Rapid reliability control of products in construction

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Abstract. The paper considers the method of rapid control of the reliability of construction products, based on the approximation of the mathematical expectation and variance of the process of changing the technical parameters of products over time by linear functions. When constructing the method of rapid reliability control in the normal mode, we will proceed only from an analysis of the external manifestation of the product wear process, that is, an analysis of the nature of the change in the technical parameters of the products over time. Therefore, we restrict ourselves to considering such methods of preliminary tests, which involve measuring the values of the technical parameters of each product of a sample of volume \( n \) at some points in time \( t \). The order of preliminary studies and rapid control is considered as an example of a special case when the change in the technical parameters is approximated by a random process with a normal distribution density. A criterion is proposed by which at the stage of preliminary tests the question of the applicability of this control method to a specific type of product is solved. Formulas are found for calculating the characteristics of the rapid control plan for construction products for given risks of the supplier and customer.

1. Introduction
The paper discusses the method of rapid reliability control of products in normal mode (i.e., in the mode specified in the technical specifications for this type of product). The construction of such a control method can be divided into the following stages: conducting and planning preliminary tests; preliminary studies; development of recommendations for determining the characteristics of a reliability control plan.

2. Preliminary tests
The purpose of the preliminary tests is to experimentally obtain information about the nature of the wear process.

When constructing the method of rapid reliability control in the normal mode, we will proceed only from an analysis of the external manifestation of the product wear process, that is, an analysis of the nature of the change in the technical parameters of the products over time. Therefore, we restrict ourselves to considering such methods of preliminary tests, which involve measuring the values of the technical parameters of each product of a sample of volume \( n \) at some points in time \( t_0, t_1 \ldots, t_l \) (\( t_0 = 0; t_l = t_g \), where \( t_g \) means the time for which reliability is calculated). The value of the volume of preliminary tests \( n \) depends on the reliability presented to their results. The question of choosing the number of products \( n \) for preliminary tests will be discussed below. For the sake of simplicity of
reasoning, we consider the control of products whose operability is determined by one technical parameter $y(t)$.

In technical conditions, often the failure of the product by the parameter $y(t)$ means that the value $\frac{y(0) - y(t)}{y(0)} \leq x_0$ has taken a value less than or equal to some given value $x_0$. In the future, it will be convenient to introduce instead of $y(t)$ the parameter $x(t) = x_0 - \frac{y(0) - y(t)}{y(0)}$.

This is because for all products $x(0) = x_0$ and $x(t) \leq 0$ in case of failed products [5,8]. Bearing in mind that measurements cannot be made absolutely accurate, we denote the results of measuring the parameter $x$ of the $i$th product at a time $t_j (i = 0, 1, ..., n; j = 0, 1, ..., t)$ through $\tilde{x}_j^i$ and its true meaning is through $x_j^i$.

In the future, we assume that the inaccuracy of the measuring device $\Delta x$ does not depend on $x$ and $t$. It is also natural to assume that the conditional distribution density $p(\tilde{x} / x)$ of a random variable $\tilde{x}$ is described by a normal law, the mathematical expectation of which coincides with the exact value of the measured quantity $x$, and the variance is related to the inaccuracy of the measuring device $\Delta x$ by the relation $3\sigma_{\Delta x} \approx \Delta x$.

Based on the analysis of the data of preliminary tests, one or another hypothesis about the random process $x(t)$ can be stated, and, therefore, about the method of calculating the unreliability estimate $\tilde{q}(t_g) = F(X)$ for the time $t_g$ according to the test results of $X$ for some time $t_e < t_g$. Here $X$ means the measurement values of the technical parameters of $n$ products at time $t_0, t_1, ..., t_s$ ($t_0 = 0; t_s = t_e < t_g$).

In [1], the issues of rapid reliability control were considered for the case when, based on a number of values of the technical parameter $x(t)$, it is possible to predict the further behavior of products with time with the required degree of accuracy.

This article assumes that the forecast for individual implementations is not feasible. The order of preliminary studies and rapid control is considered as an example of a special case when the change in the technical parameters on the interval $[0, t_g]$ is approximated by a random process with a normal distribution density, the mathematical expectation $m(t)$ and the variance $\sigma^2(t)$ which vary in time according to a linear law.

