ABSTRACT. We study continuous mappings on the Heisenberg group that up to a time change preserve horizontal Brownian motion. It is proved that only harmonic morphisms possess this property.

1. Introduction

Let \( f : \mathbb{C} \rightarrow \mathbb{C} \) be a conformal function and let \( B(t) \) be a Brownian motion on \( \mathbb{C} \). In [10] P. Lévy proved that \( f(B(t)) \) is again Brownian motion up to a random time change. The converse is also true: if \( f \) preserve Brownian motion then it is conformal (or anti-conformal). Then Bernard, Campbell, and Davie in [2] investigated mappings \( f : \mathbb{R}^n \rightarrow \mathbb{R}^p \) and proved that a continuous mapping \( f \) preserves Brownian motion iff \( f \) is a harmonic morphism. They also considered various specific examples. In particular it turned out that \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \ (n > 2) \) preserve Brownian motion iff it is an affine map. The last relates to what is known from the works of Fuglede [5] and Ishihara [8]: a map between Riemannian manifolds is a harmonic morphism if and only if it is a horizontally conformal harmonic map. In [4] Csík and Øksendal solve more general problem: they described \( C^2 \)-mappings that map the path of one diffusion process into the path of another diffusion process.

In this paper we study continuous mappings between Heisenberg groups \( f : \mathbb{H}^n \rightarrow \mathbb{H}^p \) that preserve horizontal Brownian motion. Following the approach from [2] we proved that a continuous mapping \( f \) preserves Brownian motion on the Heisenberg group if and only if it is a harmonic morphism. Close results were obtained by Wang in [13], where images of Brownian motions on the Heisenberg group under conformal maps were studied. Finally, we should mention that [4, Theorem 1] generalizes our Theorem 4.1 in case of higher smoothness.

The paper is organized as follows. In Section 2 we provide necessary notions on the Heisenberg group and on horizontal Brownian motion. In Section 3 we revise the result on representation of the solution of the Dirichlet problem via Brownian motion. Then, in Section 4 we introduce and prove the main result.

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2. Preliminaries

2.1. The Heisenberg group. The Heisenberg group $\mathbb{H}^n$ is defined as $\mathbb{R}^{2n+1} = \mathbb{C}^n \times \mathbb{R}$ with the group law

$$(z, t) \ast (z', t') = (z + z', t + t' + 2 \text{Im} \sum_{j=1}^{n} z_j \bar{z_j'}) = (x + x', y + y', t + 2 \sum_{j=1}^{n} (y_j x_j' - x_j y_j')).$$

The vector fields

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}$$

are left invariant and form a basis of left invariant vector fields on Heisenberg group $\mathbb{H}^n$. The only non-trivial commutator relations are $[X_j, Y_j] = 4T$, $j = 1, \ldots, n$. For all $g \in \mathbb{H}^n$ horizontal distribution $H_g \mathbb{H}^n = \text{span}\{X_1(g), Y_1(g), \ldots, X_n(g), Y_n(g)\}$.

A curve $\gamma : [a, b] \to \mathbb{H}^n$ is horizontal if $\gamma'(t) \in H_{\gamma(t)} \mathbb{H}^n$ for almost every $t \in [a, b]$. Let $\gamma(t) = (\xi_1(t), \zeta_1(t), \ldots, \xi_n(t), \zeta_n(t))$, then it can be shown that $\gamma(t)$ is horizontal curve if and only if

$$\eta'(t) = 2 \sum_{j=1}^{n} (\xi_j'(t) \zeta_j(t) - \xi_j(t) \zeta_j'(t)) \quad \text{for almost every } t \in [a, b].$$

A mapping $f : U \to \mathbb{H}^p$ is called contact if $\gamma \circ f$ is horizontal curve for any horizontal curve $\gamma : [a, b] \to U$. If $f(g) = (u_1(g), v_1(g), \ldots, u_n(g), v_n(g), h(g))$ then the contact condition is equivalent to

$$X_i h = 2 \sum_{j=1}^{p} v_j X_i u_j - u_j X_i v_j$$
$$Y_i h = 2 \sum_{j=1}^{p} v_j Y_i u_j - u_j Y_i v_j,$$

for $i = 1, \ldots, n$.

