ON THE MAGNITUDE OF DARK ENERGY VOIDS AND OVERDENSITIES

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ABSTRACT

We investigate the clustering of dark energy within matter overdensities and voids. In particular, we derive an analytical expression for the dark energy density perturbations, which is valid in the linear, quasi-linear, and fully nonlinear regimes of structure formation. We also investigate the possibility of detecting such dark energy clustering through the integrated Sachs-Wolfe effect. In the case of uncoupled quintessence models, if the mass of the field is of order the Hubble scale today or smaller, dark energy fluctuations are always small compared to the matter density contrast. Even when the matter perturbations enter the nonlinear regime, the dark energy perturbations remain linear. We find that virtualized clusters and voids correspond to local overdensities in dark energy, with \( \delta_\delta/(1 + w) \sim O(10^{-5}) \) for voids, \( \delta_\rho/(1 + w) \sim O(10^{-4}) \) for supervoids, and \( \delta_\rho/(1 + w) \sim O(10^{-5}) \) for a typical virtualized cluster. If voids with radii of 100–300 Mpc exist within the visible universe, then \( \delta_\rho \) may be as large as \( 10^{-3}(1 + w) \). Linear overdensities of matter and superclusters generally correspond to local voids in dark energy; for a typical supercluster, \( \delta_\rho/(1 + w) \sim O(-10^{-5}) \). The approach taken in this work could be straightforwardly extended to study the clustering of more general dark energy models.

Subject headings: cosmology: theory — galaxies: general — large-scale structure of universe — miscellaneous — relativity

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1 Capozziello et al. 2005). It has also been suggested that the universe is not accelerating at all, but that a large local inhomogeneity prevents the SNe Ia data from being correctly interpreted in terms of a homogeneous and isotropic cosmological model (Kolb Matarrese & Riotto 2006; Moffat 2006). This said, dynamical dark energy (DDE) is by far the most popular candidate to explain the current astronomical data. In the simplest DDE models, known as quintessence, dark energy is associated with the energy density of a scalar field with a canonical kinetic structure. If dark energy does indeed exist, then astronomical observations presently provide us with only hints as to its nature. We know that today it represents about 70% of the total energy density of the universe and that its equation of state (EOS) parameter, \( w \equiv p/\rho \), is fairly close to \( -1 \); \( w = -1.0 \pm 0.1 \) for \( z < 1 \) (Riess et al. 2005). Furthermore, if matter dominates the expansion of the universe for \( z > 1.8 \), then \( w(z > 1) = -0.8^{+0.6}_{-0.1} \) (Riess et al. 2005). Hence, dark energy has negative pressure at higher redshifts with a 98% confidence. These bounds on \( w \) are entirely consistent with a pure cosmological constant. While detecting either \( w \neq -1 \) or \( dw/dz \neq 0 \) would rule out a cosmological constant, in many DDE models significant deviations from \( w = -1 \) only occur at early times and as such would be difficult to detect. The late-time, background cosmology predicted by many DDE models is therefore very similar to that of a universe with a true cosmological constant.

DDE models generally cease to mimic a cosmological constant in inhomogeneous backgrounds or when one considers cosmological perturbation theory. In particular, a number of authors have studied the effect of DDE on the formation of large-scale structure (Corasaniti et al. 2005; Hannestad & Mortsell 2002; Brookfield et al. 2006b; Doran et al. 2007). In the vast majority of these works, the energy density of dark energy is taken to be homogeneous, i.e., it is assumed that DDE does not cluster. The extent to which this assumption of homogeneity is valid has been the subject of some interest and much debate in the literature. In an inhomogeneous background, the DDE energy density and EOS parameter should exhibit some spatial variations. The key
issue of how large these variations should be is however far from settled, particularly when the matter perturbation goes nonlinear (Maor & Lahav 2005; Mota & van de Bruck 2004; Linder & White 2005; Bartelmann et al. 2006; Nunes & Mota 2006; Abramo et al. 2007). In this paper we attempt to settle this issue for uncoupled quintessence models by deriving an analytical expression for the dark energy contrast, \( \delta_{\phi} \), in the presence of a matter perturbation. Importantly, we do not constrain the matter perturbation to be small (i.e., linear).

Clustering of DDE over scales smaller than about 100 Mpc has been the subject of a number of recent articles (Caimmi 2007; Balaguera-Antolínez et al. 2006; Mainini 2005; Percival 2005; Wang 2006; Nunes et al. 2006; Balaguera-Antolínez et al. 2007). Most attention has been focussed on models in which the DDE couples to baryonic and/or dark matter, since DDE clustering is expected to be strongest in such theories (Amendola 2007; Balaguera-Antolínez et al. 2006; Mainini 2005; Percival 2005; Wang 2006; Nunes et al. 2006; Balaguera-Antolínez et al. 2007). In other works, a more phenomenological approach has been taken, and the observables associated with DDE clustering have been parameterized. In particular, it has been shown that inhomogeneities in dark energy could produce detectable signatures on the CMB (Weller & Lewis 2003; Koivisto & Mota 2006).

Recently, Dutta & Maor (2007) considered the growth of spatial DDE perturbations in the absence of any matter coupling. They considered only those circumstances where both the density contrast of matter, \( \delta_m \), and that of the DDE, \( \delta_{\phi} \), were small enough to be treated as linearized perturbations about a homogeneous and isotropic cosmological background. They studied the simplest class of quintessence models where dark energy is associated with the slow-roll of a scalar field \( \phi \) down a potential \( V(\phi) \); \( \phi \) is minimally coupled to gravity. The authors linearized the full field equations for \( \phi \), the matter, and the metric and solved the resulting system numerically. Intriguingly, they found that, at late times, a local overdensity of matter corresponded to a local underdensity, or void, of dark energy. Conversely, a void in the matter was seen to produce a local DDE overdensity. Although the DDE density contrast, \( \delta_{\phi} \), is initially very small compared to the matter density contrast, \( \delta_m \), they found that, when \( \delta_m \sim O(1), |\delta_{\phi}| \sim O(10^{-2}) \). The DDE density contrast \( \delta_{\phi} \) was also observed to be growing more quickly than \( \delta_m \) at late times. Their results suggest that DDE clustering may induce a positive correction to the value of \( (1 + \nu) \) of more than 10% at the center of an inhomogeneity with properties similar to that of the Local Supercluster (LSC). If accurate, the results of Dutta & Maor (2007) imply that any deviations from \( \nu = -1 \) would be significantly amplified by the presence of a local overdensity of matter. Furthermore, they suggest that DDE clustering might be relatively strong when the matter perturbation goes nonlinear.

Dutta & Maor (2007) studied DDE clustering when the matter perturbation is growing in the linear regime, which is only accurate when \( \delta_m \ll 1 \). They considered the evolution \( \delta_{\phi} \) for a cluster of matter with initial density profile (at \( z = 35 \)) of \( \delta_m = A \exp(-r^2/\sigma^2) \), where \( A = 0.1 \) and \( \sigma = 0.01H^{-1} \) with \( H \) being the initial value of the Hubble parameter. However, in the absence of any DDE, the linear approximation would generally cease to be accurate when \( z \approx 2.8 \); in addition, the perturbation would be expected to turn around when \( z \approx 1.0 \) and virialize when \( z \approx 0.3 \). It is therefore far from clear whether or not the sharp late-time growth in \( \delta_{\phi} \) found by Dutta & Maor (2007) is indeed a physical effect or just a result of using linearized field equations outside of the realm in which they are valid. However, one might well expect that there exists some mapping between the linear and nonlinear regimes as happens in an Einstein–de Sitter universe.

In this paper we investigate a similar problem to that considered by Dutta & Maor (2007), i.e., the clustering of uncoupled quintessence on subhorizon scales. There are, however, two important differences between our approach to this problem and the one taken by Dutta & Maor (2007):

1. First, we do not use numerical simulations, but instead use the method of matched asymptotic expansions (MAEs) to develop an analytical approximation to \( \delta_{\phi} \).

2. Second, we do not require the density contrast of the matter perturbation, \( \delta_m \), to be small. Indeed, our analysis and results remain valid even after the virialization of the matter overdensity. Our approximation for \( \delta_{\phi} \) is accurate provided that \( \delta_m \) is only nonlinear \( [\geq O(1)] \) on subhorizon scales. This provision is consistent with observations. For simplicity we take the matter perturbation to be spherically symmetric. We also require that gravity is suitably weak whenever the inhomogeneity is nonlinear, i.e., \( GM/R \ll 1 \). We restate these requirements in a rigorous fashion later; however, they are essentially equivalent to the statement that gravity is approximately Newtonian over scales smaller than \( H^{-1} \).

Since DDE clustering in the linear regime has been dealt with in great detail by Dutta & Maor (2007), our main focus in this paper is on what occurs when the matter perturbation goes nonlinear.

This paper is organized as follows. In § 2 we describe our model for an inhomogeneous spacetime and the DDE. We state the equations that must be satisfied by the metric quantities and the dark energy scalar field \( \phi \). In § 3 we introduce the method of MAEs. This method relies on the existence of locally small parameters, and we state what these are for our model and interpret them physically. We also note what constraints the smallness of these parameters places on our analysis. In § 4 we apply the method of MAEs to the evolution of DDE perturbations. We derive a simple equation for the DDE density contrast, \( \delta_{\phi} \), in terms of the peculiar velocity of matter particles, \( \nu \), and the matter density contrast, \( \delta_m \). Importantly, this equation is equally as valid for \( \delta_m \geq O(1) \) ("the nonlinear regime") as it is when \( \delta_m \ll O(1) \) ("the linear regime"). In § 5 we use our results to study the evolution of \( \delta_{\phi} \) in the linear (\( \delta_m \ll 1 \)), quasi-linear (\( \delta_m \sim O(1)-O(10) \)), and fully nonlinear (\( \delta_m \gg 1 \)) regimes. We compare our analytical results with those found numerically in the linear regime by Dutta & Maor (2007). In § 6 we consider the spatial profile of the DDE density contrast for realistic astrophysical inhomogeneities such as voids, supervoids, clusters, and superclusters. We conclude in § 7 with a discussion of our results and their observational implications. We also note how our analysis might be extended to include even more general DDE models. Throughout the paper we use units where \( c = 1 \).

2. THE MODEL

2.1. Geometrical Setup

Our aim in this paper is to derive an expression for the DDE density contrast, \( \delta_{\phi} = \delta_{\phi}^{(0)}/\delta_{\nu}^{(0)} \), inside an overdensity, or underdensity, of matter. Henceforth, we use \( \nu^{(0)} \) to represent the energy density of a component \( i \). We are particularly interested in those cases where the matter perturbation is nonlinear, i.e., \( \delta_{\phi}^{(mn)}/\delta_{\nu}^{(mn)} \geq O(1) \). Although we aim to remain suitably general in our treatment of the density perturbation, we do make the following simplifying assumptions:

1. We assume spherical symmetry. We briefly discuss the extent to which the relaxation of this assumption would affect our results in § 7 below and conclude that the qualitative nature of our findings would be unaffected.
2. We define a “physical radial coordinate,” $R$, by the requirement that a spherical surface with physical radius $R$ has surface area $4\pi R^2$. We assume that all curvature invariants are regular at $R = 0$, i.e., we do not consider those cases in which there is a central black hole. We argue below, however, that our results are still accurate even when there is a central black hole, provided they are only applied at radii that are large compared to the Schwarzschild radius of the black hole.

3. Finally, we assume that for radii smaller than some $R_0 \ll H^{-1}$ gravity is suitably weak in the inhomogeneous region; $H$ is the Hubble parameter of the background spacetime. We define what we mean by weak rigorously in §3. For radii $R > R_0$, we require that the matter density contrast is $\leq 1$. This assumption holds for most realistic models of collapsing overdensities provided that the radius of the overdense region is less than about $0.1/H$.

