Black universes with trapped ghosts

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A black universe is a nonsingular black hole where, beyond the horizon, there is an expanding, asymptotically isotropic universe. Such models have been previously found as solutions of general relativity with a phantom scalar field as a source of gravity and, without phantoms, in a brane world of RS2 type. Here we construct examples of static, spherically symmetric black-universe solutions in general relativity with a minimally coupled scalar field $\phi$ whose kinetic energy is negative in a restricted strong-field region of space-time and positive outside it. Thus in such configurations a “ghost” is trapped in a small part of space, which may in principle explain why no ghosts are observed under usual conditions.

1 Introduction

The existence of singularities is an undesired but probably inevitable feature of the classical theories of gravity. Singularities are places where general relativity or another classical theory of gravity does not work, so that the theory itself reveals the limits of its validity. Thus a full understanding of the physics of phenomena under study (origin and fate of our Universe, gravitational collapse etc.) requires either avoidance of singularities or/and modification of the corresponding classical theory, or addressing quantum effects. There have been numerous attempts on this trend, some of them suggesting that singularities inside the event horizons of black holes should be replaced with a kind of regular core ([1], see [2] for a recent review), others describing bouncing or “emergent” universes (see, e.g., [3, 4] for reviews).

In our view, of particular interest are models that combine avoidance of singularities in both black holes and cosmology, those which have been termed black universes [5, 6]. These are regular black holes (spherically symmetric ones in the known examples) where a possible explorer, after crossing the event horizon, gets into an expanding universe instead of a singularity. Thus such hypothetic configurations combine the properties of a wormhole (absence of a center, a regular minimum of the area function) and a black hole (a Killing horizon separating R and T regions). Moreover, the Kantowski-Sachs cosmology in the T region is asymptotically isotropic and approaches a de Sitter mode of expansion, which makes such models potentially viable as models of our accelerating Universe.

Precisely as traversable Lorentzian wormholes, black universes as solutions to the equations of general relativity require “exotic”, or phantom matter, i.e., matter that violates the null energy condition. This can be most easily shown in the case of spherical symmetry.

Various kinds of phantom matter are discussed in cosmology as possible dark energy candidates. Meanwhile, macroscopic phantom matter has not yet been observed. There exist theoretical arguments both pro et contra phantom fields, and the latter seem somewhat stronger, see, e.g., a discussion in [7].

In [7] it has been shown that black-universe models can be obtained without invoking phantom fields in the framework of the RS2-type [8] brane-world scenario, using the modified Einstein equations [9] describing gravity on the brane. In the black-universe solutions of [7], the role of exotic matter in the field equations is played by the “tidal” term of geometric origin, which has no reason to respect the energy conditions known for physically plausible matter fields. In such a scenario, worm-
hole solutions even without ordinary matter are also known [10].

In this paper, we would like to discuss another opportunity [11] of obtaining wormhole and black-universe configurations in the framework of general relativity, with a kind of matter which possesses phantom properties only in a restricted region of space, a strong-field region, whereas far away from it all standard energy conditions are observed. As an example of such matter, we consider static, spherically symmetric configurations of a minimally coupled scalar field with the Lagrangian

\[ L_s = \frac{1}{2} h(\phi) g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi), \]  

(1)

where \( h(\phi) \) and \( V(\phi) \) are arbitrary functions. If \( h(\phi) \) has a variable sign, it cannot be absorbed by re-definition of \( \phi \) in its whole range. Cases of interest are those where \( h > 0 \) (that is, the scalar field is canonical, with positive kinetic energy) in a weak field region and \( h < 0 \) (the scalar field is of phantom, or ghost nature) in some restricted region where, e.g., a wormhole throat can be expected. In this sense it can be said that the ghost is trapped. A possible transition between \( h > 0 \) and \( h < 0 \) in cosmology was considered in [12]. Examples of such “trapped-ghost” wormholes have been obtained in [11].

