New plans orthogonal through the block factor.
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Abstract
In the present paper we construct plans orthogonal through the block factor (POTBs). We describe procedures for adding blocks as well as factors to an initial plan and thus generate a bigger plan. Using these procedures we construct POTBs for symmetrical experiments with factors having three or more levels. We also construct a series of plans inter-class orthogonal through the block factor for two-level factors.

1 Introduction
A situation in which a treatment factor is neither orthogonal nor confounded to a nuisance factor was first explored in Morgan and Uddin (1996) in the context of nested row-column designs. They derived a sufficient condition for a treatment factor, possibly non-orthogonal to the nuisance factors, to be orthogonal to another treatment factor. They also derived a sufficient condition for optimality and constructed several series of orthogonal main effect plans (OMEPs) satisfying optimality properties. Mukherjee, Dey and Chatterjee (2002) discussed and constructed main effect plans (MEPs) on small-sized blocks, not necessarily orthogonal to all treatment factors. Their plans also satisfy optimality properties. Optimal blocked MEPs of similar type are also constructed in Das and Dey (2004). Wang (2004) constructed plans for two-level factors on blocks of size two, estimating interaction effects also.

Bose and Bagchi (2007) provided plans satisfying properties similar to those of the plans of Mukherjee, Dey and Chatterjee (2002), but requiring fewer blocks. In Bagchi (2010) the concept of orthogonality through the block factor [see Definition 2.2] is introduced. In that paper it has been shown that a plan orthogonal through the block factor (POTB) may exist in a set up, where an OMEP cannot exist. Making use of the Hadamard matrices in various way, Jacroux and his co-authors (20011, ... 2017) have come up with a number of such plans, mostly for two-level factors, many of them satisfying optimality properties. Other authors providing POTBs include Chen, Lin, Yang, and Wang (2015) and Saharay and Dutta (2016).

Preece (1966) constructed ‘BIBDs for two sets of treatments’. Subsequently several authors constructed similar combinatorial objects. Among these, the ones relevant to the present paper are ‘balanced Graeco-Latin block designs’ of Seberry (1979), ‘Graeco-Latin designs of type 1’ of Street (1981) and ‘Perfect Graeco-Latin balanced incomplete block designs (PERGOLAs)’ of Rees and Preece (1999). We note that all these combinatorial designs are, in fact, two-factor POTBs satisfying certain additional properties. We discuss these interesting combinatorial designs briefly in Section 3.

In the present paper our main objective is to provide plans in those set ups where no OMEP is available, accommodating as many factors as possible and deviating “as little as possible”
from orthogonality. We construct a few series of POTBs for symmetrical experiment with factors having three or more levels. We also define plans inter-class orthogonal through the block factor (PIOTBs) [see Definition 6.1] and construct a series of such plans.

In Section 2 we present the definition of a POTB along with its attractive features. The later sections are devoted to construction. In Section 3 we obtain a few infinite series of POTBs for symmetric experiments with four or less factors, each with five or more levels [see Theorems 3.1, 3.2 and 3.3]. In Section 4 we describe methods of recursive construction. In Section 5 we use these methods and construct two series of POTBs for three-level factors on blocks of size four [see Theorems 5.1 and 5.3]. Finally, in Section 6 we construct an infinite series of PIOTBs with orthogonal classes of small size for two-level factors [see Theorem 6.1]. Many of the plans constructed are saturated.

2 Preliminaries

We shall consider main effect plans for a symmetrical experiment with $m$ factors, laid out on blocks of constant size.

**Notation 2.1.** (a) $P$ will denote a main effect plan for a $s^m$ experiment consisting of $b$ blocks each of size $k$. $n$ will denote the total number of runs. Thus, $n = bk$.

(b) The set of levels for each factor is denoted by $S$, the set of integers modulo $s$, unless stated otherwise. $S^m$ will denote the following set of $m \times 1$ vectors. $S^m = \{(x_1, \ldots, x_m) : x_i \in S\}$.

(c) $A_i$ denotes the $i$th factor, $i = 1, 2, \ldots, m$. The vector $x = (x_1, x_2, \ldots, x_m) \in S^m$ represents a level combination or run, in which $A_i$ is at level $x_i$, $i = 1, 2, \ldots, m$.

(d) $B = \{B_j, j = 1, \ldots, b\}$ will denote the set of all blocks of $P$, Thus, $B_j \subset S^m, |B_j| = k, 1 \leq j \leq b$. Sometimes we describe a plan in terms of its blocks.

(e) The replication vector of $A_i$ is denoted by the $s \times 1$ vector $r_i$, the $p$th entry of which is the number of runs $x$ of $P$ such that $x_i = p, p \in S$. $R_i$ denotes a diagonal matrix with diagonal entries same as those of $r_i$ in the same order, $1 \leq i \leq m$.

(f) For $1 \leq i, j \leq m$, the $A_i$ versus $A_j$ incidence matrix is the $s_i \times s_j$ matrix $N_{ij}$. The $(p, q)$th entry of this matrix is $N_{ij}^{pq}(p, q)$, which is the number of runs $x$ of $P$ such that $x_i = p, x_j = q, p \in S_i, q \in S_j$. When $j = i, N_{ij} = R_i$.

(g) $L_i$ will denote the $A_i$-versus block incidence matrix, $1 \leq i \leq m$. Thus, the $(p, j)$th entry of the $L_i$ is $L_i^{pq} = |x \in B_j : x_i = p|, p \in S, 1 \leq j \leq b, 1 \leq i \leq m$.

(h) The $s \times 1$ vector $\alpha^i$ will denote the vector of unknown effects of $A_i$, $0 \leq i \leq m$.

Consider the normal equations for a plan $P$ as described above. If we eliminate the general effects and the vector of block effects from this system of equations, we get the reduced normal equation for the vectors of all (unknown) effects of all the treatment factors. This is a system of $ms$ equations, but it is convenient to view it as $m$ systems of $s$ equations each, the $i$th system equations is of the form

$$\sum_{j=1}^{m} C_{ij;j} \hat{B}_{i,j} = Q_{ij,B}.$$  \hspace{1cm} (2.1)
Here $C_{ij;B}$, $1 \leq j \leq m$ are the coefficient matrices and $Q_{i;B}$ is the vector of adjusted (for the blocks) totals for $A_i$.