2.2. Einstein’s Equations

We take the matter content of the universe to be a mix of irrotational dust and DDE, which is described by a scalar field $\phi$. For simplicity we restrict ourselves to considering only spherically symmetric spacetimes for which the most general line element in comoving coordinates is

$$ds^2 = dr^2 - U(t, r) dt^2 - R(t, r)^2 (d\theta^2 + \sin^2 \theta \, d\varphi^2).$$

We make the definitions $U(t, r) \equiv \tilde{R}^2(t, r)/Q(t, r)$ and $k(t, r) \equiv 1 - Q(t, r)$. This coordinate choice is unique up to $t \rightarrow t + t_0$ and $r \rightarrow r'(r)$. With these definitions the 2-spheres $(t, r) = \text{const}$ have surface area $4\pi R^2(t, r)$, and in this sense, $R(t, r)$ represents the “physical radial coordinate.” The energy-momentum tensor of pressureless dust is given by

$$T^{(m)}_{ab} = \text{diag}(\delta^{(m)}, 0, 0, 0),$$

and the energy-momentum tensor of the scalar field, $\phi$, is

$$T^{(\phi)}_{ab} = \partial_a \phi \partial_b \phi - g_{ab} \left[ \frac{1}{2} \partial \phi \partial \phi - V(\phi) \right].$$

The Einstein equations for this metric read

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} = \kappa \left[ T^{(m)}_{ab} + T^{(\phi)}_{ab} \right],$$

where $\kappa = 8\pi$. The $tt$-component of equation (1) gives

$$\left( R^2_r R_t + k R \right)_r = R_{,r} R^2 \delta^{(m)} + R_{,r} R^2 \kappa \left[ \frac{1}{2} \dot{\phi}^2 + \frac{Q}{2 R^2_r} \dot{\phi}^2_r + V(\phi) \right] + R R_{,r} R_{,r} \frac{\dot{Q}}{Q},$$

and the $rr$-component of the Einstein equations is

$$\frac{R_{,r} \dot{Q}}{R} = -\kappa \dot{\phi}^2 \phi_{,rr}.$$  (3)

The $\theta\theta$-component of equation (1) reads

$$\left( R^2_r R_\theta + k R \right)_\theta + Q_{,\theta} R =$$

$$-\kappa \left[ \frac{1}{2} \dot{\phi}^2 + \frac{Q}{2 R^2_r} \dot{\phi}^2_r - V(\phi) \right] R_{,r} R^2.$$  (4)

From $T^{(m)}_{\mu \nu} = 0$, it follows that

$$\kappa \delta^{(m)} = \frac{F(r) \sqrt{Q(r, t)}}{R_{,r} R^2},$$

where $F(r)$ is an arbitrary constant of integration. If we define $r$ so that $R(r, t) = r$ for some $t = t_0$, then $F(r) = 2M_M(r)[Q(r, t)]^2$, where $M_M(r)$ is the mass inside a shell of radius $r$ at $t = t_0$.

From $T^{(\phi)}_{\mu \nu} = 0$, one obtains

$$-\square \phi = V_{,\theta}(\phi).$$  (6)

Equation (6) is subject to the boundary conditions $\phi_r = 0$ at $R = 0$ and $\phi_r(t) = \lim_{r \to \infty} \phi(r, t)$, where $\phi_r(t)$ is the solution of equation (6) in the cosmological background. We make the definitions

$$2M(r, t) = \int_{r_0(t)}^r F(r) \sqrt{Q(r, t)} \, dr,$$

$$\varepsilon^{(0)} = \frac{1}{2} \dot{\phi}^2 + V(\phi),$$

$$\delta \varepsilon^{(0)} = \frac{1}{2} \dot{\phi}^2 + \frac{Q}{2 R^2_r} \dot{\phi}^2_r + V(\phi) - \varepsilon(t),$$

$$\delta P^{(0)}(r, t) = \delta \varepsilon^{(0)} - R \frac{\dot{\phi}}{R_r} \phi_r.$$
3. SMALL PARAMETERS AND THE MATCHED ASYMPTOTIC EXPANSIONS

One can think of the small-parameters approach and the matched asymptotic expansions (MAEs) as an expansion in the Newtonian potential. Although it is perhaps fairer to say that it is closest to the context of general relativity. The differences are not obvious at leading order, but at next to leading order, one will see that what appears in the equation for the acceleration is not necessarily what one would expect to appear in the nonrelativistic regime.

As stated above in § 2.1, we require that gravity is “suitably weak” inside the density perturbation. By suitably weak we mean that

$$|\delta_1| \ll 1,$$

where

$$\delta_1 = \delta v \equiv \frac{R_1 - HR}{HR}.$$  

Physically, $R_1$ is the velocity of a particle of dust in $(t, R)$ coordinates, where $R$ is the physical radial coordinate and $HR$ is the velocity that such a particle would have in the cosmological background. The small parameter $\delta_1 = \delta v$ is therefore the peculiar velocity of a matter particle with respect to the cosmological background. Recall that we use units where $c = 1$; hence, the first expression above is in fact the dimensionless quantity $\delta v / c \ll 1$. Our assumption requires that whenever $HR \gg O(1)$, we have $|\delta_1/HR| \ll 1$. This condition is equivalent to requiring that the mean matter density contrast, $\delta(R, t)$, inside the sphere with radius $R \sim O(1/H)$ or greater, is small enough that it may be treated as a linear perturbation about the cosmological background. The mean matter perturbation is however allowed to be nonlinear, $\delta \sim O(1)$, when $HR \ll 1$ provided $|R_1| \ll 1$. We also define

$$\delta_2 = \frac{1}{3} R^2 \kappa \left( c^m - c^v \right) \equiv \frac{1}{3} R^2 \kappa \left( c^m \right) \delta_m.$$  

We require $\delta_2 \ll 1$. This again require that $\delta m \gg 1$ only when $HR \ll 1$, i.e., the matter perturbation is only nonlinear on subhorizon scales. We note that $\delta_2 \sim O(R^2 \kappa^2 - H^2 R^2)$, and so generally $|\delta_1| \ll 1$ provided $\delta_2 \ll 1$.

In what follows we assume that $\delta_1$ and $\delta_2$ are small everywhere. These assumptions can be checked once one specifies initial conditions for the matter overdensity, but they generally hold very well whenever the scale of the inhomogeneous region is <0.1/H. In §§ 3.1 and 3.2 we define two overlapping regions which we refer to as the interior and the exterior.

3.1. The Interior Region

The interior region is defined by

$$\delta_1 \equiv HR \ll 1.$$  

We define the inner limit of a quantity $F(r, t)$ to be $L_{in}(F) = \lim_{\delta_1 \to 0} F(r, t)$. The inner limit can be imagined as the limit in which the cosmological background density of matter is taken to zero and the cosmological horizon is taken to infinity. In the interior we construct asymptotic approximations to quantities in the limit $HR \to 0$. We also expand in $\delta_1$ and $\delta_2$. We refer to this as the inner approximation.

3.2. The Exterior Region

In the exterior region, spacetime is required to be homogeneous and isotropic at leading order. In other words, the matter perturbation must be linear in the exterior.

$$\delta_4 \equiv |\delta_1/\delta_3| \ll 1, \quad \delta_5 = |\delta_m| \ll 1.$$  

From the form of $R_1$, it is clear that $\delta_3 \to 0$ implies $\delta_4 \to 0$. We define the exterior limit of a quantity $F(r, t)$ to be $L_{out}(F) = \lim_{\delta_3 \to 0} F(r, t)$. As in the interior limit we also expand in $\delta_1$ and $\delta_2$. We construct asymptotic approximations to quantities in the exterior region in this exterior limit and refer to them as the outer approximations.

3.3. Matching and the Intermediate Region

We are primarily concerned with the behavior of the DDE in the interior region. In § 4 we find the inner approximation to $\phi$ by solving equation (6) order by order in the interior limit. We cannot, however, apply both of the boundary conditions on $\phi$ directly to the inner approximation. This is because the condition $\lim_{r \to \infty} \phi = \phi_0(t)$ must be applied at $R = \infty$, a point that is very clearly in the exterior and not in the interior region. As a result, the inner approximation will contain ambiguous constants of integration. Fortunately, this ambiguity can be lifted by matching the inner and outer approximations to $\phi$ if there exists some intermediate region where both approximations are simultaneously valid. This matching of the inner approximation to the outer one is referred to as the method of MAEs. It relies on the fact that, in any given region, the asymptotic expansion of a quantity is unique (for a proof see Hinch 1991). Thus, if the inner and outer approximations of $\phi$ are both valid in the intermediate region, they must be equal in that region. For the method of MAE to be applicable we must, of course, require that an intermediate region exists. A necessary condition for an intermediate region to exist is that for some range of $R$,

$$HR \ll 1, \quad |3 \delta_2 / \kappa c^m(t) R^2| = |3 \delta_m(t) / \kappa| \ll 1.$$  

This becomes a necessary and sufficient condition if the only boundary of the interior region is in the exterior one and vice versa (see Fig. 1 for an illustration). If the scale of the inhomogeneity, $\delta_5$, is taken to be the largest value of $R$ for which $|3 \delta_2 / \kappa c^m(t) R^2| > 0.3$, then an intermediate region generally exists provided that $\delta_5 \leq 0.1H$.

In summary, in this section we have considered spherically symmetric backgrounds and far-from-black-hole horizons. We have basically performed an expansion in the Newtonian potential at leading order, at least as far as the evolution of scalar field perturbations are concerned. Note, however, that this method can be straightforwardly extended to allow for deviations from spherical symmetry (Shaw & Barrow 2006a, 2006b, 2006c).

To be clear, we should point out that the expansion in $\delta_1$ and $\delta_2$, which is a lot like expanding in the Newtonian potential, is not really the key to our method. That is just a simplification. The key is that we take $\delta_1 = HR \ll 1$ in the interior region and $\delta_3 \ll 1$ in the exterior region. That is, everything is linear in those two regions. We then assume that there is an intermediate region where both of these conditions hold. This implies that in the intermediate region $|\delta_1| \ll 1$ and $|\delta_2| \ll 1$. So the assumption that $\delta_1, \delta_2 \ll 1$ everywhere ensures that we have an intermediate
region. We could in fact relax this, but the calculations would become more difficult, often still doable, but in almost all physically interesting cases totally unnecessary. For instance, the relaxation to \( |\delta_1| < 1 \) and \( |\delta_2| \ll 1 \) is very straightforward and allows one to go all the way up to a black hole horizon (Shaw & Barrow 2006a, 2006b, 2006c).

4. EVOLUTION OF DYNAMICAL DARK ENERGY PERTURBATIONS

In this section we find an asymptotic approximation to the DDE density contrast, \( \delta_\phi \). The discussion is fairly technical, and readers more interested in the results than the machinery used to derive them may prefer to focus on the statement, discussion, and application of our results in § 4.3 and following.

The DDE is described by the field \( \phi \) which satisfies

\[
-\Box \phi = V_\phi(\phi).
\]

We solve this equation by constructing an asymptotic approximation to \( \phi \) in the small parameter \( \delta_1 \). We write

\[
\phi \sim \phi_\circ(t) + \delta \phi(t, r) [1 + O(\delta_1)], 
\]

where \( \delta \phi \sim O(\delta_1) \). Before solving for \( \phi \), we make the dependence of the metric on \( \delta_1 \) explicit by transforming to a new radial coordinate, \( \rho = R(t, r)/a(t) \), where \( a(t) \) is the scale factor of the FRW cosmological background. In \( (t, \rho) \) coordinates the metric is

\[
d s^2 = \frac{1 - k(r, t) - \delta_1^2(r, t)}{1 - k(r, t)} \frac{d t^2 + \delta_1(r, t)a(t) d t d \rho}{1 - k(r, t)} - a^2 V_\phi(\phi) \rho^2 (d \theta^2 + \sin^2 \theta d \varphi). \tag{9}
\]

Consider \(-\Box \phi_\circ(t)\) in this metric,

\[
-\Box \phi_\circ(t) = V_\phi(\phi_\circ) + \frac{1}{2} \frac{\dot{\phi}_\circ(t)}{\dot{a}(t) \rho^2} \frac{\partial}{\partial \rho} (\rho^2 \delta_1),
\]

where \( \dot{Q} \) is evaluated at constant \( r \) as opposed to constant \( \rho \) or \( R \) and is thus given by equation (3). It follows that \( \delta \phi \) satisfies

\[
-\Box \delta \phi(1 + O(\delta_1)) = \left[ V_\phi(\phi) - V_\phi(\phi_\circ) \right] + \frac{1}{2} \frac{\kappa \dot{\phi}_\circ^2(t)}{a(t) \rho^2} (\rho^2 \delta_1), \tag{10}
\]

We now solve for \( \delta \phi \) in both the interior and exterior regions.