For any $g = (z, t) \in \mathbb{H}^n$ define the Korányi norm

$$\rho(g) = (|z|^2 + t^2)^{\frac{1}{2}}$$

and the Korányi metric $\rho(g_1, g_2) = \rho(g_2^{-1} * g_1)$.

If we are given an absolutely continuous curve $\tilde{\gamma}(t) = (\xi_1(t), \zeta_1(t), \ldots, \xi_n(t), \zeta_n(t)) : [a, b] \to \mathbb{R}^{2n}$, then by defining

$$\eta(t) := \eta(a) + 2 \sum_{j=1}^{n} \int_{a}^{t} \xi_j'(s) \zeta_j(s) - \xi_j(s) \zeta_j'(s) \, ds$$

we obtain a horizontal curve $\gamma = (\tilde{\gamma}, \eta)$, which is called the horizontal lift of $\tilde{\gamma}$.

2.2. Horizontal Brownian motion on the Heisenberg group. Let $B(t) = \{B_1^1(t), B_1^2(t), \ldots, B_n^1(t), B_n^2(t)\}$ be a Brownian motion in $\mathbb{R}^{2n}$ starting at 0. Consider the Lévi area integral

$$S(t) = 2 \sum_{j=1}^{n} \int_{0}^{t} B_j^2(s) \, dB_j^1(s) - B_j^1(s) \, dB_j^2(s)$$
Then the process $\dot{W}(t) = (B(t), S(t))$, which could be viewed as a horizontal lift of $B(t)$, is the solution of a system of stochastic differential equations

$$d\dot{W}_{2k-1}(t) = dB_k^1(t), \quad d\dot{W}_{2k}(t) = dB_k^2(t), \quad k = 1, \ldots, n,$$

$$d\dot{W}_{2n+1}(t) = 2 \sum_{j=1}^{n} B_j^1(t) dB_j^1(t) - B_j^1(t) dB_j^2(t).$$

As a consequence $\dot{W}(t)$ is a Markov process with generator $\frac{1}{2} \Delta_H$, where

$$\Delta_H = \sum_{j=1}^{n} X_j^2 + Y_j^2$$

Let $g$ be a given point in $\mathbb{H}^n$, then the horizontal Brownian motion starting at this point is defined as $W(t) = g \ast \dot{W}(t)$.

We will need the following Itô formula for horizontal Brownian motion, see [1, lemma 3.2].

**Lemma 2.1** (Itô formula). Let $f \in C^2(\mathbb{H}^n; \mathbb{R})$ and $W(t)$ be a horizontal Brownian motion in $\mathbb{H}^n$. Then

$$f(W(t)) = f(W(0)) + \int_0^t \nabla_H f(W(s)) \cdot dB(s) + \frac{1}{2} \int_0^t \Delta_H f(W(s)) \, ds.$$

We are going to use the following lemma for 1-dimensional Brownian motion.

**Lemma 2.2** ([3] Problem 1, p. 45). If $e : [0, \infty] \to \mathbb{R}^d$ and $\sigma(t) = \int_0^t |e|^{2}(s) \, ds$, then $a(s) = \int_0^{\sigma^{-1}(s)} e(q) \cdot dB(q)$ is 1-dimensional Brownian motion.

And we will use the following time change formula for Itô integrals.

**Theorem 2.3** ([11] Theorem 8.5.7, p. 156). Suppose $c(s, \omega)$ and $a(s, \omega)$ are $s$-continuous almost surely, $a(0, \omega) = 0$ a.s., and that $E|a_i| < \infty$. Let $B(s)$ be a $d$-dimensional Brownian motion and $d$-vector $v(s, \omega)$ be bounded and $s$-continuous. Define

$$\tilde{B}(s) = \int_0^{a(t)} \frac{\sqrt{c(s)}}{c(a(r))} dB(s).$$

Then $\tilde{B}(s)$ is a Brownian motion and

$$\int_0^{a(t)} v(s) \cdot dB(s) = \int_0^{t} v(a(r)) \sqrt{a'(r)} \cdot d\tilde{B}(r) \quad a. \, s.,$$

where $a'(r)$ is the derivative of $a(r)$ w.r.t. $r$, so that

$$a'(r) = \frac{1}{c(a(r))} \quad \text{for almost all } r, \quad a. \, s.$$

3. **Brownian motion and the Dirichlet problem**

A function $u : \mathbb{H}^n \to \mathbb{R}$ is called harmonic if $\Delta_H u = 0$. In this section we follow [7] and [6] Theorem 2.1] to obtain

