Zeros of Riemann’s Zeta Functions in the Line \( z = 1/2 + it_0 \)

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Abstract

It was found that, in addition to trivial zeros in points \( z = -2N \), \( N = 1, 2, \ldots \), natural numbers), the Riemann’s zeta function \( \zeta(z) \) has zeros only on the line \( \{ z = 1/2 + it_0 \}; t_0 \) is real \}. All zeros are numerated, and for each number, \( N \), the positions of the non-overlap intervals with one zero inside are found. The simple equation for the determination of centers of intervals is obtained. The analytical function \( \eta(z) \), leading to the possibility fix the zeros of the zeta function \( \zeta(z) \), was estimated. To perform the analysis, the well-known phenomenon, phase-slip events, is used. This phenomenon is the key ingredient for the investigation of dynamical processes in solid-state physics, for example, if we are trying to solve the TDGLE (time-dependent Ginzburg-Landau equation).

Keywords Superconductivity · phase-slip events · time-dependent Ginzburg-Landau equation · Riemann’s zeta-function

1 Introduction

Investigations of Josephson effect, current flow in narrow superconducting strips [1], and dynamical states [2] in superconductors with use of TDGLE (time-dependent Ginzburg-Landau equation) lead to the necessity to deal with an important phenomenon: phase-slip events. It is interesting that the study of the distribution of zeros for Riemann’s \( \zeta \) function (see below Eqs. (15–17)) also requires an analysis of the same phenomenon. It means that there exists a deep internal connection between all these problems. It turns out that the Euler \( \Gamma \) function is also the essential ingredient of the scenario.

Riemann’s \( \zeta \) function and Euler \( \Gamma \) function appear in many physical problems. For example, the study of the spin-orbit interaction in inhomogeneous superconductors state reveals the existence of the infinite set of nontrivial exact relations for Euler \( \{ \psi \} \) function [3]; these relations are essential for the evaluation of the critical temperature. These equations contain the Bernoulli numbers \( B_n \). In general, these problems appear as the consequence of mentioned internal ties and arise while the temperature technique and analytical continuation are used in the theory of superconductivity. We are trying to analyze these ties. We hope that the investigation of phase-slip events with the use of the distribution of zeros for Riemann’s \( \zeta \) function will lead to new results, which are important not only in mathematics. They will provide a deep understanding of the phenomena mentioned above, especially for the case of strong suppression of superconductivity by an external field.

In order to investigate the zeros of \( \zeta(z) \), it is more convenient to use the function \( \Xi(z) \), which can be presented in the symmetrical form as follows:

\[
\Xi(z) = \frac{z(z-1)}{4} \left[ \zeta(z) \Gamma(z/2) \pi^{-z/2} + \zeta(1-z) \Gamma((1-z)/2) \pi^{-(1-z)/2} \right]
\]

where \( \Gamma(z) \) is Euler gamma function. The function \( \Xi(z) \) is the entire function of \( z \). As for the zeros of \( \zeta(z) \) and \( \Xi(z) \), they coincide on the strip. If we put \( z = 1/2 + it \), then \( \Xi(z) \) is even a function of \( t \) with real coefficients in its Taylor expansion in powers of \( t^2 \) [4]. The function of \( \Xi \) is real on the lines \( \{ z = \frac{1}{2} + it_0 \}; t_0 \) is real; \( z = \nu; \nu \) is real \}. The line \( \{ z = 1/2 + it_0 \} \) is Stock’s line for \( \Xi \) function.

2 Main Equations

Each analytical function \( f(z) \) can be presented in the form as follows:

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\[ f(x) = |f(x)|\exp(i\chi) \]  

If \( f(x) \neq 0 \) on some close counter \( C \), then we are dealing with the following quantization rule:

\[ \frac{\partial \chi}{\partial x} = 2\pi \left(N_\text{z} - N_\text{p}\right) \]

where \( N_\text{z} \) is a number of zeros, \( N_\text{p} \) is a number of poles inside the counter.

For \( z = 1/2 + \nu + it_0 \) and \( \nu \gg 1 \), we have the following:

\[ \zeta(z) = 1 + \frac{1}{\sqrt{2\nu}} \left(\cos(t_0\ln2) - \sin(t_0\ln2)\right) \]

From Eqs. (1, 4), we obtain that the integral of type (3) (see Fig. 1) for \( \Xi \) function is equal to two integrals type (3) on the counter \{DABD\}. Points \{A, F\} and \{B, E\} are symmetrically relative to the reflection over the \{D, D_1\} axis. The pair of zeros of zeta function placed in point \((t_0^{(k)}, \nu; t_0^{(k)}, -\nu)\) is not giving any contribution to the phase change during the integration over the \{L, L_1\}; the asymptotics (4) is canceling such a contribution by integration over the line \{A, B\}. Both these circumstances make impossible an appearance of phase-slip events in solid-state physics [3]. Here we use the ideas from the consideration of phase-slip events in solid-state physics [4]. More specifically, we define an analytical function \( \eta(t_0, \nu) \) as follows:

\[ \eta(t_0, \nu) = \eta_1(t_0, \nu) + i\eta_2(t_0, \nu) \]

