\chi}(j) \varepsilon_j(n),
\]

(4.25)

with the Gauss sum (2.25), and

\[
L^{(k)}(1, \chi) = (-1)^k \sum_{j=1}^{M} \chi(j) \gamma_k(a, M).
\]

(4.26)

Finally we recall the well-known Lerch formula

\[
\zeta'(0, x) = \log \frac{\Gamma(x)}{\sqrt{2\pi}},
\]

(4.27)

from which Kummer’s Fourier series is shown to be equivalent to the functional equation for the Riemann zeta-function [32, p. 108], [10, pp. 168–175]. Corresponding to (4.24), we have

\[
\frac{\partial^k}{\partial s^k} \zeta(0, x) = k! \sum_{a+b+c+d=k} \frac{(-\log 2\pi)^a}{a!} \frac{\Gamma(b)(1)(\pi i/2)^c}{b!} \frac{c!d!}{c!d!} \times i((-1)^c + 1) \ell_1^{(d)}(x) + \ell_1^{(d)}(1-x)).
\]

(4.28)

Hence

\[
\zeta'(0, x) = -(-\log 2\pi + \Gamma'(1))(\ell_1(x) - \ell_1(1-x)) + \frac{\pi i}{2} i(\ell_1(x) + \ell_1(1-x))
\]

\[
- i(\ell_1'(x) - \ell_1'(1-x)).
\]

(4.29)
Substituting (1.16) and (1.17),
\[ \ell'_1(x) = -\sum_{n=1}^{\infty} \frac{\log n}{n} e^{2\pi i n x}, \]
and invoking (4.27), we deduce Kummer’s Fourier series
\[ \log \frac{\Gamma(x)}{\sqrt{2\pi}} = \sum_{n=1}^{\infty} \left( \frac{1}{2n} \cos 2\pi n x + \frac{\gamma + \log 2\pi n}{\pi n} \sin 2\pi n x \right). \]

(4.30)

**Remark 4.** In this paper we stick to the fact that the Lerch zeta-function is a boundary function of the polylogarithm (1.1). However, it is a boundary function of the Hurwitz–Lerch zeta-function [55, p. 121] for \( x \) non-integral, \( s \in \mathbb{C}, |z| < 1 \),
\[ \Phi(z, s, x) = \sum_{n=0}^{\infty} \frac{z^n}{(n+x)^s}. \]
with \( x = 1 \) on the circle of convergence \( |z| = 1 \). Its direct restriction to a boundary function leads to the Hurwitz–Lerch transcendent [55, p. 122]
\[ L(\xi, s, x) = \sum_{n=0}^{\infty} \frac{e^{2\pi in\xi}}{(n+x)^s}. \]

(4.32)

One may continue on this more general aspect. We are indebted to the referee for this remark and thanks are due to scrutinization of an earlier draft of the paper.

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**References**

[1] S. Akiyama. Refinement of the class number formula for quadratic fields. RIMS Kokyuroku 886 (1994), 170–177.

[2] B. C. Berndt. Periodic Bernoulli numbers, summation formulas and applications. Theory and application of Special Functions. Academic Press, New York, 1975, pp. 143–189.

[3] B. C. Berndt. Character analogues of the Poisson and Euler-Maclaurin summation formula with applications. J. Number Theory 7 (1975), 413–445.

[4] B. C. Berndt. Classical theorems on quadratic residues. Enseign. Math. (2) 22 (1976), 261–304.

[5] B. C. Berndt. Ramanujan’s Notebooks, Part I. Springer, New York, 1985.

[6] W. E. Briggs. The irrationality of \( \gamma \) or of sets of similar constants. Norske Vid. Selsk. Forh. 34 (1961), 25–28.

[7] R. Campbell. Les Intégrales Éulérienne et leurs Applications, Étude Approfondie de la Fonction Gamma. Dunod, Paris, 1966.

[8] K. Chakraborty, S. Kanemitsu and T. Kuzumaki. Finite expressions for higher derivatives of the Dirichlet \( L \)-function and the Deninger \( R \)-function. Hardy-Ramanujan J. 32 (2009), 38–53.

[9] K. Chakraborty, S. Kanemitsu and T. Kuzumaki. Seeing the invisible: around generalized Kubert functions. Ann. Univ. Sci. Budapest. Sect. Comput. 47 (2018), 185–195.

[10] K. Chakraborty, S. Kanemitsu and H. Tsukada. Vistas of Special Functions, II. World Scientific, Singapore, 2009.

[11] T. Chatterjee and S. S. Khurana. Shifted Euler constants and a generalization of Euler–Stieltjes constants. J. Number Theory 204 (2019), 185–210.
T. Chinburg. Derivatives of $L$-functions at $s = 0$ (after Stark, Tate, Bienenfeld and Lichtenbaum). Compos. Math. 48 (1983), 119–127.