**Theorem 3.1.** Let $U$ be an open set in $\mathbb{H}^n$ and $\varphi$ be a bounded continuous function on $\partial U$. Let $S_U$ be the first exit time from $U$. Define function

$$u(g) = E(\varphi(W(S_U)) \mid W(0) = g), \quad g \in U.$$
Then $u$ is harmonic in $U$. Moreover, if $U$ is a regular domain, then function (3) solves the Dirichlet problem

$$
\begin{align*}
\Delta_B u &= 0, \quad \text{in } U, \\
v|_{\partial U} &= \varphi \quad \text{on } \partial U.
\end{align*}
$$

The point $g_0 \in \partial U$ is said to be a regular point if $P(S_U = 0 \mid W(0) = g_0) = 1$ and $U$ is regular open set if all points of $\partial U$ are regular.

**Lemma 3.2.** If $u$ is defined by (3), then for any $g \in U$ and stopping time $s_0 \leq S_U$ we have

$$
u(g) = E(u(W(s_0)) \mid W(0) = g).$$

**Proof.** Let $\mathcal{B}_{s_0}$ be the $\sigma$-algebra of events previous to $s_0$, then, by conditioning by $\mathcal{B}_{s_0}$ we obtain

$$
u(g) = E\left[ E(\varphi(W(S_U)) \mid \mathcal{B}_{s_0}) \mid W(0) = g \right].$$

On the other hand the Markov property gives

$$E(\varphi(W(S_U)) \mid \mathcal{B}_{s_0}) = E(\varphi(W(S_U)) \mid W(0) = W(s_0))$$

So

$$
u(g) = E\left[ E(\varphi(W(S_U)) \mid W(0) = W(s_0)) \mid W(0) = g \right] = E(\nu(W(s_0)) \mid W(0) = g).$$

\[ \square \]

**Lemma 3.3.** Let $B(g_0, \rho_0) \subset U$. Then the law of $W(S_{B(g_0, \rho_0)})$ knowing $W(0) = g_0$ is

$$P(W(S_{B(g_0, \rho_0)}) \in d\sigma(g) \mid W(0) = g_0) = \frac{2^{n-2}(\Gamma(\frac{1}{2}))^2}{\pi^{n+1}\rho_0^{2n}} \frac{2|z - z_0|^2}{\|\nabla \rho^4\|(g_0 - g_0^* g)} d\sigma(g),$$

where $g = (z, t)$, $g_0 = (z_0, t_0)$, $d\sigma(g)$ is the euclidean area element on $\partial B(g_0, \rho_0)$, and $\|\nabla \rho^4\|(z, t) = (16|z|^6 + 4t^2)^{\frac{1}{2}}$.

**Proof.** Let $h$ be a bounded continuous function on $\partial B(g_0, \rho_0)$. Consider the Dirichlet problem

$$\begin{align*}
\Delta_B v &= 0, \quad \text{in } B(g_0, \rho_0), \\
v|_{\partial B(g_0, \rho_0)} &= h \quad \text{on } \partial B(g_0, \rho_0).
\end{align*}$$

Then there exists a unique solution of (4), and the value in the center $g_0$ could be calculated via

$$v(g_0) = \int_{\partial B(g_0, \rho_0)} h(g) \, d\mu_{g_0}^{B(g_0, \rho_0)}(g)$$

with

$$d\mu_{g_0}^{B(g_0, \rho_0)}(g) = \frac{2^{n-2}(\Gamma(\frac{1}{2}))^2}{\pi^{n+1}\rho_0^{2n}} \frac{|z - z_0|^2}{(4|z - z_0|^6 + (t - t_0 - 2 \text{Im } \sum_{j=1}^{n} z^*_j z_j^2)^{\frac{1}{2}})} d\sigma(g),$$

see [3] Theorem 7.2.9].