3. Unbiased estimation of reliability

The task of preliminary studies is to establish to what extent the calculated value $\tilde{q}(t_g) = F(X)$ and the experimental $q_e$ estimate of the unreliability of the batch, obtained from the data of preliminary tests, can differ, so that it can be confirmed on the consistency of the hypothesis about the nature of the process $x(t)$ to the experimental data.

Obviously, possible deviations $\tilde{q}(t_g)$ from the true value of the probability of failure $q(t_g)$, as well as from $q_e$, can be caused by the following reasons: the presence of a statistical error related to the fact that not the entire batch is tested, but selection from it; inaccuracy of the measuring device; due to the fact that the hypothesis expressed after preliminary tests about the nature of wear of products is incorrect, i.e., the calculation performed according to statistics $\tilde{q}(t_g) = F(X)$ contains a systematic error. Depending on the magnitude of the systematic error, which cannot be taken into account in advance, the question of the possibility of using statistics $\tilde{q}(t_g) = F(X)$ to solve the problem of building reliability of products of this type is solved [6,7,9].

To estimate the magnitude of this error, we first study the nature of the statistical spread and the effect of measurement inaccuracies on the $q(t_g)$ estimate.

For this purpose, we will assume that the process $x(t)$ really has a normal distribution density for any fixed $t$, the parameters of which $m(t)$ and $\sigma^2(t)$ satisfy the following relations:

$$m(t) = x_0 + V(t - t_0), \quad \sigma^2(t) = D(t - t_0)$$  \hspace{1cm} (1)
and under this assumption we find the conditional distribution function \( F_n(\tilde{q} \mid q) \) of the estimates \( q \) provided that the true reliability of the lot is \( \tilde{q} \). Then the function \( F_n(\tilde{q} \mid q) \) will describe the desired scatter of the estimates \( \tilde{q} \), since we excluded the statistical error.

### 4. Results

In equation (1), \( x_0 \) means the mathematical expectation of the process \( x(t) \) at \( t = t_0 \), and \( V \) and \( D \) are unknown coefficients.

Note that the unknown coefficients \( V \) and \( D \) included in equation (1), under the assumptions made about the nature of the process \( x(t) \), can be estimated from the results of a single measurement of sample technical parameters [6].

Denoting by \( t_e = t_l \) the moment of measurement (\( t_e = t_g \)) and taking into account equation (1), we write the estimates of the coefficients \( V \) and \( D \)

\[
\tilde{V} = \frac{x_0 + \tilde{m}(t_e)}{t_e}, \quad \tilde{D} = \frac{\tilde{\sigma}^2(t_e)}{t_e}
\]

where \( \tilde{m}(t_e) \) and \( \tilde{\sigma}^2(t_e) \) are defined in [2].

\[
\tilde{m}(t_e) = \frac{1}{n} \sum_{i=1}^{n} \tilde{x_i}(t_e)
\]

\[
\tilde{\sigma}^2(t_e) = \frac{1}{n} \sum_{i=1}^{n} (\tilde{x_i}(t_e) - \tilde{m}(t_e))^2
\]

Using equations (1) - (4), an estimate of the unreliability of a batch based on the results of sampling tests can be represented as the following:

\[
\tilde{q}(t_g) = \Phi \left[ -\frac{km(t_e) - (k-1)x_0}{\sqrt{k\tilde{\sigma}(t_e)}} \right]
\]

where \( k = t_g / t_e \) – test acceleration coefficient.

The Laplace function is represented by the equation \( \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left\{ -\frac{x^2}{2} \right\} dx \).

Equation (5) will be exact if individual implementations of the process \( x(t) \) are monotone functions \( t \). In the remaining cases, equation (5) is approximate, since it does not take into account the fact that some products can, in principle, fail until time \( t_e \) and then again become operational at \( t = t_g \). Such an approximation can be used when \( x(t) \) is a process with weak mixing [1].