4.1. Inner Approximation

We note that \( \frac{1}{2} \kappa \dot{\phi}_\circ^2 R^2 \lesssim O(\delta_1^2) \) and that \( |k| \sim O(\delta_3, \delta_1, \delta_2) \), so that

\[
-\frac{\rho^2}{a} \frac{\partial}{\partial t} (a^2 \delta \phi, \rho) + (\rho^2 \delta \phi, \rho) \sim \left[ V_\phi(\phi) - V_\phi(\phi_\circ) \right] a^2 \rho^2
\]

\[
+ \dot{\phi}_\circ(t)a(\rho^2 \delta_1, \rho) + O(\delta_1 \delta_\phi, \delta_2 \delta_\phi, \delta_3^2 \delta \phi). \tag{11}
\]

In many cases \( \delta \phi \) is small enough so that

\[
[V_\phi(\phi) - V_\phi(\phi_\circ)]a^2 \rho^2 \sim \delta \phi(0, \rho) R^2 \delta \phi,
\]

and \( V_\phi(\phi_\circ) \sim O(H^2) \) so that \( \delta \phi(t) \) evolves over cosmological timescales. In these cases it is clear that

\[
[V_\phi(\phi) - V_\phi(\phi_\circ)]a^2 \rho^2 \sim \delta \phi(0, \rho) R^2 \delta \phi \sim O(\delta_1, \delta_2, \delta_3). \tag{12}
\]

A physically viable DDE theory for which this requirement does not hold would be difficult to construct. For instance, if this condition did not hold, then even if, at one instant, \( V_\phi(\phi_\circ) \sim O(H^2) \) or smaller, as must be required for \( \phi \), to evolve over cosmological timescales, shortly afterward small changes in \( \phi \) would cause \( V_\phi(\phi_\circ) \) to grow to be many orders of magnitude greater than \( H^2 \). This would result in \( \phi \) evolving over timescales that are much shorter than the Hubble time. The only DDE theories that we can imagine that could accommodate this and still be compatible with observations would involve \( \phi \) being held almost completely fixed at a minimum of \( V(\phi) \), in which case the DDE would be almost entirely indistinguishable from a cosmological constant.

To leading order in the interior, we would actually be justified in setting \( a(t) = \text{const} \) as \( \dot{a} \dot{\phi} = HR = \delta_1 \ll 1 \). However, since it might not be completely clear that these terms may be ignored and because we can solve the \( \delta \phi \) equation without having to ignore them, we continue to include them.

We define \( d \eta = dt/a(t) \) and \( u = a \dot{\phi} \) and note that \( \rho^2 a_{\eta \eta}/a = H R^2 + H^2 R^2 \sim O(\delta_1) \), so that

\[
\begin{aligned}
&\left[ -u_{\eta \eta} + \frac{1}{\rho^2} \left( \rho^2 u_{,\rho} \right) \right] [1 + O(\delta_1, \delta_2, \delta_3)] \\
&\sim \dot{\phi}_\circ(t) \frac{1}{\rho^2} [a(t) \rho^2 \delta_1], \tag{12}
\end{aligned}
\]
which has as a solution
\[
\delta \phi(t, \rho) \sim -\frac{1}{2} \int_0^\infty \rho^2 \, d\rho' \int_{-1}^1 d\eta \frac{f(\eta(t) - X, \rho')}{a(t)X(\rho', \rho, s)} + \mathcal{F}(\eta - \rho) + \mathcal{G}(\eta + \rho) + O(\delta_1 \delta_\rho \delta_2 \delta_\phi, \delta_1 \delta_\phi),
\]
where
\[
f(\eta, \rho) = \left[ \phi(t, a(t))/|\rho^2| |\rho^2 \phi(t, \rho)| \right] \text{ and } X(\rho, \rho', s) = (\rho^2 + \rho'^2 - 2\rho \rho' \delta)^{1/2}.
\]
\(\mathcal{F}\) and \(\mathcal{G}\) represent waves in the scalar field and we discuss them further below. To go further, we make the following reasonable assumptions about the behavior of \(\delta_1\).

We assume that there exists some \(R_{-2}\), in the interior region, such that for \(R > R_{-2}\) in the interior region, \(\delta_1\) decreases faster than \(1/R^2\) as \(t \to \infty\); \(f\) then decreases faster than \(1/R^3\) for \(R > R_{-2}\). This assumption ensures that dominant contributions to the integral in equation (13) come from values of \(\rho'\) that are well inside the interior region. Note that generally \(f(\eta, \rho)\) varies over conformal timescales of order \(1/aH\) and \(\delta_1/\delta_\rho \delta_\phi\). Provided that \(\delta_1\) is not momentarily zero, these scales are much larger than \(X = \sqrt{2}R_{-2}/a\), so we can Taylor expand \(f_I(\eta - X, \rho')\) as
\[
f(\eta - X, \rho') = f(\eta, \rho') + f_{2,0}(\eta, \rho')X + O(f_{0,2}X^2),
\]
where the neglected terms are \(O(\delta_1 \delta_\rho \delta_2 \delta_\phi)\) times \(f(\eta, \rho')\). After some algebra and integration we then arrive at
\[
\delta \phi(t, R)[1 + O(\delta_1, \delta_\rho, \delta_\phi)] \sim \tilde{\phi}_\rho(t) \int_{-1}^1 \delta_1(t, R') \, dR' + \mathcal{F}(\eta - \rho) + \mathcal{G}(\eta + \rho)
\]
where the previous assumption relating to the Taylor expansion of \(f\) can be seen to be equivalent to the statement that \(\delta_1(t, R)R^2 \ll \delta_1\). This assumption will generally break down if there is some initial instant, \(t = t_i\); say, when \(\delta_1 = 0\). At \(t \gg t_i\), however, \(\delta_1 R^2 \ll \delta_1\) is equivalent to \(\delta_1 \delta_\rho \delta_\phi \ll 1\), which is certainly the case in the interior region.

Generally, the functions \(\mathcal{F}\) and \(\mathcal{G}\) are related to the initial conditions on \(\delta \phi\). Because we have required \(R_{-2} \ll H^{-1}\), our analysis will break down at early times, since generally \(HR_{-2} \to \infty\) as \(t \to 0\). As a result, we cannot generally determine \(\mathcal{F}\) and \(\mathcal{G}\) by simply specifying some initial conditions on \(\delta \phi\), at \(t = t_i\); say, and applying them to the interior approximation, since the interior approximation may not be valid when \(t = t_i\). The requirement that \(\delta \phi(R = 0)\) can however be applied to the interior approximation to give \(\mathcal{G}(\eta) = -\mathcal{G}(\eta)\). To the order at which we work, \(\mathcal{F}(\eta)\) itself should be fixed by matching to the outer approximation.

### 4.2. Exterior Solution and Matching

In order for \(\phi_c = \lim_{t \to -\infty} \phi_c(t, R)\) we must have \(\delta \phi(t, R) \to 0\) as \(R \to \infty\). In the exterior region, spacetime is FRW to leading order in \(\delta_1\) and \(\delta_\rho\), and the equation describing it for the leading-order behavior of \(u = a(t)\delta \phi\) reads
\[
- \frac{\dot{u}}{\dot{a}} + \frac{1}{\rho^2} \left( \rho^2 \dot{u}_\rho \right) + \left[ V'(\phi_c) - \dot{H} - H^2 \right] a^2 u
+ \frac{1}{2} \kappa \delta \phi^2 a^2 \mu_{\mu, \rho} + \frac{\phi_c(t) a^2}{\rho^2} (\rho^2 \phi) = 0,
\]
where \(d\eta = dt/a(t)\). Solving this equation is far from straightforward. Fortunately, however, our main interest in not in how \(\delta \phi\) behaves in the exterior region but in the interior region where the perturbation in the matter density can be nonlinear, and so it is not necessary to solve equation (15) in the exterior region so long as we know how its solutions behave in some intermediate region where \(\delta_1 = HR \ll 1\). We require that \(\delta \phi \to 0\) as the matter perturbation is removed, i.e., \(\delta_1 \to 0\). We have also required that \(\delta_1\) drop off faster than \(1/R\) for \(R > R_{-2}\), where \(R_{-2}\) is in the interior region, i.e., \(HR_{-2} < \ll 1\). In this intermediate region, the solutions of equation (15) for which \(\delta \phi \to 0\) as \(R \to \infty\) have the form
\[
\delta \phi(t, R) \sim \phi_c(t) \int_{C(t)}^R \delta_1(t, R') \, dR' + \frac{K(\eta)}{R} + O(\delta_1 \delta_\rho \delta_2 \delta_\phi, \delta_1 \delta_\phi),
\]
where \(K(\eta) \sim O(1/H)\) or greater. We have assumed that \(t\) is larger compared to any initial instant when one or more of \(\delta_1 = 0\), \(\delta_\phi = 0\), or \(\delta \phi = 0\) hold. With these assumptions, matching to the interior region gives
\[
K(\eta) = \mathcal{F}(\eta) + \mathcal{G}(\eta) = 0,
\]
where
\[
\frac{2\mathcal{F}(\eta)}{\mathcal{A}(\eta)} = -\frac{\dot{\phi}_\rho(t)}{\mathcal{C}(t)} \int_{C(t)}^\infty \delta_1(t, R') \, dR'.
\]
That we have required a drop off of \(\delta_1\) means that, to leading order, we can set \(C = \infty\). This implies that \(2\mathcal{F}(\eta)/\mathcal{A}(\eta)\) is subleading order in the interior approximation and so may be neglected.

Thus, to leading order in the interior region we find
\[
\delta \phi(t, R) \sim \phi_c(t) \int_{C(t)}^R \delta v(R', t) \, dR' [1 + O(\delta_1, \delta_\rho, \delta_\phi)],
\]
where \(\delta v = R, t - HR\) is the peculiar radial velocity of the matter particles relative to the expansion of the background universe. This approximation is valid in the interior region at late times compared to any initial instance when \(\delta \phi = 0\) and/or \(\delta \phi = 0\), which is equivalent to \(|\delta \phi| R^2 \ll |\delta \phi| R \ll |\delta v|\). Note that, to this order, the surfaces of constant \(\phi\) are surfaces of constant
\[
t_\phi = t - \int_R^\infty \delta v \, dR',
\]
We use this asymptotic approximation for \(\delta \phi\) to evaluate the DDE density contrast on surfaces of constant \(t\) in the matter rest frame. In the rest frame of the matter particles,
\[
\varepsilon^{(\phi)} = \frac{1}{2} \phi^2 + \frac{Q}{2} \phi^2 + V(\phi),
\]
and the local inhomogeneity in \(\varepsilon^{(\phi)}\) is therefore
\[
\delta \varepsilon^{(\phi)} \sim \phi_c^2(t) \left\{ -3H \int_{C(t)}^R \delta v(R', t) \, dR' + [\delta v(R, t)]^2 \right\} + \int_R^\infty \left[R_{\mu} - (\dot{H} + H^2)R\right] dR'.
\]
where \(\delta v\) and \(\delta v\) are the corrections terms times smaller than the leading-order term. Since \(\phi_c^2 = (1 + w)\varepsilon^{(\phi)}\), we have
\[
\delta \phi \sim (1 + w) \left\{ -3H \int_{C(t)}^R \delta v(R', t) \, dR' + [\delta v(R, t)]^2 \right\} + \int_R^\infty \left[R_{\mu} - (\dot{H} + H^2)R\right] dR',
\]
where $R_m$ is the acceleration of the matter particles in $(t, R)$ coordinates. Equation (9) gives

$$R_m R - HR^2 - H^2 R^2 \sim -\frac{\delta M}{R} + O(\delta_3^2, \delta_1^2, \delta_2^2),$$

where $\delta M(R, t) = M(R, t) - \frac{1}{2} \Omega_m H^2 R^3 = \frac{1}{2} \Omega_m H^2 R^3 \delta$ is the mass contrast at time $t$ inside the sphere with physical radius $R$.

4.3. Discussion

In this section we found that the DDE density contrast can be expressed, to leading order, in terms of the radial peculiar velocity, $\delta v$, the mean density contrast, $\delta$, and the unperturbed DDE equation of state parameter $w$:

$$\delta_\phi \sim (1 + w) \left[ 3H \int_R^\infty \delta v(R', t) dR' + (\delta v)^2 + \frac{1}{2} \Omega_m H^2 \int_R^\infty \delta(R', t) R' dR' \right].$$

This expression for $\delta_\phi$ is our main result, and it is valid for both linear and nonlinear subhorizon matter perturbations. In order to evaluate the above expression, one must simply specify the peculiar velocity, $\delta v$, of the matter particles and the matter density contrast $\delta$. This exterior is valid in the interior region at late times compared to any initial instance when $\delta \phi = 0$ and/or $\delta = 0$, i.e., $|\delta v| R^2 \ll |\delta| R \ll |\delta v|$.