P. Chowla. On the class number of real quadratic field. J. Reine Angew. Math. 230 (1968), 51–60.

S. Chowla. The nonexistence of nontrivial relations between the roots of a certain irreducible equation. J. Number Theory 2 (1970), 120–123; The Collected Papers of Sarvadaman Chowla, III. CRM, Montreal, 1999, pp. 1182–1185.

H. Davenport. Multiplicative Number Theory, 2nd edn. Springer, Berlin, 1982.

C. Deninger. On the analogue of Chowla–Selberg formula for real quadratic fields. J. Reine Angew. Math. 351 (1984), 171–191.

L. E. Dickson. History of Number Theory. Chelsea, New York, 1952.

K. Dilcher. Generalized Euler constants for arithmetical progressions. Math. Comp. 59 (1992), 259–282; S21–S24.

K. Dilcher. On generalized gamma function related the Laurent coefficients of the Riemann zeta function. Aequationes Math. 48 (1994), 55–85.

M. W. Frazier. An Introduction to Wavelets through Linear Algebra. Springer, New York, 1999.

M. Fujiwara. On linear relations between roots of unity. RIMS Kokyuroku 334 (1978), 71–73.

T. Funakura. On Kronecker’s limit formula for Dirichlet series with periodic coefficients. Acta Arith. 55 (1990), 59–73.

F. Gramain and M. Weber. Computing an arithmetic constant related to the ring of Gaussian integers. Math. Comp. 44 (1985), 241–250; S13–S16.

M. Hashimoto, S. Kanemitsu and M. Toda. On Gauss’ formula for $\psi$ and finite expressions for the $L$-series at 1. J. Math. Soc. Japan 60 (2008), 219–236.

S. Kanemitsu. On evaluation of certain limits in closed form. Number Theory (Proc. Int. Conf. on Number Theory, Université, Laval). Eds J. M. de Koninck and C. Levesque. de Gruyter, Berlin, 1987, pp. 459–474.

S. Kanemitsu, T. Kuzumaki and J. Urbanowicz. On congruences for certain sums of E. Lehmer’s type. Hardy–Ramanujan J. 37 (2015), 1–28.

S. Kanemitsu, H. L. Li and N. L. Wang. Weighted short-interval character sums. Proc. Amer. Math. Soc. 139 (2011), 1521–1532.

S. Kanemitsu and H. Tsukada. Contributions to the Theory of Zeta-Functions: the Modular Relation Supremacy. World Scientific, Singapore, 2014.

S. Kanemitsu and H. Tsukada. Vistas of Special Functions, World Scientific, Singapore, 2017.

S. Kanemitsu, J. Urbanowicz and N. L. Wang. On some new congruences for generalized Bernoulli numbers. Acta Arith. 155 (2012), 247–258.

J. Knopfmacher. Generalized Euler constants. Proc. Edinb. Math. Soc. 21 (1978), 25–32.

J. C. Lagarias and W. C. W. Li. The Lerch zeta function IV. Hecke operators. Res. Math. Sci. 3 (2016), pp. 33–39.

A. Laurinčikas and R. Garunkstis. The Lerch Zeta-Function. Kluwer Academic, Dordrecht, 2002.

D. H. Lehmer. Euler constants for arithmetic progressions. Acta Arith. 27 (1975), 125–142; Selected Papers of D. H. Lehmer, Vol. II. Charles Babbage Research Center, Manitoba, 1981, pp. 591–608.

M. Lerch. Essais sur le calcul de nombre des classes de formes quadratiques binaires aux coefficients entiers. Acta Math. 29 (1905), 333–424.

M. Lerch. Essais sur le calcul de nombre des classes de formes quadratiques binaires aux coefficients entiers. Acta Math. 30 (1906), 203–293.

L. Lewin. Structural Properties of Polylogarithms. American Mathematical Society, Providence, RI, 1980.

F.-H. Li and S. Kanemitsu. Special Functions and Analysis of Differential Equations. CRC Press, Chapman & Hall, Ch. 8. To appear, 2020.

H.-L. Li, M. Hashimoto and S. Kanemitsu. Structural elucidation of Eisensteins formula. Sci. China. 53 (2010), 2341–2350.

W.-B. Li, H.-Y. Li and J. Mehta. Around the Lipschitz summation formula. Math. Probl. Eng. 2020 (2020), Article ID 5762823.

H.-L. Li, F.-H. Li, N.-L. Wang and S. Kanemitsu. Number Theory and Its Applications, II. World Scientific, Singapore, 2017.
[45] F.-H. Li, N. L. Wang and S. Kanemitsu. Number Theory and Its Applications. World Scientific, Singapore, 2013.