Taking into account the specific form (5) of the statistics \( \tilde{q}(t_g) \), we write the conditional distribution of interest to us in the form:

\[
F_n(\tilde{q} \mid q) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma(t_e)}}\exp \left\{ -\frac{[x_i(t_e) - m(t_e)]^2}{\sigma^2(t_e)} \right\} dx_1 \times \cdots \times \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma(t_e)}}\exp \left\{ -\frac{[x_i(t_e) - \tilde{m}(t_e)]^2}{\tilde{\sigma}^2(t_e)} \right\} d\tilde{x}_i
\]

\[
0 \leq \Phi \left[ -\frac{km(t_e) - (k-1)x_0}{\sqrt{k\tilde{\sigma}(t_e)}} \right] < \tilde{q}
\]

Simplifying equation (6) by performing integration over variables \( x_i \) (\( i = 1, 2, \ldots, n \)):

\[
F_n(\tilde{q} \mid q) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma(t_e)}}\exp \left\{ -\frac{[x_i(t_e) - m(t_e)]^2}{\sigma^2(t_e)} \right\} \exp \left\{ -\frac{[x_i(t_e) - \tilde{m}(t_e)]^2}{\tilde{\sigma}^2(t_e)} \right\} dx_1 \times \cdots \times \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma(t_e)}}\exp \left\{ -\frac{[x_i(t_e) - m(t_e)]^2}{\sigma^2(t_e)} \right\} \exp \left\{ -\frac{[x_i(t_e) - \tilde{m}(t_e)]^2}{\tilde{\sigma}^2(t_e)} \right\} d\tilde{x}_i
\]

\[
0 \leq \Phi \left[ -\frac{km(t_e) - (k-1)x_0}{\sqrt{k\tilde{\sigma}(t_e)}} \right] < \tilde{q}
\]

To calculate the integral (8), we find the differential distribution function \( \omega(\tilde{m}(t_e), \sigma^2(t_e)) \) of the random variables \( \tilde{m}(t_e) \) and \( \sigma^2(t_e) \). According to [2], the probability density \( \omega(\tilde{m}(t_e), \sigma^2(t_e)) \) has the following form:

\[
\omega(\tilde{m}(t_e), \sigma^2(t_e)) = \frac{1}{\sqrt{2\pi\tilde{\sigma}(t_e)^2}}\exp \left\{ -\frac{[\tilde{m}(t_e) - m(t_e)]^2}{\tilde{\sigma}^2(t_e)} \right\} \times \frac{c}{\sqrt{2\pi\sigma^2(t_e)}} \exp \left\{ -\frac{c}{2\sigma^2(t_e)} \right\}
\]

\[
\tilde{q} = \Phi \left( -\frac{km(t_e) - (k-1)x_0}{\sqrt{k\tilde{\sigma}(t_e)}} \right)
\]
where $\lambda = \frac{n-1}{2}, c = \frac{n-1}{\sigma_e^2 + \sigma^2(t_e)}$

For large values of $n$ ($n > 30$), the x-square distribution, which obeys the estimate $\sigma^2(t_e)$, can be replaced by a normal one [2], that is

$$\frac{c}{2\pi \sqrt{\lambda}} \left( \sigma^2(t_e) \right)^{\frac{1}{2}} \exp \left\{ -\frac{c}{2} \sigma^2(t_e) \right\} \approx \frac{1}{\sqrt{2\pi}} \frac{c}{\sigma^2(t_e)} \exp \left\{ -\frac{c^2}{2\sigma^2(t_e)} \cdot \frac{(2n - 3)^2}{2} \right\}$$

(10)

We introduce the notation $\xi = km(t_e) - (k - 1)x_0$ and $\eta = \sqrt{k}\sigma^2(t_e)$. Taking into account equations (9) and (10), it is easy to write the probability density $g(\xi, \eta)$ of random variables $\xi$ and $\eta$:

$$g(\xi, \eta) = \frac{1}{\sqrt{2\pi \sigma_e} \exp \left\{ -\frac{(\xi - \bar{\xi})^2}{2\sigma_e^2} \right\} \frac{1}{\sqrt{2\pi \sigma_q}} \exp \left\{ -\frac{(\eta - \bar{\eta})^2}{2\sigma_q^2} \right\}$$

(11)

Here $\xi = km(t_e) - (k - 1)x_0, \bar{\eta} = \sqrt{(2n - 3)/c} k, \sigma_\xi^2 = \frac{\sigma^2(t_e) + \sigma^2}{n}, \sigma_\eta^2 = \frac{k}{2c}$.