Note that $H$ and $w$ will depend on the mass of the field $m$, so in that sense, equation (18) is not completely model independent. However, all the model-dependent terms come from the homogeneous and isotropic background cosmology. So the expression for $\delta_\phi$ is model independent provided $m R_c \ll 1$. This is true if $m \sim O(H)$ and $HR_c \ll 1$, as assumed.

The key point is that large deviations in $\delta_\phi$ from the prediction using a $\Lambda$CDM background could only occur if $\delta v$ and $\delta m$ were predicted to change by an order of magnitude or more. As far as we know, however, structure formation is compatible with $\Lambda$CDM over the scales of clusters, and so it seems unlikely that such large deviations occur.

As we see below, the DDE perturbation is generally small compared to the matter perturbation, so to leading order one may neglect any DDE perturbations when evaluating $\delta$ and $\delta v$. It is interesting to note that the sign of $\delta_\phi$ depends on the relative magnitudes of $\delta$ and $\delta v$. From equation (18) one deduces that the formation of local overdensities or local voids of dark energy arise from two competing effects:

**Drag effect.**—Associated with deviations from $\delta v / HR = 0$. If one has an overdense region, then it expands more slowly. In agreement with Dutta & Maor (2007), we find this drag effect to produce a local underdensity of dark energy.

**Pull effect.**—Associated with deviations from $\delta m = 0$. An overdensity of matter pulls the dark energy (and everything else) toward it. So matter and fields tend to clump around it. As a result, the pull effect means that an overdensity of matter creates an overdensity of dark energy.

So there is one effect that pulls the dark energy in and another that pushes it out. As we see below, in the linear regime the drag effect grows more quickly than the pull effect, so one gets a dark energy void. The onset of nonlinear structure formation, however, causes $|\delta v / HR|$ to reach a maximum and then to tend to 1 at late times, whereas $\delta m$ just keeps on growing. So the pull effect dominates at late times, so one ends up with an overdensity of dark energy.

We have assumed above that the mass of the DDE scalar field is small compared to the inverse length scale of the cluster. This is provided $\delta_\phi$ in uncoupled DDE theories is independent of the details of theory describing the dark energy.

In § 5 we use equation (18) to evaluate $\delta_\phi$, in both the linear and nonlinear regimes. We find that $|\delta_\phi(R = 0)| \sim O(\Omega_m H^2 R^3 \delta_\text{clust})$, where $R_c$ is the radial scale of the cluster and $\delta_\text{clust}$ is the mean matter density contrast in $R \leq R_c$.

5. CLUSTERING OF DYNAMICAL DARK ENERGY

5.1. Dynamical Dark Energy Clustering in the Linear Regime

Although some of our main aims in this work is to study DDE clustering in the nonlinear regime, our analysis is also perfectly valid in the linear regime, i.e., when $\delta m \ll 1$, provided that the matter inhomogeneity is subhorizon at the instant when we wish to evaluate $\delta_\phi$. We found above that $\delta_\phi$ is given by equation (18). In the linear regime $|\delta v| \ll HR$, and so the term proportional to $\delta v^2$ in equation (18) is small compared to the other two terms and should be neglected. The matter density contrast is $\delta m (r, t)$, and we define $\delta_\phi (r, t)$ to be its Fourier transform. In the linear regime (in comoving coordinates),

$$\delta_m = -\nabla \cdot \delta v,$$

and so,

$$\delta_{\phi k} = -\frac{i H f^\phi(a) a \delta_\phi}{k},$$

where $f(a) = d \ln \delta / d \ln a$ and $\delta_\Phi$ is the Fourier transform of $\delta v$. We recognize that $3 \Omega_m H^2 \delta R^2 / \delta m R^2 = \frac{3}{2} \Omega_m H^2 \delta \Phi / \partial R$, where $\delta \Phi$ is the leading-order perturbation to the Newtonian potential and is given by

$$\nabla^2 \delta \Phi = 4 \pi G \delta m.$$

The Fourier transform of $\delta(R, t) R$ is therefore $a^2 \delta_\Phi / k$, and the Fourier transform of the DDE density contrast, $\delta_\phi$, in the linear regime is given by

$$\delta_\phi^{(\text{lin})} \sim \frac{3}{2} (1 + w) g(a) \frac{\Omega_m a^2 H^2 \delta_\phi}{k^2},$$

where

$$g(a) = \frac{2 f(a)}{\Omega_m} - 1.$$

Motivated by observations, we assume $(1 + w)$ to be small at late times and $(1 - \Omega_m)$ to be small at early times. To a first approximation then we approximate $f(a)$ by its $\Lambda$CDM value. At late times a very good approximation to $f(a)$ in the $\Lambda$CDM model is $f_{\Lambda\text{CDM}}(a)$ (Peebles 1980; Lahav et al. 1991). Note that corrections to this fitting formulae occur for quintessence-like models (Wang & Steinhardt 1998). However, these are generally too small to greatly affect our results. Hence, we take the $\Lambda$CDM expression for simplicity. Even when it is not acceptable to approximate $f(a)$ by its $\Lambda$CDM expression, we do not expect this approximation to greatly alter the qualitative nature of our results or the order of magnitude of $\delta_\phi$.

Transforming back into real space we have

$$\delta_\phi^{(\text{lin})} \sim \frac{(1 + w) g(a) \Omega_m a^2 H^2}{2} \int_r^\infty r'^2 \delta(t', r') dr'.$$
Our expression for $\delta^{(\text{lim})}_{\phi, k}$ is independent of the mass of $\phi$, because we have assumed that $m_\phi \sim O(H)$ in the linear regime and that the matter perturbation is small compared with $H^{-1}$. The mass of the scalar field therefore has only a negligible effect on the scales over which DDE clustering occurs. It is clear that the sign of $\delta_{\phi, k}$ is the same as the sign of $-g(a) \delta_k$. At late times $g(a) > 0$, and so a linear local overdensity of matter ($0 < \delta_m < 1$) corresponds to a DDE void. Similarly, there is a DDE overdensity at late times (in the linear regime) if there is a local void of matter. If an initial mean density perturbation $\delta_k(z_i)$ begins to collapse at $z = z_i$, when $a = a_i = 1$, $\Omega_m = \Omega_{mi}$, and $H = H_i$, we have

$$
\delta_k \approx \frac{1}{3\Omega_{mi}^2 + 2} \left[ 3\Omega_{mi}^{0.4} \left( \frac{\Omega_m}{\Omega_{mi}} \right)^{0.2} a + 2 \left( \frac{\Omega_{mi}}{\Omega_m} \right)^{0.5} a^{-3/2} \right] \delta_k(z_i),
$$

and so

$$
\delta^{(\text{lim})}_{\phi, k} \sim \frac{9(1 + w)\Omega_{mi}H_i^2}{6\Omega_m^{0.4} + 4} \left[ \Omega_{mi}^{0.4} (2\Omega_m^{0.4} - 1) \left( \frac{\Omega_m}{\Omega_{mi}} \right)^{0.2} 
- \frac{8}{3} \left( \frac{\Omega_{mi}}{\Omega_m} \right)^{0.5} \left( \frac{1 + z}{1 + z_i} \right)^{5/2} \delta_k(z_i) \right] \frac{k^2}{2}. 
$$

Our expression for $\delta_{\phi,k}$ was derived under the assumption that $|\delta| R^2 \ll |\delta| R \ll |\delta|$. As a result, it will break down as we approach $z = z_i$. More precisely, we find that our approximation is valid at $R = 0$ for $a \gg a_i [1 + 2H_i R_{z}(z_i)]$, i.e., $z \ll z_{\text{app}} \equiv z_i [1 - 2H_i R_{z}(z_i)]$, where $H_i$ is the initial value of $H$ and $R_{z}(z_i)$ is the initial value of $R_{z}$. From this expression we can see that a matter overdensity corresponds initially to a DDE overdensity ($\delta^{(\text{lim})}_{\phi} > 0$); however, this DDE overdensity becomes a DDE void at some $z = z_{\text{crit}}$, and $\delta^{(\text{lim})}_{\phi} < 0$ for $z < z_{\text{crit}}$. We see that $z_{\text{crit}}$ is given by

$$
\frac{1 + z_{\text{crit}}}{1 + z} \approx \Omega_m^{0.6} (z_{\text{crit}}) \left[ \frac{6 - 3\Omega_m^{0.4}(z_{\text{crit}})}{8} \right]^{2/5},
$$

where we have taken $\Omega_{mi} \approx 1$. If $\Omega_m(z_{\text{crit}}) = 1$, which should be a good approximation for $1.8 \leq z \lesssim 3000$, we have $z_{\text{crit}} = (3/8)^{2/5}(1 + z_i) - 1$. For this expression to be valid we must have $z_{\text{crit}} < z_{\text{app}}$ which with $\Omega_m(z_{\text{crit}}) \approx 1$ holds for $H_iR_{z}(z_i) \lesssim 0.24$. For inhomogeneities that are larger than this at $z = z_i$ we do not expect $z_{\text{crit}}$ to be an accurate approximation to the redshift when the DDE density contrast changes sign. Note that $z_{\text{crit}}$ does not depend on the size of the inhomogeneity, although, of course, we must have $z_{\text{crit}} < z_{\text{app}}$ for our analysis to be valid, which implies that $z = z_{\text{crit}}, H_iR_{z}(z_i) \lesssim 0.2$. If $\Omega_m = 1$ exactly, then $\delta^{(\text{lim})}_{\phi, k} / (1 + w) \rightarrow \text{const}$ at late times. If $\Omega_m < 1$, then $|\delta^{(\text{lim})}_{\phi, k} / (1 + w)|$ grows like

$$
|\delta^{(\text{lim})}_{\phi, k} / (1 + w)| \approx 2\Omega_m^{0.2} - \Omega_m^{0.2},
$$

at late times in the linear regime. This implies that $|\delta^{(\text{lim})}_{\phi, k} / (1 + w)|$ tends to a constant in the linear regime if $\Omega_m = 1$ and is growing for $\Omega_m < 1$. At late times in the linear regime, then, $|\delta^{(\text{lim})}_{\phi, k} / (1 + w)| \propto \Omega_m^{0.4} a^{-1}$. If we take $\Omega_m = 0.27$ today, then $|\delta^{(\text{lim})}_{\phi, k} / (1 + w)|$ is predicted to grow faster than $\delta_m$ when $z = 0.23$. Dutta & Maor (2007) also found that at late times in the linear regime, $|\delta^{(\text{lim})}_{\phi, k} / (1 + w)|$ grows faster than $\delta_m$. If this behavior were to continue into the nonlinear regime, it would, of course, imply that $|\delta^{(\text{lim})}_{\phi, k} / (1 + w)|$ would ultimately grow to be very large and indeed dominate over $\delta_m$. As we show in this paper, however, before this can happen the matter perturbation goes nonlinear and this slows the growth of $|\delta^{(\text{lim})}_{\phi, k} / (1 + w)|$.

We conclude our analysis of the linear regime by noting that equation (21) implies that at late times,

$$
\delta^{(\text{lim})}_{\phi}(r = 0, t) \approx -\frac{3C_\phi(1 + w)}{2} \left( 2\Omega_m^{0.6} - \Omega_m \right) H^2 (ar_{z})^2 \delta(r_{z} - t),
$$

where $r_{z} = \text{the smallest value of } r \text{ for which } \partial \ln \delta / \partial \ln t \geq -2$ and $C_\phi \propto O(1)$ depends on the precise form of $\delta$.

We have shown that in the linear regime $|\delta_{\phi}|$ at the center of the inhomogeneity is always smaller than $|\delta(r_{z} - t)|$ by a factor of about $(1 + w)(Har_{z})^2$; $Har_{z}$ is roughly equal to the physical size of the inhomogeneity as a fraction of the horizon size. If the initial density perturbation is Gaussian, i.e.,

$$
\delta_{\phi}(z_i) = \delta_0 \exp\left(-r^2/r_c^2\right),
$$

then

$$
\delta_{\phi}(r = 0, t) \approx -\frac{3\delta_0 r_c^2}{4r_c^2} \sqrt{\text{erf} \left( \frac{r}{r_c} \right)} - \frac{2}{r_c} e^{-r/r_c^2}.
$$

at late times.