For a fixed $i$, we can eliminate $\hat{\alpha}_i, j \neq i$ from (2.1) and get

the reduced normal equation for $\hat{\alpha}_i$ as $C_{i;i} \hat{\alpha}_i = Q_{i;i}$. \hfill (2.2)

We omit the expressions for the quantities $C_{ij;B}$, $C_{i;i}$, $Q_{i;B}$ and $Q_{i;i}$ above. Those are not necessary here and are available in Bagchi and Bagchi (2020), for instance. With this background we present a few definitions.

**Definition 2.1.** An $m$-factor MEP is said to be ‘connected’ if $\text{Rank}(C_{i;i}) = s - 1$, for every $i = 1, 2, \ldots, m$.

**Definition 2.2.** [Bagchi (2010)] Fix $i \neq j, 1 \leq i, j \leq m$. The factors $A_i$ and $A_j$ are said to be orthogonal through the block factor (OTB) if

$$kN_{ij} = L_i(L_j)'.$$ \hfill (2.3)

We denote this by $A_i \perp_{bl} A_j$.

A plan $P$ is said to be a plan orthogonal through the block factor (POTB) if $A_i \perp_{bl} A_j$ for every pair $(i, j)$, $i \neq j, i, j = 1, \ldots, m$.

**Remark 2.1:** Condition (2.3) is equivalent to equation (7) of Morgan and Uddin (1996) in the context of nested row-column designs.

Let us try to see the implications of orthogonality through the block factor. Let $SS_{i;all}$ (respectively $SS_{i;B}$) denote sum of squares for $A_i$, adjusted for all other factors (respectively the block factor). The following results are known.

**Theorem 2.1.** Consider a plan $P$. Fix $i \in \{1, \ldots, m\}$.

(a) [Bagchi (2010)] If for $j \neq i$ $A_i \perp_{bl} A_j$, then

(i) $C_{ij;B} = 0$ and (ii) $\text{Cov}(l'\hat{\alpha}_i, m'\hat{\alpha}_j) = 0$, for $l'1_s = 0 = m'1_s$.

(b) [Bagchi (2020)] Further, $A_i \perp_{bl} A_j, \forall j \neq i$ is necessary and sufficient for the following.

(i) $C_{i;i} = C_{ii;B}$ and (ii) $SS_{i;i} = SS_{i;0}$ with probability 1.

**Discussion:** Theorem 2.1 says the following about the inference on the factors of a connected main effect plan. The inference on a factor $A_i$ depends only on the relationship between $A_i$ and the block factor if and only if $A_i$ is orthogonal to every other treatment factor through the block factor. Moreover, the data analysis of a POTB is very similar to the data analysis of a block design with $s$ treatments.

It is well-known that the orthogonal MEP obtained from an orthogonal array is the best possible MEP in the sense that the estimates have the maximum precision among all MEPs in the same set up. The same cannot be said about an POTB since its performance also depends on the relationships of the treatment factors with the block factor. In the next theorem a guideline for the search for a ‘good’ POTB is provided. We omit the proof which can be obtained by going along the same lines as in the proofs of Lemma 1 and Theorem 1 of Mukherjee, Dey and Chatterjee (2002). [See Shah and Sinha (1989) for definitions, results and other details about standard optimality criteria]
Theorem 2.2. Suppose a connected POTB $\rho^*$ satisfies the following condition. For a factor $A_i$ and a non-increasing optimality criterion $\phi$, $L_i$ is the incidence matrix of a block design $d$ which is $\phi$-optimal in a certain class of connected block designs with $s$ treatments and $b$ blocks of size $k$ each. Then, $\rho^*$ is $\phi$-optimal in a similar class of connected $m$-factor MEPs in the same set-up as $\rho^*$ for the inference on $A_i$.

In particular, using the well-known optimality results of Kiefer (1975) and Takeuchi (1961) we get the following result.

Corollary 2.1. Suppose $\rho^*$ is a connected POTB. Fix $i \in \{1, \ldots, m\}$.

(a) If $L_i$ is the incidence matrix of a BIBD, then, for the inference on $A_i$, $\rho^*$ is universally optimal in the class of all $m$-factor connected MEP containing $\rho^*$.

(b) If $L_i$ is the incidence matrix of a group divisible design satisfying $\lambda_2 = \lambda_1 + 1$, then $\rho^*$ is E-optimal in the class of all $m$-factor connected MEP containing $\rho^*$, for the inference on $A_i$.

In view of the above result, we introduce the following term.

Definition 2.3. A connected POTB is said to be balanced if each of its factors form a BIBD with the block factor, that is $L_i$ is the incidence matrix of a BIBD for each $i$, $1 \leq i \leq m$.

We now present a small example of a balanced POTB on six blocks of size two each. It has two factors, each with four levels 0,1,2,3.

Example 1 [Bagchi and Bose (2007)] :

| Blocks → | B₁ | B₂ | B₃ | B₄ | B₅ | B₆ |
|---------|----|----|----|----|----|----|
| Factors ↓ | A₁ | 0  | 2  | 1  | 3  | 0  |
| A₂      | 1  | 3  | 0  | 2  | 2  | 1  |

3 Construction of plans with a small number of factors

We shall now proceed to construct POTBs for a symmetric experiment. Most of the constructions are of recursive type, in the sense that from a given initial plan we generate a plan by adding blocks and/or factors.

Definition 3.1. Consider an initial plan $P₀$ for an $s^m$ experiment as described in Notation 2.1. For $B \in B$ and $v \in S^m$, $B + v$ will denote the following set of $k$ runs. $B + v = \{x + v, x \in B\}$. Here $x + v = [x_i + v_i : 1 \leq i \leq m]'$, where the addition in each co-ordinate is modulo $s$.

By the plan generated from $P₀$ by adding $S$ we shall mean the plan (for the same experiment) having the set of blocks $\{B + u1_m : u \in S, B \in B\}$. The new plan $P$ will be denoted by $P₀ \oplus S$.

We shall now proceed to construction. We begin with plans with a small set of factors. Let $S^+$ denote $S \cup \{\infty\}$. The following rule will define addition in $S^+$.

$$u + \infty = \infty = \infty + u, \ u \in S. \quad (3.4)$$
Theorem 3.1. Suppose $s$ is an integer $\geq 5$. Then POTBs with block size two exists for the following experiments.

(a) For an $s^2$ experiment a POTB $\mathcal{P}$ on $2s$ blocks exists. In the case $s = 5$, $\mathcal{P}$ is balanced.