Now, let $v$ be the solution of (4). Then $v \in C^2(B(g_0, \rho_0))$ and we can apply the Itô formula (making use $\Delta_B v = 0$):

$$v(W(s)) = v(W(0)) + \sum_{j=1}^{n} \int_0^s X_j v(W(t)) \, dB_j(t) + \int_0^s Y_j v(W(t)) \, dB_j^2(t)$$

where $X_j$ and $Y_j$ are given in (5).
Then almost surely

\[ h(W(S_{B(g_0,\rho_0)})) = \lim_{s \to S_{B(g_0,\rho_0)}} v(W(s)) \]

\[ = v(W(0)) + \sum_{j=1}^{n} \int_{0}^{S_{B(g_0,\rho_0)}} X_j v(W(t)) \, dB_j^1(t) + \int_{0}^{S_{B(g_0,\rho_0)}} Y_j v(W(t)) \, dB_j^2(t). \]

It follows that

\[ E(h(W(S_{B(g_0,\rho_0)})) \mid W(0) = g_0) = v(g_0). \]

Thus, combining the last equation with (5) and noting that \( h \) was arbitrary we get the result.

**Lemma 3.4.** Let \( u \) be a bounded function such that for any \( g_0 \in U \), any \( \rho_0 \leq \varepsilon \) sufficiently small, we have the mean value property

\[ u(g_0) = \int_{\partial B(g_0,\rho_0)} h(g) \, d\mu_{g_0}^{B(g_0,\rho_0)}(g). \]

Then \( u \) is \( C^\infty \) function and satisfies \( \Delta H u = 0 \) in \( U \).

**Proof.** Let \( g_0 \in U \) and \( \varepsilon \) be small, then by the Taylor formula

\[ u(g) = u(g_0) + P_2(u, g_0)(g) + O((\rho(g_0^{-1} * g))^3). \]

Now we place (7) inside (6). Due to symmetry all first order terms and second order terms with mixed derivatives will give 0. So we have

\[ u(g_0) = u(g_0) + \frac{1}{2} \sum_{j=1}^{n} (X_j^2 u(g_0) + Y_j^2 u(g_0)) \int_{\partial B(g_0,\rho_0)} x_1^2 \, d\mu_{g_0}^{B(g_0,\rho_0)}(g) + O(\varepsilon^3) \]

or

\[ \frac{1}{2} \sum_{j=1}^{n} (X_j^2 u(g_0) + Y_j^2 u(g_0)) \cdot \frac{1}{\varepsilon^2} \cdot \int_{\partial B(g_0,\rho_0)} x_1^2 \, d\mu_{g_0}^{B(g_0,\rho_0)}(g) = o(1). \]

The integral in the last equation is of order \( \varepsilon^2 \). Thus we obtain \( \Delta H u = 0 \) in \( U \).

**Lemma 3.5.** For \( t > 0 \), the function \( g \mapsto P_g(S_U \leq t) \) is lower semicontinuous on \( \mathbb{H}^n \):

\[ \liminf_{g \to g_0} P_g(S_U \leq t) \geq P_{g_0}(S_U \leq t) \]

**Lemma 3.6.** If \( g_0 \in \partial U \) is a regular point then

\[ \lim_{g \to g_0} E(\varphi(W(S_U))) \mid W(0) = g = \varphi(g_0). \]

**Proof.** Let \( g_0 \in \partial U \) be a regular point. For \( r > 0 \), let \( s_r \) be the exit time from \( B(g_0, r) \) for \( W(t) \).

First we will prove that

\[ \lim_{g \to g_0} P_g(S_U < s_r) = 1. \]

For any \( g \in B(g_0, r) \) we have \( P_g(s_r > 0) = 1 \). Moreover, for any \( \varepsilon > 0 \) there exist \( \tau > 0 \) such that for any \( g \in B(g_0, \frac{r}{2}) \) holds \( P_g(s_r < \tau) < \varepsilon \). Fix \( \varepsilon > 0 \) and let \( \tau \) be
such that the above is true. Then we have

\[
P_g(S_U \leq s_r) = P_g(S_U \leq s_r, s_r \geq \tau) + P_g(S_U \leq s_r, s_r < \tau) = P_g(S_U \leq \tau) + P_g(S_U \leq s_r, s_r < \tau) - P_g(S_U \leq \tau, s_r < \tau) \geq P_g(S_U \leq \tau) - P_g(s_r < \tau) \geq P_g(S_U \leq \tau) - \varepsilon.
\]

Now making use above inequality and the applying lemma 3.5 and the regularity of \(g_0\) we derive

\[
\limsup_{g \to g_0} P_g(S_U < s_r) \geq \liminf_{g \to g_0} P_g(S_U < s_r) \geq \liminf_{g \to g_0} P_g(S_U < \tau) - \varepsilon \geq P_g(S_U < \tau) - \varepsilon = 1 - \varepsilon.
\]

Since \(\varepsilon > 0\) was arbitrary, we obtain (3).