We plot the evolution of $\delta_{\phi}(r = 0)$ and $\delta_{\phi}(r = 0)$ for a Gaussian initial matter density perturbation in the linear regime in Figure 2. We have assumed that $\delta_{\phi}(r = 0, z_i) = 0.02$ at $z_i = 30$, and $H_i = 0.03$, which corresponds to $\delta_{\phi}(r = 0, z_i = 0) = 0.3$ and $Har_{z} = 0.10$ today, i.e., $r_{z} = ar_{c} \approx 31 H^{-1} \text{ Mpc}$. For this choice of inhomogeneity, our approximation for $\delta_{\phi}$ is only good for $1 + z \lesssim 0.94(1 + z_i) \approx 29$. Just as Dutta & Maor (2007) saw in their numerical simulations, we find that close to $z = z_i$, $\delta_{\phi}(1 + w) > 0$. It then becomes negative, and at late times it continues to grow more negative, and it appears as if it might overtake $\delta_{m}$ at some point in the future. However, as we show below, this does not occur and $|\delta_{\phi}|$ remains small at all times. Note that the apparent increase in $\delta_{\phi}$ at early times is due in part to the fact that our approximation breaks down as one approaches $z = z_i$. If one wishes to study the evolution of $\delta_{\phi}$ all the way back to $z = z_i$, the numerical approach taken by Dutta & Maor (2007) provides accurate results. Our real focus in this work is, however, not on what occurs at very early times in the linear regime, but how the DDE density contrast evolves when the matter inhomogeneity goes nonlinear. We consider this below.

5.2. Dynamical Dark Energy Clustering in the Quasi-Linear Regime

We begin our investigation of how $\delta_{\phi}$ evolves when the matter perturbation exits the linear regime by considering a matter
perturbation that is in the weakly nonlinear or quasi-linear regime, i.e., \(-1 < \delta \leq 10\). Equations (3) and (5) give \(M(r, t)/R \sim M(r)/R + O(\delta_i \delta_j, \delta_i \delta_k, \delta_j \delta_k)\). We suppose that at some initial time, \(t_i, R_i = HR_j\), and \(R = r; M(r)\) is then the mass inside the shell with radius \(r\) at \(t_i\). We write
\[
M(r) = \frac{\Omega_m(t_i)}{2} R^2 \rho^3 [1 + \delta(r)],
\]
where \(\delta(r)\) is interpreted as the initial mean matter density contrast. In the full nonlinear regime, which we consider in § 5.3, we are only able to evaluate \(R(r, t)\) and hence, via equation (18), \(\delta_0\) analytically in a matter-dominated background (\(\Omega_m = 1\)). In many cases of interest, however, \(\delta_m \sim O(1)\), e.g., superclusters and voids. Although the linear approximation fails for such objects, a good leading approximation to the true mean density contrast of matter, \(\delta\), in the range \(-1 < \delta \leq 10\) is given by
\[
1 + \delta(r, t) \approx \left[1 - 2\bar{\delta}_{\text{lin}}(r, t)/3\right]^{-3/2},
\]
where \(\bar{\delta}_{\text{lin}}\) is the linear mean density contrast. This approximation improves as \(\Omega_m \rightarrow 0\) and is exact for \(\Omega_m = 0\). When this approximation holds we say that we are in the quasi-linear regime.

In this section we find \(\delta_\phi(\delta)\) for inhomogeneities in the quasi-linear regime. We are again assuming that any deviations from the ΛCDM model for structure formation due to the background DDE evolution are subleading order, i.e., \((1 - \Omega_m)(1 + w)\) is small. If this is not the case, then our expression for \(\delta_\phi\) in terms of \(\delta\) and \(\delta v\) is still valid, but the evolution of \(\bar{\delta}\) and \(\delta v\) would change; even still, the order of magnitude of \(\delta_\phi\) would not be greatly effected.

The peculiar velocity, \(\delta v = R, t - HR\), is related to \(\bar{\delta}(r, t)\); thus,
\[
\delta v = -\frac{\bar{\delta}_1 R}{3(1 + \delta)} \approx -\frac{1}{2} H R f(a) \left[(1 + \bar{\delta})^{2/3} - 1\right],
\]
where, as above, \(f(a) \equiv \Omega_m^{0.6}\) at late times. In the quasi-linear regime, the physical radius, \(R(r, t)\), of a shell with mass \(M(r)\) is given by
\[
R = a \left[1 - \frac{2\bar{\delta}_{\text{lin}}(r, t)}{3}\right]^{1/2} \frac{2M(r)}{a_t^2 \Omega_m H^2} \left[1 + \delta(r)\right]^{1/3}.
\]

Using equation (18), we find that the dark energy density contrast, \(\delta_\phi\), in the quasi-linear regime is well approximated by \(\delta_\phi(\delta)\), where
\[
\delta_\phi(\delta) \equiv \frac{(1 + w)\Omega_m H^2}{2} \times \int_R^\infty dR' R' \left\{\frac{3f(a)}{\Omega_m} \left[(1 + \bar{\delta})^{2/3} - 1\right] - \bar{\delta}\right\} + \frac{(1 + w)\Omega_m H^2}{4} \left[f(a)^2 / \Omega_m\right] \left[(1 + \bar{\delta})^{2/3} - 1\right]^2.
\]

For \(0.27 \leq \Omega_m \leq 1\), the first term in equation (25) is negative definite for all positive values of \(\delta\) for which the quasi-linear approximation holds \((0 < \delta \leq 10)\) and positive definite for \(-1 < \delta < 0\). The second term in equation (25) is clearly positive definite, but vanishes at \(R = 0\). The relative magnitude of the two terms depends on the choice of initial density profile. However, at \(R = 0\) it is clear that an overdensity of matter, \(\delta > 0\), corresponds to a dark energy void, \(\delta_\phi(R = 0, t) < 0\). Similarly, a void of matter corresponds to a local DDE overdensity at \(R = 0\). Figures 3 and 4 respectively show the evolution of \(\delta_\phi(1 + w)\) and \(\delta\) at \(R = 0\) for a typical supercluster and a typical void. For both the supercluster and the void we assume an initial density contrast profile that is flat for \(r < r_c\) and drops as \(1/r^2\) for \(r > r_c\). We take the initial instant to be in the far past, i.e., \(z_i \rightarrow \infty\). The supercluster is taken today to have a central density contrast of
\( \delta(R = 0) = 2 \), and \( r = r_c \) to corresponds to a physical radius of \( R_c = 15 \, h^{-1} \, \text{Mpc} \), which are appropriate for an object such as the local supercluster. For the void we take typical values of \( \delta(R = 0) = -0.93 \) today and \( R_c = 15 \, h^{-1} \, \text{Mpc} \). In each case we have taken \( \Omega_m(z = 0) = 0.27 \). It is clear that in both cases \( \delta / (1 + w) \) grows faster than \( \delta_m \); however, the DDE density contrast remains much smaller than \( \delta_m \) at all times. Note that \( \delta \) evolves monotonically for objects in both the linear and quasilinear regimes.

### 5.3. Dynamical Dark Energy Clustering in the Nonlinear Regime

The quasi-linear approximation to \( \delta \) is valid for \( -1 \leq \delta \leq 10 \); however, it breaks down for large positive values of the matter density contrast, \( \delta \). To go further we evaluate \( \delta \) in the fully nonlinear regime. To do this we need to calculate both the evolution of the peculiar velocity, \( \delta v \), and that of the mean density contrast mass contrast, \( \bar{\delta} \). The nonlinear evolution of the matter overdensity is significantly more simple to calculate analytically in a matter-dominated universe (\( \Omega_m = 1 \)). Fortunately, when the matter density contrast is large today \( \delta \gtrsim 100 \), the evolution of both \( R \) and \( v^{(m)} \) in the inhomogeneous region are well approximated by the \( \Omega_m = 1 \) solution. This is because the epoch of dark energy domination begins sometime after the cluster has turned around.

If \( \Omega_m = 1 \), we are effectively dealing with a Tolman-Bondi (Tolman 1934; Bondi 1947) background at leading order. We therefore have analytical solutions for \( R \) and \( r \) in terms of \( r \) and a parameter \( \eta \),

\[
R = \frac{1 + \delta(r)}{2\delta(r)} r (1 - \cos \eta), \quad (26)
\]

\[
t = t_0(r) + \frac{1 + \delta(r)}{2H_0 \delta^{2/3}(r)} (\eta - \sin \eta), \quad (27)
\]
where
\[ t_\delta(r) = t_\delta + \frac{1 + \delta(r)}{2H\delta_\delta^2(r)} \left[ \eta_\delta(r) - \sin \eta_\delta(r) \right] \]
\[ \sim t_\delta \left( \frac{\delta(r)}{5} + O(\delta^2) \right). \]  
(28)

\[ \eta_\delta(r) = \cos^{-1} \left[ \frac{1 - \delta(r)}{1 + \delta(r)} \right] \sim 2\delta_\delta^{1/2}(r) - \frac{2}{3}\delta_\delta^{3/2}(r) + O\left(\delta_\delta^{5/2}\right). \]

We assume that at early times (\( \delta > 1.8 \)), \( \Omega_m \approx 1 \). These solutions then provide an excellent approximation to the true early-time evolution of the inhomogeneity. These solutions also provide a good approximation at late times, provided we are dealing with an overdensity of matter that turns around prior to the onset of dark energy domination (\( \delta \approx 100 \) today). Assuming that we are dealing with such an overdensity, we find

\[ \frac{\delta M}{R^2} = \frac{1}{2} \Omega_m H^2 R \left[ \frac{9(a/a_0)^3(\eta - \sin \eta)^2}{2(t/t_0)^2(1 - \delta_\delta)(1 - \cos \eta)^2} - 1 \right] \]
\[ = \frac{1}{2} \Omega_m H^2 R \delta. \]  
(29)

When the background is matter dominated, \((a/a_0)^3(t/t_0)^2 = 1\) and \( \Omega_m = 1 \). In the nonlinear regime \( \delta_\delta = t_\delta(r)/t_\delta \ll 1 \), and terms proportional to \( \delta_\delta \) in the above expression can be dropped.

We have so far modeled the matter as being a spherically symmetric, pressureless perfect fluid. This approximation neglects the random motion of the matter particles and other effects due to the breakdown of spherical symmetry. These are small prior to turnaround, but thereafter act to slow down the collapse of the inhomogeneity. Without any such corrections, the matter perturbation would eventually collapse to a point. When these effects are included, however, the inhomogeneity virializes and relaxes to a steady state. In a matter-dominated background, each shell of constant \( M \) ceases to collapse and virializes when its radius, \( R_{\text{vir}} \), is about one-half of the radius at which it turned around, \( R_m \). N-body simulations suggest that \( R_m/R_{\text{vir}} \approx 1.8 \) (see Hamilton et al. 1991).

Since the process of virialization is not accounted for in our matter model, we must account for it in some fashion if we are to make accurate predictions. This is generally done via the rather ad hoc process of manually halting the collapse when each shell reaches its virial radius, \( R_{\text{vir}} \); in this model one takes \( R_{\text{vir}} = R_m/2 \); this occurs at \( \eta = 3\pi/2 \). While this provides a reasonably good approximation for the magnitude of the density contrast, \( \delta_\delta \), the peculiar velocity, \( \delta v \), is discontinuous at the instant of virialization (\( \eta = 3\pi/2 \)). This ad hoc procedure therefore gives widely inaccurate predictions for the peculiar velocity, \( \delta v = v - HR \) (see Fig. 5). Since our formula for \( \delta_\delta \) depends on both \( \delta v \) and \( \delta \) we require a more realistic model for virialization in the context of spherical collapse. Such an improved model was proposed recently by Shaw & Mota (2008). In this improved spherical collapse model, it is assumed that \( h_{\text{SC}} \equiv \delta v/HR \) is a function of \( \delta_\delta \), i.e., \( h_{\text{SC}} = h_{\text{SC}}(\delta) \); \( R \) is still parameterized according to equation (26), and it is found that

\[ t = t_\delta(r) + \frac{1 + \delta(r)}{2H\delta_\delta^2(r)} T(\tau), \]

where \( \tau = \eta - \sin \eta \). The function \( T(\tau) \) is then fixed so that, prior to turnaround, when deviations from spherical symmetry are small, \( T(\tau) \approx \tau \). For larger values of \( \delta (\gtrsim 15) \), \( T(\tau) \) is chosen so that \( h_{\text{SC}}(\delta) \) agrees, under certain reasonable and common assumptions, with the behavior that is seen in the \( N \)-body simulations of Hamilton et al. (1991). Shaw & Mota (2008) provide the following fitting formula for \( T(\eta) \):

\[ T(\eta) = \tau + \frac{3.468(\tau_\eta - \tau)^{-1/2} \exp\left[-15(\tau_\eta - \tau)/\tau_\eta\right]}{1 + 0.8(\tau_\eta - \tau)^{1/2} - 0.4(\tau_\eta - \tau)}, \]  
(30)

where \( \tau_\eta = 5.516 \). In this model \( R_m/R_{\text{vir}} \approx 0.59 \) which compares more favorably with the value of 0.56 suggested by \( N \)-body simulations (Hamilton et al. 1991) than the value of 0.50 that is generally used. In this improved model, \( \delta_\delta \) is still given by equation (18), and for \( \Omega_m = 1 \), the peculiar velocity is given by

\[ \frac{\delta v}{HR} = \frac{R_\delta}{HR} - 1 = \sqrt{\frac{1 + \delta}{2}} \sin \eta \frac{d\tau}{(1 - \cos \eta)^{1/2} d\tau} - 1, \]

and the density contrast is given by

\[ 1 + \delta = \frac{9T(\eta)^2}{2(1 - \cos \eta)^3}. \]