(b) (i) For an $s^4$ experiment a POTB $\mathcal{P}_1$ on $4s$ blocks exists. If $s = 10$, then $\mathcal{P}_1$ is $E$-optimal for the inference on each factor.

(ii) Moreover, if $s \geq 9$, there exists a POTB $\mathcal{P}_2$ with the same parameters as $\mathcal{P}_1$, but non-isomorphic to the same. If $s = 9$, $\mathcal{P}_2$ is balanced.

(c) A POTB $\mathcal{P}$ for a $(s + 1)^4$ experiment with $6s$ blocks exists, whenever $n \geq 7$.

Proof: In each case, we present the blocks of an initial plan $\mathcal{P}_0$. The required plan is $\mathcal{P}_0 \oplus S$ [see Definition 3.1]. Here $a, b, c, d$ are distinct members of $S \setminus \{0\}$. That the final plan is a POTB can be verified by straightforward computation. Proofs for the optimality properties are presented.

(a) The blocks of $\mathcal{P}_0$ are given below.

| Blocks $\rightarrow$ | $B_1$ | $B_2$ |
|----------------------|--------|--------|
| Factors $\downarrow$ | $A_1$  | a      | -a     |
|                     | $A_2$  | b      | -a     |

If $s = 5$, taking $a = 1, b = 2$ we get a balanced POTB.

(b) (i) The blocks $B_l$, $l = 1, \ldots, 4$ of $\mathcal{P}_0$ are as follows.

| Blocks $\rightarrow$ | $B_1$ | $B_2$ | $B_3$ | $B_4$ |
|----------------------|--------|--------|--------|--------|
| Factors $\downarrow$ | $A_1$  | 0      | a      | -a     |
|                     | $A_2$  | a      | -a     | 0      |
|                     | $A_3$  | 0      | b      | -a     |
|                     | $A_4$  | b      | -a     | 0      |

If $n = 10$, we take $a = 1$ and $b = 3$. Then for every $i = 1, \ldots, 4$, $L_i$ is the incidence matrix of a group divisible design with five groups, the $j$th group being the pair of levels $\{j, j + 5\}$ $j = 0, \ldots, 4$, satisfying $\lambda_1 = 0$ and $\lambda_2 = 1$. This plan is, therefore, $E$-optimal for the inference on all the four factors by the result of Corollary 2.1 (b).

(b) (ii) The blocks $B_l$, $l = 1, \ldots, 4$ of $\mathcal{P}_0$ are as follows.

| Blocks $\rightarrow$ | $B_1$ | $B_2$ | $B_3$ | $B_4$ |
|----------------------|--------|--------|--------|--------|
| Factors $\downarrow$ | $A_1$  | a      | -a     | b      |
|                     | $A_2$  | b      | -a     | a      |
|                     | $A_3$  | c      | -c     | d      |
|                     | $A_4$  | d      | -c     | b      |

By taking $a = 1, b = 2, c = 3$ and $d = 4$ in the case $s = 9$, we get a balanced POTB.

(c) The set of levels for each factor is $S^+$. The blocks $B_l$, $l = 1, \ldots, 6$ of the initial plan are as follows.
We now list a few combinatorial structures in the literature which are actually balanced POTBs (for symmetrical or asymmetrical experiments).

(a) **Balanced Graco-Latin block design** defined and constructed in Seberry (1979) have two factors.

(b) **Graco-Latin block design of type 1** of Street (1981) are also two-factor balanced POTBs satisfying $N_{12} = J$.

(c) **Perfect Graeco-Latin balanced incomplete block designs (PERGOLAs)** defined and discussed extensively in Rees and Preece (1999) are two-factor balanced POTBs satisfying

$$N_{12}N'_{12} = N'_{12}N_{12} = fI_n + gJ_n,$$

where $f, g$ are integers.

(d) **Mutually orthogonal BIBDs** defined and constructed by Morgan and Uddin (1996) are multi-factor balanced POTBs.

**Remark 3.1:** The definition of neither balanced Graco-Latin block designs nor of mutually orthogonal BIBDs include condition (3.5). However, it is interesting to note that all these designs constructed so far do satisfy this condition. One would, therefore, suspect that this condition is implicit in the definition. We have, however, found a balanced POTB which does not satisfy this condition, as is shown below.

**Theorem 3.2.** Let $s$ be a positive integer $\geq 5$. Then

(a) there exists a symmetric POTB $P$ with three factors each having $s + 1$ levels on $b = 6s$ blocks of size two.

(b) In the case $s = 5$, we get a Balanced POTB. The restriction to any two of the factors reduces it to a PERGOLA, except that condition (3.5) is not satisfied.

**Proof:** (a) Let $S^+$ be the set of levels for each factor. Consider an initial plan $P_0$ with the set of factors $\{A_0, A_1, A_2\}$ and $B = \{B_{ij}, i = 1, 2, j = 0, 1, 2\}$, where $B_{ij}$’s are as shown in the table below. The required plan $P = P_0 \oplus S$.

| Blocks $\to$ | $B_{10}$ | $B_{11}$ | $B_{12}$ | $B_{20}$ | $B_{21}$ | $B_{22}$ |
|-------------|----------|----------|----------|----------|----------|----------|
| Factors $\downarrow$ | $A_0$ | $\infty$ | 0 | -1 | 1 | 0 | 1 | $\infty$ | 0 | 1 | 2 | 0 | 2 |
|              | $A_1$ | 0 | 1 | $\infty$ | 0 | -1 | 1 | 0 | 2 | $\infty$ | 0 | 1 | 2 |
|              | $A_2$ | -1 | 1 | 0 | 1 | $\infty$ | 0 | 1 | 2 | 0 | 2 | $\infty$ | 0 |

That $P$ satisfies (2.3) follows by straightforward verification.

(b) Let $s = 5$. One can verify that the incidence matrices satisfy the following.
\[
N_{ij} = \begin{bmatrix}
0 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 1 & 1 & 2 \\
2 & 2 & 2 & 1 & 1 & 2 \\
2 & 1 & 2 & 2 & 2 & 1 \\
2 & 1 & 1 & 2 & 2 & 2 \\
2 & 2 & 1 & 1 & 2 & 2 \\
\end{bmatrix}, \quad i, j = 0, 1, 2. \quad (3.6)
\]

Moreover, \( L_i(L_i)' = 8I_6 + 2J_6, \ i = 0, 1, 2. \quad (3.7)\)

We see that each \( L_i \) is the incidence matrix of a BIBD with parameters \((v = 6, b = 30, r = 10, k = 2, \lambda = 2)\). Thus, by Definition 2.3 \( P \) is a balanced POTB. However, \( N_{ij} \) does not satisfy (3.5), \( i \neq j, \ i, j = 0, 1, 2. \) □

Next we construct a series of balanced POTBs using finite fields. We first introduce the following notation.