Then on the Stock’s line, we obtain the following:

\[ 2e^{\eta_0(t_0)} \cos(\phi(t_0)) + \eta_1(t_0) = \frac{D_1 + iD_2}{1 - \sqrt{2}\cos(t_0\ln2) + i\sqrt{2}\sin(t_0\ln2)} + \frac{D_1 - iD_2}{1 - \sqrt{2}\cos(t_0\ln2) - i\sqrt{2}\sin(t_0\ln2)} \]

On the Stock’s line, the function \( \eta_1(t_0, \nu) \) satisfies the condition as follows:

\[ |\eta_1| < \pi/2 \]

Inside the range \( \nu \gg 1 \), we obtain the following equations for \( \eta_1, \eta_2 \):

\[ \eta_1 = -\sin(t_0\ln2)/2^{1/2+\nu}, \quad \eta_2 = -\cos(t_0\ln2)/2^{1/2+\nu} \]

In addition, at the zero point with number \( N \), we have two equations on the Stock’s line as follows:

\[ \frac{1}{\pi} \left(\phi\left(t_0^{(N)}\right) + \eta_1(t_0^{(N)})\right) = N - 3/2, \]

Note that in the points \( \{t_0, \ln2 = 2\pi N, \quad N \neq 0, \nu = 1/2\} \), we have \( D_1 = D_2 = 0 \) [5].

Let us consider the region \( t_0 \gg 1 \) for the calculation of the distribution of zeros of the \( \Xi \) function. In this region, we obtain the following:

\[ \Xi(z) = \frac{(\pi^2 + 1/4)\pi^{1/2}e^{-\pi t_0^2/4}}{4t_0} \left(\zeta(z)\exp(\phi(t)) + \zeta(1-z)\exp(-\phi(t))\right) \]

All zeros of function \( \Xi \) coincide with zeros of expression in brackets of Eq. (7). Here we use the ideas from the consideration of phaseslip events in solid-state physics [6]. More specifically, we define an analytical function \( \eta(t_0, \nu) \) as follows:

\[ \eta(t_0, \nu) = \eta_1(t_0, \nu) + i\eta_2(t_0, \nu) \]

Then on the Stock’s line, we obtain the following:

\[ 2e^{\eta_0(t_0)} \cos(\phi(t_0)) + \eta_1(t_0) = \frac{D_1 + iD_2}{1 - \sqrt{2}\cos(t_0\ln2) + i\sqrt{2}\sin(t_0\ln2)} + \frac{D_1 - iD_2}{1 - \sqrt{2}\cos(t_0\ln2) - i\sqrt{2}\sin(t_0\ln2)} \]

Fig. 1 A single asterisk indicates a first zero of \( \zeta \) function, counter \( C_0 \{L, L_1, L_2, L_3\} \), counter \( C_1 \{F, D, A, B, D_1, E\} \)
Equations (8, 11, 13, 14) allow us to estimate the location of the interval, where the zero point with number $N$ is placed.

\[
\left. \frac{\partial \eta_1(t_0)}{\partial t_0} \right|_{t_0^{(N)}} = 0
\]  

(14)

Fig. 2 Functions $\{\eta_1, \eta_2\}$ in the interval $76.8 < t_0 < 84.2$; point with a single asterisk indicates zeros of $\zeta$ function with $\{N=20,21,22; t_0 = 77.1448,79.3373,82.9104\}$

Fig. 3 Functions $\{\eta_1, \eta_2\}$ in the interval $122.4 < t_0 < 127.9$; point with a single asterisk indicates zeros of $\zeta$ function with $\{N=40,41,42; t_0 = 122.94674,124.2568,127.5167\}$
Note that there exists a large parameter $t_0 \gg 1$ in the limit $\ln(t_0)$. It leads to the drastic increase of the first term in Eq. (10) relative the second term with an increase in $\nu$; as a result, the existence of a pair of zeros $(\nu, -\nu)$ with $\nu \neq 0$ is impossible. The ratio of these two terms is $\exp(\nu \ln(t_0/2\pi))$. Equations (6, 13) allow us get the exact expression for the zeros of the $\Xi$

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Fig. 4 Functions $\{\eta_1, \eta_2\}$ in the interval $122.4 < t_0 < 123.3$; point with a single asterisk indicates zeros of $\zeta$ function with $\{N = 40; t_0 = 122.94674\}$

Fig. 5 Functions $\{\eta_1, \eta_2\}$ in the interval $498 < t_0 < 502$; point with a single asterisk indicates zeros of $\zeta$ function with $\{N = 269,270,271; t_0 = 498.5809,500.30905,501.6045\}$
The Table contains the values of 29 first zeros of the zeta function, including some special cases, with the use of the perturbation theory. The similar situation takes place in the third interval in the vicinity of a zero of the zeta function with the number \( N = 122.9467 \). The existence of the large parameter, \( \ln t_0 \), allows to analyze Eqs. (10, 13, 14) with the use of the perturbation theory and estimate the functions \( \eta_1(t_0^{(N)}), \eta_2(t_0^{(N)}) \).