Figure 5 shows \( h_{\text{SC}} \equiv 1 - \delta v/HR \) as a function of \( \delta \) in both the improved spherical collapse model and the standard one (with virialization occurring suddenly at \( R = R_{\text{vir}} \)). Note that in the standard spherical collapse model (SCM), \( h_{\text{SC}} \) continues to grow until \( \delta \approx 140 \) and then drops sharply; whereas in the improved model \( h_{\text{SC}} \) reaches a maximum when \( \delta \approx 18 \). Note also that the maximum value of \( h_{\text{SC}} \) predicted by standard SCM is almost 5 times as large as the maximum value given by the improved model. Since \( \delta_\delta \) depends in part on \( \delta v^2 \) and hence \( h_{\text{SC}}^2 \), using the standard SCM instead of the improved model could lead to \( \delta_\delta \) being overestimated by as much as 2500%.
Figure 6 shows $\delta$, for $\Omega_m = 1$, as a function of the mean linear density contrast, $\delta_{\text{lin}} \propto a$, in both the improved model and the standard one with virialization put in by hand. We use our modified spherical collapse model to find the evolution of $\delta_{\text{lin}}$. We find that the DDE density contrast in the nonlinear regime is given by

$$
\delta_{\text{lin}}(R, t) \sim \frac{\Omega_m H^2 (1 + w)}{2} \int_R^{\infty} \delta(G(\bar{\delta}; \Omega_m) R') dR'
+ H^2 (1 + w) R^2 \bar{h}(\bar{\delta})^2,
$$

where

$$
G(\bar{\delta}; \Omega_m) = \frac{-6h(\bar{\delta})}{\delta} + 1.
$$

When $\delta \ll 1$, $G(\bar{\delta}) \sim -1 - 2(\Omega_m^{-0.4} - 1)$, and deep in the nonlinear regime, when $\bar{\delta} \gg 1$, $G(\bar{\delta}) \sim 1$. In the $\Omega_m = 1$ case we find that the following fitting formula is accurate to within 2%:

$$
G(\bar{\delta}; \Omega_m = 1, \alpha = 1) = \tanh \left\{ -0.9218 + \ln(\bar{\delta}) \right\} 
\times \left( \frac{195 + 0.543\delta^6}{403 + \delta^6} - \frac{0.38\sin^3[\ln(\bar{\delta}) - 2.49]}{[\ln(\bar{\delta}) - 2.49]^3} \right). \tag{32}
$$

We note that when $\Omega_m = 1$, $G(\bar{\delta}) < 0$ for $\bar{\delta} \leq 10.6$ and is otherwise positive. In a cluster with a core overdensity $\geq 20$, $\delta_{\text{lin}}$ is therefore generally positive at $R = 0$, i.e., there is DDE overdensity at $R = 0$, whereas for clusters with a core overdensity $\leq 10 - 20$, there is generally a DDE void around $R = 0$. Even in the deep nonlinear regime, the central DDE overdensity is still surrounded by a void, with $\delta_{\text{lin}}$ becoming negative when $\delta(R) \approx 10 - 20$. We see that $|\delta_{\text{lin}}| \sim O(\Omega_m (1 + w) H^2 R^2 \delta^2 / 2)$, where $R_c$ is the radius of the core of cluster (roughly defined by the smallest radius after which $\delta$ decreases faster than $1/R^2$) and $\delta_c$ is the core density contrast, i.e., the density contrast in the central core with $R = R_c$.

We use equation (31) to plot the evolution of $\delta_{\text{lin}}(R = 0)$ in the nonlinear regime. For $\delta \gg 1$, the matter overdensity in a background with $0.2 \leq \Omega_m < 1$ evolves, to a good approximation, according to

$$
1 + \delta_0(\bar{\nu}) = F(a) \left[ 1 + \delta_0(\bar{\nu}) \right],
$$

where $\delta_0 = 975/2 (1 - \cos \eta)^3$ and $F(a) = 4/(975\Omega_m H^2) \approx \Omega_m^{-0.4}$. We expect the largest contributions to $\delta_{\text{lin}}(R = 0)$ to come from regions where $\delta \gg 1$, which implies that we expect $\delta_{\text{lin}}(\bar{\nu} = \Omega_m) \approx 1$ for all $\Omega_m$. Therefore, using equation (31), we approximate

$$
\delta_{\text{lin}}(R = 0, t) \approx \frac{\Omega_m^2 H^2 (1 + w)}{2} \int_0^{\infty} \delta_{\text{lin}}(\bar{\nu})(\bar{\nu}, \Omega_m = 1) R' dR'.
$$

For simplicity, we take the implicit density profile to have a central core with $\delta(R) = \delta_i$ for $r < r_c$, and for $r > r_c$, we take $\delta_i \propto r^{-3}$. In Figure 7 we plot $\delta(R = 0)$ and $\delta_{\text{lin}}(R = 0)$ for a cluster where today $r = r_c$ corresponds to a physical radius of $1.5 \, h^{-1}$ Mpc, and the mean density contrast in $R < R_c$ today is 200. In Figure 8 we plot $\delta(R = 0)$ and $\delta_{\text{lin}}(R = 0)$ for a smaller but denser cluster with $R_c = 0.18 \, h^{-1}$ Mpc and $\delta(R = 0) = 1.6 \times 10^4$ today. These values are fairly typical of clusters such as Coma, which has been estimated to have a mean density contrast of 200 inside a radius of $1.5 \, h^{-1}$ Mpc and a core mean density contrast of about $1.6 \times 10^4$ inside a radius of $0.18 \, h^{-1}$ Mpc. We can see that in both cases $\delta_{\text{lin}}(1 + w) \sim O(10^{-2})$ today and that, as expected, it is positive. In both cases, $\delta_{\text{lin}}(1 + w)$ was negative in the past, changing sign when the mean overdensity of the core was $\delta \sim O(10)$. The core in these cases corresponds to the flat part of $\delta$, i.e., $R < R_c$. For more general density profiles, the core radius would be taken to correspond to $R \approx R_{\text{vir}}$, where $R_{\text{vir}}$ is the smallest value of $R$ for which $\delta$ decreases faster than $R^{-2}$. In all of the plots we have taken $\Omega_m(\bar{\nu} = 0) = 0.27$.

We can see that in both cases $\delta_{\text{lin}}(1 + w)$ continues to increase today. This increase is due to the continued collapse of matter onto the collapsed core. At very late times, however, this accretion will cease, and $\delta_{\text{lin}}(R = 0)(1 + w) \rightarrow \text{const}$ for all $\Omega_m$. We can see this from equation (31). At late times and assuming minimal accretion, $\delta = a^2 \Delta(R)$, for some function $\Delta(R)$, and so

$$
\frac{\delta}{1 + w} \approx \frac{4\Omega_m H^2}{a^2} \int_0^{\infty} R' \Delta(R') dR', \tag{34}
$$

and $a^2 \Omega_m H^2 = \text{const}$, which implies $\delta_{\text{lin}}(1 + w) \rightarrow \text{const}$ at late times. It is clear that $\delta_{\text{lin}}(1 + w)$ is smaller than $\delta_{\text{lin}}(R = R_{\text{vir}})$ by at a factor of about $\Omega_m H^2 R_{\text{vir}}^2$. It follows that even at late times $\delta_{\text{lin}}(1 + w) \ll 1$, as $\delta_{\text{lin}}(1 + w) \sim O(\Omega_m H^2 R_{\text{vir}}^2 \delta_{\text{lin}}) \sim O(M_{\text{vir}}^2 / R_{\text{vir}}^2)$, where $M_{\text{vir}} = M(R = R_{\text{vir}})$.

5.4. Discussion

We have seen that in all regimes (linear, quasi-linear, and fully nonlinear), the magnitude of DDE density contrast scales as $\Omega_m H^2 R_c^2 \bar{\delta}_{\text{lin}}$, where $R_c$ is the radial scale of the cluster and $\bar{\delta}_{\text{lin}}$ is the mean matter density contrast in $R < R_c$. $R_c$ is roughly defined to be equal to $R_{\text{vir}}$, which is the smallest radius for which $\delta_{\text{lin}}$ first decreases faster than $1/R^2$. If an inhomogeneity is subhorizon when it begins to collapse, we found that shortly after the collapse begins $\delta$ and $\delta_{\text{lin}}$ have the same sign, i.e., a matter overdensity
equates to a DDE overdensity, and the same for voids. However, a certain time after collapse begins, $\delta_c$ changes sign. This change of sign is related to the fact that at late times the decaying mode of the matter perturbation is negligible. When this occurs, a mean overdensity of matter results in a DDE void and vice versa. Deep in the nonlinear regime, when $\delta_c \approx 10^{-20}$, however, $\delta_c(R = 0)$ changes sign again. A large mean overdensity of matter then corresponds to an overdensity of dark energy at $R = 0$.

In both the linear and the quasi-linear regimes, $|\delta_m/(1 + w)|$ increases faster than $\delta_c$, although it is always much smaller than $\delta_c$. This behavior was also observed by Dutta & Maor (2007) in their numerical study of the linear regime, and it lead them to wonder whether $|\delta_m/(1 + w)|$ might grow to be $O(1)$ or larger in the nonlinear regime. Using our results, we have shown that the onset of nonlinear evolution for the matter perturbation slows down the growth of $|\delta_m/(1 + w)|$ and that at very late times $\delta_m/(1 + w) \to$ const and remains $\ll 1$. In §6 we consider the profile of $\delta_m/(1 + w)$ in and around typical astrophysical objects.

6. ASTROPHYSICAL APPLICATIONS

In this section we use our results to evaluate the profile of the DDE density contrast around typical clusters, superclusters, and voids of matter. We also investigate the possibility of detecting the dark energy clustering using the integrated Sachs-Wolfe (ISW) effect.

6.1. Galaxy Clusters

We begin by considering the DDE density contrast in a virialized galaxy cluster with similar properties to the Coma cluster. Navarro et al. (1997, hereafter NFW97) used high-resolution N-body simulations to derive a universal density profile for the dark matter halos of virialized structures such as galaxies and clusters. The resulting NFW97 profile is given by

$$\delta_m = \frac{\delta_c}{(R/R_c)[1 + (R/R_c)^2]},$$

where $R_c$ is the characteristic radius of the cluster, and $\delta_c$ is the core overdensity. In this section, we consider the profile of $\delta_m/(1 + w)$ in and around typical astrophysical objects.
where $R_c$ defines the scale at which $\delta_m$ changes from a $1/r$ drop-off to a $1/r^3$ one, and the characteristic density contrast $\delta_c$ is given by

$$
\delta_c = \frac{200}{3} \frac{c^2}{\ln(1 + c) - c/(1 + c)},
$$

where $c = R_{200}/R_c$ defines the concentration of the halo, and $R_{200}$ is roughly the virial radius and is defined to be the largest value of $R$ for which the mean density contrast is $\geq 200$. The mean density profile, $\bar{\delta}$, is then given by

$$
\bar{\delta} = \frac{3 \delta_c \ln(1 + R/R_c) - R_c/(1 + R/R_c)}{(R/R_c)^3}.
$$

Even though $\bar{\delta} \propto 1/R$ for $R < R_c$, the integrands in equation (31) still behave well enough for small $R$ for our analysis and our expression for $\delta_m$ to be valid and applicable. The DDE density contrast in a virialized cluster is given by equation (31). We note that since $\delta \propto \bar{\delta} \propto \Omega_m^{1/2}$ in the nonlinear regime, and $\delta$ depends on $R$ only through $R/R_c$ that, for fixed $\delta_c$, $\delta_0 \propto \Omega_m(1 + w)H^2R_c^2$.