**Notation 3.1.**

(i) \( \bigcup \) denotes an union counting multiplicity.

(ii) For a set \( A \) and an integer \( n \), \( nA \) denotes the multiset in which every member of \( A \) occurs \( n \) times.

(iii) For subsets \( A \) and \( B \) of a group \((G, +)\),

\[
A - B = \{a - b : a \in A, b \in B\}.
\]

**Notation 3.2.**

(i) \( s \) is an odd prime power. \( t = (s - 1)/2 \). \( F \) denotes the Galois field of order \( s \). Further, \( F^* = F \setminus \{0\} \) and \( F^+ = F \cup \{\infty\} \).

(ii) \( \alpha \) denotes a primitive element of \( F \).

(iii) \( C_0 \) denotes the subgroup of order \( t \) of the multiplicative group of \( F \) and \( C_1 \) the coset of \( C_0 \). Thus, \( C_0 \) is the set of all non-zero squares of \( F \), while \( C_1 \) is the set of all non-zero non-squares of \( F \).

(iv) \((i, j)\) denotes the number of ordered pairs of integers \((k,l)\) such that the following equation is satisfied in \( F \). [This notation is borrowed from the theory of cyclotomy]

\[
1 + \alpha^k = \alpha^l, \ k \equiv i, l \equiv j \pmod{2}.
\]

We present the following well-known result for ready reference. [See equations (11.6.30), (11.6.40) and (11.6.43) of Hall (1986)].

**Lemma 3.1.** The difference between the cosets of \( F^* \) can be expressed in terms of the cyclotomy numbers as follows.

\[
C_1 - C_0 = \bigcup_{k=0}^{t-1}(k,1)C_k.
\]

The following cyclotomy numbers are known.

Case 1: \( t \) odd. \((0,0) = (1,1) = (1,0) = (t-1)/2, (0,1) = (t+1)/2.\)

Case 2: \( t \) even. \((0,0) = t/2 - 1, (0,1) = (1,0) = (1,1) = t/2.\)

A series of two-factor balanced POTBs:
Theorem 3.3. Suppose $s$ is an odd prime or a prime power. Then there exists a balanced POTB $\mathcal{P}^*$ for a $(s+1)^2$ experiment on $b = 2s$ blocks of size $(s+1)/2$.

Proof: The set of levels of each factor is $F^+$. We shall present the initial plan $\mathcal{P}_0$ consisting of a pair of blocks. The required POTB is $\mathcal{P}^* = \mathcal{P}_0 \oplus F$.

Let $\delta \in C_1$. Consider three $2 \times (t + 1)$ arrays $R^{0}, R^{1}$ and $R^{2}$, the rows of which are indexed by $\{0, 1\}$ and the columns by $C_0 \cup \{0\}$. The entries of the arrays are as given below.

$$R^{0}(1,0) = R^{1}(0,0) = R^{2}(0,0) = 0 \text{ and } R^{0}(0,0) = R^{1}(1,0) = R^{2}(1,0) = \infty. \tag{3.8}$$

For $x = 0, 1$, $y \in C_0$, $R^{0}(x,y) = \delta^x y, R^{1}(x,y) = \delta^{-x} y$ and $R^{2}(x,y) = \delta^{x-1} y. \tag{3.9}$

For $i = 0, 1, 2$, let $B_i$ be the block, the runs of which are the columns of $R^i$. When $t$ is even, $B_0$ and $B_1$ constitute $\mathcal{P}_0$, while $B_0$ and $B_2$ constitute $\mathcal{P}_0$ when $t$ is odd.

Clearly block size is $t + 1 = (s + 1)/2$. To show that $\mathcal{P}^*$ satisfies the required property, we have to show that

(a) $\mathcal{P}^*$ is a POTB and (b) each factor forms a BIBD with the block factor.

Condition (b) follows from the construction in view of Lemma 3.1. So, we prove (a). Let us write $N$ for $N_{12}$. The rows and columns of $N$ are indexed by $F^+$. From (3.8) and (3.9), we see that

$$N(ii) = 0, \quad i \in F^+ \text{ and } N(\infty, i) = N(i, \infty) = 1, \quad i \in F. \tag{3.10}$$

So, we assume $i \neq j \in F$. Let $u = j - i$. Then, $N(ij)$ is the number of times $u$ appears in the multiset

$$\left\{ \begin{array}{ll}
(\delta - 1)C_0 \bigsqcup (\delta^{-1} - 1)C_0 & \text{if } t \text{ is even} \\
(\delta - 1)C_0 \bigsqcup (1 - \delta^{-1})C_0 & \text{if } t \text{ is odd}
\end{array} \right. \tag{3.11}$$

Since $-1 \in C_0$ if and only if $t$ is even, $\delta^{-1} - 1$ is in the same coset as $\delta - 1$ if and only if $t$ is odd. Therefore, the relations above together with (3.10) above imply that

$$N = J_{s+1} - I_{s+1}. \tag{3.12}$$

Now we take up $L_1L_2 = H$ (say). From (3.8) and (3.9), we see that

$$H(ii) = 0, \quad i \in F^+. \tag{3.13}$$

Further, for every $i \in F$, $H(\infty, i)$ is the replication number of $i$ in the block design generated by the initial block $\{0\} \cup C_1$. Similarly, $H(i, \infty)$ is the replication number of $i$ in the block design generated by the initial block $\{0\} \cup C_0$ if $t$ is odd and $\{0\} \cup C_1$ otherwise. Thus,

$$H(\infty, i) = H(i, \infty) = t + 1, \quad i \in F. \tag{3.14}$$

We, therefore, assume $i \neq j, i, j \in F$. Let $u = j - i$. Then, $H(ij)$ is the number of times $u$ appears in the multiset

$$\left\{ \begin{array}{ll}
((\{0\} \cup C_1) - C_0) \bigsqcup (C_1 - (\{0\} \cup C_0)) & \text{if } t \text{ is even} \\
((\{0\} \cup C_1) - C_0) \bigsqcup (C_0 - (\{0\} \cup C_1)) & \text{if } t \text{ is odd}
\end{array} \right. \tag{3.15}$$

These relations, together with (3.12), (3.13) and Lemma 3.1 imply that $H = (t + 1)(J_{s+1} - I_{s+1})$. Therefore, in view of (3.11), (2.3) follows and we are done. □
4 More on recursive construction

In this section we describe procedures for adding factors as well as blocks to an initial plan.