[46] C. Meyer. L-Reihen Quadratischer Zahlkörper. Vorlesungen, Köln, 1981–82.

[47] J. Milnor. On polylogarithms, Hurwitz zeta-functions and the Kubert identities. Enseign. Math. (2) 29 (1983), 281–322.

[48] N. Nörlund. Vorlesungen über Differenzenrechnung. Springer, Berlin, 1924.

[49] J. Okuda and K. Ueno. Relations for multiple zeta values and Mellin transform of multiple polylogarithms. Publ. RIMS, Kyoto Univ. 40 (1965), 537–564, 2004.

[50] A. Papoulis. The Fourier Integral and Its Applications, McGraw-Hill, New York, 1962.

[51] A. Schinzel, J. Urbanowicz and P. van Wamelen. Class numbers and short sums of Kronecker symbols. J. Number Theory 78 (1999), 62–84.

[52] W. Schnee. Die Funktionalgleichung der Zeta-funktion und der Dirichletischen Reihen mit periodischen Koeffizienten. Math. Z. 31 (1930), 378–390.

[53] S. Shirasaka. On the Laurent coefficients of a class of Dirichlet series. Results Math. 42 (2002), 128–138.

[54] R. A. Smith. The average order of arithmetical functions over arithmetic progressions with applications to quadratic forms. J. Reine Angew. Math. 317 (1980), 70–88.

[55] H. M. Srivastava and J.-S. Choi. Series Associated with the Zeta and Related Functions. Kluwer Academic, Dordrecht, 2001.

[56] H. M. Stark. L-functions at s = 1. IV. First derivatives at s = 0. Adv. Math. 17 (1975), 60–92.

[57] H. M. Stark. Derivatives of L-series at s = 0. Automorphic Forms, Representation Theory and Arithmetic (Bombay, 1979) (Tata Institute of Fundamental Research Studies in Mathematics, 10). Tata Institute of Fundamental Research, Bombay, 1981, pp. 261–273.

[58] J. Szmidt, J. Urbanowicz and D. Zagier. Congruences among generalized Bernoulli numbers. Acta Arith. 71 (1995), 273–278.

[59] T. Takagi. Algebraic Number Theory. Iwanami-Shoten, Tokyo, 1st edn, 1947; 2nd edn, 1971.

[60] A. Terras. Fourier Analysis on Finite Groups and Applications. Cambridge University Press, Cambridge, 1999.

[61] E. C. Titchmarsh. The Theory of Functions, 2nd edn. Oxford University Press, London, 1939.

[62] M. F. Vignéras. Facteurs gamma et équations fonctionnelles. Int. Summer School on Modular Functions, Bonn, 1976. Modular Functions of One Variable VI (Lecture Notes in Mathematics, 627). Springer, Berlin, 1977, pp. 79–103 (J.-P. Serre, Appendice, Relations entre facteurs gamma, pp. 99–103).

[63] A. Z. Walfisz. Über die summatorischen Funktionen einiger Dirichletschen Reihen. Inaugural Diss., Göttingen, 1923.

[64] A. Z. Walfisz. Über das Piltzsche Teilerproblem in algebraischen Zahlkörpern. Math. Z. 22 (1925), 153–188.

[65] N.-L. Wang. On Riemann’s posthumous fragment II on the limit values of elliptic modular functions. Ramanujan J. 24 (2011), 129–145.

[66] N.-L. Wang. Arithmetical Fourier and limit values of elliptic modular functions. Proc. Math. Sci. 128 (2018), 28.

[67] K. S. Williams. On the class number of $\mathbb{Q}(\sqrt{-p})$ modulo 16, for $p \equiv 1 \pmod{8}$ a prime. Acta Arith. 39 (1981), 381–391.

[68] K. S. Williams and J. D. Currie. Class numbers and biquadratic reciprocity. Canad. J. Math. 39 (1982), 969–988.

[69] S. Winograd. Arithmetic Complexity of Computations. SIAM, Philadelphia, 1980.

[70] A. Wintner. On Riemann’s fragment concerning elliptic modular functions. Amer. J. Math. 63 (1941), 628–634.

[71] Y. Yamamoto. Dirichlet series with periodic coefficients. Algebraic Number Theory (Papers contributed for the Kyoto Int. Symp., 1976). Ed. S. Iyanaga. Japan Society for the Promotion of Science, Tokyo, 1977, pp. 275–289.

[72] W.-P. Zhang and Z.-F. Xu. On a conjecture of the Euler numbers. J. Number Theory 127 (2007), 283–291.