Geller et al. (1999) studied the mass profile of the Coma galaxy cluster and found that over the entire range of $R < 10^{-1} h^{-1}$ Mpc it increased with $r$ at the rate predicted by the NFW profile. They found that three different observational samples were well fitted by $R_c = 0.182 \pm 0.030$, $0.167 \pm 0.029$, and $0.192 \pm 0.035 h^{-1}$ Mpc with, in all cases, $R_{200} = 1.5 h^{-1}$ Mpc. In what follows we take $R_c$ to be the mean of the three values, $R_c \approx 0.18 h^{-1}$ Mpc, and $R_{200} = 1.5 h^{-1}$ Mpc, which corresponds to $c \approx 8.3$. We plot the profile of DDE density contrast for a Coma-like cluster in Figure 9; $R = R_c$, and $R = R_{200}$ are indicated on the plots. We have taken $\Omega_m = 0.27$ in line with WMAP (Spergel et al. 2003, 2007). We note that $\delta_m$, $\delta_0$, and $\delta_0$ are indicated on the plots. We have taken $\Omega_m = 0.27$ in line with WMAP (Spergel et al. 2003, 2007). We note that $\delta_m$ is positive around $R = 0$ and also that it is very small, $\delta_m(R = 0) \approx 3 \times 10^{-5}$. As $R$ increases, $\delta_m$ becomes negative when $R \approx 1.5R_{200}$, and for $R \geq 4R_{200}$, $\delta_m$ tends to $0$ from below as $R \to \infty$ in line with our expectations.

6.2. Superclusters

Superclusters are the largest known gravitationally bound massive structures. The dark matter halos of superclusters typically have radii of about $10-25 h^{-1}$ Mpc and density contrasts of about $1 \leq \delta \leq 15$. We consider two examples of such objects: the Local Supercluster (LSC), of which our Galaxy is a part, and the Shapley supercluster (SSC) which lies about 650 Mlt-yr away. The LSC has a mean overdensity of $\delta \approx 2-3$ over a scale $\sim 15 h^{-1}$ Mpc (Hoffman 1986; Tully 1982). The SSC has been found to have an overdensity of $\delta \sim 10.3$ over a scale of $10.1 h^{-1}$ Mpc (Bardelli et al. 2000). Bardelli et al. (2000) also found that if the SSC had evolved linearly, then it would today have a linear overdensity of $\delta_{SSC, lin} \approx 1.3$ and a linear scale of $\approx 21.6 h^{-1}$ Mpc. For simplicity we model the matter density contrast of the clusters thus:

1. The clusters have a homogeneous core with physical radius $R_c$. In $R < R_c$, $\delta = \delta_0$ and $\delta_{lin} = \delta_{lin,0}$.

2. For $R > R_c$, the initial density contrast drops off as $1/r^3$, where $R = R_c(1 + \delta_0)/(1 + \delta(R))^{1/3}$.

With this choice of density contrast, the evolution of the cluster is free from both shell-crossing and shell-focusing singularities in the linear and quasi-linear regimes. For the LSC we take $\delta_0 = 2$ and $R_c = 15 h^{-1}$ Mpc, and for the SSC we take $\delta_0 = 10$ and $R_c = 10 h^{-1}$ Mpc. We additionally considered the S300 structure, in the vicinity of the SSC, which was identified by Bardelli et al. (2000). This structure has a mean overdensity of $\delta \approx 1.9$ on a scale of $24.8 h^{-1}$ Mpc (Bardelli et al. 2000); we take $\delta_0 = 2$ and $R_c = 25 h^{-1}$ Mpc for this structure.

We plot the DDE density contrast profile for the LSC, SSC, and S300 structures in Figures 10 and 11. The shape of $\delta_m$ is similar for all three objects, and the magnitude of the DDE density contrast scales as $(1 + w)\delta_0 H^2R_c^2$; it is largest for the S300 structure and smallest for the LSC. In all three cases, it is clear that $|\delta_m/(1 + w)|$ is small ($\leq 6.2 \times 10^{-5}$) and negative, which corresponds to a local DDE void.

6.3. Voids of Matter

In addition to local overdensities of matter, the universe also contains localized underdensities of matter or voids. Voids typically have similar radii to superclusters, although they can also be much larger. Since $|\delta_m/(1 + w)|$ is proportional to the square of the radius of the inhomogeneity, we expect voids to induce some of the largest DDE inhomogeneities. The classic historical example of a void is also one of the largest and was discovered in 1981 by Kirschner et al. (1981) in the constellations of Boötes.
and Corona Borealis. Recent measurements of this Boo"tes void have found it to be roughly spherical with radius 62 \( h^{-1} \) Mpc (Kirschner et al. 1987); 21 galaxies have been observed in the Boo"tes void, and its mean density contrast is estimated to be \(-0.80 < \bar{\delta}/\sqrt{80} < -0.66\) (Dey et al. 1990). The Boo"tes void is particularly large and is sometimes termed a supervoid. Data from the 2dFGRS shows that the average radii of voids in the north Galactic pole (NGP) and in the south Galactic pole (SGP) are \( 14.89 \pm 2.67 \) and \( 15.61 \pm 2.48 \) h \(^{-1}\) Mpc, respectively (Hoyle & Vogele 2004; Bolejko et al. 2005). The average mean density contrast for these voids is \( \bar{\delta}/\sqrt{80} \approx 0.75 \) in NGP and \( \bar{\delta}/\sqrt{80} \approx 0.93 \) in SGP (Hoyle & Vogele 2004).

Despite the large radii of these voids and supervoids, their scales are still very much subhorizon, and as such, our results are applicable to them. The evolution of voids is well approximated by the quasi-linear regime, and so we evaluate \( \bar{\delta}/\sqrt{80} \) using equation (25). We model both voids and supervoids as having a Gaussian initial mean density profile, i.e., \( \delta_{\text{in}} \propto \exp(-x^2/2) \), where \( R = R_c x(1 + \delta_0)/[1 + \delta(1 + w)]^{1/3} \) defines \( x \). With this choice, the evolution of the void is free from both shell-crossing and shell-focusing singularities provided \( \delta_0 \approx -0.95 \).

In Figure 12 we show \( \delta_0 \) and \( \bar{\delta} \) both for a supervoid with similar properties to the Boo"tes void (\( R_c = 62 \) h \(^{-1}\) Mpc, \( \delta_0 = -0.75 \)), and Figure 13 shows the same thing but for an average void (\( R_c = 15 \) h \(^{-1}\) Mpc, \( \delta_0 = -0.93 \)). In both cases we see, as expected, that \( \delta_0 > 0 \) and that a void in the matter distribution corresponds to a DDE overdensity. We note that the magnitude of |\( \delta_0/(1 + w) \)| is an order of magnitude larger for a 60 h \(^{-1}\) Mpc radius supervoid than for an average 15 h \(^{-1}\) Mpc void at the center of a supervoid we see that \( \delta_0 \approx 9 \times 10^{-3} (1 + w) \), whereas at the center of an average void \( \delta_0 \approx 5 \times 10^{-3} (1 + w) \). In all cases \( \delta_0 \) is both positive and very small.

Recently, there has been a great deal of discussion, both theoretical and observational, about the possibility that extremely large voids (\( R_c \approx 100–300 \) h \(^{-1}\) Mpc) exist within the visible universe (Rudnick et al. 2007; Inoue & Silk 2007). In addition, a
number of authors have speculated that we may even be living inside such an object (Schwarz & Weinhorst 2007; Conley et al. 2007; Zehavi et al. 1998; Giovanelli et al. 1999). For example, Rudnick et al. (2007) showed that both the WMAP cold spot and an observed dip in the surface brightness and number counts of extragalactic radio sources could be explained by the presence of an almost completely empty $R_c/C_25 \approx 10^5 \, h_0^{-1} \, Mpc$ at $z < 1$. Inoue & Silk (2007) showed that the ISW effect due to a void with $\delta_0 = -0.3$ and $R_c = 200-300 \, h_0^{-1} \, Mpc$ would be observed as cold spots with a temperature anisotropy $\Delta T/T \sim O(10^{-5})$, which might explain the observed large-angle CMB anomalies. If extremely large voids such as these exist, they would be associated with the largest DDE overdensities within the visible universe. In Figure 14 we plot the $\delta_\rho$ profile for both the $R_c = 10^5 \, h_0^{-1} \, Mpc$, $\delta_0 \approx -1$ object postulated by Rudnick et al. (2007) and the $R_c = 300 \, h_0^{-1} \, Mpc$, $\delta_0 \approx -0.3$ void considered by Inoue & Silk (2007). In the former case we find $\delta_\rho(R = 0)/(1 + w) \approx 2.4 \times 10^{-4}$ and in the latter $\delta_\rho(R = 0)/(1 + w) \approx 9.5 \times 10^{-4}$. These values of $\delta_\rho/(1 + w)$ are about 2 orders of magnitude larger than those associated with a typical void. However, even around such extremely large voids we still find that the DDE perturbation is small, $\delta_\rho/(1 + w) \lesssim 10^{-3}$.

6.4. ISW Effect due to DDE Clustering

We have seen that the DDE density contrast, $\delta_\rho$, is always small with magnitude $\lesssim 10^{-5}$ for average-sized structures in the universe. The integrated Sachs-Wolfe (ISW) effect is highly sensitive to the presence of dark energy. In a purely matter-dominated universe, the ISW effect vanishes entirely for linear perturbations. We see below that the DDE contribution to the ISW from a cluster with radius $R_c$ is proportional to $(HR_c)^2$, and so we consider only large (but still subhorizon) scales. On such scales, the matter perturbation is small and we are in the linear regime. The ISW effect is most easily derived in the conformal Newtonian gauge in which the metric takes the form

$$ds^2 = a^2[(1 + 2\Psi)d\tau^2 - (1 - 2\Phi)dx^2].$$

![Fig. 12. Profiles of (a) DDE and (b) matter density contrast for a supervoid. We have taken $\Omega_m = 0.27$ and modeled the supervoid on the Boötes void, $\delta(0) = -0.75$ and $R_c = 62 \, h^{-1} \, Mpc$. See text for further discussion. [See the electronic edition of the Journal for a color version of this figure.]

Fig. 13.— Same as Fig. 12, but for a typical void. We have taken $\Omega_m = 0.27$ and the void to have typical properties, $\delta(0) = -0.93$ and $R_c = 15 \, h^{-1} \, Mpc$. See text for further discussion. [See the electronic edition of the Journal for a color version of this figure.]
The temperature anisotropy due to the ISW effect is given by

\[
\delta_{\text{ISW}}(\mathbf{\hat{n}}) = 2 \int_{\tau_0}^{\tau} \Phi'(\alpha - \tau) \mathbf{\hat{n}} \cdot \mathbf{\hat{n}} d\tau,
\]

where \( \tau_0 \) is the value of \( \tau \) today, \( \tau_3 \) is its value on the surface of last scattering, and \( \Phi' = d\Phi/d\eta \). The contribution to \( \Phi' \) from the matter perturbation is

\[
\Phi_m' = \frac{3\Omega_m H^2}{2} (1 - f(a)) \delta_m,
\]

and \( f(a) \approx \Omega_m^{0.6} \) at late times, whereas the contribution to \( \Phi' \) from the DDE clustering is

\[
\Phi'_\phi = \frac{9(1 - \Omega_m)\Omega_m H^2}{4k^4} \delta_m \left[ (1 + w)[2 + 3w - f(a)] - w_p \right]
\]

\[
= -\frac{3\Omega_m H^2}{2k^2} V(\Omega_m, w, w_p) \delta_\phi,
\]

where \( w_p = dw/d\ln a \) and

\[
V(\Omega_m, w, w_p) = \frac{(1 - \Omega_m)[2 + 3w - f(a) - w_p/(1 + w)]}{[2f(a) - \Omega_m(1 - f(a))]}.
\]

It follows that

\[
\frac{\Phi'_m}{\Phi'_\phi} = -\frac{3(1 + w) H^2}{2k^2} \left[ \frac{1 - \Omega_m}{1 - \Omega_m^{0.6}} \right]
\]

\[
\times \left[ 1 + \Omega_m^{0.6} + \frac{w_p}{1 + w} - (3(1 + w)) \right],
\]

assuming that \( w_p/(1 + w) \sim O(1) \) or smaller in the recent past, then

\[
\left| \frac{\Phi'_m}{\Phi'_\phi} \right| < \frac{3(1 + w) H^2}{k^2}.
\]

We found above that the largest DDE perturbations would mostly likely be associated with large voids of matter, with \( \delta_0 \approx 9.5 \times 10^{-4}(1 + w) \) for a 300 Mpc void with central matter density contrast \( \delta_0 \approx 0.3 \) (such as that considered by Inoue & Silk 2007). In this case we would have

\[
\frac{\Phi'_m}{\Phi'_\phi} \approx -7 \times 10^{-3}(1 + w) \left[ 1.5 - 3(1 + w) + \frac{w_p}{1 + w} \right],
\]

which is at most \( O(-10^{-2}(1 + w)) \). In the absence of DDE clustering, the ISW temperature anisotropy of such a void was found to be \( |\Delta T/T| \sim O(10^{-5}) \). We therefore predict that the DDE clustering contribution to the temperature anisotropy would be at most \( O(10^{-7} (1 + w)) \). For smaller objects (with radius \( R_c \)) the DDE ISW temperature anisotropy scales as \( R_c^3 \).