Notation 4.1. Consider a subset $V$ of $S^m$.

For every $i, 1 \leq i \leq m$, let $V_i$ denote the following multiset of $|V|$ members of $S$. $V_i = \{v_i : v = (v_1, \ldots, v_m)^t \in V\}$. Similarly, $V_{ij}$ will denote the following multiset of $|V|$ members of $S \times S$. $V_{ij} = \{(v_i, v_j) : v = (v_1, \ldots, v_m)^t \in V\}$.

Definition 4.1. Consider an initial plan $P_0$ for an $s^m$ experiment as described in Notation 2.1. Let $V$ be as in Notation 4.1. By the plan $P_0 + V$ generated from $P_0$ along $V$ we shall mean the plan (for the same experiment) having the set of blocks $B + V = \{B + v : v \in V, B \in B\}$, where $B + v$ is as in Definition 3.1. Usually, $V$ will contain the 0-vector, so that the blocks of $P_0$ will also be blocks of $P_0 + V$.

The next lemma provides a few sufficient conditions on $P_0$ and $V$ so that a given pair of factors are orthogonal through the block factor in $P_0 + V$. The proof is by direct verification.

Remark 4.1: In an initial plan, say $P_0$, one or more levels of one or more factors may be absent. $P_0$ may still be a POTB if (2.3) holds (with one or more row/column of $N_{ij}$'s being null vectors) for every unordered pair of $(i, j)$. In such cases one has to choose $V$ such that all levels of all factors do appear in $P_0 + V$.

Lemma 4.1. Consider an initial plan $P_0$ for an $s^2$ experiment. For $V \subset F \times F$, consider $P_0 + V$. The following conditions on $P_0$ and $V$ are sufficient for $P_0 + V$ to be a POTB.

(a) In $P_0$ all the levels of the first factor appear and $V = \{(0, i), i \in S\}$.

(b) $P_0$ is arbitrary and $V = \{(i, j), i, j \in S\}$.

(c) $P_0$ has a pair of blocks $B_0, B_1$ each of size 2, as described below. Let $i \neq j, k \neq l \in S$. Let $x_0 = (i, i'), y_0 = (j, j'), x_1 = (k, l)'$ and $y_1 = (l, k)'$. $B_i$ consists of runs $x_i$ and $y_i$, $i = 0, 1$. $V = (u, u)$, $u \in S$.

(d) $P_0$ is a POTB in which with one or more levels of one or both factors may be absent. $V$ is such that every member of $S$ appears at least once in each $V_i, i = 1, 2$.

Our next procedure enlarges the set of factors of a given plan, while keeping the number of blocks fixed.

Definition 4.2. (a) Consider a plan $P$ as in Notation 2.1. Suppose there is another plan $P'$ having $b$ blocks of size $k$ each. We shall combine these two plans to get another one with a larger set of factors.

Let $x_{ij}$ (respectively $x_{ij}'$) denote the $j$th run in the $i$th block of $P$ (respectively $P'$), $1 \leq j \leq k, 1 \leq i \leq b$. Let $y_{ij} = \{x_{ij}, x_{ij}'\}$, $1 \leq j \leq k, 1 \leq i \leq b$. Then, the plan on $b$ blocks of size $k$ with $y_{ij}$ as the $j$th run in the $i$th block, $1 \leq j \leq k, 1 \leq i \leq b$ is said to obtained by joining the factors of $P$ and $P'$ together. The new plan will be denoted by \(P \cdot P'\).

(b) In case $P'$ is a copy of $P$ then \(P \cdot P'\) is denoted by $P^2$. For $t \geq 3$, the plan $P^t$ is defined in the same way. In this case we name the factors of $P$ and its power $P^t$ as in the notation below.
Notation 4.2. Consider a plan \( \mathcal{P} \) having a set of \( m \) factors \( \mathcal{F}_0 = \{ A, \cdots, M \} \). The set of factors of \( \mathcal{P}^t \) will be named as
\[
\mathcal{F} = \bigcup_{i=1}^{t} \mathcal{F}_i, \text{ where } \mathcal{F}_i = \{ A_i, \cdots, M_i \}.
\]

Combining Definitions 4.1 and 4.2 we get a recursive construction described below.

Definition 4.3. Consider an initial plan \( \mathcal{P}_0 \) for an \( s^m \) experiment laid on \( b \) blocks of size \( k \) each. Consider a \( p \times q \) array \( H = ((h_{ij}))_{1 \leq i \leq p, 1 \leq j \leq q} \). We now obtain a plan for an \( s^{m+q} \) experiment on \( bp \) blocks of size \( k \) using the array \( H \) as follows. We first obtain \( \mathcal{P}_0^q \) following Definition 4.2.

\[
\text{Let } v_i = [ h_{i1}, 1_{i1}^t, h_{i2}, 1_{i2}^t, \cdots, h_{iq}, 1_{iq}^t ], 1 \leq i \leq p \text{ and } V_H = \{ v_i, 1 \leq i \leq p \}.
\]

Our required plan \( \mathcal{P} \) is \( \mathcal{P}_0^q + V_H \) and it will be denoted by \( H \triangleright \mathcal{P} \). Symbolically,
\[
\mathcal{P} = H \triangleright \mathcal{P}_0 = \mathcal{P}_0^q + V_H. \tag{4.14}
\]

Our task is to find a suitable array \( H \) so that the plan \( H \triangleright \mathcal{P}_0 \) satisfies certain desirable properties. A natural choice for \( H \) is an orthogonal array of strength 2. We shall use a modification of an orthogonal array so as to accommodate a few more factors.

Definition 4.4 (Rao(1946)). Let \( m, n, t \geq 2 \) be integers and \( s \) is an integer \( \geq 2 \). Then an orthogonal array of strength \( t \) is an \( m \times N \) array, with the entries from a set \( S \) of \( s \) symbols satisfying the following. All the \( s^t \) \( t \)-tuples with symbols from \( S \) appear equally often as columns in every \( t \times N \) subarray. Such an array is denoted by \( OA(N, m, s, t) \).