For any \(\varepsilon > 0\) take \(r > 0\) so that for every \(g_1 \in B(g_0, r) \cap \partial U\) holds \(|\varphi(g) - \varphi(g_0)| < \varepsilon\). So

\[
|E_g(\varphi(W(S_U))) - \varphi(g_0)| \leq E_g(|\varphi(W(S_U))) - \varphi(g_0)|) < \varepsilon + E_g(\varphi(W(S_U))) \mid W(S_U) \not\in B(g_0, r) \cap \partial U) \leq \varepsilon + 2\max_{\partial U} |\varphi| \cdot P_g(W(S_U) \not\in B(g_0, r) \cap \partial U).
\]

Thanks (3) we find a neighbourhood of \(g_0\) so that

\[
P_g(W(S_U) \not\in B(g_0, r) \cap \partial U) = P_g(S_U < s_r) < \frac{\varepsilon}{2\max_{\partial U} |\varphi|}.
\]

That completes the proof. \(\square\)

**Proof of theorem 3.1** Lemmas 3.2, 3.3 and 3.4 ensure that function \(u\) defined by (3) is harmonic in \(U\). In the case of regular domain by lemma 3.6 \(u\) attains boundary values. \(\square\)

## 4. Brownian path preserving mappings

Let \(U\) be a domain in \(\mathbb{H}^n\). A continuous mapping \(f : U \to \mathbb{H}^p\) is said to be **Brownian path preserving** if for each \(g_0 \in U\) and for each horizontal Brownian motion \(W(t)\) defined on \((\Omega, \mathcal{F}, P)\), started from \(g_0\), there exist:

(A) a mapping \(\omega \mapsto \sigma_\omega\) on \(\Omega\) such that for each \(\omega \sigma_\omega(t)\) is a continuous strictly increasing function on \([0, S_U]\) and such that for any \(t > 0\) the mapping \(\omega \mapsto \sigma_\omega(t)\) is measurable on \(\{t < S_U\} \subset \Omega\). It is also required that for each \(s\) the random variable \(\sigma(s)\) be independent of the process \(\{W^{-1}(s) \ast W(t) : t > s\}\).

(B) a horizontal Brownian motion \(W'(t)\) defined on \((\Omega', \mathcal{F}', P')\) in \(\mathbb{H}^p\), started at \(0\) such that

(C) on \((\Omega, \mathcal{F}, P) \times (\Omega', \mathcal{F}', P')\) the stochastic process \(Z(s) = Z(\omega, \omega', s)\) defined for \(s \geq 0\) by

\[
\begin{cases}
  f(W(\sigma^{-1}(s))), & s < \sigma(S_U) = \lim_{t \to S_U} \sigma(t), \\
  f(W(\sigma(S_U))) \ast W'(s - \sigma(S_U)), & s \geq \sigma(S_U)
\end{cases}
\]

is horizontal Brownian motion started at \(f(g_0)\).
Theorem 4.1. Let $U$ be a domain in $\mathbb{H}^n$ and let $f : U \to \mathbb{H}^p$ be a non-constant continuous mapping. Then the following is equivalent:

(i) $f$ is Brownian path preserving mapping;

(ii) $f$ is harmonic morphism.

Proof. (i) $\Rightarrow$ (ii). Let $B(0, R)$ be a ball in $\mathbb{H}^p$, let $Q = f^{-1}(B(0, R))$, and let $g_0 \in Q$. Define $U_m = \{ g \in U : \rho(g) < m, \rho(g, \mathbb{H}^n \setminus U) > \frac{1}{m} \}$, $Q_m = Q \cap U_m$. Let $S_U$ be exit time from $U$ and $s_m$ be exit time from $U_m$. Let $\psi$ be exit time of $Z$ from $B(0, R)$, then $\theta := \min\{\psi, \sigma(S_U)\}$ and $\theta_m := \min\{\psi, \sigma(s_m)\}$ are stopping times. Consider a harmonic function $u : \mathbb{H}^p \to \mathbb{R}$. Then by theorem 3.1 and lemma 3.2 we have

$$u \circ f(g_0) = u(f(g_0)) = E_{f(g_0)}(u(Z(\psi))) = E_{f(g_0)}(u(Z(\theta))).$$