In the equation (11), we have $F_n(\bar{q}) = \int \int g(\xi, \eta) d\xi d\eta, (-\infty \leq -\frac{\xi}{\eta} < \Phi^{-1}(q) \equiv u)$.

Considering the available data [11,16,17], and knowing that:

$$\int_{-\infty}^{0} d\eta \int_{-\infty}^{0} g(\xi, \eta) d\xi = \Phi \left( \sqrt{(2n - 3)} \right) = 0,$$

(12)

we obtain the following equation for $\Psi(u)$: $\Psi(u) = \int_{-\infty}^{0} d\eta \int_{-\infty}^{0} g(\xi, \eta) d\xi$.

As a result of differentiating this equation with respect to $u$, we obtain the following equation:

$$\Psi'(u) = \frac{1}{\sqrt{2\pi}} \frac{\sigma_\xi^2 \eta - u^2 \sigma_q^2}{\sigma_\xi^2 + u^2 \sigma_q^2} \exp \left\{ -\frac{(\xi + \bar{\eta})^2}{2(\sigma_\xi^2 + u^2 \sigma_q^2)} \right\}$$

(13)

By using equation (12) we find that $F_n(\bar{q}) = \int_{-\infty}^{0} \Psi^{-1}(q) \Psi'(u) du$, or by introducing a new integration variable $z = \frac{\xi + \bar{\eta}}{\sqrt{\sigma_\xi^2 + u^2 \sigma_q^2}}$.

Thus after a series of calculations we have:

$$F_n(\bar{q}) = \Phi \left( \frac{\xi + \bar{\eta} \Phi^{-1}(q)}{\sqrt{\sigma_\xi^2 + \sigma_\eta^2 \Phi^{-1}(q)^2}} \right) + \Phi \left( \frac{-\bar{\eta}}{\sigma_\eta} \right)$$

(14)

The last term in (14) can be set equal to zero, since $\Phi \left( \frac{-\bar{\eta}}{\sigma_\eta} \right) = \Phi \left( -\sqrt{\frac{n - 3}{2}} \right) \approx 0$. In equation (14), we replace the quantities $\bar{\eta}, \sigma_\xi^2$ and $\sigma_\eta^2$ in accordance to equations (9) and (11), and $\bar{\xi}$ with equation (7). After simple transformations we obtain the following:

$$F_n(\bar{q}) = \Phi \left( \frac{1}{\sqrt{1 + k^2 \delta^2}} \Phi^{-1}(q) - \Phi^{-1}(q) \right)$$

(15)

In the last equation let us set $n \rightarrow \infty$. It is easy to see that, as $n \rightarrow \infty$, the estimate $\bar{q}$ differs from the true value of $q$ and is equal to $\bar{q} = \Phi \left( \frac{\Phi^{-1}(q)}{\sqrt{1 + k^2 \delta^2}} \right)$.

It follows that if instead of equation (5) we introduce new statistics $\bar{q}$ functionally related to $\bar{q}$:

$$\bar{q} = \Phi \left( \sqrt{1 + k^2 \delta^2} \Phi^{-1}(\bar{q}) \right)$$

(16)

following the consideration that $\delta$ is known, the $\bar{q}$ will be asymptotically unbiased.

In a practical study, instead of $\bar{q}$, the estimate $\delta_{\bar{q}} = \frac{\Delta x}{3\sigma_e(t_g)}$ is applied, where $\sigma^2(t_g)$ is calculated by formulas (3) and (4) as $\sigma_\xi(t_g) = \sqrt{D}$. The distribution function $P_n(q, /q)$ of the estimate $q$ is determined from equations (15) and (16) and has the form (17):

$$P_n(\bar{q}) = \Phi \left( \frac{-\Phi^{-1}(q) - \Phi^{-1}(\bar{q})}{\sqrt{k + 2 \delta^2}} \right)$$
5. Discussion

The application of the proposed method in calculating the required number of samples for construction products testing allows us to optimize this process in terms of resource costs and time to obtain adequate results. In most cases, it is necessary to take into account the requirements of the product supplier and customer, which is an important factor in the calculations. The use of existent data also allows further refinement of this method. However, this is possible only in case if the risks of the supplier and the customer are equal.