Notation 4.3. (a) The set of symbols of an \( OA(N, m, s, 2) \) is assumed to be the set of integers modulo \( s \).

(b) The array obtained by adding a column of all zeros (in the 0th position, say) to an \( OA(N, m - 1, s, 2) \) will be denoted by \( Q(N, m, s) \).

Exploring the properties of an orthogonal array of strength 2, we get the following result from the recursive construction described in Definition 4.3.

Theorem 4.1. Consider a plan \( \mathcal{P}_0 \) for an \( s^t \) experiment on \( b \) blocks of size \( k \) each. If an \( OA(N, m - 1, s, 2) \) exists, then \( \exists \) a plan \( \mathcal{P} \) with a set of \( s^{mt} \) factors on \( bn \) blocks of size \( k \) each with the following properties. Here the factors of \( \mathcal{P}_0^q \) as well as \( \mathcal{P} \) are named according to Notation 4.2.

(a) For \( P \neq Q, P, Q \in \mathcal{F}_0, P_1 \perp_{M} Q_i \) for every \( i, 0 \leq i \leq m - 1 \), if and only if \( P \perp_{M} Q \) in \( \mathcal{P}_0 \).

(b) \( P_i \perp_{b} Q_j, P, Q \in \mathcal{F}_0, i \neq j, 0 \leq i, j \leq m - 1 \).

Proof: By assumption \( Q = Q(N, m, s) \) exists. The required plan \( \mathcal{P} \) is \( Q \triangleright \mathcal{P}_0 \). Property (a) follows from the construction while (b) follows from (b) of Lemma 4.1.

Remark 4.2: Table 1 of Rees and Preece (1999) presents a number of examples of PER-GOLAs [see the statement proceeding (3.5)]. An Application of Theorem 4.1 on each of them would yield a balanced POTB for a larger set of factors.

Finally, we describe a procedure of modifying the sets of levels of factors. Specifically, given a pair of plans with the same number of factors and the same block size, we obtain a plan by merging the sets of levels of the corresponding factors of the given plans.
Definition 4.5. Consider a pair of plans $\mathcal{P}_1$ and $\mathcal{P}_2$ each having $t$ factors and blocks of size $k$. Let $S_i$ denote the set of levels of each factor of $\mathcal{P}_i$, $s_i = |S_i|$, $i = 1, 2$. We assume that $S_1 \neq S_2$. Let $U = S_1 \cup S_2$ and $u = |U|$. The plan consisting of all the blocks of $\mathcal{P}_1$ and $\mathcal{P}_2$ taken together will be viewed as a plan, say $\mathcal{P}_1 \cup \mathcal{P}_2$, for an $u^t$ experiment in the following sense.

(a) Each factor of $\mathcal{P}_1 \cup \mathcal{P}_2$ will have $U$ as the set of levels.

(b) Fix $p \in U$. Let $\mathcal{R}_p^i$ denote the set of runs of $\mathcal{P}_j$, in which the level $p$ of the $i$th factor appears, $j = 1, 2, 1 \leq i \leq t$. [Needless to mention that $\mathcal{R}_p^i = \emptyset$ if $p$ is not in $S_j$.] Then, the level $p$ of the $i$th factor of $\mathcal{P}_1 \cup \mathcal{P}_2$ appears in exactly the runs in $\mathcal{R}_p^1 \cup \mathcal{R}_p^2$, $1 \leq i \leq t$.

Remark 4.3: From Definition 4.5 we see that for $p \in U$, the replication number of level $p$ of the $i$th factor of $\mathcal{P}_1 \cup \mathcal{P}_2$ is $r_1^1(p) + r_2^2(p)$, where $r_j^i(p)$ is the replication number of level $p$ of the $i$th factor of $\mathcal{P}_j$.

For instance, in Theorem 5.1 below, Definition 4.5 is used to construct $\mathcal{P}_h$ by merging the corresponding factors of $\mathcal{P}_{1h}$ and $\mathcal{P}_{2h}$. There, $S_1 = \{0, 1\}$, while $S_2 = \{0, 2\}$. Thus, while both $\mathcal{P}_{1h}$ and $\mathcal{P}_{2h}$ are equireplicate, the replication number of level 0 of each factor of $\mathcal{P}_h$ is double of the levels 1 and 2 of the same factor.

The following result is an immediate consequence of Definition 4.5.

Lemma 4.2. Consider a pair of connected plans $\mathcal{P}_1$ and $\mathcal{P}_2$, as in Definition 4.3 (recall Definition 2.7). Then, we can say the following about the plan $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$.

(a) If both $\mathcal{P}_1$ and $\mathcal{P}_2$ are POTB, then so is $\mathcal{P}$.

(b) $\mathcal{P}$ is connected, if and only if $S_1 \cap S_2 \neq \emptyset$.

5 Construction of POTBs for three-level factors

In this section we make use of the tools described in Section 4 to generate plans for three-level factors. The factors of the initial and final plans are named in accordance with Notation 4.2.

Theorem 5.1. If $h$ is the order of a Hadamard matrix, then there exists a connected and saturated POTB $\mathcal{P}_h$ for a $3^h$ experiment in $2h$ blocks of size 4 each.

**Proof**: Let $O_4 = OA(4, 3, 2, 2)$ with $S = \{0, 1\}$. Let $\mathcal{P}_0$ be the plan consisting of a single block consisting of the four columns of $O_4$ as runs. Thus, $\mathcal{P}_0$ is an OMEP for a $2^4$ experiment.

By hypothesis $Q = Q(h, h, 2)$ exists. Let $\mathcal{P}_{1h} = Q \circ \mathcal{P}_0$ and $\mathcal{P}_{2h}$ be obtained from $\mathcal{P}_{1h}$ by replacing level 1 of every factor by the level 2. Next we construct our required plan $\mathcal{P}_h = \mathcal{P}_{1h} \cup \mathcal{P}_{2h}$ by using Definition 4.5. By construction $\mathcal{P}_h$ has $2h$ blocks of size 4 each.