Then by the Lebesgue theorem

$$E_{f(g_0)}(u(Z(\theta))) = \lim_{m \to \infty} E_{f(g_0)}(u(Z(\theta_m)))$$

$$= \lim_{m \to \infty} E_{g_0}(u \circ f(W(\sigma^{-1}(\theta_m))))$$

$$= \lim_{m \to \infty} E_{g_0}(u \circ f(W(\min\{\sigma^{-1}(\psi), s_m\}))).$$

Note that $\min\{\sigma^{-1}(\psi), s_m\}$ is the exit time from $Q_m$. By theorem 3.1 function $v_m(g) = E_g(u \circ f(W(\min\{\sigma^{-1}(\psi), s_m\})))$ is harmonic in $Q_m$. Therefore $u \circ f$ is harmonic in $Q$. Since $R$ is arbitrary $u \circ f$ is harmonic on $U$, meaning that $f$ is a harmonic morphism.

(ii) $\Rightarrow$ (i). Let $f = (f_1, f_2, \ldots, f_{2p+1}) : U \to \mathbb{H}^p$ be a harmonic morphism, then the following holds true

$$\Delta_{\mathbb{H}} f_i = 0, \quad \text{for } i = 1, \ldots, 2p + 1;$$

$$(\nabla_{\mathbb{H}} f_i, \nabla_{\mathbb{H}} f_j) = h(g) \cdot \delta_{i,j}, \quad \text{for } i, j = 1, \ldots, 2p;$$

$f$ is a contact mapping.

Define

$$\sigma(t) = \int_0^t |\nabla_{\mathbb{H}} f_1|^2(W(s)) \, ds, \quad 0 \leq t \leq S_U.$$ 

This $\sigma$ satisfies condition (A). Let $U_m$ and $s_m$ be as in the previous part of the proof, and let

$$\sigma_m(t) = \begin{cases} \sigma(t), & t \leq s_m; \\ \sigma(s_m) + t - s_m, & t > s_m. \end{cases}$$

With $W'$ as in (B) define a process $Z^m(s) = Z^m(\omega, \omega', s)$ by

$$Z^m(s) = \begin{cases} Z(s), & s < \sigma(s_m); \\ f(W(s_m)) * W'(s - \sigma(s_m)), & s \geq \sigma(s_m), \end{cases}$$

where $Z(s)$ as in (C). Then almost surely $Z^m$ is continuous for $s > 0$, and $Z^m(s) \to Z(s)$ when $m \to \infty$ almost surely for each $s$. We will prove that $Z^m(s)$ is a horizontal Brownian motion on $\mathbb{H}^p$, which will imply so is $Z(s)$.

Fix $m$. First we justify that $Z^m_j(s)$, $j = 1, \ldots, 2p$ are 1-dimensional Brownian motions.
By the Itô formula (Lemma 2.1)

\[
Z^m_1(\sigma_m(t)) = \begin{cases} 
  f_1(W(t)) = f_1(W(0)) + \int_0^t \nabla f_1(W(s)) \cdot dB(s), & t < \sigma_m; \\
  f_1(W(s_m)) + B^1_1(\sigma_m(t) - \sigma(s_m)), & t > \sigma_m,
\end{cases}
\]

and then

\[
Z^m_1(s) = \begin{cases} 
  f_1(W(0)) + \int_0^{\sigma_m^{-1}(s)} \nabla f_1(W(q)) \cdot dB(q), & \sigma_m^{-1}(s) < \sigma_m; \\
  f_1(W(s_m)) + \hat{B}^1_1(s - \sigma(s_m)), & \sigma_m^{-1}(s) > \sigma_m.
\end{cases}
\]