The paper is organized as follows. In Section 2 we present the basic equations and make some general observations. In Section 3 we obtain explicit examples of “trapped-ghost” black-universe solutions using the inverse-problem method, and Section 4 is a brief conclusion.

2 Scalar fields with a variable kinetic term

The general static, spherically symmetric metric can be written as

\[ ds^2 = A(u) dt^2 - \frac{du^2}{A(u)} - r^2(u) d\Omega^2, \]  

(2)

where we are using the so-called quasiglobal gauge \( g_{00} g_{11} = -1 \); \( A(u) \) is called the redshift function and \( r(u) \) the area function; \( d\Omega^2 = (d\theta^2 + \sin^2 \theta d\varphi^2) \)

is the linear element on a unit sphere. The metric is only formally static: it is really static if \( A > 0 \), but it describes a Kantowski-Sachs type cosmology if \( A < 0 \), and \( u \) is then a temporal coordinate. In cases where \( A \) changes its sign, regions where \( A > 0 \) and \( A < 0 \) are called R- and T-regions, respectively.

Let us specify which kinds of functions \( r(u) \) and \( A(u) \) are required for the metric (2) to describe a black universe.

1. Regularity in the whole range \( u \in \mathbb{R} \).
2. Asymptotic flatness as \( u \to +\infty \) (without loss of generality), i.e., \( r(u) \approx u, A(u) \to 1 \);
3. A de Sitter asymptotic as \( u \to -\infty \), i.e., a T-region \( (A < 0) \) where \( r(u) \approx |u|, -A(u) \sim u^2 \);
4. A single simple horizon (i.e., a simple zero of \( A(u) \)) at finite \( u \). It is an event horizon as seen from the static side, and it is the starting point of the cosmological evolution as seen from the T-region.

The existence of two asymptotic regions \( r \sim |u| \) as \( u \to \pm \infty \) requires at least one regular minimum of \( r(u) \) at some \( u = u_0 \), at which

\[ r = r_0 > 0, \quad r' = 0, \quad r'' > 0, \]  

(3)

where the prime stands for \( d/du \). (In special cases where \( r'' = 0 \) at the minimum, we inevitably have \( r'' > 0 \) in its neighborhood.)

The necessity of violating the weak and null energy conditions at such minima follows from the Einstein equations. Indeed, one of them reads

\[ 2A r''/r = -(T_t^t - T_u^u), \]  

(4)

where \( T_{\mu}^\nu \) are components of the stress-energy tensor (SET).

In an R-region \( (A > 0) \), the condition \( r'' > 0 \) implies \( T_t^t - T_u^u < 0 \); in the usual notations \( T_t^t = \rho \) (density) and \( -T_u^u = p_r \) (radial pressure) it is rewritten as \( \rho + p_r < 0 \), which manifests violation of the weak and null energy conditions. It is the simplest proof of this well-known violation near a throat of a static, spherically symmetric wormhole ([13]; see also [14]).

However, a minimum of \( r(u) \) can occur in a T-region, and it is then not a throat but a bounce in
the evolution of one of the Kantowski-Sachs scale factors (the other scale factor is \([-A(u)]^{1/2}\)). Since in a T-region \(t\) is a spatial coordinate and \(u\) temporal, the meaning of the SET components is \(-T_t^t = p_t\) (pressure in the \(t\) direction) and \(T_u^u = \rho\); nevertheless, the condition \(r'' > 0\) applied to (4) again leads to \(\rho + p_t < 0\), violating the energy conditions. In the intermediate case where a minimum of \(r(u)\) coincides with a horizon \((A = 0)\), the condition \(r'' > 0\) holds in its vicinity, along with all its consequences. Thus the energy conditions are violated near a minimum of \(r\) in all cases.

Let us now turn to the scalar field \(\phi(u)\) with the Lagrangian (1). In a space-time with the metric (2) it has the SET

\[
T_\mu^\nu = \frac{1}{2} h(u) A(u) \phi'(u)^2 \text{diag}(1, -1, 1, 1) + \delta_\mu^\nu V(u).
\]

The kinetic energy density is positive if \(h(\phi) > 0\) and negative if \(h(\phi) < 0\), so the solutions sought for must be obtained with \(h > 0\) at large values of the spherical radius \(r(u)\) and \(h < 0\) at smaller radii \(r\). It has been shown [11] that this goal cannot be achieved for a massless field \((V(\phi) \equiv 0)\).