We now show that $\mathcal{P}_h$ is a POTB. We note that by Theorem 4.1, each of $\mathcal{P}_{1h}$ and $\mathcal{P}_{2h}$ is a POTB for a $2^3h$ experiment on $h$ blocks of size 4 each. The sets of levels of each factor of them are $\{0, 1\}$ and $\{0, 2\}$ respectively. It follows from Lemma 4.2 that $\mathcal{P}_h$ is a connected POTB for an experiment with $3h$ factors, the set of levels of each factor being $\{0, 1, 2\}$. Since the available degrees of freedom for the treatment factors is $2h(4 - 1)$ which is the same as the required degrees of freedom, the plan is saturated. □

We now take $h = 2$ and present the plan $\mathcal{P}_2$ for a $3^6$ experiment on four blocks of size four each.
Table 5.1 : The plan \( p_2 \)

| Blocks | \( B_{01} \) | \( B_{02} \) | \( B_{11} \) | \( B_{12} \) |
|--------|-------------|-------------|-------------|-------------|
| Factors ↓ | \( A_1 \) | 00 11 | 00 11 | 00 22 | 00 22 |
|         | \( B_1 \) | 01 01 | 01 01 | 02 02 | 02 02 |
|         | \( C_1 \) | 01 10 | 01 10 | 02 20 | 02 20 |
|         | \( A_2 \) | 00 11 | 11 00 | 00 22 | 22 00 |
|         | \( B_2 \) | 01 01 | 10 10 | 02 02 | 20 20 |
|         | \( C_2 \) | 01 10 | 10 01 | 02 20 | 20 02 |

For the next construction we need some more notations.

**Notation 5.1.** \( O_4 \) is as in the proof of Theorem 5.1. \( T_4 \) will denote the array obtained from \( O_4 \) by replacing each 1 by 2 and \( \tilde{T}_4 \) the array obtained from \( T_4 \) by interchanging 0 and 2.

**Theorem 5.2.** A POTB for a \( 3^3 \) experiment on two blocks of size four exists.

**Proof:** Let \( B_{10} = O_4, B_{20} = T_4 \) and \( B_{02} = \tilde{T}_4 \). The set of columns of each of them constitutes an OMEP for a \( 2^3 \) experiment, the set of levels of factors being \( \{ 0, 1 \} \) for \( B_{10} \), while \( \{ 0, 2 \} \) for the other two.

Let \( \rho_1 \) (respectively \( \rho_2 \)) denote the plan consisting of the pair of blocks \( B_{10}, B_{20} \) (respectively \( B_{10}, B_{02} \)). By Lemma 4.2, each of \( \rho_1 \) and \( \rho_2 \) is a POTB for a \( 3^3 \) experiment. □

Using the pair of plans constructed above, we generate a bigger plan.

**Theorem 5.3.** (a) If there exists an \( OA(N, m, 3, 2) \), then there exists a connected POTB \( p_m \) for a \( 3^{3(2m+1)} \) experiment in \( 2^N \) blocks of size 4 each.

In particular \( p_m \) is saturated whenever \( N = 3^n \) and \( m = (3^{n-1} - 1)/2 \), for an integer \( n \geq 2 \).

(b) There exists a connected POTB for a \( 3^9 \) experiment in 6 blocks of size 4 each.

**Proof of (a):** Let the factors of \( \rho_1 \) and \( \rho_2 \) be named as \( A, B, C \) and \( \tilde{A}, \tilde{B}, \tilde{C} \) respectively. Let \( O = OA(N, m, 3, 2) \) and \( Q = Q(N, m, 3) \). We now use Definition 4.5 to generate bigger plans \( P_{1m} \) and \( P_{2m} \) as follows.

\[ P_{1m} = Q\rho_1 \text{ and } P_{2m} = O\rho_2. \]

Clearly, \( P_{1m} \) and \( P_{2m} \) are plans for \( 3^{3(m+1)} \) and \( 3^{3m} \) experiments respectively, each on \( 2N \) blocks of size 4. Following Notation 4.2, we name the factors of these plans as follows.

The factors of \( P_{1m} \) are \( A_0, B_0, C_0, A_1, B_1, C_1, \ldots A_m, B_m, C_m \)

and the factors of \( P_{2m} \) are \( \tilde{A}_1, \tilde{B}_1, \tilde{C}_1, \ldots \tilde{A}_m, \tilde{B}_m, \tilde{C}_m \).

Now we combine the factors of \( P_{1m} \) and \( P_{2m} \) following Definition 4.2(a) and thus obtain our required plan \( p_m \). Symbolically,

\[ p_m = [ P_{1m} \ P_{2m} ]. \]

By construction, \( p_m \) is a plan for \( 2m + 1 \) three-level factors on \( 2N \) blocks of size 4 each. We shall now show that it is a POTB.
Theorems 4.1 and 5.2 imply that each one of $P_{1m}$ and $P_{2m}$ is a POTB. Therefore, if we show the following relation, then we are done.

$$P_i \perp_{bl} \tilde{Q}_j, \ P, Q \in \{A, B, C\}, \ i \in I \cup \{0\}, \ j \in I, \ \text{where} \ I = \{1, \cdots, m\}. \quad (5.15)$$

To show this relation, we fix $P_i$ and $\tilde{Q}_j$ as above.

**Case 1.** $i, j \in I$ : Since $\rho_1$ and $\rho_2$ are POTBs, $(5.15)$ follows from Lemma 4.1 (d), whenever $Q \neq P$. Again, (c) of the same Lemma proves $(5.15)$ for the case $Q = P$.

**Case 2.** $i = 0, \ j \in I$ : We take $P_0$ as the first and $\tilde{Q}_j$ as the second factor. Then applying Lemma 4.1 (a) we get $(5.15)$.

Hence the proof of the first part is complete.

To prove the second part, we see that $P_m$ is saturated when $N = 2m + 1$. Now Rao (1946) has shown that an $OA(s^n, (s^n - 1)/(s - 1), s, 2)$ exists whenever $n \geq 2$. (see Theorem 3.20 of Hedayat, Sloane and Stufken (1999) for instance). Putting $s = 3$, we get the result.

**Proof of (b) :** Let $O = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$. Now the construction for the plan, say $P_1$, is just like that in Case (a). The verification is also exactly the same in Case (a) with $I = \{1\}$. □

We now present $P_1$.