To perform this estimation, we present Eq. (10) in the form of the following:

\[
e^{-\gamma(t_0)} \cos(\phi(t_0) + \eta_1(t_0)) = \mu(t_0)
\]

(15)

where \( \mu(t_0) \) is given by equation as follows:

\[
\mu(t_0) = \frac{1}{2} \left\{ e^{\phi(t_0)} \frac{D_1 + i D_2}{1 - \sqrt{2} \cos(t_0 \ln 2) + \sqrt{2} \sin(t_0 \ln 2)} + e^{-i \phi(t_0)} \frac{D_1 - i D_2}{1 - \sqrt{2} \cos(t_0 \ln 2) - i \sqrt{2} \sin(t_0 \ln 2)} \right\}
\]

(16)

The value of functions \( \eta_1, \eta_2 \) at \( t_0 \) can be estimated from the equations as follows:

\[
tg(\phi(t_0) + \eta_1(t_0)) = - \frac{\mu'(t_0)}{\mu(t_0)} - \frac{1}{\phi(t_0)} \left[ \eta_2(t_0) + \frac{\mu'(t_0)}{\mu(t_0)} \sin(\phi(t_0) + \eta_1(t_0)) \right]
\]

(17)

with the use of the perturbation theory relative to the second term on the right side of Eq. (17) and Eq. (15).

As an example, we made this procedure in the first order of the perturbation theory for three intervals \( 76.8 \leq t_0 \leq 83.2, 112.4 \leq t_0 \leq 127.9, \) and \( 498 \leq t_0 \leq 502 \). Inside these intervals, the zeros \( \{ N = 20, 21, 22 \}, \{ N = 40, 41, 42 \}, \) and \( \{ N = 269, 270, 271 \}\) are placed. Results are presented in Figs. 2, 3, 4, and 5 and in the Table 1.

There is a very interesting interval \( 122.4 \leq t_0 \leq 123.3 \) with very small values of both functions \( \eta_1, \eta_2 \). Inside this interval, a zeta function zero with number \( N = 40 \) is located at the point \( t_0 = 122.9467 \). The value of functions \( \eta_1, \eta_2 \) at this point is \( \eta_1(122.9467) = 3.154 \times 10^{-3}; \eta_2(122.9467) = -4.618 \times 10^{-3} \).

The values of the function \( \eta_1, \eta_2 \) in the interval \( 122.4 \leq t_0 \leq 123.3 \) are presented in Fig. 4 on a large scale.

The similar situation takes place in the third interval in the vicinity of a zero of the zeta function with the number \( N = 269, t_0 = 498.5809 \). Functions \( \eta_1, \eta_2 \) display fast oscillations with small amplitude in this interval near the values \( \{-0.31; -1.28\} \). It looks as some special “zeros” are “attractive resonant centers.” Definitely, there exists the final concentration of such “centers.”
3 Conclusions

The key results of the paper are presented by the equations (10), (13), and (14). Equation (13) represents a strong improvement of the well-known result described in the book [5]. Equation (10) is the proof of the Riemann’s hypothesis. We introduce two analytical functions {ϕ and η}. The exact asymptotical expression for function ϕ on the stripe 0 < Re z < 1 for t_0 >> 1 is obtained. With the use of the functions {ϕ, η}, one can create additional relation between the functions {Γ(z), ζ(z)}, and this allows us to produce a numeration of zeros of the Riemann’s ζ function. Equation (10) was solved for the large values of the parameter ln(t^2/π)/C_0/C_1 for functions {η_1, η_2; η = η_1 + iη_2}. We also obtained the solution (in the first order of the perturbation theory) for three intervals 76.8 ≤ t_0 ≤ 83.2, 122.4 ≤ t_0 ≤ 127.9, and 498 ≤ t_0 ≤ 502.

The condition (14) connected with symmetry of the Riemann’s zeta function and corresponding simultaneous equality to zero of both terms in Eq. (7) appears to be very essential.

The study of the phase-slip events is connected with the zeros, which are discussed above. Unlike all previous studies based on the use of the perturbation theory, the present analysis allows us to obtain an exact solution, without invoking any small parameter.

The connection between the numbers N of zeros of the Riemann’s ζ function inside of the abovementioned “resonant centers” and the primes is rather obvious.

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