Now we redefine the initial Brownian motion \(W\) changing its first coordinate (and, consequently the last one) after time \(s_m\): \(W(t) = W(t)\) when \(t \leq s_m\) and \(\hat{B}^1_1(t) = \hat{B}^1_1(t - s_m)\) for \(t > s_m\). Note that \(\hat{W}\) is defined on the product \(\Omega \times \Omega'\). Then

\[
Z^m_1(s) = f_1(W(0)) + \int_0^{\sigma_m^{-1}(s)} \nabla f_1(W(q)) \cdot d\hat{B}(q) \quad \text{when} \quad \sigma_m^{-1}(s) < \sigma_m.
\]

For \(s \geq \sigma(s_m)\) it holds \(s = \sigma_m^{-1}(s) + \sigma(s_m) - s_m\), and

\[
Z^m_1(s) = f_1(W(s_m)) + \hat{B}^1_1(\sigma_m^{-1}(s) - s_m)
\]

\[
= f_1(W(s_m)) + \hat{B}^1_1(\sigma_m^{-1}(s)) - \hat{B}^1_1(s_m) = f_1(W(s_m)) + \int_{s_m}^{\sigma_m^{-1}(s)} d\hat{B}^1_1(q).
\]

It follows

\[
Z^m_1(s) = f_1(W(0)) + \int_0^{\sigma_m^{-1}(s)} e(q) \cdot d\hat{B}(q),
\]

where

\[
e(q) = \begin{cases} 
  \nabla f_1(W(q)), & \text{if} \ q < s_m; \\
  e_1, & \text{if} \ q \geq s_m.
\end{cases}
\]

So, due to lemma 2.2 \(Z^m_1(s)\) is 1-dimensional Brownian motion. In the same manner we prove this fact for other horizontal coordinates \(Z^m_j(s), j = 2, \ldots, 2p\).

Now we should prove that \(Z^m_{2p+1}(s)\) is the Lévi area integral (2) of horizontal components.

So, with theorem 2.3 we have

\[
\int_0^{\sigma_m^{-1}(s)} e_j(q) \cdot d\hat{B}(q) = \int_0^s e_j(r) \frac{1}{|e|((\sigma_m^{-1}(r)))} \cdot d\hat{B}(r).
\]

Therefore

\[
dZ^m_j(s) = e_j(s) \frac{1}{|e|((\sigma_m^{-1}(s)))} \cdot d\hat{B}(s).
\]
From the last and (12) we derive

\[ Z_{2p+1}^m(\sigma_m(t)) = f_{2p+1}(W(t)) = f_{2p+1}(W(0)) + \int_0^t \nabla_W f_{2p+1}(W(s)) \cdot dB(s) \]

\[ = f_{2p+1}(W(0)) + \sum_{i=1}^p \int_0^t 2 \sum_{j=1}^p (f_{2j} X_i f_{2j-1} - f_{2j-1} X_i f_{2j}) \, dB_i^1(s) \]

\[ + (f_{2j} Y_i f_{2j-1} - f_{2j-1} Y_i f_{2j}) \, dB_i^2(s) \]

\[ = f_{2p+1}(W(0)) + 2 \sum_{j=1}^p \int_0^t f_{2j}(W(s)) \nabla_W f_{2j-1}(W(s)) \cdot dB(s) \]

\[ - f_{2j-1}(W(s)) \nabla_W f_{2j}(W(s)) \cdot dB(s). \]

So we have

\[ Z_{2p+1}^m(s) = f_{2p+1}(W(0)) + 2 \sum_{j=1}^p \int_0^s f_{2j}(W(q)) \nabla_W f_{2j-1}(W(q)) \cdot d\hat{B}(q) \]

\[ - f_{2j-1}(W(q)) \nabla_W f_{2j}(W(q)) \cdot d\hat{B}(q). \]

For \( s \geq \sigma_m(s_m) \) it holds \( s = \sigma_m^{-1}(s) + \sigma(s_m) - s_m \), and \( \hat{S}(\sigma_m^{-1}(s) - s_m) = \hat{S}(\sigma_m^{-1}(s)) - \hat{S}(s_m) \), so

\[ Z_{2p+1}^m(s) = f_{2p+1}(W(s_m)) + \hat{S}(\sigma_m^{-1}(s) - s_m) \]

\[ + 2 \sum_{j=1}^p f_{2j}(W(s_m)) \hat{B}_j^1(\sigma_m^{-1}(s) - s_m) - f_{2j-1}(W(s_m)) \hat{B}_j^2(\sigma_m^{-1}(s) - s_m) \]