Thus we seek black-universe configurations with a nonzero potential \(V(\phi)\). The Einstein-scalar equations can be written as

\[
(A r^2 h \phi')' - \frac{1}{2} A (r^2 h') \phi' = r^2 dV/d\phi, \quad (6) \\
(A' r^2)' = -2 r^2 V; \quad (7) \\
2 r'' / r = -h(\phi) \phi^2; \quad (8) \\
A (r^2)'' - r^2 A'' = 2, \quad (9) \\
-1 + A' r r' + A' r^2 = r^2 \left(\frac{1}{2} h A \phi^2 - V\right). \quad (10)
\]

Eq. (6) follows from (7)–(9), which, given the potential \(V(\phi)\) and the kinetic function \(h(\phi)\), form a determined set of equations for the unknowns \(r(u)\), \(A(u)\), \(\phi(u)\). Eq. (10) (the \(\{1\}\) component of the Einstein equations), free from second-order derivatives, is a first integral of (6)–(10) and can be obtained from (7)–(9) by excluding second-order derivatives. Moreover, Eq. (9) can be integrated giving

\[
B'(u) \equiv (A/r^2)' = 2(3m - u)/r^4, \quad (11)
\]

where \(B(u) \equiv A/r^2\) and \(m\) is an integration constant equal to the Schwarzschild mass if the metric (2) is asymptotically flat as \(u \to \infty\) \((r \approx u, A = 1 - 2m/u + o(1/u))\). If there is a flat asymptotic as \(u \to -\infty\), the Schwarzschild mass there is equal to \(-m\) \((r \approx |u|, A = 1 + 2m/|u| + o(1/u))\).

One more observation is that if the system contains a horizon and \(r(u) \sim |u|\) at large \(|u|\) in the T-region, then \(B \equiv A/r^2\) tends to a finite limit, which means that there is a de Sitter asymptotic. Indeed, under these conditions the integral of (11) evidently converges at large \(|u|\), so \(B\) tends to a constant. Furthermore, Eq. (9) can be rewritten as \(r^4 B'' + 4 r^3 r' B' = -2\), hence \(B'' < 0\) where \(B' = 0\), so \(B(u)\) cannot have a minimum (and it is this circumstance that restricts the possible kinds of global causal structure of any scalar-vacuum solutions [15]). This means that \(B(u)\) can only tend to a negative constant in a T-region.

Thus, in the Einstein-scalar system (6)–(10), any solution with a horizon and \(r \sim |u|\) as \(u \to \pm\infty\) is asymptotically de Sitter in the T-region. It describes a black universe if it is, in addition, asymptotically flat in the R-region.

## 3 Black universe models with a trapped ghost

If one specifies the functions \(V(\phi)\) and \(h(\phi)\) in the Lagrangian (1), it is, in general, very hard to solve the above equations. Alternatively, to find examples of solutions possessing particular properties, one may employ the inverse problem method, choosing some of the functions \(r(u)\), \(A(u)\) or \(\phi(u)\) and then reconstructing the form of \(V(\phi)\) and/or \(h(\phi)\). We will do so, choosing a function \(r(u)\) that can provide a black-universe solution. Then \(A(u)\) is found from (11) and \(V(u)\) from (7). The function \(\phi(u)\) is found from (8) provided \(h(\phi)\) is known; however, using the scalar field parametrization freedom, we can, vice versa, choose a monotonic function \(\phi(u)\) (which will yield an unambiguous function \(V(\phi)\)) and find \(h(\phi)\) from Eq. (8).