Let us consider some of the most perspective examples for the practical application of the proposed method.

5.1. Application example 1

Let the random process \( x(t) \) be distributed according to the normal law with the parameters \( m(t) \) and \( \sigma^2(t) \), satisfying conditions of (1). In addition, the probability of evaluating the product \( q \) and the variance \( \sigma^2(t) \) for the test batch is known. Then it is required to find such a confidence interval \( [(1 - \varepsilon)q; (1 + \varepsilon)q] \), which with probability \( \alpha \) will include the estimate \( \hat{q}, (t) \). Moreover, \( \varepsilon \) is found from the equation

\[
P_0[(1 + \varepsilon)q/t] - P_0[(1 - \varepsilon)q/t] = \alpha
\]  

(18)

This can be done using the distribution function \( P_n \). It was previously assumed that the values of \( q \) and \( \sigma^2(t) \) are known, but in practice these values are usually replaced by their estimates of \( q_e \) and \( \sigma_e^2 \), that are calculated experimentally after sampling. Applying equations for \( q_e \) and \( \sigma_e^2 \) in equation (18) and using equation (17) we obtain the following equation:

\[
\Phi \left[ \frac{\phi^{-1}((1+\varepsilon)q_e) - \phi^{-1}(q_e)}{\sqrt{n}} \right] - \Phi \left[ \frac{\phi^{-1}((1-\varepsilon)q_e) - \phi^{-1}(q_e)}{\sqrt{n}} \right] = \alpha
\]  

(19)

By solving equation (19) assuming that \( \delta \) is the inaccuracy of the measuring device, \( k \) is the acceleration coefficient, \( \alpha \) is the probability of failure-free operation and taking into account that \( \frac{q_0 - q}{q} < \varepsilon \), we find the required amount of samples \( n \).

5.2. Application example 2

From the batch of products submitted for control, it is required to determine the number of samples \( n \), which would ensure the conditions of the supplier \( \alpha \) and customer \( \beta \), and also to accept the batch of products when the estimate of its unreliability \( q \leq q_0 \). Otherwise, the party is rejected. The specific values of \( n \) and \( q \) can be found from the system of equations (20):

\[
\begin{align*}
\alpha &= 1 - \Phi \left[ \frac{1}{\sqrt{1+k\delta^2}} \phi^{-1}(q_1) - \phi^{-1}(q_0) \right] \\
\beta &= \Phi \left[ \frac{1}{\sqrt{1+k\delta^2}} \phi^{-1}(q_2) - \phi^{-1}(q_0) \right]
\end{align*}
\]  

(20)

By solving the equations system (20) above with respect to \( n \) and \( q_0 \), we obtain the following equations:

\[
q_0 = \Phi \left[ \frac{\phi^{-1}(\alpha\phi(q_2) + \phi^{-1}(\beta)\phi(q_1))}{\sqrt{1+k\delta^2}[\phi^{-1}(\alpha) + \phi^{-1}(\beta)]} \right]
\]  

(21)

\[
n = \frac{1}{\sqrt{1+k\delta^2}} \phi^{-1}(q_2) - \phi^{-1}(q_0) / \Phi^{-1}(\beta)^2
\]  

(22)

The value of parameter \( n \) that is calculated by the equation (22) should then be rounded to an integer.
Considering the case in which the risks of the supplier α and the customer β are equal, the equation (21) can be simplified:

\[ q_0 = \Phi \left[ \frac{\phi^{-1}(q_2) + \phi^{-1}(q_1)}{2\sqrt{1+k^2}} \right] \]  

(23)

6. Conclusion

Experiment planning is one of the most important parts of quality control. Reliability theory in relation to construction products allows us to scientifically substantiate the choice of conditions, evaluation criteria, as well as the sample size for testing. The obtained equation (19) can be used for planning preliminary tests for construction products if a value of unreliability \( q \) is set that needs to be estimated with probability α and accuracy ± \( \varepsilon_q \).

On further note, equations (21) and (22) are especially relevant for calculating the amount of samples \( n \) and the level of unreliability \( q_0 \) when carrying out accelerated acceptance tests where it is required to obtain the most significant \( q_0 \) at the minimum cost of used samples \( n \).

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