\[ = f_{2p+1}(W(s_m)) + \int_{s_m}^{\sigma_m^{-1}(s)} d\hat{S}(q) \]

\[ + 2 \sum_{j=1}^p \int_{s_m}^{\sigma_m^{-1}(s)} f_{2j}(W(s_m)) \, d\hat{B}_j^1(q) - \int_{s_m}^{\sigma_m^{-1}(s)} f_{2j-1}(W(s_m)) \, d\hat{B}_j^2(q) \]

\[ + 2 \sum_{j=1}^p \int_{s_m}^{\sigma_m^{-1}(s)} f_{2j}(W(s_m)) \, d\hat{B}_j^1(q) - \int_{s_m}^{\sigma_m^{-1}(s)} f_{2j-1}(W(s_m)) \, d\hat{B}_j^2(q) \]

\[ = f_{2p+1}(W(s_m)) + 2 \sum_{j=1}^p \int_{s_m}^{\sigma_m^{-1}(s)} f_{2j}(W(s_m)) + \hat{B}_j^1(q) \, d\hat{B}_j^2(q) \]

\[ - (f_{2j-1}(W(s_m)) + \hat{B}_j^1(q)) \, d\hat{B}_j^2(q). \]

From the last and (12) we derive

\[ Z_{2p+1}^m(s) = f_{2p+1}(W(0)) + \int_0^s e_{2p+1}(q) \, d\hat{B}(q), \]
where
\[ e_{2p+1}(q) = \begin{cases} 
2 \sum_{j=1}^{p} f_{2j}(W(q))\nabla_{\mathbb{H}} f_{2j-1}(W(q)) - f_{2j-1}(W(q))\nabla_{\mathbb{H}} f_{2j}(W(q)), & \text{if } q < s_m; \\
\begin{bmatrix}
2 f_2(W(s_m)) + B_1^2(q) \\
- f_1(W(s_m)) - B_1(q) \\
\vdots \\
f_{2p}(W(s_m)) + B_{p}(q)
\end{bmatrix}, & \text{if } q \geq s_m.
\end{cases} \]

Again, by theorem \([2,3]\) we have
\[
\int_0^{f(s)} e_{2p+1}(q) \cdot d\tilde{B}(q) = \int_0^{f(r)} e_{2p+1}(r) \frac{1}{|e|(|e|^{-1}(r))} \cdot d\tilde{B}(r).
\]

Therefore, taking into account \([11]\)
\[
dZ_{2p+1}^m(s) = e_{2p+1}(s) \frac{1}{|e|(|e|^{-1}(s))} \cdot d\tilde{B}(s) = 2 \sum_{j=1}^{p} Z_{2j}^m dZ_{2j-1}^m - Z_{2j-1}^m dZ_{2j}^m.
\]

Thus we have proved that \(Z^m(s) = (Z_1^m(s), Z_2^m(s), \ldots, Z_{2p+1}^m(s))\) is a horizontal Brownian motion. \(\square\)

**Theorem 4.2.** Let \(U\) be a domain in \(\mathbb{H}^n\) and let \(f : U \rightarrow \mathbb{H}^n\) be a Brownian path preserving mapping. Then \(f = \pi_b \circ \varphi_A \circ \delta_{\alpha,|U|}\), i.e. \(f\) is the restriction on \(U\) of the composition of translation, rotation, and dilatation.

**Proof.** Let \(f : U \rightarrow \mathbb{H}^n\) is a Brownian path preserving mapping. Due to theorem \([4,1]\) \(f\) is a harmonic morphism, so by \([9]\) and \([10]\) we have
\[
||D_H f(x)||^{2n+2} = |J(x, f)|,
\]
where \(D_H f\) and \(J(\cdot, f)\) are the formal horizontal differential and the formal Jacobian of \(f\). The last equation means that distortion coefficient of \(f\) equals 1. Then, by \([12]\) Theorem \([12]\) \(f\) is constant or the restriction of some M"obius transform to \(U\). It remains to note that translation, rotation, and dilatation are harmonic morphisms, but inversion is not. \(\square\)

**Remark 4.3.** In the case \(U \subset \mathbb{H}^n\) and \(p < n\) no nontrivial map \(f : U \rightarrow \mathbb{H}^p\) is contact. Therefore there are no harmonic morphisms in this situation.

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