**Table 5.2 : The plan $P_1$**

| Blocks | $B_{10}$ | $B_{20}$ | $B_{11}$ | $B_{21}$ | $B_{12}$ | $B_{22}$ |
|--------|----------|----------|----------|----------|----------|----------|
| Factors ↓ |
| $A_0$  | 00 11    | 00 22    | 00 11    | 00 22    | 00 11    | 00 22    |
| $B_0$  | 01 01    | 02 02    | 01 01    | 02 02    | 01 01    | 02 02    |
| $C_0$  | 01 10    | 02 20    | 01 10    | 02 20    | 01 10    | 02 20    |
| $A_1$  | 00 11    | 00 22    | 11 22    | 11 00    | 22 00    | 22 11    |
| $B_1$  | 01 01    | 02 02    | 12 12    | 10 10    | 20 20    | 21 21    |
| $C_1$  | 01 10    | 02 20    | 12 21    | 10 01    | 20 02    | 21 12    |
| $\tilde{A}_1$ | 00 11    | 22 00    | 11 22    | 00 11    | 22 00    | 11 22    |
| $\tilde{B}_1$ | 01 01    | 20 20    | 12 21    | 01 01    | 20 20    | 12 12    |
| $\tilde{C}_1$ | 01 10    | 20 02    | 12 21    | 01 10    | 20 02    | 12 21    |

6 Inter-class orthogonal plans

Inter-class orthogonal plans are defined in Bagchi (2019) in the context of plans without any blocking factor. Here we extend the definition to the present context - the orthogonality being through the block factor.

**Definition 6.1.** Let us consider a plan $\rho$. Suppose the set of all factors of $\rho$ can be divided into several classes in such a way that if two factors belong to different classes, then they are orthogonal through the block factor. Such a plan $\rho$ is called a “Plan Inter-class Orthogonal through the Blocks (PIOTB)” and the classes will be referred to as “orthogonal classes”.

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We shall now proceed towards the construction of a series of PIOTBs. Using the relation between orthogonal arrays of strength two and Hadamard matrices, [see Theorem 7.5 in Hedayat, Sloane and Stufken (1999), for instance], we see that a $Q(n, n, 2)$ exists whenever $n$ is the order of a Hadamard matrix.

**Theorem 6.1.** Suppose Hadamard matrices of orders $m$ and $n$ exist. Then, there exists a saturated PIOTB $P_{(m,n)}$ for a $2^{mn}$ experiment on $n$ blocks of size $m + 1$ each. There are $n$ orthogonal classes of size $m$ each.

**Proof:** By hypothesis $Q_m = Q(m, m, 2)$ exists. Let $R$ be the $m \times m + 1$ array obtained by juxtaposing a column of all-ones to $Q_m$. Let $P_0$ be the plan for a $2^m$ experiment on a single block consisting of $m + 1$ runs, which are the columns of $R$. Let us name the factors of $P_0$ as $A, B, \cdots M$. Note that the column added to $Q_m$ saves $A$ from being confounded with the block.

By hypothesis, $Q_n = Q(n, n, 2)$ exists. Let $P_{(m,n)} = Q_n \circ P_0$. Clearly, $P_{(m,n)}$ is an main effect plan for a $2^{mn}$ experiment with parameters as in the statement. By construction, no factor is confounded with the block factor. Using Theorem 4.1 and the property of $P_0$, we see that $P_n$ is interclass orthogonal with orthogonal classes $\{A_i, B_i, \cdots M_i\}, 1 \leq i \leq n$ (recall Notation 4.2). Hence the result. □

We now present the plans $P_{(4,4)}$.

**Table 6.1 : The plan $P_{4,4}$**

| Blocks | $B_1$ | $B_2$ | $B_3$ |
|--------|-------|-------|-------|
| Factors ↓ |       |       |       |
| $A_1$   | 00 00 1 | 00 00 1 | 00 00 1 |
| $B_1$   | 00 11 1 | 00 11 1 | 00 11 1 |
| $C_1$   | 01 01 1 | 01 01 1 | 01 01 1 |
| $D_1$   | 01 10 1 | 01 10 1 | 01 10 1 |
| $A_2$   | 00 00 1 | 11 11 0 | 11 11 0 |
| $B_2$   | 00 11 1 | 11 00 0 | 11 00 0 |
| $C_2$   | 01 01 1 | 10 10 0 | 10 10 0 |
| $D_2$   | 01 10 1 | 10 01 0 | 10 01 0 |
| $A_3$   | 00 00 1 | 11 11 0 | 00 00 1 |
| $B_3$   | 00 11 1 | 11 00 0 | 00 11 1 |
| $C_3$   | 01 01 1 | 10 10 0 | 01 01 1 |
| $D_3$   | 01 10 1 | 10 01 0 | 01 10 1 |
| $A_4$   | 00 00 1 | 11 11 0 | 11 11 0 |
| $B_4$   | 00 11 1 | 11 00 0 | 11 00 0 |
| $C_4$   | 01 01 1 | 10 10 0 | 01 01 1 |
| $D_4$   | 01 10 1 | 10 01 0 | 10 01 0 |

There are four orthogonal classes, which are $\{A_i, B_i, C_i, D_i\}, i = 1, 2, 3, 4$.

Finally, we present a PIOTB for three-level factors.

**Theorem 6.2.** A saturated PIOTB exists for a $3^6$ experiment on four blocks of size four each.

**Proof:** Consider the following plan $P$. It is easy to see that it is a PIOTB with non-orthogonal classes $\{P_1, P_2\}, P = A, B, C$.
Table 6.2 : Plan $\mathcal{P}$

| Blocks | $B_1$ | $B_2$ | $B_3$ | $B_4$ |
|--------|-------|-------|-------|-------|
| Factors ↓ |       |       |       |       |
| $A_1$  | 00 12 | 00 21 | 00 12 | 00 21 |
| $B_1$  | 01 02 | 02 01 | 10 20 | 20 10 |
| $C_1$  | 01 20 | 02 10 | 02 10 | 01 20 |
| $A_2$  | 01 01 | 02 02 | 01 01 | 02 02 |
| $B_2$  | 01 10 | 02 20 | 10 01 | 20 02 |
| $C_2$  | 00 11 | 00 22 | 11 00 | 22 00 |

**Remark 6.1:** A POTB for a $4^4$ experiment on 4 blocks of size 4 is well-known [can be obtained by treating a row of OA(16,5,4,2) as the block factor]. By collapsing two of the levels of each factor to one level one gets a POTB for a $3^4$ experiment on the same set up. Allowing non-orthogonality we have been able to accommodate two more three-level factors, making it saturated.

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