A simple example of the function \(r(u)\) satisfying the requirements 1–3 is (see Fig. 1):

\[
r(u) = a \frac{(x^2 + n)}{\sqrt{x^2 + n^2}}, \quad n = \text{const} > 2. \quad (12)
\]

where \(x = u/a\), and \(a > 0\) is an arbitrary constant (the minimum radius). It is the same function that led to symmetric wormhole solutions in [11] under the additional assumption \(m = 0\).
Let us now integrate Eq. (11), assuming $m > 0$. We find (see Fig. 2)

$$B(u) = \frac{3x^4 + 3x^2n(n + 1) + n^2(n^2 + n + 1)}{3a^2(x^2 + n)^3}$$

$$+ \frac{mx}{8a^2n(x^2 + n)} \left[ 3x^4(5n^2 + 2n + 1) 
+ 8nx^2(5n^2 + 2n - 1) + 3n^2(11n^2 - 2n - 1) \right]$$

$$- \frac{3m}{8a^2n^{3/2}} (5n^2 + 2n + 1) \cot^{-1} \left( \frac{x}{\sqrt{n}} \right). \quad (13)$$

The emerging integration constant is excluded by the requirement $B \to 0$ as $u \to \infty$, providing asymptotic flatness. Examples of the behavior of $B(u)$ for $m = 0.2a$ and some values of the parameter $n$ are presented in Fig. 2.

Substituting the expressions 12 and 13 into 7, taking into account that $A(u) = B/r^2$, we obtain the potential $V$ as a function of $u$ or $x = u/a$. This expression is rather bulky and will not be presented here.

To construct $V$ as an unambiguous function of $\phi$ and to find $h(\phi)$, it makes sense to choose a monotonic function $\phi(u)$. It is convenient to assume

$$\phi(u) = \frac{2\phi_0}{\pi} \arctan \frac{x}{n}, \quad \phi_0 = \frac{\pi a}{2} \sqrt{\frac{2(n - 2)}{n}}, \quad (14)$$

and $\phi$ has a finite range: $\phi \in (-\phi_0, \phi_0)$, which is common to kink configurations. Thus we have $x = u/a = n \tan(\pi \phi/2\phi_0)$, whose substitution into the expression for $V(u)$ gives $V(\phi)$ defined in this finite range. The function $V(\phi)$ can be extended to
the whole real axis, \( \phi \in \mathbb{R} \), by supposing \( V(\phi) \equiv 0 \) at \( \phi \geq \phi_0 \) and \( V(\phi) = V(-\phi_0) > 0 \) at \( \phi < -\phi_0 \). Plots of \( V(\phi) \) are presented in Fig. 3 for the same values of the parameters as in Fig. 2.

The expression for \( h(\phi) \) is found from (8) as follows:

\[
h(\phi) = \frac{(n - 2)x^2 + n^2(1 - 2n)}{a^2(n - 2)(x^2 + n)},
\]

where \( x = n \tan(\pi \phi/2\phi_0) \). The function \( h(\phi) \) given by Eq. (15) is also defined in the interval \((-\phi_0, \phi_0)\) and can be extended to \( \mathbb{R} \) by supposing \( h(\phi) \equiv 1 \) at \( |\phi| \geq \phi_0 \). The extended kinetic coupling function \( h(\phi) \) is plotted in Fig. 4. Evidently, the null energy condition is violated only where \( h(\phi) < 0 \).

4 Conclusion

It has been shown [11] that a minimally coupled scalar field may change its nature from canonical to ghost in a smooth way without creating any space-time singularities. This feature, in particular, allows for construction of wormhole models (trapped-ghost wormholes) where the ghost is present in some restricted region around the throat (of arbitrary size) whereas in the weak-field region far from it the scalar has usual canonical properties. The same model has been modified here to construct another interesting type of configurations, black universes.

It has also been found that, in the Einstein-scalar field system under study, a static, spherically symmetric configuration is inevitably a black universe if it is asymptotically flat, has a horizon, and the function \( r(u) \) grows linearly as \( u \to \pm\infty \). Though, all this can only happen if the scalar is of ghost nature at least in some part of space.

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