In conclusion, the ISW due to subhorizon DDE clustering in models where the dark energy is due to a minimally coupled scalar field with canonical kinetic term is always suppressed by a factor of about \((1 + w)(HR_c)^2\) relative to the ISW effect to the matter perturbation itself. It therefore seems unlikely that the
DDE contribution is detected against the background of the ISW effect caused by the matter.

6.5. Discussion

In this section we have applied our results to evaluate the profile and evolution of $\delta_j/(1+w)$ in the vicinity of typical galaxy clusters, superclusters, and voids. We found that, today, the largest inhomogeneities in the DDE energy density value should occur in the vicinity of supervoids such as the one in Boötes (Kirschner et al. 1981). At the center of a supervoid we predict $\delta_\phi/(1+w) \approx 1.6 \times 10^{-5}$. At the center of a galaxy cluster, supercluster, or average-sized void, we found $|\delta_\phi/(1+w)| \approx 1 - 6 \times 10^{-5}$, with $\delta_\phi > 0$ for both virialized clusters and voids and $\delta_\phi < 0$ for superclusters. Generally, $|\delta_\phi/(1+w)| \leq 10^{-4}$ for supervoids and $\leq 1 - 6 \times 10^{-5}$ for other structures. Recently, there has been a fair amount of interest in the possibility that extremely large DDE energy density will be smaller than a few parts in $10^6$. 

Given the current bounds on the dark energy (EOS) parameter, $w(z < 1) = -1 \pm 0.1$, our results imply that if dark energy is described by a canonical scalar field that does not couple to (or only couples very weakly to) matter and is minimally coupled to gravity, then its energy density today is virtually homogeneous with fluctuations no larger than one part in $10^5$ (or possibly one part in $10^4$ if $300$ Mpc voids exist). Indeed, typical fluctuations in the DDE energy density will be smaller than a few parts in $10^6$.

We found that superclusters are generally associated with small, local DDE voids, whereas most other structures (galaxies and voids of matter) are associated with local overdensities of dark energy.

7. SUMMARY AND CONCLUSIONS

In this paper we have studied how dynamical dark energy clusters over subhorizon scales in theories where the dark energy is described by a canonical scalar field with no coupling to matter. Subhorizon dark energy clustering in such models has always been expected to be small, but previous studies of it have assumed the matter perturbation to be small and have, for the most part, relied on numerical simulations. In this paper, however, we used the powerful method of matched asymptotic expansions to derive an analytic expression, equation (18), for the DDE density contrast in terms of the peculiar velocity and density contrast of the matter. Our equation for $\delta_\phi$ holds not only in the linear regime, where the matter density contrast is small, $\delta_m \ll 1$, but also in the far more interesting but far less studied quasi-linear and nonlinear regimes where $\delta_m \sim O(1)$ or larger.

There were then essentially two parts to this paper, finding an expression for $\delta_\phi$ as a function of $\delta_m$, $\delta_m$, and $H$ and then evaluating this. The evaluating part is only done approximately for models that look sufficiently like $\Lambda$CDM. So that to a first approximation we can use $\Lambda$CDM to get $H$, $\delta_v$, and $\delta_m$. Different dark energy models would give different predictions, but just from the form of $\delta_\phi$, one can see that unless very large deviations from the $\Lambda$CDM predictions of $\delta_v$, $\delta_m$, and $H$ occur, then $\delta_\phi$ will be well approximated by taking a $\Lambda$CDM background.

Our key assumption in deriving our formula for $\delta_\phi$ was that the matter density contrast is only $O(1)$ or greater on scales that are small compared to the Hubble length, $H^{-1}$. This assumption holds very well for typical structures such as galaxies, clusters, superclusters, and voids. We also made a number of simplifying assumptions:

1. We assumed spherical symmetry. Although our interest is in dark energy rather than varying constants, the subhorizon dynamics of perturbations in the DDE scalar field are very similar to those of the scalar fields in varying-constant models provided one sets the matter coupling in such theories to zero. Shaw & Barrow (2006a, 2006b, 2006c) studied the evolution of a light scalar field in an inhomogeneous background in the context of varying constants. Both spherically symmetric backgrounds and ones without any symmetries were considered, and it was found that deviations from spherical symmetry did not alter the leading-order behavior of the scalar field perturbations when the matter coupling vanishes. Therefore, while the assumption of spherical symmetry can be relaxed, we do not expect it to greatly alter our results.

2. We assumed that we were not near a black hole horizon. This assumption allowed us to take $GM/R \ll 1$ on subhorizon scales. This assumption could also be relaxed, and once again both the form of $\delta_\phi$ and its magnitude are not expected to be greatly affected. We again base this expectation on the results found for light scalars in the context of varying constants by Barrow & Shaw (2007).

3. We assumed that the mass of the scalar field, $m$, was very small compared to $1/R_c$, where $R_c$ is the length scale of the peak in the matter density contrast. This assumption could be done away with; however it would significantly complicate the analysis in the nonlinear regime. Moreover, since we have required $HR_c \ll 1$ and in most quintessence models $m \sim O(H)$, it is for most models generally true that $mR_c \ll 1$. If $mR_c \sim O(1)$ or larger, then we expect that it will result in a smaller DDE density contrast, i.e., smaller values of $|\delta_\phi|$.

One of the main motivations for this work was the recent work of Dutta & Maor (2007). They studied the clustering of a sub-class of the DDE models which we have looked at in this work. They numerically evolved the linearized field equations for both $\phi$ and the metric. In the regime where the linearized equations are valid (i.e., $\delta_m \ll 1$), our analytical results agree with theirs: shortly after the collapse begins, $\delta_\phi/(1+w)$ is positive, but then as the dying mode of an overdensity of matter becomes small, $\delta_\phi/(1+w)$ changes sign and becomes negative. At late times in the linear order an overdensity of matter corresponds to a DDE void and vice versa. We saw in this paper that this correspondence continues to hold in the quasi-linear regime, $-1 \leq \delta_m \leq 10$, which is appropriate for superclusters and voids of matter. When $\delta_m \sim O(1)$, Dutta & Maor found that $|\delta_\phi|$ could grow to be relatively large (of the order of 0.01–0.1) and that this could lead to corrections to the equation of state parameter, $w$, of the order of a few percent. However, as we noted in the introduction, the $O(1)$ values of $\delta_m$ for which this effect was found fall outside the regime where the linearized field equations which they used are actually valid. One should point out, however, that in a matter-dominated universe ($\Omega_m \sim 1$), there is a clear mapping between the linear regime of an overdensity and the exact fully developed nonlinear overdensity. Presumably, such mapping should also occur in the presence of a scalar field, although the multiple changes of sign of the DDE perturbation complicate things considerably and this relation is not straightforward. We also saw that at late times in the linear and quasi-linear regimes $|\delta_\phi/(1+w)|$ grows faster than $|\delta_m|$. If it were not for the onset of truly nonlinear behavior in the evolution of $\delta_m$, this would indeed lead to the large values of $|\delta_\phi|$ seen at late times by Dutta & Maor (2007).

In our study we found that $\delta_\phi/(1+w)$ for realistic astrophysical objects is very small, indeed it is generally $\leq O(10^{-4})$. The fast growth in $\delta_\phi$ seen by Dutta & Maor (2007) for $\delta_m \sim O(1)$ should therefore have been interpreted as resulting from applying linearized field equations outside their realm of validity, rather than a true physical effect. Furthermore, at very late times when the
matter perturbation has virialized and accretion onto the core has all but ceased, we found that $\delta_\phi/(1 + w) \to \text{const}$ and that $\delta_\phi/(1 + w) \sim O(M_{-2}/R_{-2})$, where $R_{-2}$ is the smallest radius for which $\delta$ decreases as quickly as $1/R^2$ and $M_{-2}$ is the mass inside $R_{-2}$.

While it is possible with highly nonlinear potentials, in general it seems improbable that $O(10^{-5}(1 + w))$ values of $\delta_\phi$, which in turn result in order $O(10^{-5}(1 + w))$ values for $\delta m$, will result in any pronounced nonlinear effects especially over subhorizon scales. It is fair to say, however, that the conditions might be on the brink of breaking down for a $300 \, h^{-1} \, \text{Mpc}$ void, since $HR \sim 0.1$. But for smaller objects provided the conditions we place on the potential hold (which essentially precludes small changes in $\phi$ from resulting in huge changes in $V$, $V'$, or $V''$), there is no particular reason to think that perturbations in $w$ will result in $O(1)$ or greater corrections to $\delta_\phi$. One should also point out that in the case where the scalar field is strongly coupled to matter (e.g., as in Mota & Shaw 2006, 2007; Brax et al. 2004), the dark energy perturbations might not be small, since it is natural to expect that when matter starts to infall into the nonlinear regime it will drag along the dark energy field.

We also used our analytical expression for $\delta_\phi/(1 + w)$ to calculate the integrated Sachs-Wolfe (ISW) effect induced by the DDE density perturbations. We found that the ISW due to the DDE perturbations is always much smaller than the ISW effect due to the clustering of DDE models by a factor of $O(10^{-6})$. Indeed, even the ISW effect due to the clustering of DDE near an extremely large $300 \, h^{-1} \, \text{Mpc}$ void is found to result in a temperature anisotropy no larger than $O(10^{-7}(1 + w)) \approx 10^{-8})$. For smaller objects the ISW effect due to DDE clustering scales as $R_h^6$ (where $R_h$ is the radius of the cluster). It is therefore unlikely that, if dark energy is described by a simple uncoupled quintessence theory, DDE perturbations will be detected through the ISW effect.

In summary, we have derived an analytical expression for the DDE density contrast in uncoupled quintessence models and used it to make quantitative predictions for DDE clustering. Our results should also apply to fields with very weak matter coupling. This analysis could also be straightforwardly extended to models that are coupled to matter, but we leave this to a later work. If the mass of the scalar in these models is $O(1)$, then $\delta_\phi/(1 + w)$ is, to leading order, independent of the specifics of the underlying theory. As opposed to what has been suggested elsewhere, we found that $\delta_\phi/(1 + w)$ is always small compared to the matter density contrast and that even when the matter perturbation goes nonlinear, $\delta m/(1 + w)$ remains $\lesssim 10^{-3}$ for typical astrophysical objects. If we define the radius, $R_h$, of a matter over-/underdensity by the condition that at $R = R_h$, the mean matter density contrast drops off like $1/R^2$, then we find that $[\delta_\phi/(1 + w)] \sim O(1/(1 + w)H^2R^2)$. DDE clustering could potentially be detected through the ISW effect or measurements of the peculiar velocity field of matter. Roughly, the magnitude of the DDE clustering contribution to these effects is a factor $[1 - (\Omega_m\delta_\phi/\Omega_m\delta_m)] \sim O((1 - \Omega_m)(1 + w)H^2R^2/\Omega_m) \ll 1$ times the contribution from the clustering of ordinary matter. For realistic astrophysical objects, we found that voids and supervoids correspond to local overdensities in dark energy, with $\delta_\phi/(1 + w) \sim O(10^{-5})$ for clusters and average voids and $\delta_\phi/(1 + w) \sim O(10^{-4})$ for supervoids. If voids with radii of $100 - 300 \, h^{-1} \, \text{Mpc}$ exist within the visible universe, then $\delta_\phi$ may be as large as $10^{-3}(1 + w)$. Linear overdensities of matter and superclusters generally correspond to local voids in dark energy; for a typical supercluster, $\delta_\phi/(1 + w) \sim O(10^{-3})$.

Astronomical observations indicate $w = -1 \pm 0.1$ for $z < 1$, and so we conclude that if dark energy is described by an uncoupled quintessence model, then today dark energy is almost homogeneous on subhorizon scales with perturbations generally of order $10^{-6}$; the largest perturbations will be $O(10^{-5})$--$O(10^{-3})$ and associated with very large voids of matter.

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